The Dyson-Schwinger equations and the non-perturbative solution of QED: exploring the two-photon-two-fermion irreducible vertex

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A minimal truncated set of Dyson-Schwinger equations that allow exploring the non-perturbative regime of QED is derived for a general linear covariant gauge. This minimal set includes the propagators, the the photon-fermion, and the two-photon-two-fermion vertices. If the equations for the first three quantities are exact, to build a closed set of equations, the last one is truncated ignoring further higher-order Green functions. We also show that the truncated equation reproduces the lowest order perturbative result for the two-photon-two-fermion vertex in the limit of the small coupling constant. Then, the two-photon-two-fermion irreducible vertex is studied by combining the corresponding Ward-Takahashi identity with the Dyson-Schwinger equation. The longitudinal part of this irreducible vertex is computed from the Ward-Takahashi identity, which is solved in the low energy limit when one of the photons has zero energy (soft photon limit). For this kinematical configuration, the general expression for the vertex is derived and its form factors related to the photon-fermion vertex and fermion propagator.

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Acknowledgments

A. Decomposing the connected Green’s functions

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Quantum Electrodynamics (QED) is one of the first quantum field theories studied by newcomers to the subject. Although the standard textbook approach to QED relies on perturbation theory \[1\], it can be solved with different techniques such as the Dyson-Schwinger equations (DSE) \[2,3\], which are an infinite tower of integral equations, the lattice regularized formulation of QED \[5\] via Monte Carlo simulations or using the functional renormalization group approach, see e.g. \[6\] for introduction and references. In all these cases, one can go beyond perturbation theory which reappears as a special case when the solution is given as a power series in the coupling constant. There are several features of the dynamics of a quantum field theory that cannot be explained within its perturbative solution. For example, the dynamical generation of masses, which also includes dynamical chiral symmetry breaking, can occur in QED \[7–12\], and certainly occurs in QCD \[13,14\], which together with confinement \[16,17\] cannot be understood within perturbation theory.

The Dyson-Schwinger equations are a set of integral equations relating the full set of the QED Green functions \[18\]. From a practical point of view, it is impossible to handle simultaneously all the Green functions and only a subset can be investigated at a time. In principle, by including as many as possible Green functions in the analysis of the DSE, one becomes closer and closer to the exact solution for QED. Of the DSE it is the equation for the fermion propagator, also known as the gap equation, that has been most investigated in the literature, together with the corresponding equation for the photon propagator. However, to build a solution to these equations the photon-fermion vertex needs to be provided \[18–20\].

The fermionic gap equation allows for the dynamical generation of a fermion mass if the coupling constant is sufficiently large \[18\]. On the other hand, the corresponding equation for the photon propagator has a massive photon solution if the aforementioned vertex is singular at zero momentum \[21,22\]. Indeed, the characteristics of the solutions are a function of the photon-fermion vertex used \[23,24\]. Typically, in QED this vertex is built by calling for a solution of the Ward-Takahashi identity (WTI) and adding perturbative corrections. To the best knowledge of the authors, we were not able to find in the literature any attempt to include in the analysis of QED the DSE for the vertex. In principle, the consideration of the vertex DSE would allow also for a solution of the theory that goes beyond perturbation theory. However, the DSE for the vertex also requires the knowledge of the two-photon-two-fermion vertex, whose DSE also needs further higher-order vertices to be solved. From the point of view of building a solution of QED that is written as a power series in \(\alpha\), these higher-order Green functions appear as corrections at higher powers of \(\alpha\) and for a small enough coupling constant they are, in principle, suppressed. On the other hand, by relying on an enlarged set of DSE to solve QED, at least a finite set of these would-be corrections has to be considered. Given that, herein, we aim to address non-perturbative QED through the Dyson-Schwinger equations and to keep a minimal set of Green functions to solve QED beyond perturbation theory, we will focus on the four-point correlation function that represents the two-photon-two-fermion irreducible vertex \(\Gamma^{\mu\nu}\).

The relevance of this irreducible vertex goes beyond its contribution to the solution of the DSE for QED. Indeed, \(\Gamma^{\mu\nu}\) has been considered long ago in the study of non-linear interactions between electromagnetic fields that include light by light scattering, two-quantum pair creation, and the scattering of light by an external electromagnetic field \[25,26\]. It is also responsible for the main contribution of the two-photon exchange corrections to the electron-proton and electron-hadron processes \[27,51\], that determine the nucleon and nuclear form factors, it impacts in the electro-production cross section of resonances and pions, in the \(\gamma Z\) interference in parity-violating electron scattering, etc. The two-photon-two-fermion vertex is also relevant in the computation of the muonic-hydrogen Lamb shift and hyperfine splitting of muonic atoms \[32,36\], in the calculation of the muon magnetic anomaly, see \[37\] and references therein, and for the extraction of the proton charge radius, see \[38\] for a recent review and references therein. The irreducible \(\Gamma^{\mu\nu}\) vertex is also relevant to virtual Compton scattering processes \[39,41\] and the computation and extraction of generalized Parton distributions \[42–44\]. Non-linear electrodynamics effects observed in intense laser beams \[43,46\] have also to consider multiple photon signals and, therefore, have to tack into account \(\Gamma^{\mu\nu}\). Note that, presently, there are ongoing experimental programs to study two-photon processes \[47,50\] among other effects.

In the current work, a theoretical setup based on the use of the DSE to investigate the two-photon-two-fermion vertex from first principles is built. To provide a consistent notation, besides the DSE for this irreducible vertex, we also derive the gap equations for the fermion and for the photon, together with the equation for the photon-fermion vertex. The Dyson-Schwinger equations can be solved within a perturbative type of solution for QED but also allow for a non-perturbative solution of QED.

After setting the DSE for the two-photon-two-fermion Green function, the corresponding DSE is analysed in an
approximated version that, at the lowest order in the coupling constant, reproduces the perturbative result. Besides the derivation of the DSE for \( \Gamma^{\mu\nu} \), the Ward-Takahashi identity for the two-photon-two-fermion irreducible vertex is derived and its implications are explored. We call the reader’s attention that our analysis applies to elementary vertices and not to elements of matrix with composite fermionic particles as considered e.g. in [24, 32, 51]. For these two cases, the WTI have similar structures but differ in the details. A solution of the scalar (contracted) WTI for the longitudinal component of \( \Gamma^{\mu\nu} \), relative to the photon momenta, is built. Moreover, in the low energy limit an exact solution of the WTI for the longitudinal component of \( \Gamma^{\mu\nu} \) is also provided. The solutions express the longitudinal component of \( \Gamma^{\mu\nu} \) in terms of the photon-fermion vertex form factors, of the fermion propagator functions, and derivatives. Furthermore, the usefulness of using a tensor basis to solve the WTI for the two-photon-two-fermion is discussed. As shown below, although the consideration of a tensor basis can be useful for practical calculations, the WTI is not able to determine uniquely the form factors that are associated with the basis elements. This reinforces and suggests that a combination of different results, as the Dyson-Schwinger equations can provide, is required to determine both the longitudinal and transverse parts of \( \Gamma^{\mu\nu} \).

Herein, besides the derivation of the DSE for QED, the decomposition of the connected Green functions in terms of irreducible vertices is worked out. However, given the complexity of the underlying mathematical problem, no attempt is made to solve any of the integral equations. The analysis and computation of the solutions for the DSE is a complex task that will be the subject of future publications.

The current work is organized as follows. In Sec. II we provide the definitions used throughout the paper, including the definitions of the various functional generators and some useful relations. The Dyson-Schwinger equations for the fermion propagator, for the photon propagator, and for the photon-fermion vertex are derived in Sec. III while the Ward-Takahashi identities are computed in Sec. IV. The renormalization program for QED and, in particular, for the DSE is discussed in Sec. V. The constraints due to the WTI for the longitudinal component of the two-photon-two-fermion vertex are discussed in VI. The DSE for \( \Gamma^{\mu\nu} \) is derived and studied in Sec. VII. Finally, in Sec. VIII we summarize and conclude. In the appendices, we collect several auxiliary results that are used through the main text.

II. QUANTUM ELECTRODYNAMICS - DEFINITIONS AND GENERATING FUNCTIONALS

The classical theory of electromagnetism considers the interaction of the four-potential field \( A^\mu(x) = (V(x), \vec{A}(x)) \), where \( V \) is the scalar potential and \( \vec{A} \) the vector potential, with a Dirac spinorial field \( \psi \) that is described by the Lagrangian density

\[
\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(x) \left( i \gamma^\mu D_\mu - m \right) \psi(x),
\]

where the covariant derivative and the Abelian field strength are given by

\[
D_\mu(x) = \partial_\mu + i g A_\mu(x) \quad \text{and} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu
\]

while \( g \) refers to the electric charge. The Lagrangian density \( \mathcal{L} \) is invariant under the gauge transformation

\[
\psi(x) \to e^{i \theta(x)} \psi(x), \quad \bar{\psi}(x) \to \bar{\psi}(x) e^{-i \theta(x)} \quad \text{and} \quad A_\mu(x) \to A_\mu(x) - \frac{i}{g} \partial_\mu \theta(x).
\]

The set of transformations associated with the fermion field defines the \( U(1) \) gauge symmetry group of electromagnetism. The definition of the corresponding quantum field theory requires the introduction of a gauge fixing term to build the generating functional for the Green functions \( Z[J] \). The Faddeev-Popov construction results in

\[
Z[J, \bar{\eta}, \eta] = e^{i W[J, \bar{\eta}, \eta]}
\]

\[
= \int D\bar{\psi} D\psi D\bar{\eta} D\eta \exp \left\{ i \int d^4x \left[ \mathcal{L}_{QED}(A, \bar{\psi}, \psi) + J^\mu(x) A_\mu(x) + \bar{\eta}(x) \psi(x) + \bar{\psi}(x) \eta(x) \right] \right\}
\]

where

\[
\mathcal{L}_{QED} = \mathcal{L} - \frac{1}{2g^2} (\partial A)^2
\]

and the fermion fields \( \psi, \bar{\psi} \) and the fermionic sources \( \eta, \bar{\eta} \) are Grassmann fields. In addition, \( W[J, \bar{\eta}, \eta] \) is the generating functional of the connected Green functions.

In the functional integration over the fundamental fields \( A_\mu, \psi \) or \( \bar{\psi} \), these can be replaced by functional derivatives

\[\text{1} \quad \text{For the derivation with respect to the Grassmann numbers we will consider left derivation.}\]
with respect to the sources following the rules

\[ \psi(x) \leftrightarrow \frac{\delta}{i \delta \eta(x)} , \quad \bar{\psi}(x) \leftrightarrow -\frac{\delta}{i \delta \eta(x)} \quad \text{and} \quad A^\mu(x) \leftrightarrow \frac{\delta}{\delta \mu(x)} . \]  

(6)

By introducing the classical fields

\[ A_{cl, \mu}(x) = \frac{\delta W}{\delta J^\mu(x)} , \quad \bar{\psi}_{cl}(x) = -\frac{\delta W}{\delta \eta(x)} , \quad \psi_{cl}(x) = \frac{\delta W}{\delta \bar{\eta}(x)} \]  

(7)

the generating functional for the one-particle irreducible diagrams is defined via the Legendre transformation

\[ \Gamma[A_{cl}, \bar{\psi}_{cl}, \psi_{cl}] = W[J, \eta, \bar{\eta}] - (J, A_{cl}) - (\eta, \psi_{cl}) - (\bar{\psi}_{cl}, \bar{\eta}) \]  

(8)

where

\[ (J, A_{cl}) = \int d^4x \ J^\mu(x) A_{cl, \mu}(x) , \quad (\eta, \psi_{cl}) = \int d^4x \ \eta_\alpha(x) \psi_{cl, \alpha}(x) \quad \text{and} \quad (\bar{\psi}_{cl}, \eta) = \int d^4x \ \bar{\psi}_{cl, \alpha}(x) \eta_\alpha(x) . \]  

(9)

It follows from the definitions given in Eqs (7) and (8) and the rules of functional derivation that

\[ \frac{\delta \Gamma}{\delta A_{cl, \mu}(x)} = -J^\mu(x) , \quad \frac{\delta \Gamma}{\delta \psi_{cl, \alpha}(x)} = \eta_\alpha(x) , \quad \frac{\delta \Gamma}{\delta \bar{\psi}_{cl, \alpha}(x)} = -\eta_\alpha(x) . \]  

