Uniqueness of strong solutions for SDEs with Hölder diffusions

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Abstract This paper is concerned with the Itô stochastic differential equation (SDE for short) with \(\mathbb{R}^{d \times k}\) diffusion and \(\mathbb{R}^d\) drift. We derive a uniqueness result for strong solutions and the present result is new. Precisely speaking, we prove that if the diffusion is Hölder continuous of order \(\geq \frac{1}{2}\) and the drift is continuous and monotonous, then the SDE has pathwise uniqueness.

Keywords: Hölder continuous; Uniqueness; Itô’s formula; Stochastic differential equation

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1 Introduction

Consider the following SDE

\[
\begin{aligned}
dX(t) &= b(t, X(t))dt + \sigma(t, X(t))dB(t), \\
X(t = 0) &= x,
\end{aligned}
\]

where \(b : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d\), \(\sigma : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^{d \times k}\) are Borel measurable functions and \((B(t))_{t \geq 0}\) is a \(k\)-dimensional standard Brownian motion on the classical Wiener space \((\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \geq 0})\), i.e. \(\Omega\) is the space of all continuous functions from \(\mathbb{R}_+\) to \(\mathbb{R}^d\) with locally uniform convergence topology, \(\mathcal{F}\) is the Borel \(\sigma\)-field, \(P\) is the Wiener measure, \((\mathcal{F}_t)_{t \geq 0}\) is the natural filtration generated by the coordinate process \(B(t, \omega) = \omega(t)\).

As is well known, the fundamental theory for (1.1) is developed mainly by Itô and furnishes a very important tool to construct diffusion process. In his memoir [1], Itô showed the existence and uniqueness of strong solutions for (1.1), in which, he presumed that \(b\) and \(\sigma\) satisfy a Lipschitz condition in \(x\) and that

\[
|b(t, x)| + |\sigma(t, x)| \leq C(1 + |x|).
\]

Later, the result was sharpened by a series of authors on the case of regular diffusion, including Skorokhod, Veretennikov, Flandoli, Davie, Fedrizzi, Gyöngy, Krylov, Röckner and Zhang (see [2-9]). For example, in [2], Skorokhod proved the existence of solutions for (1.1) under the condition that \(b\) and \(\sigma\) are only continuous, which yield the assumption (1.2). For \(d = k\), Veretennikov, in [3], showed

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the existence and uniqueness of solutions when \( b \) is time independent, Borel bounded measurable, and \( \sigma = I \). In [4], Flandoli, Gubinelli and Priola verified that, if \( b \in L^\infty(0,T;C^0_b(\mathbb{R}^d)), \alpha \in (0,1), d = k, \sigma = I, \) then (1.1) exists a unique strong solution, which forms a \( \mathcal{C}^{1+\beta} (\beta \in (0,\alpha)) \) diffeomorphism flow. For more details about this extension, one can consult the references and the references cited therein.

When the diffusion \( \sigma \) is singular and \( d = k = 1 \), (1.1) is also investigated in a number of classical works [2] [10] [11]. For example, in [2], assuming in addition that, \( b \) and \( \sigma \) are Hölder continuous in \( x \) of order \( \frac{1}{2} \), the author showed the uniqueness of solutions.

For general singular diffusion and multi-dimensional SDE, there are relatively few papers concerned with (1.1) and it is very difficult as well. In particular, in [12], Yamada and Watanabe argued that, if \( d = k, b \) meets to the condition

\[
|b(x) - b(y)| \leq \kappa(|x - y|), \quad \int_{0+} \kappa^{-1}(u) du = \infty \quad (1.3)
\]

and \( \sigma \) satisfies

\[
\begin{cases}
\sigma(x) = \text{diag}(\sigma_1(x_1), \sigma_2(x_2), \ldots, \sigma_d(x_d)), \\
|\sigma_i(u) - \sigma_i(v)| \leq \rho(|u - v|), \quad \int_{0+} \rho^{-2}(u) du = \infty,
\end{cases}
\quad (1.4)
\]

where \( x = (x_1, x_2, \ldots, x_d) \), \( \kappa \) is a positive increasing concave function, \( \rho \) is positive and increasing. Then the SDE has a unique solution.

