Iterative methods for computing U-eigenvalues of non-symmetric complex tensors with application in quantum entanglement

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Abstract The purpose of this paper is to study the problem of computing unitary eigenvalues (U-eigenvalues) of non-symmetric complex tensors. By means of symmetric embedding of complex tensors, the relationship between unitary eigenpairs (U-eigenpairs) of a non-symmetric complex tensor and unitary symmetric eigenpairs (US-eigenpairs) of its symmetric embedding tensor is established. An algorithm (Algorithm 3.1) is given to compute the U-eigenvalues of non-symmetric complex tensors by means of symmetric embedding. Another algorithm, Algorithm 3.2, is proposed to directly compute the U-eigenvalues of non-symmetric complex tensors, without the aid of symmetric embedding. Finally, a tensor version of the well-known Gauss-Seidel method is developed. Efficiency of these three algorithms are compared by means of various numeri-
cal examples. These algorithms are applied to compute the geometric measure of entanglement of quantum multipartite non-symmetric pure states.

**Keywords** complex tensor · symmetric embedding · U-eigenvalue · geometric measure of entanglement

**Mathematics Subject Classification (2000)** 15A18 · 15A69 · 81P40

### 1 Introduction

There exist varieties of ways to define tensor eigenvalues, e.g., Z-eigenvalue [1], H-eigenvalue [2], U-eigenvalue [3] and generalized eigenvalue [4]. The problem of computing the eigenvalues of a tensor has been proved to be NP-hard [5]. An increasing number of methods have been proposed in the last decades. Kolda et al. [6] introduced shifted symmetric high order power method (SS-HOPM) to compute Z-eigenvalues of symmetric real tensors. By means of a Jacobian semidefinite programming (SDP) relaxation method [7], Z-eigenvalues of real tensors have been computed in Nie et al. [8] and Cui et al. [9]. There are also several methods for computing Z-eigenvalues of real tensors [10, 11]. Kolda et al. [12] extended SS-HOPM to an adaptive shifted symmetric high order power method for computing generalized tensor eigenvalues. Chen et al. [13] studied the generalized tensor eigenvalue problem of tensors via homotopy methods. An adaptive gradient method for computing generalized tensor eigenpairs has been developed in [14]. Che et al. [15] proposed a neural networks method for computing the generalized eigenvalues of real tensors. Hua et al. [16] computed the largest US-eigenvalue of a symmetric complex tensor. Ni et al. [17] computed US-eigenpairs for symmetric complex tensors via the spherical optimization with complex variables. However, there are relatively few studies on computing eigenvalues of non-symmetric complex tensors.

This paper aims to propose three methods to compute U-eigenvalues of non-symmetric complex tensors. We first build a one-to-one correspondence between a U-eigenpair of a non-symmetric complex tensor and a US-eigenpair...
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of its symmetric embedding. Based on symmetric embedding, Algorithm 3.1 is proposed to compute U-eigenpairs of non-symmetric complex tensors. Due to symmetric embedding, the size of the resulting tensor used in Algorithm 3.1 is usually very large, which significantly affects the computational efficiency of Algorithm 3.1. To circumvent this difficulty, Algorithm 3.2 is proposed to compute the U-eigenpairs of non-symmetric complex tensors directly. Convergence of Algorithms 3.1 and 3.2 are established. Finally, Algorithm 3.3, a tensor version of the Gauss-Seidel method, is proposed.

Quantum entanglement was first introduced by Einstein and Schrödinger [18, 19], and it is regarded as one of the most important and fundamental notions in quantum information [20]. The geometric measure of entanglement (GME) is one of the most important and widely used measures for quantum entanglement [20–27]. The GME was first proposed by Shimony [28] in 1995 for bipartite states and generalized to multipartite states by Wei and Goldbart [29] in 2003. Mathematically, a quantum pure state can be described in terms of a tensor (or hypermatrice), and thus the problem of computing the GME of a pure state can be converted into a tensor eigenvalue computation problem [3, 16, 30–34]. Theorem 1 in [31] indicates that the GME of a symmetric pure state is equal to the largest Z-eigenvalue of the corresponding non-negative tensor. Ni et al. [3] found that for some real tensors, the largest absolute-value of Z-eigenvalue may not be equal to the entanglement eigenvalue. Hence, Ni et al. [3] introduced the concept of U-eigenvalue of complex tensors, and showed that the problem of computing the entanglement eigenvalue of a pure state is equivalent to the problem of computing the largest U-eigenvalue of the corresponding tensor. As an application, we apply Algorithms 3.1–3.3 to compute the GME of quantum pure states.

The paper is organized as follows. In Section 2, we introduce the concept of GME of multipartite pure states, U-eigenpairs of tensors, and tensor blocking. In Section 3, we propose three algorithms to compute the U-eigenvalues of non-symmetric complex tensors, and some basic theorems are also proved.
In Section 4, we present numerical examples for various non-symmetric pure states, and compare the efficiency of these three algorithms. Section 5 concludes this paper.

2 Preliminaries

This section introduces the geometric measure of entanglement of quantum multipartite pure states, U-eigenvalues of complex tensors, and tensor blocking. Interested reader may refer to [3, 31, 33, 35] for details.

2.1 Geometric measure of entanglement of multipartite pure states

Quantum states are fundamental quantities in a quantum system. Assume that \( H = \otimes_{k=1}^{m} \mathbb{C}^{n_k} \) is a tensor product space. Then a pure state in an \( m \)-partite quantum system is an element of \( H \). Let \( \{ |e^{(k)}_i \rangle : i_k = 1, 2, ..., n_k \} \) be the orthonormal basis in \( \mathbb{C}^{n_k} \). Then \( \{ |e^{(1)}_{i_1} e^{(2)}_{i_2} \cdots e^{(m)}_{i_m} \rangle : i_k = 1, 2, ..., n_k; k = 1, 2, ..., m \} \) is the orthonormal basis of \( H \). A pure state \( |\psi\rangle \in H \) can be written as

\[
|\psi\rangle := \sum_{i_1 \ldots i_m=1}^{n_1 \ldots n_m} \mathcal{X}_{i_1 \ldots i_m} |e^{(1)}_{i_1} \cdots e^{(m)}_{i_m}\rangle,
\]

(2.1)

where \( \mathcal{X}_{i_1 \ldots i_m} \in \mathbb{C} \). \( |\psi\rangle \) is called symmetric if \( \mathcal{X}_{i_1 \ldots i_m} \) remains the same for all permutations of indices \( \{i_1, ..., i_m\} \). Let

\[
|\varphi\rangle := \sum_{i_1 \ldots i_m=1}^{n_1 \ldots n_m} \mathcal{Y}_{i_1 \ldots i_m} |e^{(1)}_{i_1} \cdots e^{(m)}_{i_m}\rangle.
\]

(2.2)

be another pure state. The inner product and norm of multipartite pure states are denoted as

\[
\langle \psi | \varphi \rangle := \sum_{i_1 \ldots i_m=1}^{n_1 \ldots n_m} \mathcal{X}^*_{i_1 \ldots i_m} \mathcal{Y}_{i_1 \ldots i_m}, \quad ||| \varphi \|| := \sqrt{\langle \varphi | \varphi \rangle},
\]
where $X_{i_1...i_m}$ is the complex conjugate of $X_{i_1...i_m}$. If $|||\varphi||| = 1$, then the state $|\varphi\rangle$ is a normalized state, also called unit state.