(10)

Furthermore, the classical fields are independent fields which are translated into the relations

\[ \int d^4z \ \frac{\delta^2 W}{\delta J^\mu(x) \delta J^\nu(z)} \frac{\delta^2 \Gamma}{\delta A_{cl, \mu}(y) \delta A_{cl, \nu}(y)} = -g_{\mu\nu} \delta(x - y) , \]  

(11)

\[ \int d^4z \ \frac{\delta^2 W}{\delta \eta_\alpha(z) \delta \eta_\beta(x)} \frac{\delta^2 \Gamma}{\delta \psi_{cl, \alpha}(y) \delta \psi_{cl, \beta}(z)} = -\delta_{\alpha\beta} \delta(x - y) , \]  

(12)

\[ \int d^4z \ \frac{\delta^2 W}{\delta \bar{\eta}_\alpha(z) \delta \eta_\beta(x)} \frac{\delta^2 \Gamma}{\delta \bar{\psi}_{cl, \alpha}(y) \delta \psi_{cl, \beta}(z)} = -\delta_{\alpha\beta} \delta(x - y) , \]  

(13)

that hold in the limit where all the sources vanish. The results in Eqs (11) - (13) also connect the second derivatives of \( \Gamma \) with the inverse of the QED propagators. Indeed, in perturbation theory and at the lowest order in the coupling constant,

\[ W[J, \bar{\eta}, \eta] = -\frac{1}{2} \left( J, D^{(0)} J \right) - \left( \bar{\eta}, S^{(0)} \eta \right) + \mathcal{O}(g) , \]  

(14)

where \( D^{(0)} \) and \( S^{(0)} \) are the tree level photon and fermion propagators, respectively, i.e.

\[ \left( D^{(0)} \right)^{\mu\nu}(x - y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{1}{k^2 + i\epsilon} \left[ -g^{\mu\nu} + (1 - \xi) \frac{k^\mu k^\nu}{k^2} \right] , \]  

(15)

\[ S^{(0)}(x - y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \frac{p^\mu + m}{p^2 - m^2 + i\epsilon} \]  

(16)

for a fermion of mass \( m \). Then, it follows from Eq. (11) that

\[ \frac{\delta^2 W}{\delta J^\mu(x) \delta J^\nu(y)} \bigg|_{J=\bar{\eta}=\eta=0} = -\left( D^{(0)} \right)^{\mu\nu}(x - y) + \cdots \]  

(17)

\[ \frac{\delta^2 W}{\delta \eta_\beta(y) \delta \eta_\alpha(x)} \bigg|_{J=\bar{\eta}=\eta=0} = -\frac{\delta^2 W}{\delta \bar{\eta}_\alpha(x) \delta \eta_\beta(y)} \bigg|_{J=\bar{\eta}=\eta=0} = -\left( S^{(0)} \right)_{\alpha\beta}(x - y) + \cdots \]  

(18)

and from Eqs (11) to (13) that the second derivatives of \( \Gamma \) are the inverse of the propagators.

Finally and before proceeding further let us comment on the definition of the Landau gauge, that we take as the limit \( \xi \to 0^+ \) of the linear covariant gauges. From the formal point of view, the Landau gauge within this formalism cannot be treated since there is no inverse for the photon propagator and, therefore, some of the formal manipulations are meaningless if one sets \( \xi = 0 \) from the very beginning. In the following, the expressions for the Landau gauge are therefore derived from those of the linear covariant gauges after setting the gauge fixing parameter to zero.
III. THE DYSON-SCHWINGER EQUATIONS

The Dyson-Schwinger equations are the Green functions quantum equations of motion. They can be used to define the theory beyond perturbation theory, as they are valid for all values of the coupling constants and are an infinite tower of integral equations relating all QED Green functions. In the following, to set the notation and to prepare the discussion of the two-photon-two-fermion one-particle irreducible diagram we will derive the relevant DSE.

A. The fermion gap equation

The derivation of the DSE relies on the vanishing of the functional integration of the derivatives with respect to the fields. In particular, for the fermion propagator one starts from the identity

$$0 = \int \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi \frac{\delta}{\delta \psi_\alpha(x)} \exp\left\{ i \int d^4x \left[ \mathcal{L}_{QED}(A, \bar{\psi}, \psi) + J^\mu(x) A_\mu(x) + \bar{\eta}(x) \psi(x) + \bar{\psi}(x) \eta(x) \right] \right\}$$

and given that

$$\frac{\delta S}{\delta \psi_\alpha(x)} = \frac{\delta}{\delta \psi_\alpha(x)} \int d^4z \mathcal{L}_{QED}(z) = (i \delta - m)_{\alpha\beta} \psi_\beta(x) - g (\gamma^\mu)_{\alpha\beta} A_\mu(x) \psi_\beta(x) ,$$

see Eq. [19], it allows to rewrite Eq. [19] as

$$\left\{ \left[ i \delta - m \right]_{\alpha\beta} \frac{\delta}{\delta \eta_\beta(x)} - g (\gamma^\mu)_{\alpha\beta} \left( \frac{\delta}{\delta \eta_\beta(x)} \right) \left( \frac{\delta}{\delta J^\mu(x)} \right) + \eta_\alpha(x) \right\} Z[J, \bar{\eta}, \eta] = 0 .$$

The gap equation is obtained from this equation after taking its derivative with respect to $i \delta / \delta \eta_\theta(y)$ resulting in

$$\left\{ i \left[ \delta^2 W[J, \bar{\eta}, \eta] \right]_{\alpha\beta} \frac{\delta^2}{\delta \eta_\theta(y) \delta \eta_\theta(x)} - g (\gamma^\mu)_{\alpha\beta} \left( \frac{\delta^2 W[J, \bar{\eta}, \eta]}{\delta \eta_\theta(y) \delta \eta_\theta(x) \delta J^\mu(x)} \right) + i \delta_{\alpha\theta} \delta(\chi - y) + \cdots \right\} Z[J, \bar{\eta}, \eta] = 0 ,$$

where $\cdots$ represent terms that vanish when the sources are set to zero. Inserting the fermion propagator

$$S_{\alpha\beta}(x - y) = \left. \frac{\delta^2 W[J, \bar{\eta}, \eta]}{\delta \eta_\alpha(x) \delta \eta_\beta(y)} \right|_{J=\bar{\eta}=\eta=0}$$

and the photon propagator

$$\left. \frac{\delta^2 W[J, \bar{\eta}, \eta]}{\delta J^\mu(x) \delta J^\nu(y)} \right|_{J=\eta=\bar{\eta}=0} = - D_{\mu\nu}(x - y)$$

in Eq. [22] it becomes, after some algebra,

$$[S^{-1}(x - z)]_{\alpha\beta} = [i \delta \phi - m]_{\alpha\beta} \delta(x - z) - i g (\gamma^\mu)_{\alpha\alpha'} \int d^4y \frac{\delta^3 W}{\delta J^\mu(x) \delta \eta_{\alpha'}(y) \delta \eta_{\alpha'}(x)} [S^{-1}(y - z)]_{\beta\beta'} .$$
The gap equation can be rewritten using the decomposition of the connect three-point Green function in terms of one-particle irreducible functions, see App. A and in particular Eqs (A3) and (A4), as

\[
S^{-1}(x - y) = \left[ i \partial_x - m \right] \delta(x - y) - i g^2 \int d^4z_1 d^4z_2 \, D_{\mu\nu}(x - z_1) \left[ \gamma^\mu S(x - z_2) \Gamma^\nu(z_2, y; z_1) \right]
\]

where the photon-fermion vertex is defined as

\[
\frac{\delta^3 \Gamma}{\delta A_{\epsilon, \mu}(z) \delta \psi_{\epsilon, \beta}(y) \delta \psi_{\epsilon, \alpha}(x)} \bigg|_{J = \bar{\eta} = \eta = 0} = - g \left( \Gamma^\mu \right)_{\alpha\beta} (x, y; z)
\]

that is represented in Fig. 1 in momentum space - see also Eq. (30). The translation of the gap equation into momentum space uses the definitions

\[
S(x - y) = \int \frac{d^4p}{(2 \pi)^4} \, e^{-i p(x - y)} \, S(p),
\]

\[
D_{\mu\nu}(x - y) = \int \frac{d^4k}{(2 \pi)^4} \, e^{-i k(x - y)} \, D_{\mu\nu}(k),
\]

\[
\Gamma^\mu(x, y; z) = \int \frac{d^4p'}{(2 \pi)^4} \frac{d^4p}{(2 \pi)^4} \frac{d^4k}{(2 \pi)^4} \, e^{-i (p' x + p y + k z)} \left( 2 \pi \right)^4 \delta(p' + p + k) \, \Gamma^\mu(p', p; k)
\]

and, after some algebra, Eq. (26) becomes

\[
S^{-1}(p) = (\slashed{p} - m) - i g^2 \int \frac{d^4k}{(2 \pi)^4} \, D_{\mu\nu}(k) \left[ \gamma^\mu S(p - k) \Gamma^\nu(p - k, -p; k) \right].
\]

The equation above is represented diagramatically in Fig. 2 where the filled blob stands for full vertices or propagators, while the tree level photon-fermion vertex is represented by a dot. The Green functions appearing so far are bare quantities. The renormalization program for QED will be discussed later in Sec. V.

**B. The Dyson-Schwinger equation for the photon propagator**

Similarly as in the derivation of the fermionic gap equation, the starting point to derive the DSE for the photon propagator is the identity

\[
0 = \int DA \, D\bar{\psi} \, D\psi \, \frac{\delta}{\delta A_{\mu}(x)} \exp \left\{ i \int d^4x \left[ \mathcal{L}_{\text{QED}}(A, \bar{\psi}, \psi) + J^\mu(x) A_{\mu}(x) + \bar{\eta}(x) \psi(x) + \bar{\psi}(x) \eta(x) \right] \right\}
\]

\[
= \left\{ \Box_x g^{\mu\nu} - \left( 1 - \frac{1}{\xi} \right) \partial_x^\mu \partial_x^\nu \right\} D_{\mu\nu}(x - y) + i g \left( \gamma^\mu \right)_{\alpha\beta} \frac{\delta}{\delta \eta^\alpha(x)} \left( \frac{\delta}{\delta \bar{\eta}_\beta(x)} \right) + J^\mu(x) \right\} Z[J, \eta, \bar{\eta}].
\]

Performing the functional derivative with respect to \( \delta / \delta J^\nu(y) \) of this equation and after setting the sources to zero one gets

\[
\left[ \Box_x g^{\mu\nu} - \left( 1 - \frac{1}{\xi} \right) \partial_x^\mu \partial_x^\nu \right] D_{\mu\nu}(x - y) + i g \left( \gamma^\mu \right)_{\alpha\beta} \frac{\delta^3 W}{\delta J^\nu(y) \delta \eta^\alpha(x) \delta \bar{\eta}_\beta(x)} - g^{\mu\nu} \delta(x - y) = 0.
\]

In Minkowski space-time, the photon propagator is given by

\[
D_{\mu\nu}(x - y) = \int \frac{d^4k}{(2 \pi)^4} \, e^{-i k \cdot (x - y)} \left( - P^\perp_{\mu\nu}(k) D(k^2) - \frac{\xi}{k^2} P_{\mu\nu}^L(k) \right).
\]
For the linear covariant gauges where \( \xi \neq 0 \), Eq. (33) can be multiplied by the inverse of the photon propagator and, after taking into account the decomposition of the three-point connected Green function \( \Gamma_3 \), one arrives at

\[
(D^{-1})^{\mu\nu}(x-y) = \left[ \Box_x g^{\mu\nu} - \left( 1 - \frac{1}{\xi} \right) \partial_\mu \partial_\nu \right] \delta(x-y)

+ i g^2 \int d^4 x' d^4 x'' \text{Tr} \left[ \gamma_\mu S(x-x') \Gamma^{\nu}(x',x'';y) S(x''-x) \right]
\]

that in momentum space reads

\[
\frac{P_{\mu\nu}^\perp(k)}{D(k^2)} = k^2 P_{\mu\nu}^\perp(k) - i g^2 \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \left[ \gamma_\mu S(p) \Gamma_\nu(p,-p+k;-k) S(p-k) \right]
\]

and performing the contraction of the Lorentz indices it reduces to

\[
\frac{1}{D(k^2)} = k^2 - i \frac{g^2}{3} \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \left[ \gamma_\mu S(p) \Gamma_\nu(p,-p+k;-k) S(p-k) \right].
\]

This equation is represented diagrammatically in Fig. 3. In the derivation of Eq. (37) the photon propagator has to be invertible, a condition that is not fulfilled in the Landau gauge.