On the other hand, in order to define a diffusion process through a solution of SDE, it suffices to show the uniqueness of solutions in the sense of probability law. Yamada and Watanabe in [12] developed a theory in this direction, and they proved the fact that

\[
\text{strong solution} \iff \text{weak solution} + \text{pathwise uniqueness}. \quad (1.5)
\]

Therefore, from [2][12], the problem for the uniqueness of strong solutions is reduced to the pathwise uniqueness.

Our present work is a fellow work of [12], and in this paper, we consider the autonomous SDE

\[
\begin{aligned}
&\ dX(t) = b(X(t)) dt + \sigma(X(t)) dB(t) \\
&\ X(t = 0) = x \in \mathbb{R}^d.
\end{aligned}
\quad (1.6)
\]

with the corresponding initial data

\[
X(t = 0) = x \in \mathbb{R}^d. \quad (1.7)
\]

We will prove the pathwise uniqueness for general singular diffusion.

**Notations and conventions.** For \( x, y \in \mathbb{R}^d \), we denote by \(|x|\) the Euclidean norm of \( x \), and \( \langle x, y \rangle \) is the Euclidean inner product. \( C(\mathbb{R}^d; \mathbb{R}^{d\times k}) \) stands for the set of continuous functions from \( \mathbb{R}^d \) into \( \mathbb{R}^{d\times k} \) and \( C_b(\mathbb{R}^d; \mathbb{R}^{d\times k}) \) represents the set of continuous and bounded functions. \( D(\mathbb{R}^d) \) is the set of all smooth functions on \( \mathbb{R}^d \) with compact supports. We also denote \( D_+(\mathbb{R}^d) \) the nonnegative functions in \( D(\mathbb{R}^d) \). \( a \in \mathbb{R}^{d\times k}, |a| = \sqrt{\text{Tr}a^*a}. \quad \mathbb{R}^+ = \{ x \in \mathbb{R}, x \geq 0 \}, \delta_{i,j} \) is Kronecker delta function.

The letter \( C \) with or without subscripts will be a positive constant, whose value may vary from one place to another.
A stochastic process \((\xi_t)_{t \geq 0}\) is called \((\mathcal{F}_t)_{t \geq 0}\)-adapted if for any \(t \geq 0\), the random variable \(\xi_t\) is \(\mathcal{F}_t\)-measurable. By a (strong) solution \(X(t)\) of (1.6), (1.7) we mean a continuous \((\mathcal{F}_t)_{t \geq 0}\)-adapted stochastic process given by \(t \geq 0\) and such that

\[
X(t) = x + \int_0^t b(X(s)) ds + \int_0^t \sigma(X(s)) dB(s) \tag{1.8}
\]

holds for all \(t \geq 0\) at once with probability one. Here the continuous means for almost all \(\omega \in \Omega\), \(X(t)\) is continuous and for fixed \(\omega\), \(X(t)\) is a path (or trajectory).

2 Uniqueness of strong solutions

Initially, we state our assumption on \(b\) and \(\sigma\).

**Assumption 2.1** \((A_1)\) We assume that \(b \in C_b(\mathbb{R}^d; \mathbb{R}^d)\), satisfying that

\[
\langle b(x) - b(y), x - y \rangle \leq 0, \quad \text{for any } x, y \in \mathbb{R}^d;
\]

\((A_2)\) \(\sigma \in C(\mathbb{R}^d; \mathbb{R}^{d \times k})\), there exists a constant \(\alpha > 0\) and a continuous function \(f \in C[0, \infty)\), such that

\[
|\sigma(x) - \sigma(y)| \leq |x - y|^\alpha f(|x - y|), \quad \text{for any } x, y \in \mathbb{R}^d,
\]

where

\[
f(u) = \begin{cases} 
L u^{1-\alpha}, & \text{when } \alpha > 1, \\
L, & \text{when } \alpha \leq 1,
\end{cases}
\]

and \(L\) is a positive constant;

