TFD Approach to Bosonic Strings and $D_p$-branes

M. C. B Abdalla* and A. L. Gadelha

Instituto de Física Teórica, Unesp, Pamplona 145, São Paulo, SP, 01405-900, Brazil, *mabdalla@ift.unesp.br

I. V. Vancea

Departamento de Física Matemática, Faculdade de Filosofia Ciências e Letras de Ribeirão Preto, USP, Av. Bandeirantes, 3900, Ribeirão Preto, SP, 14040-901, Brazil

In this work we explain the construction of the thermal vacuum for the bosonic string, as well that of the thermal boundary state interpreted as a $D_p$-brane at finite temperature. In both case we calculate the respective entropy using the entropy operator of the Thermo Field Dynamics theory. We show that the contribution of the thermal string entropy is explicitly present in the $D_p$-brane entropy. Furthermore, we show that the Thermo Field approach is suitable to introduce temperature in boundary states.

I. INTRODUCTION

$D_p$-branes at finite temperature have been object of intense study. In particular, they appear as solitonic solutions of Supergravity in the non perturbative limit. In this limit, the search for an understanding of its statistical properties has attracted many researchers [1]. A plethora of results have been obtained contributing to clarify many difficulties as, for example, the thermodynamics of black holes and string gases [2].

However, in the perturbative limit of string theory, the understanding of the microscopic properties of $D_p$-branes at finite temperature still waits for a more satisfactory development. With the aim of changing this picture, a new approach using the framework of Thermo Field Dynamics (TDF) [7] was proposed [3]-[6] recently. This formulation consists in identifying the statistical average of some operator with a vacuum expectation value in quantum field theory which must take into account a temperature-dependent state. Indeed, the basic statement of TFD is, considering an operator $O$,

$$
\langle O \rangle = \frac{1}{Z(\beta)} \text{Tr} \rho O = \langle 0(\beta) | O | 0(\beta) \rangle .
$$

(1)

Here, $Z(\beta) = \text{Tr} \rho$, is the partition function, $\rho$ is the density operator and $\beta = 1/k_BT$, for $k_B$ being the Boltzmann constant and $T$ the temperature of the system. In eq. (1) the temperature dependent state is denoted by $|0(\beta)\rangle$.

The identification (1) is satisfied if we double the degrees of freedom of the system. To achieve this we need to work in an extended Hilbert space which consists of a product of two subspaces: the original system, $\mathcal{H}$, and another one, an auxiliary space denoted by a tilde, $\tilde{\mathcal{H}}$, independent and with the same characteristics of the original one. Being $\mathcal{H}$ the extended space, we have $\mathcal{H} = \mathcal{H} \otimes \tilde{\mathcal{H}}$. By the same characteristics we mean that, for any state or operator associated to the original Hilbert space we have a partner in the tilde Hilbert space. So, for all orthonormalized states $|n\rangle \in \mathcal{H}$ one has the states $|\tilde{n}\rangle \in \tilde{\mathcal{H}}$. In this sense, if $H$ is the hamiltonian operator of the original system and $\tilde{H}$ is its equivalent for the auxiliary tilde space, we may write

$$
H |n\rangle = E_n |n\rangle , \quad \tilde{H} |\tilde{n}\rangle = E_n |\tilde{n}\rangle .
$$

(2)

Here, we have introduced the notation $|n\rangle = |n, \tilde{n}\rangle = |n\rangle \otimes |\tilde{n}\rangle \in \tilde{\mathcal{H}}$. Furthermore, all operators $O$ and $\tilde{O}$ that act in the original and auxiliary Hilbert spaces commute. Under the above considerations, the solution for the identity (1) is

$$
|0(\beta)\rangle = \frac{1}{(Z(\beta))^{1/2}} \sum_n e^{-\frac{\beta E_n}{2}} |n, \tilde{n}\rangle .
$$

(3)

An interesting point of TFD is that the temperature-dependent state, can be obtained using a linear canonical transformation, i.e. a Bogoliubov transformation. To show this we will apply the formalism to bosonic strings [9].

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The concepts of TFD, as well as its applications, are explained in ref. 8 and references therein.
In Sec. 2 we present the basic concepts of the TFD approach. In Sec. 3, we construct the boundary states at finite temperature which are interpreted as thermal $D_p$-brane. In Sec. 4 we discuss the results.

