HYPERSPACES OF SMOOTH CONVEX BODIES UP TO CONGRUENCE

IGOR BELEGRADEK

Abstract. We determine the homeomorphism type of the hyperspace of positively curved $C^\infty$ convex bodies in $\mathbb{R}^n$, and derive various properties of its quotient by the group of Euclidean isometries. We make a systematic study of hyperspaces of convex bodies that are at least $C^1$. We show how to destroy the symmetry of a family of convex bodies, and prove that this cannot be done modulo congruence.

1. Introduction

A hyperspace of $\mathbb{R}^n$ is a set of compact subsets of $\mathbb{R}^n$ equipped with the Hausdorff metric, and two subsets are congruent if they lie in the same orbit of $\text{Iso}(\mathbb{R}^n)$, the group of Euclidean isometries. To avoid trivialities we assume that $n \geq 2$.

A convex body in $\mathbb{R}^n$ is a compact convex set with nonempty interior. A function is $C^{k,\alpha}$ if its $k$th partial derivatives are $\alpha$-Hölder for $\alpha \in (0,1]$ and continuous for $\alpha = 0$ where as usual $C^k$ means $C^{k,0}$, see [GT01] and [BB, Section 2] for background. A $C^{k,\alpha}$ convex body is a convex body whose boundary is a $C^{k,\alpha}$ submanifold of $\mathbb{R}^n$. Any convex body is $C^{0,1}$ because convex functions are locally Lipschitz.

There is an established framework for studying topological properties of hyperspaces, and e.g., the homeomorphism types of the following hyperspaces of $\mathbb{R}^n$ are known: convex compacta [NQS79], convex bodies [AJP13], convex polyhedra [BRZ96 Exercise 4.3.7], strictly convex bodies [Baz93], convex compacta of constant width [Baz97] [BZ06] [AJPJOn15].

One goal of this paper is to add to this list the hyperspace of $C^\infty$ convex bodies of positive Gaussian curvature.

Another goal is to study certain hyperspaces that are not closed under Minkowski sum, e.g., the results of this paper are used in [Bel] to determine the homeomorphism type of a hyperspace of $\mathbb{R}^3$ whose $\text{Iso}(\mathbb{R}^3)$-quotient is homeomorphic to the Gromov-Hausdorff space of $C^\infty$ nonnegatively curved 2-spheres.

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Let $\mathcal{K}_s$ be the hyperspace of convex compacta in $\mathbb{R}^n$ with Steiner point at the origin, and let $\mathcal{B}_p$ be the hyperspace of $C^\infty$ convex bodies in $\mathcal{K}_s$ with boundary of positive Gaussian curvature. Placing the Steiner point at the origin is mainly a matter of convenience. In particular, the space of all convex compacta $\mathcal{K}$ in $\mathbb{R}^n$ is homeomorphic to $\mathcal{K}_s \times \mathbb{R}^n$, and the orbit spaces $\mathcal{K}/\text{Iso}(\mathbb{R}^n)$, $\mathcal{K}_s/O(n)$ are homeomorphic, see (4.2) and Lemma 4.3.

Consider the Hilbert cube $Q = [-1,1]^\omega$ and its radial interior

$$\Sigma = \{(t_i)_{i \in \omega} \in Q : \sup_{i \in \omega} |t_i| < 1\}.$$ 

The superscript $\omega$ refers to the product of countably many copies of a space. We have a canonical inclusion $\Sigma^\omega \subset Q^\omega$. Clearly $Q^\omega$ and $Q$ are homeomorphic. Also $\Sigma$ is $\sigma$-compact while $\Sigma^\omega$ is not. Here is the main result of this paper.

**Theorem 1.1.** Given a point $\ast$ in $Q^\omega \setminus \Sigma^\omega$ there exists a homeomorphism $\mathcal{K}_s \to Q^\omega \setminus \{\ast\}$ that takes $\mathcal{B}_p$ onto $\Sigma^\omega$.

The proof of Theorem 1.1 verifies the assumptions of recognition theorems for spaces modelled on $Q$ and $\Sigma^\omega$ as described e.g., in [BRZ96]. This involves various convex geometry techniques, as well as a method developed in [BB].

Yet another objective of this paper is to study the topology of $\mathcal{K}_s$ and $\mathcal{B}_p$ modulo congruence, see [TW80] [ABR04] [Ant00] [AJP13] [Age16] for related results.

Set $\mathcal{K}_s = \mathcal{K}_s/O(n)$ and $\mathcal{B}_p = \mathcal{B}_p/O(n)$ with the quotient topology. Denote the principal $O(n)$-orbits in $\mathcal{K}_s$, $\mathcal{B}_p$ by $\hat{\mathcal{K}}_s$, $\hat{\mathcal{B}}_p$, respectively, and let $\hat{\mathcal{K}}_s$, $\hat{\mathcal{B}}_p$ be their $O(n)$-orbit spaces. By the Slice Theorem $\hat{\mathcal{K}}_s$ is open in $\mathcal{K}_s$ while the orbit maps $\hat{\mathcal{K}}_s \to \hat{\mathcal{K}}_s$ and $\hat{\mathcal{B}}_p \to \hat{\mathcal{B}}_p$ are principal $O(n)$-bundles. Some other properties of the spaces are summarized below.

**Theorem 1.2.**

1. $\mathcal{K}_s$ is a locally compact Polish absolute retract.
2. $\mathcal{B}_p$ is an absolute retract that is neither Polish nor locally compact.
3. Any $\sigma$-compact subset of $\mathcal{B}_p$ has empty interior.
4. $\mathcal{B}_p$ is homotopy dense in $\mathcal{K}_s$, i.e., any continuous map $Q \to \mathcal{K}_s$ can be uniformly approximated by a continuous map with image in $\mathcal{B}_p$.
5. $\hat{\mathcal{B}}_p$ and $\hat{\mathcal{K}}_s$ are contractible, while $\hat{\mathcal{B}}_p$ and $\hat{\mathcal{K}}_s$ are homotopy equivalent to $BO(n)$, the Grassmanian of $n$-planes in $\mathbb{R}^\omega$.
6. The pairs $(\hat{\mathcal{K}}_s, \hat{\mathcal{B}}_p)$ and $(Q^\omega, \Sigma^\omega)$ are locally homeomorphic, i.e., each point of $\hat{\mathcal{B}}_p$ has a neighborhood $U \subset \hat{\mathcal{K}}_s$ such that some open embedding $h: U \to Q^\omega$ takes $U \cap \hat{\mathcal{B}}_p$ onto $h(U) \cap \Sigma^\omega$.
7. There is a locally finite simplicial complex $L$ and a homeomorphism $h: \hat{\mathcal{K}}_s \to L \times Q^\omega$ that maps $\hat{\mathcal{B}}_p$ onto $L \times \Sigma^\omega$. 
That $\mathcal{K}_s$ and $\mathcal{B}_p$ are absolute retracts is essentially due to Antonyan [Ant05, Ant11]. Homotopy density of $\mathcal{B}_p$ in $\mathcal{K}_s$ is immediate from Schneider’s regularization of convex bodies, see Lemma 4.3.

Contractibility of $\hat{\mathcal{B}}_p$, $\hat{\mathcal{K}}_s$ is established geometrically in Lemma 8.2 which proves that these spaces are homotopy dense in $\mathcal{K}_s$. Therefore, they are classifying spaces for principal $O(n)$-bundles, and (5) of Theorem 1.2 follows.

The claim (6) of Theorem 1.2 exploits the $O(n)$-bundle structure and depends on Theorem 1.1.

The claim (7) follows in a standard way from (5)-(6) and the observation that $\mathcal{K}_s$ is homeomorphic to $\hat{\mathcal{K}}_s \times [0,1)$, see Lemma 5.4. One can take $L$ to be the product of $[0,1)$ with any locally finite simplicial complex that is homotopy equivalent to $BO(n)$.

Since $\mathcal{K}_s$ is contractible while $\hat{\mathcal{K}}_s$ is not, the latter is not homotopy dense in the former, i.e., there is no continuous “destroy the symmetry” map $\mathcal{K}_s \to \hat{\mathcal{K}}_s$ that would instantly push every singular $k$-disk in $\mathcal{K}_s$ into $\hat{\mathcal{K}}_s$. More precisely, such a map exists for $k \leq 1$ but not for $k = 2$, see Section 9 where we also prove a local version of this assertion.

The following questions highlight how much we do not yet know.

(a) Is the hyperspace $\mathcal{B}^\infty$ of $C^\infty$ convex bodies homeomorphic to $\Sigma^{\infty}$? Unlike $\mathcal{B}_p$, the hyperspace $\mathcal{B}^\infty$ is not convex [Bom90], and convexity was essential in our proof of strong $\mathcal{M}_2$-universality of $\mathcal{B}_p$.

(b) Are the orbit spaces $\mathcal{B}_p$ and $\mathcal{K}_s$ topologically homogeneous, i.e., do their homeomorphism groups act transitively? (This seems unlikely).

(c) Is the congruence class of the unit sphere a $\mathbb{Z}$-set in $\mathcal{K}_s$? Does it have contractible complement?

(d) Does every point $\mathcal{B}_p$, $\mathcal{K}_s$ have a basic of contractible neighborhoods? Like any AR, these orbit spaces are locally contractible, i.e., any neighborhood of a point contains a smaller neighborhood that contracts inside the original one.

(e) As we shall see in Lemma 5.5 the hyperspace $\mathcal{K}_s(\mathbb{B}^n)$ that consists of sets in $\mathcal{K}_s$ contained in $\mathbb{B}^n$ is homeomorphic to $Q$. Is there an $O(n)$-equivariant homeomorphism of $\mathcal{K}_s(\mathbb{B}^n)$ onto a countable product of Euclidean units disks (of various dimensions) where the $O(n)$-action on the product is diagonal and irreducible on each factor? Such products are considered in [Ant88, Section 1], [Wes90, p.553], [Age16, p.161].

