A 2-Component or N=2 Supersymmetric Camassa - Holm Equation

Ziemowit Popowicz\textsuperscript{a}

\textsuperscript{a} Institute of Theoretical Physics
University of Wroclaw
pl. M. Borna 9, 50 -205 Wroclaw, Poland

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Abstract

The extended N=2 supersymmetric Camassa - Holm equation is presented. It is accomplished by formulation the supersymmetric version of the Fuchssteiner method. In this framework we use two supersymmetric recursion operators of the N=2 $\alpha = -2, 4$ Korteweg - de Vries equation and constructed two different version of the supersymmetric Camassa - Holm equation. The bosonic sector of $N = 2, \alpha = 4$ supersymmetric Camassa - Holm equation contains two component generalization of this equation proposed by Chen, Liu and Zhang and as the special case two component generalized Hunter - Saxton equation considered by Aratyn, Gomes and Zimerman. As a byproduct of our analysis we defined the $N = 2$ supersymmetric Hunter - Saxton equation. The bihamiltonian structure is constructed for the supersymmetric $N = 2, \alpha = 4$ Camassa - Holm equation.
Introduction

Camassa and Holm introduced \[1\] in 1993 the integrable nonlinear partial differential equation

\[ u_t - u_{xxt} = -3uu_x + 2u_xu_{xx} + uu_{xxx} = (uu_{xx} + 1/2u_x^2 - 3/2u^2)_x \] (1)

which describes a special approximation of shallow water theory and has been extensively studied recently \[2, 3, 4, 5, 6, 7\]. It was shown that this equation possesses the bihamiltonian structure, could be solved using the inverse scattering method and has the so called peakons solutions.

The two component generalization of Camassa - Holm equation

\[ \begin{align*}
    m_t &= -um_x - 2mu_x + \rho \rho_x \\
    \rho_t &= -(\rho u)_x,
\end{align*} \] (2)

where \( m = u - u_{xx} \), has been proposed by Chen, Liu and Zhang \[3\]. This generalization, similarly to the Camassa - Holm equation, is the first negative flow of the AKNS hierarchy and possesses the interesting peakon and multi - kink solutions \[3, 4, 6\]. Moreover the system (2) is connected with the time dependent Schrödinger spectral problem \[3, 8\]. Quite recently Aratyn, Gomes and Zimerman \[4\] showed that the modification of the Schrödinger spectral problem for the equation (2) leads to two - component Hunter - Saxton equation \[9\]

\[ \begin{align*}
    u_{xxt} &= -2u_xu_{xx} - uu_{xxx} - \rho \rho_x \\
    \rho_t &= -(\rho u)_x,
\end{align*} \] (3)

In this paper we show that both mentioned generalizations are contained in the bosonic sector of \( N = 2 \) extended supersymmetric version of the Camassa - Holm equation. The idea of using extended supersymmetry for the generalization of the soliton equations appeared almost in parallel to the usage of this symmetry in the quantum field theory \[10\]. The main idea of the supersymmetry is to treat boson and fermion operators equally. In order to get supersymmetric theory we have to add to a system of \( k \) bosonic equations \( kN \) fermions and \( k(N - 1) \) boson fields \((k = 1, 2, ..N = 1, 2, ..)\) in such a way that the final theory becomes supersymmetric invariant. From the soliton point of view we can distinguish two important classes of the supersymmetric equations: the non-extended \( (N = 1) \) and extended \( (N > 1) \) cases. Consideration of the extended case may imply new bosonic equations whose properties need further investigation. This may be viewed as a bonus, but this extended case is in no way more fundamental than the non-extended one. Interestingly enough, some typical supersymmetric effects may occur in the supersymmetrical generalization of the soliton theory, compare to the classical case. We mention two of them: the ambiguity of the roots for the supersymmetric Lax operator \[11\] and appearance odd of Poisson brackets \[12\].

