The uniqueness of the helicoid in the Lorentz-Minkowski space $\mathbb{R}^3_1$

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Abstract

We prove that the Lorentzian helicoid and Enneper’s surface are the unique properly embedded maximal surfaces bounded by a lightlike regular arc of mirror symmetry.

1 Introduction

The helicoid $\mathcal{H}_0 := \{(x, y, t) \in \mathbb{R}^3 : x \tan(t) = y\}$ was first discovered by Jean Baptiste Meusnier in 1776. After the plane and the catenoid, is the third minimal surface in Euclidean space $\mathbb{R}^3$ to be known. The helicoid is a ruled surface which is also foliated by helices (its name derives from this fact). As shown in Figure 1 it is shaped like the Archimedes’ screw, but extends infinitely in all directions, see Figure 1 (a).

In analogy with minimal surfaces in $\mathbb{R}^3$, a maximal surface in 3-dimensional Lorentz-Minkowski space $\mathbb{R}^3_1 = (\mathbb{R}^3, dx^2 + dy^2 - dt^2)$ is a surface which is spacelike (the induced metric is Riemannian) and whose mean curvature vanishes. Maximal surfaces represent local maxima for the area functional, and besides of their mathematical interest, they have a significant importance in classical Relativity (see [21]). Moreover, their Gauss map is conformal and they admit a Weierstrass type representation (see equation (2)).

The relative complement of the rigid circular cylinder $C = \{(x, y, t) : x^2 + y^2 \leq 1\}$ in $\mathcal{H}_0$ is a spacelike surface when viewed in $\mathbb{R}^3_1$, and consists of two congruent (in the Riemannian and

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Lorentzian sense) simply connected domains of \( \mathcal{H}_0 \) bounded by a lightlike helix. The *Lorentzian helicoid* \( \mathcal{H} \) is defined to be the closure of the connected component of \( \mathcal{H}_0 - C \) containing \((1,0,1)\) in its boundary, see Figure 1 (b).

Amazingly, \( \text{Int}(\mathcal{H}) \) is a maximal surface. As a matter of fact, O. Kobayashi [13] proved that spacelike planes and Lorentzian helicoids are the only maximal surfaces which are also minimal surfaces with respect to the Euclidean metric on the ambient 3-space.

The boundary behavior of \( \mathcal{H} \) focuses on relevant geometrical information of the surface. Indeed, if \( X : \mathcal{M} \equiv \{ \text{Im}(z) \geq 0 \} \rightarrow \mathbb{R}^3_1 \) is a conformal embedding of the helicoid, the limit tangent planes to \( \mathcal{M} \) at points along \( \partial(\mathcal{M}) \) are lightlike and \( \partial(\mathcal{M}) \) is an integral curve of the *weighted gradient* \( \frac{1}{(t \circ X)} \nabla(t \circ X) \), where \( N \) is the Lorentzian Gauss map of \( X \) and \( \nabla \) is computed with respect to the intrinsic metric induced by \( \mathbb{R}^3_1 \) (the correction factor \( \frac{1}{(t \circ X)} \) just controls the singularities of \( \nabla(t \circ X) \) along \( \partial(\mathcal{M}) \)). These conditions are equivalent to the property that the immersion folds back at \( \partial(\mathcal{M}) \), that is to say, \( X \) extends harmonically to the double of \( \mathcal{M} \) being invariant under the mirror involution. Maximal surfaces with regular lightlike boundary and satisfying this symmetry property are said to have *lightlike boundary of mirror symmetry*, and will be written as LBMS maximal surfaces (if in addition the boundary is connected, we will use the acrostic CLBMS).

Another interesting example of properly embedded CLBMS maximal surface in \( \mathbb{R}^3_1 \) is the so called Lorentzian Enneper surface \( E_1 := \{ (x, y, t) : 32(y - t)^3 - 3(y + t) + 24(y - t)x = 0 \} \), see Figure 1 (c). Unlike the Lorentzian helicoid, \( E_1 \) has well defined lightlike tangent plane at infinity. Furthermore, the change of the tangent angle along the orthogonal projection of \( \partial(E_1) \) over \( \{ t = 0 \} \) is finite (and equal to \( 2\pi \)).

Recently, \( \mathcal{H}_0 \) has been characterized by W. H. Meeks III and H. Rosenberg [22] as the unique properly embedded non flat simply connected minimal surface in \( \mathbb{R}^3_1 \). Likewise, J. Perez [19] has proved that half of the Enneper minimal surface is the only properly embedded non flat oriented stable minimal surface bounded by a straight line and having quadratic area growth. Somehow, this paper is devoted to obtain a Lorentzian compilation of both Riemannian theorems.

We have proved the following:

**Theorem:** The Lorentzian helicoid and the Lorentzian Enneper surface are the unique properly embedded CLBMS maximal surfaces in \( \mathbb{R}^3_1 \).

The required theoretical background includes classical Calabi’s theorem [3] (see also Cheng-Yau work [6]) about complete maximal surfaces, and some basic existence and regularity properties of area maximizing surfaces in the Lorentz Minkowski space \( \mathbb{R}^3_1 \), mainly proved by Bartnik and Simon in [2].

In a first step, we obtain some regularity theorems and parabolicity criteria for maximal graphs, and use these results to control the asymptotic behavior of maximal graphs over planer wedges. Among other things, we prove that no homothetical blow down of a such graph converges to an angular region of the light cone.

Taking advantage of this analysis and Calabis’ theorem, we can derive an elementary Colding-Minicozzi theory [4] and prove that any homothetical blow down of a properly embedded CLBMS maximal surface \( S \) is a degenerated planar multigraph with a singular point. This means that any leaf of the blow down sequence converges in the \( C^1 \)-topology outside the singularity to a plane \( \Sigma_\infty \) depending neither on the leaf nor the homothetical blow down, that we call the blow down plane.

Finally, and using W.H. Meeks and Rosenberg ideas in [22], we deduce that the Gauss map of \( S \) omits the normal direction of \( \Sigma_\infty \), and that any plane parallel to \( \Sigma_\infty \) intersects \( S \) into a single arc. This reasoning strategy requires of a finiteness theorem for maximal graphs with planar boundary, whose proof has been deeply inspired by P. Li and J. Wang work [16]. The natural dichotomy between spacelike and lightlike blow down plane leads to \( S = \mathcal{H} \) and \( S = E_1 \), respectively.

The paper has been laid out as follows:
In Section 2 we introduce some terminology and the background material. A detailed description of the basic examples (Helicoid, Enneper’s surface and conjugate surfaces) is given in Section 3. Section 4 is devoted to obtaining some parabolicity criteria for maximal surfaces. In Section 5 we deal with the geometry of maximal graphs, specially those over wedge-shaped regions. We also prove the Li-Wang type finiteness theorem for maximal graphs. The deepest results are contained in Section 6, which has been devoted to the global geometry of properly embedded CLBMS maximal surfaces. We construct the blow down multigraph and prove the transversality of the surface and the blow down limit plane. Finally, in Section 7 we prove the uniqueness theorem.

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2 Notations and Preliminaries

The Riemann sphere, the complex plane, the upper half plane and the unit disc will be denoted by $\mathbb{C}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{D}$, respectively. We label $[-\infty, +\infty]$ as the extended half line $\mathbb{R} \cup \{-\infty, +\infty\}$.

The Euclidean metric and norm in $\mathbb{R}^n$ will be denoted by $\langle , \rangle_0$ and $\| \cdot \|_0$, respectively. The origin in $\mathbb{R}^n$ will be written as $O$. Given $W_1, W_2 \subset \mathbb{R}^n$ we denote by $d(W_1, W_2)$ and $d_H(W_1, W_2) := \max\{\sup\{d(w, W_j) \mid w \in W_1 \cup W_2\} : j = 1, 2\}$ the Euclidean and Hausdorff distance between $W_1$ and $W_2$, respectively.

A smooth divergent arc $\alpha(u) : [0, +\infty[ \to \mathbb{R}^n$ is defined to be sublinear with direction $v$ if $\lim_{u \to +\infty} \alpha'(u) = v$, where $u$ is the arclength parameter of $\alpha$ with respect to $\langle , \rangle_0$.

We call $\mathbb{R}^3_1$ the three dimensional Lorentz-Minkowski space ($\mathbb{R}^3 \equiv \mathbb{R}^2 \times \mathbb{R}, \langle , \rangle$), where as usual $\langle (x_1, t_1), (x_2, t_2) \rangle = (x_1, x_2)_0 - t_1 t_2$, and write $\| (x, t) \|^2 := \| x \|^2 - t^2$. A vector $v \in \mathbb{R}^3 - \{0, 0, 0\}$ is said to be spacelike, timelike or lightlike if $\| v \|^2 > 0$, $\| v \|^2 < 0$ or $\| v \|^2 = 0$, respectively. The vector $(0, 0, 0)$ is spacelike by definition. A smooth curve in $\mathbb{R}^3_1$ is defined to be spacelike, timelike or lightlike if all its tangent vectors are spacelike, timelike or lightlike, respectively. A plane in $\mathbb{R}^3_1$ is spacelike, timelike or lightlike if the induced metric is Riemannian, non degenerate indefinite or degenerate, respectively. The spacelike plane $\{t = 0\}$ will be denoted by $\Pi_0$. We often use the identification $\Pi_0 \equiv \mathbb{R}^2$ given by $(x, 0) \equiv x$. Given a spacelike plane $\Sigma \subset \mathbb{R}^3_1$, $\pi_\Sigma : \mathbb{R}^3_1 \to \Sigma$ will denote the Lorentzian orthogonal projection. If $\Sigma = \Pi_0$ we simply write $\pi$ instead of $\pi_{\Pi_0}$, and in this case

$$\pi((x, t)) := x \quad (x, t) \in \mathbb{R}^3_1.$$

For any $p = (x_0, t_0)$, we denote by $C_p := \{x \in \mathbb{R}^3_1 : \| x - p \|^2 = 0\}$ the light cone with vertex at $p$, and label $C_p^+ := C_p \cap \{t \geq t_0\}$ and $C_p^- := C_p \cap \{t \leq t_0\}$. We also set $\text{Int}(C_p) := \{x \in \mathbb{R}^3_1 : \| x - p \|^2 < 0\}$ and $\text{Ext}(C_p) := \{x \in \mathbb{R}^3_1 : \| x - p \|^2 > 0\}$, and likewise define $\text{Int}(C_p^+) := \text{Int}(C_p) \cap \{t > t_0\}$ and $\text{Int}(C_p^-) := \text{Int}(C_p) \cap \{t < t_0\}$.

Open connected subsets of surfaces are called domains. The closure of a domain is said to be a region. Throughout this paper we will deal with regions and domains $\Omega$ with regular enough boundary.

If $S$ is a manifold and $f : S \to \mathbb{R}$ is a function, the expression $\lim_{x \in S} f(x) = L$ means that $\lim_{n \to +\infty} f(x_n) = L$ for any divergent sequence $\{x_n\}_{n \in \mathbb{N}} \subset S$. Let $R^* := \{(z, w) \in (\mathbb{C} - \{0\}) \times \mathbb{C} : e^w = z\}$ denote the Riemann surface of $\log(z)$ endowed with the Riemannian metric $|dz|^2$. The map $w : R^* \to \mathbb{C}$ is a biholomorphism and $z : R^* \to \mathbb{C}^* := \mathbb{C} - \{0\}$ is the isometric universal covering of the Euclidean once punctured plane. The argument

\[^1\text{If } \Omega \text{ lies in a Riemannian surface } M, \text{ this means that } \partial(\Omega) \text{ is } C^0 \text{ and locally Lipschitzian functions in } \text{Int}(\Omega) \text{ extend continuously to } \partial(\Omega).\]

\[^2\text{If } \{x_n\}_{n \in \mathbb{N}} \subset S \text{ is divergent if no subsequence of } \{x_n\}_{n \in \mathbb{N}} \text{ converges in } S.\]
function is given by \( \arg : \mathbb{R}^* \to \mathbb{R} \), \( \arg = \text{Im}(w) \). For convenience, we add an extra point \([0]\) to \( \mathbb{R}^* \) and endow \( \mathbb{R} := \mathbb{R}^* \cup \{0\} \) with the smallest topology making \( z : \mathbb{R} \to \mathbb{C} \) open and continuous.

A proper subset \( W \subset \mathbb{R} \) homeomorphic to \( \overline{\mathbb{R}} - \{1\} \) is defined to be a (generalized) wedge if for any divergent Jordan arc \( \alpha \subset \partial(W) \), \( \alpha \cong [0, +\infty] \), either \( \theta_{\alpha} := \lim_{x\in\alpha, x\to\infty} \arg(x) = \pm\infty \) or \( z(\alpha) \) is a planar sublinear arc (hence \( \theta_{\alpha} \in \mathbb{R} \)). If \( \partial(W) \) contains two disjoint divergent Jordan arcs \( \alpha_j \), \( j = 1, 2 \) and \( \theta_{\alpha_j} \in \mathbb{R} \), \( j = 1, 2 \), we set \( \theta := |\theta_{\alpha_2} - \theta_{\alpha_1}| \in [0, +\infty[ \) the angle of \( W \). In case \( \partial(W) \) is compact or contains a unique divergent Jordan arc \( W \) is defined to be a wedge of infinite angle.

When \( z|_W : W \to z(W) \) is one to one, \( W \) and \( z(W) \subset \mathbb{C} \equiv \Pi_0 \) will be identified. The wedge \( \arg^{-1}([-\theta, \theta]) \cup \{[0]\} \) will be denoted by \( W_\theta, \theta \in [0, +\infty] \).

In what follows, \( \mathcal{M} \) will denote a differentiable surface, possibly with non empty boundary.

A map \( X : \mathcal{M} \to \mathbb{R}^3 \) is said to be pseudo spacelike if \( X \) is continuous and for any \( p \in \mathcal{M} \) there is an open neighbourhood \( U \) of \( p \) such that \( ||X(p_1) - X(p_2)|| \geq 0 \), for any \( p_1, p_2 \in U \). If in addition \( p \circ X \) is a local embedding\(^3\) then \( X \) is defined to be a pseudo spacelike immersion. If \( X : \mathcal{M} \to \mathbb{R}^3 \) is pseudo spacelike immersion and \( p \circ X \) is one to one, then \( X(\mathcal{M}) \) is said to be a pseudo spacelike graph over \( \pi(X(\mathcal{M})) \subset \Pi_0 \).

In the sequel, we write the acrostic abbreviation PS for pseudo spacelike.

Assume that \( \mathcal{M} \) is simply connected, and consider a PS immersion \( X : \mathcal{M} \to \mathbb{R}^3 \) satisfying \( O \notin \pi(\mathcal{M}) \). Basic topology guarantees the existence of an embedding \( Y : \mathcal{M} \to \mathbb{R}^3 \) satisfying \( \pi \circ Y = \pi \circ X \) (here we have identified the \( z \)-plane and \( \Pi_0 \)). Identifying \( \mathcal{M} \), \( Y = Y(M) \), \( X : W \to \mathbb{R}^3 \) is a PS immersion satisfying \( \pi \circ X = z|_W \). The function \( u = t \circ X : W \to \mathbb{R} \) is locally Lipschitzian with Lipschitz constant \( 1 \), or equivalently, \( \nabla u \) is well defined in the weak sense and \( ||\nabla u||_0 \leq 1 \). If \( W \) is a wedge of angle \( \theta \), the map \( X : W \to \mathbb{R}^3 \) (and the set \( X(\mathcal{M}) = \{(z(x), u(x)) : x \in W \}) \) is said to be a PS multigraph of angle \( \theta \).

Set \( G = \{ (x, u(x)) : x \in \Omega \} \) is a PS graph over a domain \( \Omega \subset \Pi_0 \), and call \( d_\Omega \) the inner metric in \( \Omega \) induced by \( \langle , \rangle \). The PS condition gives \( |u(x) - u(y)| \leq d_\Omega(x, y) \) for all \( x, y \in \Omega \). Thus, if \( \Omega \) is starshaped with center \( x_0 \) we have:

\[ G - \{(x_0, u(x_0))\} \subset \text{Ext}(\mathcal{C}(x_0, u(x_0))). \tag{1} \]

**Lemma 2.1** Let \( G \) be a PS graph over a region \( R \subset \Pi_0 \). The following statements hold:

(a) If \( R \) is convex and \( G \) contains a lightlike straight line \( l \), then \( G \) lies in the lightlike plane \( \Sigma_0 \) containing \( l \).

(b) If \( l_1 = [p_1, p_2] \) and \( l_2 \) are lightlike segments in \( G \) such that \([p_1, p_2] \cap l_2 \neq \emptyset \), then \( l_1 \) and \( l_2 \) lie in the same lightlike straight line.

**Proof:** To check (a), use equation (1) and observe that \( G \subset \cap_{l \subset \mathbb{R}^3} \text{Ext}(\mathcal{C}_l) = \Sigma_0 \).

To prove (b), suppose without loss of generality that \( O \in [p_1, p_2] \cap l_2 \) and consider the dilated graphs \( G_n := n \cdot G \), \( n \in \mathbb{N} \). By Ascoli-Arzela Theorem, the sequence \( \{G_n\}_{n \in \mathbb{N}} \) converges uniformly on compact subsets to an entire PS graph \( G_\infty \) containing the lightlike straight line \( l_0 \) containing \( l_1 \). From (a), \( G_\infty \) is a lightlike plane, hence \( l_2 \) lies in \( l_0 \) too. \( \square \)

A smooth immersion \( X : \mathcal{M} \to \mathbb{R}^3 \) is said to be a spacelike \( ^3 \) and \( X(\mathcal{M}) \) a spacelike surface in \( \mathbb{R}^3 \) if the tangent plane at any point is spacelike, that is to say, if the induced metric \( ds^2 := X^*\langle , \rangle \) on \( \mathcal{M} \) is Riemannian. In this case, the Gauss map \( N \) of \( X \) is well defined and takes values in the Lorentzian sphere \( \mathbb{H}^2 := \{ x \in \mathbb{R}^3_1 : \langle x, x \rangle = -1 \} \). If we attach to \( \mathcal{M} \) the conformal structure induced by \( ds^2 \), \( \mathcal{M} \) becomes a Riemann surface and \( X \) a conformal spacelike immersion. It is easy to see that spacelike immersions are PS immersions.

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\(^3\)By the Domain Invariance Theorem, this simply means that \( p \circ X \) is locally injective.
Let $\mathcal{M}$ be a Riemann surface, and let $S_X$ be a closed subset with empty interior (usually, a family of curves and points). A smooth map $X : \mathcal{M} \to \mathbb{R}^3_1$ is said to be a conformal spacelike immersion with singular set $S_X$ (and $X(\mathcal{M})$ a spacelike surface with singular set $X(S_X)$) if $X^*(g) = \lambda ds^2_0$, where $X^*(g)$ is the pull back metric, $ds^2_0$ is any conformal Riemannian metric on $\mathcal{M}$ and $\lambda$ is a function which is positive on $\mathcal{M} - S_X$ and vanishes on $S_X$. A singular point $p \in S_X$ is said to be a lightlike singularity of $X$ if $\lim_{t \to 0} g(p, X_t) = \infty$, where $\mathcal{N} : \mathcal{M} - S_X \to \mathbb{H}^2$ is the Lorentzian Gauss map of $X|_{\mathcal{M} - S_X}$. In addition $dX_p \neq 0$, $p$ is said to be a regular lightlike singularity. See the papers \cite{23, 9} for a good setting about singularities.

A spacelike immersion is said to be maximal (and $X(\mathcal{M})$ a maximal surface) if its mean curvature vanishes. A conformal maximal immersion $X : \mathcal{M} \to \mathbb{R}^3_1$ has harmonic coordinate functions and admits a Weierstrass type representation $(g, \phi_3)$:

$$X = \text{Real} \int (\phi_1, \phi_2, i\phi_3),$$

where $\phi_1 = \frac{1}{2}(1/g - g)\phi_3$, $\phi_2 = \frac{1}{2}(1/g + g)\phi_3$ and $\phi_3$ are holomorphic 1-forms without common zeroes in $\mathcal{M}$. Recall that $g = st \circ \mathcal{N}$, where $\mathcal{N} : \mathcal{M} \to \mathbb{H}^2$ is the Gauss map of $X$ and $st : \mathbb{H}^2 \to \mathbb{C}$ is the Lorentzian stereographic projection given by $st : \mathbb{H}^2 \to \mathbb{D}$, $st((x_1, x_2, t)) = (\frac{x_1}{x_3}, \frac{x_2}{x_3})$. We also set $st_0 : \mathbb{H}^2 - \{(0, 0, 0)\} \to (z \in \mathbb{C} : |z| = 1)$, $st_0(v_1, v_2, v_3) := \frac{1}{|v_3|}(v_2, -v_1)$. Note that $st_0(v) = \lim_{n \to \infty} st(w_n)$ for all $\{w_n\}_{n \in \mathbb{N}} \subset \mathbb{H}^2$ satisfying $\{\frac{w_n}{|w_n|}\}_{n \in \mathbb{N}} \to \frac{v}{|v|}$.

For more details about the Weierstrass representation of maximal surfaces see \cite{15}.

**Definition 2.1** A map $X : \mathcal{M} \to \mathbb{R}^3_1$ is defined to be a conformal maximal immersion with singular set $S_X$ if $X$ is a conformal spacelike immersion with singular set $S_X$ and $X|_{\mathcal{M} - S_X}$ is maximal. We also say that $X(\mathcal{M})$ is a maximal surface with singular set $X(S_X)$.

A conformal maximal immersion $X : \mathcal{M} \to \mathbb{R}^3_1$ with singularities has well defined Weierstrass data $(g, \phi_3)$, and since the intrinsic metric is given by $ds^2 = \frac{1}{2}(1/|g| - |g|)^2 |\phi_3|^2$, $S_X$ coincides with the analytical set $|g|^{-1}(1) \cup |\phi_3|^{-1}(0)$. If in addition all the singular points are regular and lightlike, then $\phi_3$ never vanishes and $S_X = |g|^{-1}(1)$. If $A$ is a connected component of $\mathcal{M} - S_X$, its Gaussian image $\mathcal{N}(A)$ lies in either $\mathbb{H}^2_+ := \mathbb{H}^2 \cap \{t > 0\}$ or $\mathbb{H}^2_+ := \mathbb{H}^2 \cap \{t < 0\}$. Choosing the orientation of $A$ in such a way that $\mathcal{N}(A) \subset \mathbb{H}^2_+$, we have $(|g||A| < 1$, where $g$ is the holomorphic Gauss map of $X|_A$.

Let $\mathcal{M}$ be a Riemann surface with analytical boundary. The mirror and double surface of $\mathcal{M}$ with respect to $\partial(\mathcal{M})$ will be denoted by $\mathcal{M}^*$ and $\mathcal{M} := \mathcal{M} \cup \mathcal{M}^*$, respectively. Recall that, up to natural identifications, $\partial(\mathcal{M}) = \partial(\mathcal{M}^*) = \mathcal{M} \cap \mathcal{M}^* = \{p \in \mathcal{M} : J(p) = p\}$, where $J : \mathcal{M} \to \mathcal{M}$ is the antiholomorphic involution mapping each point $p \in \mathcal{M}$ into its mirror image $p^* \in \mathcal{M}^*$.

**Definition 2.2** Let $\tilde{X} : \tilde{\mathcal{M}} \to \mathbb{R}^3_1$ be a conformal maximal immersion with regular lightlike singularities. The map $X := \tilde{X}|_{\tilde{\mathcal{M}}}$ is said to be a conformal maximal immersion with lightlike boundary of mirror symmetry (or simply, a LBMS conformal maximal immersion) if $S_{\tilde{X}} = \partial(\mathcal{M})$ and $\tilde{X} \circ J = \tilde{X}$. We also say that $X(\mathcal{M}) = \tilde{X}(\tilde{\mathcal{M}})$ is a LBMS maximal surface. We write the acrostic CLBMS to indicate that in addition $\partial(\mathcal{M})$ is connected. In terms of the Weierstrass data $(g, \phi_3)$ of $\tilde{X}$, this notion is equivalent to say that:

$$\overline{g}(g \circ J) = 1, \quad J^*(\phi_3) = -\overline{\phi_3}. \quad (3)$$

If $X$ is a proper embedding, $\mathcal{M}$ and $X(\mathcal{M})$ will be identified via $X$, and $\mathcal{M} \subset \mathbb{R}^3_1$ is said to be a properly embedded LBMS maximal surface.
It is not hard to see that $X$ is a LBMS conformal maximal surface if and only if $S_X = \partial (M)$ and $\partial (M)$ consists of integral curves of $\frac{1}{(\eta, w)^2} \nabla \langle \hat{X}, w \rangle$, where $w$ is any timelike vector and $\nabla$ is the gradient computed with respect to the intrinsic metric\footnote{Despite the degeneracy of $ds^2$ at $S_X$, the weighted gradient $\frac{1}{(N, w)^2} \nabla \langle \hat{X}, w \rangle$ extends analytically to this set.}. Indeed, assume up to a Lorentzian isometry that $w = (0, 0, 1)$, and write $\phi_3 = -if(z)dz$. Then, suppose that $S_X = \partial (M)$ and take a conformal disc $(U, z = u + iv)$ in $\hat{\mathcal{M}}$ centered at $p \in \partial (M)$ and satisfying $J(U) = U$, $z \circ J = \overline{\mathcal{T}}$. A standard computation gives $\lambda^2 \nabla(t \circ \hat{X}) = \text{Re}(f)^2 \frac{\partial \hat{X}}{\partial u} - \text{Im}(f)^2 \frac{\partial \hat{X}}{\partial v}$, where $\lambda = \| \frac{\partial \hat{X}}{\partial u} \| = \| \frac{\partial \hat{X}}{\partial v} \| = \frac{1}{2}(1/|g| - |g|)|f|$. Therefore, $\partial (M) \cap U$ is an integral curve of $\lambda^2 \nabla(t \circ \hat{X})$ if and only if $\text{Im}(f) = 0$ (that is to say, $X$ is a LBMS maximal surface). Taking into account that $(t \circ N)^2 \lambda^2$ is well defined and positive on $U$, we are done.

We will need the following basic lemma.

\textbf{Lemma 2.2} Let $X : \mathcal{M} \to \mathbb{R}^3_1$ be a conformal proper LBMS maximal immersion, and label $(g, \phi_3)$ as its Weierstrass data.

Then $dg$ and $\phi_3$ never vanish along $\partial (M)$ and $\pi \circ X : \mathcal{M} \to \Pi_0$ is a local embedding.

\textbf{Proof}: From (3), $g$ and $\phi_3$ extend by Schwarz reflection to the double surface $\hat{\mathcal{M}}$. Since $\partial (M) = [g]^{-1}(1)$ and $\partial (M)$ consists of a family of pairwise disjoint proper regular analytical curves in $\hat{\mathcal{M}}$, the harmonic function $\log(|g|)$ has no singular points on $\partial (M)$ and $dg(p) \neq 0$ for all $p \in \partial (M)$.

On the other hand, $S_X$ consists of regular lightlike singularities, hence $dX \neq 0$ on $\partial (M)$ and equation (2) gives $\phi_3(p) \neq 0$ for all $p \in \partial (M)$.

