Weak saturation stability

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Abstract

The paper studies $\text{wsat}(G, H)$ which is the minimum number of edges in a weakly $H$-saturated subgraph of $G$. We prove that $\text{wsat}(K_n, H)$ is ‘stable’ — remains the same after independent removal of every edge of $K_n$ with constant probability — for all pattern graphs $H$ such that there exists a ‘local’ set of edges percolating in $K_n$. This is true, for example, for cliques and complete bipartite graphs. We also find a threshold probability for the weak $K_{1,t}$-saturation stability.

1 Introduction

Let $G$ and $H$ be graphs (below, we refer to them as host and pattern graphs respectively). Let $F \subset G$ be a spanning subgraph of $G$. Let us call a sequence of graphs $F = F_0 \subset \ldots \subset F_m = G$ an $H$-bootstrap percolation process, if $F_i$ is obtained from $F_{i-1}$ by adding an edge that belongs to a copy of $H$ in $F_i$. $F$ is weakly $(G, H)$-saturated, if $G$ can be obtained from $F$ in an $H$-bootstrap percolation process (i.e., there exists an ordering $e_1, \ldots, e_m$ of the edges of $G \setminus F$ such that, for every $i \in [m] := \{1, \ldots, m\}$, $F \cup \{e_1, \ldots, e_i\}$ has a copy of $H$ that contains $e_i$). The smallest number of edges in a weakly $(G, H)$-saturated graph is called the weak saturation number and is denoted by $\text{wsat}(G, H)$. This notion was first introduced by Bollobás in 1968 [6].

In this paper, we study the phenomena of stability of the weak saturation number. It was observed by Korándi and Sudakov [12] that $\text{wsat}(G = K_n, K_s)$ remains the same after the deletion of each edge of $K_n$ independently with a constant positive probability (as usual, $K_n$ denotes a complete graph on $n$ vertices). We show that this stability property holds for a wider class of pattern graphs $H$, and conjecture that it actually holds for all $H$. We also find a threshold probability for the stability of the weak $(K_n, K_{1,t})$-saturation number. Before stating the results, let us recall known values of the weak saturation number when $G = K_n$. We denote by $K_{s,t}$ a complete bipartite graph with parts of size $s$ and $t$.

The exact value of $\text{wsat}(K_n, K_s)$ was achieved by Lovász [14]: if $n \geq s \geq 2$ then

$$\text{wsat}(K_n, K_s) = \binom{n}{2} - \binom{n - s + 2}{2}.$$

The value of $\text{wsat}(K_n, K_{s,t})$ for an arbitrary choice of parameters is still unknown. The most general result was obtained by Kalai [11] in 1985 and Kronenberg, Martins and Morrison [13] in 2020. They proved that

$$\text{wsat}(K_n, K_{t,t}) = (t-1)(n+1-t/2),$$
$$\text{wsat}(K_n, K_{t,t+1}) = (t-1)(n+1-t/2) + 1$$

if $t \geq 2$ and $n \geq 3t - 3$. In [13] general bounds for arbitrary choice of parameters $s, t$ were also obtained:

$$\text{wsat}(K_n, K_{s,t}) \leq (s-1)(n-s) + \binom{t}{2}$$

(1)

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Figure 1: structure of \( F_n^0 \).

if \( t > s \geq 2 \) and \( n \geq 2(s + t) - 3 \) and

\[
\text{wsat}(K_n, K_{s,t}) \geq (s - 1)(n - t + 1) + \left( \frac{t}{2} \right)
\]

if \( t > s \geq 2 \) and \( n \geq 3t - 3 \).

Notice that, for \( s = 1 \), the exact value of the weak saturation number is straightforward:

\[
\text{wsat}(K_n, K_{1,t}) = \left( \frac{t}{2} \right).
\]

Indeed, \( \text{wsat}(K_n, K_{1,t}) \) \( \leq \left( \frac{t}{2} \right) \) since we can use \( F = K_t \) to restore all edges of \( K_n \) (first, we restore all edges with one endpoint in the clique, then all the rest). Eventually, \( \text{wsat}(K_n, K_{1,t}) \) \( \geq \left( \frac{t}{2} \right) \) since any weakly \( (K_n, K_{1,t}) \)-saturated graph \( F \) must contain vertices \( v_1, \ldots, v_{t-1} \) such that, for every \( i \in [t-1] \), the number of neighbors of \( v_i \) in \( V(F) \setminus \{v_1, \ldots, v_{i-1}\} \) is at least \( t - i \). To see this, it is sufficient to consider the first distinct \( t - 1 \) vertices playing the roles of the central vertices of \( K_{1,t} \) in a bootstrap percolation process that starts on \( F \) and finishes on \( K_n \).

There are also many results about weak saturation numbers for other specific pairs of host and pattern graphs (e.g., for both \( G \) and \( H \) being complete bipartite \([16]\), for multipartite graphs \([1]\), for disjoint copies of graphs \([8]\), for hypercubes and grids \([3, 15]\)). The weak saturation number for hypergraphs has also been studied \([1, 3, 7, 8, 16, 19, 20]\).

In 2017, Korándi and Sudakov \([12]\) proved that, if \( s \geq 3 \), then \( \text{wsat}(K_n, K_s) \) is stable, i.e., for constant \( p \in (0, 1) \),

\[
\text{wsat}(G(n, p), K_s) = \text{wsat}(K_n, K_s)
\]

with high probability. Here, we denote by \( G(n, p) \) the binomial random graph on the vertex set \([n]\), where every pair of distinct \( i, j \in [n] \) is adjacent with probability \( p \) independently of the others. Hereinafter, we say that some property holds with high probability, or \( \text{whp} \), if its probability tends to 1 as \( n \to \infty \).

In this paper, we prove a transference theorem that can be used to derive such stability results. It immediately implies the result of Korándi and Sudakov as well as stability results for all complete bipartite pattern graphs (despite the fact that the exact value of \( \text{wsat}(K_n, K_{s,t}) \) is not known for almost all pairs \( s \) and \( t \)). Below, we denote by \( \delta(H) \) the minimum degree of graph \( H \). Without loss of generality, we set \( V(K_n) = [n] \).

**Theorem 1.** Let \( H \) be a graph without isolated vertices, and let \( p \in (0, 1) \), \( C \geq \delta(H) - 1 \) be constants. For every \( n \in \mathbb{N} \), let \( F_n^0 \) be a weakly \((K_n, H)\)-saturated graph containing a set of vertices \( S_n^0 \subset [n] \) with \( |S_n^0| \leq C \), such that every vertex from \([n] \setminus S_n^0 \) is adjacent to at least \( \delta(H) - 1 \) vertices of \( S_n^0 \) (see Figure 1). Then \( \text{whp} \) there exists a subgraph \( F_n \subset G(n, p) \) which is weakly \((G(n, p), H)\)-saturated and \( F_n \) has \( \min\{|E(G(n, p))|, |E(F_n^0)|\} \) edges.
This theorem implies

**Corollary 1.** Let \( p \in (0,1) \) be constant. For an arbitrary graph \( H \) without isolated vertices, whp

\[
wsat(G(n,p), H) = wsat(K_n, H),
\]

if, for every \( n \in \mathbb{N} \), there exists a minimum (having \( wsat(K_n, H) \) edges) weakly \((K_n, H)\)-saturated graph with the property described in Theorem 1.

**Proof.** Indeed, assume that the condition from Theorem 1 is satisfied. Then, it immediately implies that whp \( wsat(G(n,p), H) \leq wsat(K_n, H) \). Assume that, with a probability bounded away from 0, \( wsat(G(n,p), H) \) is strictly less than \( wsat(K_n, H) \). Since whp every pair of vertices of \( G(n,p) \) has at least \( |V(H)| \) pairwise adjacent common neighbors [17], \( wsat(G(n,p)) \) is weakly \((K_n, H)\)-saturated — a contradiction.

This corollary immediately implies stability for several pattern graphs. Notice that the graph obtained by removing a copy of \( K_{n-s+2} \) from \( K_n \) is weakly \((K_n, K_s)\)-saturated, has the structure described in Theorem 1 and has \( wsat(K_n, K_s) \) edges. Therefore, the result of Korándi and Sudakov can be immediately deduced from Corollary 1.

The constructions of Kronenberg, Martins and Morrison [13] of weakly \((K_n, K_{t,1})\)-saturated and weakly \((K_n, K_{t,t+1})\)-saturated graphs with \( wsat(K_n, K_{t,1}) \) and \( wsat(K_n, K_{t,t+1}) \) edges respectively also have the structure described in Theorem 1. Therefore, by Corollary 1 for every \( p \in (0,1) \), whp

\[
wsat(G(n,p), K_{t,1}) = wsat(K_n, K_{t,1}), \quad wsat(G(n,p), K_{t,t+1}) = wsat(K_n, K_{t,t+1}).
\]

The bounds (1), (2) imply that Corollary 1 can be also applied to pattern graphs \( K_{s,t} \) for all possible values of \( s \leq t \). To see this it is sufficient to show that, for every \( n \in \mathbb{N} \), there exists a minimum weakly \((K_n, K_{s,t})\)-saturated graph with the property described in Theorem 1. Note that due to (1) and (2) \( wsat(K_n, K_{s,t}) = (s-1)n + O(1) \), and \( s \) is the minimum degree of \( K_{s,t} \). Moreover, it is clear that there exists a constant \( C = C(s,t) \) such that, for all \( n \) large enough, \( wsat(K_n, K_{s,t}) = (s-1)n + C \). Indeed, otherwise there exist \( C_1 < C_2 \) and two infinite sequences \( \{n^1_i\}_{i \in \mathbb{N}} \) and \( \{n^2_i\}_{i \in \mathbb{N}} \) such that

\[
wsat(K_{n^1_i}, K_{s,t}) = (s-1)n^1_i + C_1 \quad \text{and} \quad wsat(K_{n^2_i}, K_{s,t}) = (s-1)n^2_i + C_2.
\]

Let us choose sufficiently large \( i \) and \( j \) such that \( n_1 := n^1_i < n^2_j := n_2 \) and let \( F_1 \) be weakly \((K_{n_1}, K_{s,t})\)-saturated. Then, the graph \( F_1 \) on \([n_2]\) obtained from \( F_1 \) by adding \( s-1 \) edges from each of the vertices from \([n_2]\) to \([n_1]\) is \((K_{n_2}, K_{s,t})\)-weakly saturated and has the number of edges \((s-1)n^2_i + C_1 < (s-1)n^2_i + C_2 \) — a contradiction. From this, the existence of a weakly \((K_n, K_{s,t})\)-saturated graph with the desired property is straightforward. Indeed, let \( n_0 \) be so large that \( wsat(K_{n_0}, K_{s,t}) = (s-1)n_0 + C \) and let \( F_0^{n_0} \) be a weakly \((K_{n_0}, K_{s,t})\)-saturated graph with \((s-1)n_0 + C \) edges. For every \( n > n_0 \), set \( S_0^n = \{n\} \) and define \( F_0^n \) to be the union of \( F_0^{n_0} \) with \( s-1 \) edges going from each of the vertices from \([n]\) to \( S_0^n \). The graph \( F_0^n \) is weakly \((K_n, K_{s,t})\)-saturated and has the desired number of edges. Therefore by Corollary 1, we get that, for every constant \( p \in (0,1) \) and all \( 1 \leq s \leq t \), whp

\[
wsat(G(n,p), K_{s,t}) = wsat(K_n, K_{s,t}).
\]

This stability result demonstrates the efficiency of Theorem 1; it can be used to prove stability even when the exact value of \( wsat(K_n, K_{s,t}) \) is unknown.

