Stability of MHD shear flows: Application to space physics

Michael S. Ruderman\(^1\) and Nikolai A. Belov\(^2\)
\(^1\)Department of Applied Mathematics, University of Sheffield, Hounsfield Road, Hicks Building, Sheffield S3 7RH, United Kingdom
\(^2\)Institute for Problems in Mechanics, Russian Academy of Sciences, Vernadsky Ave. 101, Moscow 117527, Russia
E-mail: M.S.Ruderman@sheffield.ac.uk

Abstract. Shear flows of magnetised plasmas are routinely observed in the solar atmosphere, in planetary magnetospheres, and in interplanetary space. They are also ubiquitous elements of models of remote astrophysical objects like the interacting stellar winds in binary stellar systems. Studying stability of such flows is paramount for understanding physical processes in space.

The simplest shear flow is a planar tangential discontinuity. This is a flow where all flow parameters are constant at the two sides of a particular plane, and have jumps across the plane. In addition, the magnetic field and velocity are tangential to the plane.

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1. Introduction
Shear flows of magnetised plasmas are routinely observed in the solar atmosphere, in planetary magnetospheres, and in interplanetary space. They are also ubiquitous elements of models of remote astrophysical objects like accretion disks near neutron stars and black holes, and the interacting stellar winds in binary stellar systems. Hence, studying stability of such flows is paramount for understanding physical processes in space.

The simplest shear flow is a planar tangential discontinuity. This is a flow where all flow parameters are constant at the two sides of a particular plane, and have jumps across the plane. In addition, the magnetic field and velocity are tangential to the plane.

The standard approach to studying stability of shear flows is the normal mode analysis. This analysis is based on the linearised magnetohydrodynamic (MHD) equations. Perturbations of all
variables are taken to be proportional to $\exp(-i\omega t)$, where $t$ is the time. As a result we obtain an eigenvalue problem for $\omega$. The flow is unstable if there is at least one eigenvalue with the positive imaginary part, and stable otherwise. The instability that a tangential discontinuity can suffer from is called the Kelvin-Helmholtz (KH) instability.

It turns out that the normal mode analysis is not sufficient to describe the flow behaviour for large time. If the normal mode analysis predicts stability, then no other investigation is necessary. However if it predicts instability, then the behaviour of perturbations at large time can be quite different dependent on the nature of the instability. To study the behaviour of perturbations at large time we need to introduce the notion of absolute and convective instabilities.

The instability is called absolute if an initial perturbation of the flow grows exponentially with time at any fixed spatial position. It is called convective if an initial perturbation is exponentially decays with time at any spatial fixed position. The schematic picture of time evolution of an initial perturbation in absolutely and convectively unstable flows is shown in Fig. 1. We see that in the lower panel the initial perturbation is convected so quickly that it decays at any spacial position in spite that the perturbation amplitude increases. The instability character, absolute or convective, depends on the reference frame. In the lower panel the instability is absolute in the primed reference frame moving together with the growing perturbation.

To find out if the instability is absolute or convective we have to solve the initial value problem and then to evaluate the asymptotic behavior at a fixed spatial position as $t \to \infty$. The initial value problem for a tangential discontinuity is ill-posed: the perturbation increment is unbounded because, for a perturbation harmonic in space, it is proportional to the wavenumber. This implies that we cannot study the absolute and convective instabilities of tangential discontinuities.

There are two ways to alleviate this difficulty and obtain a well-posed problem: either to take
dissipation into account, or to consider a continuous velocity profile connecting two regions with constant but different flow velocities. In both cases the normal mode analysis gives a surprising from the first sight result: the account of either of these two effects not only makes the initial value problem well-posed, but also decreases the threshold value of the velocity jump needed to make the flow unstable. This phenomenon is related to the existence of negative energy waves. It can be shown that in both cases the instability is the so-called negative energy instability rather than the Kelvin-Helmholtz instability.

In this paper we give a brief review of linear stability of MHD shear flows and then describe its applications to the stability of the Earth’s magnetopause and heliopause. In the next section we present the linearised MHD equations used in this paper and write down the linearised boundary conditions at the tangential discontinuity. In Sect. 3 we consider the Kelvin-Helmholtz instability of tangential discontinuities. In Sect. 4 we describe negative energy instability of tangential discontinuities in dissipative fluids. This kind of instability is also called the dissipative instability. In Sect. 5 we study the negative energy instability of shear flows with continuous velocity profile. Since this instability is related to the wave energy resonant absorption it is also called the resonant instability. In Sect. 6 we apply the general results concerning the stability of MHD shear flows to the heliopause stability. In Sect. 7 a similar application is made to the stability of the Earth’s magnetopause. Sect. 8 contains the summary and conclusions.

2. Governing equations and boundary conditions

In what follows we use the MHD equations, which are the mass conservation, momentum, induction and entropy equations,

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \]  
\[ \rho \left( \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) = -\nabla p + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} + \mathbf{F}_{\text{vis}} + \rho \mathbf{g}, \]  
\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \lambda \nabla^2 \mathbf{B}, \]  
\[ \frac{\partial}{\partial t} \left( \frac{p}{\rho^\gamma} \right) + \mathbf{v} \cdot \nabla \left( \frac{p}{\rho^\gamma} \right) = (\gamma - 1) \rho \gamma \left( \mathcal{L}_{\text{vis}} + \frac{\gamma^2}{\sigma} - \nabla \cdot \mathbf{q} \right). \]

Here \( \rho \) is the density, \( p \) the pressure, \( \mathbf{v} \) the velocity, \( \mathbf{g} \) the gravity acceleration, and \( \mathbf{B} \) the magnetic field; \( \mu_0 \) is permeability of free space, \( \lambda \) the coefficient of magnetic diffusion, and \( \gamma \) the ratio of specific heats. \( \mathbf{F}_{\text{vis}} \) is the viscous force. The most general expression for it is given by Braginskii [1],

\[ \mathbf{F}_{\text{vis}} = \nabla \cdot \mathbf{P}, \]

where \( \mathbf{P} \) is the viscosity tensor determined by

\[ \mathbf{P} = \eta_0 \mathbf{P}_0 + \eta_1 \mathbf{P}_1 + \eta_2 \mathbf{P}_2 - \eta_3 \mathbf{P}_3 - \eta_4 \mathbf{P}_4, \]

\[ \mathbf{P}_0 = (\mathbf{b} \mathbf{b} - \frac{1}{3} \mathbf{I}) \mathcal{Q}, \]

\[ \mathbf{P}_1 = \nabla \mathbf{v} + (\nabla \mathbf{v})^T - \mathbf{b} \mathbf{W} - \mathbf{W} \mathbf{b} + (\mathbf{b} \mathbf{b} - \mathbf{I}) \nabla \cdot \mathbf{v} + (\mathbf{b} \mathbf{b} + \mathbf{I}) \mathbf{b} \cdot \nabla (\mathbf{b} \cdot \mathbf{v}), \]

\[ \mathbf{P}_2 = \mathbf{b} \mathbf{W} + \mathbf{W} \mathbf{b} - 4(\mathbf{b} \mathbf{b}) \mathbf{b} \cdot \nabla (\mathbf{b} \cdot \mathbf{v}), \]

\[ \mathbf{P}_3 = \frac{1}{2} \left\{ (\mathbf{b} \times \nabla) \mathbf{v} + [(\mathbf{b} \times \nabla) \mathbf{v}]^T + \nabla (\mathbf{b} \times \mathbf{v}) + [\nabla (\mathbf{b} \times \mathbf{v})]^T - \mathbf{b} (\mathbf{b} \times \mathbf{W}) - (\mathbf{b} \times \mathbf{W}) \mathbf{b} \right\}, \]

\[ \mathbf{P}_4 = \mathbf{b} (\mathbf{b} \times \mathbf{W}) + (\mathbf{b} \times \mathbf{W}) \mathbf{b}, \]

\[ \mathcal{Q} = 3 \mathbf{b} \cdot \nabla (\mathbf{b} \cdot \mathbf{v}) - \nabla \cdot \mathbf{v}, \]

\[ \mathbf{W} = \nabla (\mathbf{b} \cdot \mathbf{v}) + \mathbf{b} \cdot \nabla \mathbf{v}. \]
In Eqs. (6)–(12) \( \eta_0, \ldots, \eta_4 \) are the coefficients of viscosity, \( \mathbf{l} \) is the unit tensor, \( \mathbf{b} = \mathbf{B}/B \) is the unit vector in the magnetic field direction, the symbol \( \mathbf{ab} \) indicates the dyadic product of vectors \( \mathbf{a} \) and \( \mathbf{b} \) (in particular, one of the vectors can be \( \nabla \)), and the superscript ‘\( T \)’ indicates a transposed tensor.

In the entropy equation \( \mathcal{L}_{\text{vis}} \) is the rate of mechanical energy dissipation due to viscosity given by

\[
\mathcal{L}_{\text{vis}} = P : \nabla \mathbf{v} = P_{ij} \frac{\partial v_i}{\partial x_j},
\]

(13)

It can be shown that \( \mathcal{L}_{\text{vis}} > 0 \). The quantity \( \mathbf{j} \) is the electric current density related to the magnetic field by

\[
\mathbf{j} = \frac{1}{\mu_0} \nabla \times \mathbf{B},
\]

(14)

and \( \sigma \) is the conductivity related to \( \lambda \) by \( \lambda = 1/ (\mu_0 \sigma) \). Finally, \( q \) is the thermal energy flux given by [1]

\[
q = -k_\parallel \nabla_\parallel T - k_\perp \nabla_\perp T - \kappa_\Lambda \mathbf{b} \times \nabla T,
\]

(15)

where \( k_\parallel, k_\perp \) and \( \kappa_\Lambda \) are the parallel, perpendicular and skew coefficients of thermal conduction, \( \nabla_\parallel = \mathbf{b} (\mathbf{b} \cdot \nabla) \), \( \nabla_\perp = \nabla - \mathbf{b} (\mathbf{b} \cdot \nabla) \), and \( T \) is the temperature related to \( \rho \) and \( p \) by the ideal gas law,

\[
p = \frac{k_B}{m} \rho T
\]

(16)

with \( k_B \) and \( m \) being the Boltzmann constant and the mean mass per particle, respectively.

