ESCAPE QUARTERED THEOREM AND THE CONNECTIVITY
OF THE JULIA SETS OF A FAMILY OF RATIONAL MAPS

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Abstract. In this paper, we investigate the dynamics of the following family of rational maps

\[ f_\lambda(z) = \frac{z^{2^n} - \lambda^{3^n+1}}{z^{n}(z^{2n} - \lambda^{n-1})} \]

with one parameter \( \lambda \in \mathbb{C}^* - \{ \lambda : \lambda^{2n+2} = 1 \} \), where \( n \geq 2 \). This family of rational maps is viewed as a singular perturbation of the bi-critical map \( P_{2n}(z) = z^{-n} \) if \( \lambda \neq 0 \) is small. It is proved that the Julia set \( J(f_\lambda) \) is either a quasicircle, a Cantor set of circles, a Sierpiński carpet or a degenerate Sierpiński carpet provided the free critical orbits of \( f_\lambda \) are attracted by the super-attracting cycle \( 0 \leftrightarrow \infty \). Furthermore, we prove that there exists suitable \( \lambda \) such that \( J(f_\lambda) \) is a Cantor set of circles but the dynamics of \( f_\lambda \) on \( J(f_\lambda) \) is not topologically conjugate to that of any known rational maps with only one or two free critical orbits (including McMullen maps and the generalized McMullen maps). The connectivity of \( J(f_\lambda) \) is also proved if the free critical orbits are not attracted by the cycle \( 0 \leftrightarrow \infty \). Finally we give an estimate of the Hausdorff dimension of the Julia set of \( f_\lambda \) in some special cases.

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1. Introduction. Let \( f \) be a rational map with degree \( d \geq 2 \) on the Riemann sphere \( \mathbb{C} = \mathbb{C} \cup \{ \infty \} \), where \( \mathbb{C} \) is the complex plane. Let \( f^k \) be the \( k \)-th iteration of \( f \), where \( k \in \mathbb{N} \). The Julia set of \( f \), denoted by \( J(f) \), is defined as the set of points at which the family of iterates \( \{ f^k : k \in \mathbb{N} \} \) fails to be a normal family in the sense of Montel. The complement \( \mathbb{C} \setminus J(f) \) of the Julia set is the Fatou set \( f \) and is denoted by \( F(f) \). It is easy to see that \( F(f) \) is the largest totally invariant open set and \( J(f) \) is the minimal totally invariant closed set of cardinality at least 3. The Julia set is also the closure of the set of all repelling periodic points.

A connected component of Fatou set is called a Fatou component and any Fatou component is mapped onto another under \( f \). According to Sullivan’s theorem, every Fatou component of a rational map is eventually periodic and there are five kinds of periodic Fatou components: attracting domains, super-attracting domains, parabolic domains, Siegel disks and Herman rings. A point, say \( c \), is called a critical point of \( f \) if \( f \) is not univalent in any neighborhood of \( c \) and \( f(c) \) is called a critical value of \( f \).

1.1. Background and motivation. The topology of the Julia sets of rational maps, such as the connectivity and local connectivity, is an interesting and important research subject in complex dynamics. The dynamics of the unicritical polynomial \( P_n(z) = z^n \) \((n \geq 2)\) is simple. Naturally, one wishes to study the dynamics of the rational maps nearby \( P_n \) by a small perturbation on \( P_n \). McMullen is the first one who investigated the small singular perturbation on \( P_n(z) \) (see [17]) and he considered the following family of rational maps

\[
F_{\lambda}(z) = z^n + \frac{\lambda}{z^m},
\]

where \( n \geq 2, m \geq 1 \) and \( \lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\} \). This family is referred as McMullen maps later (see [5], [21] for example), which exhibits very rich dynamical behaviors and can be viewed as a singular perturbation of \( P_n(z) \) acting on \( \mathbb{C} \). In 2005, Devaney, Look and Uminsky obtained an Escape Trichotomy Theorem for \( F_{\lambda}(z) \) according to the free critical orbits of \( F_{\lambda}(z) \) [5]. They proved that the Julia set \( J(F_{\lambda}) \) is either a Cantor set, a Cantor set of circles or a Sierpiński carpet if the free critical orbits are attracted by the infinity.

A Sierpiński carpet is a planar set which is homeomorphic to the standard Sierpiński carpet fractal. A subset \( S \) of the Riemann sphere \( \overline{\mathbb{C}} \) is a Sierpiński carpet if and only if it has empty interior and can be written as \( S = \overline{\mathbb{C}} \setminus \bigcup_{k \in \mathbb{N}} U_k \), where \( U_k \subseteq \mathbb{C}, k \in \mathbb{N}, \) are pairwise disjoint Jordan disks with \( \partial U_k \cap \partial U_j = \emptyset \) for \( k \neq j \), and \( \text{diam}(U_k) \to 0 \) as \( k \to \infty \). Furthermore, we call a compact set \( S \) in \( \mathbb{C} \) degenerate Sierpiński carpet if it satisfies all the conditions of the Sierpiński carpet except it allows that the intersection of the boundaries of complementary domains can be non-empty. In addition, a subset of the Riemann sphere \( \overline{\mathbb{C}} \) is called a Cantor set of circles (or Cantor circles in short) if it consists of uncountably many closed Jordan curves which is homeomorphic to \( \mathbb{C} \times S^1 \), where \( \mathbb{C} \) is the Cantor middle third and \( S^1 \) is the unit circle.

For Cantor circle Julia sets, the first example is due to McMullen ([17]). He proved that the Julia set of the rational map \( F_\lambda(z) = z^2 + \frac{\lambda}{z^3} \) is a Cantor set of circles if \( \lambda \neq 0 \) is small enough. As a variation of \( F_\lambda(z) \), the generalized McMullen maps \( F_{\lambda, \eta}(z) = z^n + \frac{\lambda}{z^m} + \eta \) are also investigated by many people. There exists some additional dynamical phenomenon for the family \( F_{\lambda, \eta}(z) \). We refer the reader to [9], [13], [30], [12] and the references therein. It is known that the Cantor circle Julia sets and Sierpiński carpet Julia sets can appear in McMullen family and the
generalized McMullen family (see [5, 30]). For the study of singular perturbation of unicritical polynomials, one may also refer to [27], [31], [32], [15] and [16].

It is not difficult to show that there are some generalized McMullen maps $F_{\lambda, \eta}$ with their Julia sets being Cantor circles and the dynamics of $F_{\lambda, \eta}$ in a neighborhood of the Julia set is quasiconformally conjugate to that of some McMullen maps (the corresponding Julia sets are also Cantor circles). However, by quasiconformal surgery, Haïssinsky and Pilgrim [14] proved that there are some other types of rational maps whose Julia sets are Cantor circles but the dynamics on the Julia sets are not topologically conjugate to any McMullen map on their corresponding Julia sets. Later, the specific expressions of these rational maps were given in [23].

Recently Fu and Yang [9] studied the following one-dimensional family of rational maps

$$h_\lambda(z) = \frac{z^n(z^{2n} - \lambda^{n+1})}{z^{2n} - \lambda^{3n-1}}, \quad (1)$$

where $n \geq 2$ and $\lambda \in C^* - \{\lambda : \lambda^{2n-2} = 1\}$. They proved that the Julia sets of some maps in this family are Cantor circles and each of these maps has essentially only one free critical orbit. Moreover, its dynamics on the Julia set are not topologically conjugate to any McMullen map on their corresponding Julia sets.

We are interested in the problem to find a one-dimensional family of such rational maps, such that the Julia sets of some maps in this family are Cantor circles but its dynamics is neither topologically conjugate to any McMullen map nor the rational map $h_\lambda$ on their corresponding Julia sets but for each map, there exists essentially only one free critical orbit. For this, we consider the following family of rational maps

$$f_\lambda(z) = \frac{z^{2n} - \lambda^{3n+1}}{z^n(z^{2n} - \lambda^{n-1})}, \quad (2)$$

where $n \geq 2$ and $\lambda \in \Lambda := C^* - \{\lambda : \lambda^{2n+2} = 1\}$. The rational map $f_\lambda$ degenerates to the bi-critical rational map $P_{-n}(z) = z^{-n}$ if $\lambda = 0$ or $\lambda^{2n+2} = 1$. Thus the map $f_\lambda$ with $\lambda \in \Lambda$ can be viewed as a perturbation of the simple map $P_{-n}$ if $\lambda \neq 0$ lies in a small punctured neighborhood of the origin. This perturbation is essentially different from that of McMullen maps and the family $h_\lambda$ since $f_\lambda$ keeps the dynamics of $P_{-n}$ near the orbit $0 \leftrightarrow \infty$ of $f_\lambda$ with period two.

1.2. Statement of the main results. A straightforward computation shows that $f_\lambda$ has a super-attracting periodic orbit $0 \leftrightarrow \infty$. We denote by $D_0$ and $D_\infty$, respectively, the immediate attracting basins of $0$ and $\infty$. One can observe easily that $D_0$ is always different from $D_\infty$ since a Fatou component cannot contain two different periodic points. The map $f_\lambda$ has $6n - 2$ critical points (counted with multiplicity) since the degree of $f_\lambda$ is $3n$. Note that the local degrees of $0$ and $\infty$ are both $n$, and there are $4n$ critical points other than $0$ and $\infty$, which are called the free critical points (for the definition of free critical point, see Section 2). The forward orbits of $0$ and $\infty$ are trivial since they lie in the super-attracting orbit $0 \leftrightarrow \infty$. We will show in Section 2 that the remaining $4n$ critical points behave symmetrically. So we just have essentially one free critical orbit for each $f_\lambda$. The dynamics of $f_\lambda$ are determined by the forward orbit of any free critical point $c_\lambda$. Under the assumption that the free critical orbits are attracted by the cycle $0 \leftrightarrow \infty$, we obtain the following main result.

**Theorem 1.1.** Suppose that the free critical point $c_\lambda$ is attracted by $0$ or $\infty$. Then there are following four cases.
(a) If $c_{\lambda} \in D_0 \cup D_\infty$, then $J(f_{\lambda})$ is a quasicircle.
(b) If $f_{\lambda}^j(c_{\lambda}) \in D_0 \cup D_\infty$ but $c_{\lambda} \notin D_0 \cup D_\infty$, then $J(f_{\lambda})$ is a Cantor set of circles.
(c) If $f_{\lambda}^{2k}(c_{\lambda}) \in D_0 \cup D_\infty$ for $k \geq 2$ but $f_{\lambda}^j(c_{\lambda}) \notin D_0 \cup D_\infty$ for $0 \leq j < k$, and further
   (c1) If $\partial D_0 \cap \partial D_\infty = \emptyset$, then $J(f_{\lambda})$ is a Sierpiński carpet.
   (c2) If $\partial D_0 \cap \partial D_\infty \neq \emptyset$, then $J(f_{\lambda})$ is a degenerate Sierpiński carpet.