II. BOSONIC STRINGS AT FINITE TEMPERATURE

Consider the bosonic string in the conformal gauge. The space-time has $D = 26$, the metric is lorentzian and the string spans a world-sheet that can be described by two parameters. Let $X^\mu$ be the string position vector, from the view point of the world-sheet. It can be thought as a 2-dimensional bosonic free field $X^\mu (\tau, \sigma)$, with $\tau$ and $\sigma$ the parameters of the world-sheet. This field satisfies a massless 2-dimensional Klein-Gordon equation.

We can impose periodic conditions to the solution for this equation $X^\mu (\tau, 0) = X^\mu (\tau, \pi)$ defining, in this way, the closed bosonic string. The solution that satisfies these conditions has a Fourier expansion in terms of right- and left-modes with coefficients $\alpha^i_k$ and $\beta^i_k$, respectively.

The vanishing of the energy-momentum tensor characterizes the constrained system. To avoid spurious states in the vector space of the quantized system, we choose the light-cone gauge, $X = 0$ for $\mu = 0, 1, 2, ..., 24$. In this way, we have the following commutation relations

$$[\alpha^i_k, \alpha^j_l] = [\beta^i_k, \beta^j_l] = m \delta^{ij} \delta_{m+n,0}, \quad [\alpha^i_k, \beta^j_l] = 0. \quad (4)$$

These oscillators-like operators can be redefined as

$$A^i_k = \frac{1}{\sqrt{\alpha^i_k}}, \quad A^{ij}_k = \frac{1}{\sqrt{k}} \alpha^{ij}_{-k}, \quad B^i_k = \frac{1}{\sqrt{k}} \beta^i_k, \quad B^{ij}_k = \frac{1}{\sqrt{k}} \beta^{ij}_{-k}, \quad (5)$$

for $k > 0$, in such a way that we have the vacuum defined by

$$A^i_k |0\rangle = B^i_k |0\rangle = 0, \quad (6)$$

where $|0\rangle = |0\rangle_\alpha |0\rangle_\beta$, and the subscripts $\alpha$ and $\beta$ refers to the left- and right-modes for the closed string.

To apply the TFD we need firstly to double the space. So we have an auxiliary system called “tilde system” which as was have already mentioned, the temperature-dependent state can be obtained as a Bogoliubov transformation

$$|0 (\beta)\rangle = e^{-iG} |0\rangle. \quad (10)$$

The oscillators-like operators for the doubled system satisfy the following algebra

$$[A^i_k, A^{ij}_m] = [\tilde{A}^i_k, \tilde{A}^{ij}_m] = [B^i_k, B^{ij}_m] = [\tilde{B}^i_k, \tilde{B}^{ij}_m] = \delta_{km} \delta^{ij}, \quad (8)$$

and zero for the rest. The vacuum of the total system is defined by

$$A^i_k |0\rangle = B^i_k |0\rangle = \tilde{A}^i_k |0\rangle = \tilde{B}^i_k |0\rangle = 0. \quad (9)$$

In the above expression we have used the following notation $|0\rangle = |0\rangle_\alpha |0\rangle_\beta$.

As was have already mentioned, the temperature-dependent state can be obtained as a Bogoliubov transformation

$$|0 (\beta)\rangle = e^{-iG} |0\rangle. \quad (10)$$

In the case of closed bosonic string, we have two independent original subsystems, for the left- and right-modes. So we need a transformation generator that acts on both modes. The expression for that generator is given by

$$G = G^1 = \sum_k \left( G^\alpha_k + G^\beta_k \right), \quad (11)$$

with

$$G^\alpha_k = -i \theta_k \left( A_k \cdot \tilde{A}_k - \tilde{A}^i_k \cdot A^i_k \right), \quad G^\beta_k = -i \theta_k \left( B_k \cdot \tilde{B}_k - \tilde{B}^i_k \cdot B^i_k \right). \quad (12)$$

