Partly dissipative system with multizonal initial and boundary layers

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Abstract. For a singularly perturbed parabolic - ODE system we construct the asymptotic expansion in the small parameter in the case, when the degenerate equation has a double root. Such systems, which are called partly dissipative reaction-diffusion systems, are used to model various natural processes, including the signal transmission along axons, solid combustion and the kinetics of some chemical reactions. It turns out that the algorithm of the construction of the boundary layer functions and the behavior of the solution in the boundary layers essentially differ from that ones in case of a simple root. The multizonal initial and boundary layers behaviour was stated.

1. Introduction and main results
We consider a coupled PDE-ODE system of the form

\[ \varepsilon^2 (\partial_t u - \partial_x^2 u) = F(u, v, x, t, \varepsilon), \quad (x, t) \in D := (0, 1) \times (0, T], \]  
\[ \varepsilon \partial_t v = f(u, v, x, t, \varepsilon), \quad (x, t) \in D, \]  
with homogeneous Neumann boundary conditions

\[ \partial_x u(0, t, \varepsilon) = \partial_x u(1, t, \varepsilon) = 0, \quad t \in (0, T], \]  
and initial conditions

\[ u(x, 0, \varepsilon) = u^0(x), \quad v(x, 0, \varepsilon) = v^0(x), \quad x \in [0, 1]. \]  

Here, \( \varepsilon > 0 \) is a small parameter, \( F, f \) and \( u^0, v^0 \) are sufficiently smooth functions. The initial condition \( u^0 \) is supposed to be consistent with the boundary condition. Systems (1)–(2) are called partly dissipative reaction-diffusion systems, because the diffusion term appears only in the first of these equations. Such systems are used for mathematical modeling in chemical kinetics, theoretical biology and other applied sciences.

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Our goal is to construct the asymptotics of boundary layer solutions to problem (1)–(4) in the case when the reduced equation

\[ F(u, v, x, t, 0) = 0 \]

obtained from equation (1) for \( \varepsilon = 0 \), has a double root

\[ u = \varphi(v, x, t). \]

This assumption can be reformulated in the following form.

(A1) There exist functions \( \varphi, h \) and \( F_1 \) such that

\[ F(u, v, x, t, \varepsilon) = -h(v, x, t)(u - \varphi(v, x, t))^2 + \varepsilon F_1(u, v, x, t, \varepsilon). \]

Some singularly perturbed problems with double roots of the reduced equation have been considered [1, 2, 3]. They showed that standard singular perturbation techniques developed for the case of a simple root of the reduced equation must be substantially modified in the case of a double root. The main difference is concerned with a more complicated three-zone structure of the initial and boundary layers.

In the present work it is shown that close to the initial time moment this solution behaves as a superposition of two initial layers with two different characteristic time scales

\[ \sigma = t/\varepsilon \quad \text{and} \quad \tau = t/\varepsilon^2. \quad (5) \]

Respectively, two types of corner layers appear in the vicinity of the corner points \((x, t) = (0, 0)\) and \((x, t) = (1, 0)\).

We construct the asymptotics of boundary layer solution \( u, v \) to problem (1)–(4) in the form

\[ u = \overline{u}(x, t, \varepsilon) + Su(x, \sigma, \varepsilon) + \Pi u(x, \tau, \varepsilon) + Qu(x, \xi, \varepsilon) + Ru(x, \xi, \varepsilon) \]

\[ + Pu(x, \tau, \varepsilon) + \tilde{Q}u(x, \tilde{\tau}, \varepsilon) + \tilde{P}u(x, \tilde{\tau}, \varepsilon) \]

\[ v = \overline{v}(x, t, \varepsilon) + Sv(x, \sigma, \varepsilon) + \Pi v(x, \tau, \varepsilon) + Qv(x, \xi, \varepsilon) + Rv(x, \xi, \varepsilon) \]