(f) For $\kappa > 0$ consider the hyperspaces $\mathcal{B}_{>\kappa}$, $\mathcal{B}_{\geq k}$ in $\mathcal{K}_s$ consisting of of convex bodies whose boundary has Gaussian curvature $> k$ or $\geq k$, respectively. Are they ANR, or more generally $H$-ANR for every closed
subgroup $H \leq O(n)$? All I can say is that $B_{>\kappa}/H$ and $B_{\geq\kappa}/H$ are weakly contractible as each singular sphere in $B_{\geq\kappa}/H$ contracts in $B_p/H$ and every singular disk in $B_p/H$ can be rescaled into $B_{>\kappa}$.

The structure of the paper is as follows. Section 1 describes the main results and also lists some open questions. In Section 2 we collect a number of notations and conventions. Some facts of infinite dimensional topology and convex geometry are reviewed in Sections 3 and 4, respectively. In Section 5 we classify the $Q$-manifolds encountered in the paper. Section 6 is the heart of the paper where the key claim of Theorem 1.1 is proven: $B_p$ is homeomorphic to $\Sigma^\omega$. The difficulty here is to match the tools of convex geometry with what is required by the infinite dimensional topology. The proof is finished in Section 7 via a standard argument. In Sections 8–9 we prove Theorem 1.2 and show that one can continuously destroy the symmetry of convex bodies, but one cannot do this modulo congruence.

2. Notations and conventions

Throughout the paper $\omega$ is the set of the nonnegative integers, $\mathbb{I} = [0, 1]$, $\mathbb{S}^{n-1} = \partial \mathbb{B}^n$ where $\mathbb{B}^n$ is the closed unit ball about the origin $o \in \mathbb{R}^n$. We use the following notations for hyperspaces of $\mathbb{R}^n$:

$$
\mathcal{K} = \{ \text{convex compacta in } \mathbb{R}^n \}
$$

$$
\mathcal{K}^{k \leq l} = \{ \text{sets in } \mathcal{K} \text{ of dimension at least } k \text{ and at most } l \} \quad \text{and} \quad \mathcal{K}^k = \mathcal{K}^{k \leq k}
$$

$$
\mathcal{K}_s = \{ \text{sets in } \mathcal{K} \text{ with Steiner point at } o \} \quad \text{and} \quad \mathcal{K}_s^{k \leq l} = \mathcal{K}_s^{k \leq l} \cap \mathcal{K}_s
$$

$$
\mathcal{B}^{k,\alpha} = \{ C^{k,\alpha} \text{ convex bodies in } \mathcal{K}_s \} \quad \text{and} \quad \mathcal{B}^k = \mathcal{B}^{k,0}
$$

$$
\mathcal{B}_p = \{ \text{convex bodies in } \mathcal{B}^\infty \text{ with boundary of positive Gaussian curvature} \}
$$

To stress that these are hyperspaces of $\mathbb{R}^n$ we may write $\mathcal{K}(\mathbb{R}^n)$ for $\mathcal{K}$, etc. On one occasion we discuss $\mathcal{K}_s(\mathbb{B}^n)$, the hyperspace of consisting of sets of $\mathcal{K}_s$ lying in the unit ball about the origin. Note that $\mathcal{B}_p$ equals the class $C^\infty_+$ discussed in [Sch14, Section 3.4]. Each of the above hyperspaces is $O(n)$-invariant, and we denote the principal $O(n)$-orbit, i.e., the set of points with the trivial isotropy in $O(n)$, by placing $\circ$ over the hyperspace symbol, e.g., $\hat{\mathcal{B}}_d$ is the principal orbit for the $O(n)$-action on $\mathcal{B}_p$.

We denote the $O(n)$-orbit space of a hyperspace by the same symbol made bold, e.g., $\mathcal{B}^{k,\alpha}$, $\mathcal{K}_s$, $\hat{\mathcal{B}}_p$ are the $O(n)$-orbit spaces of $\mathcal{B}^{k,\alpha}$, $\mathcal{K}_s$, $\hat{\mathcal{B}}_p$, respectively.

3. Brief dictionary of infinite dimensional topology

All definitions and notions of infinite dimensional topology that are used in this paper can be found in [BRZ96] and are also reviewed below.
Unless stated otherwise any space is assumed metrizable and separable, and any map is assumed continuous. A space is Polish if it admits a complete metric. A subspace is a subset with subspace topology. If Ω and M are spaces, then M is an Ω-manifold if each point of M has a neighborhood homeomorphic to an open subset of Ω.

A closed subset B of a space X is a Z-set if every map Q → X can be uniformly approximated by a map whose range misses B. A σZ-set is a countable union of Z-sets. An embedding is a Z-embedding if its image is a Z-set.

A subspace A ⊂ X is homotopy dense if there is a homotopy h: X × I → X with h₀ = id and h(X × (0, 1]) ⊂ A. If X is an ANR, then A ⊂ X is homotopy dense if and only if each map I^k → X with k ∈ ω and ∂I^k ⊂ B can be uniformly approximated rel boundary by maps I^k → B [BRZ96, Theorem 1.2.2].

If X is an ANR and B ⊂ X is a closed subset, then B is a Z-set if and only if X \ B is homotopy dense [BRZ96, Theorem 1.4.4], [BP75, Proposition V.2.1].

A space X has the Strong Discrete Approximation Property or simply SDAP if for every open cover U of X each map Q × ω → X is U-close to a map g: Q × ω → X such that every point of X has a neighborhood that intersects at most one set of the family \{g(Q × \{n\})\}n∈ω.

A space X has the Locally Compact Approximation Property or simply LCAP if for every open cover U of X there exists a map f: X → X that is U-close to the identity of X and such that f(X) has locally compact closure.

Let M₀ be the class of compact spaces, and M₂ be the class of spaces homeomorphic to Fσδ-sets in compacta, see [BRZ96, Exercise 3 in 2.4]. Note that X ∈ M₂ if and only if the image of any embedding of X into a Polish space is Fσδ, see [BP75, Theorem VIII.1.1].

Let C be a class of spaces, such as M₀ or M₂. A space X is C-universal if each space in C is homeomorphic to a closed subset of X.

A space X is strongly C-universal if for every open cover U of X, each C ⊂ C, every closed subset B ⊂ C, and each map f: C → X that restricts to a Z-embedding on B there is a Z-embedding f: C → X with f|B = f|B such that f, f are U-close.

A space X is C-absorbing if X is a strongly C-universal ANR with SDAP that is the union of countably many Z-sets, and also the union of a countably many closed subsets homeomorphic to spaces in C.

For example, Σ is M₀-absorbing and Σω is M₂-absorbing, see [BRZ96] Exercises 3 in 1.6 and 2.4. Let us list some properties of M₂-absorbing spaces:
• (Triangulated $\Sigma^\omega$-manifold) A space $X$ is $\mathcal{M}_2$-absorbing if and only if $X$ is an $\Sigma^\omega$-manifold if and only if $X$ is homeomorphic to $\Sigma^\omega \times L$ where $L$ is a locally finite simplicial complex [BMS6, Corollary 5.6].

• (Uniqueness) Any two homotopy equivalent $\mathcal{M}_2$-absorbing spaces are homeomorphic, see [BMS6, Theorem 3.1].

• (Z-set unknotting) If $A$, $B$ are $Z$-sets in a $\mathcal{M}_2$-absorbing space $X$, then any homeomorphism $A \to B$ that is homotopic to the inclusion of $A$ into $X$ extends to a homeomorphism of $X$ [BMS6, Theorem 3.2].

4. Convex geometry background

Our main reference for convex geometry is [Sch14]. We give $\mathbb{R}^n$ the Euclidean norm $\|v\| = \sqrt{\langle v,v \rangle}$ where $\langle u,v \rangle = \sum_{i=1}^n u_i v_i$.

The support function $h_D \colon \mathbb{R}^n \to \mathbb{R}$ of a compact convex non-empty set $D$ is defined by $h_D(v) = \sup\{ \langle x,v \rangle : x \in D \}$. Thus $h_{D+u}(v) = h_D(v) + \langle v,w \rangle$ for any $w \in \mathbb{R}^n$, hence $D$ has nonempty interior if and only if there is $w \in \mathbb{R}^n$ such that $h_D(v) + \langle v,w \rangle > 0$ for any $v \neq 0$. The function $h_D$ is sublinear, i.e., $h_D(tv) = th_D(v)$ for $t > 0$ and $h_D(v + w) \leq h_D(v) + h_D(w)$. Conversely, any sublinear real-valued function on $\mathbb{R}^n$ is the support function of a unique compact convex set in $\mathbb{R}^n$, see [Sch14, Theorem 1.7.1].

If $o \in D$ and $u \in S^{n-1}$, then $h_D(u)$ is the distance to the origin from the support hyperplane to $D$ with outward normal vector $u$. If $o \in \text{Int } D$, then $h_D(v) > 0$ for any $v \neq 0$. In summary, support functions of convex bodies whose interior contains $o$ are precisely the sublinear positive functions from $\mathbb{R}^n$ to $\mathbb{R}$.

Given a $C^{k,\alpha}$ convex body $D$ in $\mathbb{R}^n$ with $k \geq 1$, let $\nu_D \colon \partial D \to S^{n-1}$ be the Gauss map given by the outward unit normal. For $k \geq 2$ the Gaussian curvature is the determinant of the differential of $\nu_D$. It is well-known that the Gaussian curvature of a convex body is nonnegative. Note that $\nu_D$ is $C^{k-1,\alpha}$. Also $\nu_D$ is a $C^1$ diffeomorphism if and only if the Gaussian curvature is positive, and $\nu_D$ is a homeomorphism if and only if $D$ is strictly convex, i.e., $\partial D$ contains no line segments. Positive Gaussian curvature implies strict convexity.

A convex body $D$ is strictly convex if and only if $h_D$ is differentiable away from $o$, and furthermore, if $D$ is strictly convex, then the restriction of $\nabla h_D$ to $S^{n-1}$ equals $\nu_D^{-1}$ so that $h_D$ is $C^1$, see [Sch14, Corollary 1.7.3].

