THE P-ADIC ANALYTIC SUBGROUP THEOREM AND APPLICATIONS

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ABSTRACT. We prove a \( p \)-adic analogue of Wüstholz's analytic subgroup theorem. We apply this result to show that a curve embedded in its Jacobian intersects the \( p \)-adic closure of the Mordell-Weil group transversely whenever the latter has rank equal to 1. This allows us to give some theoretical justification to Chabauty techniques applied to finding rational points on curves whose Jacobian has Mordell-Weil group of rank 1.

1. Introduction

In this paper we develop a \( p \)-adic analogue to Wüstholz's analytic subgroup theorem [11] and give an application to finding rational points on curves.

Let \( G \) be a commutative algebraic group defined over the algebraic numbers and let \( V \) be a proper linear subspace of the Lie algebra of \( G(\mathbb{C}) \), spanned by algebraic vectors. Then the analytic subgroup theorem states that if \( V \) contains a vector whose image under the exponential map is an algebraic point in \( G(\mathbb{C}) \), then there exists an algebraic subgroup of \( G \), containing the said point, whose Lie algebra is a linear subspace of \( V \).

In order to state the \( p \)-adic analogue we replace the exponential map with the logarithm, since it behaves better in the non-archimedean case. Let \( V \) be a linear subspace of the \( p \)-adic Lie algebra of \( G \) spanned by algebraic vectors. Then the theorem states that if the logarithm of an algebraic point is contained in \( V \), then there exists an algebraic subgroup of \( G \), containing that point, whose Lie algebra is a linear subspace of \( V \). The precise statement is given in Section 2.

The proof of the theorem is very similar to the proof of the original result. One constructs a section of a sheaf on \( G \) which has zeros of high order on a certain finite set of algebraic points. We then proceed to show that the negation of the theorem forces this section to have many more zeros, which, using a multiplicity estimate, gives a contradiction. The main difference to the original proof is the replacement of

\[ \text{Date: October 15, 2010.} \]
certain complex analytic estimates with their $p$-adic analogues, which are stated in Proposition 13. The proof of the theorem is presented in Sections 4 to 8.

In Section 3 we give an application of the $p$-adic analytic subgroup theorem to finding rational points on curves. Let $\varphi : C \to J$ be an embedding of a curve defined over $\mathbb{Q}$ into its Jacobian. We show that if the Jacobian is simple and has Mordell-Weil rank equal to 1, then the $p$-adic completion of $C$ intersects the $p$-adic closure of $J(\mathbb{Q})$ transversally at every point in $C(\mathbb{Q})$. In particular, every rational point in $C$ has a $p$-adic neighbourhood which intersects the $p$-adic closure of the Mordell-Weil group at a single point.

Bruin and Stoll have combined the Mordell-Weil Sieve with Chabauty techniques to develop an algorithm for finding the rational points on curves whose Mordell-Weil rank is smaller than the genus (see [4], 4.4). The proof that the algorithm terminates is conditional on Stoll’s Main Conjecture [10], as well as on an additional conjecture (Conjecture 4.2 in [4]). Our results imply that when the rank of the Mordell-Weil group is equal to 1, one can slightly modify their algorithm so that its termination depends only on the Main Conjecture.

2. Notation and Main Result

Let $\overline{\mathbb{Q}}$ denote the field of algebraic numbers and let $M$ be a complete ultrametric extension of $\overline{\mathbb{Q}}$. Let $G$ be a commutative algebraic group over $\overline{\mathbb{Q}}$ and let $G_M$ be its base extension over $M$. If we denote by $t_G$ and $t_{G_M}$ the tangent spaces at the identity element $e$ of $G$ and $G_M$ respectively, we have the relation $t_{G_M} = t_G \otimes \overline{\mathbb{Q}} M$ and one has a canonical $\overline{\mathbb{Q}}$-linear embedding $\iota : t_G \to t_{G_M}$, $v \mapsto v \otimes 1$. We will use this map to identify $t_G$ with a subset of $t_{G_M}$, and to identify the space $V \otimes \overline{\mathbb{Q}} M$ with a subspace of $t_{G_M}$ for every vector space $V \subseteq t_G$. We shall call vectors that lie in the image of $\iota$ algebraic vectors.

Following Bourbaki [3, §3.7.6] one can define a logarithm map $\log : G(\mathbb{M})^* \to t_{G_M}$, where $G(\mathbb{M})^*$ is the set of all points $\gamma \in G(\mathbb{M})$ with the property that the identity $e$ is an accumulation point of the set $\{\gamma^n \mid n \in \mathbb{N}\}$. Then we have

**Theorem 1.** Assume that $\gamma \in G(\mathbb{M})^*$ is an algebraic point and that $V \subseteq t_G$ is a $\overline{\mathbb{Q}}$-linear subspace such that $\log(\gamma) \in V \otimes \overline{\mathbb{Q}} M$. Then there exists an algebraic subgroup $B \subseteq G$ defined over $\overline{\mathbb{Q}}$, such that $\gamma \in B(\overline{\mathbb{Q}})$ and $t_B \subseteq V$.

**Remark.** There are a couple of cases, where the statement of the theorem is trivial. If $V = t_G$, then one can simply choose $B = G$. If on the
other hand $\gamma$ is a torsion point, say $\gamma^m = e$ for some $m \in \mathbb{N}$, then we can pick $B$ to be the group of $m$-torsion points. Therefore the theorem gives non-trivial implications only when $V$ is a proper linear subspace, and $\gamma$ is a non-torsion point.

