TRACE EXPANSIONS AND THE NONCOMMUTATIVE RESIDUE FOR MANIFOLDS WITH BOUNDARY

GERD GRUBB AND ELMAR SCHROHE

Abstract. For a pseudodifferential boundary operator $A$ of order $\nu \in \mathbb{Z}$ and class 0 (in the Boutet de Monvel calculus) on a compact $n$-dimensional manifold with boundary, we consider the function $\text{Tr}(AB^{-s})$, where $B$ is an auxiliary system formed of the Dirichlet realization of a second order strongly elliptic differential operator and an elliptic operator on the boundary. We prove that $\text{Tr}(AB^{-s})$ has a meromorphic extension to $\mathbb{C}$ with poles at the half-integers $s = (n + \nu - j)/2$, $j \in \mathbb{N}$ (possibly double for $s < 0$), and we prove that its residue at 0 equals the noncommutative residue of $A$, as defined by Fedosov, Golse, Leichtnam and Schrohe by a different method. To achieve this, we establish a full asymptotic expansion of $\text{Tr}(A(B - \lambda)^{-k})$ in powers $\lambda^{-l/2}$ and log-powers $\lambda^{-l/2} \log \lambda$, where the noncommutative residue equals the coefficient of the highest order log-power. There is a related expansion of $\text{Tr}(Ae^{-tB})$.

The paper will appear in Journal Reine Angew. Math. (Crelle’s Journal).

1. Introduction.

In this paper we study trace expansions associated to operators in the calculus of Boutet de Monvel on a compact manifold $X$ with boundary. As an auxiliary operator, we first fix an invertible system $B$ formed by the Dirichlet realization of a strongly elliptic differential operator in the interior of $X$ and an elliptic operator on the boundary. Given a pseudodifferential boundary value problem $A$ in the Boutet de Monvel calculus, the mapping

$$s \mapsto \text{Tr}(AB^{-s}),$$

then is a holomorphic function of $s$ for large Re $s$. We show that it extends to a meromorphic function on the whole complex plane with at most double poles. Moreover, we prove that the noncommutative residue $\text{res}(A)$ of $A$ can be recovered as a residue in this expansion:

$$\text{res}(A) = \text{ord } B \cdot \text{Res}_{s=0} \text{Tr}(AB^{-s}),$$

provided that $A$ is of class zero. In particular, the right hand side does not depend on the choice of $B$.

In order to put these results into perspective, let us recall the situation of a compact manifold without boundary. In 1984, Wodzicki found a trace on the algebra $\Psi(X)$ of all classical pseudodifferential operators on a closed manifold $X$ of dimension $n > 1$. In fact, this traces lives on the quotient $\Psi(X)/\Psi^{-\infty}(X)$ modulo the regularizing elements, and it is the unique trace there — up to multiples — if $X$ is connected. (A trace on an algebra

1991 Mathematics Subject Classification. 58J42, 35S15.
\( \mathcal{A} \) is a linear map \( \tau : \mathcal{A} \to \mathbb{C} \) such that \( \tau(ab) = \tau(ba) \) for all \( a, b \in \mathcal{A} \). Wodzicki called it the noncommutative residue.

His approach was based on an analogue of (1.1): Given an arbitrary classical pseudodifferential operator \( A \) acting on sections of a vector bundle \( E \) over \( X \), he chose an invertible classical pseudodifferential operator \( P \) of sufficiently large positive order (larger than \( \text{ord} A \)), satisfying Agmon’s condition for the existence of a ray of minimal growth for the resolvent. Following Seeley [S67], this allows the construction of the complex powers \( (P + uA)^s \), \( s \in \mathbb{C} \), for \( u \in \mathbb{R} \) close to zero, and a meromorphic extension of \( \text{Tr}(P + uA)^{-s} \) to \( \mathbb{C} \). Wodzicki proved that the right hand side of the formula, below, is independent of \( P \) and defined

\[
\text{res}(A) = \lim_{u \to 0} \text{Res}_{s=1} \text{Tr}((P + uA)^{-s}).
\]

It is clear from Seeley’s results that the above expression only depends on finitely many terms in the asymptotic expansion \( \sum a_j(x, \xi) \) of the symbol \( A \) into terms \( a_j \) which are homogeneous of degree \( j \) in \( \xi \). Wodzicki showed that in fact

\[
\text{res}(A) = (2\pi)^{-n} \int_{S^*X} \text{tr}_E a_{-n}(x, \xi) \, d\sigma;
\]

here \( \text{tr}_E \) is a trace in \( \text{Hom}(E) \) and \( d\sigma \) the surface measure on \( S^*X \).

The notion ‘noncommutative residue’ is justified also by a historical reason. In the late seventies Manin [M79] and Adler [A79] in their work on the Korteweg-de Vries equation introduced a trace functional on an algebra of formal pseudodifferential operators in one dimension. The symbols were Laurent series in one variable with coefficients in \( \mathbb{C} \), and to the operator with the symbol \( \sum a_j \xi^j \) they associated the trace \( a_{-1} \), i.e., the usual (‘commutative’) residue of complex analysis. As formula (1.3) shows, Wodzicki’s residue is a natural generalization of the functional employed by Manin and Adler to the \( n \)-dimensional case.

The noncommutative residue was discovered independently by Guillemin [Gu85] in connection with his alternative proof of Weyl’s law on the asymptotic distribution of the eigenvalues of an elliptic operator. An interesting survey of these facts was given by Kassel in [K89].

For a compact manifold \( X \) with boundary \( \partial X = X' \), the situation is more complicated. As pointed out by Wodzicki, there is no trace \( \neq 0 \) on the algebra of all classical pseudodifferential operators with the Leibniz product on \( X \) unless \( \partial X = \emptyset \): This algebra is not sensitive to what happens at the boundary. A more natural choice in this context is the Boutet de Monvel algebra [BM71] of polyhomogeneous pseudodifferential boundary value problems (or pseudodifferential boundary operators, \( \psi \text{dbo}'s \) for short) of integer order. An operator in this algebra is a map \( A \) acting on sections of vector bundles \( E \) over \( X \) and \( F \) over \( X' \), given by a matrix

\[
A = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} : C^\infty(X, E) \oplus C^\infty(X', F) \to C^\infty(X, E) \oplus C^\infty(X', F)
\]

(Here \( P_+ \) is a \( \psi \text{do} \) truncated to \( X \), \( G \) is a singular Green operator (s.g.o.), \( K \) a Poisson operator, \( T \) a trace operator and \( S \) a \( \psi \text{do} \) on \( X' \).) Indeed it could be shown by Fedosov,
Golse, Leichtnam, and Schrohe [FGLS96] that there is a trace on this algebra, also called \( \text{res} \). It is given by

\[
(1.5) \quad \text{res}(A) = (2\pi)^{-n} \int_{S^*X} \left( \text{tr}_E p_{-n}(x, \xi) \right) d\sigma
\]

\[
+ (2\pi)^{1-n} \int_{S^*X'} \left[ \text{tr}_E (\text{tr}_n g)_{1-n}(x', \xi') + \text{tr}_F s_{1-n}(x', \xi') \right] d\sigma',
\]

(we have corrected a sign error in [FGLS96, (2.19)]). Here \( d\sigma' \) is the surface measure on \( S^*X' \) and \( \text{tr}_n g \) stands for the \textit{normal trace} of the symbol \( g \) of the singular Green operator \( G' \):

\[
(1.6) \quad (\text{tr}_n g)(x', \xi') = (2\pi)^{-1} \int_0^+ g(x', \xi', \xi_n) d\xi_n,
\]

it is a polyhomogeneous pseudodifferential symbol on \( X' \). Assuming that \( X \) is connected, ‘\( \text{res} \)’ turns out to be the unique continuous trace — up to multiples — on the quotient modulo the regularizing operators.

In [FGLS96], the definition of \( \text{res}(A) \) and the proof of the trace property relied completely on the symbolic level, since the analytic properties of traces of powers of \( \psi_d \)-bo’s were not known in sufficient generality. It remained an open problem to identify the noncommutative residue with a residue in the sense of (1.1) or (1.2).

Progress was possible as a result of the work of Grubb and Seeley [GS95] on weakly parametric pseudodifferential operators on closed manifolds. In fact, [GS95, Theorem 2.7] shows that, for classical pseudodifferential operators \( A \) and \( P \) of integer order, where \( P \) is elliptic with parameter in a sector in \( \mathbb{C} \), there are expansions (regardless of the magnitude of \( \text{ord} P \)):

\[
(1.7) \quad \begin{align*}
\text{(I)} \quad & \text{Tr}(A(P - \lambda)^{-k}) \sim \sum_{j \geq 0} c_j (-\lambda)^{\frac{n - \text{ord} A}{\text{ord} P} k - j} + \sum_{l \geq 0} (c'_l \log(-\lambda) + c''_l)(-\lambda)^{-l - k}, \\
\text{(II)} \quad & \text{Tr}(A e^{-tP}) \sim \sum_{j \geq 0} \tilde{c}_j t^j \frac{1 - n - \text{ord} A}{\text{ord} P} + \sum_{l \geq 0} (-c'_l \log t + c''_l)t^l, \\
\text{(III)} \quad & \Gamma(s) \text{Tr}(A P^{-s}) \sim \sum_{j \geq 0} \frac{\tilde{c}_j}{s + \frac{1 - n - \text{ord} A}{\text{ord} P} s} + \sum_{l \geq 0} \left( \frac{c'_l}{(s + l)^2} + \frac{c''_l}{s + l} \right).
\end{align*}
\]

In (I), \( k > (n + \text{ord} A)/\text{ord} P \), and \( \lambda \) goes to infinity on suitable rays in \( \mathbb{C} \setminus \mathbb{R}_+ \); in (II), \( t \to 0^+ \), and it holds when \( P \) is strongly elliptic; in (III), the sign “\( \sim \)” indicates that the left hand side is meromorphic on \( \mathbb{C} \) with pole structure as in the right hand side. The coefficients \( \tilde{c}_j, c'_l, c''_l \) are proportional to the coefficients \( c_j, c'_l, c''_l \) by universal constants. In particular,

\[
\text{Res}_{s=0} \text{Tr}(A P^{-s}) = c'_0 = c''_0.
\]

We recall moreover that the statements (I), (II) and (III) are essentially equivalent, as explained e.g. in [GS96], via transition formulas such as

\[
(1.8) \quad \begin{align*}
P^{-s} &= \frac{1}{(s-1) \cdots (s-k) 2\pi i} \int_C \lambda^{k-s} \partial^k \lambda(P - \lambda)^{-1} d\lambda = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-tP} dt, \\
e^{-tP} &= t^{-k} \frac{i}{2\pi} \int_C e^{-t\lambda} \partial^k \lambda(P - \lambda)^{-1} d\lambda = \frac{1}{2\pi i} \int_{\text{Res}_{s=0} \Gamma(s)} t^{-s} \Gamma(s) P^{-s} ds;
\end{align*}
\]
note that $\partial_k^k(P - \lambda)^{-1} = k!(P - \lambda)^{-k-1}$.

The point of view of determining the residue from formulas (1.5), together with more general expansion theorems for parameter-dependent $\psi$do’s in [GS95], is developed by Lesch in [L99], who extends the considerations to a graded algebra of logarithmically polyhomogeneous $\psi$do’s, defining higher residues as well.

In the present paper we show how the results of [GS95] can be applied to the analysis of boundary value problems in the Boutet de Monvel calculus. It is the main technical difficulty to establish expansions analogous to (I), (II), and (III) in (1.7) above. In order to describe our result, let $A$ be an arbitrary $\psi$do of order $\nu \in \mathbb{Z}$ and class zero as in (1.4); we denote $\dim E = n'$. We next choose our auxiliary operator. We let $P_1$ be a strongly elliptic second order differential operator in $E$, which has scalar principal symbol on a neighborhood of $X'$; by $P_{1,D}$ we denote its Dirichlet realization. Finally we pick a strongly elliptic second order pseudodifferential operator $S_1$ on $C^\infty(X',F)$ and set $B = \begin{pmatrix} P_{1,D} & 0 \\ 0 & S_1 \end{pmatrix}$. Then

\begin{equation}
(1.9) \quad \text{Tr}(A(B - \lambda)^{-k}) = \text{Tr}_X ((P_+ + G)(P_{1,D} - \lambda)^{-k}) + \text{Tr}_{X'} (S(S_1 - \lambda)^{-k})
\end{equation}

is well-defined for $k > (n + \nu)/2$. The behavior of $\text{Tr}_{X'} (S(S_1 - \lambda)^{-k})$ is covered by the preceding results: We have an expansion as in (I) of (1.7) with $n$ replaced by $n - 1$; moreover, the coefficient of the first logarithmic term in this expansion is proportional to the noncommutative residue of $S$ on $X'$. It therefore remains to analyze the first term. This the content of the following theorem:

**Theorem 1.1.** There are full asymptotic expansions (when $k > (n + \nu)/2$)

\begin{equation}
(1.10) \quad \text{I} \quad \text{Tr}((P_+ + G)(P_{1,D} - \lambda)^{-k}) \sim \sum_{j \geq 0} c_j (-\lambda)^{\frac{n+\nu-j}{2}} - k + \sum_{l \geq 0} (c_l' \log(-\lambda) + c_l'') (-\lambda)^{-\frac{\nu}{2} - k},
\end{equation}

\begin{equation}
\text{(II)} \quad \text{Tr}((P_+ + G)e^{-tP_{1,D}}) \sim \sum_{j \geq 0} \tilde{c}_j t^{\frac{j-n-\nu}{2}} + \sum_{l \geq 0} (-c_l' \log t + c_l'') t^\frac{\nu}{2},
\end{equation}

\begin{equation}
\text{(III)} \quad \Gamma(s) \text{Tr}((P_+ + G)P_{1,D}^{-s}) \sim \sum_{j \geq 0} \tilde{c}_j s^{\frac{j-n-\nu}{2}} + \sum_{l \geq 0} \left( \frac{c_l'}{(s + \frac{1}{2})^2} + \frac{c_l''}{s + \frac{1}{2}} \right).
\end{equation}

The coefficients $\tilde{c}_j, c_l, c_l''$ are proportional to the coefficients $c_j, c_l', c_l''$ by universal constants. In particular, $\text{Res}_{s=0} \text{Tr}((P_+ + G)P_{1,D}^{-s})$ equals $c_0' = c_0'$, and

\begin{equation}
(1.11) \quad \text{res}(P_+ + G) = \text{ord} P_1 \cdot \text{Res}_{s=0} \text{Tr}((P_+ + G)P_{1,D}^{-s}).
\end{equation}

The main effort in our proof lies in the deduction of (I) in (1.10). Besides this, we show how the specific contributions to (1.5) arise in the trace expansion of $(P_+ + G)(P_{1,D} - \lambda)^{-k}$.

**Remark 1.2.** Our method also applies to the case where $G$ is of class $r > 0$. In general, the expansion (I) will then take the form

\begin{equation}
(1.12) \quad \text{Tr}((P_+ + G)(P_{1,D} - \lambda)^{-k}) \sim \sum_{j \geq 0} c_j (-\lambda)^{\frac{n+\nu-j}{2}} - k + \sum_{l \geq -r} (c_l' \log(-\lambda) + c_l'' (-\lambda)^{-\frac{\nu}{2} - k},
\end{equation}
and the other expansions are accordingly modified. The relation (1.11) will not in general be valid. For example, if $G = K\gamma_0$, $\text{res}(G)$ is generally nonzero according to [FGLS96], whereas $G(P_{1,D} - \lambda)^{-k}$ is zero since $(P_{1,D} - \lambda)^{-1}$ maps into a space where $\gamma_0 u$ vanishes. (We denote $u|_{X'} = \gamma_0 u$.)

Let us moreover remark that the Dirichlet condition may be replaced by the Neumann condition or other coercive boundary conditions, since the resulting singular Green operators have symbol structures similar to the case studied in detail here.

Partial expansions were obtained for large classes of parameter-dependent operators in [G96] and its predecessor from 1986 (see also the appendix of [G92]), however without getting as far as the residue at zero or the logarithmic term that is interesting in the present context. Full expansions were given for some problems connected with Dirac operators in [GS95], [G99], where the efforts were concentrated on reducing the question to a study of $\psi$do’s on the boundary, to which the calculus of so-called weakly polyhomogeneous $\psi$do’s could be applied. An extension of the calculus to a class of parameter-dependent $\psi$dbo’s has been worked out in [G00] for an algebra of operators containing the strongly polyhomogeneous $\psi$dbo’s. It provides full trace expansions, but it does not include the general compositions with parameter-independent operators that we need to treat here.

An interesting point in the present investigation is that the important contribution comes from a logarithmic term, whereas the logarithmic terms are considered more as a disturbance in the problems connected with Dirac operators.

Plan of the paper: In Section 2 we write $(P_+ + G)(P_{1,D} - \lambda)^{-k}$ as a sum of five terms of different nature, reduce the problem to local coordinates at the boundary, and determine the structure of the symbols connected with the auxiliary operator as rational functions. Section 3 begins with a simple example demonstrating the basic idea of the proof; then we use Laguerre expansions to extend this idea to the two terms composed of singular Green operators. Sections 4 and 5 treat the terms composed of a $\psi$do and an s.g.o. In Section 6, the various results are collected to give the proof of Theorem 1.1. The special Laguerre expansions we use are recalled in the Appendix.

2. The structure of the compositions.

We are going to establish an asymptotic expansion for $\text{Tr}(AR^k_\mu)$, where $A = P_+ + G$ and $R_\mu = (P_{1,D} + \mu^2)^{-1}$.

Our main tool is the Boutet de Monvel calculus. We assume familiarity with the standard notions in this context, recommending the reference [G96]. Under reasonable conditions on $P_1$, one can show that $R_\mu$, acting on $C^\infty(X, E)$, is a strongly parameter-dependent polyhomogeneous element of the Boutet de Monvel calculus, of the kind of the upper left corner of the matrix in (1.4). Its pseudodifferential part simply is $((P_1 + \mu^2)^{-1})_+$, so that the usual parametrix construction gives us full information on its symbol. The analysis of the singular Green part of $R_\mu$ requires considerably more care. Knowing the structure of its symbol components will enable us, however, to also determine the symbolic structure of the $k$-th power $R^k_\mu$ and, eventually, that of $AR^k_\mu$.