Otherwise, the free critical orbits are not attracted by the cycle $0 \leftrightarrow \infty$ and $J(f_{\lambda})$ is connected.

The four cases in Theorem 1.1 happen indeed, see Figure 1 for examples (see also Section 5). We call Theorem 1.1 the Escape Quartered Theorem. As one comparison between McMullen maps and our family, the Julia set of a McMullen map cannot be a quasicircle nor a degenerate Sierpiński carpet if the free critical orbits are attracted by the cycle $0 \leftrightarrow \infty$. As another comparison between $h_{\lambda}$ and $f_{\lambda}$, the Fatou components of $f_{\lambda}$ containing the origin and the infinity are of period two, which is different from the family $h_{\lambda}$.

For the connectivity of the Julia sets under the assumption that the free critical orbits are bounded, the case of McMullen maps has been studied in [29] and [7], and the case of $h_{\lambda}$ is considered in [10].

As a consequence of Theorem 1.1, we obtain the following result.

**Theorem 1.2.** Suppose that one of the free critical orbits of $f_{\lambda}$ is attracted by the cycle $0 \leftrightarrow \infty$. Then

(a) The boundary components of all Fatou components of $f_{\lambda}$ are quasicircles;
(b) The Hausdorff dimension of $J(f_{\lambda})$ satisfies $1 < \dim_H J(f_{\lambda}) < 2$.

For the Julia set of $f_{\lambda}$ being a Cantor set of circles, one can prove that there exists a punctured region of $\lambda = 0$ in which the Julia set of $f_{\lambda}$ is a Cantor set of circles (see Section 5). This punctured region is called the McMullen domain (see Figure 2).

**Theorem 1.3.** The family $f_{\lambda}$ has a McMullen domain if and only if $n \geq 4$.

As it was mentioned above, our primary motivation of studying the family (2) is to obtain a one-dimensional family of rational maps with Cantor circle Julia sets which are neither topologically conjugate to any McMullen map nor to the above family $h_{\lambda}$ on their corresponding Julia sets. The following result shows that $f_{\lambda}$ is the desired family.

**Theorem 1.4.** Suppose that the Julia set of $f_{\lambda}$ is a Cantor set of circles. Then

(a) $f_{\lambda}$ is neither topologically conjugate to any McMullen map nor to any map $h_{\lambda}$ on their corresponding Julia sets.
(b) The Hausdorff dimension of $J(f_{\lambda})$ satisfies

$$1 + \frac{\log 3}{\log n} \leq \dim_H J(f_{\lambda}) < 2.$$

Actually, Theorem 1.4 (b) implies Theorem 1.3 in some sense.

1.3. **Organization of the paper.** This paper is organized as follows:

In Section 2 we give some preliminary results on the rational map $f_{\lambda}$, including the analysis of two kinds of symmetries and the dynamical properties of $f_{\lambda}$. We also give the proof of Theorem 1.1 (a).
Figure 1. The Julia sets of $f_\lambda$ for different $\lambda$’s when $n = 4$. Top left: $\lambda = 0.8 + 0.3i$ and $J(f_\lambda)$ is a quasicircle; Top right: $\lambda = 0.4$ and $J(f_\lambda)$ is a Cantor set of circles; Bottom left: $\lambda = 0.7$ and $J(f_\lambda)$ is a Sierpiński carpet; Bottom right: $\lambda = 0.92 + 0.01i$ and $J(f_\lambda)$ is a degenerate Sierpiński carpet.

Section 3 is devoted to giving the proofs of Theorem 1.1 (b)(c), and Theorems 1.2-1.4 under the assumption that the critical orbits are attracted by the cycle $0 \leftrightarrow \infty$.

In Section 4 we deal with the case that all the free critical orbits are not attracted by the cycle $0 \leftrightarrow \infty$ and give the proof of the connectivity of the Julia set of $f_\lambda$, which completes the proof of Theorem 1.1.

In Section 5 we study the dynamical properties of $f_\lambda$ when $\lambda \neq 0$ is real or small. Under these settings we give specific examples such that the Julia set of $f_\lambda$ is a quasicircle or a Cantor set of circles.

2. Preliminaries and basic settings.

2.1. The symmetric property. In this paper, we will need some preliminary properties of $f_\lambda$, including the symmetric distribution of critical points and the symmetric dynamical behaviors. In the following, we assume that $n \geq 2$ is an integer if not special specified. Recall that $D_{\infty}$ (respectively, $D_0$) is the immediate
Figure 2. The non-escaping loci of $f_\lambda$, where $n = 3$ and 4. Left: $n = 3$, the McMullen domain does not exist and the Julia set $J(f_\lambda)$ cannot be a Cantor set of circles; Right: $n = 4$, there is a punctured domain centered at origin which corresponds to the McMullen domain (the big white part in the center).

super-attracting basin of $\infty$ (respectively, 0) and $f_\lambda$ has a super-attracting periodic orbit $0 \leftrightarrow \infty$. Let $U \subset \mathbb{C}$ be a set and $a \in \mathbb{C} \setminus \{0\}$, we denote $aU = \{az : z \in \mathbb{C}\}$.

Lemma 2.1 (Dynamical symmetry). We have

(a) Let $\omega$ be a complex number satisfying $\omega^{2n} = 1$, then $f_\lambda(\omega z) = \omega^{-n} f_\lambda(z)$. Moreover, $f_\lambda^k(\omega z) = \omega^{(-1)^k n} f_\lambda^k(z)$ for $k \geq 1$.

(b) Let $\tau(z) = \lambda^2/z$, then $\tau \circ f_\lambda(z) = f_\lambda \circ \tau(z)$ for each $z \in \mathbb{C}$.

(c) The attracting basins $D_0$ and $D_\infty$ have $2n$-fold symmetry. That is, if $z \in D_0$ (or $D_\infty$), then $\omega z \in D_0$ (or $D_\infty$), where $\omega$ satisfies $\omega^{2n} = 1$.

Proof. (a) A direct calculation shows that $f_\lambda(\omega z) = \frac{z^{2n} - \lambda^{3n+1}}{\omega z^{2n} - \lambda^{n-1}} = \omega^{-n} f_\lambda(z)$.

By induction it is easy to see that $f_\lambda^k(\omega z) = \omega^{(-1)^k n} f_\lambda^k(z)$ for $k \geq 1$.

(b) Clearly $\tau(z) = \tau^{-1}(z)$, it thus follows that $\tau \circ f_\lambda \circ \tau(z) = \frac{\lambda^2(z)^{n}((\lambda^2/z)^{2n} - \lambda^{n-1})}{(\lambda^2/z)^{2n} - \lambda^{3n+1}} = f_\lambda(z)$.

(c) The proof of this part is straightforward and we omit the details. \qed

From Lemma 2.1, one knows that all the orbits of points with the form $\omega^j z$ ($j = 0, 1, 2, \ldots, 2n-1$) behave “symmetrically” under $f_\lambda$. For example, on the one hand, $f_\lambda^k(z)$ tends to origin (or infinity) as $k$ tends to infinity if and only if $f_\lambda^k(\omega^j z)$ tends to origin (or infinity) as $k$ tends to infinity. On the other hand, $f_\lambda^k(z)$ tends to origin (or infinity) as $k$ tends to infinity if and only if $f_\lambda^k(\lambda^2 z)$ tends to infinity (or origin) as $k$ tends to infinity. We thus obtain the result as follows.
Corollary 2.2. We have
(a) If $U$ is a Fatou component of $f_\lambda$, then $\tau(U)$ is also (here $U$ is allowed to be equal to $\tau(U)$). In particular, $\tau(D_0) = D_\infty$ and $\tau(D_\infty) = D_0$.
(b) If $U$ is a Fatou component of $f_\lambda$. Suppose that $z_0, \omega^j z_0 \in U$, where $\omega$ satisfies $\omega^{2n} = 1$ and $\omega^h \neq 1$. Then $\omega^j z_0 \in U$ for each integer $j$. In particular, $U$ has $2n$-fold symmetry and winds around the origin.
(c) If $U$ is a Fatou component of $f_\lambda$ which is different from $D_0, D_\infty$, then either $U$ has $2n$-fold symmetry and surrounds the origin or $\omega U, \omega^2 U, \cdots, \omega^{2n} U = U$ are pairwise disjoint.
(d) $J(f_\lambda)$ has $2n$-fold symmetry.

Remark. The proof of Corollary 2.2 is straightforward and one can refer to [5, Lemma 1.3] for a similar proof. Suppose that $z_0$ lies in a Fatou component $U$, while $\omega^j z_0$ does not lie in this component for some $j$. Corollary 2.2 implies that there are $2n$ such Fatou components which surround the origin (or infinity) with $2n$-fold symmetry.

2.2. The quasicircle case. A direct computation implies that
$$f'_\lambda(z) = -n \cdot \frac{z^{4n} + (\lambda^{n-1} - 3\lambda^{n+1})z^{2n} + \lambda^{4n}}{z^{n+1}(z^{2n} - \lambda^{n-1})^2}.$$ (3)

Obviously 0 is a critical point of $f_\lambda$ with multiplicity $n - 1$. Lemma 2.1 implies that $\infty$ is also a critical point of $f_\lambda$ with multiplicity $n - 1$. If $c_\lambda$ is one of the critical points other than 0 and $\infty$, then the resting $4n$ critical points (including $c_\lambda$) of $f_\lambda$ have the following form:
$$\text{Crit}(f_\lambda) = \{\omega^j c_\lambda, \omega^j \lambda^2 / c_\lambda : 0 \leq j \leq 2n - 1\},$$

where $\omega^{2n} = 1$. These $4n$ critical points are called the free critical points. It follows from Lemma 2.1 again that the properties of the Julia set of $f_\lambda$ depend only on one of the free critical orbits.

Lemma 2.3. If one of the free critical points lies in $D_0$ (or $D_\infty$), then $J(f_\lambda)$ is a quasicircle.

Proof. Suppose that the free critical point $c_\lambda \in D_0$. Then Lemma 2.1 shows that $\lambda^2 / c_\lambda \in D_\infty$. In viewed of Lemma 2.1 again, one gets that $\omega^j c_\lambda \in D_0$ and $\omega^j \lambda^2 / c_\lambda \in D_\infty$ for all $0 \leq j \leq 2n - 1$. We claim that $D_\infty$ is the unique component of $f_\lambda^{-1}(D_0)$ and $D_0$ is also the unique component of $f_\lambda^{-1}(D_\infty)$. In fact, let $d$ be the degree of the restriction of $f_\lambda$ on $D_\infty$. From the assumption that $f_\lambda : D_\infty \to D_0$ is proper and the local degree of $f_\lambda$ at $\infty$ is $n$, it follows that $d \geq n$. If $0$ has preimages other than $\infty$ in $D_\infty$, one must deduce from Lemma 2.1 that $d \geq 3n$. It is easily obtained that $d = 3n$ since the degree of $f_\lambda$ is $3n$. Hence $d = n$ or $d = 3n$.