3. Finding points on curves

Let $K$ be a number field with a non-archimedean valuation $v$, and let $K_v$ be the completion with respect to this valuation. Let $K_v^{alg}$ denote the field of all algebraic numbers in $K_v$. We are going to apply Theorem 1 in the case when the group $G$ is a simple abelian variety defined over $K$ and the linear space $V$ is one-dimensional. Then the set $G(K_v)$ is compact, which implies that $G(K_v)^* = G(K_v)$ and that the logarithm is globally defined. We have the following:

Lemma 2. Let $A$ be a simple abelian variety defined over $K$. Let $b \in t_A$ be a non-zero tangent vector and let $\gamma \in A(K)$ be a $K$-rational point of infinite order. Then the vectors $b$ and $\log \gamma$ are linearly independent over $K_v$.

Remark. We should note that if the abelian variety $A$ is absolutely simple then the lemma is a trivial corollary of Theorem 1. The difficulty lies in showing the result for a simple, but not absolutely simple, abelian variety.

Proof. We shall give a proof by contradiction. Let $A$ be a simple abelian variety which is not absolutely simple. Assume that there exists an algebraic vector $b$ and a $K$-rational point $\gamma$ such that $\log \gamma$ lies in the one-dimensional $K_v$-linear space spanned by $b$. Then Theorem 1 implies that there exists an algebraic subgroup $B$ defined over some finite Galois extension $L$, such that $\gamma \in B(L)$ and such that $t_B$ is a subspace of the $L$-linear space spanned by $b$. Let $B_0$ be the connected component at identity of this group, and let $\gamma_0$ be a multiple of $\gamma$, which lies in $B_0$. The group $B$ is one-dimensional, therefore $B_0$ is an elliptic curve. We will show that $B_0$ is defined over $K$ and thus derive a contradiction with the assumption that $A$ is simple.

The idea of the following was suggested to me by Brendan Creutz. Let $\sigma \in Gal(L/K)$. The set of points $\{\gamma_0^n : n \in \mathbb{Z}\}$ is infinite, therefore it is Zariski dense in $B_0$. The Galois automorphism induces a morphism $\sigma : A_L \rightarrow A_L$. This morphism is both open and continuous in the Zariski topology. It fixes the points $\gamma_0^n$, for all $n \in \mathbb{Z}$, therefore it fixes their Zariski closure. But the closure of the set $\{\gamma_0^n : n \in \mathbb{Z}\}$ is the curve $B_0$, therefore the morphism $\sigma$ fixes $B_0$. Since this is true for every Galois automorphism in $Gal(L/K)$, we have that $B_0$ is defined over $K$. $\square$
Let $C$ be a smooth curve defined over $K$. We assume that $C$ has at least one $K$-rational point $P$. Then we have an embedding defined over $K$ $$\varphi_P : C \rightarrow J$$ into the Jacobian $J/K$ of $C$ such that $\varphi_P(P) = e$, where $e \in J(K)$ is the identity element. Let $A/K$ be a simple abelian variety, and let $$i : C \rightarrow A$$ be a smooth non-constant morphism defined over $K$ which factors through $\varphi_P$. (Since $A$ is simple this implies that the implied map $J \rightarrow A$ is surjective.)

For any field extension $L$ of $K$, let $\Omega^1(L)$ denote the sheaf of algebraic 1-forms with coefficients in $L$ on a variety. The general theory of abelian varieties allows us to identify the cotangent space at the identity $t^*_A \otimes L$ with the space of global 1-forms $\Gamma(A, \Omega^1(L))$. We shall use that identification without further mention.

**Theorem 3.** Assume that $W := \text{span}_{K_v} \log A(K)$ is a 1-dimensional $K_v$-vector space. Then for every point $Q \in C(K_{v\text{alg}})$ there exists a 1-form $w \in W^\perp \subset \Gamma(A, \Omega^1(K_v))$ such that $i^*(w)(Q) \neq 0$.

In other words, the image of the curve $C$ under the map $i$ is transversal to the $v$-adic closure of $A(K)$ at every intersection point which is algebraic. The theorem, however, does not say anything for any possible transcendental intersection points.

**Proof.** Without loss of generality we can assume that $Q \in C(K)$ (otherwise we replace $K$ by a finite extension whose $v$-adic completion is still $K_v$). Let $t_{C,Q}$ denote the tangent space at $Q$, which is a 1-dimensional $K$-vector space. We have a map $i_* : t_{C,Q} \rightarrow t_{A,i(Q)}$, where, similarly, $t_{A,i(Q)}$ is the tangent space at $i(Q)$. Composing this map with the differential of the translation map we get a $K$-linear homomorphism $\phi : t_{C,Q} \rightarrow t_A$. Note that, since $i$ is smooth, the map $\phi$ is an injection. Let $V := \phi(t_{C,Q})$. Let $\gamma \in A(K)$ be a point of infinite order. Since $A$ is simple, Lemma 2 implies that $\log \gamma \notin V \otimes_K K_v$, therefore the spaces $V \otimes_K K_v$ and $W$ are linearly independent over $K_v$. Hence there exists a global 1-form $w \in \Gamma(A, \Omega^1(K_v))$ which does not vanish on $V$ and such that $w(W) = 0$. It is easily seen that any such form $w$ satisfies $i^*(w)(Q) \neq 0$. \hfill $\Box$

**3.1. The modified Bruin-Stoll algorithm.** Let $C/\mathbb{Q}$ be a smooth curve of genus at least two. We assume that $C$ has a rational point $P$ which defines the embedding $\varphi_P : C \rightarrow J$ into the Jacobian of $C$. We also assume that its Jacobian is simple, and that the group
$J(Q)$ has rank 1. We pick a prime $p$ such that the map $\varphi_p$ can be extended to a smooth morphism between smooth and proper $\mathbb{Z}_p$-schemes $\Phi_p : C \to J$, where $C_{q_p} := C \times Q \text{Spec} \mathbb{Q}_p$ and $J_{q_p} := J \times Q \text{Spec} \mathbb{Q}_p$ are the generic fibres of $C$ and $J$ respectively.

