Optimal Stopping for Non-linear Expectations

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Abstract

We develop a theory for solving continuous time optimal stopping problems for non-linear expectations. Our motivation is to consider problems in which the stopper uses risk measures to evaluate future rewards.

Keywords: Nonlinear expectations, Optimal stopping, Snell envelope, Stability, $g$-expectations.

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1 Introduction

We solve continuous time optimal stopping problems in which the reward is evaluated using non-linear expectations. Our purpose is to use criteria other than the expected value to evaluate the present value of future rewards. Such criteria include risk measures, which are not necessarily linear. Given a filtered probability space

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Optimal Stopping for Non-linear Expectations

Let \((\Omega, \mathcal{F}, P, \mathbf{F}) = \{\mathcal{F}_t\}_{t \in [0, T]}\) satisfying the usual assumptions, we define a filtration-consistent non-linear expectation (\(\mathbf{F}\)-expectation for short) with domain \(\Lambda\) as a collection of operators \(\{\mathcal{E}[\cdot|\mathcal{F}_t] : \Lambda \to \Lambda_t \triangleq \Lambda \cap L^0(\mathcal{F}_t)\}_{t \in [0, T]}\) satisfying “Monotonicity”, “Time-Consistency”, “Zero-one Law” and “Translation-Invariance”. This definition is similar to the one proposed in Peng [2004]. A notable example of an \(\mathbf{F}\)-satisfying “Monotonicity”, “Time-Consistency”, “Zero-one Law” and “Translation-Invariance”. This definition is convex risk measures when there are multiple priors and the

\[
Z(0) = \sup_{(\iota, \rho) \in \mathcal{I} \times \mathcal{S}_{0, T}} \mathcal{E}_i[Y_{\rho} + H_{\rho}^i]\mathbf{F}_0,
\]

in which \(\mathcal{E} = \{\mathcal{E}_i\}_{i \in \mathcal{I}}\) is a stable class of \(\mathbf{F}\)-expectations, \(\mathcal{S}_{0, T}\) is the set of stopping times that take value in \([0, T]\).

The reward process \(Y\) is a right-continuous \(\mathbf{F}\)-adapted process and for any \(\nu \in \mathcal{S}_{0, T}\), \(Y_{\nu}\) belongs to \(\Lambda^\# \triangleq \{\xi \in \Lambda | \xi \geq c, a.s.\text{ for some } c \in \mathbb{R}\}\), where \(\Lambda\) is the common domain of the elements in \(\mathcal{E}\). On the other hand, the model-dependent reward processes \(\{H^\iota\}_{i \in \mathcal{I}}\) is a family of right-continuous adapted processes with \(H_{\iota}^0 = 0\) that is consistent with \(\mathcal{E}\). We will express the solution of this problem in terms of the \(\mathcal{E}\)-upper Snell envelope \(Z^0\) of \(Y_{\nu}\), the smallest RCLL \(\mathbf{F}\)-adapted process dominating \(Y\) such that \(Z_{\iota, 0} \triangleq \{Z^0_{\iota} + H^i_{\iota}\}_{i \in [0, T]}\) is an \(\mathcal{E}_i\)-supermartingale for each \(i \in \mathcal{I}\).

The construction of the Snell envelope is not straightforward. First, for any \(i \in \mathcal{I}\), the conditional expectation \(\mathcal{E}_i[\cdot|\mathcal{F}_\nu]\), \(\xi \in \Lambda\) and \(\nu \in \mathcal{S}_{0, T}\) may not be well defined. However, we show that \(t \to \mathcal{E}_i[\cdot|\mathcal{F}_t]\) admits a right-continuous modification \(t \to \tilde{\mathcal{E}}_i[\cdot|\mathcal{F}_t]\) for any \(\xi \in \Lambda\) and that \(\tilde{\mathcal{E}}_i\) is itself an \(\mathbf{F}\)-expectation on \(\Lambda^\#\) such that \(\tilde{\mathcal{E}}_i[\cdot|\mathcal{F}_\nu]\) is well defined on \(\Lambda^\#\) for any \(\nu \in \mathcal{S}_{0, T}\). In terms of \(\tilde{\mathcal{E}}_i\) we have that

\[
Z(0) = \sup_{(i, \rho) \in \mathcal{I} \times \mathcal{S}_{0, T}} \tilde{\mathcal{E}}_i[Y_{\rho} + H_{\rho}^i]\mathbf{F}_0.
\]

Finding a RCLL modification requires the development of an upcrossing theorem. This theorem relies on the strict monotonicity of \(\mathcal{E}_i\) and other mild hypotheses, one of which is equivalent to having lower semi-continuity (i.e. Fatou’s lemma). Thanks to the right continuity of \(t \to \tilde{\mathcal{E}}_i[\cdot|\mathcal{F}_t]\), we also have an optional sampling theorem for right-continuous \(\tilde{\mathcal{E}}_i\)-supermartingales. Another important tool in finding an optimal stopping time, the dominated convergence theorem is also developed under another mild assumption.

The stability assumption we make on the family \(\mathcal{E}\) is another essential ingredient in the construction of the Snell envelope. It guarantees that the class \(\mathcal{E}\) is closed under pasting: for any \(i, j \in \mathcal{I}\) and \(\nu \in \mathcal{S}_{0, T}\) there exists a \(k \in \mathcal{I}\) such that \(\tilde{\mathcal{E}}_k[\cdot|\mathcal{F}_\nu] = \tilde{\mathcal{E}}_j[\cdot|\mathcal{F}_{\nu \vee \sigma}]|\mathcal{F}_\sigma\), for any \(\sigma \in \mathcal{S}_{0, T}\). Under this assumption it can then be seen, for example, that the collection of random variables \(\{\tilde{\mathcal{E}}_i[\cdot|\mathcal{F}_\nu + H_{\rho}^i - H_{\sigma}^i]\mathbf{F}_\sigma\}_{i \in \mathcal{I} \times \mathcal{S}_{0, T}}\) is directed upwards. When the constituents of \(\mathcal{E}\) are linear expectations, the notion of stability of this collection is given by Föllmer and Schied [2004, Definition 6.44], who showed that pasting two probability measures equivalent to \(P\) at a stopping time one will result in another probability measure equivalent to \(P\). Our result in Proposition 3.1 shows that we have the same pasting property for \(\mathbf{F}\)-expectations. As we shall see, the stability is a crucial assumption in showing that the Snell envelope is a supermartingale. This property of the Snell envelope is a generalization of time consistency, i.e.,

\[
\text{esssup}_{i \in \mathcal{I}} \tilde{\mathcal{E}}_i[\cdot|\mathcal{F}_\nu] = \text{esssup}_{i \in \mathcal{I}} \tilde{\mathcal{E}}_i \left[ \text{esssup}_{i \in \mathcal{I}} \tilde{\mathcal{E}}_i[\cdot|\mathcal{F}_\sigma]|\mathcal{F}_\nu \right], \quad \forall \nu, \sigma \in \mathcal{S}_{0, T} \text{ with } \nu \leq \sigma, \ a.s.
\]

Delbaen [2006, Theorem 12] showed in the linear expectations case that the time consistency (1.3) is equivalent to the stability.

When the reward \(t \to Y_{\iota} + H_{\iota}^i\) is “\(\mathcal{E}\)-uniformly-left-continuous” and each non-linear expectation in \(\mathcal{E}\) is convex, we can find an optimal stopping time \(\tau(\iota, 0)\) for (1.1) in terms of the Snell envelope. As a corollary we can solve the
problem
\[ \sup_{\rho \in \mathcal{S}_{0,T}} \mathcal{E}_i[Y_{\rho} + H^i_{\rho} | \mathcal{F}_0], \]
(1.4)

when \( \mathcal{E}_i[\cdot | \mathcal{F}_i] \) has among other properties strict monotonicity, lower semi-continuity, dominated convergence theorem and the upcrossing lemma. Note that although, \( \text{esssup}_{i \in I} \mathcal{E}_i[\cdot | \mathcal{F}_i] \) has similar properties to \( \mathcal{E}_i[\cdot | \mathcal{F}_i] \) (and that might lead one to think that (1.1) can actually be considered as a special case of (1.4)), the former does not satisfy strict monotonicity, the upcrossing lemma, and the dominated convergence theorem. One motivation for considering optimal stopping with multiple priors is to solve optimal stopping problems for “non-linear expectations” which do not satisfy these properties.

We show that the collection of \( g \)-expectations with uniformly Lipschitz generators satisfy the uniform left continuity assumption. Moreover, a \( g \)-expectation satisfies all the assumptions we ask of each \( \mathcal{E}_i \) for the upcrossing theorem, Fatou’s lemma and the dominated convergence theorem to hold; and pasting of \( g \)-expectations results in another \( g \)-expectation. As a result the case of \( g \)-expectations presents a non-conventional example in which we can determine an optimal stopping time for (1.1). In fact, in the \( g \)-expectation example we can even find an optimal prior \( i^* \in I \), i.e.,
\[ Z(0) = \mathcal{E}_{i^*}[Y_{\tau(0)} + H^+_{\tau(0)} | \mathcal{F}_0]. \]
(1.5)

In the second problem, the \textit{stopper} tries to find a robust optimal stopping time that attains
\[ V(0) \triangleq \sup_{\rho \in \mathcal{S}_{0,T}} \inf_{i \in I} \mathcal{E}_i[Y_{\rho} + H^i_{\rho} | \mathcal{F}_0]. \]
(1.6)

Under the “\( \triangle \)-uniform-right-continuity” assumption, we find an optimal stopping time in terms of the \( \triangle \)-lower Snell envelope. An immediate by-product is the following minimax theorem
\[ V(0) = \inf_{i \in I} \sup_{\rho \in \mathcal{S}_{0,T}} \mathcal{E}_i[Y_{\rho} + H^i_{\rho} | \mathcal{F}_0]. \]
(1.7)

Our paper was inspired by Karatzas and Zamfirescu [2006] and Karatzas and Zamfirescu [2008], which developed a martingale approach to solving (1.1) and (1.6), when \( \triangle \) is a class of linear expectations. In particular, Karatzas and Zamfirescu [2006] considered the \textit{controller-stopper} problem
\[ \sup_{\rho \in \mathcal{S}_{0,T}} \sup_{U \in \mathcal{U}} \mathbb{E}^u \left[ g(X(\rho)) + \int_0^\rho h(s, X, U_s)ds \right], \]
(1.8)
where \( X(t) = x + \int_0^t f(s, X, U_s)ds + \int_0^t \sigma(s, X)dW^U_s \). In this problem, the stability condition is automatically satisfied. Here, \( g \) and \( h \) are assumed to be bounded measurable functions. Our results on \( g \)-expectations extend the results of Karatzas and Zamfirescu [2006] from bounded rewards to rewards satisfying linear growth. Delbaen [2008], Karatzas and Zamfirescu [2003] also considered (1.1) when the \( \mathcal{E}_i \)’s are linear expectations. The latter paper made a \textit{convexity} assumption on the collection of equivalent probability measures instead of a stability assumption. On the other hand, the discrete time version of the robust optimization problem was analyzed by Föllmer and Schied [2004]. Also see Cheridito et al. [2006, Sections 5.2 and 5.3].

The rest of the paper is organized as follows: In Section 1.1 we will introduce some notations that will be used throughout the paper. In Section 2 we define what we mean by an \( \mathcal{F} \)-expectation \( \mathcal{E} \), propose some basic hypotheses on \( \mathcal{E} \) and discuss their implications such as Fatou’s lemma, dominated convergence theorem and upcrossing lemma. We show that \( t \to \mathcal{E}_i[\cdot | \mathcal{F}_t] \) admits a right-continuous modification which is also an \( \mathcal{F} \)-expectation and satisfies Fatou’s lemma and the dominated convergence theorem. This step is essential since \( \mathcal{E}_i[\cdot | \mathcal{F}_t] \), \( \nu \in \mathcal{S}_{0,T} \) may not be well defined. We also show that the optional sampling theorem holds. The results in Section 2 will be the backbone of our analysis in the later sections.
In Section 3 we introduce the stable class of \( \mathcal{F} \)-expectations and review the properties of essential extremum. In Section 4 we solve (1.2), and find an optimal stopping time in terms of the \( \mathcal{E} \)-upper Snell envelope. On the other hand, in Section 5 we solve the robust optimization problem (1.4) in terms of the \( \mathcal{E} \)-lower Snell envelope. In Section 6 we give some interpretations and remarks on our results in the previous sections. In Section 7 we determine an optimal prior \( \nu^* \in \mathcal{X} \) satisfying (1.5). Moreover, we show how the controller and stopper problem of Karatzas and Zamfirescu 2006 fits into our g-expectations framework. This lets us extend their result from bounded rewards to rewards satisfying linear growth. In this section, we also solve optimal stopping problem for quadratic g-expectations. The proofs of our results are presented in Section 8.

1.1 Notation

Throughout this paper, we fix a finite time horizon \( T > 0 \) and consider a complete probability space \((\Omega, \mathcal{F}, P)\) equipped with a right continuous filtration \( \mathcal{F} \triangleq \{ \mathcal{F}_t \}_{t \in [0,T]} \), not necessarily Brownian one, such that \( \mathcal{F}_0 \) is generated by all \( P \)-null sets in \( \mathcal{F} \) (in fact, \( \mathcal{F}_0 \) collects all measurable sets with probability 0 or 1). Let \( \mathcal{S}_{0,T} \) be the collection of all \( \mathcal{F} \)-stopping times \( \nu \) such that \( 0 \leq \nu \leq T \), a.s. For any \( \nu, \sigma \in \mathcal{S}_{0,T} \) with \( \nu \leq \sigma \), a.s., we define \( \mathcal{S}_{\nu,\sigma} \triangleq \{ \rho \in \mathcal{S}_{0,T} \mid \nu \leq \rho \leq \sigma \}, \) and let \( \mathcal{S}_{\nu,\sigma}^0 \) denote all finite-valued stopping times in \( \mathcal{S}_{\nu,\sigma} \). We let \( \mathcal{D} = \{ k2^{-n} \mid k \in \mathbb{Z}, n \in \mathbb{N} \} \) denote the set of all dyadic rational numbers and set \( \mathcal{D}_T \triangleq ([0, T] \cap \mathcal{D}) \cup \{ T \} \). For any \( t \in [0, T] \) and \( n \in \mathbb{N} \), we also define

\[
q_n^-(t) \triangleq \left( \frac{2^n t - 1}{2^n} \right) \quad \text{and} \quad q_n^+(t) = \frac{2^n t}{2^n} \wedge T.
\]

(1.9)

It is clear that \( q_n^-(t), q_n^+(t) \in \mathcal{D}_T \).

In what follows we let \( \mathcal{F}' \) be a generic sub-\( \sigma \)-field of \( \mathcal{F} \) and let \( \mathcal{B} \) be a generic Banach space with norm \( \| \cdot \|_\mathcal{B} \). The following spaces of functions will be used in the sequel.

(1) For \( 0 \leq p \leq \infty \), we define

- \( L^p(\mathcal{F}'; \mathcal{B}) \) to be the space of all \( \mathcal{B} \)-valued, \( \mathcal{F}' \)-measurable random variables \( \xi \) such that \( E(\| \xi \|_\mathcal{B}^p) < \infty \). In particular, if \( p = 0 \), \( L^0(\mathcal{F}'; \mathcal{B}) \) stands for the space of all \( \mathcal{B} \)-valued, \( \mathcal{F}' \)-measurable random variables; and if \( p = \infty \), \( L^\infty(\mathcal{F}'; \mathcal{B}) \) denotes the space of all \( \mathcal{B} \)-valued, \( \mathcal{F}' \)-measurable random variables \( \xi \) with \( \| \xi \|_\infty \triangleq \mathrm{esssup}_{\omega \in \Omega} |\xi(\omega)|_\mathcal{B} < \infty \).
- \( L^p_F([0, T]; \mathcal{B}) \) to be the space of all \( \mathcal{B} \)-valued, \( \mathcal{F} \)-adapted processes \( X \) such that \( E \int_0^T |X_t|_\mathcal{B}^p \, dt < \infty \). In particular, if \( p = 0 \), \( L^0_F([0, T]; \mathcal{B}) \) stands for the space of all \( \mathcal{B} \)-valued, \( \mathcal{F} \)-adapted processes; and if \( p = \infty \), \( L^\infty_F([0, T]; \mathcal{B}) \) denotes the space of all \( \mathcal{B} \)-valued, \( \mathcal{F} \)-adapted processes \( X \) with \( \| X \|_\infty \triangleq \mathrm{esssup}_{(t,\omega) \in [0, T] \times \Omega} |X_t(\omega)|_\mathcal{B} < \infty \).
- \( C^0_F([0, T]; \mathcal{B}) \triangleq \{ X \in L^0_F([0, T]; \mathcal{B}) : \text{ X has continuous paths} \} \).
- \( H^p_F([0, T]; \mathcal{B}) \triangleq \{ X \in L^p_F([0, T]; \mathcal{B}) : \text{ X is predictably measurable} \} \).

(2) For \( p \geq 1 \), we define a Banach space

\[
M^p_F([0, T]; \mathcal{B}) = \left\{ X \in H^0_F([0, T]; \mathcal{B}) : \| X \|_{M^p} \triangleq \left( E \left[ \left( \int_0^T |X_s|_\mathcal{B}^2 \, ds \right)^{p/2} \right] \right)^{1/p} < \infty \right\},
\]

and denote \( M^p_F([0, T]; \mathcal{B}) \triangleq \bigcap_{p \geq 1} M^p_F([0, T]; \mathcal{B}) \).

(3) We further define

\[
L_p(\mathcal{F}'; \mathcal{B}) \triangleq \left\{ \xi \in L^0(\mathcal{F}'; \mathcal{B}) : E[e^{\lambda|\xi(\omega)|}] < \infty \text{ for all } \lambda > 0 \right\},
\]

\[
C^p_F([0, T]; \mathcal{B}) \triangleq \left\{ X \in C^0_F([0, T]; \mathcal{B}) : E\left[ \exp \left\{ \lambda \sup_{t \in [0, T]} |X_t|_\mathcal{B} \right\} \right] < \infty \text{ for all } \lambda > 0 \right\}.
\]
If $d = 1$, we shall drop $\mathbb{B} = \mathbb{R}$ from the above notations (e.g., $L^p([0,T]) = L^p([0,T]; \mathbb{R})$, $L^p(\mathcal{F}_T) = L^p(\mathcal{F}_T; \mathbb{R})$). In this paper, all $\mathcal{F}$-adapted processes are supposed to be real-valued unless specifying otherwise.

## 2 F-expectations and Their Properties

We will define non-linear expectations on subspaces of $L^0(\mathcal{F}_T)$ satisfying certain algebraic properties, which are listed in the definition below.

**Definition 2.1.** Let $\mathcal{D}_T$ denote the collection of all non-empty subsets $\Lambda$ of $L^0(\mathcal{F}_T)$ satisfying:

1. $0, 1 \in \Lambda$;
2. $\Lambda$ is closed under addition and under multiplication with indicator random variables. Namely, for any $\xi, \eta \in \Lambda$ and $A \in \mathcal{F}_T$, both $\xi + \eta$ and $1_A \xi$ belong to $\Lambda$;
3. $\Lambda$ is positively solid: For any $\xi, \eta \in L^0(\mathcal{F}_T)$ with $0 \leq \xi \leq \eta$, a.s., if $\eta \in \Lambda$, then $\xi \in \Lambda$ as well.

**Remark 2.1.** (1) Each $\Lambda \in \mathcal{D}_T$ is also closed under maximization “$\vee$” and under minimization “$\wedge$”: In fact, for any $\xi, \eta \in \Lambda$, since the set $\{\xi > \eta\} \in \mathcal{F}_T$, (D2) implies that $\xi \vee \eta = \xi 1_{\{\xi > \eta\}} + \eta 1_{\{\xi \leq \eta\}} \in \Lambda$. Similarly, $\xi \wedge \eta \in \Lambda$;
2. For each $\Lambda \in \mathcal{D}_T$, (D1)-(D3) imply that $c \in \Lambda$ for any $c \geq 0$;
3. $\mathcal{D}_T$ is closed under intersections: If $\{\Lambda_i\}_{i \in I}$ is a subset of $\mathcal{D}_T$, then $\bigcap_{i \in I} \Lambda_i \in \mathcal{D}_T$; $\mathcal{D}_T$ is closed under unions of increasing sequences: If $\{\Lambda_n\}_{n \in \mathbb{N}} \subset \mathcal{D}_T$ such that $\Lambda_n \subset \Lambda_{n+1}$ for any $n \in \mathbb{N}$, then $\bigcup_{n \in \mathbb{N}} \Lambda_n \in \mathcal{D}_T$;
4. It is clear that $L^p(\mathcal{F}_T) \in \mathcal{D}_T$ for all $0 \leq p \leq \infty$.

**Definition 2.2.** An $\mathcal{F}$-consistent non-linear expectation ($\mathcal{F}$-expectation for short) is a pair $(\mathcal{E}, \Lambda)$ in which $\Lambda \in \mathcal{D}_T$ and $\mathcal{E}$ denotes a family of operators $\{\mathcal{E}[\cdot|\mathcal{F}_t] : \Lambda \ni \Lambda_t \triangleq \Lambda \cap L^0(\mathcal{F}_t)\}_{t \in [0,T]}$ satisfying the following hypothesis for any $\xi, \eta \in \Lambda$ and $t \in [0,T]$:

1. “Monotonicity (positively strict)”: $\mathcal{E}[\xi|\mathcal{F}_t] \leq \mathcal{E}[\eta|\mathcal{F}_t]$, a.s. if $\xi \leq \eta$, a.s.; Moreover, if $0 \leq \xi \leq \eta$ a.s. and $\mathcal{E}[\xi|\mathcal{F}_0] = \mathcal{E}[\eta|\mathcal{F}_0]$, then $\xi = \eta$, a.s.;
2. “Time Consistency”: $\mathcal{E}[\mathcal{E}[\xi|\mathcal{F}_t]|\mathcal{F}_s] = \mathcal{E}[\xi|\mathcal{F}_s]$, a.s. for any $0 \leq s \leq t \leq T$;
3. “Zero-one Law”: $\mathcal{E}[1_A \xi|\mathcal{F}_t] = 1_A \mathcal{E}[\xi|\mathcal{F}_t]$, a.s. for any $A \in \mathcal{F}_t$;
4. “Translation Invariance”: $\mathcal{E}[\xi + \eta|\mathcal{F}_t] = \mathcal{E}[\xi|\mathcal{F}_t] + \eta$, a.s. if $\eta \in \Lambda_t$.

We denote the domain $\Lambda$ by $Dom(\mathcal{E})$ and define

$$Dom_\nu(\mathcal{E}) \triangleq Dom(\mathcal{E}) \cap L^0(\mathcal{F}_\nu), \quad \forall \nu \in \mathcal{S}_{0,T}.$$  

For any $\xi, \eta \in Dom(\mathcal{E})$ with $\xi = \eta$, a.s., (A1) implies that $\mathcal{E}[\xi|\mathcal{F}_t] = \mathcal{E}[\eta|\mathcal{F}_t]$, a.s. for any $t \in [0,T]$, which shows that the $\mathcal{F}$-expectation $(\mathcal{E}, Dom(\mathcal{E}))$ is well-defined. Moreover, since $Dom_\nu(\mathcal{E}) = Dom(\mathcal{E}) \cap L^0(\mathcal{F}_\nu) \subset L^0(\mathcal{F}_\nu) = \mathbb{R}$, $\mathcal{E}[\cdot|\mathcal{F}_\nu]$ is a real-valued function on $Dom(\mathcal{E})$. In the rest of the paper, we will substitute $\mathcal{E}[\cdot]$ for $\mathcal{E}[\cdot|\mathcal{F}_\nu]$.

**Remark 2.2.** Our definition of $\mathcal{F}$-expectations is similar to that of $\mathcal{F}_t^\mathcal{X}$-consistent non-linear expectations introduced in Peng [2004, page 4].

**Example 2.1.** The following pairs satisfy (A1)-(A4); thus they are $\mathcal{F}$-expectations:

1. $(\{\mathcal{E}[\cdot|\mathcal{F}_t]\}_{t \in [0,T]}, L^1(\mathcal{F}_T))$: the linear expectation $E$ is a special $\mathcal{F}$-expectation with domain $L^1(\mathcal{F}_T)$;
2. $(\{\mathcal{E}_g[\cdot|\mathcal{F}_t]\}_{t \in [0,T]}, L^2(\mathcal{F}_T))$: the $g$-expectation with generator $g(t, z)$ Lipschitz in $z$ (see Peng [1997], Coquet et al. [2003] or Subsection 7.1 of the present paper);
3. $(\{\mathcal{E}_p[\cdot|\mathcal{F}_t]\}_{t \in [0,T]}, L^p(\mathcal{F}_T))$: the $g$-expectation with generator $g(t, z)$ having quadratic growth in $z$ (see Subsection 7.2 of this paper).
F-expectations can alternatively be introduced in a more classical way:

**Proposition 2.1.** Let $\mathcal{E}^\alpha : \Lambda \mapsto \mathbb{R}$ be a mapping on some $\Lambda \in \mathcal{T}$ satisfying:

(a1) For any $\xi, \eta \in \Lambda$ with $\xi \leq \eta$, a.s., we have $\mathcal{E}^\alpha[\xi] \leq \mathcal{E}^\alpha[\eta]$. Moreover, if $\mathcal{E}^\alpha[\xi] = \mathcal{E}^\alpha[\eta]$, then $\xi = \eta$, a.s.;

(a2) For any $\xi \in \Lambda$ and $t \in [0, T]$, there exists a unique random variable $\xi_t \in \Lambda_t$ such that $\mathcal{E}^\alpha[1_A \xi + \gamma] = \mathcal{E}^\alpha[1_A \xi_t + \gamma]$ holds for any $A \in \mathcal{F}_t$ and $\gamma \in \Lambda_t$.

Then $\{\mathcal{E}[\xi|\mathcal{F}_t] \triangleq \xi_t, \xi \in \Lambda\}_{t \in [0,T]}$ defines an $\mathcal{F}$-expectation with domain $\Lambda$.

**Remark 2.3.** For a mapping $\mathcal{E}^\alpha$ on some $\Lambda \in \mathcal{T}$ satisfying (a1) and (a2), the implied operator $\mathcal{E}^\alpha[\cdot|\mathcal{F}_0]$ is also from $\Lambda$ to $\mathbb{R}$, which, however, may not equal $\mathcal{E}^\alpha$. In fact, one can only deduce that $\mathcal{E}^\alpha[\xi] = \mathcal{E}^\alpha[\mathcal{E}^\alpha[\xi|\mathcal{F}_0]]$ for any $\xi \in \Lambda$.

From now on, when we say an $\mathcal{F}$-expectation $\mathcal{E}$, we will refer to the pair $(\mathcal{E}, \text{Dom}(\mathcal{E}))$. Besides (A1)-(A4), the $\mathcal{F}$-expectation $\mathcal{E}$ has the following properties:

**Proposition 2.2.** For any $\xi, \eta \in \text{Dom}(\mathcal{E})$ and $t \in [0, T]$, we have

(1) "Local Property": $\mathcal{E}[1_A \xi + 1_A \eta|\mathcal{F}_t] = 1_A \mathcal{E}[\xi|\mathcal{F}_t] + 1_A \mathcal{E}[\eta|\mathcal{F}_t]$, a.s. for any $A \in \mathcal{F}_t$;

(2) "Constant-Preserving": $\mathcal{E}[\xi|\mathcal{F}_t] = \xi$, a.s. if $\xi \in \text{Dom}_t(\mathcal{E})$;

(3) "Comparison": Let $\xi, \eta \in L^0(\mathcal{F}_\nu)$ for some $\nu \in \mathcal{S}_{0,T}$. If $\eta \geq c$, a.s. for some $c \in \mathbb{R}$, then $\xi \leq (\text{or } =) \eta$, a.s.

if and only if $\mathcal{E}[1_A \xi] \leq (\text{or } =) \mathcal{E}[1_A \eta]$ for all $A \in \mathcal{F}_\nu$.

The following two subsets of $\text{Dom}(\mathcal{E})$ will be of interest:

$\text{Dom}^+(\mathcal{E}) \triangleq \{\xi \in \text{Dom}(\mathcal{E}) : \xi \geq 0, \text{ a.s.}\}, \text{Dom}^\#(\mathcal{E}) \triangleq \{\xi \in \text{Dom}(\mathcal{E}) : \xi \geq c, \text{ a.s. for some } c = c(\xi) \in \mathbb{R}\}$. (2.1)

**Remark 2.4.** The restrictions of $\mathcal{E}$ on $\text{Dom}^+(\mathcal{E})$ and on $\text{Dom}^\#(\mathcal{E})$, namely $(\mathcal{E}, \text{Dom}^+(\mathcal{E}))$ and $(\mathcal{E}, \text{Dom}^\#(\mathcal{E}))$ respectively, are both $\mathcal{F}$-expectations: To see this, first note that $\text{Dom}^+(\mathcal{E})$ and $\text{Dom}^\#(\mathcal{E})$ both belong to $\mathcal{T}$. For any $t \in [0, T]$, (A1) and Proposition 2.2 (2) imply that for any $\xi \in \text{Dom}^\#(\mathcal{E})$

$\mathcal{E}[\xi|\mathcal{F}_t] \geq \mathcal{E}[c(\xi)|\mathcal{F}_t] = c(\xi), \text{ a.s.} \Rightarrow \mathcal{E}[\xi|\mathcal{F}_t] \in \text{Dom}^\#(\mathcal{E}),$

which shows that $\mathcal{E}[\cdot|\mathcal{F}_t]$ maps $\text{Dom}^\#(\mathcal{E})$ into $\text{Dom}^\#(\mathcal{E}) \cap L^0(\mathcal{F}_t)$. Then it is easy to check that the restriction of $\mathcal{E} = \{\mathcal{E}[\cdot|\mathcal{F}_t]\}_{t \in [0,T]}$ on $\text{Dom}^\#(\mathcal{E})$ satisfies (A1) to (A4), thus it is an $\mathcal{F}$-expectation. Similarly, $(\mathcal{E}, \text{Dom}^+(\mathcal{E}))$ is also an $\mathcal{F}$-expectation.

We should remark that restricting $\mathcal{E}$ on any subset $\Lambda'$ of $\text{Dom}(\mathcal{E})$, with $\Lambda' \in \mathcal{T}$, may not result in an $\mathcal{F}$-expectation, i.e. $(\mathcal{E}, \Lambda')$ may not be an $\mathcal{F}$-expectation.

**Definition 2.3.** (1) An $\mathcal{F}$-adapted process $X = \{X_t\}_{t \in [0,T]}$ is called an \textit{\"E-process\"} if $X_t \in \text{Dom}(\mathcal{E})$ for any $t \in [0, T]$;

(2) An $\mathcal{E}$-process $X$ is said to be an $\mathcal{E}$-supermartingale (resp. $\mathcal{E}$-martingale, $\mathcal{E}$-submartingale) if for any $0 \leq s < t \leq T$, $\mathcal{E}[X_t|\mathcal{F}_s] \leq (\text{resp. } =, \geq) X_s$, a.s.

Given a $\nu \in \mathcal{S}_{0,T}^\mathcal{F}$ taking values in a finite set $\{t_1 < \cdots < t_n\}$, if $X$ is an $\mathcal{E}$-process, (D2) implies that $X_{\nu} = \sum_{i=1}^{n} 1_{\{\nu = t_i\}} X_{t_i} \in \text{Dom}(\mathcal{E})$, thus $X_{\nu} \in \text{Dom}_{\nu}(\mathcal{E})$. Since $\{X_t^\xi \triangleq \mathcal{E}[\xi|\mathcal{F}_t]\}_{t \in [0,T]}$ is an $\mathcal{E}$-process for any $\xi \in \text{Dom}(\mathcal{E})$, we can define an operator $\mathcal{E}^\xi[\cdot|\mathcal{F}_\nu]$ from $\text{Dom}(\mathcal{E})$ to $\text{Dom}_\nu(\mathcal{E})$ by

$\mathcal{E}[\xi|\mathcal{F}_\nu] \triangleq X_{\nu}^\xi$, for any $\xi \in \text{Dom}(\mathcal{E}),$

which allows us to state a basic Optimal Sampling Theorem for $\mathcal{E}$.

**Proposition 2.3.** (Optional Sampling Theorem) Let $X$ be an $\mathcal{E}$-supermartingale (resp. $\mathcal{E}$-martingale, $\mathcal{E}$-submartingale). Then for any $\nu, \sigma \in \mathcal{S}_{0,T}^\mathcal{F}$, $\mathcal{E}[X_{\nu}|\mathcal{F}_\sigma] \leq (\text{resp. } =, \geq) X_{\nu \wedge \sigma}$, a.s.
In particular, applying Proposition 2.4 to each $\mathcal{E}$-martingale $\{\mathcal{E}[\xi|\mathcal{F}_t]\}_{t\in[0,T]}$, in which $\xi \in \operatorname{Dom}(\mathcal{E})$, yields the following result.

**Corollary 2.1.** For any $\xi \in \operatorname{Dom}(\mathcal{E})$ and $\nu, \sigma \in \mathcal{S}_{0, T}^F$, $\mathcal{E}[\mathcal{E}[\xi|\mathcal{F}_\nu]|\mathcal{F}_\sigma] = \mathcal{E}[\xi|\mathcal{F}_{\nu, \sigma}]$, a.s.

**Remark 2.5.** Corollary 2.1 extends the “Time-Consistency” (A2) to the case of finite-valued stopping times.

$\mathcal{E}[\cdot|\mathcal{F}_\nu]$ inherits other properties of $\mathcal{E}[\cdot|\mathcal{F}_t]$ as well:

**Proposition 2.4.** For any $\xi, \eta \in \operatorname{Dom}(\mathcal{E})$ and $\nu \in \mathcal{S}_{0, T}^F$, it holds that

1. “Monotonicity (positively strict)”: $\mathcal{E}[\xi|\mathcal{F}_\nu] \leq \mathcal{E}[\eta|\mathcal{F}_\nu]$, a.s. if $\xi \leq \eta$, a.s.; Moreover, if $0 \leq \xi \leq \eta$, a.s. and $\mathcal{E}[\xi|\mathcal{F}_\sigma] = \mathcal{E}[\eta|\mathcal{F}_\sigma]$, a.s. for some $\sigma \in \mathcal{S}_{0, T}^F$, then $\xi = \eta$, a.s.;
2. “Zero-one Law”: $\mathcal{E}[1_A \xi|\mathcal{F}_\nu] = 1_A \mathcal{E}[\xi|\mathcal{F}_\nu]$, a.s. for any $A \in \mathcal{F}_\nu$;
3. “Translation Invariance”: $\mathcal{E}[\xi + \eta|\mathcal{F}_\nu] = \mathcal{E}[\xi|\mathcal{F}_\nu] + \mathcal{E}[\eta|\mathcal{F}_\nu]$, a.s. if $\eta \in \operatorname{Dom}_\nu(\mathcal{E})$;
4. “Local Property”: $\mathcal{E} [1_A \xi + 1_{A'} \eta|\mathcal{F}_\nu] = 1_A \mathcal{E} [\xi|\mathcal{F}_\nu] + 1_{A'} \mathcal{E}[\eta|\mathcal{F}_\nu]$, a.s. for any $A \in \mathcal{F}_\nu$;
5. “Constant-Preserving”: $\mathcal{E}[\xi|\mathcal{F}_\nu] = \xi$, a.s., if $\xi \in \operatorname{Dom}_\nu(\mathcal{E})$.

We make the following basic hypotheses on the $\mathbf{F}$-expectation $\mathcal{E}$. These hypotheses will be essential in developing Fatou’s lemma, the Dominated Convergence Theorem and the Upcrossing Theorem.

**Hypotheses**

**(H0)** For any $A \in \mathcal{F}_T$ with $P(A) > 0$, we have $\lim_{n \to \infty} \mathcal{E}[n 1_A] = \infty$;

**(H1)** For any $\xi \in \operatorname{Dom}^+(\mathcal{E})$ and any $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{F}_T$ with $\lim_{n \to \infty} 1_{A_n} = 1$, a.s., we have $\lim_{n \to \infty} \mathcal{E}[1_{A_n} \xi] = \mathcal{E}[\xi]$;

**(H2)** For any $\xi, \eta \in \operatorname{Dom}^+(\mathcal{E})$ and any $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{F}_T$ with $\lim_{n \to \infty} 1_{A_n} = 0$, a.s., we have $\lim_{n \to \infty} \mathcal{E}[\xi + 1_{A_n} \eta] = \mathcal{E}[\xi]$.

**Remark 2.6.** The linear expectation $E$ on $L^1(\mathcal{F}_T)$ clearly satisfies (H0)-(H2). We will show that Lipschitz and quadratic $g$-expectations also satisfy (H0)-(H2) in Propositions 7.1 and 7.5 respectively.

The $\mathbf{F}$-expectation $\mathcal{E}$ satisfies the following Fatou’s Lemma and Dominated Convergence Theorem.

**Theorem 2.1.** (Fatou’s Lemma) (H1) is equivalent to the lower semi-continuity of $\mathcal{E}$: If a sequence $\{\xi_n\}_{n \in \mathbb{N}} \subset \operatorname{Dom}^+(\mathcal{E})$ converges a.s. to some $\xi \in \operatorname{Dom}^+(\mathcal{E})$, then for any $\nu \in \mathcal{S}_{0, T}^F$, we have

$$
\mathcal{E}[\xi|\mathcal{F}_\nu] \leq \lim_{n \to \infty} \mathcal{E}[\xi_n|\mathcal{F}_\nu], \quad \text{a.s.,}
$$

(2.2)

where the right hand side of (2.2) could be equal to infinity with non-zero probability.

**Remark 2.7.** In the case of the linear expectation $E$, a converse to (2.2) holds: For any non-negative sequence $\{\xi_n\}_{n \in \mathbb{N}} \subset L^1(\mathcal{F}_T)$ that converges a.s. to some $\xi \in L^0(\mathcal{F}_T)$, if $\lim_{n \to \infty} E[\xi_n] < \infty$, then $\xi \in L^1(\mathcal{F}_T)$. However, this statement may not be the case for an arbitrary $\mathbf{F}$-expectation. That is, $\lim_{n \to \infty} E[\xi_n] < \infty$ may not imply that $\xi \in \operatorname{Dom}^+(\mathcal{E})$ given that $\{\xi_n\}_{n \in \mathbb{N}} \subset \operatorname{Dom}^+(\mathcal{E})$ is a sequence convergent a.s. to some $\xi \in L^0(\mathcal{F}_T)$. (See Example 7.4 for a counterexample in the case of a Lipschitz $g$-expectation.)

**Theorem 2.2.** (Dominated Convergence Theorem) Assume (H1) and (H2) hold. Let $\{\xi_n\}_{n \in \mathbb{N}}$ be a sequence in $\operatorname{Dom}^+(\mathcal{E})$ that converges a.s. If there is an $\eta \in \operatorname{Dom}^+(\mathcal{E})$ such that $\xi_n \leq \eta$ a.s. for any $n \in \mathbb{N}$, then the limit $\xi$ of $\{\xi_n\}_{n \in \mathbb{N}}$ belongs to $\operatorname{Dom}^+(\mathcal{E})$, and for any $\nu \in \mathcal{S}_{0, T}^F$, we have

$$
\lim_{n \to \infty} \mathcal{E}[\xi_n|\mathcal{F}_\nu] = \mathcal{E}[\xi|\mathcal{F}_\nu], \quad \text{a.s.}
$$
Next, we will derive an Upcrossing Theorem for $\mathcal{E}$-supermartingales, which is crucial in obtaining an RCLL (right-continuous, with limits from the left) modification for the process $\{\mathcal{E}[\xi, \mathcal{F}_t]\}_{t \in [0, T]}$ as long as $\xi \in \text{Dom}(\mathcal{E})$ is bounded from below. Obtaining a right continuous modification is crucial, since otherwise the conditional expectation $\mathcal{E}[\xi, \mathcal{F}_t]$ may not be well defined for any $\nu \in \mathcal{S}_{0,T}$.

Let us first recall what the “number of upcrossings” is: Given a real-valued process $\xi$ that $\xi$ is an RCLL process. Then we can deduce from (D2) that

$$\nu_{2j-1}(\omega) \triangleq \min\{t \in F : t > \nu_{2j-2}(\omega), X_t(\omega) < a\} \land T \in \mathcal{S}_{0,T}^F,$$

$$\nu_{2j}(\omega) \triangleq \min\{t \in F : t > \nu_{2j-1}(\omega), X_t(\omega) > b\} \land T \in \mathcal{S}_{0,T}^F,$$

with the convention that $\min \emptyset = \infty$. Then $U_I(a, b; X(\omega))$ is defined to be the largest integer $j$ for which $\nu_{2j}(\omega) < T$. If $I \subset [0, T]$ is not a finite set, we define

$$U_I(a, b; X(\omega)) \triangleq \sup\{U_F(a, b; X(\omega)) : F \text{ is a finite subset of } I\}.$$

It will be convenient to introduce a subcollection of $\mathcal{T}_T$

$$\tilde{\mathcal{T}}_T \triangleq \{\Lambda \in \mathcal{T}_T : \mathbb{R} \subset \Lambda\}.$$

Clearly, $\tilde{\mathcal{T}}_T$ contains all $L^p(\mathcal{F}_T), 0 \leq p \leq \infty$. In particular, $L^\infty(\mathcal{F}_T)$ is the smallest element of $\tilde{\mathcal{T}}_T$ in the following sense:

**Lemma 2.1.** For each $\Lambda \in \tilde{\mathcal{T}}_T$, $L^\infty(\mathcal{F}_T) \subset \Lambda$.

**Proof:** For any $\xi \in L^\infty(\mathcal{F}_T)$, we have $-||\xi||_\infty, 2||\xi||_\infty \in \mathbb{R} \subset \Lambda$. Since $0 \leq \xi + ||\xi||_\infty \leq 2||\xi||_\infty$, a.s., (D3) implies that $\xi + ||\xi||_\infty \in \Lambda$. Then we can deduce from (D2) that $\xi = (\xi + ||\xi||_\infty) + (-||\xi||_\infty) \in \Lambda$. \hfill \Box

For any $\mathbf{F}$-adapted process $X$, we define its left-limit and right-limit processes as follows:

$$X^-_t \triangleq \lim_{n \to \infty} X_{q^-_n(t)} \quad \text{and} \quad X^+_t \triangleq \lim_{n \to \infty} X_{q^+_n(t)}, \quad \text{for any } t \in [0, T],$$

where $q^-_n(t)$ and $q^+_n(t)$ are defined in (1.9). Since the filtration $\mathbf{F}$ is right-continuous, we see that both $X^-$ and $X^+$ are $\mathbf{F}$-adapted processes.

It is now the time to present our Upcrossing Theorem for $\mathcal{E}$-supermartingales.

**Theorem 2.3.** (Upcrossing Theorem) Assume that (H0), (H1) hold and that $\text{Dom}(\mathcal{E}) \in \tilde{\mathcal{T}}_T$. For any $\mathcal{E}$-supermartingale $X$, we assume either that $X_T \geq c$, a.s. for some $c \in \mathbb{R}$ or that the operator $\mathcal{E}[\cdot]$ is concave: For any $\xi, \eta \in \text{Dom}(\mathcal{E})$

$$\mathcal{E}[\lambda \xi + (1-\lambda)\eta] \geq \lambda \mathcal{E}[\xi] + (1-\lambda)\mathcal{E}[\eta], \quad \forall \lambda \in (0, 1). \quad (2.3)$$

Then for any two real numbers $a < b$, it holds that $P(U_{\mathcal{T}_T}(a, b; X) < \infty) = 1$. Thus we have

$$P\left(X^-_t = \lim_{n \to \infty} X_{q^-_n(t)} \quad \text{and} \quad X^+_t = \lim_{n \to \infty} X_{q^+_n(t)} \quad \text{for any } t \in [0, T]\right) = 1. \quad (2.4)$$

As a result, $X^+$ is an RCLL process.

In the rest of this section, we assume that the $\mathbf{F}$-expectation $\mathcal{E}$ satisfies (H0)-(H2) and that $\text{Dom}(\mathcal{E}) \in \tilde{\mathcal{T}}_T$. The following proposition will play a fundamental role throughout this paper.

**Proposition 2.5.** Let $X$ be a non-negative $\mathcal{E}$-supermartingale. (1) Assume either that $\text{esssup}_{t \in \mathcal{T}_T} X_t \in \text{Dom}^+(\mathcal{E})$ or that for any sequence $\{\xi_n\}_{n \in \mathbb{N}} \subset \text{Dom}^+(\mathcal{E})$ convergent a.s. to some $\xi \in L^0(\mathcal{F}_T)$,

$$\lim_{n \to \infty} \mathcal{E}[\xi_n] < \infty \implies \xi \in \text{Dom}^+(\mathcal{E}). \quad (2.5)$$
Then for any \( \nu \in \mathcal{S}_{0,T} \), \( X^\nu_\cdot \) and \( X^\nu_\cdot \) both belong to \( \text{Dom}^+(\mathcal{E}) \):

(2) If \( X^\nu_t \in \text{Dom}^+(\mathcal{E}) \) for any \( t \in [0,T] \), then \( X^+ \) is an RCLL \( \mathcal{E} \)-supermartingale such that for any \( t \in [0,T] \), \( X^+_t \leq X_t \), a.s.;

(3) Moreover, if the function \( t \mapsto \mathcal{E}[X_t] \) from \( [0,T] \) to \( \mathbb{R} \) is right continuous, then \( X^+ \) is an RCLL modification of \( X \). Conversely, if \( X \) has a right-continuous modification, then the function \( t \mapsto \mathcal{E}[X_t] \) is right continuous.

Now we add one more hypothesis to the \( \mathcal{F} \)-expectation \( \mathcal{E} \):

(H3) For any \( \xi \in \text{Dom}^+(\mathcal{E}) \) and \( \nu \in \mathcal{S}_{0,T} \), \( X^\nu_{\cdot}^+ \in \text{Dom}^+(\mathcal{E}) \).

In light of Proposition 2.5 (1), (H3) holds if \( \sup_{t \in [0,T]} \mathcal{E} \xi \in \text{Dom}^+(\mathcal{E}) \) or if \( \mathcal{E} \) satisfies (2.3).

For each \( \xi \in \text{Dom}^+(\mathcal{E}) \), we define \( \xi^+ \in \text{Dom}^+(\mathcal{E}) \). Clearly \( X^\xi_t \triangleq \{ \xi^+ \}_{t \in [0,T]} \) is a non-negative \( \mathcal{E} \)-martingale. By (A2), \( \mathcal{E} X^\xi_t = \mathcal{E} \xi^+ \) for any \( t \in [0,T] \), which means that \( t \mapsto \mathcal{E} X^\xi_t \) is continuous function on \( [0,T] \). Thanks to Proposition 2.5 (2) and (H3), the process \( X^\xi_{\cdot}^+ \triangleq \lim_{n \to \infty} X^\xi_{\cdot^+} t \in [0,T] \) is an RCLL modification of \( X^\xi \). Then for any \( \nu \in \mathcal{S}_{0,T} \), we define

\[
\tilde{\mathcal{E}}[\xi|\mathcal{F}_\nu] \triangleq X^\nu_{\cdot}^+ + c(\xi)
\]

as the conditional \( \mathcal{F} \)-expectation of \( \xi \) at the stopping time \( \nu \in \mathcal{S}_{0,T} \). Since \( \sup_{t \in [0,T]} \mathcal{E} \xi \in \text{Dom}^+(\mathcal{E}) \), Lemma 2.1 (H3), (D2) as well as the non-negativity of \( X^\nu_{\cdot}^+ \) imply that

\[
\tilde{\mathcal{E}}[\xi|\mathcal{F}_\nu] \in \text{Dom}^+(\mathcal{E}),
\]

which shows that \( \tilde{\mathcal{E}}[\cdot|\mathcal{F}_\nu] \) is an operator from \( \text{Dom}^+(\mathcal{E}) \) to \( \text{Dom}^+(\mathcal{E}) \cap L^0(\mathcal{F}_\nu) \). In fact, \( \{ \tilde{\mathcal{E}}[\cdot|\mathcal{F}_\nu] \}_{t \in [0,T]} \) defines a \( \mathcal{F} \)-expectation on \( \text{Dom}^+(\mathcal{E}) \), as the next result shows.

**Proposition 2.6.** For any \( \xi \in \text{Dom}^+(\mathcal{E}) \), \( \tilde{\mathcal{E}}[\xi|\mathcal{F}_\nu] \) is an RCLL modification of \( \mathcal{E}[\xi|\mathcal{F}_\nu] \). \( \{ \tilde{\mathcal{E}}[\cdot|\mathcal{F}_\nu] \}_{t \in [0,T]} \) is an \( \mathcal{F} \)-expectation with domain \( \text{Dom}(\tilde{\mathcal{E}}) = \text{Dom}^+(\mathcal{E}) \subset \tilde{\mathcal{F}}_T \) and satisfying (H0)-(H2); thus all preceding results are applicable to \( \tilde{\mathcal{E}} \).

**Proof:** As \( \text{Dom}(\mathcal{E}) \subset \tilde{\mathcal{F}}_T \) is assumed, we see that \( \text{Dom}^+(\mathcal{E}) \) also belongs to \( \tilde{\mathcal{F}}_T \). Fix \( \xi \in \text{Dom}^+(\mathcal{E}) \). Since \( X^\xi_{\cdot}^+ \) is an RCLL modification of \( X^\xi_{\cdot} \), (A4) implies that for any \( t \in [0,T] \)

\[
\tilde{\mathcal{E}}[\xi|\mathcal{F}_\nu] = X^\nu_{\cdot}^+ + c(\xi) = \mathcal{E}[\xi^+|\mathcal{F}_\nu] + c(\xi) = \mathcal{E} \xi^+ c(\xi)|\mathcal{F}_\nu] = \mathcal{E}[\xi|\mathcal{F}_\nu], \quad \text{a.s.}
\]

Thus \( \tilde{\mathcal{E}}[\xi|\mathcal{F}_\nu] \) is actually an RCLL modification of \( \mathcal{E}[\xi|\mathcal{F}_\nu] \). Then it is easy to show that the pair \( \{ \tilde{\mathcal{E}}, \text{Dom}^+(\mathcal{E}) \} \) satisfies (A1)-(A4) and (H0)-(H2); thus it is an \( \mathcal{F} \)-expectation.

We restate Proposition 2.5 with respect to \( \tilde{\mathcal{E}} \) for future use.

**Corollary 2.2.** Let \( X \) be an \( \tilde{\mathcal{E}} \)-supermartingale such that \( \text{essinf}_{t \in [0,T]} X_t \geq c \), a.s. for some \( c \in \mathbb{R} \).

(1) If \( \sup_{t \in [0,T]} \mathcal{E} X_t \in \text{Dom}^+(\mathcal{E}) \) or if \( \sup_{t \in [0,T]} \mathcal{E} X_t \in \text{Dom}^+(\mathcal{E}) \) holds, then \( X^- \) and \( X^+_\nu \) belong to \( \text{Dom}^+(\mathcal{E}) \) for any \( \nu \in \mathcal{S}_{0,T} \):

(2) If \( X^\nu_t \in \text{Dom}^+(\mathcal{E}) \) for any \( t \in [0,T] \), then \( X^+ \) is an RCLL \( \tilde{\mathcal{E}} \)-supermartingale such that for any \( t \in [0,T] \), \( X^+_t \leq X_t \), a.s.;

(3) Moreover, if the function \( t \mapsto \tilde{\mathcal{E}}[X_t] \) from \( [0,T] \) to \( \mathbb{R} \) is right continuous, then \( X^+ \) is an RCLL modification of \( X \). Conversely, if \( X \) has a right-continuous modification, then the function \( t \mapsto \tilde{\mathcal{E}}[X_t] \) is right continuous.

The next result is the Optional Sampling Theorem of \( \tilde{\mathcal{E}} \) for the stopping times in \( \mathcal{S}_{0,T} \).
Theorem 2.4. (Optional Sampling Theorem 2) Let $X$ be a right-continuous $\tilde{E}$-supermartingale (resp. $\tilde{E}$-martingale, $\tilde{E}$-submartingale) such that $\operatorname{essinf}_{t \in \mathcal{D}_T} X_t \geq c$, a.s. for some $c \in \mathbb{R}$. If $X_\nu \in \operatorname{Dom}_\#(\mathcal{E})$ for any $\nu \in \mathcal{S}_{0,T}$, then for any $\nu, \sigma \in \mathcal{S}_{0,T}$, we have
\[
\tilde{E}[X_\nu|\mathcal{F}_\sigma] \leq (\operatorname{resp.} =, \geq) X_{\nu \wedge \sigma}, \quad \text{a.s.}
\]
Using the Optional Sampling Theorem, we are able to extend Corollary 2.3 and Proposition 2.4 to the operators $\tilde{E}[\cdot|\mathcal{F}_\nu], \nu \in \mathcal{S}_{0,T}$.

