Abstract

In this paper we use the Kazhdan grading on the symmetric algebra $S(g)$ of a semi-simple Lie algebra $g$ to introduce the Kazhdan grading into Kontsevich’s deformation quantization of the linear Poisson manifold $g^\ast$. Choosing a nilpotent element $e \in g$, we provide an alternative model of the $W$−algebra associated to the pair $(g, e)$. It is the 0−th cohomology of a flat $A_\infty$−algebra, called the Cattaneo-Felder reduction algebra, appearing in the coisotropic submanifold case.

1 Introduction

1.1 In [18] Kontsevich solved the deformation quantization problem of Poisson manifolds, proving his Formality Theorem for the $L_\infty$− algebras $T_{poly}(\mathbb{R}^k)$ of polyvector fields and $D_{poly}(\mathbb{R}^k)$ of polydifferential operators of bounded order on $\mathbb{R}^k$. The theorem states the map $U : T_{poly}(\mathbb{R}^k) \rightarrow D_{poly}(\mathbb{R}^k)$ defined by its Taylor coefficients

$U_n := \sum_{m \geq 0} \left( \sum_{\Gamma \in Q, m} \omega_\Gamma B_\Gamma \right) \tag{1}$

is an $L_\infty$− morphism and a quasi-isomorphism. Properties of this map prove that there is a bijection between the gauge equivalence classes of $*$−products on $C^\infty(\mathbb{R}^k)$ and the gauge equivalence classes of Poisson structures $\pi$ on $\mathbb{R}^k$. Kontsevich also provided an explicit formula of the $*$−product, denoted by $*_K$, associated to a Poisson structure. The result states that choosing a Poisson structure $\pi$ on $\mathbb{R}^k$, the operator $*_K : C^\infty(\mathbb{R}^k)[[\epsilon]] \times C^\infty(\mathbb{R}^k)[[\epsilon]] \rightarrow C^\infty(\mathbb{R}^k)[[\epsilon]]$ defined for $f, g \in C^\infty(\mathbb{R}^k)$ by the formula

$f *_K g := fg + \sum_{n=1}^{\infty} \epsilon^n \left( \frac{1}{n!} \sum_{\Gamma \in Q_{n,2}} \omega_\Gamma B_\Gamma( f, g ) \right) \tag{2}$

is an associative product. The formula comes from [11] and the ingredients are smaller cases of the ones therein: $\epsilon$ is a deformation parameter, $Q_{n,2}$ is a family of graphs, $\omega_\Gamma$ is real coefficient associated to each $\Gamma \in Q_{n,2}$ computed as the integral of a differential form on a compactified concentration manifold $C \subset (\mathcal{H}^+)^n \times \mathbb{R}^2$, $\mathcal{H}^+$ being the upper hyperbolic half-plane. $B_\Gamma$ is a linear bidifferential operator on $C^\infty(\mathbb{R}^k) \times C^\infty(\mathbb{R}^k)$. Details can be found in [18], [1], [9], [13]. This product was later globalised on a Poisson manifold $X$ in [12].
Cattaneo and Felder considered the case of coisotropic submanifolds $C \subset X$ in [11]. The Relative Formality Theorem proves an $L_{\infty}$–isomorphism from $T(X, C) := \lim_{\to} T(X)/I(C)^n T(X)$ ($T(X)$ being the DGLA of multivector fields on $X$ as defined by Kontsevich), the DGLA of multivector fields in an infinitesimal neighbourhood of $C$ to $\mathcal{D}(A) = \oplus_n \mathcal{D}^n(A)$ where $\mathcal{D}^n(A) := \prod_{p+q-1=n} \text{Hom}(\wedge^p \mathfrak{g}, A)$, and $A = \Gamma(C, \wedge T_X)$, the sections of this exterior algebra bundle. They associate a curved $A_{\infty}$ algebra, which in the linear Poisson case $X = g^*$, the dual of a Lie algebra $g$, is flat. Its $0$–th cohomology then accepts an associative product on it, called the Cattaneo-Felder product $*_{CF, e} : C^\infty(\mathbb{R}^r)[[\epsilon]] \times C^\infty(\mathbb{R}^r)[[\epsilon]] \to C^\infty(\mathbb{R}^r)[[\epsilon]]$. It is given by the formula $f *_{CF, e} g := f \cdot g + \sum_{n=1}^{\infty} \epsilon^n \left( \frac{\prod}{U} \sum_{\Gamma \in \mathcal{Q}^{(2)}_{n, 2}} \omega_T B^n_T(f, g) \right) $

where $\mathcal{Q}^{(2)}_{n, 2}$ is a family of graphs with two colors, a notion to be explained in § 2.1, and $\omega_T, B^n_T$ is a coefficient and bidifferential operator computed similarly to the case $C = \emptyset$ but with some substantial modifications. Here we considered $X = \mathbb{R}^k, C = \mathbb{R}^r, r < k$ as a coisotropic submanifold.

In [15] the general theory was studied in the case $X = g^*$ and $C = m^\perp$ for $m \subset g$ a Lie subalgebra. The affine case was also studied, namely $C = \chi + m^\perp$, where $\chi$ is a character of $m$. The basic objects of study was the $\epsilon$–deformed and the (non-$\epsilon$–) deformed reduction algebras associated to the data $g, m, q, \chi$, where $q$ is a supplementary of $m$ in $g$. In [3] the $\epsilon$–deformed reduction algebra, denoted as $H^0_\epsilon(m^\perp, q^\varepsilon_{m^\perp, q})$, was proved to be isomorphic to $(U_\epsilon(g)/U_\epsilon(g) m_{\chi + \rho})^m$, where $\rho \in m^*$ defined by $\rho(H) = -\omega_T \text{Tr}(adH), H \in m$, $\Gamma'$ being a short loop appearing in this situation following [14].

1.2 The systematic study of $W$–algebras began with the paper of Premet [22]. The main motivation behind understanding $W$– algebras began with the paper of Premet [22]. The main motivation behind understanding $U(g)$ of a semisimple Lie algebra $g$. Using the 1-1 correspondence of them with the primitive ideals of $U(g)$, the main line of approach is to study the finite dimensional irreducible representations of the $W$– algebra and then pass the results on $U(g)$ via Skryabin equivalence (Appendix in [22]). Thus, the central problem of the theory is to classify the finite dimensional irreducible representations of the $W$– algebra. We first review the approach of Premet (see [22, 24, 23, 25]). Let $G$ be a connected reductive Lie group and $g$ its Lie algebra. Choose a nilpotent element $e \in g$ and pick $h, f, e \in g$ forming an $sl_2$–triple with $e$. There exists a $g$–invariant bilinear form $\langle \cdot, \cdot \rangle$ such that $\langle e, f \rangle = 1$, e.g a normalization of the Killing form. Let $\chi \in g^*$ be defined for all $x \in g$ by $\chi(x) = \langle e, x \rangle$. Set $g(\iota) := \{ \xi \in g \mid [h, \xi] = i\xi \}$. Consider a skew-symmetric form $\omega_\chi$ on $g$ defined by $\omega_\chi(\xi, \eta) = \chi(\xi, \eta)$. The restriction $\omega_\chi|_{g(\iota)}$ is non-degenerate so one can pick a lagrangian subspace $\iota \subset g(\iota)$. Set $m := \Gamma \bigoplus_{i \leq -2} g(\iota)$ (so $\chi$ is a character of $m$). Define now $m_\chi := \{ \xi - \chi(\xi), \xi \in m \}$ a shift of $m$ and let $q$ be such that $q = m \oplus q$. From the PBW theorem one gets $U(g) \simeq U(g)/U(g)m_\chi$. On the quotient $U(q)$ one can define a natural $ad_\chi$–action and then define the (finite) $W$-algebra $U(g, e)$ corresponding the data $(g, e)$ as the quantum Hamiltonian reduction $(U(g)/U(g)m_\chi)^m$. The associated graded algebra $gr U(g, e)$ is naturally isomorphic to the graded (with the Kazhdan grading, to be defined later) algebra of functions on the Slodowy slice $S := e + \text{kef}(f)$, (see [22] and [17] § 1.2). Geometrically, it is a transverse slice to the adjoint orbit $\mathcal{O} := G \cdot e$.

