On the Characteristic Foliations of Metric Contact Pairs

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To John C. Wood for his sixtieth birthday

Abstract. A contact pair on a manifold always admits an associated metric for which the two characteristic contact foliations are orthogonal. We show that all these metrics have the same volume element. We also prove that the leaves of the characteristic foliations are minimal with respect to these metrics. We give an example where these leaves are not totally geodesic submanifolds.

1. Introduction

In a previous paper BH2, we have considered contact pair structures and studied some properties of their associated metrics. This notion was first introduced by Blair, Ludden and Yano BLY by the name bicontact structures, and is a special type of f-structure in the sense of Yano Y. More precisely, a metric contact pair on an even dimensional manifold is a triple (α₁, α₂, g), where (α₁, α₂) is a contact pair (see BH1) with Reeb vector fields Z₁, Z₂, and g a Riemannian metric such that

\[ g(X, Z_i) = \alpha_i(X), \text{ for } i = 1, 2, \text{ and for which the endomorphism field } \phi \text{ uniquely defined by } g(X, \phi Y) = (d\alpha_1 + d\alpha_2)(X, Y) \text{ verifies} \]

\[ \phi^2 = -Id + \alpha_1 \otimes Z_1 + \alpha_2 \otimes Z_2, \; \phi(Z_1) = \phi(Z_2) = 0. \]

Contact pairs always admit associated metrics with decomposable structure tensor \( \phi \), i.e. \( \phi \) preserves the two characteristic distributions of the pair (see BH2). In this paper, we first show that for a given contact pair all such associated metrics have the same volume element. Next we prove that with respect to these metrics the two characteristic foliations are orthogonal and minimal. We end by giving an example where the leaves of the characteristic foliations are not totally geodesic.

All the differential objects considered in this paper are assumed to be smooth.

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2. Preliminaries on metric contact pairs

In this section we gather the notions concerning contact pairs that will be needed in the sequel. We refer the reader to [Ba1, Ba2, BH1, BH2, BH3, BGK, BK] for further informations and several examples of such structures.

2.1. Contact pairs and their characteristic foliations. Recall that a pair \((\alpha_1, \alpha_2)\) of 1-forms on a manifold is said to be a contact pair of type \(\langle h, k \rangle\) if:

\[
\alpha_1 \wedge (d\alpha_1)^h \wedge \alpha_2 \wedge (d\alpha_2)^k \text{ is a volume form,}
\]

\[
(d\alpha_1)^{h+1} = 0 \text{ and } (d\alpha_2)^{k+1} = 0.
\]

Since the form \(\alpha_1\) (resp. \(\alpha_2\)) has constant class \(2h+1\) (resp. \(2k+1\)), the characteristic distribution \(\ker \alpha_1 \cap \ker d\alpha_1\) (resp. \(\ker \alpha_2 \cap \ker d\alpha_2\)) is completely integrable and determines the so-called characteristic foliation \(\mathcal{F}_1\) (resp. \(\mathcal{F}_2\)) whose leaves are endowed with a contact form induced by \(\alpha_2\) (resp. \(\alpha_1\)).

The equations

\[
\alpha_1(Z_1) = \alpha_2(Z_2) = 1, \quad \alpha_1(Z_2) = \alpha_2(Z_1) = 0,
\]

\[
i_{Z_1}d\alpha_1 = i_{Z_2}d\alpha_2 = i_{Z_2}d\alpha_1 = i_{Z_2}d\alpha_2 = 0,
\]

where \(i_X\) is the contraction with the vector field \(X\), determine completely the two vector fields \(Z_1\) and \(Z_2\), called Reeb vector fields. Notice that \(Z_i\) is nothing but the Reeb vector field of the contact form \(\alpha_i\) on each leaf of \(\mathcal{F}_j\) for \(i \neq j\).

