On degenerate planar Hopf bifurcations

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Received 24 June 2010, in final form 29 November 2010
Published 10 January 2011
Online at stacks.iop.org/JPhysA/44/065202

Abstract
Our concern is the study of the limit cycles emerging at a Hopf bifurcation. To do so, we consider an averaging method which allows us to find asymptotic expansions of the radius and frequency of the cycles. We consider a discriminant function where each root corresponds to an emerging cycle. A way of classifying the Hopf bifurcation on the basis of the number of emerging limit cycles arises from the present investigation: namely non-degenerate, first-type degenerate and second-type degenerate. We give genericity conditions for non-degenerate bifurcations up to 6-jet-equivalence, ‘typical’ forms for degenerate bifurcations, and also a sufficient condition for second-type degenerate bifurcation up to 6-jet-equivalence.

PACS numbers: 02.30.Hq, 02.30.Oz
Mathematics Subject Classification: 34C23, 37G15, 34C29

1. Introduction
The goal of this paper is the study of a degenerate Hopf bifurcation (HB) near the critical value of a one-parameter family of analytic planar vector fields in the neighborhood of an isolated singular point. The HB is usually associated with the emergence of limit cycles after or before the critical value of the bifurcation parameter. An isolated periodic orbit of a vector field on the plane is a limit cycle. The HB phenomenon has been well studied in the literature, and its main details and the implications for dynamic systems of higher dimensions via the center manifold theorem are summarized in the monograph [1]. Applications of the HB in finite-dimensional dynamical systems are often discussed in the literature [2–9]. Significant applications of the HB in fluid dynamics are now well established and can be found in [1] and references therein. In addition, the topic is extensively treated in the theory of bifurcations with higher codimension, for instance, in Turing–Hopf instabilities [10].

In this paper, we shall consider the extension of the concept of non-degeneracy and we suggest a division of the classification of degeneracy into two different types. We use the previous notion [10] of the ‘discriminant’ function and equation, each root (simple or multiple) of which is associated with an emerging limit cycle when the bifurcation parameter...
crosses the critical value $\tau_a = 0$. One way of classifying the HB on the basis of the number $C$ of roots of the discriminant equation, i.e. the number of emerging limit cycles at bifurcation, is proposed in this paper. We summarize our proposed classification in the following table, in which the number $N$ corresponds to the $(2N + 1)$-jet engendering limit cycles:

| Behavior at $\tau_a = 0$ | Standard classification | Classification on the basis of $C$ |
|--------------------------|-------------------------|-----------------------------------|
| Weak focus               | Non-degenerate          | Non-degenerate ($N = 1$)          |
| Center                   | Degenerate              | Non-degenerate ($N > 1$)          |
|                          |                         | First-type degenerate             |
|                          |                         | Second-type degenerate            |

First-type degeneracy will mean that no limit cycle surrounding the singular point emerges either after or before the critical value of the bifurcation parameter, with the possible emergence of limit cycles surrounding the point at infinity [11]. The second type of degeneracy will mean that multiple limit cycles, or semistable cycles as a limit case, emerge surrounding the singular point either at the super- or subcritical HB.

We focus our attention on such a system which, in a neighborhood of the origin, can be written in the form

$$\dot{X} = F(X, a),$$

where

$$F = \sum_{k+l=1}^{M} \left( \frac{\sigma_{kl}}{\sigma_{kl}^2} \right) x^k y^l + O(\|X\|^{M+1})$$

is the Taylor expansion of $F$ in a neighborhood of the origin, the coefficients $\sigma_{kl} = \sigma_{kl}^a (a)$ are real, and $X = (x, y)^T$. Here $a$ is a real parameter and $F$ is at least a $C^0$-function of $a$. Up to a shift of coordinates, it is enough to consider an analytic dynamical system in the form (1), in which the singular point $P_a$ of the system is fixed at the origin of coordinates. The vector field with polynomial components of degree $M$, represented in the main part in equation (2), is called the $M$-jet of equation (1) around the origin, which is usually denoted as $j_M (F)(0)$. Two analytic vector fields defined in a neighborhood of the origin are called $M$-jet equivalent if their $M$-jets coincide.

The plan of the paper is as follows. In section 2, we summarize previous results in the treatment of an HB taken basically from [10], particularly the notion of discriminant that is quoted. In section 3, we introduce the notion of ‘negligible’ coefficients in the discriminant (definition 2). Upon this definition we introduce the concept of degeneracy of the HB (definition 3), which allows us to classify the HB on the basis of the asymptotic behavior of the discriminant as the trace of the Jacobian tends to zero. The split version of the HB theorem 1 in this section describes the emergence of limit cycles in non-degenerate, first-type degenerate and second-type degenerate, respectively. We prove that, at the degenerate HB of the first type, no limit cycle surrounding the singular point emerges, allowing the emergence of limit cycles surrounding the point at infinity. At the degenerate HB of second type, multiple or semistable limit cycles emerge surrounding the singular point, while, at non-degenerate HB, a single limit cycle emerges. We conclude that only a few combinations of coefficients corresponding to odd terms in the reaction system can contribute to the emergence of limit cycles at bifurcation. We have called these combinations Hopf coefficients. These Hopf coefficients can be easily
calculated and will play a role similar to that of Lyapunov coefficients [12]. As we will see, the radius of the limit cycles emerging at the degenerate HB of the second type tends to zero with an ‘anomalous’ order as the bifurcation parameter tends to the critical value. The period of the emerging limit cycles in one-parameter bifurcation has attracted the attention of researchers (see [11]) showing that the main term in the asymptotic of the period of the emerging periodic solution characterizes the bifurcation. Our procedure gives us simultaneously an asymptotic estimate of the radius of the cycle and the frequency of the corresponding periodic orbit. Later, in section 4, we derive the corresponding genericity conditions for the non-degenerate HB up to 6-jet-equivalence. In section 5, ‘typical’ forms corresponding to each type of degenerate bifurcation are gathered. We include a sufficient condition for the degenerate HB of the second type of equation (1) valid up to 6-jet-equivalence. We offer an appendix which includes useful calculations.