The quotient $U(g)/U(g)m_\chi$ has a natural $U(g) - U(g, e)$–bimodule structure. Let $N \mapsto S(N) := (U(g)/U(g)m_\chi) \otimes_{U(g, e)} N$ be the functor from the category $U(g, e)$–Mod of left $U(g, e)$–modules to the category $U(g)$–Mod of $U(g)$–modules. The map $P : N \mapsto \text{Ann}_U(g) \text{S}(N)$ defines an equivalence of $U(g, e)$–Mod to the category of Whittaker modules of $U(g)$. In [23], the author proved that the image of $P$ consists of ideals $J$ whose associated variety in $g$ coincides with $\mathcal{O}$ and then in [24] Theorem 1.1, he proved, under a small condition on the infinitesimal character of $J$, that any such primitive ideal is of the form $\text{Ann}_U(g)(S(N))$, where $N$ is a finite dimensional representation of the $W$– algebra. This was later proved in full generality in [26].

1.3 The approach followed by Losev in [20, 21] is to close to deformation quantization and begins
with a more geometric definition of the $W-$ algebra $\mathcal{W}$ as the, specialized at $h = 1$, $G-$ invariants of the $\h-$ deformed algebra $\mathbb{K}[X][\h]$, where $X = G \times S$. This algebra is equipped with a $*$ product using the Fedosov construction (see [16]). In [20,21], Losev describes two category equivalences between suitable categories of $\mathcal{W}-$ modules and of $U(\mathfrak{g})-$ modules. Particularly in [21], the author classified the finite dimensional irreducible representations of $W-$ algebras up to a $Q/Q^0,$ action on the set of two sided ideals of finite codimension in $\mathcal{W}$, where $Q := Z_G(e, h, f)$. To do this he studied the category $HC(\mathcal{W}(g))$ of Harish-Chandra bimodules of $U(\mathfrak{g})$ whose corresponding primitive ideal's associated variety is the Zariski closure $\overline{O}$ and the category $HC^Q_{\text{fin}}(\mathcal{W})$ of finite dimensional $Q-$ equivariant bimodules of $\mathcal{W}$. The proof was based on constructing certain functors $\mathfrak{O}_{\chi}^* : HC^Q_{\text{fin}}(\mathcal{W}) \rightarrow HC_{\overline{O}}(\mathfrak{g})$ and $\mathfrak{O}_{\chi} : HC_{\overline{O}}(\mathfrak{g}) \rightarrow HC^Q_{\text{fin}}(\mathcal{W})$ respecting many primitive ideal properties on each side (see Theorem 1.3.1 and section 3.4 of [20]). Finally, the classification of finite dimensional irreducible $W-$ modules in the case $\mathfrak{g} = \mathfrak{gl}_n$ was achieved in [7, 8] using a combinatorial approach based on shifted Yangians.

1.4 Our main result (Theorem 3.1) proves that when $\mathfrak{g}$ is semisimple, there is an isomorphism between the (non-$\epsilon-$) deformed reduction algebra $H^0(\mathfrak{m}_\chi^+, d\mathfrak{m}_\chi^+)$ associated to $(\mathfrak{g}, \epsilon)$ and the $W-$ algebra $(U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}_\chi^+)^{\mathfrak{m}^*}$ with respect to the isomorphism of [5], the character $\rho \in \mathfrak{m}^*$ is missing, since $\mathfrak{m}$ is a nilpotent subalgebra and $\rho(H) := -\omega_T \text{Tr(adH)} = 0$, $\forall H \in \mathfrak{m}$ by Engel’s Theorem. In the case when $\mathcal{O}$ is principal, Theorem 3.1 recovers the Duflo algebra isomorphism $(S(\mathfrak{g})^{G,*}) \simto Z(\mathfrak{g})$ between the $G-$ invariants of $S(\mathfrak{g})$ and the center of $U(\mathfrak{g})$, as in [13] pp.43-44. Note that the (non- $\epsilon-$) deformed reduction algebra $H^0(\mathfrak{m}_\chi^+, d\mathfrak{m}_\chi^+)$ is a different from the $\epsilon-$ deformed one $H^0(\mathfrak{m}_\chi^+, d\mathfrak{m}_\chi^+ \epsilon)$. In fact, it is known that for every Lie algebra $\mathfrak{g}$, subalgebra $\mathfrak{m} \subset \mathfrak{g}$ and character $\chi$, $H^0(\mathfrak{m}_\chi^+, d\mathfrak{m}_\chi^+ \epsilon) \hookrightarrow H^0(\mathfrak{m}_\chi^+, d\mathfrak{m}_\chi^+) \hookrightarrow (U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}_\chi^+)^{\mathfrak{m}^*}$ (see Lemma 3.1 and Proposition 3.4 of [2]). The absence of the deformation parameter $\epsilon$ in this first result is not straightforward. We use suitable definitions of the Kazhdan grading, a standard feature of the $S_\mathfrak{g}$ theory of semisimple Lie algebras, for the bidifferential operators appearing in deformation quantization. The proof is close to the one for the $\epsilon-$ deformed objects found in [4]. The new feature is that we write homogeneous reduction equations, both in the definition of $H^0(\mathfrak{m}_\chi^+, d\mathfrak{m}_\chi^+)$ as well as in the proof of Theorem 3.1. In view of Premet’s definition presented in section 1.2, this provides us with a new model of the $W-$ algebra. This way, we transfer the study of $W-$ algebras to the study of $H^0(\mathfrak{m}_\chi^+, d\mathfrak{m}_\chi^+ \epsilon)$. Since we have an explicit formula of the $*$ product, one is able to compute the relations of the $*$- commutator $[P,Q]_* := P*Q - Q*P$.

2 Kazhdan grading in Deformation Quantization.

2.1. Deformation Quantization background.

2.1.1. Some notation. Let $\mathbb{K}$ be a field with $\text{char}\mathbb{K} = 0$ and $G$ a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$ of finite dimension. Let $\mathfrak{m} \subset \mathfrak{g}$ be a subalgebra with Lie group $M$, $\chi$ a character of $M$, $S(\mathfrak{g})$ the symmetric and $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$, respectively. Set $\mathfrak{m}_\chi$ to be the vector subspace of $S(\mathfrak{g})$ generated by the set $\{m + \chi(m), m \in \mathfrak{m}\}$ and denote as $S(\mathfrak{g})\mathfrak{m}_\chi$, $U(\mathfrak{g})\mathfrak{m}_\chi$, the ideal of $S(\mathfrak{g})$ and right ideal of $U(\mathfrak{g})$, respectively, generated by $\mathfrak{m}_\chi$. One has $S(\mathfrak{g}) \simeq \mathbb{K}[\mathfrak{g}^+]$, the algebra of polynomials on the dual Lie algebra $\mathfrak{g}^*$. This algebra is equipped with a natural Poisson structure defined for $x_1, x_2 \in \mathfrak{g}$ by $\{x_1, x_2\} := [x_1, x_2]$. In turn this induces a Poisson structure on $(S(\mathfrak{g})/S(\mathfrak{g})\mathfrak{m}_\chi)^{\mathfrak{m}^*}$, the invariants with respect to the adjoint action ad$\chi$, $x \in \mathfrak{m}$, on $\mathfrak{g}$, extended to $S(\mathfrak{g})$. Let $\mathfrak{m}^\perp := \{l \in \mathfrak{g}^*/l(\mathfrak{m}) = 0\}$ and $\mathbb{K}[\chi + \mathfrak{m}^\perp]$ be the Poisson algebra of $\mathfrak{m}$-invariant polynomial functions on $\chi + \mathfrak{m}^\perp$. Then $(S(\mathfrak{g})/S(\mathfrak{g})\mathfrak{m}_\chi)^{\mathfrak{m}^*} \simeq \mathbb{K}[\chi + \mathfrak{m}^\perp]^{\mathfrak{m}^*}$ as algebras. Consider $\mathfrak{g}^*$ as a Poisson manifold. Then $C^\infty(\mathfrak{g}^*) = S(\mathfrak{g})$ is a Poisson algebra with the bracket defined previously and one can directly apply Kontsevich’s results. Considering $\mathfrak{m}_\chi^+ := \{f \in \mathfrak{g}^*/f|\mathfrak{m} = -\chi\}$ as a coisotropic submanifold of $\mathfrak{g}^*$ one can apply
respectively the biquantization techniques of [13] to write the corresponding $\ast_{CF}$– product. We briefly recall some of the necessary definitions adjusted in our setting.

