Localization of 3d $\mathcal{N} = 2$ Supersymmetric Theories on $S^1 \times D^2$

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Abstract

We study three dimensional $\mathcal{N} = 2$ supersymmetric Chern-Simons-Matter theories on the direct product of circle and two dimensional hemisphere ($S^1 \times D^2$) with specified boundary conditions by the method of localization. We construct boundary interactions to cancel the supersymmetric variation of three dimensional superpotential term and Chern-Simons term and show inflows of bulk-boundary anomalies. It finds that the boundary conditions induce two dimensional $\mathcal{N} = (0, 2)$ type supersymmetry on the boundary torus. We also study the relation between the 3d-2d coupled partition function of our model and three dimensional holomorphic blocks.
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1 Introduction

After the prominent work by Pestun [1], various supersymmetric gauge theories on curved spacetimes have been investigated and rigid supersymmetries on these backgrounds have been constructed. By evaluating fixed point sets of supersymmetries combined with the method of localization, one can calculate exact partition functions of these supersymmetric theories. For example, round sphere $S^3$ [2, 3, 4] and ellipsoid $S^3_b$ [5] are investigated as suitable backgrounds of three dimensional supersymmetric theories. The superconformal index on $S^1 \times S^2$ [6, 7] is also analyzed to count the number of BPS operators and extract topological data in mathematical physics applications. These functions play important roles in studying M5-branes, IR dualities among supersymmetric theories and the AdS/CFT correspondence.

It is widely believed that these supersymmetric theories have interesting properties of the factorizations with fundamental building blocks. For examples, in three dimensional cases, it is conjectured [8] that the $\mathcal{N} = 2$ supersymmetric partition function on $S^3_b$ and the superconformal index on $S^1 \times S^2$ can be expressed as bilinears of two identical building blocks $B^\alpha(x, q)$:

$$Z_{S^3_b \text{ or } S^1 \times S^2} = \sum_\alpha B^\alpha(x, q) B^\alpha(\tilde{x}, \tilde{q}).$$

The authors of [8] have proposed some general rule to write down these universal blocks called holomorphic blocks. By the analogy of the pure Chern-Simons theory [9] and topological and anti-topological fusion in two dimensions [10], it is expected that holomorphic blocks should be partition functions of three dimensional Chern-Simons-Matter theories on a solid torus (Melvin cigar). In order to clarify the relation between their partition functions (indices) and holomorphic blocks, it is an interesting problem to construct explicitly $\mathcal{N} = 2$ supersymmetric Chern-Simons-Matter theories on a solid torus.

By considering properties of the factorization, we expect that these holomorphic blocks originate in supersymmetric theories with boundaries. When the backgrounds have boundaries, we can introduce physical degrees of freedom on them and classify what types of BPS boundaries are allowed in the supersymmetric Chern-Simons-Matter theories.

The half BPS boundary conditions have been studied for $\mathcal{N} = 1$ supersymmetric pure Chern-Simons theory [11], for supersymmetric Chern-Simons-Matter theories [12, 13].
The BPS boundary theories known at present are realized as super Wess-Zumino-Witten (WZW) models and it is an natural question to ask whether other types of boundary interactions exist or not? For example, in two dimensional $\mathcal{N} = (2, 2)$ theories, the boundary interaction is described in terms of matrix factorizations so that the effect of supersymmetric variation of the superpotential [14, 15] is cancelled. However higher dimensional analogues of the matrix factorization have not been much studied yet.

In this article, we investigate three dimensional $\mathcal{N} = 2$ supersymmetric Chern-Simons-Matter theories on the direct product of circle and two dimensional hemisphere ($S^1 \times D^2$) towards the understanding of properties of building blocks and the factorization. We impose suitable boundary conditions on $\mathcal{N} = 2$ multiplets consistent with supersymmetric transformations. Under the conditions, super Yang-Mills action and kinematic action of the chiral multiplet are written as $Q$-exact manner without surface terms and it implies these actions are invariant under the supersymmetric transformations. On the other hand, in the presence of boundary, the Chern-Simons term is not invariant under the supersymmetric transformation nor the gauge transformation. So we have to introduce some boundary term to compensate these two variations. Here we propose two possible ways to make the theory gauge invariant. One way is to lift gauge parameters to physical fields on the boundary and treat them as a chiral gauged WZW model. The other is to introduce $\mathcal{N} = (0, 2)$ theories on the boundary. They are chiral theories and have gauge anomalies which compensate gauge non-invariant terms of the three dimensional bulk theory.

