Robustness of quantum Markov chains

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If the conditional information of a classical probability distribution of three random variables is zero, then it obeys a Markov chain condition. If the conditional information is close to zero, then it is known that the distance (minimum relative entropy) of the distribution to the nearest Markov chain distribution is precisely the conditional information. We prove here that this simple situation does not obtain for quantum conditional information. We show that for tripartite quantum states the quantum conditional information is always a lower bound for the minimum relative entropy distance to a quantum Markov chain state, but the distance can be much greater; indeed the two quantities can be of different asymptotic order and may even differ by a dimensional factor.

Keywords: Markov chain, quantum information, conditional mutual information, relative entropy.

I. INTRODUCTION

From the point of view of information theory, as well as physics, it is very interesting to know when entropy or, more generally, information inequalities are saturated. For example, the basic quantities von Neumann entropy $S(A) = S(\rho_A) = -\text{Tr} \rho_A \log \rho_A$, quantum mutual information $I(A : B) = S(A) + S(B) - S(AB)$ for a bipartite state $\rho_{AB}$ and conditional mutual information $I(A : C|B) = S(AB) + S(BC) - S(B) - S(ABC)$ for a tripartite state $\rho_{ABC}$ are all non-negative; for the latter two this is known as the subadditivity and strong subadditivity of the entropy, respectively [17]. The entropy is 0 if and only if the state is pure, and the mutual information is 0 if and only if the state $\rho_{AB}$ is a product state, $\rho_{AB} = \rho_A \otimes \rho_B$.

However, in many applications it is not the case or not known that the state is exactly pure or a product, only that it is very close to being so. In such situations, there are continuity bounds on entropic quantities that one can use to quantify how small the entropy or mutual information is. Fannes’ inequality [11] states that if $\|\rho_A - \sigma_A\|_1 \leq \epsilon \leq 1/e$ (with the trace norm $\|X\|_1 := \text{Tr} |X| = \text{Tr} \sqrt{X^*X}$), then

$$|S(\rho) - S(\sigma)| \leq -\epsilon \log \epsilon + \epsilon \log d_A,$$

where $d_A$ is the dimension of the Hilbert space supporting the states. (“$\log$” in this paper is always the binary logarithm; the natural logarithm is denoted “$\ln$".) In particular, if $\rho$ has trace distance $\epsilon \leq 1/e$ to a pure state, then $S(\rho) \leq -\epsilon \log \epsilon + \epsilon \log d_A$. Recently, Alicki and Fannes [2] proved an extension of the Fannes inequality to quantum conditional entropy $S(A|B) = S(AB) - S(B)$ for bipartite states $\rho_{AB}$ and $\sigma_{AB}$: if $\|\rho_A - \sigma_A\|_1 \leq \epsilon \leq 1$, then

$$|S(A|B)_\rho - S(A|B)_\sigma| \leq -2\epsilon \log \epsilon - 2(1 - \epsilon) \log(1 - \epsilon) + 4\epsilon \log d_A.$$

The crucial observation here is that the bound only depends on $\epsilon$ and $d_A$, not $d_B$ as the bound yielded by a naive application of the original Fannes inequality. This gives an upper bound on the mutual information for a state that is at trace distance $\epsilon$ from a product state (using convexity of the trace distance, and (1) and (2) together with the triangle inequality).

Conversely, one may ask, if say the entropy of a state is small, $S(\rho) \leq \epsilon$, is it close to being pure? Indeed yes, as the following argument shows. Fix a diagonalisation of $\rho$, $\rho = \sum_{i=1}^{d_A} \lambda_i |e_i\rangle \langle e_i|$
with eigenvalues $\lambda_i$ arranged in decreasing order. Then, as $-x \log x \geq x$ for $0 \leq x \leq 1/2$,  
\[ \epsilon \geq S(\rho) = \sum_{i=1}^{d_A} -\lambda_i \log \lambda_i \geq \sum_{i=2}^{d_A} \lambda_i = 1 - \lambda_1. \]  
(3)

Hence,  
\[ \|\rho - |e_1\rangle \langle e_1||_{1} = 2(1 - \lambda_1) \leq 2\epsilon. \]  
(4)

Note however that this bound and Fannes’ inequality are not “inverse” to each other; plugging the $2\epsilon$ into the Fannes bound yields something much larger than order $\epsilon$.

Similarly, what can we say about the state when $I(A : B) \leq \epsilon$? Here, a new quantity, the relative entropy $D(\rho\|\sigma) = \text{Tr} \rho (\log \rho - \log \sigma)$, comes into play, when we observe that $I(A : B)_{\rho} = D(\rho_{AB}\|\rho_A \otimes \rho_B)$. Invoking another inequality between distance measures for states, namely Pinsker’s inequality, see [12],  
\[ D(\rho\|\sigma) \geq \left( \frac{1}{2 \ln 2} \|\rho - \sigma\|_1 \right)^2, \]  
(5)

we conclude that $\|\rho_{AB} - \rho_A \otimes \rho_B\|_1 \leq 2 \sqrt{\epsilon}$. Note that in both examples discussed, we found an explicit candidate for the closest pure/product state to the given state (as can be checked), and that the bound on the trace distance depends only on $\epsilon$, not on dimensions as in the converse Fannes-style inequalities. Third, that the relative entropy gives even tighter control on the distance due to Pinsker’s inequality.

In this paper we study the quantum conditional information. If the quantum conditional information of a tri-partite state $\rho$ vanishes, then $\rho$ obeys a quantum Markov chain condition. Here we analyze what can be said if $\rho$ has small quantum conditional information; in particular we investigate how close it is to a Markov chain state. The motivation is partly classical (e.g. cryptographic [14]), but in the quantum case a strong motivation comes from considerations of new entropy inequalities: in [18] a so-called constrained inequality for the von Neumann entropies of subsystems was found, namely a relation which is valid provided three quantum conditional mutual informations are zero. The desire to turn this into an unconstrained, universal inequality lead to speculations that if one understood the near-vanishing of these constraints, then perhaps a trade-off between the constraints and the new inequality solely in terms of entropies might be established.