**Definition 2.1** An $m$-partite pure state $|\phi\rangle$ is called separable, if it can be written as

$$|\phi\rangle := \otimes_{k=1}^{m} |\phi^{(k)}\rangle,$$

where $|\phi^{(k)}\rangle \in \mathbb{C}^{n_k}$. If an $m$-partite pure state is not separable, then it is an entangled state.

The set of all separable normalized pure states in $H$ is denoted as $\text{Separ}(H)$.

The geometric measure of entanglement of a given multipartite pure state $|\psi\rangle$ can be defined by the distance between $|\psi\rangle$ and $\text{Separ}(H)$

$$E_{G}(|\psi\rangle) := \min_{|\phi\rangle \in \text{Separ}(H)} |||\psi\rangle - |\phi\rangle||.$$

(2.3)

The minimizer of (2.3) does exist, since the objective function in (2.3) is a continuous function on a compact set in a finite dimensional space. (2.3) can be converted to a maximization problem

$$G(|\psi\rangle) = \max_{|\phi\rangle \in \text{Separ}(H)} |\langle \psi | \phi \rangle|.$$

$G(|\psi\rangle)$ is called the maximal overlap of a given $m$-partite pure state, and it is also called entanglement eigenvalue in [29]. By [92], the geometric measure of entanglement of $|\psi\rangle$ is equal to

$$E_{G}(|\psi\rangle) = \sqrt{2 - 2G(|\psi\rangle)}.$$

(2.4)

Clearly, the smaller the maximal overlap is, the larger the geometric measure of entanglement of $|\psi\rangle$ is. In quantum physics, a large geometric measure usually indicates that the state $|\psi\rangle$ is more entangled.
2.2 Complex tensors and their U-eigenpairs

For the pure state $|\psi\rangle$ defined in (2.1), one can define an $m$th-order complex tensor $X = (X_{i_1 \cdots i_m})$. In other words, each quantum pure state corresponds to a complex tensor. Hence, we can calculate the GME of a quantum state by means of its corresponding complex tensor. If $X_{i_1 \cdots i_m}$ remains the same for all permutations of indices $\{i_1, \ldots, i_m\}$, then $X$ is called symmetric. For $X, Y \in H = \mathbb{C}^{n_1 \times \cdots \times n_m}$, the inner product and norm are defined as

$$\langle X, Y \rangle := \sum_{i_1 \cdots i_m = 1}^{n_1 \cdots n_m} X_{i_1 \cdots i_m}^* Y_{i_1 \cdots i_m}, \quad ||X|| := \sqrt{\langle X, X \rangle},$$

where $X_{i_1 \cdots i_m}^*$ denotes the complex conjugate of $X_{i_1 \cdots i_m}$.

A tensor can be geometrically viewed as a multi-liner function, and it can be represented by a linear combination of outer products of vectors. Let $x^{(i)} \in \mathbb{C}^{n_i}$, $i = 1, \ldots, m$, their outer product denoted as $\otimes_{i=1}^m x^{(i)}$ is called a rank-one tensor, whose components are

$$(\otimes_{i=1}^m x^{(i)})_{i_1 \cdots i_m} := x^{(1)}_{i_1} \cdots x^{(m)}_{i_m}.$$  

When $x^{(1)} = \cdots = x^{(m)} = x \in \mathbb{C}^n$, $\otimes_{i=1}^m x$ is called a symmetric rank-one complex tensor, simply written as $x^m$.

By the notation in [3], for $T \in H = \mathbb{C}^{n_1 \times \cdots \times n_m}$, we denote the inner product between $T$ and a rank-one tensor $\otimes_{i=1}^m x^{(i)}$ by a homogenous polynomial

$$\langle T, \otimes_{i=1}^m x^{(i)} \rangle \equiv T^* x^{(1)} \cdots x^{(m)} := \sum_{i_1 \cdots i_m = 1}^{n_1 \cdots n_m} T_{i_1 \cdots i_m}^* x^{(1)}_{i_1} \cdots x^{(m)}_{i_m}.$$  

Moreover, $\langle T, \otimes_{i=1}^m x^{(i)} \rangle$ denotes a vector in $\mathbb{C}^{n_k}$, whose $i_k$-th components are

$$\langle T, \otimes_{i=1}^m x^{(i)} \rangle_{i_k} := \sum_{i_1 \cdots, i_{k-1}, i_{k+1}, \cdots, i_m = 1}^{n_1, \cdots, n_{k-1}, n_{k+1}, \cdots, n_m} T_{i_1 \cdots i_{k-1} i_{k+1} \cdots i_m}^* x^{(1)}_{i_1} \cdots x^{(k-1)}_{i_{k-1}} x^{(k+1)}_{i_{k+1}} \cdots x^{(m)}_{i_m}.$$  


We define a new complex vector \( \langle \otimes_{i=1, i \neq k}^m x^{(i)}, T \rangle := \langle T, \otimes_{i=1, i \neq k}^m x^{(i)} \rangle^* \). The definition of U-eigenpair of \( T \), introduced in [3], is given below.

