On the global rigidity of sphere packings on 3-dimensional manifolds

Xu Xu

March 16, 2017

Abstract

In this paper, we prove the global rigidity of sphere packings on 3-dimensional manifolds, which implies the uniqueness of hyperbolic structure on 4-dimensional manifolds. This is a 3-dimensional analogue of the rigidity in Andreev-Thurston Theorem conjectured by Cooper and Rivin in [5]. We also study the global rigidity of the combinatorial scalar curvature introduced by Ge and the author in [13].

MSC (2010): 52C25; 52C26
Keywords: Global rigidity; Sphere packing; Combinatorial scalar curvature

1 Introduction

To study the hyperbolic metrics on 3-manifolds, Thurston ([35], Chapter 13) introduced the notion of circle packing with prescribed intersection angles and got the Andreev-Thurston Theorem, which is composed of two parts. The first part is on the existence of circle packing for a given triangulation, saying that the admissible space of combinatorial curvature is a convex open set bounded by some hyperplanes determined by the combinatorial and topological structures of the triangulated surface. The second part is on the rigidity of circle packings, saying that the circle packing is uniquely determined by the discrete Gauss curvature (up to scaling for the Euclidean background geometry). For a proof of Andreev-Thurston Theorem, see [4, 6, 23, 32, 34, 35].

There are lots of research on related topics of Andreev-Thurston Theorem. Chow and Luo [4] once introduced a combinatorial Ricci flow to deform circle packing metrics and gave a new proof for Andreev-Thurston Theorem. Ge and the author [11, 12] introduced a new combinatorial Gauss curvature for triangulated surfaces and got a generalization of Andreev-Thurston Theorem. We further introduced several combinatorial curvature flows, including combinatorial Ricci flow, Calabi flow and Yamabe flow, to deform the
circle packing metrics, aiming at finding circle packing metrics with given combinatorial curvatures. Thurston’s circle packing metrics could be generalized to circle packing metrics with inversive distance \[3, 24\] and piecewise linear metrics \[27\]. For the rigidity theory of these metrics, we refer the readers to \[2, 14, 21, 22, 29, 30\]. For the deformation theory of these metrics, we refer the readers to \[8, 9, 14, 19, 20, 27, 36\].

To study the high dimensional analogy of the circle packings on surfaces, Cooper and Rivin \[5\] introduced the notion of sphere packing on 3-dimensional manifolds. Suppose \(M\) is a 3-dimensional closed manifold with a triangulation \(T = \{V, E, F, T\}\), where the symbols \(V, E, F, T\) represent the sets of vertices, edges, faces and tetrahedrons respectively. A sphere packing metric is a map \(r : V \rightarrow (0, +\infty)\) such that the length between vertices \(i\) and \(j\) is \(l_{ij} = r_i + r_j\) for each edge \(\{i, j\} \in E\) and the lengths \(l_{ij}, l_{ik}, l_{il}, l_{jk}, l_{jl}, l_{kl}\) determine a Euclidean or hyperbolic tetrahedron for each tetrahedron \(\{i, j, k, l\} \in T\). The nondegenerate condition restricts the space of sphere packing metrics to be an open subset of \(\mathbb{R}^{|V|}\), which is denoted as \(\Omega\) in the following.

To study the properties of sphere packing metrics, Cooper and Rivin \[5\] introduced the notion of combinatorial scalar curvature \(K_i\), which is defined as angle deficit of solid angles at a vertex \(i\)

\[
K_i = 4\pi - \sum_{\{i,j,k,l\} \in T} \alpha_{ijkl},
\]

where \(\alpha_{ijkl}\) is the solid angle at the vertex \(i\) of the tetrahedron \(\{i,j,k,l\} \in T\) and the summation is taken over all tetrahedrons with \(i\) as one of its vertices. When we are discussing in a single tetrahedron \(\{ijkl\} \in T\), we usually denote the solid angle \(\alpha_{ijkl}\) at the vertex \(i\) as \(\alpha_i\) for convenience in the following. \(K_i\) locally measures the difference between the volume growth rate of a small ball centered at vertex \(v_i\) in \(M\) and a Euclidean ball of the same radius. Cooper and Rivin’s definition of combinatorial scalar curvature is motivated by the fact that, in the smooth case, the scalar curvature at a point \(p\) locally measures the difference of the volume growth rate of the geodesic ball with center \(p\) to the Euclidean ball \([1, 26]\). From this viewpoint, Cooper and Rivin’s definition of combinatorial scalar curvature is a good candidate for 3-dimensional combinatorial scalar curvature with geometric meanings parallel to the smooth case. For convenience, we call Cooper and Rivin’s definition of combinatorial scalar curvature as CR-curvature for short in the following, if there is no confusion.

Using the combinatorial scalar curvature \(K\), Cooper and Rivin \[5\] proved the following local rigidity of sphere packing metrics.

**Theorem 1.1** \([5]\). Given a closed 3-dimensional triangulated manifold \((M, T)\), a Euclidean or hyperbolic sphere packing metric is locally determined by its combinatorial scalar curvature \(K\) (up to scaling for the Euclidean background geometry).
The global rigidity of sphere packing metrics is then left as a conjecture and the existence is left as an open problem. Glickenstein [15][16] once introduced a combinatorial Yamabe flow to study the corresponding constant curvature problem. He found that the combinatorial scalar curvature evolves according to a heat type equation along his flow and showed that the solution converges to a constant curvature metric under some nonsingular conditions. Luo [28] once introduced a combinatorial curvature flow to deform metrics of triangulated three-dimensional manifolds with boundary. The definition of curvature he used is different from that of Cooper and Rivin’s definition. Ge and the author [10][11] generalized Cooper and Rivin’s definition of combinatorial scalar curvature and introduced a combinatorial Yamabe flow to deform the sphere packing metrics aiming at finding the corresponding constant curvature sphere packing metrics on 3-dimensional triangulated manifolds.

In this paper, we prove the global rigidity of sphere packing for 3-dimensional triangulated manifolds, which was conjectured by Cooper and Rivin in [5]. We get the following main result.

**Theorem 1.2.** Given a closed connected triangulated 3-manifold \((M, T)\),

1. A Euclidean sphere packing metric on \((M, T)\) is determined by its combinatorial scalar curvature \(K : V \to \mathbb{R}\) up to scaling.

2. A hyperbolic sphere packing metric on \((M, T)\) is determined by its combinatorial scalar curvature \(K : V \to \mathbb{R}\).

Although CR-curvature is a good candidate for the 3-dimensional combinatorial scalar curvature, it has two disadvantages comparing to the smooth scalar curvature on Riemannian manifolds. The first is that it is scaling invariant with respect to the sphere packing metrics, i.e. \(K(\lambda r) = K(r)\) for \(\lambda > 0\); The second is that \(K_i\) tends to zero as the triangulation of the manifold is finer and finer. Motivated by the observations, Ge and the author [11] introduced a new combinatorial scalar curvature defined as \(R_i = \frac{K_i}{s_i^2}\) for 3-dimensional manifolds with Euclidean background geometry, which makes up the two disadvantages just mentioned if we take \(g_i = r_i^2\) as an analogue of the Riemannian metric tensor for the Euclidean background geometry. This definition can be modified to fit the case of hyperbolic background geometry. In summery, we define the combinatorial scalar curvature as

\[
R_i = \frac{K_i}{s_i^2},
\]

where \(s_i = r_i\) for the Euclidean background geometry and \(s_i = \tanh\frac{r_i}{2}\) for the hyperbolic background geometry. In the following, we call the combinatorial scalar curvature defined in (1.2) as \(R\)-curvature for short. For \(R\)-curvature, we have the following result on global rigidity.
Theorem 1.3. Given a closed triangulated 3-manifold $(M, \mathcal{T})$, $\mathcal{R}$ is a given function defined on the vertices of $(M, \mathcal{T})$.