Let us show that $\pi \circ X$ is a local embedding. Since $X|_{\mathcal{M} - \partial (M)}$ is spacelike, $(\pi \circ X)|_{\mathcal{M} - \partial (M)}$ is a local diffeomorphism, and so it suffices to deal with boundary points. Fix $p \in \partial (M)$, and up to a Lorentzian isometry, suppose $g(p) = 1$ and $\hat{X}(p) = 0$. Then take a conformal disc $(D, z)$ in $\mathcal{M}$ satisfying $z(D) = D, z(p) = 0, J(z) = \overline{\mathcal{T}}, z(D \cap \mathcal{M}) = D^\perp := D \cap \{ z \in \mathbb{C} : \text{Im}(z) \geq 0 \}$ and $\phi_3(z) = dz$. From equation (2), $\phi_3(z) = \text{zh}(z)dz$, where $h : D \to \mathbb{C}$ is holomorphic, $h(p) \neq 0$ and $h \circ J = \overline{h}$. In the sequel we call $D = D^\perp$ and call $D_{\epsilon} = \{ z \in \mathbb{C} : |z| < \epsilon \}$. By the Domain Invariance Theorem, it suffices to show that $(\pi \circ X)|_{D_{\epsilon}}$ is injective provided that $\epsilon > 0$ is small enough. Reason by contradiction and take sequences $\{ z_n \}_{n \in \mathbb{N}}, \{ w_n \}_{n \in \mathbb{N}}$, in $D^+$ converging to 0 satisfying $z_n \neq w_n, \text{Re}(z_n) = \text{Re}(w_n)$, and $\text{Re}(\int_{z_n}^{w_n} \text{zh}(z)) = 0$. Therefore we can find $\xi_n$ in the vertical segment $|z_n, w_n|$ such that $\text{Im}(\xi_n h(\xi_n)) = 0, n \in \mathbb{N}$. This contradicts that $\{ z \in D_{\epsilon} : \text{Im}(zh(z)) = 0 \} \subset \mathbb{R}$ provided $\epsilon$ is small enough and concludes the proof.\hfill \Box

The main global result about maximal surfaces was proved by Calabi\footnote{See also [6] for further generalizations.} (see also [6] for further generalizations). It asserts the following:

\textbf{Theorem 2.1 (Calabi)} Let $X : \mathcal{M} \to \mathbb{R}^3_1$ be a complete maximal immersion. Then $X(\mathcal{M})$ is a spacelike plane. The same result holds if we replace complete for proper.

\section{Basic examples}

Let $\hat{X} : \mathbb{C} \to \mathbb{R}^3_1$ be the conformal maximal immersion with regular lightlike singularities associated to the Weierstrass data $g(z) = e^{iz}, \phi_3(z) = -idz$. It is clear that $X(u, v) = (\cosh(v) \cos(u), \cosh(v) \sin(u), u)$, where $z = u + iv$. Since $\hat{X}(\overline{\mathcal{T}}) = \hat{X}(z)$ and $X = \hat{X}|_{\overline{\mathcal{T}}}$ is a proper embedding, then $\mathcal{H} := X(\overline{\mathcal{T}})$ is a properly embedded CLBMS maximal surface which has been named as the \textit{Lorentzian helicoid}, see Figure 1(b).
The conjugate surface is the universal converging of the Lorentzian catenoid with Weierstrass data \( \mathcal{M} = \{0\} \), \( g(z) = z \), and \( \phi_{3}(z) = \frac{i(2m)}{z} \). The associated immersion is

\[
X : \mathbb{C} - \{0\} \rightarrow \mathbb{R}^{3}, \quad X(m, s) = \left(\frac{1-m^2}{2m} \sin(s), \frac{m^2-1}{2m} \cos(s), \log(m)\right),
\]

where \( z = me^{i\alpha} \), and \( S_X \) is the unit circle \( \{ |z| = 1 \} \). In this case, \( S_X \) consists of regular lightlike singularities, \( X(S_X) \) is a single point, \( X(1/\tau) = -X(z) \) and \( C := X(\mathbb{U} - \{0\}) \) is an entire graph over the plane \( \Pi_0 \). Elementary characterizations of the Lorentzian catenoid can be found in [15], [4] and [5].

Consider now the data \( \mathcal{M} = \mathbb{C} \), \( g(z) = (z-i)/(z+i) \) and \( \phi_{3}(z) = i(z^2+1)dz \). The corresponding maximal immersion \( \hat{X} : \mathbb{C} \rightarrow \mathbb{R}^{3} \) is given by:

\[
\hat{X}((m, s)) = \left(-m^2 \cos(2s), \frac{1}{3}(3m \cos(s) - m^3 \cos(3s)), -\frac{1}{3}m(3 \cos(s) + m^2 \cos(3s))\right),
\]

where \( z = me^{i\alpha} \). Since \( \hat{X}(\mathbb{U}) = \hat{X}(z) \) and \( X = \hat{X}|_{\mathbb{U}} \) is a proper embedding, then \( E_1 := X(\mathbb{U}) \) is a properly embedded CLBMS maximal surface, that we call the first Enneper’s maximal surface, see Figure 1(c). \( E_1 \) contains a half line parallel to the \( x_1 \)-axis and is invariant under the reflection about this line.

The conjugate surface \( E_1^* \) is called the second Enneper’s maximal surface. Its Weierstrass data are \( \mathcal{M} = \mathbb{C} \), \( g(z) = (z-i)/(z+i) \), \( \phi_{3}(z) = -(z^2+1)dz \), and the associated immersion \( X : \mathbb{U} \rightarrow \mathbb{R}^{3} \) is given by

\[
X(m, s) = \left(m^2 \sin(2s), \frac{1}{3}(-3m \sin(s) + m^3 \sin(3s)), \frac{1}{3}(3m \sin(s) + m^3 \sin(3s))\right),
\]

where \( z = me^{i\alpha} \). In this case, \( S_X \) is the real axis and \( X(S_X) \) is the origin. Furthermore, \( X \) is not proper. Indeed, \( E_2 = X(\mathbb{U}) \) is an entire graph over \( \Pi_0 \) and \( E_2 - X(\mathbb{U}) \) consists of open lightlike half line \( \{(x_1, x_2, t) \in \mathbb{R}^{3} : x_1 = x_2 - t = 0, t > 0\} \).

![Figure 2: (a) The Lorentzian catenoid; (b) Enneper’s graph \( E_2 \)](image)

**Remark 3.1** The Lorentzian half catenoid \( C \) and Enneper’s surface \( E_1^* \) satisfy the implicit equations \( x_1^2 + x_2^4 - \sinh^2(t) = 0 \) and \( 3(x_2 - t)^4 - \frac{1}{4}(x_2 - t)^3 + 3x_1^2 + 6(x_2 - t)t = 0 \), respectively. Therefore, any blow up of these surfaces with center the origin converges in the \( C^0 \)-topology to the lightcone.

Ecker [4] proved that the Lorentzian catenoid is the unique entire maximal graph with one singular point. A similar result for \( E_2 \) can be found in Section 5 (Proposition 5.3).
4 Parabolicity of maximal surfaces in \( \mathbb{R}^3_1 \)

This section is devoted to proving some parabolicity criteria for properly immersed maximal surfaces in \( \mathbb{R}^3_1 \).

A non compact Riemann surface \( \mathcal{M} \) with non empty boundary is said to be parabolic if the only bounded harmonic function \( f \) vanishing on \( \partial(\mathcal{M}) \) is the constant function \( f = 0 \), or equivalently, if there exists a proper positive superharmonic function on \( \mathcal{M} \). Otherwise, \( \mathcal{M} \) is said to be hyperbolic.

If \( \partial(\mathcal{M}) = \emptyset \), parabolicity means that positive superharmonic functions are constant.

For instance, \( \overline{U} \) is parabolic, whereas \( \mathbb{D} \cap \overline{U} \) is hyperbolic.

Let \( g : \overline{U} \to \mathbb{C} \) be continuous on \( \overline{U} \) and holomorphic on \( U \). A divergent curve \( \alpha \subset \overline{U} \) is defined to be an asymptotic curve of \( g \) if the limit \( a := \lim_{z \to \alpha} g(z) \in \mathbb{C} \) exists. In this case, \( a \) is said to be an asymptotic value of \( g \). The following theorem summarizes some well known results on classical complex analysis.

**Theorem 4.1** Set \( g : \overline{U} \to \mathbb{C} \) continuous, holomorphic on \( U \) and omitting two finite complex values. Then:

(I) \( g \) has at most one asymptotic value, and in this case \( g|_{\overline{U}} \) has angular limits at \( \infty \).

(II) If \( [0, +\infty[ \subset \partial(U) \) and \( ]-\infty, 0[ \subset \partial(U) \) are asymptotic curves of \( g \), then the limit \( \lim_{z \to \infty} g(z) \) exists.

Given a Riemann surface with boundary \( \mathcal{M} \) and \( p \in \mathcal{M} - \partial(\mathcal{M}) \), we denote by \( \mu_p \) the harmonic measure respect to the \( p \). It is well known that \( \mathcal{M} \) is parabolic if and only if there exists \( p_0 \in \mathcal{M} - \partial(\mathcal{M}) \) such that \( \mu_{p_0} \) is full, i.e., \( \mu_{p_0}(\partial(\mathcal{M})) = 1 \). In this case \( \mu_p \) is full for any \( p \in \mathcal{M} - \partial(\mathcal{M}) \), and bounded harmonic (superharmonic) functions \( u \) on \( \mathcal{M} \) satisfy the mean property

\[
\mu \left( \int_{x \in \partial \mathcal{M}} u(x) \, d\mu_p \right) \quad \text{for any } p \in \mathcal{M}.
\]

Regions of parabolic Riemann surfaces are parabolic. Moreover, if \( \mathcal{M} \) is the union of two parabolic regions with compact intersection, then \( \mathcal{M} \) is parabolic too. See [12], [1], [7], [17] and [18] for more details. The proof of the following theorem has bee inspired by some ideas in [7].

**Theorem 4.2** Let \( X : \mathcal{M} \to \mathbb{R}^3_1 \) be a conformal proper maximal immersion with singularities, where \( \partial(\mathcal{M}) \neq \emptyset \), and suppose that there exists \( \varepsilon > 0 \) and a compact subset \( C \subset \mathcal{M} \) such that

\[
\langle X, X \rangle \geq \varepsilon \quad \text{on } \mathcal{M} - C,
\]

Then \( \mathcal{M} \) is parabolic.

**Proof:** Since parabolicity is not affected by adding compact subsets, we can suppose that \( \|X(p)\|^2 \geq \varepsilon \) on \( \mathcal{M} \).

For any \( n \in \mathbb{N} \) let \( \mathcal{M}_n := \{ p \in \mathcal{M} : \langle X, X \rangle(p) \leq n \} \). Let us see that \( \mathcal{M}_n \) is parabolic. Indeed, since \( t \circ X \) is a proper positive harmonic function on \( \mathcal{M}_n^+ = (t \circ X)^{-1}([0, +\infty[) \), \( \mathcal{M}_n^+ \) is parabolic, and likewise \( \mathcal{M}_n^- = (t \circ X)^{-1}([0, -\infty[) \) is parabolic. As \( \mathcal{M}_n^+ \cap \mathcal{M}_n^- \) is compact and \( \mathcal{M}_n = \mathcal{M}_n^+ \cup \mathcal{M}_n^- \), then we are done.

Now define \( h : \mathcal{M} \to \mathbb{R}, h(p) = \log \langle X, X \rangle(p) \). Since \( X \) is maximal, a direct computation gives that \( \Delta h = -\frac{\langle X, X \rangle^2}{\langle X, X \rangle} \leq 0 \), where \( \Delta \) is the intrinsic Laplacian and \( \mathcal{N} \) is the Lorentzian Gauss map of \( X \). Therefore \( h \) is superharmonic.
Without loss of generality, suppose there exists \( p \in M \) with \( h(p) > 0 \) (otherwise \( M = \mathcal{M}_1 \) and we have finished). Up to rescaling assume that \( h(p) = 1 \). Since \( h \) is a bounded superharmonic function on the parabolic surface \( \mathcal{M}_n \), we have

\[
1 = h(p) \geq \int_{\partial(\mathcal{M}_n)} h \, d\mu_p(n) = \int_{\partial(\mathcal{M}) \cap \mathcal{M}_n} h \, d\mu_p(n) + \int_{\{q \in \mathcal{M} : h(q) = \log(n)\}} \log(n) \, d\mu_p(n)
\]

where \( \mu_p(n) \) denotes the harmonic measure in \( \mathcal{M}_n \) respect to \( p \). Therefore

\[
1 \geq \log(\varepsilon) \int_{\partial(\mathcal{M}) \cap \mathcal{M}_n} d\mu_p(n) + \log(n) \int_{\{q \in \mathcal{M} : h(q) = \log(n)\}} d\mu_p(n).
\]

As \( 0 \leq \int_{\partial(\mathcal{M}) \cap \mathcal{M}_n} d\mu_p(n) \leq 1 \), then dividing by \( \log(n) \) and taking the limit as \( n \) goes to infinity we get

\[
\lim_{n \to \infty} \int_{\{q \in \mathcal{M} : h(q) = n\}} d\mu_p(n) \leq 0,
\]

and we have finished. Up to rescaling assume that

\[
\int_{\partial(\mathcal{M}) \cap \mathcal{M}_n} d\mu_p(n) = 0 \quad \text{and reasoning as in the preceding proof we obtain the desired conclusion.}
\]

**Corollary 4.1** Let \( X : \mathcal{M} \to \mathbb{R}^3 \) be a conformal proper maximal immersion with singularities, where \( \partial(\mathcal{M}) \neq \emptyset \), and suppose that \( X(\mathcal{M}) \) lies in a spacelike half plane.

Then \( \mathcal{M} \) is parabolic.

**Proof:** Up to scaling and Lorentzian isometry, suppose \( X(\mathcal{M}) \subset \{ t \geq 0 \} \).

From Theorem 4.2, \( \mathcal{M}_n := \{ p \in \mathcal{M} : (t \circ X)(p) \leq n \} \) is parabolic, \( n \in \mathbb{N} \). Defining now \( h = t \circ X \) and reasoning as in the preceding proof we obtain the desired conclusion. \( \square \)

## 5 Some results on maximal graphs

The space of continuous functions \( u \) on a domain \( \Omega \subset \Pi_0 \) with weak gradient satisfying \( \| \nabla u \|_0 \leq 1 \) will be denoted by \( \mathcal{C}^1_0(\Omega) \). We endow \( \mathcal{C}^1_0(\Omega) \) with the \( \mathcal{C}^0 \)-topology of the uniform convergence on compact subsets of \( \Omega \). Likewise, and for any \( k \in \mathbb{N} \cup \{ \infty \} \), \( \mathcal{C}^k_0(\Omega) \subset \mathcal{C}^1_0(\Omega) \) will denote the space of functions with continuous partial derivatives of order \( < k + 1 \), endowed with the \( \mathcal{C}^k \)-topology of the uniform convergence of \( u \) and its partial derivatives of order \( < k + 1 \) on compact subsets.

A sequence of PS graphs \( \{(x, u_n(x)) : x \in \Omega\} \), \( n \in \mathbb{N} \), is said to be convergent in the \( \mathcal{C}^k \)-topology to \( \{(x, u(x)) : x \in \Omega\} \) if \( \{u_n\}_{n \in \mathbb{N}} \to u \) in the \( \mathcal{C}^k \)-topology, \( k \in \mathbb{N} \cup \{ \infty \} \).

**Remark 5.1** If \( \Omega \) is bounded, \( \mathcal{C}^0_1(\Omega) \) lies in the Sobolev space \( \mathcal{W}^{1,2}(\Omega) \) of \( L^2 \) functions with \( L^2 \) gradient, and convergence in \( \mathcal{C}^0_1(\Omega) \) implies convergence in \( \mathcal{W}^{1,2}(\Omega) \). Therefore, Ascoli-Arzela theorem guarantees that sequences in \( \mathcal{C}^0_1(\Omega) \) admit subsequences converging in both \( \mathcal{C}^1_0(\Omega) \) and \( \mathcal{W}^{1,2}(\Omega) \).

Let \( u \in \mathcal{C}^\infty_1(\Omega) \). The associated graph \( G = \{(x, u(x)) : x \in \Omega\} \) defines a maximal surface if and only if:

\[
|\nabla u| < 1 \quad \text{and} \quad \text{div} \left( \frac{\nabla u}{\sqrt{1 - \|\nabla u\|_0^2}} \right) = 0.
\]

(4)

The conjugate function \( u^* \) is characterized by the identity

\[
\sqrt{1 - \|\nabla u\|_0^2} \, du^* = \frac{\partial u}{\partial x_2} \, dx_1 - \frac{\partial u}{\partial x_1} \, dx_2,
\]

(5)
besides an initial condition. It is well defined only if \( \frac{\partial \phi_1}{\partial x_1} - \frac{\partial \phi_2}{\partial x_2} \) is an exact 1-form (for instance, if \( \Omega \) is simply connected), and satisfies the minimal surface equation \( \text{div} (\nabla u^* / (\sqrt{1 + \|\nabla u^*\|^2})) = 0 \). Thus, \( G^* = \{(x, u^*(x)) : x \in \Omega\} \) is a minimal surface in \( \mathbb{R}^3 \). In terms of the Weierstrass representation, the conformal maximal and minimal immersions associated to \( G \) and \( G^* \) are given by

\[
X = \text{Real} \int (\phi_1, \phi_2, i\phi_3) \quad \text{and} \quad X^* = \text{Real} \int (\phi_1, \phi_2, \phi_3),
\]

respectively.

The following Lorentzian version of classical Plateau’s problem can be found in [2].

**Theorem 5.1** Let \( \gamma \subset \mathbb{R}^3 \) be a Jordan curve bounding a PS embedded surface. Then there exists a PS area maximizing disc \( S \) in \( \mathbb{R}^3 \) bounded by \( \gamma \). Furthermore, \( S \) is smooth (hence a maximal surface) except possibly on piecewise linear lightlike arcs connecting points of \( \gamma \).

This result applies to the case when \( \pi(\gamma) \) is a Jordan curve and \( |t(p) - t(q)| \leq d_\Omega(\pi(p), \pi(q)) \) for any \( p, q \in \gamma \), where \( \Omega \) is the domain bounded by \( \pi(\gamma) \) and \( d_\Omega \) is the inner distance in \( \Omega \) (see [13]). If in addition \( |t(p) - t(q)| < d_\Omega(\pi(p), \pi(q)) \) for any \( p, q \in \gamma \), \( S \) is in fact a maximal graph.

Let \( \Omega \subset \Pi_1 \) be a bounded domain and consider a sequence \( \{u_n\}_{n \in \mathbb{N}} \subset C^\infty(\Omega) \) of functions satisfying equation (\( \Omega \)). Assume that \( \{\Omega_n\}_{n \in \mathbb{N}} \to \Omega \) in the \( C^0 \) topology, where \( u \in C^\infty(\Omega) \) (see Remark 5.1). Write \( A = \{(x, y) \in \partial(\Omega)^2 : x, y \subset \Omega \} \) and \( |u(x) - u(y)| = \|x - y\|_0 \) and set \( A := U(x, y) \subset \Omega \) the singular set of \( u \). We also say that \( (x, y) \) is a singular segment of \( u \), for any \( (x, y) \in A \).

The following theorem summarizes some known results:

**Theorem 5.2** ([2]) The function \( u \) defines an area minimizing PS graph. Moreover,

(A) \( u|_{\Omega - A} \) satisfies \( (4) \) and \( \{u_n|_{\Omega - A}\}_{n \in \mathbb{N}} \to u|_{\Omega - A} \) in the \( C^\infty \)-topology,

(B) \( x_0 \in A \) if and only if there exists \( \{x_n\}_{n \in \mathbb{N}} \to x_0 \) such that \( \|\nabla u_n(x_n)\|_0 \to 1 \), and

(C) for any \( (x, y) \in A \), \( \{(z, u(z)) : z \in [x, y]\} \) is a lightlike segment.

**Remark 5.2** From Lemma 2.2, (b), different points of \( A \) determine disjoint singular segments of \( u \), hence \( A \) is a closed subset of \( \Omega \) foliated by singular segments of \( u \).

If \( \Omega \) is unbounded and \( \{\Omega_n\}_{n \in \mathbb{N}} \) is an exhaustion of \( \Omega \) by bounded domains, we set \( A := \cup_{n \in \mathbb{N}} A_n \), where \( A_n \) is the singular set of \( u|_{\Omega_n} \), \( n \in \mathbb{N} \). Since \( A_n \subset A_{n+1} \) for any \( n \), \( A \) is foliated too by inextensible segments in \( \Omega \), but in this case some of them could have infinite length.

Given \( (x, y) \in A \), we call \( \Sigma_{(x, y)} \) as the unique lightlike plane containing \( (x, u(x)) \) and \( (y, u(y)) \), and set \( \sigma_{(x, y)} \in \mathbb{R}^2 \equiv \Pi_0 \) the unitary vector for which \( \Sigma_{(x, y)} = \{(z, \langle z, \sigma_{(x, y)}\rangle_0) : z \in \Pi_0\} \). Since \( u \in W^{1,2}(\Omega) \), \( \nabla u \) is well defined almost everywhere on \( \Omega \) (that is to say, on a subset \( \Omega_0 \subset \Omega \) having the same Lebesgue measure as \( \Omega \)). Furthermore, item (A) in Theorem 5.2 implies that \( \Omega - A \subset \Omega_0 \), whereas item (C) gives that and \( \nabla u(z) = \sigma_{(x, y)} \), for any \( z \in [x, y] \cap \Omega_0 \) and \( (x, y) \in A \). Therefore, it is natural to extend \( \nabla u \) to \( \Omega \) by setting \( \nabla u(z) = \sigma_{(x, y)} \), for any \( z \in [x, y] \) and \( (x, y) \in A \). In particular, \( A = \|\nabla u\|_0^{-1}(1) \).

**Remark 5.3** Do not confuse the singular set \( A \) of the area maximizing PS graph \( u \) with the singular set \( S_X \) of a conformal maximal immersion \( X : M \to \mathbb{R}^3 \). The set \( S_X \) lies in the conformal support of \( X \) and has vanishing measure, whereas \( A \) is a closed subset in the domain of \( u \) possibly with non zero Lebesgue measure (for instance, blowing up the catenoid we obtain the lightcone).

\(^5\text{Recall that } u \text{ is locally Lipschitzian, hence extends continuously to } \Omega_0.\)
Proposition 5.1 The function \( u \) belongs to \( C_1^1(\Omega) \) and \( \{u_n\}_{n \in \mathbb{N}} \to u \) in the \( C^1 \)-topology.

Proof: Take a sequence \( \{x_n\}_{n \in \mathbb{N}} \subset \Omega \) converging to \( x_0 \in \Omega \) and such that \( \sigma := \lim_{n \to +\infty} \nabla u_n(x_n) \) exists.

Claim 1: If \( \sigma \) is spacelike then \( x_0 \in \Omega - A \) and \( \sigma = \nabla u(x_0) \).

Proof: Let \( D_0 \subset \Omega \) be a closed disc of positive radius centered in \( x_0 \). Take \( \epsilon \in \|\sigma\|_0, 1 \) and without loss of generality suppose \( \|\nabla u_n(x_n)\|_0 < \epsilon \) for all \( n \in \mathbb{N} \). Label \( u_n^* \) as the conjugate function of \( u_n|_{D_0} \) satisfying \( u_n^*(x_n) = 0 \) (well defined because \( D_0 \) is simply connected, see equation \( [5] \), and denote by \( S_n := \{(x, u_n^*(x)) : x \in D_0\} \) the associated minimal graph, \( n \in \mathbb{N} \). Standard curvature estimates for minimal graphs give that \( |K_n| \leq C_1 \) on \( D_0 \) for any \( n \in \mathbb{N} \), where \( K_n \) is the Gaussian curvature of \( S_n \) and \( C_1 \) is a constant depending only on \( d(D_0, \partial(\Omega)) > 0 \). From our hypothesis, \( \|\nabla u_n^*\|_0(x_n) < \frac{1}{\sqrt{1+\epsilon^2}} \), and taking \( \delta > \frac{1}{\sqrt{1+\epsilon^2}} \), the Uniform Graph Lemma for minimal surfaces [20] implies the existence of a smaller disc \( D \subset D_0 \) centered at \( x_0 \) such that \( \|\nabla u_n^*\|_0 < \delta \) on \( D \), for any \( n \in \mathbb{N} \). Thus, \( \|\nabla u_n\|_0 < \delta \sqrt{1+\epsilon^2} < 1 \) on \( D \) for all \( k \in \mathbb{N} \), and Barnik-Simon results [2] give that \( \{u_n\}_{k \in \mathbb{N}} \to u|_D \) in the \( C^\infty \)-topology and \( u|_D \) satisfies the maximal surface equation, (that is to say, \( D \subset \Omega - A \)). In particular, \( \sigma = \nabla u(x_0) \) and we are done.

Claim 2: If \( \sigma \) is lightlike, then \( x_0 \in A \) and \( \sigma = \nabla u(x_0) \).

Proof: It is clear that \( x_0 \in A \) (see Theorem \ref{thm5.2} (B)). Consider \( \{\mu_n\}_{n \in \mathbb{N}} \to 0, \mu_n > 0 \), and define \( \Omega_n := \frac{1}{\mu_n} \cdot (\Omega - x_n) \) and \( v_n : \Omega_n \to \mathbb{R}, v_n(y) := \frac{1}{\mu_n} (\mu_n y + x_n) - u_n(x_n) \), \( n \in \mathbb{N} \).

Let us show that \( \{v_n\}_{n \in \mathbb{N}} \) converges, up to subsequences, in the \( C^0 \)-topology to \( v : \Pi_0 \to \mathbb{R}, v(y) = (y, \langle \sigma, y \rangle_0) \).

Since \( v_n \) lie in \( C_0^1(\Omega_n) \) and vanish at the origin, \( n \in \mathbb{N} \), Remark \ref{rem5.1} gives that, up to subsequences, \( \{v_n\}_{n \in \mathbb{N}} \to v_0 \) in the \( C^1 \)-topology, where \( v_0 \in C^1_0(\Pi_0) \). Call \( G \) as the entire graph defined by \( v_0 \), and for any bounded domain \( \Omega' \subset \Pi_0 \) label \( A_{\Omega'} \) as the singular set of \( v|_{\Omega'} \). If \( A_{\Omega'} = \emptyset \) for any \( \Omega' \), Calabi’s theorem would imply that \( v_0 \) is a linear map defining a spacelike plane, and Theorem \ref{thm5.2} gives that \( \{v_n\}_{n \in \mathbb{N}} \to v_0 \) in the \( C^\infty \)-topology. In particular, \( \sigma = \lim_{n \to +\infty} \nabla v_n(0) \) would be spacelike, a contradiction. Therefore \( G \) contains a lightlike straight line, and from Lemma \ref{lem2.1} it must be a lightlike plane. Notice that \( G \) is foliated by singular straight lines of \( v \), hence from Claim 1 we can infer that \( \{|\nabla v_n|_0\}_{n \in \mathbb{N}} \to 1 \) in the \( C^0 \)-topology.

In the sequel we will assume that \( \Omega \) is simply connected (otherwise, replace \( \Omega \) for a small enough disc centered at \( x_0 \)). Let \( v_n^* : \Omega_n \to \mathbb{R} \) denote the conjugate function of \( v_n \) with initial condition \( v_n^*(0) = 0 \), and label \( S_n \) as its associated minimal graph. Let \( \Pi_n \) denote the tangent plane of \( S_n \) at \( 0 \), i.e., the plane passing through \( 0 \) and orthogonal in the Euclidean sense to the vector \( \sqrt{1 - \|\nabla v_n\|_0^2} (\nabla v_n^*, 1) (0) = (\nabla u_n^*, \sqrt{1 - \|\nabla u_n\|_0^2}) (x_n) \), where \( \nabla u_n^*(x_n) = (-\frac{\mu_n}{\partial y}, \frac{\mu_n}{\partial x})(x_n) \).

The limit plane \( \Sigma = \lim_{n \to +\infty} \Pi_n \) is orthogonal to \( (\sigma^*, 0) \), where \( \sigma^* = (w_2, w_1) \) provided that \( \sigma = (w_1, w_2) \).

From standard curvature estimates for minimal graphs and the Uniform Graph Lemma [20], we can find a sequence \( \{S_n\}_{n \in \mathbb{N}} \) of graphs over \( \Sigma \), \( S_n \subset S_n \) for any \( n \), converging to \( \Sigma \) in the \( C^\infty \)-topology as graphs over \( \Sigma \).