Let $Q \in C(Q)$. Then, according to Theorem 3, there exists a global $p$-adic 1-form $w \in \Gamma(C, \Omega^1(Q_p))$, such that its corresponding 1-form $\varphi_p^{-1}w$ on $J_{q_p}$ annihilates $\log J(Q)$. We fix one such form. Let $t$ be a local parameter on $P$ which gives a local parameter $\bar{t}$ on the reduction of $C$ as well. Then one has the expansion $w = (a_0 + a_1t + \ldots)dt$, where one can assume that $a_i \in \mathbb{Z}_p$. We define $v(w) := v(a_0)$. This definition does not depend on the choice of the local parameter.

Let $J^1(Q_p)$ be the kernel of the reduction map $J(Q_p) \to J(F_p)$, and let $J^{n+1}(Q_p) := P^nJ^1(Q_p)$, where we consider $J^1(Q_p)$ as a formal group. We have maps $\varphi_p^n : C(Q_p) \to J(Q_p)/J^n(Q_p)$ defined in the obvious way. Then we have

**Proposition 4.** If $p \geq 3$ and $n \geq v(w)+1$, then the preimage of $\varphi_p^n(Q)$ contains a single rational point.

**Proof.** The proof is a slight modification of the proof of Proposition 6.3 in [9]. Let $r_1, \ldots, r_g$ be local parameters of $J$ at $\varphi_p(Q)$ such that their reduction gives local parameters on the special fiber of $J$. Then there exists an index $i$ such that $\varphi_p^n r_i$ and its reduction give local parameters on $C$ and $C \times \text{Spec} F_p$ respectively. Without loss of generality we can assume that $i = 1$. Then one has the representation

$$w = (a_0 + a_1r_1 + \ldots)dr_1.$$ 

Since the residue class of $\varphi_p^n(Q)$ consists precisely of the points for which $\max\{|r_1|, \ldots, |r_g|\} \leq p^{-n}$, we have that $|r_1| \leq p^{-n}$ for all points in the preimage of $\varphi_p^n(Q)$. Let $r := p^{-n}r_1$.

Then, the logarithm corresponding to $w$ is given by

$$\lambda_w(r) = a_0p^n r + \frac{a_1p^{2n}}{2} r^2 + \cdots + \frac{a_mp^{n(m+1)}}{m+1} r^{m+1} + \cdots$$

According to the Chabauty theory (see [9] for details) we know that all the rational points lying in the preimage of $\varphi_p^n(Q)$ correspond to zeros of $\lambda_w(r)$ for $|r| \leq 1$. It is clear that if $v\left(\frac{a_mp^{n(m+1)}}{m+1}\right) > v(p^n a_0)$ for all $m \geq 1$, then this function will have a single solution $r = 0$. One can easily check that for $p \geq 3$ and $n \geq v(a_0)+1$ this is precisely the case. This proves the proposition. \qed

The Mordell-Weil Sieve involves constructing a certain finite abelian group $G$ together with a subset $X_G \subset G$ and a group homomorphism $\psi : J(Q) \to G$ such that $\varphi(G) \subseteq \psi^{-1}(X_G)$. It is expected that one
can construct \( G \) in such a way so that the last relation becomes an equality of sets (see [4] for details). In practice one picks a number \( N \) which kills all elements in \( G \), and only considers the quotient map \( \psi_N : J(\mathbb{Q})/NJ(\mathbb{Q}) \to G \). Then we have the following algorithm for finding the rational points on \( C \):

1. Fix a prime of smooth reduction \( p \geq 3 \), and compute \( G_p := J(\mathbb{F}_p) \), as well as \( X_p := \varphi_p(C(\mathbb{F}_p)) \). Compute \( G \) and \( X_G \) using the Mordell-Weil Sieve and pick \( N \) to be a multiple of the exponent of \( G_p \times G \).
2. Find all the residue classes of \( J(\mathbb{Q})/NJ(\mathbb{Q}) \) which map into \( X_p \times X_G \subset G_p \times G \). If there are no such classes then terminate.
3. For each of the residue classes we have found in Step 2, find the representative with smallest canonical height in \( J(\mathbb{Q}) \) and check if it comes from a rational point on the curve. Let \( A \) be the set of all rational points which we have found in this way.
4. For each rational point \( Q \in A \), find a global analytic 1-form \( w_Q \) which does not vanish on \( Q \) but such that \( \varphi_p^{-1}w \) vanishes on \( J(\mathbb{Q}) \). Compute \( v_Q := v(w_Q) \). Let \( n \) be the maximum of all such \( v_Q \)s.
5. Set \( G_p := J(\mathbb{Q}_p)/J^n(\mathbb{Q}_p), X_p := \varphi_p^n(C(\mathbb{Q}_p)) \setminus \varphi_p^n(A) \). Fix a new choice for \( G, X_G \) and \( N \) such that \( N \) is a multiple of the exponent of \( G_p \times G \).
6. Go to Step 2.