In order to show the trace expansions we then rely on the concepts of polyhomogeneous parameter-dependent symbol classes; references here are [G96] and [GS95]. Let us just emphasize one aspect: When symbols depend on a parameter $\mu$ (running in a sector $\Gamma$ of $\mathbb{C} \setminus \{0\}$), we distinguish between a weak and a strong version of the homogeneity property $p(x, t\xi, t\mu) = t^m p(x, \xi, \mu)$. When it holds for all $\xi, \mu, t$ with $|\xi|^2 + |\mu|^2 \geq 1$,
$t \geq 1$, $(\xi, \mu) \in \mathbb{R}^n \times (\Gamma \cup \{0\})$, $p$ is said to be strongly homogeneous (of degree $m$). When it holds for all $\xi, \mu, t$ with $|\xi| \geq 1$, $t \geq 1$, $(\xi, \mu) \in \mathbb{R}^n \times \Gamma$, $p$ is said to be weakly homogeneous or just homogeneous. In the weakly homogeneous case, one must further specify which behavior is assumed in the noncompact set $\{(\xi, \mu) \mid |\xi| \leq 1, |\mu| \geq 1\}$. Strong resp. weak poly-homogeneity of a symbol means that it has an asymptotic expansion in strongly resp. weakly homogeneous terms of decreasing orders, with specific requirements on the remainder estimates. Briefly explained, strong polyhomogeneity means that the new variable $\mu$ enters on a par with the cotangent variables $\xi$, so that both $\mu$-derivatives and $\xi_j$-derivatives give decrease in the estimates in terms of $\langle \xi, \mu \rangle$. Moreover, by [G96, (3.3.21)],

\[
(2.1) \quad W = \{z \in \mathbb{C} \mid |z| \leq r \text{ or } |\arg z| \leq \frac{\pi}{2} + \varepsilon\}
\]

(for some $\varepsilon > 0$) such that when $\mu^2 \in W$, the inverses $Q_\mu = (P_1 + \mu^2)^{-1}$ (on $\tilde{X}$) and $R_\mu = (P_{1,D} + \mu^2)^{-1}$ (on $X$) exist as bounded operators in $L_2$ and are $O(\mu^{-2})$ on rays in $W$. Moreover, by [G96, (3.3.21)],

\[
(2.2) \quad R_\mu = Q_{\mu,+} + G_\mu,
\]

where $Q_{\mu,+} = (Q_\mu)_+$ is the restriction of $Q_\mu$ to $X$ and $G_\mu$ is a singular Green operator, described in detail in Lemma 2.4 below. Then

\[
(2.3) \quad R^k_\mu = (Q_{\mu,+} + G_\mu)^k = (Q^k_\mu)_+ + G^{(k)}_\mu, \quad \text{with} \quad G^{(k)}_\mu = (Q_{\mu,+})^k - (Q^k_\mu)_+ + \text{pol}(G_\mu, Q_{\mu,+});
\]

here pol($G_\mu, Q_{\mu,+}$) is a linear combination of compositions of the (non-commuting) factors $G_\mu$ and $Q_{\mu,+}$ with $k$ factors in each term, at least one of them being $G_\mu$. The symbol structure of $G^{(k)}_\mu$ will be given in Proposition 2.5. So, writing $(Q^k_\mu)_+ = Q^k_{\mu,+}$,

\[
(2.4) \quad AR^k_\mu = (P_+ + G)(Q^k_{\mu,+} + G^{(k)}_\mu) = (PQ^k_\mu)_+ + L(P, Q^k_\mu) + P_+ G^{(k)}_\mu + GG^k_{\mu,+} + GG^{(k)}_\mu.
\]

Here, $L(P, Q^k_\mu) = (PQ^k_\mu)_+ - P_+ Q^k_{\mu,+}$ is the leftover term in the composition of $P_+$ and $Q^k_{\mu,+}$. All these operators are trace class since $k > (n + \nu)/2$. There are five terms to be dealt with:

\[
(2.5) \quad \text{Tr}((PQ^k_\mu)_+), \quad -\text{Tr}(L(P, Q^k_\mu)), \quad \text{Tr}(P_+ G^{(k)}_\mu), \quad \text{Tr}(GG^k_{\mu,+}), \quad \text{Tr}(GG^{(k)}_\mu);
\]
they will now be analyzed in detail.

The first term in (2.5) is covered by [GS95, Th. 2.1], applied to the extensions of $P$ and $Q^k_{\mu}$ to $\tilde{X}$, when we integrate the resulting kernel formula over $X$. It remains to consider the other four. It will be relatively easy to deal with the effects caused away from the boundary:

2.4 Contributions from the interior.

We shall first show that the only interesting contributions stem from a neighborhood of the boundary: Using a partition of unity $1 = \sum_{i=1}^I \theta_i(x)$ subordinate to a cover of $X$ by coordinate patches $(\Omega_j)_{j=1}^J$ (with trivializations of the bundles) in such a way that any four of the functions $\theta_i$ are supported in one of the coordinate patches (see e.g. [GK93, Lemma 1.5]), we can localize any of the operators in (2.5). For those that are products $A_1A_2$, we write

\begin{equation}
A_1A_2 = \sum_{i,j,l,m \leq I} \theta_i A_1 \theta_j A_2 \theta_m,
\end{equation}

which reduces the problem to the consideration of compositions $A_1' A_2'$ where both factors act in one coordinate patch $\Omega$ and map into compactly supported functions there. We can assume that the principal symbol of $P_1$ is a scalar times the identity matrix in those patches that meet the boundary.

If $\Omega$ is disjoint from the boundary, the factor $\theta_l \sigma^{(k)}_{\mu} \theta_m$ coming from the third and the fifth expression in (2.5) is strongly polyhomogeneous and of order $-\infty$. By [G96, Theorem 2.5.11], a strongly polyhomogeneous operator of order $-r$ is continuous from $H^{s,\mu}(X,E)$ to $H^{s+r,\mu}(X,E)$ for $s > -\frac{1}{2}$, hence its composite with an operator of order $\nu$ is trace class when $r > n + \nu$ with a trace norm that is $O(\mu^{n+\nu-r})$ for $\mu \to \infty$ in $\Gamma$,

\begin{equation}
\Gamma = \{ \mu \in \mathbb{C} \mid |\arg \mu| \leq \frac{\pi}{4} + \frac{\varepsilon}{2} \}
\end{equation}

(cf. (2.1)). So the terms with the factor $\theta_l \sigma^{(k)}_{\mu} \theta_m$ are trace class operators whose traces are $O(\mu^{-N})$ for any $N$, hence do not contribute to the trace expansions.

For the second expression in (2.5), we can assume that the $\theta_i$ form a partition of unity on $\tilde{X}$, so that we can write

\[
L(P_+, Q^k_{\mu}) = (PQ^k_{\mu})_+ - P_+ Q^k_{\mu} + \sum_{i,j,l,m \leq I} (\theta_i P \theta_j \theta_l Q^k_{\mu} \theta_m)_+ - \sum_{i,j,l,m \leq I} (\theta_i P \theta_j)_+ (\theta_l Q^k_{\mu} \theta_m) = \sum_{i,j,l,m \leq I} L(\theta_i P \theta_j, \theta_l Q^k_{\mu} \theta_m).
\]

In the local coordinates for the coordinate patch $\Omega$ where $\theta_i, \theta_j, \theta_l, \theta_m$ are supported,

\[
L(\theta_i P \theta_j, \theta_l Q^k_{\mu} \theta_m) = G^+(\theta_i P \theta_j) G^- (\theta_l Q^k_{\mu} \theta_m),
\]

by [G96, (2.6.27)]. Here, when $\Omega$ is disjoint from $X'$, $G^- (\theta_l Q^k_{\mu} \theta_m)$ is a strongly polyhomogeneous s.g.o. of order $-\infty$, so the product is trace class with $O(\mu^{-N})$ estimates for any $N$. 
In the fourth expression in (2.5), when $\Omega$ is disjoint from $X'$, the operator $\theta_j G \theta_j$ is a singular Green operator of order $-\infty$, hence has a $C^\infty$ kernel, and identifies with a negligible $\psi$do $P_{ij,\pm}$. Here

$$\theta_j G \theta_j Q^k_{\mu,\pm} \theta_m = \mathcal{P}_{ij,\pm} \theta_i Q^k_{\mu,\pm} \theta_m = (\mathcal{P}_{ij,\pm} \theta_i Q^k_{\mu} \theta_m)_+ - L(\mathcal{P}_{ij,\pm}, \theta_i Q^k_{\mu} \theta_m).$$

The $L$-term gives $O(\mu^{-N})$ estimates just like the preceding case. For $\mathcal{P}_{ij,\pm} \theta_i Q^k_{\mu} \theta_m$ we have from the calculus of \cite{GS95} for $\psi$do's on $\tilde{X}$ that it belongs to $S^{-\infty,-2\kappa}$ there, hence has a diagonal kernel expansion

$$\sum_{r \in \mathbb{N}} c_r(x) \mu^{-2\kappa - r} \text{ for } \mu \to \infty \text{ in } \Gamma,$$

which, when integrated over $X$, gives the trace expansion $\sum_{r \in \mathbb{N}} c_r \mu^{-2\kappa - r}$.

So we see that all the interior contributions from the second to fifth term in (2.5) have trace expansions that are either void or have the form (2.8), with no logarithms.

We can therefore focus on a neighborhood of the boundary where we can work in local coordinates on $\mathbb{R}^n_+$. We shall use the notation in (2.5), omitting explicit mention of the $\theta_i$'s or coordinate changes.

2.6 The symbol of $Q^k_{\mu}$.

Let us first recall the symbol structure of $Q^k_{\mu}$. For a boundary patch where $P_1$ has principal symbol $p_{1,2}I$ with a scalar $p_{1,2}$, the symbol of $Q_{\mu}$ has the following form in local coordinates:

$$q(x, \xi, \mu) \sim \sum_{l \in \mathbb{N}} q_{-2-l}(x, \xi, \mu), \text{ with }$$

$$q_{-2}(x, \xi, \mu) = (p_{1,2}(x, \xi) + \mu^2)^{-1}I,$$

$$q_{-2-J}(x, \xi, \mu) = \sum_{2 \leq m \leq 2J + 1} \frac{r_{J,m}(x, \xi)}{(p_{1,2}(x, \xi) + \mu^2)^m}, \text{ for } J > 0;$$

here the $r_{J,m}$ are $n' \times n'$-matrices ($n' = \dim E$) of homogeneous polynomials in $\xi$ of degree $2m - J$ with smooth coefficients (cf. \cite{S67, (1)–(1a)}).

For $Q^k_{\mu}$ we find from the composition rules for $\psi$do's that the symbol $q^{\circ k}$ satisfies:

$$q^{\circ k}(x, \xi, \mu) \sim \sum_{l \in \mathbb{N}} q_{-2k-l}^{\circ k}(x, \xi, \mu), \text{ with }$$

$$q_{-2k}^{\circ k}(x, \xi, \mu) = (p_{1,2}(x, \xi) + \mu^2)^{-k}I,$$

$$q_{-2k-J}^{\circ k}(x, \xi, \mu) = \sum_{k+1 \leq m \leq 2J+k} \frac{r_{k,J,m}(x, \xi)}{(p_{1,2}(x, \xi) + \mu^2)^m}, \text{ for } J > 0;$$

the $r_{k,J,m}$ being matrices of homogeneous polynomials in $\xi$ of degree $2m - 2k - J$. 
2. The symbols of $G^\pm(Q^k_\mu)$.

The second term in (2.5) is $L(P, Q^k_\mu)$ which can be written $L(P, Q^k_\mu) = G^+(P)G^-(Q^k_\mu)$, cf. [G96, (2.6.25–26)]. We shall now study the structure of $G^-(Q^k_\mu)$. For simplicity of notation we shall write $x'$ instead of $(x', 0)$.

The polynomial

\[ p_{1.2}(x', \xi) + \mu^2 = a(x')\xi_n^2 + b(x', \xi')\xi_n + c(x', \xi') + \mu^2 \]

has two roots with respect to $\xi_n$,

\[ i\kappa^+(x', \xi', \mu) \in \mathbb{C}_+ \quad \text{and} \quad -i\kappa^-(x', \xi', \mu) \in \mathbb{C}_-, \]

strongly homogeneous of degree 1 with $\text{Re} \kappa^\pm > 0$ when $\mu^2 \in W$, cf. (2.1), so

\[ p_{1.2}(x', \xi) + \mu^2 = a(x')(\xi_n - i\kappa^+(x', \xi', \mu))(\xi_n + i\kappa^-(x', \xi', \mu)) \]

\[ = a(x')(\kappa^+(x', \xi', \mu) + i\xi_n)(\kappa^-(x', \xi', \mu) - i\xi_n). \]

Then we have formulas (by decomposition of rational functions of $\xi_n$ in simple fractions)

\[ \frac{1}{p_{1.2}(x', \xi) + \mu^2} = a(x')^{-m}\left( \sum_{1 \leq j \leq m} \frac{a_{m,j}^+(x', \xi', \mu)}{(\kappa^+(x', \xi', \mu) + i\xi_n)^j} + \sum_{1 \leq j \leq m} \frac{a_{m,j}^-(x', \xi', \mu)}{(\kappa^-(x', \xi', \mu) - i\xi_n)^j} \right), \]

where the $a_{m,j}^\pm(x', \xi', \mu)$ are strongly homogeneous of degree $j - 2m$ in $(\xi', \mu)$.

Recall that when a rational function $f(\xi_n)$ with poles in $\mathbb{C} \setminus \mathbb{R}$ is decomposed as $f = f^+ + f^- + f'$, where $f^\pm$ have poles in $\mathbb{C}_\pm = \{ \xi_n \in \mathbb{C} \mid \text{Im} \xi_n \geq 0 \}$ and are $O(\langle \xi_n \rangle^{-1})$, and $f'$ is polynomial, then the projections $h^+, h^-, h_1 - h_2$ (cf. [G96, Prop. 2.2.2]), are simply the mappings $h^+: f \mapsto f^+, h^-: f \mapsto f^- + f'$, $h_1: f \mapsto f^-$. (2.14) gives for $m = k$ the decomposition $q_{2k}^\pm = h^+q_{2k} - h^-q_{2k};$ here $h^-q_{2k} = h_1q_{2k}$.

We can also apply $h^+$ and $h^-$ to the lower order terms in $q^\pm_{2k}$. Since $q_{2k-1 \pm}(x', \xi, \mu)$ is a proper rational function of $\xi_n$ with poles $i\kappa^+$ and $-i\kappa^-$ of order $\leq 2J + k$ for $J \geq 0$ (cf. (2.10)), and all coefficients as well as $\kappa^+$ and $\kappa^-$ are strongly polynormogeneous in $(\xi', \mu)$, one finds by decomposition in simple fractions that $q_{2k-1 \pm}(x', \xi, \mu)$ is the sum of a proper rational function $h^+q_{2k-1 \pm}(x', \xi, \mu)$ of $\xi_n$ with poles $i\kappa^+$ and a proper rational function $h^-q_{2k-1 \pm}(x', \xi, \mu)$ of $\xi_n$ with poles $-i\kappa^-$. (as described in detail in (2.15) below); the coefficients are strongly homogeneous in $(\xi', \mu)$ and the poles are of orders $\leq 2J + k$. The functions $q_{2k-1 \pm}$ are homogeneous of degree $-2k - J$ in $(\xi, \mu)$. Thus we find:

**Proposition 2.1.** For $x = (x', 0)$, denoted $x'$, there is a decomposition of $q^\pm_{2k}$ into a part $q^\pm_{2k,+} = h^+q^\pm_{2k}$ with poles in $\mathbb{C}_+$ and a part $q^\pm_{2k,-} = h^-q^\pm_{2k}$ with poles in $\mathbb{C}_-$ (as functions of $\xi_n$) of the following form:

\[ q^\pm_{2k} = q^\pm_{2k,+} + q^\pm_{2k,-}, \quad q^\pm_{2k,(x', \xi, \mu)} = \sum_{J \geq 0} q^\pm_{2k-1}(x', \xi, \mu) \sim \sum_{J \geq 0} q^\pm_{2k-1}(x', \xi, \mu), \]

\[ q^\pm_{2k-1}(x', \xi, \mu) = q^\pm_{2k-1}(x', \xi, \mu) + q^\pm_{2k-1}(x', \xi, \mu), \]

\[ q^\pm_{2k-1}(x', \xi, \mu) = \sum_{1 \leq j \leq 2J} \frac{r^\pm_{j,k}(x', \xi', \mu)}{(\kappa^\pm(x', \xi', \mu) \pm i\xi_n)^j}, \]
where the \( n' \times n' \)-matrix formed numerators \( r_{k,j}^\pm(x',\xi',\mu) \) are strongly homogeneous in \((\xi',\mu)\) of degree \( j - 2k - J \).

Moreover, in the Taylor expansion of \( q^{\circ k} \),

\[
(2.16) \quad \sum_{l \in \mathbb{N}} \frac{x'}{l!} \partial_{x_n}^l q^{\circ k}(x',0,\xi,\mu),
\]

the terms satisfy:

\[
\partial_{x_n}^l q^{\circ k} = \partial_{x_n}^l q^{\circ k,+} + \partial_{x_n}^l q^{\circ k,-}, \quad \partial_{x_n}^l q^{\circ k,\pm}(x',\xi,\mu) \sim \sum_{j \geq 0} \partial_{x_n}^j q^{\circ k,\pm}_{-2k-J}(x',\xi,\mu),
\]

\[
(2.17) \quad \partial_{x_n}^l q^{\circ k}_{-2k-J}(x',\xi,\mu) = \partial_{x_n}^l q^{\circ k,+}_{-2k-J}(x',\xi,\mu) + \partial_{x_n}^l q^{\circ k,-}_{-2k-J}(x',\xi,\mu),
\]

\[
\partial_{x_n}^l q^{\circ k,\pm}_{-2k-J}(x',\xi,\mu) = \sum_{1 \leq j \leq 2J+k+l} \frac{r_{k,j}^{l,\pm}(x',\xi',\mu)}{(p_{1,2}(x,\xi) + \mu^2)^m},
\]

where the numerators \( r_{k,j}^{l,\pm}(x',\xi',\mu) \) are strongly homogeneous of degree \( j - 2k - J \).

The indication “\( \circ k \)” will be left out when \( k = 1 \).

**Proof.** (2.15) has already been shown. For (2.17) we note as a starting point that (2.10) implies

\[
\partial_{x_n} q^{\circ k}_{-2k-J}(x,\xi,\mu) = \sum_{k+1 \leq m \leq 2J+k} \left( \frac{\partial_{x_n} r_{k,J,m}}{(p_{1,2} + \mu^2)^m} - \frac{r_{k,J,m} \partial_{x_n} p_{1,2}}{(p_{1,2} + \mu^2)^{m+1}} \right)
\]

\[
= \sum_{k+1 \leq m \leq 2J+k+1} \frac{r_{k,J,m}^l (x,\xi)}{(p_{1,2}(x,\xi) + \mu^2)^m},
\]

and hence, successively

\[
\partial_{x_n}^l q^{\circ k}_{-2k-J}(x,\xi,\mu) = \sum_{k+1 \leq m \leq 2J+k+l} \frac{r_{k,J,m}^l (x,\xi)}{(p_{1,2}(x,\xi) + \mu^2)^m},
\]

for all \( l \), with homogeneous polynomials \( r_{k,J,m}^l (x,\xi) \) of degree \( 2m - 2k - J \) in \( \xi \). At \( x_n = 0 \) we decompose these expansions just as above, which leads to formulas as in (2.15), except that the index \( j \) runs up to \( 2J + k + l \). □

The symbol-kernel and symbol of \( G^- (Q^k_\mu) \) are found from the \( h^- \)-projection of the symbol of \( Q^k_\mu \) by the formulas in [G96, Th. 2.6.10 and 2.7.4]. (We recall the relation between an s.g.o. symbol \( g(\xi_n,\eta_n) \) of class 0 and the associated symbol-kernel \( \tilde{g}(x_n, y_n) \): (2.18)

\[
\tilde{g}(x_n, y_n) = r_{x_n}^+ r_{y_n}^+ \mathcal{F}_{\xi_n \to x_n}^{-1} \mathcal{F}_{\eta_n \to y_n}^{-1} g(\xi_n, \eta_n), \quad g(\xi_n, \eta_n) = \mathcal{F}_{x_n \to \xi_n} \mathcal{F}_{y_n \to \eta_n} e_{x_n}^+ e_{y_n}^+ \tilde{g}(x_n, y_n);
\]

here \( r^+ \) denotes restriction from \( \mathbb{R} \) to \( \mathbb{R}_+ \), and \( e^+ \) denotes extension by 0.) As a variant of (A.7), \( \mathcal{F}_{\xi_n \to z_n}^{-1} (\kappa^- - i\xi_n)^{-j} = H(z_n) \frac{1}{(z_n - 1)!} z_n^j e^{-z_n \kappa^-} \). Then [G96, (2.6.44–45)] give by direct application to (2.15):
Proposition 2.2. If the coefficients in $P_1$ are independent of $x_n$ near $x_n = 0$, then the singular Green operator $G^-(Q^k)$ has symbol-kernel $\tilde{g}^{0k,-}(x', x_n, y_n, \xi', \mu)$ (defined for $x_n$ and $y_n \geq 0$) and symbol $g^{0k,-}(x', \xi', \xi_n, \eta_n, \mu)$ derived from the symbol of $q^{0k}$ at $x_n = 0$, cf. (2.15):

\begin{equation}
\tilde{g}^- (q^{0k}) = \tilde{g}^{0k,-}(x', x_n, y_n, \xi', \mu) \sim \sum_{J \in \mathbb{N}} \tilde{g}^{0k,-}_{-1-2k-J}(x', x_n, y_n, \xi', \mu), \text{ with }
\end{equation}

\begin{equation}
\tilde{g}^{0k,-}_{-1-2k-J}(x', x_n, y_n, \xi', \mu) = \sum_{1 \leq j \leq 2J+k} r^{-1}_{k,J,j}(x', \xi', \mu) \frac{1}{(j-1)!} (x_n + y_n)^{j-1} e^{-(x_n+y_n)\kappa^-},
\end{equation}

\begin{equation}
g^- (q^{0k}) = g^{0k,-}(x', \xi', \xi_n, \eta_n, \mu) \sim \sum_{J \in \mathbb{N}} g^{0k,-}_{-1-2k-J}(x', \xi', \xi_n, \eta_n, \mu), \text{ with }
\end{equation}

\begin{equation}
g^{0k,-}_{-1-2k-J}(x', \xi', \xi_n, \eta_n, \mu) = \sum_{1 \leq j \leq 2J+k} r^{-1}_{k,J,j}(x', \xi', \mu) \sum_{0 \leq j' \leq j-1} \frac{1}{(j-1-j')!} x_n^{j-j'} e^{-x_n\kappa^-} \frac{1}{(j-1-j)!} y_n^{j-1-j'} e^{-y_n\kappa^-},
\end{equation}

since the $r^{-1}_{k,J,j}$ are strongly homogeneous of degree $j - 2k - J$ in $(\xi', \mu)$, the degree of $g^{0k,-}_{-1-2k-J}(x', \xi', \xi_n, \eta_n, \mu)$ is $-1 - 2k - J$.

