Bounce cosmology in generalized modified gravities

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We investigate the bounce realization in the framework of generalized modified gravities arising from Finsler and Finsler-like geometry. In particular, the richer intrinsic geometrical structure is reflected in the appearance of extra degrees of freedom in the Friedmann equations that can drive the bounce. We examine various Finsler and Finsler-like constructions. In the cases of general very special relativity as well as of Finsler-like gravity on the tangent bundle we show that a bounce cannot be easily obtained. However, in the Finsler-Randers space the induced scalar anisotropy can fulfill the bounce conditions and bouncing solutions are easily obtained. Finally, for the general class of theories that include a nonlinear connection a new scalar field is induced, leading to a scalar-tensor structure that can easily drive a bounce. These features reveal the capabilities of Finsler and Finsler-like geometry.

I. INTRODUCTION

Bounce cosmologies offer an alternative view of the early universe [1–6] (for a review see [7]). Historically, this idea belongs to Tolman who in 1930’s first suggested the possibility of a re-expansion of a closed universe which has already collapsed to an extremely dense state [8]. Since then, various bouncing models have been proposed within an effort for a systematic explanation of the origin of our universe.

The main advantage of bouncing cosmology is that it provides a way of solving the singularity problem which appears in the standard cosmological paradigm. The singularity (Big Bang) is replaced with a smooth transition from contraction to expansion (Big Bounce). In this sense, bounce cosmology offers an opportunity of obtaining a more continuous picture of the early universe. The efficiency of bouncing models in solving basic cosmological problems in comparison with inflationary scenarios is visualized via the wedge diagram introduced in [9].

In general, the realization of a bounce requires a violation of the null energy condition. This can be achieved with the introduction of extra degrees of freedom which are added ad hoc into the Lagrangian [4,10]. The violation of null energy condition needs to be handled with care, in order not to spoil the usual thermal history and the sequence of epochs after the bounce. Nevertheless, such violations can easily be acquired from modified [7] or quantum gravity [12]. In particular, they can be easily acquired in the Pre-Big-Bang [13] and the Ekpyrotic [14,15] models, in gravity actions with higher order corrections [16,17]; in \( f(R) \) gravity [18,19], in \( f(T) \) gravity [20], in braneworld scenarios [21,22], in non-relativistic gravity [23,25], in Galileon theory [26,27], in massive gravity [28], in Lagrange modified gravity [29], in loop quantum cosmology [30,32], etc. Moreover, a non-singular bounce model which supports magnetogenesis at the inflationary epoch is presented in [33].

Among modified gravity theories, an interesting class is that of gravitational models based on Finsler and Finsler-like geometries. These are natural extensions of Riemannian geometry in which the physical quantities may directly depend on observer 4-velocity, and this velocity-dependence reflects the Lorentz-violating character of the kinematics. Such a property is called dynamic anisotropy [34,45]. Additionally, Finsler and Finsler-like geometries are strongly connected to the effective geometry within anisotropic media [46, 47] and naturally enter the analogue gravity program [48]. These features suggest that Finsler and Finsler-like geometries may play an important role within quantum gravity physics. The dependence of the metric tensor and other quantities on the position coordinates of the base-manifold and the directional/velocity variables of the tangent space suggest that the natural geometrical framework for the description of these models is the tangent bundle of a smooth manifold. Finally, in the case where there is no velocity-dependence, Finsler geometry becomes Riemannian.

The intrinsic geometrical space-time dynamical anisotropy of Finsler geometry (not to be confused with the spatial anisotropy that may exist also in Riemannian geometry, as for instance in Bianchi cases) is included in the geometry of space-time as an intrinsic field (variable) which influences its geometrical and physical concepts. Hence, it can give us the form of anisotropy as a hypothetical field, the anisotropion, which produces this...
deviation from isotropy. This appears in the Friedmann equations and Lorentz violations \[49, 52\], and thus the anisotropy arises as a property of Finslerian spacetime \[49, 50, 53, 54\].

In the present work we are interested in investigating the bounce realization in the framework of modified gravity related to Finsler and Finsler-like geometries. In particular, we desire to see how the new features of Finsler geometry can drive bouncing solutions, and to examine the evolution of the intrinsic anisotropy during the bounce. In some bouncing scenarios the anisotropy decreases in the contracting phase and remains quite small during the bounce, in agreement with the current observational data \[4\]. On the other hand, there are also scenarios where the reduction of anisotropy in the contracting phase is followed by its exponential growth during the bounce, mainly due to quantum fluctuations of curvature \[55\]. Finally, we mention that nonsingular bounces are also possible to be generated in models which spontaneously violate Lorentz symmetry \[51\]. In this framework Lorentz symmetry violations lead to interactions with anisotropies \[56\]. Hence, we can establish a connection between anisotropic fields and nonsingular bounce. In summary, we can depict all the above form of connections in the diagram of Fig. 1

\[\text{anisotropy} \rightarrow \text{Lorentz violation} \rightarrow \text{nonsingular bounce} \]

**FIG. 1:** Connections of anisotropy, nonsingular bounce and Lorentz violation.

The plan of this work is the following: In Section II we first describe the basic conditions for a bounce realization and we briefly review Finsler geometry and gravity. Then we examine the bounce realization in general very special relativity and Finsler-Randers models. In Section III we study the case of Finsler-like gravity on a tangent bundle, while in Section IV we analyze bouncing solutions from scalar tensor theory on the fiber bundle. Finally, in Section V we present the summary and the conclusions.

## II. BOUNCE FROM FINSLER GRAVITY

In this section we are interested in studying the bounce realization in the framework of Finsler gravity. We start by describing the conditions for a bounce realization, and we provide the basics of Finsler geometry and gravity. Then we proceed to the examination of the bounce realization in specific models, such as general very special relativity and Finsler-Randers ones.

