The revival-collapse phenomenon in the quadrature field components of the two-mode multiphoton Jaynes-Cummings model

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In this paper we consider a system consisting of a two-level atom in an excited state interacting with two modes of a radiation field prepared initially in \( l \)-photon coherent states. This system is described by two-mode multiphoton (i.e., \( k_1, k_2 \)) Jaynes-Cummings model (JCM). For this system we investigate the occurrence of the revival-collapse phenomenon (RCP) in the evolution of the single-mode, two-mode, sum and difference quadrature squeezing. We show that there is a class of states for which all these types of squeezing exhibit RCP similar to that involved in the corresponding atomic inversion. Also we show numerically that the single-mode squeezing of the first mode for \( (k_1, k_2) = (3, 1) \) provides RCP similar to that of the atomic inversion of the case \( (k_1, k_2) = (1, 1) \), however, sum and difference squeezing give partial information on that case. Moreover, we show that single-mode, two-mode and sum squeezing for the case \( (k_1, k_2) = (2, 2) \) provide information on the atomic inversion of the single-mode two-photon JCM. We derive the rescaled squeezing factors giving accurate information on the atomic inversion for all cases. The consequences of these results are that the homodyne and heterodyne detectors can be used to detect the RCP for the two-mode JCM.

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I. INTRODUCTION

The simplest model in quantum optics is the Jaynes-Cummings model (JCM), in which a radiation field interacts with a single two-level atom \[.\] This system has become experimentally realizable with the Rydberg atoms in high-Q microwave cavities (e.g., see \[.\]). In the framework of the rotating wave approximation many of interesting effects have been reported for JCM. The most important phenomenon is the revival-collapse phenomenon (RCP), which occurs in the evolution of the atomic inversion \( \langle \hat{\sigma}_z(T) \rangle \), i.e., instead of displaying steady Rabi oscillations in the case of a classical field coupled to the atom \[.\], there is an initial collapse of these oscillations followed by regular revivals that slowly become broader and eventually overlap \[.\]. This indicates that RCP is a pure quantum mechanical effect \[.\]. For more details about the RCP of the JCM the reader
can consult, e.g. [4].

The JCM has been extended to include multimode fields, e.g. [6, 7, 8], multilevel atoms [9] and multatom interactions [10]. Two-mode JCM (TJCM) has taken a considerable interest and studied from different points of view, e.g. [6, 8, 11, 12]. RCP of the TJCM is rather complicated compared to that of the single-mode JCM in the sense that the revival series is compact and each revival is followed by secondary revival. Furthermore, for strong intensities the locations of the revival patterns in the time interaction "domain" are independent of the intensities (see Fig. 1). Such behaviour has been partially explained in [11], however, an investigation for the occurrence of the secondary revivals is given in [6].

Quite recently a new technique is developed for discussing how within the single-mode multiphoton JCM the RCP of the atomic inversion of the standard (i.e. single-photon) JCM is manifested in the evolution of the quadrature squeezing of the field [13]. Two approaches have been adopted for such analysis, namely, natural and numerical-simulation approaches. For natural approach it has been shown that there is a class of states whose squeezing factors can directly include information on the corresponding atomic inversion. However, the numerical-simulation approach has been given to show that the evolution of the quadrature squeezing of the three-photon JCM reflects the RCP involved in the $\langle \hat{\sigma}_z(T) \rangle$ of the single-photon JCM for the same initial field state. Moreover, we have deduced a general form for the higher-order squeezing factor, which can give information on the atomic inversion of the single-photon JCM [14] and two-photon JCM [15].

In this paper we apply the technique given in [13] to TJCM for investigating the occurrence of the RCP in the quadrature squeezing and how can be connected with the atomic inversion. For convenience we assume that the radiation fields are initially prepared in $l$-photon coherent states [16, 17, 18] and the atom is in an excited atomic state. Needless to say that the situation for the TJCM is more complicated than that of the single-mode JCM. For instance, there are different types of quadrature squeezing such as single-mode, two-mode, sum and difference squeezing. Moreover, the strong entanglement between the two bosonic systems over the atomic system making the investigation is rather complicated. In spite of these difficulties we have obtained many interesting results, e.g. for all types of squeezing there is a class of states for which squeezing factors can directly give the corresponding atomic inversion. Additionally, using numerical technique we have shown that when $k_1 + k_2 = 4$ (cf. [11]) the $Y$-quadrature squeezing factor of the particular types can provide RCP similar to that exhibited in the evolution of the atomic inversion of the standard (i.e., $k_1 = k_2 = 1$) TJCM or single-mode two-photon JCM based on the values of $k_j$. We have to stress that the nonclassical squeezing for TJCM has been studied by several authors,
e.g., see [19], and it will not be considered in the present paper. Finally, the results given here and in [13, 14, 15] show that the RCP occurred in \( \langle \hat{\sigma}_z(T) \rangle \) can be detected using techniques similar to those used for quadrature squeezing, e.g. homodyne detector [20], nonlinear homodyne detector [21] and multiport homodyne detector [22]. It is worth mentioning that in cavity QED, the homodyne detector technique has been applied to the single Rydberg atom and one-photon field for studying the field phase evolution of the regular JCM [23]. Quite recently similar setup is given for induced measurement and quantum computation with atoms in optical cavities [24]. Moreover, the progress in both of the trapped ions [25] and micromaser [26] is promising to produce the phenomena discussed in the paper. We conclude this part by drawing the attention to that the first experimentally observed squeezed states are of the two-mode type [27].

