Schwarzschild-type solution in an effective gravitational theory with local Galilean invariance

R. R. Cuzinatto\textsuperscript{a}, P. J. Pompeia\textsuperscript{a,b,c}, M. de Montigny\textsuperscript{a,d}, F.C. Khanna\textsuperscript{a,e}

\textsuperscript{a}Theoretical Physics Institute, University of Alberta, Edmonton, Alberta, Canada T6G 2J1
\textsuperscript{b}Instituto de Física Teórica, Universidade Estadual Paulista, Rua Pamplona 145, 01405-000, São Paulo, SP, Brazil
\textsuperscript{c}Comando-Geral de Tecnologia Aeroespacial, Instituto de Fomento e Coordenação Industrial, Praça Mal. Eduardo Gomes 50, 12228-901, São José dos Campos, SP, Brazil
\textsuperscript{d}Campus Saint-Jean, University of Alberta, Edmonton, Alberta, Canada T6C 4G9
\textsuperscript{e}TRIUMF, 4004, Westbrook Mall, Vancouver, British Columbia, Canada V6T 2A3

Abstract

We construct a Schwarzschild-type exact external solution for a theory of gravity admitting local Galilean invariance. In order to realize the Galilean invariance we need to adopt a five-dimensional manifold. The solution for the gravitational field equations obeys a Birkhoff-like theorem. Three classic tests of general relativity are analyzed in detail: the perihelion shift of the planet Mercury, the deflection of light by the Sun, and the gravitational redshift of atomic spectral lines. The Galilean version of these tests exhibits an additional parameter $b$ related to the fifth-coordinate. This constant $b$ can be estimated by a comparison with observational data. We observe that the Galilean theory is able to reproduce the results traditionally predicted by general relativity in the limit of negligible $b$. This shows that the tests are not specifically Lorentz invariant.

Keywords: Galilean covariance; General relativity; Schwarzschild solution
PACS: 11.30.-j; 11.30.Cp; 04.20.-q; 04.90.+e

\textsuperscript{1}rcuzin@phys.ualberta.ca, ppompeia@phys.ualberta.ca, montigny@phys.ualberta.ca, khanna@phys.ualberta.ca
1 Introduction

In recent papers, we have examined various aspects of Galilean covariance in a flat spacetime (for instance, see Ref. [1]), previously discussed in Refs. [2]–[4]. This formalism allows us to express physical theories in a tensor form by employing a 5-dimensional flat manifold characterized by a metric tensor (given in Eq. (2)). Therefore, as mentioned in these early papers, the reduction from this extended space can be done to the (3+1) Galilean spacetime or Minkowski spacetime, thereby providing a unified framework to treat both kinematics. Among our motivations for the project of extending the study on Galilean symmetry to gravitation is the following statement (p. 92 of Ref. [5]): “... general covariance does not imply Lorentz invariance - there are generally covariant theories of gravitation that allow the construction of inertial frames at any point in a gravitational field, but satisfy Galilean relativity rather than special relativity in these frames.” Our intent is to examine various consequences of replacing local Lorentz invariance with local Galilean invariance. Traditionally, an obstacle to the implementation of Galilean invariance in a four-dimensional manifold is the absence of a metric tensor satisfying this requirement locally [6], but as we will discuss, a metric structure exists if we extend the spacetime to (4 + 1) dimensions. As far as we know, a systematic study of gravitational theories which admit Galilean invariance has never been performed before.

Let us begin by briefly describing the Galilean-covariant formalism (more details can be found in Refs. [2]–[4]). The first step consists in constructing Lorentz-covariant action functionals defined on a flat Galilean spacetime, defined as a (4 + 1)-dimensional flat Minkowski manifold with light-cone coordinates, defined by Eq. (3). A Galilean five-vector, such as \( x, x^4, x^5 \), transforms under Galilean boosts as follows:

\[
\begin{align*}
x' &= x - v x^4, \\
x'^4 &= x^4, \\
x'^5 &= x^5 - v \cdot x + \frac{1}{2} v^2 x^4,
\end{align*}
\] (1)

where \( v \) is the relative velocity. The first two lines of Eq. (1) define the well-known pure Galilean transformations in (3 + 1) dimensions.

Note that a general Galilean transformation, in the (4 + 1) Galilean spacetime, has a form similar to the Poincaré transformation, \( x'^\mu = \Lambda^\mu_{\nu} x^\nu + a^\nu \). The tensor methods utilized in the Galilean-covariant formalism are based on the scalar product, \( A \cdot B = A \cdot B - A_4 B_5 - A_5 B_4 \), which is invariant under the transformations in Eq. (1). This suggests the following Galilean metric in order to describe the flat Galilean spacetime:

\[
\eta_{\mu\nu} = \begin{pmatrix}
1_{3 \times 3} & 0 & 0 \\
0 & 0 & -1 \\
0 & -1 & 0
\end{pmatrix}.
\] (2)
This defines light-cone coordinates which, through the change of coordinates,
\[
\bar{x} = x, \quad \bar{x}^4 = \frac{x^4 + x^5}{\sqrt{2}}, \quad \bar{x}^5 = \frac{x^4 - x^5}{\sqrt{2}},
\]
(3)
can convert the metric, Eq. (2), into its diagonal form:
\[
\bar{\eta}_{\mu\nu} = \text{diag}(1, 1, 1, -1, 1).
\]
(4)
Therefore, dimensional reductions from (4 + 1) theories can lead to either a Lorentz-covariant or a Galilean-covariant theory in (3 + 1) dimensions [3].

The projection from the extended manifold onto the (3 + 1)-dimensional spacetime can be defined as
\[
(x, t) \mapsto x^\mu = (x^1, \ldots, x^5) \equiv \left( x, ct, \frac{S}{c} \right),
\]
where \(x\) is the position vector, \(t\) is the time, \(c\) is a parameter with the dimensions of velocity, and \(S\) is the extra coordinate. The five-momentum
\[
p_\mu = i\partial_\mu = (i\nabla, ic, i\partial S) = \left( p, \frac{E}{c}, mc \right),
\]
(5)
where \(p\) is the three-momentum, \(E\) the energy and \(m\) the mass, suggests that the additional coordinate, \(S\), may be seen as being conjugate to the mass. Clearly, the theory in the (4 + 1)-dimensional manifold is Lorentz-invariant since the two metrics are equivalent. It is through the definition of the physical quantities that the reduction to a (3 + 1)-dimensional spacetime leads to a Galilean theory. From a group-theoretical point of view, this dimensional reduction is motivated by the fact that the Poincaré group in (4 + 1) dimensions contains, as a subgroup, the extended Galilei group of (3 + 1)-dimensional spacetime. Indeed, the Poincaré group of (4 + 1) spacetime is generated by 15 elements (10 Lorentz transformations, \(M_{\mu\nu}\), and 5 translations, \(P_\mu\), where \(\mu, \nu = 1, \ldots, 5\)). The 11 generators of the extended Galilei group consist of the 3 rotations \((M_{12}, M_{23}, M_{31})\), 3 Galilean boosts \((M_{41}, M_{42}, M_{43})\), 4 spacetime translations \((P_1, P_2, P_3, P_4)\), and 1 central charge, \(P_5\), which is the mass.

The Galilean symmetry is important because it is the kinematic symmetry which underlies non-relativistic, or low-energy, phenomena [7]. By simply taking low-velocity approximations of relativistic, Poincaré-invariant, expressions, we do not necessarily obtain theories with suitable symmetry properties. Moreover, such approximations may not provide a complete description, as we will illustrate hereafter. Also, a careful consideration of Galilean invariance proves that some concepts or properties usually thought to be specifically relativistic can in fact be shared with non-relativistic systems (e.g. the spin of particles [8]). Hereafter, we will reach a similar conclusion with the classic tests of general relativity (GR).

These points are illustrated clearly by the theory of Galilean electromagnetism [9, 10]. Firstly, the electric field \(E\) and magnetic field \(B\) transform,
under a Lorentz transformation, as follows:

\[ E' = \gamma(E + v \times B) + (1 - \gamma) \frac{v \cdot E}{v^2}, \]

\[ B' = \gamma(B - \frac{1}{c^2} v \times E) + (1 - \gamma) \frac{v \cdot B}{v^2}, \]

where the parameter \( v \) represents the relative velocity, and \( \gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}} \) is the Lorentz factor. An inconsistent approximation (see remarks in Ref. [9]) is given by the relations:

\[ E' = E + v \times B, \]  \hspace{1cm} (6)

\[ B' = B - \frac{1}{c^2} v \times E. \]  \hspace{1cm} (7)

For a recent and clear discussion, see Ref. [10]. Moreover, Le Bellac and Lévy-Leblond found that there exists not one but two consistent Galilean limits of electromagnetism; the “electric” and the “magnetic” limits [9].

It is noteworthy to mention that Galilean invariance and the equations above are not simply a theoretical figment, since they describe concrete physical systems. Indeed, the two Galilean limits of the Maxwell’s equations, given above, are the basis of electroquasistatics and magnetoquasistatics [11]. In turn, they are connected to electrohydrodynamics and magnetohydrodynamics, which have many applications in geophysics, astrophysics, physics of fluids and plasmas, engineering, etc.

