Non-exponential long-range interaction of magnetic impurities in spin-orbit coupled superconductors.

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The Ruderman-Kittel-Kasuya-Yosida (RKKY) interaction of magnetic impurities in a superconductor exponentially decreases when the distance between them is larger than the superconductor’s coherence length, because this interaction is mediated by quasiparticles which have a gap in their energy spectrum. At the same time, the spin-singlet superconducting condensate was always assumed to stay neutral to magnetic impurities. Due to a spin-orbit coupling (SOC), however, Cooper pairs gain an admixture of spin-triplet correlated states which provide for a link between impurity spins and an s-wave condensate. It is shown that its static Goldstone mode mediates the $1/r^2$ interaction of the spins in two-dimensional (2D) systems. This effect is considered within two models: of a clean 2D s-wave superconductor with the strong Rashba SOC and of a bilayer system combining a 2D Rashba coupled electron gas and an s-wave superconducting film. The predicted long-range interaction can have a strong effect on a spin order in superconductor-magnetic impurity systems that are expected to host Majorana fermions.

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Introduction - RKKY interaction of localized spins in metals $[1,2]$ is carried by spin excitations. A localized spin, by coupling to spins of conduction electrons through the exchange interaction, creates one-particle spin excitations close to the Fermi surface. They, in turn, exert influence upon the spin of an impurity placed at some distance. This results in an effective exchange interaction between impurity spins. Recently, the RKKY interaction has attracted a renewed attention in connection with the search of Majorana quasiparticles localized at 1D and 2D magnetic impurity systems on the surface of superconductors $[3,4]$. In this context the presence of a strong SOC has been found very important for the formation of the topological phase in superconductors and superconductor proximity systems $[5-13]$. For example, in spin-orbit coupled superconductors the RKKY interaction contains the Dzyaloshinskii-Moryia term $[13,15]$ which favors a spiral spin order in impurity spin chains.

In superconductors having a gap in the quasiparticle spectrum the RKKY interaction exponentially decreases with the distance between impurities $[16]$. Such a behavior was also found in spin orbit coupled superconductor proximity systems $[14]$. In both cases the interaction range is restricted by the Cooper pair coherence length. This is because at low energies only evanescent quasiparticle waves can propagate in a gapped superconductor. While such virtual quasiparticles mediate interaction of impurity spins, the condensate of singlet Cooper pairs does not participate in this process. This is obviously true, if SOC is absent. At the same time, in the presence of strong SOC the role of the singlet condensate should be revised. In this Letter the superconducting condensate contribution to the localized spins interaction will be studied for a helical 2D system, that is a system with strong Rashba SOC $[17]$. Such a system may be a topological s-wave superconductor on the surface of a 3D topological insulator (TI), or a non-topological superconductor with strong enough Rashba SOC. It will be shown that in such systems SOC provides a coupling of impurity spins to the condensate. Therefore, the condensate gains the ability to mediate the interaction between magnetic impurities. Since this interaction propagates through the condensate, it is important to distinguish between intrinsic superconductors and proximity induced superconducting systems. In the latter case the problem is more complicated, because magnetic impurities may be placed in the normal spin-orbit coupled system, while interaction between them is mediated by the superconductor. However, it will be shown that in both cases at large distances the interaction decreases algebraically, as $1/r^2$. Such a 2D dipole-dipole interaction is provided by the presence in the superconducting condensate of the static Goldstone mode with a zero wave-vector.

Two models will be considered. The first one represents a 2D clean helical superconductor with the isotropic electron-electron BCS interaction. In the second model a superconducting film makes a contact with the surface of a 3D TI and a pair of magnetic impurities are placed onto this surface. Both, the film and TI surface are assumed to be dirty systems. The spin dynamic of impurities will be ignored. Therefore, their spins are static classical spins $S \gg \hbar$. Also, the exchange interaction between spins of impurities and conduction electrons will be treated perturbatively by taking into account only its leading orders.

Two-dimensional superconductor - Let us consider a two-dimensional superconductor with an isotropic attractive electron-electron interaction. A pair of magnetic impurities is placed at the points $\mathbf{r}_1$ and $\mathbf{r}_2$. Their spins interact with conduction electrons according to the ex-
change Hamiltonian $H_{\text{int}} = \sum_{\nu} J_{ij} S_i^\nu \sigma_i^\nu$, where $\nu = 1, 2$ and $\sigma_i^\nu = \psi_i^\dagger(\nu) \sigma^\nu \psi_i(\nu)$, with $\sigma^\nu$ denoting Pauli matrices ($j = x, y, z$). The field operators $\psi_i(\nu)$ are vectors defined in the Nambu basis as $\psi = (\psi^\uparrow, \psi^\downarrow, \psi^\dagger \downarrow, -\psi^\dagger \uparrow)$. The interaction between impurity spins is given by a second-order correction to the free energy due to $H_{\text{int}}$:

$$U_{12} = -\int_0^\beta d\tau Z_{12} Z_{22}^{-1} \langle \text{Tr}[\sigma_1(\tau)\sigma_2^m(0)] \rangle, \quad (1)$$

where $\beta = 1/T$ ($T$ is the temperature), $Z_{12} = J_{ij} S_i^1 S_j^2$, and the angular brackets denote the thermodynamic average over unperturbed states. The trace in $\text{(1)}$ is taken over spin and Nambu variables and $T$ is the Matsubara time ordering operator. We set $\hbar = 1$ and the Boltzmann constant $k_B = 1$. The unperturbed Hamiltonian is given by the sum $H = H_0 + V$ of the one-particle Hamiltonian $H_0$ and the two-particle attractive interaction $V$. $H_0 = \sum_k \psi_k^\dagger \hat{H}_{0k} \psi_k$ represents a spin-orbit coupled 2D electron gas, where $\psi_k$ are the electron field operators in the wave-vector representation and $\hat{H}_{0k}$ is given by

$$\hat{H}_{0k} = \tau_3 (\epsilon_k - \mu) + \tau_3 h_k \sigma$$

(2)

where $\epsilon_k = k^2/2m$ and $\mu$ are the electron band energy and the chemical potential, respectively. The Pauli matrices $\tau_i$, $i = 0, 1, 2, 3$, operate in the Nambu space, where $\tau_0$ is the unit matrix. The second term in $\hat{H}_{0k}$ represents the Rashba SOC whose spin-orbit field is given by $h_k^\uparrow = -\alpha k y$ and $h_k^\downarrow = \alpha k x$. This interaction results in splitting of the conduction band into two helical bands with opposite helicities $\gamma = \pm 1$, so that the average electron's spins in these bands are parallel to $\gamma \vec{h}_k$.

In each of the bands electrons have the same Fermi velocity $v_F = \sqrt{2e/m + \alpha^2}$ and different state densities $n_{F\gamma} = (m/2\pi)(1 - \gamma^2/v_F^2)$. The two-particle interaction $V$ is assumed to be independent on wave-vectors of interacting particles, except for the high-energy cutoff $\omega_c$ near the Fermi surface. Therefore, it is given by

$$V = \frac{g}{2} \sum_{k,k',q} \langle \psi_{k'}^\dagger \tau_3 \psi_{k+q} \rangle \langle \psi_{k'}^\dagger \tau_3 \psi_{k} \rangle \langle \psi_{k}^\dagger \tau_3 \psi_{k+q} \rangle,$$

(3)

where $g < 0$ is a coupling constant.

The correlator $K^{ij} = -\langle \text{Tr}[\sigma_1(\tau)\sigma_2^m(0)] \rangle$ of spin densities in Eq. $(1)$ depends on the impurity coordinate difference $\mathbf{r} \equiv (\mathbf{r}_1 - \mathbf{r}_2)$. Therefore, it may be Fourier transformed with respect to $\mathbf{r}$. The so transformed correlator, which is integrated over the imaginary time $\tau$ in Eq. $(1)$, will be denoted as $K^{ij}_{\mathbf{q}}$, where $\mathbf{q}$ is the wave-vector. It may be written in the form

$$K^{ij}_{\mathbf{q}} = K_{\text{RKKY}}^{ij} + \Gamma^{ij}_{\mathbf{q}} C^{ij}_{\mathbf{q}} - \Gamma^{-ij}_{\mathbf{q}} - \Gamma^{ij}_{\mathbf{q}}.$$

(4)

The first term results in a usual RKKY interaction, which has been previously calculated for $s$-wave superconductors, with and without SOC [14, 10]. This term takes account of quasiparticles as mediators of the RKKY interaction, but ignores the role of the condensate. In contrast, in the second term the condensate is involved into the spin-spin interaction through the Cooper pairing channel represented by the two-particle propagator $C_{\mathbf{q}}$. The latter is given by a sum of ladder Feynman diagrams, where the attraction $V$ plays a role of a perturbation. The vertices $\Gamma^{ij}_{\mathbf{q}}$ and $\Gamma^{-ij}_{\mathbf{q}}$ in turn, guarantee the coupling of the Cooperon to spins placed at points $\mathbf{r}_1$ and $\mathbf{r}_2$. It is important that $C_{\mathbf{q}}$ diverges as $q^{-2}$ at $q \to 0$ due to the static Goldstone mode, which stems from the invariance of the system with respect to a static uniform shift of the order parameter phase. As it will be shown, this singularity results in a long-range interaction of impurity spins.

