ON THE CLASSIFICATION OF THE SPECTRA LLY STABLE STANDING WAVES OF THE HARTREE PROBLEM

VLADIMIR GEORGEV AND ATANAS STEFANOV

Abstract. We consider the fractional Hartree model, with general power non-linearity and space dimension. We construct variationally the “normalized” solutions for the corresponding Choquard-Pekar model - in particular a number of key properties, like smoothness and bell-shapedness are established. As a consequence of the construction, we show that these solitons are spectrally stable as solutions to the time-dependent Hartree model.

In addition, we analyze the spectral stability of the Moroz-Van Schaftingen solitons of the classical Hartree problem, in any dimensions and power non-linearity. A full classification is obtained, the main conclusion of which is that only and exactly the “normalized” solutions (which exist only in a portion of the range) are spectrally stable.

1. Introduction

We consider the Cauchy problem for the Hartree equation

$$\begin{cases}
    iu_t + (-\Delta)^\beta u - v|u|^{p-2}u = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d \\
    (-\Delta)^{\alpha/2} v = |u|^p, & u(0, x) = u_0(x).
\end{cases}$$

(1.1)

Here, the operator $(-\Delta)^\beta$ is defined via Fourier multiplier with $|2\pi \xi|^{2\beta}$, see the relevant definition below in Section 2. Unless otherwise indicated in a particular place, the values of the parameters will be henceforth as follows: $\beta \in (0, 1], d \geq 1, p > 1, \alpha \in (0, d)$. Resolve the elliptic equation

$$v = (-\Delta)^{-\alpha/2} |u|^p = I_\alpha |u|^p = I_\alpha \ast |u|^p, I_\alpha(x) = \frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\pi^{d/2} \alpha^\alpha |x|^{d-\alpha}}.$$

We obtain the system

$$\begin{cases}
    iu_t + (-\Delta)^\beta u - c_{d,\gamma} [\cdot]^{-\gamma} \ast |u|^p |u|^{p-2}u = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d \\
    u(0, x) = u_0(x),
\end{cases}$$

(1.2)

where we have introduced the parameter $\gamma := d - \alpha \in (0, d)$.

We will be interested in the properties of standing wave solutions $u(t, x) = e^{i\omega t}\phi(x)$, with $\phi > 0$. Clearly, $\phi = \phi_{p,\omega}$ will then satisfy the profile equation

$$(-\Delta)^\beta \phi - c_{d,\gamma} [\cdot]^{-\gamma} \ast |\phi|^p |\phi|^{p-2}\phi = \omega \phi, \quad x \in \mathbb{R}^d.$$

1991 Mathematics Subject Classification. Primary 35Q55, 35P10; Secondary 42B37, 42B35.

Key words and phrases. semilinear, ground states, Schrödinger equation, Klein-Gordon equation.

Georgiev is supported in part by INDAM, GNAMPA - Gruppo Nazionale per l’Analisi Matematica, la Probabilita e le loro Applicazioni, by Institute of Mathematics and Informatics, Bulgarian Academy of Sciences and Top Global University Project, Waseda University. Stefanov is partially supported by NSF-DMS, Applied Mathematics program under grants # 1313107 and # 1614734.

1Sometimes we shall use the notation $|\nabla| = \sqrt{-\Delta}$. 

arXiv:1702.03374v1 [math.AP] 11 Feb 2017
The equation (1.3) is (a fractional) version of the well-known Choquard equation. This is a good point for us to review some of the developments in the classical theory for this model.

1.1. The classical Hartree-Choquard-Pekar model. As one expects, most of the work was done in the classical context, \( \beta = 1 \), for the Hartree-Choquard-Pekar system (for \( \alpha \in (0, d) \))

\[
(1.4) \quad iu_t - \Delta u - I_\alpha[|u|^p]|u|^{p-2}u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d.
\]

The standing wave solutions of the form \( e^{-it}\varphi \) satisfy

\[
(1.5) \quad -\Delta \varphi + \varphi - I_\alpha[|\varphi|^p]|\varphi|^{p-2}\varphi = 0.
\]

The question for existence of localized solutions for (1.5) has been well-studied in the last thirty years or so, mostly for special values of the parameters. For example, the case of \( p = 2 \) has been studied in [12, 13, 15] by the variational approach, and in [20] by ODE techniques. The case \( \gamma = d - 2 \geq 1 \) and \( 2 \leq p < (2d - \gamma)/(d - 2) \), was previously considered in [5] by introducing the constraint minimizer similar to the one of Section 3 below.

Quite recently, a general classification result for such solutions was put forward in [14] and a complete proof was presented in [16]. The following theorem is a summary of the results presented in Theorems 1, 2, 3 in [16].

**Theorem 1.** Let \( d \geq 1, \alpha \in (0, d) \) and \( p \in (1, \infty) \).

Assuming \( \frac{d-2}{2d-\gamma} < \frac{1}{p} < \frac{d}{2d-\gamma} \), there is a solution \( \varphi \in H^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \) of (1.5). Moreover, these solutions are found in the form \( \varphi = t_0\Phi \), where \( \Phi \) is a minimizer of the following optimization problem

\[
(1.6) \quad \inf_{u \neq 0} \frac{\int_{\mathbb{R}^d}(|\nabla u(x)|^2 + |u(x)|^2)dx}{\langle I_\alpha[|u|^p], |u|^p \rangle^{1/p}}
\]

and the scalar \( t_0 \) is selected so that \( \int_{\mathbb{R}^d}(|\varphi(x)|^2 + |\varphi(x)|^2)dx = \langle I_\alpha[|\varphi|^p], |\varphi|^p \rangle \).

In addition, there exists \( x_0 \in \mathbb{R}^d \) and a decreasing function \( \rho : \mathbb{R}_+ \to \mathbb{R}_+ \), so that \( \varphi(x) = \pm \rho(|x - x_0|) \). Finally, \( \varphi \) satisfies the Pohozaev’s identity

\[
(1.7) \quad (d - 2) \int_{\mathbb{R}^d} |\nabla \varphi|^2 + d \int_{\mathbb{R}^d} |\varphi|^2 = \frac{2d - \gamma}{p} \langle I_\alpha[|\varphi|^p], |\varphi|^p \rangle.
\]

In the complementary range: \( \frac{1}{p} \leq \frac{d-2}{2d-\gamma} \) or \( \frac{1}{p} \geq \frac{d}{2d-\gamma} \), the only regular and localized solution of (1.5) is \( u = 0 \).

We note that functions in the form \( u(x) = \rho(|x|) \), where \( \rho : \mathbb{R}_+ \to \mathbb{R}_+ \) is a decreasing and vanishing at infinity, are called bell-shaped. An equivalent way of expressing the same property is the equality \( u = u^* \), where \( u^* \) is the decreasing rearrangement in the sense of Riesz. In any case, the statement of Theorem 1 implies that the only solutions of (1.5) are translates of bell-shaped functions.

1.2. The Klein-Gordon-Hartree model. We also consider the related Klein-Gordon-Hartree model\(^2\) but we have preferred to just consider the classical

\[
(1.8) \quad u_{tt} - \Delta u + u - I_\alpha[|u|^p]|u|^{p-2}u = 0.
\]

\(^2\)Here, we could have considered the more general fractional version of the model, similar to (1.1). Since we can only present a complete stability classification only in the case \( \beta = 1 \), we prefer to state only the classical form, (1.8).
Standing wave solutions in the form $u(t, x) = e^{i\omega t} \Psi$ must of course satisfy the elliptic PDE (1.9)

$$- \Delta \Psi + (1 - \omega^2) \Psi - I_\alpha |\Psi|^p |\Psi|^{p-2} \Psi = 0,$$

which is of course closely related to (1.5), provided $|\omega| < 1$, which we assume henceforth. A simple rescaling argument, together with Theorem 1, allows us to conclude that there are bell-shaped solutions of (1.9) in the form

$$\Psi(x) = (1 - \omega^2)^{\frac{d-2}{2d-\gamma}} \varphi(x \sqrt{1 - \omega^2}),$$

where $\varphi$ is the set of solutions described in Theorem 1.

1.3. **Main results.** Our results concern both the fractional model (1.2) and the more classical version (1.4). More precisely, we are interested in the existence properties of solitary waves for (1.2), that is whether and under what conditions, one obtains nice ground state solutions of (1.3).

**1.3.1. The fractional Choquard equation - existence and stability.** This calls for a generalization of Theorem 1 above, at least in the existence part of it. We have the following existence result.

**Theorem 2.** Let $\beta \in (0, 1], \gamma \in (0, d)$ and $p > 1$. Assume in addition the relationship

$$0 < (p-2)d + \gamma < 2\beta. \tag{1.11}$$

Then, there exists a solution of (1.3), $\phi$, namely a solution of a constrained minimization problem (3.1) below. Moreover, $\phi$ is bell-shaped.

Note that the inequality $0 < (p-2)d + \gamma$ is exactly equivalent to the requirement $\frac{1}{p} < \frac{d}{2d - \gamma}$ from Theorem 1. The other inequality however, say for the classical case $\beta = 1$, is $p < 2 + \frac{2\gamma}{d}$, which is a strict subset of the requirement $\frac{d-2}{2d-\gamma} < \frac{1}{p}$ imposed in Theorem 1. So, we do not seem to get all the solitary waves in this way, more on this point below. In fact, this brings us to our second object of interest, namely the stability of the waves constructed in Theorem 2. It turns out that the waves constructed in Theorem 2 are spectrally stable as solutions of (1.2). More precisely, we have the following result.

**Theorem 3.** Let $p > 2$. Then, the ground states $\phi$ constructed in Theorem 2 are spectrally stable as solutions of (1.2).

**Remark:** The condition $p > 2$ appears to be of a technical nature and it is likely removable, if one knows extra information about the waves constructed in Theorem 2 - similar to Lemma 2 and 3 below.

The waves constructed in Theorem 2 are constructed as the minimizers of the problem

$$\inf_{\|u\|_{L^2} = \lambda} E(u) \text{ (dubbed “normalized solutions” in [16])},$$

where the energy functional is given by

$$E(u) := \frac{1}{2} \int_{\mathbb{R}^d} \left[ |\nabla u(x)|^2 + |u(x)|^2 \right] dx - \frac{c_{d, \gamma}}{2p} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x)|^p |u(y)|^p}{|x-y|^{\gamma}} dx dy.$$

They turn out to be spectrally stable, per the claim of Theorem 3. It so happens these are all the stable solitary waves there are, at least in the classical case $\beta = 1$, as we discuss now.
1.3.2. The classical Choquard equation - classification of the stable ground states. In the classical Choquard case, (1.5), we provide a full description of the localized and regular solutions, given in Theorem 1. A natural question is then: which of these waves are spectrally stable as solutions of (1.4)? The full classification is provided in the following theorem.

**Theorem 4.** Let \( d \geq 1, \alpha \in (0, d) \) and \( p \in (1, \infty) \), so that \( \frac{d-2}{d+\alpha} < \frac{1}{p} < \frac{d}{d+\alpha} \).