The purpose of this paper is to construct an effective theory of gravity by performing a slight modification to the principle of general covariance that follows from the equivalence principle. Quoting Weinberg once again (p. 91 of Ref. [5]): “It [the Principle of General Covariance] states that a physical equation holds in a gravitational field, if two conditions are met: 1. The equation holds in the absence of gravitation; that is it agrees with the laws of special relativity when the metric tensor \( g_{\alpha\beta} \) equals the Minkowski tensor \( \eta_{\alpha\beta} \), and when the affine connection \( \Gamma^{\alpha}_{\beta\gamma} \) vanishes; 2. The equation is generally covariant; that is, it preserves its form under a general coordinate transformation \( x \rightarrow x' \).” Henceforth, we propose to replace the condition 1 by, “1’. The equation holds in the absence of gravitation; that is, it agrees with the laws of Galilean relativity when the metric tensor \( g_{\alpha\beta} \) equals the Galilean tensor \( \eta_{\alpha\beta} \), Eq. (2), and when the affine connection \( \Gamma^{\alpha}_{\beta\gamma} \) vanishes.” The condition 2 is kept without change. This new version of the principle of general covariance, with conditions 1’ and 2, gives rise to a general (globally) covariant theory of gravity that is locally Galilean invariant. It is in this sense that the expression “Galilean covariant” should be understood henceforth. We shall refer to this new theory of gravity as the Galilean gravity.

We call the Galilean gravity an “effective theory” in the sense that we expect it to be valid in the same range of distances (in the tests described in Section 3) as the post-Newtonian approximation (see Chap. 9 of Ref. [3]). Nevertheless, we will obtain an exact solution using local Galilean invariance (see Section 2) rather than the expansion typically encountered with the post-Newtonian
approximation. It should be emphasized that, although we work in a \((4 + 1)\)-dimensional manifold, our approach and purpose are entirely different from other 5-dimensional theories (e.g. Kaluza-Klein models, de Sitter universe, or the ‘bulk’ utilized in string theory).

In 1916, Schwarzschild published a solution of Einstein equations for the static spherically symmetric gravitational field \([12]-[14]\). In Section 2, we discuss an analogous solution in the Galilean context, called ‘Galilei-Schwarzschild solution’. This solution is exact, unlike other effective theories that ensue from approximations or expansions. Our solution contains the same event horizon as the familiar Schwarzschild metric. We also find the Galilean analogue of the Birkhoff theorem. In Section 3 predictions of this solution are confronted with three classic tests of GR. In Section 3.1 we find that the Galilei-Schwarzschild solution describes the perihelion shift of Mercury as does its counterpart of GR. In Section 3.2 we observe that it predicts the deflection of light rays by massive bodies like our Sun. In Section 3.3 we obtain a similar result for the gravitational shift of atomic spectral lines. The results of these tests depend on an additional parameter \(b\), which is related to the fifth-coordinate. This constant \(b\) can be calculated by comparing with observational data. We find that the Galilean theory can reproduce the results predicted by GR in the limit \(b \approx 0\). Thus the tests are not specifically Lorentz invariant. Section 4 contains concluding remarks.

2 Galilei-Schwarzschild solution

The principle of equivalence states that in the presence of a gravitational field, we can choose, at every spacetime point, a locally inertial coordinate system where the laws of nature take the same form as in a non-accelerated Cartesian coordinate system in the absence of gravitation. As mentioned in p. 91 of Ref. [5], this can be restated as the Principle of General Covariance, which requires the equations of Physics to preserve their form under general coordinate transformations, meaning that these laws are properly written in a tensorial form.

As we build the 5D Galilean theory for gravity upon the same requirement of general covariance demanded by GR, it is only natural that the field equations for \(g_{\mu \nu}\) in the context of our model are formally the same as in the usual GR. The manifold is equipped with a metric tensor \(g_{\mu \nu}\) which satisfies

\[
G_{\mu \nu} \equiv R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = \kappa T_{\mu \nu},
\]

where the Ricci tensor components,

\[
R_{\mu \rho} \equiv \nabla_{\mu} \nabla_{\rho} - \nabla_{\rho} \nabla_{\mu} + \Gamma_{\rho \nu \tau}^{\rho} \Gamma_{\mu \nu \tau}^{\nu} - \Gamma_{\mu \nu \tau}^{\nu} \Gamma_{\rho \nu \tau}^{\nu},
\]

are given in terms of the Christoffel symbols,

\[
\Gamma_{\mu \nu}^{\rho} = \frac{1}{2} g^{\rho \sigma} \left( \partial_{\mu} g_{\nu \sigma} + \partial_{\nu} g_{\mu \sigma} - \partial_{\sigma} g_{\mu \nu} \right).
\]
2.1 Galilean metric in curved spacetime

In GR, in order to derive the Schwarzschild solution, we begin with a 4-dimensional metric tensor of the form

\[ (g_{\mu\nu}) = \begin{pmatrix}
    e^{\lambda(r,t)} & 0 & 0 & 0 \\
    0 & r^2 & 0 & 0 \\
    0 & 0 & r^2 \sin^2 \theta & 0 \\
    0 & 0 & 0 & -e^{\nu(r,t)}
\end{pmatrix}. \] (11)

It exhibits spherical symmetry in the spatial sector. In the relativistic theory, the functions $\lambda(r,t)$ and $\nu(r,t)$ are calculated by (i) substituting this ansatz into Eqs. (8); (ii) considering $T_{\mu\nu} = 0$ in Eq. (8), corresponding to the solution outside the massive source of the field; and (iii) imposing the boundary condition $\lim_{r \to \infty} \lambda = \lim_{r \to \infty} \nu = 0$, in order to recover the Minkowski space as a limiting case, since far from the source the spacetime should be flat. The Birkhoff theorem assures that $\lambda = \lambda(r)$ and $\nu = \nu(r)$ (see, e.g. Ref. [14]).

Just as Eq. (3) converts the flat Galilean metric of Eq. (2) to the Minkowski metric given in Eq. (4), the Galilean version of the Schwarzschild solution should be related by a similar change of coordinates to Eq. (11), with one trivial additional component,

\[ (\bar{g}_{\mu\nu}) = \begin{pmatrix}
    e^{\lambda} & 0 & 0 & 0 & 0 \\
    0 & \bar{r}^2 & 0 & 0 & 0 \\
    0 & 0 & \bar{r}^2 \sin^2 \bar{\theta} & 0 & 0 \\
    0 & 0 & 0 & -e^{\nu} & 0 \\
    0 & 0 & 0 & 0 & 1
\end{pmatrix}. \] (12)

(A more general ansatz exhibiting spherical symmetry would be

\[ (\bar{g}_{\mu\nu}) = \begin{pmatrix}
    e^{\lambda} & 0 & 0 & 0 & 0 \\
    0 & \bar{r}^2 & 0 & 0 & 0 \\
    0 & 0 & \bar{r}^2 \sin^2 \bar{\theta} & 0 & 0 \\
    0 & 0 & 0 & -e^{-\nu} & 0 \\
    0 & 0 & 0 & 0 & e^{\mu}
\end{pmatrix}. \] (13)

This reduces to Eq. (12) when we take $\mu = 0$. This choice is not only adequate because of its simplicity but also due to the fact that it leads to a Schwarzschild-type of metric. Other solutions could be obtained with a non-trivial $\mu$. The metric in Eq. (12) leads to a Minkowski-like metric in $(4 + 1)$ dimensions under the constraint $\lim_{r \to \infty} \lambda = \lim_{r \to \infty} \nu = 0$. The relativistic limit of a Galilean theory is obtained from a light-cone-type transformation in spherical coordinates,

\[ r = \bar{r}, \; \theta = \bar{\theta}, \; \varphi = \bar{\varphi}, \; x^4 = \frac{\bar{x}^4 + \bar{x}^5}{\sqrt{2}}, \; x^5 = \frac{\bar{x}^4 - \bar{x}^5}{\sqrt{2}}. \] (14)

Therefore, in order to retrieve the general Galilean metric, we must transform Eq. (12) to the unprimed reference frame through the usual transformation rule
for a tensor of rank two,

$$g_{\mu\nu} = \frac{\partial \bar{x}^\rho}{\partial x^\mu} \frac{\partial \bar{x}^\sigma}{\partial x^\nu} \bar{g}^\rho\sigma$$  \hspace{1cm} (15)$$

and the metric in Eq. (12) becomes

$$g_{\mu\nu} = \begin{pmatrix}
e^\lambda & 0 & 0 & 0 & 0 \\
0 & r^2 & 0 & 0 & 0 \\
0 & 0 & r^2 \sin^2 \theta & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2}(-e^\nu + 1) & \frac{1}{2}(-e^\nu - 1) \\
0 & 0 & 0 & \frac{1}{2}(-e^\nu - 1) & \frac{1}{2}(-e^\nu + 1)
\end{pmatrix}.  \hspace{1cm} (16)$$

Henceforth we slightly generalize this metric as follows:

$$g_{\mu\nu} = \begin{pmatrix}e^{\lambda(r,x^4,x^5)} & 0 & 0 & 0 & 0 \\
0 & r^2 & 0 & 0 & 0 \\
0 & 0 & r^2 \sin^2 \theta & 0 & 0 \\
0 & 0 & 0 & f(r,x^4,x^5) & g(r,x^4,x^5) \\
0 & 0 & 0 & g(r,x^4,x^5) & h(r,x^4,x^5)
\end{pmatrix}.  \hspace{1cm} (16)$$

