Jam phases in two-dimensional cellular automata model of traffic flow

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Abstract

The jam phases in a two-dimensional cellular automata model of traffic flow are investigated by computer simulations. Two different types of the jam phases are found. The spatially diagonal long-range correlation obeys the power law at the low-density jam configurations. The diagonal correlation exponentially decays at the high-density jam. The exponent of the short-range correlation in the diagonal direction is introduced to define the transition between these two phases. We also discuss the stability of the jams.

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I. INTRODUCTION

The investigation of traffic flow has been based mainly on the methods of fluid dynamics. For example it has been studied with the Burgers equation in one-dimensional cases. Many attempts have been done to apply cellular automata modeling to fluid for computational simplicity. In recent years, considerable interests have been developed also in the cellular automata modeling of traffic flow. One of the simplest models of traffic flow in one-way expressways is the rule-184 elementary cellular automaton [1], which is a simple asymmetric exclusion rule. In spite of simplicity of the model, it shows a phase transition from a freely-moving phase at the low vehicle density to a jamming phase at the high vehicle density. More realistic models accounting a variety of speed of cars and effects of blockages have been investigated in one-dimensional models [2–5]. 1/f fluctuation has been observed in both actual expressways [6] and models [3,7].

Two-dimensional cellular automata models, however, have less direct connection to real traffic flow problems. It seems to be an abstract model to a traffic system in a whole city or an expressway network. One of the simplest two-dimensional traffic models has been investigated by Biham, Middleton and Lovine (BML) [8]. They found a sharp transition between a freely-moving phase at the low vehicle density and a jamming phases at the high density. The model has been extended to take the probability of changing directions of cars into account [9]. Nagatani has studied the effect of a traffic accident or a stagnant street to the growth of traffic jams [10,11]. He has also investigated the model with two-level crossings [12].

These studies with cellular automata models have not been concerning only the actual traffic problems or traffic light controls; Attentions also have been paid to the phase transitions and the self-organized behaviors on such simple systems. Most works have investigated the phase transition of the system. However, few have paid attention to the structure and stability of the jam configurations.

In this paper we are interested mainly in the spatial correlation in the jamming phase of
the BML model, a simple model for two-dimensional traffic flow, where cars are represented as right and up arrows exclusively distributed on a square lattice and controlled by a traffic light. As will be seen in later sections, there are two types of the jam phases. We discuss the spatial correlations of these two phases and define the transition point. We also discuss the stability of the jams in both phases.

This paper is organized as follows: We describe the BML model in §2, where the occurrence of the phase transition from the freely-moving phase to the jam phase is mentioned as well. Spatial correlations in the jam phases are discussed in §3. In §4 we investigate the diagonal correlations. The transition point between two type of jams is defined using the short range correlation. The stability of the jam configurations is studied by applying a perturbation which disturbs the jam configuration in §5. The distribution of the lasting time of perturbations is discussed. Section 6 is devoted to summary and discussion.

II. THE MODEL

The model we study is the same as the model-I of Biham, Middleton and Levine [8]. Cars are distributed on a square lattice of \(N \times N\) sites with periodic boundary conditions both on horizontal and vertical directions. Each car is represented as an arrow directed up or right. The model is, therefore, a three state cellular automata with empty sites, up-directed and right-directed cars as inner states of each site. A traffic light controls the dynamics, such that the right arrows move only at odd time steps and the up arrows move at even time steps (In the original BML model, right arrows move at even steps and up ones at odd steps.). At odd time steps, each right arrow moves one site to its right neighborhood if and only if the destination is empty. The corresponding rule is applied for up arrows. The dynamics and the periodic boundary conditions guarantee the conservation of the number of cars for each column and row; There are \(2N\) conservation rules.

The density of right (up) cars is given by

\[
p_\rightarrow = \frac{n_\rightarrow}{N^2},
\]  

(2.1)
where \( n \rightarrow (n\uparrow) \) denotes the number of right (up) arrows. We examine the isotropic case where \( p\rightarrow = p\uparrow = p/2 \) following Biham, Middleton and Levine. They reported the existence of the transition point \( p = p_c \sim 0.35 \) (Nagatani has investigated the transition point to be just under 0.4 in the large system limit \( N \rightarrow \infty \) [13]). Below the transition point, all cars move freely and the average velocity is \( \bar{v} = 1 \). All cars are blocked and the average velocity vanishes above the transition point. Such a sharp transition occurs because the right and up arrows block each other. That is in contrast with one-dimensional models, where the average velocity goes down to zero gradually with the increasing density above the transition point. In the following sections we show results of the \( N = 128 \) simulations.

