Two-dimensional state sum models and spin structures

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Abstract

The state sum models in two dimensions introduced by Fukuma, Hosono and Kawai are generalised by allowing algebraic data from a non-symmetric Frobenius algebra. Without any further data, this leads to a state sum model on the sphere. When the data is augmented with a crossing map, the partition function is defined for any oriented surface with a spin structure. An algebraic condition that is necessary for the state sum model to be sensitive to spin structure is determined. Some examples of state sum models that distinguish topologically-inequivalent spin structures are calculated.

1 Introduction

State sum models are of interest in statistical mechanics [1], random matrices [2], solid state physics [3] and quantum gravity [4], with the topologically-invariant state sum models playing an important role in these subjects. A number of features of state sum models appear in their simplest form in two dimensions and a motivation for this work is that a careful examination of the structures might lead to important lessons for various generalisations of the models, for example to higher dimensions.

The purpose of this paper is to develop the theory of state sum models in two dimensions by generalising the algebraic framework pioneered by Fukuma, Hosono and Kawai [5]. The generalisation consists of determining the state sum model coefficients using a diagrammatic calculus. Thus the original state sum models are called naive and the generalised ones diagrammatic. Diagrammatic state sum models were introduced by Lauda and Pfeiffer [6]; their models are related to one class of models described here (the curl-free ones in §4.1).
An FHK model is a naive state sum model that is topologically-invariant. The model is determined by algebraic data that correspond to a symmetric special Frobenius algebra. The construction is generalised here to the case of special Frobenius algebras that are not necessarily symmetric. This requires the use of a diagrammatic calculus in the definition of the state sum model. The most general case requires no additional data and defines a partition function for a disk. These state sum models are called planar if in addition they are invariant under Pachner moves. These models also determine a partition function for a sphere in a construction called a spherical state sum model. The diagrams involved in all these models are planar, and so do not require a crossing map.

Diagrammatic state sum models can be generalised to all surfaces by specifying a crossing map in addition to the Frobenius algebra. A set of axioms for the crossing is given; these, together with the axioms for a spherical state sum model, ensure the state sum model is an invariant of the surface with a spin structure. Such state sum models are therefore called spin. Some aspects of this construction are the subject of independent work by Novak and Runkel [7, 8].

If a spin state sum model does not actually depend on the spin structure then it is called topological and the partition function is a topological invariant of the surface. Particular examples of topological state sum models can be made by adding an additional axiom, the curl-free condition. The FHK state sum models can be seen as a special case of these by working with the canonical crossing map.

Thus there is the following hierarchy of models:

$$\text{FHK} \subset \text{topological} \subset \text{spin} \rightarrow \text{spherical}.$$ 

In this hierarchy, the FHK models are examples of topological models, which are in turn examples of spin models. Each spin model determines a spherical model by ignoring the crossing map.

The construction of naive state sum models is reviewed in §2 giving the equivalence of an FHK state sum model with a symmetric special Frobenius algebra. Over $\mathbb{C}$ or $\mathbb{R}$, these are direct sums of matrix algebras and the partition function for an oriented surface is calculated. The diagrammatic calculus is described in detail in §3 for the disk and sphere, using planar diagrams. The diagrammatic models are extended to all surfaces in §4, leading to the axioms for the spin state sum models.

The curl-free condition simplifies the models considerably, giving a class of models that is more accessible to the reader. These examples are analysed first, in §4.1. Finally, the spin models are investigated in detail in §4.2 in
A patch of a triangulated surface. Each triangle inherits the orientation induced by the overall orientation of the surface.

Figure 1

which an algebraic condition that determines whether the model depends on spin structure is determined. Some examples are computed showing that the topologically-inequivalent spin structures on a surface can be distinguished by the models.

A categorical perspective on the state sum model axioms is given in §5 with a brief discussion of category-theoretic generalisations of the models.

2 Naive state sum models

This section reviews the construction of state sum models according to the work of Fukuma, Hosono and Kawai [5], with the calculation of examples. These state sum models are called naive state sum models to distinguish them from the generalisation to diagrammatic ones in §3.

The idea of a state sum model is to calculate a quantum amplitude for a given triangulated manifold, possibly with a boundary. These amplitudes are numbers in a field $k$, for which the main examples of interest here are $k = \mathbb{R}$ or $\mathbb{C}$. A surface $\Sigma$ is a two-dimensional compact manifold, orientable but not necessarily closed. The surfaces are triangulated, and since they are compact, the number of vertices, edges and triangles is finite. The orientation of $\Sigma$ induces an orientation on each triangle. This means a triangle has a specified cyclic order of its vertices and these orientations are glued together coherently to preserve the overall orientation of the surface (see figure 1a).

A naive state sum model on an oriented triangulated surface $\Sigma$ has a set of quantum amplitudes for each vertex, edge and triangle. These are glued together using the rules of quantum mechanics (a superposition of all states) to give an overall amplitude to $\Sigma$.

Each edge on a triangle is associated with one of a finite set of states $S$ and...
Symmetry. Invariance under a rotation by \(\pi\) implies that the model cannot distinguish between the left- and right-hand sides of the equation above. As a consequence, the matrix used to glue the two triangles together must respect \(B^{ab} = B^{ba}\).

The partition function \(Z_{abc}\) is constructed from the constants \(C\) associated to each triangle and matrices \(B\) associated to each interior edge.

Figure 2

The quantum amplitude for the oriented triangle with edge states \(a, b, c \in S\) is \(C_{abc} \in k\), as shown in figure 1b. These amplitudes are required to satisfy invariance under rotations,

\[
C_{abc} = C_{bca} = C_{cab},
\]

which is to say they must respect the cyclic symmetry of an oriented triangle. If the orientation is reversed then the amplitude is \(C_{bac}\) and therefore not necessarily equal to \(C_{abc}\).

The triangles are glued together using a matrix \(B^{ab}\) associated to each edge of the triangulation not on the boundary (an interior edge). Since the formalism for naive state sum models does not distinguish the two triangles meeting at the edge (see figure 2a) then one must require symmetry,

\[
B^{ab} = B^{ba}.
\]

Note that this condition is relaxed in §3, together with a modification of the cyclic symmetry (1).

Finally, each interior vertex has amplitude \(R \in k\). This is a slight generalisation of the formalism presented in [5], where \(R = 1\) was assumed.

All the data needed to calculate the amplitude of a surface is now defined. Each edge in each triangle has a variable \(a \in S\). For a given value of each of these variables, the amplitude of a triangle \(t\) is \(A(t) = C_{abc}\) (with \(a, b, c\) the three variables on the three edges), and likewise the amplitude of an edge \(e\) is \(A(e) = B^{ab}\). The amplitude of the surface is called the partition function and is given by the formula that involves summing over the states on all interior
edges,

\[ Z(\text{boundary states}) = R^V \sum_{\text{interior states}} \left( \prod_{\text{triangles } t} A(t) \prod_{\text{interior edges } e} A(e) \right), \tag{3} \]

with \( V \) the number of interior vertices. For example, the amplitude of the triangulated disk of figure 2b is

\[ Z_{abc} = RC_{e'dc} C_{af'e} C_{fbd'} B^{d'd} B^{e'e} B^{f'f}, \tag{4} \]

using the Einstein summation convention for each paired index. The resulting partition function depends on the boundary data \( a,b,c \), which are not summed.

The formalism can be interpreted in terms of linear algebra. The states \( a \in S \) correspond to basis elements \( e_a \) of a vector space \( A \). The amplitude \( C_{abc} \) is the value of a trilinear form \( C : A \times A \times A \to k \) on basis elements, \( C(e_a, e_b, e_c) = C_{abc} \). The form \( C \) can also be viewed as a linear map on the tensor product, \( C : A \otimes A \otimes A \to k \). Similarly, \( B = e_a \otimes e_b B^{ab} \in A \otimes A \). This element can be viewed as a bilinear form on \( A^* \), i.e., \( B : A^* \times A^* \to k \), with matrix elements \( B^{ab} = B(e^a, e^b) \), using the dual basis elements \( e^a \). This linear algebra perspective means it is possible to regard state sum models as isomorphic if they are related by a change of basis; this is used below.

The bilinear form \( B \) can be used to raise indices; thus, using the definition \( C_{abc} = C_{abd} B^{dc} \) there is a multiplication map \( m : A \otimes A \to A \) with components

\[ m(e_a \otimes e_b) = C_{abc} e_c. \tag{5} \]

The notations \( m(e_a \otimes e_b) = e_a \cdot e_b \) will be used interchangeably. The state sum model data can also be used to determine a distinguished element of \( A \),

\[ \beta = m(B) = e_a \cdot e_b B^{ab}. \tag{6} \]

Throughout it is assumed the data for the state sum model are non-degenerate: \( R \neq 0, B(\cdot, a) = 0 \Rightarrow a = 0 \) and \( C(\cdot, \cdot, a) = 0 \Rightarrow a = 0 \). This means \( B \) has an inverse \( B^{-1} = B_{ab} e^a \otimes e^b \in A^* \otimes A^* \). This is defined by

\[ B_{ac} B^{cb} = \delta^b_a. \tag{7} \]

This determines a bilinear form on \( A \) with components \( B^{-1}(e_a, e_b) = B_{ab} \) and can be used to lower indices. Note that this discussion of the formalism in terms of linear algebra does not depend on the symmetry of \( B \), and these definitions will also be used in later sections where the symmetry of \( B \) is dropped.

A topological state sum is one for which the partition function of a surface is independent of the triangulation. This is made precise by the following definition.
Definition 2.1. A state sum model is said to be topological if $Z(M) = Z(M')$ whenever $M$ and $M'$ are two closed oriented triangulated surfaces on which the state sum model is defined and there is a piecewise-linear homeomorphism $f: M \rightarrow M'$ that preserves the orientation.

Any two triangulations of a surface are connected by a sequence of the two Pachner moves, shown in figures 3a and 3b or their inverses. For a closed manifold this result is proved in [9, 10]. (In fact this result can be extended to a manifold with boundary [11], but this result is not used here.) Thus it is sufficient to check for each Pachner move that the partition functions for the disk on the two sides of the move are equal.

In the case of topological state sum models there is a connection between the vector space $A$ and a Frobenius algebra. Recall that a Frobenius algebra is a finite-dimensional associative algebra $A$ with unit $1 \in A$ and a linear map $\varepsilon: A \rightarrow k$ that determines a non-degenerate bilinear form $\varepsilon \circ m$ on $A$. The linear map $\varepsilon$ is called the Frobenius form. A Frobenius algebra is called symmetric if $\varepsilon \circ m$ is a symmetric bilinear form. Let $B \in A \otimes A$ be the inverse of $B^{-1} = \varepsilon \circ m$ according to (7). Then the Frobenius algebra is called special if $m(B)$ is a non-zero multiple of the identity element.

A naive state sum model that obeys the Pachner moves is the type of model discussed by Fukuma, Hosono and Kawai, and so these are called FHK state sum models. The following result is a more precisely-stated version of their result in [5].

Theorem 2.1. Non-degenerate naive state sum model data determine an FHK state sum model if and only if the multiplication map $m$, the bilinear form $B$ and the distinguished element $\beta$ determine on $A$ the structure of a symmetric special Frobenius algebra with identity element $1 = R\beta$.