### A. Bounce conditions

Let us start by discussing the basic requirements for a bouncing solution. For the moment we consider the ordinary Friedmann-Robertson-Walker (FRW) geometry with metric

\[
[g_{\mu\nu}(x)] = \text{diag} \left( -1, \frac{a^2(t)}{1 - kr^2}, a^2(t)r^2, a^2(t)r^2 \sin^2 \theta \right),
\]

with \(a(t)\) the scale factor and \(k = -1, 0, +1\) corresponding to open, flat and closed spatial geometry respectively. As usual, in such a geometry the general field equations of any theory give rise to the Friedmann and Raychaudhuri equations, which can be written in a compact form as

\[
H^2 = \frac{8\pi G}{3} \rho_{\text{tot}} - \frac{k}{a^2},
\]

\[
\dot{H} = -4\pi G (\rho_{\text{tot}} + P_{\text{tot}}) + \frac{k}{a^2},
\]

where \(G\) is the Newton’s constant, \(H = \dot{a}/a\) is the Hubble function, and with dots denoting derivatives with respect to the cosmic time \(t\). In the above expressions \(\rho_{\text{tot}}\) and \(P_{\text{tot}}\) are respectively the total energy density and pressure of the universe, which include matter, radiation, dark energy and any other gravitational or geometrical contribution that a theory or scenario may have.

In order to obtain a bounce realization we need a contracting universe, namely with \(H < 0\), succeeded by an expanding universe, namely with \(H > 0\), and hence from continuity we deduce that at the bounce point we must have \(H = 0\). Furthermore, one can see that at the bounce point and around it we must have \(\dot{H} > 0\). Observing the form of the general Friedmann and Raychaudhuri equations \[2, 3\], and focusing on the physically more interesting flat case, we deduce that the above requirements can be fulfilled if

\[
\rho_{\text{tot}} = 0
\]

exactly at the bounce point, and if additionally the null energy condition is violated around the bounce point, namely if

\[
\rho_{\text{tot}} + P_{\text{tot}} < 0
\]

(in the case of a non-flat universe the bounce can be driven by the curvature term without null energy condition violation \[7\]). Therefore, in order to obtain a bounce one needs to construct theories in which the extra contributions to the total energy density and pressure are such that the null energy condition is violated around the bounce point and requirement \[4\] holds, and moreover the total energy becomes zero exactly at the bounce point and condition \[4\] holds. As we see in the following, scenarios based on Finsler gravity can fulfill these necessary conditions.

### B. Finsler gravity

We first briefly review the basics of Finsler gravity, since this lies in the center of the investigation of the present work.
Finsler gravity is a geometrical extension of general relativity, where the role of the metric is played by the real-valued fundamental function \( F(x, y) \), defined on the tangent bundle \( TM \) over a smooth spacetime manifold \( M \). Variable \( y \) is an element of the tangent space of \( M \) at a point \( x \) (we have suppressed indices for convenience). The distance of two elements of the tangent space of \( M \) at a point \( x \) is defined as \( ds = F(x, dx) \). We consider the following properties to hold:

1. \( F \) is continuous on \( TM \) and smooth on \( \bar{TM} \equiv TM \setminus \{0\} \) i.e. the tangent bundle minus the null section.
2. \( F \) is positively homogeneous of first degree on its second argument:
   \[
   F(x, ky) = kF(x, y), \quad k > 0.
   \] (6)
3. The form
   \[
   f_{\mu\nu}(x, y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^\mu \partial y^\nu}
   \] (7)
defines a non-degenerate matrix on \( TM \) minus the null set \( \{(x, y) \in TM | F(x, y) = 0\} \):
   \[
   \text{det} [f_{\mu\nu}] \neq 0.
   \] (8)
Using homogeneity condition (6) it can be shown that:
   \[
   F^2(x, y) = |f_{\mu\nu}(x, y)y^\mu y^\nu|,
   \] (9)
and therefore \( f_{\mu\nu}(x, y) \) can play the role of the metric for the vector space spanned by \( y \). When studying gravity, the metric \( f_{\mu\nu}(x, y) \) is considered to be of Lorentzian signature \((-, +, +, +)\).

C. General very special relativity on cosmology

A particularly interesting Finslerian cosmological model is elaborated in the framework of the so-called general very special relativity on cosmology [55]. The metric function takes the form
   \[
   F(x, y) = (g_{\mu\nu}(x)y^\mu y^\nu)^{(1-b)/2}(n_\kappa y^\kappa)^b,
   \] (10)
where \( g_{\mu\nu}(x) \) is the ordinary FRW metric [1]. Expression (10) is a direct cosmological generalization of the general very special relativity description, where the line-element is
   \[
   ds = (\eta_{\mu\nu}dx^\mu dx^\nu)^{(1-b)/2}(n_\kappa dx^\kappa)^b,
   \] (11)
with \( [\eta_{\mu\nu}] = \text{diag}(-1, 1, 1, 1) \), which is invariant under transformations generated by the deformation \( DISIM_0(2) \) of the Lorentz subgroup \( ISIM(2) \) [57][58]. The one-form \( n_\kappa \) is called “spurionic field”. We mention that the parameter \( b \) quantifies the deviation from Riemannian geometry, i.e. the Lorentz violation in the gravitational sector. Parametrized post-Newtonian (PPN) analysis [59] and use of solar system data provides the most stringent constraints on it, and thus Gravity Probe B put an upper bound at \( 10^{-7} \) [60].

The Riemannian osculating approach is followed, namely \( g_{\mu\nu}(x) = f_{\mu\nu}(x, y(x)) \), where \( y(x) \) is the tangent vector to the cosmological fluid’s (matter fluid) flow lines. As usual the matter fluid is described by the energy-momentum tensor of the perfect fluid:
   \[
   T_{\mu\nu} = \rho_m g_{\mu\nu} + (\rho_m + P_m)y_\mu y_\nu,
   \] (12)
where \( \rho_m \) is the energy density and \( P_m \) the pressure. The field equations for this construction are then:
   \[
   L_{\mu\nu} = \frac{1}{2}Lg_{\mu\nu} = -8\pi GT_{\mu\nu},
   \] (13)
where \( L_{\mu\nu} \) is the Ricci tensor for the metric \( g_{\mu\nu}(x) \) and \( L = g^{\mu\nu}L_{\mu\nu} \).