### III. TWO TYPES OF TRAFFIC JAM

Two types of traffic jam configurations are found in the BML model through the simulation. Figures 1 and 2 show typical traffic jam configurations. We investigate the structures, especially the spatial correlations, of these two configurations.

In the low-density region, a single global cluster of jam is oriented from the lower-left corner to the upper-right one. The backbone of the jam lays diagonally and branches of the jam spread horizontally and vertically like a herringbone. It takes long time to reach the jam state starting from random initial configurations. So the jam state at the low density is far from random configurations and well self-organized. The situation seems to have similarity with traffic jams caused by traffic accidents in the countryside.

In the high density region, on the other hand, patches of small local clusters of jam cover the whole system. There are no apparent global structure. Starting from random initial configurations, it needs only short time to get all cars stopped. Thus randomness initially given is expected to remain. This type of configurations seems to model chronic traffic jams in big cities. An escape from a jam only means catching up the tail of another jam in such a situation.
We investigate spatial correlation functions to study more detailed characteristics of jam configurations. We first define the distribution function of right-directed (up-directed) cars:

\[ \rho_d(\vec{r}) = \sum_i \delta(\vec{r} - \vec{R}_{d,i}), \quad (3.1) \]

where \( d = \rightarrow \) or \( \uparrow \) denotes the directions and

\[ \delta(x) = \begin{cases} 1 & x = 0, \\ 0 & \text{otherwise}. \end{cases} \quad (3.2) \]

The coordinates \( \vec{R}_{\rightarrow,i} \) (\( \vec{R}_{\uparrow,i} \)) are those of the \( i \)-th right (up) cars. The correlation functions are defined as

\[ C_{dd'}(\vec{r}) = \frac{1}{n_d} \left\langle \sum_{\vec{x}} \rho_d(\vec{x}) \rho_{d'}(\vec{x} + \vec{r}) \right\rangle. \quad (3.3) \]

The symbol \( \langle \rangle \) denotes the sample average, namely the average over the jam configurations started from different random initial configurations. The correlation function \( C_{dd'}(\vec{r}) \), therefore, stands for the probability to find a \( d' \)-directed arrow at the relative position \( \vec{r} \) from a \( d \)-directed arrow. If the correlation between the same directed cars vanishes, \( C_{dd}(\vec{r}) \) goes down to the density \( p_d (\vec{r} \neq 0) \). We also define the normalized correlation function

\[ \tilde{C}_{dd}(\vec{r}) = \frac{C_{dd}(\vec{r}) - p_d}{1 - p_d}. \quad (3.4) \]

The contour maps of the normalized correlation functions \( \tilde{C}_{dd}(\vec{r}) \) are shown in Figs. 3, 4 and 5. At the low density (Fig. 3), the correlation spreads diagonally over the system size. The diagonally spreading spatial correlation corresponds to the jam structure shown in Fig. 1. Increasing of the density weakens the diagonal correlation. The strong diagonal correlation still remains at the intermediate density (Fig. 4). The decay of the diagonal correlation causes the vibration in the anti-diagonal direction. At the high density (Fig. 5), at last, the correlation is suppressed reflecting randomness.

IV. DEFINING THE TRANSITION POINT

The spatial correlation in the diagonal direction seems to be a key to characterize the two types of jams. The diagonal correlation \( \tilde{C}_{dd}^{\text{diag}}(r) \) is defined as the spatial correlation \( \tilde{C}_{dd}(\vec{z}) \)
between diagonally separated arrows with the relative position $\vec{z} = r(\hat{x} + \hat{y})$, where circumflex identifies unit vector. At the low-density jam, one can find the diagonal correlation obeys the power law (Fig. 5). At the high-density jam, the long-range diagonal correlation dies out more rapidly than power laws (Fig. 7).