However, there are graphs \( H \) such that minimum weakly \((K_n, H)\)-saturated graphs do not satisfy the condition from Theorem 1. For such graphs, the same stability result can not be proven using Corollary 1.

For example, consider \( H \) being the, so called, \( t \)-barbell graph consisting of two vertex-disjoint \( t \)-cliques together with a single edge that has an endpoint in each clique. It is clear that \( wsat(K_n, H) \leq \left(\frac{3}{2}\right)n^2 + \frac{t}{2t} \) for \( n \) divisible by \( t \) since the disjoint union of \( n/t \) \( t \)-cliques is weakly \((K_n, H)\)-saturated. Nevertheless,
any subgraph of \( K_n \) with the property described in Theorem 1 has at least \((t-2)n + O(1)\) vertices. Therefore, it is not minimum possible, and Corollary 1 is not applicable.

Nevertheless, some extra work (it is very technical, so we omit the details) is required to show that the union of disjoint cliques has minimum possible number of edges and that, since whp \( G(n, p) \) contains a \( K_t \)-factor (see [10]), we get stability for \( t \)-barbell graph as well. In some sense, the situation covered by Theorem 1 is less pleasant. Indeed, if a graph on \([n]\) has a bounded maximum degree, then whp \( G(n, p) \) contains its isomorphic copy as a spanning subgraph (see [2]). So, for such weakly saturated graphs, stability is straightforward. In Theorem 2 we consider an opposite scenario — \( F_o^t \) has vertices with degrees \( n - O(1) \) that are not likely to be in \( G(n, p) \). Having that in mind, we conjecture that, for any constant \( p \in (0, 1) \) and every graph \( H \), whp \( \text{wsat}(G(n, p), H) = \text{wsat}(K_n, H) \).

Let us now switch to the case \( p = o(1) \).

Korándi and Sudakov [12] claim that their result can be easily extended to the range \( n^{-\epsilon(s)} \leq p \leq 1 \). It can be seen that the same is true for Theorem 1 with \( \epsilon \) depending only on \( H \). However, for smaller \( p \), Theorem 1 or Corollary 1 may not hold.

Korándi and Sudakov [12] pose the following question: what is the exact probability range where \( \text{whp} \ \text{wsat}(G(n, p), K_s) = \text{wsat}(K_n, K_s) \)? In 2020, Bidgoli et al. [5] proved the existence of threshold probability for this stability property. Moreover, they obtained bounds on the threshold:

- There exists \( c \) such that, if \( p < cn^{\frac{-1}{2t}}(\ln n)^{\frac{-1}{2t-1}} \), then whp \( \text{wsat}(G(n, p), K_s) \neq \text{wsat}(K_n, K_s) \),
- If \( p > n^{-\frac{1}{2t}}(\ln n)^2 \), then whp \( \text{wsat}(G(n, p), K_s) = \text{wsat}(K_n, K_s) \).

In this paper, we estimate the threshold probability for the weak \( K_{1,t} \)-saturation stability property

\[ \text{wsat}(G(n, p), K_{1,t}) = \text{wsat}(K_n, K_{1,t}) \]

\textbf{Theorem 2.} Let \( t \geq 3 \). Denote \( p(n, t) = n^{-\frac{1}{2t}}[\ln n]^{-\frac{1}{2t-1}} \).

- There exists \( c > 0 \) such that, if \( \frac{1}{n^2} < p < cp(n, t) \), then whp \( \text{wsat}(G(n, p), K_{1,t}) \neq \text{wsat}(K_n, K_{1,t}) \).
- There exists \( C > 0 \) such that, if \( p > C p(n, t) \), then whp \( \text{wsat}(G(n, p), K_{1,t}) = \text{wsat}(K_n, K_{1,t}) \).

Note that Theorem 2 does not cover the case \( t = 2 \) as well as \( p = O(1/n^2) \). But these cases are much easier. Below we consider them separately.

First, if \( p < \frac{Q}{n^2} \) for some constant \( Q > 0 \), then whp \( G(n, p) \) consists of isolated vertices and isolated edges (it simply follows from Markov’s inequality applying to the number of \( P_3 \) in \( G(n, p) \)). Therefore, whp there are no copies of \( K_{1,t-1} \) for \( t \geq 3 \) in \( G(n, p) \), and there are no weakly \( (G(n, p), K_{1,t}) \)-saturated subgraphs other than the entire graph. So, whp there is stability only if the number of edges of the graph is exactly \( \binom{t}{2} \). The latter property holds with probability \( \binom{\binom{t}{2}}{\binom{t}{2}} \binom{t}{2} \left(1 - \frac{Q}{n^2}\right)^{\binom{t}{2} - \binom{t}{2}} \). Therefore, it tends to 0 when \( p \ll \frac{1}{n^2} \) and it is bounded away both from 0 and from 1 when \( \frac{Q}{n^2} < p < \frac{Q}{n^2} \) for some \( 0 < q < Q \).

The case \( t = 2 \) is also trivial. Clearly, for a graph \( G \) on \([n]\),

\[ \text{wsat}(G, K_{1,2}) = \text{wsat}(K_n, K_{1,2}) = \binom{t}{2} = 1 \]

if and only if \( G \) has exactly one non-empty connected component. Using the standard first and second moment methods (see, e.g., [11], Chapter 1), it can be proven that, for every \( \epsilon > 0 \),

- If \( p > (1 + \epsilon) \frac{\ln n}{2n} \), then whp \( G(n, p) \) contains a unique non-empty component;
- If \( \frac{1}{n^2} \ll p < (1 - \epsilon) \frac{\ln n}{2n} \), then whp \( G(n, p) \) contains at least two non-empty connected components;
- If \( \frac{Q}{n^2} < p < Q \) for some \( 0 < q < Q \), then whp all edges in \( G(n, p) \) are disjoint and, arguing as above, we get that stability happens only if the graph contains \( \binom{t}{2} \) is 1 edges;
- If \( p \ll \frac{1}{n^2} \), then whp \( G(n, p) \) is empty.
Therefore,

1. if \( p > (1 + \varepsilon)\frac{\ln n}{2n} \), then whp \( \text{wsat}(G(n,p), K_{1,t}) = \text{wsat}(K_n, K_{1,t}) \);

2. if \( \frac{1}{n^r} \ll p < (1 - \varepsilon) \frac{\ln n}{2n} \), then whp \( \text{wsat}(G(n,p), K_{1,t}) \neq \text{wsat}(K_n, K_{1,t}) \);

3. if \( \frac{1}{n^r} < p < \frac{Q}{n^r} \) for some \( 0 < Q < Q \), then

\[
P \left[ \text{wsat}(G(n,p), K_{1,t}) = \text{wsat}(K_n, K_{1,t}) \right] = \frac{1}{n^r} + o(1) = \frac{1}{n^r} p(1 - p)(\frac{1}{2})^{-1} + o(1)
\]

is bounded away both from 0 and 1,

4. if \( p \leq \frac{1}{n^r} \), then whp \( \text{wsat}(G(n,p), K_{1,t}) = 0 \neq \text{wsat}(K_n, K_{1,t}) \).

The structure of the paper is the following. In Section 2, we prove Theorem \( \text{I} \). In Section 3, we prove Theorem \( \text{II} \).

2 Proof of Theorem \( \text{I} \)

We denote \( d = d(H) - 1 \), \( r = |V(H)| \). First, for convenience, let us state an equivalent modification of Theorem \( \text{I} \) which at first sight seems to be weaker. However, it is equivalent, and it is easier to prove this version. In Section 2.1, we prove the equivalence of the two statements, and then prove the modified version.

Lemma 1. Let \( H \) be an arbitrary fixed graph without isolated vertices, \( p \in (0, 1) \), \( C \geq d \) be constants. For every \( n \in \mathbb{N} \), let \( F_n = K_n \) be a weakly \( (K_n, H) \)-saturated graph containing a set of vertices \( S_n \) of size at most \( C \) such that every vertex of \( [n] \setminus S_n \) is adjacent to \textit{exactly} \( d \) vertices of \( S_n \) and there are no edges between vertices of \( [n] \setminus S_n \). Then whp there exists a subgraph \( F_n \subset G(n, p) \) which is weakly \( (G(n,p), H) \)-saturated and \( F_n \) has the same number of edges as \( F_n \).

2.1 Proof of equivalence

Clearly, Theorem \( \text{I} \) implies Lemma \( \text{I} \). Let us prove that the opposite implication is also true.

We first notice that \( \text{wsat}(K_n, H) \leq \left(\frac{1}{2}\right) + d(n-r) \). Indeed, we can construct a weakly \((K_n, H)\)-saturated subgraph with at most many edges in the following way. Let \( F_0 \) be a weakly \((K_r, H)\)-saturated subgraph on \( [r] \), the first \( r \) vertices of \( K_n \). The desired graph \( F_1 \) has all the same edges as \( F_0 \) on \( [r] \) and from every other vertex there are exactly \( d \) edges to \( F_0 \). The existence of a bootstrap percolation process that starts on \( F_0 \) and finishes on \( K_n \) is straightforward: first restore all edges of \( K_r \), then restore all edges going to \( [r] \) and finally restore all the remaining edges.

We assume that Lemma \( \text{I} \) holds. The constructed graph \( F_1 \) satisfies the conditions of this lemma. Then, whp there exists a weakly \((G(n,p), H)\)-saturated subgraph \( F_n \) such that

\[
|E(F_n)| \leq \left(\frac{r}{2}\right) + d(n-r) = dn + \left(\frac{r}{2}\right) - d(\frac{r}{2} - r).
\]

Let \( F_0 \) be a graph that satisfies the condition of Theorem \( \text{I} \). If \( |E(F_0)| \geq |E(F_1)| \), then a subgraph of \( G(n, p) \) obtaining by adding \( \min \\{ |E(F_0)|, |E(G(n,p))| \} - |E(F_0)| \) edges to \( F_0 \) is weakly \((G(n,p), H)\)-saturated and \( F_n \) has \( \min \\{ |E(F_0)|, |E(G(n,p))| \} \) edges.

