Linear Temporal Justification Logics with Past Operators

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Abstract. In this paper we present various temporal justification logics involving both past and future time modalities. We combine Artemov’s logic of proofs with linear temporal logic (with both past and future operators), and establish its soundness and completeness. Then we investigate several principles describing the interaction of justification and time.

1 Introduction

Linear temporal logics of knowledge are useful for reasoning about situations where the knowledge of an agent is changed over time [8, 16, 17]. The temporal component in such systems is usually interpreted over a discrete linear model of time with finite past and infinite future; in this case \((\mathbb{N}, <)\) can be chosen as the flow of time (for a logic of knowledge and branching time see [25]). And the knowledge component is typically modeled using the modal logic \(S5\).

This paper continues the study of temporal justification logics from [5, 6]. Temporal justification logic is a new family of temporal logics of knowledge in which the knowledge of agents is modeled using a justification logic. Justification logics are modal-like logics that provide a framework for reasoning about epistemic justifications (see [3, 4] for a survey). The language of multi-agent justification logics extends the language of propositional logic by justification terms and expressions of the form \([t]_i \varphi\), with the intended meaning “\(t\) is agent \(i\)’s justification for \(\varphi\).” The Logic of Proofs \(LP\) was the first logic in the family of justification logics, introduced by Artemov in [1, 2]. The logic of proofs is a justification counterpart of the modal epistemic logic \(S4\).

It is known that linear temporal logic with only future time operators is weak to fully express some properties of systems, such as unique initial states and synchrony (cf. [16, 10]). Neither of the temporal justification logics of [5] and [6] contains past time operators in their languages. The aim of this paper is to add past time operators to the temporal justification logic of [6], and to study principles describing the interaction of justifications and time.

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2 Language

In the following, let $h$ be a fixed number of agents, $\text{Ag} = \{1, \ldots, h\}$ the set of all agents, $\text{Const}$ a countable set of justification constants, $\text{Var}$ a countable set of justification variables, and $\text{Prop}$ a countable set of atomic propositions.

The set of justification terms $\text{Tm}$ is defined inductively by

$$t ::=} c \ | \ x \ | \ !t \ | \ t + t \ | \ t \cdot t,$$

where $c \in \text{Const}$ and $x \in \text{Var}$.

The set of formulas $\text{Fml}$ is inductively defined by

$$\varphi ::=} P \ | \ \bot \ | \ \varphi \rightarrow \psi \ | \ \bigcirc \varphi \ | \ \Box \varphi \ | \ \varphi U \psi \ | \ \varphi S \psi \ | \ [t]i\varphi,$$

where $i \in \text{Ag}$, $t \in \text{Tm}$, and $P \in \text{Prop}$. The temporal operators $\bigcirc, \Box, U, S$ are respectively called next (or tomorrow), weak previous (or weak yesterday), until, and since. An until formula is a formula of the form $\varphi U \psi$ for some formulas $\varphi$ and $\psi$, and a justification assertion is a formula of the form $[t]i\varphi$ for some formula $\varphi$ and term $t$.

We use the following usual abbreviations:

$$\neg \varphi ::= \varphi \rightarrow \bot \quad \top ::= \bot \quad \varphi \land \psi ::= \neg \neg \varphi \lor \neg \psi$$

$$\varphi \leftrightarrow \psi ::= (\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi)$$

$$\bigcirc \varphi ::= T U \varphi \quad \Box \varphi ::= \neg \bigcirc \neg \varphi$$

$$\Diamond \varphi ::= \bigcirc S \varphi \quad \blacklozenge \varphi ::= \neg \bigcirc \neg \varphi.$$

The temporal operators $\Box, \bigcirc, \Diamond, \blacklozenge$ are respectively called strong previous, always from now on (or henceforth), sometime (or eventuality), has-always-been, and once.

Associativity and precedence of connectives, as well as the corresponding omission of brackets, are handled in the usual manner.

Subformulas are defined as usual. The set of subformulas $\text{Sub}(\chi)$ of a formula $\chi$ is inductively given by:

$$\text{Sub}(P) ::= \{P\}$$

$$\text{Sub}(\bot) ::= \\{\bot\}$$

$$\text{Sub}(\varphi \rightarrow \psi) ::= \{\varphi \rightarrow \psi\} \cup \text{Sub}(\varphi) \cup \text{Sub}(\psi)$$

$$\text{Sub}(\varphi U \psi) ::= \{\varphi U \psi\} \cup \text{Sub}(\varphi) \cup \text{Sub}(\psi)$$

$$\text{Sub}(\varphi S \psi) ::= \{\varphi S \psi\} \cup \text{Sub}(\varphi) \cup \text{Sub}(\psi)$$

$$\text{Sub}([t]i\varphi) ::= \{[t]i\varphi\} \cup \text{Sub}(\varphi)$$

For a set $S$ of formulas, $\text{Sub}(S)$ denotes the set of all subformulas of the formulas from $S$.

The combined language of justification logic and temporal logic allows for expressing some properties of systems that are not expressible in the known logics of knowledge and time. For example,
– “t justifies \( \varphi \) for agent \( i \) until \( \psi \) holds” can be expressed by \( ([t]_{i} \varphi) U \psi \).
– “t justifies \( \varphi \) for agent \( i \) since \( \psi \) holds” can be expressed by \( ([t]_{i} \varphi) S \psi \).
– “t is agent \( i \)'s conclusive evidence that \( \varphi \) is true” can be expressed by \( \Box[\text{t}][\text{i}][\varphi] \) or even by \( \Box [\text{t}][\text{i}][\varphi] \land \Box [\text{t}][\text{i}][\varphi] \).
– “If agent \( i \) knows that \( \varphi \) for reason \( t \), then \( \varphi \) is always true” can be expressed by \( [\text{t}][\text{i}][\varphi] \rightarrow \Box \varphi \).
– “Agent \( i \) will have not forgotten her justification \( t \) for \( \varphi \) by tomorrow, providing she possesses the justification now” can be expressed by \( [\text{t}][\text{i}][\varphi] \land \Box [\text{t}][\text{i}][\varphi] \).
– “Agent \( i \) will learn that \( t \) is a justification for \( \varphi \) tomorrow, but she does not know it now” can be expressed by \( \neg [\text{t}][\text{i}][\varphi] \land \Box [\text{t}][\text{i}][\varphi] \).

More connections between justification and time will be explored in Sections 8, 10, and 11.

3 Axioms

The axiom system for temporal justification logic consists of three parts, namely propositional logic, temporal logic, and justification logic.

Propositional Logic

For propositional logic, we take

1. all propositional tautologies (Taut)

as axioms and the rule modus ponens, as usual:

\[
\frac{\vdash \varphi, \vdash \varphi \rightarrow \psi}{\vdash \psi} \quad \text{(MP)}.
\]

Temporal Logic

For the temporal part, we use a system of [12, 14, 15] and [20, 19, 10] with axioms

Axioms for the future operators:

2. \( \Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi) \) (\( \Box \)-k)
3. \( \Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi) \) (\( \Box \)-k)
4. \( \Box \neg \varphi \leftrightarrow \neg \Box \varphi \) (funk)
5. \( \Box(\varphi \rightarrow \Box \varphi) \rightarrow (\varphi \rightarrow \Box \varphi) \) (ind)
6. \( \varphi U \psi \rightarrow \Box \psi \) (U1)
7. \( \varphi U \psi \leftrightarrow \psi \lor (\varphi \land \Box(\varphi U \psi)) \) (U2)

Axioms for the past operators:

8. \( \Diamond(\varphi \rightarrow \psi) \rightarrow (\Diamond \varphi \rightarrow \Diamond \psi) \) (\( \Diamond \)-k)
9. \( \Diamond(\varphi \rightarrow \psi) \rightarrow (\Diamond \varphi \rightarrow \Diamond \psi) \) (\( \Diamond \)-k)
10. \( \Box \varphi \rightarrow \Box \varphi \) (sw)
11. $\Diamond \Box \bot$ (initial)
12. $\Box (\varphi \to \Box \varphi) \to (\varphi \to \Box \varphi)$ ($\Box$-ind)
13. $\varphi S \psi \to \Diamond \psi$ ($S1$)
14. $\varphi S \psi \leftrightarrow \psi \lor (\varphi \land \Box (\varphi S \psi))$ ($S2$)

Axioms for the interaction of the future and past operators:

15. $\varphi \to \Box \Diamond \varphi$ (FP)
16. $\varphi \to \Diamond \Box \varphi$ (PF)

and rules

\[ \vdash \varphi \quad \vdash \Diamond \varphi \quad \vdash \Box \varphi \quad \vdash \Box \Diamond \varphi \quad \vdash \Diamond \Box \varphi \]

Let $\text{LTL}^P$ denote the axiomatic system given by the above axioms and rules.

Justification Logic

Finally, for the justification logic part, we use a multi-agent version of the Logic of Proofs [2, 7, 13, 26] with axioms

17. $[t]_i(\varphi \to \psi) \to ([s]_i \varphi \to [t \cdot s]_i \psi)$ (application)
18. $[t]_i \varphi \to [t + s]_i \varphi$, $[s]_i \varphi \to [t + s]_i \varphi$ (sum)
19. $[t]_i \varphi \to \varphi$ (reflexivity)
20. $[t]_i \varphi \to [t]_i [t]_i \varphi$ (positive introspection)

and the iterated axiom necessitation rule

\[ \vdash [c_{j_n}]_{i_n} \ldots [c_{j_1}]_{i_1} \varphi \in \text{CS} \quad \text{(iax-nec)} \]

where the constant specification $\text{CS}$ is a set of formulas of the form

$[c_{j_n}]_{i_n} \ldots [c_{j_1}]_{i_1} \varphi$

where $n \geq 1$, $i_1, \ldots, i_n$ are arbitrary agents, $c_{j_n}, \ldots, c_{j_1}$ are justification constants, and $\varphi$ is an axiom instance of propositional logic, temporal logic, or justification logic. Moreover, a constant specification $\text{CS}$ should be downward closed in the sense that whenever $[c_{j_n}]_{i_n} \ldots [c_{j_1}]_{i_1} \varphi \in \text{CS}$, then $[c_{j_{n-1}}]_{i_{n-1}} \ldots [c_{j_1}]_{i_1} \varphi \in \text{CS}$ for $n > 1$.

Definition 3.1. A constant specification $\text{CS}$ for a justification logic $L$ is axiomatically appropriate provided, for every axiom instance $\varphi$ of $L$ and for every $n \geq 1$, and every $i_1, \ldots, i_n \in \text{Ag}$, $[c_{j_n}]_{i_n} \ldots [c_{j_1}]_{i_1} \varphi \in \text{CS}$ for some justification constants $c_{j_n}, \ldots, c_{j_1}$. 
Remark 3.2. It is perhaps worth noting that the temporal justification logics of \([5, 6]\) are formalized using the following axiom necessitation rule

\[
[c]_i \varphi \in CS \quad \vdash \quad [c]_i \varphi \quad (ax\text{-}nec).
\]

We prefer \((i\text{ax-nec})\) to \((ax\text{-}nec)\) because the iterated axiom necessitation rule enables us to prove the internalization property (see Section 9). All the results of this paper, except the results of Section 9, continue to hold if the logics are formalized by the rule \((ax\text{-}nec)\).

For a given constant specification \(CS\), we use \(LPLTL_{CS}^P\) to denote the Hilbert system given by the axioms and rules for propositional logic, temporal logic, and justification logic as presented above. We write \(\vdash_{CS} \varphi\) if a formula \(\varphi\) is derivable in \(LPLTL_{CS}^P\).

The definition of derivation from a set of premises is standard. A formula \(\varphi\) is derivable from the set of assumptions \(\Gamma\), written \(\Gamma \vdash_{CS} \varphi\), iff \(\varphi\) is in \(\Gamma\), or is one of the axioms of \(LPLTL_{CS}^P\), or follows from derivable formulas through applications of the rules \((\text{MP})\), \((ax\text{-}nec)\), and necessitation, where necessitation rules can be applied only to derivations without assumptions. In other words:

\[
\begin{align*}
\varphi \in \Gamma & \quad \Rightarrow \quad \Gamma \vdash_{CS} \varphi, \\
\varphi \in \text{Axiom} & \quad \Rightarrow \quad \Gamma \vdash_{CS} \varphi, \\
\Gamma \vdash_{CS} \chi \quad \Delta \vdash_{CS} \psi & \quad \Rightarrow \quad \Gamma, \Delta \vdash_{CS} \psi, \\
[c]_i \varphi \in CS & \quad \Rightarrow \quad \Gamma \vdash [c]_i \varphi.
\end{align*}
\]

Note that the Deduction Theorem holds in \(LPLTL_{CS}^P\). It is easy to show that:

\[\Gamma \vdash_{CS} \varphi\] iff there exist \(\psi_1, \ldots, \psi_n \in \Gamma\) such that \(\vdash_{CS} (\psi_1 \land \cdots \land \psi_n) \rightarrow \varphi\).

Temporal justification logic \(LPLTL\) of \([6]\) is a fragment of \(LPLTL_{CS}^P\) without past operators \(\oplus\) and \(S\), and without axioms and rules involving past operators.

The axiomatization for linear time temporal logic given in \([12, 14, 15, 18]\) includes the following axioms

\[
\begin{align*}
\square \varphi & \rightarrow (\varphi \land \square \varphi), \\
\Box \varphi & \rightarrow (\varphi \land \Box \varphi).
\end{align*}
\]

The following lemma shows that we do not need these axioms since in our formalization \(\square\) and \(\Box\) are defined operators.

Lemma 3.3. The following formulas are provable in \(LTL^P\):

1. \(\Box \varphi \rightarrow (\varphi \land \Box \varphi)\).
2. \(\Box \varphi \rightarrow \Box \varphi\).
3. $\Box \varphi \rightarrow (\varphi \land \Box \Box \varphi)$.
4. $\Box \varphi \rightarrow \Box \Box \varphi$.

In item 1, $(MP)$ is the only rule that is used in the derivation.

Proof. 1. $\Box \varphi$ stands for $\neg (\top \U \neg \varphi)$. Hence from $(U2)$ we get

$$\neg \varphi \lor \Box (\top \U \neg \varphi) \rightarrow \top \U \neg \varphi.$$ 

Taking the contrapositive yields

$$\neg (\top \U \neg \varphi) \rightarrow (\neg \varphi \lor \Box (\top \U \neg \varphi)).$$

By propositional reasoning and $(fun)$ we get

$$\neg (\top \U \neg \varphi) \rightarrow (\varphi \land \Box (\top \U \neg \varphi)),$$

which is

$$\Box \varphi \rightarrow (\varphi \land \Box \Box \varphi).$$

2. From item 1 and propositional reasoning we get

$$\Box \varphi \rightarrow \varphi \quad (1)$$
$$\Box \varphi \rightarrow \Box \Box \varphi \quad (2)$$

From $(1)$ and $(\Box \text{-nec})$ we get

$$\Box (\Box \varphi \rightarrow \varphi)$$

which, in turn, using $(\Box \text{-k})$ and propositional reasoning gives

$$\Box \Box \varphi \rightarrow \Box \varphi.$$ 

By propositional reasoning and using $(2)$ we obtain

$$\Box \varphi \rightarrow \Box \varphi.$$ 

3. Similar to item 1.

4. Similar to item 2.

Lemma 3.4. The following formulas are provable in $LTL^P$:

1. $\Box \varphi \rightarrow \neg (\top 1)$.
2. $(\Box (\varphi_1 \lor \ldots \lor \varphi_n) \leftrightarrow (\Box \varphi_1 \lor \ldots \lor \Box \varphi_n))$.
3. $(\Box (\varphi_1 \lor \ldots \lor \varphi_n) \leftrightarrow (\Box \varphi_1 \lor \ldots \lor \Box \varphi_n))$.
4. $(\Box (\varphi_1 \lor \varphi_2 \lor \ldots \lor \varphi_{n-1} \lor \varphi_n) \leftrightarrow (\Box \varphi_1 \lor \Box \varphi_2 \lor \ldots \lor \Box \varphi_{n-1} \lor \Box \varphi_n))$.
5. $(\Box (\varphi_1 \land \ldots \land \varphi_n) \leftrightarrow (\Box \varphi_1 \land \ldots \land \Box \varphi_n))$.
6. $(\Box (\varphi_1 \land \ldots \land \varphi_n) \leftrightarrow (\Box \varphi_1 \land \ldots \land \Box \varphi_n))$.
7. $(\Box (\varphi_1 \lor \ldots \lor \varphi_n) \leftrightarrow (\Box \varphi_1 \lor \ldots \lor \Box \varphi_n))$.

Lemma 3.5. The following formulas are provable in $LTL^P$:
Lemma 3.6. The following rules are derivable in LTL$^P$:

$$
\begin{align*}
\vdash \varphi \to \psi & \quad \vdash \varphi \to \Box \varphi \\
\vdash \varphi \to \psi & \quad \vdash \Box \varphi \\
\vdash \varphi \to \Box \psi & \quad \vdash \varphi \to \Box \Box \varphi \\
\end{align*}
$$

Lemma 3.7. The following rules are derivable in LTL$^P$:

$$
\begin{align*}
\chi \to \neg \psi \land \Box \chi & \quad \chi \to \neg \psi \land \Box \chi \\
\chi \to \neg \varphi \land \Box \chi & \quad \chi \to \neg \varphi \land \Box \psi \\
\chi \to \neg \psi \land \Box (\chi \lor (\neg \varphi \land \neg \psi)) & \quad \chi \to \neg \psi \land \Box (\chi \lor (\neg \varphi \land \neg \psi)) \\
\end{align*}
$$

(\mathcal{U} \text{-} R) \quad \text{(S-R)}

4 Maximal consistent sets

All the results of this section hold for extensions of LTL$^P$, i.e. LPLTL, LPLTL$^P$, and all extensions introduced in Sections 8, 9, 11. Let $L$ be an extension of LTL$^P$, and let $\vdash_{CS}$ denote derivability in $L_{CS}$, where CS is a constant specification for $L$.

For a formula $\chi$, let

$$
A_\chi := \text{Sub}(\chi) \cup \text{Sub}(\top \land \psi),
$$

$$
\text{Sub}^+(\chi) := A_\chi \cup \{\neg \psi \mid \psi \in A_\chi\}.
$$

Definition 4.1. Let CS be a constant specification for $L$.

