Inhomogeneous Diophantine approximation of some Hurwitzian numbers

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Abstract

We continue the work of Takao Komatsu, and consider the inhomogeneous approximation constant $L(\theta, \phi)$ for Hurwitzian $\theta$ and $\phi \in \mathbb{Q}(\theta) + \mathbb{Q}$. The current work uses a compactness theorem to relate such inhomogeneous constants to the homogeneous approximation constants. Among the new results are: a characterization of such pairs $\theta, \phi$ for which $L(\theta, \phi) = 0$, consideration of small values of $n^2 L(e^{2\pi s}, \phi)$ for $\phi = (r\theta + m)/n$, and the proof of a conjecture of Komatsu.

1 Introduction

The inhomogeneous approximation constant for a pair of real numbers $\theta, \phi$ (with $\phi \notin \mathbb{Z}\theta + \mathbb{Z}$) is

$$L(\theta, \phi) = \liminf_{|q| \to \infty} \left\{ |q| ||q\theta - \phi|| : q \in \mathbb{Z} \right\},$$
where \( \|x\| \) denotes the distance from the real number \( x \) to the nearest integer. Minkowski proved that when \( \theta \) is irrational, \( L(\theta, \phi) \leq 1/4 \) holds for all \( \phi \). Grace [8] used regular simple continued fractions to construct \( \theta \) with \( L(\theta, 1/2) = 1/4 \). Further historical details on these and related results can be found in Koksma [9]. In the middle of the twentieth century there was substantial work related to these inhomogeneous approximation constants and also to the associated inhomogeneous Markoff values. Reference [6] contains a good overview of this work and has a comprehensive list of references.

In the last decade, interest in these problems was rekindled by the authors of [3, 4, 6], and continued with articles by Christopher Pinner [19] and Takao Komatsu [10, 11, 12, 13]. In particular, Komatsu used several different types of continued fractions to compute the inhomogeneous constants when \( e^{1/s} \) (for positive integer \( s \)) is paired with various \( \phi \) in \( \mathbb{Q}\theta + \mathbb{Q} \). In this article we make use of the “relative rationality” of these pairs \( \theta, \phi \) to show how the technically simpler ideas of Grace [8] and regular simple continued fractions can be used to unify and extend Komatsu’s results.

Perron [18, Section 32] defines an arithmetic progression of order \( m \) to be a polynomial of degree \( m \) with rational coefficients that is a function from \( \mathbb{N} \) to \( \mathbb{N} \). The real number \( \theta \) is a Hurwitzian number of order \( m \) if there exists a finite number of arithmetic progressions \( f_1(x), \ldots, f_R(x) \) of order at most \( m \) (and at least one has order \( m \)) such that

\[
\theta = [b_0; b_1, \ldots, b_n, f_1(1), \ldots, f_R(1), f_1(2), \ldots, f_R(2), \ldots].
\]

We use Perron’s convenient notation

\[
\theta = [b_0; b_1, \ldots, b_n, \left( f_1(i), \ldots, f_R(i) \right)_{i=1}^{\infty}].
\]

Quadratic irrationals are the Hurwitzian numbers of order 0. For a nonzero integer \( k \), \( e^{2/k} \) and \( \tanh(1/k) \) are examples of Hurwitzian numbers of order 1. In 1714 Roger Cotes found the continued fraction expansion of \( e \):

\[
e = [2; 1, 2, 1, 1, 4, 1, 1, 6, \ldots] = [2; \left( 1, 2j, 1 \right)_{j=1}^{\infty}].
\]

Euler (1737) proved this is indeed the continued fraction of \( e \), and also that for integers \( s \geq 2 \),

\[
e^{1/s} = [1; \left( (2j - 1)s - 1, 1, 1 \right)_{j=1}^{\infty}]
\]

and

\[
\tanh(1/s) = [0; \left( (2j - 1)s \right)_{j=1}^{\infty}].
\]
In correspondence with Hermite, Stieljes described the continued fraction of \( e^{2/k} \) for odd \( k \):

\[
e^2 = \left[ 7; \left( 3j - 1, 1, 1, 3j, 12j + 6 \right)_{j=1}^\infty \right],
\]

and for integers \( s \geq 1 \),

\[
e^{2/(2s+1)} = \left[ 1; \left( 3(2s + 1)j + s, 6(2s + 1)(2j + 1), 3(2s + 1)j + 5s + 2, 1, 1 \right)_{j=0}^\infty \right].
\]

For references and insight into the proofs, we refer the reader to \([2, 17, 18]\), with an additional comment on the continued fraction of \( \alpha = \coth(1/s) \).

Since \( \alpha \) is the result of applying a linear fractional transformation with integer coefficients to \( \beta = e^{2/s} \), an algorithm of G. N. Raney \([20]\) (also reported in \([1]\)) can be used to relate the continued fractions of \( \alpha \) and \( \beta \).