Let us now write down the boundary conditions at a tangential discontinuity. We assume that the equation of a perturbed discontinuity is \( f(x, y, z, t) = 0 \) in Cartesian coordinates \( x, y, z \).

If an elementary fluid volume is at this surface at the initial moment of time, then it remains on this surface forever. Let \( x = x(t), y = y(t), z = z(t) \) be the equations of this volume trajectory. Then we have an identity

\[
f(x(t), y(t), z(t), t) \equiv 0.
\]

Differentiating this identity with respect to time we obtain

\[
\frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} + \frac{\partial f}{\partial t} = 0.
\]

(17)

Taking into account that the derivatives of \( x(t), y(t) \) and \( z(t) \) are equal to the components of the elementary volume velocity, we rewrite this equation as

\[
\mathbf{v} \cdot \nabla f + \frac{\partial f}{\partial t} = 0.
\]

Since \( \nabla f = \mathbf{n} |\nabla f| \) where \( \mathbf{n} \) is the normal vector to the surface of tangential discontinuity, it follows from this equation that the normal component of the velocity is continuous at the discontinuity. Introducing the jump of a function \( g(x, y, z, t) \) through the discontinuity,

\[
\left[ g(\mathbf{r}) \right] = \lim_{\varepsilon \to 0} [g(\mathbf{r} + \varepsilon \mathbf{n}) - g(\mathbf{r} - \varepsilon \mathbf{n})]
\]

where \( \mathbf{r} = (x, y, z) \) is the position vector, and the point \( \mathbf{r} \) in this expression is on the surface of tangential discontinuity, we eventually write the kinematic boundary condition as

\[
\left[ \mathbf{v}_n \right] = 0,
\]

(18)

where \( \mathbf{v}_n = \mathbf{n} \cdot \mathbf{v} \).

In the presence of magnetic field and viscosity the stress tensor is given by

\[
\mathbf{T} = - \left( p + \frac{B^2}{2\mu_0} \right) \mathbf{l} + \frac{\mathbf{BB}}{\mu_0} + \mathbf{P}.
\]
Since at the tangential discontinuity \( \mathbf{n} \cdot \mathbf{B} = 0 \), the stress at the surface of tangential discontinuity is

\[
\mathbf{n} \cdot \mathbf{T} = - \left( p + \frac{\mathbf{B}^2}{2\mu_0} \right) \mathbf{n} + \mathbf{n} \cdot \mathbf{P}.
\]

The stress has to be continuous at the surface of tangential discontinuity, which gives us the dynamic boundary condition,

\[
\left[ \left( p + \frac{\mathbf{B}^2}{2\mu_0} \right) \mathbf{n} - \mathbf{n} \cdot \mathbf{P} \right] = 0. \tag{19}
\]

In an inviscid fluid this condition reduces to the continuity of the total pressure, kinetic plus magnetic,

\[
\left[ p + \frac{\mathbf{B}^2}{2\mu_0} \right] = 0. \tag{20}
\]

In what follows we study only the linear stability of MHD shear flows, so that we use the linearised equations. We assume that the equilibrium flow is stationary. In a dissipative fluid we, in general, need external forces and energy sources to maintain a stationary flow. We assume that such forces and energy sources are implicitly present. Linearising Eqs. (1)–(4) we obtain

\[
\rho \frac{\partial \mathbf{v}'}{\partial t} + \nabla \cdot (\rho_0 \mathbf{v}' + \mathbf{v}_0') = 0, \tag{21}
\]

\[
\rho_0 \left( \frac{\partial \mathbf{v}'}{\partial t} + (\mathbf{v}_0 \cdot \nabla) \mathbf{v}' + (\mathbf{v}' \cdot \nabla) \mathbf{v}_0 \right) + \rho' (\mathbf{v}_0 \cdot \nabla) \mathbf{v}_0 = -\nabla p' + \frac{1}{\rho_0} \left[ (\nabla \times \mathbf{B}') \times \mathbf{B}_0 + (\nabla \times \mathbf{B}_0) \times \mathbf{B}' \right] + \mathbf{F}_{\text{vis}}' + \rho' \mathbf{g}, \tag{22}
\]

\[
\frac{\partial \mathbf{B}'}{\partial t} = \nabla \times (\mathbf{v}_0 \times \mathbf{B}' + \mathbf{v}' \times \mathbf{B}_0) + \lambda \nabla^2 \mathbf{B}', \tag{23}
\]

\[
\rho \frac{\partial (p' - c_S^2 \rho')}{\partial t} + \rho_0' \mathbf{v}_0 \cdot \nabla \left( \frac{p' - c_S^2 \rho'}{\rho_0} \right) + \rho_0' \mathbf{v}' \cdot \nabla \frac{p_0}{\rho_0} + \gamma \rho_0'^{-1} \rho_0' \mathbf{v}_0 \cdot \nabla \frac{p_0}{\rho_0} = (\gamma - 1) \left( \mathbf{L}_{\text{vis}}' + \frac{2\mathbf{j}' \cdot \mathbf{j}_0}{\sigma} - \nabla \cdot \mathbf{q}' \right), \tag{24}
\]

where \( c_S \) is the sound speed determined by \( c_S^2 = \gamma p_0 / \rho_0 \), and the subscript ‘0’ and the prime indicate an equilibrium quantity and the perturbation respectively. Under the assumption that \( T_0 \) is constant the expression for \( \mathbf{q}' \) is given by Eq. (15) with \( T' \) substituted for \( T \). We do not give the expressions for \( \mathbf{F}_{\text{vis}}' \) and \( \mathbf{L}_{\text{vis}}' \). They will be given later for particular cases. The linearized ideal gas law takes the form

\[
p' = k_B m \rho_0 T' + \frac{k_B m}{m} \rho' T_0. \tag{25}
\]

To derive the linearized boundary conditions we assume that the unperturbed tangential discontinuity coincides with the \( z = 0 \) plane, while the equation of the perturbed discontinuity can be written as \( z = \zeta(x, y, t) \). Then, in particular, \( w_0 = 0 \), where \( w \) is the \( z \)-component of the velocity, and

\[
\mathbf{n} = \frac{\mathbf{e}_z - \nabla \zeta}{\sqrt{1 + |\nabla \zeta|^2}}.
\]
where $e_z$ in the unit vector in the $z$-direction. Then the linearized kinematic boundary condition obtained from Eq. (18) takes the form

$$\left[ w' - v_0 \cdot \nabla \zeta \right] = 0. \quad (26)$$

To relate $v'$ and $\zeta$ we use the linearized Eq. (17). As a result we obtain

$$w' = \frac{\partial \zeta}{\partial t} + v_0 \cdot \nabla \zeta. \quad (27)$$

This equation is valid at both sides of the discontinuity, so that we can use it instead of Eq. (26).

Note that both Eqs. (26) and Eq. (27) are valid not at $z = \zeta$, but at $z = 0$.

Using the fact that $n' = -\nabla \zeta$ we write the linearised dynamic boundary condition obtained from Eq. (19) as

$$\left[ \left( p' + \frac{B_0 \cdot B'}{\mu_0} \right) e_z - e_z \cdot P' - \left( p_0 + \frac{B_0^2}{2\mu_0} \right) \nabla \zeta + P_0 \cdot \nabla \zeta + \zeta \left[ e_z \frac{\partial}{\partial z} \left( p_0 + \frac{B_0^2}{2\mu_0} \right) - e_z \cdot \frac{\partial P_0}{\partial z} \right] \right] = 0. \quad (28)$$

This equation is also valid at $z = 0$. In an inviscid fluid this equation is simplified to

$$\left[ \left( p' + \frac{B_0 \cdot B'}{\mu_0} \right) + \zeta \frac{\partial}{\partial z} \left( p_0 + \frac{B_0^2}{2\mu_0} \right) \right] = 0, \quad (29)$$

where we have taken into account that Eq. (20) is valid for the equilibrium quantities.

The linearized MHD equations and boundary conditions presented in this section will be used in what follows to study the stability of MHD shear flows.

### 3. Stability of MHD tangential discontinuities

In this section we study the stability of planar MHD tangential discontinuity in an ideal fluid. It is assumed that the unperturbed discontinuity coincides with the $z = 0$ plane, and the equilibrium quantities are constant above and below this plane, while they, in general, have jumps through the discontinuity. The equilibrium velocity and magnetic field are parallel to the $xy$-plane.