Proposition 4 guarantees that the only rational points of \( C \) mapped to \( \varphi_p^n(A) \) are those coming from \( A \), which implies that if the algorithm terminates, then the union of all sets \( A \) which we find in Step 3 will be equal to \( C(\mathbb{Q}) \). This algorithm is in practice equivalent to the one given by Bruin and Stoll. Theorem 3, however, guarantees that one can always perform Step 4, hence the termination of the algorithm depends only on Stoll’s Main Conjecture.

4. Reduction to the semistable case

We are going to prove Theorem 1 by reducing it to a special case. We will need the following definition. Let \( G \) be a commutative algebraic group of dimension \( n \) defined over \( \overline{\mathbb{Q}} \), and let \( V \subseteq t_G \) be a \( d \)-dimensional \( \overline{\mathbb{Q}} \)-linear subspace (we allow \( d = n \)). We set \( \tau(G,V) := d/n \), if \( \dim G \geq 1 \), and \( \tau(G,V) := 1 \), otherwise. The pair \( (G,V) \) is called \textbf{semistable}, if for all proper quotients \( \pi : G \to G' \) we have

\[
\tau(G,V) \leq \tau(G',V'),
\]
where $V' = \pi_*(V)$. Since $\tau(G', V')$ can take only finitely many possible values, it follows that every pair $(G, V)$ has a semistable quotient. It is also easy to see that if $\tau(G, V) = 0$ or $\tau(G, V) = 1$, then the pair $(G, V)$ is semistable.

The following statement is a special case of Theorem 1.

**Theorem 5.** Let $V$ be a proper linear subspace of $t_G$, such that $(G, V)$ is semistable. Then $\log \gamma \notin V \otimes \mathbb{Q}M$ for any algebraic non-torsion point $\gamma \in G(M)^*$.  

**Lemma 6.** Theorem 5 implies Theorem 1.

**Proof.** Let $G$ be a commutative algebraic group and let $V$ be a linear subspace of $t_G$. We proceed by induction on $\dim G$.

If the pair $(G, V)$ is semistable, then Theorem 5 together with the remark in Section 2 trivially imply Theorem 1. In particular, Theorem 1 is true whenever $\dim G \leq 1$.

Assume $(G, V)$ is not semistable and that Theorem 1 is true for all commutative algebraic groups with dimension less than $\dim G$. Let $\gamma \in G(M)^*$ be an algebraic point such that $\log \gamma \in V \otimes \mathbb{Q}M$. Let $\pi : G \to G'$ be a quotient to a semistable pair $(G', V')$, where $V' = \pi_*(V)$. Then $1 > \tau(G, V) > \tau(G', V')$, hence $\dim G' > \dim V'$ and $V'$ is a proper linear subspace of $t_G$. Theorem 5 implies that the only algebraic points in $G'(M)^*$, whose logarithm lies in $V' \otimes \mathbb{Q}M$, are the torsion points. On the other hand $\pi(\gamma) \in G'(M)^*$, since $\pi$ is continuous. The commutativity of the diagram

\[
\begin{array}{ccc}
G(M)^* & \xrightarrow{\pi} & G'(M)^* \\
\downarrow \log & & \downarrow \log \\
t_G \otimes \mathbb{Q}M & \xrightarrow{\pi_*} & t_{G'} \otimes \mathbb{Q}M
\end{array}
\]

implies that $\log \pi(\gamma) \notin V' \otimes \mathbb{Q}M$, therefore, by the inductive hypothesis $\pi(\gamma)$ is a torsion point. Let $k$ be the order of $\pi(\gamma)$, and let $\gamma' = k\gamma$. Then $\gamma' \in H(M)$, where $H := \ker \pi$. If $t_H \subseteq V$, then we pick the algebraic group $B$ consisting of the points $B(\overline{\mathbb{Q}}) = \{\theta \in G(\overline{\mathbb{Q}}) : \theta^k \in H(\overline{\mathbb{Q}})\}$. Clearly $\gamma \in B(\overline{\mathbb{Q}})$ and $t_B = t_H \subseteq V$, hence Theorem 1 is true. If on the other hand $t_H \not\subseteq V$, then, using the inductive hypothesis, we can apply Theorem 1 to $H$, $\gamma'$ and the space $t_H \cap V$. Since $\log \gamma' \in (t_H \cap V) \otimes \mathbb{Q}$, there exists an algebraic subgroup $B_1$ of $H$, such that $t_{B_1} \subseteq t_H \cap V \subseteq V$ and $\gamma' \in B_1(\overline{\mathbb{Q}})$. We then pick $B$ such that $B(\overline{\mathbb{Q}}) = \{\theta \in G(\overline{\mathbb{Q}}) : \theta^k \in B_1(\overline{\mathbb{Q}})\}$. This group satisfies the properties prescribed in Theorem 1. This concludes the proof. \qed
5. Preliminaries

In this section we introduce some notation and standard results which are going to be needed for the proof of Theorem 5.

We shall prove Theorem 5 by contradiction. We assume that there exist a linear subspace \( V \subset t_G \) and an algebraic non-torsion point \( \gamma \in G(M)^* \) such that the pair \((G, V)\) is semistable and \( \log \gamma \in V \otimes M \).

We will denote \( n = \dim G, \ d = \dim V \). Let \( \Gamma \) be the group generated by \( \gamma \) in \( G(M) \).

