High-precision Asteroseismology in a Slowly Pulsating B Star: HD 50230

Tao Wu1,2,3 and Yan Li1,2,3,4

1 Yunnan Observatories, Chinese Academy of Sciences, 396 Yangfangwang, Guandu District, Kunming, 650216, People’s Republic of China; wutao@ynao.ac.cn
2 Key Laboratory for the Structure and Evolution of Celestial Objects, Chinese Academy of Sciences, 396 Yangfangwang, Guandu District, Kunming, 650216, People’s Republic of China
3 Center for Astronomical Mega-Science, Chinese Academy of Sciences, 20A Datun Road, Chaoyang District, Beijing, 100012, People’s Republic of China
4 University of Chinese Academy of Sciences, Beijing 100049, People’s Republic of China

Received 2018 May 30; revised 2019 June 17; accepted 2019 June 17; published 2019 August 14

Abstract

The slowly pulsating B star HD 50230, which is in fact a hybrid B-type pulsator, has been observed by CoRoT for at least 137 days. Nearly equidistant period spacing patterns are found among eight modes that are extracted from the oscillation spectrum with more than 500 frequencies. However, it is thought to be most likely accidental by Szewczuk et al. In the present work, we analyze the eight modes in depth with the \( \chi^2 \)-matching method. Based on the best-fitting model for CoRoT data, we find that they can be well explained as sequences of consecutive dipolar \((l, m) = (1, 0)\). The period discrepancies between observations and the best-fitting model are within 100 s except for the outlier, which is up to 300 s. Based on the calculated \( \chi^2 \)-minimization models, we find that, for pure g-mode oscillations, the buoyancy radius, \( A_0 \), can be precisely measured with the \( \chi^2 \)-matching method between observations and calculations. It represents the “propagation time” of the g-mode from the stellar surface to the center. It is of \( A_0 = 245.78 \pm 0.59 \) \( \mu \)Hz with a precision of 0.24\%. In addition, we also find that HD 50230 is a metal-rich \((Z_{\text{init}} = 0.034–0.043)\) star with a mass of \( M = 6.15–6.27 \, M_\odot \). It is still located in the hydrogen-burning phase with central hydrogen of \( X_C = 0.298–0.316 \) (or \( X_C = 0.306^{+0.002}_{-0.005} \)); therefore, it has a convective core with a radius of \( R_C = 0.525–0.536 \, R_\odot \) (or \( R_C = 0.531^{+0.005}_{-0.006} \, R_\odot \)). In order to interpret the structure of the observed period spacing pattern well, the convective core overshooting \((f_{\text{ov}} = 0.0175–0.0200)\) and the extra diffusion mixing \((\log D_{\text{mix}} = 3.7–3.9)\) should be taken into account in theoretical models.

Key words: asteroseismology – stars: fundamental parameters – stars: individual (HD 50230) – stars: interiors – stars: oscillations (including pulsations)

1. Introduction

Slowly pulsating B stars (hereafter, SPB stars) are the upper main-sequence stars of intermediate masses \((2.5 \sim 8 \, M_\odot)\); for more descriptions, see, e.g., Aerts et al. 2010). Their effective temperatures range from about 11,000 to 22,000 K with spectral types between B3 and B9. Their pulsation pattern represents the characteristic of nonradial, high-order, low-degree, multiperiodic g-mode oscillations with a period of about 0.5–3 days (Aerts et al. 2010). The oscillations of SPB stars are thought to be excited by the \( \kappa \)-mechanism, i.e., the opacity is enhanced due to the ionization of iron-group elements, also called the Z-bump, at a temperature of about \( 200,000 \) K \((\log T \sim 5.3)\); see, e.g., Aerts et al. 2010; Moravveji et al. 2016).

Recently, more and more SPB stars have been observed and abundant seismological data have been collected (e.g., Degroote et al. 2010, 2012; Pápics et al. 2012, 2014, 2015, 2017; Moravveji et al. 2015, 2016; Triana et al. 2015; Briquet et al. 2016; Bowman et al. 2018; Buysschaert et al. 2018; Zhang et al. 2018; Pedersen et al. 2019) via ground- and space-based missions, such as CoRoT (e.g., Baglin et al. 2006), Kepler (e.g., Borucki et al. 2010; Gilliland et al. 2010; Koch et al. 2010), K2 (e.g., Haas et al. 2014; Howell et al. 2014), and the Transiting Exoplanet Survey Satellite (TESS; e.g., Ricker et al. 2015). Some SPB stars have extremely slow rotations, such as KIC 10526294 with a rotational period of \( P_{\text{rot}} \sim 188 \) days (e.g., Pápics et al. 2014; Moravveji et al. 2015); some of them are moderate or faster rotators, such as KIC 7760680 with a rotational frequency of \( f_{\text{rot}} \sim 0.48 \) day\(^{-1}\) (Moravveji et al. 2016) and KIC 3459297, KIC 6352430A, KIC 4930889 \((A')\), KIC 9020774, and KIC 11971405 with rotational frequencies of \( f_{\text{rot}} \sim 0.63, 0.64, 0.74, 1.06, \) and 1.62 day\(^{-1}\), respectively (see Pápics et al. 2017, refer to Table 14). In addition, some SPB stars have both rotation and magnetic field—for instance, \( \zeta \) Cassiopeiae has a moderate rotation period of \( P_{\text{rot}} = 5.370447 \pm 0.000078 \) days and a weak magnetic field whose intensity is about 100–150 G (see Briquet et al. 2016); HD 43317 has a lower rotation period of \( P_{\text{rot}} = 0.897673 \) days and a magnetic field of \( B_p = 1312 \pm 332 \) G (see Buysschaert et al. 2018).

Recently, Moravveji et al. (2015) made detailed model analyses for the ultra-slow rotating SPB star KIC 10526294. Finally, they found that the profile of elements, i.e., the shape of the buoyancy frequency \((N)\) beyond and near the convective core can be constrained by comparing theoretical models with the observed 19 dipole prograde \((l, m) = (1, 0)\) g-modes through period spacing pattern (i.e., period spacing versus period). In the process, they also found that the exponentially decaying diffusion in convective core overshooting \((f_{\text{ov}} = 0.017–0.020)\) is better than the corresponding step function formulation; in addition, an extra diffusion mixing \((\log D_{\text{mix}} = 1.75–2.00)\) is necessary to explain the sine-like structure in the observed period spacing pattern in this target.
Moravveji et al. (2016) analyzed 37 clearly identified dipole prograde \((l, m) = (1, +1)\) g-modes in KIC 7760680. Similar to Moravveji et al. (2015) in KIC 10526294, they also found that the exponentially decaying diffusion in convective core overshooting is better than the corresponding step function formulation. In addition, they found that the convective core overshooting \((f_{\text{osc}} \approx 0.024 \pm 0.001)\) can coexist with moderate rotation \((f_{\text{rot}} \sim 0.48 \text{ day}^{-1})\). In order to interpret the sine-like structure of period spacing pattern, the extra diffusion mixing is also necessary in theoretical models. But they found that the optimal extra diffusion mixing coefficient of \(\log D_{\text{mix}} \approx 0.75 \pm 0.25\) is notably smaller than that of KIC 10526294 \((\log D_{\text{mix}} = 1.75 \pm 2.00)\), in that KIC 7760680 rotates faster and also has an older age compared to KIC 10526294.

