THE ORLICZ VERSION OF THE $L_p$ MINKOWSKI PROBLEM ON $S^{n-1}$ FOR $-n < p < 0$

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1. Introduction

The scalar product on $\mathbb{R}^n$ is denoted by $\langle \cdot, \cdot \rangle$, and the corresponding Euclidean norm is denoted by $\| \cdot \|$. A convex body $K$ in $\mathbb{R}^n$ is a compact convex set that has non-empty interior. We write $K_0^n$ ($K_0^n$) to denote the family of convex bodies with $o \in K$ ($o \in \text{int} K$). The $k$-dimensional Hausdorff measure normalized in a way such that it coincides with the Lebesgue measure on $\mathbb{R}^k$ is denoted by $\mathcal{H}^k$. The angle (spherical distance) of $u, v \in S^{n-1}$ is denoted by $\angle(u,v)$.

For any $x \in \partial K$, $\nu_K(x) \subset S^{n-1}$ (“the Gauß map”) is the family of all unit exterior normal vectors at $x$. For a Borel set $\omega \subset S^{n-1}$, $\nu_K^{-1}(\omega)$ is the Borel set of $x \in \partial K$ with $\nu_K(x) \cap \omega \neq \emptyset$. An $x \in \partial K$ is called smooth if $\nu_K(x)$ consists of a unique vector, and in this case, we use $\nu_K(x)$ to denote this unique vector, as well. It is well-known that $\mathcal{H}^{n-1}$ almost all $x \in \partial K$ is smooth (see, e.g., Schneider [83]), and let $\partial'K$ denote the family of smooth points of $\partial K$.

The surface area measure $S_K$ of $K$ is a Borel measure on the unit sphere $S^{n-1}$ of $\mathbb{R}^n$, defined, for a Borel set $\omega \subset S^{n-1}$ by

$$S_K(\omega) = \mathcal{H}^{n-1}(\nu_K^{-1}(\omega)) = \mathcal{H}^{n-1}(\{x \in \partial K : \nu_K(x) \cap \omega \neq \emptyset\})$$

(see, e.g., Schneider [83]).

As one of the cornerstones of the classical Brunn-Minkowski theory, the Minkowski’s existence theorem characterizes surface area measures, and states that the solution is unique up to translation. The regularity of the solution has been also well investigated, see e.g., Lewy [58], Nirenberg [77], Cheng and Yau [20], Pogorelov [80], and Caffarelli [14, 15].

The surface area measure of a convex body has a clear geometric significance. In [64], Lutwak showed that there is an $L_p$ analogue of the surface area measure (known as the $L_p$-surface area measure). For a convex compact set $K$ in $\mathbb{R}^n$, let $h_K$ be its support function:

$$h_K(u) = \max\{\langle x, u \rangle : x \in K\} \quad \text{for} \quad u \in \mathbb{R}^n,$$

where $\langle \cdot, \cdot \rangle$ stands for the Euclidean scalar product.

Let $K_0^n$ denote family of convex bodies in $\mathbb{R}^n$ containing the origin $o$. Note that if $K \in K_0^n$, then $h_K \geq 0$. If $p \in \mathbb{R}$ and $K \in K_0^n$, then the $L_p$-surface area measure is defined by

$$dS_{K,p} = h_K^{1-p} dS_K$$

where for $p > 1$ the right hand side is assumed to be a finite measure. In particular, if $p = 1$, then $S_{K,1} = S_K$, and if $p < 1$ and $\omega \subset S^{n-1}$ Borel, then

$$S_{K,1}(\omega) = \int_{\nu_K^{-1}(\omega)} \langle x, \nu_K(x) \rangle^{1-p} d\mathcal{H}^{n-1}(x).$$
In recent years, the $L_p$-surface area measure appeared in, e.g., \cite{7,10,12,14,15,21,24,29,30,31,32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49,50,51,52,53,54,55,56,57,58,59,60,61,62,63,64,65,66,67,68,69,70,71,72,73,74,75,76,77,78,79,80}. In \cite{42}, Lutwak posed the associated $L_p$-Minkowski problem for $p \geq 1$ which extends the classical Minkowski problem. In addition, the $L_p$-Minkowski problem for $p < 1$ was publicized by a series of talks by Erwin Lutwak in the 1990’s, and appeared in print in Chou, Wang \cite{22} for the first time.

Besides discrete measures, an important special type are Borel measures $\mu$ on $S^{n-1}$, which have a density with respect to $\mathcal{H}^n$.

\begin{equation}
    d\mu = f \, d\mathcal{H}^n
\end{equation}

for some non-negative $L_1$ function $f$ on $S^{n-1}$. In this case the $L_p$-Minkowski problem amounts to solving the Monge-Ampère type equation

\begin{equation}
    h^{1-p} \det(\nabla^2 h + hI) = f
\end{equation}

where $h$ is the unknown non-negative (support) function on $S^{n-1}$ to be found, $\nabla^2 h$ denote the (covariant) Hessian matrix of $h$ with respect to an orthonormal frame on $S^{n-1}$, and $I$ is the identity matrix.

The case $p = 1$, namely the classical Minkowski problem, was solved by Minkowski \cite{7} in the case of polytopes, and in the general case by Alexandrov \cite{2}, and Fenchel and Jessen \cite{23}. The case $p > 1$ and $p \neq n$ was solved by Chou, Wang \cite{22}, Guan, Lin \cite{33} and Hug, Lutwak, Yang, Zhang \cite{49}; Zhu \cite{100} investigated the dependence of the solution on $p$ for given target measure. We note that the solution is unique if $p > 1$ and $p \neq n$, and unique up to translation if $p = 1$. In addition, if $p > n$, then the origin lies in the interior of the solution $K$, however, if $-n(n-2) < p < n$, then possibly the origin lies on the boundary of the solution $K$ even if (2) holds for a positive continuous $f$ (see Chou, Wang \cite{22}, Hug, Lutwak, Yang, Zhang \cite{49} and Bianchi, Böröczky, Colesanti, Yang \cite{6}).

The goal of this paper is to discuss the $L_p$-Minkowski problem for $p < 0$ where the case $p = 0$ is the so called logarithmic Minkowski problem see, e.g., \cite{10,13,22,24,29,30,31,32,35,37,38,39,40,41,42,43,46,47,48,49,50,51,52,53,54,55,56,57,58,59,60,61,62,63,64,65,66,67,68,69,70,71,72,73,74,75,76,77,79,80}. Additional references regarding the $L_p$-Minkowski problem and Minkowski-type problems can be found in, e.g., \cite{19,22,23,24,25,26,27,47,48,49,50,51,53,54,55,56,57,58,59,60,61,62,63,64,65,66,67,68,69,70,71,72,73,74,75,76,77,78,79,80,81,82,83,84,85,86,87,88,89,90,91,92,93,94,95,96,97,98}. Applications of the solutions to the $L_p$-Minkowski problem can be found in, e.g., \cite{3,4,13,21,23,24,26,27,49,50,51,52,53,54,55,56,57,58,59,60,61,62,63,64,65,66,67,68,69,70,71,72,73,74,75,76,77,78,79,80,81,82,83,84,85,86,87,88,89,90,91,92,93,94,95,96,97,98}. We note that if $p < 1$, then non-congruent $n$-dimensional convex bodies may give rise to the same $L_p$-surface area measure, see Chen, Li, Zhu \cite{18} for examples when $0 < p < 1$, Chen, Li, Zhu \cite{17} for examples when $p = 0$ and Chou, Wang \cite{22} for examples if $p < 0$.

If $0 < p < 1$, then the $L_p$-Minkowski problem is essentially solved by Chen, Li, Zhu \cite{18}, proving that any finite Borel measure on $S^{n-1}$ not concentrated on a great subsphere is the $L_p$-surface area measure of a convex body $K \in \mathcal{K}_0^n$. This result was slightly strengthened by Bianchi, Böröczky, Colesanti, Yang \cite{6}.

The case $p = 0$ concerns the cone volume measure $V_K$ of a convex body $K$ in $\mathbb{R}^n$ containing the origin. Chen, Li, Zhu \cite{17} proved that if the Borel $\mu$ on $S^{n-1}$ satisfies the so called the subspace concentration condition, then $\mu$ is a cone volume measure. In particular, we have the following where we say that a measurable function $f$ on $S^{n-1}$ is essentially positive if $\mathcal{H}^{n-1}(\{u \in S^{n-1} : f(u) \leq 0\}) = 0$ where the empty set is of measure zero.

**Theorem 1.1** (Chen, Li, Zhu). For $n \geq 2$ and $p = 0$, if the function $f$ in (2) is in $L_1(S^{n-1})$ and essentially positive, then (2) has a solution in Alexandrov sense; namely, $f \, d\mathcal{H}^{n-1} = dS_{K,0}$ for a convex body $K \in \mathcal{K}_0^n$.

We note that one can show that if $\mu$ in Theorem 1.1 is invariant under a closed subgroup $G$ of $O(n)$, then $K$ can be chosen to be invariant under $G$. 
However, characterization of the cone volume measure is still not known in general. Contrasting the sufficient condition provided by [17], Böröczky, Hegedüs [9] presented some necessary condition for a measure on $S^{n-1}$ being a cone volume measure ($L_0$ surface area measure).

For all $p < 0$, the only known results seems to be the following one due to Zhu [99]:

**Theorem 1.2 (Zhu).** For $p < 0$ and $n \geq 2$, any discrete measure measure on $S^{n-1}$ not concentrated on a closed hemisphere such that any $n$ vectors in the support are linearly independent in $\mathbb{R}^n$ is the $L_p$-surface area measure of convex body in $\mathbb{R}^n$.

If $-n < p < 0$, then Chou, Wang [22] solves the case when the measure $\mu$ in question has a density $f$ with respect to Haar measure $\mathcal{H}^{n-1}$ on $S^{n-1}$ and $f$ is bounded and bounded away from zero, which result is slightly generalized by Bianchi, Böröczky, Colesanti, Yang [6] allowing that $f$ is in $L_{\frac{n}{n+p}}$.

**Theorem 1.3 (Chou, Wang, Bianchi, Böröczky, Colesanti, Yang).** For $n \geq 1$ and $-n < p < 0$, if the function $f$ in (2) is in $L_{\frac{n}{n+p}}(S^{n-1})$ and essentially positive, then (2) has a solution in Alexandrov sense; namely, $f \, d\mathcal{H}^{n-1} = dS_{K,p}$ for a convex body $K \in \mathcal{K}_0^n$. In addition, if $f$ is invariant under a closed subgroup $G$ of $O(n)$, then $K$ can be chosen to be invariant under $G$.

The critical case $p = -n$ is exceptional and corresponds to the misterious centro-affine curvature (see Ludwig [62] or Stancu [86]), which is equi-affine invariant. Partial results on the $L_{-n}$-Minkowski problem are due to for example Ivaki [51], Jian, Lu, Zhu [54], Li [59], Zhu [97].

The so called Orlicz version of the $L_p$ Minkowski problem generalizes the Monge-Ampère equation (2) on $S^{n-1}$, and considers the equation

$$\varphi(h) \det(\nabla^2 h + h\text{Id}) = f$$

for suitable function $\varphi$ replacing $t \mapsto t^{1-p}$.

If $p > 1$, then the Orlicz $L_p$ Minkowski problem is solved by Haberl, Lutwak, Yang, Zhang [36] for even measures, and by Huang, He [46] in general. If $0 < p < 1$, then the Orlicz $L_p$ Minkowski problem is due to Jian, Lu [53] if $0 < p < 1$. We note that Orlicz versions of the so called $L_p$ dual Minkowski are considered recently by Gardner, Hug, Weil, Xing, Ye [28], Gardner, Hug, Xing, Ye [29], Xing, Ye, Zhu [91] and Xing, Ye [92].

As usual in the case of Orlicz versions of Minkowski type problems, we can only provide a solution up to a constant factor.

