Pseudogap and Amplitude Fluctuations in High Temperature Superconductors

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Amplitude fluctuations of the pairing field are responsible together with phase fluctuations for the pseudogap phenomena in high temperature superconductors. Here we present the more detailed theory of the amplitude and phase fluctuations approach in the framework of a fermionic pairing model. New experimental comparisons are presented for the specific heat of the cuprate LSCO confirming the generality of this phenomenological approach. The strong decrease of amplitude fluctuations near optimal doping induces the illusion of a “quantum critical point”, which in fact does not exist since the pseudogap energy scale is always different from zero even in the overdoped regime.

One major problem concerning high temperature superconductors is to establish the correct phase diagram temperature versus hole or electron doping. A related question is the interpretation of the anomalous behaviour above the critical temperature. This regime is called pseudogap\textsuperscript{[1,2]} because it contains effects similar to superconductivity like a partial suppression of electronic density of states. The pseudogap region starts below a temperature $T^\ast$ where observable quantities deviate from Fermi liquid behaviour and seems to be present well inside the superconducting phase.

Until now two interpretations seem to emerge: the first point of view assumes the existence of quantum critical point (QCP) sitting in the middle of the superconducting regime\textsuperscript{[2]}. This quantum critical point is due to an (unknown) order parameter which is competition with superconductivity. The experimental evidences for a QCP are however not convincing until now. Moreover effects of quantum fluctuations are suppressed by superconductivity at low temperature. Loram and Tallon\textsuperscript{[4]} have analyzed many experimental data like specific heat or scanning tunneling experiments and extract the value of an external energy scale $E_g$. In their analysis, they assume the presence of an external gap and extract its value by fitting experiments.

The second group of approaches tends to analyse the consequence of pairing and superconductivity on the phase diagram\textsuperscript{[2,3,4]}. The pseudogap phase is then due to precursor effects of superconductivity or amplitude pairing fluctuations for example. With this point of view, one does not need the existence of the QCP. However, there is still the need to explain the characteristic pseudogap energy found $E_g$ by Loram and Tallon\textsuperscript{[4]}.

It is perhaps important to remind that $T^\ast$ is a crossover temperature that can be directly seen in experiments where observable quantities deviate from Fermi liquid behaviour. $E_g$ in the sense of ref.\textsuperscript{[4]} is a phenomenological energy scale extracted by fitting experiments according to a simple theory of gapped electrons.

In a recent letter\textsuperscript{[6]}, the importance of treating both amplitude and phase fluctuations in a pairing model has been shown. Using this method, it is possible to fit accurately experiments and to extract all the phase diagram of cuprates except the antiferromagnetic regime. By expanding observables around the average amplitude, one can compute thermodynamic quantities like specific heat or magnetic spin susceptibility. Two regions have to be distinguished (see Fig. 1): for a relatively small temperature interval $T_c < T < T_\phi$ the phase of $\psi$ is still correlated in space over some coherence length $\xi$ (the Kosterlitz-Thouless coherence length in 2$d$). Thus, in this regime, observables are governed by correlated phase fluctuations described by the XY-model. For $T_\phi < T < T^\ast$, phases of $\psi$ are essentially uncorrelated ($\xi$ is on the order of the lattice constant), but $|\psi|$ is still fluctuating and non-zero, signaling local pair fluctuations. This explains the wide hump between $T_c$ and temperature $T^\ast$ seen in specific heat experiments\textsuperscript{[7]}, the depression of the spin susceptibility\textsuperscript{[10]} and the persistence of the pseudogap for $T < T^\ast$.

One important result is the extracted phase diagram: by fitting the experiments it is possible to get the phase diagram

![Phase Diagram](image.png)

FIG. 1: Schematic phase diagram of cuprates. The pseudogap region of the copper oxides phase diagram lies between the critical temperature $T_c$ and a temperature $T^\ast$. The temperature $T_\phi$ has been reported in several experiments\textsuperscript{[8,9]} and shows the onset of phase correlations or superconducting fluctuations. The energy scale $E_g$ has also been found in many experiments like specific heat\textsuperscript{[2]} or transport\textsuperscript{[3]}. $V_0$ and $T_0$ are the model parameters. $V_0$ is the superfluid stiffness and $T_0$ the mean-field pairing temperature.
stiffness $V_0$, the mean-field temperature $T_0$ and the energy scale $E_g$ versus doping.