(1) In the case of Euclidean background geometry, if $\mathcal{R} \equiv 0$, there exists at most one sphere packing metric $\mathcal{r} \in \Omega$ with combinatorial $R$-curvature 0 up to scaling; If $\mathcal{R} \leq 0$ and $\mathcal{R} \not\equiv 0$, there exists at most one sphere packing metric $\mathcal{r} \in \Omega$ with combinatorial $R$-curvature $\mathcal{R}$.

(2) In the case of hyperbolic background geometry, if $\mathcal{R} \leq 0$, there exists at most one sphere packing metric $\mathcal{r} \in \Omega$ with combinatorial $R$-curvature $\mathcal{R}$.

We further extend our definition of combinatorial scalar curvature $R$ to combinatorial $\alpha$-curvature, which is defined as

$$R_{\alpha,i} = \frac{K_i}{s_i^\alpha}$$

for $\alpha \in \mathbb{R}$. For the $\alpha$-curvature, we have the following global rigidity.

Theorem 1.4. Given a closed triangulated 3-manifold $(M, \mathcal{T})$ and $\mathcal{R}$ is a given function defined on the vertices of $(M, \mathcal{T})$.

(1) In the case of Euclidean background geometry, if $\alpha \mathcal{R} \equiv 0$, there exists at most one sphere packing metric $\mathcal{r} \in \Omega$ with combinatorial $\alpha$-curvature $\mathcal{R}$ up to scaling. If $\alpha \mathcal{R} \leq 0$ and $\alpha \mathcal{R} \not\equiv 0$, there exists at most one sphere packing metric $\mathcal{r} \in \Omega$ with combinatorial $\alpha$-curvature $\mathcal{R}$.

(2) In the case of hyperbolic background geometry, if $\alpha \mathcal{R} \leq 0$, there exists at most one sphere packing metric $\mathcal{r} \in \Omega$ with combinatorial $\alpha$-curvature $\mathcal{R}$.

Obviously, Theorem 1.4 generalizes Theorem 1.2 and Theorem 1.3.

The paper is organized as follows. In Section 2 we recall some basic facts on sphere packing metrics for 3-dimensional triangulated manifolds and give some useful lemmas for the extension of functions. In Section 3 we give a proof for Theorem 1.2. In Section 4 we study the global rigidity of $R$-curvature and give a proof for Theorem 1.3. In Section 5 we generalize the results to $\alpha$-curvature.

2 Preliminary

2.1 Admissible space for sphere packing metrics

Suppose $M$ is a 3-dimensional connected closed manifold with a triangulation $\mathcal{T} = \{V,E,F,T\}$, where the symbols $V, E, F, T$ represent the sets of vertices, edges, faces and
tetrahedrons respectively. A sphere packing metric is a map \( r : V \to (0, +\infty) \) such that the length between vertices \( i \) and \( j \) is \( l_{ij} = r_i + r_j \) for each edge \( \{i, j\} \in E \), and the lengths \( l_{ij}, l_{ik}, l_{il}, l_{jk}, l_{jl}, l_{kl} \) determine a Euclidean or hyperbolic tetrahedron for each tetrahedron \( \{i, j, k, l\} \in T \). We can take sphere packing metrics as points in \( \mathbb{R}_N^N \), \( N \) times Cartesian product of \((0, \infty)\), where \( N = |V| \) denotes the number of vertices. And we use the notion \( C(V) \) to denote the set of functions defined on the set of vertices \( V \). If \( r \) is a sphere packing metric on \((M, T)\), we can image that there is a 2-dimensional sphere \( S_i \) with radius \( r_i \) attached to each vertex \( i \) with \( i \) as center. And if the vertices \( i \) and \( j \) are adjacent, the two spheres \( S_i \) and \( S_j \) are externally tangent.

As \( l_{ij} = r_i + r_j \), it is obviously that the triangle inequalities are all satisfied on the faces. However, in order that \( l_{ij}, l_{ik}, l_{il}, l_{jk}, l_{jl}, l_{kl} \) determine a Euclidean or hyperbolic tetrahedron, there should be some nondegenerate conditions. It is found \([5, 15]\) that, for the Euclidean background geometry, Descartes circle theorem, also called Soddy-Gossett theorem, can be used to describe the critical degenerate case. For the Euclidean background geometry, the admissible space for a tetrahedron \( \{ijkl\} \in T \) is

\[
\Omega_{ijkl}^E = \{ (r_i, r_j, r_k, r_l) \in \mathbb{R}_4^4 | Q_{ijkl}^E > 0 \},
\]

where

\[
Q_{ijkl}^E = \left( \frac{1}{r_i} + \frac{1}{r_j} + \frac{1}{r_k} + \frac{1}{r_l} \right)^2 - 2 \left( \frac{1}{r_i^2} + \frac{1}{r_j^2} + \frac{1}{r_k^2} + \frac{1}{r_l^2} \right). \tag{2.2}
\]

For the hyperbolic background geometry, we also have Soddy-Gossett theorem, which says that four geodesic circles in \( \mathbb{H}^2 \) of radii \( r_i, r_j, r_k, r_l \) are externally tangent if

\[
Q_{ijkl}^H = (\coth r_i + \coth r_j + \coth r_k + \coth r_l)^2 - 2 (\coth^2 r_i + \coth^2 r_j + \coth^2 r_k + \coth^2 r_l) + 4 = 0. \tag{2.3}
\]

Analogous to the Euclidean case, the admissible space of a tetrahedron \( \{ijkl\} \in T \) to be a hyperbolic tetrahedron is given by

\[
\Omega_{ijkl}^H = \{ (r_i, r_j, r_k, r_l) \in \mathbb{R}_4^4 | Q_{ijkl}^H > 0 \}, \tag{2.4}
\]

where \( Q_{ijkl}^H \) is given in \((2.3)\). There is also a version of Soddy-Gossett theorem for the spherical background geometry. We suggest the readers to \([25, 31]\) for Descartes circle theorem and general Soddy-Gosset theorems with different background geometries. In this paper, we concentrate on the Euclidean and hyperbolic cases.

Cooper and Rivin \([5]\) called the tetrahedrons generated in this way conformal and proved that a tetrahedron is a Euclidean conformal tetrahedron if and only if there exists a unique sphere tangent to all of the edges of the tetrahedron. Moreover, the point of tangency with the edge \( \{i, j\} \) is of distance \( r_i \) to \( i \). They got the following useful lemma on the admissible space of sphere packing metrics for a single tetrahedron.
Lemma 2.1 [5]. For a Euclidean or hyperbolic tetrahedron \( \{ijkl\} \in T \), the admissible spaces \( \Omega^E_{ijkl} \) and \( \Omega^H_{ijkl} \) are simply connected open subsets of \( \mathbb{R}^4_{>0} \).

They further pointed out that \( \Omega^E_{ijkl} \) is not convex. However, for a triangulated 3-manifold \((M, T)\), we know little about the admissible space

\[
\Omega = \cap_{\{ijkl\} \in T} \Omega_{ijkl} = \{ r \in \mathbb{R}^N_{>0} | Q_{ijkl} > 0, \forall \{ijkl\} \in T \}, \tag{2.5}
\]

except that it is an open subset of \( \mathbb{R}^N_{>0} \). Here \( \Omega_{ijkl} \) denotes the Euclidean or hyperbolic admissible space for a tetrahedron \( \{ijkl\} \in T \) according to the background geometry and \( \mathbb{R}^4_{>0} \) is naturally embedded in \( \mathbb{R}^N_{>0} \).

