

Gravity In A Model Of Multi Branes Alternated Between Minkowski And AdS Spacetime In The Fifth Dimension

Yun-Song Piao\textsuperscript{a,c}, Xinmin Zhang\textsuperscript{a} and Yuan-Zhong Zhang\textsuperscript{b,c}

\textsuperscript{a}Institute of High Energy Physics, Chinese Academy of Sciences, P.O. Box 918(4), Beijing 100039, China
\textsuperscript{b}CCAST (World Lab.), P.O. Box 8730, Beijing 100080
\textsuperscript{c}Institute of Theoretical Physics, Chinese Academy of Sciences, P.O. Box 2735, Beijing 100080, China

Abstract

We construct in this paper a multi-brane model with Minkowski and AdS spacetime alternated in the fifth dimension and study the gravity on the brane of our universe. We will show that in this model there are both the graviton-like resonant state which generates the “quasi-localized gravity” and the resonant states of the massive KK mode. This model gives rise to gravity which deviates from the Newton gravitational potential at small and large scale which, however can be shown to be consistent with the observations in the absence of fine-tuning of the model parameters.

\footnote{Mailing address in China, Email address: yspiao@itp.ac.cn}
As motivated by the string theory [1] [2], there has been a lot of studies in the recent years on brane models. These models could provide a solution to problem of the hierarchy between the Plank scale and the electroweak scale [3] [4]. Furthermore various generalizations of models in [3] [4], such as the multi-brane models [5] [6] and Crystal Universe model [7] [8], have also been proposed and studied in the literature. Phenomenologically these models provide a way of placing the visible sector on a positive tension brane and give rise to many predictions [9] on neutrino mixing and Dark Matter. Usually the branes are supposed to live in the AdS spacetime. Recently the authors of Refs. [5] and [10] have proposed multi-brane models with “quasi-localized gravity” where five dimensional (5D) Minkowski region is embedded in AdS region and the gravity is mediated not by a normalized graviton state but by a resonant state. In the model of Ref. [11], a finite Minkowski region is surrounded by a AdS region and the resonant modes modify gravity at small scale.

In this paper we study a brane setup which is a generalization of the multi-brane model considered in [8] with Minkowski and AdS spacetime alternated in the fifth dimension, then calculate the gravitational potential between two unit masses on positive tension brane. In our model, there is not only a zero mode resonant state which generates the ”quasi-localized gravity”, but also exist many massive resonant states of the massive KK mode, which modify Newtonian gravitational potential at different length scales.

Our 5D model is shown in Fig. 1 which consists of four parallel 3-branes [1] with Minkowski and AdS spacetimes alternated and a $z_2$ symmetry in the fifth dimension $y$. Due to the $z_2$ symmetry we consider $y \geq 0$. Two 3-branes are located at $y = y_1$ and $y = y_2$ and the action of this brane setup is

\begin{equation}
S_{\text{bulk}} = \int d^4x \int \sqrt{-g} \left( \frac{1}{2\kappa^2} R - \tilde{\Lambda} \right) dy \quad \text{with} \quad \tilde{\Lambda} = \begin{cases} 0 & \text{for } 0 \leq y_1 \\
\Lambda & \text{for } y_1 \leq y \leq y_2 \end{cases} \quad \text{and} \quad y \geq y_2
\end{equation}

\begin{equation}
S_{\text{brane}} = -\int d^4x \left( \sqrt{-g_1} V_1 + \sqrt{-g_2} V_2 \right),
\end{equation}

where $\kappa$ sets the 5D fundamental scale, $g_1$ and $g_2$ are the induced metrics on the branes located at $y = y_1$ and $y = y_2$ respectively, and the corresponding brane tensions denoted by $V_1$ and $V_2$ are $\Lambda/2k$ and $-\Lambda/2k$. The 5D metric ansatz which respects 4D Poincare invariance is given by

\begin{equation}
ds^2 = e^{-2\sigma(y)}\eta_{\mu\nu} dx^\mu dx^\nu + dy^2,
\end{equation}

where the warp function $\sigma(y)$ is essentially a conformal factor that rescales the 4D component of the metric: $\sigma(y) = ky$ for $y_1 \leq y \leq y_2$ and $\sigma(y)$ equal to $ky_1$ and $ky_2$ for $0 \leq y \leq y_1$ and $y_2 \leq y$ respectively, where $k = \sqrt{\frac{-\kappa^2 \Lambda}{6}}$ is effectively the bulk curvature in the region between the two 3-branes located at $y = y_1$ and $y = y_2$. 