**Definition 2.2** For an \( m \)th-order tensor \( T \in H = C^{n_1 \times \cdots \times n_m} \), a tuple \( \{ \lambda, (x^{(1)}, \ldots, x^{(m)}) \} \) with \( \lambda \in \mathbb{R}, x^{(i)} \in C^{n_i}, i = 1, \ldots, m \) is called a U-eigenpair of \( T \) if \( \lambda \) and the rank-one tensor \( \otimes_{i=1, i \neq k}^m x^{(i)} \) are solutions of the following system of equations:

\[
\begin{align*}
\langle T, \otimes_{i=1, i \neq k}^m x^{(i)} \rangle &= \lambda x^{(k)}^*, \\
\langle \otimes_{i=1, i \neq k}^m x^{(i)}, T \rangle &= \lambda x^{(k)}, \\
\lambda &\in \mathbb{R}, ||x^{(i)}|| = 1, i = 1, 2, \ldots, m,
\end{align*}
\]

(2.5)

In fact, (2.5) is equivalent to

\[
\begin{align*}
\langle T, \otimes_{i=1, i \neq k}^m x^{(i)} \rangle &= \lambda x^{(k)}^*, \\
\lambda &\in \mathbb{R}, ||x^{(i)}|| = 1, i = 1, 2, \ldots, m,
\end{align*}
\]

(2.6)

or

\[
\begin{align*}
\langle \otimes_{i=1, i \neq k}^m x^{(i)}, T \rangle &= \lambda x^{(k)}, \\
\lambda &\in \mathbb{R}, ||x^{(i)}|| = 1, i = 1, 2, \ldots, m,
\end{align*}
\]

(2.7)

When \( S \) is a symmetric tensor, we call the tuple \( \{ \lambda, x \} \) a US-eigenpair of \( S \) if the scalar \( \lambda \) and the vector \( x \) satisfy

\[
\begin{align*}
\langle S, \otimes_{i=1}^{m-1} x \rangle &= \lambda x^*, \\
\lambda &\in \mathbb{R}, x \in C^n, ||x|| = 1.
\end{align*}
\]

(2.8)

or

\[
\begin{align*}
\langle \otimes_{i=1}^{m-1} x, S \rangle &= \lambda x, \\
\lambda &\in \mathbb{R}, x \in C^n, ||x|| = 1.
\end{align*}
\]

(2.9)
Given a tensor $T \in H = \mathbb{C}^{n_1 \times \cdots \times n_m}$, a rank-one complex tensor $\otimes_{i=1}^m x^{(i)}$ is called the best complex rank-one approximation to $T$ if it is the minimizer of the optimization problem

$$\min_{x^{(i)} \in \mathbb{C}^{n_i}, ||x^{(i)}|| = 1} ||T - \otimes_{i=1}^m x^{(i)}||. \quad (2.10)$$

It is proved in [3,36] that for a symmetric $m$th-order complex tensor $S$, its best symmetric complex rank-one approximation is also the best complex rank-one approximation, in other words, for a symmetric complex tensor $S$, the optimization problem (2.10) reduces to the following one

$$\min_{x \in \mathbb{C}^{n}, ||x|| = 1} ||S - \otimes_{i=1}^m x||. \quad (2.11)$$

As pointed out by [33], (2.10) is equivalent to the following maximization problem

$$\left\{ \begin{array}{l}
\max \langle T, \otimes_{i=1}^m x^{(i)} \rangle \\
\text{s.t. } ||x^{(i)}|| = 1, \quad x^{(i)} \in \mathbb{C}^{n_i}, i = 1, \ldots, m. \end{array} \right. \quad (2.12)$$

Moreover, let $(z^{(1)}, \ldots, z^{(m)})$ be a solution to (2.12). It is shown in [33] that the largest U-eigenvalue of the tensor $T$ is actually $\max \langle T, \otimes_{i=1}^m z^{(i)} \rangle$ and $(z^{(1)}, \ldots, z^{(m)})$ are the corresponding U-eigenvector. Also, the rank-1 tensor $\otimes_{i=1}^m z^{(i)}$ is the best complex rank-one approximation of $T$.

The following result has been proved in [33].

**Theorem 2.1** Assume that $T$ is the corresponding tensor of a multipartite pure state $|\psi\rangle$ under an orthonormal basis as in (2.1). Let $\lambda_{\text{max}}$ be the largest U-eigenvalue of $T$. Then

1) $G(|\psi\rangle) = \lambda_{\text{max}},$

2) $E_G(|\psi\rangle) = \sqrt{2 - 2\lambda_{\text{max}}}.$

This theorem makes it possible to investigate the geometric measure of entanglement of multipartite pure states by the U-eigepairs of complex tensors.
2.3 Tensor blocking

We introduce how to block a complex tensor, in analog to the real tensor case proposed in [35]. Let $T \in \mathbb{C}^{n_1 \times \cdots \times n_m}$ be an $m$th-order tensor. If $a$ and $b$ are integers with $a \leq b$, then let $a : b$ denote the row vector $[a, a + 1, \cdots, b]$.

Blocking the tensor $T$ is the act of partitioning its index range vectors $1 : n_1, 1 : n_2, \cdots, 1 : n_m$ in the following way. For each $k = 1, \cdots, m$, let $r^{(k)}$ denote the row vector $[a, a + 1, \cdots, b]$. Partition $r^{(k)}$ into $b_k$ blocks as

$$r^{(k)} = [r_1^{(k)}, \cdots, r_{b_k}^{(k)}].$$

(2.13)

Denote $\rho_i^{(k)} = \sum_{j=1}^{i-1} \text{length}(r_j^{(k)})$ for each $i = 1, \cdots, b_k$. Then $T$ can be regarded as a $b_1 \times \cdots \times b_m$ block tensor. Let $i = \{i_1, \cdots, i_m\}$ where $1 \leq i_k \leq b_k$. The $i$-th block is denoted as

$$T[i] = T[i_1, \cdots, i_m].$$

To be specific, let $j = \{j_1, \cdots, j_m\}$, where $j_k = 1, \cdots, \text{length}(r_{i_k}^{(k)})$. Then the $j$-th entry of the subtensor $T[i]$ is

$$[T[i]]_j = [T[i]]_{j_1 \cdots j_m} = T_{(\rho_{i_1}^{(1)} + j_1) \cdots (\rho_{i_m}^{(m)} + j_m)}.$$

(2.14)

Definition 2.3 Let $T \in \mathbb{C}^{n_1 \times \cdots \times n_m}$ be an $m$th-order tensor, $p = \{p_1, \cdots, p_m\}$ be a permutation of $1 : m$. The $p$-transpositional tensor of $T$ denoted by $T^{<p>}$ is defined as

$$T^{<p>}_{p(j)} = T_j,$$

where $j = \{j_1, \cdots, j_m\}$, $p(j) = \{j_{p_1}, \cdots, j_{p_m}\}$ is a $p$-transposition of $j$.

Lemma 2.1 Let $T \in \mathbb{C}^{n_1 \times \cdots \times n_m}$ be a $b_1 \times \cdots \times b_m$ block tensor defined by the partition (2.13). Let $p = \{p_1, \cdots, p_m\}$ be a permutation of $1 : m,$
i = \{i_1, ..., i_m\}. Then

\[(T^{<p>})_{[i]} = (T^{<p>})_{[i]}\]

The proof of Lemma 2.1 is essentially the same as that of Lemma 2.1 in [35], hence it is omitted.

