ON THE HOMOTOPY COLIMIT DECOMPOSITION FOR QUOTIENTS OF MOMENT-ANGLE COMPLEXES AND ITS APPLICATIONS

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Abstract. In this paper we prove that the quotient of any real or complex moment-angle complex by any closed subgroup in the naturally acting compact torus on it is equivariantly homotopy equivalent to the homotopy colimit of a certain toric diagram. For any partial quotient this homotopy equivalence rises to an equivariant homeomorphism generalizing the well-known Davis-Januszkiewicz construction for quasitoric manifolds and small covers. We prove formality of the corresponding Borel construction space under the natural assumption on the group action in the complex case leading to the new description of the equivariant cohomology for the quotients by any coordinate subgroups. We deduce the Toral Rank Conjecture for the partial quotient of the moment-angle complex over any skeleton of the simplex by the diagonal circle action from previously known results, and give an explicit construction of partial quotients having arbitrary torsion in integral cohomology.

1. Introduction

Geometry and topology of moment-angle complexes and manifolds, and of quasitoric manifolds and small covers introduced in the seminal paper [17] is one of key points of study in toric topology. The class of partial quotients is a wide family of topological spaces arising in toric topology, which includes all moment-angle complexes on one side, and on other side, all quasitoric manifolds and small covers. The term “partial quotient” was introduced in [2] for the quotient space of the complex moment-angle complex \( Z_K = (D^2, S^1)^K \) by any freely acting subtorus in \( T^m = (S^1)^m \), where \( K \) is a simplicial complex on the vertex set \( [m] = \{1, 2, \ldots, m\} \). Note that in [22] the quotient of the complex moment-angle complex \( Z_K \) by an arbitrary closed subgroup (that is, a quasitorus) in \( T^m \) acting freely on \( Z_K \) was called a partial quotient. In this paper a partial quotient of the (real or complex, respectively) moment-angle complex \((D^d, S^{d-1})^K\), \( d = 1, 2\), is the corresponding quotient by the action of any freely acting closed subgroup \( H_d \) from \((G_d^m)^m\), where \( G_1 := \mathbb{Z}/2\mathbb{Z} \) and \( G_2 := S^1 \) (see [17]).

The notion of a polyhedral product introduced in [7] is an instance of a colimit for a certain diagram of topological spaces over a small category \( \text{cat} K \). The categorical approach to polyhedral products [35] includes the homotopy equivalence of any moment-angle complex [35] and of any quasitoric manifold [42] to the homotopy colimits of toric diagrams in the terminology of [32]. This elegant approach has several applications. For example, it implies that any quasitoric manifold is rationally formal space [36] and that any complex Davis-Januszkiewicz space is formal [35].

Moment-angle complexes and quasitoric manifolds have already found numerous valuable applications in homotopy theory [9, 13, 18, 25, 26, 30–32], cobordism theory [15, 16, 33], hyperbolic geometry [12], combinatorial commutative algebra [8, 34]. Unlike these two particularly important families of partial quotients, geometry and topology of general partial quotients is still far from being well-understood. Several authors attacked the problem of describing cohomology rings of general partial quotients [14, 21]. For example, a complete and rigorous argument giving the multiplicative structure in the cohomology ring \( H^*(Z_K/H; \mathbb{Z}) \) of any partial quotient was given only recently in [22].

In addition to moment-angle complexes and quasitoric manifolds, there are two particularly interesting classes of partial quotients introduced recently. The first class appeared in [29] and contains all moment-angle complexes and pullbacks from linear models [17]. A Hochster type formula for the respective cohomology groups and, moreover, astable decomposition similar to that of polyhedral products [7] were proven in this class [29]. The second class introduced in [28] consists of the quotients for the moment-angle complex over the \( k \)-skeleton \( \Delta^k_m \) of the \((m-1)\)-dimensional simplex \( \Delta^{m-1} \) with the set \([m]\) of vertices by the action of the diagonal circle subgroup \( S^1_k \) from \( T^m \), where \( k, m \geq 0 \) are any integers. A sophisticated homotopy decomposition was proved for any element of this class in [28].

The first main result of this paper is given as follows (for the precise definition of the right-hand side see [22]).

Theorem 1. For any \( d = 1, 2 \), any closed subgroup \( H_d \) in \( G_d^m \) and any simplicial complex \( K \) on \([m]\) there is the homotopy equivalence of spaces

\[
(D^d, S^{d-1})^K / H_d \simeq \text{hocolim} \ G_d^m / (G_d^I / H_d),
\]

where \( H_d \cdot G_d^I \) is the subgroup generated by \( G_d^I := \prod_{i \in I} G_d \) and \( H_d \) in \( G_d^m \).

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The proof of Theorem 1 is given in §2. Theorem 1 gives a generalization of the homotopy colimit decompositions 37, 3 to the quotient of \((D^d, S^{d-1})_K\) by any closed subgroup in \(G^m_d\). The above homotopy equivalence is shown to be equivariant, and in the particular case of a partial quotient it rises to an equivariant homeomorphism (Theorem 2.2). This result leads to the explicit \(L_d\)-CW-approximation for quotients of moment-angle complexes, where \(L_d := G^m_d/H_d\). The last result generalizes (see Corollary 2.19) the well-known Davis-Januszkiewicz construction 17 to the case of arbitrary partial quotients.

The second main result of this paper is as follows.

**Theorem 2.** For any \(d = 1, 2\), any closed subgroup \(H_d\) in \(G^m_d\) and any simplicial complex \(K\) on \([m]\) one has the following isomorphism of integral cohomology rings

\[
H^*_d((D^d, S^{d-1})_K/H_d) \simeq H^*(\text{colim } B(G^1_D/(G^1_D \cap H_d))).
\]

In order to prove Theorem 2 we construct the homotopy equivalence between the diagram for the Borel construction of the \(L_d\)-action on \((D^d, S^{d-1})_K/H_d\) and \(B(G^1_D/(G^1_D \cap H_d))\). We introduce the class of closed subgroups \(H_d\) in \(G^m_d\) satisfying a certain condition (see Condition 4.1) and show that it is strictly wider than the class of closed subgroups (in \(G^m_d\)) freely acting on \((D^d, S^{d-1})_K\). For any element \(H_d\) from this new class we deduce the commutation rule for the respective \(L_d\)-equivariant cohomology (Corollary 4.11) generalizing the limit description for the Stanley-Reisner ring \(\mathbb{Z}[K]\). For \(d = 2\) we prove formality of the respective Borel construction (Theorem 4.13) by following a similar argument to 35. As a further application of these results, we give a nice formula for the \(L_d\)-equivariant cohomology ring of the quotient \((D^d, S^{d-1})_K/G^1_D\) for the moment-angle complex by the action (not necessarily free) of the coordinate subgroup \(G^1_D\) for arbitrary \(I \subseteq [m]\) in terms of Stanley-Reisner rings (Corollary 4.3). Another application is a new proof of degeneration (at the second page) for the Eilenberg-Moore spectral sequence (Proposition 4.19) for the fiber inclusion of the respective Borel construction. This generalizes the well-known case of partial quotients (for example, see 22).

The third and final main result of this paper is as follows.

**Theorem 3.** Let \(G\) be any finitely generated Abelian group. Then there exist a simple polytope \(P \subseteq \mathbb{R}^n\) with \(m\) facets and a circle subgroup \(H \subseteq T^m\) such that \(H\) acts freely on the moment-angle manifold \(\mathbb{Z}_P\) and \(H^*(\mathbb{Z}_P/H)\) contains \(G\) as a direct summand.

We prove Theorem 3 by using the Hochster type formula from 29. Furthermore, we show that the Toral Rank Conjecture holds for the class of partial quotients for moment-angle complexes over the skeleta of simplices by the action of the diagonal circle subgroup by examining the respective decomposition from 23. We finish the paper by a list of some related open problems.

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2. Homotopy colimit decomposition for quotients of moment-angle complexes

Unless explicitly stated otherwise, in this paper the cohomology groups of a topological space are the singular cohomology groups with integral coefficients. Given a simplicial complex \(K\) on the vertex set \([m] := \{1, 2, \ldots, m\}\), the objects of \(K\) together with the initial object (an empty set) form a small category \(\text{cat} K\) with the arrows induced by the natural inclusions of subsets from \([m]\). The formula \(I \in \text{cat} K\) means that \(I\) is an object of \(\text{cat} K\), that is, either \(I = \emptyset\) or \(I \subseteq K\) holds.