This must obey the following constraints, which ensure the relativistic limit,
and follow from Eq. (15) for $g_{44}$ and $g_{55}$:

$$\frac{1}{2} (f - h) = 0,  \hspace{1cm} (17)$$

$$\frac{1}{2} (f + h - 2g) = 1.  \hspace{1cm} (18)$$

Let us justify this statement. In the limit $r \to \infty$, the gravitational field should become negligible, so that the metric tensor reduces to the flat Galilean metric, Eq. (2). The transformation, Eq. (14), allows us to obtain the relativistic limit. Since Eq. (14) is valid in the limit $r \to \infty$, it can be applied everywhere. This assumption is consistent with the principle of minimal coupling, a prescription used in GR to express the laws of physics valid in flat spacetime in a covariant fashion. Under the transformation in Eq. (14), the metric becomes

$$(g_{\mu\nu}) \to \begin{pmatrix}(g_{\mu\nu})_{4 \times 4} & 0 \\
0 & 1
\end{pmatrix}.$$  \hspace{1cm} (19)$$

where $(g_{\mu\nu})_{4 \times 4}$ is the standard Schwarzschild metric of GR. In this case, (i) the component $g_{55}$ of the transformed metric, $g_{55} = \frac{1}{2}(\bar{g}_{44} - 2\bar{g}_{45} + \bar{g}_{55})$, should be equal to 1; and (ii) the component $g_{45} = \frac{1}{2}(\bar{g}_{44} - \bar{g}_{55})$ would have to be null. These facts lead to Eqs. (17) and (18) for the metric, Eq. (16). Under these constraints it assumes the form:

$$g_{\mu\nu} = \begin{pmatrix}e^{\lambda(r,x^4,x^5)} & 0 & 0 & 0 & 0 \\
0 & r^2 & 0 & 0 & 0 \\
0 & 0 & r^2 \sin^2 \theta & 0 & 0 \\
0 & 0 & 0 & f(r,x^4,x^5) & -1 + f(r,x^4,x^5) \\
0 & 0 & 0 & -1 + f(r,x^4,x^5) & f(r,x^4,x^5)
\end{pmatrix}.  \hspace{1cm} (19)$$

This is the ansatz employed in the following for the Galilean metric.
2.2 Static spherical vacuum solution

The field equations in vacuum, for which $T_{\mu \nu} = 0$ in Eq. (8), reduce to

$$R_{\mu \nu} = 0. \tag{20}$$

In this section, we analyze the vacuum field equations for which the arbitrary functions in the metric tensor do not depend on the $x^4$-coordinate; that is,

$$\lambda = \lambda (r, x^5); \quad f = f (r, x^5). \tag{21}$$

We call this metric a static solution, by analogy with the GR context where the coordinate $x^4$ is the time. Henceforth, we denote the derivatives as follows:

$$f' = \partial_r f; \quad \dot{f} = \partial_5 f. \tag{22}$$

We can calculate the explicit form of the Einstein field equations, Eq. (20), for the metric in Eq. (19) with the simplifying hypothesis, Eq. (21), after obtaining the inverse matrix $g^{\mu \nu}$, the corresponding Christoffel symbols, Eq. (10), and finally the Ricci tensor, Eq. (9). The resulting non-zero Ricci components are

$$R_{11} = e^\lambda \left[ \frac{2 (1-f) \dot{\lambda} + (-f) (-1+2f) \left( \dot{\lambda}^2 + 2 \dot{\lambda} \right)}{4 (1-2f)^2} \right. +$$

$$\left. + \left[ \frac{\lambda'}{r} + \frac{2 f'^2 + (-1+2f) (f' \lambda' - 2 f'')}{2 (1-2f)^2} \right] \right], \tag{23}$$

$$R_{14} = -\frac{2 \dot{f} f' + (-1+2f) \left( \dot{\lambda} f' - 2 \dot{f} \right)}{4 (1-2f)^2}, \tag{24}$$

$$R_{15} = \frac{2 \dot{f} f' + (-1+2f) \left( \dot{\lambda} f' - 2 \dot{f} \right)}{4 (1-2f)^2} + \dot{\lambda} = \left( \frac{\lambda}{r} - R_{14} \right), \tag{25}$$

$$R_{22} = 1 + \frac{r e^{-\lambda}}{2} \left[ \lambda' - \frac{2 f'}{1-2f} \right], \tag{26}$$

$$R_{33} = R_{22} \sin^2 \theta, \tag{27}$$

$$R_{44} = \frac{2 f \left[ f'^2 - (-1+2f) \dot{f} \right] + (-f) (-1+2f) \dot{\lambda} - e^{-\lambda} \left[ \frac{2 f'^2 + (-1+2f) (f' \lambda' - 2 f'')}{4 (1-2f)} + \frac{f'}{r} \right]}{4 (1-2f)^2}, \tag{28}$$
\[
R_{45} = \frac{2(-1 + f) \left[ \ddot{f}^2 + (-1 + 2f) \dot{f} \right] + (1 - f) (-1 + 2f) \dot{f} \dot{\lambda} - e^{-\lambda} \left[ \frac{2f'^2 + (-1 + 2f) (f'\lambda' - 2f'')}{4(1 - 2f)} + \frac{f'}{r} \right]}{4(1 - 2f)^2}, \quad (29)
\]

\[
R_{55} = \frac{2f \left[ \ddot{f}^2 + (-1 + 2f) \dot{f} \right] + (2 - f) (-1 + 2f) \dot{f} \dot{\lambda} - (1 + 2f)^2 \left( \dot{\lambda}^2 + 2\ddot{\lambda} \right)}{4(1 - 2f)^2}
- e^{-\lambda} \left[ \frac{2f'^2 + (-1 + 2f) (f'\lambda' - 2f'')}{4(1 - 2f)} + \frac{f'}{r} \right]. \quad (30)
\]

The condition in Eq. (20) requires that each component be equal to zero. Then, \( R_{14} = 0 \) in Eq. (24). By using this information in Eq. (25), we find
\[
\dot{\lambda} = 0. \quad (31)
\]

Therefore, the function \( \lambda(r, x^5) \) depends solely on \( r \): \( \lambda = \lambda(r) \).

From Eqs. (24) and (25), and the fact that \( \dot{\lambda} = 0 \), together with the requirement that \( 1 - 2f \neq 0 \), we find
\[
\ddot{f}' - (-1 + 2f) \dot{f}' = 0. \quad (32)
\]

Also, since the Ricci components are equal to zero, we must have \( R_{44} + R_{55} - 2R_{45} = 0 \), so that Eqs. (28) to (30) lead to
\[
\ddot{f}^2 - (-1 + 2f) \ddot{f} = 0. \quad (33)
\]

It is important to observe that, with Eqs. (31), (32) and (33), we find that Eqs. (28) to (30), for the \( x^4 \)-independent solution, reduce to
\[
R_{11} = \frac{\lambda'}{r} + \frac{2f'^2 + (-1 + 2f) (f'\lambda' - 2f'')}{2(1 - 2f)^2} = 0, \quad (34)
\]
\[
R_{22} = 1 + \frac{re^{-\lambda}}{2} \left[ \lambda' - \frac{2f'}{r} + \frac{f'}{1 - 2f} \right] = 0, \quad (35)
\]
\[
R_{33} = R_{22} \sin^2 \theta = 0, \quad (36)
\]
\[
R_{44} = R_{45} = R_{55} = -e^{-\lambda} \left[ \frac{2f'^2 + (-1 + 2f) (f'\lambda' - 2f'')}{}{4(1 - 2f)} + \frac{f'}{r} \right] = 0. \quad (37)
\]

With the condition \( 1 - 2f \neq 0 \), we can write Eq. (37) as follows:
\[
\frac{2f'^2 + (-1 + 2f) (f'\lambda' - 2f'')}{2(1 - 2f)^2} = -\frac{2f'}{r(1 - 2f)}.
\]

This can be substituted into Eq. (34) and it leads to the relation
\[
\lambda' - \frac{2f'}{1 - 2f} = 0, \quad (38)
\]
which is valid for \( r \neq 0 \).

If we take the derivative of Eq. (38) with respect to \( x^5 \), and use Eq. (31), we obtain

\[
(1 - 2f) \dot{f}'' + 2 \dot{f} f' = 0.
\] (39)

This relation, together with Eq. (32), allows us to prove that \( f \) does not depend on \( x^5 \). Indeed, multiplying Eq. (32) by 2 and subtracting from (39) gives

\[
(-1 + 2f) \dot{f}' = 0.
\]

This implies that either

\[
\dot{f}' = 0, \tag{40}
\]
or

\[-1 + 2f = 0. \tag{41}
\]

This last possibility is prevented by consistency of the theory since the factor \(-1 + 2f\) appears in the denominator of the Ricci components, Eqs. (23) to (30).

