C*-ALGEBRAS ASSOCIATED WITH HILBERT C*-QUAD MODULES OF FINITE TYPE

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Abstract. A Hilbert C*-quad module of finite type has a multi structure of Hilbert C*-bimodules with two finite bases. We will construct a C*-algebra from a Hilbert C*-quad module of finite type and prove its universality subject to certain relations among generators. Some examples of the C*-algebras from Hilbert C*-quad modules of finite type will be presented.

1. Introduction

G. Robertson–T. Steger [23] have initiated a certain study of higher dimensional analogue of Cuntz–Krieger algebras from the viewpoint of tiling systems of 2-dimensional plane. After their work, A. Kumjian–D. Pask [11] have generalized their construction to introduce the notion of higher rank graphs and its C*-algebras. Since then, there have been many studies on these C*-algebras by many authors (see for example [6], [7], [11], [22], [18], [23], etc.).

In [12], the author has introduced a notion of C*-symbolic dynamical system, which is a generalization of a finite labeled graph, a λ-graph system and an automorphism of a unital C*-algebra. It is denoted by \( (A, \rho, \Sigma) \) and consists of a finite family \( \{\rho_\alpha\}_{\alpha \in \Sigma} \) of endomorphisms of a unital C*-algebra \( A \) such that \( \rho_\alpha(Z_A) \subseteq Z_A, \alpha \in \Sigma \) and \( \sum_{\alpha \in \Sigma} \rho_\alpha(1) \geq 1 \) where \( Z_A \) denotes the center of \( A \). It provides a subshift \( \Lambda_\rho \) over \( \Sigma \) and a Hilbert C*-bimodule \( H_\rho^A \) over \( A \) which gives rise to a C*-algebra \( O_\rho \) as a Cuntz-Pimsner algebra ([12], cf. [8], [16], [21]). In [13] and [14], the author has extended the notion of C*-symbolic dynamical system to C*-textile dynamical system which is a higher dimensional analogue of C*-symbolic dynamical system.

A C*-textile dynamical system \( (A, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa) \) consists of two C*-symbolic dynamical systems \( (A, \rho, \Sigma^\rho) \) and \( (A, \eta, \Sigma^\eta) \) with common unital C*-algebra \( A \) and commutation relations \( \kappa \) between the endomorphisms \( \rho_\alpha, \alpha \in \Sigma^\rho \) and \( \eta_a, a \in \Sigma^\eta \). A C*-textile dynamical system provides a two-dimensional subshift and a multi structure of Hilbert C*-bimodules that has multi right actions and multi left actions and multi inner products. Such a multi structure of Hilbert C*-bimodule is called a Hilbert C*-quad module. In [14], the author has introduced a C*-algebra associated with the Hilbert C*-quad module of C*-textile dynamical system. It is generated by the quotient images of creation operators on two-dimensional analogue of Fock Hilbert module by module maps of compact operators. As a result, the C*-algebra has been proved to have a universal property subject to certain operator relations of generators encoded by structure of C*-textile dynamical system ([14]).

1991 Mathematics Subject Classification. Primary 46L35; Secondary 46L08.

Key words and phrases. C*-algebras, Hilbert C*-modules, symbolic dynamics, Cuntz algebras, Cuntz-Krieger algebras.
In this paper, we will generalize the construction of the $C^*$-algebras of Hilbert $C^*$-quad modules of $C^*$-textile dynamical systems. Let $\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2$ be unital $C^*$-algebras. Assume that $\mathcal{A}$ has unital embeddings into both $\mathcal{B}_1$ and $\mathcal{B}_2$. A Hilbert $C^*$-quad module $\mathcal{H}$ over $(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$ is a Hilbert $C^*$-bimodule over $\mathcal{A}$ with $\mathcal{A}$-valued right inner product $\langle \cdot \mid \cdot \rangle_{\mathcal{A}}$ which has a multi structure of Hilbert $C^*$-bimodules over $\mathcal{B}_i$ with right actions $\varphi_i$ of $\mathcal{B}_i$ and left actions $\phi_i$ of $\mathcal{B}_1$ and $\mathcal{B}_2$-valued inner products $\langle \cdot \mid \cdot \rangle_{\mathcal{B}_i}$ for $i = 1, 2$ satisfying certain compatibility conditions. A Hilbert $C^*$-quad module $\mathcal{H}$ is said to be of finite type if there exist a finite basis $\{u_1, \ldots, u_M\}$ of $\mathcal{H}$ as a Hilbert $C^*$-right module over $\mathcal{B}_1$ and a finite basis $\{v_1, \ldots, v_N\}$ of $\mathcal{H}$ as a Hilbert $C^*$-right module over $\mathcal{B}_2$ such that

$$\sum_{i=1}^{M} \langle u_i \mid \phi_2(\langle \xi \mid \eta \rangle_{\mathcal{B}_2})u_i \rangle_{\mathcal{B}_1} = \sum_{k=1}^{N} \langle v_k \mid \phi_1(\langle \xi \mid \eta \rangle_{\mathcal{B}_1})v_k \rangle_{\mathcal{B}_2} = \langle \xi \mid \eta \rangle_{\mathcal{A}}$$

for $\xi, \eta \in \mathcal{H}$ (see [26] for the original definition of finite basis of Hilbert module). For a Hilbert $C^*$-quad module, we will construct a Fock space $F(\mathcal{H})$ from $\mathcal{H}$, which is a 2-dimensional analogue to the ordinary Fock space of Hilbert $C^*$-bimodules (cf. [10], [21]). We will then define two kinds of creation operators $s_\xi, t_\xi$ for $\xi \in \mathcal{H}$ on $F(\mathcal{H})$. The $C^*$-algebra on $F(\mathcal{H})$ generated by them is denoted by $T_{F(\mathcal{H})}$ and called the Toeplitz quad module algebra. We then define the $C^*$-algebra $C_{F(\mathcal{H})}$ associated with the Hilbert $C^*$-quad module $\mathcal{H}$ by the quotient $C^*$-algebra of $T_{F(\mathcal{H})}$ by the ideal generated by the finite rank operators. We will then prove that the $C^*$-algebra $O_{F(\mathcal{H})}$ for a $C^*$-quad module $\mathcal{H}$ of finite type has a universal property in the following way.

**Theorem 1.1** (Theorem 5.17). Let $\mathcal{H}$ be a Hilbert $C^*$-quad module over $(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$ of finite type with a finite basis $\{u_1, \ldots, u_M\}$ of $\mathcal{H}$ as a Hilbert $C^*$-right module over $\mathcal{B}_1$ and a finite basis $\{v_1, \ldots, v_N\}$ of $\mathcal{H}$ as a Hilbert $C^*$-right module over $\mathcal{B}_2$. Then the $C^*$-algebra $O_{F(\mathcal{H})}$ generated by the quotients $[s_\xi], [t_\xi]$ of the creation operators $s_\xi, t_\xi$ for $\xi \in \mathcal{H}$ on the Fock spaces $F(\mathcal{H})$ is canonically isomorphic to the universal $C^*$-algebra $O_{\mathcal{H}}$ generated by operators $S_1, \ldots, S_M, T_1, \ldots, T_N$ and elements $z \in \mathcal{B}_1, w \in \mathcal{B}_2$ subject to the relations:

$$\sum_{i=1}^{M} S_i S_i^* + \sum_{k=1}^{N} T_k T_k^* = 1, \quad S_j^* T_l = 0,$$

$$S_i^* S_j = \langle u_i \mid u_j \rangle_{\mathcal{B}_1}, \quad T_k^* T_l = \langle v_k \mid v_l \rangle_{\mathcal{B}_2},$$

$$z S_j = \sum_{i=1}^{M} S_i \langle u_i \mid \phi_1(z)u_j \rangle_{\mathcal{B}_1}, \quad z T_l = \sum_{k=1}^{N} T_k \langle v_k \mid \phi_1(z)v_l \rangle_{\mathcal{B}_2},$$

$$w S_j = \sum_{i=1}^{M} S_i \langle u_i \mid \phi_2(w)u_j \rangle_{\mathcal{B}_1}, \quad w T_l = \sum_{k=1}^{N} T_k \langle v_k \mid \phi_2(w)v_l \rangle_{\mathcal{B}_2}$$

for $z \in \mathcal{B}_1, w \in \mathcal{B}_2, i, j = 1, \ldots, M, k, l = 1, \ldots, N$.

The eight relations of the operators above are called the relations (H). As a corollary we have

**Corollary 1.2** (Corollary 5.18). For a $C^*$-quad module $\mathcal{H}$ of finite type, the universal $C^*$-algebra $O_{\mathcal{H}}$ generated by operators $S_1, \ldots, S_M, T_1, \ldots, T_N$ and elements $z \in \mathcal{B}_1, w \in \mathcal{B}_2$ subject to the relations (H) does not depend on the choice of the finite bases $\{u_1, \ldots, u_M\}$ and $\{v_1, \ldots, v_N\}$. 
The paper is organized in the following way. In Section 2, we will define Hilbert $C^*$-quad module and present some basic properties. In Section 3, we will define a $C^*$-algebra $O_{F(H)}$ from Hilbert $C^*$-quad module $H$ of general type by using creation operators on Fock Hilbert $C^*$-quad module. In Section 4, we will study algebraic structure of the $C^*$-algebra $O_{F(H)}$ for a Hilbert $C^*$-quad module $H$ of finite type. In Section 5, we will prove, as a main result of the paper, that the $C^*$-algebra $O_{F(H)}$ has the universal property stated as in Theorem 1.1. A strategy to prove Theorem 1.1 is to show that the $C^*$-algebra $O_{F(H)}$ is regarded as a Cuntz-Pimsner algebra for a Hilbert $C^*$-bimodule over the $C^*$-algebra generated by $\phi_1(B_1)$ and $\phi_2(B_2)$. We will then prove the gauge invariant universality of the $C^*$-algebra (Theorem 5.16). In Section 6, we will present K-theory formulae for the $C^*$-algebra $O_{F(H)}$. In Section 7, we will give examples. In Section 8, we will formulate higher dimensional analogue of our situations and state a generalized proposition of Theorem 1.1 without proof.

Throughout the paper, we will denote by $\mathbb{Z}_+$ the set of nonnegative integers and by $\mathbb{N}$ the set of positive integers.

2. Hilbert $C^*$-quad modules

Throughout the paper we fix three unital $C^*$-algebras $A, B_1, B_2$ such that $A \subset B_1, A \subset B_2$ with common units. We assume that there exists a right action $\psi_i$ of $A$ on $B_i$ so that

$$b_i \psi_i(a) \in B_i \quad \text{for} \quad b_i \in B_i, a \in A, i = 1, 2,$$

which satisfies

$$\|b_i \psi_i(a)\| \leq \|b_i\|\|a\|, \quad b_i \psi_i(aa') = b_i \psi_i(a) \psi_i(a')$$

for $b_i \in B_i, a, a' \in A, i = 1, 2$. Hence $B_i$ is a right $A$-module through $\psi_i$ for $i = 1, 2$. Suppose that $H$ is a Hilbert $C^*$-bimodule over $A$, which has a right action of $A$, an $A$-valued right inner product $\langle \cdot | \cdot \rangle_A$ and a $*$-homomorphism $\phi_A$ from $A$ to the algebra of all bounded adjointable right $A$-module maps $\mathcal{L}_A(H)$ satisfying

(i) $\langle \cdot | \cdot \rangle_A$ is linear in the second variable.
(ii) $\langle \xi | \eta a \rangle_A = \langle \xi | \eta \rangle_A a$ for $\xi, \eta \in H, a \in A$.
(iii) $\langle \eta \xi \rangle_A^* = \langle \eta \xi \rangle_A^*$ for $\xi, \eta \in H$.
(iv) $\langle \xi | \xi \rangle_A \geq 0$, and $\langle \xi | \xi \rangle_A = 0$ if and only if $\xi = 0$.

A Hilbert $C^*$-bimodule $H$ over $A$ is called a Hilbert $C^*$-quad module over $(A; B_1, B_2)$ if $H$ has a further structure of a Hilbert $C^*$-bimodule over $B_i$ for each $i = 1, 2$ with right action $\psi_i$ of $B_i$ and left action $\phi_i$ of $B_i$ and $B_i$-valued right inner product $\langle \cdot | \cdot \rangle_B$ such that for $z \in B_1, w \in B_2$

$$\phi_1(z), \phi_2(w) \in \mathcal{L}_A(H)$$

and

$$[\phi_1(z)\xi \phi_2(w) = \phi_1(z)[\xi \phi_2(w)], \quad [\phi_2(w)\xi] \phi_1(z) = \phi_2(w)[\xi \phi_1(z)],$$

$$\xi \phi_1(z \psi_1(a)) = [\xi \phi_1(z)]a, \quad \xi \phi_2(w \psi_2(a)) = [\xi \phi_2(w)]a,$$

for $\xi \in H, z \in B_1, w \in B_2, a \in A$ and

$$\phi_A(a) = \phi_1(a) = \phi_2(a) \quad \text{for} \quad a \in A$$

where $A$ is regarded as a subalgebra of $B_i$. The left action $\phi_i$ of $B_i$ on $H$ means that $\phi_i(b_i)$ for $b_i \in B_i$ is a bounded adjointable operator with respect to the inner
product $\langle \cdot | \cdot \rangle_{B_i}$ for each $i = 1, 2$. The operator $\phi_i(b_i)$ for $b_i \in B_i$ is also adjointable with respect to the inner product $\langle \cdot | \cdot \rangle_A$. We assume that the adjoint of $\phi_i(b_i)$ with respect to the inner product $\langle \cdot | \cdot \rangle_{B_i}$ coincides with the adjoint of $\phi_i(b_i)$ with respect to the inner product $\langle \cdot | \cdot \rangle_A$. Both of them coincide with $\phi_i(b_i^*)$. We assume that the left actions $\phi_i$ of $B_i$ on $H$ for $i = 1, 2$ are faithful. We require the following compatibility conditions between the right $A$-module structure of $H$ and the right $A$-module structure of $B_i$ through $\psi_i$:

$$\langle \xi | \eta a \rangle_{B_i} = \langle \xi | \eta \rangle_{B_i} \psi_i(a) \quad \text{for } \xi, \eta \in H, a \in A, \quad i = 1, 2. \quad (2.4)$$

We further assume that $H$ is a full Hilbert $C^\ast$-bimodule with respect to the three inner products $\langle \cdot | \cdot \rangle_A$, $\langle \cdot | \cdot \rangle_{B_1}$, $\langle \cdot | \cdot \rangle_{B_2}$ for each. This means that the $C^\ast$-algebras generated by elements $\{\langle \xi | \eta \rangle_A | \xi, \eta \in H\}$, $\{\langle \xi | \eta \rangle_{B_1} | \xi, \eta \in H\}$ and $\{\langle \xi | \eta \rangle_{B_2} | \xi, \eta \in H\}$ coincide with $A$, $B_1$ and $B_2$ respectively.

For a vector $\xi \in H$, denote by $\|\xi\|_A$, $\|\xi\|_{B_1}$, $\|\xi\|_{B_2}$ the norms $\|\langle \xi | \eta \rangle_A\|_A^2$, $\|\langle \xi | \eta \rangle_{B_1}\|_{B_1}$, $\|\langle \xi | \eta \rangle_{B_2}\|_{B_2}$ induced by the right inner products respectively. By definition, $H$ is complete under the above three norms for each.

**Definition.**

(i) A Hilbert $C^\ast$-quad module $H$ over $(A; B_1, B_2)$ is said to be of general type if there exists a faithful completely positive map $\lambda_i : B_i \rightarrow A$ for $i = 1, 2$ such that

$$\lambda_i(b_i \psi_i(a)) = \lambda_i(b_i) a \quad \text{for } b_i \in B_i, a \in A, \quad (2.5)$$

$$\lambda_i(\langle \xi | \eta \rangle_{B_i}) = \langle \xi | \eta \rangle_A \quad \text{for } \xi, \eta \in H, \quad i = 1, 2. \quad (2.6)$$

(ii) A Hilbert $C^\ast$-quad module $H$ over $(A; B_1, B_2)$ is said to be of finite type if there exist a finite basis $\{u_1, \ldots, u_M\}$ of $H$ as a right Hilbert $B_1$-module and a finite basis $\{v_1, \ldots, v_N\}$ of $H$ as a right Hilbert $B_2$-module, that is,

$$\sum_{i=1}^M u_i \varphi_1(\langle u_i | \xi \rangle_{B_1}) = \sum_{k=1}^N v_k \varphi_2(\langle v_k | \xi \rangle_{B_2}) = \xi, \quad \xi \in H \quad (2.7)$$

such that

$$\langle u_i | \phi_2(w) u_j \rangle_{B_1} \in A, \quad i, j = 1, \ldots, M, \quad (2.8)$$

$$\langle v_k | \phi_1(z) v_l \rangle_{B_2} \in A, \quad k, l = 1, \ldots, N \quad (2.9)$$

for $w \in B_2$, $z \in B_1$ and

$$\sum_{i=1}^M \langle u_i | \phi_2(\langle \xi | \eta \rangle_{B_2}) u_i \rangle_{B_1} = \langle \xi | \eta \rangle_A, \quad (2.10)$$

$$\sum_{k=1}^N \langle v_k | \phi_1(\langle \xi | \eta \rangle_{B_1}) v_k \rangle_{B_2} = \langle \xi | \eta \rangle_A \quad (2.11)$$

for all $\xi, \eta \in H$. Following [26], $\{u_1, \ldots, u_M\}$ and $\{v_1, \ldots, v_N\}$ are called finite bases of $H$ respectively.

