Closed-form solutions of Lucas-Uzawa model with externalities via partial Hamiltonian approach

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Abstract

In this paper, we establish multiple closed-form solutions for all the variables in the Lucas-Uzawa model with externalities for the case with no parameter restrictions as well as for cases with specific parameter restrictions. These multiple solutions are derived with the help of the results derived in Naz et al (2016); Naz and Chaudhry (2017). This multiplicity of solutions is new to the economic growth literature on Lucas-Uzawa model with externalities. After finding solutions for the Lucas-Uzawa model with externalities, we use these solutions to derive the growth rates of all the variables in the system which enables us to fully describe the dynamics of the model. The multiple solutions can potentially explain why some countries economically overtake other countries even though they start from the same initial conditions.

Keywords: Economic growth; Multiplicity; Lucas-Uzawa model with externalities; Partial Hamiltonian approach; Current-value Hamiltonian.

1 Introduction

Over the last few decades, two strands of the literature looking at economic growth have been brought together to explain the factors that affect long run growth. So the neoclassical models of economic growth, which were based on the assumption of competitive markets, were adapted to incorporate the theories of human capital, which looked at the private benefits of education. The first way in which education was incorporated into economic growth theories was by incorporating human capital as a factor input which has a direct impact on output (see Mankiw et al \cite{1} and Romer \cite{2}). The second way in which education was incorporated into economic growth models was to include knowledge accumulation into growth models through human capital accumulation or through research and development activities (See Romer \cite{2,3}, Lucas \cite{4} and Uzawa \cite{5}).

What differentiated many of these models from traditional neoclassical growth models was the way in which knowledge accumulation was used to endogenize the factors that explain long run economic growth. These models extended the early growth models which were characterized by exogenous productivity shocks
and the way endogeneity was introduced was through the presence of knowledge externalities (See Romer [2]) in the economy-wide production function which lead to increasing returns to scale. Another way of introducing endogenous growth was developed by Lucas [4] in the well-know Lucas-Uzawa model in which human capital was incorporated into the production function in which the average level of human capital in the economy affects individual firm level output but is not taken into account in each firm's profit maximizing decisions. Tamura [6] also developed a model of economic growth which analyzed a human capital externality in the production of human capital itself.

The basic Lucas-Uzawa model was solved using simplification methods: Some authors have solved the basic Lucas-Uzawa model using dimension reduction techniques (see Benhabib and Perli [7], Caballe and Santos [8]) or time elimination methods (see Mulligan and Sala-i-Martin [9]), but these methods fail to provide explicit solutions for the dynamics of the original models. Other authors have used methods like parameter restrictions or hypergeometric functions to find closed-form solutions of the original Lucas-Uzawa model (see Xie [10]; Boucekkine and Ruiz-Tamarit [11]). Recently Naz et al [12] established closed-form solutions for the basic Lucas-Uzawa model by using their newly developed partial Hamiltonian approach [13, 14] with a specific parameter restriction. The partial Hamiltonian approach partial Hamiltonian approach [13, 14] is a significant contribution in Lie group theoretical methods. The interested reader is referred to read some interesting papers utilizing Lie group theoretical methods in solving real world applications (e.g. [18] - [24]). Naz and Chaudhry [15] then constructed completely new multiple closed-form solutions for the basic Lucas-Uzawa model with no parameter restrictions using partial Hamiltonian approach [13, 14]. Moreover, Naz and Chaudhry [15] also compared the closed-form solutions derived via partial Hamiltonian approach with the solutions derived from the classical approach.

Ruiz-Tamarit [16] have derived only one closed-form solution of Lucas-Uzawa model with externalities under the assumption that the capital share equals the inverse of the intertemporal elasticity of substitution i.e. \( \sigma = \beta \). Hiraguchi [17] transformed the Lucas-Uzawa model with externalities into the basic Lucas-Uzawa model and then derived a unique closed-form solution directly from Boucekkine and Ruiz-Tamarit [11] for fairly general values of the parameters of the model.

One important issue that arises is the role of growth models like the Lucas-Uzawa model in explaining differences in cross country growth rates especially between countries that start from the same initial conditions. Though the Lucas-Uzawa model illustrates the key role of human capital accumulation for endogenous growth, the basic model may not explain how some countries overtake other countries over time. This why the issue of multiplicity of equilibrium paths is important in the context of growth models in general and the Lucas-Uzawa model in particular.

This paper is distinct from the previous literature in that it establishes multiple closed-form solutions for the Lucas-Uzawa model with externalities. This is the first time in the literature that multiple closed-form solutions have been
found for the Lucas-Uzawa model with externalities and this multiplicity is critical in understanding how certain countries may overtake other countries in the growth process. The closed-form solutions derived in this paper are for the case with no parameter restrictions and also for the cases with particular parameter restrictions. We also derive the growth rates of all the variables in the model and an interesting feature of our result is that while one solution yields static growth rates, the other solutions provide dynamic growth rates. In the long run, all the growth rates approach the same steady state growth rate.

The layout of the paper is as follows. In Section 2, we provide an overview of the Lucas-Uzawa model with externalities based on the transformation by Hiraguchi [17]. In Section 3, the closed-form solutions of Lucas-Uzawa model with externalities are derived for the no parameter restriction case. The characteristics of balanced growth path and growth rates of all the variables in the model are analyzed in Section 4. In Section 5, we derive the closed-form solutions of the model in the commonly discussed case where $\sigma = \beta$. Finally, our conclusions are presented in Section 6.

