On the Fock representation of the $q$-commutation relations

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Abstract

We consider the C*-algebra $\mathcal{R}^q$ generated by the representation of the $q$-commutation relations on the twisted Fock space. We construct a canonical unitary $U (= U(q))$ from the twisted Fock space to the usual Fock space, such that $U\mathcal{R}^qU^*$ contains the extended Cuntz algebra $\mathcal{R}^0$, for all $q \in (-1, 1)$. We prove the equality $U\mathcal{R}^qU^* = \mathcal{R}^0$ for $q$ satisfying:

$$q^2 < 1 - 2|q| + 2|q|^4 - 2|q|^9 + \cdots + 2(-1)^k|q|^{k^2} + \cdots.$$

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1 Introduction and statement of results

In this paper we study the C*-algebra generated by the representation on the twisted Fock space of the $q$-commutation relations. These relations, introduced by Greenberg [6] and Bożejko and Speicher [2], provide an interpolation depending on a parameter $q \in (-1, 1)$ between the bosonic and the fermionic commutation relations (which correspond to $q = 1$ and $q = -1$, respectively). For $q$ in [-1,1], a representation of the $q$-commutation relations is of the form:

$$c(\xi)c(\eta)^* - qc(\eta)^*c(\xi) = \langle \xi \mid \eta \rangle I, \quad \xi, \eta \in \mathcal{H},$$

(1.1)

where $\mathcal{H}$ is a separable Hilbert space and $c(\cdot)$ is linear with values operators on some Hilbert space $\mathcal{K}$ (called the space of the representation). The Fock representation of these relations is the one uniquely determined, up to unitary equivalence, by the following condition: there exists a vacuum vector $\Omega$ in the space of the representation, that is cyclic for the C*-algebra generated by $\{c(\xi)\mid \xi \in \mathcal{H}\}$, and such that $c(\xi)\Omega = 0$ for every $\xi$ in $\mathcal{H}$. The uniqueness of the Fock representation is easy to show, but the proof of its existence is not at all trivial (see [2], [5], [8]); the construction giving this representation will be briefly reviewed in Section 2.1 below.

We shall consider the case when the Hilbert space $\mathcal{H}$ of (1.1) has finite dimension $d \geq 2$; $d$ will be fixed throughout the whole paper. Choosing an orthonormal basis of $\mathcal{H}$, we see that the representations of the $q$-commutation relations come to those of the universal unital C*-algebra generated by $d$ elements $a_1, \ldots, a_d$, that satisfy:

$$a_ia_j^* - qa_j^*a_i = \delta_{i,j}I, \quad 1 \leq i, j \leq d. \quad (1.2)$$

We shall denote, following [7], this universal C*-algebra by $E^q$ ($= E^q(d)$). Also, we shall use the following notations: the Fock representation of the $q$-commutation relations, viewed as a representation of $E^q$, will be denoted by $\Phi_q$, and its space will be denoted by $T_q$ (and called the twisted Fock space); we shall put

$$A_i = \Phi_q(a_i) \in \mathcal{L}(T_q), \quad 1 \leq i \leq d, \quad (1.3)$$

and we shall denote by $\mathcal{R}^q$ the C*-algebra generated by $A_1, \ldots, A_d$ in $\mathcal{L}(T_q)$. Equivalently, $\mathcal{R}^q = \Phi_q(E^q)$; this C*-algebra will be our main object of investigation.

For $q = 0$, we have that $a_1, \ldots, a_d$ of (1.2) are the adjoints of $d$ isometries with mutually orthogonal ranges, hence $E^0$ is the well-known extension by the compacts of the Cuntz algebra $O_d$ ([3]); moreover, $\Phi_0 : E^0 \rightarrow \mathcal{L}(T_0)$ is precisely the canonical representation of $E^0$. 

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on the usual Fock space \( T_0 = \mathcal{T} = \bigoplus_{n=0}^{\infty} \left( (\mathbb{C}^d)^{\otimes n} \right) \). It is known that \( \Phi_0 \) is faithful, hence (if we consider that the “non-deformed case” is \( q = 0 \)), both \( \mathcal{E}^q \) and \( \mathcal{R}^q \) can be viewed as deformations of the extension by the compacts of \( \mathcal{O}_d \).

In order to distinguish the case \( q = 0 \), we shall write \( v_1, \ldots, v_d \) for the \( a_1, \ldots, a_d \) of (1.2) corresponding to this case (to be very rigorous, we should have written in (1.2) \( a_{i,q} \) instead of \( a_i \), and then \( v_i \) would be defined as \( a_{i,0} \); however, the value of \( q \) which is considered will always be clear, and we preferred to keep the notations simple). We denote the projection \( \sum_{i=1}^d v_i^* v_i \in \mathcal{E}^0 \) by \( p \). Similarly, we shall write \( V_1, \ldots, V_d \) for the \( A_1, \ldots, A_d \) of (1.3) corresponding to \( q = 0 \), and put \( P = \sum_{i=1}^d V_i^* V_i \); then \( V_1, \ldots, V_d \) are annihilation operators on the (non-deformed) Fock space \( \mathcal{T} = \mathcal{T}_0 \) (formula (2.2) below), and \( P \) is the projection onto the orthogonal complement of the vacuum vector.

