Gluon condensate and the vacuum structure of QCD

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Abstract

Phenomenological evidence and analytic approximations to the QCD ground state suggest a complex gluon condensate structure. Exclusion of elementary fermion excitations by the generation of infinite mass corrections is a consequence. In addition the existence of vacuum condensates in unbroken non-abelian gauge theories, endows SU(3) and higher order groups with a non-trivial structure in the manifold of possible vacuum solutions, which is not present in SU(2). This may be related to the existence of particle generations.

1 Vacuum condensates in QCD. Phenomenological and theoretical evidence

There is now ample phenomenological evidence for the existence of a non-trivial structure in the QCD vacuum, containing both quark\footnote{\[1\]} and gluon\footnote{\[2\]} condensates. A good description of many hadronic quantities is obtained in the framework of the QCD sum rules\footnote{\[3\]}, using as input the vacuum expectation values \(\langle \overline{q} q \rangle\) and \(\langle F^a_{\mu \nu} F^a_{\mu \nu} \rangle\).

Analytical approximations to the QCD ground state also provide theoretical evidence for the existence of the condensates and, in addition, supply
some additional information on the nature of the condensates. In Ref.[4], for
example, an exact path-integral representation of the ground state is used
to obtain a systematic expansion in which even the leading term contains
non-perturbative information. For the gluon sector of QCD (in the temporal
gauge), the leading term is

$$\Psi_0 \{ A \} = \exp \left( -\frac{1}{2} \int d^3x B^\alpha_k(x) \left( \frac{1}{\sqrt{R(A(x)) R(A(x))}} \right)^{\alpha\alpha'} B^{\alpha'}_{k'}(x) \right)$$

(1)

with

$$B^\alpha_i = \epsilon_{ijk} \left( \partial_j A^\alpha_k - \frac{g}{2} f_{\alpha\beta\gamma} A^\beta_j A^\gamma_k \right)$$

(2)

$$R(A)^{\alpha\alpha'}_{nmn'} = \epsilon_{nmn'} \left( \partial_m \delta^{\alpha\alpha'} - gf_{\alpha\gamma\alpha'} A^\gamma_m \right)$$

(3)

In the long-wavelength limit, the $\Psi_0 \{ A \}$ of Eq.(1) bears some resemblance to
the ansatz proposed by Greensite[5], however, the power dependence on the
chromomagnetic fields is different. By considering either fast varying poten-
tials $A^\alpha_k$ or constant field configurations, one sees that (1) interpolates nic ely
between an abelian-type vacuum for the high frequencies and a configuration
of random magnetic fluxes for the constant fields.

The non-perturbative nature of this analytical approximation to the (gluon
sector) vacuum is made apparent when, for example, one considers the effect
of long-wavelength (constant $A^\alpha_k$) field fluctuations on the effective mass and
propagation of elementary fermions. This was already briefly mentioned in
[4] and is described in some detail in Sect.2.

In addition some information may be gathered from (1) concerning the
group-theoretical structure of the QCD vacuum. This, however, is likely to
be more general than the approximation described in Eq.(1) and to depend
only on the octet structure of the operators that appear in the composite
condensates $\langle A^\mu_a A_{a\mu} \rangle$ and $\langle F^{\mu\nu}_{a\mu} F_{a\mu\nu} \rangle$. A gluonic ground state, of the type of
Eq.(1), defines a probability distribution for chromomagnetic field fluctua-
tions around zero mean in such a way that

$$\langle A^\mu_a \rangle = \langle F^{\mu\nu}_{a\mu} \rangle = 0$$

(4)