(iii) A Hilbert $C^\ast$-quad module $H$ over $(A; B_1, B_2)$ is said to be of strongly finite type if it is of finite type and there exist a finite basis $\{e_1, \ldots, e_M\}$ of $B_1$ as a right $A$-module through $\psi_1 \circ \lambda$ and a finite basis $\{f_1, \ldots, f_N\}$ of $B_2$ as a right $A$-module.
through $\psi_2 \circ \lambda_2$. This means that the following equalities hold:

\[
z = \sum_{j=1}^{M'} e_j \psi_1(\lambda_1(e_j^* z)), \quad z \in \mathcal{B}_1, \tag{2.12}
\]

\[
w = \sum_{i=1}^{N'} f_i \psi_2(\lambda_2(f_i^* w)), \quad w \in \mathcal{B}_2. \tag{2.13}
\]

We note that for a Hilbert $C^*$-quad module of general type, the conditions (2.6) imply

\[
\|\langle \xi \mid \eta \rangle_{\mathcal{A}}\| \leq \|\lambda_i(1)\| \|\langle \xi \mid \eta \rangle_{\mathcal{B}_i}\|, \quad \xi \in \mathcal{H}.
\]

Put $C_i = \|\lambda_i(1)\|^{\frac{1}{2}} > 0$ so that $\|\xi\|_{\mathcal{A}} \leq C_i \|\xi\|_{\mathcal{B}_i}$. Hence the identity operators from the Banach spaces $(\mathcal{H}, \| \cdot \|_{\mathcal{B}_i})$ to $(\mathcal{H}, \| \cdot \|_{\mathcal{A}})$ are bounded linear maps. By the inverse mapping theorem, there exist constants $C_i'$ such that $\|\xi\|_{\mathcal{B}_i} \leq C_i' \|\xi\|_{\mathcal{A}}$ for $\xi \in \mathcal{H}$. Therefore the three norms $\| \cdot \|_{\mathcal{A}}$, $\| \cdot \|_{\mathcal{B}_i}$, $i = 1, 2$, induced by the three inner products $\langle \cdot \mid \cdot \rangle_{\mathcal{A}}$, $\langle \cdot \mid \cdot \rangle_{\mathcal{B}_i}$, $i = 1, 2$ on $\mathcal{H}$ are equivalent to each other.

**Lemma 2.1.** Let $\mathcal{H}$ be a Hilbert $C^*$-quad module $\mathcal{H}$ over $(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$. If $\mathcal{H}$ is of finite type, then it is of general type.

**Proof.** Suppose that $\mathcal{H}$ is of finite type with finite bases $\{u_1, \ldots, u_M\}$ of $\mathcal{H}$ as a right Hilbert $\mathcal{B}_1$-module and $\{v_1, \ldots, v_N\}$ of $\mathcal{H}$ as a right Hilbert $\mathcal{B}_2$-module as above. We put

\[
\lambda_1(z) = \sum_{k=1}^{N} (v_k \mid \phi_1(z) v_k)_{\mathcal{B}_2} \in \mathcal{A}, \quad z \in \mathcal{B}_1,
\]

\[
\lambda_2(w) = \sum_{i=1}^{M} (u_i \mid \phi_2(w) u_i)_{\mathcal{B}_1} \in \mathcal{A}, \quad w \in \mathcal{B}_2.
\]

They give rise to faithful completely positive maps $\lambda_i : \mathcal{B}_i \rightarrow \mathcal{A}, i = 1, 2$. The equalities (2.10) (2.11) imply that

\[
\lambda_i(\langle \xi \mid \eta \rangle_{\mathcal{B}_i}) = \langle \xi \mid \eta \rangle_{\mathcal{A}}, \quad \text{for } \xi, \eta \in \mathcal{H}, i = 1, 2. \tag{2.14}
\]

It then follows that

\[
\lambda_i(\langle \xi \mid \eta \rangle_{\mathcal{B}_i}, \psi_i(a)) = \lambda_i(\langle \xi \mid \eta \rangle_{\mathcal{B}_i}, a) = \langle \xi \mid \eta \rangle_{\mathcal{A}} a = \lambda_i(\langle \xi \mid \eta \rangle_{\mathcal{B}_i}) a, \quad a \in \mathcal{A}.
\]

Since $\mathcal{H}$ is full, the equalities (2.5) hold. \hfill \Box

**Lemma 2.2.** Suppose that a Hilbert $C^*$-quad module $\mathcal{H}$ of finite type is of strongly finite type with a finite basis $\{e_1, \ldots, e_{M'}\}$ of $\mathcal{B}_1$ as a right $\mathcal{A}$-module through $\psi_1 \circ \lambda_1$ and a finite basis $\{f_1, \ldots, f_{N'}\}$ of $\mathcal{B}_2$ as a right $\mathcal{A}$-module through $\psi_2 \circ \lambda_2$. Let $\{u_1, \ldots, u_M\}$ and $\{v_1, \ldots, v_N\}$ be finite bases of $\mathcal{H}$ satisfying (2.7). Then two families $\{u_i \varphi_1(e_j) \mid i = 1, \ldots, M, j = 1, \ldots, M'\}$ and $\{v_k \varphi_2(f_l) \mid k = 1, \ldots, N, l = 1, \ldots, N'\}$ of $\mathcal{H}$ form bases of $\mathcal{H}$ as right $\mathcal{A}$-modules respectively.

**Proof.** For $\xi \in \mathcal{H}$, by the equalities

\[
\xi = \sum_{i=1}^{M} u_i \varphi_1(\langle u_i \mid \xi \rangle_{\mathcal{B}_1}), \quad \langle u_i \mid \xi \rangle_{\mathcal{B}_1} = \sum_{j=1}^{M} e_j \psi_1(\lambda_1(e_j^* \langle u_i \mid \xi \rangle_{\mathcal{B}_1})),
\]
it follows that

\[
\xi = \sum_{i=1}^{M} \sum_{j=1}^{M'} u_i \varphi_1 \left( \sum_{j=1}^{M'} \varepsilon_j \psi_1 (\lambda_1 (e_j^r (u_i \mid \xi)_{B_i})) \right) \\
= \sum_{i=1}^{M} \sum_{j=1}^{M'} u_i \varphi_1 (e_j) \cdot \lambda_1 (e_j^r (u_i \mid \xi)_{B_i}) \\
= \sum_{i=1}^{M} \sum_{j=1}^{M'} u_i \varphi_1 (e_j) \cdot \lambda_1 ((u_i \varphi_1 (e_j) \mid \xi)_{B_i}) \\
= \sum_{i=1}^{M} \sum_{j=1}^{M'} u_i \varphi_1 (e_j) \cdot (u_i \varphi_1 (e_j) \mid \xi)_A.
\]

We similarly have

\[
\xi = \sum_{k=1}^{N} \sum_{l=1}^{N'} v_l \varphi_2 (f_l) \cdot (v_l \varphi_2 (f_l) \mid \xi)_A.
\]

\[\Box\]

We present some examples.

**Examples.**

1. Let \( \alpha, \beta \) be automorphisms of a unital \( C^* \)-algebra \( A \) satisfying \( \alpha \circ \beta = \beta \circ \alpha \). Define right actions \( \psi_i \) of \( A \) on \( B_i \) by

\[
b_1 \psi_1 (a) = b_1 \alpha (a), \quad b_2 \psi_2 (a) = b_2 \beta (a)
\]

for \( b_i \in B_i, a \in A \). We set \( B_1 = B_2 = A \). We put \( \mathcal{H}_{\alpha, \beta} = A \) and equip it with Hilbert \( C^* \)-quad module structure over \((A; A, A)\) in the following way. For \( \xi = x, \xi' = x' \in \mathcal{H}_{\alpha, \beta} = A, a \in A, z \in B_1 = A, w \in B_2 = A \), define the right \( A \)-module structure and the right \( A \)-valued inner product \( \langle \cdot \mid \cdot \rangle_A \) by

\[
\xi \cdot a = xa, \quad \langle \xi \mid \xi' \rangle_A = x^* x'.
\]

Define the right actions \( \varphi_1 \) of \( B_1 \) with right \( B_1 \)-valued inner products \( \langle \cdot \mid \cdot \rangle_{B_1} \) and the left actions \( \phi_i \) of \( B_i \) by setting

\[
\xi \varphi_1 (z) = xa (z), \quad \xi \varphi_2 (w) = x \beta (w),
\]

\[
\langle \xi \mid \xi' \rangle_{B_1} = \alpha^{-1} (x^* x'), \quad \langle \xi \mid \xi' \rangle_{B_2} = \beta^{-1} (x^* x'),
\]

\[
\phi_1 (z) \xi = \beta (\alpha (z)) x, \quad \phi_2 (w) \xi = \alpha (\beta (w)) x.
\]

It is straightforward to see that \( \mathcal{H}_{\alpha, \beta} \) is a Hilbert \( C^* \)-quad module over \((A; A, A)\) of strongly finite type.

2. We fix natural numbers \( 1 < N, M \in \mathbb{N} \). Consider finite dimensional commutative \( C^* \)-algebras \( A = \mathbb{C}, B_1 = \mathbb{C}^N, B_2 = \mathbb{C}^M \). The right actions \( \psi_i \) of \( A \) on \( B_i \) are naturally defined as right multiplications of \( \mathbb{C} \). The algebras \( B_1, B_2 \) have the ordinary product structure and the inner product structure which we denote by \( \langle \cdot \mid \cdot \rangle_N \) and \( \langle \cdot \mid \cdot \rangle_M \) respectively. Let us denote by \( \mathcal{H}_{M, N} \) the tensor product \( \mathbb{C}^M \otimes \mathbb{C}^N \). Define the right actions \( \varphi_i \) of \( B_i \) with \( B_i \)-valued right inner products
It is straightforward to see that for $z$ be the standard basis of $C^{1}$ of strongly finite type, which are studied in [15].

for $z \in B_{1}, \xi, \xi' \in C^{N}, w \in B_{2}, \eta, \eta' \in C^{M}$. Let $e_{i}, i = 1, \ldots, M$ and $f_{k}, k = 1, \ldots, N$ be the standard bases of $C^{M}$ and that of $C^{N}$ respectively. Put the finite bases

$$u_{i} = e_{i} \otimes 1 \in \mathcal{H}_{M,N}, \quad i = 1, \ldots, M,$$

$$v_{k} = 1 \otimes f_{k} \in \mathcal{H}_{M,N}, \quad i = 1, \ldots, N.$$ 

It is straightforward to see that $\mathcal{H}_{M,N}$ is a Hilbert $C^{*}$-quad module over $(C; C^{N}, C^{M})$ of strongly finite type.

3. Let $(A, \rho, \eta, \Sigma^{c}, \Sigma^{n}, \kappa)$ be a $C^{*}$-textile dynamical system which means that for $j \in \Sigma^{n}, l \in \Sigma^{c}$ endomorphisms $\eta_{j}, \rho_{l}$ of $A$ are given with commutation relations $\eta_{j} \circ \rho = \rho \circ \eta_{j}$ if $\kappa(l, j) = (i, k)$. In [14], a Hilbert $C^{*}$-quad module $\mathcal{H}_{\kappa,n}^{\rho}$ over $(A; B_{1}, B_{2})$ from $(A, \rho, \eta, \Sigma^{c}, \Sigma^{n}, \kappa)$ is constructed (see [14] for its detail construction). The two triplets $(A, \rho, \Sigma^{c})$ and $(A, \eta, \Sigma^{n})$ are $C^{*}$-symbolic dynamical systems ([12]), that yield $C^{*}$-algebras $\mathcal{O}_{\rho}$ and $\mathcal{O}_{\eta}$ respectively. The $C^{*}$-algebras $B_{1}$ and $B_{2}$ are defined as the $C^{*}$-subalgebra of $\mathcal{O}_{\rho}$ generated by elements $T_{j}yT_{j}^{*}, j \in \Sigma^{n}, y \in A$ and that of $\mathcal{O}_{\eta}$ generated by $S_{k}yS_{k}^{*}, k \in \Sigma^{c}, y \in A$ respectively. Define the maps $\psi_{i}: A \rightarrow B_{i}, i = 1, 2$ by

$$\psi_{1}(a) = \sum_{j \in \Sigma^{n}} T_{j}aT_{j}^{*}, \quad \psi_{2}(a) = \sum_{l \in \Sigma^{c}} S_{l}aS_{l}^{*}, \quad a \in A$$

which yield the right actions of $A$ on $B_{i}, i = 1, 2$. Define the maps $\lambda_{i}: B_{i} \rightarrow A, i = 1, 2$ by

$$\lambda_{1}(z) = \sum_{j \in \Sigma^{n}} T_{j}^{*}zT_{j}, \quad \lambda_{2}(w) = \sum_{l \in \Sigma^{c}} S_{l}^{*}wS_{l}, \quad z \in B_{1}, w \in B_{2}.$$ 

Put $e_{j} = T_{j}yT_{j}^{*} \in B_{1}, j \in \Sigma^{c}$. Let $z = \sum_{j \in \Sigma^{n}} T_{j}z_{j}T_{j}^{*}$ be an element of $B_{1}$ for $z_{j} \in A$ with $T_{j}^{*}T_{j}z_{j}T_{j}^{*} = z_{j}$. As $\lambda_{1}(e_{j}^{*}z) = \lambda_{1}(T_{j}^{*}z_{j}T_{j}^{*}) = z_{j}$, one sees

$$z = \sum_{j \in \Sigma^{n}} T_{j}T_{j}^{*}T_{j}z_{j}T_{j}^{*} = \sum_{j \in \Sigma^{n}} T_{j}^{*}T_{j}^{*}\psi_{1}(z_{j}) = \sum_{j \in \Sigma^{n}} e_{j}\psi_{1}(\lambda_{1}(e_{j}^{*}z)).$$

We similarly have by putting $f_{l} = S_{l}S_{l}^{*} \in B_{2},$

$$w = \sum_{l \in \Sigma^{c}} f_{l}\psi_{2}(\lambda_{2}(f_{l}^{*}w)) \quad \text{for } w \in B_{2}.$$ 

We see that $\mathcal{H}_{\kappa,n}^{\rho}$ is a Hilbert $C^{*}$-quad module of strongly finite type. In particular, two nonnegative commuting matrices $A, B$ with a specification $\kappa$ coming from the equality $AB = BA$ yield a $C^{*}$-textile dynamical system and hence a Hilbert $C^{*}$-quad module of strongly finite type, which are studied in [15].
3. Fock Hilbert $C^*$-quad modules and creation operators

In this section, we will construct a $C^*$-algebra from a Hilbert $C^*$-quad module $\mathcal{H}$ of general type by using two kinds of creation operators on Fock space of Hilbert $C^*$-quad module. We first consider relative tensor products of Hilbert $C^*$-quad modules and then introduce Fock space of Hilbert $C^*$-quad modules which is a two-dimensional analogue of Fock space of Hilbert $C^*$-bimodules. We fix a Hilbert $C^*$-quad module $\mathcal{H}$ over $(\mathcal{A}; B_1, B_2)$ of general type as in the preceding section. The Hilbert $C^*$-quad module $\mathcal{H}$ is originally a Hilbert $C^*$-right module over $\mathcal{A}$ with $\mathcal{A}$-valued inner product $\langle \cdot | \cdot \rangle_\mathcal{A}$. It has two other structure of Hilbert $C^*$-bimodules, the Hilbert $C^*$-bimodule $(\phi_1, \mathcal{H}, \varphi_1)$ over $B_1$ and the Hilbert $C^*$-bimodule $(\phi_2, \mathcal{H}, \varphi_2)$ over $B_2$ where $\phi_i$ is a left action of $B_i$ on $\mathcal{H}$ and $\varphi_i$ is a right action of $B_i$ on $\mathcal{H}$ with $B_i$-valued right inner product $\langle \cdot | \cdot \rangle_{B_i}$ for each $i = 1, 2$. This situation is written as in the figure:

$$
\begin{array}{c}
\phi_2 \\
\downarrow \\
B_1 \xrightarrow{\phi_1} \mathcal{H} \xleftarrow{\varphi_1} B_1 \\
\uparrow \\
\varphi_2 \\
B_2
\end{array}
$$

We will define two kinds of relative tensor products

$$
\mathcal{H} \otimes_{B_1} \mathcal{H}, \quad \mathcal{H} \otimes_{B_2} \mathcal{H}
$$

as Hilbert $C^*$-quad modules over $(\mathcal{A}; B_1, B_2)$. The latter one should be written vertically as

$$
\mathcal{H} \otimes_{B_2} \mathcal{H}
$$

rather than horizontally $\mathcal{H} \otimes_{B_2} \mathcal{H}$. The first relative tensor product $\mathcal{H} \otimes_{B_1} \mathcal{H}$ is defined as the relative tensor product as Hilbert $C^*$-modules over $B_1$, where the left $\mathcal{H}$ is a right $B_1$-module through $\varphi_1$ and the right $\mathcal{H}$ is a left $B_1$-module through $\phi_1$. It has a right $B_1$-valued inner product and a right $B_2$-valued inner product defined by

$$
\langle \xi \otimes_{B_1} \zeta | \xi' \otimes_{B_1} \zeta' \rangle_{B_1} := \langle \zeta | \phi_{B_1}((\xi | \xi')_{B_1}) \zeta' \rangle_{B_1}, \\
\langle \xi \otimes_{B_1} \zeta | \xi' \otimes_{B_1} \zeta' \rangle_{B_2} := \langle \zeta | \phi_{B_1}((\xi | \xi')_{B_1}) \zeta' \rangle_{B_2}
$$

respectively. It has two right actions, $\text{id} \otimes \varphi_1$ from $B_1$ and $\text{id} \otimes \varphi_2$ from $B_2$. It also has two left actions, $\phi_1 \otimes \text{id}$ from $B_1$ and $\phi_2 \otimes \text{id}$ from $B_2$. By these operations $\mathcal{H} \otimes_{B_1} \mathcal{H}$ is a Hilbert $C^*$-bimodule over $B_1$ as well as a Hilbert $C^*$-bimodule over $B_2$. It also has a right $\mathcal{A}$-valued inner product defined by

$$
\langle \xi \otimes_{B_1} \zeta | \xi' \otimes_{B_1} \zeta' \rangle_{\mathcal{A}} := \lambda_1((\xi \otimes_{B_1} \zeta | \xi' \otimes_{B_1} \zeta')_{B_1})(= \lambda_2((\xi \otimes_{B_1} \zeta | \xi' \otimes_{B_1} \zeta')_{B_2})).
$$
a right $\mathcal{A}$-action $\mathrm{id} \otimes a$ for $a \in \mathcal{A}$ and a left $\mathcal{A}$-action $\phi_{\mathcal{A}} \otimes \mathrm{id}$. By these structure $\mathcal{H} \otimes_{\mathcal{B}_1} \mathcal{H}$ is a Hilbert $C^*$-quad module over $(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$.

\[
\begin{array}{c}
\phi_2 \otimes \mathrm{id} \\
\downarrow \\
\phi_1 \otimes \mathrm{id}
\end{array}
\]

\[
\begin{array}{c}
\mathcal{B}_2 \\
\mathcal{H} \otimes \mathcal{B}_1 \mathcal{H} \\
\mathcal{B}_1
\end{array}
\]

\[
\begin{array}{c}
\mathrm{id} \otimes \varphi_2 \\
\uparrow
\end{array}
\]

\[
\begin{array}{c}
\mathcal{B}_2
\end{array}
\]

We denote the above operations $\phi_1 \otimes \mathrm{id}, \phi_2 \otimes \mathrm{id}, \mathrm{id} \otimes \varphi_1, \mathrm{id} \otimes \varphi_2$ still by $\phi_1, \phi_2, \varphi_1, \varphi_2$ respectively. Similarly we consider the other relative tensor product $\mathcal{H} \otimes_{\mathcal{B}_2} \mathcal{H}$ defined by the relative tensor product as Hilbert $C^*$-modules over $\mathcal{B}_2$, where the left $\mathcal{H}$ is a right $\mathcal{B}_2$-module through $\varphi_2$ and the right $\mathcal{H}$ is a left $\mathcal{B}_2$-module through $\varphi_2$. By a symmetric discussion to the above, $\mathcal{H} \otimes_{\mathcal{B}_2} \mathcal{H}$ is a Hilbert $C^*$-quad module over $(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$. The following lemma is routine.