We investigate the exponents of the power law in the diagonal correlation. It can be estimated easily for the low-density jam as

$$\tilde{C}_{\text{dd}}^{\text{diag}}(r) \sim r^{-\beta}, \quad \beta \sim 0.1.$$  \hspace{1cm} (4.1)

On the other hand, the correlation in the high-density jam does not obey the power law. So we approximate the short-range correlation of the power law and define the effective exponent for the high-density jam. Actually we fit the short-range data $(\log r, \log \tilde{C}_{\text{dd}}^{\text{diag}}(r))$, $(1 \leq r \leq n < N/2)$ to a linear function by the method of least squares. Namely, parameters $\beta$ and $\gamma$ are chosen to minimize a squared deviation

$$\chi_n(\beta, \gamma)^2 = \sum_{r=1}^{n} \left( \log \tilde{C}_{\text{dd}}^{\text{diag}}(r) - \beta \log r - \gamma \right)^2,$$  \hspace{1cm} (4.2)

for fixed $n$. Then maximize the range $n$ under the condition

$$\left( \frac{1}{n} \min_{n} \chi_n(\beta, \gamma)^2 \right)^{1/2} < \epsilon,$$  \hspace{1cm} (4.3)

where $\epsilon$ is an adequate constant, which all data in the low-density phase are fit with.

Figure 8 shows the dependence of the exponent $\beta$ on the density $p$ for $N = 64, 128$ and 200 systems. One can find a clear transition between low-density and high-density regions at $p_t = p \sim 0.52$. The low-density jam above the transition point $p_c$, namely $p_c < p < p_t$, shows the power-law diagonal correlation. The exponent of the correlation is $\beta \sim 0.1$. At the high-density jam above the second transition point, $p > p_t$, the diagonal correlation decays. The exponent $\beta$ for the short-range correlation increases in proportional to the density $p$ in the high-density region. The values of the exponent hardly depend on the system size. The values of the exponent themselves, however, seem not to be meaningful for the high-density region, because they depend on the value of $\epsilon$ in Eq. 4.3.
The diagonal correlation can be fit also with exponential functions \( \tilde{C}^{\text{diag}}_{dd}(r) \propto \exp(-r/\xi) \) in for large \( r \). The correlation length \( \xi \) exceeds the system size \( N \) in the low density jam region (Fig. 9). This causes the power-law dependence of the diagonal correlation. The correlation length suddenly decreases to \( \xi \ll N \) near the transition point \( p_t \). The estimated correlation lengths contain errors due to statistical errors and finite-size effect especially in the high-density jam phase.

V. STABILITY OF THE JAM

Finally we investigate the difference of the stability of the jam configurations in both phases mentioned above. For this end, we perturb the jam state to study the stability of the phase after the jamming state appears. In the center of jams, there are blockade pairs with an up arrow blocking the forehead of a right one or a right arrow blocking an up one. To perturb the jam phase, we remove one of such blockade pairs to the tailend of jams leaving a pair of vacant sites in the center of the jam. Then the jamming state melts using this vacant sites. All cars stop again after some steps \( t \) in the unit of a cycle of traffic light. We observe the distribution \( P(t) \) defined as

\[
P(t) = \frac{n(t)}{\sum_{t=0}^{\infty} n(t)},
\]

where \( n(t) \) is a number of events that the effect of the perturbation remains until \( t \).

Let us briefly describe the actual simulation on the model to estimate the distribution \( P(t) \). For the given values of \( N \) and \( p \), we first give an initial random state using an adequate pseudo-random number generator. Starting from this state the system runs until a jam occurs. Then the perturbation mentioned above is applied, namely removing a randomly chosen blockade pair to the tailend of the jam. For example, we consider the case that a blockade is a pair of a right arrow on \((i, j)\)-site and an up on \((i+1, j)\)-site. Then we start to find an empty site from the left-nearest site to the right-next-nearest of \((i, j)\)-site leftward. If \((i-k, j)\)-site is empty, it is marked as a candidate of the destination for the right arrow
on \((i, j)\)-site to move. The same procedure is applied also to the up arrow on \((i + 1, j)\)-site, searching an empty site downward and marking a candidate. If a pair of vacant sites are marked as the destination, each one of the blockade pair is removed to the marked sites. If there is no available pair of empty sites, another blockade pair is tried to find for removing. After removing a blockade pair, the system runs until the next jam occurs. Then the next perturbation is given. For one initial random state, 10,000 such perturbations are applied repeatedly. These 10,000 data are obtained in one run of the simulation. The detailed method to remove a blockade pair would not affect the results qualitatively.

