Restricted Minimum Condition in Reduced Commutative Rings

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Abstract. We say that a commutative ring $R$ satisfies the restricted minimum (RM) condition if for all essential ideals $I$ in $R$, the factor $R/I$ is an Artinian ring. We will focus on Noetherian reduced rings because in this setting known results for RM domains generalize well. However, as we will show, RM rings need not be Noetherian and may have nilpotent elements. One of the classic results in the theory of RM rings is that for Noetherian domains the RM condition corresponds to having Krull dimension at most one. We will show that this can be generalized to reduced Noetherian rings, thus proving that affine rings corresponding to curves are RM. We will give examples showing that the assumption that the ring is reduced is not superfluous. In the second part, we will study CDR domains, i.e., domains where for any two ideals $I, J$ the inclusion $I \subseteq J$ implies that $I$ is a multiple of $J$. We will prove that CDR domains are RM and this will allow us to give a new characterization of Dedekind domains. Examples of RM rings for various classes of rings will be given. In particular, we will show that a ring of polynomials $R[x]$ is RM if and only if $R$ is a reduced Artinian ring. And we will study the relation between RM rings and UFDs.

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1. Introduction

Unlike general lattice, the descending chain condition on right ideals for an associative unital ring implies the ascending chain condition, as is shown by the classical Hopkins–Levitzki theorem [4, Thm. 15.20]. For the purposes of this article, it is enough to consider the commutative version due to Akizuki.

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It can be proved as a special case of Hopkins–Levitzki but it is historically older. Among other sources, a proof can be found in Ref. [6].

**Theorem 1.** (Akizuki) A commutative ring $R$ is Artinian iff it is Noetherian and all prime ideals are maximal.

In particular, an Artinian ring has a composition series. In some settings, the Artinian condition is too strong. Reduced commutative rings are Artinian only in the trivial case, i.e., if they are a finite product of fields. It is well known that a module is Artinian iff all its factors are Artinian. The restricted minimum condition generalizes Artinian modules, requiring only *some* of the factors to be Artinian.

This concept was first used during the 1950s in the work of Cohen [6], who studied commutative rings whose factors by non-zero ideals are Artinian rings. Ornstein [14] generalised this concept to non-commutative rings. In the 2000s, Király studied this condition for group rings [12].

We will call the domains with this condition *RM domains*. This concept proved itself to be a useful tool in the study of Dedekind domains as it allowed both for proofs of new theorems and simpler proofs of known facts—see Ref. [6] for details. We will use this method to prove that a domain is a containment-division ring (CDR) iff it is a Dedekind domain [Thm. 12].

By Akizuki, all factors of RM domains are also Noetherian, hence RM domains are Noetherian too. Cohen proved that domains of finite rank are RM. For local domains, the converse is also true [6, Thm. 10, 9]. He also observed the following corollary of Akizuki’s theorem:

**Corollary 2.** A Noetherian domain is RM iff its Krull dimension is at most one, i.e., non-zero prime ideals are maximal

However, in many aspects Cohen’s definition is insufficient. If such a commutative ring has zero-divisors, it has to be Artinian [6, Cor. 1]. We also see that such rings are not closed under finite products.

In the module-theoretic setting, a module is said to satisfy the restricted minimum condition if its factors by *essential* submodules are Artinian modules. A submodule is essential if it has a non-zero intersection with all non-zero submodules. This concept was studied, among others, by Refs. [1,9,11,13].

**Definition.** Let $R$ be a commutative ring.

We say that $R$ satisfies the restricted minimum condition, or that it is an *RM ring*, if all factors by essential ideals are Artinian rings.

Equivalently, $R$ is an RM ring iff it is an RM module when considered as a module over itself.

The class of RM rings is easily seen to be closed under factors and finite products. Any finitely generated module over an RM ring is RM. See Ref. [11] for details.

Important examples of RM rings (even in a non-commutative case) are Noetherian hereditary rings [5, Thm. 2.5]. However, an RM ring needs not to
be Noetherian [Sect. 2.2] and can have infinite global homological dimension [Example 16].