**Theorem 1.4.** For $n \geq 2$, $-n < p < 0$ and monotone increasing continuous function $\varphi : [0, \infty) \to [0, \infty)$ satisfying $\varphi(0) = 0$,

$$\liminf_{t \to 0^+} \frac{\varphi(t)}{t^{1-p}} > 0$$

$$\int_1^\infty \frac{1}{\varphi(t)} \, dt < \infty,$$

if the essentially positive function $f$ is in $L_{\frac{n}{n+p}}(S^{n-1})$, then there exists $\lambda > 0$ and a convex body $K \in \mathcal{K}_0^n$ with $V(K) = 1$ such that

$$\lambda \varphi(h) \det(\nabla^2 h + h\text{Id}) = f$$

holds for $h = h_K$ in the Alexandrov sense; namely, $\lambda \varphi(h_K) \, dS_K = f \, d\mathcal{H}^{n-1}$. In addition, if $f$ is invariant under a closed subgroup $G$ of $O(n)$, then $K$ can be chosen to be invariant under $G$.

We note that the origin may lie on $\partial K$ for the solution $K$ in Theorem 1.4.
We observe that Theorem 1.4 readily yields Theorem 1.3 as if \(-n < p < 0\), \(f \in L_{\frac{n}{n+p}}(\mathbb{S}^{n-1})\) is essentially positive and \(\lambda h_{K}^{1-p} dS_K = f d\mathcal{H}^{n-1}\) for \(K \in \mathbb{K}_0^n\) and \(\lambda > 0\), then \(h_{K}^{1-p} dS_{\tilde{K}} = f d\mathcal{H}^{n-1}\) for \(\tilde{K} = \lambda^{\frac{1}{p-n}} K\).

In Section 2 we sketch the proof of Theorem 1.4 and the structure of the paper.

2. SKETCH OF THE PROOF OF THEOREM 1.4

To sketch the argument leading to Theorem 1.4 first we consider the case when \(-n < p \leq -(n-1)\) and \(\varphi(t) = t^{1-p}\), and \(\tau_1 \leq f \leq \tau_2\) for constants \(\tau_2 > \tau_1 > 0\). We set \(\psi(t) = \frac{1}{\varphi(t)} = t^{p-1}\) for \(t > 0\), and define \(\Psi : (0, \infty) \to (0, \infty)\) by

\[
\Psi(t) = \int_t^{\infty} \psi(s) ds = \frac{1}{p} t^p,
\]

which is a strictly convex function.

Given a convex body \(K\) in \(\mathbb{R}^n\), we set

\[
\Phi(K, \xi) = \int_{S^{n-1}} \Psi(h_{K-\xi}) f d\mathcal{H}^{n-1},
\]

which is a strictly convex function of \(\xi \in \text{int } K\). As \(f > \tau_1\) and \(p \leq -(n-1)\), there is a (unique) \(\xi(K) \in \text{int } K\) such that

\[
\Phi(K, \xi(K)) = \min_{\xi \in \text{int } K} \Phi(K, \xi).
\]

The statement is proved in Proposition 4.2, but the conditions \(f > \tau_1\) and \(p \leq -(n-1)\) are actually used in the preparatory statement Lemma 4.1.

Using \(p > -n\) and the Blaschke-Santaló inequality (see Lemma 4.4 and the preparatory statement Lemma 3.3), one verifies that there exists a convex body \(K_0\) in \(\mathbb{R}^n\) with \(V(K_0) = 1\) minimizing \(\Phi(K, \xi(K))\) over all convex bodies \(K\) in \(\mathbb{R}^n\) with \(V(K) = 1\).

Finally a variational argument proves that there exists \(\lambda_0 > 0\) such that \(f d\mathcal{H}^{n-1} = \lambda_0 \varphi(h_{K_0}) S_{K_0}\). A crucial ingredient (see Lemma 5.2) is that as \(\psi\) is \(C^1\) and \(\psi' < 0\), \(\Phi(K_t, \xi(K_t))\) is a differentiable function of \(K_t\) for suitable variation \(K_t\) of \(K_0\).

In the general case, when still keeping the condition \(\tau_1 \leq f \leq \tau_2\), but only knowing about \(\varphi\) that \(-n < p < 0\), \(\varphi[0, \infty) \to [0, \infty)\) is continuous and increasing satisfying \(\varphi(0) = 0\),

\[
\liminf_{t \to 0^+} \frac{\varphi(t)}{t^{1-p}} > 0 \quad \text{and} \quad \int_1^{\infty} \frac{1}{\varphi(t)} dt < \infty,
\]

we meet two main obstacles. On the one hand, even if \(\varphi(t) = t^{1-p}\) but \(0 < t < -(n-1)\), it may happen that for a convex body \(K\) in \(\mathbb{R}^n\), the infimum of \(\Phi(K, \xi)\) for \(\xi \in \text{int } K\) is attained when \(\xi\) tends to the boundary of \(K\). On the other hand, the possible lack of differentiability of \(\varphi\) (or equivalently of \(\psi\)) destroys the variational argument.

Therefore we approximate \(\psi\) by smooth functions, and also make sure that the approximating functions are large enough near zero to ensure that the minimum of the analogues of \(\Phi(K, \xi)\) as a function of \(\xi \in \text{int } K\) exists for any convex body \(K\).

After Section 3 proves some preparatory statements, Section 4 introduces the suitable analogue of the energy function \(\Phi(K, \xi(K))\), and Section 5 provides the variational formula for an extremal body for the energy function. We prove Theorem 1.4 if \(f\) is bounded and bounded away from zero in Section 6 and finally in full strength in Section 7.
3. Some preliminary estimates

In this section, we prove the simple but technical estimates Lemmas 3.1 and 3.3 that will be used in various settings during the main argument.

Lemma 3.1. For $\delta \in (0, 1)$, $A, \tilde{N} > 0$ and $q \in (-n, 0)$, let $\tilde{\psi} : (0, \infty) \to (0, \infty)$ satisfy that $\tilde{\psi}(t) \leq R t^q$ for $t \in (0, \delta]$ and $\int_{\delta}^{\infty} \tilde{\psi} \leq A$. If $t \in (0, \delta)$ and $\tilde{N}_0 = \max \{ \frac{\tilde{N}}{|q|}, A \delta^q \}$, then $\tilde{\Psi}(t) = \int_{\delta}^{\infty} \tilde{\psi}$ satisfies

$$\tilde{\Psi}(t) \leq \tilde{N}_0 t^q.$$

Proof. We observe that if $t \in (0, \delta)$, then

$$\tilde{\Psi}(t) \leq \int_{t}^{\delta} \tilde{\psi}(s) ds + A \leq \tilde{N} \int_{t}^{\delta} s^q ds + A = \frac{\tilde{N}}{|q|} (t^q - \delta^q) + A \leq t^q \max \left\{ \frac{\tilde{N}}{|q|}, A \delta^q \right\}. \quad \text{Q.E.D.}$$

We write $B^n$ to denote the Euclidean unit ball in $\mathbb{R}^n$, and set $\kappa_n = H^n(B^n)$. For a convex body $K$ in $\mathbb{R}^n$, let $\sigma(K)$ denote its centroid, which satisfies (see Schneider [83])

$$(6) \quad -(K - \sigma(K)) \subseteq n(K - \sigma(K)).$$

Next if $o \in \text{int } K$, then the polar of $K$ is

$$K = \{ x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \forall y \in K \} = \{ tu : u \in S^{n-1} \text{ and } 0 \leq t \leq h_K(u)^{-1} \}.$$ 

In particular, the Blaschke-Santaló inequality $V(K) V((K - \sigma(K))^*) \leq V(B^n)^2$ (see Schneider [83]) yields that

$$(7) \quad \int_{S^{n-1}} h_{K - \sigma(K)} d\mathcal{H}^{n-1} \leq \frac{n V(B^n)^2}{V(K)}.$$ 

As a preparation for the proof of Lemma 3.3, we need the following statement about absolute continuous measures. For $t \in (0, 1)$ and $v \in S^{n-1}$, we consider that spherical strip

$$\Xi(v, t) = \{ u \in S^{n-1} | (u, v) \leq t \}.$$

Lemma 3.2. If $f \in L_1(S^{n-1})$ and

$$\varrho_f(t) = \sup_{v \in S^{n-1}} \int_{\Xi(v, t)} |f| d\mathcal{H}^{n-1}$$

for $t \in (0, 1)$, then we have $\lim_{t \to 0^+} \varrho_f(t) = 0$.

Proof. We observe that $\varrho_f(t)$ is decreasing, therefore the limit $\lim_{t \to 0^+} \varrho_f(t) = \delta \geq 0$ exists. We suppose that $\delta > 0$, and seek a contradiction.

Let $\mu$ be the absolute continuous measure $d\mu = |f| d\mathcal{H}^{n-1}$ on $S^{n-1}$. According to the definition of $\varrho_f$, for any $k \geq 2$, there exists some $v_k \in S^{n-1}$ such that $\mu(\Xi(v_k, \frac{1}{k}) \geq \delta/2$. Let $v \in S^{n-1}$ be an accumulation point of the sequence $\{v_k\}$. For any $m \geq 2$, there exists $\alpha_m > 0$ such that $\Xi(u, \frac{1}{2m}) \subset \Xi(v, \frac{1}{m})$ if $u \in S^{n-1}$ and $\angle(u, v, v_k) \leq \alpha_m$. Since for any $m \geq 2$, there exists some $k \geq 2m$ such that $\angle(v_k, v) \leq \alpha_m$, we have $\mu(\Xi(v, \frac{1}{m})) \geq \mu(\Xi(v_k, \frac{1}{k})) \geq \delta/2$. We deduce that $\mu(v^+ \cap S^{n-1}) = \mu(\cap_{m \geq 2} \Xi(v, \frac{1}{m})) \geq \delta/2$, which contradicts $\mu(v^+ \cap S^{n-1}) = 0$. \quad Q.E.D.

Lemma 3.3. For $\delta \in (0, 1)$, $\tilde{N} > 0$ and $q \in (-n, 0)$, let $\tilde{\Psi} : (0, \infty) \to (0, \infty)$ be a monotone decreasing continuous function such that $\tilde{\Psi}(t) \leq R t^q$ for $t \in (0, \delta]$ and $\lim_{t \to \infty} \tilde{\Psi}(t) = 0$, and let $\tilde{f}$ be a non-negative function in $L_{-\frac{n}{n+p}}(S^{n-1})$. Then for any $\zeta > 0$, there exists a $D_\zeta$ depending on $\zeta$, $\tilde{\Psi}$, $\delta$, $\tilde{N}$, $q$ and $\tilde{f}$ such that if $K$ is a convex body in $\mathbb{R}^n$ with $V(K) = 1$ and $\text{diam } K \geq D_\zeta$ then

$$\int_{S^{n-1}} (\tilde{\Psi} \circ h_{K - \sigma(K)}) \tilde{f} d\mathcal{H}^{n-1} \leq \zeta.$$
Proof. We may assume that $\sigma(K) = o$. Let $R = \max_{x \in K} \|x\|$, and let $v \in S^{n-1}$ such that $Rv \in K$. It follows from (5) that $-\frac{R}{n}v \in \hat{K}$.