In the following we shall show that $d = n$ is impossible. Assume by contradiction that $d = n$. By the argument above it follows from $c_\lambda \in D_0$ that the $2n - 1$ free critical points $\{\omega^j c_\lambda : 1 \leq j \leq 2n - 1\}$ also lie in $D_0$. From the definition of $f_\lambda$ (see (2)), one can conclude that $f_\lambda(c_\lambda)$ has at least $2n$ preimages $\{\omega^j c_\lambda : 0 \leq j \leq 2n - 1\}$ contained in $D_0$ (counted with multiplicity). Therefore $d \geq 2n$, reaching a contradiction with the assumption $d = n$. It then follows that $d = 3n$ and $D_\infty$ is the unique component of $f_\lambda^{-1}(D_0)$. By the similar argument as before, we can prove that $D_0$ is also the unique component of $f_\lambda^{-1}(D_\infty)$.

Note that all the critical points are attracted by 0 and $\infty$. We claim that there are no Fatou components other than $D_0$ and $D_\infty$ for $f_\lambda$. Indeed, firstly if there
were either (super-)attracting periodic basins or parabolic periodic basins which are different from \( D_0 \) and \( D_\infty \), then those periodic basins should contain at least one critical point, which is impossible. Finally, if there were either Herman rings or Siegel disks, then \( J(f_\lambda) \) would contain at least one critical value. Clearly this is also impossible since all the critical points lie in \( D_0 \) and \( D_\infty \). Thus \( f_\lambda \) has only two Fatou components \( D_0, D_\infty \) and \( f_\lambda \) is hyperbolic. According to [3, p.102], \( J(f_\lambda) \) is a quasicircle. This ends the proof of Lemma 2.3 and hence Theorem 1.1(a). □

2.3. Some dynamical properties in the non-quasicircle case. In the following we investigate the case that the free critical points fail to lie in \( D_0 \) and \( D_\infty \). The first preimage of \( D_0 \) is denoted by \( A_\infty = f_\lambda^{-1}(D_0) \setminus D_\infty \) and the first preimage of \( D_\infty \) is denoted by \( A_0 = f_\lambda^{-1}(D_\infty) \setminus D_0 \). It is not difficult to see that \( A_0 \neq \emptyset \) and \( A_\infty \neq \emptyset \). By Lemma 2.1, one can know that \( c_\lambda \in A_0 \) if and only if \( \lambda^2 / c_\lambda \in A_\infty \). To show that \( J(f_\lambda) \) is a Cantor set of circles, one needs the following result.

**Lemma 2.4.** If one of the free critical points \( c_\lambda \) lies in \( A_0 \) (or \( A_\infty \)), then both \( D_0 \) and \( D_\infty \) are simply connected and \( A_0, A_\infty \) are annuli surrounding the origin with \( 2n \)-fold symmetry.

**Proof.** Note that \( 0 \in D_0 \) stays at the super-attracting periodic orbit \( 0 \leftrightarrow \infty \) with period 2. One may take a small disk \( \Delta_0 = \{ z : |z| < r \} \) such that \( \partial \Delta_0 \) is disjoint from the orbits of all free critical points. For simplicity, let \( z_k = f_\lambda^{2k}(0) \in f_\lambda^{2k}(D_0) \) for \( k = 0, 1 \). Let \( \Delta_m \) denote the connected component of \( f_\lambda^{-m}(\Delta_0) \) containing \( z_k \) for \( k = m (\text{mod} \ 2) \). Clearly \( \Delta_0 \subset D_0 \) and \( \Delta_1 \subset D_\infty \). It then follows that \( \Delta_{2l+k} \subset \Delta_{2(l+1)+k} \subset \cdots \subset D_k \) (where \( D_1 := D_\infty \)), therefore \( D_0 = \bigcup_{l \geq 0} \Delta_{2l} \) and \( D_\infty = \bigcup_{l \geq 0} \Delta_{2l+1} \). Note that \( \Delta_0 \) is simply connected and \( f_\lambda^{2l} : \Delta_{2l} \setminus \{0\} \rightarrow \Delta_0 \setminus \{0\} \) is a covering map with degree \( n^{2l} \). By the Riemann-Hurwitz formula, we obtain that

\[
\chi(\Delta_{2l} \setminus \{0\}) + \delta(\Delta_{2l} \setminus \{0\}) = n^{2l} \chi(\Delta_0 \setminus \{0\}),
\]

where \( \chi(\cdot) \) denotes the Euler characteristic and \( \delta(\cdot) \) denotes the number of critical points (counted with multiplicity). Since none of free critical points of \( f_\lambda^{2l} \) lie in \( \Delta_{2l} \setminus \{0\} \) and \( \chi(\Delta_0 \setminus \{0\}) = 0 \), then \( \chi(\Delta_{2l} \setminus \{0\}) = \emptyset \). This means that \( \Delta_{2l} \setminus \{0\} \) is an annulus and then \( \Delta_{2l} \) is simply connected. Hence \( D_0 = \bigcup_{l \geq 0} \Delta_{2l} \) is simply connected. By the similar argument, \( D_\infty = \bigcup_{l \geq 0} \Delta_{2l+1} \) is also simply connected.

We then invoke Corollary 2.2 (2n-fold symmetry of Fatou component) to obtain that \( A_\infty \) has only one or \( 2n \) components. Suppose that \( A_\infty \) has \( 2n \) components, then \( 0 \) has \( 5n \) preimages (counted with multiplicity) since each component of \( A_\infty \) is mapped to \( D_0 \) by degree 2 and there are \( 2n \) such components in this case. This is impossible since the degree of \( f_\lambda \) is 3n by the definition. Thus \( A_\infty \) consists of one component and it is connected. In view of the Riemann-Hurwitz formula for \( f_\lambda : A_\infty \rightarrow D_0 \), one can deduce that \( \chi(A_\infty) + 2n = 2n \chi(D_0) \). Together with Lemma 2.1, this implies that \( \chi(A_\infty) = 0 \) and \( A_\infty \) is an annulus surrounding the origin with \( 2n \)-fold symmetry. The completely similar arguments can also be applied to \( A_0 \). Hence \( D_\infty \) is a simply connected domain and \( A_0 \) is an annulus surrounding the origin with \( 2n \)-fold symmetry. □

Recall that the Sierpiński carpet is a planar set which is compact, connected, locally connected, nowhere dense, and has the property that any two complementary domains are bounded by disjoint simple closed curves ([28]). As a kind of Julia sets, the Sierpiński carpets attract much interest recently. One may refer to [4], [2], [34], [11], [25] and the references therein.
In this subsection, we will consider a sufficient and necessary condition when the Julia set of \( f_\lambda \) is a Sierpiński carpet by studying the “escaping times” of the free critical points. By Lemmas 2.3 and 2.4 (see also Theorem 1.1), when \( J(f_\lambda) \) is a quasicircle or a Cantor set of circles, then the free critical points must belong to \( D_0 \cup D_\infty \) and \( f_\lambda^{-1}(D_0 \cup D_\infty) - (D_0 \cup D_\infty) \). Therefore, we should exclude these cases.

In the following we will see that the first preimages of \( D_0 \) and \( D_\infty \) may not be annuli. We write \( U_0^1 = f_\lambda^{-1}(D_0) - D_\infty \) and \( U_\infty^1 = f_\lambda^{-1}(D_\infty) - D_0 \) respectively. For each \( k \geq 2 \), let \( U_0^k \) and \( U_\infty^k \) be the preimages of \( U_0^{k-1} \) and \( U_\infty^{k-1} \) respectively. In the rest of this subsection, we also assume that one of the free critical points \( c_\lambda \) belongs to \( U_0^k \) for some \( k \geq 2 \). From the dynamical symmetry stated in Corollary 2.2, it can be deduced that \( \lambda^2/c_\lambda \in U_\infty^k \) and \( f_\lambda \) is \( n \)-to-1 on \( D_0 \) and \( D_\infty \). Note that there are no critical points in \( U_0^1 \). The first preimage \( U_0^1 \) of \( D_0 \) has \( 2n \)-symmetric Fatou components and it cannot be an annulus. If it is proved that \( D_0 \) is a Jordan disk and all components of its preimages are simply connected, then it is easy to show that the Julia set is a Sierpiński carpet.

**Lemma 2.5.** *If one of the free critical points \( c_\lambda \) lies in \( U_0^k \) (or \( U_\infty^k \)) for some \( k \geq 2 \), then all Fatou components of \( f_\lambda \) are simply connected. Furthermore \( J(f_\lambda) \) is compact, connected, locally connected and nowhere dense.*

**Proof.** By the similar argument of Lemma 2.4, one can obtain that the immediate super-attracting basins \( D_0 \) and \( D_\infty \) are simply connected. Just as the stated above, the preimage \( U_\infty^0 = f_\lambda^{-1}(D_0) - D_\infty \) of \( D_0 \) has \( 2n \)-symmetric Fatou components, all of which are simply connected because \( f_\lambda \) maps each one of them onto \( D_0 \) conformally. Since \( c_\lambda \in U_\infty^k \) for \( k \geq 2 \), each component of \( U_0^j \) is simply connected for \( 1 \leq j \leq k-1 \). Moreover, the components in \( U_0^1 \) consists of at least \( 2n \) such components which are located symmetrically around the origin for each \( j \).

Let \( V \) be a simply connected component in \( U_\infty^{k-1} \). If \( V \) contains no critical orbits, then all the components of the preimages of \( V \) are simply connected. If \( V \) contains some free critical value and one component \( P \) of \( f_\lambda^{-1}(V) \) is not simply connected, then \( P \) contains at least two free critical points. But, in view of the symmetry of Fatou components, there are \( 2n-1 \) distinct Fatou components \( \omega^l P \), where \( \omega^{2n} = 1 \) and \( 1 \leq l \leq 2n-1 \), such that there are at least two free critical points in each component of \( \omega^l P \). This implies that \( f_\lambda \) has at least \( 4n \) free critical points, which is impossible. Hence, if there is a free critical value being to \( V \), then all components of \( f_\lambda^{-1}(V) \) are also simply connected. This means that all components in \( U_0^k \) are simply connected.