\textsuperscript{2} This model can easily be generalized to include more than four 3-branes.
Let’s consider the linearized gravitational perturbations about the background metric in Eq.(3):

\[ ds^2 = (e^{-2\sigma(y)}\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu + dy^2. \]  

Following Ref. [12], we work in the transverse-traceless gauge, \( \partial^\mu h_{\mu\nu} = h_\mu^\mu = 0 \), and do not consider the metric fluctuations \( h_{55} \) and \( h_{5\mu} \). It is useful to define a conformal coordinate

\[
    z = \begin{cases}
        e^{k(y_1 - 1)/k} & \text{for } 0 \leq y \leq y_1 \\
        e^{k(y_2 - 1)/k} & \text{for } y_1 \leq y \leq y_2 \\
        e^{k(y_2 - 1)/k} & \text{for } y \geq y_2.
    \end{cases}
\]  

(5)

We use the ansatz \( h_{\mu\nu} = e^{ip\cdot x}\exp\left(-\frac{\sigma(z)}{2}\right)\Psi_m(z)\epsilon_{\mu\nu} \) with \( \epsilon_{\mu\nu} \) being a constant polarization tensor and \( m^2 = -p^2 \) is an effective 4D mass. The function \( \Psi_m(z) \) obeys the following Schrödinger-like equation which is derived from the linearized Einstein equation:

\[
    \left[-\frac{1}{2}\partial_z^2 + V(z)\right]\Psi_m(z) = \frac{1}{2}m^2\Psi_m(z),
\]  

(6)

where the potential is given by

\[
    V(z) = \frac{15k^2}{8[g(z)]^2} \left[ \theta(z - z_1) - \theta(z - z_2) \right] - \frac{3k}{4g(z)} \left[ \delta(z - z_1) - \delta(z - z_2) \right],
\]  

(7)

and

\[
    g(z) = \begin{cases}
        kz_1 + 1 & \text{for } 0 \leq z \leq z_1 \\
        kz + 1 & \text{for } z_1 \leq z \leq z_2 \\
        kz_2 + 1 & \text{for } z \geq z_2
    \end{cases}
\]  

(8)

with \( z_1 = z(y_1) \) and \( z_2 = z(y_2) \).

Solution to the Schrödinger-like equation in Eq.(6) is a combination of plane waves in the Minkowski region and Bessel functions in the AdS region:

\[
    \Psi_m = \begin{cases}
        A \cos(mz) & \text{for } 0 \leq z \leq z_1 \\
        \sqrt{k^{-1}g(z)} \left[ B_1 N_2 \left(\frac{m}{k}g(z)\right) + B_2 J_2 \left(\frac{m}{k}g(z)\right) \right] & \text{for } z_1 \leq z \leq z_2 \\
        C_1 \cos(mz) + C_2 \sin(mz) & \text{for } z \geq z_2
    \end{cases}
\]  

(9)

where \( J \) and \( N \) are Bessel functions. The constants \( A, B_1 \) and \( B_2 \) obey two relations obtained from junction conditions at the positive tension brane at \( y = y_1 \). Similarly, the relations between \( B_1, B_2 \) and \( C_1, C_2 \) are given by the junction conditions at \( y = y_2 \).

Considering the normalized condition \( \int |\Psi_m^2(z)|dz = 1 \) and making use of the asymptotic behaviour of \( \Psi \) at \( z \to \infty \), we have,

\[
    \pi \left( C_1^2 + C_2^2 \right) = 1.
\]  

