The $C^*$-algebra of the variable Mautner group

Hedi Regeiba

Received: 22 April 2022 / Accepted: 12 September 2022 / Published online: 24 September 2022
© Tusi Mathematical Research Group (TMRG) 2022

Abstract
Let $\mathbb{M} = P \times M$ be a variable Mautner group. We describe the $C^*$-algebra $C^*(\mathbb{M})$ of $\mathbb{M}$ in terms of an algebra of operator fields defined over $P \times \mathbb{C}^2$.

Keywords $C^*$-algebras · Variable groups · Algebras of operator fields · Fourier transform

Mathematics Subject Classification 22D25 · 22D10 · 43A10 · 43A20

1 Introduction
The notion of a variable group $G = P \times G$ has been introduced in [4]. Here, $G$ is a locally compact topological Hausdorff space and $P$ is a compact Hausdorff space. For every $p \in P$, we have a group multiplication $\cdot_p$ defined on $G$, such that the mappings

$$(p; x, y) \mapsto x \cdot_p y$$

and

$$(p, x) \mapsto x^{-1}_p$$

are continuous mappings on $P \times G \times G \to G$ resp. on $P \times G \to G$.

We say that the variable group is unimodular if there exists a positive Radon measure $dx$ on $G$, which is also a left and right Haar measure for the groups $G_p := (G, \cdot_p)$ for every $p \in P$.

Communicated by Michael Frank.

Hedi Regeiba
rejaibahedi@gmail.com

1 Faculté des Sciences de Gabés, Université de Gabés, Cité Erriadh Zrig (Laboratory Code: LR 17 ES 11), 6072 Gabès, Tunisia
A variable unimodular group defines the family \( \{ L^1(G_p) \}_{p \in P} \) of \( L^1 \)-algebras \( L^1(G_p) \) with respect to the Haar measures \( dx \). Let \( L^1(G) \) be the Banach space \( L^1(G, dx) \) of \( \mathbb{C} \)-valued, \( dx \)-integrable functions on \( G \), with norm \( \| f \|_1 = \int_G |f(x)|dx \). Then for every \( p \in P \), we have the convolution (see [4])

\[
F_p \ast F'_p(x) := \int_G F(y)F'_p(y^{-1} \cdot_p x)dy, \quad F, F' \in L^1(G), \ x \in G.
\]

We denote by \( C(P, L^1) \) the involutive Banach algebra of all continuous mappings \( F : P \to L^1(G) \) equipped with the product

\[
(F, F') \to (P \ni p \to F_p \ast F'_p(p)), \quad F, F' \in C(P, L^1)
\]

and the involution

\[
F^*(p) := F(p)^*, \quad p \in P, \ F, F' \in C(P, L^1).
\]

The \( C^* \)-hull of the algebra \( C(P, L^1) \) of the unimodular variable group \( \mathbb{G} \) will be denoted by \( C^*(\mathbb{G}) \). This \( C^* \)-algebra is then a subfield of the field of \( C^* \)-algebras \( (C^*(G_p))_{p \in P} \).

The classical Mautner group \( M_\theta \) is the semi-direct product

\[
M = M_\theta = \mathbb{R} \ltimes \mathbb{C}^2,
\]

where the reals act on the abelian group \( \mathbb{C}^2 \) by

\[
t \cdot (a, b) := (e^{it}a, e^{i\theta t}b), \quad t \in \mathbb{R}, \ a, b \in \mathbb{C}
\]

and where \( \theta \) is a fixed irrational number. The group \( M_\theta \) is a non type I solvable Lie group of minimal dimension and therefore the unitary dual cannot be described.

In this paper, we consider \( P = [-1, 1] \) and \( M_\theta \) as a limit group of a family \( (M_{p, \theta}, p \neq 0, \). \)

Here \( M_{p, \theta} = (M, \cdot_p), p \in P, \) is the space \( M = \mathbb{R} \times \mathbb{C}^2 \) equipped with the multiplication \( \cdot_p \), where

\[
(t, (a, b)) \cdot_p (t', (a', b')) = (t + t', (e^{-p+i\vartheta}a + a'e^{-(-p+i\vartheta)}b + b')).
\]

The groups \( M_{p, \theta}, p \neq 0 \) are unimodular exponential solvable Lie groups.

The \( C^* \)-algebra of \( M_{p, \theta} \) is determined by Sudo in [7] for fixed \( p \in P \), but in this paper, we analyze the behavior of \( C^*(M_{p, \theta}) \) as \( p \) tends to 0. The main result is Theorem 3.10, which characterizes the \( C^* \)-algebra of the variable group \( \mathbb{M} \).

Let me thank here Professor Jean Ludwig for his suggestions and his help during the preparation of this paper.
2 The variable group $\mathcal{M}$

**Definition 2.1** Let $P$, and $M$, be the topological spaces

\[ P = [-1, 1], \]
\[ M = \mathbb{R} \times \mathbb{C}^2. \]

Fix a real number $\theta$.

We define the variable group $\mathcal{M} = \mathcal{M}_\theta, \mathcal{M} = P \times M$ by giving for any $p \in P$ the group multiplication $\cdot_p$ on the set $M$ through the formula (1.1).

In particular, for $p = 0$, we recover our Mautner group $M_\theta$.

