Unifying Semantic Foundations for Automated Verification Tools in Isabelle/UTP

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Abstract

The growing complexity and diversity of models used in the engineering of dependable systems implies that a variety of formal methods, across differing abstractions, paradigms, and presentations, must be integrated. Such an integration relies on unified semantic foundations for the various notations, and co-ordination of a variety of automated verification tools. The contribution of this paper is Isabelle/UTP, an implementation of Hoare and He’s Unifying Theories of Programming, a framework for unification of formal semantics. Isabelle/UTP permits the mechanisation of computational theories for diverse paradigms, and their use in constructing formalised semantic models. These can be further applied in the development of verification tools, harnessing Isabelle/HOL’s powerful proof automation facilities. Several layers of mathematical foundations are developed, including lenses to model variables and state spaces as algebraic objects, alphabetised predicates and relations to model programs, including algebraic and axiomatic semantics, and UTP theories to encode computational paradigms. We illustrate our approach with a variety of proof tools, and in particular develop a verification tool for the formal state machine notation, RoboChart.

Keywords: Formal Methods, Automated Verification, Theorem Proving, Formal Semantics, Lenses, Unifying Theories of Programming, Hoare Logic, Reactive Systems, State Machines, Isabelle/HOL

1. Introduction

Unifying Theories of Programming [43] (UTP) is a framework for capturing, unifying, and integrating formal semantics of programming languages using predicate and relational calculus. It aims to provide a coherent structure for the intellectual discipline of computer science, by characterising the various programming languages it has produced, their foundational computational paradigms, and the various approaches to presentation of formal semantics [43]. In one axis, UTP allows us to study the plethora of computational paradigms — such as functional, imperative, concurrent [43, 55], real-time [62], and hybrid dynamical systems [28, 27] — using “UTP theories”: semantic components that allow us to construct heterogeneous semantic models. In another axis, it allows us to characterise and link different presentations of semantics, such as axiomatic semantics, like Hoare logic, refinement calculus, and separation logic, with operational semantics, such as big-step and small-step, and also algebraic semantics.

The UTP employs Helner’s “programs-as-predicates” approach [36, 38], where a program is modelled as a relational predicate over initial and final, or intermediate, observations of the state. The observation space may consist of both the internal state of a computer, such as its program variables and objects, and also modelled quantities of the real-world, such as time and physical interactions. The UTP approach provides a universal domain for both deterministic programs and non-deterministic specifications. These can be formally related through refinement, $S \subseteq P$, which states that $P$ is refined by $S$, and corresponds to universally closed implication. Moreover, the embedding of programs into logic means that formal verification, using the aforementioned axiomatic calculi, can be aided by automated theorem provers and SMT solvers.

Though the original emphasis of UTP is on programming languages, it can also be used to provide semantics for high-level modelling languages and diagrammatic notations. The use of UTP, crucially, enables us to provide various semantic models that account for different computational paradigms, and yet are linked through a common foundation. With adequate tool support, UTP can therefore allow us to answer
Figure 1: Integration of Heterogeneous Notations using UTP

the substantial challenge of integrating formal methods [56, 34, 10], by enabling us to co-ordinate a diverse number of notations, each invoking its own unique modelling paradigms, and thus with distinct formal semantics. In particular, UTP allows us to consider each of these paradigms as individual building-blocks for a semantic model, which in turn allows us form theoretical links and translations between them. Such an integration of formal methods is a crucial part of tackling the inherent semantic heterogeneity in cyber-physical systems and autonomous robots [19]. This, as the top of Figure 1 illustrates, includes notations as diverse as differential equations, control law diagrams, state machines, and C code.

A recent example of a heterogeneous modelling notation is RoboChart [47, 14], a diagrammatic language for modelling the controllers for autonomous mobile robots with a UTP semantics. RoboChart includes a formal state machine notation that can be considered a subset of UML/SysML state machine diagrams enriched with time and probability constructs. Each state machine has a well defined interface describing the events that are externally visible. The behaviour of states and transitions is described using a formal action language that corresponds to a subset of the Circus modelling language [55]. The notation supports real-time constraints, through delays, timeouts and deadlines, and also probabilistic choices, to express uncertainty. State machines can also be composed with hybrid dynamical models of the real-world, in order to characterise a robot’s sensors and actuators. The UTP semantics for RoboChart, therefore, needs to include formalisation of all these different paradigms.

In previous work [48], model checking facilities for RoboChart have been developed and applied in verification. This provides a valuable automated technique for model development, which allows detection of problems during the early development stages. However, explicit state model checking is limited to checking finite state models. In practice this means that data types must be abstracted with a small number of elements. In order to exhaustively check the potentially very large or infinite state space of many robotic applications, symbolic techniques, like theorem proving, are required. The development of such a theorem prover requires an axiomatic semantics for RoboChart, which should be consistent with the operational semantics used by the model checker, in order to ensure that the respective verification results can be meaningfully combined. Moreover, for theorem proving to be practically applicable, like model checking, automation is highly desirable. In summary, tool support for UTP allows a comprehensive solution for integration and automation of diverse formal methods.

The contribution of this paper is the theoretical and practical foundations of a tool for building UTP-
based verification tools called Isabelle/UTP. We envisage a framework that can unify the different paradigms and semantic models needed for modelling heterogeneous systems, and provide facilities for constructing automated verification tools from them. As illustrated in Figure 1, this involves constructing a number of foundational layers. The general framework of the UTP allows us to develop theories for building-block computational paradigms, like concurrency [55], real-time [62], and object orientation [60]. These theories can then be composed, in various ways, to assign denotational semantics to different languages. From this denotational semantics, UTP allows us to derive different presentations, including axiomatic, operational, and algebraic semantics. Finally, we can use this in the construction or validation of automated verification tools, which finally allows us to provide unifying support for the diverse notations on the top.

Concretely, Isabelle/UTP allows us to construct and compose UTP theories, apply them to them give denotational semantics to a given language, derive semantic models, such as axiomatic verification calculi, and then define proof tactics for automated verification. The Isabelle system [52] provides a wealth of facilities for both embedding various domain-specific languages, and automating proof [7]. It has previously been called “the Eclipse of formal methods” [66], since, like the Eclipse programming environment, it provides all the technical facilities needed to develop plugins for performing analysis on mathematical models of software and hardware. The research question that this paper answers is: how can Isabelle’s automated proof technology be brought to bear in verifying a UTP-based language like RoboChart?

The answer to this question is a new and enriched theoretical foundation for the UTP, that nevertheless seeks to conserve the capabilities envisioned by Hoare and He [43]. To provide this foundation, we draw inspiration from several related frameworks, notably Back’s refinement calculus [3], which encapsulates an explicit model of program state that is amenable to mechanisation, and also generalises concepts like substitution and assignment. We also employ and extend Foster’s work on lenses [24], which complements Back’s work with an algebraic semantics for program variables and observation spaces. Lenses allow us to model different observable regions of a program’s state, and show how they are related through properties like independence and containment. They, in particular, overcome textbook UTP’s reliance on syntactic properties of predicate constructions [20, 21], such as free variables, that has hampered proof automation in several previous mechanisation efforts [54, 68, 32], but without loss of practical generality.

Upon our algebraic foundation of observation spaces, we construct the UTP’s relational program model. This includes an expression model that is extensible, type-checkable by Isabelle, and inherently amenable to proof automation. Lenses allow us to characterise meta-logical properties, such as dependence upon a particular variable, variable substitution, and extension of an expression’s observation space. We then specialise expressions to predicates, and then relations and provide a rich set of mechanically verified algebraic theorems, including the famous “laws of programming” [44], which form the basis for axiomatic semantics. We also support symbolic execution of relational programs, and verification using the UTP characterisation of Hoare logic [43], both of which illustrate the practicability of our tool.

We then demonstrate the mechanisation of UTP theories within the relational model, that allows us to support a hierarchy of advanced computational paradigms, and so supported heterogeneous verification tools. Our work significantly advances previous contributions [68] with UTP theories whose observation spaces are typed by Isabelle, and links to established libraries of algebraic theorems [6, 2] that facilitate efficient derivation of programming laws. In particular, we develop the theory hierarchy as far as the theory of reactive designs [26, 29], which integrates the computational paradigms of state-based non-deterministic specification, imperative programming, and concurrency, and forms the semantic foundation of Circus [55] and related reactive languages. We develop a set of operators for reactive programming from this theory, and show how the reactive laws of programming can be derived. Moreover, we also show how the application of these laws can be optimised so that Isabelle can automatically perform efficient rewrite proofs.

Finally, we use our theory of reactive programs to develop a verification tool for RoboChart. With it, state machines can be verified against properties formalised in a refinement statement, such as deadlock freedom. We mechanise the state machine meta-model, including its data types, well-formedness constraints, and validation support. We use a UTP reactive designs to provide a dynamic semantics, based on guarded iteration [17]. This semantics can then be used to perform verification of infinite-state systems by theorem proving, with the help of a verified induction theorem, and Isabelle/HOL’s automated proof facilities [7]. Our denotational approach is extensible, and further mechanised UTP theories can account for real-time [62],
probability [9], and other paradigms [28]. Our work therefore also serves as a template for building automated verification tools with Isabelle/UTP.

In §2 we describe background materials for our work. Following this, in sections 3 to 7, we give an exposition of our contributions. The paper overview given in Figure 2 shows how the various foundational parts of Isabelle/UTP are connected, and the sections in which they are documented. In §3 we describe how state spaces and variables in a program can be modelled algebraically using lenses. In §4 we describe the core of Isabelle/UTP, first defining its expression model in §4.1, predicates in §4.2, and the relational program model in §4.6 with proof support and mechanised laws of programming. In §5 we utilise this relational program model, to build tools for symbolic evaluation and verification using Hoare calculus. In §6 we describe how different computational paradigms can be captured and mechanised using Isabelle/UTP theories, and illustrate this using a UTP theory for concurrent and reactive programs. In §7 we use our UTP theory of reactive programs to build a verification tool for state machines. Finally in §8, we conclude.

This paper is an extension of two conference papers [33, 25]. We extend and refine the material on modelling state spaces using lenses from [33], and give a fuller account of their use in the foundations of Isabelle/UTP. We extend [25] with further materials on how UTP theories are mechanised, building on the results given in [33]. We stress that all the definitions, theorems, and proofs in this paper are mechanically validated in Isabelle/HOL. On the whole, we simply present the core theorems without proofs, and therefore refer the interested reader to a number of companion reports [30, 31]. Though our work is primarily based on Isabelle/HOL, we prefer to use more traditional mathematical notations [63, 43] in this work, since we believe this makes the results more accessible.

2. Background and Related Work

In this section we review some preliminary material, and highlight related work.

2.1. Unifying Theories of Programming

The last decade of the twentieth century saw several seminal works on integrating formal methods in order to allow their insertion into the software engineering lifecycle [10] across its diverse notations [56, 34]. These works emphasise the centrality of unified formal semantics [38, 37] for the various models and artefacts. An important development in this direction is Hoare and He’s Unifying Theories of Programming [42, 43, 16] (UTP), which draws together results from several intellectual streams, notably Hehner’s predicative programming [36, 38], Tarski’s relational calculus [64], Dijkstra’s guarded command language [17], and the refinement calculi of Back [3] and Morgan [50].

UTP supports formalisation of computational semantic domains that are used to give denotational semantics to a variety of programming and modelling languages. It employs alphabetised binary relations to model programs as predicates [36, 37] relating the initial values of variables \( x \) to their later values \( x' \). UTP divides variables into two classes: (1) program variables, that model data, and (2) observational variables, that encode additional semantic structure. For example, \( \text{clock} : \mathbb{N} \) is a variable to record the passage of time. Unlike a program variable, it makes no sense to assign values to \( \text{clock} \), as this would model arbitrary time travel. Therefore, observational variables are constrained using healthiness conditions, which are encoded as idempotent functions on predicates. For example, application of \( HT(P) \triangleq (P \land \text{clock} \leq \text{clock}') \) results in a healthy predicate that specifies there is no reverse time travel.
The observational variables and healthiness conditions give rise to a subset of the alphabetised relations called a UTP theory, which can be used to justify the fundamental theorems of a computational paradigm. A UTP theory is the set of fixed points of the healthiness condition: \([H]_\alpha \triangleq \{ P \mid H(P) = \bot \}\). A set of signature operators can then be defined, under which the theory’s healthiness conditions are closed, and are thus guaranteed to construct programs that satisfy these theorems. UTP theories allow us to model a variety of paradigms beyond simple imperative programs, such as concurrency [43, 55], real-time [62], object orientation [60], hybrid [28, 27], and probabilistic systems [9].

The use of relational calculus means that the UTP lends itself to automated program verification using refinement \(\subseteq P\): program \(P\) satisfies specification \(S\). Since both \(S\) and \(P\) are specified in formal logic, and refinement equates to reverse implication, we can utilise interactive and automated theorem proving technology for verification. This allows application of tools like Isabelle/HOL to program verification, which is the goal of our tool, Isabelle/UTP [33].

2.2. Isabelle/HOL

Isabelle/HOL [52] is a proof assistant for Higher Order Logic (HOL). It consists of the Pure meta-logic, and the HOL object logic. Pure provides a term language, polymorphic type system, syntax translation framework for extensible parsing and pretty printing, and an inference engine. The jEdit-based IDE allows \(\LaTeX\)-like term rendering using Unicode. An Isabelle theory consists of type declarations, definitions, and theorems, which are usually proved by composition of existing theorems. Theorems have the form of \(\xi \longrightarrow \chi\), where \(\xi\) is an assumption, and \(\chi\) is the conclusion. Proof in Isabelle is conducted by invocation of tactics, which provide various mechanisms for automating proof steps. The simplifier tactic, simp, rewrites terms using equational theorems of the form \(f(x_1 \cdots x_n) \equiv y\), where the right-hand side is smaller than the left-hand side. The auto tactic combines simp with deductive reasoning using introduction, elimination, and destruction rules. Isabelle also has the powerful sledgehammer proof method [7] which invokes external first-order automated theorem provers and SMT solvers on a proof goal, and internally verifies their results using tactics like simp and metis, which is a first-order resolution prover. Additional proof tactics and strategies can be constructed using the ML programming language.

The HOL object logic implements an ML-like functional programming language founded on an axiomatic set theory similar to ZFC. HOL is purely definitional: mathematical libraries of definitions and theorems are constructed purely by application of the foundational axioms in the proof kernel. This afford a highly principled approach to mechanised mathematics. HOL provides inductive datatypes, recursive functions, and records. Several basic types are provided, including sets (\(\mathcal{P} A\)), total functions (\(A \rightarrow B\)), numbers (\(\mathbb{N}\), \(\mathbb{Z}\), \(\mathbb{R}\)), and lists. Parametric types are written by precomposing the type name, \(\tau\), with the type variables \(\alpha_1, \cdots, \alpha_n\), for example \([\mathbb{N}]\text{list}\). A notion of type specialisation allows two types to be unified if one can be expressed by instantiating the type variables of the other. For example, \([\mathbb{N}]\text{list}\) specialises \([\alpha]\text{list}\), where \(\alpha\) is a type parameter.

Types in Isabelle/HOL correspond to non-empty sets. The fundamental command for defining new types is the typedef command, which creates a type, which is potentially parametric, based on a non-empty subset of an existing type. For example, if \(\text{prime} : \mathbb{N} \rightarrow \mathbb{B}\) is a predicate that specifies whether or not a given number is prime, then we can create the type definition using the subset \(\{ n : \mathbb{N} \mid \text{prime}(n) \}\), which requires that we can exhibit at least one prime number. Upon this foundation several specialised type definition commands exist. The datatype command creates an algebraic data type with a finite number of constructors. The record command creates a record type with a number of fields.

Isabelle also provides a number of facilities helpful in mechanisation of algebraic structures and specifications. Locales [5] provide axiomatic specifications of an algebraic structure in terms of (1) a number of abstract sort types, (2) a collection of signature operators over the sorts, (3) axioms that constrain the behaviour of the operators, and (4) theorems that follow from the axioms. For example, a semigroup structure can be specified in terms of (1) a single sort \(A\), (2) a binary operator \((\cdot : A \rightarrow A \rightarrow A)\), (3) an associativity axiom \((x \cdot y) \cdot z = x \cdot (y \cdot z)\), and (4) any theorems that follow. With such a structure defined, we can

\[\begin{aligned}
\text{The square brackets are not used in Isabelle; we add them for readability.}
\end{aligned}\]
instantiate it with a particular concrete set of types and functions, provided that those functions can be proven to obey the axioms. For example, we could instantiate the semigroup locale with \([A]\text{list}\) and the list concatenation function \((xs \land ys)\), which is associative. Locales can also be arranged in a hierarchy, with child locales adding further operators and axioms. A related notion is type classes, which restrict locales to a single type sort, and use the polymorphic overloading function of Isabelle to characterise the algebraic operators. This for instance allows us to overload + with several definitions for different types.

### 2.3. Mechanising UTP

Mechanisation of the UTP involves constructing a tool that can aid the construction and verification of UTP theories, and put these to work in program verification. Most existing mechanisations \([53, 54, 20]\) semantically embed the UTP into an existing logic, usually HOL, which allows them harness the latter’s existing proof facilities. The exception is the \(U \cdot (TP)^2\) tool \([11, 12]\), a bespoke proof assistant for the UTP in the early stages of development. A semantic embedding of the UTP requires that we first model its fundamental model of computation: alphabetised relations \([43, \text{Chapter } 2]\). The model must both be faithful to the UTP, so that its pen-and-paper results can be mechanised, whilst also providing access to automated proof facilities of the host logic.

In the UTP book \([43]\), an alphabetised relation is a pair \((\alpha P, P)\), where \(\alpha P\) is a set of variable declarations, of the form \(x : t\), and \(P\) is a predicate containing no free variables other than those in \(\alpha P\) \([43]\). Intuitively, the alphabet consists of the selection of system properties that we wish to observe. As in the Z notation \([63]\), a variable can be primed \(x'\), meaning that it is an output (or “after”) variable. Then, the alphabet of a relation can be decomposed into the set of inputs and outputs, respectively: \(\alpha P = \text{in} \alpha P \cup \text{out} \alpha P\).

Alphabets play a very important role in UTP by providing information that is used by programming operators to calculate the semantics of compositions. They are used in Chapter 2 to determine whether two sequentially executing programs \(P ; Q\) are compatible by ensuring that \(\text{out} \alpha P = \text{in} \alpha Q\). In Chapter 7 (Concurrency), alphabets are used to determine whether two concurrently executing programs, \(P \parallel Q\), manipulate disjoint regions of the memory through a proviso \(\alpha P \cap \alpha Q = \emptyset\). Moreover, alphabets are used to specify observational variables, and are thus important in giving a structure to UTP theories.

The earliest UTP mechanisation, created by Nuka and Woodcock \([53]\), develops a deep embedding of the alphabetised relational calculus in ProofPower-Z, a proof assistant that builds on HOL. They develop an abstract syntax tree for predicates, a semantic domain, and an interpretation function. They utilise a semantic domain of the form \(\text{Pred} \triangleq (\mathcal{P} \text{Var}) \times (\mathcal{P} \text{Binding})\), which is a product of an alphabet, that is a set of variable names, and a set of possible observations (\(\text{Binding}\)), which is modelled as a partial function from variables to values: \(\text{Binding} \triangleq (\text{Var} \to \text{Value})\). This naturally means it is necessary to define types for both variables (\(\text{Var}\)) and values (\(\text{Value}\)). The set of observations is equivalent to a predicate over the observation space of type \(\text{Binding} \to \mathbb{B}\).

The need for an abstract syntax tree for predicates is implied by UTP’s definition of alphabetised relations \([43]\), which depends upon a notation of free variables, a syntactic property of predicates. However, as pointed out by \([54]\), fixing an abstract syntax tree runs against the UTP doctrine of having an extensible syntax to support various notations with a unified semantics. Consequently, Oliveira’s mechanisation of UTP in ProofPower-Z \([54]\) omits an abstract syntax tree, and instead defines operators directly on the semantic domain, which takes a similar form to \(\text{Pred} \triangleq [53]\), resulting in a shallow embedding. The absence of an abstract syntax tree means that there is no longer a concept of free variables, which is sometimes used in the side conditions of programming laws. They solve this problem by defining a notion of “unrestriction”, a semantic notion that a predicate does not depend on particular variables, an idea we adopt in Isabelle/UTP.