\[ + Pv(x, \tau, \varepsilon) + \tilde{Q}v(x, \tilde{\tau}, \varepsilon) + \tilde{P}v(x, \tilde{\tau}, \varepsilon), \quad (6) \]

\[ (7) \]

where the terms are the partial sum in some arbitrary orders of \( \varepsilon \),

\[ \xi = x/\varepsilon^{3/4}, \quad \tilde{\xi} = (1 - x)/\varepsilon^{3/4}, \quad (8) \]

and variables \( \sigma, \tau \) are defined by formulas (5). Functions \( \overline{u}, \overline{v} \) are regular (or outer) asymptotics, whereas all other terms describe initial, boundary and corner layers. In particular, \( Su, Sv, \Pi u, \Pi v \) are initial layers describing the behaviour of the solution close to the initial time moment \( t = 0 \). Note that \( S- \) and \( \Pi- \)terms vary on different time scales corresponding to variables \( \sigma \) and \( \tau \), respectively. Moreover, these layers have qualitatively different decay estimates

\[ |Su(x, \sigma, \varepsilon)| + |Sv(x, \sigma, \varepsilon)| \leq c_0 e^{-\kappa_0 \sigma} \]

\[ |\Pi u(x, \tau, \varepsilon)| + |\Pi v(x, \tau, \varepsilon)| \leq c_0 \Pi(\tau, \varepsilon) \]

where

\[ \Pi_\kappa(\tau, \varepsilon) := \frac{\sqrt{\varepsilon} \exp(-\sqrt{\varepsilon}\kappa \tau)}{1 + \sqrt{\varepsilon} - \exp(-\sqrt{\varepsilon}\kappa \tau)} \]
is a special etalon function, and $c_0 > 0$ and $\kappa_0 > 0$ are some constants. Four other functions $Q_u, Q_v, \tilde{Q}_u, \tilde{Q}_v$ describe the behaviour of the solution close to the end points of the interval $0 \leq x \leq 1$. They will satisfy the estimate

$$\left| Q_u(\xi, t, \varepsilon) \right| + \left| Q_v(\xi, t, \varepsilon) \right| + \left| \tilde{Q}_u(\xi, t, \varepsilon) \right| + \left| \tilde{Q}_v(\xi, t, \varepsilon) \right| \leq c_0 e^{-\kappa_0 \xi}.$$ 

Finally, functions $R_u, R_v, P_u, P_v, \tilde{R}_u, \tilde{R}_v, \tilde{P}_u, \tilde{P}_v$ provide a description of the corner layers in the vicinity of the corner points $(x, t) = (0, 0)$ and $(x, t) = (1, 0)$. They satisfy the estimates

$$\left| R_u(\xi, \sigma, \varepsilon) \right| + \left| R_v(\xi, \sigma, \varepsilon) \right| + \left| \tilde{R}_u(\xi, \sigma, \varepsilon) \right| + \left| \tilde{R}_v(\xi, \sigma, \varepsilon) \right| \leq c_0 e^{-\kappa_0 (\xi + \sigma)}$$

$$\left| P_u(\xi, \tau, \varepsilon) \right| + \left| P_v(\xi, \tau, \varepsilon) \right| + \left| \tilde{P}_u(\xi, \tau, \varepsilon) \right| + \left| \tilde{P}_v(\xi, \tau, \varepsilon) \right| \leq c_0 \Omega_{\kappa_0}(\tau, \varepsilon) e^{-\kappa_0 \xi}.$$ 

To construct a sequence of approximate solutions $U_n(x, t, \varepsilon), V_n(x, t, \varepsilon)$ to problem (1)–(4) we need some other natural assumptions.

