Supremacy of quantum theory over supra-quantum models of communication

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Quantum correlations can provide dramatic advantage over the corresponding classical resources in several communication tasks. However a broad class of probabilistic theories exists that attributes greater success than quantum theory in many of these tasks by allowing supra-quantum correlations in ‘space-like’ and/or ‘time-like’ paradigms. Here we propose a communication task involving three spatially separated parties where one party (verifier) aims to verify whether the bit strings possessed by the other two parties (terminals) are equal or not. We call this task authentication with limited communication, the restrictions on communication being: (i) the terminals cannot communicate with each other, but (ii) each of them can communicate with the verifier through single use of channels with limited capacity. Manifestly, classical resources are not sufficient for perfect success of this task. Moreover, it is also not possible to perform this task with certainty in several supra-quantum theories. Surprisingly quantum resources can achieve the perfect winning strategy. The proposed task thus stands apart from all previously known communication tasks as it exhibits quantum supremacy over supra-quantum strategies.

Advent of quantum information theory identifies useful applications of quantum mechanics over its classical counterpart in several mathematical as well as information theoretic protocols. It admits extremely efficient algorithms, such as Shor’s factoring algorithm [1], advantageous communication tasks, viz. quantum teleportation [2] and quantum super dense coding [3], qualitatively superior cryptographic protocols, eg. the BB-84 key distribution protocol [4], effective algorithms in a distributed computational task known as communication complexity [5], and also shows provable advantage in randomness processing [6–8]. Development of quantum error-correcting codes and fault-tolerant quantum computation [9] make many of these noble concepts realizable in experiment [10–13]. However, in many cases, it is notoriously hard to find which particular feature(s) of quantum theory like coherent superposition, continuity of state space, non-classical correlations, viz. non-locality/entanglement/quantum discord accounts for quantum supremacy in a particular task. A more general mathematical modeling of an operational theory is possible under the framework of generalized probability theories (GPTs) which incorporates several non-classical features of quantum theory and thus manifests many of the aforesaid advantageous protocols [14–18]. For example, in the distributed computing setting, where several spatially separated computing devices are allowed to exchange limited communications in order to perform some computational task, quantum nonlocal correlations can provide surprising advantages [19, 20]. Interestingly, in such cases, one can come up with more dramatic correlations that satisfy the relativistic causality or more broadly no-signaling (NS) principle but at the same time exhibit supremacy over the quantum correlations – Popescu-Rohrlich (PR) correlation is one such celebrated example in the bipartite setting [21]. Such stronger correlations exhibit weird phenomena as reflected in violation of several physical and information theoretic principles [22–26]. On the other extreme, a different toy theory is also possible that contains only local correlations but can supersede quantum theory in certain communication task by allowing stronger ‘time-like’ correlations. Such an anomalous behavior has been reported very recently by the name of hypersignaling (HS) phenomena [27].

Existence of such supra-quantum toy theories thus provoke an important question: what makes quantum theory special in operational sense? In other words, does there exist some task(s) where quantum theory outperforms these supra-quantum toy theories? Answer to this question is partially known from the perspective of computational power of a physical theory [28]. It has been shown that several beyond-quantum models of computation are trivial, i.e., the set of reversible transformations consists entirely of single-bit gates, and not even classical computation is possible [29]. However it is known that the class of functions computable with classical physics exactly coincides with the class computable quantum mechanically, and the quantum exponential speed-up over classical computation for a range of problems, such as factoring, is based upon the strong believe about persistence of polynomial hierarchy [30]. In the present work we approach the question of quantum supremacy over the supra-quantum theories from a different outlook – from the perspective of communication task. Interestingly we find that there
exists a communication task that can perfectly be won in quantum theory but the success probability of this task is limited not only in classical theory but also in HS model and PR model. Our task involves three spatially separated parties, where two parties are given random two-bit strings. The third party acts as a verifier who has to verify whether these strings are identical or not. The first two parties cannot communicate with each other but can encode their messages in the state of some physical system and consequently send it to the verifier. However information carrying capacity, namely the Holevo capacity, of these physical systems are limited by unity. We call this task authentication with limited communication (ALC). Naturally the question arises which feature of quantum theory makes it quintessential for perfectly winning the ALC task even though it allows limited correlations in space-like and time-like paradigms compared to supra-quantum GPTs. At this point we note that though PR theory is more radical than quantum mechanics in allowing joint state space structure and hence stronger nonlocal correlations but it is conservative in comparison to the later one to allow measurement in entangled bases. The HS model is the other extreme: it allows more general kind of measurements than quantum theory but grants only local correlations. We then consider other two theories, namely Hybrid model and frozen model, lying in between PR theory and HS model. These two theories allow entangled kind of states as well as measurements in entangled bases. However we show that perfect success of ALC is not possible even in those models. This indicates that the perfect success of ALC in quantum theory depends on the more intricate structure of the theory. To apprehend this intricate nature we define the ALC task in a generic convex model of operational theories also known as GPT framework.

The origin of the convex operational framework dates back to 1960s with the aim to investigate axiomatic derivations of the Hilbert space formalism of quantum theory from operational postulates [31–34]. Recently the approach has gained renewed interest from researchers in quantum information theory exploring the information theoretic foundations of quantum theory [35–40].

A GPT is specified by a list of system types, together with composition rules specifying which system type describes the combination of several other types. In a GPT, each system is described by a state ω which specifies outcome probabilities for all measurements that can be performed on it. A complete representation of the state is achieved by listing the outcome probabilities for measurements belonging to ‘fiducial set’ [35]. The set of possible states Ω of a given system type is a compact and convex set embedded in positive convex cone \( V_+ \) of some real vector space \( V \). Convexity of \( Ω \) assures that any statistical mixture of states is a valid state. The extremal points of the set \( Ω \) that do not allow any convex decomposition in terms of other states are called pure states or states of maximal knowledge.

An effect \( e \) is a linear functional on \( Ω \) that maps each state onto a probability, i.e., \( e : Ω \rightarrow [0, 1] \), with \( e(ω) \) bearing the interpretation of successfully filter the effect \( e \) on the system state \( ω \). The set of linear functionals \( Ω^* \) is embedded in the dual cone \( (V^*)_+ \). The unit effect \( u \) is defined as, \( u(ω) = 1, \forall ω \in Ω \). A \( d \)-outcome measurement is specified by a collection of \( d \) effects \( M = \{ e_j \mid \sum_j e_j = u \} \) such that \( \sum_j e_j(ω) = 1 \) for all valid states \( ω \). A set of state \( \{ ω_i \} \) is perfectly distinguishable in a single shot measurement if there exists some measurement \( M = \{ e_j \} \) such that \( e_j(ω_i) = δ_{ij} \).