2 Overview of Lucas-Uzawa model with externalities [17]

In this section, we provide an overview of the transformed Lucas-Uzwa model in which in Hiraguchi [17] reduced the Lucas-Uzawa model with externalities to the basic model.

2.1 Model setup

The representative agent’s utility function is defined as

$$\max_{c,k,h,u} \int_0^{\infty} \frac{c^{1-\sigma} - 1}{1 - \sigma} e^{-\rho t}, \sigma \neq 1$$

subject to constraints on physical capital and human capital:

$$\dot{k}(t) = \gamma k^\beta u^{1-\beta} h^{1-\beta+\theta} - \pi k - c, \quad k_0 = k(0)$$

$$\dot{h}(t) = \delta (1 - u) h, \quad h_0 = h(0)$$

where $1/\sigma$ is the constant elasticity of intertemporal substitution, $\rho > 0$ is the discount factor, $\beta$ is the elasticity of output with respect to physical capital, $\gamma > 0$ is the technological levels in the good sector, $\pi > 0$ is the depreciation rate for physical capital, $\delta > 0$ is the technological levels in the education sector, $k$ is the physical capital, $h$ is the human capital, $c$ is per capita consumption and $u$ is the fraction of labor allocated to the production of physical capital. In the equation above, the parameter $\theta$ illustrates the human capital externality (see Lucas (1988) and Tamura (1991)). It is important to mention here that Hiraguchi [17] considered a version of the Lucas-Uzwa model with externalities in which there was no depreciation of physical capital i.e. $\pi = 0$. 

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The current value Hamiltonian function for this problem is defined as

\[ H(t, c, k, \lambda) = \frac{c^{1-\sigma} - 1}{1-\sigma} + \lambda[\gamma k^\beta u^{1-\beta} h^{1-\beta + \theta} - \pi k - c] + \mu \delta(1 - u) h, \]  

(3)

where \( \lambda(t) \) and \( \mu(t) \) are costate variables. The transversality conditions are

\[ \lim_{t \to \infty} e^{-\rho t} \lambda(t) k(t) = 0, \quad \lim_{t \to \infty} e^{-\rho t} \mu(t) h(t) = 0. \]  

(4)

Pontrygin’s maximum principle yields the following necessary first order conditions for optimal control:

\[ \lambda = c^{-\sigma}, \]  

(5)

\[ u^\beta = \frac{\gamma (1 - \beta) k^\beta h^{-\beta + \theta}}{\delta \mu}, \]  

(6)

\[ \dot{k}(t) = \gamma k^\beta u^{1-\beta} h^{1-\beta + \theta} - \pi k - c, \]  

(7)

\[ \dot{h}(t) = \delta(1 - u) h, \]  

(8)

\[ \dot{\lambda} = -\lambda \gamma \beta u^{1-\beta} k^\beta - 1 h^{1-\beta + \theta} + \lambda(\rho + \pi), \]  

(9)

\[ \dot{\mu} = \mu(\rho - \delta) - \frac{\mu \delta \lambda}{1-\beta} u. \]  

(10)

The growth rates for the variables \( c \) and \( u \) after simplification take the following form:

\[ \frac{\dot{c}}{c} = \frac{\beta \gamma}{\sigma} u^{1-\beta} k^\beta - 1 h^{1-\beta + \theta} - \frac{\rho + \pi}{\sigma}, \]  

(11)

\[ \frac{\dot{u}}{u} = \frac{(\delta + \pi)(1 - \beta) + \delta \theta}{\beta} - \frac{c}{k} + \left( \frac{1 - \beta + \theta}{1 - \beta} \right) \delta u. \]  

(12)

### 2.2 Transformed first-order conditions [17]

Introducing the transformations, put forward by Hiraguchi [17],

\[ \phi = \frac{1 - \beta + \theta}{1 - \beta}, \quad h^* = h^\phi, \quad \delta^* = \delta \phi, \quad \mu^* = \mu \phi^{-1} h^{1-\phi}, \]  

(13)

the system of equations [3]-[12] takes the following form:

\[ \lambda = c^{-\sigma}, \]  

(14)

\[ \left( \frac{h^* u}{k} \right)^\beta = \frac{\gamma (1 - \beta)}{\delta^*} \frac{\lambda}{\mu^*}. \]  

(15)
\[ \dot{k} = \gamma k^\beta (uh^*)^{1-\beta} - \pi k - c, \]  
\[ \dot{h}^*(t) = \delta^*(1 - u)h^*, \]  
\[ \dot{\lambda} = -\lambda \beta \gamma \left( \frac{h^* u}{k} \right)^{1-\beta} + \lambda (\rho + \pi), \]  
\[ \dot{\mu}^* = \mu^*(\rho - \delta^*), \]  
\[ \frac{\dot{c}}{c} = \beta \gamma \left( \frac{h^* u}{k} \right)^{1-\beta} - \frac{\rho + \pi}{\sigma}, \]  
\[ \frac{\dot{u}}{u} = \frac{(\delta^* + \pi)(1 - \beta)}{\beta} - \frac{c}{k} + \delta^* u. \]

The transversality conditions transform to
\[ \lim_{t \to \infty} e^{-\rho t} \lambda(t)k(t) = 0, \lim_{t \to \infty} e^{-\rho t} \mu^*(t)h^*(t) = 0. \]

2.3 Transformed model [17]

Based on the transformation discussed above, Hiraguchi [17] considered the following new growth problem with the variable \( h^* \):

\[ \text{Max}_{c,k,h^*,u} \int_0^\infty \frac{c^{1-\sigma} - 1}{1-\sigma} e^{-\rho t}, \sigma \neq 1 \]  
subject to constraints on physical capital and human capital:
\[ \dot{k}(t) = \gamma k^\beta u^{1-\beta} h^{*1-\beta+\theta} \rho k - c, \quad k_0 = k(0) \]
\[ \dot{h}(t) = \delta^* h^*(1 - u)h, \quad h^*_0 = h^*(0), \]

where \( \phi = \frac{1-\beta+\theta}{1-\beta} \), \( h^* = h^\phi \), \( \delta^* = \delta \phi \).