There are many different methods of supersymmetrization of the classical equations. The most popular one is to use the gradation arguments in which to each dependent and independent variables, in a given equation, we associate some weights.
We then replace the bosonic fields by superboson fields. For example, for the extended \( N = 2 \) supersymmetric case we consider the supersymmetric analog of dependent variable \( u(x,t) \), which can be thought as

\[
\Phi(x, t, \theta_1, \theta_2) = v(x, t) + \theta_1 \xi_1(x, t) + \theta_2 \xi_2(x, t) + \theta_2 \theta_1 u(x, t)
\]

where \( \theta_1 \) and \( \theta_2 \) are two anticommuting variables while \( \xi_1 \) and \( \xi_2 \) are Grassmann valued functions and \( v \) is an additional "new" bosonic field. In the next step using the usual and supersymmetric derivatives

\[
D_1 = \partial_{\theta_1} + \theta_1 \partial_x, \quad D_2 = \partial_{\theta_2} + \theta_2 \partial_x, \quad D_1 D_2 + D_2 D_1 = \partial
\]

we consider the most general assumption on the supersymmetric version of the given classical system in such a way that to preserve the given gradation of the supersymmetric equation. In the last step, we assume, that our supersymmetric generalization should possesses some special properties as for example, the existence of bihamiltonian structure or Lax operator.

However this procedure can not be applied to the Camassa - Holm equation, because this equation does not preserve gradation with respect to weight. In order to overcome this problem, we supersymmetrize the Fuchssteiner method \cite{7}, in which the hereditary operator, responsible for Camassa - Holm hierarchy is constructed out of two different hereditary recursion operators. We introduce the supersymmetry to the theory, considering the supersymmetric analog of these two operators.

The paper is organized as follows. In the first section we summarize the Fuchssteiner method. In the second section, using the supersymmetric recursion operators, which creates two different supersymmetric \( N = 2 \) Korteweg - de Vries equation, we constructed two different supersymmetric version of Camassa - Holm equation. As a byproduct of our analysis we define the extended \( N = 2 \) version of the supersymmetric Hunter - Saxton equation. In the same section we investigate the bosonic sector of this equation. In the last section we describe the bihamiltonian structure of the supersymmetric Camassa - Holm equation.

**Camassa - Holm Equation**

In order to construct the Camassa - Holm equation, we briefly describe the method used by Fuchssteiner \cite{7}. This method based on the following observation. If we have two different hereditary recursion operator \( R_1 \) and \( R_2 \) and one of them, for example \( R_2 \), is invertible then \( R = R_1 R_2^{-1} \) is also the hereditary recursion operator. Therefore this operator can be used to construct some new integrable hierarchy of equations. Let us present this method for the Camassa - Holm equation.

First let us consider the hereditary recursion operator for the Korteweg de Vries hierarchy

\[
R = c \partial^2 + \lambda (\partial u \partial^{-1} + u).
\]

where \( c \) and \( \lambda \) are an arbitrary constants. This operator generates the hierarchy of integrable equations in which the first member is

\[
\frac{\partial u}{\partial t} = Ru_x = cu_{xxx} + 3\lambda uu_x
\]
For $c = 1$ and $\lambda = 2$ we have famous Korteweg - de Vries equation.

The second recursion operator can be extracted from the first operator shifting function $u \to u + \gamma$ in recursion operator (6) where $\gamma$ is a constant.

Indeed after shifting the function $u$ in (6) the $R$ operator transforms to

$$R(u + \gamma) = R_1 + R_2 = (c_1 \partial^2 + \lambda (\partial u \partial^{-1} + u)) + (c_2 \partial^2 + \lambda \gamma).$$

where $c = c_1 + c_2$. It appeared that the recursion operator $R = R_1 R_2^{-1}$ generates new hierarchy of integrable equations

$$u_{tn} = (R_1 R_2^{-1})^n u_x$$

Assuming that $u = R_2 v = c_2 v_{xx} + \lambda \gamma v$ the first member of the hierarchy is

$$\lambda \gamma v_t + c_2 v_{xxx} = (c_1 v_{xx} + \lambda c_2 v_{xx} v + \frac{\lambda c_2}{2} v_x^2 + \frac{3 \lambda^2 \gamma}{2} v^2)_x.$$ 

Now assuming that $c_1 = 0, c_2 = \gamma = 1$ and $\lambda = -1$ the last equation is exactly the Camassa - Holm equation. The second choice $\gamma = 0, c_1 = 0, \lambda = -1$ leads us to the Hunter - Saxton equation

$$v_t = -(v_{xx} v + \frac{1}{2} v_x^2)_x$$

The bihamiltonian structure of equation (1) has been constructed in [1]

$$m_t = J_2 \frac{\delta H_1}{\delta m} = (c_1 \partial^3 + \lambda (\partial m + m \partial)) \frac{\delta H_1}{\delta m} = J_1 \frac{\delta H_2}{\delta m} = (c_2 \partial^3 + \lambda \gamma \partial) \frac{\delta H_2}{\delta m}$$

where $m = c_2 v_{xx} + \lambda \gamma v$ and $H_1 = \frac{1}{2} \int dx (c_2 v_{xx} + \lambda \gamma v)v, H_2 = \frac{1}{2} \int dx (2 c_1 \lambda v_{xx} v^2 + 3 c_1 v_{xx} v + c_2 \lambda v_x^2 v + 3 \gamma \lambda^2 v^3)$.