Moreover, if \(\alpha < 1\), we presume in addition that

\((A_3)\) For any \(x, y \in \mathbb{R}^d\),

\[
\sum_{i=1}^d \sum_{l=1}^k |\sigma_{i,l}(x) - \sigma_{i,l}(y)|^2 - \sum_{i,j=1}^d \sum_{l=1}^k \frac{(x_i - y_i)(x_j - y_j)}{|x - y|^2} (\sigma_{i,l}(x) - \sigma_{i,l}(y))(\sigma_{j,l}(x) - \sigma_{j,l}(y)) \leq 0.
\]

**Theorem 2.2.** Consider the Itô SDE (1.6), (1.7), with \(\sigma\) and \(b\) meeting \((A_1) - (A_3)\). If \(\alpha \geq \frac{1}{2}\), then (1.6), (1.7) has a unique strong solution.

**Proof.** To reach this aim, let us consider two solutions \(X(t)\) and \(Y(t)\) of the SDE (1.6) associated to the same \((k\text{-dimensional})\) Brownian motion \(B(t)\) and the different initial data:

\[
\begin{cases}
X(t) = x + \int_0^t b(X(s)) ds + \int_0^t \sigma(X(s)) dB(s), \\
Y(t) = y + \int_0^t b(Y(s)) ds + \int_0^t \sigma(Y(s)) dB(s).
\end{cases} \tag{2.1}
\]

First, we show when \(\alpha \geq 1\), the strong solution is unique.

Due to the assumption \((A_1)\) and \((A_2)\), if one applies the Itô rule (see [13] for example) to \(|X(t) - Y(t)|^2\), then it leads to

\[
\mathbb{E}|X(t) - Y(t)|^2 - |x - y|^2
\]
\[
\begin{align*}
&= 2 \mathbb{E} \int_0^t \langle X(s) - Y(s), b(X(s)) - b(Y(s)) \rangle ds \\
&\quad + 2 \mathbb{E} \int_0^t \langle X(s) - Y(s), (\sigma(X(s)) - \sigma(Y(s))) dB(s) \rangle \\
&\quad + \mathbb{E} \int_0^t \langle (\sigma(X(s)) - \sigma(Y(s))) dB(s), (\sigma(X(s)) - \sigma(Y(s))) dB(s) \rangle \\
&\quad \leq \mathbb{E} \int_0^t \sum_{i=1}^d \sum_{j=1}^k |\sigma_{ij}(X(s)) - \sigma_{ij}(Y(s))|^2 ds \\
&\quad \leq C \int_0^t \mathbb{E} |X(s) - Y(s)|^2 ds. \quad (2.2)
\end{align*}
\]

From (2.2), if we set \( \mathbb{E}|X(t) - Y(t)|^2 = z(t) \), then
\[
z(t) \leq C \int_0^t z(s) ds + z(0). \quad (2.3)
\]

The Grönwall inequality uses, it is clear from (2.3) to get
\[
z(t) \leq \exp\{Ct\} z(0).
\]

Thus
\[
\mathbb{E}|X(t) - Y(t)|^2 \leq \exp\{Ct\} |x - y|^2.
\]

In particular, when \( x = y \), then
\[
\mathbb{E}|X(t) - Y(t)|^2 = 0,
\]
therefore
\[
X(t) = Y(t), \quad a.s..
\]

So we complete the proof for the case of \( \alpha \geq 1 \).

Next, we will verify the situation of \( \frac{1}{2} < \alpha < 1 \). In view of (1.5) and [2], it is sufficient to demonstrate \( X(t) \) has a unique sample trajectory and we shall prove this assertion in two cases respectively: \( \frac{1}{2} < \alpha < 1 \) and \( \alpha = \frac{1}{2} \).

Case 1: \( \frac{1}{2} < \alpha < 1 \).

