Local density of States in Two-Dimensional Nano-Structured Superconducting Systems with Superconductor-Normal Metal Interfaces

Saoto Fukui\textsuperscript{1}, Zhen Wang\textsuperscript{1}, and Masaru Kato\textsuperscript{2}

\textsuperscript{1} Shanghai Institute of Microsystem and Information Technology, Chinese Academy of Sciences, 865 Changning Road, Shanghai 200050, China
\textsuperscript{2} Department of Physics and Electronics, Osaka Prefecture University, Gakuencho, Nakamozu, Sakai, Osaka, 599-8531, Japan

E-mail: saoto@mail.sim.ac.cn

Abstract. In nano-structured superconductors, it is important to consider a discreteness of energy levels instead of continuous energy levels due to the quantum confinement effect. This discreteness causes an appearance of many peaks in a density of state (DOS). In this paper, the effect of the discreteness of energy level or the effect of the quantum confinement on the DOS in superconducting systems with superconductor / normal metal (S/N) interfaces are focused. In particular, in nano-sized finite systems, a local density of state (LDOS) becomes strongly spatial dependent because of non-uniform spatial distributions of a gap energy and a pair amplitude. We investigate the pair amplitude and the LDOS by solving the Bogoliubov-de Gennes equations self-consistently. The proximity effect leads to a penetration of the pair amplitude into the normal metal region. Then, the pair amplitude in the normal metal decays non-monotonously because of the effect of the nano-sized finite system. On the other hand, the spatial-averaged LDOS plots as a function of the energy have many peaks in both superconductor and normal metal regions. Also, a contribution of the normal metal to the superconductor causes an appearance of peaks of the LDOS at the energy below the gap energy. In the SNS junction, when width of the normal metal increases, these peaks at the energy below the gap energy appear clearly.

1. Introduction
A nano-structured superconductor has many different properties from those of a bulk superconductor. An important property is the quantum confinement effect. In the nano-structured system, electron states are restricted strongly and discreteness of energy levels becomes more important than continuous energy levels. The discreteness of energy levels causes an oscillation of the superconducting quantity such as a critical temperature, a density of state (DOS), a gap energy, and so on, as a function of a system size. Oscillations of superconducting quantities are reported theoretically and experimentally \cite{1, 2, 3, 4}.

An energy-dependence of the DOS is also related to the discreteness of energy levels closely. In the nano-structured superconductor, the DOS has a many peaks as a function of the energy because of the discreteness of energy levels \cite{5, 6}. This peak structure of the DOS is expected to affect several superconducting properties. An example is a tunnel current in a superconductor / normal metal / superconductor (SNS) junction. The tunnel effect is related to the DOS of the...
quasi-particle in the superconductor directly. Effects of the discreteness of energy levels on the DOS and the tunnel effect in junction systems are not reported in detail.

We investigate the DOS in two-dimensional nano-structured superconducting system with superconductor / normal metal (S/N) interfaces theoretically. In the nano-structured finite system, the DOS depends on the space because of spatial dependences of the gap energy and a pair amplitude. So, the local density of state (LDOS) is considered as a spatial distribution of the DOS. We solve the Bogoliubov-de Gennes (BdG) equations self-consistently and obtain wave functions of quasi-particles in each energy level. Using their wave functions, the pair amplitude and the LDOS can be calculated numerically.

2. Method

In order to investigate electronic states, the microscopic Bogoliubov-de Gennes (BdG) equations are solved [6, 7]. The BdG equations are given by,

\[
\begin{align*}
\frac{1}{2m} \left( \frac{\hbar}{i} \nabla - eA(r) \right)^2 - \mu + V(r) & \quad u_n(r) + \Delta(r) u_n(r) = E_n u_n(r), \quad (1a) \\
\frac{1}{2m} \left( \frac{\hbar}{i} \nabla + eA(r) \right)^2 - \mu + V(r) & \quad v_n(r) + \Delta^*(r) v_n(r) = E_n v_n(r), \quad (1b)
\end{align*}
\]

where \( u_n(r) \) and \( v_n(r) \) are \( n \)-th wave functions of electron-like and hole-like quasi-particles, \( E_n \) is a \( n \)-th energy eigenvalue measured with respect to a chemical potential \( \mu \), \( \Delta(r) \) is the gap energy, \( m \) is an electron effective mass, \( e \) is an electron charge, \( c \) is a light velocity, and \( A \) is a magnetic vector potential. \( V(r) \) is a potential energy for the S/N interface, which is given by,

\[
V(r) = \begin{cases} V_0 & \text{(Location at the S/N interface)} \\ 0 & \text{(Otherwise)} \end{cases}
\]

