Finiteness of $\pi_1$-sensitive Hofer–Zehnder capacity and equivariant loop space homology

Urs Frauenfelder and Andrei Pajitnov

Abstract. We prove that if $M$ is a closed, connected, oriented, rationally inessential manifold, then the Hofer–Zehnder capacity of the unit disk bundle of the cotangent bundle of $M$ is finite.

Mathematics Subject Classification. 53D40, 55P35, 55N91.

Keywords. Symplectic capacity, Equivariant loop space homology, Symplectic homology, Inessential manifolds.

1. Introduction

In this note we prove the following theorem.

Theorem A:. Assume that $M$ is a closed, connected, oriented, rationally inessential manifold. Then the $\pi_1$-sensitive Hofer–Zehnder capacity of the unit disk bundle of its cotangent bundle is finite.

It immediately follows from Theorem A that under its assumptions the Hofer–Zehnder capacity of the unit disk bundle of the cotangent bundle of $M$ is finite. We show that a contractible periodic orbit bounds the Hofer–Zehnder capacity which implies the stronger result that the $\pi_1$-sensitive Hofer–Zehnder capacity becomes finite. Here, we recall that a manifold is called rationally inessential if the image of its fundamental class under the classifying map to the classifying space of its fundamental group vanishes in homology with rational coefficients.

To prove our main result we use a theorem of Irie. This theorem tells us that if the fundamental class of $M$ vanishes in equivariant loop space homology, then the $\pi_1$-sensitive Hofer–Zehnder capacity is finite. The intuition behind Irie’s result is that the period of the killer of the fundamental class of $M$ in equivariant symplectic homology which by a famous result...
of Viterbo coincides with equivariant loop space homology bounds the $\pi_1$-sensitive Hofer–Zehnder capacity. We then show taking advantage of a deep result of Goodwillie that for a rationally inessential manifold the fundamental class of $M$ vanishes in equivariant homology.

Irie’s theorem holds as well in the nonequivariant setup. It cannot directly be applied to the nonequivariant homology of the free loop space, since the fundamental class of $M$ never vanishes in nonequivariant loop space homology. However, it can well happen that it vanishes when one twists the coefficients. This has been examined in [3]. In particular, it is shown there that if there exists a nontrivial invariant homomorphism from $\pi_2(M)$ to any finite cyclic group, then the $\pi_1$-sensitive Hofer–Zehnder capacity of the unit disk bundle of its cotangent bundle is finite. Here, invariant refers to the action of the fundamental group of $M$ on $\pi_2(M)$. It is interesting to note that equivariant and nonequivariant methods lead to quite different topological conditions which guarantee finiteness.

Without the assumption that $M$ is rationally inessential, Theorem A is wrong in general. An example is the $n$-dimensional torus $T^n$. If one endows the torus $T^n = \mathbb{R}^n / \mathbb{Z}^n$ with its standard flat metric and considers the Hamiltonian which corresponds to kinetic energy for the flat metric, periodic orbits project down to closed geodesics. However, there are no contractible closed geodesics on the flat torus. Therefore, the $\pi_1$-sensitive Hofer–Zehnder capacity of the unit disk bundle of the cotangent bundle of the torus is infinite. But the torus is the classifying space of its fundamental group $\pi_1(T^n) = \mathbb{Z}^n$ and hence not rationally inessential.

2. The Hofer–Zehnder capacity

We briefly recall the definition and general properties of the Hofer–Zehnder capacity. A comprehensive reference is the book by Hofer and Zehnder [14].

Assume that $(W, \omega)$ is a symplectic manifold. A smooth function $H : W \to \mathbb{R}$ gives rise to a vector field $X_H \in \Gamma(TW)$ implicitly defined by

$$dH = \omega(\cdot, X_H)$$

which is referred to as the Hamiltonian vector field of $H$.

**Definition 2.1.** A compactly supported, smooth function $H : W \to [0, \infty)$ is called *simple*, if it meets the following conditions

(i) There exists an open non-empty subset $U \subset W$, such that $H|_U = \max H$.