Note that \( U_0^k \) contains no critical values. Each component of the preimages of \( D_0 \) is simply connected. By the similar argument, each component of the preimages of \( D_\infty \) is also simply connected. From the symmetry in Corollary 2.2, it follows that each Fatou component of \( f_\lambda \) is simply connected. Then the Julia set \( J(f_\lambda) = \mathbb{C} - \bigcup_{j \geq 0} f_\lambda^{-j}(D_0 \cup D_\infty) \) is compact and connected. Since \( J(f_\lambda) \) is not equal to the whole Riemann sphere and has no interior points, it is nowhere dense. By [18, Theorem 19.2], the Julia set is locally connected since \( f_\lambda \) is hyperbolic and its Julia set is connected.

**3. The escaping case.** In this section, we will give the proofs of Theorems 1.1-1.4 (The connectivity of the Julia sets in Theorem 1.1 will be delayed to Section 4). Some proofs are based on the results obtained in the last section.
3.1. The Cantor circle case. Since the quasicircle case has been considered in the last section, we now first consider the Cantor circle case.

Proof of Theorem 1.1 (b). By Lemma 2.4, one can deduce that $A_\infty$ is contained in the component of $\mathbb{C} - A_0$ containing 0. We claim that the closures of the following sets: $D_0, A_\infty, A_0$ and $D_\infty$, are mutually disjoint. To see this, since $f_\lambda(D_0) = D_\infty$, $f_\lambda(D_\infty) = D_0$ and $D_0 \cap D_\infty = \emptyset$, it follows easily that $A_0 \cap A_\infty = \emptyset$. One can see that $A_0$ and $A_\infty$ are two annuli separating 0 from $\infty$. Therefore $\overline{D_0} \cap \overline{D_\infty} = \emptyset$. Applying the fact that $f_\lambda(\overline{A_\infty}) = f_\lambda(\overline{D_\infty}) = \overline{D_0}$ and $f_\lambda(\overline{A_0}) = f_\lambda(\overline{D_0}) = \overline{D_\infty}$, we obtain that $\overline{A_0} \cap \overline{A_\infty} = \emptyset, \overline{A_0} \cap \overline{D_\infty} = \emptyset$ and $\overline{D_0} \cap \overline{A_\infty} = \emptyset$. Note that $A_\infty$ is contained in the component of $\mathbb{C} - A_0$ containing the origin. It means that $\overline{D_0} \cap \overline{A_0} = \emptyset$ and $\overline{D_\infty} \cap \overline{A_\infty} = \emptyset$.

Let $V = \mathbb{C} \setminus (D_0 \cup D_\infty)$. Then it is clear that $V$ is the closed set between $D_0$ and $D_\infty$. Suppose that $V_1, V_2, V_3$ are the closed sets between $D_\infty$ and $A_0, A_0$ and $A_\infty, A_\infty$ and $D_0$, respectively (see the bottom picture of Figure 3).

Taking a Jordan curve $\gamma$ in $A_0 \subset V$ such that it is smooth and surrounds the origin. Again we claim that the preimage of $\gamma$ in $V_3$ is a smooth Jordan curve surrounding the origin. Indeed, since none of critical values belongs to $V$, the preimage of $\gamma$ in $V_3$ has finitely many smooth Jordan curves. If one of them, denoted by $\gamma_3$, does not wind around the origin, then $\int(\gamma_3)$ is a simply connected domain in $V_3$, where $\text{int}(\Gamma)$ (resp. $\text{ext}(\Gamma)$) denotes the bounded (resp. unbounded) component of $\mathbb{C} - \Gamma$ for a Jordan closed curve $\Gamma \subset \mathbb{C}$. Then it follows that the image $\int(\gamma) = \int(f_\lambda(\gamma))$ is also a simply connected domain in $V$. Thus one gets a contradiction because $\gamma \subset V$ surrounds the origin. This implies that the preimages of $\gamma$ in $V_3$ are all smooth Jordan curves separating the origin from the infinity. Arranging $f_\lambda^{-1}(\gamma)$ such that it has two components in $V_3$, then the annular region between these two Jordan curves will contain either zeros or poles, which is impossible. Then $f_\lambda^{-1}(\gamma) \cap V_3$ is a smooth Jordan curve surrounding the origin. We denote it by $\eta$ for simplicity. Similarly, one can find a Jordan curve $\xi$ in $D_\infty$ such that $\xi = f_\lambda(\eta)$ winds around the origin.

Note that the Jordan disk $\int(\eta)$ is compactly contained in the Jordan disk $\int(\xi)$. Then $f_\lambda$ maps $\int(\eta)$ onto $\text{ext}(\gamma)$ properly with degree $n$ and maps $\text{ext}(\gamma)$ onto $\int(\xi)$ as covering with degree $n$. Hence the map $f_\lambda^2 : \int(\eta) \to \int(\xi)$ is a polynomial-like mapping with degree $n^2$. By Douady and Hubbard’s Straightening Theorem [8, p.296], $f_\lambda^2 : \int(\eta) \to \int(\xi)$ is quasiconformally equivalent to the uncritical polynomial $P_{n^2}(z) = z^{n^2}$. It is well known that the Julia set of $P_{n^2}$ is the unit circle. Hence the boundary $\partial D_0$ is a quasicircle. Similarly, the boundary $\partial D_\infty$ is also a quasicircle.

Since the super-attracting cycle $0 \leftrightarrow \infty$ attracts all the critical points of $f_\lambda$, it means that all the preimages of $\partial D_0$ and $\partial D_\infty$ are quasicircles. It is evident that $V_1 \cup V_2 \cup V_3$ contains all the preimages of $\partial D_0$ and $\partial D_\infty$. Then all of them surround the origin by a similar argument as above.

From the previous argument, we know that each $f_\lambda : V_i \to V$ is a covering map in $n$-to-1 fashion ($1 \leq i \leq 3$) (see also the bottom picture of Figure 3). The Julia set of $f_\lambda$ is $J(f_\lambda) = \bigcap_{k \geq 0} f_\lambda^{-k}(V)$. Let $g_i : V \to V_i$ be the inverse branch of $f_\lambda : V_i \to V$ ($1 \leq i \leq 3$). Then each component of $J(f_\lambda)$ has the form $J_{i_1i_2\cdots i_k} = \bigcap_{k \geq 1} g_{i_k} \circ \cdots \circ g_{i_2} \circ g_{i_1}(V)$, where $i_k \in \{1, 2, 3\}$ for each $k \in \mathbb{N}$. By virtue of the construction of $J(f_\lambda)$, we know that each component $J_{i_1i_2\cdots i_k}$ is a compact set separating 0 from $\infty$. 
Figure 3. The above and below pictures illustrate the mapping relations of $h_\lambda$ (see (1)) and $f_\lambda$ respectively when $D_0$ contains one of the free critical values but contains no free critical points. One can observe clearly that $f_\lambda$ and $h_\lambda$ are not topologically conjugate on their corresponding Julia sets.

Let $L = J' \cup J''$, where $J' = J_{1,3,1,3,...}$ and $J'' = J_{3,1,3,1,...}$ are the boundaries $\partial D_\infty$ and $\partial D_0$ respectively. Since $V_i$ is contained in $V$, the identity map $\text{id}: V_i \rightarrow V$ is not homotopic to a constant map. We consider the following two cases.

**Case 1.** The forward orbit of the Julia component $J_{i_1i_2...i_k,...}$ is contained in $\text{int}(V)$. Then $J_{i_1i_2...i_k,...}$ is a Jordan curve by [19, Lemma 2.4, Case 2] and [19, Proposition Case 2].

**Case 2.** The Julia component $J_{i_1i_2...i_k,...}$ is eventually iterated onto $J'$ or $J''$ under the map $f_\lambda$. Then $J_{i_1i_2...i_k,...}$ is a quasicircle by the above argument. In either case, $J_{i_1i_2...i_k,...}$ is always a Jordan curve.
On the one hand, we have already shown that all the Julia components of $f_\lambda$ are Jordan curves (actually quasicircles). On the other hand, there is an isomorphism between the dynamics on the Julia components $J_{i_1, \ldots, i_k}$ and the one-sided shift on the space of 3 symbols $\Sigma_3 = \{1, 2, 3\}^\mathbb{N}$. Moreover, the Julia set $J(f_\lambda)$ is homeomorphic to $\Sigma_3 \times \mathbb{S}^1$, where $\mathbb{S}^1 := \{z : |z| = 1\}$. This means that $J(f_\lambda)$ is a Cantor set of circles. This finishes the proof of Theorem 1.1 (b).

3.2. The carpet case. In this subsection, we consider the case that the free critical points enter into $D_0 \cup D_\infty$ by at least two iterates. In this case we prove that the Julia set of $f_\lambda$ is either a Sierpiński carpet or a degenerate Sierpiński carpet.

Proof of Theorem 1.1 (c). The proof will be divided into three main steps.

Step 1. All Fatou components of $f_\lambda$ are Jordan disks. To see this, it is sufficient to show that the boundary $\partial D_0$ is a Jordan curve by Lemma 2.5 and Corollary 2.2. Note that $\partial D_0$ is connected and locally connected. There are at most countably many Jordan disks in the complement $\mathbb{C} \setminus D_0$. Let us assume that $W_\infty$ is the component of $\mathbb{C} \setminus D_0$ such that it contains the infinity. Then $\partial W_\infty$ is a Jordan curve. In what follows we shall prove that $f_\lambda^{-1}(W_\infty) \subset D_0$.

Clearly we have $\infty \in D_\infty \subset W_\infty$. Therefore, we only need to show that $f_\lambda^{-1}(\infty) \subset D_0$. Applying Lemma 2.1 and Corollary 2.2, the 2n poles $f_\lambda^{-1}(\infty) \setminus \{0\}$ belong either to the same Fatou component $U_0$ surrounding the origin or are contained in 2n distinct Fatou components of $f_\lambda$. For the first case, we assume that $f_\lambda^{-1}(\infty) \not\subset U_0$. It then follows that the Fatou component $U_0$ must separate $D_0$ from $W_\infty$. This is a contradiction since $\partial W_\infty \subset \partial D_0$. For the second case, there are 2n distinct components of $\mathbb{C} \setminus (\overline{D_0} \cup \overline{W_\infty})$. We denote them by $V_0, V_1, \ldots, V_{2n-1}$. Then each of these 2n distinct components contains at least one Fatou component. Since $f_\lambda^{-1}(0) \setminus \{\infty\} \subset \bigcup_{i=0}^{2n-1} \tau(V_i) \subset W_\infty$, where $\tau(z) = \lambda^2/z$, one obtains that $f_\lambda(V_0) = W_\infty$ for each $i$ and $f_\lambda(\bigcup_{i=0}^{2n-1} \tau(V_i)) = \partial W_\infty \subset \partial D_0$. It is easy to see that each point of $\partial W_\infty \subset \partial D_\infty$ has at least 2n preimages on $\partial D_0$. This is a contradiction since the degree of $f_\lambda : \partial D_\infty \to \partial D_0$ is n. Hence we have $f_\lambda^{-1}(W_\infty) \subset D_0$ and $W_\infty = \mathbb{C} \setminus \overline{D_0}$.