First we assume that the fluid is incompressible. In that case we use equation

$$\nabla \cdot v' = 0 \quad (30)$$

instead of Eq. (21), and we do not need Eq. (24). Since the equilibrium quantities are independent of $t$, $x$ and $y$, we can take perturbations of all quantities proportional to $\exp(i k \cdot r - i \omega t)$, where the wave vector $k$ is parallel to the $xy$-plane. As a result we reduce the system of Eqs. (22), (23) and (30) to the system of ordinary differential equations. Eliminating all variables from this system in favor of $w'$ we arrive at

$$\frac{d^2 w'}{dz^2} - k^2 w' = 0. \quad (31)$$

In addition we have

$$p' + \frac{B_0 \cdot B'}{\mu_0} = i \rho_0 \frac{[(\omega - k \cdot v_0)^2 - (k \cdot V_A)^2]}{k^2 (\omega - k \cdot v_0)} \frac{dw'}{dz}, \quad (32)$$
where $V_A = B_0/\sqrt{\mu_0 \rho_0}$ is the Alfvén velocity. Using this result we rewrite the dynamic boundary condition (29) as

$$
\left[ \rho_0 \left( \frac{(\omega - k \cdot v_0)^2 - (k \cdot V_A)^2}{\omega - k \cdot v_0} \right) \frac{dw'}{dz} \right] = 0,
$$

(33)

where we have taken into account that, in the absence of gravity, the derivative of the total pressure with respect to $z$ is continuous as the tangential discontinuity. Using Eq. (27) we write the kinematic boundary condition as

$$
\left[ \frac{w'}{\omega - k \cdot v_0} \right] = 0.
$$

(34)

It follows from Eq. (31) that

$$
w' = \begin{cases} 
  w_1 e^{kz}, & z < 0, \\
  w_2 e^{-kz}, & z > 0.
\end{cases}
$$

(35)

Substituting this result in Eq. (33) and Eq. (34) we obtain the system of two linear homogeneous algebraic equations with respect to $w_1$ and $w_2$. This system has a non-trivial solution only if its determinant is zero. This gives the dispersion equation

$$
(\rho_1 + \rho_2)\omega^2 - 2k \cdot (\rho_1 v_{01} + \rho_2 v_{02}) \omega + \rho_1 ((k \cdot v_{01})^2 - (k \cdot V_{A1})^2) + \rho_2 ((k \cdot v_{02})^2 - (k \cdot V_{A2})^2) = 0,
$$

(36)

where the subscripts ‘1’ and ‘2’ refer to the regions $z < 0$ and $z > 0$ respectively. The discriminant of this equation is equal to

$$
(\rho_1 + \rho_2)(\rho_1 (k \cdot V_{A1})^2 + \rho_2 (k \cdot V_{A2})^2) - \rho_1 \rho_2 |k \cdot (v_{01} - v_{02})|^2.
$$

(37)

When it is positive the roots of Eq. (36) are real and the perturbation is neutrally stable. However when the discriminant is negative, there are two complex conjugate roots and the perturbation grows exponentially. Hence, the instability criterion is

$$
[k \cdot (v_{01} - v_{02})]^2 > \frac{(\rho_1 + \rho_2)(\rho_1 (k \cdot V_{A1})^2 + \rho_2 (k \cdot V_{A2})^2)}{\rho_1 \rho_2}.
$$

(38)

We see that the instability criterion is independent of the magnitude of $k$, however it does depend on its direction. The tangential discontinuity is unstable as soon as the instability criterion (38) is satisfied for at least one $k$. In particular, when $B_{01} \parallel B_0$ and $v_{01} - v_{02}$ is not parallel to $B_{01}$, all perturbations with $k \perp B_{01}$ are unstable for any value of $|v_{01} - v_{02}|$. On the other hand, when $B_{01} \parallel B_0 \parallel (v_{01} - v_{02})$, Eq. (38) reduces to

$$
|v_{01} - v_{02}|^2 > V_{KH}^2 = \frac{(\rho_1 + \rho_2)(\rho_1 V_{A1}^2 + \rho_2 V_{A2}^2)}{\rho_1 \rho_2}.
$$

(39)

The quantity $V_{KH}$ is the so-called Kelvin-Helmholtz threshold velocity, and the instability that takes place when Eq. (39) is satisfied is the Kelvin-Helmholtz (KH) instability. The stability of an MHD tangential discontinuity in an ideal incompressible fluid was first studied by Syrovatskii [2] (see also Chandrasekhar [3]).

Later Fejer [4] generalized this analysis to take compressibility into account. He obtained the following dispersion equation written in the reference frame where $v_{01} = 0$:

$$
(\rho_1/m_1)V_{A1}^2 k^2 \cos^2(\phi - \psi_1) - \omega^2 + (\rho_2/m_2)V_{A2}^2 k^2 \cos^2(\phi - \psi_2) - (\omega - U k \cos \phi)^2 = 0,
$$

(40)
where \( U = |v_{02}| \), \( \psi_1 \) and \( \psi_2 \) are the angles between \( k \) and \( B_{01} \) and \( B_{02} \) respectively, and \( \phi \) is the angle between \( k \) and \( v_{02} \). The quantities \( m_1 \) and \( m_2 \) are given by

\[
m_1^2 = k^2 - \frac{\omega^4}{\omega^2(c_{S1}^2 + V_{A1}^2) - c_{S1}^2 V_{A1}^2 k^2 \cos^2(\phi - \psi_1)},
\]

\[
m_2^2 = k^2 - \frac{(\omega - U k \cos \phi)^4}{(\omega - U k \cos \phi)^2(c_{S2}^2 + V_{A2}^2) - c_{S1}^2 V_{A1}^2 k^2 \cos^2(\phi - \psi_1)}.
\]

The quantities \( m_1 \) and \( m_2 \) have to satisfy the conditions \( \Re(m_1) > 0 \) and \( \Re(m_2) > 0 \), where \( \Re \) indicates the real part of a quantity.

Equation (40) can be reduced to the tenth-order algebraic equation. Written in the dimensionless form this equation contains 8 parameters, which makes its comprehensive analysis extremely difficult. Usually only particular cases with the reduced number of parameter are studied. One such particular case will be considered in Sect. 6.

If we rewrite the dispersion equation (36) or a more general equation (40) in terms of \( \omega/k \), then we obtain equations that are independent of \( k \). Then it follows that the instability increment is proportional to \( k \), and thus its is unbounded. This implies that the initial value problem for an MHD tangential discontinuity is ill-posed. In the next two sections we will discuss how to modify the equilibrium flow to make the initial value problem well-posed.

4. Negative energy waves and dissipative instabilities

4.1. Derivation of governing equation

In this section we consider the stability problem for an MHD tangential discontinuity in a viscous fluid. It is well known that tangential discontinuities cannot exist in viscous fluids: viscosity smears out the discontinuity. To overcome this difficulty Ruderman and Goossens [5] considered an MHD tangential discontinuity in a fluid viscous at one side of the discontinuity and inviscid at the other side. They also assumed that the fluid is incompressible. The third assumption that they made concerns the form of the viscosity tensor. The coefficients \( \eta_0, \ldots, \eta_4 \) in Eq. (6) depend on the dimensionless parameter \( \tau_i \omega_i \), where \( \tau_i \) is the collisional time and \( \omega_i \) the cyclotron frequency of the ions. When \( \tau_i \omega_i \ll 1 \) the viscosity tensor becomes isotropic, i.e. independent of the magnetic field direction, and reduces to a relatively simple form,

\[
P_{ij} = \eta_0 \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right).
\]

Then

\[
e_z \cdot \mathbf{P}' = \eta_0 \left( \frac{\partial v'}{\partial z} + \nabla w' \right).
\]

Using this result, assuming that the fluid is viscous in the region \( z < 0 \) and inviscid in the region \( z > 0 \), and recalling that the equilibrium quantities are constant in \( z < 0 \) and \( z > 0 \), we reduce Eq. (28) to two boundary conditions at \( z = 0 \),

\[
p'_1 + \frac{B_{01} \cdot B'_1}{\mu_0} - 2\eta_0 \frac{\partial w'}{\partial z} = p'_2 + \frac{B_{02} \cdot B'_2}{\mu_0},
\]

\[
\frac{\partial v'_1}{\partial z} + \nabla_\perp w'_1 = 0,
\]

where \( v'_1 = v' - w' e_z \) and \( \nabla_\perp = \nabla - e_z \partial / \partial z \). We use the reference frame where the fluid in the region \( z < 0 \) is at rest, while the fluid in the region \( z > 0 \) is moving with the speed \( U_0 > 0 \) in
In Eq. (47), we obtain the following equation valid in the boundary layer,

\[ w_1' = \frac{\partial \zeta}{\partial t}, \quad w_2' = \frac{\partial \zeta}{\partial t} + U_0 \frac{\partial \zeta}{\partial x}. \tag{46} \]

Now we take perturbations of all variables proportional to \( \exp(ik \cdot r - i\omega t) \), where the wave vector \( k \) is parallel to the \( xy \)-plane. Then after long but straightforward calculation we obtain from the system of Eqs. (22), (23) and (30)

\[ p_1' + \frac{B_{01}B_{x1}'}{\mu_0} = A_1(t, k)e^{kz}, \quad p_2' + \frac{B_{02}B_{x2}'}{\mu_0} = A_2(t, k)e^{-kz}. \tag{47} \]

In addition, we obtain the equation for \( w_2' \),

\[ \frac{\partial^2 w_2'}{\partial t^2} + 2ik_x U_0 \frac{\partial w_2'}{\partial t} - k_x^2 (U_0^2 - V_{A1}^2) w_2' = \frac{k}{\rho_{02}} \left( \frac{\partial A_2}{\partial t} + iU_0 k_x A_2 \right). \tag{48} \]

Using the second equation in Eq. (46) we derive from Eq. (48) that

\[ A_2 = \frac{\rho_{02}}{k} \left[ \frac{\partial^2 \zeta}{\partial t^2} + 2ik_x U_0 \frac{\partial \zeta}{\partial t} - k_x^2 (U_0^2 - V_{A2}^2) \zeta \right]. \tag{49} \]

Now we proceed to studying the motion in the region \( z < 0 \). Let us introduce the Reynolds number \( R_c = \rho_0 V_{A1}/k \eta_0 \) and assume that \( R_c \gg 1 \). Then viscosity is only important in a thin boundary layer of thickness of the order of \( R_c^{-1/2}/k \) near the discontinuity, while the motion outside the dissipative layer can be described by the ideal MHD equations. This analysis inspired us to use matched asymptotic expansions to study the motion in the region \( z < 0 \). To describe the motion in the boundary layer we introduce the stretching variable \( Z = R_c^{1/2} z \). Then, using Eq. (47), we obtain the following equation valid in the boundary layer,

\[ \frac{\partial^2 w_1'}{\partial t^2} + k_x^2 V_{A1}^2 w_1' = - \frac{k}{\rho_01} \frac{\partial A_1}{\partial t} e^{R_c^{1/2}kZ} + \frac{\bar{\eta}_0}{\rho_01} \left( \frac{\partial^2 w_1'}{\partial Z^2} - R_c^{-1} k^2 w_1' \right), \tag{50} \]

where \( \bar{\eta}_0 = R_c \eta_0 \). Using Eq. (30) to eliminate \( w_1' \) from Eq. (45) we obtain the boundary condition for \( w_1' \) valid at \( Z = 0 \),

\[ \frac{\partial^2 w_1'}{\partial Z^2} - R_c^{-1} k^2 w_1' = 0. \tag{51} \]

