Non-Archimedean Compactification of a Topological Space

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Contents

0 Introduction 1
1 Ultrafilters 4
2 Maximal Ideals 11
3 Characters 14
4 Automatic Continuity Theorem 18
5 Gel'fand Theory 20
6 Ground Field Extensions 23
Acknowledgements 28
References 28

0 Introduction

In this paper, we do not assign to the compactness the Hausdorffness or to a compactification of a topological space the condition that the relative topology coincides with the original topology. A compactification of a topological space $X$ is a compact topological space $Y$ endowed with a continuous map $\iota: X \to Y$ whose image $\iota(X)$ is dense in $Y$. We deal with the ordinary compactification, which we call a faithful compactification, in §5. The Stone-Čech compactification of a topological space $X$ is the universal Hausdorff compactification $SC(X)$ of $X$, i.e. a compact Hausdorff topological space $SC(X)$ endowed with a canonical continuous map $\iota: X \to SC(X)$ with respect to which $SC(X)$ has the following universal property: For a compact Hausdorff topological space $Y$ and a continuous map $f: X \to Y$, there uniquely exists a continuous map $SC(f): SC(X) \to Y$ such that $f = SC(f) \circ \iota$. Since the closed interval $[-1, 1] \subset \mathbb{R}$ is a compact Hausdorff topological
space, the universality implies that any real-valued bounded continuous function on $X$ is uniquely extended to a continuous function on $SC(X)$ through $\iota$. Moreover by Urysohn’s lemma, a compact Hausdorff topological space, which is obviously a normal topological space, is canonically embedded in a direct product of copies of the closed interval $[-1, 1]$, which is again compact and Hausdorff by Tychonoff’s theorem. Therefore the extension property for a real-valued bounded continuous function implies the universality of the Stone-Čech compactification. It follows that the Stone-Čech compactification $SC(X)$ and the structure continuous map $\iota: X \to SC(X)$ are presented in the following way: Denote by $C_{bd}(X, \mathbb{R})(1) \subset C_{bd}(X, \mathbb{R})$ the collection $C(X, [-1, 1])$ of real-valued bounded continuous maps on $X$ which take values in $[-1, 1]$, and consider the evaluation map

$$\iota: X \to [-1, 1]^{C_{bd}(X, \mathbb{R})(1)}$$

$$x \mapsto (f(x))_{f \in C_{bd}(X, \mathbb{R})(1)}.$$

By the definition of the direct product topology, $\iota$ is continuous. The closure $SC(X)$ of the image of $\iota$ is a compact Hausdorff topological space endowed with the restriction $\iota: X \to SC(X)$, and it obviously satisfies the extension property for a real-valued bounded continuous function. Note that one has another canonical presentation of the Stone-Čech compactification using the Gel’fand transform. Namely, set $SC(X) := \mathcal{M}_C(C_{bd}(X, \mathbb{C}))$, where $C_{bd}(X, \mathbb{C})$ is the commutative $C^*$-algebra of complex-valued bounded continuous functions on $X$ and $\mathcal{M}_C$ is the functor of the Gel’fand transform, [Dou] 2.21. The evaluations at points of $X$ give a canonical continuous map $\iota: X \to SC(X)$. Gel’fand-Naimark theorem, [Dou] 4.29, guarantees that the pull-back homomorphism

$$\iota^*: C(SC(X), \mathbb{C}) \to C_{bd}(X, \mathbb{C})$$

$$f \mapsto f \circ \iota$$

is an isometric isomorphism, and hence $(SC(X), \iota)$ satisfies the extension property for a complex-valued bounded continuous function on $X$, which is equivalent to the extension property for a real-valued bounded continuous function.

How about the extension property for a $k$-valued bounded continuous function for a non-Archimedean field $k$? Is there the universal compactification of a topological space $X$ with the extension property for a $k$-valued bounded continuous function? Consider the case $k$ is a finite field endowed with the trivial valuation or a local field. Since the ring $k^\circ \subset k$ of integral elements is a compact Hausdorff topological space, the Stone-Čech compactification satisfies the extension property for a $k$-valued bounded continuous function. Moreover in the case, the construction of the compactification analogous to that with the presentation as a closed subspace of the direct product of copies of the closed interval works. Denote by $C_{bd}(X, k)(1) \subset C_{bd}(X, k)$ the collection $C(X, k^\circ)$ of $k$-valued bounded continuous maps on $X$ which take values in $k^\circ$, and consider the evaluation map

$$\iota_k: X \to (k^\circ)^{C_{bd}(X, k)(1)}$$

$$x \mapsto (f(x))_{f \in C_{bd}(X, k)(1)}.$$
It is obvious that $t_k$ is continuous and the closure $SC_k(X)$ of the image of $t_k$ is a totally disconnected compact Hausdorff topological space endowed with the restriction $t_k: X \to SC_k(X)$. It is a totally disconnected Hausdorff compactification of $X$ satisfying the extension property for a $k$-valued bounded continuous function, Definition 3.6 and Proposition 3.7. Consider the general case. The ring $k^\circ \subset k$ of integral elements is not necessarily compact, and hence such a construction is not valid. Even the Stone-Čech compactification might not satisfy the extension property for a $k$-valued bounded continuous function.

Recall that there is another construction of the Stone-Čech compactification by the Gel'fand transform. The counterpart of the Gel'fand transform for the non-Archimedean commutative $C^\ast$-algebra $C_{bd}(X, k)$ is Berkovich's spectrum $BSC_k(X) := \mathcal{M}(C_{bd}(X, k))$, [Ber] 1.2, which is a compact Hausdorff topological space endowed with the canonical continuous map $t_k: X \to BSC_k(X)$. Since the counterpart of Gel'fand-Mazur theorem, [Dou] 2.31, never holds in the non-Archimedean case, the compactification $BSC_k(X)$ might contains a point which is not necessarily $k$-rational. Thus a $k$-valued bounded continuous function on $X$ might not be extended to a $k$-valued continuous function on $BSC_k(X)$.

The aim of this paper is to specify the topological structure of the compactification $BSC_k(X)$ of a topological space $X$, and we prove that $BSC_k(X)$ coincides with the universal totally disconnected Hausdorff compactification $TDC(X)$ of $X$, i.e. the totally disconnected Hausdorff compactification of $X$ which satisfies the following universal property: For a totally disconnected compact Hausdorff topological space $Y$ and a continuous map $f: X \to Y$, there uniquely exists a continuous map $BSC_k(f): BSC_k(X) \to Y$ such that $f = BSC_k(f) \circ t_k$. See Theorem 3.3 and its corollaries. In addition if $k$ is a finite field or a local field, then the two compactifications $SC_k(X)$ and $BSC_k(X)$ coincide, Proposition 3.8 and Corollary 3.9. We listed the one-to-one correspondences which appear in this paper:

- $SC_k(X) = \text{the universal compactification of } X \text{ with the extension property for a } k\text{-valued bounded continuous function}$
- $TDC(X) = \text{the universal totally disconnected Hausdorff compactification of } X$
- $UF(X) = \text{the space of ultrafilters of the Boolean algebra } \text{CO}(X) \text{ of clopen subsets of } X$
- $\text{Max}(C_{bd}(X, k)) = \text{the space of maximal ideals of } C_{bd}(X, k)$
- $\text{Sp}(C_{bd}(X, k)) = \text{the space of closed prime ideals of } C_{bd}(X, k)$
- $BSC_k(X) = \text{the space of characters of } C_{bd}(X, k) \text{ over } k$
They are homeomorphisms in fact, but we will not deal with all of the continuity. For example, we will verify that the composition $BSC_k(X) \to UF(X)$ is a homeomorphism. This characterisation of Berkovich’s spectrum $BSC_k(X)$ is purely topological, and it follows the compactification $BSC_k(X)$ is independent of the choice of the non-Archimedean base field $k$. It yields the significant corollaries. First, if $k$ is a finite field or a local field, the compactification $BSC_k(X)$ of $X$, being homeomorphic to the universal totally disconnected Hausdorff compactification $TDC(X)$, satisfies the extension property for a $k$-valued bounded continuous function, Corollary 3.10 as the Stone-Čech compactification does for a real-valued bounded continuous function. Secondly, if $k$ is a local field, the non-Archimedean commutative $C^*$-algebra $C_{bd}(X, k)$ satisfies the weak automatic continuity theorem, Theorem 4.5. Namely, for a Banach $k$-algebra $\mathcal{A}$, any injective $k$-algebra homomorphism $f: C_{bd}(X, k) \hookrightarrow \mathcal{A}$ whose image is closed is always continuous. In particular the automatic continuity theorem gives a criterion for the continuity of a faithful linear representation of the non-Archimedean commutative $C^*$-algebra $C_{bd}(X, k)$ on a Banach space, Corollary 4.6. Thirdly, if $X$ is totally disconnected and Hausdorff, the non-Archimedean generalised Stone-Weierstrass theorem immediately gives the non-Archimedean Gel’fand theory for totally disconnected Hausdorff compactifications whose structure continuous map is a homeomorphism onto the image, Theorem 5.8. We call such a compactification a faithful totally disconnected Hausdorff compactification. The non-Archimedean Gel’fand theory here is the canonical contravariantly functorial one-to-one corresponding between the collection $\mathcal{C}(X)$ of equivalence classes of a faithful totally disconnected Hausdorff compactifications of $X$ and the set $\mathcal{C}'(X)$ of closed $k$-subalgebra of $C_{bd}(X, k)$ separating points of $X$.

$$\mathcal{C}(X) \leftrightarrow \mathcal{C}'(X)$$

$$[f: X \hookrightarrow Y] \mapsto \text{Im}(f^\#: C(Y, k) \hookrightarrow C_{bd}(X, k))$$

$$[X \hookrightarrow BSC_k(X) \rightarrow \mathcal{M}_k(\mathcal{A})] \leftrightarrow (\mathcal{A} \subset C_{bd}(X, k)).$$

Finally, the ground field extension $BSC_K(X) \to BSC_k(X)$ induced by the extension $C_{bd}(X, k) \hookrightarrow C_{bd}(X, K)$ of the scalar for an extension $K/k$ of complete valuation fields is a homeomorphism, Corollary 6.2. There is another ground field extension induced by the universality of the complete tensor product in the categories of Banach $k$-algebra. We will see the difference of those two ground field extension in Theorem 6.6.

## 1 Ultrafiltres

Throughout this paper, let $X$ be a topological space. In order to interpret a maximal ideal of the Banach $k$-algebra $C_{bd}(X, k)$ in §2 we introduce the notion of the Boolean algebra $CO(X)$ of clopen subsets of $X$, and the space $UF(X)$ of ultrafiltres of $CO(X)$ in this section.

**Definition 1.1.** A subset $U \subset X$ is said to be clopen if it is closed and open. Denote by $CO(X) \subset 2^X$ the collection of clopen subsets of $X$. 

4
Example 1.2. The total set $X$ itself and the empty set $\emptyset$ is clopen subset of $X$. The following relations between $\text{CO}(X)$ and the connectedness of $X$ hold:

(i) The underlying set of $X$ is the empty set if and only if $\text{CO}(X) = \{\emptyset\}$;

(ii) Suppose $X$ is not empty. Then $X$ is connected if and only if $\text{CO}(X) = \{\emptyset, X\}$;

(iii) Suppose $X$ is locally compact and Hausdorff. Then $X$ is totally disconnected if and only if $\text{CO}(X)$ forms the open basis of $X$; and

(iv) The topology of $X$ is discrete if and only if $\text{CO}(X) = 2^X$.

Concerning the criterion (iii), we will deal with an analogous criterion in Lemma 1.19.

Definition 1.3. A Boolean algebra is data $(A, \lor, \land, \neg)$ of a set $A$, binary operations $\lor$ and $\land$ on $A$, and an unary operation $\neg$ on $A$ satisfying the following:

(i) The operations $\lor$ and $\land$ are associative, i.e. one has

$$a \lor (b \lor c) = (a \lor b) \lor c, \quad a \land (b \land c) = (a \land b) \land c$$

for any $a, b, c \in A$;

(ii) The operations $\lor$ and $\land$ are commutative, i.e. one has

$$a \lor b = b \lor a, \quad a \land b = b \land a$$

for any $a, b, c \in A$;

(iii) The operations $\lor$ and $\land$ are distributive, i.e. one has

$$a \lor (b \land c) = (a \lor b) \land (a \lor c), \quad a \land (b \lor c) = (a \land b) \lor (a \land c)$$

for any $a, b, c \in A$; and

(iv) There is an element $\bot \in A$ such that

$$a \lor \bot = a, \quad a \lor (\neg a) = \bot, \quad a \land (\neg \bot) = a, \quad a \land (\neg a) = \bot$$

for any $a \in A$.