Consider any solution \( \phi \) of (1.5) guaranteed by Theorem 1. Then, the solution \( e^{-it}\phi \) of the time dependent Hartree problem (1.4) is spectrally stable if and only if

\[
\Gamma := 2 - \gamma - (p - 2)d = 2 + \alpha - (p - 1)d \geq 0.
\]

More specifically,

- If \( d = 1, 2 \), the MVS waves are stable if and only if \( 1 + \frac{\alpha}{d} < p \leq 1 + \frac{2+\alpha}{d} \) and unstable in the complementary range \( 1 + \frac{2+\alpha}{d} < p < \infty \). The instability presents itself as a simple growing mode.
- If \( d \geq 3 \), the MVS waves are stable if \( 1 + \frac{\alpha}{d} < p \leq 1 + \frac{2+\alpha}{d} \) and unstable in the complementary range \( 1 + \frac{2+\alpha}{d} < p < 1 + \frac{3+\alpha}{d+2} \). The instability presents itself as a simple growing mode.

A few remarks are in order.

1. Note that the statement in Theorem 4 agrees well with Theorem 2. In particular, we find in Theorem 4 that the only stable solitons for the Hartree model are the normalized solutions - that is, those obtained in the range \( \Gamma = 2 + \alpha - (p - 1)d \geq 0 \).
2. Some of the instability results have been previously established by other methods. In particular, in the case \( d = 3 \) and in the optimal range \( \frac{5+\alpha}{3} < p < 3 + \alpha \), strong instability was established in [2], see also a very recent extension of these results to Hartree models with potentials in [3]. In these works, the authors employ a virial identity type arguments, which show that there exists data arbitrarily close to the soliton, for which the solution blows up in finite time.
3. Note that in the limit \( \alpha \to 0+ \), we recover the stability results for the NLS model (with power non-linearity of order \( q = 2p - 1 \)). This is indeed the case, since (formally as \( \alpha \to 0+ \)) one obtains stability for \( 1 < p < 1 + \frac{2}{d} \), which is equivalent to \( q = 2p - 1 \in (1, 1 + \frac{4}{d}) \), the well-known Schrödinger result.
4. In the case \( \Gamma = 0 \) or equivalently \( p = 1 + \frac{2+\alpha}{d} \), we discover that there is an extra pair of elements in the generalized kernel of \( L_+ \), \( gKer[L_+] \). Indeed, using \( p \) as a bifurcation parameter, one sees that starting from \( p > 1 + \frac{2+\alpha}{d} \), there is a pair of stable/unstable eigenvalues which approaches the origin and it turns into a pair of purely imaginary eigenvalues for \( p < 1 + \frac{2+\alpha}{d} \). At \( p = 1 + \frac{2+\alpha}{d} \), this pair introduces extra two dimensions in \( gKer[L_+] \). This is very similar to the pseudo-conformal symmetry for the standard Schrödinger equation, which arises only for \( p = 1 + \frac{4}{d} \). We thus conjecture that there is an extra symmetry, for this particular case, which generates this extra algebraic multiplicity of the zero eigenvalue.

Our next result is a complete characterization of the spectral stability for the waves \( e^{i\omega t}\Psi \) of (1.9) in the Klein-Gordon-Hartree context.

**Theorem 5.** Let \( d \geq 1, \alpha \in (0, d), \omega \in (-1, 1), p \in (1, \infty) \) and \( \frac{d-2}{d+\alpha} < \frac{1}{p} < \frac{d}{d+\alpha} \).
Let $\varphi$ is a MVS solution of (1.5), which exists in the specified range of $p$ according to Theorem 1. Then, the solution $e^{i\omega t}\varphi(x)$ described in (1.10) of the time dependent Klein-Gordon-Hartree problem (1.8) is spectrally stable if and only if

$$\Gamma > 0, \sqrt{\frac{p-1}{p-1+\Gamma}} = \sqrt{\frac{p-1}{2+\alpha-(p-1)(d-1)}} < |\omega| < 1.$$ 

More precisely,

- If $d = 1, 2$, the MVS waves are unstable, if $1 + \frac{2+\alpha}{d} < p < \infty$ or $p \in (1 + \frac{\alpha}{d}, 1 + \frac{2+\alpha}{d}]$ and $0 \leq |\omega| < \sqrt{\frac{p-1}{2+\alpha-(p-1)(d-1)}}$. Equivalently, the waves are stable, exactly when

$$p \in (1 + \frac{\alpha}{d}, 1 + \frac{2+\alpha}{d}), \sqrt{\frac{p-1}{2+\alpha-(p-1)(d-1)}} < |\omega| < 1.$$ 

- If $d \geq 3$, the waves are unstable for $p \in (1 + \frac{2+\alpha}{d}, 1 + \frac{2+\alpha}{d-2})$ or $p \in (1 + \frac{\alpha}{d}, 1 + \frac{2+\alpha}{d}]$ and $0 \leq |\omega| < \sqrt{\frac{p-1}{2+\alpha-(p-1)(d-1)}}$. Equivalently, stability occurs exactly for

$$p \in (1 + \frac{\alpha}{d}, 1 + \frac{2+\alpha}{d}), \sqrt{\frac{p-1}{2+\alpha-(p-1)(d-1)}} < |\omega| < 1.$$ 

2. Preliminaries

The Fourier transform and its inverses are taken to be in the form

$$\hat{\hat{f}}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x \cdot \xi}dx, f(x) = \int_{\mathbb{R}^d} \hat{\hat{f}}(\xi)e^{2\pi i x \cdot \xi}d\xi.$$ 

The operators $(-\Delta)^\beta$ are defined through their multipliers (acting on Schwartz functions $f \in \mathcal{S}$) as follows

$$(-\Delta)^\beta f(\xi) = |2\pi\xi|^{2\beta}\hat{\hat{f}}(\xi).$$ 

Note that sometimes, we will use instead the Zygmund operator $|\nabla| := \sqrt{-\Delta}$. We make heavy use of the symmetric decreasing rearrangements of a function $f$, denoted by $f^\ast$. This is a classical object, see for example [11], Chapter 3. In that regard, recall that $\|f^\ast\|_{L^p} = \|f\|_{L^p}$, for $1 \leq p \leq \infty$. In addition, we make use of the classical inequality

$$\int_{\mathbb{R}^d} f(x)g(x)dx \leq \int_{\mathbb{R}^d} f^\ast(x)g^\ast(x)dx,$$

for any non-negative functions $f, g$ decaying sufficiently rapidly at infinity (see Theorem 3.4 in [11]). If $f$ is a strictly symmetric decreasing function then we have equality in (2.1) only if $g = g^\ast$ a.e. A more sophisticated version of (2.1) is the Riesz’s rearrangement inequality (see Theorem 3.7, [11])

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} f(x)g(x-y)h(y)dxdy \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} f^\ast(x)g^\ast(x-y)h^\ast(y)dxdy.$$ 

If one of the functions is in fact strictly symmetric decreasing, then equality in (2.2) is possible if the other two functions are a fixed translate of a symmetric decreasing function.

The Polya-Szego inequality states that $\|\nabla u\|_{L^2(\mathbb{R}^d)} \geq \|\nabla u^\ast\|_{L^2(\mathbb{R}^d)}$. Here, we present an extension of this inequality for fractional gradients. This is a relatively recent result.
fact, there is a proof of this fact in [4], using completely monotone maps. Here we present a simpler proof, based on the representation of $|\nabla|^\beta f$ in terms of averages of the standard heat kernel operators $e^{t\Delta}$.

**Proposition 1.** Let $\beta \in (0, 1]$, $d \geq 1$. Then, for all functions $u \in \dot{H}^\beta$, we have that its decreasing rearrangement $u^* \in \dot{H}^\beta$ and moreover

$$\| |\nabla|^\beta u\|_{L^2(\mathbb{R}^d)} \geq \| |\nabla|^\beta u^*\|_{L^2(\mathbb{R}^d)}.$$  

In addition, equality is achieved if and only if there exists $x_0 \in \mathbb{R}^d$ and a decreasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$, so that $u(x) = \rho(|x - x_0|)$.

**Note:** The classical Polya-Szego inequality is the particular case $\beta = 1$.

**Proof.** Let $\beta < 1$ and define

$$c_\beta := \int_0^{\infty} \frac{1 - e^{-y}}{y^{1+\beta}} dy.$$  

Setting $y = 4\pi^2|\xi|^2t$, we have the representation

$$(2\pi|\xi|)^{2\beta} = \frac{1}{c_\beta} \int_0^{\infty} \frac{1 - e^{-4\pi^2|\xi|^2t}}{t^{1+\beta}} dt.$$  

Equivalently

$$|\nabla|^{2\beta} = \frac{1}{c_\beta} \int_0^{\infty} \frac{1 - e^{t\Delta}}{t^{1+\beta}} dt.$$  

Since $e^{t\Delta}f = K_t*f$ and $K_t(x) = (4\pi t)^{-d/2}e^{-|x|^2/(4t)}$ is strictly symmetric decreasing, we have by (2.2) that $\langle e^{t\Delta}u, u \rangle = \langle K_t*u, u \rangle \leq \langle K_t*u^*, u^* \rangle = \langle e^{t\Delta}u^*, u^* \rangle$ and equality is achieved only if $u(x) = \rho(|x - x_0|)$ for a decreasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ and $x_0 \in \mathbb{R}^d$. Thus,

$$\| |\nabla|^\beta u\|^2 = \langle |\nabla|^{2\beta}u, u \rangle = \frac{1}{c_\beta} \int \frac{\langle u, u \rangle - \langle e^{t\Delta}u, u \rangle}{t^{1+\beta}} dt \geq \frac{1}{c_\beta} \int \frac{\langle u^*, u^* \rangle - \langle e^{t\Delta}u^*, u^* \rangle}{t^{1+\beta}} dt = \langle |\nabla|^{2\beta}u^*, u^* \rangle = \| |\nabla|^\beta u^*\|^2.$$  

Moreover, equality is possible, only if $u(x) = \rho(|x - x_0|)$, as explained above. \qed

### 3. Existence and properties of the solutions to the fractional Hartree model

For $\lambda > 0$, introduce the optimization problem

$$\min_{\int_{\mathbb{R}^d} |u(x)|^2 dx = \lambda} \left\{ E(u) := \frac{1}{2} \| |\nabla|^\beta u\|_{L^2(\mathbb{R}^d)}^2 - \frac{c_d}{2p} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x)|^p|u(y)|^p}{|x-y|^\gamma} dx dy \right\}$$

At least formally, one can see that the associated Euler-Lagrange equation is exactly (1.3). We are now ready to proceed with the proof of the existence result in Theorem 2.
3.1. Existence of solutions for the constrained minimization problem. More precisely, we have the following.

**Proposition 2.** Let $\beta \in (0, 1], \gamma \in (0, d)$ and $p > 1$ and the relation (1.11) holds. Then, the optimization problem (3.1) has a bell-shaped solution $\varphi$. Moreover, for every solution $u_0$ of (3.1), there exists $x_0 \in \mathbb{R}^d$, so that $u_0 = \pm \rho(|x - x_0|)$, where $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ is a decreasing and vanishing function. Finally, for every $\lambda > 0$, $E_\lambda = \inf_{\|u\|^2 = \lambda} E(u) = E(\varphi) < 0$.