In [7] it was proved that, for \( |q| < \sqrt{2} - 1 \), \( \mathcal{E}^q \simeq \mathcal{E}^0 \) and \( \Phi_q \) is faithful. The proof involved finding a positive element \( \rho \in \mathcal{E}^0 \) that satisfies the equation:

\[
\rho^2 = p + q \sum_{i,j=1}^d (v_i \rho v_j)^*(v_j \rho v_i), \tag{1.4}
\]

and then showing that \( a_i \to v_i \rho, \ 1 \leq i \leq d \), gives an isomorphism between \( \mathcal{E}^q \) and \( \mathcal{E}^0 \). We shall consider the analogue on the Fock space of (1.4), i.e.:

\[
R^2 = P + q \sum_{i,j=1}^d (V_i RV_j)^*(V_j RV_i), \tag{1.5}
\]

(\( P, V_1, \ldots, V_d \) defined in the preceding paragraph, \( R \in \mathcal{L}(\mathcal{T}) \) unknown). Of course, if \( |q| < \sqrt{2} - 1 \), and if \( \rho \) is the solution of (1.4) given by [7], then \( \Phi_q(\rho) \) satisfies (1.5); in addition, \( \Phi_q(\rho) \) leaves invariant each subspace of \( \mathcal{T} \) spanned by tensors of a given length (this follows immediately from the fact that \( \rho \) can be obtained by doing iterations in (1.4), starting with \( \rho_1 = p \)).

We shall prove the following:

**Theorem 1**. For every \(-1 < q < 1\), there exists a unique positive operator \( R \in \mathcal{L}(\mathcal{T}) \) which satisfies (1.5) and leaves invariant each subspace of \( \mathcal{T} \) spanned by tensors of a given length.

**Theorem 2**. For every \(-1 < q < 1\), there exists a canonical unitary \( U : \mathcal{T}_q \to \mathcal{T} \) which intertwines \( R \in \mathcal{L}(\mathcal{T}) \) defined above with \( \left( \sum_{i=1}^d A_i^* A_i \right)^{1/2} \in \mathcal{L}(\mathcal{T}_q) \). Moreover, we have that

\[
UA_i U^* = V_i R, \quad 1 \leq i \leq d. \tag{1.6}
\]
3° For every $-1 < q < 1$, the C*-algebra $UR^q U^* \subset L(T)$ contains $R^0$.

4° For $q$ satisfying:

$$q^2 < 1 - 2|q| + 2|q|^4 - 2|q|^9 + \cdots + 2(-1)^k|q|^{2k} + \cdots$$

(1.7)

the inclusion $UR^q U^* \subset R^0$ also holds, and hence $R^q$ is isomorphic to the extension by the compacts of the Cuntz algebra.

The inequality (1.7) gives for $|q|$ a bound of around 0.44. Calculations by computer indicate that Proposition 5.2 of the paper, which has the last part of the Theorem as a corollary, actually works (and gives $UR^q U^* = R^0$) for $|q|$ up to a bound somewhere between 0.455 and 0.47.

The above theorem also gives some information on $R^q$ for larger values of $q$ - for instance that $R^q$ contains the compact operators on the twisted Fock space $T_q$ for all the values of the parameter. (We suspect this was known by people working on the problem, although it has not yet appeared in writing.)

The paper is subdivided into sections as follows: in Section 2 we review some basic facts about the twisted Fock space, and fix our notations. In Section 3 we introduce the canonical unitary $U : T_q \to T$ and prove the first two assertions of the above Theorem. In Section 4 we show that $UR^q U^* \supseteq R^0$, and in Section 5 we prove the opposite inclusion for $q$ satisfying (1.7).

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2 Preliminaries

2.1 The Fock representation If $\xi_1, \ldots, \xi_d$ is an orthonormal basis of the space $H$ appearing in (1.1) (which is fixed throughout the paper, and has finite dimension $d \geq 2$), then an orthonormal basis of the Fock space on $H$, $T = C \bigoplus (\bigoplus_{n=1}^{\infty} H^\otimes n)$, is:

$$\{\Omega\} \cup \{\xi_{i_1} \otimes \cdots \otimes \xi_{i_n} \mid n \geq 1, 1 \leq i_1, \ldots, i_n \leq d\};$$

(2.1)
\( \Omega \) in (2.1) (the vacuum vector) is 1 in the first summand, \( \mathbb{C} \), in the expression of \( \mathcal{T} \). For every \( n \geq 0 \) we shall denote by \( \mathcal{V}_n \subset \mathcal{T} \) the subspace spanned by tensors of length \( n \) in (2.1) \( (\mathcal{V}_0 = \mathbb{C} \Omega, \text{by convention}; \text{clearly } \dim \mathcal{V}_n = d^n, \ n \geq 0, \text{ and } \mathcal{T} = \bigoplus_{n=0}^{\infty} \mathcal{V}_n, \text{ orthogonal direct sum}) \). The annihilation operators \( \mathcal{V}_1, \ldots, \mathcal{V}_d \) involved in equation (1.5) are determined by

\[
\mathcal{V}_i \Omega = 0, \quad \mathcal{V}_i (\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) = \delta_{i,i_1} \xi_{i_2} \otimes \cdots \otimes \xi_{i_n}; \quad \text{(2.2)}
\]

their adjoints are the corresponding creation operators:

\[
\mathcal{V}_i^* \Omega = \xi_i, \quad \mathcal{V}_i^* (\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) = \xi_i \otimes \xi_{i_1} \otimes \cdots \otimes \xi_{i_n}.
\]