- A set $\Gamma$ of formulas is called $L_{CS}$-consistent (or simply CS-consistent) if $\Gamma \not\vdash_{CS} L$.
- A set $\Gamma$ of formulas is called maximal if it has no $L_{CS}$-consistent proper extension of formulas.
- A set $\Gamma \subseteq \text{Sub}^+(\chi)$ is called $\chi$-maximal if it has no $L_{CS}$-consistent proper extension of formulas from $\text{Sub}^+(\chi)$.

Let $\text{MCS}_\chi$ denote the set of all $\chi$-maximally $L_{CS}$-consistent subsets of $\text{Sub}^+(\chi)$. Note that $\text{MCS}_\chi$ is a finite set.

Let $\text{MCS}$ denote the set of all maximally $L_{CS}$-consistent sets, and for $\Gamma \in \text{MCS}$, let

$$
\mathcal{T} := \Gamma \cap \text{Sub}^+(\chi).
$$

Lemma 4.2.

$$
\text{MCS}_\chi = \{\mathcal{T} \mid \Gamma \in \text{MCS}\}.
$$
Proof. (1) Let $\Delta \in \text{MCS}_X$. Then $\Delta$ can be extended to a maximal CS-consistent set $\Gamma \in \text{MCS}$. It is easy to show that $\Delta = \Gamma \cap \text{Sub}^*(\chi)$, and thus $\Delta = T$. 

(2) It is sufficient to show that for each $\Gamma \in \text{MCS}$ the set $T$ is CS-consistent and $\chi$-maximal. The CS-consistency of $T$ follows from the CS-consistency of $\Gamma$. In order to show the $\chi$-maximality of $T$, suppose towards a contradiction that $T$ has a CS-consistent proper extension $\Sigma \subseteq \text{Sub}^*(\chi)$. Let $\varphi \in \Sigma \setminus T$. Thus $\varphi \notin \Gamma$, and hence $\lnot \varphi \in \Gamma$. Since $\varphi \in \text{Sub}^*(\chi)$ we can distinguish the following cases:

- $\varphi \in A_\chi$. In this case $\lnot \varphi \in \text{Sub}^*(\chi)$, and hence $\lnot \varphi \in T \subseteq \Sigma$, which contradicts $\varphi \notin \Sigma$.
- $\varphi = \lnot \psi$ and $\psi \in A_\chi$. In this case $\lnot \varphi = \lnot \lnot \psi \in \Gamma$, and hence $\psi \in \Gamma$. Thus $\psi \in T \subseteq \Sigma$, which contradicts $\lnot \psi \in \Sigma$.

Lemma 4.3. Let $T \in \text{MCS}_X$.

1. If $T \Vdash_{\text{CS}} \varphi$, then $\lnot \varphi \in \text{Sub}^*(\chi)$, and hence $\lnot \varphi \in T \subseteq \Sigma$, which contradicts $\varphi \notin \Sigma$.
2. $\varphi \in A_\chi$ and $\varphi \notin T$, then $\lnot \varphi \in T$.
3. If $\varphi \in \text{Sub}^*(\chi)$ and $T \Vdash_{\text{CS}} \varphi$, then $\psi \in T$.
4. $\psi \in A_\chi$, $\varphi \in T$, and $\lnot \psi \in \Sigma$, which contradicts $\lnot \psi \in \Sigma$.

Proof. The proof of all items are standard. $\square$

Definition 4.4. The relation $R_\bigcirc$ on $\text{MCS}_X$ is defined as follows:

$XR_\bigcirc Y$ iff there exists $\Gamma, \Delta \in \text{MCS}$ such that $X = T$ and $Y = \Delta$ and $\{ \varphi \mid \bigcirc \varphi \in \Gamma \} \subseteq \Delta$.

Notation. The notation $XR_\bigcirc Y [T, \Delta]$ means that $X, Y \in \text{MCS}_X$, $\Gamma, \Delta \in \text{MCS}$, $X = T$, $Y = \Delta$ and $\{ \varphi \mid \bigcirc \varphi \in \Gamma \} \subseteq \Delta$.

Note that for $\Gamma, \Delta \in \text{MCS}$ if $\{ \varphi \mid \bigcirc \varphi \in \Gamma \} \subseteq \Delta$, then $T R_\bigcirc \Delta$. Hence, if $XR_\bigcirc Y [T, \Delta]$, then $T R_\bigcirc \Delta$. In addition, it is easy to show that if $XR_\bigcirc Y [T, \Delta]$, then $\{ \varphi \mid \bigcirc \varphi \in \Gamma \} \subseteq \Delta$ (see item 1 of Lemma 4.2).

From the above definition we immediately get the following lemma.

Lemma 4.5. The relation $R_\bigcirc$ is serial. That is for each $X \in \text{MCS}_X$, there exists $Y \in \text{MCS}_X$ such that $XR_\bigcirc Y$.

Proof. For $X \in \text{MCS}_X$, by Lemma 4.2 there exists $\Gamma \in \text{MCS}$ such that $X = T$. Let

$\Delta := \{ \varphi \mid \bigcirc \varphi \in \Gamma \}$.

We prove that $\Delta \in \text{MCS}$, and so $T R_\bigcirc \Delta$ as desired. We first show that $\Delta$ is CS-consistent. If $\Delta$ is not CS-consistent, then

$\vdash_{\text{CS}} \varphi_1 \land \ldots \land \varphi_n \rightarrow \bot$

for some $\bigcirc \varphi_1, \ldots, \bigcirc \varphi_n \in \Gamma$. Thus

$\vdash_{\text{CS}} \bigcirc \varphi_1 \land \ldots \land \bigcirc \varphi_n \rightarrow \bot$. 

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Hence $\bigcirc \bot \in \Delta$. Since $\vdash_{\text{CS}} \bigcirc \bot \rightarrow \bot$, we have $\bot \in \Delta$ which is a contradiction.

In order to show the maximality of $\Delta$, suppose towards a contradiction that $\Delta$ has a $\text{CS}$-consistent proper extension $\Sigma$. Let $\varphi \in \Sigma \setminus \Delta$. Thus $\varphi \notin \Delta$, and hence $\neg \bigcirc \varphi \in \Gamma$. From the latter it follows that $\bigcirc \neg \varphi \in \Gamma$, and hence $\neg \varphi \in \Delta \subseteq \Sigma$ which is a contradiction.

\begin{lemma}
Let $XR_\bigcirc Y [\overline{T}, \overline{\Delta}]$.
\begin{enumerate}
\item $\bigcirc \varphi \in \Gamma$ iff $\varphi \in \Delta$.
\item $\varphi \in \Gamma$ iff $\bigcirc \varphi \in \Delta$.
\item $\varphi \in \Gamma$ iff $\bigcirc \bigcirc \varphi \in \Delta$.
\end{enumerate}
\end{lemma}

\begin{proof}
1. The proof of the only if direction follows from the definition of $R_\bigcirc$. For the if direction, suppose that $\varphi \in \Delta$, and suppose towards a contradiction that $\bigcirc \varphi \notin \Gamma$. Thus $\neg \bigcirc \varphi \in \Gamma$, and hence $\bigcirc \neg \varphi \in \Gamma$. Since $\{ \varphi \mid \bigcirc \varphi \in \Gamma \} \subseteq \Delta$, we get $\neg \varphi \in \Delta$, which would contradict the assumption.

2. If $\varphi \in \Gamma$, then by the axiom (FP) we get $\bigcirc \bigcirc \varphi \in \Gamma$, and hence by $\{ \varphi \mid \bigcirc \varphi \in \Gamma \} \subseteq \Delta$ we get $\bigcirc \varphi \in \Delta$. For the converse, suppose that $\bigcirc \varphi \in \Delta$. Then, by item 1, $\bigcirc \bigcirc \varphi \in \Gamma$. Assume to obtain a contradiction that $\varphi \notin \Gamma$. Then $\neg \varphi \in \Gamma$, and by the axiom (FP) we get $\bigcirc \neg \varphi \in \Gamma$. Since $\vdash_{\text{CS}} \bigcirc \neg \varphi \rightarrow \neg \bigcirc \varphi$, we arrive at a contradiction $\neg \bigcirc \varphi \in \Gamma$.

3. The only if direction is obtained from item 2 and axiom (sw). For the converse suppose $\bigcirc \varphi \in \Delta$. By item 1, it follows that $\bigcirc \bigcirc \varphi \in \Gamma$. Assume to obtain a contradiction that $\varphi \notin \Gamma$. Then $\neg \varphi \in \Gamma$, and hence $\bigcirc \neg \varphi \in \Gamma$. Since $\vdash_{\text{CS}} \bigcirc \neg \varphi \rightarrow \neg \bigcirc \varphi$, we arrive at a contradiction $\neg \bigcirc \varphi \in \Gamma$.
\end{proof}

\begin{definition}
Given $X \in \text{MCS}_\chi$, $X$ is called initial if $\bigcirc \bot \in X$.
\end{definition}

\begin{lemma}
Given $X \in \text{MCS}_\chi$, $X$ is not initial iff there exists $Y \in \text{MCS}_\chi$ such that $Y R_\bigcirc X$.
\end{lemma}

\begin{proof}
Suppose that $X \in \text{MCS}_\chi$ is not initial. There is $\Gamma \in \text{MCS}$ such that $X = \overline{T}$. Let $A := \{ \bigcirc \varphi \mid \varphi \in \Gamma \}$. We first prove that $A$ is $\text{CS}$-consistent. If $A$ is not $\text{CS}$-consistent, then

$\vdash_{\text{CS}} \bigcirc \varphi_1 \land \ldots \land \bigcirc \varphi_n \rightarrow \bot$

for some $\varphi_1, \ldots, \varphi_n \in \Gamma$. Using item 1 of Lemma 4.4 we have

$\vdash_{\text{CS}} \bigcirc \bigcirc \varphi_1 \land \ldots \land \bigcirc \bigcirc \varphi_n \rightarrow \bigcirc \bot$.

By axiom (PF)

$\vdash_{\text{CS}} \varphi_1 \land \ldots \land \varphi_n \rightarrow \bigcirc \bot$.

Hence $\bigcirc \bot \in \Gamma$. Since $\bigcirc \bot \in \text{Subf}^+(\chi)$, we have $\bigcirc \bot \in \overline{T}$, contradicting our assumption that $\overline{T}$ is not initial. Thus $A$ is $\text{CS}$-consistent, and it can be extended to a maximally $\text{CS}$-consistent set $\Delta \in \text{MCS}$. It is easy to show that $\Delta R_\bigcirc \overline{T}$. Finally put $Y : = \overline{\Delta}$.  

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Conversely, suppose that there exists \( Y \in \text{MCS}_X \) such that \( YR \in X \). Let \( YR \in X \) \( \bar{\Delta} \). Suppose towards a contradiction that \( \Diamond \bot \in \bar{T} \). By item 3 of Lemma 4.10, we get \( \bot \in \Delta \), which is a contradiction. \( \square \)

**Lemma 4.9.** If \( \bar{T} \) is initial, then \( \Diamond \varphi \not\in \Gamma \) for all formulas \( \varphi \).

*Proof.* Suppose \( \bar{T} \) is initial. Then \( \Diamond \bot \in \Gamma \). Now the result follows from the fact that \( \vdash \text{CS} \Diamond \bot \rightarrow \Diamond \varphi \). \( \square \)

The following lemma provides another helpful characterization for the relation \( R \in \). In fact, this characterization is used as the definition of \( R \in \) in [10].

**Lemma 4.10.** Let \( X,Y \in \text{MCS}_X \). Then

\[
XR \in Y \iff \nabla_{\text{CS}} X \land \neg \Diamond X \land Y.
\]

*Proof.* (\( \Rightarrow \)) Suppose that \( XR \in Y \), and thus \( XR \in Y \) \( \bar{T}, \bar{\Delta} \) for some \( \Gamma, \Delta \in \text{MCS} \).

By item 1 of Lemma 4.9, we get \( \Diamond X \land \bar{\Delta} \in \Gamma \), and hence \( \Diamond \bar{T} \land \Diamond X \land \bar{\Delta} \in \Gamma \). Therefore,

\[
\nabla_{\text{CS}} \neg (\Diamond \bar{T} \land \Diamond X \land \bar{\Delta})
\]

and hence

\[
\nabla_{\text{CS}} \Diamond T \land \neg \Diamond X \land Y.
\]

(\( \Leftarrow \)) Suppose \( \nabla_{\text{CS}} \Diamond X \land \neg Y \land X \). Thus there is \( \Gamma \in \text{MCS} \) such that \( \Diamond X \land Y \land Y \in \Gamma \). Thus \( X \in \bar{T} \), which immediately implies \( X = \bar{T} \), by the \( \chi \)-maximality of \( X \). Now let \( \Delta := \{ \varphi \mid \Diamond \varphi \in \Gamma \} \). From the proof of Lemma 4.5, it follows that \( \Delta \in \text{MCS} \) and \( \bar{T}R_\in X \). On the other hand, if \( \varphi \in Y \), then \( \Diamond \varphi \in \Gamma \), and hence \( \varphi \in \bar{\Delta} \). Thus \( Y \in \bar{\Delta} \), which immediately implies \( Y = \bar{\Delta} \), by the \( \chi \)-maximality of \( Y \).

Therefore, \( XR \in Y \). \( \square \)

**Lemma 4.11.** Let \( X,Y \in \text{MCS}_X \).

\[
XR \in Y \iff \nabla_{\text{CS}} Y \land \Diamond X \land Y.
\]

*Proof.* The proof follows from Lemma 4.10 and the following fact

\[
\vdash \text{CS} \varphi \rightarrow \Diamond \psi \iff \vdash \text{CS} \psi \rightarrow \Diamond \varphi.
\]

\( \square \)

**Lemma 4.12.** Let \( X,Y \in \text{MCS}_X \), \( XR \in Y \), and \( \varphi \in A_X \).

1. If \( X \vdash \text{CS} \Diamond \varphi \), then \( \varphi \in Y \).
2. If \( X \vdash \neg \Diamond \varphi \), then \( \neg \varphi \in Y \).

*Proof.* 1. Suppose toward a contradiction that \( \varphi \notin Y \). Thus, by Lemma 4.3

\[
\neg \varphi \in Y. \text{ Since } X \vdash \text{CS} \Diamond \varphi, \text{ we have } \vdash \text{CS} \Diamond X \rightarrow \Diamond \varphi. \text{ Hence } \vdash \text{CS} \Diamond X \rightarrow \Diamond \neg \varphi.
\]

Therefore \( \vdash \text{CS} \Diamond X \rightarrow \Diamond \neg \varphi \).

which would contradict \( XR \in Y \).
Lemma 4.13. Let $X \in \text{MCS}_\chi$ and let $R_\bigcirc(X) := \{Y \in \text{MCS}_\chi \mid XR_\bigcirc Y\}$. We have

$$\vdash_{cs} \bigwedge X \rightarrow \bigcirc \bigvee \{ \bigwedge Y \mid Y \in R_\bigcirc(X) \}.$$

Proof. By Lemma 4.10, for all $X,Y \in \text{MCS}_\chi$ we have

$$(\text{not } XR_\bigcirc Y) \implies \vdash_{cs} \bigwedge X \rightarrow \neg \bigcirc \bigwedge Y.$$

Thus

$$\vdash_{cs} \bigwedge X \rightarrow \bigwedge \{ \neg \bigcirc \bigwedge Y \mid Y \in \text{MCS}_\chi \text{ and not } XR_\bigcirc Y \}$$

and hence

$$\vdash_{cs} \bigwedge X \rightarrow \neg \bigvee \{ \bigcirc \bigwedge Y \mid Y \in \text{MCS}_\chi \text{ and not } XR_\bigcirc Y \}. \tag{3}$$

We also have (cf. [16, Lemma 4.1])

$$\vdash_{cs} \bigvee \{ \bigwedge Y \mid Y \in \text{MCS}_\chi \}. \tag{4}$$

From (2) by ($\bigcirc$-nec) we get

$$\vdash_{cs} \bigcirc \bigvee \{ \bigwedge Y \mid Y \in \text{MCS}_\chi \}.$$

By item 2 of Lemma 3.4 we get

$$\vdash_{cs} \bigvee \{ \bigcirc \bigwedge Y \mid Y \in \text{MCS}_\chi \}.$$

Hence

$$\vdash_{cs} \bigwedge X \rightarrow \bigvee \{ \bigcirc \bigwedge Y \mid Y \in \text{MCS}_\chi \},$$

from which it follows that

$$\vdash_{cs} \bigwedge X \rightarrow \bigvee \{ \bigcirc \bigwedge Y \mid Y \in \text{MCS}_\chi \text{ and } XR_\bigcirc Y \} \lor \bigvee \{ \bigcirc \bigwedge Y \mid Y \in \text{MCS}_\chi \text{ and not } XR_\bigcirc Y \}.$$ 

By (3) we infer

$$\vdash_{cs} \bigwedge X \rightarrow \bigvee \{ \bigcirc \bigwedge Y \mid Y \in \text{MCS}_\chi \text{ and } XR_\bigcirc Y \}$$

and thus, by item 2 of Lemma 3.4 we get

$$\vdash_{cs} \bigwedge X \rightarrow \bigcirc \bigvee \{ \bigwedge Y \mid Y \in R_\bigcirc(X) \}.$$ 

$\square$

Lemma 4.14. Let $X \in \text{MCS}_\chi$ and let $R^{-1}_\bigcirc(X) := \{Y \in \text{MCS}_\chi \mid YR_\bigcirc X\}$. We have

$$\vdash_{cs} \bigwedge X \rightarrow \bigcirc \bigvee \{ \bigwedge Y \mid Y \in R^{-1}_\bigcirc(X) \}.$$
Proof. First note that if $X$ is initial, i.e. $\bot \in X$, then by Lemma 4.8 the set $R^{-1}(X)$ is empty, and moreover we have

$$\vdash \text{cs} \land X \rightarrow \langle \land Y \mid Y \in \emptyset \rangle,$$

which implies that

$$\vdash \text{cs} \land X \rightarrow \langle \land Y \mid Y \in \emptyset \rangle.$$

Now suppose $X$ is not initial, and thus by Lemma 4.8 there is $Y_0 \in \text{MCS}_\chi$ such that $Y_0R_{\land}X$. By Lemma 4.11, for all $X, Y \in \text{MCS}_\chi$ we have

$$(\text{not } YR_{\land}X) \implies \vdash \text{cs} \land X \rightarrow \neg \land Y.$$

