Application of DRBEM for 2D sine-Gordon equation

Nagehan Alsoy-Akgün

Department of Mathematics, Van Yüzüncü Yıl University, Van, Turkey

ABSTRACT
This research paper introduces an application of dual reciprocity boundary element method (DRBEM) for the solution of sine-Gordon equation (SGE) in two-space dimension. Initially, the time derivatives are expanded using central difference schemes. After inserting the finite difference approximations into the governing equation, the pattern of the modified Helmholtz equation is obtained. The fundamental solution of modified Helmholtz equation is employed in the integral equation formulation. The inhomogeneous terms of the equation cause a domain integral in the integral equation formulation which leads to loss of advantage of the method. The DRBEM provides to transform the domain integral into the boundary integral by approximating the inhomogeneous term of the equation with thin plate spline ($r^2 \ln r$). First, code validation of the procedure is done using a test problem and then proposed method is applied for several cases involving line and ring solitons to display its capacity to treat the problem. Presented numerical results are observed to be in good agreement with other numerical results available in the literature.

1. Introduction
The soliton model can be used for describing for many physical phenomena. A soliton is a solitary wave and it behaves like a particle such that whose shape, amplitude and velocity are not changed after interacting or colliding with another soliton [1]. There are many causes of soliton formation. One of them is the balance between the dispersion and nonlinearity [1,2]. One of these models is SGE, which is a nonlinear partial differential equation with a nonlinear sine source term and its solution has the soliton like structure [3]. This equation has great importance in physical phenomena and some important applications include numerical forecasting, optical fibres, nonlinear optics, quantum field theory, etc. [4–12]. Particularly, two-dimensional SGE can be used for identifying Josephson-junction model in quantum tunnelling [13].

Time-dependent nonlinear SGE in two space variables is modelled by

$$z_{tt} + \nu z_t = \nabla^2 z - \chi(x_1, x_2) \sin(z), \quad (x_1, x_2) \in D,$$

$$t \geq 0,$$ (1)

where $z = z(x_1, x_2, t)$ and $D = \{ (x_1, x_2) : |x_1| \leq L_1, |x_2| \leq L_2 \}$. Initial conditions associated with Equation (1) will be assumed to be of the form

$$z(x_1, x_2, 0) = w_1(x_1, x_2), \quad (x_1, x_2) \in D,$$
$$z_t(x_1, x_2, 0) = w_2(x_1, x_2), \quad (x_1, x_2) \in D,$$ (2)

and Neumann boundary conditions

$$\frac{\partial z}{\partial n}(x_1, x_2, t) = 0, \quad (x_1, x_2) \in \Gamma, \quad t \geq 0.$$ (3)

Here, the function $\chi(x_1, x_2)$ named as Josephson current density, the parameter $\nu$ is a positive real number and it is known as dissipative term $w_1(x_1, x_2)$ and $w_2(x_1, x_2)$ functions are wave modes or kinks and velocity, respectively. $\Gamma$ is the boundary of the domain $D$ and $n$ is the unit outward normal to the boundary. When $\nu = 0$, Equation (1) is reduced to the undamped sine-Gordon equation, while $\nu > 0$ reduces Equation (1) to the damped sine-Gordon equation.

In the related literature, it is seen that both numerical and exact solutions can be obtained for two-dimensional SG equation. Hirota method was used to obtain the exact solution of two-dimensional SGE in [14] and to obtain soliton solutions for more general nonlinear equation in [15]. Another study for the exact solution of two- and three-dimensional time-dependent SGE was presented using Backlund method in [16]. Particular solutions of the SGE were obtained using Lamb’s method in [17–19]. To date, although many exact solutions of SGE have been found, numerical methods are needed since there are still unsolved soliton solutions. When looking at the studies for the numerical solution, it is seen that many different methods were used. Finite difference method (FDM) is used by Djidjeli et al. in [20], by Duncan in [21], by Wong in [12], by Minzoni et al. in [9], by Bratsos in [22,23], by Cui in [24] and by Liang...
et al. in [25]. Also, Jiwar in [26] and Pekmen [13] used differential quadrature method (DQM) where the intervals are discretized using Chebyshev–Gauss–Lobatto points. Another numerical study with DQM was presented by Shukla et al. in [27] where modified cubic B-spline was used as a basis function. Asgari and Hosseini considered SGE in [28]. In this article, Fourier-spectral method was used for the solution of spatial variables and fourth-order time stepping schemes were used for the time derivatives. Another work with Fourier-spectral method was presented by Jiang et al. in [29]. In this paper, the fourth-order average vector field method was utilized for the discretization in time. Time-space pseudo-spectral method was applied for the solution of one-dimensional SGE using Chebyshev–Gauss–Lobatto points in [30]. In [3], the same method was used for the solution of two-dimensional SGE by presenting the stability analysis of the method. Another numerical study for the SGE was done with spectral method using Legendre wavelets as basis in [31]. In addition, mesh-free methods were applied for the solution of the SGE presented in [32–35]. Dehghan and Shokri used radial basis functions for the solution of two-dimensional SGEs in [32], Cheng and Liew used mesh-free reproducing kernel particle Ritz method in [33] and Li et al. used element-free Galerkin method in [34]. Lingde Su used localized method of approximate particular solutions in [35]. Other presented numerical studies for the SGE were given by Argyris et al. in [36] using finite element method, by Sheng et al. in [37] using split cosine scheme, by Bratsos in [38] using method of lines, by Han and Zhang in [39] using operator splitting method, by Suarez in [40] using Chebyshev split-step scheme, by Adak and Natarajan in [41] using virtual element method, by Guo et al. in [42] using local Kriging meshless method and by Jiwari in [43] using barycentric rational interpolation and local radial basis functions based numerical algorithms, etc.

