GERBES AND QUANTUM FIELD THEORY

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1. DEFINITIONS AND AN EXAMPLE

A gerbe can be viewed as a next step in a ladder of geometric and topological
objects on a manifold which starts from ordinary complex valued functions and in
the second step of sections of complex line bundles.

It is useful to recall the construction of complex line bundles and their connec-
tions. Let $M$ be a smooth manifold and $\{U_\alpha\}$ an open cover of $M$ which trivializes
a line bundle $L$ over $M$. Topologically, up to equivalence, the line bundle is com-
pletely determined by its Chern class which is a cohomology class $[c] \in H^2(M, \mathbb{Z})$.
On each open set $U_\alpha$ we may write $2\pi ic = dA_\alpha$, where $A_\alpha$ is a 1-form. On the
overlaps $U_{\alpha\beta} = U_\alpha \cap U_\beta$ we can write

\begin{equation}
A_\alpha - A_\beta = f_{\alpha\beta}^{-1} df_{\alpha\beta},
\end{equation}

at least when $U_{\alpha\beta}$ is contractible, where $f_{\alpha\beta}$ is a circle valued complex function on
the overlap. The data $\{c, A_\alpha, f_{\alpha\beta}\}$ defines what is known as a (representative of
a) Deligne cohomology class on the open cover $\{U_\alpha\}$. The 1-forms $A_\alpha$ are the local
potentials of the curvature form $2\pi ic$ and the $f_{\alpha\beta}$’s are the transition functions of the
line bundle $L$. Each of these three different data defines separately the equivalence
class of the line bundle but together they define the line bundle with a connection.

The essential thing is here that there is a bijection between the second integral
cohomology of $M$ and the set of equivalence classes of complex line bundles over
$M$. It is natural to ask whether there is a geometric realization of integral third
(or higher) cohomology. In fact, gerbes provide such a realization. Here we shall
restrict to a smooth differential geometric approach which is by no means most
general possible, but it is sufficient for most applications to quantum field theory.
However, there are examples of gerbes over orbifolds which do not need to come
from finite group action on a manifold, which are not covered by the following
definition.

For the examples in this article it is sufficient to adapt the following definition.
A gerbe over a manifold $M$ (without geometry) is simply a principal bundle $\pi : P \to M$ with fiber equal to $PU(H)$, the projective unitary group of a Hilbert space $H$. The Hilbert space may be either finite or infinite dimensional.

The quantum field theory applications discussed in this article are related to the chiral anomaly for fermions in external fields. The link comes from the fact that the chiral symmetry breaking leads in the generic case to projective representations of the symmetry groups. For this reason, when modding out by the gauge or diffeomorphism symmetries, one is led to study bundles of projective Hilbert spaces. The anomaly is reflected as a nontrivial characteristic class of the projective bundle, known in mathematics literature as the Dixmier-Douady class.

In a suitable open cover the bundle $P$ has a family of local trivializations with transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to PU(H)$, with the usual cocycle property

$$g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$$

on triple overlaps. Assuming that the overlaps are contractible, we can choose lifts $\hat{g}_{\alpha\beta} : \hat{U}_{\alpha\beta} \to U(H)$, to the unitary group of the Hilbert space. However,

$$\hat{g}_{\alpha\beta}\hat{g}_{\beta\gamma}\hat{g}_{\gamma\alpha} = f_{\alpha\beta\gamma},$$

where the $f$’s are circle valued functions on triple overlaps. They satisfy automatically the cocycle property

$$f_{\alpha\beta\gamma}f_{\alpha\beta\delta}^{-1}f_{\alpha\gamma\delta}^{-1} = 1$$

on quadruple overlaps. There is an important difference between the finite and infinite dimensional cases. In the finite dimensional case the circle bundle $U(H) \to U(H)/S^1 = PU(H)$ reduces to a bundle with fiber $\mathbb{Z}/N\mathbb{Z} = \mathbb{Z}_N$, where $N = \dim H$. This follows from $U(N)/S^1 = SU(N)/\mathbb{Z}_N$ and the fact that $SU(N)$ is a subgroup of $U(N)$. For this reason one can choose the lifts $\hat{g}_{\alpha\beta}$ such that the functions $f_{\alpha\beta\gamma}$ take values in the finite subgroup $\mathbb{Z}_N \subset S^1$.

The functions $f_{\alpha\beta\gamma}$ define an element $a = \{a_{\alpha\beta\gamma\delta}\}$ in the Čech cohomology $H^3(U, \mathbb{Z})$ by a choice of logarithms,

$$2\pi ia_{\alpha\beta\gamma\delta} = \log f_{\alpha\beta\gamma} - \log f_{\alpha\beta\delta} + \log f_{\alpha\gamma\delta} - \log f_{\beta\gamma\delta}.$$

In the finite dimensional case the Čech cocycle is necessarily torsion, $Na = 0$, but not so if $H$ is infinite dimensional. In the finite dimensional case (by passing to a good cover and using the Čech - de Rham equivalence over real or complex numbers) the class is third de Rham cohomology constructed from the transition functions is necessarily zero. Thus in general one has to work with Čech cohomology to preserve torsion information. One can prove:

**Theorem.** The construction above is a one-to-one map between the set of equivalence classes of $PU(H)$ bundles over $M$ and elements of $H^3(M, \mathbb{Z})$.