\( V_0 \) is a constant. The gap energy \( \Delta(r) \) is given by,

\[
\Delta(r) = g \sum_n \frac{|E_n| E_n}{E_n} u_n(r)v_n^*(r)(1 - 2f(E_n)). \tag{3}
\]

\( g \) is an interaction constant, \( f(E_n) = 1/[\exp(E_n/k_B T) + 1] \) is the Fermi distribution, \( k_B \) is the Boltzmann constant, and \( T \) is a temperature. \( E_c \) is a cutoff energy, which is defined as a maximum energy where an attractive interaction is valid. \( A(r) \) and \( \Delta(r) \) are obtained at the same time self-consistently by solving the BdG equations (1a) and (1b), the gap equation (3), and the Maxwell equation is performed. The Maxwell equation is given by,

\[
\nabla \times (\nabla \times A(r) - H(r)) = \frac{4\pi}{c} j(r). \tag{4}
\]

\( H(r) \) is an external magnetic field and \( j(r) \) is a current density given by,

\[
j(r) = \frac{e\hbar}{2mi} \sum_n \left[ f(E_n) u_n^*(r) \left( \nabla - \frac{ie}{\hbar c} A(r) \right) u_n(r) + (1 - f(E_n)) v_n(r) \left( \nabla - \frac{ie}{\hbar c} A \right) v_n^*(r) - \text{H. C.} \right]. \tag{5}
\]

H. C. represents the Hermitian conjugate. Moreover, in the nano-structured superconductor, the number of electrons is important. In order to conserve the number of electrons, the chemical potential is determined by a conservation rule of a particle number,

\[
N_e = 2 \int \sum_n \left( |u_n(r)|^2 f(E_n) + |v_n(r)|^2 (1 - f(E_n)) \right) \, d^2r, \tag{6}
\]
where \( N_e \) is the total number of electrons.

In order to solve these equations self-consistently, the finite element method (FEM) is used. In this study, we consider two-dimensional systems. The system is separated into small triangular elements. Coordinates of three nodes in one triangular element are defined as \((x_i, y_i)\), \((x_j, y_j)\), and \((x_k, y_k)\). In the \( e \)-th triangular element, \( u_n(\mathbf{r}) \), \( v_n(\mathbf{r}) \), \( A(\mathbf{r}) \), \( \Delta(\mathbf{r}) \), and \( V(\mathbf{r}) \) are expanded with an area coordinate \( \zeta^e_i(x, y) \) which are given by,

\[
u_n(\mathbf{r}) = \sum_e \sum_i \zeta_i^e(\mathbf{r}) u_{ni}^e, \quad v_n(\mathbf{r}) = \sum_e \sum_i \zeta_i^e(\mathbf{r}) v_{ni}^e, \quad \Delta(\mathbf{r}) = \sum_e \sum_i \zeta_i^e(\mathbf{r}) \Delta_i^e, \quad A(\mathbf{r}) = \sum_e \sum_i \zeta_i^e(\mathbf{r}) A_i^e, \quad V(\mathbf{r}) = \sum_e \sum_i \zeta_i^e(\mathbf{r}) V_i^e,
\]

where \( i = 1, 2, 3 \) is a node number of the triangular element. The area coordinate is defined as,

\[
\zeta_i^e(x, y) = \frac{1}{2S_e} (a_i + b_i x + c_i y), \quad (i = 1, 2, 3)
\]

where,

\[
a_i = x_j y_k - x_k y_j, \quad b_i = y_j - y_k, \quad c_i = x_k - x_j,
\]

and \( S_e \) is an area in the \( e \)-th element. Using equations (7), the BdG equations (1a) and (1b), the gap equation (3), the Maxwell equation (4), and the conservation rule (6) can be rewritten by,

\[
\begin{align*}
\sum_j [P_{ij}^e(A) + P_{ij}^2e(A)] u_{nj}^e + \sum_j Q_{ij}^e(\Delta) v_{nj}^e &= E_n \sum_j I_{ij}^e u_{nj}^e, \quad (10a) \\
\sum_j [-P_{ij}^e(A) + P_{ij}^{2e}(A)] v_{nj}^e + \sum_j Q_{ij}^{e*}(\Delta) u_{nj}^e &= E_n \sum_j I_{ij}^e v_{nj}^e, \quad (10b) \\
\sum_j I_{ij}^e \Delta_j^e &= g \sum_{i_1, i_2} I_{ni_1 i_2}^e \sum_n u_{ni_1}^e v_{ni_2}^e (1 - 2f(E_n)), \quad (10c)
\end{align*}
\]