The tangent bundle of a manifold \(M\) endowed with a contact pair can be split in different ways. For \(i = 1, 2\), let \(T\mathcal{F}_i\) be the subbundle of \(TM\) determined by the characteristic foliation of \(\alpha_i\). \(T\mathcal{G}_i\) the subbundle whose fibers are given by \(\ker d\alpha_1 \cap \ker \alpha_1 \cap \ker \alpha_2 \) and \(\mathbb{R}Z_1, \mathbb{R}Z_2\) the line bundles determined by the Reeb vector fields. Then we have the following splittings:

\[
TM = T\mathcal{F}_1 \oplus T\mathcal{F}_2 = T\mathcal{G}_1 \oplus T\mathcal{G}_2 \oplus \mathbb{R}Z_1 \oplus \mathbb{R}Z_2
\]

Moreover we have \(T\mathcal{F}_1 = T\mathcal{G}_1 \oplus \mathbb{R}Z_2\) and \(T\mathcal{F}_2 = T\mathcal{G}_2 \oplus \mathbb{R}Z_1\).

Notice that \(d\alpha_1\) (resp. \(d\alpha_2\)) is symplectic on the vector bundle \(T\mathcal{G}_2\) (resp. \(T\mathcal{G}_1\)).

Example 2.1. Take \((\mathbb{R}^{2h+2k+2}, \alpha_1, \alpha_2)\) where \(\alpha_1\) (resp. \(\alpha_2\)) is the Darboux contact form on \(\mathbb{R}^{2h+1}\) (resp. on \(\mathbb{R}^{2k+1}\)).

This is also a local model for all contact pairs of type \(\langle h, k \rangle\) (see [Ba1, BH1]). Hence a contact pair manifold is locally a product of two contact manifolds.

2.2. Contact pair structures. We recall now the definition of contact pair structure introduced in [BH2] and some basic properties.

Definition 2.2. A contact pair structure on a manifold \(M\) is a triple \((\alpha_1, \alpha_2, \phi)\), where \((\alpha_1, \alpha_2)\) is a contact pair and \(\phi\) a tensor field of type \((1, 1)\) such that:

\[
\phi^2 = -Id + \alpha_1 \otimes Z_1 + \alpha_2 \otimes Z_2, \quad \phi(Z_1) = \phi(Z_2) = 0
\]

where \(Z_1\) and \(Z_2\) are the Reeb vector fields of \((\alpha_1, \alpha_2)\).

It is easy to check that \(\alpha_i \circ \phi = 0\) for \(i = 1, 2\), that the rank of \(\phi\) is equal to \(\dim M - 2\), and that \(\phi\) is almost complex on the vector bundle \(T\mathcal{G}_1 \oplus T\mathcal{G}_2\).

Since we are also interested on the induced structures, we recall that the endomorphism \(\phi\) is said to be decomposable if \(\phi(T\mathcal{F}_i) \subset T\mathcal{F}_i\), for \(i = 1, 2\). This condition is equivalent to \(\phi(T\mathcal{G}_i) = T\mathcal{G}_i\). In this case \((\alpha_1, Z_1, \phi)\) (resp. \((\alpha_2, Z_2, \phi)\)) induces,
on every leaf of $\mathcal{F}_2$ (resp. $\mathcal{F}_1$), a contact form with structure tensor the restriction of $\phi$ to the leaf.

2.3. Metric contact pairs. On manifolds endowed with contact pair structures it is natural to consider the following kind of metrics:

**Definition 2.3 ([BH2])**. Let $(\alpha_1, \alpha_2, \phi)$ be a contact pair structure on a manifold $M$, with Reeb vector fields $Z_1$ and $Z_2$. A Riemannian metric $g$ on $M$ is called:

1. **compatible** if $g(\phi X, \phi Y) = g(X, Y) - \alpha_1(X)\alpha_1(Y) - \alpha_2(X)\alpha_2(Y)$ for all vector fields $X$ and $Y$,

2. **associated** if $g(X, \phi Y) = (d\alpha_1 + d\alpha_2)(X, Y)$ and $g(X, Z_i) = \alpha_i(X)$, for $i = 1, 2$ and for all vector fields $X, Y$.

An associated metric is compatible, but the converse is not true.

**Definition 2.4 ([BH2])**. A **metric contact pair** (MCP) on a manifold $M$ is a quadruple $(\alpha_1, \alpha_2, \phi, g)$ where $(\alpha_1, \alpha_2, \phi)$ is a contact pair structure and $g$ an associated metric with respect to it. The manifold $M$ is called a MCP manifold.

