Black hole information and Reeh-Schlieder theorem

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ABSTRACT: The Reeh-Schlieder theorem, with the time-slice axiom of quantum field theory, may be used to recover the information which falls into a black hole.
1 Introduction

The black hole information problem [1] is roughly described as follows. We throw something into a black hole. Eventually the black hole evaporates. Naively it seems that the process of Hawking radiation [2] is independent of what we throw into the black hole except for its mass, charge, and angular momentum. Therefore the information is lost.

However, what do we mean by “information which is thrown into a black hole”? One may consider that it is a state vector associated to the matter which is thrown into the black hole. This is a non-relativistic picture of quantum information.

In relativistic quantum field theory, the situation is more subtle. It is not easy to see how to make precise the concept of “information localized in some region”. Indeed, the theorem by Reeh and Schlieder [3] may be interpreted as a fact that “information” cannot be localized precisely. Then we lose a clear distinction between “information inside the black hole” and “information outside the black hole”. This idea has appeared as a construction of the black hole interior by using the degrees of freedom outside of it [4–6].

The purpose of this paper is to try to make the preservation of information in the black hole evaporation more precise in this context. From the beginning we assume the existence of a state in which the time evolution is unitary. Then, we add perturbation to this situation by introducing additional matter which is thrown into the black hole, and formulate how that information is recovered after the evaporation of the black hole. Therefore, we are going to argue that there is no information loss in the perturbation of a state which is unitary. A big black hole may be constructed by adding perturbation to a small black hole by throwing a small amount of matter step by step, so if our proposal is correct, the information problem may be reduced to that of a Planck scale black hole, which obviously requires a full theory of quantum gravity. As we will see, our argument generalizes the fact that there is no information loss in black holes in AdS spacetime because there is no information loss in the dual CFT.
2 Reeh-Schlieder theorem

In this section we review the Reeh-Schlieder theorem. In flat Minkowski space, a rigorous proof and an excellent review are already available in [7] and [8]. In curved backgrounds, we only give sketchy discussions without trying to give a complete justification (see [9–12] for some developments).

2.1 Flat Minkowski space

Let \( \phi(x) \) represent local operators of the theory (or more precisely operator valued tempered distribution), let \( \phi(f) = \int \text{d}^dx f(x) \phi(x) \) represent smeared operators for (Schwartz) test functions \( f(x) \) on \( \mathbb{R}^{1,d-1} \), and let \( \mathcal{A}(\mathcal{O}) \) be the algebra of operators generated by \( \phi(f) \) for test functions \( f(x) \) which have support in the open set \( \mathcal{O} \subset \mathbb{R}^{1,d-1} \). We may more simply say that \( \mathcal{A}(\mathcal{O}) \) is the set of all operators in the region \( \mathcal{O} \subset \mathbb{R}^{1,d-1} \) of the Minkowski spacetime \( \mathbb{R}^{1,d-1} \).

Let \( \tilde{\mathcal{H}} \) be a subspace of the total Hilbert space \( \mathcal{H} \) with the following property. Consider operators \( \exp(\epsilon H) \) where \( H \) is the Hamiltonian and \( \epsilon > 0 \) is a positive real number. Then, a state vector \( |\Psi\rangle \) is defined to be an element of \( \tilde{\mathcal{H}} \) if \( |\Psi\rangle \) is in the domain of definition of the operator \( \exp(\epsilon H) \) for some \( \epsilon \). This basically means that \( |\Psi\rangle \) has a finite norm \( \langle \Psi | \exp(2\epsilon H) | \Psi \rangle < \infty \). A sufficient condition is that the state \( |\Psi\rangle \) only contains eigenstates of \( H \) bounded by some upper bound \( E_0 \). However, for our applications, it is important to notice that this is not a necessary condition.

Let us fix a state \( |\Psi\rangle \in \tilde{\mathcal{H}} \). The Reeh-Schlieder theorem states that the space of states of operators

\[
\{a|\Psi\rangle; \ a \in \mathcal{A}(\mathcal{O})\}
\]

is dense in the Hilbert space \( \mathcal{H} \). Namely, any state in the Hilbert space is well approximated by a state of the form \( a|\Psi\rangle \) to an arbitrary good accuracy.