Note that the element $\bot \in A$ in the condition (iv) is unique, and call it the identity with respect to $\lor$. Also call $\neg$ the identity with respect to $\land$.

Note that a Boolean algebra admits the canonical structure of a commutative $\mathbb{F}_2$-algebra setting $a + b := (a \lor b) \land \neg(a \land b) = (a \land \neg b) \lor (b \land \neg a)$ and $a \cdot b := a \land b$ for each $a, b \in A$. Such an $\mathbb{F}_2$-algebra satisfies $a^2 = a$ for any $a \in A$. Note that one has $a \lor b = a + b + a \cdot b = (a + 1) \cdot (b + 1) - 1$ and $\neg a = 1 - a$ for any $a, b \in A$. The correspondence $(A, \lor, \land, \neg) \leadsto (A, +, \cdot)$ is one-to-one between Boolean algebra structures on $A$ and commutative $\mathbb{F}_2$-algebra structures on $A$ with respect to which each element of $A$ is an idempotent.
Example 1.4. Let $P \subset 2^X$ be a non-empty family of subsets of $X$ which is stable under finite unions, finite intersections, and taking the complements. Then $(P, \cup, \cap, X\setminus \cdot)$ is a Boolean algebra which admits the identity $\emptyset \in P$ with respect to $\cup$.

Definition 1.5. Call $(CO(X), \cup, \cap, X\setminus \cdot)$ the Boolean algebra generated by clopen subsets of $X$. We simply write $CO(X)$ instead of $(CO(X), \cup, \cap, X\setminus \cdot)$ for short.

Definition 1.6. Let $(A, \lor, \land, \neg)$ be a Boolean algebra. A subset $F \subset A$ is said to be a filtre of $(A, \lor, \land, \neg)$ if it satisfies the following:

(i) $\neg \bot \in F$;

(ii) $a \land b \in F$ for any $a, b \in F$; and

(iii) $a \lor b \in F$ for any $a \in A$ and $b \in F$.

A filtre $F \subset A$ is said to be an ultrafiltre if $F \neq A$ and there is no filtre of $(A, \lor, \land, \neg)$ which does not coincide with $A$ and which properly contains $F$. Denote by $UF(A, \lor, \land, \neg)$ the set of ultrafiltres of $(A, \lor, \land, \neg)$, and endow it the topology given in the following way: A subset $\mathcal{U} \subset UF(A, \lor, \land, \neg)$ is said to be open if for any $F \in \mathcal{U}$, there is some $a \in A$ such that $a \in F$ and $G \in UF(A, \lor, \land, \neg)$ containing $a$.

Definition 1.7. Set $UF(X) := UF(CO(X))$.

Remark 1.8. If $X$ is discrete, then $UF(X)$ coincides with the space of set-theoretical ultrafiltres of $X$, and satisfies the universality of the Stone-Čech compactification $SC(X)$ of $X$.

Example 1.9. For a point $x \in X$, consider the filter

$$F(x) := \{ U \in CO(X) \mid x \in U \} \subset CO(X).$$

Then $F(x) \subset CO(X)$ is an ultrafiltre. Call such an ultrafiltre a principal ultrafiltre.

Be careful that an element of $UF(X)$ is a collection of clopen subsets of $X$, and is not a set-theoretical ultrafiltre of the family $2^X$, [Kur] 1.VII. We will deal with the relation between $UF(X)$ and the topological properties of $X$ in Lemma 1.10.

Lemma 1.10. In the situation in Definition 1.6 for a subset $S \subset A$, set

$$Fil_S := \{ (a_1 \land \cdots \land a_n) \lor b \mid n \in \mathbb{N}, a_1, \ldots, a_n \in S, b \in A \} \subset A.$$ 

Then $Fil_S$ is the smallest filter containing $S$. In addition if $a_1 \land \cdots \land a_n \neq \bot$ for any $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in S$, one has $Fil_S \neq A$. Note that the wedge of an empty family is defined as $\neg \bot \in A$ here.
Proof. The first assertion is obvious. In order to verify the second assertion, it suffices to show that $\perp \notin \text{Fil}_S$. Assume $\perp \in \text{Fil}_S$. Then there are a non-negative integer $n \in \mathbb{N}$, elements $a_1, \ldots, a_n \in S$, and an element $b \in A$ such that $(a_1 \land \cdots \land a_n) \lor b = \perp$. Set $a := a_1 \land \cdots \land a_n \in A$. Then one has

$$a = a \lor \perp = (a \lor b) \land (a \lor \neg b) = (a \lor \neg b) \land (a \lor (a \lor \neg b)) = (a \lor \neg b) \land (a \lor \neg b) \lor (a \lor (a \lor \neg b)) = (a \lor \neg b) \lor (a \lor (a \lor \neg b)) = \perp,$$

and it contradicts the condition of $S$. Thus $\perp \notin \text{Fil}_S$. □

Lemma 1.11. In the situation in Definition 1.6, for a filter $F \subset A$, there is an ultrafilter containing $F$. In particular, $\text{U}F(A, \lor, \land, \neg)$ is empty if and only if $A$ is the trivial Boolean algebra $(\{\perp\}, \text{id}, \text{id}, \text{id})$.

Proof. Use Zorn’s lemma against the family of filters containing $F$. An increasing union of filters is a filter. □

Proposition 1.12. In the situation in Definition 1.6, a subset $F \subset A$ is an ultrafilter if and only if it satisfies the following:

(i)’ $\perp \notin F$;

(ii)’ $a \land b \in F$ for any $a, b \in F$;

(iii)’ $a \lor b \in F$ for any $a \in A$ and $b \in F$;

(iv)’ $a \in F$ or $\neg a \in F$ for any $a \in A$.

Proof. The conditions (ii)’ and (iii)’ are the same with the conditions (ii) and (iii) in Definition 1.6. To begin with, suppose $F$ satisfies the conditions (i)’-(iv)’. The condition (i)’ guarantees that $\perp \notin A$. Take a filter $\mathcal{G} \subset A$ properly containing $F$. Take an element $a \notin \mathcal{G} \setminus F \neq \emptyset$. The condition (iv)’ guarantees that $\neg a \in \mathcal{G} \subset F$, and the condition (ii) in Definition 1.6 implies $\perp = a \land (\neg a) \in F$. Therefore for any $b \in A$, one has $b = b \lor \perp = b \lor \neg a \lor a \in \mathcal{G}$ by the condition (iii) in Definition 1.6 and hence $\mathcal{G} = A$. It follows $\mathcal{F}$ is an ultrafilter. On the other hand, suppose $\mathcal{F}$ is an ultrafilter. The conditions (ii)’ and (iii)’ hold by the conditions (ii) and (iii) in Definition 1.6. Assume $\perp \notin F$. Then for any $a \in A$, one has $a = a \lor \perp = a \lor \neg a \in \mathcal{F}$ by the condition (iii)’, and hence $\mathcal{F} = A$. It contradicts the assumption that $\mathcal{F}$ is an ultrafilter, and therefore $\perp \notin F$; the condition (i)’. Assume there is an element $a \in A$ such that $a \notin F$ and $\neg a \notin F$. Set

$$\mathcal{G} := \text{Fil}_{\mathcal{F} \cup S}$$

and then one has

$$\mathcal{G} = \{ (a \land b) \lor c \mid b \in \mathcal{F}, \ c \in A \} \subset A.$$
Indeed the right hand side is a subfilter of \( G \) containing \( a = (a \land (\neg \bot)) \lor \bot \), and for an
element \( b \in F \), one has
\[
\begin{align*}
b = (\bot \land b) & = (a \lor (\neg a)) \land c = (a \land b) \lor ((\neg a) \land b) = (\bot \lor (a \land b)) \lor ((\neg a) \land b) \\
& = ((a \land b) \lor (\neg (a \land b))) \lor (a \land b) \lor ((\neg a) \land b) \\
& = ((a \land b) \lor (a \land b)) \lor ((\neg a) \land b) = (a \land b) \lor ((\neg a) \land b) \\
& = (a \land b) \lor ((\neg \bot) \land b) = (a \land b) \lor b \in G.
\end{align*}
\]
Since \( F \) is an ultrafilter, one obtains \( G = A \). In particular \( \bot \in G \), and hence there are
elements \( b \in F \) and \( c \in A \) such that \( (a \land b) \lor c = \bot \). It follows that
\[
\begin{align*}
\bot &= (a \land b) \lor c = ((a \land b) \lor (a \land b)) \lor c = (a \land b) \lor ((a \land b) \lor c) \\
& = (a \land b) \lor \bot = a \land b
\end{align*}
\]
and therefore
\[
\neg a = (\neg a) \lor \bot = (\neg a) \lor (a \land b) = ((\neg a) \lor a) \land ((\neg a) \lor b) \\
& = (\bot \land ((\neg a) \lor b) = (\neg a) \lor b \in F.
\]
It contradicts the condition that \( \neg a \notin F \). We conclude that \( a \in F \) or \( \neg a \in F \) for any
\( a \in A \): the condition (iv)”.

Similarly with set-theoretical ultrafilters of a topological space, we will verify that
the space \( UF(X) \) gives the criterion for the compactness and the Hausdorffness of \( X \) in Lemma 1.19. Be careful that there are many differences between the notion of a set-
theoretical ultrafilter and an ultrafilter here.

**Proposition 1.13.** The map \( \mathcal{F}(\cdot) : X \to UF(X) \) is continuous and its image is dense.

**Proof.** The continuity is obvious because the pre-image of the open subset \( \{ \mathcal{F} \in UF(X) \mid U \in \mathcal{F} \} \) for a clopen subset \( U \in CO(X) \) is the clopen subset \( U \subset X \) itself. In order to
prove the density, take an ultrafilter \( \mathcal{F} \in UF(X) \) and a clopen subset \( U \in UF(X) \) with
\( U \in \mathcal{F} \). In particular \( U \neq \emptyset \), and hence the intersection of the image of \( \mathcal{F}(\cdot) \) and the
open neighbourhood \( \{ \mathcal{F} \in UF(X) \mid U \in \mathcal{F} \} \) of \( x \) contains the non-empty image of \( U \subset X \)
by \( \mathcal{F}(\cdot) \). \( \Box \)

**Definition 1.14.** A point \( x \in X \) is said to be a cluster point of an ultrafilter \( \mathcal{F} \in UF(X) \)
if \( \mathcal{F} \) contains all clopen neighbourhood of \( x \).

**Example 1.15.** A point \( x \in X \) is a cluster point of the principal ultrafilter \( \mathcal{F}(x) \in UF(X) \)
by definition.

Unlike a set-theoretical ultrafilter, the existence of a cluster point assigns a strong
restriction of an ultrafilter. An ultrafilter consists of open subsets, and hence has much
more information of the topology of \( X \) than a set-theoretical ultrafilter does.
Lemma 1.16. If an ultrafilter $F \in UF(X)$ has a cluster point, then $F$ is a principal ultrafilter.

Proof. Let $x \in X$ be a cluster point of $F$. Then $F$ contains the principal ultrafilter $F(x)$ by definition, and hence coincides with $F(x)$ by the maximality of an ultrafilter. □

Lemma 1.17. The set of cluster points of an ultrafilter $F \in UF(X)$ coincides with the intersection $\bigcap F$ of all clopen subsets belonging to $F$.

Proof. For a cluster point $x \in X$, one has $x \in \bigcap F(x) = \bigcap F$. For a point $x \in \bigcap F$, assume there is a clopen neighbourhood $U \in CO(X)$ of $x$ such that $U \notin F$. Then one obtains $X \setminus U \in F$, and it contradicts the condition $x \in \bigcap F$. Thus $x$ is a cluster point of $F$. □

Lemma 1.18. If $X$ is an infinite set endowed with the discrete topology, the space $UF(X)$ of ultrafilters contains a non-principal ultrafilter.

This is obvious if we use the fact that the space $UF(X)$ of ultrafilters of $X$ satisfies the universality of the Stone-Čech compactification $SC(X)$ of $X$. Moreover the stronger fact for the cardinality. That is, $\# UF(X) = 2^{2^{\# X}}$ in this situation. We do not use those facts here, and hence we give an alternative proof.