Note that the equation (2.2)

\[
g(X, \phi Y) = (d\alpha_1 + d\alpha_2)(X, Y)
\]
determines completely the endomorphism $\phi$. So we can talk about a metric $g$ **associated to a contact pair** $(\alpha_1, \alpha_2)$ when $g(X, Z_i) = \alpha_i(X)$, for $i = 1, 2$, and the endomorphism $\phi$ defined by equation (2.2) verifies (2.1).

**Theorem 2.5 ([BH2])**. For a MCP $(\alpha_1, \alpha_2, \phi, g)$, the tensor $\phi$ is decomposable if and only if the characteristic foliations $\mathcal{F}_1, \mathcal{F}_2$ are orthogonal.

Using a standard polarization on the symplectic vector bundles $T\mathcal{G}_i$ (see Section 2.1), one can see that for a given contact pair $(\alpha_1, \alpha_2)$ there always exist a decomposable $\phi$ and a metric $g$ such that $(\alpha_1, \alpha_2, \phi, g)$ is a MCP (see [BH2]). This can be stated as:

**Theorem 2.6 ([BH2])**. For a given contact pair on a manifold, there always exists an associated metric for which the characteristic foliations are orthogonal.

Let $(\alpha_1, \alpha_2, \phi, g)$ be a MCP on a manifold with decomposable $\phi$. Then $(\alpha_i, \phi, g)$ induces a contact metric structure on the leaves of the characteristic foliation $\mathcal{F}_j$ of $\alpha_j$, for $i \neq j$ (see [BH2]).

**Example 2.7.** As a trivial example one can take two metric contact manifolds $(M_i, \alpha_i, g_i)$ and consider the MCP $(\alpha_1, \alpha_2, g_1 \oplus g_2)$ on $M_1 \times M_2$. The characteristic foliations are given by the two trivial fibrations.

**Remark 2.8.** To get more examples of MCP on closed manifolds, one can imitate the constructions on flat bundles and Boothby-Wang fibrations given in [BH3] and adapt suitable metrics on the bases and fibers of these fibrations. See also Example 3.5 below which concerns a nilpotent Lie group and its closed nilmanifolds.

3. Minimal foliations

Given any compatible metric $g$ on a manifold endowed with a contact pair structure $(\alpha_1, \alpha_2, \phi)$ of type $(h, k)$, with Reeb vector fields $Z_1$ and $Z_2$, one can
construct a local basis, called $\phi$-basis. On an open set, on the orthogonal complement of $Z_1$ and $Z_2$, choose a vector field $X_1$ of length 1 and take $\phi X_1$. Then take the orthogonal complement of $\{Z_1, Z_2, X_1, \phi X_1\}$ and so on. By iteration of this procedure, one obtains a local orthonormal basis
$$\{Z_1, Z_2, X_1, \phi X_1, \cdots, X_{h+k}, \phi X_{h+k}\},$$
which will be called $\phi$-basis and is the analog of a $\phi$-basis for almost contact structures, or $J$-basis in the case of an almost complex structure $J$.

If $\phi$ is decomposable and $g$ is an associated metric, since the characteristic foliations are orthogonal, it is possible to construct the $\phi$-basis in a better way. Starting with $X_1$ tangent to one of the characteristic foliations, which are orthogonal, with a slight modification of the above construction, we obtain a $\phi$-basis
$$\{Z_1, X_1, \phi X_1, \cdots, X_h, \phi X_h, Z_2, Y_1, \phi Y_1, \cdots, Y_k, \phi Y_k\}$$
such that $\{Z_1, X_1, \phi X_1, \cdots, X_h, \phi X_h\}$ is a $\phi$-basis for the induced metric contact structures on the leaves of $\mathcal{F}_2$, and $\{Z_2, Y_1, \phi Y_1, \cdots, Y_k, \phi Y_k\}$ is a $\phi$-basis for the leaves of $\mathcal{F}_1$.

