STURM–PICONE THEOREM FOR FRACTIONAL NONLOCAL EQUATIONS

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ABSTRACT. In this paper, we establish a generalization of Sturm–Picone comparison theorem for a pair of fractional nonlocal equations:

\[ (-\text{div}(A_1(x)\nabla))^{s} u = C_1(x)u \text{ in } \Omega, \]
\[ u = 0 \text{ on } \partial\Omega, \]

and

\[ (-\text{div}(A_2(x)\nabla))^{s} v = C_2(x)v \text{ in } \Omega, \]
\[ v = 0 \text{ on } \partial\Omega, \]

where \( \Omega \subset \mathbb{R}^n \) is an open bounded subset with smooth boundary, \( 0 < s < 1 \), \( A_1, A_2 \) are real symmetric and positive definite matrices on \( \Omega \) with continuous entries on \( \overline{\Omega} \) and \( C_1, C_2 \in C(\overline{\Omega}) \).

1. INTRODUCTION

In this paper, we are interested to generalize Sturm–Picone comparison theorem for a pair of fractional nonlocal equations:

\[ (-\text{div}(A_1(x)\nabla))^{s} u = C_1(x)u \text{ in } \Omega, \]
\[ u = 0 \text{ on } \partial\Omega, \]

(1.1)

and

\[ (-\text{div}(A_2(x)\nabla))^{s} v = C_2(x)v \text{ in } \Omega, \]
\[ v = 0 \text{ on } \partial\Omega, \]

(1.2)

where \( \Omega \subset \mathbb{R}^n \) is an open bounded subset with smooth boundary, \( 0 < s < 1 \), \( A_1, A_2 \) are real symmetric and positive definite matrices on \( \Omega \) with continuous entries on \( \overline{\Omega} \) and \( C_1, C_2 \in C(\overline{\Omega}) \). The nonlocal fractional operator \((-\text{div}(A(x)\nabla))^s u\) where \( A \) is a real symmetric matrix is defined next.

Let us recall briefly the earlier developments on this subject which have played important roles in the qualitative theory of differential equations. In 1836, Sturm [29] established the first important comparison theorem which deals with a pair of linear ODEs

\[ lx \equiv (p_1(t)x'(t))' + q_1(t)x(t) = 0, \]
(1.3)

\[ Ly \equiv (p_2(t)y'(t))' + q_2(t)y(t) = 0, \]
(1.4)

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on a bounded interval \((t_1, t_2)\), where \(p_1, p_2, q_1, q_2\) are real-valued continuous functions and \(p_1(t) > 0, p_2(t) > 0\) on \([t_1, t_2] \subset (0, \infty)\). The original Sturm’s comparison theorem \cite{29} reads as

**Theorem 1.1.** (Sturm’s comparison theorem) Suppose \(p_1(t) = p_2(t)\) and \(q_1(t) > q_2(t)\), \(\forall t \in (t_1, t_2)\). If there exists a nontrivial real solution \(y\) of \((1.3)\) such that \(y(t_1) = 0 = y(t_2)\), then every real solution of \((1.3)\) has at least one zero in \((t_1, t_2)\).

In 1969, Picone \cite{24} modified Sturm’s theorem. The modification reads as

**Theorem 1.2.** (Sturm–Picone theorem) Suppose that \(p_2(t) \geq p_1(t)\) and \(q_1(t) \geq q_2(t)\), \(\forall t \in (t_1, t_2)\). If there exists a nontrivial real solution \(y\) of \((1.4)\) such that \(y(t_1) = 0 = y(t_2)\), then every real solution of \((1.3)\) unless a constant multiple of \(y\) has at least one zero in \((t_1, t_2)\).

In 1962, Leighton \cite{19} proved a comparison theorem to the above pair of Equations \((1.3)-(1.4)\). He showed that Sturm and Sturm-Picone theorems may be regarded as special cases of this theorem. In order to prove his theorem, he defined the quadratic functionals associated with \((1.3)-(1.4)\) as follows:

\[
J(u) = \int_{t_1}^{t_2} [p_2(t)(u'(t))^2 - q_2(t)(u(t))^2] dt,
\]

where the domain \(D\) of \(J\) is defined to be the set of all real-valued functions \(u \in C^1([t_1, t_2])\) such that \(u(t_1) = u(t_2) = 0\) \((t_1, t_2\) are consecutive zeros of \(u\)). The variation of \(J(u)\) is defined as \(V(u) = J(u) - J(u)\), i.e.,

\[
V(u) = \int_{t_1}^{t_2} [(p_2(t) - p_1(t))(u'(t))^2 + (q_1(t) - q_2(t))(u(t))^2] dt.
\]