1Triangulations are allowed to be degenerate, that is, two simplexes can intersect in more than one face. It is allowable to use a degenerate triangulation that can be subdivided by Pachner moves into a non-degenerate one.
Proof. The proof begins by showing that the data determine a symmetric Frobenius algebra. The first Pachner move, shown in figure 3a, can be written

$$C_{ab}^e C_{ecd} = C_{bc}^e C_{aed}$$

and is equivalent to associativity of the multiplication. To see this note that using the notation (5) of a multiplication, \((e_a \cdot e_b) \cdot e_c = C_{ab}^e C_{ec}^f e_f\) and \(e_a \cdot (e_b \cdot e_c) = C_{bc}^e C_{ae}^f e_f\); hence, the identity (8) is \(B^{-1}(e_a \cdot (e_b \cdot e_c), e_d) = B^{-1}((e_a \cdot e_b) \cdot e_c, e_d).\) Since the bilinear form \(B^{-1}\) is non-degenerate this is equivalent to having an associative multiplication \(m.\) A linear functional can be defined by setting \(\varepsilon(x) = B^{-1}(x, 1).\) The cyclic symmetry (1) implies that \(B^{-1}(x \cdot y, z) = B^{-1}(x, y \cdot z)\) and so \(\varepsilon(x \cdot y) = B^{-1}(x \cdot y, 1) = B^{-1}(x, y),\) which is non-degenerate and symmetric.

The move in figure 3b requires the partition function of the disk (4) to equal \(C_{abc}.\) This is equivalent to

$$C_{ab}^e = RC_{ed}^f C_{fa}^d B_{ff'}$$

using associativity. For non-degenerate \(C,\) and rewriting \(C_{ab}^e = C_{ab}^e \delta_b^h,\) this is equivalent to

$$\delta_b^h = RC_{d}^h C_{fb}^d B_{ff'}$$

Recognising that \(\beta = B_{ff'} C_{ff'}^d e_d,\) expression (10) implies that \(R\beta\) must be the unit element for multiplication, and hence \(A\) is an algebra; it is therefore a symmetric special Frobenius algebra. It is worth noting that the non-degeneracy of \(C\) is necessary here, as without it the algebra need not even be unital.

Conversely, given a symmetric Frobenius algebra with linear functional \(\varepsilon,\) this defines a non-degenerate and symmetric bilinear form \(B^{-1} = \varepsilon \circ m\) with property (1). The fact that the algebra is unital implies that \(C\) is non-degenerate. Finally, associativity and the property \(R\beta = 1\) guarantee the Pachner moves are satisfied, meaning the state sum model created is an FHK model.

It is worth noting that having \(\beta\) proportional to the identity is a non-trivial restriction on Frobenius algebras. For the cases \(k = \mathbb{R}\) or \(\mathbb{C}\) of interest in this paper the Frobenius algebras, and hence the state sum models, are easily classified. The results for the symmetric Frobenius algebras in this
section are stated here, with the proof of the classification given in a more general context in theorem 3.2 of [3].

Let $M_n(C)$ denote the algebra of $n \times n$ matrices over $C$. An FHK state sum model over the field $C$ is isomorphic, by a change of basis, to one in which the algebra is a direct sum of matrix algebras,

$$A = \bigoplus_{i=1}^{N} M_{n_i}(C).$$  \hspace{1cm} (11)

The Frobenius form on an element $a = \oplus_i a_i$ is defined using the matrix trace on each factor:

$$\varepsilon(a) = R \sum_{i=1}^{N} n_i \text{Tr}(a_i).$$ \hspace{1cm} (12)

For the real case, the classification uses the division rings $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ regarded as algebras over $\mathbb{R}$; these are denoted $\mathbb{R}$, $\mathbb{C}_R$ and $\mathbb{H}_R$, and the dimension of the division ring $D$ as an $\mathbb{R}$-algebra is denoted $|D|$; thus $|\mathbb{R}| = 1$, $|\mathbb{C}_R| = 2$, $|\mathbb{H}_R| = 4$. The imaginary unit in $\mathbb{C}$ is denoted $i$ and the corresponding units for the quaternions $\hat{i}$, $\hat{j}$ and $\hat{k}$. The real part of a quaternion is defined as $\text{Re}(t + x\hat{i} + y\hat{j} + z\hat{k}) = t$ and the conjugate by $(t + x\hat{i} + y\hat{j} + z\hat{k})^* = t - x\hat{i} - y\hat{j} - z\hat{k}$. The $n \times n$ matrices with entries in $D$ are denoted $M_n(D)$ and are algebras over $\mathbb{R}$.

An FHK state sum model over the field $\mathbb{R}$ is isomorphic by a change of basis to one in which

$$A = \bigoplus_{i=1}^{N} M_{n_i}(D_i), \quad \text{with } D_i = \mathbb{R}, \mathbb{C}_R, \text{ or } \mathbb{H}_R. \hspace{1cm} (13)$$

The Frobenius form is defined by

$$\varepsilon(a) = R \sum_{i=1}^{N} |D_i| n_i \text{Re Tr}(a_i). \hspace{1cm} (14)$$

The fact that these formulas do determine Frobenius algebras is proved here.

**Lemma 2.1.** The equations (12) and (14) determine symmetric Frobenius forms such that $R\beta = 1$.

**Proof.** That (12) determines a symmetric Frobenius form follows from the fact that $\text{Tr}(xy)$ is a non-degenerate symmetric bilinear form on $M_n(C)$.

For (14) there are three separate cases to handle: $M_n(D)$ for $D = \mathbb{R}$, $\mathbb{C}_R$ and $\mathbb{H}_R$. The bilinear form $\text{Re Tr}(xy)$ reduces to $\text{Tr}(xy)$ in the first case.
and this is non-degenerate on $\mathbb{M}_n(\mathbb{R})$. In the $D = \mathbb{C}_R$ case, Re Tr$(xy) = 0$ and Re Tr$(x\dot{y}) = 0$ implies that Tr$(xy) = 0$. So Re Tr$(xy) = 0$ for all $y \in \mathbb{M}_n(\mathbb{C}_R)$ implies that $x = 0$. Thus Re Tr$(xy)$ is a non-degenerate form. Finally, a similar proof works for $D = \mathbb{H}_R$. In all these cases the bilinear form determined by Re Tr is symmetric.

Let $k = \mathbb{C}$. A basis for (11) is given by elementary matrices $\{e_{lm}^j\}_{l,m=1,\ldots,n_i}$ satisfying $\epsilon_{lm}^j_{rs} = \delta_{ir}\delta_{ms}$. Then

$$B = \frac{1}{R} \sum_{i,lm} \frac{1}{n_i} e_{lm}^i \otimes e_{ml}^i;$$

(15)

as can be verified by applying the identity $B^{-1}B = 1$ to the above expression and using equation (12). Let $1 = \oplus_1 1_i$; noticing $\sum_{lm} \epsilon_{lm} e_{lm} = n_i 1_i$, it is straightforward to conclude that $\beta = m(B) = R^{-1} 1$.

Suppose now that $k = \mathbb{R}$ and let $A$ be as in (13). Choose as a basis for the $i$-th component of $A$ either $\{e_{lm}^i\}$, $\{e_{lm}^i, i e_{lm}^i\}$ or $\{e_{lm}^i, i e_{lm}^i, j e_{lm}^i, k e_{lm}^i\}$ according to $D_i = \mathbb{R}$, $\mathbb{C}_R$ or $\mathbb{H}_R$, respectively. The element $B$ associated with (14) will then take the form

$$B = \frac{1}{R} \sum_{i,lm} \sum_{w_i} \frac{1}{|D_i|n_i} w_i e_{lm}^i \otimes \mathbb{R} w_i^\ast e_{ml}^i, \quad w_i = \begin{cases} 1 & (D_i = \mathbb{R}) \\ 1, i & (D_i = \mathbb{C}_R) \\ 1, i, j, k & (D_i = \mathbb{H}_R) \end{cases}$$

(16)

Since the product $w_i w_i^\ast = 1$ for all $i$ then $\sum_{lm,w_i} w_i e_{lm}^i w_i^\ast e_{ml}^i = n_i |D_i| 1_i$. The identity $m(B) = R^{-1} 1$ is therefore satisfied.

The partition function for a surface can now be calculated for these examples. Let $\Sigma_g$ denote an oriented surface of genus $g$. Gluing two triangles together gives the partition function (8) of the disk with four boundary edges labelled with states $a, b, c, d$ which is equal to $\varepsilon(e_a \cdot e_b \cdot e_c \cdot e_d)$. Gluing these boundary edges to make the sphere $\Sigma_0$ by identifying the states $a, d$ and $b, c$ results in the partition function

$$Z(\Sigma_0) = R^2 \varepsilon(e_a \cdot e_b \cdot e_c \cdot e_d) B^{ad} B^{bc} = R \varepsilon(1).$$

(17)

Gluing opposite edges results in the torus

$$Z(\Sigma_1) = R \varepsilon(e_a \cdot e_b \cdot e_c \cdot e_d) B^{ac} B^{bd} = R \varepsilon(z)$$

(18)

with $z = e_a \cdot e_b \cdot e_c \cdot e_d B^{ac} B^{bd}$. The surface $\Sigma_g$ for $g > 0$ can be constructed from a disk with $4g$ boundary edges as presented in figure [4]. This results in the partition function

$$Z(\Sigma_g) = R \varepsilon(z^g)$$

(19)
valid for all \( g \). Although a specific orientation was picked when constructing expression \([19]\) the result is actually independent of orientation. Such a symmetry of the partition function is to be expected as it is easy to show orientation-reversing homeomorphisms exist for closed surfaces. Alternatively, this invariance can be proved directly through the partition function. For example, for the torus the two possible partition functions corresponding to two different orientations are given by expression \([18]\) and \( Z'(\Sigma_1) = R \varepsilon (e_d \cdot e_c \cdot e_b \cdot e_a) B^{ac} B^{bd} \). By relabelling \((d, c, b, a) \rightarrow (a, b, c, d)\) and using the bilinear form symmetry it is established the two invariants are indeed equal.

The classification of FHK state sum models gives an explicit expression for \( Z(\Sigma_g) \). This is based on the following calculations for the partition function in the case of simple algebras. For \( A = \mathcal{M}_n(\mathbb{C}) \), choose as a basis the elementary matrices \( \{ e_{lm} \}_{l,m=1,n} \). Then for a Frobenius form \([12]\) the element \( z \) is given by

\[
 z = R^{-2}n^{-2} \sum_{lm,rs} e_{lm} e_{rs} e_{ml} e_{sr} = R^{-2}n^{-2}1.
\]

This gives the partition function

\[
 Z(\Sigma_g, \mathcal{M}_n(\mathbb{C})) = R^{2-2g}n^{2-2g}, \tag{20}
\]

a result also found in \([3]\). The same conclusion holds for \( \mathcal{M}_n(\mathbb{R}) \), now with \( R \in \mathbb{R} \). For the case of \( \mathcal{M}_n(\mathbb{C}_R) \), the element \( z \) again takes the form \( z = R^{-2}n^{-2}1 \) but it produces a new partition function

\[
 Z(\Sigma_g, \mathcal{M}_n(\mathbb{C}_R)) = 2R^{2-2g}n^{2-2g} \tag{21}
\]

due to the extra factor of \(|\mathbb{C}_R| = 2\) present in the Frobenius form \([14]\). Further details of this calculation are explained in the more general example \([4.4]\).