Lemma 4.1. Let $k \geq 2$ and $\alpha \in \mathbb{I}$. For $D \in \mathcal{K}(\mathbb{R}^n)$ the following are equivalent

1. $D$ is a $C^{k,\alpha}$ convex body and $\partial D$ has positive Gaussian curvature,
2. $h_D|_{S^{n-1}}$ is $C^{k,\alpha}$ and $\nabla h_D|_{S^{n-1}}$ has no critical points.
Proof. Let us show (1) \(\Rightarrow\) (2). Since \(\partial D\) is \(C^{k,\alpha}\), the Gauss map \(\nu_D\) is \(C^{k-1,\alpha}\). Nonvanishing of the Gaussian curvature of \(\partial D\) means that \(\nu_D\) is a \(C^1\) diffeomorphism, and hence a \(C^{k-1,\alpha}\) diffeomorphism by the inverse function theorem, see [BHS05, Theorem 2.1]. Now \(\nabla h_D|_{S^{n-1}} = \nu_D^{-1}\), implies that \(h_D|_{S^{n-1}}\) is \(C^{k,\alpha}\) and \(\nabla h_D|_{S^{n-1}}\) has no critical points.

To show (2) \(\Rightarrow\) (1) first note that the assumption \(h_D|_{S^{n-1}}\) is \(C^{k,\alpha}\) and the homogeneity of \(h_D\) shows that \(h_D\) is \(C^{k,\alpha}\) on \(\mathbb{R}^n\setminus\{0\}\), and in particular, is differentiable there. The latter implies that every support hyperplane meets \(D\) in precisely one point, see [Sch14, Corollary 1.7.3], and in particular \(D\) has nonempty interior and \(\nu_D\) is a homeomorphism. As was mentioned above, \(\nabla h_D|_{S^{n-1}} = \nu_D^{-1}\), so using the assumptions we conclude that \(\nu_D^{-1}\) is a \(C^{k-1,\alpha}\) diffeomorphism and \(\partial D\) is a \(C^{k-1,\alpha}\) submanifold. These two statements in fact imply that \(\partial D\) is \(C^{k,\alpha}\), see e.g., the proof of [Gho12, Lemma 5.4]. Finally, non-vanishing of the Gaussian curvature is equivalent to \(\nu_D\) being an immersion. \(\square\)

The set of \(C^{k,\alpha}\) convex bodies of positive Gaussian curvature is convex under scaling and Minkowski addition, see [Gho12, Proposition 5.1] for \(\alpha = 0\), and [BJ17] in general.

We make heavy use of the map \(s\colon K(\mathbb{R}^n) \to C(S^{n-1})\) given by \(s(D) = h_D|_{S^{n-1}}\) which enjoys the following properties:

- \(s\) is an isometry onto its image, where as usual the domain has the Hausdorff metric and the co-domain has the metric induced by the \(C^0\) norm [Sch14, Lemma 1.8.14].
- the image of \(s\) is closed [Sch14 Theorem 1.8.15] and convex [Sch14 pp.45 and 48] in \(C(S^{n-1})\).
- \(s\) is Minkowski linear [Sch14, Section 3.3], i.e., \(s(aD + bK) = as(D) + bs(K)\) for any nonnegative \(a, b\) and any \(D, K \in K\).

The Steiner point is a map \(s\colon K(\mathbb{R}^n) \to \mathbb{R}^n\) given by

\[
s(D) = \frac{1}{\text{vol}(\mathbb{R}^n)} \int_{S^{n-1}} u h_D(u) \, du
\]

which has the following properties:

- \(s\) is Lipschitz [Sch14, p.66, Section 1.8],
- \(s\) is invariant under rigid motions, i.e., \(s(gD) = gs(D)\) for any \(g \in \text{Iso}(\mathbb{R}^n)\) [Sch14, p.50, Section 1.7],
- \(s\) is Minkowski linear, i.e., \(s(aD + bK) = as(D) + bs(K)\) for any positive reals \(a, b\) and \(D, K \in K(\mathbb{R}^n)\).
- \(s(D)\) lies in the relative interior of \(D\) [Sch14 p.315, Section 5.2.1],
- if \(D\) is a point, then \(s(D) = D\) [Sch14, p.50, Section 1.7],
• $s$ is the only continuous Minkowski linear, $\text{Iso}(\mathbb{R}^n)$-invariant map from $\mathcal{K}(\mathbb{R}^n)$ to $\mathbb{R}^n$ [Sch14 Theorem 3.3.3].

Thus the hyperspace $\mathcal{K}_s$ of convex compacta in $\mathbb{R}^n$ with Steiner point at $o$ is an $O(n)$-invariant closed convex subset of $\mathcal{K}$ and the map

\[(4.2) \quad \mathcal{K} \to \mathbb{R}^n \times \mathcal{K}_s\]

sending $D$ to $(s(D), D - s(D))$ is a homeomorphism.

**Lemma 4.3.** The retraction $r: \mathcal{K} \to \mathcal{K}_s$ given by $r(D) = D - s(D)$ is equivariant under the homomorphism $\text{Iso}(\mathbb{R}^n) \to O(n)$ given by $\phi \to \phi - \phi(0)$, and descends to a homeomorphism $\bar{r}: \mathcal{K} / \text{Iso}(\mathbb{R}^n) \to \mathcal{K}_s / O(n)$.

**Proof.** The equivariance of $r$ and bijectivity of $\bar{r}$ is straightforward from the properties of the Steiner point and the fact that any isometry of $\mathbb{R}^n$ can be written as $x \to Ax + b$ for some unique $A \in O(n)$ and $b \in \mathbb{R}^n$. Bijectivity of $\bar{r}$ implies that the inclusion $i: \mathcal{K}_s \to \mathcal{K}$ descends to $\bar{r}^{-1}$, and since $r$, $i$ are continuous, so are $\bar{r}$, $\bar{r}^{-1}$. $\Box$

**Lemma 4.4.** If $t > 0$ and $D \in \mathcal{K}_s$ with $h_D|_{S^{n-1}} \in C^\infty$, then $D + t \mathbb{B}^n \in \mathcal{B}_p$.

**Proof.** Set $B = t \mathbb{B}^n$. From $h_{D+B} = h_D + h_B$ we conclude that $h_{D+B}$ is $C^\infty$. The equality $\nu_{D+B}^{-1} = \nabla h_{D+B}|_{S^{n-1}}$ implies that $\nu_{D+B}^{-1}$ is $C^\infty$. To show that $D + B$ is in $\mathcal{B}_p$, let us check that $\nu_{D+B}$ is a $C^\infty$ diffeomorphism, which by the inverse function theorem is equivalent to $\nu_{D+B}^{-1}$ having no critical points, i.e.,

$$\text{Hess} h_{D+B} = \text{Hess} h_D + \text{Hess} h_B$$

is positive definite. The Hessian of any convex $C^2$ function is nonnegative definite which applies to $h_D$ while $\text{Hess} h_B$ is positive definite e.g., because $\nabla h_{1\mathbb{B}^n}(x) = t x/\|x\|$ is a diffeomorphism on $S^{n-1}$. $\Box$

A key tool in this paper is the *Schneider’s regularization* which is immediate from the remark after the proof of [Sch14 Theorem 3.4.1].

**Lemma 4.5.** There is a continuous map $\rho: \mathcal{K}_s \times \mathbb{I} \to \mathcal{K}_s$ such that $\rho(\cdot, 0)$ is the identity map, and for each $t > 0$ the map $\rho(\cdot, t)$ is $O(n)$-equivariant and has image in $\mathcal{B}_p$.

**Proof.** Fix a nonnegative function $\psi \in C^\infty(\mathbb{R})$ with support in $[1, 2]$ and such that $\int_\mathbb{R} \psi = 1$. For $t \in (0, 1]$ and $D \in \mathcal{K}$ let $\rho(D) = T(D) + t \mathbb{B}^n$ where $T(D)$ is the convex set with support function

$$h_{T(D)}(x) = \int_{\mathbb{R}^n} h_D(x + z\|x\|) \psi(\|z\|/t) t^n \; dz.$$
Note that $h_{\rho(D)} = h_{T(D)} + h_{t_{B^n}}$ and $h_{t_{B^n}}(x) = t \|x\|$. It follows from [Sch14, Theorem 3.4.1] that $h_{T(D)}$ is $C^\infty$ on $\mathbb{R}^n \setminus \{0\}$ and $T$ is $O(n)$-equivariant and continuous on $\mathcal{K}_s \times \mathcal{I}$. These properties are clearly inherited by $\rho$. The image of $\rho$ is in $\mathcal{B}_p$ by Lemma 4.4. □

Another useful operation is what we call $(\varepsilon, u)$-truncating of a $C^1$ convex body $K$, where $\varepsilon \in \mathcal{I}$ and $u \in S^{n-1}$, defined as the convex set $K_{u,\varepsilon}$ obtained by removing all points of $K$ that lie in the open $\varepsilon$-neighborhood of the support hyperplane to $K$ with the normal vector $u$, and then subtracting the Steiner point of the result. For each $u$ the map $K_s \times \mathcal{I} \to K_s$ that sends $(K, \varepsilon)$ to $K_{u,\varepsilon}$ is continuous. If $\varepsilon$ is smaller than the length of the projection of $K$ onto the line span$\{u\}$, then $K_{u,\varepsilon}$ is not $C^1$.

5. The hyperspace of $\mathbb{R}^n$ of convex compacta of dimension $\geq 2$

In this section we determine the homeomorphism types of $K_s$ and $K_s^{2\leq n}$ which is straightforward but apparently not in the literature. Our interest in $K_s^{2\leq n}$ stems from the fact that $\mathcal{K}_s^{2\leq 3}(\mathbb{R}^3)$ can be identified with the Gromov-Hausdorff space of convex surfaces, see [Bel].

**Lemma 5.1.** $K_s$ is homeomorphic to $Q \times [0,1)$.

**Proof.** By [NQS79] the hyperspace $K$ is homeomorphic to a once-punctured Hilbert cube, which in turn is homeomorphic to $Q \times [0,1)$, see [BP75, p.118] or [Cha76, Theorem 25.1]. Since $\mathbb{R}^n \times K_s$ is homeomorphic to $K$, we conclude that $\mathcal{K}_s$ is infinite dimensional and locally compact. Since $\mathcal{K}$ is an AR that retracts onto $K_s$, the latter is an AR.

