On the Tensor product of archimedean $d$-algebras

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Abstract
The aim of this work is to prove that the Riesz tensor product of two archimedean $d$-algebras is itself a $d$-algebra.

1 Introduction
Since Fremlin introduce the Riesz Tensor product of two vector lattices in \cite{8}, several authors refined his construction, we cite namely \cite{9, 10, 15}. Recently, the Riesz Tensor product regain interest with the work of Azouzi, Ben Amor and Jaber (see \cite{2}) and separately Buskes and Wicksted (see \cite{7}). In fact, they proved that the Riesz (Fremlin) tensor product of archimedean $f$-algebras is itself an $f$-algebra. The aim of this paper is to prove that the Riesz tensor product of archimedean $d$-algebras is itself a $d$-algebra.

2 Preliminaries
A Riesz space $A$ is called a lattice ordered algebra (briefly an $\ell$-algebra), if $A$ is an associative algebra such that the multiplication and the order structure are compatible. That is if $a$ and $b$ are positives in the $\ell$-algebra $A$, then so is $ab$. The $\ell$-algebra $A$ is called $d$-algebra after Birkhoff and Pierce in \cite{4} if for every $a$ and $b$ in $A$ and every positive element $c$ in $A$ we have

\[(a \lor b)c = ac \lor bc\]
\[c(a \lor b) = ca \lor cb\]

Which means that multiplication by a positive element is a lattice homomorphism.
The $\ell$-algebra $A$ is called an almost-$f$-algebra, if for every $a$ and $b$ in $A$ we have

$$a \wedge b = 0 \implies a \wedge b = 0$$

The $\ell$-algebra $A$ is called $f$-algebra, if for every $a$ and $b$ in $A$ and every positive element $c$ in $A$ we have

$$a \wedge b = 0 \implies ac \wedge b = ca \wedge b = 0$$

This definition goes back another time to the fundamental paper [4] of Birkhoff and Pierce. It is quite obvious that all $f$-algebras are $d$-algebras but the converse is far from being true as shows the next example (see [11]).

Example 1. $\mathbb{R}^2$ equipped with the following multiplication

$$(a_1, b_1) (a_2, b_2) = (a_1 b_1, a_1 b_2)$$

is an archimedean $d$-algebra but fails to be an $f$-algebra nor an almost-$f$-algebra.

It is clear then that $d$-algebras form a wider class of $\ell$-algebras than the class of $f$-algebras. Even $d$-algebras fail to be commutative as shows also the previous example while $f$-algebras as almost-$f$-algebras are, one can ask if other properties of $f$-algebras can be extended to $d$-algebras.

We recall that the universal completion of a Riesz space $E$ can be equipped with a product that made it an $f$-algebra. Throughout this paper $E^u$ will always denote the universal completion of the Riesz space $E$ and the multiplication on it will be denoted by juxtaposition.

For the elementary theory of vector lattices and $\ell$-algebras and for unexplained terminology we refer to [1, 3, 11, 14, 16].

3 Disjointness preserving bilinear maps

Let $E$, $F$ and $G$ be archimedean Riesz spaces. A bilinear map $b : E \times F \to G$ is called disjointness preserving if

$$x \perp y \implies b(x, z) \perp b(y, z)$$
$$u \perp v \implies b(t, u) \perp b(t, v)$$

for all $t \in E$ and $z \in F$.

A positive disjointness preserving bilinear map is called an $\ell$-bimorphism.

A Meyer-type theorem is still valid for order bounded disjointness preserving bilinear maps that is:

**Theorem 1** (Kusraev and Tabuev). Let $E$, $F$ and $G$ be archimedean Riesz spaces. Let $b : E \times F \to G$ be an order bounded disjointness preserving bilinear map then there exist $b^+$, $b^-$ and $|b|$ $\ell$-bimorphisms such that

$$|b(x, y)| = |b(|x|, |y|)| = |b(|x|, |y|)| = |b(|x|, |y|)| = |b(|x|, |y|)| = |b(|x|, |y|)| = |b(|x|, |y|)| = |b(|x|, |y|)|$$

for all $x \in E$ and $y \in F$.$b^+(x, y) = (b(x, y))^+$ and $b^-(x, y) = (b(x, y))^-$ for all $x \in E^+$ and $y \in F^+$. 


