Traversable Wormholes Construction in 2+1 Dimensions

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Abstract: We study traversable Lorentzian wormholes in the three-dimensional low energy string theory by adding some matter source involving a dilaton field. It will be shown that there are two-different types of wormhole solutions such as BTZ and black string wormholes depending on the dilaton backgrounds, respectively. We finally obtain the desirable solutions which confine exotic matter near the throat of wormhole by adjusting NS charge.

Keywords: Traversable Wormhole, String Theory.
1. Introduction

Since Morris and Thorne have verified the realistic possibilities of constructing a traversable wormhole spacetime and traveling through it in the theoretical context of the general relativity [1], many scientists have had great interests in this curious and weird object even though they still wonder if it can be used for a miraculous but wishful transportation of interstellar journey in our world or backward travel to past world.

Topologically, wormhole spacetimes are the same as that of black holes, but a minimal surface called *throat of wormhole* is maintained in time evolution, and then a traveler can pass through it in both directions. To support a throat of wormhole, an extra matter source called *exotic matter* should be added to the Einstein’s equation, which inevitably violates energy conditions including weak(WEC), strong(SEC), and dominant energy conditions (DEC). It is believed that the fact that the energy density of every kinds of matter should be non-negative everywhere is restricted to the classical system. As for *exotic matter*, it might let us give up all powerful theorems such as the singularity theorem and the positive mass theorem that require some types of energy conditions in the classical theory of gravity. Visser et al. showed that traversable wormholes necessarily require a violation of the averaged null energy condition (ANEC) and can be supported by arbitrarily small amount of *exotic matter* [2]. And many authors intensively have studied on various aspects of the energy condition of traversable wormholes in Refs. [3, 4, 5, 6].

On the other hand, the string theory unlike the general relativity is described by the gravity coupled to dilaton field and additional gauge fields carrying Neve Schwarz(NS), Ramond(R) charges, and so on. Especially in 2+1 dimensions, the geometric solutions are well-known Bañados-Teitelboim-Zanelli(BTZ) black hole [7] for the constant dilaton field, and the other is the black string solution for the logarithmic dilaton which is dual
to the constant dilaton \[8\]. Now, it will be interesting to study a traversable wormhole in the the string theory, following the conventional recipe by introducing an additional exotic matter source to support a throat of wormhole. However, in string theory compared to the conventional Einstein theory, the consistency of the equation of motion makes the additional matter source extend non-trivially along with the dilaton field, which will be shown in later. So we shall assume that the total action can be written in the form of the string theory with an additional matter action as

\[
S_{total} = S_{string}(g, \phi, B_{\mu\nu}) + S_{M}(f, g, \phi),
\]

(1.1)

where \( g \) is a metric, \( \phi \) is a dilaton field, \( B_{\mu\nu} \) is an antisymmetric tensor field, and \( f \) is a matter field. The additional matter part \( S_{M} \) contributes to each equation of motion for \( g_{\mu\nu} \) and \( \phi \) to form a wormhole, which determines the desirable wormhole solutions depending on some matter distributions. Note that the traversable wormholes in three-dimensional Einstein gravity with a cosmological constant are intensively studied in Ref. \[9\].

In this paper, some general conditions of traversable wormholes are presented in Sec. 2. In Sec. 3, we shall obtain traversable wormhole solutions in the three-dimensional low energy string action by adding the additional matter term involving dilaton field. The various solutions of traversable wormholes for two-different types of dilaton backgrounds are presented and the exotic behaviors of the corresponding matter distributions are shown in Sec. 4. Finally, in Sec. 5 some discussions and comments on our results are given.