Thus

$$\vdash \text{cs} \land X \rightarrow \land \{ \land Y \mid Y \in \text{MCS}_\chi \text{ and not } YR_{\land}X \},$$

and hence

$$\vdash \text{cs} \land X \rightarrow \land \{ \land Y \mid Y \in \text{MCS}_\chi \text{ and not } YR_{\land}X \}.$$

Therefore

$$\vdash \text{cs} \land X \rightarrow \neg \lor \{ \land Y \mid Y \in \text{MCS}_\chi \text{ and not } YR_{\land}X \} \lor \langle \land Y \mid Y \in \emptyset \rangle.$$  \hfill (5)

From (4) by $(\land \text{- nec})$ we get

$$\vdash \text{cs} \land \langle \land Y \mid Y \in \text{MCS}_\chi \rangle,$$

and thus by item 3 of Lemma 3.4 we have

$$\vdash \text{cs} \lor \{ \land Y \mid Y \in \text{MCS}_\chi \text{ and } Y \neq Y_0 \} \lor \langle \land Y \mid Y \in \emptyset \rangle.$$  \hfill (6)

By (5) and (6) we infer

$$\vdash \text{cs} \land X \rightarrow \lor \{ \land Y \mid Y \in \text{MCS}_\chi \text{ with } YR_{\land}X \text{ and } Y \neq Y_0 \} \lor \langle \land Y \mid Y \in \emptyset \rangle.$$  \hfill (7)

and thus by item 2 of Lemma 3.4

$$\vdash \text{cs} \land X \rightarrow \lor \{ \land Y \mid Y \in \text{MCS}_\chi \text{ with } YR_{\land}X \text{ and } Y \neq Y_0 \} \lor \langle \land Y \mid Y \in \emptyset \rangle.$$  \hfill (8)

Thus, by the axiom $(\text{sw})$, we get

$$\vdash \text{cs} \land X \rightarrow \lor \{ \land Y \mid Y \in \text{MCS}_\chi \text{ with } YR_{\land}X \text{ and } Y \neq Y_0 \} \lor \langle \land Y \mid Y \in \emptyset \rangle.$$  \hfill (9)

Therefore, by item 2 of Lemma 3.4 we get

$$\vdash \text{cs} \land X \rightarrow \lor \{ \land Y \mid Y \in \text{MCS}_\chi \text{ with } YR_{\land}X \},$$

and thus

$$\vdash \text{cs} \land X \rightarrow \langle \land Y \mid Y \in R^{-1}_\land(X) \rangle.$$  \hfill $\Box$

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Definition 4.15. A finite sequence \( (X_0, X_1, \ldots, X_n) \) of elements of \( \text{MCS}_\chi \) is called a \( \varphi U \psi \)-sequence starting with \( X \) if

1. \( X_0 = X \),
2. \( X_j \vDash X_{j+1} \), for all \( 0 \leq j < n \),
3. \( \psi \in X_n \),
4. \( \varphi \in X_j \), for all \( 0 \leq j < n \).

Lemma 4.16. For every \( X \in \text{MCS}_\chi \), if \( \varphi U \psi \in X \), then there exists a \( \varphi U \psi \)-sequence starting with \( X \).

Proof. Let \( X = \overline{T} \), for some \( \Gamma \in \text{MCS} \). Suppose \( \varphi U \psi \in X \) and there exists no \( \varphi U \psi \)-sequence starting with \( X \). Since \( \varphi U \psi \in \text{Subf}^\chi(\chi) \), we have \( \varphi U \psi \in A_\chi \), and thus \( \varphi, \psi \in A_\chi \). We first show that:

\[
-\psi \in X \text{ and } \varphi \in X.
\] (7)

Suppose \( \psi \in X \). Then the sequence \( (X) \) would be a \( \varphi U \psi \)-sequence starting with \( X \), contradicting our assumption. Thus \( \psi \notin X \), and hence by Lemma \ref{lem:4.3} we get \( -\psi \in X \). On the other hand, from \( \varphi U \psi \in T \) it follows that \( \psi \lor (\varphi \land \varphi U \psi) \in \Gamma \). From this we can immediately deduce by \( -\psi \in \Gamma \) that \( \varphi \in \Gamma \), and hence \( \varphi \in X \).

Let \( T_\varphi \) be the smallest set of elements of \( \text{MCS}_\chi \) such that

1. \( X \in T_\varphi \);
2. for each \( Z \in \text{MCS}_\chi \), if there is \( Y \in T_\varphi \) such that \( Y R \varphi Z \) and \( \varphi \in Z \), then \( Z \in T_\varphi \).

First we show that for all \( Z \in T_\varphi \) such that \( Z \neq X \) and \( \varphi \in Z \) there is \( Y \in T_\varphi \) such that \( Y R \varphi Z \). In order to prove this let

\[
T = \{ Z \in T_\varphi \mid Z \neq X, \varphi \in Z, \text{ and there is no } Y \in T_\varphi \text{ such that } Y R \varphi Z \}. \]

If \( T \neq \emptyset \), then \( T_\varphi \setminus T \) is a proper subset of \( T_\varphi \) that satisfies properties 1 and 2. This contradicts the fact that \( T_\varphi \) is the smallest set with properties 1 and 2. Thus \( T = \emptyset \).

From the definition of \( T_\varphi \) and \ref{lem:4.3}, it is not difficult to show that \( \varphi \in Z \) for all \( Z \in T_\varphi \). Thus it follows that for all \( Z \in T_\varphi \) there exists \( X_0, \ldots, X_n \in T_\varphi \), for \( n \geq 0 \), such that \( X_0 = X, X_n = Z, \varphi \in X_0 \cap \ldots \cap X_n \) and \( X_0 R_\varphi \ldots R_\varphi X_n \).

Now we claim that \( -\psi \in Z \), for all \( Z \in T_\varphi \). First note that, by \ref{lem:4.3}, \( -\psi \notin X \). For \( Z \in T_\varphi \) such that \( Z \neq X \), there exists \( X_0, \ldots, X_n \in T_\varphi \), for \( n \geq 0 \), such that \( X_0 = X, X_n = Z, \varphi \in X_0 \cap \ldots \cap X_n \), and \( X_0 R_\varphi \ldots R_\varphi X_n \). Thus \( \psi \notin Z \), since otherwise \( (X_0, \ldots, X_n) \) would be a \( \varphi U \psi \)-sequence starting with \( X \), contradicting our assumption. Therefore, by Lemma \ref{lem:4.3} \( -\psi \in Z \). This completes the proof of the claim.

Let

\[
\rho := \bigvee \{ \bigwedge Y \mid Y \in T_\varphi \}. \]

Using the above claim we get \( \text{i-cs} \rho \rightarrow -\psi \).
Let $Y \in T_U$ and $Z \in \text{MCS}_\chi$ such that $Y R_O Z$. We have either $\varphi \in Z$ or $\varphi \notin Z$. If $\varphi \in Z$, then by property 2 we have $Z \in T_U$, and hence $\vdash_{CS} \land Z \rightarrow \rho$. If $\varphi \notin Z$, then $\neg \varphi \in Z$. In addition, $\psi \notin Z$, since otherwise we get a $\varphi U \psi$-sequence starting with $X$. Thus $\neg \psi \in Z$, and hence $\vdash_{CS} \land Z \rightarrow \neg \varphi \land \neg \psi$.

Thus, for each $Y \in T_U$ and each $Z \in \text{MCS}_\chi$ such that $Y R_O Z$, we have

$$\text{either } \vdash_{CS} \land Z \rightarrow \rho \text{ or } \vdash_{CS} \land Z \rightarrow \neg \varphi \land \neg \psi,$$

and hence,

$$\vdash_{CS} \land Z \rightarrow \rho \lor (\neg \varphi \land \neg \psi). \tag{8}$$

By Lemma 4.13 for each $Y \in T_U$ we have

$$\vdash_{CS} \land Y \rightarrow \circ (\land Z_1 \lor \ldots \lor \land Z_n)$$

such that $Z_i \in \text{MCS}_\chi$ and $Y R_O Z_i$, for $i = 1, \ldots, n$. By (8), we get

$$\vdash_{CS} \land Y \rightarrow \circ (\rho \lor (\neg \varphi \land \neg \psi)),$$

for each $Y \in T_U$. Thus $\vdash_{CS} \rho \rightarrow \circ (\rho \lor (\neg \varphi \land \neg \psi))$. Using (U-R), we obtain $\vdash_{CS} \rho \rightarrow (\varphi U \psi)$. Since $X \in T_U$, this implies $\vdash_{CS} \land X \rightarrow (\varphi U \psi)$, which contradicts the assumption $\varphi U \psi \in X$.

\[\square\]

**Definition 4.17.** A finite sequence $(X_0, X_1, \ldots, X_n)$ of elements of $\text{MCS}_\chi$ is called a $\varphi S \psi$-sequence ending with $X$ if

1. $X_n = X$,
2. $X_j R_O X_{j+1}$, for all $0 \leq j < n$,
3. $\psi \in X_0$,
4. $\varphi \in X_j$, for all $0 < j \leq n$.

**Lemma 4.18.** For every $X \in \text{MCS}_\chi$, if $\varphi S \psi \in X$, then there exists a $\varphi S \psi$-sequence ending with $X$.

*Proof.* Let $X = T$, for some $\Gamma \in \text{MCS}$. Suppose $\varphi S \psi \in X$ and there exists no $\varphi S \psi$-sequence ending with $X$. Since $\varphi S \psi \in \text{SubF}(\chi)$, we have $\varphi, \psi \in A_\chi$. From this, similar to the proof of Lemma 4.13 it is proved that $\varphi, \neg \psi \in X$.

Let $T_S$ be the smallest set of elements of $\text{MCS}_\chi$ such that

1. $X \in T_S$;
2. for each $Y \in \text{MCS}_\chi$, if there is $Z \in T_S$ such that $Y R_O Z$ and $\varphi \in Y$, then $Y \in T_S$.

Similar to the proof of Lemma 4.13 it is proved that for all $Y \in T_S$ there exists $n \geq 0$ and there exists $X_0, \ldots, X_n \in T_S$ such that $X_0 = Y$, $X_n = X$, and $X_0 R_O \ldots R_O X_n$. Moreover, it is not difficult to show that $\neg \psi, \varphi \in Y$, for all $Y \in T_S$.

Let

$$\rho := \bigvee \{ \land Z \mid Z \in T_S \}.$$
We have $\vdash_{\text{CS}} \rho \rightarrow \neg \psi$. In addition, for each $Z \in T_S$ and each $Y \in \text{MCS}_x$ with $Y R_0 Z$, we have

$$\text{either } \vdash_{\text{CS}} Y \rightarrow \rho \text{ or } \vdash_{\text{CS}} Y \rightarrow \neg \varphi \land \neg \psi,$$

and hence,

$$\vdash_{\text{CS}} Y \rightarrow \rho \lor (\neg \varphi \land \neg \psi). \quad (9)$$

By Lemma 4.14 for each $Z \in T_S$ we have

$$\vdash_{\text{CS}} Z \rightarrow (\diamond \bigwedge Y_1 \lor \ldots \lor \bigwedge Y_n)$$

such that $Y_i \in \text{MCS}_x$ and $Y_i R_0 Z$, for $i = 1, \ldots, n$. By (9), we get

$$\vdash_{\text{CS}} Z \rightarrow (\rho \lor (\neg \varphi \land \neg \psi)),$$

for each $Z \in T_S$. Thus $\vdash_{\text{CS}} \rho \rightarrow (\diamond (\rho \lor (\neg \varphi \land \neg \psi)))$. Using ($S \cdot R$), we obtain $\vdash_{\text{CS}} \rho \rightarrow \neg (\varphi S \psi)$. Since $X \in T_S$, this implies $\vdash_{\text{CS}} X \rightarrow \neg (\varphi S \psi)$, which contradicts the assumption $\varphi S \psi \in X$. \hfill $\Box$

**Definition 4.19.** An infinite sequence $(X_0, X_1, \ldots)$ of elements of $\text{MCS}_x$ is called acceptable (for $\text{LCS}$) if 

1. $X_n R_0 X_{n+1}$ for all $n \geq 0$, and 
2. for all $n$, if $\varphi U \psi \in X_n$, then there exists $m \geq n$ such that $\psi \in X_m$ and $\varphi \in X_k$ for all $k$ with $n \leq k \leq m$. 
3. $\boxbot \in X_0$ (i.e. $X_0$ is initial).

**Lemma 4.20.** Every finite sequence $(X_0, X_1, \ldots, X_n)$ of elements of $\text{MCS}_x$ with $\boxbot \in X_0$ and $X_j R_0 X_{j+1}$, for all $0 \leq j < n$, can be extended to an acceptable sequence.

**Proof.** In order to fulfill the requirements of Definition 4.19, we shall extend the sequence $(X_0, X_1, \ldots, X_n)$ by the following steps.

Suppose $\varphi U \psi \in X_0$. Then either $\psi \in X_0$ or $\neg \psi \in X_0$. In the former case the requirement is fulfilled for the formula $\varphi U \psi$ in $X_0$, and we go to the next step. In the latter case, using axiom ($U 2$), $X_0 \vdash_{\text{CS}} \varphi \land \diamond (\varphi U \psi)$. Since $X_0 R_0 X_1$, by Lemma 4.12 we get $\varphi U \psi \in X_1$.

We can repeat this argument for $X_i$ for $1 \leq i \leq n$. We find that the requirement for $\varphi U \psi \in X_0$ is either fulfilled in $(X_0, X_1, \ldots, X_n)$ or we get $\varphi U \psi \in X_n$ and $\varphi \in X_i$ for $1 \leq i \leq n$. In the latter case, by Lemma 4.16 there exists a sequence $(X_n, X_{n+1}, \ldots, X_{n+m})$ such that $\varphi \in X_i$ for $n \leq i < n + m$, $\psi \in X_{n+m}$, and $X_i R_0 X_{i+1}$ for $n \leq i < n + m$. This gives a finite extension of the original sequence that satisfies the requirement imposed by $\varphi U \psi \in X_0$.

In the next step we repeat this argument for the remaining $U$-formulas at $X_0$. Eventually we obtain a finite sequence that satisfies all requirements imposed by $U$-formulas at $X_0$.

We may move on to $X_1$ and apply the same procedure. It is clear that by iterating the above argument to all $U$-formulas of all elements $X_i$ of the sequence
(X_0, X_1, \ldots, X_n, \ldots), including U-formulas of new elements X_i for i > n, we obtain in the limit a (finite or infinite) sequence that extends (X_0, X_1, \ldots, X_n) and satisfies conditions 1–3 of Definition 4.20. If the resulting sequence is finite, then by seriality of R_\circ, it can be extended to an infinite sequence, and in each step of this extension we can repeat the above argument to fulfill the obligations arising from the U-formulas. Thus, we finally get an acceptable sequence that extends (X_0, X_1, \ldots, X_n).

**Corollary 4.21.** For every X ∈ MCS_X, there is an acceptable sequence containing X.

**Proof.** Given X ∈ MCS_X, since ⊥ U ∈ X, by Lemma 4.18 there exists a τ S ⊥ -sequence (X_0, X_1, \ldots, X_n) ending with X, i.e. X_n = X, X_jR_\circ X_{j+1}, for all 0 ≤ j < n, and ⊥ U ∈ X_0. By Lemma 4.20 this sequence can be extended to an acceptable sequence containing X.

Let (X_0, X_1, \ldots) be an acceptable sequence of elements of MCS_X. Then there exist \Gamma_0, \Gamma_1, \ldots ∈ MCS and \Delta_1, \Delta_2, \ldots ∈ MCS such that

\[
X_0R_\circ X_1 [\Gamma_0, \Delta_1], X_1R_\circ X_2 [\Gamma_1, \Delta_2], X_2R_\circ X_3 [\Gamma_2, \Delta_3], \ldots.
\]

Note that for each i ≥ 1 we have X_{i-1}R_\circ X_i [\Gamma_{i-1}, \Delta_i], and thus X_0 = \Gamma_0 and X_j = \Gamma_j = \Delta_j for all j > 0.

**Lemma 4.22.** Let (X_0, X_1, \ldots) be an acceptable sequence of elements of MCS_X, let n ≥ 0, and let X_{i-1}R_\circ X_i [\Gamma_{i-1}, \Delta_i], for i ≥ 1.

1. (a) If \varphi S \psi ∈ X_n, then there exists m ≤ n such that \psi ∈ X_m and \varphi ∈ X_k for all k with m < k ≤ n.
   (b) If \varphi S \psi ∈ Sub^+(\chi) and there exists m ≤ n such that \psi ∈ X_m and \varphi ∈ X_k for all k with m < k ≤ n, then \varphi S \psi ∈ X_n.
2. If \square \varphi ∈ X_n, then \varphi ∈ X_m for some m ≥ n.
3. (a) If \square \varphi ∈ X_n, then \varphi ∈ X_m for all m ≥ n.
   (b) If \square \varphi ∈ Sub^+(\chi) and \varphi ∈ X_m for all m ≥ n, then \square \varphi ∈ X_n.
4. If \varphi \psi ∈ X_n, then \varphi ∈ X_m for some m ≤ n.
5. (a) If \exists \varphi ∈ X_n, then \varphi ∈ X_m for all m ≤ n.
   (b) If \exists \varphi ∈ Sub^+(\chi) and \varphi ∈ X_m for all m ≤ n, then \exists \varphi ∈ X_n.
6. If n > 0 and \varphi S \psi ∈ X_n, then \varphi ∈ X_{n-1}.
7. If n > 0 and \boxempty \varphi ∈ X_n, then \varphi ∈ X_{n-1}.

**Proof.** 1. The proof involves a routine induction on n. We prove item (a). The proof of (b) is similar.
Suppose n = 0 and \varphi S \psi ∈ X_0. Since \varphi S \psi ∈ \Gamma_0, using axiom (S2) we have either \psi ∈ \Gamma_0 or \varphi ∧ S(\varphi S \psi) ∈ \Gamma_0. In the former case, we get \psi ∈ X_0 and we are done. By Lemma 4.19 the latter case cannot happen.
Suppose \psi \psi ∈ X_n, and then using axiom (S2) we have either \psi ∈ \Delta_n or \varphi ∧ S(\varphi S \psi) ∈ \Delta_n. In the former case, we get \psi ∈ X_n and we are done. In the latter case, we have \varphi ∈ \Delta_n, and thus \varphi ∈ X_n. On the
In this section we introduce interpreted systems based on Fitting-models as semantics for temporal justification logic LPLTL\(^P\).