2. Preliminaries

Let $\delta_a$ and $\tau_a$ be the determinant and the trace of the Jacobian matrix $J_a$, respectively, at the origin of the function $F$ in equation (2). Besides, up to a shift of coordinates, $J_a$ can be taken to satisfy $\sigma_{10}^a = \sigma_{01}^a = \tau_a / 2$ and $\sigma_{00}^a = -\sigma_{11}^a = \Lambda_a / 2$, where $\Lambda_a = +\sqrt{4\delta_a - \tau_a^2} > 0$, whenever equation (3) holds. We assume that $a$ varies in an open small neighborhood $U$ of the point $a^*$ at which the trace vanishes to change its sign. We assume that the function $a \mapsto \tau_a$ is a homeomorphism between $U$ and a neighborhood $V$ of $\tau_a = 0$, instead of the standard transversality condition $\tau_a'(a^*) \neq 0$. Consequently, the parameter $\tau_a$, $|\tau_a| \ll 1$, can be considered as the intrinsic bifurcation parameter, and $\tau_a = 0$ is the critical value. The HB appears provided the inequality

$$\tau_a^2 - 4\delta_a < 0$$

holds for any value $\tau_a$ in some neighborhood of $\tau_a = 0$. Thus, redefining $U$ if necessary, we can take the (positive) frequency $\omega_a$ given by

$$\omega_a^2 := \delta_a \geq \delta_{\text{min}} > 0.$$  

Further, if $\tau_a < 0$ (respect. $\tau_a > 0$), the origin is a stable (respect. unstable) focus. The bifurcation is subcritical (supercritical) if the emergence of a limit cycle is expected for the negative (positive) values of $\tau_a$ close enough to zero. From equation (3) $\sigma_{01}^a \cdot \sigma_{10}^a < 0$ follows and we do not lose generality assuming $\sigma_{01}^f < 0$.

2.1. Averaging Hopf periodic solutions

Let us quote in this subsection some results about the procedure in the study of an HB proposed in [10]. There, the authors proposed an algorithm allowing the transformation of this system showing an HB into a second-order differential equation representing a weakly nonlinear oscillator in normal form. The required transformation is analytical and nonlinear in general, but it was proved there that it is enough to consider the linear part of this transformation to obtain the equation of the oscillator preserving the required accuracy. The main idea of this procedure is that the transformation of variables can be taken ‘close’ to an appropriate linear transformation in a neighborhood of the origin.

**Definition 1.** We say that the diffeomorphisms $\mathcal{H}$ and $\Gamma$ have a contact at the origin of order $S \in \mathbb{N}$, $S \geq 1$, if

$$\mathcal{H}(X) - \Gamma X = O(\|X\|^{S+1})$$

as $\|X\| \to 0$. 

3
More precisely, if we consider an invertible transformation of variables between neighborhoods of the origin \( Y = \mathcal{H}(X) \) having a contact of order \( S \) with the non-singular matrix \( \Gamma \), then the inverse \( \mathcal{H}^{-1} \) has a contact of order \( S \) with \( \Gamma^{-1} \). We would like to substitute \( \mathcal{H} \) with an equivalent simpler one, say \( \Gamma \), in the transformation of the system (1) into a second-order differential equation. So we shall take \( \Gamma \) to be equal to \( M \) in equation (2). Thus, \( Y = \mathcal{H}(X) \) represents a change of variables such that every solution \((x(t), y(t))^T\) to equation (1) is transformed into \( Y = (z(t), \dot{z}(t))^T \) provided that equation (3) and the matrix \( \Gamma \) are any non-trivial linear combination of the pair

\[
\Gamma_1 = \begin{pmatrix} 1 & 0 \\ \sigma_{10} & \sigma_{01} \end{pmatrix}; \quad \Gamma_2 = \begin{pmatrix} 0 & 1 \\ \sigma_{10} & \sigma_{01} \end{pmatrix}.
\]  

The function \( z \) satisfies the following second-order equation:

\[
\ddot{z} - \tau_a \dot{z} + \delta_a z = \Pi_2[\Gamma(\Psi(\Gamma^{-1} Y))] + O(\|Y\|^M),
\]  

with \( \Pi_2 \) being the standard projector over the second component and \( \Psi \) gathers the nonlinear terms in the Taylor expansion of the right-hand side in equation (2).

Let us now focus on equation (7). We shall look for an oscillation with positive and small, but finite, amplitude \( \varepsilon \). The small parameter \( \varepsilon \) will be connected with the small bifurcation parameter \( \tau_a \) and will be defined later. Substituting \( z(t) = \varepsilon \varsigma(t) \) into equation (7), we obtain the equation of a weakly nonlinear oscillator in normal form:

\[
\ddot{\varsigma} - \tau_a \dot{\varsigma} + \delta_a \varsigma = \varepsilon G(\varsigma, \dot{\varsigma}; \varepsilon).
\]  

Then, to each periodic solution to equation (1) will correspond a non-trivial periodic solution to equation (8), and vice versa. Let us consider the Krylov–Bogoliubov averaging method [13, 14] to derive an asymptotic expansion to the solution of equation (8). To do so, we introduce the new variables \( r = r(t) \) (amplitude) and \( \theta = \theta(t) \) (phase) defined as follows:

\[
\varsigma = r \cos(\omega_a t + \theta)
\]

\[
\dot{\varsigma} = -r \omega_a \sin(\omega_a t + \theta),
\]

so the corresponding averaged equations are

\[
\dot{r} = -\frac{1}{2\pi \omega_a} \int_0^{2\pi} \sin \phi (-\tau_a \omega_a r \sin \phi + \varepsilon G(r \cos \phi, -r \omega_a \sin \phi; \varepsilon)) \, d\phi
\]

\[
\dot{\theta} = -\frac{1}{2\pi \omega_a \rho} \int_0^{2\pi} \cos \phi (-\tau_a r \omega_a \sin \phi + \varepsilon G(r \cos \phi, -r \omega_a \sin \phi; \varepsilon)) \, d\phi,
\]

where \( \phi = \omega_a t + \theta \). Thus,

\[
\dot{r} = \frac{r}{2} \{\tau_a - p(r; \varepsilon)\}
\]

\[
\dot{\theta} = q(r; \varepsilon)
\]

where

\[
p(r; \varepsilon) = \frac{\varepsilon}{\pi \omega_a \rho} \int_0^{2\pi} \sin \phi G(r \cos \phi, -r \omega_a \sin \phi; \varepsilon) \, d\phi
\]

\[
q(r; \varepsilon) = -\frac{\varepsilon}{2\pi \omega_a \rho} \int_0^{2\pi} \cos \phi G(r \cos \phi, -r \omega_a \sin \phi; \varepsilon) \, d\phi.
\]
Let us now quote some important properties about the analytic functions \( p(r; \varepsilon) \) and \( q(r; \varepsilon) \) above. Note that \( p(r; \varepsilon) / r^2 \) and \( q(r; \varepsilon) / r^2 \) have a finite limit as \( r \to 0 \). Moreover, the Taylor expansions of \( p(r; \varepsilon) \) and \( q(r; \varepsilon) \) must not contain odd powers of \( r \), and

\[
p(r; \varepsilon) = p_3 \varepsilon^2 r^2 + p_5 \varepsilon^4 r^4 + \cdots
\]

(17)
in which \( p_i = p_i(\tau_a) \).