3 Iterative methods for computing U-eigenpairs of non-symmetric complex tensors

In this section, we first introduce how to embed a non-symmetric complex tensor \(A\) into a symmetric complex tensor \(S\), and illustrate the relationship between the U-eigenpairs of \(A\) and the US-eigenpairs of \(S\). Then we propose three iterative algorithms to compute the eigenpairs of a non-symmetric tensor.

3.1 The extended embedding operation

**Definition 3.1** Let \(A \in \mathbb{C}^{n_1 \times \cdots \times n_m}\), \(i = \{i_1, ..., i_m\}\). The \textit{sym}() operator means to construct an \(n \times \cdots \times n\) block tensor denoted by \(S = \text{sym}(A) \in \mathbb{C}^{n_1 \times \cdots \times n_m}\), \(n = n_1 + \cdots + n_m\), and the \(i\)-th block \(S_{[i]} \in \mathbb{C}^{n_{i_1} \times \cdots \times n_{i_m}}\) is defined as

\[S_{[i]} = \begin{cases} A^{<i>}, & \text{if } i \text{ is a permutation of } 1:m, \\ 0, & \text{else.} \end{cases}\]

\(S = \text{sym}(A)\) is called the symmetric embedding of \(A\).

**Theorem 3.1** Assume that \(A \in \mathbb{C}^{n_1 \times \cdots \times n_m}\), \(S = \text{sym}(A) \in \mathbb{C}^{n_1 \times \cdots \times n}\), \(n = n_1 + \cdots + n_m\), then \(S\) is symmetric.

**Proof** The proof is similar to Lemma 2.2 in [35].

For example, let \(A \in \mathbb{C}^{3 \times 4 \times 5}\), \(S = \text{sym}(A) \in \mathbb{C}^{12 \times 12 \times 12}\). Then

\[S_{[ij]} = 0 \in \mathbb{C}^{n_{i} \times n_{j} \times n_{k}}, \text{ for all } i, j, k \in \{1, 2, 3\}, \text{ if } i = j \text{ or } i = k \text{ or } j = k,\]
and
\[ S_{[123]} = A_{<123>}, \ S_{[132]} = A_{<132>}, \ S_{[213]} = A_{<213>}, \]
\[ S_{[231]} = A_{<231>}, \ S_{[312]} = A_{<312>}, \ S_{[321]} = A_{<321>}. \]

The following theorem illustrates the relationship between the U-eigenpairs of a complex non-symmetric tensor \( A \) and the US-eigenpairs of its symmetric embedding \( S = \text{sym}(A) \).

**Theorem 3.2** Let \( A \in \mathbb{C}^{n_1 \times \cdots \times n_m} \), \( S = \text{sym}(A) \), \( n = n_1 + \cdots + n_m \). Assume that \( \lambda_S \) is a nonzero US-eigenvalue of \( S \) associated with the US-eigenvector \( x \in \mathbb{C}^n \). We partition \( x \) as \( x = (x^{(1)})^T, \ldots, (x^{(m)})^T, x^{(i)} \in \mathbb{C}^{n_i} \) for all \( i = 1 : m \). Then the following hold:

(a) For \( i = 1, \ldots, m \), \( \|x^{(i)}\| = \frac{1}{\sqrt{m}} \), i.e., all \( x^{(i)} \) have the same norm \( \frac{1}{\sqrt{m}} \).

(b) Let \( \lambda_A = (\sqrt{m/n})^m \lambda_S \). Then \( \lambda_A \) is a U-eigenvalue of \( A \) associated with the U-eigenvector \( \{\sqrt{m}x^{(1)}, \ldots, \sqrt{m}x^{(m)}\} \).

**Proof** (a) Since \( \lambda_S \) is a US-eigenvalue of \( S \) associated with the US-eigenvector \( x \in \mathbb{C}^n \), then we have
\[
\langle S, x^{m-1}\rangle_{[i]} = \lambda_S x^{(i)*}, \quad i = 1, \ldots, m.
\] (3.1)

By the definition of the inner product of complex tensors, we have
\[
\langle S, x^{m-1}\rangle_{[i]} = \sum_{i_2, \ldots, i_m=1}^{m} \langle S_{[i_2 \ldots i_m]}, x^{(i_2)} \cdots x^{(i_m)}\rangle
\]
\[
= \sum_{[i_2, \ldots, i_m] \in p(1,2,\ldots,i-1,i+1,\ldots,m)} \langle S_{[i_2 \ldots i_m]}, x^{(i_2)} \cdots x^{(i_m)}\rangle
\]
\[
= \sum_{[i_2, \ldots, i_m] \in p(1,2,\ldots,i-1,i+1,\ldots,m)} \langle A_{<i_2 \ldots i_m>}, x^{(i_2)} \cdots x^{(i_m)}\rangle
\]
\[
= (m-1)!\langle A, x^{(1)} \cdots x^{(i-1)}x^{(i+1)} \cdots x^{(m)}\rangle
\] (3.2)

Comparing the right-hand sides of (3.1) and (3.2), we have that
\[(m - 1)! \langle A, x^{(1)} \cdots x^{(i-1)} x^{(i+1)} \cdots x^{(m)} \rangle = \lambda_S x^{(i)^*}. \quad (3.3)\]

It follows that
\[(m - 1)! \langle A, x^{(1)} \cdots x^{(m)} \rangle = \lambda_S (x^{(i)^*}, x^{(i)}). \quad (3.4)\]

On the other hand, there is
\[
\lambda_S = \langle S, x^m \rangle = \sum_{i_1, \cdots, i_m = 1}^m \langle S_{[i_1, \cdots, i_m]}, x^{(i_1)} \cdots x^{(i_m)} \rangle = m! \langle A, x^{(1)} \cdots x^{(m)} \rangle. \quad (3.5)
\]

Since \(\lambda_S \neq 0\), by (3.4) and (3.5), we have that
\[
\langle x^{(i)^*}, x^{(i)} \rangle = \frac{1}{m}, \quad \text{i.e.} \quad \|x^{(i)}\| = \frac{1}{\sqrt{m}} \quad (3.6)
\]
for all \(i = 1 : m\).

(b) According to (3.3) and (3.6), we have \(\|\sqrt{m}x^{(i)}\| = 1\), and
\[
\langle A, (\sqrt{m}x^{(1)})(\sqrt{m}x^{(i-1)})(\sqrt{m}x^{(i+1)})(\sqrt{m}x^{(m)}) \rangle = \frac{(\sqrt{m})^m \lambda_S}{m!} \sqrt{m}x^{(i)^*}. \quad (3.7)
\]
By the definition of the U-eigenvalue of complex tensors, it follows that
\[
\lambda_A = \frac{(\sqrt{m})^m}{m!} \lambda_S \quad (3.8)
\]
is a U-eigenvalue of \(A\) associated with the eigenvectors \(\{\sqrt{m}x^{(1)}, \cdots, \sqrt{m}x^{(m)}\}\).