From Eq. (38), we find the relation

\[
\lambda + \ln (1 - 2f) = \ln D,
\]

where \( D \) depends on \( x^5 \) only. We use \( \ln D \) for convenience. Indeed, this equation relates \( \lambda \) and \( f \) as follows:

\[
e^\lambda = \frac{D}{1 - 2f}. \tag{41}
\]

If we substitute Eqs. (38) and (41) into Eq. (35), we obtain

\[
(rf)'' = \frac{1 - D}{2} = E,
\]

where \( E \) is a function of \( x^5 \) only. Therefore,

\[
rf = Er + F,
\]

with \( F \) depending on \( x^5 \) only. Since \( E = (1 - D)/2 \), we can write

\[
f = \frac{1 - D}{2} + \frac{F}{r}. \tag{42}
\]

Up to this point, we have \( f = f(r, x^5) \), \( D = D(x^5) \) and \( F = F(x^5) \). By differentiating Eq. (42) with respect to \( r \) and \( x^5 \), we find

\[
\dot{f}' = -\frac{\dot{F}}{r^2}. \tag{43}
\]

From Eqs. (43) and (40), we conclude that

\[
\dot{F} = 0, \tag{44}
\]
so that $F$ is constant. Therefore,

$$
\dot{f} = -\frac{\dot{D}}{2}.
$$

(45)

By substituting Eq. (40) into Eq. (32), we find

$$
\dot{f}f' = 0.
$$

Since Eq. (42) prevents $f'$ from being zero, this equation gives

$$
\dot{f} = 0.
$$

(46)

Note that with Eq. (46), the constraint in Eq. (33) is automatically satisfied. In addition, Eq. (45) gives

$$
\dot{D} = 0,
$$

(47)

so that $D$ is also a constant.

According to Eqs. (31) and (46), we see that

$$
\lambda = \lambda(r), \quad f = f(r),
$$

thereby obtaining the following theorem:

**Galilei-Birkhoff theorem**: A static spherically vacuum solution is necessarily $x^5$-independent.

This theorem is subjected to the restriction specified in Eq. (13); that is, our ansatz for the Galilean metric is the one consistent with Eq. (12).

Since the metric of the flat Galilean space-time, Eq. (2), is symmetric under the exchange of $x^4$ and $x^5$, we obtain the usual Birkhoff theorem for the time $x^4$; that is, if we begin with a solution independent of $x^5$, then the solution is necessarily static.

The metric is determined up to $D$ and $F$, which are both constant, due to Eqs. (47) and (44), respectively. By inserting Eq. (42) into Eq. (41), we obtain

$$
e^{\lambda} = \frac{1}{1 - \frac{2(F/D)}{r}}.
$$

(48)

The constant $D$ is determined by the behavior of the metric at large distances from the source. In order to ensure that the metric reduces to the Galilean metric, Eq. (2), in the limit $r \rightarrow \infty$, we must impose $\lim_{r \rightarrow \infty} f(r) = 0$. From Eq. (42), this leads to

$$
D = 1.
$$

(49)

The constant $F$ is determined from the relativistic limit in the weak-field approximation. The light-cone transformation, Eq. (13), and Eq. (15) give $g_{44} = (\tilde{g}_{44} + 2\tilde{g}_{45} + \tilde{g}_{55})/2$. For the Galilean metric, Eq. (16), subject to the constraints in Eqs. (17) and (18), we find

$$
g_{44} = \frac{1}{2} (f + h + 2g) = \frac{1}{2} [f + f + 2(-1 + f)] = -1 + \frac{2F}{r},
$$

(50)

10
where we have used Eqs. (42) and (49). In the weak-field approximation of GR, the component $g_{44}$ is written in the perturbed form as

$$g_{44} = -1 - 2\phi_N,$$

(51)

where

$$\phi_N = -\frac{GM}{c^2 r} = -\frac{m}{r}$$

(52)

is the Newtonian potential, $G$, the gravitational constant, $M$, the mass of the source, and $m = \frac{GM}{c^2}$, the geometrical mass. From Eqs. (50) and (51), we conclude that

$$F = m.$$  

(53)

We conclude this discussion by stating that the metric of the vacuum static solution is completely specified by Eqs. (42), (48), (49) and (53). This metric is given by

$$(g_{\mu\nu}) = \begin{pmatrix}
\frac{1}{1 - \frac{2m}{r}} & 0 & 0 & 0 & 0 \\
0 & \frac{r^2}{1 - \frac{2m}{r}} & 0 & 0 & 0 \\
0 & 0 & r^2 \sin^2 \theta & 0 & 0 \\
0 & 0 & 0 & \frac{m}{r} & -1 + \frac{m}{r} \\
0 & 0 & 0 & -1 + \frac{m}{r} & \frac{m}{r}
\end{pmatrix}.$$  

(54)

With this result, the Galilean trivial metric in spherical coordinates is obtained in the limit $r \to \infty$ and it has a well defined relativistic limit. Eq. (54) will be called the Galilei-Schwarzschild solution. Notice that the event horizon, at $r = 2m$, is the same as in the relativistic context. However, this is of no concern for the applications discussed henceforth since the numerical values of the radii will be much larger than this critical value.

### 3 Three classic tests of Galilean gravity

In view of the Galilei-Birkhoff theorem, the Galilei-Schwarzschild metric, Eq. (54) is the only Schwarzschild-like metric consistent with Galilean covariance. The resulting line element,

$$ds^2 = \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 + \frac{m}{r} \left( dx^4 \right)^2 + \frac{m}{r} \left( dx^5 \right)^2 + 2 \left( -1 + \frac{m}{r} \right) dx^4 dx^5,$$

(55)

describes the invariant interval on the curved manifold in which the motion of particles and light rays will take place. In this section, we shall study these motions in order to test the Galilei-Schwarzschild solution. Along the lines of Ref. [8], obtaining results from Galilean gravity that are identical to results from GR means that these are not specifically relativistic in the Lorentz sense; in order the concept of curved manifold is sufficient to explain these effects without relying on the underlying Lorentz metric.
3.1 Planetary motion

The motion of particles is described by the geodesic equations,

$$\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{ds} \frac{dx^\rho}{ds} = 0.$$ 

For the Galilei-Schwarzschild solution, these equations become

$$\frac{d^2 r}{ds^2} - \frac{m}{r^2} \frac{1}{(1 - \frac{2m}{r})} \left( \frac{dr}{ds} \right)^2 - \left( 1 - \frac{2m}{r} \right) r \left( \frac{d\theta}{ds} \right)^2$$

$$- \left( 1 - \frac{2m}{r} \right) r \sin^2 \theta \left( \frac{d\phi}{ds} \right)^2 + m \frac{1}{2r^2} \left( 1 - \frac{2m}{r} \right) \left( \frac{dx^4}{ds} + \frac{dx^5}{ds} \right)^2 = 0,$$ (56)

$$\frac{d^2 \theta}{ds^2} + \frac{2}{r} \frac{dr}{ds} \frac{d\theta}{ds} - \sin \theta \cos \theta \left( \frac{d\phi}{ds} \right)^2 = 0,$$ (57)

$$\frac{d^2 \phi}{ds^2} + \frac{2}{r} \frac{dr}{ds} \frac{d\phi}{ds} + 2 \cot \theta \frac{d\theta}{ds} \frac{d\phi}{ds} = 0,$$ (58)

$$\frac{d^2 x^4}{ds^2} + \frac{m}{r^2} \frac{1}{(1 - \frac{2m}{r})} \frac{dr}{ds} \left( \frac{dx^4}{ds} + \frac{dx^5}{ds} \right) = 0,$$ (59)

$$\frac{d^2 x^5}{ds^2} + \frac{m}{r^2} \frac{1}{(1 - \frac{2m}{r})} \frac{dr}{ds} \left( \frac{dx^4}{ds} + \frac{dx^5}{ds} \right) = 0.$$ (60)

We show that the motion lies in a plane as it happens in classical mechanics for a central force [12]. With an appropriate orientation of the axis, we can choose the initial conditions to be

$$\theta_0 = \frac{\pi}{2}, \quad \left( \frac{d\theta}{ds} \right)_0 = 0,$$

for some initial value of $s$. This choice implies that the motion of the test particle starts at the ecliptic plane with zero initial azimuthal velocity. It means also that Eq. 57 gives a zero initial elevation acceleration, $\left( \frac{d^2 \theta}{ds^2} \right)_0 = 0$. Thus, at an infinitesimal proper instant later, $\theta_{\Delta s} = \frac{\pi}{2}$ and $\left( \frac{d\theta}{ds} \right)_{\Delta s} = 0$, and similarly after another $\Delta s$, and so on. As a result, the motion is confined to the plane $\theta = \frac{\pi}{2}$.

