Geometry of effective gauge fields

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Abstract. There is a recent proposal for effective gauge theories that does not involve a background metric which makes it attractive for quantum gravity. The basic building blocks are new spaces of effective connections and the coarse graining maps which relate them. One space of effective connections can be coarse grained to another one assigned to a “more macroscopic situation” and that hence captures less information about the system. We study the geometry of the spaces of effective connections and the coarse graining maps. They have the structure of a fiber bundle where the macroscopic space of effective connections is the base space, the total space is the space of effective connections with more information about the system, and the projection is the coarse graining map. More over, the fiber bundle turns out to be principal and to have a preferred section.

1. Introduction
One physical system can be observed at different scales. Say that you perform a set of measurements at one scale. Another physicist performs a bigger set of measurements that gives him knowledge about the systems behavior at a smaller scale. Both sets of measurements are incomplete, but each physicist can make predictions about his measurements using an effective theory that works at its scale. Working at a smaller scale means that one can coarse grain to infer the results of measurements made at a bigger scale. Also an effective theory at a smaller scale can make predictions about those measurements.

There are important physical systems for which the concept of scale can not be used to describe a set of measurements to say if it includes another such set of measurements. Any system that includes gravitational phenomena is in this category of systems because a scale is defined through a metric which is a dynamical variable. Thus, a definition of effective theories and coarse graining requires an “extension of the concept of scale.” There is a recent proposal of effective theories and coarse graining for gauge theories [1, 2].

In this work we study the geometry of the structures introduced in the cited work. More specifically, we study the geometry of the spaces of effective connections and the coarse graining maps which relate the different spaces. Our coarse graining models the lowering of resolution in observation, and the loss of information caused by this.

We obtain a structure in which there are several microscopic configurations which are confused when only macroscopic observations are available. Geometrically we have a fiber bundle in which the elements of the base space are macroscopic configurations. The total space is composed by the microscopic configurations which are organized as fibers of these configurations that are identified by the macroscopic observations. Furthermore, we find that the fiber bundle is principal and that it has a preferred section.
2. C-flat connections
Given a manifold \( M \), a \( k \)-cell \( c^k_\alpha \) is a subset of \( M \) such that is the embedding of an open polyhedron in \( \mathbb{R}^k \), \( k \leq \dim[M] \), plus an orientation, as is shown in figure 1.

A cellular decomposition \( C \) of \( M \) is a set of cells \( C = \{c_\alpha\} \) such that

(i) \( \bigcup_\alpha c_\alpha = M \),
(ii) \( c_\alpha \cap c_\beta = \emptyset \) if \( \alpha \neq \beta \).

**Definition 1** Given a principal fiber bundle \( \Phi = (E, \varphi, M) \) with a cellular decomposition \( C \) of its base space, and an auxiliary connection \( A_0 \), which is flat on the closure of the \( c^{\dim M}_\alpha \) cells, a \( C \)-flat connection \( A \in \mathcal{A}_{C^{\text{flat}}, A_0} \) is defined through its holonomies in the following way:

\[
h(l, A) = \prod h(l_i, A),
\]

where \( l = \prod l_i \), \( l_i \subset c_\alpha \), with \( c_\alpha \in C \).

The possible holonomies inside the closed cell \( c_\alpha \) are:

(i) \( h(l_i, A) = h(l_i, A_0) \) if \( l_i \subset c_\alpha \),
(ii) \( h(l_i, A) = h(l_{i1}, A_0)^{-1} g(\beta, \gamma) h(l_{i0}, A_0) \),

where \( l_i(0) \in c_\beta, l_i(1) \in c_\gamma; \ c_\beta, c_\gamma \subset c_\alpha \), and \( g(\beta, \gamma) = g^{-1}(\gamma, \beta) \).

What \( g(\beta, \gamma) \) represents is the action of the bundle group over the fiber of an arbitrary element \( p_0 \) of \( c_\alpha \).