**Proof.** (Theorem 2) First, we show that the constrained minimization problem (3.1) is well-posed. That is, the quantity $E(u)$ is bounded from below, when $u$ obeys the constraint $\int_{\mathbb{R}^d} |u(x)|^2 dx = \lambda > 0$. To this end, note that we can interpret the potential energy term (or Hartree interaction term) as follows

$$\int_{\mathbb{R}^d x \mathbb{R}^d} \frac{|u(x)|^p |u(y)|^p}{|x - y|^\gamma} dxdy = \langle | \cdot |^{-\gamma} * |u|^p, |u|^p \rangle.$$ 

Thus, by Hölder’s and the Hardy-Littlewood-Sobolev inequalities, we have

$$\langle | \cdot |^{-\gamma} * |u|^p, |u|^p \rangle \leq \|u\|_{2p \prime}^{2p} \| \cdot |^{-\gamma} \|_{L^\frac{d}{\gamma}, \infty} = C_d \|u\|_{L^{2p \prime}}^{2p},$$

with $r = \frac{2d}{\gamma}$. Denote $q = pr'$. One can check that $q \geq 2$ is equivalent to the constraint $(p - 2)d + \gamma \geq 0$, which is one of the requirements in (1.11). By Sobolev embedding and the Gagliardo-Nirenberg’s inequalities, we have

$$\|u\|_{L^{pr'}} = \|u\|_{L^2} \leq C_d \|u\|_{H^s} \leq C_d \|u\|_{L^\frac{s}{\beta}}^s \|u\|_{L^2}^{1 - \frac{s}{\beta}},$$

where $s = d(\frac{1}{2} - \frac{1}{q})$, provided $s < \beta$ (still to be verified under (1.11)). In turn, this yields

$$c_{d, \gamma} \langle | \cdot |^{-\gamma} * |u|^p, |u|^p \rangle \leq C_d \|\nabla\|^{\beta}u\|_{L^2}^{2p} \|u\|_{L^2}^{2p - 2p},$$

so we have

$$c_{d, \gamma} \langle | \cdot |^{-\gamma} * |u|^p, |u|^p \rangle \leq C_{d, \lambda} \|\nabla\|^{\beta}u\|_{L^2}^{2p}.$$ 

Now, the right-hand side of the constraint (1.11) ensures exactly that $\frac{2ps}{\beta} < 2$, so in particular $s < \beta$ (since $p > 1$), which was required earlier. Hence, by Young’s inequality

$$E(u) \geq \frac{1}{2} \|\nabla\|^{\beta}u\|^{2} - C_{d, \lambda} \|\nabla\|^{\beta}u\|^{2p} \geq M_{d, \lambda},$$

which is the desired control from below of the cost functional $J$. Introduce

$$E_\lambda = \inf_{u^2(x)dx = \lambda} \frac{1}{2} \|\nabla\|^{\beta}u\|^{2} - \frac{c_{d, \gamma}}{2p} \int_{\mathbb{R}^d x \mathbb{R}^d} \frac{u^p(x)u^p(y)}{|x - y|^\gamma} dxdy,$$

which we know from our previous arguments exists.

Next, we discuss the existence and the other properties of the constrained minimizers. We work with a fixed $\lambda$, so we omit the superscript in $\phi^\lambda$. Take a minimizing sequence, say $u_n$, and $\lim_n E(u_n) = E_\lambda$. We have by the Polya-Szegö inequality, (2.3)

$$\|\nabla\|^{\beta}u_n\|_{L^2(\mathbb{R}^d)} \geq \|\nabla\|^{\beta}u_n^\ast\|_{L^2(\mathbb{R}^d)}.$$
In addition, we have by the Riesz rearrangement inequality
\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u_n(x)|^p |u_n(y)|^p}{|x-y|^\gamma} \, dx \, dy = \langle | \cdot |^{-\gamma} * u_n^p, u_n^p \rangle \leq \langle | \cdot |^{-\gamma} (u_n^*)^p, (u_n^*)^p \rangle =
\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(u_n^*)^p(x)(u_n^*)^p(y)}{|x-y|^\gamma} \, dx \, dy.
\]
Combining the last two estimates tells us that \( E(u_n) \geq E(u_n^*) \), while \( \|u_n^*\|_L^2 = \lambda \). Hence, \( \lim_n E(u_n^*) = E \lambda \) and \( u_n^* \) is uniformly bounded sequence in \( H^1(\mathbb{R}^d) \).

Moreover, \( u_n^* \) are now bell-shaped functions in the unit sphere of \( L^2 \), so they have a weakly convergent subsequence (denoted again \( u_n^* \)), converging weakly in \( L^2 \) to say \( \phi \), a bell-shaped function. By the lower semi-continuity of the norm with respect to weak convergence, \( \|\phi\|^2 \leq \lambda \) and also (note that \( |\nabla|^\beta u_n^* \) converges weakly to \( |\nabla|^\beta \phi \))
\[
\text{(3.3)} \quad \liminf \| |\nabla|^\beta u_n^* \|_L^2 \geq \| |\nabla|^\beta \phi \|_L^2.
\]

We also have that for every \( x : |x| > 0 \),
\[
\lambda = \int_{\mathbb{R}^d} |u_n^*(y)|^2 \, dy \geq \int_{|y|<|x|} |u_n^*(y)|^2 \, dy \geq c_d \cdot | |u_n^*(x)|^2 |,
\]
whence \( |u_n^*(x)| \leq C_d |x|^{-d/2} \) for every \( x \in \mathbb{R}^d, x \neq 0 \).

It follows that \( \{ u_n^* \} \) is a compact sequence in any \( L^q, q > 2 \) (Rellich-Kondrashov’s), hence we can assume (after taking subsequences) \( \lim_n \| u_n^* - \phi \|_{L^q} = 0 \) for any \( q > 2 \). As a consequence, we claim that
\[
\text{(3.4)} \quad \lim_n \int_{\mathbb{R}^2} \frac{(u_n^*)^p(x)(u_n^*)^p(y)}{|x-y|^\gamma} \, dx \, dy = \int_{\mathbb{R}^2} \frac{\phi^p(x) \phi^p(y)}{|x-y|^\gamma} \, dx \, dy.
\]
Indeed, denoting \( K(u,v) := \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x)|^p |v(y)|^p}{|x-y|^\gamma} \, dx \, dy \), we have by triangle inequality and the Hardy-Littlewood-Sobolev inequalities displayed earlier
\[
|K(u_n^*, u_n^*) - K(\phi, \phi)| \leq |K(u_n^*, u_n^*) - K(\phi, u_n^*)| + |K(\phi, u_n^*) - K(\phi, \phi)| =
|K(u_n^* - \phi, u_n^*)| + |K(\phi, u_n^* - \phi)| \leq C \| u_n^* - \phi \|_{L^q} \| u_n^* \|_{L^q} + \| \phi \|_{L^q}.
\]
for \( q = pr' = \frac{2dp}{2d-\gamma} > 2 \). Clearly now \( \lim_n K(u_n^*, u_n^*) = K(\phi, \phi) \).

All in all, it follows that \( E(\phi) \leq E \lambda \). Let us now show that under the constraint \( |1,1] \), we have that \( E \lambda < 0 \). To that end, take a test function, say \( \varphi : \int_{\mathbb{R}^d} \varphi^2(y) \, dy = \lambda, \varepsilon : \varepsilon << 1 \) and set \( u_\varepsilon = \varepsilon^{d/2} \varphi(\varepsilon x) \). Clearly \( \| u_\varepsilon \|_L^2 = \lambda \), so it satisfies the constraint. On the other hand
\[
E(u_\varepsilon) = \frac{1}{2} \| |\nabla|^\beta u_\varepsilon \|_L^2 - \frac{c_d \gamma}{2p} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u_\varepsilon(x)|^p |u_\varepsilon(y)|^p}{|x-y|^\gamma} \, dx \, dy =
\varepsilon^{2p} \frac{1}{2} \| |\nabla|^\beta \varphi \|_L^2 - \varepsilon^{(p-2)d+\gamma} \frac{c_d \gamma}{2p} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\varphi(x)^{p+1} |\varphi(y)|^{p+1}}{|x-y|^\gamma} \, dx \, dy.
\]
Clearly, since \( (p-2)d + \gamma < 2 \), we have that for small enough \( \varepsilon \) the potential term \( K(\varphi, \varphi) \) dominates and hence \( I_\lambda < 0 \).
We are now ready to prove that $\phi$ is a minimizer. We need to show that $\|\phi\|_{L^2}^2 = \lambda$. Assume that $\|\phi\|_{L^2}^2 < \lambda$. Then, there is $\mu > 1$, so that $\|\mu\phi\|_{L^2}^2 = \lambda$. Hence
\[
E_\lambda \leq E(\mu\phi) = \mu^2 \left[ \frac{1}{2} \|\nabla|^{\beta}\phi\|^2 - \mu^{2p-2} \frac{c_{d,\gamma}}{2p} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\phi^p(x)\overline{\phi^p(y)}}{|x-y|^\gamma} dxdy \right] \leq \mu^2 E_\lambda,
\]
a contradiction, since $E_\lambda < 0$. This means $\|\phi\|_{L^2}^2 = \lambda$, but we will show now that $E(\phi) \leq E_\lambda$. Indeed, by (3.3) and (3.4),
\[
E(\phi) = \lim_{n \to \infty} E(u_n^\ast) \geq E(\phi).
\]
From here, we may conclude that $E(\phi) = E_\lambda$, otherwise $E(\phi) < E_\lambda$, a contradiction with the definition of $E_\lambda$. Thus, $\phi$ is a minimizer.

Note that in addition, this last equality implies $\liminf \|\nabla|^{\beta}u_n^\ast\|_{L^2} = \|\nabla|^{\beta}\phi\|_{L^2}$, which in addition to the weak convergence $|\nabla|^{\beta}u_n^\ast \to |\nabla|^{\beta}\phi$ allows us to conclude $\lim_n \|\nabla|^{\beta}u_n^\ast - |\nabla|^{\beta}\phi\|_{L^2} = 0$. So, in the end, it turns out that the minimization sequence converges strongly to the minimizer $\phi$.

Now that we have established the existence of the constrained minimizers, we proceed to our next result which concerns the Euler-Lagrange equation and explicit calculations of various quantities associated with the energy functional $E(\phi)$.

3.2. The Euler-Lagrange equation and scaling relations. For convenience, we introduce the positive parameter
\[
\Gamma = \Gamma_{\gamma,\beta,d,p} := 2\beta - \gamma - d(p - 2),
\]
which appears often in the subsequent formulas.

**Theorem 6.** Under the assumption of Theorem 2 for the parameters, a constrained minimizer $\phi_\lambda$ as minimizer of (3.1) satisfies the Euler-Lagrange equation (1.3). Moreover, there are the identities
\[
\begin{aligned}
(3.5) & \quad \phi_\lambda(x) = \lambda^{\frac{\mu+1}{\mu-1}} \phi(\lambda^{\frac{\mu}{\mu-1}} x), \\
(3.6) & \quad E_\lambda = \lambda^{1 + \frac{2\beta(p-1)}{p-1}} E_1, \\
(3.7) & \quad J_\lambda = \||\nabla|^{\beta}\phi_\lambda\|^2 = \frac{2(\gamma + d(p - 2))}{\Gamma} (-E_1) \lambda^{1 + \frac{2\beta(p-1)}{p-1}}, \\
(3.8) & \quad K = c_{d,\gamma} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\phi^p(x)\phi^p(y)}{|x-y|^\gamma} dxdy = \frac{4\beta p}{\Gamma} (-E_1) \lambda^{1 + \frac{2\beta(p-1)}{p-1}}, \\
(3.9) & \quad \omega = \left(1 + \frac{2\beta(p-1)}{\Gamma} \right) E_1 \lambda^{\frac{2\beta(p-1)}{p-1}}.
\end{aligned}
\]
Finally, there is the positivity of the (self-adjoint) linearized operator
\[
\mathcal{L}f := (-\Delta)^{\beta} f - \omega f - p(c_{d,\gamma} \cdot |\gamma| * [\phi^{p-1}])\phi^{p-1} - (p - 1)(c_{d,\gamma} \cdot |\gamma| * [\phi])\phi^{p-2} f,
\]
on the co-dimension one subspace $\{\phi\}^\perp$. That is,
\[
(3.10) \quad \langle \mathcal{L}h, h \rangle \geq 0, h \in \{\phi\}^\perp.
\]

Otherwise, one gets the impossible inequality $E(\phi) < E_\lambda$. 

