TWO RESULTS ON THE LOGARITHMIC COTANGENT COMPLEX

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ABSTRACT. In this paper we give two results on the logarithmic cotangent complex: we construct logarithmic analogues to the complex of Lichtenbaum and Schlessinger and to Quillen’s fundamental spectral sequence.

André-Quillen homology first appeared in dimension 0 and 1 implicitly in [5]. Subsequently, Lichtenbaum and Schlessinger in [6] introduce the second homology module as the homology of a simple complex. The general definition was then given by André [1] and Quillen [10] using simplicial methods. The fact that the second André-Quillen homology module agrees with that of Lichtenbaum and Schlessinger was proved in [2, 15.12]. Due to its simplicity, the complex of Lichtenbaum and Schlessinger is still useful if we do not need higher homology modules. It is for example the only one used in the entire book [7].

A logarithmic version of the cotangent complex was introduced by Gabber (see [8]). Since the second homology module of the logarithmic cotangent complex detects Kato’s log regularity [4], a logarithmic analogue of the complex of Lichtenbaum and Schlessinger would be desirable. This is the first purpose of this paper. The second one is to give a logarithmic analogue of Quillen’s fundamental spectral sequence [10, Theorem 6.8]

\[ E^2_{p,q} = H_{p+q}(S^q\mathbb{L}_B|C) \Rightarrow \text{Tor}_{p+q}^{C}(B,B), \]

for a surjective ring homomorphism \( C \to B \).

We will use the notation on the logarithmic setting as in [4, Sections 2 and 3], and on the logarithmic cotangent complex as in [4, Section 4]. For instance, a prelog ring \((A,M,\alpha)\) (or simply \((A,M)\) if there is no confusion) consists of a commutative ring \(A\), a commutative monoid \(M\) and a multiplicative homomorphism of monoids \(\alpha: M \to A\). When \(m \in M\), we sometimes write also \(m\) for its image \(\alpha(m)\) in \(A\). A homomorphism of prelog rings

\[ f = (f^A, f^M): (A, M, \alpha_A) \to (B, N, \alpha_B) \]

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1The same day we are preparing this paper to upload it to arXiv, it has appeared in that site the paper “A Hochschild-Kostant-Rosenberg theorem and residue sequences for logarithmic Hochschild homology” by Federico Binda, Tommy Lundemo, Doosung Park and Paul Arne Østvær (arXiv:2209.11182) where the authors give an analogous spectral sequence in the case of Hochschild homology, that is, when \( C \to B \) is the diagonal homomorphism of a homomorphism. In that paper, they even compute all the terms \( E^{2q}_{pq} \), while we treat only the terms with \( q < 2 \).

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is a homomorphism of rings \( f^2: A \to B \) together with a homomorphism of monoids \( f^\#: M \to N \) such that \( \alpha_B f^\# = f^2 \alpha_A \).

1. A complex to compute \( H_2 \)

Let \((A,M) \to (B,N)\) be a homomorphism of prelog rings. Let \((A,M) \to (R,P_0) \to (B,N)\) be a factorization with \( R \to B \) and \( h: P_0 \to N \) surjective homomorphisms, where \( P_0 = M \oplus \mathbb{N}^X \) (with \( h \) extending \( M \to N \)) and \( R = \mathbb{Z}[P_0] \odot \mathbb{Z}[M] A[Y] = A[X \cup Y] \) (with \( R \to B \) extending \( A \to B \)).

We consider \( W_0 = \ker(P_0^{\text{gp}} \to N^{\text{gp}}) \), \( J = \ker(Z[P_0] \to Z[N]) \) and \( I = \ker(R \to B) \).

Let \[
\begin{array}{c}
0 \to V \to G \xrightarrow{\pi} J \to 0
\end{array}
\]
be an exact sequence of \( \mathbb{Z}[P_0] \)-modules with \( G \) a free \( \mathbb{Z}[P_0] \)-module,

\[
\begin{array}{c}
0 \to U \to F \xrightarrow{\tau} I \to 0
\end{array}
\]
and exact sequence of \( R \)-modules with \( F \) a free \( R \)-module containing \( R \otimes \mathbb{Z}[P_0] G \) as a direct summand and \( \tau \) extending \( G \xrightarrow{\pi} J \to I \), and

\[
\begin{array}{c}
0 \to W_1 \to Q_1 \to W_0 \to 0
\end{array}
\]
an exact sequence of \( \mathbb{Z} \)-modules with \( Q_1 \) a free \( \mathbb{Z} \)-module.

Let \( U_0 = \{ \tau(x)y - \tau(y)x \in U \mid x,y \in F \} \), which is an \( R \)-submodule of \( U \), and \( V_0 = \{ \pi(x)y - \pi(y)x \in V \mid x,y \in G \} \) which is a \( \mathbb{Z}[P_0] \)-submodule of \( V \). The \( R \)-module \( U/U_0 \) is in fact a \( B \)-module since \( IU \subset U_0 \) and similarly, \( V/V_0 \) is a \( \mathbb{Z}[N] \)-module.

Consider the following commutative diagram of \( B \)-modules

\[
\begin{array}{cccccc}
B \otimes_{\mathbb{Z}[N]} V/V_0 & \xrightarrow{\beta_2} & B \otimes_{\mathbb{Z}} W_1 & \xrightarrow{D_2} & B \otimes_{\mathbb{Z}} Q_1 & \xrightarrow{D_1} & B \otimes_{\mathbb{Z}} P_0^{\text{gp}} / M^{\text{gp}} \\
B \otimes_{\mathbb{Z}[P_0]} G = B \otimes_{\mathbb{Z}[N]} G/JG & \xrightarrow{\beta_1} & B \otimes_{\mathbb{Z}} Q_1 & \xrightarrow{D_1} & B \otimes_{\mathbb{Z}} P_0^{\text{gp}} / M^{\text{gp}} \\
B \otimes_{\mathbb{Z}[P_0]} \Omega_{[P_0][Z[M]]} & \xrightarrow{\beta_0} & B \otimes_{\mathbb{Z}} P_0^{\text{gp}} / M^{\text{gp}} \\
F/IF & \xrightarrow{\delta_1} & B \otimes_{R} \Omega_{R/A}
\end{array}
\]

(1.1)

where the homomorphisms are defined as follows:

- \( \delta_2, d_2 \) and \( D_2 \) are induced by the inclusions \( U \to F, V \to G \) and \( W_1 \to Q_1 \) respectively.
- \( \delta_1 \) is the composition \( F/IF \to I/I^2 \to B \otimes_{R} \Omega_{R/A} \), where the first map is the one induced by \( \tau \) and the second one by the canonical derivation \( R \to \Omega_{R/A} \). The map \( d_1 \) is analogously defined while \( D_1 \) is induced by the composition \( Q_1 \to W_0 \to P_0^{\text{gp}} \to P_0^{\text{gp}} / M_0^{\text{gp}} \).
• The homomorphism $\beta_0$ is the one of [4, Definition 3.9]: $\beta_0(1 \otimes dp) := h(p) \otimes \bar{p}$.