**Definition 5.1.** A frame is a tuple \((S, R_1, \ldots, R_h)\) where

1. \(S\) is a non-empty set of states;
2. each \(R_i \subseteq S \times S\) is a reflexive and transitive relation.

A run \(r\) on a frame is a function from \(\mathbb{N}\) to states, i.e., \(r: \mathbb{N} \rightarrow S\). A system \(\mathcal{R}\) is a non-empty set of runs.

Given a run \(r\) and \(n \in \mathbb{N}\), the pair \((r, n)\) is called a point.
Definition 5.2. Given a frame \((S, R_1, \ldots, R_h)\), a CS-evidence function for agent \(i\) is a function
\[\mathcal{E}_i : S \times Tm \rightarrow \mathcal{P}(Fml)\]
satisfying the following conditions. For all terms \(s, t \in Tm\), all formulas \(\varphi, \psi \in Fml\), all \(v, w \in S\), and all \(i \in Ag\):

1. \(\mathcal{E}_i(v, t) \subseteq \mathcal{E}_i(w, t)\), whenever \(R_i(v, w)\); (monotonicity)
2. if \([t]_i \varphi \in \mathcal{CS}\), then \(\varphi \in \mathcal{E}_i(w, t)\); (constant specification)
3. if \(\psi \rightarrow \varphi \in \mathcal{E}_i(w, t)\) and \(\varphi \in \mathcal{E}_i(w, s)\), then \(\psi \in \mathcal{E}_i(w, t \cdot s)\); (application)
4. \(\mathcal{E}_i(w, s) \cup \mathcal{E}_i(w, t) \subseteq \mathcal{E}_i(w, s + t)\); (sum)
5. if \(\varphi \in \mathcal{E}_i(w, t)\), then \([t]_i \varphi \in \mathcal{E}_i(w, !t)\). (positive introspection)

Definition 5.3. An interpreted system for LPLTL\(_{CS}^P\) (or for CS) is a tuple
\[\mathcal{I} = (\mathcal{R}, S, R_1, \ldots, R_h, \mathcal{E}_1, \ldots, \mathcal{E}_h, \nu)\]
where

1. \((S, R_1, \ldots, R_h)\) is a frame;
2. \(\mathcal{R}\) is a system on that frame;
3. \(\mathcal{E}_i\) is a CS-evidence function for agent \(i\) for \(1 \leq i \leq h\);
4. \(\nu : S \rightarrow \mathcal{P}(Prop)\) is a valuation.

Definition 5.4. Given an interpreted system
\[\mathcal{I} = (\mathcal{R}, S, R_1, \ldots, R_h, \mathcal{E}_1, \ldots, \mathcal{E}_h, \nu),\]
a run \(r \in \mathcal{R}\), and \(n \in \mathbb{N}\), we define truth of a formula \(\varphi\) in \(\mathcal{I}\) at point \((r, n)\) inductively by

\[\mathcal{I}, r, n) \models P \text{ iff } P \in \nu(r(n))\]
\[\mathcal{I}, r, n) \not\models \bot\]
\[\mathcal{I}, r, n) \models \varphi \rightarrow \psi \text{ iff } (\mathcal{I}, r, n) \not\models \varphi \text{ or } (\mathcal{I}, r, n) \models \psi,\]
\[\mathcal{I}, r, n) \models [\top] \varphi \text{ iff } n = 0 \text{ or } (\mathcal{I}, r, n - 1) \models \varphi,\]
\[\mathcal{I}, r, n) \models [\top] \varphi \text{ iff } (\mathcal{I}, r, n + 1) \models \varphi,\]
\[\mathcal{I}, r, n) \models \varphi S \psi \text{ iff there is some } m \leq n \text{ such that } (\mathcal{I}, r, m) \models \psi\]
\[\text{and } (\mathcal{I}, r, k) \models \varphi \text{ for all } k \text{ with } m < k \leq n,\]
\[\mathcal{I}, r, n) \models \varphi U \psi \text{ iff there is some } m \geq n \text{ such that } (\mathcal{I}, r, m) \models \psi\]
\[\text{and } (\mathcal{I}, r, k) \models \varphi \text{ for all } k \text{ with } n \leq k < m,\]
\[\mathcal{I}, r, n) \models [t] \varphi \text{ iff } \varphi \in \mathcal{E}_i(r(n), t) \text{ and } (\mathcal{I}, r', n') \models \varphi\]
\[\text{for all } r' \in \mathcal{R} \text{ and } n' \in \mathbb{N} \text{ such that } R_i(r(n), r'(n')).\]

As usual, we write \(\mathcal{I} \models \varphi\) if for all \(r \in \mathcal{R}\) and all \(n \in \mathbb{N}\), we have \((\mathcal{I}, r, n) \models \varphi\). Further, we write \(\models_{CS} \varphi\) if \(\mathcal{I} \models \varphi\) for all interpreted systems \(\mathcal{I}\) for CS.
Definition 5.5. Given a set of formulas $\Gamma$ and a formula $\varphi$, the (local) consequence relation is defined as follows: $\Gamma \triangledown \varphi$ iff for all interpreted systems $I = (R, \ldots)$ for CS, for all $r \in R$, and for all $n \in \mathbb{N}$, if $(I, r, n) \models \psi$ for all $\psi \in \Gamma$, then $(I, r, n) \not\models \varphi$.

From the above definitions it follows that:

- $(I, r, n) \not\models \varphi$ iff $(I, r, n + m) \models \varphi$ for some $m \leq n$,
- $(I, r, n) \models \varphi$ iff $(I, r, n + m) \models \varphi$ for some $m \leq n$, 
- $(I, r, n) \models \varphi$ iff $(I, r, n - m) \models \varphi$ for all $m \leq n$,
- $(I, r, n) \models \varphi$ iff $n > 0$ and $(I, r, n - 1) \models \varphi$.

It is sometime convenient to use the following truth conditions for since and until formulas, which are clearly equivalent to the corresponding conditions given in Definition 5.4.

- $(I, r, n) \not\models \varphi \mathcal{S} \psi$ iff there is some $m$ with $n \geq m \geq 0$ such that $(I, r, n - m) \models \psi$ and $(I, r, n - k) \not\models \varphi$ for all $k$ with $0 \leq k < m$,
- $(I, r, n) \not\models \varphi \mathcal{U} \psi$ iff there is some $m \geq 0$ such that $(I, r, n + m) \models \psi$ and $(I, r, n + k) \not\models \varphi$ for all $k$ with $0 \leq k < m$.

Remark 5.6. Note that $(I, r, n) \models w_{\bot}$ iff $n = 0$. Thus $w_{\bot}$ expresses the property “the time is 0.” Similarly, $\otimes_{\bot}$, where $\otimes_{\bot}$ is the iteration of $\otimes$, $m$ times, expresses the property “the time is $m$.”

Let “time $= m$” abbreviate $w_{\bot}$ and true$^{m}$(\varphi) abbreviate time $= m \not\models \varphi$.

It is easy to show that

- $(I, r, n) \not\models \models \varphi \models \psi$ iff $n = m$
- and $(I, r, n) \models \models \varphi \models \psi$ iff $n = m$

Thus, true$^{m}$(\varphi) expresses that “\varphi is true at time $m$.”

Remark 5.7. The interpreted systems are originally formulated by means of the notions of local and global states (see e.g. [8, 16]). Now I aim to define the interpreted systems for LPLTL$^P$, using the notions of local and global states, so that it more closely matches the original definition of interpreted systems given in [8].

Suppose that at any point in time the system is in some global state, defined by the local states of the agents and the state of other objects of interest (which is referred to as the “environment”). Let $L$ be some set of local states. Informally, an agent’s local state captures all the information available to her at a given moment of time. A global state is a $(h + 1)$-tuple $(l_e, l_1, \ldots, l_h) \in L^{h+1}$, where $l_e$ is the state
of environment and $l_i$ is the local state of agent $i$ for $i = 1, \ldots, h$. Now in order to define the interpreted systems for LPLTL$^P$ using the notions of local and global states, it is enough to put the set of states $S := L^{h+1}$. As before a run $r$ is a function from time to global states, i.e., $r : \mathbb{N} \to L^{h+1}$, and a system is a set $R$ of runs. The definitions of CS-evidence functions, interpreted systems, and truth are as before. Note that here $(l_e, l_1, \ldots, l_h) \mathcal{R}_i (l'_e, l'_1, \ldots, l'_h)$ means “the local state $l'_i$ is epistemically possible for agent $i$ in the local state $l_i$.”

It is worth noting that the semantics given by Bucheli in [5] for temporal justification logic employs global states. However, there is a minor difference between Bucheli’s semantics and ours. Since he modeled the knowledge part of the temporal justification logic by a justification counterpart of the modal logic $S5$, he defines indistinguishability relations $\sim_i$ between points, for each agent $i$, which are clearly equivalence relations. In contrast to his formulation, our temporal justification logic is based on a justification counterpart of the modal logic $S4$, and thus we naturally make use of reflexive and transitive accessibility relations $R_i$ for each agent $i$.

### 6 Soundness and Completeness of LPLTL$^P$

The soundness proof for LPLTL$^P_{CS}$ is a straightforward combination of the soundness proofs for temporal logic and justification logic by induction on the derivation.

**Theorem 6.1 (Soundness).** For each formula $\varphi$ and finite set of formulas $\Gamma$,

$$\Gamma \vdash_{CS} \varphi \quad \text{implies} \quad \Gamma \models_{CS} \varphi.$$  

For the completeness proof we employ the canonical model construction.

**Definition 6.2.** The $\chi$-canonical interpreted system

$$\mathcal{I} = (\mathcal{R}, S, R_1, \ldots, R_h, \mathcal{E}_1, \ldots, \mathcal{E}_h, \nu)$$

for LPLTL$^P_{CS}$ is defined as follows:

1. $\mathcal{R}$ consists of all mappings $r : \mathbb{N} \to \text{MCS}_X$ such that $(r(0), r(1), \ldots)$ is an acceptable sequence;
2. $S := \text{MCS}_X$;
3. $XR_iY$ iff for all $\Delta \in \text{MCS}$ such that $Y = \overline{\Delta}$ there exists $\Gamma \in \text{MCS}$ such that $X = \overline{T}$ and $\{ \varphi \mid [t], \varphi \in \Gamma \text{ for some } t \} \subseteq \Delta$;
4. $\mathcal{E}_i(X, t) := \{ \varphi \mid [t], \varphi \in \Gamma \in \text{MCS} \mid X = \overline{T} \}$;
5. $\nu(X) := \text{Prop} \cap X$.

Note that $S = \text{MCS}_X = \{ r(n) \mid r \in \mathcal{R}, n \in \mathbb{N} \}$.

**Lemma 6.3.** The $\chi$-canonical interpreted system

$$\mathcal{I} = (\mathcal{R}, S, R_1, \ldots, R_h, \mathcal{E}_1, \ldots, \mathcal{E}_h, \nu)$$

for LPLTL$^P_{CS}$ is an interpreted system for LPLTL$^P_{CS}$.  

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Proof. It is not difficult to show that each $R_i$ is reflexive and transitive. We now have to show that each $E_i$ satisfies the conditions of Definition 5.2. We only show here the (monotonicity) condition for $E_i$. The proof for other conditions is easy. Suppose that $X, Y \in S$, $XR, Y$, and $\varphi \in E_i(X, t)$. We have to show that $\varphi \in E_i(Y, t)$. Let $Y = \overrightarrow{A}$, for $A \in \text{MCS}$. We have to show that $[t]_i \varphi \in A$. Since $XR, Y$, there exists $I \in \text{MCS}$ such that $X = \overrightarrow{I}$ and $\{ \varphi \mid [t]_i \varphi \in A \text{ for some } t \} \subseteq A$. Since $\varphi \in E_i(X, t)$, we get $[t]_i \varphi \in A$, and thus $[t]_i \varphi \in A$. Therefore, $[t]_i \varphi \in A$, as desired.

Lemma 6.4 (Truth Lemma). Let $L = \text{LPLTL}^p$, let $CS$ be a constant specification for $L$, and let $I = (R, S, R_1, \ldots, R_n, E_1, \ldots, E_n)$ be the $\chi$-canonical interpreted system for $\text{LPLTL}^p_{CS}$. For every formula $\psi \in \text{Subf}^+(\chi)$, every run $r$ in $R$, and every $n \in \mathbb{N}$ we have:

$$(I, r, n) \models \psi \iff \psi \in r(n).$$

Proof. As usual, the proof is by induction on the structure of $\psi$. Let $r(i - 1)R_\Diamond r(i)$, $i \geq 1$, for $i \geq 1$. We only show the following cases:

$\psi = \varnothing \varphi$.

$(\Rightarrow)$ Suppose that $(I, r, n) \models \varnothing \varphi$ and $(I, r, n + 1) \models \varphi$, by the induction hypothesis, iff $\varphi \in r(n + 1)$ iff $\varphi \in \Delta_{n+1}$, by Lemma 4.6. Suppose $\varphi \in I_n$ iff $\varnothing \varphi \in r(n)$. Hence $\varphi \in \Delta_n$, which is a contradiction.

$(\Leftarrow)$ Suppose that $(I, r, n) \models \varnothing \varphi$ and $\varnothing \varphi \not\in r(n)$. Then $n = 0$ or $(I, r, n - 1) \models \varphi$.

- Suppose $n = 0$. Since $r(0)$ is initial, $\varnothing \top \models r(0)$. Since $\vdash_{LCS} \varnothing \top \rightarrow \varnothing \varphi$, by Lemma 4.3, we get $\varnothing \varphi \in r(0)$. The latter clearly contradicts the assumption $\varnothing \varphi \not\models r(0)$.

- Suppose $n > 0$ and $(I, r, n - 1) \models \varphi$. By the induction hypothesis $\varphi \in r(n - 1)$. Since $r(n - 1)R_\Diamond r(n)$, we have $\varphi \in I_{n-1}$. Hence, by Lemma 4.6, $\varnothing \varphi \in \Delta_n$. By axiom (5w), we get $\varnothing \varphi \in \Delta_n$, which is a contradiction.

$\psi = \psi_1 \mathcal{U} \psi_2$.

$(\Rightarrow)$ If $(I, r, n) \models \psi_1 \mathcal{U} \psi_2$, then $(I, r, m) \models \psi_2$ for some $m \geq n$, and $(I, r, k) \models \psi_1$ for all $k$ with $n \leq k < m$. By the induction hypothesis we get $\psi_2 \in r(m)$, and $\psi_1 \in r(k)$ for all $k$ with $n \leq k < m$. We have to show $\psi_1 \mathcal{U} \psi_2 \in r(n)$, which follows by induction on $m$ as follows:

- Base case $m = n$. Since $\psi_2 \in r(n) = r(m)$ and $\vdash_{LCS} \psi_2 \rightarrow (\psi_1 \mathcal{U} \psi_2)$, by Lemma 4.3, we get $\psi_1 \mathcal{U} \psi_2 \in r(n)$.

- Suppose $m > n$. It follows from the induction hypothesis that $\psi_1 \mathcal{U} \psi_2 \in r(n+1)$. Since $r(n)R_\Diamond r(n+1)$, and hence $\psi_1 \mathcal{U} \psi_2 \in \Delta_{n+1}$. Thus, by Lemma 4.6, $\varnothing (\psi_1 \mathcal{U} \psi_2) \in I_n$. Now suppose towards a contradiction that $\psi_1 \mathcal{U} \psi_2 \not\in I_n$. Hence $\neg (\psi_1 \mathcal{U} \psi_2) \in I_n$. By axiom (U2),

$$\vdash_{LCS} \neg (\psi_1 \mathcal{U} \psi_2) \rightarrow \neg \psi_2 \wedge (\neg \psi_1 \vee \neg (\psi_1 \mathcal{U} \psi_2)).$$
and thus
\[ \vdash_{\text{CS}} \neg (\psi_1 \cup \psi_2) \land \psi_1 \rightarrow \neg \circ (\psi_1 \cup \psi_2), \]

Thus, \( \neg \circ (\psi_1 \cup \psi_2) \in \Gamma_n \), which is a contradiction. Thus, \( \psi_1 \cup \psi_2 \in \Gamma_n \) and hence \( \psi_1 \cup \psi_2 \in r(n) \).

\((\Leftarrow)\) If \( \psi_1 \cup \psi_2 \in r(n) \), then since \( (r(0), r(1), \ldots, r(n + 1), \ldots) \) is an acceptable sequence there exists \( m \geq n \) such that \( \psi_2 \in r(m) \), and \( \psi_1 \in r(k) \) for all \( k \) with \( n \leq k < m \). By the induction hypothesis we obtain \( (I, r, m) \models \psi_2 \), and \( (I, r, k) \models \psi_1 \) for all \( k \) with \( n \leq k < m \). Thus \( (I, r, n) \models \psi_1 \cup \psi_2 \).

\((\Rightarrow)\) If \( (I, r, n) \models \psi_1 \cup \psi_2 \), then \( (I, r, m) \models \psi_2 \) for some \( m \leq n \), and \( (I, r, k) \models \psi_1 \) for all \( k \) with \( m < k \leq n \). By the induction hypothesis, \( \psi_2 \in r(m) \), and \( \psi_1 \in r(k) \) for all \( k \) with \( m < k \leq n \). We want to show that \( \psi_1 \cup \psi_2 \in r(n) \). We prove it by induction on \( m \) as follows.

- Base case \( m = n \). Since \( \psi_2 \in r(n) = r(m) \) and \( \vdash_{\text{CS}} \psi_2 \rightarrow (\psi_1 \cup \psi_2) \), we obtain \( \psi_1 \cup \psi_2 \in r(n) \).
- Suppose \( m < n \). Since \( r(n-1)R_C r(n) \upharpoonright \overline{T_{n-1}, \Delta_n} \), it follows from the induction hypothesis that \( \psi_1 \cup \psi_2 \in r(n-1) \), and hence \( \psi_1 \cup \psi_2 \in r(n) \). Thus, by Lemma 6.3, \( \bigcirc (\psi_1 \cup \psi_2) \in \Delta_n \). Now suppose towards a contradiction that \( \psi_1 \cup \psi_2 \not\in \Delta_n \). Hence \( \neg (\psi_1 \cup \psi_2) \in \Delta_n \). By axiom \((S2)\),

\[ \vdash_{\text{CS}} \neg (\psi_1 \cup \psi_2) \rightarrow \neg (\psi_1 \cup \neg \bigcirc (\psi_1 \cup \psi_2)), \]

and thus
\[ \vdash_{\text{CS}} \neg (\psi_1 \cup \psi_2) \land \psi_1 \rightarrow \neg \bigcirc (\psi_1 \cup \psi_2) \],

Thus, \( \neg \bigcirc (\psi_1 \cup \psi_2) \in \Delta_n \), which is a contradiction.