Another preferred numerical solution procedure for the solution of problems modelled with partial differential equations is the boundary element method (BEM) which has found an important place among the other numerical methods. In the BEM solution procedure, the differential equation is transformed into an integral equation that contains only boundary integrals [44]. Comparing to the domain discretization methods, the BEM eliminates the need of the discretization of the whole domain which causes large quantities of data. Thus the dimension of the problem can be reduced and obtained smaller linear systems need less computational effort. On the other hand, the use of BEM as a solution technique of a differential equation depends on the existence of the fundamental solution of that differential equation. However, obtaining the fundamental solution of a differential equation such as nonlinear and time-dependent equations is often not straightforward. Therefore the method should be extended for these cases but this has some difficulties. When this differential equations are transformed into the boundary integral equation, the domain integrals occur and so the method loses its advantage. Thus DRBEM was developed to deal with the difficulty in BEM caused by the domain integral [45].

DRBEM is based on the idea of transforming domain integrals to boundary integrals using fundamental solutions of the SGE. All terms in the original governing equation apart from the Laplace operator are treated as nonhomogeneous terms and expanded with the help of interpolating functions [45]. The DRBEM idea can also be extended to other differential equations for which fundamental solutions exist or can be obtained. In [46], DRBEM was built by the utilization of the fundamental solution of modified Helmholtz equations and used for the solution of many partial differential equations. DRBEM based on the fundamental solution of Laplace equation was used as numerical technique for the solution of SGE by Dehghan and Mirzaei in [47,48]. Two-dimensional SGE was solved by using constant and linear boundary elements in [47] and [48], respectively. In these studies, inhomogeneous terms were approximated by using linear radial basis functions $1 + r$.

The main purpose of this study is to show that DRBEM which was built using the fundamental solution of modified Helmholtz equation is suitable as a numerical solution procedure for the solution of the SGE. In this work, DRBEM is used for the solution of SGE after converting it into a modified Helmholtz equation. To obtain the modified Helmholtz equation, the time derivatives are first approximated with central finite difference schemes. Then these approximations are inserted into the governing equation and the governing equation is written in the form of nonhomogeneous modified Helmholtz equation. This enables us to use more information from the original governing equation. Inhomogeneous terms are approximated using thin plate spline $r^2 \ln r$. Constant boundary elements are used for the discretization of the boundary of the domain and thus obtained small size algebraic system can be solved effectively at a reasonable cost. In addition, the requirement for special attention caused by the geometric discontinuity of the boundary nodes that arise in the use of other element types such as linear elements has been eliminated by using constant boundary elements. The code validation of method is carried out by solving a test problem which has an exact solution. Later DRBEM is utilized for both line and ring solitons. Several examples are used to evaluate the capability of the present procedure and the consequences show that the proposed method captures the behaviour of each types of soliton very well.
2. Mathematical formulation

In order to solve SGE numerically, DRBEM based on fundamental solution of modified Helmholtz equation is adopted. Thus as an initial step, it is necessary to write the original equation in modified Helmholtz equation form. The transformation procedure begins by approximating time derivatives in equation (1). Thus the approximations are

\[ z_t = \frac{z_{t+1} - 2z(t) + z_{t-1}}{\tau^2}, \]

\[ z_t = \frac{z_{t+1} - z_{t-1}}{2\tau}, \]

where \( z(t) = (x_1, x_2, t_1), t_1 = \xi t \) and \( t \) is the time step. Relaxing the solution \( z \) at three time levels using the parameters \( 0 < \sigma_1, \sigma_2, \sigma_3 < 1, \sigma_1 + \sigma_2 + \sigma_3 = 1 \) as

\[ z(t) = \sigma_1 z(t\xi) + \sigma_2 z(t\xi) + \sigma_3 z(t\xi+1) \]

and substituting approximations (4)–(6) into Equation (1), we have

\[ \nabla^2 (\sigma_1 z(t\xi) + \sigma_2 z(t\xi) + \sigma_3 z(t\xi+1)) = \frac{z_{t+1}(t\xi) - 2z(t\xi) + z_{t-1}(t\xi)}{\tau^2} + \frac{z(t\xi) - z_{t-1}(t\xi)}{2\tau} + \psi(z(t\xi)), \]

where

\[ \psi(z(t\xi)) = \chi(x_1, x_2) \sin(z(t\xi)). \]

After rearranging Equation (7), it can be written in the form of modified Helmholtz equation for \( z(t\xi+1) \)

\[ \nabla^2 z(t\xi+1) - \left( \frac{1}{\tau^2 \sigma_3} \right) z(t\xi+1) = -\frac{\sigma_1}{\sigma_3} \nabla^2 z(t\xi) - \sigma_2 \nabla^2 z(t\xi) - \frac{2}{\tau^2 \sigma_3} z(t\xi) + \left( \frac{1}{\tau^2} - \frac{\nu}{2\tau} \right) z(t\xi) + \psi(z(t\xi)), \]

in which the inhomogeneous term contains \( z(t\xi) \) and \( z(t\xi) \) from the previous solutions. The terms \( \nabla^2 z(t\xi) \) and \( \nabla^2 z(t\xi) \) at the inhomogeneity can be expressed in terms of known values of \( z \). This can be done as follows: first, Equation (8) is rewritten by taking \( \sigma_3 = 1 \) which implies that \( \sigma_1 = \sigma_2 = 0 \) and

\[ \nabla^2 z(t\xi) = \left( \frac{1}{\tau^2} + \frac{\nu}{2\tau} \right) z(t\xi) = -\frac{2}{\tau^2} z(t\xi) + \left( \frac{1}{\tau^2} - \frac{\nu}{2\tau} \right) z(t\xi) + \psi(z(t\xi)). \]

Then, Equation (9) is rewritten for \( \xi = 0 \) and \( \xi = 1 \) as

\[ \nabla^2 z(0) = \left( \frac{1}{\tau^2} + \frac{\nu}{2\tau} \right) z(0) - \frac{2}{\tau^2} z(0) \]

and

\[ \nabla^2 z(1) = \left( \frac{1}{\tau^2} + \frac{\nu}{2\tau} \right) z(1) - \frac{2}{\tau^2} z(0) \]

respectively. Next, Equation (8) is rewritten by taking \( \xi = 2 \) as

\[ \nabla^2 z(2) = \left( \frac{1}{\tau^2} + \frac{\nu}{2\tau} \right) z(2) - \frac{2}{\tau^2} z(1) + \left( \frac{1}{\tau^2} - \frac{\nu}{2\tau} \right) z(0) + \psi(z(1)). \]