Buysschaert et al. (2018) modeled HD 43317 via 16 identified modes that are retrograde modes with \((l, m) = (1, -1)\) and \((2, -1)\) and one distinct prograde \((2, 2)\) mode and constrained the surface strength of the magnetic field \((B_g = 1312 \pm 332 \text{ G})\), except for precisely determining its fundamental parameters. On the other hand, they found that the magnetic field caused a suppression of near-core mixing in this star and that it has a lower convective core overshooting parameter \((f_{\text{osc}} = 0.004^{+0.014}_{-0.002})\). These previous works (Moravveji et al. 2015, 2016; Buysschaert et al. 2018) give us so many new insights into seismically modeled SPB stars and help us understand the interior structure of SPB stars on a higher level, such as the connection among the extra diffusive mixing, the shape of the buoyancy frequency \((N)\) and the structure of the period spacing pattern, as well as the relationship and/or interaction among convective core overshooting, rotation, and the magnetic field.

However, there are still significant discrepancies between observed periods and modeled calculated ones. The maximum period discrepancy is up to about 500 s for both KIC 10526294 (see Figure 4 of Moravveji et al. 2015) and KIC 7760680 (see Figure 8 of Moravveji et al. 2016). For HD 43317, the period discrepancy between observations and the best-fitting model is up to thousands of seconds (see Buysschaert et al. 2018, Table 3). So, what factors cause such large period discrepancy between the observations and the best-fitting models? Up until now, the question has remained unanswered.

In the present work, we make detailed seismic analyses of a unique SPB star, HD 50230, in order to probe its interior structures. HD 50230 has a larger mass \((7–8 M_\odot);\) Degroote et al. (2010), which is close to the upper limit of the mass range of SPB stars \((2.5–8 M_\odot)\) and is larger than that of the other analyzed SPB stars (refer to, e.g., Moravveji et al. 2015, 2016; Buysschaert et al. 2018). It also has a higher metallicity \((\log Z/Z_\odot = 0.3)\) and stays in a binary system (for a more detailed introduction about HD 50230, see Section 2).

### 2. HD 50230

HD 50230 is a metal-rich \((\log Z/Z_\odot = 0.3 \text{ dex}; Z_\odot = 0.02)\) young massive star with a spectral type of B3V and a visual magnitude of 8.95 (e.g., Degroote et al. 2010, 2012; Szewczuk et al. 2014). It is the primary component of a binary system with a rotational velocity \((V_{\text{eq}} \sin i) = 6.9 \pm 1.5 \text{ km s}^{-1}\). The effective temperature is of \(T_{\text{eff}} = 18,000 \pm 1500 \text{ K}\), and surface gravity to be of \(\log g = 3.8 \pm 0.3\) (c.g.s. units; refer to Degroote et al. 2012). For the effective temperature of the secondary component, Degroote et al. (2012) give an upper limit of \(T_{\text{eff}, 2} \leq 16,000 \text{ K}\) when assuming surface gravity \(\log g \approx 4\) (c.g.s. units).

Based on CoRoT observations of \(\sim 137\) days, Degroote et al. (2010) extracted eight peaks, which are listed in Table 1, from the oscillation power spectra. These peaks are almost uniformly spaced in period with the mean spacing of \(9418 \text{ s}\) and a deviation of about \(200 \text{ s}\). Degroote et al. (2010) interpreted them as consecutive radial orders, \(n\), with the same spherical harmonic degree, \(l\), and azimuthal order, \(m\). Based on the assumption of \(l = 1\) and \(m = 0\), Degroote et al. (2010) found the mass of HD 50230 to be of \(7–8 M_\odot\), with an overshooting extension of \(\epsilon_\text{ov} \geq 0.2 H_p\) \((H_p\text{ pressure scale height})\) by the step overshooting description in the radiative regions adjacent to the convective core (see Szewczuk et al. 2014 for an overview). In addition, Degroote et al. (2010) suggested that the HD 50230 stays in the middle of the main sequence, and about 60% of the initial hydrogen in its center has already been consumed.

Motivated by the detection of period spacings, Degroote et al. (2012) reanalyzed the observations. They extracted 566 frequencies from the oscillation power spectra, including high- and low-order g-mode and low-order p-mode. Most of them are not clearly identified. Based on those extracted frequencies, they obtained a rotational splitting of p-mode to be of \(\Delta f_{\text{obs}} = 0.044 \pm 0.007 \text{ day}^{-1}\) via the analysis of the autocorrelation of the periodogram between 10 and 15 day\(^{-1}\). Finally, Degroote et al. (2012) suggested that the rotational effects should be taken into account in modeling when interpreting the small deviations from the uniform period spacings despite the fact that the surface rotational velocity has merely an order of magnitude of \(10 \text{ km s}^{-1}\).

Recently, Szewczuk et al. (2014) reanalyzed the light curves of HD 50230 and extracted 515 frequencies to try to interpret the oscillation spectra in depth and to reidentify the detected frequencies. They found three series modes nearly uniformly spaced in periods from these extracted frequencies. But they have some common modes. The largest series has 11 frequencies and includes 8 frequencies that Degroote et al. (2010, 2012) determined (see Figure 1 of Szewczuk et al. 2014). But, finally, the 11 frequencies are thought to have different degrees \(l\) and azimuthal orders \(m\).

Assuming the equatorial velocity to have an order of magnitude of \(V_{\text{eq}} \sim 10 \text{ km s}^{-1}\) (Szewczuk et al. 2014), and assuming the radius be of \(R \sim 5 R_\odot\), the corresponding rotational frequency is about \(\Omega_{\text{rot}} \equiv V_{\text{eq}}/R \sim 0.040 \text{ day}^{-1}\), which is far smaller than the oscillation frequencies \(\nu \sim 1 \text{ day}^{-1}\) (see Table 1), i.e., \(\Omega_{\text{rot}} \ll \nu\). This indicates that the perturbation theory of rotational splitting for low-degree, high-order modes is available for the slow rotating star HD 50230.

Based on the rotational splitting of p-mode \(\Delta f_{\text{obs}} = 0.044 \pm 0.007 \text{ day}^{-1}\), and the first-order approximation of rotational splitting of low-degree, high-order modes (Dziembowski & Goode 1992; Christensen-Dalsgaard 2003; Aerts et al. 2010):

\[
\delta \omega_{\text{rot,nlin}}^1 \approx m \Omega_{\text{rot}} s \left(1 - \frac{1}{L}ight), \text{ for g mode;}
\]

\[
\delta \omega_{\text{rot,nlin}}^1 \approx m \Omega_{\text{rot},s}\, \text{ for p mode,}
\]

where \(L = l(l + 1)\), the corresponding rotational splitting of g-mode is about \(\Delta f_{\text{obs}, g,m=1} \approx 0.022 \pm 0.0035 \text{ day}^{-1}\) for dipolar modes. Correspondingly, the second-order approximation term
of the low-degree, high-order mode is about $\Delta f_{\text{obs, } g,m=1}^{II} \approx 1.2 \times 10^{-5}$ day$^{-1}$ (the second-order approximation term $\Delta f_{\text{II}}^{\alpha}$ is derived from Dziembowski & Goode 1992). The second-order approximation term can be ignored compared to the first-order approximation term $\Delta f_{\text{obs, } g,m=1}^{I}$ for target HD 50230. In addition, combining the rotational splitting of p-mode and Equation (1), we obtain the rotational frequency of HD 50230 to be of $\Omega_{\text{rot,s}} \approx 0.044 \pm 0.007$ day$^{-1}$.