In QED the knowledge of the fermion-photon vertex determines univocally the fermion and the photon propagator by solving Eqs. (31) and (37), respectively. Apparently, these equations do not depend on the gauge but, as discussed below for the vertex Dyson-Schwinger equation, the vertex itself has a non-trivial dependence on the gauge fixing parameter that feeds the propagator equations and, in this way, they become functions of \( \xi \) themselves.

C. A Dyson-Schwinger equation for the photon-fermion vertex

As described previously, starting from Eq. (21) and taking an additional functional derivative one arrives at the gap equation (22). A Dyson-Schwinger equation for the vertex can be derived from this last equation by evaluating its functional derivative with respect to \( \delta_i \delta J^\mu(w) \) and taking into account those terms not represented explicitly in Eq. (22). Then, setting the sources to zero, the vertex equation reads

\[
\left[ i \theta_x - m \right]_{\alpha\beta'} \frac{\delta^4 W}{\delta J^\nu(w) \delta \eta^\beta(y) \delta \tilde{\eta}^{\beta'}(x)} + ig (\gamma^\mu)_{\alpha\beta'} \frac{\delta^4 W}{\delta J^\nu(w) \delta J^\mu(x) \delta \eta^\beta(y) \delta \tilde{\eta}^{\beta'}(x)} - g (\gamma^\mu)_{\alpha\beta'} S_{\beta\beta'}(x-y) D_{\mu\nu}(x-w) = 0
\]

with the fermion propagator defined in Eq. (23) and the photon propagator settled in Eq. (24). Note that besides the propagators, the equation requires also the knowledge of the two-photon-two-fermion connected Green function. The decomposition of the Green functions that appear in (38) in terms of one-particle irreducible functions is discussed in App. A; see, in particular, Eqs (A2), (A3), (A5) and the definitions (A6) for the decomposition of the three-point function and Eqs (A7), (A10) and the definitions (A8), (A9) for the decomposition of the four-point function. In momentum space the Dyson-Schwinger equation for the vertex reads

\[
\Gamma^\mu(p,-p-k;k) = \gamma^\mu + i g^2 \int \frac{d^4 q}{(2\pi)^4} D_\xi(q)
\]

\[
\left\{ \left[ \gamma^\xi S(p-q) \Gamma^\mu(p-q,-p-k+q;k) S(p+k-q) \Gamma^\xi(p+k-q,-p-k;q) \right] + \left[ \gamma^\xi S(p-q) \Gamma^\xi(p-q,-p-k;q,k) \right] \right\}.
\]

FIG. 3: The photon gap equation
In the derivation of the last expression the gap equation (31) was used to simplify the final result. Furthermore, the original Dyson-Schwinger equation includes a boson propagator that needs to be canceled out to arrive at Eq. (39). We recall that there is no inverse for the photon propagator in the Landau gauge and, therefore, the cancelation of the propagator is possible only for the linear covariant gauges. The diagramatic representation of Eq. (39) is given in Fig. 4.

The dependence on the gauge parameter $\xi$ of the vertex can be seen explicitly in Eq. (39) plugging the tensor structure of the photon propagator, see Eq. (34), and using also the Ward-Takahashi identities (44) and (45). Then, the DSE (39) becomes

$$\Gamma^\mu(p, -p - k; k) = \gamma^\mu + ig^2 \int \frac{d^4q}{(2\pi)^4} \left\{ -D(q^2) \left( \gamma \zeta S(p - q) \Gamma^\mu(p - q, -p - k + q; k) S(p + k - q) \Gamma^\xi(p + k - q, -p - k; q) \right) 
+ \left[ \gamma \zeta S(p - q) \Gamma^\xi(p - q, -p - k; q, k) \right] \right\} + \left( \frac{D(q^2)}{q^2} - \frac{\xi}{q^4} \right) \left( [\gamma S(p - q)] \Gamma^\mu(p - q, -p - k + q; k) S(p + k - q) \right) \left( S^{-1}(p + k) - S^{-1}(p + k - q) \right) 
+ \left[ \gamma S(p - q) \right] \left( \Gamma^\mu(p, -p - k; k) - \Gamma^\mu(p + q, -p - k - q; k) + 2 S^{-1}(p) S(p + q) \Gamma^\mu(p + q, -p - k - q; k) \right) - 2 \Gamma^\mu(p, -p - k; k) S(p + k) S^{-1}(p + k + q) \right\} . \quad (40)$$

It follows that, at the lowest order in the coupling constant, the solution of the equation is $\Gamma^\mu = \gamma^\mu$ recovering the tree level vertex. Moreover, the naive power count of the argument that appears in the integration anticipate possible infrared divergences that, in principle, can be less severe for the Landau gauge where $\xi = 0$. Note also that if one uses the perturbative result $D(q^2) = 1/q^2$ the second term that appears in the integral vanishes for $\xi = 1$ (Feynman gauge).

**IV. WARD-TAKAHASHI IDENTITIES**

For QED the Ward-Takahashi identities can be derived by taking the variation of the generating functional (4) with respect to an infinitesimal gauge transformation (3) that results in

$$\frac{1}{g\zeta} \partial_{x}^\mu \left( \delta Z[J, \bar{\eta}, \eta] \right) + \frac{1}{g} \left( \partial_{x}^\mu J_{\mu}(x) \right) \left( \delta Z[J, \bar{\eta}, \eta] \right) + \eta_{\alpha}(x) \left( \frac{\delta Z[J, \bar{\eta}, \eta]}{\delta \eta_{\alpha}(x)} \right) + \left( \frac{\delta Z[J, \bar{\eta}, \eta]}{\delta \bar{\eta}_{\alpha}(x)} \right) \eta_{\alpha}(x) = 0 . \quad (41)$$
By differentiating this equation and after setting all the sources to zero, one can arrive at the usual Ward-Takahashi identity for the fermion-photon vertex

\[
\frac{1}{g} \square_x \partial^\mu_x \left( i \delta J^\mu (x) \frac{\delta^2 Z}{\delta \eta_{\beta}(z) \delta \bar{\eta}_{\gamma}(y)} \right) + \delta(x-y) \left( \frac{\delta^2 Z}{\delta \eta_{\beta}(z) \delta \bar{\eta}_{\gamma}(y)} \right) - \delta(x-z) \left( \frac{\delta^2 Z}{\delta \eta_{\beta}(x) \delta \bar{\eta}_{\gamma}(y)} \right) = 0 \quad (42)
\]

and, after further differentiation, to the Ward-Takahashi identity for the two-photon-two-fermion vertex

\[
\frac{1}{g} \square_x \partial^\mu_x \left( i \delta J^\mu (w) \frac{\delta^2 Z}{\delta \eta_{\beta}(z) \delta \bar{\eta}_{\gamma}(y)} \right) + \frac{1}{g} \left( \partial^\nu \delta(x-w) \right) \left( \frac{\delta^2 Z}{\delta \eta_{\beta}(z) \delta \bar{\eta}_{\gamma}(y)} \right) + \delta(x-y) \left( \frac{\delta^2 Z}{\delta J^\nu (w) \delta \eta_{\beta}(z) \delta \bar{\eta}_{\gamma}(x)} \right) - \delta(x-z) \left( \frac{\delta^2 Z}{\delta J^\nu (w) \delta \eta_{\beta}(x) \delta \bar{\eta}_{\gamma}(y)} \right) = 0 \quad (43)
\]

The identities given in Eqs (42) and (43) can be written in terms of one-particle irreducible Green functions and in momentum space following the usual rules, see (28) - (30) for example, and taking into account the results and definitions given in App. A. Indeed, from a straightforward algebra Eq. (42) becomes

\[
k_\mu \Gamma^\mu(p, -p - k; k) = S^{-1}(p + k) - S^{-1}(p) \quad (44)
\]

while Eq. (43) results in

\[
k_\mu \Gamma^{\mu\nu}(p, -p - k - q; k, q) = \Gamma^\nu(p, -p - q; q) - \Gamma^\nu(p + k, -p - k - q; q) + 2 S^{-1}(p) S(p + k) \Gamma^\nu(p + k, -p - k - q; q) - 2 \Gamma^\nu(p, -p - q; q) S(p + q) S^{-1}(p + k + q) \quad (45)
\]

Let us pause for a moment and comment what we obtained above, the expressions in momentum space are derived for the linear covariant gauges that profit from the existence of an inverse photon propagator. Also, we call the reader’s attention to the factor two that appears in the last two terms of Eq. (45) that, to the best knowledge of the authors, does not appear in previous versions of the Ward-Takahashi identity found in the literature, see e.g. [29, 30]. The factor two comes from the contribution of the fermionic part in Eq. (11) and, therefore, is associated with the gauge transformation of the fermionic fields. From the viewpoint of an effective coupling of a fermion to the electromagnetic field, which in principle can contain higher-order terms in $A_\mu$ also via the Maxwell tensor $F^{\mu\nu}$, this factor of two does not show up if the electric charge of the particle vanishes. In principle, the Ward-Takahashi identity can have terms coming from the new effective interactions, but the lowest order term contribution is represented in after replacing the two with a one. In this sense, for example, the Ward-Takahashi identity distinguishes between a charged particle, let us say a proton, and a chargeless composite particle, as a neutron.

The second and third terms in the Ward-Takahashi identity are proportional to fermion propagators associated with different momenta, respectively $p + k$ and $p + q$. These terms become dominant if the corresponding momenta refer to on-shell fermions and, in these cases, the identity is well approximated by

\[
k_\mu \Gamma^{\mu\nu}(p, -p - k - q; k, q)_{p + k \text{ on shell}} \approx 2 S^{-1}(p) S(p + k) \Gamma^\nu(p + k, -p - k - q; q) \quad (46)
\]

\[
k_\mu \Gamma^{\mu\nu}(p, -p - k - q; k, q)_{p + q \text{ on shell}} \approx -2 \Gamma^\nu(p, -p - q; q) S(p + q) S^{-1}(p + k + q) \quad (47)
\]

The on-shell approximations can be used to simplify the analysis of the Dyson-Schwinger equations discussed below for 4-point and above for 3-point DSE.

V. RENORMALIZATION

So far the Dyson-Schwinger equations involve only bare quantities being the Green functions, the mass or the coupling constant. One has to go through the renormalization procedure to write the equations in terms of physical quantities. In this section, the use of the index “phys” refers to physical quantities, while the bare quantities have no index.

---

2 In an effective field theory (EFT) approach, the operators $\mathcal{O}_i$ in $\mathcal{L}$ are classified according to their mass dimension and appear as $\mathcal{O}_i/\Lambda^\alpha$, where $\Lambda$ is a typical mass scale associated with the theory and $\alpha$ refers to the mass dimension of the operator. Our reasoning applies not to the expansion in the coupling constant but to the expansion in powers of $1/\Lambda$. Note that they are not necessarily independent.
For QED the renormalization constants, generically named \( Z \), are

\[
A_\mu = Z_A^{\frac{1}{3}} A_\mu^{(\text{phys})}, \quad \psi = Z_2^{\frac{1}{2}} \psi^{(\text{phys})}, \quad g = \frac{Z_1}{Z_2} g^{(\text{phys})}, \quad m = \frac{Z_0}{Z_2} m^{(\text{phys})} \quad \text{and} \quad \xi = Z_3 \xi^{(\text{phys})}. \tag{48}
\]

These definitions imply in the following relations between bare and renormalized propagators

\[
D_{\mu\nu}(k^2) = Z_3 D_{\mu\nu}^{(\text{phys})}(k^2) \quad \text{and} \quad S(p) = Z_2 S^{(\text{phys})}(p) \tag{49}
\]

and, from the definitions Eqs (A4) and (A8), one has

\[
\Gamma_\mu = \frac{\Gamma_\mu^{(\text{phys})}}{Z_1} \quad \text{and} \quad \Gamma_{\mu\nu} = \frac{Z_2}{Z_1^2} \Gamma_{\mu\nu}^{(\text{phys})}. \tag{50}
\]

On the other hand, the Ward identities \((44)\) and \((45)\) imply that

\[
\Gamma_\mu = \frac{\Gamma_\mu^{(\text{phys})}}{Z_2} \quad \text{and} \quad \Gamma_{\mu\nu} = \frac{\Gamma_{\mu\nu}^{(\text{phys})}}{Z_2} \tag{51}
\]

and, therefore,

\[
Z_1 = Z_2. \tag{52}
\]

The renormalization program for QED requires three renormalization constants, from three renormalization conditions computed from the fermion gap equation \((51)\), that provide \( Z_1 = Z_2 \) and \( Z_0 \), and the equivalent photon gap equation \((37)\), which determines \( Z_3 \). Note that with the above definitions, the coupling constant is renormalized by \( \sqrt{Z_3} \) and the combination \( g^2 D(k^2) \) is independent of the renormalization scale.