(A2) The equation

$$g(v, x, t) := f(\varphi(v, x, t), v, x, t, 0) = 0$$

has a solution $v = \tau_0(x, t) \in C(D)$ such that

$$\partial_v g(\tau_0(x, t), x, t) < 0 \quad \text{for all} \quad (x, t) \in D.$$ 

(A3) The inequalities

$$h(\tau_0(x, t), x, t) > 0 \quad \text{and} \quad F_1(\varphi(\tau_0(x, t), x, t), \tau_0(x, t), x, t, 0) > 0$$

are fulfilled for $(x, t) \in \overline{D}$.

(A4) Let $g(v, x, 0) \neq 0$ for

$$(v, x) \in \{ (v, x) : (v - \tau_0(x, 0))(v - v^0(x)) \leq 0 \quad \text{and} \quad v \neq \tau_0(x, 0), \ x \in [0, 1] \}.$$ 

(A5) Let

$$h(v, x, 0) > 0 \quad \text{and} \quad A(v, x) := F_1(\varphi(v, x, 0), v, x, 0, 0) - \partial_v \varphi(v, x, 0)g(v, x, 0) > 0$$

for $(v, x) \in \{ (v, x) : (v - \tau_0(x, 0))(v - v^0(x)) \leq 0 \}, \ x \in [0, 1]$. 

(A6) Let

$$u^0(x) > \varphi(v^0(x), x, 0)$$

for $x \in [0, 1]$. 

(A7) The inequalities

$$\partial_v \varphi(\tau_0(x, t), x, t) > 0 \quad \text{and} \quad \partial_u f(\varphi(\tau_0(x, t), x, t), \tau_0(x, t), x, t, 0) > 0$$

are fulfilled for $(x, t) \in \overline{D}$.

Our main result is summarized in the following theorem.
Theorem 1. Suppose that functions $F, f, u^0$ and $v^0$ are $C^2$-smooth. Moreover, suppose (A1)–(A7). Then, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there exists a solution $u(x, t, \varepsilon), v(x, t, \varepsilon)$ to problem (1)–(4) and for each $(x, t) \in D$ we have

$$
\lim_{\varepsilon \to +0} u(x, t, \varepsilon) = \varphi(\varpi_0(x, t), x, t) \quad \text{and} \quad \lim_{\varepsilon \to +0} v(x, t, \varepsilon) = \overline{\varpi}_0(x, t).
$$

Furthermore, if the functions $F, f, u^0$ and $v^0$ are $C^{n+2}$-smooth for some $n \geq 0$, then using the algorithm described in Section 2 one can construct approximate solution $U_n(x, t, \varepsilon), V_n(x, t, \varepsilon)$ such that

$$
\text{sup}_{(x, t) \in D} \left( |u(x, t, \varepsilon) - U_n(x, t, \varepsilon)| + |v(x, t, \varepsilon) - V_n(x, t, \varepsilon)| \right) = O\left(\varepsilon^{(n+1)/2}\right) \quad \text{for} \quad \varepsilon \to +0.
$$

2. Asymptotics of the solution to problem (1)–(4)

To construct the asymptotics we write the right-hand side of equation (1) in the following form

$$
F = \mathcal{F} + SF + PF + RF + PF' + QF + RF + \tilde{PF},
$$

where

$$
\mathcal{F} = F(\overline{u}(x, t, \varepsilon), \overline{v}(x, t, \varepsilon), x, t, \varepsilon),
$$

$$
SF = \left[ F(\overline{u}(x, t, \varepsilon) + Su(x, \sigma, \varepsilon), \overline{v}(x, t, \varepsilon) + Sv(x, \sigma, \varepsilon), x, t, \varepsilon) - F(\overline{u}(x, t, \varepsilon), \overline{v}(x, t, \varepsilon), x, t, \varepsilon) \right]_{t = \varepsilon^{3/2} \sigma, \sqrt{\varepsilon}}.
$$

$$
QF = \left[ F(\overline{u}(x, t, \varepsilon) + Qu(x, \xi, \varepsilon), \overline{v} + Sv, x, t, \varepsilon) - F(\overline{u}(x, t, \varepsilon), \overline{v}, x, t, \varepsilon) \right]_{x = \varepsilon^{3/4} \xi}.
$$