In this paper, we will show that the phase diagram of cuprates has indeed an energy scale $E_g$ similar to the one found in [4]. However this pseudogap energy scale is due the presence of a pairing attraction and controlled by amplitude fluctuations. When these two effects are taken into account, they produced a very large $E_g$ in the underdoped regime where the superfluid density is low and fluctuations are large. In the maximum $T_c$ regime, $E_g$ becomes rapidly small because amplitude fluctuations decrease rapidly. The interaction strength is also decreasing when overdoping. This creates the illusion of a QCP in the middle of the superconducting region.

Our approach has the advantage that the only strong assumption is that high temperature superconductivity is caused by a fermionic pairing attraction which seems to be likely the case. All other assumptions are only technical and related to the calculations.

Various experimental observations can indeed be interpreted in terms of fluctuations of the pairing field $\psi = |\psi|e^{i\phi}$, and that two temperature regions have to be distinguished (see Fig. ??): for a relatively small temperature interval $T_c < T < T_\phi$ the phase of $\psi$ is still correlated in space over some coherence length $\xi$ (the Kosterlitz-Thouless coherence length in 2d) whereas the amplitude $|\psi|$ is almost constant. Thus, in this regime, observables are governed by correlated phase fluctuations described by the XY-model. For $T_\phi < T < T^*$, phases of $\psi$ are essentially uncorrelated ($\xi$ is on the order of the lattice constant), but $|\psi|$ is still non-zero, signaling independent fluctuating local pairs. This explains the wide hump seen in specific heat experiments [1], the depression of the spin susceptibility [11] and the persistence of the pseudogap for $T < T^*$. Moreover, a magnetic field that destroys this pseudogap has to break fluctuating pairs and must therefore be much higher than the one which suppresses phase coherence and thus superconductivity [11].

Our approach has a major difference with the Emery and Kivelson phase fluctuations scenario [4] of the pseudogap regime: our calculations show that phase fluctuations decrease rapidly. The interaction strength is also decreasing when overdoping. This creates the illusion of a QCP in the middle of the superconducting region.

We are not primarily interested in the origin of pairing between electrons but in the consequences of pairing on the phase diagram or physical quantities like specific heat or magnetic susceptibility. We assume that there is an attractive force between fermions, and we base our calculations on a $d$-wave attractive Hubbard model

$$H = - \sum_{(i,j)\sigma} t c_{i\sigma}^\dagger c_{j\sigma} - U \sum_i Q^\dagger_i(i) Q_i(i)$$

with a hopping $t$ between nearest neighbour sites $i$ and $j$ on a square lattice. The interaction favours the formation of onsite $d$-wave pairs since

$$Q^\dagger_i(i) = \sum_j D_{ij} Q^\dagger_{ij}$$

where $D_{ij} = 1, (-1)$ for $i$ being the nearest neighbour site of $j$ in horizontal (vertical) direction.

$$Q^\dagger_{ij} = (c_{i\uparrow}^\dagger c_{j\downarrow}^\dagger - c_{i\downarrow}^\dagger c_{j\uparrow}^\dagger) / \sqrt{2}$$

is an operator creating a singlet pair on neighbouring sites. Decoupling the interaction with the help of a Stratonovich-Hubbard transformation, the partition function $Z = \text{Tr} e^{-\beta H}$ is then

$$Z = Z_n \int D\psi \left< \sum_{i} \left( e^{\beta E_{\psi}} \psi(i) \right) \right>_{H_n}$$

where

$$\psi = \psi(i, \tau) = |\psi(i, \tau)|e^{i\phi(i, \tau)}$$

FIG. 2: Amplitude of the pairing field above $T_c$ on a $80 \times 80$ array by cluster Monte Carlo simulations: dark color represents large amplitude and light colors low amplitude.
and $H_0 = - \sum_{(i,j)\sigma} t \, c_{i\sigma}^\dagger c_{j\sigma}$ is the non-interacting part. The trace over the fermionic operators can be evaluated yielding

$$Z = \int D\psi \, e^{-\int_0^\beta d\tau [\sum_i \frac{1}{2} |\psi|^2 + \text{Tr} \ln c]}.$$  \hspace{1cm} (6)

Here $G$ is a Nambu matrix of one-electron Green functions for fermions interacting with a given, space and time dependent pairing field $\psi(i,\tau)$. The Green functions are solution of Gorkov’s equations (see [13]).