\((\Leftarrow)\) Suppose \( \psi_1 \cup \psi_2 \in r(n) \). By Lemma 6.3, there is \( m \leq n \) such that \( \psi_2 \in r(m) \), and \( \psi_1 \in r(k) \) for all \( k \) with \( m < k \leq n \). By the induction hypothesis, \( (I, r, m) \models \psi_2 \) and \( (I, r, k) \models \psi_1 \) for all \( k \) with \( m < k \leq n \), and thus \( (I, r, n) \models \psi_1 \cup \psi_2 \) as desired.

\((\Rightarrow)\) If \( (I, r, n) \models [t] \varphi \), then \( \varphi \in E_i(r(n), t) \). Thus, by the definition of \( E_i \), \( [t] \varphi \in I \), where \( r(n) = \overline{T} \), and hence \( [t] \varphi \in r(n) = \overline{T} \).

\((\Leftarrow)\) If \( [t] \varphi \in r(n) \), then \( [t] \varphi \in \bigcap \{ r(n) : \overline{T} \} \). Thus, \( \varphi \in E_i(r(n), t) \). Now suppose that \( r(n)R_{r'}(n') \) and let \( r'(n') = \overline{T} \), for some \( \Delta \in \text{MCS} \). By the definition of \( R_{r'} \), there is \( \Gamma \in \text{MCS} \) such that \( r(n) = \overline{T} \) and \( \{ \varphi : [t] \varphi \in I \text{ for some } t \in \Delta \} \subseteq \Delta \). We find \( \varphi \in \Delta \). By the induction hypothesis, we get \( (I, r', n') \models \varphi \). Since \( r' \) and \( n' \) were arbitrary, we conclude that \( (I, r, n) \models [t] \varphi \).

**Theorem 6.5 (Completeness).** Let \( L = LPTL^p \) and let \( \text{CS} \) be a constant specification for \( L \). For each formula \( \chi \),

\[ \vdash_{\text{CS}} \chi \text{ implies } \vdash_{\text{CS}} \chi. \]

**Proof.** Suppose that \( \not\vdash_{\text{CS}} \chi \). Thus, \( \{ \neg \chi \} \) is an \( \text{CS} \)-consistent set. Therefore, there exists \( \Gamma \in \text{MCS} \) with \( \neg \chi \in \Gamma \). Let \( \overline{T} = \Gamma \cap \text{Subf}^+(\chi) \). By Corollary 6.21, there is an acceptable sequence containing \( \Gamma \), say \( (X_0, X_1, \ldots) \) where \( \overline{T} = X_n \), for some

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n ≥ 0. Define the run \( r \) as follows \( r(i) := X_i \). The run \( r \) is in the system \( \mathcal{R} \) of the \( \chi \)-canonical model \( \mathcal{I} \) for Lcs. Since \( \chi \not\models r(n) \), by the Truth Lemma, \( (\mathcal{I}, r, n) \not\models \chi \). Therefore, \( \not\models_{\text{LCS}} \chi \).

**Theorem 6.6 (Completeness).** Let \( L = \text{LPLTL}^P \) and let \( \text{CS} \) be a constant specification for \( L \). For each formula \( \chi \) and finite set of formulas \( T \),

\[
T \models_{\text{LCS}} \chi \quad \text{implies} \quad T \not\models_{\text{LCS}} \chi .
\]

**Proof.** Suppose that \( T \not\models_{\text{LCS}} \chi \). Thus, \( \not\models_{\text{LCS}} \land T \rightarrow \chi \). By Theorem 6.5 there is an interpreted system \( \mathcal{I} = (\mathcal{R}, \ldots), \ r \in \mathcal{R} \), and \( n \in \mathbb{N} \) such that \( (\mathcal{I}, r, n) \not\models \land T \) and \( (\mathcal{I}, r, n) \not\models \chi \). Therefore, \( T \not\models_{\text{LCS}} \chi \).

### 7 Another semantics for LPLTL^P

In this section we present another semantics based on Mkrtchyan models [22] for LPLTL^P. These models are indeed the interpreted systems with singleton system of runs.

**Definition 7.1.** An \( \text{LPLTL}_{\text{CS}}^P \)-model is a tuple \( \mathcal{M} = (r, S, E_1 \ldots, E_h, \nu) \) where

1. \( S \) is a non-empty set of states;
2. \( r : \mathbb{N} \rightarrow S \) is a run on \( S \);
3. \( E_i \) is a \( \text{CS} \)-evidence function for agent \( i \), for \( 1 \leq i \leq h \), that satisfies conditions 2–5 of Definition 5.2;
4. \( \nu : S \rightarrow \wp(\text{Prop}) \) is a valuation.

Given an \( \text{LPLTL}_{\text{CS}}^P \)-model \( \mathcal{M} = (r, S, E_1 \ldots, E_h, \nu) \) and \( n \in \mathbb{N} \), we define truth of a formula \( \varphi \) in \( \mathcal{M} \) at state \( r(n) \) inductively by

\[
(M, r(n)) \models P \text{ iff } P \in \nu(r(n)),
\]

\[
(M, r(n)) \not\models \bot
\]

\[
(M, r(n)) \models \varphi \rightarrow \psi \text{ iff } (M, r(n)) \not\models \varphi \text{ or } (M, r(n)) \models \psi,
\]

\[
(M, r(n)) \models \bigcirc \varphi \text{ iff } n = 0 \text{ or } (M, r(n - 1)) \models \varphi,
\]

\[
(M, r(n)) \models \bigcirc \bigcirc \varphi \text{ iff } (M, r(n + 1)) \models \varphi,
\]

\[
(M, r(n)) \models \varphi S \psi \text{ iff there is some } m \leq n \text{ such that } (M, r(m)) \models \psi
\]

\[
\text{and } (M, r(k)) \models \varphi \text{ for all } k \text{ with } m < k \leq n,
\]

\[
(M, r(n)) \models \varphi U \psi \text{ iff there is some } m > n \text{ such that } (M, r(m)) \models \psi
\]

\[
\text{and } (M, r(k)) \models \varphi \text{ for all } k \text{ with } n \leq k < m,
\]

\[
(M, r(n)) \models [t] \varphi \text{ iff } \varphi \in E_i(r(n), t) \text{ and } (M, r(n)) \models \varphi.
\]

We write \( \mathcal{M} \models \varphi \) if \( (M, r(n)) \models \varphi \) for all \( n \in \mathbb{N} \).
From the above definitions it follows that:

\[
\begin{align*}
(M, r(n)) & \models \Diamond \varphi \text{ iff } (M, r(m)) \models \varphi \text{ for some } m \geq n, \\
(M, r(n)) & \models \Box \varphi \text{ iff } (M, r(m)) \models \varphi \text{ for all } m \geq n, \\
(M, r(n)) & \models \Diamond \varphi \text{ iff } (M, r(m)) \models \varphi \text{ for some } m \leq n, \\
(M, r(n)) & \models \Box \varphi \text{ iff } (M, r(m)) \models \varphi \text{ for all } m \leq n, \\
(M, r(n)) & \models s \varphi \text{ iff } n > 0 \text{ and } (M, r(n-1)) \models \varphi,
\end{align*}
\]

Definition 7.2. Given a set of formulas \( \Gamma \) and a formula \( \varphi \), the (local) consequence relation is defined as follows: \( \Gamma \models CS \varphi \) iff for all \( LPLTL^P_{CS} \)-models \( M = (r, \ldots) \), and for all \( n \in \mathbb{N} \), if (\( M, r(n) \)) \models \psi \) for all \( \psi \in \Gamma \), then (\( M, r(n) \)) \models \varphi.

Remark 7.3. As in Remark 5.6, if we let \( \text{time} = m \) abbreviate \( \otimes m \square \) and \( \text{true}_m(\varphi) \) abbreviate \( \text{time} = m \rightarrow \varphi \), then we have \( M \models \text{true}_m(\varphi) \) iff \( M, m \models \varphi \).

Theorem 7.4 (Soundness). For each formula \( \varphi \) and finite set of formulas \( \Gamma \),

\( \Gamma \models CS \varphi \) implies \( \Gamma \models CS \varphi \).

Definition 7.5. Let \( (X_0, X_1, \ldots) \) be an acceptable sequence of elements of \( MCS_\chi \) for \( LPLTL^P_{CS} \). The \( \chi \)-canonical model \( M = (r, S, E_1, \ldots, E_h, \nu) \) for \( CS \) with respect to \( (X_0, X_1, \ldots) \) is defined as follows:

1. \( S := \{X_0, X_1, \ldots\} \).
2. \( r(n) := X_n \).
3. \( E_t(X_n, t) := \{ \varphi \mid \varphi \in \Gamma \cap \{ \nu \in \mathbb{N} \mid X = \overline{t} \} \} \).
4. \( \nu(X_n) := \text{Prop} \cap X_n \).

Lemma 7.6. The \( \chi \)-canonical model \( M = (r, S, E_1, \ldots, E_h, \nu) \) for \( CS \) with respect to an acceptable sequence \( (X_0, X_1, \ldots) \) is an \( LPLTL^P_{CS} \)-model.

Proof. Similar to the proof of Lemma 6.3.

Lemma 7.7 (Truth Lemma). Let \( M = (r, S, E_1, \ldots, E_h, \nu) \) be the \( \chi \)-canonical model for \( CS \) with respect to an acceptable sequence \( (X_0, X_1, \ldots) \). For every formula \( \psi \in \text{Sub}^*(\chi) \), and every \( n \in \mathbb{N} \) we have:

\( (M, r(n)) \models \psi \) iff \( \psi \in r(n) \).

Proof. As usual, the proof is by induction on the structure of \( \psi \). Let \( r(i-1)R_\circ r(i) [T_{i-1} \rightarrow \Delta_i] \), for \( i \geq 1 \). Since the proof is similar to the proof of Lemma 6.4, we only show the following case:
Proof. Suppose that \( \Gamma \vdash \varphi \). Thus, \( \{ \neg \varphi \} \) is a \( CS \)-consistent set. Therefore, there exists \( \Gamma \in \text{MCS} \) with \( \neg \varphi \in \Gamma \). Let \( T = \Gamma \cap \text{Sub}^\ast (\varphi) \). By Corollary 7.21, there is an acceptable sequence containing \( T \), say \( T_0, T_1, \ldots, T_n, T_{n+1}, \ldots \) where \( n \geq 0 \) and \( T_n = T \). Construct the \( \varphi \)-canonical model \( M \) for \( CS \) with respect to this acceptable sequence. Since \( \varphi \notin \Gamma_n \), by the Truth Lemma, \( (M, r(n)) \models \varphi \). Therefore, \( M \not\models \varphi \).

\[ \square \]

Theorem 7.9 (Completeness). For each formula \( \varphi \) and finite set of formulas \( \Gamma \),
\[ \Gamma \models CS \varphi \quad \text{implies} \quad \Gamma \not\models CS \varphi. \]

Proof. Suppose that \( \Gamma \not\models CS \varphi \). Thus, \( \not\models CS \land \Gamma \rightarrow \varphi \). By Theorem 7.8 there is an \( LPLTL_{CS}^P \)-model \( M = (r, \ldots) \), and \( n \in \mathbb{N} \) such that \( (M, r(n)) \models \land \Gamma \) and \( (M, r(n)) \not\models \varphi \). Therefore, \( \Gamma \not\models CS \varphi \).

\[ \square \]

8 Connecting principles

In \( LPLTL_{CS}^P \), epistemic and temporal properties do not interact. In this section we study some principles that create a connection between justifications and temporal modalities. We assume the language for terms to be augmented in the obvious way.

1. \( \square [t] \varphi \rightarrow [\uparrow t], \square \varphi \) \hspace{2cm} (generalize)
2. \( [t], \square \varphi \rightarrow [\downarrow t], \square \varphi \) \hspace{2cm} (\( \square \)-access)
3. \( [t], \square \varphi \rightarrow [t], \square \varphi \) \hspace{2cm} (\( \square \)-access)
4. \( [t], \square \varphi \rightarrow [\Rightarrow t], \square \varphi \) \hspace{2cm} (\( \square \)-right)
5. \( \square [t], \varphi \rightarrow [\leftarrow t], \square \varphi \) \hspace{2cm} (\( \square \)-left)
6. \( \Box [t], \varphi \rightarrow [\downarrow p], \varphi \) \hspace{2cm} (\( \Box \)-generalize)
7. \( [t], \Box \varphi \rightarrow [\downarrow p], \varphi \) \hspace{2cm} (\( \Box \)-access)
8. \( [t], \Box \varphi \rightarrow [t], \Box \varphi \) \hspace{2cm} (\( \Box \)-access)
9. \( [t], \Box \varphi \rightarrow [t], \Box \varphi \) \hspace{2cm} (\( \Box \)-right)
10. \( [t], \Box \varphi \rightarrow [\leftarrow p], \varphi \) \hspace{2cm} (\( \Box \)-left)
11. \( \Box [t], \varphi \rightarrow [\leftarrow p], \Box \varphi \) \hspace{2cm} (\( \Box \)-left)

Principles 1–5 were first proposed by Bucheli in [5] from which the name of the axioms are also taken\(^1\). A few remarks on these principles are in order:

\(^1\) The principle \( [t], \Box \varphi \rightarrow [\leftarrow t], \varphi \) is called (\( \Box \)-access) in [6].
(generalize) This principle says that if you have a fixed piece of evidence that always supports a proposition, then you have evidence that this proposition is always true. The term operator \( \uparrow \) converts permanent evidence for a proposition to evidence for knowing that this proposition is always true.

(\(\Box\)-access) This principle says that if you have evidence that a proposition is always true, then at every point in time you are able to access this information. The term operator \( \downarrow \) makes the evidence accessible in every future point in time. This principle is a counterpart of the axiom \( K_i \Box \varphi \rightarrow \Box K_i \varphi \) in the logics of knowledge and time, which is valid in the interpreted systems with perfect recall (where an agent retains the knowledge of previous times), but does not characterize it, see [16].

(\(\Diamond\)-access) This principle is similar to the valid formula \( \Box \varphi \rightarrow \Diamond \varphi \) augmented by justifications. In fact, if you have evidence that a proposition is always true, then you have evidence that it is true tomorrow, and the term operator \( \downarrow \) constructs such an evidence.

(\(\Diamond\)-right) This principle says that agents do not forget evidence once they have gathered it and can “take it with them”. The term operator \( \Rightarrow \) carries evidence through time. This principle is a counterpart of the axiom \( K_i \Diamond \varphi \rightarrow \Diamond K_i \varphi \) in the logics of knowledge and time, which characterizes the synchronous systems (where each agent always knows the time) with perfect recall, see [16].

(\(\Diamond\)-left) This principle implies some form of conditional prediction. The term operator \( \Leftarrow \) predicts future evidence for knowledge. This principle is a counterpart of the axiom \( \Diamond K_i \varphi \rightarrow K_i \Diamond \varphi \) in the logics of knowledge and time, which characterizes the synchronous systems with no learning (where an agent’s knowledge can not increase over time), see [16].

The connecting principles involving past operators are the dual of those involving future operators, and thus the meaning of the term operators with subscript \( P \) can be guessed straightforwardly.

Let us show that a version of (\(\Diamond\)-access) is derivable from (\(\Box\)-access) and (\(\Diamond\)-left).

**Lemma 8.1.** Let \( Ax \) contains axioms (\(\Box\)-access) and (\(\Diamond\)-left). For every agent \( i \), formula \( \varphi \) and term \( t \) there is a term \( s(t) \) such that

\[
\vdash_{LPLTL^P(Ax)} [t]_i \Box \varphi \rightarrow [s(t)]_i \Diamond \varphi.
\]
Proof. Construct the following proof in LPLTL<sup>P</sup>(Ax)<sub>ϕ</sub> where Ax contains axioms (□-access) and (○-left).

1. \([t], \Box \varphi \rightarrow [\parallel t], \varphi\) \hspace{1cm} \text{instance of axiom (□-access)}
2. \([\parallel t], \varphi \rightarrow [\parallel t], \varphi\) \hspace{1cm} \text{Lemma 8.3 item 2}
3. \([\parallel t], \varphi \rightarrow [\leq t], \Box \varphi\) \hspace{1cm} \text{instance of axiom (○-left)}
4. \([t], \Box \varphi \rightarrow [\leq t], \Box \varphi\) \hspace{1cm} \text{from 1–3 by propositional reasoning}

Finally put \(s(t) := \leq t\).

\(\square\)

8.1 Semantics

Now we present a semantics for LPLTL<sup>P</sup>(Ax) based on Mktychev models. In the next section these models will be extended to interpreted systems.