It immediately follows from Eqs. (46), (50) and (51) that

\[ A_1 = -\frac{\rho_01}{k} \left( \frac{\partial^2 \zeta}{\partial t^2} + k_x^2 V_{A2}^2 \zeta \right). \tag{52} \]

Outside the boundary layer we neglect viscosity and obtain

\[ \frac{\partial^2 w_1'}{\partial t^2} + k_x^2 V_{A2}^2 w_1' = - \frac{k}{\rho_01} \frac{\partial A_1}{\partial t} e^{kz}. \tag{53} \]

It follows from Eqs. (52) and (53) that

\[ w_1' = \frac{\partial \zeta}{\partial t} e^{kz}. \tag{54} \]
Let us look for the solution to Eqs. (50) in the form

$$ w'_1 = \left( \frac{\partial \zeta}{\partial t} + w \right) e^{R_{\xi}^{-1/2}kZ} \tag{55} $$

It follows from Eq. (46) that $\dot{w} = 0$ at $Z = 0$. In addition, it follows from the matching condition with the external solution given by Eq. (54) that $\dot{w} \to 0$ as $Z \to -\infty$. It is straightforward to show that $\dot{w} \to 0$ as $t \to \infty$ on the time scale $kV_{A1}$. In what follows we are only interested in slow processes with the time scale $R_{\xi}kV_{A1}$, so we can safely take $\dot{w} = 0$. Then we obtain from Eq. (55) that

$$ \frac{\partial w'_1}{\partial z} = k \frac{\partial \zeta}{\partial t} \tag{56} $$

at $z = 0$. Substituting Eqs. (49), (52) and (56) in Eq. (44) we obtain the governing equation for $\zeta$,

$$ (\rho_0 + \rho_2) \frac{\partial^2 \zeta}{\partial t^2} + 2i\rho_2k_z U_0 \frac{\partial \zeta}{\partial t} + \rho_2k_z^2 (U_c^2 - U_0^2) \zeta = -2\eta k^2 \frac{\partial \zeta}{\partial t}, \tag{57} $$

where

$$ U_c^2 = \frac{\rho_0 V_{A1}^2 + \rho_2 V_{A2}^2}{\rho_0 + \rho_2} \tag{58} $$

Equation (57) is written for the Fourier transform of $\zeta$ with respect to $x$ and $y$. Making the inverse Fourier transform we obtain the equation written in physical variables,

$$ (\rho_0 + \rho_2) \frac{\partial^2 \zeta}{\partial x^2} + 2\rho_2 U_0 \frac{\partial^2 \zeta}{\partial t \partial x} + \rho_2 (U_0^2 - U_c^2) \frac{\partial^2 \zeta}{\partial x^2} = 2\eta \nabla^2 \frac{\partial \zeta}{\partial t}. \tag{59} $$

4.2. Negative energy instability

Multiplying Eq. (59) by $\partial \zeta/\partial t$ and integrating the result with respect to $x$ and $y$ yields

$$ \frac{dE}{dt} = -D, \tag{60} $$

where

$$ E = \frac{1}{2} \iiint \left[ (\rho_0 + \rho_2) \left( \frac{\partial \zeta}{\partial t} \right)^2 + \rho_2 (U_0^2 - U_c^2) \left( \frac{\partial \zeta}{\partial x} \right)^2 \right] dx \, dy, \tag{61} $$

$$ D = 2\eta_0 \iiint \left[ \left( \frac{\partial^2 \zeta}{\partial t \partial x} \right)^2 + \left( \frac{\partial^2 \zeta}{\partial t \partial y} \right)^2 \right] dx \, dy. \tag{62} $$

Here the integration limits are $\pm \infty$ for perturbations decaying at infinity, and $\pm 1/2L_x$, $\pm 1/2L_y$ for periodic perturbations with the periods with respect to $x$ and $y equal to L_x$ and $L_y$ respectively.

Consider the solution to Eq. (59) in the form $\zeta = \zeta_0 \Re[e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t}]$. Substituting this expression in Eq. (59) we obtain the dispersion equation

$$ (\rho_0 + \rho_2)\omega^2 - 2\rho_2 U_0 \omega k_x + \rho_2 (U_0^2 - U_c^2) k_x^2 = -2i\eta k^2 \omega. \tag{63} $$

It is easy to see that the ratio of the right hand side of this equation to its left-hand side is of the order of $R_{\xi}^{-1} \ll 1$. This observation inspires us to use the regular perturbation method to find the approximate solution to Eq. (63). As a result we obtain

$$ \omega = \omega_{\pm} \equiv C_{\pm} k \mp i\chi_{\pm} k^2, \tag{64} $$
Note that Eqs. (64) and (65) are only valid when $U_0 < V_{KH}$, i.e. when there is no KH instability.

It follows from Eq. (65) that $C_+ > 0$, $\chi_+ > 0$, and the wave mode with $\omega = \omega_+$ is stable. The wave mode with $\omega = \omega_-$ is also stable when $C_- < 0$, however it is unstable when $C_+ > 0$. This condition reduces to

$$U_0 > U_c.$$  \hspace{1cm} (66)

This is the criterion for the onset of dissipative instability. Since, in accordance with Eq. (58), $U_c < V_{KH}$, the dissipative instability occurs for the values of the shear velocity $U_0$ smaller than those needed for the onset of the KH instability. Hence, contrary to our intuition, dissipation destabilises tangential discontinuities.

The quantity $E$ can be considered as the wave energy. Since $D > 0$, it follows from Eq. (61) that dissipation causes the wave energy to decrease. Let us calculate $E$ for the wave modes with $\omega = \omega_{\pm}$. Since the imaginary part of $\omega$ is much smaller than its real part, we can take $\omega \approx C_{\pm} k$. Then we easily obtain

$$E_{\pm} = \pm 2\pi^2 k_x k_y^{-1} \zeta_0 C_{\pm} \sqrt{\rho_0, \rho_0 (V_{KH}^2 - U_0^2)}.$$  \hspace{1cm} (67)

Now we assume that $\zeta_0$ is a slowly varying function of time. We see that $E_{\pm} > 0$, so that the decrease of $E_+$ caused by dissipation results in the decrease of the wave amplitude $\zeta_0$. When $U_0 < U_c$ we have $E_- > 0$, and the amplitude of the mode with $\omega = \omega_-$ also decreases due to dissipation. However $E_- < 0$ when $U_0 > U_c$. Hence, when $E_-$ decreases due to dissipation, its absolute value increases. As a result, the wave amplitude $\zeta_0$ increases.

Obviously the wave energy is frame-dependent. The wave mode is unstable in the presence of dissipation when its energy is negative in the reference frame where the fluid domain where the dissipation takes place is at rest. In the problem studied in this section this is just the reference frame where the fluid below the discontinuity is at rest.

When $U_0 = 0$ the wave mode with $\omega = \omega_+$ propagates in the positive $x$-direction, and the wave mode with $\omega = \omega_-$ propagates in the negative $x$-direction. These two modes are called forward and backward propagating respectively. We see that the backward propagating mode becomes a negative energy wave as soon as the shear velocity $U_0$ is so large that the mode phase velocity changes the sign and the mode propagates in the positive $x$-direction.

Waves with negative energy are called negative energy waves. We see that the dissipative instability of MHD tangential discontinuities is related to the existence of negative energy waves.

Ruderman et al. [6] considered another limiting case where $\tau_0 \omega_i \gg 1$, which is satisfied in sufficiently rarefied plasmas in presence of strong magnetic field. In this case the first term in the expression for the viscosity tensor (6) strongly dominates four other terms, so that we can take $\mathbf{P} \approx \eta_0 \mathbf{P}_0$. Similarly the first term in the expression for the heat flux (15) strongly dominates two other terms, so that we can take $\mathbf{q} \approx -\kappa_0 \nabla T$. Viscosity and thermal conduction of this form are called strongly anisotropic. Interestingly, in contrast to isotropic viscosity and thermal conduction, strongly anisotropic viscosity and thermal conduction do not prevent the existence of velocity and temperature discontinuities. Ruderman et al. [6] studied dissipative instabilities of MHD tangential discontinuities in fluids with strongly anisotropic viscosity and thermal conduction and arrived at the same conclusion as Ruderman and Goossens [5]: dissipation destabilizes MHD tangential discontinuities.

Equation (64) shows that the increment of dissipative instability is proportional to $k^2$. However this result was obtained in the long-wavelength approximation. Really, the condition
where \( V \) is the velocity of the flow. In Cartesian coordinates \( x, y, z \), \( (x, y) \) describes the flow perturbations in the region \( z \geq 0 \) and \( (x, y, z) \) in the region \( z \leq 0 \). The density is equal to \( \rho_0 \) for \( z < 0 \) and \( \rho_0^2 \) for \( z > 0 \). There is also a shear flow in the \( x \)-direction with the magnitude \( U_0 \) given by

\[
U_0 = \begin{cases} 
U_m, & z \leq -h, \\
-U_m(z/h), & -h < z < 0, \\
0, & z \geq 0.
\end{cases}
\]  

(68)

It is assumed that the fluid is incompressible. Then the linearised hydrodynamic equations describing the flow perturbations in the region \( z < 0 \) can be reduced to one equation for the \( z \)-component of the velocity perturbation, \( \nabla^2 u' = 0 \). As a result, taking perturbations of all quantities proportional to \( \exp(ik \cdot r - i\omega t) \) with \( k \) parallel to the \( xy \)-plane, we obtain the system of ordinary differential equations that can be solved analytically. Using this analytical solution and the boundary conditions at the discontinuity at \( z = 0 \) we obtain the dispersion equation

\[
2\alpha^4 + 2(\Delta Q - \alpha)\Omega^2 + (Q - W)\Omega + QW = 0,
\]  

(69)

where

\[
\alpha = kh, \quad s = \frac{\rho_0}{\rho_0^2}, \quad \Delta = \frac{s - 1}{s + 1}, \quad \Omega = \frac{\omega h}{V_m(1 + \Delta)},
\]  

(70)

\[
Q = \frac{2\alpha - 1 + e^{-2\alpha}}{2(1 + \Delta)}, \quad W = \frac{\alpha^2(1 - \Delta)}{M_A^2(1 + \Delta)^2}, \quad M_A^2 = \frac{\mu_0 \rho U_m^2}{B_0^2},
\]  

\( M_A \) being the Alfvén Mach number. The dispersion equation (69) was studied analytically and, for illustration, numerically. The main results are the following. The flow becomes unstable when \( M_A > 1/(1 + s) \). In the dimensional variables the instability criterion takes the form

\[
U_m^2 > U_c^2 \equiv \frac{s}{(1 + s)^2}V_{\text{HK}}^2,
\]  

(71)

where \( V_{\text{HK}} \) is given by Eq. (39) with \( V_{A1} = 0 \). If we take \( h = 0 \) we obtain a particular case of MHD tangential discontinuity with the magnetic-free fluid at one side. Such a discontinuity is unstable when \( U_m > V_{\text{HK}} \). Since \( U_c < V_{\text{HK}} \), it follows that the presence of the transitional layer where the shear velocity linearly decreases from its maximum value to zero destabilizes the flow.

