Critical traveling wave solution in the Meinhardt type model

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Abstract. The traveling waves problem for the singularly perturbed semilinear parabolic equations is considered in the paper. It is shown that the corresponding problem for a singularly perturbed ODE system can be reduced to a certain problem of lower dimension using a splitting transformation based on the technique of slow and fast integral manifolds. The Meinhardt model is considered as an example.

1. Introduction

Travelling waves play a fundamental role in many mathematical equations. We can write a number models of interest in the general form

$$\frac{\partial u}{\partial t} = \varepsilon D\frac{\partial^2 u}{\partial x^2} + F(u),$$  \hspace{1cm} (1)

$u \in R^n, \; x \in R, \; \geq 0, \; \varepsilon$ is a small positive parameter. Here, $D$ is the constant diagonal matrix $D = diag(D_1, D_2, ..., D_n)$, $F(u)$ is the sufficiently smooth and bounded vector-function.

Equations of these types are widely used as mathematical models of biology, chemistry and physics, and many examples in phenomena resembling travelling waves have been found empirically.

A travelling wave is a solution which is of the form $u(x; t) = u(\xi)$ with $\xi = x - st$ for some wave speed $s \in R$ and which satisfies the following ODEs:

$$-su' = \varepsilon Du'' + P(u),$$ \hspace{1cm} (2)

where $(\cdot)'$ refers to differentiation with respect to $\xi$.

Our main goal is the reduction of (2) to the equation of lower order using the splitting transformation [4, 5].

2. Splitting transformation

Consider the differential system that is linear with respect to fast variables

$$\dot{x} = f(x, t, \varepsilon) + F(x, t, \varepsilon)y,$$  \hspace{1cm} (3)

$$\varepsilon \dot{y} = g(x, t, \varepsilon) + G(x, t, \varepsilon)y,$$ \hspace{1cm} (4)
where \( x \in \mathbb{R}^m, y \in \mathbb{R}^n, t \in \mathbb{R} \). We assume that the eigenvalues \( \lambda_i(x,t) \) of the matrix \( G(x,t,0) \) have the property \( \text{Re}\lambda_i(x,t) \leq -2\gamma < 0 \), in \( t \in \mathbb{R}, x \in \mathbb{R}^m \), and that the matrix- and vector-functions \( f, g, F \) and \( G \) are continuous and bounded as well as their partial derivatives with respect to the arguments \( t \in \mathbb{R}, x \in \mathbb{R}^m, \varepsilon \in [0, \varepsilon_0] \).

When these assumptions hold, the system (3)-(4) has a slow integral manifold

\[
y = h(x,t,\varepsilon) = h_0(x,t) + \varepsilon h_1(x,t) + \ldots.
\]

Noting that

\[
\frac{dy}{dt} = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}(f + Fh),
\]

on using the first of (3)-(4)), the functions \( h_i \) can be derived from the invariance equation

\[
\varepsilon \frac{\partial h}{\partial t} + \varepsilon \frac{\partial h}{\partial x}(f + Fh) = g + Gh.
\]

Suppose we have the following representations

\[
F(x,t,\varepsilon) = \sum_{j \geq 0} \varepsilon^j F_j(x,t), \quad G(x,t,\varepsilon) = \sum_{j \geq 0} \varepsilon^j G_j(x,t),
\]

\[
f(x,t,\varepsilon) = \sum_{j \geq 0} \varepsilon^j f_j(x,t), \quad g(x,t,\varepsilon) = \sum_{j \geq 0} \varepsilon^j \xi j(x,t).
\]

Here, \( G_0 = G_0(x,t) \) plays the role of the matrix \( B(x,t) \). The formulae for the coefficients of the asymptotic expansions of slow integral manifold \( h = h(x,t,\varepsilon) \) take the form

\[
h_0 = G_0^{-1}\xi_0,
\]

\[
h_k = G_0^{1-1}(\varepsilon) F_k \frac{\partial h_{k-1}}{\partial t} + \sum_{p=0}^{k-1} \frac{\partial h_p}{\partial x} (f_{k-1-p} + \sum_{j=0}^{k-1-p} F_j h_{k-p-1-j})
\]

