Hadron Masses in Strong Magnetic Fields

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Hadron masses under strong magnetic fields are studied. In the presence of strong magnetic fields exceeding the QCD energy scale $eB \gtrsim \Lambda_{QCD}^2$, SU(6) = SU(3)$_{flavor}$ ⊗ SU(2)$_{spin}$ symmetry of hadrons is explicitly broken so that the quark components of hadrons differ from those with zero or weak magnetic fields $eB \lesssim \Lambda_{QCD}^2$. Also, squeezing of hadrons by strong magnetic fields affects the hadron mass spectrum. We develop a quark model which appropriately incorporates these features and analytically calculate various hadron masses including mesons, baryons and those with strangeness.

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Introduction.— Quantum Chromodynamics (QCD) exhibits highly non-trivial behaviors in the presence of strong magnetic fields exceeding the QCD energy scale $eB \gtrsim \Lambda_{QCD}^2$ [1]. Indeed, lattice simulations are available without the notorious sign problem and many interesting phenomena are observed such as (inverse-)magnetic catalysis [2,3], anisotropy in the quark-antiquark potential [4] and non-trivial $eB$-dependence in hadron masses [5], while their physical interpretation is still in intense discussions. On the experimental side, such strong magnetic fields are realized in the peripheral collisions of relativistic heavy ions and possibly in the interior of neutron stars. Thus, it is also phenomenologically important to improve our understanding of QCD under strong magnetic fields.

In this letter, we discuss hadron masses under strong magnetic fields. Recent lattice study [6] calculated $\pi, \rho$ meson masses under strong magnetic fields and observed that (i) not only charged meson masses $M_{\pi^+,\mu^+}$ but also neutral meson masses $M_{\pi^0,\mu^0}$ depend on the strength of magnetic fields $eB$: (ii) $M_{\pi^+,\mu^+}$ increase as $\sqrt{eB}$ while $M_{\pi^0,\mu^0}$ show weak $eB$-dependence i.e., increase only slowly; (iii) there is a mass hierarchy $M_{\mu^0} \sim M_{\pi^0} \sim M_{\pi^+} > M_{\mu^+} > M_{\pi^0}$ for $eB \gtrsim \Lambda_{QCD}^2$. Two phenomenological studies, a relativistic Hamiltonian technique [7] and the NJL model [8], tried to explain these non-trivial behaviors so far. However, there is no consensus on the physical reason why there is such non-trivial behaviors.

The purpose of this letter is to present a simple analytical model which clearly explains the physics of the non-trivial behaviors in hadron masses under strong magnetic fields. We also apply the model to predict other hadron masses including baryons and hadrons with strangeness.

Model Description.— Before presenting the model, let us first clarify what are the essential properties for describing hadron masses under strong magnetic fields.

(a) Quark degrees of freedom: Since the typical energy scale of the system is now characterized by strong magnetic fields $eB \gtrsim \Lambda_{QCD}^2$, the internal structure of hadrons i.e., the quark degrees of freedom should be explicitly treated to describe hadron masses.

(b) Explicit breaking of SU(6) symmetry: SU(6) = SU(3)$_{flavor}$ ⊗ SU(2)$_{spin}$ symmetry of hadrons is the key concept in describing hadron masses with zero magnetic field [4,6]. However, in the presence of strong magnetic fields, this SU(6) symmetry must be broken. This is because quarks under strong magnetic fields form Landau levels and their masses depend on the spin and the electric charge. As a result, the quark components of hadrons under strong magnetic fields differ from those with zero or weak magnetic fields $eB \lesssim \Lambda_{QCD}^2$.