Now, Leighton’s theorem reads as follows:

**Theorem 1.3.** (Leighton’s theorem) Suppose there exists a nontrivial real solution \(u\) of \(Lu = 0\) in \((t_1, t_2)\) such that \(u(t_1) = u(t_2) = 0\) and \(V(u) \geq 0\), then every real solution of \(Lv = 0\) unless a constant multiple of \(u\) has at least one zero in \((t_1, t_2)\).

It is easy to see that Theorems \ref{thm1.1} and \ref{thm1.2} are special cases of Leighton’s theorem. We point out that the proof of Leighton’s theorem heavily depends on a lemma so-called Leighton’s variational lemma, which is stated as follows:

**Lemma 1.4.** (Leighton’s variational lemma) If there exists a function \(u \in D\), not identically zero, such that \(J(u) \leq 0\), then every real solution of \(Lv = 0\) except a constant multiple of \(u\) vanishes at some point of \((t_1, t_2)\).

We refer to a very recent work \cite{15}, where the authors consider a pair of equations of the form

\[
(p(u' + su))' + rp(u' + su) + qu = 0
\]
on a finite interval, where \(1/p, r, s\) and \(q\) are real integrable functions. They established a generalization of Leighton’s comparison theorem for these equations and as special cases, they provide a generalization of a Sturm-Picone-type theorem and a generalization of a Sturm-type separation theorem.
We mention that most of the above comparison theorems have been extended to a pair of linear elliptic partial differential equations of type

\begin{equation}
lu \equiv \sum_{i,j=1}^{n} D_i(a_{ij}D_ju) + cu = 0. \tag{1.6}
\end{equation}

\begin{equation}
Lv \equiv \sum_{i,j=1}^{n} D_i(A_{ij}D_jv) + Cv = 0, \tag{1.7}
\end{equation}

in \( \Omega \subset \mathbb{R}^n \), where \( \Omega \) is a bounded domain with smooth boundary, \( a_{ij}, A_{ij}, c, C \) are real and continuous on \( \overline{\Omega} \) and the matrices \( a_{ij} \) and \( A_{ij} \) are symmetric and positive definite in \( \Omega \).

In 1955, Hartman and Wintner \[16\] extended Sturm–Picone theorem (Theorem 1.2) to (1.6)–(1.7) and their theorem reads as follows:

**Theorem 1.5.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain whose boundary has a piecewise continuous unit normal. Suppose \( a_{ij} - A_{ij} \) is positive semidefinite and \( C \geq c \) on \( \Omega \). If there exists a nontrivial solution \( u \) of \( lu = 0 \) in \( \Omega \) such that \( u = 0 \) on \( \partial \Omega \), then every solution of \( Lv = 0 \) vanishes at some point of \( \Omega \).

In 1965, Clark and Swanson \[4\] obtained a analog of Leighton’s theorem (Theorem 1.3) using the variation of \( lu \), which is defined as

\[ V(u) = \int_{\Omega} \left[ \sum_{i,j=1}^{n} (a_{ij} - A_{ij}) D_iuD_ju + (C - c)u^2 \right] dx. \]

Their theorem reads as follows:

**Theorem 1.6.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain whose boundary has a piecewise continuous unit normal. Suppose \( a_{ij} - A_{ij} \) is positive semidefinite and \( C \geq c \) on \( \Omega \). If there exists a nontrivial solution \( u \) of \( lu = 0 \) in \( \Omega \) such that \( u = 0 \) on \( \partial \Omega \) and \( V(u) \geq 0 \), then every solution of \( Lv = 0 \) vanishes at some point of \( \Omega \).

Again, it is easy to see that Theorem 1.5 is a special case of Theorem 1.6 and the proof of Theorem 1.6 depends on the following \( n \)-dimensional version of Lemma 1.4.

Let us define the quadratic functional associated with (1.7):

\[ M(u) = \int_{\Omega} \left[ \sum_{i,j=1}^{n} (A_{ij}D_iuD_ju - Cu^2) \right] dx, \]

where the domain \( \mathcal{D} \) of \( M \) is defined to be the set of all real-valued continuous functions on \( \overline{\Omega} \) which vanish on the boundary and have uniformly continuous first partial derivatives in \( \Omega \).

