A REVIEW OF HARDY INEQUALITIES

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Abstract

We review the literature concerning the Hardy inequality for regions in Euclidean space and in manifolds, concentrating on the best constants. We also give applications of these inequalities to boundary decay and spectral approximation.

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1 Introduction

Let $H$ be a non-negative second order elliptic operator acting in $L^2(U)$ subject to Dirichlet boundary conditions, where $U$ is a region in $\mathbb{R}^N$ or in a Riemannian manifold. Also let $d$ be a positive function on $U$ which is continuous and satisfies $|\nabla d| \leq 1$. Traditionally one takes $d(x)$ to be the distance of $x \in U$ from the boundary $\partial U$, but another possibility is that $d(x)$ is the distance from any closed subset of $M \setminus U$ if $U$ is embedded in some larger Riemannian manifold $M$.

We say that $H$ satisfies a weak Hardy inequality with respect to $d$ if there exists a constant $c > 0$ and a constant $a \geq 0$ such that

$$\int_U \frac{|f|^2}{d^2} \leq c^2 \left( Q(f) + a\|f\|^2 \right)$$

is valid for all $f \in C_c^\infty(U)$, and hence for all $f$ in the domain of the quadratic form $Q$ of $H$. The infimum of all possible $c$ in (1) is then called the weak Hardy constant. We say that $H$ satisfies a strong Hardy inequality if (1) holds with $a = 0$, in which case the minimum possible $c$ is called the strong Hardy constant.

There are also $L^p$ and higher order analogues of the above notion, which we mention briefly later in this review.

In section 2 we describe the method of geodesic integrals for proving Hardy inequalities in higher dimensions. Section 3 describes a method ultimately due to Jacobi, while Section 4 gives various miscellaneous results.
We then turn to the applications of the HI to the proof of boundary decay. It was shown in [21] that Hardy’s inequality can be used to prove the $L^2$ boundary decay of eigenfunctions without any further assumptions. This in turn leads to the possibility of controlling the rate of convergence of the eigenvalues when the region $U$ is approximated by a family $U_{\varepsilon}$ of slightly smaller regions. Very recently progress has been made on this problem, [24], and we are able to announce bounds on the rate of convergence which are sharp in a certain sense.

Our main results on boundary decay, Theorems 11 and 12, may be regarded as $L^2$ analogues of much stronger pointwise bounds on eigenfunctions given in [3, 13, 34]. Note however that our bounds depend only on the validity of (1), hold for all functions in the domains of the operators, not just for eigenfunctions, and have rather precise constants.

If we abandon interest in the precise value of the constant, and choose $d$ to be the Euclidean distance from an arbitrary point of $U$, then it may be seen that our results are related to Morrey space estimates. These have been of considerable importance in the theory of elliptic operators, and recently in the proof of heat kernel bounds, and we refer the reader to [2, 3, 32] for further details.

2 Geodesic integrals

The first method which we describe depends upon the one-dimensional case, which is the only one Hardy actually studied. We refer to [40] for an exhaustive study, which involves generalizations to the variable coefficient case of the original formula

$$\int_0^\infty \frac{|f(x)|^2}{x^2}dx \leq 4 \int_0^\infty |f'(x)|^2dx$$

valid for all $f \in C^\infty_c(0, \infty)$, and hence for all $f \in W^{1,2}_0(0, \infty)$.

Let $H := -\Delta_{DIR}$ in the Hilbert space $L^2(U)$ where $U$ is a bounded region in $\mathbb{R}^N$. For every unit vector $u \in S^{N-1}$ and $x \in U$ let

$$d_u(x) := \min\{|t| : x + tu \notin U\}$$

if the set of such $t$ is non-empty, and put $d_u(x) := +\infty$ otherwise. We define the (harmonic) mean distance of $x$ from $\partial U$ by

$$m(x)^{-2} := |S^{N-1}|^{-1} \int_{S^{N-1}} d_u(x)^{-2}dS(u). \quad (2)$$

It is easy to prove that $d(x) \leq m(x)$ for all $x \in U$.

