A note on the complexity of K-METRIC DIMENSION

Yannick Schmitz, Duygu Vietz, and Egon Wanke

Heinrich-Heine-Universität Düsseldorf, Germany
yannick.schmitz@hhu.de, duygu.vietz@hhu.de, egon.wanke@hhu.de

Abstract

Two vertices $u,v \in V$ of an undirected connected graph $G = (V,E)$ are resolved by a vertex $w$ if the distance between $u$ and $w$ and the distance between $v$ and $w$ are different. A set $R \subseteq V$ of vertices is a $k$-resolving set for $G$ if for each pair of vertices $u,v \in V$ there are at least $k$ distinct vertices $w_1, \ldots, w_k \in R$ such that each of them resolves $u$ and $v$. The $k$-Metric Dimension of $G$ is the size of a smallest $k$-resolving set for $G$. The decision problem $k$-METRIC DIMENSION is the question whether $G$ has a $k$-resolving set of size at most $r$, for a given graph $G$ and a given number $r$. In this paper, we proof the NP-completeness of $k$-METRIC DIMENSION for bipartite graphs and each $k \geq 2$.

1 Introduction

The metric dimension of graphs has been introduced in the 1970s independently by Slater [Slater75] and by Harary and Melter [Harary76]. We consider simple undirected and connected graphs $G = (V,E)$, where $V$ is the set of vertices and $E \subseteq \{(u,v) \mid u,v \in V, u \neq v\}$ is the set of edges. Such a graph has metric dimension at most $r$ if there is a vertex set $R \subseteq V$ such that $|R| \leq r$ and $\forall u,v \in V, u \neq v$, there is a vertex $w \in R$ such that $d(w,u) \neq d(w,v)$, where $d(u,v)$ is the distance (the length of a shortest path in an unweighted graph) between $u$ and $v$. The metric dimension of $G$ is the smallest integer $r$ such that $G$ has metric dimension at most $r$.

If $d(w,u) \neq d(w,v)$, for three vertices $u,v,w$, we say that $u$ and $v$ are resolved or distinguished by vertex $w$. If every pair of vertices is resolved by at least one vertex of a vertex set $R$, then $R$ is a resolving set or metric generator for $G$. In certain applications, the vertices of a resolving set are also called landmark nodes or anchor nodes. This is a common naming, particularly in the theory of sensor networks.

The metric dimension finds applications in various areas, including network discovery and verification [Bueno05], geographical routing protocols [Lee06], combinatorial optimization [Steger04], sensor networks [Hei12], robot navigation [Karp96] and chemistry [Cecchin00, Hay77].

There are several algorithms for computing a minimum resolving set in polynomial time for special classes of graphs, for example trees [Cecchin00, Karp96], wheels [Harary75], grid graphs [Merrill84], $k$-regular bipartite graphs [Bunne10], amalgamation of cycles [Bunne11] and outerplanar graphs [Dussel12]. The approximability of the metric dimension has been studied for bounded degree, dense and general graphs in [Hei12]. Upper and lower bounds on the metric dimension are considered in [Chapman08, Cheng09] for further classes of graphs.

In this paper, we consider the $k$-Metric Dimension for some positive integer $k$. A set $R \subseteq V$ of vertices is a $k$-resolving set for $G$ if for each pair of vertices $u,v \in V$ there are at least $k$ vertices $w_1, \ldots, w_k \in R$ such that each of them resolves $u$ and $v$. The $k$-Metric Dimension of $G$ is the size of a smallest $k$-resolving set for $G$. The $k$-METRIC DIMENSION problem was introduced by Estrada-Moreno et al. in [Estrada13]. The 1-metric dimension is simply called metric dimension. The 2-metric dimension is also called fault-tolerant metric dimension and was introduced in [Hei08].

Estrada-Moreno et al. analysed the $(k,t)$-METRIC DIMENSION [Estrada16]. The $(k,t)$-METRIC DIMENSION is the $k$-METRIC DIMENSION, with the addition, that the distance between two vertices $u,v$ of $G$ is defined as the minimum of $d(u,v)$ and $t$. Therefore, if $t$ is set to the diameter of $G$, the $(k,t)$-METRIC DIMENSION is the same as the $k$-METRIC DIMENSION. Estrada-Moreno et al. showed the NP-completeness of $(k,t)$-METRIC DIMENSION for odd values of $k$.

