Nucleon quasi-Parton Distributions in the large $N_c$ limit

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In this letter, we investigate the nucleon quasi-parton distribution functions in the chiral quark soliton model. We derive a set of sum-rules depending on the velocity of the nucleon and on the Dirac matrix defining the distribution functions. We present numerical results for the isosinglet unpolarized distribution, in which we find that the anti-quark distribution breaks the positivity condition at nucleon velocities of $v \approx 0.99 \left( P_N \approx 7.0 M_N \right)$ and smaller. We found that, for the isosinglet unpolarized case, a large nucleon momentum is required for the quasi-parton distribution to get close enough to the usual parton distribution function.

I. INTRODUCTION

The parton distribution functions (PDFs) are of great importance as they provide information about the underlying structure of the hadrons. In 2013 Xiangdong Ji [1] suggested that one can approach the PDFs asymptotically from the Euclidean region, starting from nucleon matrix elements of bilinear quark (gluon) operators with fields separated by space-like distance (the quasi-parton distribution function (quasi-PDF)), and then boosting the nucleon state to large momentum. It has been followed by numerous researches to elaborate the idea and important lattice results have been obtained [2–10]. The detailed status on both the theory and the lattice simulations can be found in the recent review [11] and in the community white paper, Ref. [12].

Switching our viewpoint from the lattice, one may utilise the chiral effective models to provide the initial values of the QCD evolution for the (quasi-)PDFs at a low renormalization point. For the model calculations which were recently made in the context of the quasi-PDFs, we refer to [13–16].

A sound model calculation of the PDFs has to fulfill the key criteria such as the sum-rules and the positivity. Diakonov et al. [17, 18] achieved such satisfactory description of the nucleon PDFs adopting the chiral quark soliton model [19]. Here, we closely follow those works where the authors already in year 1997 introduced the object identical to the quasi-distributions but concentrated on the limiting case to study the usual nucleon PDFs [18]. Here we simply compute them for the case where the nucleon has finite momentum to investigate the properties of the quasi-PDFs such as the sum-rules and the momentum evolutions.

In this letter, we focus on the leading-twist quasi-PDFs, the isosinglet unpolarized $u(x,v) + d(x,v)$ and the isovector polarized $\Delta u(x,v) - \Delta d(x,v)$ distributions with the nucleon velocity $v$ and the corresponding ones for the anti-quarks. They are the leading components in the large $N_c$ approximation, and of particular interest as being related to the fundamental sum-rules.

Firstly, we dedicate a section to sketch the model. Next we derive the model expressions of the quasi-PDFs from the definitions of the quark and anti-quark quasi-densities. In the following section we calculate the first and next order Mellin moments and obtain the corresponding sum-rules. After deriving the interpolation formula for the quasi-PDFs we present the numerical results on the isosinglet unpolarized quark and anti-quark distributions and discuss their characteristics. Finally, we come to the conclusion where we summarize the present work and provide future perspectives.

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$^*$ Instead of the nucleon momentum $P^+$, we take the nucleon velocity $v$ as the parameter describing approach to the light-cone. We find it much simpler and transparent to express our formulae and to discuss the results. Conversion can be easily made with help of equation: $P^+ = M_N v / \sqrt{1 - v^2}$. 

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II. NUCLEONS AS A SOLITON IN THE LARGE $N_c$ LIMIT

We begin by describing the model framework briefly. The detailed procedure and the formulae can be found in the original paper Ref. [19] or in reviews Refs. [20, 21].

Our starting point is the following effective action in the large $N_c$,

$$
\exp(iS_{\text{eff}}[\pi(x)]) = \int D\psi D\bar{\psi} \exp \left( i \int d^4x \bar{\psi}(i\gamma^\mu \partial_\mu - MU^\gamma)\psi \right),
$$

$$
U^\gamma(x) = \frac{1 + \gamma_5}{2} U(x) + \frac{1 - \gamma_5}{2} U(x)^\dagger, \quad U(x) = \exp(i\pi^a(x)x^a). \tag{2}
$$

Here, $\psi$ is the quark field and $M$ is the dynamical quark mass, which is, in general, momentum dependent. Note that the above expression can be derived form the QCD at low renormalization point where the vacuum is dominated by the instanton configurations [22, 23]. In such a picture, the model scale is naturally given by the inverse of the average instanton size $1/\rho \sim 600$ MeV and the dynamical quark mass is momentum dependent. It is a reasonable approximation to switch off such momentum dependence by assuming the low quark virtuality. This makes the model calculations dramatically easier with still reasonable results in most cases, we simply need to introduce an artificial regularization scheme as a payoff.