Let $z$ be a point in $\partial W_\infty$. Note that $f_\lambda^{-1}(\overline{W_\infty}) \subset \overline{D_0}$ and $z \in \partial W_\infty$ is not any exceptional point, we get that $\partial D_0 \subset J(f_\lambda) = J(f_\lambda^2) = \bigcup_{j \geq 0} f_\lambda^{-j}(z) \subset \partial W_\infty$ and $\partial W_\infty \subset \partial D_0$. It means that $\partial D_0 = \partial W_\infty$ is a Jordan curve. Indeed, by applying a standard argument and noting that $f_\lambda$ is hyperbolic, it is easy to obtain that $\partial D_0$ is a quasicircle (for a proof, see [25, Lemma 3.3]).

According to the Böttcher theorem, there exist a unique conformal map $\phi : D_\infty \to \mathbb{C} \setminus \overline{D}$ tangent at $\infty$ and another unique conformal map $\varphi : D_0 \to \mathbb{D}$ tangent at 0 such that $\phi \circ f_\lambda(z) = (\varphi(z))^{-n}$, where $\mathbb{D} = \{z : |z| < 1\}$. Given an angle $\theta \in [0, 1)$, the external ray in $D_\infty$ with angle $\theta$ is denoted by $\gamma(\theta) = \{z \in D_\infty : \phi(z) = re^{2\pi i \theta}, r > 1\}$. The external ray $\gamma(\theta)$ is called landing at $z$ if $\gamma(\theta) \to z \in \partial D_\infty$ as $r \to 1$. Suppose that $D$ is a component of $f_\lambda^{-k}(D_\infty)$. A component $t \subset D$ of $f_\lambda^{-k}(\gamma(\theta))$ is called an internal ray in $D$. When $z \in \partial D_\infty$ is a landing point of $\gamma(\theta)$, $f_\lambda^{-k}(z) \in \partial D$ is also a landing point of $f_\lambda^{-k}(\gamma(\theta))$. Note that $f_\lambda(\gamma(\theta)) = \gamma(n \theta)$ and $\partial D_\infty$ is locally connected. From Carathéodory’s theorem we know that all external rays land and each point of $\partial D_\infty$ is a landing point of an external ray.
Step 2. The Julia set of $f_\lambda$ is a Sierpiński carpet if $\partial D_0 \cap \partial D_\infty = \emptyset$. Up to now, we have proved that all Fatou components of $f_\lambda$ are Jordan disks. The following argument will be divided into three cases.

Case 1. Suppose that $P$ and $Q$ are two components of $U_0^k$ (or $U_\infty^k$) satisfying $\overline{P} \cap \overline{Q} \neq \emptyset$, where $k \geq 1$. We have $f_\lambda^{\circ (k+1)}(P) = f_\lambda^{\circ (k+1)}(Q) = D_\infty$ (or $f_\lambda^{\circ k}(P) = f_\lambda^{\circ k}(Q) = D_\infty$). There are two internal rays $l_1, l_2$ in $P$ and $Q$, respectively, such that both of them land at a point $z \in \overline{P} \cap \overline{Q}$ and $\gamma = f_\lambda^{\circ (k+1)}(l_1) = f_\lambda^{\circ (k+1)}(l_2)$ (or $\gamma = f_\lambda^{\circ k}(l_1) = f_\lambda^{\circ k}(l_2)$) is a landing ray in $D_\infty$ which lands at $f_\lambda^{\circ (k+1)}(z)$ (or $f_\lambda^{\circ k}(z)$).

This implies that $z$ is a critical point of $f_\lambda^{\circ (k+1)}$ (or $f_\lambda^{\circ k}$), which is a contradiction since all critical points are attracted by the cycle $0 \leftrightarrow \infty$.

Case 2. Suppose that $P$ and $Q$ are two components of $U_0^i$ and $U_\infty^j$ (or $U_0^i$ and $U_\infty^j$) respectively such that $\overline{P} \cap \overline{Q} \neq \emptyset$, where $0 \leq i < j$. Then $f_\lambda^{\circ j}(P) = D_0$ or $D_\infty$, and $f_\lambda^{\circ j}(Q) = D_\infty$. When $f_\lambda^{\circ j}(P) = D_0$ and $f_\lambda^{\circ j}(Q) = D_\infty$, it follows that $f_\lambda^{\circ j}(\overline{P} \cap \overline{Q}) \neq \emptyset$ and then $\partial D_0 \cap \partial D_\infty \neq \emptyset$, yielding a contradiction. When $f_\lambda^{\circ j}(P) = D_\infty$ and $f_\lambda^{\circ j}(Q) = D_\infty$, by a similar argument as Case 1, we also have a contradiction.

Case 3. Suppose that $P$ and $Q$ are two components of $U_0^i$ and $U_\infty^j$ respectively, where $(i, j) \neq (0, 0)$. Since $\partial D_0 \cap \partial D_\infty = \emptyset$, it follows that $\overline{P} \cap \overline{Q} = \emptyset$.

The above arguments imply that the closures of each two different Fatou components of $f_\lambda$ are disjoint. By Lemma 2.5, the Julia set $J(f_\lambda)$ is a Sierpiński carpet. This finishes the proof of Theorem 1.1 (c1).

Step 3. The Julia set of $f_\lambda$ is a degenerate Sierpiński carpet if $\partial D_0 \cap \partial D_\infty \neq \emptyset$. By Lemma 2.5 and Step 1, the Julia set of $f_\lambda$ is compact, connected, locally connected, nowhere dense and all Fatou components of $f_\lambda$ are Jordan disks. On the other hand, the assumption that $\partial D_0 \cap \partial D_\infty \neq \emptyset$ implies that $J(f_\lambda)$ cannot be a Sierpiński carpet. This means that $J(f_\lambda)$ is a degenerate Sierpiński carpet. This finishes the proof of Theorem 1.1 (c2).

3.3. The regularity of the boundaries of Fatou components. Let $f$ be a rational map with degree at least two. If the orbits of all critical points of $f$ tend to the attracting or super-attracting periodic cycles, then $f$ is called hyperbolic.

Proof of Theorem 1.2. (a) The statement has been included in the proof of Theorem 1.1 (see Lemma 2.3, §3.1 and §3.2). For the case of Cantor circles, one may also refer to [24, Corollary 1.7].

(b) If the free critical points are attracted by the cycle $0 \leftrightarrow \infty$, then $f_\lambda$ is hyperbolic. According to [26, p.745], the Julia set of $f_\lambda$ has Hausdorff dimension strictly less than two. Since $f_\lambda$ has two Fatou components $D_0$ and $D_\infty$ which are Jordan domains, we have $\dim_H J(f_\lambda) \geq 1$.

Since $f_\lambda^{\circ 2}$ has two fixed super-attracting basins $D_0$, $D_\infty$, and $f_\lambda^{\circ 2}$ is neither conjugate to a Blaschke product nor a quotient of a Blaschke product, it means that $\dim_H (\partial D_0) > 1$ by [20]. Note that $\partial D_0 \subset J(f_\lambda^{\circ 2}) = J(f_\lambda)$. We have $1 < \dim_H J(f_\lambda) < 2$. This ends the proof of Theorem 1.2.

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1 Sullivan only proves that hyperbolic Julia sets have zero area in [26]. However, from his proof one can obtain that the Hausdorff dimension is strictly less than two directly.
3.4. Existence of McMullen domain. For McMullen maps \( F_\lambda(z) = z^n + \lambda/z^n \), the McMullen domain exists if and only if \( n \geq 3 \). For our family \( f_\lambda \), we will show that the McMullen domain exists if and only if \( n \geq 4 \).

Proof of Theorem 1.3. Suppose that \( J(f_\lambda) \) is a Cantor set of circles. Then it follows that \( D_0 \) and \( D_\infty \) must be both simply connected and all other Fatou components (other than \( D_0 \) and \( D_\infty \)) are annuli which separate the origin from the infinity. Lemma 2.4 yields that there are two of the annular Fatou components containing both \( 2n \) free critical points. Using the Riemann-Hurwitz’s formula, one can get that the first preimages of \( D_0 \) and \( D_\infty \) contain all of these free critical points. Furthermore, none of the free critical points belongs to \( D_0 \) and \( D_\infty \) since \( J(f_\lambda) \) is a Cantor set of circles (not a quasicircle).

From the proof of Theorem 1.1 (b), we have the bottom picture of Figure 3. The conformal moduli of annuli satisfy \( \text{mod}(V_1) = \text{mod}(V_2) = \text{mod}(V_3) = \text{mod}(V)/n \) because \( V_j \) is mapped onto \( V \) under \( f_\lambda \) by \( n \)-to-1 for \( j = 1, 2, 3 \). Note that \( V_1, V_2, V_3 \) are essentially contained in \( V \) and \( V \setminus (V_1 \cup V_2 \cup V_3) \neq \emptyset \). From Grotzsch’s modulus inequality, it means that

\[
\text{mod}(V_1) + \text{mod}(V_2) + \text{mod}(V_3) = \frac{3}{n} \text{mod}(V) < \text{mod}(V),
\]

which holds if and only if \( n \geq 4 \). The proof of Theorem 1.3 is complete.

3.5. This family is different from known ones. In this subsection we show that the dynamics of \( f_\lambda \) is neither conjugate to \( h_\lambda \) nor to any McMullen maps. Moreover, we study the Hausdorff dimension of \( f_\lambda \) when \( J(f_\lambda) \) is a Cantor set of circles.

Proof of Theorem 1.4. (a) On the one hand, from the proof of Theorem 1.1 (b) we know that the dynamics on the set of Julia components of \( f_\lambda \) is isomorphic to the one-sided shift on three symbols \( \Sigma_3 := \{1, 2, 3\}^\mathbb{N} \). But it is known that the dynamics on the set of Julia components of McMullen map \( F(z) = z^n + \lambda/z^n \) is conjugate to the one-sided shift on only two symbols \( \Sigma_2 := \{1, 2\}^\mathbb{N} \). It follows that \( f_\lambda \) cannot be topologically conjugate to \( f \) on their corresponding Julia sets (see Figure 4 for an example).

On the other hand, from the proof of Theorem 1.1 (b), we know that \( f_\lambda \) has a super-attracting cycle \( D_0 \leftrightarrow D_\infty \) with period 2. However \( h_\lambda \) has only two fixed super-attracting domains \( D_0 \) and \( D_\infty \). Therefore \( f_\lambda \) cannot be topologically conjugate to \( h_\lambda \) on their corresponding Julia sets (see also Figure 3).