The decision problem $k$-METRIC DIMENSION is defined as follows.
The complexity of $k$-Metric Dimension has only been investigated for very few graph classes, such as trees and other simple graph classes. For general graph classes, $k$-Metric Dimension is assumed to be NP-complete if $k$ is given as part of the input. The decision problem $1$-Metric Dimension is known to be NP-complete, see [GJ79]. A proof can be found in [KRR96]. In this paper, we show the NP-completeness of $k$-Metric Dimension for bipartite graphs and each $k \geq 2$ by an alternative approach to [YER17], whose proof unfortunately is incorrect and does not offer any simple correction options.

## 2 The NP-completeness of $k$-Metric Dimension

In this section, $k$-Metric Dimension is shown to be NP-complete for bipartite graphs and each $k \geq 2$ by a reduction from $3$-Dimensional $k$-Matching, which is defined as follows.

### 3-Dimensional $k$-Matching ($3$DM)

**Instance:** A set $S \subseteq A \times B \times C$, where $A$, $B$ and $C$ are disjoint sets of the same size $n$.

**Question:** Does $S$ contain a $k$-matching, i.e. a subset $M$ of size $k \cdot n$ such that each element of $A$, $B$ and $C$ is contained in exactly $k$ triples of $M$?

For $k = 1$, the $3$D1M problem is the well-known NP-complete $3$-Dimensional Matching ($3$DM) problem, see [GJ79]. The next theorem shows that $3$D$k$M is also NP-complete for each $k \geq 2$.

**Theorem 1.** $3$D$k$M is NP-complete for each $k \geq 2$.

**Proof.** The $3$D$k$M problem is obviously in NP, because it can be checked in polynomial time whether a selection of triples from $S$ is a $k$-matching.

The NP-hardness is shown by a reduction from $3$DM. Let

$$A = \{a_1, \ldots, a_n\}, \quad B = \{b_1, \ldots, b_n\},$$

$$C = \{c_1, \ldots, c_m\}, \quad \text{and} \quad S = \{s_1, \ldots, s_m\}$$

be an instance for $3$DM. Without loss of generality, $n$ is assumed to be a multiple of $(k - 1)$, that is $n = r(k - 1)$ for a positive integer $r$. If this is not the case, then expand $A$, $B$ and $C$ by at most $k - 2$ elements each and $S$ by at most $k - 2$ triples, which cover every additional element exactly once and none of the originally given elements.

Now consider the following instance for $3$D$k$M defined by

$$A' = A \cup \{a_{n+1}, \ldots, a_{3n}\}, \quad B' = B \cup \{b_{n+1}, \ldots, b_{3n}\},$$

$$C' = C \cup \{c_{n+1}, \ldots, c_{3m}\}, \quad \text{and} \quad S' = S \cup R \cup T$$

where $R = \{(a_i, b_i, c_i) | n + 1 \leq i \leq 3n\}$ and $T \subseteq A' \times B' \times C'$. Set $T$ is a set with $3n(k - 1)$ triples, which will be defined later.

The set $A'$, $B'$ and $C'$ is the set $A$, $B$ and $C$ respectively, each expanded by additional $2n$ elements. Set $S'$ is the set $S$ expanded by the $2n$ triples of $R$ and the $3n(k - 1)$ triples of $T$.

Let $U = A \cup B \cup C$ and $U' = A' \cup B' \cup C'$. The $2n$ triples of $R$ cover each element of $U' \setminus U$ exactly once and no element of $U$. Set $T$ will be defined such that its $3n(k - 1)$ triples cover each element of $U'$ exactly $k - 1$ times. Each triple of $T$ will have exactly one element from $U$ and two elements from $U' \setminus U$. 


If $M$ is a matching for $U$ then $M \cup R \cup T$ is obviously a $k$-matching for $U'$ for any $k \geq 2$. Any $k$-matching $M'$ for $U'$ contains all triples from $R$ and $T$, because otherwise it is not possible to cover the elements of $U' \setminus U$ at least $k$ times. The triples of $T$ cover the elements of $U'$ exactly $k - 1$ times. That is, if $M'$ is a $k$-matching for $U'$ then $M = M' \setminus (R \cup T)$ is a matching for $U$.

The set $T$ of triples can be easily defined with the help of a set

$$T_{p,q} \subseteq (A \times B) \cup (A \times C) \cup (B \times C)$$

of tuples defined by

$$T_{p,q} = \bigcup \{ (a_i, b_j) \mid i \in \{p, \ldots, p + q - 1\}, j \in \{p + q, \ldots, p + 2q - 1\} \}$$

These $3q^2$ tuples cover each element of

$$\{a_p, \ldots, a_{p+2q-1}, b_p, \ldots, b_{p+2q-1}, c_p, \ldots, c_{p+2q-1}\}$$

exactly $q$ times. There are

- $q^2$ tuples between the elements of $\{a_p, \ldots, a_{p+q-1}\}$ and $\{b_p, \ldots, b_{p+2q-1}\}$,
- $q^2$ tuples between the elements of $\{b_p, \ldots, b_{p+q-1}\}$ and $\{c_p, \ldots, c_{p+2q-1}\}$,
- $q^2$ tuples between the elements of $\{c_p, \ldots, c_{p+q-1}\}$ and $\{a_p, \ldots, a_{p+2q-1}\}$.