We introduce the hedgehog ansatz for the pion field,

$$
\pi^a(x)x^a = \frac{x^a x^a}{|x|} P(r). \tag{3}
$$

$P(r)$ depends only on the distance $r = |x|$ and is called the pion profile function. Introducing the ansatz, we treat the quark-pion interaction by a mean-field approach. The Dirac equation can be solved to obtain the quark spectra

$$
H\Phi_n(x, t) = E_n\Phi_n(x, t), \tag{4}
$$

with the Dirac Hamiltonian,

$$
H(U) = -i\gamma^\mu \partial_\mu + M\gamma^0 U^\gamma. \tag{5}
$$

Note that, the hedgehog ansatz [3] breaks the individual rotational symmetries in the total angular momentum $J$ and the isospin $\tau$. Instead we have the grandspin $K = J + \tau$ as good quantum number as well as the parity $P = (-1)^{K, K+1}$, so called the hedgehog symmetry. Among the Dirac spectra, there exists a distinct level with quantum number $K^P = 0^+$ that emerges from the upper continuum as the pion mean-field is turned on. We label this as the bound level. The baryon number of the nucleon is given by the $N_c$ quarks in the bound level. When the pion mean-field gets even stronger, eventually the bound level falls into the negative continuum and the Skyrme picture of the nucleon is applied.

By calculating the nucleon-nucleon correlation function and passing it to the large Euclidean time, one can find the classical soliton energy,

$$
M_{cl} = N_c E_{\text{level}} + N_c \sum_{occ} (E_n - E^0_n)^{\text{reg}}. \tag{6}
$$

In the above equation, the first term corresponds to the contribution of the $N_c$ quarks in the bound level and the second term corresponds to that of the continuum part with the regularization ($\text{reg}$). For the continuum part, the summation is over the negative (occupied) energy levels. One can write this in terms of the summation over the upper continuum (non-occupied) levels by using the traceless nature of the Dirac Hamiltonian. We identify the classical soliton energy as the nucleon mass in this work, $M_N = M_{cl}$.

III. NUCLEON QUASI-PARTON DISTRIBUTIONS IN THE $\chi$QSM

Following Ref. [13], we define the quark quasi-densities in the nucleon as \[\dagger\]

$$
D_f(x, v) = \frac{1}{2E_N} \int \frac{d^4k}{(2\pi)^3} \delta \left( x - \frac{k^3}{P_N} \right) \int d^3x \ e^{-ik \cdot x} \langle N_v | \bar{\psi}_f \left( \frac{x}{2}, t \right) \Gamma \psi_f \left( \frac{x}{2}, t \right) | N_v \rangle, \tag{7}
$$

\[\dagger\] The quasi-densities of Ref. [18] coincide with quasi-distributions introduced afterwards in Ref. [1].
and for the antiquarks

$$D_f(x,v) = \frac{1}{2E_N} \int \frac{d^3k}{(2\pi)^3} \delta \left( x - \frac{k^3}{P_N} \right) \int d^3x \ e^{-ik \cdot x} \langle N_v | \text{Tr} \left[ \Gamma \psi_f \left( -\frac{x}{2}, t \right) \bar{\psi}_f \left( \frac{x}{2}, t \right) \right] | N_v \rangle.$$  \quad (8)

In the above formulae the path ordered exponential is assumed. $E_N$ and $P_N$ are the energy and momentum of the nucleon moving with the velocity $v$

$$E_N = \frac{M_N}{\sqrt{1 - v^2}}, \quad P_N = \frac{M_N v}{\sqrt{1 - v^2}},$$

and $| N_v \rangle$ is the corresponding nucleon state. The spin matrix $\Gamma$ depends on the particular distribution one is interested in. For example, for the distribution of quarks polarized along or against the direction of the nucleon velocity one can use:

$$\Gamma = \gamma^0 \frac{1 \pm \gamma_5}{2} \quad \text{or} \quad \Gamma = \gamma^3 \frac{1 \pm \gamma_5}{2},$$

or any of their linear combinations with proper normalization.