### 3. Physical Inputs and Modeling

#### 3.1. Physical Inputs

In the present work, our theoretical models were computed by the Modules of Experiments in Stellar Astrophysics (MESA), which was developed by Paxton et al. (2011). It can be used to calculate both the stellar evolutionary models and their corresponding oscillation information (Paxton et al. 2013, 2015). We adopt the package "pulse" of version "v6208" to make our calculations for both stellar evolutions and oscillations (for more detailed descriptions refer to Paxton et al. 2011, 2013, 2015, 2016, 2018). The package "pulse" is a test suite example of MESA in the directory of "$\text{SMESA_DIR}/star/test_suite/pulse.""

The module of the pulsation calculation is based on the ADIPLS code, which was developed by Christensen-Dalsgaard (2008) and added into MESA by the MESA team (for more information, refer to Paxton et al. 2011, 2013, 2015, 2016, 2018).

Based on the default parameters, we adopt the OPAL opacity table GS98 (Grevesse & Sauval 1998) series. We choose the Eddington gray-atmosphere $T-\tau$ relation as the stellar atmosphere model, and treat the convection zone by the standard mixing-length theory (MLT) of Cox & Giuli (1968) with mixing-length parameter $\alpha_{\text{MLT}} = 2.0$.

In the previous asteroseismic modeling of SPB stars, Moravveji et al. (2015, 2016) reported that the exponentially decaying diffusive description is better than a step function formulation for treating the convective core overshooting. Here, we also adopt the theory of Herwig (2000) to treat the convective overshooting in the core. The overshooting mixing diffusion coefficient $D_{\text{ov}}$ exponentially decreases with distance, which extends from the outer boundary of the convective core with the Schwarzschild criterion:

$$D_{\text{ov}} = D_{\text{conv, } 0} \exp \left( \frac{-\Delta z}{f_{\text{ov}} H_{P,0}} \right)$$

where $D_{\text{conv, } 0}$ and $H_{P,0}$ are the MLT derived diffusion coefficient near the Schwarzschild boundary and the corresponding pressure scale height at that location, respectively. $\Delta z$ is the distance in the radiative layer away from that location. $f_{\text{ov}}$ is an adjustable parameter (for more detailed descriptions, refer to Herwig 2000; Paxton et al. 2011).

In addition, the element diffusion, semiconvection, thermohaline mixing, and the mass-loss were not included in the theoretical models.

The previous works (Degroote et al. 2010; Moravveji et al. 2015, 2016) suggested that extra diffusion mixing ($D_{\text{mix}}$) is necessary in theoretical models for interpreting the deviations of period spacings. In the present work, we also take it into account in the theoretical models.

Similar to the works of Moravveji et al. (2015, 2016), we also set the initial hydrogen mass fraction $X_{\text{init}} = 0.71$ which takes from the Galactic B-star standard (Nieva & Przybilla 2012). Therefore, the adjustable parameter of element compositions reduce to one, initial metal mass fraction $Z_{\text{init}}$ or helium mass fraction $Y_{\text{init}}$. They follow the relation: $X_{\text{init}} + Y_{\text{init}} + Z_{\text{init}} = 1$.

#### 3.2. Modeling and Finding the Best-Fitting Model

According to the works described above, there are four initial input parameters, stellar mass ($M$), initial metal mass fraction ($Z_{\text{init}}$), overshooting parameter in convective core ($f_{\text{ov}}$), and extra diffusion mixing (log $D_{\text{mix}}$) in theoretical models. The ranges and corresponding steps of the initial input parameters are listed in Table 2. According to the study of Degroote et al. (2010, 2012) and Szewczuk et al. (2014), we preliminarily set stellar mass $M \in [6.6, 8.2] M_{\odot}$ with a step of 0.2 $M_{\odot}$, initial metal mass fraction $Z_{\text{init}} \in [0.010, 0.040]$ with a step of 0.005, overshooting parameter $f_{\text{ov}} \in [0.010, 0.035]$ with a step of 0.005, and the extra diffusion mixing coefficient log $D_{\text{mix}} \in [2.0, 4.5]$ with a step of 0.5, respectively (Grid A hereafter).

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**Table 1**

**Summary of Observations for HD 50230**

| Fre. ID | $n_g$ | $\nu_i \pm \sigma_{\nu_i}$ | $P_i \pm \sigma_{P_i}$ | $\Delta P_i \pm \sigma_{\Delta P_i}$ |
|--------|-------|-----------------------------|--------------------------|-----------------------------------|
| $f_{028}$ | $-6$ | 1236.85 $\pm$ 0.04 | 69854.873 $\pm$ 6.777 | 9241.379 $\pm$ 20.692 |
| $f_{101}$ | $-7$ | 1092.34 $\pm$ 0.09 | 79096.252 $\pm$ 19.551 | 9640.178 $\pm$ 20.301 |
| $f_{006}$ | $-8$ | 973.67 $\pm$ 0.02 | 88756.430 $\pm$ 5.468 | 9522.432 $\pm$ 6.852 |
| $f_{005}$ | $-9$ | 879.31 $\pm$ 0.02 | 98258.862 $\pm$ 6.705 | 9186.452 $\pm$ 10.451 |
| $f_{011}$ | $-10$ | 804.13 $\pm$ 0.02 | 100445.314 $\pm$ 8.017 | 9562.853 $\pm$ 12.436 |
| $f_{016}$ | $-11$ | 738.41 $\pm$ 0.02 | 110048.166 $\pm$ 9.508 | 9233.798 $\pm$ 14.590 |
| $f_{003}$ | $-12$ | 684.40 $\pm$ 0.02 | 126241.964 $\pm$ 11.067 | 9430.037 $\pm$ 33.819 |
| $f_{002}$ | $-13$ | 636.83 $\pm$ 0.05 | 135672.000 $\pm$ 31.956 | ... |

**Notes.** Observations include frequencies ($\nu_i$), associated periods ($P_i$) and period spacings ($\Delta P_i$), and the corresponding observational uncertainties ($\sigma_{\nu_i}$, $\sigma_{P_i}$, and $\sigma_{\Delta P_i}$) respectively. The radial order ($n_g$) is decided from the best-fitting model.

* These observational frequencies and the corresponding observational uncertainties ($\nu_i \pm \sigma_{\nu_i}$) come from Degroote et al. (2012, Table A.2). Periods ($P_i$) and period spacings ($\Delta P_i$) are derived from the frequencies ($\nu_i$). Similar to Table 1 of Degroote et al. (2012), the uncertainties of periods and period spacings ($\sigma_{P_i}$ and $\sigma_{\Delta P_i}$) correspond to 3$\sigma_{\nu_i}$ in the table.
Based on the above initial input parameters, we use MESA to calculate the corresponding stellar models and oscillation frequencies. Similar to the work of Wu & Li (2016, 2017a, 2017b), we will use a $\chi^2$-matching method to compare the observations and models to search the best-fitting model. In the process, we also merely use the asteroseismic information (periods and/or period spacings) to constrain the theoretical models. First, we decide on the $\chi^2$-minimization model (CMM hereafter) from every evolutionary track, as shown in Figure 1. Finally, we determine the best-fitting model from the selected CMMs. The $\chi^2$ is defined as

$$
\chi^2_\rho = \frac{1}{N} \sum_{i=1}^{N} \left( \frac{P_{\rho_i}^{\text{obs}} - P_{\rho_i}^{\text{mod}}}{\sigma_{P_{\rho_i}^{\text{obs}}}^{\text{obs}}} \right)^2,
$$

$$
\chi^2_{\Delta \rho} = \frac{1}{N-1} \sum_{i=1}^{N-1} \left( \frac{\Delta P_{\Delta \rho_i}^{\text{obs}} - \Delta P_{\Delta \rho_i}^{\text{mod}}}{\sigma_{\Delta P_{\Delta \rho_i}^{\text{obs}}}^{\text{obs}}} \right)^2,
$$

$$
\chi^2 = \frac{1}{2N-1} \left[ N \chi^2_{\rho} + (N-1) \chi^2_{\Delta \rho} \right],
$$

where $N=8$, the superscripts “obs” and “mod” represent observations and theoretical calculations, respectively. $P_{\rho_i}$ and $\Delta P_{\Delta \rho_i}$ are the oscillation period and period spacing, respectively. $\sigma_{P_{\rho_i}^{\text{obs}}}^{\text{obs}}$ and $\sigma_{\Delta P_{\Delta \rho_i}^{\text{obs}}}^{\text{obs}}$ denote their corresponding observational errors, which are listed in Table 1.