Since $\lim_{t \to \infty} \tilde{\Psi}(t) = 0$ and $\tilde{f}$ is in $L_1(S^{n-1})$ by the Hölder inequality, we can choose $r \geq 1$ such that

$$
\tilde{\Psi}(r) \int_{S^{n-1}} \tilde{f} \, d\mathcal{H}^{n-1} < \frac{\zeta}{2}.
$$

We partition $S^{n-1}$ into the two measurable parts

$$
\Xi_0 = \{ u \in S^{n-1} : h_K(u) \geq r \},
\Xi_1 = \{ u \in S^{n-1} : h_K(u) < r \}.
$$

Let us estimate the integrals over $\Xi_0$ and $\Xi_1$. We deduce from (8) that

$$
\int_{\Xi_0} (\tilde{\Psi} \circ h_K) \tilde{f} \, d\mathcal{H}^{n-1} \leq \frac{\zeta}{2}.
$$

Next we claim that

$$
\Xi_1 \subset \Xi\left(v, \frac{nr}{R}\right).
$$

For any $u \in \Xi_1$, we choose $\eta \in \{-1, 1\}$ such that $\langle u, \eta v \rangle \geq 0$, thus $\frac{2R}{n}v \in K$ yields that $r > h_K(u) \geq \langle u, \frac{2R}{n}v \rangle$. In turn, we conclude (10). It follows from (10) and Lemma 3.2 that for the $L_1$ function $f = \tilde{f} \frac{n}{n+q}$, we have

$$
\int_{\Xi_1} \tilde{f} \frac{n}{n+q} \leq \varrho_f \left(\frac{nr}{R}\right).
$$

To estimate the decreasing function $\tilde{\Psi}$ on $(0, r)$, we claim that if $t \in (0, r)$ then

$$
\tilde{\Psi}(t) \leq \frac{\tilde{\Psi}R}{r^q} t^q.
$$

We recall that $r \geq 1 > \delta$. In particular, if $t \leq \delta$, then $\tilde{\Psi}(t) \leq \frac{\tilde{\Psi}R}{r^q} t^q$ yields (12). If $t \in (\delta, r)$, then using that $\tilde{\Psi}$ is decreasing, (12) follows from

$$
\tilde{\Psi}(t) \leq \tilde{\Psi}(\delta) \leq \frac{\tilde{\Psi}R}{r^q} t^q \leq \frac{\tilde{\Psi}R}{r^q} \frac{n}{n+q} t^q.
$$

Applying first (12), then the Hölder inequality, after that the Blaschke-Santaló inequality (7) with $V(K) = 1$ and finally (11), we deduce that

$$
\int_{\Xi_1} (\tilde{\Psi} \circ h_K) \tilde{f} \, d\mathcal{H}^{n-1} \leq \frac{\tilde{\Psi}R}{r^q} \int_{\Xi_1} h_K^{-|q|} \tilde{f} \, d\mathcal{H}^{n-1}
\leq \frac{\tilde{\Psi}R}{r^q} \left(\int_{\Xi_1} h_K^{-n} \, d\mathcal{H}^{n-1}\right)^{\frac{|q|}{n}} \left(\int_{\Xi_1} \tilde{f} \frac{n}{n+q} \, d\mathcal{H}^{n-1}\right)^{\frac{n}{n+q}}
\leq \frac{\tilde{\Psi}R}{r^q} \left(nV(B^n)\right)^{\frac{|q|}{n}} \varrho_f \left(\frac{nr}{R}\right)^{\frac{n+q}{n}}.
$$

Therefore after fixing $r \geq 1$ satisfying (8), we may choose $R_0 > r$ such that

$$
\frac{\tilde{\Psi}R}{r^q} \left(nV(B^n)\right)^{\frac{|q|}{n}} \varrho_f \left(\frac{nr}{R_0}\right)^{\frac{n+q}{n}} \leq \frac{\zeta}{2}.
$$
by Lemma 3.3. In particular, if \( R \geq R_0 \), then
\[
\int_{\mathbb{R}^1} (\tilde{\Psi} \circ h_K) \tilde{f} d\mathcal{H}^{n-1} \leq \frac{\zeta}{2}.
\]
Combining this estimate with (9) shows that setting \( D_\zeta = 2R_0 \), if \( \text{diam} K \geq D_\zeta \), then \( R \geq R_0 \), and hence \( \int_{S^{n-1}} (\tilde{\Psi} \circ h_K) \tilde{f} d\mathcal{H}^{n-1} \leq \zeta \). Q.E.D.

4. **The extremal problem related to Theorem 1.4 when \( f \) is bounded and bounded away from zero**

For \( 0 < \tau_1 < \tau_2 \), let the real function \( f \) on \( S^{n-1} \) satisfy
\[
\tau_1 < f(u) < \tau_2 \quad \text{for} \quad u \in S^{n-1}.
\]
In addition, let \( \varphi : [0, \infty) \to [0, \infty) \) be a continuous monotone increasing function satisfying \( \varphi(0) = 0 \),
\[
\liminf_{t \to 0^+} \frac{\varphi(t)}{t^{1-p}} > 0 \quad \text{and} \quad \int_1^\infty \frac{1}{\varphi(t)} dt < \infty.
\]
It will be more convenient to work with the decreasing function \( \psi = 1/\varphi : (0, \infty) \to (0, \infty) \), which has the properties
\[
\limsup_{t \to 0^+} \frac{\psi(t)}{t^{p-1}} < \infty
\]
\[
\int_1^\infty \psi(t) dt < \infty.
\]
We consider the function \( \Psi : (0, \infty) \to (0, \infty) \) defined by
\[
\Psi(t) = \int_t^\infty \psi(s) ds,
\]
which readily satisfies
\[
\Psi' = -\psi, \quad \text{and hence} \quad \Psi \text{ is convex and strictly monotone decreasing},
\]
\[
\lim_{t \to \infty} \Psi(t) = 0.
\]
According to (14), there exist some \( \delta \in (0, 1) \) and \( \mathcal{H} > 1 \) such that
\[
\psi(t) < \mathcal{H}t^{p-1} \quad \text{for} \quad t \in (0, \delta).
\]

As we pointed out in Section 2, we smoothen \( \psi \) using convolution. Let \( \eta : \mathbb{R} \to [0, \infty) \) be a non-negative \( C^\infty \) ”approximation of identity” with \( \text{supp} \eta \subset [-1, 0] \) and \( \int_\mathbb{R} \eta = 1 \). For any \( \varepsilon \in (0, 1) \), we consider the non-negative \( \eta_\varepsilon(t) = \frac{1}{\varepsilon} \eta(\frac{t}{\varepsilon}) \) satisfying that \( \int_\mathbb{R} \eta_\varepsilon = 1 \) and \( \text{supp} \eta_\varepsilon \subset [-\varepsilon, 0] \), and define \( \theta_\varepsilon : (0, \infty) \to (0, \infty) \) by
\[
\theta_\varepsilon(t) = \int_\mathbb{R} \psi(t - \tau) \eta_\varepsilon(\tau) d\tau = \int_{-\varepsilon}^0 \psi(t - \tau) \eta_\varepsilon(\tau) d\tau.
\]
As \( \psi \) is monotone decreasing and continuous on \((0, \infty)\), the properties of \( \eta_\varepsilon \) yield
\[
\theta_\varepsilon(t) \leq \psi(t) \quad \text{for} \quad t > 0 \quad \text{and} \quad \varepsilon \in (0, 1)
\]
\[
\theta_\varepsilon(t_1) \geq \theta_\varepsilon(t_2) \quad \text{for} \quad t_2 > t_1 > 0 \quad \text{and} \quad \varepsilon \in (0, 1)
\]
\[
\theta_\varepsilon \quad \text{tends uniformly to} \quad \psi \quad \text{on any interval with positive endpoints as} \quad \varepsilon \quad \text{tends to zero}.
\]
Next, for any \( t_0 > 0 \), the function \( l_{t_0} \) on \( \mathbb{R} \) defined by

\[
l_{t_0}(t) = \begin{cases} 
\psi(t) & \text{if } t \geq t_0 \\
0 & \text{if } t < t_0
\end{cases}
\]

is bounded, and hence locally integrable. For the convolution \( l_{t_0} * \eta_\varepsilon \), we have that \( (l_{t_0} * \eta_\varepsilon)(t) = \theta_\varepsilon(t) \) for \( t > t_0 \) and \( \varepsilon \in (0, 1) \), thus

\[
\theta_\varepsilon \text{ is } C^1 \text{ for each } \varepsilon \in (0, 1).
\]

As it is explained in Section 2, we need to modify \( \psi \) in a way such that the new function is of order at least \( t^{-(n-1)} \) if \( t > 0 \) is small. We set

\[
q = \min\{p, -(n-1)\},
\]

and hence (19) and \( \delta \in (0, 1) \) yields that

\[
\theta_\varepsilon(t) \leq \psi(t) < \mathbb{N} t^{q-1} \text{ for } t \in (0, \delta) \text{ and } \varepsilon \in (0, \delta).
\]

Next we construct \( \tilde{\theta}_\varepsilon : (0, \infty) \to (0, \infty) \) satisfying

\[
\tilde{\theta}_\varepsilon(t) = \theta_\varepsilon(t) \leq \psi(t) \text{ for } t \geq \varepsilon \text{ and } \varepsilon \in (0, \delta) \\
\tilde{\theta}_\varepsilon(t) \leq \mathbb{N} t^{q-1} \text{ for } t \in (0, \delta) \text{ and } \varepsilon \in (0, \delta) \\
\tilde{\theta}_\varepsilon(t) = \mathbb{N} t^{q-1} \text{ for } t \in (0, \frac{\varepsilon}{2}] \text{ and } \varepsilon \in (0, \delta)
\]

\( \tilde{\theta}_\varepsilon \) is \( C^1 \) and is monotone decreasing.

It follows that

\( \tilde{\theta}_\varepsilon \) tends uniformly to \( \psi \) on any interval with positive endpoints as \( \varepsilon \) tends to zero.

To construct suitable \( \tilde{\theta}_\varepsilon \), first we observe that the conditions above determine \( \tilde{\theta}_\varepsilon \) outside the interval \( (\frac{\varepsilon}{2}, \varepsilon) \), and \( \tilde{\theta}_\varepsilon(\varepsilon) < \mathbb{N} \varepsilon^{q-1} \). Writing \( \Delta \) to denote the degree one polynomial whose graph is the tangent to the graph of \( t \mapsto \mathbb{N} t^{q-1} \) at \( t = \varepsilon/2 \), we have \( \Delta(t) < \mathbb{N} t^{q-1} \) for \( t > \varepsilon/2 \) and \( \Delta(\varepsilon) < 0 \). Therefore we can choose \( t_0 \in (\frac{\varepsilon}{2}, \varepsilon) \) such that \( \tilde{\theta}_\varepsilon(\varepsilon) < \Delta(t_0) < \mathbb{N} \varepsilon^{q-1} \). We define \( \tilde{\theta}_\varepsilon(t) = \Delta(t) \) for \( t \in (\frac{\varepsilon}{2}, t_0) \), and construct \( \tilde{\theta}_\varepsilon \) on \( (t_0, \varepsilon) \) in a way that such that \( \tilde{\theta}_\varepsilon \) stays \( C^1 \) on \( (0, \infty) \). It follows from the way \( \tilde{\theta}_\varepsilon \) is constructed follows that \( \tilde{\theta}_\varepsilon(t) \leq \mathbb{N} t^{q-1} \) also for \( t \in [\frac{\varepsilon}{2}, \varepsilon] \).

In order to ensure a negative derivative, we consider \( \psi_\varepsilon : (0, \infty) \to (0, \infty) \) defined by

\[
(\psi_\varepsilon(t) = \tilde{\theta}_\varepsilon(t) + \frac{\varepsilon}{1 + t^2}
\]

for \( \varepsilon \in (0, \delta) \) and \( t > 0 \). This \( C^1 \) function \( \psi_\varepsilon \) has the following properties:

\[
\psi_\varepsilon(t) \leq \psi(t) + \frac{1}{1 + t^2} \text{ for } t \geq \varepsilon \text{ and } \varepsilon \in (0, \delta) \\
\psi_\varepsilon(t) < 0 \text{ for } t > 0 \text{ and } \varepsilon \in (0, \delta) \\
\psi_\varepsilon(t) < 2\mathbb{N} t^{q-1} \text{ for } t \in (0, \delta) \text{ and } \varepsilon \in (0, \delta) \\
\psi_\varepsilon(t) > \mathbb{N} t^{q-1} \text{ for } t \in (0, \frac{\varepsilon}{2}) \text{ and } \varepsilon \in (0, \delta)
\]

\( \psi_\varepsilon \) tends uniformly to \( \psi \) on any interval with positive endpoints as \( \varepsilon \) tends to zero.