The geodesic equations, Eqs. (56) to (60), are then simplified to:

$$\frac{d^2 r}{ds^2} - \frac{m}{r^2} \frac{1}{(1 - \frac{2m}{r})} \left( \frac{dr}{ds} \right)^2 - \left( 1 - \frac{2m}{r} \right) r \left( \frac{d\phi}{ds} \right)^2$$

$$+ \frac{1}{2} \frac{m}{r^2} \left( 1 - \frac{2m}{r} \right) \left( \frac{dx^4}{ds} + \frac{dx^5}{ds} \right)^2 = 0,$$ (61)

$$\frac{d^2 \phi}{ds^2} + \frac{2}{r} \frac{dr}{ds} \frac{d\phi}{ds} = 0,$$ (62)
\[
\frac{d^2 x^4}{ds^2} + \frac{m}{r^2} \left(1 - \frac{2m}{r}\right) \frac{dr}{ds} \left( \frac{dx^4}{ds} + \frac{dx^5}{ds} \right) = 0, \quad (63)
\]

\[
\frac{d^2 x^5}{ds^2} + \frac{m}{r^2} \left(1 - \frac{2m}{r}\right) \frac{dr}{ds} \left( \frac{dx^4}{ds} + \frac{dx^5}{ds} \right) = 0. \quad (64)
\]

If we multiply Eq. (62) by \( r^2 \), we find

\[
r^2 \frac{d\phi}{ds} = h, \quad (65)
\]

where \( h \) is a constant related to the conserved angular momentum of the particle. Similarly, by adding Eqs. (63) and (64), and by multiplying the result by \( \left(1 - \frac{2m}{r}\right) \), we are led to

\[
\frac{dx^4}{ds} + \frac{dx^5}{ds} = \frac{k}{\left(1 - \frac{2m}{r}\right)}, \quad (66)
\]

where \( k \) is another constant. By subtracting Eq. (63) from Eq. (64), we find

\[
\frac{dx^4}{ds} - \frac{dx^5}{ds} = b, \quad (67)
\]

where \( b \) is constant. From Eqs. (66) and (67), we obtain

\[
\frac{dx^4}{ds} = \frac{1}{2} \left[ \frac{k}{\left(1 - \frac{2m}{r}\right)} + b \right], \quad (68)
\]

and

\[
\frac{dx^5}{ds} = \frac{1}{2} \left[ \frac{k}{\left(1 - \frac{2m}{r}\right)} - b \right]. \quad (69)
\]

By substituting Eqs. (65) and (66) into Eq. (61), it follows that

\[
\frac{d^2 r}{ds^2} - \frac{m}{r^2} \left(1 - \frac{2m}{r}\right) \left( \frac{dr}{ds} \right)^2 - \left(1 - \frac{2m}{r}\right) \frac{h^2}{r^3} + 2 \frac{m}{2} \frac{k^2}{r^4} = 0.
\]

As in the classic Kepler problem, we can simplify the integration processes by considering \( r \) as a function of \( \varphi \) instead of \( s \). If we change the variable \( r \) to

\[
u = \frac{1}{r},
\]

it is possible to rewrite the last differential equation as

\[
\frac{d^2 u}{d\varphi^2} + \frac{m}{\left(1 - 2mu\right)} \left( \frac{du}{d\varphi} \right)^2 + \left(1 - 2mu\right) u - \frac{1}{2} \frac{k^2}{h^2} \frac{m}{\left(1 - 2mu\right)} = 0. \quad (70)
\]

The term proportional to \( \left( \frac{du}{d\varphi} \right)^2 \) can be expressed in another form. For this, we use the constraint

\[
g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = -1, \quad (71)
\]
which leads to

\[
\left( \frac{du}{d\varphi} \right)^2 + (1 - 2mu) \left[ u^2 + \frac{1}{h^2} \left( 1 + \frac{b^2}{2} \right) \right] - \frac{1}{2} \frac{k^2}{h^2} = 0. \tag{72}
\]

By inserting Eq. \((72)\) into Eq. \((70)\), we find the orbital equation,

\[
\frac{d^2u}{d\varphi^2} + u - \frac{m}{h^2} - 3mu^2 - \frac{m b^2}{h^2} = 0. \tag{73}
\]

The first four terms are the usual ones obtained in the standard 4-dimensional GR. The extra term should provide non-relativistic corrections.

**Remark 1. Relativistic (Lorentzian) Limit.** The projection of a Galilean theory to a Lorentzian theory can be performed by letting

\[
(x, t) \mapsto x^\mu = (x^1, \ldots, x^5) \equiv \left( x, \frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}} \right).
\]

This is the case when, from Eq. \((3)\), we have \(\bar{x}^4 = x^4\) and \(\bar{x}^5 = 0\). Therefore, in order to check consistency of our theory with the relativistic case, we should take \(x^4 = x^5\), which implies \(\frac{dx^4}{ds} = \frac{dx^5}{ds}\). In this case, Eqs. \((68)\) and \((69)\) impose \(b = 0\) and Eq. \((73)\) reduces simply to:

\[
\frac{d^2u}{d\varphi^2} + u - \frac{m}{h^2} - 3mu^2 = 0.
\]

This is the orbital equation obtained in standard GR for the Schwarzschild solution. So it is verified that the geodesic equation also has a good relativistic limit.

**Remark 2. Large distances.** If the large distance case is considered then the terms proportional to \(u^2\) in Eq. \((73)\) should be neglected. Thus

\[
\frac{d^2u}{d\varphi^2} + u - \frac{m}{h^2} - \frac{mb^2}{h^2} = 0. \tag{74}
\]

The first three terms are the usual terms obtained by Newtonian gravitation. The additional term, proportional to \(b^2\), is open for interpretation. If agreement with measurements are to be obtained, then either \(b\) should be negligibly small (which would leave us with the usual GR result) or \(b\) should be included in a renormalized value of \(m\). In Section 3.2 we shall use \(b \ll 1\).

### 3.1.1 The perihelion shift

Let us rewrite Eq. \((73)\) as follows:

\[
\frac{d^2u}{d\varphi^2} + u = \frac{m}{h^2} + 3mu^2, \tag{75}
\]
with
\[\frac{1}{h^2} = \frac{1}{\bar{h}^2} \left( 1 + \frac{b^2}{2} \right).\] (76)

The orbital equation (75) is formally exactly as predicted by GR, the only difference being the redefinition \( h \rightarrow \bar{h} \). Then, we know beforehand that our Galilean version of Schwarzschild solution will predict a perihelion shift for the orbit of the planets in the way of GR.

The usual procedure is to obtain a solution of Eq. (75) through an iterative procedure taking \( u \simeq u^{(0)} + u^{(1)} \) [13]. The zero-order is the unperturbed solution of Eq. (74), or Eq. (75) with \( 3mu^2 = 0 \). It is utilized as a source term for the differential equation of \( u^{(1)} \), i.e., we shall write \( 3mu^2 = 3m \left( u^{(0)} \right)^2 \). The integration constants are the eccentricity of the orbit \( e \) and an arbitrary initial value \( \varphi_0 \) for the azimuthal angle. The constant \( e \) is related to the major axis \( a = r_{\text{max}} \) by
\[a = \frac{L}{1 - e}, \] (77)
where
\[L = \frac{m}{h^2}, \] (78)
is the semi-latus rectum of the orbit. Considering orbits of small eccentricity (like the ones of Mercury), the terms of order \( e^2 \) and higher may be neglected and the solution of the orbital equation becomes
\[u \simeq \frac{m}{\bar{h}^2} \left[ 1 + e \cos \left( \varphi - \varphi_0 - \Delta \varphi_0 \right) \right],\]
where
\[\Delta \varphi_0 = \frac{3m^2}{\bar{h}^2} \varphi. \] (79)

Thus, while the planet rotates through an angle \( \varphi \), the perihelion shifts by a fraction of the revolution angle given by
\[\frac{\Delta \varphi_0}{\varphi} = \frac{3m^2}{h^2} = \frac{3m}{L} = \frac{3m}{a \left( 1 - e \right)},\]
where we have employed Eqs. (78) and (77). After a full revolution, \( \varphi = 2\pi \), we have
\[\Delta \varphi_0 = 6\pi \frac{m^2}{h^2} = 6\pi \frac{m}{L} = 6\pi \frac{GM}{c^2 a \left( 1 - e \right)}. \] (80)

### 3.1.2 Numerical analysis for the planet Mercury

Here, the quantities of interest are expressed in terms of orbital parameters of the planet under consideration and the geometrical mass of the Sun. Let us consider the planet Mercury, with the following orbital data [15]:
\[a = 57.91 \times 10^6 \text{ km}, \quad e = 0.2056.\]
The numerical value of the geometrical mass of the Sun $m$ is calculated from the observational data:

$$m = \frac{GM_{\text{Sun}}}{c^2} = \frac{132712 \times 10^6 \text{ km}^3/\text{s}^2}{(299792.458)^2 \text{ (km/s)}^2} = 1.4766 \text{ km}, \quad (81)$$

where $GM_{\text{Sun}} = 132712 \times 10^6 \text{ km}^3/\text{s}^2$ [15].