Proof. Fix $\lambda$. By scaling, one sees that the solution $\phi_\lambda$ of (3.1) can be represented by the following formula

$$
\phi_\lambda(x) = \lambda^{\frac{p-1}{2}} \phi_1(\lambda \frac{x}{\lambda^{p-1}}),
$$

From here, a short computation shows that it suffices to prove the results for the case $\lambda = 1$.

So, fix $\lambda = 1$. Let $\phi = \phi_1$ be a minimizer for (3.1). For any $\delta > 0$, consider $u_\delta = \phi + \delta h$, with $h$ real-valued. We have that

$$
E\left(\frac{u_\delta}{\|u_\delta\|}\right) \geq E_1.
$$

Note that

$$
\|u_\delta\| = \sqrt{\|\phi\|^2 + 2\delta \langle \phi, h \rangle + O(\delta^2)} = 1 + \delta \langle \phi, h \rangle + O(\delta^2).
$$

We have

$$
\frac{1}{2} \|\nabla^\beta u_\delta\|^2 = \frac{1}{2} J - \delta(-(-\Delta)^\beta \phi, h) + J\langle \phi, h \rangle + O(\delta^2)
$$

and

$$
-\frac{c_{d,\gamma}}{2p\|u_\delta\|^{2p}} \int_{\mathbb{R}^2} \frac{u_\delta^p(x)u_\delta^p(y)}{|x - y|^{\gamma}} \, dx \, dy = -\frac{1}{2p} K + \delta \left[ \langle \phi, h \rangle K - \langle (c_{d,\gamma} \cdot |^{-\gamma} \ast \phi^p) \phi^{p-1}, h \rangle \right] + O(\delta^2).
$$

Taking into account that

$$
(3.11) \quad \frac{1}{2} J - \frac{1}{2p} K = E_1,
$$

we conclude

$$
\delta \langle (-\Delta)^\beta \phi - (c_{d,\gamma} \cdot |^{-\gamma} \ast \phi^p) \phi^{p-1} + (K - J)\phi, h \rangle + O(\delta^2) \geq 0.
$$

Since this is true for all $\delta \in \mathbb{R}$ and for all test functions $h$, we conclude that $\phi$ satisfies

$$
(-\Delta)^\beta \phi - (c_{d,\gamma} \cdot |^{-\gamma} \ast \phi^p) \phi^{p-1} + (K - J)\phi = 0,
$$

which is the Euler-Lagrange equation (1.3), with a scalar $\omega = J - K$. Finally, there is the Pohozaev’s identity, which we derive in the following way. Set $z_\mu(x) = \mu^{d/2} \phi(\mu x)$. Clearly, since $\int_{\mathbb{R}^d} z_\mu^2(x) \, dx = \int \phi^2(x) \, dx = 1$, $z_\mu$ satisfies the constraint of (3.1). Now

$$
E(z_\mu) = \frac{\mu^{2\beta}}{2} J - \mu^{\gamma + d(p-2)} \frac{1}{2p} K.
$$

Since the scalar valued function $\mu \rightarrow E(z_\mu)$ achieves its minimum at $\mu = 1$, we must have $\frac{dE(z_\mu)}{d\mu}|_{\mu=1} = 0$. This relation yields the Pohozaev’s identity

$$
(3.12) \quad \beta J - \frac{\gamma + d(p-2)}{2p} K = 0.
$$
Combining (3.11) and (3.12), we obtain the formulas
\[
K = \frac{4\beta p}{\Gamma} (-E_1)
\]
\[
J = \frac{2(\gamma + d(p-2))}{\Gamma} (-E_1)
\]
\[
\omega = J - K = 2\frac{2\beta p - \gamma - d(p-2)}{\Gamma} E_1 = (1 + \frac{2\beta(p-1)}{\Gamma}) E_1.
\]
Thus, we arrive at the statements of (3.7), (3.8), (3.9). Clearly, \(K > 0, J > 0, \omega < 0\), since \(E_1 < 0\).

We now establish the coercivity of \(\mathcal{L}\) on the co-dimension one subspace \(\{\phi\}^\perp\). To that end, note that for every test function \(h\), the function
\[
g(\delta) := E \left( \frac{\phi + \delta h}{\|\phi + \delta h\|} \right)
\]
has a minimum at \(\delta = 0\). In fact, the Euler-Lagrange equation (1.3) is nothing but a rephrased version of the necessary condition for a minimum \(g'(0) = 0\). Given that \(g\) achieves its minimum at \(\delta = 0\), one has a second necessary condition for minimum, namely \(g''(0) \geq 0\). We will exploit this fact to our advantage in order to deduce (3.10). In order to simplify the computations (and to reflect the fact that the coercivity of \(\mathcal{L}\) is only over \(\{\phi\}^\perp\) anyway), we take \(h : \langle h, \phi \rangle = 0, \|h\| = 1\). Note that under this restriction
\[
\|\phi + \delta h\| = (1 + \delta^2)^{1/2} = 1 + \frac{\delta^2}{2} + O(\delta^3).
\]
Next, taking into account that \(\phi\) satisfies (1.3), we write
\[
g(\delta) = \frac{1}{2} \|\nabla^\beta (\phi + \delta h)\|^2 - c_{d,\gamma} \int_{\mathbb{R}^2} \frac{(\phi + \delta h)^p(x)(\phi + \delta h)^p(y)}{|x - y|^\gamma \|\phi + \delta h\|^{2p}} dxdy = \]
\[
= g(0) + \frac{\delta^2}{2} \left[ \langle ((-\Delta)^\beta + K - J)h, h \rangle - p(c_{d,\gamma} \cdot |^{-\gamma} (\phi^{p-1}h)\phi^{p-1}, h) \right] -
\]
\[
- (p - 1) \langle c_{d,\gamma} \cdot |^{-\gamma} (\phi^{p})\phi^{p-2}h, h \rangle + o(\delta^2).
\]
Recall that \(\omega = J - K\). Since \(g(\delta) \geq g(0)\) for all small enough \(\delta\), it follows that the operator \(\mathcal{L}\) defined by
\[
\mathcal{L}f = (-\Delta)^\beta f - \omega f - p(c_{d,\gamma} \cdot |^{-\gamma} (\phi^{p-1})\phi^{p-1})\phi^{p-1} - (p - 1)(c_{d,\gamma} \cdot |^{-\gamma} (\phi^{p}))\phi^{p-2} f
\]
satisfies \(\langle \mathcal{L}h, h \rangle \geq 0\), which is exactly (3.10). \(\square\)

3.3. The linearized problem and spectral properties of the self-adjoint part. We impose the ansatz\(^3\) \(u = \phi_\lambda + \epsilon v, \) where \(v\) is necessarily complex valued field. We have
\[
(| \cdot |^{-\gamma} |u|^p)u = (| \cdot |^{-\gamma} |\phi + \epsilon v|^p)\phi + \epsilon v|^{p-2}(\phi + \epsilon v) = \]
\[
= (| \cdot |^{-\gamma} \phi^{p})\phi^{p-1} + \epsilon(p(| \cdot |^{-\gamma} (\phi^{p-1}v))\phi^{p-1} + (| \cdot |^{-\gamma} \phi^{p})[(p - 2)\phi^{p-2}v + \phi^{p-2}v]) +
\]
\[+ o(\epsilon).\]

\(^3\)We suppress the super index \(\phi_\lambda\) in what follows, but we would like to keep \(\phi_\lambda\) dependent upon the parameter \(\lambda\).
Setting \( u = e^{i\omega t}[\phi + \epsilon v] = e^{i\omega t}[\phi + \epsilon(Re v + iIm v)] \) in (1.2) and ignoring \( o(\epsilon) \), we obtain the following linearized system

\[
\begin{align*}
-\partial_t v_2 + (\Delta)^{1/2} v_1 - \omega v_1 - p(c_{d,d}\cdot |\gamma| \cdot |x|^{p-1}v_1) &= - (p-1) (c_{d,d} \cdot |\gamma| \cdot |x|^{p-1}) \phi^{p-2} v_1 = 0 \\
\partial_t v_1 + (\Delta)^{1/2} v_2 - \omega v_2 - (c_{d,d}\cdot |\gamma| \cdot |x|^{p-1}) \phi^{p-2} v_2 &= 0
\end{align*}
\]

for \( v_1 = Re v, v_2 = Im v \). As is customary, we adopt the notation

\[
L_+ = L = (\Delta)^{1/2} - \omega - p(c_{d,d} |\gamma| |x|^{p-1}) \phi^{p-1} - (p-1) (c_{d,d} |\gamma| |x|^{p-1}) \phi^{p-2}
\]

\[
L_- = (\Delta)^{1/2} - \omega - (c_{d,d} |\gamma| |x|^{p-1}) \phi^{p-2}
\]

\[
\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, L = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}
\]

so that we can rewrite the eigenvalue problem (1.1) in the Hamiltonian form

\[
\tilde{v}_t = \mathcal{J} L \tilde{v}.
\]

We will show that \( L \) is a self-adjoint operator, at least for \( p > 2 \). Indeed, one can apply the KLMN theorem (see Theorem X.17 in [19]) for the operators

\[
L_+ = (\Delta)^{1/2} - \omega - p(c_{d,d} |\gamma| |x|^{p-1}) \phi^{p-1} - (p-1) (c_{d,d} |\gamma| |x|^{p-1}) \phi^{p-2}
\]

where

\[
V_1(f) = (|\cdot|^{-\gamma} \cdot |x|^{p-1}) \phi^{p-1}, \quad V_2(f) = (|\cdot|^{-\gamma} \cdot |x|^{p}) \phi^{p-2} f.
\]

The check of the assumptions of the KLMN theorem follow from the simple Sobolev estimate

\[
|\langle |\cdot|^{-\gamma} g_1, g_2 \rangle| \leq C \|g_1\|_{L^{2d/(d+\alpha)}} \|g_2\|_{L^{2d/(d+\alpha)}},
\]

applied for \( g_1 = g_2 = \phi^{p-1} f \). In this way we find\(^6\)

\[
|\langle V_1(f), f \rangle| \leq C \|f\|_{L^2}.
\]

For the operator \( V_2 \) we observe that \( |\cdot|^{-\gamma} \cdot |x|^{p} \in L^\infty \), so we find

\[
|\langle V_2(f), f \rangle| \leq C \|f\|_{L^2}.
\]

We are in position to conclude that \( L_+ \) are self-adjoint operators, whence \( L \) is self-adjoint as well. On the other hand, \( J \) is clearly skew-symmetric. By Weyl’s criterion, both operators \( L_{\pm} \), have absolutely continuous spectrum, which fills the interval \([-\omega, \infty)\), which verifies the spectral gap condition at zero, since \( \omega < 0 \) by virtue of Theorem 6. In addition, we have verified in Theorem 6 that \( L_+ \) has at most one negative eigenvalue. We have the following lemma, regarding the spectral properties of \( L \).