The previous theorem was proved by Kusraev and Tabuev in [12] and presented without proof in Theorem 33.2 in [6]. The next theorem also stated and proved by Kusraev and Tabuev in [12], we present here an alternative proof.

Theorem 2. Let $E$, $F$ and $G$ be archimedean Riesz spaces. Let $b : E \times F \to G$ be an order bounded disjointness preserving bilinear map then there exists (essentially unique) order bounded disjointness preserving operator $b^\ominus : E \otimes F \to G$ such that

$$b(x, y) = b^\ominus(x \otimes y)$$

for all $x \in E$ and $y \in F$.

Moreover

$$(b^+)^\ominus = (b^\ominus)^+, \ (b^-)^\ominus = (b^\ominus)^- \text{ and } (|b^\ominus|) = (|b|)^\ominus$$

Proof. Let $b : E \times F \to G$ be an order bounded disjointness preserving bilinear map. According to theorem 3.4 in [12], there exist three $\ell$-bimorphisms $b^+$, $b^-$ and $|b|$ such that

$$b = b^+ - b^- \quad \text{and} \quad |b| = b^+ + b^-$$

From Theorem in [8], there exist three $\ell$-homomorphisms $T$, $S$ and $U$ from $E \otimes F$ to $G$ such that

$$b^+(x, y) = S(x \otimes y), \quad b^-(x, y) = T(x \otimes y), \quad |b|(x, y) = U(x \otimes y)$$

for all $x \in E$ and $y \in F$. Thus, $S + T = U$ on $E \otimes F$. Since $E \otimes F$ is ru-complete in $E \otimes F$ (see theorem in [8]) then for any $w \in E \otimes F$, there is some $x \in E^+$ and $y \in F^+$ such that for all $\varepsilon > 0$, there exists an $v_\varepsilon \in E \otimes F$ such that

$$|w - v_\varepsilon| \leq \varepsilon x \otimes y$$

This with the fact that $U$ is an $\ell$-homomorphism lead to

$$U(|w - v_\varepsilon|) = |U(z) - U(v_\varepsilon)| \leq \varepsilon U(x \otimes y)$$

which implies that

$$|U(w) - (S + T)(v_\varepsilon)| \leq \varepsilon U(x \otimes y)$$

Hence, $(S + T)(v_\varepsilon)$ ru-converges to $U(w)$. But since $(S + T)(v_\varepsilon)$ ru-converges to $(S + T)(w)$, it follows that $S + T = U$ on $E \otimes F$. Using twice lemma 5.2 in [5], we obtain that $S - T$ is an order bounded disjointness preserving operator. Put $b^\ominus = S - T$. Then $b^\ominus(x \otimes y) = b(x, y)$ for all $x \in E$ and $y \in F$. Moreover, $|S - T|$ coincides with $U$ on $E \otimes F$, so we can prove using the same techniques of ru-denseness that $U = |S - T|$ on $E \otimes F$. It follows from it that $|T - S| = |T + S|$ that is $T \perp S$. Since $b^\ominus = S - T$, then

$$(b^+)^\ominus = (b^\ominus)^+, \ (b^-)^\ominus = (b^\ominus)^- \quad \text{and} \quad (|b^\ominus|) = (|b|)^\ominus$$

and our proof comes to an end. \qed
4 The Riesz tensor product of d-algebras

From now, we shall impose as a blanket assumption that all real Riesz spaces under consideration in the sequel are archimedean.

The aim of this part is to prove that the tensor product of archimedean d-algebra is itself a d-algebra. In order to achieve this goal, we will use the next theorem due to Kusraev and Tabuev in [13]. It states that:

**Theorem 3 (Kusraev and Tabuev).** Every order bounded disjointness preserving bilinear map \( b : E \times F \to G \) is representable as the product

\[
b(x, y) = S(x)T(y) \quad (x \in E, \ y \in F),
\]

where one of the operators \( S : E \to G^\alpha \) or \( T : F \to G^\alpha \) is order bounded and disjointness preserving, while the other operator is a lattice homomorphism.

We begin our study with this technical proposition

**Proposition 1.** Let \( E, F \) and \( G \) be Riesz spaces. Let \( S \) be in \( \mathcal{L}^b(E) \) and \( T \) be in \( \mathcal{L}^b(F) \), two order bounded disjointness preserving operators then the order bounded bilinear map. Let \( b : E \times F \to G \) be an \( \ell \)-bimorphism then

\[
b(x, y) = b(S(x), T(y))
\]

is order bounded disjointness preserving bilinear map. Moreover

\[
b^+(x, y) = b(S^+(x), T^+(y)) + b(S^-(x), T^-(y)) \quad \text{and} \\
b^-(x, y) = b(S^+(x), T^-(y)) + b(S^-(x), T^+(y))
\]

for all \( x \) in \( E \) and \( y \) in \( F \).