2. Preliminary: Traversable Wormhole

In this section, we shall briefly discuss some conditions for forming a traversable wormhole and expressions of energy-momentum tensors in the proper reference frame. A traversable wormhole can be constructed by assuming a metric ansatz of spherically symmetric form as

\[
(ds)^2 = -e^{2\Phi(r)}dt^2 + \frac{d^2r}{1 - b(r)/r} + r^2d^2\varphi,
\]

(2.1)

where \( \Phi(r) \) is a redshift function and \( b(r) \) is a shape function of a traversable wormhole. One of the wormhole conditions \[1\] restricts the shape function \( b(r) \) in the metric (2.1) to satisfy a flaring-out condition as

\[
[r b'(r) - b(r)]|_{r_{throat}} < 0,
\]

(2.2)

where a prime denotes a differentiation with respect to \( r \), and a throat of wormhole is defined as

\[
[1 - b(r)/r]|_{r_{throat}} = 0.
\]

(2.3)

Note that the restriction of the redshift function contains realistic concepts of the traversable wormhole conditions for constructing wormholes and traveling through them. Since there exists no horizon in the wormhole geometry, \( \Phi(r) \) should be finite everywhere, and \( |\Phi(r)| < 1 \) and \( |\Phi'(r)| \lesssim [\text{earth gravity}] \) for a comfortable journey through the wormhole.
In our model, the vacuum solutions are well-known black hole solutions called BTZ and black string solutions. The relevant matter action denoted by \( S_M \) should be added in order for creating a traversable wormhole. To simplify subsequent calculations and physical interpretations, we switch the coordinates to set of new orthonormal basis called proper reference frame, which describes observers who always remain at rest in the coordinate system satisfying

\[
(ds)^2 = g_{\mu\nu} dx^\mu dx^\nu = \eta_{\hat{\alpha}\hat{\beta}} dx^{\hat{\alpha}} dx^{\hat{\beta}}
\]

(2.4)

with

\[
e^t = e^{-\Phi(r)\hat{e}_t}, \quad e^r = \left(1 - \frac{b(r)}{r}\right)^{1/2} e^r, \quad e^{\hat{\phi}} = r^{-1} e^{\hat{\phi}},
\]

(2.5)

where \( \text{diag}(\eta_{\hat{\alpha}\hat{\beta}}) = (-1,1,1) \). In this frame, we define the energy-momentum tensor \( T^M_{\mu\nu} \) as

\[
T^M_{tt} = \rho(r), \quad T^M_{rr} = -\tau(r), \quad T^M_{\hat{\phi}\hat{\phi}} = p(r),
\]

(2.6)

where \( \rho \) is the density of mass-energy, \( \tau \) is the tension for the radial direction, and \( p \) is the pressure for the angular direction. Note that these energy-momentum tensors are exactly determined by the metric solutions satisfying the appropriate wormhole conditions as discussed above.

3. Construction of Stringy Traversable Wormhole

Although the metric is coupled to dilaton field and gauge field in the string theory, it seems to be difficult to construct traversable wormhole without considering an additional (exotic) matter, which means that those fields do not play any role in constructing the traversable wormhole. So we will consider additional matter by adding matter action to the original string action, then the total action is given as

\[
S = \frac{1}{2\pi\ell} \int d^3x \sqrt{-g} e^{-2\phi} \left[ R + 4(\nabla\phi)^2 + \frac{2}{\ell^2} - \frac{1}{12} H^2 \right] + S_M(g_{\mu\nu}, f, \phi),
\]

(3.1)

where \( \phi \) is a dilaton and \( H \) is a NS field strength with \( H = dB \). Note that the stringy action in Eq. (3.1) has two dual invariant solutions: one is the BTZ black hole for the constant dilaton, and the other is the black string solution for the dual solution of the constant dilaton (the logarithmic dilaton) when there exists no additional matter fields, \( S_M = 0 \).