The paper is prepared in the following order. In section 2 we give the basic relations and equations including the model and the definition of squeezing. In sections 3–5 we investigate single-mode squeezing, two-mode squeezing and sum-difference squeezing, respectively. In section 6 the main conclusions are summarized.

II. BASIC RELATIONS AND EQUATIONS

In this section we give the basic relations and equations, which will be used throughout the paper. Precisely, we write down the Hamiltonian of the system under consideration, its wave function and the definition of quadrature squeezing.

The Hamiltonian controlling the TJCM in the rotating wave approximation is [12]:

\[
\frac{\hat{H}}{\hbar} = \omega_1 \hat{a}_1^\dagger \hat{a}_1 + \omega_2 \hat{a}_2^\dagger \hat{a}_2 + \frac{1}{2} \omega_a \hat{\sigma}_z + g (\hat{a}_1^k \hat{a}_2^{k^2} \hat{\sigma}_+ + \hat{a}_1^{k^1} \hat{a}_2^{k^2} \hat{\sigma}_-),
\]

where \( \hat{\sigma}_\pm \) and \( \hat{\sigma}_z \) are the Pauli spin operators; \( \omega_j, (j = 1, 2) \) and \( \omega_a \) are the frequencies of the cavity modes \( \hat{a}_j \) and the atomic frequency, respectively; \( g \) is the atom-field coupling constant and \( k_j \) is the transition parameter of the \( j \)th mode. The derivation of the Hamiltonian (1) from the first principle is given in [28].

We restrict the investigation to the exact resonance case \( k_1 \omega_1 + k_2 \omega_1 = \omega_a \). For evaluating the dynamical state of (1) we define two operators \( \hat{F}_1 \) and \( \hat{F}_2 \) as

\[
\hat{F}_1 = \omega_1 \hat{a}_1^\dagger \hat{a}_1 + \omega_2 \hat{a}_2^\dagger \hat{a}_2 + \frac{1}{2} \omega_a \hat{\sigma}_z, \quad \hat{F}_2 = g (\hat{a}_1^{k_1} \hat{a}_2^{k_2} \hat{\sigma}_+ + \hat{a}_1^{k_1} \hat{a}_2^{k_2} \hat{\sigma}_-).
\]

It is easy to prove that \( \hat{F}_1 \) and \( \hat{F}_2 \) are constants of motion. This fact leads to that the evolution of the mean-photon number of the modes and the atomic inversion of the system include information
on each other. In the interaction picture the unitary evolution operator of the Hamiltonian takes the form

\[ \hat{U}_I(T, 0) = \exp(-i \frac{T \hat{F}}{g}) \]

\[ = \cos(T \hat{D}) - i \frac{\sin(T \hat{D})}{g \hat{D}} \hat{F}, \quad (3) \]

where

\[ T = gt, \quad \hat{D}^2 = \hat{a}_1^{k_1} \hat{a}_2^{k_2} \hat{a}_1^{\dagger k_1} \hat{a}_2^{\dagger k_2} \hat{\sigma}_+ \hat{\sigma} - + \hat{a}_1^{\dagger k_1} \hat{a}_2^{\dagger k_2} \hat{a}_1^{k_1} \hat{a}_2^{k_2} \hat{\sigma}_- \hat{\sigma} +. \quad (4) \]

For the sake of generalization we consider the \( j \)th mode is initially prepared in the \( l \)-photon coherent states having the form

\[ |\psi_j(0)\rangle = \sum_{n=0}^{\infty} C_n^{(j)} |l_j n\rangle, \quad C_n^{(j)} = \exp(-\frac{1}{2} |\alpha_j|^2) \frac{\alpha_j^n}{\sqrt{n!}}. \quad (5) \]

where \( l_j \) are parameters their values will be specified in the text. Also throughout the paper we consider \( \alpha_j \) are real. States (5) can be obtained from \( l \)th harmonic generation using Brandt-Greenberg operators. We proceed by considering that the atom is initially in the excited state \( |+\rangle \). Therefore, the total initial state of the system is

\[ |\Psi(0)\rangle = |\psi_1(0)\rangle \bigotimes |\psi_2(0)\rangle \bigotimes |+. \quad (6) \]

From (3) and (6) the dynamical state vector of the system can be evaluated as

\[ |\Psi(T)\rangle = \sum_{n,m=0}^{\infty} C_{n,m} \left[ \cos(T \Lambda_{n,m}) |+, l_1 n, l_2 m\rangle - i \sin(T \Lambda_{n,m}) |-, l_1 n + k_1, l_2 m + k_2\rangle \right], \quad (7) \]

where \(-\rangle\) denotes ground atomic state, \( C_{n,m} = C_n^{(1)} C_m^{(2)} \) and

\[ \Lambda_{n,m} = \sqrt{\frac{(l_1 n + k_1)!(l_2 m + k_2)!}{(l_1 n)!(l_2 m)!}}. \quad (8) \]

The atomic inversion associated with (7) is

\[ \langle \hat{\sigma}_z(T) \rangle = \sum_{n,m} C_{n,m}^2 \cos(2T \Lambda_{n,m}). \quad (9) \]
FIG. 1: The atomic inversion $\langle \hat{\sigma}_z(T) \rangle$ against the scaled time $T$ when the optical cavity modes are initially prepared in the coherent states for $(k_1, k_2, \alpha_1, \alpha_2) = (1, 1, 5, 5)$.