Proof. Since $X$ is an infinite set, there is a set-theoretical injective map $f : \mathbb{N} \hookrightarrow X$. For each natural number $n \in \mathbb{N}$, set $U_n := \{f(m) \mid m \in \mathbb{N}, m \geq n\} \subset X$. Since $U_0 \supset U_1 \supset \cdots$ and $U_n \neq \emptyset$, the family $\{U_n \mid n \in \mathbb{N}\} \subset CO(X) = 2^X$ is contained in an ultrafilter $F \in UF(X)$ by Lemma 1.10 and Lemma 1.11. Since $f$ is injective, one has $\bigcap F \subset \bigcap_n U_n = \emptyset$, and hence $F$ has no cluster point. Therefore the ultrafilter $F$ is not principal. □

Lemma 1.19. Suppose $CO(X)$ forms an open basis of $X$. Then the following hold:

(i) $X$ is compact if and only if every ultrafilter has at least one cluster point;

(ii) $X$ is Hausdorff if and only if every ultrafilter has at most one cluster point; and

(iii) $X$ is a totally disconnected compact Hausdorff topological space if and only if every ultrafilter has precisely one cluster point.

Note that in the assumption that $CO(X)$ forms an open basis of $X$, $X$ is Hausdorff if and only if $X$ is totally disconnected, and therefore the criteria (i) and (ii) immediately imply the criterion (iii).

Proof. If $X$ is compact, an ultrafilter has a cluster point because the intersection $\bigcap F$ is non-empty by the finite-intersection property of the compact topological space $X$. On the other hand, suppose every ultrafilter has at least one cluster point. Assume $X$ is compact. Since $CO(X)$ forms an open basis of $X$, there is a clopen covering $\mathcal{U} \subset CO(X)$ of $X$ which has no finite subcovering. In particular, the complements $\mathcal{V} := \{U \in CO(X) \mid X \setminus U \in \mathcal{U}\}$ satisfies $\bigcap \mathcal{V} = \emptyset$ and any finite intersection of clopen subsets in $\mathcal{V}$ is non-empty. Then
by Lemma 1.10 and Lemma 1.11, there is an ultrafilter $F \in UF(X)$ containing $\mathcal{V}$. One has $\mathcal{F} \subseteq \bigcap \mathcal{V} = \emptyset$, and it contradicts the assumption that every ultrafilter has at least one cluster point. Thus $X$ is compact.

If $X$ is Hausdorff, then obviously the continuous map $\mathcal{F}(\cdot): X \rightarrow UF(X): x \mapsto \mathcal{F}(x)$ is injective because $CO(X)$ forms an open basis of $X$. Suppose every ultrafilter has at most one cluster point. Assume $X$ is not Hausdorff. There are two distinct points $x, y \in X$ such that any clopen neighbourhoods of $x$ and $y$ have the non-empty intersection. In other words, one has $U \cap V \neq \emptyset$ for any $(U, V) \in \mathcal{F}(x) \times \mathcal{F}(y)$. Take a clopen neighbourhood $U \in \mathcal{F}(x)$ of $x$. By the argument above, one has $X \setminus U \notin \mathcal{F}(y)$, and hence $U \notin \mathcal{F}(y)$. It follows $\mathcal{F}(x) \subseteq \mathcal{F}(y)$, and therefore $\mathcal{F}(x) = \mathcal{F}(y)$ by the maximality of an ultrafilter. It follows that both $x$ and $y$ are two distinct cluster points of $\mathcal{F}(x) = \mathcal{F}(y)$, and it contradicts the assumption that every ultrafilter has at most one cluster point. Thus $X$ is Hausdorff.

\[\square\]

**Corollary 1.20.** In the situation in Lemma 1.19, the following hold:

(i) $X$ is compact if and only if the continuous map $\mathcal{F}(\cdot): X \rightarrow UF(X): x \mapsto \mathcal{F}(x)$ is surjective;

(ii) $X$ is Hausdorff if and only if the continuous map $\mathcal{F}(\cdot)$ is injective; and

(iii) $X$ is a totally disconnected compact Hausdorff topological space if and only if the continuous map $\mathcal{F}(\cdot)$ is a homeomorphism.

Note that in the criteria (i) and (iii), the continuous map $\mathcal{F}(\cdot)$ is obviously open by the definition of the topology of $UF(X)$.

We verify that $UF(X)$ has the universality of the universal totally disconnected Hausdorff compactification $TDC(X)$ of $X$.

**Proposition 1.21.** The space $UF(X)$ is a totally disconnected compact Hausdorff topological space.

**Proof.** For a clopen subset $U \in CO(X)$, one has

$$UF(X) = \{ \mathcal{F} \in UF(X) \mid U \in \mathcal{F} \} \cup \{ \mathcal{F} \in UF(X) \mid X \setminus U \in \mathcal{F} \},$$

and hence $CO(UF(X))$ forms an open basis of $UF(X)$. Therefore it suffices to show that $UF(X)$ is compact and Hausdorff.

Assume $UF(X)$ is not compact. There is a clopen covering $\mathcal{U} \subset CO(UF(X))$ of $UF(X)$ which has no finite subcovering. In particular, the complements $\mathcal{V} := \{ U \in CO(UF(X)) \mid X \setminus U \in \mathcal{U} \}$ satisfies $\mathcal{Y} = \emptyset$ and any finite intersection of clopen subsets belonging to $\mathcal{Y}$ is non-empty. Since the map $\mathcal{F}(\cdot)$ is continuous, the pull-back $\mathcal{F}(\cdot)^* \mathcal{V} := \{ \mathcal{F}(\cdot)^{-1}(V) \mid V \in \mathcal{V} \}$ is a non-empty collection of clopen covering of $X$ satisfying that $\bigcap \mathcal{F}(\cdot)^* \mathcal{V} = \emptyset$ and any finite intersection of clopen subsets belonging to $\mathcal{F}(\cdot)^* \mathcal{V}$ is non-empty. Then by Lemma 1.10 and Lemma 1.11 there is an ultrafilter $\mathcal{F} \in UF(X)$
containing $\mathcal{F}(\cdot)^\forall$. Since $\mathcal{U}$ covers $UF(X)$, there is a clopen subset $U \in \mathcal{U}$ containing $\mathcal{F}$. The pre-image $V \in \mathcal{F}(\cdot)^\forall$ of the complement $UF(X) \setminus U \in \mathcal{V}$ is contained in the ultrafiltre $\mathcal{F}$ because $\mathcal{F}(\cdot)^\forall \subset \mathcal{F}$. By the definition of the topology of $UF(X)$, there is a clopen subset $W \in \mathcal{F}$ such that $W \in \mathcal{G}$ implies $\mathcal{G} \in U$ for an ultrafiltre $\mathcal{G} \in UF(X)$. Since $V, W \in \mathcal{F}$, one has $V \cap W \in \mathcal{F}$ and hence $U \cap W \neq \emptyset$. Take a point $x \in U \cap V \subset X$. Since $V = \mathcal{F}(\cdot)^{-1}(UF(X) \setminus U)$, one has $\mathcal{F}(x) \notin U$, but it contradicts the condition $x \in W \subset \mathcal{F}(\cdot)^{-1}(U)$. Thus $UF(X)$ is compact.

Take two distinct ultrafiltres $\mathcal{F}, \mathcal{G} \in UF(X)$. Since $\mathcal{F} \neq \mathcal{G}$, there is a clopen subset $U \in CO(X)$ such that $U$ is contained in precisely one of them. Simultaneously, the complement $X \setminus U$ is contained in precisely one of them, and the one containing $U$ does not contain $X \setminus U$. Therefore the partition

$$
UF(X) = \{ \mathcal{F} \in UF(X) \mid U \in \mathcal{F} \} \cup \{ \mathcal{F} \in UF(X) \mid X \setminus U \in \mathcal{F} \}
$$

by clopen subsets of $UF(X)$ separates $\mathcal{F}$ and $\mathcal{G}$, and thus $UF(X)$ is Hausdorff.

**Corollary 1.22.** One has $UF(X) \cong UF(UF(X))$.

**Proposition 1.23.** The space $UF(X)$ satisfies the universality of the universal totally disconnected Hausdorff compactification $TDC(X)$ of $X$ with respect to the continuous map $\mathcal{F}(\cdot) : X \to UF(X)$, i.e. for a totally disconnected compact Hausdorff topological space $Y$ and a continuous map $f : X \to Y$, there uniquely exists a continuous map $UF(f) : UF(X) \to Y$ such that $f = UF(f) \circ \mathcal{F}(\cdot)$.

**Proof.** The uniqueness of the continuous extension $UF(f)$ is trivial because the image of $X$ is dense in $UF(X)$ and $Y$ is Hausdorff. We define $UF(f)$. Take an ultrafiltre $\mathcal{F} \in UF(X)$, and set $UF(f)_\mathcal{F} := \{ U \in CO(Y) \mid f^{-1}(U) \in \mathcal{F} \}$. Then it is obvious that $UF(f)_\mathcal{F} \in UF(Y)$, and denote by $UF(f)(\mathcal{F}) \in Y$ the unique cluster point of the totally disconnected compact Hausdorff topological space $Y$. One obtains a well-defined map $UF(f) : UF(X) \to Y$. The pre-image of a clopen subset $U \in CO(Y)$ by $UF(f)$ is the clopen subset $\{ \mathcal{F} \in UF(X) \mid f^{-1}(U) \in \mathcal{F} \} \subset UF(X)$, and hence $UF(f)$ is continuous. For a point $x \in X$, take a clopen neighbourhood $U \in CO(Y)$ of $f(x) \in Y$. Since $f$ is continuous, the pre-image $f^{-1}(U) \subset X$ is a clopen neighbourhood of $x \in X$, and hence one has $f^{-1}(U) \in \mathcal{F}(x)$. It follows $U \in UF(f)_\mathcal{F}(x)$, and therefore $f(x) \in Y$ is the cluster point of $UF(f)_\mathcal{F}(x)$. Thus $UF(f)(\mathcal{F}(x)) = f(x)$.

**Corollary 1.24.** The correspondence $UF$ determines a functor from the category $Top$ of topological spaces to the full subcategory of totally disconnected compact Hausdorff topological spaces which is the left adjoint functor of the inclusion of the full subcategory.

### 2 Maximal Ideals

Throughout this paper, let $k$ be a complete valuation field of rank 1. In this section, we determine the maximal spectrum $\text{Max}(C_{bd}(X, k))$ of the commutative Banach $k$-algebra
\(C_{bd}(X, k)\) of \(k\)-valued bounded continuous maps on \(X\), using the space \(UF(X)\) of ultrafilters introduced in §1

**Definition 2.1.** Denote by \(C_{bd}(X, k)\) the Banach \(k\)-algebra of \(k\)-valued bounded continuous functions on \(X\) endowed with the supremum norm. For a prime ideal \(m \subset C_{bd}(X, k)\), set

\[
\mathcal{F}_m := \{ U \in CO(X) \mid 1_U \notin m \} \subset CO(X).
\]

**Proposition 2.2.** For a prime ideal \(m \subset C_{bd}(X, k)\), the subset \(\mathcal{F}_m \subset CO(X)\) is an ultrafilter.

**Proof.** We verify the conditions (i)’-(iv)’ in Proposition[1,12] Since \(1_\emptyset = 0 \in m\), one has \(\emptyset \notin \mathcal{F}_m\): the condition (i)’. Take a clopen subset \(U \in CO(X)\). If \(1_U \notin m\), then \(U \in \mathcal{F}_m\) by definition. Suppose \(1_U \in m\). Since \(1 \notin m\), one has \(1_{X \setminus U} = 1 - 1_U \notin m\), and hence \(X \setminus U \in \mathcal{F}_m\). It follows \(U \in \mathcal{F}_m\) or \(X \setminus U \in \mathcal{F}_m\) for any \(U \in CO(X)\): the condition (iv)’. Take clopen subsets \(U, V \in \mathcal{F}_m\). Then one has \(1_U, 1_V \notin m\) by definition and hence \(1_{U \cap V} = 1_U 1_V \notin m\). Therefore \(U \cap V \in \mathcal{F}_m\): the condition (ii)’. Finally take clopen subsets \(U \in CO(X)\) and \(V \in \mathcal{F}_m\). One obtains \(1_V \notin m\) by definition, and \(1_{X \setminus V} \in m\) by the condition (iv)’. Therefore \(1_{U \cup V} = 1 - 1_{X \setminus (U \cup V)} = 1 - 1_{X \setminus U} 1_{X \setminus V} \notin m\), and hence \(U \cup V \in \mathcal{F}_m\): the condition (iii)’. Thus \(\mathcal{F}_m \subset CO(X)\) is an ultrafilter. \(\Box\)

**Example 2.3.** For a point \(x \in X\), consider the ideal

\[
m_x := \{ f \in C_{bd}(X, k) \mid f(x) = 0 \} \subset C_{bd}(X, k).
\]

Then \(m_x \subset C_{bd}(X, k)\) is a maximal ideal, and one has \(\mathcal{F}_{m_x} = \mathcal{F}(x)\).

