MODULAR DIFFERENTIAL EQUATIONS FOR CHARACTERS OF RCFT

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Abstract. We discuss methods, based on the theory of vector-valued modular forms, to determine all modular differential equations satisfied by the conformal characters of RCFT; these modular equations are related to the null vector relations of the operator algebra. Besides describing effective algorithmic procedures, we illustrate our methods on an explicit example.

1. Introduction

Differential equations are arguably one of the most important tools of theoretical physics. They appear in many guises, like equations of motion, conservation laws, etc. The usual approach to a physical theory is to deduce the governing differential equations starting from some basic theoretical considerations or experimental observations, and then to investigate the theory by solving these equations under different circumstances. But, while this is by far the most common situation, there are cases where one can determine the quantities of interest by some different method, without the knowledge of the differential equations themselves.

Conformal characters of RCFT provide an interesting example of this phenomenon. It is known [2, 22, 23] that these quantities satisfy differential equations of a very special kind, so-called modular equations, related to the null vector relations of the chiral operator algebra [7]. By analyzing the representation theory of the latter, one could find enough null vector relations to make the corresponding system of modular equations completely determined, whose unique solution should therefore give the conformal characters. In many examples this procedure does indeed work. But in some circumstances we don’t have enough information about the operator algebra, the most extreme case being when even its existence is unknown, while we still have enough information to determine the would-be characters, at least partially, e.g. using the fact that they form a vector-valued modular form for a suitable automorphy factor of weight 0, which can in turn be determined from the would-be fusion rules. In this situation, one faces the following question: can we determine from the knowledge of the characters all the modular equations that they satisfy, and from this information infer the null vectors of the operator algebra? As to the second question, the precise relation of null vectors to modular equations has been settled in work of Gaberdiel-Keller [13] and of Zhu [28]. In this note we want to address the question of how to determine, for a given character vector, all modular differential equations satisfied by it. Our answer is based on the machinery of vector-valued modular forms and their invariant differential operators, as developed in joint work with Terry Gannon [3, 4]; the relevant concepts and results are reviewed in the first sections. As we shall see, we can give an effective computational answer, which we illustrate on the well-known example of the Ising model. We conclude with some general comments on the applications of our results.

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2. Vector-valued modular forms

Recall [1, 6] that the classical modular group \( \Gamma = \text{SL}_2(\mathbb{Z}) \) of 2\times2 integer matrices with unit determinant acts on the complex upper half-plane \( H = \{ \tau \in \mathbb{C} \mid \text{Im} \tau > 0 \} \) by fractional linear transformations

\[
\tau \mapsto \gamma \tau = \frac{a \tau + b}{c \tau + d},
\]

for \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \). It is well known that the so-called modular curve \( X(\Gamma) \), the one-point compactification of the quotient \( H/\Gamma \) obtained by adjoining the cusp \( \tau = i\infty \), is a Riemann-surface of genus 0 [18, 26].

In case \( d \) is a positive integer, an automorphy factor of rank \( d \) for \( \Gamma \) is a map \( \varrho : \Gamma \times H \to \text{GL}_d(\mathbb{C}) \) that satisfies

\[
\varrho(\gamma_1 \gamma_2, \tau) = \varrho(\gamma_1, \gamma_2 \tau) \varrho(\gamma_2, \tau)
\]

for all \( \gamma_1, \gamma_2 \in \Gamma \) and \( \tau \in H \), and is holomorphic as a function of \( \tau \); it is flat of weight \( w \in \mathbb{R} \) if the expression

\[
\varrho(\gamma, \tau) \left( \frac{d(\gamma \tau)}{d\tau} \right)^{w/2} = \varrho(\gamma, \tau) (c\tau + d)^{-w}
\]

is independent of \( \tau \) [14]. According to the above definition, a weight 0 automorphy factor is nothing but a homomorphism from \( \Gamma \) to \( \text{GL}_d(\mathbb{C}) \), i.e. a \( d \)-dimensional matrix representation of \( \Gamma \). From a geometric point of view, an automorphy factor determines a holomorphic vector bundle over the modular curve \( X(\Gamma) \), making obvious how to define direct sums and tensor products of automorphy factors. Note that the direct sum of two flat automorphy factors is flat only if the weights of the summands equal each other.

Given an automorphy factor \( \varrho \) of rank \( d \), an automorphic form for \( \varrho \) is a meromorphic map \( \mathcal{X} : H \to \mathbb{C}^d \) which satisfies the transformation rule

\[
\mathcal{X}(\gamma \tau) = \varrho(\gamma, \tau) \mathcal{X}(\tau)
\]

for all \( \gamma \in \Gamma \) and \( \tau \in H \); clearly, automorphic forms are the meromorphic sections of the corresponding vector bundle. An automorphic form \( \mathcal{X}(\tau) \) is called weakly holomorphic if it is holomorphic in the upper half-plane \( H \), and has only a finite order pole at \( \tau = i\infty \), meaning that its Puiseux series (in terms of the local uniformizer) involves only finitely many negative powers; if there are no negative powers at all, then \( \mathcal{X}(\tau) \) is holomorphic, and it is a cusp form if it vanishes in the limit \( \tau \to i\infty \). Another common appellation for weakly holomorphic forms, spread across much of the literature, is vector-valued modular form [4, 8, 17], emphasizing their multicomponent nature. We shall denote by \( \mathcal{M}(\varrho) \) the set of weakly holomorphic forms for the automorphy factor \( \varrho \); obviously, these form a linear space over \( \mathbb{C} \).