To investigate the evolution of quadrature squeezing we evaluate the general form for the different moments of the $\hat{a}_j^\dagger$ and $\hat{a}_j$ for (7) when $l_1 = l_2 = 1$ as

$$
\langle \hat{a}_1^{s_{1}'}(T)\hat{a}_1^{s_{2}'}(T)\hat{a}_2^{s_{3}'}(T)\hat{a}_2^{s_{4}'}(T) \rangle = \sum_{n,m=0}^{\infty} C_{n+s_1',m+s_2'} C_{n+s_2',m+s_4'} \left[ \cos(TA_{n+s_1',m+s_2'}) \cos(TA_{n+s_2',m+s_4'}) \right.
$$

$$
\times \sqrt{(n+s_1'!)(n+s_2'!)(m+s_2'!)(m+s_4'!)} \left. \sqrt{(n+k_1+1)!/(n+k_1)!} \sin(TA_{n+s_1',m+s_2'}) \sin(TA_{n+s_2',m+s_4'}) \sqrt{(n+k_1+1)!/(n+k_1)!} \right]
$$

(10)

where $s_j', j = 1, 2, 3, 4$ are positive integers. Also we define two quadratures $\hat{X}$ and $\hat{Y}$, which denote the real (electric) and imaginary (magnetic) parts of the radiation field. Assuming that
these quadratures satisfy the following commutation rule:

\[ [\hat{X}, \hat{Y}] = \frac{i \hat{d}}{2}, \tag{11} \]

where \( \hat{d} \) may be c-number or operator. The uncertainty relation related to the commutation rule (11) is

\[ \langle (\Delta \hat{X})^2 \rangle \langle (\Delta \hat{Y})^2 \rangle \geq \frac{\langle |\hat{d}|^2 \rangle}{16}, \tag{12} \]

where \( \langle (\Delta \hat{X})^2 \rangle = \langle \hat{X}^2 \rangle - \langle \hat{X} \rangle^2 \) and similar form can be given for \( \langle (\Delta \hat{Y})^2 \rangle \). The system is said to be squeezed in the \( X \)-quadrature if

\[ S(T) = 2\langle (\Delta \hat{X}(T))^2 \rangle - \frac{1}{2}\langle |\hat{d}| \rangle \leq 0. \tag{13} \]

The equality sign in (13) holds for minimum-uncertainty states. Similar definition can be given for the \( Y \)-quadrature (defining a \( Q \)-factor). As we mentioned in the Introduction we study the evolution of four types of quadrature squeezing: single-mode, two-mode, sum and difference squeezing. The object of such study is to follow the possible occurrence of the RCP in the evolution of the squeezing factors and the conditions required for such occurrence. Also we try to find which type of squeezing factors can fit well information on the evolution of the atomic inversion. These issues will be discussed in the following sections.

III. SINGLE-MODE SQUEEZING

In this section we study the occurrence of the RCP in the evolution of the single-mode squeezing factors for TJCM. The single-mode squeezing factors \( S_j(t) \) and \( Q_j(t) \) for the \( j \)th-mode when the quadratures \( \hat{X} \) and \( \hat{Y} \) are defined in the standard form, can be expressed as

\[ S_j(T) = \langle \hat{a}_j^\dagger(T)\hat{a}_j(T) \rangle + \text{Re}\langle \hat{a}_j^2(T) \rangle - 2\left(\text{Re}\langle \hat{a}_j(T) \rangle \right)^2, \tag{14} \]

\[ Q_j(T) = \langle \hat{a}_j^\dagger(T)\hat{a}_j(T) \rangle - \text{Re}\langle \hat{a}_j^2(T) \rangle - 2\left(\text{Im}\langle \hat{a}_j(T) \rangle \right)^2. \]

In the following parts we investigate the natural and numerical approaches in a greater details.

A. Natural approach

In this part and throughout the paper the natural approach is given for the standard TJCM. This approach is based on the fact that \( \hat{F}_1 \) is a constant of motion and hence the quantities
FIG. 2: The single-mode squeezing factors against the scaled time $T$ when the modes are initially prepared in the coherent light with $(\alpha_1, \alpha_2) = (5, 5)$ and for $(k_1, k_2) = (3, 1)$ (a), $(1, 3)$ (b) and $(2, 2)$ (c).