We fix an embedding \( \phi : G \rightarrow \mathbb{P}^N \) of \( G \) into an \( N \)-dimensional projective space such that \( \phi(\Gamma) \cap \{X_N = 0\} = \emptyset \). (We will use \( X_0, \ldots, X_N \) to denote the coordinates in \( \mathbb{P}^N \).) We can always choose such an embedding. Indeed, let \( \psi : G \rightarrow \mathbb{P}^N \) be an arbitrary embedding. Then the composition of \( \psi \) with a projective automorphism which sends some hyperplane defined over a sufficiently large extension of the field of definition of \( \Gamma \) to the hyperplane \( \{X_N = 0\} \) will give us the desired morphism \( \phi \).

We shall identify \( G \) with \( \phi(G) \). We denote \( \bar{U} := \bar{G} \cap \{X_N \neq 0\} \), and \( U := \bar{U} \cap G \). Here \( \bar{G} \) is the Zariski closure of \( G \) in \( \mathbb{P}^N \). The restriction morphism \( \mathcal{O}_G(\bar{U}) \rightarrow \mathcal{O}_G(U) = \mathcal{O}_G(U) \) is an injection. Therefore we shall identify \( \mathcal{O}_G(\bar{U}) \) with a subset of \( \mathcal{O}_G(U) \).

The proof of the following lemma is given in [11, Section 2]

**Lemma 7.** There exists a Zariski open set \( U' \subset G \times G \) such that \( \Gamma \times \Gamma \subset U' \) and such that the group law is represented on \( U' \) by a single set of bi-homogeneous polynomials \( E_0, \ldots, E_N \in \mathbb{Q}[X_0, \ldots, X_N, X'_0, \ldots, X'_N] \) of bi-degree \( b \), whose coefficients are algebraic integers.

We fix such a set of polynomials and denote \( \mathbf{E} := (E_0 : \cdots : E_N) \).

**Remark.** One can show that for an appropriate embedding \( \phi \) the group operation can be given by bi-homogenous polynomials of bi-degree 2. However, since our results are not effective, we are not going to need this fact.

From now on we shall assume that \( G, \ V, \ \gamma \) and \( \mathbf{E} \) are defined over a fixed number field \( K \) with a fixed embedding into \( \mathbb{Q} \). We will abuse notation by denoting the tangent space at \( e \) of \( G \) again by \( t_G \). It is now a \( n \)-dimensional \( K \)-linear space. The embedding \( K \hookrightarrow \mathbb{Q} \hookrightarrow M \) gives rise to a non-archimedean valuation \( v \) on \( K \). Let \( K_v \) be the completion of \( K \) with respect to that valuation. Then \( \log \gamma \in t_{G_{K_v}} = t_G \otimes_K K_v \).

We take the height of a point in \( \mathbb{P}^N(\mathbb{Q}) \) to be its projective height. This means that if \( \alpha \in \mathbb{P}^N(L) \) for some number field \( L, \alpha = (\alpha_0, \ldots, \alpha_N) \), then
\[
h(\alpha) := \sum_{w \in M_L} \log(\max_{i} |\alpha_i|_w),
\]
where the sum runs over the set \(M_L\) consisting of all places of \(L\). Here we define \(|x|_w := |N_{Lw/Q_w}(x)|^{1/[L:Q]}\), where \(w'\) is the unique place of \(Q\) such that \(w'|w\). The height \(h(P)\) of a homogenous polynomial \(P\) of degree \(D\) is the height of its coefficients taken as a point in \(\mathbb{P}^D(Q)\), where \(A = \binom{N+D}{N} - 1\). The following lemma is proved in [8, Proposition 5]:

**Lemma 8.** Let \(\alpha_1, \ldots, \alpha_k \in G(K)\). Then there exist constants \(c_1, c_2\) such that

\[
h(\alpha_1^{n_1} \cdots \alpha_k^{n_k}) \leq c_1 + c_2 (\sum_i |n_i|)^2, \text{ for all } (n_i) \in \mathbb{Z}^k.
\]

### 6. Differentiation

We fix a basis \(\partial_1, \ldots, \partial_n\) of \(t_G\). It is a standard fact in the theory of group varieties that for every \(\partial \in t_G\) there exists a unique translation-invariant derivation \(D(\partial)\). This means that we have a morphism of sheaves of \(K\)-linear spaces \(D(\partial) : \mathcal{O}_G \rightarrow \mathcal{O}_G\) such that if \(U\) is a Zariski-open subset of \(G\), \(f, g \in \mathcal{O}_G(U)\), \(\alpha \in U(\overline{Q})\) and \(c \in K\) then

1. \(D(\partial)c = 0\);
2. \(D(\partial)fg = fD(\partial)g + gD(\partial)f\);
3. \(D(\partial)f(\alpha) = D(\partial)(f \circ T_\alpha)(e) = \partial(f \circ T_\alpha)\),

where \(T_\alpha(\beta) := \alpha \beta\) is the translation-by-\(\alpha\) morphism. Since \(G\) is commutative, we also have

4. \(D(\partial)D(\partial')f = D(\partial')D(\partial)f\)

for any two vectors \(\partial, \partial' \in t_G\). From now on we shall use the same notation for a vector in \(t_G\) and for its corresponding derivation. For any linear space \(W \subseteq t_G\), \(K[W]\) will denote the ring of differential operators with coefficients in \(K\) generated by derivations in \(W\).

Let \(\Delta_1, \ldots, \Delta_d\) be a basis of \(V\). We denote

\[
\mathcal{D}(\infty) := \{\Delta \in K[V] : \Delta = \Delta_1^{t_1} \cdots \Delta_d^{t_d}, \ t_i \geq 0\}.
\]

For any differential operator \(\Delta \in \mathcal{D}(\infty)\), \(\Delta = \Delta_1^{t_1} \cdots \Delta_d^{t_d}\), we denote \(|\Delta| := t_1 + \cdots + t_d\). For any non-negative integer \(T\) we set

\[
\mathcal{D}(T) := \{\Delta : |\Delta| \leq T\}.
\]

Despite the notation, the sets \(\mathcal{D}(\infty)\) and \(\mathcal{D}(T)\) depend on the choice of basis of \(V\). We shall later fix a convenient basis to work with.