**Lemma 1** We have

$$\frac{N}{4m^2} \leq H$$
in the sense of quadratic forms. If $\lambda_1$ is the smallest eigenvalue of $H$ then

$$\lambda_1 \geq \frac{N}{4\mu^2}$$

where the quasi-inradius $\mu$ of $U$ is defined by

$$\mu := \sup \{ m(x) : x \in U \}.$$  

Proof  See [18] or [17, Th. 1.5.3].

Applications of the above lemma depend on making assumptions on $U$ which enable one to bound $m(x)$ above by some multiple of $d(x)$. The first of these is folklore and seems not to have been written down explicitly until very recently; see [36, 37] and the next section for alternative proofs.

**Theorem 2** If $U$ is a convex subset of $\mathbb{R}^N$ then

$$\frac{1}{d^2} \leq 4H$$

in the sense of quadratic forms.

Proof  If $a$ is the point of $\partial U$ closest to $x$ then we can obtain the relevant upper bound of $m(x)$ by computing an appropriate integral over the supporting hyperplane at $a$. See [23, Exercise 5.7].

The following lemma is typical of a variety of methods of obtaining crude upper bounds on $m(x)$. The hypothesis is valid not only for regions with Lipschitz boundaries, but also for a variety of regions with fractal boundaries, such as the Koch snowflake region in $\mathbb{R}^2$.

**Lemma 3** Suppose that there is a constant $k$ such that for each $a \in \partial U$ and each $\alpha > 0$ there exists a ball $B$ disjoint from $U$ with centre $b$ and radius $\beta \geq k\alpha$, where $|b - a| = \alpha$. Then there exists constants $c_0, c_1$ such that $m(x) \leq c_0 d(x)$ and hence

$$\frac{1}{d^2} \leq c_1 H$$

in the sense of quadratic forms.

Proof  See [18], [17, Th. 1.5.4] and [1, Th. 3].

The condition of Lemma 3 is not satisfied for regions satisfying a uniform exterior power-like cusp condition. In such cases one may prove a modified Hardy inequality using Lemma 1, namely

$$\int_U \frac{|f|^2}{d^\gamma} \leq c^2 \left( Q(f) + a\|f\|^2 \right)$$  \hfill (3)

for some $0 < \gamma < 2$; see [28, p 369]. See also [26, Th. 3.2, 3.3] where a similar situation arises for locally Euclidean manifolds with fractal boundaries.
A procedure closely related to the idea of this section was developed for regions in Riemannian manifolds independently by Croke and Derdzinski, [14], and Donnelly, [30]. The integrals over straight lines were replaced by integrals over geodesics, so the formulation involves the geodesic flow on the unit sphere bundle of the manifold. However, both papers are concerned with obtaining lower bounds on the bottom eigenvalue, much as in Lemma 1, rather than Hardy’s inequality.

We mention in passing that there is no requirement that one should assign equal weights to every direction in Euclidean space. In some cases one obtains a better constant in the Hardy inequality by taking an average over a few directions which are well adapted to the region in question.

3 The Classical Method

The following method goes back to Jacobi, and was used by Barta and Kasue to obtain lower bounds on the first eigenvalue, [1, 33]. It is the easy half of a theorem of Allegretto, Moss and Piepenbrink characterising the bottom of the spectrum of a Schrödinger operator in terms of the existence of positive distributional solutions of the eigenvalue equation, [15, p. 23]. Assume that

\[ Hf(x) := -\sum \frac{\partial}{\partial x_i} \left\{ a_{i,j}(x) \frac{\partial f}{\partial x_j} \right\} \]

where \( a(x) \) is a non-negative \( C^1 \) real symmetric matrix-valued function and \( f \in C^2_c(U) \). Then \( H \) is a non-negative symmetric operator and we can use the same symbol to denote its Friedrichs extension.

**Lemma 4** Let \( \phi \) be a positive \( C^2 \) function on \( U \) and let \( V \) be a continuous function on \( U \) such that

\[ -\sum \frac{\partial}{\partial x_i} \left\{ a_{i,j}(x) \frac{\partial \phi}{\partial x_j} \right\} \geq V \phi. \]

Then we have

\[ H \geq V \]

in the sense of quadratic forms.

**Proof** See [17, Th. 4.2.1].

The conditions of the above lemma can be weakened to allow a distributional inequality.