There are similar formulas for the symbol-kernel $\tilde{g}^+(q^{0k})$ and symbol $g^+(q^{0k})$ of $G^+(Q^k)$, where $\kappa^-$ and $r^{-1}_{k,J,j}$ are replaced by $\kappa^+$ and $r^+_{k,J,j}$.

If the coefficients in $P_1$ are not independent of $x_n$ near $x_n = 0$, the symbols of $G^\pm(Q^k)$ contain moreover the contributions from the terms in the Taylor expansion (2.16) of $q^{0k}$ in $x_n$ at $x_n = 0$.

Proposition 2.3. When the coefficients in $P_1$ depend on $x_n$,

\begin{equation}
g^\pm (q^{0k})(x', \xi, \eta_n, \mu) = g^{0k,\pm}(x', \xi, \eta_n, \mu) \sim \sum_{j \in \mathbb{N}} \frac{1}{j!} \partial^j_{\xi_n} g^\pm_{\xi_n}(x', 0, \xi, \mu),
\end{equation}

where $g^\pm$ of the $x_n$-independent symbols are found as in (2.19), now using the formulas (2.17). In particular, in the term of homogeneity degree $-1 - 2k - J$, the number of powers of $\kappa^+ + i\xi_n$ and $\kappa^--i\eta_n$ taken together is at most $2J+k+1$, so the structure is

\begin{equation}
g^{0k,\pm}(x', \xi, \eta_n, \mu) \sim \sum_{J \in \mathbb{N}} g^{0k,\pm}_{-1-2k-J}(x', \xi, \eta_n, \mu), \text{ with }
\end{equation}

\begin{equation}
g^{0k,\pm}_{-1-2k-J}(x', \xi, \eta_n, \mu) = \sum_{j,j' \geq 1, \ j+j' \leq 2J+k+1} s^{0k,\pm}_{k,J,j,j'}(x', \xi', \mu) (\kappa^+ + i\xi_n)^j (\kappa^- + i\eta_n)^{j'},
\end{equation}

with the $s^{0k,\pm}_{k,J,j,j'}$ strongly homogeneous of degree $j + j' - J - 2k - 1$.

Proof. We apply [G96, Th. 2.7.4], which gives the general formula (2.20). Now look at the powers that occur in the resulting denominators. As noted in (2.17), $\partial^j_{x_n}$ augments the
range of the powers in $q_{2k-J}$ to $m \leq 2J + k + l$, hence in the s.g.o. derived from this, the
powers go up to $2J + k + l + 1$. The application of $\nabla^J \xi_n$ furthermore augments the range
of the powers to $m \leq 2J + k + 2l + 1 = 2(J + l) + k + 1$. Since the resulting term has homogeneity degree $-2k - J - l - 1$, we find (2.21) by collecting the terms by degrees of homogeneity. □

2.4 The symbol of $G^{(k)}$.  
As a first step we shall describe the singular Green operator $G_\mu$ in (2.2) in a form convenient for the present purposes: 
The resolvent $R_\mu$ exists for $\mu^2 \in W$ (e.g. by variational theory), and the solution operator for the fully nonhomogeneous Dirichlet problem

$$(2.22) \quad \left( P_1 + \mu^2 \right)^{-1} = \left( R_\mu \quad K_\mu \right)$$

exists since $\gamma_0$ has a convenient right inverse. (Recall that $\gamma_0u = u|_{X'}$.)

We can assume that a normal coordinate $x_n$ and a normal derivative $D_n$ ($= -i\partial_{x_n}$) have been chosen in a neighborhood of $\partial X = X'$ so that, with $\varrho = \{\gamma_0, \gamma_0D_n\}$, we have Green’s formula for $P_1$:

$$(2.23) \quad (P_1u, v)_X - (u, P_1^*v)_X = (\varrho_0u, \varrho v)_{X'}, \quad \mathcal{A} = \begin{pmatrix} \varrho_{00}(x', D') & i\varrho(x') \\ i\varrho(x') & 0 \end{pmatrix};$$

here $\varrho_{00}$ is a first order differential operator on $X'$ and $\varrho(x')$ is the coefficient of $D_n^2$ for $P_1$ at $X'$. Define the nullspace

$$(2.24) \quad Z^{s}_{\mu, +} = \{u \in H^s(X) \mid (P_1 + \mu^2)u = 0 \text{ on } X\}.$$

Since $P_1 + \mu^2$ is invertible on $\tilde{X}$, one can show that the functions $u \in Z^{s}_{\mu, +}$ are uniquely determined by their Cauchy data $\varrho u$. In fact, the Poisson operator

$$(2.25) \quad K^{+}_\mu = -r^+ Q_\mu \left( \tilde{\gamma}_0^*, D^*_n\tilde{\gamma}_0^* \right) A : H^{s-\frac{1}{2}}(X') \times H^{s-\frac{3}{2}}(X') \to H^s(X)$$

acts as an inverse of $\varrho$: $Z^{s}_{\mu, +} \to H^{s-\frac{1}{2}}(X') \times H^{s-\frac{3}{2}}(X')$ (cf. Seeley [S69] or e.g. [G96, Ex. 1.3.5]; here $\tilde{\gamma}_0^*$ is the adjoint of the restriction operator $\gamma_0$ from functions on $\tilde{X}$ to functions on $X'$). The operator $C^{+}_\mu = \varrho K^{+}_\mu$ is a pseudodifferential projection in $H^{s-\frac{1}{2}}(X') \times H^{s-\frac{3}{2}}(X')$, the Calderón projector. Clearly, it is strongly polyhomogeneous.

By (2.22) there is also a mapping $K_\mu$ from $H^{s-\frac{1}{2}}(X')$ to $Z^{s}_{\mu, +}$ giving the solution $u = K_\mu \varphi$ of the semi-homogeneous Dirichlet problem $(P_1 + \mu^2)u = 0, \gamma_0u = \varphi$. Composition with $\gamma_0D_n$ leads to the $\psi$do $P_\mu = \gamma_0 \psi D_n K_\mu$, often called the Dirichlet-to-Neumann operator. By [G71, Sect. 6], the blocks in $C^{+}_\mu = \begin{pmatrix} C^{+}_{\mu, 00} & C^{+}_{\mu, 01} \\ C^{+}_{\mu, 10} & C^{+}_{\mu, 11} \end{pmatrix}$ are elliptic, and

$$(2.26) \quad P_\mu \sim \left( C^{+}_{\mu, 01} \right)^{\circ(-1)}(I - C^{+}_{\mu, 00})$$

where $(C^{+}_{\mu, 01})^{\circ(-1)}$ is a parametrix of $C^{+}_{\mu, 01}$; $P_\mu$ is strongly polyhomogeneous.
Now the solution operator $R_\mu : f \mapsto u$ for the other semihomogeneous Dirichlet problem $(P_1 + \mu^2)u = f$, $\gamma_0 u = 0$, is described as follows: Let $v = Q_{\mu,+}f$, then $z = u - v$ solves

$$(P_1 + \mu^2)z = 0, \quad \gamma_0 z = -\gamma_0 Q_{\mu,+}f.$$  

Here $g_z = \{-\gamma_0 Q_{\mu,+}f, -P_\mu \gamma_0 Q_{\mu,+}f\}$, so $z$ must equal $K_\mu^+$ of this. Thus

$$u = v + z = Q_{\mu,+}f + K_\mu^+ \{-\gamma_0 Q_{\mu,+}f, -P_\mu \gamma_0 Q_{\mu,+}f\}$$

$$= Q_{\mu,+}f + r^+ Q_{\mu} D^*_n \tilde{\gamma}_0^* \left( A_{00}(x', D') \begin{pmatrix} 0 \\ i\alpha(x') \end{pmatrix} \right) \left( \begin{pmatrix} I \\ P_\mu \end{pmatrix} \right) \gamma_0 Q_{\mu,+}f$$

$$= Q_{\mu,+}f + r^+ Q_{\mu} D^*_n \tilde{\gamma}_0^* S_{\mu}\gamma_0 Q_{\mu,+}f + r^+ Q_{\mu} D^*_n \tilde{\gamma}_0^* i\alpha_0 Q_{\mu,+}f,$$

where $S_{\mu}$ is the $\varphi$do $A_{00} + i\alpha P_\mu$ on $X'$.

So if we define

$$K_{0,\mu} = r^+ Q_{\mu} \tilde{\gamma}_0^*, \quad K_{1,\mu} = r^+ Q_{\mu} D^*_n \tilde{\gamma}_0^* \quad \text{(Poisson operators)},$$

$$T_{0,\mu} = \gamma_0 Q_{\mu,+} \quad \text{(trace operator of order } -2 \text{ and class } 0),$$

then $G_\mu$ is a sum of two terms, each composed of a Poisson operator ($K_{0,\mu}$ resp. $K_{1,\mu}$), a $\varphi$do on $X'$ ($S_\mu$ resp. $i\alpha$) and a trace operator $T_{0,\mu}$; all strongly polyhomogeneous:

$$G_\mu = K_{0,\mu} S_\mu T_{0,\mu} + K_{1,\mu} i\alpha T_{0,\mu}.$$  

The symbols $k_0(x', \xi, \mu)$, $k_1(x', \xi, \mu)$ and $t_0(x', \xi, \mu)$ of the Poisson and trace operators $K_{0,\mu}$, $K_{1,\mu}$ and $T_{0,\mu}$ in local coordinates are easy to find from (2.15)–(2.17) (with $k = 1$), by use of the rules in [G96] (see in particular the introduction to Section 2.3 there). When the coefficients in $P_1$ are independent of $x_n$ near $x_n = 0$, then

$$k_0 \sim \sum_{J \in \mathbb{N}} k_{0,-2-J}, \text{ with } k_{0,-2-J} = \sum_{1 \leq j \leq 2J+1} \frac{r_{1,2J+1}^{+}}{(\kappa^{+} + i\xi_{n})^{j}};$$

$$k_1 \sim \sum_{J \in \mathbb{N}} k_{1,-1-J}, \text{ with } k_{1,-1-J} = \sum_{1 \leq j \leq 2J+1} \frac{\kappa^{+} r_{1,2J+1}^{+}}{(\kappa^{+} + i\xi_{n})^{j}},$$

$$t_0 \sim \sum_{J \in \mathbb{N}} t_{0,-2-J}, \text{ with } t_{0,-2-J} = \sum_{1 \leq j \leq 2J+1} \frac{r_{1,2J+1}^{-}}{(\kappa^{-} - i\xi_{n})^{j}}.$$  

The $r_{1,2J+1}^{\pm}(x', \xi', \mu)$ are strongly homogeneous of degree $j - J - 2$. The $r_{1,2J+1}^{\pm}(x', \xi', \mu)$ are strongly homogeneous of degree $j - J - 1$, found from the $r_{1,2J+1}^{\pm}$ by use of the fact that $i\xi_{n}(\kappa^{+} + i\xi_{n})^{-j} = (\kappa^{+} + i\xi_{n})^{1-j} + \kappa^{+}(\kappa^{+} + i\xi_{n})^{-j}$.  

When the coefficients in $P_1$ depend on $x_n$, one can use [G96, Lemma 2.7.3] as in the proof of Proposition 2.3 above to see that the formulas for the Poisson and trace symbols have a similar structure, only with $r_{1,2J+1}^{\pm}$ replaced by other strongly homogeneous functions of degree $j - J - 2$ resp. $j - J - 1$.  

In the compositions needed to find the symbol of (2.28), we note that in the homogeneous term of degree \(-3 - J\) (order \(-2 - J\)) in for example \(K_{0,\mu}S_{\mu}T_{0,\mu}\), it is terms from \(k_{0,-2-J'}\) and \(t_{0,-2-J''}\) with \(J' + J'' \leq J\) that enter. The general structure of the symbol of \(G_\mu\) is then:

\[
(2.30) \quad g(x', \xi', \xi_n, \eta_n, \mu) \sim \sum_{J \in \mathbb{N}} g_{-3-J}(x', \xi', \xi_n, \eta_n, \mu), \quad \text{with} \quad g_{-3-J}(x', \xi', \xi_n, \eta_n, \mu) = \sum_{j \geq 1, j' \geq 1, j + j' \leq 2J+2} \frac{s_{J,j,j'}(x', \xi', \mu)}{(\kappa^+(x', \xi', \mu) + i\eta_n)^n (\kappa^-(x', \xi', \mu) - i\eta_n)^{j'}},
\]

with strongly homogeneous symbols \(s_{J,j,j'}\) of degree \(-3 - J + j + j'\).

We have shown:

**Lemma 2.4.** The singular Green operator \(G_\mu\) in \(R_\mu = Q_{\mu,+} + G_\mu\) has the form (2.28), with strongly polyhomogeneous entries. In local coordinates, the symbols of the Poisson operators \(K_{0,\mu}\) and \(K_{1,\mu}\) have homogeneous terms that are rational functions of \(\xi_n\) with the pole \(\im \kappa^+(x', \xi', \mu)\), and the symbol of the trace operator \(T_{0,\mu}\) has homogeneous terms that are rational functions of \(\xi_n\) with the pole \(-\im \kappa^-(x', \xi', \mu)\), as described in detail in (2.29)ff. The symbol structure of \(G_\mu\) is described in (2.30)ff.

Moreover, \(G^{(k)}_{\mu}\) has the same kind of symbol structure as \(G_\mu\):

**Proposition 2.5.** \(G^{(k)}_{\mu}\) in (2.3) is a strongly polyhomogeneous singular Green operator of order \(-2k\) (degree \(-1 - 2k\)), and the homogeneous terms \(g^{(k)}_{-1-2k-J}(x', \xi', \xi_n, \eta_n, \mu)\) in its symbol \(g^{(k)}\) of \(g^{(k)}_{-1-2k-J}(x', \xi', \xi_n, \eta_n, \mu) = \sum_{J \in \mathbb{N}} g^{(k)}_{-1-2k-J}(x', \xi', \xi_n, \eta_n, \mu)\) are of the form

\[
(2.31) \quad g^{(k)}_{-1-2k-J}(x', \xi', \xi_n, \eta_n, \mu) = \sum_{j \geq 1, j' \geq 1, j + j' \leq 2J+k+1} \frac{s_{k,J,j,j'}(x', \xi', \mu)}{(\kappa^+(x', \xi', \mu) + i\eta_n)^n (\kappa^-(x', \xi', \mu) - i\eta_n)^{j'}},
\]

with \(s_{k,J,j,j'}\) strongly homogeneous of degree \(-1 - 2k - J + j + j'\).

**Proof.** For the term \((Q_{\mu,+})^k - (Q_{\mu,+})^k\) we have:

\[
(2.32) \quad (Q_{\mu,+})^k - (Q_{\mu,+})^k = (Q_{\mu,+})^k - (Q_{\mu,+}^2)^{k-2} + (Q_{\mu,+}^2)^{k-2} - (Q_{\mu,+}^3)^{k-3} + (Q_{\mu,+}^3)^{k-3} + \cdots = G^+(Q_{\mu})G^-(Q_{\mu})(Q_{\mu,+})^{k-2} - G^+(Q_{\mu})G^-(Q_{\mu})(Q_{\mu,+})^{k-3} - \cdots.
\]

By Proposition 2.3, the terms in the symbol of \(G^+(Q_{\mu})G^-(Q_{\mu})\) are compositions of the form

\[
(\kappa^+ + i\eta_n)^{r_1} (\kappa^- - i\eta_n)^{r_2}, \quad \text{and} \quad (\kappa^- - i\eta_n)^{r_3}.
\]

The composition rule in the normal variable is recalled in (3.7) below; the composition in tangential variables follows standard \(\psi\)do rules. As a result, we get sums of expressions

\[
(2.33) \quad \frac{r_1}{(\kappa^+ + i\eta_n)^{r_1}} \frac{r_2}{(\kappa^- - i\eta_n)^{r_2}},
\]
since \( \frac{1}{(\kappa^+ - i\eta_n)^{1/2}} \circ n \frac{1}{(\kappa^- + i\xi_n)^{1/2}} \) gives a strongly polyhomogeneous symbol on \( \mathbb{R}^{n-1} \). Next, when an s.g.o. with symbol of the form (2.33) is composed with \( Q_{\mu,+} \), we must calculate

\[
(2.34) \quad h_{\eta_n}^- \left[ \frac{1}{(\kappa^+ + i\xi_n)^{1/2}} \left( \frac{1}{(\kappa^- - i\eta_n)^{1/2}} \right) \right]
\]

(as recalled in (3.9) below), which is seen to give symbols of the form (2.33) by decomposition of the function of \( \eta_n \) in simple fractions. So indeed, the contribution from (2.32) has homogeneous terms of the form (2.33).

Thus also the contributions from \( \text{pol}(G_{\mu}, Q_{\mu,+}) \) are of the asserted form.

