Quantifying Three-Party Optical Coherence: Definition, Constraints, and Experimental Confirmation

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Abstract: We introduce a quantification of three-party coherence of a classical paraxial optical beam. The traditional basis-independent optical quantity called degree of polarization is determined to be the desired quantitative two-party coherence measure when appropriately generalized. A set of fundamental constraint relations is derived among three principal beam two-party coherences. The constraint relations can be geometrically interpreted and visualized as tetrahedra nested within a coherence cube. A novel measure of three-party coherence is defined based on the constraints. We are reporting completed experimental tests and confirmations of the constraints as well as measurement of three-party coherence. The close relation between the multi-vector-space tensorial natures of a paraxial light beam and analogous multiparty quantum states suggests that our results can also apply to the quantification of multiparty quantum coherences. Our analysis opens a new way to study multiparty coherence in classical optics or quantum wave physics.

Introduction: It is sometimes remarked that one of the features showing the departure of quantum mechanics from the classical world is the coherent superposition of states \[1\], but this overlooks the essential role that state superposition plays in every linear classical field theory, with electromagnetism as a chief example. A classical linear field theory is restricted by the same principles of linear vector space algebra as is quantum theory. Moreover, coherence itself is a concept that arose in connection with classical wave theory, principally in optics from the times of Huygens, Young and Fresnel (e.g., see \[2,3\]). However, the well-known historical development and understanding of classical optics did not produce a unique meaning for quantification of coherence - one can associate different kinds of coherence with differently named interferometers, for example.

In the quantum realm the issue of coherence quantification seems to have been entirely overlooked at first. However, proposals for quantification of quantum mechanical coherence have very recently been made (see \[4,5\]). Attention to details associated with more than two-party coherence is still developing, and experimental records of multi-party coherence quantification have been lacking in both classical and quantum contexts. In the following discussion we will address coherence quantification and basis dependence, and present an experimental demonstration, all in the context of three-party optical coherence. Because of the commonality of linear tensor space structures, our definitions and quantifications will apply to polarizable wave systems generally, whether classical or quantum.

Accompanying the recent search for coherence measures, two key issues remain largely open: (a) characterization(s) of multiparty coherence and (b) specification(s) of constraints on multiparty coherences. In the general case quantum states are multi-vector-space tensors, a feature widely shared among classical wave fields (see examples in optics: \[10,11\]), which emphasizes the need for analyses of multiple and multiparty coherence. Preliminary characterizations have been proposed based on entropic considerations in terms of sub-additivity \[12\] and in analysis of distributed monogamy and polygamy relations \[13\]. Here we approach the issue of multiparty coherence from a purely classical optical perspective associated with a specific polarization coherence, and do not enter the topic of higher-order quantum polarization properties, comprehensively treated by Söderholm, et al. \[14\].

We will adopt an attitude toward coherence itself, in the following sense. Recent work on the quantum side has been approached with a “resource attitude”, which is natural in regard to potential applications. A basis dependence has been recognized as applying to the quantifications adopted with that attitude. Here we begin with an “intrinsic content” point of view, the traditional view that coherence refers to an intrinsic property of a field or state. This view implies that coherence refers to something about properties that are inherent, e.g., the ability to exhibit interference \[15\], independent of a basis chosen by an external party to undertake observation. Measurement of interference, in a particular basis, is only a way to exploit some or all of the available coherence. The fact that a coherence measurement is basis dependent suggests a limitation of the observer, not a limitation on intrinsic coherence. We note that a long-known optical measure, degree of polarization \[2\], is already a suitable basis-independent measure of vector coherence for consideration under this point of view.
While noting our optical orientation, we want to emphasize that the term “polarization” is appropriate in more than one context in physics, but it always indicates a relationship between two independent attributes of a physical system (as highlighted in [16]). Historically in optics “degree of polarization” has measured the concentration or the “alignment” of one attribute against the other, and the term completely polarized means that the optical field exhibits a perfect alignment. This is in the sense that the entire optical amplitude aligns with a single vector direction, say \( \hat{u} \), in experimental lab space. It is the same as the validity of the expression

\[
E_0(r_\perp, t) = \hat{u} E_u(r_\perp, t)
\]  

(1)

for the transverse field vector. That is, one needs the discrete two-dimensional vector space \{\( \hat{x}, \hat{y} \)\} for \( \hat{u} \), and continuous linear \{\( t_2 \)\} function spaces for \( r_\perp \) and \( t \), for the total field amplitude \( E_u(r_\perp, t) \), and they are thus factorable, tensor-separable, as shown in the field state \( E(r_\perp, t) \) in [1].