In section II we review, as a model, the classical case, where it turns out that the conditional mutual information is exactly the minimum relative entropy distance between the distribution and the closest Markov chain distribution. In section III we formulate the analogous quantum problem, which we analyse in the rest of the paper: section IV presents several simplifications of the question – we prove continuity of the minimum relative entropy, and that it is lower bounded by the quantum conditional mutual information, and some useful formulas for later numerical and analytical evaluation of the quantity. Then, in section V we specialise to pure states: we relate the minimum relative entropy to the so-called entanglement of purification, and for a large family of states show upper and lower bounds of matching order. These results are then used in section VI to provide examples of states for which the minimum relative entropy is much larger than the quantum conditional mutual information, and also ones where the dimension enters explicitly, showing that the classical and the quantum case are very different indeed.
II. CLASICAL CASE

In the classical case, Markov chain distributions are used to define the conditional mutual information. A classical distribution \( P_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) \) form a Markov chain denoted as \( X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow \ldots \rightarrow X_n \) if the distribution can be written as

\[
P_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) = P_{X_1}(x_1)P_{X_2|X_1}(x_2|x_1) \cdots P_{X_n|X_{n-1}}(x_n|x_{n-1})
\]

(6)

If we take any three linked variables in the above Markov chain i.e. \( X_{\alpha-1}, X_{\alpha}, X_{\alpha+1} \) then the conditional distribution of \( P_{X_{\alpha+1}|X_{\alpha} \ldots X_1}(x_{\alpha+1}|x_{\alpha} \ldots x_1) \) depends only on \( X_{\alpha} \) and \( X_{\alpha+1} \) is conditionally independent of \( X_{\alpha-1} \), given \( X_{\alpha} \). Consider three random variables \( X, Y \) and \( Z \) which form a Markov chain \( X \rightarrow Y \rightarrow Z \). The probability distribution for this system is

\[
P_{XYZ}(xyz) = P_{XY}(xy)P_{Z|Y}(z|y)
\]

(7)

Aside, we define the conditional mutual information as,

\[
I(X : Z|Y) = \sum_{x, y, z} P_{XYZ}(xyz) \log \frac{P_{XZ|Y}(xz|y)}{P_{X|Y}(x|y)P_{Z|Y}(z|y)}
\]

(8)

Note that throughout this section we use the convention \( 0 \log 0 = 0 \). (This is justified by looking at the behavior of \( x \log x \) as \( x \to 0 \).) This conditional mutual information is equal to zero if and only if for all \( x, y \) and \( z \),

\[
\frac{P_{XZ|Y}(xz|y)}{P_{X|Y}(x|y)P_{Z|Y}(z|y)} = 1
\]

(9)

Therefore,

\[
P_{XZ|Y}(xz|y) = P_{X|Y}(x|y)P_{Z|Y}(z|y)
\]

(10)

Hence a classical Markov chain distribution is characterized by zero conditional mutual information. The classical case is characterized by an exact correspondence between the conditional mutual information and the relative entropy distance to the set of Markov chains: for any joint distribution \( P_{XYZ} \) of three random variables \( X, Y, Z \),

\[
I(X : Z|Y) = \min \{ D(P||Q) : Q \text{ Markov} \}.
\]

(11)

It can be shown that the Markov chain required to minimise this quantity is

\[
Q_{XYZ}(xyz) = P_{Y}(y)P_{X|Y}(x|y)P_{Z|X}(z|x).
\]

(12)

**Proof** Imagine a joint probability distribution \( Q_{XYZ} \) that also forms a general Markov chain:

\[
Q_{XYZ}(xyz) = Q_{Y}(y)Q_{X|Y}(x|y)Q_{Z|Y}(z|y).
\]

(13)

We can write the probability distribution of \( P_{XYZ} \) as follows

\[
P_{XYZ}(xyz) = P_{XYZ}(xyz) = P_{Y}(y)P_{Z|Y}(z|y)P_{X|YZ}(x|yz),
\]

(14)
therefore the relative entropy between the two probability distributions is
\[ D(P||Q) = \sum_{xyz} P_{XYZ}(xyz) \log \frac{P_Y(y)P_{Z|Y}(z|y)P_{X|Y,Z}(x|yz)}{Q_Y(y)Q_{Z|Y}(z|y)Q_{X|Y}(x|y)}. \] (15)

Since we have a product of logarithms we can represent the relative entropy as such,
\[ D(P||Q) = \sum_{xyz} P_{XYZ}(xyz) \left( \log \frac{P_Y(y)}{Q_Y(y)} + \log \frac{P_{Z|Y}(z|y)}{Q_{Z|Y}(z|y)} + \log \frac{P_{X|Y,Z}(x|yz)}{Q_{X|Y}(x|y)} \right). \] (16)

On inspection of the final term we can use the following equivalence
\[ \frac{P_{X|Y,Z}(x|yz)}{Q_{X|Y}(x|y)} = \frac{P_{XYZ}(xyz)}{P_Y(y)Q_{X|Y}(x|y)} = \frac{P_{Z|XY}(z|xy)}{P_{Z|Y}(z|y)} \frac{P_{X|Y}(x|y)}{Q_{X|Y}(x|y)}. \] (17)

Observing that the first two terms of eq. (16) are relative entropy terms, we have
\[ D(P||Q) = D(P_Y||Q_Y) + D(P_{Z|Y}(z|y)||Q_{Z|Y}(z|y)) \]
\[ + D(P_{X|Y}(x|y)||Q_{X|Y}(x|y)) + \sum_{xyz} P_{XYZ}(xyz) \log \frac{P_{Z|XY}(z|xy)}{P_{Z|Y}(z|y)} \frac{P_{X|Y}(x|y)}{Q_{X|Y}(x|y)}. \] (18)

Note that the only terms that depend on the distribution of \( Q \) are the first three relative entropy terms. Since relative entropy is non-negative and \( D(S||T) = 0 \) if and only if \( S = T \), the Markov chain that provides the minimum relative entropy between \( P \) and \( Q \) can be achieved by setting these terms to zero which gives the required distribution in (12). This concludes the proof.