The current value Hamiltonian for this problem is
\[ H^*(t, c, u, k, h^*, \lambda, \mu^*) = \frac{c^{1-\sigma} - 1}{1-\sigma} + \lambda [\gamma k^\beta (uh^*)^{1-\beta} - \pi k - c] + \mu^* \delta^* (1 - u)h^*, \]

where \( \mu^* = \mu^\phi^{-1} h^{1-\phi} \). It is straightforward to show that the first order conditions and transversality conditions of this new problem are the same as given in (14)-(22).

Let \( \bar{c}, \bar{u}, \bar{k}, \bar{h}^* \) be the equilibrium values for consumption \( c \), the fraction of labor allocated to the production of physical capital \( u \), physical capital \( k \) and human capital \( h^* \). Let \( g_c, g_k, g_{h^*} \) and \( g_u \) be the growth rates of the per capita
consumption $c$, physical capital $k$, human capital $h^*$, the fraction of labor allocated to the production of physical capital $u$. In long run, for $\rho < \delta^* < \rho + \delta^* \sigma$, the balanced growth path (BGP) satisfies the following conditions stated (see e.g. Naz et al (2016); Naz and Chaudhry (2017)):

$$g_c = g_k = g_{h^*} = \frac{\delta^* - \rho}{\sigma}, \quad g_u = 0$$

$$\bar{u} = \frac{\rho + \delta^*(\sigma - 1)}{\delta \sigma}, \quad \bar{u} \in [0, 1]$$

$$\bar{c} = \frac{\delta^* + \pi(1-\beta)}{\beta} - \frac{\delta^* - \rho}{\sigma} = \xi > 0,$$

$$\bar{k} = \frac{\rho + \delta^*(\sigma - 1)}{\delta^* \sigma} \left(\frac{\beta \gamma}{\delta(1-\beta) + \pi(1-\beta)}\right)^{\frac{1}{1-\beta}}.$$

The characteristics of the BGP for the Lucas-Uzawa model with externalities are given in following proposition:

**Proposition 1:** Let $g_c, g_k, g_h$ and $g_u$ be the growth rates of the per capita consumption $c$, physical capital $k$, human capital $h$, the fraction of labor allocated to the production of physical capital $u$. Let $\rho(1-\beta) < \delta(1-\beta + \theta) < \rho(1-\beta) + \delta(1-\beta + \theta)$, then the system reaches the BGP and the following statements are valid:

i. $g_c = g_k = \frac{(\delta - \rho)(1-\beta) + \delta \theta}{\sigma(1-\beta)}$, $g_h = \frac{(\delta - \rho)(1-\beta) + \delta \theta}{\sigma(1-\beta + \theta)}$ and $g_u = 0$,

ii. $\bar{u} \in [0, 1]$ and $\bar{u} = \frac{(\rho - \delta + \delta \theta)(1-\beta) + \delta \theta (\sigma - 1)}{\delta \sigma (1-\beta)}$,

iii. $\bar{c} = \frac{\delta(1-\beta + \theta) + \pi(1-\beta)}{\beta(1-\beta)} - \frac{\delta(1-\beta + \theta) - \rho(1-\beta)}{\sigma(1-\beta)} = \xi$,

iv. $\frac{\bar{k}}{\bar{h}^*} = \sqrt[\frac{1}{1-\beta}]{\frac{\rho - \delta + \delta \theta (1-\beta) + \delta \theta (\sigma - 1)}{\delta \sigma (1-\beta + \theta) + \pi(1-\beta)}}$.

**Proof:** The results directly follow by using $\phi = \frac{1-\beta + \theta}{1-\beta}$, $h^* = h^\phi$, $\delta^* = \delta \phi$ in (26). We need to prove $\xi > 0$.

$$\xi = \frac{\delta(1-\beta + \theta) + \pi(1-\beta)^2}{\beta(1-\beta)} - \frac{\delta(1-\beta + \theta) - \rho(1-\beta)}{\sigma(1-\beta)}$$

$$= \frac{\sigma \delta(1-\beta + \theta) + \sigma \pi(1-\beta)^2 - \beta \delta(1-\beta + \theta) + \beta \rho(1-\beta)}{\sigma \beta(1-\beta)}$$

$$> \frac{[\delta(1-\beta + \theta) - \rho(1-\beta)][\delta(1-\beta + \theta) + \pi(1-\beta)](1-\beta)}{\delta \sigma \beta(1-\beta + \theta)} > 0$$

(27)

and this completes the proof.

In the next section, we establish the closed-form solutions of Lucas-Uzawa model with externalities by utilizing the results of basic Lucas-Uzawa model.
presented in [12, 15]. Then we check all the derived closed-form solutions to ensure that they satisfy all the properties of the BGP.

3 Closed-form solutions for Lucas-Uzawa-model with externalities

In this section, we derive multiple closed-form solutions for the Lucas-Uzawa model with externalities by utilizing the results of the basic Lucas-Uzawa model presented in [12, 15]. If we let $[k, h^*, c, u, \lambda, \mu^*]$ denote the closed-form solution to the transformed problem, a closed-form solution for the original model also exists and is given by $[k, h, c, u, \lambda, \mu] = [k, (h^*)^{\frac{1}{\sigma}}, c, u, \phi \mu^* (h^*)^{\sigma - 1}]$ (see e.g. [17]).