In the large $N_c$ limit the nucleon matrix element entering Eqs. (7) and (8) can be expressed in terms of the mean-field Green’s function [18] which can be computed as:

$$\langle N_v | \{ \psi(\vec{x}_1, t_1) \bar{\psi}(\vec{x}_2, t_2) \} | N_v \rangle = -S[\vec{v}] \left[ \Theta(t_2 - t_1) \sum_{\text{occ}} \Phi_n(\vec{x}_1) \Phi_n^\dagger(\vec{x}_2) \gamma_0 \exp(-iE_n(t'_1 - t'_2)) \right.$$

$$\left. - \Theta(t_1 - t_2) \sum_{\text{noocc}} \Phi_n(\vec{x}_1) \Phi_n^\dagger(\vec{x}_2) \gamma_0 \exp(-iE_n(t'_1 - t'_2)) \right] S^{-1}[\vec{v}],$$

where $\sum_{\text{occ}}$ and $\sum_{\text{noocc}}$ represent the summation over the occupied and the non-occupied Dirac levels, respectively. The $t'$ and $\vec{x}'$ are Lorentz transforms of the space-time coordinates:

$$\vec{x}'_{1,2} = \vec{x}_{1,2} - \vec{v} t_{1,2}, \quad t'_{1,2} = t_{1,2} + \vec{v} \cdot \vec{x}_{1,2} \sqrt{1 - v^2}.$$  \quad (10)

Eventually, $S[\vec{v}]$ is the Lorentz transformation matrix acting on the quark spinor indices:

$$S[\vec{v}] = \exp \left( \frac{1}{4} [\gamma^0, \gamma^3] \omega \right), \quad \text{th}(\omega) = v.$$  \quad (11)

Using Eq. (9) for Eqs. (7) and (8) and representing the quark states in the momentum space, we obtain the expressions for the quasi-densities in the large $N_c$ limit:

$$D_i(x,v) = N_c M_N v \sum_{\text{occ}} \int \frac{d^3k}{(2\pi)^3} \delta(k^3 + vE_n - vM_N x) \left[ \Phi_n^\dagger(\vec{k}) (1 + v\gamma^0 \gamma^3) \gamma_0 \Gamma_i \Phi_n(\vec{k}) \right],$$

$$\dot{D}_i(x,v) = -N_c M_N v \sum_{\text{oocc}} \int \frac{d^3k}{(2\pi)^3} \delta(k^3 + vE_n + vM_N x) \left[ \Phi_n^\dagger(\vec{k}) (1 + v\gamma^0 \gamma^3) \gamma_0 \Gamma_i \Phi_n(\vec{k}) \right],$$

in terms of quark orbitals [4] in the meson mean-field. The index $i$ above combines the flavour and helicity indices of quarks. Using the anti-commutativity of the quark fields at equal time, it is possible to represent the densities equivalently by the summation over the non-occupied Dirac levels,

$$\bar{D}_i(x,v) = -N_c M_N v \sum_{\text{noocc}} \int \frac{d^3k}{(2\pi)^3} \delta(k^3 + vE_n - vM_N x) \left[ \Phi_n^\dagger(\vec{k}) (1 + v\gamma^0 \gamma^3) \gamma_0 \Gamma_i \Phi_n(\vec{k}) \right],$$

$$\bar{D}_i(x,v) = N_c M_N v \sum_{\text{noocc}} \int \frac{d^3k}{(2\pi)^3} \delta(k^3 + vE_n + vM_N x) \left[ \Phi_n^\dagger(\vec{k}) (1 + v\gamma^0 \gamma^3) \gamma_0 \Gamma_i \Phi_n(\vec{k}) \right].$$

Using the anti-commutativity of the quark fields at equal time, it is possible to represent the densities equivalently by the summation over the non-occupied Dirac levels,
In the above expressions, we traced out the colour space and obtained the overall factor of $N_c$. Notice as well that the factor of nucleon mass $M_N$ is also order of $N_c$. One has to keep in mind that it is necessary to take the vacuum subtraction $-(H \rightarrow H_0)$ where $H_0$ is the free Dirac Hamiltonian. This will be consistently omitted in the following formulae for brevity and will be mentioned if required.