Properties of $s$ reviewed in Section 4 imply that $\mathcal{K}_s$ is homeomorphic to $s(\mathcal{K}_s)$ which is a closed convex subset of $C(S^{n-1})$. Now [BRZ96, Theorem 5.1.3] classifies closed convex subsets in linear metric spaces that are infinite-dimensional locally compact absolute retracts as spaces homeomorphic to $Q \times [0,1)$ and $Q \times \mathbb{R}^k$, $k \geq 0$. These spaces are pairwise non-homeomorphic, see [BP75, p.116, Theorem III.7.1] or [Cha76, Theorem 25.1]. Since $\mathbb{R}^n \times \mathcal{K}_s$ is homeomorphic to $K$, which in turn is homeomorphic to $Q \times [0,1)$, the hyperspace $\mathcal{K}_s$ cannot be homeomorphic to $Q \times \mathbb{R}^k$, and hence it must be homeomorphic to $Q \times [0,1)$. □

The homeomorphism $\mathcal{K} \to \mathbb{R}^n \times \mathcal{K}_s$ given by (4.2) is dimension preserving, so it restricts to the homeomorphism $K_s^{2\leq n} \to \mathbb{R}^n \times K_s^{2\leq n}$.

**Lemma 5.2.**

1. $K_s^{0\leq 1}$ is a Z-set in $\mathcal{K}_s$.

2. The one-point compactification of $K_s^{0\leq 1}$ is a Z-set in the one-point compactification of $\mathcal{K}_s$ which is is homeomorphic to $Q$. 

(3) $\mathcal{K}^{0 \leq 1}_s$ homeomorphic to $\mathbb{R}^n/\{\pm 1\}$, the open cone over $RP^{n-1}$.

(4) $\hat{\mathcal{K}}^{2 \leq n}_s$ is a contractible $Q$-manifold which is obtained from $Q$ be deleting a $Z$-set homeomorphic to the suspension over $RP^{n-1}$.

**Proof.** Note that $\mathcal{K}^{0 \leq 1}_s$ consists of line segments (of possibly zero length) with Steiner point at the origin. Hence $\mathcal{K}^{0 \leq 1}_s$ is homeomorphic to the quotient space $\mathbb{R}^n/\{\pm 1\}$. Also $\mathcal{K}^{0 \leq 1}_s$ is $Z$-set because any map $f: Q \to \mathcal{K}(\mathbb{R}^n)$ can be approximated by $f_\varepsilon$, where $f_\varepsilon(q)$ is the $\varepsilon$-neighborhood of $f(q)$, which clearly has dimension $n \geq 2$. As $\mathcal{K}^{0 \leq 1}_s$ is closed and noncompact in $\mathcal{K}_s$, the one-point compactification of $\mathcal{K}^{0 \leq 1}_s$ embeds into the one-point compactification of $\mathcal{K}_s$, into which $\hat{\mathcal{K}}^{2 \leq n}_s$ projects homeomorphically to $Q$ because the inclusion of a space into its one-point compactification is an open map. Since $\mathcal{K}_s$ is homeomorphic to the once-punctured copy of $Q$, its one-point compactification is homeomorphic to $Q$. Any point of $Q$ is a $Z$-set, so the one-point compactification takes any $Z$-set in $\mathcal{K}_s$ to a $Z$-set in $Q$ (because a map $Q \to Q$ can be first pushed off the added point, and then pushed off the $Z$-set inside $\mathcal{K}_s$). The one-point compactification of $\mathcal{K}^{0 \leq 1}_s$ is a $Z$-set in $Q$ homeomorphic to $SRP^{n-1}$, the suspension over the real projective $(n-1)$-space. Thus $\hat{\mathcal{K}}^{2 \leq n}_s$ is homeomorphic to the complement in $Q$ of a $Z$-set homeomorphic to $SRP^{n-1}$. □

**Remark 5.3.** Any two homeomorphic $Z$-sets in $Q$ are ambiently homeomorphic [Cha76 Theorem 11.1], and in particular, the part (4) of Lemma 5.2 uniquely describes $\hat{\mathcal{K}}^{2 \leq n}_s$ up to homeomorphism. Note that the one-point compactification of $\mathcal{K}^{0 \leq 1}_s$ can be moved by an ambient homeomorphism to a face of $Q$ because any closed subset of a face is clearly a $Z$-set.

Chapman showed [Cha76 Theorem 21.2] that homotopy equivalent $Q$-manifolds become homeomorphic after multiplying by $[0, 1)$, and hence for the products of $Q$-manifolds with $[0, 1)$ their homeomorphism and homotopy classifications coincide. A commonly used unpublished result of Wong, see [Cur78 p.275], says that a $Q$-manifold $M$ is homeomorphic to $M \times [0, 1)$ if and only if for each compact subset $A$ in $M$ there is a proper homotopy $f_t: M \to M$ such that $f_1$ is the identity and $f_t(M) \subset M \setminus A$ for some $t > 1$.

**Lemma 5.4.** If $M$ is $\hat{\mathcal{K}}^{k \leq l}_s$ or $\hat{\mathcal{K}}^{k \leq l}_s$, then $M$ is homeomorphic to $M \times [0, 1)$.

**Proof.** Suppose $M = \hat{\mathcal{K}}^{k \leq l}_s$ and $f_t(K) = tK$ where $K \in M$ and $t \geq 1$. The diameter of $tK$ is $t$ times the diameter of $K$. Since $o \notin M$, any $K \in M$ has positive diameter, which is bounded above on any compact subset $A$ of $M$. Thus $f_T(M)$ is disjoint from $A$ for some $T > 1$. The map $(K, t) \to tK$ is proper on $M \times [0, T]$. Hence by Wong’s result $M$ is homeomorphic to $M \times [0, 1)$. The same holds for $M = \hat{\mathcal{K}}^{k \leq l}_s$ because scaling commutes with the $O(n)$-action,
so that \( f_t \) descends to a homotopy of the orbit spaces that eventually pushes \( M \) off any compact subset.

Let \( K_s(\mathbb{B}^n) \) be the hyperspace consisting of sets in \( K_s \) contained in \( \mathbb{B}^n \). It is compact by the Blaschke selection theorem.

**Lemma 5.5.** \( K_s(\mathbb{B}^n) \) is homeomorphic to \( \mathbb{Q} \) and \( \mathbb{K}_s^{2 \leq n}(\mathbb{B}^n) \) is homeomorphic to \( \mathbb{Q} \times [0, 1) \).

**Proof.** Note that \( \mathcal{S} \) maps \( K_s(\mathbb{B}^n) \) onto a compact convex subset of \( C(\mathbb{S}^{n-1}) \) which is infinite-dimensional, which can be seen, e.g., by embedding \( \mathbb{Q} \) into \( K_s(\mathbb{B}^n) \) and then rescaling to embed it into \( K_s(\mathbb{B}^n) \). Any compact convex infinite-dimensional subset of a Banach space is homeomorphic to \( \mathbb{Q} \) [BP75, p.116, Theorem III.7.1], so \( K_s(\mathbb{B}^n) \) is homeomorphic to \( \mathbb{Q} \). Now \( K_s^{0 \leq 1}(\mathbb{B}^n) \) is homeomorphic to the cone over \( RP^{n-1} \), and in particular it is contractible, and hence has the shape of a point. It is also a \( Z \)-set in \( K_s(\mathbb{B}^n) \): contract the image of \( Q \to K_s(\mathbb{B}^n) \) by rescaling and then take \( \varepsilon \)-neighborhood to push it off \( K_s^{0 \leq 1}(\mathbb{B}^n) \). Two \( Z \)-sets in \( \mathbb{Q} \) the same shape if and only of their complement are homeomorphic [Cha76, Theorem 25.1], so \( K_s^{0 \leq 1}(\mathbb{B}^n) \) is homeomorphic to a once-punctured Hilbert cube, and hence to \( \mathbb{Q} \times [0, 1) \). □

### 6. On Hyperspaces Homeomorphic to \( \Sigma^\omega \)

Let us try to isolate the conditions on a hyperspace that would make it homeomorphic to \( \Sigma^\omega \). Throughout the section we fix \( \alpha \in I \) and let \( D \) denote an arbitrary hyperspace of \( \mathbb{R}^n \) satisfying \( \mathcal{B}_p \subset D \subset \mathcal{B}^{1, \alpha} \).

**Lemma 6.1.** If \( \mathcal{B}_p \subset X \subset K_s \), then \( X \) is an AR that is homotopy dense in \( K_s \). This applies to \( X = D \).

**Proof.** By Schneider’s regularization \( \mathcal{B}_p \) is dense in \( K_s \). The map \( \mathcal{S} \) homeomorphically takes \( K_s \) and \( \mathcal{B}_p \) to convex subsets of \( C(\mathbb{S}^{n-1}) \). By [BRZ96, Exercises 12 and 13 in Section 1.2] any dense convex subset of a set in a linear metric space is homotopy dense. Thus \( \mathcal{B}_p \) is homotopy dense in \( K_s \), and hence so is \( X \). Any homotopy dense subset of an AR is an AR [BRZ96, Proposition 1.2.1]. This applies to \( X \) because \( K_s \) is an AR; in fact, \( K_s \) is homeomorphic to a once-punctured Hilbert cube [NQS79]. □

**Lemma 6.2.** \( D \) has SDAP.

**Proof.** Lemma 6.1 shows homotopy density of \( \mathcal{B}^{1, \alpha} \) in \( K_s \). Also \( \mathcal{B}^{1, \alpha} \) is homotopy negligible in \( K_s \) because if we fix \( u \in \mathbb{S}^{n-1} \) and a map \( Q \to K_s \), then there is \( \varepsilon > 0 \) such that for any \( D \in f(Q) \) the result of \( (\varepsilon, u) \)-truncating of \( D \) is not \( C^1 \). Since \( K_s \) is locally compact, it has LCAP, and [BRZ96, Exercise
12h in Section 1.3] implies that any homotopy dense and homotopy negligible subset of an ANR with LCAP has SDAP. Thus $B^{1,\alpha}$ has SDAP. By [BRZ96, Exercise 4 in Section 1.3] every homotopy dense subset of an ANR with SDAP has SDAP, and this applies to $\mathcal{D}$. □

Lemma 6.3. If $\alpha > 0$, then $\mathcal{D}$ is $\sigma Z$, and it lies in a $\sigma Z$-subset of $\mathcal{K}_s$.