3.1 Closed-form solutions using fairly general parameters values

Naz and Chaudhry [15] established three sets of closed-form solutions for the basic Lucas-Uzawa model with the help of their newly developed partial Hamiltonian approach for fairly general values of the parameters. We derive a solution for the dynamical system of ODEs (14)-(21) directly from [15] in terms of a variable $z(t) = \frac{u(t) h^*(t)}{k(t)}$.

The first set of solutions for all the economic variables of the dynamical system of ODEs (14)-(21), satisfying the transversality condition (22), is (see [15]):

$$
c(t) = c_0 e^{-\frac{(\rho - \delta^* - \gamma)}{\sigma} t},
$$

$$
k(t) = k_0 e^{-\frac{(\rho - \delta^* + \delta^* \sigma)}{\sigma} t},
$$

$$
u(t) = \frac{\rho - \delta^* + \delta^* \sigma}{\delta^* \sigma} = \bar{u},
$$

$$
h^*(t) = h^*_0 e^{-\frac{(\rho - \delta^*)}{\sigma} t},
$$

$$
\lambda(t) = c_0 e^{(\rho - \delta^*) t},
$$

$$
\mu^* = c_1 e^{(\rho - \delta^*) t},
$$

$$
\bar{z} = \left( \frac{\delta^* + \pi}{\beta \gamma} \right)^{\frac{1}{1-\beta}},
$$

provided $\rho < \delta^* < \rho + \delta^* \sigma, c_0 = \left( \frac{(1-\beta) \gamma}{\beta \delta^* \sigma} \right)^{\frac{1}{2}}$, $k_0 = \frac{\delta^* + \pi(1-\beta)}{\beta \sigma} - \frac{\delta^*}{\sigma}, h^*_0 = \bar{z} b_0 u$.

The second set of solutions for all the economic variables of the dynamical system of ODEs (14)-(21), satisfying the transversality condition (22), is (see
The system of ODEs (14)-(21), satisfying the transversality condition (22), is (see [15]):

\[
c(t) = c_0 z_0^\beta e^{-\frac{(\sigma+\pi+\delta)}{\sigma} t} z^{-\frac{\beta}{\sigma}},
\]

\[
k(t) = \left( \frac{k_0}{c_0 z_0^\beta} - F(t) \right) c_0 z_0^\beta z(t)^{-1} e^{\frac{(\sigma+\pi+\delta)}{\sigma} t} z^{-\frac{\beta}{\sigma} + \beta},
\]

\[
h^*(t) = \frac{k_0^*}{z_0^* c_0 z_0^{-1} - (\rho + \pi - \pi \sigma) k_0 z_0^{-1} + \beta \gamma (1 - \sigma) k_0} \left[ \sigma c_0 z_0^\beta e^{-\frac{(\sigma+\pi+\delta)}{\sigma} t} z^{-\frac{\beta}{\sigma} + \beta} \right],
\]

\[
\beta \gamma (1 - \sigma) - (\rho + \pi - \pi \sigma) z^{\beta-1} \left( \frac{k_0}{c_0 z_0^\beta} - F(t) \right) c_0 z_0^\beta e^{\frac{(\sigma+\pi+\delta)}{\sigma} t},
\]

\[
\lambda(t) = c^0 z_0^{-\beta} e^{(\rho - \delta^*) t} z^{\beta},
\]

\[
\mu^*(t) = c_1 e^{(\rho - \delta^*) t},
\]

where

\[
F(t) = \int_0^t z(t)^{-\frac{\beta}{\sigma}} e^{-\frac{(\sigma+\pi+\delta)}{\sigma} t} z^{-\frac{\beta}{\sigma} + \beta} dt,
\]

\[
z(t) = \frac{z_0}{[(z_0^{1-\beta} - z_0^{1-\beta}) e^{-\frac{(\sigma+\pi+\delta)}{\sigma} t} z_0^{1-\beta} + z_0^{1-\beta} t]^{\frac{1}{\beta}}},
\]

\[
\lim_{t \to \infty} F(t) = \frac{k_0}{c_0 z_0^\beta}.
\]

\[
\rho < \delta^* < \rho + \delta^* \sigma, \quad \xi = \frac{\delta^* + \pi - \pi \beta}{\beta} - \frac{\delta^* - \rho}{\sigma} > 0,
\]

\[
c_0 z_0^\beta = \left( \frac{c_1 \delta^*}{(1 - \beta) \gamma} \right)^{\frac{1}{\beta}},
\]

\[
\gamma (1 - \beta) (\rho - \delta^* + \delta^* \sigma)
\]

\[
= \frac{u_0}{k_0} \left[ \sigma c_0 z_0^{-1} - (\rho + \pi - \pi \sigma) k_0 z_0^{-1} + \beta \gamma (1 - \sigma) k_0 \right],
\]

\[
\bar{z} = \left( \frac{\beta \gamma}{\delta^* + \pi} \right)^{\frac{1}{\beta - 1}}.
\]