A sketch of the proof goes as follows. Suppose that the statement of the Reeh-Schlieder theorem does not hold. Then there exists a state \( |\chi\rangle \) which is orthogonal to all of \( a|\Psi\rangle \),

\[
\langle \chi | \phi(x_1) \phi(x_2) \cdots \phi(x_n) | \Psi \rangle = 0, \quad x_i \in \mathcal{O} \ (i = 1, \cdots, n).
\]

Now, notice that the above matrix elements can be written as

\[
\langle \chi | \phi(x_1) \phi(x_2) \cdots \phi(x_n) | \Psi \rangle = \langle \chi | e^{-i x_1 P} \phi(0) e^{i(x_1-x_2) P} \phi(0) \cdots e^{i(x_{n-1}-x_n) P} \phi(0) e^{i x_n P} | \Psi \rangle.
\]

where \( P = P^\mu = (H, \vec{P}) \) is the four-momentum operator, and \( xP = x_\mu P^\mu = -x^0 H + \vec{x} \cdot \vec{P} \). We analytically continue \( x_i \) as \( x_i \to z_i = x_i - iy_i \) in such a way that vectors \( -y_1 \) and \( (y_i - y_{i+1}) \) are inside the forward light cone. Notice that in the analytic continuation we have factors \( e^{-y_1 P} \), \( e^{y_i - y_{i+1}) P} \), and \( e^{y_n P} \) in the above equation. The factors \( e^{-y_1 P} \) and \( e^{y_i - y_{i+1}) P} \) are bounded operators with exponential damping of high energy modes. Therefore, the analytic continuation is safe as far as these factors are concerned. The only factor which is dangerous is \( e^{y_n P} \), which is unbounded and exponentially large for high energy modes because \( -y_n \) is in the forward light cone. However, the spectral
condition $H \geq |\vec{P}|$ implies that $y_n P \leq (|y_0| + |\vec{y}_n|)H$. By definition of $|\Psi\rangle$, there exists $\epsilon$ such that $e^{\epsilon H}|\Psi\rangle$ is finite. Therefore, as long as $y_n$ satisfies $(|y_0| + |\vec{y}_n|) < \epsilon$, we can make sense of the state vectors $e^{y_n P-\epsilon H}|\Psi\rangle$ because $e^{y_n P-\epsilon H}$ is bounded. Hence we can safely perform the analytic continuation in this region.

Thus we see that

$$f(z_1, \cdots, z_n) = \langle \chi | \phi(z_1)\phi(z_2)\cdots\phi(z_n)|\Psi\rangle$$

(2.4)

is holomorphic in the region where $\text{Im} \ z_1 = -y_1$ and $-\text{Im}(z_i - z_{i+1}) = y_i - y_{i+1}$ are inside the forward light cone and $(|y_0| + |\vec{y}_n|) < \epsilon$. Moreover, it is zero when $\text{Im} \ z_i = 0$. Roughly speaking, such a function is analytically continued to be zero everywhere; see [7, 8] for more rigorous treatment. By the analytic continuation, we conclude that $f(x_1, \cdots, x_n)$ is zero not only when $x_i \in \mathcal{O}$, but for any $x_i$ in the Minkowski space.

In particular, let us take an open set $\mathcal{O}_{\text{cauchy}}$ which is a tubular neighborhood of a Cauchy surface of the spacetime such that it contains $\mathcal{O}$; see Figure 1. An example is given by $\mathcal{O}_{\text{cauchy}} = \{(t, \vec{x}); \ t_1 < t < t_2\}$. Now we have the following axioms of quantum field theory. (See the axioms F. Completeness and G. “Time-slice Axiom” of Section II.1.2 of [13]. See also Theorem 4-5 of [7].) The axioms say that the algebra $\mathcal{A}(\mathcal{O}_{\text{cauchy}})$ acts irreducibly on the Hilbert space $\mathcal{H}$ and hence we can generate a dense set in the Hilbert space by acting $\mathcal{A}(\mathcal{O}_{\text{cauchy}})$ to $|\Psi\rangle$.1 By the above analytic continuation, $|\chi\rangle$ is still orthogonal to this dense space and hence $|\chi\rangle = 0$. This completes the sketch of the proof.

There is an interesting corollary of the Reeh-Schlieder theorem. Let $\mathcal{O}_A$ be an open set of the spacetime and suppose there exists another open set $\mathcal{O}_B$ which is space-like separated from $\mathcal{O}_A$. Then we have $a|\Psi\rangle \neq 0$ for any $a \neq 0$. The proof is as follows. Suppose $a|\Psi\rangle = 0$. Then for any $b \in \mathcal{A}(\mathcal{O}_B)$ we get $ab|\Psi\rangle = \pm ba|\Psi\rangle = 0$. The sign $\pm$ depends on whether the operators are bosonic/fermionic. (Here we have assumed that $|\Psi\rangle$ has a definite fermion parity and hence without loss of generality we can assume that $a$ and $b$ also has definite fermion parities.) Now $b|\Psi\rangle$ spans a dense subset of the Hilbert space, and hence $ab|\Psi\rangle = 0$ for any $b$ implies that $a = 0$. This completes the proof.