\[
\begin{align*}
\sum_j R_{ij}^e(u, v) A_{ijx}^e + \sum_j S_{ij}^e A_{ijy}^e &= T_{ijx}^e(u, v) - U_{ijx}^e, \quad (10d) \\
\sum_j R_{ij}^e(u, v) A_{ijy}^e - \sum_j S_{ij}^e A_{ijy}^e &= T_{ijy}^e(u, v) + U_{ijy}^e, \quad (10e) \\
N_e &= 2 \sum_{ij} I_{ij}^e \sum_n [f(E_n) u_{ni}^e v_{nj}^e + (1 - f(E_n)) v_{ni}^e u_{nj}^e]. \quad (10f)
\end{align*}
\]

Here, we define integrals \( I_{ij}^e, I_{i_1 i_2 i_3}^e, I_{i_1 i_2 i_3 i_4}^e, J_{i_1 i_2}^{\alpha i}, J_{i_1 i_2}^{\alpha i}, \) and \( K_{i_1 i_2}^{\alpha \alpha \alpha} \) as following forms,

\[
\begin{align*}
I_{ij}^e &= \int \zeta_i^e(\mathbf{r}) \zeta_j^e(\mathbf{r}) d\mathbf{r}, \quad I_{i_1 i_2 i_3}^e = \int \zeta_i^e(r) \zeta_j^e(r) \zeta_k^e(r) d\mathbf{r}, \quad I_{i_1 i_2 i_3 i_4}^e = \int \zeta_i^e(r) \zeta_j^e(r) \zeta_k^e(r) \zeta_l^e(r) d\mathbf{r}, \\
J_{i_1 i_2}^{\alpha i} &= \int \frac{\partial \zeta_i^e(r)}{\partial \alpha_i} d\mathbf{r}, \quad J_{i_1 i_2}^{\alpha i} = \int \frac{\partial \zeta_i^e(r)}{\partial \alpha_i} \zeta_j^e(r) \zeta_k^e(r) d\mathbf{r}, \quad J_{i_1 i_2}^{\alpha \alpha \alpha} = \int \frac{\partial \zeta_i^e(r)}{\partial \alpha_i} \frac{\partial \zeta_j^e(r)}{\partial \alpha_i} d\mathbf{r},
\end{align*}
\]

where \( \alpha_i = x, y \). Using equation (11), we also define coefficients \( P_{ij}^{1e}(A), P_{ij}^{2e}(A), \) \( Q_{ij}^e(\Delta) \), \( R_{ij}^e(u, v) \), \( S_{ij}^e \), \( T_{ijx}^e(u, v) \), and \( U_{ijx}^e \) as,

\[
\begin{align*}
P_{ij}^{1e}(A) &= \frac{\hbar^2}{2m} \sum_{\alpha} K_{ij}^{\alpha \alpha \alpha} + \frac{e^2}{2mc^2} \sum_{i_1 i_2} \sum_{\alpha} I_{i_1 i_2}^{e \alpha} A_{i_1}^e A_{i_2}^e - \mu I_{ij}^e + \sum_{i_1} I_{ij}^e V_{i_1}^e.
\end{align*}
\]
and the Fermi wave vector in the superconductor $k_{\text{F}}$. In the S/N interface as $d/\xi_0$ superconductor. In the SNS junction, we define a width of the normal metal perpendicular to systems are fixed to $10\xi_0 \times 5\xi_0$, where $\xi_0$ is the coherence length at zero temperature in the bulk superconductor. Red and green represent regions of the superconductor and the normal metal, respectively.