At this point, it could be helpful to review the case of classical scalar modular forms [1, 6, 15, 18, 20, 25]. These are automorphic forms for the rank one automorphy factors

\[
\rho_{2k}(\gamma, \tau) = (c\tau + d)^{2k}
\]

of weight 2\( k \), defined for any integer \( k \). There exist holomorphic forms only for \( k > 1 \), classical examples being the Eisenstein series [1, 25]

\[
E_{2k}(q) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,
\]
where
\begin{equation}
\sigma_k(n) = \sum_{d \mid n} d^k
\end{equation}
is the $k$-th power sum of the divisors of the integer $n$, and $B_k$ denotes the $k$-th Bernoulli number. Actually, any holomorphic form may be expressed uniquely as a polynomial in $E_4(\tau)$ and $E_6(\tau)$. There are no cusp forms for $k < 6$, and there is a unique one (up to a multiplicative constant) for $k = 6$, the famous discriminant form
\begin{equation}
\Delta(\tau) = \frac{E_4(\tau)^3 - E_6(\tau)^2}{1728} = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.
\end{equation}
The expression of $\Delta$ as an infinite product shows that the discriminant form doesn’t vanish on the upper half-plane $\mathbb{H}$, and this makes it possible to construct weakly holomorphic forms for arbitrary $k$ as suitable quotients of Eisensteins by powers of $\Delta$. In particular, the so-called Hauptmodul $[1, 6, 18]$
\begin{equation}
J(q) = \frac{E_4(\tau)^3}{\Delta(\tau)} - 744 = q + \sum_{n=1}^{\infty} c(n) q^n = q^1 + 196884q + 21493760q^2 + \ldots,
\end{equation}
is invariant under $\Gamma$, i.e. $J(\gamma \tau) = J(\tau)$ for all $\gamma \in \Gamma$, is holomorphic in $\mathbb{H}$, and has a first order pole at the cusp, hence it is a weakly holomorphic form for $\rho_0$, and every element of $\mathcal{M}(\rho_0)$ can be expressed as a univariate polynomial in $J(\tau)$. Finally, we note that the Eisenstein series $E_2(\tau)$, defined by Eq.(2.6) for $k = 1$, is equal to the logarithmic derivative of the discriminant form
\begin{equation}
E_2(\tau) = \frac{1}{2\pi i} \frac{d(\ln \Delta)}{d\tau}.
\end{equation}
While obviously holomorphic, $E_2(\tau)$ is not a modular form, since it does not satisfy the transformation rule Eq.(2.4).

A major simplification follows from the observation that the theory for automorphy factors of non-zero weight may be reduced to the case of zero weight via the so-called weight shifting trick. Indeed, thanks to the fact that the discriminant form doesn’t vanish on the upper half-plane, it is meaningful to consider arbitrary fractional powers of $\Delta$. This allows one to associate to the flat automorphy factor $\rho$ of weight $w$ the automorphy factor
\begin{equation}
\rho_0(\gamma, \tau) = \rho(\gamma, \tau) \left( \frac{\Delta(\tau)}{\Delta(\gamma \tau)} \right)^{w/12}
\end{equation}
of weight 0. If $X \in \mathcal{M}(\rho)$ is a weakly holomorphic form for $\rho$, then
\begin{equation}
X_0(\tau) = \Delta(\tau)^{-w/12} X(\tau)
\end{equation}
is a weakly holomorphic form for $\rho_0$, providing a one-to-one correspondence between $\mathcal{M}(\rho)$ and $\mathcal{M}(\rho_0)$. Consequently, we can restrict our attention to forms for automorphy factors of weight 0, without any loss of generality. What is more, in many important applications the only automorphy factors that actually show up are of weight 0. For example, the character vector of a RCFT, formed by the genus one characters of the primary fields, is a vector-valued modular form [3], its weight 0 automorphy factor being the celebrated modular representation determined by the fusion rules and conformal weights of the primary fields via Verlinde’s theorem [27, 24].

For the above reasons, from now on we shall only consider flat automorphy factors of weight 0, i.e. matrix representations $\rho : \Gamma \to \text{GL}_d(\mathbb{C})$. We make the following technical assumptions on $\rho$ [3, 4]:
(1) \( g^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) equals the identity matrix;
(2) there exists a real diagonal matrix \( \Lambda \), called the exponent matrix, such that
\[
(2.13) \quad g \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \exp(2\pi i \Lambda)
\]
Note that \( \Lambda \) is not unique, since only the fractional part of its diagonal elements are determined by Eq. (2.13), but not the integer parts.

If \( \mathcal{X}(\tau) \) is an automorphic form for \( g \), then (2.13) implies, taking into account Eq. (2.4), that the map \( \exp(-2\pi i \Lambda \tau) \mathcal{X}(\tau) \) is periodic in \( \tau \) with period 1: as a consequence, it may be expanded into a Fourier series (the \( q \)-expansion of \( \mathcal{X} \))
\[
(2.14) \quad q^{-\Lambda} \mathcal{X}(q) = \sum_{n \in \mathbb{Z}} \mathcal{X}[n] q^n,
\]
where \( q = \exp(2\pi i \tau) \) and \( \mathcal{X}[n] \in \mathbb{C}^d \). Note that \( \mathcal{X} \) is weakly holomorphic precisely when its \( q \)-expansion contains only finitely many negative powers of \( q \).

For any automorphy factor \( g \), multiplication by \( J(\tau) \) takes the linear space \( \mathcal{M}(g) \) to itself: in other words, \( \mathcal{M}(g) \) is a \( \mathbb{C}[J] \)-module, which may be shown to be free of finite rank. Under suitable circumstances (that are always met in practice), there exists a \( d \)-by-\( d \) matrix \( \Xi(\tau) \), whose columns freely generate \( \mathcal{M}(g) \), and such that the asymptotic relation
\[
(2.15) \quad \Xi(q) \to q^{\Lambda - 1} \quad \text{as} \quad q \to 0
\]
holds for a suitable exponent matrix \( \Lambda \). In such case, we call \( \Xi(\tau) \) a fundamental matrix for \( g \), and the limit
\[
(2.16) \quad \mathcal{X} = \lim_{q \to 0} \left( q^{-\Lambda} \Xi(q) - q^{-1} \right)
\]
whose existence follows from Eq. (2.15), the corresponding characteristic matrix [4].