Expanding (6) in power of $\nabla \psi$, $Z$ can be written as a functional integral involving an action $S[\psi]$ for a field $\psi$ that changes slowly in space and that can be taken time-independent:

$$S[\psi] = S_0(|\psi|) + S_1(\nabla \psi)$$  \hspace{1cm} (7)

where $S_0$ is a local functional of $\psi$:

$$S_0(|\psi|) = V \frac{|\psi|^2}{U} - \frac{2}{\beta} \sum q \log [2 \cosh \beta E_q/2]$$  \hspace{1cm} (8)

and $S_1 = c \int d^3 \nabla |\nabla \psi|^2 / 2$ can be considered as the deformation or kinetic energy where $c$ is a constant. The quasi-particle energy is $E_q = \sqrt{|\psi|^2 + (\varepsilon_q - \mu)^2}$.

Now we would like to compute thermodynamic observables such as energy $U = \langle S \rangle_S$, specific heat and spin susceptibility. The point is that we want to keep the XY universality class of the transition and to describe the ferromagnetic character of the system: in the limit of high density or weak interaction, the superconductor should be described by a BCS like mean field theory whereas in the low density limit with strong interaction the transition becomes XY like. Our goal is to derive a theory that describes these two regimes and, of course, the intermediate regime.

Our main strategy will be to neglect amplitude correlations since simulations show that they are weak between different sites $i,j$: $\langle |\psi_i| |\psi_j| \rangle - \langle |\psi| \rangle^2 \approx 0$ since the amplitude is always positive and cannot show any critical behaviour. In this spirit, two different approaches are possible:

First approach: the amplitude is fixed but still temperature dependent, and is determined by a suitable variational equation. Fluctuations from the XY model are kept as well as the amplitude weight coming from the Jacobian of the cartesian to polar coordinate transformation.

Second approach: the energy is expanded around the average amplitude. Higher powers of amplitude fluctuations are neglected. Here the local coupling between phase and amplitude is kept, and amplitudes are allowed to fluctuate.

II. VARIATIONAL METHOD

The integration of the partition function can be expressed in polar coordinates using the transformation:

$$D\psi = \prod_i \int_{-\infty}^{+\infty} d\psi_i \, d\psi_i^* = \prod_i \int_0^{+\infty} d|\psi_i| \int_0^{2\pi} d\phi_i \langle 0 | \prod_i |\phi_i \rangle$$  \hspace{1cm} (9)

Rewriting the free energy $F$ in terms of a constant amplitude $|\psi|$ yields

$$F = -\frac{1}{\beta} \log \int D\phi \, e^{-\beta (S_0(|\psi|) - \log(|\psi|) V/\beta + S_1)}$$  \hspace{1cm} (10)

where the Jacobian $|\psi_1|$ of the polar transformation is put into the exponential, and $V$ is the volume. Taking the derivative of $F$ with respect to $|\psi|$ and equating it to zero leads to the self-consistent equation:

$$\frac{\partial S_0(|\psi|)}{\partial |\psi|} - \frac{V}{|\psi|} + c |\psi| \left( \langle |\nabla e^{i\phi}|^2 \rangle_{\phi} \right)_{S_1} = 0.$$  \hspace{1cm} (11)

Evaluating equation (11) and multiplying it with $|\psi|^2 / 2$

$$\frac{|\psi|^2}{U} - \frac{|\psi|^2}{W} \int_{-\mu}^{W-\mu} d\xi \frac{\tanh(\sqrt{2^2 + |\psi|^2/(2T)})}{\sqrt{2^2 + |\psi|^2}} - \frac{1}{2\beta} + c |\psi|^2 \left( \langle |\nabla e^{i\phi}|^2 \rangle_{\phi} \right)_{S_1} = 0.$$  \hspace{1cm} (12)