(ii) The only critical values of $H$ are 0 and $\max H$.

Pictorially a simple function looks like a pot turned upside down.

**Definition 2.2.** A simple function $H$ is called *admissible*, if its Hamiltonian vector field $X_H$ has no nonconstant orbit of period less than or equal to one.

We are now in position to define the Hofer–Zehnder capacity of the symplectic manifold $(W, \omega)$

$$c_{HZ}(W, \omega) := \sup \{ \max H \mid H : W \to [0, \infty) \text{ admissible} \}.$$
The Hofer–Zehnder capacity has the following properties:

(i) Assume there exists a symplectic embedding \((W_1, \omega_1) \hookrightarrow (W_2, \omega_2)\) between symplectic manifolds \((W_1, \omega_1)\) and \((W_2, \omega_2)\), then
\[
c_{HZ}(W_1, \omega_1) \leq c_{HZ}(W_2, \omega_2).
\]

(ii) If \(a \neq 0\), then
\[
c_{HZ}(W, a\omega) = |a|c_{HZ}(W, \omega).
\]

(iii) For \(r > 0\) let
\[
B_r := \{z \in \mathbb{C}^n : |z| < r\}
\]
be the \(r\)-ball, and
\[
Z_r := \{z \in \mathbb{C}^n : |z_1| < r\}
\]
be the \(r\)-cylinder, where \(\mathbb{C}^n\) is endowed with its standard symplectic structure, then
\[
c_{HZ}(B_r) = r^2 \pi = c_{HZ}(Z_r).
\]

Properties (i) and (ii) are straightforward, where property (iii) is highly nontrivial. It implies Gromov’s nonsqueezing result [11].

\section*{Theorem 2.3}
Assume that \(r < r'\), then there exists no symplectic embedding \(B_{r'} \hookrightarrow Z_r\).
\qed

Another application of the Hofer–Zehnder capacity is Struwe’s almost sure existence theorem.

\section*{Theorem 2.4}
Assume that \((W, \omega)\) is a compact symplectic manifold with connected boundary such that
\[
c_{HZ}(W, \omega) < \infty.
\]
If \(H : W \to \mathbb{R}\) is a smooth function satisfying \(H|_{\partial W} > \epsilon\) for some \(\epsilon > 0\) such that the interval \((-\epsilon, \epsilon)\) consists of regular values of \(H\), then for almost every \(c \in (-\epsilon, \epsilon)\) there exists a periodic orbit on the energy level set \(H^{-1}(c)\).
\qed

There is the following variant of the Hofer–Zehnder capacity. Instead of asking just for nonconstant periodic orbits in the definition of admissible Hamiltonian, one can ask more generally for contractible periodic orbits. The variant of the Hofer–Zehnder capacity is denoted by \(c_{Hz}^0(W, \omega)\) and referred to as the \(\pi_1\)-sensitive Hofer–Zehnder capacity. There is the following obvious inequality
\[
c_{HZ}(W, \omega) \leq c_{Hz}^0(W, \omega).
\]

Moreover, one has the following variant of Struwe’s theorem, which guarantees not just periodic orbits on almost every energy level but more excitingly contractible periodic orbits, namely
Theorem 2.5. Under the assumptions of Theorem 2.4, assume that
\[ c_0^0_{HZ}(W, \omega) < \infty. \]
Then for almost every \( c \in (-\epsilon, \epsilon) \) there exists a contractible periodic orbit on the energy level \( H^{-1}(c) \). \( \square \)

In view of Theorem 2.5, a challenging question is to find conditions which guarantee the finiteness of the \( \pi_1 \)-sensitive Hofer–Zehnder capacity. A first tool to do that is the displacement energy.