Varying (3.1) with respect to metric, dilaton, and antisymmetric tensor fields yields the equations of motion

\[
G_{\mu\nu} + 2\nabla_\mu \nabla_\nu \phi - \frac{1}{4} H_{\alpha\beta\gamma} H^{\alpha\beta\gamma} - \frac{1}{2} g_{\mu\nu} \left[ 4\Box \phi - 4(\nabla\phi)^2 + \frac{4}{\ell^2} - \frac{1}{12} H^2 \right] = T^M_{\mu\nu},
\]

(3.2)

\[
R - 4(\nabla\phi)^2 + 4\Box \phi + \frac{4}{\ell^2} - \frac{1}{12} H^2 = F^M,
\]

(3.3)

\[
\nabla_\mu(e^{-2\phi} H^{\mu\nu\rho}) = 0,
\]

(3.4)

respectively, where \( T^M_{\mu\nu} \sim \delta S_M/\delta g^{\mu\nu} \) and \( F^M \sim \delta S_M/\delta \phi \) are energy-momentum tensor and dilaton scalar source from \( S_M \). The NS-field equation (3.4) is exactly solved as \( H^{\mu\nu\rho} = \)
\(Q e^{2\phi} \varepsilon^{\mu\nu\rho}\) where \(Q\) is a NS charge. Therefore, the equations of motion (3.2) and (3.3) are rewritten in the new coordinate system (2.5) as

\[G_{\hat{t}\hat{t}} + 2\nabla_{\hat{t}}\nabla_{\hat{t}}\phi - \frac{1}{2} g_{\hat{t}\hat{t}} \left( 4 \Box \phi - 4 (\nabla \phi)^2 + \frac{4}{\ell^2} - \frac{1}{2} Q^2 e^{4\phi} \right) = \rho,\]  

(3.5)

\[G_{\hat{r}\hat{r}} + 2\nabla_{\hat{r}}\nabla_{\hat{r}}\phi - \frac{1}{2} g_{\hat{r}\hat{r}} \left( 4 \Box \phi - 4 (\nabla \phi)^2 + \frac{4}{\ell^2} - \frac{1}{2} Q^2 e^{4\phi} \right) = -\tau,\]  

(3.6)

\[G_{\hat{\phi}\hat{\phi}} + 2\nabla_{\hat{\phi}}\nabla_{\hat{\phi}}\phi - \frac{1}{2} g_{\hat{\phi}\hat{\phi}} \left( 4 \Box \phi - 4 (\nabla \phi)^2 + \frac{4}{\ell^2} - \frac{1}{2} Q^2 e^{4\phi} \right) = p,\]  

(3.7)

\[R - 4(\nabla \phi)^2 + 4\Box \phi + \frac{4}{\ell^2} + \frac{1}{2} Q^2 e^{4\phi} = F^M,\]  

(3.8)

where the non-vanishing components of the Einstein tensor are calculated as

\[G_{\hat{t}\hat{t}} = \frac{b' - b}{2r^3},\]  

(3.9)

\[G_{\hat{r}\hat{r}} = \frac{1 - b/r}{r} \Phi',\]  

(3.10)

\[G_{\hat{\phi}\hat{\phi}} = \left( 1 - \frac{b}{r} \right) \left[ \Phi'' + \Phi'^2 + \frac{b - br'}{2r(r - b)} \Phi' \right].\]  

(3.11)

Since Eq. (3.9) is independent of the redshift function \(\Phi(r)\), it will determine a shape function \(b(r)\) easily. However, the other components in the Einstein tensors are associated with the redshift function \(\Phi(r)\) that should be consistently solved to satisfy some appropriate physical conditions as discussed above. In three dimensions, since the \((\hat{r}\hat{r})\)-component (3.10) only depends upon the differentiation of the redshift function \(\Phi(r)\), the wormhole conditions that \(\Phi(r)\) should be regular everywhere and sufficiently small for a comfortable journey determine the vanishing or constant redshift solution, while this is somewhat different from that in \(d \geq 4\) dimensions. Mathematically, partial integration of Eqs. (3.6) and (3.10) tells us that the integration of \(\Phi'(r)\) diverges at the throat of wormholes since it includes a term of \(\ln(r - b)\), and it violates a finiteness of the redshift function \(\Phi(r)\). Furthermore, since we expect \(|\Phi'(r)| \ll 1\) for no tidal forces to human body to travel comfortably, the regular solution of \(\Phi(r)\) everywhere is only possible when \(\Phi(r) = \Phi_0 = \text{constant}\).