1When there are conserved gauge charges such as the electric charges of QED, the statement is expected to hold for each superselection sector in which the total charge is fixed. In the following, we always work in such a superselection sector.
2.2 Remark on intuitive understanding and its surprising failure

One intuitive way to understand the Reeh-Schlieder theorem may be as follows, and it is used in the standard understanding of the properties of the Rindler space. Let $A$ and $B$ be spatial regions without overlap at a fixed time. We denote the causal diamonds of them in the spacetime as $O_A$ and $O_B$. We may try to associate Hilbert spaces $H_A$ and $H_B$ to the regions $A$ and $B$, and the Hilbert space of the region $A \sqcup B$ may be thought to be factorized as $H_{A\sqcup B} = H_A \otimes H_B$ which follows from the intuition from lattice regularization (at least in the absence of gauge fields). We further suppose that these Hilbert spaces are all finite dimensional, again from the intuition from lattice in which each site has qubits.

Let us take a state $|\Psi_{A\sqcup B}\rangle \in H_{A\sqcup B}$. The Reeh-Schlieder theorem may be modeled in this finite dimensional setting by saying that for any state $|\Phi_{A\sqcup B}\rangle$, we can find a unique operator $a \in \mathcal{A}(O_A)$ which acts on the Hilbert space $H_A$ such that $|\Psi_{A\sqcup B}\rangle = a|\Phi_{A\sqcup B}\rangle$. Here in the right hand side, $a$ is an abbreviation of $a \otimes 1_B$ where $1_B$ is the identity operator acting on $H_B$.

We can also slightly rephrase it as follows. For any $b \in \mathcal{A}(O_B)$ which acts on $H_B$, we define a state as $|\Phi_{A\sqcup B}\rangle = b|\Psi_{A\sqcup B}\rangle$. Then we can find $a \in \mathcal{A}(O_A)$ such that $b|\Psi_{A\sqcup B}\rangle = a|\Phi_{A\sqcup B}\rangle$. The role of $A$ and $B$ can be interchanged. This is possible, in the finite dimensional case, if and only if dim $H_A = \text{dim} H_B$ and that $|\Psi_{A\sqcup B}\rangle$ is given as

$$
|\Psi_{A\sqcup B}\rangle = \sum_{ij} c_{ij} |\psi^i_A\rangle \otimes |\psi^j_B\rangle \quad (2.5)
$$

where $|\psi^i_A\rangle$ and $|\psi^j_B\rangle$ are basis vectors of $H_A$ and $H_B$ respectively, and the coefficients $c_{ij}$ have the maximal rank as a matrix $(c_{ij})$. In such a case, we may say that the state $|\Psi_{A\sqcup B}\rangle$ is fully entangled, following [8].

The above understanding is standard in the Rindler space. In this case, we take $A = \{ \vec{x}; \; x^1 < 0 \}$ and $B = \{ \vec{x}; \; x^1 > 0 \}$. See the left side of Figure 2. The vacuum state $|\Omega\rangle$ satisfies the condition $|\Omega\rangle \in \mathcal{H}$ of the Reeh-Schlieder theorem, and hence, intuitively, the regions $A$ and $B$ are fully entangled. More explicitly, in the picture of the tensor factorization of the Hilbert space $H_A \otimes H_B$, the vacuum is schematically given as $|\Omega\rangle \propto \sum_i e^{-\pi \omega_i} |i\rangle_A \otimes |i\rangle_B$, where $|i\rangle_A$ and $|i\rangle_B$ are eigenmodes of the boost operator $K$, and $\omega$ is the (negative of) the eigenvalue of $K$ on $|i\rangle_B$ ($|i\rangle_A$). See e.g. [14] for a review. This is the origin of the thermal properties of the Rindler space after tracing out the $H_A$.

However, the above intuition should not be trusted too much, as is well known to experts (see e.g. [8] and references therein). We give a rather surprising demonstration of its failure by using the right side of Figure 2. We devide the space into three regions

$$
A = \{ \vec{x}; \; x^1 < 0 \}, \quad B = \{ \vec{x}; \; 0 < x^1 < \ell \}, \quad C = \{ \vec{x}; \; x^1 > \ell \} \quad (2.6)
$$

for some $\ell > 0$. Consider a state $|\Psi_{A\sqcup B\sqcup C}\rangle$ for which the finite dimensional version of the Reeh-Schlieder theorem is supposed to hold. Then, for any $b \in \mathcal{A}(O_B)$, we can find an operator $a \in \mathcal{A}(O_A)$ such that

$$
a|\Psi_{A\sqcup B\sqcup C}\rangle = b|\Psi_{A\sqcup B\sqcup C}\rangle. \quad (2.7)
$$
By a small computation and Shur’s lemma, one can see that this is possible for the tensor-factorized finite dimensional Hilbert space only if

$$|\Psi_{A\cup B\cup C}\rangle = \left( \sum_{ij} c_{ij} |\psi_i^A\rangle \otimes |\psi_j^B\rangle \right) \otimes |\Psi_C\rangle,$$