$\langle \hat{\sigma}_z(T) \rangle$, $\langle \hat{a}_1(T)\hat{a}_1(T) \rangle$ and $\langle \hat{a}_2(T)\hat{a}_2(T) \rangle$ can carry information on each other. Thus the squeezing factors (14) can give information on $\langle \hat{\sigma}_z(T) \rangle$ when

$$\langle \hat{a}_j(T) \rangle = 0, \quad \langle \hat{a}_2^2(T) \rangle = 0$$

simultaneously. This situation can be established when the $j$th mode is initially prepared in three-photon states [29], four-photon states [30] and so on. Also for particular values of the parameter $l$ the $l$-coherent state [5] can fulfill conditions (15). For such type of initial states one can easily prove that

$$\langle \hat{\sigma}_z(T) \rangle = 2\langle \hat{a}_j(0)\hat{a}_j(0) \rangle + 1 - 2S_j(T).$$

Expression (16) shows that the atomic inversion can be readout from the quadrature squeezing.

B. Numerical simulation

In this part and throughout the paper numerical-simulation approach is applied to TJCM when $k_1 + k_2 > 2$ and the modes are initially prepared in coherent light, i.e. $l_1 = l_2 = 1$ in [5]. Now the object here is to discuss the possibility of obtaining RCP in the evolution of the single-mode squeezing factors similar to that of the atomic inversion of the standard TJCM, i.e. $\langle \hat{\sigma}_z(T) \rangle_{k_1=k_2=1}$. The procedures related to this technique are given in [7] and we briefly explain them for the first
mode. From \textbf{14} RCP may occur in $S_j(T)$ (or $Q_j(T)$) only when the evolution of the $\text{Re}(\hat{a}_j(T))$ (or $\text{Im}(\hat{a}_j(T))$) are close to zero (i.e. steady state) since these quantities are squared. Thus when the probability amplitudes $C_{n,m}$ are real $\text{Im}(\hat{a}_j(T)) = 0$ and consequently the RCP can likely occur in the evolution of $Q_j(T)$. Additionally, when $k_1 + k_2 > 2$, $\langle \hat{a}_j(T) \hat{a}_j(T) \rangle$ exhibits chaotic behaviour.

From numerical data for this case we can consider $\langle \hat{a}_j(T) \hat{a}_j(T) \rangle \approx \langle \hat{a}_j(0) \hat{a}_j(0) \rangle$. This means that the occurrence of the RCP (if it is so) in $Q_j(T)$ is related to the quantity $\text{Re}(\hat{a}_j^2(T))$. Consequently, we compare the form of $\text{Re}(\hat{a}_j^2(T))$ with that of $\langle \hat{\sigma}_z(T) \rangle_{k_1=k_2=1}$. Now we give a closer look at $\langle \hat{a}_j^2(T) \rangle$, which from \textbf{10} has the form:

$$
\langle \hat{a}_j^2(T) \rangle = \alpha_j^2 \sum_{n,m=0}^{\infty} P(n)P(m) \left[ \cos(T\Lambda_{n+2,m}) \cos(T\Lambda_{n,m}) + \sqrt{\frac{(n+k_1+1)(n+k_2+2)}{(n+1)(m+2)}} \sin(T\Lambda_{n+2,m}) \sin(T\Lambda_{n,m}) \right].
$$

(17)

where $P(n) = (C_n^{(1)})^2$ is the photon-number distribution for the coherent light. In the strong-intensity regime, i.e. $\tilde{n}_j = \langle \hat{a}_j^\dagger(0) \hat{a}_j(0) \rangle = |\alpha_j|^2 >> 1$, and finite values of the transition parameters $k_j$ we can apply the harmonic approximation technique \textbf{4, 11}. This technique is based on the fact that the photon-number distribution of the coherent light is Poissonian with a sharp peak at $n = \tilde{n}$ and hence the terms which contribute effectively to the summation in (17) are those for which $n \approx \tilde{n}$. As a result of this fact the square root in the second line of (17) tends to unity and (17) reduces to

$$
\langle \hat{a}_j^2(T) \rangle \approx \tilde{n}_1 \sum_{n,m=0}^{\infty} P(n)P(m) \cos[T(\Lambda_{n+2,m} - \Lambda_{n,m})].
$$

(18)

Regardless of the prefactor $\tilde{n}_1$ in \textbf{18}, the comparison between \textbf{11} of the case $(k_j,l_j) = (1,1)$ and \textbf{18} leads to that both expressions exhibit quite similar dynamical behaviour only when the arguments of cosines in the two expressions are comparable. Therefore, we seek the proportionality factor $\mu_1$, say, which can be evaluated from the following expression

$$
\mu_1 = \frac{\Lambda_{n+2,m} - \Lambda_{n,m}}{2\sqrt{(n+1)(m+1)}}.
$$

(19)
FIG. 3: The modified squeezing factor for the cases \((k_1, k_2) = (3, 1)\) (a) and \((k_1, k_2) = (2, 2)\) (b) when the modes are initially prepared in the coherent light with \(\alpha_1 = \alpha_2 = 5\).