The characteristic class in $H^3(M, \mathbb{Z})$ of a $PU(H)$ bundle is called the Dixmier-Douady class.
First example. Let $M$ be an oriented Riemannian manifold and $FM$ its bundle of oriented orthonormal frames. The structure group of $FM$ is the rotation group $SO(n)$ with $n = \dim M$. The spin bundle (when it exists) is a double covering $Spin(M)$ of $FM$, with structure group $Spin(n)$, a double cover of $SO(n)$. Even when the spin bundle does not exist there is always the bundle $\text{Cl}(M)$ of Clifford algebras over $M$. The fiber at $x \in M$ is the Clifford algebra defined by the metric $g_x$, i.e., it is the complex $2^n$ dimensional algebra generated by the tangent vectors $v \in T_x(m)$ with the defining relations

$$\gamma(u)\gamma(v) + \gamma(v)\gamma(u) = 2g_x(u,v).$$

The Clifford algebra has a faithful representation in $N = 2^{\lfloor n/2 \rfloor}$ dimensions ($\lfloor x \rfloor$ is the integral part of $x$) such that

$$\gamma(a \cdot u) = S(a)\gamma(u)S(a)^{-1}$$

where $S$ is an unitary representation of $Spin(n)$ in $\mathbb{C}^N$. Since $Spin(n)$ is a double cover of $SO(n)$, the representation $S$ may be viewed as a projective representation of $SO(n)$. Thus again, if the overlaps $U_{\alpha\beta}$ are contractible, we may choose a lift of the frame bundle transition functions $g_{\alpha\beta}$ to unitaries $\hat{g}_{\alpha\beta}$ in $H = \mathbb{C}^N$. In this case the functions $f_{\alpha\beta\gamma}$ reduce to $\mathbb{Z}_2$ valued functions and the obstruction to the lifting problem, which is the same as the obstruction to the existence of spin structure, is an element of $H^2(M,\mathbb{Z}_2)$, known as the second Stiefel-Whitney class $w_2$. The image of $w_2$ with respect to the Bockstein map (in this case, given by the formula (5)) gives a 2-torsion element in $H^3(M,\mathbb{Z})$, the Dixmier-Douady class.

Another way to think of a gerbe is the following (we shall see that this arises in a natural way in quantum field theory). There is a canonical complex line bundle $L$ over $PU(H)$, the associated line bundle to the circle bundle $S^1 \to U(H) \to PU(H)$. Pulling back $L$ by the local transition functions $g_{\alpha\beta} \to PU(H)$ we obtain a family of line bundles $L_{\alpha\beta}$ over the open sets $U_{\alpha\beta}$. By the cocycle property (2) we have natural isomorphisms

$$L_{\alpha\beta} \otimes L_{\beta\gamma} = L_{\alpha\gamma}.$$

We can take this as a definition of a gerbe over $M$ : A collection of line bundles over intersections of open sets in an open cover of $M$, satisfying the cocycle condition (6). By (6) we have a trivialization

$$L_{\alpha\beta} \otimes L_{\beta\gamma} \otimes L_{\gamma\alpha} = f_{\alpha\beta\gamma} \cdot 1,$$

where the $f$’s are circle valued functions on the triple overlaps. By the Theorem we conclude that indeed the data in (6) defines (an equivalence class of) a principal $PU(H)$ bundle.

If $L_{\alpha\beta}$ and $L'_{\alpha\beta}$ are two systems of local line bundles over the same cover, then the gerbes are equivalent if there is a system of line bundles $L_{\alpha}$ over open sets $U_{\alpha}$ such that

$$L'_{\alpha\beta} = L_{\alpha\beta} \otimes L_{\alpha}^* \otimes L_{\beta}$$
on each $U_{\alpha\beta}$. 
A gerbe may come equipped with geometry, encoded in a Deligne cohomology class with respect to a given open covering of $M$. The Deligne class is given by functions $f_{\alpha\beta\gamma}$, 1-forms $A_{\alpha\beta}$, 2-forms $F_\alpha$, and a global 3-form (the Dixmier-Douady class of the gerbe) $\Omega$, subject to the conditions

$$
\begin{align*}
    dF_\alpha &= 2\pi i \Omega \\
    F_\alpha - F_\beta &= dA_{\alpha\beta} \\
    A_{\alpha\beta} - A_{\alpha\gamma} + A_{\beta\gamma} &= f_{\alpha\beta\gamma}^{-1} df_{\alpha\beta\gamma}.
\end{align*}
$$

(9)

2. GERBES FROM CANONICAL QUANTIZATION

Let $D_x$ be a family of self-adjoint Fredholm operators in a complex Hilbert space $H$ parametrized by $x \in M$. This situation arises in quantum field theory for example when $M$ is some space of external fields, coupled to Dirac operator $D$ on a compact manifold. The space $M$ might consists of gauge potentials (modulo gauge transformations) or $M$ might be the moduli space of Riemann metrics. In these examples the essential spectrum of $D_x$ is both positive and negative and the family $D_x$ defines an element of $K^1(M)$. In fact, one of the definitions of $K^1(M)$ is that its elements are homotopy classes of maps from $M$ to the space $\mathcal{F}_*$ of self-adjoint Fredholm operators with both positive and negative essential spectrum. In physics applications one deals most often with unbounded hamiltonians, and the operator norm topology must be replaced by something else; popular choises are the Riesz topology defined by the map $F \mapsto F/(|F|+1)$ to bounded operators or the gap topology defined by graph metric.