Applying the above geometrical construction in a cosmological framework we consider the spurionic field to be parallel to the velocity of the comoving observer, namely
   \[
   n_\kappa = (n(t), 0, 0, 0).
   \] (14)
As a simple model, in [56] the following approximations we imposed
   \[
   n(t) \approx At + B
   \]
   \[
   A \to 0
   \]
   \[
   B \to 0,
   \] (15)
since \( n(t) \), parametrized by \( A, B \), needs to be suitably small in order to be consistent with the observational small bound on \( b \). For these choices, the Ricci tensor components for the metric function (10) are calculated as [56]
   \[
   L_{00} = 3\frac{a}{a} + 3\frac{A b \dot{a}}{B a} + O( A^2)
   \]
   \[
   L_{11} = -\frac{a\ddot{a} + 2a\dot{a}^2 + 2k}{1 - kr^2} + 5\frac{A}{B} \frac{b}{1 - kr^2} + O(A^2)
   \]
   \[
   L_{22} = -r^2(a\ddot{a} + 2a\dot{a}^2 + 2k) - 5\frac{A}{B} br^2 a\ddot{a} + O(A^2)
   \]
   \[
   L_{33} = -r^2(a\ddot{a} + 2a\dot{a}^2 + 2k) \sin^2\theta - 5\frac{A}{B} br^2 a\ddot{a} \sin^2\theta
   \] (16)
   \[+ O(A^2). \]
Therefore, using the above, we result to the following generalization of the Friedmann equations:
   \[
   H^2 + \frac{k}{a^2} + 2\frac{A}{B} b H = \frac{8\pi G}{3} \left[ \rho_m - 2\frac{A}{B} b P_m \left( t + \frac{B}{A} \ln B \right) \right]
   \] (17)
   \[
   \dot{H} + H^2 + \frac{Ab}{B} H = -\frac{4\pi G}{3} \left[ (\rho_m + 3P_m) + 4 \ln(At + B)b(\rho_m + P_m) \right].
   \] (18)
Unfortunately, as one can see, the above Friedmann equations do not accept a bounce solution. One could still try to construct a model with a different approximation than [15] of [56], however such a detailed investigation of a new construction lies beyond the scope of the present work. Hence, in the following subsection we examine the case of another Finslerian construction, where bounce realization is possible.
D. Bounce in Finsler-Randers Space

Let us now consider a different Finslerian construction, namely the Finsler-Randers (FR) space [61,62]. In this space a Lagrangian metric function is given by

\[ F(x, y) = a(x, y) + u_\mu y^\mu, \quad ||u_\mu|| \ll 1, \quad (19) \]

where \( a(x, y) = \sqrt{g_{\kappa\lambda}(x)y^\kappa y^\lambda} \) and \( g_{\kappa\lambda}(x) \) is the FRW metric \([1]\), with \( \kappa, \lambda, \mu \in \{0, 1, 2, 3\} \).

In this cosmological model an important role is played by the variation of anisotropy \( Z_t \). In the case of the FRW geometry \([1]\) the modified Friedmann equations of the generalized form of the FR-type cosmology have been studied in \([62]\), and are written as

\[ H^2 = \frac{8\pi G}{3} p_m - HZ_t - \frac{k}{a^2}, \quad (20) \]

\[ \dot{H} = -4\pi G (p_m + m_p) + \frac{1}{4} HZ_t + \frac{k}{a^2}. \quad (21) \]

In these expressions, we have defined the variation of anisotropy \( Z_t \) as \( Z_t = u_0 \), namely as the derivative of the time component of the unit vector \( u_\alpha \) \([62]\). This variation affects the form of geometry as can be seen from relations \([20], [21]\), and at the limit \( Z_t \to 0 \) we recover the ordinary Friedmann equations of general relativity. Finally, we have considered the matter sector to correspond to a perfect fluid with energy density and pressure \( p_m \) and \( p_m \) respectively.

Observing the form of two Friedmann equation \([20], [21]\) we can define the effective energy density and pressure of geometrical origin as

\[ \rho_{FR} \equiv -\frac{3}{8\pi G} HZ_t \quad (22) \]

\[ p_{FR} \equiv \frac{5}{16\pi G} HZ_t. \quad (23) \]

Therefore, the total energy density and pressure respectively become \( \rho_{tot} = p_{tot} + P_{FR} \) and \( P_{tot} = P_m + P_{FR} \), and the Friedmann equations take the usual form of \([2], [3]\). Hence, we can now easily examine what are the conditions in order to fulfill the bounce requirements \([4]\) and \([5]\). Firstly, from \([4]\) we deduce that for the flat universe exactly at the bounce point \( \rho_m \) must be zero (\( \rho_{FR} \) becomes also zero exactly at the bounce point, since \( H = 0 \)). This is a usual assumption in many bouncing models and it is expected to be fulfilled in the early universe. Taking this into account, we moreover see that condition \([5]\) implies that around the bouncing point \( \rho_{FR} + P_{FR} < 0 \) and thus that \( HZ_t > 0 \). Hence, we deduce that the above requirements can be fulfilled if we suitably choose the variation of anisotropy \( Z_t \).

In order to provide a specific example we focus on a flat FRW geometry \((k = 0)\) and we consider a bouncing scale factor of the form

\[ a(t) = a_0(1 + Bt^2)^{1/3}, \quad (24) \]

where \( a_0 \) is the scale factor value at the bounce, while \( B \) is a positive parameter which determines how fast the bounce takes place. In this case time varies between \(-\infty \) and \(+\infty \), with \( t = 0 \) the bouncing point, and where away from the bounce one obtains the usual expansion behavior. Moreover, we consider that the matter sector is absent in the early universe. Inserting these into \([20]\) we immediately find that

\[ Z_t = -\frac{2Bt}{3(1 + Bt^2)}. \quad (25) \]

Hence, it is this \( Z_t \), that comes from the Finslerian modification of the geometry, that generates the bouncing scale factor \([24]\). Moreover, we remark that the variation of anisotropy actually determines the physically important quantity \( B \) in \([24]\).