If \( |E(F_0)| < |E(F_1)| \), then there are at most \( a \) vertices with degree more than \( d \) in \( F_0 \) outside \( S_n \). We can add them to \( S_n \) and then the size of this set will be bounded from above by \( C + a \). So, \( F_n \) satisfies the condition of Lemma \( \text{I} \) with \( C := C + a \). Therefore, whp there exists a weakly \((G(n,p), H)\)-saturated subgraph with \( |E(F_0)| \) edges. \( \square \)
Below, we prove Lemma 1. Let us outline the proof. In Section 2.2 we describe sufficient properties of a spanning subgraph \( G \) of \( K_n \) that allow to find a weakly \((G, H)\)-saturated subgraph with the same number of edges as in a weakly \((K_n, H)\)-saturated subgraph. In Section 2.3 we prove that these properties are indeed sufficient for the described transference property. In Section 2.4 we prove that whp \( G(n, p) \) has the described properties and, thus, finish the proof of Lemma 1.

Let us now switch to the proof of Lemma 1. Assume that the requirements of the lemma hold. We also suggest that \( n \) is even to avoid overloading with floor and ceiling functions notations. This does not affect the proof anyhow.

Let us denote the vertices of \( H \) as \( w_1, \ldots, w_r \), where \( w_1 \) is a vertex with degree \( d + 1 \) and \( w_2, \ldots, w_{d+2} \) are its neighbours. Let \( H' \) be obtained from \( H \) by deleting the vertices \( w_1, w_2 \) and let \( H'' \) be obtained from \( H \) by deleting the edge \( \{w_1, w_2\} \) (but preserving the vertices \( w_1, w_2 \)).

In what follows, for a graph \( F \) and its vertex \( v \), we denote by \( N_F(v) \) the set of all neighbours of \( v \) in \( F \).

### 2.2 Sufficient properties

We want to find in \( G(n, p) \) a subgraph having similar structure to a weakly saturated subgraph in \( K_n \). However, it is not immediate since whp all vertices in \( G(n, p) \) have degrees \( \Theta p (1 + o(1)) \) which is far away from \( n - O(1) \). Nevertheless, we can find a clique \( K \) in \( G(n, p) \) of size \( \Theta (\ln n) \), and first reconstruct the edges of the clique. For that, we fix in \( K \) a weakly saturated spanning subgraph with the minimum possible number of edges and the desired structure. In other words, we choose a subset \( S \subset K \) playing the role of \( S' \) in \( K \). After reconstructing the edges of \( K \) we might hope that it is sufficient to use \( d \) edges of \( G(n, p) \) at every vertex outside \( K \) to reconstruct all the other edges of \( G(n, p) \). The properties that allow to do this are described below.

We start from distinguishing several subsets of \([n]\) that we use to describe the properties. Everywhere below, by \( G[V] \) we denote the induced subgraph of \( G \) on the vertex set \( V \) (where \( V \subset V(G) \)).

Let \( c > 0 \). Let \( G \) be a graph on the vertex set \([n]\). Let

- \( G_1 = G|_{[n]/2} \), \( G_2 = G|_{[n]\setminus[n]/2} \);
- \( K \subset V_1 := V(G_1) \) be a set of size \( k \geq c \ln n \),
  \( S \) be a subset of \( K \) of size \( |S_1| \),
  \( D \) be a subset of \( S \) of size \( d \) (clearly, the requirements in Lemma 1 imply that \( |S_1| \geq d \));
- \( Z \) be the set of all common neighbours of \( D \) in \( V(G_2) \);
- \( Z = Z_1 \cup Z_2 \cup Z_3 \) be a partition of \( Z \);
- \( R \) be a set of \( r \) vertices from \( K \setminus S \) and \( T \subset Z \) be the set of all common neighbors from \( Z \) of vertices from \( R \).

For vertices \( u, v \) of \( G \) and a subgraph \( \tilde{H} \cong H' \) of \( G \), we call the tuple \((v, u, \tilde{H})\) \( H\)-completable in \( G \) (see Figure 2), if there exists an embedding \( f \) (we call it \((v, u, \tilde{H})\)-embedding) from \( H \) to \( G|_{V(H)\cup\{v, u\}} \) such that \( f(w_1) = v \), \( f(w_2) = u \) and \( f \) maps \( H' \) to \( \tilde{H} \), i.e. the graph with the set of vertices \( V(\tilde{H}) \) and the set of edges \( \{(f(x), f(y)), (x, y) \in E(H')\} = E(\tilde{H}) \) equals \( \tilde{H} \). In plain words, it means that we are able to immediately reconstruct the edge \( \{u, v\} \). For a vertex \( v \) of \( G \) and a subgraph \( \tilde{H} \cong H' \) of \( G \), we call the pair \((v, \tilde{H})\) \( H\)-completable in \( G \), if there exists an embedding \( f \) (we call it \((v, \tilde{H})\)-embedding) from \( H|_{V(H)\setminus\{w_1\}} \) to \( G|_{V(\tilde{H})\cup\{v\}} \) such that \( f(w_1) = v \) and \( f \) maps \( H' \) to \( \tilde{H} \). Intuitively, it means that nothing prevents us from adding edges from \( v \) to some other vertices \( w \) (such that \( (v, w, \tilde{H}) \) is \( H\)-completable in \( G \)). Although we do not know whether such \( w \) exists, it is not 'the fault' of \( v \). Let \((v, u, \tilde{H})\) be \( H\)-completable in \( G \) and \( V \subset V(\tilde{H}) \) have exactly \( d \) vertices. We call the tuple \((v, u, \tilde{H}, V)\) \( H\)-completable in \( G \), if there exists a \((v, u, \tilde{H})\)-embedding \( f \) such that \( f \) maps \( N_H(w_1) \setminus \{w_2\} \) onto \( V \). In Figure 2, \( \tilde{V} \) is shown in violet color. Similarly, for an \( H\)-completable \((v, \tilde{H})\) and \( V \subset V(\tilde{H}) \) of size \( d \), we call the tuple
Let us now desribe the desired properties. We define two properties of $G$. The first one is used to reconstruct all the edges but those between $S \setminus D$ and $[n] \setminus (K \cup Z)$. All the other edges are reconstructed using the second property.

For a vertex $v$ of $G$, a subgraph $\tilde{H} \cong H'$ of $G$ and a set of vertices $\tilde{V} \subset V(\tilde{H})$ of size $d$, denote by $U_v(\tilde{H}, V)$ the set of all neighbours $u$ of $v$ in $Z_2$ such that the tuple $(v, u, \tilde{H}, \tilde{V})$ is $H$-completable.

Let us say that the tuple $(G; K, S, D, Z_1, Z_2, Z_3, T)$ satisfies the first $H$-saturation property (or, simply, the first property), if (note that, among the members of the tuple, $G$ is a graph, and all the others are sets of vertices; recall also that $Z_1 \cup Z_2 \cup Z_3$ is a partition of $Z$)

1. $K$ induces a clique in $G$;
2. for any adjacent (in $G$) pair $v_1, v_2 \in Z \setminus T$, there exists a copy $H_{v_1,v_2}$ of $H'$ in $G|_T$ such that $(v_1, v_2, H_{v_1,v_2})$ is $H$-completable in $G$;
3. for every $v \notin K \cup Z$ there exists $H_v \cong H'$ inside $Z_1$ and $V_v \subset V(H_v)$ such that $(v, H_v, V_v)$ is $H$-completable in $G$;
4. for any pair $v_1, v_2 \notin K \cup Z$, there exists a copy of $H'$ in $G|_{U_{v_1}(H_{v_1,v_2}) \cup U_{v_2}(H_{v_1,v_2})}$;
5. for any $v \notin K \cup Z$, $u \in (K \setminus S) \cup Z_2 \cup Z_3$, there exists a copy of $H'$ in $G|_{U_v(H_v, V_v) \cap N_G(u)}$;
6. for any $v \notin K \cup Z$, $u \in Z_1$, there exists a copy of $H'$ in $G|_{Z_3 \cap N_G(v) \cap N_G(u)}$;
7. for any $u \notin S$, there exists a copy of $H'$ in $G|_{Z \cap N_G(u)}$.

Finally, let $V(G_2) = V^1 \cup V^2 \cup V^3$ be a partition. Let us say that the tuple $(G; V^1, V^2, V^3, S, D)$ satisfies the second $H$-saturation property (or, simply, the second property), if

1. there exists $H_D \subset G|_{D \cup V^3}$ such that $H_D \cong H'$ and, for every $v \in S \setminus D$, $u \in [V(G_1) \setminus S] \cup V^3$, there is a copy of $H'$ inside $G|_{U_v \cap N_G(u)}$, where $U_v$ is the set of all neighbors $z$ of $v$ in $V^2$ such that $(v, z, H_D, D)$ is $H$-completable.
2. for every $v \in S \setminus D$, $u \in V^1 \cup V^2$, there is a copy of $H'$ inside $G|_{V^3 \cap N_G(u) \cap N_G(v)}$.

Figure 2: $H$-completable tuples $(v, u, \tilde{H})$ and $(v, u, \tilde{H}, \tilde{V})$. Black edges are edges mapped to edges of $H$, dashed edges are other edges of $G$, the blue dashed edge can be immediately reconstructed in a bootstrap percolation process.
2.3 Proof for a non-random graph

Assume that tuples \((G; K, S, D, Z_1, Z_2, Z_3, T)\) and \((G; V^1, V^2, V^3, S, D)\) satisfy the first and the second property respectively.

Clearly, in \(G\), there are at least \(\binom{k^n_2}{2} + d(n - k) > |E(F^1_n)|\) (for \(n\) large enough, since \(F^1_n\) has \(dn + O(1)\) edges due to the requirements in Lemma 1 and \(k \geq c \ln n\)) edges. Let us prove that there exists a weakly \((G, H)\)-saturated graph with \(|E(F^1_n)|\) edges for sufficiently large \(n\). Clearly, it is sufficient to prove the existence of a weakly \((G, H)\)-saturated graph with at most \(|E(F^1_n)|\) edges.

Without loss of generality, assume that, for every \(n\), \(|V(S^n_1)\) = \(C\) (we can add \(C - |V(S^n_1)\) vertices of \([n]\) \(\setminus S^n_1\) to \(S^n_1\), and then we will preserve conditions of Lemma 1). We can also assume that, for every \(n\), \(F^1_n\) has the minimum number of edges among all graphs that satisfy conditions of Lemma 1 and have \(|V(S^n_1)\) = \(C\).

Let us now construct a spanning subgraph \(F \subset G\) with at most \(|E(F^1_n)|\) edges. After that, we will prove that this graph is weakly \((G, H)\)-saturated.

Let us first define those edges of \(F\) that belong to \(K\). Let \(\varphi\) be a bijection from \(K\) to \(V(K_k)\) such that \(S \subset K\) is mapped onto \(S_k\). Then, we construct a graph on \(K\) (and we let \(F|_K\) to be exactly this graph) isomorphic to \(F^1_k\) such that \(\varphi\) is an isomorphism of \(F|_K\) and \(F^1_k\) (i.e., an edge \(\{u, v\}\) belongs to \(F|_K\) iff \(\varphi(u), \varphi(v)\) belongs to \(F^1_k\)).