The space $\mathcal{F}_*$ is homotopy equivalent to the group $G = U_1(H)$ of unitary operators $g$ in $H$ such that $g - 1$ is a trace-class operator. This space is a classifying space for principal $U_{res}$ bundles where $U_{res}$ is the group of unitary operators $g$ in a polarized complex Hilbert space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ such that the off-diagonal blocks of $g$ are Hilbert-Schmidt operators. This is related to Bott periodicity. There is a natural principal bundle $P$ over $G = U_1(H)$ with fiber equal to the group $\Omega G$ of based loops in $G$. The total space $P$ consists of smooth paths $f(t)$ in $G$ starting from the neutral element such that $f^{-1} df$ is smooth and periodic. The projection $P \to G$ is the evaluation at the end point $f(1)$. The fiber is clearly $\Omega G$. By Bott periodicity, the homotopy groups of $\Omega G$ are shifted from those of $G$ by one dimension, i.e.,

$$
\pi_n \Omega G = \pi_{n+1} G.
$$

The latter are zero in even dimensions and equal to $\mathbb{Z}$ in odd dimensions. On the other hand, it is known that the even homotopy groups of $U_{res}(\mathcal{H})$ are equal to $\mathbb{Z}$ and the odd ones vanish. In fact, with a little more effort one can show that the embedding of $\Omega G$ to $U_{res}(\mathcal{H})$ is a homotopy equivalence, when $\mathcal{H} = L^2(S^1, H)$, the polarization being the splitting to nonnegative and negative Fourier modes and the action of $\Omega G$ is the pointwise multiplication on $H$ valued functions on the circle $S^1$.

Since $P$ is contractible, it is indeed the classifying bundle for $U_{res}$ bundles. Thus we conclude that $K^1(M) = \text{the set of homotopy classes of maps } M \to G = \text{the set of equivalence classes of } U_{res}$ bundles over $M$. The relevance of this fact in quantum
field theory follows from the properties of representations of the algebra of canonical anticommutation relations (CAR). For any complex Hilbert space $H$ this algebra is the algebra generated by elements $a(v)$ and $a^*(v)$, with $v \in H$, subject to the relations

$$a^*(u)a(v) + a(v)a^*(u) = 2 \langle v, u \rangle,$$

where the Hilbert space inner product on the right is antilinear in the first argument, and all other anticommutators vanish. In addition, $a^*(u)$ is linear and $a(v)$ antilinear in its argument.

An irreducible Dirac representation of the CAR algebra is given by a polarization $H = H_+ \oplus H_-$. The representation is characterized by the existence of a vacuum vector $\psi$ in the fermionic Fock space $\mathcal{F}$ such that

$$a^*(u)\psi = 0 = a(v)\psi \text{ for } u \in H_-, \ v \in H_+.$$  

A theorem of D. Shale and W.F. Stinespring says that two Dirac representations defined by a pair of polarizations $H_+, H'_+$ are equivalent if and only if there is $g \in U_{res}(H_+ \oplus H_-)$ such that $H'_+ = g \cdot H_+$. In addition, in order that a unitary transformation $g$ is implementable in the Fock space, i.e., there is a unitary operator $\hat{g}$ in $\mathcal{F}$ such that

$$\hat{g}a^*(v)\hat{g}^{-1} = a^*(gv) \ , \forall v \in H,$$

and similarly for the $a(v)$’s, one must have $g \in U_{res}$ with respect to the polarization defining the vacuum vector. This condition is both necessary and sufficient.

The polarization of the 1-particle Hilbert space comes normally from a spectral projection onto the positive energy subspace of a Hamilton operator. In the background field problems one studies families of Hamilton operators $D_x$ and then one would like to construct a family of fermionic Fock spaces parametrized by $x \in M$. If none of the Hamilton operators has zero modes, this is unproblematic. However, the presence of zero modes makes it impossible to define the positive energy subspace $H_+(x)$ as a continuous function of $x$. One way out of this is to weaken the condition for the polarization: Each $x \in M$ defines a Grassmann manifold $Gr_{res}(x)$ consisting of all subspaces $W \subset H$ such that the projections onto $W$ and $H_+(x)$ differ by Hilbert-Schmidt operators. The definition of $Gr_{res}(x)$ is stable with respect to finite rank perturbations of $D_x/|D_x|$. For example, when $D_x$ is a Dirac operator on a compact manifold then $(D_x - \lambda)/|D_x - \lambda|$ defines the same Grassmannian for all real numbers $\lambda$ because in each finite interval there are only a finite number of eigenvalues (with multiplicities) of $D_x$. From this follows that the Grassmannians form a locally trivial fiber bundle $Gr$ over families of Dirac operators.

