Minimization of the Difference between the Theoretical Mean of the Rayleigh Probability Density Function and the Mean Obtained from its Plot

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Abstract In this study, the difference between the mean of the Rayleigh Probability Density Function and the mean obtained from the graph of Rayleigh Probability Density Function is minimized by changing the coefficient in the equation yielding the mean. By using various numbers of data and K values, Rayleigh Probability Density Function is plotted with the means mentioned above.

Keywords Rayleigh Probability Density Function, Line of Sight, No Line of Sight

1. Introduction

Rayleigh Probability Density Function is used for the cases in which there is NLOS between transmitters and receivers for the communication networks and channel modeling. When there occur phase differences between multipath signals arriving the receiver, fading takes place. Rayleigh distribution function is used for multipath fading in modeling of change of voltage or power that will be received[1-8]. In Rayleigh modeling, in contrast to the Ricean, there is no any specified direction, that is, signals coming from any direction are assumed to have equal probability.

Moreover Rayleigh Modeling is used for noise analysis. If a signal coming to a receiver via reflection becomes so greater than the direct signal that it suppresses that, in this case this type of channel modeling is done by means of Rayleigh Probability Density Function.

The aim of our study is to find an expression which results in numerical values that are very close to the actual values. For this purpose after much iteration we have obtained a new coefficient that gives the desired results.

2. Theory

Rayleigh Probability Density Function changes with respect to a single parameter, either standard deviation or [K[9-15] where K is the power of direct signal divided by the power of coming signal via reflection and expressed as

\[ K = \frac{P_{\text{los}}}{P_{\text{multipath}}} \]  

(1)

\[ K = \frac{1}{2} \sigma^2 \]  

(2)

\[ P_{\text{los}} = 0 \text{ Watt} = 1 \text{ dBW} \]  

(3)

K characterizes the environment. K values are normalized when using in the equation(4).

Rayleigh Probability Density Function is expressed as

\[ f_z(z) = 2z K e^{-\frac{z^2}{K}} \]  

(4)

When K represents the Power

\[ L = K/10 \]  

(5)

and if it represents the voltage

\[ L = K/20 \]  

(6)

is used. For both cases,

\[ S = 10^L. \]  

(7)

Normalized Rayleigh Probability Density Function can be written as

\[ f_z(z) = 2z S e^{-\frac{z^2}{S}} \]  

(8)

S is obtained through the equations (5), (6), (7) and used in equation (8) to obtain \( f_z(z) \).

Mean of this function is

\[ f_z(z)(\text{mean}) = M[z] = \int_0^\infty z f_z(z) \, dz \]  

(9)

And theoretical mean can be written as,

\[ TM = M[z] = \left(\frac{1}{\sqrt{2\pi}}\right) \sqrt{\frac{\pi}{K}} = 1.2533 \sigma \]  

(10)

And the variance is written as

\[ \sigma^2 = M[z^2] - M^2[z] = (0.2146/K) \]  

(11)
And the Standard deviation is
\[ \sigma = 0.4632/\sqrt{K} \] (12)

Table 1 shows the actual and theoretical means and percent error of theoretical mean.

| K voltage (dB) | AM Actual mean | TM theoretical mean | Error(%) |
|----------------|----------------|---------------------|----------|
| 10             | 41             | 49.83               | 5.16     |
| 12             | 36             | 44.44               | 4.93     |
| 14             | 33             | 39.58               | 3.84     |
| 16             | 29             | 35.28               | 3.67     |
| 18             | 26             | 31.44               | 3.18     |
| 20             | 23             | 28.02               | 2.93     |

### 3. Modification of Theoretical Mean Expression

In equation (10) the coefficient \( \frac{1}{2} \) is not suitable. Because there is a big difference between the actual and theoretical curves when theoretical mean is used for the plot of Rayleigh Probability Density Function. Instead we suggest a change in equation (10) to achieve more reasonable results. After many iterations the coefficient \( \frac{1}{2} \) is replaced by \( \frac{1}{2.3589} \) to minimize the difference between the theoretical mean(TM) and the actual mean (AM) which is obtained from the graph of Rayleigh Probability Density Function.

Since TM being a function of K is the theoretical mean, if and only if K is used instead of K in Rayleigh Probability Density Function its actual mean becomes TM. This is illustrated in Table 4.

In Table 3 peak values corresponding to AM and TM voltage differences (errors) are given.