Oliveira’s mechanisation \([54]\) omits an explicit formalisation of types, which is often needed for alphabet declarations, and so Zeyda and Cavalcanti later extended that work with a type model and typing relation \([68]\). This proves to be essential for mechanising UTP theories, and their work develops a hierarchy of such theories, from imperative programs through to reactive programs. Two related shortcomings of this work, though, is the lack of support for native type checking and proof automation. Since the typing relation is formalised, it is necessary to prove typing side conditions, which hampers proof. This problem is alleviated in a following work \([32]\), which provides an additional “shallow layer” of the predicate model that harnesses
the Isabelle type system to discharge typing provisos. However, the work also places an additional burden on the user of defining their own Value type. Proof is also hampered by the need to discharge alphabet provisos. For example, a sequential composition \( P : Q \) is well-formed only if \( \text{out}_{\alpha}(P) = \text{in}_{\alpha}(Q) \), a property which is not automatically checked and must be proved.

A different approach to the works above is taken by Feliachi et al. [20, 21], who develop a UTP theorem prover in Isabelle/HOL. Like [53] and [54], they model predicates as sets of observations \( \mathbb{P} \mathbb{S} \). However, \( \mathbb{S} \) is not Binding, but any definable type in Isabelle/HOL. Generally, they use record types to encode observation spaces, where each field corresponds to a variable, and therefore proof automation is greatly improved in their tool. Their predicate type, \([\alpha] \text{predicate}\) is parametrised by an observation space type \( \alpha \). Moreover, by convention, the record type declarations double as alphabets, so that the type checker is directly harnessed in checking alphabet provisos. Specifically, the type \([\alpha] \text{relation} \triangleq [\alpha \times \alpha] \text{predicate}\), denotes a homogeneous relation whose alphabet is \( \alpha \). Sequential composition has type \([\alpha] \text{relation} \to [\alpha] \text{relation} \to [\alpha] \text{relation}\), and so relational compositions with incompatible alphabets are rejected.

Moreover, their approach can also harness type inference to automatically assign a polymorphic alphabet type to a relation. In textbook UTP, the assignment operator, \( x :=_A v \), is decorated with an explicit alphabet \( A \), which must be provided in order to infer the alphabet. This annotation is often omitted for convenience in [43], but this can only be done when the alphabet can be unambiguously inferred. By harnessing type inference, we can omit such explicit decorations, whilst obtaining relations that are well-formed by construction.

In addition to overcoming the problems with alphabets and types, [20, 21] also overcomes the need for an explicitly formalised universe, Value, as required in the other works [53, 54, 68, 32]. Since any type in HOL is required to have an a priori fixed cardinality, it is impossible to construct a type universe into which any HOL type can be injected. This is not a problem for [20, 21], as predicates are parametric and an observation space type can make use of any HOL type for its variables. This problem is also alleviated in a further extension to the other works [69], so that we can have a distinguished type universe, but at the expense of extending HOL with additional axioms.

In our development of Isabelle/UTP, we broadly follow the direction taken by [20, 21]. In particular we use the type system for alphabet inference as this greatly improves automation. However, we adopt several ideas from the other works [54, 68, 32], notably unrestriction, because the variable model in [20, 21] does not permit us to form such properties of variables. In order to understand this, we need to look more closely at how observation spaces should be modelled. It turns out that these two families of mechanisms represent two unique approaches to modelling observation, which we consider next.

2.4. Observation Space Modelling

As we have seen, the principle difference between the UTP mechanisations is the approach to modelling the observation space \( \mathbb{S} \). These distinct approaches have been highlighted before in [61], which provides a general comparison. They identify four often employed models of observation, or state, in mechanised program models, namely (1) state as functions, (2) tuples, (3) records, and (4) abstract types, of which the first and third seem the most prevalent.

The first approach models observation as a function \( \text{Var} \to \text{Value} \), for suitable value and variable types. This approach is taken by [53, 54, 32, 18, 69], and requires a deep model of variables and values, in which concepts such as names and typing are first-class. This provides a highly expressive model with few limitations on possible manipulations [32]. However, [61] highlights two obstacles: (1) the machinery required for deep reasoning about program values is heavy and a priori limits possible constructions, and (2) explicit variable naming requires one to consider syntactic issues like substitution and \( \alpha \)-renaming. Whilst [69] effectively mitigates (1), by axiomatically introducing a value universe, the complexities associated with (2) remain. Once names and types are treated as first-class, it becomes necessary to replicate a large portion of the underlying proof system meta-logic, which requires great effort and can hit proof efficiency. Nevertheless, the approach seems necessary to model dynamic creation of variables, as required, for example, in modelling memory heaps in separation logic [13, 18].

The alternative approach uses records to model state; a technique often used by verification tools in Isabelle [1, 20, 21, 2]. In particular, [20] uses this approach to create their semantic embedding of the UTP
in Isabelle/HOL. A variable in this kind of model is represented simply by a record field. These can be abstractly represented using pairs of field-query and update functions, $f_i$ and $f_i$-upd. The set of variable names (Var) therefore does not need to encoded in this approach. As shown in [20, 21, 2], this approach greatly simplifies automation of program verification in comparison to the former functional approach through directly harnessing, rather than replicating, the polymorphic type system and automated proof tactics. However, the expense is a loss of flexibility compared to the functional approach, particularly in regards to decomposition of state spaces. Moreover, those employing records seldom provide general support for syntactic concepts like free variables and substitution.

Isabelle/UTP employs an approach which generalises both these models by abstractly characterising the behaviour of state and variables algebraically using lenses. Consequently, it can achieve both a high-level of proof automation like [20, 21], but also support for syntax-like queries, such as unrestricted and substitution. Lenses were originally created as an abstraction for bidirectional programming and solving the view-update problem [23]. They abstract different views on a data space, and allow their manipulation independently of the context. A lens consists of two functions: get that extracts a view from a larger source, and put that puts back an updated view. [22] gives a detailed study of the algebraic lens laws for these functions. Combinators are also provided for composing lenses [24, 23]. They have been practically applied in the Boomerang language for transformations on textual data structures.

Our lens approach is indeed related to the state-space solution in [61] of using Isabelle locales to characterise a state type abstractly and polymorphically. A difference though is their use of explicit names, where our lenses are nameless. We also note that a very similar concept to lenses exists in Back’s refinement calculus [3, Chapter 5], which substantially pre-dates the work on lenses. Here, get is called val, the “access function”, and put is called set, the “update function”, but the three lens axioms listed above are all present. A difference is that, unlike for lenses, [3] does not provide explicit models for the variable axioms.

3. Algebraic Observation Spaces

In this section, we present our theory of lenses, which provides an algebraic semantics for states and observation space modelling in Isabelle/UTP. Although some core definitions like the lens laws and composition operator are well known [24, 23, 22], we introduce several novel operators, like summation and quotient, and relations like independence and equivalence. This enables an algebraic approach to modelling of state spaces that is strongly related to the work of Back [3]. This section also constitutes a substantial expansion of materials in [33, Section 3] with a more precise presentation and several novel results.

As we have observed in §2.3, program variables are often modelled as opaque syntactic objects that necessitate a mixing of syntactic meta-logic concerns and program logic. Here, we will use lenses [24, 23] to model variables as abstract behavioural semantic entities. Rather than fixing a particular model for variables, we simply require that the chosen model satisfies the lens axioms. Our contention is, therefore, that lenses provide a unifying solution to state space modelling.

All definitions, theorems, and proofs in this section may be found in our Isabelle/HOL mechanisation [30].

3.1. Signature

Lenses are a well-known mathematical concept, but here we will generalise them and recast several mathematical results. A lens is used, intuitively, to view and manipulate a $V$-shaped region of a larger source type $S$, as illustrated in Figure 3. We introduce lenses axiomatically as two-sorted algebraic structures.

Definition 3.1 (Lenses). A lens is a quadruple $\langle V \mid S \mid get : S \rightarrow V \mid put : S \rightarrow V \rightarrow S \rangle$, where $V$ and $S$ are non-empty sets called the view type and source type, respectively, and get and put are total functions. We write $V \equiv S$ to denote the type of lenses with source type $S$ and view type $V$, and subscript get and put with the name of a particular lens. We define $\text{create}x \equiv put \{ \varepsilon s \mid s \in S \} v$, which constructs an arbitrary source using Hilbert’s choice operator ($\varepsilon$) and then puts $v$ into it.

\footnote{\textit{Boomerang home page: http://www.seas.upenn.edu/~harmony/}}
The \( \text{get} \) function views the current value of the region, and \( \text{put} \) updates it. Intuitively, we will use these structures to model sequences of queries and updates on the state space \( S \) in §4. Each variable in a program can be represented by an individual lens, with operators like assignment utilising \( \text{get} \) and \( \text{put} \) to manipulate the corresponding region of program memory. The \( \text{create} \) function is usually axiomatised as a separate element of the lens signature, but here we prefer to define it in terms of \( \text{put} \), which depends only upon \( S \) being non-empty. For the purpose of example, we describe lenses for record types.

**Definition 3.2** (Record Lens). Consider the definition of a new record type, \( R \triangleq \langle f_1 : \tau_1, \ldots, f_n : \tau_n \rangle \), which consists of \( n \) fields each having a corresponding type. Each field yields a function: \( f_i : R \rightarrow \tau_i \) which queries the current value of a field. Moreover, we can update the value of field \( f_i \) in \( r : R \) with \( k : \tau_i \) using \( r(f_i := k) \).

We can construct a lens for each field using \( \text{get} \) and \( \text{put} \). More specifically, we will use these \( \text{get} \) and \( \text{put} \) to manipulate the data within a particular region of program memory. The crucial characteristic of lenses is that they allow manipulation of the data within a particular region of program memory.

**Definition 3.3** (Cartesian Product Lenses).

\[
\begin{align*}
\text{fst}^{S_1, S_2} & \triangleq \langle S_1 \mid S_1 \times S_2 \mid \lambda(x, y) \cdot x \cdot \lambda(x, y) z \cdot (z, y) \\
\text{snd}^{S_1, S_2} & \triangleq \langle S_2 \mid S_1 \times S_2 \mid \lambda(x, y) \cdot y \cdot \lambda(x, y) z \cdot (x, z) 
\end{align*}
\]

The subscripted source types are necessary in order to specify the product type; we will omit them when they can be determined from the context. Lenses \( \text{fst} \) and \( \text{snd} \) allow us to focus on the first and second element of a product type, respectively. Their \( \text{get} \) functions project out these elements, and the \( \text{put} \) functions replace the first and second elements with the given value \( z \). A final example is the total function lens:

**Definition 3.4.** \( \text{fun}^{A, B}_k \triangleq \langle A \mid A \rightarrow B \mid \lambda f \cdot f(k) \rangle \mid \lambda f \cdot v \cdot (\lambda x \cdot \text{if } x = k \text{ then } v \text{ else } f(x)) \)

The total function lens \( \text{fun}^{A, B}_k \) views the output of a function \( f : A \rightarrow B \) associated with a given input value \( k : A \). The \( \text{get} \) function simply applies \( f \) to \( k \), and the \( \text{put} \) function associates a new output \( v \) with \( k \). The function lens allows us to also employ the “state as functions” approach [54, 61].

### 3.2. Axiomatic Basis

The crucial characteristic of lenses is that they allow manipulation of the data within a particular region of the source type, without affecting the data stored elsewhere. As we have previously shown, lenses therefore provide an effective method for giving behavioural semantics to program variables. The use of lenses to model variables depends on \( \text{get} \) and \( \text{put} \) behaving according to a set of axioms.

**Definition 3.5** (Total Lenses). A total lens obeys the following axioms:

\[
\begin{align*}
\text{get}(\text{put } s \cdot v) & = v \quad \text{(PutGet)} \\
\text{put}(\text{put } s \cdot v') v & = \text{put } s \cdot v \quad \text{(PutPut)} \\
\text{put } s \cdot (\text{get } s) & = s \quad \text{(GetPut)}
\end{align*}
\]

We write \( V \rightarrow S \) for the set of total lenses with view type \( V \) and source type \( S \), and \( * \rightarrow S \) for the set of total lenses with any view type, whose source type is \( S \).
We mechanise this algebraic structure in Isabelle using locales [5]. Total lenses are usually called “very well-behaved” lenses in the literature [24, 23], but we believe “total” is more descriptive of the behaviour, since it is always possible to meaningfully project a view from a source. Axiom PutGet states that if a source has been constructed by application of put s v, then a matching get returns the injected value, v. Axiom PutPut states that a later put overrides an earlier one, so that the previously injected value \( v' \) is replaced by \( v \). Finally, Axiom GetPut states that for any source element \( s \), if we extract the view element and then update the original using it then we get precisely \( s \) back.

We will now demonstrate that every field of a record yields a total lens.

**Theorem 3.1.** For any field \( f_i \) of a record type \( R \), the record lens \( \text{rec}^R_i \) forms a total lens.

**Proof.** For illustration, we prove each lens axiom in turn.

1. **PutGet:** \( \text{get} \) \( (\text{put} \ s \ v) = (\lambda s \bullet f_i \ s)(s(f_i := v)) = f_i(s(f_i := v)) = v \)
2. **PutPut:** \( \text{put} \ s \ v' \ \overset{\text{def}}{=} (\lambda s \bullet s(f_i := v))(s(f_i := v')) = s(f_i := v) = \text{put} \ s \ v \)
3. **GetPut:** \( \text{put} \ s \ (\text{get} \ s) = (\lambda s \bullet s(f_i := v))(f_i) = s(f_i := f_i) = s \)

Consequently, the record lens is indeed a total lens.

We can similarly show that \( \text{fst}, \ \text{snd}, \ \text{and fun} \) are total lenses.

**Theorem 3.2.** For any \( A, B, \) and \( k \in A , \ \text{fst}^{A,B}, \ \text{snd}^{A,B}, \ \text{and fun}^k_{A,B} \) are total lenses.

Whilst both PutGet and PutPut are satisfied for most useful state-space models we can consider, this is not the case for GetPut. For example, if we consider a lens that projects the valuation of an element \( x : A \) from a partial function \( f : A \to B \), then get is only meaningful when \( x \in \text{dom}(f) \). Since get is total, it must return a value, but this will be arbitrary and therefore placing it back into \( f \) will alter its domain. We can therefore also consider the set of partial lenses, that do not satisfy GetPut, but we will not consider this further for this paper in order to retain focus on the Isabelle/UTP core, but simply observe that total lenses are a useful, but not universal solution for state space modelling.

A useful subset of the total lenses are bijective lenses, as identified in [23, 22].

**Definition 3.6 (Bijective Lenses).** A bijective lens satisfies the axiom PutGet and the identity

\[
\text{put} \ s \ (\text{get} \ s') = s'
\]

We write \( X : \mathcal{V} \iff \mathcal{S} \), for \( X : \mathcal{V} \implies \mathcal{S} \), if \( X \) is a bijective lens.

Axiom StrongGetPut states that application of \( \text{get} \ s' \) retrieves a view that is to sufficiently large to entirely replace any preexisting source \( s \) with \( s' \). In other words, a bijective lens declares a kind of isomorphism between \( \mathcal{V} \) and \( \mathcal{S} \), so that \( \mathcal{V} \) must contain precisely the same level of information as is present in \( \mathcal{S} \). In state space terms, bijective lenses allow us to switch between equivalently expressive state space models. Clearly, any bijective lens is also a total lens, since StrongGetPut is more general than GetPut.

In order to illustrate this, an example bijective lens is defined below.

**Definition 3.7.** \( \text{swap}_{S_1,S_2} \overset{\text{def}}{=} (S_1 \times S_2 \mid S_2 \times S_1 \mid \lambda(x, y) \bullet (y, x) \mid \lambda s \ (x, y) \bullet (y, x)) \)

The \( \text{swap} \) lens commutes the elements of a product space \( S_1 \times S_2 \). Clearly then there is no loss of information associated with the get function. We prove that this lens satisfies axiom StrongGetPut below.

**Theorem 3.3.** The \( \text{swap} \) lens is a bijective lens.

**Proof.** We prove that both PutGet and StrongGetPut hold.

1. **PutGet:** \( \text{get} \ (\text{put} \ s \ v) = \text{get} \ (\text{put} \ s \ (v_1, v_2)) = (\lambda(x, y) \bullet (y, x))(v_2, v_1) = (v_1, v_2) = v \)
2. **StrongGetPut:** \( \text{put} \ s \ (\text{get} \ s') = \text{put} \ s \ (\text{get} \ s'_1, s'_2) = \text{put} \ s \ (s'_2, s'_1) = (s'_1, s'_2) = s' \)

\(^3\text{All omitted proofs can be found in our Isabelle/HOL mechanisation [30].}\)
3.3. Independence

So far we have considered the behaviour of individual lenses, but programs reference several variables and so it is necessary to coordinate the semantics of the corresponding lenses. One of the most important relationships between lenses is independence of their corresponding views, which is illustrated in Figure 4. We formally characterise independence below.

**Definition 3.8 (Independent Lenses).** Lenses $X : V_1 \implies S$ and $Y : V_2 \implies S$ are independent, written $X \bowtie Y$, provided they satisfy the following laws:

\[
\begin{align*}
\text{LI1} & : \quad \text{put}_X (\text{put}_Y s) v = \text{put}_Y (\text{put}_X s) v \\
\text{LI2} & : \quad \text{get}_X (\text{put}_Y s) v = \text{get}_X s \\
\text{LI3} & : \quad \text{get}_Y (\text{put}_X s) u = \text{get}_Y s
\end{align*}
\]

Lenses $X$ and $Y$, which share the same source type but not necessarily the same view type, are independent provided that their \text{put} functions commute (LI1), and their respective \text{get} functions are not influenced by the corresponding \text{put} functions (LI2, LI3). If $X$ and $Y$ are variables, then independence means that we can reorder assignments to them. We can, for example, show that $\text{fst} \bowtie \text{snd}$ using the calculation below.

**Theorem 3.4.** $\text{fst} \bowtie \text{snd}$

**Proof.** We first prove that $\text{put}_\text{fst}$ commutes with $\text{put}_\text{snd}$ (LI1) by evaluation:

\[
\begin{align*}
\text{put}_\text{fst}(\text{put}_\text{snd} s v) u &= \text{put}_\text{fst}(\text{put}_\text{snd}(s_1, s_2) v) u \\
&= \text{put}_\text{fst}(s_1, v) u = (u, v) = \text{put}_\text{snd}(u, s_2) v \\
&= \text{put}_\text{snd}(\text{put}_\text{fst}(s_1, s_2) u) v = \text{put}_\text{snd}(\text{put}_\text{fst} s u) v
\end{align*}
\]

Similarly, we can also prove LI2 by evaluation:

\[
\text{get}_\text{fst}(\text{put}_\text{snd} s v) = \text{get}_\text{fst}(\text{put}_\text{snd}(s_1, s_2) v) = \text{get}_\text{fst}(s_1, v) = s_1 = \text{get}_\text{fst}(s_1, s_2) = \text{get}_\text{snd} s
\]

Finally, LI3 follows by a symmetric proof.

A further, illustrative, result is the meaning of independence in the total function lens:

**Theorem 3.5.** $\text{fun}_a \bowtie \text{fun}_b \iff a \neq b$

Two instances of the total function lens are independent if, and only if, the parametrised inputs $a$ and $b$ are different. This reflects the intuition of a function – every input is associated with a distinct output. This kind of independence property is also present in [3], though it is characterised as an assumption of variables with different names, rather than as a predicate to be used in assumptions of a modelled variable.

3.4. Combinators

In this section, we define an algebraic structure for lenses, beginning with some basic lenses.

**Definition 3.9 (Basic Lenses).** $0_S \triangleq \{(\emptyset) | S | \lambda s \cdot \emptyset | \lambda s v \cdot s \} \quad 1_S \triangleq (S | S | \lambda s \cdot s | \lambda s v \cdot v)$

**Theorem 3.6 (Basic Lenses Closure).** $0$ and $1$ are total lenses.

The $0$ lens is a nominal lens that has a unitary view type, $\{\emptyset\}$. Consequently, for any element of the source type, it always views the same value $\emptyset$. It cannot be used to either observe or change a source, and it is therefore entirely ineffectual in nature. Conversely, the $1$ lens view, with type $S \implies S$, views the entirety of the source. The $0$ and $1$ lenses serve no practical purpose, but are useful units for the other functions.

**Definition 3.10 (Lens Summation).**

\[
X + Y \triangleq \langle V_X \times V_Y | S_X | \lambda s \cdot (\text{get}_X s, \text{get}_Y s) | \lambda s (v_1, v_2) \cdot \text{put}_Y (\text{put}_X s v_1) v_2 \rangle \quad \text{if } S_X = S_Y
\]
Theorem 3.7 (Summation Closure). If $X$ and $Y$ are total independent lenses, then $X + Y$ is a total lens.

Lens sum allows us to simultaneously manipulate two regions of the source $S$ characterised by $X : V_1 \rightarrow S$ and $Y : V_2 \rightarrow S$. Consequently, the view type is the product of the two constituents: $V_1 \times V_2$. The get function applies both constituent get functions in parallel. The put function applies the constituent put functions, but in sequence since we are in the function domain. We require that $X \cong Y$, since manipulation of two overlapping regions could have unexpected results, and then also the order of the put functions is irrelevant. The sum operator can be used to characterise concurrent updates to the state space.