2.1.2. Kontsevich’s construction. Denote by $Q_{n,2}$, $n \in \mathbb{N}^+$ the set of all admissible graphs $\Gamma$, meaning graphs with the following properties: The set $V(\Gamma)$ of vertices of $\Gamma$ is the disjoint union of two ordered sets $V_1(\Gamma)$ and $V_2(\Gamma)$, isomorphic to $\{1, \ldots, n\}$ and $\{1, 2\}$ respectively. Their elements are called type I vertices, for $V_1(\Gamma)$, and type II vertices, for $V_2(\Gamma)$. The set $E(\Gamma)$ of edges in the graph is finite. Each edge starts from a type I vertex and ends to a vertex of type I or type II allowing no loops or double edges. All elements of $E(\Gamma)$ are oriented and the set of edges $S(r)$ starting from $r \in V_1(\Gamma)$ is ordered. This induces an order on $E(\Gamma)$, the one compatible with the order on $V_1(\Gamma)$, and $S(r), r \in V_1(\Gamma)$.

To the data of a Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ and a graph $\Gamma \in Q_{n,2}$ one associates a bidifferential operator $B_{\Gamma}$ as follows: Suppose $\mathfrak{g}$ is $k$– dimensional and $\{x_1, \ldots, x_k\}$ is a basis of $\mathfrak{g}$. Let $L : E(\Gamma) \rightarrow \{x_1, \ldots, x_k\}$ be a labeling function. Fix a vertex $r \in \{1, \ldots, n\}$. If $\mathrm{card}(S(r)) \neq 2$, set $B_{\Gamma} = 0$. If $\mathrm{card}(S(r)) = 2$, let $S(r) = \{c^1_r, c^2_r\}$ be the ordered set of edges leaving $r$. Associate the bracket $[L(c^1_r), L(c^2_r)]$ to $r$. To each vertex $1, 2 \in V_2(\Gamma)$ associate respectively a function $F, G \in S(\mathfrak{g})$, and to the $p^{th}$– edge of $S(r)$, associate the partial derivative w.r.t the coordinate variable $L(c^p_r)$. This derivative acts on the function or bracket associated to $v \in V_1(\Gamma) \cup V_2(\Gamma)$ where the edge $e^p_r$ arrives. Since $E(\Gamma) \subset V_1(\Gamma) \times (V_1(\Gamma) \cup V_2(\Gamma))$, let $(p, m) \in E(\Gamma)$ represent an oriented edge of $\Gamma$ from $p$ to $m$. The operator associated to a $\Gamma \in Q_{n,2}$ and $(\mathfrak{g}, [\cdot, \cdot])$ is

$$B_{\Gamma}(F, G) = \sum_{L : E(\Gamma) \rightarrow \{1, \ldots, k\}} \left[ \prod_{r=1}^{\#(V_1(\Gamma))} \prod_{\delta \in E(\Gamma), \delta = (r, r)} \partial_{L(\delta)} \right] [L(c^1_r), L(c^2_r)] \times \prod_{\delta \in E(\Gamma), \delta = (-, 1)} \partial_{L(\delta)} (F) \times \prod_{\delta \in E(\Gamma), \delta = (-, 2)} \partial_{L(\delta)} (G).$$

We drop the definition of the coefficient $\omega_\Gamma$ in [2] since it is not central for the paper and can be found in the given references, e.g. [13, 15]. In deformation quantization of a (linear) Poisson manifold, one trivially has a single choice of variables for every edge in a graph $\Gamma \in Q_{n,2}$ since for $e \in E(\Gamma)$, $L(e)$ determines a basis variable of $\mathfrak{g}$. In the case of biquantization with $X = \mathfrak{g}^\ast$ and $C = \mathfrak{m}^\perp$ respectively we consider two colors; each edge of a colored graph $\Gamma$ carries a color, either $(\ast)$ or $(\dagger)$. Double edges are not allowed, meaning edges with the same color, source and target. Suppose $\mathfrak{q}$ is a supplementary of $\mathfrak{m}$, i.e $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{q}$, $\{m_1, \ldots, m_i\}$ is a basis of $\mathfrak{m}$ and $\{q_1, \ldots, q_r\}$ a basis of $\mathfrak{q}$. We identify spaces $\mathfrak{q}^\ast \simeq \mathfrak{g}^\ast/\mathfrak{m}^\ast \simeq \mathfrak{m}^\perp$. For $e \in E(\Gamma)$, let $c_e \in \{\ast, \dagger\}$ be its color. Let $L : E(\Gamma) \rightarrow \{m_1, \ldots, m_t, q_1, \ldots, q_r\}$, satisfying $L(e) \in \{m_1, \ldots, m_t\}$ if $c_e = \ast$ and $L(e) \in \{q_1, \ldots, q_r\}$ if $c_e = \dagger$ be a 2-colored labeling function. This way, the dual basis variables $\{m^1_1, \ldots, m^*_i\}$ of $\mathfrak{m}^\ast$ are associated to the color $(\ast)$ and dual basis variables $\{q^1_1, \ldots, q^*_r\}$ of $\mathfrak{q}^\ast$ are associated to $(\dagger)$. Graphically, the color $(\ast)$ will be represented with a dotted edge and the color $(\dagger)$ will be represented with a straight edge. The corresponding formulas (4) and (2) need some modifications in biquantization (in the Lie algebra case); for $F, G \in S(\mathfrak{q})$, one has to use the 2-colored labeling function $L$ that we just described. From now on, all graphs, their associated operators $B_{\Gamma}$ and coefficients $\omega_\Gamma$ are colored. We denote by $Q_{n,2}^{(i)}$ the family of admissible graphs with $i$ colors.

2.1.3. Reduction algebras. We now need to describe some particular graphs (see [15] § 1.3, 1.6 and [2] § 2.3). They are colored graphs with only one type II vertex and with an edge colored by $(\dagger)$ with no end. Denote this edge as $e_\infty$. As usually in the literature, the point moving on the horizontal axis of a biquantization diagramm is associated with a function $F \in S(\mathfrak{q})$. Set $Q_{n,1}^\infty$ to be the family of such graphs with $n$ type I vertices, namely graphs of the categories 1 and 3 in the following definition.
Definition 2.1

1. Bernoulli. The Bernoulli type graphs with \( i \) type I vertices, \( i \in \mathbb{N}, i \geq 2 \), will be denoted by \( B_i \). They have \( 2i \) edges, \( i \) of them pointing to the type II vertex, and leave an edge towards \( \infty \). These conditions imply the existence of a vertex \( s \in V_1(\Gamma) \) that receives no edge, called the root.

2. Wheels. The wheel type graphs with \( i \) type I vertices, \( i \in \mathbb{N}, i \geq 2 \), will be denoted by \( W_i \). They derive the function \( F \) \( i \) times, have \( 2i \) edges and leave no edge to \( \infty \).

3. Bernoulli attached to a wheel. Graphs of this type with \( i \) type I vertices, \( i \in \mathbb{N}, i \geq 4 \), will be denoted by \( BW_i \). They have \( i - 1 \) edges towards the type II vertex and leave an edge to \( \infty \). For an \( W_m \)– type graph \( W_m \) attached to a \( B_2 \)– type graph \( B_3 \), we will write \( B_3W_m \in B_3W_m \). Obviously \( B_3W_m \subset BW_{2+m} \).