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The invariance equation for the fast integral manifold \( H = H(v,z,t,\varepsilon) \) \cite{4,5} takes the form

\[
\varepsilon \frac{\partial H}{\partial t} + \varepsilon \frac{\partial H}{\partial v} [f(v,t,\varepsilon) + F(v,t,\varepsilon)h(v,t,\varepsilon)] + \frac{\partial H}{\partial z} [G(v + \varepsilon H, t, \varepsilon)]
\]

\[
- \varepsilon \frac{\partial h}{\partial x} (v + \varepsilon H, t, \varepsilon) F(v + \varepsilon H, t, \varepsilon) z = f(v + \varepsilon H, t, \varepsilon) - f(v,t,\varepsilon)
\]

\[
+ F(v + \varepsilon H, t, \varepsilon) (z + h(v + \varepsilon H, t, \varepsilon)) - F(v,t,\varepsilon) h(v,t,\varepsilon).
\]

Setting \( \varepsilon = 0 \), we obtain

\[
\frac{\partial H_0}{\partial z} G_0(v,t) z = F_0(v,t) z.
\]

It is possible to represent \( H_0(v,t,z) \) in the form \( H_0(v,t,z) = D_0(v,t) z \), where the matrix \( D_0(v,t) \) satisfies the equation

\[
D_0(v,t) G_0(v,t) = F_0(v,t),
\]

and, therefore,

\[
H_0(v,z,t) = F_0(v,t) G_0^{-1}(v,t) z.
\]
Our main goal is the construction of the transformation

\[ x = v + \varepsilon H(v, z, t, \varepsilon), \]

\[ y = z + h(x, t, \varepsilon), \]  

which reduces the original system (3), (4) to the form such that

\[ v \]

There exists a solution \((v(t), \varepsilon(t))\) of (8), (9) with the initial condition \(v(t_0) = v_0, z(t_0) = z_0\), such that

\[ x(t) = v(t) + \varepsilon H(v(t), z(t), t, \varepsilon), \quad y(t) = z(t) + h(x(t), t, \varepsilon). \]  

(10)

It is sufficient to show that (10) holds for \(t = t_0\). Setting \(t = t_0\) in (10) we obtain

\[ x_0 = v_0 + \varepsilon H(v_0, z_0, t_0, \varepsilon), \quad y_0 = z_0 + h(x_0, t_0, \varepsilon), \]

and, therefore, \(z_0 = y_0 - h(x_0, t_0, \varepsilon)\).

For \(v_0\) we have the equation

\[ v_0 = x_0 - H(v_0, z_0, t_0, \varepsilon) = V(v_0), \]  

(11)

which has a unique solution for any \(x_0 \in \mathbb{R}^n\) and fixed \(z_0\) and \(t_0\), where

\[ \|z_0\| = \|y_0 - h(x_0, t_0, \varepsilon)\| \leq \rho_1 \]

for some \(\rho_1\).

The following statement is true. Any solution \(x = x(t, \varepsilon), y = y(t, \varepsilon)\) of system (3), (4) with the initial condition \(x(t_0, \varepsilon) = x_0, y(t_0, \varepsilon) = y_0\), can be represented in form of (10).

This statement means that (3), (4) can be reduced to the form (8), (9) by the splitting transformation (6), (7). Thus, system (3), (4) was split into two subsystems, the first of which is independent and contains a small parameter in a regular manner. Note that the initial value \(v_0\) can be calculated from (11) in the form of an asymptotic expansion:

\[ v_0 = v_{00} + \varepsilon v_{01} + \varepsilon^2 v_{02} + \ldots. \]

For example, \(v_{00} = z_0, v_{01} = -H(x_0, z_{00}, t_0, 0)\), where \(z_{00} = y_0 - h(x_0, t_0)\).