**Lemma 2.4.** The map

\[
\mathcal{F}: \text{Spec}(C_{bd}(X, k)) \rightarrow UF(X)
\]

\[
m \mapsto \mathcal{F}_m
\]

is anti-order preserving with respect to the inclusions. Since an ultrafilter is maximal with respect to the inclusions of non-trivial filters, it implies \(\mathcal{F}_{m_1} = \mathcal{F}_{m_2}\) for any closed prime ideals \(m_1 \subset m_2 \subset C_{bd}(X, k)\).

**Proof.** Trivial by the definition of \(\mathcal{F}\). \(\Box\)

**Definition 2.5.** Let \(R\) be a commutative topological ring. Denote by \(Sp(R) \subset \text{Spec}(R)\) the subset of closed prime ideals and by \(\text{Max}(R) \subset \text{Spec}(R)\) the subset of maximal ideals.

Note that for a Banach \(k\)-algebra \(\mathcal{A}\), one has \(\text{Max}(\mathcal{A}) \subset Sp(\mathcal{A})\) by [BGR] 1.2.4/5, but the converse inclusion does not hold in general. For example, a Banach \(k\)-algebra \(\mathcal{A}\) which is an integral domain and is not a field, such as the Tate algebra \(k\{T\}\), has a non-maximal closed ideal \([0] \subset \mathcal{A}\).
Proposition 2.6. The restriction $\mathcal{F} : Sp(C_{bd}(X, k)) \to UF(X)$ is injective.

Proof. Take two closed prime ideals $m_1, m_2 \in Sp(C_{bd}(X, k))$, and suppose $\mathcal{F}_{m_1} = \mathcal{F}_{m_2}$. It suffices to show that $m_1 \subset m_2$. Take an element $f \in m_1$. For a positive number $\epsilon > 0$, set $U_\epsilon := \{x \in X \mid |f(x)| < \epsilon\}$, and then $U_\epsilon \subset X$ is a clopen subset. Indeed $U_\epsilon$ is open by the continuity of $f$. For any $x \in X \setminus U_\epsilon$, there is an open neighbourhood $U \subset X$ of $x$ such that $|f(y) - f(x)| < \epsilon$ for any $y \in U$. Then for any $y \in U$, one has $|f(y)| = |f(x)| \geq \epsilon$ because $|f(y) - f(x)| < \epsilon \leq |f(x)|$, and hence $U \subset X \setminus U_\epsilon$. It follows $X \setminus U_\epsilon \subset X$ is open. Set $f_\epsilon := (1 - 1_{U_\epsilon})f \in C_{bd}(X, k)$. Since $f \in m_1$, one has $f_\epsilon \in m_1$. Since the absolute value of $f_\epsilon + 1_{U_\epsilon} \in C_{bd}(X, k)$ at each point in $X$ has the lower bound $\min[\epsilon, 1]$, and hence its inverse is a bounded continuous function on $X$. It implies that $f_\epsilon + 1_{U_\epsilon}$ is invertible in $C_{bd}(X, k)$, and therefore $1_{U_\epsilon} \notin m_1$. It follows $U_\epsilon \in \mathcal{F}_{m_1} = \mathcal{F}_{m_2}$, and hence $1 - 1_{U_\epsilon} = 1_{X \setminus U_\epsilon} \in m_2$. Thus $f_\epsilon = (1 - 1_{U_\epsilon})f \in m_2$, and the inequality $\|f - f_\epsilon\| = \|1_{U_\epsilon}f\| \leq \epsilon$ guarantees $f \in m_2$ by the closedness of $m_2$. □

Corollary 2.7. One has $Sp(C_{bd}(X, k)) = Max(C_{bd}(X, k))$.

Proof. The assertion immediately follows from Lemma 2.4 and Proposition 2.6 □

Proposition 2.8. The restriction $\mathcal{F} : Max(C_{bd}(X, k)) \to UF(X)$ is bijective.

Proof. If $X = \emptyset$, the both hand sides are the empty sets, and hence we may and do assume $X \neq \emptyset$. By Proposition 2.6 it suffices to verify the surjectivity. Take an ultrafilter $\mathcal{F} \in UF(X)$. Set

$$ m := \left\{ f \in C_{bd}(X, k) \mid \inf_{U \in \mathcal{F}} \sup_{x \in U} |f(x)| = 0 \right\} \subset C_{bd}(X, k). $$

The subset $m \subset C_{bd}(X, k)$ is obviously an ideal, and since $|1(x)| = 1$ for any $x \in X \neq \emptyset$, one has $1 \notin m$. The map

$$ \| \cdot \|_{\mathcal{F}} : C_{bd}(X, k) \to [0, \infty) $$

$$ f \mapsto \inf_{U \in \mathcal{F}} \sup_{x \in U} |f(x)| < \|f\| $$

is a contraction map as a map between metric spaces, and hence is continuous. Since $[0] \subset [0, \infty)$ is closed, $m$ is a closed ideal. For bounded continuous functions $f, g \in C_{bd}(X, k)$, suppose that $fg \in m$. Assume $f \notin m$, and we prove that $g \in m$. If $g = 0$, then $g \in m$. Therefore we may and do assume $g \neq 0$. Since $f \notin m$, there is some positive number $\epsilon > 0$ such that the subset

$$ V := \left\{ x \in X \mid |f(x)| < \epsilon^{1/2} \right\}, $$

which is clopen by a similar calculation with that in the proof of 2.6 does not belong to $\mathcal{F}$. Since $\mathcal{F}$ is an ultrafilter, one has $X \setminus V \in \mathcal{F}$ and hence $U \setminus V = U \cap (X \setminus V) \in \mathcal{F}$ for a point $x \in U \setminus V$, the inequality $|g(x)| = |f(x)|^{-1} |f(x)g(x)| < \epsilon^{1/2}$ implies $\sup_{x \in U} |g(x)| < \epsilon$. Therefore $\|g\|_{\mathcal{F}} = 0$, and thus $g \in m$. We conclude that $m$ is a closed prime ideal, and is a maximal ideal by Corollary 2.7. It is obvious that $\mathcal{F}_m = \mathcal{F}$. □
3 Characters

In this section, we study the relation between the Hausdorff compactifications $SC_k(X)$ and $BSC_k(X)$ referred in §1, the maximal spectrum $Max(C_{bd}(X,k))$, and the ultrafiltres $UF(X)$. Note that the bijective map $Max(C_{bd}(X,k)) \rightarrow UF(X)$ in §2 is a homeomorphism with respect to the Zariski topology of $Max(C_{bd}(X,k))$ in fact, but we do not use this fact in this paper.

**Definition 3.1.** Denote by $BSC_k(X)$ Berkovich’s spectrum, the space $M_k(C_{bd}(X,k))$ of the equivalence classes of characters, [Ber] 1.2, of the commutative Banach $k$-algebra $C_{bd}(X,k)$ and by $t_k : X \rightarrow BSC_k(X)$ the evaluation map

$$t_k : X \rightarrow BSC_k(X)$$

$$x \mapsto (t_k(x) : f \mapsto |f(x)|).$$

Berkovich’s spectrum $BSC_k(X)$ is a compact Hausdorff topological space by [Ber] 1.2.1, and the map $t_k$ is continuous by the definition of the topology of $BSC_k(X)$.

**Definition 3.2.** For a point $x \in BSC_k(X)$, denote by $supp(x) \in Sp(C_{bd}(X,k)) = Max(C_{bd}(X,k))$ the support of $x$, i.e. $supp(x) = \{ f \in C_{bd}(X,k) \mid |f(x)| := x(f) = 0 \}$, and by $supp$ the map

$$supp : BSC_k(X) \rightarrow Max(C_{bd}(X,k))$$

$$x \mapsto supp(x),$$

which is continuous with respect to the Zariski topology of $Max(C_{bd}(X,k))$.

**Theorem 3.3.** The composition $\mathcal{F}_{supp} := \mathcal{F} \circ supp : BSC_k(X) \rightarrow UF(X)$ is a homeomorphism.

**Proof.** To begin with, we prove the bijectivity. Since $\mathcal{F}$ is bijective, it suffices to show that supp is bijective. The surjectivity of supp is proved in the proof of [Ber] 1.2.1. Before proving the injectivity of supp, we study for a maximal ideal $m \in Max(C_{bd}(X,k))$ the relation between the quotient seminorm $\| \cdot + m \|$ at $m$ and the map $\| \cdot \|_{\mathcal{F}_m}$ defined in the proof of Proposition 2.8. For a bounded continuous function $f \in C_{bd}(X,k)$, one has

$$\|f + m\| = \inf_{g \in m} \|f - g\| \geq \inf_{g \in m} \|f - g\|_{\mathcal{F}_m} = \inf_{g \in m} \|f\|_{\mathcal{F}_m} = \|f\|_{\mathcal{F}_m}. \$$

Assume $\|f + m\| > \|f\|_{\mathcal{F}_m}$. Take a positive number $r \in (\|f\|_{\mathcal{F}_m}, \|f + m\|)$. Set

$$U := \{ x \in X \mid |f(x)| > r \},$$

and then $U \subset X$ is clopen by a similar calculation with that in the proof of Proposition 2.6. If $U \in \mathcal{F}_m$, then one has

$$\|f\|_{\mathcal{F}_m} = \inf_{V \in \mathcal{F}_m} \sup_{x \in V} |f(x)| \geq \inf_{V \in \mathcal{F}_m} \sup_{x \in U \cap V} |f(x)| \geq \inf_{V \in \mathcal{F}_m} r = r$$
and hence it contradicts the condition $r < ||f||_{\mathcal{F}_m}$. It follows $U \notin \mathcal{F}_m$, and therefore $1_U \in m$. One obtains
\[ ||f + m|| \leq ||f - 1_Uf|| = ||1_Uf|| \leq r, \]
and hence it contradicts the condition $r > ||f + m||$. It follows $||f + m|| = ||f||_{\mathcal{F}_m}$.

Next, we prove that the map $||\cdot||_{\mathcal{F}_m}$ is a bounded multiplicative seminorm on $C_{bd}(X, k)$. It is obviously a bounded power-multiplicative seminorm, and it suffices to show the multiplicity. Take two bounded continuous maps $f, g \in C_{bd}(X, k)$ and assume $||fg||_{\mathcal{F}_m} < ||f||_{\mathcal{F}_m}||g||_{\mathcal{F}_m}$. In particular $f, g \notin m$. Take a sufficiently small positive number $\epsilon > 0$ such that the conditions $r \geq ||f||_{\mathcal{F}_m} - \epsilon$ and $s \geq ||g||_{\mathcal{F}_m} - \epsilon$ for positive numbers $r, s > 0$ imply $rs > ||fg||_{\mathcal{F}_m}$. Set
\[
V_1 := \{ x \in X \mid |f(x)| > ||f||_{\mathcal{F}_m} - \epsilon \}
\]
and $V_2 := \{ x \in X \mid |g(x)| > ||g||_{\mathcal{F}_m} - \epsilon \}$,
and then $V_1, V_2 \subset X$ are clopen by a similar calculation with that in the proof of Proposition 2.6. If $V_1 \notin \mathcal{F}_m$, then one has $X \setminus V_1 \in \mathcal{F}_m$, but the inequality
\[
||f||_{\mathcal{F}_m} \leq \sup_{x \in X \setminus V} |f(x)| \geq ||f||_{\mathcal{F}_m} - \epsilon
\]
contradicts the condition $\epsilon > 0$. Therefore $V_1 \in \mathcal{F}_m$. Similarly one obtains $V_2 \in \mathcal{F}_m$, and hence $V_1 \cap V_2 \in \mathcal{F}_m$. Then the inequality
\[
||fg||_{\mathcal{F}_m} = \inf_{W \in \mathcal{F}_m} \sup_{x \in W} |f(x)g(x)| \geq \inf_{W \in \mathcal{F}_m} \sup_{x \in V_1 \cap V_2 \cap W} |f(x)||g(x)|
\]
\[
\geq \inf_{W \in \mathcal{F}_m} ((||f||_{\mathcal{F}_m} - \epsilon)(||g||_{\mathcal{F}_m} - \epsilon)) > \inf_{W \in \mathcal{F}_m} ||fg||_{\mathcal{F}_m}
\]
implies the assumption $||fg||_{\mathcal{F}_m} < ||f||_{\mathcal{F}_m}||g||_{\mathcal{F}_m}$ is false. We conclude that the map $||\cdot||_{\mathcal{F}_m}$ is a bounded multiplicative seminorm, and hence corresponds to a point in Berkovich’s spectrum $BSC_k(X)$.