**Lemma 1.7.** (\( n \)-dimensional version of Leighton’s variational lemma) If there exists \( u \in \mathcal{D} \) not identically zero such that \( M(u) \leq 0 \), then every solution \( v \) of \( Lv = 0 \) vanishes at some point of \( \Omega \).

In recent years, there have been a good amount of research works on the fractional Laplace equations dealing with existence, multiplicity and regularity questions, see for instance \[1, 6, 7, 10, 11, 13, 20, 23, 25, 26, 27, 33\] and many other papers but to the best of our knowledge, there are not many results available which deal with the qualitative behavior of the solutions such as Sturm-Picone theorem. We
refer to a very recent paper \cite{8} which deals with qualitative behaviours of fractional equations. Very recently, an attempt is also made to generalize the Leighton’s variational lemma for a class of fractional Laplace equations. More precisely, the following lemma is established in \cite{34}.

Lemma 1.8. \cite{34} Let $2s < n < 4s$, $0 < s < 1$. Let $a \in L^\infty(\Omega)$. If there exists a function $u \in X_0$ not identically zero such that $j(u) \leq 0$, then every solution $v$ of

\begin{equation}
(-\Delta)^s v = a(x)v \text{ in } \Omega; \quad v = 0 \text{ in } \mathbb{R}^n \setminus \Omega,
\end{equation}

eexcept a constant multiple of $u$ vanishes at some point of $\Omega$, where

\begin{equation}
j(u) = \int_{\mathbb{R}^n} \left[ |(-\Delta)^{\frac{s}{2}} u|^2 - a(x)u^2 \right] dx.
\end{equation}

In the above works, we have defined the fractional Laplacian of $u$ in P.V. integral sense, see Section 3 \cite{7} for the details and

\begin{equation}
X_0 = \{ g \in X : g = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \},
\end{equation}

where $X$ denotes the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to $\Omega$ of any function $g$ in $X$ belongs to $L^2(\Omega)$ and the map

\begin{equation}
(x, y) \mapsto \frac{g(x) - g(y)}{|x - y|^{n+2s}} \in L^2(\mathbb{R}^{2n} \setminus (C \Omega \times C \Omega)),
\end{equation}

where $C \Omega = \mathbb{R}^n \setminus \Omega$.

Since the classical proof of Sturm-Picone theorem for a pair of ordinary differential equations/systems as well as elliptic partial differential equations steadily rests on Leighton’s lemma, so it has been extended in different directions and applied to establish oscillation as well as nonoscillation theorems to differential equations. There are several interesting papers on this subject but for the sake of brevity, we list a few works. For instance, see the works of Jaros et al. \cite{17}, Komkov \cite{18}, Dosly and Jaros \cite{9}, see \cite{32} for a generalization of Leighton’s variational lemma for nonlinear differential equations and the earlier developments on this area. The results of the author \cite{32} are used and extended to more general equations by A. Tiryaki \cite{30, 31} and in his other papers. For a Sturmian comparison and oscillation theorems for a class of half-linear elliptic equations, we refer to \cite{36} and the references cited therein. Motivated by the above research works and by an increasing interest on fractional Laplace equations and related existence and qualitative questions in recent years, it is natural to ask the following question:

\textit{Is there any generalization of Sturm–Picone theorem for a pair of fractional nonlocal equations (1.1), (1.2)?}

In this paper, we answer the above question affirmatively. More precisely, we establish a generalization of Sturm–Picone theorem for a pair of equations (1.1) and (1.2). Firstly, we obtain Leighton’s variational lemma for fractional nonlocal equations by defining the suitable quadratic functional associated with the equation and then using Leighton’s variational lemma, we establish the generalization of Sturm–Picone theorem.

The plan of this paper is as follows. Section 2 deals with the briefs on the fractional nonlocal equations. In Section 3, we state and prove Leighton’s variational lemma and establish a generalization of Sturm–Picone theorem to fractional nonlocal equations. A few remarks are a part of Section 4.
2. Fractional nonlocal equations

Let us recall the very useful briefs on fractional nonlocal equations, see [3, 28] for the details.