As we see the second hamiltonian operator is the same as for the Korteweg - de Vries equation and is connected with the Virasoro algebra. For centerless Virasoro algebra $c_1 = 0$ we have additional conserved Hamiltonian $2 \int \sqrt{m}$ which is the Casimir for this algebra as well. Using the first Hamiltonian operator to this quantity we obtain the Harry Dym type equation of motion

$$m_t = J_1 \frac{1}{\sqrt{m}}$$

which reduces to the Harry Dym equation when $c_2 = 1, \gamma = 0$.

**N=2 Supersymmetric Camassa - Holm and Hunter - Saxton equation**

We have four different integrable extended $N = 2$ supersymmetric extensions of the Korteweg - de Vries equation. Three of them are connected with the supersymmetric Virasoro algebra [13] while fourth is connected with the odd version of supersymmetric Virasoro algebra [12]. The first three mentioned equations are

$$\Phi_t = - J_2 \frac{\delta H_1}{\delta \Phi} = (D_1 D_2 \partial + 2 \partial \Phi + 2 \Phi \partial - D_1 \Phi D_1 - D_2 \Phi D_2) \frac{\delta H_1}{\delta \Phi} = (\Phi_{xx} + 3 \Phi (D_1 D_2 \Phi) + \frac{(\alpha - 1)}{2} (D_1 D_2 \Phi^2 + \alpha \Phi^3)_x$$
where \( H_1 = \frac{1}{2} \int dx d\theta d\theta_2 (\Phi \Phi (D_1 D_2 \Phi) + \frac{1}{3} \Phi^3) \) and parameter \( \alpha \) can take three different values 1, 4, -2. This system has been extensively studied from different point of view in many papers.

The properties of these generalizations are different for different values of \( \alpha \). The Lax operator for \( \alpha = 4 \) has two different roots \([11]\). For \( \alpha = 4 \) case instead the bihamiltonian formulation, we deal with the inverse first hamiltonian structure \([11]\), while for \( \alpha = 1 \) case we have the nonstandard Lax operator \([14]\) and higher order nonlocal recursion operator \([15]\).

The hereditary recursion operator for \( \alpha = 4 \) and \( \alpha = -2 \) has been constructed in \([11]\)

\[
\begin{align*}
R_4 &= J_2 J_{1,-1} = c D_1 D_2 - \lambda (2 \partial \Phi \partial^{-1} - (D_1 \Phi) D_1 \partial^{-1} - (D_2 \Phi) D_2 \partial^{-1}) \\
R_{(-2)} &= J_2 J_{1,-2} \\
J_2 &= c D_1 D_2 \partial - \lambda (2 \partial \Phi + 2 \Phi \partial - D_1 \Phi D_1 - D_2 \Phi D_2) \\
J_{1,4} &= \partial \\
J_{1,(-2)} &= c D_1 D_2 \partial^{-1} - \lambda (\partial^{-1} D_1 \Phi D_1 \partial^{-1} + \partial^{-1} D_2 \Phi D_2 \partial^{-1})
\end{align*}
\]

Here \( J_2 \) is the second Hamiltonian operator, which is connected with the extended \( N = 2 \) supersymmetric version of Virasoro algebra, \( J_{1,4} \) is the first hamiltonian operator for \( \alpha = 4 \) while \( J_{1,-2} \) is an inverse operator of the first Hamiltonian operator for \( \alpha = -2 \) case.

In the next we will use these operators in order to adopt the Fuchssteiner method to the supersymmetric case. For this pourposes we consider these operators individually.

\textbf{A.)} \( \alpha = 4 \).