Once more, the superscripts denote left- and right-modes. The dot means the euclidian scalar product in the transverse space-time. With that generator we obtain explicitly the temperature dependent state as
\[ |0(\beta)\rangle = \prod_k [\cosh(\theta_k)]^{-2T \delta^{ij}} e^{\tan(\theta_k)(\bar{A}^i_k, A^i_k)} e^{\tan(\theta_k)(\bar{B}^i_k, B^i_k)} |0\rangle \]
\[ = \prod_k [u_k]^{-2T \delta^{ij}} e^{u_k (\bar{A}^i_k, A^i_k)} e^{u_k (\bar{B}^i_k, B^i_k)} |0\rangle. \]

with \( u_k = u(\theta_k) = \cosh(\theta_k) \) and \( v_k = v(\theta_k) = \sinh(\theta_k) \). The transformations for the \( A \) operators are given as follows:
\[ A^i_k(\theta) = e^{-iG} A^i_k e^{iG} = u_k A^i_k - v_k \bar{A}^i_k, \]
\[ \bar{A}^i_k(\theta) = e^{-iG} \bar{A}^i_k e^{iG} = u_k \bar{A}^i_k - v_k A^i_k, \]

and similarly for the \( B \) operators.

Observing the structure of thermal-dependent state (10)-(14) and the expressions for the transformed operators (15)-(16), we can introduce the so called tilde conjugation rules which maps the operators living in the original subspace in to those of the auxiliary (tilde) space and vice-versa
\[ (AB)^\sim = \bar{A} \bar{B}, \]
\[ (c_1 A + c_2 B)^\sim = (c_1^* \bar{A} + c_2^* \bar{B}), \]
\[ (A^\dagger)^\sim = \bar{A}^\dagger, \]
\[ (\bar{A}^\dagger)^\sim = A, \]
\[ (|0(\theta))\rangle^\sim = |0(\theta)\rangle, \]
\[ (\langle 0(\theta)|)^\sim = \langle 0(\theta)|, \]

with \( A \) and \( B \) bosonic operators and \( c_1, c_2 \in \mathbb{C} \).

Now it is possible to define the hamiltonian of the total system. First we note that as the auxiliary system independes of the original one, the introduction of the former does not change the dynamics of the latter, in such a way that the total hamiltonian, denoted by \( \hat{H} \), does not contain terms like \( A \bar{A} \). Furthermore, as the tilde system is identical to the original one, its dynamics has to be the same. These conditions means that the Heisenberg equations for the free bosonic fields and its tilde counterpart must be the same. Joining these statements together with the tilde conjugation rules we find the total hamiltonian, \( \hat{H} \)
\[ \hat{H} = H - \bar{H}. \]

The hamiltonian operator that satisfies the condition for the extended system is given by
\[ \hat{H} = H - \bar{H} = \sum_{n>0} (n A_n^\dagger \cdot A_n + n B_n^\dagger \cdot B_n) - \sum_{n>0} (n \bar{A}_n^\dagger \cdot \bar{A}_n + n \bar{B}_n^\dagger \cdot \bar{B}_n). \]

When the approach is applied to equilibrium systems the transformation generator commutes with the hamiltonian of the total system. A direct implication of this statement is that the total hamiltonian is invariant under transformation generated by (11).

The Bogoliubov transformation is canonical, in such a way that the commutation relations, for the oscillators-like operators, remain unchanged. The vacuum of the transformed system is just the temperature-dependent state and for this reason it is called thermal vacuum.

III. BOSONIC \( D_\beta \)-BRANES AT FINITE TEMPERATURE

\( D_\beta \)-branes are extended objects that can have open strings endpoints attached to them\(^2\). This can be realized if we impose Neumann boundary conditions in the parallel directions and Dirichlet boundary conditions in the transversal directions. For an open string with one end (\( \sigma = 0 \) for instance) attached to the object, we have
\[ \partial_\sigma X^a|_{\sigma=0} = 0, \quad a = 1, \ldots, p, \]
\[ X^i|_{\sigma=0} = x^i, \quad i = p + 1, \ldots 24, \]

\(^2\)For the boundary states construction of \( D_\beta \)-brane we follow Ref. 10.
where \( a = 1, \ldots, p \) are the parallel directions to the \( D_p \)-brane and \( i = p + 1, \ldots, 24 \) are the transversal ones. This description is in the open string channel. A conformal transformation can be used to map these conditions, imposed for the open string solution, to the following conditions imposed to the closed string:

\[
\begin{align*}
\partial_{\tau} X^a |_{\tau = 0} &= 0, \quad a = 1, \ldots, p, \quad \text{(25)} \\
X^i |_{\tau = 0} &= y^i, \quad i = p + 1, \ldots, 24. \quad \text{(26)}
\end{align*}
\]