Its variable Lie algebra $P \times \mathfrak{m}$ is given by

\[ P \times \mathfrak{m} = P \times \mathbb{R} \times \mathbb{C}^2 = P \times (\mathbb{R} T + \mathbb{C} U + \mathbb{C} V) \]
\[ T := (1, 0, 0), \quad U = (0, 1, 0), \quad V = (0, 0, 1), \]

and the Lie brackets

\[ [T, U] = (i + p) U, \]
\[ [T, V] = (i\theta - p) V. \]

2.1 The $C^*$-algebra of $\mathcal{M}$

**Definition 2.2** We let

\[ C_\infty(\mathcal{M}) = \{ F : P \times M \to \mathbb{C} | F \text{ smooth with compact support} \}, \]
\[ C_c(\mathcal{M}) = \{ F : P \times M \to \mathbb{C} | F \text{ continuous with compact support} \}, \]
\[ L^1(\mathcal{M}) = C(P, L^1) = \overline{C_\infty(\mathcal{M})}_{||\cdot||_1}, \]

where

\[ ||F||_1 = \sup_{p \in P} ||F(p)||_1, \quad F \in C_\infty(\mathcal{M}). \]

Also define the $C^*$-norm $||\cdot||_{C^*}$ on $C_\infty(\mathcal{M})$ by

\[ ||F||_{C^*} = \sup_{p \in P} ||F(p)||_{C^*}, \quad F \in C_\infty(\mathcal{M}). \]

This definition gives us the usual $C^*$-algebra of the involutive Banach algebra $L^1(\mathcal{M})$, which is given as the completion $\overline{L^1(\mathcal{M})}_{||\cdot||_{C^*}}$ of the algebra $L^1(\mathcal{M})$ for the norm $||\cdot||_{C^*}$.

For $F \in C_\infty(\mathcal{M}), \xi \in C_c(M)$ and $p \in P$, we then have that
This gives us the left regular representation $\Lambda$ of $\mathbb{M}$ on the Hilbert space $L^2(M)$:

$$\Lambda^p(F)\xi := F \ast_p \xi, \quad F \in L^1(\mathbb{M}), \quad \xi \in L^2(M).$$

**Remark 2.3** Denote for $p \in P$ the canonical projection $\mu_p : C^*(\mathbb{M}) \to C^*(M_p)$ defined by

$$\mu(a) := a(p), \quad a \in C^*(\mathbb{M}).$$

These mappings $\mu_p$ are surjective on $L^1(\mathbb{M})$. Indeed, if we take any $F \in L^1(M_p)$, then we can consider $F$ also as a (constant) element $\overline{F}$ of $L^1(\mathbb{M})$, where $\overline{F}(p) = F, \ p \in P$, and then

$$\mu_p(\overline{F}) = F.$$ 

Let $I_p$ the kernel of the linear mapping $\mu_p$ then $I_p \cap L^1(\mathbb{M})$ in $L^1(\mathbb{M})$ is a closed ideal of the algebra $L^1(\mathbb{M})$ and

$$L^1(\mathbb{M})/I_p \cap L^1(\mathbb{M}) \simeq L^1(M_p),$$

since $\mu_p$ is surjective. Hence, we also have that the mapping

$$\mu_p : C^*(\mathbb{M}) \to C^*(M_p), \quad p \in P, \quad (2.1)$$

is surjective.

**2.2 The dual space of $\mathbb{M}$**

It is well known (see [4]) that the spectrum $\widehat{\mathbb{M}}$ of the variable group $\mathbb{M} = P \times M$ is given by

$$\widehat{\mathbb{M}} = \bigcup_{p \in P} \widehat{M}_p.$$

**Definition 2.4** Let for $(p, \ell') \in P \times C^2$,

$$\kappa_p^{\ell'} := \text{ind}_C^{M_p} \chi_{\ell'},$$

where $C = \{0\} \times C^2 \subset M$ and $\chi_{\ell'}(c) = e^{i(c,\ell')}, \ c \in C.$
The Hilbert space \( \mathcal{H}_\ell^p \) of the representation \( \pi^p_\ell \) is the space of functions

\[
\mathcal{H}_\ell^p := \left\{ \xi : M \to \mathbb{C}; \xi \text{ measurable}, \right. \\
\xi(m \cdot c) = \chi_\ell(-c)\xi(m); m \in M, c \in \mathbb{C}, \\
\left. \|\xi\|_2^2 = \int_{\mathbb{R}} |\xi(t, 0, 0)|^2 \, dt < \infty \right\} \\
\simeq L^2(\mathbb{R}).
\]

The group \( M_p \) acts then by left translation on this space.

**Definition 2.5** For \( a \in C^*(\mathbb{M}) \) define the operator field \( \hat{a} \) on \( P \times \mathbb{C}^2 \) by

\[
\hat{a}(p, \ell') := \pi^p_\ell(a(p)), \quad \ell' \in (\mathbb{C}^2)^*, p \in P.
\]

We define the Fourier transform on \( C^*(\mathbb{M}) \) by \( \mathcal{F}(a) := \hat{a} \).

Let us compute for \( p \in P, \ell' \in (\mathbb{C}^2)^* \) and \( F \in L^1(M_p) \) the operator \( \pi^p_\ell(F) \in B(L^2(\mathbb{R})) \).