Another important operator is lens composition, which is a variant of an operator described in [23].

Definition 3.11. $X \triangleright Y \overset{\Delta}{=} \langle V_X | S_Y | \text{get}_Y \circ \text{get}_X | \lambda s v \bullet \text{put}_Y s (\text{put}_X (\text{get}_Y s) v) \rangle$ if $S_X = V_Y$

Theorem 3.8 (Composition Closure). If $X$ and $Y$ are total lenses, then $X \triangleright Y$ is a total lens.

A lens composition, $X \triangleright Y$, for $X : A \rightarrow B$ and $Y : B \rightarrow C$ chains together two lenses. When $X$ characterises an $A$-shaped region of $B$, and $Y$ characterises a $B$-shaped region of $C$, overall lens $X \triangleright Y$ characterises an $A$-shaped region of $C$. It is useful when we have a state space composed of several individual state components, and we wish to select a variable of an individual component. We exemplify this below.

Example 3.1. In an object oriented program we may have $m \in \mathbb{N}$ objects whose states are characterised by lenses $a_i : O_i \rightarrow S$, for $i \in \{1, m\}$, where each $O_i$ characterises the respective object state. An object $a_k$ has $n \in \mathbb{N}$ attributes, characterised by lenses $x_j : \tau_j \rightarrow O_k$ for $j \in \{1, n\}$. We can select one of these attributes from the global state context by the composition $x_j \triangleright a_k : \tau_j \rightarrow S$.

Lens composition obeys a number of useful algebraic properties, as shown below.

Theorem 3.9 (Lens Composition Laws). If $X$, $Y$, and $Z$ are total lenses then the following identities hold:

\[
X \triangleright (Y \triangleright Z) = (X \triangleright Y) \triangleright Z \quad X \triangleright 1 = 1 \triangleright X = X \quad X \triangleright 0 = 0 \triangleright X = 0
\]

Lens composition is associative and has $1$ as its left and right units, making it a monoid. Moreover, $0$ is a left and right annihilator, since if the view is reduced to $\emptyset$ then no further information can be extracted. With the help of lens composition, we can now also prove some algebraic laws for lens sum.

Theorem 3.10 (Lens Sum). If $X$ and $Y$ are independent total lenses then the following identities hold:

\[
\text{fst} \triangleright (X + Y) = X \quad \text{snd} \triangleright (X + Y) = Y \quad \text{swap} \triangleright (X + Y) = (Y + X) \quad (X + Y) \triangleright Z = (X \triangleright Z) + (Y \triangleright Z)
\]

The first two identities show that fst and snd composed with + yield the left- and right-hand side lenses, respectively. The third shows that swap indeed commutes +. The fourth shows that \triangleright distributes from the right through lens sum. We next show some independence properties for the operators introduced so far.

Theorem 3.11 (Independence). If $X$, $Y$, and $Z$ are total lenses then the following identities hold:

\[
0 \bowtie X \\
X \bowtie Y \Leftrightarrow Y \bowtie X \\
Y \bowtie Z \Rightarrow (X \bowtie Y) \bowtie Z \\
(X \bowtie Z) \bowtie (Y \bowtie Z) \Leftrightarrow X \bowtie Y \\
X \bowtie Z \land Y \bowtie Z \Rightarrow (X + Y) \bowtie Z
\]

The $0$ lens is independent from any lens, since it has no manipulative power. Lens independence is a symmetric relation, as expected. Lens composition preserves independence: if $Y \bowtie Z$ then composing $X$ with $Y$ still yields a lens independent of $Z$. $X$ and $Y$ composed with a common lens $Z$ are independent if, and only if, $X$ and $Y$ are themselves independent. Finally, lens plus also preserves independence: if both $X$ and $Y$ are independent of $Z$, then also $X + Y$ is independent of $Z$. 

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3.5. Observational Order and Equivalence

A lens $X$ can view a larger region than another lens $Y$, with the implication that $Y$ is fully dependent on $X$. For example, it is clear that in Example 3.1 each object fully possesses each of its attributes, and likewise each attribute lens $x_j \downarrow o_k$ is fully dependent on lens $o_k$. We can formalise this using the lens order.

**Definition 3.12** (Lens Order). $(X \preceq Y) \triangleq (S_X = S_Y \land (\exists Z : V_X \Rightarrow V_Y \bullet X = Z \downarrow Y))$

A lens $X : V_1 \Rightarrow S$ is narrower than a lens $Y : V_2 \Rightarrow S$ provided that they share the same source, and there exists a total lens $Z : V_1 \Rightarrow V_2$, such that $X$ is the same as $Z \downarrow Y$. In other words, the behaviour is defined by firstly viewing the source using $Y$, and the secondly viewing a subregion of this using $Z$. An order characterises the size of a lens’s aperture: how much of the source a lens can view. For example, we can prove that $x_j \downarrow o_k \preceq o_k$, by setting $Z = x_j$ in Definition 3.12. The lens order relation is a preorder.

**Theorem 3.12.** For any $S$, $(\ast \longmapsto S, \preceq)$ forms a preorder, that is, $\preceq$ is reflexive and transitive. Furthermore, the least element of $\ast \longmapsto S$ is $0_S$, and the greatest element is $1_S$.

Clearly, $0$ is the narrowest possible lens since it allows us to view nothing, and $1$ is the widest lens, since it views the entire source. We can prove the following intuitive theorem for sublens.

**Theorem 3.13.** If $X, Y$ are total lenses and $X \preceq Y$, then the following identities hold:

\[
\begin{align*}
\text{put}_Y(\text{put}_X s v) u &= \text{put}_Y s u \\
\text{get}_X(\text{put}_Y s v) &= \text{get}_X(\text{put}_Y s' v)
\end{align*}
\]

(LS1)

(LS2)

Law LS1 is a generalisation of Axiom PutPut: a later put$_{Y}$ overrides an earlier put$_{X}$ when $X \preceq Y$. Law LS2 states when viewing an update on $Y$ via a narrower lens $X$ we can ignore the valuation of the original source, since the update replaces all the relevant information. We can now use these results to prove a number of ordering theorems for lens compositions.

**Theorem 3.14** (Lens Order Laws). If $X$, $Y$, and $Z$ are total lenses then they satisfy the following laws:

\[
\begin{align*}
X \downarrow Y &\preceq Y \\
X \preceq Y \land Y \bowtie Z &\Rightarrow X \bowtie Z \\
X \bowtie Z \land Y \preceq Z &\Rightarrow (X + Y) \preceq (X + Z) \\
X \bowtie Y &\Rightarrow X + Y \preceq Y + X \\
X \bowtie Y \land X \bowtie Z \land Y \bowtie Z &\Rightarrow X + (Y + Z) \preceq (X + Y) + Z
\end{align*}
\]

As we observed, composition of $X$ and $Y$ yields a narrow lens than $Y$ (LO1). Independence is preserved by the ordering, since a subregion of a larger independent region is also clearly independent (LO2). Lens plus also preserves the ordering in its right-hand component (LO3). Moreover, lens plus is commutative with respect to $\preceq$ (LO4), and also associative, assuming appropriate independence properties (LO5). From these laws, and utilising the preorder theorems, we can prove various useful corollaries, such as

\[
X \bowtie Y \Rightarrow X \preceq (X + Y)
\]

which shows that $X$ is, of course, narrower than $X + Y$, and

\[
X \preceq Z \land Y \preceq Z \land X \bowtie Y \Rightarrow (X + Y) \preceq Z
\]

which shows that plus preserves the order in both its arguments. We can also induce an equivalence relation on lenses using the lens order.

**Definition 3.13** (Lens Equivalence). $(X \simeq Y) \triangleq (X \preceq Y \land Y \preceq X)$

**Theorem 3.15** (Lens Equivalence Relation). For any $S$, $(\ast \longmapsto S, \simeq)$ forms a setoid, that is, $\simeq$ is an equivalence relation on the set $\ast \longmapsto S$ — it is reflexive, symmetric, and transitive.
Proof. Reflexivity and transitivity follow by Theorem 3.12, and symmetry follows by definition.

This \(\approx\) relation is different to equality (\(X = Y\)), since it requires only that the two sources types are the same, whilst the view types can be different. This makes this relation much more useful for evaluating observational equivalence between two lenses which have apparently differing views, and yet characterise precisely the same region. This is reflected by the following set of algebraic laws.

**Theorem 3.16** (Lens Algebra). If \(X, Y,\) and \(Z\) are all total lenses then they satisfy the following identities:

\[
\begin{align*}
X + (Y + Z) &\approx (X + Y) + Z & \text{if } X \bowtie Y, X \bowtie Z, Y \bowtie Z \\
X + 0 &\approx X \\
X + Y &\approx Y + X & \text{if } X \bowtie Y
\end{align*}
\]

Lenses thus form a partial commutative monoid [18], also known as a separation algebra [13], where \(X + Y\) is effectively defined only when \(X \bowtie Y\), which is a separateness relation.

We can also use lens equivalence to give the following alternative characterisation of bijective lenses.

**Theorem 3.17.** \(X\) is a bijective lens if, and only if, \(X \approx 1\)

A bijective lens is equivalent to \(1\), meaning that its view type contains exactly the same amount of information as the source type. We can use this fact to describe the partitioning of a state space by two independent lenses: \(X + Y \approx 1\). This characterises the notion of lens complements [23], where the view types of \(X\) and \(Y\) represent complementary regions of the source. Similarly, we can use bijective lenses to establish an equivalence relation between lenses:

**Theorem 3.18.** If \(X\) and \(Y\) are total lenses, then \((X \approx Y) \iff (\exists Z : \forall X \iff \forall Y \cdot X = Z \bowtie Y)\).

An equivalence between \(X\) and \(Y\) can be established if we can find a bijective lens which when composed with \(Y\), yields \(X\). Lens \(Z\) can be viewed as an adaptor from one lens to the other.

3.6. Mechanised State Spaces

The algebraic theory of lenses described in this section is mechanised in Isabelle/HOL [30]. We will now show how it provides the basis for state spaces in our program verification framework in Isabelle/UTP.

Manual construction of state spaces using the lens combinators is tedious and so we also implement an Isabelle/HOL command for automatically creating a new state space with the following form:

\[
\text{alphabet} (\alpha_1, \ldots, \alpha_k) S = [([\beta_1, \ldots, \beta_m] T +) \upharpoonright x_1 : \tau_1 \cdots x_n : \tau_n]
\]

It creates a new state space type \(S\), and a lens for each of the variables \(x_i : \tau_i\). It can be used to describe a concrete state space for a program. For brevity, we often abbreviate the \texttt{alphabet} command by the syntax

\[
(\alpha_1, \ldots, \alpha_k) S \triangleq (\beta_1, \ldots, \beta_m) T + [x_1 : \tau_1, \cdots, x_n : \tau_n]
\]

when used in mathematical definitions. Internally, the command performs the following steps:

1. generate a record space type \(S\) with \(n\) fields, which optionally extends an existing state space \(T\);
2. generate a lens \(x_i\) for each of the fields using the record lens \(\text{rec}_R\);
3. automatically prove that each lens \(x_i\) is a total lens;
4. automatically prove an independence theorem \(x_i \bowtie x_j\) for each pair \(i, j \in \{1 \cdots n\}\) such that \(i \neq j\);
5. generate lenses \texttt{base}s and \texttt{more}s that characterise the “base part” and “extension part”, respectively;
6. automatically prove a number of independence and equivalence properties.
We will now elaborate on each of these steps in detail.

The new record type $S$ yields an auxiliary type $(\alpha_1, \cdots, \alpha_k, \phi) S$-ext with additional type parameter $\phi$ that characterises future extensions. In particular, the non-extended type $(\alpha_1, \cdots, \alpha_k)$ is characterised by $(\alpha_1, \cdots, \alpha_k, \text{unit}) S$-ext, where unit is a distinguished singleton type. This extensible record type is isomorphic to a product of three basic component types:

$$([\beta_1, \cdots, \beta_m] T \times (\tau_1 \times \cdots \times \tau_n)) \times \phi$$

These characterise, respectively, the part of state space described by $T$, the part described by the $n$ additional fields, and the extension part $\phi$. In the case that the state space does not extend an existing type, we can set $(\beta_1, \cdots, \beta_m) T = \text{unit}$.

For each field, the command generates a lens $x_i : \tau_i \Rightarrow (\alpha_1, \cdots, \alpha_k, \phi) S$-ext using the record lens, and prove total lens and independence theorems. Each of these lenses is polymorphic over $\phi$, so that they can be applied to the base type and any extension thereof, in the style of inheritance in object oriented data structures. As we show in §6, this polymorphism allows us to characterise a hierarchy of UTP theories.

In addition to the field lenses, we create two special total lenses:

- $\text{base}_S : (\alpha_1, \cdots, \alpha_k) S \Rightarrow (\alpha_1, \cdots, \alpha_k, \phi) S$-ext, which characterises the base part; and
- $\text{more}_S : \phi \Rightarrow (\alpha_1, \cdots, \alpha_k, \phi) S$-ext, which characterises the extension part.

The base part consists of only the inherited fields and those added by $S$. We automatically prove a number of theorems about these special lenses:

- $\text{base}_S \bowtie \text{more}_S$: the base and extension parts are independent;
- $\text{base}_S + \text{more}_S \approx 1$: they partition the entire state space;
- for $i \in \{1 \cdots n\}$, $x_i \preceq \text{base}_S$: each variable lens is part of the base;
- $\text{base}_S \approx \text{base}_T + \left(\sum_{i \in \{1 \cdots n\}} x_i\right)$: the base is composed of the parent’s base and the variable lenses;
- $\text{more}_T \approx \left(\sum_{i \in \{1 \cdots n\}} x_i\right) + \text{more}_S$: the parent’s extension is composed of the variable lenses and the child’s extension part.

These theorems support the Isabelle/UTP laws of programming, which we elaborate in the next section.

4. Mechanising the UTP Relational Calculus

In this section we will describe the core of Isabelle/UTP, including its expression model, meta-logical operators, predicate calculus, and relational calculus, building upon our algebraic model of state spaces. A direct result is an expressive model of relational programs which can be used in proving fundamental algebraic laws of programming [44], and for formal verification (§4). Moreover, the relational model is foundational to the mechanisation of UTP theories, and thus advanced computational paradigms, such as reactive programming, which we consider in §6.

4.1. Expressions

Expressions are the basis of all other program and model objects in Isabelle/UTP, in that every such artefact is a specialisation of the expression type. An expression language typically includes literals, variables, and function symbols, all of which are also accounted for here.

We model expressions as functions on the observation space: $[A, S]uexpr \cong (S \rightarrow A)$, where $A$ is the return type, which is a similar approach to that taken in [3]. There is no inductively defined abstract syntax tree for expressions, but simply a collection of definitions that allow composition of expressions using
HOL functions. Our model therefore technically represents a shallow embedding \cite{8,20}, but it nevertheless supports expression of a variety of syntax-like constraints using lenses.

A major advantage of this model is that we need not preconceive of all expression constructors, but can add them by definition. Using Isabelle/HOL’s type definition (\texttt{typedef}) mechanism we obtain both a new type \([A,S]uexpr\), and a function, \([=]:[A,S]uexpr \to S \to A\), that obtains the characteristic function from an expression. We also obtain the following foundational extensionality principle:

**Theorem 4.1** (Expression Equality). \(e = f \iff (\forall s \bullet \llbracket e \rrbracket s = \llbracket f \rrbracket s)\)

We can demonstrate that two expressions \(e\) and \(f\) are equal provided that their characteristic functions yield the same value for any state space \(s\). We now show how the core calculus for expressions is constructed.

**Definition 4.1** (Expression Constructors). Assume types \(A, B, C,\) and \(S\). Then we declare the constants:

\[
\begin{align*}
\text{var} &: (A \to S) \to [A,S]uexpr \\
\text{lit} &: A \to [A,S]uexpr \\
\text{cond} &: [B,S]uexpr \to [A,S]uexpr \to [A,S]uexpr \to [A,S]uexpr \\
\text{uop} &: (A \to B) \to [A,S]uexpr \to [B,S]uexpr \\
\text{bop} &: (A \to B \to C) \to [A,S]uexpr \to [B,S]uexpr \to [C,S]uexpr
\end{align*}
\]

and assign these constants the following semantic definitions:

\[
\begin{align*}
\llbracket \text{var} \ s \rrbracket s &= \lambda \cdot \text{get}_s s \\
\llbracket \text{lit} \ k \rrbracket s &= \lambda \cdot k \\
\llbracket \text{cond} \ b \ u_1 \ u_2 \rrbracket s &= \lambda \cdot \text{if} \ [b]_s \ s \text{ then } [u_1]_s \text{ else } [u_2]_s \\
\llbracket \text{uop} \ f \ u \rrbracket s &= \lambda \cdot f ([u]_s) \\
\llbracket \text{bop} \ g \ u \ v \rrbracket s &= \lambda \cdot g ([u]_s) ([v]_s)
\end{align*}
\]

where \(x: A \Rightarrow S\) is a lens; \(k: A\) is a HOL constant; \(f: A \to B\) and \(g: A \to B \to C\) are functions; and \(b: [B,S]uexpr, u, u_1, u_2: [A,S]uexpr,\) and \(v: [B,S]uexpr\) are expressions.

The operator \(\text{var} x\) represents a variable expression, and returns the present value in the state characterised by lens \(x\). For convenience, we assume that \(x, y, z,\) and decorations thereof, are lenses, and will often use them directly as variable expressions without explicitly using \(\text{var}\). We will also use \(v\) to denote the \(\text{1}\) lens; this is effectively a special variable for the entire state.

The operator \(\text{lit} k\) represents a literal, or alternatively an arbitrary lifted HOL value, and corresponds to a constant function expression. We use the notation \(\llbracket \cdot \rrbracket\) to denote a literal \(k\). As well as lens-based variables, which are used to model program variables, expressions can also contain HOL variables, which are orthogonal and constant with respect to the program variables. HOL variables in literal constructions can be used to represent meta-variables, for example \(\llbracket \cdot \rrbracket\), which are something necessary in proofs about programs. We use the decorations \(x, y,\) and \(z\) to denote logical variables in expressions.

Operator \(\text{cond} \ b \ u_1 \ u_2\) denotes a conditional expression; if \(b\) is true then it returns \(u_1\), otherwise \(u_2\). It evaluates the boolean expression \(b\) under the incoming state, and chooses the expression based on this. We will use the notation \(e < u_1 > u_2\) as a short hand for it.

Operators \(\text{uop}\) and \(\text{bop}\) lift HOL functions into the expression space by a pointwise lifting. With them we can transparently use HOL functions as UTP expression functions, for instance the summation \(e + f\) is denoted by \(\text{bop}(+) \ s f\). Moreover, it is often possible to lift theorems from the underlying operators to the expressions themselves, which allows us to reuse the large library of HOL algebraic structures in Isabelle/UTP. For instance, if we know that \((A, +, 0)\) is a monoid, then also we can show that for any \(S,\) \(([A,S]uexpr, \text{bop}(+), \text{lit}0)\) forms a monoid. For convenience, we therefore often overload mathematically defined functions as expression constructs without further comment. In particular, we will often overload \(\cdot\) as both equivalence of two expressions \((e = f)\), and an expression of equality within an expression \((x = 5)\).
4.2. Predicate Calculus

A predicate is an expression with a Boolean return type, \([S]\upred \triangleq [\mathcal{B}, \mathcal{S}]\upred\), so that predicates are a subtype of expressions. Then, the majority of predicate calculus operators can be obtained by pointwise lifting of the equivalent operators in HOL. In order to notationally distinguish the latter, in the following definitions we subscript them with a \(H\).

**Definition 4.2** (Predicate Calculus Operators).

\[
\begin{align*}
\text{true} & \triangleq \text{lit true}_x, \\
\text{false} & \triangleq \text{lit false}_x, \\
(\neg P) & \triangleq \text{uop}_x (\neg x) P, \\
(P \land Q) & \triangleq \text{bop}_x (\land x) P Q, \\
(P \lor Q) & \triangleq \text{bop} (\lor x) P Q, \\
(P \Rightarrow Q) & \triangleq (\neg P \lor Q).
\end{align*}
\]

The propositional calculus operators \((\neg, \land, \lor, \Rightarrow)\) are simply defined by pointwise lifting. We define two pairs of quantifiers, since we have both logical variables and lenses in Isabelle/UTP. Quantifiers \(\forall\) and \(\exists\) universally and existentially quantify lens \(x\) by universally quantifying over possible values, \(v\), and updating \(x\) with them. The dual quantifiers \(\forall\) and \(\exists\), which are notationally emboldened, quantify a logical variable in a parametric predicate \(P(x)\) by a direct lifting of the corresponding HOL quantifier. The universal closure, \([P]\), universally quantifies every variable in the alphabet of \(P\) using the state variable \(v\). The refinement relation \(P \sqsubseteq Q\) is then defined as a HOL predicate, requiring that \(Q\) implies \(P\) in every state \(s\).