(c) Strong deformation of hadrons: Without strong magnetic fields, the typical volume of a hadron is solely determined by the confinement of QCD and is roughly given by $(1/\Lambda_{QCD})^3$. In the presence of strong magnetic fields, however, hadrons are strongly squeezed in the transverse direction $\langle r \rangle \sim 1/\sqrt{eB}$ not by QCD but by strong magnetic fields and so the typical volume would be given by $(1/\sqrt{eB})^2 \times (1/\Lambda_{QCD})$. As a result, the typical distance between quarks inside a hadron $|r_{eq}|$ rapidly decreases as the magnetic field gets stronger. This reduction of $|r_{eq}|$ leads to the following consequences: (c-1) The mass contribution from the long-range interaction between quarks i.e., the confinement potential of QCD $\propto |r_{eq}|$ decreases. (c-2) The short-range interaction between quarks, one-gluon exchange potential for example, becomes weaker due to the asymptotic freedom of QCD.

When describing hadron masses under strong magnetic fields, one should take into account all of the essential properties (a)-(c). We are going to develop a simple analytical model which incorporates all of the properties: A quark model under strong magnetic fields (a) whose quark components of hadrons respect the explicit breaking of the SU(6) symmetry of hadrons (b) and which includes the confinement potential of QCD (c-1) and neglects the short-range interaction between quarks (c-2).

Let us consider a Hamiltonian $H$, for a single quark with an electric charge $q$ and a current quark mass $m$, described by

$$H(r) = \alpha \cdot (-i \nabla - qA(r)) + \beta V(r),$$

$$A(r) = \frac{1}{2}Bre_\theta,$$
Here, we take cylindrical coordinates \( r = \sqrt{x^2 + y^2}, \theta = \arctan(y/x) \). \( \alpha \) and \( \beta \) are the Dirac matrices. The vector potential \( \mathbf{A} \) is chosen to realize a constant magnetic field along the \( z \)-axis \( \mathbf{B} = Be_z \). The potential \( V \) contains not only the mass term \( m^2 \) but also the term \( \sigma_1^z r^2 + \sigma_2^z z^2 \), which phenomenologically represents the linear confinement of QCD: \( \sigma_\parallel, \sigma_\perp \) are the string tension of QCD in the transverse and the longitudinal direction with respect to the magnetic field, respectively. Notice that the chiral symmetry is explicitly broken by the potential \( V \). We also remark that the detailed choice of the confinement potential in \( V \) does not affect our qualitative results. The advantage of this particular choice of \( V \) (Eq. (3)) is that the Dirac equation \( i\partial_\tau \psi = H\psi \) is analytically solvable.

By solving the Dirac equation, one can analytically show that the lowest energy level \( M \), which we shall call constituent quark mass, and the probability density \( \rho \equiv \psi^\dagger \psi \) of a single quark in the s-wave state are given by

\[
M = \sqrt{2\sqrt{(qB/2)^2 + \sigma_\perp^2 + \sigma_\parallel^2 + m^2 - qBs}} \quad (4)
\]

and

\[
\rho = |N|^2 e^{-\sqrt{(qB/2)^2 + \sigma_\perp^2 + \sigma_\parallel^2 + m^2 - qBs}} \frac{1}{(M + \sqrt{m^2 + \sigma_\perp^2 + \sigma_\parallel^2})^2} \quad (5)
\]

Here, \( s = 1 \) for spin \( \uparrow \), \(-1 \) for spin \( \downarrow \) and \( N \) is the normalization constant.

Let us discuss the \( eB \)-dependence of Eq. (4) in detail. For strong magnetic fields \( qB \gtrsim \sigma_\perp \), we obtain

\[
M \sim \sqrt{m^2 + \sigma_\parallel} |qB| - qBs \quad \text{for} \quad qB \gtrsim \sigma_\perp. \quad (6)
\]