Now one has to take certain combinations of (12) and (13) to get the desired quasi-parton distribution functions. The leading distributions at large $N_c$ are isoscalar unpolarized $(u+d)$ and isovector polarized $(\Delta u - \Delta d)$ ones. For the isosinglet unpolarized quasi-distribution one has to take average over the flavour space and sum up the polarizations:

$$\sum_f q_f(x,v) = N_c M_N v \sum_{occ} \int \frac{d^3 k}{(2\pi)^3} \delta(k^3 + v E_n - v M_N x) \Phi^d_n(\vec{k})(1 + v \gamma^0 \gamma^3) \gamma^0 \Gamma \Phi_n(\vec{k}).$$  \hspace{1cm} (16)

In this expression, we absorbed the anti-quark distribution using the identity $\bar{q}(x,v) = -q(-x,v)$. Thus the variable $x$ ranges from negative to positive infinity.

For the isovector polarized quark and anti-quark quasi-distribution, we obtain the following with the identity $\Delta \bar{q}(x,v) = \Delta q(-x,v)$.

$$\Delta q_f(x,v) = -\frac{2}{3}(T_3)_{ff} N_c M_N v \sum_{occ} \int \frac{d^3 k}{(2\pi)^3} \tau^3 \delta(k^3 + v E_n - v M_N x) \Phi^u_n(\vec{k})(1 + v \gamma^0 \gamma^3) \gamma^0 \Gamma \gamma^3 \Phi_n(\vec{k}).$$  \hspace{1cm} (17)

Here, $T_3 = \text{diag}(1/2, -1/2)$ is the proton isospin matrix. Again, the above expressions can be equally represented as a summation over the non-occupied states with opposite sign.

The Dirac matrix $\Gamma$ in (16) and (17) is any linear combination of $\gamma^0$ and $\gamma^3$. Note that there is no unique definition of the quasi-PDFs, one can use any Dirac structures as far as they provide the correct limit to the usual PDFs. In the next section, we work with the both $\gamma^0$ and $\gamma^3$ to examine the sum-rules.

### IV. SUM-RULES

Discussing the sum-rules, it is more convenient to use the frequency representation instead of the discrete summation using the following identity

$$\sum_{occ} |n\rangle \langle n| \exp(-iz^0E_n) = \int_{-\infty}^{E_{level} + \epsilon} dw \delta(w - H) \exp(-iz^0w).$$  \hspace{1cm} (18)

The leading Mellin moment of the distribution (12) can be written as

$$\int_{-\infty}^{\infty} dx \sum_f q(x,v) = N_c \int_{-\infty}^{E_{level} + \epsilon} dw \text{Sp}[\delta(w - H)(1 + v \gamma^0 \gamma^3) \gamma^0 \Gamma] - (H \rightarrow H^0)$$

$$= N_c \text{Sp}[\Theta(-H + E_{level} + \epsilon)(1 + v \gamma^0 \gamma^3) \gamma^0 \Gamma] - (H \rightarrow H^0).$$  \hspace{1cm} (19)

One can show that the term proportional to $\gamma^0 \gamma^3$ inside the trace vanishes with the hedgehog symmetry. Then we are left with the following trace

$$\text{Sp}[\Theta(-H + E_{level} + \epsilon) - \Theta(-H_0)],$$  \hspace{1cm} (20)

which is nothing but the baryon number $B$ as it counts the number of the quarks occupying the negative energy levels and the bound level subtracted by that of the negative levels with the vacuum Hamiltonian. Thus we obtain

$$\int_0^{\infty} dx (q(x,v) - \bar{q}(x,v)) = \begin{cases} N_c B & \Gamma = \gamma^0 \\ v N_c B & \Gamma = \gamma^3. \end{cases}$$  \hspace{1cm} (21)

Similarly, from the next-order Mellin moment, we obtain the momentum sum-rule:

$$\int_0^{\infty} dx x(q(x,v) + \bar{q}(x,v)) = \begin{cases} 1 & \Gamma = \gamma^0 \\ v & \Gamma = \gamma^3. \end{cases}$$  \hspace{1cm} (22)
Deriving the momentum sum-rule we use the identity $\text{Sp}[\Theta(-H + E_{\text{level}} + \epsilon)\gamma^0\gamma^3P^3] = 0$, which can be proven using the saddle point equation of the effective action \[17\]. In that case, the chiral field should minimize the energy functional and the sum-rule is only satisfied strictly with such solution. Note the nucleon momentum is soley carried by the quarks being the only effective degrees of freedom in the model picture.