The other elementary notion of a GPT is transformation. A transformation \( T \) maps states to states, i.e., \( T : Ω \rightarrow Ω \). Similarly as effects, they also have to be linear in order to preserve statistical mixtures. Under a valid transformation the total probability cannot increase, but can decrease in general.

The framework of GPTs does not assume, a priori, that all mathematically well-defined states, transformations and measurements can actually be physically implemented. For example, not all elements of \( Ω^* \) are required to be allowed effects in the theory. Indeed, the set of physically allowed effects \( E \) may be a strict subset of \( Ω^* \). A theory in which all elements of \( Ω^* \) are allowed effects is called ‘dual’. The property of duality is often assumed as a starting point in derivations of quantum theory, and is usually referred to as the ‘no-restriction hypothesis’. According to this hypothesis any mathematically well defined measurement should be physically allowed [38, 41].

GPT framework also considers composite systems with local state spaces (say) \( Ω_1 \) and \( Ω_2 \). Such a composition must be constructed in accordance with NS principle that prohibits instantaneous communication between two spatially separated locations. Under another less intuitive assumption called tomographic locality [42], the state space of the composite system lives in the vector space \( V_1 \otimes V_2 \). We denote the composite state space as \( Ω = Ω_1 \otimes Ω_2 = (V_1 \otimes V_2)_1^+ \), where \( (V_1 \otimes V_2)_1^+ \) denotes the normalized positive cone with normalization given by the order unit \( u_1 \otimes u_2 \in V_1^+ \otimes V_2^+ \). There is no unique choice for the positive cone, but it lies within the two extremes, \( (V_1 \ominus \min V_2)_+ := \{ \sum_j a_{ij} ω_1^j \otimes ω_2^i \mid a_{ij} ∈ R_{>0} \} \) and \( (V_1 \otimes \max V_2)_+ := (V_1^+ \otimes \min V_2^+_+) \). While the local state spaces are simplexes, which is the case for classical probability theory with discrete event space, the choice of tensor product is unique [43]. The quantum mechanical tensor product is neither the minimal one...
A GPT system can be used as classical information carrier. In a generic communication protocol a sender aims to send some classical information $x$ appearing with probability $p(x)$ to a spatially separated receiver. Sender encodes the classical message $x$ into some GPT state $\omega_x$ and sends the encoded system to the receiver who decode the message by performing some measurement $M = \{e_y \in \mathcal{E} \mid \sum e_y = u\}$. Given a message $x \in \mathcal{X}$ the probability of getting the outcome $y \in \mathcal{Y}$ is $p(y|x) := e_y(\omega_x)$ and the mutual information $I(X : Y) := \sum_{x,y} p_{xy} \log_2[p_{xy}/p(x)p(y)]$ quantifies the amount of classical information transmitted through such a protocol. The Holevo capacity $\mathcal{H}(\Omega)$, for a system type with state space $\Omega$, is defined as the maximum of $I(X : Y)$, over all probability distributions $p(x)$, all encoding strategies, and all decoding measurements \[44]\]. The measurement dimension $d_m(\Omega)$ of a GPT system is defined as the maximum number of states that can be perfectly distinguished in a single measurement \[45\]. It may be possible to encode arbitrary large amount (even unbounded amount) of classical information in some GPT system. However the Holevo capacity is always bounded by the measurement dimension, i.e., $\mathcal{H}(\Omega) \leq \log_2 d_m(\Omega)$. Subsequently these two concepts will be used while defining the ALC task.

The ALC task can be presented as a game involving three spatially separated players. Alice and Bob are two non-communicating players who are given random two-bit strings $x \in \{0, 1\}^2$ and $y \in \{0, 1\}^2$, respectively. Charlie is the verifier whose goal is to verify whether the strings given to Alice and Bob are identical or not. If there is no restriction on the amount of communications that Alice and Bob can convey to the verifier then there is no reason to not accomplishing the goal with perfect success. However the game has to be played under restricted communication scenario. Each of the players can encode their respective message in the state of some GPT system and subsequently send the system to the verifier through memoryless channels (see Fig. 1). The Holevo capacity of each system is bounded by unity. In other words the measurement dimension of the GPT system cannot be more than 2. Though the players cannot communicate with each other, they are allowed to make their respective encoding systems correlated, i.e., they can use some composite state $\omega_{AB} \in \Omega_A \otimes \Omega_B$, where $\Omega_A$ and $\Omega_B$ denote Alice’s and Bob’s state spaces, respectively. While in classical theory, this implies that the players can use only some classical correlation, in quantum theory they can use entangled states and in supra-quantum GPTs even more generic composite states can be used. For decoding, the verifier performs a two outcome measurement on the composite state space and depending on the measurement result he tries to authenticate whether $x = y$ or not.

An asymptotic version of the ALC task has already been studied in quantum theory by the name quantum fingerprinting \[46\], which was originally introduced by Yao to address a particular model of communication complexity namely simultaneous message passing model \[47\]. There Alice and Bob are given two random $n$-bit strings. Charlie has to answer whether their strings are equal or not while minimizing the amount of information that Alice and Bob send to him. We consider the simplest non-asymptotic version of the task with $n = 2$ with a prior limitation on the amount of communications. While the goal in \[46\] was to establish an exponential quantum-classical gap for the equality problem in the simultaneous message passing model, here our aim is to establish quantum supremacy in communication task not only over the classical theory but also over some supra-quantum GPT models of communication. Subsequently, we study the ALC task in different theories.