Before proceeding any further, let us write the Dyson-Schwinger equations in their renormalized form where all quantities are finite. To simplify the notation, from now on, in the renormalized equations we will omit the suffix \((\text{phys})\). The renormalized fermion gap equation \((51)\) reads

\[
S^{-1}(p) = Z_2 \bar{\psi} - Z_0 m - i g^2 \int \frac{d^4k}{(2\pi)^4} D_{\mu\nu}(k) \left[ \gamma^\mu S(p-k) \Gamma^{\nu}(p-k, -p; k) \right] \\
= Z_2 \bar{\psi} - Z_0 m - i g^2 \Sigma(p), \tag{53}
\]

while the renormalized photon gap equation \((37)\) is given by

\[
\frac{1}{D(k^2)} = Z_3 k^2 - i \frac{g^2}{3} \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left[ \gamma_\mu S(p) \Gamma^{\mu}(p, -p+k; -k) S(p-k) \right] \\
= Z_3 k^2 - i \frac{g^2}{3} k^2 \Pi(k^2), \tag{54}
\]

where \( \Sigma(p) \) and \( \Pi(k^2) \) are the fermion and the photon self-energies, respectively. Then, the renormalized Dyson-Schwinger equation for the vertex \((30)\) is

\[
\Gamma^\mu(p, -p-k; k) = Z_2 \gamma^\mu + i g^2 \int \frac{d^4q}{(2\pi)^4} D_{\zeta\zeta'}(q) \\
\{ \left[ \gamma^\zeta S(p-q) \Gamma^\mu(p-q, -p-k+q; k) S(p+k-q) \Gamma^{\zeta'}(p+k-q, -p-k; q) \right] \\
+ \left[ \gamma^\zeta S(p-q) \Gamma^{\zeta'}(p-q, -p-k; q, k) \right] \} \\
= Z_2 \gamma^\mu + i g^2 \Delta \Gamma^\mu(p, k). \tag{55}
\]

It is convenient to introduce some extra notation and write the inverse of the fermion propagator as

\[
S^{-1}(p) = A(p^2) \bar{\psi} - B(p^2). \tag{56}
\]
and, therefore,

$$S(p) = Z(p^2) \frac{\not{p} + M(p^2)}{p^2 - M^2(p^2) + i\epsilon}$$

(57)

with $Z(p^2) = 1/A(p^2)$ and $M(p^2) = B(p^2)/A(p^2)$. It is also common practice to write the fermion self-energy as

$$\Sigma(p) = \Sigma_v(p^2) \not{p} + \Sigma_s(p^2)$$

(58)

that allows to split the fermion gap equation into a vectorial and a scalar part as follows

$$A(p^2) = Z_2 - i \frac{g^2}{p^2} \int \frac{d^4k}{(2\pi)^4} D_{\mu\nu}(k) \text{Tr} \left[ \not{\gamma}^{\mu} S(p-k) \Gamma^\nu(p-k, -p; k) \right] = Z_2 - i \frac{g^2}{p^2} \Sigma_v(p^2)$$

(59)

$$B(p^2) = Z_0 m + i g^2 \int \frac{d^4k}{(2\pi)^4} D_{\mu\nu}(k) \text{Tr} \left[ \gamma^\mu S(p-k) \Gamma^\nu(p-k, -p; k) \right] = Z_0 m + i \frac{g^2}{p^2} \Sigma_s(p^2)$$

(60)

One way to define the renormalization constants is choosing renormalization conditions implemented using Eqs (57), (59) and (60). Thus, demanding that

$$A(\mu^2) = 1, \quad B(\mu^2) = m \quad \text{and} \quad D(\mu^2) = \frac{1}{\mu^2}$$

(61)

it comes that

$$Z_2 = 1 + i \frac{g^2}{m} \Sigma_v(\mu^2)$$

(62)

$$Z_0 = 1 - i \frac{g^2}{2m} \Sigma_s(\mu^2)$$

(63)

$$Z_3 = 1 + i \frac{g^2}{3} \Pi(\mu^2)$$

(64)

and the renormalized equations for the propagators are

$$A(p^2) = 1 - i \frac{g^2}{m} \left( \Sigma_v(p^2) - \Sigma_v(\mu^2) \right)$$

(65)

$$B(p^2) = m + i \frac{g^2}{m} \left( \Sigma_s(p^2) - \Sigma_s(\mu^2) \right)$$

(66)

$$\frac{1}{D(k^2)} = k^2 \left( 1 - i \frac{g^2}{3} \left( \Pi(k^2) - \Pi(\mu^2) \right) \right)$$

(67)

Here, we have used the same renormalization scale for the fermion and the photon, but one could use instead different mass scales for the renormalization of fermionic and bosonic fields.

VI. THE TWO-PHOTON-TWO-FERMION ONE-PARTICLE IRREDUCIBLE VERTEX

The one-particle irreducible two-photon-two-fermion vertex, see Eq. (AS) for the definition,

$$\Gamma^{\mu\nu}(x, y; z, w) = -\frac{1}{g^2} \frac{\delta^4 \Gamma}{\delta A_{\mu}(z) \delta A_{\nu}(w) \delta \psi_{\mu}(y) \delta \psi_{\nu}(x)}$$

$$= \int \frac{d^4p'}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} e^{-i(p'x + py + kz + qw)} (2\pi)^4 \delta(p' + p + k + q) \Gamma^{\mu\nu}(p', p; k, q)$$

(68)

is, in momentum space, symmetric under interchange of $(\mu, k) \leftrightarrow (\nu, q)$ and can be decomposed in longitudinal $\Gamma^{\mu\nu}_L$ and orthogonal $\Gamma^{\mu\nu}_T$ components relative to the photon momenta, i.e.

$$\Gamma^{\mu\nu} = \Gamma^{\mu\nu}_L + \Gamma^{\mu\nu}_T$$

and

$$k_\mu \Gamma^{\mu\nu}_T(p', p; k, q) = q_\nu \Gamma^{\mu\nu}_T(p', p; k, q) = 0$$

(69)

The Ward-Takahashi identity (45) constrains only the longitudinal part of $\Gamma^{\mu\nu}$. The diagramatic representation of Eq. (68) can be seen in Fig. 5.
A. The longitudinal component of $\Gamma^\mu\nu$ from the Ward-Takahashi identity

1. Solving the contracted Ward-Takahashi identity

The Ward-Takahashi identity \cite{15} calls for the fermion-photon vertex whose general Dirac and Lorentz’s tensor structure is complex \cite{22, 54, 57} making the construction of a solution for $\Gamma^\mu\nu$ that verifies the identity rather difficult to achieve. In principle, it is possible to consider a tensor basis to describe $\Gamma^\mu\nu$ and use it to solve the Ward-Takahashi identity. For on-shell fermions, this irreducible vertex requires eighteen different tensors and, therefore, eighteen scalar form factors \cite{58, 59}. However, if the fermions are off-shell, the total number of form factors describing $\Gamma^\mu\nu$ is larger and building a solution to the Ward-Takahashi identity \cite{15} becomes difficult. On the other hand, by contracting (45) with $q^\nu$ and considering the photon-fermion vertex Ward-Takahashi identity (44) one arrives at the scalar identity

$$k_\mu q_\nu \Gamma^\mu\nu(p, -p - q - k; k, q) = S^{-1}(p + q) + S^{-1}(p + k) - 3S^{-1}(p) - 3S^{-1}(p + q + k)$$

$$+ 2S^{-1}(p) \left(S(p + q) + S(p + k)\right) S^{-1}(p + q + k)$$

(70)

that suggests the following ansatz

$$\Gamma^\mu\nu_L(p, -p - q - k; k, q) = \frac{k_\mu}{k^2} \left\{ \Gamma^\nu(p, -p - q; q) - \Gamma^\nu(p + k, -p - k - q; q) \right. \right.$$ 

$$\left. + 2S^{-1}(p) \Gamma^\nu(p + k, -p - k - q; q) \right. \right.$$ 

$$\left. - 2\Gamma^\nu(p, -p - q; q) S(p + q) S^{-1}(p + k + q) \right. \right.$$ 

$$+ a \frac{q^\nu}{q^2} \left\{ \Gamma^\mu(p, -p - k; k) - \Gamma^\mu(p + q, -p - k - q; k) \right. \right.$$ 

$$\left. + 2S^{-1}(p) \Gamma^\mu(p + q, -p - k - q; k) \right. \right.$$ 

$$\left. - 2\Gamma^\mu(p, -p - k; k) S(p + k) S^{-1}(p + k + q) \right. \right.$$ 

$$+ b \frac{k_\mu}{k^2} \frac{q^\nu}{q^2} \left(k_\alpha q_\beta \Gamma^{\alpha\beta}(p, -p - q - k; k, q) \right).$$

(71)

The solution of Eq. (70) requires $a = -b = 1$. This solution of the Ward-Takahashi identity suggests that $\Gamma^\mu\nu$ is singular for $k = 0$ or $q = 0$ but this is not the case. Indeed, setting either $k = 0$ or $q = 0$ it comes that the r.h.s. of the double contraction (70) vanishes. Furthermore, for $k = 0$ (or $q = 0$) on the r.h.s. of (71) the first term (second term) and the last term inside the curly brackets also vanish. Assuming that the terms inside the curly brackets are regular and can be written as a Taylor series, it follows that the series starts at order $k$ for the first term, at order $q$ for the second, and at order $kq$ for the last term in (71). This result ensures that the above solution has no poles at zero momentum. Note, however, that the Ward-Takahashi identity and its solution (71) have singular terms that are associated with the poles of the fermion propagators $S(p + q)$ and $S(p + k)$.
2. Low momenta solution of the Ward-Takahashi identity

The discussion at the end of the last section suggests that it is possible to build a solution to the Ward-Takahashi identity for the two-photon-two-fermion vertex for low momenta. Let us now consider such identity \([54]\) at \(q = 0\). For this kinematics, the fermion-photon vertex appearing in the r.h.s. of the identity requires only its soft photon limit \(\Gamma^\mu(p, -p; 0)\). Borrowing the notation of \([57]\) that relies on the Ball-Chiu tensor decomposition of the vertex \([54]\), it follows that

\[
\Gamma^\mu(p, -p; 0) = \lambda_1(p^2, p^2, 0) \gamma^\mu + 4 \lambda_2(p^2, p^2, 0) \not{p} \not{p} - 2 \lambda_3(p^2, p^2, 0) p^\mu
\]

For \(q = 0\) the WTI \([45]\) reads

\[
k_\mu \Gamma^{\mu\nu}(p, -p - k; k, 0) = \Gamma^\nu(p, -p; 0) - \Gamma^\nu(p + k, -p - k; 0) + 2 S^{-1}(p) S(p + k) \Gamma^\nu(p + k, -p - k; 0)
- 2 \Gamma^\nu(p, -p; 0) S(p) S^{-1}(p + k)
= \Gamma^\nu(p + k, -p - k; 0) - \Gamma^\nu(p, -p; 0) + 2 S^{-1}(p) \left( \frac{\partial S(p)}{\partial \zeta} k_\zeta \right) \Gamma^\nu(p + k, -p - k; 0)
- 2 \Gamma^\nu(p, -p; 0) S(p) \left( \frac{\partial S^{-1}(p)}{\partial \zeta} k_\zeta \right) + \cdots
\]

where in the last expression the fermion propagators were expanded as a power series in \(k\) and only the lowest order term in this momentum was considered. Doing the expansion of the first term in the r.h.s. it follows that

\[
k_\mu \Gamma^{\mu\nu}(p, -p - k; k, 0) = k_\mu \left( \Gamma_0 g^{\mu\nu} + \Gamma_1 g^{\mu\nu} \not{p} + \Gamma_2 p^\mu \gamma^\nu + \Gamma_3 \gamma^\mu p^\nu + \Gamma_4 p^\mu p^\nu + \Gamma_5 \not{p} p^\mu p^\nu \right)
\]

and, therefore, a lowest momenta solution of the Ward-Takahashi identity reads

\[
\Gamma_L^{\mu\nu}(p, -p - k; k, 0) = \Gamma_0 g^{\mu\nu} + \Gamma_1 g^{\mu\nu} \not{p} + \Gamma_2 p^\mu \gamma^\nu + \Gamma_3 \gamma^\mu p^\nu + \Gamma_4 p^\mu p^\nu + \Gamma_5 \not{p} p^\mu p^\nu
\]

where the \(\Gamma_i\) are functions of \(\lambda_1, \lambda_2, \lambda_3\) and of the fermion propagator functions \(A, B\) and derivatives. The expressions for these form factors are given in App. \([13]\).