where $c_{ij}$ has the maximal rank and $|\Psi_C\rangle$ is a state vector of $\mathcal{H}_C$. Namely, $A$ and $B$ are fully entangled, and there is no entanglement between $A \cup B$ and $C$. However, we can repeat the same argument for $B$ and $C$ by using

$$b|\Psi_{A\cup B\cup C}\rangle = c|\Psi_{A\cup B\cup C}\rangle.$$

and also for $A$ and $C$ by using

$$c|\Psi_{A\cup B\cup C}\rangle = a|\Psi_{A\cup B\cup C}\rangle.$$

Then we conclude that $A$ is fully entangled with $B$, $B$ is fully entangled with $C$, and $C$ is fully entangled with $A$, which is impossible! This clearly shows that the oversimplification of the Hilbert space $\mathcal{H}$ by the tensor factorization in terms of $\mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_C$ can fail in the context of information theory.

It is quite tempting to think about the implications of the above discussion for the “entanglement monogamy problem” of the firewall paradox [15] and some of its proposed resolutions (e.g. [4–6, 16–19]). See [14] for a review. The Reeh-Schlieder theorem was discussed in this context in [5]. We do not investigate this problem further to avoid dangerous statements, but the above discussion suggests a possibility that “relativistic quantum information theory” rather than just an ordinary non-relativistic quantum information theory might play a role in black hole information problems.

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2It is tempting to formally call $\mathcal{H}_A$ as “the degrees of freedom inside the event horizon”, $\mathcal{H}_B$ as “the degrees of freedom in the late Hawking radiation”, and $\mathcal{H}_C$ as “the degrees of freedom in the early Hawking radiation”, and claim that they are fully entangled with each other by the Reeh-Schlieder theorem, despite the fact that it is impossible in the tensor-factorized finite dimensional Hilbert space. Of course these formal definitions are not exactly what is meant in the firewall paradox, but it suggests that a careful definition of various concepts can be important.
2.3 Curved backgrounds

We do not try to give a proof of the Reeh-Schlieder theorem on curved backgrounds (see [9–12] for results in this direction). However, we would like to make a few comments.

One of the essential points in the proof of the Reeh-Schlieder theorem was the analytic continuation of matrix elements like (2.4). This analytic continuation was made possible by the following two facts; (i) The energy of any system is bounded from below so that \( e^{-\tau H} \) is a bounded operator for any positive \( \tau \); (ii) We choose state vectors \( |\Psi\rangle \in \widetilde{H} \) such that \( e^{\epsilon H} \Psi \) has a finite norm for some positive \( \epsilon \).

On curved manifolds, there are no translation symmetries and hence the above proof does not apply straightforwardly. However, even in curved backgrounds, we still expect that the “energy” (in some appropriate sense) has a lower bound.

How about the condition that there exists \( \epsilon > 0 \) such that \( e^{\epsilon H} |\Psi\rangle \) has a finite norm? To get some intuition about this problem, let us ask the following simpler question. We treat the metric as background fields. Suppose that the background metric has a flat space form in the limit \( t \to \pm \infty \). In a finite spacetime region, the metric deviates from the flat space. Now, we can prepare a state \( |\Omega_{t=-\infty}\rangle \) in the Heisenberg picture such that it is indeed in the vacuum state in the limit \( t \to -\infty \). This does not coincide with the state \( |\Omega_{t=+\infty}\rangle \) which goes to the vacuum in the limit \( t \to +\infty \). In free field theory, these two are related by using the Bogoliubov transformation of creation and annihilation operators as in the Hawking’s computation of the black hole radiation [2]. Now we ask the question of whether \( |\Omega_{t=-\infty}\rangle \) satisfies the condition of the Reeh-Schlieder theorem in the region \( t \to +\infty \) with respect to the Hamiltonian \( H_{t=+\infty} \).

To make the question much simpler, let us investigate the above question in the case of the harmonic oscillator with a time-dependent Hamiltonian

\[
H(t) = \frac{1}{2} \left( \left( \frac{dx}{dt} \right)^2 + \omega(t)^2 x^2 \right),
\]

where \( \omega(t) > 0 \) is time-dependent with \( \omega(t) \to \omega_{\pm} \) for \( t \to \pm \infty \). The operator \( x(t) \) can be written by using creation and annihilation operators as

\[
x(t) \to \begin{cases} 
\frac{1}{\sqrt{2\omega_+}} (ae^{-i\omega_-t} + a^\dagger e^{i\omega_-t}) & t \to -\infty \\
\frac{1}{\sqrt{2\omega_+}} (be^{-i\omega_+t} + b^\dagger e^{i\omega_+t}) & t \to +\infty
\end{cases}
\]

where \([a, a^\dagger] = 1 \text{ and } [b, b^\dagger] = 1\) as usual. The point of the time dependence of \( \omega(t) \) is that these creation/annihilation operators at \( t \to \pm \infty \) are related by a nontrivial Bogoliubov transformation as

\[
a = \alpha b - \beta b^\dagger, \quad a^\dagger = \alpha^* b^\dagger - \beta^* b.
\]