In addition to the binary operators \(\land\) and \(\lor\), we also define the corresponding indexed operators.

**Definition 4.3** (Indexed Operators).

\[
\begin{align*}
\forall_{i \in A} P(i) & \triangleq_\mathcal{E} \left( \lambda s \cdot \forall_{i \in A} [P(i)]_\mathcal{E} s \right), \\
\exists_{i \in A} P(i) & \triangleq_\mathcal{E} \left( \lambda s \cdot \exists_{i \in A} [P(i)]_\mathcal{E} s \right).
\end{align*}
\]

The definitions are by lifting of the underlying HOL operators, and therefore we obtain the following theorem.

**Theorem 4.2.** For any \(S\), \([S]\upred, \land, \lor, \neg, \false, \true) forms a complete Boolean algebra, that is:

- \((S]\upred, \land, \lor, \neg)\) is a complete lattice with infimum \(\lor\), supremum \(\land\), top element \(\true\), and bottom \(\false\).

As usual, via the Knaster-Tarski theorem, for any monotonic function \(F : [S]\upred \rightarrow [S]\upred\) we can describe the least and greatest fixed points, \(\mu F\) and \(\nu F\), which in UTP are called the weakest and strongest fixed points, and obey the usual fixed point laws. We can also algebraically characterise the UTP variable quantifiers using Cylindric Algebra [39], which axiomatises the quantifiers of first-order logic.

**Theorem 4.3.** For any \(S\), \((S]\upred, \land, \lor, \neg, \false, \true, \exists, =)\) forms a Cylindric Algebra; the following laws are satisfied for total lenses \(x, y, z\):

\[
\begin{align*}
(\exists x \cdot \false) & = \false \quad \text{(C1)} \\
(\exists x \cdot P) & \sqsubseteq P \quad \text{(C2)} \\
(\exists x \cdot (P \land (\exists x \cdot Q))) & = (((\exists x \cdot P) \land (\exists x \cdot Q)) \quad \text{(C3)} \\
(\exists x \cdot \exists y \cdot P) & = (\exists y \cdot \exists x \cdot P) \quad \text{(C4)} \\
(x = x) & = \true \quad \text{(C5)} \\
(y = z) & = (\exists x \cdot y = x \land x = z) \quad \text{if } x \sqsupset y, x \sqsupset z \quad \text{(C6)} \\
\false & = \left( (\exists x \cdot x = y \land P) \land (\exists x \cdot x = y \land \neg P) \right) \quad \text{if } x \sqsupset y \quad \text{(C7)}
\end{align*}
\]
From this algebra, the usual laws of quantification can be derived [39]. These laws illustrate the difference in expressive power between HOL and UTP variables. For the former, we cannot pose meta-logical questions like whether two variable names x and y refer to the same region, such as may be the case if they are aliased. For this kind of property, we can use lens independence \( x \varpropto y \), as required by laws C6 and C7.

Lenses can model both individual variables, and also larger regions of the state space formed by summation of several \((x + y)\). We can therefore model finite sets of variables, and thus quantification over multiple variables such as \( \exists x, y, z \cdot P \), which is represented as \( \exists x + y + z \cdot P \), and then prove the following laws.

**Theorem 4.4.** If \( A \) and \( B \) are total lenses, then the following identities hold:

\[
\begin{align*}
(\exists A + B \cdot P) &= (\exists A \cdot \exists B \cdot P) & A \varpropto B \quad &\text{(Ex1)} \\
(\exists B \cdot \exists A \cdot P) &= (\exists A \cdot P) & \text{if } B \preceq A \quad &\text{(Ex2)} \\
(\exists A \cdot P) &= (\exists B \cdot Q) & \text{if } A \preceq B \quad &\text{(Ex3)}
\end{align*}
\]

Ex1 shows that quantifying over two disjoint sets of variables equates to quantification over both. Disjointness of variable sets is modelled by requiring that the corresponding lenses are independent. Ex2 shows that quantification over a larger lens subsumes a smaller lens. Finally Ex3 shows that if we quantify over two lenses that identify the same subregion then those two quantifications are equal.

We now have a complete set of operators and laws for the predicate calculus; next we will show how proof can be mechanically automated.

### 4.3. Automated Proof

The approach to define the operators of the predicate calculus using pointwise lifted HOL operators allows us to harness Isabelle/HOL’s automated proof tactics. The strategy is to transfer an equality or refinement conjecture about UTP predicates to an equivalent one about HOL predicates using the interpretation function \([e]_\mathcal{S}\). The resulting proof tactic, which we called rel-auto, employs the following steps to attempt discharge a goal of the form \( e = f \):

1. apply a transfer rule, such as Theorem 4.1, to obtain a goal of the form \([e]_\mathcal{S} \circ s = [f]_\mathcal{S} \circ s\);
2. evaluate \([e]_\mathcal{S} \circ s\) and \([f]_\mathcal{S} \circ s\) to HOL terms using interpretation rules, such as Definitions 4.1 and 4.2;
3. apply the auto tactic to invoke automated simplification and deduction.

The strategy is not always successful, and so residual proof obligations may be left afterwards. In this case, it is possible to apply the sledgehammer [7] tactic on the resulting proof obligations, or alternatively the nitpick counterexample generator. We permit the set of semantic interpretation rules used by step (2) to be expanded by additional definitional equations, so that the proof strategy is extensible. The example calculation below shows how the proof strategy can be used for a simple predicative conjecture.

**Example 4.1 (Proof Strategy Rewrites).**

\[
\begin{align*}
((x = 1 \land y = x + x) \Rightarrow y > 2) &= \text{true} \\
\equiv (\forall s \cdot [(x = 1 \land y = x + x) \Rightarrow y > 2]_\mathcal{S} \circ s = [\text{true}]_\mathcal{S} \circ s) & \text{Theorem 4.1} \\
\equiv (\forall s \cdot (\text{get}_x s = 1 \land \text{get}_y s = \text{get}_x s + \text{get}_x s \Rightarrow \text{get}_y s > \text{get}_x s) = \text{true}_n) & \text{Definitions 4.1 and 4.2} \\
\equiv (\forall s \cdot (\text{get}_x s = 1 \land \text{get}_y s = 1 + 1 \Rightarrow \text{get}_y s > 1) = \text{true}_n) & \text{get}_x\text{ equality} \\
\equiv ((2 > 1) = \text{true}_n) & \text{predicate calculus} \\
\equiv \text{true}_n & \text{arithmetic}
\end{align*}
\]

The crucial observations from this example are that (1) UTP variables are compiled to expressions of the form \( \text{get}_x s \), and (2) all other operators simply become their HOL equivalents. Of course, \( x \) and \( y \) can be any lens into the state space \( \mathcal{S} \), which is an arbitrary type.

The proof procedure is enhanced when \( \mathcal{S} \) is constructed using the alphabet command of §3.6, because we can enumerate all the field lenses. Given a state space \([(x_1 : \tau_1, \cdots, x_n : \tau_n)]\), we can eliminate the \( s \) state
space variable in a proof goal by replacing it with a tuple of HOL variables \((x_1 \cdots x_n)\). This, in turn, means that we can eliminate each lens expression and replace it with a corresponding HOL variable, so that the example above, when transferred, simply becomes

\[(x = 1 \land y = x + x) \Rightarrow y > 2) = \text{true}_e\]

that is, every UTP variable has been replaced by a corresponding HOL variable, which means that we can directly and without reference to lenses invoke HOL proof procedures. This approach means that we have both the additional expressivity and fidelity afforded by lenses, and the proof automation of Isabelle/HOL.

In the next section we explore some of the advantages of using lenses rather than HOL variables directly.

### 4.4. Meta-Logic

Lenses allow us to treat variables as semantic objects that can be checked for independence, ordered, and composed in various ways. As we have noted, we can consider such manipulations as meta-logical with respect to the predicate. We will now add further specialised meta-logical queries for expressions.

Often we want to check which parts of the state space an expression depends on, for example to support laws of programming and verification calculi. Though this is often characterised syntactically using free variables, here we follow [54] and characterise it semantically with the “unrestriction” operator.

**Definition 4.4 (Unrestriction).**

\[-\sharp - : (A \implies S) \rightarrow [B; S]uexpr \rightarrow B\]

\[(x \sharp e) \triangleq (\forall (s, k) \bullet [e]_e (\text{put}_x s k) = [\text{set}_e] s)\]

Intuitively, \(e\) is unrestricted by lens \(x\), written \(x \sharp e\), provided that \(e\)’s valuation does not depend on \(x\). Specifically, the effect of \(e\) evaluated under state \(s\) is the same if we change the value of \(x\). It is thus a sufficient notion to formalise the meta-logical provisos for the laws of programming. Below are some key laws for establishing whether an expression is unrestricted by a variable.

**Theorem 4.5 (Unrestriction Laws).** If \(x\) and \(y\) are total lenses, then the following laws hold:

\[
\begin{align*}
\text{hit} & \quad x \triangleright y \quad x \triangleright u \quad x \triangleright u \quad x \triangleright v \quad x \triangleright u \quad 0 \triangleright u \quad x \triangleright y \quad x \triangleright y \\
\text{var} & \quad x \triangleright y \quad x \triangleright P \quad x \triangleright P \quad x \triangleright (\exists y \bullet P) \quad x \triangleright (\forall y \bullet P) \quad x \triangleright (\forall y \bullet P)
\end{align*}
\]

Expression \(\text{hit}k\) does not depend on the state since it always returns \(k\); consequently it is unrestricted by any lens \(x\). Expression \(\text{var}x\) is unrestricted by any lens \(y\) that is independent from \(x\). The laws for \(\text{uop}f\) and \(\text{bop}f\) require simply that the component expressions are unrestricted by lens \(x\). No expression is restricted by the \(0\) lens, since it characterises none of the state-space. Unrestriction is preserved by the lens order \(\preceq\).

An expression \(u\) is unrestricted by the summation of lenses \(x\) and \(y\) provided that \(x\) and \(y\) are independent, and the expression is unrestricted by both \(u\) and \(v\). Again, we note that + can effectively be used to group lenses in order to characterise a set of variables.

Aside from checking for use of variables, UTP theories often require that the state space of a predicate can be extended with additional variables. An example of this is the variable block operator that adds a new local variable. In Isabelle/UTP, alphabets are implicitly characterised by state space types, rather than explicitly as sets of variables. Consequently, we perform alphabet extensions by manipulation of the underlying state space using lenses. We define the following operators to extend a state space.

**Definition 4.5 (Alphabet Extrusion).**

\[-\oplus_a : [A; S_1]uexpr \rightarrow (S_1 \implies S_2) \rightarrow [A; S_2]uexpr\]

\[[c \oplus_a a]_e \triangleq (\lambda s \bullet [c]_e (\text{get}_a s))\]
Alphabet extrusion, \( e \oplus a \), uses the lens \( a : S_1 \implies S_2 \) to extend the state space of \( e : [A,S_1]uexpr \) to become \( S_2 \). The lens effectively describes how one state space can be embedded into another. For example, let us assume we have a state space composed of two sub-regions: \( S \triangleq A \times B \). Then, a predicate \( P : [A]upred \), that acts on state space \( A \), can be coerced to one acting on \( S \) by an alphabet extrusion: \( P \oplus \text{fst} : [S]upred \). Naturally, we know that the resulting predicate is unrestricted by the \( B \) region.

**Theorem 4.6 (Alphabet Extrusion Laws).**

\[
\begin{align*}
\text{lit } k \oplus a &= \text{lit } k \\
\text{var } x \oplus a &= \text{var } (x \triangleright a) \\
\text{var } f (u \oplus a) &= \text{var } f (u \ominus a) \\
\text{bop } g (u \oplus v) &= \text{bop } g (u \ominus v) \\
\end{align*}
\]

Alphabet extrusion has no effect on a literal expression, other than to change its type, because it refers to no lenses. A variable constructor has its lens augmented by composing it with the alphabet lens \( a \). The extrusion simply distributes through unary and binary expressions.

### 4.5. Substitutions

Deterministic programs can be modelled as total functions \( \sigma, \rho : S \rightarrow S \) that transform an initial state to a final state. Such programs are conceptually substitutions, since they define the valuation for each program variable in terms of a current value. The simplest substitution, \( id \triangleq (\lambda x \cdot x) \), leaves the state unchanged. We define operators for querying and updating substitutions below.

**Definition 4.6 (Substitution Query and Update).**

\[
[[\sigma]]_s \triangleq (\lambda s \cdot \text{get}_s (\sigma(s))) \quad \sigma(x \mapsto e) \triangleq (\lambda s \cdot \text{put}_s (\sigma(s)) ([e]_s s))
\]

Substitution query \( \langle \sigma \rangle_s x : [A,S]uexpr \) returns the expression associated with \( x : A \implies S \) in \( \sigma : S \rightarrow S \) by composition of the latter with \( \text{get}_x \). Substitution update assigns the expression \( e \) to the lens \( x \) in \( \sigma \). The definition constructs a function that inputs the state \( s \), evaluates \( e \) with respect to \( s \), calculates the state space updated by \( \sigma \), and then uses \( \text{put} \) to update the value of \( x \) in \( \sigma(s) \) with the evaluated expression. We can then introduce the short-hand

\[
[x_1 \mapsto e_1, \cdots, x_n \mapsto e_n] \triangleq id(x_1 \mapsto e_1, \cdots, x_n \mapsto e_n)
\]

which constructs a substitution in \( n \) variables that assigns the expression \( e_i \) to each variable \( x_i \). This notation will allow to model multiple variable assignment, and also evaluation contexts for programs and expressions. We also prove a number of laws about substitutions constructed from maplets.

**Theorem 4.7.** If \( x \) and \( y \) are total lenses, then the following identities hold:

\[
\begin{align*}
\langle \sigma(x \mapsto e) \rangle_s x &= e \quad \text{(SB1)} \\
\sigma(x \mapsto \sigma) &= \sigma \quad \text{(SB2)} \\
\sigma(x \mapsto y) &= \sigma \quad \text{(SB3)} \\
\sigma(x \mapsto e, y \mapsto f) &= \sigma(y \mapsto f, x \mapsto e) \quad \text{if } x \triangleleft y \quad \text{(SB4)} \\
\sigma(x \mapsto e, y \mapsto f) &= \sigma(y \mapsto f) \quad \text{if } x \cong y \quad \text{(SB5)}
\end{align*}
\]

The majority of these reflect the intuition of substitution queries and updates. Law SB4 states that two substitution maplets, \( x \mapsto e \) and \( y \mapsto f \), are commutable provided that \( x \) and \( y \) are independent. Similarly, SB5 states that a later assignment for \( y \) overrides an earlier assignment for \( x \) when \( y \) is a wider lens than \( x \). This is, of course, true in particular when \( x = y \).

Substitutions can be sequentially composed using the composition operator \( \rho \circ \sigma \triangleq (\lambda x \cdot \rho(\sigma(x))) \), meaning \( \rho \) after \( \sigma \). Conditional choices can be expressed using the following construct.

**Definition 4.7 (Conditional Substitution).** \( \sigma \triangleright b \triangleright \rho \triangleq (\lambda s \cdot \text{if } [[b]]_s \text{ then } \sigma(s) \text{ else } \rho(s)) \)
A conditional substitution is equivalent to σ when b is true, and ρ otherwise. The definition evaluates b under the incoming state s, and then chooses which substitution to apply based on this. Substitutions can be applied to an expression using the following operator.

**Definition 4.8 (Substitution Application).**

\[
\begin{align*}
- \mapsto &: (S \to S) \to [A, S]expr \to [A, S]expr \\
\llbracket \sigma \mapsto e \rrbracket_s &\triangleq (\lambda s \cdot \llbracket e \rrbracket_s(\sigma(s))) 
\end{align*}
\]

Application of a substitution σ to an expression e simply evaluates e in the context of state σ(s). We can also model the classical syntax for substitution, \( P[v/x] \triangleq [x_1 \mapsto v_1] \vdash P \), and prove the substitution laws.

**Theorem 4.8 (Substitution application laws).**

\[
\begin{align*}
\sigma \mapsto \text{var } x &= \langle \sigma \rangle_s x & (SA1) \\
\sigma(x \mapsto_s e) \mapsto u &= \sigma \mapsto u & \text{if } x \not\in u \quad (SA2) \\
\sigma \mapsto \text{uop } v &= \text{uop } f (\sigma \mapsto v) & (SA3) \\
\sigma \mapsto \text{bop } u v &= \text{bop } f (\sigma \mapsto u)(\sigma \mapsto v) & (SA4) \\
(\exists y \cdot P)[e/x] &= (\exists y \cdot P[e/x]) & \text{if } x \not\in y, y \not\in e \quad (SA5) \\
\text{id } \mapsto e &= e & (SA6) \\
\sigma \mapsto \rho \mapsto e &= (\rho \circ \sigma) \mapsto e & (SA7) \\
\rho(x \mapsto_s e) \circ \sigma &= (\rho \circ \sigma)(x \mapsto_s \sigma \mapsto e) & (SA8) \\
\sigma(x \mapsto_s e) \bullet b \bullet \rho(x \mapsto_s f) &= (\sigma \bullet b \bullet \rho)(x \mapsto (e \bullet b \bullet f)) & (SA9)
\end{align*}
\]

Application of σ to a variable x is the valuation of x in σ (SA1). A substitution maplet for an unrestricted variable can be removed (SA2). Substitutions distribute through both unary (SA3) and binary (SA4) operators. A singleton substitution for variable x can pass through an existential quantification over y provided that x and y are independent, and e is unrestricted by y (SA5), which prevents variable capture. Application of id has no effect (SA6), and application of two substitutions can be expressed by their composition (SA7). SA8 shows that when σ is composed with another substitution composed of maplets \( x \mapsto_s e \), it is simply applied to the expression e of every such maplet. SA9 shows how two substitutions with matching maplets can be conditionally composed, by distributing the conditional.

As for expressions, we define an operator to extend the alphabet of a substitution.

**Definition 4.9 (Substitution Alphabet Extrusion).** \( \sigma \oplus a \triangleq (\lambda s \cdot \text{put}_a s(\sigma(\text{get}_a s))) \)

We use the lens \( a : S_1 \to S_2 \) to coerce \( \sigma : S_1 \to S_1 \) to the state space \( S_2 \). The resulting substitution first obtains an element of \( S_1 \) from the incoming state \( s : S_2 \) using \( \text{get}_a \), applies the substitution to this, and then places the updated state back into \( s \) using \( \text{put}_a \). This is the essence of a framed computation; the parts of \( s \) outside of the view of \( a \) are unchanged.

### 4.6. Relational Calculus

A relation is a predicate on a product space \( S_1 \times S_2 \), specifically, where \( S_1 \) and \( S_2 \) are the state spaces before and after execution, respectively. All laws that have been proved for expressions and predicates therefore hold for relations. We define types for both heterogeneous relations, \([S_1, S_2]\text{urel} \triangleq [S_1 \times S_2]\text{upred},\) and homogeneous relations \([S]\text{hetrel} \triangleq [S, S]\text{urel}.\) Operators \text{true} and \text{false} can be specialised in this relational setting, and stand for the most and least non-deterministic relations. Due to their special role, we will use the notation \text{true} and \text{false} to explicitly refer to these relational counterparts.

In common with formal languages like Z and B, UTP uses the notional convention for variables that x is the initial value and \( x’ \) is its final value. The former can be denoted by the lens composition \( x \mapsto \text{fst} \), and the latter by \( x \mapsto \text{snd} \), where x is actually the name of a lens of type \( \tau \to S \). In this presentation we define the operators below for lifting variables, expressions, and substitutions into the product space.
Definition 4.10 (Pre- and Postcondition Lifting).

\[
\begin{align*}
\mathbf{x}^* & \triangleq \mathbf{x} \uplus \mathbf{fst} \quad \mathbf{x}^* \triangleq \mathbf{x} \uplus \mathbf{snd} \quad e^* & \triangleq e \uplus e_1, \mathbf{fst} \quad e^* & \triangleq e \uplus e_2, \mathbf{snd} \quad \sigma^* & \triangleq \sigma \uplus \sigma_1, \mathbf{fst} \quad \sigma^* & \triangleq \sigma \uplus \sigma_2, \mathbf{snd}
\end{align*}
\]

The $\uplus$ and $\uplus$ lift a lens, expression, or substitution, into the first and second components of a product state space $S_1 \times S_2$. Operator $e^*$ lifts an expression $e : [A,S_1]uexpr$ to an expression on the product state space $S_1 \times S_2$, for any given $S_2$. If $e$ is a predicate on the state variables, then $e^*$ is a predicate on the initial state, that is a precondition. Similarly, $e^*$ constructs a postcondition with state space $S_1 \times S_2$ from $e : [A,S_2]uexpr$. The analogous operators $\sigma^*$ and $\sigma^*$ lift a substitution to the product space.