The third set of solutions for all the economic variables of the dynamical system of ODEs (14)-(21), satisfying the transversality condition (22), is (see
φ

original variables of the model by using the transformations

The closed-form solutions (28), (29) and (30) can be expressed in terms of the

where

\[ u(t) = \frac{\left( \frac{(\delta^* + \pi)(1-\beta)}{\beta} \right) \frac{k_0}{c_0 z_0} - \delta^* u_0 G(t) - \delta^* u_0 \left[ \frac{k_0}{c_0 z_0^2} - F(t) \right]}{\left( \frac{(\delta^* + \pi)(1-\beta)}{\beta} \right) \frac{k_0}{c_0 z_0} - \delta^* u_0 G(t) - \delta^* u_0 \left[ \frac{k_0}{c_0 z_0^2} - F(t) \right]} \]

\[ \lambda(t) = c_0 z_0^2 e^{(\rho - \delta^*)t} z^\beta, \]

\[ \mu^*(t) = c_1 e^{(\rho - \delta^*)t}, \]

where

\[ \rho < \delta^* < \rho + \delta^* \sigma, \quad \frac{\delta^* + \pi - \pi \beta}{\beta} - \frac{\delta^* - \rho}{\sigma} > 0, \]

\[ F(t) = \int_0^t z(t) \frac{k_0}{c_0 z_0} e^{-\left( \frac{\delta^* + \pi - \pi \beta}{\beta} - \frac{\delta^* - \rho}{\sigma} \right) t} dt, \]

\[ G(t) = \int_0^t z(t) \frac{k_0}{c_0 z_0} e^{-\left( \frac{\delta^* + \pi - \pi \beta}{\beta} - \frac{\delta^* - \rho}{\sigma} \right) t} dt, \] (30)

\[ z(t) = \frac{z_0}{\left( z_1 \beta - z_0 (1-\beta) \right) e^{-\frac{1}{\sigma} \left( \frac{k_0}{c_0 z_0} - \frac{k_0}{c_0 z_0^2} \right) t} + z_0} \]

\[ c_0 z_0^2 = \left( \frac{c_1 \delta^*}{1-\beta} \right)^{\beta}, \]

\[ \lim_{t \to \infty} F(t) = \frac{k_0}{c_0 z_0}, \]

\[ \lim_{t \to \infty} \left( \frac{(\delta^* + \pi)(1-\beta)}{\beta} + \delta^* u_0 \right) \frac{k_0}{c_0 z_0} - \delta^* u_0 G(t) = 0, \]

\[ \lim_{t \to \infty} G(t) = \left( \frac{(\delta^* + \pi)(1-\beta)}{\beta} + \delta^* u_0 \right) \lim_{t \to \infty} F(t), \]

\[ \bar{z} = \left( \frac{\beta \gamma}{\delta^* + \pi} \right)^{\frac{1}{\beta}}. \]

The closed-form solutions (28), (29) and (30) can be expressed in terms of the original variables of the model by using the transformations \( \phi = \frac{1-\beta + \theta}{1-\beta} \), \( h^* = \)
$h^\phi$, $\delta^* = \delta \phi$, $\mu^* = \mu \phi^{-1} h^{1-\phi}$. This completes the closed-form solutions of Lucas-Uzawa model with externalities with no parameter restrictions.

### 3.2 Comparison of the multiple closed-form solutions

Since this is the first time in the literature that multiple closed-form solutions have been obtained for the Lucas-Uzawa model with externalities, it is useful to compare these results.

The first thing to note is that the values of consumption $c$, physical capital stock $k$, and the costate variables $\lambda$ and $\mu$ in solution (28) are different from the values of these variables in solutions (29) and (30). Next, a comparison of the closed-form solutions (29) and (30) shows that the expressions for consumption $c$, physical capital stock $k$, and the costate variables $\lambda$ and $\mu$ are the same in both solutions. On the other hand, the expressions for the fraction of labor devoted to physical capital, $u$, and the level of human capital, $h^*$, are different.

These results are important in the context of understanding differences between economies transitioning towards their long run equilibria. Our results show that countries that start with the same initial conditions can have significant differences as they progress towards their long run equilibria. For example, our results show that countries can have the same levels of consumption and capital stock, but may differ significantly in their levels of human capital. Or countries may have the same levels of consumption but may differ significantly in their levels of physical and human capital. These differences are critical in understanding differences between countries during the development process.

In terms of the previous literature, Hiraguchi [17] derived only one solution for the model without parameter restrictions which was similar to (30), and in this solution $F(t)$ and $G(t)$ were computed in terms of the hypergeometric functions. Hiraguchi [17] also claimed that there existed no other solutions for the Lucas-Uzawa model with externalities. Contrary to this claim and for the first time in the literature, we have established multiple closed-form solutions for the Lucas-Uzawa model with externalities in the case where there are no parameter restrictions.

### 4 Characteristics of balanced growth path

In this section, we discuss the growth rates of the key variables in our model for each of the solutions we have obtained. We then compare the long run equilibrium values of these growth rates.

#### 4.1 Growth rates for each of the solutions

First of all, we analyze the simple solution (28). The growth rates for per capita consumption $c$, physical capital $k$, human capital $h$, the fraction of labor allocated to the production of physical capital $u$, costate variables $\mu$ and $\lambda$ for solution (28) after using $\phi = \frac{1-\delta+\theta}{1-\beta}$, $h^* = h^\phi$, $\delta^* = \delta \phi$, $\mu^* = \mu \phi^{-1} h^{1-\phi}$, (and
after some simplifications) take the following forms:

\[
\frac{\dot{c}}{c} = \frac{(\delta - \rho)(1 - \beta) + \delta \theta}{\sigma(1 - \beta)},
\]

\[
\frac{\dot{k}}{k} = \frac{(\delta - \rho)(1 - \beta) + \delta \theta}{\sigma(1 - \beta)},
\]

\[
\frac{\dot{h}}{h} = \frac{(\delta - \rho)(1 - \beta) + \delta \theta}{\sigma(1 - \beta + \theta)},
\]

\[
\frac{\dot{u}}{u} = 0,
\]

\[
\frac{\dot{\lambda}}{\lambda} = \frac{(\rho - \delta)(1 - \beta) - \delta \theta}{1 - \beta},
\]

\[
\frac{\dot{\mu}}{\mu} = \frac{(\rho - \delta)(1 - \beta) - \delta \theta)(\sigma(1 - \beta + \theta) - \theta)}{\sigma(1 - \beta)(1 - \beta + \theta)}.
\]