III. FINSLER-LIKE GRAVITY ON A TANGENT BUNDLE

In this Section we are interested in examining whether a bounce can be realized from Finsler-like gravity on a tangent bundle. Generally, we will use the term Finsler-like for any metric theory in which the various structures may depend on a set of internal variables apart from the position or external ones which we denote as \( x^\alpha \) through this work. Geometrical extensions of general relativity on the tangent bundle \( TM \) of a smooth manifold \( M \) have been presented in the bibliography \([63,64]\). In the following we focus our interest on a tangent bundle equipped with a Finslerian Sasaki-type metric:

\[ g = g_{\mu\nu}(x, y) dx^\mu \otimes dx^\nu + v_{\alpha\beta}(x, y) dy^\alpha \otimes dy^\beta, \quad (26) \]

where \( x^\mu \) are the coordinates on the base manifold, with \( \kappa, \lambda, \mu, \nu, \ldots = 0, 1, 2, 3 \), and \( y^\alpha \) are the fiber coordinates, with \( \alpha, \beta, \ldots, \theta = 0, 1, 2, 3 \). On the total space \( TTM \) of \( TM \), the adapted basis is \( \{\delta_\mu, \delta_\alpha\} \) and its dual is given by \( \{dx^\mu, dy^\alpha\} \), and the following definitions hold:

\[ \delta_\mu = \frac{\delta}{\delta x^\mu} = \frac{\partial}{\partial x^\mu} - N^\alpha_\mu(x, y) \frac{\partial}{\partial y^\alpha} \]

\[ \delta_\alpha = \frac{\partial}{\partial y^\alpha} \]

\[ \delta y^\alpha = dy^\alpha + N^\alpha_\mu(x, y) dx^\mu, \quad (27) \]

where \( N^\alpha_\mu(x, y) \) are the coefficients of a nonlinear connection on \( TM \). This connection is defined by a splitting of the total space \( TTM \) of \( TM \) into an h-subspace \( HTM \) spanned by \( \{\delta_\mu\} \) and a v-subspace \( VTM \) spanned by \( \{\delta_\alpha\} \) \([63]\). The tangent space of \( TM \) is thus a Whitney sum of the h-subspace and v-subspace, namely

\[ TTM = HTM \oplus VTM. \quad (28) \]

One can now introduce the \( d \)-connection \( \mathcal{D} \) as a covariant linear differentiation rule that preserves h-space and v-space:

\[ \mathcal{D}_{\delta_\mu} \delta_\nu = L^\mu_{\nu\kappa}(x, y) \delta_\mu \quad \mathcal{D}_{\delta_\mu} \delta_\nu = C^\mu_{\nu\lambda}(x, y) \delta_\mu \quad (29) \]

\[ \mathcal{D}_{\delta_\mu} \delta_\alpha = L^\mu_{\alpha\kappa}(x, y) \delta_\alpha \quad \mathcal{D}_{\delta_\mu} \delta_\alpha = C^\mu_{\alpha\beta}(x, y) \delta_\alpha. \quad (30) \]
A canonical $d$–connection is a linear connection that is compatible with the metric [26] and it preserves under parallel translation the horizontal and vertical subspaces $HTM$ and $VTM$ [63]. It can be uniquely defined if one demands that it only depends on $g_{\mu\nu}, v_{\alpha\beta}$ and $N^a_{\mu}$, and moreover that the connection coefficients $L^\mu_{\nu\kappa}$ and $C^a_{\mu\nu}$ are symmetric on the lower indices. In this case, its coefficients turn out to be [67]:

\[ L^\mu_{\nu\kappa} = \frac{1}{2} g^{\rho\sigma} \left( \delta_\kappa g_{\rho\sigma} + \delta_\rho g_{\kappa\sigma} - \delta_\sigma g_{\kappa\rho} \right) \]

\[ L^a_{\nu\kappa} = \frac{1}{2} v^{\alpha\nu} \left( \delta_\kappa v_{\beta\gamma} - v_{\beta\gamma} \delta_\kappa v_{\lambda\sigma} - v_{\beta\sigma} \delta_\kappa v_{\lambda\gamma} - v_{\beta\gamma} \delta_\kappa v_{\lambda\sigma} \right) \]

\[ C^\mu_{\nu\gamma} = \frac{1}{2} g^{\rho\sigma} \delta_\gamma g_{\rho\sigma} \]

\[ C^a_{\nu\gamma} = \frac{1}{2} v^{\delta\nu} \left( \delta_\gamma h_{\delta\beta} + \delta_\beta h_{\delta\gamma} - \delta_\gamma v_{\delta\beta} \right). \]  

(31)

Now, the curvature of the nonlinear connection is defined as

\[ \Omega^a_{\nu\kappa} = \frac{\delta N^a_{\nu}}{\delta x^\kappa} - \frac{\delta N^a_{\kappa}}{\delta x^\nu}, \]  

(32)

and the space at hand is equipped with various Ricci curvature tensors such as:

\[ \tilde{R}_{\mu\nu} = \delta_\kappa L^\kappa_{\mu\nu} - \delta_\nu L^\kappa_{\mu\kappa} + L^\rho_{\mu\nu} L^\kappa_{\rho\kappa} - L^\rho_{\mu\kappa} L^\kappa_{\rho\nu} \]  

(33)

\[ S_{\alpha\beta} = \delta_\gamma C^\gamma_{\alpha\beta} - \delta_\beta C^\gamma_{\alpha\gamma} + C^\epsilon_{\alpha\gamma} C^\gamma_{\epsilon\beta} - C^\epsilon_{\alpha\gamma} C^\gamma_{\beta\epsilon}. \]  

(34)

Hence, the generalized Ricci scalar curvature reads as

\[ \mathcal{R} = g^{\mu\nu} \tilde{R}_{\mu\nu} + v^{\alpha\beta} S_{\alpha\beta} \equiv \tilde{R} + S. \]  

(35)

One can now write a Hilbert-like action, namely [63–66]

\[ S_{TM} = \frac{1}{16\pi G} \mathcal{S}_H + \mathcal{S}_M \]

\[ \equiv \frac{1}{16\pi G} \int d^8 \mathcal{U} \sqrt{\det G} \mathcal{L}_H + \int d^8 \mathcal{U} \sqrt{\det G} \mathcal{L}_M, \]  

(36)

with

\[ d^8 \mathcal{U} = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dy^0 \wedge dy^1 \wedge dy^2 \wedge dy^3, \]  

(37)

where the gravitational part of the action $\mathcal{S}_H$ is constructed by the gravitational Lagrangian

\[ \mathcal{L}_H = \mathcal{R} = (\tilde{R} + S), \]  

(38)

and the matter action $\mathcal{S}_M$ by the matter Lagrangian $\mathcal{L}_M$.

