Quantum Computing with Global One-and Two-Qubit Gates

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Abstract

We present generalized and improved constructions for simulating quantum computers with a polynomial slowdown on lattices composed of qubits on which certain global versions of one- and two-qubit operations can be performed.

1 Introduction

In [9] we have shown that usual quantum circuits can be efficiently simulated on several regions of lattices composed of qubits on which instead of individual addressing of qubits we can perform certain global versions of one and two-qubit gates. The Hamiltonian of the global version of a two-qubit gate acting on a pair of a qubit is the (weighted) average of the translates of the Hamiltonian of the two-qubit gate. In [9] translations by lattice vectors were considered. A brief discussion and references to experimental results regarding implementation of global gates can also be found in [9]. Our simulation requires a constant blowup of space and the slowdown is polynomial.

Global two-qubit Hamiltonians considered in [9] and in this paper are special cases of Hamiltonians built from pairwise interactions. Results regarding efficient simulation of all such Hamiltonians using a fixed one with connected interaction graph and local one-qubit operations can be found e. g. in [8] (or in [13] and [12] in the context of NMR quantum computers). In [11], the sets of operations which are implementable as products of global gates from various collections are described. In the latter manuscript efficiency is not addressed: products of arbitrary length are allowed.

Methods not requiring individual addressability of local spins are given in a series of papers by Benjamin and his collaborators. In [2] it is shown how

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different levels of addressability assumptions can be dispensed with in the case of a chain of spins. Multidimensional generalisations and issues related to physical implementations including error correction techniques are discussed in [1, 4, 3]. In an intermediate proposal, given in [2], spins only at even and odd positions have to be distinguished, but there is also a method that – like the proposals in [9] and the present paper – does not require that distinction either. Note however, that the endpoints of the chain play an essential role in the proposal given in [2]. In this paper symmetry receives a more attentive treatment and we do not make use of edge effects. The most closely related work to the present paper is probably [15], where translation and reflection symmetric global operations in a chain of 5-level local systems were considered.

In this paper we strengthen the results of [9] in two directions. First, we extend the method of [9] to more general models of global gates. In these models, the global Hamiltonians are built from translates of local ones by elements of groups more general than translation groups. In case of multidimensional lattices these groups may include not only translations but also transformations like rotations and reflections. Most notably, we propose constructions for the case of hypercubic lattices and global gates acting simultaneously on pairs of qubits at various Euclidean distances. In contrast to [9], the constructions in the present paper do not exploit that the domain containing the lattice points we are working with has borders.

Another direction of improvement is restricting the set of global two-qubit gates needed for the simulation. We show that efficient simulation is possible with global one-qubit gates and global versions of two-qubit gates whose Hamiltonians are diagonal in the computational basis. As the translates of such gates commute, in these cases the global gate is just the simultaneous action of the local translates. We note that some of the global gates in [9] are not of this form and therefore it seems to be much harder to implement them in practice.

It is also natural to restrict the ”distances” of pairs of qubits which we let global gates act on. The most natural restriction would be allowing only next-neighbour interactions (e.g., distance 1). Results with next-neighbor interactions can be found in [14] and [15]. The latter paper presents a method which works with next-neighbor interactions in lattices composed of 5-level systems rather than qubits. The method of [14] works with a chain of qubits but it exploits that the chain has endpoints. It is not known if there is a version of our method using only global two-qubit gates with distance 1. In the one-dimensional case we can present a version in which the global two-qubit gates have distance at most 22. It works with qubits and does not make use of the border of the chain. Furthermore, our method also works if there is certain imperfection in the global gates.

The constructions of the present paper require an initial state where qubits at certain positions are set to one while the others are set to zero. Obviously, form the all-zero state such state cannot be achieved using global gates. In [15] this problem is circumvented using certain techniques such as running several reflection symmetric simulations at sufficiently large distances simultaneously. Such techniques seem to be applicable in the case of 2-level (qubit) systems as
well – maybe using a bigger system of global two-qubit gates in the initialization phase.

The structure of this paper is the following. The model of global gates is described in Section 2. Our main theorem is stated in Section 3, where also combinatorial notions describing the schemes making efficient simulation possible are introduced. Section 5 is devoted to examples for such schemes, while the proof of our main result can be found in Section 4. We present a construction for simulation using only global two-qubit gates with distances between 1 and 22 in Section 6.

2 Global gates

In this section we introduce some notation and describe a general mathematical model for global one- and two-qubit gates.

Throughout the paper we assume that $G$ is a transitive permutation group on a possibly infinite set $\Omega$ and $D$ is a subset of $\Omega$. For $p \in \Omega$ and $g \in G$ we denote by $p^g$ the image of $p$ under $g$. The permutation action of $G$ induces an equivalence relation $\sim$ on $D \times D$: $(p, q) \sim (p', q')$ if there is an element $g \in G$ such that $p' = p^g$ and $q' = q^g$. We denote by $O$ the set of the equivalence classes of $\sim$ different from the diagonal $\{(p, p) \mid p \in D\}$. For $(p, q) \in D \times D$ the equivalence class containing $(p, q)$ is denoted by $C\,(p, q)$.

We encourage the reader to consider the following instructive class of examples. Here $\Omega = R^s$, the $s$-dimensional Euclidean space and $G$ is the Euclidean group $E(s)$ which consists of isometries of $R^s$. For every subset $D \subseteq R^s$, two pairs $(p, q), (p', q')$ of $D \times D$ are equivalent under the action of $E(s)$ if and only if the distances $|p - q|$ and $|p' - q'|$ are the same. As a consequence, the classes of $O$ can be indexed by positive real numbers (distances) $\delta$ and $(p, q) \in C_\delta$ if and only if $|p - q| = \delta$. (Of course, depending on the choice of $D$, the class $C_3$ may be empty for certain $\delta$’s. We shall refer to this geometric example in intuitive explanations of abstract notions and arguments.

To define global gates, we assume that $D$ is finite with $|D| = n$ and we have a configuration of $n$ qubits sitting at each element of $D$. The Hilbert space of the pure states over these $n$ qubits is $C^2^{|D|}$. The elements of the standard basis are indexed by the functions $a : D \rightarrow \{0, 1\}$. In order to shorten notation, for $p \in D$ we also write $a_p$ for the value $a(p) \in \{0, 1\}$.

As a part of our model we introduce a balance function $W : D \times D \rightarrow R_{>0}$ on the pairs of $D$. This function will make it possible to model effects like imperfection of global gates, namely that the tool performing a global gate (e.g., a laser beam or a magnetic field) may act with different strength on pairs or singletons of qubits at different positions. Of course we could take $W$ to be constant. This choice would express invariance of global gates under the action of $G$ perfectly. In order to shorten notation we will also denote $W(p, p)$ by $W(p)$.

Below we give a formal definition of the model of global gates. The first part is devoted to describe how global Hamiltonians are built from local ones.
Then global gates are obtained from global Hamiltonians in the usual way. As in Section 4 we shall make use of the formalism introduced here in the context of arbitrary operations, in the first part we do not assume Hermiticity of operations. That is, by an operation we mean an arbitrary linear transformation or matrix of the appropriate dimension. However, the reader not interested in the details given in Section 4 may think merely of Hamiltonians first.