**Classical theory:** To perform the ALC task in classical theory, the players can undergo the following naive protocol: both Alice and Bob send the first bit of their strings and the verifier answers $x = y$, if he obtains identical bits, otherwise answers $x \neq y$. The average success probability under this strategy is $3/4$. However the players can follow more general strategies— pure, mixed or shared. A pure strategy can be defined as a tuple $(E_A, E_B, D)$,
while $E_A : \{0,1\}^2 \mapsto \{0_A, 1_A\}$ and $E_B : \{0,1\}^2 \mapsto \{0_B, 1_B\}$ are some encoding strategies for Alice and Bob, respectively, and $D : \{0_A, 1_A\} \times \{0_B, 1_B\} \mapsto \{0,1\}$ is some decoding strategy for the verifier. A mixed strategy is a tuple $(P_{E_A}, P_{E_B}, P_D)$, where $P_D$ denotes distribution over $Z$. They can use classical correlation to get a shared strategy $(\lambda_{E_A}E_B P_D)$, where $\lambda$ is shared randomness between Alice and Bob and in general $\lambda_{E_A}E_B \neq P_{E_A}P_{E_B'}$. However to obtain the optimal strategy it is enough to search over all possible pure strategies \cite{48}. The optimal classical success turns out to be $13/16$ with one of the following strategies: Alice’s (Bob’s) encoding:– $00 \mapsto 0_{A(B)}; 01, 10, 11 \mapsto 1_{A(B)}$; verifier’s decoding:– $x = y$, if obtained identical bits from Alice and Bob, otherwise $x \neq y$. The above observation leads us to the following proposition.

**Proposition 1.** There exist no perfect classical strategy for the ALC task, neither pure, nor mixed, nor shared.

**Quantum theory:** In this case both Alice and Bob can use state of a qubit to encode their messages, i.e., the encoding state space is the set of density operators $D(C^2)$ acting on the Hilbert space $C^2$ which is isomorphic to unit sphere in $\mathbb{R}^3$. A general encoding strategy for Alice is a mapping, $E_A^i : x \mapsto \rho^i_A \in D(C^2_A)$ and similarly for Bob, $E_B^i : y \mapsto \rho^i_B \in D(C^2_B)$. Verifier performs a two outcome positive operator valued measurement (POVM) $M \equiv \{M_0, M_1 | M_i > 0, i \in \{0,1\}; M_0 + M_1 = I_4\}$ on the composite system $C^2_A \otimes C^2_B$ and answers $x = y$ while $M_0$ clicks, otherwise answers $x \neq y$. However for uncorrelated (product states) quantum strategies we have the following no-go result.

**Lemma 1.** There is no perfect quantum uncorrelated strategy for the ALC task.

**Proof.** Since $x$ and $y$ are given randomly, Alice and Bob can obtain the strings in 16 different possible ways. In 4 cases the strings are identical and in other cases they are different. For perfect uncorrelated strategy the subspace spanned by the product states for identical strings must be orthogonal to the subspace spanned by the product states for different strings, i.e., $\text{Tr}[\rho^i_A \otimes \rho^j_B M_i] = 1$ for $x = y$ and 0 otherwise, while $\text{Tr}[\rho^i_A \otimes \rho^j_B M_{i \neq 1}] = 1$ for $x \neq y$, and 0 otherwise. But it is not possible to satisfy the requirement in $C^2 \otimes C^2$.

Interestingly, if the players start their protocol with two-qubit entangled state then the ALC task can be perfectly won.

**Proposition 2.** There exists a perfect quantum entangled strategy for the ALC task.

**Proof.** Let Alice and Bob share a two qubit singlet state $|\psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B)$. Consider the mapping $\{0,1\}^2 \mapsto k$, with $k \in \{0, 1, 2, 3\}$ as follows $00 \mapsto 01, 01 \mapsto 10, 10 \mapsto 11 \mapsto 3$. Whenever Alice (Bob) obtains a string $x$ ($y$) she (he) applies $\sigma_k (\sigma_k)$ on her (his) part of the singlet state and sends that part to the verifier, where $\sigma_0 = I$ and rest are Pauli matrices. Verifier obtains the state $\sigma_k \otimes \sigma_k |\psi^-\rangle_{AB}$ and performs the measurement, $M \equiv \{|\psi^-\rangle_{AB}, 1 - |\psi^-\rangle_{AB}\}$. Whenever $k = k'$, verifier gets the state $|\psi^-\rangle_{AB}$, otherwise he gets one of the rest three Bell states. Hence this protocol gives perfect success probability.

**Remark:** Note that for perfect quantum strategy both the entangled state for encoding and the measurement in entangled basis for decoding have been used. Furthermore, after the protocol verifier only knows whether Alice’s and Bob’s string are identical or not but no other information about the individual strings is revealed to him.

**Square bit theory:** This particular toy model of GPT allows more generic state space structure than qubit state space. The two dimensional state space $S$ is the collection of all vectors $(x, y, 1) \in \mathbb{R}^3$, with $-1 \leq x + y \leq 1, -1 \leq x - y \leq 1$, where $T$ denotes transposition. Shape of the state space turns out to be a square with four pure (extremal) states,

$$
\omega_0 := (1, 0, 1)^T, \quad \omega_1 := (0, 1, 1)^T, \\
\omega_2 := (-1, 0, 1)^T, \quad \omega_3 := (0, -1, 1)^T.
$$

Specifying the outcome probability rule for the effect $e$ on state $\omega$ as $\text{Tr}[e^T \omega] \geq 0$, leads to the following four extremal effects,

$$
e_0 := (1, 1, 1)^T, \quad e_1 := (-1, 1, 1)^T, \\
e_2 := (-1, -1, 1)^T, \quad e_3 := (1, -1, 1)^T.
$$

The condition $\text{Tr}[e^T \omega] \leq 1, \forall \omega$ and $\forall e$, implies a normalization factor 1/2. The set of reversible channels for the system $S$ turns out to be a finite group of symmetries (the dihedral group of order eight $D_8$ containing four rotations and four reflections), explicitly given by,

$$
U(S) = \{U^k_e : k = 0, \ldots , 3, s = \pm\},
$$

$$
U^k_e = \begin{pmatrix}
\cos \frac{k\pi}{2} & -s \sin \frac{k\pi}{2} & 0 \\
\sin \frac{k\pi}{2} & s \cos \frac{k\pi}{2} & 0 \\
0 & 0 & 1
\end{pmatrix}.
$$