Solution (28) yields constant growth rates (31) for all the variables of the model. The growth rates of the per capita consumption \(c\), physical capital \(k\) and human capital \(h\) are positive provided \((\delta - \rho)(1 - \beta) + \delta \theta > 0\).

The growth rates of all the variables for closed-form solution (29) after simplifications are

\[
\frac{\dot{c}}{c} = \frac{(\delta - \rho)(1 - \beta) + \delta \theta}{\sigma(1 - \beta)} - \frac{\beta z}{\sigma z},
\]

\[
\frac{\dot{k}}{k} = \frac{\pi(1 - \beta)^2 + \delta(1 - \beta + \theta)}{\beta(1 - \beta)} - \frac{\zeta}{\sigma z},
\]

\[
\frac{\dot{h}}{h} = \frac{1 - \beta + \theta}{\sigma z(t)^{\beta - 1} + \beta \gamma(1 - \sigma) - (\rho + \pi - \pi \sigma)z(t)^{\beta - 1}} \left[ \frac{\sigma z(t)^{\beta - 1} - \beta \gamma(1 - \sigma) - (\rho + \pi - \pi \sigma)z(t)^{\beta - 1}}{\sigma z(t)^{\beta - 1} + \beta \gamma(1 - \sigma) - (\rho + \pi - \pi \sigma)z(t)^{\beta - 1}} \right] \frac{\dot{c}}{c} + \frac{\dot{k}}{k} - \frac{\dot{h}}{h} + \frac{\dot{z}}{z},
\]

\[
\frac{\dot{u}}{u} = \frac{\dot{k}}{k} - \frac{(1 - \beta + \theta)\dot{h}}{h} + \frac{\dot{z}}{z},
\]

\[
\frac{\dot{\lambda}}{\lambda} = \frac{(\rho - \delta)(1 - \beta) - \delta \theta}{1 - \beta} + \beta \frac{\dot{z}}{z},
\]

\[
\frac{\dot{\mu}}{\mu} = \frac{(\rho - \delta)(1 - \beta) - \delta \theta)(\sigma(1 - \beta + \theta) - \theta)}{\sigma(1 - \beta)(1 - \beta + \theta)} + \frac{\delta^* + \pi - \pi \beta}{\beta} - \frac{\delta^* - \rho}{\sigma},
\]

\[\xi = \frac{\delta^* + \pi - \pi \beta}{\beta} - \frac{\delta^* - \rho}{\sigma} \]
where

\[
\dot{z} = \frac{(z_0^{1-\beta} - \bar{z}^{1-\beta}) \gamma \bar{z}^{1-\beta} e^{-(1-\beta)(\pi + \delta) + \delta \theta t}}{\bar{z}^{\beta-1} + (z_0^{1-\beta} - \bar{z}^{1-\beta}) e^{-(1-\beta)(\pi + \delta) + \delta \theta t}}.
\]

(33)

It can be shown that for solution (30) the dynamic growth rates for consumption \(c\), physical capital stock \(k\), the fraction of labor allocated to the production of physical capital \(u\) and the costate variables \(\lambda\) and \(\mu\) will be also equal to the expressions for these variables given in (32). The growth rate for human capital \(h\) for solution (30) is complex and is omitted.

4.2 Comparison of the growth rates

First, it is important to point out that we obtain a static growth rate (31) for one solution and dynamic growth rates (32) for the other solutions. Next, it is important to see what happens to these growth rates in the long run.

In order to determine the long run growth rates, one needs to start by looking at the long run value of \(\dot{z}\). It is clear from equation (33) that \(\dot{z}\) approaches zero as \(t \to \infty\) which means that the rate of growth of \(z\) decreases asymptotically as we approach the steady state.

This means that in the long run: (i) the growth rates of consumption \(c\) and the physical capital \(k\) decrease over time and approach \(\frac{(\delta - \rho)(1-\beta) + \delta \theta}{\sigma(1-\beta)}\) as \(t \to \infty\); (ii) the growth rate of human capital decreases over time and approaches \(\frac{(\delta - \rho)(1-\beta) + \delta \theta}{\sigma(1-\beta+\theta)}\) as \(t \to \infty\); (iii) the growth rate of the fraction of labor allocated to the production of physical capital \(u\) approaches zero as \(t \to \infty\); (iv) The growth rate of costate variable \(\lambda\) equals \(\frac{\alpha - \delta(1-\beta) - \delta \theta}{\sigma(1-\beta+\theta)}\) as \(t \to \infty\); and (v) The growth rate of costate variable \(\mu\) equals \(\frac{((\alpha - \delta) (1-\beta) - \delta \theta)(\sigma(1-\beta) + \theta)}{\sigma(1-\beta)(1-\beta+\theta)}\) as \(t \to \infty\).

Our results imply that even though the growth rates of the key variables in the model differ between solutions, in the long run, the static growth rates of all the variables in (31) and the dynamic growth rates of all the variables in (32) reach the same value. Or, in other words, even though the key variables may differ in terms of growth between countries in the short run, in the long run all the economies reach the same steady state growth rates.