Here we restrict to \( \phi \in \mathbb{Q} \theta + \mathbb{Q} \) (where \( \phi \notin \mathbb{Z} \theta + \mathbb{Z} \)). By definition, \( L(\theta, \phi_1) = L(\theta, \phi_2) \) when \( \phi_1 - \phi_2 \in \mathbb{Z} \theta + \mathbb{Z} \), and so it suffices to assume that \( \phi \) is in reduced form:

\[
\phi = \frac{r\theta + m}{n} \quad \text{and} \quad n \geq 2, \quad \gcd(r, m, n) = 1 \quad \text{and} \quad 0 \leq r, m < n.
\]

The integer \( n \) will be called the reduced denominator of \( \phi \).

## 2 Connections with homogeneous approximation

In this section we consider \( \theta \) of the form

\[
\theta = [c_0; c_1, \ldots, c_{n_1}, a_0, c_{n_1+1}, \ldots, c_{n_1+n_2}, a_1, c_{n_1+n_2+1}, \ldots],
\]

where \( \lim_{i \to \infty} a_i = \infty \), \( n_j \geq 0 \), and \( \{c_i\} \) is a bounded sequence.

We use standard results on simple continued fractions that can be found for example in \([5, 15, 18, 21]\). Our principal reference is \([15]\) Chapter 1.

Let \( \theta = [b_0; b_1, b_2, \ldots] \) be the simple continued fraction of the real number \( \theta \). For \( i \geq 0 \), \( P_i = (p_i, q_i) \) is called the \( i \)-th convergent of \( \theta \) if \( q_i > 0 \) and \([b_0; b_1, b_2, \ldots, b_i] = p_i/q_i \) in reduced form. Then \( P_0 = (b_0, 1) \), and using \( P_{-1} = (1, 0) \) we have

\[
P_{i+1} = b_{i+1} P_i + P_{i-1} \quad \text{for all} \quad i \geq 0,
\]

and

\[
q_i \| q_i \theta - p_i \| q_i \theta \| = \mu_i^{-1}, \quad \text{where} \quad \mu_i := [b_{i+1}; b_{i+2}, \ldots] + [0; b_i, \ldots, b_1].
\]
(Refer to Theorem 1 in [15, page 2] and Corollary 3 in [15, page 5].)

For $\theta$ of the form in (4), the subscripts $I_j$ for which $b_{I_j+1} = a_j$ will be referred to as leaping subscripts with associated leapers $L_j = (p_{I_j}, q_{I_j})$. The name is appropriate since from (6) the rational number given by a leaper yields a very efficient rational approximation to $\theta$ as compared with the approximations using earlier convergents. This terminology was used by Komatsu in [14] in a slightly different context.

**Theorem 2.1.** Let $\phi = (r\theta + m)/n$ be in reduced form. If there exists an integer $g$ such that
\[ gP_i \equiv (m, -r) \pmod{n} \] holds for infinitely many convergents of $\theta$ then
\[ n^2 L(\theta, \phi) \leq g^2 \left( \limsup_{i \to \infty} \{ \mu_i : gP_i \equiv (m, -r) \pmod{n} \} \right)^{-1}. \]

Moreover, if (7) holds for infinitely many leapers, then $L(\theta, \phi) = 0$.

**Proof.** Let $\{i_j\}$ be the infinite sequence for which $gP_{i_j} \equiv (m, -r) \pmod{n}$. Then for each $j$ there exist integers $R_j, S_j$ such that $gP_{i_j} = (m + nR_j, nS_j - r)$:
\[ n^2 |(S_j - r/n)(S_j\theta - \phi - R_j)| = |nS_j - r| |(nS_j - r)\theta - (m + R_jn)| \]
\[ = g^2 |q_{i_j}| |q_{i_j}\theta - p_{i_j}|; \]
by (6),
\[ n^2 |(S_j - r/n)(S_j\theta - \phi - R_j)| = g^2 / \mu_{i_j}. \] (8)

Therefore,
\[ n^2 L(\theta, \phi) = n^2 \liminf_{|q| \to \infty} \{ |q| \|q\theta - \phi\| : q \in \mathbb{Z} \} \]
\[ \leq n^2 \liminf_{j \to \infty} \{ |S_j| \|S_j\theta - \phi - R_j\| \} \]
\[ = n^2 \liminf_{j \to \infty} \{ |S_j - r/n| \|S_j\theta - \phi - R_j\| \} \]
\[ = g^2 \liminf_{j \to \infty} \{ 1/\mu_{i_j} \}. \]

Since (7) is a congruence modulo $n$, we may assume $1 \leq g < n$, giving
\[ L(\theta, \phi) \leq \liminf_{j \to \infty} \{ 1/\mu_{i_j} \} \leq \liminf_{j \to \infty} \{ 1/b_{i_j} \}. \]