We can visualize this in an easier way by requiring that the following diagram commutes:

\[
\begin{array}{ccc}
\varphi^{-1}(l(0)) & \xrightarrow{h(l, A)} & \varphi^{-1}(l(1)) \\
h(l_0, A_0) & \downarrow & h_{-1}(l_1, A_0) \\
\varphi^{-1}(p_0) & \xrightarrow{g(\beta, \gamma)} & \varphi^{-1}(p_0)
\end{array}
\]

It is clear by the previous definition that once a space of \( C \)-flat connection is chosen, there shall exist an equivalence relation between neighboring paths in the sense that they produce the same holonomy; this is because such holonomy only depends on the first and last cell for every \( l_i \).

We can define a lattice \( L(C) \) that contains one representative of each of the mentioned path classes.

![Figure 1](image.png)
(i) For each cell, we choose an arbitrary point inside it; we call this point the **Barycenter** of the cell.

(ii) If two cells are neighbors, this means $c_\alpha \subset \partial c_\beta$ or $c_\beta \subset \partial c_\alpha$, we put in correspondence their respective barycenters with **links** having the orientation resulting of going from the high-dimensional cell to the low-dimensional cells which form its bound. We choose one representative from each class of such links. In the case simplicial cellular decompositions $L(C)$ is the oneskeleton of the barycentric subdivision, $L = Sd(C)^1$. For a general cellular decomposition this idea can be visualized in the figure 2.

![Figure 2. How to do the auxiliary cellular decomposition.](image)

So far, the definition of a $C$-flat connection depends on the cellular decomposition and on the choice of a particular auxiliary connection. But of course, we are interested in knowing which choices result in essentially different notions.

**Definition 2** $A_{C,\text{flat},A_0} \sim A_{C,\text{flat},A_0'}$ if there exists a gauge transformation $f$ such that

$$f^*(A_{C,\text{flat},A_0}) = A_{C,\text{flat},A_0'}.$$  

**Theorem 1** If $(f, \tilde{f})$ is an automorphism of the fiber bundle $\Phi$ with cellular decomposition $C$, then

$$f^*(A_{C,\text{flat},A_0}) = A_{f^{-1}(C),\text{flat},f^*(A_0)}.$$  

**Corollary 1** $A_{C,\text{flat},A_0} \sim A_{C,\text{flat},A_0'} \iff A_0 \sim A_0'$, in words, $A_0$ is gauge related to $A_0'$.

We note that the theorem also includes the case when the cells are moved by $f$.

### 3. Coarse graining and refining

A **refinement** $C_2$ of a cellular decomposition $C_1$ is a cellular decomposition such that $\forall c_\alpha \in C_1$,

$$c_\alpha = \bigcup_{i=1}^n c_{\beta_i}, \quad \{c_{\beta_i}\} \subset C_2,$$

we represent this as $C_1 \leq C_2$.

We consider two cellular decompositions $C_1 \leq C_2$. A choice of embeddings $E_i : L(C_i)^1 \rightarrow M$ can be used to pull back bundles and connections from the continuum to the lattice $L(C_i)$. We note that the restriction of $E_i^* : \mathcal{A}_M \rightarrow A_{L(C_i)}$ to the set $A_{C_i,\text{flat}}$, namely $E_i^*|_{A_{C_i,\text{flat}}}$, is a isomorphism of fiber bundles.

Let us define the projection map $\pi$ as

$$\pi = (E_1^*|_{A_{C_1,\text{flat}}})^{-1} \circ E_1^*|_{A_{C_2,\text{flat}}}.$$
In general there exist more than one projection of this kind, how many depends on the cellular decompositions. Also, we can define the inclusion map \( \bar{\iota} \) as

\[
A_{C_2}^{\text{flat}}, \quad \subset \quad \overline{\mathcal{A}_M} \supset \quad A_{C_1}^{\text{flat}},
\]

\[
\pi_2 \left|_{A_{C_2}^{\text{flat}}} \right., \quad \downarrow \quad \overline{\mathcal{A}_L(C_2)} \quad \uparrow \quad \pi_1 \left|_{A_{C_1}^{\text{flat}}} \right.,
\]

\[
i = (\pi_2 \left|_{A_{C_2}^{\text{flat}}} \right.)^{-1} \circ \pi_1 \left|_{A_{C_1}^{\text{flat}}} \right.,
\]

It is independent of the choice of embedding \( E_2 \).