**Proof.** Let \( y \in F \) and \( bp_y(x) = b(S^+(x), T^+(y)) + b(S^-(x), T^-(y)) \) and \( bm_y(x) = b(S^+(x), T^-(y)) + b(S^-(x), T^+(y)) \) for all \( x \). It is clear that \( bp_y \) and \( bm_y \) are linear and order bounded. Pick now \( x_1 \) and \( x_2 \) in \( E \) such that \( |x_1| \wedge |x_2| = 0 \). Since \( S^\pm(x_i) \perp S^\pm(x_j) \) for all \( i \) and \( j \) in \( \{1, 2\} \), then

\[
b(S^\pm(x_i), T^\pm(y)) \perp b(S^\pm(x_j), T^\pm(y))
\]

This fact with lemma 5.2 in [5] yield to that \( bp_y \) and \( bm_y \) are disjointness preserving and \( bp_y \perp bm_y \). We prove an analogous result by fixing an \( x \) in \( E \) and considering \( bp_x \) and \( bm_x \) which make an end to our prove.

Since the Riesz tensor product induces an \( \ell \)-bimorphism on \( E \times F \), the next corollary follows immediately from the previous lemma.

**Corollary 1.** Let \( E \) and \( F \) be Riesz spaces. Let \( S \) be in \( \mathcal{L}_b(E) \) and \( T \) be in \( \mathcal{L}_b(F) \), two order bounded disjointness preserving operators then the order bounded bilinear map then

\[
b : E \times F \to E \otimes F \\
(x, y) \mapsto S(x) \otimes T(y)
\]
is order bounded disjointness preserving bilinear map. Moreover
\[ b^+(x, y) = S^+(x) \otimes T^+(y) + S^-(x) \otimes T^-(y) \] and
\[ b^-(x, y) = S^+(x) \otimes T^-(y) + S^-(x) \otimes T^+(y) \]
for all \( x \) in \( E \) and \( y \) in \( F \).

At this point, we gathered all ingredients we need to prove the main theorem of this work.

**Theorem 4.** Let \((A_1, \star)\) and \((A_2, \star)\) be two \( d \)-algebras, then the Riesz Tensor product \( A_1 \otimes A_2 \) can be equipped with a \( d \)-algebra product that extends the product on the algebraic tensor product \( A_1 \otimes A_2 \).

**Proof.** Since products on \( d \)-algebras are order bounded disjointness preserving bilinear map, then, according to Corollary 3.3 in [13], there exist an \( \ell \)-homomorphism \( S_1 \) and an order bounded disjointness preserving operator \( T_1 \) (respectively \( S_2 \) and \( T_2 \)) from \( A_1 \) to its universally completion \( A_1^u \) (respectively from \( A_2 \) to \( A_2^u \)) such that
\[ x_1 \star y_1 = S_1(x_1)T_1(y_1) \]
for all \( x_1, y_1 \) in \( A_1 \). Put
\[ \sigma : A_1 \times A_2 \rightarrow A_1^u \overset{\otimes}{\otimes} A_2^u \]
\[ (x_1, x_2) \mapsto S_1(x_1) \otimes S_2(x_2) \]
and
\[ \tau : A_1 \times A_2 \rightarrow A_1^u \overset{\otimes}{\otimes} A_2^u \]
\[ (y_1, y_2) \mapsto T_1(y_1) \otimes T_2(y_2) \]
From Theorem 4.2 in [8] and Theorem [2] it follows that there exist an \( \ell \)-homomorphisms \( \bar{\sigma} \) and an order bounded disjointness preserving bilinear map \( \bar{\tau} \) such that:
\[ \bar{\sigma} : A_1 \overset{\bar{\otimes}}{\otimes} A_2 \rightarrow A_1^u \overset{\otimes}{\otimes} A_2^u \]
\[ \bar{\tau} : A_1 \overset{\bar{\otimes}}{\otimes} A_2 \rightarrow A_1^u \overset{\otimes}{\otimes} A_2^u \]
and \( \bar{\sigma}(x_1 \otimes x_2) = \sigma(x_1, x_2) \) and \( \bar{\tau}(y_1 \otimes y_2) = \tau(y_1, y_2) \) for every \( x_1 \) and \( y_1 \) in \( A_1 \) and \( x_2 \) and \( y_2 \) in \( A_2 \).