All entries in Eq. $(1)$ can be expressed through the two-particle correlator $\Pi_{lm}^{ij}(\mathbf{q})$ which is defined as

$$\Pi_{lm}^{ij}(\mathbf{q}) = \frac{T}{4} \sum_{\omega_n} \langle \text{Tr}[G_{\mathbf{k}}(\omega_n)\sigma^l \tau^i G_{\mathbf{k}+\mathbf{q}}(\omega_n)\sigma^m \tau^j] \rangle,$$

(5)

where $\sigma^0$ denotes the unit matrix in the spin space and $G_{\mathbf{k}}(\omega_n) = (i\omega_n - \hat{H}_{0k} - \tau_3 \Delta)\sigma^0$ is the Matsubara Green’s function. The order parameter $\Delta$ in $G_{\mathbf{k}}(\omega_n)$ is determined within the BCS formalism with the attractive interaction $\hat{V}$. The Green functions in Eq. $(5)$ consist of two parts which are associated with one of the helical bands. It will be assumed below that SOC is much stronger than $\Delta$. In this case the main contribution in Eq. $(5)$ is given by the terms where both Green functions enter with equal helicities. The details of the calculation are presented in “Supplemental Material”.

In the case of a clean electron gas the functions $K_{\text{RKKY}}^{ij}$, $\Gamma^{ij}_{\mathbf{q}}$ and $C_{\mathbf{q}}$ in Eq. $(1)$ may be expressed as

$$\Gamma^{ij}_{\mathbf{q}} = 2\Pi_{00}^{0i}(\mathbf{q}) \quad K_{\text{RKKY}}^{ij} = 4\Pi_{00}^{ij}(\mathbf{q})$$

and

$$C_{\mathbf{q}} = -\frac{g}{1 - g\Pi_{00}^{0i}(\mathbf{q})}.$$

(6)

(7)

Note that the nondiagonal "il0" superscript in $\Pi_{00}^{0i}(\mathbf{q})$ signals that $\Gamma^{ij}_{\mathbf{q}}$ couples the spin projection $i$ to a singlet Cooper pair. At $\mathbf{q} = 0$ the denominator of Eq. $(7)$ turns to $q^2/\nu_F^2$. By expanding $\Pi_{02}^{0i}(\mathbf{q})$ over small $q \ll \Delta/\nu_F$ we arrive to

$$C_{\mathbf{q}} = -\frac{a}{q^2} \frac{16}{\nu_F^2} (N_{F+} + N_{F-})^{-1},$$

(8)

where

$$\frac{1}{a} = \pi T \sum_{\omega_n} \frac{1}{(\omega_n^2 + \Delta^2)^{3/2}}.$$

(9)

The vertex $\Gamma^{ij}$ can be expressed from Eqs. $(6,7)$ as

$$\Gamma^{ij}_{\mathbf{q}} = -\frac{i}{4\alpha} q^2 \Delta v_F (N_{F+} - N_{F-}),$$

(10)
where $\tilde{q}^i = e^{i\theta^i q^i}$. By substituting Eqs. (8) and (10) in Eq. (4) we finally obtain

$$K_q^{ij} - K_R^{ij} = -\frac{q^2}{2} \frac{\Delta^2 (N_{F_+} - N_{F_-})^2}{a(N_{F_+} + N_{F_-})}. \quad (11)$$

It is seen from this equation that two helical bands tend to compensate each other, while the above expression reaches its maximum in the case when there is only a single helical band on the Fermi surface, for example, in the case of Dirac electrons on the surface of a topological insulator.

**Interaction between magnetic impurities and spin orders -** By Fourier transforming Eq. (11) to the coordinate representation and substituting into Eq. (11) we obtain the condensate mediated interaction between magnetic impurities placed at the sites $r_1$ and $r_2$:

$$U_{12} = \frac{\beta}{r} \left(2(Z_1\hat{r})(Z_2\hat{r}) - r^2 Z_{[1][2]} \right), \quad (12)$$

where $\beta = (\Delta^2 / 2\pi a)(N_{F_+} - N_{F_-})^2 (N_{F_+} + N_{F_-})^{-1}$, $r^3 = e^{i\theta^i r^i}$, with $r^1 = r_1^1 - r_2^1$ and $Z_{[1][2]} = (Z^x_{1(2)}, Z^y_{1(2)})$.

Eq. (12) is valid at $r \gg v_F / \Delta$. For a chain of magnetic atoms, whose spins are perpendicular to the chain direction, the interaction is antiferromagnetic, while it is ferromagnetic for spins which are parallel to the chain. For a ferromagnetically ordered two-dimensional array of spins the long-range interaction results in the mean field $B$ which depends on the shape of the system. For example, in the center of a rectangle whose sides are $L_x$ and $L_y$ this field may be expressed from Eq. (12) by summing $U$ over all thermodynamically averaged spins $\langle S_{2x} \rangle$, with the spin $S_1$ fixed in the rectangle center. Since mostly distant spins contribute to this sum, one may replace the summation by integration with the spin density $n_i$. By writing the result in the form $\sum_m U_{1m} = S_{12} B_x$ the mean field $B_x$ is obtained as

$$B_x = 2n_i \beta J^2 \langle S_{2x} \rangle \left(\arctan \frac{L_y}{L_x} - \arctan \frac{L_x}{L_y} \right), \quad (13)$$

where the exchange interaction is taken as $J_{ij} = \delta_{ij} J$.