The classical perturbation theory [14] gives a uniform \( O(\varepsilon) \)-estimation for the difference between the corresponding solutions to equation (8) and to the average systems, but only on the time scale \( 1/\varepsilon \). In our scenario, we may expect that the amplitude of any solution to equation (8) starting in the region of attraction of the limit cycle can be uniformly expanded by the average solution uniformly for \( t > 0 \) (see subsection 3.2). Thus, the coefficients \( (p_{2j+1})_{j=1}^{D+1} \) in equation (17) will determine the emergence of cycles up to \( (2D + 1) \)-jet-equivalence (in fact, up to \( 2(D + 1) \)-jet-equivalence). It can be expected that we could manage these coefficients by taking \( D \) appropriate independent relations involving the parameters in the system (1). This scenario is called a codimension-\( D \) bifurcation.

3. Classification and theorems

Let us introduce a concept that will be essential for the classification of an HB.

**Definition 2.** Let \( p_{2j+1} \) be a coefficient in the formal development equation (17), which is derived from the formal \( \infty \)-jet of \( F \). It shall be called negligible if satisfying

\[
|p_{2j+1}| \leq K_s |\tau_a|
\]

(18)
for a certain constant \( K_s > 0 \) as \( \tau_a \to 0 \). The function \( p(r; \varepsilon) \) in equation (17) is said to be negligible if for all \( s \in \mathbb{N} \), the coefficient \( p_{2s+1} \) is negligible.

Naturally, \( p_{2s+1} \equiv 0 \) is a negligible coefficient, or \( p \equiv 0 \) is also negligible. For instance, \( p \equiv 0 \) if in the formal \( j \infty(F)(0) \), the non-vanishing terms have even degree. Moreover, it may occur that \( p_{2s+1} \equiv 0 \) even if there are non-zero coefficients with degree \( 2s + 1 \) in the formal Taylor development on the right-hand side in equation (1). As we will see in the next section, negligible terms have no influence in the generation of limit cycles. Besides, with this notion we are able to give an improved version of proposition 3 in [10] as follows.

**Proposition 1.** If the function \( p(r; \varepsilon) \) is non-negligible, there must exist a positive integer \( N \) and a positive real value \( r_0 = r_0(\tau_a) \) such that \( p(r; \varepsilon) \) has the non-trivial Taylor expansion:

\[
p(r; \varepsilon) = \chi e^{2N_n} r_0^{-2N} + O(e^{2N_n} r_0^{-2N+2})
\]

(19)
where \( \chi = +1 \) or \(-1 \). In addition, the behavior of the factor \( r_0^{-2N} \) as \( \tau_a \to 0 \) obeys the following alternative: either

\[
\lim_{\tau_a \to 0} r_0^{-2N} = r_*^{-2N} > 0
\]

(20)
or, for a given \( \gamma \), \( 0 < \gamma < 1 \),

\[
r_0^{-2N} = O(|\tau_a|^{\gamma}) \quad \text{as} \quad \tau_a \to 0.
\]

(21)

As in [14], the symbol \( O_\gamma \) in equation (21) means a sharp estimate, that is, \( r_0^{-2N} = O(|\tau_a|^{\gamma}) \) and \( r_0^{-2N} \neq o(|\tau_a|^{\gamma}) \) as \( \tau_a \to 0 \). We recall that the bifurcation is supercritical (respect. subcritical) if \( \chi = +1 \) (respect. \( \chi = -1 \)). In the supercritical case, the root \( r_0 \) appears for \( \tau_a > 0 \) (respect. \( \tau_a < 0 \)), so the limit in equation (20) or the order relation in equation (21) should be considered as \( \tau_a \to 0^+ \) (respect. \( \tau_a \to 0^– \)). Consequently, if we
assume that \( p (r; \varepsilon) \) is non-negligible and also that \( r_0 \) in equation (19) has the property in equation (20), then there is a positive root \( \rho \) to the discriminant equation

\[
p (r; \varepsilon) - \tau_a = 0 \quad (22)
\]
either for each positive value or for each negative value of \( \tau_a \) close enough to zero. Furthermore, up to the leading term, the root to equation (22) has the form

\[
\rho = \left( \frac{|\tau_a|}{\varepsilon^{2N}} \right)^{1/2} (r_* + O(|\tau_a|)) + O(\varepsilon^2). \quad (23)
\]

Let us assume the existence of a (finite) positive root equation (23) such that equation (20) holds. Then, the small parameter \( \varepsilon \) can be taken as

\[
\varepsilon^{2N} = |\tau_a|. \quad (24)
\]
From equation (24) it follows that equation (23) can now be written as

\[
\rho = r_* + O(|\tau_a|^{1/N}). \quad (25)
\]
If \( r_0 \) in equation (19) has the property in equation (21), we may proceed similarly as we do to obtain equation (24), to get

\[
\varepsilon^{2N} = |\tau_a|^{1-\gamma}. \quad (26)
\]
Moreover, if \( r_0^{2N} = r_L |\tau_a|^\gamma + O(|\tau_a|^\gamma) \) as \( \tau_a \to 0 \) for a certain positive number \( r_L \), then equation (23) can be rewritten as

\[
\rho = r_L + O(|\tau_a|^{(1-\gamma)/N}). \quad (27)
\]

As we shall see, equations (18), (20) and (21) provide evidence of the reason for differentiation in an HB we are suggesting here.

**Definition 3.** We shall say that the HB is first-type degenerate if \( p \) is negligible (see definition 2). Let \( N \) be given as in equation (19). The bifurcation shall be called second-type degenerate, if there exists a number \( \gamma, 0 < \gamma < 1 \), such that equation (21) holds. The HB shall be called non-degenerate, provided equation (20).