(10)
Therefore, $A$ can be formulated as

\[
\frac{1}{\pi A^2} \left[ N_2 \left( \frac{m}{k} g_1 \right) J_1 \left( \frac{m}{k} g_1 \right) - N_1 \left( \frac{m}{k} g_1 \right) J_2 \left( \frac{m}{k} g_1 \right) \right] \left( \frac{g_1}{g_2} \right) = 
\]

\[
= \left[ J_1 \left( \frac{m}{k} g_1 \right) \cos (mz_1) + J_2 \left( \frac{m}{k} g_1 \right) \sin (mz_1) \right]^2 \left[ N_1^2 \left( \frac{m}{k} g_2 \right) + N_2^2 \left( \frac{m}{k} g_2 \right) \right] 
- 2 \left[ J_1 \left( \frac{m}{k} g_1 \right) \cos (mz_1) + J_2 \left( \frac{m}{k} g_1 \right) \sin (mz_1) \right] \left[ N_1 \left( \frac{m}{k} g_1 \right) \cos (mz_1) + N_2 \left( \frac{m}{k} g_1 \right) \sin (mz_1) \right] 
\]

\[
\left[ N_1 \left( \frac{m}{k} g_2 \right) J_1 \left( \frac{m}{k} g_2 \right) + N_2 \left( \frac{m}{k} g_2 \right) J_2 \left( \frac{m}{k} g_2 \right) \right] 
+ \left[ N_1 \left( \frac{m}{k} g_1 \right) \cos (mz_1) + N_2 \left( \frac{m}{k} g_1 \right) \sin (mz_1) \right]^2 \left[ J_1 \left( \frac{m}{k} g_2 \right) + J_2 \left( \frac{m}{k} g_2 \right) \right].
\] (11)

Now we calculate the static gravitational potential between two unit masses placed on the $z = z_1$ positive tension brane at a distance $r$ from each other. This potential is generated by the exchange of the massive modes. Following Ref. [12] (also see Ref. [13] for a more extensive discussion), we have

\[
V(r) = -G_5 \int_0^\infty dm \frac{e^{-mr}}{r} \Psi^2_m(z = z_1) 
= -G_5 \int_0^\infty dm \frac{e^{-mr}}{r} A^2 \cos^2(mz_1),
\] (12)

where $G_5$ is 5D Newton constant.

In Fig.2 and Fig.3 we plot $A^2 \cos^2(mz_1)$ vs $mz_1$, from which one can see that for $mz_1 \gg 1$, $A^2 \cos^2(mz_1)$ is a periodic function and independent of the value of $z_2/z_1$. Eq.(12) shows that the massive KK modes effectively modify Newton law on the positive tension brane at various length scale.

We now present results in the case of $kz_1 \gg 1$. Conveniently we rewrite the integral in (12) into three parts:

\[
V(r) = V_1(r) + V_2(r) + V_3(r) 
= -G_5 \int_0^{z_1^{-1}} dm \frac{e^{-mr}}{r} A^2 \cos^2(mz_1) - G_5 \int_{z_1^{-1}}^{z_2^{-1}} dm \frac{e^{-mr}}{r} A^2 \cos^2(mz_1) 
- G_5 \int_{z_2^{-1}}^\infty dm \frac{e^{-mr}}{r} A^2 \cos^2(mz_1).
\] (13)

The constant $A$ in the first term of right-handed side of Eq.(13) i.e. $V_1(r)$ can be evaluated by using the series expansion of Bessel functions with the arguments $mg_1/k$ and $mg_2/k$:

\[
A^2 \cos^2(mz_1) \approx \frac{1}{\pi^2} \left[ \frac{1}{64} \left( \frac{z_2}{z_1} \right)^3 \left( \frac{mz_1}{2} \right)^2 \right].
\] (14)
Substituting (14) into (13), we get
\[
V_1(r) \simeq -\frac{2G_N}{\pi r} \int_0^{r_1/z_1} dx \frac{1}{1+x^2} e^{-xr_1},
\] (15)
where \( x = \frac{3}{2}mz_1(z_2/z_1)^3 \equiv mr_1 \) and \( G_N = G_5/2z_1 \) is the renormalized 4D Newton constant.

We see that \( V(r) = -G_N/r \) for \( r \ll r_1 \), i.e., the 4D Newton law in this case, however \( V(r) = -G_N r_1 \pi r \) for \( r \gg r_1 \) which has the form of Newton law of 5D gravity with a renormalized gravitational constant. This result is similar to that of Ref.[5], except for the different normalization of \( G_N \) and definition of \( r_1 \). This is because in the limit of \( z_1 \to 0 \) our model approaches to the GRS model[5].