Using this basis and the formula for the volume form on contact metric manifolds (see [Bl], for example), one can easily show the following:

**Proposition 3.1.** On a manifold endowed with a MCP $(\alpha_1, \alpha_2, \phi, g)$ of type $(h, k)$, with a decomposable $\phi$, the volume element of the Riemannian metric $g$ is given by:

$$(3.1) \quad dV = \frac{(-1)^{h+k}}{2^{h+k}h!k!} \alpha_1 \wedge (d\alpha_1)^h \wedge \alpha_2 \wedge (d\alpha_2)^k.$$

A direct application of the minimality criterion of Rummler (see [R] page 227) to the volume form on a MCP manifold yields the following result:

**Theorem 3.2.** On a MCP manifold $(M, \alpha_1, \alpha_2, \phi, g)$ with decomposable $\phi$, the characteristic foliations are minimal.

**Proof.** Recall the minimality criterion of Rummler: let $\mathcal{F}$ be a $p$-dimensional foliation on a Riemannian manifold and $\omega$ its characteristic form (i.e. the $p$-form which vanishes on vectors orthogonal to $\mathcal{F}$ and whose restriction to $\mathcal{F}$ is the volume of the induced metric on the leaves). Then $\mathcal{F}$ is minimal iff $\omega$ is closed on $T\mathcal{F}$ (i.e. $d\omega(X_1, \ldots, X_p, Y) = 0$ for $X_1, \ldots, X_p$ tangent to $\mathcal{F}$).

Let $\mathcal{F}_i$ be the characteristic foliation of $\alpha_i$. As the volume element of the Riemannian metric $g$ is given by (3.1), the characteristic form of $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) is, up to a constant, $\alpha_2 \wedge (d\alpha_2)^k$ (resp. $\alpha_1 \wedge (d\alpha_1)^h$). But these forms are closed by the contact pair condition, and then the criterion applies directly. □

Since every manifold endowed with a contact pair always admits an associated metric with decomposable $\phi$, we have a statement already proved in [BK]:

**Corollary 3.3.** On every manifold endowed with a contact pair there exists a metric for which the characteristic foliations are orthogonal and minimal.

**Remark 3.4.** Although a contact pair manifold is locally a product of two contact manifolds (see Section 2.1), an associated metric for which the characteristic foliations are orthogonal is not necessary locally a product as in Example 2.7. Here is an interesting case:
Example 3.5. Let us consider the simply connected 6-dimensional nilpotent Lie group $G$ with structure equations:

$$d\omega_3 = d\omega_6 = 0, \quad d\omega_2 = \omega_5 \wedge \omega_6,$$

$$d\omega_1 = \omega_3 \wedge \omega_4, \quad d\omega_4 = \omega_3 \wedge \omega_5, \quad d\omega_5 = \omega_3 \wedge \omega_6,$$

where the $\omega_i$'s form a basis for the cotangent space of $G$ at the identity.

The pair $(\omega_1, \omega_2)$ is a contact pair of type $(1, 1)$ with Reeb vector fields $(X_1, X_2)$, the $X_i$'s being dual to the $\omega_i$'s. The characteristic distribution of $\omega_1$ (resp. $\omega_2$) is spanned by $X_2, X_5$ and $X_6$ (resp. $X_1, X_3$ and $X_4$).

The left invariant metric

$$(3.2) \quad g = \omega_1^2 + \omega_2^2 + \frac{1}{2} \sum_{i=3}^{6} \omega_i^2$$

is associated to the contact pair $(\omega_1, \omega_2)$ with decomposable structure tensor $\phi$ given by $\phi(X_6) = X_5$ and $\phi(X_4) = X_3$.

The characteristic foliations have minimal leaves. Moreover the leaves tangent to the identity of $G$ are Lie subgroups isomorphic to the Heisenberg group.

Notice that these foliations are not totally geodesic since $g(\nabla X_5, X_6) \neq 0$ and $g(\nabla X_3, X_6) \neq 0$, where $\nabla$ is the Levi-Civita connection of this metric. So the metric $g$ is not locally a product.

Since the structure constants of the group are rational, there exist lattices $\Gamma$ such that $G/\Gamma$ is compact. Since the MCP on $G$ is left invariant, it descends to all quotients $G/\Gamma$ and we obtain closed nilmanifolds carrying the same type of structure.

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