The second important result is that now only perturbations with \( k < k_{\text{max}} \) are unstable. As a result the instability increment is bounded and the initial value problem is well-posed.
5.2. Flow with continuous velocity and magnetic field

Let us now discuss the stability properties of MHD flows with continuous velocity and magnetic field. In this case a new phenomenon, called resonant absorption, appears. As it is well known, Alfvén waves can propagate in conducting fluids permeated by magnetic field [10]. When a conducting fluid is compressible, two additional wave modes exist: fast and slow magnetosonic waves (e.g. Goedbloed and Poedts [11]). Resonant absorption occurs when, in an inhomogeneous conducting fluid, global wave motion is locally in resonance with either Alfvén or slow magnetosonic waves at a particular magnetic surface. The resonance causes energy to build up in the vicinity of a resonance magnetic surface, at the expense of the energy in the global motion. As a result the energy of the global motion decreases. When the equilibrium is static, this energy decrease causes the global wave to decay. However when the equilibrium is steady, i.e. there is flow, negative energy waves can exist. As we have already seen in the previous section, the negative energy wave amplitude grows when its energy decreases, which can result in the instability. The negative energy wave instability related to resonant absorption is also called resonant instability.

In what follows we consider resonant instability of a shear flow of an incompressible fluid. In that case we can only have the Alfvén resonance. It occurs at the magnetic surface where the phase velocity of a global wave matches the local Alfvén velocity. This only can occur when there is the continuous variation of the Alfvén velocity. This is why we did not have the Alfvén resonance in the problem studied in the previous subsection.

The resonant instability of a shear flow of incompressible fluid was first studied by Hollweg et al. [12]. In what follows we give slightly modified version of their analysis. Let us consider a flow with the flow velocity in the $x$-direction of Cartesian coordinates $x$, $y$, $z$. The velocity magnitude is a function of $z$. The equilibrium magnetic field is also in the $x$-direction, and its magnitude as well as the fluid density are functions of $z$.

We assume that the equilibrium quantities vary only in the slab $|z| < a$, while they are constant in the regions $z < -a$ and $z > a$, where they are marked by the subscripts ‘1’ and ‘2’. The total equilibrium pressure is constant,

$$p_0 + \frac{B_0^2}{2\mu_0} = 0. \quad (72)$$

In what follows we use the perturbation of the total pressure,

$$P' = p' + \frac{B_0 B'_x}{\mu_0}. \quad (73)$$

Now we use the system of linearised ideal MHD equations for an incompressible fluid, which consists of Eqs. (22), (23) and (30) with $F_{\text{vis}} = 0$ and $\lambda = 0$. Eliminating all variables from this system in favour of $P'$ and $w'$ we obtain

$$(\omega - U_0 k_x) \frac{dP'}{dz} = i \rho_0 \left[ (\omega - U_0 k_x)^2 - V_A^2 k_x^2 \right] w', \quad (74)$$

$$\rho_0 \left[ (\omega - U_0 k_x)^2 - V_A^2 k_x^2 \right] \frac{d}{dz} \left( \frac{w'}{\omega - U_0 k_x} \right) = -ik^2 P'. \quad (75)$$

If there is $z_0$ such that $\omega = k_x U_0(z_0)$, then $z_0$ is a singular point of the system of Eqs. (74) and (75). However, this is only an apparent singularity. Really, introducing $\bar{w} = w'/(\omega - U_0 k_x)$ and eliminating $P'$ from Eqs. (74) and (75) we obtain the equation

$$\frac{d}{dz} \rho_0 \left[ (\omega - U_0 k_x)^2 - V_A^2 k_x^2 \right] \frac{d\bar{w}}{dz} = k^2 \rho_0 \left[ (\omega - U_0 k_x)^2 - V_A^2 k_x^2 \right] \bar{w}. \quad (76)$$
We see that \( z_0 \) is a regular point of this equation, so that \( \tilde{w} \) is a regular function at \( z_0 \). Then it follows that \( w' \) and \( P' \) are also regular functions at \( z_0 \).

In what follows we only consider long-wavelength perturbations satisfying \( k\alpha \ll 1 \). Then it follows from Eq. (74) that \( dP'/dz = \mathcal{O}(k\alpha) \) when \( |z| < a \), and we can take \( P' \approx \text{const} \) in this region. This approximation was first used by Hollweg and Young [13] to study resonant damping of surface waves propagating along a thin transitional layer. Using this assumption we immediately obtain from Eq. (75) that

\[
w' = (\omega - U_0k_x) \left( -\frac{ik^2P'}{\rho_0} \int_{-a}^{z} \frac{d\tilde{w}}{[(\omega - U_0k_x)^2 - V_A^2k_x^2]} + w_0' \right),
\]

where \( w_0' \) is a constant. The integral in this expression is divergent if there is \( z_A \in (-a,a) \) such that

\[
|\omega - k_xU_0(z_A)| = k_xV_A(z_A),
\]

and \( z > z_A \). This condition means that, at \( z = z_A \), the absolute value of the Doppler-shifted frequency of the global wave coincides with the local Alfvén frequency \( V_Ak_x \). As a result there is resonance between the global wave and the local Alfvén waves at \( z = z_A \), which causes the global wave energy to be transferred to the local Alfvén waves in the vicinity of \( z_A \). The point \( z_A \) is called the Alfvén resonant position. We assume that there is at most one Alfvén resonant position for any value of \( k \).

To evaluate the integral in Eq. (77) when the condition (78) is satisfied we notice that, when taking perturbations of all variables proportional to \( e^{-i\omega t} \), we implicitly assumed that we made the Laplace transform. The Laplace transform is an analytic function of \( \omega \) only in the half-plane \( \omega = \Im(\omega) > \Im(\tilde{w}) \), where \( \Im \) indicates the imaginary part of a quantity. When studying instabilities we have to take \( \tilde{w} \) larger than the instability increment. We will see in what follows that the condition \( ak \ll 1 \) implies that the instability increment is much smaller than \( |\omega_r| = |\Re(\omega)| \), so that we can take \( |\omega_i| \ll |\omega_r| \). Then, assuming that all the equilibrium quantities are analytic functions of \( z \) for \( z \in (-a,a) \), we obtain that Eq. (78) is satisfied at \( z = z_A + iz_i \), where

\[
z_i = \frac{\omega_i}{k_x\Delta_A}, \quad \Delta_A = \frac{d(U_0 \pm V_A)}{dz} \bigg|_{z=z_A}.
\]

Now we formally take \( \omega_i \to 0 \), so that \( z_i \to 0 \). To avoid singularity in the integral in Eq. (77) we deform the integration contour in such a way that the new contour consists of two intervals, \((a, z_A - \epsilon)\) and \((z_A + \epsilon, a)\), and a semi-circle of radius \( \epsilon \) centred at \( z_A \), where \( 0 < \epsilon \ll a \). The semi-circle is below the \( z \)-axis if \( z_i > 0 \), and above the \( z \)-axis if \( z_i < 0 \). Then Eq. (77) reduces to the approximate expression

\[
w' = (\omega_r - U_0k_x) \left( -\frac{\pi k^2P'}{2k_x\rho_A|\Delta_A|} + \frac{ik^2P'}{\rho_0} \int_{-a}^{z} \frac{dz}{[(\omega_r - U_0k_x)^2 - V_A^2k_x^2]} + w_0' \right),
\]

where \( \mathcal{P} \) indicates the principal Cauchy part of the integral and \( \rho_A = \rho_0(z_A), U_A = U_0(z_A) \). Recall that this expression is only valid for \( z > z_A \). When \( z < z_A \) the first term in the brackets is absent. In particular, \( w'(-a) = w_0'(\omega_r - U_0k_x) \).

