Intertwining Operators for the Central Extension of the Yangian Double

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Abstract

We continue the investigation of the central extended Yangian double $K$. In this paper we study the intertwining operators for certain infinite dimensional representations of $\hat{DY}(\mathfrak{sl}_2)$, which are deformed analogs of the highest weight representations of the affine algebra $\hat{\mathfrak{sl}}_2$ at level 1. We give bosonized expressions for intertwining operators, verify that they generate an algebra isomorphic to Zamolodchikov–Faddeev algebra for the $SU(2)$-invariant Thirring model. We compose from them $L$-operators by Miki’s prescription and verify that they coincide with $L$-operators constructed from universal $R$-matrix.

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1 Introduction

The Yangian $Y(g)$ was introduced by V.G. Drinfeld $[D1]$. As a Hopf algebra it is a deformation of the universal enveloping algebra $U(g[u])$ of $g$-valued polynomial currents where $g$ is a simple Lie algebra. Recently, it was understood that in physical applications one needs the double of the Yangian $[S2]$. In $[K]$ the double of the Yangian was described quite explicitly in terms of Drinfeld’s construction of a quantum double $[D3]$. After adding to the Yangian a derivation operator, the same quantum double construction produces a central extension of the Yangian double $[K]$. The latter algebra has a richer representation theory than just the Yangian double has. For example, infinite dimensional representations which are deformed analogs of the highest weight representations of the affine algebra can be constructed. This type of representations appears to be very important in understanding the structure of conformal field theories associated with affine Lie algebras and the structure of the quantum integrable models associated with deformation of the universal enveloping affine algebras. In the paper $[DFJMN]$ and then in the book $[JM]$ the representation theory of the infinite dimensional representations of $U_q(\mathfrak{sl}_2)$ has been used in order to solve completely the quantum spin 1/2 lattice XXZ model. A complete solution of the quantum model means the possibility to write down explicitly an analytical integral representation for arbitrary form-factor of any local operator in the model.

Some years before an approach to relativistic integrable massive field theories have been developed in the works by F.A. Smirnov $[S1]$. This approach did not use representation theory of the hidden non-abelian symmetry of the model which ensures its integrability, but the ideas of phenomenological bootstrap. For some quantum integrable models such as Sin-Gordon, SU(2)-invariant Thirring, etc. models the bootstrap program have led to a complete solution. Understanding the fact that the dynamical non-abelian symmetry algebra for SU(2)-invariant Thirring model is the Yangian double (see $[S2]$ and reference therein) opens a possibility to describe the structure of the model (local operators, Zamolodchikov–Faddeev operators which create particles, etc.) in terms of the representation theory of the Yangian double. It appears that it should be central extended Yangian double.

On the other hand, a free field approach to massive integrable field theories has been developed in $[L]$. The starting point there has been the Zamolodchikov–Faddeev (ZF) algebra $[Z2]$ for the operators which create and annihilate particles in the model and describe the local operators. The author of this paper was able to construct a free field representation for ZF operators using an ultraviolet regularization of the Fock modules. Our approach shows that there exists possibility to construct a free field representation of the ZF operators in massive integrable field theories where only a minor regularization in the definition of these operators will be necessary (see $[K3]$–$[K11]$).

The paper is organized as follows. After the definition of $\widehat{DY}(\mathfrak{sl}_2)$ in section 2, an infinite dimensional representations of this algebra are constructed in terms of free field in section 3. The next two sections are devoted to the free field representation of the intertwining operators and to their commutation and normalization relations. In section 6 an universal $R$-matrix description of the central extended Yangian double and Ding–Frenkel isomorphism $[DF]$ between the “new” $[D2]$ and “RLL” formulations $[FRT, RS]$ of $\widehat{DY}(\mathfrak{sl}_2)$ is given $[K1, K]$. In the last section we consider a free field construction of the $L$-operators corresponding to the central extension of the Yangian double at level 1 (Miki’s formulas $[M]$) and verify that they coincide with $L$-operators constructed from universal $R$-matrix. Some explicit calculations are gathered in Appendix.

2 Central Extension of $\widehat{DY}(\mathfrak{sl}_2)$

The Yangian $Y(\mathfrak{sl}_2)$ is a Hopf algebra generated by the elements $e_k, f_k, h_k$, $k \geq 0$, subject to the relations

\[
[h_k, h_l] = 0, \quad [e_k, f_l] = h_{k+l},
\]

\[
[h_0, e_l] = 2e_l, \quad [h_0, f_l] = -2f_l,
\]

\[
[h_{k+1}, e_l] - [h_k, e_{l+1}] = h\{h_k, e_l\},
\]

\[
[h_{k+1}, f_l] - [h_k, f_{l+1}] = -h\{h_k, f_l\},
\]

\[
[e_{k+1}, e_l] - [e_k, e_{l+1}] = h\{e_k, e_l\},
\]

\[
[e_{k+1}, f_l] - [e_k, f_{l+1}] = h\{e_k, f_l\},
\]

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\[ [f_{k+1}, f_l] - [f_k, f_{l+1}] = -\hbar \{f_k, f_l\}, \quad (2.1) \]

where \( \hbar \) is a deformation parameter and \( \{a, b\} = ab + ba \). The coalgebra structure is uniquely defined by the relations

\[
\begin{align*}
\Delta(e_0) &= e_0 \otimes 1 + 1 \otimes e_0, & \Delta(h_0) &= h_0 \otimes 1 + 1 \otimes h_0, & \Delta(f_0) &= f_0 \otimes 1 + 1 \otimes f_0, \\
\Delta(e_1) &= e_1 \otimes 1 + 1 \otimes e_1 + h h_0 \otimes e_0, & \Delta(f_1) &= f_1 \otimes 1 + 1 \otimes f_1 + h f_0 \otimes h_0, \\
\Delta(h_1) &= h_1 \otimes 1 + 1 \otimes h_1 + h h_0 \otimes h_0 - 2 f_0 \otimes e_0.
\end{align*}
\]

In the form (2.1) the Yangian \( Y(sl_2) \) appeared in \([D]\) in so called “new” realization which is a deformed analog of the loop realization of the affine algebras. In this paper we study representations and intertwining operators for the central extension \( \hat{DY}(sl_2) \) of the quantum double of \( Y(sl_2) \). The Hopf algebra \( \hat{DY}(sl_2) \) was introduced in \([R]\). It can be described as a formal completion of algebra with generators \( d \), central element \( c \) and \( e_k, f_k, h_k, k \in \mathbb{Z} \), gathered into generating functions

\[
e^\pm(u) = \pm \sum_{k \geq 0} e_k u^{-k-1}, \quad f^\pm(u) = \pm \sum_{k \geq 0} f_k u^{-k-1}, \quad h^\pm(u) = 1 \pm \hbar \sum_{k \geq 0} h_k u^{-k-1}, \quad (2.2)
\]

with the relations

\[
[d, e(u)] = \frac{d}{du} e(u), \quad [d, f(u)] = \frac{d}{du} f(u), \quad [d, h^\pm(u)] = \frac{d}{du} h^\pm(u),
\]

\[
\begin{align*}
e(u)e(v) &= \frac{u - v + \hbar}{u - v - \hbar} e(v)e(u) \\
f(u)f(v) &= \frac{u - v - \hbar}{u - v + \hbar} f(v)f(u) \\
h^+(u)e(v) &= \frac{u - v + \hbar}{u - v - \hbar} e(v)h^+(u) \\
h^+(u)f(v) &= \frac{u - v - \hbar - hc}{u - v + \hbar - hc} f(v)h^+(u) \\
h^-(u)f(v) &= \frac{u - v - \hbar}{u - v + \hbar} f(v)h^-(u) \\
h^+(u)h^-(v) &= \frac{u - v + \hbar}{u - v - \hbar} \left( \frac{u - v - \hbar - hc}{u - v + \hbar - hc} h^-(v)h^+(u) \right) \\
[e(u), f(v)] &= \frac{1}{\hbar} \left( \delta(u - (v + \hbar c))h^+(u) - \delta(u - v)h^-(v) \right) \quad (2.3)
\end{align*}
\]

where

\[
\delta(u - v) = \sum_{n+m=-1} u^n v^m, \quad \delta(u - v)g(u) = \delta(u - v)g(v).
\]

This algebra admits a filtration

\[
\ldots \subset C_{-n} \subset \ldots \subset C_{-1} \subset C_0 \subset C_1 \ldots \subset C_n \ldots \subset C \quad (2.4)
\]

defined by the conditions \( \deg e_k = \deg f_k = \deg h_k = k; \deg \{x \in C_m\} \leq m \). Then \( \hat{DY}(sl_2) \) is a formal completion of \( C \) with respect to filtration (2.4).