Let us now pick a \( q \in (-1, 1) \). One defines recursively a \( q \)-inner product \( < \cdot | \cdot >_q \) on the subspaces \( \mathcal{V}_n \subset \mathcal{T}, \ (n \geq 0) \), as follows: on \( \mathcal{V}_0 \), \( < \cdot | \cdot >_q \) is determined by \( < \Omega | \Omega >_q = 1 \); then for every \( n \geq 1 \), one puts:

\[
< \xi_{i_1} \otimes \cdots \otimes \xi_{i_n} | \xi_{j_1} \otimes \cdots \otimes \xi_{j_n} >_q = \\
= \sum_{k=1}^{n} q^{k-1} \delta_{i_1,j_k} < \xi_{i_2} \otimes \cdots \otimes \xi_{i_n} | \xi_{j_1} \otimes \cdots \otimes \widehat{\xi_{j_k}} \otimes \cdots \otimes \xi_{j_n} >_q
\]

(where \( 1 \leq i_1, \ldots, i_n, j_1, \ldots, j_n \leq d \), and the hat on \( \xi_{j_k} \) means that \( \xi_{j_k} \) is deleted from the tensor). We shall denote \( \mathcal{V}_n \), considered with the \( q \)-inner product, by \( \mathcal{V}_{n,q} \). The natural basis of \( \mathcal{V}_n \), consisting of tensors of length \( n \) from (2.1), will be no longer orthogonal for \( < \cdot | \cdot >_q \) (unless \( q=0 \) or \( n \leq 1 \)); the point is, however, that the Gramm matrix \( \Gamma_n \) of \( q \)-inner products of elements from this basis remains positive and non-degenerate \( (2, 3, 5) \). Hence one can define the Hilbert space \( \mathcal{T}_q = \bigoplus_{n=0}^{\infty} \mathcal{V}_{n,q} \) (orthogonal direct sum); this is the twisted Fock space. The operators \( A_1, \ldots, A_d \) of (1.3) act on \( \mathcal{T}_q \) by:

\[
A_i \Omega = 0, \quad A_i (\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) = \sum_{k=1}^{n} q^{k-1} \delta_{i,i_k} \xi_{i_1} \otimes \cdots \otimes \widehat{\xi_{i_k}} \otimes \cdots \otimes \xi_{i_n};
\]

their adjoints are the corresponding creation operators:

\[
A_i^* \Omega = \xi_i, \quad A_i^* (\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) = \xi_i \otimes \xi_{i_1} \otimes \cdots \otimes \xi_{i_n}.
\]

An important role in what follows will be played by the operator

\[
M = \sum_{i=1}^{d} A_i^* A_i \in \mathcal{L}(\mathcal{T}_q). \quad \text{(2.3)}
\]

Note that \( M \) leaves invariant every subspace \( \mathcal{V}_{n,q} \subset \mathcal{T}_q \) spanned by tensors of length \( n \). We shall denote by \( M_n \in \mathcal{L}(\mathcal{V}_{n,q}) \) the operator induced by \( M \) on \( \mathcal{V}_{n,q} \), and by \( [M_n] \in \text{Mat}_{d^n}(\mathbb{C}) \)
the matrix of $M_n$ with respect to the natural basis $\{\xi_{i_1} \otimes \cdots \otimes \xi_{i_n} \mid 1 \leq i_1, \ldots, i_n \leq d\}$ of $V_{n,q}$ (ordered lexicographically, for instance). It is important to make distinction between $M_n$ and $[M_n]$, since, due to the non-orthogonality of the natural basis of $V_{n,q}$, the matrix $[M_n]$ is generally non-selfadjoint, although $M_n$ itself is positive.

In general, if $X$ is in either of $L^1(V_{n,q})$, $L^0(V_{n,q}, V_n)$, $L^{-1}(V_n, V_{n,q})$ ($n \geq 0$, $-1 < q < 1$), we shall denote by $[X]$ its matrix with respect to the natural basis of its domain and codomain. We have the usual rules $[\alpha X + \beta Y] = \alpha [X] + \beta [Y]$, $[XY] = [X][Y]$, but computing $[X^*]$ needs a correction with the Gramm matrix $\Gamma_n$ of $q$-inner products of vectors from the natural basis of $V_{n,q}$:

$$[X^*] = \begin{cases} \Gamma_n^{-1}[X]^*\Gamma_n & \text{if } X \in L^1(V_{n,q}) \\ \Gamma_n^{-1}[X]^* & \text{if } X \in L^0(V_{n,q}, V_n) \\ [X]^*\Gamma_n & \text{if } X \in L^{-1}(V_n, V_{n,q}) \end{cases} \quad (2.4)$$

(where $[X]^*$ is the conjugated-transpose of the matrix $[X]$).