From this result it is simple to show that conditional mutual information can be achieved. Since we know the relative entropy terms in eq. (18) are zero if we use \( Q \) as the minimising Markov chain:
\[ D(P||Q) = \sum_{xyz} P_{XYZ}(xyz) \log \frac{P_{Z|XY}(z|xy)}{P_{Z|Y}(z|y)}. \] (19)

Using the following equivalence
\[ P_{Z|XY}(z|xy) = \frac{P_{XYZ}(xyz)}{P_{XY}(xy)} \frac{P_{Y}(y)}{P_{X|Y}(xy)} = \frac{P_{Z|XY}(x|z)y}{P_{Z|Y}(z|y)}. \] (20)

We can substitute this into (19) to produce the final result
\[ D(P||Q) = \sum_{xyz} P_{XYZ}(xyz) \log \frac{P_{XZ|Y}(x|z|y)}{P_{X|Y}(x|y)P_{Z|Y}(z|y)} = I(X : Z|Y). \] (21)

III. QUANTUM ANALOGUE

A quantum analogue of (short) Markov chains, i.e. quantum states of some tripartite system \( ABC \) with a suitably defined Markov property, was first proposed by Accardi and Frigerio [1]. In finite Hilbert space dimension, which will be the case we will consider in the present paper, this property reads as follows: \( \mu_{ABC} \) is a quantum Markov state if there exists a quantum channel, i.e. a completely positive and trace preserving (c.p.t.p.) map \( T : \mathcal{B}(B) \rightarrow \mathcal{B}(B) \otimes \mathcal{B}(C) \) such that
\( \mu_{ABC} = (\text{id}_A \otimes T)\mu_{AB} \), with \( \mu_{AB} = \text{Tr}_C \mu_{ABC} \). In [19] it was shown that this Markov condition is equivalent to vanishing conditional mutual information,

\[
I(A : C|B)_\mu = 0,
\]

just as in the classical case, and in [13] the most general form of such states was given, as follows: system \( B \) has a direct sum decomposition into tensor products,

\[
B = \bigoplus_j b_j^L \otimes b_j^R,
\]

such that

\[
\mu_{ABC} = \bigoplus_j p_j \mu_{AB_j}^{(j)} \otimes \mu_{b_j^R C}^{(j)}.
\]

Note that this precisely generalises eq. (7). We introduce the notation \( \delta \) for the direct sum decomposition of \( B \). Note that we can always think of \( \mathcal{H}_B \) as being a subspace of a larger Hilbert space \( \mathcal{H}_\hat{B} \) (for which inclusion we use the shorthand \( B \hookrightarrow \hat{B} \)). This doesn’t change the fact that a state is a Markov chain state or not, but it leads to more possibilities of decomposing the ambient Hilbert space as a sum of products as in eq. (23). In other words, in a larger space there is a larger set of quantum Markov chains. This latter is evidently going to be relevant when comparing a given state \( \rho_{ABC} \) to the class of Markov chain states: in general we will have to admit that all three systems \( A, B, C \) are subspaces of larger Hilbert spaces, and we have to take into account the Markov states on the extended system.

Now we go on to develop some formalism to deal with these embeddings: If we have the embedded system \( B \hookrightarrow \hat{B} \) then we define \( \delta = (B \hookrightarrow \hat{B} = \bigoplus_j B_j) \) as both the isometric embedding and the orthogonal decomposition of the embedding system. For a specific such direct sum decomposition \( \delta \) we introduce \( \tau = \tau_\delta \) for the family of tensor product decompositions of the \( B_j \), which are, w.l.o.g., embeddings \( \tau_\delta \equiv (\tau_j : B_j \hookrightarrow b_j^L \otimes b_j^R) \). Note that this latter only gives us increased flexibility: we could as well demand that each \( \tau_j \) is actually a unitary isomorphism between \( B_j \) and \( b_j^L \otimes b_j^R \), because one can always blow up the spaces \( B_j \), extending the isometry to a unitary.

This brings us to the main question of this paper: for given state \( \rho_{ABC} \), to find

\[
\Delta(\rho) := \inf \{ D(\rho||\mu) : \mu \text{ Markov} \},
\]

and to compare it to \( I(A : C|B)_\rho \). The remainder of this paper will be devoted to a study of the properties of this function. To be precise, we would like to consider \( \mu \) to be a Markov state on a tripartite system \( \hat{A}\hat{B}\hat{C} \), with \( A \subset \hat{A}, B \subset \hat{B} \) and \( C \subset \hat{C} \) (with \( \rho \) understood to be also a state on \( \hat{A}\hat{B}\hat{C} \) via these embedding), which is why above we have to use the infimum, since the dimension of \( \hat{A}\hat{B}\hat{C} \) is unbounded. This appears to be necessary for the reason that the decompositions as in eq. (24) depend on the dimension of \( B \).

We will show below (in the next section) that w.l.o.g. \( A = \hat{A} \) and \( C = \hat{C} \), and \( \dim \hat{B} \leq d_B^1 \), so that the infimum is actually a minimum. We also show lower bounds on \( \Delta \) comparing it to \( I(A : C|B) \), in particular exhibiting examples of states \( \rho \) with \( \Delta(\rho) \gg I(A : C|B)_{\rho} \).

\section{IV. General Properties of \( \Delta \)}

Here we show that the problem of determining the minimum relative entropy to a Markov state is really only a minimisation over decompositions of the type (23) for \( \hat{B} \). For given dimensions of the quantum system, there is only a finite number number of decomposition types.
Therefore we need to perform a finite-dimensional optimisation for each decomposition (some of which we can perform explicitly) and choose the global minimum.

**Proposition 1** The optimal state for given direct sum and tensor decomposition we denote \( \omega[\delta, \tau] \) describing the specific direct sum as \( \delta \) and the chosen tensor decomposition for that direct sum \( \tau = \tau_\delta \). We obtain \( \omega[\delta, \tau] \) by the following procedure: first, with the subspace projections \( P_j \) onto \( \mathcal{B}_L^j \otimes \mathcal{B}_R^j \subset \mathcal{B} \), let

\[
\omega[\delta] := \bigoplus_j (\mathbb{1}_{AC} \otimes P_j) \rho(\mathbb{1}_{AC} \otimes P_j) = \bigoplus_j q_j \omega_{\mathcal{A}^{(j)}_L \mathcal{B}_R^{(j)} C},
\]

(26)

where for each part \( j \) of the direct sum for the given decomposition, we project system \( \mathcal{B} \) via the corresponding projections \( P_j \) to produce \( \omega_{\mathcal{A}^{(j)}_L \mathcal{B}_R^{(j)} C} \) with corresponding probability \( q_j \). Then, form the reduced states

\[
\sigma_{\mathcal{A}^{(j)}_L} = \text{Tr}_{\mathcal{B}_R^{(j)}} \omega_{\mathcal{A}^{(j)}_L \mathcal{B}_R^{(j)} C} \quad \text{and} \quad \chi_{\mathcal{B}_R^{(j)} C} = \text{Tr}_{\mathcal{A}^{(j)}_L} \omega_{\mathcal{A}^{(j)}_L \mathcal{B}_R^{(j)} C},
\]

and let

\[
\omega[\delta, \tau] := \bigoplus_j q_j \sigma_{\mathcal{A}^{(j)}_L} \otimes \chi_{\mathcal{B}_R^{(j)} C}.
\]

(27)

With these definitions, it is easy to work out that

\[
D(\rho||\omega[\delta, \tau]) = -S(\rho) + H(\mathcal{Q}) + \sum_j q_j \left( S(\sigma_{\mathcal{A}^{(j)}_L}) + S(\chi_{\mathcal{B}_R^{(j)} C}) \right).
\]

(28)

Then, among all Markov states with decomposition (23) of \( \mathcal{B} \), \( \omega[\delta, \tau] \) is the one with smallest relative entropy, given by eq. (28).