Let \( \{e^1_i, e^2_i\} \) be the ordered set of edges leaving the vertex \( l \in V_1(\Gamma) \) of a colored graph \( \Gamma \in Q_{m,1}^\infty \).
For such a \( \Gamma \) and using the notation \( m^i_\Gamma := \partial_i \), let \( B_\Gamma : S(q) \rightarrow S(q) \otimes m^* \), \( F \rightarrow B_\Gamma(F) = \sum_{i=1}^l B_i(F) \cdot m^i_\Gamma \) where

\[
B_i(F) = \sum_{L: E(\Gamma) \rightarrow \{m_1, \ldots, m_q\}} \prod_{r=1}^n \left( \prod_{e \in E(\Gamma), e=(\cdot,r)} \partial_{L(e)} \left[ L(e^1_i), L(e^2_i) \right] \right) \times \prod_{e \in E(\Gamma), e=(l,F)} \partial_{L(e)} F \tag{4}
\]

Definition 2.2 \cite{13} Let \( \mathfrak{g} \subset \mathfrak{g} \) be a subalgebra, \( \chi \) a character of \( \mathfrak{m} \) and \( \mathfrak{q} = \mathfrak{m} \oplus \mathfrak{q} \).

a) Let \( d^{(e)}_{m^\chi,q} : S(q)[e] \rightarrow S(q)[e] \otimes m^* \) be the differential operator \( d^{(e)}_{m^\chi,q} = \sum_{i=1}^\infty e^i d^{(i)}_{m^\chi,q} \) where

\[
d^{(i)}_{m^\chi,q} = \prod_{e \in E(\Gamma) : e=(l,F)} \omega_{\Gamma} B_{\Gamma} \quad \text{the } \epsilon- \text{ deformed reduction algebra over } \chi + m^+, \text{ denoted as}
\]

\[
H^0(\epsilon)(m^\chi, d^{(e)}_{m^\chi,q}), \text{ is the vector space of solutions } F(\epsilon) \in S(q)[e] \text{ of the equation}
\]

\[
d^{(e)}_{m^\chi,q}(F(\epsilon)) = 0, \tag{5}
\]

equipped with the \( *_{CF,\epsilon} \) product.

b) Let \( d_{m^\chi,q} : S(q) \rightarrow S(q) \otimes m^* \) be the differential operator \( d_{m^\chi,q} = \sum_{i=1}^\infty d^{(i)}_{m^\chi,q} \) where

\[
d^{(i)}_{m^\chi,q} = \prod_{e \in E(\Gamma) : e=(l,F)} \omega_{\Gamma} B_{\Gamma} \quad \text{the (non-} \epsilon)- \text{ reduction algebra over } \chi + m^+, \text{ } H^0(\chi, d^{(e)}_{m^\chi,q}), \text{ is the vector space of solutions } F \in S(q) \text{ of the equation}
\]

\[
d^{(e)}_{m^\chi,q}(F) = 0, \tag{6}
\]
equipped with the $*_{CF}$- product, meaning the product constructed without a deformation parameter $\epsilon$.

In the sequence, $H^0(\mathfrak{m}^\perp_{\chi}, d_{\mathfrak{m}^\perp_{\chi}, \mathfrak{q}})$ will be just referred to as the reduction algebra. For a homogeneous polynomial $G \in S(\mathfrak{q})$ of degree $p$ with respect to the ordinary polynomial degree, we will write $\deg_{\mathfrak{q}}(G) = p$. Similarly we consider the $\epsilon-$ degree $\deg_\epsilon$ for elements of $S(\mathfrak{q})[\epsilon]$ and differential operators $\epsilon B_T$ on $S(\mathfrak{q})[\epsilon]$. For $F \in S(\mathfrak{q})[\epsilon]$, set $\deg_{\mathfrak{q}}(F) := \deg_{\mathfrak{q}}(F) + \deg_\epsilon(F)$. We consider the corresponding notions of degree also for differential operators $d_{\mathfrak{m}^\perp_{\chi}, \mathfrak{q}}^{(i)}$ on $S(\mathfrak{q})[\epsilon]$, e.g. $\deg_\epsilon(\epsilon d_{\mathfrak{m}^\perp_{\chi}, \mathfrak{q}}^{(i)}) = i$. Let now $F(\epsilon) = F_0 + \epsilon F_1 + \cdots + \epsilon^n F_n$, $F_i \in S(\mathfrak{q})$. Using the $\deg_\epsilon$ of the terms $\epsilon d_{\mathfrak{m}^\perp_{\chi}, \mathfrak{q}}^{(i)}$, $\epsilon F_j$ to write down $\epsilon-$ homogeneous equations of $\deg_\epsilon = 1, 2, \ldots$, the defining equation \([2]\) gives a system of linear partial differential equations, namely

$$\forall p \in \mathbb{N}^*, \quad \sum_{i=1}^{p} d_{\mathfrak{m}^\perp_{\chi}, \mathfrak{q}}^{(i)}(F_{p-i}) = 0$$

(7)

or equivalently $d_{\mathfrak{m}^\perp_{\chi}, \mathfrak{q}}^{(1)}(F_0) = 0, d_{\mathfrak{m}^\perp_{\chi}, \mathfrak{q}}^{(1)}(F_1) + d_{\mathfrak{m}^\perp_{\chi}, \mathfrak{q}}^{(2)}(F_0) = 0, d_{\mathfrak{m}^\perp_{\chi}, \mathfrak{q}}^{(1)}(F_2) + d_{\mathfrak{m}^\perp_{\chi}, \mathfrak{q}}^{(2)}(F_1) + d_{\mathfrak{m}^\perp_{\chi}, \mathfrak{q}}^{(3)}(F_0) = 0$

etc. By \([15]\), only the colored graphs $\Gamma \in \mathbb{Q}_{2n+1,1}, \forall n \in \mathbb{N}$, have non-zero contribution to the differential $d_{\mathfrak{m}^\perp_{\chi}, \mathfrak{q}}^{(i)}$. Thus \([7]\) becomes

$$\forall p \in \mathbb{N}_0, \quad \sum_{i=0}^{p} d_{\mathfrak{m}^\perp_{\chi}, \mathfrak{q}}^{(2i+1)}(F_{2(p-i)}) = 0$$

(8)

or equivalently $d_{\mathfrak{m}^\perp_{\chi}, \mathfrak{q}}^{(1)}(F_0) = 0, d_{\mathfrak{m}^\perp_{\chi}, \mathfrak{q}}^{(1)}(F_2) + d_{\mathfrak{m}^\perp_{\chi}, \mathfrak{q}}^{(3)}(F_0) = 0, d_{\mathfrak{m}^\perp_{\chi}, \mathfrak{q}}^{(1)}(F_4) + d_{\mathfrak{m}^\perp_{\chi}, \mathfrak{q}}^{(3)}(F_2) + d_{\mathfrak{m}^\perp_{\chi}, \mathfrak{q}}^{(5)}(F_0) = 0$

etc. Thus every element of $H^0(\mathfrak{m}^\perp_{\chi}, d_{\mathfrak{m}^\perp_{\chi}, \mathfrak{q}}^{(i)})$ is written as $F(\epsilon) = \sum_{i=0}^{n} F_{2i} \epsilon^{2i}, n \in \mathbb{N}$. Turning the system \([5]\) into a homogeneous system is only possible using $\deg_\epsilon$ and thus works only for $H^0(\mathfrak{m}^\perp_{\chi}, d_{\mathfrak{m}^\perp_{\chi}, \mathfrak{q}}^{(i)})$ and not for $H^0(\mathfrak{m}^\perp_{\chi}, d_{\mathfrak{m}^\perp_{\chi}, \mathfrak{q}}^{(i)})$. In fact, the system \([5]\) is much more complicated. Let $\mathfrak{g}_\epsilon$ be the Lie algebra over $\mathbb{K}[\epsilon]$ with Lie bracket for $X, Y \in \mathfrak{g}$ defined as $[X, Y]_\epsilon := \epsilon[X, Y]$. Set $U(\epsilon)(\mathfrak{g})$ to be the universal enveloping algebra $U(\mathfrak{g}_\epsilon)$ over the ring $\mathbb{K}[\epsilon]$ and consider the corresponding ideal $U(\epsilon)(\mathfrak{g})\mathfrak{m}_{\chi}$ with the previous notation. In \([3]\) it was proved that there is a non-canonical isomorphism of associative algebras,

$$\beta_{\mathfrak{q},(\epsilon)} \circ \frac{\partial}{\partial \mathfrak{q}_{(\epsilon)}} \circ T_1^{-1} T_2 : H^0(\mathfrak{m}^\perp_{\chi}, d_{\mathfrak{m}^\perp_{\chi}, \mathfrak{q}}^{(i)}) \xrightarrow{\sim} (U(\epsilon)(\mathfrak{g})/U(\epsilon)(\mathfrak{g})\mathfrak{m}_{\chi})^m,$$