Corollary 2.3. For any $\xi \in \operatorname{Dom}_\#(\mathcal{E})$ and $\nu, \sigma \in \mathcal{S}_{0,T}$, we have
\[
\tilde{E}[\tilde{E}[\xi|\mathcal{F}_\nu]|\mathcal{F}_\sigma] = \tilde{E}[\xi|\mathcal{F}_{\nu \wedge \sigma}], \quad \text{a.s.}
\] (2.9)

Proof: Since $(\tilde{E}, \operatorname{Dom}_\#(\mathcal{E}))$ is an $\mathcal{F}$-expectation by Proposition 2.6 for any $\xi \in \operatorname{Dom}_\#(\mathcal{E})$, (A2) implies that the RCLL process $\tilde{X}_\xi \triangleq \{\tilde{E}[\xi|\mathcal{F}_t]\}_{t \in [0,T]}$ is an $\tilde{E}$-martingale. For any $t \in [0,T]$, Proposition 2.8 and 2.2 (2) show that
\[
\tilde{X}_\xi^t = \tilde{E}[\xi|\mathcal{F}_t] \geq \tilde{E}[\xi|\mathcal{F}_t] = \mathcal{E}[\xi|\mathcal{F}_t] = c(\xi), \quad \text{a.s.,}
\]
which implies that $\operatorname{essinf}_{t \in [0,T]} \tilde{X}_\xi^t \geq c(\xi)$, a.s. Then (2.7) and Theorem 2.4 give rise to (2.9). \qed

Proposition 2.7. For any $\xi, \eta \in \operatorname{Dom}_\#(\mathcal{E})$ and $\nu \in \mathcal{S}_{0,T}$, it holds that

1. “Strict Monotonicity”: $\tilde{E}[\xi|\mathcal{F}_\nu] \leq \tilde{E}[\eta|\mathcal{F}_\nu], \text{a.s. if } \xi \leq \eta, \text{a.s.; Moreover, if } \tilde{E}[\xi|\mathcal{F}_\nu] = \tilde{E}[\eta|\mathcal{F}_\nu], \text{a.s. for some } \sigma \in \mathcal{S}_{0,T}, \text{then } \xi = \eta, \text{a.s.;}

2. “Zero-one Law”: $\tilde{E}[1_A|\mathcal{F}_\nu] = 1_A \tilde{E}[\xi|\mathcal{F}_\nu], \text{a.s. for any } A \in \mathcal{F}_\nu;

3. “Translation Invariance”: $\tilde{E}[\xi + \eta|\mathcal{F}_\nu] = \tilde{E}[\xi|\mathcal{F}_\nu] + \eta, \text{a.s. if } \eta \in \operatorname{Dom}_\#(\mathcal{E});

4. “Local Property”: $\tilde{E}[1_A \xi + 1_A \eta|\mathcal{F}_\nu] = 1_A \tilde{E}[\xi|\mathcal{F}_\nu] + 1_A \tilde{E}[\eta|\mathcal{F}_\nu], \text{a.s. for any } A \in \mathcal{F}_\nu;

5. “Constant-Preserving”: $\tilde{E}[\xi|\mathcal{F}_\nu] = \xi, \text{a.s., if } \xi \in \operatorname{Dom}_\#(\mathcal{E}).$

Remark 2.8. Corollary 2.3, Proposition 2.7 (2) and 2.8 imply that for any $\xi \in \operatorname{Dom}_\#(\mathcal{E})$ and $\nu \in \mathcal{S}_{0,T}$,
\[
\mathcal{E}[1_A \xi] = \tilde{E}[1_A \xi] = \tilde{E}[\tilde{E}[1_A \xi]|\mathcal{F}_\nu] = \tilde{E}[1_A \tilde{E}[\xi|\mathcal{F}_\nu]] = \tilde{E}[\tilde{E}[\xi|\mathcal{F}_\nu]], \quad \forall A \in \mathcal{F}_\nu.
\] (2.10)

In light of Proposition 2.2 (3), $\tilde{E}[\xi|\mathcal{F}_\nu]$ is the unique element (up to a $\mathcal{P}$-null set) in $\operatorname{Dom}_\#(\mathcal{E})$ that makes (2.10) hold. Therefore, we see that the random variable $\mathcal{E}[\xi|\mathcal{F}_\nu]$ defined by (2.10) is exactly the conditional $\mathcal{F}$-expectation of $\xi$ at the stopping time $\nu$ in the classical sense.

In light of Corollary 2.3 and Proposition 2.7, we can generalize Fatou’s Lemma (Theorem 2.1) and the Dominated Convergence Theorem (Theorem 2.2) to the conditional $\mathcal{F}$-expectation $\tilde{E}[\cdot|\mathcal{F}_\nu], \nu \in \mathcal{S}_{0,T}$.

Proposition 2.8. (Fatou’s Lemma 2) Let $\{\xi_n\}_{n \in \mathbb{N}}$ be a sequence in $\operatorname{Dom}_\#(\mathcal{E})$ that converges a.s. to some $\xi \in \operatorname{Dom}_\#(\mathcal{E})$ and satisfies $\operatorname{essinf} \xi_n \geq c$, a.s. for some $c \in \mathbb{R}$, then for any $\nu \in \mathcal{S}_{0,T}$, we have
\[
\tilde{E}[\xi|\mathcal{F}_\nu] \leq \lim_{n \to \infty} \tilde{E}[\xi_n|\mathcal{F}_\nu], \quad \text{a.s.,}
\] (2.11)

where the right hand side of (2.11) could be equal to infinity with non-zero probability.

Proposition 2.9. (Dominated Convergence Theorem 2) Let $\{\xi_n\}_{n \in \mathbb{N}}$ be a sequence in $\operatorname{Dom}_\#(\mathcal{E})$ that converges a.s. and that satisfies $\operatorname{essinf} \xi_n \geq c$, a.s. for some $c \in \mathbb{R}$. If there is an $\eta \in \operatorname{Dom}_\#(\mathcal{E})$ such that $\xi_n \leq \eta$ a.s. for any $n \in \mathbb{N}$, then the limit $\xi$ of $\{\xi_n\}_{n \in \mathbb{N}}$ belongs to $\operatorname{Dom}_\#(\mathcal{E})$ and for any $\nu \in \mathcal{S}_{0,T}$, we have
\[
\lim_{n \to \infty} \tilde{E}[\xi_n|\mathcal{F}_\nu] = \tilde{E}[\xi|\mathcal{F}_\nu], \quad \text{a.s.}
\] (2.12)

Proof of Proposition 2.8 and 2.9: In the proofs of Theorem 2.1 and Theorem 2.2, we only need to replace $\{\xi_n\}_{n \in \mathbb{N}}$ and $\mathcal{E}[\cdot|\mathcal{F}_\nu]$ by $\{\xi_n - c\}_{n \in \mathbb{N}}$ and $\tilde{E}[\cdot|\mathcal{F}_\nu]$ respectively. Instead of (A1), (A3) and (A4), we apply Proposition 2.7 (1)-(3). Moreover, since (A2) is only used on $\operatorname{Dom}_\#(\mathcal{E})$ in the proofs of Theorem 2.1 and Theorem 2.2, we can substitute Corollary 2.3 for it. Eventually, a simple application of Proposition 2.7 (3) yields (2.11) and (2.12). \qed
3 Collections of F-Expectations

In this section, we will show that pasting of two F-expectations at a given stopping time is itself an F-expectation. Moreover, pasting preserves (H1) and (H2). We will then introduce the concept of a stable class of F-expectations, which are collections closed under pasting. We will solve the optimal stopping problems introduced in ([1.1]) and ([1.2]) over this class of F-expectations. Before we show the pasting property of F-expectations, we introduce the concept of convexity for an F-expectation and give one of the consequences of having convexity:

Definition 3.1. An F-expectation E is called “positively-convex” if for any \( \xi, \eta \in \text{Dom}^+(E) \), \( \lambda \in (0,1) \) and \( t \in [0,T] \)

\[
E[\lambda \xi + (1-\lambda)\eta | F_t] \leq \lambda E[\xi | F_t] + (1-\lambda)E[\eta | F_t], \quad \text{a.s.}
\]

Lemma 3.1. Any positively-convex F-expectation satisfies (H0). Moreover, an F-expectation E is positively-convex if and only if the implied F-expectation \( (\tilde{E}, \text{Dom}^#(E)) \) is convex, i.e., for any \( \xi, \eta \in \text{Dom}^#(E) \), \( \lambda \in (0,1) \) and \( t \in [0,T] \)

\[
\tilde{E}[\lambda \xi + (1-\lambda)\eta | F_t] \leq \lambda \tilde{E}[\xi | F_t] + (1-\lambda)\tilde{E}[\eta | F_t], \quad \text{a.s.}
\]

Proposition 3.1. Let \( \mathcal{E}_i, \mathcal{E}_j \) be two F-expectations with the same domain \( \Lambda \in \tilde{\Omega}_T \) and satisfying (H1)-(H3). For any \( \nu \in S_{0,T} \), we define the pasting of \( \mathcal{E}_i, \mathcal{E}_j \) at the stopping time \( \nu \) to be the following RCLL F-adapted process

\[
\mathcal{E}_{\nu,i,j}[\xi | F_t] \triangleq 1_{\{\nu \leq t\}} \tilde{E}_j[\xi | F_{\nu\wedge \sigma} | F_{\sigma}] + 1_{\{\nu > t\}} \tilde{E}_i[\tilde{E}_j[\xi | F_{\nu\vee \sigma} | F_{\sigma}] | F_t], \quad \forall t \in [0,T]
\]

for any \( \xi \in \Lambda^# = \{ \xi \in \Lambda : \xi \geq c, \text{a.s. for some } c = c(\xi) \in \mathbb{R} \} \). Then \( \mathcal{E}_{\nu,i,j} \) is an F-expectation with domain \( \Lambda^\# \in \tilde{\Omega}_T \) and satisfying (H1) and (H2). Moreover, if \( \mathcal{E}_i \) and \( \mathcal{E}_j \) are both positively-convex, \( \mathcal{E}_{\nu,i,j} \) is convex in the sense of ([3.1]).

In particular, for any \( \sigma \in S_{0,T} \), applying Proposition 2.7 (4) and (5), we obtain

\[
\mathcal{E}_{\nu,i,j}[\xi | F_\sigma] = 1_{\{\nu \leq \sigma\}} \tilde{E}_j[\xi | F_\sigma] + 1_{\{\nu > \sigma\}} \tilde{E}_i[\tilde{E}_j[\xi | F_{\nu\wedge \sigma} | F_{\sigma}] | F_\sigma] = 1_{\{\nu \leq \sigma\}} \tilde{E}_j[\xi | F_\sigma] + 1_{\{\nu > \sigma\}} \tilde{E}_i[\tilde{E}_j[\xi | F_{\nu\vee \sigma} | F_{\sigma}] | F_\sigma], \quad \text{a.s.,}
\]

where we used the fact that \( \{\nu > \sigma\} \in \mathcal{F}_{\nu \vee \sigma} \) thanks to [Karatzas and Shreve 1991, Lemma 1.2.16].

Remark 3.1. Pasting may not preserve (H0). From now on, we will replace the (H0) assumption by the positive convexity, which implies the former and which is an invariant property under pasting thanks to the previous two results. Positive convexity is also important in constructing an optimal stopping time of \( \mathcal{E} \) (see Theorem 4.1).

All of the ingredients are in place to introduce what we mean by a stable class of F-expectations. As we will see in Lemma 4.2, stability assures that the essential supremum or infimum over the class can be approximated by an increasing or decreasing sequence in the class.

Definition 3.2. A class \( \mathcal{E} = \{ \mathcal{E}_i \}_{i \in I} \) of F-expectations is said to be “stable” if

1. All \( \mathcal{E}_i, i \in I \) are positively-convex F-expectations with the same domain \( \Lambda \in \tilde{\Omega}_T \) and they satisfy (H1)-(H3);
2. \( \mathcal{E} \) is closed under pasting: namely, for any \( i, j \in I, \nu \in S_{0,T} \), there exists a \( k = k(i,j,\nu) \in I \) such that \( \mathcal{E}_{\nu,i,j} \) coincides with \( \mathcal{E}_k \) on \( \Lambda^\# \).

We shall denote \( \text{Dom}(\mathcal{E}) \triangleq \Lambda^\# \), thus \( \text{Dom}(\mathcal{E}) = \text{Dom}^#(\mathcal{E}_i) \in \tilde{\Omega}_T \) for any \( i \in I \). Moreover, if \( \mathcal{E}' = \{ \mathcal{E}_i \}_{i \in I'} \) satisfies (2) for some non-empty subset \( I' \) of \( I \), then we call \( \mathcal{E}' \) a stable subclass of \( \mathcal{E} \), clearly \( \text{Dom}(\mathcal{E}') = \text{Dom}(\mathcal{E}) \).

Remark 3.2. The notion of “pasting” for linear expectations was given by [Föllmer and Schied 2004, Definition 6.41]. The counterpart of Proposition 3.1 for the linear expectations, which states that pasting two probability measures equivalent to \( \rho \) in another probability measure equivalent to \( P \), is given by [Föllmer and Schied 2004, Definition 6.41].
Lemma 6.43\cite{2004}. Note that in the case of linear expectations, (H1), (H2) and the convexity are trivially preserved because pasting in that case gives us a linear expectation. On the other hand, the notion of stability for linear expectations was given by Föllmer and Schied \cite[Definition 6.44]{2004}. The stability is also referred to as “fork convexity” in stochastic control theory, “m-stability” in stochastic analysis or “rectangularity” in decision theory (see the introduction of Delbaen \cite{2006} for details).

Example 3.1. (1) Let \( \mathcal{P} \) be the set of all probability measures equivalent to \( P \), then \( \mathcal{P} \triangleq \{ E_Q \}_{Q \in \mathcal{P}} \) is a stable class of linear expectations; see Föllmer and Schied \cite[Proposition 6.45]{2004}.

(2) Consider a collection \( \mathcal{U} \) of admissible control processes. For any \( U \in \mathcal{U} \), let \( \mathcal{P}^U \) be the equivalent probability measure defined via Karatzas and Zamfirescu \cite[(5)]{2008} (or Karatzas and Zamfirescu \cite[(2.5)\]}), then \( \mathcal{U} \triangleq \{ E_{P^U} \}_{U \in \mathcal{U}} \) is a stable class of linear expectations; see Subsection 7.3 of the present paper.

(3) For any \( M > 0 \), a family \( \mathcal{E}_M \) of convex Lipschitz \( g \)-expectations with Lipschitz coefficient \( K_g \leq M \) is an example of stable class of non-linear expectations; see Subsection 7.1 of this paper.

The following Lemma gives us a tool for checking whether a random variable is inside the domain \( \text{Dom}(\mathcal{E}) \) of a stable class \( \mathcal{E} \).

Lemma 3.2. Given a stable class \( \mathcal{E} \) of \( \mathcal{F} \)-expectations, a random variable \( \xi \) belongs to \( \text{Dom}(\mathcal{E}) \) if and only if \( c \leq \xi \leq \eta \), a.s. for some \( c \in \mathbb{R} \) and \( \eta \in \text{Dom}(\mathcal{E}) \).

Proof: Consider a random variable \( \xi \). If \( \xi \in \text{Dom}(\mathcal{E}) \), since \( \text{Dom}(\mathcal{E}) = \text{Dom}^\#(\mathcal{E}_i) \) for any \( i \in \mathcal{I} \), we know that there exists a \( c = c(\xi) \in \mathbb{R} \) such that \( \xi \geq c(\xi) \), a.s.

On the other hand, if \( c \leq \xi \leq \eta \), a.s. for some \( c \in \mathbb{R} \) and \( \eta \in \text{Dom}(\mathcal{E}) \), it follows that \( 0 \leq \xi - c \leq \eta - c \), a.s. Since \( \text{Dom}(\mathcal{E}) \in \mathcal{D}_T \), we see that \( -c, c \in \mathbb{R} \subset \text{Dom}(\mathcal{E}) \). Then (D2) shows that \( \eta - c \in \text{Dom}(\mathcal{E}) \) and thus (D3) implies that \( \xi - c \in \text{Dom}(\mathcal{E}) \), which further leads to that \( \xi = (\xi - c) + c \in \text{Dom}(\mathcal{E}) \) thanks to (D2) again. \(\square\)

We end this section by reviewing some basic properties of the essential supremum and essential infimum (for their definitions, see e.g. Neveu \cite[Proposition VI-1-1]{1973}, or Föllmer and Schied \cite[Theorem A.32]{2004}).

Lemma 3.3. Let \( \{ \xi_j \}_{j \in \mathcal{J}} \) and \( \{ \eta_j \}_{j \in \mathcal{J}} \) be two families of random variables of \( L^0(\mathcal{F}) \) with the same index set \( \mathcal{J} \).

(1) If \( \xi_j \leq (\leq) \eta_j \), a.s. for any \( j \in \mathcal{J} \), then \( \text{esssup}_{j \in \mathcal{J}} \xi_j \leq (\leq) \text{esssup}_{j \in \mathcal{J}} \eta_j \), a.s.

(2) For any \( A \in \mathcal{F} \), it holds a.s. that \( \text{esssup}_{j \in \mathcal{J}} (\mathbb{1}_A \xi_j + \mathbb{1}_A \eta_j) = \mathbb{1}_A \text{esssup}_{j \in \mathcal{J}} \xi_j + \mathbb{1}_A \text{esssup}_{j \in \mathcal{J}} \eta_j \). In particular, \( \text{esssup}_{j \in \mathcal{J}} (\mathbb{1}_A \xi_j) = \mathbb{1}_A \text{esssup}_{j \in \mathcal{J}} \xi_j \), a.s.

(3) For any random variable \( \gamma \in L^0(\mathcal{F}) \) and any \( \alpha > 0 \), we have \( \text{esssup}_{j \in \mathcal{J}} (\alpha \xi_j + \gamma) = \alpha \text{esssup}_{j \in \mathcal{J}} \xi_j + \gamma \), a.s.

Moreover, (1)-(3) hold when we replace \( \text{esssup}_{j \in \mathcal{J}} \) by \( \text{essinf}_{j \in \mathcal{J}} \).

4 Optimal Stopping with Multiple Priors

In this section, we will solve an optimal stopping problem in which the objective of the stopper is to determine an optimal stopping time \( \tau^* \) that satisfies

\[
\sup_{(i,p) \in \mathcal{I} \times S_0,T} \mathcal{E}_i [Y_p + H_i^p] = \sup_{i \in \mathcal{I}} \mathcal{E}_i [Y_{\tau^*} + H_{\tau^*}],
\]

(4.1)

where \( \mathcal{E}_i \) is a stable class of \( \mathcal{F} \)-expectations, \( Y \) is a primary reward process and \( H_i \) is a model-dependent cumulative reward process. (We will outline the assumptions on the reward processes below.) To find an optimal stopping time, we shall build a so-called “\( \mathcal{E} \)-upper Snell envelope”, which we will denote by \( Z^0 \), of the reward process \( Y \). Namely, \( Z^0 \) is the smallest RCLL \( \mathcal{F} \)-adapted process dominating \( Y \) such that \( Z^0 + H_i \) is an \( \mathcal{E}_i \)-supermartingale...
for any \(i \in \mathcal{I}\). We will show under certain assumptions that the first time \(Z^0\) meets \(Y\) is an optimal stopping time for (4.1).

We start by making some assumptions on the reward processes: Let \(\mathcal{E} = \{\mathcal{E}_i\}_{i \in \mathcal{I}}\) be a stable class of \(\mathbf{F}\) - expectations accompanied by a family \(\mathcal{H} \triangleq \{H^i\}_{i \in \mathcal{I}}\) of right-continuous \(\mathbf{F}\)-adapted processes that satisfies

\[(S1)\] For any \(i \in \mathcal{I}\), \(H^i_0 = 0\), a.s. and

\[H^i_{\nu,\rho} \triangleq H^i_\rho - H^i_\nu \in \text{Dom}(\mathcal{E}), \quad \forall \nu, \rho \in \mathcal{S}_0, \text{a.s.} \tag{4.2}\]

Moreover, if no member of \(\mathcal{E}\) satisfies (2.5), we assume that there exists a \(j \in \mathcal{I}\) such that

\[\zeta^i \triangleq \text{esssup}_{s,t \in \mathcal{D}_T; s < t} H^j_{s,t} \in \text{Dom}(\mathcal{E}). \tag{4.3}\]

\[(S2)\] There exists a \(C_H < 0\) such that for any \(i \in \mathcal{I}\), \(\text{essinf}_{s,t \in \mathcal{D}_T; s < t} H^i_{s,t} \geq C_H\), a.s.

\[(S3)\] For any \(\nu \in \mathcal{S}_0, T\) and \(i,j \in \mathcal{I}\), it holds for any \(0 \leq s < t \leq T\) that \(H^k_{s,t} = H^i_{\nu \land s, \nu \land t} + H^j_{\nu \lor s, \nu \lor t}\), a.s., where \(k = k(i,j,\nu) \in \mathcal{I}\) is the index defined in Definition 3.2 (2).

**Remark 4.1.** (1) For any \(i \in \mathcal{I}\), (S2) and the right-continuity of \(H^i\) imply that except on a null set \(N(i)\)

\[H^i_{s,t} \geq C_H, \quad \text{for any } 0 \leq s < t \leq T, \quad \text{thus } H^i_{\nu,\rho} \geq C_H, \quad \forall \nu, \rho \in \mathcal{S}_0, \text{with } \nu \leq \rho, \text{ a.s.} \tag{4.4}\]

(2) If (4.3) is assumed for some \(j \in \mathcal{I}\), we can deduce from the right-continuity of \(H^j\) and (4.4) that except on a null set \(N\)

\[C_H \leq H^j_{s,t} \leq \zeta^j, \quad \text{for any } 0 \leq s < t \leq T, \quad \text{thus } C_H \leq H^j_{\nu,\rho} \leq \zeta^j, \quad \forall \nu, \rho \in \mathcal{S}_0, \text{with } \nu \leq \rho, \text{ a.s.} \]

Then Lemma 3.3 implies that (4.2) holds for \(j\). Hence we see that (4.3) is a stronger condition than (4.2).

(3) Since \(H^i, H^j\) and \(H^k\) are all right-continuous processes, (S3) is equivalent to the following statement: It holds a.s. that

\[H^k_{s,t} = H^i_{\nu \land s, \nu \land t} + H^j_{\nu \lor s, \nu \lor t}, \quad \forall 0 \leq s < t \leq T. \tag{4.5}\]

Now we give an example of \(\mathcal{H}\).

**Lemma 4.1.** Let \(\{h^i\}_{i \in \mathcal{I}}\) be a family of progressive processes satisfying the following assumptions:

\[(h1)\] For any \(i \in \mathcal{I}\) and \(\nu, \rho \in \mathcal{S}_0, T\) with \(\nu \leq \rho\), a.s., \(\int_0^T h^i_t \, dt \in \text{Dom}(\mathcal{E})\). Moreover, if no member of \(\mathcal{E}\) satisfies (2.2), we assume that there exists a \(j \in \mathcal{I}\) such that \(\int_0^T |h^j_t| \, dt \in \text{Dom}(\mathcal{E})\).

\[(h2)\] There exists a \(c < 0\) such that for any \(i \in \mathcal{I}\), \(h^i_t \geq c, dt \times dP\)-a.s.

\[(h3)\] For any \(\nu \in \mathcal{S}_0, T\) and \(i,j \in \mathcal{I}\), it holds for any \(0 \leq s < t \leq T\) that \(h^k_t = h^i_{\nu \land t} + h^j_{\nu \lor t}\), dt \times dP\)-a.s., where \(k = k(i,j,\nu) \in \mathcal{I}\) is the index defined in Definition 3.2 (2).

Then \(\{H^i_t \triangleq \int_0^t h^i_s \, ds, t \in [0, T]\}_{i \in \mathcal{I}}\) is a family of right-continuous \(\mathbf{F}\)-adapted processes satisfying (S1)-(S3).

**Standing assumptions on \(Y\) in this section.**

Let \(Y\) be a right-continuous \(\mathbf{F}\)-adapted process that satisfies

\[(Y1)\] For any \(\nu \in \mathcal{S}_0, T\), \(Y_\nu \in \text{Dom}(\mathcal{E})\).
(Y2) \( \sup_{(i, \rho) \in I \times S_0, T} E_i [Y^i_n] < \infty \), where \( Y^i \triangleq \{Y_t + H^i_t\}_{t \in [0, T]} \). Moreover, if no member of \( \mathcal{E} \) satisfies (2.5), then we assume that
\[
\zeta_Y \triangleq \esssup_{(i, \rho, t) \in I \times S_0, T \times \mathcal{D}_T} \tilde{E}_i [Y^i_t | F_t] \in \text{Dom}(\mathcal{E}).
\]

(Y3) \( \text{essinf}_{t \in \mathcal{D}_T} Y_t \geq C_Y \), a.s. for some \( C_Y < 0 \).

Remark 4.2. (1) For any \( i \in I \), (A4) and (2.8) imply that \( E_i \) satisfies (2.5) if and only if \( \tilde{E}_i \) satisfies the following: Let \( \{\xi_n\}_{n \in \mathbb{N}} \subset \text{Dom}(\mathcal{E}) \) be a sequence converging a.s. to some \( \xi \in L^0(\mathcal{F}_T) \). If \( \inf_{n \in \mathbb{N}} \xi_n \geq c \), a.s. for some \( c \in \mathbb{R} \), then \( \lim_{n \to \infty} \tilde{E}_i[\xi_n] < \infty \) implies \( \xi \in \text{Dom}(\mathcal{E}) \). The proof of this equivalence is similar to that of Corollary 2.2.

(2) It is clear that (4.6) implies \( \sup_{(i, \rho) \in I \times S_0, T} E_i [Y^i_\rho] < \infty \).

(3) In light of (Y3) and the right-continuity of \( Y \), it holds except on a null set \( N \) that
\[
Y_t \geq C_Y, \quad \forall t \in [0, T], \quad \text{thus} \quad Y_{\nu} \geq C_Y, \quad \forall \nu \in S_0, T.
\]
Then for any \( i \in I \), Remark 4.1 (1) implies that except on a null set \( \tilde{N}(i) \)
\[
Y^i_{\nu} = Y_{\nu} + H^i_{\nu} \geq C_s \triangleq C_Y + C_H, \quad \forall \nu \in S_0, T.
\]

The following lemma states that the supremum or infimum over a stable class of \( \mathcal{F} \)-expectations can be approached by an increasing or decreasing sequence in the class.

Lemma 4.2. Let \( \nu \in S_0, T \) and \( \mathcal{U} \) be a non-empty subset of \( S_{\nu, T} \) such that
\[
\rho_1 1_A + \rho_2 1_{A^c} \in \mathcal{U}, \quad \forall \rho_1, \rho_2 \in \mathcal{U}, \quad \forall A \in \mathcal{F}_\nu.
\]
Let \( \{X(\rho)\}_{\rho \in \mathcal{U}} \subset \text{Dom}(\mathcal{E}) \) be a family of random variables, indexed by \( \rho \), such that for any \( \nu, \sigma \in \mathcal{U}, \ 1_{(\nu = \sigma)} X(\nu) = 1_{(\nu = \sigma)} X(\sigma), \ a.s. \), then for any stable subclass \( \mathcal{E}' = \{E_i\}_{i \in I} \) of \( \mathcal{E} \), there exist two sequences \( \{(i_n, \rho_n)\}_{n \in \mathbb{N}} \) and \( \{(i'_n, \rho'_n)\}_{n \in \mathbb{N}} \) in \( I \times \mathcal{U} \) such that
\[
\esssup_{(i, \rho) \in I \times \mathcal{U}} \tilde{E}_i [X(\rho) + H^i_{\nu, \rho} | \mathcal{F}_\nu] = \lim_{n \to \infty} \uparrow \tilde{E}_{i_n} [X(\rho_n) + H^i_{\nu, \rho_n} | \mathcal{F}_\nu], \quad \text{a.s.,}
\]
\[
\essinf_{(i, \rho) \in I \times \mathcal{U}} \tilde{E}_i [X(\rho) + H^i_{\nu, \rho} | \mathcal{F}_\nu] = \lim_{n \to \infty} \downarrow \tilde{E}_{i'_n} [X(\rho'_n) + H^i_{\nu, \rho'_n} | \mathcal{F}_\nu], \quad \text{a.s.}
\]

For any \( \nu \in S_0, T \) and \( i \in I \), let us define
\[
Z(\nu) \triangleq \esssup_{(i, \rho) \in I \times S_0, T} \tilde{E}_i [Y_{\rho} + H^i_{\nu, \rho} | \mathcal{F}_\nu] \in \mathcal{F}_\nu \quad \text{and} \quad Z^i(\nu) \triangleq Z(\nu) + H^i_{\nu}.
\]
Clearly, taking \( \rho = \nu \) above yields that
\[
Y_{\nu} \leq Z(\nu), \quad \text{a.s.}
\]

The following two Lemmas give the bounds on \( Z(\nu), Z^i(\nu), i \in I \), and show that they all belong to \( \text{Dom}(\mathcal{E}) \).

Lemma 4.3. For any \( \nu \in S_0, T \) and \( i \in I \)
\[
Z(\nu) \geq C_s \quad \text{and} \quad Z^i(\nu) \geq C_Y + 2C_H, \quad \text{a.s.}
\]
Moreover, if no member of \( \mathcal{E} \) satisfies (2.5), then we further have
\[
Z(\nu) \leq \zeta_Y - C_H \quad \text{and} \quad Z^i(\nu) \leq \zeta_Y - C_H + H^i_{\nu}, \quad \text{a.s.,}
\]
where \( \zeta_Y - C_H \) and \( \zeta_Y - C_H + H^i_{\nu} \) both belong to \( \text{Dom}(\mathcal{E}) \).
Lemma 4.4. For any $\nu \in \mathcal{S}_{0,T}$ and $i \in \mathcal{I}$, both $Z(\nu)$ and $Z^i(\nu)$ belong to $\text{Dom}(\mathcal{E})$.

In the next two propositions, we will see that the $\mathcal{F}$-adapted process $\{Z(t)\}_{t \in [0,T]}$ has an RCLL modification $Z^0$, and that both $\{Z^i(t)\}_{t \in [0,T]}$ and $Z^{i,0} \triangleq \{Z^{i,0}_t + H^i_t\}_{t \in [0,T]}$ are $\tilde{E}_i$-supermartingales for any $i \in \mathcal{I}$.

Proposition 4.1. For any $\nu, \sigma \in \mathcal{S}_{0,T}$ and $\gamma \in \mathcal{S}_\nu$, we have

$$Z(\nu) = Z(\sigma), \quad \text{a.s. on } \{\nu = \sigma\},$$

(4.14)

$$\text{esssup}_{i \in \mathcal{I}} \tilde{E}_i[Z(\gamma) + H^\gamma_i]_{\mathcal{F}_\nu} = \text{esssup}_{(i, \rho) \in \mathcal{I} \times \mathcal{S}_{0,T}} \tilde{E}_i[Y_{\rho} + H^{i,\rho}_i]_{\mathcal{F}_\nu} \leq Z(\nu), \quad \text{a.s.}$$

(4.15)

Proposition 4.2. Given $i \in \mathcal{I}$, for any $\nu, \rho \in \mathcal{S}_{0,T}$ with $\nu \leq \rho$, a.s., we have

$$\tilde{E}_i[Z^i(\rho)|\mathcal{F}_\nu] \leq Z^i(\nu), \quad \text{a.s.}$$

(4.16)

In particular, $\{Z^i(t)\}_{t \in [0,T]}$ is an $\tilde{E}_i$-supermartingale. Moreover, the process $\{Z(t)\}_{t \in [0,T]}$ admits an RCLL modification $Z^0$. The process $Z^{i,0} \triangleq \{Z^{i,0}_t + H^i_t\}_{t \in [0,T]}$ is also an $\tilde{E}_i$-supermartingale.

We call $Z^0$ the “$\mathcal{E}$-upper Snell envelope” of the reward process $Y$. From (4.11) and their right-continuity, we see that $Z^0$ dominates $Y$ in the following sense:

Definition 4.1. We say that process $X$ “dominates” process $X'$ if $P(X_t \geq X'_t, \forall t \in [0,T]) = 1$.

Remark 4.3. (1) If $X$ dominates $X'$, then $X_{\nu} \geq X'_{\nu}$, a.s. for any $\nu \in \mathcal{S}_{0,T}$.

(2) Let $X$ and $X'$ be two right-continuous $\mathcal{F}$-adapted processes. If $P(X_t \geq X'_t) = 1$ holds for all $t$ in a countable dense subset of $[0,T]$, then $X$ dominates $X'$.

The following theorem demonstrates that $Z^0$ is the smallest RCLL $\mathcal{F}$-adapted process dominating $Y$ such that $Z^{i,0}$ is an $\tilde{E}_i$-supermartingale for any $i \in \mathcal{I}$.

Proposition 4.3. The process $Z^0$ dominates the process $Y$. Moreover, for any $\nu \in \mathcal{S}_{0,T}$ and $i \in \mathcal{I}$, we have $Z^0_\nu, Z^{i,0}_\nu \in \text{Dom}(\mathcal{E})$ and

$$Z^0_\nu = Z(\nu), \quad Z^{i,0}_\nu = Z^i(\nu), \quad \text{a.s.}$$

(4.17)

Furthermore, if $X$ is another RCLL $\mathcal{F}$-adapted process dominating $Y$ such that $X^i \triangleq \{X_t + H^i_t\}_{t \in [0,T]}$ is an $\tilde{E}_i$-supermartingale for any $i \in \mathcal{I}$, then $X$ also dominates $Z^0$.

As a consequence of Proposition 4.3 and (4.12), we have for any $\nu \in \mathcal{S}_{0,T}$ and $i \in \mathcal{I}$ that

$$Z^0_\nu \geq C_\nu, \quad Z^{i,0}_\nu \geq C_Y + 2C_H, \quad \text{a.s.}$$

(4.18)

In what follows, we first give approximately optimal stopping times. This family of stopping times will be crucial in finding an optimal stopping time for (4.11).

Definition 4.2. For any $\delta \in (0,1)$ and $\nu \in \mathcal{S}_{0,T}$, we define

$$\tau_\delta(\nu) \triangleq \inf \{t \in [\nu, T] : Y_t \geq \delta Z^0_t + (1-\delta)(C_Y + 2C_H)\} \wedge T \in \mathcal{S}_{0,T}$$

and

$$J_\delta(\nu) \triangleq \text{esssup}_{i \in \mathcal{I}} \tilde{E}_i[Z_{\tau_\delta(\nu)} + H^i_{\nu,\tau_\delta(\nu)}]_{\mathcal{F}_\nu}.$$

Remark 4.4. (1) For any $\delta \in (0,1)$ and $\nu \in \mathcal{S}_{0,T}$, the right-continuity of $Y$ and $Z^0$ implies that $\{\tau_\delta(t)\}_{t \in [0,T]}$ is also a right-continuous process. Moreover, since $Y_T = Z(T) = Y_T$, a.s., we can deduce from (Y3) that $Y_T > \delta Z^0_T + (1-\delta)(C_Y + 2C_H)$. Then the right-continuity of processes $Y$ and $Z^0$ implies that

$$Y_{\tau_\delta(\nu)} \geq \delta Z^0_{\tau_\delta(\nu)} + (1-\delta)(C_Y + 2C_H), \quad \text{a.s.}$$

(2) For any $\nu \in \mathcal{S}_{0,T}$, we can deduce from (4.17) and (4.13) that

$$J_\delta(\nu) = \text{esssup}_{i \in \mathcal{I}} \tilde{E}_i[Z^0_{\tau_\delta(\nu)} + H^i_{\nu,\tau_\delta(\nu)}]_{\mathcal{F}_\nu} = \text{esssup}_{i \in \mathcal{I}} \tilde{E}_i[Z(\tau_\delta(\nu)) + H^i_{\nu,\tau_\delta(\nu)}]_{\mathcal{F}_\nu} \leq Z(\nu) = Z^0_\nu, \quad \text{a.s.}$$

(4.19)
The following two results show that for any $\delta \in (0, 1)$, the family $\{J_\delta(\nu)\}_{\nu \in \mathcal{S}_{0,T}}$ possesses similar properties to $\{Z(\nu)\}_{\nu \in \mathcal{S}_{0,T}}$.

**Lemma 4.5.** For any $\delta \in (0, 1)$ and $\nu \in \mathcal{S}_{0,T}$, we have $J_\delta(\nu) \in \text{Dom}(\mathcal{E})$. And for any $\sigma \in \mathcal{S}_{0,T}$, $J_\delta(\nu) = J_\delta(\sigma)$, a.s. on $\{\nu = \sigma\}$.

**Proposition 4.4.** Given $\delta \in (0, 1)$, the following statements hold:

1. For any $i \in \mathcal{I}$, $\{J_\delta^i(t) \triangleq J_\delta(t) + H^i_t\}_{t \in [0,T]}$ is an $\mathcal{E}_t$-supermartingale;
2. $\{J_\delta(t)\}_{t \in [0,T]}$ admits an RCLL modification $J^{\delta,0}$ such that the process $J^{\delta,i,0} \triangleq \{J^i_{\delta,t} + H^i_t\}_{t \in [0,T]}$ is an $\mathcal{E}_t$-supermartingale for each $i \in \mathcal{I}$;
3. For any $\nu \in \mathcal{S}_{0,T}$, $J^{\delta,0}_\nu \in \text{Dom}(\mathcal{E})$ and $J^{\delta,0}_\nu = J_\delta(\nu)$, a.s.

Fix $\nu \in \mathcal{S}_{0,T}$. The right continuity of $Z^0$ and (4.18) imply that the stopping times $\tau_\delta(\nu)$ are increasing in $\delta$. Therefore, we can define the limiting stopping time

$$\tau(\nu) \triangleq \lim_{\delta \searrow 0} \tau_\delta(\nu). \quad (4.20)$$

To show that $\tau(0) \in \mathcal{S}_{0,T}$ is an optimal stopping time for (4.11), we need the family of processes $\{Y^i\}_{i \in \mathcal{I}}$ to be uniformly continuous from the left over the stable class $\mathcal{E}$.

**Definition 4.3.** The family of processes $\{Y^i\}_{i \in \mathcal{I}}$ is called “$\mathcal{E}$-uniformly-left-continuous” if for any $\nu, \rho \in \mathcal{S}_{0,T}$ with $\nu \leq \rho$, a.s. and for any sequence $\{\rho_n\}_{n \in \mathbb{N}} \subset \mathcal{S}_{0,T}$ increasing a.s. to $\rho$, we can find a subsequence $\{n_k\}_{k \in \mathbb{N}}$ of $\mathbb{N}$ such that

$$\lim_{k \to \infty} \text{esssup}_{i \in \mathcal{I}} \left| \mathcal{E}_t \left[ Y_{t \rho_{n_k}^i} + H^i_{\rho_{n_k}} \mid \mathcal{F}_\nu \right] - \mathcal{E}_t \left[ Y_{\rho \mid \mathcal{F}_\nu} \right] \right| = 0, \quad \text{a.s.} \quad (4.21)$$

The next theorem shows that $\tau(\nu)$ is not only the first time when process $Z^0$ meets the process $Y$ after $\nu$, but it is also an optimal stopping time after $\nu$. The assumption that the elements of the stable class $\mathcal{E}$ are convex (see (3.1)) becomes crucial in the proof of this result.

**Theorem 4.1.** Assume that $\{Y^i, i \in \mathcal{I}\}$ is “$\mathcal{E}$-uniformly-left-continuous”. Then for each $\nu \in \mathcal{S}_{0,T}$, the stopping time $\tau(\nu)$ defined by (4.20) satisfies

$$Z(\nu) = \text{esssup}_{i \in \mathcal{I}} \mathcal{E}_t \left[ Y_{\tau(\nu)_i} + H^i_{\nu,\tau(\nu)} \mid \mathcal{F}_\nu \right] = \text{esssup}_{i \in \mathcal{I}} \mathcal{E}_t \left[ Z(\tau(\nu)) + H^i_{\nu,\tau(\nu)} \mid \mathcal{F}_\nu \right] = \text{esssup}_{i \in \mathcal{I}} \mathcal{E}_t \left[ Z(\rho) + H^i_{\nu,\rho} \mid \mathcal{F}_\nu \right], \quad \text{a.s.} \quad (4.22)$$

for any $\rho \in \mathcal{S}_{0,T}$ and $\tau(\nu) = \tau_1(\nu) \triangleq \inf \left\{ t \in [\nu, T] : Z^0_t = Y_t \right\}, \quad \text{a.s.}$

In particular, taking $\nu = 0$ in (4.22), one can deduce from (2.8) that $\tau(0) = \inf \left\{ t \in [0, T] : Z^0_t = Y_t \right\}$ satisfies

$$\sup_{(i, \rho) \in \mathcal{I} \times \mathcal{S}_{0,T}} \mathcal{E}_t \left[ Y_{\rho} + H^i_{\rho} \right] = \sup_{(i, \rho) \in \mathcal{I} \times \mathcal{S}_{0,T}} \mathcal{E}_t \left[ Y_{\rho} + H^i_{\rho} \right] = Z(0) = \sup_{i \in \mathcal{I}} \mathcal{E}_t \left[ Y_{\tau(0)} + H^i_{\tau(0)} \right] = \sup_{i \in \mathcal{I}} \mathcal{E}_t \left[ Y_{\tau(0)} + H^i_{\tau(0)} \right].$$

Therefore, we see that $\tau(0)$, the first time the Snell envelope $Z^0$ meets the process $Y$ after time $t = 0$, is an optimal stopping time for (4.11).

## 5 Robust Optimal Stopping

In this section we analyze the *robust* optimal stopping problem in which the stopper aims to find an optimal stopping time $\tau_*$ that satisfies

$$\sup_{\rho \in \mathcal{S}_{0,T}} \inf_{i \in \mathcal{I}} \mathcal{E}_i \left[ Y^i_{\tau_*} \right] = \inf_{i \in \mathcal{I}} \mathcal{E}_i \left[ Y^i_{\tau_*} \right], \quad (5.1)$$
where $\mathcal{E} = \{\mathcal{E}_i\}_{i \in I}$ is a stable class of $\mathcal{F}$-expectations and $Y^i = Y + H^i$, $i \in I$ is the model-dependent reward process introduced in (4.11). (We will modify the assumptions on the reward processes shortly). In order to find an optimal stopping time we construct the lower and the upper values of the optimal stopping problem at any stopping time $\nu \in \mathcal{S}_{0,T}$, i.e.,

$$V(\nu) \triangleq \esssup_{\rho \in \mathcal{S}_{0,T}} \left( \essinf_{i \in I} \mathcal{E}_i[Y_{\rho} + H_{\nu,\rho}^i | \mathcal{F}_\nu] \right), \quad \overline{V}(\nu) \triangleq \essinf_{i \in I} \left( \esssup_{\rho \in \mathcal{S}_{\nu,T}} \mathcal{E}_i[Y_{\rho} + H_{\nu,\rho}^i | \mathcal{F}_\nu] \right).$$

With the so-called "$\mathcal{E}$-uniform-right-continuity" condition on $\{Y^i\}_{i \in I}$, we can show that at any $\nu \in \mathcal{S}_{0,T}$, $V(\nu)$ and $\overline{V}(\nu)$ coincide with each other (see Theorem 5.1). We denote the common value, the value of the robust optimal stopping problem, as $V(\nu)$ at $\nu$. We will show that up to a stopping time $\tau(0)$ (see Lemma 5.2), at which we have $V(\tau(0)) = Y_{\tau(0)}$, a.s., the stopped value process $\{V(\tau(0) \wedge t)\}_{t \geq 0}$ admits an RCLL modification $V^0$. The main result in this section, Theorem 5.2 shows that the first time $V^0$ meets $Y$ is an optimal stopping time for (5.1).

**Standing assumptions on $\mathcal{H}$ and $Y$ in this section.** Let us introduce

$$R^i(\nu) \triangleq \esssup_{\rho \in \mathcal{S}_{0,T}} \mathcal{E}_i[Y_{\rho} + H_{\nu,\rho}^i | \mathcal{F}_\nu], \quad \text{for any } i \in I \text{ and } \nu \in \mathcal{S}_{0,T}.$$ 

To adapt the results we obtained for the family $\{Z(\nu)\}_{\nu \in \mathcal{S}_{0,T}}$ to each family $\{R^i(\nu)\}_{\nu \in \mathcal{S}_{0,T}}$, $i \in I$, we assume that $\mathcal{H} = \{H^i\}_{i \in I}$ is a family of right-continuous $\mathcal{F}$-adapted processes satisfying (S2), (S3) and,

**(S1')** For any $i \in I$, $H^i_0 = 0$, a.s. and (4.2) holds. If $\mathcal{E}_i$ does not satisfy (2.5), then we assume that $\zeta^i = \esssup_{s,t \in \mathcal{D}_{\tau(0)}} H^i_{\nu,\rho} \in \text{Dom}(\mathcal{E})$.

On the other hand, we assume that $Y$ is a right-continuous $\mathcal{F}$-adapted process that satisfies (Y1), (Y3) and

**(Y2')** For any $i \in I$, $\sup_{\rho \in \mathcal{S}_{0,T}} \mathcal{E}_i[Y^i_{\rho}] < \infty$. If $\mathcal{E}_i$ does not satisfy (2.5), then we assume that $\esssup_{(\rho,t) \in \mathcal{S}_{0,T} \times \mathcal{D}_{\tau}} \mathcal{E}_i[Y^i_{\rho} | \mathcal{F}_t] \in \text{Dom}(\mathcal{E})$.

We also assume that for any $i \in I$, $Y^i$ is “quasi-left-continuous” with respect to $\mathcal{E}_i$: for any $\nu, \rho \in \mathcal{S}_{0,T}$ with $\nu \leq \rho$, a.s. and for any sequence $\{\rho_n\}_{n \in \mathbb{N}} \subset \mathcal{S}_{0,T}$ increasing a.s. to $\rho$, we can find a subsequence $\{n_k = n_k^i\}_{k \in \mathbb{N}}$ of $\mathbb{N}$ such that

$$\lim_{n \to \infty} \mathcal{E}_i[Y^i_{\rho_n} + H^i_{\rho_n} | \mathcal{F}_\nu] = \mathcal{E}_i[Y^i_{\rho} | \mathcal{F}_\nu], \quad \text{a.s.} \quad (5.2)$$

**Remark 5.1.** (S1') and (Y2') are analogous to (S1) and (Y2) respectively while the quasi-left-continuity (5.2) is the counterpart of the $\mathcal{E}$-uniform-left-continuity (4.21). It is obvious that (S1') implies (S1) and that (4.21) gives rise to (5.2). Moreover, (4.16) implies (Y2'): In fact, for any $i \in I$, one can deduce from (4.16) that

$$C_* \leq \esssup_{(\rho,t) \in \mathcal{S}_{0,T} \times \mathcal{D}_T} \mathcal{E}_i[Y^i_{\rho} | \mathcal{F}_t] \leq \esssup_{(\rho,t) \in I \times \mathcal{S}_{0,T} \times \mathcal{D}_T} \mathcal{E}_i[Y^i_{\rho} | \mathcal{F}_t], \quad \text{a.s.}$$

Then Lemma 5.2 implies that $\esssup_{(\rho,t) \in \mathcal{S}_{0,T} \times \mathcal{D}_T} \mathcal{E}_i[Y^i_{\rho} | \mathcal{F}_t] \in \text{Dom}(\mathcal{E})$, and it follows that $\sup_{\rho \in \mathcal{S}_{0,T}} \mathcal{E}_i[Y^i_{\rho}] < \infty$. \hfill \Box

Fix $i \in I$. Applying Lemma 4.2, 4.7, 4.4, 4.19, Proposition 4.2, Proposition 4.3 and Theorem 4.1 to the family $\{R^i(\nu)\}_{\nu \in \mathcal{S}_{0,T}}$, we obtain

**Proposition 5.1.** (1) For any $\nu \in \mathcal{S}_{0,T}$, $R^i(\nu)$ belongs to $\text{Dom}(\mathcal{E})$ and satisfies

$$C_Y \leq Y_{\nu} \leq \esssup_{\rho \in \mathcal{S}_{0,T}} \mathcal{E}_i[Y_{\rho} + H_{\nu,\rho}^i | \mathcal{F}_\nu] = R^i(\nu), \quad \text{a.s.}, \quad \text{thus } C_* \leq Y^i_{\nu}, \quad \text{a.s.} \quad (5.3)$$

(2) For any $\nu, \sigma \in \mathcal{S}_{0,T}$ and $\gamma \in \mathcal{S}_{\nu,T}$, we have

$$R^i(\nu) = R^i(\sigma), \quad \text{a.s. on } \{\nu = \sigma\}, \quad (5.4)$$

$$\mathcal{E}_i[R^i(\gamma) + H_{\nu,\gamma}^i | \mathcal{F}_\nu] \leq \esssup_{\rho \in \mathcal{S}_{\nu,T}} \mathcal{E}_i[Y_{\rho} + H_{\nu,\rho}^i | \mathcal{F}_\nu] \leq R^i(\nu), \quad \text{a.s.} \quad (5.5)$$
The process \( \{R(t)\}_{t \in [0,T]} \) admits an RCLL modification \( R^{i,0} \), called “\( E \) Snell envelope”, such that \( \{R^{i,0} + H^i\}_{t \in [0,T]} \) is an \( E \)-supermartingale and that for any \( \nu \in S_{0,T} \)
\[
R^{i,0}_\nu = R^i(\nu), \quad \text{a.s.}
\] (5.6)

(4) For any \( \nu \in S_{0,T} \), \( \tau^i(\nu) \triangleq \inf\{t \in [\nu,T] : R^{i,0}_t = Y_t \} \) is an optimal stopping time with respect to \( \mathcal{E}^i \) after time \( \nu \), i.e., for any \( \gamma \in S_{\nu,\tau^i(\nu)} \),
\[
R^i(\nu) = \tilde{E}_i[Y_{\tau^i(\nu)} + H^i_{\nu,\tau^i(\nu)}|\mathcal{F}_\nu] = \tilde{E}_i[R^i(\gamma) + H^i_{\nu,\gamma}|\mathcal{F}_\nu], \quad \text{a.s.}
\] (5.7)

**Corollary 5.1.** For any \( \nu \in S_{0,T} \), both \( \underline{V}(\nu) \) and \( \overline{V}(\nu) \) belong to \( \text{Dom}(\mathcal{E}) \).

**Proof:** Fix \((l, \rho) \in I \times S_{\nu,T} \), for any \( i \in I \), (4.7), (4.4) and Proposition 2.7 (5) imply that
\[
\tilde{E}_i[Y_{\tau^i(l)} + H^i_{\nu,\tau^i(l)}|\mathcal{F}_\nu] \geq \tilde{E}_i[C_Y + C_H|\mathcal{F}_\nu] = C_*, \quad \text{a.s.}
\]
Taking the essential infimum over \( i \in I \) on the left-hand-side yields that
\[
C_* \leq \text{essinf}_{i \in I} \tilde{E}_i[Y_{\tau^i(l)} + H^i_{\nu,\tau^i(l)}|\mathcal{F}_\nu] \leq \text{esssup}_{\rho \in S_{\nu,T}} \left( \text{essinf}_{i \in I} \tilde{E}_i[Y_{\tau^i(l)} + H^i_{\nu,\tau^i(l)}|\mathcal{F}_\nu] \right) = \overline{V}(\nu) \leq \underline{V}(\nu) = \text{essinf}_{i \in I} R^i(\nu) \leq R^l(\nu), \quad \text{a.s.}
\]
Since \( R^l(\nu) \in \text{Dom}(\mathcal{E}) \) by Proposition 5.1 (1), a simple application of Lemma 3.2 proves the corollary.

As we will see in the next lemma since the stable class \( \mathcal{E} \) is closed under pasting (see Definition 3.2 (2)), \( \underline{V}(\nu) \) can be approximated by a decreasing sequence that belongs to the family \( \{R^i(\nu)\}_{i \in I} \).

**Lemma 5.1.** For any \( \nu \in S_{0,T} \), there exists a sequence \( \{i_n\}_{n \in \mathbb{N}} \subset I \) such that
\[
\underline{V}(\nu) = \text{essinf}_{i \in I} R^i(\nu) = \lim_{n \to \infty} R^{i_n}(\nu), \quad \text{a.s.}
\] (5.8)

Thanks again to the stability of \( \mathcal{E} \) under pasting, the infimum of the family \( \{\tau^i(\nu)\}_{i \in I} \) of optimal stopping times can be approached by a decreasing sequence that belongs in the family. As a result the infimum is also a stopping time.