For \( \varepsilon \in (0, \delta) \), we also consider the \( C^2 \) function \( \Psi_\varepsilon : (0, \infty) \to (0, \infty) \) defined by

\[
\Psi_\varepsilon(t) = \int_t^\infty \psi_\varepsilon(s) \, ds,
\]

and hence (21) yields

\[
\lim_{t \to \infty} \Psi_\varepsilon(t) = 0
\]

(22)

(23) \[ \Psi_\varepsilon' = -\psi_\varepsilon, \] thus \( \Psi_\varepsilon \) is strictly decreasing and strictly convex.
For $\varepsilon \in (0, \delta)$, Lemma 3.1 and (21) imply that setting
\[ A = \int_\delta^\infty \psi(t) + \frac{1}{1 + t^2} dt, \]
we have
\[ (24) \quad \Psi_\varepsilon(t) \leq R_0 t^q \quad \text{for} \quad R_0 = \max\left\{ \frac{2k}{|q|}, \frac{A}{\delta^q} \right\} \quad \text{and} \quad t \in (0, \delta). \]

On the other hand, if $\varepsilon \in (0, \delta)$ and $t \in (0, \frac{\varepsilon}{2})$, then
\[ (25) \quad \Psi_\varepsilon(t) \geq \int_t^{\varepsilon/2} R s^{q-1} ds = \frac{\varepsilon}{|q|} (t^q - (\varepsilon/2)^q) \geq \frac{\varepsilon}{|q|} (t^q - (2t)^q) = R_1 t^q \quad \text{for} \quad R_1 = \frac{1 - 2^q}{|q|} > 0. \]

According to (21), we have $\lim_{\varepsilon \to 0^+} \psi_\varepsilon(t) = \psi(t)$ and $\psi_\varepsilon(t) \leq \psi(t) + \frac{1}{1 + t^2}$ for any $t > 0$, therefore Lebesgue’s Dominated Convergence Theorem implies
\[ (26) \quad \lim_{\varepsilon \to 0^+} \Psi_\varepsilon(t) = \Psi(t) \quad \text{for any} \quad t > 0. \]

It also follows from (21) that if $t \geq \varepsilon$, then
\[ (27) \quad \Psi_\varepsilon(t) = \int_t^\infty \psi \leq \int_t^\infty \psi(s) + \frac{1}{1 + s^2} ds = \Psi(t) + \frac{\pi}{2}. \]

For any convex body $K$ and $\xi \in \text{int} K$, we consider
\[ \Phi_\varepsilon(K, \xi) = \int_{S^{n-1}} (\Psi_\varepsilon \circ h_{K-\xi}) f d\mathcal{H}^{n-1} = \int_{S^{n-1}} \Psi_\varepsilon(h_K(u) - \langle \xi, u \rangle) f(u) d\mathcal{H}^{n-1}(u). \]

Naturally, $\Phi_\varepsilon(K)$ depends on $\psi$ and $f$, as well, but we do not signal these dependences.

We equip $K^n_0$ with the Hausdorff metric, which is the $C_\infty$ metric on the space of the restrictions of support functions to $S^{n-1}$. For $v \in S^{n-1}$ and $\alpha \in [0, \frac{\pi}{2}]$, we consider the spherical cap
\[ \Omega(v, \alpha) = \{ u \in S^{n-1} \langle u, v \rangle \geq \cos \alpha \}. \]

We write $\pi : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ the radial projection. In particular, if $\pi$ is restricted to the boundary of a $K \in K^n_0$, then this map is Lipschitz. Another typical application of the radial projection is to consider, for $v \in S^{n-1}$, the composition $x \mapsto \pi(x + v)$ as a map $v^+ \to S^{n-1}$ where
\[ (28) \quad \text{the Jacobian of} \quad x \mapsto \pi(x + v) \quad \text{at} \quad x \in v^+ \quad \text{is} \quad (1 + \|x\|^2)^{-n/2}. \]

The following Lemma 4.1 is the statement where we apply directly that $\psi$ is modified to be essentially $t^q$ if $t$ very small.

**Lemma 4.1.** Let $\varepsilon \in (0, \delta)$, and let $\{K_i\}$ be a sequence of convex bodies tending to a convex body $K$ in $\mathbb{R}^n$, and let $\xi_i \in \text{int} K_i$ such that $\lim_{i \to \infty} \xi_i = x_0 \in \partial K$. Then
\[ \lim_{i \to \infty} \Phi_\varepsilon(K_i, \xi_i) = \infty. \]

**Proof.** We may assume that $\lim_{i \to \infty} \xi_i = x_0 = 0$. Let $v \in S^{n-1}$ be an exterior normal to $\partial K$ at $0$, and choose some $R > 0$ such that $K \subset RB^n$. Therefore we may assume that $K_i - \xi_i \subset (R + 1)B^n$, $h_{K_i}(v) < \varepsilon/8$ and $\|\xi_i\| < \varepsilon/8$ for all $\xi_i$, thus $h_{K_i-\xi_i}(v) < \varepsilon/4$ for all $i$.

For any $\zeta \in (0, \frac{\pi}{2})$, there exists $I_\zeta$ such that if $i \geq I_\zeta$, then $\|\xi_i\| \leq \zeta/2$ and $\langle y, v \rangle \leq \zeta/2$ for all $y \in K_i$, and hence $\langle y, v \rangle \leq \zeta$ for all $y \in K_i - \xi_i$. For $i \geq I_\zeta$, any $y \in K_i - \xi_i$ can be written in the form $y = sv + z$ where $s \leq \zeta$ and $z \in v^+ \cap (R + 1)B^n$, thus if $\zeta(v, u) = \alpha \in [\zeta, \frac{\pi}{2})$ for $u \in S^{n-1}$, then we have
\[ (29) \quad h_{K_i-\xi_i}(u) \leq (R + 1) \sin \alpha + \zeta \cos \alpha \leq (R + 2) \alpha. \]
We set $\beta = \frac{\varepsilon}{4(R+2)}$, and for $\zeta \in (0, \beta)$, we define

$$\Omega_{\zeta} = \Omega(v, \beta) \setminus \Omega(v, \zeta).$$

In particular, as $\Psi_\varepsilon(t) \geq \mathcal{N}_1 t^q$ for $t \in (0, \frac{\varepsilon}{2})$ according to (25), if $u \in \Omega_{\zeta}$, then (29) implies

$$\Psi_\varepsilon(h_{K_{-\xi_1}}(u)) \geq \gamma(\langle v, u \rangle)^q$$

for $\gamma = \mathcal{N}_1(R + 2)^q$.

The function $x \mapsto \pi(x + v)$ maps $B_\zeta = v^\perp \cap ((\tan \beta) B^n \setminus (\tan \zeta) B^n)$ bijectively onto $\Omega_{\zeta}$, while $\beta < \frac{1}{8}$ and (28) yield that the Jacobian of the map is $x \mapsto \pi(x + v)$ is at least $2^{-n}$ on $B_\zeta$.

Since $f > \tau_1$ and $\langle v, \pi(x + v) \rangle \leq 2x$ for $x \in B_\zeta$, if $i \geq I_\zeta$, then

$$\Phi_\varepsilon(K_{i}, \xi_1) = \int_{S^{n-1}} \Psi_\varepsilon(h_{K_{i}, -\xi_1}(u)) f(u) \, d\mathcal{H}^{n-1}(u) \geq \int_{\Omega_{\zeta}} \tau_1 \gamma(\langle v, u \rangle)^q \, d\mathcal{H}^{n-1}(u) \geq \frac{\tau_1 \gamma}{2^{n+|q|}} \int_{B_\zeta} \|x\|^q \, d\mathcal{H}^{n-1}(x) = \frac{(n-1)K_{n-1} \tau_1 \gamma}{2^{n+|q|}} \int_{\tan \zeta} t^{\gamma q + n-2} \, dt.$$

As $\zeta > 0$ is arbitrarily small and $q \leq 1 - n$, we conclude that $\lim_{i \to \infty} \Phi_\varepsilon(K_{i}, \xi_1) = \infty$. Q.E.D.

Now we single out the optimal $\xi \in \mathrm{int} K$.

**Proposition 4.2.** For $\varepsilon \in (0, \delta)$ and a convex body $K$ in $\mathbb{R}^n$, there exists a unique $\xi(K) \in \mathrm{int} K$ such that

$$\Phi_\varepsilon(K, \xi(K)) = \min_{\xi \in \mathrm{int} K} \Phi_\varepsilon(K, \xi).$$

In addition, $\xi(K)$ and $\Phi_\varepsilon(K, \xi(K))$ are continuous functions of $K$, and $\Phi_\varepsilon(K, \xi(K))$ is translation invariant.

**Proof.** Let $\xi_1, \xi_2 \in \mathrm{int} K$, $\xi_1 \neq \xi_2$, and let $\lambda \in (0, 1)$. If $u \in S^{n-1} \setminus (\xi_1 - \xi_2)^\perp$, then $\langle u, \xi_1 \rangle \neq \langle u, \xi_2 \rangle$, and hence the strict convexity of $\Psi_\varepsilon$ (see (23)) yields that

$$\Psi_\varepsilon(h_K(u) - \langle u, \lambda \xi_1 + (1 - \lambda) \xi_2 \rangle) > \lambda \varphi_\varepsilon(h_K(u) - \langle u, \xi_1 \rangle) + (1 - \lambda) \varphi_\varepsilon(h_K(u) - \langle u, \xi_2 \rangle),$$

thus $\Phi_\varepsilon(K, \xi)$ is a strictly convex function of $\xi \in \mathrm{int} K$ by $f > \tau_1$.

Let $\xi_i \in \mathrm{int} K$ such that

$$\lim_{i \to \infty} \Phi_\varepsilon(K, \xi_i) = \inf_{\xi \in \mathrm{int} K} \Phi_\varepsilon(K, \xi).$$

We may assume that $\lim_{i \to \infty} \xi_i = x_0 \in K$, and Lemma 4.1 yields $x_0 \in \mathrm{int} K$. Since $\Phi_\varepsilon(K, \xi)$ is a strictly convex and continuous function of $\xi \in \mathrm{int} K$, $x_0$ is the unique minimum point of $\xi \mapsto \Phi_\varepsilon(K, \xi)$, which we denote by $\xi(K)$ (not signalling the dependence on $\varepsilon$, $\psi$ and $f$).

Readily $\xi(K)$ is translation equivariant, and $\Phi_\varepsilon(K, \xi(K))$ is translation invariant.

For the continuity of $\xi(K)$ and $\Phi_\varepsilon(K, \xi(K))$, let us consider a sequence $\{K_i\}$ of convex bodies tending to a convex body $K$ in $\mathbb{R}^n$. We may assume that $\xi(K_i)$ tends to a $x_0 \in K$.

For any $y \in \mathrm{int} K$, there exists an $I$ such that $y \in \mathrm{int} K_i$ for $i \geq I$. Since $h_{K_i}$ tends uniformly to $h_K$ on $S^{n-1}$, we have that

$$\limsup_{i \to \infty} \Phi_\varepsilon(K_i, \xi(K_i)) \leq \lim_{i \to \infty} \Phi_\varepsilon(K_i, y) = \Phi_\varepsilon(K, y).$$

Again Lemma 4.1 implies that $x_0 \in \mathrm{int} K$. It follows that $h_{K_i, -\xi(K_i)}$ tends uniformly to $h_{K-x_0}$, thus

$$\Phi_\varepsilon(K, x_0) = \lim_{i \to \infty} \Phi_\varepsilon(K_i, \xi(K_i)) \leq \lim_{i \geq I} \Phi_\varepsilon(K_i, y) = \Phi_\varepsilon(K, y).$$
In particular, \( \Phi_\varepsilon(K, x_0) \leq \Phi_\varepsilon(K, y) \) for any \( y \in \text{int } K \), thus \( x_0 = \xi(K) \). In turn, we deduce \( \xi(K_i) \) tends to \( \xi(K) \), and \( \Phi_\varepsilon(K_i, \xi(K_i)) \) tends to \( \Phi_\varepsilon(K, \xi(K)) \). Q.E.D.

Since \( \xi \mapsto \Phi_\varepsilon(K, \xi) = \int_{S^{n-1}} \Psi_\varepsilon(h_K(u) - \langle u, \xi \rangle) f(u) \, d\mathcal{H}^{n-1}(u) \) is maximal at \( \xi(K) \in \text{int } K \) and \( \Psi' = -\psi' \), we deduce

**Corollary 4.3.** For \( \varepsilon \in (0, \delta) \) and a convex body \( K \) in \( \mathbb{R}^n \), we have

\[
\int_{S^{n-1}} u \, \psi_\varepsilon \left( h_K(u) - \langle u, \xi(K) \rangle \right) f(u) \, d\mathcal{H}^{n-1}(u) = o.
\]

For a closed subgroup \( G \) of \( O(n) \), we write \( \mathcal{K}^{n,G}_{(0)} \) to denote the family of \( K \in \mathcal{K}^n_{(0)} \) invariant under \( G \).