**Proof** The relative entropy between a general state \( \rho_{ABC} \) and a general quantum Markov state \( \mu_{ABC} \) is, with given decompositions \( \delta \) and \( \tau \),

\[
D(\rho_{ABC}||\mu_{ABC}) = -S(\rho_{ABC}) - \text{Tr}(\rho_{ABC} \log \mu_{ABC}),
\]

(29)

where the general quantum Markov state is defined as

\[
\mu_{ABC} = \bigoplus_j p_j \mu_{\mathcal{A}^{(j)}_L} \otimes \mu_{\mathcal{B}_R^{(j)} C}.
\]

(30)

Therefore we can calculate the logarithm of \( \mu_{ABC} \),

\[
\log \mu_{ABC} = \bigoplus_j \left( \log p_j (\mathbb{1}_{AC} \otimes P_j) + \log (\mu_{\mathcal{A}^{(j)}_L} \otimes \mu_{\mathcal{B}_R^{(j)} C}) \right) =: \bigoplus_j L_j,
\]

(31)

where \( P_j \) are the subspace projections from \( \mathcal{B} \) onto \( \mathcal{B}_L^j \mathcal{B}_R^j \). For clarity we assume \( \rho \) indicates the state over all parties unless otherwise indicated.

\[
\text{Tr} \rho_{ABC} \log \mu_{ABC} = \sum_j \text{Tr} \left( (\mathbb{1}_{AC} \otimes P_j) \rho(\mathbb{1}_{AC} \otimes P_j) L_j \right).
\]

(32)
Now \((\mathbb{1}_{AC} \otimes P_j)\rho(\mathbb{1}_{AC} \otimes P_j) = q_j \omega^{(j)}_{Ab_j^L b_j^R C}\) with \(\text{Tr} \omega^{(j)}_{Ab_j^L b_j^R C} = 1\). Therefore,

\[
\begin{align*}
\text{Tr} \rho_{ABC} \log \mu_{ABC} &= \sum_j \text{Tr} q_j \omega^{(j)}_{Ab_j^L b_j^R C} \log p_j (\mathbb{1}_{AC} \otimes P_j) + \sum_j \text{Tr} q_j \omega^{(j)}_{Ab_j^L b_j^R C} \log (\mu^{(j)}_{Ab_j^L} \otimes \mu^{(j)}_{b_j^R C}) \\
&= \sum_j q_j \log p_j + \sum_j q_j \text{Tr} \omega^{(j)}_{Ab_j^L b_j^R C} \log (\mu^{(j)}_{Ab_j^L} \otimes \mu^{(j)}_{b_j^R C}) \\
&= -H(q) - D(q\|\mu) \\
&- \sum_j q_j \left( S(\sigma^{(j)}_{Ab_j^L}) + S(\chi^{(j)}_{b_j^R C}) + D(\sigma^{(j)}_{Ab_j^L} || \mu^{(j)}_{Ab_j^L}) + D(\chi^{(j)}_{b_j^R C} || \mu^{(j)}_{b_j^R C}) \right), \quad (35)
\end{align*}
\]

where \(\sigma^{(j)}_{Ab_j^L} = \text{Tr}_{b_j^R C} \omega^{(j)}_{Ab_j^L b_j^R C}\) and \(\chi^{(j)}_{b_j^R C} = \text{Tr}_{Ab_j^L} \omega^{(j)}_{Ab_j^L b_j^R C}\). For a given decomposition of system \(\hat{B}\), the subspace projections \(P_j\) and hence \(q_j\) and \(\omega^{(j)}_{Ab_j^L b_j^R C}\) are fixed. Since we want to minimise the relative entropy we want to maximise the quantity \(\text{Tr} \rho_{ABC} \log \mu_{ABC}\). Therefore to maximise the first relative entropy term we set \(p_j = q_j\). For the sum we consider each \(i\) individually and only have freedom of setting the last two relative entropy terms to zero. Therefore we can maximise this expression by setting \(p^{(j)}_{Ab_j^L} = \sigma^{(j)}_{Ab_j^L}\) and \(\mu^{(j)}_{b_j^R C} = \chi^{(j)}_{b_j^R C}\). This concludes the proof. \(\square\)

A nice observation is that the relative entropy of interest can be decomposed into two relative entropies, as follows:

\[
D(\rho||\omega[\delta]) = D(\rho||\omega[\delta]) + D(\omega[\delta]||\omega[\delta, \tau]) \\
= D(\rho||\omega[\delta]) + \sum_j q_j I(\text{Ab}_j^L : b_j^R C)_{\omega^{(j)}}. \quad (36)
\]

Note that this result has an important consequence for the infimum defining \(\Delta(\rho)\): we only need to worry about embedding \(B\) into a larger system \(\hat{B}\); \(A\) and \(C\) can, w.l.o.g., stay the same.

**Theorem 2** The infimum of eq. (25) is achieved on a decomposition \(\hat{B} = \bigoplus_{i=1}^{k \leq d_B^2} b_i^L \otimes b_i^R\), with \(\dim b_i^L, \dim b_i^R \leq d_B\). In particular, because it is one of a continuous function over a compact domain, the infimum is actually a minimum. Also, it means that \(\Delta(\rho)\), as the minimum of a continuous function over a compact domain, is itself a continuous function of its argument \(\rho\).

The reader may wish to skip the rather lengthy and somewhat technical proof of this theorem; note however that in it some notation is introduced which is referred to later.