Here \( \alpha \) and \( \beta \) are constants which are determined by solving the equation of motion \( \frac{d^2}{dt^2} x(t) + \omega(t)^2 x(t) = 0 \) and comparing the limits \( t \to \pm \infty \). But the explicit computation is not necessary for our purposes. For the commutation relations \([a, a^\dagger] = 1 \text{ and } [b, b^\dagger] = 1\) to be consistent, we have \(|\alpha|^2 - |\beta|^2 = 1\).
The state $|\Omega_{t=-\infty}\rangle$ is defined by $a|\Omega_{t=-\infty}\rangle = 0$. This means $(\alpha b - \beta b^\dagger)|\Omega_{t=-\infty}\rangle = 0$ and hence we get

$$|\Omega_{t=-\infty}\rangle = C \sum_{k\geq 0} \left( \frac{\beta}{\alpha} \right)^k \sqrt{\frac{(2k)!}{2^{2k}k!}} \cdot |2k\rangle_{t=+\infty}$$

where $|n\rangle_{t=+\infty} := (n!)^{-1/2}(b^\dagger)^n|\Omega_{t=+\infty}\rangle$, and $C$ is an overall normalization constant. By Stirling formula we have

$$\sqrt{\frac{(2k)!}{2^{2k}k!}} \sim (\pi k)^{-1/2} \quad (k \gg 1).$$

Also we introduce $\beta/\alpha = e^{-\eta+i\theta}$, where $\eta > 0$ and $\theta$ are real numbers. The $\eta$ is positive because $|\alpha|^2 - |\beta|^2 = 1$. Then we get

$$|\Omega_{t=-\infty}\rangle \sim C \sum e^{(-\eta+i\theta)k} (\pi k)^{-1/2} |2k\rangle_{t=+\infty}$$

for large $k$. Now it is clear that this state satisfies the condition that $\exp(\epsilon H_{t=+\infty})|\Omega_{t=-\infty}\rangle$ has a finite norm as long as $\epsilon < \eta/2\omega_+$. Notice that $\eta/\omega_+$ is like an inverse temperature in a rough sense.

In the above discussion we assumed that the Hamiltonian is time-independent in the region $t \to \pm \infty$. However, what we have seen is essentially the fact that Bogoliubov transformations do not spoil the condition of the Reeh-Schlieder theorem (at least in the above simple harmonic oscillator). The effects of backgrounds are that, intuitively speaking, we have successive Bogoliubov transformations as the time evolves. We expect that this is a general feature. An intuition is that the background fields affect low energy modes but not high energy modes (as far as the background fields are smooth so that the Fourier transformation of the background fields are exponentially suppressed at high energies), and the condition of the Reeh-Schlieder theorem is only about the high energy modes.

One may think that black holes are fundamentally different because it has the space-time singularity. However, just the presence of the singularity may not imply the violation of the Reeh-Schlieder theorem. To argue this point, let us recall how the Hartle-Hawking state [20] is defined.

The metric of a Schwarzschild black hole is

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega_{d-2}^2$$

where $f(r)$ is a function of $r$ which is positive in the region $r > r_s$, negative in $0 < r < r_s$, and has a singularity at $r = 0$. The $f(r)$ becomes zero at the horizon $r = r_s$ with $f'(r_s) \neq 0$. We introduce the tortoise coordinate $r_*$ and the Kruskal coordinates $U$ and $V$ as

$$r_* = \int \frac{dr}{f(r)}$$

$$U = -\exp \left( \frac{1}{2} f'(r_s)(-t + r_*) \right)$$

$$V = \exp \left( \frac{1}{2} f'(r_s)(t + r_*) \right).$$

$$\text{– 7 –}$$
By using them, the metric becomes
\[
ds^2 = -\frac{4f(r)}{f'(r_s)^2 \exp(f'(r_s)r_s)}dUdV + r^2d\Omega^2_{d-2}. \tag{2.21}\]
The $r$ is determined from $U$ and $V$ as $\exp(f'(r_s)r_s) = -UV$, and one can check that this metric is smooth in the region $r > 0$. This is the metric of a two-sided black hole.