Let \(\alpha \in G(K)\) and let \(P \in K[X_0, \ldots, X_N]\) be a homogenous polynomial of degree \(D\). Assume that \(Q\) is another such polynomial with the
same degree and such that $Q(\alpha) \neq 0$. We define the order of $P$ along $V$ at the point $\alpha$ to be the smallest number $t$ such that there exists \( \Delta \in \mathcal{D}(t) \) with

\[
\Delta \frac{P}{Q}(\alpha) \neq 0
\]

We denote the order by $\text{ord}_{V,\alpha} P$. One can check that this definition depends neither on $Q$, nor on the choice of basis for $V$.

Let $P$ be a homogenous polynomial of degree $D$ and let \( \Delta \in K[V] \). Then 

\[
\frac{P \circ E(X_0, \ldots, X_N, Y_0, \ldots, Y_N)}{Y_N^{bD}},
\]

when considered as a function of $Y$, induces a rational function $f(X_0, \ldots, X_N) \in \mathcal{O}_G(U)$. We define

\[
P_\Delta(X_0, \ldots, X_N) := (\Delta f(X_0, \ldots, X_N))(e)
\]

Then $P_\Delta$ is a homogenous polynomial of degree $bD$ and it is not difficult to show that $\text{ord}_{V,\alpha} P_\Delta \geq \text{ord}_{V,\alpha} P - |\Delta|$. This allows us to study the multiplicity of $P$ at a point using the polynomials $P_\Delta$.

If $P$ is any polynomial we write $|P|_v$ for the maximum of the absolute values of its coefficients. We shall later need an estimate of $|P_\Delta|_v$ in terms of $|P|_v$. We are going to show next that after an appropriate choice of basis for $V$ such an estimate becomes trivial. More precisely, let us call a basis $\Delta_1, \ldots, \Delta_d \in V$, nice, if the following property holds:

For any homogenous polynomial $P$ with $|P|_v \leq 1$ and any $\Delta \in \mathcal{D}(\infty)$ one has $|P_\Delta|_v \leq 1$.

**Lemma 9.** The linear space $V$ has a nice basis.

**Proof.** Let $\Delta'_1, \ldots, \Delta'_d$ be any basis of $V$. The $K$-algebra $\mathcal{O}_G(U)$ is finitely generated. Pick a set of generators $f_1, \ldots, f_k \in \mathcal{O}_G(U)$ which contains the functions $\frac{X_i}{X_N}$ for all $i = 0, \ldots, N - 1$ and such that $|f_j(e)|_v \leq 1$ for all $j = 1, \ldots, k$. Then there exist polynomials $F'_{ij} \in K[T_1, \ldots, T_k]$ such that $\Delta_i f_j = F'_{ij}(f_1, \ldots, f_k)$. Pick $\Delta_i = c_i \Delta'_i$ where $c_i$ is any non-zero integer such that $|c_i F'_i|_v \leq 1$ for all $j$. We are going to show that this basis has the required property.

Let $F \in K[T_1, \ldots, T_k]$ be such, that $|F|_v \leq 1$. It is then easy to show that $|\Delta F(f_1, \ldots, f_k)(e)|_v \leq 1$ for any choice of $\Delta = \Delta'_1 \cdots \Delta'_d$.

Let now $P$ be a homogenous polynomial of degree $D$. Then we have the representation

\[
f(X_0, \ldots, X_N) := \frac{P \circ E(X_0, \ldots, X_N, Y_0, \ldots, Y_N)}{Y_N^{bD}} = \sum_j A_{ij} X_0^{f_0} \cdots X_N^{f_N},
\]
where $A_J \in \mathcal{O}_G(U)$. Since the coefficients of $E$ are algebraic integers and $|P|_v \leq 1$, it follows that $f$, when considered as a polynomial in $X_0, \ldots, X_N; \frac{Y_0}{Y_N}, \ldots, \frac{Y_{N-1}}{Y_N}$, has $|f|_v \leq 1$. Hence, since the functions $Y_i/Y_N$ lie in the set $\{f_1, \ldots, f_k\}$, one has a representation $A_J = \sum I A_I^J f_I^1 \cdots f_I^k$, where $A_I^J \in K$, $|A_I^J|_v \leq 1$. By the observation in the previous paragraph it follows that $|\Delta A_J(e)|_v \leq 1$ for all $\Delta \in D(\infty)$. Since

$$P_\Delta(X_0, \ldots, X_N) = \sum_J \Delta A_J(e) X_0^{j_0} \cdots X_N^{j_N},$$

one concludes that $|P_\Delta|_v \leq 1$ for all $\Delta \in D(\infty)$. □

From now on we fix a nice basis $\Delta_1, \ldots, \Delta_d$ of $V$.