**Theorem 2** Given the projection map and inclusion map defined above, \( A_{C_2}^{\text{flat}} \), is the total space of a principal fiber bundle over \( A_{C_1}^{\text{flat}} \), with a preferred section.

**Example.** Let us take the disc with the following cellular decompositions, \( C_1, C_2 \), with their corresponding auxiliary decompositions, \( L(C_1), L(C_2) \).

![Figure 3. The two simplest cellular decomposition of the disc.](image)

Given that the path equivalence \( A_{C_2}^{\text{flat}}, A_0 \) is finite dimensional and can be parameterized in terms of the group elements using the auxiliary connection \( A_0 \) and an auxiliary fiber. We parameterize \( A \in A_{C_1}^{\text{flat}} \), as \( A = (g_1, g_2, g_3, g_4) \in G^4 \), and \( A' \in A_{C_2}^{\text{flat}} \), as \( A' = (g_1', g_2', g_3', g_4', g_5, g_6, g_7, g_8) \in G^8 \). Then the inclusion map/preferred section is

\[
i(A) = (g_1, g_2, g_2, g_3, Id, Id, g_4),
\]

and the three different projections depending on the choice of embedding are

\[
\pi_1(A') = (g_1', g_2', g_3', g_4'(g_5')^{-1}g_6),
\]

\[
\pi_2(A') = (g_1', g_3', g_4'(g_6)^{-1}, g_8(g_7)^{-1}),
\]

\[
\pi_3(A') = (g_1', g_4', g_5'(g_6)^{-1}g_7, g_8).
\]

We will see that in this example the fiber is \( G^4 \) for any \( \pi_i \). Given \( A_0 = (g_1^0, g_2^0, g_3^0, g_4^0) \in A_{C_1}^{\text{flat}}, \)

\[
\pi_1^{-1}(A_0) = \{(g_1^0, g_2^0, g_3, g_4, g_3, g_7(g_8)^{-1}g_4, g_7, g_8)\}.
\]
We see that the fiber has four free group parameters and it is homeomorphic to $G^4$. Therefore $(\mathcal{A}_{C^2\text{-flat}}, \pi_1, \mathcal{A}_{C^1\text{-flat}})$ is a fiber bundle with a $G^4$ fiber.

Now let us take an element $A'_0 = (g_1^0, \ldots, g_8^0) \in \pi^{-1}(A_0)$; and let $\nu \in G^4$ be an arbitrary element of the group. Consider the function $F : G^4 \times \mathcal{A}_{C^2\text{-flat}}, \longrightarrow \mathcal{A}_{C^2\text{-flat}},$ defined by

$$F(\nu, A'_0) = (g_1^0, g_2^0, \nu_1 g_3^0, \nu_2 g_4^0, g_5^0, \nu_3 g_6^0, \nu_3 g_7^0 \nu_4^{-1}, g_8^0 \nu_4^{-1}) .$$

We can see that $\pi_1(A'_0) = \pi_1(F(\nu, A'_0))$, therefore the map $F(\nu, A'_0)$ preserves fibers; furthermore, it gives a bijection between each fiber and $G^4$. Then $(\mathcal{A}_{C^2\text{-flat}}, \pi_1, \mathcal{A}_{C^1\text{-flat}})$ is a principal fiber bundle.

Similarly, it can be shown that $\pi_2$ and $\pi_3$ are projections in the same sense as $\pi_1$.

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References
[1] Manrique E, Oeckl R, Weber A and Zapata J (in Preparation) Effective gauge theories and coarse graining for loop quantization.
[2] Zapata J (in this proceedings) Effective loop quantized theories.
[3] Isham C J 2002 *Modern Differential Geometry for Physicists*, second (London: World Scientific Publishing)
[4] Nakahara M 1990 *Geometry, Topology, and Physics* (New York: IOP Publishing Ltd.)
[5] Baez J and Munian J 1994 *Gauge Fields, Knots and Gravity* (Singapore: World Scientific Publishing)
[6] Arnold V I 1978 *Mathematical Methods of Classical Mechanics* (New York: Springer-Verlag)