For \( \ell' = (\ell'_1, \ell'_2) \in (\mathbb{C}^2)^* \simeq \mathbb{C}^2, m = (t, c) \in M \) and \( \xi \in L^2(\mathbb{R}), s \in \mathbb{R} \) we have that

\[
\pi^p_\ell(t, c\xi)(s) = \xi((t, c)^{-1}p \cdot (s, 0, 0)) \\
= \xi((s - t, -(t - s) \cdot c)(s, 0, 0)) \\
= e^{i((t-s)\cdot c, \ell')} \xi(s - t).
\]

Hence

\[
\pi^p_\ell(F)\xi(s) = \int_M F(g)\pi^p_\ell(g)\xi(s) \\
= \int_{\mathbb{R} \times \mathbb{C}^2} F((t, c)) e^{i((t-s)\cdot c, \ell')} \xi(s-t) \, dc \, dt \\
= \int_{\mathbb{R} \times \mathbb{C}^2} F((s - t, c)) e^{i((-t-s)\cdot c, \ell')} \xi(t) \, dc \, dt \\
= \int_{\mathbb{R} \times \mathbb{C}^2} F((s - t, c)) e^{i(c, (t-s)\cdot \ell')} \xi(t) \, dc \, dt \\
= \int_{\mathbb{R}} \hat{F}^2(s - t, t \cdot \ell') \xi(t) \, dt,
\]

where
\[ t \cdot (\ell_1, \ell_2) := (e^{(-i+\vartheta)p} \ell_1, e^{(-i\vartheta-p)} \ell_2), \]

\[ \hat{F}^2(s, \ell) := \int_{\mathbb{C}^2} F(s, c)e^{i(c, \ell)} dc, \quad \ell = (\ell_1, \ell_2) \in \mathbb{C}^2, \quad p \in P, \quad t \in \mathbb{R}. \]

**Remark 2.6** We see

- that for \( p \neq 0 \), \( \ell = (\ell_1, \ell_2) \) with \( \ell_1 \cdot \ell_2 \neq 0 \), the operators \( \hat{a}(p, \ell) \) are compact and that for \( \ell \neq (0, 0) \), the representations
  
  \[ C^*(\mathbb{M}) \rightarrow \mathcal{B}(L^2(\mathbb{R})) : a \rightarrow \hat{a}(p, \ell), \]

  are irreducible, since \( M_p \) is then an exponential Lie group,

- that for \( p = 0 \) with \( \ell_1 \cdot \ell_2 \neq 0 \) the representations \( \pi^p_\ell \) are irreducible too, since then the stabilizer of \( \ell \) is the subgroup \( C \).

- that for \( p = 0 \) with \( \ell_1 \neq 0 \) and \( \ell_2 = 0 \) or \( \ell_1 = 0 \) and \( \ell_2 \neq 0 \) the representations \( \pi^p_\ell \) are not irreducible, since then the stabilizer of \( \ell \in \mathbb{C}^2 \) is the subgroup

\[ S^0_\ell = \{ (m, c) | m \in 2\pi \mathbb{Z}, \ c \in C \}, \]

respectively.

\[ S^0_\ell = \{ (m, c) | m \in \frac{2\pi}{\theta} \mathbb{Z}, \ c \in C \}. \]

Hence the representations

\[ \pi^0_{\omega, \ell} := \text{ind}_{S^0_\ell}^{M_p} \chi_{\omega, \ell} \]

where \( \omega \in \mathbb{R}/\mathbb{Z} \) (respectively, in \( \mathbb{R}/2\pi \theta \mathbb{Z} \)) and

\[ \chi_{\omega, \ell}(m, c) := e^{i(m\omega+(c, \ell))}, \ m \in 2\pi \mathbb{Z}, \ c \in C, \]

respectively, \( m \in \frac{2\pi}{\theta} \mathbb{Z}, \ c \in C, \)

are irreducible.

- If \( \ell = (0, 0) \) and \( p \in P \), then the representation \( \pi^p_{(0,0)} \) is the left regular representation of \( M_p \) on \( L^2(M_p/C) \) and, therefore, equivalent to the direct integral over the space of unitary characters \( \text{Ch}^p := \{ \chi_r \}, \ r \in \mathbb{R} \) of \( M_p \), where

\[ \chi_r(t, c) := e^{itr}, \quad (t, c) \in M. \]
2.3 Disintegration of the induced representation

We define the Fourier transform $\mathcal{F} : L^2(M) \to L^2(\mathbb{R} \times \mathbb{C}^2)$ by

$$\mathcal{F}(\xi)(\ell)(s) := \int_{\mathbb{C}^2} \xi(s, c)e^{i\langle c, \ell \rangle} dc.$$

We can disintegrate the left regular representation $(\Lambda^p, L^2(M_p))$ of $M_p$ into a direct integral of irreducibles using the Fourier transform $\mathcal{F}$. Indeed, we have for $\xi \in C_c(M_p)$ and $\ell \in \mathbb{C}^2$ that the function

$$s \mapsto \mathcal{F}(\xi)(p, s, \ell), \quad (s \in \mathbb{R})$$

is contained in the Hilbert space of the representation $\pi^p_\ell$.