• The exact sequences

\[
0 \rightarrow V \rightarrow G \rightarrow J \rightarrow 0,
\]

\[
0 \rightarrow W_1 \rightarrow Q_1 \rightarrow W_0 \rightarrow 0
\]

give a diagram of homomorphisms of $\mathbb{Z}[N]$-modules

\[
\begin{array}{cccc}
V/JV & \rightarrow & G/JG & \rightarrow & J/J^2 & \rightarrow & 0 \\
\downarrow \beta_1' & & \downarrow \beta_1' & & \downarrow \beta_1' & & \downarrow v_h \\
0 & \rightarrow & \mathbb{Z}[N] \otimes_{\mathbb{Z}} W_1 & \rightarrow & \mathbb{Z}[N] \otimes_{\mathbb{Z}} Q_1 & \rightarrow & \mathbb{Z}[N] \otimes_{\mathbb{Z}} W_0 & \rightarrow & 0 \\
\end{array}
\]

where $v_h$, which is defined by

\[
\lambda(p - p') \mapsto \lambda h(p') \otimes p(p')^{-1}
\]

(whenever $\lambda \in \mathbb{Z}$, $h(p) = h(p')$), is the map of [4, Definition 3.1] (see [4, Remark 3.2]). Since $G/JG = G \otimes_{\mathbb{Z}[P]} \mathbb{Z}[N]$ is a projective $\mathbb{Z}[N]$-module, we have a commutative diagram

\[
\begin{array}{cccc}
V/JV & \rightarrow & G/JG & \rightarrow & J/J^2 & \rightarrow & 0 \\
\downarrow \beta_1' & & \downarrow \beta_1' & & \downarrow \beta_1' & & \downarrow v_h \\
0 & \rightarrow & \mathbb{Z}[N] \otimes_{\mathbb{Z}} W_1 & \rightarrow & \mathbb{Z}[N] \otimes_{\mathbb{Z}} Q_1 & \rightarrow & \mathbb{Z}[N] \otimes_{\mathbb{Z}} W_0 & \rightarrow & 0 \\
\end{array}
\]

We define $\beta_1 = B \otimes_{\mathbb{Z}[N]} \beta_1'$. Since $V_0 \subset JG$, $f(V_0/JV) = 0$ and then $\beta_1'(V_0/JV) = 0$ by the injectivity of $g$. So $\beta_1'$ gives a map $\tilde{\beta}_1 : V/V_0 \rightarrow \mathbb{Z}[N] \otimes_{\mathbb{Z}} W_1$ and we define $\beta_2 = B \otimes_{\mathbb{Z}[N]} \tilde{\beta}_1$.

• Finally, $\alpha_2, \alpha_1, \alpha_0$ are the obvious maps.

The commutativity of the square $D_1 \beta_1 = \beta_0d_1$ follows from the commutativity of the squares

\[
\begin{array}{cccc}
G/JG & \rightarrow & J/J^2 & \\
\downarrow \beta_1' & & \downarrow v_h & \\
\mathbb{Z}[N] \otimes_{\mathbb{Z}} Q_1 & \rightarrow & \mathbb{Z}[N] \otimes_{\mathbb{Z}} W_0 \\
\end{array}
\]

and

\[
\begin{array}{cccc}
J/J^2 & \rightarrow & \mathbb{Z}[N] \otimes_{\mathbb{Z}[P]} \Omega_{Z[P] \otimes Z[M]} & \\
\downarrow v_h & & \downarrow & \\
\mathbb{Z}[N] \otimes_{\mathbb{Z}} W_0 & \rightarrow & \mathbb{Z}[N] \otimes_{\mathbb{Z}} P_0^{\text{gp}} / M_0^{\text{gp}} \\
\end{array}
\]

where the right map in this square is defined similarly to $\beta_0$.

The commutativity of the remaining squares is clear.
We have $\delta_1 \delta_2 = 0$, $d_1 d_2 = 0$, $D_1 D_2 = 0$.

**Definition 1.1.** We define the complex $LS_{(B,N)}((A,M))$ as follows: $(LS_{(B,N)}((A,M)))_i$ is the pushout of $(\alpha_i, \beta_i)$ for $i \in \{0, 1, 2\}$ and zero for $i \notin \{0, 1, 2\}$, and the differential is the one induced by the maps $\delta_i$, $d_i$, $D_i$.

**Proposition 1.2.** In this context, we consider the complex $K := (T \otimes \mathbb{Z} W_1 \xrightarrow{D_2} T \otimes \mathbb{Z} Q_1 \xrightarrow{D_1} T \otimes \mathbb{Z} P_{0}^{gp}/M^{gp})$ for any $\mathbb{Z}$-module $T$. Then:

(i) $H_2(K) = \text{Tor}_{1}^{\mathbb{Z}}(T, \ker(M^{gp} \to N^{gp}))$

(ii) We have an exact sequence $0 \to T \otimes \mathbb{Z} \ker(M^{gp} \to N^{gp}) \to H_1(K) \to \text{Tor}_{1}^{\mathbb{Z}}(T, N^{gp}/\text{Im}(M^{gp} \to N^{gp})) \to 0$.

(iii) $H_0(K) = T \otimes \mathbb{Z} N^{gp}/\text{Im}(M^{gp} \to N^{gp})$

**Proof.** We have $H_2(K) = \ker(T \otimes \mathbb{Z} W_1 \to T \otimes \mathbb{Z} Q_1)$. Applying $T \otimes -$ to the exact sequence of $\mathbb{Z}$-modules

$$
0 = \text{Tor}_{1}^{\mathbb{Z}}(T, Q_1) \to \text{Tor}_{1}^{\mathbb{Z}}(T, W_0) \to T \otimes \mathbb{Z} W_1 \to T \otimes \mathbb{Z} Q_1 \to T \otimes \mathbb{Z} W_0 \to 0
$$

and then $H_2(K) = \text{Tor}_{1}^{\mathbb{Z}}(T, W_0)$.