**Lemma 1.** For \( p > 2 \), the self-adjoint operator \( L_+ = L \) has exactly one negative eigenvalue, while \( L_- \geq 0 \).

As an immediate consequence of Lemma 1, the matrix operator \( L \) has exactly one negative eigenvalue.

**Proof.** (Lemma 1) Regarding \( L \), we only need to verify that it does indeed have a negative eigenvalue. This is easily seen by testing the quantity \( \langle L \phi, \phi \rangle \). Indeed, taking into account the Euler-Lagrange equation (1.3), we compute

\[
L \phi = (\Delta)^{1/2} \phi - \omega \phi - p(c_{d,d} \cdot |\gamma| \cdot |x|^{p-1}) \phi^{p-1} - (p-1) (c_{d,d} \cdot |\gamma| \cdot |x|^{p-1}) \phi^{p-2} = - (2p-2) (c_{d,d} \cdot |\gamma| \cdot |x|^{p-1}) \phi^{p-1}.
\]

\(^6\)recall that \( \phi \) is a bell - shaped function
Thus,
\[
\langle L \phi, \phi \rangle = -(2p - 2)c_{d, \gamma} \int_{\mathbb{R}^d} (| \cdot |^{-\gamma} \ast [\phi^p]) \phi^p = -(2p - 2)c_{d, \gamma} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\phi^p(x)\phi^p(y)}{|x - y|^{\gamma}} dx dy < 0.
\]

Next, we wish to show that \( L \) is a non-negative operator. Note that the Euler-Lagrange equation (1.3) is nothing but
\[
L - \langle \phi \rangle = 0,
\]
where \( \phi > 0 \). In particular zero is an eigenvalue for \( L \) and it remains to show that it is at the bottom of its spectrum. Assume that this is not the case, hence \( L \) has a negative eigenvalue, say \( -\sigma^2 \) and assume, without loss of generality that it is the smallest such eigenvalue. In particular,
\[
-\sigma^2 = \inf_{\|\psi\|=1} \langle L \psi, \psi \rangle.
\]
The corresponding eigenfunction, say \( \varphi \) can be constructed as a minimizer of the minimization problem (3.14). More precisely, upon introducing the bell-shape function
\[
V(x) := (c_{d, \gamma} | \cdot |^{-\gamma} \ast \varphi^p) \varphi^p - 2,
\]
we have
\[
\begin{align*}
\langle L \psi, \psi \rangle &= \|\nabla^2 \psi\|^2 - \int_{\mathbb{R}^d} V(x)\psi^2(x) dx \rightarrow \min \\
\int_{\mathbb{R}^d} \psi^2(x) dx &= 1.
\end{align*}
\]
We will now show that the eigenfunction \( \varphi \) satisfies \( \varphi = \varphi^* \) and as such is a positive function. To that end, by Proposition 1 we have
\[
\|\nabla^2 \psi\|^2 \geq \|\nabla^2 [\varphi^*]\|^2
\]
Next, applying (2.1) and observing that \((h^2)^* = (h^*)^2\), we obtain
\[
\int_{\mathbb{R}^d} V(x)(\psi(x))^2 dx \leq \int_{\mathbb{R}^d} V(x)(\psi^*(x))^2 dx,
\]
while the constraint \( 1 = \int_{\mathbb{R}^d} \psi^2(x) dx = \int_{\mathbb{R}^d} (\psi^*(x))^2 dx \) remains satisfied. Thus, \( \langle L \psi, \psi \rangle \geq \langle L \psi^*, \psi^* \rangle \). It follows that the solution \( \varphi \) of (3.13), which must exists, is bell-shaped and in particular \( \varphi > 0 \). But if such eigenfunction corresponds to a negative eigenvalue \( -\sigma^2 \), then it must be perpendicular to the eigenfunction \( \phi \) corresponding to eigenvalue zero. However, both \( \phi > 0 \), \( \varphi > 0 \), a contradiction. It follows that \( L \geq 0 \).

At this point, we are essentially ready to consider the stability of these waves, more precisely the eigenvalue problem (3.13). We will postpone these considerations to Section 4.7. This is done in the interest of presenting an unified approach for the classical case of MVS waves and then for the fractional waves. The approach for the fractional case turns out to be pretty similar, we outline the details in Section 4.7.

4. Classification of the stability of the ground states for the Hartree and Klein-Gordon-Hartree models: proof of Theorems 4 and 5

We start with the proof of Theorem 4. Henceforth, the assumptions made in Theorem 4 are in force. Recall \( \Gamma = 2 - \gamma - (p - 2)d \). Consider the linearization of the solutions of the time-dependent Hartree model (1.4) around the ground states constructed in Theorem 1 and the Klein-Gordon-Hartree model (1.8), around the ground state constructed in (1.9).
4.1. The linearized problem for the Hartree model (1.2). As before, we take the ansatz
\[ u = e^{-it} [\varphi + \epsilon v] = e^{-it} [\varphi + \epsilon (\Re v + i \Im v)] = e^{-it} [\varphi + \epsilon (v_1 + iv_2)] \]
in (1.2) and ignoring \( O(\epsilon^2) \), leads us to the following linearized system
\[
\begin{align*}
-\partial_t v_2 - \Delta v_1 + v_1 - p I_\alpha [\varphi^{p-1} v_1] \varphi^{p-1} - (p-1) I_\alpha [\varphi^p] \varphi^{p-2} v_1 &= 0, \\
\partial_t v_1 - \Delta v_2 + v_2 - I_\alpha [\varphi^p] \varphi^{p-2} v_2 &= 0.
\end{align*}
\]
As is customary, we adopt the notation
\[
L_+ = -\Delta + 1 - p I_\alpha [\varphi^{p-1} v_1] \varphi^{p-1} - (p-1) I_\alpha [\varphi^p] \varphi^{p-2},
\]
\[
L_- = -\Delta + 1 - I_\alpha [\varphi^p] \varphi^{p-2},
\]
\[ \mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}. \]
so that we can rewrite the eigenvalue problem (1.1) in the Hamiltonian form
\[ \vec{v}_t = \mathcal{J} \vec{L} \vec{v} \]
Note that \( \mathcal{J} \) is clearly skew-symmetric. Next we derive the linearized problem for the Klein-Gordon-Hartree model (1.8).

4.2. Linearized problem for the Klein-Gordon-Hartree model (1.8). As in the Schrödinger case, take
\[ u = e^{i\omega t} [\Psi + \epsilon v] = e^{i\omega t} [\Psi + \epsilon (\Re v + i \Im v)] = e^{i\omega t} [\Psi + \epsilon (v_1 + iv_2)] \]
and plug this in (1.8). After ignoring \( O(\epsilon^2) \) terms and taking real and imaginary parts, we arrive at
\[
\begin{align*}
\partial_t v_1 - 2 \omega \partial_t v_2 + (1 - \omega^2) v_1 - \Delta v_1 - p I_\alpha [\Psi^{p-1} v_1] \Psi^{p-1} - (p-1) I_\alpha [\Psi^p] \Psi^{p-2} v_1 &= 0, \\
\partial_t v_2 + 2 \omega \partial_t v_1 + (1 - \omega^2) v_2 - \Delta v_2 - I_\alpha [\Psi^p] \Psi^{p-2} v_2 &= 0.
\end{align*}
\]
In order to bring the eigenvalue problem (4.2) to a form similar to (4.1), recall (1.10). In accordance with that, we rescale the variables as follows
\[ v_j(t, x) = e^{i\lambda \sqrt{1-\omega^2} v_j(x \sqrt{1-\omega^2})}, \quad j = 1, 2, \]
so that the eigenvalue problem (4.2) is transformed into the standard form
\[
\begin{align*}
\lambda^2 V_1 - 2\lambda \frac{\omega}{\sqrt{1-\omega^2}} V_2 + L_+[V_1] &= 0, \\
\lambda^2 V_2 + 2\lambda \frac{\omega}{\sqrt{1-\omega^2}} V_1 + L_-[V_2] &= 0.
\end{align*}
\]
Introducing the skew-symmetric matrix \( \mathcal{J}_\omega := \begin{pmatrix} 0 & -\frac{\omega}{\sqrt{1-\omega^2}} \\ -\frac{\omega}{\sqrt{1-\omega^2}} & 0 \end{pmatrix} \), we can rewrite the relevant eigenvalue problem in the compact form
\[
\begin{pmatrix} 0 & I_2 \\ -I_2 & -\mathcal{J}_\omega \end{pmatrix} \begin{pmatrix} L & 0 \\ 0 & I_2 \end{pmatrix} \vec{W} = \lambda \vec{W}, \quad \vec{W} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}.
\]
\[ \text{We suppress the super index } \phi^\lambda \text{ in what follows, but we would like to keep } \phi^\lambda \text{ dependent upon the parameter } \lambda \]
4.3. Spectral information about the operators $L_{\pm}$. Here, we shall need to summarize the results in [16] about the properties of the MVS solutions $\varphi$ of (1.5).

Lemma 2. (Theorem 4 and Lemma 6.7, [16]) The MVS solution $\varphi$ of (1.3) satisfy

\begin{equation}
\lim_{|x| \to \infty} \frac{I_\alpha[\varphi^p]}{I_\alpha(x)} = \int_{\mathbb{R}^d} \varphi^p.
\end{equation}

For $p \geq 2$, $\varphi$ has exponential decay at $\pm \infty$, while for $p < 2$, there is the relation

\begin{equation}
\varphi(x) = \frac{1}{|x|^q} \left( \frac{\Gamma(\frac{d-\alpha}{2})}{\Gamma(\frac{d}{2}) \pi^{d/2} 2^\alpha} \int_{\mathbb{R}^d} \varphi^p \right)^{\frac{1}{p}} + O \left( \frac{1}{|x|^q} \right)
\end{equation}

for large $|x|$. In addition,

\begin{equation}
J[\varphi] := \int_{\mathbb{R}^d} |\nabla \varphi(x)|^2 dx = \frac{d(p-2) + \gamma}{2d - \gamma - p(d-2)} \|\varphi\|_{L^2}^2,
\end{equation}

\begin{equation}
K[\varphi] := \langle I_\alpha[\varphi^p], \varphi \rangle = c_d, \gamma \int_{\mathbb{R}^d} \varphi^p(x) \varphi(x) |x-y|^{-\gamma} dxdy = \frac{2p}{2d - \gamma - p(d-2)} \|\varphi\|_{L^2}^2,
\end{equation}

\begin{equation}
E[\varphi] = \frac{1}{2} J - \frac{1}{2p} K = -\frac{\Gamma}{2(2d - \gamma - p(d-2))} \|\varphi\|_{L^2}^2.
\end{equation}

Note: By the assumptions in the existence theorem $2d - \gamma - p(d-2) > 0$, so $\text{sgn}(E) = -\text{sgn}(\Gamma)$.

Proof. The formula (4.4) appears in Theorem 4 in [16]. The formula (4.5) is a combination of the last statement of Theorem 4 and the final remark in the proof of Lemma 6.7. The formulas (4.6), (4.7) and (4.8) are just a elementary consequence of the Pohozaev’s identity (1.7), together with the relation $\int_{\mathbb{R}^d} \|\nabla u(x)\|^2 + |u(x)|^2 dx = \langle I_\alpha[u^p], |u|^p \rangle$, which follows from (1.5) by taking dot product with $\varphi$. \hfill \square

In addition to (4.4) and (4.5), we will need more precise information on the behavior of $I_\alpha[\varphi^p]$, for the case $p < 2$. This is provided in the following lemma.