**Definition 8.2.** Let \(CS\) be a constant specification for LPLTL<sup>P</sup>(Ax). An LPLTL<sup>P</sup>(Ax) CS-model is a tuple \(M = (r, S, E_1, \ldots, E_h, \nu)\) where \(S\) is a non-empty set of states, \(r\) is a run on \(S\), \(\nu\) is a valuation, and the \(CS\)-evidence functions \(E_1, \ldots, E_h\) should satisfy conditions 2–5 of Definition 4.2 and the following conditions depending on axioms in Ax. For all \(n \in \mathbb{N}\), all terms \(s, t \in T_m\), all formulas \(\varphi, \psi \in Fml\), and all \(i \in A_g:\)

1. if \(\varphi \in E_i(r(m), t)\) for all \(m \geq n\), then \(\Box \varphi \in E_i(r(n), \parallel t)\). \hspace{1cm} \text{(generalize-)}
2. if \(\Box \varphi \in E_i(r(n), t)\), then \(\varphi \in E_i(r(m), \parallel t)\) for all \(m \geq n\). \hspace{1cm} \text{(□-access-)}
3. if \(\Box \varphi \in E_i(r(n), t)\), then \(\Box \varphi \in E_i(r(n), \down t)\). \hspace{1cm} \text{(□-access-)}
4. if \(\Box \varphi \in E_i(r(n), t)\), then \(\varphi \in E_i(r(n + 1), \Rightarrow t)\). \hspace{1cm} \text{(○-right-)}
5. if \(\varphi \in E_i(r(n + 1), t)\), then \(\Box \varphi \in E_i(r(n), \leq t)\). \hspace{1cm} \text{(○-left-)}
6. if \(\varphi \in E_i(r(m), t)\) for all \(m \leq n\), then \(\Box \varphi \in E_i(r(n), \parallel p t)\). \hspace{1cm} \text{(□-generalize-)}
7. if \(\Box \varphi \in E_i(r(n), t)\), then \(\varphi \in E_i(r(m), \parallel p t)\) for all \(m \leq n\). \hspace{1cm} \text{(□-access-)}
8. if \(\Box \varphi \in E_i(r(n), t)\), then \(\Box \Box \varphi \in E_i(r(n), \down p t)\). \hspace{1cm} \text{(□□-access-)}
9. if \(\Box \varphi \in E_i(r(n), t)\) and \(n > 0\), then \(\varphi \in E_i(r(n - 1), \Rightarrow p t)\). \hspace{1cm} \text{(□-right-)}
10. if \(\Box \varphi \in E_i(r(n), t)\) and \(n > 0\), then \(\varphi \in E_i(r(n - 1), \Rightarrow p t)\). \hspace{1cm} \text{(□-right-)}
11. if \(\varphi \in E_i(r(n - 1), t)\), then \(\Box \varphi \in E_i(r(n), \leq p t)\). \hspace{1cm} \text{(□-left-)}

**Lemma 8.3.** Let \((\overline{t_0}, \overline{t_1}, \ldots)\) be an acceptable sequence of elements of \(\text{MCS}_\chi\).

1. If \(\Box \varphi \in \Gamma_n\), then \(\varphi \in \Gamma_m\) for all \(m \geq n\).
2. If \(\Diamond \varphi \in \overline{t_m}\), then \(\varphi \in \overline{t_m}\) for some \(m \geq n\).
3. If \(\Box \Diamond \varphi \in \Gamma_n\), then \(\varphi \in \Gamma_m\) for some \(m \leq n\).
4. \(\Box \varphi \in \Gamma_n\) iff \(\varphi \in \Gamma_m\) for all \(m \leq n\).

**Proof.** 1. Suppose \(\Box \varphi \in \Gamma_n\) and there is \(m \geq n\) such that \(\varphi \notin \Gamma_m\). If \(m = n\), then by Lemma 8.3 we get \(\varphi \notin \Gamma_n\), which is a contradiction. Now suppose \(m > n\). Then, by Lemma 8.3 \(\Box \varphi \in \Gamma_n\). By Lemma 14.6 \(\Box \varphi \in \Gamma_{n+1}\). By repeating the argument, we get \(\Box \varphi \in \Gamma_m\), and hence \(\varphi \in \Gamma_m\) which is a contradiction.
2. Suppose \(\Diamond \varphi \in \overline{t_m}\). Then \(\tau U \varphi \in \overline{t_m}\). Since \((\overline{t_0}, \overline{t_1}, \ldots)\) is an acceptable sequence, there exists \(m \geq n\) such that \(\varphi \in \overline{t_m}\).

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3. Suppose $\Theta \varphi \in \Gamma_n$. Then $\tau \Sigma \varphi \in \Gamma_n$. Then either $\varphi \in \Gamma_n$ or $\tau \land \otimes (\tau \Sigma \varphi) \in \Gamma_n$. In the former case, we are done. In the latter case, by Lemma 1.6 we have $\tau \Sigma \varphi \in \Gamma_{n-1}$. Again from $\tau \Sigma \varphi \in \Gamma_{n-1}$ it follows that either $\varphi \in \Gamma_{n-1}$ or $\tau \land \otimes (\tau \Sigma \varphi) \in \Gamma_{n-1}$. In the former case, we are done. In the latter case, by Lemma 1.6 we have $\tau \Sigma \varphi \in \Gamma_{n-2}$. By repeating this argument, we finally get either $\varphi \in \Gamma_0$ or $\otimes (\tau \Sigma \varphi) \in \Gamma_0$. In the former case, we are done. In the latter case, by Lemma 3.3, we get a contradiction since $T_0$ is initial.

4. Suppose $\forall \varphi \in \Gamma_n$ and there is $m \leq n$ such that $\varphi \notin \Gamma_m$. If $m = n$, then by Lemma 3.3 we get $\varphi \in \Gamma_n$, which is a contradiction. Now suppose $m < n$. Then, by Lemma 3.3, $\forall \varphi \in \Gamma_n$. By Lemma 4.6, $\forall \varphi \in \Gamma_{n-1}$. By repeating the argument, we get $\exists \varphi \in \Gamma_m$, and hence $\varphi \in \Gamma_m$ which is a contradiction.

For the converse suppose that $\varphi \in \Gamma_m$ for all $m \leq n$, and $\forall \varphi \notin \Gamma_n$. Thus $\neg \exists \varphi \in \Gamma_n$, and hence $\Theta \neg \varphi \in \Gamma_n$. By clause 3 above, $\neg \varphi \in \Gamma_m$ for some $m \leq n$, which would contradict the assumption. □

**Theorem 8.4 (Soundness and completeness).** Let $CS$ be a constant specification for $LPLTL^P(Ax)$.

1. Suppose that (generalize) $\in Ax$. If $\varphi$ is provable in $LPLTL^P(Ax)_{CS}$, then $M \models \varphi$ for all $LPLTL^P(Ax)_{CS}$-models $M$.

2. Suppose that (generalize) $\notin Ax$. $\varphi$ is provable in $LPLTL^P(Ax)_{CS}$ iff $M \models \varphi$ for all $LPLTL^P(Ax)_{CS}$-models $M$.

**Proof.** The proof of soundness of $LPLTL^P(Ax)$, for arbitrary $Ax$, is straightforward.

Now suppose (generalize) is not in $Ax$. The proof of completeness of $LPLTL^P(Ax)$ is similar to the proof of Theorem 7.8 by constructing a canonical model. Let $M = (r, E_1, \ldots, E_h, \nu)$ be the $\chi$-canonical model for $CS$ with respect to an acceptable sequence $(T_0, T_1, \ldots)$ for $LPLTL^P(Ax)_{CS}$. Truth Lemma can be proved as before. The only new part is to show that $M$ is an $LPLTL^P(Ax)_{CS}$-model. We only check the details for ($\square$-access) and ($\exists$-generalize), the case of other principles are straightforward.

Let us show that the $\chi$-canonical model of $LPLTL^P(Ax)$, where ($\square$-access) is in $Ax$, satisfies the ($\exists$-generalize) condition of Definition 8.2. Suppose $\square \varphi \in E_i(r(n), \parallel t)$. We want to show that $\varphi \in E_i(r(m), \parallel t)$ for all $m \geq n$. It is enough to show that $[\parallel t]_i \varphi \in \Gamma_n$ for all $m \geq n$. From Definition 7.3, we get $[t]_i \varphi \in \Gamma_n$. From axiom ($\square$-access), we get $\square [\parallel t]_i \varphi \in \Gamma_n$. Thus, by Lemma 8.3, we have $[\parallel t]_i \varphi \in \Gamma_m$ for all $m \geq n$.

Let us now show that the $\chi$-canonical model of $LPLTL^P(Ax)$, where ($\exists$-generalize) is in $Ax$, satisfies the ($\exists$-generalize-$E$) condition of Definition 8.2. Suppose $\varphi \in E_i(r(m), t)$ for all $m \leq n$, and thus $[t]_i \varphi \in \Gamma_n$ for all $m \leq n$. By Lemma 8.3, we get $\exists [t]_i \varphi \in \Gamma_n$. From axiom ($\exists$-generalize), we have $[\parallel p]_i \varphi \in \Gamma_n$. Thus, $\exists \varphi \in E_i(r(n), \parallel p t)$ as desired.

We leave the completeness of $LPLTL^P(Ax)$, where $Ax$ contains (generalize), as an open problem. In Sections 8.2 and 8.3 we achieve the completeness of logics involving (generalize) by changing the justification logic part of $LPLTL^P$. 28
8.2 LPLTL$^P$ with indexed application operators

In this section we formalize temporal justification logics with indexed application operators, denoted by LPLTL$^I$. Terms and formulas of temporal justification logics with indexed application operators are constructed by the following mutual grammar:

$$t ::= c | x | t | t + t | t \cdot t,$$

$$\varphi ::= P | \bot | \varphi | \bigcirc \varphi | \bigotimes \varphi | \varphi \cup \varphi | \varphi S \varphi | [t]_i \varphi.$$ 

Axioms and rules of LPLTL$^I$ are exactly the same as for LPLTL$^P$, except that axiom (application) is replaced by the following axiom

$$-[t]_i(\varphi \rightarrow \psi) \rightarrow ([s]_i \varphi \rightarrow [t \cdot \varphi]_i \psi).$$

Interpreted systems for LPLTL$^I_{CS}$ and LPLTL$^I_{CS}$-models are defined as in Definitions 5.3 and 7.1 respectively with the difference that condition (application) of Definition 5.2 is replaced by the following condition:

$$- \text{ if } \varphi \rightarrow \psi \in \mathcal{E}_i(w, t) \text{ and } \varphi \in \mathcal{E}_i(w, s), \text{ then } \psi \in \mathcal{E}_i(w, t \cdot \varphi, s).$$

The notions of LPLTL$^I_{CS}$-validity is defined as usual. The proof of soundness and completeness theorems for annotated justification logics with respect to their models is similar to that of LPLTL$^P$.

**Theorem 8.5.** Let $CS$ be a constant specification for LPLTL$^I$. The formula $\varphi$ is provable in LPLTL$^I_{CS}$ iff $M \models \varphi$ for all LPLTL$^I_{CS}$-models $M$.

In order to prove completeness of logics involving axiom (generalize), we need to change the notion of subformula. The following definition is inspired by the work of Marti and Studer [21].

**Definition 8.6.** The set of subformulas, denoted by $\text{Subf}$, is defined by induction on the rank of formulas as follows:

- $\text{Subf}(P) := \{P\}$, and $\text{Subf}(\bot) := \{\bot\}$.
- $\text{Subf}(\ast \varphi) := \{\ast \varphi\} \cup \text{Subf}(\varphi)$, where $\ast \in \{\bigcirc, \bigotimes\}$.
- $\text{Subf}(\varphi \ast \psi) := \{\varphi \ast \psi\} \cup \text{Subf}((\varphi \cup \text{Subf}(\psi))$, where $\ast \in \{\rightarrow, S, \cup\}$.
  - $\text{Subf}([x]_i \varphi) := \{[x]_i \varphi\} \cup \text{Subf}(\varphi)$.
  - $\text{Subf}([c]_i \varphi) := \{[c]_i \varphi\} \cup \text{Subf}(\varphi)$.
  - $\text{Subf}([t + s]_i \varphi) := \{[t + s]_i \varphi\} \cup \text{Subf}([t]_i \varphi) \cup \text{Subf}([s]_i \varphi)$.
  - $\text{Subf}([s \cdot t]_i \psi) := \{[s \cdot t]_i \psi\} \cup \text{Subf}([s]_i (\varphi \rightarrow \psi)) \cup \text{Subf}([t]_i \varphi)$.
  - $\text{Subf}([\lceil t]]_i \varphi) := \{[\lceil t]]_i \varphi\} \cup \text{Subf}(\varphi)$.
  - $\text{Subf}([\lfloor t\rfloor]_i \varphi) := \{[\lfloor t\rfloor]_i \varphi\} \cup \text{Subf}(\varphi)$.
- $\text{Subf}([\uparrow t]_i \varphi) := \{[\uparrow t]_i \varphi\} \cup \text{Subf}(\varphi)$, $\varphi$ is not an $\square$-formula.
- $\text{Subf}([\uparrow t]_i \square \varphi) := \{[\uparrow t]_i \square \varphi\} \cup \text{Subf}([t]_i \varphi)$.
- $\text{Subf}([\downarrow t]_i \varphi) := \{[\downarrow t]_i \varphi\} \cup \text{Subf}([t]_i \square \varphi)$.

2 The indexed application operators were first suggested by Renne [23].
we aim to prove completeness of LPLTL. To keep the notation simple, let LPLTL

Moreover, we extend the subformula relation Sub by transitivity.

8.3 Completeness for (generalize)

As before let LPLTL(Ax) denote the result of adding axioms from Ax to LPLTL.

To keep the notation simple, let $L_{gen}^\text{m}$ denote LPLTL((generalize)). In this section we aim to prove completeness of $L_{gen}^\text{m}$.

For a formula $\chi$, let

$$B_\chi := \text{Subf}(\chi) \cup \text{Subf}(\top \alpha \land \neg \phi \land \square [t_i] \phi \mid [\top t_i] \phi \in \text{Subf}(\chi))$$

$$\cup \text{Subf}(\square [\parallel t_i] \phi \mid [\parallel t_i] \phi \in \text{Subf}(\chi)) \cup \text{Subf}(\parallel [\parallel t_i] \phi \mid [\parallel t_i] \phi \in \text{Subf}(\chi))$$

$$\cup \text{Subf}(\parallel [\rightarrow p t_i] \phi \mid [\rightarrow p t_i] \phi \in \text{Subf}(\chi)) \cup \text{Subf}(\parallel [\rightarrow p t_i] \phi \mid [\rightarrow p t_i] \phi \in \text{Subf}(\chi)),$$

and

$$\text{Subf}^+(\chi) := B_\chi \cup \{\neg \phi \mid \phi \in B_\chi\}.$$ 

Let $\text{MCS}_\chi^0$ denote the set of all $\chi$-maximally $L_{CS}$-consistent subsets of $\text{Subf}^+(\chi)$.

For $\Gamma \in \text{MCS}$, let

$$\overline{\Gamma} := \Gamma \cap \text{Subf}^+(\chi).$$

Note that all the results of Section 4 are valid if $\text{Subf}^+(\chi)$ is replaced by $\text{Subf}^+(\chi)$, and $\sim_{CS}$ is replaced by $\sim_{L_{CS}}$. Since the proofs of the results of Section 4 have been given in details, we only outline the necessary changes here while omitting the proofs.

**Lemma 8.7.**

$$\text{MCS}_\chi^0 = \{\overline{\Gamma} \mid \Gamma \in \text{MCS}\}.$$ 

**Proof.** Similar to the proof of Lemma 4.2. □

\textsuperscript{3} For simplicity we use the same symbol $\overline{\Gamma}$ as in Section 4.
Lemma 8.8. Let $\mathcal{T} \in \text{MCS}^\dagger_{\chi}$. 

1. If $\phi \in B_{\chi}$ and $\phi \notin \mathcal{T}$, then $\neg \phi \in \mathcal{T}$.
2. If $\phi \in \text{Subf}^+(\chi)$ and $\mathcal{T} \vdash_{\chi} \phi$, then $\phi \in \mathcal{T}$.
3. If $\psi \in \text{Subf}^+(\chi)$, $\phi \in \mathcal{T}$ and $\vdash_{\chi} \phi \rightarrow \psi$, then $\psi \in \mathcal{T}$.

Proof. The proof of all items are standard. \hfill $\Box$

Lemma 8.9. If either $\phi \cup \psi \in \text{Subf}^+(\chi)$ or $\phi \downarrow \psi \in \text{Subf}^+(\chi)$, then $\phi, \psi \in B_{\chi}$.

Proof. We first show that if $\phi \cup \psi \in \text{Subf}^+(\chi)$, then $\phi, \psi \in B_{\chi}$. There are three cases:

1. $\phi \cup \psi \in \text{Subf}(\chi)$. Clearly $\phi, \psi \in \text{Subf}(\chi)$, and hence $\phi, \psi \in B_{\chi}$.
2. $\phi = \top, \psi = \neg [t]_{\sigma} \in \text{Subf}(\chi)$. Then from $\neg [t]_{\sigma} \in \text{Subf}(\neg [t]_{\sigma})$, it follows that $\psi \in \text{Subf}(\chi)$. Thus $\phi, \psi \in B_{\chi}$.
3. $\phi \cup \psi \in \text{Subf}(\neg [t]_{\sigma})$ such that $\neg [t]_{\sigma} \in \text{Subf}(\chi)$. In this case $\phi \cup \psi \in \text{Subf}(\sigma)$. Since $\sigma \in \text{Subf}(\chi)$, we get $\phi \cup \psi \in \text{Subf}(\chi)$ and we reduce to case 1.

Now we show that if $\phi \downarrow \psi \in \text{Subf}^+(\chi)$, then $\phi, \psi \in B_{\chi}$. There are three cases:

1. $\phi \downarrow \psi \in \text{Subf}(\chi)$. Clearly $\phi, \psi \in \text{Subf}(\chi)$, and hence $\phi, \psi \in B_{\chi}$.
2. $\phi = \top, \psi = \Box_{[t]_{\sigma}}$. Then $\phi, \psi \in B_{\chi}$ as desired.
3. $\phi \downarrow \psi \in \text{Subf}(\top \neg [t]_{\sigma})$ such that $\neg [t]_{\sigma} \in \text{Subf}^+(\chi)$. In this case $\phi \downarrow \psi \in \text{Subf}(\neg [t]_{\sigma})$, and hence $\phi \downarrow \psi \in \text{Subf}(\neg [t]_{\sigma})$. Since $\neg [t]_{\sigma} \in \text{Subf}(\neg [t]_{\sigma})$, we get $\phi \downarrow \psi \in \text{Subf}(\chi)$ and we reduce to case 1. \hfill $\Box$

Using Lemma 8.9 it is not difficult to show the following results (the proofs are similar to the proofs of Lemmas 8.10 and 8.11 and thus are omitted here).

Lemma 8.10. For every $\mathcal{T} \in \text{MCS}^\dagger_{\chi}$, if $\phi \cup \psi \in \mathcal{T}$, then there exists a $\phi \cup \psi$-sequence starting with $\mathcal{T}$.

Lemma 8.11. For every $\mathcal{T} \in \text{MCS}^\dagger_{\chi}$, if $\phi \downarrow \psi \in \mathcal{T}$, then there exists a $\phi \downarrow \psi$-sequence ending with $\mathcal{T}$.

Corollary 8.12. For every $\mathcal{T} \in \text{MCS}_{\chi}$, there is an acceptable sequence containing $\mathcal{T}$.