Finally, for \( \mathcal{M}_n(\mathbb{H}_R) \) a calculation shows that \( z = 4^{-1}R^{-2}n^{-2}1 \).
details can be found in example\textsuperscript{4,5}. The partition function reads

\[ Z(\Sigma_g, M_n(H_R)) = 2^{2-2g} R^{2-2g} n^{2-2g}. \]  

(22)

Given the information gathered above, the most general form of an invariant from a symmetric Frobenius algebra can be stated.

**Theorem 2.2.** Let $A$ be a symmetric special Frobenius algebra over the field $k = \mathbb{C}$ or $\mathbb{R}$, as in theorem 2.1. The topological invariant $Z(\Sigma_g)$ constructed from $A$ and an orientable surface $\Sigma_g$ is

\[ Z(\Sigma_g) = R^{2-2g} \sum_{i=1}^{N} n_i^{2-2g} \]  

(23)

if $k = \mathbb{C}$ or

\[ Z(\Sigma_g) = R^{2-2g} \sum_{i=1}^{N} f(i, g)n_i^{2-2g}, \quad f(i, g) = \begin{cases} 1 & (D_i = \mathbb{R}) \\ 2 & (D_i = \mathbb{C}_R) \\ 2^{2-2g} & (D_i = \mathbb{H}_R) \end{cases} \]  

(24)

if $k = \mathbb{R}$.

Another example of a Frobenius algebra is given by the complex group algebra. Recall an algebra can be built from any finite group $G$ by taking formal linear combinations of the group elements. This algebra, denoted $\mathbb{C}G$, has elements $f = \sum_{h \in G} f(h)h$, $f(h) \in \mathbb{C}$ and product defined pointwise according to

\[ (f \cdot f')(h) = \sum_{l \in G} f(l)f'(l^{-1}h). \]  

(25)

A Frobenius form is $\varepsilon(f) = R|G|f(1)$, where $|G|$ is the order of the group. This form is the unique symmetric special Frobenius form such that $R\beta = 1$. The Peter-Weyl decomposition \textsuperscript{12} gives an isomorphism with a complex matrix algebra satisfying the conditions of theorem 2.1. The general form of the invariant associated with the group algebra is therefore

\[ Z(\Sigma_g) = R^{2-2g} \sum_{i \in I} (\dim i)^{2-2g}, \]  

(26)

where each $i$ labels an irreducible group representation, a result that is given for a Lie group in \textsuperscript{13}. Expression (26) agrees with the results of \textsuperscript{5} when $R = 1$. 


3 Planar and spherical state sum models

3.1 Planar models

A more general algebraic framework can be used for state sum models if a more sophisticated method to define the model is employed. In this generalisation some of the conditions on the data of a naive state sum model are relaxed.

This new framework uses a diagrammatic calculus to determine the combinatorics of the partition function. The algebraic data is again non-degenerate $C$, $B$ and $R$, the generalisation being the replacement of the symmetry requirements (1) and (2) on $C_{abc}$ and $B_{ab}$ with the one equation

$$C_{abc}B_{cd} = B_{de}C_{eab}.$$  

(27)

The matrix $B_{ab}$ is no longer required to be symmetric. Since it is non-degenerate it has an inverse $B_{ab}^{-1}$ defined by (7). Using the inverse, equation (27) can equivalently be written as either of the two equations

$$C_{eab}B_{dc}B_{de} = C_{abc} = C_{bce}B_{ad}B_{ed}.$$  

(28)

Note that if $B$ is symmetric, condition (28) reduces to cyclicity as presented in (1).

It is necessary to understand how to write the building blocks, the maps $C$, $B$ and $B^{-1}$, in diagrammatic form. This is depicted below:

[Diagram showing $C_{abc}$, $B_{ab}$, and $B_{ab}^{-1}$]

The defining relation (7) between $B$ and its inverse is translated into the snake identity

$$B_{ac}B_{cb} = \delta_a^b.$$  

Either side of (27) can be taken as the definition of $C_{ab}^d$, the components of a multiplication map $m$ in equation (5). The diagrammatic counterpart is below. Similar expressions are used to define a vertex with two or three legs pointing upwards.

[Diagram showing $C_{ab}^d$]
Equation (28) can now be easily described – note that keeping track of the index order is essential:

\[
\begin{array}{c}
\begin{array}{cc}
\begin{array}{c}
 a \\
 b \\
 c \\
 a \\
 b \\
 c
\end{array}
& \equiv
\begin{array}{c}
 a \\
 b \\
 c \\
 c \\
 a \\
 b
\end{array}

\end{array}
\end{array}
\rightharpoonup
C_{eab}B_{de}B^{de} = C_{abc} = C_{bce}B_{ad}B^{ed}.
\]

The data \(C_{abc}\) and \(B_{ab}\) together with the vertex amplitude \(R \in k\) determine a new type of state sum model called a diagrammatic state sum model. This is a generalisation of the naive state sum model construction of §2. Starting with a triangulation of a compact subset \(M \subset \mathbb{R}^2\), a state sum model for \(M\) is constructed from the planar graph \(G\) formed by the dual vertices and dual edges. Given fixed states on the boundary edges of \(M\), the graph is evaluated to give the quantum amplitude \(|G| \in k\). Simple examples for the \(M\) consisting of one and two triangles are shown in figure 5.

Note that for the state sum model to be well-defined, the dual edges on the boundary have to be pointing either upwards or downwards. Due to the identities for \(C\) and \(B\), the interior of the graph can be moved by a homeomorphism (fixing the boundary) to any convenient graph in order to construct the required algebraic expression. Thus on figure 5b the left-hand vertex has been perturbed so that both vertices correspond to the multiplication map. Note that the plane \(\mathbb{R}^2\) is considered to have a standard orientation, so that \(M\) is an oriented manifold. The general formula for the partition function of this diagrammatic state sum model is

\[
Z(M) = R^V|G|
\]

with \(V\) the number of interior vertices.

Now the Pachner moves are introduced. A Pachner move preserves the boundary of a triangulation and it is assumed that the corresponding dual
edges do not change in a neighbourhood of the boundary, so remain either upward or downward-pointing.

**Definition 3.1.** A planar state sum model is a diagrammatic state sum model for any compact $M \subset \mathbb{R}^2$ satisfying the Pachner moves.

The planar state sum models depend on the details of the diagram in the neighbourhood of the boundary. Thus the partition function of a disk is no longer symmetric under cyclic permutations of the boundary edges, but has a more refined mapping property that generalises (28). Note that this is why the diagrammatic state sum models escape the conclusion of §2 that $B$ is symmetric for the naive models. These mappings of boundaries and the boundary data are not studied further in this paper. It will be assumed that any mapping of surfaces is the identity mapping in a neighbourhood of the boundary.

The result below is a refinement of theorem 2.1 and its proof develops the properties of the graphical calculus.

**Theorem 3.1.** Non-degenerate diagrammatic state sum model data determine a planar state sum if and only if the multiplication map $m$, the bilinear form $B$ and the distinguished element $\beta = m(B)$ determine on $A$ the structure of a special Frobenius algebra with identity element $1 = R\beta$.

**Proof.** The proof of theorem 2.1 will be followed very closely. The essential difference relies on the translation of Pachner moves into the new diagrammatic model.

Suppose that $(C, B, R)$ is the data for a planar state sum model. As before, define $A$ to be the vector space spanned by $S$. Consider the 2-2 move depicted in figure 3a. Its graphical counterpart is given below.

The power of the graphical calculus lies in the ability to simplify algebraic manipulations. Using first the non-degeneracy of $B$ by contracting each side with $B_{ea}$ and second the definition of the multiplication components, one can
simplify the identity above to obtain
\[
\begin{array}{c}
\begin{array}{c}
\text{d} \\
\text{a b c} \\
\text{d}
\end{array} \\
\begin{array}{c}
\text{a b c} \\
\text{d}
\end{array}
\end{array}
\]

The multiplication map is therefore associative, as in theorem 2.1.

Next, (27) implies
\[
\text{B}^{-1}(e_a \cdot e_b, e_c) = \text{B}^{-1}(e_a, e_b \cdot e_c).
\]

This means that a functional \( \varepsilon : A \to k \) can be defined by \( \varepsilon(x) = B^{-1}(x, 1) \). However, there are no additional symmetry requirements that \( \varepsilon \) must obey.

To simplify the exposition of the 1-3 Pachner move, a 2-2 move was performed on the two left-most triangles of figure 3b. The relation

\[
R \beta = 1.
\]

is obtained. It was simplified using the definition of multiplication components and associativity. The 1-3 Pachner move predicts the expression above must equal \( C_{bc}^a \). Since \( C \) is assumed to be non-degenerate one concludes the highlighted new element, \( R \beta \) with \( \beta = e_a \cdot e_b B^{ab} \), must satisfy \( R \beta = 1 \). Since \( A \) has a unit it is an algebra and is therefore a special Frobenius algebra.

Conversely, given a special Frobenius algebra with multiplication \( m \) and a linear functional \( \varepsilon \), a non-degenerate bilinear form is defined by \( B^{-1} = \varepsilon \circ m \), with property (28). As previously stated, the fact the algebra is unital implies the non-degeneracy of \( C \), while associativity and the relation \( R \beta = 1 \) guarantee invariance under Pachner moves. The diagrammatic state sum model created is therefore planar.

A point that is worth noting from the proof is that in the diagrammatic calculus, a power of \( R \) is associated to every closed region in the diagram. If
the diagram comes from a triangulation, then the closed regions are dual to
the vertices of a triangulation.

It is also worth noting that having $\beta$ proportional to the identity is a
non-trivial restriction on Frobenius algebras. The following arguments show
that this condition implies the algebra must be separable. Note that some
presentations of these state sum models \[6, 14\] assume from the outset the
algebra is of this type. There are a number of equivalent definitions of the
separability condition; the most convenient one for the purpose of this work
is as follows \[15\], where the vector space $A \otimes A$ is a bimodule over $A$ with
the actions $x \triangleright (u \otimes v) = (x \cdot u) \otimes v$ and $(u \otimes v) \triangleleft x = u \otimes (v \cdot x)$.

**Definition 3.2** (Separable algebra). An algebra $A$ is called separable if there
exists $t \in A \otimes A$ such that $x \triangleright t = t \triangleleft x$ for all $x \in A$ and $m(t) = 1 \in A$.

The relevance of this definition to the state sum models is given in the
following lemma.

**Lemma 3.1.** A special Frobenius algebra is a separable algebra.

**Proof.** Define $R \in k$ by $\beta = R^{-1}1$. Using the basis $\{e_a\}$ of the Frobenius
algebra $A$ with Frobenius form $\varepsilon$, define $B_{ab} = \varepsilon(e_a \cdot e_b)$, $B^{ab}B_{bc} = \delta^a_c$, and set
$t = Re_a \otimes e_b B^{ab}$. Then the identity $\varepsilon(y \cdot e_a) e_b B^{ab} = y$ for all $y \in A$ follows.
Using this identity twice, one finds $\varepsilon(y \cdot x \cdot e_a) e_b B^{ab} = y \cdot x = \varepsilon(y \cdot e_a) e_b \cdot x B^{ab}$.
Then, the non-degeneracy of $\varepsilon$ guarantees that $x \triangleright t = t \triangleleft x$ for all $x \in A$.
Also, $m(t) = R \beta = 1$. \[Q.E.D.\]

For a field $k$ of characteristic zero, separability for an algebra is equivalent
to it being both finite dimensional and semisimple \[15, 16\]. Therefore, if
$k = \mathbb{R}$ or $\mathbb{C}$ these Frobenius algebras are easily classified.