This completes the proof. \(\square\)

### 3.2 Iterative methods

When an \(n_1 \times \cdots \times n_m\) tensor \(A\) is non-symmetric, we can use Algorithm 4.1 in [17] to compute the US-eigenpairs of its symmetric embedding \(S = \text{sym}(A)\),
and obtain the U-eigenpairs of the non-symmetric complex tensor \( A \) through Theorem 3.2. Hence, we have the following algorithm.

**Algorithm 3.1** Computing the U-eigenpairs of an \( n_1 \times \cdots \times n_m \) non-symmetric complex tensor \( A \).

**Step 1 (Initial step):** Let \( S = \text{sym}(A) \), and \( n = n_1 + \cdots + n_m \). Choose a starting point \( x_0 \in \mathbb{C}^n \) with \( \|x_0\| = 1 \), and \( 0 < \alpha_S \in \mathbb{R} \). Let \( \lambda_0 = S^*x_0^m \).

**Step 2 (Iterating step):**

\[
\hat{x}_k = \lambda_{k-1}Sx_{k-1}^{m-1} + \alpha_Sx_{k-1}, \\
x_k = \hat{x}_k / \|\hat{x}_k\|, \\
\lambda_k = S^*x_k^m.
\]

**return:** 
US-eigenvalue \( \lambda_S = |\lambda_k| \), US-eigenvector \( x = (\frac{\hat{x}_k}{\sqrt{\sum_{j=1}^{m} \|\hat{x}_k\|^2}})^{1/m}x_k \).

Let \( x = (x^{(1)}\top, \ldots, x^{(m)}\top)\top \in \mathbb{C}^{n_1} \), for all \( i = 1 : m \).

**U-eigenvalue** \( \lambda_A = (\frac{\sqrt{m}}{m!})\lambda_S \).

**U-eigenvector** \( \{\sqrt{m}x^{(1)}, \ldots, \sqrt{m}x^{(m)}\} \).

Given a non-symmetric tensor \( A \), the size of the symmetric embedding tensor \( S = \text{sym}(A) \) is much larger that of the tensor \( A \) itself. This affects the computational efficiency of Algorithm 3.1 proposed above. Motivated by this, we propose a new iterative algorithm.

**Algorithm 3.2** Computing the U-eigenpairs of an \( m \)th-order non-symmetric complex tensor \( A \).

**Step 1 (Initial step):** Choose starting points \( \hat{x}_0^{(i)} \in \mathbb{C}^{n_i} \) with \( \|\hat{x}_0^{(i)}\| \neq 0 \) for all \( i = 1 : m \). Let \( x_0^{(i)} = \hat{x}_0^{(i)} / \sqrt{\sum_{j=1}^{m} \|\hat{x}_0^{(j)}\|^2} \) for all \( i = 1 : m \), \( \lambda_0 = \langle A, x_0^{(1)} \cdots x_0^{(m)} \rangle \). Choose \( 0 < \alpha_A \in \mathbb{R} \).

**Step 2 (Iterating step):**
for $k = 1, 2, \cdots$, do
  for $i = 1, 2, \cdots, m$, do
    $\hat{x}^{(i)}_k = \lambda_{k-1} A x^{(1)}_{k-1} \cdots x^{(i-1)}_{k-1} x^{(i+1)}_{k-1} \cdots x^{(m)}_{k-1} + \alpha_{x^{(i)}_k} x^{(i)}_{k-1} \cdots x^{(m)}_{k-1} + \alpha_{x^{(i)}_k} x^{(i)}_{k-1} \cdots x^{(m)}_{k-1}$.
  end for.
  for $i = 1, 2, \cdots, m$, do
    $x^{(i)}_k = \frac{\hat{x}^{(i)}_k}{\sqrt{\sum_{j=1}^{m} ||x^{(j)}_k||^2}}$.
  end for.
end for.

$\lambda_k = \langle A, x^{(1)}_k \cdots x^{(m)}_k \rangle$.

return:

$U$-eigenvalue $\lambda_A = (\sqrt{m})^m |\lambda_k|$. 
$U$-eigenvector $\{\sqrt{m} (|\lambda_k| |\lambda_k|^{1/m} x^{(1)}_k), \ldots, \sqrt{m} (|\lambda_k|^{1/m} x^{(m)}_k)\}$.

The following Theorem establishes the relationship between Algorithms 3.1 and 3.2.

**Theorem 3.3** Let $A \in \mathbb{C}^{n_1 \times \cdots \times n_m}$, $S = \text{sym}(A)$, $\alpha_S = m! (m - 1)! \alpha_A$, $n = n_1 + \cdots + n_m$. Choose points $\hat{x}^{(i)}_0 \in \mathbb{C}^{n_i}$ with $||\hat{x}^{(i)}_0|| \neq 0$ for all $i = 1 : m$. Let $x^{(i)}_0 = \hat{x}^{(i)}_0 / \sqrt{\sum_{j=1}^{m} ||x^{(j)}_0||^2}$ for all $i = 1 : m$. Let $x^\top = (x^{(1)}^\top, \ldots, x^{(m)}^\top)^\top$.

Assume that $x_0$ is the starting point of Algorithm 3.1, $\lambda_{S_k}$ and $x_k$ are obtained by the $k$-th iteration of Algorithm 3.1. Assume that $\{x^{(1)}_0, \ldots, x^{(m)}_0\}$ is the starting point of Algorithm 3.2, $\lambda_{A_k}$ and $\{x^{(1)}_k, \ldots, x^{(m)}_k\}$ are obtained by the $k$-th iteration of Algorithm 3.2. Then $x^\top = (x^{(1)}_k^\top, \ldots, x^{(m)}_k^\top)^\top$, $\lambda_{S_k} = m! \lambda_{A_k}$.

Proof Partition $x_k$ as $x_k = (x^{(1)}_k^\top, \ldots, x^{(m)}_k^\top)^\top$ with $x^{(i)}_k \in \mathbb{C}^{n_i}$. Similar to (3.2), we have

$$\langle S, x^{m-1}_k | | = (m - 1)! \langle A, x^{(1)}_k \cdots x^{(i-1)}_k x^{(i+1)}_k \cdots x^{(m)}_k \rangle. \quad (3.9)$$

$$\lambda_{S_k} = \langle S, x^m_k | = m! \langle A, x^{(1)}_k \cdots x^{(m)}_k \rangle. \quad (3.10)$$
In the following, we will use the mathematical induction to prove Theorem 3.3.