We consider the following fractional nonlocal equation

\[
(-\text{div}(A(x)\nabla))^{s} u = C(x)u \text{ in } \Omega,
\]
(2.1)

where \(\Omega \subset \mathbb{R}^{n}, 0 < s < 1\) is an open bounded subset with smooth boundary, \(A\) is real symmetric and positive definite matrix with continuous entries on \(\Omega\), \(C \in C(\overline{\Omega})\).

By using the \(L^{2}\)-Dirichlet eigenvalues and eigenfunctions \((\lambda_{k}, \phi_{k})_{k=0}^{\infty}\), \(\phi_{k} \in H_{0}^{1}(\Omega)\) of \(L = (-\text{div}(A(x)\nabla))^{s}\), we can define the fractional powers \(L^{s}u = (-\text{div}(A(x)\nabla))^{s}u\), \(0 < s < 1\), for \(u\) in the domain \(\text{Dom}(L^{s}) \equiv \mathcal{H}^{s}\) in a natural manner, where

\[
\mathcal{H}^{s} = \begin{cases} 
H^{s}(\Omega), & \text{when } 0 < s < \frac{1}{2}, \\
H_{0}^{s}(\Omega), & \text{when } s = \frac{1}{2}, \\
H_{0}^{1}(\Omega), & \text{when } \frac{1}{2} < s < 1.
\end{cases}
\]

The spaces \(H^{s}(\Omega)\) and \(H_{0}^{s}(\Omega), s \neq \frac{1}{2}\) are the classical fractional Sobolev spaces which are given by \(C_{0}^{\infty}(\Omega)\) under the norm

\[
||u||_{H^{s}(\Omega)}^{2} = ||u||_{L^{2}(\Omega)}^{2} + [u]_{H^{s}(\Omega)}^{2},
\]

where

\[
[u]_{H^{s}(\Omega)}^{2} = \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))^{2}}{|x - y|^{n+2s}} \, dx \, dy, \quad 0 < s < 1.
\]

The space \(H_{0}^{s}(\Omega)\) is called Lions-Magenes space which is defined as follows:

\[
H_{0}^{s}(\Omega) := \left\{ u \in L^{2}(\Omega) \mid [u]_{H^{s}(\Omega)} < \infty, \text{ and } \int_{\Omega} \frac{(u(x))^{2}}{\text{dist}(x, \partial \Omega)} < \infty \right\},
\]

see Chapter 1 [21] and [22] for the details. Following [3], if \(u(x) = \sum_{k=0}^{\infty} \lambda_{k}^{s} \phi_{k}(x)\), \(x \in \Omega\), then

\[
L^{s}u(x) = \sum_{k=0}^{\infty} \lambda_{k}^{s} \phi_{k}(x).
\]

One can see that \(u = 0\) on \(\partial \Omega\) and equivalently, we have the semigroup formula

\[
L^{s}u(x) = \frac{1}{\Gamma(-s)} \int_{0}^{\infty} (e^{-tL}u(x) - u(x)) \frac{dt}{t^{1+s}},
\]
(2.2)

where \(\{e^{-tL}u\}_{t>0}\) is the heat diffusion semigroup generated by \(L\) with the Dirichlet boundary conditions and \(\Gamma\) is the Gamma function. Now, as it is already known that (see [28]) the fractional operators can be described as Dirichlet-to-Neumann maps for an extension problem in the spirit of the extension problem for the fractional Laplacian on \(\mathbb{R}^{n}\) of [2]. More precisely, let \(U = U(x, y) : \Omega \times (0, \infty) \rightarrow \mathbb{R}\) be the solution of the following degenerate elliptic equation with \(A_{2}\) weight:

\[
div(y^{a}B(x)\nabla U) = 0 \text{ in } \Omega \times (0, \infty),
\]
(2.3) \[
U = 0 \text{ on } \partial \Omega \times [0, \infty),
\]
\[
U(x, 0) = u(x) \text{ on } \Omega,
\]
where

\[(2.4) \quad B(x) := \begin{bmatrix} A(x) & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \text{ and } a := 1 - 2s \in (-1, 1).\]

Then we have

\[-\frac{1}{2s} \lim_{y \to 0^+} y^a U_y(x, y) = -\lim_{y \to 0^+} \frac{U(x, y) - U(x, 0)}{y^s} = c_s L^s u(x), \quad x \in \Omega.\]

Also, there are explicit formulas for \(U\) in terms of the semigroup \(e^{-tL}\), see Theorem 2.5 [3].