**Table 3.** Actual and Theoretical Peak Values and the difference between them

| K (dB) | K1 (dB) | Actual peak value of \( f_z(z) \) (%) | theoretical peak value of \( f_z(z) \) (%) | Voltage difference |
|--------|---------|--------------------------------------|------------------------------------------|-------------------|
| 10     | 7.13    | 1.5252                               | 1.4503                                   | 0.0749            |
| 12     | 9.29    | 1.7112                               | 1.6398                                   | 0.0714            |
| 14     | 11.13   | 1.9199                               | 1.8240                                   | 0.0959            |
| 16     | 13.13   | 2.1545                               | 2.0688                                   | 0.0857            |
| 18     | 15.13   | 2.4174                               | 2.3317                                   | 0.08423           |
| 20     | 17.13   | 2.7117                               | 2.6049                                   | 0.1068            |

Equation (14) can be used for finding K1 by replacing it in place of K.
\[ f_1(z) = 2 z K1 exp(-z^2 K1) \] (13)
\[ MM = M1[z] = \left( \frac{1}{2.3589} \right) \sqrt{\pi K} \] (14)

Where MM is the modified mean. As can be seen from Table1, and Table 2 for K=10 dB AM=41, TM=49.83 MM=42 is obtained. This shows that the equation (14) yields a value which is very close to the actual mean whereas the theoretical one (TM) is far from it.

Since the variance and standard deviation values are derived from the mean, an erroneous mean causes wrong results in variance and standard deviation[16-23]. Therefore to obtain minimum difference between the actual and theoretical mean, we changed the coefficient value several times. As a result we reduced the percent error from 5.16 percent to 0.32 percent. The average of x number of data can be expressed as
\[ MM_x = \frac{\text{Data x}}{171} MM_{171} \] (15)

Where MM\(_{171}\) is the average value of 171 data.
Table 4 shows the variations of $f_2(z)$ and $f_1(z)$.

| K(dB) | $f_2(z) = 2z \exp(-z^2S)$ | $f_1(z) = 2z S \exp(-z^2S_1)$ |
|-------|-----------------------------|-------------------------------|
| 10    | 6.32 $z \exp(-z^23.16)$    | 7.13 4.54 $z \exp(-z^22.27)$ |
| 12    | 7.96 $z \exp(-z^23.98)$    | 9.29 5.82 $z \exp(-z^22.91)$ |
| 14    | 10.02 $z \exp(-z^25.01)$   | 11.13 7.20 $z \exp(-z^23.60)$ |
| 16    | 12.61 $z \exp(-z^26.30)$   | 13.13 9.06 $z \exp(-z^24.53)$ |
| 18    | 15.88 $z \exp(-z^27.94)$   | 15.13 11.41 $z \exp(-z^25.7)$ |
| 20    | 20 $z \exp(-z^210)$        | 17.13 14.37 $z \exp(-z^27.18)$ |

Actual, theoretical, and modified Rayleigh Probability Density Functions are plotted according to different K and data values in Fig. 1 through Fig. 6.
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**Table 5.** The mean values of \( f_1(z) \), \( f_2(z) \), and their values around the means

| K=20 dB, for 171 data AM | Data number | 22 | 23 (mean) | 24 | 25 |
|--------------------------|------------|----|-----------|----|----|
| Probability value | 2.70224936 | 2.71177808 | 2.71029179 | 2.69828373 |
| K=20 dB, for 171 data MM | Data number | 23 | 24 (mean) | 25 | 26 |
| Probability value | 2.63034350 | 2.63552781 | 2.63075854 | 2.61649395 |
| K=20 dB, for 171 data TM | Data number | 26 | 27 (mean) | 28 | 29 |
| Probability value | 2.29304064 | 2.29894371 | 2.29814705 | 2.29090434 |

**Table 6.** The mean values of \( f_1(z) \), \( f_2(z) \), and their values around the means

| K=10 dB, for 171 data AM | Data number | 40 | 41 (mean) | 42 | 43 |
|--------------------------|------------|----|-----------|----|----|
| Probability value | 1.52477789 | 1.52529002 | 1.52388460 | 1.52061275 |
| K=10 dB, for 171 data MM | Data number | 41 | 42 (mean) | 43 | 44 |
| Probability value | 1.49019207 | 1.49053635 | 1.48909413 | 1.48591223 |
| K=10 dB, for 171 data TM | Data number | 47 | 48 (mean) | 49 | 50 |
| Probability value | 1.29257103 | 1.29305214 | 1.29236036 | 1.29052165 |