We take now $p \in P$ and some $\xi, \eta \in L^2(M)$. Then for $m = (t, c) \in M$, we have that (forgetting about constants like the number $\pi$):

$$\langle \Lambda^p(m)\xi, \eta \rangle = \int_M \xi(m^{-1}p, u, b)\overline{\eta(u, b)}dudb$$

$$= \int_\mathbb{R} \int_{\mathbb{C}^2} \xi(u - t, -(t - u) \cdot c + b)\overline{\eta(u, b)}dudb$$

Plancherel

$$= \int_\mathbb{R} \int_{\mathbb{C}^2} e^{-i\langle (u-t) \cdot c, \ell \rangle} \xi(u - t, \ell)\overline{\eta(u, \ell)}dud\ell$$

$$= \int_{\mathbb{C}^2} \langle \pi^p_\ell(g)(\xi(\ell)), \eta(\ell) \rangle_2 d\ell.$$

This shows that

$$\Lambda^p = \int_{\mathbb{C}^2} \pi^p_\ell d\ell. \quad (2.4)$$

**Remark 2.7** Since $M_p$ is a solvable and amenable group then $\Lambda^p$ is injective.

Hence, the Fourier transform on $C^*(\mathbb{M})$ is an injective mapping.

**Definition 2.8** Let $A$ be an involutive, semi-simple Banach algebra and let $C^*(A)$ be its $C^*$-algebra. We say that $A$ is *star regular*, if the canonical mapping from $\text{Prim}(C^*(A)) \to \text{Prim}^*(A)$ defined by

$$I \to I \cap A$$

is a homeomorphism for the Jacobson topologies.

It had been shown in [1] that every connected locally compact group $G$ of polynomial growth (in particular the group $M_0$) is $*$-regular. Furthermore, according to [2] the groups $M_{p,0}, p \neq 0$, are also $*$-regular. In particular this implies that the closure for the Jacobson topology of $I_1 = \ker_{L^1(M_0)}(\pi) \subseteq \text{Prim}^*(M_0)$ is the primitive ideal $\ker_{C^*(M_0)}(\pi)$. 
2.4 The topology of the orbit space

**Definition 2.9** We let

\[
\mathcal{E}^0 = \mathbb{R} \times \{ (e^{(-i+p)t}\ell_1, e^{-i\theta-p}t\ell_2), \ t \in \mathbb{R} \}, \ \ell = (\ell_1, \ell_2) \in \mathbb{C}^2, \ p \in P \setminus \{ 0 \},
\]

\[
\mathcal{E}^0 = \mathbb{R} \times \{ (e^{-it}\ell_1, e^{-i\theta t}\ell_2), \ t \in \mathbb{R} \} = \mathbb{R} \times \mathbb{T}^2,
\]

\[
\mathcal{E}(\mathbb{M}) = \{ \mathcal{E}^0, \ \ell \in \mathbb{C}^2, \ p \in P \},
\]

the quasi-orbit space.

We define the Kirillov–Pukanszky mapping \( K : \mathcal{E}(\mathbb{M}) \to \text{Prim}(\mathbb{M}) \) by

\[
K(\mathcal{E}^p) := (I^p_k),
\]

where for \( p \in P \) and \( \ell \in \mathbb{C}^2 \) or \( \ell = s \in \mathbb{R} \) the symbol \( I^p_\ell = \ker(\pi^p_\ell) \) is the primitive ideal given by the quasi-orbit of \( \ell \). Indeed by the formula (2.3)

\[
\ker(\pi^p_\ell) = \left\{ F \in L^1(M_p) | \widehat{F}^2(s, (f, g)) = 0, \ (s, (f, g)) \in \mathcal{E}^p_\ell \right\}.
\]

**Remark 2.10** The mapping \( K : \mathcal{E}(\mathbb{M}) \to \text{Prim}(\mathbb{M}) \) is bijective.

**Theorem 2.11** The variable group \( \mathbb{M} \) is *-regular.

**Proof** It suffices to apply the proof of Theorem 4 in [1], since the mapping \( \mathcal{E}^p_\ell \rightarrow I^p_\ell \) from \( \mathcal{E}(\mathbb{M}) \) into \( \text{Prim}(M_p), p \in P \), is bijective. \( \square \)

**Theorem 2.12** The Kirillov–Pukanszky mapping \( K \) is a homeomorphism of the quasi-orbit space \( \mathcal{E}(\mathbb{M}) \) onto \( \text{Prim}(\mathbb{M}) \).

**Proof** Let \( (p_k, (u_k, f_k, g_k)) \) be a converging sequence in \( (P \times \mathfrak{m}^*) \) with limit point \( (p, (u, f, g)) \). Take \( F \in L^1(\mathbb{M}) \). Then, \( F(p_k) \in L^1(\mathbb{R} \times \mathbb{C}^2) \) converges in \( L^1 \)-norm to \( F(p) \).