If the bundle $Gr$ has a global section $x \mapsto W_x$ then we can define a bundle of Fock space representations for the CAR algebra over the parameter space $M$. However, there are important situations when no global sections exist. It is easier to explain the potential obstruction in terms of a principal $U_{res}$ bundle $P$ such that $Gr$ is an associated bundle to $P$.

The fiber of $P$ at $x \in M$ is the set of all unitaries $g$ in $H$ such that $g \cdot H_+ \in Gr_x$ where $H = H_+ \oplus H_-$ is a fixed reference polarization. Then we have

$$Gr = P \times_{U_{res}} Gr_{res},$$
where the right action of $U_{\text{res}} = U_{\text{res}}(H_+ \oplus H_-)$ in the fibers of $P$ is the right multiplication on unitary operators and the left action on $Gr_{\text{res}}$ comes from the observation that $Gr_{\text{res}} = U_{\text{res}}/(U_+ \times U_-)$ where $U_\pm$ are the diagonal block matrices in $U_{\text{res}}$. By a result of N. Kuiper, the subgroup $U_+ \times U_-$ is contractible and so $Gr$ has a global section if and only if $P$ is trivial.

Thus in the case when $P$ is trivial we can define the family of Dirac representations of the CAR algebra parametrized by $M$ such that in each of the Fock spaces we have a Dirac vacuum which is in a precise sense close to the vacuum defined by the energy polarization. However, the triviality of $P$ is not a necessary condition.

Actually, what is needed is that $P$ has a prolongation to a bundle $\hat{P}$ with fiber $\hat{U}_{\text{res}}$. The group $\hat{U}_{\text{res}}$ is a central extension of $U_{\text{res}}$ by the group $S^1$.

The Lie algebra $\hat{u}_{\text{res}}$ is as a vector space the direct sum $u_{\text{res}} \oplus i\mathbb{R}$, with commutators

\begin{equation}
[X + \lambda, Y + \mu] = [X, Y] + c(X, Y),
\end{equation}

where $c$ is the Lie algebra cocycle

\begin{equation}
c(X, Y) = \frac{1}{4} \text{tr} \epsilon[\epsilon, X][\epsilon, Y].
\end{equation}

Here $\epsilon$ is the grading operator with eigenvalues $\pm 1$ on $H_\pm$. The trace exists since the off diagonal blocks of $X, Y$ are Hilbert-Schmidt.

The group $\hat{U}_{\text{res}}$ is a circle bundle over $U_{\text{res}}$. The Chern class of the associated complex line bundle is the generator of $H^2(U_{\text{res}}, \mathbb{Z})$ and is given explicitly at the identity element as the antisymmetric bilinear form $c/2\pi i$ and at other points on the group manifold through left-translation of $c/2\pi i$. If $P$ is trivial, then it has an obvious prolongation to the trivial bundle $M \times \hat{U}_{\text{res}}$. In any case, if the prolongation exists we can define the bundle of Fock spaces carrying CAR representations as the associated bundle

$$\mathcal{F} = \hat{P} \times \hat{U}_{\text{res}} \mathcal{F}_0,$$

where is $\mathcal{F}_0$ is the fixed Fock space defined by the same polarization $H = H_+ \oplus H_-$ used to define $U_{\text{res}}$. By the Shale-Stinespring theorem, any $g \in U_{\text{res}}$ has an implementation $\hat{g}$ in $\mathcal{F}_0$, but $\hat{g}$ is only defined up to phase, thus the central $S^1$ extension.

The action of the CAR algebra in the fibers is given as follows. For $x \in M$ choose any $\hat{g} \in \hat{P}_x$. Define

$$a^*(v) \cdot (\hat{g}, \psi) = (\hat{g}, a^*(g^{-1}v)\psi),$$

where $\psi \in \mathcal{F}_0$ and $v \in H$; similarly for the operators $a(v)$. It is easy to check that this definition passes to the equivalence classes in $\mathcal{F}$. Note that the representations in different fibers are in general inequivalent because the transformation $g$ is not implementable in the Fock space $\mathcal{F}_0$.

The potential obstruction to the existence of the prolongation of $P$ is again a 3-cohomology class on the base. Choose a good cover of $M$. On the intersections $U_{\alpha\beta}$ of the open cover the transition functions $g_{\alpha\beta}$ of $P$ can be prolonged to functions $\hat{g}_{\alpha\beta} : U_{\alpha\beta} \to \hat{U}_{\text{res}}$. We have

\begin{equation}
\hat{g}_{\alpha\beta} \hat{g}_{\beta\gamma} \hat{g}_{\gamma\alpha} = f_{\alpha\beta\gamma} \cdot 1,
\end{equation}

where $f_{\alpha\beta\gamma}$ is the 3-cocycle.

\[f_{\alpha\beta\gamma} = \frac{1}{4} \text{tr} \epsilon[\epsilon, X][\epsilon, Y][\epsilon, Z].\]
for functions $f_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \to S^1$, which by construction satisfy the cocycle property (4). Since the cocycle is defined on a good cover, it defines an integral Čech cohomology class $\omega \in H^3(M, \mathbb{Z})$.