**Lemma 3.1.** Let $\mathcal{H}_i = \mathcal{H}, i = 1, 2, 3$. The correspondences

\[
(\xi_1 \otimes_{\mathcal{B}_1} \xi_2) \otimes_{\mathcal{B}_2} \xi_3 \in (\mathcal{H}_1 \otimes_{\mathcal{B}_1} \mathcal{H}_2) \otimes_{\mathcal{B}_2} \mathcal{H}_3 \rightarrow \xi_1 \otimes_{\mathcal{B}_1} \left(\xi_2 \otimes_{\mathcal{B}_2} \xi_3\right) \in \mathcal{H}_1 \otimes_{\mathcal{B}_1} \left(\mathcal{H}_2 \otimes_{\mathcal{B}_2} \mathcal{H}_3\right),
\]

\[
(\xi_1 \otimes_{\mathcal{B}_2} \xi_2) \otimes_{\mathcal{B}_1} \xi_3 \in (\mathcal{H}_1 \otimes_{\mathcal{B}_2} \mathcal{H}_2) \otimes_{\mathcal{B}_1} \mathcal{H}_3 \rightarrow \xi_1 \otimes_{\mathcal{B}_2} \left(\xi_2 \otimes_{\mathcal{B}_1} \xi_3\right) \in \mathcal{H}_1 \otimes_{\mathcal{B}_2} \left(\mathcal{H}_2 \otimes_{\mathcal{B}_1} \mathcal{H}_3\right)
\]

yield isomorphisms of Hilbert $C^*$-quad modules respectively.

We write the isomorphism class of the former Hilbert $C^*$-quad modules as $\mathcal{H}_1 \otimes_{\mathcal{B}_1} \mathcal{H}_2 \otimes_{\mathcal{B}_2} \mathcal{H}_3$ and that of the latter ones as $\mathcal{H}_1 \otimes_{\mathcal{B}_2} \mathcal{H}_2 \otimes_{\mathcal{B}_1} \mathcal{H}_3$ respectively.

Note that the direct sum $\mathcal{B}_1 \oplus \mathcal{B}_2$ has a structure of a pre Hilbert $C^*$-right module over $\mathcal{A}$ by the following operations: For $b_1 \oplus b_2, b_1' \oplus b_2' \in \mathcal{B}_1 \oplus \mathcal{B}_2$ and $a \in \mathcal{A}$, set

\[
(b_1 \oplus b_2)\psi_a(a) := b_1 \psi_1(a) \oplus b_2 \psi_2(a) \in \mathcal{B}_1 \oplus \mathcal{B}_2,
\]

\[
\langle b_1 \oplus b_2 | b_1' \oplus b_2' \rangle_A := \lambda_1(b_1^*b_1') + \lambda_2(b_2^*b_2') \in \mathcal{A}.
\]

By (2.5) the equality

\[
\langle b_1 \oplus b_2 | (b_1' \oplus b_2')\psi_a(a) \rangle_A = \langle b_1 \oplus b_2 | b_1' \oplus b_2' \rangle_A \cdot a
\]

holds so that $\mathcal{B}_1 \oplus \mathcal{B}_2$ is a pre Hilbert $C^*$-right module over $\mathcal{A}$. We denote by $\mathcal{F}_0(\mathcal{H})$ the completion of $\mathcal{B}_1 \oplus \mathcal{B}_2$ by the norm induced by the inner product $\langle \cdot | \cdot \rangle_A$. It has right $\mathcal{B}_r$-actions $\varphi_1$ and left $\mathcal{B}_r$-action $\phi_1$ by

\[
(b_1 \oplus b_2)\varphi_1(z) = b_1z \oplus 0, \quad (b_1 \oplus b_2)\varphi_2(w) = 0 \oplus b_2w,
\]

\[
\varphi_1(z)(b_1 \oplus b_2) = zb_1 \oplus 0, \quad \varphi_2(w)(b_1 \oplus b_2) = 0 \oplus wb_2
\]

for $b_1 \oplus b_2 \in \mathcal{B}_1 \oplus \mathcal{B}_2, z \in \mathcal{B}_1, w \in \mathcal{B}_2$.

We denote the relative tensor product $\mathcal{H} \otimes_{\mathcal{B}_1} \mathcal{H}$ and elements $\xi \otimes_{\mathcal{B}_1} \eta$ by $\mathcal{H} \otimes_i \mathcal{H}$ and $\xi \otimes_i \eta$ respectively for $i = 1, 2$. Let us define the Fock Hilbert $C^*$-quad module as a two-dimensional analogue of the Fock space of Hilbert $C^*$-bimodules. Put
\(\Gamma_0 = \{\emptyset\}\) and \(\Gamma_n = \{(i_1, \ldots, i_n) \mid i_j = 1, 2\}, n = 1, 2, \ldots,\). We set

\[
F_1(\mathcal{H}) = \mathcal{H}, \\
F_2(\mathcal{H}) = (\mathcal{H} \otimes_1 \mathcal{H}_0) \oplus (\mathcal{H} \otimes_2 \mathcal{H}), \\
F_3(\mathcal{H}) = (\mathcal{H} \otimes_1 \mathcal{H} \otimes_1 \mathcal{H}) \oplus (\mathcal{H} \otimes_1 \mathcal{H} \otimes_2 \mathcal{H}) \\
\quad \oplus (\mathcal{H} \otimes_2 \mathcal{H} \otimes_1 \mathcal{H}) \oplus (\mathcal{H} \otimes_2 \mathcal{H} \otimes_2 \mathcal{H}),
\]

\[\ldots\]

\[
F_n(\mathcal{H}) = \oplus_{(i_1, \ldots, i_{n-1}) \in \Gamma_{n-1}} \mathcal{H} \otimes_{i_1} \mathcal{H} \otimes_{i_2} \cdots \otimes_{i_{n-1}} \mathcal{H}
\]

as Hilbert \(C^*\)-bimodules over \(\mathcal{A}\). We will define the Fock Hilbert \(C^*\)-module \(F(\mathcal{H})\) by setting

\[
F(\mathcal{H}) := \overline{\oplus_{n=0}^{\infty} F_n(\mathcal{H})}
\]

which is the completion of the algebraic direct sum \(\oplus_{n=0}^{\infty} F_n(\mathcal{H})\) of the Hilbert \(C^*\)-right module over \(\mathcal{A}\) under the norm \(\|\xi\|_A\) on \(\oplus_{n=0}^{\infty} F_n(\mathcal{H})\) induced by the \(\mathcal{A}\)-valued right inner product on \(\oplus_{n=0}^{\infty} F_n(\mathcal{H})\). Then \(F(\mathcal{H})\) is a Hilbert \(C^*\)-right module over \(\mathcal{A}\). It has a natural left \(\mathcal{B}_1\)-action defined by \(\phi_i\) for \(i = 1, 2\).

For \(\xi \in \mathcal{H}_n\) we define two operators

\[
s_\xi : F_n(\mathcal{H}) \rightarrow F_{n+1}(\mathcal{H}), \quad n = 0, 1, 2, \ldots,
\]

\[
t_\xi : F_n(\mathcal{H}) \rightarrow F_{n+1}(\mathcal{H}), \quad n = 0, 1, 2, \ldots
\]

by setting for \(n = 0,\)

\[
s_\xi(b_1 \oplus b_2) = \xi \varphi_1(b_1), \quad b_1 \oplus b_2 \in \mathcal{B}_1 \oplus \mathcal{B}_2, \\
t_\xi(b_1 \oplus b_2) = \xi \varphi_2(b_2), \quad b_1 \oplus b_2 \in \mathcal{B}_1 \oplus \mathcal{B}_2,
\]

and for \(n = 1, 2, \ldots,\)

\[
s_\xi(\xi_1 \otimes_{i_1} \cdots \otimes_{i_{n-1}} \xi_n) = \xi \otimes_{i_1} \xi_1 \otimes_{i_1} \cdots \otimes_{i_{n-1}} \xi_n, \\
t_\xi(\xi_1 \otimes_{i_1} \cdots \otimes_{i_{n-1}} \xi_n) = \xi \otimes_{i_2} \xi_1 \otimes_{i_1} \cdots \otimes_{i_{n-1}} \xi_n
\]

for \(\xi_1 \otimes_{i_1} \cdots \otimes_{i_{n-1}} \xi_n \in F_n(\mathcal{H})\) with \((\pi_1, \ldots, \pi_{n-1}) \in \Gamma_{n-1}.

**Lemma 3.2.** For \(\xi \in \mathcal{H}\) the two operators

\[
s_\xi : F_n(\mathcal{H}) \rightarrow F_{n+1}(\mathcal{H}), \quad n = 0, 1, 2, \ldots,
\]

\[
t_\xi : F_n(\mathcal{H}) \rightarrow F_{n+1}(\mathcal{H}), \quad n = 0, 1, 2, \ldots
\]

are both right \(\mathcal{A}\)-module maps.

**Proof.** We will show the assertion for \(s_\xi\). For \(n = 0\), we have for \(b_1 \oplus b_2 \in \mathcal{B}_1 \oplus \mathcal{B}_2\) and \(a \in \mathcal{A},\)

\[
s_\xi((b_1 \oplus b_2)\psi_1(a)) = \xi \varphi_1(b_1\psi_1(a)) = (\xi \varphi_1(b_1))a = (s_\xi(b_1 \oplus b_2))a.
\]

For \(n = 1, 2, \ldots,\) we have

\[
s_\xi((\xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_{n-1}} \xi_n)a)
\]

\[
=s_\xi(((\xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_{n-1}} (\xi_n)a)) = \xi \otimes_{i_1} \xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_{n-1}} (\xi_n a)
\]

\[
=(\xi \otimes_{i_1} \xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_{n-1}} \xi_n)a = [s_\xi(\xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_{n-1}} \xi_n)]a.
\]
It is clear that the two operators \( s_\xi, t_\xi \) yield bounded right \( \mathcal{A} \)-module maps on \( F(\mathcal{H}) \) having its adjoints with respect to the \( \mathcal{A} \)-valued right inner product on \( F(\mathcal{H}) \). The operators are still denoted by \( s_\xi, t_\xi \) respectively. The adjoints of \( s_\xi, t_\xi : F(\mathcal{H}) \to F(\mathcal{H}) \) with respect to the \( \mathcal{A} \)-valued right inner product on \( F(\mathcal{H}) \) map \( F_{n+1}(\mathcal{H}) \) to \( F_n(\mathcal{H}), n = 0, 1, 2, \ldots \).

**Lemma 3.3.**

(i) For \( \xi, \xi' \in \mathcal{H} = F_1(\mathcal{H}) \), we have

\[
 s_\xi^* \xi' = \langle \xi | \xi' \rangle_{B_1} \oplus 0 \in B_1 \oplus B_2,
 t_\xi^* \xi' = 0 \oplus \langle \xi | \xi' \rangle_{B_2} \in B_1 \oplus B_2.
\]

(ii) For \( \xi \in \mathcal{H} \) and \( \xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_n} \xi_{n+1} \in F_{n+1}(\mathcal{H}), n = 1, 2, \ldots \), we have

\[
 s_\xi^* (\xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_n} \xi_{n+1}) = \begin{cases} \phi_1(\langle \xi | \xi_1 \rangle_{B_1}) \xi_2 \otimes_{i_2} \cdots \otimes_{i_n} \xi_{n+1} & \text{if } i_1 = 1, \\ 0 & \text{if } i_1 = 2, \end{cases}
 t_\xi^* (\xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_n} \xi_{n+1}) = \begin{cases} 0 & \text{if } i_1 = 1, \\ \phi_2(\langle \xi | \xi_1 \rangle_{B_2}) \xi_2 \otimes_{i_2} \cdots \otimes_{i_n} \xi_{n+1} & \text{if } i_1 = 2. \end{cases}
\]

**Proof.** We will show the assertions (i) and (ii) for \( s_\xi^* \). (i) For \( b_1 \oplus b_2 \in B_1 \oplus B_2 \), we have

\[
 \langle b_1 \oplus b_2 | s_\xi^* \xi' \rangle_{\mathcal{A}} = \langle \xi \phi_1(b_1) | \xi' \rangle_{\mathcal{A}} = \lambda_1(b_1^*(\xi | \xi')_{B_1} = \langle b_1 \oplus b_2 | \langle \xi | \xi' \rangle_{B_1} \oplus 0 \rangle_{\mathcal{A}}
\]

so that \( s_\xi^* \xi' = \langle \xi | \xi' \rangle_{B_1} \oplus 0 \).

(ii) For \( \xi_1 \otimes_{j_1} \cdots \otimes_{j_{n-1}} \xi_n \in F_n(\mathcal{H}) \) with \( n = 1, 2, \ldots \) we have

\[
 \langle \xi_1 \otimes_{j_1} \cdots \otimes_{j_{n-1}} \xi_n | s_\xi^* (\xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_n} \xi_{n+1}) \rangle_{\mathcal{A}} = \lambda_1(\langle \xi_1 \otimes_1 \xi_2 \otimes_{i_2} \cdots \otimes_{i_n} \xi_{n+1})_{B_1} = \lambda_1(\langle \xi | \xi_1 \rangle_{B_1}) \xi_2 \otimes_{i_2} \cdots \otimes_{i_n} \xi_{n+1} \rangle_{B_1}
\]

\[
 = \begin{cases} \langle \xi_1 \otimes_{j_1} \cdots \otimes_{j_{n-1}} \xi_n | \phi_1(\langle \xi | \xi_1 \rangle_{B_1}) \xi_2 \otimes_{i_2} \cdots \otimes_{i_n} \xi_{n+1} \rangle_{\mathcal{A}} & \text{if } i_1 = 1, \\ 0 & \text{if } i_1 = 2. \end{cases}
\]

Denote by \( \bar{\phi}_i \) the left actions of \( B_i, i = 1, 2 \) on \( F_n(\mathcal{H}) \) and hence on \( F(\mathcal{H}) \) respectively. They satisfy the following equalities

\[
 \bar{\phi}_1(z)(b_1 \oplus b_2) = z b_1 \oplus 0, \quad \bar{\phi}_2(w)(b_1 \oplus b_2) = 0 \oplus w b_2,
\]

\[
 \bar{\phi}_1(z)(\xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_n} \xi_n) = (\phi_1(z)\xi_1) \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_n} \xi_n,
\]

\[
 \bar{\phi}_2(w)(\xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_n} \xi_n) = (\phi_2(w)\xi_1) \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_n} \xi_n
\]

for \( z \in B_1, w \in B_2, b_1 \oplus b_2 \in B_1 \oplus B_2 \) and \( \xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_n} \xi_n \in F_n(\mathcal{H}) \). More generally let us denote by \( \mathcal{L}_A(\mathcal{H}) \) and \( \mathcal{L}_A(F(\mathcal{H})) \) the \( \mathcal{C}^* \)-algebras of all bounded adjointable right \( \mathcal{A} \)-module maps on \( \mathcal{H} \) and on \( F(\mathcal{H}) \) with respect to their right \( \mathcal{A} \)-valued inner products respectively. For \( L \in \mathcal{L}_A(\mathcal{H}) \), define \( \overline{L} \in \mathcal{L}_A(F(\mathcal{H})) \) by

\[
 \overline{L}(b_1 \oplus b_2) = 0 \quad \text{for } b_1 \oplus b_2 \in B_1 \oplus B_2 \subset F_0(\mathcal{H}),
\]

\[
 \overline{L}(\xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_n} \xi_n) = (L \xi_1) \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_n} \xi_n
\]

for \( \xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_n} \xi_n \in F_n(\mathcal{H}) \).
Lemma 3.4. Both the maps $\tilde{\phi}_i : B_i \rightarrow \mathcal{L}_A(F(\mathcal{H}))$ for $i = 1, 2$ are faithful $*$-homomorphisms.

Proof. By assumption, the $*$-homomorphisms $\phi_i : B_i \rightarrow \mathcal{L}_A(\mathcal{H}), i = 1, 2$ are faithful, so that the $*$-homomorphisms $\phi_i : B_i \rightarrow \mathcal{L}_A(F(\mathcal{H})), i = 1, 2$ are both faithful. □

Lemma 3.5. For $\xi, \zeta \in \mathcal{H}, z \in B_1, w \in B_2, L \in \mathcal{L}_A(\mathcal{H})$ and $c, d \in \mathbb{C}$, the following equalities hold on $F(\mathcal{H})$:

\begin{align}
  s_{c\xi + d\zeta} &= cs\xi + ds\zeta, & t_{c\xi + d\zeta} &= ct\xi + dt\zeta, \quad (3.1) \\
  s_{L\xi} &= \overline{T}_s\phi_1(z), & t_{L\xi} &= \overline{T}_t\phi_2(w), \quad (3.2) \\
  s^*_\xi T_s\xi &= \phi_1(\langle \xi \mid L\xi \rangle_{B_1}, & t^*_\xi T_t\xi &= \phi_2(\langle \xi \mid L\xi \rangle_{B_2}). \quad (3.3)
\end{align}

Proof. The equalities (3.1) are obvious. We will show the equalities (3.2) and (3.3) for $s_\xi$. We have for $b_1 + b_2 \in B_1 \oplus B_2$

\[ s_{L\xi}(b_1 + b_2) = [L\xi(b_1)]b_2 = [L\xi(b_2)]b_1 = [L\xi(b_2)](b_1 + b_2) \]

\[ = [L\xi(b_1)](b_1 + b_2) = [L\xi(b_1)] = [L\xi(b_2)] = [L\xi(b_2)] \]

so that $s_{L\xi}(z) = \overline{T}_s\phi_1(z)$ on $F_n(\mathcal{H}), n = 0, 1, \ldots$. Hence the equalities (3.2) hold.