In view of (2.2) and (2.3), Equation (2.1) turns out the following

\[\text{div}(y^a B(x) \nabla U) = 0 \text{ in } \Omega \times (0, \infty),\]

\[U = 0 \text{ on } \partial \Omega \times [0, \infty),\]

\[-\frac{1}{2s} \lim_{y \to 0^+} y^a U_y(x, y) = c_s C(x)u \text{ on } \Omega,\]

where \(U\) and \(u\) are related by \(U(x, 0) = u(x) \in \Omega\) and \(B\) is defined in (2.4).

Using Theorem 2.5 [3], one can see the existence of a unique weak solution to (2.4), which is given below:

Let \(u \in \mathcal{H}^s, C \in L^\infty(\Omega)\), then there exists a unique weak solution \(U \in H^1_0(\Omega \times (0, \infty), y^a dX)\) of (2.5), where \(B\) and \(a\) are defined as above. More precisely, for each \(\phi \in H^1_0(\Omega \times (0, \infty), y^a dX),\)

\[(2.6) \quad \int_{\Omega} \int_{0}^{\infty} y^a B(x) \nabla U \nabla \phi \, dx \, dX = c_s \int_{\Omega} C(x) u(x) \phi(x, 0) \, dx.\]

Let us recall the following boundary regularity of the solution of (2.1), see Theorem 1.5 [3] for the details.

(i) Suppose that \(2(2s - 1)^+ < n < 4s\), and \(\Omega\) is a \(C^1\) domain and that \(A(x)\) is continuous in \(\overline{\Omega}\). Then \(u \in C^{0, \alpha}(\overline{\Omega})\), for \(\alpha = 2s - \frac{n}{2}\).

(ii) Suppose that \(s > \frac{1}{2}, n < 2(2s - 1), \Omega\) is a \(C^{1, \alpha}(\Omega)\) domain and \(A(x)\) is in \(C^{0, \alpha}(\Omega)\), for \(\alpha = 2s - n \in (0, 1)\). Then \(u \in C^{1, \alpha}(\overline{\Omega})\).

3. STURM-PICONE THEOREM

In this section, we state and prove Sturm-Picone theorem.

Let \(\mathcal{D} := \{U \in H^1_0(\Omega \times (0, \infty), y^a dX) : U(x, 0) = u(x)\}\) and let us define the quadratic functional associated with (2.5):

\[(3.1) \quad M(U) = \int_{\Omega} \int_{0}^{\infty} \sum_{i,j} y^a B_{ij} D_i D_j U \, dX - 2sc_s \int_{\Omega} C(x)(U(x, 0))^2 \, dx, \quad U \in \mathcal{D}.\]

The following is an important lemma to establish Sturm-Picone theorem.

**Lemma 3.1.** (Leighton’s Variational Lemma) Let \(s, \Omega\) and \(A\) be defined as in (i) and (ii). If there exists \(U \in \mathcal{D}\) not identically zero such that \(M(U) \leq 0\), then every solution \(v\) of (2.5) vanishes at some point of \(\overline{\Omega} \times [0, \infty)\). Also, if \(C \in C(\overline{\Omega}), C > 0\), then every nontrivial solution of (2.1) vanishes at some point of \(\Omega\).
Proof. We will prove this lemma by the method of contradiction. Suppose there exists a solution \( v \) of (2.5) such that \( v \neq 0 \) on \( \Omega \times [0, \infty) \). For \( U \in \mathcal{D} \), let us define

\[
X^i = vD_i \left( \frac{U}{v} \right), \quad Y^i = \frac{1}{v} \sum_j y^a B_{ij} D_j v, \quad i = 1, 2, \cdots, n.
\]

\[
X^{n+1} = vD_g \left( \frac{U}{v} \right), \quad Y^{n+1} = \frac{1}{v} \sum_j y^a B_{(n+1)j} D_j v = \frac{1}{v} y^a D_y v.
\]

\[
G(U, v) = \sum_{i,j} y^a B_{ij} X^i X^j + \sum_i D_i (U^2 Y^i).
\]

Now, one can establish the following identity in \( \Omega \times [0, \infty) \):

\[
G(U, v) = \sum_{i,j} y^a B_{ij} D_i U D_j v + \frac{U^2}{v} L v,
\]

where \( L v = \text{div}(y^a B(x) \nabla v) \). Indeed,

\[
\sum_{i,j} y^a B_{ij} X^i X^j + \sum_i D_i (U^2 Y^i) = \frac{1}{v^2} \sum_{i,j} y^a B_{ij} (v D_i U - UD_i v)(v D_j U - UD_j v)
\]

\[
+ \frac{2U}{v} \sum_{i,j} y^a B_{ij} D_i U D_j v + \frac{U^2}{v^2} \sum_{i,j} (v D_i (y^a B_{ij} D_j v) - y^a B_{ij} D_i v D_j v).
\]