Equations (10) and (11) are substituted into Equation (12) which becomes

\[ \nabla^2 z(3) = \left( \frac{1}{\tau^2} + \frac{\nu}{2\tau} \right) z(3) - \frac{2}{\tau^2} z(2) + \left( \frac{1}{\tau^2} - \frac{\nu}{2\tau} \right) z(0) + \psi(z(2)). \]

Equation (13) can be written in a general form as

\[ \nabla^2 z(t\xi) = \left( \frac{1}{\tau^2} + \frac{\nu}{2\tau} \right) z(t\xi) = -\frac{\sigma_1}{\sigma_3} \nabla^2 z(t\xi) - \sigma_2 \nabla^2 z(t\xi) - \frac{2}{\tau^2 \sigma_3} z(t\xi) + \left( \frac{1}{\tau^2} - \frac{\nu}{2\tau} \right) z(t\xi) + \psi(z(t\xi)), \]

The solution procedure starts with the solution of Equation (10) by considering it as a modified Helmholtz equation as

\[ \nabla^2 z(1) = \left( \frac{1}{\tau^2} + \frac{\nu}{2\tau} \right) z(1) - \frac{2}{\tau^2} z(0) + \psi(z(0)). \]
information at $-\tau$. This difficulty is easily overcome by the initial conditions given in (2) as

$$z(0) = w_1(x_1,x_2)$$

and

$$z(-1) = w_1(x_1,x_2) - \tau w_2(x_1,x_2).$$

Then, Equation (11) is solved by considering it as a modified Helmholtz equation as

$$\nabla^2 z^{(2)} - \left(\frac{1}{\tau^2} + \frac{v}{2\tau}\right) z^{(2)} = \frac{2}{\tau^2} z^{(1)} + \left(\frac{1}{\tau^2} - \frac{v}{2\tau}\right) z^{(0)} + \psi(z^{(1)})$$

by using the known values of $z^{(0)}$ and $z^{(1)}$. Then the iterative solution begins with the solution of Equation (14) by taking $\xi = 2$ and by using the known values of $z$ at $\xi = -1, 0, 1, 2$. This process continues up to the stopping criteria is satisfied.

3. DRBEM formulation of the problem

In the previous section, the iterative form of the sine-Gordon equation is written as the modified Helmholtz equations given in (15), (18) and (14). Thus the fundamental solution of modified Helmholtz equation $z^* = \frac{1}{2\pi}K_0(\lambda r)$ can be used to transform the domain integrals caused by the inhomogeneous terms to the boundary integrals. Here, $K_0(\lambda r)$ is the second kind modified Bessel function of order 0.

First, for simplicity let us write Equations (15), (18) and (14) in the compact form as

$$\nabla^2 z^{(1)} - \lambda^2 z^{(1)} = b_1(x_1,y_1),$$

$$\nabla^2 z^{(2)} - \lambda^2 z^{(2)} = b_2(x_1,y_2),$$

$$\nabla^2 z^{(n+1)} - \lambda^2 z^{(n+1)} = b_n(x_1,y_2),$$

where

$$b_1(x_1,y_2) = -\frac{2}{\tau^2} z^{(0)} + \left(\frac{1}{\tau^2} - \frac{v}{2\tau}\right) z^{(-1)} + \psi(z^{(1)}),$$

$$b_2(x_1,y_2) = -\frac{2}{\tau^2} z^{(1)} + \left(\frac{1}{\tau^2} - \frac{v}{2\tau}\right) z^{(0)} + \psi(z^{(1)}),$$

$$b_3(x_1,y_2) = -\frac{\sigma_1}{\sigma_3^2} \left[\frac{\sigma_2}{\sigma_3^2} z^{(0)} - \frac{1}{\tau^2} \frac{1}{\sigma_3^2} \sigma_2 \tau \psi(z^{(2)}) - \frac{1}{\tau^2} \frac{1}{\sigma_3^2} \sigma_2 \tau \psi(z^{(3)})\right] + \frac{1}{\sigma_3^2} \left[\frac{\sigma_2}{\sigma_3^2} z^{(1)} + \frac{1}{\tau^2} \frac{1}{\sigma_3^2} \sigma_2 \tau \psi(z^{(2)}) - \frac{1}{\tau^2} \frac{1}{\sigma_3^2} \sigma_2 \tau \psi(z^{(3)})\right] - \frac{1}{\sigma_3^2} \left[\frac{\sigma_2}{\sigma_3^2} \psi(z^{(2)}) + \sigma_2 \psi(z^{(1)}) - \psi(z^{(1)})\right].$$

Here

$$\lambda_1^2 = \lambda_2^2 = \frac{1}{\tau} \left(\frac{1}{\tau} + \frac{v}{2}\right), \quad \lambda_3^2 = \frac{1}{\sigma_3^2} \left(\frac{1}{\tau} + \frac{v}{2}\right).$$

Then, $(x_{1j}, x_{2j})$ for $j = 1, \ldots, N_1 + N_2$ collocation points are chosen in the boundary and in the domain. Here, $N_2$ is the number of the constant boundary elements and the collocation points are taken in the middle of the each elements. The other collocation points $N_1$ are taken in the domain of the problem.