The important point in Eq. (6) is that the constituent quark mass \( M \) increases as \( \sqrt{2|qB|} \) for \( qB < 0 \) while \( M \) stays almost constant for \( qB > 0 \). Notice that Eq. (6) is independent of the transverse string tension \( \sigma_\perp \) because hadrons are now strongly squeezed in the transverse direction \( \langle r \rangle \sim 1/|qB| \) by the strong magnetic field (see Eq. (3)) and the mass contribution from the transverse confinement of QCD \( \sim \sigma_\perp \langle r \rangle \) becomes negligible. We also note that Eq. (6) is a slight modification of the naive lowest Landau energy for a charged fermion, \( E_{LLL} = \sqrt{m^2 + |qB|} - qBs \), due to the longitudinal confinement of QCD \( \sigma_\parallel \neq 0 \). On the other hand, for weak magnetic fields \( qB \ll \sigma_\perp \), where hadrons are weakly squeezed by magnetic fields, we have

\[
M \sim M(B = 0) - \frac{qBs}{2M(B = 0)} \quad \text{for} \quad qB \ll \sigma_\perp. \quad (7)
\]

Here, \( M(B = 0) = \sqrt{m^2 + 2\sigma_\perp + \sigma_\parallel} \) is the constituent quark mass at \( B = 0 \). Equation (7) is nothing but the Zeeman splitting formula with the \( g \)-factor \( g = 2 \).

The constituent mass \( M \) of various quarks \( u \uparrow, u \downarrow, d \uparrow, d \downarrow, s \uparrow, s \downarrow \) can now be computed by Eq. (4) and the result is plotted in Fig. 1 as a function of the strength of magnetic fields \( eB \). In Fig. 1 we have set \( \sigma_\parallel = \sigma_\perp = (200 \text{ MeV})^2 \approx \Lambda_{QCD}^2 \) and \( m_u = m_d = 0 \text{ MeV}, m_s = 350 \text{ MeV} \) in order to reproduce the typical value of the constituent quark mass at \( B = 0 \): \( m_u, m_d \sim 350 \text{ MeV}, m_s \sim 500 \text{ MeV} \). Here, we have implicitly assumed that the string tension \( \sigma_\perp, \sigma_\parallel \) do not depend on magnetic fields and thus they are always spherically symmetric \( \sigma_\perp = \sigma_\parallel \) and constant. We remark that this simplification does not change our qualitative results so much, while some studies suggest that the gluon dynamics could be modified i.e., the string tension \( \sigma_\perp, \sigma_\parallel \) might vary in the presence of strong magnetic fields through quark loop corrections [3] [12].

Figure 1 clearly illustrates the explicit breaking of SU(6) symmetry of hadrons under strong magnetic fields. Indeed, we find a mass hierarchy

\[
M_{u\uparrow} > M_{d\uparrow} > M_{u\downarrow} \gtrsim M_{d\downarrow} \sim M_{u\uparrow} \quad (8)
\]

for strong magnetic fields \( eB \gtrsim \sigma \sim \Lambda_{QCD}^2 \). There, the constituent quark mass of \( u \uparrow, s \uparrow, \bar{d} \uparrow \) increases as \( \sqrt{2|qB|} \) because \( qB < 0 \) while it stays almost constant for \( s \downarrow, d \downarrow, u \uparrow \) because \( qB > 0 \) as is explained in Eq. (6). Thus, the mass hierarchy \( M_{u\uparrow}, M_{s\uparrow}, M_{d\uparrow} > M_{u\downarrow}, M_{d\downarrow}, M_{u\uparrow} \) appears. For the low-lying quarks \( u \uparrow, d \downarrow, s \downarrow \), the splitting \( M_{u\uparrow} \gtrsim M_{d\downarrow} \approx M_{s\uparrow} \) arises due to the current quark mass difference \( m_s \gtrsim m_u = m_d \). For the heavy quarks \( qu < 0 \) quarks \( u \uparrow, d \uparrow, s \uparrow \), the splitting \( M_{u\uparrow} > M_{d\uparrow} \sim M_{s\uparrow} \) appears due to the electric charge difference \( |q_u| = 2e/3 > |q_d| = |q_s| = e/3 \). Here, the current quark mass difference becomes unimportant because the constituent quark mass of \( qu < 0 \) quarks is largely determined by the electric charge only \( M_{qsu} \sim \sqrt{2|qB|} \). On the other hand, for weak magnetic fields \( eB \ll \sigma \sim \Lambda_{QCD}^2 \),
The constituent quark mass of $u \uparrow, d \downarrow, s \downarrow$ decreases as $eB$ increases and its magnitude is slightly larger for $u \uparrow$ than for $d \downarrow, s \downarrow$ (see Eq. (7)). This is the consequence of the deformation of hadrons by magnetic fields: Hadrons are squeezed by magnetic fields and so the mass contribution from the QCD confinement potential $\sim e_B |r_{qq}|$ decreases. Since $|r_{qq}|$ becomes smaller for larger electric charge $q$, we have a stronger reduction of $M_{uu}$ than $M_{ud}, M_{us}$.