For a one-qubit operation or $2 \times 2$ matrix $M$, whose rows and columns are indexed by 0 and 1, and an element $p \in D$ we write $M^p$ for the $n$-qubit operation which acts as $M$ on the qubit at position $p$:

$$M^p_{a,b} = \begin{cases} M_{a_p,b_p} & \text{if } a_s = b_s \text{ for every } s \in D \setminus \{p\}, \\ 0 & \text{otherwise.} \end{cases}$$

We will refer to $M^p$ as a local one-qubit operation at position $p$. The global one-qubit operation corresponding to $M$ is

$$M^p = \sum_{p \in D} W(p) M^p.$$  

Similarly, for a 2-qubit operation or $2^2 \times 2^2$ matrix $M$ (rows and columns indexed by 00, 01, 10, 11) and a pair of elements $p \neq q \in D$, the local two-qubit operation $M^{(p,q)}$ at position $(p, q)$ is defined as

$$M^{(p,q)}_{a,b} = \begin{cases} M_{(a_p,a_q),(b_p,b_q)} & \text{if } a_s = b_s \text{ for every } s \in D \setminus \{p, q\}, \\ 0 & \text{otherwise}, \end{cases}$$

and for a $C \in \mathcal{O}$ the global two-qubit operation is

$$M^{C} = \sum_{(p,q) \in C} W(p,q) M^{(p,q)}.$$  

A two-qubit global Hamiltonian is a matrix of the form $H^{C}$ where $H$ is an Hermitian 2-qubit operation and a global 2-qubit gate is an operation of the form $\exp(-iH^{C})$ where $H^{C}$ is a global two-qubit Hamiltonian. Global one-qubit gates are of the form $\exp(-iH^{p})$ where $H$ is an Hermitian one-qubit operation.

Note that global one-qubit gates can be interpreted as parallel execution of local operations since

$$\exp(-iH^{p}) = \prod_{p \in D} \exp(-iW(p)H^{p}),$$

where the product on the right hand side can be taken in an arbitrary order of the terms. The analogous statement for global two-qubit operations is not true in general. Only if for all $(p, q), (p', q') \in C$ the local operators $H^{(p,q)}$ and $H^{(p',q')}$ commute (this happens for example if $H$ is diagonal in the standard basis) can $\exp(-iH^{C})$ be decomposed as $\prod_{(p,q) \in C} \exp(-iW(p,q)H^{(p,q)})$.  

4
3 Simulation results

An intuitive description of our method for simulating a "usual" quantum computation with global gates acting on $D$ is the following. We will designate two disjoint subsets $P$ and $R$ of $D$. The qubits at positions in $R$ will serve as reference points. Before and after each logical step of the simulation, their value will be set to one. Similarly, the values of the qubits at positions outside $P$ and $R$ will be set to zero. The subset $P$ will be the workspace. The qubits at positions in $P$ can take arbitrary values and in each logical step we simulate a local one- or two-qubit gate at certain (pairs of) positions in $P$. To be more precise, each logical step will approximate such a gate.

By approximation we mean approximation in terms of the operator norm. For an operator $U$ on the Hilbert space $\mathbb{C}^2$ we denote by $\|U\|$ the operator norm of $U$: $\|U\| = \sup_{\|x\|=1} |Ux|$. Note that $\|AB\| \leq \|A\| \cdot \|B\|$. Furthermore, if $\|A_1\|, \ldots, \|A_N\|, \|B_1\|, \ldots, \|B_N\| \leq 1$ (this holds in particular if $A_i$ and $B_i$ are unitary operators), then we have

$$\|A_1 \cdots A_N - B_1 \cdots B_N\| \leq \sum_{j=1}^{N} \|A_j - B_j\|. \tag{1}$$

We say that $A$ $\epsilon$-approximates $B$ if $\|A - B\| \leq \epsilon$. Equation (1) implies that in order to $\epsilon$-approximate a circuit consisting of $\ell$ gates it is sufficient to $\epsilon/\ell$-approximate each gate occurring in the circuit.

Below is a very informal – metaphoric – description of our simulation method. In this description qubits correspond to two-state ("up" or "down") switches placed at positions in $D$. The global one-qubit gates are be modelled by the ability of flipping the state of the switches simultaneously. The additional device corresponding to the two-qubit gates with distance $\delta$ is the following. For every $p \in D$, this tool commits the change of switch at $p$ if there is a position $q$ at distance $\delta$ such that the original state of the switch at $q$ was "up". The value of the commitment is defined as the number of such points $q$. After a sequence of commitment operations the change is performed at positions where it has been committed by all of the commitments.

We work with states where the switches at points in $R$ will be always "up", switches at points in $P$ can be both "up" and "down", while the rest are always "down". The goal is to be able switching the state of a specific switch in $p \in P$ by keeping the state of the others using the operations described above. We also assume that this must be done performing the global flip operation followed by a sequence of commitments which depend only on the position $p$ and not on the particular state of the switches in $P$. Furthermore, we require that for committed flips, the total value of commitments does not depend either on the particular state. To perform this we need switches at $r_1, \ldots, r_k$ in $R$ such that in any case, commitment operations with distances $|p - r_1|, \ldots, |p - r_k|$ commit the flip only at position $p$. That is, we need to be able to uniquely locate point $p \in P$ in such a strong sense just by the distances of it form points of $R$. We shall use the term "addressable" to express this property.
We stress that the description above is only metaphoric, looking for any direct connection with the reality would be misleading. The actual and accurate description of the simulation method can be found in the rather technical Section 4. Below we give the formal definitions.

We call a pair \((P, R)\) of disjoint subsets of \(D\) a \textit{workspace–base scheme}. We say that the workspace–base scheme \((P, R)\) admits the function \(a : D \rightarrow \{0, 1\}\) (or \(a\) is admissible if \((P, R)\) is clear from the context) if \(a_r = 1\) for every \(r \in R\) and \(a_q = 0\) for every \(q \in D \setminus (P \cup R)\). We say that \(p \in P\) is addressable by \(r_1, \ldots, r_k \in R\) if for every \(k + 1\)-tuple \((p', r'_1, \ldots, r'_k) \in D \times (P \cup R)^k\), the property that for each \(i = 1, \ldots, k\) either \((p', r'_i) \in C_{(p, r_i)}\) or \((r'_i, p') \in C_{(p, r_i)}\) for each \(i = 1, \ldots, k\) implies \(p' = p\) and \(r'_1, \ldots, r'_k \in R\). We say that \(p \in P\) is \(k\)-addressable if there exist \(r_1, \ldots, r_k \in R\) such that \(p\) is addressable by \(r_1, \ldots, r_k\). We call \((P, R)\) \(k\)-addressable if every \(p \in P\) is \(k\)-addressable.