For any given \( \varepsilon > 0 \), let us introduce for \( s \geq 0 \) a cut-off function
\[
\varphi_{\varepsilon}(s) = \begin{cases} 
-s^{2-2\alpha} + (2 - 2\alpha) \frac{s^{2-2\alpha}}{2-2\alpha} + (2\alpha - 1)\varepsilon^{2-2\alpha}, & s \in [\varepsilon, \infty), \\
0, & s \in (0, \varepsilon). 
\end{cases} \quad (2.4)
\]

It is easy to see that \( \varphi_{\varepsilon}(s) \) is convex and twice differentiable, with
\[
\varphi'_{\varepsilon}(s) = \begin{cases} 
-(2 - 2\alpha)s^{1-2\alpha} + \frac{2-2\alpha}{2-2\alpha}, & s \in [\varepsilon, \infty), \\
0, & s \in (0, \varepsilon), 
\end{cases} \quad (2.5)
\]
and
\[
\varphi''_{\varepsilon}(s) = \begin{cases} 
-(2 - 2\alpha)(1 - 2\alpha)s^{-2\alpha}, & s \in [\varepsilon, \infty), \\
0, & s \in (0, \varepsilon). 
\end{cases} \quad (2.6)
\]
Let $\rho$ be a regular kernel, i.e.

$$\rho \in D_+(\mathbb{R}^d) \quad \text{and} \quad \int_{\mathbb{R}^d} \rho(x) dx = 1,$$

and for any $\delta > 0$, we define

$$\rho_\delta(x) = \frac{1}{\delta^d} \rho\left(\frac{x}{\delta}\right).$$

Set $g(x) = |x|$ and

$$\varphi_{\delta, \varepsilon}(x) \equiv ((\varphi \circ g) * \rho_\delta)(x),$$

then $\varphi_{\delta, \varepsilon} \in D_+(\mathbb{R}^d)$.

Using Itô’s rule, we readily have

$$\varphi_{\delta, \varepsilon}(X_t - Y_t) = \varphi_{\delta, \varepsilon}(x - y) + I_1(t) + I_2(t) + I_3(t) + I_4(t),$$

where

$$I_1(t) = \int_0^t \langle \nabla \varphi_{\delta, \varepsilon}(X(s) - Y(s)), b(X(s)) - b(Y(s)) \rangle ds,$$

$$I_2(t) = \int_0^t \langle \nabla \varphi_{\delta, \varepsilon}(X(s) - Y(s)), (\sigma(X(s)) - \sigma(Y(s)))dB(s) \rangle,$$

$$I_3(t) = \frac{1}{2} \sum_{i,j=1}^d \sum_{l=1}^k \int_0^t (\varphi_{\varepsilon}''(g)h_{i,j,l})*\rho_\delta(X(s) - Y(s))(\sigma_{il}(X) - \sigma_{il}(Y))(\sigma_{jl}(X) - \sigma_{jl}(Y))ds,$$

$$I_4(t) = \frac{1}{2} \sum_{i,j=1}^d \sum_{l=1}^k \int_0^t (\varphi_{\varepsilon}'(g)u_{i,j,l})*\rho_\delta(X(s) - Y(s))(\sigma_{il}(X) - \sigma_{il}(Y))(\sigma_{jl}(X) - \sigma_{jl}(Y))ds,$$

where

$$h_{i,j,l}(x) = \frac{x_i x_j}{|x|^2}, \quad u_{i,j,l}(x) = \frac{\delta_{i,j}}{|x|} - \frac{x_i x_j}{|x|^3}.$$

For any positive real number $\theta > 0$, denoting by the stopping time

$$\tau_\theta = \inf\{ t \mid |X(t) - Y(t)| > \theta \} \quad \text{and} \quad |X(\infty) - Y(\infty)| = \infty.$$

If one replaces $t$ by $t \wedge \tau_\theta$, then the Itô rule uses for $\varphi_{\delta, \varepsilon}(X(t \wedge \tau_\theta) - Y(t \wedge \tau_\theta))$ (for more details in this direction see [14]), it leads to

$$\varphi_{\delta, \varepsilon}(X(t \wedge \tau_\theta) - Y(t \wedge \tau_\theta)) = \varphi_{\delta, \varepsilon}(x - y) + I_1(t \wedge \tau_\theta) + I_2(t \wedge \tau_\theta) + I_3(t \wedge \tau_\theta) + I_4(t \wedge \tau_\theta). \quad (2.9)$$