The first and second terms are the amplitude contribution and leads to the BCS gap equation if other contributions are neglected. The third comes from the Jacobian and implies that the amplitude is never zero. The last term is the expectation value of the energy $U_{xy}$ in the XY model with a constant dimensionless coupling $K$

$$K = \frac{V_0}{T} \frac{|\psi|^2}{|\psi|^2}.$$  \hspace{1cm} (13)

where $V_0$ is the zero temperature phase stiffness, and $|\psi_0|$ is the zero temperature amplitude. This contribution characterises the influence of the phase fluctuations. $U_{xy}(K)$ is a monotonic decreasing function with an inflexion point at $T_c$. Solutions of equation (13) are reliable for all temperatures except for $T \ll T_c$. However they are only expected to be accurate at $T_c$ if the average amplitude is large and not varying to much with temperature.

III. AVERAGE VALUE METHOD

Since simulations show that amplitude correlations are weak, we can expand the energy around the average amplitude where it is not coupled to the phase. The energy is then

$$U = \langle S \rangle_S \approx S_0(|\psi|) + \langle S_1 \rangle_S$$  \hspace{1cm} (14)
Here, $S_0(\langle |\psi| \rangle)$ is the first term of an expansion of the average $\langle S_0(\langle |\psi| \rangle) \rangle$ around $\langle |\psi| \rangle$.

$$\langle S \rangle_S = \left( S_0(\langle |\psi| \rangle) + c_1 \delta |\psi| + c_2 \frac{\delta |\psi|^2}{2} + ... \right)$$

$$= S_0(\langle |\psi| \rangle) + \sum_{n=1}^{\infty} \frac{c_n}{n!} \langle \delta |\psi|^{2n} \rangle$$

(15)

where $\delta |\psi| = |\psi| - \langle |\psi| \rangle$. Assuming that amplitudes have gaussian fluctuations around their average value, we can use the Wick’s theorem:

$$\langle (|\psi| - \langle |\psi| \rangle)^{2n} \rangle = (2n + 1)!! (\langle |\psi|^2 \rangle - \langle |\psi|^2 \rangle^2)^n$$

This last identity together with equation (15) leads to

$$\langle S \rangle_S = \sum_{n=1}^{\infty} \frac{c_n}{n!} (2n + 1)!! (\langle |\psi|^2 \rangle - \langle |\psi|^2 \rangle^2)^n$$

(16)

Higher corrections term are proportional to powers of the square of the standard deviation $\langle |\psi|^2 \rangle - \langle |\psi|^2 \rangle^2$. Although amplitude fluctuations are large, the standard deviation is half of the average amplitude, powers of $\langle |\psi|^2 \rangle - \langle |\psi|^2 \rangle^2$ are small compared to the average amplitude itself. This is why we only take the first term of the expansion which already contains the main thermodynamic features. Additional terms would just add more fluctuations and small corrections to the results.

For simplicity, averages are computed using a normalised Ginzburg-Landau action $S_{GL}$ (see [14]) whose potential part $U_{GL}$ is equal to the first two terms of the expansion of $S_0$ with respect to $\beta |\psi|$:

$$S_{GL}[\psi] = k_B V_0 \int d^q \tau (U_{GL} + S_1)$$

(17)

where

$$U_{GL} = \eta^2 (t^2 \tilde{\psi}^2 + \frac{1}{2} \tilde{\psi}^4),$$

(18)

and $\sim T/T_0 - 1$ is the reduced temperature, $\tilde{\psi} = \psi/\psi(T=0)$ is the reduced field, $T_0$ is the mean field BCS pairing temperature. $S_{GL}$ is normalised with a lattice spacing $\varepsilon$.