For instance, if \( p \equiv 0 \) in equation (17), the bifurcation shall be first-type degenerate. Further, the existence of at least one non-negligible \( p_{2s+1} \) derived from the formal \( \infty \)-jet of \( F \), no matter how large the number \( s \) is, implies that the HB will not be first-type degenerate.

**Remark 1.** In definition 3 we assume \( \gamma > 0 \) since \( F \) in equation (2) is a continuous function of \( \tau_a \) in a vicinity of \( \tau_a = 0 \), and equation (21) holds. Furthermore, from equation (26) it follows that \( 1 - \gamma > 0 \).

It is easy to see, from definition 3, that any type of degenerate HB implicitly implies that the system shows a center at the critical value of the bifurcation parameter. Nevertheless, this property will not characterize degeneracy.

**Theorem 1 (Hopf bifurcation).** Let us assume that equation (3) holds. Then, one of the following possibilities arises.

(i) Non-degeneracy at HB: if equation (22) has a root equation (25) with property (20) for positive (respect. negative) values of the bifurcation parameter \( \tau_a \) but sufficiently close to zero, then a single limit cycle to the system in equation (1) emerges. Furthermore, the limit cycle is orbitally asymptotically stable (respect. unstable) if and only if the bifurcation is supercritical (respect. subcritical). The radius of the emerging cycle is \( r = O_\varepsilon(|\tau_a|^{1/2N}) \), while the frequency is \( \omega = \omega_0 + O(|\tau_a|^{1/N}) \) as \( \tau_a \to 0 \).
(ii) First-type degenerate HB: if $p$ is negligible (definition 2), then none of the cycles surrounding the singular point bifurcate from this point.

(iii) Second-type degenerate HB: if equation (22) has a root with property (21) for sufficiently close to zero values of the parameter $\tau_a$, then the emergence can be assured of at least one limit cycle to the system in equation (1), the radius of which has order $r = O_{\varepsilon}(|\tau_a|^{(1-\gamma)/2N})$, while the frequency is $\varpi = \omega_a + O(|\tau_a|^{(1-\gamma)/N})$ as $\tau_a \to 0$.

**Proof** (Non-degenerate case). The proof follows from equations (22) and (13). Let the main term in equation (19) have property (20). Then, a single root to equation (22) tends to zero as $\tau_a \to 0$. This fact means that a single limit cycle emerges at bifurcation. The existence of further different roots to equation (22) is not excluded, but if any other appears, it determines a limit cycle that does not vanish in a vicinity of $\tau_a = 0$, so this cycle ‘persists’ along the bifurcation. As follows from equations (33)–(35), we obtain the order of the radius of the limit cycle. The order of the frequency is determined in subsection 3.3.

When the bifurcation occurs in a one-parameter family of vector fields whose first non-zero derivatives at the origin have order $2N + 1$, $N > 1$, it is called [11] a generalization of the Andronov–Hopf. Other higher codimension HBs, for instance, the Bautin bifurcation, are often called generalized [15]. In our formulation, both higher codimension HBs are gathered into the non-degenerate type. Consequently, a necessary but not sufficient condition for the emergence of a limit cycle at bifurcation is the existence of a non-zero odd-order term in expansion (2).

**Proof** (First-type degenerate HB). From equations (18) and (22) it follows that none of the roots to equation (22) tend to zero as $\tau_a \to 0$. More precisely, if any root exists, it tends to infinity. Taking into account only the leading terms, we may write

$$p_{2k+1} = \tilde{\alpha}_k \tau_a^{1+\mu_k}$$

where $\mu_k \geq 0$. Let $\tilde{\alpha}_N$ be the first non-zero coefficient, and take $\epsilon^{2N} = \tau_a$ (if the HB is supercritical), so equation (17) is

$$p = \epsilon^{2N} \sum_{n=N}^{\infty} \tilde{\alpha}_k \tau_a^{1+\mu_k} e^{2(n-N)} r^{2n} = \tau_a^2 \sum_{n=N}^{\infty} \tilde{\alpha}_k \tau_a^{\mu_k+(n-N)/N} r^{2n}.$$ 

Hence, if there is a root $r_0$ to equation (22), then $r_0 \to +\infty$ as $\tau_a \to 0$. □

**Remark 2.** At the first-type degenerate HB, further limit cycles may persist in a neighborhood of the critical value of the bifurcation parameter. Moreover, limit cycles surrounding the point at infinity may emerge in the so-called HB at the infinity [11] (see subsections 5.3 and 5.4).

**Proof** (Second-type degenerate HB). The proof for the first-type non-degenerate HB can be repeated, but taking into account the small parameter from equation (26). Note that in this case more than a single positive root to equation (22) may appear tending to zero as $\tau_a \to 0$.

**Proposition 2.** Consider a root $\rho$ to equation (22), which corresponds to a limit cycle $L$. This cycle is asymptotically stable or unstable if the number $dp/\partial r(\rho)$ is negative or positive, respectively.
Proof. In the calculation of \( \frac{dp}{dr} (\rho) \) we can assume that \( \rho = r_0 \) as in equation (25), or \( \rho = r_L \) in equation (27), in accordance with the type of bifurcation. Due to the continuity argument near \( \tau_a = 0 \) in equations (25) or (27), and the fact that \( \frac{dp}{dr} (r_0) \) (or \( \frac{dp}{dr} (r_L) \)) does not vanish, the assertion follows. \( \square \)

3.1. The Hopf coefficients

Let us take \( M \geq 2 \) in equation (2) and assume that \( H \) has a contact with \( \Gamma_1 \) of order \( M \) at the origin. Thus,

\[
\dot{Y} = \sum_{1 \leq k+l \leq M} \Gamma \left( \frac{\sigma_1^{1}}{\sigma_2^{1}} \right) (\mu_{11} z + \mu_{12} \dot{z})^k (\mu_{21} z + \mu_{22} \dot{z})^l + O((\|Y\|^M))
\]

where \( \Gamma = (\gamma_{ij}) \) and \( \Gamma^{-1} = (\mu_{ij}) \). For instance, if we take \( \Gamma = \Gamma_1 \) in equation (6), the quantities \( \mu_{11} = 1, \mu_{12} = 0, \mu_{21} = \tau_a \Lambda_0^{-1}, \mu_{22} = -2 \Lambda_0^{-1} \) are the components of \( \Gamma_1^{-1} \). So the right-hand side of equation (7) can be written as

\[
G(z, \dot{z}) = \sum_{2 \leq k+l \leq M} R_{kl} (\mu_{11} z + \mu_{12} \dot{z})^k (\mu_{21} z + \mu_{22} \dot{z})^l + O((\|Y\|^M))
\]

where \( R_{kl} = \gamma_{21} \sigma_1^{1} + \gamma_{22} \sigma_2^{1} \). Thus,

\[
G(z, \dot{z}) = \sum_{2 \leq k+l \leq M} H_{kl} z^k (\dot{z})^l + O((\|Y\|^M)).
\]