Let us return to the universal $U_{res}$ bundle $P$ over $G = U_1(H)$. In this case the prolongation obstruction can be computed relatively easily. It turns out that the 3-cohomology class is represented by the de Rham class which is the generator of $H^3(G, \mathbb{Z})$. Explicitly,

$$\omega = \frac{1}{24\pi^2} \text{tr} (g^{-1} dg)^3.$$  

Any principal $U_{res}$ bundle over $M$ comes from a pull-back of $P$ with respect to a map $f : M \to G$, so the Dixmier-Douady class in the general case is the pull-back $f^* \omega$.

The line bundle construction of the gerbe over the parameter space $M$ for Dirac operators is given by the observation that the spectral subspaces $E_{\lambda\lambda'}(x)$ of $D_x$, corresponding to the open interval $]\lambda, \lambda'[\mathbb{R}$ in the real line, form finite rank vector bundles over open sets $U_{\lambda\lambda'} = U_\lambda \cap U_\lambda'$. Here $U_\lambda$ is the set of points $x \in M$ such that $\lambda$ does not belong to the spectrum of $D_x$. Then we can define, as top exterior power,

$$L_{\lambda\lambda'} = \bigwedge^{\text{top}} (E_{\lambda\lambda'})$$

as the complex vector bundle over $U_{\lambda\lambda'}$. It follows immediately from the definition that the cocycle property (6) is satisfied.

**Example 1. Fermions on an interval.** Let $K$ be a compact group and $\rho$ its unitary representation in a finite-dimensional vector space $V$. Let $H$ be the Hilbert space of square-integrable $V$ valued functions on the interval $[0, 2\pi]$ of the real axis. For each $g \in K$ let $\text{Dom}_g \subset H$ be the dense subspace of smooth functions $\psi$ with the boundary condition $\psi(2\pi) = \rho(g)\psi(0)$. Denote by $D_g$ the operator $-i\frac{d}{dx}$ on this domain. The spectrum of $D_g$ is a function of the eigenvalues $\lambda_k$ of $\rho(g)$, consisting of real numbers $n + \frac{\log(\lambda_k)}{2\pi i}$ with $n \in \mathbb{Z}$. For this reason the splitting of the 1-particle space $H$ to positive and negative modes of the operator $D_g$ is in general not continuous as function of the parameter $g$. This leads to the problems described above. However, the principal $U_{res}$ bundle can be explicitly constructed. It is the pull-back of the universal bundle $P$ with respect to the map $f : K \to G$ defined by the embedding $\rho(K) \subset G$ as $N \times N$ block matrices, $N = \dim V$. Thus the Dixmier-Douady class in this example is

$$\omega = \frac{1}{24\pi^2} \text{tr} (\rho(g)^{-1} d\rho(g))^3.$$  

**Example 2. Fermions on a circle.** Let $H = L^2(S^1, V)$ and $D_A = -i(\frac{d}{dx} + A)$ where $A$ is a smooth vector potential on the circle taking values in the Lie algebra $\mathfrak{k}$ of $K$. In this case the domain is fixed, consisting of smooth $V$ valued functions on the circle. The $\mathfrak{k}$ valued function $A$ is represented as a multiplication operator through the representation $\rho$ of $K$. The parameter space $\mathcal{A}$ of smooth vector potentials is flat, thus there cannot be any obstruction to the prolongation problem. However, in quantum field theory one wants to pass to the moduli space $\mathcal{A}/\mathcal{G}$ of gauge potentials. Here $\mathcal{G}$ is the group of smooth based gauge transformations, i.e., $\mathcal{G} = \Omega K$. Now the moduli space is the group of holonomies around the circle, $\mathcal{A}/\mathcal{G} = K$. Thus we are in
a similar situation as in Example 1. In fact, these examples are really two different
realizations of the same family of self-adjoint Fredholm operators. The operator
\(D_A\) with \(k = \text{holonomy}(A)\) has exactly the same spectrum as \(D_k\) in Example 1.
For this reason the Dixmier-Douady class on \(K\) is the same as before.

The case of Dirac operators on the circle is simple because all the energy po-
larizations for different vector potentials are elements in a single Hilbert-Schmidt
Grassmannian \(\text{Gr}(H_+ \oplus H_-)\) where we can take as the reference polarization the
splitting to positive and negative Fourier modes. Using this polarization, the bun-
dle of fermionic Fock spaces over \(\mathcal{A}\) can be trivialized as \(\mathcal{F} = \mathcal{A} \times \mathcal{F}_0\). However,
the action of the gauge group \(\mathcal{G}\) on \(\mathcal{F}\) acquires a central extension \(\hat{\mathcal{G}} \subset \hat{L}K\),
where \(LK\) is the free loop group of \(K\).

The Lie algebra cocycle determining the central
extension is
\[
(17) \quad c(X, Y) = \frac{1}{2\pi i} \int_{S^1} \text{tr}_\rho XDY,
\]
where \(\text{tr}_\rho\) is the trace in the representation \(\rho\) of \(K\). Because of the central extension,
the quotient \(\mathcal{F}/\hat{\mathcal{G}}\) defines only a projective vector bundle over \(\mathcal{A}/\hat{\mathcal{G}}\), the Dixmier-
Douady class being given by (16).