It is evident from this equation that the ferromagnetic ordering is possible only at $L_x > L_y$.

$U_{12}$ in Eq. (12) does not contain the Dzyaloshinskii-Moryia interaction. In fact, this interaction appears only in higher orders with respect to $1/\hbar k_F$. Such small corrections were neglected above. On the other hand, the Dzyaloshinskii-Moryia term might appear also in the case when, due to SOC, the attractive electron-electron interaction $V$ in Eq. (3) contains a spin-dependent term. This possibility has not been investigated here.

**Proximity system-** From the practical point of view it is important to consider a system where superconductivity is induced in a spin-orbit coupled normal metal due to a close proximity to a superconductor. In this case a popular approach is to introduce in Hamiltonian of the normal system a term which looks as the superconducting order parameter. In some cases such an approach is justified. It is definitely can not be used in the studied here problem, because the selfconsistency condition for the order parameter is determined by an electron-electron attractive interaction inside the superconductor, not the normal metal. Therefore, if a pair of magnetic impurities interacts through the condensate, the superconductor must explicitly be considered as a mediator for this interaction. A strategy for the solution of this problem may be to calculate the spin density ($\sigma_{ij}^q$) which is induced at the point $r_2$ by a magnetic impurity paced in the point $r_1$, and vice versa. Then, the interaction energy of these spins can be expressed as

$$U_{12} = J_{ij} \langle S_{1}^i \sigma_{12}^j + S_{2}^j \sigma_{21}^i \rangle. \quad (14)$$

The induced spin densities must be calculated by taking into account a perturbation of the condensate by magnetic impurities. It is important that in contrast to Eq. (1), where the interaction energy is determined by a two-particle Green function, in Eq. (14) the spin densities are calculated through the one-particle Matsubara Green’s functions $G(\omega_n, r, r')$.

Let us consider a bilayer system consisting of a superconducting layer and an adjacent 2D normal metal. The latter is formed by Dirac electrons on the surface of a 3D TI. Their Hamiltonian is represented by the second term in Eq. (2). A pair of magnetic impurities on the surface of TI adds the Zeeman term $\sigma \vec{Z}(r)$, where $\vec{Z} = Z_1 \delta (r - r_1) + Z_2 \delta (r - r_2)$. For studying such an inhomogeneous system we will use a theory based on semiclassical Green’s functions. These functions are defined separately in both layers of the considered bilayer system and satisfy respective semiclassical equations [15, 20]. In addition, there is a boundary condition at the interface between layers. The semiclassical function is defined as

$$g_{\vec{k}}(r, \omega) = i \int d\xi \tau_3 G_{\vec{k}}(r, \omega_n) \quad (15)$$

where $\xi = E_{\vec{k}} - \mu$, $r = (r + r')/2$, $\vec{k}$ is the wave vector associated with the Fourier transform of $G(\omega_n, r, r')$ with respect to $r - r'$ and $\vec{k} \equiv k / \hbar$. The electron energy $E_{\vec{k}}$ is given either by a parabolic $k$-dependence (in the superconducting layer), or by a linear function $h_{\vec{k}}$ in TI. The equation for $g_{\vec{k}}(r, \omega)$ is obtained by expanding the Dyson equation with respect to Fermi wavelengths, which are small in comparison with other characteristic lengths. The corresponding procedure is well described in literature [21, 22]. It will be assumed below that the elastic scattering time on impurities $\tau \ll \Delta^{-1}$ and the corresponding mean free path is much shorter than the lengths of spatial variations of the semiclassical Green functions. In this case these functions are almost isotropic with respect to $\vec{k}$ and it is possible to obtain closed, so called, Usadel [20, 23] equations for their
isotropic parts $g_{S(N)}(r, \omega)$, where subscripts N and S denote the normal and superconducting layers, respectively. For the S-layer such an equation can be written in the standard form

$$D_S \nabla g_S \nabla g_S - [\omega \tau_3 + \hat{\Delta} + g_S] = 0,$$

where $\hat{\Delta} = \text{Re} \Delta(r) 2 - \text{Im} \Delta(r) \tau_1$. The equation for TI can be obtained by a projection of $g_{Nk}$ onto the upper helix band, if the chemical potential $\mu > 0$ and $\mu \gg 1/\tau$ [24 [20]. Accordingly, $g_{Nk}$ takes the form $g_{Nk} = g_{N\ell_0}(1 + \sigma n)/2$, where $n = \hbar k/\hbar$ and $g_{N\ell_0}$ is a spin independent function. Its angular average $g_N$ satisfies the equation

$$D_N \nabla g_N \nabla g_N - [\omega \tau_3 + T_N g_S; g_N] = 0,$$

where $\nabla^* = \nabla + i[A(r) \tau_3, *]$ and $A'(r) = e^{ijz} Z'(r) / \alpha$. $D_S$ and $D_N$ are electron diffusion coefficients in the superconductor and normal layers, respectively. The last term in Eq. (17) originates from the self-energy associated with a tunnel coupling of 2D electrons to the superconducting layer, where $T_{NS}$ is a corresponding tunneling parameter [21]. From the superconductor’s side the coupling to the normal layer is provided by the boundary condition (BC) [28]

$$D_{Sgs} \nabla z g_S = -\gamma_{SN}[g_S, g_N],$$

where the $z$-axis is directed from the $N$-layer to the $S$-layer, $g_S$ is taken at $z = 0$ and $\gamma_{NS}$ can be expressed in terms of the interface resistance.