But then for almost every \( u \in \mathbb{R} \), we have that the \( L^1(\mathbb{C}^2) \)-limit of the sequence \( (F(p_k, u)) \) is the \( L^1 \)-function \( F(p, u) \). Hence, the \( L^\infty(\mathbb{C}^2) \) sequence \( (\widehat{F}^2(p_k, u)) \) converges uniformly to \( (\widehat{F}^2(0, u)) \). This shows that for any infinite subset \( S \subset \mathbb{N} \) and every \( F \in \bigcap_{s \in S} (I^p_s(u, f_s, g_s)) \), we have that

\[
\widehat{F}^2(p, u, (M_p) \cdot (f, g)) = \{ 0 \},
\]

which means that \( F \in I^p_{(f, g)} \). We have shown that \( \bigcap_{s \in S} (I^p_s(u, f_s, g_s)) \subset (I^p_{(f, g)}) \).

If \( (f, g) = 0 \), then we use the sequence of numbers

\[
\left( \int_{\mathbb{R}} \widehat{F}^2(p, (v, f_k, g_k)) e^{-2\pi iuv} \, dv \right)_k,
\]

which converges to \( \widehat{F}^{1,2}(p, u, 0, 0) \).
This again shows that \( \bigcap_{s \in S} P^a_{(u, f, s)} \subseteq P^a_{(0, 0, 0)} \).

It suffices now to use the fact that the variable group \( M \) is \( * \)-regular.

For the other direction, we apply [6] Theorem 4.1. resp. the proof of Theorem 2.12.

\[ \square \]

### 3. Limit conditions

**Definition 3.1** For \( \alpha > 0 \) let \( M_\alpha \) be the multiplication operator on \( L^2(\mathbb{R}) \) with the characteristic function of the interval \( [-\alpha, \alpha] \).

Choose for any \( \delta \in [0, 1] \) an \( \varepsilon_\delta > 0 \) such that \( \lim_{\delta \to 0} \frac{\delta}{\varepsilon_\delta} = 0 \).

Let

\[ R^\delta_j := \frac{j}{\varepsilon_\delta}, \quad j \in \mathbb{Z}, \delta \in [-1, 1] \setminus \{0\}. \]

For any \( \delta \in [0, 1] \setminus \{0\} \) and \( j \in \mathbb{Z} \), let \( M^\delta_j \) be the multiplication operator of \( L^2(\mathbb{R}) \) with the characteristic function of the interval

\[ I^\delta_{\langle j \rangle} := \begin{bmatrix} R^\delta_j & R^\delta_{j+1} \end{bmatrix} = \begin{bmatrix} j & j + 1 \end{bmatrix} = I^\delta_0 + R^\delta_j. \]

Let now \( \delta \to r(\delta) \) be a positive function defined on \( [0, 1] \setminus \{0\} \), such that \( \lim_{\delta \to 0} r(\delta) = +\infty, \lim_{\delta \to 0} \frac{r(\delta)}{R^\delta_i} = 0 \). Then, for any \( \delta \in [0, 1] \setminus \{0\} \) and for all \( j \in \mathbb{Z} \), let \( N^\delta_j \) be the multiplication operator of \( L^2(\mathbb{R}) \) with the characteristic function of the interval

\[ J^\delta_{\langle j \rangle} := \begin{bmatrix} R^\delta_j - r(\delta), R^\delta_{j+1} + r(\delta) \end{bmatrix}, \quad j \in \mathbb{Z}. \]

Let for \( z = t + iu \in \mathbb{C}, p \in P, \) and \( \ell = (\ell'_1, \ell'_2) \in \mathbb{C}^2 \)

\[ z \odot_p \ell := (e^{pt+iu}\ell'_1, e^{-pt+iu}\ell'_2). \]

Let for \( a \in C^*(\mathbb{M}), p \in P, \delta \in [0, 1] \setminus \{0\} \) and \( \ell \in \mathbb{C}^2 \)

\[ \sigma^a_\ell(p) := \sum_{j \in \mathbb{Z}} M^a_j \circ R^0_{p, \ell} \circ (a) \circ N^\delta_j. \quad (3.1) \]

We observe that for any \( a \in C^*(\mathbb{M}), p \in P \setminus \{0\}, \) and \( \xi \in L^2(\mathbb{R}) \), we have that
\[
\|\sigma_\ell^p(a)\xi\|_2^2 = \sum_{j \in Z} \|M_j^p \circ \pi_0^p_{R_j \ell} \circ (a) \circ N_j^p(\xi)\|_2^2 \\
\leq \|a\|_{C^2}^2 \left( \sum_{j \in Z} \|N_j^p(\xi)\|_2^2 \right) \\
\leq \|a\|_{C^2}^2 \left( \sum_{j \in Z} \| (M_{j-1}^p + M_j^p + M_{j+1}^p)(\xi)\|_2^2 \right) \\
\leq 9 \|a\|_{C^2}^2 \|\xi\|_2^2.
\]

**Proposition 3.2** For any \(a \in C^*(\mathbb{M})\), we have that

\[
\lim_{\delta \to 0} \sup_{\ell \in C^2_{\mathbb{M}}} \|\sigma_\ell^p(a) - \pi_\ell^p(a)\|_{op} = 0.
\]