as required by Lorentz and SU(3) invariance, while quantities like $\langle A^\mu_a A_{a\mu} \rangle$
and $\langle F^{\mu\nu}_{a\mu} F_{a\mu\nu} \rangle$ may be different from zero. The integration measure that con-
trols the ground state fluctuations is $|\Psi_0 \{ A \}|^2 \prod_x dA^\alpha_k(x)$. The domain of
integration of this measure is the space of all possible ground state fluctuations, which should be defined in such a way as to preserve the symmetries of the theory. In the temporal gauge, one may, by space rotations, transform the gauge potential $A_\mu^a(x)$ at each point $x$ into a triplet of SU(n) orthogonal directions. Then, the directions to include in the domain of integration are those that can be obtained from this triplet by arbitrary SU(n) rotations. If the gauge group is SU(2), these three orthogonal directions in SU(2) space may, by SU(2) transformations, be brought to any other orthogonal directions. This is because the Lie algebra of SU(2) coincides with the Lie algebra of SO(3). It means that, to implement full SU(2)-symmetry, all possible directions have to be included in the domain of integration. Hence the vacuum is unique.

For SU(3), however, the situation is different. By SU(3) transformations one cannot rotate an octet vector and, even less, a triplet of octets, to all possible directions in $R^8$. This is the well-known fact that the SU(3) group has a non-trivial structure in the octet space. To preserve SU(3) symmetry it suffices to include in the vacuum the fluctuating directions that can be reached from a particular one by SU(3) transformations. Denoting by $VO_8$ the space of orthogonal frames in $R^8$ we conclude that there are as many non-equivalent ways to construct ground states compatible with SU(3) invariance as there are SU(3) orbits in $VO_8$. Conversely, if one lacks a specific reason to prefer any particular orbit, one might instead say that the gluonic vacuum has a larger SO(8) symmetry, because SO(8) is the smallest group that rotates any three-frame in $VO_8$ into any other. Therefore the possible vacuum structures correspond to representations of the homogeneous coset space SO(8)/SU(3). This reasoning holds independently of the particular form of the ground state measure density $|\Psi_0\{A\}|^2$. The consequences of the SO(8) structure of a QCD vacuum with gluon condensate are explored in Sect.3.

Greensite and Feynman, inspired by the form of the abelian ground state and the requirements of gauge invariance, have conjectured a form

$$\Psi_0\{A\} = \exp \left( - \int d^3x d^3y \text{Tr} (B(x) \cdot S_{xy} \cdot B(y) \cdot S_{yx}) f(|x - y|) \right)$$

for the (gluonic) QCD ground state. $S_{xy}$ is the gauging factor

$$S_{xy} = \exp \left( -\frac{i}{2} \int_x^y A_i(\xi)d\xi^i \right)$$
and for the kernel \( f(|x - y|) \), which contains all the non-trivial dynamical information, these authors have discussed a few requirements. However comparing Eq.(3) with the leading path-integral approximation of Eq.(1) the conclusion is that a simple coordinate dependence for the kernel is unlikely. Instead one obtains a complex dependence on the dynamical variables which seems difficult to guess from qualitative considerations. One also notices that Eq.(1) may be written as

\[
\Psi_0 \{ A \} = \exp \left( -\frac{1}{2} \int d^3x \xi(x) \cdot \xi(x) \right) \tag{6}
\]

where

\[
\xi_k^\alpha(x) = \left( \left( \frac{1}{R(A(x))} \right)^{1/4} \right)^{\alpha\alpha'} B_k^{\alpha'}(x) = \frac{1}{\pi \sqrt{2}} \int_0^\infty d\lambda \lambda^{-1/4} \frac{1}{\lambda + K_k^R} \cdot B(x) \tag{7}
\]

the last equality being obtained from a standard representation for fractional powers of positive operators. Eq.(7) puts into evidence the highly non-local nature of the effective QCD coordinates.

## 2 Gluon condensate and the effective mass of elementary fermion excitations

As will be seen later on, the nature of the gauge group determines the multiplicity of vacuum configurations, the SU(2) and SU(3) groups behaving differently in this aspect. However, dynamical features like the coupling constant dependence of the condensates and their effect on the propagation of elementary excitations are expected to depend mostly on the non-Abelian character and not so much on the order of the gauge group. Therefore, for simplicity, calculations in this section will be carried out for the SU(2) group.