The subspace of \(\mathbb{C}^{2^n}\) spanned by the basis elements corresponding to the admissible functions is obviously isomorphic to \(\mathbb{C}^{2^{|P|}}\). Its elements (of length 1) are called admissible vectors (or states, respectively). The remaining basis elements will span the orthogonal complement of the subspace of admissible states. The vectors (or states) in the latter subspace will be called inadmissible. Our first (and main) goal is to produce unitary operations which (approximately) preserve the subspace of admissible vectors and, restricted to this subspace, (approximately) act as local one-qubit gates. In other words, if we adopt an order of the basis where the first \(2^{|P|}\) basis elements correspond to the admissible functions and the rest correspond to the inadmissible functions, we will produce operations whose matrices are (approximately) block diagonal and have upper left \(2^{|P|} \times 2^{|P|}\) block which is (approximately) the same as the matrix of a local one-qubit gate acting on a qubit at position \(p \in P\). Then, having the local one-qubit gates at hand in order to get a universal quantum computer on on the admissible states, it is sufficient to use a unitary operator which has an all-pair entangling Hamiltonian on the admissible states.

We shall achieve our main goal by taking commutators of certain global one-qubit Hamiltonians with Hamiltonians of the form \(A^C\) at the Lie algebra level, where \(C \in \mathcal{O}\) and \(A\) is the Hamiltonian of the so-called \textit{controlled phase shift} operation. The entries of the matrix of this two-qubit Hamiltonian are all zero except in the lower right corner (corresponding to the basis element indexed by 11):

\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]  

(2)

The corresponding global two-qubit \textit{gates} are

\[
\exp\left(\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -T_i
\end{pmatrix}\right) \begin{pmatrix} C_{(p, r)} \end{pmatrix}, \quad (T \in \mathbb{R}, p \in P, r \in R),
\]  

(3)
where $i$ stands for the imaginary unit $\sqrt{-1}$.

We shall approximate the local one-qubit gates by products of global two-qubit gates of the form (3) and global one-qubit gates of the form

$$\exp \left( \left( \begin{array}{cc} 0 & z \\ -z & 0 \end{array} \right)^{0} \right), \quad (z \in \mathbb{C}). \quad (4)$$

In Section 4 we shall prove the following.

**Theorem 1** Assume that for every $p', p'', q', q'' \in D$ we have $\frac{W(p', q')}{W(p'', q'')} \leq w$. Assume further that $(P, R)$ is a workspace–base scheme on $D$. Then, for every $k$-addressable $p \in P$, and for every $0 < \epsilon < 1$, every operation which acts on admissible states as a one-qubit gate at $p$ with Hamiltonian of norm bounded by a constant can be $\epsilon$-approximately by a product of $(nw/\epsilon)^{O(k^2)}$ global gates of type (3) and (4).

If the "distance" between two points $p, q \in P$ cannot occur as a "distance" between a point $p'$ in $P$ and another point $r$ in $R$, that is

$$(C_{p,q} \cup C_{q,p}) \cap C_{p',r} = \emptyset \quad \text{for every } p, q, p' \in P \text{ and } r \in R,$$

then the global gate

$$\exp \left( \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -T_i \end{array} \right)^{C_{p,q}} \right) \quad (6)$$

acts on the admissible states as a unitary operator whose Hamiltonian entangles the qubits at $p$ and $q$ (and also the pairs of qubits in $P$ with the same "distance"). Global gates of this type, together with the simulated one-qubit gates on admissible states, using the selective decoupling techniques (see e.g. [12, 13, 8]), give an efficient simulation of any quantum circuit. We obtain the following.

**Corollary 2** Assume that for every $p', q', q'', q'' \in D$ we have $\frac{W(p', q')}{W(p'', q'')} \leq w$. Assume further that $(P, R)$ is a $k$-addressable workspace–base scheme on $D$ with additional property (5). Then, for every $0 < \epsilon < 1$, the action of any quantum circuit of length $\ell$ on admissible states can be $\epsilon$-approximated by a product of $(nw/\epsilon)^{O(k^2)}$ global gates of type (3), (4) and (6).

Here by a quantum circuit we mean a circuit built from 2-qubit gates, that is a sequence of unitary 2-qubit operations. As 2-qubit gates are universal (see e.g. [7]), any circuit in the sense of [6] can be efficiently simulated as well.

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1We are indebted to an anonymous referee for suggesting the argument above. This has made it possible to replace a much more complicated argument – and a somewhat weaker statement – in a preliminary version of this paper.
4 Extracting local gates by taking commutators

Our main tool will be taking commutators with global two-qubit Hamiltonians of the form $A^C$, where the matrix $A$ is as in (2). This will be the tool analogous to the commitment operation in the metaphoric description. See Proposition 7 for the effect of an appropriate sequence of such commutations. It is easy to see that, for every $C \in \mathcal{O}$, $A^C$ is a diagonal matrix whose entry at position $a, a$ is
\[ A^C_{a,a} = \sum_{(p,q) \in C} W(p,q). \]

It follows that for arbitrary $a, b : D \to \{0, 1\}$,
\[ [A^C, E_{a,b}] = \left( \sum_{(p,q) \in C} W(p,q) - \sum_{(p,q) \in C} W(p,q) \right) E_{a,b} \]
where $E_{a,b}$ is the $2^{|D|} \times 2^{|D|}$ elementary matrix having one at position $a, b$ and zero elsewhere. For $a : D \to \{0, 1\}$ we denote by $\partial_p a$ the function which takes value zero at $p$ and coincides with $a$ on $D \setminus \{p\}$. That is, $\partial_p a$ is obtained from $a$ by deleting the $p^{th}$ bit. Commutators of $A^C$ with elementary matrices of the form $E_{a,\partial_p a}$ turn out to be of particular interest. We have $[A^C, E_{a,\partial_p a}] = 0$ if $a_p = 0$ and
\[ [A^C, E_{a,\partial_p a}] = \left( \sum_{(p,q) \in C} W(p,q) + \sum_{(q,p) \in C} W(q,p) \right) E_{a,\partial_p a} \quad (7) \]
if $a_p = 1$.

For $C \in \mathcal{O}$ we define a modified balance function $W_C$ on $D \times D \setminus \{(p,p) | p \in D\}$ by
\[ W_C(p,q) = \begin{cases} 0 & \text{if } (p,q) \notin C \text{ and } (q,p) \notin C, \\ W(p,q) & \text{if } (p,q) \in C \text{ and } (q,p) \notin C, \\ W(q,p) & \text{if } (q,p) \in C \text{ and } (p,q) \notin C, \\ W(p,q) + W(q,p) & \text{if } (p,q) \in C \text{ and } (q,p) \in C. \end{cases} \]
Iterated application of (7) gives the following two lemmas.

Lemma 3 Let $p \in D$, $C_1, \ldots, C_k \in \mathcal{O}$, and $a : D \to \{0, 1\}$ with $a_p = 1$. Assume further that there is no $k$-tuple $q_1', \ldots, q_k' \in D$ such that $a_{q_1'} = \ldots = a_{q_k'} = 1$ and either $(p,q_i') \in C_i$ or $(q_i',p) \in C_i$ holds for every $i = 1, \ldots, k$. Then
\[ [A^{C_1}, \ldots, [A^{C_k}, E_{a,\partial_p a}]] = 0 \]
Lemma 4 Let \( p \in D, C_1, \ldots, C_k \in \mathcal{O}, \) and \( a : D \to \{0, 1\} \) with \( a_p = 1 \). Let \( Q \subseteq D \setminus \{p\} \) such that \( a_q = 1 \) for every \( q \in Q \) and \( q_1, \ldots, q_k \in Q \) such that \( (p, q_i) \in C_i \) or \( (q_i, p) \in C_i \) for \( i = 1, \ldots, k \). Assume further that if for a \( k \)-tuple \( q'_1, \ldots, q'_k \in D \) we have \( a_{q'_1} = \ldots = a_{q'_k} = 1 \) and either \( (p, q'_i) \in C_i \) or \( (q'_i, p) \in C_i \) holds for every \( i = 1, \ldots, k \), then \( q'_1, \ldots, q'_k \in Q \). Then

\[
[A^{C_1}, \ldots, [A^{C_k}, E_{a, \partial_q a}]] = W_1 \cdots W_k E_{a, \partial_p a},
\]

where \( W_i = \sum_{q \in Q} W_{C_i}(p, q) \).