The comultiplication in \( \hat{DY}(sl_2) \) is given by the relations

\[
\Delta(h^\pm(u)) = \sum_{k=0}^{\infty} (-1)^k (k+1) \hbar^{2k} \left( f^\pm(u + \hbar - \delta_{\pm, \hbar c_1}) \right)^k h^\pm(u) \otimes h^\pm(u - \delta_{\pm, \hbar c_1}) (e^\pm(u + \hbar - \delta_{\pm, \hbar c_1}))^k,
\]
Concrete calculations are exhibited in $[K]$. 

The quantum double $\hat{D}^3$ together with (2.5) defines the Hopf algebra $\langle c,d \rangle$. Finally, the central extension $U_\kappa$ has the following description.

One can note first that $\hat{D}^\pm$ do not contain the unit. Then the central extension $\hat{U}_\kappa$ is again a Hopf algebra if we put $\hat{c} \rightarrow \hat{c} + \hat{h}$, $\hat{h} \rightarrow \hat{h} - \hat{c}$ as well as $h \rightarrow h + \hat{c}$, $\hat{h} \rightarrow \hat{h} - \hat{c}$ in the “new” realization. Concrete calculations are exhibited in $[K]$.

Before passing to the next section where we will give a free field realization of the commutation relations (2.3) at the level $c = 1$ we would like to add two remarks.

One can note first that the Yangian $\hat{Y}^-$ is again a Hopf algebra if we put

$$\Delta(d) = d \otimes 1 + 1 \otimes d.$$
where $b$ is an arbitrary complex number, will move the commutation relations (2.3) into more symmetric form

$$
\tilde{h}^+(u)\tilde{h}^-(v) = \frac{u - v + \tilde{h} + hc/2}{u - v - \tilde{h} + hc/2} \frac{u - v - \tilde{h} - hc/2}{u - v + \tilde{h} - hc/2} \tilde{h}^-(v)\tilde{h}^+(u),
$$

$$
\tilde{h}^\pm(u)\tilde{\epsilon}(v) = \frac{u - v + \tilde{h} \pm hc/4}{u - v - \tilde{h} \pm hc/4} \tilde{\epsilon}(v)\tilde{h}^\pm(u),
$$

$$
\tilde{h}^\pm(u)\tilde{\epsilon}_0(v) = \frac{u - v - \tilde{h} \pm hc/4}{u - v + \tilde{h} \pm hc/4} \tilde{\epsilon}_0(v)\tilde{h}^\pm(u),
$$

$$
\tilde{\epsilon}(u)\tilde{f}(v) - \tilde{f}(v)\tilde{\epsilon}(u) = \frac{1}{\tilde{h}} \left( \delta \left( u - v - \frac{\tilde{h}}{2} \right) \tilde{h}^+ \left( u - \frac{\tilde{h}}{4} \right) \right)

- \delta \left( u - v + \frac{\tilde{h}}{2} \right) \tilde{h}^- \left( v - \frac{\tilde{h}}{4} \right)
$$

For quantum affine algebras such a renormalization have been initiated by the condition of consistency with Cartan involution. There is no such motivation in our case so we do not use this replacement any more.

### 3 Bosonization Formulas for Level 1

Let $\mathcal{H}$ be the Heisenberg algebra generated by free bosons with zero modes $a_{\pm n}$, $n = 1, 2, \ldots$, $a_0$, $p$ with commutation relations

$$
[a_n, a_m] = n\delta_{n+m,0}, \quad [p, a_0] = 2.
$$

Let $V_i, i = 0, 1$ be the formal power series extensions of the Fock spaces

$$
V_i = \mathbb{C}[[a_{-1}, \ldots, a_{-n}, \ldots]] \otimes (\otimes_{n \in \mathbb{Z}^1} \mathbb{C} e^{n a_0}) \quad (3.1)
$$

with the action of bosons on these spaces

$$
a_n = \text{the left multiplication by } a_n \otimes 1 \quad \text{for } n < 0,
$$

$$
a_n = [a_n, \cdot] \otimes 1 \quad \text{for } n > 0,
$$

$$
e^{n_1 a_0} (a_{-j_k} \cdots a_{-j_1} \otimes e^{n_2 a_0}) = a_{-j_k} \cdots a_{-j_1} \otimes e^{(n_1 + n_2) a_0},
$$

$$
e^{n} (a_{-j_k} \cdots a_{-j_1} \otimes e^{n a_0}) = u^{2n} a_{-j_k} \cdots a_{-j_1} \otimes e^{n a_0}.
$$

The $\text{End}V_i$-valued generating functions (fields) $\mathbf{K}$

$$
e(u) = \exp \left( \sum_{n=1}^{\infty} \frac{a_n}{n} [(u - \tilde{h})^n + u^n] \right) e^{p a_0} u^p \exp \left( -\sum_{n=1}^{\infty} \frac{a_n}{n} u^{-n} \right),
$$

$$
f(u) = \exp \left( -\sum_{n=1}^{\infty} \frac{a_n}{n} [(u + \tilde{h})^n + u^n] \right) e^{-p a_0} u^{-p} \exp \left( \sum_{n=1}^{\infty} \frac{a_n}{n} u^{-n} \right),
$$

$$
\tilde{h}^-(u) = \exp \left( \sum_{n=1}^{\infty} \frac{a_n}{n} [(u - \tilde{h})^n - (u + \tilde{h})^n] \right),
$$

$$
\tilde{h}^+(u) = \exp \left( \sum_{n=1}^{\infty} \frac{a_n}{n} [(u - \tilde{h})^{-n} - u^{-n}] \right) (u - \tilde{h})^{-p} \quad (3.2)
$$

satisfy commutation relations (2.3) with $c = 1$. 

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In terms of generating functions

\[ a_+(z) = \sum_{n \geq 1} \frac{a_n}{n} z^{-n} - p \log z, \quad a_-(z) = \sum_{n \geq 1} \frac{a_n}{n} z^n + \frac{a_0}{2}, \]  

(3.3)

\[ a(z) = a_+(z) - a_-(z), \quad \phi_\pm(z) = \exp a_\pm(z), \]  

(3.4)

\[ [a_+(z), a_-(y)] = -\log(z-y), \quad |y| < |z|, \]  

(3.5)

a free filed realization (3.2) looks as follows:

\[ e(u) = \phi_-(u-h)\phi_-(u)\phi_+^{-1}(u), \quad f(u) = \phi_+^{-1}(u+h)\phi_-^{-1}(u)\phi_+(u), \]  

\[ h^+(u) = \phi_+(u-h)\phi_+^{-1}(u), \quad h^-(u) = \phi_-(u-h)\phi_-^{-1}(u+h), \]  

(3.6)

\[ e^{\gamma d}\phi_\pm(u) = \phi_\pm(u+\gamma)e^{\gamma d}, \quad e^{\gamma d}(1 \otimes 1) = 1 \otimes 1. \]

One can note that in the classical limit (\( \hbar \to 0 \)) the bosonization formulas goes to the Frenkel-Kac (homogeneous) realization of the affine algebra \( \hat{sl}_2 \) at level 1 [FK]

\[ e(u) = \exp \left( \sum_{n=1}^{\infty} \frac{2a_n}{n} u^n \right) e^{a_0} u^p \exp \left( -\sum_{n=1}^{\infty} \frac{a_n}{n} u^{-n} \right), \]

\[ f(u) = \exp \left( -\sum_{n=1}^{\infty} \frac{2a_n}{n} u^n \right) e^{-a_0} u^{-p} \exp \left( \sum_{n=1}^{\infty} \frac{a_n}{n} u^{-n} \right), \]

\[ \kappa(u) = \kappa^+(u) - \kappa^-(u) = \sum_{n=1}^{\infty} 2a_n u^{n-1} + \sum_{n=1}^{\infty} a_n u^{-n-1} + pu^{-1}. \]  

(3.7)

Let us discuss now the structure of the representation spaces \( V_i \), \( i = 0, 1 \). In the case of affine algebra \( \hat{sl}_2 \) or quantum (trigonometric) deformation of the affine algebra \( U_q(\hat{sl}_2) \) there is a finite number of Chevalley generators which generate all the infinite dimensional algebra by means of the multiple commutators. It appears also that Chevalley generators coincide with certain components of the currents that form the loop (the new) realization of affine (quantum affine) algebra. This gives a combinatorial description of the representations \( V_i \), \( i = 0, 1 \) with highest weight vectors \( u_0 = 1 \otimes 1 \), \( u_1 = 1 \otimes e^{a_0/2} \) in terms of successive finite-dimensional modules of the two embedded copies of \( U_q(\hat{sl}_2) \).