Consider the diagram of exact rows and columns defining the lower row (1.2)

$$
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & \ker(M^{gp} \to N^{gp}) & M^{gp} & \text{Im}(M^{gp} \to N^{gp}) & 0 \\
0 & W_0 & P_{0}^{gp} = M^{gp} \oplus \mathbb{Z}^X & N^{gp} & 0 \\
0 & W_0/\ker(M^{gp} \to N^{gp}) & \mathbb{Z}^X & N^{gp}/\text{Im}(M^{gp} \to N^{gp}) & 0 \\
0 & 0 & 0 & 0
\end{array}
$$

Since the lower row is exact,

$$
\text{Tor}_{i}^{\mathbb{Z}}(T, W_0/\ker(M^{gp} \to N^{gp})) = \text{Tor}_{i+1}^{\mathbb{Z}}(T, N^{gp}/\text{Im}(M^{gp} \to N^{gp}))
$$

and this last module vanishes for $i + 1 \geq 2$ given that $\mathbb{Z}$ is a principal ideal domain. So, from the left column we deduce

$$
H_2(K) = \text{Tor}_{1}^{\mathbb{Z}}(T, W_0) = \text{Tor}_{1}^{\mathbb{Z}}(T, \ker(M^{gp} \to N^{gp})).$$
Consider now the diagram

\[
\begin{array}{cccccc}
0 & \rightarrow W_1 & \rightarrow Q_1 & \xrightarrow{\varphi} & W_0 & \rightarrow 0 \\
& & \downarrow & & & \\
& & \sigma & & & \\
0 & \rightarrow W_0 / \ker(M^{gp} \rightarrow N^{gp}) & \rightarrow \mathbb{Z}^X & \rightarrow N^{gp} / \text{Im}(M^{gp} \rightarrow N^{gp}) & \rightarrow 0 & \rightarrow 0
\end{array}
\]

We have exact sequences

\[(1.3) \quad 0 \rightarrow \varphi^{-1}(\ker(M^{gp} \rightarrow N^{gp})) \rightarrow Q_1 \rightarrow \mathbb{Z}^X \rightarrow N^{gp} / \text{Im}(M^{gp} \rightarrow N^{gp}) \rightarrow 0,
\]

and

\[0 \rightarrow W_1 \rightarrow \varphi^{-1}(\ker(M^{gp} \rightarrow N^{gp})) \rightarrow \ker(M^{gp} \rightarrow N^{gp}) \rightarrow 0.
\]

Therefore we have a vertical exact sequence of (horizontal) complexes

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
W_1 & \rightarrow Q_1 & \rightarrow \mathbb{Z}^X & \rightarrow N^{gp} / \text{Im}(M^{gp} \rightarrow N^{gp}) & \rightarrow 0 \\
& \varphi^{-1}(\ker(M^{gp} \rightarrow N^{gp})) & \rightarrow Q_1 & \rightarrow \mathbb{Z}^X & \rightarrow N^{gp} / \text{Im}(M^{gp} \rightarrow N^{gp}) \\
& \ker(M^{gp} \rightarrow N^{gp}) & \rightarrow 0 & \rightarrow 0 & \rightarrow 0 \\
0 & 0 & 0 & 0 & 0
\end{array}
\]
Applying $T \otimes \mathbb{Z}$ we obtain a diagram of (horizontal) complexes

\[\begin{array}{cccc}
E': & T \otimes \mathbb{Z} W_1 & \longrightarrow & T \otimes \mathbb{Z} Q_1 & \longrightarrow & T \otimes \mathbb{Z} Z^X \\
E: & T \otimes \mathbb{Z} \varphi^{-1}(\ker(M^\text{gp} \to N^\text{gp})) & \longrightarrow & T \otimes \mathbb{Z} Q_1 & \longrightarrow & T \otimes \mathbb{Z} Z^X \\
E'': & T \otimes \mathbb{Z} \ker(M^\text{gp} \to N^\text{gp}) & \longrightarrow & & \phantom{\longrightarrow} & \\
& & & & & 0
\end{array}\]

(1.4)

where the columns correspond to degrees 2, 1 and 0 respectively (from left to right).

We have

\[H_n(E) = \text{Tor}_n^\mathbb{Z} \left(T, N^\text{gp} / \text{Im}(M^\text{gp} \to N^\text{gp})\right)\]

for $n = 0, 1$ by the exactness of (1.3), and the fact that $Q_1$ and $Z^X$ are free $\mathbb{Z}$-modules.

On the other hand, $W_0 / \ker(M^\text{gp} \to N^\text{gp})$ is a free $\mathbb{Z}$-module by the lower row of (1.2), and then the exact sequence

\[0 \longrightarrow \varphi^{-1}(\ker(M^\text{gp} \to N^\text{gp})) \longrightarrow Q_1 \xrightarrow{\sigma} W_0 / \text{Im}(M^\text{gp} \to N^\text{gp}) \longrightarrow 0\]

splits. Therefore

\[0 \longrightarrow T \otimes \mathbb{Z} \varphi^{-1}(\ker(M^\text{gp} \to N^\text{gp})) \longrightarrow T \otimes \mathbb{Z} Q_1\]

is injective and then $H_2(E) = 0$.

Consider the left column of (1.4) and let

\[\Lambda := \ker \left[T \otimes \mathbb{Z} \varphi^{-1}(\ker(M^\text{gp} \to N^\text{gp})) \to T \otimes \mathbb{Z} \ker(M^\text{gp} \to N^\text{gp})\right] = \text{Im} \left[T \otimes \mathbb{Z} W_1 \to T \otimes \mathbb{Z} \varphi^{-1}(\ker(M^\text{gp} \to N^\text{gp}))\right] = \text{Im}(T \otimes \mathbb{Z} W_1 \to T \otimes \mathbb{Z} Q_1).\]
We have a vertical exact sequence of (horizontal) complexes

\[
\begin{array}{ccccccccc}
0 & 0 & 0 \\
\downarrow & & & & & & & & \\
\tilde{E}' & \Lambda & \rightarrow & T \otimes \mathbb{Z} Q_1 & \rightarrow & T \otimes \mathbb{Z} X \\
\downarrow & & & & & & & & \\
E & T \otimes \mathbb{Z} \varphi^{-1}(\ker(M_{gp}^{sp} \rightarrow N_{gp}^{sp})) & \rightarrow & T \otimes \mathbb{Z} Q_1 & \rightarrow & T \otimes \mathbb{Z} X \\
\downarrow & & & & & & & & \\
E'' & T \otimes \mathbb{Z} \ker(M_{gp}^{sp} \rightarrow N_{gp}^{sp}) & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\]

and taking the homology exact sequence

\[
0 = H_2(E) \rightarrow H_2(E'') = H_1(\tilde{E}') = H_1(E) \rightarrow H_1(E) = \text{Tor}_1^Z(T, N_{gp}^{sp} / \text{Im}(M_{gp}^{sp} \rightarrow N_{gp}^{sp})) \rightarrow H_1(E'') = 0
\]

we obtain the desired exact sequence

\[
0 \rightarrow T \otimes \mathbb{Z} \ker(M_{gp}^{sp} \rightarrow N_{gp}^{sp}) \rightarrow H_1(K) \rightarrow \text{Tor}_1^Z(T, N_{gp}^{sp} / \text{Im}(M_{gp}^{sp} \rightarrow N_{gp}^{sp})) \rightarrow H_1(E'') = 0.
\]

Finally, \( H_0(K) = H_0(E) = T \otimes \mathbb{Z} N_{gp}^{sp} / \text{Im}(M_{gp}^{sp} \rightarrow N_{gp}^{sp}) \).