Though there were objects defined in this subsection (or in the Introduction) which depend implicitly on the parameter $q$, but do not have this dependence reflected in their notations, we hope that this will not create any confusion in what follows. The next list may also be of some help.

| Depend on $q$ | Don’t depend on $q$ (correspond to $q = 0$) |
|---------------|------------------------------------------|
| $a_1, \ldots, a_d$ | $v_1, \ldots, v_d$ |
| $T_q = \bigoplus_{n=0}^{\infty} V_{n,q}$ | $T = \bigoplus_{n=0}^{\infty} V_n$ |
| $A_1, \ldots, A_d$ | $V_1, \ldots, V_d$ |
| $\bigoplus_{n=0}^{\infty} M_n = M = \sum_{i=1}^{d} A_i^* A_i$ | $P = \sum_{i=1}^{d} V_i^* V_i$ |
| $\Gamma_n$, $n \geq 0$ | |
| $U = \bigoplus_{n=0}^{\infty} U_n$ (see Def. 3.2) | |
| $R = \bigoplus_{n=0}^{\infty} R_n$ (see Def. 3.3) | |

### 2.2 The actions of symmetric groups

Let $n \geq 1$ be an integer, and let $S_n$ be the group of all permutations of $\{1, \ldots, n\}$. For $1 \leq k \leq l \leq n$, we shall denote the cycle

$$\begin{pmatrix} k & k+1 & \cdots & l-1 & l \\ k+1 & k+2 & \cdots & l & k \end{pmatrix} \in S_n \text{ by } (k \rightarrow l) \text{ if } k = l, \text{ then } (k \rightarrow l) \text{ is by convention the unit of } S_n.$$ For every $-1 < q < 1$, we have a natural representation by invertible operators $\pi_{n,q} : S_n \to L(V_{n,q})$, determined by:

$$\pi_{n,q}(s)(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) = \xi_{s^{-1}(i_1)} \otimes \cdots \otimes \xi_{s^{-1}(i_n)}. \quad (2.5)$$

$\pi_{n,q}$ extends to a representation of $C[S_n] = C^*(S_n)$ on $V_{n,q}$, still denoted by $\pi_{n,q}$. This is
generally not a $*$-representation; however, let us point out that since invertible elements of $C^*(S_n)$ must map to invertible operators on $V_{n,q}$, it is true that the spectrum of $\pi_{n,q}(x)$ is contained in the one of $x$, for every $x$ in $C^*(S_n)$. This will be useful for studying the spectrum of $M$, since, as shown by a moment’s reflection, we have:

$$M_n = \pi_{n,q}\left(\sum_{k=1}^{n} q^{k-1}(1 \to k)\right), \quad n \geq 1.$$

(2.6)

3 The canonical unitary $U : \mathcal{T}_q \to \mathcal{T}$

3.1 Lemma For every $q \in (-1,1)$ and $n \geq 1$ we have

$$\Gamma_n = \begin{pmatrix} \Gamma_{n-1} & 0 & \ldots & 0 \\ 0 & \Gamma_{n-1} & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & \Gamma_{n-1} \end{pmatrix} [M_n]$$

(3.1)

(equality in $\text{Mat}_{d^n}(C)$, with $\Gamma_n, \Gamma_{n-1}, [M_n]$ as in Section 2.1).

Proof Consider the representation $\pi_{n,q} : C^*(S_n) \to \mathcal{L}(V_{n,q})$ defined in Section 2.2. From Lemma 3 of [8] it follows that $\pi_{n,q}\left(\sum_{s \in S_n} q^{\text{inv}(s)}s\right)$ has matrix $\Gamma_n$ in the natural basis of $V_{n,q}$, where $\text{inv}(s)$ is the number of inversions of $s \in S_n$. Similarly, the matrix of $\pi_{n,q}\left(\sum_{t \in S_n, t(1) = 1} q^{\text{inv}(t)}t\right)$ is

$$\begin{pmatrix} \Gamma_{n-1} & 0 & \ldots & 0 \\ 0 & \Gamma_{n-1} & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & \Gamma_{n-1} \end{pmatrix}.$$  

Taking also (2.6) into account, we see that (3.1) is implied by:

$$\sum_{s \in S_n} q^{\text{inv}(s)}s = \left(\sum_{t \in S_n, t(1) = 1} q^{\text{inv}(t)}t\right)\left(\sum_{k=1}^{n} q^{k-1}(1 \to k)\right);$$

but this in turn comes out, exactly as in Proposition 1 of [8], from the fact that every $s \in S_n$ can be uniquely decomposed as a product $t(1 \to k)$ with $k \geq 1$, $t \in S_n$, $t(1) = 1$, and that in this decomposition we have $\text{inv}(s) = \text{inv}(t) + (k - 1)$. QED

3.2 Proposition and Definition Let $q$ be in $(-1,1)$. Let $U_0 : V_{0,q} \to V_0$ be defined by $U_0 \Omega = \Omega$, and then define recursively, for $n \geq 1$:

$$U_n = (I \otimes U_{n-1})M_n^{1/2} : V_{n,q} \to V_n.$$

(3.2)
(In the last formula, \( I \otimes U_{n-1} \in L(\mathcal{V}_{n,q}, \mathcal{V}_n) \) sends the tensor \( \xi_1 \otimes \cdots \otimes \xi_n \in \mathcal{V}_{n,q} \) into \( \xi_1 \otimes (U_{n-1}(\xi_2 \otimes \cdots \otimes \xi_n)) \in \mathcal{V}_n \).) Then \( U_n \) is unitary, for every \( n \geq 0 \), and thus \( U = \bigoplus_{n=0}^{\infty} U_n \) is a unitary between \( \mathcal{T}_q \) and \( \mathcal{T} \).