In the case of \( DY(\hat{sl}_2) \) the second copy of \( sl_2 \) is missing, and, on the other hand, the action of the components of the currents \( e_k \) or \( f_k \) onto any element of the representation spaces \( V_i \) produces infinite sum of the elements of the same sort. For example,

\[ e_{-1} 1 \otimes 1 = \exp \left( \sum_{n=1}^{\infty} \frac{a_n h^n}{n} \right) \otimes e^{a_0}. \]

It means that in case of the central extension of the Yangian double we have no analogous combinatorial description of the representation spaces \( V_0 \) and \( V_1 \). Nevertheless these spaces are properly defined together with an action of \( DY(\hat{sl}_2) \) on them.

One can show that there exists an outer automorphism \( \nu \) of \( DY(\hat{sl}_2) \) such that \( V_i \) is isomorphic to \( V_1^{\nu \cdot 1} \), \( i = 0, 1 \).

To construct intertwining operators we need also the finite-dimensional evaluation modules for \( DY(\hat{sl}_2) \). The following formulas can be found in [KT]

\[ h^+(u)v_{\pm, z} = \frac{u - z \pm h}{u - z} v_{\pm, z}, \quad |z| < |u|, \]

\[ h^-(u)v_{\pm, z} = \frac{u - z \pm h}{u - z} v_{\pm, z}, \quad |z| > |u|, \]
\[ e^\pm(u)v_{+,z} = 0, \quad f^\pm(u)v_{-,z} = 0, \]
\[ e^+(u)v_{-,z} = \frac{1}{u-z}v_{+,z}, \quad f^+(u)v_{+,z} = \frac{1}{u-z}v_{-,z}, \quad |z| < |u|, \]
\[ e^-(u)v_{-,z} = \frac{1}{u-z}v_{+,z}, \quad f^-(u)v_{+,z} = \frac{1}{u-z}v_{-,z}, \quad |z| > |u|, \]

where \( v_{\pm, z} \) denotes the elements of the evaluation module \( V_z = \{ V \otimes \mathbb{C}[z, z^{-1}] \} \) and \( V = \{ v_+, v_- \} \).

### 4 Intertwining Operators

Let us define the following intertwining operators

\[ \Phi^{(i)}(z) : V_i \rightarrow V_{1-i} \otimes V_z, \quad \Phi^{*(i)}(z) : V_i \otimes V_z \rightarrow V_{1-i}, \tag{4.1} \]
\[ \Psi^{(i)}(z) : V_i \rightarrow V_z \otimes V_{1-i}, \quad \Psi^{*(i)}(z) : V_z \otimes V_i \rightarrow V_{1-i} \tag{4.2} \]

which commute with the action of the Yangian double

\[ \Phi^{(i)}(z)x = \Delta(x)\Phi^{(i)}(z), \quad \Phi^{*(i)}(z)\Delta(x) = x\Phi^{*(i)}(z), \]
\[ \Psi^{(i)}(z)x = \Delta(x)\Psi^{(i)}(z), \quad \Psi^{*(i)}(z)\Delta(x) = x\Psi^{*(i)}(z), \quad \forall x \in D\hat{\mathfrak{y}(sl_2)}. \tag{4.3} \]

The components of the intertwining operators are defined as follows

\[ \Phi^{(i)}(z)v = \Phi^{(i)}_+(z)v \otimes v_+ + \Phi^{(i)}_-(z)v \otimes v_-, \quad \Phi^{*(i)}(z)(v \otimes v_\pm) = \Phi^{*(i)}_\pm(z)v, \]
\[ \Psi^{(i)}(z)v = v_+ \otimes \Psi^{(i)}_+(z)v + v_- \otimes \Psi^{(i)}_-(z)v, \quad \Psi^{*(i)}(z)(v_\pm \otimes v) = \Psi^{*(i)}_\pm(z)v, \]

where \( v \in V_z \).

According to the terminology proposed in [JM] we call the intertwining operators \((4.1)\) of type I operators and the operators \((4.2)\) type II ones. Let us also fix the normalization of the intertwining operators which yields the dependence of these operators on the index \( i \)

\[ \Phi^{(i)}(z)u_i = (-z)^{-i/2}u_{1-i} \otimes v_{\varepsilon_i} + \cdots, \]
\[ \Psi^{*(i)}(z)(v_{\varepsilon_{1-i}} \otimes u_i) = (-z)^{-i/2}u_{1-i} + \cdots, \quad \varepsilon_0 = -, \quad \varepsilon_1 = +, \tag{4.4} \]

where we denote
\[ u_0 = 1 \otimes 1 \quad \text{and} \quad u_1 = 1 \otimes e^{\mathfrak{a}_0/2} \]

and dots in \((4.4)\) mean the terms containing positive powers of \( \hbar \). The normalization \((4.4)\) is not standardootnote{In contrast to the normalization of the intertwining operators for the quantum affine algebra \( U_q(\widehat{\mathfrak{g}(sl_2)}) \), the r.h.s. of \((4.4)\) contains both positive and negative powers of the spectral parameter. The terms with negative powers of the spectral parameter disappear in the classical limit \( \hbar \rightarrow 0 \).} but it allows us to write down precise expressions for

\[ \Phi_{\varepsilon} = \Phi^{(0)}_{\varepsilon} \oplus \Phi^{(1)}_{\varepsilon} : V_0 \oplus V_1 \rightarrow V_1 \oplus V_0 \quad \text{and} \quad \Psi_{\varepsilon} = \Psi^{(0)}_{\varepsilon} \oplus \Psi^{(1)}_{\varepsilon} : V_0 \oplus V_1 \rightarrow V_1 \oplus V_0 \]

without dependence on the index \( i \).

Let \( \eta_{\varepsilon}(z) \) be the following \text{End} \( V_i \)-valued function

\[ \eta_{\varepsilon}(z) = \lim_{K \rightarrow \infty} (2\hbar K)^{-p/2} \prod_{k=0}^K \frac{\phi_{\varepsilon}(z - 2k\hbar)}{\phi_{\varepsilon}(z - \hbar - 2k\hbar)}. \tag{4.5} \]

We have the following
Proposition 1. Intertwining operators (4.1) have the free field realization:

\[
\Phi_-(z) = \phi_-(z + h)\eta_+^{-1}(z), \quad (4.6)
\]

\[
\Phi_+(z) = \Phi_-(z)f_0 - f_0\Phi_-(z) = -h \int_C \frac{du}{2i\pi(u - z)(u - z - h)} :\Phi_-(z)f(u):, \quad (4.7)
\]

\[
\Phi_\varepsilon^{(i)}(z) = \varepsilon(-1)^i\Phi_\varepsilon(z - h), \quad (4.8)
\]

\[
\Psi_+(z) = \phi_-^{-1}(z)\eta_+(z), \quad (4.9)
\]

\[
\Psi_\varepsilon^{(i)}(z) = \varepsilon(-1)^{-i}\Psi_\varepsilon^{(i)}(z - h), \quad \varepsilon = \pm, \quad (4.10)
\]

where contours \(C\) and \(\tilde{C}\) are such that the points \(z + h, 0 (z, 0)\) should be inside the contour \(C (\tilde{C})\) while the point \(z, \infty (z + h, \infty)\) should be outside \(C (\tilde{C})\) respectively.

To prove this proposition we need the following partial knowledge of the coalgebra structure of the central extended Yangian double \(\widetilde{DY}(\mathfrak{sl}_2)\) (see also (2.5))

\[
\Delta h^+(u) = h^+(u) \otimes h^+(u - hc_1) + F \otimes E, \\
\Delta h^-(u) = h^-(u) \otimes h^-(u) + F \otimes E, \\
\Delta e(u) = e(u) \otimes 1 + F \otimes E, \\
\Delta f(u) = 1 \otimes f(u) + F \otimes E', \quad (4.12)
\]

and the action of \(\widetilde{DY}(\mathfrak{sl}_2)\) onto evaluation module (3.8). In (4.12) \(E, F, E'\) and \(F'\) are generated by the elements \(\{e_k, f_k, h_{\pm}^\pm, \{f_k, h_{\pm}^\pm\}\) respectively.