Only some of the non-zero \( H_{mn} \) in equation (29) corresponding to such pairs \((m, n)\) for which \( K_{mn} = \int_0^{2\pi} \cos^m \phi \sin^{n+1} \phi \, d\phi \neq 0 \) may contribute to the appearance of a non-zero term \( p_{2N+1} \) in equation (19).

Let us introduce the following.

Definition 4. We shall call the Hopf coefficient of degree \((2N + 1)\) to the coefficient \( p_{2N+1} \) in expansion (17), which is an algebraic combination of coefficients \( H_{kl} \) in equation (29) provided that \( k + l = 2N + 1 \).

From equations (9), (10), (29) and (15), it directly follows that

\[
p_3 = -\frac{1}{4} \left( 3a_o^2 H_{03} + H_{21} \right)
\]

is the Hopf coefficient of third degree, and

\[
p_5 = -\frac{1}{8} \left( 5a_o^4 H_{05} + 2a_o^2 H_{23} + H_{41} \right)
\]

is the Hopf coefficient of fifth degree. Other Hopf coefficients can be derived by forward calculations. We recall that only the coefficients of \( \Gamma \) and the coefficients of the \( M \)-jet are required to calculate the Hopf coefficients. In accordance with equation (28), if the Jacobian matrix has the form in section 2 and being \( \Gamma = \Gamma_1 \), we have the numbers

\[
R_{mn} = \frac{1}{2} \left( \tau_a \sigma_1^{1} - \Lambda_o \sigma_2^{2} \right).
\]

In section 5, we show different examples of the degenerate HB within the class of polynomial vector fields of at most the sixth degree, while the \( H_{jk} \) in equations (30) and (31) can be found in the appendix.
3.2. Asymptotic expansions to the solutions

Going back in the variable substitutions, we could derive a uniform asymptotic expansion to the amplitude of each solution of equation (1) valid on $[0, +\infty]$. We shall use this fact to identify limit cycles in the former system through the ones in the averaged system. For instance, to the periodic solution $\Theta(t) = (x(t), y(t))$ which generates the limit cycle, we have

$$\Theta(t) = u_1(t) (|\tau_a|)^\kappa + O(|\tau_a|^{2\kappa}),$$

$$\nabla(t) = v_1(t) (|\tau_a|)^\kappa + O(|\tau_a|^{2\kappa}),$$

where $\kappa = \frac{1}{2N}$ if the HB is non-degenerate, or $\kappa = \frac{1-\gamma}{2N}$ at the second-type degenerate HB. Here

$$\left( \begin{array}{c} u_1(t) \\ v_1(t) \end{array} \right) = \frac{r_c}{r_1 - 1} \left( \begin{array}{c} \cos(\sigma t) \\ -\omega_a \sin(\sigma t) \end{array} \right),$$

with frequency

$$\sigma = \omega_a + q (\rho, |\tau_a|^\kappa)$$

and period $T = \frac{2\pi}{\omega_a}$. In equation (35) we shall take $r_c$ as $r_e$ or $r_L$ corresponding to whether the HB is non-degenerate or not. Note that, up to the leading terms, the expansions to the solutions in equations (33) and (34) are uniform, since the functions in equation (35) are bounded for $t > 0$. Moreover, to any solution starting in the region of attraction of a cycle correspond the terms on the right-hand side of equation (35) which are bounded for $t > 0$, so leading to a uniform expansion of the amplitude.

3.3. On the period of the limit cycles

The period of the emerging limit cycles in the non-degenerate or second-type degenerate HB can be determined from the formula for the frequency, given in equation (36). To do so, it is necessary to consider equations (12), (16) and (29). Up to 6-jet-equivalence, the expansion results

$$q = -\frac{1}{8\omega_a}\epsilon^2 \rho^2 \left( 3H_{30} + \omega_a^2 H_{12} \right) + \frac{1}{2} \epsilon^2 \rho^2 \left[ 5H_{50} + \omega_a^2 H_{32} + \omega_a^4 H_{14} \right] + O(\epsilon^6).$$

The coefficients $H_{ij}$ in the above formula can be calculated in a similar way as the others in equations (A.1). The procedure yields the following estimate of the frequency as $\tau_a \to 0$:

$$q = O (|\tau_a|^\kappa)$$

for the emerging cycle. We remark on the fact that, in the last formula, the order is not necessarily sharp. The exponent $\kappa$ depends on the asymptotic behavior of the $H_{ij}$ (see section 5).

4. Genericity conditions for the non-degenerate HB

Equations (A.1) in the appendix give the expressions $H_{mn}$ for up to 6-jet-equivalence for equation (1). Now, we are ready to give genericity conditions for the appearance of the non-degenerate HB in the system (1) featured by a codimension-1 or -2 bifurcations, or equivalently, up to 4- or 6-jet-equivalence.
Theorem 2. For the appearance of a non-degenerate HB in a polynomial system equation (1) up to the 4-jet-equivalence, with Jacobian at the origin taken as in section 2 and satisfying condition (3), it is necessary and sufficient that
\[ p_3(0) = -\frac{1}{4} (3\omega_2^2 H_{35} + H_{23}) \mid_{\tau_a=0} \neq 0. \] (38)

The HB is supercritical if \( p_3(0) > 0 \), or subcritical provided that \( p_3(0) < 0 \). The radius of the limit cycle as \( \tau_a \to 0 \) is \( O(\tau_a^{1/2}) \).

Proof. The result follows from equation (15), theorem 1 and the coefficient \( p_3 \) in equation (30). The last assertion follows from equation (24).

The corresponding formulation for the system (1) in HB with higher codimension is the following.