Lemma 3. Let $p \in (2 - \gamma/d, 2)$. Then,

\begin{equation}
\frac{I_\alpha[\varphi^p]}{I_\alpha(x)} = \left[ \int_{\mathbb{R}^d} \varphi^p + O(|x|^{-\min(2,\alpha)}) \right]
\end{equation}

for large $|x|$.

Remark: Note that the error term $O(|x|^{-\min(2,\alpha)})$ is not necessarily sharp for all $p \in (2 - \gamma/d, 2]$, but it is rather an upper bound, which suffices for our purposes.

Proof. We start with the relation

\[ |x-y|^{-\gamma} = |x|^{-\gamma} \left( 1 - \gamma \frac{\langle x, y \rangle}{|x|^2} + O(|y|^2/|x|^2) \right). \]

Using $\int_{|y| \leq |x|/2} y \varphi^p |y| dy = 0$, we get

\begin{equation}
\int_{|y| \leq |x|/2} \frac{\varphi^p(|y|)dy}{|x-y|^{-\gamma}} = |x|^{-\gamma} \int_{|y| \leq |x|/2} \varphi^p(|y|)dy + O \left( \frac{1}{|x|^{2+\gamma}} \int_{|y| \leq |x|/2} |y|^2 \varphi^p(y)dy \right).
\end{equation}
Note that for large \( y \),
\[
|y|^2 \varphi^p(y) \leq \frac{C}{|y|^{\frac{2}{d} - 2}},
\]
which is integrable, provided \( \frac{2}{d} - 2 > d \) or \( p > 2 - 2\gamma/(d + 2 + \gamma) \). Thus, we have that the error term in (4.10) is \( O(|x|^{-2 - \gamma}) \).

On the other hand, if \( 2 - \frac{\gamma}{d} < p \leq 2 - 2\gamma/(d + 2 + \gamma) \), we may estimate
\[
\frac{1}{|x|^{2 + \gamma}} \int_{|y| \leq |x|/2} |y|^2 \varphi^p(y)dy \leq \frac{C}{|x|^{\gamma + \alpha}} \int_{|y| \leq |x|/2} |y|^\alpha \varphi^p(y)dy.
\]
Since for large \( y \), we have
\[
|y|^\alpha \varphi^p(y) \leq \frac{C}{|y|^{\frac{2}{d} - \alpha}},
\]
which is integrable when \( \frac{2}{d} - \alpha > d \) or \( p > \frac{2(d + \alpha)}{\gamma + \alpha + d} = 2 - \frac{2}{d} \), which is the full range of interest according to Theorem 1. Thus, the error term in (4.10) is now \( O(|x|^{-\gamma - \alpha}) \). This finishes the proof of (4.9) and Lemma 3.

Our next lemma provides the self-adjointness of \( L_{\pm} \) as well as a description of the absolutely continuous spectrum.

**Lemma 4.** The linearized operators \( L_{\pm} \) with domains \( D(L_{\pm}) = H^2(\mathbb{R}^d) \), are self-adjoint. In addition, for \( p \geq 2 \), \( \sigma_{a.c.}(L_{+}) = \sigma_{a.c.}(L_{-}) = [1, \infty) \). For \( p < 2 \) however, \( \sigma_{a.c.}(L_{-}) = [0, \infty) \), while \( \sigma_{a.c.}(L_{+}) = [2 - p, \infty) \).

**Proof.** Let us first go through the easy cases \( p \geq 2 \). The self-adjointness in this case is an easy matter, since one can apply the KLMN theorem (see Theorem X.17 in [19]) for the operators
\[
L_+ = -\Delta + 1 - pV_1 - (p - 1)V_2, \quad L_- = -\Delta + 1 - V_2,
\]
where
\[
V_1(f) = c_{d,\gamma}(\cdot |^{-\gamma} \ast \varphi^{p-1}[f])\varphi^{p-1}, \quad V_2(f) = c_{d,\gamma}(\cdot |^{-\gamma} \ast \varphi^p)\varphi^{-2}f.
\]
We need to estimate \( \langle V_j f, f \rangle, j = 1, 2 \). We have
\[
|\langle |^{-\gamma} \ast g_1, g_2 \rangle| = c |(-\Delta)^{-\alpha/4} g_1, (-\Delta)^{-\alpha/4} g_2| \leq C \|g_1\|_{L^{2d/(d+\alpha)}} \|g_2\|_{L^{2d/(d+\alpha)}}.
\]
Applying this to \( g_1 = g_2 = \varphi^{p-1}f \) yields
\[
\langle V_1 f, f \rangle \leq C \|f\|^2 \|\varphi^{p-1}\|^2_{L^{\frac{2d}{d-\alpha}}} = C \|f\|^2 \|\varphi\|^2_{L^{\frac{2d}{d-\alpha}}} \|\varphi^{2(p-1)}\|^2_{L^{\frac{2d}{d-\alpha}}}.
\]
Since by construction \( \varphi \in L^q, q \in [2, \infty] \) and \( \frac{2d(\gamma-1)}{\alpha} > 2 \), we conclude that \( V_1 \) satisfies the requirements of KLMN theorem and it is an admissible perturbation of the self-adjoint operator \(-\Delta - \omega\).

Regarding \( V_2 \), we have by Hardy-Littlewood-Sobolev inequality
\[
|\langle V_2 f, f \rangle| \leq C \|f\|_{L^\infty} \|\varphi\|_{L^{\frac{2d}{d-\alpha}}} \|\varphi^{p-2}f\|^2_{L^2} \leq C \|\varphi\|^p_{L^{\frac{2d}{d-\alpha}}} \|f\|^2_{L^2}.
\]
This is also enough by KLMN, since \( \frac{2d}{\alpha} \geq \frac{2d(\gamma-1)}{\alpha} > 2 \). Thus, the self-adjointness of \( L_{\pm} \) in the case \( p \geq 2 \) follows by KLMN. The argument for \( V_2 \) however is limited to \( p \geq 2 \), because otherwise \( \varphi^{p-2} \) is actually unbounded as \( |x| \to \infty \) and the argument above clearly fails.

\footnote{Recall that by assumption for the existence of \( \varphi \): p > 1 + \frac{\alpha}{d} \}

\footnote{This argument actually works for all p > 1 + \frac{\alpha}{d} and it is not limited to p \geq 2.}
Assume $p < 2$. Let us consider first the self-adjointness of $L_-$. Since $L_- = -\Delta + 1 - I_\alpha[\varphi^p] \varphi^{p-2}$, clearly this is not a potential that decays at $\infty$. In fact, we will show that

$$I_\alpha[\varphi^p] \varphi^{p-2} = 1 + O(|x|^{-\min(2,\alpha)}).$$

Indeed, according to (4.5)

$$\varphi^{p-2}(x) = \frac{|x|^{d-\alpha} \Gamma(\alpha/2) \pi^{d/2} \alpha^{d/2}}{\Gamma((d-\alpha)/2) \int \varphi^p} [1 + O(|x|^{-\min(2,\alpha)})].$$

Thus, according to (4.4),

$$I_\alpha[\varphi^p] \varphi^{p-2}(x) = \frac{I_\alpha[\varphi^p]}{I_\alpha(x) \int \varphi^p} [1 + O(|x|^{-\min(2,\alpha)})] = 1 + O(|x|^{-\min(2,\alpha)}).$$

It follows that

$$L_- = -\Delta + 1 - I_\alpha[\varphi^p] \varphi^{p-2} = -\Delta + G(x),$$

where $G$ is a smooth and bounded function, with $|G(x)| \leq C(1 + |x|^{\min(2,\alpha)})^{-1}$. It follows that $L_- = L_-^\ast$. In addition, $\sigma_{\text{ess}}(L_-) = [0, \infty)$.

Regarding $L_+$, we have that for $p < 2$,

$$L_+ = -\Delta + 1 - pV_1 - (p-1)V_2 = -\Delta + (2-p) - pV_1 - (p-1)G(x),$$

whence $L_+$ is also self-adjoint, with $\sigma_{\text{ess}}(L_+) = [2-p, \infty)$.

Next, we discuss the point spectrum of the operators $L_\pm$. We have the following result.

**Lemma 5.** The linearized operator $L_\pm$ has exactly one negative eigenvalue. On the other hand, $L_- \geq 0$ with an eigenvalue at zero. The eigenvalue at zero is simple, with eigenfunction $\varphi$.

**Proof.** Let us first establish the claims regarding $L_-$. Clearly $L_-[\varphi] = 0$, as this is simply (4.5). Thus zero is an eigenvalue for $L_-$, with eigenfunction $\varphi$. Assuming that $L_-$ has a negative eigenvalue will lead to a contradiction. Indeed, pick the bottom of the spectrum for $L_-$. By the results in Lemma 4 (and the description of the structure of $L_-$), it will necessarily have a positive eigenfunction, say $\psi_0$. But then, $\psi_0 \perp \varphi$ as eigenfunctions corresponding to different eigenvalues, a contradiction. Thus, zero is the bottom of the spectrum. The simplicity of the bottom of the spectrum (in this case the zero eigenvalue) is also well-known by the Sturm oscillation argument.

We now turn our attention to $L_+$. First, it is easy to see that

$$L_+ \varphi = -\Delta \varphi + \varphi - pI_\alpha[\varphi^p] \varphi^{p-1} - (p-1)I_\alpha[\varphi^p] \varphi^{p-1} = -(2p-2)I_\alpha[\varphi^p] \varphi^{p-1}.$$

Thus,

$$\langle L_+ \varphi, \varphi \rangle = -(2p-2) \langle I_\alpha[\varphi^p], \varphi^p \rangle < 0.$$

This shows that $L_+$ has at least one negative eigenvalue. It remains to show that $L_+$ is positive on a codimension one subspace.

To that end, consider the minimizer $\Phi$ of the optimization problem (1.6). Consider a perturbation of $\Phi$ in the form $u_\epsilon = \Phi + \epsilon h$, for a real-valued function $h$ and $\epsilon : |\epsilon| << 1$. 