**Proof** There exists for any \(F \in L^1(\mathbb{M})\), for which \(\hat{F}^2 \in C_c^\infty(\mathbb{R} \times \mathbb{C}^2)\), some \(\varphi \in C_c(\mathbb{R}, \mathbb{R}_{\geq 0})\) and \(\psi \in C_c(\mathbb{C}^2 \times \mathbb{C}^2, \mathbb{R}_{\geq 0})\), such that

\[
|\hat{F}^2(u, \ell) - \hat{F}^2(u, \ell')| \leq \varphi(u)|\ell - \ell'| + \psi(\ell, \ell'), \quad u \in \mathbb{R}, \ \ell, \ell' \in \mathbb{C}^2.
\]

Now, since \(\varphi\) and \(\psi\) are compactly supported, there exists \(C > 0\) such that \(\varphi(u) = 0\) whenever \(|u| > C\) (resp. \(\psi(\ell, \ell') = 0\) if \(|\ell| > C\) and \(|\ell'| > C\)). Hence for any \(j \in \mathbb{Z}\), if \(u \in I_{\delta}(j)\) then \(\varphi(u - t) \neq 0\) implies that for \(\delta = p - p_0\) small enough, we have that \(t \in I_{\delta}(j - 1) \cup I_{\delta}(j) \cup I_{\delta}(j + 1)\).

Therefore

\[
M_j^p \circ \pi_\ell^p(F) = M_j^p \circ \pi_\ell^p(F) \circ N_j^p, \quad \ell \in \mathbb{C}^2, \ p \in P.
\]

We then have for \(\ell = (\ell_1, \ell_2) \in \mathbb{C}^2\), that

\[
(\sigma_\ell^p(F) - \pi_\ell^p(F))(\xi)(u) = \sum_{j \in Z} M_j^p \circ \pi_0^p_{R_j \ell} \circ (a) \circ N_j^p - \left( \sum_{j \in Z} M_j^p \right) \circ \pi_\ell^p(F)(\xi)(u) \\
= \sum_{j \in Z} (M_j^p \circ \pi_0^p_{R_j \ell} \circ (F) - \pi_\ell^p(F))(\xi)(u) \\
= \sum_{j \in Z} M_j^p \circ \pi^p_{(e^{it_2} \ell_1, e^{it_1} \ell_2)}(F - \pi_\ell^p(F))(\xi)(u) \\
= \sum_{j \in Z} 1_{I_{\delta}(j)}(u) \left( \int_{\mathbb{R}} (\hat{F}^2(u - t, (R_j^p \oplus t) \odot \ell_1) \circ \ell_2 - \hat{F}^2(u - t, (t + it) \odot \ell_2)\xi(t)dt \right).
\]

Furthermore for any \(\ell = (\ell_1, \ell_2) \in \mathbb{C}^2, \ p \in P \setminus \{0\}, \ t, u \in I_{\delta}(j)\) and \(j \in \mathbb{Z}\), we have that
The statement then is a consequence of the density in $L^1(M)$ of the subspace generated by the set \( \{ F \in L^1(M) \mid \hat{F}^2 \in C_c(\mathbb{R} \times \mathbb{C}^2) \} \).
**Corollary 3.3** For any \( a \in C^\ast(\mathbb{M}) \), such that \( a(0) = 0 \), we have that
\[
\lim_{p \to 0} \| \Lambda'(a) \|_{op} = 0.
\]

**Proof** Let \( a \in C^\ast(\mathbb{M}) \) such that \( a(0) = 0 \), then \( \sigma'_\varepsilon(a) = 0 \). By the previous proposition
\[
\lim_{p \to 0} \| \Lambda'(a) \|_{op} = 0.
\]

\[\square\]

### 3.1 \( p_0 \) different from 0

We observe that for \( p, p_0 \) different from 0, the groups \( M_p \) and \( M_{p_0} \) are isomorphic. Indeed the mappings \( h_{p_0, p} \) defined by
\[
h_{p_0, p}(s, c) := \left( \frac{p_0}{p} s, c \right), \quad (s, c) \in M,
\]
is an isomorphism from \( M_{p_0} \) onto \( M_p \) since for \((s, c), (s', c') \in M\) we have that
\[
h_{p_0, p}((s, c) \cdot (s', c')) = h_{p_0, p}(s + s', e^{-p_0 s'} \cdot c + c')
\]
\[
= \left( \frac{p_0}{p}(s + s'), e^{-p_0 s'} \cdot c + c' \right),
\]
and
\[
h_{p_0, p}(s, c) \cdot h_{p_0, p}(s', c') = \left( \frac{p_0}{p} s, c \right) \cdot \left( \frac{p_0}{p} s', c' \right)
\]
\[
= \left( \frac{p_0}{p}(s + s'), e^{-(p_0 - p) s'} c + c'' \right)
\]
\[
= \left( \frac{p_0}{p}(s + s'), e^{-p_0 s'} c + c' \right).
\]

Take \( F \in L^1(\mathbb{M}), p \in P \setminus \{0\} \), such that the function
\[
(p, s, c) \to \hat{F}^2(p, s, \ell)
\]
is contained in \( C^\infty_c(P \times \mathbb{R} \times \mathbb{C}^2) \). Then,
\[
\left( \pi'_\varepsilon(F) - \pi'^{p_0}_\varepsilon(F) \right)(\xi)(s) = \int_{\mathbb{R}} \left( \hat{F}^2(p, s - t, t \cdot \ell) - \hat{F}^2(p_0, s - t, t \cdot \ell_0) \right) \xi(t)dt.
\]

Therefore,
We see in this way that if for some \( a \in C^*(\mathbb{M}) \) and \( p_0 \in P\setminus\{0\} \), we have that
\[
a(p_0) = 0,
\]
then automatically
\[
\lim_{p \to p_0} \|a(p) - a(p_0)\| = 0. \tag{3.4}
\]

### 3.2 A \( C^* \)-condition

**Definition 3.4** Let \( CB(P) \) be the algebra of operator fields \((\Phi(p, \ell) \in B(L^2(\mathbb{R})))_{p \in P, \ell \in C^2}\), for which the mapping
\[
(p, \ell) \mapsto \Phi(p, \ell)
\]
is strongly continuous on \( P \).