Next we evaluate the 3d-2d coupled index on $S^1 \times D^2$ in terms of supersymmetric localization and try to relate the 3d-2d index to holomorphic blocks to understand the structure of the bilinearity. In the cases of Abelian gauge theories, we find that 3d-2d indices reproduce holomorphic blocks in the particular choice of the fugacity.

This paper is organized as follows. In section 2, we explain super Yang-Mills action and kinematic action of the chiral multiplet and introduce consistent boundary conditions for $\mathcal{N} = 2$ multiplets. In section 3, we discuss BPS boundary interactions for superpotential terms (three dimensional analogue of the matrix factorization) and the Chern-Simons term. We also show the restriction of the $\mathcal{N} = 2$ supersymmetric transformation to the boundary torus leads to an $\mathcal{N} = (0, 2)$ supersymmetric theory in two dimensions. In sections 4 and 5, we evaluate one-loop determinants of three dimensional $\mathcal{N} = 2$ (vector and chiral) multiplets and two dimensional boundary $\mathcal{N} = (0, 2)$ (vector, chiral, and Fermi) multiplets to study the relation between the 3d-2d indices on $S^1 \times D^2$ and holomorphic blocks in three dimensions. In section 6, we study several topics by illustrating concrete
examples; first we consider an $S^1$-uplift version of the $\mathcal{N} = (2,2)$ hemisphere partition function of $\mathbb{CP}^N$-model on $D^2$ and point out the index of this model is related to the K-theoretic $J$-function of $\mathbb{CP}^N$, in other words, the $q$-deformed Whittaker function. The second model is the $U(N)$ SQCD and we study the structure of the index of this SQCD and its connection to K-theoretic vortex partition functions and surface operators. Third one is the gauge/Bethe correspondence. The last topic is the action of Wilson loops and vortex loops on the 3 dim index. The last section is devoted to summary and discussion.

2 $\mathcal{N} = 2$ supersymmetric theory on $S^1 \times D^2$

In this section, we will construct supersymmetric gauge theories on $S^1 \times D^2$ and introduce supersymmetric boundary conditions for the hemisphere. The construction of the supersymmetry and the Lagrangian are parallel to the $S^1 \times S^2$ case [7].

The hemisphere $D^2$ with radius $r$ is specified by the set of coordinates $(\vartheta, \varphi)$ with $0 \leq \vartheta \leq \frac{\pi}{2}, 0 \leq \varphi \leq 2\pi$ and the boundary of the hemisphere is defined by $\vartheta = \frac{\pi}{2}$. The circle $S^1$ is also parameterized by the coordinate $\tau$ with $0 \leq \tau \leq \beta r$ and $\beta r$ is the length of the perimeter of $S^1$. We can write the metric of $S^1 \times D^2$;

\[ ds^2 = r^2 d\vartheta^2 + r^2 \sin^2 \vartheta d\varphi^2 + d\tau^2. \]

In the following, we use $\mu$'s ($\mu = 1, 2, 3$) for superscripts and subscripts in the curved space with $1 = \vartheta, 2 = \varphi$ and $3 = \tau$. On the other hand, we take the symbol $\hat{a}$ ($a = 1, 2, 3$) for variables in the local Lorentz frame.

Now we construct supersymmetry in the curved space that is realized by conformal Killing spinors $\epsilon, \bar{\epsilon}$. These spinors should satisfy the set of equations;

\[ \nabla_\mu \epsilon = \frac{1}{2r} \gamma_\mu \gamma_3 \epsilon, \quad \nabla_\mu \bar{\epsilon} = -\frac{1}{2r} \gamma_\mu \gamma_3 \bar{\epsilon}. \]

The solutions of these equations are given by

\[ \epsilon = e^{\frac{2}{r}} \epsilon^{(2)} e^{-\frac{1}{r} \gamma^2 \vartheta} e^{\frac{1}{r} \gamma^3 \varphi} \epsilon_o^{(2)}, \quad \bar{\epsilon} = e^{-\frac{2}{r}} \bar{\epsilon}^{(2)} e^{\frac{1}{r} \gamma^3 \varphi} e^{-\frac{1}{r} \gamma^3 \varphi} \bar{\epsilon}_o^{(2)}, \]

where we choose the constant spinors $\epsilon_o^{(2)} = \gamma_3 \epsilon_o^{(2)} = (\epsilon_o, 0)^T$ and $\bar{\epsilon}_o^{(2)} = -\gamma_3 \bar{\epsilon}_o^{(2)} = (0, \bar{\epsilon}_o)^T$ so that the component of the Killing vector $\bar{\epsilon} \gamma^\mu \epsilon$ along the $\vartheta$-direction vanishes.