Now let $T'$ be the set of tuples defined by

$$T' = \bigcup_{i=0}^{r-1} T_{n+1+i2(k-1), k-1}, \text{ with } r = \frac{n}{k-1}.$$  

$T'$ contains $r3(k-1)^2 = \frac{n}{k-1} \cdot 3(k-1)^2 = 3n(k-1)$ tuples. It is the union of $r = \frac{n}{k-1}$ sets $T_{p,q}$ where index $p$ is running from $n+1$ to $3n+1-2(k-1)$ in steps of width $2(k-1)$ and $q = k - 1$. These tuples of $T'$ cover each element of $U' \setminus U$ exactly $(k - 1)$ times.

In the last step, the $3n(k-1)$ tuples of $T'$ are expanded to $3n(k-1)$ triples for $T$, by including each element from $U$ to exactly $k - 1$ tuples from $T'$, such that each generated triple is from the set $A' \times B' \times C'$. Each tuple from $T'$ is extended by exactly one element from $U$. The result is the set $T$ of triples with the required properties. This transformation can obviously be done in polynomial time, see also Example 1.

$$\square$$

Example 1. Let $A = \{a_1, \ldots, a_4\}$, $B = \{b_1, \ldots, b_4\}$, $C = \{c_1, \ldots, c_4\}$ and

$$S = \{(a_1,b_1,c_1), (a_1,b_2,c_3), (a_2,b_3,c_3), (a_2,b_4,c_1), (a_3,b_1,c_2), (a_4,b_3,c_4)\}$$

be an instance for 3DM. The triple $(a_1,b_2,c_3), (a_2,b_4,c_1), (a_3,b_1,c_2), (a_4,b_3,c_4)$ form a 3-dimensional matching and thus a solution for 3DM.

It follows the construction of an instance for 3DkM for $k = 4$ as defined in the proof of Theorem 1. Integer $n$ has to be a multiple of $k - 1 = 3$. To ensure this, $A$ is extended by $a_5$ and $a_6$, $B$ is extended by $b_5$ and $b_6$, $C$ is extended by $c_5$ and $c_6$ and $S$ is extended by $(a_5,b_5,c_5)$ and $(a_6,b_6,c_6)$. Now $n = 6$ and $r = \frac{n}{k-1} = 2$.

Then $A' = \{a_1, \ldots, a_{18}\}$, $B' = \{b_1, \ldots, b_{18}\}$, $C' = \{c_1, \ldots, c_{18}\}$ and $R = \{(a_i,b_i,c_i) \mid i = 7, \ldots, 18\}$. Set $T'$ is defined as $T' = T_{7,3} \cup T_{13,3}$. Finally, set $S'$ is defined as

$$S' = S \cup R \cup T,$$

where, for example,

$$T_{7,3} = \{(a_7, b_10), (a_7, b_11), (a_7, b_12), (a_8, b_10), (a_8, b_11), (a_8, b_12), (a_9, b_10), (a_9, b_11), (a_9, b_12), (b_7, c_10), (b_7, c_11), (b_7, c_12), (b_8, c_10), (b_8, c_11), (b_8, c_12), (b_9, c_10), (b_9, c_11), (b_9, c_12), (c_7, a_{10}), (c_7, a_{11}), (c_7, a_{12}), (c_8, a_{10}), (c_8, a_{11}), (c_8, a_{12}), (c_9, a_{10}), (c_9, a_{11}), (c_9, a_{12})\}.$$
Figure 1: This graphic illustrates the transformation from 3DM to 3DkM for \( k = 4 \) as explained in Example 1. The drawing on the top left visualizes an instance with 6 triples in \( S \) that cover the elements \( \{a_1, \ldots, a_4, b_1, \ldots, b_1, c_1, \ldots, c_4\} \). The triples are indicated by 6 red and 2 black lines, each covering 3 elements. Set \( T \) contains a matching indicated by the red lines. Each set \( A, B \) and \( C \) is extended by two element \( a_5, a_6, b_5, b_6 \) and \( c_5, c_6 \) respectively, and set \( S \) is extended by two triples \( (a_5, b_5, c_5), (a_6, b_6, c_6) \), such that the number of elements in the new sets \( A, B \) and \( C \) is a multiple of \( (k - 1) = 3 \). These two triples are indicated by green lines. The drawing in the middle right visualizes the \( 2 \cdot 6 = 12 \) triples of \( R \) indicated by black lines. The drawing at the bottom visualizes the 54 tuples of \( T' = T_{7,3} \cup T_{13,3} \), also indicated by black lines, each covering 2 elements. The set \( T \) is formed from set \( T' \) by adding each element of \( A, B \) and \( C \) to \( k - 1 = 3 \) tuples of \( T' \). For the sake of clarity, only the triples from \( T \) for the elements \( a_1, b_1 \) and \( c_1 \) are shown in the figure. These triples are indicated by blue lines.