According to Theorem 11 in [2], \( A_1^u \overset{\otimes}{\otimes} A_2^u \) is an \( f \)-algebra. We can then define an order bounded disjointness preserving bilinear map \( \pi \) as:
\[ \pi : A_1 \overset{\bar{\otimes}}{\otimes} A_2 \times A_1 \overset{\bar{\otimes}}{\otimes} A_2 \rightarrow A_1^u \overset{\otimes}{\otimes} A_2^u \]
\[ (u, v) \mapsto \bar{\sigma}(u)\bar{\tau}(v) \]
We affirm that the range of \( \pi \) is included in \( A_1 \overset{\bar{\otimes}}{\otimes} A_2 \). In fact, \( A_1 \) and \( A_2 \) are respectively Riesz subspace of \( A_1^u \) and \( A_2^u \). Then \( A_1 \overset{\bar{\otimes}}{\otimes} A_2 \) is the Riesz subspace of \( A_1^u \overset{\otimes}{\otimes} A_2^u \) generated by the algebraic tensor product \( A_1 \otimes A_2 \) (Corollary 4.5 in [8]). Then, there exist \( I_1, J_1 \) and \( K_1 \) (and respectively \( I_2, J_2 \) and \( K_2 \)) finite sets, such that, for every \( u \) and \( v \) in \( A_1 \overset{\bar{\otimes}}{\otimes} A_2 \)
\[ u = \bigvee_{i \in I_1} \bigwedge_{j \in J_1} \sum_{k \in K_1} x^{(i,j,k)}_1 \otimes x^{(i,j,k)}_2 \]
and

\[ v = \bigvee \bigwedge_{i' \in I_2, j' \in J_2, k' \in K_2} \sum_{K_2} y_1^{(i',j',k')} \otimes y_2^{(i',j',k')} \]

where \( x_1^{(i,j,k)} \) and \( y_1^{(i',j',k')} \) are in \( A_1 \) and \( x_2^{(i,j,k)} \) and \( y_2^{(i',j',k')} \) are in \( A_2 \). Theorem 2 yields to

\[ \bar{\sigma}(u) = \bigvee \bigwedge \sum_{i \in I_1, j \in J_1, k \in K_1} \sigma(x_1^{(i,j,k)} \otimes x_2^{(i,j,k)}) \]

\[ \bar{\tau}^+(v) = \bigvee \bigwedge \sum_{i' \in I_2, j' \in J_2, k' \in K_2} \tau^-(y_1^{(i',j',k')} \otimes y_2^{(i',j',k')}) \]

and

\[ \bar{\tau}^-(v) = \bigvee \bigwedge \sum_{i' \in I_2, j' \in J_2, k' \in K_2} \tau^+(y_1^{(i',j',k')} \otimes y_2^{(i',j',k')}) \]

Then \( \pi(u,v) \) is composed by finite infima, superma and sums of \( \sigma(x_1^{(i,j,k)} \otimes x_2^{(i,j,k)}) \tau^\pm(y_1^{(i',j',k')} \otimes y_2^{(i',j',k')}) \) which, according to Corollary 1, are equal to

\[ S_1(x_1^{(i,j,k)})T_1^\pm(y_1^{(i',j',k')}) \otimes S_2(x_2^{(i,j,k)})T_2^\pm(y_2^{(i',j',k')}) \]

These terms are all in \( A_1 \otimes A_2 \) which implies that \( \pi(u,v) \) is in \( A_1 \bar{\otimes} A_2 \) for every \( u \) and \( v \) in \( A_1 \bar{\otimes} A_2 \). The order bounded disjointness preserving bilinear map \( \pi \) extends the product on the algebraic tensor product of \( A_1 \otimes A_2 \) to \( A_1 \bar{\otimes} A_2 \) as desired.

It remains to prove that this multiplication is associative. But since it is associative on the algebraic tensor product, a similar and straightforward calculus than the latter one leads to the desired conclusion and our proof comes to an end.

It remains open the fact if whether or not the Riesz Tensor product of almost-\( f \)-algebra is itself an almost-\( f \)-algebra.

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