We will equate a completely polarization-coherent field with complete tensor separability in this way. Moreover, going forward we will compute degree of separability \( S \) in place of degree of polarization because separability is more general. In the following Section we will see that paraxial beams, with their several vector-space degrees of freedom, enjoy several distinct coherence quantifications in terms of separability \( S \), including the quantification suggested by the traditional degree of polarization.

Multiple Coherences: Initially optical coherence studies focused on correlations within a single degree of freedom (e.g., temporal coherence or spatial coherence - see [17]), but usually a physical system is characterized by more than just one attribute (degree of freedom), so two of them must be selected to allow the systematic quantification of separability \( S \). Any two can be selected, so for a given multi-attribute system the quantification \( S \) will typically have more than one numerical value. The nature of the definition ensures that one has \( 1 \geq S \geq 0 \) for all of them. Developments promoted by applications of classical optical entanglement [18, 29] have called attention to the desirability of exploring coherence across multiple degrees of freedom [30], and so-called “hidden coherences” were exposed [31].

To enter questions about several-party degrees of separability and their relationships to each other we recall that vector-space features associated with degrees of freedom are common among quantum states, and we adopt here the familiar simplification of Dirac notation. Thus we now re-write (1) in that way, to obtain a multi-component optical field for a paraxial [32] optical beam characterized by three degrees of freedom:

\[
|E\rangle = E_0 [\alpha |G_x\rangle \otimes |F_x\rangle \otimes |x\rangle + \beta |G_y\rangle \otimes |F_y\rangle \otimes |y\rangle],
\]

(2)

and we will mostly omit tensor product symbols hereafter. Here we mean the bracketed factor to be unit-normalized so again \( E_0^2 \) stands for the total intensity of the light beam, and \( \alpha \) and \( \beta \) are the coefficients controlling the intensities of the separate \( |x\rangle \) and \( |y\rangle \) components: \( |\alpha|^2 + |\beta|^2 = 1 \). The symbols \( G_\mu \) and \( F_\nu \), with \( \mu, \nu = x \) or \( y \), represent dependence on the transverse coordinate \( r_\perp \) and time \( t \), respectively, consistent with paraxial beam structure.

We introduce each of the bras and kets as unit normalized: \( \langle x|x\rangle = 1 \) and \( \langle x|y\rangle = 0 \), etc., and \( \langle G_\mu|G_\nu\rangle = \int G_\mu^* G_\nu \, dr_\perp = 1 \) and for the time dependence a statistical average \( \langle F_\mu|F_\nu\rangle = \langle F_\mu^* (t) F_\nu (t) \rangle = 1 \). In this way \( G_\mu (r_\perp) \) and \( F_\mu (t) \) are unit-normalized functions. An important point is that the components of those functions are not generally orthogonal, so we allow them arbitrarily assignable overlaps: \( \langle G_x|G_y\rangle = \delta \) and \( \langle F_x|F_y\rangle = \gamma \), with both \( |\delta| \leq 1 \) and \( |\gamma| \leq 1 \) as guaranteed by the Schwarz inequality.

We will now denote each of the three vector spaces with a letter label for convenience: the traditional polarization vector components \( \hat{x}, \hat{y} \) form one space we label as \( a \), the spatial functions \( G_x, G_y \) occupy another one labelled \( b \), and the temporal functions \( F_x, F_y \) occupy the third space labelled \( c \).

We will employ the outer product \( |E\rangle \langle E| \) to obtain \( \mathcal{W} \), the classical field’s equivalent of a density matrix. We divide by \( |E_0|^2 \) to obtain the unit normalized form:

\[
\mathcal{W} = \frac{|E\rangle \langle E|}{|E_0|^2}.
\]

(3)

Then tracing over the \( b \) and \( c \) spaces leads to a \( 2 \times 2 \) unit-normalized reduced density matrix in the \( a \) space:

\[
\mathcal{W}_a = \begin{bmatrix}
W_{xx} & W_{xy} \\
W_{yx} & W_{yy}
\end{bmatrix} = \begin{bmatrix}
|\alpha|^2 & \alpha \beta^* \delta \gamma \\
\alpha^* \beta \delta^* \gamma^* & |\beta|^2
\end{bmatrix}.
\]

(4)

This is what is traditionally called the optical polarization coherence matrix, which allows the long-known expression [33] for the field’s “degree of polarization” to be written immediately for space \( a \)’s separability:

\[
S_a = \sqrt{1 - \frac{4Det \mathcal{W}_a}{(Tr \mathcal{W}_a)^2}}.
\]

(5)

Independence of the particular basis \( \{x, y\} \) chosen to use for writing \( |E\rangle \) is assured because of the basis independence of determinant and trace.