**Theorem 1.3.** The homology modules of the logarithmic cotangent complex of Gabber can be computed as the homology of the complex \( \text{LS}_{(B,N)}|_{(A,M)} \).

**Proof.** We keep the notation as in the beginning of this section. Let \( \text{LS}_{Z[N]}|_{Z[M]} \) be the complex of \( Z[N] \)-modules

\[
V/V_0 \rightarrow G/JG \rightarrow Z[N] \otimes Z[P_0] \Omega Z[P_0]|_{Z[M]},
\]

\( \text{LS}_{B|A} \) the complex of \( B \)-modules

\[
U/U_0 \rightarrow F/I F \rightarrow B \otimes R \Omega R|_{A}
\]

and \( D \) the complex of \( Z \)-modules

\[
W_1 \rightarrow Q_1 \rightarrow P_0^{\text{sp}} / M_{\text{sp}}.
\]

By [6], for any \( B \)-module \( T \), \( H_*(\text{LS}_{Z[N]}|_{Z[M]} \otimes Z[N]|T) \) and \( H_*(\text{LS}_{B|A} \otimes_B T) \) do not depend on the choices, and similarly \( H_*(D \otimes T) \) by Proposition 1.2.

By definition, we have a commutative diagram of complexes

\[
\begin{array}{ccccccc}
B \otimes Z[N] & \xrightarrow{\beta} & B \otimes Z D \\
\downarrow{\alpha} & & \downarrow{\epsilon} \\
\text{LS}_{B|A} & \xrightarrow{\gamma} & \text{LS}_{(B,N)}|_{(A,M)}
\end{array}
\]
where

By diagram chasing, we deduce an exact sequence

\[ 0 \to T \otimes_{\mathbb{Z}[N]} \text{LS}_{\mathbb{Z}[N][\mathbb{Z}[M]]} T \otimes_{\mathbb{B}} \text{LS}_{\mathbb{B}[A]} \to \text{coker}(T \otimes \alpha) \to 0 \]

Therefore we have a commutative diagram of complexes for any \( B \)-module \( T \)

\[
\begin{array}{ccc}
0 & \to & T \otimes_{\mathbb{Z}} D \to T \otimes_{\mathbb{B}} \text{LS}_{(B,N)((A,M))} \to \text{coker}(T \otimes \varepsilon) \to 0 \\
\end{array}
\]

Since \( \alpha_0, \alpha_0 \) are split injective, so are \( \varepsilon_1, \varepsilon_0 \), and then both lines are exact except that

\[ T \otimes_{\mathbb{Z}} (\text{LS}_{\mathbb{Z}[N][\mathbb{Z}[M]]})_2 = T \otimes_{\mathbb{Z}[N]} V/V_0 \to T \otimes_{\mathbb{B}} U/U_0 = T \otimes_{\mathbb{B}} (\text{LS}_{\mathbb{B}[A]})_2 \]

and

\[ T \otimes_{\mathbb{Z}} D_2 = T \otimes_{\mathbb{Z}} W_1 \to T \otimes_{\mathbb{B}} (\text{LS}_{(B,N)((A,M))})_2 \]

are not necessarily injective. We deduce a commutative diagram of exact rows

\[
\begin{array}{ccc}
H_2(T \otimes_{\mathbb{Z}[N]} \text{LS}_{\mathbb{Z}[N][\mathbb{Z}[M]]}) & \to & H_2(T \otimes_{\mathbb{B}} \text{LS}_{\mathbb{B}[A]}) \to H_2(\text{coker}(T \otimes \alpha)) \to \\
\uparrow & & \uparrow \\
H_2(T \otimes_{\mathbb{Z}} D) & \to & H_2(T \otimes_{\mathbb{B}} \text{LS}_{(B,N)((A,M))}) \to H_2(\text{coker}(T \otimes \varepsilon)) \to \end{array}
\]

\[
\begin{array}{ccc}
\to & H_1(T \otimes_{\mathbb{Z}[N]} \text{LS}_{\mathbb{Z}[N][\mathbb{Z}[M]]}) & \to \cdots \to H_0(\text{coker}(T \otimes \alpha)) \to 0 \\
\downarrow & & \downarrow \\
\to & H_1(T \otimes_{\mathbb{Z}} D) & \to \cdots \to H_0(\text{coker}(T \otimes \varepsilon)) \to 0 \\
\end{array}
\]

By diagram chasing, we deduce an exact sequence

\[
H_2(T \otimes_{\mathbb{Z}[N]} \text{LS}_{\mathbb{Z}[N][\mathbb{Z}[M]]}) + H_2(T \otimes_{\mathbb{B}} \text{LS}_{\mathbb{B}[A]}) + H_2(T \otimes_{\mathbb{Z}} D) + H_2(T \otimes_{\mathbb{B}} \text{LS}_{(B,N)((A,M))}) + \\
\to H_1(T \otimes_{\mathbb{Z}[N]} \text{LS}_{\mathbb{Z}[N][\mathbb{Z}[M]]}) + \cdots \to H_0(T \otimes_{\mathbb{B}} \text{LS}_{(B,N)((A,M))}) + 0.
\]

By [2] Proposition 15.12 and Proposition 1.2, this exact sequence takes the form

\[
\begin{array}{ccc}
H_2(\mathbb{Z}[M], \mathbb{Z}[N], T) & \to & H_2(A, B, T) \oplus H_2(T \otimes_{\mathbb{Z}} D) \to H_2(T \otimes_{\mathbb{B}} \text{LS}_{(B,N)((A,M))}) \to \\
\to & H_1(\mathbb{Z}[M], \mathbb{Z}[N], T) & \to H_1(A, B, T) \oplus H_1(T \otimes_{\mathbb{Z}} D) \to H_1(T \otimes_{\mathbb{B}} \text{LS}_{(B,N)((A,M))}) \to \\
\to & H_0(\mathbb{Z}[M], \mathbb{Z}[N], T) & \to H_0(A, B, T) \oplus H_0(T \otimes_{\mathbb{Z}} D) \to H_0(T \otimes_{\mathbb{B}} \text{LS}_{(B,N)((A,M))}) \to 0
\end{array}
\]

where

- \( H_2(D \otimes_{\mathbb{Z}} W) = \text{Tor}_2^D(T, \ker(M^{sp} \to N^{sp})) \),
- \( H_1(D \otimes_{\mathbb{Z}} W) \) appears in an exact sequence

\[ 0 \to T \otimes_{\mathbb{Z}} \ker(M^{sp} \to N^{sp}) \to H_1(D \otimes_{\mathbb{Z}} W) \to \text{Tor}_1^D(T, \ker(M^{sp} \to N^{sp})) \to 0, \]