Expression (19) can be re-expressed as

\[
\mu_1 = \sqrt{\frac{(m+k_2)(m+k_1)}{(n+1)(m+1)}} \left\{ \frac{(2n+3)k_1+k_2^2}{2\sqrt{(n+2)(n+1)[\sqrt{(n+k_1+2)(n+k_1+1)}+\sqrt{(n+2)(n+1)}]}} \right\} \\
= \frac{\frac{k_1-3}{2} \frac{k_2-1}{2} \prod_{j=0}^{k_1} (1 + \frac{k_1-j}{n})}{\prod_{j'=0}^{k_2} (1 + \frac{k_2-j'}{n})} \left\{ \frac{k_2}{m} \right\} \left\{ \frac{\frac{2}{3} k_1 + \frac{k_2^2}{2}}{(1+\frac{3}{n})\sqrt{(1+\frac{3}{n})(1+\frac{3}{2n})}(1+\frac{1}{n})} + \sqrt{(1+\frac{3}{n})(1+\frac{3}{n})} \right\}^{\frac{1}{2}}
\]

(20)

In the framework of the harmonic approximation (i.e. \(\epsilon/\bar{n} \to 0\) where \(\epsilon\) is an arbitrary finite number) the second part of (20) reduces to

\[
\mu_1 \approx \frac{k_1}{2} \frac{k_1-3}{2} \frac{k_2-1}{2} \frac{k_2}{m}.
\]

(21)

Expression (21) shows that there are three cases can provide RCP in the evolution of the \(Q_1(T)\), which are: \((k_1, k_2, \mu_1) = (3, 1, 3/2)\) and \((k_1, k_2, \mu_1) = (1, 3, 1/2), (2, 2, 1)\). For the latter cases the values of \(\bar{n}\) and \(\bar{m}\) have to be comparable. Information about these cases has been shown in Figs. 2(a)–(c) for given values of the parameters. It is obvious that we have three different shapes of the RCP. Form Fig. 2(a) revivals and secondary revivals are remarkable having shapes similar to those of the \(\langle \hat{\sigma}_z(T) \rangle_{k_1=k_2=1}\) (compare to Fig. 1). Fig. 2(b) includes revivals and collapses but their forms are different from those of the \(\langle \hat{\sigma}_z(T) \rangle_{k_1=k_2=1}\). Nevertheless, in Fig. 2(c) the RCP is
systematic, i.e. revivals are compact and occur periodically with period $\pi$. Actually, this behaviour is quite typical with the evolution of the atomic inversion8 of the single-mode two-photon JCM, i.e. $\langle \hat{\sigma}_z(T) \rangle_{k_1=2,k_2=0}$. This can be analytically realized by applying the harmonic approximation to the argument of the cosine in (18) for the case $(k_1,k_2) = (2,2)$ as

$$\Lambda_{n+2,m} - \Lambda_{n,m} = \sqrt{(m+1)(m+2)} \left[ \sqrt{(n+3)(n+4)} - \sqrt{(n+1)(n+2)} \right]$$

$$\simeq \sqrt{(m+1)(m+2)} \left[ \bar{n} + \frac{7}{2} - \bar{n} - \frac{3}{2} \right]$$

$$= 2\sqrt{(m+1)(m+2)}. \tag{22}$$

The last line in (22) is typical the argument of the cosine of the $\langle \hat{\sigma}_z(T) \rangle_{k_1=2,k_2=0}$, e.g. see equation (12) in [31] for $\chi = 0$.

Now we deduce the rescaled squeezing factors for the cases $(k_1,k_2) = (3,1)$ and $(2,2)$, which can give $\langle \hat{\sigma}_z(T) \rangle_{k_1=k_2=1}$ and $\langle \hat{\sigma}_z(T) \rangle_{k_1=2,k_2=0}$, respectively. From Fig. 2(a) and the discussion given above we can write the rescaled squeezing factor for $(k_1,k_2) = (3,1)$ as

$$V_1(T) = \frac{\langle \hat{n}_1(0) \rangle - Q_1(\frac{2}{3}T)}{\langle \hat{n}_1(0) \rangle}. \tag{23}$$

Similarly the rescaled squeezing factor for the case $(k_1,k_2) = (2,2)$ is

$$V'_1(T) = \frac{\langle \hat{n}_1(0) \rangle - Q_1(T)}{\langle \hat{n}_1(0) \rangle}. \tag{24}$$

Expression (23) and (24) have been depicted in Figs. 3(a) and (b) for the values of the interaction parameters as those given for Figs. 2(a) and (b), respectively. Comparison between Fig. 1 and Fig. 3(a) demonstrates our conclusion. Also Fig. 3(b) provides completely typical shape as that of $\langle \hat{\sigma}_z(T) \rangle_{k_1=2,k_2=0}$ (see Fig. 2 in [32]).

### IV. TWO-MODE SQUEEZING

In this section, we use procedures similar to those given in section 3 to investigate the RCP in the evolution of the two-mode squeezing for TJCM. Starting with the two-mode squeezing factors,
which can be expressed as

\[ S_2(T) = \text{Re}\langle \hat{a}_2^\dagger(T)\hat{a}_1(T) + \hat{a}_1(T)\hat{a}_2(T) \rangle - 2\text{Re}\langle \hat{a}_1(T) \rangle \text{Re}\langle \hat{a}_2(T) \rangle \]

\[ + \sum_{j=1}^{2} \left[ \text{Re}\langle \hat{a}_j^\dagger(T)\hat{a}_j(T) \rangle + \text{Re}\langle \hat{a}_j^2(T) \rangle - 2 \left( \text{Re}\langle \hat{a}_j(T) \rangle \right) \right]^2, \tag{25} \]

\[ Q_2(T) = \text{Re}\langle \hat{a}_2^\dagger(T)\hat{a}_1(T) - \hat{a}_1(T)\hat{a}_2(T) \rangle - 2\text{Im}\langle \hat{a}_1(T) \rangle \text{Im}\langle \hat{a}_2(T) \rangle \]

\[ + \sum_{j=1}^{2} \left[ \langle \hat{a}_j^\dagger(T)\hat{a}_j(T) \rangle - \text{Re}\langle \hat{a}_j^2(T) \rangle - 2 \left( \text{Im}\langle \hat{a}_j(T) \rangle \right)^2 \right]. \]

We discuss the natural and numerical approaches for two-mode squeezing in the following parts.