**Proof** The fact that \( U_n \) is unitary is equivalent to

\[
[U_n]^*[U_n] = \Gamma_n
\]

(by relation (2.4) and using Lemma 3.1 we get that this equals:)

\[
\Gamma_n[M_{n/2}^1] \Gamma_n^{-1} = \Gamma_n[M_{n/2}^1 M_{n/2}^{-1} M_{n/2}^1] = \Gamma_n.
\]

3.3 Remark The construction of the above unitary \( U \) would still work if \( d \) (the dimension of the separable Hilbert space \( \mathcal{H} \) of (1.1)) were infinite. In this case, the spaces \( (\mathcal{V}_n)_{n=0}^{\infty} \) and \( (\mathcal{V}_{n,q})_{n=0}^{\infty} \) we are working with would of course no longer be finite dimensional. However, for every sequence \( \alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots) \) of non-negative integers with \(|\alpha| = \sum_{i=1}^{\infty} \alpha_i < \infty\), let us denote by \( \mathcal{V}_\alpha \) (respectively \( \mathcal{V}_{\alpha,q} \)) the subspace of \( \mathcal{T} \) (respectively \( \mathcal{T}_q \)) generated by the tensors \( \xi_{i_1} \otimes \cdots \otimes \xi_{i_n} \) having the property that among \( i_1, \ldots, i_n \) there are \( \alpha_1 \) of 1, \( \alpha_2 \) of 2, \( \alpha_3 \) of 3, . . . . Then for every \( n \geq 0 \) we have \( \mathcal{V}_n = \bigoplus_{|\alpha|=n} \mathcal{V}_\alpha \), \( \mathcal{V}_{n,q} = \bigoplus_{|\alpha|=n} \mathcal{V}_{\alpha,q} \), orthogonal direct sums, and each space \( \mathcal{V}_\alpha \) and \( \mathcal{V}_{\alpha,q} \) is finite dimensional even when \( d \) is infinite. Moreover, one can rewrite the construction of the above \( U : \mathcal{T}_q \to \mathcal{T} \) by defining a
family of unitaries \((U_\alpha : V_{\alpha,q} \to V_{\alpha})_\alpha\), by induction on \(|\alpha|\), and then putting \(U = \bigoplus_\alpha U_\alpha\). (When doing so, \(M_n\) of formula (3.2) should be replaced by \(M_\alpha \in L(V_{\alpha,q})\), determined by

\[
M_\alpha(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) = \sum_{k=1}^n q^{k-1} \xi_{k} \otimes \xi_{i_1} \otimes \cdots \otimes \xi_{k-1} \otimes \xi_{i_{k+1}} \otimes \cdots \otimes \xi_{i_n},
\]

for \(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n} \in V_{\alpha,q}\). This way of proving the preceding proposition still works for \(d = \infty\), (but in what follows, we are only concerned with the case of finite \(d\)).

3.4 Definition and Proposition Let \(q\) be in \((-1,1)\), and define:

\[
R = U M^{1/2} U^* \in L(T). \tag{3.4}
\]

Then: 1° We have

\[
UA_i U^* = V_i R, \quad 1 \leq i \leq d, \tag{3.5}
\]

and as a consequence \(V_i R, \ldots, V_d R\) satisfy the \(q\)-commutation relations.

2° \(R\) is the unique positive operator on the (non-deformed) Fock space \(T\) which satisfies equation (1.5), and which leaves invariant each subspace of \(T\) spanned by tensors of a given length.

**Proof** 1° We pick \(1 \leq i \leq d\) and check the equality \(A_i^* = M^{1/2} U^{-1} V_i^* U\) (obviously equivalent to (3.5)) on vectors of the natural basis of \(T_q\). Note first that \(A_i^* \Omega = \xi_i = M^{1/2} U^{-1} V_i^* U \Omega\) (the second equality following from the facts, easy to check, that \(U \xi_i = \xi_i\) and that \(M_1\) is the identity on the space \(V_{1,q}\)). Next, for a tensor \(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n} \in T_q\) we have:

\[
M^{1/2} U^{-1} V_i^* U \xi_{i_1} \otimes \cdots \otimes \xi_{i_n} = M^{1/2} U_{n+1}^{-1} V_i^* U_{n+1} \xi_{i_1} \otimes \cdots \otimes \xi_{i_n}.
\]

But \(M^{1/2} U_{n+1}^{-1} = (I \otimes U_n)^{-1}\) (by (3.2)), which implies that the coincidence of \(A_i^*\) and \(M^{1/2} U^{-1} V_i^* U\) on \(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}\) is equivalent to:

\[
(I \otimes U_n) A_i^* \xi_{i_1} \otimes \cdots \otimes \xi_{i_n} = V_i^* U_{n+1} \xi_{i_1} \otimes \cdots \otimes \xi_{i_n}.
\]

The last equality is, however, obvious from the definitions of \(A_i^*\) and \(V_i^*\).