Interestingly the supersymmetric Korteweg - de Vries equation for this value of \( \alpha \) is not a first member of hierarchy \( \Phi_{t_n} = R^n \Phi \). The first member is \( \Phi_t = ((D_1 D_2 \Phi) + \Phi^2)_x \) while the second is our supersymmetric Korteweg - de Vries equation. However let us use the supersymmetric recursion operator \( R_4 \) to the construction of the supersymmetric analog of the Camassa - Holm recursion operator. In order to find the analog of \( R_2 \) operator let us shift the \( \Phi \) superfunction to \( \Phi \rightarrow \Phi + \gamma \) in \( R_4 \) obtaining

\[
R_4(\Phi + \gamma) = R_1 + R_2
\]

where

\[
\begin{align*}
R_1 &= c_1 D_1 D_2 - \lambda (2 \partial \Phi \partial^{-1} - (D_1 \Phi) D_1 \partial^{-1} - (D_2 \Phi) D_2 \partial^{-1}) \\
R_2 &= c_2 D_1 D_2 - 2 \lambda \gamma.
\end{align*}
\]

and \( c = c_1 + c_2 \). Obviously \( R_2 \) is the hereditary operator and we can consider the hierarchy of equations generated by

\[
\Phi_{t_n} = (R_1 R_2^{-1})^n \Phi_x
\]
Assuming that $\Phi = R_2 \Upsilon = c_2(D_1D_2 \Upsilon) - 2\lambda \gamma \Upsilon$ we obtain that the first member in this hierarchy is

$$c_2(D_1D_2 \Upsilon_t) - 2\lambda \gamma \Upsilon_t = c_1(D_1D_2 \Upsilon)_x - 2\lambda(\Phi \Upsilon)_x + \lambda(D_2 \Phi)(D_2 \Upsilon) + \lambda(D_1 \Phi)(D_1 \Upsilon).$$

(19)

or explicitly as

$$c_2(D_1D_2 \Upsilon_t) - 2\lambda \gamma \Upsilon_t = (c_1(D_1D_2 \Upsilon) - 2c_2 \Upsilon(D_1D_2 \Upsilon) + 4\gamma \lambda^2 \Upsilon^2 + c_2 \lambda(D_2 \Upsilon)(D_1 \Upsilon))_x.$$  

(20)

It is our supersymmetric $N = 2, \alpha = 4$ Comasa - Holm equation.

Let us compute the bosonic sector of the previous equation where all odd function disappear. Assuming that $\Upsilon = v + \theta_2 \theta_1 u$ in this sector, we obtain

$$\begin{align*}
(c_2u - 2\gamma \lambda v)_t &= (c_1u + 4\gamma \lambda^2 v^2 - 2c_2 \lambda v u)_x, \\
(-c_2v_{xx} - 2\gamma \lambda u)_t &= (-c_2\lambda u^2 + 2c_2 \lambda v_{xx} v - c_1v_{xx} v + c_2 \lambda v_x^2 + 8\gamma \lambda^2 vu)_x
\end{align*}$$

(21)

Now introducing new function $\rho = c_2u - 2\gamma \lambda v$ our system (21) can be rewritten as

$$\begin{align*}
\rho_t &= \frac{1}{c_2}(c_1\rho + 2c_1\gamma \lambda v - 2c_2 \lambda v u)_x \\
(-c_2v_{xx} - \frac{4\gamma^2 \lambda^2}{c_2} v)_t &= \left(\frac{2c_1 \gamma \lambda}{c_2}\rho + \frac{4c_1 \gamma^2 \lambda^2}{c_2} v - c_1v_{xx} v + 2c_2 \lambda v v_{xx} + c_2 \lambda v_x^2 + \frac{12\gamma^2 \lambda^3}{c_2} v^2 - \lambda \frac{\rho^2}{c_2}\right)_x
\end{align*}$$

(22)

Thus we obtained even more general generalization of two - component Camassa - Holm equation than equation (2). Interestingly our equation (22) contains equation (2) as a special case in which $c_1 = 0, c_2 = -1, \lambda = \frac{1}{2}, \gamma = 1$.

For $\gamma = c_1 = 0, \lambda = \frac{1}{2}, c_2 = -1$ our system of equation (22) reduces to the two component version of Hunter - Saxton (3).

From that reason the equation (20) when $\lambda = \frac{1}{2}, c_1 = 0$

$$(D_1D_2 \Upsilon)_t = \frac{1}{2}((D_2 \Upsilon)(D_1 \Upsilon) - 2\Upsilon(D_1D_2 \Upsilon))_x.$$  

(23)

could be considered as the $N = 2$ supersymmetric Hunter - Saxton equation.