3. Trace calculations for terms in \( G^+(P)G^-(Q^k_{\mu}) \) and \( GG^k_{\mu} \).

In the next three sections we analyze the traces of the operators \( G^+(P)G^-(Q^k_{\mu}) \), \( P_+ G^k_{\mu} \), \( GQ_{\mu,+}^k \) and \( GG^k_{\mu} \) in the localized situation. We consider in detail the case where the fiber dimension \( n' \) of \( E \) equals 1; it serves as a model, and the results in general are found by adding finitely many terms of this kind. For the right hand side factor in each composition we insert the expansion of the symbol in homogeneous terms. The remainders \( G^-(Q^k_{\mu}) - OPG(\sum_{J < J_0} g^{< k}_{-1-2k-J}) \), \( G^k_{\mu} - OPG(\sum_{J < J_0} g^{< k}_{-1-2k-J}) \), \( Q_{\mu,+} - OP(\sum_{J < J_0} q^{< k}_{-2k-J}) \), are bounded from \( L_2(\mathbb{R}^n_+) \) to \( H^{2k-\nu}(\mathbb{R}^n_+) \) uniformly for \( \mu \) on rays (cf. [G96, Theorems 2.5.6, 2.5.11]), so the compositions of these with \( G^+(P), P_+ \) or \( G \) have norm \( O(\mu^{-J_0}) \) as operators from \( L_2(\mathbb{R}^n_+) \) to the space of functions in \( H^{2k-\nu}(\mathbb{R}^n_+) \) with support in a fixed compact set. Since \( 2k - \nu > n \), they are trace class with trace norms that are \( O(\mu^{-J_0}) \):

\[
(3.1) \quad \|G^+(P)|G^-(Q^k_{\mu}) - OPG(\sum_{J < J_0} g^{< k}_{-1-2k-J})\|_\text{Tr}, \quad \|P_+[G^k_{\mu}] - OPG(\sum_{J < J_0} g^{< k}_{-1-2k-J})\|_\text{Tr}, \quad \|G[G^k_{\mu}] - OPG(\sum_{J < J_0} g^{< k}_{-1-2k-J})\|_\text{Tr}, \quad \|G[Q_{\mu,+} - OP(\sum_{J < J_0} q^{< k}_{-2k-J})]\|_\text{Tr}
\]

are all \( O(\mu^{-J_0}) \).

Here we can take \( J_0 \) arbitrarily large, so it remains to analyze the traces of \( G^+(P) \) \( OPG(g^{< k}_{-1-2k-J}) \), \( P_+ \) \( OPG(g^{< k}_{-1-2k-J}) \), \( G \) \( OPG(g^{< k}_{-1-2k-J}) \) and \( G \) \( OPG(q^{< k}_{-2k-J}) \) for each \( J \).

In the following, we operate with symbols in \( (x', y') \)-form and in \( y' \)-form as well as \( x' \)-form (cf. [G96], p. 141, Sect. 2.4)]; the advantage is that

\[
(3.2) \quad g_1(x', \xi, \eta_n) \circ g_2(y', \xi, \eta_n) = g_1(x', \xi, \eta_n) \circ_n g_2(y', \xi, \eta_n),
\]

with no further terms; this may be reduced to \( x' \)-form by [G96, Th. 2.4.6].
We shall use the fact that a singular Green operator $G = OPG(g(x', y', \xi', \xi_n, \eta_n))$ on $\mathbb{R}^n_{+}$ of order $< 1 - n$ and class 0 with compact $(x', y')$-support is trace class (in view of the rapid decrease of the symbol-kernel $\tilde{g}$ (cf. (2.18)) for $x_n, y_n \to \infty$), and that its trace can be calculated by a diagonal integral of its kernel $K_G(x, y)$:

\[
K_G(x, y) = (2\pi)^{-n-1} \int_{\mathbb{R}^{n+1}} e^{i(x'-y')\cdot\xi'} + i x_n \xi_n - i y_n \eta_n g(x', y', \xi', \xi_n, \eta_n) \, d\xi' d\xi_n d\eta_n
\]

(3.3)

\[
\text{Tr} \ G = \int_{\mathbb{R}^n_{+}} K_G(x, x) \, dx = (2\pi)^{-n} \int_{\mathbb{R}^n_{+} \times \mathbb{R}^{n-1}} \tilde{g}(x', x', x_n, x_n, \xi') \, d\xi' dx.
\]

(This and the following holds also when $G$ is in $x'$-form or $y'$-form, vanishing for large $x'$ resp. $y'$.) One defines the normal trace $\text{tr}_n g$ by:

\[
\text{tr}_n g(x', y', \xi') = \int_0^\infty \tilde{g}(x', y', x_n, x_n, \xi') \, dx_n = \frac{1}{2\pi} \int g(x', y', \xi', \xi_n, \eta_n) \, d\xi_n,
\]

(3.4)

it is a $\psi$do symbol on $\mathbb{R}^{n-1}$. $(\text{tr}_n g(x', \xi'))$ is called $\tilde{g}(x', \xi')$ in [G92, G96] and $(\text{tr}_n g)(x', \xi', D_n)$ in [FGLS96].) Denoting by $\text{tr}_n G$ the $\psi$do it defines, we obtain for its trace:

\[
\text{Tr}'(\text{tr}_n G) = (2\pi)^{n-1} \int_{\mathbb{R}^{2n-2}} \int_0^\infty \tilde{g}(x', x', x_n, x_n, \xi') \, dx_n d\xi' dx' = \text{Tr} \ G.
\]

(3.5)

Here we write $\text{Tr}'$ to indicate the trace of an operator defined over $\mathbb{R}^{n-1}$.

When $g(x', y', \xi, \eta_n, \mu)$ is strongly polyhomogeneous and has order $\nu < 1 - n$ and compact $(x', y')$-support (or is in $x'$-form or $y'$-form with compact $x'$-support resp. $y'$-support), then the operator it defines is trace class with an expansion

\[
\text{Tr} \ OPG(g) \sim \sum_{r \in \mathbb{N}} c_r \mu^{n-1+\nu-r} \text{ for } \mu \to \infty \text{ in } \Gamma,
\]

(3.6)

without logarithmic terms. This holds by the proof of [G96, Th. 3.3.10ff.] and its 1986 predecessor, also recalled in [G92, App.:] the reason is simply that $\text{tr}_n g$ (or $\tilde{g}$) is a $\psi$do symbol of order $\nu$ and regularity $+\infty$ so that the homogeneous terms in its symbol give the terms in (3.6).

**Example 3.1.** To explain the basic point of our strategy, we consider the simple case where $P = 0$ and $G$ and $P_1$ are replaced by operators with $x$-independent symbols, $G = OPG(g(\xi, \eta_n))$ and $P_1 = 1 - \Delta = \text{OP}(p_1(\xi))$, with

\[
g(\xi, \eta_n) = \frac{c(\xi')}{(\sigma(\xi') + i\xi_n)(\sigma(\xi') - i\eta_n)}, \quad p_1(\xi) = |\xi|^2 + 1, \quad \sigma(\xi') = [\xi'];
\]

here $c(\xi')$ is a $\psi$do symbol on $\mathbb{R}^{n-1}$ of order $\nu + 1$, whereby $G$ is of order $\nu$; $[\xi']$ denotes a positive $C^\infty$ function equal to $|\xi'$ for $|\xi'| \geq \frac{1}{2}$. We assume $\nu < 2 - n$. 
The resolvent $R_\mu = (P_1 + \mu \gamma_0)^{-1}$ is of the form $Q_{\mu, +} + G_{1, \mu}$, where $G_{1, \mu} = -\mu \gamma_0 Q_{\mu, +}$ with $K_\mu$ denoting the Poisson operator $K_\mu: v \mapsto u$ solving the Dirichlet problem $(P_1 + \mu^2)u = 0$, $\gamma_0 u = v$. It is easily checked that $K_\mu$ has the symbol-kernel $e^{-x_n \kappa}$ and symbol $(\kappa + i\xi_n)^{-1}$, with $\kappa = \langle (\xi', \mu) \rangle$. Thus (cf. [G96, Theorem 2.6.1] for composition rules), $R_\mu = \text{OP}(q)_+ + \text{OPG}(g_1)$ with symbols

$$q(\xi, \mu) = \frac{1}{(\kappa + i\xi_n)(\kappa - i\xi_n)} = \frac{1}{2\kappa(\kappa + i\xi_n)} + \frac{1}{2\kappa(\kappa - i\xi_n)},$$

$$g_1(\xi, \eta_n, \mu) = \frac{1}{\kappa + i\xi_n} q(\xi', \eta_n, \mu) = \frac{-1}{2\kappa(\kappa + i\xi_n)(\kappa - i\eta_n)}.$$

Consider $GG_\mu$. Composition with respect to the normal variable (in fact the full composition since the symbols are independent of $x'$) and application of $\text{tr}_n$ have the form

$$g(\xi', \xi_n, \eta_n) \circ_n g_1(\xi', \xi_n, \eta_n, \mu) = \frac{1}{(2\pi)^2} \int g(\xi', \xi_n, \eta_n) g_1(\xi', \xi_n, \eta_n, \mu) d\xi_n,$$

$$\text{tr}_n (g \circ_n g_1) = \frac{1}{(2\pi)^2} \int g(\xi', \xi_n, \eta_n) g_1(\xi', \xi_n, \eta_n, \mu) d\xi_n d\xi_n,$$

which in the considered case gives

$$\text{tr}_n (g \circ_n g_1) = \frac{1}{(2\pi)^2} \int \frac{c(\xi')}{(\sigma + i\xi_n)(\sigma - i\xi_n)} \frac{-1}{2\kappa(\kappa + i\xi_n)(\kappa - i\xi_n)} d\xi_n d\xi_n = \frac{-c}{2\kappa(\sigma + \kappa)^2},$$

by residue calculus. This is a product of symbols on $\mathbb{R}^{n-1}$ that are weakly polyhomogeneous according to [GS95], with $c \in S^{\nu + 1, 0}$, $\kappa^{-1}$ and $(\sigma + \kappa)^{-1} \in S^{-1, 0} \cap S^{0, -1}$, so $\text{tr}_n (g \circ_n g_1) \in S^{\nu - 2, 0} \cap S^{\nu + 1, -3}$. When $\psi(x') \in C^\infty_0 (\mathbb{R}^{n-1})$, the operator $\psi \text{OP}^1 (\text{tr}_n g \circ g_1) = \psi \text{tr}_n (GG_\mu)$ is trace class since $\nu - 2 < -n$ (in fact $\nu - 2 < 1 - n$ would suffice). By (3.5) and [GS95, Th. 2.1],

$$\text{Tr}'(\psi \text{tr}_n GG_\mu) = \text{Tr}(\psi GG_\mu) \sim \sum_{l=0}^\infty a_l \mu^{n - 1 + \nu - 2 - l} + \sum_{l=0}^\infty (a_l' \log \mu + a_l'') \mu^{3 - l};$$

note that there is no log-term at the power $-2$.

Now consider $GQ_{\mu, +}$. The composition rules are here

$$g(\xi', \xi_n, \eta_n) \circ_n q(\xi', \xi_n, \mu) + = h_{\eta_n}^- [g(\xi', \xi_n, \eta_n) q(\xi', \xi_n, \mu)],$$

$$\text{tr}_n (g \circ_n q_+) = \frac{1}{2\pi} \int h_{\eta_n}^- [g(\xi', \xi_n, \eta_n) q(\xi', \xi_n, \mu)] d\xi_n = \frac{1}{2\pi} \int g(\xi', \eta_n, \eta_n) q(\xi', \eta_n, \mu) d\eta_n,$$

(since $h_{\eta_n}^+ [g(\xi', \xi_n, \eta_n) q(\xi', \xi_n, \mu)] |_{\xi_n = \eta_n}$ is meromorphic in $\eta_n$ with no poles in $\mathbb{C}_-$ and is $O(\eta_n^{-2})$ for $\eta_n \to \infty$ in $\mathbb{C}$). In the considered case this gives:

$$\text{tr}_n (g \circ_n q_+) = \frac{1}{(2\pi)^2} \int \frac{c(\xi')}{(\sigma + i\xi_n)(\sigma - i\xi_n)} \left[ \frac{1}{2\kappa(\kappa + i\xi_n)} + \frac{1}{2\kappa(\kappa - i\xi_n)} \right] d\xi_n = \frac{c}{2\kappa(\sigma + \kappa)\sigma},$$
by residue calculus. Then since \( \sigma^{-1} \in S^{-1,0} \), we find that \( \text{tr}_n(g \circ_n q_+) \in S^{\nu-2,0} \cap S^{\nu,-2} \).

By [GS95, Th. 2.1],

\[
(3.10) \quad \text{Tr}'(\psi \text{tr}_n GQ_{\mu,+}) = \text{Tr}(\psi GQ_{\mu,+}) \sim \sum_{l=0}^{\infty} b_l \mu^{n-1+\nu-2-l} + \sum_{l=0}^{\infty} (b'_l \log \mu + b''_l) \mu^{-2-l}.
\]

In this case there is a log-term at the power \(-2\). The proof of [GS95, Th. 2.1] allows an analysis of the coefficient \(b'_0\) in (3.10). It shows that \(b'_0 = (2\pi)^{1-n} \int_{S^{n-1}} s(x', \xi') \, d\sigma'\), where \(s(x', \xi')\) is the coefficient of \(\mu^{-2}\) in the expansion of \(\psi \text{tr}_n(g \circ_n q_+)\) into decreasing powers of \(\mu\), and the subscript denotes the component of homogeneity \(1-n\) with respect to \(\xi'\). In our case, we have \(\psi \text{tr}_n(g \circ_n q_+)=\psi c(2\kappa(\kappa+\sigma)\sigma)^{-1} \sim \psi c(2\sigma)^{-1} \mu^{-2} + \ldots\) (lower order in \(\mu\)). Since \(\text{tr}_n(\psi g)=\psi c(2\sigma)^{-1}\), we obtain the statement for \(\psi G\) on the noncommutative residue in Theorem 1.1.

In the general case we shall use expansions of the \(\mu\)-independent symbols in Laguerre functions (with the pole \(i\sigma\)), which together with the structure of the symbols of \(Q_\mu\) and \(G_\mu\) as rational functions with well-behaved poles allow termwise residue calculus like in the example. The general outcome is that the compositions with \(\mu\)-dependent s.g.o.s do not contribute to the relevant logarithmic term, whereas the compositions with \(Q_\mu^k\) do so. It is important to show that the results we find are preserved after summation of the Laguerre expansions. The reader is kindly advised to consult the Appendix, where we have collected some facts on the Laguerre expansions used here.

In the present section we study the terms in \(G^+(P)G^-(Q_\mu^k)\) and \(GG_\mu^k\). These two compositions are treated in similar ways; we give the details for the latter which contains the most general types of singular Green operators.

In the local coordinate system, \(G = \text{OPG}(g(x', \xi', \xi_n, \eta_n))\), where we can expand the symbol \(g(x', \xi', \xi_n, \eta_n)\) in a convergent series in terms of Laguerre functions of \(\xi_n\) and \(\eta_n\):

\[
(3.11) \quad g(x', \xi', \xi_n, \eta_n) = \sum_{l,m \in \mathbb{N}} 2\sigma c_{lm}(x', \xi') \tilde{\varphi}_l'(\xi_n, \sigma) \tilde{\varphi}_m(\eta_n, \sigma), \quad \sigma = [\xi']
\]

with a rapidly decreasing double sequence of \(\psi\)-do symbols \(c_{lm}(x', \xi')\) of order \(\nu\) with compact \(x'\)-support. (Note that we use the non-normed Laguerre functions \(\tilde{\varphi}_l', \) cf. (A.4).)

For \(g_{-1-2k-J}^{(k)}\) we have the structure described in Proposition 2.5.

We shall use formulas (3.4)–(3.5) to calculate the trace of \(G \text{OPG}(g_{-1-2k-J}^{(k)})\). For convenient composition rules, we assume that the right hand factor is given in \(y'\)-form, with symbol \(g_{-1-2k-J}^{(k)}(y', \xi, \eta_n, \mu)\) (recall (3.2)). The general rule for passage from \(x'\)-form to \(y'\)-form (cf. [G96, (2.4.30 iii)]) shows that this symbol has the same structure as the \(x'\)-form described in Proposition 2.5. Then

\[
\text{Tr} \ G \text{OPG}(g_{-1-2k-J}^{(k)}) = \text{Tr} \ G \text{OPG}(g(x', \xi, \eta_n) \circ_n g_{-1-2k-J}^{(k)}(y', \xi, \eta_n, \mu)), \quad g \circ_n g_{-1-2k-J}^{(k)} = \sum_{l,m \in \mathbb{N}} g_{l,m,J}(x', y', \xi, \eta_n, \mu) \with\ g_{l,m,J}(x', y', \xi, \eta_n, \mu) = 2\sigma c_{lm}(x', \xi') \tilde{\varphi}_l'(\xi_n, \sigma) \tilde{\varphi}_m(\eta_n, \sigma) \circ_n g_{-1-2k-J}^{(k)}(y', \xi, \eta_n, \mu).
\]
Termwise, we have by (3.5),

$$\text{Tr OPG}(g_{l,m,j}(x', y', \xi, \eta_n, \mu)) = \text{Tr OPG}'(s_{l,m,j}(x', y', \xi', \mu)),$$

where

$$s_{l,m,j}(x', y', \xi', \mu) = \text{tr}_n g_{l,m,j}(x', y', \xi, \eta_n, \mu).$$

The crucial step is now to calculate the normal trace of the s.g.o. arising from composing \(\bar{\phi}'(\xi_n, \sigma)\bar{\phi}_m(\eta_n, \sigma)\) with the fractions in (2.31). In view of (3.7), we must calculate

$$\text{tr}_n(\bar{\phi}'(\xi_n, \sigma)\bar{\phi}_m(\eta_n, \sigma) \circ_n \frac{1}{(\kappa^+ + i\xi_n)^j(\kappa^- - i\eta_n)^{j'}})$$

$$= \text{tr}_n(\bar{\phi}'(\xi_n, \sigma)) \left[ \frac{1}{2\pi i} \int \varphi_m(\xi_n, \sigma) \frac{1}{(\kappa^+ + i\xi_n)^j} d\xi_n \right] \frac{1}{(\kappa^- - i\eta_n)^{j'}}$$

$$= \frac{1}{2\pi} \int \frac{\varphi_m(\xi_n, \sigma)}{(\kappa^+ + i\xi_n)^j} d\xi_n \cdot \frac{1}{2\pi} \int \varphi'(\xi_n, \sigma) \frac{1}{(\kappa^- - i\eta_n)^{j'}} d\xi_n;$$

a product of two \(\psi\)do symbols on \(\mathbb{R}^{n-1}\).

The resulting formulas are worked out in the following lemma.

**Lemma 3.2.** One has for all \(m \geq 0\) and \(j \geq 1\):

$$\frac{1}{2\pi} \int \frac{\varphi'_m(\xi_n, \sigma)}{(\kappa^+(x', \xi', \mu) + i\xi_n)^j} d\xi_n = \sum_{m' \geq 0, \lvert m' - m \rvert < j} (\kappa^+)^{1-j} a_{jm'} \frac{\sigma - \kappa^+)^m}{(\sigma + \kappa^+)^{m'+1}},$$

$$\frac{1}{2\pi} \int \frac{\varphi_m(\xi_n, \sigma)}{(\kappa-(x', \xi', \mu) - i\xi_n)^j} d\xi_n = \sum_{m' \geq 0, \lvert m' - m \rvert < j} (\kappa^-)^{1-j} a_{jm'} \frac{\sigma - \kappa^-)^m}{(\sigma + \kappa^-)^{m'+1}},$$

with universal constants \(a_{jm'}\) that are \(O(m^j)\) for fixed \(j\). The resulting expressions are weakly polyhomogeneous \(\psi\)do symbols belonging to \(S^{-j,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \Gamma) \cap S^{0,-j}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \Gamma)\).