Since all the equilibrium quantities are constant in the regions \( z < -a \) and \( z > a \), the solution to Eqs. (74) and (75) in these regions is straightforward,

\[
w' = w_1e^{kz}, \quad P' = \frac{i\rho_0 \left[(\omega - U_0k_x)^2 - V_A^2k_x^2\right] w_1e^{kz}}{k(\omega - U_0k_x)}, \quad (z < -a),
\]
Comparing Eqs. (80)–(82) we obtain the dispersion equation

\[
\frac{w_2}{\omega_r - U_{02}k_x} - \frac{w_1}{\omega_r - U_{01}k_x} = -k^2 P' e^{ak} \left( \frac{\pi}{2k_x \rho_A |\Delta_A|} \right) \left( \frac{1}{\rho_0} \left[ (\omega - U_{02}k_x)^2 - V_{A2}^2 k_x^2 \right] \right) + i P \int_{-a}^{a} \frac{dz}{\rho_0 \left[ (\omega_r - U_{02}k_x)^2 - V_{A2}^2 k_x^2 \right]} .
\]  

(83)

Recalling that \( P' \) is constant for \(|z| < a\) we can obtain another expression for the left-hand side of this equation from Eqs. (81) and (82),

\[
\frac{w_2}{\omega_r - U_{02}k_x} - \frac{w_1}{\omega_r - U_{01}k_x} = ik P' e^{ak} \left( \frac{1}{\rho_0} \left[ (\omega - U_{02}k_x)^2 - V_{A2}^2 k_x^2 \right] \right) + \frac{1}{\rho_0} \left[ (\omega - U_{01}k_x)^2 - V_{A1}^2 k_x^2 \right] .
\]  

(84)

Comparing Eqs. (83) and (84) we obtain the dispersion equation

\[
\frac{1}{\rho_0} \left[ (\omega - U_{02}k_x)^2 - V_{A2}^2 k_x^2 \right] + \frac{1}{\rho_0} \left[ (\omega - U_{01}k_x)^2 - V_{A1}^2 k_x^2 \right] = \frac{i \pi k}{2k_x \rho_A |\Delta_A|} \left( \omega_r - U_{02}k_x \right) - k P \int_{-a}^{a} \frac{dz}{\rho_0 \left[ (\omega_r - U_{02}k_x)^2 - V_{A2}^2 k_x^2 \right]} .
\]  

(85)

It is straightforward to estimate that the ratio of the right-hand side of this equation to its left-hand side is of the order of \( ak \ll 1 \). Hence we can use the regular perturbation method to calculate \( \omega \). In the first order approximation we obtain \( \omega = k_x V_\pm \), where

\[
V_\pm = \frac{\rho_0 U_{01} + \rho_0 U_{02} \pm \sqrt{\rho_0 \rho_2 V_{KH}^2 - (U_{02} - U_{01})^2}}{\rho_0 + \rho_2} ,
\]  

(86)

and we have assumed that \(|U_{02} - U_{01}| < V_{KH}\), so that there is no KH instability. Proceeding to the next order approximation we eventually arrive at

\[
\omega = k_x V_\pm + k k_x (i \Gamma_\pm + S_\pm) ,
\]  

(87)

where

\[
\Gamma_\pm = \frac{1}{4 \rho_A |\Delta_A| H_\pm (U_A - V_\pm)} , \quad S_\pm = \frac{1}{2 H_\pm} P \int_{-a}^{a} \frac{dz}{\rho_0 \left[ (V_\pm - U_0)^2 - V_{A2}^2 \right]} ,
\]  

\[
H_\pm = \frac{V_\pm - U_{02}}{\rho_0 \left[ (V_\pm - U_{02})^2 - V_{A2}^2 \right] + \frac{V_\pm - U_{01}}{\rho_0 \left[ (V_\pm - U_{01})^2 - V_{A1}^2 \right]} .
\]  

(88)

(89)

The wave mode is unstable if \( \Gamma_\pm > 0 \). Using Eq. (85) in the first order approximation and (86) it is straightforward to show that \( H_+ > 0 \) and \( H_- < 0 \). Then the forward propagating mode is unstable when \( V_+ < U_A \), i.e. when it propagates in the negative \( x \)-direction in the reference frame moving with the velocity \( U_A \). The backward propagating mode is unstable when \( V_- > U_A \), i.e. when it propagates in the positive \( x \)-direction in the reference frame moving with the velocity \( U_A \). It can be shown that in both cases the unstable mode has negative energy in the reference frame moving with the velocity \( U_A \).
frame moving with the velocity $U_A$. Since dissipation just takes place at $z = z_A$, we see that the instability condition is the same as in the previous section: a mode becomes unstable when it changes the propagation direction due to the effect of flow in the reference frame where the dissipation region is at rest. It turns out that this is a general condition for the negative energy instability (see, e.g., Ostrovskii et al. [14]).

Note that, in contrast to the dissipative instability, the condition that a mode has negative energy is not enough for the onset of the resonant instability. The second condition that also has to be satisfied is that there is a resonant position in the interval $(-a, a)$, i.e. that there is $z_A$ satisfying Eq. (78). It is straightforward to see that this equation reduces to $V_+ = U_0(z_A) - V_A(z_A)$ for the forward propagating wave, and to $V_- = U_0(z_A) + V_A(z_A)$ for the backward propagating wave.

Let us consider one example. Suppose that both $U_0$ and $V_A$ are linear functions of $z$, while $\rho_0$ is constant,

$$U_0 = \frac{U_{01} + U_{02}}{2} + \frac{z(U_{02} - U_{01})}{2a}, \quad V_A = \frac{V_{A1} + V_{A2}}{2} + \frac{z(V_{A2} - V_{A1})}{2a}. \quad (90)$$

We also assume that $U_{01} < U_{02}$. The equation $V_+ = U_0(z_A) - V_A(z_A)$ reduces to

$$V_{A1} + V_{A2} + \sqrt{V_{KH}^2 - (U_{02} - U_{01})^2} = (z_A/a)(U_{02} - U_{01} + V_{A1} - V_{A2}),$$

where now $V_{KH}^2 = 2(V_{A1}^2 + V_{A2}^2)$. This equation has the solution $z_a \in (-a, a)$ only when

$$U_{02} - U_{01} + V_{A1} - V_{A2} > V_{A1} + V_{A2} + \sqrt{V_{KH}^2 - (U_{02} - U_{01})^2}. \quad (91)$$

A necessary condition to satisfy this inequality is $U_{02} - U_{01} > 2V_{A2}$, which is compatible with $U_{02} - U_{01} < V_{KH}$ only if $V_{A1} > V_{A2}$. Let us assume that this condition is satisfied. Then Eq. (91) reduces to $U_{02} - U_{01} > U_c$, where

$$U_c = V_{A1} + V_{A2}. \quad (92)$$

It is straightforward to see that $U_c < V_{KH}$. It is also straightforward to show that $V_+ < U_A$ when $U_{02} - U_{01} > U_c$, so that the mode is unstable.

A similar analysis shown that the backward propagating mode can be unstable only if $V_{A1} < V_{A2}$. Then the instability condition is the same. Hence, summarizing, the condition for the onset of the resonant instability is

$$U_{02} - U_{01} > U_c. \quad (93)$$

The unstable wave mode is the forward propagating mode when $V_{A1} > V_{A2}$, and the backward propagating mode when $V_{A1} < V_{A2}$.

In accordance with Eq. (87) the increment of the resonant instability is proportional to $kk_z$. However, this result was obtained in the long wavelength approximation. If we extend the analysis to perturbations with arbitrary wavelength, we obtain the instability saturation. Moreover, this result is valid even when $U_{02} - U_{01} > V_{KH}$. Hence, the initial value problem for a shear flow with continuous velocity profile is well-posed.

6. Heliopause stability

The solar system is moving with the supersonic speed of about 26 km/s with respect to the local interstellar medium. The sun produces a permanent supersonic flow of plasma with the speed of about 500 km/s called the solar wind. The collision of two supersonic flows results in a region of their interaction called the heliospheric interface. The heliospheric interface is bounded
by the bow shock at the interstellar side, and by the termination shock at the solar wind side. Inside the heliospheric interface there is a surface called heliopause. The heliopause separates the flow of the interstellar plasma decelerated at the bow shock from the flow of the solar wind plasma decelerated at the termination shock. A schematic picture of the heliospheric interface is shown in Fig. 2. The model of the heliospheric interface with two shocks was first suggested and developed by Baranov et al. [15] in the approximation of infinitely thin interface. Then Baranov et al. [16, 17] improved this model by taking the finite thickness of the heliospheric interface into account. After that it was further developed by the effort of many researchers (see the review papers by Baranov [18, 19]).

![Diagram of the heliospheric interface]

**Figure 2.** The schematic picture of the heliospheric interface. The lines with arrows are the streamlines.

The problem of the heliopause stability is very important for understanding physical processes in the heliospheric interface and proper interpretation of the observational data. The heliopause instability can result in turbulization of the plasma flow in the heliospheric interface, which can seriously affect the propagation of electromagnetic waves from the interstellar medium to the inner heliosphere. To our knowledge the first review of different forms of instabilities that can operate at the heliopause was given by Fahr et al. [20]. After that the problem of the heliopause stability remains an important problem in the heliospheric physics.

In the absence of magnetic filed the heliopause is unconditionally unstable. However, in accordance with the observational data, there is magnetic field in the interstellar medium with the magnitude of a few microgauss (µG). This magnetic field can stabilize the heliopause. Fahr et al. [20] applied the stability analysis for MHD tangential discontinuities to the near flanks of the heliopause (θ ≲ 30°, see Fig. 2), where the plasma flow can be considered as approximately incompressible. They carried out the local analysis assuming that the wavelengths of perturbations are much smaller than the characteristic size of the heliospheric interface. This assumption enabled them to consider the heliopause as a plane tangential discontinuity.

Ruderman and Fahr [21] generalized the analysis by Fahr et al. [20] and took the compressibility into account. As a result their analysis is not restricted to the near flanks of the heliopause and is applicable to the whole heliopause. In their analysis Ruderman and
Fahr [21] used the dispersion equation (40) assuming that there is magnetic field only at one side of the discontinuity, i.e. $B_{02} = 0$. To simplify the analysis they took into account that the ratio of plasma densities at the two sides of the heliopause is small, $\rho_{02}/\rho_{01} \sim 0.01$, and used the regular perturbation method with this ratio as a small parameter. Their main results can be summarized as follows. The magnetic field can stabilize a part of the heliopause only if it is parallel to the flow velocity. The typical values of the magnetic field magnitude and plasma number density in the interstellar medium are $B_{01} \approx 3$ $\mu$G and $n \approx 10^5$ m$^{-3}$. This gives $V_{\text{Al}} \approx 21$ km/s. Using these values Ruderman and Fahr [21] found $V_{\text{KH}} \approx 210$ km/s. The speed of the solar wind compressed at the termination shock at the heliopause grows when $\theta$ increases, and it does not exceed 75 km/s for $\theta \lesssim 30^\circ$ (e.g. Baranov et al. [17]). Thus, in accordance with the analysis by Ruderman and Fahr [21], the interstellar magnetic field can stabilize the near flanks of the heliopause corresponding to $\theta \lesssim 30^\circ$.