For $\xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_{n-1}} \xi_n \in F_n(\mathcal{H})$, we have

\[ s_{L\xi_1}(b_1 + b_2) = s_{L\xi_1}(L\xi_1(b_1)) = \langle \xi_1 \mid L\xi_1(b_1) \rangle_{B_1} \otimes 0 \\
= \langle \xi_1 \mid L\xi_1 \rangle_{B_1} b_1 \otimes 0 = \phi_1(\langle \xi_1 \mid L\xi_1 \rangle_{B_1})(b_1 + b_2) \]

so that $s_{L\xi_1}(z) = \overline{T}_s\phi_1(z)$ on $F_n(\mathcal{H})$ for $n = 0, 1, 2, \ldots$. Hence the equalities (3.3) hold. □

The $C^*$-subalgebra of $\mathcal{L}_A(F(\mathcal{H}))$ generated by the operators $s_\xi, t_\xi$ for $\xi \in \mathcal{H}$ is denoted by $\mathcal{T}_F(\mathcal{H})$ and is called the Toeplitz quad module algebra for $\mathcal{H}$.

Lemma 3.6. The $C^*$-algebra $\mathcal{T}_F(\mathcal{H})$ contains the operators $\overline{T}_1(z), \overline{T}_2(w)$ for $z \in B_1, w \in B_2$.

Proof. By (3.3) in the preceding lemma, one sees

\[ s^*_\xi s_\xi = \overline{T}_1(\langle \xi \mid \xi \rangle_{B_1}, \quad t^*_\xi t_\xi = \overline{T}_2(\langle \xi \mid \xi \rangle_{B_2}, \quad \zeta, \xi \in \mathcal{H}. \]
Since $\mathcal{H}$ is a full $C^*$-quad module, the inner products $\langle \zeta \mid \xi \rangle_{B_1}, \langle \zeta \mid \xi \rangle_{B_2}$ for $\zeta, \xi \in \mathcal{H}$ generate the $C^*$-algebras $B_1, B_2$ respectively. Hence $\hat{\phi}_1(B_1), \hat{\phi}_2(B_2)$ are contained in $\mathcal{T}_{F(\mathcal{H})}$.

\begin{lemma}
There exists an action $\gamma$ of $\mathbb{R}/\mathbb{Z} = \mathbb{T}$ on $\mathcal{T}_{F(\mathcal{H})}$ such that
\[\gamma_r(s_\xi) = e^{2\pi \sqrt{-1} r} s_\xi, \quad \gamma_r(t_\xi) = e^{2\pi \sqrt{-1} r} t_\xi, \quad \xi \in \mathcal{H},\]
\[\gamma_r(\hat{\phi}_1(z)) = \hat{\phi}_1(z), \quad z \in B_1, \quad \gamma_r(\hat{\phi}_2(w)) = \hat{\phi}_2(w), \quad w \in B_2\]
for $r \in \mathbb{R}/\mathbb{Z} = \mathbb{T}$.
\end{lemma}

\begin{proof}
We will first define a one-parameter unitary group $u_r, r \in \mathbb{R}/\mathbb{Z} = \mathbb{T}$ on $F(\mathcal{H})$ with respect to the right $A$-valued inner product as in the following way.

For $n = 0 : u_r : F_0(\mathcal{H}) \rightarrow F_0(\mathcal{H})$ is defined by
\[u_r(b_1 \oplus b_2) = b_1 \oplus b_2 \quad \text{for } b_1 \oplus b_2 \in B_1 \oplus B_2.\]

For $n = 1, 2, \ldots : u_r : F_n(\mathcal{H}) \rightarrow F_n(\mathcal{H})$ is defined by
\[u_r(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = e^{2\pi \sqrt{-1} r} \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n\]
for $\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n \in F_n(\mathcal{H})$. We therefore have a one-parameter unitary group $u_r$ on $F(\mathcal{H})$. We then define an automorphism $\gamma_r$ on $\mathcal{L}_A(F(\mathcal{H}))$ for $r \in \mathbb{R}/\mathbb{Z}$ by
\[\gamma_r(T) = u_r T u_r^* \quad \text{for } T \in \mathcal{L}_A(F(\mathcal{H})), \quad r \in \mathbb{R}/\mathbb{Z}.\]

It then follows that for $b_1 \oplus b_2 \in B_1 \oplus B_2$
\[\gamma_r(s_\xi)(b_1 \oplus b_2) = u_r s_\xi u_r^*(b_1 \oplus b_2) = u_r(\xi_1 \otimes b_1)) = e^{2\pi \sqrt{-1} r} s_\xi(b_1 \oplus b_2),\]
and for $\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n \in F_n(\mathcal{H})$
\[\gamma_r(s_\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = e^{2\pi \sqrt{-1} r} u_r(\xi \otimes \xi_1 \cdots \otimes \xi_n)\]
for $\xi_1 \otimes \cdots \otimes \xi_n \in F_n(\mathcal{H})$. It is direct to see that
\[\gamma_r(\hat{\phi}_1(z)) = \hat{\phi}_1(z), \quad \gamma_r(\hat{\phi}_2(w)) = \hat{\phi}_2(w), \quad \text{for } z \in B_1, w \in B_2.\]

It is also obvious that $\gamma_r(\mathcal{T}_{F(\mathcal{H})}) = \mathcal{T}_{F(\mathcal{H})}$ for $r \in \mathbb{R}/\mathbb{Z}$. \hfill $\square$

Denote by $J(\mathcal{H})$ the $C^*$-subalgebra of $\mathcal{L}_A(F(\mathcal{H}))$ generated by the elements
\[\mathcal{L}_A(\bigoplus_{n=0}^{\text{finite}} F_n(\mathcal{H})).\]

The algebra $J(\mathcal{H})$ is a closed two-sided ideal of $\mathcal{L}_A(F(\mathcal{H}))$.

\begin{definition}
The $C^*$-algebra $\mathcal{O}_{F(\mathcal{H})}$ associated to the Hilbert $C^*$-quad module $\mathcal{H}$ of general type is defined by the quotient $C^*$-algebra of $\mathcal{T}_{F(\mathcal{H})}$ by the ideal $\mathcal{T}_{F(\mathcal{H})} \cap J(\mathcal{H})$.

We denote by $[x]$ the quotient image of an element $x \in \mathcal{T}_{F(\mathcal{H})}$ under the ideal $\mathcal{T}_{F(\mathcal{H})} \cap J(\mathcal{H})$. We set the elements of $\mathcal{O}_{F(\mathcal{H})}$
\[s_\xi = [s_\xi], \quad t_\xi = [t_\xi], \quad \Phi_1(z) = [\hat{\phi}_1(z)], \quad \Phi_2(w) = [\hat{\phi}_2(w)]\]
for $\xi \in \mathcal{H}$ and $z \in B_1, w \in B_2$. By the preceding lemmas, we have
Proposition 3.8. The $C^*$-algebra $\mathcal{O}_{F(H)}$ is generated by the family of operators $S_\xi T_\xi$ for $\xi \in \mathcal{H}$. It contains the operators $\Phi_1(z), \Phi_2(w)$ for $z \in \mathcal{B}_1, w \in \mathcal{B}_2$. They satisfy the following equalities

$$S_\xi + d_\xi = cS_\xi + dS_\xi, \quad T_\xi = cT_\xi + dT_\xi,$$

$$S_{\Phi_1(z')\xi \varphi_1(z)} = \Phi_1(z')S_\xi \Phi_1(z), \quad T_{\Phi_1(z')\xi \varphi_2(w)} = \Phi_1(z')T_\xi \Phi_2(w),$$

$$S_{\Phi_2(w')\xi \varphi_1(z)} = \Phi_2(w')S_\xi \Phi_1(z), \quad T_{\Phi_2(w')\xi \varphi_2(w)} = \Phi_2(w')T_\xi \Phi_2(w),$$

$$S_\xi T_\xi = \Phi_1(\langle \xi | \xi \rangle_{B_1}), \quad T_\xi S_\xi = \Phi_2(\langle \xi | \xi \rangle_{B_2})$$

for $\xi, \zeta \in \mathcal{H}$, $c, d \in \mathbb{C}$ and $z, z' \in \mathcal{B}_1, w, w' \in \mathcal{B}_2$.

4. The $C^*$-algebras of Hilbert $C^*$-quad modules of finite type

In what follows, we assume that a Hilbert $C^*$-quad module $\mathcal{H}$ is of finite type. In this section, we will study the $C^*$-algebra $\mathcal{O}_{F(H)}$ for a Hilbert $C^*$-quad module $\mathcal{H}$ of finite type. Let $\{u_1, \ldots, u_M\}$ be a finite basis of $\mathcal{H}$ as a Hilbert $C^*$-right module over $\mathcal{B}_1$ and $\{v_1, \ldots, v_N\}$ a finite basis of $\mathcal{H}$ as a Hilbert $C^*$-right module over $\mathcal{B}_2$.

Keep the notations as in the previous section. We set

$$s_i = s_{u_i} \quad \text{for} \quad i = 1, \ldots, M \quad \text{and} \quad t_k = t_{v_k} \quad \text{for} \quad k = 1, \ldots, N.$$  

(4.1)

By (2.7) and Lemma 3.5 we have for $\xi \in \mathcal{H}$

$$s_\xi = \sum_{i=1}^{M} s_i \bar{\phi}_1(u_i | \xi)_{B_1}, \quad t_\xi = \sum_{k=1}^{N} t_k \bar{\phi}_2(v_k | \xi)_{B_2}.$$  

(4.2)

Let $P_n$ be the projection on $F(\mathcal{H})$ onto $F_n(\mathcal{H})$ for $n = 0, 1, \ldots$ so that $\sum_{n=1}^{\infty} P_n = 1$ on $F(\mathcal{H})$.

Lemma 4.1. For $\xi, \zeta \in \mathcal{H}$, we have

(i) $s_\xi t_\zeta P_n = 0$ for $n = 1, 2, \ldots$ and hence $s_\xi t_\zeta = s_\xi t_\zeta P_0$.

(ii) $t_\xi s_\zeta P_n = 0$ for $n = 1, 2, \ldots$ and hence $t_\xi s_\zeta = t_\xi s_\zeta P_0$.

Proof. (i) For $n = 1, 2, \ldots$, we have

$$s_\xi t_\zeta(\xi_1 \otimes i_1 \otimes i_{1-1}, \xi_n) = s_\xi(\xi_2 \otimes i_1 \otimes i_{1-1}, \xi_n) = 0.$$  

(ii) is similarly shown to (i). \qed

Define two projections on $F(\mathcal{H})$ by

$$P_s = \text{The projection onto } \bigoplus_{n=0}^{\infty} \sum_{(i_1, \ldots, i_n) \in \Gamma_n} \mathcal{H} \otimes_1 \mathcal{H} \otimes_{i_1} \mathcal{H} \otimes_{i_2} \cdots \otimes_{i_n} \mathcal{H},$$

$$P_t = \text{The projection onto } \bigoplus_{n=0}^{\infty} \sum_{(i_1, \ldots, i_n) \in \Gamma_n} \mathcal{H} \otimes_2 \mathcal{H} \otimes_{i_1} \mathcal{H} \otimes_{i_2} \cdots \otimes_{i_n} \mathcal{H}.$$  

Lemma 4.2. Keep the above notations.

$$\sum_{i=1}^{M} s_i s_i^* = P_1 + P_s \quad \text{and} \quad \sum_{k=1}^{N} t_k t_k^* = P_1 + P_t.$$  

(4.3)
Hence
\[ \sum_{i=1}^{M} s_i s_i^* + \sum_{k=1}^{N} t_k t_k^* + P_0 = 1_{F(H)} + P_1. \] (4.4)

Proof. For \( \xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_{n-1}} \xi_n \in F_n(H) \) with \( 2 \leq n \in \mathbb{N} \), we have
\[ s_i s_i^* (\xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_{n-1}} \xi_n) \]
\[ = \begin{cases} u_i \otimes_1 \phi_1((u_i | \xi_1)_n) \xi_2 \otimes_{i_2} \cdots \otimes_{i_{n-1}} \xi_n & \text{if } i_1 = 1, \\ 0 & \text{if } i_1 = 2. \end{cases} \]
As \( u_i \otimes_1 \phi_1((u_i | \xi_1)_n) \xi_2 \otimes_{i_2} \cdots \otimes_{i_{n-1}} \xi_n \) and \( \sum_{i=1}^{M} u_i \varphi_1((u_i | \xi_1) B_i) = \xi_1 \), we have
\[ \sum_{i=1}^{M} s_i s_i^* (\xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_{n-1}} \xi_n) = \begin{cases} \xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_{n-1}} \xi_n & \text{if } i_1 = 1, \\ 0 & \text{if } i_1 = 2. \end{cases} \]
and hence
\[ \sum_{i=1}^{M} s_i s_i^* |_{\oplus_{n=2}^{\infty} F_n(H)} = P_1 |_{\oplus_{n=2}^{\infty} F_n(H)}. \]
For \( \xi \in F_1(H) = H \), we have \( s_i s_i^* \xi = s_i((u_i | \xi) B_i) = 0 \) and \( \sum_{i=1}^{M} u_i \varphi_1((u_i | \xi) B_i) = \xi \) so that
\[ \sum_{i=1}^{M} s_i s_i^* \xi = \sum_{i=1}^{M} u_i \varphi_1((u_i | \xi) B_i) = \xi \]
and hence
\[ \sum_{i=1}^{M} s_i s_i^* |_{F_1(H)} = 1_{F_1(H)}. \]
As \( s_i s_i^* (b_1 \oplus b_2) = 0 \) for \( b_1 \oplus b_2 \in B_1 \oplus B_2 \), we have
\[ \sum_{i=1}^{M} s_i s_i^* |_{B_0(H)} = 0. \]
Therefore we conclude that
\[ \sum_{i=1}^{M} s_i s_i^* = P_1 + P_1 \quad \text{and similarly} \quad \sum_{k=1}^{N} t_k t_k^* = P_1 + P_1. \]
As \( P_1 + P_1 + P_0 + P_1 = 1_{F(H)} \), one obtains (4.4). \( \square \)

We set the operators
\[ S_i = S_{u_i} (= [s_i]) \quad \text{for } i = 1, \ldots, M, \quad \text{and} \]
\[ T_k = T_{v_k} (= [t_k]) \quad \text{for } k = 1, \ldots, N. \]
in the \( C^* \)-algebra \( \mathcal{O}_{F(H)} \). As two operators \( \sum_{i=1}^{M} s_i s_i^* \) and \( \sum_{k=1}^{N} t_k t_k^* \) are projections by (4.3), so are \( \sum_{i=1}^{M} S_i S_i^* \) and \( \sum_{k=1}^{N} T_k T_k^* \). Since \( P_1 - P_0 \in J(H) \), the identity (4.4) implies
\[ \sum_{i=1}^{M} S_i S_i^* + \sum_{k=1}^{N} T_k T_k^* = 1. \] (4.5)
Therefore we have
Theorem 4.3. Let \( \mathcal{H} \) be a Hilbert \( C^* \)-quad module over \((\mathcal{A}; B_1, B_2)\) of finite type with finite basis \(\{u_1, \ldots, u_M\}\) as a right \(B_1\)-module and \(\{v_1, \ldots, v_N\}\) as a right \(B_2\)-module. Then we have

(i) The \( C^* \)-algebra \( \mathcal{O}_{F(\mathcal{H})} \) is generated by the operators \( S_1, \ldots, S_M, T_1, \ldots, T_N \) and the elements \( \Phi_1(z), \Phi_2(w) \) for \( z \in B_1, w \in B_2 \).

(ii) They satisfy the following operator relations:

\[
\sum_{i=1}^M S_i S_i^* + \sum_{k=1}^N T_k T_k^* = 1, \quad S_j^* T_l = 0, \quad (4.6) \\
\Phi_1(z) S_j = \sum_{i=1}^M S_i \Phi_1((u_i \mid u_j)_{B_1}), \quad T_l \Phi_2((v_k \mid v_l)_{B_2}) = \sum_{k=1}^N T_k T_l = \Phi_2((v_k \mid \Phi_1(z) v_l)_{B_2}), \quad (4.7)
\]

\[
\Phi_2(w) S_j = \sum_{i=1}^M S_i \Phi_2((u_i \mid \Phi_2(w) u_j)_{B_1}), \quad T_l \Phi_2((v_k \mid \Phi_2(w) v_l)_{B_2}) = \sum_{k=1}^N T_k \Phi_2((v_k \mid \Phi_2(w) v_l)_{B_2}), \quad (4.8)
\]

for \( z \in B_1, w \in B_2, i, j = 1, \ldots, M, k, l = 1, \ldots, N \).

(iii) There exists an action \( \gamma \) of \( \mathbb{R}/\mathbb{Z} = \mathbb{T} \) on \( \mathcal{O}_{F(\mathcal{H})} \) such that

\[
\gamma_r(s_i) = e^{2\pi \sqrt{-1} r} s_i, \quad \gamma_r(T_k) = e^{2\pi \sqrt{-1} T_k}, \\
\gamma_r(\Phi_1(z)) = \Phi_1(z), \quad \gamma_r(\Phi_2(w)) = \Phi_2(w)
\]

for \( r \in \mathbb{R}/\mathbb{Z} = \mathbb{T}, i = 1, \ldots, M, k = 1, \ldots, N, \) and \( z \in B_1, w \in B_2 \).

Proof. (i) The assertion comes from the equalities (4.2).

(ii) The first equality of (4.6) is (4.5). As the projection \( P_0 \) belongs to \( J(H) \), Lemma 4.1 ensures us the second equality of (4.6). The equalities (4.7) come from (3.7). For \( z \in B_1 \) and \( j = 1, \ldots, M \), we have \( \Phi_1(z) u_j = \sum_{i=1}^M u_i \phi_1((u_i \mid \phi_1(z) u_j)_{B_1}) \) so that

\[
\bar{\phi}_1(z) s_j = s_{\phi_1(z) u_j} = \sum_{i=1}^M s_{u_i} \bar{\phi}_1((u_i \mid \phi_1(z) u_j)_{B_1})
\]

which goes to the first equality of (4.8). The other equalities of (4.8) and (4.9) are similarly shown.

(iii) The assertion is direct from Lemma 3.7. \( \square \)

The action \( \gamma \) of \( \mathbb{T} \) on \( \mathcal{O}_{F(\mathcal{H})} \) defined in the above theorem (iii) is called the gauge action.