State space for composition of two such square bits $S \otimes S$ is a convex set in $\mathbb{R}^9$. The states $\Omega$ and the normalized effects $E$ thus can be represented by vectors in $\mathbb{R}^9$. A convenient representation can be given by $3 \times 3$ real matrices rather than vectors in $\mathbb{R}^9$. Any bipartite
composition naturally includes 16 factorized extremal states and 16 factorized extremal effects given by,

\[ \Omega_{4i+j} := \omega_i \otimes \omega_j^T, \quad E_{4i+j} := e_i \otimes e_j^T, \]

where \( i, j \in \{0, 1, 2, 3\} \). One can also introduce non factorized matrices that play the role of entangled states and effects. Such an entangled state (effect) must be compatible with all factorized effects (states). Explicit calculation shows that one can have 8 such entangled states \( \{\Omega_i\}_{i=16}^{23} \) and 8 such entangled effects \( \{E_i\}_{i=16}^{23} \) that satisfy the requirement \( \text{Tr}[E_j^T \Omega_i] \geq 0 \) for any \( i \in [0, 15] \) and \( j \in [16, 23] \), and for any \( i \in [16, 23] \) and \( j \in [0, 15] \). While considering the bipartite theory containing entangled states and entangled effects the general consistency requirement must be fulfilled, i.e., all the considered effects must give positive probabilities on all the considered states. For the two square-bit theory following four consistent composite models are possible [27]:

1. **PR model**: All the 24 states \( i \in [0, 23] \); only the 16 factorized effects \( j \in [0, 15] \);
2. **HS model**: Only the 16 factorized states \( i \in [0, 15] \); all the 24 effects \( j \in [0, 23] \);
3. **Hybrid models**: Only 2 entangled states and effects are included (along with factorized states and effects): (a) \( i \in [0, 15] \cup [20, 22] \) and \( j \in [0, 15] \cup [20, 22] \); (b) \( i \in [0, 15] \cup [21, 23] \) and \( j \in [0, 15] \cup [21, 23] \);
4. **Frozen Models**: Only one entangled state and effect is included (along with factorized states and effects), i.e. \( i \in [0, 15] \cup \{i'\} \) and \( j \in [0, 15] \cup \{j'\} \) with \( i' = j' \in [16, 23] \).

All these four models, like quantum theory, satisfy the no-restriction hypothesis. Since PR model consists of only factorized effects it allows more generic bipartite states and hence stronger nonlocal correlation than quantum theory resulting in violation of several principles [22–26]. HS model is the other extreme– allows only factorized states and hence effects are more general than quantum. Clearly, HS model allows only local correlations and hence satisfies all bipartite principles involving space-like separated correlations. However it violates no-hypersignaling principle which imposes the restriction that signaling capacities of composite systems must be additive on signaling capacities of component subsystems [27].

While performing the ALC task in HS model, Alice and Bob can follow some product state encoding whereas the verifier has more freedom to choose the decoding measurement. On the other hand in PR model Alice and Bob have more freedom for the encoding strategy while the verifier’s decoding strategy is restricted. However there is no perfect strategy in any of these two models. At this point it seems that Hybrid model and Frozen model may provide perfect success for ALC task as they allow entangled states as well as entangled effects. However we find that even in these two models it is not possible to win the ALC game with perfect success, which leads us to the following proposition (see Appendix for proof).

**Proposition 3.** There exist no perfect strategy for the ALC task in HS model, in PR model, in Frozen model, and in Hybrid model.

While investigating generalized NS correlations, a number of games have been studied where supra-quantum NS correlations outperform optimal quantum winning strategies [20, 24, 49–54]. There also exist games where quantum resources are as good as generalized NS correlations [55–57]. On the other hand, it has been shown that even a generalized probabilistic local model can outperform quantum theory by allowing stronger ‘time-like’ correlation [27]. The ALC task, proposed in this work, is a notable exception from all these games: it can be won perfectly in quantum theory while several supra-quantum models having stronger ‘space-like’ and ‘time-like’ correlations do not provide a perfect strategy.

PR model stands as a testimony that presence of nonlocal correlation (and hence presence of steerable/entangled state) is not a sufficient requirement to win the ALC game in a GPT. Naturally the question arises: is nonlocality necessary for perfect winning of the ALC task? Interestingly the answer is negative. We find that one can perfectly win the ALC task in Spekkens’ toy-bit theory which is a local theory by construction [58].

The winning protocol in toy-bit theory is analogous to the quantum entangled protocol (detailed in Appendix). However the toy-bit theory is not a perfect GPT in true sense as it only allows some particular convex mixtures as valid states. Furthermore the elementary system of toy-bit theory does not satisfy the no restriction hypothesis [41].

On the other hand, from the example of Hybrid and Frozen models it is evident that even simultaneous presence of entangled states and entangled effects is not enough for perfect success of ALC in a GPT. One possible reason may be that the reversible dynamics in those theories are too restricted.

At this point the question remains whether entangled/steerable states are necessary for perfect winning strategy of ALC task in an arbitrary GPT model. Answer to this question is not very obvious in general as the example of HS model suggests that information carrying capacity of composite systems can be
super-additive. Therefore a more relevant question is whether entangled/steerable states are necessary for perfectly winning the ALC task in GPTs that respect the no-hypersignaling principle. This possibility requires further investigation. Another interesting research direction is to look for GPTs other than quantum theory that can achieve perfect success in ALC task while satisfying no-restriction hypothesis.

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SUPPLEMENTARY MATERIAL

ALC IN CLASSICAL WORLD

Classical theory arises as a special case of generalized probability theories (GPTs). State space of a classical system having measurement dimension $d_m$ is a $(d_m - 1)$ simplex. The restriction on communication in the ALC task compels Alice and Bob to encode their messages in 1-simplex which geometrically represents a line segment.

As already mentioned Alice and Bob can follow encoding strategies that are pure, mixed or shared. However due to convexity it will be sufficient to consider only the pure encoding strategies for finding the optimal success in ALC task. For one such strategy we first show how to calculate the average success probability.

| Encoding | Charlie’s Decoding | $P_{avg}$ |
|----------|--------------------|-----------|
| **Partitioning** | **Type** | **Partitioning** | **Type** | | |
| 00 → $0_A$ | (i) | 01 → $0_B$ | (i) | | 0_A0_B → $x \neq y$ | 3/4 |
| 01 | | 00 | | | 0_A1_B → $x \neq y$ | |
| 10 | | 10 | | | 1_A0_B → $x \neq y$ | |
| 11 | | 11 | | | 1_A1_B → $x \neq y$ | |
| 00 → $0_A$ | (i) | 00 → $0_B$ | (i) | | 0_A0_B → $x \neq y$ | 3/4 |
| 01 | | 10 | | | 0_A1_B → $x \neq y$ | |
| 10 | | 01 | | | 1_A0_B → $x \neq y$ | |
| 11 | | 11 | | | 1_A1_B → $x \neq y$ | |

Table I. Non optimal classical strategies.