Finally, for the isovector polarized distribution, we obtain the Bjorken sum-rule,

$$\int_0^\infty dx \left( \Delta u(x,v) - \Delta d(x,v) + \Delta \bar{u}(x,v) - \Delta \bar{d}(x,v) \right) = \begin{cases} v g_A & \Gamma = \gamma^0 \\ g_A & \Gamma = \gamma^3 \end{cases},$$

(23)

where the nucleon axial charge $g_A$ has the following expression

$$g_A = -\frac{N_c}{3} \int_{-\infty}^{E_{\text{level}}+\epsilon} dw \text{Sp}[\delta(w-H)\tau^3\gamma^0\gamma^3\gamma_5] - (H \to H_0).$$

(24)

Equations (21), (22) and (23) are the generalized version of the usual baryon number-, momentum-, and Bjorken sum-rules of the nucleon. Note that two choices of the Dirac matrix $\Gamma = \gamma^0$ and $\gamma^3$ results in different velocity dependence for the sum-rules. Clearly, in the limit $v \to 1$ they become the usual sum-rules.

Although the proofs here are given within the model framework, the sum-rules can be understood from more general point of view. Taking the Mellin moments, the non-locality of the quark bilinear in the matrix elements is lifted and in general we are left with the charges of the symmetry currents. For example, the zeroth component of the vector current is just the number density. With the conservation of the vector current, the spatial component($\gamma^3$) should be proportional to the ‘velocity’ by virtue of the continuity. For the momentum sum-rule, different choices of the operator $\Gamma = \gamma^0$ and $\gamma^3$ correspond to taking different components of the energy-momentum tensor matrix element, $\sim T^{00}$ and $T^{33}$, respectively.

In general case the momentum sum rule (22) has the following form (see, e.g. \[15\]):

$$\int_0^\infty dx x(q(x,v) + \bar{q}(x,v)) = \begin{cases} M_Q^2 v M_Q^2 - c^0(0) \frac{1-x^2}{v} & \Gamma = \gamma^0 \\ \Gamma = \gamma^3 \end{cases}.$$

(25)

where $M_Q^2$ is the momentum fraction carried by the quarks in the nucleon, $c^0(t)$ is the form factor of the energy-momentum tensor. The form factor $c^0(t)$ is zero in the chiral quark-soliton model (as it appears in the next to leading order in the instanton density, see \[24\]), also $M_Q^2 = 1$ in the model. Taking this into account, one can reproduce Eq. (22) in the chiral quark-soliton model.

In the case of the Bjorken sum-rule, $\Gamma = \gamma^3$ corresponds to the third component of the nucleon spin $S^3$ times the axial charge $g_A$ whereas the $\Gamma = \gamma^0$ case can be related to $\vec{S} \cdot \vec{v} g_A$.

V. QUASI-PARTON DISTRIBUTIONS IN TERMS OF THE PION MEAN-FIELD

While the quasi-PDFs \[16\] and \[17\] can be computed directly by calculating the Dirac spectra, it often takes significant computational time and techniques. Here we suggest taking further approximation which makes the numerical computation pretty simple and is feasible to inspect the structure of the divergences.

One can expand the real part of the effective action \[1\], taking $pM(U-1)/(p^2 + M^2)$ as the small expansion parameter where $p$ is the characteristic pion momentum. Such expansion is valid for three limiting cases: when the pion field is small, when the pion momentum is small $p \ll M$, and when the pion momentum is large $p \gg M$. This is called the interpolation formula \[19\] and provides a reasonable result compared to the full calculation even only when the lowest order is considered, for example look Ref. \[17\].