Since \( B_{ij} \) is symmetric, so (3.3) reduces to the RHS of (3.2). Now, since from (2.5), \( L v = 0 \) in \( \Omega \times (0, \infty) \), so from (3.1) and (3.2), it follows that

\[
\int_{\Omega} \int_0^\infty \sum_{i,j} y^a B_{ij} D_i U D_j v dX = \int_{\Omega} \int_0^\infty \left[ \sum_{i,j} y^a B_{ij} X^i X^j + \sum_i D_i (U^2 Y^i) \right] dX.
\]

Since \( U \) vanishes on \( \partial \Omega \times [0, \infty) \), so by Green’s theorem and third equation of (2.5), we get

\[
\int_{\Omega} \int_0^\infty \sum_i D_i (U^2 Y^i) dX = 2sc_a \int_{\Omega} C(x)(U(x, 0))^2 dx.
\]

Now, (3.4) and (3.5) yields that

\[
\int_{\Omega} \int_0^\infty \sum_{i,j} y^a B_{ij} D_i U D_j U dX - 2sc_a \int_{\Omega} C(x)(U(x, 0))^2 dx = \int_{\Omega} \int_0^\infty \sum_{i,j} y^a B_{ij} X^i X^j.
\]

Since \( (B_{ij}) \) is positive definite so from (3.1) and (3.6), we get \( M(U) \geq 0 \), and equality holds if and only if \( X^i = 0 \) for each \( i = 1, 2, 3, \cdots, n, n + 1 \), i.e., \( U \) is a constant multiple of \( v \). But this is not possible, since \( U = 0 \) on \( \partial \Omega \times [0, \infty) \) while \( v \neq 0 \) on \( \Omega \times [0, \infty) \), and therefore \( M(U) > 0 \), which is a contradiction. This implies that there exists \( (x, y_1) \in \Omega \times [0, \infty) \) such that \( v(x, y_1) = 0 \).

Now, by using the heat kernel, we can see that the solution of (2.5) can be represented in terms of Poisson’s kernel, i.e.,

\[
v(x, y) = c_s \int_\Omega P_y^s(x, z) C(z) u(z) dz,
\]
where \( u \) satisfies Equation (2.1) and \( P_y^s(x, z) \) is the Poisson kernel, which is given by

\[
P_y^s(x, z) = \frac{y^{2s}}{4s\Gamma(s)} \int_0^\infty e^{-\frac{t^2}{4}} W_t(x, z) \frac{dt}{t^{1+s}},
\]

see, pp. 777 [3] for the details. In (3.8), \( W_t(x, z) \) is the distributional heat kernel for \( L = -\text{div}(B(x)\nabla) \) with the Dirichlet boundary condition, which is given by

\[
W_t(x, z) = \sum_{k=0}^\infty e^{-t\lambda_k} \phi_k(x)\phi_k(z) = W_t(z, x), \quad t > 0, \quad x, z \in \Omega.
\]

From [5], it is clear that \( W_t(x, z) > 0, \forall t > 0 \) and \( x, z \in \Omega \). From (3.7), (3.8) and (3.9), it follows that \( v(x, y_1) = 0 \) implies that either \( y_1 = 0 \) or \( u \) changes sign in \( \Omega \) and in both cases, \( u \) vanishes at some point of \( \Omega \). This completes the proof of the lemma.

**Remark 3.2.** It will be of interest to remove the sign condition on \( C \) for the second part of the lemma.

Let us consider a pair of nonlocal equations:

\[
(-\text{div}(A_1(x)\nabla))^s u = C_1(x) u \quad \text{in } \Omega,
\]

\[
u = 0 \quad \text{on } \partial\Omega,
\]

and

\[
(-\text{div}(A_2(x)\nabla))^s w = C_2(x) w \quad \text{in } \Omega,
\]

\[
w = 0 \quad \text{on } \partial\Omega,
\]

where \( \Omega \subset \mathbb{R}^n \) is an open bounded subset with smooth boundary, \( 0 < s < 1 \), \( A_1, A_2 \) are real symmetric and positive definite matrices on \( \Omega \) with continuous entries on \( \overline{\Omega} \) and \( C_1, C_2 \in C(\overline{\Omega}) \).