Let k=0. By the assumption of $x_0$ and $\{x_0^{(1)}, x_0^{(2)}, \cdots, x_0^{(m)}\}$, we have that

$$
\bar{x}_0^{(i)} = x_0^{(i)} \quad (i = 1, 2, \cdots, m),
$$

$x_0 = (x_0^{(1)\top}, \cdots, x_0^{(m)\top})\top$.

By (3.10),

$$
\lambda_{S_0} = m! \langle A, \bar{x}_0^{(1)} \cdots \bar{x}_0^{(m)} \rangle = m! \langle A, x_0^{(1)} \cdots x_0^{(m)} \rangle = m! \lambda_{A_0}.
$$

Hence, the results holds for $k = 0$. Assume that the result holds for $k - 1$. That is

$$
\bar{x}_{k-1}^{(i)} = x_{k-1}^{(i)} \quad (i = 1, 2, \cdots, m),
\bar{x}_{k-1} = (x_{k-1}^{(1)\top}, \cdots, x_{k-1}^{(m)\top})\top, \lambda_{S_{k-1}} = m! \lambda_{A_{k-1}}.
$$

Partition $\bar{x}_k$ as $\bar{x}_k = \{\bar{x}_k^{(1)}, \cdots, \bar{x}_k^{(m)}\}$ with $\bar{x}_k^{(i)} \in \mathbb{C}^n$. Then by Algorithm 3.1, we have

$$
\bar{x}_k = \lambda_{S_{k-1}} \bar{x}_{k-1} + \alpha_S \bar{x}_{k-1} = \lambda_{S_{k-1}} (\bar{x}_{k-1})^* + \alpha_S \bar{x}_{k-1}.
$$

Then,

$$
\bar{x}_k^{(i)} = \lambda_{S_{k-1}} \langle S, x_{k-1}^{(m-1)\top} \rangle + \alpha_S \bar{x}_{k-1}^{(i)}
\quad = \lambda_{S_{k-1}} (m-1)! \langle A, x_{k-1}^{(1)} \cdots x_{k-1}^{(i-1)} x_{k-1}^{(i+1)} \cdots x_{k-1}^{(m)\top} \rangle + \alpha_S \bar{x}_{k-1}^{(i)}
\quad = m! (m-1)! \lambda_{A_{k-1}} (A, x_{k-1}^{(1)} \cdots x_{k-1}^{(i-1)} x_{k-1}^{(i+1)} \cdots x_{k-1}^{(m)\top} \rangle + \alpha_A x_{k-1}^{(i)}
\quad = m! (m-1)! \bar{x}_k^{(i)}.
$$

Hence,

$$
\bar{x}_k = m! (m-1)! (\bar{x}_k^{(1)\top}, \bar{x}_k^{(2)\top}, \cdots, \bar{x}_k^{(m)\top})\top.
$$
It follows that
\[ x_k = \hat{x}_k = \frac{\left( \hat{x}_k^{(1)\top}, \hat{x}_k^{(2)\top}, \cdots, \hat{x}_k^{(m)\top} \right)\top}{\sqrt{\sum_{j=1}^{m} \|\hat{x}_k^{(j)}\|^2}} = (\hat{x}_k^{(1)}\top, \hat{x}_k^{(2)}\top, \cdots, \hat{x}_k^{(m)}\top)\top. \]

By (3.10),
\[ \lambda_{S_k} = m! \langle A, \hat{x}_k^{(1)} \cdots \hat{x}_k^{(m)} \rangle = m! \langle A, x_k^{(1)} \cdots x_k^{(m)} \rangle = m\lambda_k. \]

Hence, the result holds for \( k \). This completes the proof. \( \square \)

Remark 1. The convergence of Algorithm 3.1 has already been proved in [17]. By Theorem 3.3, Algorithm 3.2 is also convergent.

Remark 2. In [32] an algorithm is proposed to compute the U-eigenpairs of non-symmetric complex tensors. The difference between it and Algorithm 3.2 is that they use different normalization conditions. To be specific, in Algorithm 3.2 the normalization condition is 
\[ x_k^{(i)} = \frac{\hat{x}_k^{(i)}}{\sqrt{\sum_{j=1}^{m} \|\hat{x}_k^{(j)}\|^2}}, \]
while the normalization condition used in the algorithm of [32] is 
\[ x_k^{(i)} = \frac{\hat{x}_k^{(i)}}{\|\hat{x}_k^{(i)}\|}. \]
This is a key difference as the new normalization condition enables us to establish the convergence of Algorithm 3.2.

Inspired by the well-known Gauss-Seidel method, we propose the following algorithm which may improve computational efficiency of Algorithm 3.2.

**Algorithm 3.3** Tensor Gauss-Seidel method for computing U-eigenpairs of non-symmetric complex tensors.

**Step 1 (Initial step):** Choose starting points \( x_0^{(i)} \in \mathbb{C}^{n_i} \) with \( \|x_0^{(i)}\| = 1 \) for all \( i = 1 : m \). Let \( \lambda_0 = \langle A, x_0^{(1)} \cdots x_0^{(m)} \rangle \). Choose \( 0 < \alpha_A \in \mathbb{R} \).

**Step 2 (Iterating step):**

for \( k = 1, 2, \cdots, \) do

for \( i = 1, 2, \cdots, m, \) do

\[ \hat{x}_k^{(i)} = \lambda_{k-1} A \hat{x}_k^{(i-1)*} \cdots \hat{x}_k^{(i-2)*} \hat{x}_k^{(i-1)*} x_{k-1}^{(i+1)*} \cdots x_{k-1}^{(m)*} + \alpha_A x_{k-1}^{(i)}, \]

\[ x_k^{(i)} = \hat{x}_k^{(i)}/\|\hat{x}_k^{(i)}\|. \]
end for.

\[
\lambda_k = \mathbf{A}^* \mathbf{x}_k^{(1)} \cdots \mathbf{x}_k^{(m)}.
\]

end for.

return:

U-eigenvalue \( \lambda_A = |\lambda_k| \), U-eigenvector \( \mathbf{x}^{(i)} = (\frac{\lambda_A}{\lambda_k})^{1/m} \mathbf{x}_k^{(i)}. \)

4 Numerical examples

In this section, we present numerical studies of using Algorithms 3.1-3.3 to find the geometric measure of entanglement of pure quantum states. These algorithms can find the maximum U-eigenvalue of a complex tensor. Then by Theorem 2.1 we can calculate the geometric measure of entanglement of the corresponding pure state. As Algorithm 3.1 is based on tensor symmetric embedding, the size of the symmetric tensor obtained is usually very large, which may significantly reduce the computational efficiency in applications. In the first three examples, we will compare the computational efficiency of these three algorithms. It can be observed that when the size of quantum states increases, Algorithms 3.2 and 3.3 are much more efficient than Algorithm 3.1. Thus, in other examples, we only apply Algorithms 3.2 and 3.3.