Therefore, $L(\theta, \phi) = 0$ when there are infinitely many leapers satisfying (7).
Theorem 2.1 was implicit in Grace’s work [8]. We illustrate its usefulness by proving that for any integer \( k \geq 3 \), \( L(e^{2/k}, (e^{2/k} + 1)/2) = 0 \). This was proved by Komatsu for even \( k \) in [11, Theorem 3.1]. From (1) and (2), we note that the sequence of convergents for \( e^{2/k} \) is completely periodic modulo 2. In fact, for odd \( k = 2s + 1 \), the modulo 2 sequence of convergents of \( e^{2/k} \) has period 
\[(1, 1), (s + 1, s), (1, 1), (0, 1), (1, 0), (1, 1), (s, s + 1), (1, 1), (0, 1), (1, 0), \]
where the leapers are congruent to \((1, 1), (s + 1, s), (s, s + 1) \) modulo 2. Since these are all of the congruence classes modulo 2, \( L(e^{2/k}, \phi) = 0 \) for all \( \phi \) whose reduced denominator is 2. On the other hand, for even \( k = 2s \) the modulo 2 period for the convergents of \( \theta = e^{1/s} \) is
\[(1, 1), (s, s + 1), (s + 1, s), (1, 1), (0, 1), (1, 0), \]
where every leaper is congruent to \((1, 1) \) (mod 2). Again \( L(\theta, \phi) = 0 \) for \( \phi = (\theta + 1)/2 \).

Lemma 2.2. Let \( \theta = [b_0; b_1, b_2, \ldots] \) be irrational and \( \phi = (r\theta + m)/n \) be in reduced form. For any nonzero integer \( S \), set
\[
\lambda(S) := \left| S - \frac{r}{n} \right| \| S\theta - \phi \|,
\]
and let \( R \) be the nearest integer to \( S\theta - \phi \). If \( 0 < n^2 \lambda(S) < 1 \) then there exist integers \( i, g \) with \( g \) invertible modulo \( n \) such that either
\[
(m + Rn, Sn - r) = gP_i \quad \text{and} \quad n^2 \lambda(S) = \frac{g^2}{\mu_i} \tag{11}
\]
or
\[
b_{i+1} \neq 1 \quad \text{and} \quad n^2 \lambda(S) \geq g^2(1 - w) \quad \text{for some} \quad 0 \leq w \leq [0; b_{i+1}] \tag{12}
\]
Moreover, if \( n^2 \lambda(S) < 1/2 \), then (11) must hold.

Proof. Define the integers \( M := m + Rn \) and \( N := Sn - r \). Then calculation gives
\[
|N| |N\theta - M| = n^2 \lambda(S).
\]
Since \( 0 < n^2 \lambda(S) < 1 \), then \( N \neq 0 \) and \( M/N \) is a rational that satisfies
\[
\left| \theta - \frac{M}{N} \right| < \frac{1}{N^2}.
\]
By Theorem 10 in [15 page 16], there exist integers \( i, g \) such that either \((M, N) = g\mathcal{P}_i\) or \(b_{i+1} \neq 1\) and \((M, N) = g(d\mathcal{P}_i + \mathcal{P}_{i-1})\) where \( d \) equals 1 or \( b_{i+1} - 1 \). In either case, \( \gcd(g, n) \) must divide both \( M \) and \( N \), and so each of \( r, m \). The fact that \( \phi \) is reduced therefore implies \( g \) is invertible modulo \( n \). In addition, by Corollary 2 in [15 page 11], if \( n^2 \lambda(S) < 1/2 \) then \((M, N) = g\mathcal{P}_i\). Also, if \((M, N) = g\mathcal{P}_i\) then (5) yields \( n^2 \lambda(S) = g^2/\mu_i \), which is (11). It remains to prove \((M, N) = g(d\mathcal{P}_i + \mathcal{P}_{i-1})\) implies (12).

We note that \( \lambda \) is invertible modulo \( n \) if and only if \( d \mathcal{P}_i + \mathcal{P}_{i-1} \) is reduced therefore implies \( g \) is invertible modulo \( n \). In addition, by Corollary 2 in [15 page 11], if \( n^2 \lambda(S) < 1/2 \) then \((M, N) = g\mathcal{P}_i\). Also, if \((M, N) = g\mathcal{P}_i\) then (5) yields \( n^2 \lambda(S) = g^2/\mu_i \), which is (11). It remains to prove \((M, N) = g(d\mathcal{P}_i + \mathcal{P}_{i-1})\) implies (12).

\[
\begin{align*}
\lambda(S) &= |N| |N \theta - M| \\
&= g^2 (q_j \pm q_{j-1}) |(q_j \pm q_{j-1}) \theta - (p_j \pm p_{j-1})| \\
&= g^2 (q_j \pm q_{j-1}) |(q_{j-1} \theta - p_{j-1}) \pm (q_j \theta - p_j)| \\
&= g^2 (q_j \pm q_{j-1}) |\|q_{j-1} \theta\| \mp |q_j \theta\| |
\end{align*}
\]

since the differences \( q_k \theta - p_k \) alternate in sign. Then (6) implies

\[
\lambda(S) = g^2 \left( 1 \pm \frac{q_{j-1}}{q_j} \right) \left( \frac{q_j}{q_{j-1} \mu_{j-1}} \mp \frac{1}{\mu_j} \right). 
\quad (13)
\]

