Co-Poisson structures on polynomial Hopf algebras

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Abstract The Hopf dual $H^\circ$ of any Poisson Hopf algebra $H$ is proved to be a co-Poisson Hopf algebra provided $H$ is noetherian. Without noetherian assumption, unlike it is claimed in literature, the statement does not hold. It is proved that there is no nontrivial Poisson Hopf structure on the universal enveloping algebra of a non-abelian Lie algebra. So the polynomial Hopf algebra, viewed as the universal enveloping algebra of a finite-dimensional abelian Lie algebra, is considered. The Poisson Hopf structures on polynomial Hopf algebras are exactly linear Poisson structures. The co-Poisson structures on polynomial Hopf algebras are characterized. Some correspondences between co-Poisson and Poisson structures are also established.

Keywords Poisson algebra, co-Poisson coalgebra, Poisson Hopf algebra, co-Poisson Hopf algebra

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1 Introduction

Poisson structure naturally appears in classical/quantum mechanics, in mathematical physics, and in deformation theory. It is an important algebra structure in Poisson geometry, algebraic geometry and non-commutative geometry. There is much research in the related subjects.

Co-Poisson structure is a dual concept of Poisson structure in categorical point of view. It arises also in mathematics and mathematical physics naturally as explained in the next two paragraphs.

Let $G$ be a Lie group and $O(G)$ be its algebra of functions. A Lie group $G$ is said to be a Poisson Lie group if $O(G)$ is a Poisson Hopf algebra. It is well known that the category of connected and simply-connected Lie groups is equivalent to the category of finite-dimensional Lie algebras. In this case, $O(G)$ is identified with the Hopf dual $U(g)^\circ$ of the universal enveloping algebra $U(g)$, where $g$ is the corresponding Lie algebra of $G$. The Poisson counterpart of this fact holds also, namely, the category of connected and simply-connected Poisson Lie groups is equivalent to the category of finite-dimensional Lie bialgebras (see [6, Theorem 3.3.1] or [2, Theorem 1]). However, the Lie bialgebra structures on any Lie algebra $g$ are in one-to-one correspondence with the co-Poisson Hopf structures on $U(g)$ (see [1, Proposition 6.2.3]).

On the other hand, to quantize a Lie group or Lie algebra one should equip it with an extra structure, namely, a Poisson Lie group structure or Lie bialgebra structure, respectively. Therefore, co-Poisson structure naturally appears in the theory of quantum groups and in mathematical physics.

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If $G$ is a connected and simply-connected Poisson Lie group and $\mathfrak{g}$ is the corresponding Lie bialgebra, then the Poisson Hopf structure on $U(\mathfrak{g})^\circ \cong \mathcal{O}(G)$ is the dual of the co-Poisson Hopf structures on $U(\mathfrak{g})$. In [14], Oh and Park proved that the dual Hopf algebra $U(\mathfrak{g})^\circ$ of $U(\mathfrak{g})$ is a Poisson Hopf algebra for any finite-dimensional Lie bialgebra $\mathfrak{g}$. In fact, in general, as stated in [6, Proposition 3.1.5] earlier, the dual Hopf algebra of any co-Poisson Hopf algebra is a Poisson Hopf algebra. A complete proof is given in a recent paper by Oh [13, Theorem 2.2]. The dual proposition that the dual Hopf algebra of any Poisson Hopf algebra is a co-Poisson Hopf algebra is also stated in [6, Proposition 3.1.5]. Unfortunately, this claim is not true in general as showed in our Example 3.6. Under an additional assumption that the algebra is noetherian, we prove the statement is true in Proposition 3.5.

We prove that there is no nontrivial Poisson Hopf structure on the universal enveloping algebra of a non-abelian Lie algebra in Proposition 2.5. So, we turn to consider in latter sections the abelian case, i.e., the polynomial Hopf algebra $A = k[x_1, x_2, \ldots, x_d]$, viewed as the universal enveloping algebra of an abelian Lie algebra of dimension $d$. The Poisson Hopf structures on $A = k[x_1, x_2, \ldots, x_d]$ are exactly linear Poisson structures on $A$ (see Proposition 5.3). By establishing a reciprocity law between two linear maps of $A \to A \otimes A$ for $A = k[x_1, x_2, \ldots, x_d]$ (see Proposition 4.3), all co-Poisson coalgebra and co-Poisson Hopf algebra structures on $A = k[x_1, x_2, \ldots, x_d]$ are described in Theorem 4.7 and Proposition 5.4, respectively. In particular, the co-Poisson coalgebra structures on $A = k[x, y]$ are given by the linear maps $I : A \to k(x \otimes y - y \otimes x)$ (see Proposition 4.8). By using the algebra of divided power series, the co-Poisson coalgebra structures on $A = k[x_1, x_2, \ldots, x_d]$ are showed to be in one-to-one correspondence with the Poisson algebra structures on $A = k[[x_1, x_2, \ldots, x_d]]$, the algebra of formal power series in Theorem 5.8.

The paper is organized as follows. The definitions of (co-)Poisson (co)algebras are recalled in Section 2. Some preliminary results and examples are also given in Section 2. In Section 3, we establish some dual properties between co-Poisson structures and Poisson structures. In Section 4, we characterize co-Poisson coalgebra structures on polynomial Hopf algebra. In Section 5, we characterize co-Poisson Hopf structures on polynomial Hopf algebras.

Convention. Let $k$ be a base field. All vector spaces, algebras, coalgebras, and Hopf algebras are over $k$. All linear maps mean $k$-linear. Unadorned $\otimes$ means $\otimes_k$.

Let $V$ be a vector space. Let $t_n : V^\otimes n \to V^\otimes n$ ($n \in \mathbb{N}^+$) be the linear map given by $v_1 \otimes \cdots \otimes v_n \mapsto v_n \otimes v_1 \otimes \cdots \otimes v_{n-1}$. For convenience, let $\circ = 1 + t_3 + t_3^2$.

Suppose $(C, \Delta, \varepsilon)$ is a coalgebra where $\Delta$ is the comultiplication and $\varepsilon$ is the counit. We frequently use the sigma notation

$$\Delta(\varepsilon) = \sum c_1 \otimes c_2 \quad \text{and} \quad (\Delta \otimes 1)\Delta(\varepsilon) = \sum c_1 \otimes c_2 \otimes c_3,$$

where $\sum$ is often omitted in the computations.

Let $\Delta^{(2)} = (\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta : C \to C \otimes C \otimes C$, and $\Delta' = \Delta - t_2 \circ \Delta$ be the cocommutator.

Suppose $(A, \mu, \eta)$ is an algebra where $\mu$ is the multiplication and $\eta$ is the unit. For any $a, b \in A$, $[a, b] = ab - ba$ is the commutator.

2 Poisson structures and co-Poisson structures

2.1 Poisson algebras and Poisson Hopf algebras

Definition 2.1 (See [8,17]). An algebra $A$ equipped with a linear map $\{-, -\} : A \otimes A \to A$ is called a Poisson algebra if

1. $A$ with $\{-, -\} : A \otimes A \to A$ is a Lie algebra;
2. $\{-, c\} : A \to A$ is a derivation with respect to the multiplication of $A$ for all $c \in A$, i.e., $\{ab, c\} = a\{b, c\} + \{a, c\}b$ for all $a, b \in A$.

It should be noted that we do not assume that $A$ is commutative in general. As showed in [3, Theorem 1.2], if $A$ is prime and not commutative, then any nontrivial Poisson structure $\{-, -\}$ on $A$ is the