As we shall see later, the numerical matrices \( \mathcal{X} \) and \( \Lambda \) determine the fundamental matrix completely, and provide a useful parametrization of the different automorphy factors of weight 0.

By definition, the fundamental matrix \( \Xi(\tau) \) satisfies the transformation rule
\[
(2.17) \quad \Xi(\gamma \tau) = g(\gamma) \Xi(\tau)
\]
and for each \( \mathcal{X} \in \mathcal{M}(g) \) there exists a vector \( j\mathcal{X} \in \mathbb{C}[J]^d \) with components that are polynomials in the Hauptmodul \( J(\tau) \), such that
\[
(2.18) \quad \mathcal{X}(\tau) = \Xi(\tau) j\mathcal{X}.
\]
The vector \( j\mathcal{X} \) is called the polynomial representation of the vector-valued modular form \( \mathcal{X} \), and it will play an important role later.

The fundamental matrix \( \Xi(\tau) \) is the most important piece of data needed to describe vector-valued modular forms for an automorphy factor of weight 0, so the question is how could one determine it. This will be achieved through the consideration of the invariant differential operators to be discussed in the next section, which are also of primary interest in the study of modular differential equations.

### 3. Invariant Differential Operators

Suppose that \( g \) is a flat automorphy factor of rank \( d \) and weight \( w \), and that \( \mathcal{X} \in \mathcal{M}(g) \) is a vector-valued modular form for \( g \). Except in case \( w = 0 \), the \( \tau \) derivative of \( \mathcal{X}(\tau) \) fails to be a vector-valued modular form, but this can be cured by the introduction of a suitable correction term. Indeed, the expression
\[
(3.1) \quad D_w \mathcal{X} = \frac{1}{2\pi i} \frac{d}{d\tau} - \frac{w}{12} E_2(\tau) \mathcal{X}(\tau)
\]
is easily shown to be a vector-valued modular form for the automorphy factor

\[ g'(\gamma, \tau) = g(\gamma, \tau) \frac{d(\gamma \tau)}{d\tau} \]

of weight \( w + 2 \). Here

\[ E_2(\tau) = \frac{1}{2\pi i} \frac{d(\ln \Delta)}{d\tau} = 1 - 24q - 72q^2 - \ldots \]

is the logarithmic derivative of the discriminant form \( \Delta(\tau) \), cf. Eq.(2.10).

An important consequence of (3.3) is the equivariance relation

\[ D_{w+12n} (\Delta^n X) = \Delta^n D_w X, \]

valid for any \( n \in \mathbb{R} \), which allows the use of the weight shifting trick Eq.(2.12) discussed in the previous section; if \( X \in \mathcal{M}(\rho) \), then

\[ (D_w X)_n = D_0 X_0. \]

When defining higher powers of \( D_w \), one should take into account that it increases the weight by 2; as a result, one should use the recurrence relation

\[ D_{w+2n} = D_w \circ D_w^n, \]

Note that \( D_w^n \) increases the weight by 2. To get a differential operator of order \( n \) that maps \( \mathcal{M}(\rho) \) to itself, one has to multiply \( D_w^n \) by a scalar modular form of weight \( -2n \). A suitable choice is

\[ d_n(\tau) = E_4(\tau)^{n_3} E_6(\tau)^{n_2} \Delta(\tau)^{n_\infty}, \]

with \( n_k \) denoting the value of \( n \) modulo \( k \), and

\[ n_\infty = \frac{n + 2n_3 + 3n_2}{6}. \]

With the above choice of prefactors, the operators

\[ \nabla_n = d_n(\tau) D_w^n \]

are invariant scalar differential operators, which means that they act on vector-valued modular forms component-wise, and that they map \( \mathcal{M}(\rho) \) to itself for each automorphy factor \( \rho \). What is more, any invariant scalar differential operator may be expressed as a linear combination of the \( \nabla_n \)-s, with coefficients that are polynomials in the Hauptmodul\(^1\). In particular, this is true for the products \( \nabla_n \circ \nabla_m \). Some of the relevant multiplication rules read

\[ \nabla_n \circ \nabla_m = \nabla_{n+m} \]

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\(^1\)This follows from the observation that, for a positive integer \( n \), any weakly holomorphic scalar form of weight \(-2n\) is the product of \( d_n(\tau) \) with a form of weight 0, i.e. \( \mathcal{M}(\rho\cdot-2n) = d_n(\tau) \mathcal{M}(\rho)_0 \). In other words, \( d_n(\tau) \) generates \( \mathcal{M}(\rho\cdot-2n) \) as a \( \mathbb{C}[J] \)-module.
\( \nabla_1 \circ \nabla_1 = (J - 984) \nabla_2 - \frac{1}{6} (5J + 264) \nabla_1 \)
\( \nabla_1 \circ \nabla_2 = (J + 744) \nabla_3 - \frac{2}{3} (J - 984) \nabla_2 \)
\( \nabla_1 \circ \nabla_3 = (J - 984) \nabla_4 - \frac{1}{5} (J + 744) \nabla_3 \)
\( \nabla_1 \circ \nabla_4 = \nabla_5 - \frac{1}{4} (J - 984) \nabla_4 \)
\( \nabla_1 \circ \nabla_5 = (J - 984) (J + 744) \nabla_6 - \frac{1}{36} (7J - 1704) \nabla_5 \)