Taking into account (4.12) we can obtain from (4.13) for \(x = h^+(u), h^-(u), e(u)\) and \(f_0\) the following equations for \(\Phi_\varepsilon^{(i)}(z)\)

\[
\Phi_\varepsilon^{(i)}(z)h^+(u) = \frac{u - z - 2h}{u - z - h}h^+(u)\Phi_\varepsilon^{(i)}(z), \quad (4.13)
\]

\[
\Phi_\varepsilon^{(i)}(z)h^-(u) = \frac{u - z - h}{u - z}h^-(u)\Phi_\varepsilon^{(i)}(z), \quad (4.14)
\]

\[
\Phi_\varepsilon^{(i)}(z)e(u) = e(u)\Phi_\varepsilon^{(i)}(z), \quad (4.15)
\]

\[
\Phi_\varepsilon^{(i)}(z) = \Phi_\varepsilon^{(i)}(z)f_0 - f_0\Phi_\varepsilon^{(i)}(z). \quad (4.16)
\]

Analogous formulas for the dual operators \(\Phi^\varepsilon(z)\) are

\[
\Phi^\varepsilon^{(i)}(z)h^+(u) = \frac{u - z - h}{u - z}h^+(u)\Phi^\varepsilon^{(i)}(z), \quad (4.17)
\]

\[
\Phi^\varepsilon^{(i)}(z)h^-(u) = \frac{u - z}{u - z + h}h^-(u)\Phi^\varepsilon^{(i)}(z), \quad (4.18)
\]

\[
\Phi^\varepsilon^{(i)}(z)e(u) = e(u)\Phi^\varepsilon^{(i)}(z), \quad (4.19)
\]

\[
\Phi^\varepsilon^{(i)}(z) = f_0\Phi^\varepsilon^{(i)}(z) - \Phi^\varepsilon^{(i)}(z)f_0. \quad (4.20)
\]

Comparing the above formulas we obtain (4.8). With this choice of the \(z\)-independent factor \((-1)^i\) in (4.8) we will have the same normalization condition for the dual intertwining operators as for the operator \(\Psi_\varepsilon^{(i)}\)

\[
\Phi^\varepsilon^{(i)}(z)(u_i \otimes v_{\varepsilon_1}) = (-z)^{-i/2}u_{1-i} + \cdots, \quad \varepsilon_0 = -, \quad \varepsilon_1 = +. \quad (4.21)
\]

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It is clear that solution to the equations (4.13)–(4.15) can be found in terms of exponential functions of bosons. 

Equation (4.13) can be satisfied by the exponent

$$\exp \left( \sum_{n=1}^{\infty} \frac{a-n}{n} (z + h)^n \right) e^{\alpha_0/2}$$

while (4.14)–(4.15) have a solution (4.5) which can be rewritten using Stirling formula in the following form

$$\lim_{N \to \infty} (2h)^{p/2} \left( \frac{\Gamma \left( \frac{h + \frac{z}{2h}}{2h} \right)}{\Gamma \left( \frac{z}{2h} \right)} \right)^p \prod_{k=0}^{N} \exp \left( - \sum_{n=1}^{\infty} \frac{a_n}{n} \left[ (z - 2kh)^{-n} - (z - h - 2kh)^{-n} \right] \right).$$  \hspace{1cm} (4.17)

For example, in order to verify that operator (4.6) is a solution to the equations (4.15) one should substitute

$$\exp \left( \sum_{n=1}^{\infty} \frac{a-n}{n} (z + h)^n \right) e^{\alpha_0/2} \lim_{N \to \infty} (2h)^{p/2} \left( \frac{\Gamma \left( \frac{h + \frac{z}{2h}}{2h} \right)}{\Gamma \left( \frac{z}{2h} \right)} \right)^p \prod_{k=0}^{N} \exp \left( - \sum_{n=1}^{\infty} \frac{a_n}{n} \left[ (z + h + 2kh)^{-n} - (z + 2h + 2kh)^{-n} \right] \right),$$  \hspace{1cm} (4.20)

which produces different from (5.10) rule of normal ordering, but the same commutation relation (5.1). To fix a unique solution we should add to the defining equations (4.13)–(4.15) some more information. This will be a correspondence with analogous bosonization formulas for the intertwining operators of the highest weight modules for the quantum affine algebra [JM]. A limit to rational (Yangian) case from trigonometric (quantum affine algebra) one means the replacement

$$\zeta = q^{4(\frac{x}{4})}$$

and then sending $q^4 \to 1$. In this limit

$$\prod_{n=0}^{\infty} (1 - q^{4(x+n)}) \sim (1 - q^4)^{1-x} \Gamma^{-1}(x) \prod_{n=0}^{\infty} (1 - q^{4(1+n)}) \).$$

The normal ordering relation of type I intertwining operators for quantum affine algebra $U_q(\hat{\mathfrak{sl}}_2)$ produces a factor

$$\prod_{n=0}^{\infty} \frac{(1 - q^{2+4n}\zeta_1/\zeta_2)}{(1 - q^{2+4n}\zeta_1/\zeta_2)}$$

which goes to a necessary ratio of $\Gamma$-functions \hspace{1cm} \hspace{1cm} (5.10) up to some divergent factor in the limit $q^4 \to 1$. The choice of the solution (4.21) corresponds to the choice of the intertwining operators for the quantum affine algebra which have a “good” analytical properties in the limit $\eta \to \infty$.

Note also that in the classical limit $\hbar \to 0$, our intertwining operators coincide with those of the affine algebra $\mathfrak{sl}_2$ in homogeneous Frenkel-Kac realization because of the sort of the Stirling formula

$$\lim_{\hbar \to 0} \sum_{k=0}^{\infty} \left( \frac{1}{z - 2kh} - \frac{1}{z - h - 2kh} \right) = \frac{1}{2z}.$$
5 Commutation Relations for the Intertwining Operators

Let us formulate the following

**Proposition 2.** Type I and type II intertwining operators satisfy following commutation and normalization relations

\[ \Phi_{\varepsilon_2}(z_2)\Phi_{\varepsilon_1}(z_1) = R_{\varepsilon_1\varepsilon_2}(z_1 - z_2)\Phi_{\varepsilon_1}(z_1)\Phi_{\varepsilon_2}(z_2), \quad (5.1) \]

\[ \Psi^*_{\varepsilon_2}(z_1)\Psi^*_{\varepsilon_2}(z_2) = -R_{\varepsilon_1\varepsilon_2}(z_1 - z_2)\Psi^*_{\varepsilon_2}(z_2)\Psi^*_{\varepsilon_1}(z_1), \quad (5.2) \]

\[ \Phi_{\varepsilon_1}(z_1)\Psi^*_{\varepsilon_2}(z_2) = \tau(z_1 - z_2)\Psi^*_{\varepsilon_2}(z_2)\Phi_{\varepsilon_1}(z_1), \quad (5.3) \]

\[ g \sum_{\varepsilon} \Phi_{\varepsilon}^{(1-i)}(z)\Phi_{\varepsilon}^{(i)}(z) = \text{id}, \quad (5.4) \]

\[ g\Phi_{\varepsilon_1}^{(1-i)}(z)\Phi_{\varepsilon_2}^{(i)}(z) = \delta_{\varepsilon_1\varepsilon_2} \text{id}, \quad (5.5) \]

\[ g^{-1}\Psi_{\varepsilon_1}^{(1-i)}(z)\Psi_{\varepsilon_2}^{(i)}(z) = \frac{\delta_{\varepsilon_1\varepsilon_2}}{z_1 - z_2} + o(z_1 - z_2), \quad (5.6) \]

where the $R$-matrix is given by

\[ R(z) = r(z)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b(z) & c(z) & 0 \\ 0 & c(z) & b(z) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (5.7) \]

and

\[ r(z) = \frac{\Gamma\left(\frac{1}{2} - \frac{z}{2\hbar}\right)\Gamma\left(1 + \frac{z}{2\hbar}\right)}{\Gamma\left(\frac{1}{2} + \frac{z}{2\hbar}\right)\Gamma\left(1 - \frac{z}{2\hbar}\right)}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(z) & c(z) & 0 \\ 0 & c(z) & b(z) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (5.8) \]

\[ \tau(z) = -\cot\frac{\pi z}{2\hbar}, \quad b(z) = \frac{z}{z + \hbar}, \quad c(z) = \frac{\hbar}{z + \hbar}, \quad g = \sqrt{\frac{2\hbar}{\pi}}. \]

The commutation relations between $\Phi^{(i)}(z)$ and $\Psi^{(i)}(z)$ follows from (5.1)–(5.3) and the identifications (4.8) and (4.11).

One can easily check that $R$-matrix (5.7) satisfy the unitary and crossing symmetry conditions

\[ R(z)R(-z) = 1, \quad (C \otimes \text{id})R(z)(C \otimes \text{id}) = R^t(z - h) \]

with charge conjugation matrix

\[ C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (5.9) \]

and $R^t(z)$ means transposition with respect to the first space.