Second, for every vertex \(v \in Z\), keep the edges of \(G\) going from \(v\) to \(D\) (there are \(d\) of them). Moreover, for every vertex outside \(K \cup Z\), keep specific \(d\) edges of \(G\) going from \(v\) to \(Z\) (so many edges exist due to the condition of the first property). The choice of edges between \([n]\) \(\setminus (K \cup Z)\) and \(Z\) will be explained later.

Clearly,

\[|E(F)| = |E(F^1_n)| + \left| E\left(F^1_k|_{S^1_k}\right) \right| - \left| E\left(F^1_n|_{S^1_n}\right) \right|. \tag{3} \]

Let us prove that \(\left| E\left(F^1_k|_{S^1_k}\right) \right| = |E(F^1_n|_{S^1_n})|\) for \(n\) large enough. It is enough to prove that there exists \(N \in \mathbb{N}\) such that, for every \(n_1, n_2 \geq N\), \(\left| E\left(F^1_n|_{S^1_n}\right) \right| = \left| E\left(F^1_n|_{S^1_n}\right) \right|\). Assume the contrary: for every \(N \in \mathbb{N}\), there exist \(n_1 > n_2 \geq N\) such that \(\left| E\left(F^1_{n_1}|_{S^1_{n_1}}\right) \right| > \left| E\left(F^1_{n_2}|_{S^1_{n_2}}\right) \right|\). Let \(N\) be large enough. Since \(|S^1_{n_1}| = |S^1_{n_2}| = C\), we get that \(\left| E(F^1_{n_1}) \right| > \left| E(F^1_{n_2}) \right| + d(n_1 - n_2)\). Since \(F^1_{n_2}\) is weakly \((K_{n_2}, H)\)-saturated, we get that a graph on \([n_1]\) obtained from \(F^1_{n_2}\) by adding \(d\) edges from each vertex of \([n_1] \setminus [n_2]\) to \(F^1_{n_2}\) is weakly \((K_{n_1}, H)\)-saturated. This contradicts with the minimality of the number of edges in \(F^1_{n_1}\).

From (3), we get that \(|E(F)| = |E(F^1_n)|\) for large \(n\).

Now let us show that \(F\) is weakly \((G, H)\)-saturated and, on the way, specify the edges from \([n]\) \(\setminus (K \cup Z)\) to \(Z\).

We first sequentially add the following bunch of edges to \(F\): edges inside \(K\), edges from \(K\) to \(Z\), edges inside \(T\), edges between \(Z\) and \(T\), edges inside \(Z\).

1. Here, we restore the edges of \(G\) that are inside \(K\). This is straightforward since \(K\) is a clique (by the condition of the first property), \(F|_K \cong F^1_k\) and there exists a bootstrap percolation process that starts on \(F^1_k\) and finishes on \(K_k\). Let \(F_1 = F \cup G|_K\).

2. Let us restore the edges of \(G\) between \(K\) and \(Z\). Edges between \(Z\) and \(D\) are already in \(F_1\). Consider \(u \in K \setminus D, v \in Z\). Let \(K'\) be a set of \(r - d - 2\) vertices of \(K \setminus D \cup \{u\}\). Then, for any graph \(\tilde{H} \cong H'\) on the vertex set \(K' \cup D\) such that \(N_H(w_1)\) is mapped onto \(D\) (such a mapping exists since \(K\) induces a clique in \(G\)), the tuple \((v, u, \tilde{H})\) is \(H\)-completable in \((V(F_1), E(F_1) \cup \{u, v\})\) since \(v\) is adjacent to every vertex from \(D\) in \(G\). So, we can restore \((u, v)\). Let \(F_2\) be obtained from \(F_1\) by adding all edges from \(G\) between \(K\) and \(Z\).

3. Let us switch to edges that are entirely in \(T\). Consider \(u, v \in T\). The edges inside \(R\) and between \((u, v)\) and \(R\) are already in \(F_2\) (since \(T \subset Z, R \subset K\), and the edges inside \(K\) and between \(K\) and \(Z\) are already restored). Let \(\tilde{H} \cong H'\) be inside \(R\) (recall that \(R\) is a clique). Then \((v, u, \tilde{H})\)
is $H$-completable in $(V(F_2), E(F_2) \cup \{u, v\})$ and, therefore, we are able to restore $\{u, v\}$. We get $F_3 = F_2 \cup G_{\lnot \tau}$.

4) Let us restore edges between $Z \setminus T$ and $T$. Consider $v \in Z \setminus T$, $u \in T$. Since $u$ is adjacent to all vertices from $R \cup D$, $v$ is adjacent to all vertices in $D$, $|R| = r$ and $|D| = d$, we get that there exists $H \cong H'$ in $F_3 |_{R \cup D}$ such that $(v, u, H)$ is $H$-completable in $(V(F_3), E(F_3) \cup \{u, v\})$. Let $F_4$ be obtained from $F_3$ by adding all edges from $G$ between $Z \setminus T$ and $T$.

5) Let us restore the remaining edges inside $Z$. Consider adjacent (in $G$) $v_1, v_2 \in Z \setminus T$. By the condition $3$ of the first property, there is a copy of $H$ inside $G_{\lnot \tau} = F_4 |_{\lnot \tau}$ such that $(v_1, v_2, H_{v_1, v_2})$ is $H$-completable. Since edges from $F_4$ and between $\{v_1, v_2\}$ and $T$ are already in $F_4$, we get that $(v_1, v_2, H_{v_1, v_2})$ is $H$-completable in $(V(F_4), E(F_4) \cup \{v_1, v_2\})$ and we can restore $\{v_1, v_2\}$. We get $F_5 = F_4 \cup G_{\lnot Z}$.

Next we restore edges that are entirely outside $K \cup Z$.

6) Let $v \notin K \cup Z$. Notice that we have to specify $d$ edges going from $v$ to $Z$ in $F$. Let us do that. By the condition $3$ of the first property, there exists a copy $H_u \subset G_{\lnot Z}$ of $H'$ such that $(v, H_u)$ is $H$-completable. We specify $V_r$.

Let us now restore edges between $v$ and $U_v(H_v, V_v)$. The edges inside $H_v \subset Z$ and between $v$ and $V_v$ are already in $F_5$, so by the definition of $U_v(H_v, V_v)$, we can restore all edges between $v$ and $U_v(H_v, V_v)$. Let $F_6$ be obtained from $F_5$ by adding all edges between every $v \notin K \cup Z$ and $U_v(H_v, V_v)$.

7) Here, we consider edges that have both vertices outside $K \cup Z$. Consider $v_1, v_2 \notin K \cup Z$. Edges from $v_1$ and $v_2$ to $U_{v_1}(H_{v_1}, V_{v_1}) \cup U_{v_2}(H_{v_2}, V_{v_2})$ and edges inside $U_{v_1}(H_{v_1}, V_{v_1}) \cap U_{v_2}(H_{v_2}, V_{v_2})$ are already in $F_6$. By the condition $3$ of the first property, there is a copy of $H'$ inside $G_{\lnot \tau} = F_6 |_{\lnot \tau}$ such that $(v_1, v_2, H_{v_1, v_2})$ is $H$-completable in $(V(F_6), E(F_6) \cup \{v_1, v_2\})$ and we can restore $\{v_1, v_2\}$. Let $F_7 = F_6 \cup G_{\lnot [n] \setminus (K \cup Z)}$.

It remains to restore only edges between $K \cup Z$ and $[n] \setminus (K \cup Z)$.

8) Let us restore all edges between $(K \cup Z) \setminus S$ and $[n] \setminus (K \cup Z)$. Let $v \notin K \cup Z$.

First, let $u \in (K \setminus S) \cup Z_2 \cup Z_3$. The edges inside $U_v(H_v, V_v) \cap N_G(u) \subset Z$, edges from $u$ to $N_G(u) \cap Z_2$ and edges from $v$ to $U_v(H_v, V_v)$ are already in $F_7$. By the condition $3$ of the first property, there is a copy of $H'$ inside $U_v(H_v, V_v) \cap N_G(u)$. So, we can restore the edge between $u$ and $v$.

Second, let $u \in Z_1$. The edges from $v$ to $Z_3$ are just restored. The edges inside $Z_3$ and the edges between $u$ and $Z_3$ are already in $F_7$. By the condition $3$ of the first property, there is a copy of $H'$ inside $Z_1 \cap N_G(u) \cap N_G(v)$, so we can restore $\{u, v\}$.

$F_8$ is obtained from $F_7$ by adding all edges of $G$ between $(K \cup Z) \setminus S$ and $[n] \setminus (K \cup Z)$.

It remains to restore only edges between $S$ and $[n] \setminus (K \cup Z)$.

9) Here, we restore edges between $D$ and $[n] \setminus (K \cup Z)$. Let $v \in D$, $u \in [n] \setminus (K \cup Z)$. Then edges between $u$ and $Z$, edges between $v$ and $Z$ and edges inside $Z$ are in $F_8$, and $Z \subset N_G(v)$. By the condition $3$ of the first property, there is a copy of $H'$ inside $Z \cap N_G(u)$. So, we can restore $\{u, v\}$. Let $F_9$ be obtained from $F_8$ by adding all edges of $G$ between $D$ and $[n] \setminus (K \cup Z)$.

It remains to restore edges between $S \setminus D$ and $[n] \setminus (K \cup Z)$. Consider $v \in S \setminus D$. By the definition of $U_v$ (given in the condition $3$ of the second property), the edges from $v$ to $U_v$ can be restored immediately, as edges inside $H_D$, edges between $U_v$ and $H_D$ and edges from $v$ to $D$ are in $F_9$. For $u \in (V(G_1) \setminus S) \cup V^3$, by the condition $3$ of the second property, there is a copy of $H'$ inside $U_v \cap N_G(u)$. Edges inside $U_v \cap N_G(u) \subset V^2$ and edges between $\{u, v\}$ and $U_v \cap N_G(u) \subset V^2$ are already restored, so $\{u, v\}$ can be restored. Finally, consider $u \in V^3 \cup V^2$. The edges between $S$
and $V^3$ have just been restored. By the condition \[2\] of the second property, there is a copy of $H'$ inside $V^3 \cap N_G(u) \cap N_G(v)$, so we can restore $\{u, v\}$ as well.

### 2.4 Random graph has the properties

Let us first recall some results on the distribution of small subgraphs in the binomial random graph.

Given a graph $Y$, it is well known that the number of subgraphs isomorphic to $Y$ in $G(n, p)$ is well-concentrated around its expectation. In particular, Janson’s inequality implies that (see, e.g., [9, Theorem 2.14]) the probability that $G(n, p)$ does not contain an isomorphic copy of $K_\ell$ ($\ell$ is a positive integer constant) is at most $e^{-\Omega(n^2)}$. By the union bound, we get

**Claim 1.** Let $\varepsilon > 0$. Whp, for any subset $A \subset [n]$ such that $|A| \geq \varepsilon n$, there exists a copy of $K_\ell$ in $G(n, p)|_{\bar{A}}$.

Since $K_\ell$ contains as a subgraph any graph on $\ell$ vertices, we get that the statement of Claim 1 is also true for any graph $Y$.