- \( H_0(D \otimes_{\mathbb{Z}} W) = T \otimes_{\mathbb{Z}} \text{coker}(M^{sp} \to N^{sp}) \).
Let \((A, M) \xrightarrow{\varphi} (F, R) \xrightarrow{\psi} (B, N)\) be a factorization in the category of simplicial prelog rings with \(\varphi\) a free cofibration and \(\psi\) a trivial fibration as in \([4\text{, Section 4}]\). We have a commutative diagram of complexes

\[
\begin{array}{c}
T \otimes_{\mathbb{Z}[R]} \Omega_{\mathbb{Z}[R][\mathbb{Z}]M} \\
\downarrow \\
T \otimes_{\mathbb{Z}[N]} \text{LS}_{\mathbb{Z}[N][\mathbb{Z}]M} \\
\downarrow \\
T \otimes_F \Omega_{F|A} \\
\downarrow \\
T \otimes_B \text{LS}_{B|A} \\
\end{array}
\begin{array}{c}
\rightarrow T \otimes_{\mathbb{Z}} R^{sp}/M^{sp} \\
\rightarrow T \otimes_{\mathbb{Z}} D \\
\rightarrow T \otimes_B L_{(B, N)|(A, M)} \\
\end{array}
\]

inducing a morphism from the exact sequence of \([4\text{, Theorem 4.3}]\) into the above exact sequence, and giving isomorphisms in two of each three terms (by \([2\text{, Proposition 15.12}]\) the ones corresponding to the oblique maps, and by direct inspection the one associated to the oblique upper right map, since we can take as \(P_{0}^{gp}, Q, P_{1}^{gp}\) of the proof of \([4\text{, Theorem 4.3}]\)). Therefore, the remaining homomorphisms

\[
H_i(T \otimes_B \text{LS}_{(B, N)|(A, M)}) \longrightarrow H_i(T \otimes_B \text{LS}_{(B, N)|(A, M)}), \quad \text{for } i = 0, 1, 2,
\]

are also isomorphisms. \(\square\)

2. A spectral sequence

Let \(C \to B\) be a surjective homomorphism of rings. We have a convergent spectral sequence \([11\text{, Theorem 6.8}]\)

\[
E^2_{p, q} = H_{p+q}(S^q \Lambda^q_{B|C}) \Rightarrow \text{Tor}^C_{p+q}(B, B),
\]

which when \(A \to B\) is a flat homomorphism and \(C = B \otimes_A B\) takes the form

\[
E^2_{p, q} = H_p(\Lambda^q_{B|A}) \Rightarrow \text{HH}^p_{p+q}(B|A)
\]

where \(\text{HH}_k(B|A) = \text{Tor}^{B^{gp} \otimes_A B}_{p+q}(B, B)\) is the Hochschild homology of the \(A\)-algebra \(B\) and \(S^q, \Lambda^q\) denote symmetric and exterior powers respectively. We will give here an analogous spectral sequence for the logarithmic cotangent complex.

Let \((C, Q) \to (B, N)\) be a homomorphism of prelog rings such that \(C \to B\) and \(Q \to N\) are surjective. Let \((C, Q) \to (R, P) \to (B, N)\) be a free cofibration - trivial fibration factorization as in \([4\text{, Section 4}]\) with \(R_0 = C\) and \(P_0 = Q\). Let \(I = \ker(\pi : R \otimes_C B \to B), J = \ker(\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[N] \to \mathbb{Z}[N])\) and \(T = \ker((P \oplus_Q N)^{sp} \to N^{sp})\).

Let $D^1_{(B,N)\|C,Q)}$ be the simplicial $R \otimes_C B$-module defined by the pushout

$$
\begin{array}{ccc}
U & \xrightarrow{\beta} & B \otimes \mathbb{Z} T \\
\downarrow{\alpha} & & \downarrow \\
I & \xrightarrow{} & D^1_{(B,N)\|C,Q)}
\end{array}
$$

where $U = (R \otimes_C B) \otimes \mathbb{Z}[P] \otimes \mathbb{Z}[Q] \otimes \mathbb{Z}[N] J$, $\beta$ is the homomorphism induced by the map $J \to B \otimes \mathbb{Z} T$ of the proof of \textit{[4, Proposition 3.1]} and the map $\pi: R \otimes_C B \to B$, and the homomorphism $\alpha$ is the obvious one.

**Proposition 2.1.** In this situation, we have an exact sequence

$$
\begin{array}{cccccccc}
\cdots & \xrightarrow{} & H_n(U) & \xrightarrow{} & \text{Tor}_{C}^n(B,B) & \oplus & W_n & \xrightarrow{} & H_n(D^1_{(B,N)\|C,Q)}) & \xrightarrow{} & \\
& & H_{n-1}(U) & \xrightarrow{} & \cdots & & H_1(D^1_{(B,N)\|C,Q)}) & \xrightarrow{} & 0
\end{array}
$$

where

$$W_n = \begin{cases} 
\ker(Q^g \to N^g) \otimes \mathbb{Z} B & \text{if } n = 1, \\
\text{Tor}_{1}^n(\ker(Q^g \to N^g), B) & \text{if } n = 2, \\
0 & \text{if } 1 \neq n \neq 2.
\end{cases}$$

**Proof.** If $P = Q \oplus N^Y$, $R = C[X \cup Y]$, then $J$ is the ideal of $\mathbb{Z}[P] \otimes \mathbb{Z}[Q] \otimes \mathbb{Z}[N] = \mathbb{Z}[N][Y]$ generated by $Y$, $I$ is the ideal of $R \otimes_C B = B[X \cup Y]$ generated by $X \cup Y$, and the map $\alpha$ in the diagram is (split) injective. Thus, we have an exact sequence

$$
\cdots \to H_n(U) \to H_n(I) \oplus H_n(B \otimes \mathbb{Z} T) \to H_n(D^1_{(B,N)\|C,Q)}) \to H_{n-1}(U) \to \cdots.
$$

Since $0 \to I \to R \otimes_C B \to B \to 0$ is exact and $H_n(R \otimes_C B) = \text{Tor}_{n}^C(B,B)$, we have $H_n(I) = \text{Tor}_{n}^C(B,B)$ if $n > 0$ and $H_0(I) = 0$ (in fact, $I_0 = 0$). We also have $J_0 = 0$ and then $H_0(U) = 0$.