2.1. Algebraic preparation and definitions of some diagrams. The proof of the following lemma is straight-forward.

**Lemma 2.1.** Consider the following commutative diagram of abelian group homomorphisms, where each of \(a, a', c, c'\) is mono:

\[
\begin{array}{ccc}
A' & \xrightarrow{a'} & B' \\
\downarrow a & & \downarrow b' \\
A & \xrightarrow{a} & B \\
\downarrow c & & \downarrow d' \\
C & \xrightarrow{c} & D
\end{array}
\]
Then there is the following commutative diagram of group homomorphisms, where any row is exact:

\[
\begin{array}{ccc}
1 & \longrightarrow & A' \\
1 & \longrightarrow & A \\
1 & \longrightarrow & C'
\end{array}
\]

\[
\begin{array}{ccc}
1 & \longrightarrow & B' \\
1 & \longrightarrow & B \\
1 & \longrightarrow & D'
\end{array}
\]

\[
\begin{array}{ccc}
1 & \longrightarrow & B'/A' \\
1 & \longrightarrow & B/A \\
1 & \longrightarrow & D'/C'
\end{array}
\]

Following [17], we make use of the notation:

\[
G_d := \begin{cases} 
\mathbb{Z}/2\mathbb{Z}, & d = 1, \\
S^1, & d = 2, \\
\mathbb{C}, & d = 2, \\
\mathbb{Z}, & d = 2.
\end{cases}
\]

We call any subgroup isomorphic to \(G_d^m\) for some \(d = 1, 2\) and \(m \geq 0\) a torus.

Throughout the paper we use the standard formalism of polyhedral products, see [3]. Let \(H_d\) be any closed subgroup in \(G_d^m\). By a slight abuse of the notation we identify the group \(G_d^m\) with the isomorphic coordinate subgroup \((G_d, 1)^I = \prod_{i \in I} G_d\) with \(I \subseteq [m]\).

Proposition 2.2. For any \(I \subseteq J \in \text{cat } K\) the following diagram

\[
\begin{array}{ccc}
1 & \longrightarrow & G_d^m/(G_d^I \cap H_d) \\
1 & \longrightarrow & G_d^I \cap H_d \\
1 & \longrightarrow & G_d^I/((G_d^I \cap H_d) \\
\end{array}
\]

is commutative and has exact rows.

Proof. There is the following short exact sequence of groups

\[
\begin{array}{ccc}
1 & \longrightarrow & G_d^I \cap H_d \\
1 & \longrightarrow & H_d \\
\end{array}
\]

so that \(\varphi_I(H_d) \subseteq G_d^m/G_d^I\) holds. Clearly, (2) is functorial with respect to \(I \in \text{cat } K\) (that is, the upper face of the diagram (3) below is commutative). One has an isomorphism

\[
(G_d^m/G_d^I)/\varphi_I(H_d) \simeq G_d^m/(H_d \cdot G_d^I).
\]

Then one constructs three out of four cubes of the following commutative diagram with exact rows and columns by applying Lemma [2.1] to the upper left cube:

\[
\begin{array}{ccc}
1 & \longrightarrow & G_d^I \cap H_d \\
1 & \longrightarrow & G_d^I \\
1 & \longrightarrow & G_d^I/((G_d^I \cap H_d) \\
\end{array}
\]

There are two different ways to define the right bottom cube of this diagram by applying Lemma [2.1]. A simple check verifies that these two cubes coincide. So the above diagram is well-defined and commutative. The necessary diagram (1) is then given by restricting (3) to the lowest face.

Definition 2.3. Define \(S_d, \kappa(G_d^m/H_d), Q_d\) to be the (cat \(K\))-diagrams of topological spaces such that the arrow corresponding to \(I \subseteq J \in \text{cat } K\) is given by left, central and right-most columns of the diagram (1), respectively, where \(\kappa(G_d^m/H_d)\) is the constant (cat \(K\))-diagram corresponding to \(G_d^m/H_d\).

In what follows we write \(Q_d \equiv Q_d(K, H)\) to indicate dependency of the diagram \(Q_d\) on \(K\) and \(H\) and use the similar notation for \(S_d\) and \(\kappa(G_d^m/H_d)\) if it is necessary.
Remark. Let $D \in \text{Top}$ be any diagram over a small category $\mathcal{C}$ with values in the category Top of topological spaces. Suppose that any object of $D$ is a torus and any its arrow is a group homomorphism. Then $D$ is called a toric diagram ([22]). The diagrams $S_d$, $\kappa(G^m_d/H_d)$, $Q_d$ are toric diagrams.

Corollary 2.4. There is the following sequence of (cat $K$)-diagram morphisms

$$
\kappa(1) \longrightarrow S_d \longrightarrow \kappa(G^m_d/H_d) \longrightarrow Q_d \longrightarrow \kappa(1),
$$
given by $\mathcal{I}$. Objectwise $\mathcal{I}$ is a short exact sequence of groups. The diagram $\mathcal{S}$ is cofibrant.

Proof. The homomorphism $S(I \to J): G^m_d/(G^m_d \cap H_d) \to G^m_d/(G^m_d \cap H_d)$ has a trivial kernel for any $I \subseteq J \subseteq K$, so $S_2$ is cofibrant by [22, Lemma 4.10, p.134]. The remaining claims are clear. 

We will make use of the next basic properties of tori.

Proposition 2.5. (i) Any closed subgroup $H_d$ of the torus $G^m_d$ is a quasitorus, that is, is isomorphic to the direct product of a finite set of tori and of a compact complex torus.

(ii) For any closed subgroup $H_d$ of the torus $G^m_d$ the natural exact sequence of groups

$$1 \to H_d \to G^m_d \to G^m_d/H_d \to 1,$$

splits iff $d = 1$ or $H_d$ is connected;

(iii) Quotient of a torus by any its closed subgroup is isomorphic to a torus.

2.2. Homotopy colimit description for $(D^d, S^{d-1})^K/H_d$ and subgroup arrangements. In what follows, throughout the paper we use the standard formalism of homotopy colimits for diagrams with values in (pointed) topological spaces. We refer to the sources [1], [19], [20], [42] for the foundations of the corresponding theory. Unless explicitly stated otherwise, throughout the paper we consider limits and colimits (as well as the corresponding homotopy analogues) over small categories cat$^{op}$ $\mathcal{K}$ and cat $\mathcal{K}$ and with values in the category of compactly generated Hausdorff topological spaces Top only, respectively. Often in the text below we refer to a (cat $K$) or cat$^{op}$ $\mathcal{K}$-diagram $D$ by referring to its objects $D(I)$ (as a function on $I$) for brevity.

Let $K$ be a simplicial complex on $[m]$. Recall that there is the (cat $K$)-diagram $(D^d, S^{d-1})^I$ of topological spaces $\mathcal{I}$ given by the maps

$$(D^d, S^{d-1})^I \to (D^d, S^{d-1})^J, \quad (D^d, S^{d-1})^I := \prod_{i \in I} D^d_i \times \prod_{j \in [m] \setminus I} S^{d-1}_j \subseteq \prod_{i = 1}^m D^d_i,$$

induced by the identity map $\text{Id}: D^d \to D^d$ and by the embedding of the boundary map $S^{d-1} = \partial D^d \to D^d$, where $I \subseteq J \subseteq \text{cat} \mathcal{K}$. The respective colimits

$$\mathcal{R}Z_K = (D^1, S^0)^K := \text{colim} (D^1, S^0)^I, \quad Z_K = (D^2, S^1)^K := \text{colim} (D^2, S^1)^I,$$

are called the real and complex moment-angle complex, respectively ([3]). Furthermore ([3]), the natural quotient homomorphisms

$$G^m_d/G^1_d \to G^m_d/G^1_d,$$

for any $I \subseteq J \subseteq \text{cat} \mathcal{K}$, form another (cat $K$)-diagram $G^m_d/G^1_d$ of spaces.