**B.) $\alpha = -2$.**

In contrast to the previous case, now the supersymmetric Korteweg - de Vries equation is a first member of the hierarch $\Phi_t = R_{(-2)} \Phi_x$. Similarly to the $\alpha = 4$ case we can shift the superfunction $\Phi \to \Phi + \gamma$ in $R_{-2}$ in order to obtain $R_2$ operator. However then we obtain that operator $R_2$ contains the $\Phi$ superfunction and therefore will be not considered here. The next choice is to use the $R_2$ operator from $\alpha = 4$ case. In this situation we obtain an integro - differential supersymmetric equation and from that reasons we will not consider such possibility.

From the other side, we can try to choose $R_2 = \gamma - \partial^2$ operator, the same as in the classical situation, in order to consider the hierarchy of equation generated by

$$\Phi_t = R_{(-2)} R_2^{-1} \Phi_x$$

(24)
where now $\Phi = R_2 \Upsilon = \gamma \Upsilon - \Upsilon_{xx}$.

Assuming that $\lambda = \frac{1}{2}$ and $c = 1$ the first equation reads

\begin{equation}
(\gamma \Upsilon - \Upsilon_{xx})_t = \frac{1}{2} \left( 4(D_1 D_2 \Upsilon) \Upsilon_{xx} - 2 \Upsilon_{xx} + \Upsilon_{xx} \Upsilon_x^2 - 4 \gamma (D_1 D_2 \Upsilon) \Upsilon - \gamma (D_2 \Upsilon)(D_1 \Upsilon)(D_1 D_2 \Upsilon) - \gamma \Upsilon_{xx} \Upsilon^2 - \gamma \Upsilon_x^2 \Upsilon - 2 \gamma^3 \Upsilon^3 + \gamma (D_2 \Upsilon)(D_1 \Upsilon) \right)_x + \end{equation}

\begin{equation}
\frac{1}{4} \left( (D_2 \Upsilon_{xx})(D_2 \Upsilon) \Upsilon_{xx} - 2 (D_2 \Upsilon_{xxx})(D_1 \Upsilon) - (D_2 \Upsilon_{xx})(D_2 \Upsilon_x) \Upsilon_x - (D_2 \Upsilon_{xx})(D_1 \Upsilon_x)(D_1 D_2 \Upsilon) + (D_2 \Upsilon_{xx})(D_1 \Upsilon)(D_1 D_2 \Upsilon_x) - (D_2 \Upsilon)(D_1 \Upsilon_{xx})(D_1 D_2 \Upsilon_x) + 2 (D_1 \Upsilon_x)(D_1 \Upsilon) \Upsilon_{xxx} + \gamma ((D_2 \Upsilon)(D_1 \Upsilon)_x(D_1 D_2 \Upsilon) - 3 \gamma (D_2 \Upsilon_x)(D_2 \Upsilon) \Upsilon_x + (D_1 \Rightarrow D_2, D_2 \Rightarrow - D_1) \right). \end{equation}

It is our supersymmetric $N = 2, \alpha = -2$ Camassa - Holm equation.

The bosonic sector in which $\Upsilon = v + \theta_1 \theta_2 u$ read

\begin{equation}
(\gamma v - v_{xx})_t = \frac{1}{2} \left( 4 v_{xx} u - 2 v_{xx} + v_{xx} v_x^2 - \gamma v_{xx} v^2 - \gamma v_x^2 v + \gamma v^3 - 4 \gamma v u \right)_x, \end{equation}

\begin{equation}
(\gamma u - u_{xx})_t = \frac{1}{2} \left( 2 u_{xx} u - 2 u_{xx} + u_x^2 - 3 \gamma u^2 - 2 v_{xxx} v_x u - 2 v_{xxx} v_x u - 3 v_{xx}^2 + 2 v_{xx} v_x u_x - 2 \gamma v_{xx} v u + \gamma v_x^2 + v_x^2 - 2 v_x v u - \gamma v_x^2 u_x + \gamma^2 v^2 u \right)_x + v_{xxxx} v_x + 2 v_{xxx} v_x u - 3 \gamma v_{xx} v_x u. \end{equation}

For $v = 0$ this system reduces to the classical Camassa - Holm equation (1) while for $v \neq 0$ gives us new generalization of Camassa - Holm equation.