$$\eta := \varepsilon/\xi_0,$$

where $\xi_0$ is the mean field coherence length at zero temperature, and $V_0$ is the zero temperature phase stiffness. Contrary to the variational method, amplitude and phase are still locally coupled through $S_1$. The energy becomes

$$U \approx S_0(\langle |\psi| \rangle_{GL}) + \langle S_1 \rangle_{GL}$$

(19)

where

$$S_0(\langle |\psi| \rangle) = \langle V \langle |\psi|^2 \rangle / U \rangle - \frac{2}{\beta} \sum_q \log[2 \cosh \beta E_q / 2]$$
where the amplitude \(|\psi|\) is replaced by \(|\bar{\psi}|\) in the variational approach and by \(\langle |\psi| \rangle\) in the average value approach. The entropy \(S\) for a fermionic system in the presence of a gap \(|\psi|\) is a universal function of the ratio \(|\psi|/T\):

\[
S(|\psi|/T) = -k_B \sum_k f_k \log(f_k) \tag{23}
\]

where the Fermi distribution is \(f_k = \frac{1}{e^{\beta E_k} + 1}\) and the energy \(E_k = \sqrt{\xi_k + |\psi|^2}\). By using the entropy, it is not necessary to perform the sum over \(k\) each time one wants to evaluate the specific heat. It is then sufficient to take the derivative of the entropy.

**Normalisation of the specific heat:** \(\gamma_0\) is 1 at high temperature since it is divided by the Sommerfeld constant \(\gamma_n\). The phase contribution is normalised as:

\[
\frac{C_1}{\gamma_n} = \frac{k_B}{\xi_0^3\gamma_n N k_B} \tag{24}
\]

where \(C_1\) is the specific heat per volume \(V = N \xi_0^3\), and \(C_1^{(s)}/(N k_B)\) is the specific heat per number of lattice sites coming from the simulations. Experiments give \(\gamma_n \approx 26 \text{ mJ K}^{-1} \text{mol}^{-1} = 252 \text{ J K}^{-1} \text{m}^{-3}\). For the fit of Fig. 6 using the reasonable value \(\xi_0 \approx 16 \text{ Å}\), we get the dimensionless constant \(\alpha = k_B/(\xi_0^3 \gamma_n) \approx 13.5\).

**A. Specific heat for the variational method (d-wave)**

Using the variational method the specific heat \(C\) is now

\[
\gamma = \gamma_0 \langle |\bar{\psi}| \rangle + \gamma_1 \tag{25}
\]

where the amplitude \(|\bar{\psi}|\) is now solution of the amplitude equation \(\text{[11]}\). The dimensionless constant is \(\alpha \approx 13.5\).
| Doping (x) | $T_c$ [K] | $V_0$ [K] | $T_0$ [K] | $\alpha$ |
|-----------|-----------|-----------|-----------|---------|
| 0.92      | 92.9      | 261.9     | 124.3     | 13.7    |
| 0.87      | 93.7      | 178.0     | 143.3     | 14.3    |
| 0.80      | 88.8      | 121.8     | 158.9     | 10.6    |
| 0.73      | 69.3      | 111.0     | 177.6     | 13.7    |
| 0.67      | 60.8      | 88.9      | 193.0     | 18.1    |
| 0.57      | 55.9      | 76.1      | 213.5     | 20.6    |
| 0.48      | 47.7      | 57.6      | 241.5     | 16.0    |
| 0.43      | 31.5      | 37.3      | 251.2     | 9.0     |

**TABLE I:** Extracted parameters $V_0, T_0, \alpha$ from the fits of YBCO specific heat. (see Figure 7).

In Fig. 6, the experimental specific heat of YBa$_2$Cu$_3$O$_{6.73}$ [1] is fitted using the variational method reproducing the double peak structure: a sharp peak below $T_\phi$ coming from phase fluctuations and a wide hump below $T^*$ rounded by amplitude fluctuations. The crossover temperature $T_\phi$, where phases become random, corresponds to the temperature where $\gamma_1$ is less than approximately 2% of the normal specific heat. In amplitude equation (11), a 2 dimensional density of states $D(\varepsilon) = 1/W$ is used with $W = 5000K$, $\mu = 0.25W$ and $U = 959K$. These parameters give $T_0 \approx 200K$ and $\psi_0 \approx 2.14T_0$ in agreement with experiments [17]. The other parameters are $V_0 = 72K$ and $\eta = 5$.