‡ For its definition and discussion of its properties see, e.g. Ref. \[24\]
Accordingly, the isosinglet unpolarized distribution can be expanded as below

\[ \sum f q_f(x, v) = \sum_{m=0}^{\infty} \left[ \sum f q_f(x, v) \right]^{(m)}, \tag{26} \]

with

\[ \left[ \sum f q_f(x, v) \right]^{(m)} = \frac{N_c M_N}{T} \text{Im} \text{Tr} \left[ (i\dot{\phi} + M U^{-\gamma_5})(-1)^m \left[ (-\partial^2 - M^2 + i\epsilon)^{-1} iM(\dot{\theta} U^{-\gamma_5}) \right]^m \right. \]

\[ \left. (-\partial^2 - M^2 + i\epsilon)^{-1} \delta(i\phi - xM_N) \right] - (U \rightarrow 1), \tag{27} \]

where \( \text{Tr} \) is the functional trace and \( T \) denotes the Euclidean time which drops in the final result. Here we defined the quasi-light vectors for brevity

\[ \bar{n} = (1, 0, 0, -v), \tag{28} \]
\[ n = (1, 0, 0, -1/v). \tag{29} \]

The \( m = 0 \) term in (27) is trivially zero. The leading order \( m = 1 \) can be written as follows in the momentum representation

\[ \left[ \sum f q_f(x, v) \right]^{(1)} = -N_c M_N M^2 \text{Im} \int \frac{d^4k}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} \left[ (p + k)^2 - M^2 + i\epsilon)^{-1} (p^2 - M^2 + i\epsilon)^{-1} \right. \]

\[ \delta(n \cdot p - xM_N)(\bar{n} \cdot k) \text{Sp}[\tilde{U}(\vec{k})^\dagger \tilde{U}(\vec{k})], \tag{30} \]

where \( k^\mu = (0, \vec{k}) \) and the Fourier transform of the pion mean-field is defined as follows

\[ \tilde{U}(\vec{k}) = \int d^3xe^{-i\vec{k} \cdot \vec{x}} [U(\vec{x}) - 1]. \tag{31} \]

To calculate the momentum \( p^- \)-integral, we use the Sudakov decomposition,

\[ \int \frac{d^4p}{(2\pi)^4} = \frac{1}{2} \int \frac{dp^+}{2\pi} \frac{dp^-}{2\pi} \frac{d^2p_\perp}{(2\pi)^2}. \tag{32} \]

We first integrate over \( p^+ \) using the \( \delta \)-function and then perform the \( p^- \) integral. We present a detailed analysis on the poles in \( p^- \) in Appendix A. We check that the quark-loop momentum integral and the limit \( v \rightarrow 1 \) are commutable, i.e. correct limit of the quasi-PDFs to PDFs is achieved. We also observe that the remaining integration over \( p_\perp \) is logarithmically divergent. In Refs. [17, 18] the authors introduced the Pauli-Villars as one of the ‘good’ regularization method preserving the required properties of the PDFs, such as the positivity and sum-rules. For the numerical calculation of the quasi-PDFs, we follow the same strategy: the Pauli-Villars regularization with single subtraction.

\[ IV. \text{ NUMERICAL RESULTS AND DISCUSSIONS} \]

For the numerical results, we focus only on the isosinglet unpolarized quasi-distribution \( u(x, v) + d(x, v) \) with \( \Gamma = \gamma^0 \). While the bound level contribution is calculated exactly by solving the Dirac Hamiltonian, we use the interpolation formula (26) and (27) up to \( m = 2 \) for the continuum part. The full calculation including the isovector polarized with the both Dirac matrices \( \Gamma = \gamma^0 \) and \( \Gamma = \gamma^3 \) will be covered in another publication.

For simplicity in the computation procedure, we use the following ansatz for the pion mean-field

\[ P(r) = -2 \text{arctg} \left( \frac{r}{r_0} \right), \tag{33} \]
with $r_0 \approx 1.0/M$. This choice of the pion mean-field has been used to successfully describe various nucleon observables and typical deviation on the results between using the ansatz (33) and the self-consistent solution which minimizes the energy functional is known to be up to around 10% [17–19, 21].