In order to complete the proof of Theorem 5 we will need to apply a multiplicity estimate. We state here one estimate, given in [2]:

**Theorem 10.** Let $(G, V)$ be a semistable pair and let $\gamma \in G(M)$. We fix an embedding of $G$ into projective space, and a Zariski open subset $U$ as in Section 5. Let $\gamma_0 \in \Gamma$. We fix a basis $\Delta_1, \ldots, \Delta_d$ of $V$. There exists an effectively computable constant $c > 0$ with the following property. Let $S_0, T_0$ and $D_0$ be non-negative integers such that

$$S_0 T_0^d > c D_0^n.$$

Assume in addition that there exists a homogenous polynomial $P$ of degree $D_0$, which does not vanish identically on $G$ and such that

$$P_{\Delta_1^{t_1} \cdots \Delta_d^{t_d}}(\gamma_0^s) = 0$$

for all $0 \leq s \leq S_0$ and all non-negative integers $t_1, \ldots, t_d$ with $0 \leq t_1 + \cdots + t_d \leq T_0$. Then $\gamma_0$ is a torsion point.

One could also use the more general Philippon multiplicity estimate [7] of which the previous theorem is an easy consequence (See Chapter 11, Corollary 4.2 in [6]).

7. Proof of Theorem 5

There exists a system of $v$-adic open neighbourhoods of $e$ in $G(K_v)$ consisting of subgroups of $G(K_v)$. Hence there exists an open (and closed) subgroup $W_0 \subset U(K_v) \cap \{\alpha \in \mathbb{P}^N(K_v): |X_0/\cdots/X_N(\alpha)|_v \leq 1 \text{ for } i = 0, \ldots, N-1\}$ where the logarithm is a local isomorphism. Since $\gamma \in G(K_v)^\times$, there exists $k \in \mathbb{N}$ such that $\gamma^k \in W_0$. Since $\log \gamma = \frac{1}{k} \log(\gamma^k)$, in order to prove Theorem 5 one only needs to show that $\log(\gamma^k) \notin V \otimes K_v$. Therefore, without loss of generality, we can assume that $\gamma \in W_0$. 
Let $D, T, S, l$ be nonnegative integers. Since our result is not effective we do not need to keep track of all the constants appearing in our estimates. Therefore, in order to simplify notation, we will use the letter $c$ to denote a sufficiently large positive number, which can change from line to line, and which does not depend on $D, T, S$ and $l$. Let $\gamma_1 := \gamma^p$, where $p$ is the prime which is extended by $v$, and let $\gamma' = \gamma_1^l$. We denote the group generated by $\gamma'$ by $\Gamma'$, and we also set $\Gamma'(S) := \{\gamma^s : 0 \leq s \leq S\}$.

The main steps of the proof are contained in the following three propositions. We defer the proofs of those propositions to the next section.

**Proposition 11** (Auxiliary polynomial). Assume that the following inequality holds

$$D^n \geq 2n(\deg K)(T + n)^d(S + 1)$$

Then there exists a homogeneous polynomial $P$ with integer coefficients and degree $D$, which does not vanish on $G$, such that for all $\Delta \in D(T/2)$ and all $\alpha \in \Gamma'(S)$ we have

(a). $\ord_{V, \alpha} P_\Delta \geq T/2$.

(b). $h(P_\Delta) \leq c(D + T) \log(D + T) + cD(lS)^2$.

To any polynomial $P$ satisfying the conditions given in the proposition and any $\Delta \in D(\infty)$ we associate a function $f_\Delta = \frac{P_\Delta}{c_\Delta X_N^D}$, where $c_\Delta \in K$ is any coefficient of $P_\Delta$ such that $|P_\Delta|_v = |c_\Delta|_v$.

**Lemma 12.** The functions $f_\Delta$ have the following properties:

(a). $|f_\Delta(\alpha)|_v \leq 1$ whenever $|X_i(\alpha)/X_N(\alpha)|_v \leq 1$ for all $i$;

(b). If $X_N(\alpha) \neq 0$, then $h(f_\Delta(\alpha)) \leq \log\left(\frac{N + bD}{N}\right) + h(P_\Delta) + bDh(\alpha)$.

**Proof.** Part (a) is trivial. Part (b) is easily shown using elementary height estimates. \qed

**Proposition 13** (Upper bound). Let $\Delta \in D(T/2)$, let $0 \leq s \leq lS$, and let $P$ be the polynomial in Proposition 11. Then

$$\log|f_\Delta(\gamma^s_1)|_v \leq -cST$$

**Proposition 14** (Lower bound). Let $\Delta \in D(T/2)$, let $0 \leq s \leq lS$ and let $P$ be the polynomial in Proposition 11. Then either $P_\Delta(\gamma^s_1) = 0$, or

$$\log|f_\Delta(\gamma^s_1)|_v \geq -c(lS)^2D - c(D + T)\log(D + T).$$

**Proof of Theorem 5.** Choose $l = S^2$, $T = S^{5n+2}$, $D = \lfloor(S^{5n+2d+2})^{1/n}\rfloor$. Then we can pick $S$ large enough so that the inequality (1) is satisfied. Choose a polynomial $P$ with the properties given in Proposition 11.
Let $\Delta \in \mathcal{D}(T/2)$, $0 \leq s \leq lS$. Then, according to Proposition 13, for $S$ large enough we have

\[
\log |f_\Delta(\gamma_1^s)|_v \leq -cST \leq -cS^{5n+3}.
\]

On the other hand Proposition 14 implies that either $P_\Delta(\gamma_1^s) = 0$ or

\[
-\log |f_\Delta(\gamma_1^s)|_v \leq c(lS)^2 D + c(D + T) \log(D + T) \\
\leq cS^{5+5d+\frac{2d+2}{n}} + c(S^{5d+\frac{2d+2}{n}} + S^{5n+2}) \log S.
\]

At this point we use the assumption that the linear space $V$ is a proper subspace of $t_G$. This means that $d \leq n - 1$, hence we get the estimate

\[
-\log |f_\Delta(\gamma_1^s)|_v \leq cS^{5n+2} + c(S^{5n-2} + S^{5n+2}) \log S \\
\leq cS^{5n+2.5}.
\]

However, if $S$ is large enough, the estimates (2) and (3) cannot simultaneously be satisfied, hence $P_\Delta(\gamma_1^s) = 0$ for all non-negative $s \leq lS$ and all $\Delta \in \mathcal{D}(T/2)$.