**Proof** The proof has two parts – first, that the direct sum decomposition \(\delta\) may be taken to have only \(d_B^2\) terms, and second, that each direct summand may be embedded into a space of not more than \(d_B \times d_B\) dimensions. These two arguments are quite independent of each other; we start with the first.

1. For given embedding \(B \hookrightarrow \hat{B}\) and decomposition of \(\hat{B}\), we have

\[
\omega[\delta] = \bigoplus_j (\mathbb{1}_{AC} \otimes P_j)\rho_{ABC}(\mathbb{1}_{AC} \otimes P_j) = \bigoplus_j (\mathbb{1}_{AC} \otimes P_J P)\rho_{ABC}(\mathbb{1}_{AC} \otimes PP_J), \quad (37)
\]

with the projector \(P\) of \(\hat{B}\) onto \(B\). Note that the operators \(P_J P\) form a complete Kraus system:

\[
\sum_j (P_J P)^\dagger (P_J P) = \sum_j PP_J P = P = \mathbb{1}_B. \quad (38)
\]
Hence the operators $M_j = PP_j P$ form a POVM on $B$, and introducing an auxiliary register $J$ with orthogonal states $|j⟩$ to reflect the direct sum, $ω[δ]$ is equivalent, up to local isometries, to the state

$$
Ω = \sum_j (1_{AC} \otimes \sqrt{M_j}) ρ_{ABC}(1_{AC} \otimes \sqrt{M_j}) \otimes |j⟩⟨j|_J \tag{39}
$$

At the same time, the embedding $τ_j$ can be reinterpreted as a family of isometries $τ_j : B \hookrightarrow b^L \otimes b^R$ controlled by the content $j$ of the $J$-register (note that we may, w.l.o.g., assume that the $τ_j$ all map into the same tensor product space), so that the state after the action of $τ$ is

$$
Ω_{Ab^L b^R C J K} = \sum_j (1_{AC} \otimes τ_j \sqrt{M_j}) ρ_{ABC}(1_{AC} \otimes \sqrt{M_j} τ_j^†) \otimes |j⟩⟨j|_J. \tag{40}
$$

In this notation, our formula (28) can be rewritten as

$$
D(\rho∥ω[δ, τ]) = −S(\rho) + S(J) + S(Ab^L|J) + S(b^R C|J). \tag{41}
$$

Now, to reduce the number of POVM elements (i.e., entries of the $J$-register with non-zero probability amplitude), we invoke a theorem of Davies [10] on extremal POVMs: One looks at all real convex-decomposed into extremal ones, i.e., is just a special case of Caratheodory’s lemma). On the other hand, the all-ones vector can be Davies’ theorem states that its extremal points have at most all-ones vector is eligible, and that this set is compact and convex – in fact, it is a polytope, and Davies’ theorem states that its extremal points have at most $d_{2J}^2$ non-zero entries (actually, this is just a special case of Caratheodory’s lemma). On the other hand, the all-ones vector can be convex-decomposed into extremal ones, i.e.,

$$
∀j \quad M_j = \sum_k r_k λ_j^{(k)} M_j, \tag{42}
$$

with extremal vectors $(λ^{(k)}_j)_j$ and positive reals $r_k$ with $\sum_k r_k = 1$. In operational terms, the POVM $(M_j)$ is equivalent to choosing $K = k$ with probability $r_k$ and then measuring the POVM $(λ_j^{(k)} M_j)$. This means that we can extend the state $Ω$ above to

$$
Ω_{Ab^L b^R C J K} = \sum_{jk} r_k (1_{AC} \otimes τ_j \sqrt{λ_j^{(k)} M_j}) ρ_{ABC}(1_{AC} \otimes \sqrt{λ_j^{(k)} M_j} τ_j^†) \otimes |j⟩⟨j|_J \otimes |k⟩⟨k|_K, \tag{43}
$$

of which it can be readily verified that tracing over $K$ gives eq. (40). Then, by the concavity of the von Neumann entropy, $S(J) ≥ S(J|K)$ and by the way we constructed the POVMs,

$$
S(AB^L|J) = S(AB^L|JK), \quad S(b^R C|J) = S(b^R C|JK). \tag{44}
$$

Hence, eq. (41) is lower bounded by

$$
−S(\rho) + S(J|K) + S(AB^L|JK) + S(b^R C|JK), \tag{45}
$$

and there exists a value $k$ of $K$ for which

$$
D(\rho∥ω[δ, τ]) ≥ −S(\rho) + S(J|K = k) + S(AB^L|JK = k) + S(b^R C|JK = k). \tag{46}
$$

But for each $K = k$, the right hand side is a relative entropy with a Markov state referring to the POVM $(λ_j^{(k)} M_j)$; it can be lifted, by Naimark’s theorem, to an orthogonal measurement on a
larger space \( \hat{B} \). It is clear that w.l.o.g. \( B_j \) has dimension at most \( d_B \): the state \( \omega^{(j)}_{ABC} \) is supported in \( B_j \) on a subspace of dimension at most \( d_B \).

2. Now for the second part: looking at eq. (28), we see that once \( \delta \) is fixed, we have states \( \omega^{(j)}_{ABC} \) and we need to find, for each \( j \) individually, a decomposition/embedding \( \tau_j : B_j \to b_j^R \otimes b_j^R \) that minimises the term \( S(\sigma^{(j)}_{ABj}) + S(\chi^{(j)}_{RjC}) \) in eq. (28). Dropping the index \( j \) for now, since we will keep it fixed, let us introduce a purification \(|\phi\rangle_{ABCD} \) of \( \omega_{ABC} \); then, with the isometric embedding \( \tau : B \to b^L \otimes b^R \) implicit and the slight abuse of notation

\[
|\phi\rangle_{AB-b^RCD} := (\mathbb{1}_{ACD} \otimes \tau)|\phi\rangle_{ABCD},
\]

our task is to minimise, over all choices of \( \tau \),

\[
S(\rho^L) + S(b^R C) = S(\rho^L) + S(AD\rho^L).
\]