The weighted residual formulations of Equations (19)–(21) can be obtained by multiplying them with the fundamental solution of the modified Helmholtz equations $p_i^* = K_0(\lambda r)$ ($i = 1, 2, 3$) and by integrating over the domain as

$$\int_D \left(\nabla^2 z^{(1)} - \lambda^2 z^{(1)} - \lambda^2 z^{(1)}\right) p_i^* dD = \int_D b_1 p_i^* dD,$$

$$\int_D \left(\nabla^2 z^{(2)} - \lambda^2 z^{(2)} - \lambda^2 z^{(2)}\right) p_i^* dD = \int_D b_2 p_i^* dD,$$

$$\int_D \left(\nabla^2 z^{(n+1)} - \lambda^2 z^{(n+1)} - \lambda^2 z^{(n+1)}\right) p_i^* dD = \int_D b_n p_i^* dD.$$

By applying the Greens second identity to the resulting weighted residual statements, the left-hand side of the governing equations (26)–(28) is transformed into boundary integral equations as in [44]

$$\epsilon_j z^{(1)} + \int_\Gamma \left(\frac{\partial p_j^*}{\partial n} z^{(1)} - \frac{\partial p_j^*}{\partial n} \frac{\partial z^{(1)}}{\partial n}\right) d\Gamma = \int_D b_1 p_j^* dD,$$

$$\epsilon_j z^{(2)} + \int_\Gamma \left(\frac{\partial p_j^*}{\partial n} z^{(2)} - \frac{\partial p_j^*}{\partial n} \frac{\partial z^{(2)}}{\partial n}\right) d\Gamma = \int_D b_2 p_j^* dD,$$

$$\epsilon_j z^{(n+1)} + \int_\Gamma \left(\frac{\partial p_j^*}{\partial n} z^{(n+1)} - \frac{\partial p_j^*}{\partial n} \frac{\partial z^{(n+1)}}{\partial n}\right) d\Gamma = \int_D b_n p_j^* dD,$$

where $\epsilon_j$ ($j = 1, \ldots, N_2 + N_1$) are coefficients and their values change depend on the location of the collocation points $(x_{1j}, y_{1j})$. For constant elements, $\epsilon_j$ are defined as

$$\epsilon_j = \begin{cases} 1, & (x_{1j}, x_{2j}) \in D \\ \frac{1}{2}, & (x_{1j}, x_{2j}) \in \Gamma. \end{cases}$$

In order to transform the domain integrals caused by the inhomogeneities to the boundary integrals, the functions $b_i(x_1,y_2)$ ($i = 1, 2, 3$) are expanded as

$$b_1(x_1,y_2) = \sum_{s=1}^{N_2 + N_1} \alpha_{1s}(t) f_s(x_1,y_2),$$

$$b_2(x_1,y_2) = \sum_{s=1}^{N_2 + N_1} \alpha_{2s}(t) f_s(x_1,y_2),$$

$$b_3(x_1,y_2) = \sum_{s=1}^{N_2 + N_1} \alpha_{3s}(t) f_s(x_1,y_2).$$
where \( f_i(x, y) = r_i^2 \ln r_i \) and \( r_i = \sqrt{(x_1 - x_is)^2 + (x_2 - x_is)^2} \). The values of functions \( f_i \) can be represented as \( \hat{f}_i \) at the collocation point for \( j = 1, \ldots, N_t + N_i \) and \( F \) can be constructed as a \((N_t + N_i) \times (N_t + N_i)\) square matrix such that \( F(j, s) = \hat{f}_i \). Therefore, \( \alpha_{is} \) (\( i = 1, 2, 3 \)) can be written as

\[
\alpha_{1s}(t) = \sum_{m=1}^{N_t + N_i} E_{m}(t)b_{1m}(t), \quad \alpha_{2s}(t) = \sum_{m=1}^{N_t + N_i} E_{m}(t)b_{2m}(t), \quad \alpha_{3s}(t) = \sum_{m=1}^{N_t + N_i} E_{m}(t)b_{3m}(t),
\]

where \( E = F^{-1} \) and \( b_{im}(t) = b(x_{1m}, x_{2m}, t), \ i = 1, 2, 3 \).

The set of functions \( f_i \) provide the following relationship:

\[
\begin{align*}
\nabla^2 \hat{z}_{1s} - \lambda \hat{z}_{1s} &= f_s, \\
\nabla^2 \hat{z}_{2s} - \lambda \hat{z}_{2s} &= f_s, \\
\nabla^2 \hat{z}_{3s} - \lambda \hat{z}_{3s} &= f_s,
\end{align*}
\]

where \( \hat{z}_{is}, (i = 1, 2, 3) \) are the particular solutions of the corresponding equations. The particular solutions and their normal derivatives can be given as

\[
\hat{z}_s = \begin{cases} 
\frac{4}{\lambda^4} (K_0(\lambda r_s) + \log r_s) - r_s^2 \log r_s - \frac{4}{\lambda^4}, & r_s \neq 0 \\
\frac{4}{\lambda^4} (\gamma + \log \frac{r_s}{2}) - \frac{4}{\lambda^4}, & r_s = 0
\end{cases}
\]

and

\[
\hat{\alpha}_s = \begin{cases} 
-\frac{4}{\lambda^4} - \frac{2}{\lambda^2} \frac{r_s}{\lambda^4} + \frac{4}{\lambda^4} K_1(\lambda r_s), & r_s \neq 0 \\
\left( \frac{\partial x_1}{\partial n} + r_s \frac{\partial x_2}{\partial n} \right), & r_s = 0
\end{cases}
\]

where \( r_{x_1} = x_1 - x_{1s}, r_{x_2} = x_2 - x_{2s} \) and \( n = \left( \frac{\partial x_1}{\partial n}, \frac{\partial x_2}{\partial n} \right) \). \( \gamma = 0.57721566490153 \) is Euler constant and \( K_1(x) \) is the second kind Bessel function of order 1.

Inserting Equations (38)–(40) into the corresponding inhomogeneous terms in Equations (32)–(34), we obtain the modified Helmholtz operators in the domain integrals of Equations (29)–(31). Then using the Green’s second identity for the right-hand side of the equations, we have

\[
e_j \hat{z}_{1j}^{(1)} + \int_{\Gamma} \left( \frac{\partial p_s^e}{\partial n} \hat{z}_{1s}^{(1)} - p_1 \frac{\partial \hat{z}_{1s}}{\partial n} \right) \, d\Gamma = \sum_{i=1}^{N_t + N_i} \alpha_{is}(t) \left[ e_j \hat{z}_{1j} + \int_{\Gamma} \left( \frac{\partial p_s^e}{\partial n} \hat{z}_{1s} - p_1 \frac{\partial \hat{z}_{1s}}{\partial n} \right) \, d\Gamma \right],
\]

(41)