(9)

where the operators $T_1, T_2$ are entirely described by wheel type Kontsevich graphs, $\beta_{\mathfrak{q},(\epsilon)} : S(\mathfrak{q})[\epsilon] \rightarrow U(\epsilon)(\mathfrak{g})/U(\epsilon)(\mathfrak{g})\mathfrak{m}_{\chi}$ is the quotient symmetrization map and $q(Y) := \det_{\mathfrak{g}} \left( \frac{\sinh \frac{\mathfrak{m} Y}{2}}{\mathfrak{m} Y} \right)$, $Y \in \mathfrak{g}$. The proof was based entirely on deformation quantization techniques and the idea of translating into equations, the concentrations of configuration spaces needed to solve a Stokes equation. The idea behind the Stokes argument comes from Kontsevich’s formality theorem and associativity for his $*$- product in \([18]\).

2.2. Necessary Lemmata

2.2.1. Kazhdan degree. Our goal is to describe the $W-$ algebra via the reduction algebra construction. As explained in \([2]\) it is not straightforward to eliminate the deformation parameter
$\epsilon$ from the isomorphism \[\boxed{\epsilon},\] meaning that \(\boxed{\epsilon}\) does not imply \(H^0(m^+_{\mathfrak{X}}, d_{m^+,\mathfrak{q}}) \simeq (U(\mathfrak{g})/U(\mathfrak{g})m_{\chi})^m\).

To prove an isomorphism analogous to \(\boxed{\epsilon}\) but without the $\epsilon-$ parameter, we need to recall the Kazhdan grading of $\text{S}(\mathfrak{g})$. This grading is extensively used by Premet and Losev and is defined as follows; let $\text{S}(\mathfrak{g}) = \bigoplus_{n \in \mathbb{Z}} \text{S}(\mathfrak{g})[n]$ be the standard polynomial grading on $\text{S}(\mathfrak{g})$. The \(\text{adh}-\) action extends uniquely to a derivation on $\text{S}(\mathfrak{g})$ which we denote with the same symbol. Define $\text{S}^n(\mathfrak{g})(i) := \{x \in \text{S}^n(\mathfrak{g})/\text{ad}(x) = ix, i \in \mathbb{Z}\}$. The Kazhdan grading on $\text{S}(\mathfrak{g})$ is $\text{S}(\mathfrak{g}) = \bigoplus_{n \in \mathbb{Z}} \text{S}(\mathfrak{g})[n]$ where $\text{S}(\mathfrak{g})[n] := \bigoplus_{i+j=\eta} \text{S}^i(\mathfrak{g})(i)$. Similarly, a Kazhdan filtration can be defined on $U(\mathfrak{g})$ letting $U_0(\mathfrak{g}) \subset U_1(\mathfrak{g}) \subset \cdots \subset U_n(\mathfrak{g}) \subset \cdots$ be the PBW filtration of $U(\mathfrak{g})$ and setting $U_n(\mathfrak{g})(i) := \{x \in U_n(\mathfrak{g})/\text{ad}(x) = ix\}$. Then the Kazhdan filtration on $U(\mathfrak{g})$ is a $\mathbb{Z}-$ indexed filtration with $F_p U(\mathfrak{g}) := \{x \in U_j(\mathfrak{g})(i)/ i + 2j = p\}$ meaning the subspace spanned by all such $x$. The Kazhdan degree of a homogeneous element $P \in S(\mathfrak{g})$ is denoted by $\text{deg}_K(P)$. In what follows we study the Kazhdan degree aspect of the reduction algebra $H^0(m^+_{\mathfrak{X}}, d_{m^+,\mathfrak{q}})$ and prove that $H^0(m^+_{\mathfrak{X}}, d_{m^+,\mathfrak{q}}) \simeq (U(\mathfrak{g})/U(\mathfrak{g})m_{\chi})^m$ providing an alternative model of the $\mathfrak{W}-$ algebra.

### 2.2.2. Kazhdan degree for graphs.

From now on, $\mathfrak{g}$ will be a semisimple Lie algebra unless otherwise stated. Recall that $\mathfrak{I} \subset \mathfrak{g}(-1)$ is a lagrangian subspace with respect to the restriction $\omega_{\chi}|_{\mathfrak{g}(-1)}$, $m := \bigoplus \sum_{i \leq -2} \mathfrak{g}(i)$, and $\mathfrak{q}$ is a vector subspace such that $\mathfrak{g} = m \oplus \mathfrak{q}$. We consider the same notation with $\boxed{22}$, §3. Let $\mathfrak{J}_X = \text{kerade}$. For $x \in \mathfrak{g}$, denote as $wt(x)$ the weight of $x$ in the adh-- decomposition of $\mathfrak{g}$, i.e $[h, x] = wt(x)x$. Let $x_1, \ldots, x_r$ be a basis of $\mathfrak{J}_X$ and $x_1, \ldots, x_r, x_{r+1}, \ldots, x_m$ a basis of $\mathfrak{p}_c := \sum_{2 \leq 0} \mathfrak{g}(i)$. Denote by $y_{r+1}, \ldots, y_m$ the corresponding basis of the parabolic subalgebra $\mathfrak{b} = \sum_{n \leq -2} \mathfrak{g}(i)$ such that $([y_i, x_j], e) = \delta_{ij}$, $r+1 \leq i, j \leq m$. Complete $x_i, y_i$ to a basis of $\mathfrak{g}$ by fixing a Witt basis $z_1, \ldots, z_k, z_{k+1}, \ldots, z_{2k}$ of $\mathfrak{g}(-1)$ that is, $[z_i, z_j] = [z_{i+s}, z_{j+s}] = 0$, $[z_i, z_{j+s}] = \delta_{ij}f$, $i, j = 1, \ldots, s$ and such that $z_1, \ldots, z_s \in \mathfrak{I}$, $z_{1+s}, \ldots, z_{2s} \in \mathfrak{q}$. For $(\mathfrak{a}, \mathfrak{b}) \in \mathbb{Z}_+^m \times \mathbb{Z}_+^r$ let $x^\mathfrak{a}y^\mathfrak{b} = x_1^{a_1} \cdots x_m^{a_m} \cdot z_1^{b_1} \cdots z_{2k}^{b_s}$. We also denote as $1_{\mathfrak{Z}_i}$ the $m-$ tuple $(0, \ldots, 0, 1, \ldots, 0)$ with the unity in the $i-$th position. When we write $1_{\mathfrak{Z}_{i+s}}$ we refer to an $s-$ tuple with unity in the $i-$th position. With the above notation \(\text{deg}_K(x^\mathfrak{a}y^\mathfrak{b}) = wt(x^\mathfrak{a}y^\mathfrak{b}) + 2 \text{deg}_\mathfrak{q}(x^\mathfrak{a}y^\mathfrak{b}) = \sum_{i=1}^m a_i(wt(x_i) + 2) + \sum_{i=1}^s b_i\).

The following Lemma computes the Kazhdan degrees of the operators in the differential $d_{m^+,\mathfrak{q}}$.