**Lemma 5.2.** For any \( \nu \in S_{0,T} \), there exists a sequence \( \{i_n\}_{n \in \mathbb{N}} \subset I \) such that
\[
\tau(\nu) \triangleq \text{essinf}_{i \in I} \tau^i(\nu) = \lim_{n \to \infty} \tau^{i_n}(\nu), \quad \text{a.s.}, \quad \text{thus } \tau(\nu) \in S_{\nu,T}.
\]

The family of stopping times \( \{\underline{V}(\nu)\}_{\nu \in S_{0,T}} \) will play a critical role in this section. The next lemma basically shows that if \( \tilde{E}_j \) and \( \tilde{E}_k \) behave the same after some stopping time \( \nu \), then \( R^{k,0} \) and \( R^{l,0} \) are identical after \( \nu \):

**Lemma 5.3.** For any \( i, j \in I \) and \( \nu \in S_{0,T} \), let \( k = k(i, j, \nu) \in I \) as in Definition 3.2. For any \( \sigma \in S_{\nu,T} \), we have
\[
R^{k,0}_{\sigma} = R^k(\sigma) = R^l(\sigma) = R^{l,0}_{\sigma}, \quad \text{a.s.}
\]

**Proof:** For any \( \rho \in S_{\sigma,T} \), applying Proposition 2.7 (5) to \( \tilde{E}_i \), we can deduce from (3.5) and (3.3) that
\[
\tilde{E}_k[Y_{\rho} + H^k_{\rho}|\mathcal{F}_\sigma] = \tilde{E}_k[Y_{\rho} + H^j_{\rho}|\mathcal{F}_\sigma] = \tilde{E}_k[Y_{\rho} + H^j_{\rho}|\mathcal{F}_{\nu \wedge \sigma}] = \tilde{E}_j[Y_{\rho} + H^j_{\rho}|\mathcal{F}_{\nu \wedge \sigma}], \quad \text{a.s.}
\]
Then (5.6) implies that
\[
R^{k,0}_{\sigma} = R^k(\sigma) = \text{esssup}_{\rho \in S_{\sigma,T}} \tilde{E}_k[Y_{\rho} + H^k_{\rho}|\mathcal{F}_\sigma] = \text{esssup}_{\rho \in S_{\sigma,T}} \tilde{E}_j[Y_{\rho} + H^j_{\rho}|\mathcal{F}_\sigma] = R^l(\sigma) = R^{l,0}_{\sigma}, \quad \text{a.s.,}
\]
which proves the lemma.

We now introduce the notion of the uniform right continuity of the family \( \{Y^i\}_{i \in I} \) over the stable class \( \mathcal{E} \). With this assumption on the reward processes, we can show that at any \( \nu \in S_{0,T} \), \( \underline{V}(\nu) = \overline{V}(\nu) \), a.s., thus the robust optimal stopping problem has a value \( V(\nu) \) at \( \nu \).
Definition 5.1. The family of processes \( \{Y^i\}_{i \in I} \) is called \( \mathcal{E} \)-uniformly-right-continuous if for any \( \nu \in S_{0,T} \) and for any sequence \( \{\nu_n\}_{n \in \mathbb{N}} \subset S_{\nu,T} \) decreasing a.s. to \( \nu \), we can find a subsequence \( \{\nu_n\}_{n \in \mathbb{N}} \) (we still denote it by \( \{\nu_n\}_{n \in \mathbb{N}} \)) such that

\[
\lim_{n \to \infty} \esssup_{i \in I} \left| \tilde{E}_i[Y^i_{\nu_n}|\mathcal{F}_\nu] - Y^i_\nu \right| = 0, \quad \text{a.s.}
\]

Theorem 5.1. Suppose that \( \{Y^i\}_{i \in I} \) is \( \mathcal{E} \)-uniformly-right-continuous. Then for any \( \nu \in S_{0,T} \), we have

\[
\nabla_\nu(t) = \essinf_{i \in I} \tilde{E}_i\left[Y^i_{\nu}\right] + H^i_{\nu}(t) \geq V(\nu), \quad \text{a.s.}
\]

(5.9)

We will denote the common value by \( \nabla_\nu(t) = \essinf_{i \in I} \tilde{E}_i\left[Y^i_{\nu}\right] + H^i_{\nu}(t) \). Observe that \( \nabla(0) \) is optimal for the robust optimal stopping problem in (5.1).

Standing assumption on \( Y \) for the rest of this section. We assume that the family of processes \( \{Y^i\}_{i \in I} \) is \( \mathcal{E} \)-uniformly-right-continuous”.

Proposition 5.2. For any \( \nu \in S_{0,T} \), we have \( V(\nabla(\nu)) = Y^\nu(\nu), \) a.s.

Note that \( \nabla(\nu) \) may not be the first time after \( \nu \) when the value of the robust optimal stopping problem is equal to the primary reward. Actually, since the process \( \{V(t)\}_{t \in [0,T]} \) is not necessarily right-continuous, \( \inf\{t \in [\nu, T] \mid V(t) = Y_t\} \) may not even be a stopping time. We will address this issue in the next two results.

Proposition 5.3. Given \( i \in I \), for any \( \nu, \rho \in S_{0,T} \) with \( \nu \leq \rho \), a.s., we have

\[
\essinf_{k \in I} \tilde{E}_k[V^k(\rho)|\mathcal{F}_\nu] \leq V^i(\nu), \quad \text{a.s.}
\]

(5.10)

where \( V^i(\nu) \overset{\Delta}{=} V(\nu) + H^i_{\nu} \in \text{Dom}(\mathcal{E}) \). Moreover if \( \rho \leq \nabla(\nu) \), a.s., then

\[
\tilde{E}_i[V^i(\rho)|\mathcal{F}_\nu] \geq V^i(\nu), \quad \text{a.s.}
\]

(5.11)

In particular, the stopped process \( \{V^i(\nabla(\nu) \wedge t)\}_{t \in [0,T]} \) is an \( \tilde{E}_i \)-submartingale.

Now we show that the stopped value process \( \{V(\nabla(0) \wedge t)\}_{t \in [0,T]} \) admits an RCLL modification \( V^0 \). As a result, the first time when the process \( V^0 \) meets the process \( Y \) after time \( t = 0 \) is an optimal stopping time of the robust optimal stopping problem.

Theorem 5.2. Assume that for some \( i' \in I \), \( \zeta' = \esssup_{s,t \in \mathbb{D}_Y; s < t} H^{i'}_{s,t} \in \text{Dom}(\mathcal{E}) \) and that there exists a concave F-expectation \( \mathcal{E}' \) (not necessarily in \( \mathcal{E} \)) satisfying (H0) and (H1) such that

\[
\text{Dom}(\mathcal{E}') \supset \{-\xi : \xi \in \text{Dom}(\mathcal{E})\} \text{ and for every } \tilde{E}_i\text{-submartingale } X, \ -X \text{ is an } \mathcal{E}'\text{-supermartingale.}
\]

(5.12)

We also assume that for any \( \rho \in S_{0,T} \), there exists a \( j = j(\rho) \in I \) such that \( \esssup_{t \in \mathbb{D}_T} \tilde{E}_j[Y^j_{\rho}|\mathcal{F}_t] \in \text{Dom}(\mathcal{E}) \).

(1) Then the stopped value process \( \{V(\nabla(0) \wedge t)\}_{t \in [0,T]} \) admits an RCLL modification \( V^0 \) (called “\( \mathcal{E} \)-lower Snell envelope” of \( Y \)) such that for any \( \nu \in S_{0,T} \)

\[
V^0_\nu = V(\nabla(0) \wedge \nu), \quad \text{a.s.}
\]

(5.13)

(2) Consequently,

\[
\tau_\nu \overset{\Delta}{=} \inf\{t \in [0,T] : V^0_t = Y_t\}
\]

(5.14)

is a stopping time. In fact, it is an optimal stopping time of (5.1).
6 Remarks on Sections 4 & 5

Remark 1.

Let \( \mathcal{E} = \{ \mathcal{E}_i \}_{i \in \mathcal{I}} \) be a stable class of \( \mathbf{F} \)-expectations. For any \( \xi \in \text{Dom}(\mathcal{E}) \) and \( \nu \in \mathcal{S}_{0,T} \), we define

\[
\overline{\mathcal{E}}[\xi | \mathcal{F}_\nu] \triangleq \text{esssup}_{i \in \mathcal{I}} \overline{\mathcal{E}}_i[\xi | \mathcal{F}_\nu] \quad \text{and} \quad \underline{\mathcal{E}}[\xi | \mathcal{F}_\nu] \triangleq \text{essinf}_{i \in \mathcal{I}} \underline{\mathcal{E}}_i[\xi | \mathcal{F}_\nu]
\]

as the maximal and minimal expectation of \( \xi \) over \( \mathcal{E} \) at the stopping time \( \nu \). It is worth pointing out that the \( \overline{\mathcal{E}} \) is not an \( \mathcal{F} \)-expectation on \( \text{Dom}(\mathcal{E}) \) simply because \( \overline{\mathcal{E}}[\xi | \mathcal{F}_t] \) may not belong to \( \text{Dom}(\mathcal{E}) \) for some \( \xi \in \text{Dom}(\mathcal{E}) \) and \( t \in [0,T] \). However, we shall see in the same example, \( \overline{\mathcal{E}} \) does not satisfy (H2) while \( \underline{\mathcal{E}} \) does not satisfy (H1); thus we do not have a dominated convergence theorem for either \( \overline{\mathcal{E}} \) or \( \underline{\mathcal{E}} \). Note also that \( \underline{\mathcal{E}} \) may not even be convex.

Our results in Sections 4 and 5 can be interpreted as a first step in extending the results for the optimal stopping problem \( \sup_{\rho \in \mathcal{S}_{0,T}} \mathcal{E}_i[\rho] \), in which \( \mathcal{E}_i \) \( (i \in \mathcal{I}) \) is an \( \mathbf{F} \)-expectation satisfying the assumptions (H1)-(H3), to optimal stopping problems for other non-linear expectations, such as \( \overline{\mathcal{E}} \) and \( \underline{\mathcal{E}} \), which may fail to satisfy these assumptions.

Example 6.1. Consider a probability space \( ([0, \infty), \mathcal{B}[0, \infty), \mathbf{F} = \{ \mathcal{F}_t \}_{t \geq 0}, \mathbf{P}) \) be a filtered probability space in which \( \mathbf{P} \) is defined by

\[
P(A) \triangleq \int_A e^{-x} dx, \quad \forall A \in \mathcal{B}[0, \infty).
\]

We assume that the filtration \( \mathbf{F} \) satisfies the usual hypothesis. Let \( \mathcal{P} \) denote the set of all probability measures equivalent to \( \mathbf{P} \). For any \( n \in \mathbb{N} \), we define a \( \mathcal{P}_n \in \mathcal{P} \) by

\[
P_n(A) \triangleq n \int_A e^{-nx} dx, \quad \forall A \in \mathcal{B}[0, \infty).
\]

As discussed in Example 5.1, \( \mathcal{E} = \{ E_Q \}_{Q \in \mathcal{P}} \) is a stable class. For any \( h > 0 \), one can deduce that

\[
1 = \sup_{Q \in \mathcal{P}} E_Q[1] \geq \overline{\mathcal{E}}[1_{[0,h]}] = \sup_{Q \in \mathcal{P}} E_Q[1_{[0,h]}] \geq \sup_{n \in \mathbb{N}} E_{P_n}[1_{[0,h]}] = \sup_{n \in \mathbb{N}} P_n[0, h] = \lim_{n \to \infty} (1 - e^{-nh}) = 1,
\]

where we used the fact that \( E_Q = E_Q \) for any \( Q \in \mathcal{P} \) since \( E_Q[\xi | \mathcal{F}_t] \) is an RCLL process for any \( \xi \in L^1([0, \infty), \mathcal{B}[0, \infty), \mathbf{P}) \). Hence, we have

\[
\overline{\mathcal{E}}[1_{[0,h]}] = 1, \quad \forall h > 0,
\]

which implies that \( \overline{\mathcal{E}} \) does not satisfy strict monotonicity.

Moreover, \( \overline{\mathcal{E}} \) does not satisfy (H2). For \( \xi = 0 \), \( \eta = 1 \) and \( A_n = [0, \frac{1}{n}] \), \( n \in \mathbb{N} \), it follows that

\[
\lim_{n \to \infty} \frac{1}{n} \overline{\mathcal{E}}[\xi + 1_{A_n}] = \lim_{n \to \infty} \overline{\mathcal{E}}[1_{[0,\frac{1}{n}]}] = 1 \neq 0 = \sup_{Q \in \mathcal{P}} E_Q[0] = \overline{\mathcal{E}}[0] = \overline{\mathcal{E}}[\xi].
\]

On the other hand, it is simple to see that \( \underline{\mathcal{E}}[1_{[h, \infty)}] = 0 \) for any \( h > 0 \), which means that \( \underline{\mathcal{E}} \) does not satisfy strict monotonicity either. Furthermore, \( \underline{\mathcal{E}} \) does not satisfy (H1). For \( \xi = 1 \) and \( A_n = [\frac{1}{n}, \infty) \), \( n \in \mathbb{N} \), we have that

\[
\lim_{n \to \infty} \frac{1}{n} \underline{\mathcal{E}}[1_{A_n} \xi] = \lim_{n \to \infty} \underline{\mathcal{E}}[1_{[\frac{1}{n}, \infty)}] = 0 \neq 1 = \inf_{Q \in \mathcal{P}} E_Q[1] = \underline{\mathcal{E}}[1] = \underline{\mathcal{E}}[\xi]. \quad \square
\]

Although it does not satisfy strict monotonicity, \( \underline{\mathcal{E}} \) is almost an \( \mathbf{F} \)-expectations on \( \text{Dom}(\mathcal{E}) \) as we will see next.

Proposition 6.1. For any \( t \in [0,T] \), \( \underline{\mathcal{E}}[\cdot | \mathcal{F}_t] \) is an operator from \( \text{Dom}(\mathcal{E}) \) to \( \text{Dom}(\mathcal{E}) \triangleq \text{Dom}(\mathcal{E}) \cap L^0(\mathcal{F}_t) \). Moreover, the family of operators \( \{ \underline{\mathcal{E}}[\cdot | \mathcal{F}_t] \}_{t \in [0,T]} \) satisfies (A2)-(A4) as well as

\[
\underline{\mathcal{E}}[\xi | \mathcal{F}_t] \leq \underline{\mathcal{E}}[\eta | \mathcal{F}_t], \quad \text{a.s. for any } \xi, \eta \in \text{Dom}(\mathcal{E}) \text{ with } \xi \leq \eta, \quad \text{a.s.}
\]

(6.1)
6. Remarks on Sections 4 & 5

Remark 2.

We have found that the first time \( \tau(0) \) when the Snell envelope \( Z^0 \) meets the process \( Y \) is an optimal stopping time for (4.1), while the first time \( \tau_Y \) when the process \( V^0 \) meets the process \( Y \) is an optimal stopping time for (5.1). It is natural to ask whether \( \tau(0) \) (resp. \( \tau_Y \)) is the minimal optimal stopping time of (4.1) (resp. (5.1)). This answer is affirmative when \( \mathcal{E} \) is a singleton. Let \( \mathcal{E} \) be a positively-convex \( \mathbb{F} \)-expectation satisfying (H1)-(H3) and let \( Y \) be a right-continuous \( \mathbb{F} \)-adapted process satisfying (Y1), (Y3) and the following

\[
\sup_{\rho \in S_{0,T}} \mathcal{E}[Y_\rho] < \infty; \text{ if } \mathcal{E} \text{ does not satisfy (2.3), then } \operatorname{esssup}_{(\rho, t) \in S_{0,T} \times \mathcal{F}_T} \tilde{\mathcal{E}}[Y_\rho | \mathcal{F}_t] \in \text{Dom}^\#(\mathcal{E}).
\]

(Note that we have here merged the cumulative reward process \( H \) into the primary reward process \( Y \).) If \( \tau \in S_{0,T} \) is an optimal stopping time for (4.1), i.e. \( \sup_{\rho \in S_{0,T}} \mathcal{E}[Y_\rho] = \mathcal{E}[Y_\tau] \), Proposition 4.2 and (4.17) imply that

\[
\sup_{\rho \in S_{0,T}} \mathcal{E}[Y_\rho] = \sup_{\rho \in S_{0,T}} \tilde{\mathcal{E}}[Y_\rho] = Z(0) \geq \tilde{\mathcal{E}}[Z(\tau)] = \tilde{\mathcal{E}}[Z^0_\tau] = \mathcal{E}[Z^0_\tau] \geq \mathcal{E}[Y_\tau] = \sup_{\rho \in S_{0,T}} \mathcal{E}[Y_\rho],
\]

thus \( \mathcal{E}[Z^0_\tau] = \mathcal{E}[Y_\tau] \). The second part of (A1) then implies that \( Z^0_\tau = Y_\tau \), a.s. Hence \( \tau(0) \leq \tau \), a.s., which means that \( \tau(0) \) is the minimal stopping time for (4.1).

However, this is not the case in general. Let \( \tilde{\mathcal{E}} = \{ \mathcal{E}_i \}_{i \in I} \) be a stable class of \( \mathbb{F} \)-expectations and let \( Y \) be a right-continuous \( \mathbb{F} \)-adapted process satisfying (Y1)-(Y3). We take \( H^i \equiv 0 \) for any \( i \in I \). If \( \tau \in S_{0,T} \) is an optimal stopping time for (4.1), i.e. \( \sup_{(i, \rho) \in I \times S_{0,T}} \mathcal{E}_i[Y_\rho] = \sup_{i \in I} \mathcal{E}_i[Y_\tau] \), (4.15) and (4.17) then imply that

\[
\sup_{(i, \rho) \in I \times S_{0,T}} \mathcal{E}_i[Y_\rho] = \sup_{i \in I} \mathcal{E}_i[Y_\tau] = \tilde{\mathcal{E}}[Y_\tau] = \sup_{i \in I} \mathcal{E}_i[Y_\tau],
\]

thus \( \tilde{\mathcal{E}}[Z^0_\tau] = \sup_{i \in I} \mathcal{E}_i[Z^0_\tau] = \sup_{i \in I} \mathcal{E}_i[Y_\tau] = \mathcal{E}[Y_\tau] \). However, this may not imply that \( Z^0_\tau = Y_\tau \), a.s. since \( \tilde{\mathcal{E}} \) does not satisfy strict monotonicity as we have seen in Example 6.1.

Now we further assume that \( Y \) satisfies (Y2'), if \( \tau' \in S_{0,T} \) is an optimal stopping time for (5.1), i.e. \( \sup_{\rho \in S_{0,T} \cap I} \inf_{i \in I} \mathcal{E}_i[Y_\rho] = \inf_{i \in I} \mathcal{E}_i[Y_\tau'] \), (5.10) and Theorem 5.1 imply that

\[
\sup_{\rho \in S_{0,T} \cap I} \inf_{i \in I} \mathcal{E}_i[Y_\rho] = \sup_{\rho \in S_{0,T} \cap I} \inf_{i \in I} \tilde{\mathcal{E}}_i[Y_\rho] = \inf_{i \in I} \inf_{\rho \in S_{0,T} \cap I} \mathcal{E}_i[Y_\rho] \geq \inf_{i \in I} \inf_{\rho \in S_{0,T} \cap I} \mathcal{E}_i[Y_\rho],
\]

thus \( \tilde{\mathcal{E}}[V(\tau')] = \inf_{i \in I} \mathcal{E}_i[V(\tau')] = \inf_{i \in I} \mathcal{E}_i[Y_\tau'] = \mathcal{E}[Y_\tau'] \). However, this may not imply that \( V(\tau') = Y_\tau' \), a.s. since \( \tilde{\mathcal{E}} \) does not satisfy strict monotonicity, which we have also seen in Example 6.1 (If \( V(\tau') \) were a.s. equal to \( Y_\tau' \), for any \( i \in I \), applying (4.15) to singleton \( \{ \mathcal{E}_i \} \), we would deduce from (4.13) and Lemma 5.3 that

\[
\begin{align*}
V^0_{\tau' \land \tau_Y} &= V(\tau' \land \tau_Y) = \tilde{V}(\tau' \land \tau_Y) = \operatorname{essinf}_{i \in I} R^i(\tau' \land \tau_Y) = \operatorname{essinf}_{i \in I} \left( 1_{\{\tau' \leq \tau_Y\}} R^i(\tau') + 1_{\{\tau' > \tau_Y\}} R^i(\tau_Y) \right) \\
&= 1_{\{\tau' \leq \tau_Y\}} \operatorname{essinf}_{i \in I} R^i(\tau') + 1_{\{\tau' > \tau_Y\}} \operatorname{essinf}_{i \in I} R^i(\tau_Y) = 1_{\{\tau' \leq \tau_Y\}} \tilde{V}(\tau_Y) + 1_{\{\tau' > \tau_Y\}} \tilde{V}(\tau_Y) \\
&= 1_{\{\tau' \leq \tau_Y\}} V(\tau') + 1_{\{\tau' > \tau_Y\}} V(\tau_Y) = 1_{\{\tau' \leq \tau_Y\}} V(\tau') + 1_{\{\tau' > \tau_Y\}} V(\tau_Y) \\
&= 1_{\{\tau' \leq \tau_Y\}} Y_{\tau'} + 1_{\{\tau' > \tau_Y\}} Y_{\tau_Y} = Y_{\tau' \land \tau_Y}, \text{ a.s.,}
\end{align*}
\]

which would further imply that \( \tau_Y = \tau' \land \tau_Y \), a.s., thus \( \tau_Y \leq \tau' \), a.s.)
7 Applications

In this section, we take a $d$-dimensional Brownian motion $B$ on the probability space $(\Omega, \mathcal{F}, P)$ and consider the Brownian filtration generated by it:

$$\mathbf{F} = \{ \mathcal{F}_t \equiv \sigma(B_s; s \in [0, t]) \cup \mathcal{N} \}_{t \in [0, T]},$$

where $\mathcal{N}$ collects all $P$-null sets in $\mathcal{F}$.

We also let $\mathcal{P}$ denote the predictable $\sigma$-algebra with respect to $\mathbf{F}$.

7.1 Lipschitz $g$-Expectations

Suppose that a “generator” function $g = g(t, \omega, z) : [0, T] \times \Omega \times \mathbb{R}^d \to \mathbb{R}$ satisfies

\[
\begin{aligned}
(i) & \quad g(t, \omega, 0) = 0, \quad dt \times dP\text{-a.s.} \\
(ii) & \quad g \text{ is Lipschitz in } z \text{ for some } K_g > 0: \text{ it holds } dt \times dP\text{-a.s. that } \\
& \quad |g(t, \omega, z_1) - g(t, \omega, z_2)| \leq K_g |z_1 - z_2|, \quad \forall z_1, z_2 \in \mathbb{R}^d.
\end{aligned}
\]

For any $\xi \in L^2(\mathcal{F}_T)$, it is well known from Pardoux and Peng [1990] that the backward stochastic differential equation (BSDE)

$$\Gamma_t = \xi + \int_t^T g(s, \Theta_s) ds - \int_t^T \Theta_s dB_s, \quad t \in [0, T]$$

admits a unique solution $(\Gamma^{\xi, g}, \Theta^{\xi, g}) \in C^2_T([0, T]) \times \mathcal{H}^2_T([0, T]; \mathbb{R}^d)$ (for convenience, we will denote (7.3) by BSDE$(\xi, g)$ in the sequel), based on which Peng [1997] introduced the so-called “$g$-expectation” of $\xi$ by

$$\mathcal{E}_g[\xi|\mathcal{F}_t] \overset{\Delta}{=} \Gamma^{\xi, g}_t, \quad t \in [0, T].$$

To show that the $g$-expectation $\mathcal{E}_g$ is an $\mathbf{F}$-expectation with domain $\text{Dom}(\mathcal{E}_g) = L^2(\mathcal{F}_T)$, we first note that $L^2(\mathcal{F}_T) \subseteq \mathcal{F}_T$. The (strict) Comparison Theorem for BSDEs (see e.g. Peng [1993, Theorem 35.3]) then shows that (A1) holds for the family of operators $\{\mathcal{E}_g[\cdot|\mathcal{F}_t] : L^2(\mathcal{F}_T) \to L^2(\mathcal{F}_t)\}_{t \in [0, T]}$, while the uniqueness of the solution $(\Gamma^{\xi, g}, \Theta^{\xi, g})$ to the BSDE$(\xi, g)$ implies that the family $\{\mathcal{E}_g[\cdot|\mathcal{F}_t]\}_{t \in [0, T]}$ satisfies (A2)-(A4) (see e.g. Peng [2004, Lemma 36.6] and Coquet et al. [2002, Lemma 2.1]). Therefore, $\mathcal{E}_g$ is an $\mathbf{F}$-expectation with domain $\text{Dom}(\mathcal{E}_g) = L^2(\mathcal{F}_T)$.

Moreover, the generator $g$ characterizes $\mathcal{E}_g$ in the following ways:

1. If the generator $g$ is convex (resp. concave) in $z$, i.e., it holds $dt \times dP$-a.s. that

$$g(t, \lambda z_1 + (1 - \lambda) z_2) \leq (\text{resp.} \geq) \lambda g(t, z_1) + (1 - \lambda) g(t, z_2), \quad \forall \lambda \in (0, 1), \quad \forall z_1, z_2 \in \mathbb{R}^d,$$

then $\mathcal{E}_g[\cdot|\mathcal{F}_t]$ is a convex (resp. concave) operator on $L^2(\mathcal{F}_T)$ for any $t \in [0, T]$, thanks to the Comparison Theorem for BSDEs. (see e.g. El Karoui et al. [1997] or Peng [2004, Proposition 5.1]).

2. Let $\tilde{g}$ be another generator satisfying (7.2). If it holds $dt \times dP$-a.s. that

$$g(t, z) \geq \tilde{g}(t, z), \quad \forall z \in \mathbb{R}^d,$$

the Comparison Theorem for BSDEs again implies that for any $\xi \in L^2(\mathcal{F}_T)$ and $t \in [0, T]$

$$\mathcal{E}_g[\xi|\mathcal{F}_t] \geq \mathcal{E}_{\tilde{g}}[\xi|\mathcal{F}_t], \quad a.s.$$

(6.7)

3. $g^{-}(t, \omega, z) \overset{\Delta}{=} -g(t, \omega, -z), \quad (t, \omega, z) \in [0, T] \times \Omega \times \mathbb{R}^d$ also satisfies (7.2). Its corresponding $g$-expectation $\mathcal{E}_{g^{-}}$ relates to $\mathcal{E}_g$ in that for any $\xi \in L^2(\mathcal{F}_T)$ and $t \in [0, T]$

$$\mathcal{E}_{g^{-}}[\xi|\mathcal{F}_t] = -\mathcal{E}_g[-\xi|\mathcal{F}_t], \quad a.s.$$  

(7.7)

(In fact, multiplying both sides of BSDE$(-\xi, g)$ by $-1$, we see that the pair $(-\Gamma^{-\xi, g}, -\Theta^{-\xi, g})$ solves the BSDE$(\xi, g^{-})$.)

To show that the $g$-expectation $\mathcal{E}_g$ satisfies (H0)-(H3), we need two basic inequalities it satisfies.
Lemma 7.1. Let \( g \) be a generator satisfying (7.2).

1. For any \( \xi \in L^2(F_T) \), we have

\[
\left\| \sup_{t \in [0,T]} \left| \mathcal{E}_g[\xi|\mathcal{F}_t] \right| \right\|_{L^2(F_T)} + \left\| \Theta^{\xi,g} \right\|_{L^2(\nu \oplus \mathbb{R}^d)} \leq C(e^{(K_1+K_2)^T} \|\xi\|_{L^2(F_T)}),
\]

where \( C \) is a universal constant independent of \( \xi \) and \( g \).

2. For any \( \mu \geq K_g \) and \( \xi, \eta \in L^2(F_T) \), it holds a.s. that

\[
|\mathcal{E}_g[\xi|\mathcal{F}_t] - \mathcal{E}_g[\eta|\mathcal{F}_t]| \leq \mathcal{E}_{g_{\mu}}[|\xi - \eta|\mathcal{F}_t], \quad \forall t \in [0,T],
\]

where the generator \( g_{\mu} \) is defined by \( g_{\mu}(z) \overset{\triangle}{=} \mu|z|, z \in \mathbb{R}^d \).

**Proof:** A simple application of Briand et al. (2000, Proposition 2.2) yields (1). On the other hand, (2) is a mere generalization of Peng (2004, Proposition 3.7, inequality (60)) by taking into account the continuity of processes \( \mathcal{E}_g[\xi|\mathcal{F}_t] \) and \( \mathcal{E}_{g_{\mu}}[\xi|\mathcal{F}_t] \) for any \( \xi \in L^2(F_T) \).

**Proposition 7.1.** Let \( g \) be a generator satisfying (7.2). Then \( \mathcal{E}_g \) satisfies (H0)-(H3).

**Remark 7.1.** Since \( \mathcal{E}_g[\xi|\mathcal{F}_t] \) is a continuous process for any \( \xi \in L^2(F_T) \), we see from (2.6) that \( \tilde{\mathcal{E}}_g[\cdot|\mathcal{F}_T] \) is just a restriction of \( \mathcal{E}_g[\cdot|\mathcal{F}_T] \) to \( L^2(\nu,\mathcal{F}_T) \overset{\triangle}{=} \{ \xi \in L^2(F_T) : \xi \geq c, \text{ a.s. for some } c = c(\xi) \in \mathbb{R} \} \) for any \( \nu \in \mathcal{S}_{0,T} \).

Thanks to Proposition 7.1, all results on \( \mathcal{F} \)-expectations \( \mathcal{E} \) and \( \tilde{\mathcal{E}} \) in Section 2 are applicable to \( g \)-expectations. In the following example we deliver the promise we made in Remark 2.7. This example indicates that for some \( g \)-expectations, \( \lim_{n \to \infty} \mathcal{E}_g[\xi_n] < \infty \) is not a sufficient condition for \( \lim_{n \to \infty} \xi_n \in \text{Dom}^{+}(\mathcal{E}_g) = L^{2,+}(F_T) \overset{\triangle}{=} \{ \xi \in L^2(F_T) : \xi \geq 0, \text{ a.s.} \} \) given that \( \{\xi_n\}_{n \in \mathbb{N}} \) is an a.s. convergent sequence in \( \text{Dom}^{+}(\mathcal{E}_g) \).

**Example 7.1.** Consider a probability space \((\Omega, \mathcal{F}, \mathbb{P}, \lambda)\), where \( \lambda \) is the Lebesgue measure on \([0,1]\). We define a generator \( \tilde{g}(z) \overset{\triangle}{=} -|z|, z \in \mathbb{R}^d \). For any \( n \in \mathbb{N} \), it is clear that the random variable \( \{\xi_n(\omega) \overset{\triangle}{=} \omega^{-\frac{1}{2+\frac{1}{2}}}\}_{\omega \in [0,1]} \in L^{2,+}(\mathcal{F}_T) = \text{Dom}^{+}(\tilde{g}) \). Proposition 2.2 (2) then implies that

\[
0 = \mathcal{E}_{\tilde{g}}[0] = \mathcal{E}_{\tilde{g}}[\xi_n] = \Gamma_0 \xi_n - \int_0^T |\Theta_s^{\xi_n,\tilde{g}}|ds - \int_0^T \Theta_s^{\xi_n,\tilde{g}}dB_s \leq \xi_n - \int_0^T \Theta_s^{\xi_n,\tilde{g}}dB_s.
\]

Taking the expected value of the above inequality yields that

\[
0 \leq \mathcal{E}_g[\xi] \leq \mathcal{E}[\xi] - \int_0^T \Theta_s^{\xi,\tilde{g}}dB_s \leq 0 \leq \mathcal{E}_g[\xi] = \mathcal{E}[\xi] - \int_0^T \Theta_s^{\xi,\tilde{g}}dB_s < 2.
\]

(7.8)

Since \( \{\xi_n\}_{n \in \mathbb{N}} \) is an increasing sequence, we can deduce from (A1) and (7.8) that \( 0 \leq \lim_{n \to \infty} \mathcal{E}_g[\xi_n] \leq 2 \). However, \( \lim_{n \to \infty} \uparrow \xi_n = \{\omega^{-\frac{1}{2}}\}_{\omega \in [0,1]} \) does not belong to \( L^{2,+}(\mathcal{F}_T) = \text{Dom}^{+}(\tilde{g}) \).

Similar to Proposition 5.1, pasting two \( g \)-expectations at any stopping time generates another \( g \)-expectation.

**Proposition 7.2.** Let \( g_1, g_2 \) be two generators satisfying (7.2) with Lipschitz coefficients \( K_1 \) and \( K_2 \) respectively. For any \( \nu \in \mathcal{S}_{0,T} \), we define the pasting of \( \mathcal{E}_{g_1}, \mathcal{E}_{g_2} \) at \( \nu \) to be the following continuous \( \mathcal{F} \)-adapted process

\[
\mathcal{E}^\nu_{g_1, g_2}[\xi|\mathcal{F}_t] \overset{\triangle}{=} 1_{\{\nu \leq t\}} \mathcal{E}_{g_2}[\xi|\mathcal{F}_t] + 1_{\{\nu > t\}} \mathcal{E}_{g_1}[\xi|\mathcal{F}_t|\mathcal{F}_t], \quad \forall t \in [0,T]
\]

(7.9)

for any \( \xi \in L^2(F_T) \). Then \( \mathcal{E}^\nu_{g_1, g_2} \) is exactly the \( g^\nu \)-expectation \( \mathcal{E}_{g^\nu} \) with

\[
g^\nu(t, \omega, z) \overset{\triangle}{=} 1_{\{\nu(\omega) \leq t\}} g_2(t, \omega, z) + 1_{\{\nu(\omega) > t\}} g_1(t, \omega, z), \quad (t, \omega, z) \in [0,T] \times \Omega \times \mathbb{R}^d,
\]

(7.10)

which is a generator satisfying (7.2) with the Lipschitz coefficient \( K_1 \lor K_2 \).
Fix $M > 0$, we denote by $\mathcal{G}_M$ the collection of all convex generators $g$ satisfying (5.2) with Lipschitz coefficient $K_g \leq M$. Proposition 5.2 shows that the family of convex $g$-expectations $\mathcal{E}_M \triangleq \{E_g\}_{g \in \mathcal{G}_M}$ is closed under the pasting (7.9). To wit, $\mathcal{E}_M$ is a stable class of $g$-expectations in the sense of Definition 5.2. In what follows we let $\mathcal{G}'$ be a non-empty subset of $\mathcal{G}_M$ such that $\mathcal{E}' \triangleq \{E_g\}_{g \in \mathcal{G}'}$ is closed under pasting. Now we make the following assumptions on the reward processes:

**Standing assumptions on the reward processes in this subsection.** Let $Y$ be a continuous $\mathcal{F}$-adapted process with

$$
\zeta_Y \triangleq \left( \text{esssup}_{t \in D_T} Y_t \right)^+ \in L^2(\mathcal{F}_T)
$$

and satisfying (Y3). Moreover, for any $g \in \mathcal{G}'$, we suppose that the model-dependent cumulative reward process is in the form of

$$
H_t^g \triangleq \int_0^t h^g_s ds, \quad \forall t \in [0, T],
$$

where $\{h^g_t, t \in [0, T]\}_{g \in \mathcal{G}'}$ is a family of predictable processes satisfying the following assumptions:

- (h1) There exists a $c' < 0$ such that for any $g \in \mathcal{G}'$, $h^g_t \geq c'$, $dt \times dP$-a.s.

- (h2) The random variable $\omega \mapsto \int_0^T h^g(t, \omega) dt$ belongs to $L^2(\mathcal{F}_T)$ with $h^g(t, \omega) \triangleq \left( \text{esssup}_{g \in \mathcal{G}'} h^g_s(\omega) \right)^+$ (the essential supremum is taken with respect to the product measure space $([0, T] \times \Omega, \mathcal{F}, \lambda \times P)$, where $\lambda$ denotes the Lebesgue measure on $[0, T]$).

- (h3) For any $\nu \in \mathcal{S}_{0,T}$ and $g_1, g_2 \in \mathcal{G}'$, it holds for any $0 \leq s < t \leq T$ that

$$
h^g_{t-s} = 1_{\{\nu \leq t\}} h^g_t + 1_{\{\nu > t\}} h^g_s, \quad dt \times dP\text{-a.s.},
$$

where $g^\nu$ is defined in (7.10).

Then the triple $(\mathcal{E}', \mathcal{H}', \zeta' \triangleq \{H^g\}_{g \in \mathcal{G}'}, Y)$ satisfies all the conditions stated in Section 4 and 5. Thus we can carry out the optimal stopping theory developed for $g$-expectations to $(\mathcal{E}'', \mathcal{H}'', Y)$ as we will see next.

**Theorem 7.1.** The stable class $\mathcal{E}'$ satisfies (5.12), the family of processes $\mathcal{H}'$ satisfies (S1') (thus (S1) see Remark 5.7), (S2) and (S3), while the process $Y$ satisfies (Y1), (4.5) (thus (Y2'), again by Remark 5.7 and (Y3)). Moreover, the family of processes $\{Y^\rho_t \triangleq Y_t + H^\rho_t, t \in [0, T]\}_{g \in \mathcal{G}'}$ is both $\mathcal{E}'$-"uniformly-left-continuous" (thus satisfies (5.2), see also Remark 5.7) and $\mathcal{E}'$-"uniformly-right-continuous".

### 7.2 Existence of an Optimal Prior in (4.1) for $g$-Expectations

For certain collections of $g$-expectations, we can even determine an optimal generator $g^*$, i.e., we can find a generator $g^*$ such that

$$
\mathcal{E}_{g^*}[Y^\rho_{\tau(0)}] = \sup_{g \in \mathcal{G}} \mathcal{E}_g[Y^\rho_{\tau(0)}] = \sup_{(g, \rho) \in \mathcal{G} \times \mathcal{S}_{0,T}} \mathcal{E}_g[Y^\rho_{\tau}],
$$

where the optimal stopping time $\tau(0)$ is defined as in Theorem 4.1.

Let $S$ be a separable metric space with metric $| \cdot |_S$ such that $S$ is a countable union of non-empty compact subsets. We denote by $\mathcal{S}$ the Borel $\sigma$-algebra of $S$ and take $\mathcal{H}_F^\mathcal{S}([0, T]; S)$ as the space of admissible control strategies. For any $U \in \mathcal{H}_F^\mathcal{S}([0, T]; S)$, we define the generator

$$
g_{\nu}(t, \omega, z) \triangleq g^\rho(t, \omega, z, U_t(\omega)),
$$

where the function $g^\rho(t, \omega, z, u) : [0, T] \times \Omega \times \mathbb{R}^d \times S \mapsto \mathbb{R}$ satisfies:
7.2 Existence of an Optimal Prior in \([\mathcal{F}^t]\) for \(g\)-Expectations

\((g^0) 1\) \(g^o\) is \(\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)\)-\(\mathcal{S}/\mathcal{B}(\mathbb{R})\)-measurable.

\((g^0) 2\) It holds \(dt \times dP\)-a.s. that

\[
g^o(t, \omega, 0, u) = 0, \quad \forall u \in S.
\]

\((g^0) 3\) \(g^o\) is Lipschitz in \(z\): For some \(K_0 > 0\), it holds \(dt \times dP\)-a.s. that

\[
|g^o(t, \omega, z_1, u) - g^o(t, \omega, z_2, u)| \leq K_0|z_1 - z_2|, \quad \forall z_1, z_2 \in \mathbb{R}^d, \quad \forall u \in S.
\]

\((g^0) 4\) \(g^o\) is convex in \(z\): It holds \(dt \times dP\)-a.s. that

\[
g^o(t, \omega, \lambda z_1 + (1 - \lambda) z_2, u) \leq \lambda g^o(t, \omega, z_1, u) + (1 - \lambda) g^o(t, \omega, z_2, u), \quad \forall \lambda \in (0, 1), \quad \forall z_1, z_2 \in \mathbb{R}^d, \quad \forall u \in S.
\]

Now fix a non-empty subset \(\mathcal{U}\) of \(\mathcal{H}^\mathcal{F}_T([0, T]; S)\) that preserves “pasting”, i.e., for any \(\nu \in \mathcal{S}_{0, T}\) and \(U^1, U^2 \in \mathcal{U}\),

\[
U^1_\nu(\omega) \overset{\Delta}{=} 1_{\{\nu(\omega) \leq t\}} U^2_\nu(\omega) + 1_{\{\nu(\omega) > t\}} U^1_\nu(\omega), \quad (t, \omega) \in [0, T] \times \Omega,
\]

also belongs to \(\mathcal{U}\). Then it is easy to check that \(\{\mathcal{E}_{g_\nu}\}_{\nu \in \mathcal{U}} \subset \mathcal{E}_{K_0}\) forms a stable class of \(g\)-expectations.

Let \(Y\) still be a continuous \(\mathcal{F}\)-adapted process satisfying \((7.11)\) and \((Y3)\). For any \(U \in \mathcal{U}\), assume that the model dependent reward process has a density which is given by

\[
h^U_{\nu}(\omega) \overset{\Delta}{=} h(t, \omega, U_\nu(\omega)), \quad (t, \omega) \in [0, T] \times \Omega,
\]

where \(h(t, \omega, u) : [0, T] \times \Omega \times S \to \mathbb{R}\) is a \(\mathcal{P} \otimes \mathcal{S}/\mathcal{B}(\mathbb{R})\)-measurable function satisfying the following assumptions:

\((h1)\) For some \(c < 0\), it holds \(dt \times dP\)-a.s. that \(h(t, \omega, u) \geq c\) for any \(u \in S\).

\((h2)\) The random variable \(\omega \mapsto \int_0^T \hat{h}(t, \omega) dt\) belongs to \(L^2(\mathcal{F}_T)\) with \(\hat{h}(t, \omega) \overset{\Delta}{=} \left(\text{esssup}_{U \in \mathcal{U}} h^U_{\nu}(\omega)\right)^+\) (the essential supremum is taken with respect to the product measure space \(([0, T] \times \Omega, \mathcal{P}, \lambda \times P)\), where \(\lambda\) denotes the Lebesgue measure on \([0, T]\)).

It is easy to see that \(\{h^U_{\nu}, t \in [0, T]\}_{\nu \in \mathcal{U}}\) is a family of predictable processes satisfying \((\hat{h}1)-(\hat{h}3)\). Hence, we can apply the optimal stopping theory developed for \(\mathcal{F}\)-expectations to the triple \((\{\mathcal{E}_{g_\nu}\}_{\nu \in \mathcal{U}}, \{h^U\}_{U \in \mathcal{U}}, Y)\) thanks to Theorem \([4.1]\). Now let us construct a so-called Hamiltonian function

\[
H(t, \omega, z, u) \overset{\Delta}{=} g^o(t, \omega, z, u) + h(t, \omega, u), \quad (t, \omega, z, u) \in [0, T] \times \Omega \times \mathbb{R}^d \times S.
\]

We assume that for any \((t, \omega, z) \in [0, T] \times \Omega \times \mathbb{R}^d\), there exists a \(u = u^*(t, \omega, z) \in S\) such that

\[
\sup_{u \in S} H(t, \omega, z, u) = H(t, \omega, z, u^*(t, \omega, z)),
\]

(7.14)

(This is valid, for example, when the metric space \(S\) is compact and the mapping \(u \mapsto H(t, \omega, z, u)\) is continuous.) Then it can be shown (see Beneš [1970], Lemma 1 or Elliott [1982], Lemma 16.34) that the mapping \(u^* : [0, T] \times \Omega \times \mathbb{R}^d \to S\) can be selected to be \(\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)\)-\(\mathcal{S}/\mathcal{B}(\mathbb{R})\)-measurable.

The following theorem is the main result of this subsection.

**Theorem 7.2.** There exists a \(U^* \in \mathcal{U}\) such that

\[
\sup_{(U, \rho) \in \mathcal{U} \times \mathcal{S}_{0, T}} \mathcal{E}_{g_\rho} [Y^U_\rho] = \mathcal{E}_{g_{U^*}} [Y^U_{\tau(0)}],
\]

where the stopping time \(\tau(0)\) is as in Theorem \([4.1]\).
7.3 The Cooperative Game of Karatzas and Zamfirescu [2006] Revisited

In this subsection, we apply the results of the last subsection to extend the results of Karatzas and Zamfirescu [2006]. Let us first recall their setting:

- Consider the canonical space $\left(\Omega, \mathcal{F}\right) = (C([0, T]; \mathbb{R}^d), \mathcal{B}(C([0, T]; \mathbb{R}^d)))$ endowed with Wiener measure $P$, under which the coordinate mapping process $B(t, \omega) = \omega(t)$, $t \in [0, T]$ becomes a standard $d$-dimensional Brownian motion. We still take the filtration $\mathbf{F}$ generated by the Brownian motion $B$ (see (7.1)) and let $\mathcal{P}$ denote the predictable $\sigma$-algebra with respect to $\mathbf{F}$.

- It is well-known (see e.g. Elliott [1982, Theorem 14.6]) that given a $x \in \mathbb{R}^d$, there exists a pathwise unique, strong solution $X(\cdot)$ of the stochastic equation

$$X(t) = x + \int_0^t \sigma(s, X) dB_s, \quad t \in [0, T],$$

where the diffusion term $\sigma(t, \omega)$ is a $\mathbb{R}^{d \times d}$-valued predictable process satisfying:

$$(\sigma 1) \int_0^T |\sigma(t, \bar{0})|^2 dt < \infty$$

and $\sigma(t, \omega)$ is nonsingular for any $(t, \omega) \in [0, T] \times \Omega$.

$$(\sigma 2)$$ There exists a $K > 0$ such that for any $\omega, \bar{\omega} \in \Omega$ and $t \in [0, T]$

$$\|\sigma^{-1}(t, \omega)\| \leq K \quad \text{and} \quad |\sigma_{ij}(t, \omega) - \sigma_{ij}(t, \bar{\omega})| \leq K \|\omega - \bar{\omega}\|^*_{ij} \quad \forall 1 \leq i, j \leq n$$

with $\|\omega\|^* \triangleq \sup_{s \in [0, t]} |\omega(s)|$.

Let $f(t, \omega, u) : [0, T] \times \Omega \times S \mapsto \mathbb{R}^d$ be a $\mathcal{P} \otimes \mathcal{G}/\mathcal{B}(\mathbb{R}^d)$-measurable function such that

(i) For any $u \in S$, the mapping $(t, \omega) \mapsto f(t, \omega, u)$ is predictable (i.e. $\mathcal{P}$-measurable);

(ii) With the same $K$ as in (7.15),

$$|f(t, \omega, u)| \leq K \left(1 + \|\omega\|^*\right), \quad \forall (t, \omega, u) \in [0, T] \times \Omega \times S.$$  

(7.16)

For any $U \in \widetilde{\mathcal{U}} \triangleq \mathcal{H}_P^0([0, T]; S)$, Karatzas and Zamfirescu [2006, page 166] shows that

$$\frac{dP_U}{dP} \triangleq \exp \left\{ \int_0^T \langle \sigma^{-1}(t, X)f(t, X, U_t), dB_t \rangle - \frac{1}{2} \int_0^T |\sigma^{-1}(t, X)f(t, X, U_t)|^2 dt \right\},$$

defines a probability measure $P_U$.

The objective of Karatzas and Zamfirescu [2006] is to find an optimal stopping time $\tau^* \in \mathcal{S}_{0,T}$ and an optimal control strategy $U^* \in \mathcal{U}$ that maximizes the expected reward

$$E_U \left[ \varphi(X(\rho)) + \int_0^\rho h(s, X, U_s) ds \right]$$

over $(\rho, U) \in \mathcal{S}_{0,T} \times \mathcal{U}$. Here $\varphi : \mathbb{R}^d \mapsto \mathbb{R}$ is a bounded continuous function, and $h(t, \omega, u) : [0, T] \times \Omega \times S \mapsto \mathbb{R}$ is a $\mathcal{P} \otimes \mathcal{G}/\mathcal{B}(\mathbb{R})$-measurable function such that $|h(t, \omega, u)| \leq K$ for any $(t, \omega, u) \in [0, T] \times \Omega \times S$ (with the same $K$ that appears in (7.15)).

Corollary 8 of Karatzas and Zamfirescu [2006] shows that under (7.16), the process

$$\mathcal{Z}(t) \triangleq \text{esssup}_{(U, \rho) \in \mathcal{U} \times \mathcal{S}_{0,T}} E_U \left[ \varphi(X(\rho)) + \int_t^\rho h(s, X, U_s) ds \bigg| \mathcal{F}_t \right], \quad t \in [0, T]$$

admits an RCLL modification $\tilde{Z}^0$, and the first time processes $\tilde{Z}^0$ and $\{\varphi(X(t))\}_{t \in [0, T]}$ meet with each other, i.e. $\tau(0) \triangleq \inf \{ t \in [0, T] \mid \tilde{Z}^0_t = \varphi(X(t)) \}$, is an optimal stopping time. That is,

$$\sup_{(U, \rho) \in \mathcal{U} \times \mathcal{S}_{0,T}} E_U \left[ \varphi(X(\rho)) + \int_0^\rho h(s, X, U_s) ds \right] = \sup_{U \in \mathcal{U}} E_U \left[ \varphi(X(\tau(0))) + \int_0^{\tau(0)} h(s, X, U_s) ds \right].$$  

(7.17)
Moreover, if for any \( (t, \omega, z) \in [0, T] \times \Omega \times \mathbb{R}^d \), there is a \( u^*(t, \omega, z) \in S \) which is \( \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)/\mathcal{G} \)-measurable such that
\[
\sup_{u \in S} \overline{H}(t, \omega, z, u) = H(t, \omega, z, u^*(t, \omega, z))
\] (7.18)
with \( \overline{H}(t, \omega, z, u) \triangleq \sigma^{-1}(t, \omega)f(t, \omega, u, z) + h(t, \omega, u, (t, \omega, z, u) \in [0, T] \times \Omega \times \mathbb{R}^d \times S \), then there further exists an optimal control strategy \( U^* \in \mathcal{U} \) (see Karatzas and Zamfirescu 2006, Section 8) such that
\[
\sup_{(U, \rho) \in \mathcal{U} \times S_{0,T}} E_U \left[ \varphi \left( X(\rho) \right) + \int_0^\rho h(s, X, U_s)ds \right] = E_{U^*} \left[ \varphi \left( X(\tau) \right) + \int_0^{\tau(0)} h(s, X, U^*_s)ds \right].
\] (7.19)

In the main result of this subsection, we will show that the assumption of Karatzas and Zamfirescu 2006 that \( \varphi \) and \( h \) are bounded from above by constants can be relaxed and replaced by linear-growth conditions. This comes, however, at the cost of strengthening the assumption stated in (7.10).

**Proposition 7.3.** With the same \( K \) as in (7.15), we assume that
\[
-K \leq \varphi(x) \leq K|z|, \quad \forall x \in \mathbb{R}^d
\] (7.20)
and that for a.e. \( t \in [0, T] \)
\[
|f(t, \omega, u)| \leq K \quad \text{and} \quad -K \leq h(t, \omega, u) \leq K||\omega||_{\mathcal{P}}, \quad \forall (u, \omega) \in \Omega \times S.
\] (7.21)

Then the process \( \{\tilde{Z}(t)\}_{t \in [0, T]} \) has a RCLL modification \( \tilde{Z}^0 \), and the first time \( \tau(0) \) when the process \( \tilde{Z}^0 \) meets the process \( \{\varphi(X(t))\}_{t \in [0, T]} \) is an optimal stopping time; i.e., it satisfies (7.17). Moreover, if there exists a measurable mapping \( u^*: [0, T] \times \Omega \times \mathbb{R}^d \rightarrow S \) satisfying (7.13), then there exists an optimal control strategy \( U^* \in \mathcal{U} \) such that (7.10) holds.

### 7.4 Quadratic \( g \)-Expectations

Now we consider a quadratic \( g \)-generator \( \hat{g} = \hat{g}(t, \omega, z) : [0, T] \times \Omega \times \mathbb{R}^d \rightarrow \mathbb{R} \) that satisfies
\[
\begin{cases}
(i) & \hat{g}(t, \omega, 0) = 0, \ dt \times dP\text{-a.s.} \\
(ii) & \text{For some } \kappa > 0, \ \text{it holds } dt \times dP\text{-a.s. that} \\
& \left| \frac{\partial \hat{g}}{\partial z}(t, \omega, z) \right| \leq \kappa(1 + |z|), \ \forall z \in \mathbb{R}^d. \\
(iii) & \hat{g} \text{ is convex in } z \text{ in the sense of } (65).
\end{cases}
\] (7.22)

Note that under (ii), (i) is equivalent to the following statement: It holds \( dt \times dP\text{-a.s.} \) that
\[
|\hat{g}(t, \omega, z)| \leq \kappa \left( |z| + \frac{1}{2} |z|^2 \right), \quad \forall z \in \mathbb{R}^d.
\] (7.23)

In fact, it is clear that (7.23) implies (i). Conversely, for \( dt \times dP\text{-a.s.} \) \( (t, \omega) \in [0, T] \times \Omega \), one can deduce that for any \( z \in \mathbb{R}^d \), \( |\hat{g}(t, \omega, z)| = |\hat{g}(t, \omega, z) - \hat{g}(t, \omega, 0)| = \int_0^1 \frac{\partial \hat{g}}{\partial z}(t, \lambda z)\lambda d\lambda \leq \kappa \int_0^1 (1 + \lambda |z|)|z| d\lambda = \kappa(|z| + \frac{1}{2} |z|^2).