(42)

\[
e_j \hat{z}_{2j}^{(2)} + \int_{\Gamma} \left( \frac{\partial p_s^e}{\partial n} \hat{z}_{2s}^{(2)} - p_2 \frac{\partial \hat{z}_{2s}}{\partial n} \right) \, d\Gamma = \sum_{i=1}^{N_t + N_i} \alpha_{2j}(t) \left[ e_j \hat{z}_{2j} + \int_{\Gamma} \left( \frac{\partial p_s^e}{\partial n} \hat{z}_{2s} - p_2 \frac{\partial \hat{z}_{2s}}{\partial n} \right) \, d\Gamma \right].
\]

(43)

Equations (41)–(43) can be described as the matrix-vector form by using the coefficient matrices \( H_i \) and \( G_i \) as

\[
H_1 \hat{z}_1^{(1)} - G_1 \frac{\partial \hat{z}_1^{(1)}}{\partial n} = (H_1 \hat{Z}_1 - G_1 \hat{Q}_1) \alpha_1, \quad (44)
\]

\[
H_2 \hat{z}_2^{(2)} - G_2 \frac{\partial \hat{z}_2^{(2)}}{\partial n} = (H_2 \hat{Z}_2 - G_2 \hat{Q}_2) \alpha_2, \quad (45)
\]

\[
H_3 \hat{z}_3^{(3)} - G_3 \frac{\partial \hat{z}_3^{(3)}}{\partial n} = (H_3 \hat{Z}_3 - G_3 \hat{Q}_3) \alpha_3, \quad (46)
\]

where their components for \( i = 1, 2, 3 \) are

\[
H_i(j, s) = e_j \delta_{is} + \frac{1}{2\pi} \int_{\Gamma_s} \frac{\partial K_0(\lambda r_{is})}{\partial n} \, d\Gamma_s, \quad \text{and} \quad G_i(j, s) = -\frac{1}{2\pi} \int_{\Gamma_s} K_0(\lambda r_{is}) \, d\Gamma_s
\]

such that \( \Gamma_s \) is the sth boundary element and \( \delta_{is} = \begin{cases} 1, & s = j \\
0, & s \neq j \end{cases} \).

Also, the dimension of the \( \alpha_i \) vectors for \( i = 1, 2, 3 \) are \((N_t + N_i) \times 1\) and they are defined as

\[
\alpha_i = F^{-1}(b(x_{1i}, x_{2i})). \quad (47)
\]

\( \hat{Z}_1, \hat{Z}_2, \hat{Z}_3, \hat{Q}_1, \hat{Q}_2, \) and \( \hat{Q}_3 \) are all square matrices with the dimensions of \((N_t + N_i) \times (N_t + N_i)\) and they are constructed by using the vectors \( \hat{Z}_1, \hat{Z}_2, \hat{Z}_3, \hat{Q}_1, \hat{Q}_2, \) and \( \hat{Q}_3 \) as column vectors, respectively.

4. Implementation of the numerical method

In this section, we report the numerical simulations obtained by using the DRBEM for the solution of two-dimensional sine-Gordon equation. First, the test problem is solved by using DRBEM for code validation and the results are compared with the exact solution. Then five problems of the line and ring solitons are solved as examples.
considered in order to present numerical solutions of equation (1) with the initial conditions (2) and Neumann boundary conditions (3).

4.1. Test problem

The problem defined by sine-Gordon equation \[32,43,49\]

\[
z_{tt} + \nu z_t = \nabla^2 z - \chi(x_1, x_2) \sin(z), \quad (x_1, x_2) \in D, \quad t \geq 0 \tag{48}
\]

is solved as a test problem. Here, \(z = z(x_1, x_2, t)\) and \(D = \{(x_1, x_2) : |x_1| \leq 7, |x_2| \leq 7\}, \chi = 1 \) and \( \nu = 0 \). Initial conditions will be considered as

\[
z(x_1, x_2, 0) = 4 \tan^{-1}(\exp(x_1 + x_2)), \quad (x_1, x_2) \in D,
\]

\[
z_t(x_1, x_2, 0) = - \frac{4 \exp(x_1 + x_2)}{1 + \exp(2(x_1 + x_2))}, \quad (x_1, x_2) \in D \tag{49}
\]

and Neumann boundary conditions are

\[
z_{x_1}(x_1, x_2, t) = \frac{4 \exp(x_1 + x_2 + t)}{\exp(2t) + \exp(2(x_1 + x_2))},
\]

\[
x_1 = -7, x_1 = 7, \quad |x_2| \leq 7, \quad t \geq 0,
\]

\[
z_{x_2}(x_1, x_2, t) = \frac{4 \exp(x_1 + x_2 + t)}{\exp(2t) + \exp(2(x_1 + x_2))},
\]

\[
x_2 = -7, x_2 = 7, \quad |x_1| \leq 7, \quad t \geq 0. \tag{50}
\]

The exact solution of equations (48)–(50) is

\[
z(x_1, x_2, t) = 4 \tan^{-1}(\exp(x_1 + x_2 - t)). \tag{51}
\]

Table 1. \(L_{\infty}, L_2\)-errors and the related convergence orders for test problem.

| \(\tau\) | \(L_2\)-error | Order | \(L_{\infty}\)-error | Order |
|----------|----------------|-------|----------------------|-------|
| 0.1000   | 7.01281 \times 10^{-1} | 4.34761 \times 10^{-1} | 4.34761 \times 10^{-1} | 1.99 |
| 0.0500   | 1.75527 \times 10^{-1} | 2.00  | 1.09769 \times 10^{-1} | 1.99 |
| 0.0250   | 4.37972 \times 10^{-2} | 2.00  | 2.77182 \times 10^{-2} | 1.99 |
| 0.0125   | 1.09329 \times 10^{-2} | 2.00  | 7.07024 \times 10^{-3} | 1.98 |

The exact and DRBEM solutions for \(t = 5\) and \(t = 7\) are drawn in the left and middle columns of Figure 1, respectively. Also, the computational results and exact values at the points located \(x_2 = x_1\) are presented at the right column of Figure 1 for the same time levels. From these figures, it is observed that exact
Figure 2. The numerical results at $t = 0, 3, 7, 10, 15$ and $20$ when $\nu = 0$ for Example 1.