Now take a point $x \in BSC_k(X)$. Since $||\cdot||_{\mathcal{F}_{\text{supp}(x)}}$ coincides with the quotient seminorm $||\cdot||_{+\text{supp}(x)}$, one has $|f(x)| \leq ||f||_{\mathcal{F}_{\text{supp}(x)}}$ for any $f \in C_{bd}(X, k)$. It follows that $x$ determines a bounded multiplicative norm of the complete residue field at the point of $BSC_k(X)$ corresponding to the bounded multiplicative seminorm $||\cdot||_{\mathcal{F}_{\text{supp}(x)}}$, and hence $x = ||\cdot||_{\mathcal{F}_{\text{supp}(x)}}$, because Berkovich’s spectrum of a Banach $k$-algebra whose underlying $k$-algebra is a field and whose norm is multiplicative is a singleton by [Ber] 1.3.4(i). Thus $x$ is presented by its image of the composition $\mathcal{F}_{\text{supp}}$, and hence $\mathcal{F}_{\text{supp}}$ is injective.

We verify the continuity of $\mathcal{F}_{\text{supp}}$. Take a clopen subset $U \in CO(X)$, and set $\mathcal{U} := \{ \mathcal{F} \in UF(X) \mid U \in \mathcal{F} \}$. The pre-image of $\mathcal{U}$ by $\mathcal{F}_{\text{supp}}$ is the subset
\[
\{ x \in BSC_k(X) \mid U \in \mathcal{F}_{\text{supp}(x)} \} = \{ x \in BSC_k(X) \mid 1_U \notin \text{supp}(x) \}.
\]
Because it is open by the definition of the topology of Berkovich’s spectrum. Therefore \( \mathcal{F}_{\text{supp}} \) is continuous. On the other hand, take a bounded continuous function \( f \in C_{\text{bd}}(X, k) \) and an open subset \( I \subset [0, \infty) \). Consider the open subset \( \mathcal{V} := \{ x \in \text{BSC}_k(X) \mid |f(x)| \in I \} \). The pre-image of \( \mathcal{V} \) by the evaluation map \( \iota_k : X \to \text{BSC}_k(X) \) is the clopen subset \( V := \{ x \in X \mid |f(x)| \in I \} \). The image of \( \mathcal{V} \) by \( \mathcal{F}_{\text{supp}} \) is the subset

\[
\{ \mathcal{F} \in \text{UF}(X) | \| f \|_{\mathcal{F}} \in I \} = \{ \mathcal{F} \in \text{UF}(X) | V \in \mathcal{F} \} \subset \text{UF}(X),
\]

and it is open by the definition of the topology of the space of ultrafilters. Therefore \( \mathcal{F}_{\text{supp}} \) is an open map, and it completes the proof. \( \square \)

**Corollary 3.4.** Berkovich’s spectrum \( \text{BSC}_k(X) \) is the universal totally disconnected Hausdorff compactification \( \text{TDC}(X) \) of \( X \).

**Corollary 3.5.** The image of the evaluation map \( X \to \text{BSC}_k(X) \) is dense.

As a consequence in the case \( k \) is a finite field or a local field, we verify that Berkovich’s spectrum \( \text{BSC}_k(X) \) coincides with the universal compactification \( \text{SC}_k(X) \) of \( X \) with the extension property for a \( k \)-valued bounded continuous function.

**Definition 3.6.** Denote by \( C_{\text{bd}}(X, k)_{(1)} \subset C_{\text{bd}}(X, k) \) the collection \( C(X, k^\circ) \) of \( k \)-valued bounded continuous functions on \( X \) which take values in the ring \( k^\circ \subset k \) of integral elements, and consider the evaluation map

\[
\iota_k : X \to (k^\circ)^{C_{\text{bd}}(X, k)_{(1)}}
\]

\[
x \mapsto (f(x))_{f \in C_{\text{bd}}(X, k)_{(1)}}.
\]

By the definition of the direct product topology, \( \iota_k \) is continuous. The closure \( \text{SC}_k(X) \) of the image of \( \iota_k \) is a totally disconnected compact Hausdorff topological space endowed with the restriction \( \iota_k : X \to \text{SC}_k(X) \) because \( k^\circ \) is a totally disconnected compact Hausdorff topological space.

**Proposition 3.7.** The totally disconnected Hausdorff compactification \( \text{SC}_k(X) \) satisfies the extension property for a \( k \)-valued bounded continuous function, i.e. for a \( k \)-valued bounded continuous function \( f \in C_{\text{bd}}(X, k) \), there uniquely exists a \( k \)-valued bounded continuous function \( \text{SC}_k(f) \in C_{\text{bd}}(\text{SC}_k(X), k) \) such that \( f = \text{SC}_k(f) \circ \iota_k \). Moreover one has \( \| f \| = \| \text{SC}_k(f) \| \).

**Proof.** The uniqueness of the extension and the norm-preserving property is obvious because \( \iota_k(X) \subset \text{SC}_k(X) \) is dense and \( k \) is Hausdorff. We construct the extension \( \text{SC}_k(f) \).

It is obvious that there is an invertible element \( a \in k^\times \) such that \( \| f \| \leq |a| \) by the definition of the supremum norm and by the multiplicativity of the norm of \( k \). For a point \( x = (x_x)_{x \in C_{\text{bd}}(X, k)_{(1)}} \in \text{SC}_k(X) \), set \( \text{SC}_k(f)(x) := ax_{x \cdot f} \in ak^\circ \subset k \). The value \( \text{SC}_k(f)(x) \in k \) is independent of the choice of an invertible element \( a \in k^\times \). Indeed, take two invertible
elements $a_1, a_2 \in k^\times$ and suppose $\|f\| \leq \min\{|a_1|, |a_2|\}$. For a point $y \in X$, one has $t_k(y)_{a_1^{-1}f} = a_1^{-1}f(y)$ and $t_k(y)_{a_2^{-1}f} = a_2^{-1}f(y)$. It follows that $a_1t_k(y)_{a_1^{-1}f} = a_2t_k(y)_{a_2^{-1}f} \in k$. Since the image $t_k(X) \subset SC_k(X)$ is dense, one obtains $a_1x_{a_1^{-1}f} = a_2x_{a_2^{-1}f} \in k$, and therefore the value $SC_k(f)(x) \in k$ is independent of the choice of $a$. In the calculation above, one acquires that $SC_k(f) \circ t_k = f$. The continuity of $SC_k(f)$ is obvious because it is the restriction of the canonical projection. □

**Proposition 3.8.** Suppose $k$ is a finite field endowed with the trivial valuation or a local field, i.e. a complete discrete valuation field whose residue field is finite. The totally disconnected Hausdorff compactification $SC_k(X)$ is the universal totally disconnected Hausdorff compactification $TDC(X)$ of $X$.

Remark that the condition of the base field $k$ is easily removed when we assume the base topological space $X$ is compact. The analysis of continuous functions on a compact topological space is quite classical. Concerning the relation between the topological condition that $X$ is compact and the algebraic and analytic condition that $k$ is a finite field or a local field, see Theorem 6.6.

**Proof.** It suffices to verify that a totally disconnected compact Hausdorff topological space $Y$ admits a homeomorphism onto the image into a direct product of copies of $k^\circ$. Define a map $\phi$ in the following way:

\[
\phi: Y \rightarrow (k^\circ)^{\text{CO}(Y)}
\]

\[
y \mapsto (1_U(y))_{U \in \text{CO}(Y)}.\]

It is injective because $Y$ is totally disconnected and Hausdorff, and is continuous because the pre-image of the clopen subset given by the condition that the $U$-entry is 1 (or 0) is the clopen subset $U \subset X$ (resp. $X \setminus U \subset X$). In order to prove that $\phi$ is an open map onto the image, take a clopen subset $U \subset \text{CO}(Y)$. The image $\phi(U) \subset \phi(Y)$ is the intersection of $\phi(Y)$ and the open subset given by the condition that the $U$-entry is contained in the open neighbourhood $k^\times \subset k$ of 1. Thus $\phi$ is a homeomorphism onto the image. □

Now we obtain many direct corollaries about Berkovich’s spectrum $BSC_k(X)$. Note that we will see the converse of following corollaries in Theorem 6.6.

**Corollary 3.9.** In the situation in Definition 3.8 one has $SC_k(X) \cong BSC_k(X)$.

**Corollary 3.10.** In the situation in Definition 3.8 the compactification $BSC_k(X)$ of $X$ satisfies the extension property for a $k$-valued bounded continuous function.

**Corollary 3.11.** In the situation in Definition 3.8 the canonical homomorphism $C(BSC_k(X), k) \rightarrow C_{bd}(X, k)$ is an isometric isomorphism of Banach $k$-algebras.

**Corollary 3.12.** In the situation in Definition 3.8 Berkovich’s spectrum $BSC_k(X)$ consists of $k$-rational points and any maximal ideal of $C_{bd}(X, k)$ is of codimension 1.
Proof. The assertion immediately follows from the non-Archimedean generalised Stone-Weierstrass theorem for the compact Hausdorff topological space and the ring $C(BSC_k(X), k) \cong k$ for any $a$ associated with the Set theory because the existence of such a discontinuous homomorphism is independent of the axiom of ZFC and depends on the negation of the continuum hypothesis. Now for a good class of Banach function algebra, a weakened version of the automatic continuity theorem, Theorem 4.5, is easily verified in the following way:

Lemma 4.1. Let $\mathcal{A}$ be a commutative Banach $k$-algebra. For a maximal ideal $m \in \mathcal{A}$ of codimension 1, the canonical projection $\mathcal{A} \to \mathcal{A}/m \cong k$ gives the decomposition $\mathcal{A} = k \oplus m$ as the orthogonal direct sum, i.e. the one has

$$\|a + g\| = \max(\|a\|, \|g\|)$$

for any $a \in k$ and $g \in m$.

Proof. Since the composition $k \inj \mathcal{A} \surj \mathcal{A}/m$ is a bijective $k$-linear homomorphism, one obtains the decomposition $\mathcal{A} = k \oplus m$ as the direct sum of $k$-vector spaces. Take an element $f \in \mathcal{A}$, and denote by $f(m) \in k$ the image of $f$ in the quotient $\mathcal{A}/m \cong_k k$. In order to prove the orthogonality of the direct sum $\mathcal{A} = k \oplus m$, it suffices to show that $\|f\| = \max(\|f(m)\|, \|f - f(m)\|)$. The inequality $\leq$ is obvious. If $\|f(m)\| \neq \|f - f(m)\|$, the equality follows from the general property of a non-Archimedean norm, and hence we may and do assume $\|f(m)\| = \|f - f(m)\|$ without loss of generality. If $f(m) = 0$, then one has $\|f - f(m)\| = 0$ and therefore $f = f(m) + (f - f(m)) = 0$. Suppose $f(m) \neq 0$. Assume $\|f\| < \|f(m)\|$. Then one has $\|f(m)^{-1}f\| < 1$, and hence

$$f - f(m) = -f(m)(1 - f(m)^{-1})f) \in k \mathcal{A}^{\times} = \mathcal{A}^{\times}$$

4 Automatic Continuity Theorem

The norm of the commutative Banach $k$-algebra $C_b(X, k)$ is power-multiplicative, and we proved in §4 that a maximal ideal of $C_b(X, k)$ is of codimension 1 when $k$ is a finite field or a local field. Such a Banach $k$-algebra is contained in a quite specific class of Banach function algebras, [BGR] 3.8.3. One of the important classical problem for an Archimedean Banach function algebra is Kaplansky Conjecture, or the automatic continuity problem in other words, and we want to consider the analogous question in the non-Archimedean case. When is any injective $k$-algebra homomorphism $\phi: C_b(X, k) \inj \mathcal{A}$ to a Banach $k$-algebra continuous? The problem in the Archimedean case is deeply related with the Set theory because the existence of such a discontinuous $k$-algebra homomorphism is independent of the axiom of ZFC and depends on the negation of the continuum hypothesis. Now for a good class of Banach function algebra, a weakened version of the automatic continuity theorem, Theorem 4.5, is easily verified in the following way:

Lemma 4.1. Let $\mathcal{A}$ be a commutative Banach $k$-algebra. For a maximal ideal $m \in \mathcal{A}$ of codimension 1, the canonical projection $\mathcal{A} \to \mathcal{A}/m \cong_k k$ gives the decomposition $\mathcal{A} = k \oplus m$ as the orthogonal direct sum, i.e. the one has

$$\|a + g\| = \max(\|a\|, \|g\|)$$

for any $a \in k$ and $g \in m$.