**Proof.** If \(j = 1\), then since the integrand is \(O(\xi_n^{-2})\) for \(|\xi_n| \to \infty\),

$$\frac{1}{2\pi} \int \frac{\varphi'_m(\xi_n, \sigma)}{\kappa^+(x', \xi', \mu) + i\xi_n} d\xi_n = \frac{1}{2\pi} \int \frac{\sigma + i\xi_n)^m}{\sigma - i\xi_n)^{m+1}} \frac{1}{\kappa^n - i\kappa^n} d\xi_n = \text{Res}_{\xi_n = i\kappa^n} \frac{(\sigma + i\xi_n)^m}{(\sigma - i\xi_n)^{m+1}} \frac{1}{\kappa^n - i\kappa^n} d\xi_n$$

$$= \text{Res}_{\xi_n = i\kappa^n} \left( \frac{(\sigma + i\xi_n)^m}{(\sigma - i\xi_n)^{m+1}} \cdot \frac{1}{\kappa^n - i\kappa^n} \right)_{\xi_n = i\kappa^n} = \left( \frac{(\sigma + \kappa^+)^m}{(\sigma + \kappa^+)^{m+1}} \right)_{\xi_n = i\kappa^n},$$

where \(L_+\) is a positively oriented curve around \(i\kappa^n + \mathbb{C}_+\). By the rules in [GS95] (more specifically, Th. 1.16 and 1.23 there), \((\sigma - \kappa^+)^{m+1}\) is weakly polyhomogeneous (wphg) belonging to \(S^{-1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \Gamma) \cap S^{0,-1}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \Gamma)\) (recall (2.7)). For a larger value of \(j\) we use (A.7) and the third line in (A.5) (conjugated), noting that

$$\xi_n^{j-1} \partial_n^{j-1} f(\xi_n) = \sum_{0 \leq j' \leq j-1} c_{jj'} (\partial_n \xi_n)^{j'} f(\xi_n),$$

with universal constants \(c_{jj'}\). This gives the following calculation:

$$\frac{1}{2\pi} \int \frac{\varphi'_m(\xi_n, \sigma)}{(\kappa^+ + i\xi_n)^j} d\xi_n = \frac{1}{2\pi} \int \frac{\varphi'_m(\xi_n, \sigma)}{(\kappa^+ + i\xi_n)^j} d\xi_n$$

$$= \frac{1}{2\pi} \int \frac{1}{(j-1)!} \left[ (-i\partial_n)^{j-1} \varphi'_m(\xi_n, \sigma) \frac{1}{\kappa^+ + i\xi_n} d\xi_n \right.$$

$$= \frac{(-i)^{j-1}}{(j-1)!} \partial_n^{j-1} \varphi'_m(\xi_n, \sigma) \right|_{\xi_n = i\kappa^n} = \left( \frac{(-\kappa^+)^{j-1}}{(j-1)!} \partial_n^{j-1} \frac{1}{\kappa^+ + i\xi_n} \frac{1}{(\kappa^- - i\eta_n)^{j'}} d\xi_n \right)_{\xi_n = i\kappa^n}$$
The pseudodifferential symbols statements hold for symbols constructed in the following.

Proof. The contribution from each term in the finite expansion (2.31) is treated by Lemma (3.19) Tr\( \sim \frac{(-\kappa^+)^{1-j}}{(j-1)!} \sum_{0 \leq j' \leq j-1} c_{jj'} (\partial_\xi \xi_n) j' \tilde{\varphi}_m (\xi_n, \sigma) \right|_{\xi_n = i\kappa^+}

(3.18)

\[ = \sum_{m' \geq 0, |m'-m|<j} (\kappa^+)^{1-j} a_{jm'} \tilde{\varphi}_m (\xi_n, \sigma) \right|_{\xi_n = i\kappa^+} \]

\[ = \sum_{m' \geq 0, |m'-m|<j} (\kappa^+)^{1-j} a_{jm'} \frac{(\sigma - \kappa^+)^{m'}}{m'+1} \]

with some universal constants \(a_{jm'}\) that are \(O(m^j)\). (Observe that in the third line of (A.5), the fact that \(\tilde{\varphi}'_{k-1}\) in the right hand side has coefficient \(-\frac{1}{2}k\) assures that no terms with negative index are introduced when we apply the formula repeatedly to a \(\tilde{\varphi}'_m\) with \(m \geq 0\).) In view of the observations made further above on \(\frac{(\sigma - \kappa^+)^m}{(\sigma + \kappa^+)^m} \), and the composition rules of [GS95], this is a 

\[\text{wpgh} \psi\text{do symbol in } S^{-j,0} \cap S^{0,-j}.\]

This proves the statements for the first line in (3.15).

The second line in (3.15) follows by conjugation, replacing \(\kappa^+\) by \(\kappa^-\). \(\square\)

Remark 3.3. One has moreover that the \(\partial_x\) derivatives belong to \(S^{-j-r,0} \cap S^{0,-j-r}\) for all \(r\) (such symbols are called special parameter-dependent of degree \(-j\) in [G99]). Similar statements hold for symbols constructed in the following.

We then find:

Proposition 3.4. The pseudodifferential symbols \(s_{l,m,J}(x',y',\xi',\mu)\) defined in (3.13) are in \(S^{\nu-2k-J,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \Gamma) \cap S^{\nu+1,-1-2k-J}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \Gamma)\).

Moreover, the sum over \(l,m\) converges in all symbol norms, so that also the symbol

\[s_{J}(x',y',\xi',\mu) = \sum_{l,m \in \mathbb{N}} s_{l,m,J}(x',y',\xi',\mu)\]

\[= \text{tr}_n(g(x',\xi',\xi_n,\eta_n) \circ_n g_{-1-2k-J}^{(k)}(y',\xi',x_n,\eta_n,\mu))\]

is in \(S^{\nu-2k-J,0} \cap S^{\nu+1,-1-2k-J}\).

Finally, there is an expansion for \(\mu \to \infty\) in \(\Gamma\):

\[\text{Tr}(G_{OPG}(g_{-1-2k-J}^{(k)}(y',\xi,\eta_n,\mu))) = \text{Tr}' \text{OP}'(s_{J}(x',y',\xi',\mu))\]

\(\sim \sum_{l=0}^{\infty} A_{J,l} n^{\nu-2k-J-l} + \sum_{l=0}^{\infty} (a_{J,l}^2 \log \mu + a_{J,l}^3) \mu^{-1-2k-J-l}\).

Proof. The contribution from each term in the finite expansion (2.31) is treated by Lemma 3.2. For \(s_{k,J,j'}(x',\xi',\mu)\) and \(2\sigma c_{lm}\) replaced by 1, this gives a symbol in \(S^{-j-j',0} \cap S^{0,-j-j'}\). The factors are taken into account as follows: Since \(s_{k,J,j'}(x',\xi',\mu)\) is strongly homogeneous of degree \(-1-2k-J+j+j'\), it is in \(S^{-1-2k-J+j+j',0} \cap S^{0,-1-2k-J+j+j'}\) so multiplication by it gives a symbol in \(S^{-1-2k-J,0} \cap S^{0,-1-2k-J}\). Next, \(2\sigma c_{lm} \in S^{\nu+1,0}\), so multiplication by it gives a symbol in \(S^{\nu-2k-J,0} \cap S^{\nu+1,-1-2k-J}\). This shows the statement on the \(s_{m,l,J}\).
For the summation in \( l, m \), we use that the symbols \( c_{lm}(x', \xi') \) are rapidly decreasing in \( l \) and \( m \) in all symbol norms. Observe moreover, that the new coefficients introduced in the formulas in (3.15) are polynomially bounded in \( l \) and \( m \), for each \( j \) and \( j' \), and that the same will hold for derivatives in view of the formulas (with \( \kappa = \kappa^\pm \)),

\[
\begin{align*}
\partial_{\xi_j} (\frac{\sigma - \kappa}{\sigma + \kappa})^m &= m(\frac{\sigma - \kappa}{\sigma + \kappa})^{m-1} \frac{\kappa \partial_{\xi_j} \sigma - \sigma \partial_{\xi_j} \kappa}{(\sigma + \kappa)^2}, \\
\partial_{\xi_j} (\frac{\sigma - \kappa}{\sigma + \kappa})^m &= m(\frac{\sigma - \kappa}{\sigma + \kappa})^{m-1} - \sigma \partial_{\xi_j} \kappa
\end{align*}
\]

(3.21)

that can be continued to a study of higher derivatives again giving expressions with polynomial coefficients in \( m \). Using these facts together, one finds that summation in \( l, m \) converges in each symbol norm. This shows the second assertion.

Since the trace norm is dominated by a finite set of symbol norms, the summation now also converges in trace norm, so by (3.13),

\[
\text{Tr}' \text{OP}'(s_{(J)}) = \sum_{l,m} \text{Tr}' \text{OP}'(s_{l,m,J}) = \sum_{l,m} \text{Tr} \text{OPG}(g_{l,m,J}) = \text{Tr} \text{OPG}(g \circ_n g_{-1-2k-\lambda})
\]

for the last equality sign we used that the sum of s.g.o.s likewise converges in trace norm. Now we can apply [GS95, Th. 2.1] to OP'(s_{(J)}) in \( n - 1 \) dimensions, which shows that the trace has an asymptotic expansion (3.20). \( \square \)

Note that the expansions (3.20) are of the type (1.10) (I) with \( -\lambda = \mu^2 \), and that they do not contribute to the coefficient \( c_0' \) there, not even for \( J = 0 \), since the first logarithmic term in (3.20) has \( \mu^2 \) to the power \( -k - \frac{1}{2} - \frac{d}{2} \).

A similar result holds for the operators \( G^+(P) \) \( \text{OPG}(g_{-1-2k-\lambda}) \), except that the root in \( \mathbb{C}^+ \) is now \( i\kappa^- \); we shall not write a separate statement on this.

**Remark 3.5.** One can fairly easily allow \( P_1 \) to be of higher order (or non-scalar) with constant multiplicity of the roots (in \( \xi_n \)) of the (determinant of the) principal symbol at \( x_n = 0 \), since the symbol of \( G_{(k)}^{(k)} \) then has a decomposition in rational functions in terms of these poles depending continuously on \( (\xi', \mu) \); this just requires a little more residue calculus (with several poles instead of one in \( \mathbb{C}_\pm \)). For the general case where multiplicities may vary with \( (\xi', \mu) \), one can probably instead obtain the desired results by use of contour integrals around all the poles in \( \mathbb{C}_\pm \), respectively. We leave out these generalizations here, since the method is already complicated to explain in the case of the two poles \( \kappa^+ \) and \( \kappa^- \).

**4. Trace calculations for terms in \( P_+ G_{(k)}^{(k)} \).**

We now turn to the operator \( P_+ G_{(k)}^{(k)} \) in (2.5), in the localized version. Here \( P_+ G_{(k)}^{(k)} = P_+ G_{(k)}^{(k)} \theta \) for some \( \theta \in C^\infty_{(0)}(\mathbb{R}_+^d) \) (\( C^\infty \) functions with compact support in \( \mathbb{R}_+^d \)). Some manipulations with the effect of \( P \) are needed before we decompose the right hand factor in homogeneous terms. We can assume that \( P \) is in \( (x', y_n) \)-form, \( P = \text{OP}(p(x', y_n, \xi)) \).

Since \( p \) satisfies the transmission condition, each term in the Taylor expansion in \( y_n \) can be decomposed:

\[
p(x', y_n, \xi) = \sum_{0 \leq l < l_0} \frac{1}{l!} \partial_{y_n}^l p(x', 0, \xi) y_n^l + \tilde{p}_{l_0}(x', y_n, \xi) y_n^{l_0}, \quad \text{with}
\]

\[
\partial_{y_n}^l p(x', 0, \xi) = p_{(l)}'(x', \xi) + p_{(l)}''(x', \xi);
\]

(4.1)
here \( \tilde{p}_{l_0} \) is a \( \psi \)do symbol of order \( \nu \), \( p'_{(l)} \) is a differential operator symbol of order \( \nu \) (void if \( \nu < 0 \)) and \( p''_{(l)} \) is a \( \psi \)do symbol of order \( \nu \) that is \( O(\langle \xi_n \rangle^{-1}) \) in \( \xi_n \) (cf. (A.6)). We take as usual \( 2k - \nu > n \).

**Lemma 4.1.** Each of the terms in the decomposition

\[
(4.2) \quad P^\pm \Gamma \Omega^{(k)} = P^\pm \Gamma \Omega^{(k)} \theta
= \sum_{l < l_0} \frac{1}{l!} \text{OP}(p'_{(l)}(x', \xi)) \pm x_n \Gamma \Omega^{(k)} \theta + \text{OP}(p''_{(l)}(x', \xi)) \pm x_n \Gamma \Omega^{(k)} \theta
+ \text{OP}(\tilde{p}_{l_0}(x', y_n, \xi)) \pm x_n \Gamma \Omega^{(k)} \theta
\]

is trace class.

The trace norm of \( \text{OP}(\tilde{p}_{l_0}(x', y_n, \xi)) \pm x_n \Gamma \Omega^{(k)} \theta \) is \( O(\mu^{n+1+\nu+2k-l_0}) \), for each \( l_0 \in \mathbb{N} \) with \( l_0 > n - 2k + \nu_+ \), \( \nu_+ = \max\{\nu, 0\} \).

For each \( l \), the trace of the operator \( \text{OP}(p'_{(l)}(x', \xi)) \pm x_n \Gamma \Omega^{(k)} \theta \) has an asymptotic expansion in a series without logarithms

\[
(4.3) \quad \text{Tr} \text{OP}(p'_{(l)}(x', \xi)) \pm x_n \Gamma \Omega^{(k)} \theta \sim \sum_{r \in \mathbb{N}} c_r \mu^{n-1+\nu-2k-l-r}, \text{ for } \mu \to \infty \text{ in } \Gamma.
\]

**Proof.** The decomposition (4.2) follows directly from (4.1), and all the operators are trace class since they are of the form \( \Gamma \Omega^{(k)} \theta = (\theta \Gamma \Omega^{*})^{*} \), where \( \Gamma \) and \( \Omega^{*} \) are singular Green operators of order \( \nu - 2k < -n \) and class 0.

For the second statement, we recall the rule

\[
(4.4) \quad x_n \text{OPG}(g) = \text{OPG}(\widetilde{D}_{\xi_n} g)
\]

(cf. e.g. [G96, Lemma 2.4.3]); thus \( x_n \Gamma \Omega^{(k)} \theta \) is a singular Green operator that is strongly polyhomogeneous of order \( -2k + l_0 \) and class 0. In the terminology of [G96], it is of regularity \( +\infty \), whereas \( \text{OP}(\tilde{p}_{l_0}) \), being of order \( \nu \) and \( \mu \)-independent, is of regularity \( \nu \). The composed operator \( \text{OP}(\tilde{p}_{l_0}) + x_n \Gamma \Omega^{(k)} \) is then an s.g.o. of order \( \nu - 2k - l_0 \), class 0 and regularity \( \nu \), and so is its adjoint, and they map \( L_2(\mathbb{R}_+^n) \) into \( H^{2k+l_0-\nu, \mu}(\mathbb{R}_+^n) \) with norm \( O(\langle \mu \rangle^{-\nu} + 1) \) (cf. [G96, Ths 2.5.11, 2.7.6]). Then the trace norm of \( \text{OP}(\tilde{p}_{l_0}) + x_n \Gamma \Omega^{(k)} \theta \) is \( O(\langle \mu \rangle^{n+1-2k-l_0+\nu_+}) \).

For the third statement we note that since \( \text{OP}(p'_{(l)}) \) is a differential operator of order \( \nu \), and the factor \( x_n \) has the effect described in (4.4), the composite is a strongly polyhomogeneous singular Green operator of order \( \nu - 2k - l \). Then its trace has an expansion as in (3.6); this gives (4.3). \( \square \)

It remains to study the terms

\[
(4.5) \quad \frac{1}{l!} \text{OP}(p''_{(l)}(x', \xi)) \pm x_n \Gamma \Omega^{(k)} \theta.
\]

Fix \( l \) and denote for simplicity

\[
(4.6) \quad \text{OP}(p''_{(l)}(x', \xi)) = P'' = \text{OP}(p''(x', \xi)), \quad x_n \Gamma \Omega^{(k)} \theta = \text{OPG}(g').
\]
Here we expand the symbol of $P''$ in a convergent series as in (A.6):

\begin{equation}
(4.7) \quad p''(x', \xi', \xi_n) = \sum_{m \in \mathbb{Z}} b_m(x', \xi') \hat{\varphi}_m(\xi_n, \sigma),
\end{equation}

where $b_m$ is rapidly decreasing in $S^{\nu+1}$. Concerning $g'$, we note that $D_{\xi_n} g^{(k)}$ has a structure like that of $g^{(k)}$ except that the summation in homogeneous terms starts with $J = l$; this must moreover be composed with $\theta$, which again gives a symbol with the same structure. We can take it in $y'$-form, considering its expansion in strongly homogeneous terms:

\begin{equation}
(4.8) \quad g'(y', \xi, \eta_n, \mu) \sim \sum_{l \leq J < \infty} g'_{-1 - 2k - J}(y', \xi, \eta_n, \mu);
\end{equation}

here the $J$'th term is of the form in (2.31) (with $x'$ replaced by $y'$). As in the analysis around (3.1) it suffices to study $\text{Tr} P''_+ \text{OPG}(g'_{-1 - 2k - J})$ for each $J$.

Now we proceed as in Section 3, writing

\begin{equation}
(4.9) \quad \text{Tr} P''_+ \text{OPG}(g'_{-1 - 2k - J}) = \text{Tr} \text{OPG}(p''(x', \xi) + \circ_n g'_{-1 - 2k - J}(y', \xi, \eta_n, \mu))
\end{equation}

where

\begin{equation}
\begin{aligned}
p'' & \circ_n g'_{-1 - 2k - J} = \sum_{m \in \mathbb{Z}} g_{m,J}(x', y', \xi, \eta_n, \mu) \quad \text{with} \\
g_{m,J} & = b_m(x', \xi') \hat{\varphi}_m(\xi_n, \sigma) + \circ_n g'_{-1 - 2k - J}(y', \xi, \eta_n, \mu).
\end{aligned}
\end{equation}

The composition rule is: $p(\xi_n) \circ_n g'(\xi_n, \eta_n) = h^+_\xi_n [p(\xi_n)g'(\xi_n, \eta_n)]$.

Termwise, we have by (3.5),

\begin{equation}
(4.10) \quad \text{Tr} \text{OPG}(g_{m,J}(x', y', \xi, \eta_n, \mu)) = \text{Tr} \text{OP}'(s_{m,J}(x', y', \xi', \mu)),
\end{equation}

where

\begin{equation}
\begin{aligned}
s_{m,J}(x', y', \xi, \eta_n, \mu) & = \text{tr}_n g_{m,J}(x', y', \xi, \eta_n, \mu).
\end{aligned}
\end{equation}

So the crucial step is to calculate the normal trace of the s.g.o. arising from composing $\hat{\varphi}_m(\xi_n, \sigma)$ as a $\psi$do symbol with the fractions in (2.31). Here

\begin{equation}
(4.11) \quad s_{j,j',m}(x', \xi', \mu) = \text{tr}_n (\hat{\varphi}_m(\xi_n, \sigma) + \circ_n \frac{1}{(\kappa^+ - i \xi_n)^{1/2} (\kappa^- - i \xi_n)^{1/2}})
\end{equation}

\begin{equation}
= \text{tr}_n (h^+ [\hat{\varphi}_m(\xi_n, \sigma) \frac{1}{(\kappa^+ - i \xi_n)^{1/2}}] \frac{1}{(\kappa^- - i \xi_n)^{1/2}}) d\xi_n
\end{equation}

\begin{equation}
= \frac{1}{2\pi} \int h^+ [\hat{\varphi}_m(\xi_n, \sigma) \frac{1}{(\kappa^+ - i \xi_n)^{1/2}}] \frac{1}{(\kappa^- - i \xi_n)^{1/2}} d\xi_n
\end{equation}

\begin{equation}
= \frac{1}{2\pi} \int \hat{\varphi}_m(\xi_n, \sigma) \frac{1}{(\kappa^+ - i \xi_n)^{1/2}} \frac{1}{(\kappa^- - i \xi_n)^{1/2}} d\xi_n,
\end{equation}

where we used that $h^- [\hat{\varphi}_m(\xi_n, \sigma)] \frac{1}{(\kappa^+ - i \xi_n)^{1/2}} \frac{1}{(\kappa^- - i \xi_n)^{1/2}}$ is holomorphic on $\mathbb{C}_+$ and $O(\xi_n^{-2})$ for $|\xi_n| \to \infty$.