Figure 3. The contour lines at $t = 0, 3, 7, 10, 15$ and $20$ when $\nu = 0$ for Example 1.
solutions agree very well with the computational results obtained by DRBEM as time advances.

4.2. Line solutions

4.2.1. Example 1: Superposition of two line solitons
In the first problem, the numerical solution of Equation (1) is obtained for \( \chi(x_1, x_2) = 1 \) and \( \nu = 0 \) such that this case is called superposition of two line solitons. Initial conditions are considered as

\[
\begin{align*}
    w_1(x_1, x_2) &= 4 \tan^{-1}(\exp(x_1)) + 4 \tan^{-1}(\exp(x_2)), \\
    w_2(x_1, x_2) &= 0, \quad -6 \leq x_1, x_2 \leq 6. \tag{52}
\end{align*}
\]

The numerical results are given at \( t = 0, 3, 7, 10, 15 \) and 20 in Figures 2 and 3. The computations are done by using \( N_B + N_I = 80 + 400 \) and \( \tau = 0.05 \). Initially \((t = 0)\), there are two orthogonal line solitons.

Figure 4. Effect of \( \nu \) on the contourlines at \( t = 2 \) and \( t = 4 \) for Example 1.

Figure 5. The numerical results at \( t = 0, 4, 7, 10 \) and 14 when \( \nu = 0 \) for Example 2.
occur which are parallel to the diagonal $x_2 = -x_1$. As $t$ increases, these two orthogonal line solitons move away from each other in the direction of $x_2 = x_1$ without losing their form. When $t = 7$, a deformation is observed on the graphs. All behaviours seen in the graphs are in agreement with those published in [27,32,38,43,47] and [49]. In addition, the effect of dissipative term is evaluated at $t = 2$ and $t = 4$ when $\nu = 0$ and $\nu = 0.5$ by giving the contourlines. The results are presented in Figure 4, for $\nu = 0$ and $\nu = 0.5$, by using full and dashed contours, respectively. From the figures, the delaying effect of dissipative term on the propagation of the solitons is seen clearly. As time increases, this delaying effect of dissipative term becomes more evident. This conclusion agrees with [47] and [38].

Figure 6. The numerical results at $t = 0, 6, 12$ and $18$ when $\nu = 0$ for Example 3.

Figure 7. The contourlines at $t = 0, 6, 12$ and $18$ when $\nu = 0.05$ for Example 3.
Figure 8. The numerical results at $t = 0, 2.8, 5.6, 8.4, 11.2$ and $12.8$ when $\nu = 0.05$ for Example 4.

Figure 9. The contour lines at $t = 0, 2.8, 5.6, 8.4, 11.2$ and $12.8$ when $\nu = 0.05$ for Example 4.
4.2.2. Example 2: Perturbation of a line soliton
In this example, the proposed method is used for the perturbation of a single soliton which is calculated for 
\( \chi(x_1, x_2) = 1 \) and \( \nu = 0 \). Initial conditions are considered as

\[
\begin{align*}
    w_1(x_1, x_2) &= 4 \tan^{-1}(\exp(x_1 + 1 - 2 \text{sech}(x_2 + 7)) - 2 \text{sech}(x_2 - 7))), \\
    w_2(x_1, x_2) &= 0, \quad -7 \leq x_1, x_2 \leq 7.
\end{align*}
\]

The numerical results are obtained by using \( N_B + N_I = 100 + 625, \tau = 0.5 \) and are presented in terms of 
\( \sin(z/2) \) at \( t = 0, 2, 4, 7, 10 \) and \( 14 \) in Figure 5. It is seen that two symmetrical dents occur at \( t = 0 \) and as time increases, these dents move towards each other. At \( t = 7 \), they reach each other and an interaction occurs between them. After that they are moving away from each other. Looking at the surfaces given for \( t = 10 \) and \( t = 14, \) it can be concluded that there is no change on the shape of dents after the collision. If the obtained results are compared with the results in [32,38,47] and [49], it can be seen that the agreement of these results is very well.

4.2.3. Example 3: Line soliton in an inhomogeneous medium
In the next problem, the numerical solution of Equation (1) is obtained for 
\( \chi(x_1, x_2) = 1 + \text{sech}^2 \sqrt{x_1^2 + x_2^2} \) and \( \nu = 0.05 \) which is called perturbation of a line soliton. Initial conditions are considered as

\[
\begin{align*}
    w_1(x_1, x_2) &= 4 \tan^{-1}(\exp \left( \frac{x_1 - 3.5}{0.954} \right)), \\
    w_2(x_1, x_2) &= 0.639 \text{sech} \left( \frac{x_1 - 3.5}{0.954} \right), \quad -7 \leq x_1, x_2 \leq 7.
\end{align*}
\]

Figure 10. The numerical results at \( t = 0, 1.6, 3.2, 4.8, 6.4, 8, 9.6 \) and \( 11.2 \) when \( \nu = 0 \) for Example 5.
Figures 6 and 7 show the results of the problem at time levels \( t = 0, 6, 12, 18 \) in terms of \( \sin(z/2) \) using \( N_B + N_I = 92 + 529 \) and \( \tau = 0.4 \) for \( \nu = 0.05 \). At \( t = 0 \), the soliton is in the form of a straight line which is parallel to the \( x_2 \) axis. As time increases, it moves slowly in \( x_1 \) direction and a curl caused by inhomogeneity of the medium occurs in its straightness. These results show that a good agreement is obtained with those given in [27,32,43,47] and [49]. Also when these results are compared with the results given for \( \nu \) in [38], it can be expressed that the propagation of the line soliton is slowing down.