3. Tensor basis for \(\Gamma^{\mu\nu}\) and the solution of the Ward-Takahashi identity

The low energy solution of the Ward-Takahashi identity \([45]\) discussed in the previous subsection relates the longitudinal component of \(\Gamma^{\mu\nu}\) with the photon-fermion vertex and the propagators for particular kinematics. In principle, this can be done for general kinematics by considering the general parameterization of the photon-fermion vertex itself. This latter vertex is described by twelve form factors that are associated with eight orthogonal form factors \(\tau_i\), relative to the photon momentum, and four longitudinal form factors \(\lambda_i\). Following \([54]\), the photon-fermion vertex can be written as

\[
\Gamma^\mu(p_2, p_1; q) = \sum_{i=1}^{8} \lambda_i(p_1^2, p_2^2, q^2) L_{\mu}^{(i)}(p_1, p_2, q) + \sum_{i=1}^{8} \tau_i(p_1^2, p_2^2, q^2) T_{\mu}^{(i)}(p_1, p_2, q)
\]

where \(L_{\mu}^{(i)}\) and \(T_{\mu}^{(i)}\) define a tensor basis and

\[
q^\mu T_{\mu}^{(i)}(p_1, p_2, q) = 0
\]

There are several possible choices for the basis. Here, we will use the basis considered in \([57, 60, 61]\) that takes the Ball-Chiu choice for the longitudinal operators \([54]\) and the Kızılersu-Reenders-Pennington basis \([56]\) for the orthogonal
operators, which is free of kinematical singularities. Thus, the tensor operator basis is

\[ L_\mu^{(1)}(p_1, p_2, q) = \gamma_\mu, \]  
\[ L_\mu^{(2)}(p_1, p_2, q) = (\gamma_1 - \gamma_2)(p_1 - p_2)_\mu, \]  
\[ L_\mu^{(3)}(p_1, p_2, q) = (p_1 - p_2)_\mu, \]  
\[ L_\mu^{(4)}(p_1, p_2, q) = \sigma_{\mu\nu}(p_1 - p_2)_\nu, \]

and where \( \sigma_{\mu\nu} = \frac{1}{2} [\gamma_{\mu}, \gamma_{\nu}] \). The inverse of the fermion propagator requires two functions of the momenta that are defined in Eq. (50). Writing the vertex as in Eq. (75), using the tensor basis (68) - (84) and the fermion decomposition (50), it is possible to compute the r.h.s. of the WTI (45) in terms of the form factors \( \lambda_i, \tau_i, A \) and \( B \). The full expression is lengthy and to simplify the computation of the solution of the WTI the r.h.s. of (45) is decomposed into a linear part, that disregard the contributions that include the fermion propagators, and the two terms that require the functions associated with the propagators. For the linear part, the r.h.s. of the Ward-Takahashi is given by

\[
\Gamma^\mu(p, p - q; q) = \Gamma^\mu(p + k, p - k - q; q) = \\
= p^\mu \left\{ 2L_3 - 2\lambda_3 + \tau_1 q^2 + T_1 [(q \cdot k) + (q \cdot p) - q^2] - T_7 [2(q \cdot k) + 2(q \cdot p) + q^2] \right\} \\
+ k^\mu \left\{ 2L_3 + T_1 [(q \cdot k) + (q \cdot p) - q^2] - T_7 [2(q \cdot k) + 2(q \cdot p) + q^2] \right\} \\
+ q^\mu \left\{ -\lambda_3 - \tau_1 (q \cdot p) + T_1 [(q \cdot k) + (q \cdot p)] - \frac{1}{2} T_7 [2(q \cdot k) + 2(q \cdot p) + q^2] \right\} \\
+ \gamma^\mu \left\{ \lambda_1 - L_1 + \tau_6 [2(q \cdot p) + q^2] - T_6 [2(q \cdot k) + 2(q \cdot p) + q^2] + \tau_3 q^2 - T_3 q^2 \right\} \\
+ \bar{p} \left\{ p^\mu \left[ 4\lambda_2 - 4L_2 - 2\tau_2 q^2 + 2T_2 [(q \cdot k) + (q \cdot p) - q^2] \right] \\
+ k^\mu \left[ 2T_2 [(q \cdot k) + (q \cdot p) - q^2] - 4L_2 \right] \\
+ q^\mu \left[ 2\lambda_2 - 2L_2 - 2\tau_2 (q \cdot p) + 2T_2 [(q \cdot k) + (q \cdot p)] \right] \right\} \\
+ \bar{k} \left\{ k^\mu \left[ 2T_2 [(q \cdot k) + (q \cdot p) - q^2] \right] \\
+ q^\mu \left[ 2T_2 [(q \cdot k) + (q \cdot p)] \right] \right\} \\
+ \bar{q} \left\{ p^\mu \left[ 2\lambda_2 - 2L_2 + T_2 [(q \cdot k) + (q \cdot p) - q^2] - \tau_2 (q \cdot p) + 2T_6 - 2\tau_6 \right] \\
+ k^\mu \left[ T_2 [(q \cdot k) + (q \cdot p) - q^2] + 2T_6 - 2L_2 \right] \\
+ q^\mu \left[ \lambda_2 - L_2 + T_2 [(q \cdot k) + (q \cdot p) - 2T_7 - 2\tau_6 + T_3 - \tau_3] \right] \right\} \\
+ i \epsilon^{\mu\nu\alpha\beta} \gamma_\zeta \gamma_5 p_\alpha q_\beta \left\{ \tau_8 - T_8 \right\}
\]
The above expressions were computed using FeynCalc \[64–66\] software package that runs on Mathematica \[67\]. Similar but much longer expressions can be derived for the two remaining terms. These two terms saturate the WTI when the fermions are on-shell, give a contribution to all the terms appearing in Eq. \[90\], and generate a Dirac pseudo-scalar term not seen in the linear term. The new Dirac pseudo-scalar term contribution reads

\[
i \epsilon^{\mu \nu \alpha \beta} k_\alpha p_\beta q_\zeta \left\{ -T_8 \right\}
\]

\[
+ \sigma^{\mu \zeta} p_\zeta \left\{ 2L_4 - \lambda_4 + \tau_7 [2(q \cdot p) + q^2] - T_7 [2(q \cdot k) + 2(q \cdot p) + q^2] \right\}
\]

\[
+ \sigma^{\mu \zeta} k_\zeta \left\{ 2L_4 - T_7 [2(q \cdot k) + 2(q \cdot p) + q^2] \right\}
\]

\[
+ \sigma^{\mu \zeta} q_\zeta \left\{ L_4 - \frac{1}{2} \lambda_4 + \frac{1}{2} \tau_7 [2(q \cdot p) + q^2] - \frac{1}{2} T_7 [2(q \cdot k) + 2(q \cdot p) + q^2] + \tau_7 - T_7 \right\}
\]

\[
+ \sigma^{\nu \zeta} p_\zeta q_\nu \left\{ p^\mu \left[ T_4 [(q \cdot k) + (q \cdot p) - q^2] + \frac{1}{2} \tau_4 q^2 + 2 \tau_7 - 2 T_7 \right]
\]

\[
+ k^\mu \left[ T_4 [(q \cdot k) + (q \cdot p) - q^2] - 2 T_7 \right]
\]

\[
+ q^\mu \left[ T_4 [(q \cdot k) + (q \cdot p) - q^2] - \frac{1}{2} \tau_4 (q \cdot p) + \tau_7 - T_7 \right] \}
\]

\[
+ \sigma^{\nu \zeta} k_\zeta q_\nu \left\{ p^\mu \left[ T_4 [(q \cdot k) + (q \cdot p) - q^2] - 2 T_7 \right]
\]

\[
+ k^\mu \left[ T_4 [(q \cdot k) + (q \cdot p) - q^2] - 2 T_7 \right]
\]

\[
+ q^\mu \left[ T_4 [(q \cdot k) + (q \cdot p) - T_7] \right] \}
\]

(90)

where

\[
\lambda_i \equiv \lambda_i \left( (p + q)^2, p^2, q^2 \right), \quad (91)
\]

\[
\tau_i \equiv \tau_i \left( (p + q)^2, p^2, q^2 \right), \quad (92)
\]

\[
L_i \equiv \lambda_i \left( (p + k + q)^2, (p + k)^2, q^2 \right), \quad (93)
\]

\[
T_i \equiv \tau_i \left( (p + k + q)^2, (p + k)^2, q^2 \right) \quad (94)
\]

and

\[
\epsilon^{\mu \nu \alpha \beta} = \begin{cases} +1, & \text{if } \{\mu \nu \alpha \beta\} \text{ is an even permutation of } \{0123\}, \\ -1, & \text{if } \{\mu \nu \alpha \beta\} \text{ is an odd permutation of } \{0123\}, \\ 0, & \text{otherwise.} \end{cases} \quad (95)
\]

The above expressions were computed using FeynCalc \[64–66\] software package that runs on Mathematica \[67\]. Similar but much longer expressions can be derived for the two remaining terms. These two terms saturate the WTI when the fermions are on-shell, give a contribution to all the terms appearing in Eq. \[90\], and generate a Dirac pseudo-scalar term not seen in the linear term. The new Dirac pseudo-scalar term contribution reads

\[
i \epsilon^{\mu \nu \alpha \beta} k_\alpha p_\beta q_\zeta \left\{ D_{nlin}(p + k) \left( A(p^2) A((p + k)^2) \left( T_5 + \frac{1}{2} T_7 [2(qk) + 2(qp) + q^2] - L_4 \right) + B(p^2) B((p + k)^2) T_8 \right) \right.
\]

\[
- D_{nlin}(p + q) \left( \tau_5 \left[ B((p + k)^2) A((p + k + q)^2) - A((p + k)^2) B((p + k + q)^2) \right]
\]

\[
- \frac{2}{A(p^2) p^2 - B^2(p^2)} \right\},
\]

(96)

where

\[
D_{nlin}(p) = \frac{2}{A^2(p^2) p^2 - B^2(p^2)},
\]

(97)

but the remaining terms result in rather lengthy expressions that prevent us from writing them explicitly.

A general solution for the WTI can be built if a complete tensor basis, compatible with the boson symmetry and with the discrete symmetries of QED, is considered. As discussed in \[58, 59\], for on-shell fermions a complete
tensor basis for $\Gamma^{\mu\nu}$ has 18 different elements. However, for off-shell fermions further operators compatible with the requirements of parity and charge conjugation [54] conservation are allowed and, therefore, the number of basis elements becomes prohibitive to handle. Furthermore, from the point of view of computing the form factors that are associated with the operators of the tensor basis, the number of constraints coming from the WTI are smaller than the number of, in principle, independent form factors resulting in an underdetermined system of equations with multiple solutions.