Proposition 2.6. ([37, 3] Proposition 8.1.5, p.316) The diagram $(D^d, S^{d-1})^I$ is cofibrant. One has the following homotopy equivalences:

$$(D^d, S^{d-1})^K \simeq \text{hocolim} G^m_d/G^1_d \simeq \text{colim} (D^d, S^{d-1})^I.$$

Example 2.7. Let $m = 2$, $d = 2$, $K = \{\{1\}, \{2\}\}$. Then $Z_K = S^3$. In this case, the homotopy colimit of $G^m_d/G^1_d$ over cat $\mathcal{K}$ is obtained by gluing the boundary components of the cylinder $I^1 \times T^2$ to two disjoint copies of $S^1$ by two different coordinate projections from $T^2 := (S^1)^2$ to $S^3$.

Let $H_d$ be any closed subgroup of the torus $G^m_d$. The subset $(D^d, S^{d-1})^I$ of $(D^d)^m$ is $G^m_d$-invariant with respect to the natural $G^m_d$-action on $(D^d)^m$. Hence, the subset $(D^d, S^{d-1})^I$ is $H_d$-invariant in $(D^d)^m$ for any $I \subseteq \text{cat} \mathcal{K}$. Then there are the induced embeddings of the orbit spaces

$$\text{(5) } (D^d, S^{d-1})^I/H_d \to (D^d, S^{d-1})^I/H_d,$$

where $I \subseteq J \subseteq \text{cat} \mathcal{K}$, forming the (cat $K$)-diagram $(D^d, S^{d-1})^I/H_d$ of spaces.

Proposition 2.8. The (cat $K$)-diagram $(D^d, S^{d-1})^I/H_d$ is cofibrant. One has

$$\text{(6) } (D^d, S^{d-1})^K/H_d = \text{colim}(D^d, S^{d-1})^I/H_d.$$

Proof. Any morphism in the diagram $(D^d, S^{d-1})^I/H_d$ is a closed immersion. Hence, the first claim follows by [22, Lemma 4.10, p.134]. The second claim follows from commutation of colimits by representing the quotient by $H_d$-action as the respective colimit.
Proposition 2.9. One has
\[ \text{hocolim } S_d \simeq \bigcup_{t \in K} G^t_d/(G^t_d \cap H_d) \subseteq G^m_d/H. \]

**Proof.** The claim follows trivially from cofibrancy of \( S_d \) by Corollary 2.4 due to the Projection Lemma from [12].

Denote by
\[ \pi_t: (D^d, S^{d-1})^t / G^m_d \rightarrow G^m_d/(G^t_d \cdot H_d), \]
the map induced by the identity map \( \text{Id}: S^d \rightarrow S^d \) and by the projection \( D^d \rightarrow 1 \). Notice that the equality
\[ \pi_t(gx) = \varphi_t(g)\pi_t(x), \]
holds for any \( g \in G^m_d \) and any \( x \in (D^d, S^{d-1})^t \). Hence, \( \pi_t \) is an equivariant map with respect to the respective \((H_d)\)- and \((\varphi_t(H_d))\)-actions. Therefore, the map \( \pi_t \) induces the map of the quotient spaces
\[ \tilde{\pi}_t: (D^d, S^{d-1})^t / H_d \rightarrow G^m_d/(G^t_d \cdot H_d). \]

**Proposition 2.10.** (i) The restriction
\[ G^m_d/H_d \rightarrow G^m_d/(G^t_d \cdot H_d), \]
of \( \tilde{\pi}_t \) to \( G^m_d/H_d \) is the trivial principal fiber bundle with the fiber \( G^t_d/(H_d \cap G^t_d) \).

(ii) The map \( \tilde{\pi}_t \) is the trivial fiber bundle associated with the principal fiber bundle from (i) under the natural action of \( G^t_d/(H_d \cap G^t_d) \) on \( (D^d)^t/(H_d \cap G^t_d) \), where \( (D^d)^t := \bigoplus_{i \in I} D^d_i \). The fiber of \( \tilde{\pi}_t \) is equal to \( (D^d)^t/(H_d \cap G^t_d) \).

**Proof.** By Proposition 2.4, the following sequence of groups
\[ 1 \rightarrow G^t_d/(H_d \cap G^t_d) \rightarrow G^m_d/H_d \rightarrow G^m_d/(G^t_d \cdot H_d) \rightarrow 1, \]
is exact. By Proposition 2.4 (ii), (iii), every group from the sequence (6) is a torus, and in particular, is a Lie group, and the short exact sequence (6) of groups splits. Hence, there exists a section of the short exact sequence (6). Therefore, (6) defines a trivial fiber bundle. This proves (i).

Let
\[ G := G^t_d/(H_d \cap G^t_d), \quad X := G^m_d/H_d, \quad Y := (D^d)^t/(H_d \cap G^t_d). \]
The group \( G \) acts on \( X \) by left translations as its subgroup, see [12]. The natural embedding \( G^t_d \rightarrow (D^d)^t \) is \( G^t_d \)-equivariant. Hence, this embedding induces the \( G \)-action on \( Y \). Recall that the standard action of \( G \) on \( X \times Y \) given by the formula \( g(x, y) := (gx, g^{-1}y) \) for \( x \in X \), \( y \in Y \) and \( g \in G \), has the orbit space \( X \times_G Y := (X \times Y)/G \).

Denote by \( d \times t' \) and \( t \times t' \) arbitrary elements of \( (D^d, S^{d-1})^t \) and \( G^m_d \), respectively, where \( d \in (D^d)^t \); \( t \in G^t_d \); \( t' \in G^m_d \). Then, for instance, the natural \( G^m_d \)-action on \( (D^d, S^{d-1})^t \) is given by the formula
\[ g \cdot (d \times t') = ((\varphi_r|_{(r)}(g) \cdot d) \times (\varphi_t(g) \cdot t')), \quad g \in G^m_d. \]

Consider the map
\[ \Psi: X \times_G Y \rightarrow (D^d, S^{d-1})^t / H_d, \]
\[ \left[ [(t \times t')_Hd, [d]_{Hd \cap G^t_d}] \right] \rightarrow [(t \cdot d) \times t']_Hd, \]
where, for instance, \([t \times t']_Hd\) denotes the \( H_d \)-orbit in \( X \) represented by \( t \times t' \). Any element from \( G^m_d \times (D^d)^t \) representing the same \( G \)-orbit in \( X \times_G Y \) as on the left hand side of (7) has the form \((gh \cdot (t \times t'), g^{-1}r \cdot d)\), where \( g \in G^t_d; h \in H_d; r \in H_d \cap G^t_d \). We compute the value of \( \Psi \) on this element as follows:
\[ \Psi \left( [(gh \cdot (t \times t'))_Hd, [g^{-1}r \cdot d]_{Hd \cap G^t_d}] \right)_G = \left[ (\varphi_r|_{(r)}(gh) \cdot t \cdot r \cdot d) \times (\varphi_t(gh)) \right]_Hd = \left[ (\varphi_r|_{(r)}(h) \cdot t \cdot r \cdot d) \times (\varphi_t(h)) \right]_Hd = \left[ rh \cdot (t \cdot d) \times t' \right]_Hd.
\]
In the second equality we use the identity \( \varphi_r(g) = 1 \) which follows easily from the definition of \( \varphi_t \). Hence, the map \( \Psi \) is well defined.

Consider the map
\[ \Phi: (D^d, S^{d-1})^t / H_d \rightarrow X \times_G Y, \]
\[ [d \times t']_Hd \mapsto \left[ [1 \times t']_Hd, [d]_{Hd \cap G^t_d} \right]_G. \]
Any element from \((D^d, S^{d-1})^I\) representing the same \(H_d\)-orbit in \((D^d, S^{d-1})^I/H_d\) as on the left hand side of (9) has the form \(h \cdot (d \times t')\), where \(d \in (D^d)^I; h \in H_d; t' \in G_d^{|m|\setminus I}\). We compute the value of \(\Phi\) on this element as follows:

\[
\Phi[h \cdot (d \times t')]_{H_d} = \left[\left(1 \times \varphi_I(h) \cdot t', [\varphi_{|m|\setminus I}(h) \cdot d]_{H_d \cap G_d^J}\right)\right]_G = \left[\left(1 \times \varphi_I(h) \cdot t', [d]_{H_d \cap G_d^J}\right)\right]_G.
\]

Hence, the map \(\Phi\) is well defined.