The distributions \(P(t)\) at the low-density jam are shown in Fig. 10. The effect of the perturbation remains for long time. It propagates along the sequences of blockade pairs which compose the center of the jam. So the distribution \(P(t)\) has a peak corresponding to the system size (Fig. 10).

At the high-density jam (Fig. 11), on the other hand, the effect of the perturbation vanishes in shorter times than the low-density cases. The distribution \(P(t)\) decays in the large \(t\) region. The decay rate of \(P(t)\) is not clear at this stage.

VI. DISCUSSION

We studied the properties of the jam in a two-dimensional cellular automata model of traffic flow. The basic model treated here is the same as the model-I of Biham, Middleton and Levine [8]. We found two types of the jam (Fig. 12). At the low density above the transition \((p_c < p < p_t)\), the spatially diagonal long-range correlation appears. The backbone of the jam, sequences of blockade pairs, lays diagonally through the whole system. The branches of the jam spread horizontally and vertically. Thus the jam configurations are well-organized. We call this type of jams as self-organized jam. At the high-density jam \((p > p_t)\), on the other hand, local clusters of jam cover the whole system and no global structure remains. We call this type of jams as random jam.

The spatial correlations were studied to characterize these two type of jam configurations.
The spatially diagonal correlation obeys the power law at the low-density jam. Spatial correlations decay with increment of the density. We defined the short range exponent of the diagonal correlation for the high-density jam. The transition point $p_t$ could be defined by the change in the values of the exponent.

The long-range diagonal correlations decay exponentially reflecting the randomness of the system. The correlation length $\xi$ exceeds the system size $N$ in the low-density jam phase ($p < p_t$). It is expected that the correlation length remain finite in the $N \to \infty$ limit. We, however, have no clear evidence of the finite correlation length in the limit. Another possibility is that the correlation length diverges in the thermodynamic limit. In this case, it is expected that the two transition points $p_c$ and $p_t$ coincide with each other, and the low-density jam phase will be observed only just at $p_c$.

The stability of these jam configurations was investigated by applying perturbations to the jam configurations. We investigated the distribution $P(t)$ of the time that the effect of the perturbation remains. The distribution $P(t)$ at the low density above the transition has a peak corresponding to the system size reflecting the long-range correlation. Detailed analysis of the distribution $P(t)$ is discussed elsewhere.

The power-law relations of the diagonal correlation remind us of the self-organized critical properties of the jam configurations [14]. The jam configurations seem to be self-organized well at the low-density jam. At the high density, however, the initial randomness remains and thus the jam configurations are not self-organized. We have no explanation on the origin of the power law and the values of the exponent at present.

The occurrence of the jam may be affected by boundary conditions and isotropy especially at the low density. The effect of anisotropy ($p_\rightarrow \neq p_\uparrow$) has been studied by Nagatani [13]. He pointed out that strong anisotropy prevents the system from the jam. The effect of another isotropy is still unclear, namely the effect of the equality of the horizontal and vertical system sizes.

Cellular automata models of traffic flow are not restricted to concern traffic flow problems. They are simplified abstract models of exclusion processes. On one-dimensional models
analogies to the ballistic deposition have been found \cite{8,15,17}. Although any connection of the two-dimensional models to physical systems have never been discussed so far, studies on these systems are expected to clarify behaviors of complex systems.

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REFERENCES

[1] S. Wolfram, Rev. Mod. Phys. 55, 601 (1983).

[2] K. Nagel and M. Schreckenberg, J. Phys. I France 2, 2221 (1992).

[3] K. Nagel and H. J. Herrmann, *Deterministic models for traffic jam*, HLRZ preprint 46/93 (1993).

[4] A. Schadschneider and M. Schreckenberg, J. Phys. A26, L679 (1993).

[5] S. Yukawa, M. Kikuchi and S. Tadaki, *Dynamical phase transition in one dimensional traffic flow model with blockage*, J. Phys. Soc. Jap. 63 (1994) in press.

[6] T. Musha and H. Higuchi, Jap. J. Appl. Phys. 15, 1271 (1976).

[7] M. Takayasu and H. Takayasu, Fractals 1, 860 (1993).

[8] O. Biham, A. A. Middleton and D. Levine, Phys. Rev. A46, 6124 (1992).

[9] J. A. Cuesta, F. C. Martínez, J. M. Molera and A. Sánchez, Phys. Rev. E48, 4175 (1993).