Below, we use a notion of $(X, Y)$-extension introduced by Spencer in [18]. Let $x \in \mathbb{N}$ and $X = \{\omega_1, \ldots, \omega_x\}$ be a set of $x$ vertices called roots. Let $Y$ be a graph on $\{\omega_1, \ldots, \omega_y\}$, $y > x$. Then a graph $\bar{Y}$ on $\{\bar{\omega}_1, \ldots, \bar{\omega}_y\}$ is called $(X, Y)$-extension of $\bar{X} = (\omega_1, \ldots, \omega_x)$, if, for distinct $i \in [y], j \in [y] \setminus [x]$, the presence of the edge $\{\omega_i, \omega_j\}$ in $Y$ implies the presence of the edge $\{\bar{\omega}_i, \bar{\omega}_j\}$ in $\bar{Y}$.

In [18], it is proven (by a straightforward application of another Janson's inequality, [9, Theorem 2.18 (i)]) that $|z|$ does not have an $(X, Y)$-extension with probability at most $e^{-\Omega(n)}$. By the union bound, this observation implies the following.

**Claim 2.** Let $\varepsilon > 0$, $\bar{n} \in (\varepsilon n, n]$ be a sequence of positive integers. Then whp

- for any pair $u, v \notin [\bar{n}]$ of adjacent in $G(n, p)$ vertices, there exists a copy $H_{uv}$ of $H'$ in $G(n, p)|_{[\bar{n}]}$ such that $(v, u, H_{uv})$ is $H$-completable in $G(n, p)$;
- for any $v \in [n] \setminus [\bar{n}]$, there exists a copy $H_v$ of $H'$ in $G(n, p)|_{[\bar{n}]}$ such that $(v, H_v)$ is $H$-completable in $G(n, p)$.

Let $b \in \mathbb{N}$. The number of common neighbors of $[b]$ in $G(n, p)$ has binomial distribution with parameters $n - b$ and $p^b$. By the Chernoff bound, this number is smaller than $\frac{1}{2}p^b(n - b)$ with probability at most $e^{-\Omega(n)}$. By the union bound, we get the following.

**Claim 3.** Let $\varepsilon > 0$, $\bar{n} \in (\varepsilon n, n]$ be a sequence of positive integers, $b \in \mathbb{N}$. Then whp, any subset of $[n] \setminus [\bar{n}]$ of size at most $b$ has at least $\frac{1}{2}p^b n$ common neighbors in $G(n, p)|_{[\bar{n}]}$.

Now, let us prove that there exist $c > 0$ and sets $K, S, D, Z_1, Z_2, Z_3, T, V^1, V^2, V^3$ such that

- the tuple $(G(n, p), c, K, S, D, Z_1, Z_2, Z_3, T)$ whp satisfies the first $H$-saturation property,
- the tuple $(G(n, p), V^1, V^2, V^3, S, D)$ whp satisfies the second $H$-saturation property.

Let us start with the first $H$-saturation property. We will at the same time define the parameters and prove that whp each condition (out of 7 from the definition of the first property) holds for these parameters.

1. Since $G_1 \overset{d}= G(n/2, p)$, whp, in $G_1$, there is a clique of size at least $c \ln n$ for some positive constant $c$ (see [9, Theorem 7.1]). Let $K$ be this clique of size $k \geq c \ln n$. So, Condition 1 of the first property holds whp.

Let

- $S$ be a set of $|S_0|$ vertices of $K$,
• $D$ be a set of $d$ vertices of $S$.

• $Z$ be the set of all common neighbours of $D$ from $V(G_2)$.

Notice that edges between $D$ and $V(G_2)$ do not depend on the choice of $K, S$ and $D$. Then, $|Z|$ has binomial distribution with parameters $n/2$ and $p^d$. Therefore, $\Pr[\frac{2}{3}p^d n < |Z| < \frac{4}{3}p^d n]$ (say, by Chebyshev’s inequality).

2. Let $R$ be an arbitrary set of $r$ vertices in $Z \setminus S$. Then $T$ is the set of all common neighbours of $D \cup R$ in $V(G_2)$.

Edges between $V(G_2)$ and $D \cup R$ do not depend on the choice of $D$ and $R$. Then, $|T|$ has binomial distribution with parameters $n/2$ and $p^{r+d}$. Therefore, $\Pr[\frac{2}{3}p^{r+d} n < |T| < \frac{4}{3}p^{r+d} n]$.

Notice that edges between $T$ and $Z \setminus T$ and edges inside $T$ are independent of the choice of $Z$ and $T$, so, conditioned on $Z$ and $T$, they have independent Bernoulli distributions. Then, by Claim 2, whp for every adjacent in $G(n,p)$ pair $v_1, v_2 \in Z \setminus T$ there exists a copy $H_{v_1,v_2} \cong H'$ inside $G(n,p)$, such that $(v_1, v_2, H_{v_1,v_2})$ is $H$-completable.

3. Let $Z = Z_1 \cup Z_2 \cup Z_3$ be an edge partition of $Z$ (the sizes of the parts differ by at most one). Then $\Pr[|Z_1| = \Omega(n)$ and edges inside $Z_1$ and between $Z_1$ and $[n] \setminus (K \cup Z)$ do not depend on the choice of $K, Z$ and $Z_1$. Notice that, if $v \notin K \cup Z$, there exists $H_v \cong H'$ in $G(n,p)|Z_v$, such that $(v, H_v)$ is $H$-completable. It implies that there is some copy $H \cong H|_{G(n,p)}$ such that $v_1$ is mapped onto $v$ and $H'$ is mapped onto $H_v$. Clearly, it implies the existence of $V_v \subset V(H_v \setminus \{v\})$ such that $(v, H_v, V_v)$ is $H$-completable.

4. Recall that, for a vertex $v \notin K \cup Z$, we denote by $U_v(H_v, V_v)$ the set of all neighbours $u$ of $v$ in $Z_2$ such that the tuple $(v, u, H_v, V_v)$ is $H$-completable.

Set $\hat{U}_v := U_v(H_v, V_v)$. Fix $v_1, v_2 \notin K \cup Z$. Notice that if $u \in Z_2$ is a common neighbour of $V(H_{v_1}) \cup V(H_{v_2}) \cup \{v_1, v_2\}$ then $u$ lies in $\hat{U}_{v_1} \cap \hat{U}_{v_2}$. Notice that $V(H_{v_i}) \cup V(H_{v_j}) \subset Z_1$ and $|V(H_{v_i}) \cup V(H_{v_j})| \leq 2r$. Edges between vertices of $[n] \setminus (K \cup Z)$ and vertices of $Z_2$ are independent of the choice of these sets. Edges between $Z_1$ and $Z_2$ are independent of the choice of $Z_1$ and $Z_2$ as well, and, moreover, $Z_2 = \Omega(n)$. So, applying Claim 3, we get that $\Pr[|\hat{U}_{v_1} \cap \hat{U}_{v_2}| = \Omega(n)$. By Claim 1, whp there is a copy of $H'$ in every $\hat{U}_{v_1} \cap \hat{U}_{v_2}$.

5. Fix $v \notin K \cup Z$, $u \in (K \setminus S) \cup Z_2 \cup Z_3$. Notice that edges between $([n] \setminus (K \cup Z)) \cup (K \setminus S) \cup Z_2 \cup Z_3 = [n] \setminus S \setminus Z_1$ and $Z_2$ are independent of the choice of $S, K, Z, Z_1, Z_2, Z_3$ and so are edges between $Z_1$ and $Z_2$. By the Chernoff bound, with probability $1 - e^{-\Omega(n)}$, vertices from $V(H_v) \cup \{u, v\}$ have $\Omega(n)$ common neighbours in $Z_2$. Since there are at most $n^2$ choices of $v \notin K \cup Z$, $u \in (K \setminus S) \cup Z_2 \cup Z_3$, by the union bound, we get that whp there are $\Omega(n)$ common neighbours in $Z_2$ for each element of $\{V(H_v) \cup \{u, v\} \mid v \notin K \cup Z, u \in (K \setminus S) \cup Z_2 \cup Z_3\}$. Notice that, if $u' \in Z_2$ is common neighbour of $V(H_v) \cup \{u, v\}$, then $u' \in \hat{U}_v \cap N_{G(n,p)}(u)$. So, whp, for every $v \notin K \cup Z$, $u \in (K \setminus S) \cup Z_2 \cup Z_3$, $|\hat{U}_v \cap N_{G(n,p)}(u)| = \Omega(n)$. By Claim 1, whp there is a copy of $H'$ in every $\hat{U}_v \cap N_{G(n,p)}(u)$.

6. Notice that edges between $([n] \setminus (K \setminus Z)) \cup Z_1$ and $Z_3$ do not depend on the choice of these sets, and whp $|Z_3| = \Omega(n)$, therefore whp any $v \notin K \cup Z, u \in Z_1$ have $\Omega(n)$ common neighbours in $Z_3$ by Claim 3. So, by Claim 1, whp there is a copy of $H'$ in $Z_3 \cap N_{G(n,p)}(u) \cap N_{G(n,p)}(v)$ for every $u \in Z_1, v \notin K \cup Z$.

7. Notice that, if $u \notin S$, then edges between $u$ and $Z$ do not depend on the choice of $Z$ and $u$. If $u \notin S$ then, by the Chernoff bound, with probability $1 - e^{-\Omega(n)}$, $u$ has $\Omega(n)$ neighbors in $Z$. Since there are at most $n$ choices of $u \notin S$, by the union bound, we get that whp, every vertex $u \notin S$ has at least $\Omega(n)$ neighbors in $Z$. So, we can apply Claim 1 and get that whp there is a copy of $H'$ in $N_{G(n,p)}(u) \cap Z$ for every $u \notin S$.

Now let us prove the second property. Let $V(G_2) = V^1 \cup V^2 \cup V^3$ be an equal partition.
1. Let us find a copy of $H'$ such that $N_H(w_1) \setminus \{w_2\}$ is mapped onto $D$ and other vertices of this copy lie in $V^1$. Notice that edges between $D$ and $V^1$ and inside $V^1$ have independent Bernoulli distributions. Recall that $D$ induces a clique in $G(n, p)$ and $|V^1| = \Omega(n)$. So, Claim 3 implies the existence (whp) of $\Omega(n)$ common neighbors of $D$ in $V^1$. By Claim 4 there exists an $r$-clique in the set of all common neighbors of $D$. This immediately implies the existence of the desired $H_D$.

Notice that all common neighbours of $V(H_D) \cup \{v\}$ in $V^2$ lie in $\hat{U}_v$ for every $v \in S \setminus D$ as $D \subset N_{G(n, p)}(v)$. Edges between $D \cup V^1 \cup (V(G_1) \setminus S) \cup V^3$ and $V^2$ do not depend on the choice of these sets. So, by Claim 3 whp, for every $u \in V(G_1) \setminus S \cup V^3$, there are $\Omega(n)$ common neighbours of $V(H_D) \cup \{v, u\}$ in $V^2$ (and so $|\hat{U}_v \cap N_{G(n, p)}(u)| = \Omega(n)$). By Claim 4 whp, for any $v \in S \setminus D$, $u \in (V(G_1) \setminus S) \cup V^3$, there is a copy of $H'$ in $\hat{U}_v \cap N_{G(n, p)}(u)$.