On the other hand, if we consider the matter action \(S_M(f, g)\) independent of the dilaton field instead of choosing \(S_M(f, g, \phi)\) in Eq. (3.1), the theory has no wormhole solution due to the inconsistency of the equations of motion as far as we are interested in the BTZ wormhole. More precisely, the equations of motion is given by using the solution for the NS-field,

\[R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \phi + \frac{1}{2} Q^2 e^{4\phi} g_{\mu\nu} = T^M_{\mu\nu},\]  

(3.12)

\[R - 4(\nabla \phi)^2 + 4\Box \phi + \frac{4}{\ell^2} + \frac{1}{2} Q^2 e^{4\phi} = 0.\]  

(3.13)

At first sight, the exotic matter affects the geometry in Eq. (3.12), however, as seen from Eq. (3.13), the curvature scalar is constant for the constant dilaton background even in the
presence of the exotic matter, whereas it is not generally constant for the assumed wormhole metric, Eq. (2.1). If the (non-rotating) BTZ solution is connected with the wormhole solution in the appropriate finite region, then one should consider the vanishing dilaton background. Therefore, the consistent formulation for traversable wormholes requires some modification on the additional matter action.

4. Specific Traversable Wormhole Solutions

To study a traversable wormhole alters the direction of solving field equations from the usual way of gravitational system in that the matter fields are determined by an appropriate wormhole metric ansatz. Once we construct functions of metric $b(r)$ and $\Phi(r)$ so as to shape the wormhole by the physical speculations, then we naturally obtain the corresponding exotic or normal matter distributions. Here, we shall exhibit some specific wormhole solutions of some physical interests.

4.1 Constant dilaton background (BTZ Wormhole)

A simple solution in the constant dilaton field $\phi = 0$ with vanishing redshift function following the argument in Sec. 2,

$$b(r) = b_0, \quad \Phi(r) = 0,$$

(4.1)

determines the distributions of the additional matter source and the dilaton scalar source as

$$\rho = -\frac{b_0}{2r^3} - \frac{1}{4}Q^2 + \frac{2}{\ell^2},$$

$$\tau = -p = \frac{2}{\ell^2} - \frac{1}{4}Q^2,$$

$$F_M = \frac{4}{\ell^2} - \frac{b_0}{r^3} + \frac{1}{2}Q^2.$$  

(4.2)

The specific dimensionless parameter $\zeta$ defined as $\zeta \equiv (\tau - \rho)/|\rho|$ characterizes how the exotic or normal matters are distributed, and the exoticity at or near throat of the wormhole requires to be non-negative, $\zeta > 0$. From Eq. (4.2), the exoticity reads $|\rho|\zeta = b_0/2r^3 > 0$ everywhere since $r \geq r_{\text{throat}} > 0$. The exoticity with respect to the radial coordinate $r$ is shown in Fig. and it converges to zero at infinity, which is somewhat tricky to thread an another spacetime with the wormhole in the finite region by imposing the appropriate vacuum boundary condition.

On the other hand, another class of solution of the shape function $b(r)$ induced by the NS charge $Q$ is obtained as

$$b(r) = b_0 + \frac{1}{4}Q^2r^3, \quad \Phi(r) = 0,$$

(4.3)

by choosing the additional matter fields and the dilaton scalar source as

$$\rho = \frac{2}{\ell^2} - \frac{b_0}{2r^3}.$$

(4.4)
Figure 1: The exoticity $|\rho|\zeta$ for $b(r) = b_0$ for $b_0 = 1$. The exotic matter is distributed everywhere, and the asymptotic behavior shows that it goes to zero as $r$ increases.

$$\tau = -p = \frac{2}{\ell^2} - \frac{1}{4}Q^2,$$

$$F^M = \frac{4}{\ell^2} - \frac{b_0}{r^3} + Q^2.$$  \hspace{1cm} (4.4)