For any \( \xi \in L^r(F_T), \) Briand and Hu 2008, Corollary 6 (where we take \( g = \alpha \), thus \( \alpha(t) \equiv \frac{\lambda}{2} \) and \( \beta, \gamma = (0, 2\kappa) \)) shows that the quadratic BSDE(\( \xi, \hat{g} \)) admits a unique solution \( (\Gamma^\xi, \Theta^\xi) \in C^p_F([0, T]) \times M^p_F([0, T]; \mathbb{R}^d) \). Hence we can correspondingly define the “quadratic” \( g \)-expectation of \( \xi \) by
\[
E^\xi_{\hat{g}}[\xi|F_t] \triangleq \Gamma^\xi, \quad t \in [0, T].
\]

To show that the quadratic \( g \)-expectation \( E^\xi_{\hat{g}} \) is an \( F \)-expectation with domain \( \text{Dom}(E^\xi_{\hat{g}}) = L^r(F_T) \), we first note that \( L^r(F_T) \in \tau_{F_T} \) (Clearly, \( L^r(F_T) \) satisfies (D1) and (D3) and \( \mathbb{R} \subset L^r(F_T) \)). For any \( \xi, \eta \in L^r(F_T), A \in F_T, \lambda > 0 \), we have
\[
E[\xi^{\lambda A}] \leq E[\eta^{\lambda A}] < \infty \quad \text{and} \quad E[\xi^{\lambda A}] \leq E[\eta^{\lambda A}] \leq \frac{1}{2} E[\xi^{2\lambda A}] + \frac{1}{2} E[\eta^{2\lambda A}] < \infty, \quad \text{thus (D2) also holds for } L^r(F_T).
\]

Similar to the Lipschitz \( g \)-expectation case, the uniqueness of the solution \( (\Gamma^\xi, \Theta^\xi) \) to the quadratic BSDE(\( \xi, \hat{g} \)) implies that the family of operators \( \{E^\xi_{\hat{g}}[\cdot|F_t] : L^r(F_T) \rightarrow L^r(F_t)\}_{t \in [0, T]} \) satisfies (A2)-(A4) (cf. Peng 2004, Lemma 36.6 and Coquet et al. 2002, Lemma 2.1), while a comparison theorem for quadratic BSDEs (see e.g. Briand and Hu 2008, Theorem 5) and the following proposition show that (A1) also holds for the family \( \{E^\xi_{\hat{g}}[\cdot|F_t]\}_{t \in [0, T]} \).
Proposition 7.4. Let \( \hat{g} \) be a quadratic generator satisfying (7.22). For any \( \xi^1, \xi^2 \in L^r(F_t) \), if \( \xi^1 \geq \xi^2 \), a.s., then it holds a.s. that
\[
\Gamma^\xi_1, \widehat{g} \geq \Gamma^\xi_2, \widehat{g}, \quad \forall t \in [0, T]. \tag{7.24}
\]
Moreover, if \( \Gamma^\xi_1, \nu \geq \Gamma^\xi_2, \nu \), a.s. for some \( \nu \in S_{0,T} \), then
\[
\xi^1 = \xi^2, \quad \text{a.s.} \tag{7.25}
\]

Therefore, the quadratic \( g \)-expectation \( \mathcal{E}_g \) is an \( F \)-expectation with domain \( \text{Dom}(\mathcal{E}_g) = L^r(F_T) \). Similar to the Lipschitz \( g \)-expectation case, the convexity (7.22)(iii) of the quadratic generator \( \hat{g} \) as well as Theorem 5 of Briand and Hu (2008) determine that \( \mathcal{E}_g[\cdot|F_t] \) is a convex operator on \( L^r(F_T) \) for any \( t \in [0, T] \). Hence, \( \mathcal{E}_g \) satisfies (H0) thanks to Lemma 3.1. To see \( \mathcal{E}_g \) also satisfying (H1)-(H3), we need the following stability result.

Lemma 7.2. If \( \xi_n \to \xi \), a.s. and \( E[e^{\lambda|\xi|}] + \sup_{n \in \mathbb{N}} E[e^{\lambda|\xi_n|}] < \infty \) for any \( \lambda > 0 \), then
\[
\lim_{n \to \infty} \left[ \sup_{t \in [0,T]} \left| \mathcal{E}_g[\xi_n|F_t] - \mathcal{E}_g[\xi|F_t] \right| \right] = 0. \tag{7.26}
\]

Proof: Taking \( f_n \equiv g \) and \( f = g \) in Proposition 7 of Briand and Hu (2008) yields that
\[
\lim_{n \to \infty} E \left[ \exp \left\{ p \sup_{t \in [0,T]} \left| \mathcal{E}_g[\xi_n|F_t] - \mathcal{E}_g[\xi|F_t] \right| \right\} \right] = 0, \quad \forall p \geq 1.
\]
Then (7.26) follows since \( E \left[ \sup_{t \in [0,T]} \left| \mathcal{E}_g[\xi_n|F_t] - \mathcal{E}_g[\xi|F_t] \right| \right] \leq E \left[ \exp \left\{ \sup_{t \in [0,T]} \left| \mathcal{E}_g[\xi_n|F_t] - \mathcal{E}_g[\xi|F_t] \right| \right\} \right] \) for any \( n \in \mathbb{N} \). \( \square \)

Proposition 7.5. Let \( \hat{g} \) be a quadratic generator satisfying (7.22). Then the quadratic \( g \)-expectation \( \mathcal{E}_g \) satisfies (H0)-(H3).

Similar to Remark 7.4, since \( \mathcal{E}_g[\cdot|F] \) is a continuous process for any \( \xi \in L^r(F_T) \), we see from (2.6) that \( \widetilde{\mathcal{E}}_g[\cdot|F] \) is just a restriction of \( \mathcal{E}_g[\cdot|F] \) to \( L^r(F_T) \equiv \{ \xi \in L^r(F_T) : \xi \geq c, \text{a.s. for some } c \in \mathbb{R} \} \) for any \( \nu \in S_{0,T} \). Therefore, all results on \( F \)-expectations \( \mathcal{E} \) and \( \widetilde{\mathcal{E}} \) in Section 2 work for quadratic \( g \)-expectations.

The next result, which shows the existence of an optimal stopping time for a quadratic \( g \)-expectation, is the main result of this subsection.

Theorem 7.3. Let \( \hat{g} \) be a quadratic generator satisfying (7.22). For any right-continuous \( F \)-adapted process \( Y \) with \( \xi_Y \triangleq \left( \operatorname{esssup}_{t \in D_T} Y_t \right)^+ \in L^r(F_T) \) and satisfying (Y3), we have
\[
\sup_{\rho \in S_{0,T}} \mathcal{E}_g[Y_{\rho}] = \mathcal{E}_g[Y_{\tau(0)}],
\]
where \( \tau(0) \) is as in Theorem 4.1.

8 Proofs

8.1 Proofs of Section 2

Proof of Proposition 2.1. For any \( \xi \in \Lambda \) and \( t \in [0, T] \), let us define \( \mathcal{E}[\xi|F] \triangleq \xi_t \). We will check that the system \( \{ \mathcal{E}[\xi|F_t], \xi \in \Lambda \}_{t \in [0, T]} \) satisfies (A1)-(A4); thus it is an \( F \)-expectation with domain \( \Lambda \).

1) For any \( \eta \in \Lambda \) with \( \xi \leq \eta \), a.s., we set \( A \triangleq \{ \mathcal{E}[\xi|F_T] > \mathcal{E}[\eta|F_T] \} \in F_t \), thus \( 1_A \mathcal{E}[\xi|F_T] \geq 1_A \mathcal{E}[\eta|F_T] \). It follows from (a1) and (a2) that
\[
\mathcal{E}^{\omega}[1_A \mathcal{E}[\xi|F_T]] \geq \mathcal{E}^{\omega}[1_A \mathcal{E}[\eta|F_T]] = \mathcal{E}^{\omega}[1_A \eta] \geq \mathcal{E}^{\omega}[1_A \xi] = \mathcal{E}^{\omega}[1_A \mathcal{E}[\xi|F_T]],
\]
which shows that $\mathcal{E}^o\left[1_A\mathcal{E}[\xi|\mathcal{F}_t]\right] = \mathcal{E}^o\left[1_A\mathcal{E}[\eta|\mathcal{F}_t]\right]$. Then the “strict monotonicity” of (a1) further implies that $1_A\mathcal{E}[\xi|\mathcal{F}_t] = 1_A\mathcal{E}[\eta|\mathcal{F}_t]$, a.s., thus $P(A) = 0$, i.e., $\mathcal{E}[\xi|\mathcal{F}_t] \leq \mathcal{E}[\eta|\mathcal{F}_t]$, a.s.

Moreover, if $0 \leq \xi \leq \eta$, a.s. and $\mathcal{E}[\xi|\mathcal{F}_0] = \mathcal{E}[\eta|\mathcal{F}_0]$, applying (a2) with $A = \Omega$ and $\gamma = 0$, we obtain

$$
\mathcal{E}^o[\xi] = \mathcal{E}^o[\mathcal{E}[\xi|\mathcal{F}_0]] = \mathcal{E}^o[\mathcal{E}[\eta|\mathcal{F}_0]] = \mathcal{E}^o[\eta].
$$

Then the strict monotonicity of (a1) implies that $\xi = \eta$, a.s., proving (A1).

2) Let $0 \leq s \leq t \leq T$, for any $A \in \mathcal{F}_s \subset \mathcal{F}_t$ and $\gamma \in \Lambda_s \subset \Lambda_t$, one can deduce that

$$
\mathcal{E}^o\left[1_A\mathcal{E}[\xi|\mathcal{F}_t]\right] + \gamma = \mathcal{E}^o\left[1_A\mathcal{E}[\xi|\mathcal{F}_t] + \gamma\right] = \mathcal{E}^o\left[1_A\mathcal{E}[\xi] + \gamma\right].
$$

Since $\mathcal{E}[\mathcal{E}[\xi|\mathcal{F}_s]|\mathcal{F}_s] \in \mathcal{F}_s$, (a2) implies that $\mathcal{E}[\xi|\mathcal{F}_s] = \xi_s = \mathcal{E}[\mathcal{E}[\xi|\mathcal{F}_s]|\mathcal{F}_s]$, proving (A2).

3) Fix $A \in \mathcal{F}_t$, for any $\tilde{A} \in \mathcal{F}_t$ and $\gamma \in \Lambda_t$, we have

$$
\mathcal{E}^o\left[1_{\tilde{A}}\mathcal{E}[\xi|\mathcal{F}_t]\right] + \gamma = \mathcal{E}^o\left[1_{\tilde{A} \cap \Lambda}\mathcal{E}[\xi|\mathcal{F}_t]\right] = \mathcal{E}^o\left[1_{\tilde{A} \cap \Lambda}\xi + \gamma\right] = \mathcal{E}^o\left[1_{\tilde{A}}\mathcal{E}[\xi] + \gamma\right].
$$

Since $1_A\mathcal{E}[\xi|\mathcal{F}_s] \in \mathcal{F}_s$, (a2) implies that $\mathcal{E}[1_A\xi|\mathcal{F}_s] = 1_A\mathcal{E}[\xi|\mathcal{F}_s]$, proving (A3).

4) For any $A \in \mathcal{F}_t$ and $\eta, \gamma \in \Lambda_t$, (D2) implies that $1_{A \cap \eta} + \gamma \in \Lambda_t$, thus we have

$$
\mathcal{E}^o\left[1_A\mathcal{E}[\xi|\mathcal{F}_t] + \eta\right] = \mathcal{E}^o\left[1_A\mathcal{E}[\xi|\mathcal{F}_t] + (1_A\gamma + \gamma)\right] = \mathcal{E}^o\left[1_A\mathcal{E}[\xi + (1_A\gamma + \gamma)]\right] = \mathcal{E}^o\left[1_A(\xi + \gamma)\right].
$$

Then it follows from (a2) that $\mathcal{E}[\xi + \eta|\mathcal{F}_t] = \mathcal{E}[\xi|\mathcal{F}_t] + \eta$, proving (A4). □

**Proof of Proposition 2.2** (1) For any $A \in \mathcal{F}_t$, using (A3) twice, we obtain

$$
\mathcal{E}[1_A\xi + 1_{A \cap \eta}|\mathcal{F}_t] = 1_A\mathcal{E}[1_A\xi + 1_{A \cap \eta}|\mathcal{F}_t] + 1_{A \cap \eta}\mathcal{E}[1_A\xi + 1_{A \cap \eta}|\mathcal{F}_t] = \mathcal{E}[1_A\mathcal{E}[1_A\xi + 1_{A \cap \eta}|\mathcal{F}_t] + 1_{A \cap \eta}\mathcal{E}[1_A\mathcal{E}[1_A\xi + 1_{A \cap \eta}|\mathcal{F}_t]] = \mathcal{E}[1_A\xi + 1_{A \cap \eta}||\mathcal{F}_t]] = \mathcal{E}[1_A\xi + 1_{A \cap \eta}||\mathcal{F}_t]], \ a.s.
$$

(2) Applying (A3) with a null set $A$ and $\xi = 0$, we obtain $\mathcal{E}[0|\mathcal{F}_t] = 1_A\mathcal{E}[0|\mathcal{F}_t] = 1_A\mathcal{E}[0|\mathcal{F}_t] = 0$, a.s. If $\xi \in Dom(\mathcal{E})$, (A4) implies that $\mathcal{E}[\xi|\mathcal{F}_t] = \mathcal{E}[0 + \xi|\mathcal{F}_t] = \mathcal{E}[0|\mathcal{F}_t] + \xi = \xi$, a.s.

(3) If $\xi \leq \eta$, a.s., (A1) directly implies that for any $A \in \mathcal{F}_t$, $\mathcal{E}[1_A\xi] \leq \mathcal{E}[1_A\eta]$. On the other hand, suppose that $\mathcal{E}[1_A\xi] \leq \mathcal{E}[1_A\eta]$ for any $A \in \mathcal{F}_t$. We set $\tilde{A} \equiv \{\xi > \eta\} \subset \mathcal{F}_t$, thus $1_A\xi \geq 1_{\tilde{A}}\eta \geq 1_{\tilde{A}}\eta \geq c \land 0$, a.s. Using (A1) we see that $\mathcal{E}[1_A\xi] \geq 1_{\tilde{A}}\eta$; hence $\mathcal{E}[1_A\xi] = \mathcal{E}[1_{\tilde{A}}\eta]$. Then (A4) implies that

$$
\mathcal{E}[1_A\xi - c \land 0] = \mathcal{E}[1_{\tilde{A}}\xi - c \land 0] = \mathcal{E}[1_A\eta] - c \land 0 = \mathcal{E}[1_A\eta] - c \land 0.
$$

Applying the second part of (A1), we obtain that $1_A\xi - c \land 0 = 1_{\tilde{A}}\eta - c \land 0$, a.s., which implies that $P(\tilde{A}) = 0$, i.e., $\xi \leq \eta$, a.s. □

**Proof of Proposition 2.3** We shall only consider the $\mathcal{E}$-supermartingale case, as the other cases can be deduced similarly. We first show that for any $s \in [0, T]$ and $\nu \in S^F_{\nu, T}$

$$
\mathcal{E}[X_{\nu}|\mathcal{F}_s] \leq X_{\nu \land s}, \ a.s. \tag{8.1}
$$

To see this, we note that since $\{\nu \leq s\} \subset \mathcal{F}_s$, (A3) and Proposition 2.2 (2) imply that

$$
\mathcal{E}[X_{\nu}|\mathcal{F}_s] = 1_{(\nu > s)}\mathcal{E}[X_{\nu}|\mathcal{F}_s] + 1_{(\nu \leq s)}\mathcal{E}[X_{\nu}|\mathcal{F}_s] = \mathcal{E}[1_{(\nu > s)}X_{\nu \land s}|\mathcal{F}_s] + \mathcal{E}[1_{(\nu \leq s)}X_{\nu \land s}|\mathcal{F}_s] = 1_{(\nu > s)}\mathcal{E}[X_{\nu \land s}|\mathcal{F}_s] + 1_{(\nu \leq s)}X_{\nu \land s}, \ a.s. \tag{8.2}
$$

Suppose that $\nu_s \equiv \nu \lor s$ takes values in a finite subset $\{t_1 < \cdots < t_n\}$ of $[s, T]$. Then (A4) implies that

$$
\mathcal{E}[X_{\nu_s}|\mathcal{F}_t_{n-1}] = \mathcal{E}[1_{(\nu_s = t_n)}X_{t_n}|\mathcal{F}_t_{n-1}] + \sum_{i=1}^{n-1} 1_{(\nu_s = t_i)}X_{t_i}, \ a.s.
$$
Since \( \{ \nu_s = t_n \} = \{ \nu_s > t_{n-1} \} \in \mathcal{F}_{t_{n-1}} \), (A3) shows that
\[
\mathcal{E} [ 1_{\{ \nu_s = t_n \}} X_{t_n} | \mathcal{F}_{t_{n-1}} ] = 1_{\{ \nu_s = t_n \}} \mathcal{E} [ X_{t_n} | \mathcal{F}_{t_{n-1}} ] \leq 1_{\{ \nu_s = t_n \}} X_{t_{n-1}}, \quad \text{a.s.}
\]

Thus it holds a.s. that \( \mathcal{E} [ X_{\nu_s} | \mathcal{F}_{t_{n-1}} ] \leq 1_{\{ \nu_s > t_{n-2} \}} X_{t_{n-1}} + \sum_{i=1}^{n-2} 1_{\{ \nu_s = t_i \}} X_{t_i} \). Applying \( \mathcal{E} [| \cdot | \mathcal{F}_{t_{n-2}}] \) on both sides, we can further deduce from (A2)-(A4) that
\[
\mathcal{E} [ X_{\nu_s} | \mathcal{F}_{t_{n-2}} ] = \mathcal{E} \left[ \mathcal{E} [ X_{\nu_s} | \mathcal{F}_{t_{n-1}} ] | \mathcal{F}_{t_{n-2}} \right] \leq 1_{\{ \nu_s > t_{n-2} \}} \mathcal{E} [ X_{t_{n-1}} | \mathcal{F}_{t_{n-2}} ] + \sum_{i=1}^{n-2} 1_{\{ \nu_s = t_i \}} X_{t_i}
\]
\[
\leq 1_{\{ \nu_s > t_{n-2} \}} X_{t_{n-1}} + \sum_{i=1}^{n-2} 1_{\{ \nu_s = t_i \}} X_{t_i} = 1_{\{ \nu_s > t_{n-3} \}} X_{t_{n-2}} + \sum_{i=1}^{n-3} 1_{\{ \nu_s = t_i \}} X_{t_i}, \quad \text{a.s.}
\]

Inductively, it follows that \( \mathcal{E} [ X_{\nu_s} | \mathcal{F}_{t_i} ] \leq X_{t_i}, \) a.s. Applying (A2) once again, we obtain
\[
\mathcal{E} [ X_{\nu_s} | \mathcal{F}_s ] = \mathcal{E} \left[ \mathcal{E} [ X_{\nu_s} | \mathcal{F}_{t_i} ] | \mathcal{F}_s \right] \leq \mathcal{E} [ X_{t_i} | \mathcal{F}_s ] \leq X_s, \quad \text{a.s.},
\]
which together with (S.2) implies that
\[
\mathcal{E} [ X_{\nu} | \mathcal{F}_s ] \leq 1_{\{ \nu > s \}} X_s + 1_{\{ \nu \leq s \}} X_{\nu \wedge s} = X_{\nu \wedge s}, \quad \text{a.s.}, \quad \text{proving (S.1)}.
\]

Let \( \sigma \in \mathcal{S}_{0,T}^F \) taking values in a finite set \( \{ s_1 < \cdots < s_m \} \), then
\[
\mathcal{E} [ X_{\nu} | \mathcal{F}_\sigma ] = \sum_{j=1}^{m} 1_{\{ \sigma = s_j \}} \mathcal{E} [ X_{\nu} | \mathcal{F}_{s_j} ] \leq \sum_{j=1}^{m} 1_{\{ \sigma = s_j \}} X_{\nu \wedge s_j} = X_{\nu \wedge \sigma}, \quad \text{a.s.} \quad \square
\]

**Proof of Proposition 2.3**: Given \( \xi \in \text{Dom}(\mathcal{E}) \), we let \( \nu \in \mathcal{S}_{0,T}^F \) take values in a finite set \( \{ t_1 < \cdots < t_n \} \).

1) For any \( \eta \in \text{Dom}(\mathcal{E}) \) with \( \xi \leq \eta, \) a.s., (A1) implies that
\[
\mathcal{E} [ \xi | \mathcal{F}_\nu ] = \sum_{i=1}^{n} 1_{\{ \nu = t_i \}} \mathcal{E} [ \xi | \mathcal{F}_{t_i} ] \leq \sum_{i=1}^{n} 1_{\{ \nu = t_i \}} \mathcal{E} [ \eta | \mathcal{F}_{t_i} ] = \mathcal{E} [ \eta | \mathcal{F}_\nu ], \quad \text{a.s.}
\]

Moreover, if \( 0 \leq \xi \leq \eta, \) a.s. and \( \mathcal{E} [ \xi | \mathcal{F}_\sigma ] = \mathcal{E} [ \eta | \mathcal{F}_\sigma ], \) a.s. for some \( \sigma \in \mathcal{S}_{0,T}^F \), we can apply Corollary 2.1 to obtain
\[
\mathcal{E} [ \xi ] = \mathcal{E} [ \mathcal{E} [ \xi | \mathcal{F}_\sigma ] ] = \mathcal{E} [ \mathcal{E} [ \eta | \mathcal{F}_\sigma ] ] = \mathcal{E} [ \eta ].
\]

The second part of (A1) then implies that \( \xi = \eta, \) a.s., proving (1).

2) For any \( A \in \mathcal{F}_\nu \), it is clear that \( A \cap \{ \nu = t_i \} \in \mathcal{F}_{t_i} \) for each \( i \in \{ 1, \cdots, n \} \). Hence we can deduce from (A3) that
\[
\mathcal{E} [ 1_A \xi | \mathcal{F}_\nu ] = \sum_{i=1}^{n} 1_{\{ \nu = t_i \}} \mathcal{E} [ 1_A \xi | \mathcal{F}_{t_i} ] = \sum_{i=1}^{n} \mathcal{E} [ 1_{\{ \nu = t_i \} \cap A \xi | \mathcal{F}_{t_i} ] = \sum_{i=1}^{n} 1_{\{ \nu = t_i \} \cap A \mathcal{E} [ \xi | \mathcal{F}_{t_i} ]
\]
\[
= 1_A \sum_{i=1}^{n} 1_{\{ \nu = t_i \}} \mathcal{E} [ \xi | \mathcal{F}_{t_i} ] = 1_A \mathcal{E} [ \xi | \mathcal{F}_\nu ], \quad \text{a.s., \quad proving (2)}.
\]

3) For any \( \eta \in \text{Dom}_\nu(\mathcal{E}) \), since \( 1_{\{ \nu = t_i \}} \eta \in \text{Dom}_{t_i}(\mathcal{E}) \) for each \( i \in \{ 1, \cdots, n \} \), (A3) and (A4) imply that
\[
\mathcal{E} [ \xi + \eta | \mathcal{F}_\nu ] = \sum_{i=1}^{n} 1_{\{ \nu = t_i \}} \mathcal{E} [ \xi + \eta | \mathcal{F}_{t_i} ] = \sum_{i=1}^{n} \left( \mathcal{E} [ 1_{\{ \nu = t_i \}} \xi + 1_{\{ \nu = t_i \}} \eta | \mathcal{F}_{t_i} ] = \sum_{i=1}^{n} \left( \mathcal{E} [ 1_{\{ \nu = t_i \}} \xi | \mathcal{F}_{t_i} ] + 1_{\{ \nu = t_i \}} \eta \right) \right)
\]
\[
= \sum_{i=1}^{n} 1_{\{ \nu = t_i \}} \mathcal{E} [ \xi | \mathcal{F}_{t_i} ] + \eta = \mathcal{E} [ \xi | \mathcal{F}_\nu ] + \eta, \quad \text{a.s., \quad proving (3)}.
\]
The proof of (4) and (5) is similar to that of Proposition 2.2 (1) and (2) by applying the just obtained “Zero-one Law” and “Translation Invariance”.

**Proof of Theorem 2.1**  
(H1) is an easy consequence of the lower semi-continuity (2.2). In fact, for any $\xi \in Dom^+(\mathcal{E})$ and any $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{F}_T$ with $\lim_{n \to \infty} 1_{A_n} = 1$ a.s., $\{A_n\}_{n \in \mathbb{N}}$ is an increasing sequence converging to $\xi$. Then applying the lower semi-continuity with $\nu = 0$ and using (A1), we obtain $\mathcal{E}[\xi] \leq \lim_{n \to \infty} \mathcal{E}[1_{A_n} \xi] \leq \mathcal{E}[\xi]$; so (H1) follows.

On the other hand, to show that (H1) implies the lower semi-continuity, we first extend (H1) as follows: For each deterministic stopping time $\tau$, we extend (H1) as follows: For any $\xi \in Dom^+(\mathcal{E})$ and any $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{F}_T$ with $\lim_{n \to \infty} 1_{A_n} = 1$, a.s., it holds for any $t \in [0, T]$ that

$$\lim_{n \to \infty} \uparrow \mathcal{E}[1_{A_n} \xi | \mathcal{F}_t] = \mathcal{E}[\xi | \mathcal{F}_t], \quad a.s. \quad (8.3)$$

In fact, by (A1), it holds a.s. that $\{\mathcal{E}[1_{A_n} \xi | \mathcal{F}_t]\}_{n \in \mathbb{N}}$ is an increasing sequence bounded from above by $\mathcal{E}[\xi | \mathcal{F}_t]$. Hence, $\lim_{n \to \infty} \uparrow \mathcal{E}[1_{A_n} \xi | \mathcal{F}_t] \leq \mathcal{E}[\xi | \mathcal{F}_t]$, a.s. Assuming that $\lim_{n \to \infty} \uparrow \mathcal{E}[1_{A_n} \xi | \mathcal{F}_t] < \mathcal{E}[\xi | \mathcal{F}_t]$ with a positive probability, we can find an $\varepsilon > 0$ such that the set $A_\varepsilon = \{\lim_{n \to \infty} \uparrow \mathcal{E}[1_{A_n} \xi | \mathcal{F}_t] \leq \mathcal{E}[\xi | \mathcal{F}_t] - \varepsilon\} \in \mathcal{F}_t$ still has positive probability. Hence for any $n \in \mathbb{N}$, we have

$$1_{A_n} \mathcal{E}[1_{A_n} \xi | \mathcal{F}_t] \leq 1_{A_n} \lim_{n \to \infty} \uparrow \mathcal{E}[1_{A_n} \xi | \mathcal{F}_t] \leq 1_{A_n} (\mathcal{E}[\xi | \mathcal{F}_t] - \varepsilon), \quad a.s. \quad (8.4)$$

Then (A1)-(A4) imply that

$$\mathcal{E}[1_{A_n} \xi + \varepsilon] = \mathcal{E}[1_{A_n} \xi] + \varepsilon = \lim_{n \to \infty} \mathcal{E}[1_{A_n} \xi] + \varepsilon \leq \mathcal{E}[1_{A_n} \xi + \varepsilon 1_{A_n}] \leq \mathcal{E}[1_{A_n} \xi + \varepsilon],$$

thus $\mathcal{E}[1_{A_n} \xi + \varepsilon] = \mathcal{E}[1_{A_n} \xi + \varepsilon 1_{A_n}]$. Then the second part of (A1) implies that $1_{A_n} \xi + \varepsilon = 1_{A_n} \xi + \varepsilon 1_{A_n}$, a.s., which can hold only if $P(A_n) = 0$. This results in a contradiction. Thus $\lim_{n \to \infty} \uparrow \mathcal{E}[1_{A_n} \xi | \mathcal{F}_t] = \mathcal{E}[\xi | \mathcal{F}_t]$, a.s., proving (8.3).

Next, we show that (2.2) holds for each deterministic stopping time $\nu = t \in [0, T]$. For any $j, n \in \mathbb{N}$, we define $A_n^j = \cap_{k=n}^{\infty} \{|\xi - \xi_k| < 1/j\} \in \mathcal{F}_T$. (A1) and (A4) imply that for any $k \geq n$

$$\mathcal{E}[1_{A_n^j} \xi | \mathcal{F}_t] \leq \mathcal{E}[1_{\{|\xi - \xi_k| < 1/j\}} \mathcal{F}_t] \leq \mathcal{E}[\xi_k + 1/j | \mathcal{F}_t] = \mathcal{E}[\xi_k | \mathcal{F}_t] + 1/j, \quad a.s. \quad (8.5)$$

Hence, except on a null set $N_n^j$, the above inequality holds for any $k \geq n$. As $k \to \infty$, it holds on $\left(N_n^j\right)^c$ that

$$\mathcal{E}[1_{A_n^j} \xi | \mathcal{F}_t] \leq \lim_{k \to \infty} \mathcal{E}[\xi_k | \mathcal{F}_t] + 1/j.$$
which completes the proof. □

**Proof of Theorem 2.2** We first show an extension of (H2): For any $\xi, \eta \in \text{Dom}^+(E)$ and any $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{F}_T$ with $\lim_{n \to \infty} \downarrow 1_{A_n} = 0$, a.s., it holds a.s. that

$$\lim_{n \to \infty} \downarrow E[\xi + 1_{A_n}\eta|\mathcal{F}_k] = E[\xi|\mathcal{F}_k], \quad \text{a.s.} \quad (8.6)$$

In fact, by (A1), it holds a.s. that $\{E[\xi + 1_{A_n}\eta|\mathcal{F}_k]\}_{n \in \mathbb{N}}$ is a decreasing sequence bounded from below by $E[\xi|\mathcal{F}_k]$. Hence, $\lim_{n \to \infty} \downarrow E[\xi + 1_{A_n}\eta|\mathcal{F}_k] \geq E[\xi|\mathcal{F}_k]$, a.s. Assume that $\lim_{n \to \infty} \downarrow E[\xi + 1_{A_n}\eta|\mathcal{F}_k] > E[\xi|\mathcal{F}_k]$ with a positive probability, then we can find an $\varepsilon > 0$ such that the set $A'_\varepsilon = \{ \lim_{n \to \infty} \downarrow E[\xi + 1_{A_n}\eta|\mathcal{F}_k] \geq E[\xi|\mathcal{F}_k] + \varepsilon \} \in \mathcal{F}_k$ still has positive probability. For any $n \in \mathbb{N}$, (A4) implies that

$$1_{A'_\varepsilon} E[\xi + 1_{A_n}\eta|\mathcal{F}_k] \geq 1_{A'_\varepsilon} \lim_{n \to \infty} \downarrow E[\xi + 1_{A_n}\eta|\mathcal{F}_k] \geq 1_{A'_\varepsilon} E[\xi|\mathcal{F}_k] + \varepsilon = 1_{A'_\varepsilon} E[\xi|\mathcal{F}_k], \quad \text{a.s.}$$

Applying (A1)-(A3), we obtain

$$E[1_{A'_\varepsilon} \xi + 1_{A_n} 1_{A'_\varepsilon} \eta] = E[E[1_{A'_\varepsilon} \xi + 1_{A_n} 1_{A'_\varepsilon} \eta|\mathcal{F}_k]] = E[1_{A'_\varepsilon} E[\xi + 1_{A_n}\eta|\mathcal{F}_k]] \geq E[1_{A'_\varepsilon} E[\xi + \varepsilon|\mathcal{F}_k]]$$

Thus $E[1_{A'_\varepsilon} \xi] = E[1_{A'_\varepsilon} (\xi + \varepsilon)]$. Then the second part of (A1) implies that $P(A'_\varepsilon) = 0$, which yields a contradiction.

Therefore, $\lim_{n \to \infty} \downarrow E[\xi + 1_{A_n}\eta|\mathcal{F}_k] = E[\xi|\mathcal{F}_k]$, a.s., proving (8.6).

Since the sequence $\{\xi_n\}_{n \in \mathbb{N}}$ is bounded above by $\eta$, it holds a.s. that $\xi = \lim_{n \to \infty} \xi_n \leq \eta$, thus (D3) implies that $\xi \in \text{Dom}(E)$. Then Fatou’s Lemma (Theorem 2.1) implies that for any $\nu \in \mathcal{S}_{0,T}^I$

$$E[\xi|\mathcal{F}_\nu] \leq \lim_{n \to \infty} E[\xi_n|\mathcal{F}_\nu], \quad \text{a.s.} \quad (8.7)$$

On the other hand, we first fix $t \in [0, T]$. For any $j, n \in \mathbb{N}$, define $A^j_n \triangleq \cap_{k=n}^\infty \{ \xi - k < 1/j \} \in \mathcal{F}_T$. Then one can deduce that for any $k \geq n$

$$E[\xi_k|\mathcal{F}_t] \leq E[1_{A^j_n}(\xi + 1/j) + 1_{(A^j_n)^c}\eta|\mathcal{F}_t] \leq E[\xi + 1/j + 1_{(A^j_n)^c}(\eta - \xi)|\mathcal{F}_t], \quad \text{a.s.}$$

Hence, except on a null set $N^j_n$, the above inequality holds for any $k \geq n$. As $k \to \infty$, it holds on $N^c_n$ that

$$\lim_{k \to \infty} E[\xi_k|\mathcal{F}_t] \leq E[\xi + 1/j + 1_{(A^j_n)^c}(\eta - \xi)|\mathcal{F}_t].$$

Since $\xi \in \mathcal{L}^0(\mathcal{F}_T)$ and $\xi_n \to \xi$, a.s. as $n \to \infty$, it is clear that $\lim_{n \to \infty} \uparrow 1_{A^j_n} = 1$, a.s. Then (8.6) and (A4) imply that except on a null set $N^j_0$, we have

$$\lim_{n \to \infty} \downarrow E[\xi + 1/j + 1_{(A^j_n)^c}(\eta - \xi)|\mathcal{F}_t] = E[\xi + 1/j|\mathcal{F}_t] = E[\xi|\mathcal{F}_t] + 1/j.$$
which together with (8.7) proves the theorem.

□

**Proof of Theorem 2.3** Let $F = \{ t_1 < t_2 < \cdots < t_d \}$ be any finite subset of $D_T$. For $j = 1, \cdots, d$, we define $A_j = \{ \nu_j < T \} \in \mathcal{F}_{\nu_j}$, clearly, $A_j \supset A_{j+1}$. Let $d' = \left\lfloor \frac{d}{2} \right\rfloor$, one can deduce that $U_F(a, b; X) = \sum_{j=1}^{d'} 1_{A_j}$ and that
\[
1_{\cup_{j=1}^{d'} (A_{j+1} \setminus A_j)}(X_T - a) \geq 1_{\cup_{j=1}^{d'} (A_{j+1} \setminus A_j)}(a) \geq 1_{\{X_T < a\}}(X_T - a) = -(a - X_T)^+.
\]
Since $X_T \in \text{Dom}(\mathcal{E})$ and $L^\infty(\mathcal{F}_T) \subset \text{Dom}(\mathcal{E})$ (by Lemma 2.1), we can deduce from (D2) that
\[
(b - a)U_F(a, b; X) - (a - X_T)^+ = \sum_{j=1}^{d'} 1_{A_j}(b - a) + 1_{\{X_T < a\}}(X_T - a) \in \text{Dom}(\mathcal{E}).
\]
Then Proposition 2.4 (1)-(3) and Proposition 2.3 imply that
\[
\mathcal{E}[\mathcal{E}[\mathcal{E}[(b - a)U_F(a, b; X) - (a - X_T)^+ | \mathcal{F}_{\nu_{2d'}}] - a] \leq (b - a) \sum_{j=1}^{d'} 1_{A_j} + 1_{\cup_{j=1}^{d'} (A_{j+1} \setminus A_j)}(X_{\nu_{2d'}} - a), \ a.s.
\]
Applying $\mathcal{E}[-|F_{\nu_{2d'}}]$ to the above inequality, using Proposition 2.4 (1)-(3) and Proposition 2.3 once again, we obtain
\[
\mathcal{E}[(b - a)U_F(a, b; X) - (a - X_T)^+ | F_{\nu_{2d'}}] \leq (b - a) \sum_{j=1}^{d'} 1_{A_j} + 1_{\cup_{j=1}^{d'} (A_{j+1} \setminus A_j)}(X_{\nu_{2d'}} - a), \ a.s.,
\]
where we used the fact that $X_{\nu_{2d'}} > b$ on $A_{2d'}$ in the first inequality and the fact that $X_{\nu_{2d'-1}} < a$ on $A_{2d'-1}$ in the last inequality. Similarly, applying $\mathcal{E}[-|F_{\nu_{2d'-2}}]$ to the above inequality yields that
\[
\mathcal{E}[(b - a)U_F(a, b; X) - (a - X_T)^+ | F_{\nu_{2d'-2}}] \leq (b - a) \sum_{j=1}^{d'-1} 1_{A_j} + 1_{\cup_{j=1}^{d'-1} (A_{j+1} \setminus A_j)}(X_{\nu_{2d'-2}} - a), \ a.s.,
\]
Iteratively applying $\mathcal{E}[-|F_{\nu_{2d'-3}}]$, $\mathcal{E}[-|F_{\nu_{2d'-4}}]$ and so on, we eventually obtain that
\[
\mathcal{E}[(b - a)U_F(a, b; X) - (a - X_T)^+] \leq 0.
\]
We assume first that $X_T \geq c$, a.s. for some $c \in \mathbb{R}$. Since $(a - X_T)^+ \leq |a| + |c|$, it directly follows from (A4) that
\[
0 \geq \mathcal{E}[(b - a)U_F(a, b; X) - (a - X_T)^+] \geq \mathcal{E}[(b - a)U_F(a, b; X)] - (|a| + |c|).
\]
Let \( \{ F_n \}_{n \in \mathbb{N}} \) be an increasing sequence of finite subsets of \( \mathcal{D}_T \) with \( \cup_{n \in \mathbb{N}} F_n = \mathcal{D}_T \), thus \( \lim_{n \to \infty} U_{F_n}(a, b; X) = U_{\mathcal{D}_T}(a, b; X) \). Fix \( M \in \mathbb{N} \), we see that

\[
\lim_{n \to \infty} \mathbb{1}_{\{ U_{F_n}(a, b; X) > M \}} = \mathbb{1}_{\{ U_{\mathcal{D}_T}(a, b; X) > M \}}. \tag{8.10}
\]

For any \( n \in \mathbb{N} \), we know from (8.9) that \( \mathcal{E}[(b - a)M \mathbb{1}_{\{ U_{F_n}(a, b; X) > M \}}] \leq \mathcal{E}[(b - a)U_{F_n}(a, b; X)] \leq |a| + |c| \), thus Fatou’s Lemma (Theorem 2.1) implies that

\[
\mathcal{E}[(b - a)M \mathbb{1}_{\{ U_{\mathcal{D}_T}(a, b; X) = \infty \}}] \leq \mathcal{E}[(b - a)M \mathbb{1}_{\{ U_{\mathcal{D}_T}(a, b; X) > M \}}] \leq \lim_{n \to \infty} \mathcal{E}[(b - a)M \mathbb{1}_{\{ U_{F_n}(a, b; X) > M \}}] \leq |a| + |c|. \tag{8.11}
\]

On the other hand, if \( \mathcal{E}[\cdot] \) is concave, then we can deduce from (8.8) that

\[
0 \geq \mathcal{E}[(b - a)U_F(a, b; X) - (a - X_T)^+] \geq \frac{1}{2} \mathcal{E}[2(b - a)U_F(a, b; X)] + \frac{1}{2} \mathcal{E}[-2(a - X_T)^+].
\]

Mimicking the arguments in (8.10) and (8.11), we obtain that

\[
\mathcal{E}[(b - a)2M \mathbb{1}_{\{ U_{\mathcal{D}_T}(a, b; X) = \infty \}}] \leq -\mathcal{E}[-2(a - X_T)^+],
\]

where \( -2(a - X_T)^+ = \mathbb{1}_{\{ X_T < a \}} 2(X_T - a) \in \text{Dom}(\mathcal{E}) \) thanks to (D2). Also note that (A1) and Proposition 2.4 (5) imply that \( \mathcal{E}[-2(a - X_T)^+] \leq \mathcal{E}[0] = 0 \).

Using (H0) in both cases above yields that \( P(U_{\mathcal{D}_T}(a, b; X) = \infty) = 0 \), i.e., \( U_{\mathcal{D}_T}(a, b; X) < \infty \), a.s. Then a classical argument (see e.g. Karatzas and Shreve [1991, Proposition 1.3.14]) shows that

\[
P(\text{both } \lim_{s \searrow t, s \in \mathcal{D}_T} X_s \text{ and } \lim_{s \searrow t, s \in \mathcal{D}_T} X_s \text{ exist for any } t \in [0, T]) = 1.
\]

This completes the proof. \( \square \)

**Proof of Proposition 2.5:** We can deduce from (2.4) that except on a null set \( N \)

\[
X^-_T = \lim_{n \to \infty} X_{q_n^{-}}(t) \leq \text{esssup}_{s \in \mathcal{D}_T} X_s \text{ and } X^+_T = \lim_{n \to \infty} X_{q_n^+}(t) \leq \text{esssup}_{s \in \mathcal{D}_T} X_s \text{ for any } t \in [0, T], \tag{8.12}
\]

thus \( X^-_\nu = \lim_{n \to \infty} X_{q_n^{-} \nu}(t) \leq \text{esssup}_{s \in \mathcal{D}_T} X_s \) and \( X^+_\nu = \lim_{n \to \infty} X_{q_n^+ \nu}(t) \leq \text{esssup}_{s \in \mathcal{D}_T} X_s \text{ for any } \nu \in \mathcal{S}_{0, T}. \tag{8.13} \)

**Proof of (1):** Case I. For any \( \nu \in \mathcal{S}_{0, T} \), if esssup \( X_s \in \text{Dom}^+(\mathcal{E}) \), (D3) and (8.13) directly imply that both \( X^-_{\nu} \) and \( X^+_{\nu} \) belong to \( \text{Dom}(\mathcal{E}) \).

Case II. Assume that \( \mathcal{E} \) satisfies (2.5). For any \( n \in \mathbb{N} \), since \( X \) is an \( \mathcal{E} \)-supermartingale and since \( q_n^{-}(\nu), q_n^+ \nu(\nu) \in \mathcal{S}^{\mathcal{E}}_{0, T} \), Corollary 2.1 and Proposition 2.3 imply that

\[
\mathcal{E}[X_{q_n^{-} \nu}(\nu)] = \mathcal{E}[\mathcal{E}[X_{q_n^{-} \nu}(\nu)|\mathcal{F}_{q_{n+1}^{-}}(\nu)] \leq \mathcal{E}[X_{q_n^+ \nu}(\nu)] \leq X_0 \text{ and }
\]

\[
\mathcal{E}[X_{q_n^+ \nu}(\nu)] = \mathcal{E}[\mathcal{E}[X_{q_n^+ \nu}(\nu)|\mathcal{F}_{q_{n+1}^+}(\nu)] \leq \mathcal{E}[X_{q_n^{-} \nu}(\nu)] \leq X_0.
\]

Hence, \( \{ \mathcal{E}[X_{q_n^{-} \nu}(\nu)] \}_{n \in \mathbb{N}} \) is an increasing non-negative sequence and \( \{ \mathcal{E}[X_{q_n^+ \nu}(\nu)] \}_{n \in \mathbb{N}} \) is a decreasing non-negative sequence, both of which are bounded from above by \( X_0 \in [0, \infty) \). (2.5) and (8.13) then imply that both \( X^-_{\nu} \) and \( X^+_{\nu} \) belong to \( \text{Dom}(\mathcal{E}) \), proving statement (1).

**Proof of (2):** Now suppose that \( X^+_T \in \text{Dom}^+(\mathcal{E}) \) for any \( t \in [0, T] \). First, we shall show that for \( t \in [0, T] \) and \( A \in \mathcal{F}_t \)

\[
\mathcal{E}[\mathbb{1}_A X^+_T] = \lim_{n \to \infty} \mathcal{E}[\mathbb{1}_A X_{q_n^+}(t)]. \tag{8.14}
\]
Since the distribution function $x \mapsto P\{X_t^+ \leq x\}$ jumps up at most on a countable subset $S$ of $[0, \infty)$, we can find a sequence $\{K_j\}_{j=1}^{\infty} \subset [0, \infty) \setminus S$ increasing to $\infty$. Fix $m, j \in \mathbb{N}$, (A1)-(A3) imply that for any $n \geq m$

\[
\mathcal{E}\left[1_A 1\{X_{q_n(t)}^- < K_j\} (X_{q_n(t)}^+ \wedge K_j)\right] = \mathcal{E}\left[1_A 1\{X_{q_n(t)}^- < K_j\} X_{q_n(t)}^+\right] \geq \mathcal{E}\left[1_A 1\{X_{q_n(t)}^- < K_j\} X_{q_n(t)}^+\right] = \mathcal{E}\left[1_A 1\{X_{q_n(t)}^- < K_j\} X_{q_n(t)}^+\right].
\]

Since $K_j \notin S$, $P\{X_t^+ = K_j\} = 0$, one can easily deduce from (8.12) that $\lim_{n \to \infty} 1\{X_{q_n(t)}^- < K_j\} = 1\{X_t^+ < K_j\}$, a.s. (In fact, for almost every $\omega \in \{X_t^+ < K_j\}$ (resp. $\{X_t^+ > K_j\}$), there exists an $N(\omega) \in \mathbb{N}$ such that $X_{q_n(t)}^- < (\text{resp.} >) K_j$ for any $n \geq N(\omega)$, which means $\lim_{n \to \infty} 1\{X_{q_n(t)}^- < K_j\}(\omega) = 1$ (resp. $0 = 1\{X_t^+ < K_j\}(\omega)$). Applying the Dominated Convergence Theorem (Theorem 2.2) twice, we obtain

\[
\mathcal{E}\left[1_A 1\{X_t^+ < K_j\} X_t^+\right] = \mathcal{E}\left[1_A 1\{X_t^+ < K_j\} (X_t^+ \wedge K_j)\right] = \lim_{n \to \infty} \mathcal{E}\left[1_A 1\{X_{q_n(t)}^- < K_j\} (X_{q_n(t)}^+ \wedge K_j)\right] \geq \lim_{n \to \infty} \mathcal{E}\left[1_A 1\{X_{q_n(t)}^- < K_j\} X_{q_n(t)}^+\right] = \mathcal{E}\left[1_A 1\{X_t^+ < K_j\} X_t^+\right].
\]

Since $\lim_{j \to \infty} 1\{X_t^+ < K_j\} = 1$, a.s., the Dominated Convergence Theorem again implies that

\[
\mathcal{E}\left[1_A X_t^+\right] = \lim_{j \to \infty} \mathcal{E}\left[1_A 1\{X_t^+ < K_j\} X_t^+\right] \geq \lim_{n \to \infty} \mathcal{E}\left[1_A 1\{X_{q_n(t)}^- < K_j\} X_{q_n(t)}^+\right] = \mathcal{E}\left[1_A X_{q_n(t)}^+\right],
\]

which leads to that $\mathcal{E}\left[1_A X_t^+\right] \geq \lim_{n \to \infty} \mathcal{E}\left[1_A X_{q_n(t)}^+\right]$. Fatou’s Lemma (Theorem 2.1) gives the reverse inequality, thus proving (8.14). Since $X$ is an $\mathcal{E}$-supermartingale, using (8.14), (A2) and (A3), we obtain

\[
\mathcal{E}\left[1_A X_t^+\right] = \lim_{n \to \infty} \mathcal{E}\left[1_A X_{q_n(t)}^+\right] = \lim_{n \to \infty} \mathcal{E}\left[\mathcal{E}\left[1_A X_{q_n(t)}^+|\mathcal{F}_t\right]\right] = \lim_{n \to \infty} \mathcal{E}\left[1_A \mathcal{E}\left[X_{q_n(t)}^+|\mathcal{F}_t\right]\right] \leq \mathcal{E}\left[1_A X_t\right]
\]

for any $A \in \mathcal{F}_t$, which further implies that $X_t^+ \leq X_t$, a.s. thanks to Proposition 2.2 (3).

Next, we show that $X^+$ is an $\mathcal{E}$-supermartingale: For any $0 \leq s < t \leq T$, it is clear that $q_n^+(s) \leq q_n^+(t)$ for any $n \in \mathbb{N}$. For any $A \in \mathcal{F}_s$, (A3) and Corollary 2.1 imply that for any $n \in \mathbb{N}$

\[
\mathcal{E}\left[1_A X_{q_n^+(s)}\right] \geq \mathcal{E}\left[1_A \mathcal{E}\left[X_{q_n^+(t)}|\mathcal{F}_{q_n^+(s)}\right]\right] = \mathcal{E}\left[1_A \mathcal{E}\left[X_{q_n^+(t)}|\mathcal{F}_{q_n^+(s)}\right]\right] = \mathcal{E}\left[1_A X_{q_n^+(t)}\right].
\]

As $n \to \infty$, (8.14), (A2) and (A3) imply that

\[
\mathcal{E}\left[1_A X_{s}^+\right] = \lim_{n \to \infty} \mathcal{E}\left[1_A X_{q_n^+(s)}\right] \geq \lim_{n \to \infty} \mathcal{E}\left[1_A X_{q_n^+(t)}\right] = \mathcal{E}\left[1_A X_t^+\right] = \mathcal{E}\left[\mathcal{E}[1_A X_t^+|\mathcal{F}_s]\right] = \mathcal{E}\left[1_A \mathcal{E}[X_t^+|\mathcal{F}_s]\right].
\]

Then Proposition 2.2 (3) implies that $X_t^+ \geq \mathcal{E}[X_t^+|\mathcal{F}_s]$, a.s., thus $\{X_t^+\}_{t \in [0,T]}$ is an RCLL $\mathcal{E}$-supermartingale.