In the Schrödinger formulation, the Hamiltonian involves products of fields and functional derivatives at the same point. Regularization of these ill-defined quantities is needed. The simplest way is to use a lattice cut-off, with the ultraviolet (continuum) limit obtained when the lattice spacing \( a \to 0 \). As shown in [4] for the quantum theory constructed from (1) the existence of a finite mass gap implies a running coupling constant behavior \( g^2(a) \sim \frac{c}{\log a} \). Therefore \( g(a) \to 0 \) when \( a \to 0 \) which justifies the use of
the small noise Wentzell-Freidlin technique to analyze the mass-gap. In the lattice regularization one makes the substitutions

\[ gaA_i^\alpha(x) \to \theta^\alpha(x, x + \vec{v}) = \theta_i^\alpha(x) \]  

(8)

\[ ga^2B_i^\alpha(x) \to \beta_i^\alpha(x) = \frac{1}{4} \gamma_{ijk} \left( \theta^\alpha(x + \vec{j}, x + \vec{j} + \vec{k}) - \frac{1}{2} f_{\alpha\beta\gamma} \theta^\beta(x, x + \vec{j}) \theta^\gamma(x, x + \vec{k}) \right) \]  

(9)

\[ (D_i)_{\alpha\beta} v^\beta(x) \to \frac{1}{a} (D_i)_{\alpha\beta} v^\beta(x) = \frac{1}{a} \left( \frac{1}{2} (v^\alpha(x + \vec{i}) - v^\alpha(x - \vec{i})) - f_{\alpha\gamma\delta} \theta^\gamma(x, x + \vec{i}) v^\delta(x) \right) \]  

(10)

where \( \gamma_{ijk} = (\text{sign}i)(\text{sign}j)(\text{sign}k) \epsilon_{ijk} \) and \( \vec{i} \) denotes the unit lattice vector along the \( i \)-direction. Then (11) becomes

\[ \Psi_0 \{ \theta \} = \exp \left( -\frac{1}{2 \pi g^2} \sum_x \int_0^\infty d\lambda \lambda^{-\frac{1}{2}} \beta^\alpha_k(x) \left( \frac{1}{\lambda + \mathcal{R}_x \cdot \mathcal{R}_x} \right)^{\alpha\gamma} \beta^\gamma_k(x) \right) \]  

(11)

with \( \mathcal{R}(\theta)^{\alpha\alpha'} = \epsilon_{mnm} D_m (\theta)^{\alpha\alpha'} \) and a standard integral representation\[8\] was used for the fractional power of the operator.

To study the long-wavelength contribution to the vacuum condensates, consider the case of constant gauge potentials. Because \( M_{ij} = \sum_\alpha \theta_i^\alpha \theta_j^\alpha \) is a symmetric matrix it may diagonalized by a space rotation and in the new coordinates the three SU(2) vectors \( \theta_i^0, \theta_i^2, \theta_i^3 \) are orthogonal. Without losing generality, SU(2) coordinates may be chosen such that

\[ \theta_1^0 = (a_1, 0, 0) \; \theta_2^0 = (0, a_2, 0) \; \theta_3^0 = (0, 0, a_3) \]  

(12)

\[ \beta_1^0 = (-a_2a_3, 0, 0) \; \beta_2^0 = (0, -a_3a_1, 0) \; \beta_3^0 = (0, 0, -a_1a_2) \]  

(13)

Then (11) becomes \( \Psi_0 \{ \theta \} = \exp (\sigma (a_1, a_2, a_3)) \) with

\[ \sigma (a_1, a_2, a_3) = -\frac{N}{2 \pi g^2} \int_0^\infty d\lambda \lambda^{-\frac{1}{2}} \left\{ \left[(a_1, a_2, a_3)^2 + (a_1, a_2, a_3)^2 + (a_1, a_2, a_3)^2 \right] \right. \right. \]  

\[ + \lambda (a_1^2 + a_2^2 + a_3^2) + a_1^2 (a_1^2 + a_2^2) + a_2^2 (a_1^2 + a_2^2) \]  

\[ + \left. a_3^2 (a_1^2 + a_2^2) \right\} + a_1^2 (a_2^2 + a_3^2 + a_2^2 a_3^2) \]  

\[ \times \left. 4(a_1 a_2 a_3)^2 + \lambda (\lambda + a_1^2 + a_2^2 + a_3^2) \right\}^{-1} \]  