Finally we apply Theorem 10 where we set $S_0 := lS$, $T_0 := T/2$, $D_0 := D$ and $\gamma_0 := \gamma_1$. Since $(lS)T^d \sim D^n S^\frac{1}{2}$, if we apply the theorem for large enough $S$ we get that $\gamma_1$ is a torsion point. This, however, contradicts the assumption that $\gamma$ is a non-torsion point, which completes the proof of the theorem. \qed

8. Proofs of the main propositions

8.1. The auxiliary polynomial. We are only going to give a sketch of the proof of Proposition 11. It is very similar to the proof of Lemma 4.1 in [11], the only difference being that the heights of the points in $\Gamma'(S)$ are bounded above by $c(lS)^2$ instead of $cS^2$. Condition (1) of the proposition is implied by having $P_\Delta(\gamma^s) = 0$ for all $\Delta \in \mathcal{D}(T)$ and all $s$, $0 \leq s \leq S$. Each of those equations is a linear equation for the coefficients of $P$ and there are roughly $ST^d$ such equations. The space of polynomials of degree $D$ modulo polynomials vanishing on $G$ has dimension roughly $D^n$. Therefore if $D^n \geq cST^d$ there exists a polynomial satisfying Condition (1). Siegel’s Lemma tells us that we can pick the polynomial to have integer coefficients and a certain upper bound of the height, determined by the height of the system of linear equations. A not very difficult (but long) computation shows that the height of the coefficients of each equation is bounded above by $c(D + T) \log(D + T) + h(\gamma^s)$. Using Lemma 8 we see that the second term is bounded above by $c(lS)^2$. Applying Siegel’s Lemma one shows the
existence of a polynomial $P$ with small height which satisfies Condition (1). It is then not difficult to show that condition (2) is also satisfied.

8.2. The upper bound. Let $f$ be a $v$-adic analytic function of one variable. We will use the notation $\|f\|_r := \max_{|z|_v \leq r} |f(z)|_v$. We shall call a power series $f(z) = \sum_{n=1}^{\infty} a_n z^n$ normal, if

1. $\lim_{n \to \infty} |a_n|_v = 0$, and
2. $|a_n|_v \leq 1$.

Those conditions are equivalent to

1. $f(z)$ is defined and analytic on $|z|_v \leq 1$.
2. $\|f\|_1 \leq 1$.

The following theorem is due to Mahler [1, Appendix, Theorem 14]:

**Theorem 15.** Let $f(z)$ be a normal function which has zeros at $x_1, \ldots, x_n \in K_v$ of multiplicities $d_1, \ldots, d_n$ respectively. Assume $|x_i|_v \leq r$, where $0 < r < 1$, and let $d = d_1 + \cdots + d_n$. Then given any $x \in K_v$ such that $|x|_v \leq r$ and any $k = 0, 1, 2, \ldots$ we have

$$|f^{(k)}(x)|_v \leq r^{d-k}$$

**Proof of Proposition 13.** Let $w := \log \gamma$, and let $\phi(z) := f_\Delta(\exp(zw))$. (Here exp is the inverse of restriction of the the logarithm function to $W_0$.) Since $\gamma \in W_0$ we have that $\phi$ is defined and analytic for all $|z|_v \leq 1$. For any $\alpha \in W_0$, since $\left| \frac{X}{N} (\alpha) \right|_v \leq 1$ for all $i$, Lemma 12(a) implies that $|f_\Delta(\alpha)|_v \leq 1$. We conclude that $\|\phi\|_1 \leq 1$, hence $\phi$ is normal.

By Proposition 11 (1), it follows that the order of $\phi$ at points $0, pl, 2pl, \ldots, plS$ is at least $T/2$. Let $r = |p|_v$. Then, since $0 < r < 1$ and $|spl|_v \leq r$, applying Theorem 15 we conclude that for any integer $s$ we have

$$|f_\Delta(\gamma_s^t)|_v = |\phi(ps)|_v \leq r^{ST/2}.$$ 

Taking logarithms we obtain the desired result. 

8.3. The lower bound.

**Proof of Proposition 14.** We are going to prove this proposition by means of Liouville’s inequality [5, Lemma D.3.3]. Assume that $P_\Delta(\gamma_s^t) \neq 0$. 

Using Proposition 11, Lemma 12(b) and Lemma 8 we estimate

\[ h(f(\Delta(\gamma_1^s))) \leq bDh(\gamma_1^s) + h(P_\Delta) + \log \left( \frac{N + bD}{N} \right) \]

\[ \leq cs^2D + c(D + T) \log(D + T) + cD(lS)^2 + c \log D \]

\[ \leq c(lS)^2D + c(D + T) \log(D + T) \]

Since we have assumed that \( P_\Delta(\gamma_1^s) \neq 0 \), Liouville’s inequality implies that

\[ [K_v : \mathbb{Q}_p] \log |f(\Delta(\gamma_1^s))|_v \geq -[K : \mathbb{Q}]h(\Delta(\gamma_1^s)) \]

Combining this inequality with the previous estimate proves the assertion. \( \square \)

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THE P-ADIC ANALYTIC SUBGROUP THEOREM AND APPLICATIONS

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