For \( x := [0; b_j, \ldots, b_1] \) and \( y := [0; b_{j+1}, b_{j+2}, \ldots] \),

\[
\mu_{j-1} = \frac{1}{x} + y \quad , \quad \mu_j = \frac{1}{y} + x,
\]

and \( q_{j-1}/q_j = x \) by Theorem 4 in [15 page 6]. Putting these into (13) yields

\[
\lambda(S) = g^2 \left( 1 \pm x \right) \left( \frac{1}{1 + xy} \mp \frac{y}{1 + xy} \right) = g^2 \left( 1 \mp \frac{1 + y}{1 + xy} \right)
\]

where \( 0 \leq x \leq [0; b_j] \) and \( 0 < y < [0; b_{j+1}] \). When the upper sign holds (that is, when \( j = i \)), \( x \geq 0 \) and \( y \leq 1 \) yield \( n^2 \lambda(S) \geq g^2 (1 - y) \) and \( w = y \) satisfies conclusion (12). Analogously, \( w = x \) can be used for the lower sign. \( \square \)

**Theorem 2.3.** Let \( \theta \) be as in (4) and \( \phi = (r\theta + m)/n \) be in reduced form. Then \( L(\theta, \phi) = 0 \) if and only if there exist infinitely many leapers \( L_j \) such that \( g_j L_j \equiv (m, -r) \) (mod \( n \)) for an integer \( g_j \) that is invertible modulo \( n \).
Proof. Let \( \{S_k\} \) be an infinite sequence of nonzero integers such that
\[
L(\theta, \phi) = \lim_{k \to \infty} |S_k| \|S_k \theta - \phi\| .
\]
If \( L(\theta, \phi) = 0 \), then
\[
0 = L(\theta, \phi) = \lim_{k \to \infty} \left| S_k - \frac{r}{n} \right| \|S_k \theta - \phi\| = \lim_{k \to \infty} \lambda(S_k) .
\]
Restricting to \( k \) satisfying \( n^2 \lambda(S_k) < 1/2 \), for each such \( k \) Lemma 2.2 implies there exist \( i_k \) and invertible \( g_k \) modulo \( n \) such that (11) holds. Then
\[
b_{i_k+1} + 2 \geq \mu_i = \frac{g_k^2}{n^2 \lambda(S_k)} \to \infty .
\]
The condition on \( \{c_i\} \) in (4) implies \( i_k \) is a leaping subscript for sufficiently large \( k \).

The converse was proved in Theorem 2.1. \( \square \)

Corollary 2.4. Let \( \theta \) be as in (4) and \( \phi = (r\theta + m)/n \) be in reduced form. If \( L(\theta, \phi) = 0 \) then \( L(\theta, g \phi) = 0 \) for every integer \( g \) that is not a multiple of \( n \). In particular, for any \( n \geq 2 \) there exists \( m/n \) such that \( L(\theta, m/n) = 0 \) if and only if \( L(\theta, m_1/n) = 0 \) for all \( m_1 \in \mathbb{Z}, m_1/n \notin \mathbb{Z} \).

Proof. Let \( g \) be an integer that is not a multiple of \( n \). By Theorem 2.3 \( L(\theta, \phi) = 0 \) implies there exist infinitely many leapers \( \mathcal{L}_{j_k} \) such that \( g_k \mathcal{L}_{j_k} \equiv (m, -r) \pmod{n} \) for some invertible \( g_k \pmod{n} \), and so
\[
g_k g \mathcal{L}_{j_k} \equiv (gm, -gr) \pmod{n} \quad \text{for all } k .
\]
Setting \( d := \gcd(g, n) \) and \( h := g/d \) this implies
\[
g_k h \mathcal{L}_{j_k} \equiv (hm, -hr) \pmod{n/d} \quad \text{for all } k .
\]
Since \( g_k h \) is invertible modulo \( n/d \), from Theorem 2.3 we obtain \( L(\theta, g \phi) = 0 \). \( \square \)

Henceforth, we’ll restrict consideration to a slight generalization of \( c^{2/k} \); namely,
\[
\theta = [a_0; c_1, \ldots, c_{n_1}, a_1, c_{n_1+1}, \ldots, c_{n_1+n_2}, a_2, \ldots] , \quad \text{where } \lim_{i \to \infty} a_i = \infty
\]
and either \( \{c_i\} \) is a finite sequence or \( \limsup_{i \to \infty} c_i = 1 \).