The proof of Proposition 2 is based on the normal ordering relations

\[ \Phi_-(z_2)\Phi_-(z_1) = (2\hbar)^{1/2} \frac{\Gamma\left(\frac{1}{2} + \frac{z_1 + z_2}{2\hbar}\right)}{\Gamma\left(\frac{1}{2} + \frac{z_1 - z_2}{2\hbar}\right)} \Phi_-(z_2)\Phi_-(z_1), \quad (5.10) \]

\[ \Psi^*_+(z_1)\Psi^*_+(z_2) = (2\hbar)^{1/2} \frac{\Gamma\left(\frac{1}{2} - \frac{z_1 + z_2}{2\hbar}\right)}{\Gamma\left(-\frac{z_1 - z_2}{2\hbar}\right)} \Psi^*_+(z_1)\Psi^*_+(z_2), \quad (5.11) \]

\[ \Psi^*_-(z_2)\Phi_-(z_1) = (2\hbar)^{-1/2} \frac{\Gamma\left(\frac{1}{2} + \frac{z_1 - z_2}{2\hbar}\right)}{\Gamma\left(1 + \frac{z_1 - z_2}{2\hbar}\right)} \Psi^*_+(z_2)\Phi_-(z_1), \quad (5.12) \]

\[ \Phi_-(z)f(u) = \frac{1}{u - z} :\Phi_-(z)\Phi_-(z) : = -\sum_{k=0}^{\infty} \frac{u^k}{z^{k+1}} :\Phi_-(z)\Phi_-(z) : , \]
The goal of this section is to describe the notion of the universal Yangian double have been obtained in [KT, K] by investigating the canonical pairing in question will be satisfied as formal series identities with respect to all spectral parameters instead of the operator $\Phi$ we have to insert there an operator $\Phi$, $\Psi$, $\Psi^*$ and $\Psi^*$ will be obtained in $\text{sl}_2$-matrix for the central extended Yangian double have been obtained in [KT, K] by investigating the canonical pairing in $\text{sl}_2$. This $\mathcal{R}$-matrix is

$$f(u)\Phi_-(z) = \frac{1}{u-z-\hbar} \Phi_-(z) f(u) = \sum_{k=0}^{\infty} \frac{(z+\hbar)^k}{u^{k+1}} \Phi_-(z) f(u); \quad (5.13)$$

$$\Psi^*(z) e(v) = \frac{1}{v-z-\hbar} \Psi^*(z) e(v) = -\sum_{k=0}^{\infty} \frac{\hbar^k}{(z+\hbar)^{k+1}} \Psi^*(z) e(v); \quad (5.14)$$

For example, to check three terms relations like

$$\Phi_+(z_2) \Phi_-(z_1) = r(z) [c(z) \Phi_+(z_1) \Phi_-(z_2) + b(z) \Phi_-(z_1) \Phi_+(z_2)]$$

and

$$\Phi_+(z_2) \Phi_+(z_1) = r(z) \Phi_+(z_1) \Phi_+(z_2) \quad (z = z_1 - z_2)$$

we have to insert there an operator $\Phi_+(z, u) = \Phi_-(z) f(u) - f(u) \Phi_-(z)$ instead of the operator $\Phi_+(z)$. Then using normal ordering relations (5.13) we can easily see that the relations in question will be satisfied as formal series identities with respect to all spectral parameters $u$, $z_1$ and $z_2$. Commutativity of type I and type II intertwining operators to the scalar factor $\tau(z_1 - z_2)$ (5.3) follows from the commutativities of the operators $\Phi^{(i)}(z)$ and $\Psi^{(i)}(z)$ with currents $e(u)$ and $f(v)$ respectively. The proof of the identities (5.4) and (5.5) is given in the Appendix.

6 Universal $\mathcal{R}$-Matrix Formulation of $\text{DY}(\text{sl}_2)$

The goal of this section is to describe $\text{DY}(\text{sl}_2)$ in “RLL” formalism [FR1, RS]. The essential part of this construction is the notion of the universal $\mathcal{R}$-matrix. The universal $\mathcal{R}$-matrix for the central extended Yangian double have been obtained in [KT, K] by investigating the canonical pairing in $\text{DY}(\text{sl}_2)$. This $\mathcal{R}$-matrix is

$$\mathcal{R} = \mathcal{R}_+ \cdot \mathcal{R}_0 \cdot \exp(hc \otimes d) \cdot \mathcal{R}_-; \quad (6.1)$$

where

$$\mathcal{R}_+ = \prod_{k \geq 0} \exp(-he_1 \otimes f_{-k-1}) = \exp(-he_0 \otimes f_{-1}) \exp(-he_1 \otimes f_{-2}) \cdots ,$$

$$\mathcal{R}_- = \prod_{k \geq 0} \exp(-hf_0 \otimes e_{-k-1}) = \cdots \exp(-hf_1 \otimes e_{-2}) \exp(-hf_0 \otimes e_{-1}),$$

$$\mathcal{R}_0 = \prod_{n \geq 0} \exp \left( \sum_{u=v}^{n} \frac{d}{du} \ln h^+(u) \otimes \ln h^-(v + \hbar + 2n\hbar) \right).$$

Here a residue operation $\text{Res}$ is defined as

$$\text{Res}_{u=v} \left( \sum_{i \geq 0} a_i u^{-i-1} \otimes \sum_{k \geq 0} b_k v^k \right) = \sum_{i \geq 0} a_i \otimes b_i.$$
In the evaluation representation $\pi(z)$ of $DY(\frak{sl}_2)$ in the space $V_z$ the operators $e_k, f_k, h_k, c, k \in \mathbb{Z}$ act as follows
\[ e_k v_{+,z} = f_k v_{-,z} = cv_{\pm,z} = 0, \quad e_k v_{-,z} = z^k v_{+,z}, \quad f_k v_{+,z} = z^{-k} v_{-,z}, \quad h_k v_{\pm,z} = \pm z^k v_{\pm,z}. \]

With this action the $L$-operators (see \[FR\] for $U_q(\frak{g})$ case)
\[ L^-(z) = (\pi(z) \otimes \text{id}) R \exp(-hc \otimes d), \quad L^+(z) = (\pi(z) \otimes \text{id}) \exp(hd \otimes c) (R^{21})^{-1} \quad (6.2) \]
appear in the Gauss decomposed form
\[ L^-(z) = \begin{pmatrix} 1 & \frac{hf^-(z)}{1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (k^-(z - \hbar))^{-1} & 0 \\ 0 & k^-(z) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \hbar e^-(z) & 1 \end{pmatrix}, \quad (6.3) \]
\[ L^+(z) = \begin{pmatrix} 1 & \frac{hf^+(z - ch)}{1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (k^+(z - \hbar))^{-1} & 0 \\ 0 & k^+(z) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \hbar e^+(z) & 1 \end{pmatrix}. \quad (6.4) \]

To obtain the diagonal part of the operators $L^+(z)$ the property of operation $\text{Res}$ have been used
\[ \text{Res}_{u=v} \left( \frac{d}{du} \sum_{i \geq 0} a_i u^{-i-1} \otimes \sum_{k \geq 0} b_k v^k \right) = - \text{Res}_{u=v} \left( \sum_{i \geq 0} a_i u^{-i-1} \otimes \frac{d}{dv} \sum_{k \geq 0} b_k v^k \right). \]

From formulas (5.3) and (6.4) the following representation of the fields $k^\pm(z)$ can be obtained
\[ k^-(z) = \prod_{n=0}^\infty \frac{h^-(z + 2n \hbar + 2n \hbar)}{h^-(z + 2n \hbar)}, \quad k^+(z) = \prod_{n=0}^\infty \frac{h^+(z - \hbar - 2n \hbar)}{h^+(z - \hbar)}. \quad (6.5) \]

To proceed further we have to calculate the universal $\mathcal{R}$-matrix on the tensor product of two evaluation representations ($z = z_1 - z_2$)
\[ \tilde{R}^-(z) = (\pi(z_1) \otimes \pi(z_2)) R \exp(-hc \otimes d) = (\text{id} \otimes \rho(z_2)) L^-(z_1) = \rho^-(z) \overline{R}(z) \]
\[ \tilde{R}^+(z) = (\pi(z_1) \otimes \pi(z_2)) \exp(hd \otimes c) (R^{21})^{-1} = (\text{id} \otimes \rho(z_2)) L^+(z_1) = \rho^+(z) \overline{R}(z), \]
where $\overline{R}(z)$ is given by (5.8) and
\[ \rho^\pm(z) = \left[ \frac{\Gamma\left(\mp \frac{z}{2\hbar}\right) \Gamma\left(1 \pm \frac{z}{2\hbar}\right)}{\Gamma^2\left(\frac{1}{2} \pm \frac{z}{2\hbar}\right)} \right]^{\mp1}. \]

For the $R$-matrices $\tilde{R}^\pm(z)$ the conditions of unitary and crossing symmetry look as follows
\[ R^+(z) R^-(z) = 1, \quad (C \otimes \text{id}) \tilde{R}^\pm (C \otimes \text{id})^{-1} = \tilde{R}^{\mp1} (-z - \hbar) \quad (6.6) \]
with charge conjugation matrix $C$ defined by (5.3).