In the Example 1 (and 2) above the complex line bundles can be con-
structed quite explicitly. Let us study the case \(K = SU(n)\). Define \(U_\lambda \subset K\) as the set
of matrices \(g\) such that \(\lambda\) is not an eigenvalue of \(g\). Select \(n\) different points \(\lambda_j\)
on the unit circle such that their product is not equal to 1. We assume that the
points are ordered counter clock-wise on the circle. Then the sets \(U_j = U_{\lambda_j}\) form
an open cover of \(SU(n)\). On each \(U_j\) we can choose a continuous branch of the
logarithmic function \(\log : U_j \to \text{su}(n)\). The spectrum of the Dirac operator \(D_g\)
with the holonomy \(g\) consists of the infinite set of numbers \(\mathbb{Z} + \text{Spec}(-i \log(g))\). In
particular, the numbers \(\mathbb{Z} - i \log \lambda_j\) do not belong to the spectrum of \(D_g\). Choosing
\(\mu_k = -i \log \lambda_k\) as an increasing sequence in the interval \([0, 2\pi]\) we can as well
define \(U_j = \{x \in M | \mu_j \notin \text{Spec}(D_x)\}\). In any case, the top exterior power of the
spectral subspace \(E_{\mu_j, \mu_k}(x)\) is given by zero Fourier modes consisting of the spectral
subspace of the holonomy \(g\) in the segment \([\lambda_j, \lambda_k]\) of the unit circle.

3. INDEX THEORY AND GERBES

Gauge and gravitational anomalies in quantum field theory can be computed
by Atiyah-Singer index theory. The basic setup is as follows. On a compact even
dimensional spin manifold \(S\) (without boundary) the Dirac operators coupled to
vector potentials and metrics form a family of Fredholm operators. The parameter
space is the set \(\mathcal{A}\) of smooth vector potentials (gauge connections) in a vector
bundle over \(S\) and the set of smooth Riemann metrics on \(S\). The family of Dirac
operators is covariant with respect to gauge transformations and diffeomorphims of
\(S\), thus we may view the Dirac operators parametrized by the moduli space \(\mathcal{A}/\mathcal{G}\) of
gauge connections and the moduli space \(\mathcal{M}/\text{Diff}_0(S)\) of Riemann mertics. Again,
in order that the moduli spaces are smooth manifolds one has to restrict to the
based gauge transformations, i.e., those which are equal to the neutral element
in a fixed base point in each connected component of \(S\). Similarly, the Jacobian
of a diffeomorphism is required to be equal to the identity matrix at the base.
passing to the quotient modulo gauge transformations and diffeomorphisms we obtain a vector bundle over the space

\[ S \times \mathcal{A}/\mathcal{G} \times \mathcal{M}/\text{Diff}_0(S). \]

Actually, we could as well consider a generalization in which the base space is a fibering over the moduli space with model fiber equal to \( S \), but for simplicity we stick to (18).

According to Atiyah-Singer index formula for families, the K-theory class of the family of Dirac operators acting on the smooth sections of the tensor product of the spin bundle and the vector bundle \( V \) over (18) is given through the differential forms

\[ \hat{A}(R) \wedge ch(V), \]

where \( \hat{A}(R) \) is the A-roof genus, a function of the Riemann curvature tensor \( R \) associated to the Riemann metric,

\[ \hat{A}(R) = \det^{1/2} \left( \frac{R/4\pi i}{\sinh(R/4\pi i)} \right), \]

and \( ch(V) \) is the Chern character

\[ ch(V) = \text{tr} e^{F/2\pi i}, \]

where \( F \) is the curvature tensor of a gauge connection. Here both \( R \) and \( F \) are forms on the infinite-dimensional base space (18). After integrating over the fiber \( S \),

\[ \text{Ind} = \int_S \hat{A}(R) \wedge ch(V), \]

we obtain a family of differential forms \( \phi_{2k} \), one in each even dimension, on the moduli space.

The (cohomology classes of) forms \( \phi_{2k} \) contain important topological information for the quantized Yang-Mills theory and for quantum gravity. The form \( \phi_2 \) describes potential chiral anomalies. The chiral anomaly is a manifestation of gauge or reparametrization symmetry breaking. If the class \( [\phi_2] \) is nonzero, the quantum effective action cannot be viewed as a function on the moduli space. Instead, it becomes a section of a complex line bundle \( DET \) over the moduli space.

Since the Dirac operators are Fredholm (on compact manifolds), at a given point in the moduli space we can define the complex line

\[ DET_x = \bigwedge^{\text{top}} (\ker D^+_x) \otimes \bigwedge^{\text{top}} (\text{coker} D^+_x) \]

for the chiral Dirac operators \( D^+_x \). In the even dimensional case the spin bundle is \( \mathbb{Z}_2 \) graded such that the grading operator \( \Gamma \) anticommutes with \( D_x \). Then \( D^+_x = P_-D_xP_+ \) where \( P_{\pm} = \frac{1}{2} (1 \pm \Gamma) \) are the chiral projections. \( \bigwedge^{\text{top}} \) means the operation on finite dimensional vector spaces \( W \) taking the exterior power of \( W \) to \( \dim W \).