Let \( \gamma_n(s) : [-L_n', L_n'] \to \Pi_0 \) be the arc-length parameterized inextensible arc in \( S_n \cap \Pi_0 \) satisfying \( \gamma_n(0) = 0 \). Write \( \gamma_n = (\gamma_n, 0) \), and note that \( \{[L_n', L_n]\}_{n \in \mathbb{N}} \to \mathbb{R} \) and \( \{\gamma_n(s)\}_{n \in \mathbb{N}} \to \gamma_0 \) in the \( C^\infty \)-topology, where \( \gamma_0 : \mathbb{R} \to \mathbb{R}^2 \) is given by \( \gamma_0(s) = s \sigma \).

For each \( n \in \mathbb{N} \), call \( G_n \) the maximal graph determined by \( v_n \) and set \( \alpha_n : [-L_n', L_n'] \to G_n, \alpha_n(s) := (\gamma_n(s), v_n(s)) \), where \( v_n(s) := v_n(\gamma_n(s)) \). It is not hard to see that \( \{\alpha_n(s)\}_{n \in \mathbb{N}} \to (\sigma, 1) \) in the \( C^\infty \) topology. Indeed, since \( \alpha_n'(s) = (\gamma_n'(s), \gamma_n'(s), v_n(\gamma_n(s))) \), it suffices to check that \( \{\nabla v_n(\gamma_n(s))\}_{n \in \mathbb{N}} \to \sigma \). Taking into account that \( v_n^*(\gamma_n(s)) = 0 \) for any \( s \), we have that
\[ \langle \nabla v_n^*(\gamma_n(s)), \gamma_n'(s) \rangle_{\theta^0} = 0, \] and so, \( \nabla v_n(\gamma_n(s)) = \lambda_n(s)\gamma_n'(s). \) As \( \{\|v_n\|_0\}_{n \in \mathbb{N}} \to 1 \) in the \( \mathcal{C}^0 \)-topology on \( \Pi_0 \), then \( \lambda_n(s) \to 1 \) uniformly on compact subsets of \( \mathbb{R} \). But \( \{\nabla v_n(0)\}_{n \in \mathbb{N}} \to \sigma \) implies \( \lambda_n(0) = 1 \), hence \( \{\lambda_n(s)\}_{n \in \mathbb{N}} \to 1 \) and \( \nabla v_n(\gamma_n(s))_{n \in \mathbb{N}} \to \sigma \). As a consequence, \( \{\alpha_n\}_{n \in \mathbb{N}} \) converges to the lightlike straight line \( \alpha_0 : \mathbb{R} \to G, \alpha_0(s) = (s\sigma, s) \), hence \( G \) is the lightlike plane containing \( \alpha_0 \) and \( v = v_0 \), proving our assertion.

To finish the claim, take a closed disc \( D \subset \Omega \) of positive radius centered at \( x_0 \), and without loss of generality, suppose \( \{x_n, n \in \mathbb{N}\} \subset D \). Label \( u_n := \max\{|u_n(x) - u(x)|, x \in D\} \), and as above define \( v_n, w_n : \Omega_n \to \mathbb{R} \) by \( v_n(y) = \frac{1}{n}(u_n(\mu_n y + x_n) - u_n(x_n)), \) \( w_n(y) = \frac{1}{n}(u_n(\mu_n y + x_n) - u(x_n)), \) \( n \in \mathbb{N} \). We know that \( \{v_n\}_{n \in \mathbb{N}} \to v \) and \( \{w_n\}_{n \in \mathbb{N}} \to w \) in the \( \mathcal{C}^0 \)-topology, where \( v, w : \Pi_0 \to \mathbb{R} \) are PS maps.

If \( [x, y] \subset \mathcal{A} \) be the inextensible singular segment of \( u \) containing \( x_0 \), then the PS graph \( G' := \{(y, w(y)) : y \in \mathbb{R}^2\} \) contains the straight line passing through \( O \) and parallel to the lightlike vector \( (y - x, u(y) - u(x)) \). From Lemma 2, \( G' \) is the lightlike plane parallel to this vector, that is to say, \( w(y) = (y, \nabla u(x_0))_{\theta^0} \), for any \( y \in \mathbb{R}^2 \).

Setting \( D_n := \frac{1}{\mu_n}(D - x_n) \), the graphs \( G_n := \{(y, v_n(y)) : y \in D_n\} \) and \( G'_n := \{(y, w_n(y)) : y \in D_n\} \) satisfy \( d_H(G_n, G'_n) \leq 2, n \in \mathbb{N} \), hence \( d_H(G, G') \leq 2 \). This implies that \( G \) and \( G' \) must be parallel and \( \sigma = \nabla u(x_0) \), which proves the claim.

Claims 1 and 2 imply that \( \{\|\nabla u_n - \nabla u\|_0\}_{n \in \mathbb{N}} \to 0 \) in the \( \mathcal{C}^0 \) topology, hence it remains to show that \( \nabla u \) is continuous on \( \Omega \). From Theorem 2, \( \nabla u \) is continuous on \( \Omega - \mathcal{A} \), and since the vector \( \sigma(x, y) \) depends continuously on \( (x, y) \in \mathcal{A} \), then \( \nabla u \) is continuous on \( \mathcal{A} \) too. Therefore, it suffices to prove that \( \lim_{n \to \infty} \nabla u_k(y_k) = \nabla u(x_0) \), provided that \( \{y_k\}_{k \in \mathbb{N}} \subset \Omega - \mathcal{A} \) and \( \lim_{k \to +\infty} y_k = x_0 \in \partial(\mathcal{A}) \). To see this, use Claim 1 to find a divergent sequence \( \{n_k\}_{k \in \mathbb{N}} \) in \( \mathcal{N} \) such that \( \|\nabla u_{n_k}(y_k) - \nabla u(x_0)\|_0 < 1/k, \) for any \( k \in \mathbb{N} \). From Claim 2, \( \lim_{k \to +\infty} \nabla u_{n_k}(y_k) = \nabla u(x_0) \), and so \( \lim_{k \to +\infty} \nabla u(y_k) = \nabla u(x_0) \), concluding the proof.

\[ \square \]

5.1 Asymptotic behaviour of maximal multigraphs of finite angle

Set \( G = \{(z(x), u(x)) : x \in W\} \) a PS multigraph over a wedge \( W \subset \mathcal{R} \) of finite angle. Let \( d_W \) be the intrinsic distance in \( W \) induced by \( |z|^2 \), and fix \( x_0 \in W \). Since \( \|u\|_0 \leq 1 \) and \( \lim_{x \to \infty} \frac{d_W(z(x), 0)}{\|z(x)\|_0} = 1 \), then \( \lim sup_{x \to \infty} \frac{|u(x)|}{\|z(x)\|_0} \leq 1 \) and \( \tau^+(r) := \min\{\frac{x}{\tau(z(x))} : \|z(x)\|_0 = r\} \in [-1, 1] \) for any \( r > 0 \).

Define \( \tau^+(G) := \lim_{r \to -\infty} \tau^+(r), \tau^-(G) := \tau^+(-G) \) and \( \tau(G) = \min\{\tau^+(G), \tau^-(G)\} \).

Likewise, \( \tau_0^+(G) := \lim_{r \to -\infty} \tau_0^+(r), \tau_0^-(G) := \tau_0^+(-G), \) and \( \tau_0(G) = \min\{\tau_0^+(G), \tau_0^-(G)\} \).

These numbers lie in \([0, 2]\) and give different measures of the asymptotic closeness between \( G \) and the light cone. Obviously, \( \tau^+(G) \geq \tau_0^+(G), \tau^-(G) \geq \tau_0^-(G) \).

For \( \theta \in [0, +\infty[ \), call

\[ \Xi(\theta) := \inf \{\tau(G) : G \text{ is a maximal multigraph over a wedge of angle } \theta\} \]

\[ \Xi(\theta) := \inf \{\tau_0(G) : G \text{ is a maximal multigraph over a wedge of angle } \theta\} \]

and notice that \( \Xi(\theta) \geq \Xi(\theta) \).

If \( G' \subset G \), the monotonicity formulae \( \tau(G') \leq \tau(G) \) and \( \tau_0(G') \leq \tau_0(G) \) hold straightforwardly.

As a consequence, \( \Xi(\theta') \leq \Xi(\theta) \) and \( \Xi(\theta') \leq \Xi(\theta) \) provided that \( \theta' \leq \theta \).

**Lemma 5.1** \( \Xi(\theta) > 0 \) for any \( \theta \in [0, +\infty[ \).

**Proof:** Since \( \tau^+(G) = \tau^-(-G) \), we have

\[ \Xi(\theta) = \inf \{\tau^+(G) : G \text{ is a maximal multigraph over a wedge of angle } \theta\} > 0. \]
On the other hand, any multigraph of angle $\theta$ contains, up to a translation, a graph over the wedge $W_{\theta'}$, for any $\theta' < \frac{\theta}{2}$. From the above monotonicity argument if suffices to prove that

$$\inf\{\tau^+(G) : G \text{ is a graph over } W_{\theta}\} > 0$$

for any $\theta \in [0, \pi]$. Reason by contradiction, and assume that there exists $\theta \in [0, \pi]$ and sequence of maximal graphs $\{G_n\}_{n \in \mathbb{N}}$ over $W_{\theta}$ such that $\lim_{n \to +\infty} \tau^+(G_n) = 0$. Write $G_n = \{(x, u_n(x)) : x \in W_{\theta}\}$, and without loss of generality suppose $u_n((0,0)) = 0$, $n \in \mathbb{N}$. From equation (1) and up to scaling depending on $n$, we can also assume that

$$u_n(x)/\|x\| \in [1 - \tau^+(G_n), 1], \text{ for all } x \in W_{\theta} \cap \{\|x\| \geq 0\}.$$  \hspace{1cm} (7)

Define $v : W_{\theta} \to \mathbb{R}$, $v(x) = \|x\|_0$, and let us see that

$$\lim_{n \to +\infty} \sup\{\|\nabla u_n - \nabla v\|_0 : x \in W_{\theta'} \cap \{\|x\| \geq 0\}\} = 0,$$

for any $\theta' \in [0, \theta]$. Indeed, reason by contradiction and suppose there is a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $W_{\theta'} \cap \{\|x\| \geq 1\}$ such that, and up to subsequences, $\|\nabla u_n(x_n) - \nabla v(x_n)\|_0 \geq \epsilon > 0$. Call $v_n(y) := \frac{1}{\|x_n\|_0} u_n(\|x_n\|_0 y)$, for each $n \in \mathbb{N}$. The fact that $\{\tau^+(G_n)\}_{n \in \mathbb{N}} \to 0$, and Proposition 5.1 imply that $\{v_n\}_{n \in \mathbb{N}} \to v$ in the $C^1$-topology on $W_{\theta'}$, contradicting that $\|\nabla v_n(\frac{x}{\|x\|_0}) - \nabla v(\frac{x}{\|x\|_0})\|_0 \geq \epsilon > 0$ for all $n \in \mathbb{N}$ and proving our assertion.

Set $g_n = \frac{1}{1+\sqrt{-1\|\nabla u_n\|^2}} \nabla u_n$ the holomorphic Gauss map of $G_n$. Rewriting the above limit in polar coordinates, we infer that $\lim_{n \to +\infty} \sup\{|g_n| \in [-\theta', \theta'] \times [1, +\infty] = 0,$

for any $\theta' \in [0, \theta]$. Therefore, fixing $\theta' \in [0, \theta]$ and $\epsilon \in [0, \min(\frac{1}{\pi}, \frac{\theta}{2})]$, we can find $n_0 \in \mathbb{N}$ large enough in such a way that $|\Im(\log(g_{n_0})(se^{i\xi})) - \xi + \pi/2| < \epsilon$ and $|g_{n_0}(se^{i\xi})| > 1 - \epsilon > \frac{1}{\pi}$, for every $s \geq 1$ and $\xi \in [-\theta', \theta']$. An intermediate value argument gives that $C_3 := \{x \in W_{\theta'} \cap \{\|x\| \geq 1\} : |\Im(\log(g_{n_0})(x)) = \delta\}$ is non compact and $C_4 \|\partial W_{\theta'} \cap \{\|x\| \geq 1\}\} \subset \{\|x\| = 1\}$, for any $\delta \in -\theta' - \pi/2 + 2\epsilon, \theta' - \pi/2 - 2\epsilon$.

Choose $\delta$ in such a way that $dg_{n_0}$ never vanishes along $\partial C_3$, and take a divergent regular arc $\alpha_\delta \subset C_3$. As $\log(g_{n_0})$ is strictly monotone on $\alpha_\delta$, then $\lim_{x \to \alpha_\delta} g_{n_0}(x) = -r_3 se^{i\delta}$, $1 - \epsilon \leq r_3 \leq 1$. In other words, $r_3 e^{i\delta}$ is an asymptotic value of $g_{n_0}$ at the unique end of $G_{n_0}$. This argument works for infinitely many $\delta' \in -\theta' - \pi/2 + 2\epsilon, \theta' - \pi/2 - 2\epsilon$ different from $\delta$, and so $g$ has infinitely many asymptotic values at the end. This contradicts the parabolicity of $G_{n_0}$ (see for instance Corollary 4.11 and Theorem 4.13) and proves the lemma.

**Corollary 5.1** $\Xi_0(\theta) > 0$ for any $\theta \in [\pi, +\infty[$.

**Proof**: Reasoning as in the preceding lemma, if suffices to prove that

$$\inf\{\tau^+_0(G) : G \text{ is a maximal graph over } W_{\theta}\} > 0$$

for any $\theta \in [\pi/2, \pi[$.

Assume there is a sequence of graphs $\{G_n\}_{n \in \mathbb{N}}$ over $W_{\theta}$, $\theta \in [\frac{\pi}{2}, \pi[$, such that $\{\tau^+_0(G_n)\}_{n \in \mathbb{N}} \to 0$.

From Lemma 5.1 we can suppose that $0 \leq \tau^+_0(G_n) < 2\tau^+_0(G_n) < \Xi(\theta) \leq \tau^+(G_n)$ for any $n \in \mathbb{N}$.

For every $n \in \mathbb{N}$ write $G_n = \{(x, u_n(x)) : x \in W_{\theta}\}$, take $c_n \in \tau^+_0(G_n), 2\tau^+_0(G_n)$, and call $V_n := \{(x, t) \in \mathbb{R}^2 : \|t\| > 1 - c_n \|x\|_0\}$, $V^+_n = V_n \cap \{t \geq 0\}$.

We can find divergent sequences $\{r_k\}_{k \in \mathbb{N}}$ in $[0, +\infty[$ and $\{x_k\}_{k \in \mathbb{N}}$ in $W_{\theta}$ such that $\lim_{k \to +\infty} \frac{u_n(x_k)}{\|x_k\|_0} \leq 1 - \Xi(\theta)$, $G_n \cap \pi^{-1}(\{x \in \mathbb{R}_0^+ : \|x\|_0 = r_k\}) \subset V^+_n$ and $r_k < \|x_k\|_0 < r_{k+1}$ for any $k \in \mathbb{N}$.

As a consequence, $G_n := G_n - V^+_n$ is unbounded and any connected component of $G_{n}'$ is compact. For each $\xi \in [\frac{\pi}{2} - \theta, -\frac{\pi}{2} + \theta]$ call $H_n(\xi)$ as the closed half space tangent to $\partial(V^+_n)$ at
Lemma 5.2

A divergent spacelike arc \( c \subset \mathbb{R}^3_+ \) is said to be an upward (resp., downward) lightlike ray if, up to removing a compact subarc, \( \pi(c) \) is a closed half line and there exists \( p \in \pi^{-1}(\pi(c)) \) such that \( \lim_{x \in c - \infty} d(x, l_c) = 0 \), where \( l_c = C^+_p \cap \pi^{-1}(\pi(c)) \) (resp., \( l_c = C^-_p \cap \pi^{-1}(\pi(c)) \)).

The existence of lightlike rays imposes some restrictions on the geometry of maximal surfaces. We start with the following lemma.

Lemma 5.2

Let \( W \subset \mathbb{R} \) be a wedge of angle \( \theta \in [0, +\infty) \), write \( \partial(W) = L_1 \cup L_2 \), where \( L_1 \) and \( L_2 \) are divergent arcs with the same initial point, and call \( \theta_j = \lim_{x \in L_j} \arg(x) \), \( j = 1, 2 \). Let \( c_0 \subset W \) be an arc such that \( z(c_0) \) is a half line and \( \lim_{x \in \pi(c_0) - \infty} \arg(x) = \xi \in (\theta_1, \theta_2] \).

If \( X : W \rightarrow \mathbb{R}^3 \) is a maximal multigraph and \( c := X(c_0) \) is a lightlike ray then \( c \) is sublinear with direction \( v_c \in \{ \frac{1}{\sqrt{2}}(e^{i\xi}, 1), \frac{1}{\sqrt{2}}(e^{i\xi}, -1) \} \).

Proof: Take into account that any blow down of \( c \) with center \( O \) is a lightlike half line in \( C_0 \) and use Proposition 5.1

Proposition 5.2

Let \( N \subset \mathbb{R}^3 \) be a properly embedded maximal multigraph of finite angle \( \theta > 0 \), and assume that \( \partial(N) \) can be split into two proper sublinear arcs \( l_1 \) and \( l_2 \) with lightlike directions \( v_1 \) and \( v_2 \), respectively.

Then \( v_1 = \pm v_2 \), and \( \lim_{x \in N - \infty} g(x) = s \tau_0(w) \), where \( g \) is the holomorphic Gauss map of \( N \) and \( w = -\frac{1}{(\tau_0') v_1} \). In particular, the underlying complex structure of \( N \) is parabolic.

Proof: As usual \( \arg : N \rightarrow \mathbb{R} \), \( \arg := \text{Im}(\log(\pi(N))) \) represents a branch of the argument. Write \( \theta_j = \lim_{x \in l_j} \arg(x), \ j = 1, 2 \), and suppose without loss of generality that \( \theta_1 < \theta_2 \). Fix a compact arc \( l_0 \subset \partial(N) \). From the definition of multigraph, it is not hard to construct a foliation \( \mathcal{F}(\xi, u) : (\theta_1, \theta_2) \times [0, +\infty) \rightarrow N \) satisfying:

(i) \( l_\xi := \mathcal{F}(\xi, \cdot) \) is a proper arc with initial point in \( l_0 \), for any \( \xi \in (\theta_1, \theta_2] \), and \( l_{\theta_j} = l_j \) up to a compact subset, \( j = 1, 2 \).

(ii) For any \( \epsilon > 0 \), there is a closed disc \( D(\epsilon) \subset \Pi_0 \) such that \( \pi(l_{\xi}) - D(\epsilon) \) is a half line for any \( \xi \in (\theta_1 + \epsilon, \theta_2 - \epsilon) \).

(iii) \( \mathcal{F}(\xi, u) \) is the Euclidean arclength parameter of \( \pi(l_{\xi}) \), \( \xi \in (\theta_1, \theta_2) \), and

\[ \lim_{u \rightarrow +\infty} \max\{|\arg(\mathcal{F}(\xi, u)) - \xi| : \xi \in [\theta_1, \theta_2]\} = 0. \]

Let \( F \subset [\theta_1, \theta_2] \) be the closure of \( F_0 := \{ \xi \in [\theta_1, \theta_2] : \xi \text{ is a lightlike ray} \} \). Since blow downs of lightlike rays are lightlike half straight lines and \( F_0 \) is dense in \( F \), any blow down of \( N_F := \mathcal{F}(F \times [0, +\infty]) \) with center the origin is a closed countable collection of angular regions in \( C_0 \) (some of them could be lightlike rays). This argument and Proposition 5.1 show that \( l_\xi \) is a sublinear arc with lightlike direction \( v_\xi \in \{ \frac{1}{\sqrt{2}}(e^{i\xi}, 1), \frac{1}{\sqrt{2}}(e^{i\xi}, -1) \} \), for every \( \xi \in F \).

Let us see that \( F \) is a compact totally disconnected set. Reason by contradiction and suppose there exists a closed interval \( J \subset F \) of length \( |J| > 0 \). Then, any blow down of \( N_J := \mathcal{F}(J \times [0, +\infty]) \) with center \( O \) is an angular region of \( C_0 \) of positive angle. Thus, \( \tau(N_J) = 0 \), which contradicts Lemma 5.1 and proves our assertion.

\[ W \subset C_0 \text{ is said to be an angular region if either } W = \pi^{-1}(W_0) \cap C_0^+ \text{ or } W = \pi^{-1}(W_0) \cap C_0^-, \text{ where } \theta \in [0, 2\pi]. \]
Claim 1: If \( \{p_n\}_{n \in \mathbb{N}} \subset N \) is divergent and \( \lim_{n \to \infty} \arg(p_n) = \xi \in F \), then \( \lim_{n \to \infty} g(p_n) = s_0(w_\xi) \), where \( w_\xi = -\frac{1}{i(\xi)}v_\xi \).

Proof: Call \( \mu_n := d(p_n,l_\xi) \) and take \( q_n \in l_\xi \) satisfying \( \|p_n - q_n\|_0 = \mu_n, \ n \in \mathbb{N} \). Set \( G := \arg^{-1}([\xi - \delta, \xi + \delta]), \ \delta \in ]0, \pi[ \), and consider \( G_n := \frac{1}{\lambda_n}(G - q_n) \), where \( \lambda_n := \max\{\mu_n, 1\} \), for any \( n \in \mathbb{N} \).

Since \( \{q_n\}_{n \in \mathbb{N}} \) is divergent, \( \{G_n\}_{n \in \mathbb{N}} \) converges in the \( C^0 \)-topology to either an entire graph over \( \Pi_0 \) (if \( \xi \notin \{\theta_1, \theta_2\} \)) or a graph over a closed half plane \( H \subset \Pi_0 \) (if \( \xi \in \{\theta_1, \theta_2\} \)), and in the second case \( H \) is bounded by a straight line parallel to \( d(p_v), \ j \in \{1, 2\} \). Anyway, Lemma 2.1 gives that either \( G_\infty \) is a lightlike plane or it is a lightlike half plane bounded by a lightlike straight line. The claim follows from Proposition 5.1. □

The closure \( I \) of a connected component of \( [\theta_1, \theta_2] - F \) is defined to be a good component of \( [\theta_1, \theta_2] \), and as above we call \( N_I := F(I \times [0, +\infty[) \).

Claim 2: If \( I = [\xi_1, \xi_2] \) is a good component of \( [\theta_1, \theta_2] \), then the limit \( w_I := \lim_{x \in N_I} g(x) \) exists. In particular, \( w_I = w_{\xi_1} = w_{\xi_2} \).

Proof: Define \( \mathcal{H}_I = \{x \in \mathbb{R}_1^2 : \|x\| \leq 1\} \), and let us show that \( l_{\xi} \cap \mathcal{H}_I \) is compact for every \( \xi \in [\xi_1, \xi_2] \). It suffices to check that any divergent subarc \( l'_{\xi} \subset l_{\xi} \) satisfying that \( \pi(l'_{\xi}) \subset \{se^{i\xi} : s \geq 0\} \) intersects \( \mathcal{H}_I \) in a compact set. Assume that \( l'_{\xi} \cap \mathcal{H}_I \neq \emptyset \) (otherwise we are done), and note that \( l'_{\xi} \) can not lie in \( \mathcal{H}_I \) because otherwise \( l'_{\xi} \) would be a lightlike ray. Therefore, \( l'_{\xi} \cap \partial(\mathcal{H}_I) \neq \emptyset \). Since \( l'_{\xi} \) has slope \( <1 \) and the hyperbola \( \pi^{-1}(\{se^{i\xi} : s \geq 0\}) \cap \mathcal{H}_I \) is timelike, then \( l'_{\xi} \) and \( \partial(\mathcal{H}_I) \) meet at a unique point, proving our assertion.

Thus we can find an smooth proper arc in \( c \subset N_I - \mathcal{H}_I \) with no endpoints in \( N_I \) satisfying that \( \arg(c) \) is monotone and \( \arg(c) = [\xi_1, \xi_2] \). Set \( N'_I \subset N_I \) the simply connected region bounded by \( c \) and disjoint from \( \partial(N_I) \), and note that \( N'_I \) has parabolic underlying conformal structure (use that \( N'_I \cap \mathcal{H}_I = \emptyset \) and Theorem 4.2). Split \( c \) into two divergent arcs \( c_1 \) and \( c_2 \) with the same initial point. From Claim 1 and up to relabeling, we have that \( \lim_{x \in c_j} g(x) = w_{\xi_j}, \ j = 1, 2 \). Theorem 4.1 can be applied to deduce that \( w_{\xi_1} = w_{\xi_2} \), and \( \lim_{x \in N'_I} g(x) = w_{\xi_1} \).

To finish, take a compact arc \( c_0 \subset N_I \) intersecting \( \partial(N_I) \) and \( \partial(N'_I) \) at a unique point, and call \( N''_I, \ j = 1, 2 \) as the two regions in which \( c_0 \) splits \( N_I - \text{Int}(N'_I) \). Since \( N''_I \) is contained in a spacelike half space, then it is parabolic too (see Corollary 4.1), and reasoning as above \( \lim_{x \in N''_I} g(x) = w_{\xi_1} \), completing the proof. □

Claim 2 and a connection argument give that \( w := w_I \) does not depend on the good component \( I \) of \( [\theta_1, \theta_2] \). Therefore, \( \lim_{x \in N} g(x) = w, \ w = -\frac{1}{i(\xi_j)}v_j, \ j = 1, 2 \), and \( v_1 = v_0 = \pm v_2 = \pm v_3 \). By Privalov’s theorem, the ideal boundary of \( N \) has vanishing harmonic measure, and so \( N \) is parabolic. This concludes the proof. □

Corollary 5.2 Let \( N \) be as in Proposition 5.2 but allowing one (or both) of the \( l_j \)'s to be a lightlike arc. Then \( v_1 = \pm v_2 \) and \( \lim_{x \in N} g(\partial(N)) = s_0(w) \), where \( w = -\frac{1}{i(\xi_1)}v_1 \).

Moreover, if in addition \( N \) is a CLBMS maximal surface then its underlying complex structure is parabolic.

Proof: Consider a properly embedded embedded maximal multigraph \( N' \in \mathbb{R}_1^2 \) contained in \( N - \partial(N) \) and with the same angle of \( N \). If \( \partial(N') \) is close enough to \( \partial(N) \), \( \partial(N') \) can be also split into two proper spacelike subarcs \( l'_1 \) and \( l'_2 \) with directions \( v_1 \) and \( v_2 \), respectively, and Proposition 5.2 applies to \( N' \). Since \( N' \) is any arbitrary region of \( N \) satisfying these conditions, the first part of the corollary easily holds. For the second part, take a conformal parameterization \( X : \mathcal{M} \to \mathbb{R}_1^3 \). .
of \( N \) and note that \( g \) has well defined limit at the end of \( \mathcal{M} \). The parabolicity of \( \mathcal{M} \) follows from Privalov’s theorem once again.

\[ \square \]

**Proposition 5.3** Let \( G \) be an entire PS graph which is a maximal surface except on a closed lightlike half line \( l \subset G \). Then \( G \) is congruent in the Lorentzian sense to \( \mathbb{E}_2 \).

**Proof:** Up to a Lorentzian isometry, put \( l = \{(x_1, x_2, t) \in \mathbb{R}^3_1 : x_1 = x_2 - t = 0, t \geq 0\} \).

Set \( G_0 = G - l \) and \( l_0 = l - \{0\} \). From Riemann’s uniformization theorem, \( G_0 \) is conformally equivalent to either \( \mathbb{C} \) or \( U \). Since \( g \) is not constant and \( |g| < 1 \) on \( G_0 \), then necessarily \( G_0 \equiv \mathbb{U} \).

Label \( X : \mathbb{U} \to \mathbb{R}^3_1 \) as the associated conformal parameterization of \( G_0 \).

**Claim 1:** If \( \{z_n\}_{n \in \mathbb{N}} \subset \mathbb{U} \) and \( \{X(z_n)\}_{n \in \mathbb{N}} \to p_0 \in l_0 \cup \{\infty\} \) then \( \lim_{n \to \infty} g(z_n) = 1 \).