In order to confirm that our solutions satisfy the conditions for the BGP (given in Proposition 1), it is also straightforward to show using l’Hôpital rule that for the closed-form solutions (28), (29) and (30), that

\[
\lim_{t \to \infty} u(t) = \bar{u}.
\]

(34)

Also, for the closed-form solutions (28), (29) and (30) the ratios \(\frac{c(t)}{k(t)}\) and \(\frac{h(t)}{k(t)}\) equal the values given in Proposition 1. For the sake of brevity, we are omitting these calculations.
5 Lucas-Uzawa model with externalities for the \( \sigma = \beta \) case

In this Section, we consider the special case when \( \sigma = \beta \) which is a simplifying assumption that is commonly used in the economic growth literature. We start by deriving the closed-form solutions for this special case and then discuss the balanced growth path associated with this solution.

5.1 Closed-from solution for the \( \sigma = \beta \) case

We obtain two closed-form solutions for the Lucas-Uzawa model with externalities under the restriction \( \sigma = \beta \). The first closed-form solution for all variables is obtained by taking \( \sigma = \beta \) in (28) and also using the transformations

\[
\phi = \frac{1 - \beta + \theta}{1 - \beta}, \quad h^* = h^\phi, \quad \delta^* = \delta^\phi, \quad \mu^* = \mu^\phi - 1 h^{1 - \phi},
\]

and is given by

\[
c(t) = c_0 e^{-\frac{(\rho - \delta)(1 - \beta)}{\beta(1 - \beta)}}, \quad k(t) = k_0 e^{-\frac{(\rho - \delta)(1 - \beta)}{\beta(1 - \beta)}}.
\]

\[
u(t) = \bar{u} = \frac{(\rho - \delta)(1 - \beta + \theta)}{\delta(1 - \beta + \theta)}, \quad h(t) = h_0 e^{-\frac{(\rho - \delta)(1 - \beta)}{\beta(1 - \beta + \theta)}},
\]

\[
\lambda(t) = c_0 \beta e^{-\frac{(\rho - \delta)(1 - \beta)}{\beta(1 - \beta + \theta)}}, \quad \mu(t) = \left(1 - \beta + \theta\right)h_0^\phi c_1 e^{-\frac{(\rho - \delta)(1 - \beta + \theta)}{\beta(1 - \beta + \theta)}},
\]

\[
z^* = \left(\frac{\delta + \gamma}{\beta \gamma}\right)^{-1},
\]

provided \( \delta(1 - \beta + \theta) < \rho, c_0 = \left(\frac{\gamma}{c_1 \delta(1 - \beta + \theta) z^*}\right)^{\frac{1}{\beta}}, c_0 \frac{\rho + \pi(1 - \beta)}{\beta} > 0, h_0 = \frac{\bar{z} h_0}{\bar{u}}.\]

The second closed-form solution for all variables is obtained by taking \( \sigma = \beta \) in
and is given by

\[ c(t) = c_0 z_0 e^{\frac{\rho - \delta (1 - \beta + \theta)}{\beta (1 - \beta)} t} z(t)^{-1}, \]

\[ k(t) = k_0 z_0 e^{\frac{\rho - \delta (1 - \beta + \theta)}{\beta (1 - \beta)} t} z(t)^{-1}, \]

\[ h(t) = h_0 e^{\frac{\rho - \delta (1 - \beta + \theta)}{\beta (1 - \beta + \theta)} t}, \]

\[ u(t) = \bar{u} = \left( \rho - \delta (1 - \beta + \theta) \right) \frac{1}{(1 - \beta)} \]

\[ \lambda(t) = c_0 e^{-\beta} z_0 e^{\frac{\rho - \delta (1 - \beta + \theta)}{\beta (1 - \beta + \theta)} t} z(t)^{\beta}, \]

\[ \mu(t) = \frac{1 - \beta + \theta}{1 - \beta} h_0 e^{\frac{\rho - \delta (1 - \beta + \theta)}{\beta (1 - \beta + \theta)} t}, \]

provided

\[ \delta(1 - \beta + \theta) < \rho, \quad c_0 = \left( \frac{\gamma}{c_1 \delta (1 - \beta + \theta) z_0} \right)^\frac{1}{\beta}, \]

\[ \frac{c_0}{k_0} = \frac{\rho + \pi (1 - \beta)}{\beta} > 0, \quad h_0 = \frac{z_0 k_0}{u} \]

\[ z(t) = \frac{\bar{z} z_0}{\left( \gamma z_0^{1-\beta} - z_0^{1-\beta} e^{-\frac{(1-\beta)(\rho + \pi z_0^{1-\beta})}{\beta}} + z_0^{1-\beta} \right)^{1+\pi}}. \]

It is important to mention here that the closed-form solution for all the variables obtained by taking \( \sigma = \beta \) in (36) is the same as given in (30).

In the previous literature, Ruiz-Tamarit [16] established one closed-form solution which was equal to our first solution (35) by utilizing the classical approach. Our methodology has established two closed-form solutions ((35) and (36)) for the \( \sigma = \beta \) case and one of these solutions (36) is completely new to the literature.