Any ideal containing a regular commutative element (i.e., an element that is not a zero divisor) is an essential ideal. We see that for a domain our definition is equivalent to Cohen’s. Hence the study of RM domains is essentially a study of Noetherian domains of dimension at most one. We will see that Corollary 2 generalizes to reduced rings [Thm. 5], hence the affine rings corresponding to algebraic curves all possess the RM property [Cor. 6].

Over the last decade, there was significant progress in the theory of modules over RM domains, generalizing some of the known results about torsion self-small abelian groups, see Refs. [1,2]. Some of these results were later generalized to commutative rings by Kosan and Žemlička [13].

**Theorem 3.** [13, Thm 3.7] The following are equivalent for a commutative ring $R$.

(i) $R$ is an RM ring.

(ii) If $M$ is a singular module, then $M = \bigoplus_{P \in \text{Max}(R)} M[P]$, where $M[P]$ denotes the sum of all finite-length submodules of $M$ whose composition factors are isomorphic to $R/P$.

(iii) $R/\text{Soc}(R)$ is Noetherian and every self-small singular module is finitely generated.

The aim of this paper is to study RM rings in terms of their internal properties, continuing the work of Cohen with the aid of results from the general theory of RM modules, mainly [13]. We will be particularly interested in rings without non-zero nilpotent elements. We accompany the theory with constructions of several examples, focusing on rings of polynomials and their factors.

In Sect. 2, we will generalize Cohen’s characterization of RM domains to reduced Noetherian rings. We will show that the ring $R[x]$ of polynomials is RM iff $R$ is reduced Artinian. And we will construct an example of a reduced RM ring that is not Noetherian.

In Sect. 3, we will study the relation of RM domains with various important classes of domains such as Dedekind domains, unique factorization domains (UFDs) and principal ideal domains (PIDs). We will prove that containment-division domains are RM domains. Then we will utilize this fact in proving that Dedekind domains can be characterized as CDR domains. We will show that GCD domains and domains of Laurent polynomials are RM domains iff they are principal ideal domains.

### 2. Reduced RM Rings

From now on, the word ring will always mean commutative ring. For an ideal $I$, the intersection of all prime ideals containing $I$ is called the radical of $I$, denoted by $\sqrt{I}$. A ring is reduced if it has no non-zero nilpotent elements, i.e., $\sqrt{0} = 0$. A ring is a *domain* if it has at least two elements and the zero element is the only zero divisor, i.e., the zero ideal is prime.
In this section, we study the RM condition for reduced RM rings. We start by recalling some of the theory needed. In Sect. 2.1 we will present our results on Noetherian reduced rings. In Sect. 2.2 we construct an example of a reduced RM ring that is not Noetherian.

For most of this section, we will study rings using their lattice of ideals. Let $R$ be a ring with an ideal $I$. If $E/I$ is an essential ideal in $R/I$ then $E$ is essential in $R$. Since factors by prime ideals are domains, it follows that all ideals containing a prime ideal are essential. Minimal primes may or may not be essential:

**Example 4.** In domains, the unique minimal prime ideal, i.e., the zero ideal, is not essential.

Minimal primes contain the nilradical $\sqrt{0}$. If it is a non-zero ideal, it is an essential ideal, hence also minimal primes are essential.

In $\mathbb{Z}_5[x]/(x^2)$ the nilradical is the minimal prime ideal. In $\mathbb{Z}_6[x]/(x^2)$ the nilradical is not prime, consider $[x + 2]_{(x^2)} \cdot [x + 3]_{(x^2)} = [5x]_{(x^2)}$, so its minimal primes strictly contain an essential ideal.

For a prime ideal $P \leq R$, its *height* is the supremum of lengths of chains of prime ideals strictly included in $P$. The *Krull dimension of $R$*, abbreviated as $Kdim(R)$, is the supremum of heights of its primes. The Krull dimension of an ideal $I$ is $Kdim(R/I)$.

Fields have dimension zero. Dedekind domains that are not fields have dimension one. If $R$ is a Noetherian ring and $R[x]$ its ring of polynomials, then $Kdim(R[x]) = Kdim(R) + 1$ [8, Cor. 10.13].

The Krull dimension of RM rings is at most one. Take any prime ideal $P$ in an RM ring $R$. RM rings are closed under taking factors, so $R/P$ is an RM domain thus all non-zero prime ideals are maximal. Hence, the chain of primes containing $P$ can have length at most one.