(14)

\( N \) is the number of sites in the regularizing lattice. The state \( \Psi_0 \{ \theta \} = \exp (\sigma (a_1, a_2, a_3)) \), as it stands, is not normalizable. This has two origins. First, as one sees from (13), there still is a gauge freedom on the planes
where one of the arguments vanish. This is corrected by integrating not
on $da_1 da_2 da_3$ but on the components of the chromomagnetic field. This is
equivalent to introduce the Jacobian factor $\sqrt{a_1^2 a_2^2 a_3^2}$. Even then the state is
not normalizable unless one restricts the large potential fluctuations. This is
done by multiplying the integration measure by $\exp\{-\mu (a_1^2 + a_2^2 + a_3^2)\}$ with
$\lim_{\mu \to 0}$ being taken in the end. Expectation values of operators are therefore
obtained from

$$<O> = \lim_{\mu \to 0} \frac{\int \sqrt{a_1^2 a_2^2 a_3^2} e^{-\mu (a_1^2 + a_2^2 + a_3^2)} da_1 da_2 da_3 \Psi_0 \{\theta\} \cdot O \cdot \Psi_0 \{\theta\}}{\int \sqrt{a_1^2 a_2^2 a_3^2} e^{-\mu (a_1^2 + a_2^2 + a_3^2)} da_1 da_2 da_3 \Psi_0^2 \{\theta\}}$$  \hspace{1cm} (15)$$

and because $g(a) \to 0$ when $a \to 0$, these integrals may be evaluated by
asymptotic expansion methods, namely the Laplace method. The minimum
of $\sigma(a_1, a_2, a_3)$ is zero and is obtained when any two of its arguments vanish.
Fixing each turn one of the arguments and applying the Laplace method in
the plane of the other two arguments, one obtains three similar contributions.
For example for fixed $a_3$ one computes the second derivatives in the plane
$(a_1, a_2)$

$$\sigma(0, 0, a_3) = 0$$

$$\frac{\partial}{\partial a_1} \sigma(0, 0, a_3) = \frac{\partial}{\partial a_2} \sigma(0, 0, a_3) = 0$$

$$\frac{\partial^2}{\partial a_1^2} \sigma(0, 0, a_3) = \frac{\partial^2}{\partial a_2^2} \sigma(0, 0, a_3) = \frac{2}{2\pi a_3^2}$$

$$\frac{\partial^2}{\partial a_1 \partial a_2} \sigma(0, 0, a_3) = 0$$  \hspace{1cm} (16)$$

Using $\int_0^\infty z^k \exp(-\alpha z^2) dz = \Gamma\left(\frac{k+1}{2}\right)/(2^{k+1} \alpha^{(k+1)/2})$ and the generalized Laplace
method one obtains for the normalization integral at small $g$

$$\int d\nu(a_1, a_2, a_3) \Psi_0^2 \{\theta\} = 3 \int_0^\infty da_3 \left(\frac{1}{2\mu^2 |a_3| + \mu}\right)^2 |a_3| e^{-\mu a_3^2} \sim -9 \left(\frac{2\mu^2}{N}\right)^2 \ln \mu$$  \hspace{1cm} (17)$$

where I have denoted by $d\nu(a_1, a_2, a_3)$ the measure

$$d\nu(a_1, a_2, a_3) = \sqrt{a_1^2 a_2^2 a_3^2} e^{-\mu (a_1^2 + a_2^2 + a_3^2)} da_1 da_2 da_3$$  \hspace{1cm} (18)$$