When the dimensions of the kernel and cokernel of \( D_x \) are constant the formula (20) defines a smooth complex line bundle over the moduli space. In the case of varying dimensions a little extra work is needed to define the smooth structure.
The form $\phi_2$ is the Chern class of $\text{DET}$. So if $\text{DET}$ is nontrivial, gauge covariant quantization of the family of Dirac operators is not possible.

One can also give a geometric and topological meaning to the chiral symmetry breaking in Hamiltonian quantization and this leads us back to gerbes on the moduli space. Here we have to use an odd version of the index formula (19). Assuming that the physical space-time is even dimensional, at a fixed time the space is an odd dimensional manifold $S$. We still assume that $S$ is compact. In this case the integration in (19) is over odd dimensional fibers and therefore the formula produces a sequence of odd forms on the moduli space.

The first of the odd forms $\phi_1$ gives the spectral flow of a 1-parameter family of operators $D_{x(s)}$. Its integral along the path $x(t)$, after a correction by the difference of the eta invariant at the end points of the path, in the moduli space, gives twice the difference of positive eigenvalues crossing over to the negative side of the spectrum minus the flow of eigenvalues in the opposite direction. The second term $\phi_3$ is the Dixmier-Douady class of the projective bundle of Fock spaces over the moduli space. In the Examples 1 and 2 the index theory calculation gives exactly the form (16) on $K$.

**Example** Consider Dirac operators on the 3-dimensional sphere $S^3$ coupled to vector potentials. Any vector bundle on $S^3$ is trivial, so let $V = S^3 \times \mathbb{C}^N$. Take $\text{SU}(N)$ as the gauge group and let $\mathcal{A}$ be the space of 1-forms on $S^3$ taking values in the Lie algebra $\text{su}(N)$ of $\text{SU}(N)$. Fix a point $x_s$ on $S^3$, the 'South Pole', and let $\mathcal{G}$ be the group of gauge transformations based at $x_s$. That is, $\mathcal{G}$ consists of smooth functions $g: S^3 \to \text{SU}(N)$ with $g(x_s) = 1$. In this case $\mathcal{A}/\mathcal{G}$ can be identified as $\text{Map}(S^2, \text{SU}(N))$ times a contractible space. This is because any point $x$ on the equator of $S^3$ determines a unique semicircle from the South Pole to the North Pole through $x$. The parallel transport along this path with respect to a vector potential $A \in \mathcal{A}$ defines an element $g_A'(x) \in \text{SU}(N)$, using the fixed trivialization of $V$. Set $g_A(x) = g_A'(x)g_A'(x_0)^{-1}$ where $x_0$ is a fixed point on the equator. The element $g_A(x)$ then depends only on the gauge equivalence class $[A] \in \mathcal{A}/\mathcal{G}$. It is not difficult to show that the map $A \mapsto g_A$ is a homotopy equivalence from the moduli space of gauge potentials to the group $\mathcal{G}_2 = \text{Map}_{x_0}(S^2, \text{SU}(N))$, based at $x_0$. When $N > 2$ the cohomology $H^5(\text{SU}(N), \mathbb{Z}) = \mathbb{Z}$ transgresses to the cohomology $H^3(\mathcal{G}_2, \mathbb{Z}) = \mathbb{Z}$. In particular, the generator

$$\omega_5 = \left(\frac{i}{2\pi}\right)^3 \frac{2}{5!}\text{tr}(g^{-1}dg)^5$$

of $H^5(\text{SU}(N), \mathbb{Z})$ gives the generator of $H^3(\mathcal{G}_2, \mathbb{Z})$ by contraction and integration,

$$\Omega = \int_{S^2} \omega_5.$$

4. GAUGE GROUP EXTENSIONS

The new feature for gerbes associated to Dirac operators in higher than one dimensions is that the gauge group, acting on the bundle of Fock spaces parametrized by vector potentials, is represented through an abelian extension. On the Lie algebra level this means that the Lie algebra extension is not given by a scalar cocycle.
as in the one dimensional case but by a cocycle taking values in an abelian Lie algebra. In the case of Dirac operators coupled to vector potentials the abelian Lie algebra consists of a certain class of complex functions on \( \mathcal{A} \). The extension is then defined by the commutators

\[
[(X, \alpha), (Y, \beta)] = ([X, Y], \mathcal{L}_X \beta - \mathcal{L}_Y \alpha + c(X, Y))
\]

where \( \alpha, \beta \) are functions on \( \mathcal{A} \) and \( \mathcal{L}_X \beta \) denotes the Lie derivative of \( \beta \) in the direction of the infinitesimal gauge transformation \( X \). The 2-cocycle property of \( c \) is expressed as

\[
c([X, Y], Z) + \mathcal{L}_X c(Y, Z) + \text{cyclic permutations of } X, Y, Z = 0.
\]

In the case of Dirac operators on a 3-manifold \( S \) the form \( c \) is the Mickelsson-Faddeev cocycle

\[
c(X, Y) = \frac{i}{12\pi^2} \int_S \text{tr} \rho A \wedge (dX \wedge dY - dY \wedge dX).
\]

The corresponding gauge group extension is an extension of \( \text{Map}(S, G) \) by the normal subgroup \( \text{Map}(\mathcal{A}, S^1) \). As a topological space, the extension is the product

\[
\text{Map}(\mathcal{A}, S^1) \times_{S^1} P,
\]

where \( P \) is a principal \( S^1 \) bundle over \( \text{Map}(S, G) \).