(b) If \( J(f_\lambda) \) is a Cantor set of circles, there are four closed annuli \( V_1, V_2, V_3 \) and \( V = \overline{\mathbb{C}} \setminus (D_0 \cup D_\infty) \), which are introduced in the proof of Theorem 1.1 (b). Their boundaries are Jordan curves and we have (see also Figure 3)

- \( J(f_\lambda) \subset V_1 \cup V_2 \cup V_3 \subset V \);
- \( V_1, V_2 \) and \( V_3 \) are closed annuli between \( D_\infty \) and \( A_0, A_0 \) and \( A_\infty, A_\infty \) and \( D_0 \) respectively; and
- \( f_\lambda : V_i \to V \) is a coverings map with degree \( n \) for \( i = 1, 2, 3 \).

According to [14, Corollary 2.1], we have

\[
\dim_H J(f_\lambda) = \dim_H \bigcap_{j=0}^{\infty} f_\lambda^{-j}(V) \geq 1 + \frac{\log 3}{\log n}.
\]
ESCAPE QUARTERED THEOREM AND THE CONNECTIVITY OF JULIA SETS

Figure 4. The Julia sets of $f_\lambda$ with $n = 4$, $\lambda = 0.4$ and $F(z) = z^3 + 0.01/z^3$. Both of them are Cantor circles. But $f_\lambda$ and $F$ are not topologically conjugate on their corresponding Julia sets.

On the other hand, we have $\dim_H J(f_\lambda) < 2$ since $f_\lambda$ is hyperbolic. This finishes the proof of Theorem 1.4.

4. The connectivity of the Julia sets. In this section, we consider the case that the orbits of all free critical points are not attracted by the super-attracting cycle $0 \leftrightarrow \infty$. In this case, we will prove the connectivity of the Julia set of $f_\lambda$ (the last statement of Theorem 1.1) and the idea is similar to [10].

Lemma 4.1. If one of the free critical points $c_\lambda$ is not attracted by the cycle $0 \leftrightarrow \infty$, then $f_\lambda$ has no multiply-connected (actually infinitely connected) attracting Fatou components or parabolic Fatou components.

Proof. Based on the argument in the last section, we know that both $D_0$ and $D_\infty$ are simply connected and all the iterated preimages $f_\lambda^{-l}(D_0)$ and $f_\lambda^{-l}(D_\infty)$ are also simply connected for all $l \in \mathbb{N}$. Assume that there exists a periodic attracting basin or parabolic basin $U$ with period $p$ associating a periodic point $z_0$, which is different from $D_0$ and $D_\infty$. Then the period of $z_0$ is $p$ but $z_0 \not\in \{0, \infty\}$. Without loss of generality, we assume that $U$ contains at least a critical point $c_\lambda$ (see [18, §§8-10]). Obviously $f_\lambda^p(U) = U$. Note that all the components of $p$-periodic attracting or parabolic basin have the same connectivity. It means that the connectivity of $f_\lambda^j(U)$ ($1 \leq j \leq p - 1$) is the same as that of $U$. We thus only need to study the connectivity of $U$. Let $k$ be the number of points in $U \cap C_\lambda$, where $C_\lambda$ is the finite set consisting of the free critical points of $f_\lambda$. Specifically, we have $C_\lambda = \{\omega^j c_\lambda, \omega^j \lambda^2/c_\lambda : 0 \leq j \leq 2n - 1\}$.

We claim that $k$ satisfies $1 \leq k \leq 2$. Otherwise, $2 < k \leq 4n$. If $2 < k < 2n + 2$, by Lemma 2.1 and Corollary 2.2, then $\omega^j U = U$ for each $j \in \mathbb{N}$ and there are at least such $2n$ free critical points $\omega^j c_\lambda$ ($0 \leq j \leq 2n - 1$) contained in $U$ by the assumption that $c_\lambda \in U$. On the one hand, taking a Jordan curve $\gamma \subset U$ such that it surrounds the origin and $\omega^j \gamma = \gamma$ for each $j \in \mathbb{N}$. Again Lemma 2.1 implies that $f_\lambda^{\text{np}}(\gamma)$ must surround the origin for every $n \in \mathbb{N}$. On the other hand, from the assumption that $U$ is different from $D_0$ and $D_\infty$, it follows that for each $\gamma \subset U$, $f_\lambda^{\text{np}}(\gamma) \to z_0$ as $n \to \infty$. Note that $z_0$ is different from 0 and $\infty$, we conclude that $f_\lambda^{\text{np}}(\gamma)$ cannot...
wind around the origin for any \( n \in \mathbb{N} \). We thus obtain a contradiction. When \( 2n + 2 \leq k \leq 4n \), there are 4n free critical points \( \omega^j c_\lambda, \omega^j \lambda^2 / c_\lambda \) (\( 0 \leq j \leq 2n - 1 \)) in \( U \). One can obtain a contradiction by applying a similar argument as above. This ends the proof of the claim.

In the following, we divide the argument into the following two cases.

**Case 1.** If \( k = 1 \), then \( U \) must be simply connected. To see this, we consider two subcases. When \( U \) is a periodic attracting basin, let \( z_0 \in U \) be the periodic attracting point with period \( p \) and define \( z_j = f^j_\lambda(z_0) \in f^j_\lambda(U) \) for \( 1 \leq j < p \). Let \( B_0 \) be a small disk centered at \( z_0 \) such that \( f^p_\lambda(B_0) \subset B_0 \) and \( \partial B_0 \) does not contain any points in the critical orbits of \( f_\lambda \). For \( l \geq 0 \) and \( 0 \leq j < p \), one uses \( B_{lp+j} \) to denote the connected component of \( f^{-(lp+j)}(B_0) \) containing \( z_{p-j} \) (where \( z_p = z_0 \)). Then for \( 0 \leq j < p \), we have

\[
B_j \subset B_{p+j} \subset B_{2p+j} \subset \cdots \subset B_{lp+j} \subset \cdots .
\]  

(4)

Since each \( B_{lp+j} \) contains at most one critical point and \( \partial B_0 \) does not contain any points in the critical orbits of \( f_\lambda \), it follows from the Riemann-Hurwitz formula that \( B_{lp+j} \) is simply connected. Noting \( U = \bigcup_{l \geq 0} B_{lp} \) and the formula (4), we know that \( U \) is simply connected.

Suppose that \( U \) is a periodic parabolic basin. Let \( z_0 \in \partial U \) be the parabolic periodic point such that \( f^{pj}_\lambda(z) \) is attracted by \( z_0 \) for each \( z \in U \). Let \( z_j = f^j_\lambda(z_0) \in f^j_\lambda(\partial U) \) for \( 1 \leq j < p \). By virtue of the dynamics of \( f_\lambda \) on the parabolic basins, we may take a small disk \( B_0 \subset U \) such that \( z_0 \in \partial B_0, B_0 \subset U \cup \{z_0\}, f^p_\lambda(B_0) \subset B_0 \cup \{z_0\} \) and \( \partial B_0 \) does not contain any points in the critical orbits of \( f_\lambda \). By the argument similar to that of the attracting case, one can deduce that \( U \) is also simply connected.

**Case 2.** If \( k = 2 \), then \( U \) contains two distinct free critical points, say \( c_1 \) and \( c_2 \). If \( c_2 = \omega^{j_0} c_1 \), Lemma 2.1 and Corollary 2.2 imply that the 2n free critical points \( \omega^j c_1 \) (\( 0 \leq j \leq 2n - 1 \)) belong to \( U \), which is a contradiction. We thus obtain that \( c_2 = \omega^{j_0} \tau(c_1) \) for some \( j_0 \in \mathbb{N} \).

In the following we claim that \( \deg(f_\lambda|U) = 3 \). On the one hand, if \( \deg(f_\lambda|U) > 3 \), then for each \( z \in f_\lambda(U) \), Lemma 2.1 implies that \( f_\lambda \) has more than 3n preimages (counting multiplicity), which is impossible since \( \deg(f_\lambda) = 3n \). On the other hand, recall that \( z_0 \in U \) is a periodic point with period \( p \), we conclude that there are at least 3 preimages of \( f_\lambda(z_0) \) which lie in \( U \) (By a similar argument, there are at least 3 preimages of \( f_\lambda(z_0) \in f_\lambda(\partial U) \) in \( \partial U \) for the parabolic case). Indeed, it is easy to see that \( \omega^j U \) (\( 0 \leq j \leq 2n - 1 \)) contains 4n free critical points. Lemma 2.1 and Corollary 2.2 imply that \( \omega^j \tau(U) \) (\( 0 \leq j \leq 2n - 1 \)) are 2n different Fatou components containing such 4n free critical points. Therefore \( U = \omega^{j_0} \tau(U) \) and hence \( z_0 = \omega^{j_0} \tau(z_0) \) for some \( j_0 \).

Suppose that \( z_1 \in U \) satisfies \( f_\lambda(z_1) = f_\lambda(z_0) \). Then

\[
f_\lambda(\omega^{j_0} \tau(z_1)) = \omega^{-n_0} \tau(f_\lambda(z_1)) = \omega^{-n_0} \tau(f_\lambda(z_0))
= \omega^{-n_0} \tau(f_\lambda(\omega^{j_0} \tau(z_0))) = \omega^{-n_0} \tau(\omega^{-n_0} \tau(f_\lambda(z_0)))
= \omega^{-n_0} \omega^{-n_0} \tau(f_\lambda(z_0)) = \tau^{n_0} f_\lambda(z_0) = f_\lambda(z_0).
\]

For simplicity, we set \( z_2 = \omega^{j_0} \tau(z_1) \). If \( z_1 = z_2 \), then \( z_1 = \pm z_0 \). Lemma 2.1 (2n-fold symmetry of Fatou components) rules out the case that \( z_1 = -z_0 \). In fact, assume
that \( z_1 = -z_0 \), it is clear that \( U \) surrounds the origin and satisfies \( \omega^i U = U \) for every \( i \in \mathbb{N} \), which is impossible because \( U \) is different from \( D_0 \). Then \( z_2 = z_1 = z_0 \), which shows that the free critical point \( c_3 = z_0 \) and then its local degree is at least 3, thus \( \text{deg}(f_3) = 3 \). If \( z_1 \neq z_0 \), it then follows from the above argument that \( z_1 \neq z_2 \). Therefore \( z_0, z_1 \) and \( z_2 \) are three distinct points. The condition that \( U = \omega^{3n} \tau(U) \) means \( z_2 \in U \), which implies \( \text{deg}(f_{3n}) = 3 \) and the claim is proved.

By a similar argument as above, we take \( B_0 \) such that it satisfies \( B_0 = \omega^{3n} \tau(B_0) \). This implies that the two critical points \( c_1 \) and \( c_2 = \omega^{3n} c_1 \) of \( f_\lambda \) are contained in \( B_1 \) at the same time, where \( B_1 \) denotes the preimage of \( B_0 \) under \( f_\lambda \) which is contained in the \( p \)-periodic attracting or parabolic cycle. By using the Riemann-Hurwitz formula, we can use a standard process to prove that the periodic attracting or parabolic basin \( U \) is simply connected. The proof is complete.