Lemma 5 Let \((P, R)\) be the workspace–base scheme, \( a : D \to \{0, 1\} \), \( p \in P \) and \( q \in D \). Assume that \( p \) is addressable by \( r_1, \ldots, r_k \in R \). Under these assumptions, if \( a \) is admissible then

\[
[A^{C_1}, \ldots, [A^{C_k}, E_{a, \partial_q a}]] = \begin{cases} 
W_1 \cdots W_k E_{a, \partial_q a} & \text{if } q = p \text{ and } a_p = 1, \\
0 & \text{otherwise},
\end{cases}
\]

where \( C_i = C_{(p, r_i)} \) and \( W_i = \sum_{r \in R} W_{C_i}(p, r) \) for \( i = 1, \ldots, k \). If \( a \) is not admissible but \( \partial_q a \) is admissible then

\[
[A^{C_1}, \ldots, [A^{C_k}, E_{a, \partial_q a}]] = 0.
\]

Proof. Assume that either \( a \) or \( \partial_q a \) is admissible and that the commutator is nonzero. Then \( a_q = 1 \), and, by Lemma 3 there exist \( r'_1, \ldots, r'_k \in D \) such that \( a_{r'_1} = \ldots = a_{r'_k} = 1 \) and either \((q, r'_i) \in C_i \) or \((r'_i, q) \in C_i \). Since \( a \) or \( \partial_q a \) is admissible, \( r'_i \in P \cup R \) and from the definition of workspace–base schemes it follows that \( q = p \) and \( r'_i = r_i \) for \( i = 1, \ldots, k \). This shows the second statement and part of the first one. The rest follows from Lemma 4.

For an elementary one-qubit global operation we have the following.

Lemma 6 Let \((P, R)\) be a workspace–base scheme, \( a, b : D \to \{0, 1\} \), and assume that \( p \in P \) is addressable by \( r_1, \ldots, r_k \). Suppose further that either \( a \) or \( b \) is admissible. Then, for the \( 2 \times 2 \) matrix

\[
B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]

we have

\[
[A^{C_1}, \ldots, [A^{C_k}, B_{a,b}^\circ]] = W(p)W_1 \cdots W_k B_{a,b}^p,
\]

where \( C_i = C_{(p, r_i)} \) and \( W_i = \sum_{r \in R} W_{C_i}(p, r) \) for \( i = 1, \ldots, k \).

Proof. We denote by \( C \) the commutator on the left hand side of the asserted equality. For every \( q \in D \) we have \( B^q = \sum_{a \mid a_q = 1} E_{a, \partial_q a} \). Therefore

\[
B^\circ = \sum_{q \in D} W(q) \sum_{a \mid a_q = 1} E_{a, \partial_q a} = \sum_a \sum_{q \mid a_q = 1} W(q) E_{a, \partial_q a}
\]
From this equality, using Lemma 5, we infer

\[ C_{a,b} = \begin{cases} W(p) \prod_{i=1}^{k} W(p, r_i) & \text{if } b = \partial_p(a) \text{ and } a_p = 1, \\ 0 & \text{otherwise}, \end{cases} \]

whenever either \( a \) or \( b \) is admissible. From the latter equality the assertion follows because \( B_{a,b}^p \) is one if \( a_p = 1, b = \partial_p a \); and zero otherwise. \( \square \)

For a global one-qubit Hamiltonian we obtain the following.

**Proposition 7** Assume that \((P, R)\) is a workspace–base scheme on \(D\), and let

\[ X = \begin{pmatrix} 0 & z \\ -\overline{z} & 0 \end{pmatrix}, \]

where \( z \) is a complex number. Suppose further that \( p \in P \) is addressable by \( r_1, \ldots, r_k \in R \). Then for every pair \( a, b : D \to \{0, 1\} \) such that either \( a \) or \( b \) is admissible, we have

\[ [-i A^{C_1}, \ldots, [-i A^{C_k}, X]]_{a,b} = W(p) W_1 \cdots W_k \tilde{X}_{a,b}, \]

where \( C_i = C_{(p, r_i)} \), \( W_i = \sum_{r \in R} W_{C_i}(p, r) \), and

\[ \tilde{X} = \begin{pmatrix} 0 & (-i)^k z \\ -i^k \overline{z} & 0 \end{pmatrix}. \]

**Proof.** Observe that \( X = zB - \tau B^\dagger \) where \( B \) is the same \( 2 \times 2 \) matrix as in Lemma 6. Hence, using also that the matrices \( A^{C_i} \) are self-adjoint, we obtain

\[
\begin{align*}
[-i A^{C_1}, \ldots, [-i A^{C_k}, X]] &= (-i)^k [A^{C_1}, \ldots, [A^{C_k}, X]] \\
&= z(-i)^k [A^{C_1}, \ldots, A^{C_k}, B^\circ] - \tau(-i)^k [A^{C_1}, \ldots, A^{C_k}, B^\circ]\] \\
&= z(-i)^k [A^{C_1}, \ldots, A^{C_k}, B^\circ] - \tau(-i)^k [A^{C_1}, \ldots, A^{C_k}, B^\circ]^\dagger
\end{align*}
\]

Using Lemma 6, this equality gives

\[
\begin{align*}
[-i A^{C_1}, \ldots, [-i A^{C_k}, X]]_{a,b} &= z(-i)^k W(p) W_1 \cdots W_k B_{a,b}^p \\
&\quad - \tau(-i)^k W(p) W_1 \cdots W_k B_{a,b}^{1p} \\
&= W(p) W_1 \cdots W_k \tilde{X}_{a,b},
\end{align*}
\]

whenever either \( a \) or \( b \) is admissible. \( \square \)

Proposition 7 is used to prove Theorem 1, i.e., to show that the local one-qubit \textit{gates} can be efficiently approximated. We need the following fact on approximation of an operator whose Hamiltonian is a commutator. For the proof, see e.g., [9].
Fact 8 There is an absolute constant $c > 0$, such that
\[
\left\| \left( \exp\left( -\frac{i}{\sqrt{N}} X^{-1} \right) \cdot \exp\left( -\frac{i}{\sqrt{N}} Y^{-1} \right) \cdot \exp\left( -\frac{i}{\sqrt{N}} X \right) \cdot \exp\left( -\frac{i}{\sqrt{N}} Y \right) \right)^N - \exp\left( [iX, -iY] \right) \right\| < c \cdot M^3 N^{-\frac{1}{2}}
\]
for any $N > M^2$, where $X$ and $Y$ are Hermitian operators on the Hilbert space $\mathbb{C}^{2^n}$ and $M = \max\{\|X\|, \|Y\|, 1\}$.