We can now define the main programming operators of the relational calculus.

Definition 4.11 (Programming Operators).

\[
\begin{align*}
(P ; Q) & \triangleq (\exists v_0 \cdot P[v_0/\mathbf{v}^*] \land Q[v_0/\mathbf{v}^*]) \\
\Pi & \triangleq (\mathbf{v}^* = \mathbf{v}^*) \\
\langle \sigma \rangle & \triangleq (\mathbf{v}^* = \sigma(\mathbf{v}^*))
\end{align*}
\]

These broadly follow the common definitions given in relational calculus and the UTP book [43], but with subtle differences due to our use of lenses. Relational composition $P ; Q$ existentiality quantifies a HOL meta-variable $v_0$ that stands for the intermediate state between $P$ and $Q$. It is substituted as the final state of $P$ (using $\mathbf{v}^*$), and the initial state of $Q$; the resulting predicates are then conjoined. Composition is a heterogeneous operator of type $[S_1, S_2]urel \rightarrow [S_2, S_3]urel \rightarrow [S_1, S_3]urel$.

Relational identity ($\Pi$), or skip, equates the initial state with the final state, leaving all variables unchanged. $\langle \sigma \rangle$ is a generalised assignment operator constructed using a substitution $\sigma : S \rightarrow S$. Its definition states that the final state is equal to the initial state with $\sigma$ applied. The substitution $\sigma$ can be constructed as a set of maplets, so that the singleton assignment $x := v$ can be expressed as $\langle x \mapsto v \rangle$ and a multiple variable assignment as $\langle x_1 \mapsto v_1, \ldots, x_n \mapsto v_n \rangle$. Since $x$ can be any kind of lens in an assignment, we can encode array assignment $a[i] := v$, which updates the index $i$ in array $a$ to have value $v$, whilst leaving the remainder of $a$ unchanged. An array over type $A$, with size $n + 1$, can be modelled as a total function with a finite domain: $\{0 \cdots n\} \rightarrow A^4$. The array lens $a[i]$ can then be denoted using $\mathbf{fun}^{[0 \cdots n], A \uplus i}$.

Conditional $P \triangleq b \triangleright Q$ states that if $b$ is true then behave like $P$, otherwise behave like $Q$. In the UTP book [43] the fact that $b$ acts on initial variables is a syntactic convention, whereas here this well-formedness condition is imposed by construction using $b^*$. For completeness, we use this operator, and the fact that all predicates, including relations, form a complete lattice, to define the while loop operator $b \triangleright P$. We use the strongest fixed-point, $\nu$, as we will use it for partial correctness verification in §5. We now have all the operators of a simple imperative programming language, and can prove the laws of programming [44, 43].

Theorem 4.9 (Laws of Programming).

\[
\begin{align*}
(P ; Q) ; R & = P ; (Q ; R) \quad \text{(LP1)} \\
\Pi ; P & = P ; \Pi \quad \text{(LP2)} \\
\text{false} ; P & = P ; \text{false} = \text{false} \quad \text{(LP3)} \\
\langle \sigma \rangle ; P & = \sigma^* \downarrow P \quad \text{(LP4)} \\
\langle \sigma \rangle \triangleright b \triangleright \rho & = \langle \sigma \downarrow b \triangleright \rho \rangle \quad \text{(LP5)} \\
(P ; b \triangleright Q) ; R & = P ; R \triangleright b \triangleright P ; R \quad \text{(LP10)}
\end{align*}
\]

The majority of these are standard, and therefore we select only a few for commentary. LP10 would normally require as a proviso that $b$ is an expression in initial variables only, but in our setting this fact follows by construction. LP4 is a generalisation of the forward assignment law: an assignment by $\sigma$ followed

---

4In Isabelle/HOL, such finite types can be constructed using the Numeral package in the HOL-Library extension.
by relation $P$ is equivalent to $\sigma$ applied to the initial variables of $P$. The more traditional formulation \cite{44,43}, $x := e \; ; \; P = P[e/x]$, is an instance of this law. Similarly, LP5 is a generalised conditional assignment law which combines $\sigma$ and $\rho$ into a single conditional substitution. All the other assignment laws can be proved, but we need some additional properties for relational substitutions, which are shown below.

**Theorem 4.10** (Relational Substitutions).

$$\sigma^* \uparrow (P \; ; \; Q) = (\sigma^* \uparrow P) \; ; \; Q \quad (RS1)$$

$$\sigma^* \uparrow (P \; ; \; Q) = P \; ; (\sigma^* \uparrow Q) \quad (RS2)$$

$$\sigma^* \uparrow (P \bowtie b \bowtie Q) = (\sigma^* \uparrow P) \bowtie (\sigma \uparrow b) \bowtie (\sigma^* \uparrow Q) \quad (RS4)$$

RS1 shows that a precondition substitution applies only to the first element of a sequential composition, and RS2 is its dual. RS3 shows that precondition substitution applied to an assignment can be expressed as $\sigma^*$ applied to a conditional distributes through all three arguments. The combination of Theorems 4.8, 4.9, and 4.10 allows us to prove the classical assignment laws \cite{44}.

**Theorem 4.11** (Assignment Laws).

$$\begin{align*}
\begin{array}{ll}
x := e \; ; \; P = P[e^* / x^*] & x, y, z := e, f, g = y, x, z := f, e, g \\
x := x = I & x := e ; (P \bowtie b \bowtie Q) = P[e^* / x^*] \bowtie b[e/x] \bowtie Q[e^* / x^*] \\
x, y := e, y = x := e & x := e \bowtie b \bowtie x := f = x := (e \bowtie b) f
\end{array}
\end{align*}$$

Using these laws we can collapse any sequential and conditional composition of assignments into a single assignment, as shown in \cite{44}. In addition to the deterministic programming operators, we can formulate the following specification operators below.

**Definition 4.12** (Specifications).

$$\prod_{(i) \in A} \bullet P(i) \triangleq \bigvee_{(i) \in A} \bullet P(i) \quad P^o \triangleq P \uplus_p \text{swap}$$

A non-deterministic choice over an indexed relation, $\prod_{(i) \in A} \bullet P(i)$, can simply be expressed using the predicative disjunction operator. We note that, by Theorem 4.2, any set of relations $[S_i, S_j]_{rel}$ forms a complete lattice, where $\prod$ is the least upper bound. As usual, we write $P \cap Q$ for the simpler binary case. The relational converse $P^o$, inverts the direction of a relation by application of $\text{swap}$ as an alphabet extrusion. This swaps the first and second components of the product state space.

We will now explore the algebraic properties that our relational model satisfies, beginning with quantales.

**Theorem 4.12.** For any $S$, the set of homogeneous relations $[S]_{hrel}$ form a unital quantale $([S]_{hrel}, \sqsubseteq, ;, \sqcap)$, that is $([S]_{hrel}, \sqsubseteq)$ is a complete lattice, and in addition the following identities hold:

$$\begin{align*}
(P ; Q) ; R &= P ; (Q ; R) \\
P ; \left(\prod_{(i) \in A} \bullet Q(i)\right) &= \prod_{(i) \in A} \bullet P ; Q(i) \\
(P \cap Q) ; Q &= \prod_{(i) \in A} \bullet P(i) ; Q
\end{align*}$$

A corollary of this result is that UTP relations over $S$ also form Kleene Algebra with $P^* = (\nu X \bullet P ; X \sqcap \sqcap)$. This allows us to link to a large body of well known results in algebraic automated verification \cite{2}. We also show that UTP relations for a Relation Algebra \cite{64}; a key algebraic structure for program specifications.

**Theorem 4.13.** For any $S$, $([S]_{hrel}, \sqcap, \sqcup, \neg, \text{false}, \text{true}, ;, \sqcap, \sqcup, \neg, \text{false}, \text{true})$ forms a Relation Algebra, that is $([S]_{hrel}, \sqcap, \sqcup, \neg, \text{false}, \text{true})$ forms a Boolean algebra, $([S]_{hrel}, ;, \sqcap)$ is a monoid, and the following identities hold:

$$\begin{align*}
P^o &= P \\
(P ; Q)^o &= P^o ; Q^o \\
(P \sqcup Q)^o &= P^o \sqcup Q^o \\
(P \sqcap Q) \sqcup \neg Q &= \neg Q
\end{align*}$$

We have therefore now constructed the relational calculus, and shown that its operators satisfy a large number of important algebraic structures, which validates its correctness. In the next section, we will use these theorems to construct symbolic execution and verification facilities for relational programs.
5. Automated Verification of Relational Programs

In this section we demonstrate how the programming foundations established in §3 and §4 can be applied to automated program analysis. We show how concrete programs can be modelled, symbolically executed, and automatically verified using mechanically validated operational and axiomatic semantics.

5.1. Encoding Programs

Imperative programs can be encoded using the operators given in §4, and a concrete state space with the required variables. For example, we can describe the factorial algorithm as shown below:

**Example 5.1** (Factorial Program). We define a state space, $\text{sfact} \triangleq [x : \mathbb{N}, y : \mathbb{N}]$, consisting of two lenses $x$ and $y$. We can then define a program for computing factorials as follows:

\[
\begin{align*}
\text{pfact} : \mathbb{N} & \rightarrow [\text{sfact}] hrel \\
\text{pfact}(X) & \triangleq (x := X; y := 1; (x > 1) \odot (y := y \ast x; x := x - 1))
\end{align*}
\]

Constant $\text{pfact}$ is a function taking a natural number $X$ as input and producing a program of type $[\text{sfact}] hrel$ that computes the factorial. The given value is assigned to UTP variable $x$, and 1 is assigned to $y$. Then, the program iteratively multiplies $y$ by $x$, and decrements $x$. In the final state, $y$ will have the factorial. 

We use this as a running example for the Isabelle/UTP symbolic execution and verification components.

5.2. Symbolic Execution

The UTP book [43] characterises a small-step operational semantics for the relational imperative program languages using a step relation: $(s, P) \rightarrow (t, Q)$. Here, we adopt a similar idea to describe a reduction semantics for relational programs, and use it to perform symbolic execution. We begin by describing the execution of a program $P$ started in a state context $\Gamma$.

**Definition 5.1** (Program Evaluation). $(\Gamma \models P) \triangleq (\Gamma) ; P$ where $P : [S] hrel$ and $\Gamma : S \rightarrow S$.

The meaning of $\Gamma \models P$ is that program $P$ is executed in the context of variable context $\Gamma$, which gives assignments to the variables of $S$, for example $[x \mapsto 5, y \mapsto \text{true}]$. The expression $\Gamma \models I$ denotes a program that has terminated in state $\Gamma$. The definition of the operator is very simple, because of our encoding of relational program evaluation.

**Theorem 5.1** (Operational Reduction Rules).

\[
\begin{align*}
\Gamma \models (P ; Q) & \rightarrow (\Gamma \models P) ; Q \\
(\Gamma \models I) & ; P \rightarrow \Gamma \models P \\
\Gamma \models (\sigma) & \rightarrow (\sigma \circ \Gamma) \models I \\
(\Gamma \mid b) & \rightarrow \text{true} \\
(\Gamma \mid b \odot Q) & \rightarrow \Gamma \models P \\
(\Gamma \mid b) & \rightarrow \text{false} \\
(\Gamma \mid b \odot Q) & \rightarrow \Gamma \models Q \\
(\Gamma \mid b) & \rightarrow \text{true} \\
(\Gamma \mid b \odot P) & \rightarrow \Gamma \models P ; b \odot P \\
(\Gamma \mid b) & \rightarrow \text{false} \\
(\Gamma \mid b \odot P) & \rightarrow \Gamma \models I
\end{align*}
\]

The arrow $P \rightarrow Q$ actually denotes an equality predicate $(P = Q)$; we use the arrow notation to aid in comprehension, and to emphasise the left-to-right nature. The first two rules on the top line handle sequential composition. The first.pushes an evaluation into the first argument of a composition $P : Q$. The second rule states that when the first argument of a sequential composition has terminated $(I)$, execution moves on to the second argument $(P)$. The third rule handles assignments by creating a new context by precomposing the assignment $\sigma$ with the current context.
The second and third lines deal with conditional and while loop iteration, respectively. In all these rules, the context \( \Gamma \) is applied to the condition \( b \) as a substitution. If the result is \textit{true}, the conditional chooses the first branch \( P \), and the while loop makes a copy of the loop body. If the result is \textit{false}, the conditional chooses the second branch \( Q \), and the while loop terminates.

Using these theorems, we can perform symbolic execution of programs in Isabelle/UTP. To achieve this we load the Isabelle simplifier with both the equations of Theorem 5.1 and also those of Theorem 4.8, the latter of which allows us to apply substitutions. We also utilise the pointwise lifted semantics given in \S4.1 to evaluate expressions using the built-in HOL functional definitions and simplification laws.

In addition, we need the equations of Theorem 4.7 to evaluate and normalise substitutions. However, Law SB4, which allows reordering of substitution maplets, is symmetric and therefore cannot be directly used as it would cause the simplifier to loop. The issue is that ledes do not \textit{apriori} have a total order that can be used to reorder them. Nevertheless, Law SB4 is important to enable a canonical representation of concrete substitutions. Consequently, we extend the simplifier with a “simpoc” [52], a specialised meta-logical simplification procedure that sorts substitution maplets lexicographically using the syntactic lens names. Thus, during symbolic evaluation the variable context will always order the variables lexicographically.

Applying the Isabelle simplifier with these laws, we can symbolically execute programs. Below, we give an example execution of the factorial program from Example 5.1. We do this by using the simplifier to evaluate the term \( id \models pfact(4) \), where \( id \) encodes an arbitrary initial assignment for all the variables.

**Example 5.2 (Factorial Symbolic Execution).**

\[
\begin{align*}
id & \models pfact(4) \\
& = id \models x := 4 ; y := 1 ; (x > 1) \implies (y := y \ast x ; x := x - 1) \\
& = [x \mapsto_4 4] \models y := 1 ; (x > 1) \implies (y := y \ast x ; x := x - 1) \\
& = [x \mapsto_4 4, y \mapsto_1 1] \models (x > 1) \implies (y := y \ast x ; x := x - 1) \\
& = [x \mapsto_4 4, y \mapsto_1 1] \models (y := y \ast x ; x := x - 1) ; (x > 1) \implies (y := y \ast x ; x := x - 1) \\
& = [x \mapsto_4 4, y \mapsto_4 4] \models x := x - 1 ; (x > 1) \implies (y := y \ast x ; x := x - 1) \\
& = [x \mapsto_2 2, y \mapsto_4 12] \models (x > 1) \implies (y := y \ast x ; x := x - 1) \\
& = [x \mapsto_1 1, y \mapsto_4 24] \models (x > 1) \implies (y := y \ast x ; x := x - 1) \\
& = [x \mapsto_1 1, y \mapsto_2 44] \models id \quad \Box
\end{align*}
\]

In this case, the program terminates in final state \( [x \mapsto_1 1, y \mapsto_4 24] \), but in the case of a non-terminating program the simplifier will loop. Such a symbolic execution engine is a useful tool for simulation of programs. In the next section we show how we can also verify programs.

**5.3. Program Verification**

A Hoare triple \( \{ b \} P \{ c \} \) is an assertion that, if program \( P \) is started in a state satisfying predicate \( b \), then all final states satisfy predicate \( c \). The UTP book [43] shows how this assertion can be encoded using a refinement statement. The definition in Isabelle/UTP is given below.

**Definition 5.2 (Hoare Triple).** \( \{ b \} P \{ c \} \triangleq (b^* \implies c^*) \sqsubseteq P \)

This definition states that the Hoare triple is valid when \( P \) is a refinement of the relational specification \( b^* \Rightarrow c^* \). An alternative characterisation is given in terms of our reduction relation is given below.

**Theorem 5.2.** If \( \{ b \} P \{ c \} \) then \( \forall (\Gamma_1, \Gamma_2) \bullet \Gamma_1 \upharpoonright b \land (\Gamma_1 \models P) \rightarrow (\Gamma_2 \models id) \Rightarrow \Gamma_2 \upharpoonright c \)
This is an example of a linking theorem that relates two semantic presentations, in this case how the Hoare triple is related to the operational semantics. The satisfaction of \( \{ b \} P \{ c \} \) implies that, for any initial state assignment \( \Gamma_1 \) satisfying precondition \( b \), if \( P \) terminates with final state assignment \( \Gamma_2 \), then \( \Gamma_2 \) satisfies \( c \). Thus we have formally related the operational and axiomatic semantics for relational programs.

From Definition 5.2 we can also prove, as theorems, the following Hoare calculus laws:

**Theorem 5.3 (Hoare Calculus Laws).**

\[
\begin{align*}
    p & \Rightarrow \sigma \Uparrow q & & \{ p \} Q_1 \{ s \} \{ s \} Q_2 \{ r \} & & \{ b \land p \} S \{ q \} \quad \{ \neg b \land p \} T \{ q \} & & \{ p \land b \} S \{ p \} \\
    \{ p \} \langle \sigma \rangle \{ q \} & & \{ p \} Q_1 ; Q_2 \{ r \} & & \{ p \} S \triangleleft b \triangleright T \{ q \} & & \{ p \} b \oplus P \{ \neg b \land p \}
\end{align*}
\]

The majority of these laws are standard. However, the formulation of the assignment law is more general than the standard law, \( \{ p[e/x] \} x := e \{ p \} \). It avoids the aliasing problem of classical Hoare logic since substitution depends on semantic independence of lenses, rather than syntactic inequality of variable names.

We can use these laws to automatically verify our factorial program. We require, as usual, that the program is annotated with loop invariants. The proof strategy, implemented in the Isabelle/UTP tactic `hoare-auto`, executes the following steps:

1. Combine all composed assignments using Theorem 4.11.
2. Apply the Hoare logic laws of Theorem 5.3 as deduction rules.
3. Perform any substitutions in the resulting predicates using Theorems 4.7 and 4.8.
4. Apply the `rel-auto` tactic of §4.3 to each resulting predicate.

The result is a set of HOL predicates that characterise the verification conditions for the program. We exemplify this with the factorial program below.

**Theorem 5.4.** \( \{ \text{true} \} \text{pfact}(X) \{ y = X! \} \)

**Proof.** Application of `hoare-auto` yields the following verification conditions:

1. \( x > 1 \land y \ast x! = X! \Rightarrow y \ast x \ast (x - 1)! = X! \)
2. \( x \leq 1 \land y \ast x! = X! \Rightarrow x = X! \)

In this case, both proof obligations can be discharged using `sledgehammer`. \[Q.E.D.\]

Thus we have shown how Isabelle/UTP can be applied to practical verification of relational programs. In the next section we will consider verification of languages based on advanced computational paradigms.

## 6. Advanced Verification Tools

In this section we show how advanced verification tools can be constructed using Isabelle/UTP theories. We first describe how UTP theories are characterised in Isabelle/UTP, and illustrate their utility in §6.1. We then describe a small reactive programming language in §6.2, outline a hierarchy of UTP theories for characterising reactive programs in §6.3, and assign our language a denotational semantics and algebraic theorems in §6.4, which we utilise in §7 to provide a denotational semantics and verification tool for RoboChart [48]. Finally in §6.5, we show how proof support for UTP theories can be optimised for efficient automated reasoning. Our mechanisation adds further rigour to these theories, and enables their use in verification.
6.1. Isabelle/UTP Theories

The relational program model we constructed in §4 is powerful, but not without limitations. It is well known, for instance, that basic relations cannot adequately characterise non-terminating behaviour [16]. Furthermore, several programming paradigms, such as real-time, concurrency, and object orientation, require a richer semantic model with more observable information [16, 60, 62].

This semantic enrichment is facilitated by UTP theories. Additional semantic information is expressed by adding special observational variables, which encode quantities of a program or model, and invariants that restrict their domain, called healthiness conditions. The observational variables can be used to define specialised operators for a particular computational paradigm, such as a delay or deadline operator for a real-time language. The healthiness conditions allow us to impose well-formedness invariants over these observational variables, and allow us to prove algebraic theorems that characterise the healthy elements. As usual, healthiness conditions are represented as idempotent predicate transformers, that is, functions on relations over the observational variables.

We characterise UTP theories in Isabelle/UTP as follows:

**Definition 6.1.** An Isabelle/UTP theory is a pair \((\alpha\mathcal{T}, \mathcal{H})\), such that \(\alpha\mathcal{T}\) is an observation space type that is parametric in a vector of type variables \(\alpha\), \(\alpha : [\alpha\mathcal{T}]\mathcal{hrel} \rightarrow [\alpha\mathcal{T}]\mathcal{hrel}\) is a healthiness condition, and \(\mathcal{H}\) is idempotent: \(\mathcal{H} \circ \mathcal{H} = \mathcal{H}\).