### 2.1. Reduced Noetherian Rings

As observed by Cohen, any RM domain is necessarily Noetherian. This is not true for rings with zero divisors, as shown by Example 9. A Noetherian RM domain is either field or it has dimension one, and these are also sufficient conditions. We will now generalize this observation to reduced rings. The assumption that the ring is reduced is not superfluous, as shown by Example 8.

**Theorem 5.** Suppose $R$ is a Noetherian reduced ring.

Then $R$ is an RM ring iff $Kdim(R) \leq 1$.

**Proof.** We have already seen that any RM ring has dimension at most one. To prove the other implication, we will show that minimal prime ideals are not essential. Thus, any factor by an essential ideal is a Noetherian ring of dimension at most zero, hence it is Artinian by the Akizuki theorem.

Because $R$ is a reduced ring, the zero ideal is equal to the intersection of all prime ideals. The dimension of $R$ is finite, so it is enough to consider minimal primes in this intersection. Because $R$ is Noetherian, there are
only finitely many minimal primes. ([10, Thm. 88]). We will denote them $P_1, \ldots, P_n$.

We have $0 = \bigcap_{i \leq n} P_i$. We will show that for any $i \leq n$, the ideal $P_i$ is not essential.

Set $Q_i := \bigcap_{j \neq i} P_j$. Because $Q_i \cap P_i = 0$, it is enough to show that $Q_i \neq 0$.

If $Q_i = 0$ then $Q_i \subseteq P_i$. Since $P_i$ is a prime ideal, this implies that there is $j \in \{1, \ldots, n\}$ such that $j \neq i$ and $P_j \subseteq P_i$. However, $P_i$ is a minimal prime so $P_i = P_j$, contradicting $i \neq j$. □

Recall that $I \leq R$ is a radical ideal iff the factor $R/I$ is a reduced ring.

Using the Hilbert’s Basis theorem, we obtain the following consequence of Theorem 5, showing a relation between the class of RM rings and algebraic curves.

**Corollary 6.** Let $R$ be a Noetherian ring and $R[x]$ be a ring of polynomials in finitely many variables over $R$. Take a radical ideal $I \leq R[x]$ of dimension at most one.

Then the factor $R[x]/I$ is an RM ring.

Take a ring $R$ and consider $R[x]$, its ring of polynomials. By the Hilbert Basis Theorem, $R[x]$ is Noetherian iff $R$ is. We now aim to characterize when $R[x]$ is an RM ring. Recall that a reduced ring is Artinian iff it is a finite product of fields.

**Proposition 7.** The ring $R[x]$ of polynomials is an RM ring iff $R$ is a reduced Artinian ring.

**Proof.** Take a field $F$. Then $F[x]$ has dimension 1. $F[x]$ is Noetherian and reduced, hence by Proposition 5, an RM ring.

For any two rings $S$ and $T$, there is an isomorphism $S[x] \times T[x] \cong (S \times T)[x]$. Because RM rings are closed under finite products, we conclude that if $R$ is a finite product of fields, then $R[x]$ is an RM ring.

For the opposite direction, suppose that $R[x]$ is an RM ring. If $R$ contains a proper essential ideal $E$, then $E[x]$, the ideal of polynomials with coefficients in $E$, is a proper essential ideal in $R[x]$. By the isomorphism $R[x]/E[x] \cong (R/E)[x]$, we conclude that $(R/E)[x]$ is an Artinian ring. However, rings of polynomials are never Artinian as shown by the chain of ideals $(x) \supseteq (x^2) \supseteq (x^3) \supseteq \cdots \supseteq 0$.

We see that $R$ has no proper essential ideals. It follows that $Soc(R) = R$. Otherwise, there would be a maximal ideal containing the socle. But such an ideal is clearly proper and essential. We see that $R$ is semisimple, hence a finite direct of fields. □

**Example 8.** Let $F$ be a field and set $R := F[x]/(x^2)$.

By Proposition 7, $F[x]$ is an RM ring. Because $x^2$ is a regular element in $F[x]$, we conclude that $R$ is an Artinian ring, hence $Kdim(R) = 0$. The ring of polynomials $R[y]$ is then Noetherian of dimension 1.