As discussed in the previous section, we adopt the Pauli-Villars scheme with single subtraction to tame the logarithmic divergences in the quark-loop momentum integral. The Pauli-Villars mass $M_{PV}$ is determined by using the pion decay constant,

$$F_\pi^2 = \frac{N_c M^2}{4\pi^2} \ln \frac{M_{PV}^2}{M^2}. \quad (34)$$

Using $F_\pi = 93$ MeV and $M = 350$ MeV, we obtain $M_{PV} = 560$ MeV. At the same time, we obtain the nucleon mass $M_N \approx 1.15$ GeV which will be used in the numerical analysis.

In Fig. 1, the bound level (left) and continuum (right) contributions to the isosinglet unpolarized distribution are displayed for different $v$ values. While the bound level part shows mild dependence on the nucleon velocity, the continuum part changes significantly rapidly at small $x$ region. Already for $v = 0.999$, which corresponds to $P_N \approx 22M_N$ in the nucleon momentum, the distribution is a half of the $v = 1$ distribution at $x = 0$. Such distinct behaviour of the bound and continuum contribution leads to an interesting consequence. See Fig. 2 where the total quark (left) and anti-quark (right) distributions are shown. Note that the anti-quark distributions become negative at the nucleon velocity $v = 0.99$ ($P_N \approx 7.0M_N$) and smaller. As the bound level negatively contributes to the anti-quark distribution, the role of the positive and sizable contribution from the continuum part is essential to guarantee the positiveness for the usual PDFs [17, 18]. In our case, the positiveness is not required for the quasi-PDFs as the nucleon is off the light-cone where the probability interpretation is not valid. Indeed we observe that the positivity condition is not satisfied. In Ref. [13], using the diquark spectator model, the authors also found a breakdown of the Soffer positivity condition for the quark quasi-PDFs.

We check numerically that the baryon number sum-rule is well satisfied. On the other hand for the momentum sum-rule, we obtain $M_2 \approx 0.95$, which is 5% deviated from the correct value. The reason is simple: we use the ansatz (33) which is not the true solution which minimizes the action. Note that we derived the momentum sum-rule using the equation of motion for the chiral field $\delta U_{S_{\text{eff}}} = 0$. Thus we expect that the sum-rule will be fulfilled when we use the self-consistent pion mean-field [25]. Nevertheless we check that the $v$-dependence of the both baryon number and momentum sum-rules [21] and [22] are satisfied: they are independent on $v$ for $\Gamma = \gamma_0$.

![FIG. 1. The bound level (left) and continuum (right) parts of the isosinglet unpolarized distribution. For the continuum part, $\bar{q}(x,v) = q(x,v)$.](image)

Several studies in framework of the diquark model [15] [16] and studies of renormalon contribution to quasi-PDFs [26] show that the quasi-PDFs differ strongly from PDFs in the vicinity of $x = 1$. In our model we also observe the strong difference between quasi-PDFs and PDFs in the vicinity of $x = 1$. The corresponding ratio quasi-PDF($x$)/PDF($x$) $\sim \exp(\text{const}(1-v)N_c)$ is exponentially large for $x \sim 1$ in the large $N_c$ limit.
VII. SUMMARY AND OUTLOOK

We demonstrated how to calculate the nucleon quasi-parton distribution functions within the framework of the chiral quark soliton model. We found the sum-rules for the isosinglet unpolarized and isovector polarized distributions which generalize the usual ones for the nucleon PDFs. It is intriguing that the operators defining the matrix elements $\Gamma = \gamma^0, \gamma^3$ exhibit different nucleon velocity dependencies of the sum-rules. Although they are derived in a model approach, we argued that they can be understood in the general context, taking the notion of the local QCD operators and their components. Finally we discussed the numerical results on the quark and anti-quark isosinglet unpolarized distributions. We observed that in particular, the continuum contribution at small $x$ has sharp dependence on $v$ which consequently leads the anti-quark distribution to be negative at $v \sim 0.99$ ($P_N \approx 7.0 M_N$) and smaller. We would like to spotlight the observation that the nucleon is required to have quite large momentum so that its isosinglet unpolarized quasi-PDFs to be close enough to the usual PDFs, due to the rapidly varying continuum contribution with respect to the nucleon velocity.