\[
T_{13,3} = \left\{ \begin{array}{l}
(a_{10}, b_{10}, c_{10}), (a_{11}, b_{11}, c_{11}), (a_{12}, b_{12}, c_{12}), (a_{13}, b_{13}, c_{13}) \\
(a_{14}, b_{14}, c_{14}), (a_{15}, b_{15}, c_{15}), (a_{16}, b_{16}, c_{16})
\end{array} \right\}, \\
T = \left\{ \begin{array}{l}
(a_{7}, b_{10}, c_{1}), (a_{8}, b_{11}, c_{2}), (a_{9}, b_{12}, c_{3}), (a_{10}, b_{13}, c_{4}) \\
(a_{11}, b_{14}, c_{5}), (a_{12}, b_{15}, c_{6}), (a_{13}, b_{16}, c_{7})
\end{array} \right\}.
\]

see also Figure 1.

**Theorem 2.** \( k \)-MD is NP-complete for bipartite graphs \( G \) and each \( k \geq 2 \).

**Proof.** The \( k \)-MD problem is obviously in NP, because it can be checked in polynomial time whether a set of vertices is a \( k \)-resolving set.

The NP-hardness is proven by a reduction from 3D(k-1)M. Let \( A = \{a_1, \ldots, a_n\}, B = \{b_1, \ldots, b_n\}, C = \{c_1, \ldots, c_n\}, S = \{s_1, \ldots, s_m\} \) be an instance \( I \) for 3D(k-1)M where \( k \geq 2 \).
and \( n > k \). The aim is to define a graph \( G = (V, E) \) and a number \( x \) such that \( G \) has a \( k \)-resolving set of size \( x \) if and only if instance \( I \) has a \((k-1)\)-matching.

Graph \( G \) is defined as follows, see also Figure 2. It has a vertex \( a_i \), \( b_i \) and \( c_i \) for \( i = 1, \ldots, n \) and a vertex \( s_i \) for \( i = 1, \ldots, m \). Graph \( G \) additionally contains vertices denoted by \( a_0, b_0, c_0, v_0, v_A, v_B, v_C \) and \( d_1, \ldots, d_{m'} \) where \( m' = \lceil \log(m) \rceil \).

1. Each vertex \( a_i \), \( 0 \leq i \leq n \), is connected with
   - (a) vertex \( v_A \),
   - (b) vertex \( v_0 \), and
   - (c) vertex \( s_j \), \( 1 \leq j \leq m \) if and only if triple \( s_j \) contains element \( a_i \).

2. Each vertex \( b_i \), \( 0 \leq i \leq n \), is connected with
   - (a) vertex \( v_B \),
   - (b) vertex \( v_0 \), and
   - (c) vertex \( s_j \), \( 1 \leq j \leq m \) if and only if triple \( s_j \) contains element \( b_i \).

3. Each vertex \( c_i \), \( 0 \leq i \leq n \), is connected with
   - (a) vertex \( v_C \),
   - (b) vertex \( v_0 \), and
   - (c) vertex \( s_j \), \( 1 \leq j \leq m \) if and only if triple \( s_j \) contains element \( c_i \).

4. Each vertex \( d_i \), \( 1 \leq i \leq m' \), is connected with
   - (a) vertex \( v_0 \) and
   - (b) vertex \( s_j \), \( 1 \leq j \leq m \), if and only if the \( i \)-th bit of the binary representation of \( j \) is 1.