Extremization of the total action $S_{TM}$ with respect to the metric components $g_{\mu\nu}$ and $v_{\alpha\beta}$ leads to the following field equations [59],

\[ \tilde{R}_{(\mu\nu)} - \frac{1}{2} (\tilde{R} + S) g_{\mu\nu} = 8\pi GT_{\mu\nu} \]  

(39)

\[ S_{\alpha\beta} - \frac{1}{2} (\tilde{R} + S) v_{\alpha\beta} = 8\pi G Y_{\alpha\beta}, \]  

(40)

where we have defined $T_{\mu\nu} = -\frac{2}{\sqrt{\det G}} \delta (L \mathcal{V} \mathcal{L}_M / \delta g^{\mu\nu})$ and $Y_{\alpha\beta} = -\frac{2}{\sqrt{\det G}} \delta (L \mathcal{V} \mathcal{L}_M / \delta \mathcal{L}_M)$. Applying these field equations in the FRW metric [1], focusing on the flat case, and assuming the usual matter perfect fluid [12], one obtains the following modified Friedmann equations [50]:

\[ H^2 = \frac{8\pi G}{3} \rho_m - \frac{1}{6} S \]  

(41)

\[ \dot{H} + 2H^2 = -\frac{4\pi G}{3} (\rho_m + 3p_m) - \frac{1}{6} S, \]  

(42)

where due to the imposed symmetries all quantities depend only on time.

From the form of the two Friedmann equations [41, 42] we can see that we obtain extra contributions that reflect the Finsler-like structure of the tangent bundle. In particular, these induce an effective energy density and pressure of geometrical origin as

\[ \rho_S \equiv -\frac{1}{16\pi G} S \]  

(43)

\[ P_S \equiv \frac{1}{16\pi G} S. \]  

(44)

Hence, the total energy density and pressure respectively become $\rho_{tot} = \rho_m + \rho_S$ and $P_{tot} = P_m + P_S$, and the Friedmann equations acquire the usual form of [2, 3]. Thus, we can examine what are the conditions in order to fulfill the bounce requirements [4] and [4]. Concerning [4], we deduce that for the flat universe exactly at the bounce point we must have $S = 16\pi G \rho_m$, while [4] requires $\rho_m + 3p_m < 0$ (since according to [43, 44] $P_S + \rho_S = 0$). Therefore, we conclude that in the case of a flat universe and for standard matter a bounce cannot be obtained in the scenario at hand.

Nevertheless, a bounce could still be possible with the addition of extra fields, e.g. [50], but still one has to be careful with the constraints imposed to $S$ via [40]. For example, if we consider the trivial case where $Y_{\alpha\beta} = 0$ then the trace of [40] gives

\[ S = -2R. \]  

(45)

We assume that the extra field can be modeled to a perfect fluid as in [12], with energy density and pressure $\rho_{eff}$ and $P_{eff}$ respectively, and thus the Friedmann equation [41] takes the form

\[ H^2 = \frac{8\pi G}{3} (\rho_m + \rho_{eff}) - \frac{1}{6} S. \]  

(46)

Substituting [45] to [46]1 gives $3H^2 + 2\dot{H} + 8\pi G (\rho_m + \rho_{eff})/3 = 0$. This relation implies that in order for an extra field with trivial $Y_{\alpha\beta}$ to induce a bounce solution for our spatially flat metric it would need to have $\rho_{eff} < 0$, which is undesirable from a physical point of view.

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1 In our case $R$ reduces to the ordinary flat FRW Ricci scalar curvature of general relativity, due to the fact that the metric components $g_{\mu\nu}(x)$ do not depend on the $y$ variables, as was shown in [29].
IV. BOUNCE FROM SCALAR TENSOR THEORY ON THE FIBER BUNDLE

In this section we investigate the bounce generation in theories which include scalar tensor sectors on the fiber bundle. These constructions are very general, with very rich structure and behavior, which reveals the significant capabilities of Finsler-like geometry. We first present the basics of this construction and then we proceed to the investigation of two explicit scenarios.

A. The model

We consider a fibered space over a pseudo-Riemannian spacetime manifold $M$ of the form $M \times \{\phi^{(1)}\} \times \{\phi^{(2)}\}$, where $\phi^{(1)}, \phi^{(2)}$ stand for the fiber coordinates. Under coordinate transformations on the base manifold, fiber coordinates behave like scalars. Moreover, the space is equipped with a nonlinear connection with coefficients $N^\alpha_\mu(\chi^\nu, \phi^{(\beta)})$, where $\mu, \nu$ take the values from 0 to 3 and $\alpha, \beta$ take the values 1 and 2 \cite{53}. Its adapted bases for the tangent and cotangent spaces are $\{\delta_\mu = \partial_\mu - N_\mu^\beta \partial_\phi(\beta), \partial_{\phi(\mu)}\}$, where a summation is implied over the possible values of $\beta$, and $\{dx^\mu, \delta\phi^{(\alpha)} = d\phi^{(\alpha)} + N_\mu^\alpha dx^\mu\}$ with a summation implied over the possible values of $\mu$. The metric structure of the space is defined as \cite{53}

$$G = g_{\mu\nu}(x) dx^\mu \otimes dx^\nu + v_{(a)(b)}(x) \delta\phi^{(a)}(x) \otimes \delta\phi^{(b)}(x). \quad (47)$$

The metric coefficients for the fiber coordinates are set as $v_{(0)(0)} = v_{(1)(1)} = \phi(x^\nu) \equiv \tilde{\phi}$ and $v_{(0)(1)} = v_{(1)(0)} = 0$. Note that the function $\tilde{\phi}$ is clearly a scalar under coordinate transformations. The detailed investigation of the above construction has been performed in \cite{53}, where a metrical d-connection has been introduced and its curvature and torsion tensor coefficients have been calculated. Additionally, the Raychaudhuri equations for the model have been derived in \cite{54}.