Equation (8) is an essential relation to construct the proper quark components of hadrons under strong magnetic fields. This is summarized in Table I. For example, the quark components of $\rho^+$ meson, which is a composite of $u, d$ quarks and has total electric charge $Q = 1$ and total angular momentum $J = 1$, is given by $u \uparrow d \downarrow$. Indeed, the other $Q = 1, J = 1$ states such as $u \downarrow d \uparrow$ are heavier than $u \uparrow d \downarrow$ because of the mass hierarchy Eq. (8). Simply speaking, the proper quark components of hadrons under strong magnetic fields are determined by maximizing the number of $qs > 0$ quarks, whose constituent quark mass stays almost constant, and by minimizing the sum of the electric charge $\sum_{q < 0} |q|$ of $qs < 0$ quarks, whose constituent quark mass increases as $\sqrt{2|q_B|}$.

Now, we are ready to compute hadron mass $M_H$ under strong magnetic fields. By using the constituent quark mass $M_{quark}$ (Eq. (8)) and the proper quark components of hadrons displayed in Table I and by neglecting the short-range interaction between quarks, we have

$$ M_H = \sum_{\text{quark} \in H} M_{\text{quark}}. $$

The important point in Eq. (9) is that the $eB$-dependence of hadron mass $M_H$ is largely determined by the number of $qs < 0$ quarks and that $M_H$ increases as $\sum_{qs < 0} \sqrt{2|q_B|}$. The mass formula Eq. (9) is appropriate for strong magnetic fields $eB \gtrsim \sigma \sim \Lambda^2_{\overline{\text{QCD}}}$ because it incorporates all the essential properties (a)-(c). Although Eq. (9) roughly reproduces the physical hadron masses even for weak magnetic fields $eB \lesssim \sigma \sim \Lambda^2_{\overline{\text{QCD}}}$, in order to get a more precise description of hadron masses even for these regions, we have to take into account some other properties which Eq. (9) does not incorporate: The restoration of the SU(6) symmetry of hadrons, the short-range interaction between quarks and chiral corrections, which are essential especially for describing pion masses for weak magnetic fields.

**Hadron Masses.** — We analytically calculate various hadron masses by using Eq. (9) and the results are plotted in Fig. 2 and Fig. 3.

The masses of light mesons composed of $u, d$ quarks only are plotted in the left panel of Fig. 2. We find

$$ M_{\rho^0} \sim M_{\pi^0} > M_{\rho^\pm} \sim M_{\pi^\pm} $$

for strong magnetic fields $eB \gtrsim \sigma \sim \Lambda^2_{\overline{\text{QCD}}}$. $M_{\rho^0, \pi^0}$ increase as $\sqrt{2|q_B|}$ because $\rho^0, \pi^\pm$ contain $u \uparrow d \downarrow$ which has $qs < 0$, while there is no $qs < 0$ quark in $\rho^\pm, \pi^0$ and thus $M_{\rho^0, \pi^0}$ stay almost constant. For weak magnetic fields $eB \lesssim \sigma \sim \Lambda^2_{\overline{\text{QCD}}}$, $M_{\rho^0, \pi^0}$ decrease as $eB$ increases. This is the consequence of the deformation of hadrons by magnetic fields. For a comparison, the naive lowest Landau energies, $M_{\rho^0, \pi^0} = \sqrt{(M_{\rho^0, \pi^0}(B = 0))^2 + eB}$ and $M_{\rho^\pm, \pi^\pm} = \sqrt{(M_{\rho^\pm, \pi^\pm}(B = 0))^2 - eB}$, respectively, for $\pi^\pm$ and $\rho^\pm$ as point-like particles are plotted in black dashed lines. The deviation of our model from these lines at large $eB$ reflects the importance of the internal quark structure of hadrons. We remark that these behaviors are consistent with the lattice calculation [2].