In order to obtain the real best-fitting model, the ranges and the steps of the initial input parameter spaces are adjusted step by step during the model calculations. As shown in Table 2, from Grid A to Grid B to final Grid C, the model grids become denser and denser. Correspondingly, as shown in Figure 2 the minimum value of $\chi^2_{\text{CMM}}$ becomes smaller and smaller. Finally, more than 8000 evolutionary tracks are calculated, i.e., more than 8000 CMMs are determined.

It can be seen from Equation (3) and Figure 1 that there are three methods used to determine the CMMs. The corresponding best-fitting models are noted as models MA—$\chi^2_{\rho}$, MP—$\chi^2_{\rho}$, and MDP—$\chi^2_{\Delta \rho}$, respectively. Their fundamental parameters are listed in Table 3. For a given evolutionary track, the CMMs decided by different methods might be different, especially for that of $\chi^2_{\Delta \rho}$, as shown in Figure 1. While they are almost consistent overall for constraining the optimal parameter ranges of the target HD 50230 from those calculated models. In the present work, we will mainly analyze the combination term, i.e., $\chi^2$.

### 4. Results

The determined CMMs are shown in Figure 2 for matching goodness $\chi^2_{\text{CMM}}$ against different initial inputs: stellar mass ($M$)—panel (a), initial metal mass fraction ($Z_{\text{init}}$)—panel (b), overshooting parameter ($f_{\text{ov}}$)—panel (c), and extra diffusion mixing coefficient ($\log D_{\text{mix}}$)—panel (d), respectively. In addition, $\chi^2_{\text{CMM}}$ against period spacing $\Delta \Pi_{l=1}$ and the other fundamental parameters, such as central hydrogen $X_C$, stellar age $t_{\text{age}}$, and radius $R$, are shown in Figures 3–5.

#### 4.1. Buoyancy Radius $\Lambda_0$

The work of Wu & Li (2016) suggested that the acoustic radius $r_0$ is the only global parameter that can be accurately measured by the $\chi^2$-matching method between observed frequencies and theoretical model calculating ones for pure p-mode. This means that the pure p-mode mainly carries information about the acoustic size of the p-mode propagation cavity in stars.

Similarly, as shown in Figure 3, the CMMs converge into one point (or a narrow range) on the period spacing $\Delta \Pi_{l=1}$ and/or buoyancy radius $\Lambda_0$, which characterizes the buoyancy size of the g-mode propagation cavity (for more descriptions, see Appendix A) except for those outliers whose $\chi^2_{\text{CMM}}$ are larger than $3 \times 10^4$.

However, these CMMs distribute into a very large range of other fundamental parameters, as shown in Figure 4, such as the central hydrogen $X_C$, radius $R$, surface gravity $g$, and the mass of convective core $M_{\text{cc}}$. The distribution of CMMs on the radius of convective core $R_{\text{cc}}$ has similar behaviors with that of period spacing $\Delta \Pi_{l=1}$ and/or buoyancy radius $\Lambda_0$, but it has a slightly larger range (about $R_{\text{cc}} \sim 0.52–0.54 R_\odot$).
As shown in Figure 3, the distribution of CMMs on period spacing $\Delta \Pi$ resembles a flying bird with a pair of asymmetrical wings. Those CMMs can be roughly divided into two parts by a horizontal dashed line of $c = 3 \times 10^4$. The normal part, which has smaller $c \leq 3 \times 10^4$, constructs the "body" of the bird, while the outliers, which have larger $c > 3 \times 10^4$, have larger or smaller period spacings $\Delta \Pi$ compared with the normal part and make up the asymmetrical "wings."

It can be seen from Figure 2 that the centers of the optimal initial inputs are about 6.2 on $M_{\text{init}}$, 0.041 on $Z_{\text{init}}$, 0.01785 on $f_{\text{ov}}$, and 3.8 on log $D_{\text{mix}}$, respectively. Figure 6 intuitively illustrates the influence of extreme inputs for the value of $c$. In the figure, point size is proportional to $|\Delta \Pi|$, i.e., the larger point denotes larger discrepancy from the center value of 3.8 for the input log $D_{\text{mix}}$. The light or deep colors represent the discrepancy of $f_{\text{ov}}$ from the optimal center value of 0.01875. It can be easily found from Figures 2, 3, and 6 that those outliers possess extreme input parameters in stellar mass $M_{\text{init}}$, initial metal mass fraction $Z_{\text{init}}$, overshooting parameter $f_{\text{ov}}$, and extra diffusion coefficient log $D_{\text{mix}}$.

It can be found from Figures 3 and 6 that both $\Delta \Pi$ and $c$ are seriously affected by the extreme initial inputs. The above analysis indicates that the farther initial inputs are far away from the center (or proper) values; the larger the values of $c$ are, the larger or smaller the period spacing $\Delta \Pi$ and/or buoyancy radius $\Lambda_0$ of the CMMs will be.

As shown in Figure 3 the period spacing $\Delta \Pi$ of HD 50230 can be accurately determined from the $\chi^2$-matching method. It is $\Delta \Pi = 9038.4 \pm 21.8$ s, i.e., $\Delta \Pi = 9016.6 - 9060.2$ s, which is decided from those 65 selected better candidates whose $c$ values are smaller than 75 as shown in the smaller panel of Figure 3. Its relative precision is about 0.24%. Correspondingly, the buoyancy radius is $\Lambda_0 = 245.78 \pm 0.59 \mu$Hz, i.e., those selected better candidates distribute in the range of 245.19–246.37 $\mu$Hz on buoyancy radius $\Lambda_0$ as shown in the right panel of Figure 3.

What the buoyancy radius $\Lambda_0$ is to g-mode oscillations, the acoustic radius $\tau_0$ is to p-mode ones (for more information, see Appendix A). Both of them represent the "Propagation Time" of oscillation waves of g- and p-modes, respectively, from stellar surface to center, i.e., the "size" of oscillating cavities. The distribution of CMMs on $\Lambda_0$ and initial inputs indicates that proper inputs correspond with suitable oscillating cavities.