For further applications, we need a finer description of the admissible space \( \Omega_{ijkl} \) for a single tetrahedron \( \{ijkl\} \in T \). Geometrically, the nondegeneracy of the tetrahedron is given by the condition that any of the four spheres \( S_i, S_j, S_k, S_l \) can not go through the gap between the other three spheres when they are externally tangent to each other. This can be separated into the following four cases, each of which determines a degenerate subset of \( \mathbb{R}^4_{>0} \) respectively.

(1) The sphere \( S_i \) should not go through the gap between the other three mutually externally tangent spheres \( S_j, S_k, S_l \);

(2) The sphere \( S_j \) should not go through the gap between the other three mutually externally tangent spheres \( S_i, S_k, S_l \);

(3) The sphere \( S_k \) should not go through the gap between the other three mutually externally tangent spheres \( S_i, S_j, S_l \);

(4) The sphere \( S_l \) should not go through the gap between the other three mutually externally tangent spheres \( S_i, S_j, S_k \).

Denote the degenerate sets corresponding to the four cases, which are subsets of \( \mathbb{R}^4_{>0} \), as \( V_i, V_j, V_k, V_l \) respectively. We have the following result on the structure of the sets \( V_i, V_j, V_k, V_l \) and \( \Omega_{ijkl} \). Here and in the following, \( \Omega_{ijkl} \) denotes \( \Omega^E_{ijkl} \) or \( \Omega^H_{ijkl} \) according to the background geometry.

Theorem 2.2. Each of \( V_i, V_j, V_k, V_l \) is a connected component of \( \mathbb{R}^4_{>0} - \Omega_{ijkl} \) and they are mutually disjoint. Furthermore, the intersection of \( \Omega_{ijkl} \) with any of \( V_i, V_j, V_k, V_l \) is a connected component of \( \mathbb{R}^4_{>0} - \Omega_{ijkl} \), which is also a piecewise analytic hypersurface of \( \mathbb{R}^4_{>0} \) and a graph on \( \mathbb{R}^3_{>0} \).

Proof. We analyse the case of Euclidean background geometry with details and just give a sketch for the case of hyperbolic background geometry.
We analysis only the critical degenerate case for $V_i$. The critical degenerate case is that $r_i$ is large enough so that the sphere $S_i$ of radius $r_i$ is large enough to be externally tangent to the other three mutually externally tangent spheres $S_j, S_k, S_l$, yet small enough that its center $i$ lies in the plane determined by $j, k, l$. In fact, there should be an additional condition that the sphere $S_i$ should lie in the gap between the other three spheres $S_j, S_k, S_l$. Projecting the spheres to the plane determined by $j, k, l$, we get four externally tangent circles $C_i, C_j, C_k, C_l$. In some cases, it may happen that there are two circles externally tangent to $C_j, C_k, C_l$ simultaneously. For these cases, we should take the circle with small radius, which corresponds to the condition that the sphere $S_i$ lies in the gap between $S_j, S_k, S_l$.

Suppose $S_i$ is the sphere in the gap of the other three spheres $S_j, S_k, S_l$. Then in the critical degenerate case, by Descartes circle theorem, we have

$$Q_{ijkl} = \left(\frac{1}{r_i} + \frac{1}{r_j} + \frac{1}{r_k} + \frac{1}{r_l}\right)^2 - 2\left(\frac{1}{r_i^2} + \frac{1}{r_j^2} + \frac{1}{r_k^2} + \frac{1}{r_l^2}\right) = 0,$$

which is equivalent to the quadratic equation in $r_i$

$$Ar_i^2 + Br_i^2 + C = 0, \quad (2.6)$$

where

$$A = 2r_kr_l + 2r_kr_l + 2r_jr_kr_l - r_k^2 - r_l^2 - r_j^2 - r_k^2,$$

$$B = 2(r_k + r_l + r_j)r_k + r_l > 0,$$

$$C = - r_j^2 - r_k^2 < 0.$$

For the quadratic equation (2.6) in $r_i$, the discriminant is

$$\Delta = B^2 - 4AC = 16r_j^2 r_k^2 (r_j + r_k + r_l), \quad (2.7)$$

which is always positive for $(r_j, r_k, r_l) \in \mathbb{R}^3$. Note that

$$A = 2r_kr_l + 2r_kr_l + 2r_jr_kr_l - r_k^2 - r_l^2 - r_j^2 - r_k^2$$

$$= (\sqrt{r_jr_k} + \sqrt{r_jr_l} + \sqrt{r_kr_l})(\sqrt{r_jr_k} + \sqrt{r_jr_l} - \sqrt{r_kr_l})$$

$$= (\sqrt{r_jr_k} - \sqrt{r_jr_l} + \sqrt{r_kr_l})(- \sqrt{r_jr_k} + \sqrt{r_jr_l} + \sqrt{r_kr_l}).$$

In the case that $A > 0$, we have $\sqrt{r_jr_k}, \sqrt{r_jr_l}, \sqrt{r_kr_l}$ satisfy the triangular inequalities. As $B > 0$ and $C < 0$, the quadratic equation (2.6) has two roots with different signs and negative sum. Note that $r_i > 0$, we have

$$r_i = \frac{-B + \sqrt{\Delta}}{2A} > 0.$$
Figure 1: circle configurations (the circles with dotted line correspond to the roots of (2.6) thrown away)

The negative root corresponds to the case that the externally tangent circles $C_j, C_k, C_l$ are internally tangent to the circle $C_i$. See (a) in Figure 1 for the possible arrangements of the circles.

In the case that $A = 0$, i.e. $\sqrt{r_j r_k} + \sqrt{r_j r_l} = \sqrt{r_k r_l}$ or $\sqrt{r_j r_k} + \sqrt{r_j r_l} = \sqrt{r_k r_l}$ or $\sqrt{r_k r_l} + \sqrt{r_j r_l} = \sqrt{r_j r_k}$, we have

$$r_i = -\frac{C}{B} > 0.$$

This corresponds to the case that there is a straight line tangent to the circles $C_j, C_k, C_l$ on the same side. See (b) in Figure 1 for the possible arrangements of the circles.

In the rest cases, we have $A < 0$, $B > 0$ and $C < 0$ with discriminant $\Delta > 0$. Then the quadratic equation (2.6) have two different positive roots. This corresponds to the case that there are two circles externally tangent to $C_j, C_k, C_l$ simultaneously. This can be separated into two cases, one of which is $C_i$ lies in the gap between $C_j, C_k, C_l$ and the other is one of $C_j, C_k, C_l$ lies in the gap between the other three circles. See (c) in Figure 1 for the possible arrangements of the circles. Note that in critical degenerate case of $V_i$, the four circles with finite radii are externally tangent and the circle $C_i$ lies in the gap between the circles $C_j, C_k, C_l$. So we should take the small root of the equation (2.6) and then

$$r_i = \frac{B - \sqrt{\Delta}}{-2A} = \frac{-B + \sqrt{\Delta}}{2A} > 0.$$
Set
\[ \Omega_{jkl} = \{(r_j, r_k, r_l) \in \mathbb{R}^3 \mid \sqrt{r_j r_k} < \sqrt{r_j r_l} + \sqrt{r_k r_l}, \sqrt{r_j r_l} < \sqrt{r_j r_k} + \sqrt{r_k r_l}; \sqrt{r_k r_l} < \sqrt{r_j r_k} + \sqrt{r_j r_l}\}. \]