$$
RF = \left[ F(\overline{u}(x, t, \varepsilon) + Su(x, \sigma, \varepsilon) + Qu(x, \xi, \varepsilon) + Ru(x, \xi, \varepsilon), \overline{v} + Sv, x, t, \varepsilon) - F(\overline{u}(x, t, \varepsilon) + Su(x, \sigma, \varepsilon), \overline{v}, x, t, \varepsilon) - F(\overline{u}(x, t, \varepsilon), \overline{v}, x, t, \varepsilon) \right]_{x = \varepsilon^{3/4} \xi, t = \varepsilon \sigma}.
$$

$$
PF = \left[ F(\overline{u}(x, t, \varepsilon) + Su(x, \sigma, \varepsilon) + P\overline{u}(x, t, \varepsilon) + Qu(x, \xi, \varepsilon) + Ru(x, \xi, \varepsilon), \overline{v} + Sv, x, t, \varepsilon) - F(\overline{u}(x, t, \varepsilon), \overline{v}, x, t, \varepsilon) - F(\overline{u}(x, t, \varepsilon) + Su(x, \sigma, \varepsilon), \overline{v} + Sv, x, t, \varepsilon) - F(\overline{u}(x, t, \varepsilon), \overline{v}, x, t, \varepsilon) + F(\overline{u}(x, t, \varepsilon), \overline{v} + Sv, x, t, \varepsilon) \right]_{x = \varepsilon^{3/4} \xi, t = \varepsilon^{3/2} \sigma, \sigma = \sqrt{\varepsilon}}.
$$
(arguments of the $v$-terms are identical to the arguments of the corresponding $u$-terms, similar presentation we use for $QF, RF$ and $PF$).

Analogously, the right-hand side of equation (2) can be also decomposed by the similar way as follows

$$f = \tilde{f} + Sf + \Pi f + Qf + Pf + \tilde{Q}f + \tilde{R}f + \tilde{P}f.$$  \hfill (15)

Then, we proceed by using boundary layer function method scheme as follows. Each term in formulas (6) and (7) we write as a formal series with respect to a fractional power of parameter $\varepsilon$ (Recall that in the case of a simple root of the reduced equation one needs to work with a similar series containing integer powers of $\varepsilon$ only!) Then we insert this into equations (1), (2) and replace the right-hand sides $F$ and $f$ by formulas (9) and (15). Collecting expressions with the same power of small parameter $\varepsilon$, we obtain equations for all terms in (6) and (7).

For example, for $\overline{u}, \varpi$ we obtain the system

$$\varepsilon^2(\partial_t \overline{u} - \partial_x^2 \overline{u}) = \overline{F} := -h(\overline{u}, x, t)(\overline{u} - \varphi(\overline{u}, x, t))^2 + \varepsilon F_1(\overline{u}, \varpi, x, t, \varepsilon), \quad (x, t) \in D, \hfill (16)$$

$$\varepsilon \partial_t \varpi = \tilde{f}(\overline{u}, \varpi, x, t, \varepsilon), \quad (x, t) \in D. \hfill (17)$$

Equations for other terms can be written similarly.

Some of the terms in formulas (6) and (7) are determined from differential equations. In this case, one needs to equip the differential equations with initial and/or boundary conditions. They can be obtained if we insert expressions (6) and (7) into the boundary (3) and initial conditions (4) of the main problem. For example, first identity in (4) yields

$$\overline{u}(x, 0, \varepsilon) + Su(x, 0, \varepsilon) + \Pi u(x, 0, \varepsilon) + Qu(x, 0, \varepsilon) + Ru(x, 0, \varepsilon)$$

$$+ Pu(x, 0, \varepsilon) + \tilde{Q}u(x, 0, \varepsilon) + \tilde{R}u(x, 0, \varepsilon) + \tilde{P}u(x, 0, \varepsilon) = v^0(x).$$