C.) $\alpha = 1$.

As we mentioned earlier, the recursion operator for this case, is higher order non-local. From that reason, the $R_2$ operator obtained shifting the $\Phi$ superfunction, analogously to the previous cases, leads us to very complicated operator. If we follows in the same way as in $\alpha = 4$ case and use the the same operator $R_2$ then we obtain very complicated system also. From that reason we will not study further this case.

Bihamiltonian Structure.

We have constructed this structure for $\alpha = 4$ case only. This structure could be obtained from the recursion operator $R_4$ in a similar manner as is classical case [I]. In order to end this let us make the following observation. Notice that the first member in the hierarchy (18) could be rewritten as

\begin{equation}
M_t = J_2(M) J_{1,4}^{-1} R_2^{-1} M_x = J_2 \Upsilon = J_2 R_2^{-1} \frac{\delta H_1}{\delta \Upsilon} = J_2 \frac{\delta H_1}{\delta M} \end{equation}
where \( M = R_2 \Upsilon = c_2 (D_1 D_2 \Upsilon - \lambda \gamma \Upsilon) \) and \( H_1 = \frac{1}{2} \int dx d\theta_1 d\theta_2 ((R_2 \Upsilon) \Upsilon) \). As we see this is the second hamiltonian structure and is generated by the same supersymmetric operator which is responsible for the second hamiltonian structure of Korteweg - de Vries equation.

The first hamiltonian structure follows from the following observation. Notice that the equation (20) could be rewritten also as

\[
M_t = \partial \frac{\delta H_2}{\delta \Upsilon} = \partial R_2 R_2^{-1} \frac{\delta H_2}{\delta \Upsilon} = \partial R_2 \frac{\delta H_2}{\delta M}
\]  

(29)

where

\[
H_2 = \frac{1}{6} \int dx d\theta_1 d\theta_2 ( - 4 c_2 \lambda (D_1 D_2 \Upsilon) (\Upsilon^2 + 3 c_1 (D_1 D_2 \Upsilon) \Upsilon + 8 \gamma \lambda^2 \Upsilon^3 + 2 c_2 \lambda (D_2 \Upsilon) (D_1 \Upsilon))
\]  

(30)

Now the Hamiltonians operators \( J_2 \) and \( (c_2 D_1 D_2 \partial - \lambda \gamma \partial) \) are compatible. We have checked the compatibility using computer algebra Reduce [16] and special computer package SUSY2 [17]. Therefore we can apply these operators to the construction of recursion operator \( J_2 \partial^{-1} R_2^{-1} \). Correspondingly this operator generates an infinite sequence of conservation laws.

However in this supersymmetric structure there is a fundamental difference compare to the classical situation. The classical second hamiltonian operator of the Camassa - Holm equation has a Casimir in the form of \( \int dx \sqrt{m} \) and it is a Casimir for centerless Virasoro algebra also. The extended \( N = 2 \) centerless supersymmetric Virasoro algebra does not possesses such Casimir [18]. Hence it is impossible to start construct the supersymmetric analog of the classical hierarchy which contains the Harry Dym type equation.

**Conclusion.**

In this paper the extended \( N=2 \) supersymmetric generalization of Camassa - Holm equation was presented. It was accomplished adopting the Fuchssteiner method of the generation of the Camassa - Holm equation to the supersymmetric case. In this framework we used two different recursion operators of the \( N=2 \) supersymmetric \( \alpha = -2, 4 \) Korteweg - de Vries equation and constructed two different version of the supersymmetric Camassa - Holm equation. The bosonic sector of the \( N = 2, \alpha = 4 \) supersymmetric Camassa - Holm equation contains two component generalization of this equation proposed by Chen, Liu and Zhang and two component Hunter - Saxton equation considered by Aratyn, Gomes and Zimerman. As a byproduct of our analysis we defined the \( N = 2 \) supersymmetric Hunter - Saxton equation. We have constructed the bihamiltonian structure for the supersymmetric \( N = 2, \alpha = 4 \) Camassa - Holm equation only. For the \( \alpha = -2 \) case, the supersymmetric Korteweg - de Vries has the inverse first hamiltonian formulation. From that reasons we expect that the same may occur in the supersymmetric \( \tilde{N} = 2, \alpha = -2 \) version of the Camassa - Holm equation. However his point of view needs further investigations.
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