An equivalent approach defines \( S \) in terms of the eigenvalues \( \{\lambda_1, \lambda_2\} \) of the \( \mathcal{W} \) matrix:

\[
S_a = \left| \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2} \right|.
\]

(6)

In either case the degree of polarization coherence for the \( a \) space is obtained as

\[
S_a = \sqrt{1 - 4|\alpha|^2 |\beta|^2 [1 - |\delta|^2]}
\]

(7)

As noted, separability is a two-vector-space property and such an optical beam [2] occupying three vector
spaces can always be treated as a two-space structure by merging any two of the three spaces into one larger space. In this case the appearance of the product $\delta_\gamma$ in (7) shows that the $b$ and $c$ spaces that were traced out were treated together in this way.

One can obtain all three generic separability coherences (see related observations in [31]) in the three-space structure of $|E\rangle$ and $W$. However, to do this systematically for vector space $b$ we must first specify two orthogonal unit vectors in the $b$ space similar to $\{|x\rangle, \{y\rangle\}$ in $a$ space. We can accommodate this by introducing a new unit vector $|G_x\rangle$, explicitly defined to be orthogonal to $|G_y\rangle$, allowing $|G_y\rangle$ to remain unit-normalized when written

$$|G_y\rangle = \delta|G_x\rangle + \sqrt{1-|\delta|^2}|\tilde{G}_x\rangle,$$

(8)

so that $\langle G_x|G_y\rangle = \delta$, as specified above [31], and $|\tilde{G}_x\rangle$ is unit-normalized and orthogonal to $|G_x\rangle$, i.e., $\langle G_x|G_x\rangle = 0$. Then the paraxial beam (2) can be rearranged as

$$|E\rangle_{0} = [\alpha|x\rangle|F_x\rangle + \beta\delta|y\rangle|F_y\rangle]|G_x\rangle$$

$$+ \left[\beta\sqrt{1-|\delta|^2}|y\rangle|F_y\rangle\right]|\tilde{G}_x\rangle.$$ (9)

The entire continuous-variable transverse coordinate’s vector space has been mapped onto a two-dimensional space spanned by the orthogonal basis $\{|G_x\rangle, |\tilde{G}_x\rangle\}$. Now one can conveniently trace over the $a$ and $c$ spaces to obtain $W_b$, the coherence matrix for coherence space $b$ analogous to $W_a$ in (3) in the form:

$$W_b = \left[\begin{array}{cc} \alpha^2 + |\beta\delta|^2 & |\beta\delta|^2\sqrt{1-|\delta|^2} \\ |\beta\delta|^2\sqrt{1-|\delta|^2} & |\beta\delta|^2(1-|\delta|^2) \end{array}\right].$$ (10)

Then $S_b$ separability provides the following quantification of coherence in the transverse $b$ space:

$$S_b = \sqrt{1 - 4|\alpha\beta|^2(1 - |\delta|^2)}.$$ (11)

Similarly, the component states $|F_x\rangle$ and $|F_y\rangle$ for the temporal function $F(t)$ are in general not orthogonal, but an orthogonal basis state can be found in the same way, so $|F_y\rangle$ can be re-expressed as

$$|F_y\rangle = \gamma|F_x\rangle + \sqrt{1-|\gamma|^2}|\tilde{F}_x\rangle,$$

(12)

so that both $\langle F_y|F_y\rangle = 1$ and $\langle F_x|F_y\rangle = \gamma$ are satisfied. Consequently, the corresponding coherence matrix and degree of coherence in $c$ space are obtained as

$$W_c = \left[\begin{array}{cc} \alpha^2 + |\gamma|^2 & |\gamma|^2\sqrt{1-|\gamma|^2} \\ |\gamma|^2\sqrt{1-|\gamma|^2} & |\gamma|^2(1-|\gamma|^2) \end{array}\right],$$

(13)

and

$$S_c = \sqrt{1 - 4|\alpha\beta|^2(1 - |\gamma|^2)}.$$ (14)

This completes the derivation of the three coherences $S_a$, $S_b$, $S_c$ for a typical paraxial beam comprising three degrees of freedom and occupying three vector spaces. The coherences are distinct but not independent, as we show next.