**Proof:** Label \( \lambda_n = d(X(z_n), l) \), and let us see that \( \{g(z_n)\}_{n \in \mathbb{N}} \to 1 \) provided that \( \lim_{n \to \infty} X(z_n)/\lambda_n = \infty \). Indeed, the sequence \( \{G_n := 1/\lambda_n \cdot (G - X(z_n))\}_{n \in \mathbb{N}} \) converges in the \( C^0 \)-topology to an entire PS graph \( G_\infty \) containing a lightlike straight line parallel to \( \{x_1 = x_2 - t = 0\} \). From Lemma 2.1 we get \( G_\infty = \{x_2 - t = 0\} \). The assertion follows from Proposition 5.1.

Proposition 5.2 can be applied to the graph \( G - \pi^{-1}(\{|x_1| < \delta, x_2 > -\delta\}) \) for every \( \delta > 0 \), proving the claim. \[ \square \]

Fatou’s theorem guarantees that \( g : \mathbb{U} \to \mathbb{D} \) has well defined angular limits a. e. on \( \partial(\mathbb{U}) \equiv \mathbb{R} \), and since \( g \) is not constant, Privalov’s theorem gives that these limits are different from 1 a. e. on \( \partial(\mathbb{U}) \). Thus, Claim 1 also shows that any two sequences \( \{z_n\}_{n \in \mathbb{N}}, \{z'_n\}_{n \in \mathbb{N}} \) satisfying \( \lim_{n \to \infty} X(z_n), \lim_{n \to \infty} X(z'_n) \in l_0 \cup \{\infty\} \) converge to the same point \( z_0 \in \mathbb{R} \cup \{\infty\} \) (up to a conformal transformation we will suppose \( z_0 = \infty \)). Therefore, \( \lim_{z \to r} X(z) = 0 \) for all \( r \in \mathbb{R} \), and from equation 2 we get that \( |g| = 1 \) on \( \mathbb{R} \cup \{\infty\} \). This shows that \( X \) and \( g \) extend by Schwarz reflection to \( \mathbb{C} \) and \( \overline{\mathbb{C}} \), respectively, and \( dg \neq 0 \) on \( \partial(\mathbb{U}) \cup \{\infty\} \). The extended map \( X : \mathbb{C} \to \mathbb{R}^3_1 \) is a conformal maximal immersion with lightlike singular set \( \mathbb{R} \) and \( X(\overline{\mathbb{U}}) = G_0 \cup \{0\} \).

Set \( u := ((t - x_2) \circ X)|_{\overline{\mathbb{U}}} \) and label \( u^* \) as its harmonic conjugate.

**Claim 2:** The holomorphic map \( h : \mathbb{U} \to \mathbb{C} \), \( h := u + i u^* \), is injective and \( h(\overline{\mathbb{U}}) = \{z \in \mathbb{C} : \Re(z) \geq 0\} \).

**Proof:** From equation 1, \( G \subset \cap_{z \in \mathbb{U}} \text{Ext}(\mathbb{C}_r^z) \subset \{t - x_2 \geq 0\} \). Then, the maximum principle gives that \( G_0 \subset \{t - x_2 > 0\} \), that is to say, \( u^{-1}(0) = \mathbb{R} \). Furthermore, as \( \overline{\mathbb{U}} \) is parabolic and \( u \) is not constant (see Section 4), then \( u \) is unbounded.

Basic theory of harmonic functions says that \( u^{-1}(a) \) consists of a proper family of analytical curves meeting at equal angles at singular points of \( u, a \geq 0 \). Let us show that \( u^{-1}(a) \) consists of a unique regular analytical arc, for any \( a \geq 0 \). Indeed, otherwise we can find a region \( \Omega \subset \overline{\mathbb{U}} \) such that \( 0 \leq u|\Omega \leq a \) and \( u|\partial(\Omega) = a \), contradicting the parabolicity of \( \Omega \).

Since \( u^*|_{u^{-1}(a)} \) is one to one for any \( a \geq 0 \), then \( h \) is injective. Furthermore, \( h(\overline{\mathbb{U}}) \) is parabolic simply connected open subset of \( \{z \in \mathbb{C} : \Re(z) \geq 0\} \), and so \( h(\overline{\mathbb{U}}) = \{z \in \mathbb{C} : \Re(z) \geq 0\} \), which proves the claim. \[ \square \]

Up to a conformal transformation, we have \( h(z) = iBz, B \in \mathbb{R} - \{0\} \), and since \( dh = i\phi_3 - \phi_2 = -i(g^{-1})_z \phi_3 \), then \( \phi_3 = -B \frac{2g}{g - 1} \phi_2 \). As \( G \) has a unique topological end, then \( g^{-1}(1) = \infty \).

Moreover, \( dg \neq 0 \) along \( \mathbb{R} \cup \{\infty\} \) gives that \( g|_{\mathbb{R} \cup \{\infty\}} \) is one to one, and so \( g(z) = e^{i\theta}(z - ir)/(z + ir) \), where \( \theta, r \in \mathbb{R} \) and \( |(1-r)/(1+r)| < 1 \). Up to conformal reparameterizations, Lorentzian isometries and homotheties, these are the Weierstrass data of \( E_2 \). \[ \square \]
5.2 Finiteness of maximal graphs with planar boundary

Let $\Omega$ be a region in $\Pi_0$. A non flat maximal graph $G = \{(x,u(x)) : x \in \Omega\}$ is said to be supported on $\Omega$ if $u = 0$ on $\partial(\Omega)$.

Assume that $G = \{(x,u(x)) : x \in \Omega\}$ is supported on $\Omega$ and denote by $G(R)$ (resp., $\Omega(R)$) the intersection $G \cap (D(R) \times \mathbb{R})$ (resp., $\Omega \cap D(R)$), where $D(R) = \{x \in \Pi_0 : \|x\|_0 \leq R\}$, $R > 0$. Let $A(G(R))$ denote the area of $G(R)$ computed with the Riemannian metric induced by $\langle \cdot, \cdot \rangle$ on $G$. The spacelike condition $\|\nabla u\|_0 < 1$ gives the following trivial estimate:

$$A(G(R)) = \int_{\Omega(R)} \sqrt{1 - \|\nabla u\|_0^2} \, dx \leq A(\Omega(R)) \leq \pi R^2 \quad (8)$$

where $dx$ is the Euclidean area element in $\Pi_0$ and $A(\Omega(R))$ is the Euclidean area of $\Omega(R)$ in $\Pi_0$.

The following theorem has been inspired by Li-Wang work [10].

**Theorem 5.3** Let $\{G_i\}_{i=1}^k$ be a set of $k$ maximal simply connected graphs in $\mathbb{R}^3_1$ defined by the functions $\{u_i\}_{i=1}^k$ with disjoint supports $\{\Omega_i\}_{i=1}^k$ in $\Pi_0$. Let us assume that $\|\nabla u_i\|_0 \leq 1 - \varepsilon$, for any $i = 1, \ldots, k$, where $\varepsilon > 0$.

Then $k \leq \frac{8}{\sqrt{\varepsilon(2-\varepsilon)}}$.

**Proof**: Without loss of generality, suppose $0 \notin \bigcup_{i=1}^k \Omega_i$. Since $u|_{\partial(\Omega_i)} = 0$ and $\|\nabla u_i\|_0 \leq 1 - \varepsilon$, we get from (11) that $\|u_i(x)\| \leq (1 - \varepsilon)\|x\|_0$ on $\Omega_i$, $i = 1, \ldots, k$.

Fix $R_0 > 0$ such that $G_i(R_0) \neq \emptyset$ for $i = 1, \ldots, k$. As $\|u_i\| \leq (1 - \varepsilon)R$ on $G_i(R)$ and $A(G_i(R)) \leq \pi R^2$ for any $R \geq R_0$ and $i \in \{1, \ldots, k\}$, then for any $m \geq 1$ we obtain

$$\prod_{j=0}^{m-1} \prod_{i=1}^k \frac{\int_{G_i(2^j R_0)} |u_i| \, dx}{\int_{G_i(2^{j+1} R_0)} |u_i| \, dx} \geq \prod_{i=1}^k \frac{\int_{G_i(2^0 R_0)} |u_i| \, dx}{\int_{G_i(2^{m+1} R_0)} |u_i| \, dx} \geq \alpha \cdot ((1 - \varepsilon) 2^{3(m+1)} \pi R_0^3)^{-k},$$

where $\alpha = \prod_{i=1}^k \int_{G_i(R_0)} |u_i| \, dx$.

Hence, there exists $0 \leq t \leq m$ such that $\prod_{j=0}^{m-1} \prod_{i=1}^k \frac{\int_{G_i(2^j R_0)} |u_i| \, dx}{\int_{G_i(2^{j+1} R_0)} |u_i| \, dx} \geq 2^{-3k} \alpha^{1/(m+1)} ((1 - \varepsilon) \pi R_0^3)^{-k/(m+1)}$, and by the arithmetic means we infer that

$$\sum_{i=1}^k \frac{\int_{G_i(2^0 R_0)} |u_i| \, dx}{\int_{G_i(2^{t+1} R_0)} |u_i| \, dx} \geq \sum_{i=1}^k \frac{\int_{G_i(2^0 R_0)} |u_i| \, dx}{\int_{G_i(2^{m+1} R_0)} |u_i| \, dx}^{1/k} \geq 2^{-3k} R_0 \frac{1}{(m+1) \pi ((1 - \varepsilon) \pi R_0^3)^{-1/(m+1)}} \quad (9)$$

On the other hand, labeling $M_i = \max\{\frac{|u_i(x)|}{\int_{G_i(2^t R_0)} |u_i| \, dx} : x \in G_i(2^t R_0)\}$, $i = 1, \ldots, k$, and using that $u_i$'s are disjointly supported, we have

$$\sum_{i=1}^k \frac{\int_{G_i(2^0 R_0)} |u_i| \, dx}{\int_{G_i(2^{t+1} R_0)} |u_i| \, dx} \leq \max\{M_i : i = 1, \ldots, k\} \sum_{i=1}^k A(G_i(2^t R_0)), \quad (9)$$

and so

$$\sum_{i=1}^k \frac{\int_{G_i(2^0 R_0)} |u_i| \, dx}{\int_{G_i(2^{t+1} R_0)} |u_i| \, dx} \leq \frac{|u_{i_0}(x_0)|}{\int_{G_{i_0}(2^{t+1} R_0)} |u_{i_0}| \, dx} \pi (2^t R_0)^2, \quad (10)$$

for some $1 \leq i_0 \leq k$ and $(x_0, u_{i_0}(x_0)) \in G_{i_0}(2^t R_0)$. 

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As $\Omega_{i_0}$ is simply connected, then the conjugate minimal graph $G^*_{i_0} = \{(x, u^*_{i_0}(x)), : x \in \Omega_{i_0}\}$ of $G_{i_0}$ is well defined (see equation 5). Since $u_{i_0}$ is an harmonic function on $G^*_{i_0}$ vanishing on $\partial(G^*_{i_0})$, the mean value property for subharmonic functions gives

$$|u_{i_0}(x_0)| \leq \frac{1}{\pi(2^tR_0)^2} \int_{G^*_{i_0} \cap B(p_0, 2^tR_0)} |u_{i_0}(y)| dy,$$

where $p_0 = (x_0, u^*_{i_0}(x_0))$, $B(p_0, 2^tR_0)$ is the Euclidean ball of radius $2^tR_0$ centered at $p_0$, and $d^*y$ is the Euclidean area element on $G^*_{i_0}$.

From equation (5), $d^*y = \sqrt{1 + \|\nabla u^*_{i_0}\|^2} dx = \frac{1}{\sqrt{1 - \|\nabla u_{i_0}\|^2}} dx \leq \frac{1}{\sqrt{\varepsilon(2-\varepsilon)}} dx$. Since $G^*_{i_0} \cap B(p_0, 2^tR_0) \subset D(2^{t+1}R_0) \times \mathbb{R}$, we deduce that

$$|u_{i_0}(x_0)| \leq \frac{1}{\sqrt{\varepsilon(2-\varepsilon)}} \frac{1}{\pi(2^tR_0)^2} \int_{G^*_{i_0}(2^{t+1}R_0)} |u_{i_0}(y)| dy,$$

and from (10), $\sum_{i=1}^{k} \int_{\gamma_{i_0}(2^{t+1}R_0)|u_{i_0}|} dx \leq \frac{1}{\sqrt{\varepsilon(2-\varepsilon)}}$. Equation (9) gives

$$2^{-3k\alpha(m+1)}((1-\varepsilon)\pi R_0^3)^{-1/(m+1)} \leq \frac{1}{\sqrt{\varepsilon(2-\varepsilon)}},$$

and taking the limit as $m \to \infty$, we obtain $k \leq \frac{8}{\sqrt{\varepsilon(2-\varepsilon)}}$, concluding the proof. \hfill $\blacksquare$

6 Maximal surfaces with connected lightlike boundary of mirror symmetry.

Let us go over some basic definitions and properties, thereby fixing some notations and conventions.

Let $\mathcal{M}$ be a Riemann surface whose boundary consists of a non compact analytical arc $\Gamma$, and let $X : \mathcal{M} \to \mathbb{R}^2$ be a conformal proper CLBMS maximal embedding. As usual we identify $\mathcal{M} \equiv X(\mathcal{M}) \subset \mathbb{R}^3$ and $\Gamma \equiv X(\Gamma) \subset \mathbb{R}^2$.

From Lemma 2.2, $\pi|_{\mathcal{M}} : \mathcal{M} \to \Pi_0$ is a local embedding. Moreover, since $\Gamma$ is a regular lightlike arc then it is a smooth graph over the $t$-axis and we can write $\Gamma = \{\Gamma(s) := (\gamma(s), s) \in \mathbb{R}^3 : s \in \mathbb{R}\}$, where $s = t|_{\Gamma}$ is the Euclidean arclength of $\gamma \subset \Pi_0$. Up to a translation, we assume that $\Gamma$ passes through $q_0 := (1, 0, 0)$ (and so $\Gamma(0) = (1, 0, 0)$ and $\gamma(0) = (1, 0)$). Since spacelike planes and $\Gamma$ meet transversally at a unique point, then for any $q \in \Gamma$ we have:

$$\Gamma - \{q\} \subset \text{Int}(\mathcal{C}_q). \quad (11)$$

Let $g : \mathcal{M} \to \mathbb{T}$ denote the holomorphic Gauss map of $\mathcal{M}$, and recall that $|g|(p) = 1$ if and only if $p \in \Gamma$. Thus the argument function $\theta := \text{Im}(\log(g))$ has a well defined one to one branch along $\Gamma$. Up to a Lorentzian rotation suppose $g(q_0) = 1$, and choose the branch of $\theta$ in such a way that $\theta(q_0) = 0$. Labeling $\theta^- := \inf(\theta(\Gamma))$ and $\theta^+ := \sup(\theta(\Gamma))$, the function $\theta(s) : \mathbb{R} \to [\theta^-, \theta^+]$, $\theta(s) := \theta(\Gamma(s))$ is a diffeomorphism and provides a global parameter along $\Gamma$.

From equation (2), $\gamma'(s) = ig(s) = ie^{i\theta(s)}$ and so in complex notation $\gamma(s) = 1 + i \int_0^s e^{i\theta(x)} dx$. Up to a symmetry with respect to a timelike plane, we can assume that $\theta'(s) > 0$, hence $\lim_{s \to +\infty} \theta(s) = \theta^+$ and $\lim_{s \to -\infty} \theta(s) = \theta^-$.  

By definition, the rotation number $\theta(\mathcal{M})$ of $\mathcal{M}$ is the change of the tangent angle along $\gamma$. Obviously, $\theta(\mathcal{M}) = \theta^+ - \theta^- \in [0, +\infty]$.

The following path-lifting property is an elementary consequence of Lemma 2.2.
Given a planar curve $\beta(u) : I \to \Pi_0$, $u_0 \in I$ and $p_0 \in \pi^{-1}(\beta(u_0)) \cap M$, there exists a unique inextendible curve $\tilde{\beta} : J \to M$ in $M$ satisfying that $u_0 \in J \subset I$, $\tilde{\beta}(u_0) = s_0$ and $\pi \circ \tilde{\beta} = \beta|_J$. It is clear that either $J = I$ or at least one of the endpoints of $\tilde{\beta}$ lies in $\Gamma$. The curve $\tilde{\beta}$ is said to be the lifting of $\beta$ to $M$ with initial condition $\tilde{\beta}(u_0) = p_0$.

For each $s \in \mathbb{R}$, take an open simply connected neighbourhood $V_s$ of $\Gamma(s)$ in $M$ such that $\Gamma \cap V_s$ is connected and $\pi|_{V_s} : V_s \to \pi(V_s)$ is one to one. Then label $n(s) \in \Pi_0$ as the unit normal to $\gamma$ at $\gamma(s)$ interior to $\pi(V_s)$. Obviously, $n(s)$ does not depend on the chosen neighbourhood $V_s$.

Set $\alpha_s : [0, +\infty[ \to \Pi_0$, $\alpha_s(u) = \gamma(s) + un(s)$, and consider the lifting $\tilde{\alpha}_s : J \to M$ of $\alpha_s$ to $M$ with initial condition $\tilde{\alpha}_s(0) = \Gamma(s)$. The property $\Gamma - \{\Gamma(s)\} \subset \text{Ext}(\mathcal{G}_{\Gamma(s)})$ and equation (11) give $\tilde{\alpha}_s \cap \Gamma = \Gamma(s)$, hence from the properness of $\tilde{\alpha}_s$ we have $J = [0, +\infty[$.

On the other hand, $\gamma'(s) = ig(s) := ig(\gamma(s))$ implies that $n(s) = \pm g(s)$ (this ambiguity will be solved in the next Lemma). Taking into account that $g'(s) \neq 0$ for all $s \in \mathbb{R}$, we deduce that $\gamma$ is locally convex, and so $\gamma \cap \pi(V_s)$ lies at one side of the tangent line $r_s$ of $\gamma$ at $\gamma(s)$. Let $P^+_s$ and $P^-_s$ denote the two closed half planes in $\Pi_0$ bounded by $r_s$, and up to relabeling suppose $\gamma \cap \pi(V_s) \subset P^-_s$.

**Figure 3:** $\Gamma$, $\gamma$, $\gamma'(s)$, $g(s)$ and $n(s)$.

**Lemma 6.1** The normal vector $n(s)$ points to $P^+_s$ and $n(s) = g(s) = -i\gamma'(s)$.

As a consequence, $F_0 : \mathbb{R} \times [0, +\infty[ \to M$, $F_0(s, u) := \tilde{\alpha}_s(u)$ is a diffeomorphism.

**Proof:** Reason by contradiction and suppose there exists $s_0 \in \mathbb{R}$ such that $n(s_0)$ points to $P^-_{s_0}$. By a connection argument, in fact $n(s)$ points to $P^-_s$ for any $s \in \mathbb{R}$. Take $V_s$ as above, and without loss of generality suppose that $\pi(V_{s_0})$ is convex and contained in $P^-_{s_0}$ for all $s$ such that $\Gamma(s) \subset V_{s_0}$.

Take a segment $\zeta \subset \pi(V_{s_0})$ connecting two points $p, q \in \pi(\Gamma)$. Call $\tilde{\zeta} \subset V_{s_0}$ its corresponding lifting, and observe that $\tilde{\zeta}$ connects two points $\tilde{p}, \tilde{q} \in \Gamma \cap V_{s_0}$. The spacelike property gives $|t(q) - t(p)| < \|p - q\|_0$, which contradicts equation (11).

Therefore, $n(s)$ points to $\Pi^+_s$ for any $s$, and so $n'(s) = \kappa(s)\gamma'(s)$, $\kappa(s) > 0$. As $n(s) = \pm g(s)$, $g'(s) = \theta'(s)\gamma'(s)$ and $\theta'(s) > 0$, then $n(s) = g(s)$.

Since $F_0$ is a local diffeomorphism (we omit the details) and $F_0$ is injective (take into account the unique lifting property for $\pi|_M$), it suffices to check that $F_0$ is proper. Take a divergent sequence $\{(s_n, u_n)\}_{n \in \mathbb{N}} \subset \mathbb{R} \times [0, +\infty[$, and write $p_n := F_0(s_n, u_n)$, $n \in \mathbb{N}$. From equation (11), $p_0 \in \text{Ext}(\mathcal{G}_{\Gamma(s_0)})$ for any $n \in \mathbb{N}$, and so $\{p_n\}_{n \in \mathbb{N}}$ diverges provided that $\{s_n\}_{n \in \mathbb{N}}$ do. If $\{s_n\}_{n \in \mathbb{N}}$ is bounded, then $\{u_n\}_{n \in \mathbb{N}} \{\pi(p_{n_0})\}_{n \in \mathbb{N}}$ and $\{p_{n_0}\}_{n \in \mathbb{N}}$ diverge, concluding the proof. □

**Definition 6.1** We set $\Theta_0 : M \to \theta^-, \theta^+$, $\Theta_0(F_0(s, u)) := \theta(s)$, the argument function. For any subset $I \subset [\theta^-, \theta^+]$, we call $M^I := \Theta_0^{-1}(I) \subset M$, $\Gamma^I := M^I \cap \Gamma = \theta^{-1}(I)$ and $\gamma^I := \pi(\Gamma^I)$. 

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Note that \( \partial(M^I) = \Gamma^I \cup \overline{\alpha_{s_1}} \cup \overline{\alpha_{s_2}} \cup \Gamma^I \cup \overline{\alpha_{s_1}} \cup \overline{\alpha_{s_2}} \cup \Gamma^I \cup \overline{\alpha_{s_1}} \), provided that \( I = \theta(s_1), \theta(s_2)[, I = \theta^+, \theta(s_2)[ \) or \( I = \theta(s_1), \theta^+[, \) respectively, where \( s_1, s_2 \in \mathbb{R} \).

An open interval \( I \subset \theta^-, \theta^+ \) is said to be a good if it has length \( |I| \leq \pi \). A good interval \( I \) is said to be a tail interval if one of its endpoints lie in \( \{ \theta^-, \theta^+ \} \).

Let \( I = \theta^-, \theta(s_2)[ \) be a good interval, where \( s_1, s_2 \in \mathbb{R} \). Since the change of the tangent angle along \( \gamma^I \) is \( \leq \pi \), then \( \gamma^I \) is an embedded arc. From Lemma 6.1, the arcs \( \alpha_{s_1}, \gamma^I \) and \( \alpha_{s_2} \) are laid end to end and form an embedded divergent arc. Furthermore, \( \{ \alpha_s(0, +\infty[ \), \( s \in \{ s_1, s_2 \} \) is a foliation of the planar domain \( \Omega^I \subset \Pi_0 \) with boundary \( \pi(\partial(M^I)) \) and having \( n \) as the interior normal along \( \gamma^I \). It is clear that \( \overline{\Omega^I} \) is a wedge of angle \( \theta^2 \). \( \pi(M^I) \) and \( \pi : M^I \rightarrow \Omega^I \) is a diffeomorphism.

The case when \( I \) is a tail interval admits a similar discussion. If \( I = \theta^-, \theta(s_2)[ \) (resp., \( I = \theta(s_1), \theta^+[, \) then \( \gamma^I \) is a sublinear arc with direction \( -\gamma'(\pi) := -\operatorname{lim}_{s \rightarrow -\pi} \gamma'(s) = -ie^{-\pi} \) (resp., \( \gamma'(\pi) := \operatorname{lim}_{s \rightarrow +\pi} \gamma'(s) = ie^{\pi} \)), and \( \overline{\Omega^I} \) is the wedge of angle \( \theta^2 \). \( \pi(M^I) \) and \( \pi : M^I \rightarrow \Omega^I \) is a diffeomorphism.

**Lemma 6.2** If \( I \subset \theta^-, \theta^+ \) is a good interval then the closure of \( \Omega^I := \pi(M^I) \) is a wedge of angle \( \theta^2 \). \( \pi(M^I) \) and \( \pi : M^I \rightarrow \Omega^I \) is a diffeomorphism.

In the sequel we write \( M^I = \{ (x, u^I(x)) : x \in \Omega^I \} \), for any good interval \( I \subset \theta^-, \theta^+ \).

### 6.1 The blow down multigraph

Fix a sequence of positive real numbers \( \{ \lambda_j \}_j \in \mathbb{N} \) satisfying \( \lim_{j \rightarrow +\infty} \lambda_j = 0 \), and consider the associated blow down sequence of shrunk surfaces \( \{ M_j := \lambda_j \cdot M, \ j \in \mathbb{N} \} \).

From the conformal point of view, \( M_j = M \) and both Riemann surfaces have the same holomorphic Gauss map \( g \), thereby choosing the same branch \( \theta \) of \( \log(g) \) along \( \Gamma_j := \lambda_j \cdot \Gamma \). We also denote by \( \gamma_j := \pi \circ \Gamma_j \) and observe that \( \gamma_j = \lambda_j \cdot \gamma \). Lemma 6.1 applies to \( M_j \) and the corresponding diffeomorphism \( F_j : \mathbb{R} \times [0, +\infty[ \rightarrow M_j \) is now given by

\[
F_j(\lambda_j(s, u)) = \lambda_j F_0(s, u), \quad (s, u) \in \mathbb{R} \times [0, +\infty[.
\]

Likewise we define the argument function \( \Theta_j : M_j \rightarrow \theta^+, \theta^+[, \) the surface \( M_j \), and the curves \( \Gamma_j \Gamma_j \) and \( \gamma_j \). It is obvious that \( \Theta_j(\lambda_j, p) = \Theta_0(p) \) for any \( p \in M \), and \( M_j \). If \( I \) is a good interval, Lemma 6.2 gives that \( M_j \) is a graph over \( \Omega^I_j := \pi(M_j) = \lambda_j \cdot \Omega^I \) as well.

**Lemma 6.3** If \( I \subset \theta^-, \theta^+ \) is a good interval, then \( \{ \Omega^I_j \}_j \in \mathbb{N} \) converges in the Hausdorff distance to the domain \( \Omega^I_\infty \subset \Pi_0 \) given by:

\begin{enumerate}
\item[(i)] If \( I = \theta(s_1), \theta(s_2)[, \) \( s_1, s_2 \in \mathbb{R} \), then \( I = \theta^* \), and \( \Omega^I_\infty \) is the interior of the wedge of angle \( \theta^2 \), with boundary \( \alpha_{s_1}^0 \cup \alpha_{s_2}^0 \), where \( \alpha_{s_j}^0(u) = un(s_j) \) for any \( u \in [0, +\infty[, \ j = 1, 2 \).
\item[(ii)] If \( I = \theta^-, \theta(s_2)[, \) then \( I^* = \theta^-, \theta^+, \theta(s_2)[ \) and \( \Omega^I_\infty \) is the interior of the wedge of angle \( \theta^2 \), with boundary \( \alpha_{s_2}^0 \cup \alpha_{s_2}^0 \), where \( \alpha_{s_2}^0(u) = \theta^\gamma_\infty(\theta(s_2)[ \) for any \( u \in [0, +\infty[ \).
\item[(iii)] If \( I = \theta(s_1), \theta^*[ \), then \( I^* = \theta(s_1), \theta^+ + \frac{\pi}{2} \) and \( \Omega^I_\infty \) is the interior of the wedge of angle \( \theta^2 \), with boundary \( \alpha_{s_1}^0 \cup \alpha_{s_1}^0 \), where \( \alpha_{s_1}^0(u) = \theta^\gamma_\infty(\theta(s_1), \theta^+ + \frac{\pi}{2} \) for any \( u \in [0, +\infty[ \).
\item[(iv)] If \( I = \theta^-, \theta^+, \theta^*[ \), then \( I^* = \theta^- - \frac{\pi}{2}, \theta^+, \theta^+ \) and \( \Omega^I_\infty \) is the interior of the wedge of angle \( \theta^2 \), with boundary \( \alpha_{s_1}^0 \cup \alpha_{s_1}^0 \).
\end{enumerate}
Moreover, up to subsequences, \( \{ M^j \}_{j \in \mathbb{N}} \) converges in the \( C^0 \)-topology to a PS graph \( M^\ast \) over \( \Omega^\ast \) whose boundary \( \partial (M^\ast) := M^\ast \setminus M^\ast \) is given by:

(i) If \( I = \partial (s_1, \theta (s_2)] \) then \( \partial (M^\ast) = \hat{\alpha}^0 \cup \hat{\alpha}^0_j \), where \( \hat{\alpha}^0_j \) is a divergent arc with initial point \( O \) and projecting in a one to one way onto \( \alpha^0_j, j = 1, 2 \). 