A. Natural phenomenon

Explanations similar to those given in subsection 3.1 the two-mode squeezing factors can give direct information on the corresponding \( \langle \hat{\sigma}_z(T) \rangle \) for initial states, which satisfy simultaneously the following conditions:

\[ \langle \hat{a}_j(T) \rangle = 0, \quad \langle \hat{a}_j^2(T) \rangle = 0, \quad j = 1, 2 \tag{26} \]

We should stress that (26) requires the two modes to be initially prepared in such type of states. This is different from (15) of the single-mode squeezing, which requires the mode under consideration only to be initially in such states. In this case the two squeezing factors (25) are typical and can give \( \langle \hat{\sigma}_z(T) \rangle \). Similar to the single-mode squeezing case this can be verified when the two modes are initially prepared in the \( l \)-photon coherent states with \( l = 3, 4, \ldots \), etc. For such states one can easily prove that

\[ \langle \hat{\sigma}_z(T) \rangle = \langle \hat{a}_1^\dagger(0)\hat{a}_1(0) \rangle + \langle \hat{a}_2^\dagger(0)\hat{a}_2(0) \rangle + 1 - S_2(T). \tag{27} \]

B. Numerical simulation

Discussion similar to that given in subsection 3.2 one can easily prove that RCP can occur in \( Q_2(T) \) when \( (k_1, k_2) = (3, 1), (1, 3) \) and \( (2, 2) \). Also one can easily realize for the case \( (k_1, k_2) = (1, 3) \) that \( Q_2(T) \) exhibits RCP, which is the combination from those shown in Figs. 2(a) and (b), i.e. RCP is different from that of the \( \langle \hat{\sigma}_z(T) \rangle_{k_1=1, k_2=1} \). Nevertheless, for the case \( (k_1, k_2) = (2, 2), \)
$Q_2(T)$ can give information on the $\langle \hat{\sigma}_z(T) \rangle_{k_1=2,k_2=0}$. The rescaled squeezing factor, which is typical $\langle \hat{\sigma}_z(T) \rangle_{k_1=2,k_2=0}$, can be evaluated as
\[
V_2'(T) = \frac{\langle \hat{n}_1(0) \rangle + \langle \hat{n}_2(0) \rangle - Q_2(T)}{\langle \hat{n}_1(0) \rangle + \langle \hat{n}_2(0) \rangle}.
\] (28)

V. SUM AND DIFFERENCE SQUEEZING

In this section we investigate the occurrence of the RCP in the evolution of the sum and difference squeezing factors [33]. In these factors the intermode correlation is involved in the quadrature squeezing. We have noted for numerical-simulation approach that the technique given in section 3 is partially working for sum and difference squeezing. More illustratively, it can give the exact values for the transition parameters $k_j$ whose squeezing factors exhibit RCP but it fails to provide the correct rescaled squeezing factor. Nevertheless, this difficulty can be numerically treated. We have noted that the sum and difference squeezing give only information on the occurrence of the revivals (not secondary revivals) in $\langle \hat{\sigma}_z(T) \rangle_{k_1=1,k_2=1}$. Moreover, sum squeezing can provide $\langle \hat{\sigma}_z(T) \rangle_{k_1=2,k_2=0}$, however, difference squeezing fails. We discuss all these results in the following. For the sum squeezing we have [33]

\[
\hat{X} = \frac{1}{2}[\hat{a}_1 \hat{a}_2 + \hat{a}_1 \hat{a}_2^\dagger], \quad \hat{Y} = \frac{i}{2}[\hat{a}_1 \hat{a}_2^\dagger - \hat{a}_1 \hat{a}_2], \quad \hat{C} = \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + 1.
\] (29)

Therefore, sum-squeezing factors can be expressed as
\[
S_3(T) = \text{Re}(\hat{a}_1(T)\hat{a}_2(T)) + \langle \hat{a}_1(T)\hat{a}_1(T)\hat{a}_2(T)\hat{a}_2(T) \rangle - 2\left(\text{Re}(\hat{a}_1(T)\hat{a}_2(T))\right)^2,
\] (30)
\[
Q_3(T) = \langle \hat{a}_1^\dagger(T)\hat{a}_1(T)\hat{a}_2^\dagger(T)\hat{a}_2(T) \rangle - \text{Re}(\langle \hat{a}_1^2(T)\hat{a}_2^2(T) \rangle) - 2\left(\text{Im}(\hat{a}_1(T)\hat{a}_2(T))\right)^2.
\]