We stress that the numerical test of the momentum sum-rule is around 5% underestimated when using the ansatz [33]. The sum-rule is satisfied only when the self-consistent mean-field is used. To numerically test the momentum sum-rule with correct velocity dependencies for the both cases $\Gamma = \gamma^0$ and $\gamma^3$, it is indeed required to perform the computation using the self-consistent profile and the full calculation instead of using the interpolation formula. In the following work, we plan to provide the both polarized and unpolarized leading $N_c$ quasi distributions with the full continuum calculation evaluated at the saddle point.

Apart from the practical usage of the quasi-PDFs on the lattice, Radyushkin suggested that they have deeper theoretical ground [27, 28]. For instance, the quasi-PDF is related to the transverse momentum distributions(TMDs) and Ioffe time pseudo-parton distribution functions [14, 28]. These new ideas were already tested in the lattice simulations [29, 32]. This will be an interesting future subject to look into carefully how the transformations between various distributions are realised in the soliton picture of baryons.

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Appendix A: Determination of the poles

Let us write the $p$ integral in (30) as

$$I \equiv \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p+k)^2 - M^2 + i\epsilon} \frac{1}{p^2 - M^2 + i\epsilon} (\vec{n} \cdot p) \delta(n \cdot p - xM_N)$$

$$= \int \frac{dp^+ dp^- d^2\vec{p}}{2(2\pi)^4} \frac{1}{(p+k)^2 - M^2 + i\epsilon} \frac{1}{p^2 - M^2 + i\epsilon} (\vec{n} \cdot p) \delta(n \cdot p - xM_N).$$

Taking the limit $v \to 1$ before the integral and then performing the $p^+$ integration with the $\delta-$function, we find the following poles on the complex $p^-$ plane.

$$p_1^+ = \frac{k^3(k^3 + xM_N) + M^2_{p_k}}{k^3 + xM_N} - i\epsilon \text{sign}(k^3 + xM_N), \quad (A1)$$

$$p_2^- = \frac{M^2_p}{xM_N} - i\epsilon \text{sign}(x). \quad (A2)$$

Here the notation $M^2_{p_{k,k}} = M^2 + (\vec{p} + \ldots + \vec{k})^2$ is introduced for convenience. We obtain an important condition for the integral not to vanish,

$$\text{sign}(x)\text{sign}(k^3 + xM_N) < 0. \quad (A3)$$

When we try to integrate before taking $v \to 1$, we find the poles at

$$p_{11}^+ = \frac{1}{v-1} \left( (k_3 + xM_N)v + \sqrt{(k_3 + xvM_N)^2 + (1 - v^2)M^2_{p_k}} \right) + i\frac{1 + v}{2\sqrt{(k_3 + xvM_N)^2 + (1 - v^2)M^2_{p_k}}} \quad (A4)$$

$$p_{12}^- = \frac{1}{1 - v} \left( (k_3 + xM_N)v - \sqrt{(k_3 + xvM_N)^2 + (1 - v^2)M^2_{p_k}} \right) - i\frac{1 + v}{2\sqrt{(k_3 + xvM_N)^2 + (1 - v^2)M^2_{p_k}}} \quad (A5)$$

$$p_{21}^- = \frac{1}{1 - v} \left( xM_Nv + \sqrt{(xvM_N)^2 + (1 - v^2)M^2_p} \right) + i\frac{1 + v}{2\sqrt{(xvM_N)^2 + (1 - v^2)M^2_p}} \quad (A6)$$

$$p_{22}^- = \frac{1}{1 - v} \left( xM_Nv - \sqrt{(xvM_N)^2 + (1 - v^2)M^2_p} \right) - i\frac{1 + v}{2\sqrt{(xvM_N)^2 + (1 - v^2)M^2_p}} \quad (A7)$$

One important requirement for the integral is that we must recover the result of the corresponding integral for $v = 1$. This proper limit to the PDF is achieved by closing the contour using the upper or half semi-circle, equivalently. In any case, one of the two poles included in the contour approaches to infinity and does not contribute to the integral in the limit. Note that two zeros from the same denominator $p_{11}^+$ and $p_{12}^-$ ($p_{21}^-$ and $p_{22}^-$) are always separated by the real axis. Moreover, the poles do not cross the real axis and thus the requirement (A3) is satisfied.

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