2. Edges between $(S \setminus D) \cup V^1 \cup V^2$ and $V^3$ do not depend on the choice of these sets, so, by Claim 3 for any $v \in (S \setminus D)$, $u \in V^1 \cup V^2$, whp there are $\Omega(n)$ common neighbours of $\{u, v\}$ in $V^3$. By Claim 4 whp, for any $v \in (S \setminus D)$, $u \in V^1 \cup V^2$, there is a copy of $H'$ inside $V^3 \cap N_{G(n, p)}(u) \cap N_{G(n, p)}(v)$.

### 3 Proof of Theorem 2

Recall that $t \geq 3$ is assumed.

Let us first notice that, if $G = G_1 \cup \ldots \cup G_m$ consists of $m$ connected components $G_1, \ldots, G_m$, then

$$\text{wsat}(G, K_{1,t}) = \sum_{i=1}^{m} \text{wsat}(G_i, K_{1,t}).$$

Therefore, $\text{wsat}(G, K_{1,t})$ is at least the number of non-empty components in $G$.

Let $\frac{1}{n} \ll p \leq \frac{\ln n}{n}$ Consider $w_n > 0$ such that

- $w_n \to \infty$ as $n \to \infty$,
- $w_n \leq \frac{\ln n}{n}$ for $n$ large enough,
- $\frac{w_n}{n} \leq p$ for $n$ large enough.

Then, for $n$ large enough, $\frac{w_n}{n} \leq pm \leq \frac{\ln n}{2}$. Let $X$ be the number of isolated edges in $G(n, p)$. Since

$$\mathbb{E}X = \binom{n}{2} p(1 - p)^{2(n-2)} \sim \frac{1}{2} \exp \left[ \ln n + \ln (np) - 2pm \right] \geq \frac{w_n}{2} (1 + o(1))$$

and

$$\text{Var}X = \mathbb{E}X + \binom{n}{2} \left( \frac{n-2}{2} \right) p^2 (1 - p)^{4 + 4(n-4)} = (\mathbb{E}X)^2 = \mathbb{E}X + O \left( \frac{(\mathbb{E}X)^2 \ln n}{n} \right),$$

by Chebyshev’s inequality, we get that

$$\mathbb{P} \left( X < \frac{w_n}{3} \right) \leq \frac{\text{Var}X}{(\mathbb{E}X - w_n/3)^2} \to 0, \quad n \to \infty.$$ 

Therefore, whp

$$\text{wsat}(G(n, p), K_{1,t}) \geq w_n > \frac{t}{2} = \text{wsat}(K_n, K_{1,t}).$$

Now, let $p > \frac{\ln n}{n}$. For such $p$, for any $\varepsilon > 0$, whp there exists a connected component $G_n$ in $G(n, p)$ of size at least $(1 - \varepsilon)n$ \cite{Theorem 5.4}. Clearly, $\text{wsat}(G(n, p), K_{1,t}) \geq \text{wsat}(G_n, K_{1,t})$.

Below, for a connected graph $G$, we give a necessary and sufficient condition, in terms of the existence of a subgraph from a certain class, for the stability property $\text{wsat}(G, K_{1,t}) = \binom{t}{2}$. 


Let \( t \leq y < x \leq n \) be integers and \( G \) be a graph on \([n]\). Let \( F' \subset F \subset G \), \( V(F) = \{v_1, \ldots, v_x\} \), \( V(F') = \{v_1, \ldots, v_y\} \). Let us call \( F \) a saturating structure of length \( x \) in \( G \) with the core \( F' \), if every \( v_i, y + 1 \leq i \leq x \), sends exactly \( t - 1 \) edges to the previous vertices \( v_1, \ldots, v_{i-1} \) in \( F \), i.e. \( |NF(v_i) \cap \{v_1, \ldots, v_{i-1}\}| = t - 1 \). We call \( y \) the size of the core. The vector \( \mathbf{v} = (v_1, \ldots, v_y) \) is called a saturating ordering of \( F \).

**Claim 4.** Let \( G \) be connected.

1. If \( G \) contains a saturating structure of length \( n \) with a core \( F' \cong K_t \), then \( \text{wsat}(G, K_{1,t}) = \left(\frac{t}{2}\right) \).

2. If \( \text{wsat}(G, K_{1,t}) = \left(\frac{t}{2}\right) \) and \( G \) has at least \( \mu \) vertices with degrees at least \( t - 1 \), then \( G \) contains a saturating structure of length \( \mu \) with a core of size at most \( \left(\frac{t+1}{2}\right) \).

**Proof.** First, assume that \( G \) contains a saturating structure \( F \) of length \( n \) with a core \( F' \cong K_t \). It is clear that \( F \) is both weakly \((G, K_{1,t})\)-saturated and weakly \((K_n, K_{1,t})\)-saturated. In particular, it implies that \( G \) is weakly \((K_n, K_{1,t})\)-saturated and, therefore, \( \text{wsat}(G, K_{1,t}) \geq \text{wsat}(K_n, K_{1,t}) \). Since \( F \) is weakly \((G, K_{1,t})\)-saturated, it remains to prove that \( \text{wsat}(F, K_{1,t}) \leq \left(\frac{t}{2}\right) \). Let \( (v_1, \ldots, v_n) \) be a saturating ordering of \( F \). Then \( F' = F\{(v_1, \ldots, v_n) \cong K_t \) has exactly \( \left(\frac{t}{2}\right) \) edges. Let us show that \( F' \) is weakly \((F, K_{1,t})\)-saturated. Since each vertex of \( v_1, \ldots, v_n \) has degree \( t - 1 \) in \( F' \), we can restore all edges in \( F' \) adjacent to one of these vertices. In particular, we restore all edges going from \( v_{t+1} \) to \( v_1, \ldots, v_t \). Proceeding in this way by induction, we restore all edges of \( F \).

Now, let \( \text{wsat}(G, K_{1,t}) = \left(\frac{t}{2}\right) \) and \( G \) has at least \( \mu \) vertices with degrees at least \( t - 1 \). Let \( F' \) be a weakly \((G, K_{1,t})\)-saturated graph with \( \left(\frac{t}{2}\right) \) edges and \( y \) non-isolated vertices. Let us order these vertices \( F' \) in a way \( v_1, \ldots, v_y \) such that \( v_i \) plays the role of the \( i \)-th central vertex of \( K_{1,t} \) in a \( K_{1,t} \)-bootstrap percolation process that starts on \( F' \) and finishes on \( G \). Clearly, for every \( i \in \{t, \ldots, t+1\} \), the vertex \( v_i \) sends at least \( t - i \) edges to \( F'\{(v_1, \ldots, v_y) \). Since the total number of these edges is \( \left(\frac{t}{2}\right) \), \( F' \) cannot contain any other edge. The bound \( y \leq t + \left(\frac{t}{2}\right) = \left(\frac{t+1}{2}\right) \) follows.

Consider a \( K_{1,t} \)-bootstrap percolation process that starts on \( F' \) and finishes on \( G \). Let \( e_1, \ldots, e_m \) be edges appearing in this process sequentially that contain at least one vertex outside \( \{v_1, \ldots, v_y\} \). Let \( w_1, \ldots, w_{\mu-y} \) be vertices of \( G \) outside \( \{v_1, \ldots, v_y\} \) with degrees at least \( t - 1 \) ordered in the following way:

- Let \( i \in [m] \) be such that \( e_i \) contains \( v_1 \) and a vertex from \( \{v_1, \ldots, v_y\} \), there are exactly \( t - 2 \) edges among \( e_1, \ldots, e_{i-1} \) that contain \( v_1 \), and all of them have the second end in \( \{v_1, \ldots, v_y\} \).
- For \( j \in \{2, \ldots, \mu-y\} \), let \( i_j \in [m] \) be such that \( e_{i_j} \) contains \( v_1 \) and a vertex from \( \{v_1, \ldots, v_y, w_1, \ldots, w_j\} \), there are exactly \( t-2 \) edges among \( e_1, \ldots, e_{i_{j-1}} \) that contain \( w_j \), and all of them have the second end in \( \{v_1, \ldots, v_y, w_1, \ldots, w_{j-1}\} \).

Such an ordering exists due to the definition of the \( K_{1,t} \)-bootstrap percolation process. Then, the desired saturating structure of length \( \mu \) is obtained from \( F'\{(v_1, \ldots, v_y) \) by adding \( w_i \), \( i \in \{\mu - y \} \), with the \( t - 1 \) edges going to the previous vertices \( v_1, \ldots, v_y, w_1, \ldots, w_{\mu-1} \). \( \square \)

Now, due to Claim 4 and the fact that whp \( G(n, p) \) has at least \( n/2 \) vertices with degrees at least \( t - 1 \) (this can be proved by a straightforward application of Chebyshev’s inequality to the number of vertices with such degrees), Theorem 2 immediately follows from

**Claim 5.**

1. There exists \( c > 0 \) such that, if \( p < c p(n, t) \), then whp there is no saturating structure of length \( \lfloor n/2 \rfloor \) and with a core of size at most \( \left(\frac{t+1}{2}\right) \) in \( G(n, p) \).

2. There exists \( C > 0 \) such that, if \( p > Cp(n, t) \), then whp there exists a saturating structure of length \( n \) with a core isomorphic to \( K_t \) in \( G(n,p) \).

### 3.1 Proof of Claim 5

Let \( x = \lfloor n/2 \rfloor \), \( y = \left(\frac{t+1}{2}\right) \), \( c < e^{-(c+1)/(t-1)} \). Let \( p < c p(n, t) \).

Let \( X \) be the number of subgraphs \( F \) in \( G(n, p) \) on \( x \) vertices such that there exist \( v_1, \ldots, v_i \in V(F) \), \( i \in \{t-1, t, \ldots, y\} \) satisfying the following property:

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for every \( v \in V(F) \setminus \{v_1, \ldots, v_t\} \), there exists a set \( N_v \) of its \( t-1 \) neighbors in \( F \) such that, for every \( u \in N_v, \ v \not\in N_u \).

Clearly, an ordered saturating structure of size \( x \) with a core of size at most \( y \) is a subgraph with the above property. Therefore, it is sufficient to prove that \( P(X \geq 1) \to 0 \) as \( n \to \infty \).