(14)
Theorem 2.5. Let $\theta = [b_0; b_1, b_2, \ldots]$ be as in (14) and $\phi = (r\theta + m)/n$ be in reduced form. If $0 < n^2 L(\theta, \phi) < 1$, then there exist infinitely many non-leaping convergents $P_i \equiv (m, -r) \pmod{n}$, and

$$n^2 L(\theta, \phi) = \left( \limsup_{i \to \infty} \mu_i : P_i \equiv (m, -r) \pmod{n} \right)^{-1}. \quad (15)$$

Proof. From (14), there exists $I$ such that for $i \geq I$,

$$b_{i+1} \neq 1 \iff i \text{ is a leaping subscript}. \quad (16)$$

Let $\{S_j\}$ be an infinite sequence of nonzero integers such that

$$L(\theta, \phi) = \lim_{j \to \infty} \lambda(S_j).$$

From $0 < n^2 L(\theta, \phi) < 1$ it follows that $0 < n^2 \lambda(S_j) < 1$ holds for infinitely many $j$ and Lemma 2.2 can be applied: For each such $S_j$, we obtain a subscript $i = i_j$ such that one of the conclusions of the lemma holds. By (16), if $\lambda(S_j)$ satisfies (12) for sufficiently large $j$, then the subscript $i_j$ must be leaping. If an infinite subsequence $S_j$ were to satisfy (12) with leaping subscript $i_j$ and associated $w_j$, then

$$n^2 \lambda(S_j) \geq g_j^2 (1 - w_j) \quad \text{where} \quad w_j \leq [0; b_{i_j+1}] \to 0,$$

and we would obtain the contradiction

$$n^2 L(\theta, \phi) \geq \lim_{j \to \infty} g_j^2 \geq 1.$$

(This is similar to the argument in [21, p. 116].) Therefore, for sufficiently large $j$, $\lambda(S_j)$ satisfies (13) for some $i = i_j$ that is not leaping — else $L(\theta, \phi)$ would be zero. Since $\limsup_{i \to \infty} c_i = 1$, then $\mu_{i_j} \leq 3$ for all but finitely many $j$, and

$$\lim_{j \to \infty} \frac{g_j^2}{3} \leq \lim_{j \to \infty} \frac{g_j^2}{\mu_{i_j}} = n^2 L(\theta, \phi) < 1,$$

which implies $g_j = 1$ for all sufficiently large $j$. Therefore, $P_{i_j} \equiv (m, -r) \pmod{n}$ for infinitely many non-leaping $i_j$ and also (15) holds.

The hypothesis $L(\theta, \phi) > 0$ in Theorem 2.5 guarantees that at most finitely many leapers are congruent to $(m, -r) \pmod{n}$. It’s worth noting that $n^2 L(\theta, \phi) < 1$ implies the existence of infinitely many convergents $P_i \equiv g (m, -r) \pmod{n}$ with $g = 1$. 

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We return to the earlier question of calculating $L(e^{1/s}, \phi)$ for $\phi$ whose reduced denominator equals 2. Recall the sequence of convergents of $\theta$ is completely periodic modulo 2 with period given in (10). Since $\lim_{j \to \infty} \mu_{b_{i}+k} = 2$ for all $k \not\equiv 0 \pmod{3}$, application of Theorem 2.5 gives $L(e^{1/s}, \phi) = 1/8$ for $\phi = 1/2, e^{1/s}/2$.

3 Komatsu’s Conjecture

In Theorem 3.3 of this section we prove a generalization of the conjecture of T. Komatsu [13, p. 241] that for integers $s \geq 1$, $n^{2} L(e^{1/s}, 1/n) = 0$ or $1/2$ for all $n \geq 2$.

Proposition 3.1. Let $k, n$ be positive integers with $n \geq 2$. For any sequence of integers $\{b_{j}\}$, define a sequence $\{s_{j}\} \subset \mathbb{Z}^{k}$ inductively using any initial values $s_{0}, s_{1} \in \mathbb{Z}^{k}$, and

$$s_{j} \equiv b_{j}s_{j-1} + s_{j-2} \pmod{n} \text{ for all } j \geq 2 . \quad (17)$$

If $\{b_{j}\}$ is periodic modulo $n$, then $\{s_{j}\}$ is periodic. If $\{b_{j}\}$ is completely periodic modulo $n$, then $\{s_{j}\}$ is also completely periodic.

Proof. If $b_{i+1}, \ldots, b_{i+t}$ is a period for $\{b_{j}\}$, consider the following sequence of pairs:

$$(s_{i-1}, s_{i}), (s_{i-1+t}, s_{i+t}), (s_{i-1+2t}, s_{i+2t}), \ldots .$$

This infinite sequence eventually has a repetition modulo $n$. Because $\{s_{j}\}$ satisfies the recurrence (17) and $b_{i+1}, \ldots, b_{i+t}$ is a period for $\{b_{j}\}$, the first repetition in this sequence will identify the beginning of a period for $\{s_{j}\}$. Since $s_{i-1}$ is determined by $s_{i-1} \equiv s_{i+1} - b_{i+1}s_{i} \pmod{n}$, when $\{b_{j}\}$ is completely periodic modulo $n$, $\{s_{j}\}$ must also be completely periodic. \qed

In particular, since the partial quotient sequence of every Hurwitzian number is periodic modulo every integer $n \geq 2$, its sequence of convergents is periodic modulo $n$.