5. THE UNIVERSAL \( C^* \)-ALGEBRAS ASSOCIATED WITH HILBERT \( C^* \)-QUAD MODULES

In this section, we will prove that the \( C^* \)-algebra \( \mathcal{O}_{F(\mathcal{H})} \) associated with a Hilbert \( C^* \)-quad module of finite type is the universal \( C^* \)-algebra subject to the operator relations stated in Theorem 4.3 (ii). Throughout this section, we fix a Hilbert \( C^* \)-quad module \( \mathcal{H} \) over \((\mathcal{A}; B_1, B_2)\) of finite type with finite basis \(\{u_1, \ldots, u_M\}\) as a right Hilbert \(B_1\)-module and \(\{v_1, \ldots, v_N\}\) as a right Hilbert \(B_2\)-module as in the previous section.
Let $\mathcal{P}_H$ be the universal $*$-algebra generated by operators $S_1, \ldots, S_M, T_1, \ldots, T_N$ and elements $z \in B_1, w \in B_2$ subject to the relations:

\[
\sum_{i=1}^{M} S_i S_i^* + \sum_{k=1}^{N} T_k T_k^* = 1, \quad S_j^* T_l = 0, \quad \text{(5.1)}
\]

\[
S_i^* S_j = \langle u_i \mid \phi_1(z) u_j \rangle_{B_1}, \quad T_k^* T_l = \langle v_k \mid \phi_1(z) v_l \rangle_{B_2}, \quad \text{(5.2)}
\]

\[
z S_j = \sum_{i=1}^{M} S_i \langle u_i \mid \phi_1(z) u_j \rangle_{B_1}, \quad z T_l = \sum_{k=1}^{N} T_k \langle v_k \mid \phi_1(z) v_l \rangle_{B_2}, \quad \text{(5.3)}
\]

\[
w S_j = \sum_{i=1}^{M} S_i \langle u_i \mid \phi_2(w) u_j \rangle_{B_1}, \quad w T_l = \sum_{k=1}^{N} T_k \langle v_k \mid \phi_2(w) v_l \rangle_{B_2} \quad \text{(5.4)}
\]

for $z \in B_1, w \in B_2, i, j = 1, \ldots, M, k, l = 1, \ldots, N$. The above four relations (5.1), (5.2), (5.3), (5.4) are called the relations $(\mathcal{H})$. In what follows, we fix operators $S_1, \ldots, S_M, T_1, \ldots, T_N$ satisfying the relations $(\mathcal{H})$.

**Lemma 5.1.** The sums $\sum_{i=1}^{M} S_i S_i^*$ and $\sum_{k=1}^{N} T_k T_k^*$ are both projections.

**Proof.** Put $P = \sum_{i=1}^{M} S_i S_i^*$ and $Q = \sum_{k=1}^{N} T_k T_k^*$. By the relations (5.1), one sees that $0 \leq P, Q \leq 1$ and $P + Q = 1, PQ = 0$. It is easy to see that both $P$ and $Q$ are projections. \(\square\)

**Lemma 5.2.**

(i) For $i, j = 1, \ldots, M$ and $z \in B_1, w \in B_2$ we have

\[
S_i^* z S_j = \langle u_i \mid \phi_1(z) u_j \rangle_{B_1}, \quad S_i^* w S_j = \langle u_i \mid \phi_2(w) u_j \rangle_{B_1}.
\]

(ii) For $k, l = 1, \ldots, N$ and $z \in B_1, w \in B_2$ we have

\[
T_k^* z T_l = \langle v_k \mid \phi_1(z) v_l \rangle_{B_2}, \quad T_k^* w T_l = \langle v_k \mid \phi_2(w) v_l \rangle_{B_2}.
\]

**Proof.** (i) By (5.3), we have

\[
S_i^* z S_j = \sum_{h=1}^{M} S_i^* S_h \langle u_h \mid \phi_1(z) u_j \rangle_{B_1} = \sum_{h=1}^{M} \langle u_i \mid u_h \rangle_{B_1} \langle u_h \mid \phi_1(z) u_j \rangle_{B_1}
\]

\[
= \langle u_i \mid \sum_{h=1}^{M} u_h (u_h \mid \phi_1(z) u_j)_{B_1} \rangle_{B_1} = \langle u_i \mid \phi_1(z) u_j \rangle_{B_1}.
\]

The other equality $S_i^* w S_j = \langle u_i \mid \phi_2(w) u_j \rangle_{B_1}$ is similarly shown to the above equalities.

(ii) is similar to (i). \(\square\)

By the equalities (2.10), (2.11) we have

**Lemma 5.3.** Keep the above notations.

(i) For $w \in B_2, j = 1, \ldots M$, the element $S_j^* w S_j$ belongs to $\mathcal{A}$ and the formula holds:

\[
\sum_{j=1}^{M} S_j^* \langle \xi \mid \eta \rangle_{B_2} S_j = \langle \xi \mid \eta \rangle_{\mathcal{A}} \quad \text{for} \ \xi, \eta \in \mathcal{H}.
\]
Lemma 5.4. The following equalities for $z \in B_1$ and $w \in B_2$ hold:

(i) $$z = \sum_{i,j=1}^{M} S_i(u_i | \phi_1(z)u_j)_{B_1}S_j^* + \sum_{k,l=1}^{N} T_k(v_k | \phi_1(z)v_l)_{B_2}T_l^*, \quad (5.5)$$

$$w = \sum_{i,j=1}^{M} S_i(u_i | \phi_2(w)u_j)_{B_1}S_j^* + \sum_{k,l=1}^{N} T_k(v_k | \phi_2(w)v_l)_{B_2}T_l^*. \quad (5.6)$$

(ii) $$z^* = \sum_{i,j=1}^{M} S_i(\phi_1(z)^*u_j)_{B_1}S_j + \sum_{k,l=1}^{N} T_k(\phi_1(z)^*v_l)_{B_2}T_l^*, \quad (5.7)$$

$$w^* = \sum_{i,j=1}^{M} S_i(\phi_2(w)^*u_j)_{B_1}S_j + \sum_{k,l=1}^{N} T_k(\phi_2(w)^*v_l)_{B_2}T_l^*. \quad (5.8)$$

Proof. (i) By (5.3) and (5.4), we have

$$zw = \sum_{i,j=1}^{M} S_i(u_i | \phi_1(z)\phi_2(w)u_j)_{B_1}S_j + \sum_{k,l=1}^{N} T_k(v_k | \phi_1(z)\phi_2(w)v_l)_{B_2}T_l^*,$$

$$wz = \sum_{i,j=1}^{M} S_i(u_i | \phi_2(w)\phi_1(z)u_j)_{B_1}S_j^* + \sum_{k,l=1}^{N} T_k(v_k | \phi_2(w)\phi_1(z)v_l)_{B_2}T_l^*.$$ 

so that by (5.1)

$$z = \sum_{i,j=1}^{M} S_i(u_i | \phi_1(z)u_j)_{B_1}S_j^* + \sum_{k,l=1}^{N} T_k(v_k | \phi_1(z)v_l)_{B_2}T_l^*.$$ 

Similarly we have (5.6).

(ii) All the adjoints of $\phi_1(z), \phi_2(w)$ for $z \in B_1, w \in B_2$ by the three inner products $\langle \cdot | \cdot \rangle_{B_1}, \langle \cdot | \cdot \rangle_{B_2}$ and $\langle \cdot | \cdot \rangle_{A}$ on $\mathcal{H}$ coincide with $\phi_1(z^*), \phi_2(w^*)$ respectively. Hence the assertions are clear.

(iii) By (i) we have

$$zw = \left( \sum_{i,j=1}^{M} S_i(u_i | \phi_1(z)u_j)_{B_1}S_j^* + \sum_{k,l=1}^{N} T_k(v_k | \phi_1(z)v_l)_{B_2}T_l^* \right)$$

$$\cdot \left( \sum_{g,h=1}^{M} S_g(u_g | \phi_2(w)u_h)_{B_1}S_h^* + \sum_{m,n=1}^{N} T_m(v_m | \phi_2(w)v_n)_{B_2}T_n^* \right).$$
As $S_j^*T_m = T_i^*S_g = 0$ for any $j, g = 1, \ldots, M, l, m = 1, \ldots, N$, it follows that

$$zw = \sum_{i,j,g,h=1}^M S_i(\langle u_i | \phi_1(z)u_j \rangle_{B_1}S_j^*S_g(\langle u_g | \phi_2(w)u_h \rangle_{B_1}S_h^*$$

$$+ \sum_{k,l,m,n=1}^N T_k(\langle v_k | \phi_1(z)v_l \rangle_{B_2}T_i^*T_m(\langle v_m | \phi_2(w)v_n \rangle_{B_2}T_n^*$$

$$= \sum_{i,j,g,h=1}^M S_i(\langle u_i | \phi_1(z)u_j \rangle_{B_1}(\langle u_j | \phi_2(w)u_g \rangle_{B_1}S_h^*$$

$$+ \sum_{k,l,m,n=1}^N T_k(\langle v_k | \phi_1(z)v_l \rangle_{B_2}(\langle v_l | \phi_2(w)v_m \rangle_{B_2}T_n^*$$

$$= \sum_{i,h=1}^M S_i(\langle u_i | \phi_1(z)\sum_{g=1}^M u_g(\langle u_g | \phi_2(w)u_h \rangle_{B_1}S_h^*$$

$$+ \sum_{k,n=1}^N T_k(\langle v_k | \phi_1(z)\sum_{m=1}^N v_m(\langle v_m | \phi_2(w)v_n \rangle_{B_2}T_n^*$$

$$= \sum_{i,k,n=1}^M S_i(\langle u_i | \phi_1(z)\phi_2(w)u_h \rangle_{B_1}S_h^* + \sum_{k,n=1}^N T_k(\langle v_k | \phi_1(z)\phi_2(w)v_n \rangle_{B_2}T_n^*.$$

\[\square\]

**Lemma 5.5.** Let $p(z, w)$ be a polynomial of elements of $B_1$ and $B_2$. Then we have

(i) $p(z, w)S_j = \sum_{i=1}^M S_i z_i$ for some $z_i \in B_1$.

(ii) $p(z, w)T_l = \sum_{k=1}^N T_kw_k$ for some $w_k \in B_2$.

**Proof.** For $z \in B_1, w \in B_2$ and $i, j = 1, \ldots, M$, by putting $z_{i,j} = \langle u_i | \phi_1(z)u_j \rangle_{B_1} \in B_1$ and $w_{i,j} = \langle u_i | \phi_2(w)u_j \rangle_{B_1} \in B_1$, the relations (5.3),(5.4) imply

$$zS_j = \sum_{i=1}^M S_i z_{i,j}, \quad wS_j = \sum_{i=1}^M S_i w_{i,j}$$

so that the assertion of (i) holds. (ii) is similarly shown to (i). \[\square\]

**Lemma 5.6.** Let $p(z, w)$ be a polynomial of elements of $B_1$ and $B_2$. Then we have

(i) $S_j^*p(z, w)S_j$ belongs to $B_1$ for all $i, j = 1, \ldots, M$.

(ii) $T_l^*p(z, w)T_l$ belongs to $B_2$ for all $k, l = 1, \ldots, N$.

(iii) $S_j^*p(z, w)T_l = 0$ for all $i = 1, \ldots, M, l = 1, \ldots, N$.

(iv) $T_l^*p(z, w)S_j = 0$ for all $k = 1, \ldots, N, j = 1, \ldots, M$.  

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Proof. (i) By the previous lemma, we know
\[ p(z, w)S_j = \sum_{h=1}^{M} S_h z_h \] for some \( z_h \in B_1 \), so that
\[ S_i^* p(z, w)S_j = \sum_{h=1}^{M} S_i^* S_h z_h = \sum_{h=1}^{M} (u_h | u_h)_{B_1} z_h. \]
As \( (u_h | u_h)_{B_1} z_h \) belongs to \( B_1 \), we see the assertion. (ii) is similarly shown to (i).
(iii) As \( T_k^* S_i = 0 \), we have
\[ T_k^* p(z, w)S_j = \sum_{i=1}^{M} T_k^* S_i z_i = 0. \]
(iv) is similarly shown to (i).

We set
\[ S_{1,i} := S_i, \quad i = 1, \ldots, M, \quad S_{2,k} := T_k, \quad k = 1, \ldots, N. \] (5.9)

Put
\[ \Sigma_1 = \{(1,i) \mid i = 1, \ldots, M\}, \quad \Sigma_2 = \{(2,k) \mid k = 1, \ldots, N\}. \]

Lemma 5.7. Every element of \( \mathcal{P}_H \) can be written as a linear combination of elements of the form
\[ S_{g_1,i_1} S_{g_2,i_2} \cdots S_{g_m,i_m} b S_{h_n,j_n}^* \cdots S_{h_2,j_2}^* S_{h_1,j_1}^* \]
for some \( (g_1, i_1), (g_2, i_2), \ldots, (g_m, i_m), (h_1, j_1), (h_2, j_2), \ldots, (h_n, j_n) \in \Sigma_1 \cup \Sigma_2 \) where \( b \) is a polynomial of elements of \( B_1 \) and \( B_2 \).

Proof. The assertion follows from the preceding lemmas.

By construction, every representation of \( B_1 \) and \( B_2 \) on a Hilbert space \( H \) together with operators \( S_i, i = 1, \ldots, M, T_k, k = 1, \ldots, N \) satisfying the relations \( (\mathcal{H}) \) extends to a representation of \( \mathcal{P}_H \) on \( B(H) \). We will endow \( \mathcal{P}_H \) with the norm obtained by taking the supremum of the norms in \( B(H) \) over all such representations. Note that this supremum is finite for every element of \( \mathcal{P}_H \) because of the inequalities \( \| S_i \|, \| T_k \| \leq 1 \), which come from (5.1). The completion of the algebra \( \mathcal{P}_H \) under the norm becomes a \( C^* \)-algebra denoted by \( \mathcal{O}_H \), which is called the universal \( C^* \)-algebra subject to the relations \( (\mathcal{H}) \).

Denote by \( C^*(\phi_1(B_1), \phi_2(B_2)) \) the \( C^* \)-subalgebra of \( \mathcal{L}_A(\mathcal{H}) \) generated by \( \phi_1(B_1) \) and \( \phi_2(B_2) \).

Lemma 5.8. An element \( L \) of the \( C^* \)-algebra \( C^*(\phi_1(B_1), \phi_2(B_2)) \) is both a right \( B_1 \)-module map and a right \( B_2 \)-module map. This means that the equalities
\[ [L \xi] \phi_i(b_i) = L[\xi \phi_i(b_i)] \quad \text{for } \xi \in \mathcal{H}, b_i \in B_i \]
hold.

Proof. Since both the operators \( \phi_1(z) \) for \( z \in B_1 \) and \( \phi_2(w) \) for \( w \in B_2 \) are right \( B_i \)-module maps for \( i = 1, 2 \), any element of the \( * \)-algebra algebraically generated by \( \phi_1(B_1) \) and \( \phi_2(B_2) \) is both a right \( B_1 \)-module map and a right \( B_2 \)-module map. Hence it is easy to see that any element \( L \) of the \( C^* \)-algebra \( C^*(\phi_1(B_1), \phi_2(B_2)) \) is both a right \( B_1 \)-module map and a right \( B_2 \)-module map.
Denote by $B_0$ the $C^*$-subalgebra of $O_H$ generated by $B_1$ and $B_2$.

**Lemma 5.9.** The correspondence

$$z, w \in B_0 \rightarrow \phi_1(z), \phi_2(w) \in C^*(\phi_1(B_1), \phi_2(B_2)) \subset L_A(H)$$

(5.10)

gives rise to an isomorphism from $B_0$ onto $C^*(\phi_1(B_1), \phi_2(B_2))$ as $C^*$-algebras.

**Proof.** We note that by hypothesis both the maps

$$\phi_1 : z \in B_1 \rightarrow \phi_1(z) \in L_A(H),$$

$$\phi_2 : w \in B_2 \rightarrow \phi_2(w) \in L_A(H)$$

are injective. Denote by $P(\phi_1(B_1), \phi_2(B_2))$ the $*$-algebra on $H$ algebraically generated by $\phi_1(z), \phi_2(w)$ for $z \in B_1, w \in B_2$. Define an operator $\pi(L) \in O_H$ for $L \in P(\phi_1(B_1), \phi_2(B_2))$ by

$$\pi(L) = \sum_{i,j=1}^M S_i(u_i \mid Lu_j)_{B_1} S_j^* + \sum_{k,l=1}^N T_k(v_k \mid Lv_l)_{B_2} T_l^*.$$  

(5.11)

Let $P_2$ be the $*$-subalgebra of $P_H$ algebraically generated by $B_1$ and $B_2$. Since $\pi(\phi_1(z)) = z$ for $z \in B_1$ and $\pi(\phi_2(w)) = w$ for $w \in B_2$ and by Lemma 5.4, the map

$$\pi : P(\phi_1(B_1), \phi_2(B_2)) \rightarrow P_2 \subset B_0$$

yields a $*$-homomorphism. As $S_i^* T_k = 0$ for $i = 1, \ldots, M, k = 1, \ldots, N$, we have

$$\|\pi(L)\| = \text{Max}\{\|\sum_{i,j=1}^M S_i(u_i \mid Lu_j)_{B_1} S_j^*\|, \|\sum_{k,l=1}^N T_k(v_k \mid Lv_l)_{B_2} T_l^*\|\}.$$

We then have

$$\|\sum_{i,j=1}^M S_i(u_i \mid Lu_j)_{B_1} S_j^*\| \leq \sum_{i,j=1}^M \|u_i \mid Lu_j\|_{B_1} \leq (\sum_{i,j=1}^M \|u_i\|_{B_1} \|u_j\|_{B_1}) \|L\|$$

and similarly

$$\|\sum_{k,l=1}^N T_k(v_k \mid Lv_l)_{B_2} T_l^*\| \leq (\sum_{k,l=1}^N \|v_k\|_{B_2} \|v_l\|_{B_2}) \|L\|.$$ 

By putting $C = \text{Max}\{\sum_{i,j=1}^M \|u_i\|_{B_1} \|u_j\|_{B_1}, \sum_{k,l=1}^N \|v_k\|_{B_2} \|v_l\|_{B_2}\}$, one has

$$\|\pi(L)\| \leq C \|L\| \quad \text{for all } L \in P(\phi_1(B_1), \phi_2(B_2)).$$

Hence $\pi$ extends to the $C^*$-algebra $C^*(\phi_1(B_1), \phi_2(B_2))$ such that $\pi(C^*(\phi_1(B_1), \phi_2(B_2))) = B_0$. The equality (5.11) holds for $L \in C^*(\phi_1(B_1), \phi_2(B_2))$. 