Two-dimensional superconducting systems with (a) one superconductor / normal metal (S/N) interface and (b) two S/N interfaces. The latter is the superconductor / normal metal / superconductor (SNS) junction. Both total sizes are $10\xi_0 \times 5\xi_0$, where $\xi_0$ is the coherence length at zero temperature in the bulk superconductor. Red and green represent regions of the superconductor and the normal metal, respectively.

\begin{equation}
    P_{ij}^{2e}(A) = \frac{ie\hbar}{2mc} \sum_{\alpha} \sum_{i_1} (J_{ii_j}^{\alpha\alpha} - J_{ji_i}^{\alpha\alpha}) A_{i_1}^{\alpha \alpha},
\end{equation}

\begin{equation}
    Q_{ij}^{\Delta}(\Delta) = \sum_{i_1} \Delta_{i_1}^\alpha \Delta_{ji}^{\alpha \alpha},
\end{equation}

\begin{equation}
    R_{ij}^{\alpha}(u, v) = \sum_{\alpha} K_{ij}^{\alpha\alpha} + \frac{4\pi e^2}{mc} \sum_{i_1} \Delta_{i_1}^{\alpha \alpha} \sum_{n} \left[ f(E_n) u_{n_{i_1}}^{\alpha \alpha} u_{n_{i_2}}^{\alpha \alpha} + (1 - f(E_n)) v_{n_{i_1}}^{\alpha \alpha} v_{n_{i_2}}^{\alpha \alpha} \right],
\end{equation}

\begin{equation}
    S_{ij}^{\alpha \alpha} = K_{ij}^{\alpha \alpha} - K_{ij}^{\alpha \alpha},
\end{equation}

\begin{equation}
    T_{ij}^{\alpha \alpha}(u, v) = \frac{4\pi e\hbar}{2mc} \sum_{i_1} \sum_{n} \left[ f(E_n) u_{n_{i_1}}^{\alpha \alpha} u_{n_{i_2}}^{\alpha \alpha} + (1 - f(E_n)) v_{n_{i_1}}^{\alpha \alpha} v_{n_{i_2}}^{\alpha \alpha} \right],
\end{equation}

\begin{equation}
    U_{ij}^{\alpha \alpha} = H_0 J_{ij}^{\alpha \alpha},
\end{equation}

where $\alpha = x, y$ and $H_0$ is a magnitude of the applied magnetic field, $H(r) = (0, 0, H_0)$.

After $u_n(r), v_n(r), A(r)$, and $\Delta(r)$ are obtained, the LDOS can be calculated by,

\begin{equation}
    n(r, E) = -\sum_{n} \left[ |u_n(r)|^2 f'(E_n - E) + |v_n(r)|^2 f'(E_n + E) \right].
\end{equation}

\begin{equation}
    f'(E) = \frac{df(E)}{dE}
\end{equation}

is a derivative of the Fermi distribution [7, 8].

3. Result

In this section, we show the LDOS in two-dimensional superconducting systems with S/N interfaces. We consider two kinds of systems. One is a superconducting system with one S/N interface at the middle of the system in figure 1(a). Another is a superconducting system with two S/N interfaces in figure 1(b), which is the SNS junction. In figure 1, red and green represent regions of the superconductor and the normal metal, respectively. Total sizes in both systems are fixed to $10\xi_0 \times 5\xi_0$, where $\xi_0$ is a coherence length at a zero temperature in a bulk superconductor. In the SNS junction, we define a width of the normal metal perpendicular to the S/N interface as $d/\xi_0$, which is normalized by $\xi_0$. In the case of figure 1(b), $d/\xi_0 = 1.0$.

We set the gap energy at the zero temperature in the bulk superconductor $\Delta_0/E_c = 0.2$, and the Fermi wave vector in the superconductor $k_F\xi_0 = 3.0$, which are normalized by $E_c$ and $1/\xi_0$. Also, we set a Ginzburg-Landau parameter in a clean limit $\kappa = 0.96\lambda_L/\xi_0 = 3.0$ [9],

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Two-dimensional superconducting systems with (a) one superconductor / normal metal (S/N) interface and (b) two S/N interfaces. The latter is the superconductor / normal metal / superconductor (SNS) junction. Both total sizes are $10\xi_0 \times 5\xi_0$.}
\end{figure}
where $\lambda_L$ is a London penetration length. In this research, the magnetic field is not applied ($H_0 = 0$). Moreover, mismatches of the effective mass and the Fermi wave number between the superconductor and the normal metal are introduced by ratios,

$$ m_{\text{ratio}} \equiv m_n / m_s, \quad k_{\text{ratio}} \equiv k_{F_n} / k_{F_s}, \quad (14) $$

where $m_n$ and $m_s$ are effective masses in the normal metal and the superconductor, and $k_{F_n}$ and $k_{F_s}$ are Fermi wave numbers in the normal metal and the superconductor, respectively. We assume $m_{\text{ratio}} = k_{\text{ratio}} = 1.0$ ($m_n = m_s$ and $k_{F_n} = k_{F_s}$).