The flaring-out condition (2.2) should be satisfied for a traversable wormhole, and it tells us a simple restriction between $b_0$ and $Q$ at the throat of the wormhole $r_{\text{throat}}$ as $b_0 > Q^2r_{\text{throat}}^3/2$. Note that it is easy to show that this flaring-out condition is equivalent to the positiveness of the exoticity, $\rho|\zeta|_{r_{\text{throat}}} = (\tau - \rho)_{r_{\text{throat}}} > 0$. Another condition of the throat $r_{\text{throat}} > 0$ gives a restriction of $b_0$ such as $0 < b_0 < 4/(3\sqrt{3}Q)$. Together with the flaring-out condition and the positiveness of the throat, we obtain a condition between $b_0$ and $Q$ as

$$\frac{1}{2}Q^2r_{\text{throat}}^3 < b_0 < \frac{4}{3\sqrt{3}Q}.$$  \hspace{1cm} (4.5)

The exotic behavior of matter source can be shown by investigating the exoticity $|\rho|\zeta = \tau - \rho = (b_0 - Q^2r^3/2)/2r^3$. Obviously, there exists a specific region where the exoticity $\zeta$ is positive in the wormhole spacetimes, and it means that the exotic matter can be confined in the finite region near the throat by adding the NS charge in the shape function $b(r)$. The behavior of the exoticity $\zeta$ is shown in Fig. 2, and $\zeta$ converges to a negative finite value $-Q^2/4$ as $r$ goes to infinity unlike the case of $b(r) = b_0$. Therefore, the latter case is more desirable in that the BTZ metric or anti de-Sitter (AdS) metric can be patched in the finite region.
The exoticity $|\rho|\zeta$ for $b(r) = b_0 + \frac{1}{4}Q^2r^3$ for $b_0 = Q = 1$. The distribution of exotic matter is restricted to the finite region in the vicinity of the throat and other region is filled with the normal matters. It results from the NS charge contribution to the shape function $b(r)$.

4.2 Logarithmic dilaton background (Black String Wormhole)

The background dilaton $\phi = -\ln(r/\ell)$ is a dual solution to the $\phi = 0$ solution, and those two solutions are connected by the specific symmetry called duality [8] given by

$$
\begin{align*}
\bar{g}_{xx} &= 1, & \bar{g}_{xa} &= \frac{B_{xa}}{g_{xx}}, & \bar{g}_{ab} &= g_{ab} - \frac{1}{g_{xx}}(g_{xa}g_{xb} - B_{xa}B_{xb}), \\
\bar{B}_{xa} &= \frac{g_{xa}}{g_{xx}}, & \bar{B}_{ab} &= B_{ab} - \frac{2}{g_{xx}}g_{x[a}B_{b]x}, & \bar{\phi} &= \phi - \frac{1}{2}\ln g_{xx}, \quad (4.6)
\end{align*}
$$

where $a$ and $b$ span all directions except the compact direction $x$. Although the original theory have a duality as a fundamental symmetry, the same duality of the additional matter action is not guaranteed. However, two different dilaton backgrounds show the different asymptotic behaviors of vacuum solutions for $S_M = 0$. More precisely, the BTZ black hole($\phi = 0$) shows an asymptotic anti-de Sitter (AdS) behavior as $r$ goes infinity while the black string($\phi = -\ln(r/\ell)$) has an asymptotic flat behavior. It is expected to expose the different types of the exotic matter distribution in the wormhole spacetimes. Therefore, it will be interesting to study wormhole solutions with this different dilaton solution regardless of preservation of duality symmetry.