The following properties of the universal $R$-matrix $\mathcal{R}$ and representations of $DY(\frak{sl}_2)$

(i) Yang-Baxter relation for $\mathcal{R}$,
(ii) comultiplication rules for $\mathcal{R}$
\[ (\Delta \otimes \text{id}) \mathcal{R} = \mathcal{R}^{13} \mathcal{R}^{23}, \quad (\text{id} \otimes \Delta) \mathcal{R} = \mathcal{R}^{13} \mathcal{R}^{12}, \]
(iii) realization of the trivial representation of $DY(\frak{sl}_2)$ as submodule
\[ \mathbb{C} (v_+ \otimes v_- - v_- \otimes v_+) \leftrightarrow \pi(z + h) \otimes \pi(z) \]
Here $q$-det $A(z)$ is equal to $A_{11}(z + \hbar)A_{22}(z) - A_{12}(z + \hbar)A_{21}(z)$ and comultiplication rules are

$$\Delta' L^+(z) = L^+(z - h\bar{c}_2)\otimes L^+(z), \quad \Delta l^{ij}_+(z) = \sum_k l^{ik}_+(z) \otimes l^{kj}_+(z - h\bar{c}_1),$$

$$\Delta' L^-(z) = L^-(z)\otimes L^-(z), \quad \Delta l^{ij}_-(z) = \sum_k l^{ik}_-(z) \otimes l^{kj}_-(z),$$

where $\Delta'$ means the flipped comultiplication and $\otimes$ is matrix tensor product. Using Gauss decomposition of $L$-operators we can easily obtain from (6.10) and (6.11) the full comultiplication formulas for the currents $e^\pm(u), f^\pm(u), h^\pm(u)$ (see (5.2)).

Conversely, we can start from an algebra of coefficients of the operators $L^\pm(z)$ subjected to (6.7)–(6.9) and prove that the currents

$$e(z) = e^+(z) - e^-(z),$$
$$f(z) = f^+(z) - f^-(z),$$
$$h^\pm(z) = [k^\pm(z), h^\pm(z - h)]^{-1}$$

defined by the Gauss decompositions (6.3), (6.4) satisfy the relations of $\mathfrak{DY}(\mathfrak{sl}_2)$. These calculations (Yangian analogs of Ding–Frenkel arguments [DF]) are presented in the Appendix.

7 Miki’s Formulas and Bosonization of $L$-operators

Following the approach of the paper [M] let us compose from the intertwining operators the following $2 \times 2$ operator $\text{End}V_1$-valued matrices

$$L^+_{zz}(z) = g \Phi_+(z - \hbar) \Psi^*_+(z), \quad L^-_{zz}(z) = g \Psi^*_+(z) \Phi_+(z).$$

One can check from commutation and orthogonality relations (5.1)–(5.3) that these operators satisfy the relations (6.7)–(6.9) with $c = 1$.

It is important to point out here that the order of intertwining operators in (7.1) is strictly fixed, because only in this order the normal ordering of these operators evaluated in the same points is well defined. It follows from the fusion rules

$$\Phi_-(z_1 - h) \Psi^*_-(z_2) = (2h)^{-1/2} \frac{\Gamma \left( \frac{1}{4} - \frac{z_1 - z_2}{2h} \right)}{\Gamma \left( 1 - \frac{z_1 - z_2}{2h} \right)} \Phi_-(z_1 - h) \Psi^*_-(z_2);$$

$$\Psi^*_+(z_1) \Phi_-(z_2) = (2h)^{-1/2} \frac{\Gamma \left( \frac{1}{4} - \frac{z_1 - z_2}{2h} \right)}{\Gamma \left( 1 - \frac{z_1 - z_2}{2h} \right)} \Psi^*_+(z_1) \Phi_-(z_2 - h).$$

Also it is necessary to say that checking (6.8) or (7.1) by means of (7.1) one will never interchange the intertwining operators evaluated in the same points which is an ill defined operation.

Our goal now is to show that $L$-operators (7.1) coincide with those defined by the universal $\mathcal{R}$-matrix (6.1) and evaluated at level one modules $V_0$ and $V_1$. In order to do this we express operators (7.1) in terms of free bosons using (4.6) and (4.9). First calculate operators $L^\pm_{zz}(z)$.

$$L^+_{zz}(z) = g \Phi_-(z - \hbar) \Psi^*_+(z) = \frac{\eta_+(z)}{\eta_+(z - \hbar)};$$
$$L^-_{zz}(z) = g \Psi^*_+(z) \Phi_-(z) = \frac{\phi_-(z + \hbar)}{\phi_-(z)}.$$
Comparing (7.2) with (5.5) and (3.6) we see that
\[ L^+_-(z) = k^+(z), \quad L^-_-(z) = k^-(z), \quad (c = 1). \]

To write down the bosonized expressions for the rest elements of the \( L \)-operators we first have to obtain more reasonable for our purposes representation of the \( +^m \) components of the intertwining operators \( \Phi(z) \) and \( \Psi^*(z) \). Using definition (4.3) for \( x = f^\pm(z), e^\pm(z) \), explicit formulas of comultiplication (2.5) and (3.8) we can obtain the following relations
\[
\frac{1}{z_2 - z_1} \Phi_\pm(z_1) = \Phi_\pm(z_1) f^\pm(z_2) - \frac{z_2 - z_1 - \hbar}{z_2 - z_1} f^\pm(z_2) \Phi_\pm(z_1), \quad |z_2| \gtrapprox |z_1| \tag{7.3}
\]
\[
\frac{1}{z_2 - z_1} \Psi^*_\pm(z_1) = e^\pm(z_2) \Psi^*\pm(z_1) - \frac{z_2 - z_1 - \hbar}{z_2 - z_1} \Psi^*\pm(z) e^\pm(z_2), \quad |z_2| \gtrapprox |z_1| \tag{7.4}
\]
\[
0 = \Phi_\pm(z_1) e^\pm(z_2) - e^\pm(z_2) \Phi_\pm(z_1), \tag{7.5}
\]
\[
0 = f^\pm(z_2) \Psi^*\pm(z_1) - \Psi^*\pm(z) f^\pm(z_2), \tag{7.6}
\]
where \( f^\pm(z_2) \) and \( e^\pm(z_2) \) defined by (5.2).

Let us comment the meaning of the formulas (7.3), for example. The formulas (7.4) can be treated analogously. Expanding first equation in (7.3) in positive powers of \( z_1/z_2 \) and second one in (7.3) in positive powers of \( z_2/z_1 \) we obtain in addition to (4.10) the relations
\[
\Phi_+(z_1) = [\Phi_-(z_1), f_m] z_1^{-m} + \hbar \sum_{k=0}^{m-1} z_1^{-k-1} f_k \Phi_-(z_1),
\]
\[
\Phi_+(z_1) = [\Phi_-(z_1), f_{-m}] z_1^{m} - \hbar \sum_{k=0}^{m-1} z_1^{k} f_{-k-1} \Phi_-(z_1), \quad m = 1, 2, \ldots
\]
These formulas define infinitely many relations between the operators \( \Phi_-(z_1) \) and \( f_m, m \in \mathbb{Z} \) which can be encoded in a single relation between the operator \( \Phi_-(z_1) \) and the total current \( f(u) \)
\[
(z_2 - z_1) \Phi_-(z_1) f(z_2) = (z_2 - z_1 - \hbar) f(z_2) \Phi_-(z_1). \tag{7.7}
\]
This relation is in agreement with normal ordering rules (6.13) and therefore the components of the intertwining operators (4.10) - (4.11) satisfy the intertwining relations for all the generators of the Yangian double \( D\hat{Y}(sl_2) \).