Hence the value of $L$ for Mercury is

$$L = (1 - e^2) a = \left(1 - (0.2056)^2\right) (57.91 \times 10^6) \text{ km} = 55.462 \times 10^6 \text{ km}$$

and the perihelion shift, from Eq. (80), is

$$\Delta \varphi_0 = 1.5974 \pi \times 10^{-7} = 0.10351''.$$

This is an extremely small angle, but this is a secular effect which increases with the number of revolutions. The shift above is observed in a single Mercury-year; which corresponds to 0.241 Earth-years. So, the total shift per Earth-year is

$$\Delta \varphi_E = \frac{\Delta \varphi_0}{0.241} = \frac{0.1035''}{0.241} = 0.42946''.$$ 

If this effect is accumulated over 100 Earth-years, the total shift is

$$\Delta \varphi = 100 \Delta \varphi_E = 42.946'' \simeq 43''.$$

The conclusion is that our Galilean-Schwarzschild solution is as good as the GR Schwarzschild solution for a prediction of the perihelion shift of Mercury. The difference is that we can calculate the value for the constant $\bar{h}$ while in GR we obtain directly the value of $h$. Indeed, from the definition of the semi-latus rectum, given in Eq. (78), and the values of $m$ and $L$, we obtain

$$\bar{h}^2 = \frac{m}{L} = \frac{1.4766}{55.462 \times 10^6} = 2.6624 \times 10^{-8}.$$ 

Then, from (70), we find a constraint for the integration constant $h$ and $b$,

$$\frac{1}{h^2} = \frac{1}{\bar{h}^2} \left(1 + \frac{b^2}{2}\right) = 0.3756 \times 10^8, \quad (82)$$

so that

$$\bar{h} = \frac{1}{\sqrt{0.3756 \times 10^8}} = 1.6317 \times 10^{-4}.$$ 

Note that our results are derived by considering $c$ as the speed of light (see calculation of geometrical mass of the Sun, Eq. [21]).
3.2 The deflection of light rays

Light rays consist of massless test particles which travel at the speed of light. In special relativity, the photons move along the light-cone, following a null geodesic: $ds^2 = 0$. They are essentially relativistic particles, and they will be treated accordingly here: we will keep $ds^2 = 0$ for photon traveling in our Galilean gravity background. So our framework is a curved manifold with local Galilean invariance described by Galilei-Schwarzschild solution on which the ultra-relativistic particles will propagate. We feel justified in doing so because the photons are test particles and, by definition, test particles do not affect the geometry of the background spacetime (no matter their nature).

By taking $ds^2 = 0$ in the Schwarzschild-like spacetime, Eq. [55], and dividing the result by $d\sigma^2$ (where $\sigma$ is an appropriate invariant length), the line element becomes

$$
\frac{1}{(1 - \frac{2m}{r})} \left( \frac{dr}{d\sigma} \right)^2 + r^2 \left( \frac{d\theta}{d\sigma} \right)^2 + r^2 \sin^2 \theta \left( \frac{d\varphi}{d\sigma} \right)^2 + \frac{m}{r} \left( \frac{dx^4}{d\sigma} + \frac{dx^5}{d\sigma} \right)^2 - 2 \frac{dx^4}{d\sigma} \frac{dx^5}{d\sigma} = 0.
$$

(83)

This constraint replaces the one given by Eq. [71] for massive particles. All the equations before Eq. [71] remain valid for the propagation of light, provided that we replace the invariant length $s$ by $\sigma$; that is,

$$
\frac{d^2 r}{d\sigma^2} - m \frac{1}{r^2 (1 - \frac{2m}{r})} \left( \frac{dr}{d\sigma} \right)^2 - \left( 1 - \frac{2m}{r} \right) r \sin^2 \theta \left( \frac{d\varphi}{d\sigma} \right)^2 + \frac{1}{2} \frac{m}{r^2} \left( 1 - \frac{2m}{r} \right) \left( \frac{dx^4}{d\sigma} + \frac{dx^5}{d\sigma} \right)^2 = 0,
$$

(84)

$$
\frac{d\varphi}{d\sigma} = \frac{h}{r^2},
$$

(85)

$$
\frac{dx^4}{d\sigma} = - \frac{1}{2} \left( \frac{k}{1 - \frac{2m}{r}} + b \right),
$$

(86)

$$
\frac{dx^5}{d\sigma} = - \frac{1}{2} \left( \frac{k}{1 - \frac{2m}{r}} - b \right),
$$

(87)

where the initial conditions are $\theta_0 = \pi/2$ and $\left( \frac{d\theta}{d\sigma} \right)_0 = 0$. These conditions imply $\left( \frac{d^2 \theta}{d\sigma^2} \right)_0 = 0$ and restrict our study to the plane $\theta = \pi/2$. Therefore, Eq. [70] is still valid for the light rays. However, Eq. [72] must be modified, because it was obtained using $ds^2 \neq 0$, $g_{\mu\nu} u^\mu u^\nu = -1$, whereas we have here $ds^2 = 0$, $g_{\mu\nu} u^\mu u^\nu = 0$.

Let us rewrite the new constraint, Eq. [83], by substituting $\theta = \pi/2$, $d\theta/d\sigma = 0$, together with Eqs. [85], [86] and [87]:

$$
\left( \frac{dr}{d\sigma} \right)^2 + \left( 1 - \frac{2m}{r} \right) \left( \frac{b^2}{2} + \frac{k^2}{r^2} \right) = 0.
$$

17
Since
\[ \frac{dr}{d\sigma} = \frac{dr}{d\varphi} \frac{d\varphi}{d\sigma} = \frac{dr}{d\varphi} \frac{h}{r^2}, \]
we have
\[ \left( \frac{dr}{d\varphi} \right)^2 + \left[ \left( 1 - \frac{2m}{r} \right) \left( \frac{b^2}{2} + \frac{h^2}{r^2} \right) - \frac{k^2}{r^2} \right] \frac{r^4}{h^2} = 0. \]
This equation can be written as a function of \( u = 1/r \):
\[ \left( \frac{du}{d\varphi} \right)^2 + (1 - 2mu) \left[ u^2 + \frac{1}{h^2} \frac{b^2}{2} \right] - \frac{1}{h^2} \frac{k^2}{2} = 0, \tag{88} \]
which differs only slightly from our previous constraint, Eq. (72).
By substituting Eq. (70) into Eq. (88), one obtains
\[ \frac{d^2u}{d\varphi^2} + u - 3mu^2 + \frac{m b^2}{h^2} = 0. \tag{89} \]
If \( b \approx 0 \), we observe that Eq. (89) reduces to the equation obtained in the standard 4-dimensional GR leading to the deflection of light rays. So it is verified that the equation for the path of light in the Galilean gravity has a good relativistic limit.
In analogy with the previous subsection, we first set \( 3mu^2 = 0 \), in order to get an approximate solution of Eq. (89) by an iterative procedure, starting with a zero-order solution,
\[ u^{(0)} = \frac{1}{R} \cos(\varphi - \varphi_0) - \frac{m b^2}{h^2} \frac{R}{2}, \tag{90} \]
where \( \varphi_0 \) and \( R \) are integration constants. The interpretation of \( R \) becomes clear when we set \( \varphi_0 = 0 \), and introduce a Cartesian coordinate system \( x = r \cos \varphi \), \( y = r \sin \varphi \), with origin \( O \) at the center of the massive body, which is the source of the field.
With \( b = 0 \), we find that Eq. (90) reduces to \( R = r \cos \varphi = x \). This is a straight line parallel to the \( y \)-axis, and \( R \) is the minimum distance between the light ray and the origin \( O \). In this case, \( u^{(0)} \) does not bend the straight trajectory of the photon and there is no deflection of light. However, this is not the best possible approximation, and there is also a \( b \neq 0 \) to be taken into account.
The first approximation to Eq. (89),
\[ \frac{d^2u^{(1)}}{d\varphi^2} + u^{(1)} = - \frac{m b^2}{h^2} \frac{3m}{2} + 3m \left( u^{(0)} \right)^2, \]
becomes
\[ \frac{d^2u^{(1)}}{d\varphi^2} + u^{(1)} = - \frac{m b^2}{h^2} \frac{3m}{2} \frac{3}{R^2} \cos^2 \varphi \]
\[ + \frac{3m^3}{h^4} \left( \frac{b^4}{4} \right) - 6 \frac{m^2}{h^2} \left( \frac{b^2}{2} \right) \frac{1}{R} \cos \varphi. \]
If we consider $b << 1$, then the factor of $b^4$ can be neglected,
\[
\frac{d^2 u^{(1)}}{d\varphi^2} + u^{(1)} = \frac{3m}{R^2} \cos^2 \varphi - \frac{m}{h^2} \frac{b^2}{2} - 6m \frac{m}{h^2} \frac{1}{R} \cos \varphi.
\]
A particular solution of this differential equation is
\[
u^{(1)} = \frac{m}{R^2} \left( \cos^2 \varphi + 2 \sin^2 \varphi \right) - \frac{m}{h^2} \frac{b^2}{2} - 3m \frac{m}{h^2} \frac{1}{R} \varphi \sin \varphi.
\]
Therefore,
\[
u \simeq \nu^{(0)} + \nu^{(1)} = \frac{1}{R} \cos \varphi + \frac{m}{R^2} \left( \cos^2 \varphi + 2 \sin^2 \varphi \right) - \frac{m}{h^2} \frac{b^2}{2} \left( 2 + \frac{3m}{R} \varphi \sin \varphi \right).
\]
Notice that if $b = 0$ this equation reduces to the expression derived in GR [13]. This also means that all possible modifications predicted by the Galilean version of the theory of gravitation for the deflection of light are encapsulated in the last term of Eq. (91).