\[ (3.10) \]
\( \nabla_2 \circ \nabla_1 = (J + 744) \nabla_3 - \frac{1}{6} (5J + 264) \nabla_2 + \frac{1}{2} (J + 744) \nabla_1 \)
\( \nabla_2 \circ \nabla_2 = (J + 744) \nabla_4 - \frac{2}{3} (J + 744) \nabla_3 - \frac{1}{5} (5J + 264) \nabla_2 \)
\( \nabla_2 \circ \nabla_3 = \nabla_5 - (J + 744) \nabla_4 + \frac{1}{4} (J + 744) \nabla_3 \)
\( \nabla_2 \circ \nabla_4 = \nabla_6 - (J - 984) \nabla_5 + \frac{1}{72} (7J - 1704) \nabla_4 \)
\( \nabla_2 \circ \nabla_5 = \frac{5}{3} (5J + 264) \nabla_6 - \frac{1}{36} (5J + 264) \nabla_5 \)

\[ (3.11) \]
\( \nabla_{n+6} = \nabla_n \circ \nabla_6 \),
which follows from \( D \Delta = 0 \), and the recursion formula
\[ (3.12) \]
\( \nabla_1 \circ \nabla_n = a_n \nabla_{n+1} - b_n \nabla_n \),
where \( a_n \) and \( b_n \) denote weakly holomorphic scalar modular forms of weight 0 that can be expressed as the following univariate polynomials in the Hauptmodul:
\[ (3.13) \]
\( a_n = (J - 984)^{n^2} (J + 744)^{\frac{n(n+1)}{126}} \)
\( b_n = \frac{29}{6} (J + 744) + \frac{23}{3} (J - 984) \)

Note that, in complete accord with Eq.(3.11), the coefficients \( a_n \) and \( b_n \) depend only on the value of \( n \) modulo 6.

Since the operators \( \nabla_n \), together with the multiplication-by-\( J \) operator
\[ (3.14) \]
\( J: \mathcal{M}(\varrho) \to \mathcal{M}(\varrho) \)
\( \mathcal{X}(\varphi) \mapsto J(\varphi) \mathcal{X}(\varphi) \)
all map the space \( \mathcal{M}(\varrho) \) of weakly holomorphic forms to itself, \( \mathcal{M}(\varrho) \) is a module for the noncommutative ring \( D = \mathbb{C}[J, \nabla_1, \nabla_2, \ldots] \). This module is necessarily of finite rank, since it is already of finite rank as a \( \mathbb{C}[J] \)-module. An important result is that the ring \( D \) is generated by the operators \( J, \nabla_1, \nabla_2 \) and \( \nabla_3 \), as a consequence of the relations Eqs.(3.11) and (3.10), which allow to express any operator \( \nabla_n \) with \( n > 3 \) in terms of \( J, \nabla_1, \nabla_2, \nabla_3 \), e.g.
\[ (3.15) \]
\( \nabla_4 = \frac{1}{1728} (\nabla_2 \circ \nabla_2 - \nabla_1 \circ \nabla_3 + \frac{5}{6} (J + 744) \circ \nabla_3 + \frac{5}{2} (5J + 264) \circ \nabla_2) \).

It is clear from the multiplication rules Eq.(3.10) and the commutation relations
\[ [\nabla_1, J] = - (J - 984) \circ (J + 744) \]
\[ [\nabla_2, J] = - 2 (J + 744) \circ \nabla_1 + \frac{5}{6} (J + 744) \circ (7J - 1704) \]
\[ [\nabla_3, J] = - 3 (J - 984) \circ \nabla_2 + \frac{1}{3} (7J - 1704) \circ \nabla_1 - \frac{5}{6} (J - 984) \circ (7J + 3480) \),
that the structure of the ring \( D \) does not depend on the automorphy factor: the ring \( D \) is universal, one and the same for all automorphy factors \( \varrho \). This shows that there is a very close connection between vector-valued modular forms (for arbitrary automorphy factors) and representations of \( D \).

Let’s now turn to the determination of the fundamental matrix. Consider an automorphy factor \( \varrho \) of weight 0, with fundamental matrix \( \Xi(\varphi) \), characteristic
matrix \( \mathcal{X} \) and exponent matrix \( \mathbf{A} \). Since each operator \( \nabla_n \) maps \( \mathcal{M}(g) \) to itself, applying any \( \nabla_n \) to the fundamental matrix gives a matrix whose columns are vector-valued modular forms: consequently, for each positive integer \( n \) there exists matrices \( \mathcal{D}_n(\tau) \) for which

\[
\nabla_n \Xi(\tau) = \Xi(\tau) \mathcal{D}_n(\tau),
\]

and whose matrix elements are weakly holomorphic scalar modular forms, hence polynomials in the Hauptmodul \( J(\tau) \); these polynomials may be determined explicitly by comparing the \( q \)-expansions of both sides of Eq.\((3.17)\). For example,

\[
\mathcal{D}_1(\tau) = (J(\tau) - \mathcal{X})(\mathbf{A} - 1) + \mathbf{A}\mathcal{X} - 240(\mathbf{A} - 1)
\]
\[
\mathcal{D}_2(\tau) = (J(\tau) - \mathcal{X})(\mathbf{A} - 1)(\mathbf{A} - \frac{7}{2}) + (\mathbf{A} - \frac{1}{2})\mathbf{A}\mathcal{X} + 504(\mathbf{A} - 1)(\mathbf{A} - \frac{27}{2})
\]

and

\[
\mathcal{D}_3(\tau) = (J(\tau) - \mathcal{X})(\mathbf{A} - 1)(\mathbf{A} - \frac{7}{2})(\mathbf{A} - \frac{7}{4}) + (\mathbf{A} - \frac{1}{2})(\mathbf{A} - \frac{1}{4})\mathbf{A}\mathcal{X}
\]
\[
- 480(\mathbf{A} - 1)(\mathbf{A}^2 - \frac{140}{9} + \frac{49}{4})
\]