The Hartle-Hawking state $|\text{HH}\rangle$ is defined as follows. First we analytically continue the above metric as $U \to -\bar{Z}$ and $V \to Z$. Here $Z$ and $\bar{Z}$ are two independent complex coordinates, and $r$ is still defined by $\exp(f'(r_s)r_s) = Z\bar{Z}$ as an analytic function of $Z$ and $\bar{Z}$. Now, we restrict these coordinates to a real analytic submanifold
\[
\mathcal{M} = \{(Z, \bar{Z}, \Omega_{d-2}); \ Z = Z^*\} \tag{2.22}\]
\[
ds^2 = \frac{4f(r)}{f'(r_s)^2 \exp(f'(r_s)r_s)}|dZ|^2 + r^2d\Omega^2_{d-2}. \tag{2.23}\]
where $Z^*$ means the complex conjugate of $Z$. The geometry is completely smooth since the singularity $r = 0$ is away from this submanifold. This submanifold has a smooth metric of Euclidean signature. We perform the path integral on this submanifold, and then analytically continue the results back to the original coordinates $U$ and $V$.

More explicitly, we may compute correlation functions on the Euclidean manifold $\mathcal{M}$ and then analytically continue the result to the original Schwarzschild background.\(^3\) In this way, we get correlation functions on the Hartle-Hawking state,
\[
\langle \text{HH}|\phi(x_1)\cdots\phi(x_N)|\text{HH}\rangle. \tag{2.24}\]
From these correlation functions, the state $|\text{HH}\rangle$ itself and the entire Hilbert space $\mathcal{H}$ may be recovered by the Wightman reconstruction theorem [7] (assuming its validity on this curved spacetime).

By using the first $K$ operators to create another state as
\[
\langle \chi | = \int d^dx_1\cdots d^dx_K|\text{HH}|f_1(x_1)\phi(x_1)\cdots f_K(x_K)\phi(x_K) \tag{2.25}\]
we get matrix elements of the form
\[
\langle \chi|\phi(x_{K+1})\cdots\phi(x_N)|\text{HH}\rangle. \tag{2.26}\]

\(^3\) This analytic continuation is not Wick rotation if we want to get Wightman correlation functions instead of time-ordered correlation functions. For the Reeh-Schlieder theorem we need Wightman correlation functions. Let us recall how it is done in the case of the flat space. First we consider a Euclidean correlation function $\langle \phi(x_F^1)\cdots\phi(x_F^N) \rangle$ where the superscript E means “Euclidean”. Let $\tau_1, \ldots, \tau_N$ be the Euclidean time coordinates of $x_F^1, \ldots, x_F^N$. We analytically continue them as $\tau_i \to z_i = \tau_i + it_i$. Then, we take the time ordering of the Euclidean times $\tau_i$ (but not the Minkowski times $t_i$) as $\tau_1 > \tau_2 > \cdots > \tau_N$. After this ordering, we take $\tau_1 \to 0$, keeping $0 > \tau_2 > \cdots > \tau_N$. Then we take $\tau_2 \to 0$, and then $\tau_3 \to 0$, and so on. After taking $\tau_N \to 0$, we get the desired Wightman correlation function. Compare this process to the proof of the Reeh-Schlieder theorem. We do not try to work out the details in the case of the Hartle-Hawking state, but the idea should be similar.
This is the type of matrix elements which appeared in the proof of the Reeh-Schlieder theorem. From the beginning, the matrix elements (2.26) are obtained from analytically continued correlation functions. Therefore, it is very likely that the Reeh-Schlieder theorem holds in the Hartle-Hawking state.

In the case that the black hole is in an asymptotically AdS spacetime and the theory has a CFT dual, the Hartle-Hawking state corresponds to a thermofield state [21]

$$\langle \Psi \rangle \propto \sum e^{-\beta E_i/2} |i \rangle \otimes |i^* \rangle$$

(2.27)

where $|i \rangle$ spans the basis of the CFT Hilbert space $\mathcal{H}(S^{d-1})$ on $S^{d-1}$, and $|i^* \rangle$ are the dual basis of the Hilbert space $\mathcal{H}(\overline{S^{d-1}})$ on the orientation-flipped sphere $\overline{S^{d-1}}$. The $E_i$ is the energy eigenvalue of both $|i \rangle$ and $|i^* \rangle$, and $\beta$ is the inverse temperature of the black hole. In the CFT, this state satisfies the condition that $e^{\epsilon H} |\Psi \rangle$ has a finite norm for $\epsilon < \beta / 4$.

Therefore, the existence of the singularity itself does not seem to violate the Reeh-Schlieder theorem. In the Hartle-Hawking state the Reeh-Schlieder theorem is very likely to hold. Any smooth deformation from the Hartle-Hawking background seems to satisfy the condition of the Reeh-Schlieder theorem, because of the intuition gained from the harmonic oscillator.