Henceforth, for $\theta$ of the form in (14) we further restrict to $n \geq 2$ for which the partial quotient sequence of $\theta$ is periodic modulo $n$. If $b_{I+1}, \ldots, b_{I+T}$ is a period for the partial quotient sequence such that $P_{I+1}, \ldots, P_{I+T}$ is a period for the convergents, we define $M_{j} := \limsup_{k \to \infty} \mu_{j+kT}$ for all $j = I+1, \ldots, I+T$ and observe that

$$M_{j} = \infty \iff \limsup_{k \to \infty} b_{j+kT+1} = \infty \iff P_{j+kT} \text{ is a leaper for infinitely many } k . \quad (18)$$
Theorem 3.2. Let \( \theta \) have the form given in (14). Let \( n \geq 2 \) be such that the partial quotient sequence of \( \theta \) is periodic modulo \( n \), and \( b_{I+1}, \ldots, b_{I+T} \) and \( P_{I+1}, \ldots, P_{I+T} \) be as set up above. If \( m, r \) are integers with \( \gcd(m, r, n) = 1 \) for which there exists \( i > I \) with \( P_i \equiv (m, -r) \mod n \), we set

\[
M := \max\{ M_j : I + 1 \leq j \leq I + T \text{ and } P_j \equiv (m, -r) \mod n \}.
\]

If \( M \neq 1 \) then \( n^2 L(\theta, (m + r \theta)/n) = M^{-1} \).

Proof. Set \( \phi := (m + r \theta)/n \), and let \( j \) be such that \( 1 \leq j \leq T \), \( P_j \equiv (m, -r) \mod n \), and \( M_j = M \neq 1 \). The observation in (18) combined with Theorem 2.3 gives the conclusion for \( M = \infty \). We may therefore assume \( M \) is finite, and that by (18) at most finitely many \( P_j \equiv (m, -r) \mod n \) are leapers, that in turn gives \( L(\theta, \phi) > 0 \). Since

\[
(p_{j+kT}, q_{j+kT}) = P_{j+kT} \equiv P_j \equiv (m, -r) \mod n \quad \text{for all } k \geq 0,
\]

then

\[
n^2 \lambda(S_k) = \frac{1}{\mu_{j+kT}} \quad \text{for } S_k := (q_{j+kT} + r)/n.
\]

By Theorem 2.1,

\[
0 < n^2 L(\theta, \phi) \leq n^2 \liminf_{k \to \infty} \lambda(S_k) = \liminf_{k \to \infty} \frac{1}{\mu_{j+kT}} = \frac{1}{M} < 1,
\]

and the conclusion follows from Theorem 2.5.

\[
\square
\]

Theorem 3.3. [A generalization of Komatsu’s conjecture] Let \( \theta \) be an irrational whose continued fraction has the form given in (14), and let \( n \geq 2 \) be such that the partial quotient sequence of \( \theta \) is completely periodic modulo \( n \). If each \( n_i \in \{0, 2\} \) then

\[
n^2 L(\theta, \phi) \in \{0, 1/2\} \quad \text{for both } \phi = 1/n, \phi = -\theta/n.
\]

In particular, for every \( k \geq 2 \) and every \( n \geq 2 \)

\[
n^2 L(e^{2/k}, \phi) \in \{0, 1/2\} \quad \text{for both } \phi = 1/n, -e^{2/k}/n.
\]

Proof. The fact that \( n_i \in \{0, 2\} \) implies every \( M_j \) equals \( \infty \) or 2. By Proposition 3.1, the sequence of convergents of \( \theta \) is completely periodic modulo \( n \). If \( T \) is a period length, then

\[
P_{T-1} \equiv P_{-1} = (1, 0) \mod m \quad \text{and} \quad P_{T-2} \equiv P_{-2} = (0, 1) \mod m.
\]

Since \( M \in \{\infty, 2\} \), the conclusion follows from Theorem 3.2.

\[
\square
\]
Theorem 3.4. Let \( k \geq 1, \ n \geq 2 \). If \( \gcd(n, k) \neq 1 \) then

\[
n^2 L(e^{2/k}, 1/n) = n^2 L(e^{2/k}, -e^{2/k}/n) = 1/2.
\]

Proof. By Theorems 2.3 and 3.3 it suffices to prove that no component of any leaper of \( \theta = e^{2/k} \) is divisible by \( n \). Since \( \gcd(n, k) \neq 1 \) we first consider all sequences modulo \( k \).