Using now the relations (7.3) - (7.6) and also (4.13), (4.14) after straightforward calculations one can obtain the rest elements of \( L \)-operators
\[
L^+_- (z) = g \Phi_-(z - \hbar) \Psi^*_+(z) = \hbar k^+(z)e^+(z),
\]
\[
L^+_- (z) = g \Phi_+(z - \hbar) \Psi^*_+(z) = \hbar f^+(z - \hbar) k^+(z),
\]
\[
L^+_- (z) = g \Psi^*_+(z) \Phi_-(z) = \hbar k^-(z)e^-(z),
\]
\[
L^+_- (z) = g \Psi^*_+(z) \Phi_+(z) = \hbar f^-(z) k^-(z),
\]
\[
L^+_- (z) = g \Phi_+(z - \hbar) \Psi^*_+(z) = (k^+(z - \hbar))^{-1} + \hbar^2 f^+(z - \hbar) k^+(z)e^+(z),
\]
\[
L^+_- (z) = g \Psi^*_+(z) \Phi_+(z) = (k^-(z - \hbar))^{-1} + \hbar^2 f^-(z) k^-(z)e^-(z),
\]
where operator identities (3.12) have been used. From these formulas follow that the Gauss decomposition of the \( L^\pm(z) \)-operators (7.1) coincides with those in (3.3) and (3.4) at the level \( c = 1 \).
Let us comment the using of the relations (7.3) and (7.4). First of all multiply them by $z_2 - z_1$ to obtain
\[ \Phi_\pm(z_1) = (z_2 - z_1)\Phi_\pm(z_1)f_\pm(z_2) - (z_2 - z_1 - \hbar)f_\pm(z_2)\Phi_\pm(z_1) , \]
\[ \equiv \Phi_\pm(z_1)f_0 - f_0\Phi_\pm(z_1) , \]
\[ \Psi_+^*(z_1) = (z_2 - z_1)e^\pm(z_2)\Psi_-(z_1) - (z_2 - z_1 - \hbar)\Psi_-(z_1)e^\pm(z_2) , \]
\[ \equiv e_0 \Psi_+^*(z_1) - \Psi_+^*(z_1)e_0 . \]
(7.8) (7.9) (7.10) (7.11)
The whole set of the identities (7.7) ensures the equations (7.9) and (7.11). Let us substitute operator $\Phi_+(z_1)$ in the form (7.8) into, for example, $L^{-}_-(z_1)$. We obtain
\[ L^{-}_-(z_1) = g\Psi_+^*(z_1)\Phi_+(z_1) \]
\[ = (z_2 - z_1)k^-(z_1)f^-(z_2) - (z_2 - z_1 - \hbar)f^-(z_2)k^-(z_1) = \hbar f^-(z_1)k^-(z_1) , \]
where identity (A.10) have been used.
We can justify above formulas for comultiplication by the Miki’s relations for the $L$-operators at level $1$. Indeed, using also (7.4), (5.1)–(5.3) and identities
\[ r(z)\tau(-z) = \rho^-(z) \quad \text{and} \quad r(z)\tau(-z - \hbar) = \rho^+(z) \]
we can calculate
\[ \Phi_{\varepsilon_2}(z_2)L^{-}_e\epsilon_2(\epsilon_1, \nu, \epsilon_2) = \tilde{R}^{-}_e\epsilon_2(\epsilon_1, \nu, \epsilon_2, \nu, \epsilon_2)\Phi_{\varepsilon_2}(z_2) , \]
\[ \equiv \Phi_{\varepsilon_2}(z_2)\tilde{L}^{-}_e\epsilon_2(\epsilon_1, \nu, \epsilon_2) = \tilde{R}^{+}_e\epsilon_2(\epsilon_1, \nu, \epsilon_2, \nu, \epsilon_2)\tilde{L}^{+}_e\epsilon_2(\epsilon_1, \nu, \epsilon_2) \]
(7.12) (7.13)
which obviously coincide with those formulas which can be obtained by the definition of the intertwining operators (4.3) and (6.11) and (6.11).

8 Conclusion

To conclude let us shortly repeat the main result obtained in this paper. Starting from the free field representation of the central extension of the Yangian double at level $c = 1$ the explicit formulas for the operators which intertwine these representations have been obtained. As well as in the case of quantum affine algebras these operators appear to be divided in two types. It was shown that type II intertwining operators satisfy the commutation relations of the Zamolodchikov-Faddeev algebra for $S$-matrix coincided with kink scattering in $SU(2)$-invariant Thirring model.

\[ S_{12}(\beta) = \frac{\Gamma \left( \frac{1}{2} + \frac{\beta}{\pi i} \right) \Gamma \left( -\frac{\beta}{\pi i} \right) \left[ \beta - \pi i P_{12} \right]}{\Gamma \left( \frac{1}{2} - \frac{\beta}{\pi i} \right) \Gamma \left( \frac{\beta}{\pi i} \right) \left[ -\beta - \pi i \right]}, \]
where the deformation parameter $\hbar$ of $DY(\mathfrak{sl}_2)$ should be chosen as $\hbar = -\pi i$.

We can state (as well as it was done in [IM] for lattice XXZ model) that the physical interesting quantities of $SU(2)$-invariant Thirring model, such as form-factors of the local operators, can be calculated as traces over infinite dimensional representation spaces of $DY(\mathfrak{sl}_2)$ of the weighted product of the types I and II intertwining operators. We address the calculation of these traces to the forthcoming paper [KL].

\[ ^2 \text{In the previous version of this paper we used incorrect formulas} \]
\[ \Phi_+(z) = hf_+(z)\Phi_-(z) = hf_-(z)f^-(z + \hbar) , \]
\[ \Psi_+^*(z) = hf_+^*(z)e^+(z) = he^+(z + \hbar)\Psi_+^*(z) , \]
from which one cannot obtain the correct formulas for elements of $L$-operators $L^\pm_{++}(z)$. 

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Note added: Before this paper was resubmitted to the electronic archive the authors found that the central extension of the Yangian double associated with algebra \( A_q \) and free field representation of this algebra at level one have been obtained independently by K. Iohara and H. Kohno [15] using the methods different from ones in [6]. We thank Prof. M. Jimbo for pointing out our attention to this publication.

Appendix

A.1 Orthogonality relations for intertwining operators

Let us prove (5.5) in the simplest case \( -\varepsilon_1 = \varepsilon_2 = + \). Because of the relation

\[
\Phi^{(1-i)}_+(z_2) \Phi^{* (i)}_+(z_1) = (-1)^i \Phi_-(z_2) \Phi_-(z_1 - \hbar)
\]

\[
= (-1)^i (2\hbar)^{-1/2} (z_1 - z_2) \Gamma \left( \frac{1}{2} + \frac{z_1 - z_2}{2\hbar} \right) \Gamma \left( 1 + \frac{z_1 - z_2}{2\hbar} \right) \Phi_-(z_2) \Phi_-(z_1 - \hbar) ;
\]  \( \text{(A.1)} \)

which follows from (1.8) and (1.10) we have

\[
\Phi^{(1-i)}_+(z) \Phi^{* (i)}_+(z) = 0
\]

if we suppose that normal ordered product \( :\Phi_-(z_2) \Phi_-(z_1 - \hbar) : \) cannot produces pole in \( z_1 - z_2 \) while acting in \( V_0 \) or \( V_1 \). Here we have to be more accurate. In fact we need the following lemma.

Lemma. If two operators \( A(z_1) \) and \( B(z_2) \) satisfy the relation

\[
A(z_1)B(z_2) = (z_1 - z_2) :A(z_1)B(z_2) :;
\]

then the product \( A(z)B(z) \) acts on any finite element of the representation spaces \( V_0 \) and \( V_1 \) multiplying it by zero

\[
A(z)B(z)v = 0, \quad \forall v \in V_0 \text{ or } V_1 .
\]

The word finite in the formulation of the Lemma means finite sum of the elements of the type (3.1).

To prove (5.3) in more complicated case \( \varepsilon_1 = \varepsilon_2 = + \) we have to substitute operator

\[
\Phi^{(1-i)}_+(z_2) = \text{res}_u \Phi^{(1-i)}_+ (z_2, u) = \text{res}_u \left[ \Phi^{(1-i)}_+ (z_2) f(u) - f(u) \Phi^{(1-i)}_+ (z_2) \right]
\]

into product \( \Phi^{(1-i)}_+(z_2) \Phi^{* (i)}_+(z_1) \). Residue operation \( \text{res}_u A(v) \) means

\[
\text{res}_u A(v) = A_0, \quad \text{if} \quad A(v) = \sum_k A_k v^{-k-1} . \quad \text{(A.2)}
\]

After normal ordering all products using formulas (1.13), (A.1) we obtain

\[
g \Phi^{(1-i)}_+(z_2, u) \Phi^{* (i)}_+(z_1) = (-1)^i \frac{\Gamma \left( \frac{1}{2} + \frac{z_1 - z_2}{2\hbar} \right)}{\sqrt{2\hbar} \Gamma \left( 1 + \frac{z_1 - z_2}{2\hbar} \right)} \left( \delta(u - z_2) + o(z_1 - z_2) \right)
\]

\times :\Phi_-(z_2) f(u) \Phi_-(z_1 - \hbar) : .
Because of the $\delta$-function we can set $u = z_2$ in the normal ordered product of the operators, set $z_1 = z_2$, use the operator identity

$$:\Phi_-(z)f(z)\Phi_-(z - h): = (-1)^p$$

and apply Lemma 1 to obtain that

$$g\Phi^{(1-i)}_+(z)\Phi^{(i)}_+(z) = 1 \tag{A.3}$$

because operator $(-1)^p$ act on the spaces $V_i$ multiplying them by $(-1)^i$. Other cases can be proved analogously.