Let us consider the following strategy: given the string $x \in \{0, 1\}^2$ Alice encodes the string as $\{00\} \rightarrow 0_A$, $\{01, 10, 11\} \rightarrow 1_A$ and Bob encodes his string $y \in \{0, 1\}^2$ as $\{00\} \rightarrow 0_B$, $\{01, 10, 11\} \rightarrow 1_B$. If Charlie obtains $0_A$ and $0_B$ from Alice and Bob, respectively, he answers $x = y$, otherwise he answers $x \neq y$. Since $x$ and $y$ are random, the average success probability $P_{avg}$ turns out to be,

$$P_{avg} = \frac{1}{16} \times 1 + \frac{3}{16} \times 1 + \frac{3}{16} \times 1 + \frac{9}{16} \times \frac{2}{3} = \frac{13}{16}$$

(1)

In the above example, Alice’s (Bob’s) encoding refers to a partitioning of the strings $\{00, 01, 10, 11\}$ into two disjoint sets. Such a non-trivial partitioning can be of two types: (i) one-vs-three, (ii) two-vs-two. While in the above example both Alice and Bob follow partitioning of type (i), they can also follow other type of partitioning which result in different encodings. Also note that all the encodings where both parties follow same partitioning (say both follow type (i)) need not give same $P_{avg}$ as the success also depends on Charlie’s decoding. In Table-I we list few such encoding-decoding strategies. Calculating over all such pure strategies (a tedious but straightforward task) it turns out that classical success probability for ALC task is upper bounded by 13/16. One such optimal strategy is given by Eq.(1).

ALC IN SQUARE-BIT THEORY

**Elementary system:** The normalized state space $S$ of the elementary system takes the shape of a square with the
The unit effect \( u \) according to the rule \( \text{Tr} \) for entangled effects. Any such entangled state (effect) must give positive probability over all factorized effects (states)

\[ \text{Here the factor 2 is for normalization.} \]

One can introduce other matrices that play the role of entangled states and be represented by 3 real matrices. Any bipartite composition should include all the factorized extremal states \( \omega \)

\[ \text{The states and effects corresponding to a composition of two elementary systems can} \]

\[ \text{be represented by 3 x 3 real matrices. Any bipartite composition should include all the factorized extremal states} \]

\[ \text{and factorized extremal effects given by:} \]

\[ \Omega_{4i+j} := \omega_i \otimes \omega_j^T, \quad E_{4i+j} := 2 \left( e_i \otimes e_j^T \right), \quad i, j \in \{0, 1, 2, 3\}. \]

Here the factor 2 is for normalization. One can introduce other matrices that play the role of entangled states and entangled effects. Any such entangled state (effect) must give positive probability over all factorized effects (states) according to the rule \( \text{Tr}[E^T \Omega] \). The set of consistent normalized entangled states are given by,

\[ \Omega_{16} = \frac{1}{2} \left( \omega_1 \otimes \omega_1^T - \omega_2 \otimes \omega_2^T + \omega_2 \otimes \omega_3^T + \omega_3 \otimes \omega_3^T \right), \]

\[ \Omega_{17} = \frac{1}{2} \left( \omega_0 \otimes \omega_0^T - \omega_0 \otimes \omega_0^T + \omega_1 \otimes \omega_1^T + \omega_3 \otimes \omega_3^T \right), \]

\[ \Omega_{18} = \frac{1}{2} \left( \omega_0 \otimes \omega_0^T - \omega_1 \otimes \omega_1^T + \omega_1 \otimes \omega_2^T + \omega_2 \otimes \omega_1^T \right), \]

\[ \Omega_{19} = \frac{1}{2} \left( \omega_0 \otimes \omega_0^T - \omega_0 \otimes \omega_3^T + \omega_1 \otimes \omega_3^T + \omega_3 \otimes \omega_2^T \right), \]

\[ \Omega_{20} = \frac{1}{2} \left( \omega_0 \otimes \omega_3^T - \omega_0 \otimes \omega_0^T + \omega_1 \otimes \omega_1^T + \omega_3 \otimes \omega_3^T \right), \]

\[ \Omega_{21} = \frac{1}{2} \left( \omega_0 \otimes \omega_0^T - \omega_0 \otimes \omega_1^T + \omega_1 \otimes \omega_1^T + \omega_3 \otimes \omega_3^T \right), \]

\[ \Omega_{22} = \frac{1}{2} \left( \omega_1 \otimes \omega_1^T - \omega_2 \otimes \omega_1^T + \omega_2 \otimes \omega_2^T + \omega_3 \otimes \omega_0^T \right), \]

\[ \Omega_{23} = \frac{1}{2} \left( \omega_0 \otimes \omega_0^T - \omega_1 \otimes \omega_1^T + \omega_1 \otimes \omega_2^T + \omega_2 \otimes \omega_0^T \right), \]
and the set of consistent normalized entangled effects are given by,

\begin{align}
E_{16} &= \left( e_0 \otimes e_0^T - e_0 \otimes e_1^T + e_1 \otimes e_3^T + e_3 \otimes e_0^T \right), \\
E_{17} &= \left( e_1 \otimes e_1^T - e_2 \otimes e_2^T + e_2 \otimes e_3^T + e_3 \otimes e_1^T \right), \\
E_{18} &= \left( e_0 \otimes e_2^T - e_0 \otimes e_1^T + e_1 \otimes e_1^T + e_3 \otimes e_0^T \right), \\
E_{19} &= \left( e_0 \otimes e_0^T - e_1 \otimes e_1^T + e_1 \otimes e_2^T + e_2 \otimes e_1^T \right), \\
E_{20} &= \left( e_0 \otimes e_1^T - e_1 \otimes e_1^T + e_1 \otimes e_2^T + e_2 \otimes e_0^T \right), \\
E_{21} &= \left( e_1 \otimes e_1^T - e_2 \otimes e_2^T + e_2 \otimes e_3^T + e_3 \otimes e_1^T \right), \\
E_{22} &= \left( e_0 \otimes e_0^T - e_0 \otimes e_1^T + e_1 \otimes e_1^T + e_3 \otimes e_2^T \right), \\
E_{23} &= \left( e_0 \otimes e_3^T - e_0 \otimes e_0^T + e_1 \otimes e_0^T + e_3 \otimes e_1^T \right). 
\end{align}

(7a)
(7b)
(7c)
(7d)
(7e)
(7f)
(7g)
(7h)

The unit effect on the composite system is \( u \otimes u^T \). Note that all the entangled effects on all the entangled states do not give rise to valid probabilities (see Table-III). As shown in [27], four consistent bipartite models are possible listed in Table-II.