By virtue of (2.5),

$$I_2(t \wedge \tau_\theta) = \int_0^{t \wedge \tau_\theta} 1_{s \leq \tau_\theta} \langle \nabla \varphi_{\delta, \varepsilon}(X(s) - Y(s)), (\sigma(X(s)) - \sigma(Y(s)))dB(s) \rangle$$

is a martingale, thus if one takes expectation on both sides of (2.9), it follows that

$$\mathbb{E}\varphi_{\delta, \varepsilon}(X(t \wedge \tau_\theta) - Y(t \wedge \tau_\theta)) = \varphi_{\delta, \varepsilon}(x - y) + \mathbb{E}I_1(t \wedge \tau_\theta) + \mathbb{E}I_3(t \wedge \tau_\theta) + \mathbb{E}I_4(t \wedge \tau_\theta). \quad (2.10)$$
Thanks to (2.6) – (2.7) and the Cauchy-Schwarz inequality,
\[ ab \leq \frac{a^2 + b^2}{2}, \quad \forall \ a, b \in \mathbb{R}_+, \]
therefore
\[ \mathbb{E}I_3(t \wedge \tau_\theta) \leq C \mathbb{E} \int_0^{t \wedge \tau_\theta} \varphi''_\varepsilon(g) \ast \rho_\delta (X(s) - Y(s))|X(s) - Y(s)|^{2\alpha}ds, \]
which hints
\[ \mathbb{E}\varphi_{\delta,\varepsilon}(X(t \wedge \tau_\theta) - Y(t \wedge \tau_\theta)) \leq \varphi_{\delta,\varepsilon}(x - y) + \mathbb{E}I_1(t \wedge \tau_\theta) + \mathbb{E}I_4(t \wedge \tau_\theta) + C\mathbb{E} \int_0^{t \wedge \tau_\theta} \varphi''_\varepsilon(g) \ast \rho_\delta (X(s) - Y(s))|X(s) - Y(s)|^{2\alpha}ds \quad (2.11) \]
On the other hand, in view of Chebyshev’s inequality and Fubini’s theorem, then
\[ P(\tau_\theta \leq t)\varphi_{\delta,\varepsilon}(X(\tau_\theta) - Y(\tau_\theta)) \leq \mathbb{E}\varphi_{\delta,\varepsilon}(X(\tau_\theta \wedge t) - Y(\tau_\theta \wedge t)). \quad (2.12) \]
Combining (2.11) – (2.12), we obtain
\[ P(\tau_\theta \leq t)\varphi_{\delta,\varepsilon}(X(\tau_\theta) - Y(\tau_\theta)) \leq \varphi_{\delta,\varepsilon}(x - y) + \mathbb{E}I_1(t \wedge \tau_\theta) + \mathbb{E}I_4(t \wedge \tau_\theta) + C\mathbb{E} \int_0^{t \wedge \tau_\theta} \varphi''_\varepsilon(g) \ast \rho_\delta (X(s) - Y(s))|X(s) - Y(s)|^{2\alpha}ds. \quad (2.13) \]
For \( \varepsilon > 0 \) and \( \theta \) be fixed, by (2.5) – (2.7), assumption \( (A_1) \), and the dominated convergence, we derive from (2.13) that
\[ P(\tau_\theta \leq t)\varphi_{\varepsilon}(\theta) \leq P(\tau_\theta \leq t)\varphi_{\varepsilon}(|X(\tau_\theta) - Y(\tau_\theta)|) \]
\[ \leq \frac{1}{2} \sum_{i,j=1}^{d} \sum_{l=1}^{k} \mathbb{E} \int_0^{t \wedge \tau_\theta} \varphi'_{\varepsilon}(|X - Y|)u_{ij}(X - Y)(\sigma_{il}(X) - \sigma_{il}(Y))(\sigma_{jl}(X) - \sigma_{jl}(Y))ds \]
\[ + \varphi_{\varepsilon}(|x - y|) + C\mathbb{E} \int_0^{t \wedge \tau_\theta} \varphi''_{\varepsilon}(|X(s) - Y(s)|)|X(s) - Y(s)|^{2\alpha}ds, \quad (2.14) \]
if one lets \( \delta \) tend to zero.