In view of (2.2) and (2.3), Equations (3.10) and (3.11) turn out the following

\[
div(g^aB_1(x)\nabla U) = 0 \quad \text{in } \Omega \times (0, \infty),
\]

\[
U = 0 \quad \text{on } \partial\Omega \times [0, \infty),
\]

\[
-\frac{1}{2s} \lim_{y \to 0^+} y^a U_y(x, y) = c_s C_1(x) u \quad \text{on } \Omega,
\]

and

\[
div(g^aB_2(x)\nabla W) = 0 \quad \text{in } \Omega \times (0, \infty),
\]

\[
W = 0 \quad \text{on } \partial\Omega \times [0, \infty),
\]

\[
-\frac{1}{2s} \lim_{y \to 0^+} y^a W_y(x, y) = c_s C_2(x) w \quad \text{on } \Omega,
\]

respectively, where \( U(x, 0) = u(x), \ W(x, 0) = w(x) \) in \( \Omega \) and \( B_1 \) and \( B_2 \) are defined as follows:

\[
B_1(x) := \begin{bmatrix} A_1(x) & 0 \\ 0 & 1 \end{bmatrix}, \quad B_2(x) := \begin{bmatrix} A_2(x) & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}.
\]
Let us define the quadratic functionals associated with (3.12) and (3.13), respectively:
(3.15)
\[ M_1(U) = \int_\Omega \int_0^\infty \sum_{i,j} y^a B_{1_{ij}} D_i U D_j U dX - 2s c_{ij} \int_\Omega C_1(x) (U(x,0))^2 dx, \quad U \in \mathcal{D}. \]
(3.16)
\[ M_2(U) = \int_\Omega \int_0^\infty \sum_{i,j} y^a B_{2_{ij}} D_i U D_j U dX - 2s c_{ij} \int_\Omega C_2(x) (U(x,0))^2 dx, \quad U \in \mathcal{D} \]
and the variation is given by
\[
V(U) = M_2(U) - M_1(U)
= \int_\Omega \int_0^\infty \sum_{i,j} y^a (B_{2_{ij}} - B_{1_{ij}}) D_i U D_j U dX + 2s c_{ij} \int_\Omega (C_1(x) - C_2(x)) (U(x,0))^2 dx, \quad U \in \mathcal{D}, \ y > 0.
\]

**Theorem 3.3.** (Generalization of Leighton’s Theorem) Let \( \Omega \) be defined as in (i) and (ii). Let \( A_1, A_2 \) be real symmetric and positive definite matrices on \( \Omega \) which satisfy (i) and (ii) and \( C_1, C_2 \in C(\bar{\Omega}) \). Let \( U \) be nontrivial solution of (3.13) such that \( V(U) \geq 0 \), then every solution of (3.12) vanishes at some point of \( \Omega \times [0, \infty) \).

In addition, if \( C_1 > 0 \) in \( \Omega \), then every nontrivial solution of (3.10) vanishes at some point of \( \Omega \).

**Proof.** Since \( U \) is a nontrivial solution of (3.13), so, Green’s formula yields that \( M_2(U) = 0 \). Since \( V(U) = M_2(U) - M_1(U) \geq 0 \), i.e., \( M_1(U) \leq M_2(U) = 0 \), \( U \in \mathcal{D} \).

Now, by an application of Lemma 3.1 every solution of (3.12) vanishes at some point of \( \Omega \times [0, \infty) \). Also, every nontrivial solution of (3.10) vanishes at some point of \( \Omega \). This completes the proof.

**Theorem 3.4.** (Sturm–Picone Comparison Theorem) Let \( \Omega \) be defined as in (i) and (ii). Let \( A_1, A_2 \) be real symmetric and positive definite matrices on \( \Omega \) which satisfy (i) and (ii) and let \( C_1, C_2 \in C(\bar{\Omega}) \) with \( C_1(x) - C_2(x) \geq 0 \) on \( \Omega \). Let \( B_{2_{ij}} - B_{1_{ij}} \) be positive semidefinite and \( U \) be nontrivial solution of (3.13), then every solution of (3.12) vanishes at some point of \( \Omega \times [0, \infty) \). In addition, if \( C_1 > 0 \) in \( \Omega \), then every nontrivial solution of (3.10) vanishes at some point of \( \Omega \).