With the set of Killing spinors (2.3), supersymmetric transformation of the vector
multiplet (A.5) is expressed as
\[
\delta A_\mu = \frac{i}{2}(\bar{\epsilon}\gamma_\mu \lambda - \bar{\lambda}\gamma_\mu \epsilon),
\]
(2.4)
\[
\delta \sigma = \frac{1}{2}(\bar{\epsilon}\lambda - \bar{\lambda}\epsilon),
\]
(2.5)
\[
\delta \lambda = -\frac{1}{2}\gamma^{\mu\nu} F_{\mu\nu} \epsilon - D\epsilon + i\gamma^\mu D_\mu \sigma \epsilon + i\frac{\sigma}{r} \gamma_3 \epsilon,
\]
(2.6)
\[
\delta \bar{\lambda} = -\frac{1}{2}\gamma^{\mu\nu} F_{\mu\nu} \bar{\epsilon} + D\bar{\epsilon} - i\gamma^\mu D_\mu \sigma \bar{\epsilon} + i\frac{\sigma}{r} \gamma_3 \bar{\epsilon},
\]
(2.7)
\[
\delta D = -\frac{i}{2} \bar{\epsilon}\gamma^\mu D_\mu \lambda - \frac{i}{2} D_\mu \bar{\lambda}\gamma^\mu \epsilon + \frac{i}{2}[\bar{\epsilon}\lambda, \sigma] + \frac{i}{2}[\bar{\lambda}\epsilon, \sigma] + \frac{i}{4r}(\bar{\epsilon}\gamma_3 \lambda - \bar{\lambda}\gamma_3 \epsilon).
\]
(2.8)

The Lagrangian density \( \mathcal{L}_{vec} \) of super Yang-Mills theory is written in the \( Q \)-exact form;
\[
2\mathcal{L}_{vec} = \frac{1}{\epsilon_2 \epsilon_1} \delta_\epsilon^2 \delta_{\bar{\epsilon}} \text{Tr} \left[ \frac{1}{2} \lambda \lambda \right]
\]
\[
= \text{Tr} \left[ \frac{1}{2} F^{\mu\nu} F_{\mu\nu} + D^\mu \sigma D_\mu \sigma + D^2 + \frac{1}{r^2} \sigma^2 - e^{\mu\nu\rho} F_{\mu\nu} D_{\rho} \sigma - \frac{1}{r} e^{\mu\nu\sigma} F_{\mu\nu} \sigma + \frac{2}{r} \sigma D_3 \sigma \right]
\]
\[
+ \text{Tr} \left[ i \lambda \gamma^\mu D_\mu \bar{\lambda} + i \bar{\lambda} [\lambda, \sigma] - \frac{i}{2r} \bar{\lambda} \gamma_3 \lambda \right].
\]
(2.9)

We impose the following boundary condition for this vector multiplet at \( \vartheta = \frac{\pi}{2} \) as
\[
\sigma = 0, \quad A_1 = 0, \quad \partial_1 A_2 = 0, \quad \partial_1 A_3 = 0, \quad D_1 (D - iD_1 \sigma) = 0,
\]
\[
\lambda_1 - \lambda_2 = 0, \quad \bar{\lambda}_1 - \bar{\lambda}_2 = 0, \quad \partial_1 (\lambda_1 + \lambda_2) = 0, \quad \partial_1 (\bar{\lambda}_1 + \bar{\lambda}_2) = 0.
\]
(2.10)

Here \( \lambda = (\lambda_1, \lambda_2)^T \) and \( \bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2)^T \). This boundary condition is compatible with the supersymmetric transformation (2.8).