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We will next show that $\pi : C^*(\phi_1(\mathcal{B}_1), \phi_2(\mathcal{B}_2)) \to \mathcal{B}_0$ is injective. By (5.11), we have for $L \in C^*(\phi_1(\mathcal{B}_1), \phi_2(\mathcal{B}_2))$ and $h, h' = 1, \ldots, M$,

$$S_h^*\pi(L)S_{h'} = \sum_{i,j=1}^{M} S_h^* S_i(u_i | Lu_j)_{\mathcal{B}_i} S_j^* S_{h'},$$

$$= \sum_{i,j=1}^{M} \langle u_h | u_i \rangle_{\mathcal{B}_1} \langle u_i | Lu_j \rangle_{\mathcal{B}_1} \langle u_j | u_{h'} \rangle_{\mathcal{B}_1},$$

$$= \sum_{j=1}^{M} \sum_{i=1}^{M} \langle u_h | u_i \rangle_{\mathcal{B}_1} \langle u_i | Lu_j \rangle_{\mathcal{B}_1} \langle u_j | u_{h'} \rangle_{\mathcal{B}_1},$$

$$= \sum_{j=1}^{M} \langle u_h | Lu_j \rangle_{\mathcal{B}_1} \langle u_j | u_{h'} \rangle_{\mathcal{B}_1} = \langle u_h | Lu_{h'} \rangle_{\mathcal{B}_1}.$$

Suppose that $\pi(L) = 0$ so that $\langle u_h | Lu_{h'} \rangle_{\mathcal{B}_1} = 0$. Since

$$Lu_{h'} = \sum_{h=1}^{M} u_h \langle u_h | Lu_{h'} \rangle_{\mathcal{B}_1},$$

we see that $Lu_{h'} = 0$ so that $L = 0$. We thus conclude that $\pi$ is injective and hence isomorphic. \qed

Denote by $\phi_0 : \mathcal{B}_0 \to C^*(\phi_1(\mathcal{B}_1), \phi_2(\mathcal{B}_2))$ the inverse $\pi^{-1}$ of the $\ast$-isomorphism $\pi$ giving in the proof of the above lemma which satisfies

$$\phi_0(z) = \phi_1(z) \quad \text{for} \quad z \in \mathcal{B}_1, \quad \phi_0(w) = \phi_2(w) \quad \text{for} \quad w \in \mathcal{B}_2.$$

We put $\mathcal{F}_H^0 = \mathcal{B}_0$. For $n \in \mathbb{N}$, we denote by $\mathcal{F}_H^n$ the closed linear span of elements of the form

$$S_{g_1,i_1} \ldots S_{g_n,i_n} b S_{h_1,j_1} \ldots S_{h_2,j_2} S_{h_1,j_1}^*$$

for some $(g_1, i_1), (g_2, i_2), \ldots, (g_n, i_n), (h_1, j_1), (h_2, j_2), \ldots, (h_n, j_n) \in \Sigma_1 \cup \Sigma_2$ and $b \in \mathcal{B}_0$. Let us denote by $\mathcal{F}_H^n$ the $C^*$-subalgebra of $\mathcal{O}_H$ generated by $\cup_{n=0}^{\infty} \mathcal{F}_H^n$. By the relations (5.5) and (5.6), we see the following

**Lemma 5.10.** For $x \in \mathcal{B}_0$, the following identity holds:

$$x = \sum_{i,j=1}^{M} S_i(u_i | \phi_0(x) u_j)_{\mathcal{B}_1} S_j^* + \sum_{k,l=1}^{N} T_k(v_k | \phi_0(x) v_l)_{\mathcal{B}_2} T_l^*. \quad (5.12)$$

Hence by putting for $b \in \mathcal{B}_0$

$$b_{1,ij} = \langle u_i | \phi_0(b) u_j \rangle_{\mathcal{B}_1}, \quad i, j = 1, \ldots, M,$$

$$b_{2,kl} = \langle v_k | \phi_0(b) v_l \rangle_{\mathcal{B}_2}, \quad k, l = 1, \ldots, N,$$

we have
Lemma 5.11. For \( b \in \mathcal{B}_z \), the identity
\[
S_{g_1,i_1}S_{g_2,i_2} \cdots S_{g_n,i_n}bS_{h_1,j_1}^* \cdots S_{h_2,j_2}^*S_{h_1,j_1}^* = \sum_{i,j=1}^M S_{g_1,i_1}S_{g_2,i_2} \cdots S_{g_n,i_n}b_{1,ij}S_{h_1,j_1}^* \cdots S_{h_2,j_2}^*S_{h_1,j_1}^* + \sum_{k,l=1}^N S_{g_1,i_1}S_{g_2,i_2} \cdots S_{g_n,i_n}S_{2,k}b_{2,kl}S_{h_1,j_1}^* \cdots S_{h_2,j_2}^*S_{h_1,j_1}^*
\]
holds and induces an embedding of \( \mathcal{F}_n^H \hookrightarrow \mathcal{F}_{n+1}^H \) for \( n \in \mathbb{Z}_+ \).

Lemma 5.12. The \( C^* \)-algebra \( \mathcal{F}_H \) is the inductive limit \( \lim_{n \to \infty} \mathcal{F}_n^H \) of the sequence of the inclusions:
\[
\mathcal{F}_0^H \hookrightarrow \mathcal{F}_1^H \hookrightarrow \mathcal{F}_2^H \hookrightarrow \cdots \hookrightarrow \mathcal{F}_n^H \hookrightarrow \mathcal{F}_{n+1}^H \hookrightarrow \cdots \hookrightarrow \mathcal{F}_H.
\]
(5.13)

Let \( e^{2\pi\sqrt{-1}r} \in \mathbb{T} \) be a complex number of modulus one for \( r \in \mathbb{R}/\mathbb{Z} \). The elements
\[
e^{2\pi\sqrt{-1}r}S_i, i = 1, \ldots, M, \quad e^{2\pi\sqrt{-1}r}T_k, k = 1, \ldots, N, \quad z \in \mathcal{B}_1, \quad w \in \mathcal{B}_2
\]
in \( \mathcal{O}_H \) instead of
\[
S_i, i = 1, \ldots, M, \quad T_k, k = 1, \ldots, N, \quad z \in \mathcal{B}_1, \quad w \in \mathcal{B}_2
\]
satisfy the relations \( (\mathcal{H}) \). This implies the existence of an action on \( \mathcal{P}_H \) by automorphisms of the one-dimensional torus \( \mathbb{T} \) that acts on the generators by
\[
h_r(S_i) = e^{2\pi\sqrt{-1}r}S_i, \quad h_r(T_k) = e^{2\pi\sqrt{-1}r}T_k, \quad h_r(z) = z, \quad h_r(w) = w
\]
for \( i = 1, \ldots, M, k = 1, \ldots, N, z \in \mathcal{B}_1, w \in \mathcal{B}_2 \) and \( r \in \mathbb{R}/\mathbb{Z} = \mathbb{T} \). As the \( C^* \)-algebra \( \mathcal{O}_H \) has the largest norm on \( \mathcal{P}_H \), the action \( (h_r)_{r \in \mathbb{T}} \) on \( \mathcal{P}_H \) extends to an action of \( \mathbb{T} \) on \( \mathcal{O}_H \), still denote by \( h \). The formula
\[
a \in \mathcal{O}_H \longrightarrow \int_{r \in \mathbb{T}} h_r(a)dr \in \mathcal{O}_H
\]
where \( dr \) is the normalized Lebesgue measure on \( \mathbb{T} \) defines a faithful conditional expectation denoted by \( \mathcal{E}_H \) from \( \mathcal{O}_H \) onto the fixed point algebra \( (\mathcal{O}_H)^h \). The following lemma is routine.

Lemma 5.13. \( (\mathcal{O}_H)^h = \mathcal{F}_H \).

The \( C^* \)-algebra \( \mathcal{O}_H \) satisfies the following universal property. Let \( \mathcal{D} \) be a unital \( C^* \)-algebra and \( \Phi_1 : \mathcal{B}_1 \longrightarrow \mathcal{D}, \Phi_2 : \mathcal{B}_2 \longrightarrow \mathcal{D} \) be \( * \)-homomorphisms such that \( \Phi_1(a) = \Phi_2(a) \) for \( a \in \mathcal{A} \). Assume that there exist elements \( \tilde{S}_1, \ldots, \tilde{S}_M, \tilde{T}_1, \ldots, \tilde{T}_N \)
in $\mathcal{D}$ satisfying the relations:

$$\sum_{i=1}^{M} \hat{S}_i \hat{S}_i^* + \sum_{k=1}^{N} \hat{T}_k \hat{T}_k^* = 1, \quad \hat{S}_j^* \hat{S}_j = 0, \quad \hat{T}_j^* \hat{T}_j = 0,$$

$$\Phi_1(z) \hat{S}_j = \Phi_1((u_i \mid \phi_1(z)u_j)_{\mathcal{B}_1}), \quad \Phi_1(z) \hat{T}_j = \Phi_2((v_k \mid \phi_1(z)v_i)_{\mathcal{B}_2}),$$

$$\Phi_2(w) \hat{S}_j = \Phi_2((u_i \mid \phi_2(w)u_j)_{\mathcal{B}_1}), \quad \Phi_2(w) \hat{T}_j = \Phi_2((v_k \mid \phi_2(w)v_i)_{\mathcal{B}_2}),$$

for $z \in \mathcal{B}_1, w \in \mathcal{B}_2, i, j = 1, \ldots, M, k, l = 1, \ldots, N$, then there exists a unique $\ast$-homomorphism $\Phi : \mathcal{O}_H \rightarrow \mathcal{D}$ such that

$$\Phi(S_i) = \hat{S}_i, \quad \Phi(T_k) = \hat{T}_k, \quad \Phi(z) = \Phi_1(z), \quad \Phi(w) = \Phi_2(w)$$

for $i = 1, \ldots, M, k = 1, \ldots, N$ and $z \in \mathcal{B}_1, w \in \mathcal{B}_2$. We further assume that both the homomorphisms $\Phi_i : \mathcal{B}_i \rightarrow \mathcal{D}, i = 1, 2$ are injective. We denote by $\Phi_\circ : \mathcal{B}_\circ \rightarrow \mathcal{D}$ the restriction of $\Phi$ to the subalgebra $\mathcal{B}_\circ$. Let us denote by $\hat{\mathcal{O}}_H$ the $C^\ast$-subalgebra of $\mathcal{D}$ generated by $\hat{S}_i, \hat{T}_k, i = 1, \ldots, M, k = 1, \ldots, N$ and $\Phi_1(z), \Phi_2(w)$ for $z \in \mathcal{B}_1, w \in \mathcal{B}_2$.

**Lemma 5.14.** *Keep the above situation. The $\ast$-homomorphism $\Phi_\circ : \mathcal{B}_\circ \rightarrow \mathcal{D}$ is injective.*

**Proof.** Since the correspondence in Lemma 5.9

$$\phi_\circ : z, w, \in \mathcal{B}_\circ \rightarrow \phi_1(z), \phi_2(w) \in C^\ast(\phi_1(\mathcal{B}_1), \phi_2(\mathcal{B}_2))$$

yields an isomorphism of $C^\ast$-algebras, it suffices to prove that the correspondence

$$\phi_1(z), \phi_2(w) \in C^\ast(\phi_1(\mathcal{B}_1), \phi_2(\mathcal{B}_2)) \rightarrow \Phi_1(z), \Phi_2(w) \in \mathcal{D}$$

yields an isomorphism. Let $\mathcal{B}_\circ$ be the $C^\ast$-subalgebra of $\hat{\mathcal{O}}_H$ generated by elements $\Phi_1(z), \Phi_2(w) \in \mathcal{A}$ for $z \in \mathcal{B}_1, w \in \mathcal{B}_2$. Define an element $\hat{\pi}(L)$ of $\mathcal{D}$ for $L \in C^\ast(\phi_1(\mathcal{B}_1), \phi_2(\mathcal{B}_2))$ by setting

$$\hat{\pi}(L) = \sum_{i,j=1}^{M} \hat{S}_i \Phi_1(u_i \mid Lu_j)_{\mathcal{B}_1} \hat{S}_j^* + \sum_{k,l=1}^{N} \hat{T}_k \Phi_2(v_k \mid Lv_l)_{\mathcal{B}_2} \hat{T}_l^* \in \mathcal{D}. \quad (5.14)$$

As in the proof of Lemma 5.9, one sees that $\hat{\pi}$ gives rise to a $\ast$-homomorphism from $C^\ast(\phi_1(\mathcal{B}_1), \phi_2(\mathcal{B}_2))$ into $\mathcal{D}$. Since

$$\hat{\pi}(\phi_1(z)) = \sum_{i,j=1}^{M} \hat{S}_i \Phi_1(u_i \mid \phi_1(z)u_j)_{\mathcal{B}_1} \hat{S}_j^* + \sum_{k,l=1}^{N} \hat{T}_k \Phi_2(v_k \mid \phi_1(z)v_l)_{\mathcal{B}_2} \hat{T}_l^* = \Phi_1(z),$$

and similarly $\hat{\pi}(\phi_2(w)) = \Phi_2(w)$, it is enough to show that $\hat{\pi}$ is injective. Suppose that $\hat{\pi}(L) = 0$ for some $L \in C^\ast(\phi_1(\mathcal{B}_1), \phi_2(\mathcal{B}_2))$. By following the proof of Lemma 5.9, one sees that $\hat{S}_h \hat{\pi}(L) \hat{S}_{h'} = \Phi_1(u_h \mid Lu_{h'})_{\mathcal{B}_1}$ for all $h, h' = 1, \ldots, M$. Hence the condition $\hat{\pi}(L) = 0$ implies $\Phi_1(u_h \mid Lu_{h'})_{\mathcal{B}_1} = 0$. Since $\Phi_1$ is injective, we
Lemma 5.15. Suppose that both the \( \Phi \) have for \( \xi \in H \),

\[
L \xi = \sum_{h' = 1}^{M} L(u_{h'} | u_{h'} \xi) = \sum_{h' = 1}^{M} (L_{h'}) \xi = 0
\]

so that \( L = 0 \). Therefore \( \tilde{\pi} : C^* (\phi_1, \phi_2) \to D \) is injective. Hence the composition

\[
\Phi : \tilde{\pi} \circ \phi : B_0 \to C^* (\phi_1, \phi_2) \to D
\]

is injective.

We set

\[
\tilde{S}_{1,i} := S_i, \quad i = 1, \ldots, M, \quad \tilde{S}_{2,k} := T_k, \quad k = 1, \ldots, N.
\]

We put \( \tilde{F}_n = B_0 \). For \( n \in \mathbb{N} \), let \( \tilde{F}^n \) be the closed linear span in the \( C^* \)-algebra \( \tilde{O}_H \) of elements of the form

\[
\tilde{S}_{g_1,i_1} \tilde{S}_{g_2,i_2} \cdots \tilde{S}_{g_n,i_n} \Phi(b) \tilde{S}_{h_1,j_1} \cdots \tilde{S}_{h_2,j_2} \tilde{S}_{h_2,j_3} \tilde{S}_{h_3,j_4}
\]

for \( (g_1, i_1), (g_2, i_2), \ldots, (g_n, i_n), (h_1, j_1), \ldots, (h_n, j_n) \in \Sigma_1 \cup \Sigma_2 \) and \( b \in B_0 \). Similarly to the subalgebras \( F^n \) of \( \tilde{O}_H \), one knows that the closed linear span \( \tilde{F}^n \) is a \( C^* \)-algebra and naturally regarded as a subalgebra of \( \tilde{F}^{n+1} \) for each \( n \in \mathbb{Z}_+ \). Let us denote by \( \tilde{F}_n \) the \( C^* \)-subalgebra of \( \tilde{O}_H \) generated by \( \cup_{n=0}^{\infty} \tilde{F}^n \).

Then the \( C^* \)-algebra \( \tilde{F}_n \) is the inductive limit \( \lim_{n \to \infty} \tilde{F}^n \) of the sequence of the inclusions

\[
\tilde{F}_n \to \tilde{F}_{n+1} \to \tilde{F}_n \to \tilde{F}_{n+1} \to \tilde{F}_n \to \tilde{F}_{n+1} \to \cdots
\]

(5.16)

Lemma 5.15. Suppose that both the \( \ast \)-homomorphisms \( \Phi_i : B_1 \to \tilde{O}_H, i = 1, 2 \) are injective. Then the restriction of \( \Phi \) to the subalgebra \( \tilde{F}_n \) yields a \( \ast \)-isomorphism \( \Phi |_{\tilde{F}_n} : \tilde{F}_n \to \tilde{F}_n \).

Proof. By the universality of \( \tilde{O}_H \), the restriction of \( \Phi \) to \( \tilde{F}_n \) yields a surjective \( \ast \)-homomorphism \( \Phi |_{\tilde{F}_n} : \tilde{F}_n \to \tilde{F}_n \). It suffices to show that \( \Phi |_{\tilde{F}_n} \) is injective.