**Proof.** Since \( B_{2_{ij}} - B_{1_{ij}} \) is positive semidefinite and \( C_1(x) - C_2(x) \geq 0 \) on \( \Omega \), so \( V(U) \geq 0 \) and the proof follows from Theorem 3.3.

\[ \square \]

4. A FEW REMARKS

A few remarks concerning the qualitative behavior of the solution to fractional Laplace equations are in order:

**Remark 4.1.** Let \( \Omega(r_0) = \{ x \in \mathbb{R}^n : ||x|| \geq r_0 \} \) for some \( r_0 > 0 \) be an exterior domain in \( \mathbb{R}^n \), where \( || \cdot || \) is the usual Euclidean norm in \( \mathbb{R}^n \). Now, a very first question concerns whether one can pose (2.1) in exterior domains. If yes, then a nontrivial solution \( u \) of (2.1) (posed in exterior domains) is said to be oscillatory if the set \( \{ x \in \Omega(r_0) : u(x) = 0 \} \) is unbounded; otherwise it is called non-oscillatory, see for instance, pp. 135 [35]. Equation (2.1) is called oscillatory if all its solutions
are oscillatory. In this context, it is natural to look at the whole study of this paper in the exterior/unbounded domains.

The next remark deals with an evidence of the oscillatory behavior of the solution of (2.1) in $\mathbb{R}$.

**Remark 4.2.** Let us consider (2.1) in $\mathbb{R}$ with $A(x) = 1$ and $C = 4$, $s \in (0, 1)$ and consider the radially symmetric solutions of

$$-(\Delta)^s u + 4u = 0 \text{ in } \mathbb{R}.\quad (4.1)$$

We recall that, when $u$ is radially symmetric,

$$-(\Delta)^s u = u''(r) + (n + 1 - 2s) \frac{u'(r)}{r}, \quad u(x) = u(|x|) = u(r), \quad x \in \mathbb{R}^n,$$

see [13] for the details and by (4.2), (4.1) converts into

$$u''(r) + 2(1 - s) \frac{u'(r)}{r} + 4u = 0.\quad (4.3)$$

Now, using the standard transformation

$$u(r) = y(r)e^{-\frac{1}{4}\iint 2(1-s)dr} = y(r)r^{s-1}, \quad r > 0,$$

(4.3) is

$$y''(r) + \left(\frac{4r^2 - s^2 + s}{r^2}\right) y(r) = 0.\quad (4.4)$$

Since $s \in (0, 1)$, so $\frac{4r^2 - s^2 + s}{r^2} > 4$ and by classical Sturm’s comparison theorem, (4.4) is oscillatory. Since the above transformation is oscillation preserving, and therefore (4.3) is oscillatory. Now, it will be of interest to find out whether every solution of (4.1) and more generally, to (2.1), when posed in $\mathbb{R}^n$, is oscillatory.

In the next remark, one can also inquire on the non-oscillatory solution (eventually one signed solution) of (2.1) in $\mathbb{R}$.

**Remark 4.3.** Let us consider (2.1) in $\mathbb{R}$ with $A(x) = 1$ and $C = -1$, $s \in (0, 1)$ and consider the radially symmetric solutions of

$$-(\Delta)^s u - u = 0 \text{ in } \mathbb{R}.\quad (4.5)$$

Again by (4.2), (4.5) converts into

$$u''(r) + 2(1 - s) \frac{u'(r)}{r} - u = 0.\quad (4.6)$$

Now, using the standard transformation

$$u(r) = y(r)e^{-\frac{1}{4}\iint 2(1-s)dr} = y(r)r^{s-1}, \quad r > 0,$$

(4.6) reads as

$$y''(r) + \left(\frac{-r^2 - s^2 + s}{r^2}\right) y(r) = 0.\quad (4.7)$$

It is easy to see that $\frac{-r^2 - s^2 + s}{r^2} > \frac{r^2 - s^2 + s}{r^2}$ and we know that

$$y''(r) + \frac{1}{4r^2} y(r) = 0.\quad (4.8)$$
is non-oscillatory so by classical Sturm’s comparison theorem, (4.7) is non-oscillatory and therefore (4.6) is non-oscillatory. Now, it will be of interest to investigate whether (4.5) and more generally, (2.1), when posed in $\mathbb{R}^n$, is non-oscillatory.

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