Graph \( G \) contains additionally so-called leg vertices. These leg vertices form paths (legs) with \( \lceil k/2 \rceil \) or \( \lfloor k/2 \rfloor \) vertices. Two such legs, one with \( \lceil k/2 \rceil \) vertices and one with \( \lfloor k/2 \rfloor \) vertices, are attached to each vertex of \( L_{\text{root}} = \{v_A, v_B, v_C, v_0, d_1, \ldots, d_{m'}\} \), see Figure 2. Set \( L_{\text{root}} \) is the set of root vertices of the legs. Let \( L_v \) be the set of vertices of the two legs at vertex \( v \) and

\[
L = L_{v_A} \cup L_{v_B} \cup L_{v_C} \cup L_{v_0} \cup L_{d_1} \cup \cdots \cup L_{d_{m'}}
\]

be the set of all leg vertices of \( G \). Set \( L_{\text{root}} \) has \( 4 + m' \) vertices, each set \( L_v, v \in L_{\text{root}} \), has \( k \) vertices and \( L \) has \((4 + m')k \) vertices.

The graph \( G \) can obviously be constructed in polynomial time from instance \( I \).

First of all, let us note some properties of \( G \).

P1: \( G \) is bipartite.

P2: The distance between
   - (a) two vertices of \( \{v_B, v_B, v_C\} \) is 4,
   - (b) two vertices of \( \{d_1, \ldots, d_{m'}\} \) is 2,
   - (c) a vertex of \( \{v_B, v_B, v_C\} \) and a vertex of \( \{d_1, \ldots, d_{m'}\} \) is 3,
   - (d) vertex \( v_0 \) and a vertex of \( \{v_B, v_B, v_C\} \) is 2, and
   - (e) vertex \( v_0 \) and a vertex of \( \{d_1, \ldots, d_{m'}\} \) is 1.

P3: Every \( k \)-resolving set for \( G \) contains all vertices of \( L \). This follows from the observation that for each vertex \( v \in L_{\text{root}} \) the two vertices of \( L_v \) adjacent with \( v \) are only resolved by the \( k \) vertices of \( L_v \).
Now we will prove that $S$ has a $(k-1)$-matching for instance $I$ if and only if $G$ has a resolving set of size

$$x = (4 + m')k + 3 + (k - 1)n.$$  

"⇒": Let $M \subseteq S$ be a $(k-1)$-matching for instance $I$. The aim is to show that

$$R = L \cup \{u_0, b_0, c_0\} \cup M$$

is a $k$-resolving set for $G$ of size

$$x = (4 + m')k + 3 + (k - 1)n,$$

that is, each pair of two distinct vertices $u_1, u_2$ of $G$ is resolved by at least $k$ vertices of $U$. Here the triple $s_j$ of $M$ are considered as vertices of $G$.

Consider the following case distinctions for two vertices $u_1$ and $u_2$.

1. $u_1, u_2 \in L_v, v \in L_{\text{root}}$.
   (a) $d(u_1, v) = d(u_2, v)$. Each of the $k$ vertices of $L_v$ resolves $u_1$ and $u_2$.
   (b) $d(u_1, v) \neq d(u_2, v)$. Each of the $k$ vertices of $L_{v'}, v' \in L_{\text{root}} \setminus \{v\}$, resolves $u_1$ and $u_2$.

2. $u_1 \in L_{v_1}, u_2 \in L_{v_2}, v_1, v_2 \in L_{\text{root}}, v_1 \neq v_2$, and $d(u_1, v_1) \leq d(u_2, v_2)$. Each of the $k$ vertices of $L_{v_1}$ resolves $u_1$ and $u_2$.

Up to this point all pairs of vertices $u_1, u_2$ are considered of which both are in $L$.

3. $u_1 \in L_{v_A} \cup L_{v_B} \cup L_{v_C}$ and $u_2 \not\in L$. Each of the $k$ vertices of $L_{v_0}$ resolves $u_1$ and $u_2$.

4. $u_1 \in L_{d_1} \cup \cdots \cup L_{d_m'}$ and $u_2 \not\in L$.
   (a) $u_2 \not\in \{v_B, v_C\}$. Each of the $k$ vertices of $L_{v_A}$ resolves $u_1$ and $u_2$.
   (b) $u_2 \not\in \{v_A, v_C\}$. Each of the $k$ vertices of $L_{v_B}$ resolves $u_1$ and $u_2$.
   (c) $u_2 \not\in \{v_A, v_B\}$. Each of the $k$ vertices of $L_{v_C}$ resolves $u_1$ and $u_2$.

5. $u_1 \in L_{v_0}$ and $u_2 \not\in L$.
   (a) $u_2 \in \{v_A, a_0, \ldots, a_n\}$. Each of the $k$ vertices of $L_{v_A}$ resolves $u_1$ and $u_2$.
   (b) $u_2 \in \{v_B, b_0, \ldots, b_n\}$. Each of the $k$ vertices of $L_{v_B}$ resolves $u_1$ and $u_2$.
   (c) $u_2 \in \{v_C, c_0, \ldots, c_n\}$. Each of the $k$ vertices of $L_{v_C}$ resolves $u_1$ and $u_2$.
   (d) $u_2 \in \{d_i\} \cup \{s_j \mid \text{the } i\text{-th bit in the binary representation of } j \text{ is } 1\}$. Each of the $k$ vertices of $L_{d_i}$ resolves $u_1$ and $u_2$.