Suppose that \( \ker(\Phi |_{\tilde{F}_n}) \neq \{0\} \) and put \( I = \ker(\Phi |_{\tilde{F}_n}) \). Since \( \Phi |_{\tilde{F}_n} = \tilde{F}_n \) and \( \tilde{F}_n = \lim_{n \to \infty} \tilde{F}_n \), there exists \( n \in \mathbb{Z}_+ \) such that \( I \cap \tilde{F}_n \neq 0 \). Let us denote by \( \Sigma^n \) the set of \( n \)-tuples of \( \Sigma_1 \cup \Sigma_2 \):

\[
\Sigma^n = \{ (\mu_1, \ldots, \mu_n) | \mu_1, \ldots, \mu_n \in \Sigma_1 \cup \Sigma_2 \}
\]

For \( \mu = (\mu_1, \ldots, \mu_n) \in \Sigma^n \), denote by \( S_\mu \) the operator

\[
S_\mu = S_{\mu_1} S_{\mu_2} \cdots S_{\mu_n}
\]

where

\[
S_{\mu_m} = \begin{cases} S_{1,i} & \text{if } \mu_m = (1, i) \in \Sigma_1, \\ S_{2,k} & \text{if } \mu_m = (2, k) \in \Sigma_2, \end{cases}
\]

Any element of \( \tilde{F}_n \) is of the form

\[
\sum_{\mu, \nu \in \Sigma^n} S_\mu b_{\mu, \nu} S_\nu^* \quad \text{for some } b_{\mu, \nu} \in B_0.
\]
Hence one may find a nonzero element \( \sum_{\mu, \nu \in \Sigma^n} S_\mu b_{\mu, \nu} S_\nu^* \in I \cap I_t^\prime \). Since \( \sum_{i=1}^M S_i^* + \sum_{k=1}^N T_k^* T_k = 1 \), the equality \( \sum_{\mu, \nu \in \Sigma^n} S_\mu S_\nu^* = 1 \) hols. For any \( \omega, \gamma \in \Sigma^n \), one then sees
\[
0 \neq S_\omega^*( \sum_{\mu, \nu \in \Sigma^n} S_\mu b_{\mu, \nu} S_\nu^* ) S_\gamma \in I \cap I_t^\prime.
\]
As \( S_i^* T_k = 0 \) and \( S_i^* S_j = (u_i | u_j)_B_i \), \( T_k^* T_l = (v_k | v_l)_B_2 \) for \( i, j = 1, \ldots, M, k, l = 1, \ldots, N \), the element \( S_\omega^*( \sum_{\mu, \nu \in \Sigma^n} S_\mu b_{\mu, \nu} S_\nu^* ) S_\gamma \) belongs to \( I \cap B_2 \). By the preceding lemma, the homomorphism \( \Phi : B_2 \to \hat{\mathcal{O}}_\mathcal{H} \) is injective, so that we have \( \Phi S_\omega^*( \sum_{\mu, \nu \in \Sigma^n} S_\mu b_{\mu, \nu} S_\nu^* ) S_\gamma \neq 0 \) a contradiction. Therefore we conclude that \( \Phi |_{\mathcal{F}_\mathcal{H}} : \mathcal{F}_\mathcal{H} \to \hat{\mathcal{F}}_\mathcal{H} \) is injective and hence isomorphic. \( \square \)

The following theorem is one of the main results of the paper.

**Theorem 5.16.** Let \( \mathcal{D} \) be a unital \( C^* \)-algebra. Suppose that there exist \( * \)-homomorphisms \( \Phi_1 : \mathcal{B}_1 \to \mathcal{D}, \Phi_2 : \mathcal{B}_2 \to \mathcal{D} \) such that \( \Phi_1(a) = \Phi_2(a) \) for \( a \in \mathcal{A} \) and there exist elements \( \mathcal{S}_1, \ldots, \mathcal{S}_M, \mathcal{T}_1, \ldots, \mathcal{T}_N \) in \( \mathcal{D} \) satisfying the relations:
\[
\begin{align*}
\sum_{i=1}^M \mathcal{S}_i \mathcal{S}_i^* + \sum_{k=1}^N \mathcal{T}_k \mathcal{T}_k^* &= 1, \\
\mathcal{S}_j \mathcal{T}_1 &= \Phi_1((u_i | u_j)_B_i), \\
\mathcal{T}_k \mathcal{T}_l &= \Phi_2((v_k | v_l)_B_2),
\end{align*}
\]
for \( z \in \mathcal{B}_1, w \in \mathcal{B}_2, i, j = 1, \ldots, M, k, l = 1, \ldots, N \). Let us denote by \( \hat{\mathcal{O}}_\mathcal{H} \) the \( C^* \)-subalgebra of \( \mathcal{D} \) generated by \( \mathcal{S}_i, \mathcal{T}_k, i = 1, \ldots, M, k = 1, \ldots, N \) and \( \Phi_1(z), \Phi_2(w) \), for \( z \in \mathcal{B}_1, w \in \mathcal{B}_2 \). We further assume that the algebra \( \hat{\mathcal{O}}_\mathcal{H} \) admits a gauge action. If both the \( * \)-homomorphisms \( \Phi_i : \mathcal{B}_i \to \mathcal{A}, i = 1, 2 \) are injective, then there exists a \( * \)-isomorphism \( \Phi : \mathcal{O}_\mathcal{H} \to \hat{\mathcal{O}}_\mathcal{H} \) satisfying
\[
\Phi(S_i) = \hat{S}_i, \quad \Phi(T_k) = \hat{T}_k, \quad \Phi(z) = \Phi_1(z), \quad \Phi(w) = \Phi_2(w)
\]
(5.17)
for \( i = 1, \ldots, M, k = 1, \ldots, N \) and \( z \in \mathcal{B}_1, w \in \mathcal{B}_2 \).

**Proof.** By assumption, \( \hat{\mathcal{O}}_\mathcal{H} \) admits a gauge action, which we denote by \( \hat{h} \). Let us denote by \( (\hat{\mathcal{O}}_\mathcal{H})^\hat{h} \) the fixed point algebra of \( \hat{\mathcal{O}}_\mathcal{H} \) under the gauge action \( \hat{h} \) and by \( \hat{\mathcal{F}}_\mathcal{H} \) the \( C^* \)-subalgebra of \( \hat{\mathcal{O}}_\mathcal{H} \) defined by the inductive limit (5.16). Then it is routine to check that \( (\hat{\mathcal{O}}_\mathcal{H})^\hat{h} \) is canonically \( * \)-isomorphic to \( \hat{\mathcal{F}}_\mathcal{H} \). There exists a conditional expectation
\[
\hat{\mathcal{E}}_\mathcal{H} : \hat{\mathcal{O}}_\mathcal{H} \to \hat{\mathcal{F}}_\mathcal{H}
\]
defined by
\[
\hat{\mathcal{E}}_\mathcal{H}(x) = \int_{t \in T} \hat{h}_t(x) \, dt \quad \text{for} \ x \in \hat{\mathcal{O}}_\mathcal{H}.
\]
By the universality of the algebra $\mathcal{O}_H$ there exists a surjective *-homomorphism $\Phi$ from $\mathcal{O}_H$ to $\hat{\mathcal{O}}_H$ such that
\[
\Phi(s_i) = \hat{S}_i, \quad \Phi(T_k) = \hat{T}_k, \quad \Phi(z) = \Phi_1(z), \quad \Phi(w) = \Phi_2(w)
\]
for $i, j = 1, \ldots, M, k, l = 1, \ldots, N, z \in B_1, w \in B_2$. Then $\Phi(\mathcal{F}_H) = \hat{\mathcal{F}}_H$ and the following diagram:
\[
\begin{array}{ccc}
\mathcal{O}_H & \xrightarrow{\Phi} & \hat{\mathcal{O}}_H \\
\varepsilon_H \downarrow & & \downarrow \varepsilon_H \\
\mathcal{F}_H & \xrightarrow{\Phi|_{\mathcal{F}_H}} & \hat{\mathcal{F}}_H
\end{array}
\]
is commutative. Denote by $\Phi_o$ the restriction of $\Phi$ to the $C^*$-subalgebra $B_o$ of $\mathcal{O}_H$ generated by $z \in B_1, w \in B_2$. By assumption, both the maps $\Phi_i : B_i \rightarrow \hat{O}_H, i = 1, 2$ are injective, so that $\Phi_o : B_o \rightarrow \hat{O}_H$ is injective by Lemma 5.14. By the preceding lemma, $\Phi|_{\mathcal{F}_H} : \mathcal{F}_H \rightarrow \hat{\mathcal{F}}_H$ is an isomorphism. Since the conditional expectation $\varepsilon_H : \mathcal{O}_H \rightarrow \mathcal{F}_H$ is faithful, a routine argument shows that $\Phi$ is injective and hence isomorphic.

Therefore we have

**Theorem 5.17.** For a $C^*$-quad module $\mathcal{H}$ of finite type, the $C^*$-algebra $\mathcal{O}_{F(\mathcal{H})}$ generated by the quotients $[s_\xi], [t_\xi]$ of the creation operators $s_\xi, t_\xi$ for $\xi \in \mathcal{H}$ on the Fock spaces $F(\mathcal{H})$ is canonically isomorphic to the universal $C^*$-algebra $\mathcal{O}_H$ generated by operators $S_1, \ldots, S_M, T_1, \ldots, T_N$ and elements $z \in B_1, w \in B_2$ subject to the relations:

\[
\sum_{i=1}^M S_i S_i^* + \sum_{k=1}^N T_k T_k^* = 1, \quad S_i^* T_i = 0, \\
S_i^* S_j = \langle u_i | u_j \rangle_{B_1}, \quad T_k^* T_i = \langle v_k | v_l \rangle_{B_2}, \\
zS_j = \sum_{i=1}^M S_i \langle u_i | \phi_1(z) u_j \rangle_{B_1}, \quad zT_i = \sum_{k=1}^N T_k \langle v_k | \phi_1(z) v_l \rangle_{B_2}, \\
wS_j = \sum_{i=1}^M S_i \langle u_i | \phi_2(w) u_j \rangle_{B_1}, \quad wT_i = \sum_{k=1}^N T_k \langle v_k | \phi_2(w) v_l \rangle_{B_2},
\]

for $i, j = 1, \ldots, M, k, l = 1, \ldots, N$ and $z \in B_1, w \in B_2$.

**Proof.** Theorem 4.3 implies that the operators $S_1, \ldots, S_M, T_1, \ldots, T_N$ and the elements $\phi_1(z), \phi_2(w)$ for $z \in B_1, w \in B_2$ in $\mathcal{O}_{F(\mathcal{H})}$ satisfy the eight relations of Theorem 5.16. By Theorem 5.16, we see that the correspondences

\[
S_i \rightarrow S_i, \quad T_k \rightarrow T_k, \quad z \in B_1 \rightarrow \Phi_1(z), \quad w \in B_2 \rightarrow \Phi_2(w)
\]

for $i = 1, \ldots, M, k = 1, \ldots, N, z \in B_1, w \in B_2$ give rise to an isomorphism from $\mathcal{O}_H$ to $\mathcal{O}_{F(\mathcal{H})}$.

The eight relations of the operators above are called the relations $(\mathcal{H})$. The above generating operators $S_1, \ldots, S_M$ and $T_1, \ldots, T_N$ of the universal $C^*$-algebra $\mathcal{O}_H$ correspond to two finite basis $\{u_1, \ldots, u_M\}$ and $\{v_1, \ldots, v_N\}$ of the Hilbert $C^*$-quad module $\mathcal{H}$ respectively. On the other hand, the other $C^*$-algebra $\mathcal{O}_{F(\mathcal{H})}$ is generated by the quotients of the creation operators $s_\xi, t_\xi$ for $\xi \in \mathcal{H}$ on the Fock
spaces $F(\mathcal{H})$, which do not depend on the choice of the two finite bases. Hence we have

**Corollary 5.18.** For a C*-quad module $\mathcal{H}$ of finite type, the universal C*-algebra $\mathcal{O}_H$ generated by operators $S_1, \ldots, S_M, T_1, \ldots, T_N$ and elements $z \in B_1, w \in B_2$ subject to the relations ($\mathcal{H}$) does not depend on the choice of the finite bases $\{u_1, \ldots, u_M\}$ and $\{v_1, \ldots, v_N\}$.

### 6. K-Theory Formulae

Let $\mathcal{H}$ be a Hilbert C*-quad module over $(A; B_1, B_2)$ of finite type as in the preceding section. In this section, we will state K-theory formulae for the C*-algebra $\mathcal{O}_{F(\mathcal{H})}$. By the previous section, the C*-algebra $\mathcal{O}_{F(\mathcal{H})}$ is regarded as the universal C*-algebra $\mathcal{O}_H$ generated by the operators $S_1, \ldots, S_M$ and $T_1, \ldots, T_N$ and the elements $z \in B_1$ and $w \in B_2$ subject to the relations ($\mathcal{H}$). Let us denote by $\mathcal{B}_o$ the C*-subalgebra of $\mathcal{O}_H$ generated by elements $z \in B_1$ and $w \in B_2$. By Lemma 5.9 the correspondence

$$z, w \in B_o \rightarrow \phi_1(z), \phi_2(w) \in C^*(\phi_1(B_1), \phi_2(B_2)) \subset L_A(\mathcal{H}) \quad (6.1)$$

gives rise to a *-isomorphism from $\mathcal{B}_o$ onto $C^*(\phi_1(B_1), \phi_2(B_2))$ as C*-algebras, which is denoted by $\phi_o$. We will restrict our interest to the case when

1. $S_1, \ldots, S_M$ and $T_1, \ldots, T_N$ are partial isometries, and
2. $S_1S_1^*, \ldots, S_MS_M^*, T_1T_1^*, \ldots, T_NT_N^*$ commute with all elements of $\mathcal{B}_o$.

If the bases $\{u_1, \ldots, u_M\}$ and $\{v_1, \ldots, v_N\}$ satisfy the conditions

$$\langle u_i \mid u_j \rangle_{B_1} = 0 \text{ for } i \neq j, \quad \langle v_k \mid v_l \rangle_{B_2} = 0 \text{ for } k \neq l,$$

the condition (i) holds. Furthermore if $\phi_1(z)$ acts diagonally on $\{u_1, \ldots, u_M\}$ for $z \in B_1$ and $\phi_2(w)$ acts diagonally on $\{v_1, \ldots, v_N\}$ for $w \in B_2$, then condition (ii) holds. Recall that the gauge action is denoted by $h$ which is an action of $T$ on $\mathcal{O}_H$ such that the fixed point algebra $(\mathcal{O}_H)^h$ under $h$ is canonically isomorphic to the C*-algebra $\mathcal{F}_H$. Denote by $h$ the dual action of $h$ which is an action of $Z = \widehat{T}$ on the C*-crossed product $\mathcal{O}_H \rtimes_h T$ by the gauge action $h$ of $T$. As in the argument of [17], $\mathcal{O}_H \rtimes_h T$ is stably isomorphic to $\mathcal{F}_H$. Hence we have $K_*(\mathcal{O}_H \rtimes_h T)$ isomorphic to $K_*(\mathcal{F}_H)$. The dual action $\overline{h}$ induces an automorphism on the group $K_*(\mathcal{O}_H \rtimes_h T)$ and hence on $K_*(\mathcal{F}_H)$, which is denoted by $\sigma_*$. Then by [17] (cf. [3], [21], etc.) we have

**Proposition 6.1.** The following six term exact sequence of K-theory hold:

$$
\begin{align*}
K_0(\mathcal{F}_H) & \xrightarrow{id-\sigma_\ast} K_0(\mathcal{F}_H) \xrightarrow{} K_0(\mathcal{O}_H) \\
\uparrow & & \downarrow \\
K_1(\mathcal{O}_H) & \xleftarrow{} K_1(\mathcal{F}_H) & \xleftarrow{id-\sigma_\ast} K_1(\mathcal{F}_H).
\end{align*}
$$

We put for $x \in B_o$

$$\begin{align*}
\lambda_{1,i}(x) &= S_i^* x S_i, & i &= 1, \ldots, M, \\
\lambda_{2,k}(x) &= T_k^* x T_k, & k &= 1, \ldots, N.
\end{align*}$$
Both the families $\lambda_{1,i}$, $\lambda_{2,k}$ yield endomorphisms on $B_o$ which give rise to endomorphisms on the K-groups:

$$\lambda_{1,i} : K_0(B_o) \rightarrow K_0(B_o), \quad i = 1, \ldots, M,$$

$$\lambda_{2,k} : K_0(B_o) \rightarrow K_0(B_o), \quad k = 1, \ldots, N.$$ 

We put $\lambda_o = \sum_{i=1}^M \lambda_{1,i} + \sum_{k=1}^N \lambda_{2,k}$ which is an endomorphism on $K_0(B_o)$. Now we further assume that $K_1(\mathcal{F}_\mathcal{H}) = \{0\}$. It is routine to show that the groups $\text{Coker}(id - \sigma_C)$ in $K_0(\mathcal{F}_\mathcal{H})$ and $\text{Ker}(id - \sigma_C)$ in $K_0(\mathcal{F}_\mathcal{H})$ are isomorphic to the groups $\text{Coker}(id - \lambda_o)$ in $K_0(B_o)$ and $\text{Ker}(id - \lambda_o)$ in $K_0(B_o)$ respectively by an argument of [3]. Therefore we have

**Proposition 6.2.**

$$K_0(\mathcal{O}_\mathcal{H}) = \text{Coker}(id - \lambda_o) \quad \text{in} \quad K_0(B_o),$$

$$K_1(\mathcal{O}_\mathcal{H}) = \text{Ker}(id - \lambda_o) \quad \text{in} \quad K_0(B_o).$$

7. **Examples**

In this section, we will study the $C^*$-algebras $\mathcal{O}_\mathcal{H}$ for the Hilbert $C^*$-quad modules presented in Examples in Section 2.

1. Let $\alpha$, $\beta$ be automorphisms of a unital $C^*$-algebra $\mathcal{A}$ satisfying $\alpha \circ \beta = \beta \circ \alpha$. Let $\mathcal{H}_{\alpha,\beta}$ be the associated Hilbert $C^*$-quad module of finite type as in 1 in Section 2. It is easy to see the following proposition.

**Proposition 7.1.** The $C^*$-algebra $\mathcal{O}_{\mathcal{H}_{\alpha,\beta}}$ associated to the Hilbert $C^*$-quad module $\mathcal{H}_{\alpha,\beta}$ coming from commuting automorphisms $\alpha, \beta$ of a unital $C^*$-algebra $\mathcal{A}$ is isomorphic to the universal $C^*$-algebra generated by two isometries $U, V$ and elements $x$ of $\mathcal{A}$ subject to the following relations:

$$UU^* + VV^* = 1,$$

$$UU^*x = xUU^*, \quad VV^*x = xVV^*, \quad \alpha(x) = U^*xU, \quad \beta(x) = V^*xV$$

for $x \in \mathcal{A}$.