The second term in the right hand side of (13), i.e. \( V_2(r) \) is similar to the contribution of the continuum modes to the gravitational potential in the Randall-Sundrum model:
\[
A^2 \cos^2(mz_1) \simeq m z_2.
\] (16)
It gives rise to a correction to Newton law at short distance,
\[
V_2(r) \simeq -\frac{G_N z_1 z_2}{r^2}.
\] (17)
For \( r \ll \sqrt{z_1 z_2} \), this term will dominate the gravitational potential \( V(r) \).

Using the asymptotic expansion of Bessel functions, we obtain the constant \( A \) in \( V_3(r) \),
\[
A^2 \cos^2 (mz_1) \simeq \frac{1}{\pi} \cos^2 (mz_1).
\] (18)
Substituting (18) into (13), we have
\[
V_3(r) \simeq -\frac{2G_N}{\pi r} \int_{z_1/z_2}^{\infty} dx \ e^{-xr/z_1} \cos^2 x
\]
\[
= -\frac{G_N z_1}{r \pi r} \left[ 1 + \frac{(r/z_1)}{4 + (r/z_1)^2} \left( \frac{r}{z_1} \cos \left( \frac{2z_1}{z_2} \right) - \sin \left( \frac{2z_1}{z_2} \right) \right) \right] e^{-r/z_2}.
\] (19)
One can see that \( V(r) \simeq -\frac{G_N z_1}{2} e^{-r/z_2} \) for \( r \ll z_1 \). This is the Yukawa-type potential with a renormalized gravitational constant, however for \( r \gg z_1 \), \( V(r) \simeq -\frac{G_N z_1}{4 + (r/z_1)^2} \left[ 1 + \cos \left( 2z_1/z_2 \right) \right] \). Comparing (19) with (17) shows that for all values of \( z_1 \) and \( z_2 \), \( V_3 < V_2 \). This is different from Ref. [11] and this difference comes from the different normalization condition. In our model the contribution of massive resonant modes is not the leading contribution to the Newton gravitational potential at small scale. In Fig.4 we show the comparison between the two models at small scale.
The sum of $V_1(r)$, $V_2(r)$ and $V_3(r)$ gives rise to the gravitational potential on the brane located at $z = z_1$

\[
V(r) = -f(r) \frac{G_N}{r} \approx -2G_N \frac{r^{1/z_1}}{\pi r} \int_0^1 dx \frac{1}{1+x^2} e^{-xr/r_1} - \frac{G_N}{r} \frac{z_1 z_2}{r^2} \left[1 + \frac{(r/z_1)}{4 + (r/z_1)^2} \left(\frac{r}{z_1} \cos \left(\frac{2z_1}{z_2}\right) - \sin \left(\frac{2z_1}{z_2}\right)\right)\right] e^{-r/z_2},
\]

(20)

where $f(r)-1$ represents the deviation from the standard 4D Newton gravitation. In Fig.5 we plot $f(r)$ as a function of $r$. One can see that only in the region of $\sqrt{z_1 z_2} \ll r \ll r_1$ the gravitational potential for two unit masses on the positive tension brane is described by the 4D Newton law

\[
V(r) \approx -\frac{G_N}{r}.
\]

(21)

In the case of $r \ll \sqrt{z_1 z_2}$, the term $z_1 z_2 / r^2$ will be dominative. On the other hand for $r \gg r_1$, we have the 5D Newton gravitation with a renormalized Newton gravitational constant which is smaller than that in 4D.

In comparison with the observational data, we take $r_1 \sim 10^{28}$ cm, i.e., the present horizon size of the Universe. Defining $\sqrt{z_1 z_2} = l$ in the unit of centimeter, in Fig.6 we plot $ky_1$ and $ky_2$ as function of $l$. The updated observation [14] put a limit $l \sim 0.2$ mm, from which and Fig.6 we obtain that $ky_1 \simeq 60$ and $ky_2 \simeq 90$. As shown in Ref.[8], to have these values it does not require a strong fine tuning of the model parameters.

In summary, we have proposed a brane setup where extra dimension has a non-factorized geometry and in the fifth dimension Minkowski regions and the AdS regions are separated by branes. In our model there are both the graviton-like resonant state which generates the “quasi-localized gravity” and the resonant states of the massive KK mode. The latter one, however in comparison with the continuum modes, is not the leading contribution to Newton gravitational potential at small scale, and also differs from that in Ref. [11]. Our model in this paper provides a picture where gravitational interactions at different length scales possess different behaviors. In particular Newton coupling constant depends on the parameters of the configuration. Finally, we have shown that our model is in consistent with the updated observation without a strong fine tuning of the model parameters.