Then, in the critical degenerate case, \( r_i \) is a function of \( r_j, r_k, r_l \), which is given by
\[
- B + \frac{\sqrt{A}}{2A}, \quad (r_j, r_k, r_l) \in \Omega_{jkl};
\]
\[
- C, \quad (r_j, r_k, r_l) \in \partial \Omega_{jkl};
\]
\[
- B + \frac{\sqrt{A}}{2A}, \quad (r_j, r_k, r_l) \in \mathbb{R}^3 \setminus \overline{\Omega_{jkl}}.
\] (2.8)

Note that \( \partial \Omega_{jkl} \) is in fact defined by \( A = 0 \), it is easy to check that, as \( (r_j, r_k, r_l) \) tends to any point in \( \partial \Omega_{jkl} \), we have
\[
- B + \frac{\sqrt{A}}{2A} \to - \frac{C}{B},
\]
which implies that \( f \) is in fact a continuous and piecewise analytic function of \( r_j, r_k \) and \( r_l \). Then the degenerate set \( V_i \) is given by
\[
V_i = \{(r = (r_i, r_j, r_k, r_l) \in \mathbb{R}^4 \mid 0 < r_i \leq f(r_j, r_k, r_l)\},
\]
which is a simply connected subset of \( \mathbb{R}^4 \). Similarly, we have
\[
V_j = \{(r = (r_i, r_j, r_k, r_l) \in \mathbb{R}^4 \mid 0 < r_j \leq f(r_i, r_k, r_l)\},
\]
\[
V_k = \{(r = (r_i, r_j, r_k, r_l) \in \mathbb{R}^4 \mid 0 < r_k \leq f(r_i, r_j, r_l)\},
\]
\[
V_l = \{(r = (r_i, r_j, r_k, r_l) \in \mathbb{R}^4 \mid 0 < r_l \leq f(r_i, r_j, r_k)\}.
\]

Then we have
\[ \mathbb{R}^4 = \Omega_{ijkl} \cup V_i \cup V_j \cup V_k \cup V_l. \]

We claim that \( V_i, V_j, V_k, V_l \) are mutually disjoint. This can be proved by contradiction. Suppose \( V_i \cap V_j \neq \emptyset \) and \( r = (r_i, r_j, r_k, r_l) \in V_i \cap V_j \subset \mathbb{R}^4 \). By the geometric meaning of the critical degenerate case, we have
\[ S_{\Delta_{ij}} + S_{\Delta_{ijl}} + S_{\Delta_{ijkl}} \leq S_{\Delta_{ijkl}}, \] (2.9)
and
\[ S_{\Delta_{ijk}} + S_{\Delta_{ijkl}} + S_{\Delta_{ijkl}} \leq S_{\Delta_{ijkl}}, \] (2.10)
where \( S_{\Delta_{ijk}} \) denotes the area of the triangle \( \{ijk\} \in F \) with lengths \( l_{ij} = r_i + r_j, l_{ik} = r_i + r_k, l_{jk} = r_j + r_k \). Combining (2.9) with (2.10), we have \( S_{\Delta_{ijk}} + S_{\Delta_{ijl}} \leq 0 \), which is impossible. So we have \( V_i \cap V_j = \emptyset \). This completes the proof for the theorem with Euclidean background geometry.
The proof for the case of hyperbolic background geometry is similar. The boundary \( \partial_i \Omega_{ijkl} = \overline{\Omega}_{ijkl} \cap V_i \) of \( V_i \) in \( \mathbb{R}^4_{>0} \) is given by the following function

\[
\tanh r_i = \begin{cases} 
\frac{-B + \sqrt{\Delta}}{2A}, & (r_j, r_k, r_l) \in \mathbb{R}^3_{>0} - \partial \Omega_{jkl} \\
\frac{-C\tanh r_i}{2}, & (r_j, r_k, r_l) \in \partial \Omega_{jkl}
\end{cases}
\]

where

\[
A = 4 \tanh^2 r_j \tanh^2 r_k \tanh^2 r_l + (\tanh r_j \tanh r_k + \tanh r_j \tanh r_l + \tanh r_k \tanh r_l)^2 \\
- 2 (\tanh^2 r_j \tanh^2 r_k + \tanh^2 r_j \tanh^2 r_l + \tanh^2 r_k \tanh^2 r_l),
\]

\[
B = 2 \tanh r_j \tanh r_k \tanh r_l (\tanh r_j \tanh r_k + \tanh r_j \tanh r_l + \tanh r_k \tanh r_l) > 0,
\]

\[
C = - \tanh^2 r_j \tanh^2 r_k \tanh^2 r_l < 0,
\]

\[
\Delta = B^2 - 4AC = 16 \tanh^3 r_k \tanh^3 r_l (1 + \tanh r_k \tanh r_l) > 0,
\]

\[
\partial \Omega_{jkl} = \{ (r_j, r_k, r_l) \in \mathbb{R}^3_{>0} | A(r_j, r_k, r_l) = 0 \}.
\]

Set \( r_i = f(r_j, r_k, r_l) \), then \( f \) is obviously continuous and piecewise analytic. The corresponding degenerate set \( V_i \) is given by

\[
V_i = \{ (r_i, r_j, r_k, r_l) \in \mathbb{R}^4_{>0} | 0 < r_i \leq f(r_j, r_k, r_l) \}.
\]

The rest of the proof for the hyperbolic case is parallel to that of the Euclidean case, so we omit the details here. \( \square \)

**Remark 1.** By Theorem 2.2, the admissible space \( \Omega_{ijkl} \) of sphere packing metrics for a single tetrahedron is in fact homotopic to \( \mathbb{R}^4_{>0} \). This provides another proof for Lemma 2.1 that \( \Omega_{ijkl} \) is simply connected, which was first obtained by Cooper and Rivin in [5].

In the following, we always take \( \partial_i \Omega_{ijkl} = \overline{\Omega}_{ijkl} \cap V_i \).

### 2.2 Combinatorial scalar curvature

For a triangulated 3-manifold \((M, T)\) with sphere packing metric \( r \), Cooper and Rivin [5] defined the combinatorial scalar curvature \( K_i \) at the vertex \( i \) as angle deficit of solid angles

\[
K_i = 4\pi - \sum_{\{i,j,k,l\} \in T} \alpha_{ijkl},
\]

(2.11)

where \( \alpha_{ijkl} \) is the solid angle at the vertex \( i \) of the tetrahedron \( \{i,j,k,l\} \in T \) and the sum is taken over all tetrahedrons with \( i \) as one of its vertices. When we are discussing in a single tetrahedron \( \{ijkl\} \in T \), we usually denote the solid angle \( \alpha_{ijkl} \) at the vertex \( v_i \) as \( \alpha_i \) for convenience in the following. \( K_i \) locally measures the difference between
the volume growth rate of a small ball centered at vertex \( v_i \) in \( M \) and a Euclidean ball of the same radius. Cooper and Rivin’s definition of combinatorial scalar curvature is motivated by the fact that, in the smooth case, the scalar curvature at a point \( P \) locally measures the difference of the volume growth rate of the geodesic ball with center \( P \) to the Euclidean ball with the same radius [1, 26]. In fact, for a geodesic ball \( B(P,r) \) in an \( n \)-dimensional Riemannian manifold \((M^n, g)\) with center \( P \) and radius \( r \), we have the following asymptotical expansion for the volume of \( B(P,r) \)

\[
\text{Vol}(B(P,r)) = \omega(n) r^n \left( 1 - \frac{1}{6(n+2)} R(P) r^2 + o(r^2) \right),
\]

where \( \omega(n) \) is the volume of the unit ball in \( \mathbb{R}^n \) and \( R(P) \) is the scalar curvature of \((M, g)\) at \( P \). From this point of view, Cooper and Rivin’s definition of combinatorial scalar curvature is a good candidate for combinatorial scalar curvature with geometric meaning parallel to the smooth case.