**Second proof of Theorem 2** If we put \( \phi := d^{1/2} \) and use the fact that \( \Delta d \leq 0 \) for any convex set \( U \) then the result follows immediately from the last lemma.

The method of this section can be extended to Riemannian manifolds without difficulty. We refer to [10, 25, 1] for a variety of Hardy and Rellich type inequalities with explicit constants in Riemannian manifolds obtained in this manner. The
Theorem 5 If $U \subseteq \mathbb{R}^N$ is bounded with a $C^2$ boundary and $H := -\Delta_{DIR}$ in $L^2(U)$ then there exists $a \in \mathbb{R}$ such that

$$d^{-2} \leq 4(H + a)$$

in the sense of quadratic forms. If $U$ is convex then (4) holds for certain $a < 0$.

Proof Let $\phi$ be a positive $C^2$ function on $U$ such that $\phi(x) = d(x)^{1/2} - d(x)$ for all $x$ close enough to $\partial U$. The first statement of the theorem follows by applying Lemma 4 to $\phi$.

There are various other improvements of the strong Hardy inequality of which we mention just two. For a definitive treatment of the one-dimensional theory see [40].

Theorem 6 If $U := \{x \in \mathbb{R}^N : x_N > 0\}$ where $N > 1$ then

$$\int_U \left\{ \frac{1}{x_N^2} + \frac{1}{4x_N(x_N^2 + x_{N-1}^2)^{1/2}} \right\} |f|^2 dx \leq 4 \int_U |\nabla f|^2 dx$$

for all $f \in C^\infty_c(U)$.

Proof See [38, Sect. 2.1.6].

Theorem 7 If $U := (0, a)$ then

$$\int_U \frac{a^2 |f|^2}{x^2(a - x)^2} dx \leq 4 \int_U |f'|^2 dx$$

for all $f \in C^\infty_c(U)$.

Proof Put $\phi(x) := x^{1/2}(a - x)^{1/2}$ in Lemma 4.

4 Capacity-based methods

In this section we mention a few of the very general theorems which involve the use of capacity arguments. These have been developed in an $L^p$ context, but we only treat the case $p = 2$. If $K$ is a compact subset of $U \subseteq \mathbb{R}^N$ we define its relative capacity by

$$\text{cap}(K, U) := \inf \left\{ \int_U |\nabla f|^2 : f \in C^\infty_c(U) \text{ and } f|_K \geq 1 \right\}.$$ 

It is particularly appropriate in this conference to mention one version of the most quantitatively precise theorems of this type, due to Professor Maz’ya.
Theorem 8  If $\mu$ is a positive measure on $U$ and

$$\mu(K) \leq \beta \text{cap}(K, U)$$

for all compact subsets $K$ of $U$, then

$$\int_U |f|^2 d\mu \leq 4\beta \int_U |\nabla f|^2$$

for all $f \in C_c^\infty(U)$. Conversely the second inequality implies

$$\mu(K) \leq 4\beta \text{cap}(K, U)$$

for all compact subsets $K$ of $U$.

Proof  See [38, p.113].

Our next results are taken from a paper of Ancona, [1]. We say that $U$ is uniformly $\Delta$-regular if for all $x \in \partial U$ and all $r > 0$ the harmonic measure $w$ of $U \cap \partial B(x, r)$ in $U \cap B(x, r)$ satisfies $w \leq 1 - \beta$ on $U \cap \partial B(x, r/2)$, for some constant $\beta \in (0, 1)$ independent of $x, r$. If $N \geq 3$ this is equivalent to the uniform capacitary density condition that there exists a constant $\alpha > 0$ such that

$$\text{cap}(B(x, r) \setminus U) \geq \alpha r^{N-2}$$

for all $x \in \partial U$ and all $r > 0$.

Theorem 9  If $N \geq 2$ and $U \subseteq \mathbb{R}^N$ is uniformly $\Delta$-regular then $U$ satisfies a strong Hardy inequality with respect to the Laplace operator. If $N = 2$ then the converse is also true.

Although [1] does not provide sharp information about the size of the strong Hardy constant, it contains many more results than we have indicated above. An $L^p$ converse of Theorem 9 for $N = p > 2$ may be found in [35], using an appropriate $L^p$ Riesz capacity.