Proof of (3): If $t \mapsto \mathcal{E}[X_t]$ is right continuous, for any $t \in [0, T]$, (8.14) implies that

\[
\mathcal{E}[X_t^+] = \lim_{n \to \infty} \mathcal{E}[X_{q_n^+(t)}] = \mathcal{E}[X_t].
\]

Then the second part of (A1) imply that $X_t^+ = X_t$, a.s., which means that $X^+$ is an RCLL modification of $X$. On the other hand, if $\bar{X}$ is a right-continuous modification of $X$, we see from (2.3) that except on a null set $\bar{N}$

\[
X_t^+ = \lim_{n \to \infty} X_{q_n^+(t)} = \lim_{n \to \infty} \bar{X}_{q_n^+(t)} = \bar{X}_t = X_t, \quad \text{and} \quad \bar{X}_t = X_t^+ \text{ for any } n \in \mathbb{N}.
\]

Putting them together, it holds on $\bar{N}$ that

\[
X_t^+ = \lim_{n \to \infty} X_{q_n^+(t)} = \lim_{n \to \infty} \bar{X}_{q_n^+(t)} = \bar{X}_t = X_t \tag{8.15}
\]

Since $X$ is an $\mathcal{E}$-supermartingale, (A2) implies that for any $0 \leq t_1 < t_2 \leq T$, $\mathcal{E}[X_{t_2}] \geq \mathcal{E}[\mathcal{E}[X_{t_2}|\mathcal{F}_{t_1}]] = \mathcal{E}[X_{t_1}]$, which shows that the function $t \mapsto \mathcal{E}[X_t]$ is decreasing. Then (8.14) and (8.15) imply that for any $t \in [0, T]$

\[
\mathcal{E}[X_t] \geq \lim_{s \downarrow t} \mathcal{E}[X_s] = \lim_{n \to \infty} \mathcal{E}[X_{q_n^+(t)}] = \mathcal{E}[X^+_t] = \mathcal{E}[X_t],
\]
thus \( \lim_{s \downarrow t} \mathcal{E}[X_s] = \mathcal{E}[X_t] \), i.e., the function \( t \mapsto \mathcal{E}[X_t] \) is right continuous.

\[
\text{□}
\]

**Proof of Corollary 2.2**: Since \( \text{essinf} X_t \geq c \), a.s., we can deduce from (A4) that \( X^c \triangleq \{ X_t - c \}_{t \in [0,T]} \) is a non-negative \( \mathcal{E} \)-supermartingale. If \( \text{esssup} X_t \in \text{Dom}^{\#}(\mathcal{E}) \) (D2) implies that \( \text{esssup} X_t \in \text{Dom}^{\#}(\mathcal{E}) \) is equivalent to \( \text{esssup} X_t^c \in \text{Dom}^{\#}(\mathcal{E}) \) or if (2.3) holds, Proposition 2.5 (1) shows that for any \( \nu \in \mathcal{S}_{0,T} \), both \( (X^c)^- \nu \) and \( (X^c)^+ \nu \) belong to \( \text{Dom}^{\#}(\mathcal{E}) \). Because

\[
(X^c)^-_t = X^c_t - c \quad \text{and} \quad (X^c)^+_t = X^c_t + c, \quad \forall t \in [0,T],
\]

(D2) and the non-negativity of \((X^c)^-\), \((X^c)^+\) imply that

\[
X^c_t = (X^c)^-_t + c \in \text{Dom}^{\#}(\mathcal{E}) \quad \text{and} \quad X^c_t = (X^c)^+_t + c \in \text{Dom}^{\#}(\mathcal{E}).
\]

On the other hand, if \( X^c_t \in \text{Dom}^{\#}(\mathcal{E}) \) for any \( t \in [0,T] \), (D2) implies that the non-negative random variable \((X^c)^+_t = X^c_t - c \) belongs to \( \text{Dom}^{\#}(\mathcal{E}) \). Hence, Proposition 2.5 (2) show that \((X^c)^+_t\) is an RCLL \( \mathcal{E} \)-supermartingale such that for any \( t \in [0,T] \), \((X^c)^+ \leq X^c_t \), a.s. Then (8.10), (8.8) and (A4) imply that \( X^c \) is an RCLL \( \mathcal{E} \)-supermartingale such that for any \( t \in [0,T] \), \( X^c_t \leq X^c_t \), a.s. Moreover, if \( t \mapsto \mathcal{E}[X_t] \) is a right-continuous function (which is equivalent to the right continuity of \( t \mapsto \mathcal{E}[X^c_t] \)), then we know from Proposition 2.5 (2) that for any \( t \in [0,T] \), \( (X^c)^+_t = X^c_t \), a.s., or equivalently, \( X^c_t = X^c_t \), a.s. Conversely, if \( X \) has a right-continuous modification, so does \( X^c \), then Proposition 2.5 (2) once again shows that \( t \mapsto \mathcal{E}[X^c_t] \) is right continuous, which is equivalent to the right continuity of \( t \mapsto \mathcal{E}[X^c_t] \). This completes the proof.

\[
\text{□}
\]

**Proof of Theorem 2.3**: We shall only consider the \( \mathcal{E} \)-supermartingale case, as the other cases can be deduced easily by similar arguments. Fix \( t \in [0,T] \), we let \( \{ \nu^c_n \}_{n \in \mathbb{N}} \) be a decreasing sequence in \( \mathcal{S}_{t,T}^c \) such that \( \lim_{n \to \infty} \nu^c_n = \nu \vee t \). Since \( \text{essinf} X_t \geq c \), a.s., it holds a.s. that \( X_t \geq c \) for each \( t \in \mathcal{D}_T \). The right-continuity of the process \( X \) then implies that except on a null set \( N \), \( X_t \geq c \) for any \( t \in [0,T] \). Thus we see from (A4) that \( X^c \triangleq \{ X_t - c \}_{t \in [0,T]} \) is a non-negative \( \mathcal{E} \)-supermartingale. For any \( n \in \mathbb{N} \) and \( A \in \mathcal{F}_t \cap \mathcal{F}_{\nu^c_n} \), (A2), (A3) and Proposition 2.3 imply that

\[
\mathcal{E}[1_A X^c_n] = \mathcal{E}[\mathcal{E}[1_A X^c_n | \mathcal{F}_t]] = \mathcal{E}[1_A \mathcal{E}[X^c_n | \mathcal{F}_t]] \leq \mathcal{E}[1_A X^c_t].
\]

We also have that \( \mathcal{E}[1_A X^c_{\nu^c_n}] = \lim_{n \to \infty} \mathcal{E}[1_A X^c_n] \). The proof is similar to that of (8.11). (We only need to replace \( X^c_t \) by \( X^c_{\nu^c_n} \) and \( X^c_{\nu^c_n(t)} \) by \( X^c_{\nu^c_n} \) in the proof of (8.11) ). As \( n \to \infty \) in (8.17), (A2) and (A3) imply that

\[
\mathcal{E}[1_A X^c_t] \geq \lim_{n \to \infty} \mathcal{E}[1_A X^c_{\nu^c_n}] = \mathcal{E}[1_A X^c_{\nu^c_n}] = \mathcal{E}[\mathcal{E}[1_A X^c_{\nu^c_n} | \mathcal{F}_t]] = \mathcal{E}[1_A \mathcal{E}[X^c_{\nu^c_n} | \mathcal{F}_t]].
\]

Applying Proposition 2.2 (3), we obtain that \( \mathcal{E}[X^c_{\nu^c_n} | \mathcal{F}_t] \leq X^c_t \), a.s. Then (A4) and (2.8) imply that

\[
\tilde{\mathcal{E}}[X^c_{\nu^c_n} | \mathcal{F}_t] = \mathcal{E}[X^c_{\nu^c_n} | \mathcal{F}_t] = \mathcal{E}[X^c_{\nu^c_n} + c | \mathcal{F}_t] = \mathcal{E}[X^c_{\nu^c_n} | \mathcal{F}_t] + c \leq X^c_t + c = X^c_t, \quad a.s.
\]

Since \( \{ \nu \leq t \} \in \mathcal{F}_t \), we can deduce from (A3) and (A4) that

\[
\tilde{\mathcal{E}}[X^c_{\nu^c_n} | \mathcal{F}_t] = \tilde{\mathcal{E}}[1_{\{ \nu \leq t \}} X^c_{\nu^c_n} + 1_{\{ \nu > t \}} X^c_{\nu^c_n} | \mathcal{F}_t] = 1_{\{ \nu > t \}} \tilde{\mathcal{E}}[X^c_{\nu^c_n} | \mathcal{F}_t] + 1_{\{ \nu \leq t \}} X^c_{\nu^c_n} \leq 1_{\{ \nu > t \}} + 1_{\{ \nu \leq t \}} X^c_{\nu^c_n} = X^c_{\nu^c_n} + c = X^c_t, \quad a.s.
\]

Hence, we can find a null set \( \tilde{N} \) such that except on \( \tilde{N}^c \)

\[
\tilde{\mathcal{E}}[X^c_{\nu^c_n} | \mathcal{F}_t] \leq X^c_{\nu^c_n}, \quad \forall t \in [0,T], \quad \text{thus} \quad \tilde{\mathcal{E}}[X^c_{\nu^c_n} | \mathcal{F}_t] \leq X^c_{\nu^c_n}, \quad \forall \sigma \in \mathcal{S}_{0,T}.
\]

As a result, on \( \tilde{N}^c \)

\[
\tilde{\mathcal{E}}[X^c_{\nu^c_n} | \mathcal{F}_t] \leq X^c_{\nu^c_n}, \quad \forall t \in [0,T], \quad \text{thus} \quad \tilde{\mathcal{E}}[X^c_{\nu^c_n} | \mathcal{F}_t] \leq X^c_{\nu^c_n}, \quad \forall \sigma \in \mathcal{S}_{0,T}.
\]

\[
\text{□}
\]
8.2 Proofs of Section 3

Proof of Proposition 2.7: 1) If $\xi \leq \eta$, a.s., by (A1), it holds except on a null set $N$ that

$$
\tilde{E}[\xi|F_t] \leq \tilde{E}[\eta|F_t], \quad \forall t \in [0,T],
$$

and then the paths of $\tilde{E}[\xi|F]$ and $\tilde{E}[\eta|F]$ are all RCLL,

which implies that on $N^c$

$$
\tilde{E}[\xi|F_t] \leq \tilde{E}[\eta|F_t], \quad \forall t \in [0,T], \quad \text{thus} \quad \tilde{E}[\xi|F_T] \leq \tilde{E}[\eta|F_T].
$$

Moreover, if $\tilde{E}[\xi|F_\sigma] = \tilde{E}[\eta|F_\sigma]$, a.s. for some $\sigma \in S_{0,T}$, we can apply (2.8) and Corollary 2.3 to get

$$
\tilde{E}[\xi] = \tilde{E}[\xi] = \tilde{E}[\tilde{E}[\xi|F_\sigma]] = \tilde{E}[\tilde{E}[\eta|F_\sigma]] = \tilde{E}[\eta] = \tilde{E}[\eta].
$$

Then (A4) implies that $\tilde{E}[\xi - c(\xi)] = \tilde{E}[\xi - c(\xi)] = \tilde{E}[\eta - c(\xi)] = \tilde{E}[\eta - c(\xi)]$, clearly, $0 \leq \xi - c(\xi) \leq \eta - c(\xi)$, a.s. The second part of (A1) then implies that $\xi - c(\xi) = \eta - c(\xi)$, a.s., i.e. $\xi = \eta$, a.s., proving (1).

2) For any $A \in F_\nu$ and $\eta \in Dom_{A}(E)$, we let $\nu_n, n \in \mathbb{N}$ be a decreasing sequence in $S_{0,T}$ such that $\lim_{n \to \infty} \nu_n = \nu$, a.s. For any $n \in \mathbb{N}$, since $A \in F_{\nu_n}$ and $\eta \in Dom_{A}(E)$, Proposition 2.4 (2) and (3) imply that

$$
\tilde{E}[1_A \xi|F_{\nu_n}] = 1_A \tilde{E}[\xi|F_{\nu_n}], \quad \text{and} \quad \tilde{E}[\xi + \eta|F_{\nu_n}] = \tilde{E}[\xi|F_{\nu_n}] + \eta, \quad a.s.
$$

Then we can find a null set $N'$ such that except on $N'$

(8.18) holds for any $n \in \mathbb{N}$ and the paths of $\tilde{E}[1_A \xi|F], \tilde{E}[\xi|F]$ and $\tilde{E}[\xi + \eta|F]$ are all RCLL.

As $n \to \infty$, it holds on $(N')^c$ that

$$
\tilde{E}[1_A \xi|F_{\nu}] = \lim_{n \to \infty} \tilde{E}[1_A \xi|F_{\nu_n}] = \lim_{n \to \infty} 1_A \tilde{E}[\xi|F_{\nu_n}] = 1_A \tilde{E}[\xi|F_{\nu}],
$$

and that

$$
\tilde{E}[\xi + \eta|F_{\nu}] = \lim_{n \to \infty} \tilde{E}[\xi + \eta|F_{\nu_n}] = \lim_{n \to \infty} \tilde{E}[\xi|F_{\nu_n}] + \eta = \tilde{E}[\xi|F_{\nu}] + \eta,
$$

proving (2) and (3). Proofs of (4) and (5) are similar to those of Proposition 2.2 (1) and (2). The proofs can be carried out by applying the just obtained “Zero-one Law” and “Translation Invariance”.

8.2 Proofs of Section 3

Proof of Lemma 3.1: 1) Let $E$ be a positively-convex $F$-expectation. For any $A \in F_T$ and $n \in \mathbb{N}$, (D1) and (D2) imply that $1_A, n1_A \in Dom(E)$. Then the positive-convexity of $E$ and Proposition 2.2 (2) show that

$$
E[1_A] = E\left[-1_n(1_A)\right] \leq \frac{1}{n} E[n1_A] + (1 - \frac{1}{n})E[0] = \frac{1}{n} E[n1_A] + (1 - \frac{1}{n}) \cdot 0 = \frac{1}{n} E[n1_A].
$$

Since $P(A) > 0$, one can deduce from the second part of (A1) that $E[1_A] > 0$. Letting $n \to \infty$ in (8.19) yields that

$$
\lim_{n \to \infty} E[n1_A] = \lim_{n \to \infty} nE[1_A] = \infty,
$$

thus $E$ satisfies (H0). Moreover, for any $\xi, \eta \in Dom^+(E)$, $\xi, \eta \in [0,1)$ and $t \in [0,T]$, we can deduce from (2.8), (A4) and the positive-convexity of $E$ that

$$
\tilde{E}[\lambda \xi + (1 - \lambda)\eta|F_t] = \lambda \tilde{E}[\xi + (1 - \lambda)\eta|F_t] = \lambda \tilde{E}[\xi - c(\xi)] + (1 - \lambda) \tilde{E}[\eta - c(\eta)] + \lambda \tilde{E}[\xi - c(\xi)] + (1 - \lambda) \tilde{E}[\eta - c(\eta)] = \lambda \tilde{E}[\xi|F_t] + (1 - \lambda) \tilde{E}[\eta|F_t], \quad a.s.,
$$

which shows that $\tilde{E}$ is convex in the sense of (3.1). On the other hand, if $E$ satisfies (3.1), since $Dom^+(E) \subset Dom^+(E)$, one can easily deduce from (2.8) that $E$ is positively-convex.

Proof of Proposition 3.1: We first check that $E_{\nu}^t$ satisfies (A1)-(A4). For this purpose, let $\xi, \eta \in \Lambda^+$ and $t \in [0,T]$. 


1) If $\xi \leq \eta$, a.s., applying Proposition 2.7 (1) to $\tilde{E}_j$ yields that $\tilde{E}_j[\xi|\mathcal{F}_{\nu\land t}] \leq \tilde{E}_j[\eta|\mathcal{F}_{\nu\land t}]$, a.s. Then (A1) of $\tilde{E}_i$ and (3.3) imply that

$$E_{i,j}^\nu[\xi|\mathcal{F}_t] = \tilde{E}_i[\tilde{E}_j[\xi|\mathcal{F}_{\nu\land t}]]_{\mathcal{F}_t} \leq \tilde{E}_i[\tilde{E}_j[\eta|\mathcal{F}_{\nu\land t}]]_{\mathcal{F}_t} = E_{i,j}^\nu[\eta|\mathcal{F}_t], \quad \text{a.s.}$$

Moreover, if $0 \leq \xi \leq \eta$ a.s. and $E_{i,j}^\nu[\xi] = E_{i,j}^\nu[\eta]$ (i.e. $\tilde{E}_i[\tilde{E}_j[\xi]] = \tilde{E}_i[\tilde{E}_j[\eta]]$) by (3.3), the second part of (A1) implies that $\tilde{E}_j[\xi|\mathcal{F}_{\nu\land t}] = \tilde{E}_j[\eta|\mathcal{F}_{\nu\land t}]$, a.s. Further applying the second part of Proposition 2.7 (1), we obtain $\xi = \eta$, a.s., proving (A1) for $E_{i,j}^\nu$. 

2) Next, let $0 \leq s \leq t \leq T$ and set $\Xi_t \triangleq E_{i,j}^\nu[\xi|\mathcal{F}_t]$. Applying Proposition 2.7 (2) to $\tilde{E}_i$ and $\tilde{E}_j$, we obtain

$$E_{i,j}^\nu[\Xi_t|\mathcal{F}_s] = 1_{\{\nu \geq s\}} E_{i,j}^\nu[\Xi_t|\mathcal{F}_s] + 1_{\{\nu > s\}} E_{i,j}^\nu[\Xi_t|\mathcal{F}_s] - E_{i,j}^\nu[\Xi_t|\mathcal{F}_s] = E_{i,j}^\nu[\Xi_t|\mathcal{F}_s] + E_{i,j}^\nu[	ilde{E}_j[1_{\{\nu > s\}} \Xi_t|\mathcal{F}_s]|\mathcal{F}_s], \quad \text{a.s.,}$$

where we used the fact that $\{\nu > s\} \in \mathcal{F}_{\nu\land s}$ thanks to Karatzas and Shreve [1991], Lemma 1.2.16. Then (A3) and (A2) imply that

$$E_{i,j}^\nu[1_{\{\nu \geq s\}} \Xi_t|\mathcal{F}_s] = E_{i,j}^\nu[1_{\{\nu \geq s\}} \tilde{E}_j[\xi|\mathcal{F}_s]|\mathcal{F}_s] = 1_{\{\nu \geq s\}} E_{i,j}^\nu[\Xi_t|\mathcal{F}_s], \quad \text{a.s.} \quad (8.20)$$

On the other hand, we can deduce from (3.2) that

$$1_{\{\nu > t\}} \Xi_t = 1_{\{s < \nu \leq t\}} \tilde{E}_j[\xi|\mathcal{F}_t] + 1_{\{\nu > t\}} \tilde{E}_i[\tilde{E}_j[\xi|\mathcal{F}_t]|\mathcal{F}_t] = 1_{\{s < \nu \leq t\}} \tilde{E}_j[\xi|\mathcal{F}_t] + 1_{\{\nu > t\}} \tilde{E}_i[\tilde{E}_j[\xi|\mathcal{F}_t]|\mathcal{F}_t], \quad \text{a.s.}$$

Since both $\{s < \nu \leq t\} \cap \{\nu > t\} = \emptyset$ and $\{\nu > t\} \in \mathcal{F}_{\nu\land t}$, Proposition 2.7 (3) and (2) as well as Corollary 2.3 imply that

$$E_{i,j}^\nu[\tilde{E}_j[1_{\{\nu > t\}} \Xi_t|\mathcal{F}_t]|\mathcal{F}_t] = E_{i,j}^\nu[\tilde{E}_j[1_{\{s < \nu \leq t\}} \tilde{E}_j[\xi|\mathcal{F}_t]|\mathcal{F}_t]|\mathcal{F}_t] = E_{i,j}^\nu[\tilde{E}_j[\tilde{E}_j[\xi|\mathcal{F}_t]|\mathcal{F}_t]|\mathcal{F}_t] = E_{i,j}^\nu[\tilde{E}_j[\xi|\mathcal{F}_t]|\mathcal{F}_t], \quad \text{a.s.}$$

Taking $\tilde{E}_i[.|\mathcal{F}_s]$ of both sides as well as using (A2) and (A3) of $\tilde{E}_i$, we obtain

$$E_{i,j}^\nu[\tilde{E}_j[1_{\{\nu > t\}} \Xi_t|\mathcal{F}_t]|\mathcal{F}_s] = E_{i,j}^\nu[\tilde{E}_i[\tilde{E}_j[\xi|\mathcal{F}_t]|\mathcal{F}_s]|\mathcal{F}_s] = E_{i,j}^\nu[\tilde{E}_i[\tilde{E}_j[\xi|\mathcal{F}_t]|\mathcal{F}_s]|\mathcal{F}_s], \quad \text{a.s.,}$$

which together with (8.20) yields that

$$E_{i,j}^\nu[\tilde{E}_j[\xi|\mathcal{F}_t]|\mathcal{F}_s] = 1_{\{\nu \geq s\}} E_{i,j}^\nu[\tilde{E}_i[\tilde{E}_j[\xi|\mathcal{F}_t]|\mathcal{F}_s]|\mathcal{F}_s] = E_{i,j}^\nu[\tilde{E}_i[\tilde{E}_j[\xi|\mathcal{F}_t]|\mathcal{F}_s]|\mathcal{F}_s], \quad \text{a.s.,} \quad \text{proving (A2) for } E_{i,j}^\nu.$$ 

3) For any $A \in \mathcal{F}_t$, using (3.3), (A3) of $\tilde{E}_i$ as well as applying Proposition 2.7 (2) to $\tilde{E}_j$, we obtain

$$E_{i,j}^\nu[1_A \xi|\mathcal{F}_t] = \tilde{E}_i[1_A \tilde{E}_j[\tilde{E}_j[\xi|\mathcal{F}_{\nu\land t}]|\mathcal{F}_t]]_{\mathcal{F}_t} = 1_A \tilde{E}_i[\tilde{E}_j[\tilde{E}_j[\xi|\mathcal{F}_{\nu\land t}]|\mathcal{F}_t]|\mathcal{F}_t], \quad \text{a.s.,} \quad \text{proving (A3) for } E_{i,j}^\nu.$$ 

Similarly, we can show that (A4) holds for $E_{i,j}^\nu$ as well. Therefore, $E_{i,j}^\nu$ is an $\mathcal{F}$-expectation with domain $\Lambda^\#. \quad \text{Since } \Lambda \in \bar{\mathcal{F}}_T, \text{ i.e. } \mathbb{R} \subset \Lambda, \text{ it follows easily that } \mathbb{R} \subset \Lambda^\#, \text{ which shows that } \Lambda^\# \in \bar{\mathcal{F}}_T.$

4) Now we show that $E_{i,j}^\nu$ satisfies (H1) and (H2): For any $\xi, \eta \in \Lambda^+$ and any $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{F}_T$ with $\lim_{n \to \infty} 1_{A_n} = 1$, a.s., the Dominated Convergence Theorem (Proposition 2.9) implies that $\lim_{n \to \infty} \tilde{E}_i[1_{A_n} \xi|\mathcal{F}_T] = \tilde{E}_i[\xi|\mathcal{F}_T], \text{ a.s.}$ Furthermore, using (3.3) and applying the Dominated Convergence Theorem to $\tilde{E}_i$ yield that

$$\lim_{n \to \infty} E_{i,j}^\nu[1_{A_n} \xi|\mathcal{F}_T] = \lim_{n \to \infty} \tilde{E}_i[\tilde{E}_j[1_{A_n} \xi|\mathcal{F}_T]|\mathcal{F}_T] = \tilde{E}_i[\tilde{E}_j[\xi|\mathcal{F}_T]|\mathcal{F}_T] = E_{i,j}^\nu[\xi], \quad \text{proving (H1) for } E_{i,j}^\nu.$$ 

With a similar argument, we can show that $E_{i,j}^\nu$ also satisfies (H2).

5) If both $\tilde{E}_i$ and $\tilde{E}_j$ are positively-convex, so are $\tilde{E}_i$ and $\tilde{E}_j$ thanks to (2.8). To see that $E_{i,j}^\nu$ is convex in the sense of (3.1), we fix $\xi, \eta \in \Lambda^\#, \lambda \in (0,1)$ and $t \in [0,T]$. For any $s \in [0,T]$, we have

$$\tilde{E}_j[\lambda \xi + (1-\lambda)\eta]|\mathcal{F}_s] \leq \lambda \tilde{E}_j[\xi|\mathcal{F}_s] + (1-\lambda)\tilde{E}_j[\eta|\mathcal{F}_s], \quad \text{a.s.}$$
Since $\tilde{E}_j[\lambda \xi + (1 - \lambda) \eta | \mathcal{F}_s], \tilde{E}_j[\xi | \mathcal{F}_s]$ and $\tilde{E}_j[\eta | \mathcal{F}_s]$ are all RCLL processes, it holds except on a null set $N$ that

$$\tilde{E}_j[\lambda \xi + (1 - \lambda) \eta | \mathcal{F}_s] \leq \lambda \tilde{E}_j[\xi | \mathcal{F}_s] + (1 - \lambda) \tilde{E}_j[\eta | \mathcal{F}_s], \quad \forall s \in [0, T],$$

thus

$$\tilde{E}_j[\lambda \xi + (1 - \lambda) \eta | \mathcal{F}_{s\vee t}] \leq \lambda \tilde{E}_j[\xi | \mathcal{F}_{s\vee t}] + (1 - \lambda) \tilde{E}_j[\eta | \mathcal{F}_{s\vee t}].$$

Then (3.3) implies that

$$\mathcal{E}_{i,j}^\nu[\lambda \xi + (1 - \lambda) \eta | \mathcal{F}_t] = \tilde{E}_i[\tilde{E}_j[\lambda \xi + (1 - \lambda) \eta | \mathcal{F}_{s\vee t}] | \mathcal{F}_t] \leq \tilde{E}_i[\lambda \tilde{E}_j[\xi | \mathcal{F}_{s\vee t}] + (1 - \lambda) \tilde{E}_j[\eta | \mathcal{F}_{s\vee t}] | \mathcal{F}_t],$$

$$\leq \lambda \tilde{E}_i[\tilde{E}_j[\xi | \mathcal{F}_{s\vee t}] | \mathcal{F}_t] + (1 - \lambda) \tilde{E}_i[\tilde{E}_j[\eta | \mathcal{F}_{s\vee t}] | \mathcal{F}_t] = \lambda \mathcal{E}_{i,j}^\nu[\xi | \mathcal{F}_t] + (1 - \lambda) \mathcal{E}_{i,j}^\nu[\eta | \mathcal{F}_t], \quad a.s. \quad \square$$

### 8.3 Proofs of Section 4

**Proof of Lemma 4.1** For any $i \in \mathcal{I}$, it is clear that $H^i_0 = 0$ and that (4.2) directly follows from (h1). For any $s, t \in \mathcal{D}_T$ with $s < t$, we can deduce from (h2) that

$$H^i_{s,t} = \int_s^t h^i_r dr \geq c \int_s^t ds \geq cT, \quad a.s., \quad (8.21)$$

which implies that $\text{essinf}_{s,t\in \mathcal{D}_T:s<t} H^i_{s,t} \geq cT, \quad a.s.$ Thus (S2) holds with $C_H = cT$.

If no member of $\mathcal{E}$ satisfies (2.3), then $\int_0^T |h^i_t| dt \in \text{Dom}(\mathcal{E})$ for some $j \in \mathcal{I}$ is assumed. For any $s, t \in \mathcal{D}_T$ with $s < t$, we can deduce from (8.21) and (h2) that

$$C_H \leq H^j_{s,t} \leq \int_s^t |h^j_r| dr \leq \int_0^T |h^j_r| dr, \quad a.s. \quad (8.22)$$

which implies that $C_H \leq \text{esssup}_{s,t\in \mathcal{D}_T:s<t} H^j_{s,t} \leq \int_0^T |h^j_r| dr$ a.s. Thus Lemma 3.2 shows that $\text{esssup}_{s,t\in \mathcal{D}_T:s<t} H^j_{s,t} \in \text{Dom}(\mathcal{E})$, i.e. (4.3). Moreover, we can derive (S3) directly from (h3). \quad \square

**Proof of Lemma 4.2** For any $i, j \in \mathcal{I}'$ and $\rho_1, \rho_2 \in \mathcal{U}$, we consider the event

$$A \triangleq \left\{ \tilde{E}_i[X(\rho_1) + H^i_{\rho,\rho_1} | \mathcal{F}_s] \leq \tilde{E}_j[X(\rho_2) + H^j_{\rho,\rho_2} | \mathcal{F}_s] \right\} \in \mathcal{F}_s,$$

and define stopping times $\rho \triangleq \rho_2 1_A + \rho_1 1_{A^c} \in \mathcal{U}$ and $\nu(A) \triangleq \nu 1_A + T 1_{A^c} \in \mathcal{S}_{\nu,T}$. Since $\mathcal{E}' = \{\mathcal{E}_i\}_{i \in \mathcal{I}}$ is a stable subclass of $\mathcal{E}$, Definition 3.2 assures the existence of $k = k(i, j, \nu(A)) \in \mathcal{I}'$ such that $\tilde{E}_k^\nu = \mathcal{E}_{i,j}^{\nu(A)}$. Applying Proposition 2.7 (5) to $\tilde{E}_j$ and Proposition 2.7 (3) & (2) to $\tilde{E}_i$, we can deduce from (8.22) that for any $\xi \in \text{Dom}(\mathcal{E})$

$$\tilde{E}_k[\xi | \mathcal{F}_s] = \tilde{E}_k^{\nu(A)}[\xi | \mathcal{F}_s] = \tilde{E}_i[\tilde{E}_j[\xi | \mathcal{F}_s] | \mathcal{F}_s] = \tilde{E}_i[1_A \tilde{E}_j[\xi | \mathcal{F}_s] + 1_{A^c} \tilde{E}_j[\xi | \mathcal{F}_s] | \mathcal{F}_s] = \tilde{E}_i[1_A \tilde{E}_j[\xi | \mathcal{F}_s] + 1_{A^c} \tilde{E}_j[\xi | \mathcal{F}_s] + 1_A (1 - \tilde{E}_j[\xi | \mathcal{F}_s]) | \mathcal{F}_s] = \tilde{E}_i[1_A \tilde{E}_j[\xi | \mathcal{F}_s] + 1_{A^c} \tilde{E}_j[\xi | \mathcal{F}_s], \quad a.s. \quad (8.22)$$

Moreover, (4.5) implies that

$$H^k_{\nu,\rho} = H^i_{\nu,\rho} \wedge H^j_{\nu,\rho} \wedge H^k_{\nu,\rho} = (1_A + 1_{A^c}) H^i_{\nu,\rho}, \quad a.s.$$

Then applying Proposition 2.7 (2) to $\tilde{E}_i$ and $\tilde{E}_j$, we see from (8.22) that

$$\tilde{E}_k[X(\rho) + H^k_{\nu,\rho} | \mathcal{F}_s] = 1_A \tilde{E}_i[X(\rho) + H^i_{\nu,\rho} | \mathcal{F}_s] + 1_{A^c} \tilde{E}_i[X(\rho) + H^i_{\nu,\rho} | \mathcal{F}_s] + 1_A \tilde{E}_j[X(\rho_1) + H^j_{\nu,\rho_1} | \mathcal{F}_s] + 1_{A^c} \tilde{E}_j[X(\rho_2) + H^j_{\nu,\rho_2} | \mathcal{F}_s]$$

Similarly, taking $\rho' \triangleq \rho_1 1_{A^c} + \rho_2 1_A$ and $k' = k(i, j, \nu(A^c))$, we obtain

$$\tilde{E}_{k'}[X(\rho') + H^k_{\nu,\rho'} | \mathcal{F}_s] = \tilde{E}_i[X(\rho_1) + H^i_{\nu,\rho} | \mathcal{F}_s] \wedge \tilde{E}_j[X(\rho_2) + H^j_{\nu,\rho} | \mathcal{F}_s], \quad a.s.$$
Hence, the family \( \{ \tilde{E}_i[X(\rho) + H_{\nu,\rho}^i|F_{\rho}] \} \) is closed under pairwise maximization and pairwise minimization. Thanks to Neveu [1975, Proposition VI-1-1], we can find two sequences \( \{(i_n, \rho_n)\}_{n \in \mathbb{N}} \) and \( \{(i'_n, \rho'_n)\}_{n \in \mathbb{N}} \) in \( \mathcal{I} \times \mathcal{U} \) such that (\ref{4.10}) and (\ref{4.10'}) hold.

**Proof of Lemma 4.3.** We fix \( \nu \in \mathcal{S}_{0,T} \). For any \( (i, \rho) \in \mathcal{I} \times \mathcal{S}_{0,T} \), (\ref{4.7}), (\ref{4.7'}) and Proposition 2.7 (5) show that \( \tilde{E}_i[Y_\rho + H_{\nu,\rho}^i|F_\rho] \geq \tilde{E}_i[C_\nu|F_\rho] = C_\nu \), a.s. Taking the essential supremum over \( (i, \rho) \in \mathcal{I} \times \mathcal{S}_{0,T} \) gives
\[
Z(\nu) = \text{esssup} \left( (i, \rho) \in \mathcal{I} \times \mathcal{S}_{0,T} \right) \tilde{E}_i[Y_\rho + H_{\nu,\rho}^i|F_\rho] \geq C_\nu, \quad \text{a.s.}
\]
Then for any \( i \in \mathcal{I} \), (\ref{4.7'}) implies that \( Z^i(\nu) = Z(\nu) + H_{\nu}^i \geq C_\nu + C_H = C_Y + 2C_H, \quad \text{a.s.} \)

If no member of \( \mathcal{E} \) satisfies (\ref{2.5}) (thus (\ref{4.6}) is assumed), then for any \( (i, \rho) \in \mathcal{I} \times \mathcal{S}_{0,T} \), it holds a.s. that
\[
\tilde{E}_i[Y_\rho^i|F_t] \leq \zeta_Y, \quad \forall t \in \mathcal{D}_T.
\]
Since \( \tilde{E}_i[Y_\rho^i|F_t] \) is an RCLL process, it holds except on a null set \( N = N(i, \rho) \) that
\[
\tilde{E}_i[Y_\rho^i|F_t] \leq \zeta_Y, \quad \forall t \in [0, T], \quad \text{thus} \quad \tilde{E}_i[Y_\rho^i|F_\rho] \leq \zeta_Y.
\]
Moreover, Proposition 2.7 (3) and (\ref{4.7'}) imply that
\[
\zeta_Y \geq \tilde{E}_i[Y_\rho^i|F_\rho] = \tilde{E}_i[Y_\rho + H_{\nu,\rho}^i|F_\rho] + H_{\nu}^i \geq \tilde{E}_i[Y_\rho + H_{\nu,\rho}^i|F_\rho] + C_H, \quad \text{a.s.}
\]
Taking essential supremum over \( (i, \rho) \in \mathcal{I} \times \mathcal{S}_{0,T} \) yields that
\[
Z(\nu) = \text{esssup} \left( (i, \rho) \in \mathcal{I} \times \mathcal{S}_{0,T} \right) \tilde{E}_i[Y_\rho + H_{\nu,\rho}^i|F_\rho] \leq \zeta_Y - C_H, \quad \text{a.s.}
\]
where \( \zeta_Y - C_H \in \text{Dom}(\mathcal{E}) \) thanks to (\ref{4.10}) and (D2). Hence, for any \( i \in \mathcal{I} \), we have \( Z^i(\nu) = Z(\nu) + H_{\nu}^i \leq \zeta_Y - C_H + H_{\nu}^i \), a.s. And (\ref{4.6}) together with (D2) imply that \( \zeta_Y - C_H + H_{\nu}^i \in \text{Dom}(\mathcal{E}) \).

**Proof of Lemma 4.4.** If no member of \( \mathcal{E} \) satisfies (\ref{2.5}), then we see from Lemma 4.3 that
\[
C_\nu \leq Z(\nu) \leq \zeta_Y - C_H, \quad \text{a.s.,}
\]
and that \( \zeta_Y - C_H \in \text{Dom}(\mathcal{E}) \). Hence \( Z(\nu) \in \text{Dom}(\mathcal{E}) \) thanks to Lemma 3.2.

On the other hand, if \( \mathcal{E}_j \) satisfies (\ref{2.5}) for some \( j \in \mathcal{I} \), letting \( (X, \mathcal{I}', \mathcal{U}) = (Y, \mathcal{I}, \mathcal{S}_{0,T}) \) in Lemma 4.2, we can find a sequence \( \{(i_n, \rho_n)\}_{n \in \mathbb{N}} \) in \( \mathcal{I} \times \mathcal{S}_{0,T} \) such that
\[
Z(\nu) = \text{esssup} \left( (i, \rho) \in \mathcal{I} \times \mathcal{S}_{0,T} \right) \tilde{E}_i[Y_\rho + H_{\nu,\rho}^i|F_\rho] = \lim_{n \to \infty} \tilde{E}_{i_n}[Y_{\rho_n} + H_{\nu,\rho_n}^i|F_{\rho_n}], \quad \text{a.s.}
\]
For any \( n \in \mathbb{N} \), it follows from Definition 3.2 that there exists \( k_n = k(j, i_n, \nu) \in \mathcal{I} \) such that \( \tilde{E}_{k_n} = \mathcal{E}_{j,i_n}^\nu \). Applying Proposition 2.7 (3) to \( \tilde{E}_{k_n} \), we can deduce from (\ref{4.10}), (\ref{4.10'}) and (\ref{4.10''}) that
\[
\tilde{E}_{k_n}[Y_{\rho_n} + H_{\nu,\rho_n}^{k_n} - C_H] = \tilde{E}_{k_n}[Y_{\rho_n} + H_{\nu,\rho_n}^{k_n} + H_{\nu}^{k_n} - C_H] = \tilde{E}_{k_n}[Y_{\rho_n} + H_{\nu,\rho_n}^{k_n}] = \mathcal{E}_{j,i_n}^\nu[Y_{\rho_n} + H_{\nu,\rho_n}^i|F_{\rho_n}] = \tilde{E}_{i_n}[Y_{\rho_n} + H_{\nu,\rho_n}^i|F_{\rho_n}],
\]
which together with (Y2) shows that
\[
\lim_{n \to \infty} \mathcal{E}_{j,i_n}^\nu[Y_{\rho_n} + H_{\nu,\rho_n}^i|F_{\rho_n}] \leq \text{sup} \left( (i, \rho) \in \mathcal{I} \times \mathcal{S}_{0,T} \right) \tilde{E}_i[Y_\rho^i] - C_H < \infty.
\]
For any \( n \in \mathbb{N} \), (\ref{4.7}), (\ref{4.7'}) and Proposition 2.7 (5) imply that
\[
\tilde{E}_{i_n}[Y_{\rho_n} + H_{\nu,\rho_n}^{i_n}|F_{\rho_n}] \geq \tilde{E}_{i_n}[C_\nu|F_{\rho_n}] = C_\nu, \quad \text{a.s.}
\]
Therefore, we can deduce from Remark 4.2 (1) that
\[ Z(\nu) = \lim_{n \to \infty} \hat{E}_n [ Y_{\rho_n} + H^i_{\nu,\rho_n} | \mathcal{F}_\nu ] \in \text{Dom}(\mathcal{E}). \]

For any \( i \in \mathcal{I} \), (4.2) and (D2) imply that \( Z'(\nu) = Z(\nu) + H^i_\nu \in \text{Dom}(\mathcal{E}). \)

**Proof of Proposition 4.3.** To see (4.14), we first note that the event \( A \triangleq \{ \nu = \sigma \} \) belongs to \( \mathcal{F}_{\nu \land \sigma} \) thanks to Karatzas and Shreve [1991, Lemma 1.2.16]. For any \( i \in \mathcal{I} \) and \( \rho \in \mathcal{S}_{\nu,T} \), we define \( \rho(A) \triangleq \rho 1_A + T 1_{A^c} \), which clearly belongs to \( \mathcal{S}_{\sigma,T} \). Proposition 2.7 (2) and (3) then imply that

\[ 1_A \tilde{E}_i [ Y_\rho + H^i_{\nu,\rho} | \mathcal{F}_\nu ] = 1_A \left( \tilde{E}_i [ Y_\rho + H^i_{\nu,\rho} | \mathcal{F}_\sigma ] + H^i_\sigma \right) = 1_A \tilde{E}_i [ Y_\rho + H^i_{\nu,\rho} | \mathcal{F}_\sigma ] \]

\[ = \frac{1_A}{(i,\gamma)} \tilde{E}_{i,\gamma} [ Y_\gamma + H^i_{\sigma,\gamma} | \mathcal{F}_\sigma ] = 1_A Z(\sigma), \quad \text{a.s.} \]

Taking the essential supremum of the left-hand-side over \((i,\rho) \in \mathcal{I} \times \mathcal{S}_{\nu,T}\) and applying Lemma 3.3 (2), we obtain

\[ 1_A Z(\nu) = 1_A \text{esssup}_{(i,\rho) \in \mathcal{I} \times \mathcal{S}_{\nu,T}} \tilde{E}_i [ Y_\rho + H^i_{\nu,\rho} | \mathcal{F}_\nu ] = \text{esssup}_{(i,\rho) \in \mathcal{I} \times \mathcal{S}_{\nu,T}} \left( 1_A \tilde{E}_i [ Y_\rho + H^i_{\nu,\rho} | \mathcal{F}_\nu ] \right) \leq 1_A Z(\sigma), \quad \text{a.s.} \]

Reversing the roles of \( \nu \) and \( \sigma \), we obtain (4.14).

As to (4.15), since \( \mathcal{S}_{\sigma,T} \subset \mathcal{S}_{\nu,T} \), it is clear that

\[ \text{esssup}_{(i,\rho) \in \mathcal{I} \times \mathcal{S}_{\nu,T}} \tilde{E}_i [ Y_\rho + H^i_{\nu,\rho} | \mathcal{F}_\nu ] \leq \text{esssup}_{(i,\rho) \in \mathcal{I} \times \mathcal{S}_{\nu,T}} \tilde{E}_i [ Y_\rho + H^i_{\nu,\rho} | \mathcal{F}_\nu ] = Z(\nu), \quad \text{a.s.} \]

Letting \((X, \nu, \mathcal{T}, \mathcal{U}) = (Y, \gamma, \mathcal{I}, \mathcal{S}_{\eta,T})\) in Lemma 4.2, we can find a sequence \( \{ (i_n, \rho_n) \}_{n \in \mathbb{N}} \) in \( \mathcal{I} \times \mathcal{S}_{\gamma,T} \) such that

\[ Z(\gamma) = \text{esssup}_{(i,\rho) \in \mathcal{I} \times \mathcal{S}_{\gamma,T}} \tilde{E}_i [ Y_\rho + H^i_{\nu,\rho} | \mathcal{F}_\gamma ] = \lim_{n \to \infty} \tilde{E}_{i_n} [ Y_{\rho_n} + H^i_{\nu,\rho_n} | \mathcal{F}_\gamma ], \quad \text{a.s.} \]

Now fix \( j \in \mathcal{I} \). For any \( n \in \mathbb{N} \), it follows from Definition 3.2 that there exists a \( k_n = k(j, i_n, \gamma) \in \mathcal{I} \) such that \( \tilde{E}_{k_n} = \mathcal{E}_{j,i_n} \). Applying Proposition 2.7 (3) to \( \tilde{E}_{i_n} \), we can deduce from (3.3), (4.5) that

\[ \text{esssup}_{(i,\rho) \in \mathcal{I} \times \mathcal{S}_{\gamma,T}} \tilde{E}_i [ Y_\rho + H^i_{\nu,\rho} | \mathcal{F}_\nu ] \geq \tilde{E}_{k_n} [ Y_{\rho_n} + H^{k_n}_{\nu,\rho_n} | \mathcal{F}_\nu ] = \mathcal{E}_{j,i_n} [ Y_{\rho_n} + H^{k_n}_{\nu,\rho_n} | \mathcal{F}_\nu ] = \tilde{E}_j \left[ \tilde{E}_{i_n} [ Y_{\rho_n} + H^{k_n}_{\nu,\rho_n} | \mathcal{F}_\nu ] \right] \]

\[ = \tilde{E}_j \left[ \tilde{E}_{i_n} [ Y_{\rho_n} + H^{k_n}_{\nu,\rho_n} | \mathcal{F}_\nu ] \right] + H^{k_n}_{\nu,\rho_n} \leq \text{esssup}_{(i,\rho) \in \mathcal{I} \times \mathcal{S}_{\gamma,T}} \tilde{E}_i [ Y_\rho + H^i_{\nu,\rho} | \mathcal{F}_\nu ], \quad \text{a.s.} \]

For any \( n \in \mathbb{N} \), Proposition 2.7 (5), (4.7) and (4.4) show that

\[ C_\nu + 2C_H = \tilde{E}_{i_n} [ C_\nu ] + C_H \leq \tilde{E}_{i_n} [ Y_{\rho_n} + H^{k_n}_{\nu,\rho_n} | \mathcal{F}_\gamma ] + H^{k_n}_{\nu,\gamma} \leq Z(\gamma) + H^{k_n}_{\nu,\gamma}, \quad \text{a.s.} \]

where \( Z(\gamma) + H^i_{\nu,\gamma} \in \text{Dom}(\mathcal{E}) \) thanks to Lemma 3.3 (1.2) and (D2). Then the Dominated Convergence Theorem (Proposition 2.9) and (8.23) imply that

\[ \tilde{E}_j \left[ Z(\gamma) + H^{k_n}_{\nu,\gamma} | \mathcal{F}_\nu \right] = \lim_{n \to \infty} \tilde{E}_j [ Y_{\rho_n} + H^{k_n}_{\nu,\rho_n} | \mathcal{F}_\nu ] + H^{k_n}_{\nu,\gamma} \mid \mathcal{F}_\nu \left. \right] \leq \text{esssup}_{(i,\rho) \in \mathcal{I} \times \mathcal{S}_{\gamma,T}} \tilde{E}_i [ Y_\rho + H^i_{\nu,\rho} | \mathcal{F}_\nu ], \quad \text{a.s.} \]

Taking the essential supremum of the left-hand-side over \( j \in \mathcal{I} \), we obtain

\[ \text{esssup}_{j \in \mathcal{I}} \tilde{E}_j \left[ Z(\gamma) + H^{k_n}_{\nu,\gamma} | \mathcal{F}_\nu \right] \leq \text{esssup}_{(i,\rho) \in \mathcal{I} \times \mathcal{S}_{\gamma,T}} \tilde{E}_i [ Y_\rho + H^i_{\nu,\rho} | \mathcal{F}_\nu ], \quad \text{a.s.} \]  

(8.24)

On the other hand, for any \( i \in \mathcal{I} \) and \( \rho \in \mathcal{S}_{\gamma,T} \), applying Corollary 2.3 and Proposition 2.7 (3), we obtain

\[ \tilde{E}_i [ Y_\rho + H^i_{\nu,\rho} | \mathcal{F}_\nu ] \leq \\tilde{E}_i [ \tilde{E}_i [ Y_\rho + H^i_{\nu,\rho} | \mathcal{F}_\gamma ] + H^{i}_{\nu,\gamma} | \mathcal{F}_\nu ] \]

\[ \leq \tilde{E}_i [ Z(\gamma) + H^{i}_{\nu,\gamma} | \mathcal{F}_\nu ] \leq \text{esssup}_{i \in \mathcal{I}} \tilde{E}_i [ Z(\gamma) + H^{i}_{\nu,\gamma} | \mathcal{F}_\nu ], \quad \text{a.s.} \]
Taking the essential supremum of the left-hand-side over \((i, \rho) \in \mathcal{I} \times \mathcal{S}_{t, T}\) yields that
\[
\text{esssup}_{(i, \rho) \in \mathcal{I} \times \mathcal{S}_{t, T}} \tilde{E}_i [Y_{\rho} + H^i_{\rho, \nu}] \leq \text{esssup}_{i \in \mathcal{I}} \tilde{E}_i [Z(\gamma) + H^i_{\nu, \gamma}] , \quad \text{a.s.,}
\]
which together with (8.24) proves (4.15). □

**Proof of Proposition 4.2.** For any \(i \in \mathcal{I}\) and \(\nu, \gamma \in \mathcal{S}_{0, T}\) with \(\nu \leq \gamma\), a.s., Proposition 2.7 (3), (4.15) imply that
\[
\tilde{E}_i [Z'(\rho)] = \tilde{E}_i [Z(\rho) + H^i_{\rho, \nu}] \leq \text{esssup}_{i \in \mathcal{I}} \tilde{E}_i [Z(\nu) + H^i_{\nu, \gamma}] , \quad \text{a.s.,}
\]
which implies that \(\{Z^i(t)\}_{t \in [0, T]}\) is an \(\tilde{E}_i\)-supermartingale. Proposition 2.6 and Corollary 2.2 then show that (8.28) and Fatou’s Lemma (Theorem 2.1) imply that
\[
\text{essinf}_{t \in [0, T]} Z^i(t) \geq C_Y + 2C_H , \quad \text{a.s.}
\]
(8.25)

If \(E_j\) satisfies (2.5) for some \(j \in \mathcal{I}\), Corollary 2.2 and (8.26) imply that
\[
Z^j_{\nu} + \in Dom^\# (E_j) = Dom(E) , \quad \forall \nu \in \mathcal{S}_{0, T}, \tag{8.26}
\]
and that \(Z^{j, +}\) is an RCLL \(\tilde{E}_j\)-supermartingale such that for any \(t \in [0, T]\), \(Z^{j, +} \leq Z^j(t)\), a.s. (8.27)

Otherwise, if no member of \(\mathcal{E}\) satisfies (2.5), we suppose that (4.3) holds for some \(j \in \mathcal{I}\). Then Lemma 4.3 and (4.3) imply that for any \(t \in D_T\),
\[
C_Y + 2C_H \leq Z^j(t) = Z(t) + H^j_t \leq \zeta_Y - C_H + \zeta^j , \quad \text{a.s.}
\]
Taking essential supremum of \(Z^j(t)\) over \(t \in D_T\) yields that
\[
\text{esssup}_{t \in D_T} Z^j(t) \leq \zeta_Y - C_H + \zeta^j , \quad \text{a.s.,}
\]
where \(\zeta_Y - C_H + \zeta^j \in Dom(\mathcal{E})\) thanks to (4.3) and (D2). Hence Lemma 3.2 implies that \(\text{esssup}_{t \in D_T} Z^j(t) \in Dom(\mathcal{E}) = Dom^\# (E_j)\). Applying Corollary 2.2 and (8.26) again yields (8.26) and (8.27).

To see that \(Z^{j, +}\) is a modification of \(\{Z^j(t)\}_{t \in [0, T]}\), it suffices to show that for any \(t \in [0, T]\), \(Z^{j, +} \geq Z^j(t)\), a.s. Fix \(t \in [0, T]\). For any \((i, \nu) \in \mathcal{I} \times \mathcal{S}_{t, T}\), Definition 3.2 assures that there exists a \(k = k(j, i, t) \in \mathcal{I}\) such that \(\tilde{E}_k = \tilde{E}_{j, i, t}\) (S1) and (4.5) imply that
\[
H^k_t = H^k_{0, t} = H^j_t , \quad \text{and} \quad H^k_{t, \nu} = H^j_{t, \nu} , \quad \text{a.s.}
\]
(8.28)

For any \(n \in \mathbb{N}\), we set \(t_n \triangleq q^+_n(t)\) and define \(\nu_n \triangleq (\nu + 2^{-n}) \wedge T \in \mathcal{S}_{t, T}\). Let \(m \geq n\), it is clear that \(t_m \leq t_n \leq \nu_n\), a.s. Then Proposition 2.7 (3) implies that
\[
\tilde{E}_k [Y_{\nu_n}] \leq \tilde{E}_k [Y_{\nu_n} + H^k_{t_m, \nu_n}] + H^k_{t_m} \leq Z(t_m) + H^k_{t_m} = Z^j(t_m) + H^k_{t_m} - H^j_{t_m} , \quad \text{a.s.,}
\]
As \(m \to \infty\), (8.29) as well as the right-continuity of the processes \(\tilde{E}_k [Y_{\nu_n}]\), \(H^k\) and \(H^j\) imply that
\[
\tilde{E}_k [Y^k_{\nu_n}] \leq \lim_{m \to \infty} \tilde{E}_k [Y^k_{\nu_n}] \leq \lim_{m \to \infty} Z^j(t_m) + H^j_t - H^j_t = \lim_{m \to \infty} Z^j(t_m) = Z^{j, +} , \quad \text{a.s.}
\]
Since \(\lim_{n \to \infty} \nu_n = \nu\), a.s., the right continuity of the process \(Y^k\) implies that \(Y^k_{\nu_n}\) converges a.s. to \(Y^k_{\nu}\), which belongs to \(Dom(\mathcal{E})\) due to assumption (Y1) and (4.2). Then (4.8) and Fatou’s Lemma (Theorem 2.1) imply that
\[
\tilde{E}_k [Y^k_{\nu}] \leq \lim_{n \to \infty} \tilde{E}_k [Y^k_{\nu_n}] \leq Z^{j, +} , \quad \text{a.s.}
\]
Applying Proposition 4.3 (5) and (3) to $\tilde{E}_j$ and $\tilde{E}_i$ respectively, we can deduce from (8.33) and (8.28) that
\[ Z^{i,+}_t \geq \tilde{E}_i [Y^k_t | F_t] = \tilde{E}_i^t [Y^k_t | F_t] = \tilde{E}_i [Y^k_t | F_t] = \tilde{E}_i [Y_t + H^i_{t,v} | F_t] + H^i_t, \quad \text{a.s.} \tag{8.29} \]
Letting $(i,v)$ run throughout $\mathcal{I} \times \mathcal{S}_t$, $T$ yields that
\[ Z^{i,+}_t \geq \operatorname{esssup}_{(i,v) \in \mathcal{I} \times \mathcal{S}_t,T} \tilde{E}_i [Y_t + H^i_{t,v} | F_t] + H^i_t = Z(t) + H^i_t = Z^i(t), \quad \text{a.s.,} \]
which implies that $Z^{i,+}$ is an RCLL modification of $\{Z^i(t)\}_{t \in [0,T]}$. Correspondingly, $Z^0 \triangleq \{Z^i_{0,+} - H^i_0\}_{t \in [0,T]}$ is an RCLL modification of $\{Z(t)\}_{t \in [0,T]}$. Moreover, for any $i \in \mathcal{I}$, $Z^{i,0} \triangleq \{Z^0_t + H^i_t\}_{t \in [0,T]}$ defines an RCLL modification of $\{Z^i(t)\}_{t \in [0,T]}$, thus it is an $\tilde{E}_i$-supermartingale.

**Proof of Proposition 4.3.** For any $t \in [0,T]$, we know from (4.11) and Proposition 4.2 that $Y_t \leq Z(t) = Z^0_t$, a.s. Since the processes $Y$ and $Z^0$ are both right continuous, it follows from Remark 4.2 (2) that $Z^0$ dominates $Y$.

If $\nu \in S^F_{0,T}$ takes values in a finite set $\{t_1 < \cdots < t_n\}$, for any $\alpha \in \{1, \cdots, n\}$, we can deduce from (4.14) that
\[ 1_{\{\nu = t_n\}} Z(\nu) = 1_{\{\nu = t_n\}} Z(t_\alpha) = 1_{\{\nu = t_n\}} Z^0_{t_\alpha} = 1_{\{\nu = t_n\}} Z^0_{t_\alpha}, \quad \text{a.s.} \]
Summing the above expression over $\alpha$, we obtain
\[ Z^0_{\nu} = Z(\nu), \quad \text{a.s.} \tag{8.30} \]
For general stopping time $\nu \in S_{0,T}$, we let $\{\nu_n\}_{n \in \mathbb{N}}$ be a decreasing sequence in $S_{0,T}$ such that $\lim_{n \to \infty} \nu_n = \nu$, a.s. Thus for any $i \in \mathcal{I}$, the right-continuity of the process $Z^{i,0}$ shows that
\[ Z^{i,0}_{\nu} = \lim_{n \to \infty} Z^{i,0}_{\nu_n}, \quad \text{a.s.} \tag{8.31} \]
For any $n \in \mathbb{N}$, (8.30) and (4.12) imply that
\[ Z^{i,0}_{\nu_n} = Z^i(\nu_n) \geq C_Y + 2CH, \quad \text{a.s.} \tag{8.32} \]
If $\tilde{E}_j$ satisfies (2.3) for some $j \in \mathcal{I}$, we can deduce from (4.10) and (Y2) that
\[ \tilde{E}_j [Z^{j,0}_{\nu_n}] = \tilde{E}_j [Z^j(\nu_n)] \leq Z^j(0) = \sup_{(i,v) \in \mathcal{I} \times \mathcal{S}_0,T} \tilde{E}_i [Y + H^i_v] < \infty, \]
thus $\lim_{n \to \infty} \tilde{E}_j [Z^{j,0}_{\nu_n}] < \infty$. Then Remark 4.2 (1) implies that $Z^{j,0}_{\nu} \in \operatorname{Dom}(\mathcal{E})$.