We can now write an action as \cite{54}

$$S_G = \frac{1}{16\pi G} \int \sqrt{|\text{det } G|} L_G dx^{(N)}, \quad (48)$$

where $L_G$ is taken equal to the scalar curvature of the d-connection, and $dx^{(N)} = d^4 x \wedge d\phi^{(1)} \wedge d\phi^{(2)}$. In the special case of a holonomic basis, i.e. $[\delta_\mu, \delta_\nu] = 0$, the scalar curvature of the d-connection is

$$R = R - \frac{2}{\tilde{\phi}} \Box \tilde{\phi} + \frac{1}{4\tilde{\phi}^2} d^i \phi_\mu \phi_\mu, \quad (49)$$

where $R$ is the scalar curvature of Levi-Civita connection and $\Box$ is the d’Alembert operator with respect to it. On the other hand, in the general case one obtains the scalar curvature as

$$\tilde{R} = R - \frac{2}{\tilde{\phi}} \Box \tilde{\phi} + \frac{1}{4\tilde{\phi}^2} d^i \phi_\mu \phi_\mu + \frac{1}{\tilde{\phi}} d^i \phi_\mu \phi_\mu N_\mu^{(\alpha)}. \quad (50)$$

Additionally, we can add the matter sector too, considering the total action

$$S = \frac{1}{16\pi G} \int \sqrt{|\text{det } G|} L_G dx^{(N)} + \int \sqrt{|\text{det } g|} L_M dx^{(N)}. \quad (51)$$

Since for the determinants $\text{det } G$ and $\text{det } g$ we have the relation $\text{det } G = \tilde{\phi}^2 \text{det } g$, the above total action can be re-written as

$$S = \frac{1}{16\pi G} \int \sqrt{|\text{det } g|} \tilde{\phi} L_G dx^{(N)} + \int \sqrt{|\text{det } g|} \tilde{\phi} L_M dx^{(N)}. \quad (52)$$

In the following two subsections we study the bounce realization in the holonomic ($L_G = \mathcal{R}$) and nonholonomic ($L_G = \mathcal{R}$) basis separately.

B. Bounce in holonomic basis

Let us consider the total action (52) in the case of holonomic basis, allowing also for a potential for the scalar field, namely \cite{54}

$$S = \frac{1}{16\pi G} \int \sqrt{|\text{det } g|} \left[ \tilde{\phi} \mathcal{R} - V(\tilde{\phi}) \right] dx^{(N)} + \int \sqrt{|\text{det } g|} \tilde{\phi} L_M dx^{(N)}, \quad (53)$$

where $\mathcal{R}$ is the holonomic scalar curvature \cite{49}. We mention here that the above action belongs to the Horndeski class, and hence the resulting equations of motion are guaranteed to have up to second order derivatives \cite{11}. In particular, the field equations for the metric are extracted as

$$E_{\mu\nu} = 8\pi G T_{\mu\nu} + \frac{1}{\tilde{\phi}} \left( \nabla_\mu \nabla_\nu \tilde{\phi} - g_{\mu\nu} \Box \tilde{\phi} \right) + \frac{1}{4\tilde{\phi}^2} \left[ \frac{1}{2} g_{\mu\nu} \left( \nabla_\mu \tilde{\phi} \right)^2 - \nabla_\mu \tilde{\phi} \nabla_\nu \tilde{\phi} \right] - \frac{1}{2\tilde{\phi}} g_{\mu\nu} V, \quad (54)$$

where $E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ is the Einstein tensor, $T_{\mu\nu} = -\frac{2}{\sqrt{|g|}} \frac{\delta (\sqrt{|g|} L_M)}{\delta g^{\mu\nu}}$ is the energy-momentum tensor, and $V_\mu$ is the Levi-Civita covariant derivative, while the scalar field (extension of Klein-Gordon) equation reads as

$$\Box \tilde{\phi} = 2\tilde{\phi} (R - V^\prime) + \frac{1}{2\tilde{\phi}} \left( \nabla \tilde{\phi} \right)^2 + 32\pi G \mathcal{L}_M \phi, \quad (55)$$

with $V^\prime = dV / d\tilde{\phi}$. Note the interesting fact that in the scenario at hand we obtain an effective interaction between the scalar field and the matter sector due to the transformation from $G$-metric to $g$-metric.

Applying the above equations in the FRW metric \cite{14}, focusing on the flat case, and neglecting the matter sector, since we are interesting in the early-time bounce realization,
we obtain the following modified Friedmann equations:

\[
3H^2 = -3H \frac{\dot{\phi}}{\phi} - \frac{\dot{\phi}^2}{8\phi^2} + \frac{1}{2\phi} V \tag{56}
\]

\[
\dot{H} + H^2 = -\frac{1}{8\phi^2} \left( \frac{\dot{\phi}}{\phi} + H \frac{\ddot{\phi}}{\phi} \right) + \frac{\dot{\phi}^2}{12\phi^2} + \frac{V}{6\phi} \tag{57}
\]

\[
\ddot{\phi} + 3H \dot{\phi} = -12\phi \left( 2H^2 + \dot{H} \right) + \frac{\dot{\phi}^2}{2\phi} + 2\phi V' \tag{58}
\]

out of which two are independent.

We now proceed to show how it is possible to obtain a specific bounce in this construction. As we observe from the above equations, we may choose a specific scalar-field potential that can satisfy the general bounce conditions (4) and (5) and thus induce the bounce realization. We follow the procedure of \cite{[20, 27–29, 68]} and we first start from the desired result, that is we impose a known form of the scale factor \(a(t)\) and we apply the above steps. Since analytical solutions cannot be obtained, we numerically solve (59) and find \(\phi(t)\), and then we use (60) to find \(V(t)\). These two functions are shown in Fig. 2. Hence, from these \(\phi(t)\) and \(V(t)\) we reconstruct the potential \(V(\phi)\), which is depicted in Fig. 3. Therefore, if this \(V(\phi)\) is imposed as an input, one acquires the bounce realization, and in particular the bouncing scale factor \(a(t)\).

Let us provide an explicit example of the bounce realization. We start by inserting the desired bouncing scale factor (24) and we apply the above steps. Since analytical solutions cannot be obtained, we numerically solve (59) and find \(\phi(t)\), and then we use (60) to find \(V(t)\). These two functions are shown in Fig. 2. Hence, from these \(\phi(t)\) and \(V(t)\) we reconstruct the potential \(V(\phi)\), which is depicted in Fig. 3. Therefore, if this \(V(\phi)\) is imposed as an input, one acquires the bounce realization, and in particular the bouncing scale factor (24).