For an operator field \( \Phi \in CB(P) \) let
\[
\sigma^p(\Phi) := \sum_{j \in \mathbb{Z}} M_j^p \circ \Phi(p, R^p_j \circ \ell) \circ N_j^p, \quad P \ni p \neq 0,
\]
\[
\sigma^p(\Phi) := \oint_{C^2} \sigma^p(\Phi(p, \ell)) \, d\ell, \quad p \in P,
\]
\[
\Phi(p) := \oint_{C^2} \Phi(p, \ell) \, d\ell, \quad p \in \ell.
\]

**Remark 3.5** Let \( a \in C^*(\mathbb{M}) \). Then, as we have seen above, the operator field \((\tilde{a}(p, \ell))_{p \in P, \ell \in C^2}\) is contained in \( CB(P) \).

**Definition 3.6** Let \( \mathcal{D}^*(\mathbb{M}) = \mathcal{D}^* \) be the space consisting of all operator fields \( \Phi \) defined over \( S := P \times C^2 \) and contained in \( CB(P) \) such that:

1. \( \Phi(p) \in \widehat{C^*(M_p)}, \quad p \in P. \)
2. The field \( \Phi \) satisfies the condition
\[
\lim_{p \to 0} \|\sigma^p(\Phi) - \Phi(p, \ell)\|_{op} = 0.
\]
3. For \( p_0 \neq 0 \) in \( P \), we have that
\[
\lim_{p \to p_0} \|\Phi(p) - \Phi(p_0)\| = 0.
\]
4. The same conditions are satisfied by the field \( \Phi^* \).
The $C^*$-algebras $C^*(M_p), p \neq 0$, have been characterized as algebras of operator fields in the paper [5].

**Remark 3.7** It follows from Proposition 3.2 and from (3.4) that for every $\Phi \in C^*(\mathbb{M}) = \bigcup_{p \in \mathbb{P}} C^*(M_p)$, the operator field $\Phi$ is contained in $\mathbb{D}^*(\mathbb{M})$.

**Definition 3.8** Let $a \in C^*(\mathbb{M})$ and $\varphi \in C(P)$. Define $\varphi \cdot a$ by

$$\varphi \cdot a(p) := \varphi(p)a(p), \quad p \in P.$$  

For an operator field $(\Phi(p, \ell) \in \mathcal{B}(L^2(\mathbb{R})))_{\ell \in \mathbb{C}, p \in P},$ and $\varphi \in C(P)$, let

$$(\varphi \cdot \Phi)(p, \ell) := \varphi(p)\Phi(p, \ell), \quad \ell \in \mathbb{C}, \ p \in P.$$  

**Proposition 3.9** Any $\varphi \in C(P)$ defines a central multiplier of $\mathbb{D}^*$.

**Proof** Clearly, for every $\Phi \in D^*(\mathbb{M})$ and any $\varphi \in C(P)$, the new operator field also satisfies the conditions (1) to (4), since $\sigma^p(\varphi \cdot \Phi) = \varphi(p) \cdot \sigma^p(\Phi), p \in P, \ell \in \mathbb{C}^2$. \hfill $\Box$

**Theorem 3.10** The space $\mathbb{D}^*(\mathbb{M})$ is a $C^*$-algebra, which is isomorphic to $C^*(\mathbb{M})$.

**Proof** It is clear that $\mathbb{D}^*$ is an involutive Banach space. Let us show that condition (2) is stable under the composition of operator fields. Let $\Phi, \Phi' \in \mathbb{D}^*$.

Then $\Phi(0)$ and $\Phi'(0)$ are contained in $\mathcal{C}^*(\mathbb{M}_0)$. Since the mapping $C^*(\mathbb{M}) \to C^*(\mathbb{M}_p)$ is surjective for any $p \in P$, we find $b, b' \in C^*(\mathbb{M}_p)$, such that $\Phi(p) = b(p), \Phi'(p) = b'(p)$. Condition (2) then says that

$$\lim_{p \to 0} \|\Phi \circ \Phi'(p) - \sigma^p(\Phi \circ \Phi')\|_\text{op}$$

$$= \lim_{p \to 0} \|\Phi \circ \Phi'(p) - \sigma^p(b \circ b')\|_\text{op}$$

$$= \lim_{p \to 0} \|\Phi \circ \Phi'(p) - \sigma^p(b \cdot b')\|_\text{op}$$

$$= \lim_{p \to 0} \|\Phi \circ \Phi'(p) - b \cdot b'(p)\|_\text{op}$$

$$= \lim_{p \to 0} \|\Phi \circ \Phi'(p) - \hat{b}(p) \circ \hat{b}'(p)\|_\text{op}$$

$$= \lim_{p \to 0} \|\Phi(p) \circ \Phi'(p) - \sigma^p(\Phi) \circ \sigma^p(\Phi')\|_\text{op}$$

$$= \lim_{p \to 0} \|\Phi(p) \circ \Phi'(p) - \sigma^p(\Phi) \circ \sigma^p(\Phi')\|_\text{op}$$

$$= 0.$$  

Thus, $\mathbb{D}^*$ is a $C^*$-algebra.