Now notice that the latter quantity refers only to subsystems \( AD \) and \( b^L \), and that hence we can describe it entirely by the state \( \text{Tr}_C \phi_{ABCD} =: \vartheta_{ABD} \) and the completely positive and trace-preserving map \( T := \text{Tr}_{b^R \otimes} \) mapping density operators on \( B \) to density operators on \( b^L \) – by Stinespring’s theorem, conversely every such quantum channel can be lifted to an isometric dilation \( \tau : B \to b^L \otimes b^R \) (the system \( b^R \) would be called the environment of the channel). For fixed output system \( b^L \) the set of these quantum channels is convex and the state \( (\mathbb{1}_{AD} \otimes T)\vartheta_{ABD} \) is a linear function of the map. Hence, by the concavity of the von Neumann entropy \( S \), the smallest sum of entropies \((48)\) is attained for extremal channels, which by a theorem of Choi [6] have at most \( d_B \) operator terms in the Kraus decomposition – which translates into a dimension of at most \( d_B \) of \( b^R \). The dimensionality of the subsystem covered by the output of that channel in \( b^L \) is thus at most \( d_B \). But now we can run the same argument for \( b^R \) instead – the whole setup is symmetric, so the channel from \( B \) to \( b^R \) is also w.l.o.g. extremal, entailing \( \dim b^L \leq d_B \) (note that we fix the output dimension here to \( \leq d_B \) from the previous argument).

There is a special case of the second part of the above proof in the literature that has inspired the present argument: that is the dimension bounds in the so-called entanglement of purification [20]. There it was shown that in the problem of, for a pure state \( \omega_{ABC} \), minimising the entropy

\[
S(\rho_{AE}) = \frac{1}{2}(S(\rho_{AE}) + S(\rho_{CF})),
\]

over all isometric embeddings \( B \to EF \), one may restrict to a priori bounded dimensions \( \dim E = d_B \) and \( \dim F = d_B^2 \), or, vice versa, \( \dim E = d_B^2 \) and \( \dim F = d_B \). What is noticed above is that, apart from the generalisation to mixed states, one can apply the argument of the extremal channels twice, to get the same bound \( d_B \) on the dimensions of both \( E \) and \( F \):

**Corollary 3** The entanglement of purification,

\[
E_P(\rho_{AC}) = \inf_{B \to EF} S(\rho_{AE}),
\]

the entropy understood with respect to the state \( \phi_{AEFC} \), is attained at an embedding with dimensions \( \dim E, \dim F \leq d_B = \text{rank} \rho_{AC} \).

**Theorem 4** For any state \( \rho_{ABC} \), the quantity \( \Delta(\rho_{ABC}) \) has the following lower bound:

\[
\Delta(\rho) \geq I(A : C | B)_\rho.
\]
Proof Indeed, it is sufficient to show, for any $\rho_{ABC}$ and decomposition of $B$ as in eq. (12) with accompanying state $\omega[\delta, \tau]$, that
\[ D(\rho\|\omega[\delta, \tau]) \geq I(A : C|B)_\rho, \]
which, by eq. (17), is equivalent to
\[ H(q) + \sum_j q_j (S(\sigma_{ABj}^{(j)}) + S(\chi_{b_kC}^{(j)})) \geq S(B)_\rho + S(A|B)_\rho + S(C|B)_\rho. \]

It turns out to be convenient to introduce the following state of five registers to represent the entropic quantities in the above:
\[ \Omega_{JAB^Lb^RJC} = \sum_j q_j |j\rangle\langle j| \otimes \omega_{Ab^Lb^RC}^{(j)}, \]
observing that we may think of all $b^L_j$, $b^R_j$ as subspaces of one $b^L$, $b^R$, respectively. Then the inequality we need to prove reads
\[ S(J)_\Omega + S(AB^L|J)_\Omega + S(b^R|C|J)_\Omega \geq S(B)_\rho + S(A|B)_\rho + S(C|B)_\rho. \]
This is done by invoking standard inequalities as follows:
\begin{align*}
S(J)_\Omega + S(AB^L|J)_\Omega + S(b^R|C|J)_\Omega &= S(J)_\Omega + S(A|b^L|J)_\Omega + S(b^L|J)_\Omega + S(C|b^R|J)_\Omega + S(b^R|J)_\Omega \\
&\geq S(J)_\Omega + S(b^Lb^R|J)_\Omega + S(AB^L|J)_\Omega + S(b^L|J)_\Omega \\
&= S(Jb^Lb^R|\Omega) + S(AB^L|J)_\Omega + S(C|b^R|J)_\Omega \\
&\geq S(B)_\rho + S(A|B)_\rho + S(C|B)_\rho,
\end{align*}
where in the second line we have used ordinary subadditivity of entropy, and in the fourth line the fact that $\Omega$ is obtained from $\rho$ by a unital c.p.t.p. map on $B$; it can only increase the entropy, and, since it induces c.p.t.p. maps from $B$ to $Jb^L$ and $Jb^R$, we can use the non-decrease of the conditional entropy under processing of the condition (that’s basically strong subadditivity).

That means, for given dimensions $d_A$, $d_B$, $d_C$, we may define the continuous and monotonic real function
\[ \Delta(t; d_A, d_B, d_C) := \max \{ \Delta(\rho_{ABC}) : I(A : C|B)_\rho \leq t \}, \]
which has the property $\Delta(t; d_A, d_B, d_C) = 0$ if and only if $t = 0$ and $\Delta(t; d_A, d_B, d_C) \geq t$ (for not too large $t$, i.e. $t \leq 2 \log \min \{d_A, d_C\}$).

V. PURE STATES

Here we give some results when $\rho$ is a pure state. The entropy of the density matrix of a pure state is zero, and the minimum over $\tau_j$ of the $\frac{1}{\lambda^2}$ von Neumann entropy term in the sum in eq. (28) is the entanglement of purification of $\rho_{ABC}^{(j)}$.

Thus, we arrive at the formula
\[ \min_{\tau} D(\psi\|\omega[\delta, \tau]) = H(q) + 2 \sum_j q_j E_P(\rho_{AC}^{(j)}). \]

Using the calculation of $E_P$ for symmetric and antisymmetric states in [8], we can now show:
Theorem 5 Let $A$ and $C$ be systems of the same dimension $d$. For any pure state $\psi_{ABC}$ such that $\rho_{AC} = \text{Tr}_B\psi_{ABC}$ is supported either on the symmetric or on the antisymmetric subspace of $AC$, we have

$$S(\rho_A) \leq \Delta(\psi) \leq 2S(\rho_A).$$

Proof The upper bound can be simply derived by considering a single decomposition of system $B$ and calculating the value of $D(\psi\|\omega[\delta, \tau])$. Since $\Delta(\rho)$ is a minimum over all possible decompositions of system $B$, choosing one will immediately give an upper bound. Consider the following decomposition,

$$B = b^L \otimes b^R := B \otimes C.$$  

This gives a single term of tensor products leading to the following density matrix

$$\omega[\delta, \tau] = \rho_{AB} \otimes \rho_C,$$

therefore we have,

$$D(\psi\|\omega[\delta, \tau]) = 2E_P(\rho_{AC}).$$

A property of the entanglement of purification [20] is that if a two-party state $\rho_{AC}$ is completely supported either on the symmetric or antisymmetric subspace of $AC$ then the entanglement of purification is simply the entropy of reduced state of one of the parties [8],

$$E_P(\rho_{AC}) = S(\rho_A) = S(\rho_C).$$

Hence we prove the upper bound.