C. Bounce in nonholonomic basis

We now proceed to the investigation of the nonholonomic case, namely we consider the total action (52) with \(L_C = \hat{\mathcal{R}}\), i.e.

\[
\mathcal{S} = \frac{1}{16\pi G} \int \sqrt{|\hat{\mathcal{g}}|} \phi \hat{\mathcal{R}} dx^{(N)} + \int \sqrt{|\hat{\mathcal{g}}|} \phi L_M dx^{(N)}, \quad (61)
\]

where \(\hat{\mathcal{R}}\) is the nonholonomic scalar curvature (50). This action leads to the following equations of motion for the metric and the scalar field:

\[
E_{\mu\nu} = 8\pi G T_{\mu\nu} + \frac{1}{\phi} \left( \nabla_\nu \phi \nabla_\mu \phi - g_{\mu\nu} \Delta \phi \right) + \frac{1}{4\phi^2} \left[ \frac{1}{2} g_{\mu\nu} (\nabla \phi)^2 - \nabla_\mu \phi \nabla_\nu \phi \right] - \left( \delta^\lambda_\mu \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial_\rho \phi \right) N_\lambda \tag{62}
\]

\[
\square \phi = 2\phi \hat{\mathcal{R}} + \frac{1}{2\phi} (\nabla \phi)^2 + 32\pi G L_M \phi - \phi D^\mu N_\mu, \quad (63)
\]

where \(N_\mu \equiv \partial^{(a)}_\mu N^{(a)}_\mu\), and with \(D_\mu N^{(a)} = \delta_\mu N^{(a)} + \Gamma^{(a)}_\kappa N^{(a)}\) the d-covariant differentiation on the fiber bundle where
\[ \Gamma^{\lambda}_{\mu\nu} \] are the Christoffel symbols. We mention that the last term in (63), which reflects the internal structure of Finsler-like geometry, can be seen to act as an effective potential for the scalar field \( \phi \). Since every other quantity in (62, 63) depends on \( x^\mu \) coordinates only, this should also be the case for \( N_\lambda \) for consistency (equivalently \( \partial_{\phi(\beta)} \partial_{\phi(\alpha)} N^{(\alpha)}_{\mu} = 0 \) on shell).

Applying the above equations of motion in the FRW metric (1), focusing on the flat case, and neglecting the matter sector, since we are interesting in the early-time bounce realization, leads to the modified Friedmann equations

\[ 3H^2 = -3H \frac{\dot{\phi}}{\phi} - \frac{\dot{\phi}^2}{8\phi^2} - \frac{1}{2} \dot{\phi} N_0 \] (64)

\[ \dot{H} + H^2 = -\frac{1}{2\phi} (\dot{\phi} + H \dot{\phi}) + \frac{\dot{\phi}^2}{12\phi^2} + \frac{1}{3} \ddot{\phi} N_0 \] (65)

\[ \dot{\phi} + 3H \phi = -12\phi (2H^2 + \dot{H}) + \frac{\dot{\phi}^2}{2\phi} + \phi (N^0 + 3HN^0) \] , (66)

out of which two are independent, where as we mentioned, due to symmetries, all quantities depend only on time. Thus, in the Friedmann equations we acquire a modification reflecting the nonholonomicity of the fiber bundle of the underlying Finsler-like geometry.

Let us now show how this construction may give rise to the bounce realization. From the form of the Friedmann equations (64, 65) we deduce that we may choose a specific nonholonomic function \( N^0(t) \) that can satisfy the general bounce conditions (1) and (3) and thus induce the bounce. We first start from the desired result, that is we impose as an input a scale factor form \( a(t) \) that possesses bouncing behavior. Therefore, \( H(t) \) is known too. Eliminating \( N^0 \) from (64, 65) gives the simple differential equation

\[ \ddot{\phi} + 5H(t) \dot{\phi} + 2\dot{\phi}(t) [\ddot{H} + 3H(t)^2] = 0, \] (67)

which can be solved to provide \( \phi(t) \). Then this \( \phi(t) \) can be substituted into (64) and provide \( N^0(t) \) as

\[ N_0(t) = -6 \left[ \frac{H(t)}{\dot{\phi}(t)} + \frac{\phi(t)}{24\phi^2(t)} + \frac{H(t)^2}{\dot{\phi}(t)} \right]. \] (68)

Hence, it is this \( N_0(t) \), induced by the nonlinear connection of the Finsler-like geometry, that generates the initially given desired bouncing scale factor \( a(t) \).

We close this subsection by providing an explicit example of the bounce realization. We use as input the bouncing scale factor (24) and we apply the above steps. We numerically solve (67) and find \( \phi(t) \), and then we use (68) to find \( N_0(t) \). In Fig. 4 we depict the solution for \( N_0(t) \). Hence, if this \( N_0 \) is imposed as an input, one obtains the bounce realization, and in particular the bouncing scale factor (24).

\[ \text{V. CONCLUSIONS} \]

In this work we investigated the bounce realization in the framework of Finsler and Finsler-like gravity. Finsler and Finsler-like geometry is a natural extension of Riemannian one, where one allows that the physical quantities may directly depend on observer 4-velocity. Hence, the gravitational theory based on Finsler and Finsler-like gravity provides a gravitational modification, since it induces extra terms in the field equations. When applied in a cosmological framework, the richer intrinsic structure of Finsler and Finsler-like geometry is reflected in extra terms in the resulting modified Friedmann equations. Thus, these terms can lead to the bounce realizations.

In our analysis we considered various Finsler and Finsler-like constructions and we examined whether bouncing solutions can be obtained. As a first model we considered the so-called general very special relativity, which presents a slight Lorentz violation quantified by a single parameter and the “spurionic” one-form. As we showed, under the linear approximation this scenario cannot lead to a bounce. However, considering the Finsler-Randers space, in which the intrinsic Finslerian structure is reflected to the appearance of a new function in the Friedmann equations (the variation of anisotropy), we saw that the bounce conditions can be easily fulfilled and thus the bounce can be realized.

As a next construction we examined the Finsler-like gravity on the tangent bundle. Performing the analysis and considering the two involved curvature tensors, we extracted the Friedmann equations which contain a modification resulting from the tangent-bundle related S-curvature. Nevertheless, for simple models and standard matter, these extra terms cannot drive a bouncing solution since they cannot lead to the violation of the null energy condition.