We mechanise this algebraic structure using locales [5]. The type \(\alpha\mathcal{T}\) characterises the observations that can be made of the model. Since it can be parametric, it characterises a set of possible ground observation types, which instantiate all the type parameters. An observation space can be constructed using the `alphabet` command (§3.6), in which case it defines a set of lenses, \(\alpha_1 \cdots \alpha_n\), which are the observational variables. The healthiness conditions are encoded as total endofunctions on homogeneous relations parameterised by \(\alpha\mathcal{T}\).

If a theory has multiple healthiness conditions, then these can be composed function-wise. The simplest UTP theory is the relational theory, \(\text{Rel} \triangleq (\alpha, \lambda X \bullet X)\). Any observation type is an instance of \(\alpha\), and any relation is a fixed-point of \(\lambda X \bullet X\).

We say that a relation \(P : [\alpha\mathcal{T}]\mathcal{hrel}\) is \(\mathcal{H}\)-healthy, written \(P\) is \(\mathcal{H}\), when \(P\) is a fixed-point: \(\mathcal{H}(P) = P\). Moreover, we characterise the healthy elements of a theory by the set \([\mathcal{H}]_n \triangleq \{P \mid P\ \text{is}\ \mathcal{H}\}\). Idempotence of \(\mathcal{H}\) ensures that the UTP theory is non-empty since for any relation \(P\), \(\mathcal{H}(P) \in [\mathcal{H}]_n\). To illustrate, we formalise the UTP theory for timed relational programs introduced in §2.1.

**Example 6.1** (Timed Relations). The parametric observation space \([S]\mathcal{rt} \triangleq [\text{clock} : \mathbb{N}, \text{st} : S]\), has lenses \(\text{clock} : \mathbb{N}\), which denotes the passage of time, and \(\text{st} : S\), which denotes the user state. Healthiness function

\[HT(P) \triangleq (\text{clock}^* \leq \text{clock}^* \land P)\]

ensures that time monotonically advances. \(HT\) is clearly idempotent, since conjunction is idempotent. We define the following operator for introducing a delay for timed relations:

\[
\text{wait} : [\mathbb{N}, [S]\mathcal{rt}]}\mathcal{uexpr} \rightarrow [[S]\mathcal{rt}]\mathcal{hrel}
\]

\[
\text{wait}(n) \triangleq (\text{clock}^* = \text{clock}^* + n \land \text{st}^* = \text{st}^*)
\]

Here, \(\text{clock}\) is advanced by \(n\), and the state is unchanged. We can prove that \(\text{wait}(n)\) is \(HT\)-healthy, since it advances time. Conversely, the predicate \(\text{clock}^* = \text{clock}^* - 1\) is not healthy, since it tries to reverse time.

As in previous work [20, 68], our theories form “families”: they characterise relations on several observation spaces the the observation type is potentially polymorphic, and thus extensible with additional variables. For example, our theory in Example 6.1 is parametric in \(S\). We can characterise a subtheory relation between UTP theories:

**Definition 6.2.** \((\beta\mathcal{T}_2, \mathcal{H}_2)\) is a subtheory of \((\alpha\mathcal{T}_1, \mathcal{H}_1)\) when \(\beta\mathcal{T}_2\) specialises \(\alpha\mathcal{T}_1\), and \([\mathcal{H}_2]_n \subseteq [\mathcal{H}_1]_n\).
Subtheories allow us to arrange theories in a hierarchy with descendants that specialise the observation space with additional variables and constraints. UTP theories also include a signature: a set of operators that construct healthy elements of the theory. We say that an operator
\[ F : [[\alpha]\mathcal{T}]\text{hrel}^n \rightarrow [[\alpha]\mathcal{T}]\text{hrel} \]
in \( n \) parameters, is in the signature if \([\mathcal{H}]_n\) is closed under \( F \). Formally, we require that:
\[ \forall P_1, \cdots, P_n \cdot P_1 \text{ is } \mathcal{H} \land \cdots \land P_n \text{ is } \mathcal{H} \implies F(P_1, \cdots, P_n) \text{ is } \mathcal{H} \]

The function \( F \) can either denote a new operator defined on \([\alpha]\mathcal{T} \), or an existing operator defined over a parent observation space. For example, nondeterministic choice \( P \sqcup Q \) and sequential composition \( P ; Q \) often inhabit the signature of several theories, since they are typed by arbitrary relations and so can be instantiated by any observation space. This also means that the corresponding algebraic laws for a parent operator \( F \) can be directly applied to elements of a new subtheory. For example, sequential composition is always associative, since every theory is a subtheory of \( \text{Rel} \).

We exemplify this by giving the signature of our theory of timed relations.

**Theorem 6.1.** \([\mathcal{HT}]_n\) is closed under the following relational operators: \( I ; , \leftarrow b \rightarrow , \) and \( x := v \) when \( x \gg \text{clock} \), and so these are also within the theory signature, as demonstrated by the theorems below:

\[
\begin{array}{ccc}
\text{I is } \mathcal{HT} & P \text{ is } \mathcal{HT} & Q \text{ is } \mathcal{HT} \\
(P \sqcup Q) \text{ is } \mathcal{HT} & (P ; Q) \text{ is } \mathcal{HT} & (x := v) \text{ is } \mathcal{HT} \\
\text{wait}(n) \text{ is } \mathcal{HT} & \text{wait}(m) ; \text{wait}(n) = \text{wait}(m + n) & \end{array}
\]

Once the signature operators have been established, the final step is to prove the characteristic algebraic theorems for them. These laws effectively provide an algebraic semantics for the UTP theory, and can be used to aid in the construction of program verification tools. There are three ways to obtain such theorems in Isabelle/UTP. Firstly, we can inherit them from a parent theory by utilising Definition 6.2. Secondly, we can prove them in Isabelle/UTP. Thirdly, we can import them from an algebraic structure by proving the latter’s axioms. Thirdly, we can prove them manually using Isabelle/UTP’s proof tactics.

We exemplify the third approach for the new delay operator, using our rel-auto proof tactic (§4.3):

**Theorem 6.2 (Delay Laws).** The following laws can be proved by application of rel-auto:

\[
\begin{align*}
\text{wait}(0) & = I \\
\text{wait}(m) ; \text{wait}(n) & = \text{wait}(m + n) \\
\text{wait}(m) ; (P \leftarrow b \rightarrow Q) & = (\text{wait}(m) ; P) \leftarrow b \rightarrow (\text{wait}(m) ; Q) \\
\text{wait}(m) ; x := v & = x := v ; \text{wait}(m) \\
\text{clock} \nmid b &
\end{align*}
\]

Waiting for a zero length duration is simply a skip operation (I), sequential composition of two delays, by \( m \) and \( n \) time units, is equivalent to a single \( m + n \) delay. A delay distributes through a conditional provided that \( b \) does not refer to the clock variable. A delay commutes with an assignment provided that the delay and assigned expressions can be independently evaluated.

Algebraic theorems for UTP theories can also be obtained by linking to a variety of mechanised algebraic structures. The HOL-Algebra library [6], for example, characterises the axioms of partial orders, lattices, complete lattices, Galois connections, and the myriad of theorems that can be derived from them. A substantial advantage of the UTP approach is that algebraic theorems can often be reduced to proving properties of the underlying healthiness conditions, which allows us to obtain laws with minimal effort. We exemplify this by restricting ourselves to a particular subclass of continuous UTP theories.

**Definition 6.3.** \( \mathcal{H} \) is continuous provided that \( \mathcal{H}(\prod_{i \in A} P(i)) = \prod_{i \in A} \mathcal{H}(P(i)) \) whenever \( A \neq \emptyset \).
In a continuous theory, the healthiness condition distributes through arbitrary non-empty relational infima. A corollary of this definition is that $\mathcal{H}$ is also monotonic: $P \subseteq Q \Rightarrow \mathcal{H}(P) \subseteq \mathcal{H}(Q)$. Then, by the Knaster-Tarski theorem, also part of HOL-Algebra, we can show that $\llbracket \mathcal{H} \rrbracket_n$ forms a complete lattice under refinement $\sqsubseteq$. This allows us to import theorems for recursive and iterative programs into a UTP theory.

From the induced complete lattice, we obtain the following operators: infimum $\prod_\tau$, supremum $\bigvee_\tau$, top element $\top_\tau$, bottom element $\bot_\tau$, and least fixed-point $\mu_\tau$, which are all in the theory’s signature, and for which the usual complete lattice and fixed-point calculus theorems [4] hold. Moreover, in a continuous UTP theory, these operators can be calculated using the equational theorems given below.

**Theorem 6.3 (Continuous Theory Properties).**

\[
\begin{align*}
\top_\tau &= \mathcal{H}(\text{false}) \\
\bot_\tau &= \mathcal{H}(\text{true}) \\
\prod_\tau A &= (\top_\tau \land A = \emptyset \lor \prod A) & A \subseteq \llbracket \mathcal{H} \rrbracket_n \\
\mu_\tau F &= \mu X \bullet F(\mathcal{H}(X)) & F : [\mathcal{H}]_n \rightarrow [\mathcal{H}]_n \text{ and $F$ is monotonic}
\end{align*}
\]

These theorems demonstrate the relationship between the theory operators, and the relational ones defined in §4. The theory top and bottom elements are the relational top and bottom with $\mathcal{H}$ applied. The theory infimum $\prod_\tau A$ is $\top_\tau$ when $A$ is empty, and otherwise the relational infimum. Moreover, the least fixed-point operator can be expressed as the relational least fixed-point by precomposition of $\mathcal{H}$. The utility of these theorems is that we can construct non-deterministic choices and recursive programs using the relational operators $\land$ and $\mu$, which again allows reuse of their algebraic laws.

We now demonstrate how we can obtain such theorems for timed relations.

**Theorem 6.4.** $\text{HT}$ is continuous, since conjunction distributes through disjunction ($\prod$). We therefore obtain a complete lattice, and can calculate the top and bottom element:

\[
\top_{\text{rt}} = \text{HT}(\text{false}) = \text{false} \quad \bot_{\text{rt}} = \text{HT}(\text{true}) = (\text{clock} \leq \text{clock'})
\]

We also obtain a least fixed-point operator for constructing and reasoning about iteration, $\mu_{\text{rt}}$, and can use it to denote a timed Hoare calculus and proof tactics following the template given in §5. A concrete timed program can be modelled by creating a suitable state space, $\text{vars} \triangleq [x_1 : \tau_1 \cdots x_n : \tau_n]$, defining the program $P : [\llbracket \text{vars} \rrbracket_{\text{rt}} \llbracket \text{hrel} \rrbracket]$ using the signature operators, and proving that $P$ is $\text{HT}$. Finally, the proven Hoare logic deduction rules can be used for program verification.

We have demonstrated how a UTP theory can be mechanically validated, a set of signature operators verified, and algebraic theorems for these operators proved. Next, we apply this approach to a more substantial hierarchy of UTP theories for reactive programs.

### 6.2. Reactive Programs

The theory of reactive designs [43, 16] provides a generic semantic foundation for reactive programs. It demonstrates how UTP can be applied to integration of formal methods by semantic unification. At its core, the theory combines two computational paradigms: state- and event-based modelling. It grew out of a twenty year endeavour [67, 34, 59, 43, 54] to obtain a theory that can unify state-based notations, like Z, B, and refinement calculus, with event-based notations, like process calculi CCS, CSP, and $\pi$-calculus.

Reactive designs have notably been used to give a denotational semantics to the Circus language family [54], that integrates operators from Z, Dijkstra’s guarded command language [17], and CSP [41]. Extensions of Circus exist that utilise the theory unification facilities of UTP to characterise further paradigms like real-time [62, 65] and object-orientation [60]. In our more recent work, we have further generalised the theory of reactive designs with an abstract trace model [27, 26] that unifies all the aforementioned extensions, and also hybrid dynamical systems with continuous variables.
In this work, we use reactive designs to mechanise a denotational semantics of, and verification support for, a reactive action language, which is summarised below. We use this language in §7 to, in turn, give a denotational semantics to basic RoboChart state machines. We assume existing syntactic categories for identifiers (\texttt{ID}) and expressions (\texttt{Expr}), both of which can be provided by Isabelle/UTP.

**Definition 6.4** (Reactive Action Language).

\[
\text{Action} ::= \text{Event} \mid \text{Skip} \mid \text{Stop} \mid \text{ID} ::= \text{Expr} \mid \text{Action} \mid \text{Expr} \& \text{Action} \mid \text{Action} \sqcap \text{Action} \mid \text{do} \{\text{Expr} \rightarrow \text{Action}\} \text{ od}
\]

\[
\text{Event} ::= \text{ID} \mid \text{ID} \, ? \, \text{ID} \mid \text{ID} \, ! \, \text{Expr}
\]

In this definition, \{P\} denotes replication of \(P\) zero or more times. The language combines communication and sequential programming constructs. The basic constructs include visible events, skip, deadlock (\texttt{Stop}), assignment, and sequential composition. Events are used to communicate with the environment. An event can either be a basic synchronisation (\(a\)), an input event (\(a\,?\,x\)) which receives a value over channel \(a\) and populates variable \(x\) with it, or an output event (\(a\,!\,v\)) that sends a value over \(a\). A guarded action \(b \& P\), behaves like \(P\) if predicate \(b\) is true, and otherwise deadlocks. An external choice \(P \sqcap Q\) allows the environment to pick whether \(P\) or \(Q\) is activated, depending on their initial events offered. Finally, an action system, \texttt{do} \(b_1 \rightarrow P_1 \mid \cdots \mid b_n \rightarrow P_n \text{ od}\), iteratively executes one of the \(P_i\) actions, based on those that have a true guard. If no guard is true, the action system terminates. The iteration operator is important as it allows us to encode the transition relation in a RoboChart state machine in §7. We give an informal example of the use of this language in describing a buffer.

**Example 6.2.** \(bf := \langle \rangle\); \texttt{do true \rightarrow (inp\,?\,x \; ; \; bf := bf \mid \langle x \rangle \mid \#bf > 0 \; & \; out!(\text{head}(buf)) \; ; \; bf := tail(bf)) \text{ od}\)

The variable \(bf\), which stores the buffer contents, is initially set to the empty sequence (\(\langle \rangle\)). The buffer then repeatedly offers to input a value over the channel \(inp\), and extend \(bf\) with it, or else when the buffer is not empty (\(#bf > 0\)), output the first element of the buffer of the output channel \(out\). The \texttt{true} guard in the iteration means that this action never terminates, which is usual for reactive programs.

**6.3. Reactive Designs Theory Hierarchy**

We give an overview of the mechanisation of the reactive design hierarchy, which consists of several UTP theories, namely designs [43, 15], reactive processes [43, 27], stateful reactive designs [26], and stateful failure reactive designs [54, 29], which extends the predicative semantics of CSP with state. The observation spaces and healthiness conditions of these theories are summarised in Table 1. For reasons of space, we do not elaborate the details of the healthiness conditions, which have been thoroughly explained and presented elsewhere [43, 15, 26, 29], but concentrate on their mechanisation in Isabelle/UTP. The following list gives the intuition of the conceptual layers, which is illustrated in Figure 6.

1. **Designs** extend the relational program model with the possibility of erroneous or “divergent” behaviour. If a program diverges, it becomes unpredictable, which corresponds to unspecified behaviour.
UTP Theory | Observation Space \((T)\) | Healthiness Function \((H)\)  
|---|---|---| 
Designs \(\text{des} \triangleq [ok : \mathbb{B}]\) | \(R\) | 
Reactive Processes \(\lbrack T \rbrack rp \triangleq \text{des} + \lbrack \text{wait} : \mathbb{B}, \text{tr} : T \rbrack\) | \(H\) | 
Stateful Reactive Designs \(\lbrack S, T \rbrack \text{rsp} \triangleq \lbrack T \rbrack rp + \lbrack \text{st} : S \rbrack\) | \(\text{NSRD}\) | 
St. Failure Reactive Designs \(\lbrack S, E \rbrack \text{sfzd} \triangleq \lbrack S, \lbrack E \rbrack \text{list} \rbrack \text{rsp} + \lbrack \text{ref} : \lbrack P(E) \rbrack\) | \(\text{NCSP}\) | 

\[H1(P) \triangleq (ok^* \Rightarrow P)\] 
\[R \triangleq R2 \circ R1\] 
\[SRD1(P) \triangleq R1 \circ H1\] 
\[SRD3(P) \triangleq (P : \Pi_n)\] 
\[\text{NSRD} \triangleq \text{SRD3} \circ \text{SRD1} \circ R3_h \circ R\] 
\[CSP3(P) \triangleq (\text{Skip} : P)\] 
\[CSP4(P) \triangleq (P : \text{Skip})\] 
\[\text{Skip} \triangleq \text{NSRD}(\Pi_n)\] 
\[\text{NCSP} \triangleq \text{NSRD} \circ \text{CSP3} \circ \text{CSP4}\] 

Table 1: Reactive Designs Theories Overview

2. **Reactive Processes** introduce (1) a trace variable to record the interaction history; and (2) quiescent behaviour, where the program is awaiting interaction from the environment. Quiescence can occur when waiting for an external input, such as a message.

3. **Stateful Reactive Designs** introduce an explicit program state variable \((st)\), require that this is cannot be observed in quiescent observations, and combine designs and reactive processes to represent programs that can both quiesce and diverge. As illustrated by the red arrows in Figure 6, it is constructed by effectively embedding designs into reactive processes, which theories are otherwise disjoint.

4. **Stateful Failure Reactive Designs** additionally encode a CSP-style refusal set in every quiescent observation, which is intuitively used to indicate the events that are enabled.

The layering of UTP theories allows the reuse of laws that are proven at each level of specialisation. Here, we focus on the specialisation to stateful failure reactive designs, but reactive designs can also be specialised with real-time, probabilistic, or hybrid dynamical behaviour. There are two kinds of specialisation that can be made. The first is extension of the observation space with extra variables, which resembles inheritance, and the second is the subtheory relation of Definition 6.2. Both kinds of extension are utilised in summary Table 1. We now summarise how the layers of theory are built up in Isabelle/UTP.

The theory of designs [43] has observation space type \(\text{des}\), and healthiness condition \(H\), itself composed of \(H1\) and \(H2\) [43]. The type of \(H\) is \([\phi]\text{des-ext} \rightarrow [\phi]\text{des-ext}\), where \([\phi]\text{des-ext}\) is the polymorphic extension type (cf. §3.6). The \(\phi\) parameter means that it is applicable to any extension of observation space \(\text{des}\). The lens \(ok : \mathbb{B} \Rightarrow [\phi]\text{des-ext}\) is used as a flag that allows us to observe whether a program, or its predecessor, is proceeding normally or has diverged. Healthiness condition \(H1\) ensures that no program variables can be observed until the program has been started \((ok^*)\). \(H2\) ensures that no program can require divergence. The signature constructor for designs is \(P \vdash Q \triangleq (ok^* \land P) \Rightarrow (ok^* \land Q)\), which constructs a program with a precondition \((P)\) and a postcondition \((Q)\). The postcondition \(Q\) characterises successful and terminating observations, whilst \(P\) denotes the conditions of valid execution: if \(P\) is violated, then the design diverges.

The reactive processes [27] observation space \([T]rp\), extends \(\text{des}\) with observational variables \(\text{wait}\) and \(\text{tr}\). These are used to observe, respectively, whether a program is quiescent, that is, waiting for interaction from the environment, and the current trace history. Variable \(ok\) is inherited, and its type specialised to \(\mathbb{B} \Rightarrow \lbrack T \rbrack rp\). The type parameter \(T\), from which \(tr\) draws its type, denotes a particular trace model [29], such as a sequence of events or a piece-wise continuous function. In Isabelle/HOL, we enforce a polymorphic sort constraint on \(T\) which ensures that the core operators of trace algebra [27] are defined on \(T\), and the associated axioms proved. The theory of reactive processes does not inherit the healthiness conditions of
designs, but defines its own: $R1$ and $R2^5$. $R1$ requires that the trace variable is monotonically extended, so no event can be “undone”, and $R2$ prevents a program’s behaviour from depending on events that occurred before it began executing. Intuitively, these two healthiness conditions give rise to a special variable $tt$, which is equivalent to $tr^r - tr$, and is used to encode the traces contributed by a process [26].

Stateful reactive designs is a subtheory of reactive processes that adds a state lens $st : S \Rightarrow [S,T] rsp$. This allow characterisation of the program variables with lenses of type $\tau \Rightarrow S$. The healthiness condition is $NSRD$, which extends $R$ with $R3_b$, $SRD1$, and $SRD3^6$. $R3_b$ ensures that $st$ is unobservable in quiescent phases: if a predecessor to $P$ is quiescent (wait*), then the program behaves like $I_n$, the reactive identity, or else it behaves like $P$. The theory also includes reactive versions of the design healthiness conditions, and effectively embeds designs into reactive processes. Specifically, $SRD1$ composes $H1$ with $R1$, so that divergent behaviour allows arbitrary extension of the trace. $SRD3$ requires the the reactive design skip $I_n$ is a right unit, and is analogous to $H2$.