**No perfect strategy for ALC in Square-bit theories:** We are now in a position to prove our main result: that there exists no perfect strategy for the ALC task in HS model, in PR model, in Hybrid model, and in Frozen model. For that we first prove the following lemma.

**Lemma 2.** There is no perfect strategy for ALC task while following factorized encodings and factorized decodings.

**Proof.** Consider the mapping \( \{0, 1\}^2 \mapsto k \), with \( k \in \{0, 1, 2, 3\} \) as follows \( 00 \mapsto 0, 01 \mapsto 1, 10 \mapsto 2, 11 \mapsto 3 \). Let Alice and Bob encode their strings as \( k \mapsto \omega_k \). For this encoding, while \( x = y \) Charlie receives \( \Omega_0, \Omega_5, \Omega_10, \Omega_15 \), otherwise he receives \( \{\Omega_i \mid i \in \{0, 15\} \setminus \{0, 5, 10, 15\}\} \) (see Table-IV).

For decoding, Charlie performs some measurement,

\[ M \equiv \{M_{eq}, M_{neq} \mid M_{eq} = \sum_i p_iE_i, \ M_{neq} = \sum_j q_jE_j, \ M_{eq} + M_{neq} = u \otimes u^T\}, \tag{8} \]

where, \( p_i, q_j \geq 0; \ E_i, E_j \) are factorized effects, with \( E_i \neq E_j \).

For perfect decoding we require,

\[ \text{Tr}[M_{eq, i}^T\Omega_i] = 1, \quad \text{iff} \quad i \in \{0, 5, 10, 15\}; \tag{9a} \]
\[ \text{Tr}[M_{neq, i}^T\Omega_i] = 1, \quad \text{iff} \quad i \in \{0, 15\} \setminus \{0, 5, 10, 15\}. \tag{9b} \]

However from Table-IV it is clear that no extremal effect satisfies Eq.(9) implying no perfect strategy. This holds true for any possible factorized encoding.

| Model     | States                  | Effects                  | Transformations                                      |
|-----------|-------------------------|--------------------------|------------------------------------------------------|
| PR Model  | \( \Omega_i, i \in [0, 23] \) | \( E_j, j \in [0, 15] \) | \( \mathcal{U}(S \otimes S) \)                       |
| HS Model  | \( \Omega_i, i \in [0, 15] \) | \( E_j, j \in [0, 23] \) | \( \mathcal{U}(S \otimes S) \)                       |
| Hybrid Model | \( \Omega_i, i \in [0, 15] \cup [20, 22] \) | \( E_j, j \in [0, 15] \cup [20, 22] \) | \( \{U_k^+ \otimes U_l^+ \mid k, l = 0, 2\} \) |
| Frozen Model | \( \Omega_i, i \in [0, 15] \cup \{n\}; \ n \in [16, 23] \) | \( E_j, j \in [0, 15] \cup \{n\}; \ n \in [16, 23] \) | \( \{W_{ijkl}(U_k^0 \otimes U_l^0)\}, \text{if } n \in [16, 19] \times \{W_{ijkl}(U_k^0 \otimes U_l^0)\}, \text{if } n \in [20, 23] \} \)

Table II. Four possible bipartite models in square-bit theory. \( \mathcal{U}(S \otimes S) := \{W^i(U_k^0 \otimes U_k^0)\} \) with \( i \in \{0, 1\}; \ j, k \in \{0, 1, 2, 3\}; s_1, s_2 \in \{\pm\} \), \( W \) being the SWAP map.
Table III. The values of $\text{Tr}[E_i^T \Omega_j]$ for normalized effects $E_i$ and normalized states $\Omega_j$ are listed here, $i, j \in [0, 23]$. Note that the values in the shaded cells do not correspond to valid probability measures.

| Factorized Effects | Entangled Effects |
|--------------------|------------------|
| $E_0$ | $E_1$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ | $E_6$ | $E_7$ | $E_8$ | $E_9$ | $E_{10}$ | $E_{11}$ | $E_{12}$ | $E_{13}$ | $E_{14}$ | $E_{15}$ | $E_{16}$ | $E_{17}$ | $E_{18}$ | $E_{19}$ | $E_{20}$ | $E_{21}$ | $E_{22}$ | $E_{23}$ |
| $\Omega_0$ | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 1 |
| $\Omega_1$ | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 1 |
| $\Omega_2$ | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 1 |
| $\Omega_3$ | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 |
| $\Omega_4$ | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 |
| $\Omega_5$ | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 1 |
| $\Omega_6$ | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 |
| $\Omega_7$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 |
| $\Omega_8$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 |
| $\Omega_9$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 |
| $\Omega_{10}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| $\Omega_{11}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| $\Omega_{12}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| $\Omega_{13}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Omega_{14}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Omega_{15}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table IV. Outcome probabilities of factorized normalized effects $E_i$ on factorized normalized states $\Omega_j$, $i, j \in [0, 15]$. For the encoding considered in Lemma 2, Charlie receives the states $\{\Omega_0, \Omega_5, \Omega_{10}, \Omega_{15}\}$ while $x = y$, as shown by shaded rows. However no extremal effect satisfies the requirement (9).

| Factorized Effects | Entangled Effects |
|--------------------|------------------|
| $E_0$ | $E_1$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ | $E_6$ | $E_7$ | $E_8$ | $E_9$ | $E_{10}$ | $E_{11}$ | $E_{12}$ | $E_{13}$ | $E_{14}$ | $E_{15}$ |
| $\Omega_0$ | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| $\Omega_1$ | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 |
| $\Omega_2$ | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 |
| $\Omega_3$ | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 |
| $\Omega_4$ | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Omega_5$ | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Omega_6$ | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Omega_7$ | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Omega_8$ | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Omega_9$ | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Omega_{10}$ | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Omega_{11}$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| $\Omega_{12}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| $\Omega_{13}$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| $\Omega_{14}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Omega_{15}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

We will now extend this Lemma and proof a no-go theorem for HS model as stated in the following proposition.