4.2. Fundamental Parameters

It can be seen from Figure 4 that the distribution of CMMs on other fundamental parameters (i.e., $\chi^2_{\text{CMM}}$ against the other fundamental parameters) seems disordered compared to $\chi^2_{\text{CMM}}$ versus $\Delta \Pi$ (Figure 3). Therefore, it is very difficult to directly...
As shown in Figure 7 and Table 3 the surface gravity of HD 50230 is \( \log g = 3.722^{+0.023}_{-0.022} \) (for model MA), which is in good agreement with the spectroscopic observation of \( \log g = 3.8 \pm 0.3 \) (Degroote et al. 2012). The effective temperature is about \( T_{\text{eff}} = 14,900^{+600}_{-300} \) K, which is consistent with the spectroscopic observation of \( T_{\text{eff}} = 18,000 \pm 1500 \) K (Degroote et al. 2012) with \( 2\sigma_{\text{eff}} \).

Degroote et al. (2010, 2012) and Szewczuk et al. (2014) suggested that SPB star HD 50230 is a metal-rich (\( \log Z/Z_\odot = 0.3 \) dex; \( Z_\odot = 0.02 \) star, i.e., \( Z \approx 0.04 \). In the present work, the optimal range of better candidates in metallicty (\( Z_{\text{init}} \)) is of 0.034–0.043 (or \( Z_{\text{init}} = 0.041^{+0.002}_{-0.007} \)), which is consistent with that of the literature.

In addition, the calculations indicate that HD 50230 has a convective core. Its radius (\( R_{\text{cc}} \)) and mass (\( M_{\text{cc}} \)) are 0.525–0.536 \( R_\odot \) (or \( R_{\text{cc}} = 0.531^{+0.005}_{-0.006} R_\odot \)) and 1.004–1.063 \( M_\odot \) (or \( M_{\text{cc}} = 1.028^{+0.035}_{-0.024} M_\odot \)), respectively.

4.2.1. Statistical Analysis: Probability Distribution of Fundamental Parameters

Similar to the work of Giammichele et al. (2016), Gai et al. (2018), and Tang et al. (2018), we calculate the likelihood function in 5D parameter space

\[
L(x_1, x_2, x_3, x_4, x_5) \propto e^{-\frac{1}{2} x^2},
\]

from the merit function, \( x^2(x_1, x_2, x_3, x_4, x_5) \). For the five independent variables, four of them are the initial input parameters, i.e., stellar mass (\( M \)), initial metal mass fraction (\( Z_{\text{init}} \)), convective overshooting parameter (\( f_{\text{ov}} \)) in the center, and the extra diffusion coefficient (\( \log D_{\text{extra}} \)). The other one represents the evolutionary status of the star, such as stellar age (\( t_{\text{age}} \)), radius (\( R \)), and center hydrogen mass fraction (\( X_c \)). For a chosen parameter, such as \( x_3 \), the density of the probability function can be defined from the following integration over the full parameter range

\[
p(x_3) \propto \int L(x_1, x_2, x_3, x_4, x_5) dx_2 dx_3 dx_4 dx_5 \propto e^{-\frac{1}{2} x^2} dx_2 dx_3 dx_4 dx_5.
\]

Then, normalizing the density of the probability function, i.e., assuming that the integration of \( p(x_3)dx_3 \) over the allowed parameter range \([x_{3,\text{min}}, x_{3,\text{max}}]\) is equal to 1, to decide the normalization factor,

\[
\int p(x_3)dx_3 = 1.
\]

Based on such a statistical analysis method, the distributions of the density of the probability function for initial input parameters and the other stellar fundamental parameters are determined and shown in Figures 8 and 9, respectively.

As shown in Figure 8 the profile of the density of the probability function is similar to a \( \delta \)-like function. The probability centers on a narrow range. In order to determine their optimal values and the corresponding uncertainties, we adopt a Gaussian function to fit the density of probability for the four initial input parameters: \( M, Z_{\text{init}}, f_{\text{ov}} \), and \( \log D_{\text{extra}} \), because the parameter space of \((M, Z_{\text{init}}, f_{\text{ov}}, \log D_{\text{extra}})\) has lower resolution. However, as shown in Figure 9, we integrate the density of probability to
The depth of color represents stellar mass ($M_*$). The green lines in the zoomed panels represent $\chi^2_{\text{CMM}} = 75$.

Figure 3. $\chi^2_{\text{CMM}}$ as a function of period spacing $\Delta P$ (right) and of buoyancy radius $A_0$ (left) for all of the calculated CMMs. The smaller panels are zoomed-in images of the box in the larger panels. The depth of color represents stellar mass ($M_*$). The horizontal dashed lines represent parameters, central hydrogen ($X_c$), age (t), radius (R), effective temperature ($T_{\text{eff}}$), luminosity (L), surface gravity ($\log g$), mass, and radius of convective core ($M_*$ and $R_*$), respectively, for all of the calculated CMMs of $X_{\text{mix}} = 0.71$. The horizontal dashed lines represent $\chi^2_{\text{CMM}} = 120$ and 75, respectively. The zoomed panels are shown in Figure 5.

Figure 4. Similar to Figures 3, $\chi^2_{\text{CMM}}$ as a function of stellar fundamental parameters, central hydrogen ($X_c$), age (t), radius (R), effective temperature ($T_{\text{eff}}$), luminosity (L), surface gravity ($\log g$), mass, and radius of convective core ($M_*$ and $R_*$), respectively, for all of the calculated CMMs of $X_{\text{mix}} = 0.71$. The horizontal dashed lines represent $\chi^2_{\text{CMM}} = 120$ and 75, respectively. The zoomed panels are shown in Figure 5.

determine them for other parameters. The final decided optimal values are listed in Table 4.

4.3. Asteroseismic Analysis

The period spacings ($\Delta P$) of the observations and the best-fitting models and the differences of periods between them ($P_{\text{obs}} - P_{\text{mod}}$) are shown in Figure 10.

It can be found from the upper panel of Figure 10 that the period spacings $\Delta P$ of the best-fitting models are almost consistent with those of observations, except for the last one ($g_{13}$). As shown in the bottom panel of Figure 10 the observed periods are larger than those of model MDP, which is decided by matching period spacing $\Delta P_l$ between observations and models except for the first mode ($g_1$). For model MA, which is decided by matching both period $P_l$ and period spacing $\Delta P$, and MP, which is decided by matching period $P_l$, their periods are consistent with observations within about 100 s except for the last one ($g_{13}$) whose period is smaller than the observed one to be about 300 s.

It can be found from the upper panel of Figure 10 that the observed period spacings and the model calculated ones have different tendencies at the end of the larger period. The period spacing decreases with the increase of period for models when $P \gtrsim 1.3$ days. Based on those best-fitting models, the last observed period ($g_{13}$) seems to be an outlier for the sequence of $(l = 1, m = 0)$.

The period difference $\delta P \sim 300$ s in the last mode ($P_{\text{rot}} = 135,359.94$ s) corresponds to about $\delta f \sim -0.0014$ day$^{-1}$ (or $-0.016 \mu$Hz) in frequency. As shown in Figure 11 the rotational splitting parameters $\beta_{nl}$ are around 0.5 for all of these modes. As a matter of fact, for all of the calculated CMMs, $\beta_{nl}$ are around 0.5. They are pure g-mode. For the “outlier” corresponding mode ($g_{13}$), its rotational parameter $\beta_{nl}$ is about 0.4934. Therefore, the frequency difference of $\delta f \sim -0.0014$ day$^{-1}$ might be explained as a rotation splitting of $\Delta f_{\text{obs,g},l=1,m=1} \sim -0.0014$ day$^{-1}$, which corresponds to a rotational frequency of $\Omega_{\text{rot},l} \sim 0.0028$ day$^{-1}$. Surely such rotational frequency is far smaller than that of observation $\Omega_{\text{rot},l} \approx 0.044$ day$^{-1}$ ($=0.509 \mu$Hz, see Section 2), which is calculated from the rotational splitting of p-mode. Therefore, it is not suitable that the period difference of last mode ($g_{13}$) between observations and best-fitting models is explained as rotational splitting.
As shown in Figure 12 most of the oscillation energy is trapped in the \( \mu \)-gradient region, which is nearby the convective core, for the best-fitting model: model MA. The residual oscillation energy distributes into the outer region of the \( \mu \)-gradient region but within \( r/R \approx 0.85 \) for low-order modes. For high-order modes (see the bottom panel of Figure 12), the oscillation energy partly extends to stellar surface. According to the theory of stellar oscillations, therefore, as shown in Figure 10, the period spacings \( \Delta P \) will decrease with the increase of period \( P \) for high-order g-mode since they have larger propagation cavity compared with the low-order modes.