Consider the complex algebra $A = M_n(\mathbb{C})$ with Frobenius form
$\varepsilon(a) = \text{Tr}(xa)$ for some fixed invertible element $x \in A$. This determines the non-
degenerate bilinear form $B^{-1}(a,b) = \text{Tr}(xab)$. Let $\{e_{lm}\}_{l,m=1,n}$ be the basis
of elementary matrices such that $(e_{lm})_{rs} = \delta_{lr}\delta_{ms}$. Then $B$ must be given by

$$B = \sum_{lm} e_{lm}x^{-1} \otimes e_{ml} \in A \otimes A. \quad \text{(31)}$$

The defining equation $B^{-1}B = 1$ is satisfied since the cyclicity of the trace
guarantees $\sum_{lm} \text{Tr}(xae_{lm}x^{-1}) e_{ml} = \sum_{lm} \text{Tr}(ae_{lm}) e_{ml} = a$ for all $a \in A$.
Moreover, the distinguished element satisfies $\beta = \text{Tr}(x^{-1}1)$. This identity
follows from noticing that $p(a) = \sum_{lm} e_{lm} a e_{ml} = \text{Tr}(a)1$, where the map $p$ is
proportional to a projector $A \rightarrow A$ with the centre of $A$, $Z(A)$, as its image.
Thus our example will define a planar state sum model if $R^{-1} = \text{Tr}(x^{-1})$.
This particular example will be used to prove the theorem below.

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Theorem 3.2. A planar state sum model over the field $k = \mathbb{C}$ or $\mathbb{R}$ is isomorphic by a change of basis to one in which the algebra is a direct sum of matrix algebras over $\mathbb{C}$ or division rings $\mathbb{R}, \mathbb{C}_R, \mathbb{H}_R$ and the Frobenius form is determined by a fixed invertible element $x = \oplus x_i \in A$. For a complex algebra

$$A = \bigoplus_{i=1}^{N} \mathbb{M}_{n_i}(\mathbb{C}),$$  \quad (32)

the functional takes the form

$$\varepsilon(a) = \sum_{i=1}^{N} \Tr(x_i a_i).$$ \quad (33)

The element $x$ must satisfy the relations $R \Tr(x_i^{-1}) = 1$ for all $i = 1, \ldots, N$. For a real algebra

$$A = \bigoplus_{i=1}^{N} \mathbb{M}_{n_i}(D_i) \text{ with } D_i = \mathbb{R}, \mathbb{C}_R, \mathbb{H}_R$$ \quad (34)

the Frobenius form is given by

$$\varepsilon(a) = \sum_{i=1}^{N} \Re \Tr(x_i a_i).$$ \quad (35)

The element $x$ must satisfy the relations

$$R^{-1} = \begin{cases} \Tr(x_i^{-1}) & (D_i = \mathbb{R}) \\ 2 \Tr(x_i^{-1}) & (D_i = \mathbb{C}_R) \\ 4 \Re \Tr(x_i^{-1}) & (D_i = \mathbb{H}_R) \end{cases}$$ \quad (36)

for all $i = 1, \ldots, N$.

Proof. The classification of Frobenius forms on an algebra [6, 17] shows that any two Frobenius forms $\varepsilon, \tilde{\varepsilon}$ are related by an invertible element $x \in A$ as $\varepsilon(a) = \tilde{\varepsilon}(xa)$. Thus, for the complex case, one can write

$$\varepsilon(a) = \sum_i \Tr(x_i a_i)$$ \quad (37)

using the decomposition $x = \oplus x_i$ and lemma 2.1. Let the unit element be decomposed as $1 = \oplus_i 1_i$; from the example of a simple matrix algebra previously studied, one concludes $R \beta_i = 1_i$ with $1_i$ the unit element in $\mathbb{M}_{n_i}(\mathbb{C})$. Consequently, setting $R \beta = 1$ gives the relation $R \Tr(x_i^{-1}) = 1$. 17
As established in §2, Re Tr is a Frobenius functional for a matrix algebra over a real division ring. Thus, for an algebra (34), one can write

\[ \varepsilon(a) = \sum_i \text{Re} \text{Tr}(x_i a_i). \]  

(38)

It is easy to verify the bilinear form \( B \) associated with this Frobenius functional satisfies

\[ B = \sum_{i,lm,w_i} w_i e^i_{lm} x_i^{-1} \otimes w_i^* e^i_{ml} \]  

using the basis defined in lemma 2.1; one then finds

\[ m(B) = \sum_i \sum_{w_i} w_i \text{Tr}(x_i^{-1}) w_i^* 1_i \]  

(40)

For the identity \( R \beta = 1 \) to hold it is therefore necessary to have \( R^{-1} = \sum_{w_i} w_i \text{Tr}(x_i^{-1}) w_i^* \) for all \( i \). If \( D_i = \mathbb{R} \) or \( \mathbb{C}_\mathbb{R} \), then \( w_i^* \) and \( \text{Tr}(x_i^{-1}) \) commute, which means the expression reduces to \( R^{-1} = \text{Tr}(x_i^{-1}) \) and \( R^{-1} = 2 \text{Tr}(x_i^{-1}) \) respectively. If \( D_i = \mathbb{H}_\mathbb{R} \), the expression reduces to \( R^{-1} = 4 \text{Re} \text{Tr}(x_i^{-1}) \) – the non-real components of the trace are automatically cancelled.

As one might expect, the study of state sum models done in §2 for the disk can be regarded as a special case of theorem 3.2.

**Corollary 3.1.** An FHK state sum model on the disk over the field \( k = \mathbb{C} \) or \( \mathbb{R} \) is a planar state sum model in the conditions of theorem 3.2 where the Frobenius form is symmetric. If the algebra is of the form (32) then \( x = R \sum_i n_i 1_i \); if it is of the form (34) then \( x = R \sum_i |D_i| n_i 1_i \). The data \( A \) and \( R \) therefore uniquely determine the Frobenius form of an FHK model.

**Proof.** This is a special case of theorem 3.2 where \( \varepsilon \) must be symmetric. This means \( x \) must be a central element and can, therefore, be written as \( x = \oplus_i \mu_i 1_i \). The constants \( \mu_i \) must be in \( \mathbb{C} \) if the underlying field is \( \mathbb{C} \) or if \( D_i = \mathbb{C}_\mathbb{R} \); otherwise, they must be real numbers (recall that only real numbers commute with all the quaternions). Each of these constants must then satisfy \( R^{-1} = \mu_i^{-1} n_i \) in the complex case or \( R^{-1} = \mu_i^{-1} |D_i| n_i \) in the real one. In other words \( x = R \oplus_i n_i 1_i \) or \( x = R \oplus_i |D_i| n_i 1_i \), respectively. □

This result implies that the Frobenius form for an FHK state sum model is uniquely determined by the algebra \( A \) and the constant \( R \).
3.2 Spherical models

Suppose that $M \subset \Sigma_0$, with a chosen orientation. Then a state sum model is defined for every orientation-preserving isomorphism of $\Sigma_0 - \{p\}$ to $\mathbb{R}^2$, with $p$ the ‘point at infinity’, which should be chosen not to lie in the dual graph of the triangulation of $M$. Moving $p$ around corresponds to the spherical move [18]

$$
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{sphere_move}
\end{array}
\end{array}
$$

where the blue dot consists of a diagram that is the same on both sides of the equation. This move can be understood as making the arc on the left-hand side larger until it passes the point at infinity on the sphere, when it then re-enters the planar diagram as an arc on the right-hand side.

A sufficient condition that guarantees (41) holds for any matrix representing the spherical move is

$$B_{ca}B^{cb} = B_{ac}B^{bc}. \quad (42)$$

The meaning of (42) is easier to understand in the context of Frobenius algebras.

**Definition 3.3** (Nakayama automorphism). A Frobenius algebra has an automorphism $\sigma : A \to A$ determined uniquely by the relation $\varepsilon(x \cdot y) = \varepsilon(\sigma(y) \cdot x)$ for all $x, y \in A$.

**Lemma 3.2.** Let $A$ be a Frobenius algebra. Then the following are equivalent:

(i) Equation (42)

(ii) $\sigma^2 = \text{id}$

(iii) $B^{-1}$ decomposes into a direct sum of a symmetric and an antisymmetric bilinear form.

**Proof.** Note that equation (42) can be rewritten as $(B^{-1}B^{\text{tr}})^2 = \text{id}$, using matrix notation. The definition of $\sigma$ then implies that $\varepsilon(e_a \cdot e_b) = \varepsilon(\sigma(e_b) \cdot e_a)$ or, equivalently, $B_{ab} = \sigma_b^c B_{ca}$. By contracting both sides with $B^{ad}$ one can conclude that $\sigma_b^d = B_{ab}B^{ad}$ or, as matrices, $\sigma = B^{-1}B^{\text{tr}}$. The equivalence between (i) and (iii) is then immediate.
Suppose $B^{-1}$ is as in (iii). Then the vectors $v$ that lie in the symmetric or antisymmetric subspaces satisfy $B^{-1}v = \pm (B^{-1})^t v$. If $B^t$ is applied to this equation the identity $B^t B^{-1} v = \pm v$ is obtained, which is equivalent to $(B^t B^{-1})^2 = \text{id}$, which implies (i). On the other hand, if (42) is satisfied then $(B^t B^{-1})^2 = \text{id}$. The eigenspaces with eigenvalues $\pm 1$ give the direct sum decomposition of (iii).

For the case of triangulations of $M = \Sigma_0$ (with no boundary) the condition (42) is not required. In these cases, $\bar{z}$ in (41) is proportional to the identity matrix and so equation (41) holds for any special Frobenius algebra. For the rest of this section and in §4, only surfaces without boundary are considered and so the spherical condition is not needed. However the status of the spherical condition is addressed in a more general framework in §5.

**Definition 3.4.** A state sum model for a triangulation of $\Sigma_0$ is said to be spherical if it is determined by the data of a planar state sum model.

The partition function of a sphere can be calculated from any triangulation. The result

$$Z(\Sigma_0) = R \varepsilon(1) = \begin{cases} R \text{Tr}(x) & (k = \mathbb{C}) \\ R \text{Re Tr}(x) & (k = \mathbb{R}) \end{cases}$$

(43)

follows from the classification given by theorem 3.2. For $k = \mathbb{C}$, this result can also be written as $Z(\Sigma_0) = N \text{Tr}(x)/\text{Tr}(x^{-1})$.

### 4 Models with crossings

The diagrammatic method is extended to surfaces by the use of an immersion of the surface into $\mathbb{R}^3$. The dual of a triangulation of an oriented surface $\Sigma$ is a graph on the surface, which can be considered as a ribbon graph by taking the ribbon to be a suitable neighbourhood of the graph (called a regular neighbourhood [19]) in the surface. This ribbon graph is therefore immersed in $\mathbb{R}^3$. The state sum model partition function is evaluated by taking a suitable invariant of this ribbon graph under the equivalence relation of regular homotopy.