5 Miscellaneous results

The weak Hardy constant $c$ as defined in Section 1 was proved in [22] to be local in the sense that it is the maximum value of a certain upper semi-continuous function on the boundary, whose value at each point depends only on the geometry of the boundary around that point. Various methods of evaluating this function at different types of boundary point are described in [22].

For the remainder of this section we assume that $H := -\Delta_{DIR}$. The strong Hardy constant is a global invariant of $U$. It equals 2 for any convex set, but the condition of convexity is not necessary for this conclusion. Let

$$U_\beta := \left\{re^{i\theta} : 0 < r < 1 \text{ and } 0 < \theta < \beta \right\}.$$
Then \( U_\beta \) has strong Hardy constant 2 if and only if the internal angle \( \beta \) is less than or equal to a certain critical value \( \beta_c \sim 4.856 \) radians, \([22]\). For larger \( \beta \) the strong and weak Hardy constants are larger than 2. Similar conclusions hold for other plane regions with piecewise smooth boundaries.

If \( U \) is a simply connected region in \( \mathbb{R}^2 \) then \( U \) has strong Hardy constant at most 4 by \([1, 17, \text{Th. 1.5.10}]\). The proof of this result depends upon a fact from analytic function theory, namely Koebe’s one-quarter theorem.

There is an interesting connection between the possible constants in the strong Hardy inequality and the Minkowski dimension of the boundary, \([27]\). In two dimensions there is also a relationship with hyperbolic geometry, which we do not pursue. We say that the boundary \( \partial U \) has interior Minkowski dimension \( \alpha \) if there exist positive constants \( k_1 \) and \( k_2 \) such that

\[
k_1 \varepsilon^{N-\alpha} \leq |\{ x \in U : \text{dist}(x, \partial U) < \varepsilon \}| \leq k_2 \varepsilon^{N-\alpha}
\]

for all \( \varepsilon > 0 \). The following theorem is adapted from \([24, \text{Th. 3.3}]\). We allow \( \alpha < N - 1 \) because the theorem is applicable in manifolds, for example if \( U \) is obtained by removing a compact set \( K \) from a sphere endowed with the standard metric.

**Theorem 10** If \( \partial U \) has interior Minkowski dimension \( \alpha > N - 2 \), then the strong Hardy constant of \( U \) with respect to the Laplacian satisfies

\[
c(2 + \alpha - N) \geq 2.
\]

In most of the above lemmas we have restricted attention to Hardy inequalities in \( L^2 \). In fact many of the results have been extended to \( L^p \) with sharp constants; see \([36, 37]\) for the proofs of the following two theorems.

**Theorem 11** Let

\[
c^{-p} := \inf \left\{ \frac{\int_U |\nabla f|^p}{\int_U |f|^p} : f \in W^{1,p}_0(U) \right\}
\]

where \( 1 < p < \infty \). If \( \partial U \) is smooth then \( c \geq p/(p - 1) \). If in addition \( p = 2 \) then \( c > 2 \) if and only if the infimum is achieved by some \( f \in W^{1,2}_0(U) \).

**Theorem 12** If \( U \) is a convex set in \( \mathbb{R}^N \) and \( 1 < p < \infty \) then

\[
\int_U |f|^p \frac{d^p}{dp} \leq \left( \frac{p}{p - 1} \right)^p \int_U |\nabla f|^p
\]

for all \( f \in W^{1,p}_0(U) \).

We refer to \([1, 9, 10, 35, 38, 40, 45, 46]\) for further \( L^p \) results, since they do not yet have such direct consequences for spectral theory. We refer to \([25, 41]\) for the
analogues for higher order operators, known as Rellich inequalities, and to [12] for analogues in Orlicz spaces.

We describe some trace inequalities in [19] which may be proved using Hardy’s inequality. We assume that $H := -\Delta_{\text{DIR}}$ acting in $L^2(U)$ where $U$ is a region in $\mathbb{R}^N$. The theorems are only of interest when $U$ has infinite volume.

**Theorem 13** We have

$$\text{tr}[e^{-Ht}] \leq (2\pi t)^{-N/2} \int_U e^{-Nt/8m(x)^2} d^nx$$

for all $t > 0$, where $m$ is defined by (2).