Let us prove now the identity (5.4). Because of the fusion rule

$$\Phi_-(z - h)\Phi_-(z) = g^{-1} \cdot \Phi_-(z - h)\Phi_-(z)$$

we have

$$g \sum_k \Phi^{(1-i)}_+(z)\Phi^{(i)}_+(z) = g(-1)^i (\Phi_-(z - h)\Phi_+(z) - \Phi_+(z - h)\Phi_-(z))$$

$$= (-1)^i \text{res}_{u} \left[ \frac{2}{(u - z - h)(u + z + h)} - \frac{1}{(u - z - h)(u - z)} - \frac{1}{(u - z + h)(u - z)} \right] \times :\Phi_-(z - h)f(u)\Phi_-(z):$$

$$= (-1)^i \text{res}_{u} \delta(u - z) :\Phi_-(z)f(u)\Phi_-(z - h): = \text{id} .$$

### A.2 Ding–Frenkel Equivalence

To prove the Ding–Frenkel equivalence of new and $RLL$ realizations of $D\hat{Y}(\mathfrak{sl}_2)$ we have to insert $L$-operators in the form

$$L^+(z) = \begin{pmatrix} (k^+(z - h))^{-1} - \frac{h^2 f^+(z - hc)k^+(z)e^+(z)}{h k^+(z)e^+(z)} & h f^+(z - hc)k^+(z) \\ \frac{1}{h k^+(z)e^+(z)} & k^+(z) \end{pmatrix}$$

and

$$L^-(z) = \begin{pmatrix} (k^-(z - h))^{-1} - \frac{h^2 f^-(z)k^-(z)e^-(z)}{h k^-(z)e^-(z)} & h f^-(z)k^-(z) \\ \frac{1}{h k^-(z)e^-(z)} & k^-(z) \end{pmatrix}$$

into (5.8) and (5.7). As usual, $L^1(z)$ and $L^2(z)$ are tensor products $L(z) \otimes \text{1 and 1} \otimes L(z)$ respectively.

First, we will write the commutation relations between $k^\pm(z)$ which follows from $D_1D_2$ commutation relations. We have

$$k^\pm(z_1)k^\pm(z_2) = k^\pm(z_2)k^\pm(z_1) , \tag{A.4}$$

$$\rho(z + ch)k^-(z_1)k^+(z_2) = k^+(z_2)k^-(z_1)\rho(z) . \tag{A.5}$$

Note that non-triviality of the commutation relation in (A.3) is dictated by presence of $c$. If $c = 0$ as happen for evaluation representations then all $k^\prime$s commute. Now note that commutation relation between $h^+(z_1)$ and $h^-(z_2)$ easily follows from commutation relations (A.4)–(A.5) if we identify

$$(k^\pm(z + h))^{-1} = h^\pm(z) \tag{A.6}$$

and by making use the identity

$$\rho^\pm(z + h)\rho^\pm(z)b(z) = 1 .$$

To begin with nontrivial part of commutation relations we will consider the relation

$$b(z)D(z_1)B(z_2) + c(z)B(z_1)D(z_2) = a(z)B(z_2)D(z_1)$$
written in three cases of (5.8) and (6.7). It yields \((z = z_1 - z_2)\)

\[
\begin{align*}
\text{(6.8)} & \quad b(z)k^-(z_1)f^-(z_2)k^-(z_2) + c(z)f^-(z_1)k^-(z_1)k^-(z_2) = f^-(z_2)k^-(z_2)k^-(z_1) , \\
\text{(A.10)} & \quad b(z)k^+(z_1)f^+(z_2 - ch)k^+(z_2) + c(z)f^+(z_1 - ch)k^+(z_1)k^+(z_2) = f^+(z_2 - ch)k^+(z_2)k^+(z_1) , \\
\text{(A.11)} & \quad \rho(z + ch) \left[ b(z + ch)k^-(z_1)f^+(z_2 - ch)k^+(z_2) + c(z + ch)f^-(z_1)k^-(z_1)k^+(z_2) \right] \\
& \quad = \rho(z)k^+(z_2 - ch)k^+(z_2)k^-(z_1) . \quad \text{(A.9)}
\end{align*}
\]

Multiplying (A.7) and (A.8) from the right by \((k^+(z_1)k^+(z_2))^{-1}\), (A.3) by \((k^-(z_1))^{-1}(k^+(z_2))^{-1}\) and using in the latter case the commutation relation (A.5) to cancel the factor \(\rho(z)\) we obtain

\[
\begin{align*}
\text{(A.10)} & \quad f^-(z_2) = b(z)k^-(z_1)f^-(z_2)(k^-(z_1))^{-1} + c(z)f^-(z_1) , \\
\text{(A.11)} & \quad f^+(z_2 - ch) = b(z)k^+(z_1)f^+(z_2 - ch)(k^+(z_1))^{-1} + c(z)f^+(z_1 - ch) , \\
\text{(A.12)} & \quad f^+(z_2 - ch) = b(z + ch)k^+(z_1)f^+(z_2 - ch)(k^-(z_1))^{-1} + c(z + ch)f^-(z_1) .
\end{align*}
\]

Making comparison of (A.10) and (A.12) (in the latter relation we have to shift \(z_2 \rightarrow z_2 + ch\)) we can see that the combination

\[
f^+(z) - f^-(z) = \sum_{n \in \mathbb{Z}} f_n z^{-n-1}
\]

has “good”, namely, vertex like commutation relation with \((k^-(z))^{-1}\)

\[
(k^- (z_1))^{-1} f(z_2) = b(z) f(z_2) (k^- (z_1))^{-1} .
\]

To obtain the commutation relation \(k^+(z_1)\) with \(f(z_2)\) we have to consider another DB relation, namely,

\[
c(z) D(z_1) B(z_2) + b(z) B(z_1) D(z_2) = a(z) D(z_2) B(z_1)
\]

in the case of (5.7). It is

\[
\rho(z + ch) \left[ c(z + ch)k^-(z_1)f^+(z_2 - ch)k^+(z_2) + b(z + ch)f^-(z_1)k^-(z_1)k^+(z_2) \right] \\
= \rho(z)k^+(z_2)f^-(z_1)k^-(z_1) . \quad \text{(A.13)}
\]

Then inserting in l.h.s. of (A.13) \((k^+(z_1))^{-1}k^+(z_1)\) between \(f^+(z_2 - ch)\) and \(k^+(z_2)\), using (A.5) to cancel \(\rho(z + ch)\) and multiplying from the right by \((k^-(z_1))^{-1}(k^+(z_2))^{-1}\) we obtain

\[
c(z + ch)k^-(z_1)f^+(z_2 - ch)(k^- (z_1))^{-1} + b(z + ch)f^-(z_1) = k^+(z_2)f^-(z_1)(k^+(z_2))^{-1} . \quad \text{(A.14)}
\]

Expressing the combination \(k^-(z_1)f^+(z_2 - ch)(k^- (z_1))^{-1}\) from (A.12) via \(f^+(z_2 - ch)\) and \(f^-(z_1)\), replacing \(z_1 \rightarrow z_2, z_2 \rightarrow z_1\) in (A.14), and using the identities

\[
\frac{b(-z)}{b^2(-z) - c^2(-z)} = b(z), \quad \frac{-c(-z)}{b^2(-z) - c^2(-z)} = c(z)
\]

we obtain

\[
f^-(z_2) = b(z - ch)k^+(z_1)f^-(z_2)(k^+(z_1))^{-1} + c(z - ch)f^+(z_1 - ch) . \quad \text{(A.15)}
\]

From (A.15) and (A.11) (in the latter equation we have to shift \(z_2 \rightarrow z_2 + ch\)) we have

\[
(k^+(z_1))^{-1} f(u)(z_2) = b(z - ch)f(z_2)(k^+(z_1))^{-1} .
\]

Now fourth and fifth lines in (6.3) follow from definition (A.6) and simple identity

\[
b(z - h)b(z) = \frac{z - h}{z + h} . \quad \text{(A.16)}
\]
The relations
\[
e^{\pm}(z_2) = b(z)(k^{\pm}(z_1))^{-1}e^{\pm}(z_2)k^{\pm}(z_1) + c(z)e^{\pm}(z_1),
\]
\[
e^{\mp}(z_2) = b(z)(k^{\pm}(z_1))^{-1}e^{\mp}(z_2)k^{\pm}(z_1) + c(z)e^{\pm}(z_1)
\]
which follow from $DC$ relations yield for the combination
\[
e^{+}(z) - e^{-}(z) = e(z) = \sum_{n \in \mathbb{Z}} e_n z^{-n-1}
\]
the commutation relations with $k^{\pm}(z)$
\[
(k^{\pm}(z))^{-1}e(z_2) = b(z)^{-1}e(z_2)(k^{\pm}(z_1))^{-1}
\]
and third line in (2.3) is again a consequence of the identity (A.16).

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