We prove that \(\Psi\) and \(\Phi\) are mutually inverse maps as follows:

\[
\Phi\Psi \left(\left[t \times t'\right]_{H_d}, [d]_{H_d \cap G_d^J}\right)_G = \Phi[t \cdot d \times t']_{H_d} = \left(\left(1 \times t'\right)_{H_d}, [t \cdot d]_{H_d \cap G_d^J}\right)_G = \left(\left[t \times t'\right]_{H_d}, [d]_{H_d \cap G_d^J}\right)_G;
\]

\[
\Psi\Phi[d \times t']_{H_d} = \Psi \left(\left(1 \times t'\right)_{H_d}, [d]_{H_d \cap G_d^J}\right)_G = [d \times t']_{H_d}.
\]

Notice that there is the following diagram

\[
\begin{array}{ccc}
X \times_G Y & \xrightarrow{\Psi} & X / G \\
\downarrow & & \downarrow \\
(D^d, S^{d-1})^I/H_d & \xrightarrow{\beta} & G_d^m / (G_d^I \cdot H_d);
\end{array}
\]

where the right vertical arrow is the group isomorphism given by taking the respective quotient in (8), and the horizontal arrows are the respective projections. The diagram (11) is commutative because both compositions from the formula (11) map the element on the left side of (11) to \([t']_{H_d}\). This implies the claim about the fiber bundle from (ii). The claim about triviality of the associated fiber bundle from (ii) then follows directly from (i).

\[\square\]

**Corollary 2.11.** There is the following homeomorphism of spaces:

\[
(D^d, S^{d-1})^I/H_d \simeq (G_d^m / (G_d^I \cdot H_d)) \times ((D^d)^I / (H_d \cap G_d^I)).
\]

**Proposition 2.12.** For any closed subgroup \(H_d\) of \(G_d^m\), the quotient space \((D^d)^m/H_d\) is contractible.

**Proof.** The homotopy given by mapping \(x\) to \((1 - t)x\), where \(x \in (D^d)^I, t \in [0, 1]\), is \(H_d\)-equivariant due to the natural embedding of groups \(H_d \subseteq G_d^m \subseteq O(dm)\). Hence, it induces the deformation retraction of \((D^d)^m/H_d\) to the point.

\[\square\]

**Proposition 2.13.** The following diagram

\[
\begin{array}{ccc}
(D^d, S^{d-1})^I/H_d & \xrightarrow{f_1} & (D^d, S^{d-1})^I/H_d \\
\downarrow_{\pi_I} & & \downarrow_{\pi_I} \\
G_d^m / (G_d^I \cdot H_d) & \xrightarrow{f_2} & G_d^m / (G_d^I \cdot H_d),
\end{array}
\]

is commutative for any \(I \subseteq J \in \text{cat } K\), where \(f_1, f_2\) denote the arrows from the respective diagrams.

**Proof.** Let \((d, t', t'') \in (D^d, S^{d-1})^I, (d, d', t'') \in (D^d, S^{d-1})^J\), where \(d \in (D^d)^I; t' \in (S^{d-1})^{J\setminus I}; t'' \in (S^{d-1})^{[m]\setminus J}; d' \in (D^d)^{J\setminus I}\). We check commutativity of the above diagram as follows.

\[
f_2 \circ \pi_I [[d, t', t'']_{H_d}] = f_2([[1, t', t'']_{G_d^J \cdot H_d}]) = [[1, t', t'']_{G_d^J \cdot H_d} = [[1, 1, t'']_{G_d^J \cdot H_d},
\]

\[
\pi_I \circ f_1 [[d, t', t'']_{H_d}] = \pi_I ([[d, t', t'']_{H_d}]) = [[1, 1, t'']_{G_d^J \cdot H_d}.
\]

Hence, (12) is commutative.

\[\square\]

**Corollary 2.14.** The maps \(\pi_I\), where \(I\) runs over \(\text{cat } K\), constitute a well-defined morphism \((D^d, S^{d-1})^I/H_d \to Q_d\) of \((\text{cat } K)\)-diagrams.

**Theorem 2.15.** For any \(d = 1, 2\), any closed subgroup \(H_d\) in \(G_d^m\) and any simplicial complex \(K\) on \([m]\) there is the homotopy equivalence of spaces

\[
(D^d, S^{d-1})^K/H_d \simeq \text{hocolim} G_d^m / (G_d^I \cdot H_d),
\]

where \(H_d \cdot G_d^I\) is the subgroup generated by \(G_d^I\) and \(H_d\) in \(G_d^m\).
Proof. Any arrow from the morphism of diagrams in Corollary 2.14 is a fiber bundle projection with contractible fiber onto the respective base by Proposition 2.12. By Proposition 3.2, the Homotopy and Projection Lemmas imply that

\[(D^d, S^{d-1})^K/H_d = \text{colim} \left( (D^d, S^{d-1})^I/H_d \right) \simeq \text{hocolim} \left( Q_d = \text{hocolim} G^m_d/(H_d \cdot G^m_d) \right).\]

The proof is complete.

Example 2.16. For \( H_d = 1 \) Theorem 2.15 gives the well-known homotopy colimit description for the moment-angle complex, see Proposition 2.6.

Example 2.17. Let \( d = 2, \ m = 2, \ K = \{\{1\}, \{2\} \}, \ H_d = \mathbb{Z}/2\mathbb{Z} \) acting on \( T^2 = (S^1)^2 \) by the formula \( h(x, y) = (-x, -y) \), where \( h \) is the generator of \( H_d \). Notice that \( T^2/H_d \) is homeomorphic to \( T^2 \). Then \( \mathcal{Z}_K = S^3 \), \( \mathcal{Z}_K/H_d = \mathbb{R}P^3 \). The homotopy colimit hocolim \( G^m_d/(H_d \cdot G^m_d) \) in this case is obtained by gluing the boundary components of the cylinder \( I^1 \times T^2 \) to two disjoint copies of \( S^1 \) by the respective compositions of two different coordinate projections from \( T^2 \) to \( S^1 \) with the map \( S^1 \to S^1 \) of degree 2.

Example 2.18. Let \( H_d = G^m_d \) for some \( I_0 \subseteq [m] \). Let \( K \) be any simplicial complex on \([m]\). The (cat \( K \))-diagram \( D_1 := Q(K, H_d) \) consists of objects \( G^m_d \mid I_0, G^m_d \mid I_0 \). Let \( D_2 \) be the (cat \( K \))-diagram \( Q(K, I_0, 1) \), where \( K/I_0 := \{I \cap I_0 \mid I \in K\} \), denotes the contraction of the simplicial complex \( K \) along \( I_0 \) [3]. By the definition, one has \( D_1 = \alpha^* D_2 \), where \( \alpha: \text{cat } K \to \text{cat } (K, I_0) \), \( I \mapsto I \setminus I_0 \).

Let \( F \) be the poset morphism and \( \alpha^* D^2 \) is the pullback of the diagram \( D_2 \) along \( \alpha \). In general, the homotopy colimits of \( D_1 \) and \( D_2 \) are not homotopy equivalent, as the example of the cone \( K = \text{cone}_m \tilde{K} \) with apex at \( m \in [m] \) over \( \tilde{K} \) on \([m-1]\), and \( H_d = G^m_1 \) shows. Indeed, the join \( (D^d, S^{d-1})^K \ast pt \simeq \text{hocolim} D_1 \) is contractible, whereas hocolim \( D_2 \simeq (D^d, S^{d-1})^K \) is not contractible, in general.

Remark. Equivariant Morse theory [11] Corollary 4.11, p.150 allows to decompose \( (D^d, S^{d-1})^K/H_d \) into handle-bundles (fibrations with a fiber) over the respective orbits \( G^m_d/(G^m_d \ast H_d) \). Each of these fibrations is trivial and has a contractible fiber. This leads to the decomposition from Theorem 2.15. In what follows, we discuss a similar decomposition.

Corollary 2.19. The quotient \( (D^d, S^{d-1})^K/H_d \) is homotopy equivalent to the quotient

\[((\text{cone } K') \times (G^m_d/H_d))/\sim,\]

where \( F = I_0 \supset I_1 \supset \cdots \supset I_n \), \( F' = J_0 \supset J_1 \supset \cdots \supset J_n \) belong to the cone cone \( K' \) over the barycentric subdivision of the simplicial complex \( K \), \( I = I_n \) and \( g, g' \in G^m_d/H_d \). If the \( H_d \)-action on \( (D^d, S^{d-1})^K \) is free, then there is a homeomorphism between \( (D^d, S^{d-1})^K/H_d \) and \( ((\text{cone } K') \times (G^m_d/H_d))/\sim \).