Then, from the logarithmic dilaton $\phi = -\ln(r/\ell)$ and trivial redshift solution $\Phi(r) = 0$, the additional matter source and the dilaton scalar source are represented by solving
equations of motion (3.2) and (3.3) in the form of

\[ \rho = \frac{3}{2r^3}(b' r - b) - \frac{2}{r^2} \left( 1 - \frac{b}{r} \right) + \frac{2}{\ell^2} - \frac{Q^2 \ell^4}{4r^4}, \quad (4.7) \]

\[ \tau = -\frac{4}{r^2} \left( 1 - \frac{b_0}{r} \right) + \frac{2}{\ell^2} - \frac{Q^2 \ell^4}{4r^4}, \quad (4.8) \]

\[ p = -\frac{1}{r^2} (b' r - b) - \frac{2}{\ell^2} + \frac{Q^2 \ell^4}{4r^4}, \quad (4.9) \]

\[ F_M = \frac{3}{r^3} (b' r - b) + \frac{4}{r^2} + \frac{Q^2 \ell^4}{2r^4} - \frac{4}{r^3} (r - b). \quad (4.10) \]

At this stage, in order to obtain the shape of a wormhole, one can simply choose a shape function as \( b(r) = b_0 \) following the flaring-out condition (2.2). Then Eqs. (4.7), (4.8), (4.9), and (4.10) are reduced to

\[ \rho = \frac{b_0}{2r^3} - \frac{2}{r^2} + \frac{2}{\ell^2} - \frac{Q^2 \ell^4}{4r^4}, \quad (4.11) \]

\[ \tau = -\frac{4}{r^2} \left( 1 - \frac{b_0}{r} \right) + \frac{2}{\ell^2} - \frac{Q^2 \ell^4}{4r^4}, \quad (4.12) \]

\[ p = \frac{b_0}{r^3} - \frac{2}{\ell^2} + \frac{Q^2 \ell^4}{4r^4}, \quad (4.13) \]

\[ F_M = \frac{2}{\ell^2} + \frac{4}{r^2} + \frac{Q^2 \ell^4}{2r^4} - \frac{4}{r^3} (r - b). \quad (4.14) \]

Note that \(|\rho| = \tau - \rho = (7b_0 - 4r)/2r^3\) from Eqs. (4.11) and (4.12), the exoticity \( \zeta \) is negative within \( b_0 \leq r < 7b_0/4 \). This means that the matter distribution is exotic near the throat \( b_0 \) but normal far from the throat, which is drastically different from the previous case of \( \phi = 0 \) and \( b(r) = b_0 \). The behavior of the exoticity in \( r \) is shown in Fig. 3 especially for \( b_0 = 1, Q = 1, \) and \( \ell = 1, \) and note that the \( \zeta \) converges to zero as \( r \) goes to infinity.

On the other hand, another solution including NS charge without spoiling the flaring-out condition can be chosen as

\[ \Phi(r) = 0, \quad b(r) = b_0 - \frac{Q^2 \ell^4}{4r}, \quad (4.15) \]

which gives the additional matter source and the dilaton scalar source as

\[ \rho = \frac{b_0}{2r^3} - \frac{2}{r^2} + \frac{2}{\ell^2}, \]

\[ \tau = -\frac{4}{r^2} \left( 1 - \frac{b_0}{r} \right) + \frac{2}{\ell^2} - \frac{Q^2 \ell^4}{2r^4}, \]

\[ p = \frac{b_0}{r^3} - \frac{2}{\ell^2} - \frac{Q^2 \ell^4}{4r^4}, \]

\[ F_M = \frac{2}{\ell^2} + \frac{4}{r^2} + \frac{b_0}{r^3} + \frac{(Q^2 \ell^4)}{r^4}. \quad (4.16) \]

The throat of the wormhole is defined at \( r_{\text{throat}} = b_0/2 + \sqrt{b_0^2/4 - Q^2 \ell^4} \) for \( b_0 \geq 2Q\ell^2 \) from Eq. (4.15). It is easy to show that \( b(r) \) satisfies the flaring-out condition (2.3) everywhere.
Figure 3: The exoticity $|\rho|\zeta$ for $b(r) = b_0$ and $\phi = -\ln(r/\ell)$ for $b_0 = 1$ and $\ell = 1$. The exotic matter is restricted to the finite region $b_0 \leq r < 7b_0/4$, which is due to the nonvanishing dilaton solution which is different from the constant dilaton case.