**B. Specific heat for the average value method (d-wave)**

Following equation (19), the specific heat is given by

$$\gamma = \gamma_0 \langle |\psi| \rangle_{GL} + \gamma_1$$

(26)

where $\gamma_1$ is divided by $T_c$ instead of $T$ since $S_{GL}$ is a classical action and does not satisfy the Nernst theorem.

The average value method is compared to specific heat obtained for different doping in Fig. 7. For underdoped systems $x < 0.80$ we use simulations in $d = 2$. For the more overdoped, $x \geq 0.80$, simulations are done in $d = 3$. The parameter $\eta$ is fixed to 3. Parameters $V_0$ and $T_0$ extracted from the fits are shown in the phase diagram of Fig. 14 and in table I. Values of the phase specific heat normalisation constant $\alpha$ range from 10 to 20. The fitting procedure is done completely automatically by a random walk in the parameter space $\{V_0, T_0, \alpha\}$, until the error between experimental data and the fit is minimal. As
usual, a local minimum can be reached by using this procedure, and one can be trapped in this minimum. However, when the dimension of the parameter space is 3 as in our case, different local minima can be easily excluded and the best fit can be achieved.

The temperature \( T_0 \) is derived from figure 11 where the average amplitude is compared to the average amplitude \( \langle |\psi| \rangle \), computed for a field where phases are random, i.e. the gradient of phases is set to \( \pi/2 \):

\[
\langle |\psi| \rangle = \frac{1}{Z_r} \int d|\psi| |\psi| e^{-\frac{\gamma}{4}[|\varphi|^2 + \frac{1}{2}(|\psi|^4) + D|\psi|^2]}
\]

where \( D \) is the dimension of the system and the normalisation factor \( Z_r \) is

\[
Z_r = \int d|\psi| |\psi| e^{-\frac{\gamma}{4}[|\varphi|^2 + \frac{1}{2}(|\psi|^4) + D|\psi|^2]}
\]

C. Specific heat for \( s \)-wave symmetry

The measured specific heat of YBa\(_2\)Cu\(_{3}\)O\(_{6.76}\) \( \Gamma \) \( s \)-wave is compared to our results for the \( s \)-wave symmetry in Fig. 11 by using the average value method.

We took the following values for the two parameters: \( T_0 = 235 K \) is of the order of \( T^* \), \( V_0 = 108 K \) is of the order of \( T_c \). For the size of \( \psi \) we took the BCS \( d \)-wave relation: \( \psi(T = 0) = 2.14T_0 \) since it seems to be more in agreement with experiments \( \Gamma \) than the BCS relation of \( \psi(T = 0) = 1.76T_0 \). This choice plays only a little quantitative role.

The mean amplitude, standard deviation and phase stiffness are shown in Fig. 12 for the same parameters as in Fig. 11. The coherence length \( \xi \) is of the order the lattice constant when the temperature \( T_0 \) is reached.

Considering Fig. 11 we see that an \( s \)-wave computation fits equally well experiments as the \( d \)-wave in Fig. 12.

Hence, we cannot decide which symmetry is favored by looking at specific heat data. \( s \)-wave and \( d \)-wave symmetries are essentially different at low temperatures. \( d \)-wave specific heat has an algebraic increase with temperature whereas \( s \)-wave specific heat has an exponential behaviour. Therefore the \( s \)-wave contribution is smaller at low temperature than the \( d \)-wave. However, a lower \( s \)-wave contribution can be compensated by changing parameters \( T_0 \) and \( V_0 \).
V. DIFFERENTIAL CONDUCTANCE

The differential conductance between a normal metal and a superconductor is directly related to the density of states and the amplitude of the pairing field by the standard formula:

$$\frac{dI}{dV} = -G_{nn} \sim \frac{N_s(\xi)}{N(0)} \frac{\partial f(\xi + eV)}{\partial (eV)}$$  \hspace{1cm} (29)

where $G_{nn}$ is the differential conductance between two normal metals. The reduced energy is $\xi = \varepsilon - \mu$. The s-wave density of states is according to BCS theory:

$$N_s(\xi) = \frac{|\xi|/\sqrt{\xi^2 - \langle|\psi|^2\rangle}}{2}$$  \hspace{1cm} (30)

Of course, $N_s(\xi) = 0$ if $|\xi| < \langle|\psi|\rangle$.