For any given \( n > 0 \), the relation Eq.\((3.17)\) is nothing but a linear differential equation of order \( n \) that the fundamental matrix has to satisfy. A necessary and sufficient condition for this infinite sequence of equations to have a common solution is the relation

\[
\nabla_1 \mathcal{D}_n(\tau) + \mathcal{D}_1(\tau) \mathcal{D}_n(\tau) = a_n(\tau)\mathcal{D}_{n+1}(\tau) - b_n(\tau)\mathcal{D}_n(\tau),
\]

which follows from Eq.\((3.17)\) and the recursion relation Eq.\((3.12)\): actually, only the \( n = 1 \) and \( n = 2 \) cases of Eq.\((3.21)\) have to be considered, the remaining ones are consequences of these two. Taking into account the known expression of the matrices \( \mathcal{D}_n(\tau) \) as polynomials in the Hauptmodul, Eq.\((3.21)\) reduces to intricate algebraic relations between the matrices \( \mathbf{A} \) and \( \mathcal{X} \), whose explicit form can be found in [4]. Provided these are satisfied, one may compute the fundamental matrix by solving the first order linear differential equation

\[
\nabla_1 \Xi(\tau) = \Xi(\tau) \mathcal{D}_1(\tau),
\]

with boundary condition Eq.\((2.15)\): the solution will automatically solve the higher order equations as well. This means that \( \Xi(\tau) \) is completely determined by the two numerical matrices \( \mathbf{A} \) and \( \mathcal{X} \).

Besides \( \mathbb{D} \), another ring of interest is \( \mathbb{D}^{\text{hol}} \), spanned by linear combinations of powers of \( \mathbf{D} \) with coefficients given by holomorphic scalar modular forms, i.e. suitable polynomials in the Eisenstein series \( E_4 \) and \( E_6 \). As an algebra over \( \mathbb{C} \), the ring \( \mathbb{D}^{\text{hol}} \) is generated by the operators \( \mathbf{D} \), \( E_4 \) and \( E_6 \), where \( E_k \) denotes multiplication by the Eisenstein series \( E_k(\tau) \). The basic commutation relations connecting these generators read

\[
[E_4, E_6] = 0
\]
\[
[E_4, D] = \frac{1}{4} E_6
\]
\[
[E_6, D] = \frac{1}{2} E_4^2
\]

Note that the elements of \( \mathbb{D}^{\text{hol}} \) usually don’t preserve the weight. If an element of \( \mathbb{D}^{\text{hol}} \) happens to change the weight of all modular forms by \( 2n \), then we say that it is homogeneous of grade \( n \). This leads to a decomposition

\[
\mathbb{D}^{\text{hol}} = \bigoplus_{n=0}^{\infty} \mathbb{D}_n^{\text{hol}},
\]
the grade \(n\) homogeneous subspace \(\mathbb{D}_{n}^{\text{hol}}\) consisting of the operators that can be written as linear combinations

\[
\sum_{k=0}^{n} g_{n-k}(\tau) \mathbf{D}^{k},
\]

whose coefficients \(g_k(\tau)\) are holomorphic forms of weight \(2k\); in particular, \(g_0\) is constant, while \(g_1 = 0\). Each homogeneous subspace \(\mathbb{D}_{n}^{\text{hol}}\) is finite dimensional, the set

\[
\{ E_{a}^{b} \mathbf{D}^{n-2a-3b} \mid 0 \leq a, b, n - 2a - 3b \}
\]

providing a basis of it. It follows that the Hilbert series of \(\mathbb{D}^{\text{hol}}\) reads

\[
H_{\mathbb{D}^{\text{hol}}}(z) = \frac{1}{(1-z)(1-z^2)(1-z^3)}.
\]

4. Modular differential equations

An interesting question, with direct relevance to physics, is to determine all those invariant differential operators that annihilate a given vector-valued modular form \([12, 21, 28]\). In other words, one is interested in the annihilator

\[
\text{Ann}_{\mathbb{D}}(\mathbb{X}) = \{ \nabla \in \mathbb{D} \mid \nabla \mathbb{X} = 0 \}
\]

of the vector-valued modular form \(\mathbb{X} \in \mathcal{M}(\mathfrak{g})\); note that \(\text{Ann}_{\mathbb{D}}(\mathbb{X})\) is a (left) ideal of the ring \(\mathbb{D}\) of invariant differential operators. It is clear that, as a \(\mathbb{C}[J]\)-module, \(\text{Ann}_{\mathbb{D}}(\mathbb{X})\) is the inductive limit of the increasing sequence \(A_1 \subset A_2 \subset \cdots\), where \(A_n\) denotes the set of those elements of \(\text{Ann}_{\mathbb{D}}(\mathbb{X})\) whose order (as a differential operator) does not exceed \(n\), i.e. which can be written as a combination \(\sum_{k=0}^{n} \delta_{k}(J) \nabla_{k}\) with polynomial coefficients \(\delta_{k} \in \mathbb{C}[J]\).

The basic idea is the following: since \(\mathbb{X} \in \mathcal{M}(\mathfrak{g})\) implies \(\nabla_{n} \mathbb{X} \in \mathcal{M}(\mathfrak{g})\) for all \(n\), one has

\[
\nabla_{n} \mathbb{X} = \Xi(\tau) \mathbb{X}^{[n]}
\]

for some \(\mathbb{X}^{[n]} \in \mathbb{C}[J]^{d}\), where \(\Xi(\tau)\) is the fundamental matrix of \(\mathfrak{g}\); note that \(\mathbb{X}^{[0]} = \mathbb{J}\mathbb{X}\) is the polynomial representation of \(\mathbb{X}\), cf. Eq.\((2.18)\). Let’s now consider a syzygy \(s = (s_{0}, \cdots, s_{n})\) between the vectors \(\mathbb{X}^{[0]}, \cdots, \mathbb{X}^{[n]}\), i.e. a linear relation