After the quantization of the system the above conditions for closed string can be read, in terms of the left- and right-modes operators, as the follow operator equations:

\[
\begin{align*}
\left( A^\mu_k + S^\mu_\nu B^\nu_k \right) |B_X\rangle &= 0, \quad \left( \tilde{A}^{\mu \dagger}_k + S^{\mu}_\nu \tilde{B}^\nu_k \right) |B_X\rangle = 0, \quad \text{(27)} \\
\tilde{p}^a |B_X\rangle &= (\tilde{q}^a - y^a) |B_X\rangle = 0, \quad \text{(28)}
\end{align*}
\]

for \( k > 0 \), and \( \tilde{p}^a \) and \( \tilde{q}^a \) are the momentum and coordinate operators for the center-of-mass, respectively. \( S^{\mu \nu} \equiv (\delta^{ab}, -\delta^{ij}) \) was defined in order to obtain a compact expression. Expressions (27)-(28) define what we call boundary states denoted by \( |B_X\rangle \). The state that satisfies these expressions can be written as:

\[
|B_X\rangle = N_p \delta^{(d_\perp)} (\tilde{q} - y) \exp \left[ -\sum_{n=1}^{\infty} A^\mu_n \cdot S \cdot B^\nu_n \right] |0\rangle, \quad \text{(29)}
\]

where \( N_p \) is a normalization constant, the delta function localizes the brane in the transverse space and the vacuum is the closed bosonic string vacuum.

Following the TFD approach, the doubling of the space implies that beside the expressions (27)-(28) we have another set obtained by the tilde conjugations rules. The boundary state (29) has a partner in the tilde space in such a way that the state considered now in the extended space in given by:

\[
|\overline{B_X}\rangle = |B_X\rangle \overline{|B_X\rangle}. \quad \text{(30)}
\]

The temperature-dependent boundary state, interpreted here as a thermal \( D_p \)-brane can be obtained as before, performing the Bogoliubov transformation as:

\[
|B_X (\theta)\rangle = e^{-iG} |B_X\rangle, \quad \text{(31)}
\]

where the generator in given by expressions (11)-(12). The transformation leads us to an explicit expression for the thermal \( D_p \)-brane. Namely,

\[
|B_X (\theta)\rangle = N_p^2 \delta^{(d_\perp)} (\tilde{q} - y) \delta^{(d_\perp)} (\tilde{q} - \tilde{y}) \\
\times \exp \left\{ -\sum_{k=1}^{\infty} \left[ A^\mu_k (\theta) \cdot S \cdot B^\nu_k (\theta) + \tilde{A}^{\mu \dagger}_k (\theta) \cdot S \cdot \tilde{B}^\nu_k (\theta) \right] \right\} |0 (\theta)\rangle. \quad \text{(32)}
\]

Here, we assume that the normalization constant, as well as the momentum and coordinate operators, do not change by thermal effects. The boundary conditions that define the boundary state are now given by:

\[
\left[ A^\mu_k (\theta) + S^\mu_\nu B^{\nu \dagger}_k (\theta) \right] |B_X (\theta)\rangle = 0, \quad \left[ \tilde{A}^{\mu \dagger}_k (\theta) + S^{\mu}_\nu \tilde{B}^{\nu \dagger}_k (\theta) \right] |B_X (\theta)\rangle = 0. \quad \text{(33)}
\]

and a similar set is obtained by the tilde conjugation. Note that the thermal boundary state is invariant by the tilde conjugation rules.

**IV. TFD ENTROPY OPERATOR**

In his original paper, Takahashi and Umezawa defined an operator that when its expectation value is calculated in the thermal vacuum multiplied by the Boltzmann constant, in the Stirling approximation, results the general formula for the entropy. For this reason this operator was called the entropy operator.