Proof. Since the composition $k \inj \mathcal{A} \surj \mathcal{A}/m$ is a bijective $k$-linear homomorphism, one obtains the decomposition $\mathcal{A} = k \oplus m$ as the direct sum of $k$-vector spaces. Take an element $f \in \mathcal{A}$, and denote by $f(m) \in k$ the image of $f$ in the quotient $\mathcal{A}/m \cong_k k$. In order to prove the orthogonality of the direct sum $\mathcal{A} = k \oplus m$, it suffices to show that $\|f\| = \max(\|f(m)\|, \|f - f(m)\|)$. The inequality $\leq$ is obvious. If $\|f(m)\| \neq \|f - f(m)\|$, the equality follows from the general property of a non-Archimedean norm, and hence we may and do assume $\|f(m)\| = \|f - f(m)\|$ without loss of generality. If $f(m) = 0$, then one has $\|f - f(m)\| = 0$ and therefore $f = f(m) + (f - f(m)) = 0$. Suppose $f(m) \neq 0$. Assume $\|f\| < \|f(m)\|$. Then one has $\|f(m)^{-1}f\| < 1$, and hence

$$f - f(m) = -f(m)(1 - f(m)^{-1})f) \in k \mathcal{A}^{\times} = \mathcal{A}^{\times}$$
by [BGR] 1.2.4/4. It contradicts the fact $f - f(m) \in m$, and thus $\|f\| = |f(m)| = \max\{|f(m)|, |f - f(m)|\}$.

**Corollary 4.2.** Suppose $k$ is a finite field endowed with the trivial norm or a local field. For any maximal ideal $m \in \text{Max}(C_{bd}(X, k))$, the canonical projection $C_{bd}(X, k) \to C_{bd}(X, k)/m$ gives the decomposition $C_{bd}(X, k) = k \oplus m$ as the orthogonal direct sum of $k$-Banach spaces.

**Proof.** Straightforward from Corollary 3.12 and Lemma 4.1. □

**Corollary 4.3.** In the situation in Corollary 4.2, one has

$$\|f\| = \sup_{m \in \text{Max}(C_{bd}(X, k))} |f(m)|$$

for any $f \in C_{bd}(X, k)$, where $f(m) \in k$ is the image of $f$ in the quotient $C_{bd}(X, k)/m \cong k$ for a maximal ideal $m \in \text{Max}(C_{bd}(X, k))$. In particular the norm of $C_{bd}(X, k)$ is determined by the algebraic structure of it.

**Proof.** Since the norm of $C_{bd}(X, k)$ is power-multiplicative, one has

$$\|f\| = \sup_{x \in \text{BSC}(X)} |f(x)|$$

for any $f \in C_{bd}(X, k)$ by [Bgr] 1.3.1. The assertion follows from the facts that the support map supp: $\text{BSC}_k(X) \to \text{Sp}(C_{bd}(X, k)) = \text{Max}(C_{bd}(X, k))$ is bijective by the proof of Theorem 3.3 and that $|f(x)| = |f(\text{supp}(x))|$ by Corollary 4.2. □

**Proposition 4.4.** Suppose $k$ is a local field. Then complete norms on the underlying $k$-algebra of $C_{bd}(X, k)$ are equivalent.

**Proof.** Since $k$ is a local field, the norm of $k$ is not trivial. Therefore the boundedness and the continuity of a $k$-linear homomorphism between normed $k$-vector spaces are equivalent, and it suffices to show that the identity id: $C_{bd}(X, k) \to C_{bd}(X, k)$ is a homeomorphism with respect to the metric topologies given by an arbitrary complete norm $\|\cdot\|$ of the domain and the supremum norm $\|\cdot\|$ of the codomain. By Lemma 4.1 for the Banach $k$-algebra $(C_{bd}(X, k), \|\cdot\|)$ and Corollary 4.3 the identity id is a contraction map, and hence is continuous. Moreover, since the norm of $k$ is not trivial, the open mapping theorem holds by [BGR] 2.8.1, and therefore the identity id is an open map. Thus the identity id is a homeomorphism. □

We finish the corollaries of Theorem 3.3 introducing the automatic continuity problem for non-Archimedean commutative $C^*$-algebras. Since Corollary 4.4 deals only with complete norms, the automatic continuity problem here is weakened to include the closedness condition of the image.

**Theorem 4.5.** Suppose $k$ is a local field, and let $\mathcal{A}$ be a Banach $k$-algebra. Then any injective $k$-algebra homomorphism $\phi: C_{bd}(X, k) \to \mathcal{A}$ whose image is closed is continuous.
Proof. Since the underlying metric spaces of $C_{bd}(X, k)$ and $\mathcal{A}$ are complete, it suffices to show that $\phi$ sends a Cauchy sequence in $C_{bd}(X, k)$ to a Cauchy sequence in $\mathcal{A}$. Let $\| \cdot \|': C_{bd}(X, k) \to [0, \infty)$ the composition of $\phi$ and the norm of $\mathcal{A}$. Then since $\phi$ is a homomorphism of $k$-algebras, $\| \cdot \|'$ is a seminorm of a $k$-algebra, and in addition the injectivity of $\phi$ guarantees that $\| \cdot \|'$ is a norm of a $k$-algebra. To begin with, we verify that $\| \cdot \|'$ is a complete norm. Take a Cauchy sequence $(f_i)_{i \in \mathbb{N}} \in C_{bd}(X, k)^{\mathbb{N}}$ with respect to the norm $\| \cdot \|'$. By the definition of $\| \cdot \|'$, the sequence $(\phi(f_i))_{i \in \mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$ is a Cauchy sequence, and hence it has the unique limit $g \in \mathcal{A}$ by the completeness of $\mathcal{A}$. Since the image of $\phi$ is closed, $g$ is contained in the image of $\phi$, and hence there is an element $f \in C_{bd}(X, k)$ such that $\phi(f) = g$. One has
\[ \lim_{i \to \infty} \|f - f_i\| = \lim_{i \to \infty} \|\phi(f) - f_i\| = \lim_{i \to \infty} \|g - \phi(f_i)\| = 0, \]
and hence the Cauchy sequence $(f_i)_{i \in \mathbb{N}}$ has the limit $f \in C_{bd}(X, k)$. Thus $\| \cdot \|'$ is a complete norm. Now again take a Cauchy sequence $(f_i)_{i \in \mathbb{N}} \in C_{bd}(X, k)^{\mathbb{N}}$ with respect to the supremum norm. By Proposition 4.4, the sequence $(f_i)_{i \in \mathbb{N}}$ is a Cauchy sequence also with respect to the complete norm $\| \cdot \|'$. Therefore the image $(\phi(f_i))_{i \in \mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$ is also a Cauchy sequence by the definition of $\| \cdot \|'$. We conclude $\phi$ is continuous. 

Corollary 4.6. Suppose $k$ is a local field. Let $V$ be a $k$-Banach space and $\rho: C_{bd}(X, k) \times V \to V$ a $k$-linear representation of a $k$-algebra $C_{bd}(X, k)$. Suppose the following conditions hold:

(i) The $k$-linear operator $\rho_f: V \to V: v \mapsto \rho(f, v)$ is bounded for any $f \in C_{bd}(X, k)$;

(ii) The $k$-linear representation $\rho$ is faithful, i.e. the equality $\rho_f = 0$ implies $f = 0$ for any $f \in C_{bd}(X, k)$;

(iii) The image of the induced $k$-algebra homomorphism $\rho: C_{bd}(X, k) \to \mathcal{B}_k(V)$ is closed, where $\mathcal{B}_k(V)$ is the Banach $k$-algebra of bounded operators on $V$.

Then the $k$-linear representation $\rho: C_{bd}(X, k) \times V \to V$ is a $k$-Banach representation of a Banach $k$-algebra, i.e. the induced $k$-algebra homomorphism $\rho: C_{bd}(X, k) \to \mathcal{B}_k(V)$ is bounded.

5 Gel’fand Theory

In this section, we establish the non-Archimedean Gel’fand theory for a totally disconnected Hausdorff topological space. Before that, recall that a completely regular Hausdorff topological space, which is a topological space which can be embedded in a direct product of copies of the closed interval $[-1, 1] \subset \mathbb{R}$. A direct product of copies of the closed interval $[-1, 1]$ is a compact Hausdorff topological space. Since the compactness is not hereditary for a subspace, a completely regular Hausdorff topological space is a Hausdorff space which is not necessarily compact. On the other hand, the non-Archimedean counterpart of a completely regular Hausdorff topological space over $k$,
Lemma 5.1. The following for a topological space $X$ are equivalent:

(i) The topological space $X$ is totally disconnected and Hausdorff;

(ii) The topological space $X$ is Hausdorff, and $k$-valued bounded continuous functions separate a point and a disjoint closed subset of $X$, i.e. for a point $x \in X$ and a closed subset $F \subset X$ with $x \notin F$, there is a $k$-valued bounded continuous function $f \in C_{bd}(X,k)$ such that $f(x) = 0$ and $f(y) = 1$ for any $y \in F$;

(iii) The continuous map $\iota_k : X \to SC_k(X)$ is a homeomorphism onto the image; and

(iv) The topological space $X$ admits an embedding into a direct product of copies of the closed unit disc $k^\circ$.

Note that the conditions (ii)-(iv) seem to depend on the choice of the base field $k$ but the condition (i) is purely topological. Therefore the notion of “a non-Archimedean completely regular Hausdorff topological space” is independent of the choice of the base field.

Proof. Recall that $SC_k(X)$ is a closed subspace of a direct product of copies of the unit closed disc $k^\circ$, and hence the condition (iii) implies the condition (iv). Moreover, the condition (iii) implies the condition (i) as we referred above. Suppose the condition (i) holds. Take a point $x \in X$ and a closed subset $F \subset X$ with $x \notin F$. Since $X$ is totally disconnected and Hausdorff, there is a clopen neighbourhood $U \in CO(X)$ of $x \in X$ contained in the closed subset $X \setminus F \subset X$, and the characteristic function $1_F$ separates $x$ and $F$: the condition (ii). Suppose the condition (ii) holds. Since $X$ is Hausdorff, a point of $X$ is closed. For two distinct points $x,y \in X$, take a $k$-valued bounded continuous function $f \in C_{bd}(X,k)$ which separates $x$ and $y$. Note that the subset $\{k\} \subset [0,\infty)$ is bounded if and only if the valuation of $k$ is trivial. In particular one has $\{k\} \subset [0,\infty)$ is bounded or $\{k\} = \{0,1\} \subset [0,\infty)$ is closed. In addition, since $\|C_{bd}(X,k)\| \subset [0,\infty)$ is contained in the closure of $\{k\} \subset [0,\infty)$ by the definition of supremum norm, there is an invertible element $a \in k^\times$ such that $\|f\| \leq |a|$. Then one has $\|a^{-1}f\| \leq 1$ and the $k$-valued bounded continuous function $a^{-1}f \in C_{bd}(X,k)(1)$ separates $x$ and $y$. Therefore one has $\iota_k(x)a^{-1}f \neq \iota_k(y)a^{-1}f$ and the continuous map $\iota_k$ is injective. In order to prove that $\iota_k$ is an open map onto the image, take an open subset $U \subset X$. For a point $x \in U$, take a $k$-valued bounded continuous map $f$ such that $f(x) = 0$ and $f(y) = 1$ for any $y \in X \setminus U$. By the same argument above, there is an invertible element $a \in k^\times$ such that $\|f\| \leq |a|$. Then the pre-image by $\iota_k$ of the open subset $V \subset SC_k(X)$ given by the condition that the $(a^{-1}f)$-entry is contained in the
open neighbourhood $k\setminus\{a^{-1}\} \subset k$ of $0 \in k$ is an open neighbourhood of $x$ contained in $U$. Therefore the image $\iota_k(U)$ contains the open neighbourhood $V \cap \iota_k(X) \subset \iota_k(X)$ of $\iota_k(x)$, and thus $\iota_k(U) \subset \iota_k(X)$ is open. We conclude that $\iota_k$ is an injective open continuous map: the condition (iii).