Let us show that $\mathbb{D}^* = C^*(\mathbb{M})$. Let $\pi \in \widehat{\mathbb{D}^*}$. Then there exists a character $\psi$ of the algebra $C(P)$, such that

$$\pi(\varphi \cdot \Phi) = \psi(\varphi)\pi(\Phi), \quad \varphi \in C(P), \ \Phi \in \mathbb{D}^*.$$
Now, every character of the algebra \( C(P) \) is given by some point evaluation. Hence, there exists \( p_{\pi} \in P \) such that
\[
\pi(\varphi \cdot \Phi) = \varphi(p_{\pi}) \pi(\Phi), \quad \varphi \in C(P), \quad \Phi \in \mathbb{D}^*.
\]
Let
\[
J_{\pi} := J_{p_{\pi}} = \{ \Phi \in \mathbb{D}^* \mid \Phi(p_{\pi}) = 0 \}.
\]
Then, \( J_{\pi} \) is a closed ideal of \( \mathbb{D}^* \).

Choose a sequence bounded by 1 of functions \((\psi_k)_k \subset C([0, 1])\), such that \( \psi = 0 \) on \( U_k := \{ p' \mid |p' - p_{\pi}| \leq \frac{1}{k} \} \) and \( \psi_k(p) = 1, p \notin U_{2k}, k \in \mathbb{N} \). Then for any \( \Phi \in J_{p_{\pi}} \) we have that
\[
\lim_{k \to \infty} \psi_k \cdot \Phi = \Phi,
\]
since
\[
\| \psi_k \cdot \Phi - \Phi \| = \sup_{p \in P} \| \psi_k(p) \Phi(p) - \Phi(p) \|_{op}
\]
\[
= \sup_{|p| \leq \frac{1}{k}} \| \psi_k(p) \Phi(p) - \Phi(p) \|_{op}
\]
\[
\leq 2 \sup_{|p - p_{\pi}| \leq \frac{1}{k}} \| \Phi(p) \|_{op}
\]
\[
\to 0,
\]
since for this \( \Phi \) we have that \( \lim_{p \to 0} \Phi(p) = 0 \) by condition (2) if \( p_{\pi} = 0 \) or by condition (3) if \( p_{\pi} \neq 0 \). This shows that
\[
\pi(\Phi) = \lim_{k \to \infty} \psi_k(p_{\pi}) \pi(\Phi)
\]
\[
= 0.
\]
Hence
\[
\pi(J_{\pi}) = \{ 0 \}.
\]
It follows from the condition (1) that \( \mathbb{D}^*/J_p = \widehat{C^*(M_p)} \), since we know that already \( C^*(\mathbb{M})/I_{p_{\pi}} \approx C^*(M_{p_{\pi}}) \). Hence it follows that
\[
\pi \simeq \pi_{p_{\pi}} \circ \mu_{p_{\pi}}
\]
for some \( \pi_{p_{\pi}} \in \widehat{M_{p_{\pi}}} \).

Therefore, the subalgebra \( C^*(\mathbb{M}) \) of \( \mathbb{D}^* \) separates the irreducible representations of \( \mathbb{D}^* \) and so by the Stone–Weierstrass theorem in [3].
\[
\mathbb{D}^*(\mathbb{M}) = \widehat{C^*(\mathbb{M})}.
\]
References

1. Boidol, J., Leptin, H., Schürmann, J., Vahle, D.: Räume primitiver Ideale von Gruppenalgebren. Math. Ann. 236, 1–13 (1978)
2. Boidol, J.: *-Regularity of some classes of solvable groups. Math. Ann. 261, 477–481 (1982)
3. Dixmier, J.: Les C*-algèbres et Leurs Représentations. (French) Deuxième édition Cahiers Scientifiques, Fasc. XXIX Gauthier-Villars Éditeur, Paris 1969 xv+390 pp. 46.65
4. Leptin, H., Ludwig, J.: Unitary representation theory of exponential Lie groups. De Gruyter Expositions in Mathematics, 18. Walter de Gruyter & Co., Berlin, 1994. x+200 pp.
5. Lin, Y.-F., Ludwig, J.: The C*-algebra of ax + b-like groups. J. Funct. Anal. 259, 104–130 (2010)
6. Ludwig, J.: Dual topology of diamond groups. Journal für die reine und angewandte Mathematik 467, 67–88 (1995)
7. Sudo, T.: Structure of group C*-algebras of the generalized Mautner groups. J. Math. Kyoto Univ. (JMKYAZ) 42(2), 393–402 (2002)

Springer Nature or its licensor holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.