To illustrate this point, let us discuss the case of the Dirac scalar operators. Parity conservation requires that

$$\gamma^0 \Gamma^{\mu\nu}(p_1, p_2; k, q) \gamma^0 = \Gamma^{\mu\nu}(p_1, p_2; k, q),$$

while charge conservation asks the following condition

$$C \Gamma^{\mu\nu}(p_1, p_2; k, q) C^{-1} = \Gamma^{\mu\nu}_{T}(p_1, p_2; k, q),$$

where $T$ stands for transpose, to be fulfilled. The vertex has three independent momenta, which we take as the boson momenta $k$ and $q$ and the incoming fermion momentum $p$ and it has to be invariant under exchange of $(k, \mu) \leftrightarrow (q, \nu)$ as required by Bose symmetry. Then, for a Dirac scalar term, there are ten tensor structures that are allowed by the discrete symmetries

$$\begin{align*}
g^{\mu\nu}, \\
k^\mu q^\nu, \\
k^\mu k^\nu \pm q^\mu q^\nu, \\
p^\mu p^\nu, \\
k^\mu p^\nu \pm q^\mu p^\nu, \\
q^\mu p^\nu \pm k^\mu p^\nu,
\end{align*}$$

(100)

that give a contribution to $\Gamma^{\mu\nu}$ as

$$\begin{align*}
A_0 g^{\mu\nu} &+ A_1 k^\mu q^\nu + A_2 q^\mu k^\nu + A_3 p^\mu p^\nu \\
&+ A_4 (k^\mu k^\nu + q^\mu q^\nu) + A_5 (k^\mu p^\nu + q^\mu p^\nu) + A_6 (q^\mu p^\nu + k^\mu p^\nu) \\
&+ B_1 (k^\mu k^\nu - q^\mu q^\nu) + B_2 (k^\mu p^\nu - q^\mu p^\nu) + B_3 (q^\mu p^\nu - k^\mu p^\nu)
\end{align*}$$

(101)

where the $A_i$ and $B_i$ are Lorentz scalar form factors that are functions of the momenta that need to comply with the constraints of Bose symmetry and charge conservation. The WTI constrains only the longitudinal component and, therefore, Eq. (101) should be multiplied by the corresponding projection operator. It follows that

$$\Gamma^{\mu\nu}_L = \left( \frac{A_0}{k^2} + \frac{(qk)}{k^2} A_2 + A_4 + B_4 + \frac{(pk)}{k^2} (A_6 - B_6) \right) k^\mu k^\nu$$

$$\left( A_1 + \frac{(qk)}{k^2} (A_4 - B_4) + \frac{(pk)}{k^2} (A_5 - B_5) \right) k^\mu q^\nu$$

$$\left( \frac{pk}{k^2} A_3 + A_5 + B_5 + \frac{(qk)}{k^2} (A_6 + B_6) \right) k^\mu p^\nu$$

(102)

or

$$k_\mu \Gamma^{\mu\nu}_L = \left( (pk) A_3 + k^2 (A_5 + B_5) + (qk) (A_6 + B_6) \right) p^\nu$$

$$+ \left( A_0 + (qk) A_2 + k^2 (A_4 + B_4) + (pk) (A_6 - B_6) \right) k^\nu$$

$$+ \left( k^2 A_1 + (qk) (A_4 - B_4) + (pk) (A_5 - B_5) \right) q^\nu$$

(103)

and there are many possible solutions for $A_i$ and $B_i$ that comply with the WTI; for example, by considering only the linear part of the Ward-Takahashi identity as given in Eq. (103), only the combinations in parenthesis in (103) can be determined. The same conclusion applies to the remaining Dirac structures for $\Gamma^{\mu\nu}_L$. Our conclusion is that to determine both the longitudinal and transverse parts of $\Gamma^{\mu\nu}$, one must call for extra information.
VII. THE DYSON-SCHWINGER EQUATION FOR $\Gamma^{\mu\nu}$

As discussed in the previous section, the Ward-Takahashi identity for the two-photon-two-fermion one-particle irreducible vertex is not enough to determine $\Gamma^{\mu\nu}_L$. Moreover, the WTI does not constrain its transverse component $\Gamma^{\mu\nu}_T$. Thus, to access $\Gamma^{\mu\nu}$ we turn our attention to the Dyson-Schwinger equation for the two-photon-two-fermion Green function. This equation can be obtained in the usual way and after performing the various functional derivatives one arrives at

$$0 = (i \hat{\phi}_x - m)_{\alpha\beta} \frac{\delta^4 W}{\delta J^\rho(z) \delta J^\nu(w) \delta \eta_\gamma(y) \delta \eta_\beta(x)}$$

$$+ i g (\gamma^\mu)_{\alpha\beta} \frac{\delta^5 W}{\delta J^\rho(z) \delta J^\nu(w) \delta J^\mu(x) \delta \eta_\gamma(y) \delta \eta_\beta(x)}$$

$$- g (\gamma^\mu)_{\alpha\beta} \frac{\delta^3 W}{\delta J^\rho(z) \delta \eta_\gamma(y) \delta \eta_\beta(x)} - g (\gamma^\mu)_{\alpha\beta} \frac{\delta^2 W}{\delta J^\rho(z) \delta J^\nu(w) \delta J^\mu(x)}$$

$$+ \left( i (i \hat{\phi}_x - m)_{\alpha\beta} \frac{\delta^2 W}{\delta \eta_\gamma(y) \delta \eta_\beta(x)} - g (\gamma^\mu)_{\alpha\beta} \frac{\delta^2 W}{\delta J^\mu(x) \delta \eta_\gamma(y) \delta \eta_\beta(x)} - i \delta_{\alpha\gamma} \delta(x - y) \right) \frac{\delta^2 W}{\delta J^\rho(z) \delta J^\nu(w)}$$  \(104\)

where the term in parenthesis is the fermion gap equation \[22\] and, therefore, it vanishes. The last term in Eq. \(104\) before the one proportional to the gap equation is proportional to the three-photon vertex. In QED there are no Green functions with an odd number of photons (Furry’s theorem) and this term can also be ignored. Then, the equation to be solved reads

$$0 = (i \hat{\phi}_x - m)_{\alpha\beta} \frac{\delta^4 W}{\delta J^\rho(z) \delta J^\nu(w) \delta \eta_\gamma(y) \delta \eta_\beta(x)}$$

$$+ i g (\gamma^\mu)_{\alpha\beta} \frac{\delta^5 W}{\delta J^\rho(z) \delta J^\nu(w) \delta J^\mu(x) \delta \eta_\gamma(y) \delta \eta_\beta(x)}$$

$$- g (\gamma^\mu)_{\alpha\beta} \frac{\delta^3 W}{\delta J^\rho(z) \delta \eta_\gamma(y) \delta \eta_\beta(x)} - g (\gamma^\mu)_{\alpha\beta} \frac{\delta^2 W}{\delta J^\rho(z) \delta J^\nu(w) \delta J^\mu(x)}$$

$$+ \left( i (i \hat{\phi}_x - m)_{\alpha\beta} \frac{\delta^2 W}{\delta \eta_\gamma(y) \delta \eta_\beta(x)} - g (\gamma^\mu)_{\alpha\beta} \frac{\delta^2 W}{\delta J^\mu(x) \delta \eta_\gamma(y) \delta \eta_\beta(x)} + i \delta_{\alpha\gamma} \delta(x - y) \right) \frac{\delta^2 W}{\delta J^\rho(z) \delta J^\nu(w)}$$  \(105\)

The solution of Eq. \(105\) for the two-photon-two-fermion vertex, writing all the connect Green functions in terms of one-particle irreducible vertices, includes the five-point connect Green function that appears in the second term of the equation. Its decomposition in terms of irreducible vertices can be achieved following the type of calculations discussed in App A. For the five-point function appearing in \(105\), after performing the necessary functional derivatives and after some algebra, one arrives at

$$\delta^2 W =$$

$$= - \int d^4 u_1 d^4 u_2 d^4 v_1 d^4 v_2 d^4 v_3 \ D_{\rho\rho'}(s - v_1) D_{\nu\nu'}(w - v_2) D_{\mu\mu'}(z - v_3)$$

$$S_{\alpha\nu'}(x - u_1) S_{\beta\beta'}(u_2 - y)$$

$$\frac{\delta^3 \Gamma}{\delta A_{c,\rho'}(v_1) \delta A_{c,\nu'}(v_2) \delta A_{c,\mu'}(v_3) \delta \psi_{c,\beta'}(u_2) \delta \psi_{c,\alpha'}(u_1)}$$

$$- \int d^4 u d^4 v_1 d^4 v_2 d^4 v_3 \ D_{\rho\rho'}(s - v_1) D_{\nu\nu'}(w - v_2) D_{\mu\mu'}(z - v_3)$$

$$\frac{\delta^4 \Gamma}{\delta J^C(u) \delta \eta_\beta(y) \delta \eta_\alpha(x)}$$

$$\frac{\delta \Gamma}{\delta A_C^\rho(v_1) \delta A_C^\nu(v_2) \delta A_C^\mu(v_3) \delta A_C^\alpha(u_1)}$$
\[- \int d^4 u d^4 v_1 d^4 v_2 \quad D_{\rho\rho'}(s - v_1) D_{\nu\nu'}(w - v_2) S_{\alpha\alpha'}(x - u) \]
\[\delta^3 W \quad \delta^4 \Gamma \]
\[- \int d^4 u d^4 v_1 d^4 v_2 \quad D_{\rho\rho'}(s - v_1) D_{\mu\mu'}(z - v_2) S_{\alpha\alpha'}(x - u) \]
\[\delta^3 W \quad \delta^4 \Gamma \]
\[- \int d^4 u d^4 v_1 d^4 v_2 \quad D_{\nu\nu'}(w - v_1) D_{\mu\mu'}(z - v_2) S_{\alpha\alpha'}(x - u) \]
\[\delta^3 W \quad \delta^4 \Gamma \]
\[+ \int d^4 u d^4 v_1 \quad D_{\rho\rho'}(s - v_1) S_{\alpha\alpha'}(x - u) \]
\[\delta^3 \Gamma \]
\[\int d^4 u d^4 v_1 \quad D_{\nu\nu'}(s - v_1) S_{\alpha\alpha'}(x - u) \]
\[\delta^4 \Gamma \]
\[\int d^4 u d^4 v_1 \quad D_{\mu\mu'}(z - v_1) S_{\alpha\alpha'}(x - u) \]
\[\delta^3 \Gamma \]
\[\int d^4 u d^4 v_1 \quad D_{\mu\mu'}(z - v_1) S_{\alpha\alpha'}(x - u) \]
\[\delta^4 \Gamma \]
\[. \quad (106) \]

A direct inspection shows that this decomposition is symmetric under the interchange of any pair of the bosonic degrees of freedom as required by Bose symmetry. The three and four-point connected Green functions are written in terms of one-particle irreducible functions in Eqs (A3) and (A7), respectively; we recall to the reader the definitions given in Eqs (A3), (A7).

In order to arrive at a manageable equation, we take the decomposition of the five-point Green function (106) with some approximations. This decomposition is lengthy and requires a further Green function that was not considered so far. Moreover, to avoid the introduction of new vertices, we truncate Eq. (106) and ignore the terms proportional to the one-particle irreducible five-point Green function and the four-photon irreducible vertex, i.e. one ignores the contribution of the first two lines in the r.h.s. of the equation. Proceeding, as usual, then the Dyson-Schwinger
equation for two-photon-two-fermion one-particle irreducible vertex reads, in momentum space,

\[ \Gamma_{\mu\nu}(p, -p - k - q; k, p) = \]

\[ = Z_{\mu}(p, k) \left[ S(p + k) \Gamma_{\nu}(p + k, -p - k - q; q) \right] + Z_{\nu}(p, q) \left[ S(p + q) \Gamma_{\mu}(p + q, -p - k - q; k) \right] + i g^2 \int \frac{d^4w}{(2\pi)^4} D_{\zeta\zeta'}(w) \gamma^\zeta S(p - w) \]

\[ \{ \Gamma_{\mu}(p - w, -p - k + w; k) S(p + k - w) \]

\[ \left[ \Gamma_{\nu}(p + k - w, -p - k; w) S(p + k) \Gamma_{\nu}(p + k, -p - k - q; q) \right. \]

\[ + \Gamma_{\nu}(p + k + w, -p - k - q + w; q) S(p + k + q + w; q) \Gamma_{\mu}(p + q + w, -p - k - q; w) \]

\[ + \Gamma_{\nu}(p + k - w, -p - k - q; w, q) \left. \Gamma_{\nu}(p + k, -p - k - q; k) \right] \]

\[ + \Gamma_{\nu}(p + q - w, -p - k - q + w; k) S(p + k + q - w; k) \Gamma_{\nu}(p + k + q - w, -p - k - q; w) \]

\[ + \Gamma_{\nu}(p + q - w, -p - k - q; w, k) \]

\[ - \Gamma_{\mu\nu}(p - w, -p - k - q + w; k, q) S(p + k + q - w; k, q) \Gamma_{\nu}(p + k + q - w, -p - k - q; w) \}

\[ - \Gamma_{\mu}(p, -p - k; k) S(p + k) \Gamma_{\nu}(p + k, -p - k - q; q) \]

\[ - \Gamma_{\nu}(p, -p - q; q) S(p + q) \Gamma_{\mu}(p + q, -p - k - q; k) \]

\[ \} \]  \hspace{1cm} (107)

where

\[ Z_{\mu}(p, k) = \gamma^\mu - i g^2 \int \frac{d^4w}{(2\pi)^4} D_{\zeta\zeta'}(w) \gamma^\zeta S(p - w) \Gamma_{\mu\zeta}(p - w, -p - k; k, w) \].  \hspace{1cm} (108)

The definition of \( Z_{\mu} \) replicates the correction of the fermion self-energy after replacing \( \Gamma_{\mu} \) by the one-particle-irreducible vertex \( \Gamma_{\mu\nu} \). Equation (107) is symmetric by an interchange of the bosonic indices as required by Bose statistics. This equation can, in principle, be solved self-consistently as the computation of the r.h.s. requires the knowledge of \( \Gamma_{\mu\nu} \) itself.