Expanding in Taylor series up to order $\epsilon^2$, we obtain

\[
I[u] := \langle \nabla u, \nabla u \rangle + \langle u, u \rangle = I[\Phi] + 2\epsilon \langle -\Delta \Phi + \Phi, h \rangle + \epsilon^2 \langle -\Delta h + h, h \rangle;
\]

\[
M[u] = \langle I_\alpha[u]^p, |u|^p \rangle = M[\Phi] + \epsilon [2pI_\alpha[\Phi^p]\Phi^{p-1}, h] + \epsilon^2 [p^2 I_\alpha[\Phi^{p-1}h], \Phi^{p-1}h] + p(p-1) \langle I_\alpha[\Phi^p] \Phi^{p-2}h, h \rangle + o(\epsilon)
\]

Denote $I_0 := I[\Phi]$, $M_0 := M[\Phi]$. Recall that since $\Phi$ is a minimizer for (1.6), we will have

\[
(4.14)
\]

\[
g(\epsilon) := \frac{I[u_\epsilon]}{(M[u_\epsilon])^{1/p}} \geq \frac{I_0}{(M_0)^{1/p}} = g[0].
\]

Clearly, such a relation implies that zero is a minimum for the function $\epsilon \to g(\epsilon)$. In particular, $g'(0) = 0$. This is exactly the Euler-Lagrange equation for (1.6), which means that $\Phi$ satisfies the PDE

\[
(4.15)
\]

\[-\Delta \Phi + \Phi - \frac{I_0}{M_0} I_\alpha[\Phi^p] \Phi^{p-1} = 0.
\]

Clearly, a function in the form $\varphi = t_0 \Phi$ will satisfy (1.6), once we subject it to the normalization $\int_{\mathbb{R}^n} [\nabla \varphi(x)]^2 + |\varphi(x)|^2 dx = \langle I_\alpha[\varphi^p], \varphi^p \rangle$. That is

\[
(4.16)
\]

\[
\varphi = \left( \frac{I_0}{M_0} \right)^{\frac{1}{2p-2}} \Phi.
\]

We take advantage of the variational structure to show that the operator $L_+$ has exactly one negative eigenvalue. Indeed, since $g$ has absolute minimum at $\epsilon = 0$ and $g'(0) = 0$, it is necessary that $g''(0) \geq 0$. In order to simplify the computations, let us take a function $h$, which satisfies the orthogonality condition $\langle I_\alpha[\Phi^p] \Phi^{p-1}, h \rangle = 0$. That is $h \perp I_\alpha[\Phi^p] \Phi^{p-1}$.

By (4.15), it follows that $\langle -\Delta \Phi + \Phi, h \rangle = 0$ as well. It is now easy to see that the following expansion in powers of $\epsilon$ holds

\[
g(\epsilon) = g(0) + \frac{\epsilon^2}{M_0^{1/p}} \left[ \langle -\Delta h + h, h \rangle - \frac{I_0}{M_0} [p \langle I_\alpha[\Phi^{p-1}h], \Phi^{p-1}h \rangle + (p-1) \langle I_\alpha[\Phi^p] \Phi^{p-2}h, h \rangle] \right] + o(\epsilon^2).
\]

We see that $g''(0) \geq 0$ is equivalent to the positivity of the operator

\[
-\Delta + 1 - \frac{I_0}{M_0} [p I_\alpha[\Phi^{p-1}] \Phi^{p-1} + (p-1) I_\alpha[\Phi^p] \Phi^{p-2}] = -\Delta + 1 - p I_\alpha[\varphi^{p-1}] \varphi^{p-1} - (p-1) I_\alpha[\varphi^p] \varphi^{p-2}
\]

on the subspace $\{ I_\alpha[\Phi^p] \Phi^{p-1} \}^\perp$. We conclude that

\[
L_+ |\{ I_\alpha[\varphi^p] \varphi^{p-1} \}^\perp \geq 0,
\]

as claimed.

\[\text{\textsuperscript{10}}\text{where we have used the relation (4.16).}\]
4.4. The Basics of the Instabilities Index Counting. Now that we have established Lemma 5, we are ready to discuss the spectral stability of the waves $e^{-it}\varphi$. In fact, the eigenvalue problem (3.13) falls within the scope of the Grillakis-Shatah-Strauss (GSS) theory, see also [8, 9]. Recall that we have established that $L$ has exactly one negative eigenvalue $n(L) = 1$. In principle, in order to apply the theory, one needs to identify the kernel of the operator $L$. We have already known quite a bit about it - $\varphi \in Ker[L_-]$, while a differentiation of the Euler-Lagrange equation (1.3), in each of the variables $x_1, \ldots, x_d$ shows that $\partial x_j \varphi, j = 1, \ldots, d$ is in the kernel of $L = L_+; i.e., span \{ \partial x_j \varphi, j = 1, \ldots, d \} \subset Ker[L]$. An important problem in the theory has been to determine whether these are indeed all of the linearly independent elements of $Ker[L]$, that is - is it true that

$$\text{(4.17)} \quad Ker[L] = \text{span}\{ \partial \varphi \partial x_j, j = 1, \ldots, d \}?$$

Ground states with the property (4.17) has been referred to as non-degenerate, [4, 16, 17, 18]. Our argument goes forward even without knowledge of the non-degeneracy of $\varphi$. By the GSS theory, we have that if $Ker[L] = \text{span}\{ y_j, j = 1, \ldots, l \}$ and $J$ is invertible with $J^{-1} : Ker[L] \to [Ker[L]]^\perp$, then

$$\text{(4.18)} \quad \# \{ \lambda : \Re \lambda > 0 : Jf = \lambda f \} = n(L) - n(D) = 1 - n(D),$$

where the matrix $D \in M_{t,t}$ has entries

$$\text{(4.19)} \quad D_{ij} = \langle L\psi_j, \psi_i \rangle : L\psi_j = J^{-1} y_j,$$

where the equation $L\psi_j = J^{-1} y_j$ has a solution of $\psi_j$, since $J^{-1} : Ker[L] \to [Ker[L]]^\perp$.

4.5. Classification of the Stability for the Hartree Solitary Waves - Proof of Theorem 4. We start our considerations with a calculation, that will be useless in the sequel.

Lemma 6. $\varphi \perp Ker[L_+]$ and moreover,

$$\langle L_+^{-1} \varphi, \varphi \rangle = -\frac{\Gamma}{4(p-1)} \| \varphi \|^2_{L^2}.$$  

Proof. We take advantage of the scaling of the PDE (1.3). More precisely, introduce $\varphi_\lambda$, so that

$$\text{(4.20)} \quad \varphi = \lambda^b \varphi_\lambda(\lambda x),$$

where $\lambda > 0$ and $b$ is a parameter to be determined from the scaling. Plugging this into (1.3), we find

$$-\lambda^{b+2} \Delta \varphi_\lambda + \lambda^b \varphi_\lambda - \lambda^{b(2p-1)-(d-\gamma)} I_{\alpha} [\varphi_\lambda^p] \varphi_\lambda^{p-1} = 0.$$  

\[11\] In the standard formulation, the GSS theory requires that there is a spectral gap between the zero and the continuous spectrum of $L$. In our case, this is clearly violated in the case $p < 2$, since $\sigma_{a.c.}[L_-] = [0, \infty)$. By a remark in the argument in the original paper, this situation is also covered, in other words if the continuous spectrum just touches the zero, the statement still goes through as in the case with a spectral gap. For further justification in this case of touching, one should consult [10] as well.

\[12\] We henceforth adopt the notation $n(S)$ for a number of strictly negative eigenvalues of a self-adjoint operator/matrix $S$

\[13\] although such a statement is very likely to hold

\[14\] which is not unique, unless $Ker[L] = \{0\}$
Equating the powers $b + 2 = b(2p - 1) - (d - \gamma)$ yields $b = \frac{2 + d - \gamma}{2(p - 1)}$, which we use henceforth. Dividing by $\lambda^{b+2}$ yields the relation

\begin{equation}
- \Delta \varphi_\lambda + \lambda^{-2} \varphi_\lambda - I_\alpha[\varphi_\lambda^p] \varphi_\lambda^{p-1} = 0
\end{equation}

Taking a derivative in $\lambda$ in (4.21) yields

$$-\Delta \left[ \frac{\partial \varphi_\lambda}{\partial \lambda} \right] + \lambda^{-2} \left[ \frac{\partial \varphi_\lambda}{\partial \lambda} \right] - 2\lambda^{-3} \varphi_\lambda - pI_\alpha[\varphi_\lambda^{p-1} \frac{\partial \varphi_\lambda}{\partial \lambda}] \varphi_\lambda^{p-1} - (p - 1)I_\alpha[\varphi_\lambda^p] \varphi_\lambda^{p-2} \left[ \frac{\partial \varphi_\lambda}{\partial \lambda} \right] = 0$$

Evaluating the previous expression at $\lambda = 1$ can be interpreted as follows

$$L_+ \left[ \frac{\partial \varphi_\lambda}{\partial \lambda} \right]_{\lambda=1} = 2\varphi.$$

this shows in particular that $\varphi \perp \text{Ker}[L_+]$. In addition,

$$\langle L_+^{-1} \varphi, \varphi \rangle = \frac{1}{2} \left. \left( \frac{\partial \varphi_\lambda}{\partial \lambda} \right) \right|_{\lambda=1}, \varphi = \frac{1}{4} \left[ \frac{\partial \lambda}{\partial \lambda} \left| \varphi_\lambda^2 \right| \right]_{\lambda=1} = \frac{d - 2b}{4} \parallel \varphi \parallel^2_{L^2} = -\frac{\Gamma}{4(p - 1)} \parallel \varphi \parallel^2_{L^2}.$$

We are now in a position to consider the eigenvalue problem for the Hartree problem (3.13). With the assignment, $\bar{v} \rightarrow e^{\lambda t} \bar{v}$, we are led to consider

\begin{equation}
\mathcal{J}L\bar{v} = \lambda\bar{v}.
\end{equation}

We know that Ker$[L_-] = \text{span}[\varphi]$, while\footnote{Ker$[L_+] = \text{span}[y_1, \ldots, y_t]$. By Lemma 6, Ker$[L_-] \perp$ Ker$[L_+]$, whence} \begin{equation}
\mathcal{J}^{-1}(\text{Ker}[L]) = \mathcal{J}^{-1} \left[ \begin{array}{c} \text{Ker}[L_+] \\ \text{Ker}[L_-] \end{array} \right] = \left[ \begin{array}{c} \text{Ker}[L_-] \\ \text{Ker}[L_+] \end{array} \right] \perp \left[ \begin{array}{c} \text{Ker}[L_-] \\ \text{Ker}[L_+] \end{array} \right] = \text{Ker}[L],
\end{equation}

whence \mathcal{J}^{-1}(\text{Ker}[L]) \subset (\text{Ker}[L])^\perp as required. In addition, $L_-^{-1}$ is positive definite matrix on $\text{span}[y_1, \ldots, y_t] \subset (\text{Ker}[L_-])^\perp$. Thus, the matrix $D_t$, introduced in (1.19) has at most one negative eigenvalue. Moreover, there is a negative eigenvalue if and only if

$$D_{11} = \langle L_-^{-1} \mathcal{J}^{-1} \left( \begin{array}{c} 0 \\ \varphi \end{array} \right), \mathcal{J}^{-1} \left( \begin{array}{c} 0 \\ \varphi \end{array} \right) \rangle = \langle L_-^{-1} \varphi, \varphi \rangle < 0$$