It remains to compute $H_n(B \otimes \mathbb{Z} T)$. Since

$$P^g \oplus Q^g \cap N^g = P^g / \ker(Q^g \to N^g),$$

we have an exact sequence

$$0 \to T \to P^g / \ker(Q^g \to N^g) \to N^g \to 0 \tag{2.1}$$

From the exact sequence

$$0 \to \ker(Q^g \to N^g) \to P^g \to P^g / \ker(Q^g \to N^g) \to 0$$

we deduce $H_n(P^g / \ker(Q^g \to N^g)) = 0$ for $n \geq 2$ and an exact sequence

$$0 \to H_1(P^g / \ker(Q^g \to N^g)) \to$$

$$\ker(Q^g \to N^g) \to N^g \to H_0(P^g / \ker(Q^g \to N^g)) \to 0$$

showing that

$$H_1(P^g / \ker(Q^g \to N^g)) = \ker(Q^g \to N^g),$$

$$H_0(P^g / \ker(Q^g \to N^g)) = N^g.$$

Then, from the exact sequence (2.1) we obtain $H_n(T) = 0$ for all $n \geq 2$, $H_1(T) = \ker(Q^g \to N^g)$ and an exact sequence

$$0 \to H_0(T) \to N^g \xrightarrow{\cong} N^g \to 0.$$
In particular, $E_\infty = 0$.

By the universal coefficient theorem, we deduce

$$H_n(B \otimes Z T) = \begin{cases} H_1(T) \otimes Z B = \ker(Q^{sp} \to N^{sp}) \otimes Z B & \text{if } n = 1, \\ \Tor^p_1(\ker(Q^{sp} \to N^{sp}), B) & \text{if } n = 2, \\ 0 & \text{if } 1 \neq n \neq 2. \end{cases}$$

Example 2.2. We are going to see that $H_1(D_{(B,N)}([C,Q]))$ is the conormal module $N_{(B,N)}([C,Q])$ defined in [3] Section 3.

Let $Z[P] \otimes_{Z[Q]} Z[N] \to X \to R \otimes C B$ be a cofibration - trivial fibration factorization, and consider the derived tensor product

$$\tilde{U} = X \otimes_{Z[P] \otimes_{Z[Q]} Z[N]} J = (R \otimes C B) \otimes_{Z[P] \otimes_{Z[Q]} Z[N]} J.$$

We have an exact sequence defining $Z$

$$0 \to Z \to \tilde{U} \to U \to 0$$

and $\tilde{U}_0 = 0 = U_0$ since $J_0 = 0$. Therefore $Z_0 = 0$ and then $H_1(\tilde{U}) \to H_1(U)$ is surjective. So from Proposition 2.1 we have an exact sequence

$$H_1(\tilde{U}) \to \ker(\mathcal{C} \to B) \to H_1(D_{(B,N)}([C,Q])) \to 0$$

where $\mathcal{C} = \ker(C \to B)$.

Let us compute $H_1(\tilde{U})$. By [9, II, Theorem 6] we have a spectral sequence

$$E_{p,q}^2 = \Tor^\mathcal{C}_p(\Tor^{Z[Q]}_Z(Z[N], Z[N]), (\Tor^\mathcal{C}_* (B, B), H_* (J))_q \Rightarrow H_{p+q}(\tilde{U}).$$

Moreover, from the Tor long exact sequence associated to the exact sequence

$$0 \to H_* (J) \to \Tor^\mathcal{C}_*(Z[N], Z[N]) \to Z[N] \to 0,$$

we obtain

$$E_{p,q}^2 = \Tor^\mathcal{C}_p(\Tor^{Z[Q]}_Z(Z[N], Z[N]), (\Tor^\mathcal{C}_* (B, B), Z[N])_q$$

for $p > 0$ and an exact sequence

$$0 \to \Tor^\mathcal{C}_1(\Tor^{Z[Q]}_Z(Z[N], Z[N]), (\Tor^\mathcal{C}_* (B, B), Z[N])_q \to E_{0,q}^2 \to$$

$$\to \Tor^\mathcal{C}_q (B, B) \to (\Tor^\mathcal{C}_* (B, B) \otimes_{\Tor^{Z[Q]}_Z(Z[N], Z[N])} Z[N])_q \to 0.$$

When $p > 0$, we have then

$$E_{p,0}^2 = \Tor^\mathcal{C}_p (B, B) \otimes_{\Tor^{Z[Q]}_Z(Z[N], Z[N])} Z[N]$$

$$= \Tor^{Z[N]}_p(B, Z[N])$$

$$= 0$$

In particular, $E_{1,0}^2 = 0$ and $E_{2,0}^2 = 0$. We also have $E_{2,2}^2 = 0$, since the spectral sequence is located in the first quadrant. Therefore $E_{0,1}^3 = E_{0,1}^2$, and since $E_{0,1}^\infty =
homomorphism of prelog rings we have

Example 2.3. \( B \otimes_{\mathbb{Z}[N]} b/b^2 \rightarrow a/a^2 \oplus (\ker(Q^{sp} \rightarrow N^{sp}) \otimes_{\mathbb{Z}} B) \rightarrow H_1(D_{(B,N)}^{1}(C,Q)) \rightarrow 0 \)

where the first map is the sum of the canonical map \( B \otimes_{\mathbb{Z}[N]} b/b^2 \rightarrow a/a^2 \) and the map \( B \otimes_{\mathbb{Z}[N]} b/b^2 \rightarrow \ker(Q^{sp} \rightarrow N^{sp}) \otimes_{\mathbb{Z}} B \) of [4] Proposition 3.1. By [4] Definition 3.4, we have then

\[ H_1(D_{(B,N)}^{1}(C,Q)) = N_{(B,N)}(C,Q). \]

Note that in the case \( C = B \otimes_A B, Q = N \oplus_M N \) where \( (A, M) \rightarrow (B, N) \) is a homomorphism of prelog rings we have

\[ H_1(D_{(B,N)}^{1}(C,Q)) = \Omega_{(B,N)}(A,M). \]

Example 2.3. If \( Q = N \), then \( J = T = 0 \) and so \( D_{(B,N)}^{1}(C,Q) \) is the pushout

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
I & \rightarrow & D_{(B,N)}^{1}(C,Q)
\end{array}
\]

showing that \( H_n(D_{(B,N)}^{1}(C,Q)) = \text{Tor}_n^C(B,B) \) for \( n > 0 \). In the case where \( (A, M) \rightarrow (B, N) \) is a homomorphism of prelog rings, with \( A \rightarrow B \) flat and \( M = N \), taking \( (C, Q) = (B \otimes_A B, N \oplus_M N) \), we have \( H_n(D_{(B,N)}^{1}(C,Q)) = \text{HH}_n(B|A) \) for \( n > 0 \).