As boundary conditions, edges in the system are normal state and current doesn’t flow outside the system ($\mathbf{A} \cdot \mathbf{n} = 0$, $\mathbf{n}$ is a unit vector perpendicular to the boundary). Also, the gap energy $\Delta(\mathbf{r})$ in the normal metal region is zero.

### 3.1 Pair amplitude and LDOS in superconducting systems with one S/N interface

We show results in superconducting systems with one S/N interface in figure 1(a). At first, we show that superconductivity exists even in the normal metal. In this system, the gap energy in equation (3) cannot be used as a superconducting order parameter because the interaction constant $g$ is zero in the normal metal. Then, a pair amplitude $F(\mathbf{r})$ is used as the superconducting order parameter instead of the gap energy. A standard formation and a formation with the FEM of the pair amplitude are given by,

$$ F(\mathbf{r}) = \sum_{n}^{E_n \leq E_c} u_n(\mathbf{r})v_n^*(\mathbf{r})(1 - 2f(E_n)), \quad (15) $$

$$ \sum_{j} F_{ij}^c F_j^c = \sum_{i_1, i_2} F_{i_1 i_2}^c \sum_{n}^{E_n \leq E_c} u_{ni_1}^c v_{ni_2}^*(1 - 2f(E_n)), \quad (16) $$

where

$$ F(\mathbf{r}) = \sum_{c} \sum_{i} \zeta_i^c(\mathbf{r})F_i^c. \quad (17) $$

In figure 2, we show spatial distributions of the pair amplitude normalized by the pair amplitude at the zero temperature in the bulk superconductor, $F_0 = \Delta_0 / g$. In figure 2(a), the spatial distribution of the pair amplitude in the total system for $V(x)/E_c = 0.00$ is shown. The pair amplitude is nonzero even in the normal metal. It means that quasi-particles in the
Figure 3. Spatial-averaged LDOS (SA-DOS) as a function of the energy in superconducting systems with one S/N interface for $V(x)/E_c = 0.00$ (a), 0.30 (b), 0.50 (c), and 1.00 (d). Their lines represent SA-DOS in superconductor (red), normal metal (green), and total (blue) regions, respectively.

superconductor are penetrated in the normal metal because of the proximity effect. In general, it is known that the superconducting order parameter in the normal metal decays monotonically. However, in the finite superconductor, the superconducting order parameter tends to have a symmetric spatial distribution [6]. Then, this effect leads to a non-monotonic decrease of the pair amplitude in the normal metal. In order to see the non-monotonic decrease of the pair amplitude clearly, spatial distributions of the pair amplitude as a function of the $x$ coordinates at the middle of $y$ coordinate ($y/\xi_0 = 2.5$) are shown in figure 2(b). Each pair amplitude is calculated with different potential energies at the S/N interface, $V(x)/E_c = 0.00, 0.30, 0.50,$ and 1.00. Near the S/N interface, the pair potential in the superconductor is suppressed because of the inverse proximity effect. When the potential energy increases, the penetration of the pair amplitude into the normal metal becomes difficult. So, the pair amplitude in the superconductor near the S/N interface becomes large.

In finite systems, the DOS has a non-uniform spatial distribution as well as the pair amplitude. Then, the LDOS in equation (13) is considered instead of the DOS. In order to concentrate on the energy dependence of the LDOS, we consider a spatial-averaged LDOS (SA-DOS). The LDOS in equation (13) is integrated with respect to the space and divided by the area.

$$N_{av}(E) = \frac{1}{S} \int_S n(r, E)dr$$

Now, we calculate the SA-DOS in the superconductor, the normal metal, and the total regions. The total region is a summation of the superconductor and the normal metal regions. $S$ represents each area. The SA-DOS is normalized by $N(0) = \pi/(m^2\hbar^2)$. Figures 3 show SA-DOS as a function of the energy in the superconductor (red), the normal metal (green), and the total (blue) regions with different potential energies at the S/N interface $V(x)/E_c = 0.00(a), 0.30(b), 0.50(c),$ and 1.00(d). All SA-DOS plots have many peaks due to the discreteness of energy levels in nano-structured systems. In the superconductor region, two peaks appear in $|E/\Delta_0| < 1.0$. Because of the proximity effect, some energy states with the energy below the gap energy are
Figure 4. (a) Spatial distribution of the pair amplitude of the SNS junction in the total system for $V(x)/E_c = 0.00$. The $x$-direction width in the normal metal region is $d/\xi_0 = 1.0$. (b) Spatial distributions of the pair amplitude as a function of $x$ coordinate at the middle of $y$ ($y = 2.5\xi_0$) with $x$-direction width in the normal metal region $d/\xi_0 = 1.0$, 2.0, 3.0, and 5.0. Potential energies in all systems are $V(x)/E_c = 0.00$.