The Usadel equation for $g_S$ may be further simplified [22] by assuming that Green functions vary slowly across a thin film, whose thickness $d_S$ is much less than the superconductor’s coherence length $\sqrt{D_S / |\Delta|}$. By integrating Eq. (18) over $z$ and taking into account BC Eq. (19) we obtain the following equation for $g_S(r, z)$:

$$D_S \nabla g_S \nabla g_S - [\omega \tau_3 + \hat{\Delta} + T_{NS} g_N; g_S] = 0,$$

where $T_{SN} = D_S \gamma_{SN}/d_S$. The parameters $T_{NS}$ and $T_{SN}$ are related to each other through the equation $N_{F_N} T_{NS} = d_S N_{F_S} T_{SN}$, where $N_{F_N}$ and $N_{F_S}$ are, respectively, 2D and 3D state densities at the Fermi level in the normal metal and superconductor (in the normal state). This equation guarantees the conservation of the charge current through the NS-interface. It should be noted that the above relation between $T_{NS}$ and $T_{SN}$ means that $T_{SN} \ll T_{NS}$, because $k_{FS} d_S \gg 1$.

As was shown in [31, 33], in spin-orbit coupled superconductors a Zeeman field, which is localized within a small island, induces in its vicinity a spontaneous supercurrent. A single magnetic impurity may produce a similar effect. This supercurrent, in turn, induces a spin density due to the magnetoelectric effect which takes place in spin-orbit coupled systems [32]. The interaction of impurity spins may be calculated by substituting this spin density into Eq. (14). There are two different mechanisms which contribute in the formation of such a spin-density response. One of them involves the supercurrent that is directly produced in the Dirac gas by proximity induced Cooper pairs, in the presence of the Zeeman field $Z(r)$. This effect is controlled by a proximity induced gap $\Delta_N$, which coincides with $T_{NS}$ when $T_{NS} \ll \Delta$. As a result, the spin interaction given by Eq. (14) decreases exponentially at distances larger than the corresponding correlation length $\xi_N = \min[\sqrt{D_N / \Delta_N}, \sqrt{D_N / 2\pi k_B T}]$. The second mechanism includes several steps. An important step is a change of the superconducting order-parameter in the S-layer, due to a perturbation of the pairing function by the Zeeman field. This perturbation migrates from the normal layer through the interface barrier. The correction to $\Delta$ may be expressed in a form of a phase shift. The latter gives rise to a supercurrent in N and S layers and to a spin polarization, which results in the interaction of magnetic impurities. These calculations in detail are presented in Supplemental Material. The main result is that at large distances $r \gg \xi_N$ the interaction of spins is expressed by Eq. (12), where at $T \ll \Delta_N, \Delta$ the coefficient $\beta$ is given by

$$\beta = \frac{\mu \tau T_{NS} T_{SN} d_N}{2\pi e^2 \sqrt{\Delta D_S}}.$$  

This $\beta$ is much less than in the considered above case of a clean 2D superconductor. It is strongly reduced by the factor $d_s k_{FS} \gg 1$. This parameter enters into $T_{SN}$. The origin of such a reduction is quite clear, because the influence of SOC in the 2D gas on the condensate in the 3D superconductor film decreases at the larger thickness $d_S$.

**Conclusion** - In conclusion, in spin-orbit coupled 2D superconductors or 2D normal Dirac metals, which have a contact with thin superconducting films, the interaction of magnetic impurity spins decreases as $r^{-2}$ at distances larger than the coherence length. In fact, it has the form of the 2D dipole-dipole interaction. In contrast to the exponentially decreasing RKKY interaction, it is mediated by the Cooper pair condensate, rather than by one-particle excitations. In the case of proximity induced superconductivity in a 2D normal metal this long-range effect is suppressed, if a 3D superconductor, which serves as a source of Cooper pair correlations, is massive.