A closer look at the loop integrals shows that in almost all cases \( \Gamma_{\mu\nu} \) is combined with a photon propagator as follows

\[ D_{\zeta\zeta'}(w) \Gamma_{\mu\zeta'}(p - w, -p - k; k, w) = \]

\[ = - (P_T)_{\zeta\zeta'}(w) D(w^2) \Gamma_{\mu\zeta'}(p - w, -p - k; k, w) - \frac{\xi}{w^2} (P_L)_{\zeta\zeta'}(w) \Gamma_{\mu\zeta'}(p - w, -p - k; k, w) \]  \hspace{1cm} (109)

where

\[ (P_T)_{\zeta\zeta'}(w) = g_{\zeta\zeta'} - \frac{w_{\zeta} w'_{\zeta}}{w^2} \quad \text{and} \quad (P_L)_{\zeta\zeta'}(w) = \frac{w_{\zeta} w'_{\zeta}}{w^2} \]  \hspace{1cm} (110)

are the transverse and longitudinal momentum projection operators, respectively. It follows that the gauge-dependent part can be simplified with the help of the Ward-Takahashi identity (103). More interesting, in the Landau gauge where \( \xi = 0 \) these terms do not include the contributions that are proportional to the longitudinal part of \( \Gamma_{\mu\nu} \). In such gauge

\[ Z_{\mu}(p, k) = \gamma^\mu + i g^2 \int \frac{d^4w}{(2\pi)^4} D(w^2) \gamma^\zeta S(p - w) \Gamma_{T\zeta}(p - w, -p - k; k, w) \].  \hspace{1cm} (111)
and multiplying Eq. \[107\] by the transverse photon projector one obtains an equation that includes only the transverse part of the two-photon-two-fermion vertex. Moreover, by multiplying Eq. \[107\] with \((P_L)_{\mu\nu}(q) k, k\) one arrives at a lengthy sum rule that combines the fermion-photon and the two-photon-two-fermion vertices that is valid in the Landau gauge.

A. Exploring further the DSE for the two-photon-two-fermion vertex

In Eq. \[107\] we wrote an approximation to the bare DSE for the two-photon-two-fermion irreducible vertex that we explore further in this section. In particular, we aim to make contact with the perturbative solution of QED where all quantities appear as series in the coupling constant and where the lowest order contribution to \(\Gamma^{\mu\nu}\) is of order \(g^2\).

To proceed let us introduce the notation

\[
\Gamma^\mu(p_2, p_1; p_3) = \gamma^\mu + \Delta\Gamma^\mu(p_2, p_1; p_3),
\]

where \(\Delta\Gamma^\mu\) represents the corrections to the tree level vertex. In the perturbative solution of QED, \(\Delta\Gamma^\mu\) starts at order \(g^2\). Indeed, in the lowest order in the coupling constant, see Eq. \[99\], it is given by

\[
\Delta\Gamma^\mu(p, -p - k; k) = i g^2 \int \frac{d^4w}{(2\pi)^4} D_{\zeta\zeta'}(w) \left[ \gamma^\zeta S(p - w) \gamma^\mu S(p + k - w) \gamma^\zeta' \right].
\]

Inserting the definition \[112\] into Eq. \[107\], one arrives at

\[
\Gamma^{\mu\nu}(p, -p - k - q; k, q) = i g^2 \int \frac{d^4w}{(2\pi)^4} D_{\zeta\zeta'}(w) \left[ \gamma^\zeta S(p + q - w) \right. \\
\left. \left\{ \gamma^\nu S(p + q - w) \right\} \gamma^\mu S(p + k + q - w) \gamma^\zeta' \right.
\]

\[
+ \gamma^\mu S(p + k - w) \left[ \gamma^\zeta' S(p + k) \gamma^\nu + \gamma^\nu S(p + k + q - w) \gamma^\zeta' \right]
\]

\[
+ \gamma^\nu S(p + q - w) \Gamma^{\zeta'\nu}(p + q - w, -p - k - q; w, k)
\]

\[
+ \gamma^\mu S(p + k - w) \Gamma^{\zeta'\nu}(p + k - w, -p - k - q; w, q)
\]

\[
- \Gamma^{\zeta'\nu}(p - w, -p - k; k, w) S(p + k) \gamma^\nu
\]

\[
- \Gamma^{\nu\zeta'}(p - w, -p - q; q, w) S(p + q) \gamma^\mu
\]

\[
- \Gamma^{\mu\nu}(p - w, -p - k - q + w; k, q) S(p + k + q - w) \gamma^\zeta'.
\]

where the fermion propagators are full propagators and the contributions proportional to \(\Delta\Gamma^\mu\), that represent higher-order contributions in \(g\) were ignored. This equation shows that the perturbative expansion for \(\Gamma^{\mu\nu}\) starts at order \(g^2\) and, therefore, the lowest order approximation in the coupling constant to the two-photon-two-fermion irreducible vertex is obtained from the above equation ignoring on the r.h.s. the terms proportional to \(\Gamma^{\mu\nu}\), i.e.

\[
\Gamma^{\mu\nu}(p, -p - k - q; k, q) = i g^2 \int \frac{d^4w}{(2\pi)^4} D_{\zeta\zeta'}(w) \gamma^\zeta S(p + q - w)
\]

\[
\left\{ \gamma^\nu S(p + q - w) \gamma^\mu + \gamma^\mu S(p + k - w) \gamma^\nu \right\} S(p + k + q - w) \gamma^\zeta'.
\]

In perturbation theory, this expression represents the box diagram that gives the lowest order contribution to \(\Gamma^{\mu\nu}\).

VIII. SUMMARY AND CONCLUSIONS

The Dyson-Schwinger equations are considered to address QED beyond perturbation theory. The Dyson-Schwinger equations are an infinite tower of integral equations and only an approximate solution can be built by dealing with a
truncated version of the full set of equations. In a first approach, a minimal set of integral equations is considered. The exact and well-known equations for the two-point correlation functions are derived once more, together with the DSE for the photon-fermion vertex. Further, we also provide the renormalized versions of these three DSE.

The DSE for the photon-fermion vertex requires also the two-photon-two-fermion vertex and, to compute this latter Green function, we derive its own DSE, the corresponding Ward-Takahashi identity, and discuss the renormalization of this new equation. In all the cases, the DSE are written in terms of irreducible vertices and for arbitrary linear covariant gauges. This new equation is rather involved and it also requires further higher Green functions to compute $\Gamma^{\mu\nu}$. Then, we propose a truncated version of the exact integral equation that ignores the contribution of the three-photon-two fermion and the four-photon irreducible vertices. In this way, one arrives a closed set of integral equations that require only the fermion and photon propagators, the photon-fermion and the two-photon-two-fermion vertices that, in principle, are manageable. We also show that the truncated two-photon-two-fermion equation reproduces the perturbative result for the vertex in the limit where the coupling constant is small. From the point of view of a perturbative solution, disregarding those higher-order Green functions is equivalent to ignore contributions of order $O(g^3)$, for the three-photon-two-fermion Green function, and $O(g^4)$ for the contribution coming from the four-photon vertex.

The complexity of the integral equations prevent us from attempting to build the solution of the set of integral equations. This will be the subject of future publications. However, we explore the Ward-Takahashi identity for the two-photon-two-fermion vertex to compute its longitudinal component. We provide a solution for the scalar WTI, i.e. for the full contracted WTI, that writes the longitudinal component in terms of the fermion propagator and of the photon-fermion vertex. Furthermore, for the soft photon limit, where the momentum of one of the photons vanishes, we are able to solve exactly the WTI and write $\Gamma^{\mu\nu}_L$ in terms of form factors, that are functions of the fermion propagator and of the photon-fermion vertex, that are multiplied either by the metric tensor or by operators that require only the momentum of the incoming fermion. Moreover, we explore if the use of a tensor basis for the two-photon-two-fermion vertex can help in solving the WTI. Our analysis suggests that, although the use of a tensor basis can help in the interpretation of the contributions to this vertex, the WTI certainly provides an important constraint on $\Gamma^{\mu\nu}_L$, that connects this vertex with the photon-fermion vertex, but, in what respects to the tensor basis form factors, it does not determine uniquely the form factors. Then, the computation of $\Gamma^{\mu\nu}$ has to take into account the WTI constraint but it has to be built in in another way as, for example, solving the its own Dyson-Schwinger equation. Last but not least, the set of integral equations considered in this work allows studying the solutions in different gauges and, in this way, allows understanding the features that are gauge independent.

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Appendix A: Decomposing the connected Green’s functions

In this appendix, we discuss the decomposition of the connected Green functions that are required in the current work. Starting from the last relation given in the set of Eqs (10) and taking its second derivative with respect to $A_c$ after $\eta$, after some algebra, and after setting the sources to zero, one arrives at

$$\int d^4 u d^4 v \frac{\delta^2 \Gamma}{\delta \psi_{cl, \mu}(u) \delta \psi_{cl, \alpha}(x)} \frac{\delta^2 \Gamma}{\delta A_{cl, \nu}(z) \delta A_{cl, \nu'}(v)} \frac{\delta^3 W}{\delta J_{\mu'}(v) \delta \eta_{\beta}(y) \delta \eta_{\eta}(u)} = \int d^4 u \frac{\delta^3 \Gamma}{\delta A_{cl, \nu}(z) \delta \psi_{cl, \mu}(u) \delta \psi_{cl, \alpha}(x)} \frac{\delta^2 W}{\delta \eta_{\beta}(y) \delta \eta_{\eta}(u)}.$$  \hspace{1cm} (A1)

This equation can be solved for the connected Green function using the orthogonality relations (11) - (12) and gives

$$\frac{\delta^3 W}{\delta J_{\mu}(z) \delta \eta_{\eta}(y) \delta \eta_{\eta}(x)} = \int d^4 w_1 d^4 w_2 d^4 w_3 \frac{\delta^2 W}{\delta J_{\mu}(z) \delta J_{\mu'}(w_1) \delta \eta_{\eta'}(w_2) \delta \eta_{\eta}(x) \delta \eta_{\beta}(y) \delta \eta_{\eta}(w_3)} \frac{\delta^2 W}{\delta A_{cl, \nu}(w_1) \delta A_{cl, \nu'}(w_2) \delta \psi_{cl, \alpha}(w_3) \delta \psi_{cl, \alpha'}(w_2)}.$$  \hspace{1cm} (A2)
In terms of the full fermion propagator (23) and the full gauge boson propagator (24), the equation above can be written as

$$
\frac{\delta^4 W}{\delta J_\mu(z) \delta \eta_\beta(y) \delta \eta_\alpha(x)} =
\begin{align*}
g \int d^4w_1 d^4w_2 d^4w_3 \; D_{\mu\nu'}(z-w_1) \; S_{\alpha\alpha'}(x-w_2) \; \left(\Gamma_{\alpha'}(w_2, w_3; w_1) \; S_{\beta\beta'}(w_3-y)\right)
\end{align*}
$$

where the photon-fermion vertex was defined as

$$
\frac{\delta^4 \Gamma}{\delta A_{\epsilon\nu}(z) \delta \psi_{\epsilon}(y) \delta \psi_{\epsilon}(x)} \bigg|_{J=\bar{\eta}=\eta=0} = -g \left(\Gamma_{\alpha'}(x, y; z)\right).
$$