We remark that commutators (both group and Lie theoretic, in view of Fact 8) also play an important role in proof the Kitaev–Solovay theorem (see e.g., [10]). It turns out that decomposing unitaries as group theoretic commutators can be used to exponentially improving a sufficiently good approximation of a unitary operator by a circuit built from a given gate set.

Now we are in a position to prove Theorem 1.

Proof of Theorem 1. We show the statement for gates of type
\[
\exp\left( \begin{pmatrix} 0 & z \\ -\overline{z} & 0 \end{pmatrix}^p \right), \quad (z \in \mathbb{C}, |z| \leq 1).
\]

Then, the general case for a general one-qubit gate follows from Fact 8 and Trotter’s formula since the third basis element
\[
\iota \sigma_z = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}
\]
of the Lie algebra $su_2$ equals $\frac{1}{2}$ of the commutator of the first two ones which are
\[
\iota \sigma_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \iota \sigma_x = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.
\]

Let
\[
Z = \frac{e^{k+1}}{W(p)} \begin{pmatrix} 0 & z \\ -\overline{z} & 0 \end{pmatrix}^\circ,
\]
and for $i = 1, \ldots, k$ let $X_i = 1/W_i A^C_{(p,r_i)}$ where $W_i = \sum_{r \in R} W_{C(p,r_i)}(p,r)$. Then, by Proposition 7 on admissible states the operator $\exp(iX_1, \ldots, -iX_k, -iZ)$ acts in the same way as the required one-qubit local gate of type 8. For $i = 1, \ldots, k$ we denote the commutator $[iX_1, \ldots, -iX_k, -iZ]$ by $Y_i$. Also set $Y_{k+1} = -iZ$. Then for $i = 1, \ldots, k$ we have $Y_i = [-iX_i, Y_{i+1}]$. We have $\|Z\| = O(nw)$, $\|X_i\| = O(nw)$ and $\|Y_i\| = O((2nw)^{k-i+1})$. It follows that all the norms $\|Y_i\| (i = 2, \ldots, k+1)$, $\|X_i\| (i = 1, \ldots, k)$ are bounded by $M = O(2nw)^k$.

Set $\epsilon_1 = \epsilon$. By Fact 8 we obtain an $\epsilon/2$-approximation of $\exp Y_i$ by a product containing $2N_i$ terms of $\exp(\pm X_i)$ and $2N_i$ terms of $\exp(\pm Y_{i+1})$, where $N_i = O(M^6)/\epsilon_i$. If we substitute each factor $\exp(\pm V_{i+1})$ by approximations with error $\epsilon_{i+1} = \epsilon_i/4N_i$, the total error of the product will be bounded by $\epsilon_i$. For the sequence $\epsilon_i$ we obtain a recursion $\epsilon_{i+1} = O(M^6 \epsilon_i)$ ($i = 1, \ldots, k - 1$). That
5 Addressable workspace–base schemes

In this section we give some examples of addressable workspace–base schemes corresponding to various group actions. In order to obtain nice ("uniform") constructions, we extend the notion of addressability (and connectedness) to possibly infinite subsets $D \subseteq \Omega$ in the obvious way.

Assume that $(P, R)$ is a workspace–base scheme in $D$ with $R = \{r_1, \ldots, r_k\}$. We say that $(P, R)$ is strictly addressable if for every $p \in P$ and for every $k + 1$-tuple $(p', r_1', \ldots, r_k') \in D \times (P \cup R)^k$ and $c_{(p, r_i)} \cup c_{(r_i, p)}$ $(i = 1, \ldots, k)$ implies $p' = p$ and $r_i' = r_i$ $(i = 1, \ldots, k)$. This means that every $p \in P$ is addressable by the whole of $R$ in a strict sense where – in terms of our geometric example – the sequence of "distances" of $p$ from $r_i$ identifies $p$ and $r_1, \ldots, r_k$. We note that strict addressability implies condition 4 and therefore Corollary 2 is applicable to strictly addressable schemes.

5.1 Schemes for translation-invariant global gates

This part is devoted to constructions of strictly addressable workspace–base schemes for translation-invariant global gates in abelian groups, especially in the $s$-dimensional lattice $\mathbb{Z}^s$. In particular, we shall show that $(P_1, R_1)$ with $R_1 = \{1, 3, 9\}$ and $P_1 = 4\mathbb{Z} = \{\ldots, -12, -8, -4, 0, 4, 8, 12, \ldots\}$ is a strictly addressable scheme for translations in $\mathbb{Z}$ of density $1/4$.

The model of translations in $\mathbb{Z}$ can be interpreted as that the qubits sit in a doubly infinite chain and the group acting on the chain is given by the shifts. Assuming a constant balance function, the global gates are shift-invariant. Just like in Section 2, the actual (finite) version of the model is obtained by restricting the action of the global operation to a finite section of the chain.

For all positive integers $s$ the scheme above lifts to strictly addressable schemes $(P_s, R_s)$ where $R_s = \{(1, 0, \ldots, 0), (3, 0, \ldots, 0), (9, \ldots, 0)\}$ and $P_s = \{(4z_1, z_2, \ldots, z_s) | z_i \in \mathbb{Z}\}$. It is an open question whether $1/4$ is indeed the optimal density or there are more efficient constructions. We start with the formal definitions and give some lemmas which may be useful in constructing other addressable schemes.

Let $G$ be an abelian group. We use the additive notation for the group operations. The group $G$ acts on $\Omega = G$ by shifting: $p^g = p + g$. We set $D = G$. As $(p, q) \sim (p', q')$ iff $p' - q' = p - q$, the elements of $\mathcal{O}$ can be indexed by the elements of $G \setminus \{0\}$: $(p, q) \in c_g$ iff $p - q = g$. In other words, $c_{(p, q)} = c_{p - q}$.

Under some mild restrictions, we can lift strictly addressable schemes from factor groups as follows.

**Lemma 9** Assume that $\phi$ is a surjective homomorphism from the abelian group $G$ onto $K$, and $(Q, S = \{s_1, \ldots, s_k\})$ is a strictly addressable scheme on $K$. Let

$$
\epsilon_i = (c'M^6)^{1-i}\epsilon \text{ for some constant } c' \text{ and hence } N_i = O((c'M^6)^i).
$$

The total number of terms is $4N_1 + 4N_2 + \ldots + 4N_k = O(k(4c'M^6)^{k+1}/2) = (nw)^{O(k^2)}$. 

\[\square\]
$P = \phi^{-1}Q$. For each $i = 1, \ldots, k$ we pick a single element $r_i$ from $\phi^{-1}\{s_i\}$. Then, assuming that there exist $i, j \in \{1, \ldots, k\}$ such that $2r_i \neq 2r_j$, the scheme $(P, R = \{r_1, \ldots, r_k\})$ is a strictly addressable on $G$.