To explain the displacement energy, we first need to introduce the Hofer norm of a maybe time-dependent Hamiltonian. Suppose that \( H \in C_c^\infty(W \times [0, 1], \mathbb{R}) \) is a compactly supported Hamiltonian depending in addition on a time parameter \( t \in [0, 1] \). For each \( t \in [0, 1] \) we then obtain a compactly supported function \( H_t = H(\cdot, t) \in C_c^\infty(W, \mathbb{R}) \). The Hofer norm of \( H \) is defined as
\[
\| H \| := \int_0^1 (\max H_t - \min H_t) \, dt.
\]
If \( H \) is time independent this is just the oscillation of \( H \).

A time-dependent Hamiltonian gives rise to a time-dependent Hamiltonian vector field and we denote by \( \phi_H \) the time one map of its flow. Suppose that \( U \subset W \) is an open subset of a symplectic manifold and, therefore, itself a symplectic manifold. Then the displacement energy of \( U \) in \( W \) is defined as
\[
e(U) := \inf \{ \| H \| : \phi_H(U) \cap U = \emptyset \}.
\]
Note that this quantity depends as well on \( W \), i.e.,
\[
e(U) = e(U, W).
\]
We use the convention that the infimum of the empty set is \( \infty \). With this convention \( U \) is displaceable in \( W \) if and only if \( e(U, W) < \infty \). The following Theorem is proved in [14, Chapter 5].

Theorem 2.6. Suppose that \( U \subset W \) is an open subset and the inclusion homomorphism \( i_* : \pi_1(U) \to \pi_1(W) \) is injective. Then
\[
c_0^0_{HZ}(U) \leq e(U),
\]
i.e., if \( U \) is displaceable, its \( \pi_1 \)-sensitive Hofer–Zehnder capacity is finite. \( \square \)

3. Irie’s theorem

We first explain the notion of a Liouville domain. Suppose that \( (W, \lambda) \) is a connected exact symplectic manifold, i.e., \( \lambda \) is a one form on \( W \) with the property that \( \omega = d\lambda \) is a symplectic form. Then one can define a vector field \( Y \) on \( W \) uniquely determined by \( \lambda \) which is implicitly defined by the condition
\[
\lambda = \omega(Y, \cdot).
\]
Assume that \( W \) is compact. Because \( W \) admits an exact symplectic form, it necessarily has a boundary. We say that \( (W, \lambda) \) is a Liouville domain if \( Y \) points outside at the boundary. This implies that \( (\partial W, \lambda|_{\partial W}) \) is a contact manifold. In particular, one can define the Reeb vector field \( R \in \Gamma(T\partial W) \) implicitly by the conditions
\[
\lambda(R) = 1, \quad d\lambda|_{\partial W}(R, \cdot) = 0.
\]

To a Liouville domain one can associate its symplectic homology \( SH_\ast(W) \) as well as its equivariant symplectic homology \( SH_{S^1}^\ast(W) \). Moreover, there are inclusion homomorphisms
\[
i_\ast: H_\ast(W, \partial W) \to SH_\ast(W), \quad i_{S^1}^\ast: H_\ast(W, \partial W) \otimes H_\ast(\mathbb{CP}^\infty) \to SH_{S^1}^\ast(W).
\]

Equivariant symplectic homology as well as the inclusion homomorphisms were introduced in Viterbo’s fundamental work [18]. For a detailed discussion we refer to [5] and [2]. The following theorem is due to Irie [15].

**Theorem 3.1.** Suppose that \( i_\ast([W]) = 0 \in SH_\ast(W) \) or \( i_{S^1}^\ast([W] \times [pt]) = 0 \in SH_{S^1}^\ast(W) \) where \([pt]\) is the class of a point in \( H_\ast(\mathbb{CP}^\infty) \), then \( c_{HZ}^0(W) < \infty \).