If we multiply Eq. (91) by $rR$, we have
\[
R = r \cos \varphi + \frac{m}{R^2} \left( r \cos^2 \varphi + 2r \sin^2 \varphi \right) - \frac{m}{h^2} \frac{b^2}{2} \left[ (2R) r + (3m \varphi) r \sin \varphi \right].
\]
Then, in Cartesian coordinates, one has
\[
x = R - \frac{m}{R} \left( \frac{x^2 + 2y^2}{\sqrt{x^2 + y^2}} \right) + \frac{m}{h^2} \frac{b^2}{2} \left[ 2R \sqrt{x^2 + y^2} + \left( 3m \arctan \frac{y}{x} \right) y \right]
\]
The second term on the r.h.s gives the GR’s deviation of the light ray from the straight line $x = R$. The last term is the contribution coming from the Galilean gravity. In the limit where $y >> x$, we obtain the asymptotic equation:
\[
x \simeq R - \frac{m}{R} \left( \pm 2y \right) + \frac{m}{h^2} \frac{b^2}{2} \left[ 2R \left( \pm y \right) + \left( 3m \frac{\pi}{2} \right) y \right],
\]
because $\lim_{\alpha \to \pm \infty} (\arctan \alpha) = \pi/2$. Thus, the two possible values of $x$ are
\[
x_+ = R + \frac{2m}{R} y + \frac{m}{h^2} \frac{b^2}{2} \left( 2R + \frac{3\pi}{2} m \right) y,
\]
\[
x_- = R - \frac{2m}{R} y + \frac{m}{h^2} \frac{b^2}{2} \left( -2R + \frac{3\pi}{2} m \right) y,
\]
and the deflection is described by the angle
\[
\tan \delta \simeq \frac{x_+ - x_-}{y}.
\]
With \( \tan \alpha = \alpha + O(\alpha^3) \), the deflection angle is

\[
\delta = \frac{4m}{R} + (2mR) \frac{b^2}{h^2}.
\]  

(92)

If we take \( b = 0 \), we consistently recover the GR’s deflection angle, which acquires its maximum value when \( R \) is minimum, i.e. when the light ray coming from a distant source passes just outside the surface of the Sun. Thus, imposing that \( R \) is the radius of the Sun, \( R = 696,000 \text{ km} \) and \( m = 1.4766 \text{ km} \), we compute

\[
\frac{4m}{R} = 8.4862 \times 10^{-6} = 1.7504''.
\]  

(93)

The observations indicate that an average value for the deflection angle is about 20\% higher than this prediction:

\[
\delta = 2'' = 9.6963 \times 10^{-6}.
\]  

(94)

Therefore, we observe that the extra degree of freedom arising from the Galilean gravity, \( b \), is such that it describes this difference between GR’s prediction and observations. In other words, Eqs. (92), (93) and (94) can be used to constrain the value of the ratio \( b/h \). Indeed,

\[
\frac{b^2}{h^2} = \frac{1}{2mR} \left( \delta - \frac{4m}{R} \right) = 5.8873 \times 10^{-13}.
\]  

(95)

Now, in order to estimate the values of the parameters, consider Eq. (82), which was deduced in the context of the shift of the Mercury’s perihelion. By using the consistency of both constraints, Eqs. (82) and (95), we can determine \( b \) and \( h \). This leads to

\[
\frac{1}{h^2} \left( 1 + \frac{5.8873 \times 10^{-13}h^2}{2} \right) = 0.3756 \times 10^8,
\]

from which we find \( h \),

\[
h = \frac{1}{\sqrt{0.3756 \times 10^8 - \frac{5.8873 \times 10^{-13}}{2}}} = 1.6317 \times 10^{-4},
\]

and

\[
b = \sqrt{5.8873 \times 10^{-13}h} = 1.2520 \times 10^{-10}.
\]  

(96)

This only gives a rough estimate of the parameters, because there is no formal connection with the parameters used in our previous treatment of Mercury’s perihelion.
3.3 Gravitational redshift of spectral lines

In this section, we examine the shift of the atomic spectral lines in the presence of a gravitational field, also called the gravitational redshift.

From the Galilei-Schwarzschild line element, Eq. (55), we define

$$dτ^2 = -\frac{1}{2}ds^2$$

as the time-interval between two events with vanishing spatial separation, $dr = dθ = dφ = 0$. The minus sign comes from our choice of the signature of the metric, and the factor $1/2$ is chosen in order to allow agreement with the relativistic analog when $b = 0$.

The time-interval is related to the “coordinate time” differential, $dx^4$, and the additional Galilean differential coordinate, $dx^5$, by

$$-2dτ^2 = \frac{m}{r} (dx^4)^2 + \frac{m}{r} (dx^5)^2 + 2 \left(1 + \frac{m}{r}\right) dx^4 dx^5,$$  \hspace{1cm} (97)

where the differentials are constrained by Eqs. (68) and (69):

$$\frac{dx^4}{dτ} = \frac{1}{2} \left[ \frac{k}{(1 - 2m/r)} + b \right], \quad \frac{dx^5}{dτ} = \frac{1}{2} \left[ \frac{k}{(1 - 2m/r)} - b \right],$$

so that

$$dx^5 = dx^4 - bdτ.$$  \hspace{1cm} (98)

Note that these last three equations are very general; they are valid for massive particles, but they have exactly the same form as for massless particles, such as photons.

With Eq. (98), the time-interval in Eq. (97) can be expressed as

$$dτ^2 = \left(1 - \frac{2m}{r}\right) (dx^4)^2 - \frac{m}{r} \frac{b^2}{2} dτ^2 - \left(1 - \frac{2m}{r}\right) b dx^4 dτ.$$ \hspace{1cm} (99)

Notice that if $b = 0$, we have

$$dτ = +\sqrt{1 - \frac{2m}{r}} dx^4,$$ \hspace{1cm} (100)

which is the expected Schwarzschild solution of GR (see Ref. [13], Eq. (4.92)). The positive sign follows from the natural assumption that the proper time $τ$ should increase with the time coordinate $x^4$. With $b \neq 0$, Eq. (99) leads to

$$\left(1 + \frac{m b^2}{r^2}\right) dτ^2 + \left(1 - \frac{2m}{r}\right) b dx^4 dτ - \left(1 - \frac{2m}{r}\right) (dx^4)^2 = 0.$$  

This is a second-order equation for $dτ$. By solving for $dτ = dτ (dx^4)$, we find

$$dτ = \frac{- (1 - \frac{2m}{r}) b dx^4 \pm \sqrt{(1 - \frac{2m}{r})^2 b^2 (dx^4)^2 + 4 (1 + \frac{m b^2}{r^2}) (1 - \frac{2m}{r}) (dx^4)^2}}{2 (1 + \frac{m b^2}{r^2})}.$$  

21
In order that this expression agrees with Eq. (100) for \( b = 0 \), we must choose the plus sign in the r.h.s., which leads to
\[
d\tau = \left[ \frac{\sqrt{1 + \frac{b^2}{4}} - \frac{b}{2} \sqrt{1 - \frac{2m}{r}}}{1 + \frac{2m b^2}{r^2}} \right] \sqrt{1 - \frac{2m}{r}} \, dx^4. \tag{101}
\]

This distinction between proper time and the time coordinate gives rise to a difference between the proper frequency and the coordinate frequency of a periodic phenomenon in the curved spacetime, like the emission of electromagnetic radiation by an atom. Consider the propagation of electromagnetic waves in the limit of geometrical optics. Then, the electromagnetic field can be written as \( f = a \exp(it) \), with \( a = a(x,t) \) the wave amplitude, and the phase \( \psi = \psi(x,t) \) an eikonal function. Then the frequency of the wave can be expressed as the derivative of \( \psi \) with respect to the time, and one has a coordinate frequency, \( \omega_0 = \frac{\partial \psi}{\partial t} \), and a proper frequency, \( \omega = \frac{\partial \psi}{\partial \tau} \).

We call \( \omega_1 \) the proper frequency of the wave emitted by an atom at a point \( P_1 \). At another point, \( P_2 \), the observed proper frequency will be different, say \( \omega_2 \), once the gravitational field is not the same. They are related so that
\[
\frac{\omega_2}{\omega_1} = \frac{\left( \frac{\partial x^4}{\partial \tau} \right)_2}{\left( \frac{\partial x^4}{\partial \tau} \right)_1} = \frac{\left( \frac{\partial \psi}{\partial t} \right)_2}{\left( \frac{\partial \psi}{\partial t} \right)_1} = \frac{\omega_0 \left( \frac{\partial x^4}{\partial \tau} \right)_2}{\omega_0 \left( \frac{\partial x^4}{\partial \tau} \right)_1},
\]
with \( \omega_0 \) constant and where \( x^4 \) is the time coordinate \( t \). From Eq. (101), we find
\[
\left( \frac{dx^4}{d\tau} \right)_1 = \sqrt{1 + \frac{b^2}{4} - \frac{b}{2} \sqrt{1 - \frac{2m}{r_1}}} \sqrt{1 - \frac{2m}{r_1}}
\]
and similarly for \( \left( \frac{dx^4}{d\tau} \right)_2 \). Then, the ratio of frequencies is
\[
\frac{\omega_2}{\omega_1} = \frac{\left( \frac{\partial x^4}{\partial \tau} \right)_2}{\left( \frac{\partial x^4}{\partial \tau} \right)_1} = \frac{\left( 1 + \frac{2m b^2}{r_2} \right)}{\left( 1 + \frac{2m b^2}{r_1} \right)} \frac{\sqrt{1 + \frac{b^2}{4} - \frac{b}{2} \sqrt{1 - \frac{2m}{r_2}}} \sqrt{1 - \frac{2m}{r_2}}}{\sqrt{1 + \frac{b^2}{4} - \frac{b}{2} \sqrt{1 - \frac{2m}{r_1}}} \sqrt{1 - \frac{2m}{r_1}}}. \tag{102}
\]