4.3. Ring solutions

4.3.1. Example 4: Circular ring soliton

The circular ring solitons for the case \( \chi(x_1, x_2) = 1 \) and \( \nu = 0.05 \) are analysed with the initial conditions

\[
\begin{align*}
    w_1(x_1, x_2) &= 4 \tan^{-1} \left( \exp \left( 3 - \sqrt{x_1^2 + x_2^2} \right) \right), \\
    w_2(x_1, x_2) &= 0, \quad -7 \leq x_1, x_2 \leq 7. \quad (55)
\end{align*}
\]

Numerical computations are carried out by using \( N_B + N_I = 92 + 529 \) and \( \tau = 0.4 \) and obtained results are evaluated by giving surfaces and corresponding contourlines at \( t = 0, 2.8, 5.6, 8.4, 11.2, 12.8 \) in terms of \( \sin(z/2) \) in Figures 8 and 9, respectively. From the figures, it is observed that at the initial case \( (t = 0) \), a single ring soliton occurs and this soliton shrinks until \( t = 2.8 \). From \( t = 5.6 \) to \( t = 8.4 \), an expansion occurs on the ring soliton and this expansion phase starts with the radiation and oscillations. Finally at \( t = 11.2 \) and \( t = 12.8 \), a single ring soliton appears in the cavity which has almost the same form with the initial phase. These

Figure 11. The contourlines at \( t = 0, 1.6, 3.2, 4.8, 6.4, 8, 9.6 \) and \( 11.2 \) when \( \nu = 0 \) for Example 5.
results are good agreement with the results given in [27,32,38,43,47] and [49].

4.3.2. Example 5: Elliptic ring soliton
In this example elliptic ring soliton with \( \chi(x_1, x_2) = 1 \) and \( v = 0 \) is considered with initial conditions

\[
\begin{align*}
    w_1(x_1, x_2) &= 4 \tan^{-1} \left( \exp \left( 3 - \sqrt{(x_1 - x_2)^2/3 + (x_1 + x_2)^2/2} \right) \right), \\
    w_2(x_1, x_2) &= 0, \quad -7 \leq x_1, x_2 \leq 7. \tag{56}
\end{align*}
\]

The DRBEM results are obtained by using \( N_B + N_I = 88 + 484 \) and \( \tau = 0.4 \). Figures 10 and 11 present the results for the elliptic ring soliton as surface and corresponding contour lines, respectively, at \( t = 0, 1.6, 3.2, 4.8, 6.4, 8.0, 9.6 \) and 11.2. The elliptical circular soliton exhibits shrinkage and expansion behaviour as time progresses. A permanent change in the orientation of the major and minor axes of the ellipse creates wave pulsation. Moreover, it is observed that at \( t = 8 \), the elliptical ring soliton begins to take the form of a circular soliton, and when the time is reached \( t = 11.2 \), an almost circular ring soliton is formed. The obtained results are in good agreement with those presented in [27,38,43,47] and [50].

5. Conclusion
In this study, we have employed DRBEM used fundamental solution of modified Helmholtz equation. The proposed method has been applied to the examples of sine-Gordon equation which have both line and ring soliton solutions. The main advantage of the DRBEM is to solve the problem by using small number of points since it does not need to discretize the whole domain. Also, in this study, since the DRBEM has been constructed by using the fundamental solution of modified Helmholtz equation, it has been possible to use more information of the original equation. The obtained results have been compared with the corresponding results for damped and undamped SG equations in [27], [32,38,43,47,49] and [50]. These results reveal that good results are obtained for both line and ring solitons and the behaviour of sine-Gordon equation is very well captured with DRBEM. Therefore, it can be concluded that proposed method is applicable to find numerical solution of sine-Gordon equation.

Disclosure statement
No potential conflict of interest was reported by the author(s).

ORCID
Nagehan Alsoy-Akgün @ http://orcid.org/0000-0001-6967-0625

References
[1] Alshenawy R, Al-alwan A, Elmandouh AA. Generalized kdv equation involving Riesz time fractional derivatives: constructing and solution utilizing variational methods. J Taibah Univ Sci. 2020;14(1):314–312.
[2] Alam MN, Tunç C. Constructions of the optical solitons and other solitons to the conformable fractional Zakharov–Kuznetsov equation with power law nonlinearity. J Taibah Univ Sci. 2020;14(1):94–100.
[3] Mittal AK. A stable time-space Jacobi pseudospectral method for two-dimensional sine-Gordon equation. J Appl Math Comput. 2020;63:239–264.
[4] El-Sheikh RM, Gaballah MA. Novel solitons and periodic wave solutions for Davey–Stewartson system with variable coefficients. J Taibah Univ Sci. 2020;14(1):783–789.
[5] Barone A, Esposito F, Magee CJ, et al. Theory and applications of the sine-Gordon equation. La Riv Nuov Cimen. 1971;12:227–267.
[6] Berg L. Wave collapse in physics: principles and applications to light and plasma waves. Phys Rep. 1998;303(5–6):259–370.
[7] Deeba E, Khuri S. A decomposition method for solving the nonlinear Klein–Gordon equation. J Comput Phys. 1996;124(2):442–448.
[8] Gibbon JD, James IN, Moroz IM. The sine-Gordon equation as a model for a rapidly rotating baroclinic fluid. Phys Scr. 1979;20(3–4):402.
[9] Minzoni A, Smyth NF, Worthy AL. Pulse evolution for a two-dimensional sine-Gordon equation. Physica D. 2001;159(1–2):101–123.
[10] Sassaman R, Biswas A. Soliton perturbation theory for phi-four model and nonlinear Klein–Gordon equations. Commun Nonlinear Sci Numer Simul. 2009;14(8):3239–3249.
[11] Shen J, Tang T. Spectral and high-order methods with applications. Beijing: Science Press; 2006.
[12] Wong YS, Chang Q, Gong L. An initial-boundary value problem of a nonlinear Klein–Gordon equation. Appl Math Comput. 1997;84(1):77–93.
[13] Pekmen B, Tezer-Sezgin M. Differential quadrature solution of nonlinear Klein–Gordon and sine-Gordon equations. Comput Phys Commun. 2012;183(8):1702–1713.
[14] Hirota R. Exact three-soliton solution of the two-dimensional sine-Gordon equation. J Phys Soc Jpn. 1973;35(5):1566–1566.
[15] Gibbon J, Zambotti G. The interaction of n-dimensional soliton wave fronts. II Nuov Cimen B. 1975;28(1):1971–1996.
[16] Leibbrandt G. New exact solutions of the classical sine-Gordon equation in 2 + 1 and 3+1 dimensions. Phys Rev Lett. 1978;41(7):435–438.
[17] Grella G, Marinaro M. Special solutions of the sine-Gordon equation in 2 + 1 dimensions. Lett Nuov Cimen. 1978 (1971–1985);23(12):459–464.
[18] Christiansen PL, Olsen OH. Return effect for rotationally symmetric solitary wave solutions to the sine-Gordon equation. Phys Lett A. 1978;68(2):185–188.
[19] Zagrodzinski J. Particular solutions of the sine-Gordon equation in 2 + 1 dimensions. Phys Lett A. 1979;72(4–5):284–286.
[20] Djidjeli K, Price W, Twizell E. Numerical solutions of a damped sine-Gordon equation in two space variables. J Eng Math. 1995;29(4):347–369.
[21] Duncan D. Sympletic finite difference approximations of the nonlinear Klein–Gordon equation. SIAM J Numer Anal. 1997;34(5):1742–1760.
[22] Bratsos AG. A modified predictor corrector scheme for the two dimensional sine-Gordon equation. Numer Algor. 2006;43:295–308.