**Multi-party coherence restrictions** Each of the three coherence measures is bounded in the same way: $0 \leq S_a, S_b, S_c \leq 1$. The sum of any two coherences $S_{\text{two}}$ is thus bounded by 0 and 2, and the sum of all three lies between 0 and 3. One might expect that the difference between any two-coherence sum $S_{\text{two}}$ and the remaining coherence should be bounded by 2, but the generic structure of the light beam (2) forces this restriction to become much tighter:

$$|S_a + S_b - S_c| \leq 1,$$

(15)

quantifying the restrictions existing among the three coherences. Here $a, b, c$ can switch order freely. This set of symmetric inequalities is our first main result. It simply says that the sum of any two separabilities, upon subtracting the third, is always less than unity. A related set of inequalities for multiparty entanglement can be found in Refs. [34, 35]. The detailed proof of this set of coherence inequalities is given in the Appendix.

**FIG. 1** Coherence tetrahedra. The allowed region restricted by (15) is represented by the two tetrahedra $OABC$ and $MABC$. Point $O$ corresponds to the origin with no coherence at all, i.e., $(S_a, S_b, S_c) = (0, 0, 0)$, and point $M$ represents maximum coherence with $(S_a, S_b, S_c) = (1, 1, 1)$.
Three party coherence, which is indicated by the measure $C_{abc} = C_{b \rightarrow ca} = C_{c \rightarrow ab} = C_{a \rightarrow bc} = 1 + S_a - S_b - S_c = 0$.

(2) $S_a = S_b = S_c = 0$, which means all three degrees of freedom are incoherent (non-separable). Although incoherent individually, this case brings maximal mutual dependence among the three DoFs. That is, three incoherent DoFs can behave coherently as a whole. Therefore, there should be maximum three-party coherence, and this is also directly shown by the measure $C_{abc}$ taking its maximum value 1, i.e., $C_{abc} = C_{b \rightarrow ca} = C_{c \rightarrow ab} = C_{a \rightarrow bc} = 1 + S_a - S_b - S_c = 1$.

(3) $S_a = S_b$, $S_c = 1$, which means DoF $c$ is fully coherent and the remaining two are equally partially coherent. When $S_c = 1$, the $c$ DoF is completely independent of the remaining two, indicating zero-biased residual coherences. Therefore, there should be zero genuine three-party coherence, as indicated by the measure $C_{abc} = C_{b \rightarrow ca} = C_{a \rightarrow bc} = 1 + S_a - S_b - S_c = 0$.

These three extreme cases consistently confirm that the three-party coherence $C_{abc}$ characterizes the coherence property of all three DoFs as an interdependent unit. Therefore it can be viewed as the coherence of the electromagnetic field as a whole.

**Experimental confirmation:** Now we describe an experimental test of the coherence restriction inequalities and a measurement of the three-party optical coherence. To accomplish these two tasks, it is first needed to produce an optical beam with the three-DoF structure described in Eq. (3) and then measure all three separability coherences with respect to each optical DoF.

The general field in Eq. (3) is provided by a laser beam with two polarization components, two spatial modes and two path modes. Specifically, the two polarization components in spatial mode $a$ are still described as $|x\rangle$, $|y\rangle$. The spatial components $|G_x\rangle$, $|G_y\rangle$ in spatial $b$ are represented by superpositions of two first-order Hermite-Gaussian (HG) modes $|G_{10}\rangle \equiv HG_{10}$, $|G_{01}\rangle \equiv HG_{01}$, i.e., $|G_x\rangle = c_1|G_{10}\rangle + c_2|G_{01}\rangle$ and $|G_y\rangle = c'_1|G_{10}\rangle + c'_2|G_{01}\rangle$, where the coefficients $c_1$, $c_2$ are independent of $c'_1$, $c'_2$. For simplicity and without loss of generality, the temporal components $|F_x\rangle$, $|F_y\rangle$ in spatial $c$ are replaced by the superpositions of two path modes $|0\rangle$, $|1\rangle$, i.e., $|F_x\rangle = d_0|0\rangle + d_1|1\rangle$ and $|F_y\rangle = d'_0|0\rangle + d'_1|1\rangle$ with coefficients $d_0$, $d_1$ independent of $d'_0$, $d'_1$. It is emphasized that such a replacement doesn’t change the vector structure of the general three-DoF form described in Eq. (3).