**Proposition 4.** There is no perfect strategy for ALC task in HS model.

**Proof.** In HS model only factorized states are allowed for encoding while Charlie can perform more generalized...
decoding measurements as entangled effects are also allowed, i.e.,
\[
M \equiv \{ M_{eq}, M_{neq} | M_{eq} = \sum_i p_i E_i, M_{neq} = \sum_j q_j E_j, M_{eq} + M_{neq} = u \otimes u^T \},
\]
where, \( p_i, q_j \geq 0 \); \( E_i, E_j \) are factorized or entangled effects, with \( E_i \neq E_j \).

Whereas from Lemma-2 it follows that factorized effects do not satisfy requirement (9), from Table-III it is evident that even entangled effects are not good for perfect strategy.

While considering PR model, the encodings are factorized as well as entangled whereas the decodings are factorized only. Since local unitaries map entangled states to entangled states and factorized states to factorized states, Lemma-2 leaves open only the entangled encoding. However, in the following proposition we prove a no-go result even for such entangled encodings.

**Proposition 5.** There is no perfect strategy for ALC task in PR model.

*Proof.* Let Alice and Bob share the entangled state \( \Omega_{16} \). For encoding they can apply local reversible operations on their respective parts depending on the strings they receive. In this case eight local reversible actions are possible for each party. Under these operations the transformed states have been shown in Table-V.

Now consider an encoding as follows: for Alice 00 \( \mapsto U_0^+ \), 01 \( \mapsto U_0^- \), 10 \( \mapsto U_1^+ \), 11 \( \mapsto U_2^+ \) for Bob 00 \( \mapsto U_0^+ \), 01 \( \mapsto U_0^- \), 10 \( \mapsto U_3^- \). The encoded states have been shown in Table-V by pink whenever \( x = y \) and by yellow when \( x \neq y \). While decoding, Charlie needs to perform a two-outcome measurement such that the effect corresponding to \( x = y \) clicks only on the pink colored states and the other effect only on the yellow colored states. However such a measurement is not possible since in multiple cases the same states have been assigned two different colors. For example, under the actions \( U_0^+ \otimes U_0^+ \) (corresponds to \( x = y \)) and \( U_0^+ \otimes U_0^- \) (corresponds to \( x \neq y \)) Charlie obtains the same encoded state \( \Omega_{16} \). Considering other encodings it also turns out to be the same.

**Proposition 6.** There is no perfect strategy for ALC task in Hybrid model.

*Proof.* Consider the Hybrid models with extremal states \( \Omega_i, i \in \{0,15\} \cup \{20,22\} \) and extremal effects \( E_j, j \in \{0,15\} \cup \{20,22\} \). According to Proposition 4 factorized encodings will not give the perfect success. So, Alice and Bob can start their protocol with one of the entangled states (say) \( \Omega_{20} \). This model allows only two local reversible operations \( \{ U_0^+, U_3^- \} \) on each side. Therefore they cannot encode their four different strings reliably using two such operations and hence no perfect strategy is possible even using entangled encodings and entangled decodings. Similar argument holds true for the other Hybrid model.

**Proposition 7.** There is no perfect strategy for ALC task in Frozen model.

| Alice’s action | Alice’s action |
|----------------|----------------|
| \( U_0 \) | \( U_0 \) |
| \( U_1 \) | \( U_2 \) |
| \( U_2 \) | \( U_1 \) |
| \( U_3 \) | \( U_2 \) |

Table V. Transformation of the state \( \Omega_{16} \) under local reversible actions \( U_i^s \otimes U_j^s \)[\( \Omega_{16} \)] := \( U_i^s \Omega_{16} \left( U_j^s \right)^T \), where \( i, j \in \{0,1,2,3\} \) and \( s_1, s_2 \in \{\pm\} \). Note that under the actions \( U_i^s \otimes U_j^s \) with \( s_1 = s_2 \) the state belongs in the group \( \{16, \cdots, 19\} \) and when \( s_1 \neq s_2 \) it belongs in \( \{20, \cdots, 23\} \). Actually this fact is a generic feature: for \( s_1 = s_2 \) the groups \( G_1 \equiv \{\Omega_{16}, \Omega_{17}, \Omega_{18}, \Omega_{19}\} \) and \( G_2 \equiv \{\Omega_{20}, \Omega_{21}, \Omega_{22}, \Omega_{23}\} \) are closed while for \( s_1 \neq s_2 \), \( G_1 \leftrightarrow G_2 \). For the particular encoding strategy considered below, the encoded states have been shown by pink whenever \( x = y \) and by yellow when \( x \neq y \).
While first type corresponds to factorized states, the later one corresponds to entangled states. Four such entangled states are analogous to the four Bell states of two-qubit quantum system and pictorially they can be represented as in Fig. 2.

Proof. Consider the frozen model with extremal states $\Omega_i$, $i \in [0,15] \cup \{16\}$ and extremal effects $E_j$, $j \in [0,15] \cup \{16\}$. In this case also factorized encodings are not good (Proposition 4). On the other hand, strategy that starts with sharing entangled state is trivial in this case as this model allows only one such state. Similar reasoning also holds true for other Frozen models.

### ALC IN SPEKKENS’ TOY-BIT MODEL

This particular toy theory is based on a principle, namely knowledge balance principle (KBP), according to which in a state of maximal knowledge the amount of knowledge one possesses about the ontic state of the system must equal the amount of knowledge she/he lacks [58].

**Elementary system:** For an elementary system the number of questions in the canonical set is two, and consequently the number of ontic states is four. Denote the four ontic states as ‘1’, ‘2’, ‘3’, and ‘4’. An epistemic state is a probability distribution $\{\tilde{p} = (p_1, p_2, p_3, p_4)^T \mid p_i \in \{0,1/2\}, \ \& \ \sum_{i=1}^4 p_i = 1\}$, over the ontic states. In accordance with KBP, there exist six pure epistemic states (state with maximal knowledge) for an elementary system that are given by,

$$1 \vee 2 \equiv \left( \frac{1}{2}, \frac{1}{2}, 0, 0 \right)^T, \quad 3 \vee 4 \equiv \left( 0, 0, \frac{1}{2}, \frac{1}{2} \right)^T, \quad 1 \vee 3 \equiv \left( \frac{1}{2}, 0, \frac{1}{2}, 0 \right)^T,$$

$$2 \vee 4 \equiv \left( 0, \frac{1}{2}, 0, \frac{1}{2} \right)^T, \quad 1 \vee 4 \equiv \left( \frac{1}{2}, 0, 0, \frac{1}{2} \right)^T, \quad 2 \vee 3 \equiv \left( 0, \frac{1}{2}, 0, \frac{1}{2} \right)^T.$$  

Here the symbol ‘$\vee$’ means disjunction which reads as ‘or’. These epistemic states can be viewed as in Fig. 2.