[10] T. Nagatani, J. Phys. Soc. Jap. 62, 1085 (1993).

[11] T. Nagatani, Physica A198, 108 (1993).

[12] T. Nagatani, Phys. Rev. E48, 3290 (1993).

[13] T. Nagatani, J. Phys. Soc. Jap. 62, 2625 (1993).

[14] P. Bak, C. Tang and K. Wiesenfeld, Phys. Rev. Lett. 59, 381 (1987), Phys. Rev. A38, 364 (1988).

[15] P. Meakin, P. Ramanlal, L. M. Sander and R. C. Ball, Phys. Rev. A34, 5091 (1986).

[16] J. Krug and H. Spohn, Phys. Rev. A38 4271 (1988).

[17] S. A. Janowsky and J. L. Lebowitz, Phys. Rev. A45 618 (1992).
FIGURES

FIG. 1. A typical jamming configuration in the low-density region above the transition $p_c$. The global cluster of jam is oriented diagonally. The system size is $32 \times 32$ and $p = 408 / (32 \times 32) \sim 0.4$.

FIG. 2. A typical jamming configuration in the high-density region above the transition $p_c$. The system is covered by local clusters of jam. The system size is $32 \times 32$ and $p = 920 / (32 \times 32) \sim 0.9$.

FIG. 3. The normalized correlation functions $\tilde{C}_{dd}(\vec{r})$ at the low density. The upper diagram is of $\tilde{C}_{\rightarrow \rightarrow}(\vec{r})$ and the lower $\tilde{C}_{\uparrow \uparrow}(\vec{r})$. The correlation spreads diagonally over the system size. The high value regions at the upper-left and lower-right corners are caused by the finite-size effects. The system size is $128 \times 128$ and $p = 6552 / (128 \times 128) \sim 0.4$. The average is taken over 10 samples.

FIG. 4. The normalized correlation functions $\tilde{C}_{dd}(\vec{r})$ at the intermediate density. The upper diagram is of $\tilde{C}_{\rightarrow \rightarrow}(\vec{r})$ and the lower $\tilde{C}_{\uparrow \uparrow}(\vec{r})$. The correlation mainly spreads diagonally over the system size. There appears the vibration in the anti-diagonal direction in contrast to the low-density jam. The system size is $128 \times 128$ and $p = 6552 / (128 \times 128) \sim 0.4$. The average is taken over 10 samples.

FIG. 5. The normalized correlation functions $\tilde{C}_{dd}(\vec{r})$ at the high-density. The upper diagram is of $\tilde{C}_{\rightarrow \rightarrow}(\vec{r})$ and the lower $\tilde{C}_{\uparrow \uparrow}(\vec{r})$. The values of $\tilde{C}_{dd}(\vec{r})$ are almost zero over the whole region except the vicinity of the coordinate origin. The system size is $128 \times 128$ and $p = 14744 / (128 \times 128) \sim 0.9$. The average is taken over 10 samples.

FIG. 6. The diagonal correlation $\tilde{C}_{\text{diag}}(r)$, the normalized correlation between right arrows separated diagonally at the distance $2^{1/2}r$, are plotted for $p \leq 0.6$. The average is taken over 10 samples as the same as Figs. 3 and 5. The lines are fit with the method of least squares with $\epsilon = 0.05$ in Eq. 4.3.
FIG. 7. The diagonal correlation $\tilde{C}_{\text{diag}}(r)$, the normalized correlation between right arrows separated diagonally at the distance $2^{1/2}r$, are plotted for $p \geq 0.6$. The average is taken over 10 samples as the same as Figs. 3 and 5. The lines are fit with the method of least squares for small $r$ with $\epsilon = 0.05$ in Eq. 4.3.

FIG. 8. Dependence of the exponents $\beta$ on the density $p$.

FIG. 9. Diagonal correlation length $\xi$. It reaches the system size $N$ at $p = 0.57$.

FIG. 10. The distribution $P(t)$ at $p = 6552/(128 \times 128) \sim 0.4$ and $N = 128$. The data corresponds to the summary of 3 runs, one run means 10,000 data as mentioned in the text. It has a peak corresponding to the system size.

FIG. 11. The distribution $P(t)$ at $p > 0.6$ and $N = 128$. Each data corresponds to the summary of 3 runs, one run means 10,000 data as mentioned in the text.

FIG. 12. Schematic phase diagram of two-dimensional traffic flow model.