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The BCS Green function of a system which is represented by the Hamiltonian $H_{0k} + \tau_1 \Delta$, where the first term is given by Eq\.[\ref{eq:1}], can be written as

$$G_k(\omega_n) = \frac{1}{\omega_n^2 + (E^+)^2} \frac{(i\omega_n + \tau_1 \Delta + \xi^+ \tau_3) (1 + n \sigma) - (i\omega_n + \tau_1 \Delta + \xi^- \tau_3) (1 - n \sigma)}{2}$$ \quad (S1),



where $E^\pm = \sqrt{\Delta^2 + \xi^2 \pm \xi_0^2}$, $\xi = \xi_0 - \mu$ and $n = h \xi_0/h_k$. These functions must be substituted into the correlator given by Eq\.[\ref{eq:5}], which is the building block for the calculation of vertices and the Cooperon propagator in Eq\.[\ref{eq:6}]. The Cooperon is represented by a sum of ladder diagrams, where the electron-electron interaction $V$, given by Eq\.[\ref{eq:7}], serves as a perturbation. The dependence of this interaction on Nambu and spin variables can be represented in the form

$$V = \frac{1}{2} \sum_{k,k',q} V_{\alpha\beta,\gamma\delta}^\alpha \psi^\dagger_{k+q} \psi^\dagger_{k'} \psi_{k'} \psi_q$$ \quad (S2),

where $V_{\alpha\beta,\gamma\delta} = (g/2)(\gamma_3 \otimes \sigma^0)_{\alpha\beta}(\tau_3 \otimes \sigma^0)_{\gamma\delta}$. The Greek subscripts denote combined Nambu-spin variables. It is convenient to use, instead, vector indices by transforming the matrix $V$ to

$$V_{ab}^{ij} = \frac{1}{4} V_{\alpha\beta,\gamma\delta}(\tau_a \otimes \sigma^i)(\tau_b \otimes \sigma^j)_{\gamma\delta}$$ \quad (S3),

where $i, j = 0, x, y, z$ and $a, b = 0, 1, 2, 3$. In this representation the second term in Eq\.[\ref{eq:6}] can be written in the form

$$K_{ik}^{ij} = K_{RKKY}^{ij} = 4 \Pi^{ij}_{00}(q)C^{ij}_{0q} \Pi^{n0}_{20}(q)$$ \quad (S4),

where the function $\Pi$ is given by Eq\.[\ref{eq:9}]. Since only the singular at $q \to 0$ channel is taken into account in the Cooperon $C$, the subscripts in $\Pi$ are fixed at "02" and "20" in Eq\.[\ref{eq:9}], because this channel involves the correlator

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**Supplemental Material**

### S1. Interaction of Impurity Spins in a 2D Superconductor

The BCS Green function of a system which is represented by the Hamiltonian $H_{0k} + \tau_1 \Delta$, where the first term is given by Eq\.[\ref{eq:1}], can be written as

$$G_k(\omega_n) = \frac{1}{\omega_n^2 + (E^+)^2} \frac{(i\omega_n + \tau_1 \Delta + \xi^+ \tau_3) (1 + n \sigma) - (i\omega_n + \tau_1 \Delta + \xi^- \tau_3) (1 - n \sigma)}{2}$$ \quad (S1),



where $E^\pm = \sqrt{\Delta^2 + \xi^2 \pm \xi_0^2}$, $\xi = \xi_0 - \mu$ and $n = h \xi_0/h_k$. These functions must be substituted into the correlator given by Eq\.[\ref{eq:5}], which is the building block for the calculation of vertices and the Cooperon propagator in Eq\.[\ref{eq:6}]. The Cooperon is represented by a sum of ladder diagrams, where the electron-electron interaction $V$, given by Eq\.[\ref{eq:7}], serves as a perturbation. The dependence of this interaction on Nambu and spin variables can be represented in the form

$$V = \frac{1}{2} \sum_{k,k',q} V_{\alpha\beta,\gamma\delta}^\alpha \psi^\dagger_{k+q} \psi^\dagger_{k'} \psi_{k'} \psi_q$$ \quad (S2),

where $V_{\alpha\beta,\gamma\delta} = (g/2)(\gamma_3 \otimes \sigma^0)_{\alpha\beta}(\tau_3 \otimes \sigma^0)_{\gamma\delta}$. The Greek subscripts denote combined Nambu-spin variables. It is convenient to use, instead, vector indices by transforming the matrix $V$ to

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where $i, j = 0, x, y, z$ and $a, b = 0, 1, 2, 3$. In this representation the second term in Eq\.[\ref{eq:6}] can be written in the form

$$K_{ik}^{ij} = K_{RKKY}^{ij} = 4 \Pi^{ij}_{00}(q)C^{ij}_{0q} \Pi^{n0}_{20}(q)$$ \quad (S4),

where the function $\Pi$ is given by Eq\.[\ref{eq:9}]. Since only the singular at $q \to 0$ channel is taken into account in the Cooperon $C$, the subscripts in $\Pi$ are fixed at "02" and "20" in Eq\.[\ref{eq:9}], because this channel involves the correlator
\( \Pi_{22} \) and, hence, the \( C_{22} \) Cooperon. In its turn, this Cooperon (\( C_{22} \) is denoted as \( C \)) satisfies the equation

\[
C_{ij}^{m} = V_{22}^{ij} + V_{22}^{ij} \Pi_{22}^{m}(q) C_{ij}^{m}.
\]