This equation is rewritten in momentum space using the definitions given in (25), (26), and (27) and reads

$$
\frac{\delta^4 W}{\delta J_\mu(z) \delta \eta_\beta(y) \delta \eta_\alpha(x)} =
\begin{align*}
g \int \frac{d^4k}{(2\pi)^4} \; \frac{d^4p'}{(2\pi)^4} \; \frac{d^4p}{(2\pi)^4} \; e^{-ik\cdot z - i(\mathbf{p} - \mathbf{k}) \cdot \mathbf{y} \cdot \mathbf{p}} \; (2\pi)^4 \delta(p' - p + k) \; D_{\mu\nu'}(k) \left[J_{\nu'}(p'; -p; k) \; S(p)\right]_{\alpha\beta}.
\end{align*}
$$

The Dyson-Schwinger equation for the vertex (28) calls for the three-point connected Green function, see Eq. (A3), and the four-point connected Green with two-fermion and two-gauge boson external lines. The latter can be decomposed in terms of irreducible diagrams following the same procedure as for the three-point correlation function. In order to perform such decomposition, one starts again from the last relation in the set of Eqs (10) and takes its third derivative with respect to $A_{cl}$, twice, after $\eta$ getting

$$
0 = \int d^4u \; \frac{\delta^4 \Gamma}{\delta A_{cl,\nu}(z) \delta A_{cl,\nu'(w)} \delta \psi_{cl,\epsilon}(u) \delta \psi_{cl,\alpha}(x)} \frac{\delta^2 W}{\delta \eta_\beta(y) \delta \eta_\beta(u)}
+ \int d^4u d^4v d^4s \; \frac{\delta^4 \Gamma}{\delta A_{cl,\mu}(z) \delta A_{cl,\mu'(w)} \delta A_{cl,\mu'(v)} \delta A_{cl,\mu'(s)} \delta \psi_{cl,\epsilon}(u) \delta \psi_{cl,\alpha}(x)} \frac{\delta^2 W}{\delta J_\nu'(s) \delta J_{\nu'(v)} \delta \eta_\beta(y) \delta \eta_\beta(u)}
- \int d^4u d^4v \; \frac{\delta^4 W}{\delta \psi_{cl,\epsilon}(u) \delta \psi_{cl,\alpha}(x)} \frac{\delta^2 W}{\delta J_\nu'(v) \delta \eta_\beta(y) \delta \eta_\beta(u)}
- \int d^4u d^4v \; \frac{\delta^4 W}{\delta A_{cl,\mu}(z) \delta A_{cl,\mu'}(v) \delta \psi_{cl,\epsilon}(u) \delta \psi_{cl,\alpha}(x)} \frac{\delta^2 W}{\delta J_\nu'(v) \delta \eta_\beta(y) \delta \eta_\beta(u)}.
$$

This relation can be solved with respect to the four-point connected Green function using the orthogonality relations Eqs (11) - (13), the definition of the full fermion propagator (23), the definition of the full gauge boson propagator (24) the fermion-photon vertex as defined in (28) in combination with the decomposition of the connect three-point correlation function given in Eq. (A3). After some straightforward algebra and setting the sources to zero, it follows that the decomposition of the four-point correction function in terms of one-particle irreducible functions is given by

$$
\frac{\delta^4 W}{\delta J_{\mu}(z) \delta J_{\nu}(w) \delta \eta_\beta(y) \delta \eta_\alpha(x)} =
\begin{align*}
g^2 \int d^4u_1 d^4u_2 d^4u_3 d^4u_4 \; D_{\mu\nu'}(z-u_1) \; D_{\nu\nu'}(w-u_2) \; \left[S(x-u_3) \; \Gamma_{\nu'}(u_3, u_4; u_1, u_2) S(u_4-y)\right]_{\alpha\beta}
+ g^2 \int d^4u_1 d^4u_2 d^4u_3 d^4u_4 d^4u_5 d^4u_6 \; D_{\mu\nu'}(z-u_1) \; D_{\nu\nu'}(w-u_2) \; S(x-u_5) \; \Gamma_{\nu'}(u_5, u_6; u_4) \; S(u_6-y) \; \left[D_{\nu'}(u_4-u_3) \; \Gamma_{\nu'}(u_4-u_3) \; D_{\nu'}(u_4-u_3)\right]_{\alpha\beta}
+ g^2 \int d^4u_1 d^4u_2 d^4u_3 d^4u_4 d^4u_5 d^4u_6 \; D_{\mu\nu'}(z-u_1) \; D_{\nu\nu'}(w-u_2) \; S(x-u_3) \; \Gamma_{\nu'}(u_3, u_5; u_2) \; S(u_5-u_6) \; \Gamma_{\nu'}(u_6, u_4; u_1) \; S(u_4-y) \; \left[D_{\nu'}(u_4-u_3) \; \Gamma_{\nu'}(u_4-u_3) \; D_{\nu'}(u_4-u_3)\right]_{\alpha\beta}
+ g^2 \int d^4u_1 d^4u_2 d^4u_3 d^4u_4 d^4u_5 d^4u_6 \; D_{\mu\nu'}(z-u_1) \; D_{\nu\nu'}(w-u_2) \; S(x-u_3) \; \Gamma_{\nu'}(u_3, u_5; u_1) \; S(u_5-u_6) \; \Gamma_{\nu'}(u_6, u_4; u_2) \; S(u_4-y) \; \left[D_{\nu'}(u_4-u_3) \; \Gamma_{\nu'}(u_4-u_3) \; D_{\nu'}(u_4-u_3)\right]_{\alpha\beta}
\end{align*}
$$
where the following notation was introduced

\[ \frac{\delta^4 \Gamma}{\delta A_{ct,\mu}(x) \delta A_{ct,\nu}(y) \delta \psi_{ct,\beta}(z) \delta \psi_{ct,\alpha}(w)} = - g^2 (\Gamma_{\mu\nu})_{\alpha\beta} (w; z; x, y) , \]  

(A8)  

\[ \frac{\delta^4 \Gamma}{\delta A_{ct,\mu}(x) \delta \psi_{ct,\nu}(y) \delta A_{ct,\alpha}(z)} = - g (\Gamma_{\mu\nu}) (x, y, z) \]  

(A9)  

and the vertices with multiple bosonic lines are symmetric under interchange of boson indices. Following the same rules as those used to write Eq. (A3) in momentum space, see Eq. (A5), the decomposition of the connect four-point Green function (A7) in momentum space reads

\[ \frac{\delta^4 W}{\delta J_{\mu}(z) \delta J_{\nu}(w) \delta \bar{\eta}_{\beta}(y) \delta \bar{\eta}_{\alpha}(x)} = g^2 \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \frac{d^4 p}{(2\pi)^4} e^{-ikz -iqw -ip'x +ipy} (2\pi)^4 \delta(p' - p + q + k) D_{\mu\nu}(k) D_{\nu\nu}(q) \]

\[ \left\{ \Gamma_{\mu'\nu'\zeta}(q, k, -q - k) D_{\zeta\zeta}(p' - p) \left[ S(p') \Gamma_{\zeta}(p', -p; p - p') S(p) \right]_{\alpha\beta} \right. \]

\[ + \left. \left[ S(p') \Gamma_{\mu'}(p', -p' - q; q) S(p' + q) \Gamma_{\nu'}(p' + q, -p; k) S(p) \right]_{\alpha\beta} \right. \]

\[ + \left. \left[ S(p') \Gamma_{\nu'}(p', -p' - k; k) S(p' + k) \Gamma_{\mu'}(p - q, -p; q) S(p) \right]_{\alpha\beta} \right. \]

\[ + \left. \left[ S(p') \Gamma_{\mu'\nu'}(p', -p; q, k) S(p) \right]_{\alpha\beta} \right\} . \]  

(A10)  

Recall that Eqs (A7) and (A10) apply only for \( J = \eta = \bar{\eta} = A_c = \bar{\psi}_c = \bar{\psi}_c = 0 \). Note also that the decompositions (A7) and (A10) of the four-point connect Green function calls for the three gauge boson vertex \( \Gamma_{\mu\nu\zeta} \) that in QED vanishes as follows from the Furry theorem.

**Appendix B: The form factors for the low energy solution of the longitudinal part of \( \Gamma_{\mu\nu} \)**

The form factors that describe the solution for \( \Gamma_{\mu\nu} \) given in Eq. (15) are

\[ \Gamma_0(p, -p - k, k, 0) = -2 \lambda_3(p^2) - 4 D(p^2) \lambda_1(p^2) A(p^2) B(p^2) + O(k) , \]  

(B1)  

\[ \Gamma_1(p, -p - k, k, 0) = 4 \lambda_2(p^2) + 4 D(p^2) \lambda_1(p^2) A^2(p^2) + O(k) , \]  

(B2)  

\[ \Gamma_2(p, -p - k, k, 0) = 2 \lambda_4(p^2) - 4 D(p^2) \lambda_1(p^2) \left( 2 A(p^2) A'(p^2) p^2 - 2 B(p^2) B'(p^2) + A^2(p^2) \right) + O(k) , \]  

(B3)  

\[ \Gamma_3(p, -p - k, k, 0) = 4 \lambda_2(p^2) \]

\[ + 4 D(p^2) \left( 2 \lambda_3(p^2) A(p^2) B(p^2) - 4 \lambda_2(p^2) p^2 A^2(p^2) - \lambda_1(p^2) A^2(p^2) \right) + O(k) , \]  

(B4)  

\[ \Gamma_4(p, -p - k, k, 0) = -4 \lambda_3(p^2) \]

\[ + 8 D(p^2) \left( \lambda_3(p^2) \left( 2 A(p^2) A'(p^2) p^2 - 2 B(p^2) B'(p^2) + A^2(p^2) \right) \right) \]

\[ + \lambda_2(p^2) \left( 4 \left( A(p^2) B'(p^2) - A'(p^2) B(p^2) \right) p^2 - 2 A(p^2) B(p^2) \right) \]

\[ + \lambda_1(p^2) \left( A(p^2) B'(p^2) - A'(p^2) B(p^2) \right) \]  

\[ + O(k) , \]  

(B5)  

\[ \Gamma_5(p, -p - k, k, 0) = 8 \lambda_4(p^2) \]

\[ + 16 D(p^2) \left( \lambda_3(p^2) \left( A'(p^2) B(p^2) - A(p^2) B'(p^2) \right) \right) \]

\[ - 2 \lambda_2(p^2) \left( A(p^2) A'(p^2) - B(p^2) B'(p^2) \right) \]  

\[ + O(k) \]  

(B6)
where $A(p^2)$ and $B(p^2)$ are the functions that define the fermion propagator, see Eqs. (50) and (57), $\lambda_i(p^2)$ are the form factors appearing in (72),

$$D(p^2) = \frac{1}{A^2(p^2) p^2 - B^2(p^2)} ,$$  \hspace{1cm} (B7)

$$A'(p^2) = \frac{dA(p^2)}{dp^2} , \quad B'(p^2) = \frac{dB(p^2)}{dp^2}$$  \hspace{1cm} (B8)

and

$$\lambda_i'(p^2) = \left( \frac{\partial \lambda_i(p_1^2, p_2^2, 0)}{\partial p_1^2} + \frac{\partial \lambda_i(p_1^2, p_2^2, 0)}{\partial p_2^2} \right) \bigg|_{p_1^2 = p_2^2 = p^2} .$$ \hspace{1cm} (B9)

The form factors $\Gamma_1$ simplify considerably both when the derivative terms are subleading and/or in the chiral limit where the fermion is massless. For example, assuming that $\lambda_1$ is dominant and that the terms with derivatives of $A$ and $B$ are subleading it follows that

$$\Gamma_0(p, -p - k; k, 0) \approx -4 D(p^2) \lambda_1(p^2) A(p^2) B(p^2) ,$$  \hspace{1cm} (B10)

$$\Gamma_1(p, -p - k; k, 0) = -\Gamma_2(p, -p - k; k, 0) = -\Gamma_3(p, -p - k; k, 0) \approx 4 D(p^2) \lambda_1(p^2) A^2(p^2) ,$$  \hspace{1cm} (B11)

$$\Gamma_4(p, -p - k; k, 0) = \Gamma_5(p, -p - k; k, 0) \approx 0 .$$  \hspace{1cm} (B12)

Finally,

$$\Gamma_L^{\mu
u}(p, -p - k; k, 0) = \Gamma_0 g^{\mu\nu} + \Gamma_1 \left( g^{\mu\nu} \phi \gamma^\nu - \phi^\mu \gamma^\nu - \gamma^\mu \phi^\nu \right) .$$ \hspace{1cm} (B13)

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