In our case the expression for the entropy operator is given by
\[ K = K^\alpha + K^\beta = \sum_k \left( K_k^\alpha + K_k^\beta \right). \]  

The upper indices refer to the right- and left-modes as before. Explicitly we have

\[ K^\alpha = - \sum_k A_k^1 \cdot A_k \log \left( g \sinh^2 (\theta_k) \right) - A_k \cdot A_k^1 \log \left( 1 + g \sinh^2 (\theta_k) \right), \]

and the same for \( K^\beta \) by the change of \( A \) by \( B \) operators. Here \( g = tr \delta^3 \). As in the original case, the result is given in terms of the thermal vacuum expectation value of the number operators for the left- and right-modes. Both modes contribute by the same quantity,

\[ n_k = \langle 0 (\theta) | N_k^\alpha | 0 (\theta) \rangle = \langle 0 (\theta) | N_k^\beta | 0 (\theta) \rangle = g \sinh^2 (\theta_k) = \frac{e^{-(k_B T)^{-1} \omega_k}}{1 - e^{-(k_B T)^{-1} \omega_k}}. \]

In the case of bosonic string, the thermal vacuum expectation value for the entropy operator (34) is interpreted as the entropy of the string and we find

\[ S_{cs} = k_B \langle 0 (\theta) | K | 0 (\theta) \rangle = 2k_B \left\{ \sum_k \left[ (g + n_k) \log (1 + n_k) - n_k \log (n_k) \right] \right\}, \]

where \( n_k \) is the same given by (36). The factor two comes from the equal contribution of the right- and left-modes of the bosonic closed string.

We can perform the calculation, for the expectation value of the entropy operator in the state that represents the \( D_p \)-brane at finite temperature given by (32). In this case we find \(^3\)

\[ S_{Db} = k_B \langle B_X (\theta) | K | B_X (\theta) \rangle = -2k_B \sum_k \left\{ (1 + 2n_k) \mathcal{F}_k \log \left( \frac{n_k}{1 + n_k} \right) \right\} + \]

\[ -2k_B \sum_k \{ n_k \log (n_k) - (g + n_k) \log (1 + n_k) \}, \]

where \( \mathcal{F}_k \) is given by

\[ \mathcal{F}_k = \mathcal{N}^2 \tilde{\mathcal{N}}^2 \sum_{l_1^{24}, \ldots, l_n^{24}} \sum_{s_1^{24}, \ldots, s_n^{24}} \sum_{\mu, \sigma} (S_{24,24}^{s_1^{24}}) \cdots \times (S_{1,1}^{l_1^{24}}) \cdots (S_{1,1}^{l_n^{24}}) \cdot \cdots s_{m,\sigma}^{l_{n+1}}, \]

where we define \( \mathcal{N} \equiv N_p (F) \delta^{(d_{\perp})} (\tilde{q} - y) \) and \( \tilde{N} \equiv N_p (F) \delta^{(d_{\perp})} (\tilde{q} - \tilde{y}) \). As before, the global factor two comes from the equal contribution of the independent modes of the closed string. Note that the contribution of the second term in the above expression, is exactly the entropy of the closed bosonic string given by (37).

**V. CONCLUSIONS**

In this work we present the construction of thermal vacuum for bosonic closed string using the TFD approach. Boundary states at finite temperature were obtained from the imposition of boundary conditions at the solution of the bosonic thermal string and these thermal boundary states were interpreted as a thermal \( D_p \)-brane. Using a modified entropy operator, we obtained the entropy for these states as well for the bosonic closed string. The use of TFD approach seems to be suitable to the study of microscopic structure, in the perturbative limit at finite temperature. The result for the \( D_p \)-brane entropy presents an explicit contribution of the thermal bosonic string. We note the entropy expressions for both systems are similar to the usual entropy for bosonic particles. With the study of dimensional compacted space-time, the contributions of winding-modes will be present. Also, we note that the same analysis that has been done here in the light-cone gauge can be performed in the covariant gauge by including the “thermal ghosts” \(^{11}\). We hope that our construction can throw some light in the microscopic structure of these kind of systems and that the statistical properties of \( D_p \)-branes can be better understood and clarified.

\(^{3}\)This expression can be obtained from the entropy of a \( D_p \)-brane in an external field presented in Ref. 6, by considering a vanishing external field and a suitable choice of parameters.
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