(S5)

In order to calculate, we assume that SOC is strong, so that the spin-orbit field \( h_F \gtrsim \Delta \). In this case, the poles of the first and second terms in Eq. (S1), which correspond to opposite chiralities, are considerably shifted with respect to each other. Therefore, the main contribution in pair products of Green’s function in \( \Pi \) (Eq. (5)) is given by the functions with equal chiralities. Hence, the products of Green’s functions with opposite chiralities will be neglected. Further, we consider the situation when a distance between magnetic impurities is much larger than the Fermi wavelength. Therefore, the Green function \( G_{k+q} \), which enters in \( \Pi \), may be expanded with respect to \( q \ll k_F \). Only the leading terms of this expansion will be taken into account. With these approximations the only nonzero elements of \( \Pi_{ij}^{0}(q) \Pi_{ij}^{0}(q) \) are \( \Pi_{00}^{0}(q) \Pi_{20}^{0}(q) \). The latter enter in the definition of \( \Gamma_{i}^{0} \) in Eqs. (4) and (6). Hence, only the Cooperon \( C_{ij}^{00} \) appears in Eq. (S4). Since, according to Eqs. (S2) and (S3), \( \Pi_{00}^{0}(q) \Pi_{20}^{0}(q) \) are uniform along layers. They are determined by the gauge-invariant derivative in Eq. (17), will be treated as a perturbation and only first-order terms will be taken into account. Therefore, it is convenient to transform these equations into the Fourier representation. Within this perturbation approach the semiclassical Green functions and the order parameter can be represented in the form

\[
g_S = g_{S0} + \delta g_S ; \quad g_N = g_{N0} + \delta g_N \quad \text{and} \quad \hat{\Delta} = \Delta_0 + \delta \hat{\Delta}.
\]

(S6)

The unperturbed functions \( g_{S0} \), \( g_{N0} \) and the order parameter \( \Delta_0 \) are uniform along layers. They are determined by Eqs. (19) and (17) in the absence of magnetic impurities. Since the tunneling parameter \( T_{SN} \) is small, one may ignore a weak influence of the 2D normal gas onto the Green’s function and the order parameter of the superconducting film. Therefore, we take them the same as in a bulk superconductor, namely

\[
g_{S0} = \frac{\omega_n \tau_3 + \Delta_0 \tau_2}{\sqrt{\omega_n^2 + \Delta_0^2}} \quad \text{and} \quad \hat{\Delta}_0 = \Delta_0 \tau_2.
\]

(S7)

At the same time, the superconductor proximity effect leads to important changes in the Green’s function of the normal gas. The corresponding uniform solution \( g_{N0} \) can be obtained from the second term of Eq. (17), which must be zero. Hence, \( g_{N0} \) is given by

\[
g_{N0} = \frac{\omega_n \tau_3 + T_{SN} g_{S0}}{\sqrt{(\omega_n \tau_3 + T_{SN} g_{S0})^2}}.
\]

(S8)

This function satisfies the normalization condition \( g_{N0}^2 = 1 \). At \( \omega_n \ll \Delta \) it takes the form of Eq. (S7) with \( \Delta_0 \) substituted for \( \Delta_N \cong T_{NS} \). Hence, \( g_{N0} \) looks as the Green’s function of a superconductor, where the role of the gap is played by \( \Delta_N \).

The corrections due to the Zeeman field are obtained from linearized Eqs. (19) and (17). They are given by

\[
\delta g_S = \frac{1}{D_S q^2 + 2 \Omega_S^2} \left( T_{SN} g_{S0} |g_{S0}, \delta g_N | - g_{S0} |\delta \hat{\Delta}, g_{S0} | \right),
\]

\[
\delta g_N = - \frac{1}{D_N q^2 + 2 \Omega_N^2} \left( T_{NS} g_{N0} |\delta g_S, g_{N0}| + D_N(qA) |\tau_3, g_{N0}| \right),
\]

(S9)

where \( \Omega_S = \omega_n \tau_3 + \Delta_0 \tau_2 \) and \( \Omega_N = \omega_n \tau_3 + T_{NS} g_{S0} \). It is seen from Eq. (S8) that the term in Eq. (S9), which is associated with the Zeeman field (the second term in the second equation), is proportional to the Pauli matrix \( \tau_1 \). Then, it is easy to see that \( \delta g_N, \delta g_S \) and \( \delta \hat{\Delta} \) are also proportional to \( \tau_1 \). Accordingly, we denote \( \delta g_N = \tau_1 \delta g_N, \delta g_S = \tau_1 \delta g_S \), and \( \delta \hat{\Delta} = \tau_1 \delta \hat{\Delta} \). Therefore, let us project Eq. (S9) onto \( \tau_1 \). By resolving these equations we obtain

\[
\delta g_S = \frac{2}{D} \left( b_N \delta \Delta + 2 i T_{SN} D_N(qA) g_{N0}^{(2)} \right),
\]

\[
\delta g_N = \frac{2}{D} \left( 2 T_{NS} \delta \Delta + i b_S D_N(qA) g_{N0}^{(2)} \right),
\]

(S10)
where \( b_{N(S)} = D_{N(S)} q^2 + 2\Omega_{N(S)}^2 \), \( g_{N0}^{(2)} = \text{Tr}[\tau_2 g_{N0}] \), and \( D = b_N b_S - 4T_{SN}T_{NS} \).