**Proof.** For \( a > 0 \) which does not lie in the spectrum of the Reeb flow on \( \partial W \) we denote by \( HF_{a_*}^\ast(W) \) the Floer homology of the Liouville domain associated to a Hamiltonian which grows with slope \( a \) on the completion of the Liouville domain. If \( a < b \) the two Floer homologies are related by the inclusion homomorphism
\[
HF_{a_*}^\ast(W) \to HF_{b_*}^\ast(W)
\]
and the symplectic homology is defined as the direct limit of this directed system. Analogously equivariant symplectic homology is defined as the direct limit of the equivariant Floer homologies of the Liouville domain for Hamiltonians of increasing slope on the completion. Moreover, the inclusion homomorphism \( i_\ast \) factors through
\[
H_\ast(W, \partial W) \to HF_{a_*}^\ast(W) \to SH_\ast(W).
\]

If \( i_\ast([W]) = 0 \), then because the symplectic homology is the direct limit of the Floer homologie \( HF_{a_*}^\ast(W) \) there exists \( a > 0 \) such that the fundamental class of the Liouville domain already vanishes in \( HF_{a_*}^\ast(W) \). However, Irie proved in [15, Corollary 3.5] that this implies that the \( \pi_1 \)-sensitive Hofer–Zehnder capacity is bounded from above by \( a \).

4. Essential and inessential manifolds

In [10], M. Gromov introduced the notion of essential manifold. To recall the definition we will need some terminological conventions.

In Sects. 4, 5 and 6, the term *manifold* will mean closed connected oriented \( C^\infty \) manifold. For a connected topological space \( X \) we denote by \( \lambda_X: X \to B\pi_1(X) \) the (homotopy unique) map inducing identity isomorphism in \( \pi_1 \). Let \( R \) be one of the rings \( \mathbb{Z}, \mathbb{Q} \).
For a manifold $M$, we denote by $[M]_R$ its fundamental class in $H_\ast(M, \mathbb{R})$ and by $\langle [M] \rangle_R$ the element $(\lambda_X)_\ast([M]) \in H_\ast(B\pi_1(X), \mathbb{R})$ (the index $R$ will be omitted if the value of $R$ will be clear from the context).

**Definition 4.1.** A manifold $M$ is called $R$-essential if $\langle [M] \rangle_R \neq 0$. It is called $R$-inessential if $\langle [M] \rangle_R = 0$. We abbreviate $\mathbb{Z}$-inessential to inessential; a $\mathbb{Q}$-inessential manifold is called also rationally inessential. Similar conventions will be used for $\mathbb{Z}$-essential and $\mathbb{Q}$-essential manifolds.

The property of being essential is closely related to the geometry of the manifold. The main theorem of [10] establishes the following inequality for any essential manifold of dimension $n$:

$$\text{sys}_1(M) \leq C_n(Vol M)^{\frac{1}{n}}$$

(1)

where $\text{sys}_1(M)$ is the systolic constant, that is, the lower bound of the lengths of closed non-contractible curves in $M$.

Later on it turned out that the property of being inessential is also closely related to the geometry of the underlying manifold, namely to its macroscopic dimension and to the existence of a Riemannian metric of positive scalar curvature.

**Definition 4.2.** A manifold $M$ has macroscopic dimension less than $k$ if its universal covering $\tilde{M}$ admits a map $\phi: \tilde{M} \to X$ to a $k$-dimensional polyhedron $X$ such that the inverse images of points of $x$ are uniformly bounded. (Here, $\tilde{M}$ is endowed with a Riemannian metric, induced from $M$.) The least number $k$ such that $M$ has macroscopic dimension $\leq k$ is called macroscopic dimension of $M$, and denoted by $\dim_{md} M$. If $\dim_{md} M = \dim M$, the manifold $M$ is called macroscopically large, if not, it is called macroscopically small.

Gromov conjectured [12] that a compact manifold $M^n$ with a Riemannian metric of positive scalar curvature satisfies the inequality $\dim_{md} M \leq \dim M - 2$.

This conjecture is still open (as well as its weaker version affirming that a manifold with a Riemannian metric of positive scalar curvature is macroscopically small). Inessential manifolds provide a large source of examples (and counterexamples) to questions related to the Gromov conjecture above.