\[
\sum_{k=0}^{n} \delta_{k}(J) \mathbb{X}^{[k]} = 0
\]

with polynomial coefficients \(\delta_{k}(J) \in \mathbb{C}[J]\). Such syzygys form the syzygy module \(\text{Syz}_{n}(\mathbb{X})\), which is finitely generated and free according to Hilbert’s syzygy theorem \([10]\). Multiplying both sides of Eq.\((2.4)\) by \(\mathbb{E}(\tau)\) from the left, and taking into account the definition Eq.\((2.4)\), one gets the result that the differential operator

\[
\nabla_{s} = \sum_{k=0}^{n} \delta_{k}(J) \nabla_{k}
\]

annihilates \(\mathbb{X}\). This shows that the map \(s \mapsto \nabla_{s}\) is a module isomorphism between \(\text{Syz}_{n}(\mathbb{X})\) and \(A_{n}\). Note that this isomorphism implies that each \(A_{n}\) is a finitely generated free \(\mathbb{C}[J]\)-module. One may compute a free generating set of \(\text{Syz}_{n}(\mathbb{X})\) using standard methods of commutative algebra, and the corresponding differential operators will provide a free generating set of \(A_{n}\).

This is all that is needed if one is only interested in the modular differential equations satisfied by \(\mathbb{X}\) up to some given order. If one is interested instead in the structure of the full annihilator, then one needs to understand the relation of \(A_{n+1}\) to \(A_{n}\), at least for large enough \(n\). Luckily enough, this relation is fairly
simple. Indeed, let’s denote by $\mathfrak{X}_n$ the submodule of $\mathbb{C}[J]$ generated by the vectors $\mathfrak{X}^{[0]}, \ldots, \mathfrak{X}^{[n]}$; clearly, these submodules form an increasing sequence $\mathfrak{X}_0 \subset \mathfrak{X}_1 \subset \cdots$. But the module $\mathbb{C}[J]$ is Noetherian, hence any increasing sequence of submodules saturates, in the sense that there exists a positive integer $N$ (the saturation index) such that $\mathfrak{X}_n = \mathfrak{X}_N$ for all $n \geq N$, and in particular, $\mathfrak{X}^{[n]} \subset \mathfrak{X}_N$ for $n > N$. Since $\mathfrak{X}_N$ is generated by the vectors $\mathfrak{X}^{[0]}, \ldots, \mathfrak{X}^{[N]}$, this means that for $n > N$ there exist univariate polynomials $p_k^{(n)} \in \mathbb{C}[J]$ such that

$$
\mathfrak{X}^{[n]} = \sum_{k=0}^{N} p_k^{(n)}(J) \mathfrak{X}^{[k]}.
$$

Multiplying (from the left) both sides of this equality by the fundamental matrix $\Xi(\tau)$, we get the equality

$$
\nabla_k \mathfrak{X} = \sum_{k=0}^{N} p_k^{(n)}(J) \nabla_k \mathfrak{X},
$$

from which we conclude that the operator

$$
\mathcal{D}_n = \nabla_n - \sum_{k=0}^{N} p_k^{(n)}(J) \nabla_k
$$

belongs to $A_n$. What is more, given an element $\nabla = \sum_{k=0}^{n} f_k(J) \nabla_k \in A_n$, the combination $\nabla - f_n(J) \mathcal{D}_n$ is of order less than $n$, hence belongs to $A_{n-1}$; it follows that for $n > N$ the module $A_n$ is generated by $A_{n-1}$ and $\mathcal{D}_n$, and the full annihilator $\text{Ann}_D(\mathfrak{X})$ is generated (as a $\mathbb{C}[J]$-module) by the sequence $\mathcal{D}_N, \mathcal{D}_{N+1}, \cdots$ and a generating set of $A_N$.

What could be said about the annihilator as a left ideal of $\mathbb{D}$? Using the multiplication rules Eq.(3.10), one may show that $A_{N+3}$ generates the full annihilator. This means that every modular differential equation satisfied by $\mathfrak{X} \in \mathcal{M}(\varphi)$ is a consequence of one corresponding to an element of $A_{N+3}$. In this respect, much more is true: every modular differential equation satisfied by $\mathfrak{X} \in \mathcal{M}(\varphi)$ is a consequence of one corresponding to a generator of $A_{N+1}$, so that, in order to have full control over modular equations, it is enough to determine a generating set of the latter module.

Of course, to be able to apply the above ideas, one needs first to determine the sequence of the $\mathfrak{X}^{[n]}$'s. In principle, this computation involves transcendental operations; but, thanks to Eqs.(3.12) and (3.22), one has the simple algebraic recursion

$$
\mathfrak{X}^{[n+1]} = \frac{1}{a_n} \left\{ \mathcal{D}_1(J) + b_n - (J-984)(J+744) \frac{d}{dJ} \right\} \mathfrak{X}^{[n]}.
$$

Indeed, applying both sides of Eq.(3.12) to $\mathfrak{X}$, and taking into account Eqs.(4.2) and (3.22), one arrives at Eq.(4.8). It is straightforward to compute, starting from $\mathfrak{X}^{[0]} = \mathfrak{J} \mathfrak{X}$, the sequence of $\mathfrak{X}^{[n]}$'s by using the above recursion. Once this has been done, the whole story boils down to some more or less elementary algebraic manipulations, as explained above.