These concepts will be described in the case of smooth surfaces and immersions, for which there is a well-developed literature. As is standard in knot theory, the graphs can be described by the diagrams that result from a projection of $\mathbb{R}^3$ to $\mathbb{R}^2$ and the equivalence is a set of Reidemeister-like moves on diagrams. Then it is noted that the diagrams and their moves
Immersions of graphs in $\mathbb{R}^3$ allow for intersections. Regular homotopy thus allows a diagram undercrossing to be transformed into an overcrossing. In fact also make sense as piecewise-linear diagrams, which is more natural for triangulations. We leave it as a challenge to the reader to develop the theory using the piecewise-linear formulation of regular homotopy \[20\] from the beginning.

A smooth immersion is a map $\phi: M \to N$ having a derivative that is injective at every point. Thus an immersion is locally an embedding. A regular homotopy from $\phi_0$ to $\phi_1$ is a family of immersions $\phi_t, t \in [0, 1]$, that defines a smooth map $H(x, t) = \phi_t(x): M \times [0, 1] \to N$.

Surfaces and curves immersed in $\mathbb{R}^3$ are studied in \[21\], from which several key results are used. Let $\phi: \Sigma \to \mathbb{R}^3$ be a surface immersion and $G \subset \Sigma$ the graph dual to a triangulation of $\Sigma$. Then $\gamma = \phi|_G: G \to \mathbb{R}^3$ is an immersion of the graph $G$ and in the generic case this is an embedding, which means that there is an arbitrarily small regular homotopy to an embedding. If there is a regular homotopy $\gamma_t$ between two embedded graphs $\gamma_0$ and $\gamma_1$, then the regular homotopy can be adjusted so that $\gamma_t$ is an embedding except at a finite set of values of $t$, where there is one intersection point. As $t$ varies through one of these values, one segment of an edge of the graph passes through another (see figure 6).

The graph $\gamma$ is described by a diagram obtained by projecting $\mathbb{R}^3$ to $\mathbb{R}^2$. It is assumed that this projection is generic, so that the graph is immersed in $\mathbb{R}^2$ with transverse self-intersections of edges. Since regular homotopy allows the edges to pass through each other, there is no need to record whether the crossings are over- or undercrossings. Diagrams are thus obtained from the usual diagrams of knot theory by setting over- and undercrossings equal.

The graph $\gamma$ has a ribbon structure obtained by taking a suitably small regular neighbourhood $K$ of $\gamma$ in $\Sigma$, thus $\gamma \subset K \subset \Sigma$. The formalism is simplified if the projection to $\mathbb{R}^2$ preserves the ribbon structure of the graph. As is standard in knot theory \[22\], an embedded ribbon graph can be adjusted by a regular homotopy so that the projection of the ribbon to $\mathbb{R}^2$ is an orientation-preserving immersion. This is called ‘blackboard framing’. Then using blackboard-framed knots throughout, it is not necessary to include the ribbon in the planar diagrams.

The state sum model is defined from the diagram in the plane by augmenting the formalism for a spherical state sum with a crossing map $\lambda: A \otimes A \to \mathbb{C}$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6}
\caption{Regular homotopy. Immersions of graphs in $\mathbb{R}^3$ allow for intersections. Regular homotopy thus allows a diagram undercrossing to be transformed into an overcrossing.}
\end{figure}
Figure 7: Torus immersion. A diagrammatic state sum model for the torus created from a triangulation with two triangles.

\[ A \otimes A \text{ where one edge of the graph crosses another as shown.} \]

\[ \lambda^{cd}_{ab} \]

The partition function is calculated using the analogue of the formula (29) for the planar state sum models, with \(|\gamma|\) the invariant of the ribbon graph described above,

\[ Z(M) = R^{|\gamma|}. \tag{44} \]

An example of a planar diagram for the torus triangulated using two triangles is shown in figure 7. The middle diagram shows a projection of the graph that is not blackboard-framed but the final diagram is the result of applying a regular homotopy so that the graph is blackboard-framed.

The ribbon structure is preserved under the equivalence relation of regular homotopy. The usual Reidemeister moves for knots do not preserve the ribbon structure, so one has to use a modified set of moves for ribbon knots, described in [22, 23]. The moves for graphs are described in [24, 25] and the extension from ribbon knots to ribbon graphs is described in [26].

A diagrammatic state sum model that is invariant under these moves is called a spin state sum model – the most general state sum model with crossings considered in this paper. A diagram with \(n\) downward- and \(m\) upward-pointing legs defines a map \(\otimes^n A \to \otimes^mA\), with the convention that \(\otimes^0A = k\). Therefore, diagrams should be read bottom-to-top and the use of explicit indices has been dropped.

**Definition 4.1.** A spin state sum model is a spherical state sum model with data \((C, B, R)\) together with a crossing map \(\lambda\). The additional axioms the map \(\lambda\) obeys are

1. compatibility with \(B\),

\[ = \]

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2. compatibility with $C$,

3. the Reidemeister II move (RII),

4. the Reidemeister III move (RIII),

5. the ribbon condition,

Either side of axiom 5 defines a map, $\varphi : A \rightarrow A$, and the axioms 1 and 3 imply, via the Whitney trick [22], that $\varphi^2 = \text{id}$. Either diagram in axiom 5 is called a curl.

There are two issues to settle: the possible dependence of the state sum model on the triangulation of the surface, and on the immersion $\varphi$. The former is the easiest to resolve: since any spherical state sum model is invariant under Pachner moves the following lemma is automatically verified.

**Lemma 4.1.** The partition function of a spin state sum model is independent of the triangulation of the surface.

An interesting class of examples arises for $G$-graded algebras $A = \bigoplus_{h \in G} A_h$ where $G$ is an abelian group. Crossing maps can then be constructed from bicharacters [27]. A bicharacter $\tilde{\lambda} : G \times G \rightarrow k$ is defined by

$\tilde{\lambda}(h,jl) = \tilde{\lambda}(h,j)\tilde{\lambda}(h,l)$, \hspace{1cm} (45)  \\
$1 = \tilde{\lambda}(h,j)\tilde{\lambda}(j,h)$. \hspace{1cm} (46)

The candidate for a crossing map $\lambda$ is then determined by setting

$\lambda(a_h \otimes b_j) = \tilde{\lambda}(h,j) b_j \otimes a_h \in A_j \otimes A_h$. \hspace{1cm} (47)

With this definition, it is straightforward to conclude properties (45) and (46) of a bicharacter $\tilde{\lambda}$ are in correspondence with the crossing axioms 2 and 3 of definition 4.1. On the other hand, axiom 4 is automatically verified since $\lambda$ is $k$-valued. The remaining conditions, however, impose new constraints on a bicharacter. Write $A_h \perp A_j$ if $\varepsilon(a_h \cdot b_j) = 0$ for all $a_h \in A_h, b_j \in A_j$. 

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Lemma 4.2. A graded Frobenius algebra with a bicharacter \( \tilde{\lambda} \) determines a spin state sum model if and only if

1. For each \( h, j \in G \), either \( A_h \perp A_j \) or \( \tilde{\lambda}(h, l) = \tilde{\lambda}(l, j) \) for all \( l \in G \).

2. The Nakayama automorphism \( \sigma \) obeys \( \sigma^2 = \text{id} \).

Proof. Applying the maps in axiom 1 of definition 4.1 to \( a_h \otimes c_l \otimes b_j \) gives

\[
\tilde{\lambda}(l, j) \varepsilon(a_h \cdot b_j) c_l = \tilde{\lambda}(h, l) \varepsilon(a_h \cdot b_j) c_l,
\]

which is equivalent to condition 1.

The element \( B \) can be written as a sum of linearly independent terms as

\[
B = \sum y_m \otimes z_n,
\]

in which the gradings \( m \) and \( n \) may vary. An equivalent relation to condition 1 is that for each term \( y_m \otimes z_n \) in the sum,

\[
\tilde{\lambda}(n, l) = \tilde{\lambda}(l, m) \quad \text{for all} \quad l \in G.
\]

This can be proved by using an equivalent form of axiom 1 of definition 4.1 given by rotating both diagrams in the expression by \( \pi \). Then applying the maps on both sides of the equation to \( a_l \) gives the identity

\[
\sum \tilde{\lambda}(l, m) y_m \otimes a_l \otimes z_n = \sum \tilde{\lambda}(n, l) y_m \otimes a_l \otimes z_n.
\]

The curl on the right-hand side of axiom 5 is the map

\[
a_l \mapsto \sum \varepsilon(a_l \cdot z_n) \tilde{\lambda}(l, m) y_m.
\]

However, from (49), \( \tilde{\lambda}(l, m) = \tilde{\lambda}(n, l) \) and for the non-zero terms in (51), \( \tilde{\lambda}(l, l) = \tilde{\lambda}(l, n) \). Together these imply \( \tilde{\lambda}(l, m) = \tilde{\lambda}(l, l) \). Hence the curl is

\[
\varphi(a_l) = \tilde{\lambda}(l, l) \sum \varepsilon(z_n \cdot \sigma^{-1}(a_l)) y_m = \tilde{\lambda}(l, l) \sigma^{-1}(a_l).
\]

Since axiom 5 is equivalent to \( \varphi^2 = \text{id} \), and (46) implies \( \tilde{\lambda}(l,l)^2 = 1 \), the axioms of definition 4.1 imply that \( \sigma^2 = \text{id} \). Conversely, \( \sigma^2 = \text{id} \) together with axioms 1 to 4 imply that axiom 5 is satisfied.

The spin state sum models are analysed fully in §4.2.
4.1 Curl-free models

This section discusses a particular class of spin state sum models that do not depend on the spin structure. The data for these examples satisfy one additional axiom, $\varphi = \text{id}$. These models are called curl-free. Diagrammatically, this is

6. the Reidemeister I move (RI), \[
\begin{array}{c}
\text{Reidemeister I move} \\
\end{array}
\]

The main issue is the dependence of the partition function on the immersion. Consider a standard immersion $\varphi_0$ that is an embedding of the closed oriented surface of genus $g$ into $\mathbb{R}^3$. A triangulation of the surface $\Sigma$ can be constructed by identifying the edges of a $4g$-sided polygon, as in figure 4, and dividing it into triangles without introducing any new vertices. Let $S \subset \Sigma$ be the subset obtained by removing a disk neighbourhood of the vertex of the polygon from $\Sigma$. The embedding is such that $S$ projects to $\mathbb{R}^2$ by the immersion shown in figure 5a. The dual graph to the triangulation is shown in the figure 5b, with all of the graph vertices consolidated into one.

**Lemma 4.3.** The partition function of a curl-free state sum model for a closed surface is independent of the immersion $\varphi$.

**Proof.** Consider a ribbon graph $K \subset \Sigma$ and an immersion $\phi: \Sigma \to \mathbb{R}^3$. The immersion of the ribbon graph is moved by regular homotopy to $\psi: K \to$
that is blackboard-framed with respect to the projection $P: \mathbb{R}^3 \to \mathbb{R}^2$, $P(x, y, z) = (x, y)$. Further, a neighbourhood of the consolidated vertex in the diagram can be moved to match a neighbourhood of the vertex in figure 8b. Then each ribbon loop of $K$ can be moved independently, keeping the neighbourhood of the vertex fixed.