For a single tetrahedron \( \{ijkl\} \in T \), Cooper and Rivin got the following interesting result.

**Lemma 2.3** [5]. For a single tetrahedron \( \{ijkl\} \in T \), set

\[
S_{ijkl} = \begin{cases} 
\sum_{A \in \{i,j,k,l\}} \alpha_A r_A, & \text{Euclidean background geometry} \\
\sum_{A \in \{i,j,k,l\}} \alpha_A r_A + 2\text{Vol}, & \text{hyperbolic background geometry}
\end{cases}
\]

where \( \text{Vol} \) denotes the volume of the tetrahedron for the hyperbolic background geometry. Then

\[
dS_{ijkl} = \sum_{A \in \{i,j,k,l\}} \alpha_A dr_A = \alpha_i dr_i + \alpha_j dr_j + \alpha_k dr_k + \alpha_l dr_l. \tag{2.12}
\]

Furthermore, the Hessian of \( S_{ijkl} = \frac{\partial(\alpha_i, \alpha_j, \alpha_k, \alpha_l)}{\partial(r_i, r_j, r_k, r_l)} \) is negative semi-definite with kernel \( \{t(r_i, r_j, r_k, r_l) | t \in \mathbb{R}\} \) for the Euclidean background geometry and negative definite for the hyperbolic background geometry.

Combining Lemma 2.1 with Lemma 2.3 we have the following useful lemma on construction of functions.

**Lemma 2.4.** Given a Euclidean or hyperbolic tetrahedron \( \{ijkl\} \in T \) and \( r_0 \in \Omega_{ijkl} \),

\[
F_{ijkl}(r) = \int_{r_0}^r \alpha_i dr_i + \alpha_j dr_j + \alpha_k dr_k + \alpha_l dr_l \tag{2.13}
\]

is a well-defined locally concave function on \( \Omega_{ijkl} \). Furthermore, \( F_{ijkl}(r) \) is strictly concave on \( \Omega_{ijkl} \cap \{r_i^2 + r_j^2 + r_k^2 + r_l^2 = c\} \) for any \( c > 0 \) in the Euclidean background geometry and strictly concave on \( \Omega_{ijkl} \) in the hyperbolic background geometry.
Using Lemma 2.3, we have the following important property for the CR-curvature.

**Lemma 2.5.** \([5, 17, 33]\) Suppose \((M, T)\) is a triangulated 3-manifold with sphere packing metric \(r\), \(S\) is the total combinatorial scalar curvature, which is defined as

\[
S(r) = \begin{cases} 
\sum K_i r_i, & \text{Euclidean background geometry} \\
\sum K_i r_i - 2 \text{Vol}(M), & \text{Hyperbolic background geometry}
\end{cases}
\]  \hspace{1cm} (2.14)

Then we have

\[
dS = \sum_{i=1}^{N} K_i dr_i. \hspace{1cm} (2.15)
\]

Set

\[
\Lambda = \text{Hess}_r S = \frac{\partial(K_1, \ldots, K_N)}{\partial(r_1, \ldots, r_N)} = \begin{pmatrix} 
\frac{\partial K_1}{\partial r_1} & \cdots & \frac{\partial K_1}{\partial r_N} \\
\vdots & \ddots & \vdots \\
\frac{\partial K_N}{\partial r_1} & \cdots & \frac{\partial K_N}{\partial r_N}
\end{pmatrix}. \hspace{1cm} (2.16)
\]

In the case of Euclidean background geometry, \(\Lambda\) is symmetric and positive semi-definite with rank \(N - 1\) and the kernel \(\{tr | t \in \mathbb{R}\}\). In the case of hyperbolic background geometry, \(\Lambda\) is symmetric and positive definite.

We refer the readers to \([15, 18]\) for a nice geometrical explanation of \(\frac{\partial K_i}{\partial r_j}\). It should be emphasized that, as pointed out by Glickenstein \([15]\), the elements \(\frac{\partial K_i}{\partial r_j}\) for \(i \sim j\) may be negative, which is different from that of the two dimensional case.

### 2.3 Extension of functions

For a single Euclidean or hyperbolic tetrahedron \(\{ijkl\} \in T\), the solid angle function \(\alpha(r) = (\alpha_i(r), \alpha_j(r), \alpha_k(r), \alpha_l(r))\) is defined on the admissible space \(\Omega_{ijkl}\). However, we can extend the solid angle function by constants to be defined on the whole space \(\mathbb{R}^4_{>0}\). In fact, we have the following result.

**Lemma 2.6.** Given a Euclidean or hyperbolic tetrahedron, the solid angle function \(\alpha = (\alpha_i, \alpha_j, \alpha_k, \alpha_l)\) defined on \(\Omega_{ijkl}\) could be extended by constants to a continuous function \(\tilde{\alpha} = (\tilde{\alpha}_i, \tilde{\alpha}_j, \tilde{\alpha}_k, \tilde{\alpha}_l)\) defined on \(\mathbb{R}^4_{>0}\).

**Proof.** The extension \(\tilde{\alpha}_i\) of \(\alpha_i\) is defined to be \(\tilde{\alpha}_i(r) = 2\pi\) for \(r = (r_i, r_j, r_k, r_l) \in V_i\) and \(\tilde{\alpha}_i(r) = 0\) for \(r = (r_i, r_j, r_k, r_l) \in V_\alpha\) with \(\alpha \in \{j, k, l\}\). The extensions \(\tilde{\alpha}_j, \tilde{\alpha}_k, \tilde{\alpha}_l\) of \(\alpha_j, \alpha_k, \alpha_l\) are defined similarly.

If \(r = (r_i, r_j, r_k, r_l) \in \Omega_{ijkl}\) and \(r \to P\) for some point \(P \in \partial \Omega_{ijkl}\), then geometrically the tetrahedron \(\{ijkl\}\) tends to degenerate with the center \(v_i\) of the sphere \(S_i\) tends
to lie in the geodesic plane determined by \( v_j, v_k, v_l \) and the corresponding circle \( C_i \) is externally tangent to \( C_j, C_k, C_l \) with \( C_i \) in the gap between \( C_j, C_k, C_l \). Then we have \( \alpha_i \to 2\pi, \alpha_j \to 0, \alpha_k \to 0 \) and \( \alpha_l \to 0 \), as \( r \to P \in \partial_i \Omega_{ijkl} \). This implies that the extension \( \tilde{\alpha} \) of \( \alpha \) is continuous on \( \mathbb{R}^4_{>0} \). □

Before going on, we recall the following definition and theorem of Luo in [30].

**Definition 2.7.** A differential 1-form \( w = \sum_{i=1}^{n} a_i(x)dx^i \) in an open set \( U \subset \mathbb{R}^n \) is said to be continuous if each \( a_i(x) \) is continuous on \( U \). A differential 1-form \( w \) is called closed if \( \int_{\partial \tau} w = 0 \) for each triangle \( \tau \subset U \).

**Theorem 2.8** ([30] Corollary 2.6). Suppose \( X \subset \mathbb{R}^n \) is an open convex set and \( A \subset X \) is an open subset of \( X \) bounded by a \( C^1 \) smooth codimension-1 submanifold in \( X \). If \( w = \sum_{i=1}^{n} a_i(x)dx^i \) is a continuous closed 1-form on \( A \) so that \( F(x) = \int_{a}^{x} w \) is locally convex on \( A \) and each \( a_i \) can be extended continuous to \( X \) by constant functions to a function \( \tilde{a}_i \) on \( X \), then \( \tilde{F}(x) = \int_{a}^{x} \sum_{i=1}^{n} \tilde{a}_i(x)dx^i \) is a \( C^1 \)-smooth convex function on \( X \) extending \( F \).