**Proof.** Assume that $p' - r' = \pm(p - r)$ with $p \in P$, $p' \in G$, $r' \in P \cup R$ ($i = 1, \ldots, k$). Then $\phi(p') - \phi(r') = \pm(\phi(p) - \phi(r))$ and $\phi(p) \in Q$ and $\phi(r') \in Q \cup S$, and strict addressability of $(Q, S)$ implies $\phi(p') = \phi(p)$ and $\phi(r') = \phi(r)$ for $i = 1, \ldots, k$. From the latter facts and disjointness of $Q$ and $S$ we infer that $r' = r_i$ for $i = 1, \ldots, k$. Hence $p' - r_i = \pm(p - r_i)$ for $i = 1, \ldots, k$. If $p \neq p'$ then this is only possible if $2r_i = p + p'$ for every $i$. 

Lemma 9 gives several straightforward ways of constructing strictly addressable schemes in multidimensional lattices from one-dimensional schemes. In particular, strict addressability of the scheme $(P_1, R_1)$ given above implies that of $(P_s, R_s)$ for every $s$.

As another application of Lemma 9 we can give a simple construction for a strictly addressable scheme of density $1/14$ in $\mathbb{Z}$.

**Proposition 10** In $\mathbb{Z}_{14}$, the additive group of integers modulo 14, the scheme $\{0\}, \{1, 3, 7\}$ is a strictly addressable workspace–base scheme. As a consequence, so is the scheme $\{14z | z \in \mathbb{Z}\}, \{1, 3, 7\}$ in $\mathbb{Z}$.

**Proof.** Assume that for $p' \in \mathbb{Z}_{14}$ and for $r'_1, r'_2, r'_3 \in \{0, 1, 3, 7\}$ we have $p' - r'_1 = \pm 1, p' - r'_2 = \pm 3, p' - r'_3 = \pm 7$. If $p'$ is odd then $r'_1, r'_2, r'_3$ must be even therefore $r'_1 = r'_2 = r'_3 = 0$ and hence $\{\pm(p' - r'_i) | i = 1, 2, 3\} = \{\pm p\}$, a set containing at most 2\(3\) elements, a contradiction. Therefore $p'$ is even and $r'_1, r'_2, r'_3$ must be odd which is only possible if $\{r'_1, r'_2, r'_3\} \subseteq \{1, 3, 7\}$. On the other hand, $\{\pm(2 - 1), \pm(2 - 3), \pm(2 - 7)\} = \{1, 3, 9, 5\}, \{\pm(4 - 1), \pm(4 - 3), \pm(4 - 7)\} = \{3, 11, 1, 13\}, \{\pm(6 - 1), \pm(6 - 3), \pm(6 - 7)\} = \{5, 9, 3, 11, 13, 1\}, \{\pm(8 - 1), \pm(8 - 3), \pm(8 - 7)\} = \{7, 5, 9, 1, 13\}, \{\pm(10 - 1), \pm(10 - 3), \pm(10 - 7)\} = \{9, 5, 7, 3\}$ and none of these sets contain $\{1, 3, 7\}$, therefore $p' = 0$. As $-1$ and $-3$ are not in $\{1, 3, 7\}$, the only possibility is that $r_1 = 1, r_2 = 3$ and $r_3 = 7$. The second statement follows from Lemma 9.

We proceed with the somewhat tedious proof of strict addressability of the scheme $(P_1, R_1)$ defined above. In order to support similar but potentially more efficient constructions, we give the proof in form of some statements.

Note that strictly addressable schemes $(P, R)$ where $P$ is non-empty and $|R| \leq 2$ do not exist in any abelian group $G$. In the remainder of this subsection we consider schemes with three-element bases.

**Lemma 11** Let $G$ be an abelian group, $r_1, r_2, r_3 \in G$ such that $r_i - r_j \neq r'_i - r'_j$ for $(i, j) \neq (i', j')$. Then for every $p \in G$ such that $2p \notin \{r_1 + r_2, r_1 + r_3, r_2 + r_3\}$ for every $p' \in G$, for every $r'_1, r'_2, r'_3 \in \{r_1, r_2, r_3\}$ the condition $p - r_i = \pm(p' - r'_i)$ ($i = 1, 2, 3$) implies $p' = p, r'_1 = r_1, r'_2 = r_2$ and $r'_3 = r_3$. 


Proof. As $2p \not\in \{r_1 + r_2, r_1 + r_3, r_2 + r_3\}$, we have $p - r_i \neq \pm(p - r_j)$ for $i \neq j$. Therefore $\{r'_1, r'_2, r'_3\} = \{r_1, r_2, r_3\}$ and there exists a permutation $\pi$ of $\{1, 2, 3\}$ and $\nu_1, \nu_2, \nu_3 \in \{\pm 1\}$ such that $p - r_i = \nu_i(p' - r_i)$. There exist $i \neq j$ such that $\nu_i = \nu_j = \nu$. We have $p - r_i = \nu(p' - r_i)$ and $p - r_j = \nu(p' - r_j)$. Subtracting these equalities we obtain $r_j - r_i = \nu(r_{j'} - r_{i'})$ and hence $\{i, j\} = \{i', j'\}$.

As a consequence for the third element $k \in \{1, 2, 3\} \setminus \{i, j\}$ we have $k' = k$. If $p' \neq p$ then $p' - rk = r_k - p$ (that is, $p' = 2rk - p$) and regarding $p' - r_i$ and $p - r_j$ we are left with the following cases.

1. $\nu = -1$ and $\pi = 1$ is the identity
2. $\nu = 1$ and $\pi = (12)$
3. $\nu = -1$ and $\pi = (12)$.

In case (3) we have $p - r_i = r_j - p'$ and $p - r_j = r_i - p'$, that is, $p' = r_i + r_j - p$.

Comparing this with $p' = 2rk - p$ we obtain $2rk = r_i + r_j$ and hence $r_i - r_k = r_k - r_j$, a contradiction with our assumptions.

In case (1) we obtain $p' = 2r_k - p = 2r_j - p = 2r_k - p$, which is solvable for every $p$ if $2r_i = 2r_j = 2rk$.

In case (2) we obtain $p' = p - r_i + r_j = p - r_j + r_i = p - 2r_k$ which is solvable iff $2r_i = 2r_j$ and $2p = 2rk - r_i + r_j$.

Proposition 12 Let $G$ be an abelian group and assume that there is an epimorphism $\phi : G \to \mathbb{Z}_4$. We set $D = G, R = \{r_1, r_2, r_3\}$ and $P = \phi^{-1}(0) \setminus \{p \in G \mid 2p = r_i + r_j \text{ for some } i, j = 1, 2, 3\}$, where $\phi(R) = \{1, 3\}$, and $r_i - r_j \neq r_i - r_j'$ whenever $i \neq j$ and $(i, j) \neq (i', j')$. Then $(P, R)$ is a strictly addressable scheme in $G$.

Proof. Let $p \in P, p' \in G, r'_1, r'_2, r'_3 \in P \cup R$ such that $p' - r'_i = \pm(p - r_i)$.

Observe that $\{\phi(p - r_i) | i = 1, 2, 3\} = \{1, 3\}$ as well. As $\phi(P \cup \{r_1, r_2, r_3\}) = \{0, 1, 3\}$, this is only possible if $\phi(p')$ is even and $\phi(r'_i)$ are odd ($i = 1, 2, 3$). As $P \subseteq \phi^{-1}(0)$, this implies $\{r'_1, r'_2, r'_3\} \subseteq \{r_1, r_2, r_3\}$ whence, by Lemma 11, $p' = p$ and $r'_i = r_i$ ($i = 1, 2, 3$).