The following is an auxiliary lemma to be used in the proof of completeness.

Lemma 8.13. Let $(\mathcal{T}_0, \mathcal{T}_1, \ldots)$ be an acceptable sequence of elements of $\text{MCS}^\dagger_{\chi}$. If $\neg [t]_{\sigma} \notin \text{Subf}(\chi)$ and $\neg [t]_{\sigma} \in \Gamma_n$ for all $n \geq m$, then $\neg [t]_{\sigma} \notin \Gamma_n$.

Proof. Suppose towards a contradiction that $\neg [t]_{\sigma} \notin \Gamma_n$. Thus $\neg [t]_{\sigma} \notin \Gamma_n$, and then by (generalize) we have $\neg [t]_{\sigma} \notin \Gamma_n$. Hence $\neg [t]_{\sigma} \notin \Gamma_n$. Note that $\neg [t]_{\sigma} \notin \Gamma_n$ is an abbreviation for $\neg [t]_{\sigma} \notin \Gamma_n$. On the other hand, from $\neg [t]_{\sigma} \notin \Gamma_n$, it follows that $\neg [t]_{\sigma} \notin \Gamma_n$. Thus $\neg [t]_{\sigma} \notin \Gamma_n$. Since $(\mathcal{T}_0, \mathcal{T}_1, \ldots)$ is an acceptable sequence, there exists $m \geq n$ such that $\neg [t]_{\sigma} \notin \mathcal{T}_m$, and hence $\neg [t]_{\sigma} \notin \Gamma_n$ which contradicts the hypothesis of the Lemma. \hfill $\Box$

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Given an $L^\text{gen}$-model $\mathcal{M} = (r, S, \mathcal{E}_1, \ldots, \mathcal{E}_h, \nu)$ for $\mathsf{CS}$ and a ternary relation $B \subseteq S \times \mathsf{TM} \times \mathsf{Fml}$ and an agent $i$, we define an operator

$$\phi^B_i : \mathcal{P}(S \times \mathsf{TM} \times \mathsf{Fml}) \to \mathcal{P}(S \times \mathsf{TM} \times \mathsf{Fml})$$

for $\mathsf{CS}$ by

$$\phi^B_i(X) := \{(r(n), t, \varphi) \mid (r(n), t, \varphi) \in B \lor$$

$$\exists r, s(t = r + s \land ((r(n), r, \varphi) \in X \lor (r(n), s, \varphi) \in X)) \lor$$

$$\exists r, s, \psi(t = r \land s \land (r(n), r, \psi) \in X \land (r(n), s, \psi) \in X) \lor$$

$$\exists r, s, \psi(t = r \land \varphi = [r] \land \psi(\phi) (r(n), s, \psi) \in X) \lor$$

$$\exists r, s, \psi(t = r \land \varphi = \square \psi \land \forall m \geq n(r(m), s, \psi) \in X)\}$$

Obviously $\phi^B_i$ is monotone, i.e.

$$X \subseteq Y \implies \phi^B_i(X) \subseteq \phi^B_i(Y).$$

Therefore, $\phi^B_i$ has a least fixed point, which we denote by $\mathcal{E}^B_i$. That means $\mathcal{E}^B_i$ is the least $X \subseteq S \times \mathsf{TM} \times \mathsf{Fml}$ with $X = \phi^B_i(X)$.

**Definition 8.14.** Let $(X_0, X_1, \ldots)$ be an acceptable sequence of elements of $\mathcal{M}^\text{cs}_\chi$ for $L^\text{gen}$.

The $\chi$-canonical model $\mathcal{M} = (r, S, \mathcal{E}_1, \ldots, \mathcal{E}_h, \nu)$ for $\mathsf{CS}$ with respect to $(X_0, X_1, \ldots)$ is defined as follows:

1. $S := \{X_0, X_1, \ldots\}$.
2. $r(n) := X_n$.

Now we define relations $B_i \subseteq S \times \mathsf{TM} \times \mathsf{Fml}$ for each agent $i \in \mathsf{Ag}$ by

$$(X_n, t, \varphi) \in B_i \iff X_n \vdash_{L^\text{gen}} [t]_i \varphi$$

3. $\mathcal{E}_i(X_n, t) := \{\varphi \mid (X_n, t, \varphi) \in \mathcal{E}^B_i\}$.
4. $\nu(X_n) := \mathsf{Prop} \cap X_n$.

**Lemma 8.15.** The $\chi$-canonical model $\mathcal{M} = (r, S, \mathcal{E}_1, \ldots, \mathcal{E}_h, \nu)$ for $\mathsf{CS}$ with respect to an acceptable sequence $(X_0, X_1, \ldots)$ is an $L^\text{gen}$-model.

**Proof.** We only verify the condition (genericize-$\mathcal{E}$) of Definition 8.2. Suppose that $\varphi \in \mathcal{E}_i(r(m), t)$ for all $m \geq n$. Thus $(r(m), t, \varphi) \in \mathcal{E}^B_i$ for all $m \geq n$. Since $\mathcal{E}^B_i$ is a fixed point of $\phi^B_i$, we immediately get $(r(n), \uparrow t, \sqcap \varphi) \in \mathcal{E}^B_i$. Hence $\Box \varphi \in \mathcal{E}_i(r(n), \uparrow t)$, as desired.

**Lemma 8.16.** If $[t]_i \varphi \in \mathsf{Subf}^\uparrow(\chi)$ and $(r(n), t, \varphi) \in \mathcal{E}^B_i$, then $[t]_i \varphi \in \Gamma_n$.

**Proof.** By induction on the build-up of $\mathcal{E}^B_i$. We distinguish the following cases:

1. Base case. The case $(r(n), t, \varphi) \in B_i$ is trivial.
2. \( \exists r, s (t = r + s \land ((r(n), r, \varphi) \in E^*_1 \lor (r(n), s, \varphi) \in E^*_1)) \). Since \([t], \varphi \in \text{Subf}^+(\chi)\) we get \([r], \varphi \in \text{Subf}^+(\chi)\) and \([s], \varphi \in \text{Subf}^+(\chi)\). By I.H. we get \([r], \varphi \in \Gamma_n\) or \([s], \varphi \in \Gamma_n\). Then \([r + s], \varphi \in \Gamma_n\), and thus \([t], \varphi \in \Gamma_n\). The case where \(t = r \cdot s\) or \(t = r\) is treated similarly.

3. \( \exists r, \psi (t = \| r \land \varphi = \Box \psi \land \forall m \geq n (r(m), r, \psi) \in E^*_1) \). It is easy to show that from \([t], \varphi \in \text{Subf}^+(\chi)\) it follows that \([t], \psi \in \text{Subf}^+(\chi)\) and \([r], \psi \in \text{Subf}^+(\chi)\). By the induction hypothesis, for all \(m \geq n\) we have \([r], \psi \in \Gamma_m\). By Lemma 8.13 we get \([\| r], \psi \in \Gamma_m\), and therefore \([t], \varphi \in \Gamma_n\) as desired. \(\square\)

Lemma 8.17 (Truth Lemma). Let \(\mathcal{M} = (r, S, \mathcal{E}_1, \ldots, \mathcal{E}_h, \nu)\) be the \(\chi\)-canonical model for \(CS\) with respect to an acceptable sequence \((X_0, X_1, \ldots)\). For every formula \(\psi \in \text{Subf}^+(\chi)\), and every \(n \in \mathbb{N}\) we have:

\[ (\mathcal{M}, r(n)) = \psi \quad \text{iff} \quad \psi \in r(n). \]

Proof. As usual, the proof is by induction on the structure of \(\psi\). We show only the following case:

\( - \psi = [t], \varphi. \)

\( (\Rightarrow) \) If \((\mathcal{M}, r(n)) \models [t], \varphi\), then \((r(n), t, \varphi) \in E^*_1\). Thus, by Lemma 8.16 \([t], \varphi \in r(n)\).

\( (\Leftarrow) \) If \([t], \varphi \in r(n)\), then \((r(n), t, \varphi) \in E^*_1\). By (reflexivity), we have \(\varphi \in \Gamma_n\) and by I.H. we get \((\mathcal{M}, r(n)) \models \varphi\). We conclude \((\mathcal{M}, r(n)) \models [t], \varphi\). \(\square\)

Theorem 8.18 (Soundness and completeness). Let \(CS\) be a constant specification for \(L^{\text{gen}}\). Then we have \(\vdash_{CS} \varphi \quad \text{iff} \quad \mathcal{M} \models \varphi\) for all \(L^{\text{gen}}_{CS}\)-models \(\mathcal{M}\).

Theorem 8.19 (Soundness and completeness). Let \(Ax = \{\text{generalize}\}, (\Box\text{-left})\). Then \(L^{\text{gen}}_{Ax}\) is sound and complete with respect to all interpreted systems of \(L^{\text{gen}}_{Ax}\) satisfying \((\text{generalize-}\mathcal{E}), (\Box\text{-left-}\mathcal{E})\), and \((\Box\text{-left-}\mathcal{R})\).

Proof. We detail the proof for the soundness part. The proof of completeness is similar to the proof of Theorem 8.18.

Let \(Ax = \{\text{generalize}\}, (\Box\text{-left})\) and \(\mathcal{I} = (\mathcal{R}, S, R_1, \ldots, R_h, \mathcal{E}_1, \ldots, \mathcal{E}_h, \nu)\) be an arbitrary interpreted system for \(L^{\text{gen}}_{Ax}\). For an arbitrary \(r \in \mathcal{R}\) and \(n \in \mathbb{N}\), assume \((\mathcal{I}, r, n) \models \Box[t], \varphi\). Thus, \((\mathcal{I}, r, m) \models [t], \varphi\) for every \(m \geq n\). Hence, \(\varphi \in \mathcal{E}_i(r(m), t)\) for every \(m \geq n\). By (generalize-\mathcal{E}), we get \(\Box \varphi \in \mathcal{E}_i(r(n), \| t)\).

Now let \(r(n)R_r r'(n')\) and \(m' \geq n'\), for arbitrary \(r' \in \mathcal{R}\) and arbitrary \(n', m' \in \mathbb{N}\). By (\Box\text{-left-}\mathcal{R}) we have \(r(n+m'-n')R_r r'(m')\). On the other hand, from the assumption we have \((\mathcal{I}, r, n + m' - n') \models [t], \varphi\). Thus, \((\mathcal{I}, r', m') \models \varphi\). Since \(n' \geq m\) was chosen arbitrary we get \((\mathcal{I}, r', n') \models \Box \varphi\), and since \(r'(n')\) was chosen arbitrary, we get \((\mathcal{I}, r, n) \models [\| t], \Box \varphi\) as desired. \(\square\)

We close this section with remarking that it is quite possible to extend this completeness result to extensions of \(L^{\text{gen}}\). For example consider the logic \(L^{\text{gen}}_{Ax}\) where \(Ax = \{\text{generalize}\}, (\Box\text{-generalize})\). In order to prove completeness for \(L^{\text{gen}}_{Ax}\), redefine the operator \(\Phi^B_i\) as follows:

\[ 33 \]
\[ Φ^B_i(X) := \{(r(n), t, \varphi) \mid (r(n), t, \varphi) \in B \lor \exists r, s(t = r + s \land ((r(n), r, \varphi) \in X \lor (r(n), s, \varphi) \in X)) \lor \exists r, s, ψ(t = r \cdot ψ = [r], ψ \land (r(n), r, ψ) \in X) \lor \exists r, ψ(t = !r \land \varphi = \Box ψ \land \forall m \geq n(r(m), r, ψ) \in X) \lor \exists r, ψ(t = \diamond P r \land \varphi = \Diamond ψ \land \forall m \leq n(r(m), r, ψ) \in X)\} \]

The rest of the proof of soundness and completeness is similar to that of \( L^\text{gen} \).

9 Internalization

**Definition 9.1.** A justification logic \( L \) satisfies internalization if for each formula \( \varphi \) with \( \vdash L \varphi \) and for each agent \( i \), there exists a term \( t \) with \( \vdash L [t]_i \varphi \).

\( \text{LPLTL}^P \) satisfies a restricted form of internalization.

**Lemma 9.2.** Let \( CS \) be an axiomatically appropriate constant specification for \( \text{LPLTL}^P \). For each formula \( \varphi \) and each \( i \), if \( \vdash CS \varphi \) and \( (\text{MP}) \) and \( (\text{iax-nec}) \) are the only rules that are used in the derivation of \( \varphi \), then \( \vdash CS [t]_i \varphi \) for some term \( t \).

**Proof.** We proceed by induction on the derivation of \( \varphi \).

In case \( \varphi \) is an axiom, since \( CS \) is axiomatically appropriate, there is a constant \( c \) with \( \vdash CS [c]_i \varphi \).

In case \( \varphi \) is derived by modus ponens from \( ψ \to \varphi \) and \( ψ \), then, by the induction hypothesis, there are terms \( s_1 \) and \( s_2 \) such that \( [s_1]_i (ψ \to ϕ) \) and \( [s_2]_i ψ \) are provable. Using \( (\text{application}) \) and modus ponens, we obtain \( [s_1 \cdot s_2]_i ϕ \).

In case \( \varphi \) is \( [c_{j_n}]_{i_0} \ldots [c_{j_1}]_{i_1} ψ \), derived using \( (\text{iax-nec}) \), since \( CS \) is axiomatically appropriate, we can use \( (\text{iax-nec}) \) again to obtain \( [c_{j_{n+1}}]_i \varphi \) for some justification constant \( c_{j_{n+1}} \).

Next we shall extend \( \text{LPLTL}^P \) to obtain a justification logic with the internalization property. Although the following two formulas are provable in \( \text{LPLTL}^P \), see Lemma 23, in order to get the internalization property we need to add them as axioms:

1. \( \Box \varphi \to \Diamond \varphi \)  
   \( \text{mix1} \)
2. \( \Diamond \varphi \to \Diamond \varphi \)  
   \( \text{mix2} \)

Let \( \text{LPLTL}^\text{int} \) be the logic \( \text{LPLTL}^P \) extended by the axioms \( (\text{generalize}) \), \( (\Box \text{-generalize}) \), \( (\text{mix1}) \), and \( (\text{mix2}) \).
Theorem 9.3 (Internalization). Let CS be an axiomatically appropriate constant specification for LPLTL\textsuperscript{int}. The system LPLTL\textsuperscript{int}\textsubscript{CS} enjoys internalization.

Proof. Suppose that \( \varphi \) is provable in LPLTL\textsuperscript{int}. Let \( i \) be an arbitrary agent. We have to show that \([t]_i \varphi\) is provable in LPLTL\textsuperscript{int}\textsubscript{CS}, for some term \( t \). We proceed by induction on the derivation of \( \varphi \). We only consider the following cases:

In case \( \varphi = \Box \psi \), derived using (\( \Box \)-\text{nec}), then, by the induction hypothesis, there is a term \( s \) such that \([s] \varphi\) is provable. Now, we can use (\( \Box \)-\text{nec}) in order to obtain \( \Box [s] \psi \) and then (\text{generalize}) and modus ponens to get \([\uparrow s] \Box \psi\).

In case \( \varphi = \Diamond \psi \), derived using (\( \Diamond \)-\text{nec}), then, as above, we obtain \([\uparrow s] \Box \psi\). Since CS is axiomatically appropriate, there is a constant \( c \) such that \([c]((\Box \psi \rightarrow \Diamond \psi)\). Thus we finally conclude \([c \cdot \uparrow s] \Diamond \psi\).

In case \( \varphi = \lozenge \psi \), derived using (\( \lozenge \)-\text{nec}), then, by the induction hypothesis, there is a term \( s \) such that \([s] \varphi\) is provable. Now, we can use (\( \lozenge \)-\text{nec}) in order to obtain \( \lozenge [s] \psi \) and then (\( \lozenge \)-\text{generalize}) and modus ponens to get \([\uparrow p \uparrow s] \lozenge \psi\).

In case \( \varphi = \bigcirc \psi \), derived using (\( \bigcirc \)-\text{nec}), then, as above, we obtain \([\uparrow p \uparrow s] \lozenge \psi\). Since CS is axiomatically appropriate, there is a constant \( c \) such that \([c]((\Box \psi \rightarrow \bigcirc \psi)\). Thus we finally conclude \([c \cdot \uparrow p \uparrow s] \bigcirc \psi\). \( \Box \)

Remark 9.4. It is worth noting that there are already some known temporal justification logics that satisfy internalization, although they are formalized using only future operators. Bucheli in [5] show that, for axiomatically appropriate constant specifications, the logics LPLTL+(\text{generalize})+(\( \bigcirc \)-\text{access}) and LPLTL+(\text{generalize})+(\( \bigcirc \)-\text{access})+(\( \bigcirc \)-\text{left}) satisfy internalization.\(^4\) In [6] the authors introduced another extension of LPLTL, which was called LPLTL\textsuperscript{+} there, that satisfies internalization.

Theorem 9.5 (Internalization). Let CS be an axiomatically appropriate constant specification for LPLTL\textsuperscript{P}\textsuperscript{(Ax)} where

\[ \{(\text{generalize}), (\text{\( \equiv \)}-\text{generalize}), (\text{\( \bigcirc \)}-\text{access}), (\text{\( \bigcirc \)}-\text{access})\} \subseteq \text{Ax}. \]

Then LPLTL\textsuperscript{P}\textsuperscript{(Ax)}\textsubscript{CS} enjoys internalization.

Proof. The proof is similar to the proof of Theorem 9.3. We only consider the following cases:

In case \( \varphi = \Diamond \psi \), derived using (\( \Diamond \)-\text{nec}), then, as in the proof of Theorem 9.3 we obtain \([\uparrow s] \Box \psi\). Then, by (\( \Diamond \)-\text{access}), we get \([\downarrow \uparrow s] \Diamond \psi\).

In case \( \varphi = \bigcirc \psi \), derived using (\( \bigcirc \)-\text{nec}), then, as in the proof of Theorem 9.3 we obtain \([\downarrow \uparrow p \uparrow s] \lozenge \psi\). Then, by (\( \bigcirc \)-\text{access}), we get \([\downarrow \uparrow p \uparrow p \uparrow s] \bigcirc \psi\). \( \Box \)

In Theorems 9.3 and 9.5 we present two logics that satisfy internalization. We now prove that these two logics have the following relationship.