This exact result can be approximated by taking into account the fact that regular estimates assume \( r_1 \ll r_2 \), and that \( b \) is small (see estimate in the previous section, Eq. (96)). For instance, let us assume that the radiation observed on Earth, at \( r = r_2 = 149.6 \times 10^6 \) km, was emitted at the surface of the Sun, at \( r = r_1 = R = 696,000 \) km. Then the approximation \( r_1 \ll r_2 \) is certainly valid. Moreover, the quantity \( m/r_1 \) is very small, since \( m = 1.4766 \) km, and the functions in Eq. (102) can be approximated accordingly. Therefore,
for \(m/r_1 \ll 1\) and \(m/r_2 \ll 1\), the ratio of frequencies is

\[
\frac{\omega_2}{\omega_1} \simeq \left(1 + \frac{2m}{r_2} \frac{b^2}{4}\right) \left(1 - \frac{2m}{r_1} \frac{b^2}{4}\right) \left[\frac{\sqrt{1 + \frac{b^2}{4} - \frac{b}{2} \left(1 - \frac{m}{r_1}\right)}}{\sqrt{1 + \frac{b^2}{4} - \frac{b}{2} \left(1 - \frac{m}{r_2}\right)}}\right] \left(1 - \frac{m}{r_1}\right) \left(1 + \frac{m}{r_2}\right).
\]

If we assume that \(m/r_2 \simeq 0\), then this ratio simplifies to

\[
\frac{\omega_2}{\omega_1} \simeq \left(1 - \frac{2m}{r_1} \frac{b^2}{4}\right) \left[\frac{\sqrt{1 + \frac{b^2}{4} - \frac{b}{2} \left(1 - \frac{m}{r_1}\right)}}{\sqrt{1 + \frac{b^2}{4} - \frac{b}{2} \left(1 - \frac{m}{r_2}\right)}}\right] \left(1 - \frac{m}{r_1}\right).
\]

Furthermore, if \(b\) is also assumed to be very small, \(b \ll 1\), this equation becomes

\[
\frac{\omega_2}{\omega_1} \simeq 1 - \frac{m}{r_1} \left(1 - \frac{b}{2}\right),
\]

so that

\[
\frac{\Delta \omega}{\omega} = \frac{\omega_2 - \omega_1}{\omega_1} = -\frac{m}{R} \left(1 - \frac{b}{2}\right).
\]

When \(b = 0\), this result agrees with GR.

From Eq. (103), the frequency \(\omega_2\) measured on Earth is smaller than the proper frequency, \(\omega_1\), and we obtain that the spectral line produced on the Sun are displaced towards the red compared with the same line produced on Earth. With the values of \(m\), \(R\) and \(b\) utilized in the previous section,

\[
\frac{\Delta \omega}{\omega} = -\frac{1.4766}{696000} \left(1 - \frac{1.2520 \times 10^{-10}}{2}\right) = -2.1216 \times 10^{-10}.
\]

This is too small to be measured and the correction due to \(b\) is completely negligible. One should keep in mind that this is simply an estimate based on the parameters obtained in Eqs. (82) and (95). Accordingly, everything seems consistent with GR, which shows that the gravitational redshift is not specifically relativistic in the sense of local Lorentz invariance.

Let us emphasize that this effect can be verified by observing massive white dwarfs, or with a Mossbauer experiment designed to detect the deviation in the wavelength of the radiation emitted in the presence of the gravitational field of the Earth.

### 4 Concluding remarks

In this paper, we have constructed an exact Schwarzschild-like external solution of the gravity field equations based on local Galilean invariance. In a flat spacetime, Galilean covariance consists in writing dynamical equations in a \((4 + 1)\)
Minkowski manifold with light-cone coordinates, thereby providing a unifying treatment of relativistic and non-relativistic field theories. This is extended here to curved spacetimes. We note that our exact solution comprises an event horizon which turns out to be much smaller than the distances involved in our numerical examples. Despite being exact, the Galilei-Schwarzschild solution is said to be effective in the sense that it would not be suitable to describe near-blackhole situations, in the very strong gravity regime. We examined three GR classic tests: the perihelion shift of Mercury, the deflection of light by the Sun, and the gravitational redshift of atomic spectral lines. The Galilean theory gives results similar to the GR predictions. This means that the results remain the same, whether covariant theory of gravity is locally Galilean invariant or locally Lorentz invariant.

Possible avenues of research include: (i) the search for non-static solution (i.e. $x^4$-dependent) like Kerr-Galilei metric or Friedmann-Galilei metric. An issue with this last proposal is to determine which coordinate should be singled out as the time coordinate in order to define a cosmic time, for instance; (ii) the analysis of the Killing vectors for Galilei-Schwarzschild metric and the study of its invariants; (iii) the investigation of MOND models [16] in the context of Galilean gravity, and applications to the description of the cold dark matter and the rotation curve of spiral galaxies; (iv) the relation between our parameter $b$ and the parameters appearing in the different post-Newtonian expansions; (v) the construction of the gauge theory of gravity based on local Galilean invariance.

Acknowledgement

We acknowledge partial support by the Natural Sciences and Engineering Research Council (NSERC) of Canada. RRC and PJP acknowledge financial support from CNPq (Brazil) and thank the Department of Physics, University of Alberta, for providing the facilities. RRC thanks Prof. V. P. Frolov for the kind hospitality extended to him. The authors are grateful to Prof. V.P. Frolov, Dr. A. Zelnikov, Dr. H. Yoshino, Dr. D. Gorbonos and A. Shoom for useful comments and fruitful discussions.

References

[1] M. Kobayashi, M. de Montigny, F.C. Khanna, Phys. Lett. A 372 (2008) 3541
M. de Montigny, F.C. Khanna, F.M. Saradzhev, Ann. Phys. 323 (2008) 1191
M. Kobayashi, M. de Montigny, F.C. Khanna, J. Phys. A: Math. Theor. 40 (2007) 1117
E.S. Santos, M. de Montigny, F.C. Khanna, Ann. Phys. 320 (2005) 21
E.S. Santos, M. de Montigny, F.C. Khanna, A.E. Santana, J. Phys. A: Math. Gen. 37 (2004) 9771
[2] Y. Takahashi, Fortschr. Phys. 36 (1988) 63
    Y. Takahashi, Fortschr. Phys. 36 (1988) 83

[3] M. Omote, S. Kamefuchi, Y. Takahashi, Y. Ohnuki, Fortschr. Phys. 37 (1989) 933

[4] G. Pinski, J. Math. Phys. 9 (1968) 1927
    C. Duval, G. Burdet, H.P. Künzle, M. Perrin, Phys. Rev. 31 (1985) 1841
    E. Kapuścik, Acta Phys. Pol. B 17 (1986) 569
    H.P. Künzle, C. Duval, Relativistic and nonrelativistic physical theories on five-dimensional space-time, in Semantical Aspects of Spacetime Theories, Eds. U. Majer and H.J. Schmidt, Mannheim, BI-Wissenschaftsverlag, 113-129, 1994 (and the references therein)

[5] S. Weinberg, Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity, Wiley, New York, 1972

[6] R. Aldrovandi, A.L. Barbosa, L.C.B. Crispino, J.G. Pereira, Class. Quant. Grav. 16 (1999) 495

[7] J.M. Lévy-Leblond, Galilei group and Galilean invariance, in Group Theory and Applications, Vol. 2, Ed. E.M. Loebl, Academic Press, New York, 221-299, 1971

[8] J.M. Lévy-Leblond, Comm. Math. Phys. 6 (1967) 286

[9] M. Le Bellac, J.M. Lévy-Leblond, Nuov. Cim. 14 (1973) 217

[10] A. Horzela, E. Kapuscik, C.A. Uzes, Am. J. Phys. 61 (1993) 471
    F.S. Crawford, Am. J. Phys. 60 (1992) 109
    F.S. Crawford, Am. J. Phys. 61 (1993) 472

[11] J.R. Mercher, H.A. Hauss, Electromagnetic Fields and Energy (1998 Cambridge: Hypermedia Teaching Facility, MIT Press) Available at: \url{http://web.mit.edu/6.013/book/www}
    R. Moreau, Magnetohydrodynamics (1990, Amsterdam: Kluwer)
    A. Castellanos, Electrohydrodynamics Lecture Notes of the 7th IUTAM Summer School (1996)

[12] R. Adler, M. Bazin and M. Schiffer, Introduction to General Relativity, 2nd edition, MacGraw-Hill Book Co., New York, 1975

[13] V. de Sabbata and M. Gasperini, Introduction to Gravitation, World Scientific Publishing Co., Singapore, 1985

[14] R. D’Inverno, Introducing Einstein’s Relativity, Oxford University Press, New York, (1992)
[15] http://nssdc.gsfc.nasa.gov/planetary/factsheet/mercuryfact.html
      http://nssdc.gsfc.nasa.gov/planetary/factsheet/sunfact.html

[16] J.D. Bekenstein, Contemp. Phys. 47 (2006) 387