Embedding designs into reactive designs allows the specification of programs that have both divergent (erroneous) observations, which corresponds to violation of the precondition, and quiescent observations. The main specification operator in the reactive designs signature is $[P_1 \parallel P_2 | P_3]$, which constructs a reactive contract specification like a design, with a precondition $P_1$ and postcondition $P_3$, but also an additional part ($P_2$), called a “pericondition” to characterise quiescent observations [26]. Specifically, any $NSRD$-healthy relation corresponds to a reactive contract triple. Moreover, our previous results [26, 29] show that reactive programs can usually be reduced to reactive contracts of this from that specify the program’s behaviour using three large predicates. If this is the case then we can verify whether an implementation contract $[Q_1 \parallel Q_2 | Q_3]$ satisfies an abstract specification using the following theorem [26]:

**Theorem 6.5.** $[P_1 \parallel P_2 | P_3] \sqsubseteq [Q_1 \parallel Q_2 | Q_3]$ provided the precondition is weakened, $P_1 \Rightarrow Q_1$, the pericondition is strengthened, $Q_2 \land P_1 \Rightarrow P_2$, and the postcondition is strengthened, $Q_3 \land P_1 \Rightarrow P_3$.

This kind of contract-based verification is implemented in an Isabelle/UTP tactic called $rdes$-$refine$ [26], which invokes the rel-auto tactic described in §4.3, and is analogous to our Hoare logic tactic $hoare$-$auto$ from §5 but with a much enriched notion of specification. In this work, we do not consider reactive contracts further, but rather point the reader to companion publications [26, 29]. The crucial observation is that these contracts facilitate an axiomatic semantics, based on refinement, that can be used for verification.

The reactive designs healthiness condition $NSRD$ is continuous, and therefore we can utilise Theorem 6.3 to obtain a complete lattice, including a fixed-point operator $\mu_s$, and can use $\sqcap$ to construct non-deterministic choices. We call the top element of the lattice $Miracle$, since it denotes an unimplementable reactive program, and the bottom element $Chaos$, the most non-deterministic reactive program.

**Definition 6.5.** $Miracle \triangleq NSRD(false)$, $Chaos \triangleq NSRD(true)$

The $Miracle$ and $Chaos$ elements are defined, following Theorem 6.3, by application of $NSRD$ to false and true, respectively. We redefine the conditional operator, $P \Leftrightarrow b \triangleright Q$, using alphabetic extrusion by $st$ to ensure that $b$ may only refer to state variables, and not other variables like ok and tr, which could render the operator unhealthy. As defined, $P \Leftrightarrow b \triangleright Q$ is $NSRD$-healthy, provided that $P$ and $Q$ both are. The operator is defined in terms of the standard conditional operator from Definition 4.11, which, in particular, means that the relevant laws from Theorem 4.9 are applicable in this setting. Moreover, with the help of Theorem 6.3, we can obtain additional laws:

**Theorem 6.6.** If P is $NSRD$-healthy, then the following laws hold:

| $Miracle$ : $P = Miracle$ | $Miracle \sqsubseteq P = P$ | $P \sqsubseteq Miracle$ |
|--------------------------|-----------------|------------------|
| $Chaos$ : $P = Chaos$   | $Chaos \sqsubseteq P = Chaos$ | $Chaos \sqsubseteq P$ |

---

5Here, we omit the usually included $R3$, which is added in the reactive design theory.

6We omit $SRD2$, defined in [26], because it is not relevant for our theory and is subsumed by $SRD3$. 

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Finally, the theory of stateful failure reactive designs adds the observational variable ref to encode refusals, and two additional healthiness conditions to govern them, which are composed in NCSP. These healthiness conditions essentially ensure that ref$^*$ is only referenced in quiescent observations, and that ref is never mentioned, since a process should never depend on a predecessor’s refusal set. Since NCSP specialises NSRD, any general law of reactive designs, such as those in Theorem 6.6, are inherited. We also note that Miracle, Chaos, and $\prec b \triangleright$, remain in the signature of NCSP, and therefore all their laws are applicable in this specialised theory. We can now use this theory to denote the reactive language shown in Definition 6.4.

6.4. Denotational Semantics of Reactive Programs

In this section we will show how the mechanisation of stateful failure reactive designs can be applied to give a denotational semantics to a reactive programming language. We will define both the CSP-style communication operators, and imperative operators similar to those found in Dijkstra’s guarded command language [17]. We also prove crucial algebraic theorems, which can be used in program verification. Notably, we define the guarded iteration operator, and an induction law that can be used to support verification using refinement. We first define the following basic operators for reactive programs.

**Definition 6.6 (Reactive Programming Operators).**

\[
\langle \sigma \rangle_e \triangleq \text{NCSP}(\langle \sigma \oplus_s s \rangle)
\]

\[
\text{do}(e) \triangleq \text{NCSP} \left( \langle \mathit{ok}^* \land \mathit{tr}^* = \mathit{tr}^* \land \langle e^* \oplus_s s \rangle \notin \text{ref}^* \lor \mathit{wait}^* \land \mathit{tr}^* = \mathit{tr}^* \land \langle e^* \oplus_s s \rangle \land \mathit{st}^* = \mathit{st}^* \rangle \right)
\]

\[
b \rightarrow P \triangleq P \prec b \triangleright \text{Miracle}
\]

\[
[b] \triangleq b \rightarrow \text{Skip}
\]

\[
b \otimes P \triangleq (\mu_r X \cdot P : X \prec b \triangleright \text{Skip})
\]

where $\sigma : S \rightarrow S$, $e : [E, S]uexpr$, $b : [S]upred$, and $P : [[S, E]sfrd]hrel$.

As for the relational model, we define a generalised assignment operator $\langle \sigma \rangle_e$, which is parameterised by a substitution $\sigma$ on $S$. The definition lifts $\sigma$ into the reactive design observation space $[S, E]sfrd$ using substitution extension (see Definition 4.9). It then constructs a relational assignment, and applies NCSP to produce a reactive design. As before, we can construct a singleton assignment $x := v$ by lifting a singleton substitution: $\langle x \mapsto v \rangle_e$. Moreover the lens results in §3, including the variable models, remain valid here. For example, we can use constructs like object structures and arrays (§4.6) in this context.

The $\text{do}(e)$ operator describes an event, represented by an expression $e : [E, S]uexpr$, based on the current state $s : S$. It generalises several event operators, including CSP event prefix, which can be defined as so: $e \rightarrow P \triangleq \text{do}(e) : P$. Its definition states that the operator does not diverge ($\mathit{ok}^*$), and conjoints this with a predicate that describes the quiescent and terminated observations. In quiescent observations ($\mathit{wait}^*$), the trace remains unaltered and the event $e$ is not being refused. In terminated observations, ($\mathit{wait}^*$), the trace is finally extended by $e$, and the state remains unchanged [16].

The $b \rightarrow P$ operator denotes a “naked” guarded command [51]. Its behaviour is $P$ when $b$ is true, and miraculous otherwise, meaning it is impossible to execute. By Theorem 6.6, Miracle is a left annihilator for sequential composition, and so any behaviour following is excluded when $b$ is false. An assumption $[b]$ guards Skip with $b$, and thus holds all variables constant when $b$ is true, and is otherwise miraculous. We also define the reactive while loop $b \otimes P$, which is analogous to the relational version (Definition 4.11). $[\text{NCSP}]_{\prec}$ is closed under these three operators since they are defined only in terms of healthy elements $\prec \cdot \triangleright$, Miracle, Skip, and $\mu_r$. We can also prove a set of theorems for these operators [29].

**Theorem 6.7.** If $P$ and $Q$ are NCSP-healthy, then the following identities hold:
Definition 6.7 (Alternation). If \( i \in I \) then \( \left( b(i) \rightarrow P(i) \right) \) if \( \left( \bigcap_{i \in I} b(i) \rightarrow P(i) \right) \cap \left( \bigcap_{i \in I} \neg b(i) \right) \rightarrow Chaos \)

For brevity, we have omitted the definition of external choice which can be found elsewhere [29]. The first four of these identities are direct counterparts of relational laws in Theorem 4.9. Moreover, since these laws reduce assignment compositions to substitutions, we can also utilise Theorems 4.7 and 4.8 in this context. This, in particular, means we also obtain counterparts to all the assignment laws of Theorem 4.11.

Next, we add Dijkstra’s alternation operator [17] to our model.

**Theorem 6.8.** If, \( \forall i \in I \cdot P(i) \) is NCSP, then the following identities hold:

\[
\begin{align*}
\text{if } i \in I \cdot b(i) & \rightarrow P(i) \ 	ext{fi} = Chaos \\
\text{if } i \in I \cdot b(i) & \rightarrow P(i) \ 	ext{fi} = P(k) \triangleleft b(k) \triangleright Chaos \\
\bigvee_{i \in I} b(i) & ; \text{if } i \in I \cdot b(i) \rightarrow P(i) \ 	ext{fi} = \left( \bigcap_{i \in I} b(i) \rightarrow P(i) \right)
\end{align*}
\]

In words, (1) shows that alternation over an empty set presents no options, and so is equivalent to Chaos; (2) shows that a singleton alternation can be rewritten as a binary conditional; (3) shows that, if we assume that one of its branches is true, then an alternation degenerates to a nondeterministic choice.

We now define the guarded iteration operator as the iteration of the corresponding alternation whilst at least one of the guards remains true.

**Definition 6.8 (Iteration).** \( \text{do } i \in I \cdot b(i) \rightarrow P(i) \ 	ext{od} \triangleq \left( \bigvee_{i \in I} b(i) \right) \odot, \left( \text{if } i \in I \cdot b(i) \rightarrow P(i) \ 	ext{fi} \right) \)

We use the reactive while loop \( (b \odot P) \) to encode the operator, and can thus utilise our previous results [29] to reason about it. In keeping with the reactive programming paradigm, this while loop can pause during execution to await interaction, and it also need not terminate. However, in order to ensure that the underlying fixed point can be calculated, we assume that for all \( i \in I \), \( P(i) \) is productive [27]: that is, it produces at least one event whenever it terminates. This ensures that divergence caused by an infinite loop is avoided. Iteration is closed under \( \left[ \text{NCSP} \right]_n \), since the while loop and alternation both are. We summarise the healthy reactive programming operators in the closure theorem below.

**Theorem 6.9 (Stateful Failure Reactive Designs Signature).**

\[
\begin{array}{cccc}
\text{Miracle} & \text{Chaos} & \text{Skip} & \text{do(e)} \\
\text{is NCSP} & \text{is NCSP} & \text{is NCSP} & \text{is NCSP} \\
\begin{array}{c}
P \text{ is NCSP} \\
Q \text{ is NCSP} \\
(P : Q) \text{ is NCSP} \\
(P \triangleleft b \triangleright Q) \text{ is NCSP} \\
(P \triangleleft Q) \text{ is NCSP}
\end{array} & \\
\begin{array}{c}
\forall i \in I \cdot P(i) \text{ is NCSP} \\
I \neq \emptyset \\
\left( \bigcap_{i \in I} P(i) \right) \text{ is NCSP}
\end{array} & \\
\begin{array}{c}
\forall i \in I \cdot P(i) \text{ is NCSP} \\
\text{if } i \in I \cdot b(i) \rightarrow P(i) \ 	ext{fi} \text{ is NCSP}
\end{array}
\end{array}
\]
To complete our theory, and enable verification support for iterated reactive programs, we prove the following fundamental refinement law for iteration.

**Theorem 6.10 (Iteration Induction).** If, $\forall i \cdot P(i)$ is NCSP, then:

$$\forall i \in A \cdot P(i) \text{ is Productive} \quad S \subseteq I ; \left[ \bigwedge_{i \in A} \neg b(i) \right]$$

$$\forall i \in A \cdot S \subseteq I ; b(i) \cdot P(i) \quad \forall i \in A \cdot S \subseteq S ; b(i) \cdot P(i)$$

$$S \subseteq I ; \begin{cases} \text{do } i \in A \cdot b(i) \rightarrow P(i) \text{ od} \end{cases}$$

The law states the provisos under which an iteration, with initialiser $I$, preserves invariant $S$. These are: (1) every branch is productive; (2) if $I$ causes the iteration to exit immediately then $S$ is satisfied; (3) for any $i \in A$ if $I$ holds initially, $b(i)$ is true, and $P(i)$ executes, then $S$ is satisfied (base case); and (4) for any $i \in A$, if $S$ holds initially, $b(i)$ is true, and $P(i)$ executes, then $S$ is satisfied (inductive case). This law forms the basis for our verification strategy for state machines that we utilise in §7.

### 6.5. Proof Optimisation

We complete the discussion of UTP theories by showing how, once constructed, they can be optimised for use in automated verification tools. The approach to theory engineering we have described above has several advantages [43, 16]. There is a unified notion of refinement that can be applied across semantic domains. Operators like nondeterministic choice (?) and sequential composition (;) can occupy several theories, which facilitates generality and semantic integration. General algebraic laws can be proved, and then directly reused in more specialised UTP theories, for example reactive process and reactive design laws can be specialised to stateful failure reactive designs. The UTP approach also means that theories can be both combined and extended for a wide variety of computational paradigms and languages.

However, there is a practical downside, which is that the programming theorems, such as those in Theorem 6.7 and 6.6, depend on our showing healthiness of the constituent parameters, and therefore it is necessary to first invoke a set of closure theorems. In the context of verification, constantly proving closure can be very inefficient, particularly for larger programs. This is because Isabelle’s simplifier works best when invoked with pure equations $f(x_1, \ldots, x_n) \equiv y$ with minimal provisos.

Our solution uses the Isabelle type system to shoulder the burden of closure proof. We use Isabelle’s **typedef** mechanism, which creates a new type $T$ from a non-empty subset $A :: \mathbb{P}(U)$ of existing type $U$. For a UTP theory, we create a type with $A = [\mathcal{H}]_\mathfrak{s}$, which is a subset of the UTP relations. This then allows optimised proof for a particular UTP theory, but at the cost of generality and semantic extensibility which are more suited to the UTP relational domain.

In order to obtain the signature for the new type, we utilise the lifting package [45], whose objective is to define operators on $T$ in terms of operators on $U$, provided that $A$ is closed under each operator. Specifically, if $f$ is a signature operator in $k$ arguments, then we can create a lifted operator $\hat{f} :: T^k \rightarrow T$ using Isabelle’s **lift-definition** command [45]. This raises a proof obligation that $f \in [\mathcal{H}]_\mathfrak{s} \rightarrow [\mathcal{H}]_\mathfrak{s}$, which can be discharged by the corresponding closure theorem. Programs constructed from the lifted operators are well-formed by construction.

Finally, to lift the algebraic theorems for each lifted operator $\hat{f}$, we use the **transfer** tactic [45]. It allows us to prove theorems like $\hat{f}(P_1, \ldots, P_k) = \hat{g}(P_1, \ldots, P_k)$, where $P_i :: T$ is a free variable, by converting it to a theorem of the form $Q_i$ is $\mathcal{H} \wedge \cdots \wedge Q_k$ is $\mathcal{H} \Rightarrow f(Q_1, \ldots, Q_k) = g(Q_1, \ldots, Q_k)$. This means the closure properties of each parameter $Q_i$ can be utilised in discharging provisos of the corresponding UTP theorems, but the lifted theorems do not require them. We use this technique for our reactive program type.

Using Isabelle’s **typedef** mechanism, we define a new type $[S, E]_{\text{action}}$ to encode reactive programs, which is isomorphic to $[\text{NCSP}]_{\mathfrak{s}}$, and a subset of the type $[S, E]_{\text{sfrd}}$. We then use the lifting package to create constructors corresponding to each of the signature operators in Theorem 6.9. This means that the type system of Isabelle ensures that reactive programs constructed using these operators are healthy by construction. The final step is to transfer each of the laws given in Theorems 6.7, 6.6, 6.8, and 6.10 to the new $[S, E]_{\text{action}}$ domain. This result is a set of algebraic theorems without closure provisos, which can then be applied directly as part of our verification procedure.
In this section we have demonstrated the mechanisation of UTP, demonstrated their use in constructing a reactive program theory hierarchy, and showed how the resulting laws can be optimised for verification. In the next section we finally apply this to verification support for RoboChart state machines.

7. Automated Verification of State Machines

In this section we draw together the collected results from sections 3, 4, and 6 to construct an automated verification tool for the graphical state machine notation, RoboChart, in Isabelle/UTP. We begin by describing the RoboChart language in more detail, and then proceed to construct the tool in Isabelle.

7.1. RoboChart

RoboChart [48, 47] describes robotic systems in terms of a number of controllers that communicate using shared channels. Each controller has a well defined interface, and its behaviour is described by one or more state machines. A machine has local state variables and constants, and consists of nodes and transitions, with behaviour specified using a formal action language [55]. Advanced features such as hierarchy, shared variables, real-time constraints, and probability are supported.

A machine, GasAnalysis, is shown in Figure 7; we use it as a running example. It models a component of a chemical detector robot [40] that searches for dangerous chemicals using its spectrometer device, and drops flags at such locations. GasAnalysis is the component that decides how to respond to a sensor reading. If gas is detected, then an analysis is performed to see whether the gas is above or below a given threshold. If it is below, then the robot attempts to triangulate a position for the source location and turns toward it, and if it is above, it stops.

The interface consists of four events. The event gas is used to receive sensor readings, and turn is used to communicate a change of direction. The remaining events, resume and stop carry no data, and are used to communicate that the robot should resume its searching activities, or stop. The state machine uses four state variables: sts to store the gas analysis status, gs to store the present reading, ins to store the reading intensity, and anl to store the angle the robot is pointing. It also has a constant thr for the gas intensity threshold. RoboChart provides basic types for variables and constants, including integers, real numbers, sets, and sequences (Seq(t)). The user can also define additional types, that can be records, enumerations, or entirely abstract. For example, the type Status is an enumerated type with constructors gasD and noGas.
The behaviour is described by 6 nodes, including an initial node \((i)\); a final node \((F)\); and four states: \(\text{NoGas}\), \(\text{Analysis}\), \(\text{GasDetected}\), and \(\text{Reading}\). The transitions are decorated with expressions of the form \(\text{trigger}\{\text{condition}\}\)/\(\text{statement}\). When the event \(\text{trigger}\) happens and the guard \(\text{condition}\) is true, then \(\text{statement}\) is executed, before transitioning to the next state. All three parts can optionally be omitted. RoboChart also permits states to have entry, during, and exit actions. In our example, both \(\text{Analysis}\) and \(\text{GasDetected}\) have entry actions.

Modelling with RoboChart is supported by the Eclipse-based RoboTool\(^7\), from which Figure 7 has been captured. RoboTool automates verification via model checking using FDR4, and its extension to incorporate the verification approach presented here is ongoing work.

7.2. Static Semantics

In this section we formalise the static semantics of state machines in Isabelle/HOL, which describes the variables, transitions, and nodes. The meta-model, presented below, is based on the untimed subset of RoboChart, but note that our use of UTP ensures that our work is extensible to more advanced semantic domains \([62, 9, 27]\). For now we omit constructs concerned with interfaces, operations, shared variables, during actions, and hierarchy, and focus on basic machines.

Definition 7.1 (State Machine Meta-Model).

\[
\text{StMach ::= statemachine } ID \nameDecl{\ast} \\text{events } NameDecl{\ast} \\text{states } NodeDecl{\ast} \\
\text{initial } ID \\text{finals } ID{\ast} \\text{transitions } TransDecl{\ast}
\]

\[
\text{NameDecl ::= ID } [\text{: Type}]
\]

\[
\text{NodeDecl ::= ID } \text{entry } Action \text{ exit } Action
\]

\[
\text{TransDecl ::= ID } \text{from } ID \text{ to } ID \\text{ trigger } Event \\text{ condition } \text{Expr} \text{ action } \text{Action}
\]

A state machine is composed of an identifier, variable declarations, event declarations, state declarations, an initial state identifier, final state identifiers, and transition declarations. Each variable and event consists of a name and a type. A state declaration consists of an identifier, entry action, and exit action. A transition declaration consists of an identifier, two state identifiers for the source and target nodes, a trigger, a condition, and a body action. Transitions with no trigger are denoted by null event \((\epsilon)\) which can be hidden later.

We implement the meta-model syntax using Isabelle’s parser, and implement record types \([s, e]Node\) and \([s, e]Transition\), that correspond to the \(\text{NameDecl}\) and \(\text{TransDecl}\) syntactic categories. They are both parametric over the state-space \(s\) and event types \(e\). \(\text{Node}\) has fields \(\text{name : string, nentry : [s, e]Action, and nexit : [s, e]action}\), that contain the name, entry action, and exit action. \(\text{Transition}\) has fields \(\text{src : string, tgt : string, trig : [s, e]action, cond : [s]upred, and act : [s, e]action}\), that contain the source and target, the trigger, the condition, and the body. Triggers are denoted by actions because they may correspond to multiple possible events. We create a record type to represent a state machine.