It is proved in [7] that a rationally inessential manifold $M$ is macroscopically small if the fundamental group of $M$ is a duality group of cohomological dimension not equal to $n + 1$. Dranishnikov [8] conjectured that every rationally inessential manifold $M^n$ is macroscopically small. This conjecture was disproved by M. Marcinkowski, who constructed a rationally inessential macroscopically large manifold. The fundamental group of this manifold is a finite index subgroup of a Coxeter group. It would be interesting to have more information about the class of groups $G$ for which any rationally inessential manifold $M$ with $\pi_1(M) \approx G$ is macroscopically small.

In the two next sections, we establish a relation between the inessentiality property and Hofer–Zehnder capacity for symplectic manifolds.
5. Some properties of inessential manifolds

Let us begin with some examples of inessential manifolds.

1. If $G$ is an $R$-acyclic group (that is, $H_k(G, R) = 0$ for every $k \neq 0$) then any manifold $M$ with $\pi_1(M) \approx G$ is obviously $R$-inessential. The class of $\mathbb{Q}$-acyclic groups is rather large, it includes for example all Coxeter groups (by a theorem of Davis, see [6], p. 302). There are also many $\mathbb{Z}$-acyclic groups; for example, the famous Higman group [13] given by the following presentation

$$\langle x_i \mid x_{i+1} = [x_i, x_{i+1}] \rangle, \ i \in \mathbb{Z}/4\mathbb{Z}.$$ 

By a theorem of Baumslag et al. [4] any group can be embedded into a $\mathbb{Z}$-acyclic group.

2. Let $cdG$ denote the cohomological dimension of $G$. If $\dim M > cd\pi_1(M)$, then $M$ is clearly inessential. In particular, this is the case if $\pi_1(M)$ is a knot group.

Here are some general properties of inessential manifolds. (Recall that $R$ stands for $\mathbb{Z}$ or $\mathbb{Q}$.)

Proposition 5.1. Let $M$ be an $R$-inessential manifold, and $N$ be any manifold of dimension $> 1$. Then $M \times N$ is inessential.

Proof. Let $m = \dim M, n = \dim N$, put $G = \pi_1(M), H = \pi_1(N)$. We have

$$[M \times N] = [M] \otimes [N]$$

(using the Küneth identification of $H_m(BG) \otimes H_n(BH)$ with a submodule of $H_{m+n}(B(G \times H))$). The proposition follows. \hfill \Box

Proposition 5.2. Let $M, N$ be manifolds of same dimension $\geq 3$. Then $M \mathrel{\#} N$ is $R$-inessential if and only if both $M$ and $N$ are $R$-inessential.

Proof. Let $m = \dim M$, put $G = \pi_1(M), H = \pi_1(N)$. Observe that $B\pi_1(M \mathrel{\#} N) \sim B(G \ast H) \sim BG \vee BH$ so $H_m(B\pi_1(M \mathrel{\#} N)) \approx H_m(BG) \oplus H_M(BH)$. The class $[M \mathrel{\#} N]$ equals $[M] \oplus [N]$ with respect to this decomposition. The proposition follows. \hfill \Box

The next proposition is obvious.

Proposition 5.3. Let $M, N$ be manifolds of same dimension; assume that $M$ is $\mathbb{Q}$-inessential. Assume that there is a map $M \to N$ of non-zero degree. Then $N$ is $\mathbb{Q}$-inessential.

Proposition 5.4. Let $p : M \to N$ be a finite covering of manifolds. Then $N$ is $\mathbb{Q}$-inessential if and only if $M$ is $\mathbb{Q}$-inessential.