Figure 5. Spatial-averaged LDOS (SA-DOS) as the function of the energy in SNS junctions for $d/\xi_0 = 1.0$ (a), 2.0 (b), 3.0 (c), and 5.0 (d). Their lines represent SA-DOS in superconductor (red), normal metal (green), and total (blue) regions, respectively.

allowed in the superconductor. When the potential energy $V(x)/E_c$ increases, the proximity effect is suppressed and peaks in $|E/\Delta_0| < 1.0$ decrease. On the other hand, the SA-DOS in the normal metal region is not small in the energy below the gap energy except for $E/\Delta_0 \sim 0.0$. Near the Fermi energy ($E/\Delta_0 \sim 0.0$), the SA-DOS in the normal region is very small due to the small effect of superconductivity. This result means that Cooper pairs penetrate into the normal region and superconductivity exist weakly.

3.2. Pair amplitude and LDOS in SNS junctions
We consider the superconducting system with two S/N interfaces in figure 1(b), which is the SNS junction. We solve equations (10a)-(10f) in SNS junctions with the different $x$-direction width in the normal metal $d/\xi_0$. Then, the potential energy at S/N interface is fixed to $V(x)/E_c = 0.00$. 
At first, figures 4 show spatial distributions of the pair amplitude in the total region (a) and at the middle of the y coordinate ($y/\xi_0 = 2.5$) for $d/\xi_0 = 1.0, 2.0, 3.0,$ and $5.0$ (b). As well as the case in one S/N interface in figures 2, the pair amplitude is nonzero in the normal metal region. The pair amplitude in the normal metal region is a superposition of the pair amplitude in both sides of superconductor regions. The non-monotonous decrease in the normal metal region is also found clearly when $d/\xi_0$ is large.

Figures 5 show SA-DOS in SNS junctions as a function of the energy for $d/\xi_0 = 1.0$ (a), 2.0 (b), 3.0 (c), and 5.0 (d). As well as figures 3, their lines represent SA-DOS in the superconductor (red), the normal metal (green), and the total (blue) regions, respectively. In the case of the small $d/\xi_0$ such as figure 5 (a), a contribution of the normal metal region is small, so peaks of the SA-DOS in the superconductor region don’t appear at $|E/\Delta_0| < 1.0$ and the SA-DOS in the total region is almost equal to that in the superconductor region. When $d/\xi_0$ increases, the contribution of the normal metal region becomes large and peaks of the SA-DOS in the superconductor region appear at $|E/\Delta_0| < 1.0$ and the SA-DOS in the total region approaches to that in the normal metal region.

4. Summary

We investigate the pair amplitude and the local density of state (LDOS) in two-dimensional nano-structured superconducting systems with superconductor / normal metal (S/N) interfaces. The Bogoliubov-de Gennes equations are solved with the Maxwell equation, the gap equation, and the conservation rule of the particle number self-consistently. We focus on finite-sized superconducting systems with one S/N interface and two S/N interfaces (superconductor / normal metal / superconductor (SNS) junctions). In these systems, the pair amplitude is nonzero even in the normal metal region due to the proximity effect. Also, the non-monotonous decrease of the pair amplitude in the normal metal region is obtained, which results from the effect of the finite system. On the other hand, the spatial-averaged LDOS (SA-DOS) in the nano-sized superconducting system has many peaks due to the discreteness of energy levels. Because of the proximity effect, some peaks of the SA-DOS appear at the energy below the gap energy.

A clear difference from the bulk superconductor is the appearance of many peaks in the SA-DOS by the discreteness of energy levels. It is known that the density of state is also related to the tunnel current in SNS junctions. In general, when the voltage $V$ increases, the tunnel current is zero at $|eV| < 2\Delta_0$, and increases monotonously at $|eV| > 2\Delta_0$. However, in the nano-structured superconductor, it is expected that the tunnel current is also affected by peak structures in the SA-DOS. This is our future work.

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