2. We fix natural numbers $1 < N, M \in \mathbb{N}$. Consider finite dimensional commutative $C^*$-algebras $\mathcal{A} = \mathbb{C}$, $B_1 = \mathbb{C}^N$, $B_2 = \mathbb{C}^M$. The algebras $B_1, B_2$ have the ordinary product structure and the inner product structure which we denote by $\langle \cdot | \cdot \rangle_N$ and $\langle \cdot | \cdot \rangle_M$ respectively. Let us denote by $\mathcal{H}_{M,N}$ the Hilbert $C^*$-quad module $\mathbb{C}^M \otimes \mathbb{C}^N$ over $(\mathbb{C}; \mathbb{C}^N, \mathbb{C}^M)$ defined in 2 in Section 2. Put the finite bases

$$u_i = e_i \otimes 1 \in \mathcal{H}_{M,N}, \quad i = 1, \ldots, M \text{ as a right } B_1 \text{- module, and}$$

$$v_k = 1 \otimes f_k \in \mathcal{H}_{M,N}, \quad k = 1, \ldots, N \text{ as a right } B_2 \text{- module.}$$

We set $\Sigma^o = \{(i,k) \mid 1 \leq i \leq M, 1 \leq k \leq N\}$ and put $e_{(i,k)} = e_i \otimes f_k$, $(i, k) \in \Sigma^o$ the standard basis of $\mathcal{H}_{M,N}$. Then the $C^*$-algebra $B_o$ on $\mathcal{H}_{M,N}$ generated by $B_1$ and $B_2$ is regarded as $\mathbb{C}^M \otimes \mathbb{C}^N = B_2 \otimes B_1$. Hence

$$B_o = \sum_{(i,k) \in \Sigma^o} \mathbb{C} e_{(i,k)}.$$
Lemma 7.2. The $C^*$-algebra $\mathcal{O}_{H_{m,N}}$ is generated by operators $S_i, T_k, e_{(i,k)}; i = 1, \ldots, M, k = 1, \ldots, N$ satisfying

$$\sum_{i=1}^{M} S_i S_i^* + \sum_{k=1}^{N} T_k T_k^* = 1,$$  \hspace{1cm} (7.1)

$$S_i^* S_j = \delta_{i,j}, \quad T_k^* T_l = \delta_{k,l},$$  \hspace{1cm} (7.2)

$$e_{(i,k)} S_j = \delta_{i,j} \sum_{h=1}^{M} S_j e_{(h,k)}, \quad e_{(i,k)} T_l = \delta_{k,l} \sum_{m=1}^{N} T_l e_{(i,m)}$$  \hspace{1cm} (7.3)

for $i, j = 1, \ldots, M, k, l = 1, \ldots, N$.

Proof. It suffices to show the equalities (7.3). We have

$$e_{(i,k)} S_j = S_j (u_j | \phi(e_{(i,k)}) u_j)_{\mathcal{B}_1} = S_j (e_j \otimes 1 | (e_i \otimes f_k)(e_j \otimes 1))_{\mathcal{B}_1} = \delta_{i,j} S_j (1 \otimes f_k) = \delta_{i,j} S_j \sum_{h=1}^{M} e_{(h,k)}.$$  

The other equality of (7.3) is similarly shown. \hfill \Box

Put

$$S_{(i,k)} = e_{(i,k)} S_i, \quad T_{(i,k)} = e_{(i,k)} T_k \quad \text{for} \ (i, k) \in \Sigma^o.$$  

Then we have

Lemma 7.3.

$$e_{(i,k)} = S_{(i,k)} S_{(i,k)}^* + T_{(i,k)} T_{(i,k)}^*,$$  \hspace{1cm} (7.4)

$$S_i = \sum_{k=1}^{N} S_{(i,k)}, \quad T_k = \sum_{i=1}^{M} T_{(i,k)},$$  \hspace{1cm} (7.5)

$$\sum_{(i,k) \in \Sigma^o} S_{(i,k)} S_{(i,k)}^* + \sum_{(i,k) \in \Sigma^o} T_{(i,k)} T_{(i,k)}^* = 1,$$  \hspace{1cm} (7.6)

$$S_{(i,k)}^* S_{(i,k)} = \sum_{j=1}^{M} (S_{(j,k)} S_{(j,k)}^* + T_{(j,k)} T_{(j,k)}^*),$$  \hspace{1cm} (7.7)

$$T_{(i,k)}^* T_{(i,k)} = \sum_{l=1}^{N} (S_{(i,l)} S_{(i,l)}^* + T_{(i,l)} T_{(i,l)}^*),$$  \hspace{1cm} (7.8)

for $i = 1, \ldots, M, k = 1, \ldots, N$ and $(i, k) \in \Sigma^o$.

Proof. Since $e_{(i,k)} S_j = \delta_{i,j} e_{(i,k)} S_i$, we have

$$S_{(i,k)} S_{(i,k)}^* = e_{(i,k)} S_i S_i^* e_{(i,k)} = e_{(i,k)} \left( \sum_{j=1}^{M} S_j S_j^* \right) e_{(i,k)}$$

and similarly $T_{(i,k)} T_{(i,k)}^* = e_{(i,k)} \left( \sum_{l=1}^{N} T_l T_l^* \right) e_{(i,k)}$. Hence we have

$$e_{(i,k)} = e_{(i,k)} \left( \sum_{j=1}^{M} S_j S_j^* + \sum_{l=1}^{N} T_l T_l^* \right) e_{(i,k)} = S_{(i,k)} S_{(i,k)}^* + T_{(i,k)} T_{(i,k)}^*,$$  

for $i, j = 1, \ldots, M, k, l = 1, \ldots, N$.\hfill \Box
so that (7.4) holds. As \(1 = \sum_{(j,k) \in \Sigma^o} e_{(j,k)}\), the equality (7.6) holds. Since \(e_{(j,k)}S_i = 0\) for \(j \neq i\), we have

\[
S_i = \left( \sum_{(j,k) \in \Sigma^o} e_{(j,k)} \right) S_i = \sum_{k=1}^N S_{(i,k)}
\]

and similarly \(T_k = \sum_{i=1}^M T_{(i,k)}\), so that (7.5) holds. By (7.3), it follows that

\[
S_{(i,k)}^* S_{(i,k)} = S_i^* \sum_{j=1}^M S_j e_{(j,k)} = \sum_{j=1}^M e_{(j,k)} = \sum_{j=1}^M (S_{(j,k)} S_{(j,k)}^* + T_{(j,k)} T_{(j,k)}^*),
\]

and similarly we have

\[
T_{(i,k)}^* T_{(i,k)} = \sum_{l=1}^N (S_{(i,l)} S_{(i,l)}^* + T_{(i,l)} T_{(i,l)}^*).
\]

\[
\square
\]

**Theorem 7.4.** The \(C^*-\)algebra \(\mathcal{O}_{H_{M,N}}\) associated with the Hilbert \(C^*-\)quad module \(H_{M,N} = \mathbb{C}^M \otimes \mathbb{C}^N\) is generated by partial isometries \(S_{(i,k)}, T_{(i,k)}\) for \((i,k) \in \Sigma^o = \{(i,k) \mid i = 1, \ldots, M; k = 1, \ldots, N\}\) satisfying the relations:

\[
\sum_{(i,k) \in \Sigma^o} S_{(i,k)} S_{(i,k)}^* + \sum_{(i,k) \in \Sigma^o} T_{(i,k)} T_{(i,k)}^* = 1,
\]

\[
S_{(i,k)}^* S_{(i,k)} = \sum_{j=1}^M S_{(j,k)} S_{(j,k)}^* + T_{(j,k)} T_{(j,k)}^*,
\]

\[
T_{(i,k)}^* T_{(i,k)} = \sum_{l=1}^N (S_{(i,l)} S_{(i,l)}^* + T_{(i,l)} T_{(i,l)}^*)
\]

for \((i,k) \in \Sigma^o\).

**Proof.** By the preceding lemma, one knows that \(e_{(i,k)}, S_i, T_k\) are generated by the operators \(S_{(i,k)}, T_{(i,k)}\) so that the algebra \(\mathcal{O}_{H_{M,N}}\) is generated by the partial isometries \(S_{(i,k)}, T_{(i,k)}\), \((i,k) \in \Sigma^o\). \(\square\)

Let \(I_n\) be the \(n \times n\) identity matrix and \(E_n\) the \(n \times n\) matrix whose entries are all 1’s. For an \(M \times M\)-matrix \(C = [c_{ij}]_{i,j=1}^M\) and an \(N \times N\)-matrix \(D = [d_{kl}]_{k,l=1}^N\), denote by \(C \otimes D\) the \(MN \times MN\) matrix

\[
C \otimes D = \begin{bmatrix}
c_{11}D & c_{12}D & \cdots & c_{1M}D \\
c_{21}D & c_{22}D & \cdots & c_{2M}D \\
\vdots & \vdots & \ddots & \vdots \\
c_{M1}D & c_{M2}D & \cdots & c_{MM}D
\end{bmatrix}
\]

so that

\[
E_M \otimes I_N = \begin{bmatrix}
I_N & I_N & \cdots & I_N \\
I_N & I_N & \cdots & I_N \\
\vdots & \vdots & \ddots & \vdots \\
I_N & I_N & \cdots & I_N
\end{bmatrix}, \quad I_M \otimes E_N = \begin{bmatrix}
E_N & 0 & \cdots & 0 \\
0 & E_N & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & E_N
\end{bmatrix}.
\]
The index set \( \{(i, k) \mid i = 1, \ldots, M, k = 1, \ldots, N\} \) of the standard basis of \( \mathbb{C}^M \otimes \mathbb{C}^N \) is ordered lexicographically from left as in the following way:

\[
(1, 1), \ldots, (1, N), (2, 1), \ldots, (2, N), \ldots, (M, 1), \ldots, (M, N).
\]

Put the \( MN \times MN \) matrices

\[
A_{M,N} = E_M \otimes I_N, \quad B_{M,N} = I_M \otimes E_N
\]

and the \( 2MN \times 2MN \) matrix

\[
H_{M,N} = \begin{bmatrix} A_{M,N} & A_{M,N} \\ B_{M,N} & B_{M,N} \end{bmatrix}.
\]

Then we have

**Theorem 7.5.** The \( C^* \)-algebra \( O_{H_{M,N}} \) is isomorphic to the Cuntz-Krieger algebra \( O_{H_{M,N}} \) for the matrix \( H_{M,N} \). The algebra \( O_{H_{M,N}} \) is simple and purely infinite and is isomorphic to the Cuntz-Krieger algebra \( O_{A_{M,N}+B_{M,N}} \) for the matrix \( A_{M,N}+B_{M,N} \).

**Proof.** By the preceding proposition, the \( C^* \)-algebra \( O_{H_{M,N}} \) is isomorphic to the Cuntz-Krieger algebra \( O_{H_{M,N}} \) for the matrix \( H_{M,N} \). Since the matrix \( H_{M,N} \) is aperiodic, the algebra is simple and purely infinite. The \( n \)-th column of the matrix \( H_{M,N} \) coincides with the \( n + N \)-th column for every \( n = 1, \ldots, M \). One sees that the matrix \( A_{M,N}+B_{M,N} \) is obtained from \( H_{M,N} \) by amalgamating them. The procedure is called the column amalgamation and induces an isomorphism on their Cuntz-Krieger algebras (see [15]). □

In [15], the abelian groups

\[
\mathbb{Z}^{MN} / (A_{M,N} + B_{M,N} - I_{MN}) \mathbb{Z}^{MN}, \quad \text{Ker}(A_{M,N} + B_{M,N} - I_{MN}) \text{ in } \mathbb{Z}^{MN}
\]

have been computed by using Euclidean algorithms. For the case \( M = 2 \), they are

\[
\mathbb{Z}/(N^2 - 1)\mathbb{Z}, \quad \{0\}
\]

respectively, so that we see

\[
K_0(O_{H_{2,N}}) = \mathbb{Z}/(N^2 - 1)\mathbb{Z}, \quad K_1(O_{H_{2,N}}) = 0
\]

(see [15] for details).

3. For a \( C^* \)-textile dynamical system \((A, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)\), let \( H_{A,\rho,\eta} \) be the \( C^* \)-quad module over \((A; B_1, B_2)\) as in 3 in Section 2. The \( C^* \)-algebra \( O_{H_{A,\rho,\eta}} \) has been studied in [14].

8. Higher dimensional analogue

In this final section, we will state a generalization of Hilbert \( C^* \)-quad modules to Hilbert modules with multi actions of \( C^* \)-algebras.

Let \( A \) be a unital \( C^* \)-algebra and \( B_1, \ldots, B_n \) be \( n \)-family of unital \( C^* \)-algebras. Suppose that there exists a unital embedding

\[
i_i : A \hookrightarrow B_i
\]

for each \( i = 1, \ldots, n \). Suppose that there exists a right action \( \psi_i \) of \( A \) on \( B_i \) such that

\[
b_i \psi_i(a) \in B_i \quad \text{for} \quad b_i \in B_i, a \in A, i = 1, \ldots, n.
\]

Hence \( B_i \) is a right \( A \)-module through \( \psi_i \) for \( i = 1, \ldots, n \). Let \( \mathcal{H} \) be a Hilbert \( C^* \)-bimodule over \( A \) with a right action of \( A \), an \( A \)-valued right inner product \( \langle \cdot | \cdot \rangle_A \)
and a \( \ast \)-homomorphism \( \phi_A \) from \( A \) to \( \mathcal{L}_A(H) \). It is called a Hilbert \( C^\ast \)-module over \( (A; B_i, i = 1, \ldots, n) \) if \( H \) has a multi structure of Hilbert \( C^\ast \)-bimodules over \( B_i \) for \( i = 1, \ldots, n \) such that for each \( i = 1, \ldots, n \) there exist a right action \( \varphi_i \) of \( B_i \) on \( H \) and a left action \( \phi_i \) of \( B_i \) on \( H \) and a \( B_i \)-valued right inner product \( \langle \cdot \mid \cdot \rangle_{B_i} \), such that \( \phi_i(z_i) \in \mathcal{L}_A(H) \) and
\[
\phi_i(z_i)\psi_j(w_j) = \phi_i(z_i)\langle \psi_j(w_j) \rangle, \quad \psi_j(z_j)\psi_j(a) = [\psi_j(z_j)]a
\]
for \( \xi \in H, z_i \in B_i, w_j \in B_j, a \in A, i, j = 1, \ldots, n \) and
\[
\phi_A(a) = \phi_i(i(a)), \quad a \in A, i = 1, \ldots, n.
\]
The operator \( \phi_i(z_i) \) on \( H \) is adjointable with respect to the inner product \( \langle \cdot \mid \cdot \rangle_{B_i} \), whose adjoint \( \phi_i(z_i)^\ast \) coincides with the adjoint of \( \phi_i(z_i) \) with respect to the inner product \( \langle \cdot \mid \cdot \rangle_A \) so that \( \phi_i(z_i)^\ast = \phi_i(z_i^\ast) \). We assume that the left actions \( \phi_i \) of \( B_i \) on \( H \) for \( i = 1, 2 \) are faithful. We require the following compatibility conditions between the right \( A \)-module structure of \( H \) and the right \( A \)-module structure of \( B_i \) through \( \psi_i \):
\[
\langle \xi \mid \eta \rangle_{B_i} = \langle \xi \mid \eta \rangle_{B_i}(a), \quad \xi, \eta \in H, a \in A, i = 1, \ldots, n.
\]
We further assume that \( H \) is a full Hilbert \( C^\ast \)-bimodule with respect to the inner product \( \langle \cdot \mid \cdot \rangle_A \), \( \langle \cdot \mid \cdot \rangle_{B_i} \) for each. A Hilbert \( C^\ast \)-multi module \( H \) over \( (A; B_i, i = 1, \ldots, n) \) is said to be of general type if there exists a faithful completely positive map \( \lambda_i : B_i \rightarrow A \) for \( i = 1, \ldots, n \) such that
\[
\lambda_i(b_i \psi_i(a)) = \lambda_i(b_i) a, \quad b_i \in B_i, a \in A,
\]
\[
\lambda_i(\xi \mid \eta)_{B_i} = \lambda_i(\xi \mid \eta)_A, \quad \xi, \eta \in H, i = 1, \ldots, n.
\]
A Hilbert \( C^\ast \)-multi module \( H \) over \( (A; B_i, i = 1, \ldots, n) \) is said to be of finite type if there exists a family \( \{u^{(i)}_1, \ldots, u^{(i)}_{M(i)}\}, i = 1, \ldots, n \) of finite bases of \( H \) as a right \( B_i \)-module for each \( i = 1, \ldots, n \) such that
\[
\sum_{j=1}^{M(i)} u^{(i)}_j \varphi_i(\langle u^{(i)}_j \mid \xi \rangle_{B_i}) = \xi, \quad \xi \in H, i = 1, \ldots, n
\]
and
\[
\langle u^{(i)}_j \mid \phi_k(w_k)u^{(i)}_h \rangle_{B_i} \in A, \quad w_k \in B_k, j, h = 1, \ldots, M(i),
\]
\[
\sum_{j=1}^{M(i)} \langle u^{(i)}_j \mid \phi_k(\langle \xi \mid \eta \rangle_{B_i})u^{(i)}_j \rangle_{B_i} = \langle \xi \mid \eta \rangle_A
\]
for all \( \xi, \eta \in H, i, k = 1, \ldots, n \) with \( i \neq k \).

By a generalizing argument to the preceding sections, we may construct a \( C^\ast \)-algebra \( \mathcal{O}_{F(H)} \) associated with the Hilbert \( C^\ast \)-multi module \( H \) by a similar manner to the preceding sections, that is, the \( C^\ast \)-algebra generated by \( n \)-kinds of creation operators \( s^{(i)}_\xi, \xi \in H, i = 1, \ldots, n \) on the generalized Fock space \( F(H) \) by the ideal generated by the finite rank operators. One may show the following generalization:

**Proposition 8.1.** Let \( H \) be a Hilbert \( C^\ast \)-multi module over \( (A; B_i, i = 1, \ldots, n) \) of finite type with a finite basis \( \{u^{(i)}_1, \ldots, u^{(i)}_{M(i)}\} \) of \( H \) as a Hilbert \( C^\ast \)-right module over \( B_i \) for each \( i = 1, \ldots, n \). Then the \( C^\ast \)-algebra \( \mathcal{O}_{F(H)} \) generated by the \( n \)-kinds of creation operators on the generalized Fock spaces \( F(H) \) is canonically isomorphic to
the universal $C^*$-algebra $\mathcal{O}_\mathcal{H}$ generated by the operators $S_{1}^{(i)}, \ldots, S_{M_{(i)}}^{(i)}$ and elements $z_j \in B_j$ for $i = 1, \ldots, n$ subject to the relations:

$$\sum_{i=1}^{n} \sum_{k=1}^{M_{(i)}} S_{k}^{(i)} S_{k}^{(i)*} = 1, \quad S_{k}^{(i)*} S_{m}^{(j)} = 0, \quad i \neq j,$$

$$S_{k}^{(i)*} S_{l}^{(i)} = \langle v_{k}^{(i)} \mid v_{l}^{(i)} \rangle_{B_{i}}, \quad z_{j} S_{k}^{(i)} = \sum_{l=1}^{M_{(i)}} S_{l}^{(i)} \langle u_{l}^{(i)} \mid \phi_{j}(z_{j}) u_{k}^{(i)} \rangle_{B_{i}}.$$

for $z_j \in B_j$, $i, j = 1, \ldots, n$, $k, l = 1, \ldots, M_{(i)}$, $m = 1, \ldots, M_{(j)}$.

The proof of the above proposition is similar to the proof of Theorem 1.1.

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