**Lemma 4.4.** For \( \varepsilon \in (0, \delta) \), there exists a \( K^\varepsilon \in \mathcal{K}^{n}_{(0)} \) with \( V(K^\varepsilon) = 1 \) such that

\[
\Phi_\varepsilon(K^\varepsilon, \xi(K^\varepsilon)) \geq \Phi_\varepsilon(K, \xi(K)) \quad \text{for any } K \in \mathcal{K}^{n}_{(0)} \text{ with } V(K) = 1.
\]

In addition, if \( f \) is invariant under a closed subgroup \( G \) of \( O(n) \), then \( K^\varepsilon \) can be chosen to be invariant under \( G \).

**Proof.** We choose a sequence \( K_i \in \mathcal{K}^{n}_{(0)} \) with \( V(K_i) = 1 \) for \( i \geq 1 \) such that

\[
\lim_{i \to \infty} \Phi(K_i, \xi(K_i)) = \sup \{ \Phi(K, \xi(K)) : K \in \mathcal{K}^{n}_{(0)} \text{ with } V(K) = 1 \}.
\]

Writing \( B_1 = \kappa_{n^{-1/n}} B^n \) to denote the unit ball centred at the origin and having volume 1, we may assume that each \( K_i \) satisfies

\[
\Phi_\varepsilon(K_i, \sigma(K_i)) \geq \Phi_\varepsilon(K_i, \xi(K_i)) \geq \Phi_\varepsilon(B_1, \sigma(B_1)).
\]

According to Lemma 4.2, we may also assume that \( \sigma(K_i) = o \) for each \( K_i \).

We deduce from Lemma 3.3 [22] and [21] and [30] that there exists some \( R > 0 \) such that \( K_i \subset RB^n \) for any \( i \geq 1 \). According to the Blaschke selection theorem, we may assume that \( K_i \) tends to a compact convex set \( K^\varepsilon \) with \( o \in K^\varepsilon \). It follows from the continuity of the volume that \( V(K^\varepsilon) = 1 \), and hence \( \text{int } K^\varepsilon \neq \emptyset \). We conclude from Lemma 4.2 that \( \Phi_\varepsilon(K^\varepsilon, \xi(K^\varepsilon)) = \lim_{i \to \infty} \Phi_\varepsilon(K_i, \xi(K_i)) \).

If \( f \) is invariant under a closed subgroup \( G \) of \( O(n) \), then we apply the same argument to convex bodies in \( \mathcal{K}^{n,G}_{(0)} \) instead of \( \mathcal{K}^{n}_{(0)} \). Q.E.D.

Since \( \Phi(5) < \Phi(4), [26] \) yields some \( \tilde{\delta} \in (0, \delta) \) such that \( \Psi_\varepsilon(4) \geq \Phi(5) \) for \( \varepsilon \in (0, \tilde{\delta}) \). For future reference, the monotonicity of \( \Psi_\varepsilon \), diam\( \kappa_{n^{-1/n}} B^n \leq 4 \) and [30] yield that if \( \varepsilon \in (0, \tilde{\delta}) \), then

\[
\Phi_\varepsilon(K^\varepsilon, \sigma(K^\varepsilon)) \geq \Phi_\varepsilon(\kappa_{n^{-1/n}} B^n, \xi(\kappa_{n^{-1/n}} B^n)) \geq \int_{S^{n-1}} \Psi_\varepsilon(4) f \, d\mathcal{H}^{n-1} \geq \Psi(5) \int_{S^{n-1}} f \, d\mathcal{H}^{n-1}.
\]

5. **Variational formulae and smoothness of the extremal body when \( f \) is bounded and bounded away from zero**

In this section, again let \( 0 < \tau_1 < \tau_2 \) and let the real function \( f \) on \( S^{n-1} \) satisfy \( \tau_1 < f < \tau_2 \). In addition, let \( \varphi \) the continuous function of Theorem 1.4, and we use the notation developed in Section 4 say \( \psi : (0, \infty) \to (0, \infty) \) is defined by \( \psi = 1/\varphi \).

Now that we have constructed an extremal body \( K^\varepsilon \), we want to show that it satisfies the required differential equation in the Alexandrov sense by using a variational argument. This section provides the formulae we will need, and ensure the required smoothness of \( K^\varepsilon \).
Concerning the variation of volume, a key tool is Alexandrov’s Lemma 5.1 (see Lemma 7.5.3 in [83]). To state this, for any continuous \( h : S^{n-1} \to (0, \infty) \), we define the Alexandrov body
\[
[h] = \{ x \in \mathbb{R}^n : \langle x, u \rangle \leq h(u) \text{ for } u \in S^{n-1} \}
\]
which is a convex body containing the origin in its interior. Obviously, if \( K \in \mathcal{K}^n_{(0)} \) then \( K = [h_K] \).

**Lemma 5.1 (Alexandrov).** For \( K \in \mathcal{K}^n_{(0)} \) and continuous function \( g : S^{n-1} \to \mathbb{R} \), \( K(t) = [h_K + tg] \) satisfies
\[
\lim_{t \to 0} \frac{V(K(t)) - V(K)}{t} = \int_{S^{n-1}} g(u) \, dS_K(u).
\]

To handle the variation of \( \Phi_\varepsilon(K(t), \xi(K(t))) \) for a family \( K(t) \) is a more subtle problem. The next lemma shows essentially that if we perturb a convex body \( K \) in a way such that the support function is differentiable as a function of the time parameter for \( \mathcal{H}^{n-1} \)-almost all \( u \in S^{n-1} \), then \( \xi(K) \) changes also in a differentiable way. Lemma 5.2 is the point to use that \( \psi_\varepsilon \) is \( C^1 \) and \( \psi'_\varepsilon < 0 \).

**Lemma 5.2.** For \( \varepsilon \in (0, \delta) \), let \( c > 0 \) and \( t_0 > 0 \), and let \( K(t) \) be a family of convex bodies with support function \( h_t \), \( t \in [0, t_0) \). Assume that
\begin{enumerate}[(i)]
  \item \( |h_t(u) - h_0(u)| \leq ct \text{ for each } u \in S^{n-1} \text{ and } t \in (0, t_0) \),
  \item \( \lim_{t \to 0^+} \frac{h_t(u) - h_0(u)}{t} \) exists for \( \mathcal{H}^{n-1} \)-almost all \( u \in S^{n-1} \).
\end{enumerate}
Then \( \lim_{t \to 0^+} \frac{\xi(K(t)) - \xi(K(0))}{t} \) exists.

**Proof.** We set \( K = K(0) \). We may assume that \( \xi(K) = 0 \), and hence Lemma 4.2 yields that
\[
\lim_{t \to 0^+} \xi(K(t)) = 0.
\]

There exists some \( R > r > 0 \) such that \( r \leq h_t(u) - \langle u, \xi(K(t)) \rangle = h_{K(t)-\xi(K(t))}(u) \leq R \) for \( u \in S^{n-1} \) and \( t \in (0, t_0) \). Since \( \psi_\varepsilon \) is \( C^1 \) on \([r, R]\), we can write
\[
\psi_\varepsilon(t) - \psi_\varepsilon(s) = \psi'_\varepsilon(s)(t - s) + \eta_0(t - s)(t - s)
\]
for \( t, s \in [r, R] \) where \( \lim_{a \to 0} \eta_0(a) = 0 \). Let \( g(t, u) = h_t(u) - h_K(u) \) for \( u \in S^{n-1} \) and \( t \in [0, t_0) \). Since \( h_{K(t)-\xi(K(t))} \) tends uniformly to \( h_K \) on \( S^{n-1} \), we deduce that if \( t \in [0, t_0) \), then
\[
(32) \quad \psi_\varepsilon(h_t(u) - \langle u, \xi(K(t)) \rangle) - \psi_\varepsilon(h_K(u)) = \psi'_\varepsilon(h_K(u))(g(t, u) - \langle u, \xi(K(t)) \rangle) + e(t, u)
\]
where
\[
|e(t, u)| \leq \eta(t)|g(t, u) - \langle u, \xi(K(t)) \rangle| \quad \text{and} \quad \lim_{t \to 0^+} \eta(t) = 0.
\]
In particular, (i) yields that \( e(t, u) = e_1(t, u) + e_2(t, u) \) where
\[
|e_1(t, u)| \leq c\eta(t)t \quad \text{and} \quad |e_2(t, u)| \leq \eta(t)|\xi(K(t))|.
\]

It follows from (32) and from applying Corollary 4.3 to \( K(t) \) and \( K \) that
\[
\int_{S^{n-1}} u \left( \psi'_\varepsilon(h_0(u))(g(t, u) - \langle u, \xi(K(t)) \rangle) + e(t, u) \right) f(u) d\mathcal{H}^{n-1}(u) = o,
\]
which can be written as
\[
\int_{S^{n-1}} u \psi'_\varepsilon(h_K(u)) g(t, u) f(u) d\mathcal{H}^{n-1}(u)
\]
\[
+ \int_{S^{n-1}} u e_1(t, u) f(u) d\mathcal{H}^{n-1}(u) - \int_{S^{n-1}} u e_2(t, u) f(u) d\mathcal{H}^{n-1}(u).
\]
Since \( \psi'(s) < 0 \) for all \( s > 0 \), the symmetric matrix
\[
A = \int_{S^{n-1}} u \otimes u \psi'(h_0(u)) f(u) d\mathcal{H}^{n-1}(u)
\]
is negative definite because for any \( v \in S^{n-1} \), we have
\[
v^T Av = \int_{S^{n-1}} \langle u, v \rangle^2 \psi'(h_0(u)) f(u) d\mathcal{H}^{n-1}(u) < 0.
\]
In addition, \( A \) satisfies that there exists \( \varepsilon' \) such that
\[
\int_{S^{n-1}} u \langle u, \xi(K_t) \rangle \psi'(h_0(u)) f(u) d\mathcal{H}^{n-1}(u) = A \xi(K_t).
\]
Proof. We write \( u = \int_{S^{n-1}} u \langle u, v \rangle^2 \psi'(h_0(u)) f(u) d\mathcal{H}^{n-1}(u) \). Adding the estimate \( g(t, u) \leq \eta t \) and \( \xi(K_t) > 0 \) for constants \( \alpha_1, \alpha_2 > 0 \). Since \( \eta(t) \) tends to 0 with \( t \), if \( t \) is small, then \( \xi(K_t) = \xi(0) = 0 \) and \( \xi(0) = 0 \) for \( t = 1, 2 \). Since there exists \( \lim_{t \to 0^+} \frac{\xi(K_t)}{t} = 0 \) for \( K \) almost all \( u \in S^{n-1} \), and \( \frac{\xi(K_t)}{t} < c \) for all \( u \in S^{n-1} \) and \( t > 0 \), we conclude that
\[
\frac{d}{dt} \xi(K_t) = A^{-1} \int_{S^{n-1}} u \psi'(h_K(u)) \partial_t g(0, u) f(u) d\mathcal{H}^{n-1}(u). \quad \text{Q.E.D.}
\]
Corollary 5.3. Under the conditions of Lemma 5.2 and setting \( K = K(0) \), we have
\[
\frac{d}{dt} \Phi_\varepsilon(K(t), K(t)) \bigg|_{t=0^+} = -\int_{S^{n-1}} \frac{\partial}{\partial t} h_0(u) \bigg|_{t=0^+} \psi_\varepsilon(h_K(u) - \langle u, \xi(K) \rangle) d\mu(u).
\]
Proof. We write \( h(t, u) = h_{K(t)}(u) \) and \( \xi(t) = \xi(K(t)) \), and set \( \lim_{t \to 0^+} h(t, u) f(u) d\mathcal{H}^{n-1} = 0 \). Applying first Lebesgue’s Dominated Convergence Theorem, after that Lemma 5.2 and finally Corollary 5.3, we have
\[
\frac{d}{dt} \Phi_\varepsilon(K(t), K(t)) \bigg|_{t=0^+} = \frac{d}{dt} \int_{S^{n-1}} \Psi_\varepsilon(h(t, u) - \langle u, \xi(t) \rangle) d\mu(u) \bigg|_{t=0^+}
\]
\[
= -\int_{S^{n-1}} \partial_t h_0(u) \psi_\varepsilon(h_K(u) - \langle u, \xi(K) \rangle) d\mu(u)
\]
\[
= -\int_{S^{n-1}} \langle u, \xi'(0) \rangle \psi_\varepsilon(h_K(u) - \langle u, \xi(K) \rangle) d\mu(u)
\]
\[
= -\int_{S^{n-1}} \partial_t h_0(u) \psi_\varepsilon(h_K(u) - \langle u, \xi(K) \rangle) d\mu(u). \quad \text{Q.E.D.}
\]
Given a family \( K(t) \) of convex bodies for \( t \in [0, t_0) \), \( t_0 > 0 \), to handle the variation of \( \Phi_\varepsilon(K(t), K(t)) \) at \( K(0) = K \) via applying Corollary 5.3, we need the properties (see Lemma 5.2) that there exists \( c > 0 \) such that
\[
|h_{K(t)}(u) - h_K(u)| \leq c |t| \quad \text{for any } u \in S^{n-1} \text{ and } t \in [0, t_0)
\]
\[
\lim_{t \to 0^+} \frac{h_{K(t)}(u) - h_K(u)}{t} \quad \text{exists for } \mathcal{H}^{n-1} \text{ almost all } u \in S^{n-1}.
\]
However, even if $K(t) = [h_K + th_C]$ for $K, C \in \mathcal{K}^n_{(0)}$ and for $t \in (-t_0, t_0)$, $K$ must satisfy some smoothness assumption in order to ensure that (36) holds also for the two sided limits (problems occur say if $K$ is a polytope and $C$ is smooth).