![Figure 2](image)

Figure 2. Each box denotes an ontic state, ranging ‘1’ to ‘4’ from left to right. Epistemic states are distributions on the ontic states. For example $1 \vee 2$ denotes distribution on ontic state ‘1’ and ‘2’, i.e., on the first two boxes from left as shown in (a).

One can introduce several quantum like features, viz., convex combination, coherent superposition, in this toy theory. State transformation as well as measurement rule are defined in accordance with KBP. Transformations are given by permutations of the ontic states and can be represented as cycles. For example, the cycle $(a)(bcd)$ means $a \mapsto a$ and $b \mapsto c \mapsto d \mapsto b$. While 4-element permutation group contains 24 elements, only a few of them are compatible with KBP. Four allowed transformations that will be relevant for our purpose are,

$\begin{align*}
U_0 &= (1)(2)(3)(4), \quad U_1 = (12)(34), \\
U_2 &= (13)(24), \quad U_3 = (14)(23).
\end{align*}$

**Pairs of elementary systems:** In this case number of ontic states are 16 that are represented by $a,b$, with $a,b \in \{1,2,3,4\}$. Pure epistemic states are of two type:

**Type-1:** $(a \vee b)(c \vee d) \equiv (a.c) \vee (a.d) \vee (b.c) \vee (b.d)$; where $a,b,c,d \in 1,2,3,4$ and $a \neq b$, $c \neq d$.

**Type-2:** $(a.e) \vee (b.f) \vee (c.g) \vee (d.h)$; where $a,b,c,d,e,f,g,h \in \{1,2,3,4\}$, $a \neq b \neq c \neq d$ and $e \neq f \neq g \neq h$.

While first type corresponds to factorized states, the later one corresponds to entangled states. Four such entangled states relevant to our purpose are,

$$\begin{align*}
\psi_0 &:= (1.1) \vee (2.2) \vee (3.3) \vee (4.4), \\
\psi_1 &:= (1.2) \vee (2.1) \vee (3.4) \vee (4.3), \\
\psi_2 &:= (1.3) \vee (2.4) \vee (3.1) \vee (4.2), \\
\psi_3 &:= (1.4) \vee (2.3) \vee (3.2) \vee (4.1).
\end{align*}$$

These four states are analogous to the four Bell states of two-qubit quantum system and pictorially they can be represented as in Fig. 3.
16 ontic states.

This measurement can be visualized pictorially as in Fig. 4. 

Measurements on the pairs of elementary systems in defined as partitioning of the set of sixteen ontic states into disjoint epistemic states. A measurement defined in this way must be compatible with KBP. Every such compatible partitioning of the set of sixteen ontic states into four disjoint pure epistemic states yields a maximally informative measurement. One such measurement is $M \equiv \{S_I, S_{II}, S_{III}, S_{IV}\}$, where,

\begin{align*}
S_I &= (1.1) \lor (2.2) \lor (3.3) \lor (4.4), \\
S_{II} &= (1.2) \lor (2.1) \lor (3.4) \lor (4.3), \\
S_{III} &= (1.3) \lor (2.4) \lor (3.1) \lor (4.2), \\
S_{IV} &= (1.4) \lor (2.3) \lor (3.2) \lor (4.1).
\end{align*}

This measurement can be visualized pictorially as in Fig.4.

![Figure 4. Measurement M as defined in Eq.(14).](image)

**Perfect toy-bit protocol for ALC task**

**Encoding:** Alice and Bob start the protocol with the shared toy-bit entangled state $\psi_0$. Consider the mapping $\{0,1\}^2 \mapsto k$, with $k \in \{0,1,2,3\}$ as follows $00 \mapsto 0, 01 \mapsto 1, 10 \mapsto 2, 11 \mapsto 3$. Whenever Alice (Bob) obtains a string $x$ (y) she (he) applies $U_k$ ($U_{k'}$) defined in Eq.(12) on her (his) part of the entangled state $\psi_0$ and sends that part to Charlie. Straightforward calculation gives us,

\begin{align*}
U_0 \circ U_0[\psi_0] &= U_1 \circ U_1[\psi_0] = U_2 \circ U_2[\psi_0] = U_3 \circ U_3[\psi_0] = \psi_0, \\
U_0 \circ U_1[\psi_0] &= U_1 \circ U_0[\psi_0] = U_2 \circ U_3[\psi_0] = U_3 \circ U_2[\psi_0] = \psi_1, \\
U_0 \circ U_2[\psi_0] &= U_2 \circ U_0[\psi_0] = U_1 \circ U_3[\psi_0] = U_3 \circ U_1[\psi_0] = \psi_2, \\
U_0 \circ U_3[\psi_0] &= U_3 \circ U_0[\psi_0] = U_1 \circ U_2[\psi_0] = U_2 \circ U_1[\psi_0] = \psi_3.
\end{align*}

The notation $U_k \circ U_{k'}[\psi]$ denotes that on Alice side permutation $U_k$ and on Bob side permutation $U_{k'}$ are applied.

**Decoding:** For decoding, Charlie performs the measurement $M$ of Eq.(14). He answers $x = y$ while $S_I$ clicks, otherwise he answers $x \neq y$, resulting in perfect success.

**Remark:** The perfect toy-bit protocol for ALC task is different than the perfect quantum entangled strategy in a sense. While in quantum case Charlie performs a two outcome measurement for decoding and extracts only the information whether Alice’s and Bob’s strings are same or not, in the above case Charlie performs four-outcome measurement and hence extracts more information. However in toy-bit case also, Charlie can perform a two-outcome measurement $M' \equiv \{S_1, S_2\}$ (see Fig.5) and can make the protocol exactly same as quantum perfect protocol.
Figure 5. Measurement $M' \equiv \{ S_1, S_2 \}$.

|   | 1 | 2 | 2 | 2 |
|---|---|---|---|---|
| 2 | 2 | 2 | 1 |   |
| 2 | 2 | 1 | 2 |   |
| 2 | 1 | 2 | 2 |   |
| 1 | 2 | 2 | 2 |   |