Separating it by the above algorithm we obtain

$$\overline{u}(x, 0, \varepsilon) + Su(x, 0, \varepsilon) + \Pi u(x, 0, \varepsilon) = v^0(x), \quad 0 \leq x \leq 1,$$

$$Qu(\xi, 0, \varepsilon) + Ru(\xi, 0, \varepsilon) + Pu(\xi, 0, \varepsilon) = 0, \quad \xi \geq 0,$$

$$\tilde{Q}u(\tilde{\xi}, 0, \varepsilon) + \tilde{R}u(\tilde{\xi}, 0, \varepsilon) + \tilde{P}u(\tilde{\xi}, 0, \varepsilon) = 0, \quad \tilde{\xi} \geq 0.$$

Similarly, using the second identity in (4) and the first identity in (3) we get

$$\overline{v}(x, 0, \varepsilon) + Sv(x, 0, \varepsilon) + \Pi v(x, 0, \varepsilon) = v^0(x), \quad 0 \leq x \leq 1,$$

$$Qv(\xi, 0, \varepsilon) + Rv(\xi, 0, \varepsilon) + Pv(\xi, 0, \varepsilon) = 0, \quad \xi \geq 0,$$

$$\partial_x \overline{u}(0, t, \varepsilon) + \varepsilon^{-3/4} \partial_\xi Qu(0, t, \varepsilon) = 0, \quad 0 \leq t \leq T,$$

$$\partial_x Sv(0, \sigma, \varepsilon) + \varepsilon^{-3/4} \partial_\xi Rv(0, \sigma, \varepsilon) = 0, \quad \sigma \geq 0,$$

$$\partial_x \Pi u(0, \tau, \varepsilon) + \varepsilon^{-3/4} \partial_\tau Pu(0, \tau, \varepsilon) = 0, \quad \tau \geq 0.$$  \hfill (22)

For some boundary layer functions we also use a standard condition implying their decay to 0 for the stretched variables, tending to infinity.

The regular part of the asymptotics $\overline{u}$ and $\overline{v}$ will be constructed in the form of a power series with respect to $\sqrt{\varepsilon}$

$$\overline{u}(x, t, \varepsilon) = \sum_{i=0}^\infty \varepsilon^{i/2} \overline{u}_i(x, t), \quad \overline{v}(x, t, \varepsilon) = \sum_{i=0}^\infty \varepsilon^{i/2} \overline{v}_i(x, t).$$
Inserting these formulas into equations (16)–(17) and expanding the left- and right-hand sides into series with respect to \( \sqrt{\varepsilon} \), we obtain a recurrent sequence of equations for \( \pi_i(x, t) \) and \( \pi_i(x, t) \).

By the simple calculations we get, that \( \pi_0 \) and \( \pi_0 \) are determined due to assumption (A2), and higher order terms can be determined by means of assumption (A3).

Boundary and initial layers are also defined by proposed scheme.

For initial layers \( S_u \) and \( S_v \) we obtain the following formal system

\[
\varepsilon \partial_x S_u - \varepsilon^2 \partial_x^2 S_u = SF(x, \sigma, \varepsilon), \quad \partial_x S_v = Sf(x, \sigma, \varepsilon), \quad (x, \sigma) \in (0, 1) \times (0, \infty),
\]

where function \( SF \) is defined by (10) and \( Sf \) has the analogous definition. Similarly, for initial layers \( \Pi u \) and \( \Pi v \) we write a system

\[
\partial_t \Pi u - \varepsilon^2 \partial_t^2 \Pi u = \Pi F(x, \tau, \varepsilon), \quad \varepsilon^{-1} \partial_t \Pi v = \Pi f(x, \tau, \varepsilon), \quad (x, \tau) \in (0, 1) \times (0, \infty).
\]

Here again the right-hand side \( \Pi F \) is given by formula (11) and function \( \Pi f \) is defined by analogous expression.