For the period discrepancy of the last mode \( (g_{13}) \), based on the calculated theoretical models in the present work, there are two possibilities that might explain it. The first is that it is really an outlier, and it might not belong to the prograde mode sequence of \( (l = 1, m = 0) \). The second is that it is caused by certain possible physical mechanisms that can partly decrease the frequency of the last mode to make it consistent with the observation, such as the mode trapping. But it is not properly considered in the present theoretical models.

It can be seen from Figure 10 that there is a sine-like signal on the period discrepancy between the observations and the best-fitting models. Similar phenomena also appear in previous works, such as Moravveji et al. (2015, Figures 4, 6, A.1, and A.2), Moravveji et al. (2016, Figure 8), and Buysschaert et al. (2018, Table 3). Such sine-like signals might be caused by a thin layer in the stellar interior (see, e.g., Gough & Thompson 1990; Christensen-Dalsgaard 2003), which is not considered or improperly expressed in the present theoretical models. For the best-fitting model of HD 50230, the strength of mode trapping in the \( \mu \)-gradient region beyond the convective core might be a possible factor. On the other hand, perhaps, the present best-fitting model needs an extra mode trapping cavity to slightly revise the periods. We will investigate this question in depth in the future.

### 4.4. The Influences: Outlier, Initial Hydrogen, and Observational Uncertainties

In the above sections, we used all of the eight modes to constrain theoretical models, which are calculated with a fixed initial hydrogen \( X_{\text{init}} \). In the process, we adopt the observational uncertainties as weight factors. In this section, we will discuss the influences of different initial hydrogen, observational uncertainties, and the last mode for constraining the theoretical models and deciding the final best-fitting models.

First, our method here is similar to the above section, but we only use the former seven modes (i.e., \( g_6, g_7, \ldots, g_{12} \)) to constrain theoretical models. The last mode \( (g_{13}) \) is not...
The Astrophysical Journal, 881:86 (15pp), 2019 August 10

Wu & Li

Table 4
The Optimal Variable Range of the Fundamental Parameters of HD 50230

| Variables | Ranges | 68.3% Probability |
|-----------|--------|-------------------|
| Stellar mass $M$ ($M_\odot$) | $6.15-6.27$ ($6.21 \pm 0.06$) | $6.187 \pm 0.025^{\text{c}}$ |
| Initial metal abundance $Z_{\text{init}}$ | $0.034-0.043$ ($0.041^{+0.002}_{-0.007}$) | $0.0408 \pm 0.0009^{\text{c}}$ |
| Overshooting parameter in core $f_{\text{ov}}$ | $0.0175-0.0200$ | $0.0180 \pm 0.0014^{\text{c}}$ |
| Extra mixing parameter $\log D_{\text{mix}}$ | $3.7-3.9$ ($3.8 \pm 0.1$) | $3.800 \pm 0.045^{\text{c}}$ |
| Period spacing $\Delta \Pi_1$ (s) | $9016.6-9060.2$ ($9038.4 \pm 21.8$) | $9044.75^{+9.35}_{-9.17}$ |
| Buoyancy radius $\Lambda_0$ ($\mu$Hz) | $245.19-246.37$ ($245.78 \pm 0.59$) | $245.61^{+0.40}_{-0.25}$ |

Notes.
- The range $x_1 - x_2$ represents the minimization and maximization of the parameters for the CMMs whose $\chi^2_{\text{CMM}} \leq 75.0$. For the form of $x_{\text{fit}}$, $x$ corresponds to the value of the best-fitting model (MA) as shown in Table 3. Correspondingly, $+dx$ and $-dx$ express the discrepancy between the $x$ and $x_1$, $x_2$, respectively.
- Fitting the probability distribution density with Gaussian function and adopting $1\sigma$. The fitting results are shown in Figure 8.

---

Figure 6. $\chi^2_{\text{CMM}}$ as a function of $Z_{\text{init}}$ and $M$. The light and deep colors of points are related to $100|f_{\text{ov}} - 0.01785|$. The size of points is inversely proportional to $|\log D_{\text{mix}} - 3.8|$.

---

Considered. The corresponding best-fitting models are noted as models MA7, MP7, and MDP7, respectively. The period spacings and period differences between observations are shown in Figure 13. It can be seen from Figures 10 and 13 that the final results are almost entirely the same with or without the last mode ($g_{113}$) in observations. This indicates that eliminating the last mode from observations only partly decreases the value of $\chi^2_{\text{CMM}}$ of the best-fitting model from 58.5 (model MA) to 53.5 (model MA7). In fact, model MA and model MA7 are the same model. This means that the last mode ($g_{113}$; “outlier”) does not effectively work for constraining the theoretical models in the present work.

Second, we change the initial hydrogen $X_{\text{init}}$ from the fixed value 0.71 to 0.69, 0.70, and 0.72 and calculate the corresponding theoretical models to determine the best-fitting model. For the added calculations, their parameter resolutions are the same as the above calculations (see Table 2). Finally, more than 5500 evolutionary tracks are calculated for the added initial hydrogen $X_{\text{init}} = 0.69$, 0.70, and 0.72. Similar to the best-fitting model of model MA, we note the corresponding best-fitting models as models MAX0.69, MAX0.70, and MAX0.72, respectively. Their $\chi^2_{\text{CMM}}$ are about 62.0, 65.8, and 60.0, respectively. They are larger than that of model MA (58.5).

It can be seen from Figure 14 that the best-fitting model does not obviously become better or worse when slightly changing (increase or decrease) the initial hydrogen $X_{\text{init}}$. The periods of the best-fitting models almost overlap each other. On the other hand, their $\chi^2_{\text{CMM}}$ are on the same level. They are close to about 60.
It can be seen from Table 1 (the observations) that the observational uncertainties of periods range from 5.5 s ($g_8$) to about 32 s ($g_{13}$). The ratio between them is about 5.8. Correspondingly, in Equation (3), their weight ratio is up to about 34 for calculating the value of $\chi^2$ between the two modes.