[23] Bratsos AG. A third order numerical scheme for the two-dimensional sine-Gordon equation. Math Comput Simul. 2007;76:271–282.

[24] Cui M. High order compact alternating direction implicit method for the generalized sine-Gordon equation. J Comput Appl Math. 2010;235(3):837–849.

[25] Liang Z, Yan Y, Cai G. A Dufort–Frankel difference scheme for two-dimensional sine-Gordon equation. Discrete Dyn Nat Soc. 2014;2014:1–23.

[26] Jiwari R, Pandit S, Mittal R. Numerical simulation of two-dimensional sine-Gordon solitons by differential quadrature method. Comput Phys Commun. 2012;183(3):600–616.

[27] Shukla HS, Tamsir M, Srivastava VK. Numerical simulation of two dimensional sine-Gordon solitons using modified cubic B-spline differential quadrature method. AIP Adv. 2015;5:017121.

[28] Asgari Z, Hosseini S. Numerical solution of two-dimensional sine-Gordon and mbe models using fourier spectral and high order explicit time stepping methods. Comput Phys Commun. 2013;184(3):565572.

[29] Jiang C, Sun J, Li H, et al. A fourth-order AVF method for the numerical integration of sine-Gordon equation. Appl Math Comput. 2017;313:144–158.

[30] Mittal AK, Balyan LK. Time-space pseudospectral algorithm for numerical solution of sine/Klein–Gordon equations. AIP Conf Proc. 2019;2142:170013.

[31] Yin F, Tian T, Song J, et al. Spectral methods using legendre wavelets for nonlinear Klein/sine-Gordon equations. J Comput Appl Math. 2015;275:321–334.

[32] Dehghan M, Shokri A. A numerical method for solution of the two-dimensional sine-Gordon equation using radial basis functions. Math Comput Simul. 2008;79:700–715.

[33] Cheng R, Liew KM. Analyzing two-dimensional sine-Gordon equation with the mesh-free reproducing kernel particle Ritz method. Comput Methods Appl Mech Eng. 2012;245:132–143.

[34] Li X, Zhang S, Wang Y, et al. Analysis and application of the element-free Galerkin method for nonlinear sine-Gordon and generalized sinh-Gordon equations. Comput Math Appl. 2016;71(8):1655–1678.

[35] Su LD. Numerical solution of two-dimensional nonlinear sine-Gordon equation using localized method of approximate particular solutions. Eng Anal Bound Elem. 2019;108:95–107.

[36] Argyris J, Haase M, Heinrich JC. Finite element approximation to two-dimensional sine-Gordon solitons. Comput Methods Appl Mech Eng. 1991;86(1):1–26.

[37] Sheng Q, Khaliq AQM, Voss DA. Numerical simulation of two-dimensional sine-Gordon solitons via a split cosine scheme. Math Comput Simul. 2005;68(4):355–373.

[38] Bratsos AG. The solution of the two-dimensional sine-Gordon equation using the method of lines. J Comput Appl Math. 2007;206:251–277.

[39] Han H, Zhang Z. Split local artificial boundary conditions for the two-dimensional sine-Gordon equation on $R^2$. Commun Comput Phys. 2011;10(5):1161–1183.

[40] Suarez P, Johnson S, Biswas A. Chebyshev split-step scheme for the sine-Gordon equation in (2 + 1) dimensions. Int J Nonlinear Sci Numer Simul. 2013;14(1):69–75.

[41] Adak D, Natarajan S. Virtual element method for semi-linear sine-Gordon equation over polygonal mesh using product approximation technique. Math Comput Simul. 2020;172:224–243.

[42] Guo P, Boldbaatar A, Yi D, et al. Numerical solution of sine-Gordon equation with the local Kriging meshless method. Math Probl Eng. 2020;2020(1–10):9057387.

[43] Jiwari R. Barycentric rational interpolation and local radial basis functions based numerical algorithms for multidimensional sine-Gordon equation. Numer Methods Partial Differ Equ. 2021;37:1965–1992.

[44] Brebbia CA. The boundary element method for engineers. London: Pentech Press; 1984.

[45] Partridge PW, Brebbia CA, Wrobel LC. The dual reciprocity boundary element method. Southampton Boston: Computational Mechanics Publications; 1992.

[46] Alsoy-Akgün N. The dual reciprocity boundary element solution of Helmholtz-type equations in fluid dynamics [PhD thesis]. METU; 2013.

[47] Dehghan M, Mirzaei D. The dual reciprocity boundary element method (DRBEM) for two-dimensional sine-Gordon equation. Comput Methods Appl Mech Eng. 2008;197:476–486.

[48] Mirzaei D, Dehghan M. Boundary element solution of the two-dimensional sine-Gordon equation using continuous linear elements. Eng Anal Bound Elem. 2009;33:12–24.

[49] Cai W, Jiang C, Wang Y, et al. Structure-preserving algorithms for the two-dimensional sine-Gordon equation with Neumann boundary conditions. J Comput Phys. 2019;395:166–185.

[50] Wang JY, Huang QA. A family of effective structure-preserving schemes with second-order accuracy for the undamped sine-Gordon equation. Comput Math Appl. 2021;90:38–45.