However, $R$ contains nilpotent elements, for example $[x](x^2)$, so by the previous proposition, $R[y]$ is not an RM ring.
2.2. RM Rings Without the Maximal Condition

As discussed above, any RM domain is Noetherian. We will now show that this is not true for rings with zero divisors.

Kosan and ˇZemliˇcka showed that if \( R \) is an RM ring then \( R/Soc(R) \) is Noetherian [13, Thm. 3.4]. As a corollary, we see that an RM ring \( R \) is Noetherian iff \( Soc(R) \) is Noetherian as an \( R \)-module. Observe, that if \( Kdim(R) = 0 \) and \( R/Soc(R) \) is Noetherian then \( R \) is RM by the Akizuki theorem.

Example 9. Let \( R \) be the set of all finite and cofinite subsets of \( \mathbb{N} \).

Define addition on \( R \) as symmetric difference and define the product of two sets to be their intersection.

Then \( R \) has a structure of a reduced RM ring that is not Noetherian.

**Proof.** For any set \( S \in R \), we can define an ideal \( I(S) \) to be the set of all finite and cofinite subsets of \( S \). We see that \( R \) is not Noetherian, as we can consider the infinite ascending chain \( I(\{1\}) \subsetneq I(\{1,2\}) \subsetneq I(\{1,2,3\}) \subsetneq \cdots \subsetneq R \).

Take any ideal \( J \leq R \) and a set \( S \in J \), then the ideal \( I(S) \) is contained in \( J \). Hence simple ideals of \( R \) are the ideals generated by singletons. We conclude that \( Soc(R) \) is the ideal consisting of all finite sets.

Let \( \pi \) be the canonical projection onto \( R/Soc(R) \). If \( F \in R \) is a finite set, then \( \pi(F) = 0 \). If we take two cofinite sets \( X,Y \in R \), then \( X - Y \) is a finite set, i.e., an element of the socle; hence, all cofinite sets have the same \( \pi \)-image. The RM condition in \( R \) then follows from \( R/Soc(R) \) being the two-element ring.

\[ \square \]

3. RM Domains

A domain is Artinian iff it is a field. The concept of an RM domain gives a weaker condition. Infinite strictly descending chains of ideals are allowed, but only if the intersection of all ideals in the chain is the zero ideal.

Example 10. Let \( R \) be a principal ideal domain (PID).

Take an element \( a \in R \) and an infinite descending chain \( (a_1) \supseteq (a_2) \supseteq \cdots \supseteq (a) \). If \( a \neq 0 \) then this chain stabilizes. Take \( r_i \in R \) such that \( a_ir_i = a \). \( R \) is a PID, in particular it is Noetherian. Hence, the ascending chain \( (r_1) \subseteq (r_2) \subseteq \cdots \subseteq R \) stabilizes, so the original chain stabilizes too. We see that PIDs are RM.

Later, we will generalize this observation to containment-division domains (CDRs).

We say that a ring is *semilocal* if it has finitely many maximal ideals. As a corollary of the Akizuki theorem, any Artinian ring is semilocal. Let \( D \) be a domain that is not a field. By the Chinese-Remainder theorem, if \( D \) is semilocal then \( J(D) \neq 0 \). The converse is not true in general but holds for RM domains. If \( J(D) \neq 0 \) in an RM-domain then the Artinian factor \( D/J(D) \) is semilocal, hence \( D \) itself is semilocal.
3.1. CDR Domains

We say that a domain \( R \) is a \textit{Dedekind domain} if every non-zero proper ideal can be written as a product of finitely many (not necessarily distinct) maximal ideals. Take \( I \subseteq R \) an ideal. Because \( R \) is RM, there is only finitely many maximal ideals containing \( I \). It follows that factors of Dedekind domains have only finitely many ideals.

RM domains appeared as a generalization of Dedekind domains in several contexts. Cohen \cite{6} characterizes Dedekind domains as integrally closed RM domains. Recalling that Dedekind domains can be characterized as (Noetherian) hereditary domains, the RM condition for Dedekind domains follows from the work of Chatters \cite[Thm. 2.2]{5}. Albrecht and Breaz characterized torsion modules over RM domains, generalizing characterization of torsion modules over Dedekind domains \cite[Thm. 6]{1}.