In some important applications it is not the annihilator that one is really interested in, but rather the holomorphic annihilator

$$
\text{Ann}^{\text{hol}}(\mathfrak{X}) = \{ \nabla \in \mathbb{D}^{\text{hol}} \mid \nabla \mathfrak{X} = 0 \}.
$$

The first thing to note is that $\text{Ann}^{\text{hol}}(\mathfrak{X})$ inherits a grading from that of $\mathbb{D}^{\text{hol}}$, each homogeneous subspace $\text{Ann}^{\text{hol}}_n(\mathfrak{X}) = \text{Ann}^{\text{hol}}(\mathfrak{X}) \cap \mathbb{D}_n^{\text{hol}}$ being finite dimensional. An
important quantity related to this decomposition is the Hilbert-Poincaré-series

\[ H_X(z) = \sum_{n=0}^{\infty} \dim \left( \text{Ann}^{\text{hol}}(X)_n \right) z^n, \]

which characterizes the rate of growth, as a function of the grade \( n \), of the number of independent holomorphic operators annihilating the form \( X \). Note that \( H_X(z) \) is always majorized as a power series by \( H_{D^{2\infty}}(z) = 1 + z + 2z^2 + 3z^3 + \cdots \).

For a given grade \( n \geq 0 \), an element of \( \text{Ann}^{\text{hol}}(X)_n \) can be expressed as a linear combination of the operators \( E^a_4 E^b_6 D^{n-2a-3b} \) for \( 0 \leq a, b, n-2a-3b \). The coefficients in this linear combination satisfy a system of linear equations, whose coefficient matrix may be determined by considering the action of the operators \( E^a_4 E^b_6 D^{n-2a-3b} \) on the \( q \)-expansion of \( X(q) \), and by solving this system, one gets a basis of \( \text{Ann}^{\text{hol}}(X)_n \). While this direct approach is conceptually simple, its computational complexity grows rapidly with the grade \( n \), making it unsuitable to treat but the simplest cases.

A more effective approach is based on the following observation. Any element of the homogeneous subspace \( D^{2\infty} \) of grade \( n \) can be decomposed as

\[ \sum_{a, b > 0, \frac{a}{2} + \frac{b}{3} \leq n} C_n(a, b) E^a_4 E^b_6 D^{n-2a-3b}, \]

where the coefficients \( C_n(a, b) \) are complex numbers. If this sum annihilates the form \( X \), then so does

\[ \sum_{a, b > 0, \frac{a}{2} + \frac{b}{3} \leq n} C_n(a, b) \partial_n(\tau) E^a_4 E^b_6 D^{n-2a-3b}. \]

But the operators \( \partial_n(\tau) E^a_4 E^b_6 D^{n-2a-3b} \) don’t change the weight, hence they all belong to \( \mathbb{D} \), and being differential operators of order \( n - 2a - 3b \), they are proportional to \( \nabla_{n-2a-3b} \), the only basis element of \( \mathbb{D} \) of that order, i.e.

\[ \partial_n(\tau) E^a_4 E^b_6 D^{n-2a-3b} = h_{n, a, b} \nabla_{n-2a-3b}, \]

for some prefactors

\[ h_{n, a, b} = \frac{\partial_n(\tau) E^a_4(\tau)^a E^b_6(\tau)^b}{\partial_{n-2a-3b}(\tau)} \]

which are weight 0 scalar modular forms, hence univariate polynomials in the Hauptmodul \( J(\tau) \); the precise form of these polynomials is easy to work out. Comparing Eqs.(4.2) and (4.13), we get

\[ \partial_n(\tau) E^a_4 E^b_6 D^{n-2a-3b} X = \Xi(\tau) h_{n, a, b} X^{(n-2a-3b)} \]

hence the element Eq.(4.11) belongs to \( \text{Ann}^{\text{hol}}(X)_n \) if, and only if the following linear relation holds:

\[ \sum_{a, b > 0, \frac{a}{2} + \frac{b}{3} \leq n} C_n(a, b) h_{n, a, b} X^{(n-2a-3b)} = 0. \]

The sequence \( X^{[0]}, X^{[1]}, \ldots \) can be determined using the recursion relation Eq.(4.8), and the polynomials \( h_{n, a, b} \in \mathbb{C}[J] \) are known, so Eq.(4.16) is a linear system for the numerical coefficients \( C_n(a, b) \): solving this system, one gets a basis of \( \text{Ann}^{\text{hol}}(X)_n \). While conceptually a bit more involved, this method is much more effective than the direct approach based on the consideration of \( q \)-expansions.
5. A worked-out example: the Ising model

The Ising model [7] is the Virasoro minimal model of central charge $c = 1/2$. Its character vector is known to be

\begin{equation}
\mathbb{X} = \frac{1}{2} \begin{pmatrix} f + f_1 \\ f - f_1 \sqrt{2} \end{pmatrix},
\end{equation}

where

\begin{equation}
f(\tau) = q^{-1/48} \prod_{n=0}^{\infty} \left( 1 + q^{n+\frac{1}{2}} \right),
\end{equation}

\begin{equation}
f_1(\tau) = q^{-1/48} \prod_{n=0}^{\infty} \left( 1 - q^{n+\frac{1}{2}} \right),
\end{equation}

\begin{equation}
f_2(\tau) = \sqrt{2} q^{1/24} \prod_{n=1}^{\infty} (1 + q^n)
\end{equation}

are the classical Weber functions. These satisfy the identities

\begin{align}
f_8^2 + f_8^2 &= f_8^8, \\
f f_1 f_2 &= \sqrt{2},
\end{align}

and are related to the Hauptmodul trough

\begin{equation}
J + 744 = \left( \frac{f_2^4 - 16}{f_2^4} \right)^3 = \left( \frac{f_1^4 + 16}{f_1^4} \right)^3 = \left( \frac{f_2^4 + 16}{f_2^4} \right)^3.
\end{equation}