A 795 nm laser is directed to a spatial light modulator (SLM) to create a beam with an arbitrarily oriented first-order HG mode $|G_0\rangle = \cos \theta |G_{10}\rangle + \sin \theta |G_{01}\rangle$ with vertical polarization $|y\rangle$. Then it enters the preparation stage, as shown in Fig. 2(a), to produce an arbitrary three-DoF structured beam. It first passes a half-wave plate (HWP) to change the polarization to an arbitrary orientation, i.e., $|\phi\rangle = \cos \phi |x\rangle + \sin \phi |y\rangle$. Then the beam is split into two by a polarizing beamsplitter (PBS) where the transmission-reflection ratio depends on the rotation angle $\phi$ of the HWP in front of it. The two arms of
Similarly, the output beam of the SOC in path 0 represents the two paths are parallel without overlap. When no superposition is needed, the two paths are combined with a shift - no collimation as shown in Fig. 2(a) - so that the two outputs are parallel without overlap.

The transmitted component travels in path 0 to be directed by a 50/50 beamsplitter (BS1) to a movable mirror for an appropriate phase adjustment $\Delta$. Then it enters a spin-orbit controller (SOC) to combine with path 1 at BS2. The reflected component from the PBS travels in path 1 to enter another SOC before entering BS2.

An SOC is a modified Mach-Zehnder interferometer (see Fig. 2(b)) that manipulates the spatial modes of the beam depending on spin polarizations. For example, in path 1, the incoming signal $|0\rangle y |G_0\rangle$ enters a HWP to become $|1\rangle (\cos \phi_1 |x\rangle + \sin \phi_1 |y\rangle) |G_0\rangle$ with an arbitrary angle $\phi_1$. Then it enters the modified Mach-Zehnder interferometer, composed of two PBSs and a Dove Prism (DP). The spatial mode of the $|x\rangle$ polarization remains unchanged while that of the $|y\rangle$ component changes into an arbitrarily oriented mode controlled by the DP via a rotation angle $\theta_1$, i.e., $|G_{01}\rangle = \cos \theta_1 |G_{10}\rangle + \sin \theta_1 |G_{01}\rangle$. So in path 1, the output beam of the SOC can be described as

$$|e_1\rangle = |1\rangle (\cos \phi_1 |x\rangle |G_\theta\rangle + \sin \phi_1 |y\rangle |G_{\theta_1}\rangle).$$  \hspace{1cm} (21)

Similarly, the output beam of the SOC in path 0 can be described as

$$|e_0\rangle = |0\rangle (\cos \phi_0 |x\rangle |G_\theta\rangle + \sin \phi_0 |y\rangle |G_{\theta_0}\rangle),$$ \hspace{1cm} (22)

where the corresponding HWP in path 0 has produced the polarization $|e_0\rangle = \cos \phi_0 |x\rangle + \sin \phi_0 |y\rangle$.

As a result, the prepared beam at the output of BS2, see Fig. 2(a), is in the form

$$\frac{|E\rangle}{\sqrt{N}} = \sin \phi |e_1\rangle + \frac{\cos \phi}{2} e^{i\Delta} |e_0\rangle,$$ \hspace{1cm} (23)

where $N$ is the normalization factor, which is the intensity $I$. Since the rotation parameters $\theta, \theta_0, \phi, \phi_0, \phi_1$, and the relative phase factor $\Delta$ are independent of each other, the prepared beam (23) is a general representation of the three-DoF structure (2).

To obtain the separability coherences $S_a, S_b, S_c$, tomographic setups are employed to measure the corresponding coherence matrices $A_1, A_2, A_3$ through Stokes-like parameters [37]. The Stokes basis projection for spin polarization is realized by the standard combination of (half- and quarter-) wave plates and a polarizing beamsplitter as shown in Fig. 2(c). The projection in the spatial-mode degrees of freedom is realized by the combination of a pair of cylindrical lenses, a Dove prism and a modified Mach-Zehnder interferometer - see illustration in Fig. 2(e) and detailed description in Ref. [31].

The projection in Stokes-like basis in the path degrees of freedom is realized by BS2 together with a translation stage in path 0. In this case, the $|0\rangle$ or $|1\rangle$ basis is analyzed by simply blocking either one of the two paths. When the two paths are perfectly collimated at BS2, the two outputs are effectively in the basis $|0\rangle \pm i|1\rangle$, as shown in Fig. 2(d). The $|0\rangle \pm |1\rangle$ basis is realized by a $\Delta = \pi/2$ phase delay in path 0.