and the last expression in (17) is the asymptotic behavior for small $\mu$. Like-
\[ \int d\nu(a_1, a_2, a_3) \Psi_0^2 \{ \theta \} (a_1^2 + a_2^2 + a_3^2) \sim 3 \left( \frac{2g^2}{N} \right)^2 \frac{1}{\mu} \]
\[ \int d\nu(a_1, a_2, a_3) \Psi_0^2 \{ \theta \} (a_1 + a_2 + a_3)^2 \sim 3 \left( \frac{2g^2}{N} \right)^2 \frac{1}{\mu} \]  
(19)
\[ \int d\nu(a_1, a_2, a_3) \Psi_0^2 \{ \theta \} ((a_1 a_2)^2 + (a_2 a_3)^2 + (a_3 a_1)^2) \sim 3 \left( \frac{2g^2}{N} \right)^3 \sqrt{\frac{\pi}{\mu}} \]

Therefore for \( <a_1^2 + a_2^2 + a_3^2 > \), which is the long-wavelength contribution to \( g^2 a^2 < A_\mu A_\nu > \), one obtains
\[ <a_1^2 + a_2^2 + a_3^2 > = \lim_{\mu \to 0} \frac{1}{-3\mu \ln \mu} \]

the same result applying for \( \langle (a_1 + a_2 + a_3)^2 \rangle \). Therefore the long wavelength contribution, of the vacuum background, to these expectation values diverges for all values of the coupling constant.

These results may now be used to find the effect of the gluon background, associated to the ground state (11), on the mass and propagation of elementary fermion excitations. The eigenvalue equation for the Dirac Hamiltonian of a fermion minimally coupled to this constant background is
\[ H_1(p) \psi(p) = \left( -\alpha^i p_i + g \left( A_0^a + \alpha^i A_i^a \right) \frac{\sigma_a}{2} + \gamma^0 m \right) \psi(p) = E \psi(p) \]

To find the contribution of the background to the fermion mass consider a Lorenz frame where \( \vec{p} = 0 \). In this frame, without loss of generality, one makes \( A_0^a = 0 \) by a gauge transformation and diagonalizes \( A_i^a A_i^a \) by space rotations obtaining, as above
\[ A_1^a = \frac{2}{g} (A_1, 0, 0) \; ; \; A_2^a = \frac{2}{g} (0, A_2, 0) \; ; \; A_3^a = \frac{2}{g} (0, 0, A_3) \]

With these choices the eight eigenvalues of \( H_1 \) are
\[ \pm \sqrt{m^2 + (A_1 \pm A_2 \pm A_3)^2} \]

for all possible sign choices. Hence the background contribution to the squared mass of the single fermion excitations is \( \langle (A_1 + A_2 + A_3)^2 \rangle \), the same for all eigenstates because of the symmetry of the state (11). As seen
above, this vacuum expectation value diverges for all values of the coupling constant. Therefore single fermion excitations cannot propagate in this background.

For a composite state with overall neutral color however, the total contribution of the constant background vanishes and its effect on the mass can only come from background modifications of the many-body interactions. For a fermion-antifermion state the total energy is

\[ H_1(p_1) + H_2(p_2) + V \]

\( V \) stands for the two-body interactions and \( H_2 \) acts on the adjoint fermions by

\[ H_2(p) \bar{\psi}^\dagger (p) = \left( -\alpha^i p_i - g \left( A_0^a + \alpha^i A^a_i \right) \frac{\sigma_a}{2} + \gamma^0 m \right) \bar{\psi}^\dagger (p) \]

Hence, the leading order effect of the constant background cancels for the color neutral combination \( \bar{\psi} (p_2) \psi (p_1) \).

### 3 Group structure of the QCD vacuum with a SU(3) gluon condensate

In Sect.1 I have already explained why, in the SU(3) case, with a gluon condensate, the vacuum structure is not uniquely defined by the SU(3) symmetry, the possible vacuum structures corresponding to representations of the homogeneous coset space SO(8)/SU(3). This SO(8) being the group of rotations in the octet representation of SU(3), the imbedding of SU(3) in SO(8) is uniquely defined (see the Appendix). SU(3) is the dynamical invariance of the theory that we start with, therefore it is this group that should control all the dynamical features of the theory: selection rules, mass matrix structure, etc. The extra SO(8) symmetry emerges only as a consequence of the realization of the vacuum through the composite condensates \( \langle A_\mu^a A_{a\mu} \rangle \) and \( \langle F_{a\mu}^\mu F_{a\mu} \rangle \). Therefore the only place where the SO(8) extra quantum numbers naturally appear is on the labelling of the vacuum classes, which may however be equivalent from the point of view of QCD interactions.