According to the Whitney-Graustein theorem \cite{22}, the only invariant of an immersed circle in $\mathbb{R}^2$ under regular homotopy (in $\mathbb{R}^2$) is the Whitney degree, which is the integer that measures the number of windings of the tangent vector to the circle. This regular homotopy extends to a regular homotopy of the ribbon graph in $\mathbb{R}^2$. Then it lifts to a regular homotopy of the ribbon graph $\psi$ in $\mathbb{R}^3$, by keeping the $z$-coordinate constant in the homotopy. Therefore each loop of $K$ is regular-homotopic to the corresponding loop of figure 8b but with a number of curls. The curls can be cancelled using the ribbon condition and the move RI. Thus the partition function is the same as for $\phi_0$.

These results imply the partition function of a curl-free model is indeed a topological invariant. Let $f: \Sigma' \to \Sigma$ be a diffeomorphism. If $\Sigma$ is a triangulated surface and $\phi: \Sigma \to \mathbb{R}^3$ is an immersion, then $f$ induces a triangulation and an immersion for $\Sigma'$ such that their dual graph diagrams in the plane coincide.

Some examples of curl-free models are studied in the rest of this section. First it is shown how the naive state sum models of §2 fit within the new formalism.

**Example 4.1.** An FHK state sum model as defined in §2 is a curl-free state sum model where the choice of crossing is canonical. In other words, the map $\lambda: A \otimes A \to A \otimes A$ takes $a \otimes b \mapsto b \otimes a$.

Next, examples determined by a bicharacter are studied. Axiom $\frac{6}{\phi}$ is $\varphi(a_l) = a_l$; one can therefore conclude that $\sigma$ preserves the $G$-grading and, according to (52), obeys the eigenvector equation $\sigma(a_l) = \tilde{\lambda}(l, l) a_l$.

Explicit examples of matrix algebras that can be equipped with this type of crossing are now presented.

**Example 4.2 (Algebras $A = M_n(k)$, $k = \mathbb{R}$, $\mathbb{C}$).** Let $\varepsilon(a) = R(p - q) \text{Tr}(ua)$ with $u = \text{diag}(p, q)$ the diagonal matrix with the first $p > 0$ diagonal entries equal to $+1$ and the remaining $q = n - p > 0$ entries equal to $-1$, such that $p \neq q$. The algebra $A$ has a natural $\mathbb{Z}_2$-grading $A_0 \bigoplus A_1$. Each matrix splits into a block-diagonal and a block-anti-diagonal part.

\[
\begin{pmatrix}
  a_{p \times p} & b_{p \times q} \\
  c_{q \times p} & d_{q \times q}
\end{pmatrix} = \begin{pmatrix}
  a_{p \times p} & 0 \\
  0 & d_{q \times q}
\end{pmatrix} \oplus \begin{pmatrix}
  0 & b_{p \times q} \\
  c_{q \times p} & 0
\end{pmatrix} \in A_0 \bigoplus A_1. \quad (53)
\]
It is easy to verify there is a unique \( \mathbb{Z}_2 \)-bicharacter \( \tilde{\lambda} \) that can be constructed for this algebra for which \( \lambda \) is a curl-free crossing. Identity (45) implies \( \tilde{\lambda}(0, h) = \lambda(h, 0) = 1 \). Identity (46) implies \( \tilde{\lambda}(1, 1) = \pm 1 \) but the choice \( \tilde{\lambda}(1, 1) = 1 \) is not allowed as \( \varphi = \sigma \neq \text{id} \) would follow. The components of \( \lambda \) are therefore determined by the relation \( \tilde{\lambda}(h, j) = (-1)^{hj} \) with \( h, j = 0, 1 \). The bilinear form can be written as

\[
B = \frac{1}{R(p - q)} \sum_{lm,h} e_{lm}^h u \otimes e_{ml}^h
\]  

(54)

where the label \( h \) identifies whether the elementary matrices belong to \( A_0 \) or \( A_1 \). The standard diagram \( \gamma_g \) to be associated with a closed surface of genus \( g \) (see figure 8b) can be used to write the partition function (44) as

\[
Z(\Sigma_g) = R\varepsilon(\eta^g) \text{ with } \eta = \frac{1}{R^2(p - q)^2} \sum_{lmrs,hj} e_{lm}^h u e_{rs}^j u e_{ml}^h e_{sr}^j.
\]  

(55)

Notice that \( \sigma(a) = uau \). The simplification \( \sum_{lm,h} e_{lm}^h u e_{rs}^j u e_{ml}^h = \text{Tr}(\sigma(e_{rs}^j))1 \) follows. If \( j = 1 \), \( \text{Tr}(\sigma(e_{rs}^j)) \) vanishes since \( \sigma \) preserves the grading and block-anti-diagonal matrices are traceless. If \( j = 0 \) then \( \text{Tr}(\sigma(e_{rs}^j)) = \delta_{rs} \) and consequently \( z = R^{-2}(p - q)^{-2}1 \). Therefore, the partition function reads

\[
Z(\Sigma_g) = R^{2-2g}(p - q)^{2-2g}.
\]  

(56)

This formula does not coincide with the partition function (20) determined for \( \mathbb{M}_n(k) \) seen as an FHK state sum model, but it does reduce to it by setting \( q = 0 \). (For \( q = 0 \) the canonical crossing is the acceptable choice.)

A natural question is whether algebras with symmetric Frobenius forms, and a crossing respecting the conditions of definition 4.1 and the curl-free condition always give rise to an FHK state sum model. This is not, however, the case as it can be seen by the explicit example below.

**Example 4.3** (Algebras \( A = \mathbb{M}_n(\mathbb{C}) \)). As studied in \[27\] \( \mathbb{M}_n(\mathbb{C}) \) can be regarded as a \( \Gamma_n \)-graded algebra where the group is a direct product of two cyclic groups of order \( n \): \( \Gamma_n = \langle a \rangle \times \langle b \rangle \). This means \( \mathbb{M}_n(\mathbb{C}) \) is decomposed into \( n^2 \) components and it is natural to pick as a basis \( n^2 \) matrices that respect this decomposition. Let \( \xi \in \mathbb{C} \) be a primitive \( n \)-th root of unit and define the matrices \( X_a = \text{diag}(\xi^{n-1}, \cdots, \xi, 1) \) and \( Y_b = e_{n1} + \sum_{m=1}^{n-1} e_{mn+1} \), as in \[27\]. Then, \( X_a^i Y_b^j \) generates the \( a^i b^j \) component of the algebra and the expression

\[
\tilde{\lambda}(a^i b^j, a^{i'} b^{j'}) = \xi^{ij' - ij}
\]  

(57)
defines a bicharacter \[27\]. It must be verified the remaining conditions of lemma \[4.2\] and the Reidmeister I move hold. The latter ascertains \(\sigma(X_i^a Y_j^b) = \tilde{\lambda}(a^i b^j, a^i b^j)\). Since \(\tilde{\lambda}(a^i b^j, a^i b^j) = 1\) it follows that \(\sigma = \text{id}\) or, equivalently, that the Frobenius form is symmetric. In other words, \(\varepsilon(y) = Rn \text{Tr}(y)\).

It is necessary to verify condition \[1\] of lemma \[4.2\]. One would first show that \(\varepsilon(X_i^a Y_j^b X_i'^a Y_j'^b) = 0\) unless \(i + i' = j + j' = 0\), a fact that follows from the identity \(X_i^a Y_j^b X_i'^a Y_j'^b = \xi^{-ij'} X_i'^a Y_j'^b\) and the symmetry of \(\varepsilon\). If \(i + i' = j + j' = 0\) then the required identity for the bicharacter would reduce to \(\tilde{\lambda}(h, j) = \tilde{\lambda}(j, h^{-1})\) for all \(h, j\), which is always true.

For \(n = 1\) does \(\lambda\) coincide with the canonical crossing. The invariant created is \(Z(\Sigma_g) = R^{2-2g} n^2\), which differs from expression \[20\].

### 4.2 Spin models

The purpose of this section is to study spin state sum models and show that these are defined on a surface with a spin structure. Our ultimate objective is to introduce a crossing that distinguishes topologically-inequivalent spin structures and several examples of such algebras will be studied.

The usual notion of spin structure is defined for oriented smooth manifolds using the tangent bundle. Each immersed curve \(c\) on the manifold lifts to a curve in the frame bundle \(F\) and the spin structure \(s \in H^1(F, \mathbb{Z}_2)\) assigns to this an element \(s(c) \in \mathbb{Z}_2\). This assignment can be characterised by a skein relation on a vector space generated by curves on the manifold \[28\], a description that does not require the use of the tangent bundle (and so generalises to piecewise-linear manifolds).

On an oriented surface there is an even simpler description \[29\] of a spin structure as a quadratic form on the first homology with \(\mathbb{Z}_2\) coefficients, \(q: H_1(\Sigma, \mathbb{Z}_2) \to \mathbb{Z}_2\). The quadratic form \(q\) is defined by taking an embedded curve \(c\) to represent a cycle and setting \(q(c) = s(c) + 1 \mod 2\). The quadratic form satisfies the relation \(q(x+y) = q(x) + q(y) + x.y\), with \(x.y\) the intersection form for \(\mod 2\) homology, and so is determined by its values on a basis of \(H_1(\Sigma, \mathbb{Z}_2)\).

The immersions of a smooth surface into \(\mathbb{R}^3\) are classified in \[21\], where it is shown that there are \(2^{2g}\) inequivalent regular homotopy equivalence classes. Each immersion \(\phi: \Sigma \to \mathbb{R}^3\) determines an induced spin structure on \(\Sigma\) by pulling-back the unique spin structure on \(\mathbb{R}^3\). The induced spin structure is invariant under a regular homotopy (since the homotopy is differentiable). There are \(2^{2g}\) spin structures on an oriented surface and these classify the equivalence classes of immersions uniquely. This can be seen by explicitly constructing an immersion that corresponds to each spin structure. A spin
structure on $\Sigma$ is determined uniquely by a spin structure on the subset $S \subset \Sigma$ obtained by removing a disk. The surface $S$ can be embedded in $\mathbb{R}^3$ so that the projection to $\mathbb{R}^2$ is an immersion as in figure 8a, or a modification of it by putting a curl in any of the $2g$ ribbon loops. The spin structure is read off from this diagram: $s(c)$ is the Whitney degree mod 2 for the projection of $c$ to $\mathbb{R}^2$. For example, for the embedding $\phi_0$ each circle $c$ in figure 8b has no curls and so $q(c) = 0$. It is worth noting that this explicit construction of $q$ does not require a smooth structure and makes sense also for a piecewise-linear surface.

**Lemma 4.4.** The partition function of a spin state sum model on $\Sigma$ depends on the immersion $\phi: \Sigma \to \mathbb{R}^3$ only via the spin structure induced on $\Sigma$.