Proof. The first claim follows easily from the definition of a homotopy colimit by Theorem 2.15. Now suppose that the \( H_d \)-action on \( (D^d, S^{d-1})^K \) is free. Notice that the fiber of the fiberation \( \pi_I \) (see Proposition 2.10 (ii)) is equal to \( D^I \), and that the inclusion of the fibers corresponding to \( I \to J \) is the coordinatewise embedding \( D^I \to D^J \). Then there exists a family of homeomorphisms \( D^I \cong \Delta(I) \), \( I \in K \), agreeing on the intersections of the faces in \( K \), where \( \Delta(I) \) is the realization of the simplex \( I \). The second claim now follows by considering the restrictions of these homeomorphisms to \( \text{cat } K \), that is, the cone over the barycentric subdivision of \( K \), and gluing the obtained homeomorphisms together. (On the orbits we take the identity map.)

Let \( K = \partial P^m \) be the dual simplicial sphere to a simple polytope \( P^m \subset \mathbb{R}^n \). Suppose that the action of \( H_d \) on \( (D^d, S^{d-1})^K \) is free and that \( H_d \cong G^m_d \) holds. The quotient \( (D^d, S^{d-1})^K/H_d \) is called a small cover for \( d = 1 \) and a quasitoric manifold for \( d = 2 \), respectively [17]. Let \( G^m_d = G^m_d/H_d \) be the real or complex torus for \( d = 1 \) or \( d = 2 \), respectively. Consider the (cat \( K \))-diagram \( G^m_d/p(G^m_d) \) (see Proposition 3.2 below for the precise definition), where \( p: G^m_d \to G^m_d \) is the natural quotient homomorphism.

The following theorem was first proved for toric varieties in [12] and then generalized to quasitoric manifolds in [30]. We deduce it from Theorem 2.15 below.

Theorem 2.20. [12, 30] Let \( K, H_d \) be as above. Then there is the homotopy equivalence of spaces

\[(D^d, S^{d-1})^K/H_d \simeq \text{hocolim} G^m_d/p(G^m_d).\]
Proof. By the condition on \( H_d \), one has \( G^I_d \cap H_d = 1 \) for any \( I \in K \). Then the commutative diagram (1) takes form

\[
\begin{array}{c}
1 \\
\downarrow \\
G^I_d / \! \! / G^m_d / \! \! / H_d \\
\downarrow \\
G^m_d / \! \! / (G^I_d \cdot H_d) \\
\downarrow \\
1
\end{array}
\]

(13)

for \( I \subseteq J \in \text{cat} \, K \). Hence, (13) defines the isomorphism \( Q_d \rightarrow G^m_d / p(G^I_d) \) of (\( \text{cat} \, K \))-diagrams. Now the desired homotopy equivalence follows by Theorem 2.15. \( \square \)

Example 2.21. In the case of a quasitoric manifold or a small cover the homeomorphism from Corollary 2.19 coincides with the Davis-Januszkiewicz construction [17]. By relaxing the condition on \( H_d \cong G^m_d / - \) to the condition of only finite stabilizers of the respective action on \((D^d, S^{d-1})_K\) one obtains a homotopy equivalence in Corollary 2.19 which follows from the homoeomorphism proved for quasitoric orbifolds in [38]. Notice that in general the topological balls \((D^d, S^{d-1})_H / (d \cap G^I_d)\) and \(A(H) \cdots \supset I_s\) have different dimensions, so in general there is no obvious homeomorphism between \((D^d, S^{d-1})_K / H_d\) and hocolim \(Q_d\).

3. Equivariant homotopy colimits and G-CW-complexes

In this section we prove the strengthening of Theorem 2.15 in the equivariant setting leading to G-CW-approximation for quotients of moment-angle complexes.

Let \( G \) be a topological group.

Definition 3.1. [10] The equivariant union \( X = \operatorname{colim}_{n \geq 2} X_n \) of \( G \)-spaces \( X_n \) is called a \( G \)-complex if there is a pushout

\[
X_{n+1} = X_n \bigcup_{n \in A_n} \left( \bigcup_{\alpha \in A_n} D^{n_\alpha} \times G/H_\alpha \right),
\]

of \( G \)-spaces with the natural left \( G \)-action (left-\( G \)-action on \( G/H_\alpha \), and trivial action on \( D^{n_\alpha} \)), where

\[\varphi_n : \bigcup_{\alpha \in A_n} S^{n_\alpha}-1 \times G/H_\alpha \rightarrow X_n,\]

is \( G \)-equivariant, \( D^{n_\alpha} \) is an \( n_\alpha \)-dimensional disk and \( \{H_\alpha\}_{\alpha \in A_n} \) is a collection of closed subgroups in \( G \). If \( n_\alpha = n \) holds for any \( \alpha \in A_n \), then \( X \) is called a \( G \)-CW-complex.

The category \( G \text{-Top} \) of \( G \)-spaces (objects) and \( G \)-equivariant maps between these spaces (morphisms) has the Quillen model structure given by \( G \)-equivariant weak equivalences, \( G \)-equivariant Serre fibrations and \( G \)-equivariant retracts of \( G \)-CW-complexes [1]. Since \( \text{cat} \, K \) is a Reedy category [20], the category \( (G \text{-Top})^{\text{cat} \, K} \) has the Reedy model structure given by objectwise weak equivalences and fibrations, and cofibrations are given by morphisms \( D \rightarrow E \) such that \( D(I) \sqcup_{L_D(I)} L_E(I) \rightarrow E(I) \) is a colibration for any \( I \in \text{cat} \, K \), where

\[L_D(I) := \operatorname{hocolim}_{\text{(cat} \, K) \leq I} D(I),\]

is the natural map from the latching object of \( D \) [20], [3].

The following proposition is straightforward to prove.

Proposition 3.2. Let \( G \) be a collection of closed subgroups in \( G \) such that \( G_I \subseteq G_J \) holds for any \( I \subseteq J \in \text{cat} \, K \). Define the (\( \text{cat} \, K \))-diagram \( D \)

\[D(I) := G/G_I, \quad D(I \rightarrow J) : G/G_I \rightarrow G/G_J,\]

where \( D(I \rightarrow J) \) is the natural projection. Then \( D \) is cofibrant in \( (G \text{-Top})^{\text{cat} \, K} \) and its homotopy colimit in \( G \text{-Top} \) is given by

\[
\text{hocolim} \, D = ( \bigcup_{I \in \text{cat} \, K} D(I) \times |K_{\geq I}| ) / \sim,
\]

where by definition \( (d, \text{In}_{I \rightarrow I'}(I')) \sim (D(I, I')(d), I') \) and \( \text{In}_{I \rightarrow I'} : |K_{\geq I'}| \rightarrow |K_{\geq I}| \) is the natural embedding. Furthermore, the decomposition [14] endows hocolim \( D \) with the structure of a \( G \)-CW-complex.

Let \( G = G^m_d / H_d \). The natural \( G \)-action on \((D^d, S^{d-1})_K / H_d\) allows to consider the (\( \text{cat} \, K \))-diagram \((D^d, S^{d-1})_K / H_d\) as a (\( \text{cat} \, K \))-diagram in \( G \text{-Top} \).

Theorem 3.3. For any closed subgroup \( H_d \) in \( G^m_d \) and any simplicial complex \( K \) on \( |m| \) there is the \( G^m_d / H_d \)-equivariant homotopy equivalence

\[\quad (D^d, S^{d-1})_K / H_d \simeq \text{hocolim} \, G^m_d / (G^I_d \cdot H_d).\]
Proof. Follows from Theorem 2.15 by the standard properties of a homotopy colimit by using $G$-equivariance of all arrows in [12].

Remark. Theorem 3.3 gives an explicit $G$-CW-approximation of the quotient $(D^d, S^{d-1}K)/H_d$ with cells

$$\Delta^s(I_0 \supset \cdots \supset I_s) \times G_d^m/(H_d \cdot G_d),$$

for any closed subgroup $H_d$ in $G_d^m$ such that the corresponding $H_d$-action on $(D^d, S^{d-1}K)$ is free, the homeomorphism from Corollary 2.19 is easily shown to be $G$-equivariant.