Proposition 12 gives that $(P_1, R_1)$ is a strictly addressable scheme in the group $\mathbb{Z}_4$, where $R_1 = \{1, 3, 9\}$ and $P_1 = 4\mathbb{Z}$. In $\mathbb{Z}_{4m}$ for $m > 2$ we obtain the strictly addressable scheme $(\mathbb{Z}_{4m} \setminus \{2m + 2, 2m + 6\}, \{1, 3, 9\})$.

We shall make use of the following lemma in Section 8.

Lemma 13 Let $G$ be an abelian group and assume that there is an epimorphism $\phi : G \to \mathbb{Z}_4$. Let $D = G$ and let $r_1, r_2, r_3, r_4 \in \phi^{-1}\{1, 3\}$ such that $\phi(\{r_1, r_2, r_3\}) = \{1, 3\}, 3r_4 \neq r_1 + r_2 + r_3$, and $r_i - r_j \neq r_i - r_j'$ whenever $i \neq j$ and $(i, j) \neq (i', j')$. Then with $P = \phi^{-1}(0)$, the element $r_4$ is strictly addressable by $\{r_1, r_2, r_3\}$ with respect to the scheme $(P \cup \{r_4\}, \{r_1, r_2, r_3\})$.

Proof. Assume that for some $r'_4 \in G$, for $r'_1, r'_2, r'_3 \in P \cup \{r_1, r_2, r_3, r_4\}$ we have $r'_4 - r'_i = \pm(r_4 - r_i)$ ($i = 1, 2, 3$). We have $\{\phi(r_4 - r_i) | i = 1, 2, 3\} = \{0, 2\}$, whence also $\{\phi(r'_4 - r'_i) | i = 1, 2, 3\} = \{0, 2\}$. Since $\phi(P) = \{0\}$ and $\phi(r_1, r_2, r_3, r_4) = \{1, 3\}$ this implies that $\phi(r'_4)$ is odd and $r'_1, r'_2, r'_3 \in \{r_1, r_2, r_3\}$. There exist $\mu = \pm 1$ and $i \neq j \in \{1, 2, 3\}$ such that $r'_4 - r'_i = \mu(r_4 - r_i)$ and...
r'_4 - r'_j = \mu(r_4 - r_j). As a consequence, r'_i - r'_j = \mu(r_i - r_j), whence either 
\mu = 1, r'_i = r_i, r'_j = r_j or \mu = -1, r'_i = r_j, r'_j = r_i. In the first case we have 
r'_4 = r_4 and, by the properties of \{r_1, r_2, r_3, r_4\}, also r'_k = r_k for k = 1, 2, 3. 
Assume the second case. Then r'_i = r_i + r_j - r_4. Let k = \{1, 2, 3\} \setminus \{i, j\}. Then 
\begin{align*}
 r_i + r_j - r_4 - r'_k &= r'_i - r'_k = \nu(r_4 - r_k) \text{ for some } \nu = \pm 1. 
\end{align*}
If \nu = -1 then we obtain 
\begin{align*}
 r_i - r'_k &= r_k - r_j, 
\end{align*}
and a contradiction to the properties of \{r_1, r_2, r_3, r_4\}. If \nu = 1 then 
\begin{align*}
 2r_4 + r'_k &= r_1 + r_j + r_k. 
\end{align*}
If r'_k = r_k, then we obtain r_4 - r_i = r_i - r_4, while if r'_k = r_4 then 3r_4 = r_i + r_j + r_k = r_1 + r_2 + r_3. Both possibilities contradict 
the properties of \{r_1, r_2, r_3, r_4\}. □

5.2 Schemes for Euclidean global gates in \(Z^s\)

In this subsection we show that strictly addressable schemes \((P, R)\) exist in the 
lattice \(Z^s\) where \(|R| = \min\{3, s+1\}\) and \(P\) has density \(1/8^s\) in \(Z^s\). Here \(\Omega = \mathbb{R}^s\), 
\(D = Z^s\), \(G = E(s)\) and the orbits of \(G\) on pairs correspond to distances. This is 
of the type of our main geometric example for global gates given in Subsection 2.3. 
For \(s = 1\) see Proposition 12 gives a scheme of density \(\frac{1}{4}\). The optimality of the 
density \((1/8)^s\) (for \(s > 1\)) is an open question. Also note that the addressable 
schemes for the group \(E(s)\) are also addressable schemes for the subgroups of 
\(E(s)\), most notably for the the group of isometries preserving the lattice \(Z^s\).

Assume that \(s > 1\). We shall actually show that for an appropriate choice of 
an \(s+1\)-element subset \(R\) of \(Z^s\), and a set \(P\) consisting of almost all points of 
the set \(8Z^s = \{(8z_1, \ldots, 8z_s) | z_i \in Z\}\), the scheme \((P, R)\) is strictly addressable. 
Let \(k = s+2\) and for \(i = 1, \ldots, k\) write \(P_i\) for the vector of variables 
\(P_i = (x_{i1}, x_{i2}, \ldots, x_{is})\).

We set \(k = s+2\). A \(k\)-element subset \(S \subset \mathbb{R}^s\) is reconstructible, if any other 
\(k\)-element subset \(S' \subset \mathbb{R}^s\) which has the same distribution of pairwise distances 
among its points is actually isomorphic to \(S\) with respect to rigid motions (i.e., 
the group \(E(s)\)).

From a result of Boutin and Kemper (Theorem 2.6 in [5]) it follows that there exists a nonzero polynomial 
\(f(P_1, \ldots, P_k)\) with real coefficients such that if \(p_1, p_2, \ldots, p_k \in \mathbb{R}^s\) are points with 
\(f(p_1, \ldots, p_k) \neq 0\) then \(\{p_1, \ldots, p_k\}\) is reconstructible. The polynomial \(f\) has the additional property (Lemma 2.9, 
loc. cit.) that \(f(p_1, \ldots, p_k) = 0\) if the pairwise distances among the \(p_i\) are not 
all distinct.

If \(0 \neq g(x) \in \mathbb{R}[x]\) is a polynomial and \(H\) is an infinite subset of \(\mathbb{R}\), then 
there exists a \(c \in H\) such that \(g(c) \neq 0\). A repeated application of this simple 
fact to the Boutin-Kemper polynomial shows the existence of a subset \(R = \{r_1, \ldots, r_k-1\} \subset 4Z^s + (1, 0, \ldots, 0)\) such that the polynomial
\[
 h(P_k) := f(r_1, r_2, \ldots, r_{k-1}, P_k)
\]
is not identically zero. Moreover, we can suppose that \(r_1 \in 8Z^s + (1, 0, \ldots, 0)\) and 
\(r_2 \in 8Z^s + (5, 0, \ldots, 0)\). Let \(P\) be the set of points \(p\) from \(8Z^s\) for which 
\(h(p) \neq 0\). Note that \(P\) contains almost every point of \(8Z^s\). Also, if \(p \in P\) 
then the \(s+1\) distances \(|p - r_i|\) are all different, and the set of these distances 
uniquely determines \(p\) among the points of \(\mathbb{R}^s\).
We now show that the above pair \((P, R)\) is a strictly addressable scheme on \(D = \mathbb{Z}^4\). We note first that for every \(p \in P\) we have \(|p - r_1|^2 \equiv 1 \pmod{16}\) while \(|p - r_2|^2 \equiv 9 \pmod{16}\). Assume that we have a \(p \in P\), \(q \in \mathbb{Z}^4\) and \(R' \subset P \cup R\) such that
\[
\{|p - r|, \ r \in R\} = \{|q - r'|, \ r' \in R'\}.
\]
As for every \(p_1, p_2 \in P\) we have \(|q - p_1|^2 \equiv |q - p_2|^2 \pmod{16}\), the fact \(|p - r_1|^2 \not\equiv |p - r_2|^2 \pmod{16}\) implies that \(R'\) can not be entirely in \(P\). Also, \(q\) modulo 2 differs from \((0, \ldots, 0)\) in an odd number of coordinates while it differs from \((1, \ldots, 0)\) in an even number of coordinates or conversely. Therefore \(|q - p'|^2 \equiv |q - r|^2 \pmod{2}\) if \(p' \in P\) and \(r \in R\). This rules out the possibility that \(R'\) intersects properly both \(P\) and \(R\). It follows that \(R' \subset P\) and hence \(R' = R\).