**Lemma 4.2.** The symbol

\begin{equation}
(4.11) \quad s_{j,j',m} = \frac{1}{2\pi} \int \frac{(\sigma - i \xi_n)^m}{(\sigma + i \xi_n)^{m+1}} \frac{1}{(\kappa^+ + i \xi_n)^{1/2} (\kappa^- - i \xi_n)^{1/2}} d\xi_n
\end{equation}
satisfies for \( m \geq 0, j \) and \( j' \geq 1 \):

\[
(4.12) \quad s_{j,j',m} = \sum_{|m-m'| \leq j'' < j'} b_{jj'j''m'} (\kappa^-)^{-j''} \frac{\sigma - \kappa^-}{(\sigma + \kappa^-)^m} (\kappa^+ + \kappa^-)^{-j-j'+1+j''},
\]

where the \( b_{jj'j''m'} \) are universal constants that are \( O(m^{j'}) \) for fixed \( j, j' \). This is a weakly polyhomogeneous symbol in \( S^{-j-j',0} \cap S^{0,-j-j'} \). There is a similar formula for \( m \leq 0 \), with \( j \) and \( j' \) interchanged, \( \kappa^+ \) and \( \kappa^- \) interchanged.

**Proof.** Let \( m \geq 0 \). Using (3.17) and (A.5) to reduce the expressions \( \xi_n^{j''} \partial_{\xi_n}^{j''} \phi_m \), we find:

\[
s_{j,j',m} = \frac{1}{2\pi} \int \left( \frac{\sigma - i\xi_n}{(\sigma + i\xi_n)^{m+1}} \right) \left( \frac{1}{(\kappa + i\xi_n)^{-1}} \right) (-i\partial_{\xi_n})^{j'-1} \frac{1}{\kappa - i\xi_n} d\xi_n
\]

\[
= \frac{1}{2\pi} \int L_{\leq 0} \sum_{j'' \leq j'-1} (\kappa - i\xi_n)^{j''} \left( \frac{\sigma - i\xi_n}{(\sigma + i\xi_n)^{m+1}} \right) (-i\partial_{\xi_n})^{j'-1} \frac{1}{\kappa - i\xi_n} d\xi_n
\]

\[
= \sum_{0 \leq j'' \leq j'-1} \left( \frac{\sigma - i\xi_n}{(\sigma + i\xi_n)^{m+1}} \right) \left( \frac{\sigma - i\xi_n}{(\sigma + i\xi_n)^{m+1}} \right)^{j'-1} (-i\partial_{\xi_n})^{j'-1} \frac{1}{\kappa - i\xi_n} d\xi_n
\]

\[
= \sum_{0 \leq j'' \leq j'-1} b_{jj'j''m'} (\kappa^-)^{-j''} \frac{\sigma - \kappa^-}{(\sigma + \kappa^-)^m} (\kappa^+ + \kappa^-)^{-j-j'+1+j''},
\]

where the \( b_{jj'j''m'} \) are universal constants that are \( O(m^{j'}) \) for fixed \( j, j' \). By the rules in [GS95], this is a wphg symbol in \( S^{-j-j',0} \cap S^{0,-j-j'} \).

There is a similar calculation in case \( m < 0 \) where the integral is instead treated as a residue in \( \mathbb{C}_+ \). □

We then find:

**Proposition 4.3.** The pseudodifferential symbols \( s_{m,\gamma}(x', y', \xi', \mu) \) defined in (4.10) are in \( S^{\nu-2k-J,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \Gamma) \cap S^{\nu+1,1-2k-J}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \Gamma) \).

Moreover, the sum over \( m \) converges in all symbol norms, so that also the symbol

\[
(4.13) \quad \tilde{s}(\gamma)(x', y', \xi', \mu) = \sum_{m \in \mathbb{Z}} s_{m,\gamma}(x', y', \xi', \mu)
\]

\[
= \text{tr}_n (P''(x', \xi) + \circ_n g'_{-1-2k-J}(y', \xi', \xi_n, \eta_n, \mu))
\]

is in \( S^{\nu-2k-J,0} \cap S^{\nu+1,1-2k-J} \).

Finally,

\[
(4.14) \quad \text{Tr}(P'' \text{ OPG}(g'_{-1-2k-J}(y', \xi, \eta_n, \mu))) = \text{Tr}' \text{ OP}(\tilde{s}(\gamma)(x', y', \xi', \mu))
\]

\[
\sim \sum_{r=0}^{\infty} a_{J,r}^{4} \mu^{n-1+2k-J-r} + \sum_{r=0}^{\infty} (a_{J,r}^{5} \log \mu + a_{J,r}^{6}) \mu^{1-2k-J-r}.
\]
Proof. The first statement follows from Lemma 4.2, when the coefficients $b_m$ and the numerators in the expansions (2.31) are taken into account. The summation in $m$ is handled as in the proof of Proposition 3.4, since $b_m$ is rapidly decreasing in $m$ and the $m$-dependent coefficients arising from the compositions in Lemma 4.2 are polynomially bounded in $m$. The trace expansion also follows as in Proposition 3.4. □

Recall here that we have suppressed the indexation in $l$ referring to the Taylor expansion of $p(x', y_n, \xi)$ in $y_n$, and that $J \geq l$ for each $l$.

Again, as in the terms treated in Proposition 3.4, there is no term of the form $c\mu^{-2k}\log\mu$.

5. Trace calculations for terms in $GQ^k_{\mu, +}$

We shall finally treat the term $GQ^k_{\mu, +}$, which, as we show, gives a nontrivial contribution to the relevant log-coefficient. Again, some manipulations will be performed before expansion of the right hand factor in homogeneous terms as at the end of Section 2.

In view of the reduction to a coordinate patch, the term can be written in the form $GQ^k_{\mu, +}$ with $G = \text{OPG}(g(x', \xi, \eta_n))$, $Q^k_{\mu} = \text{OP}(q^{\circ k}(y, \xi, \mu)\theta(y_n))$, the symbols vanishing for large $x'$ and $y'$, and $\theta \in C^\infty_{(0)}(\mathbb{R}^+)$, $\theta(y_n) = 1$ near $y_n = 0$. In principle, $q$ can be thought of as determined as a parametrix of an elliptic symbol $p_1 + \mu^2$ defined for all $y_n \geq 0$, with uniform estimates. The effect of the cutoff $\theta$ can then be eliminated as follows:

Lemma 5.1. The operator $GQ^k_{\mu, +}(1 - \theta)$ is a smoothing operator with a trace expansion

$$
\sum_{r \in \mathbb{N}} c_r \mu^{n-2k-r} \text{ for } \mu \to \infty \text{ in } \Gamma.
$$

Proof. We use the old trick of “nested commutators”. With $\chi_1(x_n) = 1 - \theta(x_n)$ on $\mathbb{R}^+$, let $\chi_2, \chi_3, \ldots$ be a sequence of $C^\infty$ functions supported in $\mathbb{R}^+$ and satisfying

$$
\chi_i \chi_{i+1} = \chi_i, \text{ for all } i,
$$

in other words, $\chi_{i+1}$ is 1 on supp $\chi_i$ (but is still 0 on a neighborhood of 0). Denote $Q^k_{\mu} = Q(1)$. Now

$$
Q(1) \chi_1 = Q(1) \chi_1 \chi_2 = \chi_1 Q(1) \chi_2 + [Q(1), \chi_1] \chi_2.
$$

Here $G\chi_1 Q(1) + \chi_2 = G\chi_1 (Q(1) \chi_2) +$, where $G\chi_1$ is a $\mu$-independent operator of order $-\infty$, so that the composed operator is the restriction to $\mathbb{R}^+_n$ of an operator in $S^{-\infty, -2k}(\mathbb{R}^n \times \mathbb{R}^n, \Gamma)$; its trace has an expansion as in (5.1) by [GS95, Th. 2.1] (since $G$ maps into functions supported in a fixed compact set).

Let $Q(1)$ have the symbol $q(1)(y, \xi, \mu)$; then the commutator $Q(2) = [Q(1), \chi_1]$ has the symbol

$$
q(1)(y, \xi, \mu)(\chi_1(y_n) - \chi_1(x_n)) = q(1)(y, \xi, \mu)(y_n - x_n)\tilde{\chi}_1(x_n, y_n),
$$

where $\tilde{\chi}_1(x_n, y_n)$ is a smooth, bounded function with bounded derivatives. The $\psi$do with this symbol acts in the same way as the $\psi$do with the symbol $D\xi\eta q(1)(y, \xi, \mu)\chi_1(x_n, y_n)$, which is strongly polyhomogeneous of degree $-2k - 1$. It also has a symbol $q(2)$ in $y$-form, which is strongly polyhomogeneous of degree $-2k - 1$. So $Q(2)$ is one degree better than $Q(1)$. We can continue the argument, writing

$$
Q(i) \chi_i = Q(i) \chi_i \chi_{i+1} = \chi_i Q(i) \chi_{i+1} + [Q(i), \chi_i] \chi_{i+1}, \quad Q(i+1) = [Q(i), \chi_i],
$$

for all $i$. The first statement follows from Lemma 4.2, when the coefficients $b_m$ and the numerators in the expansions (2.31) are taken into account. The summation in $m$ is handled as in the proof of Proposition 3.4, since $b_m$ is rapidly decreasing in $m$ and the $m$-dependent coefficients arising from the compositions in Lemma 4.2 are polynomially bounded in $m$. The trace expansion also follows as in Proposition 3.4. □
in the general step, finding that $G\chi, Q_{(i)}\chi_{i+1}$ contributes with a trace expansion like (5.1) but starting with the power $\mu^{n-2k-i}$, and $Q_{(i+1)}$ is strongly polyhomogeneous of degree $-2k-i-1$. The latter gives an operator with trace norm $O(\mu^{n-\nu_i+2k-i})$ for large enough $i$, so since we can let $i \to \infty$ we get by superposition an expansion (5.1) for the original operator. □

5.a The $y_n$-independent case.

For the rest of the analysis we first consider the case where $P_1$, when in $y$-form, is independent of $y_n$ near $\{y_n = 0\}$, in the local coordinates we are using, so that we can assume that $Q^k_\mu = \text{OP}(q^k(y', \xi, \mu))$. As usual, it suffices to consider the contributions from the homogeneous terms in $q^k(y', \xi, \mu) \sim \sum_{j \in \mathbb{N}} q^k_{-2k+j}(y', \xi, \mu)$: Moreover, we expand the symbol of $G$ as in (3.11). Then

$$\text{Tr} \ G \ \text{OP}(q^k_{-2k-j}(y', \xi, \mu)) = \text{Tr} \ \text{OPG}(g(x', \xi, \eta_n) \circ_n q^k_{-2k-j}(y', \xi, \mu)), \quad \text{where}$$

$$g \circ_n q^k_{-2k-j} = \sum_{l, m \in \mathbb{N}} g_{l, m, j}(x', y', \xi, \eta_n, \mu) \text{ with}$$

$$g_{l, m, j} = 2\sigma c_{lm}(x', \xi') \tilde{\varphi}_l(\xi, \sigma) \tilde{\varphi}_m(\eta, \sigma) \circ_n q^k_{-2k-j}(y', \xi, \mu).$$

The composition rule is: $g(\xi, \eta_n) \circ_n q(\xi) = h_{\eta_n}[g(\xi, \eta_n)q(\eta_n)]$.

Termwise, we have by (3.5),

$$\text{Tr} \ \text{OPG}(g_{l, m, j}(x', y', \xi, \eta_n, \mu)) = \text{Tr}' \ \text{OP}'(\tilde{s}_{l, m, j}(x', y', \xi', \mu)), \quad \text{where}$$

$$\tilde{s}_{l, m, j}(x', y', \xi', \mu) = \text{tr}_n g_{l, m, j}(x', y', \xi, \eta_n, \mu),$$

and the crucial step is to calculate the normal trace of the s.g.o. arising from composing $\tilde{\varphi}_l(\xi, \sigma) \tilde{\varphi}_m(\eta, \sigma)$ with the fractions in (2.10) (decomposed as in (2.15)). In view of (3.9),

$$\text{tr}_n(2\sigma c_{lm}\tilde{\varphi}_l(\xi, \sigma)\tilde{\varphi}_m(\eta, \sigma) \circ_n q^k_{-2k-j}) = \frac{1}{2\pi} \int 2\sigma c_{lm}\tilde{\varphi}_l(\xi, \sigma)\tilde{\varphi}_m(\eta, \sigma) q^k_{-2k-j}(y', \xi', \mu) \ d\xi_n.$$

It is seen from the decomposition of $q^k_{-2k-j}(y', \xi', \xi_n, \mu)$ in Proposition 2.1 that the main point is to calculate the expressions

$$s^{\pm}_{j, l, m}(y', \xi', \mu) = \frac{1}{2\pi} \int \frac{(\sigma - i\xi_n)^l}{(\sigma + i\xi_n)^{l+1}} \frac{(\sigma + i\xi_n)^m}{(\sigma - i\xi_n)^{m+1}} \frac{1}{(\kappa^\pm + i\xi_n)^j} \ d\xi_n.$$

Lemma 5.2. One has for $l, m \in \mathbb{Z}$, $j \geq 1$:

For $m < l$, $s^+_{j, l, m} = 0$, $s^-_{j, l, m} = \sum_{|l-m-1-m'| < j, m' \geq 0} (k^-)^{1-j} b_{jm'} \frac{(\sigma - k^-)^{m'}}{(\sigma + k^-)^{m'+2}},$ (5.8)

For $m = l$, $s^+_{j, l, l} = \frac{1}{(\kappa^+ + \sigma)^{j/2}}$, $s^-_{j, l, l} = \frac{1}{(\kappa^- + \sigma)^{j/2}}$,

For $m > l$, $s^+_{j, l, m} = \sum_{|m-l-1-m'| < j, m' \geq 0} (k^+)^{1-j} b_{jm'} \frac{(\sigma - k^+)^{m'}}{(\sigma + k^+)^{m'+2}}$, $s^-_{j, l, m} = 0,$ (5.9)
with $b_{jm'}$ and $b'_{jm'}$ being $O(l^3 m^j)$ for fixed $j$.

The resulting symbols when $m \neq l$ are in $S^{-j-1,0} \cap S^{0,-j-1}$, whereas the symbols for $m = l$ are in $S^{-j-1,0} \cap S^{-1,-j}$.

**Proof.** Consider $s_{j,l,m}^+$. The integrand is rational with poles in $\mathbb{C} \setminus \mathbb{R}$ and is $O(\xi_n^{-3})$ for $|\xi_n| \to \infty$, so the integral can be deformed to an integral over a large closed curve in $\mathbb{C}_+$ or $\mathbb{C}_-$.

(5.9) For $m < l$, $s_{j,l,m}^+ = \frac{1}{2\pi} \int \frac{1}{(\sigma + i\xi_n) \cdot |\cdot|} d\xi_n = 0,$

since the integrand is holomorphic on $\mathbb{C}_-$.

When $m = l$, we use that there is just the simple pole $-i\sigma$ in $\mathbb{C}_-$:

(5.10) For $m = l$, $s_{j,l,l}^+ = \frac{1}{2\pi} \int \frac{1}{(\sigma + i\xi_n) \cdot |\cdot|} d\xi_n = \frac{1}{2\sigma(\kappa^+ + \sigma)^j},$

considering the integral as a residue at $-i\sigma$. This function is the product of a symbol in $S^{-j,0} \cap S^{0,-j}$ and a $\mu$-independent symbol in $S^{-1,0}$, hence it lies in $S^{-j-1,0} \cap S^{-1,-j}$.

When $m > l$, we use instead that the only pole in $\mathbb{C}_+$ is $i\kappa^+$. Writing $m-l-1 = k \geq 0$, we find:

$$
\begin{align*}
\tilde{s}_{j,l,m}^+ &= \frac{1}{2\pi} \int \frac{1}{\sigma + i\xi_n} d\xi_n \\
&= \frac{1}{2\pi} \int \frac{1}{\sigma + i\xi_n} d\xi_n \\
&= \frac{1}{2\pi} \int \frac{1}{\sigma + i\xi_n} d\xi_n \\
&= \frac{1}{2\pi} \int \frac{1}{\sigma + i\xi_n} d\xi_n,
\end{align*}
$$

very similarly to the calculations in (3.18); the $b_{jm'}$ are $O(l^3 m^j)$. This is a symbol in $S^{-j-1,0} \cap S^{0,-j-1}$.

The study of $s_{j,l,m}^-$ is similar, except that the roles of the upper and lower half-planes, as well as the roles of $m$ and $l$, are interchanged. \(\Box\)

Because of the difference between the cases, we sum over $l = m$ and $l \neq m$ separately:

**Proposition 5.3.** The pseudodifferential symbols $\tilde{s}_{l,m,J}(x', y', \xi', \mu)$ defined in (5.5) are in $S^{\nu - 2k - J, 0} \cap S^{\nu + 1, -1 - 2k - J}$ when $l \neq m$ and in $S^{\nu - 2k - J, 0} \cap S^{\nu, -2k - J}$ when $l = m$.  

Moreover, the summations over \( l \neq m \) and over \( l = m \) converge in the respective symbol norms, so that
\[
\text{tr}_n(g(x', \xi', \eta_n, \sigma_n) \circ_n q_{2k-J}^{c_k}(y', \xi', x_n, \mu)) = \tilde{s}_{(J)}(x', y', \xi', \mu) = s_{(J)}(x', y', \xi', \mu) + \tilde{s}_{(J)}(x', y', \xi', \mu), \quad \text{with}
\]

\[
(5.12) \quad s_{(J)}(x', y', \xi', \mu) = \sum_{l \in \mathbb{N}} s_{l, l}(x', y', \xi', \mu) \in S^{\nu-2k-J, 0} \cap S^{\nu-2k-J},
\]

\[
\tilde{s}_{(J)}(x', y', \xi', \mu) = \sum_{l, m \in \mathbb{N}, l \neq m} \tilde{s}_{l, m}(x', y', \xi', \mu) \in S^{\nu-2k-J, 0} \cap S^{\nu+1, -1-2k-J}.
\]

Finally,
\[
\text{Tr}(G \circ \text{OP}(q_{2k-J}^{c_k}(y', \xi, \mu))) = \text{Tr} \circ \text{OP}'(\tilde{s}'_{(J)}) + \text{Tr} \circ \text{OP}'(\tilde{s}''_{(J)})
\]
\[
\sim \sum_{l=0}^{\infty} a_{s, l}^{\text{OP}(0)} a_{s, l}^{\log \mu} a_{s, l}^{\mu-2k-J-l} + \sum_{l=0}^{\infty} a_{s, l}^{\log \mu} a_{s, l}^{\mu-2k-J-l},
\]

\[
(5.13) \quad \text{with} \quad \text{Tr} \circ \text{OP}'(\tilde{s}'_{(J)}) \sim \sum_{l=0}^{\infty} a_{s, l}^{\text{OP}(0)} a_{s, l}^{\log \mu} a_{s, l}^{\mu-2k-J-l} + \sum_{l=0}^{\infty} a_{s, l}^{\log \mu} a_{s, l}^{\mu-2k-J-l},
\]

\[
\text{Tr} \circ \text{OP}'(\tilde{s}''_{(J)}) \sim \sum_{l=0}^{\infty} a_{s, l}^{\text{OP}(0)} a_{s, l}^{\log \mu} a_{s, l}^{\mu-2k-J-l} + \sum_{l=0}^{\infty} a_{s, l}^{\log \mu} a_{s, l}^{\mu-2k-J-l}.
\]

In particular, the only contribution to a term \( c_{\mu}^{-2k} \log \mu \) comes from \( \tilde{s}'_{(0)} \).