**Theorem 14** If $U$ satisfies the regularity condition

$$d(x) \leq m(x) \leq bd(x)$$

for all $x \in U$ then

$$2^{-N} (2\pi t)^{-N/2} \int_U e^{-8\pi^2 N^2 t/d(x)^2} d^nx \leq \text{tr}[e^{-Ht}] \leq (2\pi t)^{-N/2} \int_U e^{-Nt/8b^2 d(x)^2} d^nx$$

for all $t > 0$. Hence

$$\text{tr}[e^{-Ht}] < \infty$$

for all $t > 0$ if and only if

$$\int_U e^{-t/d(x)^2} d^nx < \infty$$

for all $t > 0$.

6 Boundary estimates

The size of the constant $s$ in an inequality of the form

$$\int_U \frac{|f|^2}{d^s} < \infty$$

conveys information about the behaviour of the function $f$ near the boundary of $U$. We conjecture that it is not possible to have $s > 2$ in the inequality for any region $U$ if we only assume that $f \in \text{Dom}(Q)$, where $Q$ is the quadratic form associated with a uniformly elliptic second order operator $H$ acting in $L^2(U)$ subject to Dirichlet boundary conditions. However, if we make stronger assumptions on $f$ then one may be able to prove (5) for a larger value of $s$. The first paper with results of this type was [31], where it was assumed that $f$ was an eigenfunction of $H$. Subsequently [21] obtained better bounds for all $f \in \text{Dom}(H)$, assuming only the Hardy inequality.
Although we have concentrated on $L^2$ boundary estimates, there is a substantial literature on pointwise decay of eigenfunctions and their gradients at the boundary. Bounds of the type

$$|\phi_n(x)| \leq c_n \phi_1(x)$$  \hspace{1cm} (6)

are immediate consequences of intrinsic ultracontractivity (IU), \cite{28, 17}, in which a major ingredient of the proof is the existence of an inequality

$$\phi_1(x) \geq ad(x)^\alpha$$  \hspace{1cm} (7)

for some positive constants $a$ and $\alpha$. The proof of (7) depends in turn upon the Harnack inequality and a boundary accessibility property. The BAP was proved in \cite{28, 17} for Lipschitz domains, but Ancona and Simon commented that it holds under a suitable twisted interior cone condition, i.e. for John domains, \cite{20}, p 98. Finally Banuelos gave a detailed analysis of the relationship between (6), IU, John domains, Holder domains, NTA domains, etc. in \cite{4}.

Pointwise bounds on the gradients of the eigenfunctions $\phi_n$ of $-\Delta_{DIR}$ and of Schrödinger operators with potentials in restricted Kato classes acting in $L^2(U)$ are proved in \cite{11, 13, 34, 5, 6} in steadily increasing generality. The best upper bound is for IU domains and is in \cite{5}, while the best lower bound is for Lipschitz domains and is in \cite{6}. The inequalities are of the form

$$|\nabla \phi_n(x)| \leq c_n \phi_1(x)/d(x)$$

and

$$|\nabla \phi_1(x)| \geq c_1 \phi_1(x)/d(x),$$

the latter being for $x$ close enough to the boundary.

One may also obtain upper bounds of the form

$$|\phi_n(x)| \leq c_n d(x)^\beta$$

for explicit but non-optimal constants $c_n, \beta$ which depend only on the eigenvalue $\lambda_n$, the dimension and the constant $\alpha$ in the uniform capacitary density inequality, \cite{8}. For an open simply connected region in $\mathbb{R}^2$ the bound

$$|\phi_n(x)| \leq c_n d(x)^{1/2}$$

is proved in \cite{4, 8, 13}; in this case the power $1/2$ is sharp.

We finally present some new results on $L^2$ boundary decay, taken from \cite{24}. Let $U$ be a bounded region in $\mathbb{R}^N$ and let $H := -\Delta_{DIR}$ acting in $L^2(U)$. Let $d(x)$ denote the distance of $x$ from some closed subset of $\mathbb{R}^N \setminus U$. We make no assumptions on the boundary $\partial U$ apart from the validity of (1) for certain values of $c \geq 2$ and $a \geq 0$. We are then able to draw the following conclusions about the boundary decay of functions in the domain of $H$. We have proved in \cite{24} that the powers of $\varepsilon$ in these theorems are sharp, and conjecture that the constant $c_0$ is also sharp. Analogues of the theorems for uniformly elliptic operators in divergence form are proved in \cite{24}.  