Proof. For a convex body $K$ in $\mathbb{R}^n$ consider its orthogonal projection $\tilde{K}$ to the hyperplane $\{x \in \mathbb{R}^n : x_n = 0\}$. Let $s(\tilde{K}) \in \text{Int}(\tilde{K})$ be the Steiner point of $\tilde{K}$, consider the largest ball about $s(\tilde{K})$ that is contained in $\tilde{K}$, and let $B_K$ be the ball half that radius about $s(\tilde{K})$. Consider the portion of $\partial K$ that is the graph of a convex function on $B_K$ and precompose the function with the map $B^{n-1} \rightarrow B_K$ that is the composition of a dilation followed by the translation by $s(\tilde{K})$. The result is a convex function by $f_K : B^n \rightarrow \mathbb{R}$. It is easy to see that the map $K \rightarrow C(B^n)$ is continuous.

For $m \in \omega$ let $\Lambda_m$ be the set of functions $f \in C(\mathbb{B}^n)$ such that for every $r \in (0, \alpha)$ the $C^{1,r}$ norm of $f$ is at most $m$. Equip $\Lambda_m$ with the $C^0$ topology. A version of the Arzelà-Ascoli theorem, see [GT01, Lemma 6.36], implies that $\Lambda_m$ is compact.

Let $\hat{Z}_m = \{K \in \mathcal{K}_s : f_K \in \Lambda_m\}$ and $Z_m = \hat{Z}_m \cap \mathcal{D}$. Thus $\hat{Z}_m, Z_m$ are closed in $\mathcal{K}_s, \mathcal{D}$, respectively.

The equality $\mathcal{D} = \bigcup_m Z_m$ follows from the facts that for each $\rho \in (0, r)$ any $C^{1,\alpha}$ function on $D$ has finite $C^{1,\rho}$ norm, and the Lipschitz constant of the identity map of $C^{1,\alpha}(D)$, where the domain and the co-domain are respectively given the $C^{1,\rho}$ and $C^{1,\rho}$ norms, is bounded above independently of $\rho$, see [GT01, Lemma 6.35].

To show that $Z_m$ is a $Z$-set we start from a continuous map $f : Q \rightarrow \mathcal{D}$ and try to push it off $Z_m$ inside $\mathcal{D}$. Let $n_K$ be the outward normal vector to the graph of $f_K$ at the point that projects to $s(\tilde{K})$. A basic property of $C^1$ convex bodies is that $n_K$ varies continuously with $K$. Apply $(\varepsilon, n_K)$-truncating to $f$, and then Schneider’s $\tau$-regularization.

Since $Q$ is compact for all sufficiently small $\varepsilon$ the result of $(\varepsilon, n_K)$-truncating of each body in $f(Q)$ is not $C^1$. For small $\tau$ the result of the above procedure will have very large $C^1$ norm, and hence it will not intersect $Z_m$. (If it did, then for some $r > 0$ the $C^{1,r}$ norm would be bounded uniformly in $\delta$, and the Arzelà-Ascoli theorem would give a subsequence converging in the $C^1$ norm, but the limit is not $C^1$).

To show that $\hat{Z}_m$ is a $Z$-set in $\mathcal{K}_s$ start from a continuous map $f : Q \rightarrow \mathcal{K}_s$, push it into $\mathcal{D}$ by Schneider’s regularization, and then push it off $Z_m$ inside $\mathcal{D}$ as above. Since $Z_m = \hat{Z}_m \cap \mathcal{D}$, the resulting map will miss $\hat{Z}_m$. □
Lemma 6.3 seems to be false for $\alpha = 0$ but we have no use for this assertion hence we will not attempt to justify it.

Remark 6.4. If $D$ in Lemma 6.3 is $O(n)$-invariant, then $I_m = \bigcap_{g \in O(n)} g(Z_m)$ is a closed subset of $Z_m$, and hence a $Z$-set in $D$. The facts that $O(n)$ is compact and the $C^{1,\alpha}$ norm of $f_K$ varies continuously under slight rotations of the graph of $f_K$ easily imply that $D = \bigcup_{m \in \omega} I_m$. Similarly, $\hat{I}_m = \bigcap_{g \in O(n)} g(\hat{Z}_m)$ is an $O(n)$-invariant $Z$-set in $K_\delta$, and $\{I_m\}_{m \in \omega}$ covers $D$ because $\hat{I}_m \supset I_m$.

Lemma 6.5. $B_p$ belongs to $M_2$.

Proof. Let $B_p^{\infty}$ denote the set $B_p$ with the $C^\infty$ topology. In this topology the Gaussian curvature of any convex body in $B_p^{\infty}$ varies continuously. Thus $B_p^{\infty}$ is precisely the subset of hypersurfaces of positive Gaussian curvature in the space of all compact $C^\infty$ hypersurfaces in $\mathbb{R}^n$ equipped with the $C^\infty$ topology. The latter space is Polish, see [GBV14]. Any open subset of a Polish space is Polish, hence $B_p^{\infty}$ is Polish.

For $\gamma \in \{0, \infty\}$ let $\Gamma^\gamma(S^{n-1})$ denote the set $C^\gamma(S^{n-1})$ equipped with the $C^\gamma$ topology. Let $s^{\infty}: B_p^{\infty} \to \Gamma^\infty(S^{n-1})$ be the map that associates to a convex body its support function, i.e., $s^{\infty} = s$ as maps of sets. Similarly to $s$, the map $s^{\infty}$ is a topological embedding because the support function for sets in $B_p$ equals the distance to $o$ from the support hyperplane, and both the tangent plane and the distance to $o$ vary in the $C^\infty$ topology as $o$ lies in the interior of each set in $B_p$. Since $B_p^{\infty}$ is Polish, its homeomorphic $s^{\infty}$-image is $G_\delta$. By [BB] Lemma 5.2 the identity map $\Gamma^\infty(S^{n-1}) \to \Gamma^0(S^{n-1})$ takes any $G_\delta$ subset to a space in $M_2$. Thus $B_p$ is in $M_2$. \hfill $\Box$

Lemma 6.6. $D$ is in $M_2$ if $D \setminus B_p$ lies in a subset of $D$ that belongs to $M_2$.

Proof. $B_p$ is in $M_2$ by Lemma 6.3. Hence $D$ is the union of two subsets that belong to $M_2$, the class of absolute $F_{\alpha \delta}$ sets, i.e., their homeomorphic images in any metric space are $F_{\alpha \delta}$, and in particular, this is true in $K_\delta$. The union of two $F_{\alpha \delta}$ subset is $F_{\alpha \delta}$, so $D$ is a $F_{\alpha \delta}$ in $K_\delta$, which is complete and therefore $D$ is in $M_2$ [BP75 Theorem 1.1, p.266]. \hfill $\Box$

Lemma 6.8 below depends on the following theorem proved in [BB Theorem 5.1 and Corollary 4.9].

Theorem 6.7. Let $N$ be a smooth manifold, possibly with boundary, and let $D \subset \text{Int} \; N$ be a smoothly embedded top-dimensional closed disk that is mapped via a coordinate chart to a Euclidean unit disk. Let $l \geq 0$ be an integer and let $\mathcal{D} : C^l(N) \to C(N)$ be a continuous linear map. Given $\eta \in \mathbb{R}$ suppose there exists $h_\bullet \in C^\infty(N)$ with $\mathcal{D}h_\bullet|_D > \eta$. Let $C^\bullet_\mathcal{D}$ denote the subspace of $C^l(N)$ of
functions $u$ such that $Du|_D \geq \eta$ and $u|_{N \setminus \text{Int}D} = h$. Let $C^u = C^d \cap C^u(N)$ and $C^\infty = C^d \cap C^\infty(N)$. Let $k \geq 1$. If $f: C^k \to X$ is a continuous injective map to a Hausdorff topological space $X$, then the subspace $f(C^\infty)$ of $X$ is $\mathcal{M}_2$-universal.

**Lemma 6.8.** $B_p$ is strongly $\mathcal{M}_2$-universal.

**Proof.** Let $\hat{B}_p$ denote the hyperspace of all positively curved $C^\infty$ convex bodies in $\mathbb{R}^n$. The map $C^2$ restricts to a homeomorphism $\hat{B}_p \to \mathbb{R}^n \times B_p$. Thus the products of $\hat{B}_p$ and $B_p$ with $\mathbb{R}$ are homeomorphic.

Lemmas 6.1 and 6.2 show that $B_p$ is an AR with SDAP, and hence so is $\hat{B}_p$. By [BRZ96] Theorem 3.2.18 if $X$ is an ANR with SDAP, then $X$ is strongly $\mathcal{M}_2$-universal if and only if $X \times \mathbb{R}$ is strongly $\mathcal{M}_2$-universal. Thus it suffices to show that $B_p$ is strongly $\mathcal{M}_2$-universal.

By [BRZ96] Proposition 5.3.5 a convex AR with SDAP in a linear metric space $L$ is strongly $\mathcal{M}_2$-universal if it contains an $\mathcal{M}_2$-universal subset that is closed in $L$. Set $L = C^\infty(S^{n-1})$ with the $C^0$ norm. Recall that $\mathcal{K}$ maps $\mathcal{K}$ homeomorphically onto a convex subset of $L$. By [Cho12] Proposition 5.1 $\mathcal{s}(\hat{B}_p)$ is convex in $L$, see also [BJ17] Theorem 1.1.

To find an $\mathcal{M}_2$-universal subset of $\hat{B}_p$ we apply Theorem 6.7 to $N = [0,1]$, $D = [\frac{1}{3}, \frac{1}{2}]$, $k = l = 2$, $h(r) = r^2$, $\eta = 1$, and $D(h) = h''$. Thus

$$C^2 = \left\{ u \in C^2(\mathbb{R}) : u'' \geq 1 \text{ and } u(r) = r^2 \text{ for all } r \notin \left[\frac{1}{3}, \frac{1}{2}\right] \right\}$$

and $C^\infty = C^d \cap C^\infty(\mathbb{R})$. Define a map $f: C^2 \to \hat{B}_p$ as follows. Fix $A \in \hat{B}_p$ that is invariant under $SO(n-1)$ rotations about the $x_n$-axis and such that the portion of $\partial A$ satisfying $x_n \in [0,1]$ equals the paraboloid

$$\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n = r(x)^2 \text{ and } x_n \in [0,1]\}$$

where $r(x)$ is the distance from $x$ to the $x_n$-axis in $\mathbb{R}^n$. Let $f$ send an element $q \in C^d \cdot B_n$ to the convex body $B_n + A_q$ where $A_q$ obtained from $A$ by replacing $x_n = r^2$ with $x_n = q(r)$ in the above paraboloid portion of $\partial A$. One checks that $A_q$ has positive curvature, so that both $A_q$ and $B_n + A_q$ lie in $\hat{B}_p$. It is easy to see that $\mathcal{s}(\hat{B}_p)$ is strongly $\mathcal{M}_2$-universal.