The masses of strange mesons composed of $u, d, s$ quarks are plotted in the right panel of Fig. 2. We predict

$$ M_\phi \gtrsim M_{K_{\overline{\text{O}}^0}} \sim M_{K^\pm} > M_{\eta_s} \gtrsim M_{K_{\overline{\text{O}}^0}} \sim M_{K_{\overline{\text{O}}^0}^*} $$

for strong magnetic fields $eB \gtrsim \sigma \sim \Lambda^2_{\overline{\text{QCD}}}$. The interpretation is the same as the light meson masses for the major hierarchy $M_\phi, M_{K_{\overline{\text{O}}^0}, M_{K^\pm}, M_{\eta_s}, M_{K_{\overline{\text{O}}^0}, M_{K_{\overline{\text{O}}^0}^*}}$ appear because of the current quark mass difference $m_s \gtrsim m_u = m_d$. We also show the naive lowest Landau energies for $K^\pm, K_{\overline{\text{O}}^0}^*$ as point-like particles in black dashed lines. We again observe the deviation between our model and the lines.

The masses of light baryons (left panel of Fig. 3) and strange baryons (right panel of Fig. 3) are also investigated. The physics is the same as the meson masses: The number of $qs < 0$ quarks, the sum of the electric charge $\sum_{qs < 0} |q|$ of $qs < 0$ quarks and the number of strange
FIG. 2. (color online) Various meson masses as a function of the strength of the magnetic field $eB$. (Left) Light mesons. Black thin dashed lines are $\sqrt{(M_{\pi^\pm}(B=0))^2 + eB}$ and $\sqrt{(M_{\rho^\pm}(B=0))^2 - eB}$. (Right) Strange mesons. Black thin dashed lines are $\sqrt{(M_{K^\pm}(B=0))^2 + eB}$ and $\sqrt{(M_{K^{*\pm}}(B=0))^2 - eB}$.

FIG. 3. (color online) Various baryon masses as a function of the strength of the magnetic field $eB$. (Left) Light baryons. (Right) Strange baryons.

Quarks determine the mass hierarchy for strong magnetic fields $eB \gtrsim \sigma \sim \Lambda_{QCD}^2$. We predict mass hierarchies

$$M_{\Delta^0} > M_{\Delta^+} > M_{\Delta^-} \sim M_n \sim M_p \sim M_{\Delta^{++}}$$

for light baryons and

$$M_{\Xi^{*0}} \gtrsim M_{\Sigma^{*0}} > M_{\Xi^-} \gtrsim M_{\Sigma^-} \sim M_{\Sigma^{*+}} > M_{\Omega^-} \gtrsim M_{\Omega^-} \gtrsim M_{\Xi^0} \sim M_{\Xi^0} \sim M_{\Sigma^+}$$

for strange baryons.

Summary and Discussion. — We have studied hadron masses under strong magnetic fields. We have developed a quark model which incorporates the explicit breaking of the SU(6) symmetry and the strong deformation of hadrons, which is identified to be the essential properties to describe hadron masses under strong magnetic fields. Various hadron masses, including baryons and hadrons with strangeness, are analytically calculated by the model. In particular, the $eB$-dependence of $M_\pi, M_\rho$ are consistent with the recent lattice result [6]. The model also gives us a clear explanation why there is a non-trivial $eB$-dependence in hadron masses under strong magnetic fields: Under strong magnetic fields exceeding the QCD energy scale $eB \gtrsim \Lambda_{QCD}^2$, only the quarks in the lowest Landau level become important. In the lowest Landau level, the constituent quark mass increases as $\sqrt{2qB}$. For $qs < 0$ quarks while it stays almost constant for $qs > 0$ quarks. Also, since hadrons are strongly squeezed under strong magnetic fields and so the typical distance between quarks inside a hadron $|r_{qq}|$ rapidly decreases, the asymptotic freedom of QCD allows us to neglect the short-range interaction between quarks. From these two physical reasons, we find that $eB$-dependence of hadron masses is largely determined by the sum of constituent quark mass of $qs < 0$ quarks inside a hadron.