**Proof.** The first statement follows from Lemma 5.2, when the coefficients \( c_{lm}(x', \xi') \) and the numerators in the expansions (2.15) are taken into account. The summations in \( l \) and \( m \) are handled as in the proof of Proposition 3.4, using the rapid decrease of the \( c_{lm} \) in \( l \) and \( m \) and the polynomial bounds on the \((n, m)\)-dependent coefficients arising from the compositions in Lemma 5.2. Then the trace expansions follow as in Proposition 3.4. \( \square \)

We shall now study the log-contribution from \( \tilde{s}'_{(0)} \).

**Proposition 5.4.** We have \((\text{with} \sigma = \sigma(\xi') = [\xi'])\):
\[
(5.14) \quad \text{tr}_n(2\sigma c_{ll} \tilde{s}'_{(0)}(\xi_n, \sigma) \tilde{s}'_{(0)}(\eta_n, \sigma) \circ_n q_{2k}^{c_k})
\]
\[
= c_{ll}(x', \xi') (q_{c_k}^{c_k}(y', \xi', -i\sigma, \mu) + q_{c_k}^{c_k}(y', \xi', i\sigma, \mu)),
\]
lying in \( S^{\nu-2k, 0} \cap S^{\nu-2k} \). Summation over \( l \) gives the symbol
\[
(5.15) \quad \tilde{s}'_{(0)}(x', y', \xi', \mu) = (\text{tr}_n g)(x', \xi') \cdot (q_{c_k}^{c_k}(y', \xi', -i\sigma, \mu) + q_{c_k}^{c_k}(y', \xi', i\sigma, \mu)).
\]

**Proof.** In view of (5.6)–(5.8) and Proposition 2.1 (in particular (2.14)), the left hand side of (5.14) equals
\[
\frac{1}{2\pi} \int 2\sigma c_{ll} \tilde{s}'_{(0)}(\xi_n, \sigma) \tilde{s}'_{(0)}(\eta_n, \sigma) \sum_{1 \leq j \leq k} \left( \frac{a_{k,j}^{+}}{a_{k}^{+}(\kappa+ i\xi_n)^{j}} + \frac{a_{k,j}^{-}}{a_{k}^{-}(\kappa- i\xi_n)^{j}} \right) d\xi_n
\]
\[
= 2\sigma c_{ll} \sum_{1 \leq j \leq k} \left( \frac{a_{k,j}^{+}}{a_{k}^{+}(\kappa+ \sigma)^{j}} + \frac{a_{k,j}^{-}}{a_{k}^{-}(\kappa- \sigma)^{j}} \right)
\]
\[
= c_{ll}(q_{c_k}^{c_k}(y', \xi', -i\sigma, \mu) + q_{c_k}^{c_k}(y', \xi', i\sigma, \mu)).
\]
This shows (5.14), and (5.15) follows since
\[
\text{tr}_n g(x', \xi', \xi_n, \eta_n) = \text{tr}_n \sum_{l,m} 2\sigma c_{lm}(x', \xi') \varphi_l'(\xi_n, \sigma) \varphi_m'(\eta_n, \sigma)
\]
(5.17)
\[
= \sum_{l,m} 2\sigma c_{lm}(x', \xi')(\varphi_l', \varphi_m') = \sum_l c_l(x', \xi'),
\]
where we use that \( \frac{1}{2\pi}(\varphi_l', \varphi_m') = (\varphi_l', \varphi_m') = \frac{1}{2\pi} \delta_{lm}. \]

The contribution to the log-term in (5.13) is found as follows: Denote
\[
q_{-2k}^\alpha(x', \xi', -i\sigma, \mu) + q_{-2k}^\alpha(x', \xi', i\sigma, \mu) = \alpha(x', \xi', \mu).
\]
According to [GS95, pf. of Th. 2.1], we have to calculate (on the diagonal, where \( x' = y' \))
\[
f(x', \mu) = \int_{\mathbb{R}^{n-1}} (\text{tr}_n g)_{1-n}(x', \xi') \alpha(x', \xi', \mu) \, d\xi'
\]
(5.19)
\[
= \int_{|\xi'| \geq |\mu|} (\text{tr}_n g)_{1-n} \alpha \, d\xi' + \int_{|\xi'| \leq |\mu|} (\text{tr}_n g)_{1-n} \alpha \, d\xi' + \int_{1 \leq |\xi'| \leq |\mu|} (\text{tr}_n g)_{1-n} \alpha \, d\xi'
\]
at each \( x' \) (denoting \( (2\pi)^{-n} \, d\xi' = d\xi' \)); it is integrated afterwards in \( x' \). The integrand is homogeneous of degree \( 1 - n - 2k \) in \( (\xi', \mu) \) for \( |\xi'| \geq 1 \), so the first integral equals a constant times \( \mu^{-2k} \) (for \( |\mu| \geq 1 \)). It is seen as in [GS95] that the second integral (over \( |\xi'| \leq 1 \)) contributes to the pure powers \( c_l \mu^{-l} \). It is the third integral that produces a logarithm. To find the coefficient, we need to study \( \alpha(x', \xi', \mu) \) more closely. Since it is in \( S^{0,-2k} \), it has a “Taylor expansion at \( \infty \)” by [GS95, Th. 1.12]. More precisely this means that for \( z = \frac{1}{\mu} \) lying in \( \Gamma (2.7) \), \( z^{-2k} \alpha(x', \xi', \frac{1}{\mu}) \) has a Taylor expansion at \( z = 0 \):
\[
\alpha(x', \xi', \frac{1}{\mu}) \sim z^{2k}(\alpha_0(x', \xi') + \alpha_1(x', \xi')z + \cdots + \alpha_r(x', \xi')z^r + \cdots) \text{ for } z \to 0 \text{ in } \Gamma,
\]
which we also write
\[
\alpha(x', \xi', \mu) \sim \alpha_0(x', \xi') \mu^{-2k} + \alpha_1(x', \xi') \mu^{-2k-1} + \cdots + \alpha_r(x', \xi') \mu^{-2k-r} + \cdots
\]
for \( \mu \to \infty \) in \( \Gamma \).

Here \( \alpha_r \in S^r \), and the remainder after the \( r \)’th term is in \( S^{r+1,-2k-r-1} \).

We shall show below in Lemma 5.5 that \( \alpha_0(x', \xi') = 1 \). Then
\[
\int_{1 \leq |\xi'| \leq |\mu|} (\text{tr}_n g)_{1-n} \alpha \, d\xi' \sim \int_{1 \leq |\xi'| \leq |\mu|} (\text{tr}_n g)_{1-n}(x', \xi')(\mu^{-2k} + \alpha_1(x', \xi') \mu^{-2k-1} + \cdots) \, d\xi'.
\]
Here, since \( (\text{tr}_n g)_{1-n}(x', \eta') = t^{1-n}(\text{tr}_n g)_{1-n}(x', \eta') \) for \( |\eta'| = 1 \),
\[
\int_{1 \leq |\xi'| \leq |\mu|} (\text{tr}_n g)_{1-n}(x', \xi') \mu^{-2k} \, d\xi' \]
(5.23)
\[
= \frac{1}{(2\pi)^{n-r}} \mu^{-2k} \int_{t=1}^{t=\mu} t^{-1} dt \int_{|\eta'|=1} (\text{tr}_n g)_{1-n}(x', \eta') \, d\sigma(\eta') = \mu^{-2k} \log \mu \frac{1}{(2\pi)^{n-r}} \int_{|\eta'|=1} (\text{tr}_n g)_{1-n}(x', \eta') \, d\sigma(\eta')
\]
The functions \( \kappa \) the strong ellipticity; the precise formula for \( \kappa \) (5.25)

Proof. \( (5.31) \)

\[ \text{roots } \alpha \text{ where } \Re \alpha > 0 \text{, } b(x', \xi') \text{ is linear in } \xi', \text{ } c(x', \xi') \text{ is quadratic in } \xi'. \]

The fact that the roots \( \pm i k^\pm(x', \xi', \mu) \) with respect to \( \xi_n \) lie in \( \mathbb{C}_\pm \), respectively, when \( \mu \in \Gamma \), follows from the strong ellipticity; the precise formula for \( \kappa^\pm \) is

\[ \kappa^\pm = \frac{\mu^2}{a} + \frac{c}{a} - (b/2a)^2 \pm ib/2a. \]

The functions \( \kappa^\pm(x', \xi', \mu) \) are strongly homogeneous of degree 1 in \((\xi', \mu)\), so they have “Taylor expansions for \( \mu \to \infty \)” in \( \Gamma \) (by [GS95, Th. 1.12]):

\[ \kappa^\pm(x', \xi', \mu) \sim a(x')^{-\frac{1}{2}} \mu + g^\pm_1(x', \xi') \mu^0 + \cdots + g^\pm_r(x', \xi') \mu^{1-r} + \cdots \]

with \( g^\pm_r \in S^r \); the first coefficient is \( a^{-\frac{1}{2}} \) in view of (5.26). If we add a \( \mu \)-independent term \( t \sigma(\xi') \), \( t \in [0, 1] \), the functions \( \kappa^\pm + t \sigma \) likewise have expansions

\[ \kappa^\pm(x', \xi', \mu) + t \sigma(\xi') \sim a(x')^{-\frac{1}{2}} \mu + (g^\pm_1(x', \xi') + t \sigma(\xi')) \mu^0 + \cdots + g^\pm_r(x', \xi') \mu^{1-r} + \cdots, \]

with the same first term. Since \( \Re \kappa^\pm > 0 \), they have inverses \((\kappa^\pm + t \sigma)^{-1}\) with

\[ (\kappa^\pm(x', \xi', \mu) + t \sigma(\xi'))^{-1} \sim a(x')^{\frac{1}{2}} \mu^{-1} + \tilde{g}^\pm_1(x', \xi', t) \mu^{-2} \]

\[ + \cdots + \tilde{g}^\pm_r(x', \xi', t) \mu^{-1-r} + \cdots, \]

for \( \mu \to \infty \) in \( \Gamma \),

where the first coefficient is necessarily the inverse of the preceding first coefficient, again independent of \( t \). Now consider the decomposition (2.14) for \( m = k \):

\[ q^\pm_{-2k} \]

(5.30)

\[ q^\pm_{-2k} = \sum_{1 \leq j \leq k} a_{k,j}^\pm \frac{\alpha_{k,j}}{a(\kappa^\pm \pm i \xi_n)^j}. \]

Here the numerators \( a_{k,j}^\pm(x', \xi', \mu) \) are strongly homogeneous in \((\xi', \mu)\) of degree \( j - 2k \), hence have expansions (by [GS95, Th. 1.12])

\[ a_{k,j}^\pm(x', \xi', \mu) \sim a_{k,j0}^\pm(x', \xi') \mu^{j-2k} + a_{k,j1}^\pm(x', \xi') \mu^{j-2k-1} \]

\[ + \cdots + a_{k,jr}^\pm(x', \xi') \mu^{j-2k-r} + \cdots, \] for \( \mu \to \infty \) in \( \Gamma \).
Inserting these expansions as well as those from (5.29) in the formulas (5.30) for \( q_{-2k}^{\pm} \), we find the following expansions:

\[
q_{-2k}^{\pm}(x', \xi', \mp i\sigma(\xi'), \mu) = \sum_{1 \leq j \leq k} \frac{a_{k,j}^{\pm}(x', \xi', \mu)}{a(x')^k (\kappa^\pm(x', \xi', \mu) + t\sigma(\xi'))^j} \sim \alpha_0^\pm(x', \xi') \mu^{-2k} + \alpha_1^\pm(x', \xi', t) \mu^{-2k-1} + \dots + \alpha_r^\pm(x', \xi', t) \mu^{-2k-r} + \dots,
\]

where the first coefficient \( \alpha_0^\pm(x', \xi') \) is independent of \( t \in [0, 1] \). For \( t = 1 \) it follows from (5.18), (5.21) and (5.32) that

\[
\alpha_0(x', \xi') = \alpha_0^+(x', \xi') + \alpha_0^-(x', \xi').
\]

On the other hand, we have for \( t = 0 \) that

\[
q_{-2k}^{\pm}(x', \xi', 0, \mu) = \frac{p_{12}(x', \xi', 0) + \mu^2}{(\mu + \kappa^\pm(x', \xi', 0))^2 - k} = \mu^{-2k} + \hat{\alpha}_1(x', \xi', 0) \mu^{-2k-1} + \dots + \hat{\alpha}_r(x', \xi', 0) \mu^{-2k-r} + \dots \text{ for } \mu \to \infty \text{ in } \Gamma,
\]

with coefficient 1 for the first term. Hence in view of (5.32),

\[
\alpha_0^+(x', \xi') + \alpha_0^-(x', \xi') = 1.
\]

(5.33) and (5.34) together show the lemma. \( \square \)

5.6 The \( y_n \)-dependent case.

Finally, we consider the case where \( p_1(y, \xi) \) and \( q(y, \xi, \mu) \) do depend on \( y_n \) near \( \{y_n = 0\} \). Denote \( p_1(y', 0, \xi) = p_0(y', \xi) \) and \( q(y', 0, \xi, \mu) = q^0(y', \xi, \mu) \). Here \( (y', y_n) \) is used in a compact subset of \( \mathbb{R}^n \), but we can assume that the symbols have extensions to \( \mathbb{R}^n \) defining operators \( P_1, P_0, Q_\mu, Q_\mu^0 \) so that \( Q_\mu = (P_1 + \mu^2)^{-1}, Q_\mu^0 = (P_1^0 + \mu^2)^{-1} \). Then

\[
Q_\mu - Q_\mu^0 = Q_\mu(P_1^0 + \mu^2)Q_\mu^0 - Q_\mu(P_1 + \mu^2)Q_\mu^0 = Q_\mu(P_1^0 - P_1)Q_\mu^0,
\]

\[
(Q_\mu)^k - (Q_\mu^0)^k = (Q_\mu)^k - (Q_\mu)^k - Q_\mu(P_1^0 - P_1)Q_\mu^0 + (Q_\mu)^k - (Q_\mu^0)^k = (Q_\mu)^k - (Q_\mu^0)^k = (Q_\mu)^k - (Q_\mu^0)^k - (Q_\mu^0)^k = (Q_\mu)^k - (Q_\mu^0)^k + (Q_\mu)^k - (Q_\mu^0)^k + (Q_\mu)^k - (Q_\mu^0)^k + \dots
\]

\[
(5.35)
\]

In view of the formulas (2.10) for the symbols of \( Q_\mu \) and \( Q_\mu^0 \), we find that the symbol \( \bar{q}(y, \xi, \mu) \) of \( (Q_\mu)^k - (Q_\mu^0)^k \) has the structure of a series \( \sum_{j \in \mathbb{N}} \bar{q}_{-2k-\bar{j}}(y, \xi, \mu) \) of homogeneous terms of the form

\[
\bar{q}_{-2k-\bar{j}}(y, \xi, \mu) = \sum_{J/2+k+1 \leq m + m' \leq 2J+k+1} \frac{r_{k,J,m,m'}(y, \xi)}{(p_{12}(y, \xi) + \mu^2)^m (p_{12}^0(y, \xi) + \mu^2)^{m'}}
\]

(5.36)
with \( r_{k,l,m,m'}(y, \xi) \) denoting a homogeneous polynomial in \( \xi \) of degree \( 2(m + m') - 2k - l \).

As usual, it suffices to consider the contribution from each \( \tilde{q}_{-2k-J} \). This must be reduced to situations where the composed symbols can be calculated. We can write

\[
(5.37) \quad p_{1,2}^0(y', \xi) - p_{1,2}(y, \xi) = y_n \tilde{p}(y, \xi),
\]

where \( \tilde{p} \) is a polynomial in \( \xi \) of degree 2. Using the formula \( 1/(1-b) = \sum_{0 \leq j < N} b^j + b^N/(1-b) \), valid for \( b \neq 1 \), we write, for \( y_n \), so small that \( |y_n \tilde{p}(p_{1,2}^0 + \mu^2)| \leq c < 1, \)

\[
(5.38) \quad \frac{1}{p_{1,2} + \mu^2} = \frac{1}{p_{1,2}^0 + \mu^2} \left( \frac{p_{1,2}^0 + \mu^2}{p_{1,2}^0 + \mu^2} \right)^{-1} = \frac{1}{p_{1,2}^0 + \mu^2} \left( 1 - \frac{y_n \tilde{p}}{p_{1,2}^0 + \mu^2} \right)^{-1} = \sum_{0 \leq j < N} \frac{y_n^j \tilde{p}^j}{(p_{1,2}^0 + \mu^2)^{j+1}} + \frac{y_n^N \tilde{p}^N}{(p_{1,2}^0 + \mu^2)^N(p_{1,2}^0 + \mu^2)}.
\]

For the last term in (5.38) we observe that when \( N \) is even,

\[
\frac{\tilde{p}^N}{(p_{1,2}^0 + \mu^2)^N(p_{1,2}^0 + \mu^2)} = \tilde{p}^N/2 \left( \frac{\tilde{p}}{(p_{1,2}^0 + \mu^2)^2} \right)^{N/2} \frac{1}{p_{1,2} + \mu^2} \equiv \tilde{p}^{N/2} \tilde{p}_N,
\]

where \( \tilde{p}^{N/2} \in S^N \), whereas \( \tilde{p}_N \) belongs to \( S^{N-2} \cap S^{N-2} \).

For the study of the composed operator \( G \text{OP}(r(y, \xi)y_n^N \tilde{p}^{N/2}(y, \xi) \tilde{p}_N)(y, \xi, \mu) \), we can “disentangle” the \( \mu \)-independent factor \( r(y, \xi)y_n^N \tilde{p}^{N/2} \) from the \( \mu \)-dependent factor \( \tilde{p}_N \) as follows: We observe that the ordinary product of two pseudodifferential symbols \( p \) and \( p' \) given in \( y \)-form may be expressed as a series of compositions of derived symbols:

\[
(5.39) \quad p(y, \xi)p'(y, \xi) \sim \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial_y^\alpha p(y, \xi) \circ D_\xi^\alpha p'(y, \xi).
\]

This “disentanglement formula” (which can be deduced from the usual formula for composition of symbols in \( y \)-form by successive comparison of terms) shows how a \( \psi \text{do} \), given in the form \( \text{OP}(pp') \), may be written as a series of truly composed \( \psi \text{do} \text{s} \).

Applying (5.39), the first term gives the composite \( G \text{OP}(r(y, \xi)y_n^N \tilde{p}^{N/2}) + \text{OP}(\tilde{p}_N) \), where the order of \( G \text{OP}(r(y, \xi)y_n^N \tilde{p}^{N/2}) \) equals \( \nu + \) the order of \( r \), thanks to the fact that \( y_n^N \) reduces the order of a s.g.o. by \( N \), compensating for the factor \( \tilde{p}^{N/2} \). The factor \( \text{OP}(\tilde{p}_N) \) here gives lower order and better \( O(\mu^{-M}) \) estimates, the larger \( N \) is taken. There is also a leftover term (of the type \( GL(Q', Q'') \), that behaves similarly. In the other contributions resulting from the use of (5.39), the degree in \( y_n \) goes down, but this is compensated for by a lower degree of the right hand factor. The remainder after the terms up to \( |\alpha| = l \) in the expansion has better \( \mu \)-estimates, the larger \( l \) is taken.