### 5.2 Convergence to balanced growth path

For solution (35), the growth rates of per capita consumption \( c \), physical capital \( k \), human capital \( h \), the fraction of labor allocated to the production of physical capital \( u \), costate variables \( \mu \) and \( \lambda \) take the following forms (after some
simplifications):

\[
\dot{c} = \frac{(\delta - \rho)(1 - \beta) + \delta \theta}{\beta(1 - \beta)}, \\
\dot{k} = \frac{(\delta - \rho)(1 - \beta) + \delta \theta}{\beta(1 - \beta)}, \\
\dot{h} = \frac{(\delta - \rho)(1 - \beta) + \delta \theta}{\beta(1 - \beta + \theta)}, \\
\dot{u} = 0, \\
\dot{\lambda} = \frac{(\rho - \delta)(1 - \beta) - \delta \theta}{1 - \beta}, \\
\dot{\mu} = \frac{((\rho - \delta)(1 - \beta) - \delta \theta)(\beta - \theta)}{\beta(1 - \beta + \theta)}.
\]

For solution (36), the growth rates of the per capita consumption \(c\), physical capital \(k\), human capital \(h\), the fraction of labor allocated to the production of physical capital \(u\), costate variables \(\mu\) and \(\lambda\) take the following forms (after some simplifications):

\[
\dot{c} = \frac{(\delta - \rho)(1 - \beta) + \delta \theta}{\beta(1 - \beta)}, \\
\dot{k} = \frac{(\delta - \rho)(1 - \beta) + \delta \theta}{\beta(1 - \beta)}, \\
\dot{h} = \frac{(\delta - \rho)(1 - \beta) + \delta \theta}{\beta(1 - \beta + \theta)}, \\
\dot{u} = 0, \\
\dot{\lambda} = \frac{(\rho - \delta)(1 - \beta) - \delta \theta}{1 - \beta}, \\
\dot{\mu} = \frac{((\rho - \delta)(1 - \beta) - \delta \theta)(\beta - \theta)}{\beta(1 - \beta + \theta)},
\]

where \(\dot{z}\) is given in (33).

In both solutions we find the following: (i) the growth rates of consumption \(c\) and the physical capital \(k\) decrease over time and approach \(\frac{(\delta - \rho)(1 - \beta) + \delta \theta}{\beta(1 - \beta)}\) as \(t \to \infty\); (ii) the growth rate of human capital decreases over time and approaches \(\frac{(\delta - \rho)(1 - \beta) + \delta \theta}{\beta(1 - \beta + \theta)}\) as \(t \to \infty\); (iii) the growth rates of the per capita consumption \(c\), physical capital \(k\) and human capital \(h\) are positive provided \((\delta - \rho)(1 - \beta) + \delta \theta > 0\); (iv) the growth rate of the fraction of labor allocated to the production of physical capital \(u\) approaches zero as \(t \to \infty\); (v) the growth rate of costate variable \(\lambda\) converges to \(\frac{(\rho - \delta)(1 - \beta) - \delta \theta}{1 - \beta}\) as \(t \to \infty\) and is negative; (vi) the growth rate of the costate variable \(\mu\) converges to \(\frac{((\rho - \delta)(1 - \beta) - \delta \theta)(\beta - \theta)}{\beta(1 - \beta + \theta)}\) as \(t \to \infty\); and
(vii) the growth rate of the costate variable \( \mu \) is negative when \( \beta > \theta \) and is positive when \( \beta < \theta \).

Moreover, it is straightforward to check that both closed-form solutions (35) and (36) satisfy all properties of the BGP stated in Proposition 1.

6 Conclusions

The Lucas-Uzawa model with externalities is one of the fundamental models of endogenous growth and is used to explain how economies grow over time as they accumulate human capital. The solutions to this model in the literature focus on how the key parameters in the model, such as consumption, capital, and human capital, behave as they progress on the balanced growth path towards equilibrium or their long run equilibrium values.

But from a policymaker’s perspective, the true benefit of such a model would be to see how an economy behaves outside of equilibrium in order to be able to design policies during an economy’s transition from the short-run (when it may be in disequilibrium) to the long run. The issue that arises in much of the previous literature is that most of methodologies used to derive solutions outside of the equilibrium have yielded unique solutions. These unique solutions are important but do not adequately explain how economies starting from the same point can end up at distinctly different points in their growth trajectories as they progress towards a long run equilibrium.

We attempt to address this issue by using a newly developed methodology to derive multiple solutions for the Lucas-Uzawa model with externalities. First, we began by transforming the Lucas-Uzawa model with externalities into the basic Lucas-Uzawa model. Then we derived three closed-form solutions which are completely new to the literature from the results in [15]. The closed-from solutions presented in (28), (29) and (30) hold for fairly general values of the parameters. Also, while the previous literature, Hiraguchi [17] derived only one solution for the model without parameter restrictions which was similar to (30), and in this solution \( F(t) \) and \( G(t) \) were computed in terms of the hypergeometric functions. Hiraguchi [17] also claimed that there existed no other solutions for the Lucas-Uzawa model with externalities. Contrary to this claim and for the first time in the literature, we have established multiple closed-form solutions for the Lucas-Uzawa model with externalities in the case where there are no parameter restrictions. Moreover, for \( \sigma = \beta \), only one solution (28) was established for the model [16], we show that multiple solutions (35) and (36) exist in this case.

Our methodology yields completely new multiple closed-form solutions for the Lucas-Uzawa model with externalities and we use these solutions to derive the growth rates for all of the variables in the model. Our results are important on multiple levels: First, our solutions establish multiplicity in terms of closed-form solutions which has not been found in any of the previous economic growth literature. This is important for our understanding of the differences across countries in economic growth. Second, one of our solutions yields static
growth rates whereas the other solutions yield dynamic growth rates and in the long run all these growth rates reach the same static value. This means that these countries that experience differences in growth over the short run can still converge to the same long run growth rates.

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