Since hadron masses are the most basic quantity of hadrons, the results of this study have a wide range of applications when discussing hadron physics under strong magnetic fields. Let us see some examples. One example is decay modes of hadrons: Some decay modes are kinematically suppressed or enhanced because of the mass hierarchy under strong magnetic fields (see Fig. 2 and Fig. 3). The modification to decay modes is first suggested by [8] which discussed a suppression of $\rho$ meson decays. Our study suggests that other decay modes are also modified, for example, $K^{0} \rightarrow \pi^{+}\pi^{-}$ is suppressed so that the lifetime of $K^{0}$ may be longer. Another example is the equation of state (EoS) of nuclear matter under strong magnetic fields. This is important for the physics of neutron stars. Not only the hadron masses but also interactions between hadrons would affect the EoS. This is because hadrons are spin-aligned under strong magnetic fields as displayed in Table II and so the spin-dependent part of the hadronic interactions would change.

For the further improvement of our model, it may be important to consider the gluon dynamics: The gluon
dynamics could be modified under strong magnetic fields through quark loop corrections. As a result, the string tension \( \sigma_{\perp}, \sigma_{\parallel} \) could depend on magnetic fields \(^5\) as already mentioned. Also, it is discussed that the constituent quark mass \( M \) could vary (magnetic catalysis; see chapter 2 of \(^1\) for a review). If this is true, the constituent quark mass \( M \) acquires additional mass contribution \( \sim (\text{small number}) \times \sqrt{|q B|} \). This may explain the small splitting \( M_{\rho^+} \gtrsim M_{\pi^0} \) observed in the lattice study \(^3\). In this case, a mass splitting \( M_{u\uparrow} = M_{\bar{d}\downarrow} \gtrsim M_{d\downarrow} = M_{\bar{u}\uparrow} \) occurs due to the electric charge difference \( |q_u| > |q_d| \). Then, the ground state of \( \pi^0 \) would be given by \( \pi^0_u = \bar{d} \uparrow d \downarrow \), not by \( \pi^0_u = \bar{u} \downarrow u \uparrow \). Thus, by noting that the ground state of \( \rho^+ \) is \( u \uparrow d \uparrow \), we have \( M_{\rho^+} = M_{u\uparrow} + M_{d\downarrow} \gtrsim M_{d\downarrow} + M_{\bar{d}\uparrow} = M_{\pi^0} \).

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[1] For a recent review, see D. Kharzeev et al., Lect. Notes Phys. 871, 1 (2013).
[2] M. D’Elia and F. Negro, Phys. Rev. D 83, 114028 (2011).
[3] G. S. Bali et al., Phys. Rev. D 86, 071502 (2012); JHEP 1202, 044 (2012).
[4] F. Bruckmann et al., JHEP 1304, 112 (2013).
[5] C. Bonati et al., Phys. Rev. D 89, 114502 (2014).
[6] Y. Hidaka and A. Yamamoto, Phys. Rev. D 87, 094502 (2013).
[7] M. A. Andreichikov et al., Phys. Rev. D 87, 094029 (2013).
[8] H. Liu and M. Huang, arXiv:1408.1318.
[9] M. Gell-Mann, Phys. Lett. 8, 214 (1964).
[10] G. Zweig, CERN-TH-401; CERN-TH-412.
[11] A. DeRujula et al., Phys. Rev. D 12, 147 (1975).
[12] V. A. Miransky and I. A. Shovkovy, Phys. Rev. D 66, 045006 (2002).
[13] M. N. Chernodub, Phys. Rev. D 82, 085011 (2010).