Theorem 2.4. We have a first quadrant convergent spectral sequence

\[ E_2^{p,q} = H_{p+q}(D_{(B,N)}^{1}(C,Q)) \]

where \( E_{0,q}^2 = 0 \) for \( q = 0 \) and \( E_{p,1}^2 = H_{p+1}(L_{(B,N)}(C,Q)). \)

Proof. Consider the pushout

\[
\begin{array}{ccc}
(R \otimes_C B) \otimes_{\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[N]} & \rightarrow & B \otimes_{\mathbb{Z}} T \\
\downarrow & & \downarrow \\
I & \rightarrow & D_{(B,N)}^{1}(C,Q)
\end{array}
\]

We have already seen that \( \alpha \) is split injective, and similarly, the map

\[ (R \otimes_C B) \otimes_{\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[N]} \rightarrow I \]
is split injective for all $i > 0$.

For any $i \geq 2$, let $D^i$ be the image of the composition $I^i \hookrightarrow I \rightarrow D^i_{(B,N)|(C,Q)}$.

Applying the Ker-Coker Lemma to the following diagram for $i \geq 2$,

$$
\begin{array}{ccccccccc}
0 & \rightarrow & (R \otimes C B) \otimes \mathbb{Z}[P] \otimes \mathbb{Z}[Q] \mathbb{Z}[N] & J^i & \xrightarrow{\epsilon} & I^i & \xrightarrow{\alpha} & 0 \\
& & \downarrow & & \downarrow & & \\
0 & \rightarrow & (R \otimes C B) \otimes \mathbb{Z}[P] \otimes \mathbb{Z}[Q] \mathbb{Z}[N] & J & \xrightarrow{\beta} & I \oplus (B \otimes Z T) & \xrightarrow{\beta} & 0 \\
& & \downarrow & & \downarrow & & \\
& & I^i/(R \otimes C B) \otimes \mathbb{Z}[P] \otimes \mathbb{Z}[Q] \mathbb{Z}[N] & J^i & \rightarrow & 0 & (2.2) & \\
& & \downarrow & & \downarrow & & \\
& & \beta & \rightarrow & D^1_{(B,N)|(C,Q)} & \rightarrow & 0 & \\
\end{array}
$$

(where the vertical morphism to the left is the obvious one, the one in the middle is the inclusion $I^i \hookrightarrow I$ and the zero morphism on $B \otimes Z T$, and the square to the left is commutative since $i \geq 2$ and the homomorphism $J^2 \rightarrow B \otimes Z T$ is zero) we obtain an exact sequence

$$(R \otimes C B) \otimes \mathbb{Z}[P] \otimes \mathbb{Z}[Q] \mathbb{Z}[N] J/J^i \rightarrow I/I^i \oplus (B \otimes Z T) \rightarrow D^1_{(B,N)|(C,Q)}/D^i \rightarrow 0,$$

which we can write as a pushout:

$$
\begin{array}{cccccc}
(R \otimes C B) \otimes \mathbb{Z}[P] \otimes \mathbb{Z}[Q] \mathbb{Z}[N] J/J^i & \xrightarrow{\alpha} & B \otimes Z T \\
\downarrow & & \downarrow \\
I/I^i & \xrightarrow{\beta} & D^1_{(B,N)|(C,Q)}/D^i \\
\end{array}
$$

(2.3)

From the commutative diagram (2.2) we deduce that $\ker (I^i \rightarrow D^1_{(B,N)|(C,Q)}) = \ker(\beta \epsilon) = I^i \cap ((R \otimes C B) \otimes \mathbb{Z}[P] \otimes \mathbb{Z}[Q] \mathbb{Z}[N] J)$, identifying $(R \otimes C B) \otimes \mathbb{Z}[P] \otimes \mathbb{Z}[Q] \mathbb{Z}[N] J$ with its image in $I$ via $\alpha$, obtaining an exact sequence by definition of $D^i$

$$
0 \rightarrow I^i \cap ((R \otimes C B) \otimes \mathbb{Z}[P] \otimes \mathbb{Z}[Q] \mathbb{Z}[N] J) \rightarrow I^i \rightarrow D^i \rightarrow 0.
$$

(2.4)

Applying the Ker-Coker Lemma to the diagram of exact sequences for $1 < j < i$,

$$
\begin{array}{cccccc}
0 & \rightarrow & I^i \cap ((R \otimes C B) \otimes \mathbb{Z}[P] \otimes \mathbb{Z}[Q] \mathbb{Z}[N] J) & \xrightarrow{\alpha} & I^i & \xrightarrow{\beta} & D^i & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & I^i \cap ((R \otimes C B) \otimes \mathbb{Z}[P] \otimes \mathbb{Z}[Q] \mathbb{Z}[N] J) & \xrightarrow{\beta} & I^j & \xrightarrow{\beta} & D^j & \rightarrow & 0 \\
\end{array}
$$

we obtain an exact sequence

$$
0 \rightarrow I^i \cap ((R \otimes C B) \otimes \mathbb{Z}[P] \otimes \mathbb{Z}[Q] \mathbb{Z}[N] J)/I^i \cap ((R \otimes C B) \otimes \mathbb{Z}[P] \otimes \mathbb{Z}[Q] \mathbb{Z}[N] J) \rightarrow I^i/I^j \rightarrow D^j/D^i \rightarrow 0
$$
From the exact sequence \([2.3]\) we obtain
\[
D^i = I^i / (I^i \cap ((R \otimes C B) \otimes \mathbb{Z}[P] \otimes \mathbb{Z}[J] \otimes J)) = I^i + ((R \otimes C B) \otimes \mathbb{Z}[P] \otimes \mathbb{Z}[J] \otimes J)
\]

which is the ideal \(\tilde{I}^i\) of \((R \otimes C B) / ((R \otimes C B) \otimes \mathbb{Z}[P] \otimes \mathbb{Z}[J] \otimes J) = B[X]\) where \(\tilde{I}\) is the ideal of the simplicial ring \(B[X]\) generated by the variables \(X\). Therefore the hypotheses of \([10, \text{Theorem 8.8}]\) are verified (since \(C \to B\) and \(P \to N\) are surjective) and we obtain \(H_n(D^i) = H_n(\tilde{I}^i) = 0\) for all \(i > n\).