It is important to underline that there exists number \(K, K > 1\) such that

\[ \|z(t, \varepsilon)\| \leq K \exp(-\gamma t/\varepsilon)\|z_0\|, \quad t \geq 0. \]

This means: the solution \(x = x(t, \varepsilon), y = y(t, \varepsilon)\) of the original system (3) - (4) that satisfied the initial condition \(x(0, \varepsilon) = x_0, y(t_0, \varepsilon) = y_0\) can be represented as

\[ x(t, \varepsilon) = v(t, \varepsilon) + \varepsilon \varphi_1(t, \varepsilon), \]

\[ y(t, \varepsilon) = \bar{y}(t, \varepsilon) + \varphi_2(t, \varepsilon). \]

Thus, this solution is represented as a sum of solution which lies on the slow integral manifold, i.e.

\[ x = x(t, \varepsilon) = v(t, \varepsilon), \quad y(t, \varepsilon) = h(v(t, \varepsilon), t, \varepsilon), \]
and exponentially decreasing functions

\[\varepsilon \varphi_1(t, \varepsilon) = \varepsilon H(v(t, \varepsilon), z(t, \varepsilon), t, \varepsilon),\]

\[\varphi_2(t, \varepsilon) = z(t, \varepsilon) + h(v(t, \varepsilon)) + \varepsilon H(v(t, \varepsilon), z(t, \varepsilon), t, \varepsilon) - h(v(t, \varepsilon), t, \varepsilon).\]

Neglecting terms of order \(o(\varepsilon)\), we use the transformation

\[x = v + \varepsilon H_0(v, z, t),\]
\[y = z + h_0(x, t) + \varepsilon h_1(x, t)\]

to reduce system (3)-(4) to a nonlinear block-triangular form:

\[\dot{v} = f_0(v, t) + F_0(v, t) + \varepsilon [f_1(v, t) + F_1(v, t) h_0(v, t)] + O(\varepsilon^2),\]

\[\varepsilon \dot{z} = [G_0(v, t) + \varepsilon (G_1(v, t) + \frac{\partial G_0}{\partial x}(v, t) H_0(v, z, t)) - \frac{\partial h_0}{\partial x}(v, t) F_0(v, t)] z + O(\varepsilon^2).\]

3. Travelling wave problem decomposition

Suppose that the speed of the travelling wave is a value of order unity, i.e. \(s = O(1)\) as \(\varepsilon \to \infty\).

It is possible to rewrite (2) of form (3)-(4) with

\[f = 0,\]
\[F = I,\]
\[g = -D^{-1}P,\]
\[G = -sD^{-1}.\]

These formulae and (5) imply

\[\dot{v} = h_0(v) + \varepsilon h_1(v) + O(\varepsilon^2),\]

where

\[h_0 = G^{-1}g = \frac{1}{s}P(v)\]

and

\[h_1 = G^{-1} \frac{\partial h_0}{\partial v}(v, t) h_0(v).\]

Thus, if we find the periodic solution of (12) then we have the periodic travelling wave to the original equation (1). The situation is similar in the case of a heteroclinic or homoclinic trajectory of the equation (12).

4. Meinhardt type model

Setting \(u = (u_1, u_2)\) consider two equations

\[\frac{\partial u_1}{\partial t} = \varepsilon D_1 \frac{\partial^2 u_1}{\partial x^2} + F_1(u),\]
\[\frac{\partial u_2}{\partial t} = \varepsilon D_2 \frac{\partial^2 u_2}{\partial x^2} + F_2(u)\]

with

\[F_1 = \mu(u_2^2 - u_1),\]
\[F_2 = a - bu_2 + \frac{u_2^2}{u_1(1 + ku_2^2)}.\]

If \(0 < \varepsilon \ll \mu \ll 1\) then the corresponding planar ODE system on a slow invariant manifold possesses a canard solution/ This means that in the model under consideration there exists the critical travelling wave solution.
5. Acknowledgments
The author is supported by the Russian Foundation for Basic Research (project 20-51-53008) and the Ministry of Education and Science of the Russian Federation under the Competitiveness Enhancement Program of Samara University (2013-2020).

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