For difference squeezing the quadratures $\hat{X}$ and $\hat{Y}$ can be obtain from those in (29) by using the transformation $\hat{a}_1 \leftrightarrow \hat{a}_1^\dagger$ and consequently $\hat{C} = \hat{a}_1^\dagger \hat{a}_2 - \hat{a}_1^\dagger \hat{a}_1$. For the sake of simplicity we assume that the two modes are initially prepared in states having the same photon-number distribution with $\alpha_1 = \alpha_2$. This leads to that $\langle \hat{a}_2(T)\hat{a}_2(T) \rangle = \langle \hat{a}_1^\dagger(T)\hat{a}_1(T) \rangle$, i.e. $|\langle \hat{C}(T) \rangle| = 0$. Under these conditions the difference squeezing factors take the forms
\[
S_4(T) = \text{Re}(\langle \hat{a}_1^2(T)\hat{a}_2^2(T) \rangle) + \langle \hat{a}_1^\dagger(T)\hat{a}_1(T)\hat{a}_2^\dagger(T)\hat{a}_2(T) \rangle + \langle \hat{a}_1^\dagger(T)\hat{a}_1(T) \rangle - 2\left(\text{Re}(\langle \hat{a}_1^2(T)\hat{a}_2^2(T) \rangle)\right)^2,
\] (31)
\[
Q_4(T) = \langle \hat{a}_1^\dagger(T)\hat{a}_1(T)\hat{a}_2^\dagger(T)\hat{a}_2(T) \rangle + \langle \hat{a}_1^\dagger(T)\hat{a}_1(T) \rangle - \text{Re}(\langle \hat{a}_1^2(T)\hat{a}_2^2(T) \rangle) - 2\left(\text{Im}(\langle \hat{a}_1(T)\hat{a}_2(T) \rangle)\right)^2.
\]
FIG. 4: The rescaled squeezing factor $V_3(T)$ against the scaled time $T$ when the optical cavity modes are initially prepared in the three-photon coherent states and $(k_1, k_2, \alpha_1, \alpha_2) = (1, 1, 5, 5)$.

A. Natural phenomenon

Now we seek states that evolve with the TJCM causing the evolution of the $\langle \hat{a}^2_1(T)\hat{a}^2_2(T) \rangle$ and $\langle \hat{a}_1(T)\hat{a}_2(T) \rangle$ close to zero. This can occur if one of the two modes at least is initially in, e.g., the three-photon or four-photon states, (cf. (5)). For such states expressions (30) of sum squeezing reduce to

$$Q_3(T) = S_3(T) = \langle \hat{a}^\dagger_1(T)\hat{a}_1(T)\hat{a}^\dagger_2(T)\hat{a}_2(T) \rangle.$$  \hfill (32)
FIG. 5: The sum-squeezing factor against the scaled time $T$ when the optical cavity modes are initially prepared in the coherent states with $\alpha_1 = \alpha_2 = 5$ for $(k_1, k_2) = (3, 1)$ (a) and $(2, 2)$ (b).

In the framework of harmonic approximation the rescaled squeezing factor associated with $(32)$, which can provide the corresponding atomic inversion, is

$$V_3(T) \simeq \frac{2\langle \hat{n}_1(0) \rangle \langle \hat{n}_2(0) \rangle + \langle \hat{n}_1(0) \rangle + \langle \hat{n}_2(0) \rangle + 1 - 2S_3(T)}{\langle \hat{n}_1(0) \rangle + \langle \hat{n}_1(0) \rangle + 1}. \quad (33)$$

In Fig. 4 we have plotted $(33)$ when the modes are initially prepared in the three-photon coherent states. Actually, we have found that $V_3(T) = \langle \hat{\sigma}_z(T) \rangle$. From Fig. 4 the revivals and secondary revivals are remarkable. Additionally, the revival times of this case are three times smaller than those of the initial coherent light since we are dealing with three-photon states (compare Fig. 1 and Fig. 4).

Similarly for the difference squeezing $(31)$ we can obtain

$$\langle \hat{\sigma}_z(T) \rangle \simeq \frac{(\langle \hat{n}_1(0) \rangle + 1)^2 - Q_4(T)}{\langle \hat{n}_1(0) \rangle + 1}. \quad (34)$$

B. Numerical simulation

Similar arguments as those given in subsection 3.2 lead to that if sum-squeezing factor exhibits RCP the quantity $\text{Re}\langle \hat{a}_1^2(T)\hat{a}_2^2(T) \rangle$ is responsible for this. From $(10)$ and the harmonic approximation technique we arrive at

$$\langle \hat{a}_1^2(T)\hat{a}_2^2(T) \rangle \simeq \alpha_1^2 \alpha_2^2 \sum_{n,m=0}^{\infty} C_{n,m}^2 \cos[T(\Lambda_{n+2,m+2} - \Lambda_{n,m})]. \quad (35)$$
Now we seek the proportionality factor, which can be obtained from the following

$$\mu_2 = \frac{\Lambda_{n+2,m+2} - \Lambda_{n,m}}{2\sqrt{(n+1)(m+1)}}.$$  \hspace{1cm} (36)