Let us bound from above \( \mathbb{E} X \):

\[
\mathbb{E} X \leq \sum_{i=t-1}^{y} \binom{n}{x} \binom{x}{i} \binom{x}{t-1}^{-x-i} p^{x-y(t-1)} \leq \frac{n^n x^y}{x!} \sum_{i=t-1}^{y} \left( \frac{n}{t-1} \right)^{x-i} \]

\[
\leq \frac{yn^n x^y}{x!} (xp)^{(t-1)(x-y)} = e^x \ln n - x \ln x + (x-y)(t-1) \ln(xp) + O(\ln \ln n).
\]

Since

\[
\ln p < \ln(cp(n, t)) = -\frac{1}{t-1} \ln n - \frac{t-2}{t-1} \ln \ln n - \ln \frac{1}{c},
\]

we get that

\[
\mathbb{E} X \leq \exp \left[ y \ln n - y \ln \ln n + x - (x-y)(t-1) \ln \frac{1}{c} + O(\ln \ln n) \right] =
\]

\[
\exp \left[ \ln n \left( y + 1 - (t-1) \ln \frac{1}{c} \right) + O(\ln \ln n) \right] \to 0, \quad n \to \infty.
\]

Markov’s inequality implies \( P(X \geq 1) \to 0 \).

### 3.2 Proof of Claim 5.2

Set \( p = C p(n, t) \) where \( C \) is a large positive constant (for example, any \( C \) bigger than \( 2^{\frac{2t-1}{t(t-1)}} \) is sufficient). Since ‘containing a saturating structure of length \( n \) with a core isomorphic to \( K_t \)’ is an increasing property, it is sufficient to prove that it holds whp for this value of \( p \).

The structure of the proof is the following: at first we prove the existence of a saturating structure of size \( x = \lfloor \ln n \rfloor \) whp, then we extend this structure to size \( y = \lfloor (\ln n)^{\frac{1}{t-1}} \rfloor \) and, finally, we extended it to the desired size \( n \).

#### 3.2.1 Saturating structure of size \( x = \lfloor \ln n \rfloor \)

Let \( X \) be the number of saturating structures \( F \) of size \( x \) in \( G(n, p) \) with a core isomorphic to \( K_t \) and a saturating ordering \( (v_1, \ldots, v_x) \) such that \( v_1 < v_2 < \ldots < v_x \). Let us call such an ordering a canonical saturating ordering. Let \( S \) be the set of all such structures in \( K_n \). Then \( X = \sum_{A \in S} I_A \) where \( I_A \) indicates that \( A \) belongs to \( G(n, p) \). We have

\[
\mathbb{E} X \sim \frac{n^n x^y}{x!} p^{x-y(t-1)} \prod_{i=t-1}^{x} \binom{\frac{n}{t-1}}{x-i} p^{x-i} \tag{4}
\]

Notice that

\[
\prod_{i=t-1}^{x} \binom{\frac{n}{t-1}}{x-i} = \frac{1}{[(t-1)]^{x-t}} x! \cdots (x-1)! \]

\[
\sim \frac{1}{[(t-1)]^{x-t}} \frac{(x-1)! \cdots (x-t+1)!}{(x-t)! \cdots (x-t+1)!} \geq \frac{[(x-t+1)]^{x-t-1}}{[(t-1)]^{x-t}} \sim \frac{2\pi x}{e^{(x-t+1)(t-1)(x-t+1)}(t-1)^{x-t}} > (x-t+1)^{(x-t+1)(t-1)}.
\]

So,

\[
\mathbb{E} X \geq (1 + o(1)) \frac{n^n x^y}{\sqrt{2\pi x(x/e)^x}} (x-t+1)^{(x-t+1)(t-1)} p^{x-y(t-1)} + O(\ln \ln n).
\]
\[
\exp \left\{ \frac{t}{2} \ln n + x \left[ 1 + (t-2) \ln x + (t-1) \ln C - (t-2) \ln \ln n \right] + o(x) \right\} = \\
\exp \left\{ \ln n \left[ \frac{t}{2} + (t-1) \ln C \right] + o(\ln n) \right\}.
\]

As \( C > e^{\frac{1}{t-2}} \), we get \( \mathbf{E}X \to \infty \).

Now let us estimate \( \text{Var}\,X \). Since, for disjoint \( A, B \in S \), \( I_A, I_B \) are independent, we get

\[
\text{Var}\,X = \mathbf{E}X^2 - (\mathbf{E}X)^2 = \\
\mathbf{E} \left( \sum_{A,B \in S, V(A) \cap V(B) \neq \emptyset} I_AI_B \right) + \mathbf{E} \left( \sum_{A,B \in S, V(A) \cap V(B) = \emptyset} I_AI_B \right) - (\mathbf{E}X)^2 \leq \sum_{A,B \in S} E_{I_AI_B}.
\]

Let \( A, B \in S \) have a non-empty intersection \( W = V(A) \cap V(B) \) and let \( w_1, \ldots, w_d \) be the vertices of \( W \).

Since the number of edges in \( W \) is maximum when each \( w_i \) sends all \( \min\{t-1, i-1\} \) edges to the previous \( w_1, \ldots, w_{i-1} \), we get that \( A \) and \( B \) have at most

\[
M := \binom{t-1}{2} + (d-t+1)(t-1) I(d \geq t) + \binom{d}{2}(d-t).
\]

common edges. Notice that \( M = (d-t)(t-1) \) when \( d \geq t \). Denote by \( \text{cnt}(d,m_1,m_2) \) the number of pairs \( A,B \in S \) such that \( |V(A) \cap V(B)| = d \), \( A \cap B \) has \( m_1 \) edges inside the core of \( B \) and \( m_2 \) edges outside the core of \( B \). Then

\[
\sum_{A,B \in S, A \cap B \neq \emptyset} \mathbf{E}I_AI_B = \sum_d \sum_{m_1,m_2} \text{cnt}(d,m_1,m_2) p^{2z-m_1-m_2},
\]

where

\[
z = \binom{t-1}{2} + (x-t+1)(t-1) = (x-t/2)(t-1).
\]

Notice that

\[
\text{cnt}(d,m_1,m_2) \leq \frac{EX}{p^2} \frac{x^{n-d}}{(x-d)!} \max_{j_1,\ldots,j_x \in J(d,m_2)} \prod_{i=t+1}^{x} \binom{i-1}{t-1-\floor{j_i}} \binom{t-1}{j_i},
\]

where \( J(d,m_2) \) is the set of all tuples \((j_1,\ldots,j_x) \in \{0,1,\ldots,t-1\}^{x-t} \) such that \( j_1 + \ldots + j_x = m_2 \) and the number of non-zero \( j_i \) is at most \( d \). Clearly, for a \((j_{t+1},\ldots,j_x) \in J(d,m_2) \), we have \( \prod_{i=t+1}^{x} \binom{t-1}{j_i} \leq 2^{(t-1)d} \). Moreover,

\[
\prod_{i=t+1}^{x} \binom{i-1}{t-1-j_i} = \prod_{i=t+1}^{x} \frac{(t-1)! (i-t)!}{(t-1-j_i)! (i-t+j_i)!} = \prod_{i=t+1}^{x} \frac{(i-1)^{j_i}}{(i-t+1)^{j_i}},
\]

The function \( g(j_{t+1},\ldots,j_x) = \prod_{i=t+1}^{x} \frac{(i-1)^{j_i}}{(i-t+1)^{j_i}} \) defined on the intersection of \( \{0,1,\ldots,t-1\}^{x-t} \) with the hyperplane \( j_{t+1} + \ldots + j_x = m_2 \) achieves its maximum when

\[
\begin{cases}
    t-1, & t+1 \leq i \leq t + \left\lfloor \frac{m_2}{t-1} \right\rfloor, \\
m_2 \mod t-1, & i = t + \left\lfloor \frac{m_2}{t-1} \right\rfloor + 1, \\
0, & i > t + \left\lfloor \frac{m_2}{t-1} \right\rfloor + 1
\end{cases}
\]

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\[ \text{we get that} \quad O(\frac{1}{M}) \text{ and } d \geq 2 \]

\[ \text{Combining this with (4) and (7), we get} \]

\[
f(d, m_1, m_2) := \frac{\text{cnt}(d, m_1, m_2) p^{2z-m_1-m_2}}{(EX)^2} =
\]

\[
= O \left( \left( \frac{x}{d} \right)^d n^{-d} \frac{(t-1)^{m_2}}{(1/\lceil \frac{m_2}{t+1} \rceil)^{t-1} p^{-m_1-m_2}} \right) =
\]

\[
= O \left( \frac{x e^{2(t-1)}}{d} x^d n^{-d} p^{-m_1} \frac{(t-1)^2 e^{m_2}}{m_2 p} \right).
\]

Since \((\frac{(t-1)^2 e^{m_2}}{m_2 p})^{m_2}\) increases (as a function of \(m_2\)) on \((0, M]\) (the maximum is achieved at \(\frac{(t-1)^2}{p} \gg M\)), we get that

\[
p^{-m_1} \left( \frac{(t-1)^2 e^{m_2}}{m_2 p} \right)^{m_2} \leq p^{-m_1} \left( \frac{(t-1)^2 e^{m_2}}{M-m_1 p} \right)^{M-m_1} = p^{-M} \left( \frac{(t-1)^2 e^{m_2}}{M-m_1 p} \right)^{M-m_1}.
\]

This expression achieves its maximum when \(m_1 = M - (t-1)^2\). Since \(m_1 \leq \frac{(t-1)^2}{2}\) by the definition and \(M\) may be either large or small depending on the value of \(d\), below, we distinguish several scenarios: \(d \geq 2t - 2, t \leq d < 2t - 2\) and \(d < t\).

1. If \(d \geq 2t - 2\), then \(M \geq (3t/2 - 2)(t-1) = (t-1)^2 + \binom{t-1}{2}\). Therefore, \(M - (t-1)^2 \geq \binom{t-1}{2}\). It means that the bound to the right in (8) increases with \(m_1\) and its maximum value is achieved at \(m_1 = \binom{t-1}{2}\). Then,

\[
f(d, m_1, m_2) = O \left( \frac{x^d e^{2(t-1)}}{d n} \left( \frac{(t-1)^2 e^{m_2}}{(d+t+1)} \right)^{(t-1)(d-t+1)} p^{-M} \right)
\]

\[
= O \left( e^{d [2 \ln x + 1 - \ln d - (t-1) (\ln 2 - \ln (d-t+1)) + \ln (t-1) + 1 + \ln p]} \times e^{(t/2)(t-1) \ln p + o(1)} \right).
\]

Notice that \(\ln(d-t+1) \geq \ln \frac{d}{t}\) (since \(d \geq t\)). So,

\[
f(d, m_1, m_2) = O \left( e^{d \gamma(d) + (t/2)(t-1) \ln p + o(1)} \right),
\]

where

\[
\gamma(d) = 2 \ln x + 1 - t \ln d + (t-2) \ln \ln n + (t-1)(\ln t - \ln C + \ln(t-1) + 1 + \ln 2).
\]

Notice that \([d \gamma(d)]^t = \gamma(d) - t\) and \(\gamma\) decreases. Therefore, \(d_0 = \gamma^{-1}(t)\) is a point of global maximum of \(d \gamma(d)\). Clearly,

\[
d_0 = \left( \frac{2(t-1)t}{C} \right)^{\frac{1}{t+1}} (x^2 [\ln n]^t)^{\frac{1}{t+1}} = \left( \frac{2(t-1)t}{C} \right)^{\frac{1}{t+1}} x (1 + o(1)).
\]