Theorem 3. For the appearance of a non-degenerate HB in a polynomial system (1) up to 6-jet-equivalence, with Jacobian at the origin taken as in section 2, and satisfying condition (3), it is necessary and sufficient that either equation (38) holds or, being \( p_3 \) negligible, the inequality
\[ p_5(0) = -\frac{1}{8} (5\omega_2^4 H_{35} + \omega_2^2 H_{23} + H_{41}) \mid_{\tau_a=0} \neq 0 \] (39)
holds. If equation (38) holds, then a single limit cycle emerges at bifurcation with the radius \( O(\tau_a^{1/2}) \). If, in contrast, we have in addition that equation (39) holds, then the radius of the cycle will be \( O(\tau_a^{1/4}) \) as \( \tau_a \to 0 \). The HB is supercritical (subcritical) if either \( p_3(0) > 0 \) (\( p_3(0) < 0 \)) or, being \( p_3 \) negligible (see definition 2), the inequality
\[ p_5(0) > 0 \] (40)
holds.

Proof. In accordance with equation (15) and theorem 1, a necessary and sufficient condition for the appearance of a non-degenerate HB will be that one of the expressions in equations (30) and (31) is different from zero. In accordance with equation (15), the function \( p \) in equation (19) has at most degree 2 as a function of \( r^2 \); hence the assertion holds. The asymptotic order of the radius is a consequence of equation (24).

Remark 3. If \( p_3(0) > 0 \) and equation (40) take place together, then we are in the presence of a supercritical Bautin-type bifurcation.

The appearance of limit cycles is studied mainly using Lyapunov coefficients. The resulting normal forms given in [12] are similar to equations (13) and (17), but formulas connecting the Lyapunov coefficient \( l_k \) in the study of HB and Hopf’s \( p_{2k+1} \) are not obtained here. Nevertheless, it is expected that the genericity conditions, whether they are obtained by \( l_k \) or by \( p_n \), should hold simultaneously.

Let us now show some normal forms at the non-degenerate supercritical HB.

4.1. The Bautin bifurcation

This is a well-known polynomial vector field with degree 5 [15] depending on the parameter \( a \), which is used to be studied by taking polar coordinates (see [15, theorem 3.4]). Here we illustrate how the averaging method in the previous section works. This system features the emergence of a single limit cycle and the persistence of another one at the supercritical HB. The system can be written in complex form:
\[ \dot{\eta} = \frac{1}{2} (\alpha \eta - |\eta|^2 \eta + |\eta|^4 \eta), \] (41)
where $\eta = x + iy$, $\alpha = a + i$. Here, the trace $\tau = a$ is the bifurcation parameter, $\delta = (1 + a^2)/4$ and $\Lambda = 1$. To analyze the system using the averaging method, we first do a transformation of variables. Let us write in complex form the transformation $\Gamma$, the action of which is given by $\Gamma(\eta) = \frac{1}{2}[(1 + ia)\eta + (3 + ia)\bar{\eta}]$. So we consider the inverse transformation $\eta = \frac{1}{2}((-1 + ia)Z + (3 + ia)\bar{Z})$, where $Z = z + iz$. Making appropriate calculations (see the appendix), we obtain the Hopf coefficients $p_2 = \frac{1}{8}(4 - a^2)$ and $p_5 = -\frac{1}{16}(8 + 8a^2 + a^4) < 0$. We have a non-degenerate HB of third degree because $p_3(0) > 0$. Thus, $\nu^2 = a$. So equation (22) leads to $a^2p_5s^4 + ap_3r^2 - a = 0$, with positive solutions: $r_1 = \sqrt{2} + O(a)$ and $r_2 = a^{-1/2} - a^{1/2} + O(a^{3/2})$ provided $a > 0$. It follows the presence of two limit cycles for $a > 0$, one of them ($a^{1/2}r_1$) has radius $O_S(a^{1/2})$ as $a \to 0$, while the other ($a^{1/2}r_2$) has a non-zero radius at $a = 0$. Thus, the last cycle is not properly caused by the HB, because it persists along the bifurcation. The frequency of the emerging cycle will be $\nu = \frac{1}{2} + O(|a|)$ as we take $N = 1$.

4.2. Non-degenerate bifurcation of higher codimension

Let us consider an example [11] of a polynomial system with degree 5 showing a non-degenerate HB in the sense proposed here, the so-called generalized Andronov–Hopf bifurcation:

$$\dot{\eta} = \frac{1}{2}(\alpha\eta - v|\eta|^2\eta), \quad (42)$$

where $\eta = x + iy$, $\alpha = a + i$. This is featured by a supercritical HB with one limit cycle if $v = 1$ (subcritical if $v = -1$). The origin is a weak focus if $\tau = a = 0$. Since the Jacobian matrices in equation (42) and in the Bautin case are the same, we can use the previous calculations in subsection 4.1. By forward calculations (see the appendix), we get that $p_3$ is negligible and $p_5 = \frac{1}{16}(23 + 38a^2 + 128a^4)$. We take $\epsilon = a^{1/2}$. Further, equation (22) has a single positive root for $v = 1$: $r = \left(\frac{a}{2}\right)^{1/4} + O(a^2)$, corresponding to a limit cycle whose radius is $O_S(a^{1/4})$.

This behavior can be extrapolated for the system

$$\dot{\eta} = \frac{1}{2}(\alpha\eta - v|\eta|^{2N}\eta)$$

featured with a (codimension $N$) supercritical HB at the critical value $a = 0$, in which the emerging cycle has the radius $r = O_S(a^{1/2N})$. The frequency of the emerging cycle will be $\nu = \frac{1}{2}\sqrt{1 + a^2} + O(|a|^{1/2N})$.

5. Typical forms in the degenerate HB

The main concern in this section is the degenerate HB of the first and second type. In subsections 5.1–5.4 we will study ‘typical’ forms which are not normal forms due to the absence of genericity conditions (see [15]). Note that an emerging cycle will be considered as a multiple one whenever the corresponding root of the discriminant equation is multiple also. In subsection 5.5 we state a sufficient condition for the second-type degenerate HB up to 6-jet-equivalence.