4. Equivariant cohomology of quotients for moment-angle complexes

In this section we describe the equivariant cohomology of the quotient for any moment-angle complex by any closed subgroup $H_d$ in the respective torus $G_d^m$ (so that the $H_d$-action on $(D^d, S^{d-1}K)$ is not necessarily free) and study the classifying space $Q_d$ of all arrows in (12).

Let $N_d$ act transitively with the kernel $S_d = G_d^I/(G_d^I \cap H_d)$ by (11).

Thus, the Borel construction for the natural $L_d$-action on $(D^d, S^{d-1}K)/H_d$ takes the following form.

Corollary 4.2. For $G = L_d$, there is the following fibration in $(GTop)^{catK}$:

$$Q_d \longrightarrow BS_d \longrightarrow \kappa(BL_d). \quad (15)$$

Theorem 4.3. For any $d = 1, 2$, any closed subgroup $H_d$ in $G_d^m$ and any simplicial complex $K$ on $[m]$ one has the following isomorphism of integral cohomology rings

$$H^*_d((D^d, S^{d-1}K)/H_d) \cong H^*(\text{colim} B(G_d^m/(G_d^m \cap H_d))).$$

Proof. Follows directly from Proposition 4.1 by the Homotopy Lemma of [12].

Example 4.4. Suppose that the $H_d$-action on $(D^d, S^{d-1}K)$ is free. Then it follows from the standard properties of equivariant cohomology that the cohomology ring isomorphism (with $\mathbb{Z}$-coefficients)

$$H^*_d((D^d, S^{d-1}K)/H_d) \cong H^*_d(G_d^m/(D^d, S^{d-1}K)),$$

takes place. On the other hand, freeness of the action implies that $H_d \cap G_d^I$ is a trivial group for any $I \in \text{cat} K$.

Hence, the colimit of $BS_d$ is the Davis-Januszkiewicz space $DJ(K) = (\mathbb{Z}d^{\mathbb{R}^+}, pt)^K$ whose cohomology ring with $R_d$-coefficients is isomorphic to the Stanley-Reisner ring $R_d[K]$ [14]. Thus Theorem 4.3 gives a correct answer in this case by [10] (for $d = 1$ we take reduction of integral cohomology coefficients modulo two).

As an application of Corollary 4.2 and Theorem 4.3 we describe the Borel construction for the quotient by any coordinate subgroup in $G_d^m$ (with not necessarily free action) below.

Corollary 4.5. Let $H_d = G_d^{I_0}$ for an arbitrary fixed $I_0 \subseteq [m]$. Then for any complex $K$ the Borel construction of $L_d$-action on $(D^d, S^{d-1}K)/H_d$ is homotopy equivalent to the real or complex Davis-Januszkiewicz space, $\mathbb{R}DJ(K/I_0)$ or $DJ(K/I_0)$, for $d = 1$ or $d = 2$, respectively. Furthermore, one has the ring isomorphism:

$$H^*_d((D^d, S^{d-1}K)/H_d; R_d) \cong R_d[K/I_0].$$

Proof. Notice that the natural group isomorphism

$$H^*_d(G_d^I/(G_d^I \cap G_d^{I_0})) \cong G_d^{I \setminus I_0}, \quad (16)$$

holds for any $I \in \text{cat } K$. Hence, the following diagram

$$
\begin{array}{ccc}
G^I_d \cup G^0_d & \longrightarrow & G^I_d \\
\downarrow & & \downarrow \\
G^J_d \cup G^0_d & \longrightarrow & G^J_d \\
\end{array}
$$

where both horizontal arrows are given by (16) and the right vertical arrow is the standard embedding, is commutative for any $I \subseteq J \in \text{cat } K$. This diagram yields the isomorphism of $(\text{cat } K)$-diagrams $BS_d$ and $G^I_d \cup G^0_d$. Hence, one has

$$\text{colim } BS_d \cong \text{colim } G^I_d \cup G^0_d \cong \text{colim } G^I_d.$$

The last expression is the real or complex Davis-Januszkiewicz space by the definition for $d = 1$ or $d = 2$, respectively. This proves the first claim. The second claim then follows from the first by the standard computation for moment-angle complexes, see Example [17]. The proof is complete. □

4.2. On a certain class of quotients for moment-angle complexes. Let $K$ be any simplicial complex on $[m]$. Let $H_d$ be any closed subgroup in $G^d_\mathbb{Z}$. We introduce the following condition on the pair $(K, H_d)$.

**Condition 4.6.** For any $I \subseteq J \in \text{cat } K$ the subgroup $H_d \cap G^I_d$ maps to the subgroup $H_d \cap G^J_d$ under the natural projection $G^I_d \to G^J_d$. Or equivalently, in the following diagram there exists an upper horizontal arrow making it a commutative diagram

$$
\begin{array}{ccc}
G^I_d \cap H_d & \longrightarrow & G^J_d \\
\downarrow & & \downarrow \\
G^J_d & \longrightarrow & G^J_d \\
\end{array}
$$

where the lower horizontal arrow is the natural projection.

**Example 4.7.** For any $K$ and any $H_d$ such that $H_d$ acts freely on $(D^d, S^{d-1})_K$ both groups in the upper row of (17) are trivial. Hence, Condition 4.6 holds for any free action of $H_d$ on $(D^d, S^{d-1})_K$.

**Example 4.8.** Let $H_d = G^0_d$ for any fixed $I_0 \subseteq [m]$ and let $K$ be any simplicial complex. Notice that $G^I_d \cap H_d = G^I_d \cap G^0_d$ holds for any $I \subseteq [m]$. The natural projection $G^I_d \to G^0_d$ sends $i$-th coordinate subgroup $(G^I_d)_i$ to 1 if $i \notin I$ and acts as an identity if $i \in I$. Hence, the image of $G^I_d \cap H_d$ under this projection coincides with $G^I_d \cap H_d$. We conclude that Condition 4.6 holds for the action of a coordinate subgroup $H_d = G^0_d$ on $(D^d, S^{d-1})_K$. Notice that this action is not free, in general.

**Example 4.9.** In [29] a certain class of closed subgroups $H_d$ in $G^d_\mathbb{Z}$ acting on $(D^d, S^{d-1})_K$ was introduced. One can check that for $d = 2$, $m = 2$ and $K = \Delta^1$ the natural action of the diagonal circle $H_d = S^1_d$ on $(D^d, S^{d-1})_K$ belongs to this class and does not satisfy Condition 4.6 for $I = \{1\}$, $J = \{1, 2\}$.

4.3. Twin diagrams and equivariant cohomology. Let $D$ and $D^\vee$ be (cat $K$)- and (cat $K$)-diagrams with values in the category Top, respectively. Suppose that $D(I) = D^\vee(I)$ holds for any $I \in \text{cat } K$. Recall the following definition.

**Definition 4.10.** [35] The diagrams $D$, $D^\vee$ are called twin diagrams if the identity

$$D^\vee(J \to I') \circ D(I \to J) = D(I \cap I' \to I') \circ D^\vee(I \to I \cap I'),$$

holds for any $I, I' \subseteq J$ in $\text{cat } K$.

Recall that any (cat $K$)-diagram $D$ of pointed topological spaces gives rise to the Bousfield-Kan type cohomological (with integral coefficients) spectral sequence (see [34])

$$\lim^2 \tilde{H}^i(D) \Rightarrow \tilde{H}^{i+j}(\text{hocolim } D).$$

**Theorem 4.11.** [36] p.39, Lemma 3.8, p.41, Theorem 3.10] Suppose that a (cat $K$)-diagram $D$ is cofibrant and has a twin. Then the second page of the Bousfield-Kan spectral sequence $(E_D)_r^{s,t}$ of $D$ is concentrated at $s = 0$. In particular, $(E_D)_s^{t,0}$ collapses at the second page $r = 2$.

**Corollary 4.12.** [36] p.42, Corollary 3.12] If $D$ is cofibrant and has a twin, then one has

$$\tilde{H}^i(\text{colim } D) = \lim \tilde{H}^i(D), \lim^1 \tilde{H}^i(D) = 0, i > 0.$$
Proof. By the condition, the subgroup $H_d \cap G_d^I$ maps to the subgroup $H_d \cap G_d^J$ under the natural projection $G_d^I \to G_d^J$. Hence, there is a well-defined (cat$^{op}$ K)-diagram $S_d^J$ where $S_d^J(J \to I) : G_d^J/(H_d \cap G_d^J) \to G_d^I/(H_d \cap G_d^I)$ is induced by the natural projection $G_d^I \to G_d^J$. Define $(BS_d)^J := B(S_d^J)$. It is easy to check that the pairs $(S_d, S_d^J)$ and $(BS_d, (BS_d)^J)$ consist of twin diagrams. Clearly, $S_d, BS_d$ are cofibrant. Hence, the claim follows by Corollary 4.12.