**Proposition 5.2.** Suppose $k$ is a finite field endowed with the trivial norm or a local field. Then the evaluation map $X \to BSC_k(X)$ is a homeomorphism onto the image if and only if $X$ is totally disconnected and Hausdorff.

**Proof.** The assertion immediately follows from Corollary 3.9 and Lemma 5.1. □

**Definition 5.3.** Let $\mathcal{A} \subset C_{bd}(X,k)$ a closed $k$-subalgebra. For points $x, x' \in X$, we write $x \sim_{\mathcal{A}} x'$ if $f(x) = f(x') \in k$ for any $f \in C_{bd}(X,k)$. The binary relation $\sim_{\mathcal{A}}$ is an equivalence relation, and denote by $X/\sim_{\mathcal{A}}$ the quotient topological space $X/\sim_{\mathcal{A}}$. We say $\mathcal{A}$ separates points of $X$ if the canonical projection $X \to X/\sim_{\mathcal{A}}$ is the identity.

**Lemma 5.4.** The continuous map $\iota_k : X \to SC_k(X)$ uniquely factors through the canonical projection $X \to X/C_{bd}(X,k)$, and the induced continuous map $X/C_{bd}(X,k) \to SC_k(X)$ is injective.

**Proof.** Trivial by the definition of the equivalence relation $\sim_{C_{bd}(X,k)}$. □

**Lemma 5.5.** If $X$ is totally disconnected and Hausdorff, the Banach $k$-algebra $C_{bd}(X,k)$ separates points of $X$.

**Proof.** By Lemma 5.1 and Lemma 5.4, the canonical projection $X \to X/C_{bd}(X,k)$ is the identity. □

Now we formulate the non-Archimedean Gel’fand theory for a non-Archimedean completely regular Hausdorff topological space. Be careful that we did not assign to the notion of the compactification the condition that the structure continuous map is a homeomorphism onto the image, we have to redefine the class of suitable compactifications.

**Definition 5.6.** Suppose $X$ is totally disconnected and Hausdorff. A topological space $Y$ equipped with a continuous map $f : X \to Y$ is said to be a faithful totally disconnected Hausdorff compactification of $X$ if $Y$ is a totally disconnected compact Hausdorff topological space, if $f$ is a homeomorphism onto the image, and if the image $f(X) \subset Y$ is dense.

**Definition 5.7.** Suppose $X$ is totally disconnected and Hausdorff. For two faithful totally disconnected Hausdorff compactifications $f_1 : X \to Y_1$ and $f_2 : X \to Y_2$, we write $Y_1 \equiv_X Y_2$ if there is a homeomorphism $g : Y_1 \to Y_2$ such that $g \circ f_1 = f_2$. The binary relation $\equiv_X$ is a class-theoretical equivalence relation.

Note that the collection $\mathcal{C}(X)$ of equivalence classes is not a proper class. Indeed, any faithful totally disconnected Hausdorff compactification $f : X \to Y$ factors through the universal totally disconnected Hausdorff compactification $X \hookrightarrow TD(X)$. Since $TD(X)$ is
compact and \( Y \) is Hausdorff, the image of TD is a closed subspace containing the dense subspace \( f(X) \subset Y \), and hence the induced continuous map \( TD(X) \to Y \) is a surjective map from a compact topological space to a Hausdorff topological space. Therefore a faithful totally disconnected Hausdorff compactification is obtained as a quotient topological space of TD(X), and \( C(X) \) admits a set-theoretical representative.

**Theorem 5.8.** Suppose that \( k \) is a finite field endowed with the trivial norm or a local field, and that \( X \) is totally disconnected and Hausdorff. Then there is a canonical contravariantly functorial one-to-one correspondence between the equivalence classes \( C(X) \) of faithful totally disconnected Hausdorff compactifications of \( X \) and the family \( C'(X) \) of closed \( k \)-subalgebras of \( C_{bd}(X, k) \) separating points of \( X \).

**Proof.** The correspondences are given in the following way:

\[
\begin{align*}
C(X) & \leftrightarrow C'(X) \\
[f : X \leftrightarrow Y] & \mapsto \text{Im}(f^a : C(Y, k) \hookrightarrow C_{bd}(X, k)) \\
[X \hookrightarrow \text{BSC}_k(X) \rightarrow \mathcal{M}_k(\mathcal{A})] & \leftrightarrow (\mathcal{A} \subset C_{bd}(X, k)).
\end{align*}
\]

They are the inverses of each other by the non-Archimedean generalised Stone-Weierstrass theorem, [Ber] 9.2.5. \( \Box \)

6 Ground Field Extensions

We finish this paper showing the compatibility of the ground field extension. Be careful that there are two distinct notions of the ground field extensions. One is the ground field extension associated with the extension of the scalar of functions, and the other is the Banach-algebra-theoretical ground field extension. We will see the difference between them in Theorem 6.6.

**Proposition 6.1.** Let \( K \) and \( L \) be complete valuation fields. Then there uniquely exists a homeomorphism \( \text{BSC}_K(X) \cong \text{BSC}_L(X) \) compatible with the evaluation maps.

Note that we do not assume that the fields \( K \) and \( L \) contains the same base field \( k \), and hence it is possible to choose \( \mathbb{Q}_p \) and \( \mathbb{Q}_l \) for \( K \) and \( L \) respectively.

**Proof.** The uniqueness is obvious because the images of the evaluation maps are dense and both of them are Hausdorff. The homeomorphism is obtained factoring through \( \text{UF}(X) \) by the canonical homeomorphisms in Theorem 3.3. \( \Box \)

**Corollary 6.2.** Let \( K/k \) be an extension of complete valuation fields. Then the ground field extension \( \text{BSC}_K(X) \to \text{BSC}_L(X) \) associated with the extension \( C_{bd}(X, k) \hookrightarrow C_{bd}(X, K) \) of the scalar is a homeomorphism.

**Proof.** The ground field extension above obviously preserves the evaluation maps. \( \Box \)
Now we consider the other ground field extension, namely, the canonical $K$-algebra homomorphism $K \hat{\otimes}_k C_{bd}(X, k) \to C_{bd}(X, K)$ induced by the universal property of the complete tensor product in the category of Banach $k$-algebras. In fact, the ground field extension is not an isomorphism, and it yields a criterion for the topological property of $X$ and the valuation of $k$.

**Lemma 6.3.** The $k$-subalgebra of $C_{bd}(X, k)$ consisting of locally constant bounded functions is dense.

**Proof.** Take a bounded continuous function $f \in C_{bd}(X, k)$. If $f = 0$, then $f = 0$ is locally constant. Suppose $f \neq 0$. For a positive number $\epsilon > 0$, as we verified in the calculus in the proof of [2,6] the pre-image of an open disc of radius $\epsilon$ by $f$ is clopen. Therefore one obtains a disjoint clopen covering $\mathcal{U} \subset CO(X)$ of $X$ such that the image $f(U)$ is contained in an open disc of radius $\epsilon$. Take a representative $a_U \in U$ for each $U \in \mathcal{U}$. Though $\mathcal{U}$ is not a finite covering, the pointwise convergent infinite sum $g := \sum_{U \in \mathcal{U}} a_U 1_U : X \to k$ is a locally constant bounded continuous function with the obvious inequality $\|g\| \leq ||f||$. Moreover, one has $\|f - g\| \leq \epsilon$ by the definition of the disjoint clopen covering $\mathcal{U}$, and hence the $k$-subalgebra of locally constant functions is dense in $C_{bd}(X, k)$. \hfill \Box

**Lemma 6.4.** Suppose $k$ is spherically complete, [BGR] 2.4.1/1. Let $K/k$ be an extension of complete valuation fields. Then the Banach-algebra-theoretical ground field extension $\iota_{K/k} : K \hat{\otimes}_k C_{bd}(X, k) \to C_{bd}(X, K)$ is an isometry.

For example, an abstract field endowed with the trivial norm and a local field is spherically complete. We will use this lemma for the finite field $\mathbb{F}_p$ endowed with the trivial norm, the rational number field endowed with the trivial norm, and the $p$-adic number field $\mathbb{Q}_p$.

**Proof.** Take an element $f \in K \hat{\otimes}_k C_{bd}(X, k)$. If $f = 0$, then $\|\iota_{K/k}(f)\| = 0 = ||f||$, and hence we assume $f \neq 0$. In particular $X \neq \emptyset$ and both of $K \hat{\otimes}_k C_{bd}(X, k)$ and $C_{bd}(X, K)$ are non-zero unital Banach $k$-algebras. Therefore the norm of the bounded $K$-algebra homomorphism $\iota_{K/k}$ is 1 because the norm of $C_{bd}(X, K)$ is power-multiplicative, and $\iota_{K/k}$ is a contraction map. For the positive number $\epsilon := ||f||/2 \in (0, ||f||)$, take an element $g = \sum_{i=1}^n a_i \otimes g_i \in K \hat{\otimes}_k C_{bd}(X, k)$ with $\|f - g\| < \epsilon$ in $K \hat{\otimes}_k C_{bd}(X, k)$. Replacing the presentation $g = \sum_{i=1}^n a_i \otimes g_i$ if necessary, we may and do assume $a_i \neq 0$ for any $i = 1, \ldots, n$. By Lemma [6,3], there is a $k$-valued locally constant bounded function $g'_i \in C_{bd}(X, k)$ such that $\|g_i - g'_i\| < |a_i|^{-1} \epsilon$ for each $i = 1, \ldots, n$. In particular setting $g' := \sum_{i=1}^n a_i \otimes g'_i \in K \hat{\otimes}_k C_{bd}(X, k)$, one has

$$\|f - g'\| = \|(f - g) + (g - g')\| \leq \max \left\{ ||f - g||, \left\| \sum_{i=1}^n a_i (g_i - g'_i) \right\| \right\}$$

$$\leq \max \{ ||f - g||, \max_{i=1}^n |a_i| \|g_i - g'_i\| \} < \epsilon < ||f||,$$

and hence $\|g'\| = ||f||$. Since $k$ is spherically complete, the finite dimensional normed $k$-vector subspace $ka_1 + \cdots + ka_n \subset K$ is $k$-Cartesian by [BGR] 2.4.4/2, and hence there
is an orthogonal basis $b_1, \ldots, b_m \in K$ of $ka_1 + \cdots + ka_n$. Presenting $a_1, \ldots, a_n$ as a $k$-linear combination of $b_1, \ldots, b_m$, one obtains the presentation $g' = \sum_{i=1}^m b_i g''_i$ by the unique system $g''_1, \ldots, g''_m \in C_{bd}(X,k)$ of $k$-valued locally constant functions. For any point $x \in X$, one has

$$|\iota_{K/k}(g')(x)| = \left| \sum_{i=1}^m b_i g''_i(x) \right| = \max_{i=1}^m |g''_i(x)||b_i|$$

by the orthogonality of $b_1, \ldots, b_m$, and hence

$$|\iota_{K/k}(g')| = \sup_{x \in X} |\iota_{K/k}(g')(x)| = \sup_{x \in X} \max_{i=1}^m |b_i||g''_i(x)| = \max_{i=1}^m |b_i| \sup_{x \in X} |g''_i(x)|$$

$$= \max_{i=1}^m |b_i||g''_i| \geq ||g'||.$$

Since $\iota_{K/k}$ is a contraction map, one acquires $|\iota_{K/k}(g')| = ||g'||$. We conclude

$$|\iota_{K/k}(f - g')| \leq ||f - g'|| < \epsilon < ||f|| = ||g'|| = |\iota_{K/k}(g')||$$

and thus

$$|\iota_{K/k}(f)| = |\iota_{K/k}(f - g') + \iota_{K/k}(g')| = |\iota_{K/k}(g')| = ||g'|| = ||f||.$$

□

**Definition 6.5.** Denote by $\mathbb{F} \subset k$ the closure of the fractional field of the image of the canonical ring homomorphism $\mathbb{Z} \to k$. The fractional field of the image of $\mathbb{Z} \to k$ is $\mathbb{F}_p$ if $k$ is of characteristic $p > 0$, or is $\mathbb{Q}$ if $k$ is of characteristic $0$. In the former case, $\mathbb{F}$ is the finite field $\mathbb{F}_p$ endowed with the trivial valuation. In the latter case, $\mathbb{F}$ is the rational number field $\mathbb{Q}$ endowed with the trivial norm if $k$ is of equal characteristic $(0,0)$, or is the $p$-adic number field $\mathbb{Q}_p$ if $k$ is of mixed characteristic $(0,p)$. In particular $\mathbb{F}$ is spherically complete.