We say that $K$ is quasi-smooth if $\mathcal{H}^{n-1}(S^{n-1} \setminus \nu_K(\partial K)) = 0$; namely, the set of $u \in S^{n-1}$ that are exterior normals only at singular points has $\mathcal{H}^{n-1}$-measure zero. The following Lemma 5.4 taken from Bianchi, Böröczky, Colesanti, Yang [6] shows that (35) and (36) are satisfied even if $t \in (-t_0, t_0)$ at least for $K(t) = [h_K + th_C]$ with arbitrary $C \in \mathcal{K}^n_{(0)}$ if $K$ is quasi-smooth.

**Lemma 5.4.** Let $K, C \in \mathcal{K}^n_{(0)}$ be such that $rC \subset K$ for some $r > 0$. For $t \in (-r, r)$ and $K(t) = [h_K + th_C]$, 

(i): if $K \subset RC$ for $R > 0$, then $|h_{K(t)}(u) - h_K(u)| \leq \frac{R}{r} |t|$ for any $u \in S^{n-1}$ and $t \in (-r, r)$;

(ii): if $u \in S^{n-1}$ is the exterior normal at some smooth point $z \in \partial K$, then

$$\lim_{t \to 0} \frac{h_{K(t)}(u) - h_K(u)}{t} = h_C(u).$$

We will need the condition (36) in the following rather special setting taken from Bianchi, Böröczky, Colesanti, Yang [6].

**Lemma 5.5.** Let $K$ be a convex body with $rB^n \subset \text{int} K$ for $r > 0$, let $\omega \subset S^{n-1}$ be closed, and if $t \in [0, r)$, then let

$$K(t) = [h_K - 1_\omega] = \{x \in K : \langle x, u \rangle \leq h_K(u) - t \text{ for every } u \in \omega\}.$$

(i): We have $\lim_{t \to 0^+} \frac{h_{K(t)}(u) - h_K(u)}{t}$ exists and non-positive for all $u \in S^{n-1}$, and if $u \in \omega$, then even $\lim_{t \to 0^+} \frac{h_{K(t)}(u) - h_K(u)}{t} \leq -1$.

(ii): If $S_K(\omega) = 0$, then $\lim_{t \to 0^+} \frac{V(K(t))-V(K)}{t} = 0$.

We recall that a convex body $K$ is quasi-smooth if $\mathcal{H}^{n-1}(S^{n-1} \setminus \nu_K(\partial K)) = 0$

**Proposition 5.6.** For $\varepsilon \in (0, \delta)$, $K^\varepsilon$ is quasi-smooth.

**Proof.** We suppose that $K^\varepsilon$ is not quasi-smooth, and seek a contradiction. It follows that $\mathcal{H}^{n-1}(X) > 0$ for $X = S^{n-1} \setminus \nu_{K^\varepsilon}(\partial K^\varepsilon)$, therefore there exists closed $\omega \subset X$ such that $\mathcal{H}^{n-1}(\omega) > 0$. Since $\nu_{K^\varepsilon}^{-1}(\omega) \subset \partial K^\varepsilon$, we deduce that $S_{K^\varepsilon}(\omega) = 0$.

We may assume that $\xi(K^\varepsilon) = 0$ and $rB^n \subset K \subset RB^n$ for $R > r > 0$. As in Lemma 5.5 if $t \in [0, r)$, then we define

$$K(t) = [h_K - 1_\omega] = \{x \in K : \langle x, u \rangle \leq h_K(u) - t \text{ for every } u \in \omega\}.$$

We define $\alpha(t) = V(K(t))^{-1/n}$, and hence $\alpha(0) = 1$, and Lemma 5.5 (ii) yields that $\alpha'(0) = 0$.

We set $\tilde{K}(t) = \alpha(t) V(K(t))$, and hence $\tilde{K}(0) = K$ and $V(\tilde{K}(t)) = 1$ for all $t \in [0, r)$. In addition, we consider $\tilde{h}(t, u) = h_{K(t)}(u)$ and $\tilde{h}(t, u) = h_{\tilde{K}(t)}(u) = \alpha(t) h(t, u)$ for $u \in S^{n-1}$ and $t \in [0, r)$. Since $[h_K - th_{B^n}] \subset K(t)$, Lemma 5.4 (i) yields that $|h(t, u) - h(0, u)| \leq \frac{K}{r} |t|$ for $u \in S^{n-1}$ and $t \in [0, r)$, and hence $\alpha'(0) = 0$ implies $c > 0$ and $t_0 \in (0, r)$ such that $|\tilde{h}(t, u) - \tilde{h}(0, u)| \leq ct$ for $u \in S^{n-1}$ and $t \in [0, t_0)$. Applying $\alpha(0) = 1$, $\alpha'(0) = 0$ and Lemma 5.5 (i), we deduce that

$$\partial_1 \tilde{h}(0, u) = \lim_{t \to 0^+} \frac{\tilde{h}(t, u) - \tilde{h}(0, u)}{t} \leq -1$$

for all $u \in \omega$,
As \( \psi \) is positive and monotone decreasing, \( f > \tau_1 \) and \( \mathcal{H}^{n-1}(\omega) > 0 \), Corollary 5.3 implies that

\[
\frac{d}{dt} \Phi_\varepsilon(\tilde{K}(t), \xi(\tilde{K}(t))) \bigg|_{t=0} = - \int_{S^{n-1}} \partial_1 \tilde{h}(0, u) \cdot \psi_\varepsilon(h_K(u)) f(u) \, d\mathcal{H}^{n-1}(u) \\
\geq - \int_\omega (1) \psi_\varepsilon(R) \tau_1 \, d\mathcal{H}^{n-1}(u) > 0.
\]

Therefore \( \Phi_\varepsilon(\tilde{K}(t), \xi(\tilde{K}(t))) > \Phi_\varepsilon(K_\varepsilon, \xi(K_\varepsilon)) \) for small \( t > 0 \), which contradiction proves Proposition 5.6. Q.E.D.

For \( \varepsilon \in (0, \delta) \), we define

\[
(37) \quad \lambda_\varepsilon = \frac{1}{n} \int_{S^{n-1}} h_{K_\varepsilon - \xi(K_\varepsilon)} \cdot \psi_\varepsilon(h_{K_\varepsilon - \xi(K_\varepsilon)}) \cdot f \, d\mathcal{H}^{n-1}.
\]

**Proposition 5.7.** For \( \varepsilon \in (0, \delta) \), \( \psi_\varepsilon(h_{K_\varepsilon - \xi(K_\varepsilon)}) \cdot f \, d\mathcal{H}^{n-1} = \lambda_\varepsilon \, dS_{K_\varepsilon} \) as measures on \( S^{n-1} \).

**Proof.** We assume that \( \xi(K_\varepsilon) = 0 \), and set \( d\mu = f \, d\mathcal{H}^{n-1} \). First we claim any \( C \in \mathcal{K}_\varepsilon(0) \) satisfies

\[
(38) \quad \int_{S^{n-1}} h_C \lambda_\varepsilon \, dS = \int_{S^{n-1}} h_C \psi_\varepsilon(h_{K_\varepsilon - \xi(K_\varepsilon)}) \, d\mu(u).
\]

Let \( rC \subset K_\varepsilon \subset RC \) for \( R > r > 0 \). For \( t \in (-r, r) \), we consider we consider \( K(t) = [h_{K_\varepsilon} + th_C] \), \( \alpha(t) = V(K(t))^{-1/\varepsilon} \) and \( \tilde{K}(t) = \alpha(t) K(t) \). We have \( V(\tilde{K}(t)) = 1 \) for \( t \in (-r, r) \) and \( \alpha(0) = 1 \). Lemma 5.4 yields that

\[
\frac{d}{dt} V(K_t) \bigg|_{t=0} = \int_{S^{n-1}} h_C \, dS_{K_\varepsilon}.
\]

and hence

\[
(39) \quad \alpha'(0) = -\frac{1}{n} \int_{S^{n-1}} h_C \, dS_{K_\varepsilon}.
\]

We write \( h(t, u) = h_{K_\varepsilon}(u) \). Since \( K_\varepsilon \) is quasi-smooth by Proposition 5.6, Lemma 5.4 (i) and (ii) imply that there exists \( c > 0 \) such that if \( t \in (-r, r) \), then \( |h(t, u) - h(0, u)| \leq c|t| \) for any \( u \in S^{n-1} \), and \( \lim_{t \to 0} \frac{h(t, u) - h(0, u)}{t} = h_C(u) \) exists for \( \mathcal{H}^{n-1}\)-a.e. \( u \in S^{n-1} \). Next let \( \tilde{h}(t, u) = \alpha(t) h(t, u) = h_{K_t}(u) \) for \( u \in S^{n-1} \) and \( t \in (-r, r) \). From the properties of \( h(t, u) \) above and (39) it follows the existence of \( \tilde{c} > 0 \) such that if \( t \in (-r, r) \), then \( |\tilde{h}(t, u) - h(0, u)| \leq \tilde{c}|t| \) for any \( u \in S^{n-1} \), and

\[
\lim_{t \to 0} \frac{\tilde{h}(t, u) - h(0, u)}{t} = \partial_1 \tilde{h}(0, u) = \alpha'(0) h_{K_\varepsilon}(u) + h_C(u)
\]

for any \( u \in S^{n-1} \). As \( \Phi(\tilde{K}_t, \xi(\tilde{K}_t)) \) has a minimum at \( t = 0 \) by the extremal property of \( K_\varepsilon = \tilde{K}_0 \), Corollary 5.3 imply

\[
0 = \frac{d}{dt} \Phi(\tilde{K}_t, \xi(\tilde{K}_t)) \bigg|_{t=0} = - \int_{S^{n-1}} \partial_1 \tilde{h}(0, u) \cdot \psi_\varepsilon(h_{K_\varepsilon}(u)) \, d\mu(u)
\]

\[
= - \int_{S^{n-1}} (\alpha'(0) h_{K_\varepsilon}(u) + h_C(u)) \psi_\varepsilon(h_{K_\varepsilon}(u)) \, d\mu(u)
\]

\[
= - \int_{S^{n-1}} h_C(u) \psi_\varepsilon(h_{K_\varepsilon}(u)) \, d\mu(u) + \frac{1}{n} \int_{S^{n-1}} h_C \lambda_\varepsilon \, dS_{K_\varepsilon},
\]

and in turn we deduce (38).
Since differences of support functions are dense among continuous functions on $S^{n-1}$ (see e.g. [33]), we have

$$\int_{S^{n-1}} g\lambda \, dS_K = \int_{S^{n-1}} g \psi_\varepsilon(h_K) \, d\mu$$

for any continuous function $g$ on $S^{n-1}$. Therefore $\lambda \, dS_K = \psi_\varepsilon(h_K) \, d\mu$. Q.E.D.