**Proof.** The proof is similar to the proof of lemma 4.3, except that the curls can only be cancelled in pairs. Each curve in the graph can be moved to coincide with the curve from $\phi_0$ except that each curve contains a number of curls. These curls can be cancelled pairwise so that each curve has either one or zero curls; this is the data in the induced spin structure. \hfill $\Box$

For example, one diagram for each of the four equivalence classes for the torus are shown in figure 9. The corresponding spin structures have $(q(c_1), q(c_2)) = (0, 0), (1, 0), (0, 1), (1, 1)$ for the two embedded cycles $c_1, c_2$ forming a basis of $H_1(\Sigma, \mathbb{Z}_2)$.

Lemmas 4.1 and 4.4 imply the partition function is an invariant of a surface with spin structure. Let $f: \Sigma' \to \Sigma$ be a diffeomorphism and $\phi: \Sigma \to \mathbb{R}^3$ an immersion inducing a spin structure $s$. Then the immersion $\phi \circ f$ induces the spin structure $f^*s$ on $\Sigma'$. Note that the invariance of the partition function can also be checked directly, without using the Pachner moves, by examining the effect of Dehn twists \cite{30, 31} on the surface.

To calculate examples of spin models, an explicit formula is needed for the partition function that is manifestly an invariant. To establish this non-trivial result (theorem 4.1), the algebraic consequences of the axioms for the spin models are studied.
A straightforward first consequence is that φ as defined in 4.1 is also an isomorphism of the algebra A determined by the data (C, B, R), which is to say, \( \varphi(a \cdot b) = \varphi(a) \cdot \varphi(b) \) for all \( a, b \in A \). The diagrammatic proof of this identity can be found below.\(^2\)

\[ = \quad = \quad = \quad = \]

The next objective is to build the diagrammatic counterpart of expression (19), assigning \( Z(\Sigma_s, s) \) to an orientable surface with spin structure. It is necessary to understand the analogue of the element \( z = e_a \cdot e_b \cdot e_c \cdot e_d \cdot B_{ac}^d B_{bd} \), introduced in equation (18), in the spin model. The difference is the possible introduction of curls in the diagrams.

A useful preliminary is the study of all the possible diagrams one can associate with the cylinder topology. These maps \( A \rightarrow A \) are depicted below and denoted \( p, n_1 \) and \( n_2 \) respectively.

\[ = \quad = \quad = \]

Define two subspaces of \( A \): \( \mathcal{Z}_\lambda(A) \), the set of all elements \( a \in A \) satisfying \( m(b \otimes a) = m \circ \lambda(b \otimes a) \) for all \( b \in A \), and analogously \( \overline{\mathcal{Z}}_\lambda(A) \), the set of all elements \( a \in A \) satisfying \( m(b \otimes a) = m \circ \lambda(\varphi(b) \otimes a) \), for all \( b \in A \).

**Lemma 4.5.** The map \( R.p \) is a projector \( A \rightarrow A \) with image \( \mathcal{Z}_\lambda(A) \). Further, \( p \circ \varphi = p \) and \( n_2 = n_1 \circ \varphi = \varphi \circ n_1 \). The map \( R.n_1 \) is a projector \( A \rightarrow A \) with image \( \overline{\mathcal{Z}}_\lambda(A) \).

---

\(^2\)The existence of the isomorphism \( \varphi \) raises the question of uniqueness in the model. One could ask if compositions of \( \varphi \) and \( B \) or \( \varphi \) and \( C \) would give rise to alternative and valid spin state sum model data. It is, however, a simple exercise to verify that the original data is the only one that manifestly satisfies all the necessary axioms.
Proof. First, one must note that for all \( a \in A \), \( p(a) \in Z_\lambda(A) \).

One can then further conclude that if \( a \in Z_\lambda(A) \) then \( R.p(a) = a \).

This is enough to establish \( R.p \) as a projector onto \( Z_\lambda(A) \). The proof \( p \circ \varphi = p \) is accomplished by direct composition.

To show the identities \( \varphi \circ n_1 = n_2 = n_1 \circ \varphi \) hold one uses the fact \( \varphi \) is an algebra automorphism.
It is now shown that for all $a \in A$ the element $n_1(a)$ belongs to $\mathbb{Z}(A)$.

Finally it is established that if $a \in \mathbb{Z}(A)$ then $R.n_1(a) = a$. Then $R.n_1$ is a projector onto $\mathbb{Z}(A)$.

The spin analogues of $z$ as defined in equation (19) can be now found below. Denoted $\eta_1, \eta_2, \eta_3$ and $\chi$ they are preferred elements of the algebra – the building blocks of the spin partition functions.

It is easy to verify the identity $\eta_1 = \eta_2 = \eta_3$ holds; the notation $\eta$ is used for any of these maps. To see how the result holds note that one of the relations, $\eta_1 = \eta_2$, is trivial – it follows from $p \circ \varphi = p$. The proof for the remaining equation, $\eta_3 = \eta_1$ is depicted below.

The two non-equivalent generalisations of $z$ have the following properties.
Lemma 4.6. The elements $\eta$ and $\chi$ are central and satisfy $\eta^2 = \chi^2$.

Proof. One is able to easily conclude $\eta$ is an element of $\mathcal{Z}(A)$. Given the increasing complexity of the diagrams requiring simplification, lines being transformed have been dashed.

\[
\begin{array}{ccc}
\begin{array}{c}
\text{The first step uses multiplication associativity,}
\end{array} & \begin{array}{c}
\text{and the multiplication and}
\end{array} & \begin{array}{c}
\text{crossing compatibility a number of times.}
\end{array} \\
\begin{array}{c}
\text{The second step uses axioms (2) to (4) and reflects the}
\end{array} & \begin{array}{c}
\text{fact lines can be freely moved past each other as long}
\end{array} & \begin{array}{c}
\text{as their boundaries remains fixed.}
\end{array} \\
\begin{array}{c}
\text{The last step uses the condition } \varphi^2 = \text{id}
\end{array} & \begin{array}{c}
\text{(note the number of times the } \varphi \text{ map appears is even).}
\end{array}
\end{array}
\]

In addition, because $\eta$ is determined by a diagram closed from below (a diagram with no downward-pointing legs), $\eta a = a \eta$ for all $a \in A$.

\[
\begin{array}{ccc}
\begin{array}{c}
\text{In other words, } \eta \in \mathcal{Z}(A). \text{ Establishing the same result for } \chi \text{ is entirely}
\end{array} & \begin{array}{c}
\text{analogous.}
\end{array}
\end{array}
\]

The last and most lengthy part of the proof comes from determining a non-trivial identity: $\chi^2 = \eta^2$. To make the exposition more clear each line of the proof begins with the transformed-to-be diagram line dashed.
The partition function for a surface with spin structure can now be presented. Each handle contributes the element $\chi$ or $\eta$ depending on the spin...
structure; the partition function is thus

\[ Z(\Sigma_g, s) = R \varepsilon(\eta^{g-1} \chi^l). \]  

(58)

However, the properties described in lemma 4.6 mean that all that matters is \( l \mod 2 \) which reflects the fact the partition function is a homeomorphism invariant. The only homeomorphism invariant of a spin structure is the Arf invariant of the quadratic form, \( \text{Arf}(q) = l \mod 2 \in \mathbb{Z}_2 \). It is most convenient to express this invariant of the spin structure as a sign \( P(s) = (-1)^{\text{Arf}(q)} \) called the parity. The spin structure is called even if \( P(s) = 1 \) and odd if \( P(s) = -1 \).

These results are collected together to give the main result for this section.

**Theorem 4.1.** Let \((C, B, R, \lambda)\) be a spin state sum model. Then,

\[ Z(\Sigma_g, s) = \begin{cases} 
R \varepsilon(\eta^g) & \text{(s is even)} \\
R \varepsilon(\chi \eta^{g-1}) & \text{(s is odd)} 
\end{cases} \]  

(59)

Note that \( Z(\Sigma_0) \) is independent of the choice of \( \lambda \), as is to be expected. According to the classification of planar state sum models given in theorem 3.2, \( \eta \in Z(A) \) implies that \( \eta = \oplus_i \eta_i 1_i \) for some constants \( \eta_i \in \mathbb{R}, \mathbb{C} \) or \( \mathbb{C} \). The expression for \( \chi \) will therefore be \( \chi = \oplus_i \text{sgn}_i \eta_i 1_i \), where \( \text{sgn}_i = \pm 1 \), since \( \chi \) is also a central element and \( \eta^2 = \chi^2 \). In particular this means simple matrix algebras can at most attribute different signs to spin structures of different parity.

An algebraic condition that guarantees topologically-inequivalent spin structures cannot be distinguished is \( \eta = \chi \). It is now shown that the canonical crossing map gives rise to spin state sum models that fall into this class.

**Corollary 4.1.** Let \( \lambda: A \otimes A \rightarrow A \otimes A \) be such that \( a \otimes b \mapsto b \otimes a \). Then \( \chi = \eta \), implying the partition function does not depend on the spin structure.

**Proof.** For a crossing of the form above it is easy to conclude \( \varphi = \sigma \) where \( \sigma \) represents the Nakayama automorphism associated with the Frobenius form \( \varepsilon \). The set \( \overline{Z}_\lambda(A) \) coincides in this case with the set of elements \( a \in A \) satisfying \( ab = \sigma(b)a \) for all \( b \in A \). Recall that if an algebra \( A \) satisfies the conditions of theorem 3.1 then \( \sigma \) is an inner automorphism: \( \sigma(a) = xax^{-1} \). Then it is possible to conclude \( n_2(a) = n_1(a) \) for all \( a \in A \) by the argument

\[ n_2(a) = \sigma \circ n_1(a) = xn_1(a)x^{-1} = x\sigma(x^{-1})n_1(a) = n_1(a). \]  

(60)

The diagrammatic form of \( \eta \) and \( \chi \) implies that \( \eta = \chi \) if the maps \( n_1 \) and \( n_2 \) coincide. Then theorem 4.1 implies that the partition function does not distinguish spin parity.

\[ \square \]
Our conclusions so far do not guarantee the existence of crossing maps satisfying $\eta \neq \chi$. The last efforts in this section therefore concentrate on presenting various examples of such algebras.

**Example 4.4 (Algebras $A = M_n(\mathbb{C}_R)$).**

These algebras are naturally $\mathbb{Z}_2$-graded: $M_n(\mathbb{C}_R) = A_0 \oplus A_1$ with $A_0 = M_n(\mathbb{R})$ and $A_1 = iM_n(\mathbb{R})$. There is a unique non-trivial crossing that can be constructed from a $\mathbb{Z}_2$-bicharacter $\tilde{\lambda}$ (see example 1.2). The components of the crossing are determined by the relation $\tilde{\lambda}(h,j) = (-1)^{hj}$. If the Frobenius form is taken to be symmetric then

$$ B = (2Rn)^{-1} \sum_{lm} (e_{lm} \otimes e_{ml} - ie_{lm} \otimes ie_{ml}). $$  \hspace{1cm} (61)

Note the constant $2Rn$ arises from the definition of the Frobenius form, $\varepsilon(a) = 2Rn \text{Re} \text{Tr}(a)$. This information can be used to determine the relation

$$ \eta = (2Rn)^{-2} \left( \sum_{lmrs} e_{lm} e_{rs} e_{ml} e_{sr} \right) \left( \sum_{hj} \hat{\lambda}(h,j) \right). $$ \hspace{1cm} (62)

Further identifying the first sum as the unit element and the second one as the constant 2, one concludes that $\eta = 2(2Rn)^{-2}1$. The element $\chi$ is constructed in an analogous fashion; however, the term $\sum_{hj} \hat{\lambda}(h,j)$ is replaced with $\sum_{hj} \check{\lambda}(h,h) \hat{\lambda}(j,j) = -2$. Hence, $\chi = -2(2Rn)^{-2}1$. The invariant produced distinguishes spin structures of different parity and takes the form

$$ Z(\Sigma_g, s) = P(s) 2^{1-g} R^{2g} n^{2g}. $$  \hspace{1cm} (63)

This expression can be compared with the relation found for FHK state sums models where $Z(\Sigma_g) = 2R^{2g} n^{2g}$. Note that in this canonical case $\eta = \chi = z$ and the term $\sum_{hj} \hat{\lambda}(h,j)$ in equation (62) is equal to 4; hence $z = (Rn)^{-2}1$, as was previously remarked in 2.