In reality, the last term in (5.38) enters in (5.36) in powers, composed with terms having only powers of \( p_{1,2}^0 + \mu^2 \) in the denominator, but the pattern of the resulting operators is similar to what we have just described.

This leaves us with contributions of the form \( r(y, \xi)(p_{1,2}^0(y', \xi) + \mu^2)^{-m} \). When \( r(y, \xi) \) is disentangled from \((p_{1,2}^0(y', \xi) + \mu^2)^{-m} \) by (5.39) as above and the resulting operators are composed with \( G \), we arrive at the type of operators we have dealt with in the first part.
of this section. They do have trace expansions like the first one in (5.13), when of order \( \nu - 2k - J \). For \( J > 0 \) these do not contribute to the coefficient of \( \mu^{-2k} \log \mu \).

For \( J = 0 \), we moreover have to show that the resulting trace does not contribute to the \( \mu^{-2k} \log \mu \)-term. Here we have to deal with the principal symbol of (5.35), equal to

\[
\sum_{i=1}^{k} \frac{y_n \tilde{p}}{(p_{1,2}^0 + \mu^2)^i (p_{1,2} + \mu^2)^{k+1-i}} = \sum_{i=1}^{k} \sum_{l,m=1}^{n} y_n a_{lm} \xi_l \xi_m \left( \sum_{0 \leq j < N} \frac{y_j^{ij}}{(p_{1,2}^0 + \mu^2)^j} \right)^{k-i} + \text{rem};
\]

we have applied (5.38) to the factors \( 1/(p_{1,2} + \mu^2)^{k+1-i} \) and inserted \( \tilde{p} = \sum_{l,m} a_{lm}(y) \xi_l \xi_m \). For each \( l, m \), we can move a factor \( y_n a_{lm}(y) \xi_l \) over to \( G \) in the same way as above without changing the order and class, so we are left with a finite sum of compositions of s.g.o.’s of order \( \nu \) and class 0 with \( \psi \)do’s with symbols

\[
\frac{\xi_m}{(p_{1,2}^0 + \mu^2)^{k+1}} \left( \frac{y_n \tilde{p}}{p_{1,2}^0 + \mu^2} \right)^{j(k-i)}.\]

We decompose these rational functions of \( \xi_n \) using (2.14) and the rules

\[
\frac{i \xi_n}{p_{1,2}^0 + \mu^2} = -\frac{\kappa^+}{\kappa^+ + \kappa^-} \xi_n + \frac{\kappa^-}{\kappa^+ + \kappa^-} \frac{\xi_n^2}{p_{1,2}^0 + \mu^2} = 1 - \frac{(\kappa^+)^2}{\kappa^+ + \kappa^-} \xi_n - \frac{(\kappa^-)^2}{\kappa^+ + \kappa^-} i \xi_n,
\]

and find that the resulting simple fractions are \( O(\mu^{-2k-1}) \); in the calculation of \( \text{tr}_n \) of the compositions with the s.g.o.’s (as in Proposition 5.3) this leads to weakly polyhomogeneous \( \psi \)do symbols in \( S^{\nu+1,-2k-1} \) on \( \mathbb{R}^{n-1} \). They do not contribute to \( \mu^{-2k} \log \mu \).

We have shown:

**Proposition 5.6.** Each term in the expansion \( G((Q_\mu)^k - (Q_\mu^0)^k)_+ \approx \sum_j G \text{ OP}(\tilde{g}_{-2k-j})_+ \) has a trace expansion as in (5.13), with vanishing coefficient of \( \mu^{-2k} \log \mu \).

6. Synthesis.

We now have all the ingredients for the proof of Theorem 1.1, in the following detailed formulation:

**Theorem 6.1.** Let \( P \) be a classical \( \psi \)do of order \( \nu \in \mathbb{Z} \) defined on a neighborhood \( \tilde{X} \) of \( X \) and having the transmission property at \( X' \), and let \( G \) be a singular Green operator of order \( \nu \) and class 0 on \( X \). The operator \( P \) acts in a \( C^\infty \) vector bundle \( \tilde{E} \) over \( \tilde{X} \), and \( P_+ \) and \( G \) act in \( E = \tilde{E}|_X \).

Let \( P_1 \) be a strongly elliptic second-order differential operator in \( \tilde{E} \) (with scalar principal symbol on a neighborhood of \( X' \)) and let \( P_{1,D} \) be its Dirichlet realization on \( X \); by adding a constant morphism we can assume that \( P_1 \) and \( P_{1,D} \) have positive lower bound, so that the resolvents \( (P_1 - \lambda)^{-1} \) and \( (P_{1,D} - \lambda)^{-1} \) exist for \(-\lambda \) in a region \( W \) (2.1). We can assume that \( \tilde{X} \) is compact, without boundary.

Let \( k > (n+\nu)/2 \). Then there are full asymptotic trace expansions (1.10), with the coefficients \( c_j, c'_j, c''_j \) proportional to the coefficients \( c_j, c'_j, c''_j \) by universal constants. In (1.10 I) \( \lambda \to \infty \) on the rays in \(-W\), in (1.10 II) \( t \to 0^+ \), and (1.10 III) holds in the sense...
that the left hand side minus the terms with \( l < N \), is holomorphic for \( \text{Re} \, s > -N/2 \), for any \( N \). Here \( c_0' = c_0' \) and equals

\[
(6.1) \quad \frac{1}{2(2\pi)^n} \int_{S^*X} \text{tr}_E p_{-n}(x, \xi) \, d\sigma(\xi) \, dx + \frac{1}{2(2\pi)^{n-1}} \int_{S^*X'} \text{tr}_E (\text{tr}_n g)_{1-n}(x', \xi') \, d\sigma(\xi') \, dx'.
\]

Thus the [FGLS96] residue of \( P_+ + G \) recalled in (1.5) satisfies

\[
(6.2) \quad \text{res}(P_+ + G) = 2c_0' = \text{ord} P_1 \cdot \text{Res}_{s=0} \text{Tr}((P_+ + G)P_{1,0}^{-s}).
\]

**Proof.** Since \( P \) can be taken to be 0 outside a neighborhood of \( X \), and \( P_1 \) has to be strongly elliptic (the symbol homotopic to \( |\xi|^2 I \)), there are no topological obstructions to obtaining a compact \( \tilde{X} \).

Consider the five terms in (2.5). The four s.g.o. terms have in Section 2 been partitioned into finite sums of terms supported in coordinate patches, with errors that are either \( O(\mu^{-N}) \), any \( N \), or contribute trace expansions as in (2.8). This allowed us to work in \( \mathbb{R}^n_+ \), where we formulated the results for the model case \( \dim E = 1 \). We have shown at the end of Section 2 that the contributions from the remainder after \( J_0 \) terms in the polyhomogeneous expansions of the symbols of \( Q_{\mu,+}^k \), \( G^-(Q_{\mu}^k) \) and \( G_{\mu}^{(k)} \) give operators whose trace norms are \( O(\mu^{-N}) \) with \( N \to \infty \) for \( J_0 \to \infty \). Similar results hold for other remainders encountered along the way, such as the last term in (4.2) containing the Taylor remainder (when \( l_0 \to \infty \)), the error in cut-offs treated in Lemma 5.1, and the errors that stem from expansions (5.38) (when \( N \to \infty \)). There remain, up to a given accuracy \( O(\mu^{-M}) \) in the desired trace expansion, a finite number (depending on \( M \)) of contributions, that have been described in Propositions 3.4, 4.3, 5.3 and 5.6. It is important to observe here that the expansions stemming from the precise terms \( p_{(l)}' \) and \( p_{(l)}'' \) in the Taylor expansion (4.1) give trace expansions beginning with the monomial \( \mu^{n-1+\nu-2k-l} \) and the log-monomial \( \mu^{-1-2k-l} \log \mu \), with powers going to \( -\infty \) for \( l \to \infty \), so that any specific term gets only finitely many contributions when we let \( l_0 \to \infty \).

When \( \dim E > 1 \), the above considerations pertain to each element in the matrix compositions.

Adding all the terms, we find an expansion:

\[
\text{Tr}[-L(P, Q_{\mu}^k) + P_+ G_{\mu}^{(k)} + G Q_{\mu,+}^k + G G_{\mu}^{(k)}] \sim \sum_{j \geq 0} a_j \mu^{n+\nu-2k-j} + \sum_{l \geq 0} (a_l' \log \mu + a_l'') \mu^{-2k-l}.
\]

Here it is only \( G Q_{\mu,+}^k \) that contributes to \( a_0' \), by expressions (5.24) for the diagonal elements in each localized term. Insertion of \( \mu^2 = -\lambda \) gives

\[
(6.3) \quad \text{Tr}[-L(P, Q_{\mu}^k) + P_+ G_{\mu}^{(k)} + G Q_{\mu,+}^k + G G_{\mu}^{(k)}] \sim \sum_{j \geq 0} a_j (-\lambda)^{n+\nu-2k-j} + \sum_{l \geq 0} (\frac{1}{2} a_l' \log (-\lambda) + a_l'') (-\lambda)^{-\frac{k}{2}}.
\]

Now, as observed already in [W84] and [FGLS96], (5.24) has a meaning independent of the choice of local coordinates, so the coefficient of \( (-\lambda)^{-k} \log (-\lambda) \) satisfies

\[
(6.4) \quad \frac{1}{2} a_0' = \frac{1}{2(2\pi)^n} \int_{S^*X'} \text{tr}_E (\text{tr}_n g)_{1-n}(x', \xi') \, d\sigma(\xi') \, dx'.
\]
It remains to account for the first expression in (2.5). Here [GS95, Th. 2.1] implies an expansion of the kernel on the diagonal:

\[
K_{PQ}^k(x, x) \sim \sum_{j \geq 0} b_j(x) \mu^{n+\nu-2k-j} + \sum_{l \geq 0} (b'_l(x) \log \mu + b''_l(x)) \mu^{-2k-l},
\]

and an investigation of the proof (as in Section 5 above) shows that since

\[
q_{-2k} = \frac{1}{(p_{1,2} + \mu^2)^k} = \mu^{-2k}(1 + p_{1,2} - 2)^{-k} \sim \mu^{-2k}(1 + \varrho_2 \mu^{-2} + \varrho_4 \mu^{-4} + \ldots)
\]

with first coefficient 1, the contribution from each diagonal element to the coefficient of \(\mu^{-2k} \log \mu\) in local coordinates has the form \(\frac{1}{(2\pi)^n} \int_{|\xi| = 1} p_{-n}(x, \xi) d\sigma(\xi)\). Summation of the pieces and integration over \(X\) gives:

\[
\text{Tr}(PQ^k) + \sim \sum_{j \geq 0} b_{j,+} \mu^{n+\nu-2k-j} + \sum_{l \geq 0} (b'_l, \log \mu + b''_l) \mu^{-2k-l}.
\]

Again we can use the invariance shown in [W84] and [FGLS96] to justify writing

\[
b'_{0,+} = \frac{1}{(2\pi)^n} \int_{S^*X} \text{tr}_E p_{-n}(x, \xi) d\sigma(\xi) dx.
\]

This is turned into an expansion in powers of \(-\lambda\) as above. Collecting the contributions, we obtain (1.10), with \(c_0'\) described by (6.1).

The other expansions (1.10 II) and (1.10 III) are derived from this by the transition principles worked out e.g. in [GS96], based on the formulas (1.8). One checks from [GS96, Cor. 2.10] that \(\tilde{c}_0' = c_0'\).

\[\square\]

Appendix. Laguerre expansions.

Here we recall the basic properties of the Laguerre function systems used in [G96], and introduce a slight modification. We first recall the definitions (these and the listed properties are worked out in detail in [G96, Sect. 2.2]):

\[
\varphi_k(\xi_n, \sigma) = (2\sigma)^{\frac{1}{2}} (\frac{\sigma - i \xi_n}{\sigma + i \xi_n})^k, \quad k \in \mathbb{Z},
\]

(A.1)

\[
\varphi_k(x_n, \sigma) = (2\sigma)^{\frac{1}{2}} H(x_n)(\sigma - \partial x_n)^k (x_n e^{-x_n \sigma})/k! \text{ for } k \geq 0,
\]

\[
\varphi_k(x_n, \sigma) = \varphi_{-k-1}(-x_n, \sigma) \text{ for } k < 0.
\]

\((H(x_n)\) is the Heaviside function \(1_{x_n > 0}.)\) Here \(\sigma\) can be any positive number, and the systems \(\{\varphi_k\}_{k \in \mathbb{Z}}\) resp. \(\{\varphi_k\}_{k \in \mathbb{N}}\) are orthonormal bases of \(L_2(\mathbb{R})\) resp. \(L_2(\mathbb{R}_+)\). The \(\varphi_k\) with \(k \geq 0\) are the eigenfunctions of the (unconventional) Laguerre operator

\[
\mathcal{L}_{\sigma,+} = \sigma^{-1}(\sigma + \partial x)x(\sigma - \partial x) = -\sigma^{-1}\partial x x \partial x + \sigma x + 1
\]

in \(L_2(\mathbb{R}_+)\), with simple eigenvalues \(2(k + 1)\), and the \(\varphi_k\) with \(k < 0\) are similarly the eigenfunctions for \(\mathcal{L}_{\sigma,-}\) defined by the same expression on \(\mathbb{R}_-\).
When $u \in L_2(\mathbb{R}_+)$ is expanded in the Laguerre system $(\varphi_k)_{k \in \mathbb{N}}$, by

$$u(x_n) = \sum_{k \in \mathbb{N}} b_k \varphi_k(x_n, \sigma),$$

then $u \in \mathcal{S}(\mathbb{R}_+^*) = r^+ \mathcal{S}(\mathbb{R})$ if and only if $(b_k)_{k \in \mathbb{N}}$ is rapidly decreasing. The norming factor $\sqrt{2\sigma}$ in (A.1) can be inconvenient in some calculations, so let us also introduce a notation for the functions without it:

$$\varphi'_k(\xi_n, \sigma) = \frac{(\sigma - i\xi_n)^k}{(\sigma + i\xi_n)^{k+1}} = (2\sigma)^{-\frac{1}{2}} \hat{\varphi}_k(\xi_n, \sigma),$$

$$\varphi_k(x_n, \sigma) = (2\sigma)^{-\frac{1}{2}} \varphi_k(x_n, \sigma).$$

They satisfy

$$\begin{align*}
\partial_{\xi_n} \varphi'_k(\xi_n, \sigma) &= \frac{-i}{2\sigma} (k \varphi'_{k-1} + (2k + 1) \varphi'_k + (k + 1) \varphi'_{k+1}), \quad \text{for } k \in \mathbb{Z}, \\
i \xi_n \varphi'_k(\xi_n, \sigma) &= -\sigma \varphi'_k + 2\sigma \sum_{0 \leq j < k} (-1)^{k-j-1} \varphi'_j + (-1)^k, \quad \text{for } k \geq 0, \\
\partial_{\xi_n}(\xi_n \varphi'_k(\xi_n, \sigma)) &= -\frac{k}{2} \varphi'_{k-1} + \frac{1}{2} \varphi'_k + \frac{k+1}{2} \varphi'_{k+1}, \quad \text{for } k \in \mathbb{Z}, \\
\partial_{\sigma} \varphi'_k(\xi_n, \sigma) &= \frac{1}{2\sigma} (k \varphi'_{k-1} - \varphi'_k - (k + 1) \varphi'_{k+1}), \quad \text{for } k \in \mathbb{Z}.
\end{align*}$$

The formulas show how simple manipulations with Laguerre functions lead to expressions involving neighboring Laguerre functions.

When $p(x', 0, \xi)$ is of order $d \in \mathbb{Z}$ and satisfies the transmission condition, then

$$p(x', 0, \xi) = \sum_{0 \leq l \leq d} s_l(x', \xi') \xi_n^{l} + \sum_{k \in \mathbb{Z}} b_k(x', \xi') \varphi'_k(\xi_n, \sigma);$$

the $s_l$ are polynomials in $\xi'$ of degree $d - l$ and the $b_k$ form a sequence that is rapidly decreasing in $S^{d+1}$ for $|k| \to \infty$.

Finally, we note two useful formulas:

$$\begin{align*}
(i\partial_{\xi_n})^k \frac{1}{\sigma + i\xi_n} &= \frac{k!}{(\sigma + i\xi_n)^{k+1}}; \\
\mathcal{F}^{-1} \frac{1}{(\sigma + i\xi_n)^{k+1}} &= H(x_n) \frac{x_n^k}{k!} e^{-\sigma x_n}.
\end{align*}$$

References

[M79] M. Adler, On a trace functional for formal pseudo-differential operators and the symplectic structure of the Korteweg-Devries type equations, Invent. Math. 50 (1979), 219 - 248.

[BM71] L. Boutet de Monvel, Boundary problems for pseudo-differential operators, Acta Math. 126 (1971), 11 - 51.

[FGLS96] B. V. Fedosov, F. Golse, E. Leichtnam, E. Schrohe, The noncommutative residue for manifolds with boundary, J. Funct. Anal. 142 (1996), 1 - 31.

[G71] G. Grubb, On coerciveness and semiboundedness of general boundary problems, Isr. J. Math. 10 (1971), 32 - 95.

[G92] , Heat operator trace expansions and index for general Atiyah-Patodi-Singer problems, Comm. P. D. E. 17 (1992), 2031 - 2077.

[G96] , Functional calculus of pseudodifferential boundary problems, Progress in Math. vol. 65, Second Edition, Birkhäuser, Boston, 1996, first edition issued 1986.
[G99] _______: Trace expansions for pseudodifferential boundary problems for Dirac-type operators and more general systems, Ark. Mat. 37 (1999), 45–86.

[G00] _______, A weakly polyhomogeneous calculus for pseudodifferential boundary problems, J. Funct. Anal. (to appear), Preprint 13, Copenhagen U. Math. Dept. Preprint series 1999.

[GK93] G. Grubb and N. J. Kokholm, A global calculus of parameter-dependent pseudodifferential boundary problems in $L_p$ Sobolev spaces, Acta Math. 171 (1993), 165–229.

[GS95] G. Grubb and R. Seeley, Weakly parametric pseudodifferential operators and Atiyah-Patodi-Singer boundary problems, Invent. Math. 121 (1995), 481–529.

[GS96] _______: Zeta and eta functions for Atiyah-Patodi-Singer operators, J. Geom. An. 6 (1996), 31–77.

[Gu86] V. Guillemin, A new proof of Weyl’s formula on the asymptotic distribution of eigenvalues, Adv. Math. 102 (1985), 184–201.

[K89] C. Kassel, Le résidu non commutatif [d’après M. Wodzicki], Astérisque 177–178 (1989), 199-229; Séminaire Bourbaki, 41ème année, Expose no. 41, 1988–99.

[L99] M. Lesch, On the noncommutative residue for pseudodifferential operators with log-polyhomogeneous symbols, Ann. Global Anal. Geom. 17 (1999), 151–187.

[M79] Yu. Manin, Algebraic aspects of nonlinear equations, J. Soviet Math. 11 (1979), 1 - 122.

[S67] R. T. Seeley, Complex powers of an elliptic operator, Amer. Math. Soc. Proc. Symp. Pure Math. 10 (1967), 288–307.

[S69] ______:, Topics in pseudo-differential operators, CIME Conf. on Pseudo-Differential Operators 1968, Edizioni Cremonese, Roma, 1969, pp. 169–305.

[W84] M. Wodzicki, Spectral asymmetry and noncommutative residue (in Russian), Thesis, Steklov Institute of Mathematics, Moscow, 1984.

Department of Mathematics, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen, Denmark.

E-mail address: grubb@math.ku.dk

Universität Potsdam, Institut für Mathematik, Postfach 60 15 53, D-14415 Potsdam, Germany.

E-mail address: schrohe@math.uni-potsdam.de