3 Information recovery in black hole evaporation: A proposal

Now we would like to discuss how a small amount of information which is thrown into a black hole may be recovered by using the Reeh-Schlieder theorem. We always work in the approximation in which the background geometry is fixed and we just consider perturbation around that background. This is necessary in order to fix the gauge associated to diffeomorphisms and the concept of local operators to make sense.\(^4\)

\(^4\) Although it is not relevant for the following discussion, we would like to make a remark about how to think about them. A spatial section of the boundary of the AdS-Schwarzchild geometry is $S^{d-1} \sqcup \overline{S^{d-1}}$, where $\sqcup$ means disjoint union of the two manifolds. Therefore, we get the Hilbert space of a single theory on $S^{d-1} \sqcup \overline{S^{d-1}}$ (instead of two copies of the theory on $S^{d-1}$). By an axiom of quantum field theory, $\mathcal{H}(Y_1 \sqcup Y_2) \cong \mathcal{H}(Y_1) \otimes \mathcal{H}(Y_2)$ for any disjoint $Y_1$ and $Y_2$, and hence $\mathcal{H}(S^{d-1} \sqcup \overline{S^{d-1}}) \cong \mathcal{H}(S^{d-1}) \otimes \mathcal{H}(\overline{S^{d-1}})$. Another axiom of quantum field theory states $\mathcal{H}(\overline{Y}) \cong \mathcal{H}(Y)$, where the overline on the Hilbert space means the complex conjugate vector space. Thus, given a state $|i \rangle \in \mathcal{H}(Y)$, we naturally get a corresponding state $|i^* \rangle := \overline{|i \rangle} \in \mathcal{H}(\overline{Y})$. The overline on the spatial manifold $Y$ is basically an orientation flip, but it can be more complicated depending on whether one considers spin theory, $\text{pin}^\pm$ theory, and so on. See [22, 23] for more details.

To make it more concrete, we may use BRST quantization of gravity in which the Hilbert space is enlarged to incorporate unphysical modes. In the proof of the Reeh-Schlieder theorem, the positive definiteness of the inner product in the Hilbert space mat not be necessary. The inner product needs to be non-degenerate, which is presumably satisfied in the total unphysical Hilbert space of BRST quantization. Admittedly, the concept of algebras $\mathcal{A}(\mathcal{O})$ of not-necessary BRST invariant local operators is not so beautiful. However, the final conclusion of this section is that “information” can be stored near the region of spatial infinity $r \to \infty$. If we take the limit $r \to \infty$, hopefully the BRST transformation does not matter there because the gauge transformation is assumed to be trivial at $r \to \infty$. Alternatively, we may consider BRST invariant but not completely local operators by starting from spatial infinity, which are analogous to operators of the form $\psi(x) \exp(i \int_0^\infty A)$ in gauge theories. This may be possible if $\mathcal{O}$ contains $r \to \infty$, which is the case in the following discussion. See the right side of Figure 3.
We assume that there exists a state $|\Psi\rangle$ which is described by the geometry of the left side of Figure 3, and we further assume that the Reeh-Schlieder theorem holds in this state. The validity of this assumption is debatable, especially because of the severe backreaction of the Hawking radiation, and the author does not have strong argument for it. It would be very interesting to try to prove/disprove this assumption, but we just assume it in this paper.

More precisely, we need the following assumption. Let us take two Cauchy surfaces $\Sigma_b$ and $\Sigma_a$ as in the righ side of Figure 3. (The subscripts mean “before” and “after” the evaporation.) Next we take tubular neighborhoods of these Cauchy surfaces which we denote as $O_b$ and $O_a$ such that they are far away from the spacetime singularity. The open set $O_b$ can be taken to be outside of the event horizon if one wishes so. Moreover, we can take them so that the intersection $O_c := O_a \cap O_b$ is not empty, $O_c \neq \emptyset$. This $O_c$ is located in a region which is space-like separated from the singularity.

Then we assume that

1. There exists a Heisenberg picture state $|\Psi\rangle$ describing the situation of Figure 3.

2. The states of the form $b|\Psi\rangle$ for $b \in \mathcal{A}(O_b)$ are dense in the Hilbert space of states before the black hole evaporation.

3. The states of the form $a|\Psi\rangle$ for $a \in \mathcal{A}(O_a)$ are dense in the Hilbert space of states after the black hole evaporation.

Here the second and third assumptions are a version of the “time-slice axiom”; see the axiom G. “Time-slice Axiom” of Section II.1.2 of [13].
How the state $|\Psi\rangle$ “looks like” before the evaporation is just determined by the initial state of the gravitational collapse. For example, we may take it to be a collapsing big star.

On the other hand, how the state $|\Psi\rangle$ “looks like” after the evaporation is a highly nontrivial version of the problem which is analogous to the one considered in the harmonic oscillator as in (2.14). Basically it consists of the Hawking radiation and some Planck scale effects, but we do not need its explicit form in the following discussion.

Now we consider another state $|\Phi\rangle$ in which we throw additional matter into the black hole. We assume that the amount of the matter is small so that the backreaction to the geometry is negligible. This state is well described by a state of the form $b|\Psi\rangle$ for $b \in \mathcal{A}(\mathcal{O}_b)$ to an arbitrary good accuracy.