Let \( I, J \) be two ideals. We say that \( I \) \textit{divides} \( J \), denoted \( I \mid J \), if there is \( H \leq R \) such that \( IH = J \). A ring will be called \textit{containment-division ring}, or \textit{CDR} for short, if for any proper ideals \( I \supseteq J \), it holds that \( I \mid J \). Sometimes in the literature, CDRs are called \textit{multiplication rings}.

It is well known that Dedekind domains are CDR. In Theorem 12, we will show that this property actually characterizes Dedekind domains. This proposition sometimes appears in the literature, see Ref. \cite[Thm. 3.10.]{3}, and can be seen as a consequence of Ref. \cite[Thm. 6]{1}. We, however, aim to give a simpler proof using the notion of an RM domain.

For a domain \( R \), denote \( F \) its field of fractions. An \( R \)-submodule \( M \) of \( F \) will be called a \textit{fractional ideal} if there exists an element \( r \in R \) such that \( rM \subseteq R \). Ideals of \( R \) are fractional ideals. We say that a fractional ideal \( M \) is invertible if there exists another fractional ideal \( N \) such that \( MN = R \).

Proposition 11. \textit{CDR domains are RM domains}

\textit{Proof.} Consider a CDR domain \( R \) and its field of fractions \( F \). For any non-zero ideal \( I \) in \( R \) there exists a principal ideal \( 0 \neq (i) \) such that \((i) \subseteq I \). Because \( R \) is a CDR, there exists the ideal \( H \) such that \( HI = (i) \). The element \( i \) is regular, hence also invertible in \( F \). This means that the ideal \((i)\) is invertible. Since multiplication of ideals is associative, we see that \( I \) is invertible too.

We have seen that all non-zero ideals of \( R \) are invertible as a fractional ideals. It follows that any ideal \( I \leq R \) is finitely generated. Consider \( J \) fractional ideal, the inverse of \( I \). There exist \( a_i \in I \) and \( b_i \in J \) such that \( 1 = a_1b_1 + \cdots + a_nb_n \). If we multiply the equation by \( a \in I \), we get \( a = a_1(ab_1) + \cdots + a_n(ab_n) \). Because \( I \) is the inverse ideal to \( J \), elements \( ab_i \) lie in \( R \). We obtain that any CDR domain is Noetherian.

Now consider a decreasing chain of ideals \( R \supseteq I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I \neq 0 \) \( I_i+1 \subseteq I_i \) so by \( R \) being a CDR there exists the ideal \( H_i \) such that \( I_{i+1} = I_iH_i \).

Denote \( J_i \) the ideal of \( R \) such that \( J_iI_i = I \). We have

\[ I_iJ_i = I_{i+1}J_{i+1} = I_iH_iJ_{i+1} \]
Ideal $I_i$ is invertible so we obtain $J_i = HJ_{i+1}$ hence $J_0 \subseteq J_1 \subseteq J_2 \cdots \subseteq R$. Because $R$ is Noetherian, this chain stabilizes so $I_0 \supseteq I_1 \cdots \supseteq I$ stabilizes too.

**Theorem 12.** A domain $R$ is a Dedekind domain iff it is a CDR.

**Proof.** Suppose $R$ is a CDR domain and consider a non-zero ideal $I$ in $R$. We want to write $I$ as a product of finitely many maximal ideals. If $I$ is maximal, we are done. If $I$ is not maximal, then there exists a maximal ideal $P_1$ containing $I$. Since $R$ is a CDR there is $I_1$ such that $P_1I_1 = I$. If $I_1$ is not maximal we can find a maximal ideal $P_2$ and another ideal $I_2$ such that $P_1P_2I_2 = I$ and so on.

This process necessarily stops, giving us the desired decomposition. Suppose this is not true. Then we have an infinite descending chain $P_1 \supseteq P_1P_2 \supseteq P_1P_2P_3 \cdots \supseteq I$. Since any CDR domain is an RM domain we see that the factor $R/I$ is Artinian so the chain stabilizes.

On the other hand, suppose that $R$ is a Dedekind domain and we have two non-zero ideals $I \subseteq J$. Both ideals have decompositions into maximal ideals $I = P_1 \cdots P_n \subseteq Q_1 \cdots Q_m = J \subseteq Q_1$.