In order to analyze the influence of the observational uncertainties for the final best-fitting model, we change the
\[ \chi^2\text{-matching formulas (i.e., Equation (3)) as:} \]
\[ \Delta P_i = \frac{1}{N} \sum_{j=1}^{N} [P_{i}\text{obs} - P_{i}\text{mod}], \]
\[ \Delta \Delta P = \frac{1}{N-1} \sum_{i=1}^{N-1} |\Delta P_{i}\text{obs} - \Delta P_{i}\text{mod}|, \]
\[ \Delta \text{all} = \frac{1}{2N-1} (N \Delta P + (N-1) \Delta \Delta P), \]  

(7)
to constrain theoretical models. Here, \( \Delta P \), \( \Delta \Delta P \), and \( \Delta \text{all} \) represent the means of differences between observations and theoretical models for period \( P \), period spacing \( \Delta P \), and both of them, respectively. Correspondingly, the \( \Delta \text{all} \)-minimum model denotes model MAabs whose \( \Delta \text{all} \) is about 84.9 s. For model MA, the corresponding \( \Delta \text{all} \) is about 91.1 s. The difference of \( \Delta \text{all} \) between the two models is mainly contributed by the last period and the last period spacing since they have lower weight compared with the other modes. The period discrepancies with observations of mode \( g_5 \) decrease from about 326 s for model MA to about 235 s for model MAabs as shown in Figure 15.

The initial parameters of model MA and model MAabs are the same, except MAabs has a slightly higher initial metal abundance \( Z_{\text{init}} = 0.04375 \). In addition, their fundamental parameters are almost consistent with each other. As shown in Figure 15 the periods and period spacings for the best-fitting model (model MAabs) are not changed significantly. This indicates that the final results are not seriously changed whether using the observational uncertainties as a weight or not for determining the best-fitting model.
In addition, similarly, we eliminate the last mode \((g_{13})\) from the observations and adopt Equation (7) to determine the best-fitting model, which is named model MA7abs, similar to model MA7. The \(\Delta_{\text{all}}\) of model MA7abs is 53.3 s. Correspondingly, that of model MA7 is about 56.0 s. Models MA7 and MA7abs are located on a common evolutionary track, i.e., their initial parameters are the same. In addition, they are two adjacent models, i.e., their fundamental parameters are merely different.

As shown in the upper panel of Figure 16 the period spacings overlap with that of model MA7. For period differences \(\delta P = P_{\text{obs}} - P_{\text{mod}}\), model MA7abs is slightly larger than model MA7 overall. However, the mean values of \(|\delta P|\) are 43.8 and 48.5 s for models MA7abs and MA7, respectively.

It can be found from Figures 13, 15, and 16 that the final results are almost not affected whether using the observational uncertainties and the last mode \((g_{13})\) to constrain theoretical models or not for the present calculated models.

### 5. Summary

SPB star HD 50230 is the primary component of a binary system. It has observed about 137 days with the CoRoT satellite and more than 560 frequencies are extracted. There are eight modes identified as likely low-order g-modes with \(l = 1\) and \(m = 0\) among the extracted modes due to almost uniform period spacings among them. In addition, the period spacings periodically vary with periods. In the present work, we make model calculations and analyze the eight modes with high-precision asteroseismology. Finally, the investigation can be briefly concluded as follows:

Similar to pure p-mode oscillations, the oscillation frequencies mainly carry the information of the size of oscillation wave propagation cavity and the acoustic radius \(r_A\) is the only global parameter that can be precisely measured by the \(\chi^2\)-matching method between observed frequencies and model calculations (Wu & Li 2016), the buoyancy radius \(\Lambda_0\) also can be easily and precisely measured with a similar method for pure g-mode oscillations \((m = 0)\), as shown in Figure 3, compared to the other parameters, such as stellar age, effective temperature, and radius, which are shown in Figure 4. This is because the distribution of CMM on buoyancy radius is not sensitive for initial input parameters compared to the other fundamental parameters. Both acoustic radius and buoyancy radius represent the “propagation time” of oscillation waves from stellar surface to center for pure p- and g-modes, respectively.

Based on the calculated models, we find that the value of \(\chi^2_{\text{CMM}}\) and the distribution of CMMs on buoyancy radius \(\Lambda_0\) can be slightly affected by some extreme initial inputs. Finally, we obtain that the buoyancy radius of HD 50230 is of \(\Lambda_0 = 245.78 \pm 0.59 \mu\text{Hz}\) with a higher relative precision of 0.24%. Correspondingly, the period spacing of HD 50230 is \(\Delta \Pi_{1,1} = 9038.4 \pm 21.8\) s.

HD 50230 is a metal-rich \((Z = 0.041^{+0.002}_{-0.007})\) moderate massive star with a mass of \(M = 6.21 \pm 0.06 M_\odot\) and located on the middle phase of the main-sequence branch with an age of \(t_{\text{age}} = 61.6^{+4.0}_{-3.1}\) Myr. About 57% of the initial hydrogen is exhausted in its center \((X_C = 0.306^{+0.010}_{-0.008})\), which is close to Degroote et al.’s (2010) estimated (60%). In addition, HD 50230 has a convective core with a radius of \(R_C = 0.531^{+0.006}_{-0.005} R_\odot\). The corresponding convective core mass is \(M_{cc} = 1.028^{+0.035}_{-0.024} M_\odot\).

Based on the optimal range of stellar radius \(R = 5.50^{+0.005}_{-0.005} R_\odot\), we obtain that the rotational velocity of \(V_\text{eq} = 12.65^{+2.35}_{-2.00}\) km s\(^{-1}\) with an inclination angle of \(i = 33^{\circ \pm 1} 12\). In order to interpret the structure in the observed period spacing pattern of HD 50230, the exponentially decaying diffusive core overshooting \((f_{\text{ov}} = 0.0175 \pm 0.0200)\) and the
extra diffusive mixing (log $P_{\text{mix}} = 3.7–3.9$) should be taken into account in theoretical models. The theoretical models indicate that, for the eight modes, at least seven modes can be well explained as dipole g-modes of $(l, m) = (1, 0)$. Their period discrepancies between the observations and the best-fitting models are within 100 s. For the last mode $(g_{13})$, it is almost up to 300 s. In the present work, we still do not find a suitable interpretation for such large discrepancy, but we exclude the possibility that the discrepancy is caused by rotational splitting, i.e., it is not the mode of $(l, m) = (1, -1)$.

In the present work, the rotational effects are not taken into account in theoretical models. Degroote et al. (2012) predicted that the rotational effects should be considered when interpreting the structure in the observed period spacing pattern, since they will slightly change the period of the g-mode (Aerts & Dupret 2012). Perhaps, they could be helpful in explaining the larger period discrepancy between observation and best-fitting model. We will consider this in the next work.

This work is cosponsored by the NSFC of China (grant Nos. 11333006, 11503076, 11503079, 11773064, 11873084, and 11521303), and by Yunnan Applied Basic Research Projects (grant No. 2017B008). The authors gratefully acknowledge the computing time granted by the Yunnan Observatories Supercomputing Platform. The authors also express their sincere thanks to Pro. Guo, J.H., Dr. Zhang, Q.S., Dr. Su, J., and Dr. Chen, X.H. for their productive advice. Finally, the authors are cordially grateful to an anonymous referee for instructive advice and productive suggestions to improve this paper overall.

Appendix A

Propagation Velocity of g-mode and Buoyancy Radius

The propagation velocity of p-mode (i.e., adiabatic sound speed $c$) and the corresponding acoustic radius ($r_0$) are shown in the works of Unno et al. (1989), Christensen-Dalsgaard (2003), and Aerts et al. (2010) in detail. Here, we inspect it briefly as the background of those of g-mode as follows.