Finally we observe when the ground field extension $\iota_{k/\mathbb{F}}: k \otimes_{\mathbb{F}} C_{bd}(X,\mathbb{F}) \to C_{bd}(X,k)$ is an isomorphism. The following show that the Banach $k$-algebra $C_{bd}(X,k)$ of $k$-valued bounded continuous functions on $X$ is “naive” enough to be analysable well if and only if $X$ is compact or $k$ is sufficiently small in some sense.

**Theorem 6.6.** Suppose $X$ is totally disconnected and Hausdorff. Then the following are equivalent:

(i) The topological space $X$ is compact, or the base field $k$ is a finite field endowed with the trivial norm or a local field;

(ii) The $k$-subalgebra of $C_{bd}(X,k)$ generated by idempotents is dense;

(iii) The ground field extension $\iota_{k/\mathbb{F}}: k \otimes_{\mathbb{F}} C_{bd}(X,\mathbb{F}) \to C_{bd}(X,k)$ is an isometric isomorphism if $\mathbb{F} \neq \mathbb{Q}$, or the condition (ii) holds if $\mathbb{F} = \mathbb{Q}$;
(iv) The compactification $BSC_k(X)$ of $X$ consists of $k$-rational points if $\mathbb{F} \neq \mathbb{Q}$, or the condition (ii) holds if $\mathbb{F} = \mathbb{Q}$:

(v) The evaluation map $X \hookrightarrow BSC_k(X)$ induces an isometric isomorphism $C(BSC_k(X), k) \to C_{bd}(X, k)$ if $\mathbb{F} \neq \mathbb{Q}$, or the condition (ii) holds if $\mathbb{F} = \mathbb{Q}$; and

(vi) The compactification $BSC_k(X)$ of $X$ satisfies the extension property for a $k$-valued bounded continuous functions if $\mathbb{F} \neq \mathbb{Q}$, or the condition (ii) holds if $\mathbb{F} = \mathbb{Q}$.

In particular, by the equivalence of (i) and (vi), this is the converse of Corollary 3.10

Remark that the condition (iii) does not necessarily hold when $X$ is not compact. In particular, unlike the ground field extension $BSC_k(X) \to BSC_k(X)$ induced by the extension $C_{bd}(X, k) \hookrightarrow C_{bd}(X, K)$ of the scalar, the ground field extension $\iota_{k/k} : K \hat{\otimes} C_{bd}(X, k) \to C_{bd}(X, K)$ is not necessarily an isomorphism.

**Proof.** Suppose the condition (i) holds. Take a bounded continuous function $f \in C_{bd}(X, k)$. If $k$ is a finite field or a local field, the closed disc $\{ a \in k \mid |a| \leq ||f|| \} \subset k$ is compact. Otherwise $X$ is compact. Therefore for a positive number $\epsilon > 0$, there is a finite disjoint clopen covering $\mathcal{U} \subset CO(X)$ of $X$ such that the image $f(U) \subset k$ is contained in an open disc of radius $\epsilon$ for any $U \in \mathcal{U}$. Take a representative $a_U \in U$ for each $U \in \mathcal{U}$, and then one has $||f - \sum_{U \in \mathcal{U}} a_U 1_U|| < \epsilon$. Thus $k$-subalgebra of $C_{bd}(X, k)$ generated by idempotents is dense: the condition (ii).

Suppose the condition (ii) holds. In order to verify that the condition (iii) holds, we may and do assume $\mathbb{F} \neq \mathbb{Q}$ without loss of generality. Since the canonical $k$-algebra homomorphism $\iota_{k/\mathbb{F}} : k \hat{\otimes}_{\mathbb{F}} C_{bd}(X, \mathbb{F}) \to C_{bd}(X, k)$ is an isometry by Lemma [6.4], it suffices to show that the image of $\iota_{k/\mathbb{F}}$ is dense. An idempotent of $C_{bd}(X, k)$, which is a characteristic function on a clopen subset of $X$, is contained in the subset $C_{bd}(X, \mathbb{F}) \subset C_{bd}(X, k)$. Therefore the image of the canonical $k$-algebra homomorphism $k \otimes_{\mathbb{F}} C_{bd}(X, \mathbb{F}) \to C_{bd}(X, k)$ is dense by the condition (ii), and hence the image of $\iota_{k/\mathbb{F}}$ is dense: the condition (iii). Note that we did not use the assumption that $\mathbb{F} \neq \mathbb{Q}$, and hence the condition that the ground field extension $\iota_{k/\mathbb{F}} : k \hat{\otimes}_{\mathbb{F}} C_{bd}(X, \mathbb{F}) \to C_{bd}(X, k)$ is weaker than the condition (ii).

Suppose the condition (iii) holds. In order to verify that the condition (iv) holds, we may and do assume $\mathbb{F} \neq \mathbb{Q}$ without loss of generality. For a character $x \in BSC_k(X)$, consider the composition

$$x' : C_{bd}(X, \mathbb{F}) \hookrightarrow C_{bd}(X, k) \xrightarrow{x} k(x).$$

Since $\mathbb{F}$ is contained in $k$, the character $x'$ defines an element $x' \in BSC_{\mathbb{F}}(X)$. Recall that $\mathbb{F} = \mathbb{F}_p$ or $\mathbb{Q}_p$ now. Since $BSC_{\mathbb{F}}(X)$ consists of $\mathbb{F}$-rational points by Corollary [3.12] the image of $x'$ is contained in $\mathbb{F} \subset k$. Therefore the image of $x$ is contained in the closure of the $k$-vector subspace of $k(x)$ generated by $\mathbb{F} \subset k$, namely, the 1-dimensional vector subspace $k \subset k(x)$. It follows that $k(x)$ is the completion of the fractional field of the complete valuation field $k$, and thus $k(x) = k$: the condition (iv).
Suppose the condition (iv) holds. In order to verify that the condition (v) holds, we may and do assume \( \mathbb{F} \neq \mathbb{Q} \) without loss of generality, and hence suppose the compactification \( \text{BSC}_k(X) \) of \( X \) consists of \( k \)-rational points. Then the evaluation pairing \( \text{C}_{\text{bd}}(X,k) \times \text{BSC}_k(X) \rightarrow k: (f, x) \mapsto f(x) \) gives the Gel'fand transform \( \text{C}_{\text{bd}}(X,k) \rightarrow \text{C}(\text{BSC}_k(X), k) \), which is an isometric isomorphism by [Ber] 9.2.7(ii). The Gel'fand transform \( \text{C}_{\text{bd}}(X,k) \rightarrow \text{C}(\text{BSC}_k(X), k) \) coincides with the bounded \( k \)-algebra homomorphism induced by the evaluation map \( X \rightarrow \text{BSC}_k(X) \) by definition: the condition (v).

Suppose the condition (v) holds. In order to verify that the condition (vi) holds, we may and do assume \( \mathbb{F} \neq \mathbb{Q} \) without loss of generality, and hence suppose the evaluation map \( X \rightarrow \text{BSC}_k(X) \) induces an isometric isomorphism \( \text{C}(\text{BSC}_k(X), k) \rightarrow \text{C}_{\text{bd}}(X,k) \). Take a bounded continuous function \( f \in \text{C}_{\text{bd}}(X,k) \). The extension of \( f \) on the compactification \( \text{BSC}_k(X) \) of \( X \) is unique because the image of the evaluation map \( X \rightarrow \text{BSC}_k(X) \) is dense by Corollary 3.5 and the complete valuation field \( k \) is Hausdorff. Since the evaluation map \( X \hookrightarrow \text{BSC}_k(X) \) induces an isomorphism \( \text{C}(\text{BSC}_k(X), k) \rightarrow \text{C}_{\text{bd}}(X,k) \), there is a continuous function \( f' \in \text{C}(\text{BSC}_k(X),k) \) on the compact topological space \( \text{BSC}_k(X) \) such that the composition of the evaluation map \( X \rightarrow \text{BSC}_k(X) \) and \( f' \) coincides with \( f: X \rightarrow k \), or in other words, \( f' \) is the extension of \( f \) on \( \text{BSC}_k(X) \): the condition (vi).

Finally, suppose the condition (vii) holds. Assume \( X \) is non-compact and \( k \) is neither a finite field nor a local field. Since \( X \) is a totally disconnected non-compact Hausdorff topological space, there is an infinite disjoint clopen covering \( \mathcal{U} \subset \text{CO}(X) \) of \( X \). If the residue field \( \overline{k} \) of \( k \) is an infinite field, set \( Y := \overline{k} \) and take a set-theoretical lift \( \phi: Y \hookrightarrow k^\circ \) of the canonical projection \( k^\circ \rightarrow Y \). Otherwise, the image \( |k^\circ| \subset (0, \infty) \) is dense because \( k \) is not a finite field or a local field. Set \( Y := |k^\circ| \cap (1/2, 1) \subset (0, \infty) \), and take a set-theoretical lift \( \phi: Y \hookrightarrow k^\circ \) of the norm \( |\cdot|: k \rightarrow (0, \infty) \). Since \( Y \) is dense in \((1/2, 1)\), it is an infinite set. In both cases, endow \( Y \) with the discrete topology. Since \( Y \) is an infinite set, there is an injective map \( \psi: \mathbb{N} \hookrightarrow Y \). The composition \( \phi \circ \psi: \mathbb{N} \hookrightarrow k^\circ \) is an injective continuous map, and the image is a closed discrete subspace because \( |\phi(y) - \phi(y')| \geq 1/2 \) for any \( y, y' \in Y \). Since \( \mathcal{U} \subset \text{CO}(X) \) is an infinite covering of \( X \), there is an injective map \( \Psi: \mathbb{N} \hookrightarrow \mathcal{U} \). Then the pointwise convergent infinite sum

\[
 f := \sum_{n \in \mathbb{N}} \phi(\psi(n))1_{\mathcal{U}(n)}: X \rightarrow k
\]

determines a locally constant bounded function on \( X \). Now we use the condition (vi). Suppose \( \mathbb{F} \neq \mathbb{Q} \). There is a non-principal ultrafilitre \( \mathcal{F} \in \text{CO}(\mathbb{N}) \) by Lemma 11.8. By the conditions (vi) and \( \mathbb{F} \neq \mathbb{Q} \), there is the continuous extension \( \text{BSC}_k(f): \text{BSC}_k(X) \rightarrow k \) of \( f \). Moreover, taking a representative \( x_n \in \Psi(n) \) for each \( n \in \mathbb{N} \), one obtains a continuous map \( x: \mathbb{N} \rightarrow X \hookrightarrow \text{BSC}_k(X) \). Since \( \text{BSC}_k(X) \) is a totally disconnected compact Hausdorff topological space, the continuous extension \( \text{BSC}_k(x): \text{UF}(\mathbb{N}) \cong_X \text{BSC}_k(\mathbb{N}) \rightarrow \text{BSC}_k(X) \) of \( x \) exists. The composition \( \text{BSC}_k(f) \circ \text{BSC}_k(x): \text{UF}(\mathbb{N}) \rightarrow \text{BSC}_k(X) \rightarrow k \) is the continuous extension of the composition \( f \circ x = \phi \circ \psi: \mathbb{N} \rightarrow k \). In particular \( \text{BSC}_k(f) \circ \text{BSC}_k(x) \) is continuous at \( \mathcal{F} \in \text{UF}(\mathbb{N}) \), but it contradicts the fact that the image of \( \phi \circ \psi \) an injective map whose image is a closed discrete subspace. An injective net whose image is discrete
and closed never has a limit. Therefore one has $\mathbb{F} = \mathbb{Q}$, and the $k$-subalgebra of $C_{bd}(X, k)$ generated by idempotents is dense by the condition (vi). Take a $k$-linear combination $g = \sum_{i=1}^{n} a_i U_i \in C_{bd}(X, k)$ of idempotents with $\|f - g\| < 1$. Now the image of $g$ contains at most $n$ points, and hence there is an integer $m \in \mathbb{N}$ such that $g(x) \notin \phi_m$ for any $x \in X$ identifying the cosets $\tilde{k}$ as a family of disjoint clopen subsets of $k^\circ$ in the tautological sense. Then one has $|f(x_m) - g(x_m)| = 1$, and it contradicts the condition $\|f - g\| < 1$. Thus $X$ is compact, or $k$ is a finite field or a local field: the condition (i), which was what we wanted.

\[\square\]

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