The lower bound is a consequence of strong subadditivity of quantum entropy. We know from eq. (63) that

$$\Delta(\rho) = H(\vec{q}) + 2\sum_j q_j S(\rho_A^{(j)}) \geq H(\vec{q}) + \sum_j q_j S(\rho_A^{(j)}).$$

Note, however that

$$H(\vec{q}) + \sum_j q_j S(\rho_A^{(j)}) \geq S\left(\sum_j q_j \rho_A^{(j)}\right) = S(\rho_A).$$

Hence we have shown the lower bound and this concludes the proof. \qed

VI. EXAMPLES

In this section we examine families of states which we can use to numerically illustrate the bounds on $\Delta(\rho)$. We look at two families of examples: first, on three qubits,

Example 6 Consider the following family of three qubit states

$$|\psi(x)\rangle_{ABC} := \frac{1}{\sqrt{2}}(|\varphi_x\rangle_A|0\rangle_B|\varphi_x\rangle_C + |\varphi_{-x}\rangle_A|1\rangle_B|\varphi_{-x}\rangle_C),$$

where $|\varphi_x\rangle := \sqrt{1-x^2}|0\rangle + x|1\rangle$, and $x$ is a real parameter. Using the notation $y = \sqrt{1-x^2}$ so that $y^2 + x^2 = 1$, we can calculate the following reduced density matrices for this pure state:

$$\rho_A = \rho_C = \begin{pmatrix} y^2 & 0 \\ 0 & x^2 \end{pmatrix},$$
\[
\rho_B = \frac{1}{2} \begin{pmatrix}
1 & (y^2 - x^2)^2 \\
(y^2 - x^2)^2 & 1
\end{pmatrix}.
\] (68)

Therefore we can calculate the entropy of each single party density matrix.

\[
S(\rho_A) = S(\rho_C) = -y^2 \log y^2 - x^2 \log x^2 = H_2(x),
\]

\[
S(\rho_B) = -(y^4 + x^4) \log (y^4 + x^4) - 2x^2 y^2 \log 2x^2 y^2.
\] (70)

From theorem 5 we know that for totally symmetric or totally anti-symmetric states, \(S(\rho_A) \leq \Delta(\rho) \leq 2S(\rho_A)\). Note also that for this state \(I(A : C|B)_{\psi(x)} = S(AB) + S(BC) - S(B) = 2S(A) - S(B)\). Thus, we wish to understand the ratio

\[
\frac{S(\rho_A)}{2S(\rho_A) - S(\rho_B)}.
\] (71)

If we look at the leading order terms of the single party entropies, since \(0 < x < 1\), we know that \(x^2 \log x\) and \(x^2\) are of lower order than \(x^4\), \(x^6\), etc. Thus, only taking \(x^2 \log x\) and \(x^2\) terms,

\[
S(\rho_A) = -(1 - x^2) \log (1 - x^2) - 2x^2 \log x
\]

\[
= -\frac{2x^2 \ln x}{\ln 2} + \frac{x^2}{\ln 2} + O(x^4),
\] (72)

\[
S(\rho_B) = -(1 - 2x^2 + 2x^4) \log(1 - 2x^2 + 2x^4)
\]

\[
- 2x^2(1 - x^2)[1 + 2 \log x + \log(1 - x^2)]
\]

\[
= -\frac{4x^2 \ln x}{\ln 2} + \frac{2x^2}{\ln 2} - 2x^2 + O(x^4).
\] (73)

Inserting these expressions, we find.

\[
\frac{\Delta(\rho)}{I(A : C|B)_{\rho}} \geq \frac{S(\rho_A)}{2S(\rho_A) - S(\rho_B)} = -\frac{\ln 2}{\ln x} + O(1) \quad \text{as } x \to 0.
\] (74)

Therefore for this state we can make this quantity approach \(+\infty\) as the value of \(x\) decreases, marking a striking deviation from the classical case.

**Example 7** Another use of theorem 5 is for the pure states \(|\zeta(d)\rangle_{ABC}\) on systems \(A\) and \(C\) of dimension \(d\) and \(B\) of dimension \(d(d + 1)/2\): namely, \(|\zeta(d)\rangle\) is the purification of the completely mixed state on the symmetric subspace of \(AC\), i.e. \(\zeta_{AC} = \text{Tr}_B \zeta_{ABC}\) is proportional to the symmetric subspace projector, of rank \(d(d + 1)/2\). For this family of states, we have

\[I(A : C|B)_{\zeta(d)} = 1 + \log \frac{d}{d + 1} < 1,\] (75)

while according to our theorem,

\[\Delta(\zeta(d)) \geq S(A) = \log d.\] (76)

This example shows that not only must any bound on \(\Delta\) depend nonlinearly on \(I(A : C|B)\), but that a \(\log\)-dimensional factor is also necessary.