The ideal $Q_1$ is prime hence there is $i$ such that $P_i$ is contained in $Q_1$. Because $P_i$ is a maximal ideal, we have $P_i = Q_1$. We can assume that $i = 1$. By Ref. [8, Cor. 11.9.], ideal $Q_1$ is invertible. We get $P_2 \cdots P_n \subseteq Q_2 \cdots Q_m$.

By repeating the argument, we get $J = I \cdot Q_{n+1} \cdots Q_m$, concluding that $R$ is a CDR.

3.2. Principal Ideal Domains

In this section, we turn our attention to unique factorization domains (UFDs) and PIDs. By Example 10, PIDs are RM domains. Because a UFD needs not to be Noetherian (take a ring of polynomials in infinitely many variables over a UFD) it is clear that there exist UFDs that are not RM domains. We will show examples of Noetherian UFDs that are not RM domains and examples of RM domains that are not UFDs.

We say that a domain is a GCD if for any two regular elements the greatest common divisor exists. Recall that a Dedekind domain is a GCD iff it is a PID. This can be easily generalized to RM domains. If $R$ is a GCD and RM it is, in particular, Noetherian GCD thus it is a UFD. All UFDs are integrally closed [7, Thm 2.3] so we have proved:

**Proposition 13.** Let $R$ be a GCD. Then it is a PID iff it is an RM domain.

Király [12, Thm. 1.2] showed that a ring of Laurent polynomials over a field is an RM domain. We now prove the converse of this observation.

**Proposition 14.** Let $R$ be a domain. The ring of Laurent polynomials $R[x, x^{-1}]$ is an RM domain iff it is a PID, which is equivalent to $R$ being a field.

**Proof.** If $R$ is a field, the ring $R[x]$ of polynomials is a PID. Ring $R[x, x^{-1}]$ can be viewed as the localization of $R[x]$ at $x$ so it is also a PID, hence an RM domain.
Suppose that \( R[x, x^{-1}] \) is an RM domain. Denote \( I \) the kernel of the evaluation homomorphism \( f : R[x, x^{-1}] \to R \) sending \( x \) to the identity element in \( R \). Then \( R[x, x^{-1}]/I \cong R \). We see that \( R \) is an Artinian domain hence a field. \( \square \)

We will now utilize Corollary 6, to get some concrete examples of RM domains. Recall that \( R[x] \) is a UFD if and only if \( R \) is a UFD.

**Corollary 15.** Let \( R \) be a Dedekind domain and \( f \in R[x] \) an irreducible monic polynomial. Then \( R[x]/(f) \) is an RM ring.

**Proof.** By Theorem 12, Dedekind domain is an RM domain, hence \( Kdim(R) \leq 1 \). The ring of polynomials \( R[x] \) has dimension at most two. \( R[x] \) is a domain so non-zero prime ideals in \( R[x] \) have dimension at most one.

Dedekind domains are integrally closed. So any principal ideal generated by monic irreducible polynomials is prime \([8, \text{Cor. 4.12}]\). By Corollary 6, factors by such prime ideals are RM rings. \( \square \)

**Example 16.** The ring of integers \( \mathbb{Z} \) is a PID, hence a Dedekind domain.

Consider \( s \in \mathbb{N} \) that is not a square of an integer.

Then the ring \( \mathbb{Z}[\sqrt{-s}] \cong \mathbb{Z}[x]/(x^2 + s) \) is an RM domain.

It is known that some of these rings are not integrally closed (e.g., \( \mathbb{Z}[\sqrt{-3}] \)) so we obtain examples of RM domains that are not Dedekind.

**Example 17.**

1. The ring \( \mathbb{Z}[x] \) is a Noetherian UFD. By Proposition 7 it is not an RM domain.
2. By Example 16, the ring \( \mathbb{Z}[\sqrt{-14}] \) is an RM domain.
   
   It is not a UFD as \( 15 = 3 \cdot 5 = (1 + \sqrt{-14})(1 - \sqrt{-14}) \).
3. The ring \( \mathbb{Z}[\sqrt{-14}][x] \) is Noetherian. It is neither a UFD nor an RM domain.

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