**Acknowledgments**

This work is supported in part by National Natural Science Foundation of China under Grant Nos. 10047004 and 19835040, and also supported by Ministry of Science and Technology of China under Grant No. NKBRSF G19990754.
References

[1] P. Horava and E. Witten, Nucl. Phys. B\textbf{460} 506 (1996); B\textbf{475} 94 (1996).

[2] E. Witten, Nucl. Phys. B\textbf{471} 135 (1996); J.D. Lykken, Phys. Rev. D\textbf{54} 3693 (1996)

[3] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. lett. B\textbf{249} (1998) 263.

[4] L. Randall and R. Sundrum, Phys. Rev. Lett. \textbf{83} (1999) 3370.

[5] R. Gregory, V.A. Rubakov and S.M. Sibiryakov, \texttt{hep-th/0002072}, Phys. Rev. Lett. \textbf{84} (2000) 5928.

[6] I.I. Kogan and G.G. Ross, \texttt{hep-th/0003074}, Phys. Lett. B\textbf{485} (2000) 255.

[7] N. Kaloper, Phys. lett. B\textbf{474} (2000) 269.

I.I. Kogan, S. Mouslopoulos, A. Papazoglou and G.G Ross, \texttt{hep-th/0006030}, Nucl. Phys. B\textbf{595} (2001) 225.

[8] Y.S. Piao, X.M. Zhang and Y.Z. Zhang, \texttt{hep-th/0104020}, Phys. Lett. B\textbf{512} (2001) 1.

[9] N. Arkani-Hames, S. Dimopoulos, G. Dvali and N. Kaloper, \texttt{hep-ph/9911338}, JHEP 12 (2000) 010.

[10] C. Csaki, J. Erlich and T.J. Hollowood, Phys. Rev. Lett \textbf{84} (2000) 5932.

G. Dvali, G. Gabadadze and M. Porrati, Phys. lett. B\textbf{484} (2000) 112.

[11] J. Lykken, R.C. Myers and J. Wang, \texttt{hep-th/0006191}, JHEP 9 (2000) 009.

[12] L. Randall and R. Sundrum, Phys. Rev. Lett. \textbf{83} (1999) 4690.

[13] J. Garriga and T. Tanaka, Phys. Rev. Lett. \textbf{84} (2000) 2778.

S.B. Giddings, E. Katz and L. Randall, JHEP \textbf{3} (2000) 023.

[14] C.D. Hoyle, U. Schmidt, B.R. Heckel, E.G. Adelberger, J.H. Gundlach, D.J. Kapner and H.E. Swanson, Phys. Rev. Lett. \textbf{86} (2001) 1418
Figure 1: Illustration of our 5D model consisted of four parallel 3-branes with Minkowski and AdS spacetime alternated and a $z_2$ symmetry in the fifth dimension $y$.

Figure 2: Plot of $A^2 \cos^2(mz_1)$ (y-axis) vs $mz_1$ (x-axis) for $z_2/z_1 = 100$.

Figure 3: Plot of $A^2 \cos^2(mz_1)$ (y-axis) vs $mz_1$ (x-axis) for $z_2/z_1 = 3$. (a) and (b) are, respectively, for $mz_1 = [0, 15]$ and $mz_1 = [0, 0.2]$.  

7
Figure 4: Comparison of the $f(r)$ function in our model (the solid line) with that (the long-dash-line) of Ref. [11] where $z_c$ corresponding to $z_1$ here is taken to be $10^{-9}$cm. The x-axis is $\ln(\frac{r}{z_1})$ and $\frac{z_2}{z_1} \sim 10^{14}$ is assumed.

Figure 5: The solid line is $f(r)$ and between the two dash lines is region where the 4D Newton gravitation is approximately valid. The x-axis is $\ln(\frac{r}{z_1})$ (with $\frac{z_2}{z_1} \sim 10^{14}$).

Figure 6: Plot of $k y_2$ (the upper curve) and $k y_1$ (the lower curve) vs $l$ in the region of $10^{-12} \sim 10^{-1}$cm. The x-axis is $\ln l$. 

8