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**Theorem 15** If $f \in \text{Dom}(H)$ and $\varepsilon > 0$ then
\[
\int_{\{x:d(x)<\varepsilon\}} |f|^2 \leq c_0 \varepsilon^{2+2/c} \|(H+a)f\|_2 \|(H+a)^{1/c}f\|_2
\]
where
\[c_0 := c^{2+2/c}.
\]

**Theorem 16** If $f \in \text{Dom}(H)$ and $\varepsilon > 0$ then
\[
\int_{\{x:d(x)<\varepsilon\}} |\nabla f|^2 \leq c_1 \varepsilon^{2/c} \|(H+a)f\|_2 \|(H+a)^{1/c}f\|_2.
\]
where
\[c_1 := c^{2/c} + c^{2/c}(1+c)^{2+2/c}.
\]

**Corollary 17** If $Hf = \lambda f$ for some $\lambda > 0$, $\|f\|_2 = 1$ and $\varepsilon > 0$ then
\[
\int_{\{x:d(x)<\varepsilon\}} |f|^2 \leq c_0 \varepsilon^{2+2/c}(\lambda + a)^{1+1/c}
\]
and
\[
\int_{\{x:d(x)<\varepsilon\}} |\nabla f|^2 \leq c_1 \varepsilon^{2/c}(\lambda + a)^{1+1/c}.
\]

**Corollary 18** If $U$ is a simply connected proper subregion of $\mathbb{R}^2$ then
\[
\int_{\{x:d(x)<\varepsilon\}} |f|^2 \leq 32 \varepsilon^{5/2} \|Hf\|_2 \|H^{1/4}f\|_2
\]
and
\[
\int_{\{x:d(x)<\varepsilon\}} |\nabla f|^2 \leq 114 \varepsilon^{1/2} \|Hf\|_2 \|H^{1/4}f\|_2
\]
for all $f \in \text{Dom}(H)$ and $\varepsilon > 0$.

**Proof** We put $c = 4$ and $a = 0$ in Theorems 13 and 10.

We use the results above to consider the effect on the spectrum of $H := -\Delta_{\text{DIR}}$ of replacing the bounded region $U$ by a slightly smaller region $U_\varepsilon$ such that
\[
\{x \in U : d(x) > \varepsilon\} \subseteq U_\varepsilon \subseteq U.
\]

If $\lambda_n(U_\varepsilon)$ denote the eigenvalues of the operator $H_\varepsilon$ defined by restricting $H$ to $L^2(U_\varepsilon)$ where we again impose Dirichlet boundary conditions, then variational arguments imply that $\lambda_n(U) \leq \lambda_n(U_\varepsilon)$ for all $n$ and $\varepsilon > 0$. Our theorem below provides quantitative estimates of the difference, again only assuming (I). The first version in [21] did not obtain what we believe to be the sharp power of $\varepsilon$ given below. Pang, [43], obtained the result of Theorem 19 for $n = 1$ for simply connected plane regions by a method involving conformal mappings, improving his own earlier results in [42]. See [24] for a more general version of the theorem below, and its proof.
Theorem 19  There exist constants $c_n$ for all positive integers $n$ such that

$$\lambda_n(U) \leq \lambda_n(U_\varepsilon) \leq \lambda_n(U) + c_n \varepsilon^{2/c}.$$ 

We finally mention that there is extensive literature which compares $\lambda_n(U)$ with $\lambda_n(U \setminus K)$, where $K$ is a compact subset of $U$ which has a small capacity in a suitable sense; we believe that [14] is one of the earliest contributions to this subject, often known as the crushed ice problem. See [29] for a survey, including an explicit asymptotic formula for the difference of the eigenvalues in the limit of small $\text{Cap}(K)$ and also estimates of the difference for $n = 1$, both proved in the abstract setting of regular Dirichlet forms. See also [35], where estimates of the difference for $n = 1$ in terms of an appropriate definition of capacity are obtained in an abstract context applicable to higher order elliptic operators.

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