2° It is clear that \(R\) is positive and leaves invariant every \(V_n\) (= subspace spanned by tensors of length \(n\)). The fact that \(R\) satisfies equation (1.5) follows from (3.5) via exactly the argument preceding relation (10) of [7].

If \(\bar{R} \in L(T)\) is an operator sharing the above properties, then (as it immediately comes out of (1.5)), \(\bar{R} \Omega = 0\) and, for \(m, n \geq 1, 1 \leq i_1, \ldots, i_m, j_1, \ldots, j_n \leq d\):

\[
< \bar{R}^2(\xi_{i_1} \otimes \cdots \otimes \xi_{i_m}) | \xi_{j_1} \otimes \cdots \otimes \xi_{j_n} > =
\]
\[ \delta_{m,n} \left( \delta_{i_1,j_1} \cdots \delta_{i_m,j_n} + q < V_{j_1} \tilde{R} \xi_{i_2} \otimes \cdots \otimes \xi_{i_m} \mid V_{i_1} \tilde{R} \xi_{j_2} \otimes \cdots \otimes \xi_{j_n} > \right). \]

From the last equation it is clear that, for every \( n \geq 1 \), \( \tilde{R} \mid \mathcal{V}_{n-1} \) determines \( \tilde{R}^2 \mid \mathcal{V}_n \) (and hence \( \tilde{R} \mid \mathcal{V}_n \) too, by taking a square root). Thus an induction argument shows that \( \tilde{R} \mid \mathcal{V}_n = R \mid \mathcal{V}_n \) for every \( n \geq 0 \), and we get \( \tilde{R} = R \). QED

4 \( \mathcal{U} \mathcal{R} q \mathcal{U}^* \supseteq \mathcal{R}^0 \)

4.1 Lemma Let \( q \) be in (-1,1), let \( R \) be as in Definition 3.4, and denote, for every \( n \geq 0 \), \( R \mid \mathcal{V}_n \) by \( R_n \). Then we have

\[ \left( \frac{1}{1 - |q|} \prod_{k=1}^\infty \frac{1 - |q|^k}{1 + |q|^k} \right) I \leq R_n^2 \leq \frac{1}{1 - |q|} I \]

(inequality in \( \mathcal{L}(\mathcal{V}_n) \)), for every \( n \geq 1 \).

Proof Since \( R_n^2 \) is conjugate to \( M_n \) by \( U_n \), it suffices to prove the analogue in \( \mathcal{L}(\mathcal{V}_{n,q}) \) of (4.1), with \( R_n^2 \) replaced by \( M_n \). This comes, clearly, to showing that the spectral radii of \( M_n \) and \( M_n^{-1} \) are not greater than \( 1/(1-|q|) \) and \( (1-|q|) \prod_{k=1}^\infty \frac{1+|q|^k}{1-|q|^k} \), respectively. Recalling the considerations of Section 2.2, and dominating the spectral radius of an element \( x \in \mathcal{C}^*(S_n) \) by \( ||x|| \), we see that it will suffice to prove:

\[
\left\{ \begin{array}{l}
|| \sum_{k=1}^n q^{k-1}(1 \to k) || \leq \frac{1}{1-|q|} \\
|| (\sum_{k=1}^n q^{k-1}(1 \to k))^{-1} || \leq (1-|q|) \prod_{k=1}^\infty \frac{1+|q|^k}{1-|q|^k} .
\end{array} \right.
\]

(4.2)

The first inequality in (4.2) is obvious. The proof of the second one (which must, of course, contain a proof of the invertibility of \( \sum_{k=1}^n q^{k-1}(1 \to k) \)), is obtained from a decomposition into factors performed in the spirit of [8], Proposition 2. More precisely, one notes first the commutation relation:

\[ (1 \to j)(1 \to k) = (2 \to k)(1 \to j-1), \quad 2 \leq j \leq k \leq n, \]

which immediately implies the identity:

\[
\left( \sum_{k=1}^m q^{k-1}(1 \to k) \right) \left( I - q^{m-1}(1 \to m) \right) = \\
= (I - q^m(2 \to m)) \left( \sum_{k=1}^{m-1} q^{k-1}(1 \to k) \right), \quad 2 \leq m \leq n.
\]

(4.3)
Multiplying (4.3) by $(I - q^{n-1}(1 \rightarrow m))^{-1}$ on the right, and using induction on $m$ ($1 \leq m \leq n$), one obtains:

$$
\sum_{k=1}^{n} q^{k-1}(1 \rightarrow k) = \prod_{j=0}^{n-2} (I - q^{n-j}(2 \rightarrow n - j)) \prod_{j=1}^{n-1} (I - q^j(1 \rightarrow j + 1))^{-1}.
$$

(4.4)