since $r \geq r_{\text{throat}}$. In this case, the exotic matter exists only near the throat and the normal matter is distributed in the region of $r \geq 7b_0/8 + \sqrt{49b_0^2 - 40Q^2\ell^4}/8$, which means that the exotic matter is also confined near the throat, and those behavior is plotted in Fig. [4]. After all, as seen from Figs. (3) and (4), the wormhole geometry with the nonvanishing dilaton solution can be patched onto the black string solution at the vanishing exoticity, which is now discussed in the next section.

5. Discussions

We have studied traversable wormholes in the three-dimensional low energy string theory by adding the additional matter source involving the dilaton. The key ingredient to our traversable wormholes is that the dilaton equation should be modified by adding the dilaton scalar source $F^M$ for preserving consistencies of the equations of motion. The appropriate physical conditions such as flaring-out condition for the shape function and no horizon condition for redshift function determine the exact forms of the additional energy-momentum tensor, $T^M_{\mu\nu}$ and the dilaton scalar source $F^M$. We require the exotic matter to be confined within a small region around the throat, and it is surrounded by the non-exotic(normal) matter ($\zeta \leq 0$), which is considered as the best way to minimize the use of exotic matter [1]. Here, we can confine our wormhole solutions to the interior of a spherical region with
Figure 4: The exoticity $|\rho|\zeta$ for $b(r) = b_0 - Q^2\ell^2/4r$ and $\phi = -\ln(r/\ell)$ for $b_0 = 3$, $Q = 1$, and $\ell = 1$. The graph shows the similar behavior of $b(r) = b_0$ case for $\phi = -\ln(r/\ell)$.

a radius $r = R_S$, which is corresponding to vanishing the exotic matters $\rho$, $\tau$, $p$, and $F^M$ at all regions of $r > R_S$. Therefore, the solutions with $\rho = \tau = p = F^M = 0$ should be naturally patched onto outer regions at $r > R_S$. A simple way to thread the whole geometry is to choose wormhole spacetimes at $r < R_S$, and a (non-rotating) BTZ black hole for $\phi = 0$ or a black string for $\phi = -\ln(r/\ell)$ at $r > R_S$, as seen from Figs. (2), (3), and (4).

As for the role of the dilaton and NS charge, for the solution of $b(r) = b_0$, the confinement of the exotic matter near the throat appears only for the logarithmic dilaton case in contrast to the $\phi = 0$ case. The NS charge effect of the shape function $b(r)$ forms a charged wormhole, which confines the exotic matter within the finite region near the throat in both dilaton cases as seen in Figs. (2), (3). Therefore, the dilaton field $\phi$ and the NS charge contribution on the shape function $b(r)$ may play an important role in putting the exotic matter into a finite region in the vicinity of the throat.

On the other hand, the duality, Eq. (4.6) is considered as a symmetry that relates a certain solution to another one in the context of string theory. However, since it seems to be too difficult to find an exact form of the dual invariant exotic matter action supporting the throat of wormholes, underlying duality of this system is manifestly broken by the specific matter distribution of forming a traversable wormhole.

Finally, as for the two-dimensional wormholes, they apparently differ from the higher dimensional cases in that they have a generally covariant conformal ghost matter as an
exotic matter source and those equations of motion are exactly solvable in the conformal coordinate system. Moreover, even though the two-dimensional model is coupled to the dilaton field, it is unnecessary to introduce the dilaton scalar source because of the conformal symmetry of the equations of motion. The wormhole solutions and their dynamics from construction to collapse to black holes in two-dimensional dilaton gravity are intensively studied in Ref. [10]. However, in $d \geq 3$ dimensions, it is difficult to find an exact form of the additional exotic matter action.

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