We construct \( Su \), \( Sv \), \( \Pi u \) and \( \Pi v \) in the form

\[
Su(x, \sigma, \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^{i/2} S_i u(x, \sigma), \quad Sv(x, \sigma, \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^{i/2} S_i v(x, \sigma),
\]

\[
\Pi u(x, \tau, \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^{i/2} \Pi_i u(x, \tau), \quad \Pi v(x, \tau, \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^{i/2} \Pi_i v(x, \tau).
\]

For boundary layers \( Q u \) and \( Q v \) in the vicinity of the point \( x = 0 \) we obtain a system of equations

\[
\varepsilon^2 \partial_t Q u - \sqrt{\varepsilon} \partial_x Q u = QF(\xi, t, \varepsilon), \quad \varepsilon \partial_t Q v = Qf(\xi, t, \varepsilon), \quad (\xi, t) \in (0, \infty) \times (0, T),
\]

where stretched variable \( \xi \) is defined in (8), \( QF \) is a function given by formula (12) and \( Qf \) has analogous meaning. System (23) will be equipped with boundary condition (20) and standard decay condition.

We construct \( Q u \) and \( Q v \) in the form

\[
Q u(\xi, t, \varepsilon) = \varepsilon^{3/4} \sum_{i=0}^{\infty} \varepsilon^{i/4} Q_i u(\xi, t), \quad Q v(\xi, t, \varepsilon) = \varepsilon^{3/4} \sum_{i=0}^{\infty} \varepsilon^{i/4} Q_i v(\xi, t).
\]

To construct functions \( R u \) and \( R v \) we consider a system of equations

\[
\varepsilon \partial_x R u - \sqrt{\varepsilon} \partial_x^2 R u = RF(\xi, \sigma, \varepsilon), \quad \partial_x R v = Rf(\xi, \sigma, \varepsilon), \quad (\xi, \sigma) \in (0, \infty)^2,
\]

where the right-hand side \( RF \) is determined by formula (13) and \( Rf \) is defined by analogous expression. Similarly, we construct terms \( Pu \) and \( Pv \) considering system

\[
\partial_t P u - \sqrt{\varepsilon} \partial_x P u = PF(\xi, \tau, \varepsilon), \quad \varepsilon^{-1} \partial_t P v = Pf(\xi, \tau, \varepsilon), \quad (\xi, \tau) \in (0, \infty)^2,
\]

where \( PF \) is defined by (14) and \( Pf \) has analogous definition.

Systems (24) and (25) with boundary conditions (21) and (22) as well as with the decay conditions at infinity.
We construct $Ru, Rv, Pu$ and $Pv$ in the form

$$Ru(\xi, \sigma, \varepsilon) = \varepsilon^{3/4} \sum_{i=0}^{\infty} \varepsilon^{i/4} R_i u(\xi, \sigma),$$

$$Rv(\xi, \sigma, \varepsilon) = \varepsilon^{3/4} \sum_{i=0}^{\infty} \varepsilon^{i/4} R_i v(\xi, \sigma),$$

$$Pu(\xi, \tau, \varepsilon) = \varepsilon^{1/4} \sum_{i=0}^{\infty} \varepsilon^{i/4} P_i u(\xi, \tau),$$

$$Pv(\xi, \tau, \varepsilon) = \varepsilon^{3/4} \sum_{i=0}^{\infty} \varepsilon^{i/4} P_i v(\xi, \tau).$$

Note, that the terms marked by "tilde" in formulas (6) and (7) can be determined by the analogous problems.

To prove our main Theorem we use the asymptotic method of differential inequalities [4, 5]. We modify functions $U_n$ and $V_n$ to construct the upper and lower solutions of problem (1)–(4). Using the known comparison theorem (see for e.g. [6]) we get the existence of the solution with the asymptotics constructed.

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