Combining Theorem 2.2, Lemma 2.4, Lemma 2.6 and Theorem 2.8, we get the following useful lemma.

**Lemma 2.9.** For a Euclidean or hyperbolic tetrahedron \( \{i,j,k,l\} \in T \), the function \( F_{ijkl}(r) \) defined on \( \Omega_{ijkl} \) in (2.13) could be extended to the following function

\[
\tilde{F}_{ijkl}(r) = \int_{r_0}^{r} \tilde{\alpha}_i dr_i + \tilde{\alpha}_j dr_j + \tilde{\alpha}_k dr_k + \tilde{\alpha}_l dr_l, \tag{2.17}
\]

which is a \( C^1 \)-smooth concave function defined on \( \mathbb{R}^4_{>0} \) with

\[
\nabla_r \tilde{F}_{ijkl} = \tilde{\alpha}^T = (\tilde{\alpha}_i, \tilde{\alpha}_j, \tilde{\alpha}_k, \tilde{\alpha}_l)^T.
\]

### 3 The global rigidity for CR-curvature

Now we can prove the global rigidity of sphere packing metrics with respect to Cooper and Rivin’s combinatorial scalar curvature. We first introduce the following definition.

**Definition 3.1.** For any function \( f \in C(V) \), if there exists a sphere packing metric \( r \in \Omega = \cap_{\{ijkl\} \in T} \Omega_{ijkl} \) with \( K(r) = f \), then \( f \) is called an admissible curvature function.

Theorem 1.2 could be stated in the following form using the notion of admissible curvature function.

**Theorem 3.2.** Suppose \( \overline{K} \) is an admissible curvature function on a connected closed triangulated 3-manifold \( (M, T) \), then there exists only one admissible sphere packing metric in \( \Omega \) with CR-curvature \( \overline{K} \) (up to scaling in the case of Euclidean background geometry).
Proof. We only prove the case for Euclidean background geometry and the case for hyperbolic background geometry is proved similarly. Suppose \( r_0 \in \Omega \) is an admissible sphere packing metric. Define a Ricci potential function

\[
\tilde{F}(r) = -\sum_{\{ijkl\} \in T} \tilde{F}_{ijkl}(r) + \sum_{i=1}^{N} (4\pi - K_i) r_i,
\]

where the function \( \tilde{F}_{ijkl}(r) \) is defined by (2.17). Note that the second term in (3.1) is linear in \( r \) and well-defined on \( \mathbb{R}^N_{>0} \). Combining with Lemma 2.9, we have \( \tilde{F}(r) \) is a well-defined \( C^1 \)-smooth convex function on \( \mathbb{R}^N_{>0} \) and strictly convex on \( \mathbb{R}^N_{>0} \cap S^{N-1} \). Furthermore,

\[
\nabla_{r_i} \tilde{F} = -\sum_{\{ijkl\} \in T} \tilde{\alpha}_{ijkl} + (4\pi - K_i) = \tilde{K}_i - K_i,
\]

where \( \tilde{K}_i = 4\pi - \sum_{\{ijkl\} \in T} \tilde{\alpha}_{ijkl} \) is an extension of \( K_i = 4\pi - \sum_{\{ijkl\} \in T} \alpha_{ijkl} \).

If there are two different sphere packing metrics \( r_A \) and \( r_B \) in \( \Omega \) with the same combinatorial scalar curvature \( K \), then they are both critical points of the extended Ricci potential function \( \tilde{F}(r) \) by (3.2). It follows that

\[
\nabla \tilde{F}(r_A) = \nabla \tilde{F}(r_B) = 0.
\]

Set

\[
f(t) = \tilde{F}((1-t)r_A + tr_B) = \sum_{\{ijkl\} \in T} f_{ijkl}(t) + \sum_{i=1}^{N} (4\pi - \tilde{K}_i)[(1-t)r_{A,i} + tr_{B,i}],
\]

where

\[
f_{ijkl}(t) = -\tilde{F}_{ijkl}((1-t)r_A + tr_B).
\]

Then \( f(t) \) is a \( C^1 \)-smooth convex function on \( [0, 1] \) and \( f'(0) = f'(1) = 0 \), which implies that \( f'(t) \equiv 0 \) on \( [0, 1] \).

Note that \( r_A \in \Omega \) and \( \Omega \) is an open subset of \( \mathbb{R}^N_{>0} \), there exists \( \epsilon > 0 \) such that \( (1-t)r_A + tr_B \in \Omega \) for \( t \in [0, \epsilon] \). So \( f(t) \) is smooth on \( [0, \epsilon] \). \( f'(t) \equiv 0 \) on \( [0, 1] \) implies that \( f''(t) \equiv 0 \) on \( [0, \epsilon] \). Note that, for \( t \in [0, \epsilon] \), we have

\[
f''(t) = (r_A - r_B) \Lambda (r_A - r_B)^T,
\]

where \( \Lambda \) is the matrix defined in (2.16). By Lemma 2.5, we have \( r_A = c r_B \) for some positive constant \( c \in \mathbb{R} \). So there exists only one sphere packing metric up to scaling with the given admissible function \( \tilde{K} \) for the Euclidean background geometry. \( \square \)
Remark 2. The variational method used to prove Theorem 3.2 was first found by Y. C. de Verdière [6], and then further studied in [4]. The extension of convex function was first used in [2] to study the combinatorial curvature on triangulated surfaces with piecewise linear metric. This method was then used in the proof of the rigidity of inversive distance circle packing metric [21, 30]. These methods are then use by Ge, Jiang and the author to study the rigidity and deformation of a new combinatorial curvature for two and three dimensional triangulated manifolds [7, 8, 9, 10, 11, 12, 13, 14].

4 The global rigidity for R-curvature

In [11], Ge and the author once introduced the combinatorial scalar curvature $R_i = \frac{K_i}{s_i^2}$ for the Euclidean background geometry, which is motivated by the geometric approximation and transformation properties of the smooth scalar curvature. As mentioned in the introduction, this definition can be modified to fit the case of hyperbolic background geometry. In fact, we have the following definition of combinatorial scalar curvature on 3-dimensional manifolds.

Definition 4.1. Given a closed triangulated 3-manifold $(M, T)$ with Euclidean or hyperbolic background geometry, the combinatorial scalar curvature is defined to be

$$R_i = \frac{K_i}{s_i^2},$$

where

$$s_i(r) = \begin{cases} r_i, & \text{Euclidean background geometry} \\ \tanh \frac{r_i}{T}, & \text{hyperbolic background geometry} \end{cases}$$

Remark 3. R-curvature could be defined for spherical background geometry as $R_i = \frac{K_i}{\tan^2 \frac{r_i}{2}}$.

For R-curvature, we have the following global rigidity.

Theorem 4.2. Given a connected closed triangulated 3-manifold $(M, T)$, $\overline{R} \in C(V)$ is a given function defined on $(M, T)$.

(1) In the case of Euclidean background geometry, if $\overline{R} = 0$, there exists at most one sphere packing metric $\overline{r} \in \Omega$ with combinatorial $R$-curvature $\overline{R}$ up to scaling; if $\overline{R} \leq 0$ and $\overline{R} \neq 0$, there exists at most one sphere packing metric $\overline{r} \in \Omega$ with combinatorial $R$-curvature $\overline{R}$.

(2) In the case of hyperbolic background geometry, if $\overline{R} \leq 0$, there exists at most one sphere packing metric $\overline{r} \in \Omega$ with combinatorial $R$-curvature $\overline{R}$. 