This, together with (1.18) allows us to derive a Vakhitov-Kolokolov type criteria for the Hartree waves, namely that stability of $e^{-it} \varphi$ is equivalent to $\langle L_-^{-1} \varphi, \varphi \rangle < 0$. Using the formula for $\langle L_-^{-1} \varphi, \varphi \rangle$ in Lemma 6 we conclude that the stability occurs exactly when $\Gamma > 0$. Moreover, if $\Gamma < 0$, there is a pair (one positive and one negative) of eigenvalues $\pm \lambda$ in (1.22). By the continuity of the spectrum on $\Gamma$, we have that for $\Gamma = 0$, the pair $\pm 0$ transitions through the zero to become a pair of purely imaginary eigenvalues, so the eigenvalue problem has an extra pair of generalized eigenvalues at zero, when $\Gamma = 0$.\footnote{As it was explained above in Section 4.4 ker$[L_+]$ contains at least the vectors $\partial_j \varphi, j = 1, \ldots, d$}
4.6. Classification of the stability for the waves $e^{i\omega t} \Psi_\omega$ for the Klein-Gordon-Hartree model: Proof of Theorem 5. Much of what was established in Section 4.3 will be useful for the Klein-Gordon case as well. Indeed for the eigenvalue problem (4.3), we have

$$
\begin{pmatrix}
0 & I_2 \\
-I_2 & -J_\omega
\end{pmatrix}^{-1} \text{Ker} \left( \begin{pmatrix}
L & 0 \\
0 & I_2
\end{pmatrix} \right) = \left( J_\omega [\text{Ker}[L]] \right) \perp \left( \text{Ker}[L] \right)
$$

as established in (4.23). This is one of the requirements of GSS and it has now been verified. Similar to the arguments in Section 4.5, the portion of the matrix $D$ generated by $\text{Ker}[L_+]$ is trivially positive definite. Indeed, consider the elements of $\text{Ker}[L]$ in the form $s_j := \begin{pmatrix} y_j \\ 0 \\ 0 \end{pmatrix}$, $j = 1, \ldots, l$. Then, an easy computation shows that

$$
D_{i,j} = \langle \begin{pmatrix} L & 0 \\
0 & I_2
\end{pmatrix}^{-1} \begin{pmatrix} 0 & I_2 \\
-I_2 & -J_\omega
\end{pmatrix}^{-1} s_j, \begin{pmatrix} 0 & I_2 \\
-I_2 & -J_\omega
\end{pmatrix}^{-1} s_i \rangle = \langle \begin{pmatrix} L^{-1} & 0 \\
0 & I_2
\end{pmatrix} \left( \frac{2\omega}{\sqrt{1-\omega^2}} \begin{pmatrix} 0 \\
y_j
\end{pmatrix} \right), \left( \frac{2\omega}{\sqrt{1-\omega^2}} \begin{pmatrix} 0 \\
y_j
\end{pmatrix} \right) \rangle
$$

$$
= \frac{4\omega^2}{1-\omega^2} \langle L^{-1} y_i, y_j \rangle + \|y_j\|^2.
$$

The claim about the positivity of the portion of the matrix corresponding to these eigenvectors follows from the positivity of $L^{-1}$ on $\text{span}[y_1, \ldots, y_d] \subset \text{Ker}[L_-]^\perp$, which was previously established. Once again, the stability is found to be equivalent to the following criteria

$$
D_{11} = \langle \begin{pmatrix} L & 0 \\
0 & I_2
\end{pmatrix}^{-1} \begin{pmatrix} 0 & I_2 \\
-I_2 & -J_\omega
\end{pmatrix}^{-1} s_0, \begin{pmatrix} 0 & I_2 \\
-I_2 & -J_\omega
\end{pmatrix}^{-1} s_0 \rangle < 0,
$$

where $s_0 := \begin{pmatrix} 0 \\ \varphi \\ 0 \end{pmatrix}$. So, it remains to compute $D_{11}$. We have

$$
D_{11} = \langle \begin{pmatrix} L^{-1} & 0 \\
0 & I_2
\end{pmatrix} \left( \frac{2\omega}{\sqrt{1-\omega^2}} \begin{pmatrix} \varphi \\
0
\end{pmatrix} \right), \left( \frac{2\omega}{\sqrt{1-\omega^2}} \begin{pmatrix} \varphi \\
0
\end{pmatrix} \right) \rangle = \frac{4\omega^2}{1-\omega^2} \langle L^{-1} \varphi, \varphi \rangle + \|\varphi\|^2 = \|\varphi\|^2 \left( 1 - \frac{\Gamma \omega^2}{(p-1)(1-\omega^2)} \right),
$$

where in the last equality, we have used the formula for $\langle L_{\omega}^{-1} \varphi, \varphi \rangle$ established in Lemma 6. Since $\omega \in (-1, 1)$, we have instability ($D_{11} > 0$), if $\Gamma < 0$. If $\Gamma > 0$, we solve the inequality
1 - \frac{\Gamma^2}{(p-1)(1-\omega^2)} < 0 \text{ to obtain the necessary and sufficient condition for stability}

\[ 1 > |\omega| > \sqrt{\frac{p-1}{p-1 + \Gamma}} = \sqrt{\frac{p-1}{2 + \alpha - (p-1)(d-1)}}. \]

4.7. On the stability of the “normalized” waves for the fractional problem: Proof of Theorem 3. In order to establish the stability of the waves, we consider the eigenvalue problem (3.13), with the assignment \( \vec{v} \to e^{\lambda t} \vec{v} \). It now reads

\[ JL\vec{v} = \lambda \vec{v}. \]

It was already established that \( L \) is self-adjoint, at least for \( p > 2 \) and in addition, according to Lemma 1, \( n(L_+) = 1 \), while \( L_- \geq 0 \), with a simple eigenvalue at zero spanned by \( \phi \). Thus, we apply the instabilities index count. In fact formulas (4.18), (4.19) apply, in addition to (4.23), which shows that \( J(Ker[L]) \subset Ker[L]^\perp \). Note that we still do not know that \( \phi \) is non-degenerate, that is whether one has more than \( \partial_1 \phi, \ldots, \partial_d \phi \) in \( Ker[L] \) or not. Similarly to the classical case, we may sidestep this issue by the positivity of \( L^{-1} \) on \( (Ker[L_-])^\perp \). Hence, stability is reduced to the sign of the quantity \( D_{11} \). More specifically, since \( \phi \in Ker[L_-] \), we may label \( y_1 := \begin{pmatrix} 0 \\ \phi \end{pmatrix} \in Ker[L] \). Then, if it is indeed the case that \( \phi \in (Ker[L])^\perp \), we can compute

\[ \langle Dy_1, y_1 \rangle = D_{11} = \langle L^{-1}[\phi], \phi \rangle. \]

If we establish \( \langle L^{-1}[\phi], \phi \rangle < 0 \), this would be enough to claim that \( D \) has a negative eigenvalue and we are done, since (4.18) predicts \( 1 - 1 = 0 \) eigenvalues of (4.24) with positive real part.

For the computation of \( \langle L^{-1}[\phi], \phi \rangle \), we can apply scaling argument similar to the one presented in Lemma 6. Instead, we take derivative in \( \lambda \) in the Euler-Lagrange equation (1.3). We obtain

\[ L \left[ \frac{\partial \phi}{\partial \lambda} \right] = \frac{\partial \omega}{\partial \lambda} \phi. \]

Thus \( \phi \in (Ker[L])^\perp \) and

\[ \langle L^{-1}\phi, \phi \rangle = \frac{1}{2} \frac{\partial \omega}{\partial \lambda} \| \phi^\lambda \|_{L^2}^2. \]

By (3.5) we have \( \| \phi^\lambda \|_{L^2}^2 = \lambda \| \phi^1 \|_{L^2}^2 \), whence \( \frac{\partial \omega}{\partial \lambda} \| \phi^\lambda \|_{L^2}^2 = \| \phi^1 \|_{L^2}^2 > 0 \). According to (3.9),

\[ \frac{\partial \omega}{\partial \lambda} = (1 + \frac{2\beta(p-1)}{\Gamma}) \frac{2\beta(p-1)}{\Gamma} I_1 \lambda^\frac{2\beta(p-1)}{1} - 1 < 0, \]

since \( I_1 < 0, \Gamma > 0 \). We have thus proved Theorem 3. Note that the restriction \( p > 2 \) appeared only to satisfy technical (but important) requirements for self-adjointness of \( L \) and it is not necessary in the index computations. Thus, we expect this to be a removable, technical assumption, once we have more information about the waves \( \phi \) similar to the ones in the classical case, e.g. Theorem 1.
REFERENCES

[1] T. Cazenave and P. L. Lions, *Orbital Stability of Standing Waves for Some Nonlinear Schrodinger Equations*, Comm. Math. Phys. **85**, (1982), p. 549–561.

[2] J. Chen and B. Guo, *Strong instability of standing waves for a nonlocal Schrdinger equation*, Phys. D **227**, (2007) p. 142–148.

[3] Z. Cheng, Z. Shen and M. Yang, *Instability of standing waves for a generalized Choquard equation with potential*, J. Math. Phys. **58** (2017) 011504.

[4] R. Frank, E. Lenzmann, *Uniqueness of non-linear ground states for fractional Laplacians in R*, Acta Math. **210** (2013), no. 2, p. 261–318.

[5] Ch. Genev , G. Venkov, *Soliton and blow-up solutions to the time-dependent Schrodinger-Hartree equation* Discrete Contin. Dyn. Syst. Ser. S, **5** (2012), no. 5, 903 – 923.

[6] M. Grillakis, J. Shatah, W. Strauss *Stability theory of solitary waves in the presence of symmetry. I.* J. Funct. Anal. **74** (1987), no. 1, 160–197.

[7] J. Huang, J. Zhang, X. Li, *Stability of standing waves for the L2-critical Hartree equations with harmonic potential*, Appl. Anal. **92** (2013), no. 10, p. 2076–2083.

[8] T. M. Kapitula, P. G. Kevrekidis, B. Sandstede, *Counting eigenvalues via Krein signature in infinite-dimensional Hamiltonian systems*, Physica D, **3-4**, (2004), p. 263–282.

[9] T. Kapitula,P. G. Kevrekidis, B. Sandstede, *Addendum: "Counting eigenvalues via the Krein signature in infinite-dimensional Hamiltonian systems" [Phys. D 195 (2004), no. 3-4, 263–282] Phys. D **201** (2005), no. 1-2, 199–201.

[10] S. Lafortune, J. Lega, *, Spectral stability of local deformations of an elastic rod: Hamiltonian formalism. SIAM J. Math. Anal. **36** (2005), no. 6, p. 1726–1741.

[11] E. Lieb, M. Loss, Analysis. Second edition. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI, 2001.

[12] E.H. Lieb, *Existence and uniqueness of the minimizing solution of Choquard’s nonlinear equation*, Stud. Appl. Math. **57** (2), (1976/1977) p. 93–105.

[13] P.-L. Lions, *The Choquard equation and related questions*, Nonlinear Anal. **4** 6 (1980) p. 1063–1072.

[14] Li Ma, Lin Zhao, *Classification of positive solitary solutions of the nonlinear Choquard equation*, Arch. Ration. Mech. Anal. **195** (2) (2010), p. 455–467.

[15] G.P. Menzala, *On regular solutions of a nonlinear equation of Choquard’s type*, Proc. Roy. Soc. Edinburgh Sect. A **86** (3-4) (1980) p. 291–301.

[16] V. Moroz, J. Van Schaftingen, *Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics*, J. Funct. Anal. **265** (2013), no. 2, 1p. 153–184.

[17] V. Moroz, J. Van Schaftingen, *Existence of ground states for a class of nonlinear Choquard equations. Trans. Amer. Math. Soc. **367** (2015), no. 9, p. 6557–6579.

[18] V. Moroz, J. Van Schaftingen, *A guide to the Choquard equation, preprint*, available at https://arxiv.org/abs/1606.02158.

[19] M.Reed and B.Simon, *Methods of Modern Mathematical Physics, vol. II, Fourier Analysis and Self Adjointness*, 1975 Academic Press, New York, San Francisco , London.

[20] P. Tod, I.M. Moroz, *An analytical approach to the Schrödinger-Newton equations, Nonlinearity **12** (2) (1999), p. 201–216.

E-mail address: georgiev@dm.unipi.it

E-mail address: stefanov@ku.edu