The next logical step is to classify the possible vacuum condensate structures through the irreducible representations of SO(8) and their reduction under SU(3). The lowest dimensional representations of SO(8) are the trivial
one and those associated to the fundamental weights $\Lambda_1$, $\Lambda_2$, $\Lambda_3$ and $\Lambda_4$. The trivial $[0,0,0,0]$ representation would correspond to a vacuum without a condensate structure and therefore is of no interest. $\Lambda_2 = [0,1,0,0]$ is the 28-dimensional adjoint representation which under the SU(3) color subgroup reduces into $10 + \overline{10} + 8$. The most interesting case corresponds to the other three representations $\Lambda_1 = [1,0,0,0]$, $\Lambda_3 = [0,0,1,0]$ and $\Lambda_4 = [0,0,0,1]$ which are 8-dimensional representations which are also irreducible octets under this SU(3) subgroup. They are one-dimensional representations of the homogeneous coset space SO(8)/SU(3). Therefore one finds 3 identical vacuum structures which are however distinguished by SO(8) quantum numbers (the maximal weights or the values of the Casimir operators).

As far as SU(3) color is concerned the three vacuum backgrounds are equivalent and, if the mass matrix is SO(8)-blind, it is an example of the democratic mass matrix with all elements equal, discussed by a number of authors\cite{11} \cite{12} \cite{13}. When diagonalized it leads (in leading order) to one massive state and two massless ones. Therefore the SO(8) induced background multiplicity is suggestive of a generation type mechanism. Whether it is really at the origin of the existence of three particle generations remains to be seen.

4 Appendix. SO(8) rotations in the octet space

The antihermitean generators for the Lie algebra of SO(8) have the commutation relations

$$[M_{pq}, M_{rs}] = \delta_{qr} M_{ps} - \delta_{qs} M_{pr} - \delta_{pr} M_{qs} + \delta_{ps} M_{qr} \quad (20)$$

$p, q, r, s = 1 \cdots 8$

The generator $M_{pq}$, with matrix elements

$$(M_{pq})_{jk} = \delta_{pj} \delta_{qk} - \delta_{pk} \delta_{qj} \quad (21)$$
in the defining representation, generates rotations in the plane $(pq)$. The structure constants of a Lie algebra are the matrix elements of the adjoint representation. Therefore, from the usual structure constants $f_{abc}$ of SU(3),
one reads the representation of the antihermitean SU(3) generators $F_a$ as functions of the SO(8) rotations in the octet space

\[ \left( \frac{\lambda_a}{2} \right. \rightarrow iF_a ) \]

\[ F_1 = -M_{23} - \frac{1}{2} M_{47} + \frac{1}{2} M_{56} \]

\[ F_2 = M_{13} - \frac{1}{2} M_{46} - \frac{1}{2} M_{57} \]

\[ F_3 = -M_{12} - \frac{1}{2} M_{45} + \frac{1}{2} M_{67} \]

\[ F_4 = \frac{1}{2} M_{17} + \frac{1}{2} M_{26} + \frac{1}{2} M_{35} - \frac{\sqrt{3}}{2} M_{58} \]

\[ F_5 = -\frac{1}{2} M_{16} + \frac{1}{2} M_{27} - \frac{1}{2} M_{34} + \frac{\sqrt{3}}{2} M_{48} \]

\[ F_6 = \frac{1}{2} M_{15} - \frac{1}{2} M_{24} - \frac{1}{2} M_{37} - \frac{\sqrt{3}}{2} M_{78} \]

\[ F_7 = -\frac{1}{2} M_{14} - \frac{1}{2} M_{25} + \frac{1}{2} M_{36} + \frac{\sqrt{3}}{2} M_{68} \]

\[ F_8 = -\frac{\sqrt{2}}{2} M_{45} - \frac{\sqrt{2}}{2} M_{67} \]