With the help of \( (A_2) \) and \( (A_3) \), from (2.14), we obtain
\[ P(\tau_\theta \leq t)\varphi_{\varepsilon}(\theta) \leq \varphi_{\varepsilon}(|x - y|) + C\mathbb{E} \int_0^{t \wedge \tau_\theta} (2 - 2\alpha)(2\alpha - 1)|X(s) - Y(s)|^{2\alpha}ds \]
\[ \leq C t + \varphi_{\varepsilon}(|x - y|). \]
Especially, choose \( x = y \), then
\[ P(\tau_\theta \leq t)\varphi_{\varepsilon}(\theta) \leq C t. \quad (2.15) \]
On the other side, keeping \( \theta > 0 \) and \( t > 0 \) fixed,
\[ \varphi_{\varepsilon}(\theta) \to +\infty, \quad \text{if} \ \varepsilon \to 0, \]
if
we obtain $P(\tau_0 \leq t) = 0$ for all times. This implies $P(\tau_0 < \infty) = 0$. By letting $\theta$ tend to zero, we gain

$$P(\tau_0 < \infty) = 0,$$

i.e. the path is unique, so the strong solution is unique.

Case 2: We prove when $\alpha = \frac{1}{2}$, the solution to equation (2.1) has a unique sample trajectory.

We introduce for $\varepsilon > 0$ another cut-off function

$$\psi_\varepsilon(s) = \begin{cases} 
s \log \frac{2}{\varepsilon} - s + \varepsilon, & s \in [\varepsilon, \infty) \\
0, & s \in (0, \varepsilon).
\end{cases}$$

Then $\psi_\varepsilon(s)$ is convex and twice differentiable, with

$$\psi_\varepsilon'(s) = \begin{cases} 
\log s - \log \varepsilon, & s \in [\varepsilon, \infty), \\
0, & s \in (0, \varepsilon),
\end{cases} \quad \text{and} \quad \psi_\varepsilon''(s) = \begin{cases} \frac{1}{s}, & s \in [\varepsilon, \infty), \\
0, & s \in (0, \varepsilon).
\end{cases}$$

The argument applied in case 1 for $\varphi_\varepsilon$ from (2.7) to (2.15) adapted to $\psi_\varepsilon$ here, suggests that (2.16) is true as well. Then the proof is complete.

**Remark 2.3.** (i) Here we concentrate our discussion on non-random initial conditions, and if the initial data is random, i.e. $X_0$ is $\mathcal{F}_0$ measurable, assuming in addition that

$$\begin{cases} 
\mathbb{E}|X_0|^2 < \infty, & \text{if } \alpha \geq 1, \\
\mathbb{E}|X_0| < \infty, & \text{if } \alpha \in \left(\frac{1}{2}, 1\right), \\
\mathbb{E}|X_0|^\beta < \infty, & \text{for some } \beta > 1, \text{ if } \alpha \in \left(\frac{1}{2}, 1\right),
\end{cases}$$

then Theorem 2.1 is legitimate as well.

(ii) The present result holds mutatis mutandis for the non-autonomous system (1.1), if we supersede the assumption $(A_2)$ with $(A_2)'$ below

$$(A_2)' \sigma \in L^1_{lo}(0, \infty); C(\mathbb{R}^d; \mathbb{R}^{d \times k}),$$

there exists a constant $\alpha > 0$, a continuous function $f \in C[0, \infty)$ and a locally integrable function $0 \leq h \in L^1_{lo}[0, \infty)$, such that

$$|\sigma(t, x) - \sigma(t, y)| \leq |x - y|^{\alpha} f(|x - y|) h^{\frac{1}{2}}(t), \quad \text{for any } x, y \in \mathbb{R}_d, \ t \geq 0,$$

where $f$ is given in $(A_2)$.