The computations are implemented in Mathematica 8.0 on a Microsoft Win10 Laptop with 8GB memory and Intel(R) i5 CPU 2.40GHZ. In each example, we take 10 starting points to get the results, and obtain the maximum eigenvalue from them. Four decimal digits are presented, and two decimal digits are presented for the consumed computer time.

Example 4.1 (Example 6) Consider a non-symmetric 3-partite state

\[
|\psi\rangle = \sqrt{\frac{1}{3}} |001\rangle + \sqrt{\frac{2}{3}} |100\rangle.
\]

It corresponds a \(2 \times 2 \times 2\) non-symmetric tensor \(\mathbf{A}\) with nonzero entries \(A_{112} = \sqrt{\frac{1}{3}}, A_{211} = \sqrt{\frac{2}{3}}.\)
We apply Algorithms 3.1-3.3 to get the maximum U-eigenvalue \( \lambda_A \) of \( A \) and the geometric measure of entanglement (Abbreviated as GME) of the state \( |\psi\rangle \) respectively. The iteration is terminated when the numerical error is less than \( 10^{-9} \). The computational results are shown in Table 4.1.

| Algorithm   | \( \lambda_A \) | GME    | Time(sec) |
|-------------|-----------------|--------|-----------|
| Algorithm 3.1 | 0.8165          | 0.6058 | 2.75      |
| Algorithm 3.2 | 0.8165          | 0.6058 | 1.02      |
| Algorithm 3.3 | 0.8165          | 0.6058 | 0.23      |

Table 4.1: Computational results for Example 4.1

The GME of the state \( |\psi\rangle \) is 0.6058 with the closest product state calculated by Algorithm 3.3

\[
|\phi\rangle = |\phi_1\rangle \times |\phi_2\rangle \times |\phi_3\rangle,
\]

where

\[
|\phi_1\rangle = (0.8350 - 0.5502i)|1\rangle,
\]

\[
|\phi_2\rangle = (-0.3980 + 0.9174i)|0\rangle,
\]

\[
|\phi_3\rangle = (0.1724 - 0.9850i)|0\rangle.
\]

We can observe from Table 4.1 that Algorithms 3.1-3.3 obtain the same value of GME. However, Algorithm 3.1 takes more time than Algorithms 3.2 and 3.3.

Example 4.2 (Example 9) Consider a non-symmetric 3-partite state

\[
|\psi\rangle = \sqrt{\frac{1}{6}}(|000\rangle + |101\rangle + |012\rangle + |110\rangle + |021\rangle + |122\rangle).
\]

It corresponds a \( 2 \times 3 \times 3 \) non-symmetric tensor \( A \) with nonzero entries \( A_{111} = A_{212} = A_{123} = A_{221} = A_{132} = A_{233} = \sqrt{\frac{1}{6}} \).

We apply Algorithms 3.1-3.3 to get the maximum U-eigenvalue \( \lambda_A \) of \( A \) and the GME of the state \( |\psi\rangle \) respectively. The iteration is terminated when
the numerical error is less than $10^{-9}$. The computational results are shown in Table 4.2.

| Algorithm | $\lambda_A$ | GME  | Time (sec) |
|-----------|-------------|------|------------|
| Algorithm 3.1 | 0.5774 | 0.9194 | 5.97 |
| Algorithm 3.2 | 0.5774 | 0.9194 | 1.12 |
| Algorithm 3.3 | 0.5774 | 0.9194 | 0.19 |

Table 4.2: Computational results for Example 4.2

The GME of the state $|\psi\rangle$ is 0.9194 with the closest product state calculated by Algorithm 3.3

$$|\phi\rangle = |\phi_1\rangle \times |\phi_2\rangle \times |\phi_3\rangle,$$

where

$$|\phi_1\rangle = (-0.0336 + 0.7063i)|0\rangle + (0.2818 - 0.3240i)|1\rangle,$$
$$|\phi_2\rangle = (-0.0353 - 0.5763i)|0\rangle + (0.2567 + 0.2576i)|1\rangle - (0.4814 - 0.3187i)|2\rangle,$$
$$|\phi_3\rangle = (0.5773 + 0.0079i)|0\rangle - (0.2955 - 0.4960i)|1\rangle - (0.2818 + 0.5039i)|2\rangle.$$
It corresponds a $2 \times 2 \times 2 \times 2 \times 2$ non-symmetric tensor $A$ with nonzero entries $A_{11111} = A_{11122} = A_{12211} = A_{22121} = A_{21221} = \frac{1}{2\sqrt{2}}$, $A_{12222} = A_{21212} = -\frac{1}{2\sqrt{2}}$.

We apply Algorithms 3.1-3.3 to get the maximum U-eigenvalue $\lambda_A$ of $A$ and the GME of the state $|\psi\rangle$ respectively. The iteration is terminated when the numerical error is less than $10^{-9}$. The computational results are shown in Table 4.3.

| Algorithm  | $\lambda_A$ | GME  | Time(sec) |
|------------|-------------|------|-----------|
| Algorithm 3.1 | 0.3626      | 1.1291 | 2628.76   |
| Algorithm 3.2 | 0.3626      | 1.1291 | 14.89     |
| Algorithm 3.3 | 0.3626      | 1.1291 | 2.42      |

Table 4.3: Computational results for Example 4.3

The geometric measure of entanglement of the state $|\psi\rangle$ is 1.1291 with the closest product state calculated by Algorithm 3.3

$$|\phi\rangle = |\phi_1\rangle \times |\phi_2\rangle \times |\phi_3\rangle \times |\phi_4\rangle \times |\phi_5\rangle,$$

where

$$|\phi_1\rangle = \left( -0.1455 + 0.4361i \right)|0\rangle + \left( -0.7944 + 0.3969i \right)|1\rangle,$$
$$|\phi_2\rangle = \left( -0.0940 - 0.4500i \right)|0\rangle + \left( -0.7431 - 0.4863i \right)|1\rangle,$$
$$|\phi_3\rangle = \left( 0.4913 + 0.7398i \right)|0\rangle + \left( -0.4506 - 0.0916i \right)|1\rangle,$$
$$|\phi_4\rangle = \left( 0.8856 + 0.0669i \right)|0\rangle + \left( 0.2996 + 0.3486i \right)|1\rangle,$$
$$|\phi_5\rangle = \left( -0.1751 - 0.4251i \right)|0\rangle + \left( 0.3415 - 0.8198i \right)|1\rangle.$$

Compared to Examples 4.1 and 4.2, we increase the order of the quantum state $|\psi\rangle$, and the corresponding tensor $\text{sym}(A)$ obtained by Algorithm 5.1 is a $10 \times 10 \times 10 \times 10 \times 10$ symmetric tensor. We can observe from Table 4.3 that
although the three algorithms give us the same GME value. However, it takes Algorithm 3.1 much longer time to get the results.