As a last construction we considered theories which in-
clude scalar tensor sectors on the fiber bundle. These theories present a very rich structure revealing the capabilities of Finsler-like geometry. In particular, the nonlinear connection induces a new degree of freedom that behaves as a scalar under coordinate transformations. In a cosmological framework this scalar field appears in the Friedmann equations, and therefore its dynamics may trigger a bounce. In the case of holonomic basis we showed that the bounce can be easily obtained, and we provided the way of the reconstruction of the potential that gives rise to a desired bouncing scale factor. Similarly, in the case of nonholonomic basis we saw that the bounce can be easily realized, and we presented the reconstruction procedure of the time-coefficient related to the nonlinear connection that induces the desired bounce.

In summary, we saw that Finsler and Finsler-like geometry is a natural framework for the realization of bounce cosmology. Apart from the background evolution one should additionally investigate the various scenarios at the perturbation levels, since the process of perturbations through the bounce phase is strongly related to the subsequent development of the large scale structure and hence to observations. Such a detailed perturbation analysis lies beyond the scope of the present work and it is left for a future investigation.

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[1] T. Biswas, A. Mazumdar and W. Siegel, Bouncing universes in string-inspired gravity, JCAP 0603, 009 (2006).
[2] Y. F. Cai, T. Qiu, Y. S. Piao, M. Li and X. Zhang, Bouncing universe with quintom matter, JHEP 0710, 071 (2007).
[3] Y. F. Cai, T. t. Qiu, R. Brandenberger and X. m. Zhang, A Nonsingular Cosmology with a Scale-Invariant Spectrum of Cosmological Perturbations from Lee-Wick Theory, Phys. Rev. D 80, 023511 (2009).
[4] Y. F. Cai, D. A. Easson and R. Brandenberger, Towards a Non-Singular Bouncing Cosmology, JCAP 1208, 020 (2012).
[5] Y. F. Cai, Exploring Bouncing Cosmologies with Cosmological Surveys, Sci. China Phys. Mech. Astron. 57, 1414 (2014).
[6] R. Brandenberger and P. Peter, Bouncing Cosmologies: Progress and Problems, Found. Phys. 47, no. 6, 797 (2017).
[7] M. Novello and S. E. P. Bergliaffa, Bouncing Cosmologies, Phys. Rept. 463, 127 (2008).
[8] R. C. Tolman, On the Theoretical Requirements for a Periodic Behaviour of the Universe, Phys. Rev. 38, no. 9, 1758 (1931).
[9] A. Ijjas and P. J. Steinhardt, Bouncing Cosmology made simple, Class. Quant. Grav. 35, no. 13, 135004 (2018).
[10] T. Singh, R. Chaubej and A. Singh, Bounce conditions for FRW models in modified gravity theories, Eur. Phys. J. Plus 130, no. 2, 31 (2015).
[11] A. De Felice and S. Tsujikawa, Conditions for the cosmological viability of the most general scalar-tensor theories and their applications to extended Galileon dark energy models, JCAP 1202, 007 (2012).
[12] A. Barrau, B. Bolliet, M. Schutten and F. Vidotto, Bouncing black holes in quantum gravity and the Fermi gamma-ray excess, Phys. Lett. B 772, 58 (2017).
[13] G. Veneziano, Scale Factor Duality For Classical And Quantum Strings, Phys. Lett. B 265, 287 (1991).
[14] J. Khoury, B. A. Ovrut, P. J. Steinhardt and N. Turok, The ekpyrotic universe: Colliding branes and the origin of the hot big bang, Phys. Rev. D 64, 123522 (2001).
[15] J. Khoury, B. A. Ovrut, N. Seiberg, P. J. Steinhardt and N. Turok, From big crunch to big bang, Phys. Rev. D 65, 086007 (2002).
[16] T. Biswas, A. Mazumdar and W. Siegel, Bouncing universes in string-inspired gravity, JCAP 0603, 009 (2006).
[17] S. Nojiri and E. N. Saridakis, Phantom without ghost, Astrophys. Space Sci. 347, 221 (2013).
[18] K. Bamba, A. N. Makarenko, A. N. Myagky, S. Nojiri and S. D. Odintsov, Bounce cosmology from F(R) gravity and F(R) bigravity, JCAP 1401 (2014) 008.
[19] S. Nojiri and S. D. Odintsov, Mimetic F(R) gravity: inflation, dark energy and bounce, Mod. Phys. Lett. A 29, no. 40, 1450211 (2014).
[20] Y.-F. Cai, S.-H. Chen, J. B. Dent, S. Dutta and E. N. Saridakis, Matter Bounce Cosmology with the f(T) Gravity, Class. Quant. Grav. 28, 215011 (2011).
[21] Y. Shtanov and V. Sahni, Bouncing braneworlds, Phys. Lett. B 557, 1 (2003).
[22] E. N. Saridakis, Cyclic Universes from General Collisionless Braneworld Models, Nucl. Phys. B 808, 224 (2009).
[23] R. Brandenberger, Matter Bounce in Horava-Lifshitz Cosmology, Phys. Rev. D 80, 043516 (2009).
[24] Y. F. Cai and E. N. Saridakis, Non-singular cosmology in a model of non-relativistic gravity, JCAP 0910, 020 (2009).
[25] E. N. Saridakis, Horava-Lifshitz Dark Energy, Eur. Phys. J. C 67, 229 (2010).
[26] D. A. Easson, I. Sawicki and A. Vikman, G-Bounce, JCAP 1111, 021 (2011).
[27] T. Qiu, X. Gao and E. N. Saridakis, Towards anisotropy-free and nonsingular bounce cosmology with scale-invariant perturbations, Phys. Rev. D 88, no. 4, 043525 (2013).
[28] Y. F. Cai, C. Gao and E. N. Saridakis, Bounce and cyclic cosmology in extended nonlinear massive gravity, JCAP 1201, 048 (2012).
[29] Y.-F. Cai and E. N. Saridakis, Cyclic cosmology from Lagrange-multiplier modified gravity, Class. Quant. Grav. 28, 035010 (2011).
[30] M. Bojowald, Absence of singularity in loop quantum cosmology, Phys. Rev. Lett. 86, 5227 (2001).
[31] S. D. Odintsov and V. K. Oikonomou, Matter Bounce Loop Quantum Cosmology from F(R) Gravity, Phys. Rev. D 90, no.