To make the Cartan algebra of SU(3), \( \{F_3, F_8\} \), a subalgebra of the Cartan algebra of SO(8) it is convenient to choose this latter as

\[ \{h_i\} = \{M_{12}, M_{45}, M_{67}, M_{38}\} \] (23)

The three octet representations \([1,0,0,0], [0,0,1,0] \) and \([0,0,0,1]\) of SO(8) are also irreducible octets of the SU(3) subgroup defined in (22). The correspondence between states is the following:

\[ \Lambda_1 = [1, 0, 0, 0] \]

\[ \begin{align*}
|\frac{1}{2}, \frac{1}{2}, 1\rangle &\leftrightarrow |000\rangle \\
|\frac{1}{2}, -\frac{1}{2}, 1\rangle &\leftrightarrow |00i0\rangle \\
|110\rangle &\leftrightarrow |i000\rangle \\
|100\rangle &\leftrightarrow \frac{1}{\sqrt{2}} (|000 - i\rangle - |000i\rangle) \\
|10 - 10\rangle &\leftrightarrow |i000\rangle \\
|000\rangle &\leftrightarrow \frac{1}{\sqrt{2}} (|000 - i\rangle + |000i\rangle) \\
|\frac{1}{2}, \frac{1}{2}, -1\rangle &\leftrightarrow |00 - i0\rangle \\
|\frac{1}{2}, -\frac{1}{2}, -1\rangle &\leftrightarrow |0 - i0\rangle
\end{align*} \] (24)
\[ \Lambda_3 = [0, 0, 1, 0] \]

\[
\begin{align*}
SU(3) & \quad SO(8) \\
|1 \frac{1}{2} \frac{1}{2} \rangle & \leftrightarrow |1 \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \rangle \\
|\frac{1}{2} \frac{1}{2} \rangle & \leftrightarrow |\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \rangle \\
|110 \rangle & \leftrightarrow |\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \rangle \\
|100 \rangle & \leftrightarrow \frac{1}{\sqrt{2}} \left(|\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \rangle + |\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \rangle\right) \\
|1 - 10 \rangle & \leftrightarrow |\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \rangle \\
|000 \rangle & \leftrightarrow \frac{1}{\sqrt{2}} \left(|\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \rangle - |\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \rangle\right) \\
|\frac{1}{2} \frac{1}{2} - 1 \rangle & \leftrightarrow |\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \rangle \\
|\frac{1}{2} \frac{1}{2} - 1 \rangle & \leftrightarrow |\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \rangle
\end{align*}
\]

\[ \Lambda_4 = [0, 0, 0, 1] \]

\[
\begin{align*}
SU(3) & \quad SO(8) \\
|\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \rangle & \leftrightarrow |\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \rangle \\
|\frac{1}{2} \frac{1}{2} - 1 \rangle & \leftrightarrow |\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \rangle \\
|110 \rangle & \leftrightarrow |\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \rangle \\
|100 \rangle & \leftrightarrow \frac{1}{\sqrt{2}} \left(|\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \rangle - |\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \rangle\right) \\
|1 - 10 \rangle & \leftrightarrow |\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \rangle \\
|000 \rangle & \leftrightarrow \frac{1}{\sqrt{2}} \left(|\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \rangle + |\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \rangle\right) \\
|\frac{1}{2} \frac{1}{2} - 1 \rangle & \leftrightarrow |\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \rangle \\
|\frac{1}{2} \frac{1}{2} - 1 \rangle & \leftrightarrow |\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \rangle
\end{align*}
\]

where the SU(3) quantum numbers are \(|II_3Y\rangle\), with \(I_3 = IF_3\) and \(Y = i\frac{2}{\sqrt{3}}F_8\), and the SO(8) quantum numbers are the eigenvalues of the antihermitean generators of the Cartan algebra \(|h_i\rangle\).

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