The result (63) can be seen as a special case of the one obtained for $\varepsilon(a) = 2R(p - q) \text{Re} \text{Tr}(ua)$, $u = \text{diag}(p,q)$, $p \neq q$ and the same algebra grading:

$$ Z(\Sigma_g, s) = P(s) 2^{1-g} R^{2g} (p - q)^{2g}. $$  \hspace{1cm} (64)

To reach this conclusion note the expression for $B$ is now

$$ B = (2R(p - q))^{-1} \sum_{lm} (e_{lm} u \otimes e_{ml} - ie_{lm} u \otimes ie_{ml}). $$ \hspace{1cm} (65)
By recognising the Nakayama automorphism $\sigma$ associated with $\varepsilon$ satisfies $\sigma(a) = uau$, $\eta$ can be written as

$$\eta = (2R(p - q))^{-2} \left( \sum_{lmrs} e_{lm} \sigma(e_{rs}) e_{ml} e_{sr} \right) \left( \sum_{hj} \check{\lambda}(h, j) \right).$$  \hspace{1cm} (66)

The action of $\sigma$ separates elements of $M_n(R)$ into two types (see example 4.2): $\sigma(a) = a$ if $a$ is block-diagonal and $\sigma(a) = -a$ if $a$ is block-anti-diagonal. However, if $a$ is block-anti-diagonal $\sum_{lm} e_{lm} a e_{ml} = 0$ effectively reducing (66) to (62) with the replacement $n \to p - q$. A similar reasoning holds for $\chi$.

**Example 4.5** (Algebras $A = M_n(\mathbb{H}_\mathbb{R})$).

Consider the group of the quaternions $\hat{K} = \{1, i, j, k, -1, -i, -j, -k\} \ni w$ and define $A_w = wM_n(R)$. Then $A_w = A_{-w}$ and $A_w A_t = A_{wt}$, so that the algebra $A = M_n(\mathbb{H}_\mathbb{R})$ is graded by the quotient group $K = \hat{K}/\{\pm 1\}$, which is isomorphic to the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$. The grading is conveniently written $M_n(\mathbb{H}_\mathbb{R}) = \bigoplus_w A_w$ with $w \in \{1, i, j, k\}$.

The following table encodes the components of all the possible $K$-bicharacters $\check{\lambda}$ that would give rise to a crossing. Each triple $(\alpha, \beta, \gamma)$ with $\alpha, \beta, \gamma \in \{-1, 1\}$ determines one such bicharacter.

| $\lambda(w, t)$ | $1$ | $i$ | $j$ | $k$ |
|-----------------|-----|-----|-----|-----|
| $1$             | $1$ | $1$ | $1$ | $1$ |
| $i$             | $1$ | $\alpha \beta$ | $\alpha$ | $\beta$ |
| $j$             | $1$ | $\alpha \gamma$ | $\gamma$ | $\beta \gamma$ |
| $k$             | $1$ | $\beta$ | $\gamma$ | $\beta \gamma$ |

If the Frobenius form is taken to be symmetric then

$$B = (4Rn)^{-1} \sum_{lm, w} (w e_{lm} \otimes w^* e_{ml}).$$ \hspace{1cm} (67)

Note the constant $4Rn$ arises from the definition of the Frobenius form, $\varepsilon(a) = 4Rn \text{Re} \text{Tr}(a)$. This information can be used to determine the identity

$$\eta = (4Rn)^{-2} \left( \sum_{lmrs} e_{lm} e_{rs} e_{ml} e_{sr} \right) \left( \sum_{wt} \check{\lambda}(w, t) wt w^* t^* \right).$$ \hspace{1cm} (68)

The first sum is simply the identity element $1$. The second, with some algebraic manipulation, can be seen to satisfy

$$\sum_{wt} \check{\lambda}(w, t) wt w^* t^* = \frac{11}{2} - 2\Lambda + \frac{\Lambda^2}{2}.$$ \hspace{1cm} (69)
with \( \Lambda = \alpha + \beta + \gamma \). The element \( \chi \) is constructed in an analogous fashion; however, the term \( \sum_{\tilde w t} \tilde \lambda(w, t) w t w^* t^* \) is replaced with

\[
\sum_{\tilde w t} \tilde \lambda(w, t) w t w^* t^* - (\Lambda - 1) (\Lambda - 3) (\Lambda + 3)
\]

It is easy to verify \( \Lambda \in \{-3, -1, 1, 3\} \) (the canonical crossing corresponds to the choice \( \Lambda = 3 \)). Therefore only the crossings satisfying \( \Lambda = -1 \) distinguish spin structures. The partition functions are

\[
Z(\Sigma_g, s) = \begin{cases} 
4(Rn)^{2-2g} & (\Lambda = -3) \\
(P(s)^{2-g}(Rn)^{2-2g} & (\Lambda = -1) \\
(2Rn)^{2-2g} & (\Lambda = +1 \text{ or } +3)
\end{cases}
\]

The result (71) can be slightly generalised – in a manner identical to that described in example 4.4 – by replacing the symmetric bilinear form with \( \varepsilon(a) = 4R(p - q) \text{Re} \text{Tr}(ua), u = \text{diag}(p, q), p \neq q \). The partition functions read as (71) but with the replacement \( n \rightarrow p - q \).

The final example presents all possible crossings for some low-dimensional commutative algebras. Some of these models have the property that \( \eta \neq \pm \chi \), in contrast to the previous examples.

**Example 4.6** (Algebras \( \mathbb{C} \oplus \cdots \oplus \mathbb{C} \) for \( m = 2, 3, 4 \)).
The algebra \( A = \mathbb{C} \oplus \cdots \oplus \mathbb{C} \) is isomorphic to the group algebra of the cyclic group, \( A \cong \mathbb{C}C_m \), and this isomorphism is useful in presenting the results. The notation \( C_m = \{ e, h, \cdots, h^{m-1} \} \) is used. For \( m = 2 \) there are two possible state sum models: either \( \lambda \) is canonical, in which case the partition function is the \( n = 1 \) case of theorem 2.2 or it is the spin model given by the \( n = 1 \) case of example 4.4.

Two crossings are also compatible with \( \mathbb{C}C_3 \): one, the canonical; the other, giving rise to \( \eta = R^{-2} \left( \frac{1}{2} e + \frac{1}{5} (h + h^2) \right) \) and \( \chi = \frac{R^{-2}}{2} (h + h^2) \) and therefore to a new spin invariant:

\[
Z(\Sigma_g, s) = (1 + P(s) 2^{1-g}) R^{2-2g}.
\]

Defined according to \( \lambda(h^j \otimes h^i) = \lambda^{ij}_{op} h^o \otimes h^p \), the components of the non-trivial crossing are presented in a matrix format: \( \lambda^{ij}_{op} \) is the \( (op) \) entry of a matrix \( \lambda^{ij} \). Note axiom (1) of a crossing implies \( (\lambda^{ij})_{op} = \delta^i_p \delta^j_o \) while axiom
determines $\lambda^i = (\lambda^i)^\text{tr}$. The remaining matrices read

$$\begin{align*}
\lambda^{22} &= \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 1 \\
0 & 1 & -1
\end{pmatrix},
\lambda^{23} &= \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & -1 \\
0 & 1 & 1
\end{pmatrix}.
\end{align*}$$

Finally, the $CC_4$ case is richer in complexity. There are a total of twelve crossings allowed giving rise to the following invariants:

$$Z(\Sigma, s) = \begin{cases}
2^{2-2g} R^{2-2g} & (\eta = \chi = (2R)^{-2}e) \\
4 R^{2-2g} & (\eta = \chi = R^{-2}e) \\
P(s) 2^{2-g} R^{2-2g} & (\eta = -\chi = 2^{-1} R^{-2}e) \\
(2 + P(s) 2^{1-g}) R^{2-2g} & (\eta = \frac{R^2}{4} (3e \pm \hbar^2), \\
\chi = \frac{R^2}{4} (e \pm 3\hbar^2))
\end{cases}$$

These crossings were found by solving all of the constraints using computer algebra. The program used by the authors is available from the arXiv version of this paper as an ancillary file.

5 Categorical generalisations

The axioms for the models are motivated by the definitions of various types of categories. This means that the state sum models have abstract generalisations in a category framework. The notion of a Frobenius algebra in a monoidal category was introduced by Street [32]. In this definition the algebra is an object $A$ in the category $\mathcal{C}$. This object has duals and obeys the axioms of a Frobenius algebra. The additional axiom for a special Frobenius algebra can also be translated into the categorical language. It is convenient (but slightly less general) to assume that all of the objects in the category have duals, i.e. $\mathcal{C}$ is a pivotal category (also called a sovereign category). Thus the construction of [83] can be generalised in a straightforward way to show that a special Frobenius algebra in a pivotal category gives a categorical analogue of a planar state sum model. In this categorical analogue, the evaluation of the dual graph in (29) is determined by composition in the category instead of linear algebra.

The partition function for the corresponding spherical state sum is then

$$Z(\Sigma_0) = R^2 \dim A,$$
using $\dim A$ for the categorical dimension (quantum dimension) of $A$. The spherical symmetry arises because the pivotal subcategory of $\mathcal{C}$ generated by $A, m, 1$ is in fact a spherical category.

An example of this construction is given by starting with an object $V$ in the pivotal category $\mathcal{C}$. Then the object $A = V^* \otimes V$ is the categorical generalisation of the matrices over $V$. There is a natural multiplication map $m$ and a unit that makes $A$ a Frobenius algebra in $\mathcal{C}$. In this case, $\dim A = \dim V \dim V^*$ and $R^{-1} = \dim V$. Then the partition function reduces to

$$Z(\Sigma_0) = R \dim V^* \frac{\dim V^*}{\dim V},$$

(76)

generalising the case of $n \times n$ matrices from [43].

The spherical condition can be accommodated more generally by requiring $\mathcal{C}$ to be a spherical category. Then the spherical condition will hold for all morphisms in the category, which can be viewed as ‘defects’ [14] for the state sum model. For the example $A = V^* \otimes V$, this implies that $Z(\Sigma_0) = 1$.

The axioms of definition 4.1 for the spin state sum models are contained in the axioms for a special Frobenius algebra in a symmetric ribbon category. Thus any special Frobenius algebra in a symmetric ribbon category will determine a spin state sum model. However, we do not have any interesting examples that are more general than the matrix ones given in section 4.2.

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