**Lemma 2.3** Let $\Gamma \in \mathcal{B}_l \cup \mathcal{B}W_l$ and $F \in S(\mathfrak{q})$. Then \[\text{deg}_K(B_1(F)) = \text{deg}_K(F) + wt(L(e_{\infty})) - 2(t-1)\] \[\text{(10)}\]

**Proof.** It suffices to show this for $F = x^\mathfrak{a}y^\mathfrak{b}$. We first prove it for the simplest graph in $\mathcal{B}_l \cup \mathcal{B}W_l$, the graph of Figure 2, to explain then the general calculation. Suppose $L(e_1^j) = L(e_{\infty}) = A \in \{y_{r+1}, \ldots, y_m, z_1, \ldots, z_s\}$, $L(e_1^j) = B \in \{x_1, \ldots, x_m, z_{1+s}, \ldots, z_{2s}\}$. According to \[\boxed{3}\], the graph of Figure 2 for this labeling corresponds to the operator $B_1(F) = \sum_{A,B} [B,A] \partial_B(F)$.

![Figure 2](image-url)
Lemma 2.4
Fix a $\Gamma \in \mathcal{B}_i$ and let $F \in S(\mathfrak{g})$ where $\Gamma \in \mathcal{W}_i$. This result will also appear as a corollary of Lemma 2.8 when we compute the Kazhdan degree for bidifferential operators in the $\ast$-product. We give here a direct and alternative proof.

**Lemma 2.4** Fix a $\Gamma \in \mathcal{W}_i$ and let $F \in S(\mathfrak{g})$. Then

$$\deg_K(B_\Gamma(F)) = \deg_K(F) - 2i$$

(11)
Proof. As in Lemma 2.3 it suffices to prove the claim for \( F = x^a z^b \). Choose a vertex of type I and label it by 1. Then order the rest type I vertices following the orientation of the edges in the wheel. Put also an order in the edges leaving a vertex of type I as in Lemma 2.3. The corresponding operator is a constant coefficient operator so the symbol’s Kazhdan degree is 0. Starting from the type I vertex labeled by 1,

\[
\deg_K(B_I(x^a z^b)) = wt(x^{a - \sum_{k=1}^i L(e_k)} z^{b - \sum_{k=1}^i L(e_k)}) = wt(x^{a - \sum_{k=1}^i L(e_k)} z^{b - \sum_{k=1}^i L(e_k)}) + 2 \deg_q(x^{a - \sum_{k=1}^i L(e_k)} z^{b - \sum_{k=1}^i L(e_k)}) = \deg_K(x^a z^b) - \sum_{k=1}^i wt(L(e_k)) - 2i
\]

Checking each vertex of type I in terms of the Kazhdan degree one writes \( wt(K^2) = wt(L(e_k)) \) for \( k = 1, \ldots, i - 1 \) and \( wt(L(e_i^2)) = wt(L(e_i^1)) + wt(L(e_i^2)) \). Note that if one of these equations does not hold, then \( B_I = 0 \) by construction. This linear system implies \( \sum_{k=1}^i wt(L(e_k^1)) = 0 \).

2.2.3. Reduction equations. We turn now to the defining equation (6) of \( H^0(m_{x^k}, d_{m^k, q}) \).

Using Kazhdan degree arguments, we will write this equation into a system of homogeneous equations with respect to \( \deg_K \) and thus write (5) also for \( H^0(m_{x^k}, d_{m^k, q}) \).

Definition 2.5 With the basis chosen in § 2.2.2, let \( \alpha = (a'_m, \ldots, a'_{r+1}, a_2, \ldots, a_1, a''_1, \ldots, a''_m) \) be a multiindex, \( P^\alpha = y_{m}^{a'_m} \cdots y_{r+1}^{a'_{r+1}} x_2^{a_2} \cdots x_1^{a_1} x_1^{a''_1} \cdots x_m^{a''_m} \) denote a monomial in \( S(\mathfrak{g}) \) and \( \partial_P = \partial_{y_m}^{a'_m} \cdots \partial_{x_1}^{a''_1} \cdots \partial_{x_m}^{a''_m} \) denote the corresponding differential operator of \( P^\alpha \) on \( S(\mathfrak{g}) \). Let \( B = P^\alpha \partial_Q^\beta \) be a differential operator on \( S(\mathfrak{g}) \). Then \( B \) has Kazhdan degree \( \deg_K(B) = \deg_K(P^\alpha) - \deg_K(Q^\beta) \).

We extend this definition to operators on \( S(\mathfrak{g}) \times S(\mathfrak{g}) \).

Lemma 2.6 If \( \mathfrak{g} \) is semisimple, the defining system (4) is equivalent to (5).

Proof. Let \( F = \sum_{i=0}^p \tilde{F}_i \in S(q) \), with \( \tilde{F}_i \) homogeneous of \( \deg_K(\tilde{F}_0) = n_0 \), and \( \deg_K(\tilde{F}_i) = n_0 - i \). Then (4) is

\[
\sum_{i=0}^p d_{m^k, q}(\tilde{F}_i) = 0 \iff \sum_{k=1}^\infty \sum_{i=0}^p d_{m^k, q}(\tilde{F}_i) = 0 \quad (12)
\]

Fix a label \( L(e_\infty) \) for \( e_\infty \). By Lemma 2.3 and Definition 2.3 one can group together the operators in the differential \( d_{m^k, q} \) in terms of the number of type I vertices in the respective graphs, and thus (12) translates to the following system of equations: The only operator of Kazhdan degree \( wt(L(e_\infty)) \) is given by the graph \( \Gamma_1^1 \) of Figure 2 since \( \Gamma_1^1 \in B_1 \). Thus the homogeneous component in (12) of Kazhdan degree equal to \( n_0 + wt(L(e_\infty)) \) (the highest possible) is \( d_{m^k, q}(\tilde{F}_0) = 0 \), recovering the first equation of the system (8). The homogeneous component in (12) of the second highest Kazhdan degree, \( n_0 + wt(L(e_\infty)) - 2 \), has \( d_{m^k, q}(\tilde{F}_2) = d_{m^k, q}(\tilde{F}_1) = 0 \), which, since all weights \( \omega_\Gamma \) for \( \Gamma \in B_{2i} \cup Bw_{2i} \) are zero (13), gives \( d_{m^k, q}(\tilde{F}_2) = 0 \). With the same argument, the equation of Kazhdan degree \( n_0 + wt(L(e_\infty)) - 3 \) is \( d_{m^k, q}(\tilde{F}_3) = 0 \). Checking the homogeneous terms of degree \( n_0 + wt(L(e_\infty)) - 4 \) with the same argument we get \( d_{m^k, q}(\tilde{F}_4) + d_{m^k, q}(\tilde{F}_0) = 0 \), thus recovering the second equation.
of $\mathfrak{S}$. Similarly, for decaying Kazhdan degree, one gets $d_{m_t,q}^1(\tilde{F}_5) + d_{m_t,q}^3(\tilde{F}_1) = 0$, $d_{m_t,q}^1(\tilde{F}_6) + d_{m_t,q}^3(\tilde{F}_2) = 0$ and $d_{m_t,q}^1(\tilde{F}_7) + d_{m_t,q}^3(\tilde{F}_3) = 0$. Inductively, we regroup the terms of $F$ according to the previous equations writing $F = \sum_{i=0}^{p'} F_{2i}$, where $F_{2i} = \sum_{j=0}^{3} \tilde{F}_{4i+j}$. 

**Corollary 2.7** When $F_0$ is not homogeneous, and $F_0 = \sum_{k=1}^{q} F_{0,k}$ with $\deg_K(F_{0,k}) = r_k$, $r_1 > \ldots > r_q$, then write $F_{2i} = \sum_{k=1}^{q} F_{2i,k}$ with $\deg_K(F_{2i,k}) = r_k - 4i$. Furthermore, it is easy to see that the number $p' \in \mathbb{N}$ in the proof of Lemma 2.4 is controlled by the Kazhdan degree $r_q$ by the inequality $p' \leq \frac{r_q - 1}{4}$.