Note that \(R \cup \{p\}\) is rigid in the sense that the pairwise distances among the points are all different. The re-constructibility and rigidity of \(R \cup \{p\}\) gives that \(q = p\) and hence the claim.

### 6 A scheme with shiftable base

In this section we show that on any finite interval of \(\mathbb{Z}\) of length \(n\), a simulation of an \(|n/4|\)-qubit circuit can be done efficiently using global one-qubit gates and global two-qubit gates with Hamiltonians of the form \(A^C\) with \(O < |j| \leq 22\).

Here \(\Omega = \mathbb{Z}\) and \(G = \mathbb{Z}\) acts on \(\Omega\) by translation. Then \(O = \{C_j| j \in \mathbb{Z}\}\setminus \{C_0\}\) where \(C_j = \{(u, v) \in D \times D \mid u - v = j\}\). Let \(P = 4\mathbb{Z}\), and for every \(\ell \in \mathbb{Z}\) let \(R_\ell = \{4\ell + 1, 4\ell + 3, 4\ell + 9\}\). Just like in the argument preceding Corollary 2 it is sufficient to show how to simulate local one-qubit gates at point of \(P\) because the interaction graph of the global 2-qubit Hamiltonian \(A^{C^4}\) connects every pair of points in \(P\).

During simulation of a one-qubit operation at position \(4\ell\), the three qubits at positions from \(R_\ell\) are set to one and the qubits at position from \(\mathbb{Z} \setminus (P \cup R_\ell)\) are set to zero. This is a state admitted by the scheme \((P, R_\ell)\). By Proposition[12] \((P, R_\ell)\) is a strictly addressable scheme. In particular, \(4\ell\) and \(4\ell + 4\) are 3-addressable. Therefore, by Theorem 4 any one-qubit gate can be efficiently approximated using global one-qubit gates and global two-qubit gates with Hamiltonians of the form \(A^{C^j}\) with \(O < |j| \leq 9\).

After each step of simulation, we have a state admitted by \((P, R_\ell)\) for some \(\ell \in \mathbb{Z}\). (Initially \(\ell = 0\).) To perform another simulation step, i.e., to simulate a one-qubit gate at position \(4\ell'\), we have to produce the state admitted by \((P, R_{\ell'})\) where the state of the qubits at positions from \(P\) are left unchanged. Intuitively, the configuration \(R_\ell\) has to be shifted to \(R_{\ell'}\). Below we show a procedure doing this for \(\ell' = \ell + 1\). For general \(\ell'\) we need \(|\ell - \ell'|\) iterations of this basic procedure or its inverse.

In every intermediate step during the shifting procedure the qubits at three positions \(\{r_1, r_2, r_3\}\) are set to one while the value of the qubit at a fourth position \(r_4\) is flipped. In each step the system \(\{r_1, r_2, r_3\}, r_4\) satisfies the conditions of Lemma[13] and also \(|r_4 - r_3| \leq 22\). Therefore the one-qubit operation flipping
the value of the qubit at \( r_4 \) can be efficiently approximated using global one-qubit gates and and global two-qubit gates with Hamiltonians of the form \( A^C \) with \( O < |j| \leq 22 \). Below we give the sequence of consecutive configurations \( \{r_1, r_2, r_3\}, r_4 \).

Step 1. \( \{r_1, r_2, r_3\} = \{4\ell + 1, 4\ell + 3, 4\ell + 9\} \), \( r_4 = 4\ell + 23 \) \((r_4 : 0 \mapsto 1)\).
(Here \(|r_i - r_j|\) = \{(2, 6, 8) \cup \{22, 20, 14\}, r_1 + r_2 + r_3 = 12\ell + 16 \neq 12\ell + 6 = 3r_4\}.

Step 2. \( \{r_1, r_2, r_3\} = \{4\ell + 3, 4\ell + 9, 4\ell + 23\} \), \( r_4 = 4\ell + 1 \) \((r_4 : 1 \mapsto 0)\).
(Here \(|r_i - r_j|\) = \{(6, 14, 20) \cup \{2, 8, 22\}, r_1 + r_2 + r_3 = 12\ell + 35 \neq 12\ell + 3 = 3r_4\}.

Step 3. \( \{r_1, r_2, r_3\} = \{4\ell + 3, 4\ell + 9, 4\ell + 23\} \), \( r_4 = 4\ell + 5 \) \((r_4 : 0 \mapsto 1)\).
(Here \(|r_i - r_j|\) = \{(6, 14, 20) \cup \{2, 4, 18\}, r_1 + r_2 + r_3 = 12\ell + 35 \neq 12\ell + 15 = 3r_4\}.

Step 4. \( \{r_1, r_2, r_3\} = \{4\ell + 3, 4\ell + 5, 4\ell + 23\} \), \( r_4 = 4\ell + 9 \) \((r_4 : 1 \mapsto 0)\).
(Here \(|r_i - r_j|\) = \{(2, 18, 20) \cup \{6, 4, 14\}, r_1 + r_2 + r_3 = 12\ell + 31 \neq 12\ell + 27 = 3r_4\}.

Step 5. \( \{r_1, r_2, r_3\} = \{4\ell + 3, 4\ell + 5, 4\ell + 23\} \), \( r_4 = 4\ell + 11 \) \((r_4 : 0 \mapsto 1)\).
(Here \(|r_i - r_j|\) = \{(2, 18, 20) \cup \{8, 6, 12\}, r_1 + r_2 + r_3 = 12\ell + 31 \neq 12\ell + 23 = 3r_4\}.

Step 6. \( \{r_1, r_2, r_3\} = \{4\ell + 3, 4\ell + 5, 4\ell + 11\} \), \( r_4 = 4\ell + 23 \) \((r_4 : 1 \mapsto 0)\).
(Here \(|r_i - r_j|\) = \{(6, 8) \cup \{20, 18, 12\}, r_1 + r_2 + r_3 = 12\ell + 16 \neq 12\ell + 69 = 3r_4\}.

Steps 7–12 are the same as Steps 1–6, respectively, with \( r_1 + 2 \) in place of \( r_i \).

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