On the other hand, if no member of $\mathcal{E}$ satisfies (2.3), we suppose that (4.2) holds for some $j \in \mathcal{I}$. In light of Proposition 4.2 and Lemma 4.3, it holds a.s. that
\[ C_Y + 2CH \leq Z^j_t = Z^j_t + H^j_t \leq Z^j(t) + H^j_t \leq \zeta_Y - C_H + \zeta^j, \quad \forall t \in \mathcal{D}_T, \]
where $\zeta_Y - C_H + \zeta^j \in \operatorname{Dom}(\mathcal{E})$ thanks to (4.10), (4.3) and (D2). Since $Z^{j,0}$ is an RCLL process, it holds except on a null set $N$ that
\[ C_Y + 2CH \leq Z^j_t \leq \zeta_Y - C_H + \zeta^j, \quad \forall t \in [0,T], \quad \text{thus} \quad C_Y + 2CH \leq Z^{j,0}_t \leq \zeta_Y - C_H + \zeta^j. \tag{8.33} \]
Lemma 4.2 then implies that $Z^{j,0}_{\nu} \in \operatorname{Dom}(\mathcal{E})$. We have seen in both cases that $Z^{j,0}_{\nu} \in \operatorname{Dom}(\mathcal{E})$ for some $j \in \mathcal{I}$.

Since $Z^{j,0}$ is an RCLL $\tilde{E}_j$-supermartingale by Proposition 4.2 and the Optional Sampling Theorem (Theorem 2.4) imply that $\tilde{E}_j [Z^{j,0}_{\nu_n} | F_{\nu_n+1}] \leq Z^{j,0}_{\nu_n+1}$, a.s. for any $n \in \mathbb{N}$. Applying Corollary 2.3 and Theorem 2.4 once again, we obtain
\[ \tilde{E}_j [Z^{j,0}_{\nu_n} | F_\nu] = \tilde{E}_j [\tilde{E}_j [Z^{j,0}_{\nu_n} | F_{\nu_n+1}] | F_\nu] \leq \tilde{E}_j [Z^{j,0}_{\nu_n+1} | F_\nu] \leq Z^{j,0}_{\nu}, \quad \text{a.s.,} \tag{8.34} \]
which implies that \( \lim_{n \to \infty} \xi_j \left( Z_j^{(0)} \mid \mathcal{F}_\nu \right) \leq Z_j^{(0)}, \) a.s. On the other hand, using (8.31) and (8.32), we can deduce from Proposition 2.7 (5) and Fatou’s Lemma (Theorem 2.1) that
\[
Z_j^{(0)} = \tilde{\xi}_j \left( Z_j^{(0)} \mid \mathcal{F}_\nu \right) \leq \lim_{n \to \infty} \xi_j \left( Z_j^{(0)} \mid \mathcal{F}_\nu \right) \leq Z_j^{(0)}, \quad \text{a.s.}
\]
Then (8.30) and (4.10) imply that
\[
Z_j^{(0)} = \lim_{n \to \infty} \xi_j \left( Z_j^{(0)} \mid \mathcal{F}_\nu \right) = \lim_{n \to \infty} \xi_j \left( Z_j^{(0)} \mid \mathcal{F}_\nu \right) \leq Z_j^{(0)}, \quad \text{a.s., thus } Z_j^{(0)} \leq Z(\nu) \text{ a.s.} \quad (8.35)
\]
On the other hand, for any \((i, \rho) \in \mathcal{I} \times \mathcal{S}_{\nu,T}\) and \(n \in \mathbb{N}\), we define \(\rho_n = \Delta \rho \vee \nu_n \in \mathcal{S}_{\nu_n, T}\). Proposition 2.7 (3) implies that
\[
\tilde{E}_i \left[ Y_{\rho_n} \mid \mathcal{F}_{\nu_n} \right] = \tilde{E}_i \left[ Y_{\rho_n} + H_{\nu_n, \rho_n} \mid \mathcal{F}_{\nu_n} \right] + H_{\nu_n}^{(0)} \leq Z(\nu_n) + H_{\nu_n}^{(0)} = Z^{(0)}(\nu_n), \quad \text{a.s.}
\]
Taking \(\tilde{E}_i \left[ \cdot \mid \mathcal{F}_\nu \right]\) on both sides, we see from Corollary 2.3 that
\[
\tilde{E}_i \left[ Y_{\rho_n} \mid \mathcal{F}_{\nu_n} \right] = \tilde{E}_i \left[ Y_{\rho_n} \mid \mathcal{F}_{\nu_n} \right] \leq \tilde{E}_i \left[ Z^{(0)}(\nu_n) \mid \mathcal{F}_\nu \right], \quad \text{a.s.}
\]
It is easy to see that \( \lim_{n \to \infty} \rho_n = \rho, \) a.s. Using the right continuity of processes \(Y\) and \(H^j\), we can deduce from (4.8), Fatou’s Lemma (Proposition 2.5) and (8.30) that
\[
\tilde{E}_i \left[ Y_{\rho_n} \mid \mathcal{F}_{\nu_n} \right] \leq \lim_{n \to \infty} \tilde{E}_i \left[ Y_{\rho_n} \mid \mathcal{F}_{\nu_n} \right] \leq \lim_{n \to \infty} \tilde{E}_i \left[ Z^{(0)}(\nu_n) \mid \mathcal{F}_\nu \right] = \tilde{Z}_0^{(0)}, \quad \text{a.s.}
\]
Then Proposition 2.7 (3) again implies that
\[
\tilde{E}_i \left[ Y_{\rho} + H_{\nu, \rho} \mid \mathcal{F}_\nu \right] = \tilde{E}_i \left[ Y_{\rho} \mid \mathcal{F}_\nu \right] - H_{\nu}^{(0)} = \tilde{Z}_0^{(0)} - H_{\nu}^{(0)} = Z_{\nu}^{(0)}, \quad \text{a.s.}
\]
Taking the essential supremum over \((i, \rho) \in \mathcal{I} \times \mathcal{S}_{\nu, T}\) yields that \(Z(\nu) \leq Z_{\nu}^{(0)}, \) a.s., which in conjunction with (8.35) shows that \(Z_{\nu} = Z(\nu), \) a.s., thus \(Z_{\nu}^{(0)} \in \text{Dom}(\mathcal{E})\) by Lemma 4.4. Moreover, for any \(i \in \mathcal{I}\), we have
\[
Z_{\nu}^{(0)} = Z_{\nu}^{(0)} + H_{\nu}^{(0)} = Z(\nu) + H_{\nu}^{(0)} = Z^{(0)}(\nu), \quad \text{a.s.}
\]
thus \(Z_{\nu}^{(0)} = Z^{(0)}\) thanks to Lemma 4.4 once again. (4.17) is proved.

Now let \(X\) be another RCLL \(\mathcal{F}\)-adapted process dominating \(Y\) such that \(X^i = \Delta \{X^i + H^i_t\}_{t \in [0, T]}\) is an \(\tilde{E}_i\)-supermartingale for any \(i \in \mathcal{I}\). We fix \(t \in [0, T]\). For any \(i \in \mathcal{I}\) and \(\nu \in \mathcal{S}_{t, T}\), we let \(\{\nu_n\}_{n \in \mathbb{N}}\) be a decreasing sequence in \(\mathcal{S}_{t, T}^e\) such that \(\lim_{n \to \infty} \nu_n = \nu, \) a.s. For any \(n \in \mathbb{N}\), since \(X^i\) dominates \(Y^i\), Remark 4.3 (1) shows that \(X_{\nu_n}^i \geq Y_{\nu_n}^i, \) a.s. Then (A4), Proposition 2.6 and the Optional Sampling Theorem (Theorem 2.1) imply that
\[
\tilde{E}_i \left[ Y_{\nu_n} + H_{\nu_n}^i \mid \mathcal{F}_t \right] = \tilde{E}_i \left[ Y_{\nu_n} \mid \mathcal{F}_t \right] - H_{\nu_n}^i \leq \tilde{E}_i \left[ X_{\nu_n}^i \mid \mathcal{F}_t \right] - H_{\nu_n}^i \leq X_{\nu_n}^i - H_{\nu_n}^i = X_t, \quad \text{a.s.}
\]
The right-continuity of the processes \(Y\) and \(H^i\) shows that \(Y_{\nu} + H_{\nu, \nu}^i = \lim_{n \to \infty} \left( Y_{\nu_n} + H_{\nu_n}^i \right), \) a.s., thus it follows from (4.7), (4.4) and Fatou’s Lemma (Proposition 2.5) that
\[
\tilde{E}_i \left[ Y_{\nu} + H_{\nu, \nu}^i \mid \mathcal{F}_t \right] \leq \lim_{n \to \infty} \tilde{E}_i \left[ Y_{\nu_n} + H_{\nu_n, \nu_n}^i \mid \mathcal{F}_t \right] \leq X_t, \quad \text{a.s.}
\]
Taking the essential supremum of the left-hand-side over \((i, \nu) \in \mathcal{I} \times \mathcal{S}_{t, T}\), we can deduce from Proposition 4.2 that
\[
Z_0^i = Z(t) = \text{esssup}_{(i, \nu) \in \mathcal{I} \times \mathcal{S}_{t, T}} \tilde{E}_i \left[ Y_{\nu} + H_{\nu, \nu}^i \mid \mathcal{F}_t \right] \leq X_t, \quad \text{a.s.}
\]
Since both \(Z_0^i\) and \(X\) are RCLL processes, Remark 4.3 (2) once again shows that \(X\) dominates \(Z_0^i\). □

**Proof of Lemma 4.5** For any \(i \in \mathcal{I}\), (4.18), (4.4) as well as Proposition 2.7 (5) imply that
\[
\tilde{E}_i \left[ Z_{\tau_3(\nu)}^i + H_{\nu, \tau_3(\nu)}^i \mid \mathcal{F}_\nu \right] \geq \tilde{E}_i \left[ C_\nu + C_H \mid \mathcal{F}_\nu \right] = C_Y + 2C_H, \quad \text{a.s.}
\]
Taking the essential supremum of the left-hand-side over \(i \in \mathcal{I}\), we can deduce from (8.19) that
\[
C_Y + 2C_H \leq \operatorname{esssup}_{i \in \mathcal{I}} \tilde{E}_i[Z^0_{\tau_3(\nu)} + H^i_{\nu,\tau_3(\nu)} | \mathcal{F}_\nu] = J_\delta(\nu), \text{ a.s.}
\] (8.36)

Then Lemma 8.3.3 imply that \(J_\delta(\nu) \in \text{Dom}(\tilde{\mathcal{E}}')\). Let \(\sigma\) be another stopping time in \(\mathcal{S}_{0,T}\). In light of (8.17) and (4.11), we see that
\[
1_{\{\tau_3(\nu) = \tau_3(\sigma)\}} Z^0_{\tau_3(\nu)} = 1_{\{\tau_3(\nu) = \tau_3(\sigma)\}} Z(\tau_3(\nu)) = 1_{\{\tau_3(\nu) = \tau_3(\sigma)\}} Z(\tau_3(\sigma)) = 1_{\{\tau_3(\nu) = \tau_3(\sigma)\}} Z^0_{\tau_3(\sigma)}, \text{ a.s.}
\] (8.37)

It is clear that \(\{\nu = \sigma\} \subset \{\tau_3(\nu) = \tau_3(\sigma)\}\). Thus multiplying \(1_{\{\nu = \sigma\}}\) to both sides of (8.37) gives that
\[
1_{\{\nu = \sigma\}} Z^0_{\tau_3(\nu)} = 1_{\{\nu = \sigma\}} Z^0_{\tau_3(\sigma)}, \text{ a.s.}
\]

For any \(i \in \mathcal{I}\), applying Proposition 2.7 (2) and recalling how \(\tilde{E}_i[\cdot | \mathcal{F}_\nu]\) and \(\tilde{E}_i[\cdot | \mathcal{F}_\sigma]\) are defined in (2.6), we obtain
\[
1_{\{\nu = \sigma\}} \tilde{E}_i[Z^0_{\nu,\tau_3(\nu)} + H^i_{\nu,\tau_3(\nu)} | \mathcal{F}_\nu] = \tilde{E}_i[Z^0_{\nu,\tau_3(\nu)} + H^i_{\nu,\tau_3(\nu)} | \mathcal{F}_\sigma], \text{ a.s.}
\]
where we use the fact that \(\{\nu = \sigma\} \in \mathcal{F}_{\nu,\sigma}\) thanks to Karatzas and Shreve [1991] Lemma 1.2.16. Taking the essential supremum of both sides over \(i \in \mathcal{I}\), Lemma 8.3.3 (2) implies that
\[
1_{\{\nu = \sigma\}} J_\delta(\nu) = \operatorname{esssup}_{i \in \mathcal{I}} 1_{\{\nu = \sigma\}} \tilde{E}_i[Z^0_{\nu,\tau_3(\nu)} + H^i_{\nu,\tau_3(\nu)} | \mathcal{F}_\nu] = \operatorname{esssup}_{i \in \mathcal{I}} 1_{\{\nu = \sigma\}} \tilde{E}_i[Z^0_{\nu,\tau_3(\nu)} + H^i_{\nu,\tau_3(\nu)} | \mathcal{F}_\sigma] = 1_{\{\nu = \sigma\}} J_\delta(\sigma), \text{ a.s.}
\]
which proves the lemma.

Proof of Proposition 4.4:

Proof of 1. We fix \(i \in \mathcal{I}\) and \(\nu, \rho \in \mathcal{S}_{0,T}\) with \(\nu \leq \rho, \text{ a.s.}\). Taking \((\nu, \mathcal{T}, U) = (\rho, \mathcal{I}, \{\tau_3(\rho)\})\) and \(X(\tau_3(\rho)) = Z^0_{\tau_3(\rho)}\) in Lemma 4.2, we can find a sequence \(\{j_n\}_{n=1}^\infty\) in \(\mathcal{I}\) such that
\[
J_\delta(\rho) = \operatorname{esssup}_{j \in \mathcal{I}} \tilde{E}_j[Z^0_{\rho,\tau_3(\rho)} + H^j_{\rho,\tau_3(\rho)} | \mathcal{F}_\rho] = \lim_{n \to \infty} \tilde{E}_{j_n}[Z^0_{\rho,\tau_3(\rho)} + H^j_{\rho,\tau_3(\rho)} | \mathcal{F}_\rho], \text{ a.s.}
\]

For any \(n \in \mathbb{N}\), it follows from Definition 8.2.2 that there exists a \(k_n = k(i, j_n, \rho) \in \mathcal{I}\) such that \(\tilde{E}_{k_n} = \tilde{E}_{i,j_n}\). Applying Proposition 2.7 (3) to \(\tilde{E}_{k_n}\), we can deduce from (8.33) and (4.5) that
\[
\tilde{E}_{k_n}[Z^0_{\tau_3(\rho)} + H^{k_n}_{\nu,\tau_3(\rho)} | \mathcal{F}_\nu] = \tilde{E}_{i,j_n}[Z^0_{\rho,\tau_3(\rho)} + H^{k_n}_{\nu,\tau_3(\rho)} | \mathcal{F}_\rho] = \tilde{E}_i[\tilde{E}_{j_n}[Z^0_{\rho,\tau_3(\rho)} + H^{k_n}_{\nu,\tau_3(\rho)} | \mathcal{F}_\rho] | \mathcal{F}_\nu] = \tilde{E}_i[\tilde{E}_n[Z^0_{\rho,\tau_3(\rho)} + H^{k_n}_{\nu,\tau_3(\rho)} | \mathcal{F}_\rho] | \mathcal{F}_\nu] \leq \tilde{E}_n[Z^0_{\tau_3(\nu)} + H^{k_n}_{\nu,\tau_3(\nu)} | \mathcal{F}_\nu] \leq \tilde{E}_n[Z^0_{\tau_3(\nu)} + H^{k_n}_{\nu,\tau_3(\nu)} | \mathcal{F}_\nu] = J_\delta(\nu), \text{ a.s.}
\] (8.38)

Since \(\nu \leq \rho, \text{ a.s.}\), we see that \(\tau_3(\nu) \leq \tau_3(\rho), \text{ a.s.}\). Due to (8.17) and (4.15), we have that
\[
\tilde{E}_{k_n}[Z^0_{\tau_3(\rho)} + H^{k_n}_{\nu,\tau_3(\rho)} | \mathcal{F}_{\tau_3(\nu)}] \leq \operatorname{esssup}_{j \in \mathcal{I}} \tilde{E}_j[Z(\tau_3(\rho)) + H_{\nu,\tau_3(\rho)} | \mathcal{F}_{\tau_3(\nu)}] \leq Z(\tau_3(\nu)) = Z^0_{\tau_3(\nu)}, \text{ a.s.}
\]

Then using Corollary 2.3 and applying Proposition 2.7 (3) to \(\tilde{E}_{k_n}\), we obtain
\[
\tilde{E}_{k_n}[Z^0_{\tau_3(\rho)} + H^{k_n}_{\nu,\tau_3(\rho)} | \mathcal{F}_\nu] = \tilde{E}_{k_n} [\tilde{E}_{k_n}[Z^0_{\tau_3(\rho)} + H^{k_n}_{\nu,\tau_3(\rho)} | \mathcal{F}_{\tau_3(\nu)}] | \mathcal{F}_\nu] \
= \tilde{E}_{k_n} [\tilde{E}_{k_n}[Z^0_{\tau_3(\rho)} + H^{k_n}_{\nu,\tau_3(\rho)} | \mathcal{F}_{\tau_3(\nu)}] + H^{k_n}_{\nu,\tau_3(\rho)} | \mathcal{F}_\nu] \
\leq \tilde{E}_{k_n}[Z^0_{\tau_3(\rho)} + H^{k_n}_{\nu,\tau_3(\rho)} | \mathcal{F}_\nu] \leq \operatorname{esssup}_{j \in \mathcal{I}} \tilde{E}_j[Z^0_{\tau_3(\nu)} + H^{j}_{\nu,\tau_3(\nu)} | \mathcal{F}_\nu] = J_\delta(\nu), \text{ a.s.}
\]

which together with (8.38) shows that
\[
\tilde{E}_i[\tilde{E}_{j_n}[Z^0_{\tau_3(\rho)} + H^{j_n}_{\rho,\tau_3(\rho)} | \mathcal{F}_\rho] + H^i_{\nu,\rho} | \mathcal{F}_\nu] \leq J_\delta(\nu), \text{ a.s.}
\]

For any \(n \in \mathbb{N}\), we see from (4.18), (4.14) and Proposition 2.7 (5) that
\[
\tilde{E}_{j_n}[Z^0_{\tau_3(\rho)} + H^{j_n}_{\rho,\tau_3(\rho)} | \mathcal{F}_\rho] + H^i_{\nu,\rho} \geq \tilde{E}_{j_n}[C_Y + 2C_H | \mathcal{F}_\rho] + C_H = C_Y + 3C_H, \text{ a.s.}
\]
Then Fatou’s Lemma (Proposition 2.8) implies that
\[
\bar{E}_t\left[J_{\delta}(\rho) + H_{n,\rho}\big|\mathcal{F}_t\right] \leq \limsup_{n \to \infty} \bar{E}_t\left[J_{\delta}(\rho) + Z_{\tau_{\delta}(t_n)} + H_{n,\tau_{\delta}(t_n)}\big|\mathcal{F}_t\right] + H_{t,\rho}(\mathcal{F}_t) \leq J_{\delta}(\nu), \quad \text{a.s.}
\]
For any \( \sigma \in \mathcal{S}_{0,T} \), Lemma 4.5 and (D2) show that \( J_{\delta}(\sigma) \supseteq J_{\delta}(\sigma) + H_{\sigma} \in \text{Dom}(\mathcal{E}) \). A simple application of Proposition 2.7 (3) yields that
\[
\bar{E}_t\left[J_{\delta}(\rho)\big|\mathcal{F}_t\right] = \bar{E}_t\left[J_{\delta}(\rho) + H_{n,\rho}\big|\mathcal{F}_t\right] + H_{t,\rho} \leq J_{\delta}(\nu) + H_{t,\rho} = J_{\delta}(\nu), \quad \text{a.s.}
\]
(8.39)
In particular, when \( 0 \leq s < t \leq T \), we have \( \bar{E}_t\left[J_{\delta}(t)\big|\mathcal{F}_s\right] \leq J_{\delta}(s) \), a.s., which show that \( \{J_{\delta}(t)\}_{t \in [0,T]} \) is an \( \bar{E}_t \)-supermartingale.

**Proof of 2.** For any \( i \in \mathcal{I} \) and \( \nu \in \mathcal{S}_{0,T} \), (8.36) and (4.4) imply that
\[
J_{\delta}^i(\nu) = J_{\delta}(\nu) + H_{t,\rho} \geq C_Y + 3C_H, \quad \text{a.s.}
\]
(8.40)
In particular, \( J_{\delta}^i(T) \geq C_Y + 3C_H, \) a.s. Proposition 2.6 and Theorem 2.3 then show that \( \left\{J_{\delta}^i(t) \triangleq \limsup_{n \to \infty} J_{\delta}(q_{n,t}(t)) \right\}_{t \in [0,T]} \) defines an RCLL process. Then (8.40) implies that
\[
\text{essinf}_{t \in [0,T]} J_{\delta}^i(t) \geq C_Y + 3C_H, \quad \text{a.s.}
\]
(8.41)
If \( \mathcal{E}_j \) satisfies (2.3) for some \( j \in \mathcal{I} \), Corollary 2.2 and (8.41) imply that
\[
J_{\delta,j}^i = \limsup_{n \to \infty} J_{\delta}^i(t) \leq C_Y + 3C_H, \quad \text{a.s.,}
\]
where \( C_Y - C_H + \zeta \in \text{Dom}(\mathcal{E}) \) thanks to (4.3), (4.3) and (D2). Hence Lemma 3.2 implies that \( \text{esssup}_{t \in D_T} J_{\delta}^i(t) \in \text{Dom}(\mathcal{E}) = \text{Dom}(\mathcal{E}_j) \). Applying Corollary 2.2 and (8.41) again yields (8.42) and (8.43).

To see that \( J_{\delta,j}^i \) is a modification of \( \left\{J_{\delta}(t)\right\}_{t \in [0,T]} \), it suffices to show that for any \( t \in [0,T] \), \( J_{\delta,j}^i(t) \geq J_{\delta}^i(t) \), a.s. Fix \( t \in [0,T] \). For any \( i \in \mathcal{I} \), Definition 3.2 assures that there exists a \( k = k(j,i,t) \in \mathcal{I} \) such that \( \bar{E}_k = \mathcal{E}_j^i \). Moreover, (S1) and (4.13) imply that
\[
H_{k}^i = H_{0,t}^i + H_{i,t}^i, \quad \text{and} \quad H_{t,\tau_k(t)}^i = H_{t,\tau_k(t)}^i, \quad \text{a.s.}
\]
(8.44)
For any \( n \in \mathbb{N} \), we set \( t_n \triangleq q_{n,t}(t) \). Let \( m \geq n \), it is clear that \( t_m \leq t_n \), a.s. Then (4.17), Corollary 2.3 Proposition 2.7 (3) as well as (4.13) imply that
\[
\bar{E}_k\left[Z_{\tau_k(t_n)}^i\big|\mathcal{F}_t\right] = \bar{E}_k\left[Z_{\tau_k(t_n)}^i\big|\mathcal{F}_t\right] = \bar{E}_k\left[Z_{\tau_k(t_n)}^i\big|\mathcal{F}_t\right] + H_{t,\tau_k(t_n)}^i \leq \bar{E}_k\left[Z_{\tau_k(t_n)}^i\big|\mathcal{F}_t\right] + H_{t,\tau_k(t_n)}^i \leq \text{esssup}_{t \in \mathcal{I}} \bar{E}_k\left[Z_{\tau_k(t_n)}^i\big|\mathcal{F}_t\right] + H_{t,\tau_k(t_n)}^i \leq J_{\delta}^i(t_n) + H_{t,\tau_k(t_n)}^i, \quad \text{a.s.}
\]
As $m \to \infty$, (8.41) as well as the right-continuity of the processes $\tilde{\mathcal{E}}_k [Z_{\tau_k(t_n)}^k, \mathcal{F}_t]$, $H^k$ and $H^i$ imply that

$$\tilde{\mathcal{E}}_k [Z_{\tau_k(t_n)}^k, \mathcal{F}_t] = \lim_{m \to \infty} \tilde{\mathcal{E}}_k [Z_{\tau_k(t_m)}^k, \mathcal{F}_t] \leq \lim_{m \to \infty} J^k_\delta(t_m) + H^k_t - H^i_t = \lim_{m \to \infty} J^k_\delta(t_m) = J^k_{\delta, j^+}, \quad \text{a.s.}$$

Since $\lim_{n \to \infty} \tau_k(t_n) = \tau_\delta(t)$ a.s., the right-continuity of the process $Z_{\tau_k(t_n)}^k$ implies that $Z_{\tau_k(t_n)}^k$ converges a.s. to $Z_{\tau_\delta(t)}^k$, which belongs to $\text{Dom}(\mathcal{E})$ thanks to Proposition 4.3. Then (4.18) and Fatou’s Lemma (Theorem 2.1) imply that

$$\tilde{\mathcal{E}}_k [Z_{\tau_\delta(t)}^k, \mathcal{F}_t] \leq \lim_{n \to \infty} \tilde{\mathcal{E}}_k [Z_{\tau_k(t_n)}^k, \mathcal{F}_t] \leq J^k_{\delta, j^+}, \quad \text{a.s.}$$

Similar to (8.30), we can deduce from (8.3) and (8.41) that

$$J^k_{\delta, j^+} \geq \tilde{\mathcal{E}}_i [Z_{\tau_\delta(t)}^i + H^i_{\tau_\delta(t)} | \mathcal{F}_t] + H^i_t = J^i_\delta(t) + H^i_t = J^i_\delta(t), \quad \text{a.s.}$$

Letting $i$ run throughout $\mathcal{I}$ yields that

$$J^k_{\delta, j^+} \geq \text{esssup}_{i \in \mathcal{I}} \tilde{\mathcal{E}}_i [Z_{\tau_\delta(t)}^i + H^i_{\tau_\delta(t)} | \mathcal{F}_t] + H^i_t = J^i_\delta(t) + H^i_t = J^i_\delta(t), \quad \text{a.s.,}$$

which implies that $J^k_{\delta, j^+}$ is an RCLL modification of $\{J^i_\delta(t)\}_{t \in [0, T]}$. Correspondingly, $J^k_{\delta, j^+} = \{J^k_{\delta, j^+} + H^i_t\}_{t \in [0, T]}$ is an RCLL modification of $\{J^i_\delta(t)\}_{t \in [0, T]}$. Moreover, for any $i \in \mathcal{I}$, $J^k_{\delta, i^0} = \{J^k_{\delta, i^0} + H^i_t\}_{t \in [0, T]}$ defines an RCLL modification of $\{J^i_\delta(t)\}_{t \in [0, T]}$, thus it is an $\mathcal{E}$-supermartingale.

**Proof of 3.** Now let us show (3). Similar to (8.30), we can deduce from Lemma 4.3 that for any $\nu \in S_{0,T}^F$

$$J^\nu_{\delta, 0} = J^\delta_\delta(\nu), \quad \text{a.s.} \quad (8.45)$$

For a general stopping time $\nu \in S_{0,T}$, we let $\{\nu_n\}_{n \in \mathbb{N}}$ be a decreasing sequence in $S_{0,T}^F$ such that $\lim_{n \to \infty} \nu_n = \nu$, a.s. Thus for any $i \in \mathcal{I}$, the right-continuity of the process $J^k_{\delta, i^0}$ shows that

$$J^\nu_{\delta, i^0} = \lim_{n \to \infty} J^k_{\delta, i^0}, \quad \text{a.s.} \quad (8.46)$$

In light of (8.45) and (8.40), it holds a.s. that

$$J^\nu_{\delta, i^0} = J^\delta_i(t) \geq C_Y + 3C_H, \quad \forall t \in \mathcal{D}_T.$$ 

Since $J^k_{\delta, i^0}$ is an RCLL process, it holds except on a null set $N$ that

$$J^\delta_{\sigma, i^0} \geq C_Y + 3C_H, \quad \forall \sigma \in \mathcal{S}_{0,T}, \quad (8.47)$$

If $E_j$ satisfies (2.5) for some $j \in \mathcal{I}$, we can deduce from (8.39), (1.19) and (Y2) that

$$\tilde{\mathcal{E}}_j [J_{\nu_n}^{\delta, j^0}] = \tilde{\mathcal{E}}_j [J_{\delta}^j(\nu_n)] \leq J_{\delta}^j(0) - J_{\delta}^j(0) = Z(0) = \sup_{(r, p) \in \mathcal{I} \times \mathcal{S}_{0,T}} \tilde{\mathcal{E}}_r [Y_p + H^i_p] < \infty,$$

thus $\lim_{n \to \infty} \tilde{\mathcal{E}}_j [J_{\nu_n}^{\delta, j^0}] < \infty$. Then Remark 4.2 (1) implies that $J_{\nu_n}^{\delta, j^0} \in \text{Dom}(\mathcal{E})$.

On the other hand, if no member of $E$ satisfies (2.5), we suppose that (4.3) holds for some $j \in \mathcal{I}$. In light of (8.41), (1.19) and (1.13), it holds a.s. that

$$J_{\nu_n}^{\delta, j^0} = J_{\delta}^j(t) + H^j_t \leq Z(t) + H^j_t \leq \zeta_Y - C_H + \zeta^j, \quad \forall t \in \mathcal{D}_T,$$

where $\zeta_Y - C_H + \zeta^j \in \text{Dom}(\mathcal{E})$ thanks to (4.6), (4.3) and (D2). Since $J_{\nu_n}^{\delta, j^0}$ is an RCLL process, it holds except on a null set $N'$ that

$$J_{\nu_n}^{\delta, j^0} \leq \zeta_Y - C_H + \zeta^j, \quad \forall t \in [0, T], \quad \text{thus} \quad J_{\nu_n}^{\delta, j^0} \leq \zeta_Y - C_H + \zeta^j. \quad (8.48)$$
Then (8.47) and Lemma 3.2 imply that \( J_{\nu,j}^{\delta,0} \in \text{Dom}(\mathcal{E}) \). We have seen in both cases that \( J_{\nu,j}^{\delta,0} \in \text{Dom}(\mathcal{E}) \) for some \( j \in \mathcal{I} \).

Similar to the arguments used in (8.39) through (8.35) (with (8.35) - (8.37) replacing (8.39) - (8.34) respectively, and with (8.39) replacing (4.10)), we can deduce that

\[
J_{\nu,j}^{\delta,0} = \lim_{n \to \infty} \tilde{E}_i \left[ J_{\nu,n}^{\delta,0} \right] \mathcal{F}_n = \lim_{n \to \infty} \tilde{E}_i \left[ J_{\delta}(\nu_n) \right] \mathcal{F}_n \leq J_{\delta}(\nu), \quad \text{thus} \quad J_{\nu,j}^{\delta,0} \leq J_{\delta}(\nu), \quad \text{a.s.} \quad (8.49)
\]

The right-continuity of the process \( J_{\nu,j}^{\delta,0} \), (8.45) and (8.36) show that

\[
J_{\nu,j}^{\delta,0} = \lim_{n \to \infty} J_{\nu,n}^{\delta,0} = \lim_{n \to \infty} J_{\delta}(\nu_n) \geq C_Y + 2CH, \quad \text{a.s.}
\]

Lemma 4.5 and Lemma 3.2 thus imply that \( J_{\nu,j}^{\delta,0} \in \text{Dom}(\mathcal{E}) \). For any \( i \in \mathcal{I} \), (4.2) and (D2) show that \( J_{\nu,j}^{\delta,0} = J_{\nu}^{\delta,0} + H_j^{\nu} \in \text{Dom}(\mathcal{E}) \).

On the other hand, for any \( i \in \mathcal{I} \) and \( n \in \mathbb{N} \), it is clear that \( \nu \leq \nu_n \leq \tau_\delta(\nu_n) \), a.s. Then Corollary 2.3 (8.49) and (8.39) imply that

\[
\tilde{E}_i \left[ J_{\nu,n}^{\delta,1,0} \right] \mathcal{F}_n = \tilde{E}_i \left[ J_{\nu,n}^{\delta,0} \right] \mathcal{F}_n \leq \tilde{E}_i \left[ J_{\delta}(\nu_n) \right] \mathcal{F}_n \leq \tilde{E}_i \left[ J_{\delta}(\nu) \right] \mathcal{F}_n, \quad \text{a.s.}
\]

It is easy to see that \( \lim_{n \to \infty} \tau_\delta(\nu_n) = \tau_\delta(\nu), \) a.s. Using the right continuity of the process \( J_{\nu,j}^{\delta,0} \), we can deduce from (8.47), Fatou’s Lemma (Proposition 2.8) and (8.49) that

\[
\tilde{E}_i \left[ J_{\nu,\tau_\delta(\nu)}^{\delta,1,0} \right] \mathcal{F}_\nu \leq \lim_{n \to \infty} \tilde{E}_i \left[ J_{\nu,n}^{\delta,1,0} \right] \mathcal{F}_\nu \leq \lim_{n \to \infty} \tilde{E}_i \left[ J_{\delta}(\nu_n) \right] \mathcal{F}_\nu = J_{\nu,j}^{\delta,0}, \quad \text{a.s.}
\]

Proposition 2.7 (3) further implies that

\[
\tilde{E}_i \left[ J_{\nu,\tau_\delta(\nu)}^{\delta,0} + H_i^{\nu,\tau_\delta(\nu)} \right] \mathcal{F}_\nu = \tilde{E}_i \left[ J_{\nu,\tau_\delta(\nu)}^{\delta,1,0} \right] \mathcal{F}_\nu - H_i^{\nu} \leq J_{\nu,j}^{\delta,0} - H_i^{\nu} = J_{\nu,j}^{\delta,0}, \quad \text{a.s.}
\]

Taking the essential supremum over \( i \in \mathcal{I} \) gives

\[
J_{\delta}(\nu) = \text{esssup}_{i \in \mathcal{I}} \tilde{E}_i \left[ J_{\nu,\tau_\delta(\nu)}^{\delta,0} + H_i^{\nu,\tau_\delta(\nu)} \right] \mathcal{F}_\nu \leq J_{\nu,j}^{\delta,0}, \quad \text{a.s.,}
\]

which together with (8.39) shows that \( J_{\nu,j}^{\delta,0} = J_{\delta}(\nu) \), a.s. \( \Box \)

**Proof of Theorem 4.1** We first show that for any \( \delta \in (0,1) \) and \( \nu \in \mathcal{S}_{0,T} \)

\[
J_{\delta}(\nu) = Z_0^\delta = Z(\nu), \quad \text{a.s.} \quad (8.50)
\]

Fix \( i \in \mathcal{I} \). Lemma 3.1 indicates that \( \tilde{E}_i \) is a convex \( \mathcal{F} \)-expectation on \( \text{Dom}(\mathcal{E}) \). Since \( Z_i^{\delta,0} \) and \( J_{\nu,j}^{\delta,i,0} \) are both \( \tilde{E}_i \)-supermartingales, we can deduce that for any \( 0 \leq s < t \leq T \),

\[
\tilde{E}_i \left[ \delta Z_i^{\delta,0} + (1-\delta)J_i^{\delta,0} + H_i^{\mathcal{F}_s} \right] \mathcal{F}_s = \tilde{E}_i \left[ \delta Z_i^{\delta,0} + (1-\delta)J_i^{\delta,i,0} \right] \mathcal{F}_s \leq \delta \tilde{E}_i \left[ Z_i^{\delta,0} \right] \mathcal{F}_s + (1-\delta) \tilde{E}_i \left[ J_i^{\delta,i,0} \right] \mathcal{F}_s \leq \delta Z_i^{\delta,0} + (1-\delta)J_i^{\delta,0} + H_i^{\mathcal{F}_s}, \quad \text{a.s.,}
\]

which shows that \( \{ \delta Z_i^{\delta,0} + (1-\delta)J_i^{\delta,0} + H_i^{\mathcal{F}_s} \}_{t \in [0,T]} \) is an RCLL \( \tilde{E}_i \)-supermartingale.

Now we fix \( t \in [0,T] \) and define \( A \triangleq \{ \tau_\delta(t) = t \} \in \mathcal{F}_t \). Using Proposition 4.3 (3), Lemma 3.3 (2) as well as applying Proposition 2.7 (2) and (5) to each \( \tilde{E}_i \), we obtain

\[
1_A J_{\nu,j}^{\delta,0} = 1_A J_{\delta}(t) = 1_A \text{esssup}_{i \in \mathcal{I}} \tilde{E}_i \left[ Z_{\tau_\delta(t)}^{\delta,0} + H_i^{\tau_\delta(t)} \right] \mathcal{F}_t = \text{esssup}_{i \in \mathcal{I}} \tilde{E}_i \left[ 1_A \left( Z_\nu(t) + H_i^{\tau_\delta(t)} \right) \right] \mathcal{F}_t
\]

\[
= \text{esssup}_{i \in \mathcal{I}} \tilde{E}_i \left[ 1_A Z_i^{\delta,0} \right] \mathcal{F}_t = 1_A Z_i^{\delta,0}, \quad \text{a.s.}
\]

Then (4.11) and (4.11) imply that

\[
1_A (\delta Z_i^{\delta,0} + (1-\delta)J_i^{\delta,0}) = 1_A Z_i^{\delta,0} = 1_A Z(t) \geq 1_A Y_t, \quad \text{a.s.} \quad (8.51)
\]
Moreover, we see from the definition of $\tau_\delta(t)$ that for any $\omega \in A^c$

$$Y_s(\omega) < \delta Z^0_s(\omega) + (1 - \delta)(C_Y + 2C_H), \quad \forall s \in [t, \tau_\delta(t)(\omega))$$

(8.52)

Since both $Z^0$ and $Y$ are right-continuous processes, (8.52) and (8.56) imply that

$$Y_t \leq \delta Z^0_t + (1 - \delta)(C_Y + 2C_H) \leq \delta Z^0 + (1 - \delta)J^\delta, \quad a.s. \text{ on } A^c,$$

which in conjunction with (8.51) and Remark 4.3 (2) shows that the RCLL process $\delta Z^0 + (1 - \delta)J^\delta$ dominates $Y$, thus dominates $Z^0$ thanks to Proposition 4.3. It follows that $J^\delta$ also dominates $Z^0$. Then for any $\nu \in \mathcal{S}_{0,T}$, Proposition 4.3 (3), Remark 4.3 (1) and (4.17) imply that $J_\delta(\nu) = J^\delta, \geq Z^0, a.s.$, The reverse inequality comes from (4.15). This proves (8.50).

Next, we fix $\nu \in \mathcal{S}_{0,T}$ and set $\delta^n = \frac{n-1}{n}, n \in \mathbb{N}$. It is clear that the sequence $\{\tau_{\delta^n}(\nu)\}_{n \in \mathbb{N}}$ increasing a.s. to $\tau(\nu)$ and Proposition 4.3 (1) implies that $Y_{\tau_{\delta^n}(\nu)} \geq \delta^n Z^0_{\tau_{\delta^n}(\nu)} + (1 - \delta^n)(C_Y + 2C_H), a.s.$ Hence Proposition 2.4 (3) shows that

$$\lim_{k \to \infty} \text{esssup}_{n \in \mathbb{N}} \left| \tilde{E}_i \left[ \frac{\nu}{\delta^n} Y_{\tau_{\delta^n}(\nu)} + H^1_{\tau_{\delta^n}(\nu)} \right] - \tilde{E}_i \left[ Y_{\tau(\nu)} \right] \right| = 0, \quad a.s.$$  

(8.53)

For any $i \in \mathcal{I}$ and $k \in \mathbb{N}$, Remark 4.3 (1) implies that $Y_{\tau_{\delta^n}(\nu)} \geq \delta^n Z^0_{\tau_{\delta^n}(\nu)} + (1 - \delta^n)(C_Y + 2C_H), a.s.$ Hence Proposition 2.4 (3) shows that

$$\text{esssup}_{i \in \mathcal{I}} \left| \tilde{E}_i \left[ \frac{\nu}{\delta^n} Y_{\tau_{\delta^n}(\nu)} + H^1_{\tau_{\delta^n}(\nu)} \right] - \tilde{E}_i \left[ Y_{\tau(\nu)} \right] \right| = 0, \quad a.s.$$  

(8.54)

Taking the esssup of the left-hand-side over $\mathcal{I}$, we see from (8.50) that

$$\text{esssup}_{i \in \mathcal{I}} \left| \tilde{E}_i \left[ \frac{\nu}{\delta^n} Y_{\tau_{\delta^n}(\nu)} + H^1_{\tau_{\delta^n}(\nu)} \right] - \tilde{E}_i \left[ Y_{\tau(\nu)} \right] \right| \geq J_{\delta^n}(\nu) + \frac{1}{\delta^n}(C_Y + 2C_H) = Z(\nu), a.s.,$$

As $k \to \infty$, (8.53), (4.11) and (4.15) imply that

$$Z(\nu) \leq \text{esssup}_{i \in \mathcal{I}} \tilde{E}_i \left[ Y_{\tau(\nu)} + H^1_{\tau(\nu)} \right] \leq \text{esssup}_{i \in \mathcal{I}} \tilde{E}_i \left[ Z(\tau(\nu)) + H^1_{\tau(\nu)} \right] \leq Z(\nu), \quad a.s.,$$

which shows that

$$Z(\nu) = \text{esssup}_{i \in \mathcal{I}} \tilde{E}_i \left[ Y_{\tau(\nu)} + H^1_{\tau(\nu)} \right] = \text{esssup}_{i \in \mathcal{I}} \tilde{E}_i \left[ Z(\tau(\nu)) + H^1_{\tau(\nu)} \right], \quad a.s.$$  

(8.55)

Now we fix $\rho \in \mathcal{S}_{0,\tau(\nu)}$. For any $i \in \mathcal{I}$, Corollary 4.3 and (4.10) show that

$$\tilde{E}_i \left[ Z^i(\tau(\nu)) \right] = \tilde{E}_i \left[ Z^i(\tau(\nu)) \right] \leq \tilde{E}_i \left[ Z(\rho) \right], \quad a.s.$$  

Then Proposition 2.4 (3) implies that

$$\tilde{E}_i \left[ Z(\tau(\nu)) + H^1_{\tau(\nu)} \right] = \tilde{E}_i \left[ Z(\tau(\nu)) + H^1_{\tau(\nu)} \right] - H^1_{\tau(\nu)} \leq \tilde{E}_i \left[ Z(\rho) + H^1_{\tau(\nu)} \right], \quad a.s.$$  

Taking the essential supremum of both sides over $\mathcal{I}$, we can deduce from (4.15) that

$$\text{esssup}_{i \in \mathcal{I}} \tilde{E}_i \left[ Z(\tau(\nu)) + H^1_{\tau(\nu)} \right] \leq \text{esssup}_{i \in \mathcal{I}} \tilde{E}_i \left[ Z(\rho) + H^1_{\tau(\nu)} \right] \leq Z(\nu), \quad a.s.,$$
which together with (8.53) proves (1.22).

Finally, we will prove that $\tau(\nu) = \tau_1(\nu)$. For any $i \in \mathcal{I}$ and $k \in \mathbb{N}$, (4.17), Proposition 2.7 (3) as well as Corollary 2.3 imply that

$$
\bar{E}_i [Z_{\tau_{1/k}(\nu)}^0 + H_{i, \tau_{1/k}(\nu)}^i | \mathcal{F}_\nu] = \bar{E}_i [Z(\tau_{1/k}(\nu)) + H_{i, \tau_{1/k}(\nu)}^i | \mathcal{F}_\nu] \\
\geq \bar{E}_i [\text{esssup}_{j \in \mathcal{I}} \bar{E}_j Z(\tau(\nu)) + H_{j, \tau_{1/k}(\nu)}^j | \mathcal{F}_\nu] + H_{i, \tau_{1/k}(\nu)}^i | \mathcal{F}_\nu] \\
\geq \bar{E}_i [\text{esssup}_{j \in \mathcal{I}} \bar{E}_j Z(\tau(\nu)) + H_{j, \tau_{1/k}(\nu)}^j | \mathcal{F}_\nu] + H_{i, \tau_{1/k}(\nu)}^i | \mathcal{F}_\nu] \\
= \bar{E}_i [\bar{E}_i Z(\tau(\nu)) + H_{i, \tau(\nu)}^i | \mathcal{F}_\nu] + H_{i, \tau(\nu)}^i | \mathcal{F}_\nu], \quad \text{a.s.,}
$$

which together with (8.53) shows that

$$
\text{esssup}_{i \in \mathcal{I}} \left[ \bar{E}_i \left[ \frac{\nu}{\nu_0} Y_{\tau_{1/k}(\nu)} + H_{i, \tau_{1/k}(\nu)}^i | \mathcal{F}_\nu \right] - \bar{E}_i \left[ Y_{\tau(\nu)}^i | \mathcal{F}_\nu \right] \right] + \bar{E}_i \left[ Y_{\tau(\nu)} + H_{i, \tau(\nu)}^i | \mathcal{F}_\nu \right] \\
\geq \bar{E}_i [Z(\tau(\nu)) + H_{i, \tau(\nu)}^i | \mathcal{F}_\nu] + \frac{1}{\eta - 1} (C_Y + 2C_H), \quad \text{a.s.}
$$

As $k \to \infty$, (8.53) implies that

$$
\bar{E}_i [Y_{\tau(\nu)} + H_{i, \tau(\nu)}^i | \mathcal{F}_\nu] \geq \bar{E}_i [Z(\tau(\nu)) + H_{i, \tau(\nu)}^i | \mathcal{F}_\nu], \quad \text{a.s.}
$$

The reverse inequality follows easily from (4.11), thus (8.56) is in fact an equality. Then the second part of Proposition 2.7 (1) and (4.17) imply that

$$
Y_{\tau(\nu)} = Z(\tau(\nu)) = Z_{\tau_1(\nu)}, \quad \text{a.s.,}
$$

which shows that $\inf \{ t \in [\nu, T] : Z_t^0 = Y_t \} \leq \tau(\nu), \text{a.s.}$ For any $\delta \in (0, 1)$, since $\{ t \in [\nu, T] : Y_t \geq \delta Z_t^0 + (1 - \delta) (C_Y + 2C_H) \}$, one can deduce that

$$
\tau(\nu) \geq \inf \{ t \in [\nu, T] : Z_t^0 = Y_t \} \geq \inf \{ t \in [\nu, T] : Y_t \geq \delta Z_t^0 + (1 - \delta) (C_Y + 2C_H) \} \wedge T = \tau_\delta(\nu), \quad \text{a.s.}
$$

Letting $\delta \to 1$ yields that

$$
\tau(\nu) \geq \inf \{ t \in [\nu, T] : Z_t^0 = Y_t \} \geq \lim_{\delta \to 1} \tau_\delta(\nu) = \tau(\nu), \quad \text{a.s.,}
$$

which implies that $\tau(\nu) = \inf \{ t \in [\nu, T] : Z_t^0 = Y_t \}, \text{a.s.}$

$\square$

8.4 Proofs of Section 5

Proof of Lemma 5.1: In light of Never [1975, Proposition VI-1-1], it suffices to show that the family $\{ R^i(\nu) \}_{i \in \mathcal{I}}$ is directed downwards, i.e.,

$$
\text{for any } i, j \in \mathcal{I}, \text{there exists a } k \in \mathcal{I} \text{ such that } R^k(\nu) \leq R^i(\nu) \wedge R^j(\nu), \quad \text{a.s.}
$$

To see this, we define the event $A \triangleq \{ R^i(\nu) \geq R^j(\nu) \}$ and the stopping times

$$
\rho \triangleq \tau^i(\nu)1_A + \tau^j(\nu)1_{A^c} \in \mathcal{S}_{\nu,T} \quad \text{and} \quad \nu(A) \triangleq \nu 1_A + T 1_{A^c} \in \mathcal{S}_{\nu,T}.
$$

By Definition 3.2, there exists a $k = k(i, j, \nu(A)) \in \mathcal{I}$ such that $\bar{E}_k = \mathcal{E}^{\nu(A)}_{i,j}$. Similar to (8.22) it holds for any $\xi \in \text{Dom}(\mathcal{E})$ that

$$
\bar{E}_k [\xi | \mathcal{F}_\nu] = 1_A \bar{E}_j [\xi | \mathcal{F}_\nu] + 1_{A^c} \bar{E}_i [\xi | \mathcal{F}_\nu], \quad \text{a.s.}
$$

Moreover, (4.5) implies that

$$
H_{i, \rho}^k = H_{i, \nu(A)}^i \wedge \nu(A) \wedge \rho + H_{j, \nu(A)}^j \wedge \nu(A) \wedge \rho = 1_A H_{i, \tau^i(\nu)}^i 1_A + H_{j, \tau^j(\nu)}^j 1_{A^c}, \quad \text{a.s.}
$$

and that

$$
H_{i, \tau^k(\nu)}^k = H_{i, \nu(A)}^i \wedge \nu(A) \wedge \tau^k(\nu) + H_{j, \nu(A)}^j \wedge \nu(A) \wedge \tau^k(\nu) = 1_A H_{i, \tau^k(\nu)}^i 1_A + H_{j, \tau^k(\nu)}^j 1_{A^c}, \quad \text{a.s.}
$$
Using (8.57) twice and applying Proposition 2.7 (2) to $\tilde{\epsilon}_i$ and $\tilde{\epsilon}_j$, we can deduce from (5.7) that

$$R^k(\nu) \geq \tilde{\epsilon}_i [Y_\nu + H^k_{\nu,\rho} | F_\nu] = 1_A \tilde{\epsilon}_j [Y_\nu + H^k_{\nu,\rho} | F_\nu] + 1_{A^c} \tilde{\epsilon}_i [Y_\nu + H^k_{\nu,\rho} | F_\nu]$$

$$= \tilde{\epsilon}_i [1_A Y_{\tau^j(\nu)} + 1_A H^j_{\nu,\tau^j(\nu)} | F_\nu] + \tilde{\epsilon}_i [1_{A^c} Y_{\tau^j(\nu)} + 1_{A^c} H^j_{\nu,\tau^j(\nu)} | F_\nu]$$

$$\geq \tilde{\epsilon}_i [1_A Y_{\tau^j(\nu)} + 1_A H^j_{\nu,\tau^j(\nu)} | F_\nu] + 1_{A^c} \tilde{\epsilon}_i [Y_{\tau^j(\nu)} + H^j_{\nu,\tau^j(\nu)} | F_\nu]$$

$$= \tilde{\epsilon}_i [1_A Y_{\tau^j(\nu)} + 1_A H^j_{\nu,\tau^j(\nu)} | F_\nu] + \tilde{\epsilon}_i [1_{A^c} Y_{\tau^j(\nu)} + 1_{A^c} H^j_{\nu,\tau^j(\nu)} | F_\nu]$$

$$= \tilde{\epsilon}_i [1_A Y_{\tau^j(\nu)} + 1_A H^j_{\nu,\tau^j(\nu)} | F_\nu] + \tilde{\epsilon}_i [1_{A^c} Y_{\tau^j(\nu)} + 1_{A^c} H^j_{\nu,\tau^j(\nu)} | F_\nu]$$

which shows that $R^k(\nu) = 1_A R^j(\nu) + 1_{A^c} R^j(\nu) = R^j(\nu) \wedge R^j(\nu)$, a.s. In light of the basic properties of the essential infimum (e.g., Neveu [1975], Proposition VI-1.1), we can find a sequence $\{i_n\}_{n \in \mathbb{N}}$ in $\mathcal{I}$ such that (5.8) holds. □

**Proof of Lemma 5.2** As in the proof of Lemma 5.1, it suffices to show that the family $\{\tau^i(\nu)\}_{i \in \mathcal{I}}$ is directed downwards, i.e.,

$$\text{for any } i, j \in \mathcal{I}, \text{ there exists a } k \in \mathcal{I} \text{ such that } \tau^k(\nu) \leq \tau^i(\nu) \wedge \tau^j(\nu), \text{ a.s.} \quad (8.58)$$

To see this, we define the stopping time $\sigma \triangleq \tau^i(\nu) \wedge \tau^j(\nu) \in S_{\sigma, T}$, the event $A \triangleq \{R^i_0 \geq R^j_0\} \in \mathcal{F}_\sigma$ as well as the stopping time $\sigma(\nu) \triangleq \sigma \wedge [T \wedge \nu \in S_{\nu, T}$. By Definition 5.2 there exists a $k = k(i, j, \sigma(\nu)) \in \mathcal{I}$ such that $\tilde{\epsilon}_k = \tilde{\epsilon}_{\sigma(\nu)}$. Fix $t \in [0, T]$, similar to (8.22), it holds for any $\xi \in Dom(\mathcal{F}_\sigma)$ that

$$\tilde{\epsilon}_k[\xi|\mathcal{F}_{\sigma(t)}] = 1_A \tilde{\epsilon}_i[\xi|\mathcal{F}_{\sigma(t)}] + 1_{A^c} \tilde{\epsilon}_i[\xi|\mathcal{F}_{\sigma(t)}], \text{ a.s.} \quad (8.59)$$

Moreover, we can deduce from (4.3) that for any $\rho \in S_{\sigma(t), T}$

$$H^k_{\sigma(t), \rho} \geq H^j_{\sigma(t), \rho} + H^j_{\sigma(t), \rho} , \text{ a.s.,} \quad (8.49)$$

which together with (8.49) and Proposition 2.7 (2) imply that

$$\tilde{\epsilon}_k [Y_\rho + H^k_{\sigma(t), \rho} | \mathcal{F}_{\sigma(t)}] = 1_A \tilde{\epsilon}_i [Y_\rho + H^k_{\sigma(t), \rho} | \mathcal{F}_{\sigma(t)}] + 1_{A^c} \tilde{\epsilon}_i [Y_\rho + H^k_{\sigma(t), \rho} | \mathcal{F}_{\sigma(t)}]$$

Then applying Proposition 2.7 (3), Lemma 5.3 (2) as well as (5.6), we obtain

$$R^{k, 0}_{\sigma(t)} = R^k(\sigma \vee t) = \text{ess sup}_{\rho \in S_{\sigma(t), T}} \tilde{\epsilon}_k [Y_\rho + H^k_{\sigma(t), \rho} | \mathcal{F}_{\sigma(t)}]$$

$$= 1_A \text{ ess sup}_{\rho \in S_{\sigma(t), T}} \tilde{\epsilon}_i [Y_\rho + H^j_{\sigma(t), \rho} | \mathcal{F}_{\sigma(t)}] + 1_{A^c} \text{ ess sup}_{\rho \in S_{\sigma(t), T}} \tilde{\epsilon}_i [Y_\rho + H^j_{\sigma(t), \rho} | \mathcal{F}_{\sigma(t)}]$$

$$= 1_A R^j(\sigma \vee t) + 1_{A^c} R^j(\sigma \vee t) = 1_A R^{j, 0}_{\sigma(t)} + 1_{A^c} R^{j, 0}_{\sigma(t)}, \text{ a.s.}$$

Since $R^{i, 0}$, $R^{j, 0}$ and $R^{k, 0}$ are all RCLL processes, it holds except on a null set $N$ that

$$R^{k, 0}_{\sigma(t)} = 1_A R^{k, 0}_{\sigma(t)} + 1_{A^c} R^{k, 0}_{\sigma(t)}, \forall t \in [0, T],$$

which further implies that

$$\tau^k(\nu) = \inf \{t \in [\nu, T] : R^k_{\nu} = Y_t\} \leq \inf \{t \in [\sigma, T] : R^k_{\nu} = Y_t\}$$

$$= 1_A \inf \{t \in [\sigma, T] : R^i_{\nu} = Y_t\} + 1_{A^c} \inf \{t \in [\sigma, T] : R^j_{\nu} = Y_t\}, \text{ a.s.} \quad (8.60)$$
Since \( R_{\tau,\nu}^{l,0} = Y_{\tau,\nu} \), \( R_{\tau,\nu}^{0,0} = Y_{\tau,\nu} \), a.s. and since \( \sigma = \tau(\nu) \cap \tau(\nu) \), it holds a.s. that \( Y_{\sigma} \) is equal either to \( R_{\sigma}^{0,0} \) or to \( R_{\sigma}^{0,0} \). Then the definition of the set \( A \) shows that \( R_{\sigma}^{0,0} = Y_{\sigma} \) a.s. on \( A \) and that \( R_{\sigma}^{0,0} = Y_{\sigma} \) a.s. on \( A^{c} \), both of which further implies that

\[
1_A \inf \left\{ t \in [\sigma, T] : R_{\nu}^{0,0} = Y_t \right\} = \sigma 1_A \quad \text{and} \quad 1_A \inf \left\{ t \in [\sigma, T] : R_{\nu}^{0,0} = Y_t \right\} = \sigma 1_A, \quad \text{a.s.}
\]

Hence, we see from (8.62) that \( \tau^k(\nu) \leq \sigma = \tau(\nu) \cap \tau(\nu) \), a.s., proving (8.64). Thanks to the basic properties of the essential infimum (e.g., Neveu [1975], Proposition VI-1-1), we can find a sequence \( \left\{ j_n \right\}_{n \in \mathbb{N}} \) in \( \mathcal{I} \) such that

\[
\mathcal{I}(\nu) = \inf_{i \in \mathcal{I}} \tau^i(\nu) = \lim_{n \to \infty} \tau^{j_n}(\nu), \quad \text{a.s.}
\]

The limit \( \lim_{n \to \infty} \tau^{j_n}(\nu) \) is also a stopping time, thus we have \( \mathcal{I}(\nu) \in \mathcal{S}_{\nu,T} \).

\[\Box\]

**Proof of Theorem 6.1**: In light of Lemma 5.2 there exists a sequence \( \left\{ j_n \right\}_{n \in \mathbb{N}} \) in \( \mathcal{I} \) such that

\[
\mathcal{I}(\nu) = \lim_{n \to \infty} \tau^{j_n}(\nu), \quad \text{a.s.}
\]

Since the family of processes \( \left\{ Y^i \right\}_{i \in \mathcal{I}} \) is “\( \mathcal{E} \)-uniformly-right-continuous”, we can find a subsequence of \( \left\{ j_n \right\}_{n \in \mathbb{N}} \) (we still denote it by \( \left\{ j_n \right\}_{n \in \mathbb{N}} \)) such that

\[
\lim_{n \to \infty} \text{esssup}_{i \in \mathcal{I}} \left| \tilde{E}_i \left[ Y_{\tau^{j_n}(\nu)}^i \right] \mathcal{F}_{\mathcal{I}(\nu)}[\mathcal{F}_{\mathcal{I}(\nu)}] - Y_{\mathcal{I}(\nu)}^i \right| = 0, \quad \text{a.s.} \quad (8.61)
\]

Fix \( i \in \mathcal{I} \) and \( n \in \mathbb{N} \), we know from Definition 3.2 that there exists a \( k_n = k(i, j_n, \mathcal{I}(\nu)) \in \mathcal{I} \) such that \( \mathcal{E}_{k_n} = \mathcal{E}_{k_n}^{\mathcal{I}(\nu)} \).