These results were reconsidered by Ruderman and Brevdo [9]. In accordance with their analysis the transitional layer with the continuous velocity variation at the solar wind side of the heliopause would destabilize the flow. It follows from Eq. (71) that, for $s \approx 0.01$, $U_m \approx 0.1V_{\text{KH}} \approx 21$ km/s. Hence, if such a layer exists, then the heliopause is stable only when the solar wind speed near the heliopause is smaller than or of the order of 21 km/s, which corresponds to a small fraction of the heliopause.

7. Magnetopause stability

7.1. Resonant instability of magnetopause

The Earth magnetic field has the shape of a dipole. The pressure imposed by the solar wind distorts this magnetic field. However the Earth magnetic field prevents the direct penetration of the solar wind in the atmosphere. It stops the solar wind at the distance of about 15 Earth radii in the solar direction. As a result a magnetic cavity which is nearly free of the interplanetary plasma is formed. This cavity is called the magnetosphere, and its boundary is called the magnetopause. The magnetopause has the bullet shape with a long tail in the anti-solar direction. In simplest models of the magnetosphere the magnetopause is considered as an MHD tangential discontinuity. Studying the stability of the magnetopause is very important for understanding physical processes in the magnetosphere. In particular, the instability of the magnetopause can result in turbulization of the flow near the magnetopause, which can facilitate the indirect penetration of the solar wind in the magnetosphere.

The magnetic field in the magnetotail region is quasi-parallel to the solar wind flow, so that it can stabilise the magnetopause in this region. In the first studies of the magnetopause stability the threshold velocity of the solar wind needed for the onset of the KH instability was calculated. However later it was realised that the heliopause can be unstable at much smaller solar wind speed due to resonant instability. This possibility was studied by Ruderman and Write [22]. They studied the stability of the flow schematically shown in Fig. 3. In the equilibrium the magnetopause coincides with the $xy$-plane in Cartesian coordinates $x$, $y$, $z$. The flow velocity and magnetic field are in the $x$-direction. The flow speed is zero in the region $z < 0$ (i.e. in the magnetosphere), and is equal to constant quantity $U$ in the region $z > 0$ (this region is called the magnetosheath). The magnetic field is equal to zero in the region $z > 0$, and to constant quantity $B$ in the region $z < 0$. The pressure is equal to constant quantity $p$ in the region $z > 0$. The magnetic pressure, $B^2/2\mu_0$, in the magnetosphere is much larger than the plasma pressure. This observation enables us to neglect the plasma pressure in the region $z < 0$. The density is equal to constant quantity $\rho_1$ in the magnetosheath. In the magnetosphere it is equal to constant quantity $\rho_2$ in the magnetosheath. As a result the Alfvén speed $V_A$ also varies in the slab $-a < z < 0$. It is assumed that $V_A(z)$ is a linear function.

If we take $a \to 0$, we obtain the MHD tangential discontinuity at $z = 0$. For the particular
equilibrium considered in this section the dispersion equation (40) reduces to

\[ \frac{\rho_1(\omega^2 - V_A^2 k_x^2)}{m_1} + \frac{\rho_2(\omega - U k_x)^2}{m_2} = 0 \] (94)

with \( m_1 \) and \( m_2 \) given by

\[ m_1 = \sqrt{k^2 - \frac{\omega^2}{V_A^2}}, \quad m_2 = \sqrt{k^2 - \frac{(\omega - U k_x)^2}{c_s^2}}, \] (95)

where \( \Re(m_{1,2}) > 0 \) and \( c_s^2 = \gamma p/\rho_2 \). Introducing the dimensionless variables

\[ \Omega = \frac{\omega}{k V_A}, \quad M = \frac{U}{V_A}, \quad \beta = \frac{c_s^2}{V_A^2}, \]

rewriting Eq. (94) in the dimensionless form, squaring the obtained equation, and using the relation \( \rho_1/\rho_2 = 2\beta/\gamma \) that follows from the total pressure balance at the magnetopause, we arrive at

\[ \gamma^2(\Omega^2 - 1)(\Omega - M \cos \varphi)^4 = 4\beta(\Omega^2 - \cos^2 \varphi)^2[(\Omega - M \cos \varphi)^2 - \beta], \] (96)

where \( \varphi \) is the angle between the wavevector \( k \) and the \( x \)-direction. We have to be cautious when analysing this equation because it can contain spurious roots corresponding to the minus sign between the two terms on the left-hand side of Eq. (94). Equation (96) is a sixth-order polynomial equation containing three free parameters, \( M, \beta \) and \( \varphi \) (the ratio of specific heats \( \gamma \) can be considered as fixed). It is a difficult task to study this equation in the general case.

McKenzie [23] studied Eq. (96) in two particular cases. In both cases the left-hand side of Eq. (96) reduces to the product of quadratic and quartic polynomials. The first case corresponds to perturbations propagating along the equilibrium magnetic field (\( \varphi = 0 \)), and the second to perturbations propagating at the angle \( \varphi = \arcsin \frac{M}{\sqrt{2}} \). However studying particular cases does not provide us with the KH threshold velocity \( V_{KH} \).
To make analytical progress Ruderman and Wright [22] assumed that $\rho_1/\rho_2 \ll 1$, which is equivalent to $\beta \ll 1$. This assumption is fairly well satisfied for at least a part of the tail’s magnetopause. Their asymptotic analysis resulted in $M_{\text{KH}} = 1$, or $V_{\text{KH}} = V_A$.

When $M < 1$ and $\beta \ll 1$ Eq. (96) has only two non-spurious roots, $\Omega_+$ and $\Omega_-$, corresponding to the forward and backward propagating surface waves. If we now take the transitional layer into account but restrict our analysis to the long waves ($ak \ll 1$), then it can be shown that the backward propagating wave can be resonantly unstable. Ruderman and Wright [22] have shown that the two conditions for the onset of the resonant instability, namely, that the backward propagating wave is a negative energy wave, and that there is a resonant position $z_A \in (-a, 0)$, are satisfied when $M > M_c$ provided that $V_A(0)$ is sufficiently small, where

$$M_c = \frac{1}{\gamma} \sqrt{\frac{2\beta}{\sqrt{\gamma^2 + \cos^4 \varphi} - \cos \varphi^2}}. \quad (97)$$

We see that $M_c = O(\beta^{1/2}) \ll 1 = M_{\text{KH}}$, so that the resonant instability occurs for the flow velocity magnitude much smaller than that needed for the onset of the KH instability.

The analysis of the resonant instability of the magnetopause has been extended to include different additional effects like the finite plasma pressure in the magnetosphere, and to relax the assumption $\beta \ll 1$ (see, e.g., Erdélyi and Taroyan [24] and Taroyan and Erdélyi [25, 26, 27]). These studies confirmed that the resonant instability of the heliopause occurs for the flow velocity magnitude substantially smaller than the KH threshold.

7.2. Absolute and convective instabilities of magnetopause

The parts of the magnetopause that are situated at the solar side of the Earth are called the near flanks. While the magnetic field in the tail region of the magnetopause is almost parallel to the flow velocity, it is almost perpendicular to the flow velocity in the near flanks of the magnetopause. This implies that the near flanks should be subject to the KH instability for any velocity magnitude in the magnetosheath, in particular, because the magnetic field does not affect perturbations propagating perpendicular to the magnetic field. Then we can expect that the near flanks of the magnetopause would be destroyed by the instability, and there would be a region of highly turbulent flow instead of this part of the magnetopause. However observations onboard of satellites clearly show that the near flanks of the heliopause do exist, and they are not perturbed very much.

A viable conjecture that can resolve this apparent contradiction between the theory and observations is that the near flanks of magnetopause are only convectively unstable. In that case the unstable perturbations are convected from the near flank regions to the far flanks. Then they can be either stabilized by the quasi-parallel magnetic filed in the magnetotail, or continue to grow and eventually destroy the magnetotail at large distances from the Earth. To verify this conjecture Wright et al. [28] studied the absolute and convective instabilities of the near flanks of the magnetopause. They considered the following equilibrium flow. In Cartesian coordinates $x, y, z$ the flow velocity is in the $y$-direction, and the magnetic field is in the $z$-direction. The flow velocity in the magnetosheath ($x > 0$) is equal to the constant value $U_0$ except the boundary layer $0 < x < a$ where it monotonically decreases from $U_0$ to 0. The magnetosheath is magnetic-free ($B = 0$), and the density in the magnetosheath is equal to constant quantity $\rho_2$. The sound speed in the magnetosheath is $c_S$. The magnetopause coincides with the $yz$-plane. The density and magnetic field in the magnetosphere ($x < 0$) are constant and equal to $\rho_1$ and $B$ respectively, so that the Alfvén speed is also constant and equal to $V_A$. There is also a reflecting boundary in the magnetosphere a distance $h \gg a$ from the magnetopause. This boundary mimics the region of sharp increase of magnetic field closer to the Earth that reflects magnetosonic waves. The sketch of the equilibrium state is shown in Fig. 4.
Figure 4. The schematic picture of the flow near the near flank of magnetopause.

Only perturbations independent of $z$ were studied, so that the perturbations of all quantities were functions of $x$, $y$ and $t$. The initial value problem was solved. Using the Fourier transform with respect to $y$ and the Laplace transform with respect to $t$ reduces the system of linearised ideal MHD equations to the system of ordinary differential equations with $x$ as the independent variable. This system was solved numerically to determine the dispersion equation relating the frequency, $\omega$, and the wave number, $k$, of the normal modes. This equation has the form $D(\omega, k) = 0$. After that perturbations of all variables can be written in the form of the inverse Laplace-Fourier transform. In particular the perturbed surface of the the magnetopause is determined by the equation $x = \zeta(y, t)$, where

$$\zeta(y, t) = \int_{i\sigma-\infty}^{i\sigma+\infty} \int_{-\infty}^{\infty} T(\omega, k) e^{i(ky-\omega t)} dk d\omega.$$ (98)

Here $T(\omega, k)$ depends on the initial conditions, and $(i\sigma-\infty, i\sigma+\infty)$ is the Bromwich integration contour, and $\sigma$ is chosen in such a way that this contour is above all singularities of function $D(\omega, k)$ considered as function of $\omega$. This condition has to be satisfied for any fixed $k$. Since the instability increment is unbounded in the case of tangential discontinuity, the perturbation of such a discontinuity cannot be written in this form. However, due to the presence of the velocity boundary layer, the instability increment of the equilibrium flow considered in this subsection is bounded, $\Im(\omega) \leq \omega_{im}$ for any $k$. Hence, it is enough to take $\sigma > \omega_{im}$ in Eq. (98).