5.1. Multiple cycles in the supercritical second-type degenerate HB

In this subsection, we will give an example of a system which shows a degenerate HB of second type:

$$\dot{\eta} = (a^3 + i)\eta - a^2|\eta|^2\eta + \frac{1}{16}a|\eta|^4\eta. \quad (43)$$
Neither limit cycle surrounds the focus while the trace $\tau = 2a^3$ is negative. At the critical value $a = 0$, the origin is clearly a center to the system (43). Further, when the trace becomes positive, the singular point becomes unstable and two small limit cycles emerge due to an HB with radii $r_1 = 2\sqrt{\frac{3}{2}}a^{1/2}$ and $r_2 = 2a^{1/2}$, respectively. Both cycles emerge due to the HB and both radii have order $O_3(\tau^{1/6})$ as $\tau \to 0$. The first cycle is stable, while the second is unstable. The frequency of the emerging cycles is $\omega = \sqrt{1 + a^6 + O(|\tau a|^{1/3})}$, since $N = 1$ and $\gamma = 2/3$.

It is easy to check the above assertions by rewriting the system (43) in polar coordinates. From this example it can be concluded that, at the second-type degenerate HB, several limit cycles may emerge. With the same idea, it is possible to build polynomial dynamical systems with degree $2N + 1$ showing the emergence of $N$ different limit cycles at the super- or subcritical degenerate HB of second type.

5.2. Semistable cycles in the supercritical second-type degenerate HB

The following system shows a supercritical second-type degenerate HB:

$$\dot{\eta} = (a^3 + i) \eta - 2a^2|\eta|^2 \eta + a|\eta|^4 \eta. \tag{44}$$

The origin is a stable focus without any surrounding limit cycle while the trace $\tau = 2a^3$ is negative. At the critical value $a = 0$, the origin is clearly a center to the system (44). Further, when the trace becomes positive the singular point becomes unstable and a single small limit cycle emerges with radius $r = a^{1/2}$. This limit cycle is the $\omega$-limit set of any orbit inside the circle with the exception of the origin, but it is the $\alpha$-limit set of any orbit outside the circle. The corresponding system in polar coordinates is $\dot{r} = ar(r^2 - a), \dot{\theta} = 1$. From this example it can be concluded that, at a degenerate HB, semistable limit cycles may emerge. The radius of the limit cycle is $r = O_3(\tau^{1/6})$ as $\tau \to 0$, while the frequency of the emerging cycle is $\omega = \sqrt{1 + a^6 + O(|\tau a|^{1/3})}$, since $N = 1$ and $\gamma = 2/3$.

5.3. First-type degenerate HB without emergence of the limit cycle

(1) Let us first consider a system showing a degenerate HB of first type:

$$\dot{\eta} = (a + i) \eta - a |\eta|^2 \eta \tag{45}.$$ 

The origin is a stable focus without limit cycle while $a < 0$. At the critical value $a = 0$, the origin is a center, and it turns unstable if $a > 0$. The existence of a stable limit cycle can be noted with radius $r = 1$, but this cycle is not a consequence of the HB because it persists along the bifurcation (including $a = 0$). Note that the stability of this periodic orbit changes with the sign of $a$. The polar system is $\dot{r} = ar(1 - r^2), \dot{\theta} = 1$.

(2) We recall the fact that the persisting limit cycle may be semistable. Let

$$\dot{\eta} = (a + i) \eta - 2a|\eta|^2 \eta + a|\eta|^4 \eta \tag{46},$$

which also shows a degenerate HB of first type. The corresponding polar system is $\dot{r} = ar(1 - r^2), \dot{\theta} = 1$. So the limit cycle $r = 1$ is semistable, whose stability changes with the sign of $a$, and is still an orbit for $a = 0$.

5.4. First-type degenerate HB showing limit cycles at infinity

Let us now consider a system showing a degenerate HB of first type, but leading in this case to the so-called HB at infinity [11]:

$$\dot{\eta} = (a + i) \eta - a a^6 |\eta|^2 \eta \tag{47}.$$
Here we take $a^\beta = \text{sign}(a) \cdot |a|^\beta$. Taking polar coordinates, we obtain 
\[ \dot{r} = ar(1 - a^\beta r^2) \] and $\dot{\theta} = 1$. We conclude that, as a consequence of the degenerate HB, no limit cycle emerges surrounding the origin, but an unstable limit cycle surrounding the point at infinity, whose radius is $r = a^{-\beta/2}$, emerges supercritically, i.e. for $\tau = 2a > 0$.

5.5. A sufficient condition for the second-type degenerate HB

We consider here a degenerate HB for the system (1) up to 6-jet-equivalence. The following statement, which is inspired by the examples in subsections 5.1 and 5.2, gives a sufficient condition for the existence of the degenerate HB of second type.

**Proposition 3.** Let equation (3) hold for the system (1) at the origin, and let $j_6(F)(0)$ be such that the coefficients $p_3$ in equation (30) and $p_5$ in equation (31) are
\[ p_3 = \tau_a^\gamma Q_3 + o(\tau_a^\gamma), \quad p_5 = -\tau_a^{2\gamma-1} Q_5 + o(\tau_a^{2\gamma-1}) \]

for some $\gamma (1/2 < \gamma < 1)$ as $\tau_a \to 0$, where $Q_3, Q_5$ are both positive numbers and
\[ \Delta = Q_3^2 - 4Q_5 \geq 0. \]

Then, two different limit cycles if $\Delta > 0$, or one semistable limit cycle if $\Delta = 0$, emerge at the supercritical degenerate HB. The radius of the cycles is $O_5(\tau_a^{(1-\gamma)/2})$ as $\tau_a \to 0^+$.

**Proof.** Taking $1/2 < \gamma < 1$ in equation (21) follows equation (26); hence $\epsilon = \tau_a^{(1-\gamma)/2}$. Up to the $O(\tau_a)$ leading term in equation (22), we obtain
\[ Q_5 r^4 - Q_5 r^2 + 1 = 0 \]
which means that the algebraic equation above has either two different positive roots, or a single positive root with multiplicity 2, depending on $\Delta$. The proof follows from equations (22) and (13).

6. Conclusions

One way of classifying an HB on the basis of the number of emerging limit cycles arises from the present investigation. We propose to classify the degeneracy of the HB in (real) analytical dynamical systems near an isolated singular point on the plane by considering the number of emerging limit cycles surrounding this point. If the HB is non-degenerate, then a single limit cycle emerges. The amplitude of this cycle has the order $O(\tau_a^{1/2N})$ as $\tau_a \to 0$ for some integer $N$. In this scenario, the emerging limit cycle may coexist with other cycles which persist along the HB, as in the Bautin case. We show that no limit cycle surrounding the singular point emerges either at super- or subcritical HB if it shows a degeneracy of the first type. In this scenario, limit cycles surrounding the point at infinity may also emerge. For a degenerate HB of the second type, we found an ‘anomalous’ asymptotic order of the radius of the emerging limit cycles, and further, we give a sufficient condition to the appearance either of a couple of limit cycles or one semistable cycle of the system (1), up to 6-jet-equivalence. Finally, we quote normal forms for the non-degenerate HB, while we propose ‘typical’ forms showing different behaviors that can occur at the supercritical degenerate HB, being either of the first or second type.