When \( k = 2s \), the partial quotient sequence is completely periodic modulo \( k \) with period \( 1 \), \( s-1 \), \( 1 \), and the sequence of convergents has period \( (1,1), (s-1, s), (1,0), (1,0) \),

where the leapers are \( (1, \pm 1) \) (mod \( k \)). When \( k = 2s + 1 \), the partial quotient sequence has period \( 1, s, 0, s, 1 \) modulo \( k \), and the sequence of convergents has period

\[
(1,1), (-s,s), (1,1), (0,1), (1,-1), (-s,-s), (1,-1), (0,1), (1,0),
\]

where the leapers are either \( (1, \pm 1) \) or \( (-s, \pm s) \) (mod \( k \)). In each case we have shown that each component of every leaper is relatively prime to \( k \), and therefore cannot be divisible by \( n \). The conclusion follows from Theorems 2.3 and 3.3.

Earlier we proved \( L(e^{1/s}, \phi) = 1/8 \) for each of \( \phi = 1/2, e^{1/s}/2 \), a special case of the last result. The theorem also generalizes [13, Theorem 3], that \( n^2 L(e^{1/s}, 1/n) = 1/2 \) when \( n \) divides \( s \).

4 When is \( L(e^{1/s}, \phi) \) zero?

Theorem 4.1. Let \( s, n \) be positive integers with \( n \geq 2 \), and let \( \mathcal{L}_i = \mathcal{P}_{3i} = (P_i, Q_i) \) be the \( i \)-th leaper of \( e^{1/s} \).

(a) Then \( \{\mathcal{L}_i\} \) is a completely periodic sequence modulo \( n \) with period

\[
\mathcal{L}_0, \ldots, \mathcal{L}_{K-1}, \mathcal{L}_K, \mathcal{L}_{K-1}, \ldots, \mathcal{L}_0, \mathcal{L}_K^*, \ldots, \mathcal{L}_{K-1}^*, \mathcal{L}_K^*, \mathcal{L}_{K-1}^*, \ldots, \mathcal{L}_0^*
\]

where \( K = \lceil n/2 \rceil \) and \( \mathcal{L}_i^* := (P_i, -Q_i) \).

(b) If \( \gcd(n, 2s) = 1 \), then (19) is a minimal period for the leapers of \( e^{1/s} \) modulo \( n \).

(c) For all \( 1 \leq s < n \), the \( i \)-th leaper of \( e^{1/(n-s)} \) is \((-1)^i(Q_i, P_i) \) (mod \( n \)).
Proof. Perron [18, Section 31] proved that for $\theta = [a_0; c_1, a_1, \ldots, a_i, c_1, c_2, a_{i+1}, \ldots]$ the subsequence $P_2, P_5, \ldots, P_{3i+2}, \ldots$ of convergents of $\theta$ satisfies the second-order recurrence

$$P_{3i+2} = (a_i(c_1c_2 + 1) + c_1 + c_2)P_{3i-1} + P_{3i-4}.$$ 

Therefore, the sequence of leapers of

$$e^{1/s} = [1; (2sj - (s+1), 1, 1)]_{j=1}^{\infty}$$

satisfies the recurrence

$$L_{j+1} = A_j L_j + L_{j-1},$$

for $k := 2s$ and $A_j := (2j + 1)k$, a sequence that is completely periodic modulo $n$.

Since $A_K = (2K + 1)k \equiv 0 \pmod{n}$, then $L_{K+1} \equiv L_{K-1} \pmod{n}$.

Also, for all $j \geq 0$,

$$A_{K+j} + A_{K-j} = 2(2K + 1)k \equiv 0 \pmod{n},$$

and an inductive argument using the generating recurrence (20) yields

$$L_{K+j} \equiv L_{K-j} \pmod{n} \text{ for all } j \geq 0. \quad (21)$$

In particular, for $j = K, K + 1$,

$$L_{n-1} \equiv L_0 = (1, 1) = L_0^* \pmod{n}; L_n \equiv L_{-1} = (1, -1) = L_0^* \pmod{n};$$

again using recurrence (20) inductively,

$$L_{n+j} \equiv L_j^* \pmod{n} \text{ for all } j.$$ 

In combination with (21) this implies (19) is a period for the leapers modulo $n$.

Further, if $T$ is a period-length of the leapers, then

$$A_T L_T + L_{T-1} = L_{T+1} \equiv L_1 \equiv A_0 L_0 + L_{T-1} \pmod{n},$$

implying $(0, 0) \equiv (A_T - A_0) L_0 \equiv 2Tk (1, 1) \pmod{n}$. When $\gcd(n, k) = 1$, $T$ must be divisible by $n$. The fact that

$$L_n \equiv L_0^* \not\equiv L_0 \pmod{n}$$

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proves (19) is a minimal period for the leapers of $e^{1/s}$.