It remains to show that $(\mathcal{s} \circ f)(C^\infty)$ is closed in $L$. Note that $f(C^\infty)$ lies in a compact subset of $\mathcal{K}$, which is therefore mapped by $\mathcal{s}$ homeomorphically onto its compact image. Thus any limit point of $(\mathcal{s} \circ f)(C^\infty)$ lies in $\mathcal{s}(\mathcal{K})$ and it is enough to show that $f(C^\infty)$ is closed in the hyperspace consisting of convex bodies in $\mathbb{R}^n$ with $C^\infty$ support functions. Fix a sequence $B_n + A_{q_n}$ in $f(C^\infty)$
that converges to a convex body with $C^\infty$ support function. By construction
the limit is of the form $B^n + A_\kappa$ where $q_m \in C^\infty$ converge in the uniform $C^0$
topology to a (necessarily convex) function $\kappa$. By assumption $s(B^n + A_\kappa)$ is
$C^\infty$, and hence so is $s(A_\kappa)$. Now Lemma 4.4 implies that $B^n + A_\kappa \in \hat{B}_p$. Hence
$A_\kappa \in \hat{B}_p \quad[BJ17, Theorem 1.1]$ and therefore $\kappa \in C^\infty$. Since
$q_m(r) = r^2$ for $r \not\in [\frac{1}{3}, \frac{1}{2}]$, the same is true for $\kappa$. Set $p(r) = \frac{r^2}{2}$. Since $q_m'' \geq 1$, each function
$q_m - p$ is convex, and hence so is $\kappa - p = \lim_m q_m - p$. Since $\kappa$ is $C^\infty$, we get
$(\kappa - p)'' \geq 0$ so $\kappa'' \geq 1$, and hence $\kappa \in C^\infty$ as claimed. □

Lemma 6.9. If $B_p$ is $G_\delta$ in $D$, then $D$ is strongly $M_2$-universal.

Proof. The [BRZ96 Enlarging Theorem 3.1.5] implies that an ANR with SDAP is strongly $M_2$-universal if and only if it contains a strongly $M_2$-universal homotopy dense $G_\delta$ subset. By assumption $B_p$ is $G_\delta$ in $D$. The space $B_p$ is strongly $M_2$-universal due to Lemma 6.8. By Lemmas 6.1–6.2 the space $D$ is an ANR with SDAP and $B_p$ is homotopy dense in $X$. □

Theorem 6.10. If $\alpha > 0$ and $D \setminus B_p$ is $\sigma$-compact, then $D$ is homeomorphic to $\Sigma^\omega$, and in particular, $B_p$ is homeomorphic to $\Sigma^\omega$. 

Proof. Note that any $\sigma$-compact subspace is both $F_\sigma$ and in $M_2$. By the above lemmas $D$ is an AR with SDAP, $\sigma Z$, strongly $M_2$-universal, and is in $M_2$, so that $D$ is $M_2$-absorbing, and the only such space is $\Sigma^\omega$. □

Can $\sigma$-compactness of $D \setminus B_p$ in Theorem 6.10 can be replaced by a weaker condition that holds for examples of interest such as $B^\infty$? The following result illustrates what could go wrong.

Theorem 6.11. There is a hyperspace $D$ with $B_p \subset D \subset B^{1,\alpha}$ such that $D \setminus B_p$ embeds into the Cantor set, $B_p$ is open in $D$, and $D$ is not a topologically homogeneous, and in particular, not a $\Sigma^\omega$-manifold.

Proof. For any uncountable Polish space, such as the Cantor set, the Borel hierarchy of its subsets does not stabilize [Kec95, Theorem 22.4], so in particular, it contains a subset not in $M_2$. Use Lemma 6.12 below to embed it onto a subset $\Lambda$ of $B^{1,\alpha} \setminus B_p$ that is closed in $\Lambda \cup B_p$. Since $M_2$ is closed-hereditary, $\Lambda \cup B_p$ is not in $M_2$ and hence not a $\Sigma^\omega$-manifold. If $\Lambda \cup B_p$ were topologically homogeneous, then it would be a $\Sigma^\omega$-manifold because the $\Sigma^\omega$-manifold $B_p$ is open in $\Lambda \cup B_p$. □

The earlier results in this section imply that $D$ in Theorem 6.11 is a strongly $M_2$-universal ANR with SDAP which is also $\sigma Z$ if $\alpha > 0$.

Lemma 6.12. Any space is homeomorphic to a subset $\Lambda$ of $B^{1,\alpha} \setminus B_p$ such that $B_p$ is open in $\Lambda \cup B_p$. 

Proof. Fix $K \in \mathcal{B}^{1,\alpha}$ whose support function $h_K$ is not $C^\infty$. For $t \in \mathbb{I}$ consider the map $f_t : \mathcal{B}_p \to \mathcal{B}^{1,\alpha}$ given by $f_t(D) = tK + (1 - t)D$. By [KP91] the image of $f_t$ is in $\mathcal{B}^{1,\alpha}$, but is it not in $\mathcal{B}_p$ for $t \neq 0$ because if $h_{f_t(D)} \in \mathcal{B}_p$ then $h_K$ is a linear combination of $C^\infty$ functions. For each $t \neq 0$ the map $f_t$ is a topological embedding. (Indeed, the map is injective as we can cancel $tK$ [Sch14, p.48], and moreover if $f_t(D_1), f_t(D_2)$ are close then so are their support functions, and after subtracting $tK$ we conclude that the support functions of $(1 - t)D_1, (1 - t)D_2$ are close, and hence so are $D_1, D_2$. Since $\mathcal{B}_p$ is homeomorphic to $\Sigma_\omega$ the space $\mathcal{B}^{1,\alpha} \setminus \mathcal{B}_p$ contains a topological copy of $Q$ which must be closed in $Q \cup \mathcal{B}_p$ since $Q$ is compact. Any (separable metric) space embeds into $Q$. If $\Lambda$ is the image of such an embedding into the above copy of $Q$, then $\Lambda$ is closed in $\Lambda \cup \mathcal{B}_p$. □

7. HOMEOMORPHISMS OF PAIRS

In this section we finish the proof of Theorem 1.1 by making the homeomorphisms in Sections 5–6 compatible. This is standard but somewhat technical.

If $Y$ is a subspace of $X$, then $(X, Y)$ is a pair. A pair $(M, X)$ is $(\mathcal{M}_0, \mathcal{M}_2)$-absorbing if $(M, X)$ is strongly $(\mathcal{M}_0, \mathcal{M}_2)$-universal and $M$ contains a sequence of compact subsets $(K_l)_{l \in \omega}$ such that each $K_l \cap X$ is in $\mathcal{M}_2$ and $\bigcup_{l \in \omega} K_l$ is a $\sigma Z$-set that contains $X$. (A definition of a strongly $(\mathcal{M}_0, \mathcal{M}_2)$-universal pair can found in [BRZ96, Section 1.7] and is not essential for what follows).

Lemma 7.1. If $U$ is an open subset of a $Q$-manifold $M$, and $X$ is a homotopy dense subset of $M$ such that $X$ is a $\Sigma^\omega$-manifold and $X$ lies in a $\sigma Z$-subset of $M$, then $(U, U \cap X)$ is $(\mathcal{M}_0, \mathcal{M}_2)$-absorbing.

Proof. Any $\Sigma^\omega$-manifold is strongly $\mathcal{M}_2$-universal and any $Q$-manifold is a Polish ANR, hence by [BRZ96] Theorem 3.1.3 the pair $(U, U \cap X)$ is strongly $(\mathcal{M}_0, \mathcal{M}_2)$-universal. Since $M$ is $\sigma$-compact, so is $U$ and any $Z$-set in $M$. Thus $U \cap X$ lies in a countable union of compact $Z$-sets in $M$, and clearly the intersection of $U \cap X$ with any compact subset is in $\mathcal{M}_2$. □

Lemma 7.2. The following pairs $(M, X)$ are $(\mathcal{M}_0, \mathcal{M}_2)$-absorbing:

1. $M = V$ and $V \cap \Sigma^\omega$ where $V$ is any open subset of $Q^\omega$,
2. $M = K_s$ and $X \supset \mathcal{B}_p$ such that $X \setminus \mathcal{B}_p$ is $\sigma$-compact.

Proof. Let us verify the assumptions of Lemma 7.1

1. Since $\Sigma^\omega$ is convex and dense in $Q^\omega$, it is also homotopy dense in $Q^\omega$, see [BRZ96] Exercise 13 in 1.2]. Hence $X$ is homotopy dense in $M$. To show
that $X$ lies in a $\sigma Z$-subset of $M$ let $Q_k \subset Q$ be the set of sequences $(t_i)_{i \in \omega}$ with $|t_i| \leq 1 - \frac{1}{k}$ and

$$N_k = \{ (q_i) \in Q^\omega : q_i = 0 \text{ for } i \neq 1 \text{ and } q_1 \in Q_k \}.$$ 

Since $Q_k$ is a compact subset of the pseudo-interior of $Q$, it is a $Z$-set in $Q$. Thus $N_k$ is a $Z$-set in $Q^\omega$ and hence $M_k = V \cap N_k$ is a $Z$-set in $V$. Finally, $\Sigma = \bigcup_{k \in \omega} Q_k$ implies $\Sigma^\omega \subset \bigcup_{k \in \omega} M_k$.

(2) $B_p$ is homotopy dense in $K_s$ by Lemma 4.5, and contained in a $\sigma Z$ subset of $K_s$ by Lemma 6.3.