Therefore, \(d_0 \gamma(d_0) = \left( \frac{2(t-1) t}{C} \right)^{\frac{1}{t+1}} x t (1 + o(1))\). As \(C > 2^{\frac{1}{t+1}} t(t-1)\), we get that, for \(c > 0\) small enough, \(f(d, m_1, m_2) \leq n^{-c}\).
2. Let \( t \leq d < 2t - 2 \). Then \( \left( \frac{(t-1)^2 e}{M-m_1} \right)^{M-m_1} \leq e^{(t-1)^2} \). Therefore,

\[
f(d, m_1, m_2) = O \left( \left( \frac{x^2}{n} \right)^d p^{-M} \right) = O \left( \left( \frac{x^2}{np^{c-1}} \right)^d \right)
= O \left( \frac{\ln n}{(n \ln n)^{t-2} \ell^{t/2}} \right) = O \left( \frac{1}{n} \right).
\]

3. Finally, let us switch to the case \( d < t \). Since \( M - (t-1)^2 < 0 \), we get that \( \left( \frac{(t-1)^2 e}{M-m_1} \right)^{M-m_1} \) is maximal when \( m_1 = 0 \). Therefore,

\[
f(d, m_1, m_2) = O \left( \left( \frac{x^2}{n} \right)^d p^{-M} \right) = O \left( \frac{x^2}{np^{c-1}} \right)^d
= O \left( \frac{x^2}{n^{1-\frac{c-1}{\ell} - \frac{\ell-2}{\ell(d-1)}}} \right)^d = O \left( \frac{\ln n}{n} \right).
\]

Combining the above bounds with (5) and (6), we get, by Chebyshev's inequality, that, for \( n \) large enough,

\[
P(X = 0) \leq \frac{\text{Var} X}{(\text{EX})^2} \leq \sum_d \sum_{m_1} \sum_{m_2} f(d, m_1, m_2) \leq xz^2n^{-\varepsilon} = o(1).
\]

Therefore, whp there exists a saturating structure of size \( x \).

### 3.2.2 Saturating structure of size \( y = \left\lfloor \frac{(\ln n)^{\frac{1}{t-1}}} \right\rfloor \)

Let us divide the random graph into two parts:

\[ G_1 = G(n, p)_{\lfloor n/2 \rfloor}, \quad G_2 = G(n, p)_{\lfloor n\rfloor \setminus \lfloor n/2 \rfloor}. \]

Let \( F_0 \) be a saturating structure of size \( x = \lfloor \ln (\lfloor n/2 \rfloor) \rfloor \) in \( G_1 \) (if exists). Let \( A_0 \) be the event that \( F_0 \) exists.

Let us enlarge the saturating structure by induction.

Let \( U_1 \subset V(G_2) \) be a set of vertices connected to at least \( t-1 \) vertices of \( F_0 \), for \( i = 1, 2, \ldots \) let \( U_{i+1} \subset V(G_2) \setminus \{ U_1 \cup \cdots \cup U_i \} \) be a set of vertices connected to at least \( t-1 \) vertices of \( U_i \). If, for some \( \ell \), we get \( |V(F_0) \cup U_1 \cup \cdots \cup U_{\ell}| \geq y \), then we immediately get a saturating structure of size at least \( y \).

Let us prove that \( \ell = \left\lceil \frac{2}{\ell (t-1) \ln \ln n} \right\rceil \) is the desired value.

Set \( X_i = |U_i| \) for \( i \in [\ell] \).

Notice that edges between \( G_2 \) and \( F_0 \) do not depend on the choice of \( F_0 \) and have independent Bernoulli distributions. Therefore,

\[
E(X_1|A_0) = \left\lceil \frac{n}{2} \right\rceil P, \quad \text{Var}(X_1|A_0) = \left\lfloor \frac{n}{2} \right\rfloor P(1 - P),
\]

where

\[
P = \sum_{k=t-1}^{x} \frac{x^k}{k!} p^k (1-p)^{x-k}.
\]

We get

\[
E(X_1|A_0) \sim \frac{n}{2} \left( \frac{x}{t-1} \right) p^{t-1} (1-p)^{x-t+1} \sim \frac{n}{2} \frac{x^{t-1}}{(t-1)!} p^{t-1} = \frac{C^{t-1}}{2(t-1)!} x^{t-1}
\]

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As \( C > (2(t - 1))^{1/t} \), we get that \( \hat{C} := \frac{C_i - 1}{C_i} > 1 \). Also,

\[
\frac{\text{Var}(X_1|A_0)}{(E(X_1|A_0))^2} \sim \frac{1}{E(X_1|A_0)} \sim \frac{1}{C_i x}
\]

By Chebyshev’s inequality, for every \( \delta > 0 \),

\[
P(|X_1 - \hat{C}x| > \delta \hat{C}x|A_0) \leq (1 + o(1)) \frac{1}{\delta^2 \hat{C}x}
\]

and

\[
(1 - \delta) \hat{C} > 1
\]

Take \( \delta > 0 \) such that \( (1 - \delta) \hat{C} > 1 \). Let, for every \( i \in [\ell] \),

\[
C_i^- = \{(1 - \delta) \hat{C}^{(t-1)}\}^{-1}, \quad C_i^+ = \{(1 + \delta) \hat{C}^{(t-1)}\};
\]

\[
A_i = A_{i-1} \cap \{C_i^- x \leq X_i \leq C_i^+ x\}.
\]

Let us prove that

\[
P(C_i^- x \leq X_i \leq C_i^+ x|A_{i-1}) \geq 1 - (1 + o(1)) \frac{1}{\delta^2 C_i^- x C_i^+ x}
\]

uniformly over all \( i \in \{2, 3, \ldots, \ell\} \).

Notice that edges between \( V(G_2) \setminus \bigcup_{j=1}^t U_j \) and \( U_i \) still have independent Bernoulli distributions. Take an integer \( a \in \{C_i^- x, C_i^+ x\} \). Since, on \( A_i \), almost all vertices of \( V(G_2) \) can be included in \( U_{i+1} \) (i.e., \( |U_1 \cup \ldots \cup U_{i+1}| \leq \ln n(C_i^- + \ldots + C_i^+ ) = o(n) \)), we get

\[
E(X_{i+1}|A_i \cap \{X_i = a\}) \sim \frac{n}{2} \sum_{k=t-1}^{a} \binom{a}{k} p^k (1 - p)^{a-k} \sim \frac{n}{2} \frac{a^{t-1}}{(t-1)!} p^{t-1};
\]

\[
\frac{\text{Var}(X_{i+1}|A_i \cap \{X_i = a\})}{(E(X_{i+1}|A_i \cap \{X_i = a\}))^2} \sim \frac{1}{E(X_{i+1}|A_i \cap \{X_i = a\})}.
\]

So,

\[
E(X_{i+1}|A_i) \leq (1 + o(1)) \frac{n}{2} \frac{(C_i^+ x)^{t-1}}{(t-1)!} p^{t-1} = (1 + o(1)) \frac{C_i^+ x}{(1 + \delta)^{t-1} \hat{C}_i x}.
\]

Similarly,

\[
E(X_{i+1}|A_i) \geq (1 + o(1)) \frac{n}{2} \frac{(C_i^- x)^{t-1}}{(t-1)!} p^{t-1} = (1 + o(1)) C_i^- x \hat{C}_i x.
\]

Therefore, by Chebyshev’s inequality,

\[
P \left( \{X_{i+1} > C_i^+ x\} \cup \{X_{i+1} < C_i^- x\} \mid A_i \right) \leq \frac{\text{Var}(X_{i+1}|A_i)}{(\delta E(X_{i+1}|A_i))^2} \leq \frac{1 + o(1)}{\delta^2 C_i^- x C_i^+ x}.
\]

This finishes the proof of (10).

Notice that (9) and (10) imply

\[
P(\sim A_x) \leq o(1) + \sum_{i=2}^{\ell} P(C_i x \leq X_i \leq C_i^+ x|A_i-1) \leq o(1) + \frac{1}{\delta^2 C_i x} \sum_{i=2}^{\ell} \frac{1}{\delta^2 C_i x} \sum_{k=0}^{+\infty} \frac{1}{((1 - \delta) \hat{C})^k} \to 0, \text{ as } (1 - \delta) \hat{C} > 1.
\]
But, on $A_{\ell}$,

$$|V(F_{0})| + \sum_{i=1}^{\ell} |U_{i}| = x + \sum_{i=1}^{\ell} X_{i} \geq x + \sum_{i=1}^{\ell} C_{i}^{-1} x =$$

$$= x \left( 1 + \sum_{i=1}^{\ell} ((1 - \delta) \hat{C})^{(t-1)i-1} \right) \geq ((1 - \delta) \hat{C})^{(t-1)x} \geq$$

$$\geq ((1 - \delta) \hat{C})^{\frac{\ln n}{\ln 2}} x = \exp \left[ \frac{\ln((1 - \delta) \hat{C})}{t-1} (\ln \ln n)^{2} + \ln \ln n \right] \gg (\ln n)^{\frac{1}{2t-3}}.$$  

So, whp there exists a saturating structure of size $y$.

### 3.2.3 Saturating structure of size $n$

In this section, we prove that, for every proper $S \subset [n]$ of size at least $y$, in $G(n, p)$ there exists a vertex outside $S$ such that it is connected to at least $t - 1$ vertices of $S$. Clearly, this observation finishes the proof of Claim 5.2.

By the union bound, the probability that there exists a set $S \subset [n]$ of size $z \in [y, n/\ln n]$ such that every vertex outside $S$ has less than $t - 1$ neighbors in $S$ is at most

$$\sum_{z=y}^{[n/\ln n]} \binom{n}{z} \left( 1 - \left( \frac{z}{t-1} \right) p^{t-1} \right)^{n-z} \leq$$

$$\sum_{z=y}^{[n/\ln n]} \exp \left[ z(\ln n + 1 - \ln z) - n(1 + o(1)) \frac{z^{t-1}}{(t-1)!} p^{t-1} \right] = n \exp[-\Omega(z \ln n)] \to 0$$

since $\frac{C^{t-1}}{(t-1)!} > 1$.

Finally, the probability that there exists a proper $S \subset [n]$ of size $z > n/\ln n$ such that every vertex outside $S$ has less than $t - 1$ neighbors in $S$ is at most

$$\sum_{z=[n/\ln n]}^{n-1} \binom{n}{n-z} \left( 1 - \left( \frac{z}{t-1} \right) p^{t-1} \right)^{n-z} \leq$$

$$\sum_{z=[n/\ln n]}^{n-1} \exp \left[ (n-z) \ln n - \frac{z^{t-1}}{(t-1)!} p^{t-1} \right] = n \exp \left[ -\Omega \left( \frac{n^{t-2}}{\ln n ^{2t-3}} \right) \right] \to 0.$$  

Claim 5.2 and Theorem 2 follows.

### 4 Acknowledgements

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