Putting \(D^1 := D^1_{(B,N)((C,Q))}\), we obtain a convergent spectral sequence \([3, \text{XV, Proposition 4.1}]\)

\[
E^2_{p,q} = H_{p+q}(D^q / D^{q+1}) \Rightarrow H_{p+q}(D^1).
\]

It remains to show that \(D^1 / D^2 = L_{(B,N)((C,Q))}.\) If we consider \(R \otimes C R\) as a \(R\)-module via \(r(a \otimes b) = a \otimes br\), we have a split exact sequence of \(R\)-modules (defining \(L\))

\[
0 \to I_\omega \to R \otimes C R \to R \to 0.
\]

Applying \(- \otimes R B\), we obtain an split exact sequence

\[
0 \to I_\omega \otimes R B \to R \otimes C B \to B \to 0
\]

which proves that \(I_\omega \otimes R B = I\).

Similarly, considering the ideal \(J_\omega = \ker(\mathbb{Z}[P] \otimes \mathbb{Z}[Q] \to \mathbb{Z}[P])\), we obtain \(J_\omega \otimes \mathbb{Z}[P] \mathbb{Z}[N] = J\). Therefore

\[
I^i / I^{i+1} = (I_\omega \otimes R B)^i / (I_\omega \otimes R B)^{i+1} = I_\omega \otimes R B / I_\omega^{i+1} \otimes R B = I_\omega / I_\omega^{i+1} \otimes R B
\]

and similarly

\[
J^i / J^{i+1} = J_\omega / J_\omega^{i+1} \otimes \mathbb{Z}[P] \mathbb{Z}[N].
\]

Then the pushout \([2.3]\) for \(i = 2\) takes the form

\[
\begin{array}{ccc}
(R \otimes C B) \otimes \mathbb{Z}[P] \otimes \mathbb{Z}[Q] \otimes \mathbb{Z}[N] & \xrightarrow{\gamma} & B \otimes \mathbb{Z} T \\
\downarrow \lambda & & \downarrow \\
B \otimes R I_\omega / I_\omega^2 & \to & D^1_{(B,N)((C,Q))} / D^2
\end{array}
\]

Since the homomorphisms \(\lambda\) and \(\gamma\) can be factorized by the surjective homomorphism

\[
(R \otimes C B) \otimes \mathbb{Z}[P] \otimes \mathbb{Z}[Q] \otimes \mathbb{Z}[N] \to B \otimes \mathbb{Z}[P] \otimes \mathbb{Z}[Q] \otimes \mathbb{Z}[N] \to B \otimes \mathbb{Z}[P] \otimes \mathbb{Z}[Q] \otimes \mathbb{Z}[N]
\]

that is obtained by applying \(- \otimes \mathbb{Z}[P] \otimes \mathbb{Z}[Q] \otimes \mathbb{Z}[N]\) to the surjective morphism \(R \otimes C B \to B\), we obtain a pushout

\[
\begin{array}{ccc}
B \otimes \mathbb{Z}[P] \otimes \mathbb{Z}[Q] \otimes \mathbb{Z}[N] & \to & B \otimes \mathbb{Z} T \\
\downarrow & & \downarrow \\
B \otimes R I_\omega / I_\omega^2 & \to & D^1_{(B,N)((C,Q))} / D^2
\end{array}
\]
and additionally, since

\[ B \otimes_{\mathbb{Z}[P] \otimes \mathbb{Z}[Q]} \mathbb{Z}[N] (J_\omega / J^2_\omega \otimes_{\mathbb{Z}[P]} \mathbb{Z}[N]) = B \otimes_{\mathbb{Z}[N]} \mathbb{Z}[N] \otimes_{\mathbb{Z}[P] \otimes \mathbb{Z}[Q]} \mathbb{Z}[P] J_\omega / J^2_\omega \]

\[ = B \otimes_{\mathbb{Z}[N]} J_\omega / J^2_\omega \]

(because the \( \mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P] \)-module \( J_\omega / J^2_\omega \) is already a \( \mathbb{Z}[N] \)-module), we obtain a pushout

\[
\begin{array}{c}
B \otimes_{\mathbb{Z}[N]} (J_\omega / J^2_\omega) \\
\downarrow \\
B \otimes_R I_\omega / I^2_\omega \end{array} \longrightarrow \begin{array}{c}
\rightarrow B \otimes_{\mathbb{Z}} T \\
\rightarrow \end{array} \begin{array}{c}
\rightarrow D^1_{(B,N)((C,Q))}/D^2 \\
\end{array}
\]

Finally, we deduce \( T = \ker(P^{\operatorname{sp}}/\ker(Q^{\operatorname{sp}} \rightarrow N^{\operatorname{sp}}) \rightarrow N^{\operatorname{sp}}) = P^{\operatorname{sp}}/Q^{\operatorname{sp}} \) given that \( P^{\operatorname{sp}}_0 = Q^{\operatorname{sp}} \). Thus, the previous pushout gives us \( \mathbb{L}_{(B,N)((C,Q))} \) by definition. □

REFERENCES

[1] André, M. Méthode simpliciale en algèbre homologique et algèbre commutative. Lecture Notes in Mathematics, Vol. 32 Springer-Verlag, Berlin-New York, 1967.

[2] André, M. Homologie des Algèbres Commutatives. Grundlehren Math. Wiss., 206. Springer-Verlag, Berlin-New York, 1974.

[3] Cartan, H.; Eilenberg, S. Homological algebra. Princeton University Press, Princeton, N. J., 1956.

[4] Conde-Lago, J.; Majadas, J. Homological characterization of regularity in logarithmic algebraic geometry. J. Algebr. Geom. 31 (2022) 205-260.

[5] Grothendieck, A.; Dieudonné, J. Éléments de Géométrie Algébrique, Chapitre IV, Première Partie. Inst. Hautes Études Sci. Publ. Math. 20 (1964).

[6] Lichtenbaum, S.; Schlessinger, M. The cotangent complex of a morphism. Trans. Amer. Math. Soc. 128 (1967), 41-70.

[7] Majadas, J.; Rodicio, A.G. Smoothness, Regularity and Complete Intersection. London Math. Soc. Lect. Note Ser., 373. Cambridge Univ. Press, 2010.

[8] Olsson, M. The logarithmic cotangent complex. Math. Ann. 333 (2005), no. 4, 859–931.

[9] Quillen, D. Homotopical Algebra. Lecture Notes in Mathematics, Vol. 43. Springer-Verlag, Berlin-New York, 1967.

[10] Quillen, D. Homology of commutative rings. Mimeographed notes, MIT 1967.

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