The character vector Eq.(5.1) is a modular form for the weight 0 automorphy factor characterized by

\begin{equation}
\phi^0_{(0, -1)} = \frac{1}{2} \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix},
\end{equation}

\begin{equation}
\phi^0_{(1, -1)} = \zeta \begin{pmatrix} 1 & -1 & \sqrt{2} \zeta^3 \\ 1 & -1 & -\sqrt{2} \zeta^3 \\ \sqrt{2} & \sqrt{2} & 0 \end{pmatrix},
\end{equation}

where $\zeta = \exp\left(\frac{2\pi i}{48}\right)$. A suitable exponent matrix reads

\begin{equation}
\Lambda = \frac{1}{48} \begin{pmatrix} 47 & 23 \\ 2 & \end{pmatrix}.
\end{equation}

The fundamental matrix

\begin{equation}
\begin{pmatrix} f + f_1 \\ f - f_1 \sqrt{2} \\ \frac{f_2}{\sqrt{2}} \end{pmatrix}, \quad \begin{pmatrix} \frac{f_2^2 - f_1^2}{2} \\ \frac{f_2^2 + f_1^2}{2} - 25f + 25f_1 \\ \frac{f_2^2}{\sqrt{2}} - (25 + f_2^4) \frac{f_2}{\sqrt{2}} \end{pmatrix}, \quad \begin{pmatrix} 8(f_1^4 + f_2^4) - f_1^4 - 16f_2^4 + \frac{f_2^7}{\sqrt{2}} (f_1^4 + f_2^4) - 16f_1^4 - 16f_2^4 + \frac{f_2^7}{\sqrt{2}} (f_1^4 + f_2^4) - 16f_1^4 + 32f_1^4 \end{pmatrix}
\end{equation}
for this automorphy factor has been determined in [3], with corresponding characteristic matrix

\[
\mathcal{X} = \begin{pmatrix} 0 & 2325 & 94208 \\ 1 & 275 & -4096 \\ 1 & -25 & -23 \end{pmatrix}
\]

The character vector of the Ising model clearly equals the first column of the fundamental matrix, which implies that its polynomial representation has the simple form

\[
\mathcal{X}^0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
\]

Starting from this, it is straightforward to compute the sequence of \(X[^n]s\) using the recursion relation Eq. (4.8), leading to the result

\[
X[^0] = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \\
X[^1] = \frac{1}{48} \begin{pmatrix} 240 - J \\ 24 \\ 3 \end{pmatrix}, \\
X[^2] = \frac{1}{768} \begin{pmatrix} 3J + 1448 \\ 112 \\ -7 \end{pmatrix}, \\
X[^3] = \frac{1}{36864} \begin{pmatrix} 23648 - 51J \\ 856 \\ 107 \end{pmatrix}, \\
X[^4] = \frac{1}{1769472} \begin{pmatrix} 1275J + 288088 \\ -2080 \\ -2507 \end{pmatrix}, \ldots
\]

From this follows that the saturation index is \(N = 2\), i.e. \(X_n = X_2\) for \(n > 2\), and that there are no syzygies between the generators of \(X_2\), i.e. \(\text{Syz}_2(X)\) (hence \(A_2\) as well) is trivial. By expressing the higher \(X[^n]s\) in terms of \(X[^0], X[^1]\) and \(X[^2]\), one gets

\[
\mathcal{D}_3 = \nabla_3 - \frac{107}{2304} \nabla_1 + \frac{23}{55296} (J - 984) \\
\mathcal{D}_4 = \nabla_4 - \frac{107}{2304} \nabla_2 + \frac{23}{18432} (J - 984)
\]

Since \(A_3\) is generated by \(\mathcal{D}_3\) (because \(A_2\) is trivial), all modular equations satisfied by \(X\) are trivial consequences of the single equation \(\mathcal{D}_3 X = 0\).

As to the holomorphic annihilator \(\text{Ann}^{\text{hol}}(X)\), its Hilbert-Poincaré series reads

\[
H_X(z) = z^3 + z^4 + 2z^5 + 3z^6 + \cdots = \frac{z^3}{(1 - z)(1 - z^2)(1 - z^3)},
\]

showing clearly that \(\text{Ann}^{\text{hol}}(X)\) is generated, as an ideal of \(\mathcal{D}^{\text{hol}}\), by a single operator of grade 3, whose expression is (up to a multiplicative constant)

\[
\mathcal{D}_3(\tau)^{-1} \mathcal{D}_3 = \mathcal{D}^3 - \frac{107}{2304} \mathcal{E}_5 \mathcal{D} + \frac{23}{55296} \mathcal{E}_6,
\]

showing that, up to an irrelevant multiplicative factor, the modular equation \(\mathcal{D}_3 X = 0\) is a holomorphic equation. This is precisely the one that follows from the null vector relation for the characters of the Ising model \([13]\).

6. Summary

Differential equations satisfied by modular forms have been of interest since the time of Jacobi. The development of String Theory and two dimensional CFT has led to major advances in the application of the theory of modular forms to physics, and the important role of the modular equations satisfied by them has been clear since the early days. The work of Zhu \([28]\) and of Gaberdiel and Keller \([13]\) clarified the relation of modular equations and the structure of the operator algebra,
The present note addressed the question: given a vector-valued modular form $X$ for some weight zero automorphy factor $\rho$, determine all modular differential equations with (weakly) holomorphic coefficients that are satisfied by $X$. As we have seen, there exist effective algorithmic techniques, based on the general theory of vector-valued modular forms, that provide full control over all such modular equations.

An important possible use is to the existence problem of RCFT: in some applications (e.g.\cite{11}), one faces the question whether RCFTs with given properties (fusion rules, modular properties, torus partition function, etc.) do exist or not. In such cases one can use the above methods to check whether there exists potential character vectors that are consistent with the given data, have non-negative integral $q$-expansion coefficients and do satisfy suitable modular differential equations. If no such candidate character vectors can be found, then one can conclude that no RCFT with a consistent operator algebra exists with the given properties; on the other hand, for each consistent candidate character vector the holomorphic modular equation that it satisfies characterize the null vector relations, hence the representation theory of the would-be operator algebra.

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