For any \( t \in [0, T] \), Lemma 3.3 implies that \( R_{\nu}^{k_n,0} = R_{\nu}^{j_n,0} \), a.s. Since \( R_{\nu}^{k_n,0} \) and \( R_{\nu}^{j_n,0} \) are both RCLL processes, it holds except on a null set \( N \) that

\[
R_{\nu}^{k_n,0} \mathcal{F}_{\mathcal{I}(\nu)}[\mathcal{F}_{\mathcal{I}(\nu)}] = R_{\nu}^{j_n,0} \mathcal{F}_{\mathcal{I}(\nu)}[\mathcal{F}_{\mathcal{I}(\nu)}], \quad \forall t \in [0, T],
\]

which together with the fact that \( \mathcal{I}(\nu) \leq \tau^{k_n}(\nu) \cap \tau^{j_n}(\nu) \), a.s. implies that

\[
\tau^{k_n}(\nu) = \inf \left\{ t \in [\nu, T] : R_{\nu}^{k_n,0} = Y_t \right\} = \inf \left\{ t \in [\mathcal{I}(\nu), T] : R_{\nu}^{k_n,0} = Y_t \right\}
\]

\[
= \inf \left\{ t \in [\nu, T] : R_{\nu}^{j_n,0} = Y_t \right\} = \inf \left\{ t \in [\mathcal{I}(\nu), T] : R_{\nu}^{j_n,0} = Y_t \right\} = \tau^{j_n}(\nu), \quad \text{a.s.} \quad (8.62)
\]

Then (4.5), (8.62) and (8.3) show that

\[
R_{\nu}^{k_n}(\nu) + H_{\nu} = R_{\nu}^{k_n}(\nu) + H_{\nu}^{k_n} = \tilde{E}_{k_n} \left[ Y_{\tau^{k_n}(\nu)} + H_{\nu}^{k_n} \right] \mathcal{F}_{\mathcal{I}(\nu)}[\mathcal{F}_{\mathcal{I}(\nu)}] + H_{\nu}^{k_n}
\]

\[
= \tilde{E}_{k_n} \left[ Y_{\tau^{k_n}(\nu)} \right] \mathcal{F}_{\mathcal{I}(\nu)}[\mathcal{F}_{\mathcal{I}(\nu)}] + \tilde{E}_{k_n} \left[ H_{\nu}^{k_n} \right] \mathcal{F}_{\mathcal{I}(\nu)}[\mathcal{F}_{\mathcal{I}(\nu)}] = \tilde{E}_{k_n} \left[ Y_{\tau^{k_n}(\nu)} \right] \mathcal{F}_{\mathcal{I}(\nu)}[\mathcal{F}_{\mathcal{I}(\nu)}] + \tilde{E}_{k_n} \left[ H_{\nu}^{k_n} \right] \mathcal{F}_{\mathcal{I}(\nu)}[\mathcal{F}_{\mathcal{I}(\nu)}]
\]

\[
\leq \tilde{E}_{i} \left[ Y_{\tau^{j_n}(\nu)} \right] \mathcal{F}_{\mathcal{I}(\nu)}[\mathcal{F}_{\mathcal{I}(\nu)}] + \tilde{E}_{i} \left[ H_{\nu}^{j_n} \right] \mathcal{F}_{\mathcal{I}(\nu)}[\mathcal{F}_{\mathcal{I}(\nu)}] \leq \tilde{E}_{i} \left[ \text{esssup}_{i \in \mathcal{I}} \left| \tilde{E}_i \left[ Y_{\tau^{j_n}(\nu)} \right] \mathcal{F}_{\mathcal{I}(\nu)}[\mathcal{F}_{\mathcal{I}(\nu)}] - Y_{\mathcal{I}(\nu)}^i \mathcal{F}_{\mathcal{I}(\nu)}[\mathcal{F}_{\mathcal{I}(\nu)}] \right| \right], \quad \text{a.s.} \quad (8.63)
\]

For any \( l \in \mathcal{I} \), Proposition 27 (3), (4.6), (4.5) and (5.3) imply that

\[
\left| \tilde{E}_l \left[ Y_{\tau^{j_n}(\nu)} \right] \mathcal{F}_{\mathcal{I}(\nu)}[\mathcal{F}_{\mathcal{I}(\nu)}] - Y_{\mathcal{I}(\nu)}^i \right| \leq \tilde{E}_l \left[ \text{esssup}_{i \in \mathcal{I}} \left| \tilde{E}_i \left[ Y_{\tau^{j_n}(\nu)} \right] \mathcal{F}_{\mathcal{I}(\nu)}[\mathcal{F}_{\mathcal{I}(\nu)}] - Y_{\mathcal{I}(\nu)}^i \mathcal{F}_{\mathcal{I}(\nu)}[\mathcal{F}_{\mathcal{I}(\nu)}] \right| \right] \leq 2C_s \leq 2R_{\nu}^{l}(\nu) - 2C_s, \quad \text{a.s.}
\]

Taking the essential supremum over \( l \in \mathcal{I} \), we can deduce from (4.8) and (5.3) that

\[
C_s \leq \text{esssup}_{l \in \mathcal{I}} \left| \tilde{E}_l \left[ Y_{\tau^{j_n}(\nu)} \right] \mathcal{F}_{\mathcal{I}(\nu)}[\mathcal{F}_{\mathcal{I}(\nu)}] - Y_{\mathcal{I}(\nu)}^i \right| + Y_{\mathcal{I}(\nu)}^i \leq 3R_{\nu}^{l}(\nu) - 2C_s + H_{\nu}^{j_n}, \quad \text{a.s.}
\]
where $3R_l^i(\mathcal{I}(\nu)) - 2C_\nu + H_{\nu,\nu}(\nu) \in \text{Dom}(\mathcal{R})$ thanks to Proposition 5.1 (1), (S1') and (D2). Applying the Dominated Convergence Theorem (Proposition 2.9) and Proposition 2.7 (3), we can deduce from (8.63) and (8.64) that

$$
\nabla(\nu) = \text{essinf} R^i(\nu) \leq \beta \lim_{n \to \infty} R^{k_n}(\nu) \leq \lim_{n \to \infty} \tilde{E}[\epsilon_{i} Y_{i}^{j_n}](\nu) \mathcal{F}_{\nu} = \tilde{E}[\epsilon_{i} Y_{i}^{j_n}](\nu) \mathcal{F}_{\nu} - H_{\nu}^i
$$

Taking the essential infimum of the right-hand-side over $i \in \mathcal{I}$ yields that

$$
\nabla(\nu) \leq \text{essinf} \tilde{E}[Y_{i}^{j_n}](\nu) + H_{\nu,\nu}(\nu) \mathcal{F}_{\nu} \leq \epsilon_{i} \tilde{E}[Y_{i}^{j_n}](\nu) \mathcal{F}_{\nu}, \quad \text{a.s.}
$$

Hence, we have

$$
\nabla(\nu) = \text{essinf} \tilde{E}[Y_{i}^{j_n}](\nu) \mathcal{F}_{\nu} = \nabla(\nu) = \text{essinf} R^i(\nu) \geq Y_{\nu}, \quad \text{a.s.}
$$

where the last inequality is due to (5.5).

**Proof of Proposition 5.2** By Lemma 5.2 there exists a sequence $\{\nu_n\}_{n \in \mathbb{N}}$ in $\mathcal{I}$ such that

$$
\sigma \triangleq \underline{\mathcal{I}}(\nu) = \lim_{n \to \infty} \frac{\nu_n}{\nu}, \quad \text{a.s.}
$$

For any $n \in \mathbb{N}$, since $\sigma \leq \nu_n(\nu)$, a.s., we have

$$
\nu_n(\nu) = \inf \{t \in [\nu, T] : R^{i_n}_t = Y_t\} = \inf \{t \in [\sigma, T] : R^{i_n}_t = Y_t\} = \nu(\sigma), \quad \text{a.s.}
$$

Then (5.5) and (5.7) imply that

$$
V(\sigma) = \nabla(\sigma) \leq R^{i_n}(\sigma) = \tilde{E}[Y_{i}^{j_n}(\nu) + H_{\nu,\nu}(\nu) \mathcal{F}_{\nu}] = \tilde{E}[Y_{i}^{j_n}(\nu) \mathcal{F}_{\nu}] - H_{\nu}^i
$$

As $n \to \infty$, the "$\mathcal{R}$-uniform-right-continuity" of $\{Y^i\}_{i \in \mathcal{I}}$ implies that $V(\sigma) \leq \nu(\sigma)$, a.s., while the reverse inequality is obvious from (5.9).

**Proof of Proposition 5.3** In light of Lemma 5.1 and (5.9), there exists a sequence $\{\nu_n\}_{n \in \mathbb{N}}$ in $\mathcal{I}$ such that

$$
V(\nu) = \nabla(\nu) = \lim_{n \to \infty} \frac{R^{i_n}(\nu)}{\nu}, \quad \text{a.s.}
$$

For any $n \in \mathbb{N}$, Definition 5.2 assures a $k_n = k(i, j_n, \nu) \in \mathcal{I}$ such that $\tilde{E}_{k_n} = E_{i,j_n}$. Applying Proposition 2.7 (5) to $\tilde{E}_{i}'$, we can deduce from (5.3) and (5.5) that

$$
\tilde{E}_{k_n}[V(\nu) + H_{\nu,\nu}^j \mathcal{F}_{\nu}] \leq \tilde{E}_{k_n}[R^{j_n}(\nu) + H_{\nu,\nu}^j \mathcal{F}_{\nu}] = \tilde{E}_{i,j_n}[R^{j_n}(\nu) + H_{\nu,\nu}^j \mathcal{F}_{\nu}] = \tilde{E}_{i,j_n}[R^{j_n}(\nu) + H_{\nu,\nu}^j \mathcal{F}_{\nu}]
$$

Then Proposition 2.7 (3) and (5.5) imply that

$$
\text{essinf}_{k \in \mathcal{I}} \tilde{E}_{k}[V^k(\nu) \mathcal{F}_{\nu}] \leq \tilde{E}_{k_n}[V(\nu) + H_{\nu,\nu}^j \mathcal{F}_{\nu}] + H_{\nu}^i \leq R^{j_n}(\nu) + H_{\nu}^i, \quad \text{a.s.}
$$

As $n \to \infty$, (5.10) follows:

$$
\text{essinf}_{k \in \mathcal{I}} \tilde{E}_{k}[V^k(\nu) \mathcal{F}_{\nu}] \leq \lim_{n \to \infty} R^{j_n}(\nu) + H_{\nu}^i = V(\nu) + H_{\nu}^i = V(\nu), \quad \text{a.s.}
$$

Now we assume that $\nu \leq \rho \leq \underline{\mathcal{I}}(\nu)$, a.s. Applying Lemma 5.1 and (5.9) once again, we can find another sequence $\{\nu_n'\}_{n \in \mathbb{N}}$ in $\mathcal{I}$ such that

$$
V(\rho) = \nabla(\rho) = \lim_{n \to \infty} R^{j_n'}(\rho), \quad \text{a.s.}
$$
For any \( n \in \mathbb{N} \), Definition 5.2 assures a \( k_{n}^{i} = k(i, j_{n}^{i}, \rho) \in I \) such that \( \tilde{E}_{k_{n}^{i}} = \mathcal{E}_{i, j_{n}^{i}}^{\rho} \). Since \( \rho \leq \nu(\tau) \leq \nu(k_{n}^{i}(\tau)) \), a.s., using (5.7) with \( i = k_{n}^{i} \) and applying Proposition 2.7 (5) to \( \tilde{E}_{j_{n}^{i}} \), we can deduce from (4.3), (3.3) as well as Lemma 5.3 that

\[
V^{i}(\nu) = V(\nu) + H_{\rho}^{i}(\nu) = H_{\rho}^{i}(\nu) + H_{\rho}^{i}(\nu) = \tilde{E}_{i}^{\nu} \left[ R^{k_{n}^{i}}(\rho) + H_{\rho}^{i} \big| \mathcal{F}_{\nu} \right] = \tilde{E}_{i}^{\nu} \left[ R^{k_{n}^{i}}(\rho) + H_{\rho}^{i} \big| \mathcal{F}_{\nu} \right], \quad a.s. \quad (8.64)
\]

Then (4.8) and (5.3) imply that

\[
C_{\ast} \leq Y_{\rho}^{i} = Y_{\rho} + H_{\rho}^{i} \leq R^{k_{n}^{i}}(\rho) = R^{k_{n}^{i}}(\rho) + H_{\rho}^{i}, \quad a.s.,
\]

where \( R^{k_{n}^{i}}(\rho) \in Dom(\mathcal{E}) \) thanks to Proposition 5.1 (1), (S1') and (D2). As \( n \to \infty \) in (8.64), the Dominated Convergence Theorem (Proposition 2.9) imply that

\[
V^{i}(\nu) \leq \lim_{n \to \infty} \tilde{E}_{i}^{\nu} \left[ R^{k_{n}^{i}}(\rho) + H_{\rho}^{i} \big| \mathcal{F}_{\nu} \right] = \tilde{E}_{i}^{\nu} \left[ V(\rho) + H_{\rho}^{i} \big| \mathcal{F}_{\nu} \right] = \tilde{E}_{i}^{\nu} \left[ V^{i}(\rho) \big| \mathcal{F}_{\nu} \right], \quad a.s.,
\]

which proves (5.11).

It remains to show that \( \{V^{i}(\tilde{\tau}(0) \wedge t)\}_{t \in [0, T]} \) is an \( \tilde{E}_{i} \)-submartingale: To see this, we fix \( 0 \leq s < t \leq T \) and set \( \nu = \tilde{\tau}(0) \wedge s \), \( \rho = \tilde{\tau}(0) \wedge t \). It is clear that \( \nu \leq \rho \leq \tilde{\tau}(0) \leq \tilde{\tau}(\nu) \), a.s., hence (5.11), Corollary 2.3 and Proposition 2.7 (5) show that

\[
V^{i}(\tilde{\tau}(0) \wedge s) = V^{i}(\nu) \leq \tilde{E}_{i}^{\nu} \left[ V^{i}(\rho) \big| \mathcal{F}_{\nu} \right] = \tilde{E}_{i}^{\nu} \left[ V^{i}(\tilde{\tau}(0) \wedge t) \big| \mathcal{F}_{\nu} \right] \leq \tilde{E}_{i}^{\nu} \left[ V^{i}(\tilde{\tau}(0) \wedge t) \big| \mathcal{F}_{\nu} \right], \quad a.s.,
\]

which implies that \( \{V^{i}(\tilde{\tau}(0) \wedge t)\}_{t \in [0, T]} \) is an \( \tilde{E}_{i} \)-submartingale. \( \square \)

**Proof of Theorem 5.2:** Proof of (1).

**Step 1:** For any \( \rho, \nu \in \mathcal{S}_{0,T} \), we define

\[
\Psi^{\rho}(\nu) \overset{\Delta}{=} \text{essinf}_{i \in I} \tilde{E}_{i}^{\nu} \left[ Y_{\rho} + H_{\rho \land \nu, \rho}^{i} \big| \mathcal{F}_{\rho \land \nu} \right] + H_{\rho \land \nu}^{i} \in \mathcal{F}_{\rho \land \nu}.
\]

It follows from (4.7), (4.4), and Proposition 2.7 (5) that

\[
C_{\nu} + 2CH = \text{essinf}_{i \in I} \tilde{E}_{i}^{\nu} \left[ C_{\nu} + C_{H} \big| \mathcal{F}_{\rho \land \nu} \right] + C_{H} \leq \Psi^{\rho}(\nu) \leq \tilde{E}_{i}^{\nu} \left[ Y_{\rho} + H_{\rho \land \nu}^{i} \big| \mathcal{F}_{\rho \land \nu} \right] + H_{\rho \land \nu}^{i} \leq R^{i}(\rho \land \nu) + H_{\rho \land \nu}^{i}, \quad a.s., \quad (8.65)
\]

where \( R^{i}(\rho \land \nu) + H_{\rho \land \nu}^{i} \in Dom(\mathcal{E}) \) thanks to Proposition 5.1 (1), (S1') and (D2). Then Lemma 3.2 implies that \( \Psi^{\rho}(\nu) \in Dom(\mathcal{E}) \). Applying Proposition 2.7 (2) & (3) as well as Lemma 3.3 we can alternatively rewrite \( \Psi^{\rho}(\nu) \) as follows:

\[
\Psi^{\rho}(\nu) - H_{\rho \land \nu}^{i} = \text{essinf}_{i \in I} \left( 1_{\{\rho \leq \nu\}} Y_{\rho \land \nu} + 1_{\{\rho > \nu\}} \left( Y_{\rho} + H_{\rho \land \nu}^{i} \right) \right) \bigg| \mathcal{F}_{\rho \land \nu} \bigg] = \text{essinf}_{i \in I} \left( 1_{\{\rho \leq \nu\}} Y_{\rho} + 1_{\{\rho > \nu\}} \left( Y_{\rho} + H_{\rho \land \nu}^{i} \right) \right) \bigg| \mathcal{F}_{\rho \land \nu} \bigg], \quad a.s.
\]

Let \( \rho \in \mathcal{S}_{0,T} \). Lemma 3.3 (2) and Proposition 2.7 (2) once again imply that

\[
1_{\{\nu = \rho\}} \Psi^{\rho}(\nu) = 1_{\{\rho \leq \nu = \rho\}} Y_{\rho} + 1_{\{\rho > \nu = \rho\}} \text{essinf}_{i \in I} \left( Y_{\rho} + H_{\rho \land \nu}^{i} \right) \bigg| \mathcal{F}_{\rho \land \nu} \bigg] = 1_{\{\rho \leq \nu = \rho\}} Y_{\rho} + 1_{\{\rho > \nu = \rho\}} \text{essinf}_{i \in I} \left( Y_{\rho} + H_{\rho \land \nu}^{i} \right) \bigg| \mathcal{F}_{\rho \land \nu} \bigg] = 1_{\{\rho \leq \nu = \rho\}} Y_{\rho} + 1_{\{\rho > \nu = \rho\}} \text{essinf}_{i \in I} \left( Y_{\rho} + H_{\rho \land \nu}^{i} \right) \bigg| \mathcal{F}_{\rho \land \nu} \bigg], \quad a.s. \quad (8.66)
\]
8.4 Proofs of Section 5

Step 2: Fix $\rho \in \mathcal{S}_{0,T}$. For any $\nu \in \mathcal{S}_{0,T}$ and $\sigma \in \mathcal{S}_{0,T}$, letting $(\nu, \mathcal{I}', \mathcal{U}) = (\rho \land \sigma, \mathcal{I}, \{\rho\})$ and $X(\rho) = Y_\rho$ in Lemma 4.2, we can find a sequence $\{j_n\}_{n \in \mathbb{N}}$ in $\mathcal{I}$ such that

$$\text{esssup}_{i \in \mathcal{I}} \bar{E}_i [Y_\rho + H_{\rho \land \sigma,\rho}^j | \mathcal{F}_{\rho \land \sigma}] = \lim_{n \to \infty} \bar{E}_{j_n} [Y_\rho + H_{\rho \land \sigma,\rho}^j | \mathcal{F}_{\rho \land \sigma}], \quad \text{a.s.}$$

Definition 3.2 assures the existence of a $k_n = k(j, j_n, \rho \land \sigma) \in \mathcal{I}$ such that $\bar{E}_{j_n} = \mathcal{E}_{j_n}^{\rho \land \sigma}$. Applying Proposition 2.7 (3) to $\bar{E}_{j_n}$, we can deduce from (4.3) and (4.5) that

$$\Psi^\rho(\nu) \leq \bar{E}_{j_n} [Y_\rho + H_{\rho \land \nu,\rho}^{k_n} | \mathcal{F}_{\rho \land \nu}] + H_{\rho \land \nu}^j = \bar{E}_{j_n} [Y_\rho + H_{\rho \land \sigma,\rho}^{k_n} | \mathcal{F}_{\rho \land \sigma}] + H_{\rho \land \nu}^j = \bar{E}_{j_n} [Y_{\rho \land \nu}^k | \mathcal{F}_{\rho \land \nu}], \quad \text{a.s.}$$

For any $n \in \mathbb{N}$, Proposition 2.7 (3) & (5), (4.8) as well as (4.5) imply that

$$C_s = \bar{E}_{j_n} [C_s | \mathcal{F}_{\rho \land \sigma}] \leq \bar{E}_{j_n} [Y_{\rho \land \nu}^{k_n} | \mathcal{F}_{\rho \land \sigma}] = \bar{E}_{j_n} [Y_\rho + H_{\rho \land \sigma,\rho}^{j_n} | \mathcal{F}_{\rho \land \sigma}] + H_{\rho \land \nu}^j \leq \bar{E}_{j_n} [Y_\rho + H_{\rho \land \sigma,\rho}^{j_n} | \mathcal{F}_{\rho \land \sigma}] + H_{\rho \land \nu}^j \leq \bar{E}_{j_n} [Y_\rho + H_{\rho \land \sigma,\rho}^{j_n} | \mathcal{F}_{\rho \land \sigma}] + H_{\rho \land \nu}^j, \quad \text{a.s.},$$

where $R_{\rho \land \sigma}^j + H_{\rho \land \nu}^j \in \text{Dom}(\mathcal{E}')$ thanks to Proposition 5.1 (1), (S1') and (D2). Then the Dominated Convergence Theorem (Proposition 2.9, Corollary 2.3 and Proposition 2.7 (5) show that

$$\Psi^\rho(\nu) \leq \lim_{n \to \infty} \bar{E}_{j_n} [\Psi(\nu)] = \lim_{n \to \infty} \bar{E}_{j_n} [\Psi(\nu)] = \lim_{n \to \infty} \bar{E}_{j_n} [\Psi(\nu)] = \lim_{n \to \infty} \bar{E}_{j_n} [\Psi(\nu)], \quad \text{a.s.},$$

which implies that $\{\Psi^\rho(t)\}_{t \in [0,T]}$ is an $\bar{E}$-submartingale. Hence, $\{-\Psi^\rho(t)\}_{t \in [0,T]}$ is an $\mathcal{E}'$-supermartingale by assumption 5.12. Since $\mathcal{E}'$ satisfies (H0), (H1), (2.3) and since $\text{Dom}(\mathcal{E}') \in \hat{D}_T$ (which results from $\text{Dom}(\mathcal{E}) \in \hat{D}_T$ and 5.12), we know from Theorem 2.3 that $\Psi^\rho(\nu) = \lim_{n \to \infty} \Psi^\rho(q_n^+(t))$, $t \in [0,T]$ is an RCLL process and that

$$P\left(\Psi^\rho(\nu) = \lim_{n \to \infty} \Psi^\rho(q_n^+(t)) \right) = 1. \quad (8.68)$$

Step 3: For any $\nu \in \mathcal{S}_{0,T}$ and $n \in \mathbb{N}$, $\Psi^\rho(\nu)$ takes values in a finite set $\mathcal{D}_n \subseteq \mathcal{D}_n^{\text{fin}}$. Given an $\alpha \in \mathcal{D}_n$, it holds for any $m \geq n$ that $q_m^+(\alpha) = \alpha$ since $\mathcal{D}_m \subseteq \mathcal{D}_n^{\text{fin}}$. It follows from (8.68) that

$$\lim_{m \to \infty} \Psi^\rho(\alpha) = \lim_{m \to \infty} \Psi^\rho(q_m^+(\alpha)) = \Psi^\rho(\alpha), \quad \text{a.s.}$$

Then one can deduce from (8.68) that

$$\Psi^\rho(q_m^+(\nu)) = \sum_{\alpha \in \mathcal{D}_n^{\text{fin}}} 1_{\{\nu = \alpha\}} \Psi^\rho(\alpha) = \sum_{\alpha \in \mathcal{D}_n^{\text{fin}}} 1_{\{\nu = \alpha\}} \Psi^\rho(\alpha) = \sum_{\alpha \in \mathcal{D}_n^{\text{fin}}} 1_{\{\nu = \alpha\}} \Psi^\rho(q_m^+(\nu)) = \Psi^\rho(q_m^+(\nu)), \quad \text{a.s.}$$

Thus the right-continuity of the process $\Psi^\rho$ implies that

$$\Psi^\rho(\nu) = \lim_{n \to \infty} \Psi^\rho(\nu) = \lim_{n \to \infty} \Psi^\rho(q_n^+(\nu)), \quad \text{a.s.} \quad (8.69)$$

We have assumed that $\text{esssup}_{t \in \mathcal{D}_T} \bar{E}_j [Y_\rho^j | \mathcal{F}_t] \in \text{Dom}(\mathcal{E})$ for some $j = j(\rho) \in \mathcal{I}$. It holds a.s. that

$$\bar{E}_j [Y_\rho^j | \mathcal{F}_t] \leq \text{esssup}_{\alpha \in \mathcal{D}_T} \bar{E}_j [Y_\rho^j | \mathcal{F}_s], \quad \forall \alpha \in \mathcal{D}_T.$$

Since $\bar{E}_j [Y_\rho^j | \mathcal{F}_s]$ is an RCLL process, it holds except on a null set $\mathcal{N}$ that

$$\bar{E}_j [Y_\rho^j | \mathcal{F}_t] \leq \text{esssup}_{s \in \mathcal{D}_T} \bar{E}_j [Y_\rho^j | \mathcal{F}_s], \quad \forall t \in [0,T], \text{ thus } \bar{E}_j [Y_\rho^j | \mathcal{F}_\rho, \rho \land \nu] \leq \text{esssup}_{s \in \mathcal{D}_T} \bar{E}_j [Y_\rho^j | \mathcal{F}_s], \quad \forall n \in \mathbb{N}.$$
Then one can deduce from (8.63) and Proposition 2.7 (3) that
\[ C_Y + 2C_H \leq \Psi^\rho(q^+_n(\nu)) \leq \tilde{E}_j[Y_\rho + H^j_{\rho \land q^+_n(\nu)}|F_{\rho \land q^+_n(\nu)}] + \zeta^i = \tilde{E}_j[Y_\rho^j - H^j_{\rho \land q^+_n(\nu)}|F_{\rho \land q^+_n(\nu)}] + \zeta^i \]
\[ \leq \tilde{E}_j[Y_\rho^j - C_H|F_{\rho \land q^+_n(\nu)}] + \zeta^i = \tilde{E}_j[Y_\rho^j|F_{\rho \land q^+_n(\nu)}] - C_H \cdot \zeta^i \leq \text{esssup}_{s \in D_Y} \tilde{E}_j[Y_\rho|F_s] - C_H \cdot \zeta^i, \quad \text{a.s.,} \]
where the right hand side belongs to $\text{Dom}(\mathcal{E})$ thanks to (D2) and the assumption that $\zeta^i \in \text{Dom}(\mathcal{E})$. Hence the Dominated Convergence Theorem (Proposition 2.7.1, (8.69), (8.67)) as well as Proposition 2.7.1 (5) imply that
\[ \Psi^\rho = \lim_{n \to \infty} \Psi^\rho(q^+_n(\nu)) \in \text{Dom}(\mathcal{E}) \text{ and that} \]
\[ \Psi^\rho(\nu) \leq \lim_{n \to \infty} \tilde{E}_i[\Psi^\rho(q^+_n(\nu))|F_{\nu}] = \tilde{E}_i[\Psi^\rho_+|F_{\nu}] = \Psi^\rho_+ , \quad \text{a.s.,} \quad (8.70) \]
where in the last equality we used the fact that $\Psi^\rho_+ = \lim_{n \to \infty} \Psi^\rho(q^+_n(\nu)) \in \mathcal{F}_\nu$, thanks to the right-continuity of the filtration $\mathcal{F}$.

**Step 4:** Given $\nu \in \mathcal{S}_{n,r}$, we set
\[ \gamma = \Delta(0) \land \nu, \quad \gamma_n = \Delta(0) \land q^+_n(\nu), \quad \forall \ n \in \mathbb{N} \]
and let $\rho \in \mathcal{S}_{n,T}$. Since $\lim_{n \to \infty} 1\{\gamma \geq \gamma_n\} = 1\{\gamma = \gamma_n\}$ and since
\[ \{\gamma > \nu\} \subseteq \{q^+_n(\nu) = q^+_n(\gamma) \land \nu\}, \quad \{\gamma > q^+_n(\nu)\} \subseteq \{q^+_n(\nu) = \gamma \land q^+_n(\nu)\}, \quad \forall \ n \in \mathbb{N}, \]
one can deduce from (8.70), (8.69) and (8.67) that
\[ 1\{\gamma > \nu\} \Psi^\rho(\gamma) \leq 1\{\gamma > \nu\} \Psi^\rho_+ = 1\{\gamma > \nu\} \lim_{n \to \infty} \Psi^\rho(q^+_n(\gamma)) = \lim_{n \to \infty} 1\{\gamma > q^+_n(\nu)\} \Psi^\rho(q^+_n(\nu)) = \lim_{n \to \infty} 1\{\gamma > q^+_n(\nu)\} \Psi^\rho(q^+_n(\nu)) = 1\{\gamma > q^+_n(\nu)\} \Psi^\rho(\gamma_n), \quad \text{a.s.} \quad (8.71) \]
For any $n \in \mathbb{N}$, we see from (8.59) that
\[ V(\gamma_n) = V(\gamma_n) = \text{esssup}_{\sigma \in D_{\gamma_n}} \left( \text{essinf}_{i \in I} \tilde{E}_i[Y_\rho + H^i_{\rho \land \gamma_n,\rho}|F_{\gamma_n}] \right) \geq \text{essinf}_{i \in I} \tilde{E}_i[Y_\rho + H^i_{\rho \land \gamma_n,\rho}|F_{\gamma_n}], \quad \text{a.s.} \quad (8.72) \]
Since $\{\gamma \leq \nu\} \subseteq \{\gamma = \gamma_n\}$, Proposition 2.7.1 (2) and (3) imply that for any $i \in I$
\[ 1\{\gamma \leq \nu\} \tilde{E}_i[Y_\rho + H^i_{\rho \land \gamma_n,\rho}|F_{\gamma_n}] = \tilde{E}_i[1\{\gamma \leq \nu\} Y_\rho + H^i_{\rho \land \gamma_n,\rho}|F_{\gamma_n}] = 1\{\gamma \leq \nu\} \tilde{E}_i[Y_\rho + H^i_{\rho \land \gamma_n,\rho}|F_{\gamma_n}], \quad \text{a.s.} \quad (8.71) \]
and that
\[ \tilde{E}_i[Y_{\rho \land \gamma_n} + H^i_{\gamma_n,\rho \land \gamma_n,\rho}|F_{\gamma_n}] = \tilde{E}_i[1 \{\rho \leq \gamma_n\} Y_{\rho \land \gamma_n} + 1 \{\rho > \gamma_n\} (Y_\rho + H^i_{\rho \land \gamma_n,\rho})|F_{\gamma_n}] \]
\[ = 1 \{\rho \leq \gamma_n\} Y_{\rho \land \gamma_n} + 1 \{\rho > \gamma_n\} \tilde{E}_i[Y_\rho + H^i_{\rho \land \gamma_n,\rho}|F_{\gamma_n}] \]
\[ = 1 \{\rho \leq \gamma_n\} Y_{\rho \land \gamma_n} + 1 \{\rho > \gamma_n\} \tilde{E}_i[Y_\rho + H^i_{\rho \land \gamma_n,\rho}|F_{\gamma_n}] + 1 \{\rho > \gamma_n\} \tilde{E}_i[Y_\rho + H^i_{\rho \land \gamma_n,\rho}|F_{\gamma_n}], \quad \text{a.s.} \quad (8.71) \]
Then it follows from (8.72) and Lemma 3.3 that
\[ V(\gamma_n) \geq 1 \{\rho \leq \gamma_n\} Y_{\rho \land \gamma_n} + 1 \{\rho > \gamma_n\} \text{essinf}_{i \in I} \tilde{E}_i[Y_\rho + H^i_{\rho \land \gamma_n,\rho}|F_{\rho \land \gamma_n}] + 1 \{\rho > \gamma_n\} \text{essinf}_{i \in I} \tilde{E}_i[Y_\rho + H^i_{\rho \land \gamma_n,\rho}|F_{\rho \land \gamma_n}] \]
\[ = 1 \{\rho \leq \gamma_n\} Y_{\rho \land \gamma_n} + 1 \{\rho > \gamma_n\} \left( \Psi^\rho(\gamma_n) - H^i_{\rho \land \gamma_n} \right) + 1 \{\rho > \gamma_n\} \text{essinf}_{i \in I} \tilde{E}_i[Y_\rho + H^i_{\rho \land \gamma_n,\rho}|F_{\rho \land \gamma_n}], \quad \text{a.s.} \quad (8.71) \]
As $n \to \infty$, the right-continuity of processes $Y$ and $H^i$, (8.71), Lemma 3.3 as well as Proposition 2.7.1 (2) & (3) show that
\[ \lim_{n \to \infty} V(\gamma_n) \geq 1 \{\rho = \gamma\} Y_{\gamma} + 1 \{\rho > \gamma\} \left( \lim_{n \to \infty} \Psi^\rho(\gamma_n) - H^i_{\rho \land \gamma_n} \right) + 1 \{\rho > \gamma\} \text{essinf}_{i \in I} \tilde{E}_i[Y_\rho + H^i_{\rho \land \gamma_n,\rho}|F_{\rho \land \gamma_n}] \]
\[ = 1 \{\rho = \gamma\} Y_{\gamma} + 1 \{\rho > \gamma\} \left( \Psi^\rho(\gamma) - H^i_{\rho \land \gamma_n} \right) = 1 \{\rho = \gamma\} Y_{\gamma} + 1 \{\rho > \gamma\} \text{essinf}_{i \in I} \tilde{E}_i[Y_\rho + H^i_{\rho \land \gamma_n,\rho}|F_{\rho \land \gamma_n}] \]
\[ = \text{essinf}_{i \in I} \left( 1 \{\rho = \gamma\} Y_{\gamma} + 1 \{\rho > \gamma\} \tilde{E}_i[Y_\rho + H^i_{\rho \land \gamma_n,\rho}|F_{\gamma_n}] \right) = \text{essinf}_{i \in I} \tilde{E}_i[1 \{\rho = \gamma\} Y_{\gamma} + 1 \{\rho > \gamma\} (Y_\rho + H^i_{\rho \land \gamma_n,\rho})|F_{\gamma_n}] \]
\[ = \text{essinf}_{i \in I} \tilde{E}_i[Y_\rho + H^i_{\rho \land \gamma_n,\rho}|F_{\gamma_n}], \quad \text{a.s.} \]
Taking the essential supremum of the right-hand-side over $\rho \in \mathcal{S}_{\gamma,T}$, we obtain
\[
\lim_{n \to \infty} V(\gamma_n) \geq \text{esssup}_{\rho \in \mathcal{S}_{\gamma,T}} \left( \text{essinf}_{i \in \mathcal{I}} \tilde{\mathcal{E}}_i [Y_\rho + H_{i,\rho} I_{\mathcal{F}_i}] \right) = V(\gamma), \quad \text{a.s.} \tag{8.73}
\]

On the other hand, for any $i \in \mathcal{I}$ and $n \in \mathbb{N}$ we have that $V(\gamma_n) = \mathcal{V}(\gamma_n) = \text{essinf}_{i \in \mathcal{I}} R^i(\gamma_n) \leq R^i(\gamma_n)$, a.s. Then $(8.70)$ and the right continuity of the process $R^{i,0}$ imply that
\[
\lim_{n \to \infty} V(\gamma_n) \leq \lim_{n \to \infty} R^i(\gamma_n) = \lim_{n \to \infty} R^{i,0}_{\gamma_n} = R^i(\gamma), \quad \text{a.s.}
\]

Taking the essential infimum of $R^i(\gamma)$ over $i \in \mathcal{I}$ yields that
\[
\lim_{n \to \infty} V(\gamma_n) \leq \text{essinf}_{i \in \mathcal{I}} R^i(\gamma) = \mathcal{V}(\gamma) = V(\gamma), \quad \text{a.s.}
\]

This inequality together with (8.73) shows that $\lim_{n \to \infty} V(\gamma_n) = V(\gamma)$, a.s., which further implies that for any $\nu \in \mathcal{S}_{0,T}$ and $i \in \mathcal{I}$
\[
\lim_{n \to \infty} V^i(\tau(0) \wedge q_n^+(\nu)) = \lim_{n \to \infty} \left( V^i(\tau(0) \wedge q_n^+(\nu)) + H^i_{\tilde{\mathcal{F}}(0) \wedge q_n^+(\nu)} \right) = \mathcal{V}(\nu) \wedge H^i_{\tilde{\mathcal{F}}(0) \wedge \nu} = V^i(\tau(0) \wedge \nu), \quad \text{a.s.} \tag{8.74}
\]

**Step 5:** Proposition 5.3 shows that the stopped process $\{V^i(\tau(0) \wedge t)\}_{t \in [0,T]}$ is an $\tilde{\mathcal{E}}_{\nu}$-submartingale, thus $\{-V^i(\tau(0) \wedge t)\}_{t \in [0,T]}$ is an $\mathcal{E}^\nu$-supermartingale by (5.12). Then Theorem 2.3 implies that $V^{i,+} = \lim_{n \to \infty} V^i(\tau(0) \wedge q_n^+(t))$, $t \in [0,T]$ is an RCLL process and that
\[
P\left( V^{i,+} = \lim_{n \to \infty} V^i(\tau(0) \wedge q_n^+(t)) \text{ for any } t \in [0,T] \right) = 1.
\]

For any $\sigma, \zeta \in \mathcal{S}_{0,T}$, Lemma 5.3 and 5.4 show that
\[
1_{\{\sigma = \zeta\}} V(\sigma) = 1_{\{\sigma = \zeta\}} \mathcal{V}(\sigma) = \text{essinf}_{j \in \mathcal{I}} \left( 1_{\{\sigma = \zeta\}} R^j(\sigma) \right) = \text{essinf}_{j \in \mathcal{I}} \left( 1_{\{\sigma = \zeta\}} \mathcal{V}(\tau(0) \wedge \nu) \right) = 1_{\{\sigma = \zeta\}} V(\zeta) \quad \text{a.s.},
\]
which implies that
\[
1_{\{\sigma = \zeta\}} V^i(\sigma) = 1_{\{\sigma = \zeta\}} V(\sigma) + 1_{\{\sigma = \zeta\}} H^i_{\tau(0)} = 1_{\{\sigma = \zeta\}} V(\zeta) + 1_{\{\sigma = \zeta\}} H^i_{\tau(0)} = 1_{\{\sigma = \zeta\}} V^i(\zeta), \quad \text{a.s.} \tag{8.75}
\]
Let $\sigma \in \mathcal{S}_{0,T}^F$ take values in a finite set $\{t_1 < \cdots < t_m\}$. For any $\alpha \in \{1 \cdots m\}$ and $n \in \mathbb{N}$, since $\{\sigma = t_\alpha\} \subset \{\tau(0) \wedge q_n^+(\sigma) = \tau(0) \wedge q_n^+(t_\alpha)\}$, one can deduce from (8.75) that
\[
1_{\{\sigma = t_\alpha\}} V^i(\tau(0) \wedge q_n^+(\sigma)) = 1_{\{\sigma = t_\alpha\}} V^i(\tau(0) \wedge q_n^+(t_\alpha)), \quad \text{a.s.}
\]
As $n \to \infty$, (8.74) shows that
\[
1_{\{\sigma = t_\alpha\}} V^{i,+} = 1_{\{\sigma = t_\alpha\}} V_{t_\alpha}^{i,+} = \lim_{n \to \infty} 1_{\{\sigma = t_\alpha\}} V^i(\tau(0) \wedge q_n^+(t_\alpha)) = \lim_{n \to \infty} 1_{\{\sigma = t_\alpha\}} V^i(\tau(0) \wedge q_n^+(\sigma)) = 1_{\{\sigma = t_\alpha\}} V^i(\tau(0) \wedge \sigma), \quad \text{a.s.}
\]

Summing the above expression over $\alpha$, we obtain $V^{i,+} = V^i(\tau(0) \wedge \sigma)$, a.s. Then the right-continuity of the process $V^{i,+}$ and (8.74) imply that
\[
V^{i,+}_\nu = \lim_{n \to \infty} V^{i,+}_\nu = \lim_{n \to \infty} V^i(\tau(0) \wedge q_n^+(\nu)) = V^i(\tau(0) \wedge \nu), \quad \text{a.s.} \tag{8.76}
\]

In particular, $V^{i,+}$ is an RCLL modification of the stopped process $\{V^i(\tau(0) \wedge t)\}_{t \in [0,T]}$. Therefore,
\[
V^0_\nu = \left\{ V^{i,+}_\nu - H^i_{\tilde{\mathcal{F}}(0) \wedge \nu} \right\}_{t \in [0,T]} \quad \text{is an RCLL modification of the stopped value process $\{V(\tau(0) \wedge t)\}_{t \in [0,T]}$.}
\]
For any $\nu \in \mathcal{S}_{0,T}$, (8.76) implies that
\[
V^0_\nu = V^{i,+}_\nu - H^i_{\tilde{\mathcal{F}}(0) \wedge \nu} = V^i(\tau(0) \wedge \nu) - H^i_{\tilde{\mathcal{F}}(0) \wedge \nu} = V(\tau(0) \wedge \nu), \quad \text{a.s., proving (5.14).}
**Proof of (2).** (5.13) and Proposition (5.2) imply that $V_0^0(0) = V(\tau_0) = Y_{\tau_0}$, a.s. Hence, we can deduce from the right-continuity of processes $V^0$ and $Y$ that $\tau_0$ in (5.14) is a stopping time belonging to $\mathcal{S}_\infty(0)$ and that

$$Y_{\tau_0} = V_{\tau_0}^0 = V(\tau_0), \text{ a.s.},$$

where the second equality is due to (5.13). Then it follows from (5.11) that for any $i \in \mathcal{I}$

$$V(0) = V(\tau_0) \leq \tilde{E}_i[V^i(\tau_0)] = \tilde{E}_i[Y^i_{\tau_0}].$$

Taking the infimum of the right-hand-side over $i \in \mathcal{I}$ yields that

$$V(0) \leq \inf_{i \in \mathcal{I}} \tilde{E}_i[Y^i_{\tau_0}] \leq \sup_{\rho \in \mathcal{S}_{0,T}} \left( \inf_{i \in \mathcal{I}} \tilde{E}_i[Y^i_{\tau_0}] \right) = Y(0) = V(0),$$

which implies that $\inf_{i \in \mathcal{I}} \tilde{E}_i[Y^i_{\tau_0}] = \sup_{\rho \in \mathcal{S}_{0,T}} \inf_{i \in \mathcal{I}} \tilde{E}_i[Y^i_{\tau_0}].$ \hfill \qed

### 8.5 Proofs of Section 6

**Proof of Proposition 6.1.** Fix $t \in [0, T]$. For any $\xi \in \text{Dom}(\mathcal{E})$ and $i \in \mathcal{I}$, the definition of $\text{Dom}(\mathcal{E})$ assures that there exists a $c(\xi) \in \mathbb{R}$ such that $c(\xi) \leq \xi$, a.s. Then Proposition 2.7 (5) shows that

$$c(\xi) = \tilde{E}_i[c(\xi)|\mathcal{F}_t] \leq \tilde{E}_i[i]\mathcal{F}_t, \text{ a.s.} \quad (8.77)$$

Taking the essential infimum of the right-hand-side over $i \in \mathcal{I}$, we obtain for an arbitrary $i' \in \mathcal{I}$ that

$$c(\xi) \leq \underline{c}[\xi]|\mathcal{F}_t] \leq \tilde{E}_{i'}[\xi|\mathcal{F}_t], \text{ a.s.}$$

Since $\tilde{E}_{i'}[\xi|\mathcal{F}_t] \in \text{Dom}(\mathcal{E})$, Lemma 3.2 implies that $\underline{c}[\xi]|\mathcal{F}_t] \in \text{Dom}(\mathcal{E})$, thus $\underline{c}[\xi]|\mathcal{F}_t]$ is a mapping from $\text{Dom}(\mathcal{E})$ to $\text{Dom}(\mathcal{E}) \cap L^0(\mathcal{F}_t)$.