### 2.2.4 Kazhdan degree for the $*$-- product

The following Lemma computes the Kazhdan degrees of $*_CF$ in the linear case $X = \mathfrak{g}^*$, $\mathfrak{g}$ semisimple. Trivially it holds for $*_K$ also.

**Lemma 2.8** Let $F,G \in S(\mathfrak{g})$. If $\Gamma \in \mathcal{Q}_{n,2}$, then

$$\deg_K(B_{\Gamma}(F,G)) = \deg_K(F) + \deg_K(G) - 2n$$

(13)

**Proof.** It is $\#V_1(\Gamma) = n$ and suppose that the cardinal of the subset $E^1(\Gamma) \subset E(\Gamma)$ of all edges in $\Gamma$ between vertices of Type I, is $s$. Suppose also that $\Gamma$ has $k$ roots. Denote as $D$ the polynomial symbol of the bidifferential operator $B_{\Gamma}$ when applied to $F,G$. Then $\forall L$, and denoting as $\partial_{\Gamma} F$ the polynomial coming from the differentiation of $F$ by $B_{\Gamma}$, one has

$$\deg_K(B_{\Gamma}(F,G)) = \deg_K(D) + \deg_K(\partial_{\Gamma} F) + \deg_K(\partial_{\Gamma} G) = \sum_{k=1,\text{p.root}}^{k} \left[ \deg(L(e_{i,p}^1)) + \deg(L(e_{i,p}^2)) \right] + 2k + \deg_K(F) - \sum_{r=1}^{\# \rightarrow F} \deg(L(e_{i,r}^1)) - 2(\# \rightarrow F)$$

$$+ \deg_K(G) - \sum_{r=1}^{\# \rightarrow G} \deg(L(e_{i,r}^1)) - 2(\# \rightarrow G) = \sum_{k=1,\text{p.root}}^{k} \left[ \deg(L(e_{i,p}^1)) + \deg(L(e_{i,p}^2)) \right] + 2k + \deg_K(F) + \deg_K(G) - 2(2n-s) - \sum_{r=1}^{\# \rightarrow F} \deg(L(e_{i,r}^1)) - \sum_{r=1}^{\# \rightarrow G} \deg(L(e_{i,r}^1)).$$

The notation $\# \rightarrow F$ stands for the number of edges of $\Gamma$ deriving $F$. The expression $\sum_{r=1}^{\# \rightarrow F} \deg(L(e_{i,r}^1))$ sums the weights of those labelled edges deriving $F$, thus $i = 1$ or $2$ and $r$ runs the set of vertices carrying an edge towards $F$. The last equation means that, to prove the Lemma, it suffices to prove that $2k - 4n + 2s = -2n$ and $\sum_{p=1,\text{p.root}}^{k} [\deg(L(e_{i,p}^1)) + \deg(L(e_{i,p}^2))] = \sum_{r=1}^{\# \rightarrow F} \deg(L(e_{i,r}^1)) + \sum_{r=1}^{\# \rightarrow G} \deg(L(e_{i,r}^1)).$ For the first claim, since there are $k$ roots, there are $2k$ edges starting from a vertex that receives no edge. So there are $2(n-k)$ edges in the graph starting from a vertex of $V_1(\Gamma)$ (by definition) that receives an edge starting from another vertex of $V_1(\Gamma)$. By definition of $s$, $2(n - k) = 2s$ and the claim is proved.

For the second claim, let $e_{p-1}^r$ be the edge pointing to the vertex $p \in V_1(\Gamma)$ . Then using repeatedly the equality $\deg(L(e_{i,p-1}^r)) = \deg(L(e_{i,p}^1)) + \deg(L(e_{i,p}^2))$ the claim follows by the tree construction of the Kontsevich’s graphs in the Lie case. \n
**Remark 2.9** As a corollary of the second claim at the end of the previous proof, one has that when $\Gamma \in \mathcal{W}_i$, the sum of weights of the labels on the edges deriving the real axis, is 0.
Let be as previously defined. Then there is an associative algebra isomorphism associated to the data $(e_1^g, L, G)$. The following theorem provides a new model of the operator $3$ A NEW $W$– ALGEBRA MODEL. 11

We present below an example of the previous calculation for a graph in $\mathbb{Q}_{4,2}$. It can be used as an alternative inductive proof of Lemma $2.8$ in the sense of $[22]$.

**Example.** Consider the graph of Figure 3 with the following labelling: $F = x^d z^c, G = x^a z^b, L(e_1^g) = z_{2s}, L(e_2^g) = x_t, L(e_3^g) = z_{1+k}, L(e_4^g) = x_p$. The variables at the rest of the edges of the graphs are imposed accordingly so that the edge arriving at a type I vertex carries the bracket associated to that vertex. Suppose $[z_{2s}, [x_t, z_{1+k}, [z_{2k}, x_p]]] \in g(-2)$. Checking the weights we get $wt(x_p) + wt(x_t) - 3 = -2 \Rightarrow wt(x_p) + wt(x_t) = 1$. The symbol of the operator $B_1$ is a constant and so its Kazhdan degree is $0$. Thus

$$
\text{deg}_K(B_1(x^d z^c, x^a z^b)) = wt(x^d z^c x^a z^b) + wt(x^d -1 z^c -1 z_{1+k} x^a z^b) + wt(x^d -1 z^c z_{1+k} x^a z^b) + 2 \text{deg}_K(x^d z^c x^a z^b) - 2 \cdot 5 = \text{deg}_K(x^a d z^b c) - 8
$$

$$
\begin{align*}
\text{deg}_K(B_1(x^d z^c, x^a z^b)) &= wt([z_{2s}, [x_t, [z_{1+k}, x_p]]]) + wt(x^d -1 z^c -1 z_{1+k} x^a z^b) + 2 \text{deg}_K([z_{2s}, [x_t, [z_{1+k}, x_p]]]) \\
&= wt(x_p) + wt(x_t) - 3 + wt(x^d z^c x^a z^b) - wt(x_t) - wt(x_p) + 3 + 2 \text{deg}(x^d z^c x^a z^b) - 2 \cdot 5 + 2 = \text{deg}_K(F) + \text{deg}_K(G) - 2 \cdot 4
\end{align*}
$$

3 A new $W$– algebra model.

**The main Theorem.** The following theorem provides a new model of the $W$– algebra associated to the data $(g, e)$.

**Theorem 3.1.** Let $g$ be a semisimple Lie algebra, and $\{e, h, f\}$ an $\mathfrak{sl}_2$– triple. Let then $\chi, m_\chi, z_k, x_p$ be as previously defined. Then there is an associative algebra isomorphism

$$
\overline{\beta}_{q^\ast} \circ \partial \circ T_1^{-1} T_2 : H^0(m_{\wedge}^\perp, d_{m_{\wedge}^\perp, q}) \xrightarrow{\simeq} (U(g)/U(g)m_{\wedge})^m.
$$

**Proof.** The direction $H^0(m_{\wedge}^\perp, d_{m_{\wedge}^\perp, q}) \hookrightarrow (U(g)/U(g)m_{\wedge})^m$ works as in the proof of $[2]$ in $[3]$. Recall by $\S$ 1.4, that there are short loops in the construction, however their contribution is the character $B_\rho(H) = \rho(H) = -\omega H \text{Tr}(\text{ad} H) = 0$ since $H \in m$ and $m$ is nilpotent. The reverse direction uses instead of $\text{deg}_e$, the Kazhdan degree of the appearing operators. We omit details that can be found in $[2],[3]$. Recall that $H^0(g, d_q^\ast) = U(g)$ and let $H^0(m_{\wedge}^\perp, d_q^\ast, m_{\wedge}^\perp)$ be
the reduction space at the origin of the Cattaneo-Felder biquantization diagram for \( g^*, m^\perp_\chi \).