Proof. The if part follows from the previous proposition. To prove the only if part, let $H = \pi_1(M), G = \pi_1(N)$. The homomorphism $H \to G$ induced by $p$ is injective, and we have a commutative diagram, where both vertical arrows are finite coverings:
Recall the transfer homomorphism $T: H_*(N) \to H_*(M)$ defined on the chain level by the formula $T(\sigma) = \sum \gamma$ where the sum ranges over all lifts $\gamma$ of the singular simplex $\sigma$ to $M$. The homomorphism $p_\ast \circ T$ equals identity in $H_*(N)$. We have also $[N] = k \cdot p_\ast ([M])$, where $K$ is the degree of the covering, and $[M] = T([N])$. The commutativity of the diagram (2) together with the functoriality of the transfer homomorphism implies that

$$[M] = T([N]).$$

The result follows. \hfill $\square$

6. Applications to symplectic homology theory

Our application of the theory of inessential manifolds to symplectic homology theory uses Viterbo’s theorem (see [18] and [17], and also [1] and [16]), which identifies symplectic homology of the cotangent bundle with homology of the loop space of the underlying manifold.

Let $M$ be a manifold; in the rest of this section all homology and cohomology groups are with rational coefficients, the free loop space $\Lambda M$ is endowed with the canonical rotation action, and we endow $M$ with the trivial $S^1$-action. Denote by $\alpha$ the composition

$$H_*(M) \xrightarrow{\gamma} H_*^{S^1}(M) \xrightarrow{i_*} H_*^{S^1}(\Lambda M).$$

(3)

**Proposition 6.1.** *If $M$ is $\mathbb{Q}$-inessential, then $\alpha([M]) = 0$.*

**Proof.** Denote by $\tilde{\alpha}$ the composition of the homomorphisms in cohomology dual to (3). It suffices to prove that $\tilde{\alpha}_n : H_*^{n}(\Lambda M) \to H_*^n(M)$ is equal to 0. To this end observe that $H_*^{S^1}(M)$ is a free module over $H^*(BS^1) \approx \mathbb{Q}[u]$; therefore, the map $H_*^{S^1}(M) \to H_*^n(M)$ is isomorphic to the projection

$$\bigoplus_{i \in \mathbb{N}} H_*^{n-2i}(M) \to H_*^n(M)$$

and factors through the inclusion

$$H_*^{S^1}(M) \xrightarrow{\subset} S^{-1}H_*^{S^1}(M)$$

(where the localization is with respect to the multiplicative subset $S = \{u^k \mid k \in \mathbb{N}\}$). Thus, it suffices to prove that the following composition vanishes in degree $n$

$$H_*^{S^1}(\Lambda M) \to S^{-1}H_*^{S^1}(\Lambda M) \to S^{-1}H_*^{S^1}(M) \to H_*^n(M).$$

(4)
The sequence (4) is obviously functorial in $M$ so the classifying map $\Phi : M \to BG$ induces a commutative diagram

$$
\begin{array}{cccc}
H^*_S(\Lambda M) & \xrightarrow{\cdot} & S^{-1}H^*_S(\Lambda M) & \xrightarrow{\cdot} & S^{-1}H^*_S(M) & \xrightarrow{\cdot} & H^*(M) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
H^*_S(\Lambda BG) & \xrightarrow{\cdot} & S^{-1}H^*_S(\Lambda BG) & \xrightarrow{\cdot} & S^{-1}H^*_S(BG) & \xrightarrow{\cdot} & H^*(BG)
\end{array}
$$

The second vertical arrow is an isomorphism by Goodwillie’s theorem [9], Corollary V.3.3; therefore, the vanishing of the upper horizontal composition follows from vanishing of the homomorphism $H^n(BG) \to H^n(M)$.