A simple application of Lemma 3.3 shows that $\mathcal{E}_i$ satisfies (A3), (A4) and (0.11). Hence, it only remains to show (A2) for $\mathcal{E}_i$. Fix $0 \leq s < t \leq T$. Letting $(\nu, \mathcal{I}, \mathcal{U}) = (t, \mathcal{I}, \mathcal{T})$ and taking $X(T) = \xi$ in Lemma 4.2, we can find a sequence $\{i_n\}_{n \in \mathbb{N}}$ in $\mathcal{I}$ such that

$$\tilde{E}_i[\xi|\mathcal{F}_s] = \inf_{i \in \mathcal{I}} \tilde{E}_i[\xi|\mathcal{F}_s] = \lim_{n \to \infty} \tilde{E}_{i_n}[\xi|\mathcal{F}_s], \text{ a.s.} \quad (8.78)$$

Now fix $j \in \mathcal{I}$. For any $n \in \mathbb{N}$, it follows from Definition 3.2 that there exists a $k_n = k(j, i_n, t) \in \mathcal{I}$ such that $\tilde{E}_{k_n} = \tilde{E}_{i_n}$. Applying (3.3) yields that

$$\mathcal{E}_j[\xi|\mathcal{F}_s] \leq \tilde{E}_{k_n}[\xi|\mathcal{F}_s] = \tilde{E}_{i_n}[\xi|\mathcal{F}_s] = \tilde{E}_j[\tilde{E}_{i_n}[\xi|\mathcal{F}_s]|\mathcal{F}_s], \text{ a.s.} \quad (8.79)$$

For any $n \in \mathbb{N}$, we see from (8.77) and (8.78) that

$$c(\xi) = \tilde{E}_{i_n}[c(\xi)|\mathcal{F}_t] \leq \tilde{E}_{i_n}[\xi|\mathcal{F}_t] \leq \tilde{E}_{i_1}[\xi|\mathcal{F}_t], \text{ a.s.},$$

where $\tilde{E}_{i_1}[\xi|\mathcal{F}_t] \in \text{Dom}(\mathcal{E}_{i_1}) = \text{Dom}(\mathcal{E})$. The Dominated Convergence Theorem (Proposition 2.9) and (8.79) then imply that

$$\mathcal{E}_j[\xi|\mathcal{F}_s]|\mathcal{F}_s] = \lim_{n \to \infty} \tilde{E}_j[\tilde{E}_{i_n}[\xi|\mathcal{F}_s]|\mathcal{F}_s] \geq \mathcal{E}_j[\xi|\mathcal{F}_s], \text{ a.s.} \quad (8.80)$$

Taking the essential infimum of the left-hand-side over $j \in \mathcal{I}$, we obtain

$$\mathcal{E}_j[\xi|\mathcal{F}_s]|\mathcal{F}_s] \geq \mathcal{E}_j[\xi|\mathcal{F}_s], \text{ a.s.} \quad (8.80)$$

On the other hand, for any $i \in \mathcal{I}$ and $\rho \in \mathcal{S}_{0,T}$, applying Corollary 2.3 we obtain

$$\tilde{E}_i[\xi|\mathcal{F}_s] = \tilde{E}_i[\xi|\mathcal{F}_s]|\mathcal{F}_s] \geq \tilde{E}_i[\mathcal{E}_i[\mathcal{E}_i[\xi|\mathcal{F}_s]|\mathcal{F}_s] \geq \mathcal{E}_i[\xi|\mathcal{F}_s]|\mathcal{F}_s], \text{ a.s.}$$

Taking the essential infimum of the left-hand-side over $i \in \mathcal{I}$ yields that $\underline{c}[\xi|\mathcal{F}_s] \geq \mathcal{E}_i[\xi|\mathcal{F}_s]|\mathcal{F}_s], \text{ a.s.}$, which together with (8.80) proves (A2) for $\mathcal{E}$. \hfill \qed
8.6 Proofs of Section 7

Proof of Proposition 7.1. By (7.2), it holds $dt \times dP$-a.s. that for any $z \in \mathbb{R}^d$

$$\left| g(t, z) \right| = \left| g(t, z) - g(t, 0) \right| \leq K_g|z|, \quad \text{thus} \quad \hat{g}(t, z) = -K_g|z| \leq g(t, z).$$

Clearly, $\hat{g}$ is a generator satisfying (7.2). It is also positively homogeneous in $z$, i.e.

$$\hat{g}(t, \alpha z) = -K_g|\alpha z| = -\alpha K_g|z| = \alpha \hat{g}(t, z), \quad \forall \alpha \geq 0, \quad \forall z \in \mathbb{R}^d.$$

Then Example 10 of Peng [1997] (or Proposition 8 of Rosazza Gianin [2006]) and (7.6) imply that for any $n \in \mathbb{N}$ and any $A \in \mathcal{F}_T$ with $P(A) > 0$

$$n\mathcal{E}_g[1_A] = \mathcal{E}_g[n1_A] \leq \mathcal{E}_g[n1_A]. \quad (8.81)$$

Since $\mathcal{E}_g[1_A] > 0$ (which follows from the second part of (A1)), letting $n \to \infty$ in (8.81) yields (H0).

Next, we consider a sequence $\{\xi_n\}_{n \in \mathbb{N}} \subset L^2(\mathcal{F}_T)$ with $\sup_{n \in \mathbb{N}} |\xi_n| \in L^2(\mathcal{F}_T)$. If $\xi_n$ converges a.s., it is clear that $\xi \triangleq \lim_{n \to \infty} \xi_n \in L^2(\mathcal{F}_T)$. Applying Lemma 7.1 with $\mu = K_g$, we obtain

$$\left| \mathcal{E}_g[\xi_n] - \mathcal{E}_g[\xi] \right| \leq \mathcal{E}_{g_{\mu}}[|\xi_n - \xi|] \leq \sup_{t \in [0, T]} \mathcal{E}_{g_{\mu}}[|\xi_n - \xi|] \leq \left| \mathcal{E}_{g_{\mu}}[|\xi_n - \xi|] \right| \leq C e(K_g + K_g^2) \mathcal{E}_{g_{\mu}}[|\xi_n - \xi|] \leq L^2(\mathcal{F}_T),$$

where we used the fact that $K_{g_{\mu}} = \mu$ in the last inequality. As $n \to \infty$, thanks to the Dominated Convergence Theorem of the linear expectation $E$, we have that $\|\xi_n - \xi\|_{L^2(\mathcal{F}_T)}^2 = E|\xi_n - \xi|^2 \to 0$; thus $\lim_{n \to \infty} \mathcal{E}_g[\xi_n] = \mathcal{E}_g[\xi]$. Then (H1) and (H2) follow.

For any $\nu \in \mathcal{S}_{0, T}$ and $\xi \in L^{2, +}(\mathcal{F}_T) \triangleq \{ \xi \in L^2(\mathcal{F}_T) : \xi \geq 0, \text{a.s.} \}$, Lemma 7.1 (1) shows that $\sup_{t \in [0, T]} |\mathcal{E}_g[\xi|\mathcal{F}_t]| \in L^2(\mathcal{F}_T)$, consequently $\mathcal{E}_g[\xi|\mathcal{F}_t] \in L^2(\mathcal{F}_T)$. Since $X^\xi \triangleq \mathcal{E}_g[\xi|\mathcal{F}_t]$ is a continuous process, $X^\xi_{\nu, \Theta} = X^\xi = \mathcal{E}_g[\xi|\mathcal{F}_t] \in L^{2, +}(\mathcal{F}_T)$, which proves (H3). 

Proof of Proposition 7.2. Fix $\nu \in \mathcal{S}_{0, T}$. It is easy to check that the generator $g^\nu$ satisfies (7.2) with Lipschitz coefficient $K_1 \lor K_2$. For any $\xi \in L^2(\mathcal{F}_T)$, we set $\eta \triangleq \Gamma^\xi_{\nu, g^\nu} \in \mathcal{F}_T$ and define

$$\Theta_t \triangleq 1_{\{\nu \leq t\}} \Theta_t^{\xi, g^\nu} + 1_{\{\nu > t\}} \Theta_t^{\eta, g^\nu}, \quad \forall t \in [0, T].$$

It follows that

$$g^\nu(t, \Theta_t) = 1_{\{\nu \leq t\}} g_2(t, \Theta_t) + 1_{\{\nu > t\}} g_1(t, \Theta_t) = 1_{\{\nu \leq t\}} g_2(t, \Theta_t^{\xi, g^\nu}) + 1_{\{\nu > t\}} g_1(t, \Theta_t^{\eta, g^\nu}), \quad \forall t \in [0, T].$$

For any $t \in [0, T]$, since $\{\nu \leq t\} \in \mathcal{F}_t$, one can deduce that

$$1_{\{\nu \leq t\}} \left( \xi + \int_t^T g^\nu(s, \Theta_s)ds - \int_t^T \Theta_s dB_s \right) = 1_{\{\nu \leq t\}} \xi + \int_t^T 1_{\{\nu \leq t\}} g^\nu(s, \Theta_s)ds - \int_t^T 1_{\{\nu \leq t\}} \Theta_s dB_s$$

$$= 1_{\{\nu \leq t\}} \xi + \int_t^T 1_{\{\nu \leq t\}} g_2(s, \Theta_s^{\xi, g^\nu})ds - \int_t^T 1_{\{\nu \leq t\}} \Theta_s^{\xi, g^\nu} dB_s$$

$$= 1_{\{\nu \leq t\}} \left( \xi + \int_t^T g_2(s, \Theta_s^{\xi, g^\nu})ds - \int_t^T \Theta_s^{\xi, g^\nu} dB_s \right) = 1_{\{\nu \leq t\}} \Gamma^\xi_{t, g^\nu}, \quad \text{a.s.} \quad (8.82)$$

The continuity of processes $\int_t^T g^\nu(s, \Theta_s)ds$, $\int_t^T \Theta_s dB_s$ and $\Gamma^\xi_{t, g^\nu}$ then implies that except on a null set $N$

$$1_{\{\nu \leq t\}} \left( \xi + \int_t^T g^\nu(s, \Theta_s)ds - \int_t^T \Theta_s dB_s \right) = 1_{\{\nu \leq t\}} \Gamma^\xi_{t, g^\nu}, \quad \forall t \in [0, T].$$
Taking \( t = \nu(\omega) \) for any \( \omega \in \mathbb{N}^c \) yields that
\[
\xi + \int_{\nu}^{T} g''(s, \overline{\Theta}_s)ds - \int_{\nu}^{T} \overline{\Theta}_s dB_s = \Gamma_{\nu}^{\xi,g_2} = \eta, \quad a.s.
\] (8.83)

Now fix \( t \in [0, T] \). We can deduce from (8.83) that
\[
1_{\{\nu > t\}} \left( \xi + \int_{t}^{T} g''(s, \overline{\Theta}_s)ds - \int_{t}^{T} \overline{\Theta}_s dB_s \right) = 1_{\{\nu > t\}} \left( \eta + \int_{t}^{\nu} g''(s, \overline{\Theta}_s)ds - \int_{t}^{\nu} \overline{\Theta}_s dB_s \right)
= 1_{\{\nu > t\}} \left( \eta + \int_{\nu}^{\nu} g_1(s, \Theta^{\nu,g_1}_s)ds - \int_{\nu}^{\nu} \Theta^{\nu,g_1}_s dB_s \right), \quad a.s.
\] (8.84)

Moreover, Proposition 2.7 (5) implies that
\[
\mathcal{E}_{g_1}[\eta]_{\mathcal{F}_t} = \eta + \int_{t}^{T} g_1(s, \Theta^{\nu,g_1}_s)ds - \int_{t}^{T} \Theta^{\nu,g_1}_s dB_s = \mathcal{E}_{g_1}[\eta]_{\mathcal{F}_t} + \int_{t}^{\nu} g_1(s, \Theta^{\nu,g_1}_s)ds - \int_{t}^{\nu} \Theta^{\nu,g_1}_s dB_s,
\]
which in conjunction with (8.84) shows that for any \( t \in [0, T] \)
\[
\xi + \int_{t}^{T} g''(s, \overline{\Theta}_s)ds - \int_{t}^{T} \overline{\Theta}_s dB_s = 1_{\{\nu \leq t\}} \Gamma_{t}^{\xi,g_2} + 1_{\{\nu > t\}} \mathcal{E}_{g_1}[\Gamma_{\nu}^{\xi,g_2} | \mathcal{F}_t]
= 1_{\{\nu \leq t\}} \mathcal{E}_{g_1}[\xi]_{\mathcal{F}_t} + 1_{\{\nu > t\}} \mathcal{E}_{g_1}[\xi]_{\mathcal{F}_t} = \mathcal{E}_{g_1,g_2}[\xi]_{\mathcal{F}_t}, \quad a.s.
\]

Since \( \int_{t}^{T} g''(s, \overline{\Theta}_s)ds \), \( \int_{t}^{T} \overline{\Theta}_s dB_s \) and \( \mathcal{E}_{g_1,g_2}[\xi]_{\mathcal{F}_t} \) are all continuous processes, it holds except a null \( N' \) that
\[
\mathcal{E}_{g_1,g_2}[\xi]_{\mathcal{F}_t} = \xi + \int_{t}^{T} g''(s, \overline{\Theta}_s)ds - \int_{t}^{T} \overline{\Theta}_s dB_s, \quad \forall t \in [0, T].
\]

One can easily show that \( (\mathcal{E}_{g_1,g_2}[\xi]_{\mathcal{F}_t}, \overline{\Theta}) \in C_p([0, T]) \times \mathcal{H}_p([0, T]; \mathbb{R}^d) \). Thus the pair is the unique solution to the BSDE(\( \xi, g'' \)), namely \( \mathcal{E}_{g''}[\xi]_{\mathcal{F}_t} = \mathcal{E}_{g_1,g_2}[\xi]_{\mathcal{F}_t} \) for any \( t \in [0, T] \).

Proof of Theorem A.1: We first note that for any \( g \in \mathcal{G}' \), (2.7) implies that for every \( \mathcal{E}_g \)-submartingale \( X \), \(-X\) is an \( \mathcal{E}_g \)-supermartingale although \( g^- \) is concave (which means that \( \mathcal{E}_{g^-} \) may not belong to \( \mathcal{E}' \)). Hence, condition (A.12) is satisfied.

Fix \( g \in \mathcal{G}' \). Clearly \( H_0^g = 0 \). For any \( s, t \in \mathcal{D}_T \) with \( s < t \), we can deduce from (h1) and (h2) that
\[
C_{\mathcal{G}'}, \overset{\Delta}{=} cT \leq \int_{s}^{t} c'ds \leq \int_{s}^{t} h^g dr = H_{s,t}^g \leq \int_{s}^{t} h'(r)dr \leq \int_{0}^{T} h'(r)dr, \quad a.s.,
\] (8.85)
which implies that
\[
C_{\mathcal{G}'} \leq \essinf_{s,t \in \mathcal{D}_T; s < t} H_{s,t}^g \leq \esssup_{s,t \in \mathcal{D}_T; s < t} H_{s,t}^g \leq \int_{0}^{T} h'(r)dr, \quad a.s.,
\]
thus (S2) holds. Since \( \int_{0}^{T} h'(r)dr \in L^2(\mathcal{F}_T) \), it follows that
\[
\esssup_{s,t \in \mathcal{D}_T; s < t} H_{s,t}^g \in L^2(\mathcal{F}_T) \overset{\Delta}{=} \{ \xi \in L^2(\mathcal{F}_T) : \xi \geq c, \ a.s. \text{ for some } c \in \mathbb{R} \} = \text{Dom}(\mathcal{E}') \).
We can also deduce from (8.85) that except on a null set \(N\)
\[
C_{g,t} \leq H^g_{s,t} \leq \int_0^T h'(r)dr, \quad \forall 0 \leq s \leq t \leq T.
\]
Hence, for any \(\nu, \rho \in \mathcal{S}_0\) with \(\nu \leq \rho\), a.s., we have
\[
C_{g,t} \leq H^\rho_{\nu,\rho} \leq \int_0^T h'(r)dr, \quad a.s.,
\]
which implies that \(H^\rho_{\nu,\rho} \in L^2(\mathcal{F}_T) = \text{Dom}(\mathcal{E}')\); so we got (S1'). Moreover, (S3) directly follows from (h3).

Next, we check that the process \(Y\) satisfies (Y1) and (4.6). By (7.11) and (Y3), it holds a.s. that \(C_Y \leq Y_t \leq \zeta'_Y\) for any \(t \in \mathcal{D}_T\). The right-continuity of the process \(Y\) then implies that except on a null set \(\bar{N}\)
\[
C_Y \leq Y_t \leq \zeta'_Y, \quad \forall t \in [0, T], \quad \text{thus} \quad C_Y \leq Y_\rho \leq \zeta'_Y, \quad \forall \rho \in \mathcal{S}_0,T. \quad (8.86)
\]
Since \(\zeta'_Y \in L^2(\mathcal{F}_T)\), it follows that \(Y_\rho \in L^2(\mathcal{F}_T) = \text{Dom}(\mathcal{E}')\) for any \(\rho \in \mathcal{S}_0,T\), thus (Y1) holds. Moreover, for any \(g \in \mathcal{G}', \rho \in \mathcal{S}_0,T\) and \(t \in \mathcal{D}_T\), Proposition 22 (2), (8.86) and Lemma 7.1 (2) show that
\[
C_Y + c'T = \mathcal{E}_g[C_Y + c'T|\mathcal{F}_t] \leq \mathcal{E}_g[Y_\rho + \int_0^\rho c'ds|\mathcal{F}_t] \leq \mathcal{E}_g[Y_\rho^g|\mathcal{F}_t] \leq |\mathcal{E}_g[Y_\rho^g|\mathcal{F}_t]| \leq |\mathcal{E}_g[Y_\rho^g]| \leq |\mathcal{E}_g[Y_\rho^g[\mathcal{F}_t]| \leq \mathcal{E}_g[Y_\rho^g[\mathcal{F}_t]| \leq \sup_{t \in [0,T]} \mathcal{E}_{g_m}[\zeta'_Y + c'T - C_y - c'T], \quad a.s. \quad (8.87)
\]
Taking essential supremum of \(\mathcal{E}_g[Y_\rho^g|\mathcal{F}_t]\) over \((g, \rho, t) \in \mathcal{G}' \times \mathcal{S}_0,T \times \mathcal{D}_T\), we can deduce from (A4) that
\[
C_Y + c'T \leq \esssup_{(g, \rho, t) \in \mathcal{G}' \times \mathcal{S}_0,T \times \mathcal{D}_T} \mathcal{E}_g[Y_\rho^g|\mathcal{F}_t] \leq \sup_{t \in [0,T]} \mathcal{E}_{g_m}[\zeta'_Y + \int_0^T h'(s)ds|\mathcal{F}_t] - C_y - c'T, \quad a.s. \quad (8.87)
\]
Lemma 7.1 (1) implies that
\[
\left\| \sup_{t \in [0,T]} \mathcal{E}_{g_m}[\zeta'_Y + \int_0^T h'(s)ds|\mathcal{F}_t] \right\|_{L^2(\mathcal{F}_T)} \leq C e^{(M+M^2)T} \left\| \zeta'_Y + \int_0^T h'(s)ds \right\|_{L^2(\mathcal{F}_T)} < \infty.
\]
Hence, we see from (8.87) that \(\esssup_{(g, \rho, t) \in \mathcal{G}' \times \mathcal{S}_0,T \times \mathcal{D}_T} \mathcal{E}_g[Y_\rho^g|\mathcal{F}_t]) \in L^2(\mathcal{F}_T) = \text{Dom}(\mathcal{E}')\), which is exactly (4.6).

Now we show that the family of processes \(\{Y_\rho^g, t \in [0,T]\}_{g \in \mathcal{G}'}\) is both “\(\mathcal{E}'\)-uniformly-left-continuous” and “\(\mathcal{E}'\)-uniformly-right-continuous”. For any \(\nu, \rho \in \mathcal{S}_0,T\) with \(\nu \leq \rho\), a.s., let \(\{\rho_n\}_{n \in \mathbb{N}} \subset \mathcal{S}_\nu,T\) be a sequence increasing a.s. to \(\rho\). For any \(g \in \mathcal{G}'\), Lemma 7.1 (2) implies that
\[
\left| \mathcal{E}_g[\frac{\nu}{\nu_n}Y_{\rho_n} + H^\rho_{\rho_n}|\mathcal{F}_\nu] - \mathcal{E}_g[Y_\rho^g|\mathcal{F}_\nu] \right| \leq \mathcal{E}_{g_m}\left| \frac{\nu}{\nu_n}Y_{\rho_n} - Y_\rho - \int_{\rho_n}^\rho h^\rho(s)ds \right| \mathcal{F}_\nu \leq \mathcal{E}_{g_m}\left| \frac{\nu}{\nu_n}Y_{\rho_n} - Y_\rho + \int_{\rho_n}^\rho \tilde{h}'(s)ds \right| \mathcal{F}_\nu, \quad a.s.,
\]
where \(g_m(z) \overset{\Delta}{=} M[z], z \in \mathbb{R}^d\) and \(\tilde{h}'(t) \overset{\Delta}{=} h'(t) - c', t \in [0,T]\). Taking essential supremum of the left hand side over \(g \in \mathcal{G}'\) yields that
\[
\esssup_{g \in \mathcal{G}'} \left| \mathcal{E}_g[\frac{\nu}{\nu_n}Y_{\rho_n} + H^\rho_{\rho_n}|\mathcal{F}_\nu] - \mathcal{E}_g[Y_\rho^g|\mathcal{F}_\nu] \right| \leq \mathcal{E}_{g_m}\left| \frac{\nu}{\nu_n}Y_{\rho_n} - Y_\rho + \int_{\rho_n}^\rho \tilde{h}'(s)ds \right| \mathcal{F}_\nu, \quad a.s. \quad (8.88)
\]
Moreover, Lemma \[\text{(1)}\] implies that
\[
\left\| E_{\mathcal{G}} \left[ \int_{n-1}^{\rho} \hat{h}'(s) ds \right] \right\|_{L^2(\mathcal{F}_T)} 
\leq \sup_{t \in [0,T]} \left\| E_{\mathcal{G}} \left[ \int_{n-1}^{\rho} \hat{h}'(s) ds \right] \right\|_{L^2(\mathcal{F}_T)} 
\leq C e^{(M+M^2)T} \left\| \int_{n-1}^{\rho} \hat{h}'(s) ds \right\|_{L^2(\mathcal{F}_T)}. \tag{8.89}
\]

Since
\[
\left| \int_{n-1}^{\rho} \hat{h}'(s) ds \right| \leq \int_{n-1}^{\rho} |\hat{h}'(s)| ds \leq 2 \int_{n-1}^{\rho} |Y_\rho - Y_\rho| ds \leq 2 |Y_\rho - Y_\rho|, \quad \forall n \geq 2,
\]
the continuity of $Y$ implies that $\lim_{n \to \infty} \left( \int_{n-1}^{\rho} \hat{h}'(s) ds \right) = 0$, a.s. It also holds for any $n \geq 2$ that
\[
\left| \int_{n-1}^{\rho} \hat{h}'(s) ds \right| \leq 3 (\zeta_C' - C_Y) + \int_{0}^{T} \hat{h}'(s) ds - \zeta_C T, \quad \text{a.s.,}
\]
where the right-hand sides belongs to $L^2(\mathcal{F}_T)$. Thus the Dominated Convergence Theorem implies that
\[
\text{the sequence } \left\{ \int_{n-1}^{\rho} \hat{h}'(s) ds \right\}_{n \in \mathbb{N}} \text{ converges to 0 in } L^2(\mathcal{F}_T),
\]
which together with \[\text{(8.88)}\] and \[\text{(8.89)}\] implies that
\[
\text{the sequence } \left\{ \text{esssup}_{g \in \mathcal{G}} \left[ E_{\mathcal{G}} \left[ \int_{n-1}^{\rho} \hat{h}'(s) ds \right] \right] \right\}_{n \in \mathbb{N}} \text{ also converges to 0 in } L^2(\mathcal{F}_T).
\]

Then we can find a subsequence $\{n_k\}_{k \in \mathbb{N}}$ of $\mathbb{N}$ such that
\[
\lim_{n \to \infty} \text{esssup}_{g \in \mathcal{G}} \left[ E_{\mathcal{G}} \left[ \int_{n_k-1}^{\rho} \hat{h}'(s) ds \right] \right] = 0, \quad \text{a.s.}
\]
Therefore, the family of process $\{Y^g\}_{g \in \mathcal{G}}$ is \text{"\mathcal{G}'-uniformly-left-continuous"}. The \text{"\mathcal{G}'-uniform-right-continuity"} of $\{Y^g\}_{g \in \mathcal{G}}$ can be shown similarly. \hfill \Box

\textbf{Proof of Theorem 7.2} For any $U \in \mathcal{U}$, Theorem \text{7.1} and Proposition \text{4.2} imply that $Z^{U,0} = \left\{ Z^n_t + \int_0^t h^n_s dB_s \right\}_{\tau(t, \omega) < \infty}$ is an $\mathcal{E}_{\mathcal{G}^t}$-supermartingale. In light of the Doob-Meyer Decomposition of $g$-expectation (see e.g. Peng \text{1999}, Theorem 3.3, or Peng \text{2004}, Theorem 3.9), there exists an RCLL increasing process $\Delta^U$ null at 0 and a process $\Theta^U \in \mathcal{H}_F([0,T] ; \mathbb{R}^d)$ such that
\[
Z^n_{\tau(t, \omega) \wedge T} = Z^n_{\tau(t, \omega)} + \int_{\tau(t, \omega) \wedge t}^{T} g^n(s, \Theta^U_s) ds + \Delta^U_{\tau(t, \omega)} - \int_{\tau(t, \omega) \wedge t}^{T} \Theta^U_s dB_s, \quad t \in [0,T]. \tag{8.90}
\]

In what follows we will show that
\[
U^\ast(t, \omega) \triangleq u^\ast(t, \omega, \Theta^{U,0}_t(\omega)), \quad (t, \omega) \in [0,T] \times \Omega
\]
is an optimal control desired, where $U^0 \equiv 0$ denotes the null control.

Recall that $\tau(0) = \inf \{ t \in [0,T] \mid Z^n^0 = Y_t \}$. Taking $t = \tau(0)$ and $t = \tau(0) \wedge t$ respectively in \[\text{(8.90)}\] and subtracting the former from the latter yields that
\[
Z^{U,0}_{\tau(0) \wedge T} = Z^{U,0}_{\tau(0)} + \int_{\tau(0) \wedge t}^{T} g^n(s, \Theta^U_s) ds + \Delta^U_{\tau(0) \wedge T} - \int_{\tau(0) \wedge t}^{T} \Theta^U_s dB_s, \quad t \in [0,T], \tag{8.91}
\]
which is equivalent to
\[
Z^0_{\tau(0) \wedge T} = Z^0_{\tau(0)} + \int_{\tau(0) \wedge t}^{T} H(s, \Theta^U_s, U_s) ds + \Delta^U_{\tau(0) \wedge T} - \int_{\tau(0) \wedge t}^{T} \Theta^U_s dB_s, \quad t \in [0,T]. \tag{8.92}
\]
In particular, taking \( U = U^0 \), we obtain
\[
Z_{\tau(0)}^{U^0} = Z_{\tau(0)}^0 + \int_{\tau(0) \wedge t}^{\tau(0)} H(s, \Theta_s^{U^0}, U_s^0) ds + \Delta U_s^{U^0} d\tau - \int_{\tau(0) \wedge t}^{\tau(0)} \Theta_s dB_s, \quad t \in [0, T].
\] (8.93)

Comparing the martingale parts of (8.92) and (8.93), we see that for any \( U \in \mathfrak{U} \),
\[
\Theta_t^U = \Theta_t^{U^0}, \quad dt \times dP\text{-a.s.}
\] (8.94)
on the stochastic interval \([0, \tau(0)]\). Plugging this into (8.92) yields that
\[
Z_{\tau(0)}^{U^0} = Z_{\tau(0)}^0 + \int_{\tau(0) \wedge t}^{\tau(0)} H(s, \Theta_s^{U^0}, U_s^0) ds + \Delta U_s^{U^0} d\tau - \int_{\tau(0) \wedge t}^{\tau(0)} \Theta_s dB_s, \quad t \in [0, T].
\] (8.95)

Let us define \( g_{K_o}(z) \) and \( \hat{\Theta} \). Note that it is not necessary that \( g_{K_o} = g_U \). For any \( U \in \mathfrak{U} \), we set \( \Gamma_t = \mathcal{E}_{g_U} \left[ Z_{\tau(0)}^{U^0} \right] \) and \( \hat{\Gamma}_t = \mathcal{E}_{g_{K_o}} \left[ -\Delta U_s^{U^0} \right] \), which are the solutions to the BSDE\((Z_{\tau(0)}^{U^0}, g_U)\) and BSDE\((-\Delta U_s^{U^0}, g_{K_o})\) respectively, i.e.,
\[
\begin{align*}
\Gamma_t &= Z_{\tau(0)}^{U^0} + \int_t^T g_U(s, \Theta_s) ds - \int_t^T \Theta_s dB_s, \quad t \in [0, T], \\
\hat{\Gamma}_t &= -\Delta U_s^{U^0} + \int_t^T K_o(\hat{\Theta}_s) ds - \int_t^T \hat{\Theta}_s dB_s, \quad t \in [0, T],
\end{align*}
\]
where \( \Theta, \hat{\Theta} \in \mathcal{H}_{\mathbb{F}}([0, T]; \mathbb{R}^d) \). Applying Proposition 2.7 (5) and Corollary 2.3 we obtain that for any \( t \in [0, T] \)
\[
\Gamma_{\tau(0)} - \Gamma_{\tau(0) \wedge t} = \mathcal{E}_{g_U} \left[ Z_{\tau(0)}^{U^0} \right] - \mathcal{E}_{g_U} \left[ Z_{\tau(0)}^{U^0} \right] = \mathcal{E}_{g_U} \left[ Z_{\tau(0)}^{U^0} - Z_{\tau(0)}^U \right] - \mathcal{E}_{g_U} \left[ \frac{Z_{\tau(0)}^{U^0}}{Z_{\tau(0)}^U} \right] \mathcal{F}_{\tau(0)} = Z_{\tau(0)}^{U^0} - \Gamma_t, \quad a.s.
\]
Then the continuity of processes \( \Gamma \) and \( Z_{\tau(0)}^{U^0} \) imply that
\[
\begin{align*}
\Gamma_t - Z_{\tau(0) \wedge t}^{U^0} &= Z_{\tau(0)}^{U^0} - Z_{\tau(0)}^{U^0} \mathcal{F}_{\tau(0)} + \Gamma_{\tau(0)} - \Gamma_{\tau(0) \wedge t} = Z_{\tau(0)}^{U^0} - Z_{\tau(0)}^{U^0} \mathcal{F}_{\tau(0)} + \int_{\tau(0) \wedge t}^{\tau(0)} g_U(s, \Theta_s) ds - \int_{\tau(0) \wedge t}^{\tau(0)} \Theta_s dB_s \\
&= Z_{\tau(0)}^{U^0} - Z_{\tau(0) \wedge t}^{U^0} + \int_{\tau(0) \wedge t}^{\tau(0)} H(s, \Theta_s, U_s^0) ds - \int_{\tau(0) \wedge t}^{\tau(0)} \Theta_s dB_s \\
&= -\Delta U_s^{U^0} + \Delta U_s^{U^0} \mathcal{F}_{\tau(0)} + \int_{\tau(0) \wedge t}^{\tau(0)} \left( H(s, \Theta_s, U_s^0) - H(s, \Theta_s^{U^0}, U_s^0) \right) ds - \int_{\tau(0) \wedge t}^{\tau(0)} (\Theta_s - \Theta_s^{U^0}) dB_s, \quad t \in [0, T],
\end{align*}
\]
where we used (8.93) with \( U = U^* \) in the last inequality. Since it holds \( dt \times dP\text{-a.s.} \)
\[
H(t, \Theta_t, U_t) - H(t, \Theta_t^{U^0}, U_t) = H(t, \Theta_t, U_t) - H(t, \Theta_t^{U^0}, u^*(t, \Theta_t^{U^0})) \leq H(t, \Theta_t, U_t) - H(t, \Theta_t^{U^0}, U_t) = g^o(t, \Theta_t, U_t) - g^o(t, \Theta_t^{U^0}, U_t) \leq K_o(\Theta_t - \Theta_t^{U^0}),
\]
the comparison Theorem for BSDEs (see e.g. Peng [1997, Theorem 35.3]) implies that
\[
\hat{\Gamma}_t \geq \Gamma_t - Z_{\tau(0) \wedge t}^{U^0} - \Delta U_s^{U^0}, \quad t \in [0, T].
\]
In particular, when \( t = 0 \), we can deduce from (4.17) that
\[
\mathcal{E}_{g_{K_o}} \left[ -\Delta U_s^{U^0} \right] \geq \mathcal{E}_{g_U} \left[ Z_U(\tau(0)) \right] - Z(0).
\]
Taking supremum of the right hand side over \( U \in \mathfrak{U} \) and applying Theorem 4.1 with \( \nu = 0 \), we obtain
\[
0 \geq \mathcal{E}_{g_{K_o}} \left[ -\Delta U_s^{U^0} \right] \geq \sup_{U \in \mathfrak{U}} \mathcal{E}_{g_U} \left[ Z_U(\tau(0)) \right] - Z(0) = 0,
\]
thus $\mathcal{E}_{g_{U^*}} \left[ -\Delta_{\tau(0)}^{U^*} \right] = 0$. The strict monotonicity of $g$-expectation (see e.g. Coquet et al. 2002 Proposition 2.2(iii)) then implies that $\Delta_{\tau(0)}^{U^*} = 0$, a.s. Plugging it back to (8.91) and using (8.94), we obtain

$$Z_{\tau(0)\wedge t}^{U^*,0} = Z_{\tau(0)}^{U^*,0} + \int_{\tau(0)\wedge t}^{(0)} g_U(s, \Theta_s^{U^{(0)}})ds - \int_{\tau(0)\wedge t}^{(0)} \Theta_s^{U^{(0)}} dB_s$$

$$= Z_{\tau(0)}^{U^*,0} + \int_{t}^{(0)} g_U(s, 1_{s \leq \tau(0)} \Theta_s^{U^{(0)}})ds - \int_{t}^{(0)} 1_{s \leq \tau(0)} \Theta_s^{U^{(0)}} dB_s, \quad t \in [0, T],$$

(8.96)

which implies that

$$\mathcal{E}_{g_{U^*}} \left[ Z_{\tau(0)}^{U^*,0} \right] = \mathcal{E}_{g_{U^*}} \left[ Z_{\tau(0)\wedge t}^{U^*,0} \right] = \forall t \in [0, T],$$

namely, $\left\{ Z_{\tau(0)\wedge t}^{U^*,0} \right\}_{t \in [0, T]}$ is a $g_{U^*}$-martingale. Eventually, letting $t = 0$ in (8.96), we can deduce from (4.17) and Theorem 4.1 that

$$Z(0) = Z_0^{U^*,0} = \mathcal{E}_{g_{U^*}} \left[ Z_{\tau(0)}^{U^*,0} \right] = \mathcal{E}_{g_{U^*}} \left[ Z(\tau(0)) + \int_{0}^{(0)} h_s^{U^*} ds \right] = \mathcal{E}_{g_{U^*}} \left[ Y_{\tau(0)} + \int_{0}^{(0)} h_s^{U^*} ds \right].$$

Proof of Proposition 7.3: Because of its linearity in $z$, the primary generator

$$g^o(t, w, z, u) = \langle \sigma^{-1}(t, X(w)) f(t, X(w), u), z \rangle \quad \forall (t, w, z, u) \in [0, T] \times \Omega \times \mathbb{R}^d \times S$$

(8.97)

satisfies ($g^o2$) and ($g^o4$). Then ($g^o1$) follows from the continuity of the process $\{X(t)\}_{t \in [0, T]}$ as well as the measurability of the volatility $\sigma$ and of the function $f$. Moreover, (7.15) and (7.21) imply that for a.e. $t \in [0, T]$

$$|g^o(t, \omega, z_1, u) - g^o(t, \omega, z_2, u)| = \left| \langle \sigma^{-1}(t, X(\omega)) f(t, X(\omega), u), z_2 - z_1 \rangle \right| \leq \|\sigma^{-1}(t, X(\omega))\| \cdot |f(t, X(\omega), u)| \cdot |z_2 - z_1| \leq K^2|z_1 - z_2|, \quad \forall z_1, z_2 \in \mathbb{R}^d, \quad \forall (\omega, u) \in \Omega \times S,$$

which shows that $g^o$ satisfies ($g^o4$) with $K_o = K^2$. Clearly, $\tilde{U} = H^0([0, T]; S)$ is closed under the pasting in the sense of (7.13). Hence, we know from last section that $\left\{\mathcal{E}_{g_{U^*}}\right\}_{U \in \tilde{U}}$ is a stable class of $g$-expectations, where $g_{U^*}$ is defined in (7.12).

Fix $U \in \tilde{U}$. For any $\xi \in L^2(\mathcal{F})$, we see from (7.4) that

$$\mathcal{E}_{g_{U^*}}[\xi|\mathcal{F}_t] = \xi + \int_{t}^{T} g_U(s, \Theta_s)ds - \int_{t}^{T} \Theta_s dB_s$$

$$= \xi + \int_{t}^{T} \langle \sigma^{-1}(s, X(s)), \Theta_s \rangle ds - \int_{t}^{T} \Theta_s dB_s = \xi - \int_{t}^{T} \Theta_s dB_s, \quad t \in [0, T],$$

where $B_s^{U^*} \triangleq B_t + \int_{t}^{T} \sigma^{-1}(s, X(s,U_s)) \Theta_s ds$, $t \in [0, T]$ is a Brownian Motion with respect to $P_U$. For any $t \in [0, T]$, taking $E_U[\cdot|\mathcal{F}_t]$ on both sides above yields that

$$\mathcal{E}_{g_{U^*}}[\xi|\mathcal{F}_t] = E_U \left[ \mathcal{E}_{g_{U^*}}[\xi|\mathcal{F}_t]|\mathcal{F}_t \right] = E_U[\xi|\mathcal{F}_t] - E_U \left[ \int_{t}^{T} \Theta_s dB_s^{U^*} \bigg| \mathcal{F}_t \right] = E_U[\xi|\mathcal{F}_t], \quad a.s.$$ 

(8.98)

Hence the $g$-expectation $\mathcal{E}_{g_{U^*}}$ coincides with the linear expectation $E_U$ on $L^2(\mathcal{F})$.

Clearly, the process $Y \triangleq \left\{ \varphi(X(t)) \right\}_{t \in [0, T]}$ satisfies (Y3) since $\varphi$ is bounded from below by $-K$. We see from (7.20) that for any $t \in [0, T]$

$$Y_t = \varphi(X(t)) \leq K|X(t)| \leq K\|X\|_{T}.$$ 

Taking essential supremum of $Y_t$ over $t \in \mathcal{D}_T$ yields that

$$\zeta_T \triangleq \left( \operatorname{esssup}_{t \in \mathcal{D}_T} Y_t \right)^+ \leq K\|X\|_{T}, \quad a.s.$$ 

(8.99)
For any \( t \in [0, T] \), the Burkholder-Davis-Gundy inequality, (8.1), and as well as Fubini Theorem imply that
\[
E \left[ \left( \| X \|_{\mathcal{F}_t}^2 \right) \right] = E \left[ \sup_{s \leq t} |X(s)|^2 \right] \leq 2x^2 + 2E \left\{ \sup_{s \leq t} \int_0^s \sigma(r, X)dB_r \right\} \leq 2x^2 + 2CE \int_0^T |\sigma(s, X)|^2 ds
\]
\[
\leq 2x^2 + 4C \int_0^T |\sigma(s, \tilde{0})|^2 ds + 4CE \int_0^T |\sigma(s, X) - \sigma(s, \tilde{0})|^2 ds
\]
\[
\leq 2x^2 + 4C \int_0^T |\sigma(s, \tilde{0})|^2 ds + 4Cn^2 K^2 \int_0^T E \left[ \left( \| X \|_{\mathcal{F}_t}^2 \right) \right] ds.
\]
Then applying Gronwall’s inequality yields that
\[
E \left[ \left( \| X \|_{\mathcal{F}_T}^2 \right) \right] \leq \left( 2x^2 + 4C \int_0^T |\sigma(s, \tilde{0})|^2 ds \right) e^{4Cn^2 K^2 T} < \infty,
\] (8.100)
which together with (8.99) shows that \( \zeta'_Y \in L^2(\mathcal{F}_T) \), proving (7.11).

Next, we define a function \( h^o(t, \omega, u) \triangleq h(t, X(\omega), u), \forall (t, \omega, u) \in [0, T] \times \Omega \times S \). The continuity of the process \( \{X(t)\}_{t \in [0, T]} \) and the measurability of the function \( h \) imply that \( h^o \) is \( \mathcal{P} \otimes \mathcal{G} \otimes \mathcal{B}(\mathbb{R}) \)-measurable. We see from (7.21) that \( h^o \) satisfies \((h1)\). It also follows from (7.21) that for a.e. \( t \in [0, T] \) and for any \( \omega \in \Omega \),
\[
h^o(t, \omega) \triangleq h^o(t, \omega, U_t(\omega)) = h(t, X(\omega), U_t(\omega)) \leq K\|X(\omega)\|_{\mathcal{F}_T}^\gamma, \quad \forall U \in \mathcal{U}.
\]
Taking essential supremum of \( h^o(t, \omega) \) over \( U \in \mathcal{U} \) with respect to the product measure space \( ([0, T] \times \Omega, \mathcal{P}, \lambda \times P) \) yields that
\[
h(t, \omega) \triangleq \left( \text{esssup}_{U \in \mathcal{U}} h^o(t, \omega) \right)^{\gamma} \leq K\|X(\omega)\|_{\mathcal{F}_T}^\gamma, \quad dt \times dP\text{-a.s.,}
\]
which leads to that \( \int_0^T h(t, \omega) dt \leq KT\|X(\omega)\|_{\mathcal{F}_T}^\gamma \), a.s. Hence, (8.100) implies that \( \int_0^T h(t, \omega) dt \in L^2(\mathcal{F}_T) \), proving \((h2)\) for \( h^o \).

We can apply the optimal stopping theory developed in Section 4 to the triple \( \{\xi_{\bar{u}}\}_{U \in \mathcal{U}} \) \( \{h^o_U\}_{U \in \mathcal{U}}, Y \) and use (8.98) to obtain (7.17). In addition, if there exists a measurable mapping \( u^* : [0, T] \times \Omega \times \mathbb{R}^d \rightarrow S \) satisfying (7.18), then (8.97) indicates that for any \( (t, \omega, z) \in [0, T] \times \Omega \times \mathbb{R}^d \)
\[
\sup_{u \in S} \left( g^o(t, \omega, z, u) + h^o(t, \omega, u) \right) = \sup_{u \in S} \bar{H}(t, X(\omega), z, u) = \bar{H}(t, X(\omega), z, u^*(t, X(\omega), z)) \quad (8.101)
\]
\[
= g^o(t, \omega, z, u^*(t, X(\omega), z)) + h^o(t, \omega, u^*(t, X(\omega), z)),
\]
which shows that (7.14) holds for the mapping \( \bar{u}^*(t, \omega, z) = u^*(t, X(\omega), z), \quad (t, \omega, z) \in [0, T] \times \Omega \times \mathbb{R}^d \). Therefore, an application of Theorem 7.12 yields (7.19) for some \( U^* \in \mathcal{U} \).

**Proof of Proposition 7.4** (7.24) directly follows from Briand and Hu 2008 (Theorem 5). To see (7.25), we set \( \Delta \Gamma \triangleq \Gamma^{\xi, \tilde{\gamma}} \Delta - \Gamma^{\xi, \tilde{\gamma}} \) and \( \Delta \Theta \triangleq \Theta^{\xi, \tilde{\gamma}} - \Theta^{\xi, \tilde{\gamma}} \), then (7.22) (i) implies that
\[
d\Gamma_{\xi_1} = -\left( \tilde{g}(t, \Theta^{\xi_1}) - \tilde{g}(t, \Theta^{\xi_2}) \right) dt + \Delta \Theta_{\xi_1} dB_t = -\int_0^1 \frac{\partial \tilde{g}}{\partial z}(t, \lambda \Delta \Theta_t + \Theta^{\xi_2}) \Delta \Theta_t d\lambda dt + \Delta \Theta_t dB_t
\]
\[
= \Delta \Theta_t (-a_t dt + dB_t), \quad t \in [0, T],
\]
where \( a_t \triangleq \int_0^1 \frac{\partial \tilde{g}}{\partial z}(\lambda \Delta \Theta_t + \Theta^{\xi_2}) d\lambda, \quad t \in [0, T] \). Since \( M_F([0, T]; \mathbb{R}^d) \subset M_F^2([0, T]; \mathbb{R}^d) = \mathcal{H}_F^2([0, T]; \mathbb{R}^d) \), one can deduce from (7.22) (ii) that
\[
E \int_0^T |a_t|^2 dt \leq E \int_0^T \int_0^1 \left| \frac{\partial \tilde{g}}{\partial z}(\lambda \Delta \Theta_t + \Theta^{\xi_2}) \right|^2 d\lambda dt \leq 2K^2T + 2K^2 E \int_0^T \int_0^1 \left| \lambda \Theta^{\xi_1, \tilde{\gamma}} + (1 - \lambda) \Theta^{\xi_2, \tilde{\gamma}} \right|^2 d\lambda dt
\]
\[
\leq 2K^2T + \frac{4}{3} K^2 E \int_0^T \left( |\Theta^{\xi, \tilde{\gamma}}|^2 + |\Theta^{\xi_2, \tilde{\gamma}}|^2 \right) dt < \infty,
\]
Moreover, Doob’s inequality shows that

$$E \left[ \sup_{t \in [0, T]} \left| \int_0^t a_s dB_s \right|^2 \right] \leq 4E \left[ \int_0^T a_s dB_s \right]^2 = 4E \int_0^T |a_t|^2 dt < \infty.$$  (8.101)

Thus, we can define process $Q_t = \exp \left\{ -\frac{1}{2} \int_0^t |a_s|^2 ds + \int_0^t a_s dB_s \right\}$, $t \in [0, T]$ as well as stopping times

$$\nu_n = \inf \{ t \in [\nu, T] : Q_t \lor |\Delta \Gamma_t| > n \} \land T, \ \forall \ n \in \mathbb{N}.$$  It is clear that $\lim_{n \to \infty} \uparrow \nu_n = T$, a.s., and (8.101) assures that there exists a null set $N$ such that for any $\omega \in N^c$, $T = \nu_m(\omega)$ for some $m = m(\omega) \in \mathbb{N}$.

For any $n \in \mathbb{N}$, integrating by parts on $[\nu, \nu_n]$ yields that

$$Q_{\nu_n} \Delta \Gamma_{\nu_n} = Q_{\nu} \Delta \Gamma_{\nu} - \int_{\nu}^{\nu_n} Q_t \Delta \Theta_t dt + \int_{\nu}^{\nu_n} \Delta \Gamma_t Q_t dB_t + \int_{\nu}^{\nu_n} Q_t \Delta \Theta_t dt$$

$$= \int_{\nu}^{\nu_n} (Q_t \Delta \Theta_t + \Delta \Gamma_t Q_t dB_t),$$

which implies that $E[Q_{\nu_n} \Delta \Gamma_{\nu_n}] = 0$. Thus we can find a null set $N_n$ such that

$$\Delta \Gamma_{\nu_n}(\omega) = 0, \ \forall \omega \in N_n^c.$$  Eventually, for any $\omega \in \left\{ N \cup \left( \bigcup_{n \in \mathbb{N}} N_n \right) \right\}^c$, we have

$$\xi^1(\omega) = \Gamma_{\nu_n}^{\xi^1}(\omega) = \lim_{n \to \infty} \Gamma_{\nu_n}^{\xi^1}(\omega) = \lim_{n \to \infty} \Gamma_{\nu_n}^{\xi^2}(\omega) = \Gamma_{\nu_n}^{\xi^2}(\omega) = \xi^2(\omega).$$

**Proof of Proposition 7.5.** Let $\left\{ A_n \right\}_{n \in \mathbb{N}}$ be any sequence in $\mathcal{F}_T$ such that $\lim_{n \to \infty} \downarrow 1_{A_n} = 0$, a.s. For any $\xi, \eta \in L^{\infty}(\mathcal{F}_T)$, $\left\{ \xi \in L^{\infty}(\mathcal{F}_T) : \xi \geq 0, \ a.s. \right\}$, since

$$E \left[ e^{\lambda |\xi|} \right] < \infty \ \text{and since} \ \sup_{n \in \mathbb{N}} E \left[ e^{\lambda |\xi + 1_{A_n}|} \right] \leq E \left[ e^{\lambda |\xi + 1_{\mathbb{N}}} \right] \leq \frac{1}{2} E \left[ e^{2\lambda |\xi|} \right] + \frac{1}{2} E \left[ e^{2\lambda |\eta|} \right] < \infty$$

holds for each $\lambda > 0$, Lemma 7.2 implies that

$$0 = \lim_{n \to \infty} E \left[ \sup_{t \in [0, T]} \left| \mathcal{E}_\lambda(\xi + 1_{A_n} | \mathcal{F}_t) - \mathcal{E}_\lambda(\xi | \mathcal{F}_t) \right| \right] \geq \lim_{n \to \infty} \left| \mathcal{E}_\lambda(\xi + 1_{A_n} | \mathcal{F}_t) - \mathcal{E}_\lambda(\xi | \mathcal{F}_t) \right| \geq 0,$$

thus $\mathcal{E}_\lambda$ satisfies (H2). Similarly, we can show that (H1) also holds for $\mathcal{E}_\tilde{\lambda}$.

Moreover, for any $\nu \in \mathcal{S}_{\mathcal{F}_T}$ and $\xi \in L^{\infty}(\mathcal{F}_T)$, since the process $\Gamma^{\xi, \tilde{\lambda}}$ belongs to $\mathcal{C}^2([0, T])$, one can deduce that $\mathcal{E}_\tilde{\lambda}(\xi | \mathcal{F}_\nu) = \Gamma^{\xi, \tilde{\lambda}} \in L^{\infty}(\mathcal{F}_T)$. Then the continuity of the process $X_\xi = \mathcal{E}_\lambda[\xi | \mathcal{F}_t]$ implies that $X_\xi^{\nu, +} = X_\xi = \mathcal{E}_\lambda[\xi | \mathcal{F}_\nu] \in L^{\infty}(\mathcal{F}_T)$, which proves (H3).

**Proof of Theorem 7.3.** This Proposition is just an application of the optimal stopping theory developed in Section 4 to the singleton $\{ \mathcal{E}_\lambda \}$. Hence, it suffices to check that $Y$ satisfies (Y1), (Y2) and (1.21).

Similar to (8.80), it holds except on a null set $N$ that

$$C_\lambda \leq Y_\lambda \leq \hat{\lambda}_\lambda, \ \forall \ t \in [0, T], \ \text{thus} \ C_\lambda \leq Y_\rho \leq \hat{\lambda}_\lambda, \ \forall \ \rho \in \mathcal{S}_{\mathcal{F}_T}.$$  (8.102)

Since $\hat{\lambda}_\lambda \in L^\infty(\mathcal{F}_T)$, it holds for any $\rho \in \mathcal{S}_{\mathcal{F}_T}$ that

$$E \left[ e^{\lambda |Y_\rho|} \right] \leq E \left[ e^{\lambda |\hat{\lambda}_\lambda - C_\lambda|} \right] = e^{-\lambda C_\lambda} E \left[ e^{\lambda |\hat{\lambda}_\lambda|} \right] < \infty, \ \forall \lambda > 0.$$  (8.103)
which implies that \( Y_\rho \in L^\#((F_T) = \text{Dom}(\{E_\delta\}) \). Hence (Y1) holds.

Next, for any \( \rho \in S_{0,T} \) and \( t \in D_T \), Proposition 2.2 (2), (8.102) show that

\[
C_Y = E_\delta[C_Y|F_t] \leq E_\delta[Y_\rho|F_t] \leq E_\delta[\hat{C}_Y|F_t] = \Gamma^{\hat{C}_Y}_t \leq \sup_{t \in [0,T]} |\Gamma^{\hat{C}_Y}_t|, \ a.s.
\]

Taking essential supremum of \( E_\delta[Y_\rho|F_t] \) over \((\rho,t) \in S_{0,T} \times D_T \) yields that

\[
C_Y \leq \text{ess sup}_{(\rho,t) \in S_{0,T} \times D_T} E_\delta[Y_\rho|F_t] \leq \sup_{t \in [0,T]} |\Gamma^{\hat{C}_Y}_t|, \ a.s.
\]

Since \( \Gamma^{\hat{C}_Y}_t \in C^\infty([0,T]) \), or equivalently \( \sup \ |\Gamma^{\hat{C}_Y}_t| \in L^\#(F_T) \), we can deduce that \( \text{ess sup} \ E_\delta[Y_\rho|F_t] \in L^\#(F_T) = \text{Dom}(\{E_\delta\}) \), which together with Remark 4.2 (2) proves (Y2).

Moreover, for any \( \nu, \rho \in S_{0,T} \) with \( \nu \leq \rho \), a.s. and any sequence \( \{\rho_n\}_{n \in \mathbb{N}} \subset S_{\nu,T} \) increasing a.s. to \( \rho \), the continuity of the process \( Y \) implies that \( \frac{n}{n-1}Y_{\rho_n} \) converges to \( Y_\rho \) a.s. By (8.102), one can deduce that

\[
\sup_{n \in \mathbb{N}} E \left[ \exp \left( \lambda \frac{n}{n-1}Y_{\rho_n} \right) \right] \leq \sup_{n \in \mathbb{N}} E \left[ e^{2\lambda Y_{\rho_n}} \right] \leq E \left[ e^{2\lambda (\hat{C}_Y - C_Y)} \right] = e^{-2\lambda C_Y} E \left[ e^{2\lambda \hat{C}_Y} \right] < \infty, \ \forall \lambda > 0,
\]

which together with (8.103) allows us to apply Lemma 4.2

\[
0 = \lim_{n \to \infty} E \left[ \sup_{t \in [0,T]} \left| E_\delta \left[ \frac{n}{n-1}Y_{\rho_n}|F_t \right] - E_\delta[Y_\rho|F_t] \right| \right] \geq \lim_{n \to \infty} E \left[ \left| E_\delta \left[ \frac{n}{n-1}Y_{\rho_n}|F_t \right] - E_\delta[Y_\rho|F_t] \right| \right] \geq 0,
\]

thus \( \lim_{n \to \infty} E \left[ \left| E_\delta \left[ \frac{n}{n-1}Y_{\rho_n}|F_t \right] - E_\delta[Y_\rho|F_t] \right| \right] = 0 \). Then we can find a subsequence \( \{n_k\}_{k \in \mathbb{N}} \) of \( \mathbb{N} \) such that

\[
\lim_{n \to \infty} \left| E_\delta \left[ \frac{n_k}{n_k-1}Y_{\rho_{n_k}}|F_t \right] - E_\delta[Y_\rho|F_t] \right| = 0, \ a.s.,
\]

proving (1.21) for \( Y \).

\[ \square \]

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