15
Proof. We only prove the case of Euclidean background geometry in details and give a sketch of the proof for the case of hyperbolic background geometry.

Suppose $r_0 \in \Omega$ is an admissible sphere packing metric. Then we can define the following Ricci potential function $F(r)$ by

$$F(r) = - \sum_{ijkl \in T} F_{ijkl}(r) + \int_{r_0}^r \sum_{i=1}^N (4\pi - \bar{R}_i r_i^2) dr_i$$

(4.3)
on $\Omega$. Note that the function $F_{ijkl}(r)$ defined on $\Omega_{ijkl}$ could be extended to $\bar{F}_{ijkl}(r)$ defined by (2.17) on $\mathbb{R}^N$ by Lemma 2.9 and the second term $\int_{r_0}^r \sum_{i=1}^N (4\pi - \bar{R}_i r_i^2) dr_i$ in (4.3) can be naturally defined on $\mathbb{R}^N > 0$, then we have the following extension $\bar{F}(r)$ on $\mathbb{R}^N > 0$ of the Ricci potential function $F(r)$

$$\bar{F}(r) = - \sum_{ijkl \in T} \bar{F}_{ijkl}(r) + \int_{r_0}^r \sum_{i=1}^N (4\pi - \bar{R}_i r_i^2) dr_i,$$

As $\bar{F}_{ijkl}(r)$ is $C^1$-smooth concave by Lemma 2.9 and $\int_{r_0}^r \sum_{i=1}^N (4\pi - \bar{R}_i r_i^2) dr_i$ is a well-defined convex function on $\mathbb{R}^N$ for $\bar{R} \leq 0$, we have $\bar{F}(r)$ is a $C^1$-smooth convex function on $\mathbb{R}^N$. Furthermore,

$$\nabla_r \bar{F} = - \sum_{ijkl \in T} \bar{\alpha}_{ijkl} + 4\pi - \bar{R}_i r_i^2 = K_i - \bar{R}_i r_i^2,$$

where $\bar{K}_i = 4\pi - \sum_{ijkl \in T} \bar{\alpha}_{ijkl}$.

If there are two different sphere packing metrics $\pi_A, \pi_B \in \Omega$ with the same combinatorial $R$-curvature $\bar{R}$, then $\pi_A \in \Omega$, $\pi_B \in \Omega$ are both critical points of the extended Ricci potential $\bar{F}(r)$. It follows that

$$\nabla \bar{F} (\pi_A) = \nabla \bar{F} (\pi_B) = 0.$$

Set

$$f(t) = \bar{F}((1-t)\pi_A + t\pi_B)$$

$$= \sum_{ijkl \in F} f_{ijkl}(t) + \int_{r_0}^r (1-t)\pi_A(t) + t\pi_B(t) \sum_{i=1}^N (4\pi - \bar{R}_i r_i^2) dr_i,$$

where

$$f_{ijkl}(t) = - \bar{F}_{ijkl}((1-t)\pi_A + t\pi_B).$$

Then $f(t)$ is a $C^1$ convex function on $[0,1]$ and $f'(0) = f'(1) = 0$, which implies that $f'(t) \equiv 0$ on $[0,1]$. Note that $\pi_A$ belongs to the open set $\Omega$, there exists $\varepsilon > 0$ such that $(1-t)\pi_A + t\pi_B \in \Omega$ for $t \in [0, \varepsilon)$. So $f(t)$ is smooth on $[0, \varepsilon]$. 

16
In the case of $R \leq 0$ and $R \neq 0$, it is easy to check that $\tilde{F}(r)$ is strictly convex on $\Omega$, which implies that $f(t)$ is strictly convex on $[0, \epsilon]$ and $f'(t)$ is a strictly increasing function on $[0, \epsilon]$. Then $f'(0) = 0$ implies $f'(\epsilon) > 0$, which contradicts $f'(t) \equiv 0$ on $[0, 1]$. So there exists at most one sphere packing metric with combinatorial $R$-curvature $R$.

In the case that $R \equiv 0$, we have $f(t)$ is $C^1$ convex on $[0, 1]$ and smooth on $[0, \epsilon]$. $f'(t) \equiv 0$ on $[0, 1]$ implies that $f''(t) \equiv 0$ on $[0, \epsilon]$. Note that, for $t \in [0, \epsilon], \nabla f''(t) = (r_A - r_B) \Lambda (r_A - r_B)^T$.

By Lemma 2.5, we have $r_A = cr_B$ for some positive constant $c \in \mathbb{R}$. So there exists at most one sphere packing metric with zero $R$-curvature up to scaling.

For the case of hyperbolic background geometry, the Ricci potential $F(r)$ and its extension $\tilde{F}(s)$ are defined as

$$F(r) = -\sum_{\{ijkl\} \in T} F_{ijkl}(r) + \int_{r_0}^r \sum_{i=1}^N (4\pi - R_i \tanh^2 \frac{r_i}{2}) dr_i$$

and

$$\tilde{F}(r) = -\sum_{\{ijkl\} \in T} \tilde{F}_{ijkl}(r) + \int_{r_0}^r \sum_{i=1}^N (4\pi - \tilde{R}_i \tanh^2 \frac{r_i}{2}) dr_i$$

respectively. The proof for the hyperbolic background geometry is almost parallel to the case of Euclidean background geometry, so we omit the details here.

As a corollary of Theorem 4.2, we have the following global rigidity for the sphere packing metrics with constant combinatorial $R$-curvature.

**Corollary 4.3.** Given a connected closed triangulated 3-manifold $(M, T)$.

1. In the case of Euclidean background geometry, there exists at most one sphere packing metric with constant combinatorial $R$-curvature 0 up to scaling and there exists at most one sphere packing metric with negative constant combinatorial $R$-curvature $c \in \mathbb{R}$.

2. In the case of hyperbolic background geometry, there exists at most one sphere packing metric with nonpositive constant combinatorial $R$-curvature.

**Remark 4.** The local rigidity of sphere packing metrics with nonpositive constant $R$-curvature for Euclidean background geometry was first obtained in [11].
5 The global rigidity for combinatorial $\alpha$-scalar curvature

In [13], Ge and the author once generalized the $R$-curvature to $\alpha$-curvature for any parameter $\alpha \in \mathbb{R}$ on surfaces with Euclidean background geometry. As before, we can modify the definition to fit the hyperbolic background geometry as follows.

**Definition 5.1.** Given a closed triangulated 3-manifold $(M, T)$ with Euclidean or hyperbolic background geometry and $\alpha \in \mathbb{R}$ is a constant, the combinatorial $\alpha$-curvature is defined to be

$$R_i = \frac{K_i}{s_i^\alpha},$$

where

$$s_i(r) = \begin{cases} r_i, & \text{Euclidean background geometry} \\ \tanh \frac{r_i}{2}, & \text{hyperbolic background geometry} \end{cases}.$$ 

The definition of $\alpha$-curvature obviously generalizes the definition of CR-curvature and $R$-curvature, which are corresponding to $\alpha = 0$ and $\alpha = 2$ respectively. For this curvature, we have the following result on global rigidity.

**Theorem 5.2.** Given a connected closed triangulated 3-manifold $(M, T)$ and $\overline{R} \in C(V)$ is a given function on $M$.

1. In the case of Euclidean background geometry, if $\alpha \overline{R} \equiv 0$, there exists at most one sphere packing metric $\overline{r} \in \Omega$ with combinatorial $\alpha$-curvature $\overline{R}$ up to scaling. For $\overline{R} \in C(V)$ with $\alpha \overline{R} \leq 0$ and $\alpha \overline{R} \neq 0$, there exists at most one sphere packing metric $\overline{r} \in \Omega$ with combinatorial $\alpha$-curvature $\overline{R}$.