Definition 7.2 (State Machine Record Type).

\[
\text{record } [s, e]\text{StMach } = \begin{array}{ll}
\text{init : ID} & \text{finals : [ID]list} \\
\text{nodes : [][s, e]Node]list} & \text{transss : [][s, e]Transition]list}
\end{array}
\]

It declares four fields for the initial state identifier \((\text{init})\), final states identifiers \((\text{finals})\), nodes definitions \((\text{nodes})\), and transition definitions \((\text{transss})\), and constitutes the static semantics. Since this corresponds to the meta-model, and to ensure a direct correspondence with the parser, we do not directly use sets and maps, but only lists in our structure. We will later derive views onto the data structure above, that build on well-formedness constraints.

Below, we show how syntactic machines are translated to Isabelle definitions.

\(^7\)https://www.cs.york.ac.uk/circus/RoboCalc/robotool/
Definition 7.3 (Static Semantics Translation).

\[ \text{statemachine } s \]
\[ \text{vars } x_1 : \tau_1^n \cdots x_i : \tau_i^n \]
\[ \text{events } e_1 : \tau_1^e \cdots e_j : \tau_j^e \]
\[ \text{states } s_1 \cdots s_k \text{ initial } \text{init} \]
\[ \text{transitions } t_1 \cdots t_n \]

\[ \text{alphabet } s-alpha = x_1 : \tau_1^n \cdots x_i : \tau_i^n \]
\[ \text{datatype } s-ev = \epsilon \mid e_1 t_1^f \mid \cdots \mid e_j t_j^f \]
\[ \text{definition } \text{machine } : [s-alpha, s-ev] \text{StMach} \]
\[ \{ \text{init} = \text{init}, \}
\[ \text{where } \text{machine} = [\text{finals} = [f_1 \cdots f_m], \]
\[ \text{states} = [s_1 \cdots s_k], \]
\[ \text{transs} = [t_1 \cdots t_n]) \]
\[ \text{definition } \text{semantics} = [\text{machine}]_{\text{M}} \]

For each machine, a new observation space \( s-alpha \) is created, and lenses for each field of the form \( \tau_i^n \) is \( s-alpha \). For the events, an algebraic datatype \( s-ev \) is created with constructors corresponding to each of them. We create a distinguished event \( \epsilon \) that will be used in transitions that do not specify an explicit trigger event, which ensures that every transition has at least one visible event. This in turn ensures that the resulting reactive program is productive, as required by Theorem 6.10. The overall machine static semantics is then contained in \( \text{machine} \). We also define \( \text{semantics} \) that contains the dynamic semantics in terms of the semantic function \( [\cdot]_{\text{M}} \) that we describe in §7.3.

Elements of the meta-model are potentially not well-formed, for example specifying an initial state without a corresponding state declaration, and therefore it is necessary to formalise well-formedness. RoboTool enforces a number of well-formedness constraints \([49, 47]\), and we here formalise the subset needed to ensure the dynamic semantics given in §7.3 can be generated. We need some derived functions for this, and so we define \( \text{nnames} \triangleq \text{set} (\text{map nname (nodes)}) \), which calculates the set of node names, and \( \text{fnames} \), which calculates the set of final node names. We can now specify our well-formedness constraints.

Definition 7.4. A state machine is well-formed if it satisfies these constraints:

1. Each node identifier is distinct: \( \text{distinct (map nname (nodes))} \)
2. The initial identifier is defined: \( \text{init} \in \text{nnames} \)
3. The initial identifier is not final: \( \text{init} \notin \text{fnames} \)
4. Every transition’s source node is defined and non-final:
   \[ \forall t \in \text{transs} \bullet \text{src}(t) \in \text{nnames} \setminus \text{fnames} \]
5. Every transition’s target node is defined: \( \forall t \in \text{transs} \bullet \text{tgt}(t) \in \text{nnames} \)

We have implemented these constraints in Isabelle/HOL, along with a proof tactic called \( \text{check-machine} \) that discharges them automatically when a generated static semantics is well-formed, and ensures that crucial theorems are available to the dynamic semantics. In practice, any machine accepted by RoboTool is well-formed, and so this tactic simply provides a proof of that fact to Isabelle/HOL.

In a well-formed machine every node has a unique identifier. Therefore, using Definition 7.4, we construct two finite partial functions, \( \text{nmap} : \text{ID} \rightarrow [s, e]\text{Node} \) and \( \text{tmap} : \text{ID} \rightarrow [s, e]\text{Transition list} \), that obtain the node definition and list of transitions associated with a particular node identifier, respectively, whose domains are both equal to \( \text{nnames} \). We also define \( \text{ninit} \triangleq \text{nmap init} \), to be the definition of the initial node, and \( \text{inters} \) to be the set of nodes that are not final. Using well-formedness we can prove the following theorems.

Theorem 7.1 (Well-formedness Properties).

1. All nodes are identified: \( \forall n \in \text{set (nodes)} \bullet \text{nmap (nname(n))} = n \)
2. The initial node is defined: \( \text{ninit} \in \text{set (nodes)} \)
3. The name of the initial node is correct: \( \text{nname(ninit)} = \text{init} \)

These theorems allow us to extract the unique node for each identifier, and in particular for the initial node. Thus, Isabelle/HOL can parse a state machine definition, construct a static semantics for it, and ensure that this semantics is both well-typed and well-formed. The resulting Isabelle command is illustrated in Figure 8 that encodes the \( \text{GasAnalysis} \) state machine of Figure 7.
7.3. Dynamic Semantics

In this section we describe the behaviour of a state machine using the reactive program domain we mechanised in §6.3. The RoboChart reference semantics [49, 47] represents a state machine as a parallel composition of CSP processes that represent the individual variables and states. Variable access and state orchestration are modelled by communications between them. Here, we capture a simpler sequentialised semantics using guarded iteration, which eases verification. In particular, state variables have a direct semantics, and require no communication. The relation between these two semantics could, in the future, be formalised by an automated refinement strategy that reduces parallel to sequential composition [55].

We first define the observation space
\[ s_{rcst} = [actv: ID, r: s] \], parametrised by the state space type \( s \), and consisting of lenses \( actv: ID \Rightarrow s_{rcst} \) and \( r: s \Rightarrow s_{rcst} \). The \( actv \) lens records the currently active state, and \( r \) projects the state machine variable space. No action is permitted to refer to \( actv \), a constraint that we impose through a frame extension operator, which we now consider.

Frame extension, \( a: [P] + : [s_1, e] \) action, for \( a: s_2 \Rightarrow s_1 \) and \( P: [s_2, e] \) action, extends the state space of \( P \) using lens \( a \). It is similar to a frame in refinement calculus [51], which prevents modification of variables, but also uses the type system to statically prevent access to them. Lens \( a \) identifies a subregion \( s_2 \), which \( P \) acts upon, of the larger observation space \( s_1 \). Intuitively, \( s_2 \) specifies the set of state machine variables, and \( s_2 \) extends it with \( actv \). \( P \) can only modify variables within \( s_2 \), and others are held constant. We prove laws for this operator, which are also used in calculating the semantics.

**Theorem 7.2 (Frame Extension Laws),**

\[
a: [P; Q] + = a: [P] +; a: [Q] +; a: [e?x] + = e?(ax) +; a: [x := v] + = ax := v
\]

Frame extension distributes through sequential composition. For operators like event receive and assignment, the variable is extended by the lens \( a \) using a lens composition \( x \# a \) (see Def. 3.11), which we denote using a namespace-style operator \( (a: x) \). Specifically, it manipulates the region characterised by \( x \) within the region of \( a \).

We now describe the dynamic semantics of a state diagram using three functions.

**Definition 7.5 (Dynamic Semantics),**

\[
[M]_M \triangleq \left( \begin{array}{l}
actv := \text{init}_M ;
\text{do } N \in \text{set}(\text{inters}_M) \bullet actv = \text{name}(N) \rightarrow M \models [N]_N \od \\
\end{array} \right)
\]

\[
M \models [\text{Name}]_M \triangleq r: \text{nentry}(N) + ; (\square t \in \text{tmap}_M(\text{name}(N))) \bullet M, N \models [t]_T \\
\]

\[
M, N \models [t]_T \triangleq r: \text{cond}(t) \& \text{trig}(t) ; \text{nexit}(N) ; \text{action}(t) + ; \text{actv} := \text{tgt}(t)
\]

The function \( [\cdot]_M : [s, e] \text{SMach} \rightarrow [s]_{rcst, e} \text{action} \) calculates the overall behavioural semantics. It first sets \( actv \) to the initial node identifier, and then enters a do iteration indexed by all non-final nodes. If a
final node is selected, then the iteration terminates. In each iteration, the node \( N \) that is named by \( \text{actv} \) is selected, and the semantics for it is calculated using \( M \models [N]_N \).

When in a node, the entry action is first executed using \( \text{nentry} \), and then an external choice is presented over all transitions associated with \( N \), which are calculated using \( \text{tmap} \). The entry and exit actions do not have \( \text{actv} \) in their observation space, and therefore we apply frame extensions to them. The semantics of a transition, \( M \models [N]_N \), is guarded by the transition condition, and awaits the trigger event. Once this occurs, the exit action of \( N \) is executed, followed by the transition action, and finally \( \text{actv} \) is updated with the target node identifier.

The output of the semantics is an iterative program with one branch for every non-final state. To illustrate, we below generate the denotational semantics for the GasAnalysis state machine given in Figure 7.

Example 7.1 (GasAnalysis Dynamic Semantics).

\[
\begin{align*}
\text{actv} & := \text{InitState} ; \\
\text{do} \quad & \text{actv} = \text{InitState} \to \epsilon ; \text{r:gs} := \emptyset ; \text{r:anl} := 0 ; \text{actv} := \text{NoGas} \\
| & \text{actv} = \text{NoGas} \to \text{gas?r:gs} ; \text{actv} := \text{Analysis} \\
| & \text{actv} = \text{Analysis} \to \\
& \text{r:sts} := \text{analysis}(r:gs) ; \\
& \quad \begin{cases} \\
& \text{r:sts} = \text{noGas} & \text{r:anl} := \text{NoGas} \\
& \Box \text{r:sts} = \text{gasD} & \text{r:anl} := \text{GasDetected} \\
& \text{goreq}(\text{ins}, \text{thr}) & \text{stop} ; \text{actv} := \text{FinalState} \\
& \Box (\neg \text{goreq}(\text{ins}, \text{thr})) & \text{r:anl} := \text{location}(r:gs) ; \text{turn}(r:anl) ; \text{actv} := \text{Reading} \\
\end{cases} \\
| & \text{actv} = \text{Reading} \to \text{gas?r:gs} ; \text{actv} := \text{Analysis} \\
\end{align*}
\]

In order to yield a more concise definition, we have also applied the action equations, given in Theorem 7.2 and §6.4, as simplification laws. In particular, the frame extensions have all been expanded so that the state variables are explicitly qualified by lens \( r \).

In order to verify such state machines, we need a specialised refinement introduction law. Using our well-formedness theorem, we can specialise Theorem 6.10.

**Theorem 7.3.** The semantics of a state machine \( M \) refines a reactive invariant specification \( S \), that is \( S \subseteq [M]_M \), provided that the following conditions hold:

1. \( M \) is well-formed according to Definition 7.4;
2. the initial node establishes the invariant — \( S \subseteq M \models [\text{ninit}]_N \);
3. every non-final node preserves \( S \) — \( \forall N \in \text{inters}_M \bullet S \subseteq S ; (M \models [N]_N) \).

**Proof.** By application of Theorem 6.10, and utilising trigger productivity. \( \square \)

We now have all the infrastructure needed for verification of state machines, and in the next section we describe our verification strategy and tool.

### 7.4. Verification Strategy

In this section we use the collected results presented in the previous sections to define a verification strategy for state machines, and exemplify its use in verifying deadlock freedom. Our approach utilises Theorem 7.3 and the contractual refinement tactic, \texttt{rdes-refine} [26], which can be used to automatically show that a reactive program \( P \) satisfies (refines) a reactive contract: \( [S_1 \uparrow S_2 | S_3] \subseteq P \). We use this form to prove that every state of a state machine satisfies a given invariant. The overall workflow for description and verification of a state machine is given by the following steps:

1. parse, type check, and compile the state machine definition;
Figure 9: Selection of deadlock freedom proof obligations in Isabelle/UTP

2. check well-formedness (Definition 7.4) using the check-machine tactic;
3. calculate denotational semantics, resulting in a reactive program;
4. perform algebraic simplification and symbolic evaluation (Theorem 6.7);
5. apply Theorem 7.3 to produce sequential refinement proof obligations;
6. apply rdes-refine to each goal, which may result in residual proof obligations;
7. attempt to discharge each remaining proof obligation using sledgehammer [7].

This is effectively an elaborated version of the workflow followed by the rel-auto tactic we described in §4.3. Diagrammatic editors, like RoboTool, can be integrated with the proof tool by implementing a serialiser for the underlying meta-model. The workflow can be completely automated since there is no need to enter manual proofs, and the final proof obligations are discharged by automated theorem provers. If proof fails, Isabelle/HOL has the nitpick [7] counterexample generator that can be used for debugging. This means that the workflow can be hidden behind a graphical tool.

We can use the verification procedure to check deadlock freedom of a state machine using the reactive contract $\text{lockf} \triangleq [\text{true} \lor \exists e \in \text{ref} | \text{true}]$, an invariant specification which states that in all quiescent observations, there is always an event that is not being refused. In other words, at least one event is always enabled; this is the meaning of deadlock freedom. We can use this contract to check the GasAnalysis state machine. For a sequential machine, deadlock freedom means that it is not possible to enter a state and then make no further progress. Such a situation can occur if the outgoing transitions can all be disabled simultaneously if, for example, their guards do not cover all possibilities.

The result of applying the verification procedure up to step 5 is shown in Figure 9. At this stage, the semantics for each node has been generated, and deadlock freedom refinement conjectures need to be proved. Isabelle generates 6 subgoals, 3 of which are shown, since it is necessary to demonstrate that the invariant is satisfied by the initial state and each non-final state. The first goal corresponds to the initial state, where no event occurs and the variables gs and anl, as well as actv, are all assigned. The second goal corresponds to the Analysis state. The state body has been further simplified from the form shown in Figure 7.1, since symbolic evaluation has pushed the entry action through the transition external choice, and into the two guards. This is also the case for the third goal, which corresponds to the more complex GasDetected state.

The penultimate step applies the rdes-refine tactic to each of the 6 goals. This produces 3 subgoals for each goal, a total of 18 first-order proof obligations, and invokes the relational calculus tactic rel-auto on each of them. The majority are discharged automatically, but in this case three HOL predicate subgoals remain. One of them relates to the Analysis state, and requires that the constructors noGas and gasD of Status are the only cases for sts. If there was a third case, there would be a deadlock as the outgoing transition guards don’t cover this.

Finally, we execute sledgehammer on each of the three goals, which provides proofs and so completes the deadlock freedom check. Thus, we have engineered a fully automated deadlock freedom prover for state machines.
8. Conclusions

In this paper we have given a comprehensive exposition of the foundations of our verification tool framework, Isabelle/UTP. As we have demonstrated, Isabelle/UTP provides a unified semantic foundation for a variety of computational paradigms and programming languages, with the ability to formulate machine checked semantic models. These semantic models can then be harnessed in constructing verification tools that harness the powerful automated proof facilities of Isabelle/HOL.

We began this journey in §3 by describing how variables can be modelled as algebraic objects using lenses. This allows us to unify and compose a variety of variable models, and define generic operators for their comparison and manipulation. The lens-based model, in particular, allows us to avoid dependence on syntax, and instead build upon semantic properties like independence and containment.

We then used this algebraic observation space foundation in §4 to develop a flexible relational model including expressions, predicates, and alphabetised relations. Our fundamental expression model is a shallow embedding, which means it can directly harness the existing libraries of mathematical types and functions in Isabelle/HOL, and the associated theorems and proof strategies. We therefore provided a proof tactic for relational conjectures (rel-auto), and a library of algebraic theorems. In contrast to previous work, our expression model additionally supports syntax-like manipulations, including substitution and unrestriction, without the need for explicit modelling of variable names. It therefore offers both efficient automated proof on the one hand, and expressivity on the other hand.

We used the relational model to develop programming operators, and prove the “laws of programming” [44] as theorems, along with the axioms of several algebraic structures, like complete lattices, cylindrical algebra, and relation algebra. We then used this body of laws in §5 to provide symbolic execution and Hoare logic verification components for modelled relational programs.

We then showed, in §6, how the relational model can be specialised to particular computational paradigms with mechanised UTP theories. We showed how UTP theories are represented, in terms of observation spaces and healthiness conditions. Our theory model supports reuse of algebraic laws by a notion of inheritance, whereby a UTP theory can be extended with additional observational quantities and refined by further healthiness conditions. We showed how further laws can be obtained with links to algebraic structures.

We then used our mechanisation of UTP theories, in §6.3, to construct the reactive design theory hierarchy, which provides the semantic foundation for reactive programming languages. We used the specialised theory of stateful failure reactive designs to give a denotational semantics to a reactive programming language, with both state and interaction, and proved a number of important algebraic laws, including a verification law for iteration. We then showed how such a UTP theory can be optimised for proof automation, so that its algebraic laws can be efficiently applied to a modelled relational program.

Finally, in §7, we applied all of the aforementioned results in constructing a prototype verification tool for RoboChart state machine diagrams. We showed how the static semantics of the language can be mechanised as a collection of data types and well-formedness constraints, which Isabelle/HOL can check. We then showed how to mechanise the dynamic semantics, as an iterated system of guarded actions, using the reactive program model defined in §6. We then demonstrated a proof tactic that allows verification using invariants, and applied this to show that an example state machine is free from deadlock.

In conclusion, we believe that we have made the first steps toward realising the UTP vision [43, Chapter 0] of integrated formal methods, underpinned by unifying semantics. In many ways, this is only the beginning, as there remains a large number of computational paradigms that are not yet represented in Isabelle/UTP.

With respect to C code, one of the major features needed is a model of dynamically allocated memory addressed by pointers. We hinted back in §3.2 that we can weaken the axioms of total lenses to describe “partial lenses”. A partial lens is only defined for a subset of the possible states, which can be used to distinguish allocated and dangling pointers. We are currently using this idea to mechanise separation logic [58] in Isabelle/UTP, and hope to report on this soon.
With respect to differential equations, and hybrid dynamical systems, we have previously described an extension of the relational calculus with continuous variables [28, 27]. The generic trace model of reactive designs [27] ([§6.3]) and our extensible mechanisation of UTP theories means that we can specialise the reactive design hierarchy in a different direction to describe hybrid reactive systems, where the trace is a piecewise continuous function. We believe this can be applied, for instance, to assign a UTP semantics to a language like Hybrid CSP [35], which integrates continuous evolution with discrete CSP-style events and concurrency. Proof support for such a language depends on the ability to reason about invariants of differential equations, and so we are also integrating Platzer’s differential induction technique [57], as employed by the KeYmaera tool$(^8)$, into Isabelle/UTP. Moreover, we have previously given a UTP semantics to the dynamical systems modelling language Modelica [28], and so we will also develop proof facilities for this.

With respect to RoboChart, we will expand our semantics to handle additional features. Hierarchy, can be handled by having the actv variable hold a list of nodes, and during actions by implementing a reactive interruption operator [46]. Moreover, we are developing reasoning facilities for parallel composition and hiding to allow expression of concurrent state machines, which extends our existing work [26, 29]. This will increase verification capabilities for robotic and component-based systems, and allow us to handle asynchronous communication and shared variables, and also to mechanise the CSP reference semantics [49].

A challenge that remains is handling assumptions and guarantees between parallel components, but we believe that abstraction of state machines to invariants, using our results, can make this tractable. We will also explore other reasoning approaches, such as use of the simplifier to algebraically transform state machines to equivalent forms. Going further, we emphasise that our UTP theory hierarchy supports more advanced semantic paradigms. We will therefore develop a mechanised theory of timed reactive designs, based on existing work [62, 27], and use this to denote the timing constructs of RoboChart state machines. We are developing a UTP theory of probability [9], and will use it to handle probabilistic junctions. We also have our theory of hybrid reactive designs [28, 27], which can support state machines with hybrid dynamics.

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$^8$KeYmaera: http://symbolaris.com/info/KeYmaera.html
$^9$CyPhyAssure Project: https://www.cs.york.ac.uk/circus/CyPhyAssure/
$^{10}$RoboCalc Project: https://www.cs.york.ac.uk/circus/RoboCalc/
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