Up to this point all pairs of vertices $u_1, u_2$ are considered of which at least one of them is in $L$.

6. $u_1 \in L_{\text{root}}$ and $u_2 \not\in L$. Each of the $k$ vertices of $L_{u_1}$ resolves $u_1$ and $u_2$.

Up to this point all pairs of vertices $u_1, u_2$ are considered of which at least one of them is in $L \cup L_{\text{root}}$.

7. $u_1 = s_{i_1} \in \{s_1, \ldots, s_{m'}\}$ and $u_2 \not\in L \cup L_{\text{root}}$.
   (a) $u_2 = s_{i_2} \in \{s_1, \ldots, s_{m'}\}$. Each of the $k$ vertices of $L_{d_j}$ resolves $u_1$ and $u_2$, if the binary representation of $i_1$ and $i_2$ differs in position $j$.
   (b) $u_2 \in \{a_0, \ldots, a_n\}$, $u_2 \in \{b_0, \ldots, b_n\}$, or $u_2 \in \{c_0, \ldots, c_n\}$. Each of the $k$ vertices of $L_{v_A}$, $L_{v_B}$, or $L_{v_C}$, respectively, resolves $u_1$ and $u_2$.

Up to this point all pairs of vertices $u_1, u_2$ are considered of which at least one of them is in $L \cup L_{\text{root}} \cup \{s_1, \ldots, s_{m'}\}$.
8. $u_1 \in \{a_1, \ldots, a_n\}$ and $u_2 \not\in L \cup L_{\text{root}} \cup \{s_1, \ldots, s_{m'}\}$.
   a) $u_2 \in \{b_0, \ldots, b_n, c_0, \ldots, c_n\}$. Each of the $k$ vertices of $L_{v_A}$ resolves $u_1$ and $u_2$.
   b) $u_2 \in \{a_1, \ldots, a_n\}$. Each vertex $s_i$ for which triple $s_i$ contains $u_1$ or $u_2$ resolves $u_1$ and $u_2$. There are $2(k-1) \geq k$ such vertices for $k \geq 2$.
   c) $u_2 = u_0$. Each vertex $s_i$ for which triple $s_i$ contains $u_1$ resolves $u_1$ and $u_2$, and vertex $a_0$ resolves $u_1$ and $u_2$. Altogether these are exactly $(k-1)+1 = k$ vertices.

9. $u_1 \in \{b_1, \ldots, b_n\}$ and $u_2 \not\in L \cup L_{\text{root}} \cup \{s_1, \ldots, s_{m'}\}$. (as in case 8)

10. $u_1 \in \{c_1, \ldots, c_n\}$ and $u_2 \not\in L \cup L_{\text{root}} \cup \{s_1, \ldots, s_{m'}\}$. (as in case 8)

11. $u_1, u_2 \in \{a_0, b_0, c_0\}$. Each of the $k$ vertices of $L_{v_A}, L_{v_B}$ or $L_{v_C}$ resolves $u_1$ and $u_2$.

Now all pairs of vertices $u_1, u_2$ of $G$ are considered and it is shown that all of them are resolved by at least $k$ vertices from $R$. Note that only the vertex pairs $u_1, u_2 \in \{a_0, \ldots, a_n\}$, $u_1, u_2 \in \{b_0, \ldots, b_n\}$ and $u_1, u_2 \in \{c_0, \ldots, c_n\}$ are not already resolved by $k$ vertices of $L$. Strictly speaking, not a single vertex from $L \cup \{v_A, v_B, v_C, v_0, d_1, \ldots, d_{m'}\}$ resolves such a pair of vertices.

"⇐": Let $R \subseteq V$ be a $k$-resolving set for $G$ with $x = (4+m')k + 3 + (k-1)n$ vertices. By Property P3, $R$ contains all the $(4+m')k$ vertices of $L$. This leaves $3+(k-1)n$ vertices of $R$ that are not in $L$. Let us now consider the vertex pairs $a_0, a_i$, and $b_0, b_i$, and $c_0, c_i$ for $i = 1, \ldots, n$. The vertices of $L$ and the vertices of $\{v_A, v_B, v_C, v_0, d_1, \ldots, d_{m'}\}$ do not resolve these vertex pairs. The only way to resolve these $3n$ vertex pairs at least $k$ times with $3+(k-1)n$ vertices for $n > k \geq 2$, is to use $k-1$ vertices from $\{s_1, \ldots, s_m\}$ that form a $k-1$ matching and the three vertices $a_0, b_0, c_0$. This is the point where it is necessary that $n$ is greater than $k$.