2. In the case of hyperbolic background geometry, if $\alpha \overline{R} \leq 0$, there exists at most one sphere packing metric $\overline{r} \in \Omega$ with combinatorial $\alpha$-curvature $\overline{R}$.

Theorem 5.2 can be proved similarly to that of Theorem 4.2, using

$$\tilde{F}(r) = - \sum_{ijkl \in T} \tilde{F}_{ijkl}(r) + \int_{r_0}^{r} \sum_{i=1}^{N} (4\pi - \overline{R}_i s_i^\alpha) dr_i$$

for the Euclidean background geometry and

$$\tilde{F}(r) = - \sum_{ijkl \in T} \tilde{F}_{ijkl}(r) + \int_{r_0}^{r} \sum_{i=1}^{N} (4\pi - \overline{R}_i \tanh \frac{r_i}{2}) dr_i$$

for the hyperbolic background geometry as the extended Ricci potential functions respectively. As the proof of Theorem 5.2 is parallel to that of Theorem 4.2, we omit the details here.
Remark 5. The global rigidity result for $\alpha$-curvature, i.e. Theorem 5.2, generalizes Theorem 3.2 on the global rigidity of CR-curvature and Theorem 4.2 on the global rigidity of R-curvature. Especially, if $\alpha = 0$, the condition $\alpha \bar{R} \leq 0$ in fact has no restriction on $\bar{R}$.

Acknowledgements

The author would like to thank Kehua Su for the help of Figure 1. The research of the author is supported by National Natural Science Foundation of China under grant no. 11301402.

References

[1] A. Besse, Einstein Manifolds, Reprint of the 1987 edition, Classics in Mathematics, Springer-Verlag, Berlin, 2008.
[2] A. Bobenko, U. Pinkall, B. Springborn, Discrete conformal maps and ideal hyperbolic polyhedra. Geom. Topol. 19 (2015), no. 4, 2155-2215.
[3] P. L. Bowers, K. Stephenson, Uniformizing dessins and Belyi maps via circle packing. Mem. Amer. Math. Soc. 170 (2004), no. 805.
[4] B. Chow, F. Luo, Combinatorial Ricci flows on surfaces, J. Differential Geometry, 63 (2003), 97-129.
[5] D. Cooper, I. Rivin, Combinatorial scalar curvature and rigidity of ball packings, Math. Res. Lett. 3 (1996), 51-60.
[6] Y. C. de Verdière, Un principe variationnel pour les empilements de cercles, Invent. Math. 104(3) (1991) 655-669.
[7] H. Ge, W. Jiang, On the deformation of discrete conformal factors on surfaces. Calc. Var. Partial Differential Equations 55 (2016), no. 6, Art. 136, 14 pp.
[8] H. Ge, W. Jiang, On the deformation of inversive distance circle packings. arXiv:1604.08317 [math.GT].
[9] H. Ge, W. Jiang, On the deformation of inversive distance circle packings, II. J. Funct. Anal. 272 (2017), no. 9, 3573-3595.
[10] H. Ge, X. Xu, Discrete quasi-Einstein metrics and combinatorial curvature flows in 3-dimension, Adv. Math. 267 (2014), 470-497.
[11] H. Ge, X. Xu, A combinatorial Yamabe problem on two and three dimensional manifolds. arXiv:1504.05814v2 [math.DG].
[12] H. Ge, X. Xu, A Discrete Ricci Flow on Surfaces with Hyperbolic Background Geometry, International Mathematics Research Notices 2016; doi: 10.1093/imrn/rnw142.
[13] H. Ge, X. Xu, $\alpha$-curvatures and $\alpha$-flows on low dimensional triangulated manifolds. Calc. Var. Partial Differential Equations 55 (2016), no. 1, Art. 12, 16 pp.
[14] H. Ge, X. Xu, On a combinatorial curvature for surfaces with inversive distance circle packing metrics. arXiv:1701.01795 [math.GT].
15. D. Glickenstein, *A combinatorial Yamabe flow in three dimensions*, Topology 44 (2005), No. 4, 791-808.
16. D. Glickenstein, *A maximum principle for combinatorial Yamabe flow*, Topology 44 (2005), No. 4, 809-825.
17. D. Glickenstein, *Geometric triangulations and discrete Laplacians on manifolds*, arXiv:math/0508188v1 [math.MG].
18. D. Glickenstein, *Discrete conformal variations and scalar curvature on piecewise flat two and three dimensional manifolds*, J. Differential Geometry, 87(2011), 201-238.
19. X. D. Gu, F. Luo, J. Sun, T. Wu, *A discrete uniformization theorem for polyhedral surfaces*, arXiv:1309.4175 [math.GT]. To appear on J. Differential Geometry.
20. X. D. Gu, R. Guo, F. Luo, J. Sun, T. Wu, *A discrete uniformization theorem for polyhedral surfaces II*, arXiv:1401.4594 [math.GT].
21. R. Guo, *Local rigidity of inversive distance circle packing*, Trans. Amer. Math. Soc. 363 (2011) 4757-4776.
22. R. Guo, F. Luo, *Rigidity of polyhedral surfaces. II*, Geom. Topol. 13 (2009), no. 3, 1265-1312.
23. Z.-X. He, *Rigidity of infinite disk patterns*. Ann. of Math. (2) 149 (1999), no. 1, 1-33.
24. M. K. Hurdal, K. Stephenson, *Discrete conformal methods for cortical brain flattening*. Neuroimage, 45 (2009) S86-S98.
25. J.C. Lagarias, C.L. Mallows, A.R. Wilks, *Beyond the descartes circle theorem*, Amer. Math. Monthly 109 (4) (2002) 338-361.
26. J. Lee, T. Parker, *The Yamabe problem*, Bull. Amer. Math. Soc. (N.S.) 17 (1) (1987) 37-91.
27. F. Luo, *Combinatorial Yamabe flow on surfaces*, Commun. Contemp. Math. 6 (2004), no. 5, 765-780.
28. F. Luo, *A combinatorial curvature flow for compact 3-manifolds with boundary*. Electron. Res. Announc. Amer. Math. Soc. 11 (2005), 12-20.
29. F. Luo, *Rigidity of polyhedral surfaces, I*. J. Differential Geom. 96 (2014), no. 2, 241-302.
30. F. Luo, *Rigidity of polyhedral surfaces, III*, Geom. Topol. 15 (2011), 2299-2319.
31. J. G. Mauldon, *Sets of equally inclined spheres*, Canadian J. Math. 14 (1962) 509-516.
32. A. Marden, B. Rodin, *On Thurston's formulation and proof of Andreev's theorem*. Computational methods and function theory (Valparaíso, 1989), 103-116, Lecture Notes in Math., 1435, Springer, Berlin, 1990.
33. I. Rivin, *An extended correction to Combinatorial Scalar Curvature and Rigidity of Ball Packings, (by D. Cooper and I. Rivin)*, arXiv:math/0302069v2 [math.MG].
34. K. Stephenson, *Introduction to circle packing*. Cambridge University Press, Cambridge, 2005.
35. W. Thurston, *Geometry and topology of 3-manifolds*, Princeton lecture notes 1976, http://www.msri.org/publications/books/gt3m.
36. M. Zhang, R. Guo, W. Zeng, F. Luo, S.T. Yau, X, Gu, *The unified discrete surface Ricci flow*, Graphical Models 76 (2014), 321-339.
(Xu Xu) School of Mathematics and Statistics, Wuhan University, Wuhan 430072, P.R. China
E-mail: xuxu2@whu.edu.cn