Corollary 4.14. Suppose that $K$ and $H_d$ satisfy Condition 4.6. Then one has the following ring isomorphism:

$$
\tilde{H}_d^*(D^d, S^{d-1})^K/H_d) \cong \lim \tilde{H}^*(B(G_d^I/(G_d^I \cap H_d))).
$$

In particular, one has $\tilde{H}_d^{qd}(Z_K/H_d; \mathbb{Z}) = 0$.

Example 4.15. Let $H_d$ be a freely acting closed subgroup in $G_d^{op}$ on $(D^d, S^{d-1})^K$ (for example, $H = \{1\}$). It is well known that

$$
R_d[K] \cong \lim_{i = (i_1, \ldots, i_q) \in \text{cat } K} R_d[v_1, \ldots, v_n],
$$

holds for the Stanley-Reisner ring [3], where the arrows are the obvious monomorphisms to the polynomial ring $R_d[v_1, \ldots, v_m]$, where $\text{deg } v_j := d_j$. Therefore, the group

$$
\lim \tilde{H}^*(BG_d^I; R_d) \cong \lim_{i = (i_1, \ldots, i_q) \in \text{cat } K} (R_d[v_1, \ldots, v_n])_i,
$$

agrees with the respective component $(R_d[K])$, of the Stanley-Reisner ring (for $d = 1$ take reduction of coefficients modulo 2).

4.4. Formality of the Borel construction for the class of quotients for moment-angle complexes and Eilenberg-Moore spectral sequence. In this section we study only quotients of complex moment-angle complexes (that is, $d = 2$) and consider only cohomology with integral coefficients due to usage of the Eilenberg-Moore spectral sequences. For brevity we omit the subscript $d$ and replace $G_2$ with $T = S^1$ everywhere below. It is crucial that everywhere in §4.4 we assume that Condition 4.6 holds for the pair $(K, H)$.

Proposition 4.16. Suppose that Condition 4.6 holds for the pair $(K, H)$. Then there is a zigzag of quasiisomorphisms in $dga_2$ between $\lim H^*(BS)$ and $\lim C^*(BS)$.

Proof. One constructs the necessary zigzag of quasiisomorphisms for cofibrant (cat$^{op}$ K)-diagrams by applying [35] p.43, (4.3)] to $B(T^J/(T^J \cap H))$.

The proof of the following proposition is similar to the proof of [35] p.44, Lemma 4.7 (notice that the analogue of [35] p.42, Corollary 3.12 is given in §3.3).

Proposition 4.17. [35] Suppose that Condition 4.6 holds for the pair $(K, H)$. Then the natural homomorphism $C^*(\text{colim } BS) \to \lim C^*(BS)$ is a quasiisomorphism in $dga_2$.

The following theorem follows directly by taking the composition of quasiisomorphisms from Proposition 4.17, Lemma 4.16 and Theorem 4.13 in a similar way to [35] p.44, Theorem 4.8.

Theorem 4.18. Suppose that Condition 4.6 holds for the pair $(K, H)$. Then the differential graded algebra $C^*(\text{colim } BS)$ is formal in $dga_2$.

For a Serre fibration $p : E \to B$ with a connected fiber $F$ the Eilenberg-Moore spectral sequence $(E_2^{*,*}, d)$ of the fiber inclusion has the second page [3, p.233]

$$
E_2^{n,*} = \text{Tor}^{n,*,*}_{H(B)}(H^*(E), \mathbb{Z}),
$$

where the first grading is cohomological and the second is inner. If $B$ is simply-connected, then $(E_2^{*,*}, d)$ converges strongly to $H^*(F)$, see [3, p.233].

Proposition 4.19. Suppose that Condition 4.6 holds for the pair $(K, H)$. Then the Eilenberg-Moore spectral sequence for the fiber inclusion to the Borel construction of the L-action on $Z_K/H$ is isomorphic to

$$
\text{Tor}^{*,*}_{H^*(BL)}(\lim H^*(BS); \mathbb{Z}) \Rightarrow H^{*+1}(Z_K/H).
$$

It collapses at the second page. In particular, the associated graded algebra of $H^*(Z_K/H)$ is isomorphic to $\text{Tor}_{H^*(BL)}(\lim BS; \mathbb{Z})$.

Proof. The Eilenberg-Moore spectral sequence in question has the second page $\text{Tor}^{*,*}_{H^*(BL)}(H^*(\lim BS); \mathbb{Z})$ and converges to $\text{Tor}^{*,*}_{H^*(BL)}(C^*(\lim BS); \mathbb{Z})$. However, by formality of $BL$ and of $\lim BS$ (see Theorem 4.15), these pages coincide. Hence, this spectral sequence collapses at the second page, which proves the first claim. The second claim then follows trivially from Theorem 4.13.
Remark. Recall that the quotient \((D^d, S^{d-1})/H_d\) is called a partial quotient \([3]\) if the corresponding \(H_d\)-action is free. In the particular case of partial quotients the claim of Proposition \([1, 10]\) was previously known (see Example \([4, 13]\)). We refer to \([22]\) for the necessary bibliographical links and for the recent historical overview on the results about cohomology groups and rings of partial quotients. In comparison to the case of partial quotients, Proposition \([1, 10]\) is a new generalization of previously known results on the above Eilenberg-Moore spectral sequence for partial quotients to the case of any (not necessarily freely acting) closed subgroup \(H\) (in \(T^m\)) satisfying Condition \([4, 6]\).

Remark. Puppe’s lemma \([4]\) and cofibrancy of \(BS_d\) imply that there is the following \(L_d\)-equivariant Serre fibration:

\[
\text{hocolim } Q_d \longrightarrow \text{colim } BS_d \longrightarrow BL_d.
\]

For \(d = 2\), one can deduce that the Serre spectral sequence of the diagram of fibrations \([13]\) evaluated at \(I \in \text{cat } K\) as well as of the fibration \([18]\) collapses in the term \(E_3\), compare with \([2]\) p.115, Proposition 7.36).

5. COHOMOLOGY OF PARTIAL QUOTIENTS: CONCLUDING REMARKS AND OPEN PROBLEMS

In this section we discuss the Toral Rank Conjecture and torsion in integral cohomology of partial quotients. The next conjecture was stated in \([28]\).

**Conjecture 5.1.** Let \(X\) be a finite-dimensional CW complex. Then the inequality

\[
\text{hrk}(X) := \sum_{i \geq 0} \dim H^i(X; \mathbb{Q}) \geq 2^\text{trk}(X),
\]

holds, where \(\text{trk}(X)\) denotes the maximal rank of a torus acting almost freely on \(X\).

Recall that the Buchstaber number \(s(K)\) of a simplicial complex \(K\) is the maximal rank of a torus acting freely on \(\mathbb{Z}_K\). Following the notation from \([23]\), we denote by \(\Delta^k_m\) the \(k\)-skeleton of an \((m - 1)\)-simplex for \(m \geq 2\) and \(0 \leq k \leq m - 2\); for the computation of Buchstaber numbers for these simplicial complexes see \([24]\).

**Theorem 5.2.** For any \(m \geq 2\) and \(0 \leq k \leq m - 2\) the partial quotient \(\mathbb{Z}_{\Delta^k_m}/S^1_d\) is a rationally formal space with torsion free integral cohomology. Moreover, the following inequality holds:

\[
\text{hrk}(\mathbb{Z}_{\Delta^k_m}/S^1_d) \geq 2^{m-k-1}.
\]

**Proof.** By \([23]\) Theorem 3.9, Corollary 3.12, for any \(0 \leq k \leq m - 2\) one has

\[
\mathbb{Z}_{\Delta^k_m}/S^1_d \cong \mathbb{C}P^{k+1} \vee \mathbb{Z}_{\Delta^k_{m-1}} \vee \left(\bigvee_{i=1}^k S^{2i-1} \ast \mathbb{Z}_{\Delta^k_{m-i-1}}\right) \vee (S^{2k+1} \ast T^{m-k-2}).
\]

Moreover, due to \([27]\) Corollary 9.5] one has:

\[
\mathbb{Z}_{\Delta^k_m} \cong \bigvee_{j=k+2}^m \left(S^{k+j+1} \ast (\mathbb{C}P^{j+1})^{(n)}(\mathbb{C}P^{j+1})^{(n)}\right).
\]

This implies that the partial quotient \(\mathbb{Z}_{\Delta^k_m}/S^1_d\) is a rationally formal space with torsion free integral cohomology.