Lemma 9.6. Let CS be an axiomatically appropriate constant specification for LPLTL\textsuperscript{P}\textsuperscript{(Ax)} where \{\( (\text{mix1}), (\text{mix2}) \}\} \subseteq \text{Ax}. For every agent \( i \), formula \( \varphi \) and term \( t \) there are terms \( s_1(t) \) and \( s_2(t) \) such that

\[ \vdash_{\text{LPLTL}^{\text{P(Ax)}}\textsubscript{CS}} [t]_i \square \varphi \rightarrow [s_1(t)]_i \Box \varphi, \quad \text{and} \]

\(^4\) Note that the background logic used by Bucheli in [5] is different from LPLTL.
Thus, versions of (\(\bigcirc\)-access) and (\(\boxdot\)-access) are derivable in \(\text{LPLTL}^\text{P}(Ax)_{\text{CS}}\).

**Proof.** Since \(\text{CS} \) is axiomatically appropriate and (mix1) and (mix2) are axioms of \(\text{LPLTL}^\text{int}\), there are justification constants \(a\) and \(b\) such that \([a]_i(\bigcirc \varphi \rightarrow \bigcirc \varphi) \in \text{CS} \) and \([b]_i(\boxdot \varphi \rightarrow \boxdot \varphi) \in \text{CS} \). Thus

\[\vdash \text{LPLTL}_{\text{CS}}^\text{P} [t]_i \bigcirc \varphi \rightarrow [a \cdot t]_i \bigcirc \varphi, \quad \text{and} \]
\[\vdash \text{LPLTL}_{\text{CS}}^\text{int} [t]_i \Box \varphi \rightarrow [b \cdot t]_i \Box \varphi\]

Finally put \(s_1(t) := a \cdot t\) and \(s_2(t) := b \cdot t\). \(\Box\)

Combining Theorems 9.3, 9.5 with the results of Section 8.3, we then can obtain temporal justification logics, based on \(\text{LPLTL}^\text{I}\), that satisfy both internalization and completeness. Note that, since (mix1) and (mix2) are true in all \(\text{LPLTL}^\text{I}\)-models, the class of all models of

\(\text{LPLTL}^\text{I}(\{\text{generalize}, \text{\(\Box\)-generalize}, \text{mix1}, \text{mix2}\})\)

is the same as the class of all models of

\(\text{LPLTL}^\text{I}(\{\text{generalize}, \text{\(\bigcirc\)-generalize}\})\).

**Theorem 9.7 (Completeness and Internalization).** Let \(\text{L} \) be the logic \(\text{LPLTL}^\text{I}\) extended by either of the following set of axioms:

1. \(\{\text{generalize}, \text{\(\bigcirc\)-generalize}, \text{mix1}, \text{mix2}\}\), or
2. \(\{\text{generalize}, \text{\(\bigcirc\)-generalize}, \text{(mix1)}, \text{(mix2)}\}\).

Let \(\text{CS} \) be an axiomatically appropriate constant specification for \(\text{L} \). Then \(\text{L}_{\text{CS}}\) enjoys internalization and is sound and complete with respect to \(\text{L}_{\text{CS}}\)-models.

**Proof.** Follows from Theorems 9.3, 9.5, 8.18 \(\Box\)

10 **No forgetting and no learning**

No forgetting (or perfect recall) and no learning are two well known properties of systems that can be expressed in the language of logics of knowledge and time. It seems that the axioms (\(\bigcirc\)-access) and (\(\Box\)-access) correspond respectively to the notions of no forgetting and no learning on justifications. Let’s make this precise.

A formula \(\varphi \) is said to be stable with respect to the future if once it is true it remains true, i.e. \(\vdash \varphi \rightarrow \bigcirc \varphi \). In the framework of logics of knowledge and time, it is known that if a logic contains the axiom \(K_i \Box \varphi \rightarrow \Box K_i \varphi\), then for every formula \(\varphi\) which is stable with respect to the future it can be shown that \(\vdash K_i \varphi \rightarrow \Box K_i \varphi\), i.e. if \(\varphi\) is known at some point then it remains known at all points in the future (see [8]). Likewise, we show that logics that contain axiom (\(\bigcirc\)-access), i.e. \([t]_i \Box \varphi \rightarrow \Box [\downarrow t]_i \varphi\), have a similar property.
Theorem 10.1. Let \( Ax \supseteq \{(\Box\text{-}\text{access})\} \) and let \( L = \text{LPLTL}^P(Ax)_{CS} \) be a justification logic that satisfies internalization. If
\[
\vdash_L \varphi \rightarrow \Box \varphi,
\]
then for every term \( t \) there is a term \( s(t) \) such that
\[
\vdash_L [t]_i \varphi \rightarrow [s(t)]_i \varphi.
\]

Proof. Suppose that \( \varphi \rightarrow \Box \varphi \) is provable in \( \text{LPLTL}^P(Ax)_{CS} \), where \( Ax \supseteq \{(\Box\text{-}\text{access})\} \). Thus, by the internalization property of \( \text{LPLTL}^P(Ax)_{CS} \), we get \([r]_i(\varphi \rightarrow \Box \varphi)\) for some term \( r \). Hence, for every term \( t \), \([t]_i \varphi \rightarrow [r \cdot \varphi t]_i \Box \varphi \), and therefore by axiom \((\Box\text{-}\text{access})\) we get \([t]_i \varphi \rightarrow \Box [r \cdot \varphi t]_i \varphi\). Thus, for every term \( t \) it is enough to put \( s := [\ddownarrow_p (r \cdot \varphi t)]_i \varphi \).

Using past time operators, a similar argument can be done for no learning. A formula \( \varphi \) is said to be stable with respect to the past if once it is true it has always been true, i.e. \( \vdash \varphi \rightarrow \Box \varphi \). Using axiom \( K_i \Box \varphi \rightarrow \Box K_i \varphi \), it is easy to show that for every formula \( \varphi \) which is stable with respect to the past we have \( \vdash K_i \varphi \rightarrow \Box K_i \varphi \), i.e. if \( \varphi \) is known at some point then it has always been known at all points in the past. Note that, since \( \Box \varphi \rightarrow \Diamond \psi \) is a valid formula for every \( \psi \), \( K_i \varphi \rightarrow \Box K_i \varphi \) in turn entails \( K_i \varphi \rightarrow \Diamond K_i \varphi \), i.e. if \( \varphi \) is known at some point then it was known at some point in the past. We show that logics that contain axiom \((\Diamond\text{-}\text{access})\), i.e. \([t]_i \varphi \rightarrow \Box [\ddownarrow_p t]_i \varphi \), have a similar property.

Theorem 10.2. Let \( Ax \supseteq \{(\Diamond\text{-}\text{access})\} \) and let \( L = \text{LPLTL}^P(Ax)_{CS} \) be a justification logic that satisfies internalization. If
\[
\vdash_L \varphi \rightarrow \Diamond \varphi,
\]
then for every term \( t \) there is a term \( s(t) \) such that
\[
\vdash_L [t]_i \varphi \rightarrow [s(t)]_i \varphi.
\]

Proof. Suppose that \( \varphi \rightarrow \Diamond \varphi \) is provable in \( \text{LPLTL}^P(Ax)_{CS} \), where \( Ax \supseteq \{(\Diamond\text{-}\text{access})\} \). Thus, by the internalization property of \( \text{LPLTL}^P(Ax)_{CS} \), we get \([r]_i(\varphi \rightarrow \Diamond \varphi)\) for some term \( r \). Hence, for every term \( t \), \([t]_i \varphi \rightarrow [r \cdot \varphi t]_i \Diamond \varphi \), and therefore by axiom \((\Diamond\text{-}\text{access})\) we get \([t]_i \varphi \rightarrow \Diamond [r \cdot \varphi t]_i \varphi\). Thus, for every term \( t \) it is enough to put \( s := [\ddownarrow_p (r \cdot \varphi t)]_i \varphi \).

In the framework of logics of knowledge and time, it is known that the following principles would characterize systems with no forgetting \((\text{nf})\) and no learning \((\text{nl})\) respectively (cf. [8] [10]):
\[
\begin{align*}
- & K_i \varphi S K_i \psi \rightarrow K_i (K_i \varphi S K_i \psi) & \text{(nf)} \\
- & K_i \varphi U K_i \psi \rightarrow K_i (K_i \varphi U K_i \psi) & \text{(nl)}
\end{align*}
\]

Now let us consider the justification counterparts of the above axioms. The following principles could be considered as justification counterparts of \((\text{nf})\) and \((\text{nl})\) respectively.
In this section we explore more interactions between justification and time. Let us start with the axiom (application):

\[ [t]i(\varphi \rightarrow \psi) \rightarrow ([s]i\varphi \rightarrow [t \cdot s]i\psi). \]

This axiom says that if agent \( i \) knows \( \varphi \rightarrow \psi \) for reason \( t \) and she knows \( \varphi \) for reason \( s \), then at the same time she knows \( \psi \) for reason \( t \cdot s \). So the agent applies

\[ [t]i\varphi \rightarrow [s]i\varphi \rightarrow [t \cdot s]i\psi \]
the rule Modus Ponens (MP) in her reasoning, but this step of reasoning takes no time. Thus, at a given moment of time the agent knows all consequences of her knowledge. This is related to the Logical Omniscience Problem. This would be implausible if we expect that reasoning takes time. The same argument can be applied to the axioms (sum) and (positive introspection).

In [6] the following principles have been suggested to formalize the idea that reasoning with justifications takes time:

\[
\begin{align*}
[t]_i(\varphi \rightarrow \psi) & \rightarrow ([s]_i\varphi \rightarrow \bigcirc [t \cdot s]_i\psi), \\
[t]_i\varphi \lor [s]_i\varphi & \rightarrow \bigcirc [t + s]_i\varphi, \\
[t]_i\varphi & \rightarrow \bigcirc [t]_i[s]_i\varphi.
\end{align*}
\]

At first sight the above principles seem to be impeccable, but it is not difficult to show that they have the following implausible consequences:

\[
\begin{align*}
[t]_i\varphi & \rightarrow \bigcirc \varphi, \\
[t]_i\varphi & \rightarrow \bigcirc [t]_i\varphi, \\
[t]_i\varphi & \rightarrow \Box [t]_i\varphi, \\
[t]_i\varphi & \rightarrow \Box \varphi.
\end{align*}
\]

where \( t \in Tm \) and \( \varphi \in Fml \) are arbitrary.

In the following we study another variant of the above principles\(^5\). The logic LTL\(^J\) is defined similar to LPLTL\(^P\) with the difference that axioms of the justification part are replaced by the following axioms

1. \([t]_i(\varphi \rightarrow \psi) \rightarrow ([s]_i\varphi \rightarrow \bigcirc [t \cdot s]_i\Box \psi)\) (FP-application)
2. \([t]_i\varphi \rightarrow \bigcirc [t + s]_i\Box \varphi, \ [s]_i\varphi \rightarrow \bigcirc [t + s]_i\Box \varphi\) (FP-sum)
3. \([t]_i\varphi \rightarrow \varphi\) (reflexivity)
4. \([t]_i\varphi \rightarrow \bigcirc [t]_i\Box [t]_i\varphi\) (FP-positive introspection\(^6\))

The axiom (FP-application) says that if agent \( i \) knows \( \varphi \rightarrow \psi \) for reason \( t \) and she knows \( \varphi \) for reason \( s \), then tomorrow she will know that \( \psi \) was the case yesterday for reason \( t \cdot s \). So the agent applies the rule Modus Ponens (MP) in her reasoning, and here this step of reasoning takes time. Thus LTL\(^J\)-agents avoid the logical omniscience problem.

Next we present a semantics for LTL\(^J\) similar to LPLTL\(^P\)-models given in Section

Definition 11.1. An LTL\(^J\)-CS-model is a tuple \( M = (r, S, E_1, \ldots, E_h, \nu) \) where

1. \( S \) is a non-empty set of states;
2. \( r : \mathbb{N} \rightarrow S \) is a run on \( S \);
3. \( E_i \) is an LTL\(^J\)-CS-evidence function for each agent \( i \in Ag \);

\(^5\) Thanks to Thomas Studer for suggesting me these axioms.
\(^6\) The prefix FP in the name of these axioms comes from the first letters of ‘Future’ and ‘Past’.
4. \( \nu : S \to \mathcal{P}(\text{Prop}) \) is a valuation.

\( \text{LTL}^J_{\text{CS}} \)-evidence functions should satisfy the following conditions. For all \( n \in \mathbb{N} \), all terms \( s, t \in Tm \) and all formulas \( \varphi, \psi \in \text{Fml} \):

1. if \( [t]_i \varphi \in \text{CS} \), then \( \varphi \in \mathcal{E}_i(r(n), t) \), (constant specification)
2. if \( \varphi \to \psi \in \mathcal{E}_i(r(n), t) \) and \( \varphi \in \mathcal{E}_i(r(n), s) \), then \( \otimes \psi \in \mathcal{E}_i(r(n + 1), t \cdot s) \), (FP-application)
3. if \( \varphi \in \mathcal{E}_i(r(n), s) \cup \mathcal{E}_i(r(n), t) \), then \( \otimes \varphi \in \mathcal{E}_i(r(n + 1), s + t) \), (FP-sum)
4. if \( \varphi \in \mathcal{E}_i(r(n), t) \), then \( \otimes [t]_i \varphi \in \mathcal{E}_i(r(n + 1), !t) \). (FP-positive introspection)

Given an \( \text{LTL}^J_{\text{CS}} \)-model \( \mathcal{M} \), the truth of a formula in \( \mathcal{M} \) is defined in the same manner as in Definition 7.1. The proof of completeness is similar to the proof of Theorem 7.8 by constructing a canonical model. Note that in order to prove the completeness, conditions (2)–(4) of Definition 7.8 should be replaced by the following closure conditions:

1. If \( (r(n), t, s \cdot \psi) \in \mathbb{E} \) and \( (r(n), s, \psi) \in \mathbb{E} \), then \( (r(n + 1), t \cdot s, \psi) \in \mathbb{E} \). (FP-cl-application)
2. If \( (r(n), t, \varphi) \in \mathbb{E} \) or \( (r(n), s, \varphi) \in \mathbb{E} \), then \( (r(n + 1), s + t, \varphi) \in \mathbb{E} \). (FP-cl-sum)
3. If \( (r(n), t, \varphi) \in \mathbb{E} \), then \( (r(n + 1), !t, [t]_i \varphi) \in \mathbb{E} \). (FP-cl-positive-introspection)

Now soundness and completeness of \( \text{LTL}^J_{\text{CS}} \) is proved similar to that of \( \text{LPLTL}^P \) in Section 8.

**Theorem 11.2 (Soundness and completeness).** Let \( \text{CS} \) be a constant specification for \( \text{LTL}^J \). Then \( \vdash_{\text{LTL}^J_{\text{CS}}} \varphi \) iff \( \models \varphi \) for all \( \text{LTL}^J_{\text{CS}} \)-models \( \mathcal{M} \).

**Proof.** Soundness is straightforward. The proof of completeness is similar to the proof of Theorem 7.8 by constructing a canonical model. Truth Lemma can be proved as before. The only new part is to show that any \( \chi \)-canonical model for \( \text{CS} \) with respect to an acceptable sequence \( (\overline{T}_0, \overline{T}_1, \ldots) \) for \( \text{LTL}^J_{\text{CS}} \) is an \( \text{LTL}^J_{\text{CS}} \)-model. This is left to the reader. \( \square \)

Given a set \( \text{Ax} \) of connecting principles from Section 3 by \( \text{LTL}^J_{\text{Ax}} \) we denote the result of adding axioms from \( \text{Ax} \) to \( \text{LTL}^J \). The above completeness result can be easily extended to \( \text{LTL}^J_{\text{Ax}} \) as well.

It is not difficult to show that none of the formulas \( (10) \sim (13) \) are valid in \( \text{LTL}^J \). For example, consider the following instance of \( (10) \sim (13) \):

\[
[x], P \to \bigcirc P, \quad [x], P \to \bigcirc [x]_1 P, \quad [x], P \to \Box [x]_1 P, \quad [x], P \to \bigcirc P, \quad (14)
\]

where \( x \in \text{Var} \) and \( P \in \text{Prop} \). Let \( \mathcal{M} = (r, S, \mathcal{E}_1, \ldots, \mathcal{E}_n, \nu) \) be defined as follows:

- \( S = \{ w, v \} \).
- \( r(0) = w \) and \( r(n) = v \) for all \( n \geq 1 \).
- \( P \in \nu(w) \) and \( P \notin \nu(v) \).
\[ E_i(r(n), t) = \{ \varphi \mid (r(n), t, \varphi) \in \mathcal{E}_i^B \}, \]

where \( \mathcal{B} = \{(r(0), x, P)\} \) and \( \mathcal{E}_i^B \) is the least fixed point of \( \Phi_i^B \).

Now it is obvious that \( \mathcal{M} \) is an LTL\(_{\emptyset}^J\) model, and further none of the formulas in (14) are true in \( \mathcal{M} \) at state \( r(0) \). Thus, none of the formulas in (14) are valid in LTL\(_{\emptyset}^J\).

**Remark 11.3.** Note that since \( \varphi \rightarrow \Diamond \varphi \) is provable in LTL the following formulas trivially follows in LPLTL from the axioms (application), (sum), and (positive introspection):

\[
\begin{align*}
[t]t(\varphi \rightarrow \psi) & \rightarrow ([s]s[\varphi \rightarrow \Diamond [t \cdot s]s\psi), \\
[t]t[\varphi \lor [s]s[\varphi \rightarrow \Diamond [t + s]s\psi, \\
[t]t[\varphi \rightarrow \Diamond [!t]t[t]t[\varphi].
\end{align*}
\]

A more realistic set of axioms which do not suffer from the logical omniscience problem can be formulated as follows:

\[
\begin{align*}
[t]t(\varphi \rightarrow \psi) & \rightarrow ([s]s[\varphi \rightarrow \langle F \rangle [t \cdot s]s\psi, \\
[t]t[\varphi \lor [s]s[\varphi \rightarrow \langle F \rangle [t + s]s\varphi, \\
[t]t[\varphi \rightarrow \langle F \rangle [!t]t[t]t[\varphi.
\end{align*}
\]

where \( \langle F \rangle \varphi := \neg \varphi \lor \Diamond \varphi \). We leave the proof of completeness to possible future work.

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\[ ^7 \] Note that the closure conditions (FP-cl-application), (FP-cl-sum), and (FP-cl-positive-introspection) are used in the definition of \( \Phi_i^B \).
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