**Example 8** Consider the class of states

\[
\rho_{ABC} = \sum_j p_j |j\rangle_A \otimes |\psi_j\rangle_B \otimes |j\rangle_C,
\] (77)
characterised by an ensemble of pure states \( \{ p_j, |\psi_j\rangle \} \) on \( B \) – the states of \( A \) and \( C \) are meant to be mutually orthogonal states. For a POVM (\( M_k \)) on \( B \), and using the previous notation of \( \hat{B} = K^{b^L} b^{R} \), the optimal state is given by

\[
\omega[\delta, \tau]_{ABC} = \sum_{jk} p_j |j\rangle_A \otimes (\sqrt{M_k} |\psi_j\rangle \sqrt{M_k})_{b^L b^R} \otimes |k\rangle_K \otimes |j\rangle_C.
\] (78)

We can calculate the following using formula (41) using the fact the system is symmetric in systems \( A \) and \( C \) and \( S(\rho) = S(A) \),

\[
D(\rho||\omega[\delta, \tau]) = -S(A) + S(K) + S(A^{b^L}|K) + S(A^{b^R}|K)
\] (79)

It is fairly clear from the formula since \( S(A^{b^L}|K) + S(A^{b^R}|K) \geq 2S(A|K) \) that the optimal choice of \( b^L b^R \) is to make one trivial, the other \( B \), so that

\[
D(\rho||\omega[\delta, \tau]) = -S(A) + S(K) + 2S(A|K) = S(A|K) + S(K|A),
\] (80)

all entropies relative to the state \( \omega \). Note that \( A \) and \( K \) are essentially classical registers, so that the above is really a classical probabilistic/entropic formula for the relative entropy. It is also quite amusing to see a quantity appearing that is known as information-distance in other contexts (see e.g. [5]).

VII. CONCLUSIONS

We have investigated the relation between the quantum conditional mutual information of a three-party state, and its relative entropy distance from the set of all (short) quantum Markov chains. While the latter is always larger or equal than the former, with equality in the classical case, in general the relative entropy distance can be much larger than the conditional mutual information. We showed this by developing tools to lower bound the relative entropy distance, in particular for pure states of a special symmetric form. In the process we found many useful properties of the minimum relative entropy distance from Markov states. Our findings indicate that the characterisation of quantum Markov chains in terms of vanishing quantum conditional mutual information is not robust, or at least not at all like the classical case, or the (quantum and classical) case of ordinary mutual information. Since these lower bounds are additive for tensor products of states, this surprising and perhaps displeasing behaviour will not go away in an asymptotic limit of many copies of the state.

What we haven’t found is an upper bound of the relative entropy distance \( \Delta \) in terms of the conditional mutual information \( I(A : C|B) \); our examples above show that such a bound has to depend nonlinearly on \( I \) and it has to contain a factor proportional to the logarithm of one or more of the local dimensions. Note that if there were a bound of the form \( \Delta(\rho) \leq f(I) \log(d_A d_C) \) – in particular not depending on the dimension of \( B \) – then this would settle a question left open in [7]; namely, it would imply that the “squashed entanglement” \( E_{sq}(\rho_{AB}) \) of a bipartite state \( \rho_{AB} \) is zero if and only if the state is separable. (We are grateful to Paweł Horodecki for pointing this out to us.)

We close by pointing out that our results cast doubts on earlier ideas of two of the present authors (NL and AW), reported in [13], on how to prove a non-standard inequality for the von Neumann entropy. The heuristics given there don’t seem to bear out, in the light of the present paper; of course, the conjectured entropy inequality itself may well still be true.
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[1] L. Accardi, A. Frigerio, “Markovian cocycles”, Proc. Roy. Irish Acad., vol. 83A, no. 2, pp. 251-263, 1983.
[2] R. Alicki, M. Fannes, “Continuity of quantum conditional information”, J. Phys. A: Math. Gen., vol. 37, pp. L55-L57, 2004.
[3] C. H. Bennett, S. Wiesner, “Communication via one- and two-particle operators on Einstein-Podolsky-Rosen states”, Phys. Rev. Letters, vol. 69, 2881-2884, 1992.
[4] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, W. K. Wootters, “Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels”, Phys. Rev. Letters, vol. 70, no. 13, pp. 1895-1899, 1993.
[5] C. H. Bennett, P. Gács, M. Li, P. M. B. Vitányi, W. H. Zurek, “Information Distance”, IEEE Trans. Inf. Theory, vol. 44, no. 4, pp. 1407-1423, 1998.
[6] M.-D. Choi, “Completely positive linear maps on complex matrices”, Linear Algebra and Appl., vol. 10, pp. 285-290, 1975.
[7] M. Christandl, A. Winter, “‘Squashed entanglement’: An additive entanglement measure”, J. Math. Phys., vol. 45, no. 3, pp. 829-840, 2004.
[8] M. Christandl, A. Winter, “Uncertainty, Monogamy, and Locking of Quantum Correlations”, IEEE Trans. Inf. Theory, vol. 51, no. 9, pp. 3159-3165, 2005.
[9] T. M. Cover, J. A. Thomas, Elements of Information Theory, John Wiley & Sons, Inc., New York, 1991.
[10] E. B. Davies, “Information and Quantum Measurement”, IEEE Trans. Inf. Theory., vol. 24, pp. 596-599, 1978.
[11] M. Fannes, “A continuity property of the entropy density for spin lattice systems”, Commun. Math. Phys., vol. 31, pp. 291-294, 1973.
[12] C. A. Fuchs, J. van de Graaf, “Cryptographic distinguishability measures for quantum-mechanical states”, IEEE Trans. Inf. Theory, vol. 45, no. 4, pp. 1216-1227, 1999.
[13] P. Hayden, R. Jozsa, D. Petz, A. Winter, “Structure of states which satisfy strong subadditivity of quantum entropy with equality”, Commun. Math. Phys., vol. 246, no. 2, pp. 359-374, 2004.
[14] K. Horodecki, M. Horodecki, P. Horodecki, J. Oppenheim, “Information Theories with Adversaries, Intrinsic Information, and Entanglement”, Found. Physics, vol. 35, no. 12, pp. 2027-2040, 2005.
[15] B. Ibinson, N. Linden, A. Winter, “All inequalities for the relative entropy”, to appear in Commun. Math. Phys. e-print quant-ph/0511260, 2005.
[16] B. Ibinson, N. Linden, A. Winter, in preparation, 2006.
[17] E. H. Lieb, M. B. Ruskai, “Proof of the strong subadditivity of quantum-mechanical entropy”, J. Math. Phys., vol. 14, pp. 1938-1941, 1973.
[18] N. Linden, A. Winter, “A new inequality for the von Neumann entropy”, Commun. Math. Phys., vol. 259, no. 1, pp. 129-138, 2005.
[19] D. Petz, “Sufficiency of channels over von Neumann algebras”, Quart. J. Math. Oxford Ser. (2), vol. 39, no. 153, pp. 97-108, 1988.
[20] B. M. Terhal, M. Horodecki, D. W. Leung, D. P. DiVincenzo, “The entanglement of purification”, J. Math. Phys., vol. 43, no. 9, pp. 4286-4298, 2002.