To cover all interesting properties of the coherence re-

![FIG. 2: Experimental setup.](image-url)
striction inequalities [15], six specific representative light fields [23] are chosen for the measurements of all three coherences $S_a$, $S_b$, $S_c$ and the three-party coherence $C_{abc}$. These optical fields correspond to states with interesting entanglement properties as well. The first case we prepare is a completely separable situation of all three DoFs, i.e.,

$$|E_1\rangle = |y\rangle \otimes |G_{01}\rangle \otimes |1\rangle.$$  \hfill (24)

This beam corresponds to complete separability, i.e., full separability coherence, of each DoF with $S_a = S_b = S_c = 1$. To generate such a beam, the SLM is set to produce a $|G_{10}\rangle$ (cos $\theta = 1$) mode with vertical polarization $|y\rangle$. The HWPs in Fig. 2 (a) is set to leave the polarization state unchanged (cos $\phi = 0$). Then, it is completely reflected by the PBS to path $|1\rangle$ to enter the SOC, whose HWP is also set to keep the polarization state (i.e., cos $\phi = 0$). The reflected beam by PBS1 passes through a DP to change into vertically oriented first order Hermite-Gauss mode $G_{01}$ (cos $\theta_1 = 0$). Then the output beam after BS2 is exactly characterized $|E_1\rangle$ in (24).

The second beam we produce is a GHZ-type of entangled state with each individual DoF maximally dependent on the remaining two, i.e.,

$$|E_2\rangle = |G_{01}\rangle \otimes |y\rangle \otimes |1\rangle + |G_{10}\rangle \otimes |x\rangle \otimes |0\rangle.$$  \hfill (25)

It corresponds to the case when all DoFs are completely inseparable $S_a = S_b = S_c = 0$. It is realized with an incoming beam of $|G_{10}\rangle$ (cos $\theta = 0$) mode and vertical polarization $|y\rangle$, by setting the rotation parameters of three HWPs to be cos $\phi = \sqrt{\frac{1}{5}}$, cos $\phi_1 = 0$, cos $\phi_0 = 1$ respectively, and the rotation parameter the DP in path $|1\rangle$ to be cos $\theta_1 = 0$.

The third beam to test is a W-type state, i.e.,

$$|E_3\rangle = |G_{01}\rangle \otimes |y\rangle \otimes |0\rangle + |G_{01}\rangle \otimes |x\rangle \otimes |1\rangle + |G_{10}\rangle \otimes |y\rangle \otimes |1\rangle.$$  \hfill (26)

In this beam, all three DoFs are equally partially coherent. To generate $|E_3\rangle$, the incoming beam is produced with $|G_{01}\rangle$ (cos $\pi/2$) mode and vertical polarization $|y\rangle$. The three HWPs are set to be cos $\phi = \sqrt{\frac{1}{6}}$, cos $\phi_1 = \sqrt{\frac{1}{2}}$, cos $\phi_0 = 0$, and the two DPs are set to be cos $\theta_1 = 1$, cos $\theta_0 = 0$.

We have also tested three more representative beams with two of the three DoFs in a Bell-type state and the third DoF completely separable with the first two, i.e.,

$$|E_4\rangle = |1\rangle \otimes (|G_{01}\rangle \otimes |x\rangle + |G_{10}\rangle \otimes |y\rangle),$$  \hfill (27)

$$|E_5\rangle = |G_{01}\rangle \otimes (|y\rangle \otimes |0\rangle + |x\rangle \otimes |1\rangle),$$  \hfill (28)

$$|E_6\rangle = |y\rangle \otimes (|G_{01}\rangle \otimes |0\rangle + |G_{10}\rangle \otimes |1\rangle).$$  \hfill (29)

For these three beams, one of the three DoFs is completely coherent and the remaining two DoFs are completely incoherent. The three beams can be prepared with the same initial incoming beam with $|G_{01}\rangle$ (cos $\theta = \pi/2$) mode and vertical polarization $|y\rangle$. For state $|E_4\rangle$, the HWPs are set as cos $\phi = 0$, cos $\phi_1 = \sqrt{\frac{1}{2}}$, and the DPs are set to be cos $\theta_1 = 1$, cos $\theta_0 = any$. For state $|E_5\rangle$, the HWPs are set as cos $\phi = \sqrt{\frac{1}{5}}$, cos $\phi_1 = 1$, cos $\phi_0 = 0$, and the DPs are set to be cos $\theta_1 = any$, cos $\theta_0 = 0$. For state $|E_6\rangle$, the HWPs are set as cos $\phi = \sqrt{\frac{1}{5}}$, cos $\phi_1 = 0$, cos $\phi_0 = 0$, and the DPs are set as cos $\theta_1 = 1$, cos $\theta_0 = 0$.