(iii) Other various extensions may be made, such as generalizing $B(t)$ to a Lévy process (for $\alpha$-stable process, one can see [15, 16]), $b$ is not monotonic and so on.

**Remark 2.4.** The assumption $(A_3)$ on diffusion $\sigma$ seems to be rigid, however this hypotheses satisfies in a natural way for many models, particularly for $d = 1$. In fact, now

$$\sum_{i=1}^{d} \sum_{l=1}^{k} |\sigma_{i,l}(x) - \sigma_{i,l}(y)|^2 - \sum_{i,j=1}^{d} \sum_{l=1}^{k} \frac{(x_i - y_i)(x_j - y_j)}{|x - y|^2} (\sigma_{i,l}(x) - \sigma_{i,l}(y))(\sigma_{j,l}(x) - \sigma_{j,l}(y))$$

$$= \sum_{l=1}^{k} |\sigma_{1,l}(x) - \sigma_{1,l}(y)|^2 - \sum_{l=1}^{k} \frac{(x - y)(x - y)}{|x - y|^2} (\sigma_{1,l}(x) - \sigma_{1,l}(y))(\sigma_{1,l}(x) - \sigma_{1,l}(y))$$

$$= 0.$$

Therefore we prove the uniqueness of strong solutions for the SDE with the form below

$$dX(t) = b(t, X(t))dt + \sum_{l=1}^{k} \sigma_l(t, X(t))dB_l(t).$$
This result is a generalization from classical results in one dimensional SDE.

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References

[1] K. Itô, On stochastic differential equation, Memoirs of the American Mathematical Society, 4 (1951), 1-51.

[2] A.V. Skorohod, Studies in the theory of random processes, Addison-Wisley 1965 (Originally published in Kiev).

[3] A.J. Veretennikov, Strong solutions and explicit formulas for solutions of stochastic integral equations, Mat. Sb. (N.S.), 111 (153) (1980), 434-452.

[4] F. Flandoli, M. Gubinelli and E. Priola, Well-posedness of the transport equation by stochastic perturbation. Invent. Math., 180 (2010), 1-53.

[5] A.M. Davie: Uniqueness of solutions of stochastic differential equations, Int. Math. Res. Not. IMRN, 24 (2007), Art. ID rnm124.

[6] E. Fedrizzi and F. Flandoli, Pathwise uniqueness and continuous dependence of SDEs with non-regular drift, Stochastics, 83 (2011), 241-257.

[7] I. Gyöngy and T. Martínez, On stochastic differential equations with locally unbounded drift, Czechoslovak Math. J., 51 (126) (2001), 763-783.

[8] N.V. Krylov and M. Röckner, Strong solutions of stochastic equations with singular time dependent drift, Probab. Theory Related Fields, 131 (2005), 154-196.

[9] X. Zhang, Strong solutions of SDES with singular drift and Sobolev diffusion coefficients, Stochastic Process. Appl., 115 (2005), 1805-1818.

[10] A.K. Zvonkin, A transformation of the phase space of a diffusion process that will remove the drift, Mat. Sb. (N.S.), 93 (135) (1974), 129-149.

[11] I.V. Girsanov, An example of non-uniqueness of the solution of the stochastic equation of K. Ito, Theory of Probability and Its Applications, 7 (3) (1962), 325-331.

[12] T. Yamada, S. Watanabe, On the uniqueness of solutions of stochastic differential equations, J. Math. Kyoto Univ., 11 (1971), 155-167.

[13] D. Applebaum, Lévy Processes and Stochastic Calculus, Cambridge Studies in Advanced Mathematics 93, Cambridge Univ. Press, Cambridge, 2004.

[14] N.V. Krylov, Introduction to the theory of random process, Graduate Studies in Mathematics 43, Amer. Math. Soc., RI, 2002.

[15] H. Tanaka, M. Tsuchiya and S. Watanabe, Perturbation of drift-type for Lévy processes, J. Math. Kyoto Univ., 14 (1974), 73-92.
[16] E. Priola, Pathwise uniqueness for singular SDEs driven by stable processes, Osaka Journal of Mathematics, 49 (2) (2012) 421-447.