**Corollary 6.2.** Let $M$ be a $\mathbb{Q}$-inessential manifold. Denote by $DM$ and $SM$ the disk bundle (respectively, the sphere bundle) associated with the tangent bundle of $M$. Then

$$i^*_*(\{DM\} \times \{pt\}) = 0 \in SH^S(M).$$

*Proof.* The following diagram is commutative

$$
\begin{array}{cccc}
H_*(DM,SM) & \xrightarrow{\cdot} & SH^S_*(DM) \\
\mathcal{F} \approx & & \mathcal{V} \approx \\
H_*(M) & \xrightarrow{\cdot} & H^S_*(\Lambda M)
\end{array}
$$

(Here, $\mathcal{F}$ is the Thom isomorphism, and $\mathcal{V}$ is the Viterbo’s isomorphism [18].) The bottom arrow sends the fundamental class of $M$ to zero by Proposition 6.1; therefore, the fundamental class of $DM$ is sent to zero by the top arrow.

Combining Irie’s theorem with Corollary 6.2, we deduce a result about the finiteness of HZ capacity for tangent bundles.

**Theorem 6.3.** Let $M$ be a $\mathbb{Q}$-inessential manifold. Then the $\pi_1$-sensitive HZ capacity of $DM$ is finite.

In the next Corollary, we gathered some immediate consequences of Theorem 6.3.

**Corollary 6.4.** Let $M$ be a manifold. The $\pi_1$-sensitive HZ capacity of $DM$ is finite, if one of the conditions below holds.

1. $\pi_1(M)$ is finite.
2. $\pi_1(M)$ is $\mathbb{Q}$-acyclic.
3. $\pi_1(M)$ is free and $\dim M > 1$.
4. $M \approx N_1 \times N_2$, where $N_1, N_2$ are oriented manifolds, $N_1$ is $\mathbb{Q}$-inessential and $\dim N_2 > 0$. 

□
7. On the non-oriented case

Our results can be generalized to the case of non-oriented manifolds. In this section, all manifolds are assumed to be closed and connected, but not necessarily oriented. For such a manifold $M$ there is a twisted fundamental class $[M] \in H_n(M, w_1)$, where $n = \dim M$ and $w_1$ is the local system of groups $\mathbb{Q}$ determined by the first Stiefel–Whitney class. Let $\overline{M} \to M$ be the twofold orientation covering of $M$. The transfer homomorphism sends the twisted homology $H_*(M, w_1)$ to the usual homology of $\overline{M}$, and we have

$$T([M]) = [\overline{M}].$$

(5)

Definition 7.1. A non-oriented closed connected manifold $M$ is called $\mathbb{Q}$-inessential, if the image $[M] \in H_*(BG, w_1)$ of its twisted fundamental class $[M]$ equals 0.

Proposition 7.2. Let $M$ be a non-oriented $\mathbb{Q}$-inessential manifold. Then $\overline{M}$ is $\mathbb{Q}$-inessential.

Proof. Let $G = \pi_1(M)$, and $H = \text{Ker}(w_1 : G \to \mathbb{Z}/2\mathbb{Z})$. We have a commutative diagram

$$
\begin{array}{ccc}
\overline{M} & \xrightarrow{\lambda_M} & BH \\
\downarrow p & & \downarrow p' \\
M & \xrightarrow{\lambda_M} & BG
\end{array}
$$

and applying the functoriality of the transfer homomorphism we deduce the proposition. \qed

To deduce the finiteness of the HZ capacities for non-oriented essential manifolds we will need the following lemma.

Lemma 7.3. Let $p : W \to V$ be a finite covering of symplectic manifolds. Then

$$c_{HZ}(W) \geq c_{HZ}(V), \quad c_{HZ}^0(W) \geq c_{HZ}^0(V).$$

Proof. Let $f$ be an HZ-admissible function on $V$. Then $f \circ p$ is an HZ-admissible function on $W$, and the first inequality follows. Similarly, if $g$ is an $HZ^0$-admissible function on $V$, the function $g \circ p$ is an HZ-admissible function on $W$ (indeed, if the Hamiltonian vector field $X_{g \circ p}$ has a contractible periodic orbit $\gamma$ of period $\leq 1$ on $W$, then $p \circ \gamma$ is a contractible periodic orbit for the Hamiltonian vector field $X_g$). The result follows. \qed

Corollary 7.4. Let $M$ be a non-oriented $\mathbb{Q}$-inessential manifold. Then $c_{HZ}^0(\mathcal{D}M)$ is finite. \qed

8. On symplectic invariants of unit disc bundles

Let $M$ be a compact closed Riemannian manifold. The tangent bundle $TM$ has a natural symplectic structure, and the symplectic invariants of the unit disc bundle $\mathcal{D}M$ are, therefore, invariants of the Riemannian manifold $M$. 
The next proposition follows immediately from the definition of the HZ capacity.