The following uniqueness theorem is immediate from [BRZ96, Theorem 1.7.7].

**Lemma 7.3.** For $i \in \{1, 2\}$ let $(M_i, X_i)$ be a $(M_0, M_2)$-absorbing pair and $B_i \subset M_i$ be a closed subset. Then for any homeomorphism $f: M_1 \to M_2$ with $f(B_1) = B_2$ and $f(B_1 \cap X_1) = B_2 \cap X_2$ there exists a homeomorphism $h: M_1 \to M_2$ such that $h(X_1) = X_2$ and $h = f$ on $B_1$.

**Proof of Theorem** Let $K_s$ be homeomorphic to $Q^\omega \setminus \{\ast\}$ and $K_s^{0 \leq 1}$ is a $Z$-set in $K_s$ that is disjoint from $B_p$. Fix an arbitrary $Z$-embedding $K_s^{0 \leq 1} \to Q^\omega \setminus \{\ast\}$ whose image $B$ is disjoint from $\Sigma^\omega$, e.g., we can fix a factor in the product $Q^\omega$ and pick $B$ in the pseudo-boundary of that factor. The unknotting of $Z$-sets in $Q$-manifolds [BRZ96, Theorem 1.1.25] and Lemma 7.3 give a homeomorphism $K_s \to Q^\omega \setminus \{\ast\}$ taking $K_s^{0 \leq 1}$ to $B$, and $B_p$ to $\Sigma^\omega$. □

**Remark 7.4.** A $(Q^\omega, \Sigma^\omega)$-manifold is a pair $(M, X)$ such that any point of $M$ has a neighborhood $U$ that admits an open embedding $h: U \to Q^\omega$ with $h(U \cap X) = h(U) \cap \Sigma^\omega$. Lemmas 7.1, 7.3 imply that any $(M, X)$ as in Lemma 7.1 is a $(Q^\omega, \Sigma^\omega)$-manifold. In fact, a pair $(M, X)$ is a $(Q^\omega, \Sigma^\omega)$-manifold if and only if $M$ is a $Q$-manifold and $(M, X)$ is $(M_0, M_2)$-absorbing, see [BRZ96, Exercise 12 of Section 1.7].

### 8. Quotients of Hyperspaces

In this section we prove Theorem 1.2 and related results.

**Lemma 8.1.** Let $H \leq O(n)$ be a closed subgroup, $X$ be an $H$-invariant subspace of $K_s$ that contains $B_p$, and $X/H$ be the quotient space. Then

1. $X/H$ is a separable metrizable AR.
2. $X$ is Polish if and only if so is $X/H$.
3. $X$ is locally compact if and only if so is $X/H$.
4. If $X = K_s^{l \leq n}$ for some $l \geq 0$, then $X/H$ is Polish and locally compact.
5. If $X$ is homeomorphic to $\Sigma^\omega$, then...
(5a) any $\sigma$-compact subset of $X/H$ has empty interior,
(5b) $X/H$ is neither Polish nor locally compact.

Proof. (1) Convex hulls of finite sets with rational coordinates for $m$ a dense countable subset of $\mathcal{K}_s$. Separability and metrizability of a space is inherited by its subsets and $H$-quotients [Pal60] Proposition 1.1.12 so that $X/H$ enjoys these properties.

Antonyan [Ant05] Theorem 4.5] showed that $\mathcal{K}$ has what he called an $H$-convex structure. The structure is inherited by any convex $H$-invariant subset, such as $\mathcal{K}_s$. The unit ball $\mathbb{B}^n$ is a fixed point for the $H$-action on $\mathcal{K}_s$, hence [Ant05] Theorem 3.3] shows that $\mathcal{K}_s$ is a $H$-AE. Hence $\mathcal{K}_s$ is a $H$-AR because the identity map of any closed $H$-invariant subset extends to an $H$-retraction. Finally [Ant11] Theorem 1.1] implies that $\mathcal{K}_s/H$ is an AR.

Lemma 4.5 shows that $X/H$ is a homotopy dense subset $\mathcal{K}_s/H$, an AR, which makes $X/H$ an AR, see [BRZ96, Exercise 12 in Section 1.3].

(2) The orbit map $X \to X/H$ is a closed continuous surjection with compact preimages. For any such map the domain is Polish if and only if so is the co-domain, see the references mentioned before Theorem 4.3.27 of [Eng89].

(3) The orbit map is proper and open, so the image and the preimage of a compact neighborhood is a compact neighborhood.

(4) $\mathcal{K}_s^{\leq n}$ are homeomorphic to open subsets of $\mathcal{K}_s$ which is a $Q$-manifold, and hence so is $\mathcal{K}_s^{\leq n}$. Any $Q$-manifold is Polish and locally compact so (2)-(3) applies.

(5a) If $X/H$ contains a $\sigma$-compact subset with nonempty interior then so does $X$ because the orbit map $X \to X/H$ is proper and continuous. If $V$ is an open set in the interior of a $\sigma$-compact subset of $\Sigma^\omega$, then separability of $\Sigma^\omega$ implies that it is covered by countably many translates of $V$ and hence $\Sigma^\omega$ is $\sigma$-compact. But the product of infinitely many $\sigma$-compact noncompact spaces is never $\sigma$-compact.

(5b) That $X/H$ is not Polish follows from (2) and the remark that $\Sigma^\omega$ is $\sigma Z$ and hence not Polish by the Baire category theorem because any $Z$-set is nowhere dense. Also any locally compact (separable) space is $\sigma$-compact so we are done by (5a).

Given $X \subset \mathcal{K}_s$ let $\hat{X}$ denotes the set of points of $X$ whose isotropy subgroup in $O(n)$ is trivial. If $X$ is also $O(n)$-invariant, $\hat{X}$ denotes the orbit space $X/O(n)$. By the slice theorem $\hat{X}$ is open in $X$ [Bre72 Corollary II.5.5] and the orbit map $\hat{X} \to \hat{X}$ is a locally trivial principal $O(n)$-bundle [Bre72 Corollary II.5.8]. The classifying space for such bundles is $BO(n)$, the Grassmanian of $n$-planes in $\mathbb{R}^\omega$. 

□
Lemma 8.2. If $\hat{B}_p \subset X \subset \mathcal{K}_s$, then $X$ is homotopy dense in $\mathcal{K}_s$.

Proof. It suffices to show that any map $h : Q \to \mathcal{K}_s$ can be approximated by a map with image in $\hat{B}_p$. Since $B_p$ is homotopy dense in $\mathcal{K}_s$ we can assume that $h(Q) \subset B_p$.

Let $\{e_i\}_{1 \leq i \leq n}$ be the standard basis in $\mathbb{R}^n$. For $D \in B_p$ let $e_i(D)$ be the unique point of $\partial D$ with outward normal vector $e_i$. That $D \in B_p$ ensures continuity of the map $D \to e_i(D)$.

Pick $\varepsilon_1 > 0$ so small that if $D(\varepsilon_1)$ is the result of the $(\varepsilon_1, e_1)$-truncating of $D \in h(Q)$, then

$$D \setminus D(\varepsilon_1) \quad \text{and} \quad \bigcup_{1 < l \leq n} \{e_l(D)\}$$

have disjoint closures. The $(\varepsilon_1, e_1)$-truncating turns $D$ into a convex body $D(\varepsilon_1)$ whose flat face $F_1(D)$ has normal vector $e_1$. Continuing inductively pick $\varepsilon_k > 0$ so that if $D(\varepsilon_k)$ is the result of the $(\varepsilon_k, e_k)$-truncating of $D(\varepsilon_{k-1})$ then the three sets

$$D(\varepsilon_{k-1}) \setminus D(\varepsilon_k) \quad \bigcup_{1 \leq j < k} F_j(D) \quad \bigcup_{k < l \leq n} \{e_l(D)\}$$

have disjoint closures, and the diameter of the newly formed flat face $F_k(D)$ of $D(\varepsilon_k)$ is smaller than the minimum over $D \in h(Q)$ of the diameters of $F_{k-1}(D)$.

Compactness of $h(Q)$ implies that any small enough $\varepsilon_k$ works for all $D \in h(Q)$ at once. Set $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$. Let $h_\varepsilon : Q \to \mathcal{K}_s$ be the composition of $h$ with the map $D \to D(\varepsilon_n)$, where $D \in h(Q)$. Note that $h_\varepsilon$ converges to $h$ as $\varepsilon \to 0$. Any $\phi \in O(n)$ that preserves $D(\varepsilon_n)$ must take faces to faces, and since the faces all have different diameters $\phi$ must preserve each face and hence its normal vector, so that $\phi$ is the identity. Thus the image of $h_\varepsilon$ lies in $\hat{\mathcal{K}}_s$ which is an open subset of $\mathcal{K}_s$. Finally, using homotopy density of $B_p$ we can approximate $h_\varepsilon$ by a map with image in $B_p \cap \hat{\mathcal{K}}_s = \hat{B}_p$. \qed

Lemma 8.3. If $\hat{B}_p \subset X \subset \mathcal{K}_s$ and $\hat{X}$ is $O(n)$-invariant, then $\hat{X}$ is contractible and $\hat{X}$ is homotopy equivalent to $BO(n)$.

Proof. Lemma 8.2 gives a homotopy equivalence of $\hat{X}$ and $\mathcal{K}_s$, so that $\hat{X}$ is contractible. Hence $\hat{X}$ is the classifying space for the principal $O(n)$-bundles, and thus it is homotopy equivalent to $BO(n)$, see [Dol63, Section 7]. \qed

Remark 8.4. In Lemma 8.3 if $X$ is homeomorphic to $\Sigma^\omega$, then so is $\hat{X}$ because $\hat{X}$ is open in $X$ and homotopy equivalent $\Sigma^\omega$-manifolds are homeomorphic.
Similarly, if $X$ is $Q$-manifold, then so is $\hat{X}$, even though they might be non-homeomorphic.

**Theorem 8.5.** Suppose $\hat{B}_p \subset X \subset B^{1,\alpha}$ with $\alpha > 0$ and $\hat{X}$ is $O(n)$-invariant. If $\hat{X}$ is a $\Sigma^\omega$-manifold, then so is $\hat{X}$.