It remains to prove (c). For this, we define $\{M_j\}$ to be the sequence $M_j := (Q_j, P_j)$ where $(P_j, Q_j)$ is the $j$-th leaper of $e^{1/s}$. Then $\{M_j\}$ also satisfies the recurrence (20) with initial values $M_{-1} = (-1, 1), M_0 = (1, 1),$ and the sequence $N_j := (-1)^j M_j$ satisfies the recurrence

$$N_{j+1} = -A_j N_j + N_{j-1}.$$ 

Since $(2j + 1) 2(n - s) - (2j + 1) 2s - A_j \equiv (\text{mod } n),$ this is the recurrence for the leapers of $e^{1/(n-s)}$. \hfill \Box

In 1918, D. N. Lehmer [16] investigated the modulo $n$ period of the convergents for certain Hurwitzian numbers. More recently, C. Elsner [7] used generating functions to prove results on the period length of the modulo $n$ sequence of leapers of $e$, and Takao Komatsu [10, 11, 12, 13] found the period length of the modulo $n$ leapers of $e^{1/s}$ always divides $2n$ and it divides $n$ when $n$ is even. Both Elsner and Komatsu applied their results to homogeneous approximation over congruence classes.

**Corollary 4.2.** Let $s \geq 1, n \geq 2$ be integers, and define $\theta := e^{1/s}$. If $L(\theta, (m + r \theta)/n) = 0$, then $L(\theta, (m - r \theta)/n) = 0$.

**Proof.** The conclusion follows from Theorem 2.3 and the form of the period in (19). \hfill \Box

**Corollary 4.3.** Let $n \geq 2$ be a odd integer. Then for all $m \not\equiv 0 \pmod{n}$,

$$L(e^{2/(n+1)}, m/n) = 0 \quad \text{and} \quad L(e^{2/(n-1)}, -m e^{2/(n-1)}/n) = 0.$$

**Proof.** Since $n$ is odd, $s := (n + 1)/2$ is an integer. The first leaper of $e^{1/s}$ can be calculated using recurrence (20) with $k = n + 1$:

$$L_1 = A_0 (1, 1) + (1, -1) \equiv 2(1, 0) \pmod{n},$$

and from Theorem 4.1(c), the first leaper of $e^{2/n-1}$ is $-(0, 2)$. Therefore, Theorem 2.3 implies (22) for $m = 2$, and the conclusion follows from Corollary 2.4. \hfill \Box

**Theorem 4.4.** Let $s$ be a positive integer. If $n_1, n_2$ are relatively prime integers for which $L(e^{1/s}, 1/n_1) = L(e^{1/s}, 1/n_2) = 0$ then $L(e^{1/s}, 1/(n_1 n_2)) = 0$.  

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Proof. Since \( L(e^{1/s}, 1/n_i) = 0 \), the form of the period of the leapers of \( e^{1/s} \) yields even \( 1 \leq j_i = 2r_i \leq 2n_i \) with \( Q_{2r_i} = 0 \) (mod \( n_i \)). Using the Chinese Remainder Theorem, the system \( r \equiv r_i \) (mod \( n_i \)) has a solution \( r \) (mod \( n_1 n_2 \)), and \( Q_{2r} \equiv Q_{2r_i} \equiv 0 \) (mod \( n_i \)) for each \( i \). Therefore, we have found a subscript \( j = 2r \) such that \( Q_j \equiv 0 \) (mod \( n_1 n_2 \)), and \( L(e^{1/s}, 1/n_1 n_2) = 0 \).

**Theorem 4.5.** Let \( s \) be a positive integer and let \( n \geq 3 \) be odd. Then for any reduced \( \phi = (m + r e^{1/s})/n \) it is possible to check whether or not \( L(e^{1/s}, \phi) \) is zero in fewer than \( n/2 \) multiplications modulo \( n \). In fact, if \( n \) has \( t \) distinct prime divisors, the number of operations can be reduced to \( n/2^t \) multiplications modulo \( n \).

Proof. The form of the period in (19) allows one to conclude whether or not a leaper has the form \( g(m, -r) \) (mod \( n \)) within \( n/2 \) applications of the recurrence (20). Theorem 4.4 reduces the question to checking the period modulo each prime power divisor of \( n \).

The algorithm implicit in the proof of Theorem 4.5 can be used to verify that the following values should be added to the list given in [13, p. 241] of all values of \( s \) for which \( L(e^{1/s}, 1/n) = 0 \) (for \( n \leq 49 \)):

| \( n \) | \( s \) (mod \( n \)) |
|--------|------------------|
| 23     | 12               |
| 25     | 13, 23           |
| 29     | 15               |
| 43     | 25               |
| 47     | 11, 17, 33, 43   |
| 49     | 1, 22, 46        |

In particular, notice that (22) ensures all of \( (n, s) = (23, 12), (25, 13), (29, 15) \) must be included in the table.

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