**Acknowledgments**

The author would like to acknowledge the help given by Dr Stephen A Roberts (Thames Valley University, London) for assistance in editing the English text.
Appendix

In this appendix we include the values of the integrals which are used in the implementation of the Krylov–Bogoliubov averaging method. Note that for $m + n \leq 6$ the non-zero $K_{mn}$ in equation (29) are $K_{03} = \frac{3}{2} \pi$, $K_{21} = \frac{1}{2} \pi$, $K_{05} = \frac{5}{2} \pi$, $K_{23} = \frac{3}{2} \pi$, $K_{41} = \frac{5}{2} \pi$. We obtain the following equalities:

$$H_{03} = R_{12} \mu_{12} \mu_{22}^2 + R_{03} \mu_{12}^3 + R_{30} \mu_{12}^2 + R_{21} \mu_{12} \mu_{22}^2 = - (8 R_{03}) \Lambda_{a}^{-3},$$

$$H_{21} = 3 R_{30} \mu_{11}^2 \mu_{12} + 3 R_{30} \mu_{12}^3 \mu_{11} + 3 R_{21} \mu_{12} \mu_{22} + 2 R_{12} \mu_{11} \mu_{12} \mu_{22} + R_{12} \mu_{12} \mu_{22}^2 + 3 R_{03} \mu_{12} \mu_{22}^2 = - 2 (R_{21} \Lambda_{a}^{-3} + 2 R_{12} \tau_{a} \Lambda_{a} + 3 R_{03} \tau_{a}^{2}) \Lambda_{a}^{-3},$$

$$H_{05} = R_{32} \mu_{12}^3 \mu_{22}^2 + R_{23} \mu_{12}^2 \mu_{22} + R_{14} \mu_{12} \mu_{22}^3 + R_{05} \mu_{22}^4 + R_{30} \mu_{12} + R_{41} \mu_{12} \mu_{22}^2 = - (32 R_{05}) (\Lambda_{a}^{-5}),$$

$$H_{23} = 10 R_{30} \mu_{11}^2 \mu_{12} + 4 R_{41} \mu_{12} \mu_{22} + R_{32} \mu_{11}^2 \mu_{12} \mu_{22} + 6 R_{41} \mu_{12} \mu_{22}^2 + 6 R_{32} \mu_{11} \mu_{12} \mu_{22}^2 + 3 R_{23} \mu_{11} \mu_{12} \mu_{22}^2 + R_{23} \mu_{12} \mu_{22}^3 + 3 R_{23} \mu_{12} \mu_{22}^3 = - 8 (R_{23} \Lambda_{a}^{-3} + 4 R_{14} \tau_{a} \Lambda_{a} + 10 R_{05} \tau_{a}^2) \Lambda_{a}^{-5},$$

$$H_{41} = 2 R_{32} \mu_{11} \mu_{12} \mu_{22} + 5 R_{30} \mu_{12} + R_{41} \mu_{12} \mu_{22} + 4 R_{41} \mu_{12} \mu_{22}^2 + 4 R_{32} \mu_{11} \mu_{12} \mu_{22} + 2 R_{32} \mu_{11} \mu_{12} \mu_{22}^2 + 3 R_{23} \mu_{11} \mu_{12} \mu_{22} + 3 R_{23} \mu_{11} \mu_{12} \mu_{22}^2 + 4 R_{41} \mu_{12} \mu_{22} + 4 R_{41} \mu_{12} \mu_{22}^2 + 5 R_{41} \mu_{12} \mu_{22}^2 = - 2 (R_{41} \Lambda_{a}^{-3} + 2 R_{32} \tau_{a} \Lambda_{a} + 3 R_{23} \tau_{a}^2 \Lambda_{a} + 4 R_{14} \tau_{a} \Lambda_{a} + 5 R_{05} \tau_{a}^2) \Lambda_{a}^{-5}. \quad (A.1)$$

### A.1. Calculus from subsection 4.1

From equation (41) it directly follows that $\sigma_{11}^1 = - \frac{1}{2} = \sigma_{12}^1 = \sigma_{21}^1 = \sigma_{03}^1 = \frac{1}{2} = \sigma_{14}^1 = \sigma_{34}^1 = \sigma_{04}^1$, and $\sigma_{22}^1 = 1 = \sigma_{22}^2$. Thus, from equation (32) we obtain $R_{03} = \frac{1}{3} = R_{21}$, $R_{12} = - \frac{3}{4} = R_{30}$, $R_{50} = \frac{3}{4} = R_{14}$, $R_{41} = - \frac{1}{4} = R_{05}$, $R_{32} = \frac{1}{2}$, $R_{23} = - \frac{1}{2}$. From equations (30), (31), and (A.1) it follows that $H_{03} = - 2$, $H_{21} = \frac{a^2 - 1}{2}$, $\omega_{a}^2 = \delta = \frac{1}{4}$, $H_{05} = 8$, $H_{23} = 4(1 + 3a^2)$, $H_{41} = \frac{1}{2}(1 + a^2)^2$.

### A.2. Calculus from subsection 4.2

We obtain $\sigma_{11}^1 = \sigma_{22}^1 = 0$ if $i + j = 3$, and $\sigma_{10}^1 = - \frac{1}{2} = \sigma_{14}^1 = \sigma_{34}^1 = \sigma_{04}^1$, $\sigma_{32}^1 = - \frac{1}{2} = \sigma_{23}^1$. From equation (32), we obtain $R_{mn} = 0$, $m + n = 3$, $R_{05} = \frac{1}{2} = R_{41}$, $R_{14} = - \frac{1}{2}a$, $R_{32} = - \frac{3}{2}a$, $R_{23} = \frac{5}{2}$. Thus, $H_{05} = - 8v$, $H_{25} = - 4v \left(1 + 3a^2\right)$, $H_{41} = - 8v \left(1 + a^2\right)^2$.

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