In the introduction of this paper, we mentioned that the $k$-METRIC DIMENSION and the $(k,t)$-METRIC DIMENSION in [EMYRV16] are the same if $t$ is set to the diameter of $G$. Since the constructed graph in Theorem 2 has diameter $2 \cdot \lceil k/2 \rceil + 3$, Theorem 2 also proves the NP-completeness of $(k,t)$-METRIC DIMENSION for bipartite graphs, each $k \geq 2$ and $t \geq 2 \cdot \lceil k/2 \rceil + 3$.

References

[BBS+11] Bača, Martin ; Baskoro, Edy T. ; Salman, A. N. M. ; Saputro, Suhadi W. ; Suprijanto, Djoko: The Metric Dimension of Regular Bipartite Graphs. In: Bulletin mathématiques de la Société des sciences mathématiques de Roumanie 54 (2011), Nr. 1, S. 15–28

[BEE+05] Beervliet, Zuzana ; Eberhard, Felix ; Erlebach, Thomas ; Hall, Alexander ; Hoffmann, Michael ; Mihálik, Mateš ; Ram, L. S.: Network Discovery and Verification. In: Kratsch, Dieter (Hrsg.): Graph-Theoretic Concepts in Computer Science, Springer Berlin Heidelberg, 2005, 127–138

[CEJO00] Chartrand, Gary ; Eroh, Linda ; Johnson, Mark A. ; Oellermann, Ortrud: Resolvability in graphs and the metric dimension of a graph. In: Discrete Applied Mathematics 105 (2000), Nr. 1-3, 99–113. http://dx.doi.org/10.1016/S0166-218X(00)00198-0. – DOI 10.1016/S0166-218X(00)00198-0

[CGH08] Chappell, Glenn G. ; Gimbel, John G. ; Hartman, Chris: Bounds on the metric and partition dimensions of a graph. In: Ars Combinatoria 88 (2008)

[CPZ00] Charrand, Gary ; Poisson, Christopher ; Zhang, Ping: Resolvability and the upper dimension of graphs. In: Computers and Mathematics with Applications 39 (2000), Nr. 12, S. 19–28

[DPSL12] Díaz, Josep ; Pottonen, Olli ; Serna, Maria J. ; Leeuwen, Erik J.: On the Complexity of Metric Dimension. In: Algorithms - ESA 2012 - 20th Annual European Symposium, Ljubljana, Slovenia, September 10-12, 2012. Proceedings, 2012, 419–430

[EMRY13] Estrada-Moreno, Alejandro ; Rodríguez-Velázquez, Juan A. ; Yero, Ismael G.: The k-metric dimension of a graph. In: Applied Mathematics & Information Sciences 9 (2013), 12, Nr. 6, S. 2829–2840. http://dx.doi.org/10.12785/amis/090609. – DOI 10.12785/amis/090609
Figure 2: This graphic illustrates the transformation from 3D2M to 3-MD. The Instance $I$ consisting of $A = \{a_1, \ldots, a_4\}$, $B = \{b_1, \ldots, b_4\}$, $C = \{c_1, \ldots, c_4\}$, $S = \{s_1, \ldots, s_{12}\}$ with $s_1 = (a_2, b_1, c_1)$, $s_2 = (a_3, b_2, c_2)$, $s_3 = (a_2, b_1, c_1)$, $s_4 = (a_1, b_2, c_1)$, $s_5 = (a_4, b_3, c_2)$, $s_6 = (a_1, b_3, c_3)$, $s_7 = (a_2, b_1, c_3)$, $s_8 = (a_1, b_4, c_4)$, $s_9 = (a_3, b_2, c_2)$, $s_{10} = (a_4, b_2, c_4)$, $s_{11} = (a_4, b_3, c_1)$, $s_{12} = (a_4, b_4, c_1)$ for 3D2M is transformed into the graph $G$ and $x = (4 + 4)3 + 3 + (3 - 1)n = 35$. The set of triples $M = \{s_1, s_2, s_6, s_7, s_8, s_9, s_{11}, s_{12}\}$, indicated in the figure by the red lines, is a 2-matching for instance $I$, where $L \cup \{v_0, b_0, c_0\} \cup M$ is a 3-resolving set for $G$ of size $x$. Set $L$ is the set of vertices of the legs attached at the vertices $v_A, v_B, v_C, v_0, d_1, d_2, d_3, d_4$. In the figure, the vertices of $L$ are colored blue.