The Chern class \( c_1 \) of the bundle \( P \) is again computed by transgression from \( \omega_5 \), this time

\[
c_1 = \int_S \omega_5.
\]

In fact, we can think the cocycle \( c \) as a 2-form on the space of flat vector potentials \( A = g^{-1}dg \) with \( g \in \text{Map}(S^3, G) \). Then one can show that the cohomology classes \([c]\) and \([c_1]\) are equal.

As we have seen, the central extension of a loop group is the key to understanding the quantum field theory gerbe. Here is a short description of it starting from the 3-form (16) on a compact Lie group \( G \). First define a central extension \( \text{Map}(D, G) \times S^1 \) of the group of smooth maps from the unit disk \( D \) to \( G \), with point-wise multiplication. The group multiplication is given as

\[
(g, \lambda) \cdot (g', \lambda') = (gg', \lambda \lambda' \cdot e^{2\pi i \gamma(g, g')}),
\]

where

\[
\gamma(g, g') = \frac{1}{8\pi^2} \int_D \text{tr}_\rho g^{-1}dg \wedge dg'g^{-1},
\]

where the trace is computed in a fixed unitary representation \( \rho \) of \( G \). This group contains as a normal subgroup the group \( N \) consisting of pairs \((g, e^{2\pi i C(g)})\) with

\[
C(g) = \frac{1}{24\pi^2} \int_B \text{tr}_\rho (g^{-1}dg)^3.
\]
Here \( g(x) = 1 \) on the boundary circle \( S^1 = \partial D \), and thus can be viewed as a function \( S^2 \to G \). The 3-dimensional unit ball \( B \) has \( S^2 \) as a boundary and \( g \) is extended in an arbitrary way from the boundary to the ball \( B \). The extension is possible since \( \pi_2(G) = 0 \) for any finite-dimensional Lie group. The value of \( C(g) \) depends on the extension only modulo an integer and therefore \( e^{2\pi i C(g)} \) is well-defined.

The central extension is then defined as

\[
\hat{LG} = (\text{Map}(D, G) \times S^1)/N.
\]

One can show easily that the Lie algebra of \( \hat{LG} \) is indeed given through the cocycle (17). In the case of \( G = SU(n) \) in the defining representation, this central extension is the basic extension: The cohomology class is the generator of \( H^2(LG, \mathbb{Z}) \). In general, to obtain the basic extension one has to correct (23) and (24) by a normalization factor.

This construction generalizes to the higher loop groups \( \text{Map}(S, G) \) for compact odd dimensional manifolds \( S \). For example, in the case of a 3-manifold one starts from an extension of \( \text{Map}(D, G) \), where \( D \) is a 4-manifold with boundary \( S \). The extension is defined by a 2-cocycle \( \gamma \), but now for given \( g, g' \) the cocycle \( \gamma \) is a real valued function of a point \( g_0 \in \text{Map}(S, G) \), which is a certain differential polynomial in the Maurer-Cartan 1-forms \( g_0^{-1} dg_0, g^{-1} dg, g^{-1} dg \). The normal subgroup \( N \) is defined in a similar way, now \( C(g) \) is the integral of the 5-form \( \omega_5 \) over a 5-manifold \( B \) with boundary \( \partial B \) identified as \( D/\sim \), the equivalence shrinking the boundary of \( D \) to one point. This gives the extension only over the connected component of identity in \( \text{Map}(S, G) \), but it can be generalized to the whole group. For example, when \( S = S^3 \) and \( G \) is simple the connected components are labeled by elements of the third homotopy group \( \pi_3 G = \mathbb{Z} \).

In some cases the de Rham cohomology class of the extension vanishes but the extension still contains interesting torsion information. In quantum field theory this comes from hamiltonian formulation of global anomalies. A typical example of this phenomenon is the Witten \( SU(2) \) anomaly in four space-time dimensions. In the hamiltonian formulation we take \( S^3 \) as the physical space, the gauge group \( G = SU(2) \). In this case the second cohomology of \( \text{Map}(S^3, G) \) becomes pure torsion, related to the fact that the 5-form \( \omega_5 \) on \( SU(2) \) vanishes for dimensional reasons. Here the homotopy group \( \pi_4(G) = \mathbb{Z}_2 \) leads to the nontrivial fundamental group \( \mathbb{Z}_2 \) in each connected component of \( \text{Map}(S^3, G) \). Using this fact one can show that there is a nontrivial \( \mathbb{Z}_2 \) extension of the group \( \text{Map}(S^3, G) \).

**Related material in this Encyclopedia:** Articles 149 (Index theorems), 287 (Anomalies), 305 (Dirac fields and Dirac operators), 308 (Bosons and fermions in external fields), 322 (K-theory), 354 (Characteristic classes).

**Further reading**

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