Taking the inverse in (4.4), and after that taking norms and doing straightforward majorizations, we get the second inequality (4.2) (the norm of $(I - q^{n-j}(2 \rightarrow n - j))^{-1} = \sum_{k=0}^{\infty} q^{(n-j)k}(2 \rightarrow n - j)^k$ is dominated by $1/(1 - |q|^{n-j})$). QED

4.2 Remark $\prod_{k=1}^{\infty} \frac{1-|q|^k}{1+|q|^k}$ appearing in (4.1) is strictly positive (as it is well-known, $\sum_{k=1}^{\infty} |q|^k < \infty$ implies $\prod_{k=1}^{\infty} (1 + |q|^k) < \infty$ and $\prod_{k=1}^{\infty} (1 - |q|^k) > 0$). We take this occasion to mention the following identity (due to Gauss - see Corollary 2.10 of [1]):

$$
\prod_{k=1}^{\infty} \frac{1-|q|^k}{1+|q|^k} = \sum_{k=-\infty}^{\infty} (-1)^k q^{k^2}, \; 0 \leq q < 1.
$$

(4.5)

4.3 The proof of the inclusion contained in the title of this section is now immediate. Indeed, what we need to show is that $V_1, \ldots, V_d \in U \mathcal{R}^q U^*$. We know, from the relations (3.5), that $V_1 R, \ldots, V_d R \in U \mathcal{R}^q U^*$, where $R = UM^{1/2}U^*$ also belongs to $U \mathcal{R}^q U^*$. But now, $R$ splits as $\bigoplus_{n=0}^{\infty} R_n$, with $R_0 = 0$ on $C \Omega$ and $R_n \in \mathcal{L}(V_n)$ satisfying “the square root” of inequality (4.1) for $n \geq 1$; this immediately implies that $\text{Ker} R = C \Omega$, and that 0 is isolated in the spectrum of $R$. Denoting by $\phi$ the (continuous) function on the spectrum of $R$ which sends 0 into 0 and $\alpha \neq 0$ into $1/\alpha$, we then have that $R \phi(R)$ is the projection onto $T \ominus C \Omega$, and hence that indeed $V_i = (V_i R) \phi(R) \in U \mathcal{R}^q U^*$, $1 \leq i \leq d$. QED

5 $U \mathcal{R}^q U^* \subseteq \mathcal{R}^0$ for $q$ satisfying (1.7)

Let $q$ be in (-1,1), and let $R \in \mathcal{L}(\mathcal{T})$ be as in Definition 3.4. Due to the relations (3.4) and (3.5), it is obvious that the inclusion “$U \mathcal{R}^q U^* \subseteq \mathcal{R}^0$” is equivalent to “$R \in \mathcal{R}^0$”.

Denote now, as in Lemma 4.1, by $R_n$ the operator induced by $R$ on the invariant subspace $V_n \subset \mathcal{T}$, $(n \geq 0)$, and define, for every $n \geq 1$:

$$
X_n = R_0 \oplus R_1 \oplus \cdots R_n \oplus (R_n \otimes I) \oplus (R_n \otimes I \otimes I) \oplus \cdots \in \mathcal{L}(\mathcal{T}).
$$

(5.1)
It is immediate (from Definitions 3.2 and 3.4) that $R_0 = 0$ and $R_1$ is the identity operator on $V_1$. Moreover, for $n \geq 1$, restricting the two sides of the equation (1.5) to the subspace $V_{n+1}$ gives:

$$R_{n+1}^2 = I_{n+1} + q \sum_{i,j=1}^d V_j^* R_n V_i^* V_j R_n V_i$$  \hspace{1cm} (5.2)

(where $I_{n+1}$ is the identity operator on $V_{n+1}$, and we view $V_i \in \mathcal{L}(V_{n+1}, V_n)$, $V_j \in \mathcal{L}(V_{n}, V_{n-1})$, $V_i^* \in \mathcal{L}(V_{n-1}, V_n)$, $V_j^* \in \mathcal{L}(V_{n}, V_{n+1})$). Using all these facts, it is easy to check that the $X_n$’s defined in (5.1) satisfy:

$$X_{n+1}^2 = P + q \sum_{i,j=1}^d V_j^* X_n V_i^* V_j X_n V_i, \quad n \geq 1.$$  \hspace{1cm} (5.3)

Thus $(X_n)_{n=1}^\infty$ is the Fock representation of a sequence of iterates as considered in [7], which begins with $X_1 = P$. Note that in this particular situation, the iterates can be defined with no restriction on $q \in (-1, 1)$.

**5.1 Lemma** Let $q$ be in $(-1, 1)$, let $R = \bigoplus_{n=0}^{\infty} R_n$ be as above and denote, for every $n \geq 1$, by $\alpha_n(q)$ the smallest eigenvalue of $R_n^2$ ($\alpha_n(q) > 0$ by Lemma 4.1). Then the $(X_n)_{n=1}^\infty$ defined in (5.1) satisfy:

$$||X_{n+2} - X_{n+1}|| \leq \frac{|q|}{\sqrt{(1 - |q|)\min(\alpha_{n+1}(q), \alpha_{n+2}(q))}} ||X_{n+1} - X_n||, \quad n \geq 1.$$  \hspace{1cm} (5.4)

**Proof** The argument will consist in combining Lemma 8 of [6] with the particular form given to the iterates in (5.1). It is immediate (from (5.1)) that $||X_n - X_{n+1}|| = ||(R_n \otimes I) - R_{n+1}||$, and we shall examine the latter quantity.