The measured three separability coherences $S_a, S_b, S_c$ of the six prepared light beams are illustrated in Fig. 3 by the red dots. The three coordinates of each red dot represent the values $S_a, S_b, S_c$ respectively for a corresponding beam. Here the light beams $|E_{1,2,3,4,5,6}\rangle$ correspond to the red dots $M, O, W, A, B, C$ respectively.

**TABLE I**: Measured values of coherences $S_a, S_b, S_c$, their restriction relation, and three-party coherence for six representative light fields.

| Beams | $S_a$ | $S_b$ | $S_c$ | $S_a + S_b - S_c$ | $S_a + S_c - S_b$ | $S_b + S_c - S_a$ | $C_{abc}$ | Points | States |
|-------|-------|-------|-------|---------------------|---------------------|---------------------|----------|--------|--------|
| $|E_1\rangle$ | 0.993 $\pm$ 0.015 | 0.988 $\pm$ 0.021 | 0.995 $\pm$ 0.014 | 0.990 | 0.999 | 0.987 | 0.001 | M | Separable |
| $|E_2\rangle$ | 0.015 $\pm$ 0.016 | 0.017 $\pm$ 0.028 | 0.013 $\pm$ 0.019 | 0.014 | 0.011 | 0.019 | 0.981 | O | GHZ-type |
| $|E_3\rangle$ | 0.348 $\pm$ 0.022 | 0.319 $\pm$ 0.033 | 0.340 $\pm$ 0.020 | 0.311 | 0.368 | 0.327 | 0.632 | W | W-type |
| $|E_4\rangle$ | 0.015 $\pm$ 0.009 | 0.024 $\pm$ 0.021 | 0.991 $\pm$ 0.012 | 1.000 | 0.982 | -0.952 | 0.000 | A | Bell-type |
| $|E_5\rangle$ | 0.017 $\pm$ 0.024 | 0.983 $\pm$ 0.049 | 0.014 $\pm$ 0.023 | 0.980 | -0.953 | 0.986 | 0.014 | B | Bell-type |
| $|E_6\rangle$ | 0.989 $\pm$ 0.032 | 0.026 $\pm$ 0.052 | 0.019 $\pm$ 0.025 | -0.944 | 0.983 | 0.996 | 0.004 | C | Bell-type |

**FIG. 3**: Geometric representation of measured coherence values for all six prepared beams by the corresponding red dots. The three coordinates of each red dot represent the coherence values ($S_a, S_b, S_c$) respectively.
Specific values of each set of coherences \((S_a, S_b, S_c)\) are listed in detail in Table I for the six corresponding beams. The genuine three-party coherences \(C_{abc}\) are also obtained for the first time for each of the six beams. Apparently, the measured results confirm the generic coherence restriction relation \(|S_a \pm S_b - S_c| \leq 1\).

**Summary and Discussion:** We have approached coherence from the view of intrinsic content. By adopting a transverse basis-independence measure of coherence (by generalizing the degree of ordinary transverse polarization coherence), we have identified a set of symmetric three-coherence constraint relations that can be geometrically viewed as tetrahedral inequalities. These inequalities embody a fundamental multi-coherence law for paraxial light beams. Our analysis thus extends the traditional discussion of optical coherence into a systematic regime of multiple coherences.

The inequalities lead to expression (19) for \(C_{abc}\), which we believe to be the first non-entropic quantification of three-party coherence. It identifies the amount of coherence in terms of the separability measures associated with the optical beam as an interdependent whole system.

The first confirmation of the tetrahedral inequalities was demonstrated experimentally with three vector space degrees of freedom of a paraxial laser beam: ordinary transverse polarization, spatial mode and independent path selection. The observation of genuine three-party coherence was also achieved and shown to be consistent with the constraints we derived. The tomographic measurement in the path DoF represents a new extension of Stokes-like parameter measurements in optics.

It is important to note that the theoretical analysis relies only on the mathematical linear vector-space structure of a wave field regardless of its physical content. Thus these considerations can be directly extended from optical fields to quantum wave fields. In particular, our results apply straightforwardly to multi-qubit systems. It is expected that the coherence constraint inequalities and the associated multiparty coherence definition can be useful in new guidance for the sharing and distribution of coherence in various quantum information and communication tasks.

**Appendix - Proof of Separability Coherence Restriction:** Here we prove the tetrahedra coherence restriction relation (19) via the eigenvalues of each of the normalized coherence matrices \(\mathcal{W}_a, \mathcal{W}_b, \mathcal{W}_c\). By definition, the three corresponding separability coherences are given as

\[
S_a = \lambda_1^{(a)} - \lambda_2^{(a)},
\]

\[
S_b = \lambda_1^{(b)} - \lambda_2^{(b)},
\]

\[
S_c = \lambda_1^{(c)} - \lambda_2^{(c)},
\]

where we have assumed \(\lambda_1^{(\mu)} \geq \lambda_2^{(\mu)}, \mu = a, b, c\) without loss of generality, and have used the normalization condition \(\lambda_1^{(\mu)} + \lambda_2^{(\mu)} = 1\).