For high radial order oscillations, the oscillation equation can be approximately expressed as follows in the Cowling approximation (Christensen-Dalsgaard 2003, Equation (5.17) on page 76),

$$\frac{d^2 \xi_r}{dr^2} = \frac{\omega^2}{c^2} \left(1 - \frac{N^2}{\omega^2} \right) \left( \frac{S_l^2}{\omega^2} - 1 \right) \xi_r, \quad (8)$$

or

$$\frac{d^2 \xi_r}{dr^2} = -K(r) \xi_r, \quad (9)$$

where

$$K(r) = \frac{\omega^2}{c^2} \left(1 - \frac{N^2}{\omega^2} \right) \left(1 - \frac{S_l^2}{\omega^2} \right). \quad (10)$$

$S_l$ is the characteristic acoustic frequency

$$S_l = \frac{(l + 1)c^2}{r^2},$$

$N$ is buoyancy frequency and also called Brunt–Väisälä frequency. It is expressed as

$$N^2 \approx \frac{g \rho}{p} (\nabla_{\text{ad}} - \nabla_{\text{grad}}), \quad (11)$$

where, $\nabla_{\text{ad}}, \nabla$, and $\nabla_{\text{grad}}$ are the adiabatic temperature gradient, the temperature gradient, and $\mu$-gradient, respectively.

For high-order p-modes, typically $\omega^2 \gg N^2$, then $K$ can be approximately expressed as (Christensen-Dalsgaard 2003, Equation (5.29) on page 80)

$$K(r) \approx \frac{1}{c^2} (\omega^2 - S_l^2). \quad (12)$$

The length of the wave vector $|k|^2$ can be expressed as the sum of a radial component and a horizontal component, i.e.,

$$|k|^2 \equiv k_r^2 + k_h^2. \quad (13)$$

The radial component $k_r$

$$k_r^2 = K(r) = \frac{1}{c^2} (\omega^2 - S_l^2). \quad (14)$$

For high-order g-mode, typically $\omega^2 \ll S_l^2$ and thus $K$ can be approximately expressed as (Christensen-Dalsgaard 2003, Equation (3.55) on page 52)

$$\omega^2 = c^2 |k|^2. \quad (15)$$

According to the definition of phase velocity (for more description, refer to Unno et al. 1989, Equation (15.16) on page 116):

$$v_{\text{phase}} = \frac{\omega}{k}, \quad (16)$$

the propagation velocity of p-mode is $v_{\text{phase}, \text{p-mode}} = c$, i.e., p-mode propagates with sound speed in stars.

The definition of acoustic depth (Aerts et al. 2010, Equation (3.228) on page 219)

$$\tau(r) \equiv \int_r^R \frac{dr'}{c}, \quad (17)$$

presents the propagation time of the oscillating wave (p-mode) from stellar surface to stellar inner position $r' = r$. Correspondingly, the propagation time from stellar surface to center is called the acoustic radius

$$r_0 = \int_0^R \frac{dr'}{c}. \quad (18)$$

This represents the “size” (i.e., acoustic size) of the p-mode propagation cavity.

For high-order g-mode, typically $\omega^2 \ll S_l^2$ and thus $K$ can be approximately expressed as (Christensen-Dalsgaard 2003, Equation (5.33) on page 81)

$$K(r) \approx \frac{1}{\omega^2} (N^2 - \omega^2) \frac{l(l + 1)}{r^2}. \quad (19)$$
Similarly, for g-mode, the length of the wave vector can be expressed as

\[ |k|^2 = k_r^2 + k_\theta^2 = K(r) + k_\theta^2 = \frac{N^2}{\omega^2} \frac{l(l+1)}{r^2}, \]

i.e.,

\[ \omega^2 = \frac{N^2}{|k|^2} \frac{l(l+1)}{r^2}, \]

which is the dispersion relation of g-mode oscillations (a more detailed description refers to Unno et al. 1989, Equation (33.18) on page 286). Correspondingly, the propagation velocity of g-mode \( v_{\text{phase,g-mode}} \) is

\[ v_{\text{phase,g-mode}} = \frac{\omega}{\sqrt{l(l+1)} N}. \]  

(20)

This shows that the propagation velocity is directly proportional to the oscillation frequency \( \omega^2 \) and inversely proportional to the degree \( \sqrt{l(l+1)} \). Similarly to the definition of acoustic depth \( \tau(r) \)—Equation (17)—the buoyancy depth can be defined as

\[ \xi(r) \equiv \int_r^R \frac{dr'}{v_{\text{phase,g-mode}}} = \frac{\sqrt{l(l+1)}}{\omega^2} \int_r^R \frac{N}{r'} dr', \]  

(21)

which also represents the propagation time of the oscillation wave, but for g-mode oscillations. Correspondingly, the buoyancy radius, which presents the time of a g-mode oscillating wave propagating from stellar surface to center, is defined as

\[ \xi_0 \equiv \frac{\sqrt{l(l+1)}}{\omega^2} \int_0^R \frac{N}{r'} dr'. \]  

(22)

Compared with acoustic depth (or radius; Equations (17) and (18)), the buoyancy depth (or radius; Equations (21) and (22)) is not only related to the stellar structure \( (N \text{ and } r) \) but also related to the oscillation frequencies \( \omega \) and their degrees \( l \). Therefore, we adopt the quantities of

\[ \Lambda(r) = \int_r^R \frac{N}{r'} dr', \]  

(23)

and

\[ \Lambda_0 = \int_0^R \frac{N}{r'} dr', \]  

(24)

to replace \( \xi(r) \) and \( \xi_0 \) to characterize the buoyancy size (buoyancy depth and radius) of stars. Similar to \( \tau(r) \) and \( \tau_0 \), \( \Lambda(r) \) and \( \Lambda_0 \) are merely dependent on stellar structure and independence of the oscillations. Correspondingly, the dimensions of the improved buoyancy depth and radius are transformed to ones of angular frequency (radian per second) from ones of time. Correspondingly, the buoyancy radius \( \Lambda_0 \) can be expressed with period spacing \( \Delta \Pi_i \) as follows:

\[ \Lambda_0 = \frac{\pi}{\sqrt{l(l+1)}} \Delta \Pi_i^{-1}. \]  

(25)

Appendix B

Inlist File of Pulse in MESA (V6208)

```plaintext
&star_job ! HD49385
create_pre_main_sequence_model = .true.
kappa_file_prefix = 'gs98'
change_initial_net = .true.
new_net_name = 'o18_and_ne22.net'
/
! end of star_job namelist
&controls
initial_mass = 0.60875000D+01
initial_z = 0.43750000D-01
initial_y = 0.26625000D+00
overshoot_f_above_burn_h = 0.20000000D-01
min_D_mix = 0.50118723D+04
do_element_diffusion = .false. ! .true.
calculate_Brunt_N2 = .true.
!use_brunt_dlnRho_form = .true.
use_brunt_gradmuX_form = .true.
which_atm_option = 'Eddington_grey'
max_years_for_timestep = 0.1d6
varcontrol_target = 1d-4 ! for main-sequence stars (5d-4 for pre_main_sequence)
dH_hard_limit = 1d-3
mesh_delta_coef = 0.4
max_allowed_nz = 30000 ! maximum number of grid points allowed
max_model_number = 70000 ! negative means no maximum
xa_central_lower_limit_species(1) = 'hi'
xa_central_lower_limit(1) = 0.05
 mixing_length_alpha = 2.
set_min_D_mix = .true.
min_center_Ye_for_min_D_mix = 0.4 ! min_D_mix is only used when center
Ye > = this
dH_div_H_limit_min_H = 2d-1
dH_div_H_limit = 0.0005d0
dH_div_H_hard_limit = 1d-2
/
! end of controls namelist
```

ORCID iDs

Tao Wu @ https://orcid.org/0000-0001-6832-4325
Yan Li @ https://orcid.org/0000-0002-1424-3164

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