For a Jordan curve \( \gamma \subset \mathbb{C} \), recall that set \( \text{ext}(\gamma) \) is defined as the component of \( \mathbb{C} \setminus \gamma \) containing \( \infty \) and \( \text{int}(\gamma) \) the other. Let \( A \subset \mathbb{C} \) be an annulus. Recall that the core curve, denoted by \( \gamma_A \), of \( A \) is defined as \( \psi^{-1}(\sqrt{7}) \), where \( \psi : A \rightarrow \hat{A}_r := \{ z \in \mathbb{C} : 0 < r < |z| < 1 \} \) is a conformal isomorphism. Denoted by \( E_A \) and \( I_A \) the exterior and interior boundaries of \( A \), respectively. Since \( \gamma_A \) is a smooth Jordan curve, clearly it can separates the Riemann sphere into two disjoint disks \( A^\text{ext} := \text{ext}(\gamma_A) \) and \( A^\text{int} := \text{int}(\gamma_A) \).

For any \( z \in \hat{C} \), define the forward orbit of \( z \) under the iteration of \( f \) by \( O_f(z) := \{ f^n(z) : n \in \mathbb{N} \} \). We call that two forward orbits \( O_f(z_1) \) and \( O_f(z_2) \) disjoint if the intersection of them are empty. To show that \( J(f_\lambda) \) is connected, we need to rule out the case that \( f_\lambda \) has Herman rings. To do so, we need the following criterion which has been proved in [33, Corollary 2.2].

**Proposition 4.2.** Suppose that a rational map \( f \) has \( p \geq 1 \) fixed Herman rings \( A_0, \ldots, A_{p-1} \). Denote by \( \gamma_i \subset A_i \) the core curve whose union divides \( \mathbb{C} \) into \( p + 1 \) connected components \( W_0, W_1, \ldots, W_p \). Then \( f \) has at least \( p + 1 \) disjoint infinite critical orbits \( O_f(c_i) \) in \( J(f) \) such that \( O_f(c_i) \subset W_i \cap J(f) \), where \( c_i, 0 \leq i \leq p \) is the critical point of \( f \).

Recall that \( \Lambda = \mathbb{C}^* - \{ \lambda : \lambda^{2n+2} = 1 \} \) is defined in (2).

**Lemma 4.3.** If \( \lambda \in \Lambda \), then \( f_\lambda \) has no Herman rings.

**Proof.** Let us assume by contradiction that \( f_\lambda \) has a cycle of Herman rings \{ \( U_0, U_1, \ldots, U_{p'} = U_0 \} \). Then \( f_\lambda^{p'} \) is conjugate to the irrational rotation \( z \mapsto \mu z \) on \( U_0 \), where \( \mu = e^{2\pi i a} \) and \( a \) is an irrational number. For each \( 0 \leq j \leq p' - 1 \), \( f_\lambda : U_j \rightarrow U_{j+1} \) is conformal. According to Lemma 2.1, \( f_\lambda \) has symmetric properties and \( J(f_\lambda) \) is \( 2n \)-symmetric. We consider a new rational map obtained by semiconjugacy. Specifically, we take \( \varphi(z) = z^{2n} \) and define

\[
g_\lambda(z) = \frac{1}{z^n} \left( \frac{z - \lambda^{3n+1}}{z - \lambda^{n-1}} \right)^{2n}.
\]

Then \( \varphi \circ f_\lambda = g_\lambda \circ \varphi \). This means that the dynamics of \( f_\lambda \) is similar as that of \( g_\lambda \). In particular, \( g_\lambda \) has a cycle of Herman rings if and only if \( f_\lambda \) has.

Lemma 2.1 and Corollary 2.2 imply that the component \( U_0 \) is bounded and does not wind around the origin since \( f_\lambda \) is injective in \( U_0 \) as before. By the similar argument, one can show that \( U_j \) is bounded and does not surround the origin for each \( 1 \leq j \leq p' - 1 \). According to the semiconjugacy between \( f_\lambda \) and \( g_\lambda \) under
\( \varphi \), each of the corresponding Herman ring of \( g_\lambda \) is also bounded and does not surround the origin. Again by Lemma 2.1, we then find 2n Fatou components \( \omega^iU_0 \) \((1 \leq i \leq 2n)\) such that \( \omega^iU \cap \omega^jU = \emptyset \) for any \( i \neq j \) (mod 2n). It follows that the restriction of \( \varphi \) on \( U_0 \) is injective and \( \varphi(U_0) \) is a periodic Herman ring of \( g_\lambda \). Suppose that the period of \( \varphi(U_0) \) is \( p \) (Note that \( p \) is allowed to be equal to \( p' \) but \( p \) is a divisor of \( p' \)).

The following aim is try to obtain a contradiction by Proposition 4.2. Set \( V_0 := \varphi(U_0) \) and \( V_j = g_\lambda^{j\lambda}(V_0) \), where \( 0 < j \leq p - 1 \). In particular, \( g_\lambda(V_{p-1}) = V_0 \) and then \( \{V_0, V_1, \ldots, V_p = V_0\} \) is a \( p \)-periodic Herman rings of \( g_\lambda \). Denote by \( \tau_0(z) = \lambda^{4n}/z \).

One obtains that

\[
\tau_0 \circ g_\lambda(z) = g_\lambda \circ \tau_0(z). \tag{5}
\]

Note that \( \deg(g_\lambda) = 3n \) and \( g_\lambda(z) \) has \( 6n - 2 \) critical points (counting with multiplicity). The local degrees of 0 and \( \infty \) are both \( n \) and the local degrees of \( \lambda^{3n+1} \) and \( \lambda^{n-1} \) are both \( 2n \). Hence this leaves 2 critical points. According to (5), the two critical points can be written by \( \tilde{c}_\lambda \) and \( \tau_0(\tilde{c}_\lambda) = \lambda^{3n}/\tilde{c}_\lambda \), and the local degree of them are both 2. It is easy to see that \( V_j \) is bounded and does not surround the origin for any \( 0 \leq j \leq p - 1 \).

We now study the iteration \( g_\lambda^{op} \). It is clear that the free critical points of \( g_\lambda^{op} \) are

\[
\left( \bigcup_{j=0}^{p-1} g^{-j}(\tilde{c}_\lambda) \right) \cup \left( \bigcup_{j=0}^{p-1} g^{-j}(\lambda^{4n}/\tilde{c}_\lambda) \right).
\]

Then there exist at most \( 2p \) disjoint critical orbits of \( g_\lambda^{op} \) and such critical orbits have the following form

\[
\{O_{g_\lambda^{op}}(c_0), O_{g_\lambda^{op}}(\tau_0(c_0)), \ldots, O_{g_\lambda^{op}}(c_{p-1}), O_{g_\lambda^{op}}(\tau_0(c_{p-1}))\}.
\]

The collection of core curves \( \{\gamma_0, \gamma_1, \ldots, \gamma_p-1\} \) of the \( p \)-periodic Herman rings of \( g_\lambda \) separates \( \mathbb{C} \) into \( p + 1 \) connected components, say \( W_0, W_1, \ldots, W_p \). In the following we consider two cases.

**Case 1.** Suppose that \( \tau_0(V_0) = V_0 \). It follows from (5) that \( \tau_0(V_j) = V_j \) for all \( 0 \leq j \leq p - 1 \). For simplicity, the exterior and interior boundary components of \( V_j \) are denoted by \( E_j \) and \( I_j \) respectively, where \( 0 \leq j \leq p - 1 \). We claim that \( \tau_0(E_j) = E_j \) for each \( j \in \mathbb{N} \). To see this, recall that \( V_j \) is bounded and does not wind around the origin, we conclude that the exterior component \( E_j \) must contain the points in \( V_j \) with the largest and smallest modulus. Since the point of \( V_j \) with the largest modulus is mapped to the smallest one under \( \tau_0 \), then \( \tau_0(E_j) \neq I_j \) and hence \( \tau_0(E_j) = E_j \) for each \( j \in \mathbb{N} \). That means the claim holds. Then \( O_{g_\lambda^{op}}(c_j) \) and \( O_{g_\lambda^{op}}(\tau_0(c_j)) \) are always contained in the same component \( W_j \). By Proposition 4.2, there are at least \( p + 1 \) disjoint infinite critical orbits \( O_{g_\lambda^{op}}(c_i) \) in \( J(g_\lambda^{op}) \) such that \( O_{g_\lambda^{op}}(c_i) \subseteq W_j \cap J(g_\lambda^{op}) \) for \( g_\lambda^{op} \). However the \( 2p \) critical orbits of \( g_\lambda^{op} \) can only belong to \( p \) of \( p + 1 \) components of the collection \( \{W_0, W_1, \ldots, W_p\} \), which is impossible.

**Case 2.** Suppose that \( \tau_0(V_0) \neq V_0 \). This implies from (5) that \( \tau_0(V_j) \neq V_j \) for each \( 0 \leq j \leq p - 1 \). Therefore \( g_\lambda^{op} \) must have \( 2p \) disjoint fixed Herman rings. On the other hand, there are only at most \( 2p \) disjoint critical orbits. By Proposition 4.2, this is a contradiction.

This means that \( g_\lambda \) has no Herman rings and hence \( f_\lambda \) has neither. \( \square \)

Now we can finish the proof of Theorem 1.1.
Proof of Theorem 1.1 (The connectivity of the Julia sets). By Lemma 4.1, \( f_\lambda \) has no infinitely connected Fatou components. According to Lemma 4.3, \( f_\lambda \) has no Herman rings. This means that all periodic Fatou component of \( f_\lambda \) are simply connected. Note that the iterated preimages of such periodic Fatou components do not contain any critical points. It means that each component of such iterated preimages is also simply connected. According to [1, p. 173], we conclude that \( J(f_\lambda) \) is connected. This finishes the last statement of Theorem 1.1 and hence completes the proof of Theorem 1.1.

5. The examples of two special cases. In this section, we study the examples which correspond to the quasicircles and Cantor circles in Theorem 1.1. It turns out that they correspond to two extremal cases: \( |\lambda| \) is large and small respectively. For simplicity, we first consider the real parameter \( \lambda \).

Proposition 5.1. Let \( \lambda \in \mathbb{R} \) be a real number. Then the Julia set \( J(f_\lambda) \) is a quasicircle if and only if \( |\lambda| > 1 \).