[EMYRV16] Estrada-Moreno, Alejandro ; Yero, IG ; Rodríguez-Velázquez, JA: On the $(k, t)$-metric dimension of graphs. In: The Computer Journal (2016)

[GJ79] Garey, Michael R. ; Johnson, David S.: Computers and Intractability: A Guide to the Theory of NP-Completeness. W. H. Freeman, 1979. – ISBN 0-7167-1044-7

[Hay77] Hayat, Sakander: Computing distance-based topological descriptors of complex chemical networks: New theoretical techniques. In: Chemical Physics Letters 688 (1977), Nr. 1, 51–58. http://dx.doi.org/10.1016/j.cplett.2017.09.055. – DOI 10.1016/j.cplett.2017.09.055

[HM76] Harary, Frank ; Melter, Robert A.: On the metric dimension of a graph. In: Ars Combinatoria 2 (1976), S. 191–195

[HMP+05] Hernando, M. C. ; Mora, Mercè ; Pelayo, Ignacio M. ; Seara, Carlos ; Cáceres, José ; Puertas, María Luz: On the metric dimension of some families of graphs. In: Electronic Notes in Discrete Mathematics 22 (2005), 129–133. http://dx.doi.org/10.1016/j.endm.2005.06.023. – DOI 10.1016/j.endm.2005.06.023

[HMSW08] Hernando, M. C. ; Mora, Mercè ; Slater, Peter J. ; Wood, David R.: Fault-tolerant metric dimension of graphs. In: Convexity in Discrete Structures 5 (2008), S. 81–85
The complexity of $k$-metric dimension

Schmitz, Vietz, Wanke

[HSV12] Hauptmann, Mathias; Schmied, Richard; Viehmann, Claus: Approximation complexity of Metric Dimension problem. In: Journal of Discrete Algorithms 14 (2012), 214–222. http://dx.doi.org/10.1016/j.jda.2011.12.010. – DOI 10.1016/j.jda.2011.12.010

[HW12] Hoffmann, Stefan; Wanke, Egon: Metric Dimension for Gabriel Unit Disk Graphs Is NP-Complete. In: Algorithms for Sensor Systems, 8th International Symposium on Algorithms for Sensor Systems, Wireless Ad Hoc Networks and Autonomous Mobile Entities, ALGOSENSORS 2012, Ljubljana, Slovenia, September 13–14, 2012. Revised Selected Papers, 2012, 90–92

[IBSS10] Iswadi, H.; Baskoro, Edy T.; Salman, A.N.M.; Simanjuntak, Rinovia: The metric dimension of amalgamation of cycles. In: Far East Journal of Mathematical Sciences (FJMS) 41 (2010), Nr. 1, S. 19–31

[KRR96] Khuller, Samir; Raghavachari, Balaji; Rosenfeld, Azriel: Landmarks in Graphs. In: Discrete Applied Mathematics 70 (1996), Nr. 3, 217–229. http://dx.doi.org/10.1016/0166-218X(95)00106-2. – DOI 10.1016/0166-218X(95)00106-2

[LA06] Liu, Ke; Abu-Ghazaleh, Nael B.: Virtual Coordinates with Backtracking for Void Traversal in Geographic Routing. In: Ad-Hoc, Mobile, and Wireless Networks, 5th International Conference, ADHOC-NOW 2006, Ottawa, Canada, August 17-19, 2006, Proceedings, 2006, 46–59

[MT84] Melter, Robert A.; Tomescu, Ioan: Metric bases in digital geometry. In: Computer Vision, Graphics, and Image Processing 25 (1984), Nr. 1, 113–121. http://dx.doi.org/10.1016/0734-189X(84)90051-3. – DOI 10.1016/0734-189X(84)90051-3

[Sla75] Slater, Peter J.: Leaves of trees. In: Congressum Numerantium 14 (1975), S. 549–559

[ST04] Szabó, András; Tannier, Eric: On Metric Generators of Graphs. In: Mathematics of Operations Research 29 (2004), Nr. 2, 383–393. http://dx.doi.org/10.1287/moor.1030.0070. – DOI 10.1287/moor.1030.0070

[YER17] Yero, Ismael G.; Estrada-Moreno, Alejandro; Rodríguez-Velázquez, Juan A.: Computing the k-metric dimension of graphs. In: Applied Mathematics and Computation 300 (2017), 60–69. http://dx.doi.org/10.1016/j.amc.2016.12.005. – DOI 10.1016/j.amc.2016.12.005