6. **The proof of Theorem 4.4 when $f$ is bounded and bounded away from zero**

In this section, again let $0 < \tau_1 < \tau_2$, let the real function $f$ on $S^{n-1}$ satisfy $\tau_1 < f < \tau_2$, and let the $\varphi$ the continuous function on $[0, \infty)$ of Theorem 4.4. We use the notation developed in Section 4 and hence $\varphi : (0, \infty) \to (0, \infty)$ and $\psi = 1/\varphi$.

To ensure that a convex body is ”fat” enough in Lemma 6.2 and later, the following observation is useful:

**Lemma 6.1.** If $K$ is a convex body in $\mathbb{R}^n$ with $V(K) = 1$ and $K \subset \sigma(K) + RB^n$ for $R > 0$, then $\sigma(K) + rB^n \subset K$ for $r = \frac{\kappa_{n-1}}{2^n} R^{-(n-1)}$.

**Proof.** Let $z_0 + RroB^n$ be a largest ball in $K$. According to the Steinhagen theorem, there exists $v \in S^{n-1}$ such that

$$|\langle x - z_0, v \rangle| \leq nr_0$$

for $x \in K$.

It follows that $1 = V(K) \leq 2nr_0\kappa_{n-1}R^{-n+1}$, thus $r_0 \geq \frac{\kappa_{n-1}}{2^n} R^{-(n-1)}$. Since $\sigma(K) + \frac{r_0}{n} B^n \subset K$ by $-(K - \sigma(K)) \subset n(K - \sigma(K))$, we may choose $r = \frac{\kappa_{n-1}}{2^n} R^{-(n-1)}$ Q.E.D.

We recall (compare (37)) that if $\varepsilon \in (0, \delta_0)$ and $\xi(K^\varepsilon) = \sigma$, then $\lambda\varepsilon$ is defined by

$$\lambda\varepsilon = \frac{1}{n} \int_{S^{n-1}} hK^\varepsilon \psi_\varepsilon(hK^\varepsilon) f \, dH^{n-1}.$$  

**Lemma 6.2.** There exist $R_0 > 1$, $r_0 > 0$ and $\tilde{\lambda}_1 > \lambda_1 > 0$ depending on $f, q, \psi, R$ such that if $\varepsilon \in (0, \delta_0)$ for $\delta_0 = \min\{\delta, \frac{\pi}{2}\}$ where $\delta$ comes from (37), then $\lambda_1 \leq \lambda\varepsilon \leq \tilde{\lambda}_2$ and $\sigma(K^\varepsilon) + rB^n \subset K^\varepsilon \subset \sigma(K^\varepsilon) + R_0B^n$.

**Proof.** According to (24), there exists $\kappa_0 > 0$ depending on $q, \psi, R$ such that if $\varepsilon \in (0, \delta)$ and $t \in (0, \delta)$, then $\Psi_\varepsilon(t) \leq \kappa_0 t$. In addition, $\lim_{t \to \infty} \Psi_\varepsilon(t) = 0$ by (22), therefore we may apply Lemma 3.3. Since (31) provides the condition

$$\int_{S^{n-1}} \Psi^\varepsilon(hK^\varepsilon - \sigma(K^\varepsilon)) f \, dH^{n-1} \geq \Psi(5) \int_{S^{n-1}} f \, dH^{n-1}$$

for any $\varepsilon \in (0, \tilde{\delta})$, we deduce from Lemma 3.3 the existence of $R_0 > 0$ such that $K^\varepsilon \subset \sigma(K^\varepsilon) + R_0B^n$ for any $\varepsilon \in (0, \tilde{\delta})$. In addition, the existence of $r_0$ independent of $\varepsilon$ such that $\sigma(K^\varepsilon) + rB^n \subset K^\varepsilon$ follows from Lemma 6.1.

To estimate $\lambda\varepsilon$, we assume $\xi(K^\varepsilon) = \sigma$. Let $w_\varepsilon \in S^{n-1}$ and $\varphi_\varepsilon \geq 0$ be such that $\sigma(K^\varepsilon) = \varphi_\varepsilon w_\varepsilon$, and hence $r_0w_\varepsilon \in K^\varepsilon$. It follows that $h_{K^\varepsilon}(u) \geq r_0/2$ holds for $u \in \Omega(w_\varepsilon, \frac{\pi}{2})$, while $K^\varepsilon \subset 2R_0B^n$, $R_0 > 1$ and the monotonicity of $\psi_\varepsilon$ imply that $\psi_\varepsilon(h_{K^\varepsilon}(u)) \geq \psi_\varepsilon(2R_0) = \psi(2R_0)$ for all $u \in S^{n-1}$.

We deduce from (40) that

$$\lambda\varepsilon = \frac{1}{n} \int_{S^{n-1}} hK^\varepsilon \psi_\varepsilon(hK^\varepsilon) f \, dH^{n-1} \geq \frac{1}{n} \cdot \frac{r_0}{2} \cdot \psi(2R_0) \cdot \tau_1 \cdot H^{n-1} \left( \Omega\left(w_\varepsilon, \frac{\pi}{3}\right) \right) = \tilde{\lambda}_1.$$  

To have a suitable upper bound on $\lambda\varepsilon$, we define $\alpha \in (0, \frac{\pi}{2})$ with $\cos \alpha = \frac{r_0}{2R_0}$, and hence

$$\Omega(-w_\varepsilon, \alpha) = \left\{ u \in S^{n-1} : \langle u, w_\varepsilon \rangle \leq \frac{-r_0}{2R_0} \right\}.$$
A key observation is that if \( u \in S^{n-1} \setminus \Omega(\omega, \alpha) \), then \( \langle u, w_\varepsilon \rangle > -\frac{r_0}{2R_0} \) and \( \varepsilon \leq R_0 \) imply
\[
h_{K^\varepsilon}(u) \geq \langle u, w_\varepsilon + r_0 u \rangle \geq r_0 - \frac{r_0\varepsilon}{2R_0} \geq r_0/2,
\]
therefore \( \varepsilon < \frac{R_0}{2} \) yields
\[
(41) \quad \psi(\varepsilon)(h_{K^\varepsilon}(u)) \leq \psi(r_0/2) = \psi(r_0/2).
\]

Another observation is that \( K^\varepsilon \subset 2R_0B^n \) implies
\[
h_{K^\varepsilon}(u) < 2R_0 \quad \text{for any } u \in S^{n-1}.
\]
It follows directly from (41) and (42) that
\[
(43) \quad \int_{S^{n-1}\setminus\Omega(-\omega, \alpha)} h_{K^\varepsilon}\psi(h_{K^\varepsilon}) f d\mathcal{H}^{n-1} \leq (2R_0)\psi(r_0/2)\tau_2nK_n.
\]

However, if \( u \in \Omega(-\omega, \alpha) \), then \( \psi(h_{K^\varepsilon}(u)) \) can be arbitrary large as \( \xi(K^\varepsilon) \) can be arbitrary close to \( \partial K^\varepsilon \) if \( \varepsilon > 0 \) is small, and hence we transfer the problem to the previous case \( u \in S^{n-1} \setminus \Omega(-\omega, \alpha) \) using Corollary 4.3. First applying \( \langle u, w_\varepsilon \rangle \geq \frac{r_0}{2R_0} \) for \( u \in \Omega(-\omega, \alpha) \), then Corollary 4.3 and after that \( \langle u, w_\varepsilon \rangle \leq 1, f \leq \tau_2 \) and (41) implies
\[
\int_{\Omega(-\omega, \alpha)} \psi(h_{K^\varepsilon}(u)) f(u) d\mathcal{H}^{n-1}(u) \leq \frac{2R_0}{r_0} \int_{\Omega(-\omega, \alpha)} \langle u, w_\varepsilon \rangle \psi(h_{K^\varepsilon}(u)) f(u) d\mathcal{H}^{n-1}(u)
= \frac{2R_0}{r_0} \int_{S^{n-1}\setminus\Omega(-\omega, \alpha)} \langle u, w_\varepsilon \rangle \psi(h_{K^\varepsilon}(u)) f(u) d\mathcal{H}^{n-1}(u)
\leq \frac{2R_0}{r_0} \cdot \psi(r_0/2)\tau_2nK_n.
\]
Now (42) yields
\[
\int_{\Omega(-\omega, \alpha)} h_{K\varepsilon}\psi(h_K) f d\mathcal{H}^{n-1} \leq \frac{(2R_0)^2}{r_0} \cdot \psi(r_0/2)\tau_2nK_n,
\]
which estimate combined with (43) leads to \( \lambda \varepsilon < \frac{(2R_0)^2 + 2R_0}{r_0} \varphi'(r_0/2)\tau_2nK_n \). In turn, we conclude Lemma 6.2. Q.E.D.

Now we prove Theorem 1.4 if \( f \) is bounded and bounded away from zero.

**Theorem 6.3.** For \( 0 < \tau_1 < \tau_2 \), let the real function \( f \) on \( S^{n-1} \) satisfy \( \tau_1 < f < \tau_2 \), and let \( \varphi : [0, \infty) \rightarrow [0, \infty) \) be increasing and continuous satisfying \( \varphi(0) = 0, \lim\inf_{t \to 0^+} \frac{\varphi(t)}{t} > 0, \) and \( \int_1^\infty \frac{1}{\varphi} < \infty \). Then there exist \( \lambda > 0 \) and a \( K \in \mathcal{K}_0^n \) with \( V(K) = 1 \) such that
\[
f d\mathcal{H}^{n-1} = \lambda \varphi(h_K) dS_K,
\]
as measures on \( S^{n-1} \), and \( \Psi(t) = \int_t^\infty \frac{1}{\varphi} \) satisfies
\[
(44) \quad \int_{S^{n-1}} \Psi(h_{K-\sigma(K)}) f d\mathcal{H}^{n-1} \geq \Psi(5) \int_{S^{n-1}} f d\mathcal{H}^{n-1}.
\]
In addition, if \( f \) is invariant under a closed subgroup \( G \) of \( O(n) \), then \( K \) can be chosen to be invariant under \( G \).

**Proof.** We assume that \( \xi(K^\varepsilon) = o \) for all \( \varepsilon \in (0, \delta_0) \) where \( \delta_0 \in (0, \delta] \) comes from Lemma 6.2. Using the constant \( R_0 \) of Lemma 6.2, we have that
\[
K^\varepsilon \subset 2R_0B^n \quad \text{and} \quad h_{K^\varepsilon}(u) < 2R_0 \quad \text{for any } u \in S^{n-1} \quad \text{and} \quad \varepsilon \in (0, \delta_0).
\]
We consider the continuous increasing function $\varphi_{\varepsilon} : [0, \infty) \to [0, \infty)$ defined by $\varphi_{\varepsilon}(0) = 0$ and $\varphi_{\varepsilon}(t) = 1/\psi_{\varepsilon}(t)$ for $\varepsilon \in (0, \delta)$. We claim that

\begin{equation}
\varphi_{\varepsilon} \text{ tends uniformly to } \varphi \text{ on } [0, 2R_0] \text{ as } \varepsilon > 0 \text{ tends to zero.}
\end{equation}

For any small $\zeta > 0$, there exists $\delta_\zeta > 0$ such that $\varphi(t) \leq \zeta/2$ for $t \in [0, \delta_\zeta]$. We deduce from (21) that if $\varepsilon > 0$ is small, then $|\varphi_{\varepsilon}(t) - \varphi(t)| \leq \zeta/2$ for $t \in [\delta_\zeta, 2R_0]$. However $\varphi_{\varepsilon}$ is monotone increasing, therefore $\varphi_{\varepsilon}(t), \varphi(t) \in [0, \zeta]$ for $t \in [\delta_\zeta, 2R_0]$, completing the proof of (46).

For any $\varepsilon \in (0, \delta_0)$, it follows from Lemma 5.7 that $\psi_{\varepsilon}(h_{K_{\varepsilon}}) f dH^{n-1} = \lambda_{\varepsilon} dS_{K_{\varepsilon}}$ as measures on $S^{n-1}$. Integrating $g\varphi_{\varepsilon}(h_{K_{\varepsilon}})$ for any continuous real function $g$ on $S^{n-1}$, we deduce that

\begin{equation}
f dH^{n-1} = \lambda_{\varepsilon}\varphi(\varepsilon) dS_{K_{\varepsilon}}
\end{equation}
as measures on $S^{n-1}$.