Since \(\Sigma T^n \cong S^2 \vee \Sigma T^{n-1} \vee \Sigma^2 T^{n-1}\), for all \(n \geq 2\), it follows that \(\Sigma T^n\) is a homotopy wedge of spheres and

\[
\text{hrk}(\mathbb{Z}_{\Delta^k_m}/S^1_d) = 1 + (\text{hrk}(\mathbb{C}P^{k+1}) - 1) + (\text{hrk}(\mathbb{Z}_{\Delta^k_{m-1}}) - 1) + \sum_{i=1}^k (\text{hrk}(\mathbb{Z}_{\Delta^k_{m-i-1}}) - 1) + (\text{hrk}(\Sigma T^{m-k-2}) - 1).
\]

Observe that \(\text{hrk}(\mathbb{C}P^{k+1}) = k + 2\) and \(\text{hrk}(\Sigma T^{m-k-2}) = 2^{m-k-2}\). Moreover, \(\text{hrk}(\mathbb{Z}_{\Delta^k_{m-1}}) \geq 2^{m-1} - (k+1) = 2^{m-k-2}\) by \([39]\) Theorem 10. Therefore, we have

\[
\text{hrk}(\mathbb{Z}_{\Delta^k_m}/S^1_d) \geq k + 1 + 2^{m-k-2} + (2^{m-k-2} - 1) \geq 2^{m-k-1}.
\]

The proof is complete. \(\square\)

For a simple polytope \(P\) let \(s(P) = s(\partial P^*)\) be its Buchstaber number.

**Problem 5.3.**

(a) Does there exist a simplicial complex \(K\) on the vertex set \([m]\) and a toric subgroup \(H \subseteq T^m\) of rank \(r, 1 \leq r \leq s(K)\) acting freely on \(\mathbb{Z}_K\) such that the partial quotient \(\mathbb{Z}_K/H\) is non-formal?

(b) Does there exist a simple polytope \(P\) with \(m\) facets and a toric subgroup \(H \subseteq T^m\) of rank \(r, 1 \leq r \leq s(P)\) acting freely on \(\mathbb{Z}_P\) such that the partial quotient \(\mathbb{Z}_P/H\) is non-formal?

**Corollary 5.4.** The Toral Rank Conjecture holds in the class of all partial quotients of moment-angle complexes over the skeleta of simplices by the diagonal circle action.
Proof. By [39] Lemma 8 and [17] §7.1, if \( f_0(K) = m \) and \( \dim K = n - 1 \), then \( \text{trk}(Z_K) = m - n \). Hence, \( \text{trk}(Z_K/S^d) \leq \text{trk}(Z_K) = m - n \), and the statement follows directly from the previous theorem.\( \square \)

**Problem 5.5.** Prove that the Toral Rank Conjecture holds for all partial quotients of moment-angle complexes or provide a counterexample.

We proceed with the following theorem about torsion in integral cohomology of partial quotients.

**Theorem 5.6.** Let \( G \) be a finitely generated Abelian group. Then there exist a simple polytope \( P \subseteq \mathbb{R}^n \) with \( m \) facets and a circle subgroup \( H \subseteq T^m \) such that \( H \) acts freely on the moment-angle manifold \( Z_P \) and \( H^*(Z_P/H) \) contains \( G \) as a direct summand.

**Proof.** Let \( X \) be a Moore space for \( G \), that is, \( H^i(X) \cong G \) and \( \check{H}^i(X) = 0 \) when \( i \neq p \), for some \( p \geq 1 \). Take its arbitrary finite triangulation \( K \) and let \( K' \) be obtained from \( K \) by a stellar subdivision in a maximal simplex of \( K \). Therefore, there exists a pair of disjoint vertices, \( i \) and \( j \), in the vertex set \( [m] \) of \( K' \).

Following [11], consider the full simplex \( \Delta_{[m]} \) on the vertex set \( [m] \) and let us cut off its simplices corresponding to minimal non-faces of \( K' \), one by one. Then the nerve complex \( \tilde{K} \) of the resulting simple polytope \( P \) will be a polytopal sphere of dimension \( m - 2 \) with \( M := m + |MF(K')| \) vertices, where \( MF(K') \) denotes the set of minimal non-faces of \( K' \). Moreover, the vertices \( i \) and \( j \) of \( \tilde{K} \) remain to be disjoint, since \( \{i, j\} \in MF(K') \subseteq MF(\tilde{K}) \).

Let \( \alpha \) be a partition of \( [M] \) into \( M - 1 \) classes \( \alpha_1, \ldots, \alpha_{M-1} \), the only 2-element class being \( \{i, j\} \). Following the notation of [29], consider a map \( \lambda_0 : [M] \to \mathbb{Z}^{M-1} \), sending a vertex from the class \( \alpha_k \) to \( \tilde{e}_k \), where \( \{\tilde{e}_1, \ldots, \tilde{e}_{M-1}\} \) is a basis of the lattice \( \mathbb{Z}^{M-1} \). According to [29], this map \( \lambda_0 \) gives rise to a circle subgroup \( H_{\lambda_0} \subseteq T^M \) acting freely on the moment-angle manifold \( Z_P \).

Then the space \( \tilde{X}(\tilde{K}, \lambda_0) \cong Z_P/H \) is the corresponding partial quotient. It follows easily by construction that it is a smooth compact simply-connected manifold of dimension \( \dim Z_P - 1 = 2n + |MF(\tilde{K})| - 2 \) and that \( \tilde{K}_{\lambda_0, [M-1]} = K' \) holds. Hence, by [29] Theorem 1.2, \( H^q(Z_P/H) \) contains \( G \) as a direct summand, when \( q = p + m \). This finishes the proof.\( \square \)

**Remark.** Let \( H^*(Z_K) \) be torsion-free. Then any partial quotient of \( Z_K \) from the family of spaces introduced in [29] is also torsion-free: this follows directly from [29] Theorem 1.2.

**Example 5.7.** Let \( G = \mathbb{Z}/2\mathbb{Z} \). Take \( X \) to be \( \mathbb{R}P^2 \) and \( K \) be its minimal 6-vertex triangulation: \( K = \mathbb{R}P^2_6 \). It is well known that the set of minimal non-faces \( MF(K) \) consists of ten 3-element sets. Therefore, the 2-dimensional simplicial complex \( K' \) has \( m = 7 \) vertices and the 5-dimensional polytopal sphere \( \tilde{K} \) has \( M = 21 \) vertices. Hence \( \dim Z_P/H = 26 \) and its integral cohomology contains 2-torsion in degree \( q = 9 \).

Notice that for a non-free \( T^r \)-action the orbit space \( Z_P/T^r \) may have torsion in homology, in general. For example, see [5] p.12, Example 2.4.

**Problem 5.8.**

(a) Does there exist a simple polytope \( P \) such that \( H^*(Z_P; \mathbb{Z}) \) has torsion and \( H^*(Z_P/T^r; \mathbb{Z}) \) are free groups for all freely acting tori \( T^r \subset T^m \), \( 1 \leq r \leq s(P) \)?

(b) Does there exist a simplicial complex \( K \) such that \( H^*(Z_K; \mathbb{Z}) \) has torsion and \( H^*(Z_K/T^r; \mathbb{Z}) \) are free groups for all freely acting tori \( T^r \subset T^m \), \( 1 \leq r \leq s(K) \)?

**Problem 5.9.**

(a) Does there exist a simple polytope \( P \) such that \( H^*(Z_P; \mathbb{Z}) \) is a free group and \( H^*(Z_P/T^r; \mathbb{Z}) \) has torsion for a certain freely acting torus \( T^r \), \( 1 \leq r \leq s(P) \)?

(b) Does there exist a simplicial complex \( K \) such that \( H^*(Z_K; \mathbb{Z}) \) is a free group and \( H^*(Z_K/T^r; \mathbb{Z}) \) has torsion for a certain freely acting torus \( T^r \), \( 1 \leq r \leq s(K) \)?

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