After lengthy calculation but straightforward (36) reduces to

$$\mu_2 \simeq \frac{1}{4} \left( 2k_2 \hat{n}_1^2 \hat{n}_2^{-3} + k_2^2 \hat{n}_1^2 \hat{n}_2^{-5} + 2k_1 \hat{n}_1 \hat{n}_2^{-1} + 4k_1k_2 \hat{n}_1^2 \hat{n}_2^{-3} \right).$$ \hspace{1cm} (37)

Similar to the single-mode squeezing case when $\hat{n}_1 \simeq \hat{n}_2$ there are three cases, which can provide RCP in the evolution of the $Q_3(T)$, namely, $(k_1, k_2) = (3, 1), (1, 3)$ and $(2, 2)$. For these cases the proportionality factor is $\mu_2 = 2$. Actually, this factor cannot give the correct rescaled squeezing factor. This fact can be realized from Figs. 5(a) and (b), in which we have plotted the sum squeezing factors for $(k_1, k_2) = (3, 1)$ and $(k_1, k_2) = (2, 2)$, respectively. From these figures one can see that for the case $(k_1, k_2) = (3, 1)$ sum squeezing can give in principle information on $\langle \hat{\sigma}_z(T) \rangle_{k_1=k_2=1}$, however, for $(k_1, k_2) = (2, 2)$ it can provide the evolution of the $\langle \hat{\sigma}_z(T) \rangle_{k_1=2,k_2=0}$. From Figs. 5 and expression (30) the rescaled squeezing factor is

$$V_4(T) = \frac{\langle \hat{n}_1(0) \rangle \langle \hat{n}_2(0) \rangle - Q_3(T)}{\langle \hat{n}_1(0) \rangle \langle \hat{n}_2(0) \rangle}.$$ \hspace{1cm} (38)

Expression (38) gives $\langle \hat{\sigma}_z(T) \rangle_{k_1=k_2=1}$ and $\langle \hat{\sigma}_z(T) \rangle_{k_1=2,k_2=0}$ for $(k_1, k_2) = (3, 1)$ and $(2, 2)$, respectively.

On the other hand, for the difference squeezing we found that the RCP can be remarked in the evolution of the $Q_4(T)$ when $(k_1, k_2) = (3, 1), (1, 3)$, however, for $(k_1, k_2) = (2, 2)$ the technique fails. To demonstrate these cases we have plotted Figs. 6(a) and (b) for $Q_4(T)$ when $(k_1, k_2) = (3, 1)$ and $(2, 2)$, respectively. From Fig. 6(a) RCP is established but it is completely different from that of the $\langle \hat{\sigma}_z(T) \rangle_{k_1=k_2=1}$, however, Fig. 6(b) exhibits periodically inverted peaks, i.e. it does not exhibit RCP. Now from Fig. 6(a) and expression (31), the rescaled squeezing factor is

$$V_5(T) = \frac{\langle \hat{n}_1(0) \rangle (\langle \hat{n}_1(0) \rangle + 1) - Q_4(T)}{\langle \hat{n}_1(0) \rangle^2}.$$ \hspace{1cm} (39)

We have numerically checked (38) and (39) for the case $(k_1, k_2) = (3, 1)$ and found that they give typical behaviour as that of the $\langle \hat{\sigma}_z(T) \rangle_{k_1=k_2=1}$ except that the secondary revivals are absent. Additionally, the widths of the revival patterns of (39) are a little bit greater than those of the $\langle \hat{\sigma}_z(T) \rangle_{k_1=k_2=1}$. 
VI. CONCLUSIONS

In this paper we have discussed the possibility of including the squeezing factors of the two-mode multiphoton JCM information on the atomic inversion of the standard TJCM. In contrast to the single-mode JCM [13], we have various types of quadrature squeezing, namely, single-mode, two-mode, sum and difference squeezing. Two approaches have been applied for all these types, which are natural phenomenon and numerical simulation. Natural approach has been devoted to the standard TJCM and found that there is a class of states their squeezing factors provide the corresponding atomic inversion. For the numerical-simulation approach we have shown that for specific value of the transition parameters, in particular, \( k_1 + k_2 = 4 \) the \( Y \)-quadrature squeezing factor can provide RCP similar to that associated with the \( \langle \hat{\sigma}_z(T) \rangle_{k_1=k_2=1} \) or \( \langle \hat{\sigma}_z(T) \rangle_{k_1=2,k_2=0} \) based on the values of \( k_j \). Specifically, for \( (k_1,k_2) = (3,1) \) single-mode squeezing factor can give information on \( \langle \hat{\sigma}_z(T) \rangle_{k_1=k_2=1} \), however, sum and difference squeezing factors give partial information. On the other hand, for \( (k_1,k_2) = (2,2) \) single-mode, two-mode and sum squeezing factors give information on the \( \langle \hat{\sigma}_z(T) \rangle_{k_1=2,k_2=0} \). Also we have deduced the rescaled-squeezing factors for all these types giving information on the atomic inversion.

We conclude this paper by mentioning that the influence of the values of the atomic phases on the phenomenon under consideration is the same as that for the single-mode JCM [13]. In other words, for natural (numerical) approach the squeezing factor is sensitive (insensitive) to the
initial atomic state, i.e. natural approach can provide ”coherent trapping”\cite{34}. Also for numerical-simulation approach we have numerically checked that the RCP occurs in the quadrature squeezing only for \(k_1 + k_2 = 4\).

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