Observe that because of the obvious relations $R_n V_i = V_i (R_n \otimes I)$, $V_j^* R_n = (R_n \otimes I)V_j^*$, the equation (5.2) can be rewritten

$$R_{n+1}^2 = I_{n+1} + q(I \otimes R_n)T_{n+1}(I \otimes R_n),$$  \hspace{1cm} (5.5)

where $T_{n+1}$ is the operator induced on $V_{n+1}$ by $T = \sum_{i,j=1}^d V_j^* V_i^* V_j V_i$. Since clearly $T_{n+1} \otimes I = T_{n+2}$, the last equality gives, when tensored with $I$ on the right:

$$R_{n+1}^2 \otimes I = I_{n+2} + q(I \otimes R_n \otimes I)T_{n+2}(I \otimes R_n \otimes I).$$  \hspace{1cm} (5.6)

Thus, if in $R_{n+2}^2 - (R_{n+1} \otimes I)$ we replace $R_{n+2}^2$ using the analogue of (5.5) (for $n+2$) and $R_{n+1}^2 \otimes I$ using (5.6), we obtain, by taking norms:

$$||R_{n+2}^2 - (R_{n+1}^2 \otimes I)|| = $$
\[= |q| \||I \otimes R_{n+1})T_{n+2}(I \otimes R_{n+1} - I \otimes R_n \otimes I) + (I \otimes R_{n+1} - I \otimes R_n \otimes I)T_{n+2}(I \otimes R_n \otimes I)||
\leq |q|(|R_n| + ||R_{n+1}||)||R_{n+1} - (R_n \otimes I)||.
\]

On the other hand, \(R_{n+2}^2 \geq \alpha_{n+2}(q)I_{n+2}, R_{n+1}^2 \otimes I \geq \alpha_{n+1}(q)I_{n+2}, \) hence Lemma 8 of [7] gives that
\[||R_{n+2} - (R_{n+1} \otimes I)|| \leq \frac{1}{2\sqrt{\min(\alpha_{n+1}(q), \alpha_{n+2}(q))}} ||R_{n+2}^2 - (R_{n+1}^2 \otimes I)||.
\]
Combining this with the bound obtained for \(||R_{n+2}^2 - (R_{n+1}^2 \otimes I)|| \) (in which \(||R_n||, ||R_{n+1}|| \) are majorized, by Lemma 4.1, with \(1/\sqrt{1 - |q|} \)), we get (5.4). QED

5.2 Proposition Let \(q\) be in \((-1,1)\), and let \(R = \bigoplus_{n=0}^{\infty} R_n\) and \((\alpha_n(q))_{n=1}^{\infty}\) be as in Lemma 5.1. If \(\liminf_{n \to \infty} \alpha_n(q) > q^2/(1 - |q|)\), then \(R \in \mathcal{R}^0\) (and hence \(\mathcal{R}^q\) is isomorphic to the extension by the compacts of the Cuntz algebra).

Proof Consider the sequence \((X_n)_{n=1}^{\infty}\) defined in (5.1). From Lemma 5.1 and the ratio test it follows that \(\sum_{n=1}^{\infty} ||X_{n+1} - X_n|| < \infty\), hence this sequence is norm convergent. Each \(X_n\) is in \(\mathcal{R}^0\) (as it is clear by using (5.3) and an induction argument), hence the limit is in \(\mathcal{R}^0\), too. But the limit can only be \(R\) (indeed, it is obvious from (5.1) that \(X_n\) converges to \(R\) in the strong operator topology). QED

5.3 Corollary If \(q\) satisfies (1.7), then \(\mathcal{R}^q\) is isomorphic to the extension by the compacts of the Cuntz algebra.

Proof Since, by Lemma 4.1,
\[\liminf_{n \to \infty} \alpha_n(q) \geq \frac{1}{1 - |q|} \prod_{k=1}^{\infty} \frac{1 - |q|^k}{1 + |q|^k} = \frac{1}{1 - |q|} \sum_{k=-\infty}^{\infty} (-1)^k |q|^k,\]
the last proposition can be applied to every \(q\) satisfying (1.7). QED

5.4 Remark Truncating the series on the right-hand side of (1.7) to its first two terms leads to the bound \(|q| < \sqrt{2} - 1\) found in [7]. The only positive root of the equation
\[ q^2 = 1 - 2q + 2q^4 - 2q^9 \text{ is } 0.44005651..., \text{ hence taking four terms of the series makes us sure that (1.7) is fulfilled for } |q| \leq 0.44; \text{ the improvement obtained by considering further terms of the series is only in the sixth significant figure of the numerical value of the bound. Of course, further improvements can be obtained by giving better estimates for } \lim_{n \to \infty} \alpha_n(q). \text{ Computer aided calculations of this quantity indicate that the hypothesis of Proposition 5.2 is still fulfilled for } |q|=0.455; \text{ however, it appears that a new idea is certainly needed in order to reach, say, } 0.47. \]
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