Proof. First, one can verify that the round circle \( \mathbb{T}_{|\lambda|} = \{ z \in \mathbb{C} : |z| = |\lambda| \} \) satisfies \( f_\lambda(\mathbb{T}_{|\lambda|}) = \mathbb{T}_{|\lambda|} \). In fact, let \( z = |\lambda|e^{2\pi i \theta} \), where \( \theta \in [0, 1) \). A direct calculation shows that \( |f_\lambda(z)| = |\lambda| \), which implies that \( f_\lambda(\mathbb{T}_{|\lambda|}) \subset \mathbb{T}_{|\lambda|} \). In the following, we show that \( f_\lambda \) maps \( \mathbb{T}_{|\lambda|} \) onto \( \mathbb{T}_{|\lambda|} \). Note that \( \lambda \) is a real number satisfying \( \lambda^{2n+2} \neq 1 \) (see the definition of \( f_\lambda \) in (2)). Then \( |\lambda| \neq 1 \). There are following two cases.

Case 1. If \( |\lambda| > 1 \), then \( |\lambda|^{\frac{2n+2}{n+1}} < |\lambda| < |\lambda|^{\frac{n+1}{2n+1}} \). We conclude that \( f_\lambda \) has \( 3n \) poles (counting with multiplicity) and no zeros in \( D_{|\lambda|} = \{ z \in \mathbb{C} : |z| < |\lambda| \} \). The Argument Principle yields that \( f_\lambda(\mathbb{T}_{|\lambda|}) = \mathbb{T}_{|\lambda|} \) since \( f_\lambda(\mathbb{T}_{|\lambda|}) \) surrounds the origin \( 3n \) times (clockwise).

Case 2. If \( |\lambda| < 1 \), therefore \( |\lambda|^{\frac{n+1}{2n+1}} < |\lambda| < |\lambda|^{\frac{2n+2}{n+1}} \). It follows that \( f_\lambda \) has \( n \) poles (counting with multiplicity) and \( 2n \) zeros in \( D_{|\lambda|} \). Similarly, The Argument Principle also yields that \( f_\lambda(\mathbb{T}_{|\lambda|}) = \mathbb{T}_{|\lambda|} \) since \( f_\lambda(\mathbb{T}_{|\lambda|}) \) surrounds the origin \( n \) times (anticlockwise).

Second, we need to prove that \( J(f_\lambda) \) is a quasicircle if and only if \( |\lambda| > 1 \). From the above argument, for \( |\lambda| > 1 \), it follows that \( f_\lambda(\mathbb{T}_{|\lambda|}) = \mathbb{T}_{|\lambda|} \) and \( D_{|\lambda|} \) contains \( 3n \) poles (counting with multiplicity) and no zeros, which implies that \( f_\lambda(D_{|\lambda|}) = \overline{\mathbb{C}} - \overline{D}_{|\lambda|} \) and \( f_\lambda(\overline{\mathbb{C}} - \overline{D}_{|\lambda|}) = \overline{D}_{|\lambda|} \). This means that \( D_{|\lambda|} \subset D_0 \) and \( \overline{\mathbb{C}} - \overline{D}_{|\lambda|} \subset D_{\infty} \). Hence \( J(f_\lambda) = \mathbb{T}_{|\lambda|} \) because \( D_0 \) does not meet \( D_{\infty} \).

Assume by contradiction that \( J(f_\lambda) \) is a quasicircle if \( 0 < |\lambda| < 1 \). It then follows from \( f_\lambda(\mathbb{T}_{|\lambda|}) = \mathbb{T}_{|\lambda|} \) that \( J(f_\lambda) = \mathbb{T}_{|\lambda|} \) and \( D_0 = \overline{D}_{|\lambda|} \). Hence \( f_\lambda : D_{|\lambda|} \rightarrow \overline{\mathbb{C}} - \overline{D}_{|\lambda|} \) is a covering map with degree \( 3n \). On the other hand, \( f_\lambda \) has \( n \) poles and \( 2n \) zeros in \( D_{|\lambda|} \) when \( 0 < |\lambda| < 1 \). One thus has a contradiction, which shows that if \( 0 < |\lambda| < 1 \), \( J(f_\lambda) \) cannot be a quasicircle.

Remark. (a) It can be shown that \( \mathbb{T}_{|\lambda|} \subset J(f_\lambda) \) if the free critical orbits are attracted by \( \infty \) (or 0). Furthermore, \( \mathbb{T}_{|\lambda|} = J(f_\lambda) \) if \( |\lambda| > 1 \), where \( \lambda \in \mathbb{R} \).

(b) Actually one can show that \( J(f_\lambda) \) is a quasicircle if \( |\lambda| > 1 \) with \( \lambda \in \mathbb{C} \).

Suppose \( A(\lambda) \) and \( B(\lambda) \) are two numbers in \( \lambda \). If there is a constant \( C \geq 0 \) independent on \( \lambda \) such that \( A(\lambda) \leq CB(\lambda) \) when \( \lambda \neq 0 \) is sufficiently small, we write as \( A(\lambda) \lesssim B(\lambda) \). If \( A(\lambda) \leq B(\lambda) \) and \( B(\lambda) \leq A(\lambda) \) hold when \( \lambda \neq 0 \) is sufficiently small, we then denote \( A(\lambda) \approx B(\lambda) \).
Proposition 5.2. Suppose that n ≥ 4. If λ ≠ 0 is sufficiently small, then J(f_λ) is a Cantor set of circles.

Proof. By (3), the free critical points of f_λ have the following form
\[ c_\lambda^{2n} = \frac{3\lambda^{3n+1} - \lambda^{n-1} \pm \lambda^{-n} \sqrt{1 - 10\lambda^{2n+2} + 9\lambda^{4n+4}}}{2} = \frac{3\lambda^{3n+1} - \lambda^{n-1} \pm \lambda^{-n} (1 - 5\lambda^{2n+2} - 8\lambda^{4n+4} + O(\lambda^{4n+6}))}{2}. \]

These 4n free critical points can be classified into two categories \( \{c_1, c_2, \ldots, c_{4n}\} \) and \( \{c_1', c_2', \ldots, c_{4n}'\} \) such that, each \( c_i \) satisfies \( |c_i| < |\lambda|^{n+\frac{1}{4}} \), while each \( c_i' \) has the property \( |c_i'| > |\lambda|^{-\frac{n+1}{4}} \) when \( \lambda ≠ 0 \) is sufficiently small. Set \( a = |\lambda|^{\frac{n+1}{2}}, b = |\lambda|^{-\frac{n+1}{4}} \).

For \( z_a ∈ T_a = \{ z ∈ C : |z| = a \} \), we have
\[ |f_\lambda(z_a)| = \frac{|z_a^{2n} - \lambda^{3n+1}|}{|z_a|^n |z_a^{2n} - \lambda^{n-1}|} \leq \frac{|\lambda|^{3n+1}}{|\lambda|^{n+\frac{1}{2}} |z_a^{2n} - \lambda^{n-1}|} \leq |\lambda|^{\frac{n+3}{2}}. \]

By the similar argument, for \( z_b ∈ T_b = \{ z ∈ C : |z| = b \} \), one has
\[ |f_\lambda(z_b)| = \frac{|z_b^{2n} - \lambda^{3n+1}|}{|z_b|^n |z_b^{2n} - \lambda^{n-1}|} \geq \frac{|\lambda|^{n+1}}{|\lambda|^{n+\frac{1}{2}} |z_b^{2n} - \lambda^{n-1}|} \geq |\lambda|^{-\frac{n+1}{2}}. \]

Obviously \( n+1 < \frac{2n+3}{2n+1} < n+\frac{3}{2} \) for \( n ≥ 4 \). Let \( \alpha := \frac{3}{2} + \frac{1}{n} \) and \( \beta := \frac{1}{2} + \frac{1}{n} \). Then
\[ \frac{3n+1}{2n} < \alpha < n\beta < \frac{n+3}{2} \quad \text{and} \quad -\frac{n-1}{2} < 2n+2 - \alpha < -\beta < \frac{n-1}{2n}. \]

Define \( \Omega_0 := \{ z ∈ C : |z| < |\lambda|^\alpha \} \) and \( \Omega_1 := \{ z ∈ C : |z| > |\lambda|^{-\beta} \} \). For \( z ∈ \Omega_0 \) and sufficiently small \( |\lambda| \), a direct calculation shows that \( |z^{2n} - \lambda^{n-1}| \approx |\lambda|^{n-1} \) and \( |z^{2n} - \lambda^{3n+1}| \approx |\lambda|^{3n+1} \). Therefore, we have
\[ |f_\lambda(z)| \approx \frac{|z^{2n} - \lambda^{3n+1}|}{|z^{n}| |z^{2n} - \lambda^{n-1}|} \times \frac{|\lambda|^{3n+1}}{|\lambda|^{n+\frac{1}{2}} |z^{2n} - \lambda^{n-1}|} \approx |\lambda|^{2n+2-\alpha} = |\lambda|^{-\frac{3}{2}} + 1, \]

which implies that \( f_\lambda(T_a) ⊂ \Omega_0 ⊂ D_0 \) if \( |\lambda| \) is small enough.

Similarly, for \( z ∈ \Omega_1 \) and sufficiently small \( |\lambda| \), one can show that \( |z^{2n} - \lambda^{n-1}| \approx |\lambda|^{-2n\beta} \) and \( |z^{2n} - \lambda^{3n+1}| \approx |\lambda|^{-2n\beta} \). This means that
\[ |f_\lambda(z)| \approx \frac{|z^{2n} - \lambda^{3n+1}|}{|z^{n}| |z^{2n} - \lambda^{n-1}|} \times \frac{|\lambda|^{-2n\beta}}{|\lambda|^{-n\beta} |z^{2n} - \lambda^{n-1}|} \approx |\lambda|^{-\frac{n\beta}{2}} = |\lambda|^{\frac{n\beta}{2}} + 1. \]

Hence \( f_\lambda(T_b) ⊂ \Omega_1 ⊂ D_∞ \) if \( |\lambda| \) is small enough.

Next we only need to show that none of the free critical points are contained in \( D_0 \) and \( D_∞ \). To see this, by way of contradiction we assume that there is a free critical point, say \( c_1 \), is contained in \( D_0 \). By Lemma 2.1, one has \( c_i ∈ D_0 \) and \( c'_i ∈ D_∞ \) for every \( 1 ≤ i ≤ 2n \). From the proof of Theorem 1.1 (a) we know that \( J(f_λ) \) is quasicircle. Hence \( \text{int}(J(f_λ)) \) is mapped into \( D_∞ \) under \( f_λ \), which is impossible because \( f_λ(T_a) ⊂ D_0 \) and \( T_a ⊂ \text{int}(J(f_λ)) \).

Note that there exists a free critical point \( c_i \) such that \( f_λ(c_i) ∈ D_∞ \). However \( c_i ∉ D_0 \) when \( |\lambda| ≠ 0 \) is small enough. From Theorem 1.1 (b), it follows that \( J(f_λ) \) is a Cantor set of circles. □

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