Let \( *_1 : H^0(g^*, d_{g^*}) \times H^0(m^\perp_\chi, d_{g^*} \cdot m^\perp_\chi) \rightarrow H^0(m^\perp_\chi, d_{g^*} \cdot m^\perp_\chi) \) and \( *_2 : H^0(m^\perp_\chi, d_{g^*} \cdot m^\perp_\chi) \times H^0(m^\perp_\chi, d_{g^*} \cdot m^\perp_\chi) \) denote the \( H^0(g^*, d_{g^*}) - H^0(m^\perp_\chi, d_{g^*} \cdot m^\perp_\chi) \) bimodule structure of Cattaneo-Felder on \( H^0(m^\perp_\chi, d_{g^*} \cdot m^\perp_\chi) \). Let \( T_1 : H^0(g^*, d_{g^*}) \rightarrow H^0(m^\perp_\chi, d_{g^*} \cdot m^\perp_\chi), F \mapsto F *_1 1 \) and \( T_2 : H^0(m^\perp_\chi, d_{m^\perp_\chi} \cdot q) \rightarrow H^0(m^\perp_\chi, d_{g^*} \cdot m^\perp_\chi), G \mapsto 1 *_2 G \). Let \( G \in (U(g)/U(g)m_\chi)^m \) be on the vertical axis of the biquantization diagram of \( g^* \) and \( m^\perp_\chi \), and let \( T_2^{-1} T_1(G) = F \) be at the horizontal axis of the diagram, where \( T_1 \) denotes the restriction \( T_1|_{S(q)} \) (see Lemma 3.4 of [2]). Then if \( m + \chi(m), m \in m \) is on the vertical axis, one has \( (m + \chi(m)) *_{DK} G - G *_{DK} (m + \chi(m)) \in S(g) *_{DK} m_\chi \) implies that \( (m + \chi(m)) *_1 (1 *_2 F) = 0 \). Since \( (m + \chi(m)) *_1 1 = 0 \) (Lemma 3.3 in [2]), it is \( (m + \chi(m)) *_1 1 *_2 F = 0 \) and so one gets a Stokes equation \( \sum_{\Gamma} \int_{\Gamma_s} \sum_{\Gamma_s} d_{\omega}(s) B_{\Gamma}(F) = 0 \) letting \( F \) (as a point \( s \)) move on the horizontal axis. Indeed, the limits \( \lim_{s \to \infty} \sum_{\Gamma} \omega_{\Gamma}(s) B_{\Gamma}(F) \) and \( \lim_{s \to 0} \sum_{\Gamma} \omega_{\Gamma}(s) B_{\Gamma}(F) \) of the quantity \( (m + \chi(m)) *_1 1 *_2 F = 0 \) correspond to \( (m + \chi(m)) *_1 1 *_2 F = 0 \). The possible concentrations are as they were listed in the proof found in [5]. We recall them here in terms of possible interior and exterior graphs and then compute the total Kazhdan degree in each case. Since the function \( m + \chi(m) \) is of degree 1, it receives exactly one edge.

**Interior graphs.** Suppose there are \( k \) type I vertices and one type II vertex concentrated on the horizontal axis. For dimensional reasons (regarding the form \( \Omega_\Gamma \) integrated over the concentration manifold \( \tilde{C}_{k,1}^+ \)) of dimension \( 2k - 1 \) a possible graph in this concentration is either of \( B_k^- \) or \( BW_k^- \). Denote as \( \alpha \) the edge leaving the concentration. An interior graph is denoted by \( \Gamma_{int}^\alpha \).

**Exterior graphs.** Here one might have a \( B \)-type graph receiving at its root the edge \( \alpha \). Its own \( e_\infty \) edge derives the function \( m + \chi(m) \). In this case there can be also an infinite number of superposed \( W^- \) graphs deriving the concentration. The second possibility is to have a finite number of \( W^- \) graphs deriving the concentration. In this case \( \alpha \) derives the function \( m + \chi(m) \).

![Figure 4: The graph corresponding to \( B_{\int \alpha}^0 \) (\( \Gamma_{int}^\alpha \)).](image)

The first accepted kind of concentration is the one in Figure 4. It is the case, where there is no exterior graph while the interior graph is the one of \( d^m_{m^\perp_\chi} \). Let \( e_1^f \) be the edge deriving \( F \) and \( e_1^g = \alpha \) be the edge deriving \( m + \chi(m) \). Let \( L \) be a labelling of the edges in Figure 4 and suppose \( F = x^a z^b \). Then
\[
\deg_K(B_{\Gamma_{\text{ext}}}^0(B_{\Gamma_{\text{int}}}^1)(F)) = wt([L(e_1^a), L(e_2^b)]) + wt(x^{a-1}z \in (e_1^{x_1} \ldots e_m^{x_m})} \) e^{b-1}z \in (e_1^{x_1} \ldots e_2^{x_2})
\]

\[
+ 2 \deg_q(x^{a-1}z \in (e_1^{x_1} \ldots e_m^{x_m})} e^{b-1}z \in (e_1^{x_1} \ldots e_2^{x_2}) + 2 \deg_q([L(e_1^a), L(e_2^b)])
\]

\[
= wt(L(e_1^a)) + wt(L(e_2^b)) + wt(x^{a}z^{b}) - wt(L(e_1^a)) + 2 \deg_q(x^{a}z^{b}) - 2 \cdot 1 - 2 =
\]

\[
\deg_K(x^{a}z^{b}) + wt(L(e_1^a))
\]

where with our conventions, \( wt \cdot \deg \) and \( wt \) are straightforward computations. The Stokes equation is thus equivalent to

\[
\deg_K(B_{\Gamma_{\text{ext}}}^0(B_{\Gamma_{\text{int}}}^1)(x^{a}z^{b})) = \deg_K(x^{a}z^{b}) + wt(L(\alpha)) - 2(t - 1) - 2 \sum_{i=1}^{k} r_i.
\]

A case that requires a comment is to have one \( \Gamma_{\text{ext}}^a \in B_m \), \( k \) exterior wheel type graphs \( \Gamma_{\text{ext}}^{i,a} \in W_t \) and \( \Gamma_{\text{int}}^a \in B_m \). Then \( \alpha \) derives the root of \( \Gamma_{\text{ext}}^a \in B_m \). Then \( wt(L(\alpha)) = wt(D) \) where \( D \) is the symbol of the operator \( B_{\Gamma_{\text{ext}}}^a \). The total Kazhdan degree in the diagram is

\[
\deg_K(B_{\Gamma_{\text{ext}}}^a(B_{\Gamma_{\text{int}}}^a)(x^{a}z^{b})) = \deg_K(x^{a}z^{b}) + wt(L(\alpha)) - 2(t + m - 1) - 2 \sum_{i=1}^{k} r_i
\]

as a straightforward computation shows. The Stokes equation is thus equivalent to

\[
\sum_{\alpha} \left( \sum_{\Gamma_{\text{int}}, \Gamma_{\text{ext}}} (B_{\Gamma_{\text{int}}}^a(B_{\Gamma_{\text{ext}}}^a)(F)) \right) = 0 \iff \sum_{\alpha} \left( \sum_{\Gamma_{\text{int}}, \Gamma_{\text{ext}}} \sum_{l+k+m=0}^{\infty} (B_{\Gamma_{\text{ext}}}^m(B_{\Gamma_{\text{int}}}^k)(F)) \right) = 0
\]

for \( l = 1, \ldots n, k = 0, \ldots \infty, m = 0, \ldots \infty \). Then [19] is written as a system of homogeneous equations with respect to the total Kazhdan degree, in the same way the proof of [15] uses the total degree in the deformation parameter \( \epsilon \). These equations imply

\[
\sum_{\alpha} B_{\Gamma_{\text{ext}}}^a(B_{\Gamma_{\text{int}}}^a(F)) = 0 \iff \sum_{\alpha} \sum_{i,j} B_{\Gamma_{\text{int}}}^j(B_{\Gamma_{\text{int}}}^i)(F) = 0 \iff \sum_{i} d_{\alpha}^{(i)}(F) = 0 \iff d_{\alpha}^{(i)}(F) = 0
\]

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