(ii) If \( I = \partial (-\infty, 0] \) then \( \partial (M^\ast) = \hat{\alpha}^0 \cup \hat{\alpha}^0_{-\infty} \), where \( \hat{\alpha}^0_{-\infty} := \{ u \Gamma(-\infty) : u \in [0, +\infty[ \} \) and 
\( \Gamma(-\infty) = \lim_{u \to -\infty} \Gamma'(s) = \frac{1}{\sqrt{2}} (i e^{\theta}, 1) \).

(iii) If \( I = \partial (s_1, \theta (s_2)] \) then \( \partial (M^\ast) = \hat{\alpha}^0_{+\infty} \cup \hat{\alpha}^0_{s_1} \), where \( \hat{\alpha}^0_{+\infty} := \{ u \Gamma(+\infty) : u \in [0, +\infty[ \} \) and 
\( \Gamma(+\infty) = \lim_{u \to +\infty} \Gamma'(s) = \frac{1}{\sqrt{2}} (i e^{\theta}, 1) \).

(iv) If \( I = \partial (-\infty, 0], \theta^+ \) then \( \partial (M^\ast) = \hat{\alpha}^0_{-\infty} \cup \hat{\alpha}^0_\infty \).

Proof: For any \( s \in \mathbb{R} \) write \( \alpha^0_s := \lambda s \alpha_s \).

If \( I = \partial (s_1, \theta (s_2)] \), we have \( \partial (\Omega^j_i) = \alpha^0_{s_1} \cup \gamma^j_{s_2} \cup \alpha^0_{s_2} \), hence \( \lim_{j \to \infty} H (\partial (\Omega^j_i), \partial (\Omega^\ast)) = 0 \).

This is the case that \( \partial (\Omega^j_i) \to 0 \) as \( j \to \infty \), and proves (i). When \( I = \partial (-\infty, 0], \partial (\Omega^j_i) = \gamma^j_{+\infty} \cup \alpha^0_{s_2} \).

Taking into account that the divergent arc \( \gamma^j_j \) is sublinear with direction \( -\gamma^j_j \), hence \( \lim_{j \to +\infty} \Gamma^j_i = \hat{\alpha}^0_{-\infty} \) and \( \hat{\alpha}^0_\infty \subset \partial (M^\ast) \), proving (ii). The cases (iii) and (iv) are similar.

From Remark 5.1 and up to taking a subsequence, \( \{ M^j \}_{j \in \mathbb{N}} \) converges in the \( C^0 \)-topology to a PS graph \( M^\ast \) over \( \Omega^\ast \) that can be extended continuously to \( \Omega^\ast \).

Item (ii) is straightforward. If \( I = \partial (-\infty, 0], \theta^+ \) then \( \Gamma^j_i \) is a sublinear arc with direction \( \Gamma^j_i \), hence \( \lim_{j \to +\infty} \Gamma^j_i = \hat{\alpha}^0_{-\infty} \) and \( \hat{\alpha}^0_\infty \subset \partial (M^\ast) \), proving (ii). The cases (iii) and (iv) are similar. 

Let \( I \) denote the family of good intervals in \( \left\{ \theta^-, \theta^+ \right\} \), and set \( \mathcal{I}_0 = \left\{ a, b \in I : \left[ a, b \right] \subset \left[ \theta^-, \theta^+ \right] \right\} \).

Note that \( I \in \mathcal{I}_0 \) if and only if \( I = I^* \). That is to say, \( I \in \mathcal{I} \) if and only if \( I \) is not a tail interval.

Take a covering \( \mathcal{G} := \left\{ J_k : k \in \mathbb{N} \right\} \) of \( \left[ \theta^-, \theta^+ \right] \) by good intervals containing tail intervals, that is to say, containing intervals of the form \( \left[ \theta^-, b \right] \) provided \( \theta^- \in \mathbb{R} \), and likewise if \( \theta^+ \in \mathbb{R} \). For each \( k \in \mathbb{N} \) and from Lemma 5.3, we can find a subsequence (depending on \( k \)) of \( \{ M^j_k \}_{j \in \mathbb{N}} \) converging on compact subsets of \( \Omega^j_k \) to a PS graph. A standard diagonal process leads to a subsequence, namely \( \{ M^j_k \}_{k \in \mathbb{N}} \), such that \( \{ M^j_k \}_{k \in \mathbb{N}} \) converges in the \( C^0 \)-topology to a PS graph \( M^j_k \) on \( \Omega^j_k \) for all \( k \in \mathbb{N} \). Up to replacing \( \{ M^j_k \}_{k \in \mathbb{N}} \) by \( \{ M^j \}_{j \in \mathbb{N}} \), we can suppose that \( \{ M^j_k \}_{j \in \mathbb{N}} \to M^j_k \) for any \( k \in \mathbb{N} \) and since any \( I \in \mathcal{I} \) can be covered by finitely many intervals in \( \mathcal{G} \), we also have \( \{ M^j \}_{j \in \mathbb{N}} \to M^j \) for any \( I \in \mathcal{I} \).

Let \( \Omega^\ast \) denote the marked domain \( \{(x, I) : x \in \Omega^\ast \} \), \( I \in \mathcal{I} \), and set \( \mathcal{O} := \cup_{I \in \mathcal{I}} \Omega^j_k \). Let \( \mathcal{O} := \cup_{I \in \mathcal{I}} \Omega^j_k \), the direct sum of the topological spaces \( \{ \Omega^j_k, I \in \mathcal{I} \} \). Introduce the equivalence relation: \( (x_1, I_1) \sim (x_2, I_2) \) if and only if either \( J_1 \cap J_2 \neq \emptyset \) and \( x_1 = x_2 \in \Omega^j_k \) or \( x_1 = x_2 = 0 \).

**Definition 6.2** We set \( \mathcal{W} := \mathcal{O} / \sim \) and \( \mathcal{W}_0 := \{ (x, I) \in \mathcal{W} : I \in \mathcal{I}_0 \} \equiv \{ (x, I) \in \mathcal{W} : I \in \mathcal{I}_0 \} \equiv \left( \cup_{I \in \mathcal{I}_0} \Omega^j_k \right) / \sim \).

In the sequel we write \( [0] = \{ [0, I] \} \), and put \( \mathcal{W}^0 := \mathcal{W} \setminus \{ [0] \} \), \( \mathcal{W}_0^* := \mathcal{W}_0 \setminus \{ [0] \} \).

We can define a natural argument function \( \Theta \) : \mathcal{W} \to \{ \theta^-, \theta^+ \} \) as the limit of \( \{ \Theta_j : \mathcal{M}_j \to \theta^-, \theta^+ \} \). Indeed, take \( p = ((x, I)] \in \mathcal{W}_0 \) and an arbitrary sequence \( \{ p_j \}_{j \in \mathbb{N}} \) such that \( p_j = F_j (s_j, u_j) \in M^j_k \) and \( \lim_{j \to +\infty} p_j = (x, u^\ast_k (x)) \in M^j_k \). As the limit \( s := \lim_{j \to +\infty} s_j \) is in \( I \) only depends on \( p \) and \( \lim_{j \to +\infty} \Theta_j (p_j) = \lim_{j \to +\infty} \theta (s_j) = \theta (s) \), then we set \( \Theta \) : \( (x, I)] = \theta (s) \).

Since \( \mathcal{W}^* \) is simply connected, \( \log : \mathcal{W}^* \to \mathbb{C}, \log [((x, I))] := \log (x) \) has a well defined branch and the map \( h : \mathcal{W} \to \mathbb{R}, h ((x, I))] = (x, \log [((x, I))] ) \), \( x \neq 0 \), \( h ([0]) = [0] \) is a homeomorphism.
Moreover, we can choose the branch of \( \log : W^* \to \mathbb{C} \) in such a way that \( \Im(\log)|_{W_0} = \Theta_\infty \), that is to say, \( \arg((x, \log([[(x, I)]]) = \Theta_{\infty}([[(x, I)]) \) for any \( \{[(x, I)]\} \in W_0^* \).

We identify \( W \) with \( h(W) \) via \( h \) and consider \( W \subset R \). Up to this identification, \( z(p) = x \) provided that \( p = h([(x, I)]) \) and \( \arg|_{W_0} = \Theta_{\infty} \). Moreover, the closures of \( W^* = \arg^{-1}(\theta^- - \frac{\pi}{2}, \theta^+ + \frac{\pi}{2}) \) and \( W_0^* = \arg^{-1}(\theta^- + \pi, \theta^+) \) in \( R \) are wedges of angles \( \theta(M) + \pi \) and \( \theta(M) \), respectively. If \( \theta \in R \), we write \( W_- := \arg^{-1}([\theta^- - \frac{\pi}{2}, \theta^-]) \cup \{0\} \) and likewise \( W_+ := \arg^{-1}([\theta^+, \theta^+ + \frac{\pi}{2}]) \cup \{0\} \) provided that \( \theta^+, \theta^- \in R \). Obviously, \( W = W_0 \cup W_- \cup W_+ \) (here we are assuming \( W_\pm = \emptyset \) provided that \( \theta^\pm = \pm\infty \)).

Define \( u_\infty : W \to R \), \( u_\infty(p) := u_\infty^I(z(p)) \), where \( I \in \mathcal{I} \) is any interval satisfying \( \arg(p) \in I^* \), and call with the same name is continuous extension to \( W \). Since \( \Im(u_\infty) \leq 1 \), then the map \( \lambda : W \to R^3 \), \( \lambda(p, u_\infty(p)) := (z(p), u_\infty(p)) \) is a PS multigraph of angle \( \theta(M) + \pi \).

**Definition 6.3** \( \mathcal{X} \) is defined to be the blow down multigraph of \( M \) associated to the sequence \( \{\lambda_j\}_{j \in \mathbb{N}} \). We also say that \( \mathcal{M}_\infty := \mathcal{X}(W) \) is the blow down surface of \( M \) associated to \( \{\lambda_j\}_{j \in \mathbb{N}} \).

As \( \mathcal{X}(\{0\}) = 0 \), then equation (11) gives
\[
\mathcal{M}_\infty := \{ (z(p), u_\infty(p)) : p \in W \} \subset \mathcal{X}(\{0\}).
\]
Taking into account that \( \mathcal{M}_\infty \) is the limit set of the sequence of embedded surfaces \( \{\mathcal{M}_j\}_{j \in \mathbb{N}} \), the sheets of the multigraph \( \mathcal{X} : W \to \mathcal{M}_\infty \subset R^3 \) are ordered by height and we have:
\[
u_\infty(p) \geq u_\infty(q) \quad \text{provided that} \quad \arg(p) = \arg(q) + 2k\pi, \quad k \geq 0.
\]

Notice that \( \mathcal{X}_{\arg^{-1}(I^*)} : \arg^{-1}(I^*) \to \mathcal{M}_\infty^I \) is a homeomorphism for any good interval \( I \in \mathcal{I} \), and so \( \mathcal{M}_\infty^I \) can be identified with \( \arg^{-1}(I^*) \) via \( \lambda \). Then, it is natural to set \( W^I := \arg^{-1}(I) \subset W \) and put \( \mathcal{M}_\infty^I = \mathcal{X}(W^I) \), for any subset \( I \subset [\theta^- - \frac{\pi}{2}, \theta^+ + \frac{\pi}{2}] \). When \( I \) is connected, \( W^I \) is a wedge and \( \mathcal{X}_{\arg^{-1}(I^*)} : W^I \to R^3 \) is a PS multigraph of angle \( \{0\} \).

Label \( \mathcal{A}_X = (\mathcal{X}(W^I) \setminus \{0\}) \cap W \) as the set of singular points of \( u_\infty \) (see Theorem 5.2), where we have made the convention \( \{0\} \) in \( \mathcal{A}_X \). A point \( \xi \in \arg(W \setminus \{0\}) \) is said to be a singular angle if \( \arg^{-1}(\xi) \subset \mathcal{A}_X \). The subset of singular angles in \( \arg(W \setminus \{0\}) \) will be denoted by \( I_X \). A singular angle \( \xi \in \mathcal{A}_X \) is said to be conical if \( \mathcal{A}_X(\arg^{-1}(\xi)) \) is a lightlike half line (that is to say, \( \arg^{-1}(\xi) \) is a singular segment of \( u_\infty \). The set of conical singular angles will be denoted by \( I^c_X \). Note that \( I_X \) and \( I^c_X \) are closed subsets of \( \mathcal{X}(W \setminus \{0\}) \).

**Lemma 6.4** If \( \mathcal{A}_X \neq \emptyset \) then \( \mathcal{A}_X = \mathcal{X}(W^I) \), \( I_X \) is the convex hull of \( I^c_X \) and \( I^c_X \neq \emptyset \). Moreover:

1. If \( I_X \neq I^c_X \neq \emptyset \) then \( \mathcal{M}_\infty^{I_X - I^c_X} \subset \Sigma \), where \( \Sigma \) is the lightlike plane passing through \( O \). Moreover, \( \mathcal{M}_\infty^{I_X} \) is a half plane in \( \Sigma \) bounded by \( L := \Sigma \cap C_0 \), for any connected component \( J \) of \( I_X - I^c_X \).

2. If \( \theta^- \in R \) (resp., \( \theta^+ \in R \)) then \( \mathcal{A}_X = [\theta^- - \frac{\pi}{2}, \theta^- + \frac{\pi}{2}] \cup \{\xi_0, k\pi : k \in N, k < m\} \), (resp., \( \mathcal{A}_X = [\theta^+ + \frac{\pi}{2}, \theta^+ + \frac{\pi}{2}] \cup \{\xi_0, k\pi : k \in N, k < m\} \)), where \( m \in N \cup \{+\infty\} \) and \( \xi_0 \in [0, \pi] \). Furthermore, if \( \theta(M) < +\infty \) then \( \xi_0 = 0 \) and \( \theta(M)/\pi \) is an even integer.

3. If \( \theta^-, \theta^+ \in R \), then \( \mathcal{A}_X = \{\xi_0 - k\pi : k \in Z, m_1 < k < m_2\} \), where \( \xi_0 \in R \) and \( m_1, m_2 \in Z \cup \{-\infty, +\infty\} \).

Proof: If \( \theta(M) < +\infty \) then \( \mathcal{A}_X \) is proper and for large enough \( R > 0 \) the region \( \mathcal{M}(R) := \mathcal{M} \cap \pi^{-1}\{x \in P_0 : \|x\|_0 \geq R\} \) is a multigraph of angle \( \theta(M) + \pi \). Corollary 5.2 gives that \( \mathcal{M}(R) \) (hence \( \mathcal{M} \)) has sublinear growth over a lightlike plane and \( \theta(M) = m\pi, m \in N \). Therefore \( \mathcal{M}_\infty \) is the lightlike plane containing the lightlike straight line \( L = a_0^\infty \cup a_0^- \), \( I_X = [\theta^-, \theta^+ + \frac{\pi}{2}] \) and \( I^c_X = [\theta^- - \frac{\pi}{2}, \theta^+ + k\pi : k = 0, \ldots, m + 1] \).
In the sequel we will suppose that $\theta(M) = \infty$, and up to Lorentzian isometries, that $\theta^+ = +\infty$.

Let us see that $\xi \in I_X$ if and only if $\arg^{-1}(\xi) \cap A_X \neq \emptyset$. Indeed, suppose $\arg^{-1}(\xi) \cap A_X \neq \emptyset$ and take $p \in \arg^{-1}(\xi) \cap A_X$. From Remark 5.2, there is a divergent arc $l_p \subset V - \{0\}$ passing through $p$ such that $L_p := \mathcal{X}(l_p)$ is either a lightlike half line starting from $O$ or a complete lightlike straight line (in particular, $L_p \subset A_X$).

If $O \in L_p$ then $\xi \in I_X \subset I_X$ and we are done. If $O \notin L_p$, then $p \in A_X - \mathcal{W}^{t_k}$, $l_p \subset W - \{0\}$ and $L_p$ is a lightlike straight line. Then, consider the open half plane $H \subset \Pi_0$ satisfying $\pi(L_p) \subset H$ and $O \in \partial(H)$, and label $V_\xi$ as the connected component of $(\pi \circ \mathcal{X})^{-1}(H)$ containing $L_p$. From Lemma 2.1, $V_\xi$ lies in the lightlike plane containing $L_p$, and so $V_\xi \subset A_X$. As $\arg^{-1}(\xi) \subset V_\xi \subset A_X$ then $\xi \in I_X$, proving our assertion. As a consequence, $A_X = \mathcal{W}^{t_k}$ and $I_X \neq \emptyset$.

Assume that $I_X - I_X^\infty \neq \emptyset$ and consider $\xi \in I_X - I_X^\infty$. If $J \subset I_X - I_X^\infty$ is the connected component containing $\xi$, we have shown that $|J| = \pi$, $L_J := \partial(V_\xi) \subset \mathcal{M}_\infty^{t_k}$ is a connected component of $\mathcal{M}_\infty^{t_k}$ is a lightlike straight line and the lightlike half plane $V_\xi = \mathcal{M}_k^J$ is a connected component of $\mathcal{M}_\infty^{t_k}$. Therefore, to finish item (i) it suffices to check that the plane $\Sigma_j$ containing $\mathcal{M}_\infty^{t_k}$ (hence $L_J$) does not depend on $J$. Indeed, take two components $J_1$, $J_2 \subset I_X - I_X^\infty$, and observe that the lightlike planes $\Sigma_{j_1}$ and $\Sigma_{j_2}$ can not meet transversally because $\mathcal{M}_\infty$ is the limit set of a sequence of embedded surfaces. In the sequel we will call $\Sigma := \Sigma_j$ and $L := L_J$ provided that $I_X - I_X^\infty \neq \emptyset$ and $J \subset I_X - I_X^\infty$ is any connected component.

**Claim 1:** If $I := [\xi_2, \xi_2] \subset \arg(W^n) - I_X$ is a bounded connected component with closure $[\xi_1, \xi_2] \subset \arg(W^n)$, then $\xi_1, \xi_2 \in I_X^\infty$ and $\xi_2 - \xi_1 = k\pi$, $k \in \mathbb{N}$, $k \geq 3$. Moreover, $\mathcal{M}_\infty^J$ is an embedded multigraph whose limit tangent plane $\Sigma' \neq \xi$ at infinity is lightlike and does not depend on $I$. If in addition $I_X - I_X^\infty \neq \emptyset$, then $\Sigma' = \Sigma$ and $\partial(\mathcal{M}_\infty^J) \subset L$.

*Proof:* From item (i), $\xi_1, \xi_2 \in I_X^\infty$. Thus, $l_j := \mathcal{X}(\arg^{-1}(\xi_j))$ is a lightlike half line with initial point at the origin, $j = 1, 2$. Since $\mathcal{M}_\infty$ is the limit of a sequence of embedded maximal surfaces, it is not hard to check that $\mathcal{M}^J$ is embedded too. From Corollary 5.2, the half lines $l_1$ and $l_2$ must be parallel, hence $\xi_2 = \xi_1 + k\pi$, $k \in \mathbb{N}$. Furthermore, $\mathcal{M}_\infty^J$ has the lightlike plane $\Sigma'$ containing $l_1 \cup l_2$ as the limit tangent plane at infinity.

If $k = 1$, Lemma 2.1 implies that $\mathcal{M}_\infty^J$ lies in a lightlike plane, a contradiction. If $k = 2$, Proposition 5.3 gives that $\mathcal{M}_\infty^J$ is congruent in the Lorentzian sense to the Enneper graph $E_2$, hence from Remark 3.1, $\mathcal{M}_\infty^J$ is asymptotic to the light cone at the origin. Fix $\epsilon \in ]0, \Xi_0(2\pi)]$ (see Corollary 6.1) and up to a dilation suppose that $\mathcal{M}_\infty^J \cap \{(x, t) : 1 \leq \|x\|_0 \leq 2\}$ lies in a neighbourhood of radius $\frac{\epsilon}{2}$ of $C_0$. Since $\mathcal{M}_\infty^J \cap \{(x, t) : 1 \leq \|x\|_0 \leq 2\} \cap \{t \neq 0\} \Rightarrow \mathcal{M}_\infty^J \cap \{(x, t) : 1 \leq \|x\|_0 \leq 2\}$ uniformly, there exists $j_0 > 0$ such that $\mathcal{M}_\infty^J \cap \{(x, t) : 1 \leq \|x\|_0 \leq 2\}$ lies in a neighbourhood of radius $\epsilon$ of $C_0$ for every $j \geq j_0$. Therefore, $\tau_0(\mathcal{M}^J) \leq \epsilon < \Xi_0(2\pi)$, which is absurd and proves that $k \geq 3$.

Finally, let $I_1, I_2 \subset \arg(W^n) - I_X$ be two components as in the claim. Reasoning as in the proof of item (i), the embedded multigraphs $\mathcal{M}_\infty^{I_1}, \mathcal{M}_\infty^{I_2}$ can not meet transversally, and so $\Sigma' := \Sigma_{I_1} = \Sigma_{I_2}$. Likewise $\Sigma$ and $\mathcal{M}_\infty^J$ can not meet transversally, provided that $I_X - I_X^\infty \neq \emptyset$ and the plane $\Sigma$ makes sense, and in this case $\Sigma' = \Sigma$. □

**Claim 2:** If $J_0$ is a connected component of $I_X^\infty$ distinct of a point, then $J_0$ is a tail interval with endpoint $\theta^-$.

*Proof:* It is clear that $\mathcal{M}_\infty^{J_0} \subset C_0$, hence a connection argument gives that either $\mathcal{M}_\infty^{J_0} \subset C_0^-$ or $\mathcal{M}_\infty^{J_0} \subset C_0^+$. Let us show first that $|J_0| \leq \pi$. Reason by contradiction, and assume that $|J_0| > \pi$. Note that $\mathcal{M}_\infty^{J_0} \subset C_0$ and fix $\epsilon \in ]0, \Xi_0(|J_0|)]$ (see Corollary 5.1). From the definition of $\mathcal{M}_\infty^{J_0}$, we can find large enough $j_0 \in \mathbb{N}$ such that $\mathcal{M}_j^{J_0} \cap \{(x, t) : 1 \leq \|x\|_0 \leq 2\}$ lies in an Euclidean neighbourhood of $C_0$ for every $j > j_0$. Then, $\tau_0(\mathcal{M}_\infty^{J_0}) \leq \epsilon < \Xi_0(2\pi)$, which is absurd and proves that $|J_0| \leq \pi$. □
of radius $\epsilon$ of $C_0 \cap \{(x,t) : 1 \leq \|x\| \leq 2\}$ for any $j \geq j_0$. This shows that $\tau_0(M^{j_0}) \leq \epsilon < \Xi_0(|J_0|)$, a contradiction.

Write $J_0 = [\xi^-,\xi^+]$. Let us check that $\xi^- = \theta^-$ provided that $M^{j_0}_\infty \subset C_0^-$. Once again reason by contradiction, and suppose $\theta^- > -\frac{\pi}{2} < \xi^-$. Set $\xi_0 := \max\{\theta^- - \frac{\pi}{2}, \xi^+ - 2\pi\} < \xi^-$. Now, if $\xi_0 = \xi^+ - 2\pi$, equations (13) and (12) give that $\partial(M^{j_0}_\infty) = \partial(M^{j_0}_\infty)$, and so $\xi_0 \in I_X$ and $\partial \arg^{-1}(\xi_0) \subset C_0^-$. The same conclusion can be obtained when $\xi_0 = \theta^- - \frac{\pi}{2}$. Therefore, Claim 1 and the fact $|J_0| < 2\pi$ imply that $I_0 \cap (\arg(W^*) - I_X) = \emptyset$, that is, $I_0 \subset I_X$. Since $J_0$ is a connected component of $I_X$, item (i) gives that $\xi^- - \xi_0 \geq \pi$, $I_1 := [\xi^- - \pi, \xi^- - \xi_0] \subset I_X - I_X^c$ and $\xi^- - \pi \in I_X^-$. Furthermore, as $\xi^+ - \pi < \xi^- < \pi$ then item (i) yields that $I_2 := [\xi_0, \xi^+ - \pi] \subset I_X$. Thus, $M^{j_0}_\infty \subset C_0$, and so $M^{j_0}_\infty \subset C_0$ because $\arg^{-1}(\xi_0) \in C_0^-$. However, item (i) gives that $M^{j_0}_\infty$ is a half plane in $\Sigma$ bounded by $L$, and so $\partial \arg^{-1}(\xi^- - \pi) \subset C_0^+$, a contradiction. Likewise, the option $M^{j_0}_\infty \subset C_0^+$ leads to $\xi^+ = \theta^+$, which is contrary to the assumption $\theta^+ = +\infty$. □

It remains to check that $I_X$ is the convex hull of $I_X^c$ (this fact together with item (i) and Claim 2 imply item (ii)). Taking into account the preceding discussion, it suffices to see that $\arg(W^*) - I_X$ contains no bounded connected components.

Reason by contradiction and take such a component $I = \overline{[\xi_1, \xi_2]}$. Label $I_j = \partial \arg^{-1}(\xi_j) \subset C_0$, $j = 1, 2$, and use Claim 1 to get $\xi_2 - \xi_1 = k\pi$, $k \geq 3$, and $I_1 \cup I_2 \subset L$. Equation (14) gives that $\partial \arg^{-1}(\xi_2 - 2\pi) \subset \text{Ext}(C_0)$, and equation (13) gives that $\partial \arg^{-1}(\xi_2 - 2\pi)$ lies below $I_2$. Therefore, $I_2 \subset C_0^+$ and likewise $I_1 \subset C_0^-$. This also shows that $L = I_1 \cup I_2$, i.e., $k$ is an even integer $\geq 4$. By a similar argument, $\partial \arg^{-1}(\xi_2 + 2m\pi) = I_2$ for any integer $m \geq 1$, hence from Claims 1 and 2 we have that $I_1 = [\xi_2, \xi_2 + 2\pi] \subset I_X$. Item (i) implies that $M_{-\infty}^{l_1} = \Sigma$, and since $M_{-\infty}^{l_1}$ lies above $M_{-\infty}^{l_2}$ (see equation (13)), we deduce from equation (12) that $\partial \arg^{-1}(\xi_2 - \pi) = I_2$. This contradicts that $\overline{[\xi_1, \xi_2]} \subset \arg(W^*) - I_X$ and concludes the proof. □

If $\theta^+ = +\infty$, (resp., $\theta^- = -\infty$) label $u^+(x) := \sup\{u_{\infty}(p) : p \in z^{-1}(x)\}$, (resp., $u^-(x) := \inf\{u_{\infty}(p) : p \in z^{-1}(x)\}$), for any $x \in \Pi_0$, and call $M_{-\infty} := \{(x, u^+(x)) : x \in \Pi_0\}$ (resp., $M_{+\infty} := \{(x, u^-(x)) : x \in \Pi_0\}$) the associated graph. From Remark 5.1 and equation (12), $M_{-\infty}$ and $M_{+\infty}$ are entire PS graphs containing the origin, and so lying in $\text{Ext}(C_0)$.

Any domain in $M_{-\infty} \setminus \{0\}$ (resp., $M_{+\infty} \setminus \{0\}$) is the limit in the $C^0$ topology of a sequence of maximal graphs. Therefore, the singular segments of $u^+$ (resp., $u^-$) are either straight lines or half lines starting from the origin in $\Pi_0 \setminus \{0\}$ (see Theorem 5.2 and Remark 5.2). We will call $A^+$ (resp., $A^-$) as the singular set of $u^+$ (resp., $u^-$) in $\Pi_0 \setminus \{0\}$.