**Proposition 8.1.** Let $\gamma(M)$ denote the minimal length of a non-constant closed geodesic on $M$. Then $\gamma(M) \leq C_{HZ}(DM)$. In particular, we have the following lower bound for the HZ capacity of $DM$: $\rho(M) \leq C_{HZ}(M)$ where $\rho(M)$ is the injectivity radius of $M$. □

For a non-simply connected manifold $W$, there is a natural refinement of the definition of the Hofer–Zehnder capacities (see for example [15]).

**Definition 8.2.** Let $W$ be a symplectic manifold. Denote by $\langle \pi_1(W) \rangle$ the set of conjugacy classes of $\pi_1(W)$. Let $\Gamma \subset \langle \pi_1(W) \rangle$ be any non-empty subset of $\pi_1(W)$. A simple function $H$ is called $HZ$-admissible with respect to $\Gamma$ if the Hamiltonian vector field $X_H$ of $H$ has no non-constant periodic orbits $\gamma$ of period $\leq 1$ with $[\gamma] \in \Gamma$. Denote by $C_{HZ}^\Gamma(W)$ the upper bound of the set

$$\{\max Hq \mid H \text{ is } HZ-\text{admissible with respect to } \Gamma\}$$

We have obviously

$$C_{HZ}^\Gamma(W) \geq C_{HZ}(W)$$

for any $\Gamma$. The $\pi_1$-sensitive capacity considered above equals by definition $C_{HZ}^{\{1\}}(W)$.

**Conjecture 8.3.** Let $M$ be an inessential closed compact manifold. For any $\Gamma \subset \langle \pi_1(M) \rangle$ the capacity $C_{HZ}^\Gamma(DM)$ is finite.

The case of essential manifolds is not covered by the methods of the present paper. However, there is an obvious relation between the $HZ$ capacities and systolic constants of $M$. Namely, let $\langle \pi_1(M) \rangle'$ denote the set of all nontrivial conjugacy classes of the fundamental group. Recall that the lower bound of lengths of closed non-contractible curves in $M$ is denoted by $sys_1(M)$. We have obviously

$$sys_1(M) \leq C_{HZ}(DM, \langle \pi_1(M) \rangle').$$

In view of Gromov’s inequality (1), it is natural to suggest the following conjecture

**Conjecture 8.4.** Let $M^n$ be an essential closed Riemannian manifold of dimension $n$. Then

$$(\text{Vol} M)^{\frac{1}{n}} \leq D_n \cdot C_{HZ}(DM, \langle \pi_1(M) \rangle'),$$

where $D_n$ depends only on $n$.

**Acknowledgements**

The first author would like to thank the City University of Hong Kong and the Université de Nantes for the excellent research atmosphere during the visits in March 2015 and November 2015. The second author is grateful to the City University of Hong Kong and to the University of Augsburg for the warm hospitality during the visits (March 2015 and May 2015) that allowed us to
start and helped to continue this work. The authors thank the anonymous reviewer for helpful comments on the manuscript.

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Urs Frauenfelder
Universität Augsburg, Institut für Mathematik
Universitätsstrasse 14
86159 Augsburg
Germany
e-mail: urs.frauenfelder@math.uni-augsburg.de

Andrei Pajitnov
Laboratoire Mathématiques Jean Leray UMR 6629, Faculté des Sciences
Université de Nantes
2, rue de la Houssinière
44072 Nantes Cedex
France
e-mail: andrei.pajitnov@univ-nantes.fr