In the following we describe the proof of the inequality

\[
1 + S_a \geq S_b + S_c,
\]

and the remaining two can be proved similarly.

The eigenvectors of each of the coherence matrices \(\mathcal{W}_a, \mathcal{W}_b, \mathcal{W}_c\) can be defined correspondingly and denoted as \(|\phi_1^{(a)}\rangle, |\phi_2^{(a)}\rangle, |\phi_1^{(b)}\rangle, |\phi_2^{(b)}\rangle, |\phi_1^{(c)}\rangle, |\phi_2^{(c)}\rangle\). Then one can rewrite the unit-normalized paraxial field as

\[
|E\rangle = \sqrt{\lambda_1^{(a)}}|\phi_1^{(a)}\rangle|x_1|\phi_1^{(b)}\rangle|\phi_1^{(c)}\rangle + x_2|\phi_1^{(b)}\rangle|\phi_2^{(c)}\rangle
\]

\[
+ x_3|\phi_2^{(b)}\rangle|\phi_1^{(c)}\rangle + x_4|\phi_2^{(b)}\rangle|\phi_2^{(c)}\rangle,
\]

\[
+ \sqrt{\lambda_2^{(a)}}|\phi_2^{(a)}\rangle|y_1|\phi_1^{(b)}\rangle|\phi_1^{(c)}\rangle + y_2|\phi_2^{(b)}\rangle|\phi_2^{(c)}\rangle
\]

\[
+ y_3|\phi_1^{(b)}\rangle|\phi_1^{(c)}\rangle + y_4|\phi_2^{(b)}\rangle|\phi_2^{(c)}\rangle,
\]

with the orthonormality conditions given as

\[
\sum_{j=1}^{4} |x_j|^2 = \sum_{j=1}^{4} |y_j|^2 = 1, \quad (35a)
\]

\[
\sum_{j=1}^{4} x_j y_j^* = 0. \quad (35b)
\]

When factoring out the states \(|\phi_1^{(b)}\rangle\) and \(|\phi_2^{(b)}\rangle\) of the field \(|33\), it is straightforward to obtain eigenvalues for \(\mathcal{W}_b\) in vector space \(b\), i.e.,

\[
\lambda_1^{(b)} = \sum_{j=1,2} (\lambda_1^{(a)}|x_j|^2 + \lambda_2^{(a)}|y_j|^2), \quad (36)
\]

\[
\lambda_2^{(b)} = \sum_{j=3,4} (\lambda_1^{(a)}|x_j|^2 + \lambda_2^{(a)}|y_j|^2). \quad (37)
\]

Similarly, the eigenvalues of \(\mathcal{W}_c\) are given as

\[
\lambda_1^{(c)} = \sum_{j=1,3} (\lambda_1^{(a)}|x_j|^2 + \lambda_2^{(a)}|y_j|^2), \quad (38)
\]

\[
\lambda_2^{(c)} = \sum_{j=2,4} (\lambda_1^{(a)}|x_j|^2 + \lambda_2^{(a)}|y_j|^2). \quad (39)
\]

Then one has the sum of two separability coherences,

\[
S_b + S_c = 2 - 2\lambda_1^{(a)}(|x_2|^2 + |x_3|^2 + 2|x_4|^2)
\]

\[
- 2\lambda_2^{(a)}(|y_2|^2 + |y_3|^2 + 2|y_4|^2)
\]

\[
\leq 2 - 2\lambda_1^{(a)} \sum_{j=2}^{4} |x_j|^2 - 2\lambda_2^{(a)} \sum_{j=2}^{4} |y_j|^2
\]

\[
\leq 2 - 2\lambda_2^{(a)} \sum_{j=2}^{4} (|x_j|^2 + |y_j|^2). \quad (40)
\]

From the orthonormal conditions \(35a\) and \(35b\) it is easy to see that

\[
1 \geq |x_1|^2 + |y_1|^2, \quad (41)
\]
and when combining the two results (40) and (41), it is straightforward to obtain

\[ S_b + S_c \leq 2 - 2\lambda_2^{(a)} = 1 + S_a, \] (42)

which proves relation (33). The remaining inequalities \[16\] can be proved in exactly the same way.

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