If we look back the proof of the Reeh-Schieder theorem, what was shown there is that the states of the form $c|\Psi\rangle$ for $c \in \mathcal{A}(\mathcal{O}_c)$ are dense in the states of the form $b|\Psi\rangle$ for $b \in \mathcal{A}(\mathcal{O}_b)$, which are dense in the total Hilbert space (at least in the approximation of the fixed background geometry). Thus our state $|\Phi\rangle$ is very well described by a state of the form $c|\Psi\rangle$ to an arbitrary good accuracy. Now, because $\mathcal{O}_c \subset \mathcal{O}_a$, we have $c \in \mathcal{A}(\mathcal{O}_c) \subset \mathcal{A}(\mathcal{O}_a)$.

Then we notice that the state $c|\Psi\rangle$ makes sense as a state vector in the Hilbert space after the evaporation of the black hole. We propose that this is indeed the state vector after the black hole evaporation. Therefore, we get the pure state $c|\Psi\rangle$ without any loss of the information of the additional matter thrown into the black hole, once we assume that $|\Psi\rangle$ itself is pure. As discussed in the introduction, a big black hole may be reached by successively throwing a small amount of matter, and hence we may reduce the information problem to a Planck size black hole.

Notice that the Reeh-Schlieder theorem is just an existence proof of the operator $c \in \mathcal{A}(\mathcal{O}_c)$ and it does not tell us how the operator $c$ looks like. The fact that the region $\mathcal{O}_c$ is far away from the black hole and we are thinking about throwing something into the black hole means that this operator can be extremely complicated. One may call it a version of scrambling of the information. In any case, via the operator $c \in \mathcal{A}(\mathcal{O}_c)$ we may connect the world before and after the black hole evaporation, without requiring any radical new ideas. The Reeh-Schlieder theorem is already radical enough.

If we apply the above arguments to approximate global symmetry charges such as the baryon number of the standard model, one gets an apparent contradiction whose resolution may suggest some properties of the black hole evaporation. We leave the discussion of this point to future work.

Finally, let us notice an important analogy with the AdS/CFT correspondence. The AdS/CFT correspondence says that we can generate all states in the Hilbert space by acting boundary CFT operators to a state (in which the Reeh-Schlieder theorem holds). Indeed, the region $\mathcal{O}_c$ is near the spatial infinity $r \to \infty$ as is clear from Figure 3. The analogy becomes even more clear if we put a black hole in AdS and consider a version of the AdS/CFT dictionary between bulk operators $\phi(r,x)$ and boundary operators $\phi(x)$ given by $\lim_{r \to \infty} r^2 \partial^2 \phi(r,x) = \phi(x)$ [24, 25]. Therefore, our discussion should be essentially the same as the argument that there is no information loss in CFT and hence no information loss in AdS. The point of our discussion is that we may be able to discuss the absence of information loss without using the CFT dual, but instead by using the Reeh-Schlieder
theorem and the time-slice axiom in the bulk. Indeed, our discussion also makes sense (in
the approximation of a fixed background) in asymptotically flat spacetimes in which the
existence of a dual theory is not clear. (See e.g. [26] for an attempt towards this direction.)

By using the above understandings, the ER=EPR proposal of [17] has a very natural
justification, where ER means Einstein-Rosen bridge and EPR means Einstein-Podolsky-
Rosen entanglement. Let us discuss it in the most basic case of a two-sided AdS-Schwartzchild
black hole. We start from the boundary of one side of the black hole. Consider boundary
operators on that boundary. We slightly move them into the bulk by the dictionary men-
tioned above. Then, the bulk Reeh-Schlieder theorem says that we can generate a dense
subspace of the Hilbert space by these operators. We can move the region of the support
of these operators to the other side by going through the worm hole. Then we reach the
other boundary. This consideration implies that the state \( |\Psi\rangle \) has the following property.
For an arbitrary given operator \( a' \) on one of the boundaries, we can find an operator \( a'' \)
on the other side of the boundaries such that \( a'|\Psi\rangle \) is well approximated by \( a''|\Psi\rangle \) to an
arbitrary good accuracy. This is possible only if the two boundaries are entangled with
each other. Here it is crucial that the two sides are connected (by the worm hole), since
the proof of the Reeh-Schlieder theorem requires analytic continuation from a local region
to a tubular neighborhood of a Cauchy surface. In summary, we can say that when the
geometry is connected, the two boundaries are entangled by the Reeh-Schlieder theorem.
In this basic case the state is explicitly given by (2.27) which is clearly entangled, but the
above discussion is more general (though less quantitative). This is our interpretation of
ER=EPR.

Acknowledgments

The work of K.Y. is supported by JSPS KAKENHI Grant-in-Aid (Wakate-B), No.17K14265.

Acknowledgments

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