Next theorem is the main result of this section.

**Theorem 6.1** The following statements hold:

(i) If $\theta(M) < +\infty$ then $M_{-\infty}$ is a lightlike plane and $\theta(M) = 2k\pi$, $k \in \mathbb{N}$.

(ii) If $\theta^+ = +\infty$ (resp., $\theta^- = -\infty$) then $M_{-\infty}$ is a non timelike plane.

(iii) If $\theta^+ = +\infty$ (resp., $\theta^- = -\infty$) and $\theta^+ \in \mathbb{R}$ then $A_X = W$, $[\theta^- + \frac{\pi}{2}, +\infty] \subset I_X - I_X^c$ (resp., $[\theta^- - \frac{\pi}{2}, -\infty] \subset I_X - I_X^c$) and $M_{-\infty}^{l_1 - l_1}$ is a lightlike plane.

**Proof:** Assume that $\theta(M) < +\infty$. Like in the proof of Lemma 6.4, $I_X = \overline{I_X - I_X^c} = \arg W$, $M_{-\infty}$ is a lightlike plane and $\theta(M) = m\pi$, $m \in \mathbb{N}$. Furthermore, Lemma 6.3 gives that $\partial_{-\infty} \subset C_0^-$ and $\partial_{+\infty} \subset C_0^+$, hence $m$ is even and (i) holds.

To check (ii), we need the following claim:

**Claim 1:** If $\theta^+ = +\infty$ (resp., $\theta^- = -\infty$) then $M^+_{-\infty}$ (resp., $M^-_{+\infty}$) is a non timelike plane.
Proof: Suppose $\theta^+ = +\infty$ (the case $\theta^- = -\infty$ is similar).

Taking into account Remark 5.2 we have four possibilities: (a) $A^+ = \emptyset$, (b) $A^+$ contains a straight line as singular segment, (c) $\Pi_0$ is foliated by singular segments of $u^+$, and all of them are half straight lines starting from $O$, or (d) $u^+$ is smooth and defines a maximal graph $G$ on the interior of a wedge $W \subset \Pi_0$ with vertex at $O$.

In case (a), it is well known that $M_\infty^+$ is either a spacelike plane or a half of the Lorentzian catenoid, see [4]. In case (b), Lemma 2.1 implies that $M_\infty^+$ is the lightlike plane.

Let us see that (c) is impossible. Indeed, in this case $M_\infty^+ = \cup_{p \in M_\infty^+ - \{0\}} L_p$, where $L_p$ is a lightlike half line containing $p$ and passing through the origin, hence either $M_\infty^+ = G_0^+$ or $M_\infty^+ = C_0^-$. Fix a compact interval $I \subset \theta^-, +\infty$ of length $\theta \in [\pi, 2\pi]$ and $\epsilon \in [0, \Xi_0(\theta)]$. From the definition of $M_\infty^+$, there exists $k_0 \in \mathbb{N}$ such that $M_\infty^{2k_0\pi + I} \cap \{(x, t): 1 \leq ||x||_0 \leq 2\}$ lies in an Euclidean neighborhood of radius $\epsilon/2$ of $C_0 \cap \{(x, t): 1 \leq ||x||_0 \leq 2\}$. On the other hand, and from the definition of $M_\infty^{2k_0\pi + I}$, we can find $j_0 \in \mathbb{N}$ such that $M_{j_0}^{2k_0\pi + I} \cap \{(x, t): 1 \leq ||x||_0 \leq 2\}$ lies in an Euclidean neighborhood of radius $\epsilon/2$ of $M_{j_0}^{2k_0\pi + I} \cap \{(x, t): 1 \leq ||x||_0 \leq 2\}$, for any $j \geq j_0$. This shows that $M_{j_0}^{2k_0\pi + I} \cap \{(x, t): 1 \leq ||x||_0 \leq 2\}$ lies in an Euclidean neighborhood of radius $\epsilon$ of $C_0 \cap \{(x, t): 1 \leq ||x||_0 \leq 2\}$, for any $j \geq j_0$, and proves that $\tau_0(M_\infty^{2k_0\pi + I}) \leq \epsilon < \Xi_0(\theta)$, contradicting Corollary 5.1.

Suppose (d) holds and label as $l_j$ as the two lightlike half lines in $\partial(G)$. From Corollary 5.2 $l_1$ and $l_2$ must lie in the same lightlike straight line $L$, and since $l_1 \cup l_2 \neq L$ (otherwise from (b) $M_\infty^+$ would be a lightlike plane, impossible), we infer that $l_1 = l_2$ and $M_\infty^+$ is congruent in the Lorentzian sense to the entire Enneper graph $E_2$ (see Proposition 5.3).

Summarizing, in order to prove the claim it suffices to check that $M_\infty^+$ cannot be neither a half of the Lorentzian catenoid nor an Enneper’s graph $E_2$. But in both cases $M_\infty^+$ is asymptotic to the light cone at the origin (see Remark 3.1), and reasoning as above, we can take $\theta \in [\pi, 2\pi]$, define $I = [0, \theta]$, fix $\epsilon \in [0, \Xi_0(\theta)]$, and find $k_0 \in \mathbb{N}$ such that $\tau_0(M_\infty^{2k_0\pi + I}) \leq \epsilon < \Xi_0(\theta)$, getting a contradiction and proving the claim. \hfill \Box

Now we can prove (ii). From Claim 1, $M_\infty^+$ and $M_\infty^-$ are non timelike planes passing through the origin. If $M_\infty^+ \neq M_\infty^-$, they meet transversally, which contradicts that $M_\infty^+$ and $M_\infty^-$ lie in the limit set of a sequence of embedded surfaces.

Finally let us see (iii). We only deal with the case $\theta^- \in \mathbb{R}$, $\theta^+ = +\infty$ (the other one is similar).

Reason by contradiction and suppose $A_X \neq W$, i.e., $I_X \neq \arg(W^*)$. From Lemma 5.4 $M_\infty$ must contain an embedded maximal multigraph $M_\infty^-$, where $J = [a, +\infty] = \arg(W^*) - I_X$. Let us show that $M_\infty^-$ must be a spacelike plane. Indeed, otherwise $M_\infty^+$ is lightlike and equation (13) gives that the lightlike straight line $L := M_\infty^+ \cap C_0$ lies above $M_\infty^-$. From equation (12) we deduce that $L \cap C_0^- \subset M_\infty^-$, contradicting that $A_X \cap W^J \neq \emptyset$.

Take an arc $c \subset M_\infty^-$ projecting onto a divergent arc $\pi(c)$ in $\Pi_0$ with initial point $O$ and satisfying $\lim_{x \to +\infty} \partial(c, M_\infty^+) = 0$. Let us see that the limit $\lim_{x \to +\infty} g(x) = \text{st}(v)$, where $v \in \mathbb{H}^2$ is the Lorentzian normal to $M_\infty^+$. Indeed, take a divergent sequence $\{p_n, n \in \mathbb{N}\} \subset c$ and consider the sequence of multigraphs $\{G_n := M_\infty^+ - p_n, n \in \mathbb{N}\}$. The sequence $\{G_n\}$ converges in the $C^0$-topology to an entire PS graph $G_\infty$ over $\Pi_0$. Since $G_\infty$ lies in a half space bounded by the spacelike plane $M_\infty^+$, $G_\infty$ is an entire maximal graph (see Remark 5.2 and Lemma 2.1). By Calabi’s theorem, $G_\infty$ is a spacelike plane, and so $G_\infty = M_\infty^+$. Proposition 5.1 proves our assertion.

On the other hand, set $l$ the lightlike half line $\partial(M_\infty^+) = X(\arg^{-1}(a))$ (eventually, $a = \theta^- - \frac{\pi}{2}$, $l = \Gamma^0_\infty$ and $W^J = W$), and denote by $N_0 \subset W^J$ the proper region bounded by $c \cup l$. Consider a new proper region $N_0'' \subset N_0$ having $\partial(N_0'') = c \cup l'$, where $l' \subset \text{Int}(N_0)$ is a divergent arc close enough to $l$ in such a way that $\lim_{x \to -\infty} g(x) = \text{st}_0(w)$, where $w$ is the lightlike direction of $l$ and $g$ is the holomorphic Gauss map of $N_0''$. Since $N_0''$ lies in a half space bounded by $M_\infty^+$, it is parabolic (see Corollary 3.1). However, $\lim_{x \to -\infty} g(x) = \text{st}(v) \neq \lim_{x \to l' - \infty} g(x)$, because $\text{st}(v) \in M$ and $\text{st}_0(w) \in \partial(D)$. 25
This is contrary to Theorem 4.1 and shows that \( \mathcal{A}_W = \mathcal{W} \). In particular, \( \mathcal{M}_+^{\mathcal{T}_\infty - \mathcal{T}_L} = \mathcal{M}_\infty^{\mathcal{I}_\infty - \mathcal{I}_L} \) is the lightlike plane \( \Sigma \) given in Lemma 6.3 proving (iii).

\[ \square \]

**Definition 6.4** The plane \( \Sigma_\infty := \mathcal{M}_\infty^{\mathcal{I}_\infty - \mathcal{I}_L} \) is defined to be the limit plane of the homothetical blow down \( \{ \mathcal{M}_n \}_{n \in \mathbb{N}} \) of \( \mathcal{M} \). In the cases \( \theta^+, \theta^+ |\mathbb{R} \) or \( \theta(\mathcal{M}) < +\infty \), \( \Sigma_\infty \) coincides with \( \mathcal{M}_\infty \).

### 6.2 The transversality of \( \mathcal{M} \) and the blow down plane \( \Sigma_\infty \).

Let \( c \) be a lightlike ray in \( \mathcal{M} \) (see Definition 6.1), call \( l_c \) the lightlike half line to which \( c \) is asymptotic and write \( L_c := \pi(c) = \pi(l_c) \). Putting \( L_c = \{ x_0 + u e^\xi : u \geq 0 \} \), there is a unique real number \( \xi_c \) congruent to \( \xi \) modulo \( 2\pi \) such that \( c \in \mathcal{M}_I \), where \( J = [\xi_c - \frac{\pi}{2}, \xi_c + \frac{\pi}{2}] \).

As a consequence, the limit \( \theta_c := \lim_{x \to -\infty} \Theta_0(x) \in [\theta^-, \theta^+] \) exists and is a finite real number. The arguments \( \theta_c \) and \( \xi_c \) coincide provided that \( \theta_c \in [\theta^+, \theta^-] \). If \( \theta_c \in (\theta^+, \theta^-) \), then \( J \) is a tail interval and either \( \xi_c \in [\theta^+, \theta^+ + \frac{\pi}{2}] \) or \( \xi_c \in [\theta^- - \frac{\pi}{2}, \theta^-] \) (see Lemma 6.2).

**Lemma 6.5** If \( \mathcal{M} \) admits an upward (resp., downward) lightlike ray \( c \), then \( \theta^+ = \theta_c + \frac{\pi}{2} \) (resp., \( \theta^- = \theta_c - \frac{\pi}{2} \)) and \( \theta(\mathcal{M}) = +\infty \).

**Proof:** We only deal with the case when \( c \subset \mathcal{M} \) is an upward lightlike ray.

**Claim 1:** \( \theta_c + \frac{\pi}{2} \geq \theta^+ \).

**Proof:** Reason by contradiction, and assume that \( \theta_c + \frac{\pi}{2} < \theta^+ \). Write \( \Theta_0^{-1}(\theta_c) = F_0(s_c, \cdot) \), \( s_c \in \mathbb{R} \), and let us show that \( \tilde{\alpha}_{s_c} = F_0(s_c, \cdot) \) is an upward lightlike ray too. Indeed, since the parallel half lines \( \alpha_{s_c} \) and \( L_c \) satisfy \( \delta(\alpha_{s_c}, L_c) < +\infty \), the spacelike condition gives that \( \delta(\tilde{\alpha}_{s_c}, L_c) \leq \sqrt{2}\delta(\alpha_{s_c}, L_c) \). Taking into account that \( \tilde{\alpha}_{s_c} \) has slope \( < 1 \) and \( \tilde{\alpha}_{s_c} - \{ \Gamma(s_c) \} \subset \text{Ext}((\Gamma(s_c)) \subset L_c \). We deduce that \( \lim_{x \to \alpha_{s_c} - \infty} d(x, \tilde{C}_G(s_c)) < +\infty \), proving the assertion.

Write \( L_c := [\theta_c, \theta^+] \) and take an open half space \( H \subset \mathbb{R}^3 \) containing \( \tilde{\alpha}_{s_c} \), whose boundary plane is lightlike and parallel to \( l_c \). From equation (1), we have that \( \Gamma^{\mathcal{I}_c} \subset H \), hence \( \partial(\mathcal{M}_c) \subset H \).

Let us see that \( \mathcal{M}_c \subset H \). Indeed, label \( l_0 \in \partial(H) \) as the complete lightlike straight line for which \( \alpha_{s_c} \subset L_0 := \pi(l_0) \), let \( L' \) denote the half line \( L_0 - \alpha_{s_c} \) and set \( c' \) a connected component of \( \pi^{-1}(L') \cap \mathcal{M}_c \). Since any endpoint of \( c' \) lies in \( \partial(\mathcal{M}_c) \subset H \), property (1) gives that \( c' \subset H \), and therefore \( \mathcal{M}_c \) is disjoint from \( l_0 \).

Assume that \( \mathcal{M}_c \neq \emptyset \), and take a connected component \( A \) of \( \mathcal{M}_c \). It is clear that \( A \cap \partial(\mathcal{M}_c) = \emptyset \), and so from the maximum principle \( A \) is simply connected. Moreover, \( \pi|_A : A \to \Pi_0 \) is a proper local embedding, hence \( A \) is a graph over \( \Pi_0 \). Taking into account that \( A \cap l_0 = \emptyset \) and \( \partial(A) \subset \partial(H) \), we infer that \( G := A \cup (\partial(H) - \pi^{-1}(\pi(A))) \) is a PS entire graph over \( \Pi_0 \) containing \( l_0 \). But Lemma 6.1 gives \( G = \partial(H) \), which is absurd. This proves that \( \mathcal{M}_c \subset H \).

Let \( s_0 \) be the unique real number satisfying \( \Theta_0(\Gamma(s_0)) = \theta_c + \frac{\pi}{2} \), and call \( \Gamma_{\mathcal{S}(s_0)} \mathcal{M} \) as the tangent plane to \( \mathcal{M} \) at \( \Gamma(s_0) \) (that is to say, the lightlike plane parallel to the vector \( \Gamma(s_0) \)). Observe that \( T_{\mathcal{S}(s_0)} \mathcal{M} \) and \( \partial(\mathcal{M}) \) are parallel, and call \( H' \) as the open half space with boundary \( \partial(H') = T_{\mathcal{S}(s_0)} \mathcal{M} \) and containing \( \partial(H) \). Let \( y : \mathcal{M} \to \mathbb{R} \) denote the harmonic coordinate function vanishing on \( \partial(H') \cap \mathcal{M} \). From equation (2), the holomorphic 1-form \( dy \) (that can be reflected holomorphically to the mirror surface \( \mathcal{M}^* \)), has a zero or order \( \geq 2 \) at \( \Gamma(s_0) \). Therefore, \( \mathcal{M}_c - \partial(H') \) has at least a connected component \( A \) lying in the slab \( H' \cap \mathcal{H} \) and with boundary \( \partial(A) \subset \partial(H') \).

As before, \( A \) is a graph over \( \Pi_0 \) and \( A_0 := A \cup (\pi|_{\partial(H')})^{-1}(\Pi_0 - \pi(A)) \) is an entire PS graph over \( \Pi_0 \). Take \( p_0 \in A - \partial(A) \) and a neighborhood \( D_0 \subset A - \partial(A) \) of \( p_0 \) projecting via \( \pi \) onto a closed disc. From equation (1), \( D_0 - \{ p_0 \} \subset \text{Ext}(C_{p_0}) \) and \( \delta := d(\partial(D_0), C_{p_0}) \) is positive. As a consequence, the PSI graph \( A_0 - D_0 \) lies in \( \text{Ext}(C_{p_0}) \) and \( d(A_0 - D_0, C_{p_0}) \geq \delta > 0 \). This shows that \( A_0 \) lies in \( \{ x \in \mathbb{R}^3 : ||x|| \geq \delta \} \) up to a compact subset, and the same holds for \( A \). From Theorem 4.2 \( A \) is a parabolic. As \( \partial(A) \subset \partial(H') \) then \( A \subset \partial(H') \), which is absurd and proves the claim. \( \square \)
Claim 1 gives that \( \theta^+ \in \mathbb{R} \). As \( \mathcal{M}^c \) is, up to removing a compact subset, a multigraph, \( \partial(\mathcal{M}^c) = \Gamma_c \cup \widetilde{\alpha}_s \) and \( \Gamma_c \) and \( \widetilde{\alpha}_s \) are sublinear arcs with lightlike direction, then they have the same direction by Corollary 5.2. This shows that \( \theta^+ = \theta_c + \frac{4\pi}{3} \), concluding the first part of the lemma.

It remains to check that \( \theta^- = -\infty \). Reason by contradiction and assume \( \theta^- \in \mathbb{R} \). As above, Corollary 5.2 gives that \( \mathcal{M} \) is parabolic, hence \( \mathcal{M} \) is biholomorphic to \( \mathbb{C} \cup \{z \in \mathbb{C} : \text{Im}(z) \geq 0\} \). Let \( X : \overline{\mathbb{U}} \to \mathbb{R}^3_+ \) be a conformal maximal embedding satisfying \( X(\overline{\mathbb{U}}) = \mathcal{M} \). Set \( (\phi_3, g) \) the Weierstrass representation of \( X \), see equation (2). The holomorphic map \( g \) extends by Schwarz reflection to a meromorphic map on \( \mathbb{C} \) of finite degree \( n \), and so we can put \( g(z) = \frac{P(z)}{Q(z)} \), where \( P(z) = \sum_{j=0}^n a_j z^j \), \( Q(z) = \sum_{j=0}^n \overline{a_j} z^j \) and \( \overline{a_n} \neq a_n \neq 0 \). Since the 1-forms \( \phi_1, \phi_2 \) and \( \phi_3 \) have no common zeroes in \( \overline{\mathbb{U}} \), we get \( \phi_3 = -iBP(z)Q(dz), \quad B \in [0, +\infty] \). Up to a Lorentzian isometry, we can suppose that \( g(\infty) = 1, a_n = 1 \) and \( \theta^+ = \lim_{r \to +\infty} g(r) \) (note that \( X^{-1}(\Gamma) = \mathbb{R} \)). Therefore we have \( \frac{\theta^+}{2\pi} \in \mathbb{Z} \) and \( \theta^- = \theta^+ - 2n\pi \).

Observe that \( f_2(z) := \int_0^z \phi_2 = \int_0^z f_2(z) = \int_0^z \left( P(w)^2 + Q(w)^2 \right) dw \), \( f_1(z) := \int_0^z \phi_1 = \int_0^z f_1(z) = \int_0^z \left( P_2(w) - Q_2(w) \right) dw \) and \( f(z) := \int_0^z (\phi_2 - i\phi_3) = \int_0^z f_2(z) - \int_0^z f_1(z) \) are polynomial functions of degrees \( 2n + 1 \), \( n + 1 \) and \( 2n_0 + 1 \), respectively, where \( 0 \leq n_0 < n \). Since \( \widetilde{\alpha}_s \) is a lightlike ray with direction \((0,1,1)\), then the limits \( \lim_{z \to -1(\widetilde{\alpha}_s) = -\infty} \text{Re}(f_1(z)) \) and \( \lim_{z \to -1(\widetilde{\alpha}_s) = -\infty} \text{Re}(f(z)) \) are finite, and so \( \lim_{z \to -1(\widetilde{\alpha}_s) = -\infty} \text{Im}(\log(z)) = \frac{m_1 + \frac{\pi}{n}}{2(2n + 1)} = \frac{m_1 + \frac{\pi}{n}}{2(2n_0 + 1)} \) for suitable odd integers \( m_1 \) and \( n \). Furthermore, we know that \( \mathcal{M}^c \subset \{(x,y,t) : t - y \geq R\} \) for a suitable \( R \in \mathbb{R} \), and since \( X^{-1}(\partial(\mathcal{M}^c)) = [r, +\infty] \cup \{s \} \), where \( r = X^{-1}(\Gamma(s)) \), we infer that \( m_1 = 1 \).

Therefore, \( n - n_0 = (2n_0 + 1)(m_1 - 1) \geq 2(2n_0 + 1) \), that is to say, \( n \geq 5n_0 + 2 \).

On the other hand, \( X^{-1}(\partial(\mathcal{M}^c)) \cap \text{Re}(f_2)^{-1}(k) \) consists of two proper arcs homeomorphic to \([0, +\infty] \), for any \( k \in \mathbb{R} \). Indeed, just take into account that \( \mathcal{M}^c \) is (up to removing a compact set) a multigraph of angle \( 2\pi \) with sublinear arcs homeomorphic to \([0, +\infty] \) and \( \widetilde{\alpha}_s \) of direction \((0,1,1)\). As any divergent nodal arc of \( \text{Re}(f_2) - k \) in \( \overline{\mathbb{U}} \) is asymptotic to \( \{se^{2\pi i s} + \frac{\pi}{n}, s \geq 0\} \) for a suitable odd integer \( j \leq n \), and \( \mathcal{M}^c \) contains two such arcs for any \( k \in \mathbb{R} \), then \( \lim_{z \to -1(\widetilde{\alpha}_s) = -\infty} \text{Im}(\log(z)) = \frac{m_1 + \frac{\pi}{n}}{2(2n_0 + 1)} \), or equivalently \( n \leq 5n_0 + 2 \), which is absurd and concludes the proof.

Set \( t_M := \pi^{-1}(O) \cap M \) the intersection of \( M \) and the \( t \)-axis, and for any \( q = F_0(s,u) \in t_M \) write \( r_q = F_0(s) \times [0, u] \). Consider the simply connected surface \( S := M - \cup_{q \in t_M} \text{Re}(q) \) and fix a branch of \( \log \sigma \) on \( S \). It is clear that \( \kappa : S \to M \times \mathbb{C}, \kappa(q) = (q, f(q)) \) is an embedding, and that \( \mathcal{M} := \kappa(S) \) is a surface with piecewise analytical boundary homeomorphic to \( \mathbb{W} \setminus \{1\} \).

Set \( Y : M \to M \) the projection map \( Y(q, f(q)) = q \), and note that for any \( q \in t_M \) we have \( Y^{-1}(r_q) = r_q^+ \cup r_q^- \), where \( r_q^+ \cap r_q^- = Y^{-1}(q) = (q, \infty) \) and \( Y|_{r_q^+} : r_q^+ \to r_q, Y|_{r_q^-} : r_q^- \to r_q \) are homeomorphisms. Furthermore, the restriction of \( Y \) to \( M - \cup_{q \in t_M} Y^{-1}(r_q) = \mathcal{M} \) is one to one, and \( \partial(\mathcal{M}) = Y^{-1}(\Gamma \cup \{r_q \}) \).

As \( \mathcal{M} - Y^{-1}(t_M) \) is simply connected and \( f|_{\mathcal{M} - Y^{-1}(t_M)} \) never vanishes, \( \text{Arg} := \text{Im}(f) \) is well defined on \( \mathcal{M} - Y^{-1}(t_M) \).

**Remark 6.1** Up to a suitable choice of the branch \( f \) of \( \log \sigma \), we can assume that
\[
\lim_{x \to \theta^-} \text{Arg}(Y^{-1}(x)) - \Theta_0(x) = 0 \quad \text{provided that} \quad J \subset \theta^-, \theta^+[ \text{ is compact.}
\]
Moreover, \( \theta^- - \frac{\pi}{2} \leq \liminf_{x \to \infty} \text{Arg}(Y^{-1}(x)) \leq \limsup_{x \to \infty} \text{Arg}(Y^{-1}(x)) \leq \theta^+ + \frac{\pi}{2} \), and if \( \{x_n\}_{n \in \mathbb{N}} \subset \mathcal{M} \) is a divergent sequence satisfying \( \liminf_{n \to \infty} \text{Arg}(Y^{-1}(x_n)) \subset [\theta^-, \theta^+ + \frac{\pi}{2}] \) (resp., \( \limsup_{n \to \infty} \text{Arg}(Y^{-1}(x_n)) \subset [\theta^- - \frac{\pi}{2}, \theta^-] \)), then \( \lim_{n \to \infty} \Theta_0(x_n) = 0^+ \) (resp., \( \theta^- \)).

Standard monodromy arguments also show that:
Assume that Corollary 6.1

Lemma 6.6

Γ is monotone. For any q |J| ∪ \partial(\hat{M}) is compact, for any compact interval J ⊂ \mathbb{R} \setminus \{\theta^- - \frac{\pi}{2}, \theta^+ + \frac{\pi}{2}\}.

(i) If \{(q_n)\}_{n \in \mathbb{N}} \subset \partial(\mathcal{M}) and \{\Theta_0(Y(q_n))\}_{n \in \mathbb{N}} \to \theta^+ (resp., \theta^-), then \{\arg(q_n)\}_{n \in \mathbb{N}} \to \theta^+ + \frac{\pi}{2} (resp., \theta^- - \frac{\pi}{2}).

Denote by \mathcal{V}_r := \text{Int}(C^+_{(0,0,r)}) and \Gamma_r := \Gamma \cap \mathcal{V}_r. Since (1,0,0) \in \Gamma, equation (11) gives that \Gamma_r \neq \emptyset provided that r ≤ -1. In this case \Gamma \cap \partial(\mathcal{V}_r) consists of a unique point \Gamma(s_r) and \Gamma_r = \Gamma^{s_r}, where \mathcal{I}_r = [\theta(s_r), \theta^+]. In the sequel we will suppose r ≤ -1.

We label \mathcal{M}(r) as the connected component of \mathcal{V}_r \cap \mathcal{M} containing \Gamma_r, and write \mathcal{M}'(r) = (\mathcal{V}_r \cap \mathcal{M}) - \mathcal{M}(r). Likewise we put \mathcal{M}(r) := Y^{-1}(\mathcal{M}(r)).

Lemma 6.6 If c \subset \partial(\mathcal{V}_r) \setminus \{(0,0,r)\} is a spacelike arc and \rho is a branch of \text{Im}(\log(\pi)|_c), then \rho is monotone and \|q_2 - q_1\| > 0 provided that q_1, q_2 \in c and 0 < |\rho(q_2) - \rho(q_1)| < 2\pi.

Proof: The spacelike property gives that \rho has neither local maxima nor local minima, hence it is monotone. For any q \in \partial(\mathcal{V}_r) \setminus \{(0,0,r)\}, label I_q as the closed lightlike half line in \partial(\mathcal{V}_r) containing q. It is obvious that \partial(\mathcal{V}_r) - I_q \subset \text{Ext}(C_q), which simply means that \|q' - q\| > 0 for any q' \in \partial(\mathcal{V}_r) - I_q. If q \in c, the monotonicity of \rho yield that \{q' \in c - \{q\} : |\rho(q') - \rho(q)| < 2\pi\} \subset c - I_q \subset \text{Ext}(C_q). This concludes the proof.

Corollary 6.1 Assume that \mathcal{M} contains no upward lightlike rays, and fix r ≤ -1. Then the following statements hold:

(a) If \hat{c} \subset \hat{\mathcal{M}} is an arc and c := Y(\hat{c}) \subset \partial(\mathcal{V}_r) \setminus \{(0,0,r)\}, then \arg(\hat{c}) is monotone and \|Y(q_2) - Y(q_1)\| > 0 provided that q_1, q_2 \in \hat{c} and 0 < |\arg(q_2) - \arg(q_1)| < 2\pi.

(b) \partial(\mathcal{M}(r)) = \Gamma_r \cup \beta_r, where \beta_r is a proper divergent arc in \mathcal{M} with initial point \Gamma(s_r) and meeting \Gamma only at this point.