The selfconsistency reads

\[ \delta \Delta = \frac{g_{NF}T}{2} \sum_{\omega_n} \delta g_S. \]  

(S11)

The correction \( \delta \Delta \) to the gap can be obtained by substituting in this equation \( \delta g_S \) from Eq. (S10). We will keep only leading terms in \( T_{SN} \). By taking into account the unperturbed selfconsistency equation \( 1 = g_{NF}T \sum_{\omega_n}(1/2|\Omega_s|) \) this correction may be expressed from Eq. (S11) in the form

\[ \delta \Delta = 4T_{SN} \frac{i q A D_N}{q^2 D_S} \left( \sum_{\omega_n} g_{N0}^{(2)} \frac{\Omega_s}{\delta g_S} \right) . \]  

(S12)

This expression should be substituted into Eq. (S10). Further, \( \delta g_N \) can be used for the calculation of the spin densities \( \langle \sigma^j_{12} \rangle \) and \( \langle \sigma^j_{21} \rangle \) in Eq. (14). For example, the former is given by

\[ \langle \sigma^j_{12} \rangle = \frac{T}{2} \sum_{\omega_n, k} \text{Tr}[\sigma^j G_k(r_1, \omega_n)] = -\frac{\pi}{2} T N_{FN} \sum_{\omega_n} \int \frac{d\phi}{2\pi} \text{Tr} \left[ \tau_3 \sigma^j \frac{1 + n\sigma}{2} \delta g_N k(r_1) \right] , \]  

(S13)

where the angle \( \phi \) specifies a direction of the unit vector \( \hat{k} \). Due to the trace over spin variables, the integrand is proportional to \( n^j \), which is antisymmetric with respect to \( \hat{k} \). At the same time, the antisymmetric in \( \hat{k} \) function \( \delta g_{Nk} \) may be expressed in terms of the angular-averaged function \( \delta g_N \), according to \( \delta g_{Nk} = -2\tau\alpha \hat{k} g_{N0} \nabla \delta g_N(r_1) \) [24–26]. Within the linear in \( Z \) approximation, in the latter expression \( \nabla \) may be substituted for \( \nabla \), while in \( \delta g_N(r_1) \) only the Zeeman field \( Z_2 \delta (r - r_2) \) must be taken into account. By substituting \( \delta g_{Nk}(r_1) \) into Eq. (S13) and calculating the trace over spin and Nambu variables we obtain

\[ \langle \sigma^j_{12} \rangle = -i \pi T N_{FN} \sum_{\omega_n, q} \hat{q}^j \langle q \rangle \delta g_N e^{i q (r_1 - r_2)} . \]  

(S14)

The function \( \delta g_N \) in Eq. (S14) contains two parts, as can be seen from Eq. (S10). One of them is proportional to \( \delta \Delta \). It stems from the condensate, which is perturbed by magnetic impurities. The second term represents a direct effect of the Zeeman field, where the sole role of the superconductor is to produce the proximity gap \( \Delta_N \) in the normal film. Indeed, by neglecting the small term in \( D \), which is proportional to \( T_{SN} \), at \( T_{NS} \ll \Delta \), one obtains for \( \delta g_N \) (given by the second term in Eq. (S10) the same expression as in the case of a superconductor whose energy gap is \( \Delta_N = T_{NS} \). It follows straight from Eq. (S8), which at \( \omega_n \) and \( T_{NS} \ll \Delta \) takes the form of the semiclassical Green function of a superconductor. Therefore, this part of \( \delta g_N \) contributes to the “smooth” RKKY interaction [14] (the oscillating with \( k_F r \) term can not be treated within the semiclassical theory). The dependence on a distance between magnetic impurities is seen from the pole of \( \delta g_N \sim 1/b_N = (D_N q^2 + 2\Delta_N^2 + 2\omega_n^2)^{-1} \). This sort of \( q \)-dependence signals that the interaction between impurities exponentially decreases at \( r \gg \sqrt{D_N}/\Delta_N \). Unlike such an exponential dependence, the contribution associated with the condensate demonstrates a power-low behavior. Indeed, as can be seen from Eqs. (S12) and (S10), at \( q \rightarrow 0 \) \( \delta g_N \sim q A /q^2 \). After the substitution of such \( \delta g_N \) into Eq. (S13) and by taking into account Eq. (14) one, finally, obtains the asymptotically 2D dipole-dipole interaction given by Eq. (12).