**Table 7.** The mean values of \( f_1(z) \), \( f_2(z) \), and their values around the means

| K=20 dB, for 1701 data AM | Data number | 224 | 225 (mean) | 226 | 227 |
|--------------------------|------------|----|-----------|----|----|
| Probability value | 2.71246757 | 2.71247918 | 2.71238249 | 2.71217797 |
| K=20 dB, for 1701 data MM | Data number | 235 | 236 (mean) | 237 | 238 |
| Probability value | 2.57543251 | 2.57552531 | 2.57552503 | 2.57543207 |
| K=20 dB, for 1701 data TM | Data number | 264 | 265 (mean) | 266 | 267 |
| Probability value | 2.29939718 | 2.29941544 | 2.29936765 | 2.29925407 |
Table 8. The mean values of $f(z), f_1(z), f_2(z)$ and their values around the means K=10 dB, for 1701 data AM

| Data number | Probability value |
|-------------|-------------------|
| 398         | 1.52533995        |
| 399         | 1.52534257        |
| 400         | 1.52532591        |
| 401         | 1.52529002        |

K=10 dB, for 1701 data MM

| Data number | Probability value |
|-------------|-------------------|
| 417         | 1.45500983        |
| 418         | 1.45501578        |
| 419         | 1.45500499        |
| 420         | 1.45497749        |

K=10 dB, for 1701 data TM

| Data number | Probability value |
|-------------|-------------------|
| 469         | 1.29305056        |
| 470         | 1.29305723        |
| 471         | 1.29305214        |
| 472         | 1.29303533        |

Table 9. The mean values of $f(z), f_1(z), f_2(z)$ and their values around the means K=20 dB, for 17001 data AM

| Data number | Probability value |
|-------------|-------------------|
| 2236        | 2.71248695        |
| 2237        | 2.71248756        |
| 2238        | 2.71248709        |
| 2239        | 2.71248554        |

K=20 dB, for 17001 data MM

| Data number | Probability value |
|-------------|-------------------|
| 2361        | 2.56916919        |
| 2362        | 2.56916947        |
| 2363        | 2.56916883        |
| 2364        | 2.56916727        |

K=20 dB, for 17001 data TM

| Data number | Probability value |
|-------------|-------------------|
| 2638        | 2.29941691        |
| 2639        | 2.29941708        |
| 2640        | 2.29941659        |
| 2641        | 2.2994154         |

Table 10. The mean values of $f(z), f_1(z), f_2(z)$ and their values around the means K=10 dB, for 17001 data AM

| Data number | Probability value |
|-------------|-------------------|
| 3976        | 1.52534367        |
| 3977        | 1.52534384        |
| 3978        | 1.52534381        |
| 3979        | 1.52534359        |

K=10 dB, for 17001 data MM

| Data number | Probability value |
|-------------|-------------------|
| 4175        | 1.45292350        |
| 4176        | 1.45292351        |
| 4177        | 1.45292335        |
| 4178        | 1.45292302        |

K=10 dB, for 17001 data TM

| Data number | Probability value |
|-------------|-------------------|
| 4691        | 1.29305056        |
| 4692        | 1.29305723        |
| 4693        | 1.29305214        |
| 4694        | 1.29303533        |

Table 5-Table 10 show the actual, modified and theoretical results for some data group. It can be seen from Table 1, Table 2, and Table 5 through Table 8, both TM and MM values are greater than AM values.
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Table 11. Percent errors according to various K and data numbers

| K voltage (dB) | for 171 data | for 1701 data | for 17001 data |
|---------------|--------------|---------------|---------------|
|               | Error(%)     | Error(%)      | Error(%)      |
| 10            | 0.58         | 1.11          | 1.17          |
| 12            | 0.58         | 0.76          | 0.78          |
| 14            | 0.32         | 0.99          | 1.035         |
| 16            | 0.58         | 0.88          | 0.98          |
| 18            | 0.35         | 0.76          | 0.79          |
| 20            | 0.44         | 0.64          | 0.73          |

For a definite K value if the number of data increases percent error slightly increases as well.

4. Conclusion

In this study, to minimize the difference between the theoretical mean of Rayleigh Probability Density Function and the mean of the plot of Rayleigh function, we modified the equation (10) and obtained the equation (14) as the ultimate expression for the mean.

In actual plots and plots corresponding to theoretical means are shown together.

It is observed from the Table 11 that, for the smaller values of K, percent error increases by the number of data much more than for the larger values of K. For this particular study although the number of data increased 100 times, percent error was just only doubled.

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