Since $\lambda_1 \leq \lambda_{\varepsilon} \leq \lambda_2$ for some $\lambda_2 > \lambda_1$ independent of $\varepsilon$ according to Lemma 6.2, (45) yields the existence of $\lambda > 0$, $K \in K_0$ with $V(K) = 1$ and sequence $\{\varepsilon(m)\}$ tending to 0 such that $\lim_{m \to \infty} \lambda_{\varepsilon(m)} = \lambda$ and $\lim_{m \to \infty} K_{\varepsilon(m)} = K$. As $h_{K_{\varepsilon(m)}}$ tends uniformly to $h_K$ on $S^{n-1}$, we deduce that $\lambda_{\varepsilon(m)}\varphi(\varepsilon(m)) h_{K_{\varepsilon(m)}}$ tends uniformly to $\lambda \varphi(h_K)$ on $S^{n-1}$. In addition, $S_{K_{\varepsilon(m)}}$ tends weakly to $S_K$, thus (47) yields

\begin{equation}
f dH^{n-1} = \lambda \varphi(h_K) dS_K.
\end{equation}

We note that if $f$ is invariant under a closed subgroup $G$ of $O(n)$, then each $K^\varepsilon$ can be chosen to be invariant under $G$ according to Lemma 4.4, therefore $K$ is invariant under $G$ in this case.

To prove (44), if $\varepsilon \in (0, \delta_0)$, then (31) provides the condition

\begin{equation}
\Psi_{\varepsilon}(h_{K_{\varepsilon-\sigma(K_{\varepsilon})}}) f dH^{n-1} \geq \Psi(5) \int_{S^{n-1}} f dH^{n-1}.
\end{equation}

Now Lemma 6.2 yields that there exists $r_0 > 0$ such that if $\varepsilon \in (0, \delta_0)$, then $\sigma(K_{\varepsilon}) + r_0 B^n \subset K_{\varepsilon}$ where $0 < \delta_0 \leq \frac{\alpha(n)}{2}$. In particular, if $u \in S^{n-1}$, then $h_{K_{\varepsilon-\sigma(K_{\varepsilon})}}(u) \geq r_0$, and hence we deduce from (27) that

\begin{equation}
\Psi_{\varepsilon}(h_{K_{\varepsilon-\sigma(K_{\varepsilon})}}(u)) \leq \Psi(h_{K_{\varepsilon-\sigma(K_{\varepsilon})}}(u)) + \frac{\pi}{2}.
\end{equation}

Since $K_{\varepsilon(m)} - \sigma(K_{\varepsilon(m)})$ tends to $K - \sigma(K)$, (26) implies that if $u \in S^{n-1}$, then

\begin{equation}
\lim_{\varepsilon \to 0^+} \Psi_{\varepsilon}(h_{K_{\varepsilon-\sigma(K_{\varepsilon})}}(u)) = \Psi(h_{K-\sigma(K)}(u)).
\end{equation}

Combining (48), (49) and (50) with Lebesgue’s Dominated Convergence Theorem, we conclude (44), and in turn Theorem 6.3.

Q.E.D.

7. The proof of Theorem 1.4

Let $-n < p < 0$, let $f$ be a non-negative essentially positive function in $L_{\frac{1}{n+p}}(S^{n-1})$, and let $\varphi : [0, \infty) \to [0, \infty)$ be a monotone increasing continuous function satisfying $\varphi(0) = 0$,

\begin{equation}
\liminf_{t \to 0^+} \frac{\varphi(t)}{t^{1-p}} > 0,
\end{equation}

\begin{equation}
\int_1^\infty \frac{1}{\varphi(t)} dt < \infty.
\end{equation}

We associate certain functions to $f$ and $\varphi$. For any integer $m \geq 2$, we define $f_m$ on $S^{n-1}$ as follows:

\[ f_m(u) = \begin{cases} 
m & \text{if } f(u) \geq m, 
f(u) & \text{if } \frac{1}{m} < f(u) < m, 
\frac{1}{m} & \text{if } f(u) \leq \frac{1}{m}. \end{cases} \]
In particular, \( f_m \leq \tilde{f} \) where the function \( \tilde{f} : S^{n-1} \to [0, \infty) \) in \( L_{n^p} (S^{n-1}) \), and hence in \( L_1 (S^{n-1}) \), is defined by

\[
\tilde{f}(u) = \begin{cases} 
    f(u) & \text{if } f(u) > 1, \\
    1 & \text{if } f(u) \leq 1.
\end{cases}
\]

As in Section 4, using (52), we define the function

\[
\Psi(t) = \int_0^\infty \frac{1}{\varphi} \text{ for } t > 0.
\]

According to (51), there exist some \( \delta \in (0, 1) \) and \( \Re > 1 \) such that

\[
\frac{1}{\varphi(t)} < \Re t^{p-1} \text{ for } t \in (0, \delta).
\]

We deduce from Lemma 3.3 that there exists \( \Re_0 > 1 \) depending on \( \varphi \) such that

\[
\Psi(t) < \Re_0 t^p \text{ for } t \in (0, \delta).
\]

For \( m \geq 2 \), Theorem 6.3 yields a \( \lambda_m > 0 \) and a convex body \( K_m \in \mathcal{K}_0^n \) with \( \xi(K_m) = o \in \text{int } K_m \), \( V(K_m) = 1 \) such that

\[
\lambda_m \varphi(h_{K_m}) \, dS_{K_m} = f_m \, d\mathcal{H}^{n-1}
\]

\[
\int_{S^{n-1}} \Psi(h_{K_m - \sigma(K_m)}) \, f_m \, d\mathcal{H}^{n-1} \geq \Psi(5) \int_{S^{n-1}} f_m \, d\mathcal{H}^{n-1}.
\]

In addition, if \( f \) is invariant under a closed subgroup \( G \) of \( O(n) \), then \( f_m \) is also invariant under \( G \), and hence \( K_m \) can be chosen to be invariant under \( G \).

Since \( f_m \leq \tilde{f} \), and \( f \) tends pointwise to \( f \), Lebesgue’s Dominated Convergence theorem yields the existence of \( m_0 > 2 \) such that if \( m > m_0 \), then

\[
\frac{1}{2} \int_{S^{n-1}} f < \int_{S^{n-1}} f_m < 2 \int_{S^{n-1}} f.
\]

In particular, (56) implies

\[
\int_{S^{n-1}} \Psi(h_{K_m - \sigma(K_m)}) \tilde{f} \, d\mathcal{H}^{n-1} \geq \frac{\Psi(5)}{2} \int_{S^{n-1}} f \, d\mathcal{H}^{n-1}.
\]

We deduce from \( V(K_m) = 1 \), \( \lim_{t \to \infty} \Psi(t) = 0 \), (54), (58) and Lemma 3.3 that there exists \( R_0 > 0 \) independent of \( m \) such that

\[
K_m \subset \sigma(K_m) + R_0 B^n \subset 2R_0 B^n \text{ for all } m > m_0.
\]

Since \( V(K_m) = 1 \), Lemma 5.1 yields some \( r_0 > 0 \) independent of \( m \) such that

\[
\sigma(K_m) + r_0 B^n \subset K_m \text{ for all } m > m_0.
\]

To estimate \( \lambda_m \) from below, (60) implies that

\[
\int_{S^{n-1}} \varphi(h_{K_m}) \, dS_{K_m} \leq \varphi(2R_0) \mathcal{H}^{n-1}(\partial K_m) \leq \varphi(2R_0)(2R_0)^{n-1} nk_m,
\]

and hence it follows from (55) and (57) the existence of \( \tilde{\lambda}_1 > 0 \) independent of \( m \) such that

\[
\lambda_m = \frac{\int_{S^{n-1}} f_m \, d\mathcal{H}^{n-1}}{\int_{S^{n-1}} \varphi(h_{K_m}) \, dS_{K_m}} \geq \tilde{\lambda}_1 \text{ for all } m > m_0.
\]

To have a suitable upper bound on \( \lambda_m \) for any \( m > m_0 \), we choose \( w_m \in S^{n-1} \) and \( \varrho_m \geq 0 \) such that \( \sigma(K_m) = \varrho_m w_m \). We set \( B_m^\# = w_m^\perp \cap \text{int } B^n \) and consider the relative open set

\[
\Xi_m = (\partial K_m) \cap (\varrho_m w_m + r_0 B_m^\# + (0, \infty) w_m).
\]
If \( u \) is an exterior unit normal at an \( x \in \Xi_m \) for \( m > m_0 \), then \( x = (\xi_m + s)w_m + rv \) for \( s > 0 \), \( r \in [0, r_0) \) and \( v \in \omega_m^+ \cap S^{n-1} \), and hence \( \xi_m w_m + rv \in K_m \) yields
\[
\langle u, (\xi_m + s)w_m + rv \rangle = h_{K_m}(u) \geq \langle u, \xi_m w_m + rv \rangle,
\]
implying that \( \langle u, w_m \rangle \geq 0 \); or in other words, \( u \in \Omega(w_m, \frac{\pi}{2}) \). Since the orthogonal projection of \( \Xi_m \) onto \( \omega_m^+ \) is \( B^+ \) for \( m > m_0 \), we deduce that
\[
\tag{62} S_{K_m}(\Omega\left(w_m, \frac{\pi}{2}\right)) \geq H^{n-1}(\Xi_m) \geq H^{n-1}(B^+) = r_0^{n-1}\kappa_{n-1}.
\]
On the other hand, if \( u \in \Omega(w_m, \frac{\pi}{2}) \) for \( m > m_0 \), then \( \xi_m w_m + r_0 u \in K_m \) yields
\[
\tag{63} h_{K_m}(u) \geq \langle u, \xi_m w_m + r_0 u \rangle \geq r_0.
\]
Combining (57), (62) and (63) implies
\[
\tag{64} \lambda_m = \frac{\int_{\Omega(w_m, \frac{\pi}{2})} f_m dH^{n-1}(\Omega)}{\int_{\Omega(w_m, \frac{\pi}{2})} \varphi(h_{K_m}) dS_{K_m}} \leq \frac{2 \int_{S^{n-1}} f dH^{n-1}(\varphi(0), r_0^{n-1}\kappa_{n-1})}{\varphi(0, r_0^{n-1}\kappa_{n-1})} = \bar{\lambda}_2 \text{ for all } m > m_0.
\]

Since \( K_m \subset 2R_0B^n \) and \( \bar{\lambda}_1 \leq \lambda_m \leq \bar{\lambda}_2 \) for \( m > m_0 \) by (60), (61) and (64), there exists subsequence \( \{K_{m'}\} \subset \{K_m\} \) such that \( K_{m'} \) tends to some convex compact set \( K \) and \( \lambda_{m'} \to \lambda \) tends to some \( \lambda > 0 \). As \( o \in K_{m'} \) and \( V(K_{m'}) = 1 \) for all \( m' \), we have \( o \in K \) and \( V(K) = 1 \).

We claim that for any be continuous function \( g : S^{n-1} \to \mathbb{R} \), we have
\[
\tag{65} \int_{S^{n-1}} g \lambda \varphi(h_K) dS_K = \int_{S^{n-1}} g f dH^{n-1}.
\]
As \( \varphi \) is uniformly continuous on \([0, 2R_0]\) and \( h_{K_{m'}} \) tends uniformly to \( h_K \) on \( S^{n-1} \), we deduce that \( \lambda_{m'} \varphi(h_{K_{m'}}) \) tends uniformly to \( \lambda \varphi(h_K) \) on \( S^{n-1} \). Since \( S_{K_{m'}} \) tends weakly to \( S_K \), we have
\[
\lim_{m' \to \infty} \int_{S^{n-1}} g \lambda_{m'} \varphi(h_{K_{m'}}) dS_{K_{m'}} = \int_{S^{n-1}} g \lambda \varphi(h_K) dS_K.
\]

On the other hand, \( |g f_m| \leq \tilde{f} \cdot \max_{S^{n-1}} |g| \) for all \( m \geq 2 \), and \( g f_m \) tends pointwise to \( g f \) as \( m \) tends to infinity. Therefore Lebesgue’s Dominated Convergence Theorem implies that
\[
\lim_{m \to \infty} \int_{S^{n-1}} g f_m dH^{n-1} = \int_{S^{n-1}} g f dH^{n-1},
\]
which in turn yields (65) by (55). In turn, we conclude Theorem 1.4 by (65). Q.E.D.
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