In Fig. 13 the s-wave differential conductance normalised with $G_{nn}$ is presented using an average amplitude with the same parameters as in Figure 4 of reference [7]. Figure 13 is presented using an average amplitude with the same parameters as in Figure 4 of reference [7].

VI. EXTRACTED PHASE DIAGRAM

Values for $T_0$, $V_0$ and $T_\phi$ extracted for the specific heat in Fig. 7 are reported in Fig. 14. $T_0$ is obtained by using the comparison between amplitude from simulations and amplitude in the disordered phase as shown in Fig. 9. $T_0$ is defined as the temperature where $\langle|\psi|\rangle/T = h$ for various thresholds $h = 2\%, 3\%, ...$. The random amplitude $\langle|\psi|\rangle_r$ comes from equation 27.

It is remarkable that phase correlations above $T_c$ grow rapidly in the underdoped regime following the $T_\phi$ line, and reduce when approaching the overdoped regime. $T_\phi$ lines are similar to Nernst effect results. $T_\phi$ is also in agreement with the evolution of the temperature $T_{scf}$ (scf=superconducting fluctuations) of S. H. Naqib et al derived by transport measurements. The gradient specific heat seems to disappears more rapidly. This latter doping dependence is in better agreement with the phase diagram derived in Hall effect experiments.

In Fig. 14 the pseudogap energy scale $E_g$ is reported in the phase diagram of YBCO. The pseudogap energy scale $E_g$ is defined here as the amplitude at $T = 200$K:

$$E_g = \langle|\psi|\rangle_{T=200K}$$  \hspace{1cm} (32)

$E_g$ shows approximately the same doping dependence as the one found by Loram and Tallon [4]. However $E_g$ is not related to some additional hidden order parameter since $E_g$ is simply related to amplitude fluctuations as shown by calculations.
VII. DISCUSSION

We have shown that amplitude and phase fluctuations can explain the emergence of the pseudogap region of underdoped high temperature superconductors. Phase coherence disappears completely near a temperature $T_\phi$ above $T_c$, and therefore, for $T > T_\phi$, the pseudogap region is dominated by amplitude fluctuations. We find that the mean field temperature $T_0$ has a similar doping dependence as $T^\ast$, signaling that the pseudogap region is due to independent fluctuating pairs. The energy scale $E_g$ of the pseudogap is also derived. The large value of $E_g$ in the underdoped domain is due to a finite $T_0$ and large amplitude fluctuations. At maximum doping, $T_0$ remains of the order of $T_c$ but amplitude fluctuations are much weaker. Therefore $E_g$ vanishes linearly when approaching maximum doping. The fact that $E_g$ takes small values above maximum doping should not be interpreted as a quantum critical point but as a crossover: amplitude fluctuations are still present in the overdoped regime and $E_g$ has always a finite value.

Comparison with measured specific heat on underdoped LSCO reproduces the double peak structure like in YBCO: a sharp peak below $T_\phi$ coming from phase fluctuations and a separate wide hump below $T^\ast$ rounded by the amplitude. The spin susceptibility, related to the amplitude, recovers its normal behaviour near $T^\ast$ whereas the orbital magnetic susceptibility, related to phases, disappears near $T_\phi$. These considerations are independent of the underlying pairing mechanism, and any microscopic theory inducing pairing should lead to similar conclusions.

An important difference between overdoped and underdoped superconductors is the "separation" between effects of amplitudes and phases: in the overdoped regime, contributions of amplitude and phase are added producing only one peak in the specific heat for example. In the underdoped regime, phase correlations remain near $T_c$ and still produce a small peak whereas amplitude fluctuations extend to much larger temperature producing a separate wide hump between $T_c$ and $T^\ast$.

Further work is needed in order to include in the theory effects of the magnetic field on the pseudogap.

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