**Proof.** By local triviality of the bundle $\hat{X} \to \hat{X}$ each point of $\hat{X}$ has a neighborhood that $U$ such that $U \times O(n)$ is a $\Sigma^\omega$-manifold. Hence $U \times \mathbb{R}^d$ is a $\Sigma^\omega$-manifold where $d = \dim O(n)$.

Let us show that $U$ is a $\Sigma^\omega$-manifold. Note that $U$ is an ANR as a retract of the ANR $U \times \mathbb{R}^d$. Also $U$ satisfies SDAP by [BRZ96, Exercise 9 in section 1.3]. Strong $\mathcal{M}_2$-universality of $U$ comes from [BRZ96, Theorem 3.2.18], namely, since $U \times \mathbb{R}^d$ is strongly $\mathcal{M}_2$-universal, so is $U \times \mathbb{R}^d \times \mathbb{R}$, and hence $U$. Since $\mathcal{M}_2$ is closed-hereditary and $U \times \mathbb{R}^d$ is in $\mathcal{M}_2$, so is $U$.

**Remark 6.4** shows that $X$ is a countable union of $O(n)$-invariant $\Sigma$-sets, and hence so is $\hat{X}$ because $\hat{X}$ is $O(n)$-invariant and open, while the intersection of a $\Sigma$-set with any open is a $\Sigma$-set in that open set [BRZ96, Corollary 1.4.5].

The bundle projection $\pi: \hat{X} \to \hat{X}$ maps any $O(n)$-invariant $\Sigma$-set $\hat{Z}$ to a $\Sigma$-set $\pi(\hat{Z})$. Indeed, since $Q$ is contractible, any map $Q \to \hat{X}$ lifts to a map $Q \to \hat{X}$ which since $\hat{X}$ is open can be approximated by a map $h: Q \to \hat{X}$ that misses $\hat{Z}$. Now $\pi \circ h$ misses $\pi(\hat{Z})$ due to $O(n)$-invariance of $\hat{Z}$.

Therefore $\hat{X}$ is $\sigma Z$, and hence so is $U$, again by [BRZ96, Corollary 1.4.5]. Thus $U$ is $\mathcal{M}_2$-absorbing and hence is a $\Sigma^\omega$-manifold. Thus $\hat{X}$ is a $\Sigma^\omega$-manifold. □

**Theorem 8.6.** Suppose $\hat{B}_p \subset X \subset K_s$ and $\hat{X}$ is $O(n)$-invariant. If $\hat{X}$ is a $Q$-manifold, then so is $\hat{X}$.

**Proof.** As in the proof of Lemma 8.4 we see that $\hat{X}$ is locally compact. As in the proof of Theorem 8.5 we conclude that any point of $\hat{X}$ has an ANR neighborhood, hence $\hat{X}$ is an ANR. By Toruńczyk characterization of $Q$-manifolds among locally compact ANRs [Lor80, Theorem 1] it remains to check that any map $f: I^k \to \hat{X}$ can be approximated by a map whose image is a $\Sigma$-set. By Lemma 4.5 $B_p$ is homotopy dense in $K_s$, so we approximate $f$ by a map with image in $B_p$, which actually lies in $\hat{B}_p = B_p \cap \hat{X}$ because $\hat{X}$ is open. Since $\hat{X}$ is a $\Sigma^\omega$-manifold, any compactum in $\hat{X}$ is a $\Sigma$-set [BRZ96, Proposition 1.4.9]. □

**Theorem 8.7.** Suppose $\hat{B}_p \subset X \subset B^{1,\alpha}$ with $\alpha > 0$ and $\hat{X}$ is $O(n)$-invariant. If $\hat{X}$ is a $\Sigma^\omega$-manifold, then each point of $\hat{X}$ has a neighborhood $U$ in $\hat{K}_s$ such that there is an open embedding $h: U \to Q^\omega$ with $h(U \cap X) = h(U) \cap \Sigma^\omega$. 
Proof. As in the proof of Theorem \[8.5\] we use Remark \[6.4\] to show that \( \hat{\mathcal{K}} \) lies in the \( \sigma\mathcal{Z} \)-subset of \( \hat{\mathcal{K}} \). Now Remark \[7.4\] and Theorems \[8.5\] \[8.6\] imply that \((\hat{\mathcal{K}}, \hat{X})\) is a \((Q^\omega, \Sigma^\omega)\)-manifold. \(\square\)

**Theorem 8.8.** Let \( L \) be the product of \([0, 1)\) and any locally finite simplicial complex homotopy equivalent to \( BO(n) \). If \( \hat{\mathcal{B}} \subset X \subset \mathcal{B}^{1,\alpha} \) with \( \alpha > 0 \) such that \( \hat{X} \) is an \( O(n) \)-invariant \( \Sigma^\omega \)-manifold, then there is a homeomorphism \( h: \hat{\mathcal{K}} \to L \times Q^\omega \) that takes \( \hat{X} \) onto \( L \times \Sigma^\omega \).

**Proof.** Theorem \[8.6\] says that \( \hat{\mathcal{K}} \) is a \( Q \)-manifold, which by Lemma \[5.4\] is homeomorphic to its product with \([0, 1)\). The product of any locally finite simplicial complex with \( Q \) is a \( Q \)-manifold [BRZ96, Theorem 1.1.24]. Lemma \[8.3\] gives a homotopy equivalence of \( \hat{\mathcal{K}} \) and \( L \times Q^\omega \), and since both spaces are products of \([0, 1)\) and a \( Q \)-manifold, they are homeomorphic [Cha76, Theorem 23.1]. Also \( \hat{X}, L \times \Sigma^\omega \) are \( \Sigma^\omega \)-manifolds, see Theorem \[8.5\] and Section \[3\]. The pair \((\hat{\mathcal{K}}, \hat{X})\) is a \((Q^\omega, \Sigma^\omega)\)-manifolds by Theorem \[8.7\], and hence is \((\mathcal{M}_0, \mathcal{M}_2)\)-absorbing, see Remark \[7.4\]. As a \( \sigma \)-compact space \( L \) lies in \( \mathcal{M}_2 \), and then the proof of Lemma \[7.2\](1) shows that \((L \times Q^\omega, L \times \Sigma^\omega)\) is \((\mathcal{M}_0, \mathcal{M}_2)\)-absorbing. The claim now follows from Lemma \[7.3\]. \(\square\)

**Remark 8.9.** To make \( L \) explicit start with the standard CW structure on \( BO(n) \) given by Schubert cells, use mapping telescope to replace it by a homotopy equivalent locally finite CW complex, and triangulate the result.

9. **Deforming disks modulo congruence**

The results of Section \[8\] say little about the local structure of \( \mathcal{K} \) near the points of \( \mathcal{K} \setminus \hat{\mathcal{K}} \). For example, one wants to have better understanding of the standard stratification of \( \mathcal{K} \) by orbit type. As was mentioned in the introduction the stratum \( \hat{\mathcal{K}} \) is not homotopy dense in \( \mathcal{K} \) because they have different fundamental groups, so a singular 2-disk in \( \mathcal{K} \) cannot always be pushed into \( \hat{\mathcal{K}} \). This is possible for any path in \( \mathcal{K} \) because it lifts into \( \mathcal{K} \) [Bre72, Theorem II.6.2] and then Lemma \[8.2\] applies. The lemma below shows that under mild assumptions on a point \( x \in \mathcal{K} \setminus \hat{\mathcal{K}} \) there is a small singular disk in \( \mathcal{K} \) near \( x \) that cannot be pushed into \( \hat{\mathcal{K}} \). Thus the orbit space analog of Lemma \[8.2\] fails locally.

Let \( \pi_j \) denote the \( j \)th homotopy group, let \( \pi: \mathcal{K} \to \mathcal{K} \) be orbit map, \( I_x \) be the isotropy subgroup of \( x \) in \( O(n) \), and \( x = \pi(x) \).

**Lemma 9.1.** Let \( B \subset X \subset \mathcal{K} \) where \( X \) is \( O(n) \)-invariant. If \( x \in X \setminus \hat{X} \) and the inclusion \( I_x \to O(n) \) is \( \pi_k \)-nonzero for some \( k \geq 0 \), then any neighborhood
of $x \in X$ contains a neighborhood $U$ such that the inclusion $\hat{X} \cap U \to U$ is not $\pi_{k+1}$-injective, and in particular, $\pi_k(U, \hat{X} \cap U) \neq 0$.

**Proof.** Inside any neighborhood of $x$ in $X$ one can find a neighborhood of $x$ in the form $U = X \cap V$ where $V$ is $I_x$-invariant, convex and open in $\mathcal{K}_s$ (e.g., let $V$ be a sufficiently small ball about $x$ in the Hausdorff metric). The orbit $I_x(y)$ of any $y \in U \cap \hat{X}$ lies in $U \cap \hat{X}$. Let $f : S^k \to I_x(y)$ be a map that is $\pi_k$-nonzero when composed with the inclusion $I_x(y) \to O(n)$. Since $V$ is convex it contains the family $f_t : S^k \to V$ of singular $k$-spheres given by $f_t(v) = tf(v) + (1 - t)x$, $t \in \mathbb{I}$.

Since $\hat{X}$ is homotopy dense in $\mathcal{K}_s$ we can slightly deform $f_t$ to $h_t : S^k \to \hat{X} \cap V$ with $h_1 = f$. Define $h : B^{k+1} \to \hat{X} \cap V$ by $h(tv) = h_t(v)$. Since $f$ is $\pi_k$-nonzero in the fiber of the bundle $\pi : \hat{K}_s \to \mathcal{K}_s$ the singular $(k + 1)$-sphere map $\pi \circ h$ in $\hat{X} \cap U$ is not null-homotopic in $\hat{K}_s$.

Now $\pi \circ h$ gives a null-homotopy of the singular $(k + 1)$-sphere $\pi \circ h$ inside $V$, and hence in $U = X \cap V$ by Schneider’s regularization. □

**Remark 9.2.** The closed subgroups $H$ for which the inclusion $H \to O(n)$ is zero on all homotopy groups can be completely understood, see [Bry].

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Igor Belegradek, School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160

E-mail address: ib@math.gatech.edu