(c) If D \subset \mathcal{M}'(r) is a connected component, then D is a closed disc meeting t_M at a unique point and having \partial(D) \subset \partial(\mathcal{V}_r).

As a consequence, \mathcal{M} = \bigcup_{n \in \mathbb{N}} \mathcal{M}(r_n) provided that \{r_n\}_{n \in \mathbb{N}} \subset ]-\infty, -1[ is divergent.

Figure 4: The surface \hat{\mathcal{M}} and the projection Y.
**Proof:** Since $\mathcal{M}$ is spacelike and $\partial(\mathcal{V}_r)$ is lightlike, they meet transversally and $(\mathcal{M} \cap \partial(\mathcal{V}_r)) - \{(0,0,r)\}$ consists of a family of pairwise disjoint properly embedded analytical regular curves.

Item (a) is an elementary consequence of Lemma 6.6.

From our hypothesis and Lemma 6.6 we get $\theta^+ = +\infty$, hence from Remark 6.1 we have $\text{Arg}(\Gamma_r) = [a, +\infty], a \in \mathbb{R}$.

Let us show that $\text{Arg}(Y^{-1}(\mathcal{V}_r \cap \mathcal{M})) = [b, +\infty], b \in \mathbb{R}$. Reason by contradiction, and suppose there exists a divergent arc $\hat{c} \in \partial (Y^{-1}(\mathcal{V}_r \cap \mathcal{M}))$ homeomorphic to $[0, +\infty]$ such that $\text{Arg}(\hat{c}) = \alpha \setminus (0, \pi]$, $d \in \mathbb{R}$, $d < a - \pi$. For any $q \in \hat{c}$, let $L_q \subset \Pi_0$ denote the straight line passing through $\pi(Y(q))$ and the origin, and label $l_q$ as the connected component of $\pi^{-1}(L_q) \cap \mathcal{M}$ containing $Y(q)$. The choice of $\hat{c}$ yields that $\text{Arg}(Y^{-1}(l_q)) \cap [a, +\infty] = \emptyset$ and so $l_q \cap \mathcal{V}_r$ is disjoint from $\Gamma$. Since $\mathcal{M}$ has no upward lightlike rays, we infer that $c_q := l_q \cap \mathcal{V}_r$ is a compact arc passing through a point in $t_M$ and with endpoints in $\partial(\mathcal{V}_r) - \{(0,0,r)\}$. However, $t_M = \pi^{-1}(O) \cap \mathcal{M}$ is a closed discrete set, and therefore the point $c_q \cap t_M$ does not depend on $q \in \hat{c}$. This obviously contradicts that the family of compact curves $\{c_q, q \in \hat{c}\}$ diverge in $\mathbb{R}^3$ as $q$ diverges in $\hat{c}$.

Now we can prove (b). Indeed, first note that from the convex hull property $\mathcal{M}(r)$ contains no (closed) Jordan curves (recall that $\mathcal{M}(r)$ is simply connected, and so any such curve must bound a compact region totally contained in $\mathcal{V}_r$). Suppose there are two different divergent arcs $\hat{c}_1, \hat{c}_2$ in $\partial(\mathcal{M}(r))$ homeomorphic to $[0, +\infty]$ and disjoint from $\Gamma_r$. From the previous arguments, $\text{Arg}(\hat{c}_j) = [a_j, +\infty], j = 1, 2$, hence there are points $q_1 \in \hat{c}_1$ and $q_2 \in \hat{c}_2$ satisfying $\text{Arg}(q_1) = \text{Arg}(q_2)$. As above, set $L_j \subset \Pi_0$ and $l_j$ the straight line passing through $O$ and $\pi(Y(q_j))$ and its lifting to $\mathcal{M}$ with initial condition $Y(q_j)$, respectively. $j = 1, 2$. Let us check that $l_1 \cap l_2 = \emptyset$. Indeed, the fact $L_1 = L_2$ and the uniqueness of the lifting give that either $l_1 = l_2$ or $l_1 \cap l_2 = \emptyset$, and the first option leads to $q_1, q_2 \in l_1 = l_2$, contradicting that $q_2 \neq q_1$ is a lightlike vector.

As a consequence, $\lim_{x \in t_1 \to \infty} \Theta_0(x) \neq \lim_{x \in t_2 \to \infty} \Theta_0(x)$. However, Remark 6.1 gives that $\text{Arg}(q_1) = \lim_{x \in Y^{-1}(l_j) \to \infty} \text{Arg}(x) = \lim_{x \in L_j \to \infty} \Theta_0(x), j = 1, 2$, which contradicts that $\text{Arg}(q_2) = \text{Arg}(q_1)$ and proves (b).

To finish, consider a connected component $D \subset \mathcal{M}'(r)$. If $\partial(D)$ is not compact, we can find two divergent arcs in $\partial(D)$ homeomorphic to $[0, +\infty]$, getting a contradiction as above. Therefore, $\partial(D)$ is compact and consists of Jordan curves. Since $D$ is simply connected and $\text{Arg}$ is monotone on $Y^{-1}(\partial(D) - \cup_{q \in \partial(D)} \mathbb{R})$, then $D$ is a closed disc meeting $t_M$ at a unique point, proving (c).

Finally, take a divergent sequence $\{r_n\}_{n \in \mathbb{N}} \subset [-\infty, -1]$. To prove that $\mathcal{M} = \cup_{n \in \mathbb{N}} \mathcal{M}(r_n)$, fix an arbitrary point $q = F_0(s,u) \in \mathcal{M}$ and take $n \in \mathbb{N}$ large enough in such a way that $\Gamma(s) = F_0(s,0)$ and $q$ lie in $\mathcal{V}_{r_n}$. Since $F_0([s] \times [0, u])$ is connected and contained in $\mathcal{V}_{r_n}$, we get that $q \in F_0([s] \times [0, u]) \subset \mathcal{M}(r_n)$. Therefore $q \in \cup_{n \in \mathbb{N}} \mathcal{M}(r_n)$ and we are done.

**Lemma 6.7** Assume that $\mathcal{M}$ contains no upward lightlike rays, and fix $r \leq -1$. Then, there exists a smooth foliation $\mathcal{D}(r) := \{D_s(r) : s \in [r, +\infty[\}$ of $\mathcal{V}_r$ satisfying:

(a) $D_s(r) = \{(0,0,r)\}$, and for any $s > r$, $D_s(r)$ is a maximal disc containing $(0,0,s)$ and having $\partial(D_s(r)) \subset \partial(\mathcal{V}_r)$.

(b) $D_s(r) \cap \mathcal{M}'(r) \neq \emptyset$ if and only if $D_s(r)$ is a connected component of $\mathcal{M}'(r)$.

(c) $D_s(r) \cap \mathcal{M}(r) \neq \emptyset$ if and only if $D_s(r) \cap \partial(\mathcal{M}(r)) \neq \emptyset$, and in this case $D_s(r) \cap \mathcal{M}$ is an embedded arc lying in $\mathcal{M}(r)$ with initial point at $\Gamma_r$ and final point at $\beta_r$.

**Proof:** Consider a spacelike smooth divergent embedded arc $\delta_r \subset \partial(\mathcal{V}_r)$ with initial point $(0,0,r)$ and containing $\beta_r$.

**Claim:** There exists a smooth foliation $\mathcal{F}_r : [0, +\infty[\times S^1 \to \partial(\mathcal{V}_r)$ of $\partial(\mathcal{V}_r)$ satisfying

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7If $(0,0,r) \in \mathcal{M}$, equation 1 ensures that $\{(0,0,r)\}$ is a connected component of $\mathcal{M} \cap \partial(\mathcal{V}_r)$. 29
(i) $c_y := F_r(y, \cdot) : S^1 \to \partial(\mathcal{V}_r)$ is a Jordan curve, $y > 0$, and $c_0$ is the constant curve $c_0(\xi) = (0, 0, r)$, $\xi \in S^1$.

(ii) $c_y$ and $\delta_r$ meet at a unique point in a transversal way, $y > 0$.

(iii) $\|c_y(\xi) - c_y(\xi')\| > 0$ for any $\xi, \xi' \in S^1, \xi \neq \xi'$, and any $y > 0$.

(iv) For any connected component $D$ of $\mathcal{M}'(r)$, there is an unique $y \in [0, +\infty]$ such that $c_y = \partial(D)$.

Proof: From Lemma 6.6, any branch of $\text{Im} (\log \circ \pi)_{\delta_r}$ is monotone. Since $\pi|_{\partial(\mathcal{V}_r)}$ is injective, we deduce that $\pi(\delta_r)$ is an embedded divergent arc of spiral type with initial point at $O$. Hence, we can take a foliation $F^*_r : [0, +\infty] \times S^1 \to \Pi_0$ of $\Pi_0$ by Jordan curves $d_y := F^*_r(y, \cdot) : S^1 \to \Pi_0$ (where $F^*_r(0, \cdot)$ is constant and equal to $O$) in such a way that $d_y$ bounds a star-shaped domain centered at the origin (i.e., $d_y - \{O\}$ can be parameterized by the principal argument) and $d_y \cap \pi(\delta_r)$ consists of an unique point where both curves meet transversally, $y > 0$. Furthermore, constructing $F^*_r$ with a little care, we can ensure that for every connected component $D$ of $\mathcal{M}'(r)$, there exists an unique $y_D \in [0, +\infty]$ such that $\partial(D) = F^*_r(y_D, \cdot)$.

It suffices to define $F_r := (\pi|_{\partial(\mathcal{V}_r)})^{-1} \circ F^*_r$. Items (i), (ii) and (iv) are clear, and item (iii) follows from Lemma 6.6.

From Theorem 5.1 and item (iii), there is a unique maximal disc $D_y \subset \mathcal{V}_r$ with boundary $c_y, y \geq 0$ (we have made the convention $D_0 = \{(0, 0, r)\}$). Furthermore, since $\pi|_{D_y}$ is a local homeomorphism, $D_y$ is a graph over the planar domain bounded by $d_y, y > 0$.

The convex hull property for maximal surfaces gives $D_y \subset \mathcal{V}_r$ (even more, $D_y - c_y \subset \mathcal{V}_r - \partial(\mathcal{V}_r)$).

If $y_1 > y_2 > 0$, then $c_{y_1} > c_{y_2}$ (that is to say, $t(c_{y_1}(\xi)) > t(c_{y_2}(\xi))$ for any $\xi \in S^1$). A standard application of the maximum principle gives that $D_{y_1}$ lies above $D_{y_2}$, and so $D_{y_1} \cap D_{y_2} = \emptyset$. The smooth dependence of Plateau’s problem solutions with respect to the boundary data implies that there is a unique $D_s(r) \in \{D_y : y \in [0, +\infty]\}$ such that $(0, 0, s) \in D_s(r), s \geq r$. Furthermore, $D(r) = \{D_s(r) : s \in [r, +\infty]\}$ defines a smooth foliation of $\mathcal{V}_r$ satisfying (a) and (b).

In order to prove (c), let us see that $D_s(r) \cap \partial(\mathcal{M}(r)) \neq \emptyset$ if and only if $D_s(r) \cap \mathcal{M}(r) \neq \emptyset$. Suppose $D_s(r) \cap \mathcal{M}(r) \neq \emptyset$, and reasoning by contradiction, assume that $D_s(r) \cap \partial(\mathcal{M}(r)) = \emptyset$.

As $\partial(D_s(r)) \cap \mathcal{M}(r) = \emptyset$, then $D_s(r) \cap \mathcal{M}(r)$ is a family of piecewise analytically Jordan curves lying in the interior of both surfaces. Hence we can find compact discs $S_1 \subset \text{Int}(D_s(r))$ and $S_2 \subset \text{Int}(\mathcal{M}(r))$ with common boundary in $D_s(r) \cap \mathcal{M}(r)$ and common projection on the plane $\Pi_0$. Since both discs are graphs over $\Pi_0$, the maximum principle gives $S_1 = S_2$, and by an analytic continuation argument $D_s(r) \subset \mathcal{M}(r)$. This is absurd and shows that $D_s(r) \cap \partial(\mathcal{M}(r)) \neq \emptyset$.

Finally, assume that $D_s(r) \cap \mathcal{M}(r) \neq \emptyset$. From equations (11) and (11), $q_1 := D_s(r) \cap \Gamma = D_s(r) \cap \Gamma_r$ consists of at most one point where $D_s(r)$ and $\Gamma$ meet transversally (in case $D_s(r) \cap \Gamma = \emptyset$ we make the convention $q_1 = \emptyset$). Likewise, from (ii) in the preceding claim, $q_2 := \partial(D_s(r)) \cap \mathcal{M}(r) = c_y(r) \cap \delta_r$ is a point. If $\alpha$ is an inextendible arc in $D_s(r) \cap \mathcal{M}(r) \subset \mathcal{V}_r$, then $\alpha$ is compact and with endpoints lying in $(\partial(D_s(r)) \cap \mathcal{M}(r)) \cup (\partial(\mathcal{M}(r)) \cap D_s(r)) = \{q_1, q_2\}$. Taking into account that $D_s(r) \cap \mathcal{M}(r)$ contains no Jordan curves (reason as above), we get that $q_1 \neq \emptyset$ is a point and $\alpha$ joins $q_1$ with $q_2$, concluding (c) and the lemma.

The following theorem has been inspired by Meeks and Rosenberg ideas in [22].

**Theorem 6.2** Any plane parallel to $\Sigma_\infty$ is transverse to $\mathcal{M}$.

As a consequence, either $\Sigma_\infty$ is spacelike and $|\theta^-| = \mathbb{R}$ or $\Sigma_\infty$ is lightlike and $\theta(\mathcal{M}) = 2\pi$.

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8If $M_1$ and $M_2$ are maximal surfaces, then $M_1 \cap M_2$ consists of a family of analytical proper analytical arcs in $M_j - (\partial(M_1) \cup \partial(M_2)), j = 1, 2$, meeting equiangularly at points with the same normal.
**Proof:** Up to a Lorentzian isometry, we assume that either $\theta^+ = +\infty$ or $\theta^- = 0$ is a finite interval. In any case, Lemma 6.4 guarantees that $\mathcal{M}$ contains no upward lightlike rays, and consequently, the foliation $\mathcal{D}(r)$ in Lemma 6.7 makes sense, for any $r \in [-\infty, -1]$. From Theorem 4.11, $\theta(\mathcal{M}) = 2\pi\kappa$, $k \in \mathbb{N}$, provided that $\theta(\mathcal{M}) < +\infty$, and in case $k = 1$ we are done. Therefore, in the sequel we will only deal with the case $\theta(\mathcal{M}) \geq 4\pi$. Take $\theta_0 \in [\theta^-, \theta^+]$ such that $I_0 := \theta_0 - 2\pi, \theta_0 + 2\pi \subset [\theta^-, \theta^+]$. In case $\theta^+ = +\infty$ we also impose that $\theta_0 - 2\pi > \theta^- + \frac{\pi}{2}$. Let $s_0 \in \mathbb{R}$ be the unique real number such that $\Theta_0(\Gamma(s_0)) = \theta_0$. In the sequel we only consider $r \in [-\infty, -1]$ small enough in such a way that $\Gamma^{(0)} \subset \text{Int}(\mathcal{C}^1_{(0,0,r)}) \subset \mathcal{V}_r$.

Set $\Sigma_\infty(s)$ the plane parallel to $\Sigma_\infty$ and passing through $(0,0,s)$. The plane parallel to $\Sigma_\infty$ and passing through $(0,0,s)$.

**Claim:** There exists a divergent sequence $\{R_k\}_{k \in \mathbb{N}} \subset [-\infty, -1]$ such that $\mathcal{D}(R_k)_{k \in \mathbb{N}}$ converges in the $C^1$-topology to the foliation of $\mathbb{R}^3_1$ by planes parallel to $\Sigma_\infty$.

**Proof:** For any real number $r \leq -1$ and $n \in \mathbb{N}$ label $r(n) := \frac{r}{\lambda_n}$. Since $\mathcal{M}$ has no upward lightlike rays, then $F_0(s_0, r(n))$ and $\partial(V_0(n))$ meet at a unique point $q(n) \in \beta_{r(n)}$. Call $D(r(n))$ as the unique maximal disc in $D(r(n))$ containing $q(n)$, and let us show that $\{\lambda_n D(r(n))\}_{n \in \mathbb{N}}$ converges in the $C^1$-topology to $\Sigma_\infty \cap \mathcal{V}_r$ as graphs over $\Pi_0$.

Since $\{\mathcal{M}^0\}_{n \in \mathbb{N}}$ converges uniformly on compact subsets to the twice-covered once punctured plane $\Sigma_\infty - \{O\}$ (see Theorem 6.1), the arcs $c(n) := \lambda_n (\mathcal{M}^0 \cap \partial(V_0(n))) \subset \lambda_n \beta_{r(n)}$ converge as $n \to +\infty$ to the twice-covered Lorentzian circle $c := \Sigma_\infty \cap \partial(V_0)$ (a parabola when $\Sigma_\infty$ is lightlike). On the other hand, we know that $\lambda_n \partial(D(r(n)))$ and $c(n)$ meet only at $\lambda_n q(n)$ in a transversal way, and both components of $c(n) - \lambda_n q(n)$ converge uniformly on compact subsets as $n \to +\infty$ to $c$. Taking into account that $\lambda_n \partial(D(r(n)))$ lies in between these components, we deduce that $\{\lambda_n \partial(D(r(n)))\}_{n \in \mathbb{N}} \to c$ too. If $\Sigma_\infty$ is spacelike, $c$ is a closed curve and the assertion holds from the continuous dependence of Plateau’s problem solutions with respect to the boundary data.

Assume now that $\Sigma_\infty$ is lightlike (and $c$ is a parabola), and call $D_{r(\infty)} := \lim_{n \to \infty} \lambda_n D(r(n))$. Note that $\lambda_n D(r(n)) \subset \mathcal{V}_r$ for every $n \in \mathbb{N}$, hence $D_{r(\infty)} \subset \mathcal{V}_r$. Since $D_{r(\infty)}$ is a PS graph and $\partial(D_{r(\infty)}) = c$, equation (1) yields that $D_{r(\infty)} \subset \left(\bigcap_{r \in \mathbb{R}} \text{Ext}(\mathcal{C}^1_{r}) \right) \cap \mathcal{V}_r$. This proves that $D_{r(\infty)}$ lies in the slab bounded by $\Sigma_\infty$ and $\Sigma_\infty - \{r\}$, and as a consequence, $D_{r(\infty)}$ must contain lightlike segments (otherwise $D_{r(\infty)}$ would be a parabolic maximal graph by Corollary 5.1 and so a planar domain in $\Sigma_\infty$, absurd). From Remark 5.2, $D_{r(\infty)} \cap \Sigma_\infty$ contains a lightlike half line $L$ with initial point in $c$, and the entire PS graph $S := D_{r(\infty)} - (\Sigma_\infty \cap \pi^{-1}(\Pi_0 - \pi(D_{r(\infty)})))$ contains the straight line determined by $L$. Lemma 2.1 shows that $S = \Sigma_\infty$, proving our assertion.

Take a divergent sequence $\{r_k\}_{k \in \mathbb{N}}$ in $[-\infty, -1]$, and use a standard diagonal process to find a divergent sequence $\{n_k\}_{k \in \mathbb{N}} \subset \mathbb{N}$ such that $\{\lambda_{n_k} D(r_k(n_k))\}_{k \in \mathbb{N}} \to \Sigma_\infty$ in the $C^0$-topology as graphs over $\Pi_0$. Define $R_k := r_k(n_k), k \in \mathbb{N}$, and let us show that $\{R_k\}_{k \in \mathbb{N}}$ solves the claim.

To do this, let $I \subset \mathbb{R}$ be a compact interval, and take a sequence $\{s_k, k \in \mathbb{N}\} \subset I$ converging to $s \in I$. It suffices to check that $\{D_{s_k}(R_k)\}_{k \in \mathbb{N}} \to \Sigma_\infty(s)$ in the $C^1$-topology as graphs over $\Pi_0$.

Let us see first that $\{\lambda_{n_k} D_{s_k}(R_k)\}_{k \in \mathbb{N}} \to \Sigma_\infty$ in the $C^1$-topology. Indeed, note that $\lambda_{n_k}(0,0,s_k) \in \lambda_{n_k} D_{s_k}(R_k)$ and $\lambda_{n_k}(0,0,s_k)_{k \in \mathbb{N}} \to 0$. Hence, $\{\lambda_{n_k} D_{s_k}(R_k)\}_{k \in \mathbb{N}}$ converges in the $C^0$-topology to an entire PS graph $\Sigma_{\infty}'$ passing through $O$. As either $D_{s_k}(R_k) = D(R_k)$ or $D_{s_k}(R_k) \cap D(R_k) = \emptyset$ for any $k$, then $\Sigma_{\infty}'$ lies in one of the closed half spaces bounded by $\Sigma_\infty$. Using Calabi’s theorem, Remark 5.2 and Lemma 2.1 we infer that $\Sigma_{\infty}'$ is a non timelike plane passing through $O$, hence $\Sigma_{\infty}' = \Sigma_\infty$. From Proposition 5.1, $\{\lambda_{n_k} D_{s_k}(R_k)\}_{k \in \mathbb{N}} \to \Sigma_\infty$ in the $C^1$ topology and we are done.

Finally, let $u_k := \pi(D_{s_k}(R_k)) \to \mathbb{R}$ and $v_k := \lambda_{n_k} \pi(D_{s_k}(R_k)) \to \mathbb{R}$ be the functions determining the graphs $D_{s_k}(R_k)$ and $\lambda_{n_k} D_{s_k}(R_k)$ respectively, $k \in \mathbb{N}$. Proposition 5.4 gives that $\{\nabla v_k\}_{k \in \mathbb{N}} \to \sigma$ in the $C^0$-topology, where $\sigma$ is the gradient of the linear function defining $\Sigma_\infty$. Since $u_k(\lambda_{n_k} x) = 9$ This means that for any compact interval $I \subset \mathbb{R}$, $\{D_s(R_k)\}_{k \in \mathbb{N}} \to \Sigma_\infty(s)$ in the $C^1$-topology uniformly on $s \in I$.
λ_n u_k(x), we infer that \( \nabla v_k(\lambda_n x) = \nabla u_k(x) \) for any \( x \in \pi(D_{\lambda_k}(R_k)) \). This gives \( \{ \nabla u_k \}_{k \in \mathbb{N}} \to \sigma \) in the \( C^0 \)-topology, and so \( \lim _{k \to +\infty} D_{\lambda_k}(R_k) = \Sigma_{\infty} \) in the \( C^1 \)-topology.

To finish the theorem, reason by contradiction and assume there is \( q \in \mathbb{R}^3 \) such that \( T_q M = \Sigma_{\infty} \). Recall that the holomorphic Gauss map \( g \) of \( M \) extends to the mirror \( M^* \) of \( M \) by Schwarz reflection, and take a closed disc \( U \subseteq M \cup M^* \) containing \( q \) as interior point and no more points of \( g^{-1}(g(q)) \). Let \( m \geq 1 \) denote the multiplicity of \( g \) at \( q \), that is to say, the winding number of \( g(\partial(U)) \) around \( g(q) \). For any \( k \in \mathbb{N} \) such that \( U \subseteq \mathcal{V}_k \) and for any \( p \in U \cap M \), let \( s_k(p) \in \mathbb{R} \) denote the unique real number such that \( p \in D_{s_k(p)}(R_k) \) and call \( U_k = \cup_{p \in U \cap M} D_{s_k(p)}(R_k) \). From equation \( (\ref{equation}) \), \( U_k \subseteq \cup_{p \in U \cap M} \mathbb{R} \), and so \( I_k := \{ s_k(p) : p \in U \cap M \} \) lies in the compact interval \( I = \{ s \in \mathbb{R} : (0,0,s) \in \cup_{p \in U \cap M} \mathbb{R} \} \). In other words, \( U_k \subseteq \cup_{s \in I} D_s(R_k) \), for any \( k \in \mathbb{N} \) satisfying \( U \cap M \subseteq \mathcal{V}_k \).

Let \( g_{k,p} : D_{s_k(p)}(R_k) \to \mathbb{D} \) be the holomorphic Gauss map of \( D_{s_k(p)}(R_k) \), and set \( h_k : U \to \mathbb{C} \),

\[
h_k(p) := g(p) - g_{k,p}(p), \quad p \in U \cap M, \quad \text{and} \quad h_k(p) := g(p) - g_{k,p}(p^*), \quad p \in U \cap M^*.
\]

Labeling \( s(p) \in I \) as the unique real number such that \( p \in \Sigma_{\infty}(s(p)), p \in U \cap M \), the previous Claim gives that \( \lim _{k \to +\infty} s_k(p) = s(p) \) and \( \lim _{k \to +\infty} D_{s_k(p)}(R_k) = \Sigma_{\infty}(s(p)) \) in the \( C^1 \)-topology uniformly on \( p \in U \cap M \). Therefore, \( \lim _{k \to +\infty} h_k = g - g(q) \) uniformly on \( U \), hence for large enough \( k \) the winding number of \( h_k(\partial(U)) \) around the origin is equal to \( m \). In particular, we can find \( q_k \in (U \cap M) - \partial(U) \) such that \( h_k(q_k) = 0 \) (that is to say, \( T_{q_k} D_{s_k(q_k)}(R_k) = T_{q_k} M \), \( k \) large enough, contradicting that \( D(R_k) \) is transverse to \( M \) for any \( k \) and proving that \( T_q M \) is not parallel to \( \Sigma_{\infty} \) for any \( q \in M \).

The case when \( \Sigma_{\infty} \) is lightlike and \( \theta(M) \geq 4\pi \) can not happen (otherwise we can find \( q \in \Gamma \) such that \( T_q M \) is parallel to \( \Sigma_{\infty} \), getting a contradiction). Therefore, \( \Sigma_{\infty} \) is spacelike and \( \{ \theta^+, \theta^- \} = \mathbb{R} \) by Theorem \( \ref{thm:classification} \). This concludes the proof. \( \square \)

**Corollary 6.2** The blow down plane \( \Sigma_{\infty} \) does not depend on the blow down sequence \( \{ \lambda_n \}_{n \in \mathbb{N}} \).

**Proof:** If \( \theta(M) = 2\pi \), \( \Sigma_{\infty} \) is the limit plane of \( M \) at infinity and the corollary holds.

Assume that \( \theta(M) = +\infty \), take a new blow down sequence \( \{ \lambda'_n \}_{n \in \mathbb{N}} \) and construct the corresponding blow down PS multigraph \( X' : M'_{\infty} \to \mathbb{R}^3 \) and blow down spacelike plane \( \Sigma'_{\infty} := X'(M'_{\infty}) \).

Reason by contradiction and suppose \( \Sigma'_{\infty} \neq \Sigma_{\infty} \). Consider an interval \( I \subseteq \theta^+, \theta^+ = \mathbb{R} \) of length \( 2\pi \), fix \( p \in \Pi_l - O \) and take \( p_n \in \pi^{-1}(p) \cap (\lambda'_n \cdot M^I) \), \( p'_n \in \pi^{-1}(p) \cap (\lambda'_n \cdot M^I) \), \( n \in \mathbb{N} \) (well defined provided that \( n \) is large enough). We know that \( \lim _{n \to +\infty} \mathcal{N}(\frac{1}{\lambda'_n} p_n) = \zeta' \) and \( \lim _{n \to +\infty} \mathcal{N}(\frac{1}{\lambda'_n} p'_n) = \zeta' \), where \( \mathcal{N} \) is the Lorentzian Gauss map of \( M \) and \( \zeta \) and \( \zeta' \in \mathbb{H}^2 \) are the unitary normal vectors to \( \Sigma_{\infty} \) and \( \Sigma'_{\infty} \), respectively. By a connectedness argument, we can find \( \zeta \in \mathbb{H}^2 - \{ \zeta, \zeta' \} \) and divergent sequence \( \{ q_n \}_{n \in \mathbb{N}} \subset M^I \) such that \( \lim _{n \to +\infty} \mathcal{N}(q_n) = \zeta_0 \). As above, \( \zeta_0 \) is the unitary normal to the blow down plane \( \Sigma'_{\infty} \) associated to blow down sequence \( \{ \frac{1}{\lambda'_n} q_n \}_{n \in \mathbb{N}} \).