It can be seen from the above examples that, when the order and dimension of the quantum state increase, the size of the symmetric tensor obtained by Algorithm 3.1 is getting more and more large, which directly leads to a decrease in the efficiency of Algorithm 3.1. When we compute the GME of a larger quantum state, the efficiency of Algorithms 3.2 and 3.3 will be significantly higher than Algorithm 3.1. Hence we only compare Algorithms 3.2 and 3.3 in the following examples.

**Example 4.4** (Example 4.3) Consider a non-symmetric 3-partite state

$$|\psi\rangle = \sum_{i_1, i_2, i_3=1}^{n} \cos(i_1 - i_2 + i_3) + \sqrt{-1} \sin(i_1 + i_2 - i_3) \sqrt{n^3} / (i_1-1)(i_2-1)(i_3-1).$$

It corresponds to an $n \times n \times n$ non-symmetric tensor $A$ with

$$A_{i_1i_2i_3} = \cos(i_1 - i_2 + i_3) + \sqrt{-1} \sin(i_1 + i_2 - i_3) \sqrt{n^3}.$$

| n  | $\lambda_A$ | GME | Algo 3.2 (sec) | Algo 3.3 (sec) |
|-----|-------------|-----|----------------|----------------|
| 2   | 0.8895      | 0.4701 | 0.47           | 0.09           |
| 5   | 0.7815      | 0.6611 | 5.90           | 1.62           |
| 10  | 0.7072      | 0.7652 | 51.59          | 8.74           |
| 15  | 0.7243      | 0.7425 | 173.06         | 11.53          |
| 20  | 0.7175      | 0.7516 | 969.73         | 96.07          |
| 50  | 0.7087      | 0.7632 | 127332.83      | 2070.27        |

Table 4.4: Computational results for Example 4.4

For a range of values of $n$ from 2 to 50, we apply Algorithms 3.2 and 3.3 to get the maximum $U$-eigenvalue $\lambda_A$ of $A$ and the GME of the state $|\psi\rangle$. The iteration is terminated when the numerical error is less than $10^{-9}$. The computational results are presented in Table 4.4.
We can observe from Table 4.4 that both Algorithm 3.2 and Algorithm 3.3 can obtain the same GME value of the state $|\psi\rangle$ for different $n$. However, as $n$ increases, Algorithm 3.3 becomes more efficient than Algorithm 3.2.

**Example 4.5 (Random Examples)** In this example, we randomly generate quantum pure states, and compute their GME by means of Algorithms 3.2 and 3.3. For a range of values of order $m$ from 4 to 5, we apply both Algorithm 3.2 and Algorithm 3.3 to get the maximum U-eigenvalue $\lambda_A$ of $A$ and the GME of the state $|\psi\rangle$. The iteration is terminated when the numerical error is less than $10^{-9}$. The computational results are presented in Tables 4.5 and 4.6 respectively.

| $(n_1 \times n_2 \times n_3 \times n_4)$ | $\lambda_A$ | GME  | Algo 3.2 (sec) | Algo 3.3 (sec) |
|--------------------------------------|-----------|------|----------------|----------------|
| $3 \times 3 \times 3 \times 3$      | 0.4854   | 1.0145 | 21.40         | 2.84           |
| $5 \times 5 \times 5 \times 5$      | 0.2555   | 1.2203 | 242.13        | 66.06          |
| $2 \times 5 \times 8 \times 15$     | 0.2240   | 1.2458 | 656.30        | 131.14         |

Table 4.5: Computational results for Example 4.5 with $m = 4$

| $(n_1 \times n_2 \times n_3 \times n_4 \times n_5)$ | $\lambda_A$ | GME  | Algo 3.2 (sec) | Algo 3.3 (sec) |
|---------------------------------------------------|-----------|------|----------------|----------------|
| $2 \times 2 \times 2 \times 2 \times 2$          | 0.5475   | 0.9513 | 9.70            | 1.59           |
| $8 \times 2 \times 3 \times 5 \times 4$          | 0.2164   | 1.2519 | 720.05          | 103.35         |
| $10 \times 3 \times 15 \times 2 \times 5$        | 0.1322   | 1.3175 | 12194.04        | 1406.26        |

Table 4.6: Computational results for Example 4.5 with $m = 5$

We can observe from Tables 4.5 and 4.6 that the same GME values of the state $|\psi\rangle$ are obtained by both Algorithm 3.2 and Algorithm 3.3 within a reasonable time. When the order $m$ or dimension $n$ increases, it will take more time to find the GME of the state $|\psi\rangle$. And it is clear that Algorithm 3.3 is always more efficient than Algorithm 3.2.

We end this section with a final example.
Example 4.6 (32 (31], 32 Example 11] Given a $3 \times 3 \times 3 \times 3 \times 3 \times 2$ pure state

$$|\psi\rangle = \frac{1}{3\sqrt{2}} (|000000\rangle + |001121\rangle + |010220\rangle$$
$$+ |012011\rangle + |021210\rangle + |022101\rangle$$
$$+ |111110\rangle + |112201\rangle + |121000\rangle$$
$$+ |120121\rangle + |102020\rangle + |100211\rangle$$
$$+ |222220\rangle + |220011\rangle + |202110\rangle$$
$$+ |201201\rangle + |210100\rangle + |211021\rangle),$$

Algorithms 3.2 and 3.3 give the same GME value 1.2364, which is the same as that of 32 Example 11. The time Algorithms 3.2 takes is 1298.95 seconds, while the time Algorithms 3.3 takes is 9.70 seconds.

5 Conclusion

In this paper, we have proposed three different methods to compute U-eigenvalues of non-symmetric complex tensors and geometric measures of entanglement of non-symmetric pure states. The theory of symmetric embedding has been generalized from real tensors to complex tensors, and the relationship between U-eigenvalues of a non-symmetric complex tensor and US-eigenvalues of its symmetric embedding tensor has been established. Three algorithms have been given. Algorithm 3.1 computes the U-eigenvalues of a complex tensor by means of symmetric embedding, Algorithm 3.2 computes directly the U-eigenvalues of a complex tensor, and Algorithm 3.3 is a tensor version of the Gauss-Seidel method. The convergence of Algorithms 3.1 and 3.2 has been proved. Numerical examples are used to demonstrate Algorithms 3.1 and 3.3 and it is observed that Algorithm 3.3 is more computationally efficient than the other two algorithms. The convergence analysis of Algorithm 3.3 is our future research.
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