The study of the absolute and convective instabilities is based of calculating the asymptotic behaviour of $\zeta(y, t)$ as $t \to \infty$. The instability is absolute when $\zeta(y, t)$ grows exponentially with time, and convective when $\zeta(y, t) \to 0$ as $t \to \infty$. To calculate this asymptotic behavior Wright et al. [28] used Briggs’ [29] method first developed in plasma physics and then extended to hydrodynamics. This method is based on the extension of function $D(\omega, k)$ to the complex $k$-plane and the use of methods of complex analysis. The main result obtained by Wright et al. [28] is that, for any reasonable values of the flow parameters, the shear flow at the near flanks of the heliopause is only convectively unstable in the terrestrial reference frame.

Wright et al. [28] also studied the so-called signalling problem. This problem concerns propagation of perturbations excited by an external driver at a fixed spatial position. They found that the distance from the driver where the amplitude of a convectively unstable perturbation increases $e$-times (the so-called $e$-fold distance) is sufficiently large, so that we can expect to observe large-amplitude perturbations only in the magnetotail.
Wright et al. [28] investigated the absolute and convective instabilities for a very restricted range of parameters. Mills et al. [30] extended the analysis to a much wider range of parameters. They confirmed the conclusion made by Wright et al. [28] that the near flanks of the magnetopause are only convectively unstable. Wright et al. [31] studied the excitation of magnetic field line resonances by growing and convected wave packets, and by spatially amplifying waves excited by an external driver. They concluded that excitation of field line resonances by spatially amplifying waves is more efficient than the excitation by wave packets.

8. Summary and conclusions

In this brief review we have discussed the linear theory of stability of MHD shear flows. We started our review from considering the stability of MHD tangential discontinuities. The main conclusions that we have made are the following. The magnetic field can stabilize a tangential discontinuity, however its ability to do this depends on the mutual direction of the field and the flow velocity. In particular, the magnetic field parallel to the flow velocity can stabilize the discontinuity. In general, the tangential discontinuity becomes unstable when the velocity jump across the discontinuity exceeds the Kelvin-Helholtz (KH) instability threshold \( V_{KH} \). This threshold depends on the magnetic field magnitude and direction as well as densities at the two sides of the discontinuity. For particular values of this parameters it can be zero. In particular, \( V_{KH} = 0 \) when the magnetic field is perpendicular to the flow velocity. Another important conclusion is that the instability increment of an MHD tangential discontinuity is proportional to the wave vector, so that it is unbounded. This means that the initial value problem for the perturbed tangential discontinuity is ill-posed. This, in particular, makes studying the absolute and convective instabilities of MHD tangential discontinuities impossible.

There are two ways to make the initial value problem for a perturbed MHD shear flow well-posed: either to take dissipation into account, or to consider shear flows with continuous velocity profile. As an example of the first approach we considered the stability of an MHD tangential discontinuity in an incompressible fluid viscous at one side of the discontinuity and inviscid at the other side. The account of viscosity makes the instability increment bounded, so that the initial value problem for perturbations is now well-posed. In addition, contrary to our intuition, it destabilizes the discontinuity, so that it becomes unstable when the velocity jump is larger than \( U_c < V_{KH} \). We have shown that this phenomenon is related to the existence of negative energy waves. This kind of instability is often called the negative energy wave instability. Since it is also related to dissipation, it has the second name: the dissipative instability.

As an example of the second approach we considered an MHD shear flow in an incompressible fluid with continuous velocity profile. In this case the instability increment is also bounded and the initial value problem is well-posed. We studied two different flows. In the first flow we still have an MHD tangential discontinuity, however only the density and magnetic field have jumps across it, while the velocity is continuous. The magnetic field is present only at one side of the discontinuity, while the fluid is magnetic-free at the other side. The velocity varies from zero to its maximum value in a layer bounded at one side by the discontinuity. Once again the existence of this transitional layer not only makes the instability increment bounded, but also destabilizes the flow. But in this case this effect is not related to negative energy waves because in this problem there is no energy sink.

In the second flow there is no discontinuity, so that not only the velocity but also the density and magnetic field are continuous. The stability of this flow was studied in the long wavelength approximation, i.e. under the assumption that the perturbation wavelength is much larger than the thickness of the layer where the equilibrium quantities vary. In this approximation there are two surface waves propagating along the layer provided that the variation of the shear velocity is smallened than \( V_{KH} \). One of this two waves can become a negative energy wave. If an additional condition is satisfied, this wave is in resonance with local Alfvén waves at the resonant position.
inside the inhomogeneous layer. The resonance results in the decrease of the surface wave due to the energy transfer to the local Alfvén waves. Since the surface wave is a negative energy wave, this energy decrease causes its amplitude to grow exponentially. Since this negative energy wave instability is related to the existence of resonance, it also has the second name: the resonant instability.

In the last two sections we considered the application of theoretical results to the stability of the heliopause and magnetopause. We discussed the possibility to stabilise the heliopause by the interstellar magnetic field. The conclusion is that the typical interstellar magnetic field can only stabilise the near flanks of the heliopause. If the heliopause is modelled as an MHD tangential discontinuity, then the portion of the heliopause that can be stabilised corresponds to the polar angle $\theta \lesssim 30^\circ$. However, if we add a transitional layer from the solar wind side of the heliopause where the velocity varies continuously, than the stable part of the heliopause shrinks dramatically.

We discussed the resonant instability of the magnetopause tail regions and showed that they can become unstable for the values of the plasma velocity in the magnetosheath much smaller than $V_{KH}$. In the near flanks of the magnetopause the magnetic field is quasi-perpendicular to the velocity, so that it cannot stabilise this portion of the magnetopause. However observations show that the near flanks of the heliopause exist and remain almost unperturbed. We showed that this apparent contradiction between theory and observations can be resolved if we study the absolute and convective instabilities of the magnetopause. It turns out that the near flanks of the magnetopause are only convectively unstable. As a result, growing wave packets are convected away of this region before their amplitudes become sufficiently large. Even if perturbations are excited by an external driver at a fixed spatial position, the spatial growth rate of these perturbations is sufficiently small, so that we can expect to observe large-amplitude perturbations only in the magnetotail.

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References

[1] Braginskii S I 1965 *Review of Plasma Physics* 1 205
[2] Syrovatskii S I 1957 *Usp. Fiz. Nauk* 62 247
[3] Chandrasekhar S 1961 *Hydrodynamic and Hydromagnetic Stability* (Oxford: Clarendon Press)
[4] Fejer J A 1964 *Phys. Fluids* 7 499
[5] Ruderman M S and Goossens M 1995 *J. Plasma Phys.* 54 149
[6] Ruderman M S, Verwichte E, Erdélyi R and Goossens M 1996 *J. Plasma Phys.* 56 285
[7] Kikina N G 1967 *Sov. Phys. Acoust.* 13 184
[8] Ruderman M S, Brezov L and Erdélyi R 2004 *Proc. Roy. Soc. Lond. A* 460 847
[9] Ruderman M S and Brezov L 2006 *Astron. Astrophys.* 448 1177
[10] Alfvén H 1942 *Nature* 150 405
[11] Goedbloed J P and Poedts S 2004 *Principles of Magnetohydrodynamics* (Cambridge: University Press)
[12] Hollweg J V, Yang G, Cadez V M and Gakovic B 1990 *Astrophys. J.* 349 335
[13] Hollweg J V and Yang G 1988 *J. Geophys. Res.* 93 5423
[14] Ostrovskii L A, Rybak S A and Tsmyrion L Sh 1986 *Sov. Phys. Usp.* 29 1040
[15] Baranov V B, Krasnobaev K V and Kulikovskii A G 1970 *Sov. Phys. Dokl.* 15 791
[16] Baranov V B, Krasnobaev K V and Ruderman M S 1976 *Astrophys. Space Sci.* 41 481
[17] Baranov V B, Lebedev M G and Ruderman M S 1979 *Astrophys. Space Sci.* 66 441
[18] Baranov V B 1990 *Space Sci. Rev.* 52 89
[19] Baranov V B 2009 *Space Sci. Rev.* 142 23
[20] Fahr H J, Neutch W, Grzedzielski S, Maciek W and Ratkiewicz R 1986 *Space Sci. Rev.* 43 329
[21] Ruderman M S and Fahr H J 1993 *Astron. Astrophys.* 275 635
[22] Ruderman M S and Wright A N 1998 *J. Geophys. Res.* 103 26,573
[23] McKenzie 1970 *Planet. Space. Sci.* **18** 1
[24] Erdélyi R and Taroyan Y 2003 *J. Geophys. Res.* **108** Article No. 1043
[25] Taroyan Y and Erdélyi R 2002 *Phys. Plasmas* **9** 3121
[26] Taroyan Y and Erdélyi R 2003 *Phys. Plasmas* **10** 266
[27] Taroyan Y and Erdélyi R 2003 *J. Geophys. Res.* **108** Article No. 1301
[28] Wright A N, Mills K J, Ruderman M S and Brevdo L 2000 *J. Geophys. Res.* **105** 385
[29] Briggs R J 1964 *Electron-stream interaction with plasmas* (Cambridge, MA: MIT Press)
[30] Mills K J, Longbottom A W, Wright A N and Ruderman, M S 2000 *J. Geophys. Res.* **105** 27,685
[31] Wright A N, Mills K J, Longbottom A W and Ruderman M S 2002 *J. Geophys. Res.*, **107** Article No. 1242