On the other hand, let \( (g, \phi_3) \) denote the Weierstrass data of \( M \) (see equation \( \ref{equation} \)) and extend \( (g, \phi_3) \) by Schwarz reflection to the double \( \bar{M} \) of \( M \). Then, consider the conjugate minimal immersion \( X' : M \to \mathbb{R}^3 \) associated to the same Weierstrass data \( (g, \phi_3) \), see equation \( \ref{equation} \), and recall that the metrics induced by \( \langle ., . \rangle \) and \( \langle ., . \rangle_0 \) on \( M \) are given by \( ds^2 = \frac{1}{4} |\phi_3|^2 (|g| - |g|)^2 \) and \( ds^2 = \frac{1}{4} |\phi_3|^2 (\frac{1}{|g|} + |g|)^2 \), respectively.

Let us show that \( ds^2 \) is complete. Since the mirror involution is an isometry of \( (\bar{M}, ds^2_0) \), it suffices to check that divergent curves in \( M \) have infinite length with respect to \( ds^2_0 \). Indeed, set \( \alpha \subset M \) a divergent curve. Since the first and second coordinate functions of a maximal surface and its conjugate minimal one are the same, then the length \( L_0 \) of \( \alpha \) with respect to \( ds^2_0 \) is greater
or equal to the Euclidean length $L$ of $\pi(\alpha)$. The spacelike property of $\mathcal{M}$ and the divergence of $\alpha$ give that $L = +\infty$, and so $L_0 = +\infty$.

By Theorem 6.2, $\mathcal{N} : \mathcal{M} \rightarrow \mathbb{H}^2$ omits the values $\zeta, \zeta'$ and $\zeta_0 \in \mathbb{H}^2$, hence the Gauss map of $X^*$ omits six complex values. This contradicts Fujimoto’s theorem [10] and proves the corollary. \hfill \Box

7 The Uniqueness Theorems

In this section we prove the main results of this paper. We start with the following:

**Theorem 7.1 (Uniqueness of the Enneper surface)** The only properly embedded CLBMS maximal surface with finite rotation number is, up to Lorentzian congruence, the Enneper surface $E_1$.

**Proof:** Let $\mathcal{M}$ be a properly embedded CLBMS maximal surface with $\theta(\mathcal{M}) < +\infty$.

Since $\mathcal{M}$ is a multigraph outside a compact set, Corollary 5.2 gives that $\mathcal{M}$ is parabolic, hence conformally equivalent to $\overline{\mathbb{D}} - \{1\}$. Let $(g, \phi_3)$ denote the Weierstrass data of $\mathcal{M}$. From Theorem 6.2, the holomorphic map $g : \mathbb{D} - \{1\} \rightarrow \overline{\mathbb{D}}$ is one to one on $\partial(\mathbb{D}) - \{1\}$, and so, up to a Lorentzian isometry, we can suppose that $g(z) = z$. On the other hand, equation (2.2) leads to $\phi_3 = h(z) \frac{dz}{(z - 1)}$ (note that the mirror involution is given by $h : \mathbb{C} \rightarrow \mathbb{C}$ is a meromorphic function satisfying $h \circ J = \overline{h}$. Since the 1-forms $\phi_3$ given in (2) have no common zeroes, $h$ never vanishes on $\mathbb{D}$. Furthermore, as the unique end of $\mathcal{M}$ corresponds to $z = 1$, $h$ has no poles in $\mathbb{D}$ as well. The symmetry condition $h \circ J = \overline{h}$ gives that the zeroes and poles of $h$, if they occur, lie in $\partial \mathbb{D}$. However, Lemma 2.2 implies that $\phi_3$ never vanishes on $\partial \mathbb{D} - \{1\}$, and so $h$ must be a real number different from zero. Up to scaling, $(g, \phi_3)$ are the Weierstrass data of $E_1$ (see Section 3), concluding the proof. \hfill \Box

In the sequel we will deal with the uniqueness of properly embedded CLBMS maximal surfaces with infinite rotation number. We have been mainly inspired by Meeks-Rosenberg work [22].

Let $\mathcal{M}$ denote a properly embedded CLBMS maximal surface with $\theta(\mathcal{M}) = +\infty$. From Theorem 6.2, $|\theta^+, \theta^-| = \mathbb{R}$, $\Sigma_{\infty}$ is a spacelike plane, any plane $\Sigma$ parallel to $\Sigma_{\infty}$ meets $\mathcal{M}$ transversally. Furthermore, basic theory or harmonic functions gives that $\Sigma \cap \Sigma_{\infty}$ consist of a family of pairwise disjoint proper analytical arcs.

Label $\Sigma^+$ and $\Sigma^-$ as the two closed half spaces in $\mathbb{R}^3$ bounded by $\Sigma$. As $\Sigma$ is spacelike then $q := \Sigma \cap \Gamma$ is a single point. We set $\mathcal{M}(q) = \mathcal{M} \cap \text{Int}(\Sigma_q)$, $\mathcal{M}^+(q) := \mathcal{M}(q) \cap \Sigma^+$ and $\mathcal{M}^-(q) := \mathcal{M}(q) \cap \Sigma^-$. Note that equation (11) gives $\Gamma \subset \mathcal{M}(q)$. Since the arcs $F_0(s, \cdot)$ have slope $\leq 1$ and $\mathcal{M}$ has no lightlike asymptotic rays (see Lemma 5.5), $F_0(s, \cdot) \cap \mathcal{M}(q)$ is compact and connected for any $s \in \mathbb{R}$. We deduce that $\mathcal{M}^+(q)$ and $\mathcal{M}^-(q)$ are simply connected closed domains in $\mathcal{M}$ with connected boundary. Moreover, $\mathcal{M}^+(q) \cap \mathcal{M}^-(q) = \{q\}$, and so $\mathcal{M}(q)$ is connected too.

Consider a region $A \subset \mathcal{M}$ (in most cases we will deal with $A = \mathcal{M}$). The closure of a connected component of $\text{Int}(A) - \Sigma$ is defined to be a $\Sigma$-region of $A$.

A $\Sigma$-region $W$ of $\mathcal{M}$ is said to be a finite (resp., infinite) if $\partial(W)$ has finitely (resp., infinitely) many pairwise disjoint proper arcs. A finite $\Sigma$-region of $\mathcal{M}$ is said to be simple if it has connected boundary. Any $\Sigma$-region $W$ of $\mathcal{M}$ is parabolic (see Corollary 4.1) and simply connected, hence conformally equivalent to $\overline{\mathbb{D}} - E$, where $E \subset \partial(\mathbb{D})$ is a totally disconnected closed subset of measure zero. For convenience, we will identify $W$ and $\overline{\mathbb{D}} - E$ and call $E$ as the set of ends of $W$.

**Lemma 7.1** If $W \equiv \overline{\mathbb{D}} - E$ is an infinite $\Sigma$-region of $\mathcal{M}$ then either $\mathcal{M}^+(q) \subset W$ or $\mathcal{M}^-(q) \subset W$.

Moreover, the unique limit end $*$ of $E$ is the endpoint of $\Gamma \cap W \subset \partial(\mathbb{D}) - E$. 

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Proof: Up to a Lorentzian isometry, we will suppose that $\Sigma = \Sigma_\infty = \Pi_0$, $q = \Gamma \cap \Sigma = O$ and $\Sigma^+ = \{ t \geq 0 \}$ (hence $\Sigma^- = \{ t \leq 0 \}$). For simplicity we write $\Gamma_0$ instead of $\Gamma \cap W$.

First of all recall that $\mathcal{M}^+(O)$ and $\mathcal{M}^-(O)$ are connected, hence they lie in the $\Sigma$-regions of $\mathcal{M}$ containing $\Gamma \cap \{ t > 0 \}$ and $\Gamma \cap \{ t < 0 \}$, respectively.

Take a limit end $\ast$ of $E$ and an auxiliary point $q_0 \in \partial(\mathcal{D}) - \{ \ast \}$. Label $c$ and $c'$ as the two open arcs in $\partial(\mathcal{D}) - \{ \ast, q_0 \}$, and consider sequences $\{ c_n : n \in \mathbb{N} \} \subset \mathcal{D} \cap E$ and $\{ c'_n : n \in \mathbb{N} \} \subset \mathcal{D} \cap E$ converging to $\ast$. Without loss of generality, suppose that $\{ c_n : n \in \mathbb{N} \}$ is not a finite set of ends.

Reason by contradiction, and assume that either $\Gamma_0 = \emptyset$ or $\Gamma_0 \neq \emptyset$ and $\Gamma_0$ does not diverge to $\ast$. Then there exists a compact arc $c \subset W - \Gamma$ connecting two points of $\partial(W)$, and such that $W - c$ has a connected component with infinitely many boundary components and disjoint from $\Gamma$. Call $W'$ the closure of this component.

Since $W' \cap \mathcal{M}(O)$ is compact (just observe that $\partial(W') \cap \mathcal{M}(O) = c \cap \mathcal{M}(O)$ is compact and $W' \cap \Gamma = \emptyset$), then $\pi|_{W'} : W' \to \Pi_0$ is proper. Take $R > 0$ such that $c \subset \{ (x, t) \in \mathbb{R}^3_1 : \|x\|_0 < R \}$ and consider the connected component $W_R$ of $W' \cap \{ (x, t) \in \mathbb{R}^3_1 : \|x\|_0 \geq R \}$ with infinitely many boundary arcs. It is clear that $W_R$ is biholomorphic to $D_R - E_R$, where $D_R \subset \mathbb{R}^3$ is a closed topological disc and $E_R = \partial(D_R) \cap E$. Furthermore, without loss of generality we can suppose that $\{ c_n : n \in \mathbb{N} \} \cup \{ c'_n : n \in \mathbb{N} \} \cup \{ \ast \} \subset E_R$.

Put $\alpha_R := \partial(W_R) \cap \{ (x, t) \in \mathbb{R}^3_1 : \|x\|_0 = R \}$, and set $\sigma$ the connected component of $\partial(W_R)$ containing $\alpha_R$. Let $W_R$ denote the region in $\mathcal{M}$ bounded by $\sigma$ and disjoint from $\Gamma$ (obviously $W_R \subset W$).

Let us see that $\pi|_{W_R} : W_R \to \pi(W_R)$ is a diffeomorphism. To see this, first notice that $\pi|_{\partial(W_R)}$ is injective. Indeed, since $\pi|_{\partial(W_R)}$ is the identity map, it suffices to prove that $\pi|_{\alpha_R}$ is injective. Assume without loss that $W_R \subset \{ t \geq 0 \}$ and note that $W_R$ separates the region $\{ (x, t) \in \mathbb{R}^3_1 : \|x\|_0 \geq R, t \geq 0 \}$. Therefore, for an arc $\alpha \subset \alpha_R = W_R \cap \{ (x, t) \in \mathbb{R}^3_1 : \|x\|_0 = R \}$ there cannot be another arc $\beta \subset \alpha_R$ immediately above of below $\alpha$. Otherwise, the Euclidean normal vectors to $W_R$ along $\alpha$ and $\beta$ would lie in different hemispheres, which contradicts that the projection $\pi$ orients $W_R$. Therefore, $\pi|_{W_R} : W_R \to \pi(W_R)$ is a proper local embedding satisfying that $\pi|_{\partial(W_R)}$ is one to one. Our assertion follows from the simply connectedness of $\pi(W_R)$.

Set $\{ \Omega_n : n \in \mathbb{N} \}$ the countable family of connected components in $\pi(W_R - \text{Int}(W_R))$, and let $G_n = \{ (x, u_n(x)) : x \in \Omega_n \}$ denote the maximal graph in $\pi(W_R - \text{Int}(W_R))$ satisfying $\pi(G_n) = \Omega_n$, $n \in \mathbb{N}$. It is clear that $\Omega_n \cap \Omega_m = \emptyset$, $n \neq m$, and $u_n|_{\partial(G_n)} = 0$. The desired contradiction will follow from Theorem 5.3 provided that it is proved that $|\nabla u_n| < 1 - \epsilon$ for any $n$.

To check the last inequality, reason by contradiction and suppose there exists a sequence $\{ p_n \}_{n \in \mathbb{N}}$, where $p_n \in G_n$, such that $\{ \nabla u_n(p_n) \}_{n \in \mathbb{N}} \to 1$ (or in other words, $\{ |g(p_n)| \}_{n \in \mathbb{N}} \to 1$, where $g$ is the holomorphic Gauss map of $\mathcal{M}$). Since $G_n$ is parabolic (see Corollary 4.4) and $|g|$ subharmonic, then $|g(p_n)| \leq \int_{\partial(G_n)} |g| d\mu_{p_n}$, where $d\mu_{p_n}$ is the harmonic measure respect to a given point $p_n \in G_n$. As $\mu_{p_n}$ is a probabilistic measure (i.e., $\int_{\partial(G_n)} d\mu_{p_n} = 1$), then we can find $q_n \in \partial(G_n) \subset \Pi_0$ satisfying $|g(q_n)| \geq |g(p_n)|$. Taking into account that $\mathcal{M}$ is proper and the $G_n$’s are pairwise disjoint, we deduce that $\{ \lambda_n := \frac{1}{|g(p_n)|} \}_{n \in \mathbb{N}} \to 0$. Like in Section 4 consider the sequence $\{ \lambda'_n := \lambda'_n M \}_{n \in \mathbb{N}}$, and the corresponding blow down PS multigraph $\lambda' : \mathcal{M}'_\infty \to \mathbb{R}^3$. Assume that $\{ p_n := \frac{q_n}{|q_n|} \}_{n \in \mathbb{N}} \to p \in \Pi_0$, take an open disc $D \subset \Pi_0$ centered at $p$ of radius $\frac{1}{2}$ and call $N_n$ the closure of the connected component of $\mathcal{M}' \cap \pi^{-1}(D)$ containing $p_n$, $n \in \mathbb{N}$. From equation (11), $N_n \subset \text{Ex}(G_{p_n}) \cap \pi^{-1}(D)$, and therefore $N_n \cap \text{Int}(G_{p_n}) = \emptyset$, for any $n \in \mathbb{N}$. On the other hand, equation (11) and the fact $O \subset \Gamma$ give that $\chi'_\Gamma \subset \text{Int}(G_0)$ and prove that $N_n \cap (\lambda' \cdot \Gamma) = \emptyset$, for any $n \in \mathbb{N}$. As a consequence $\pi(N_n) = D$ and $\pi|_{N_n} : N_n \to D$ is a diffeomorphism, $n \in \mathbb{N}$.

The hypothesis $\{ |g(p_n)| \}_{n \in \mathbb{N}} \to 1$ implies that $\Xi := \lim_{n \to \infty} N_n$ contains a lightlike half line passing through $p$ (see Theorem 5.2). Therefore $\Xi$ and the plane $\lambda'(\mathcal{M}'_\infty) = \Pi_0$ (see Corollary 6.2) meet transversally at $p$. This contradicts that $\Pi_0$ lies in the limit set of the sequence of
Proposition 7.7 If $\Sigma$ is a plane parallel to $\Sigma_\infty$ then $\mathcal{M} \cap \Sigma$ consists of a proper regular arc.

Proof: Like in the preceding lemma, assume that $\Sigma = \Sigma_\infty = \Pi_0$ and $\Gamma \cap \Sigma = O$.

Claim 1: If $U \subset \mathcal{M}$ is a region disjoint from $\Gamma$ bounded by an arc lying in $\Pi_0$, then $U$ contains finitely many $\Sigma$-regions.

Proof: Since $\pi|_{\partial(U)}$ is injective, the proper local embedding $\pi|_U : U \to \pi(U)$ is a covering, and so, an homeomorphism. If $U \cap \Pi_0$ has infinitely many connected components, $U$ contains infinitely many disjointly supported maximal graphs on $\Pi_0$. Like in the proof of Lemma 7.4, Theorem 5.3 leads to a contradiction.

Let $c_0$ denote the unique proper divergent arc in $\mathcal{M} \cap \Pi_0$ meeting $\Gamma$ (that is to say, the one with initial point $O$), and set $\mathcal{M}^+$ (resp., $\mathcal{M}^-$) the region in $\mathcal{M}$ bounded by $c_0 \cup (\Gamma \cap \{t \geq 0\})$ (resp., $c_0 \cup (\Gamma \cap \{t \leq 0\})$). It is clear that $W^+ \subset \mathcal{M}^+$ and $W^- \subset \mathcal{M}^-$, where $W^+$ and $W^-$ are the $\Sigma$-regions of $\mathcal{M}$ containing $\mathcal{M}^+(O)$ and $\mathcal{M}^-(O)$, respectively.

From Lemma 7.1 and Claim 1 we can find a divergent arc $\beta^+ \subset \mathcal{M}^+$ disjoint from $\Gamma$, meeting $c_0$ just at the initial point of $\beta^+$, and meeting twice any connected component of $(\mathcal{M}^+ \cap \Pi_0) - c_0$. In a similar way we define $\beta^-$, and without loss of generality suppose $c_0 \cap \beta^+ = c_0 \cap \beta^-$. It is clear that $\beta = \beta^+ \cup \beta^-$ is a proper arc in $\mathcal{M}$ splitting $\mathcal{M}$ into two connected components. Set $\mathcal{M}_\beta$ the closure of the connected component of $\mathcal{M} - \beta$ disjoint from $\Gamma$.

Let $V$ be a $\Sigma$-region of $\mathcal{M}_\beta$, obviously non compact. $V$ is said to be a middle $\Sigma$-region of $\mathcal{M}_\beta$ if $\partial(V) \cap \beta$ is compact. Otherwise, $V$ is said to be a tail $\Sigma$-region of $\mathcal{M}_\beta$. Two different $\Sigma$-regions of $\mathcal{M}_\beta$ are said to be contiguous if they share a non compact boundary arc in $\Pi_0$. It is obvious that $\mathcal{M}_\beta \cap M^+$ (resp., $\mathcal{M}_\beta \cap \mathcal{M}^-$) contains at most one tail $\Sigma$-region, and this occurs if an only if $\partial(W^+)$ (resp., $\partial(W^-)$) contains finitely many components. Moreover, $\mathcal{M}_\beta$ contains no middle $\Sigma$-regions if and only if $\mathcal{M}_\beta \cap \Pi_0 = c_0$.

Let $t^* : \mathcal{M} \to \mathbb{R}$ denote the harmonic conjugate of the third coordinate function $t : \mathcal{M} \to \mathbb{R}$, and consider the holomorphic function $h := t + it^* : \mathcal{M} \to \mathbb{C}$.

Claim 2: If $V_0$ is a middle $\Sigma$-region of $\mathcal{M}_\beta$ then $t|_{V_0}$ is unbounded.

Proof: Suppose that $t|_{V_0}$ is bounded and without loss of generality assume that $t(V_0) \subset ]-\infty, 0]$. Since $V_0$ is parabolic (see Corollary 4.1), there is a biholomorphism $T : V_0 \to A = \{z \in \mathbb{D} - \{0\} : \arg(z) \leq \frac{\pi}{2}\}$ satisfying that $T(\partial(V_0) \cap \Pi_0) = \{z \in A : \Re(z) = 0\}$. Up to the identification $V_0 \equiv A$ via $T$, $f := e^h : A \to \mathbb{C}$ extends by Schwarz reflection to a bounded function, that we keep calling $f$, on $\mathbb{D} - \{0\}$, by Riemann’s removable singularity theorem, $f$ extends holomorphically to $\mathbb{D}$. Furthermore, $f$ has no zeroes in $\mathbb{D}$ because $t = \log(|f|) \leq 0$ is bounded, and thus $h = \log(f)$ has well defined limit at $0$. Thus, $h|_{V_0}$ is bounded and has well defined limit at its unique end.

Let $V_1$ be a middle $\Sigma$-region of $\mathcal{M}_\beta$ contiguous to $V_0$. Since $h(V_1) \subset \{z \in \mathbb{C} : \Re(z) \geq 0\}$ and $h(V_0)$ is bounded, $h(V_0 \cup V_1)$ omits infinitely many complex values, and consequently is a normal function. From the conformal point of view, $D_1 := V_0 \cup V_1$ is biholomorphic to the $\mathbb{D} - \{1\}$. Identifying $D_1 \equiv \mathbb{D} - \{1\}$, $h$ has a well defined finite limit $w_0$ along arcs $\alpha \subset V_0 \subset \mathbb{D} - \{1\}$ diverging to $1$. Basic sectorial theorems for normal functions imply that $h|_{D_1}$ has well defined finite angular limit $w_0$ at the end $1$. In particular, $t|_{V_1}$ can not have asymptotic curves with asymptotic value $\infty$, which proves that $t|_{V_1}$ is bounded. Reasoning as at the beginning of the claim, $h|_{V_1}$ is bounded and has well defined finite limit $w_0$ at its unique end. Repeating this argument for successive contiguous middle $\Sigma$-regions, we conclude that $h$ has limit $w_0$ at the end of any middle $\Sigma$-region.

Now we can finish the claim. As we are assuming that $\mathcal{M}_\beta$ contains middle $\Sigma$-regions, then there is a $\Sigma$-region $U$ of $\mathcal{M}$ with $\partial(U) \subset \Pi_0$. The parabolicity of $U$ and Claim 1 show that
\( U \) is biholomorphic to \( \overline{D} - \{ w_1, \ldots, w_k \} \), where \( \{ w_1, \ldots, w_k \} \subset \partial(D) \). Since \( t|_{\partial(U)} = 0 \), we get \( t|_{U} = 0 \), which is absurd.

From Theorem 6.2, \( Y := \frac{1}{(t \wedge \nu)^{1/2}} \nabla t^* \) never vanishes on \( M \), and so any integral curve of \( Y \) is a proper arc in \( M \) contained in a horizontal plane.\(^{10}\) Furthermore, since \( Y \) is a spacelike field and \( \Gamma \) is lightlike, the integral curves of \( Y \) are transverse to \( \Gamma \). Consider the flow \( F : \Gamma \times [0, +\infty[ \rightarrow \mathbb{R}^3 \) of \( Y \) and define \( D \subset M \) as the open subdomain that is the image \( F(\Gamma \times [0, +\infty[) \). For the sake of simplicity, we denote by \( c_s \) integral curve \( F(\Gamma(s), [0, +\infty[), \) where as usual \( \Gamma(s) = \Gamma \cap \{ t = s \}, s \in \mathbb{R} \). In other words, \( c_s \) is the connected component of \( M \cap \{ t = s \} \) meeting \( \Gamma \).

To finish the theorem, it suffices to check that \( D = M \). Reason by contradiction, and assume that \( M - D \neq \emptyset \). Therefore, \( \partial(D) = \Gamma \cup C \), where \( C \neq \emptyset \) is a collection of pairwise disjoint proper integral curves of \( Y \) disjoint from \( \Gamma \).

Since arcs in \( C \) lie in horizontal planes, we can suppose up to a translations that \( C \cap \Sigma = C \cap \Pi_0 \neq \emptyset \). Let \( c \) be a proper arc in \( C \cap \Sigma \). Fix \( p_0 \in c \) and let \( \delta : [-\epsilon, \epsilon] \rightarrow M \) be the integral curve of \( \nabla t \) with initial condition \( \delta(0) = p_0 \). For \( p_0 \in C \subset \partial(D) \), we infer that \( \delta([0, \epsilon]) \subset D \), provided that \( \epsilon \) is small enough. Write \( t(\delta(\epsilon)) = a > 0 \) and note that \( t(\delta([0, \epsilon])) = [0, a] \). For any \( s \in [0, a] \), let \( \tilde{c}_s \subset c_s \) denote the compact arc joining \( \Gamma(s) = F(\Gamma(s), 0) = \Gamma \cap c_s \) and \( t(s) \delta^{-1}(s) \). From the choice of \( c_s \), the curves \( \{ \tilde{c}_s : s \in [0, a] \} \) converge as \( s \rightarrow 0 \) uniformly on compact subsets of \( M \) to \( c_0 \cup \tilde{c} \), where \( \tilde{c} \subset C \cap \Sigma \) is a collection of proper subarcs in \( C \) and \( p_0 \in \tilde{c} \). By Lemma 7.1, \( \tilde{c} \) has finitely many connected components, one of then being a divergent subarc of \( c \) with initial point \( p_0 \).

Set \( V = (\cup_{s \in [0, a]} \tilde{c}_s) \cup c_0 \cup \tilde{c} \), and note that \( V \) is a region in \( M \) homeomorphic to a closed disc minus a finite set of boundary points and with boundary \( \partial(V) = \Gamma([0, a]) \cup \delta([0, \epsilon]) \cup \tilde{c}_s \cup c_0 \cup \tilde{c} \). Since \( \Gamma([0, a]) \cup \delta([0, \epsilon]) \cup \tilde{c}_s \) is compact and \( c_0 \cup \tilde{c} \subset \Sigma \), \( V \) contains, up to a compact set, at least one \( \Sigma \)-region of \( M_\beta \). However, \( t|_{V} \) is bounded, contradicting Claim 2 and concluding the proof.

**Theorem 7.2 (Uniqueness of the Lorentzian Helicoid)** The unique properly embedded CLBMS maximal surface with infinite rotation number is, up to Lorentzian congruence, the Lorentzian helicoid.

**Proof:** Let \( M \) be a properly embedded CLBMS maximal surface with \( \theta(M) = +\infty \). Up to isometries, suppose \( \Sigma_\infty = \Pi_0 \). From Proposition 7.1, \( h := t + it^* : M \rightarrow \mathbb{C} \) is an injective holomorphic map. Furthermore, since \( \Gamma \) is a lightlike arc of mirror symmetry, \( t^*|_\Gamma \) is constant \( (\text{without loss of generality suppose } t^*|_\Gamma = 0) \).

Let us see that \( \lim_{x \in c_s \to -\infty} t^*(x) = +\infty \) for any \( s \in \mathbb{R} \), where as above \( c_s \) is the integral curve of \( Y \) with initial condition \( \Gamma(s) \). Indeed, as \( t^*|_{c_s} \) is monotone then the limit \( r_s := \lim_{x \in c_s \to -\infty} t^*(x) \) exists, and without loss of generality, belongs to \([0, +\infty[\), for any \( s \in \mathbb{R} \). In particular

\[
\lim_{x \in c_s \to -\infty} h(x) = s + ir_s, \quad s \in \mathbb{R}.
\]

Let \( V_s \) denote the region in \( M \) bounded by \( \Gamma([0, s]) \cup c_0 \cup c_s \). The holomorphic function \( h|_{V_s} \) omits infinitely many complex values, hence from Theorem 4.1, the limits along \( c_0 \) and \( c_s \) must coincide for any \( s \in \mathbb{R} \). Therefore, \( r_s = r_\infty \) for any \( s \in \mathbb{R} \), proving our assertion.

As a consequence, \( h(M) = \overline{\mathbb{U}} - \{ \infty \} = \{ z \in \mathbb{C} : \text{Im}(z) \geq 0 \} \) and \( h : M \rightarrow \overline{\mathbb{U}} - \{ \infty \} \) is a biholomorphism. Furthermore, identifying \( M \) and \( \overline{\mathbb{U}} - \{ \infty \} \) via \( h \), we get \( \phi_3 = -1Bi \) for \( B > 0 \).

On the other hand, Theorem 6.2 gives that \( g(U) \subset \mathbb{D} - \{ 0 \} \), and so \( \log(g) : U \rightarrow \mathbb{C} \) is well defined. As \( |g|^{-1}(1) = \partial(U) \), then \( \text{Re}(\log(g)) \) only vanish on the real axis and \( \log(g)|_{\partial(U)} \) is one to one. Therefore, \( g(z) = e^{az+ib} \), where \( a, b \in \mathbb{R} \), and up to Lorentzian congruence, \( M \) is the Lorentzian helicoid. \( \square \)

\(^{10}\)\( \nabla \) is the gradient with respect to the metric \( ds^2 \) induced by \( \langle ., . \rangle \).
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