Next we consider the chiral multiplet. The supersymmetric transformation of the chiral multiplet is given by
\[
\delta \phi = \bar{\epsilon}\psi,
\]
(2.11)
\[
\delta \bar{\phi} = \epsilon\bar{\psi},
\]
(2.12)
\[
\delta \psi = i\gamma^\mu \epsilon D_\mu \phi + i\epsilon \sigma \phi + \frac{i\Delta}{r} \gamma_3 \epsilon \phi + \bar{\epsilon} F,
\]
(2.13)
\[
\delta \bar{\psi} = i\gamma^\mu \bar{\epsilon} D_\mu \bar{\phi} + i\bar{\epsilon} \sigma \bar{\phi} - \frac{i\Delta}{r} \bar{\phi} \gamma_3 \bar{\epsilon} + \bar{\bar{F}} \epsilon,
\]
(2.14)
\[
\delta F = \epsilon (i\gamma^\mu D_\mu \psi - i\psi \sigma - i\lambda \phi) + \frac{i}{2r} (2\Delta - 1) \epsilon \gamma_3 \psi,
\]
(2.15)
\[
\delta \bar{F} = \bar{\epsilon} (i\gamma^\mu D_\mu \bar{\psi} - i\bar{\psi} \sigma + i\bar{\phi} \bar{\lambda}) - \frac{i}{2r} (2\Delta - 1) \bar{\epsilon} \gamma_3 \bar{\psi}.
\]
(2.16)
The kinetic term of the chiral multiplet is also given in the \( Q \)-exact form:

\[
L_{\text{ch}} = \frac{1}{\epsilon_2\epsilon_1} \delta_{\epsilon_2} \delta_{\epsilon_1} \left( \bar{\phi} F \right) \\
= -\bar{\phi} D^\mu D_\mu \phi + \bar{\phi} \phi^2 \phi + i\bar{\phi} D \phi + \bar{F} F + \frac{1 - 2\Delta}{r} \bar{\phi} D_3 \phi + \frac{\Delta - \Delta^2}{r^2} \bar{\phi} \phi \\
- i\bar{\psi} \gamma^\mu D_\mu \psi + i\bar{\psi} \sigma \psi + \frac{i(1 - 2\Delta)}{2r} \bar{\psi} \gamma_3 \psi + i\bar{\psi} \lambda \phi - i\bar{\phi} \lambda \psi.
\] (2.17)

We can introduce the Neumann boundary condition for the chiral multiplet at \( \vartheta = \frac{\pi}{2} \) as

\[
\partial_1 \phi = 0, \quad \partial_1 \bar{\phi} = 0, \quad F = 0, \quad \bar{F} = 0, \\
\psi_1 + \psi_2 = 0, \quad \bar{\psi}_1 + \bar{\psi}_2 = 0, \quad \partial_1(\psi_1 - \psi_2) = 0, \quad \partial_1(\bar{\psi}_1 - \bar{\psi}_2) = 0,
\] (2.18)

with \( \psi = (\psi_1, \psi_2)^T \) and \( \bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2)^T \).

For the Dirichlet-type boundary, we can write down conditions:

\[
\phi = 0, \quad \bar{\phi} = 0, \quad \partial_1(ie^{\tau} e^{i\varphi} \partial_1 \phi + F) = 0, \quad \partial_1(ie^{-\tau} e^{-i\varphi} \partial_1 \bar{\phi} + \bar{F}) = 0, \\
\psi_1 - \psi_2 = 0, \quad \bar{\psi}_1 - \bar{\psi}_2 = 0, \quad \partial_1(\psi_1 + \psi_2) = 0, \quad \partial_1(\bar{\psi}_1 + \bar{\psi}_2) = 0.
\] (2.19)

The Neumann (Dirichlet) boundary condition (2.18), (2.19) is the \( S^1 \)-uplift of the Neu mann (Dirichlet) boundary condition in two dimensional \( N = (2, 2) \) theories on \( D^2 \) [16, 17] and different from the boundary condition imposed in [18].

3 The BPS boundary interactions on the torus

Under the boundary conditions (2.10) and (2.18), the Lagrangians (2.9) and (2.17) are invariant by the supersymmetric transformations generated by \( \delta_\epsilon, \delta_{\bar{\epsilon}} \). On the other hand, the supersymmetric transformations of supersymmetric Chern-Simons term and the superpotential term do not vanish and we have to introduce supersymmetric boundary interactions to cancel surface terms coming from these terms.

3.1 Three dimensional analogue of matrix factorization

In the presence of the boundary, the supersymmetric variation of the superpotential does not vanish. In two dimensional \( N = (2, 2) \) theories, the boundary term for the superpotential is cancelled by the boundary interaction which satisfies the matrix factorization [14, 15]. The three dimensional analogue of the matrix factorization is first pointed out in [19]. In this subsection, we study similar story for the three dimensional \( N = 2 \) theories on \( S^1 \times D^2 \).
The supersymmetric transformations of the superpotential induce a surface term on the boundary

$$\delta S_W = \int_{T^2} \sqrt{g} d^2 x \sum_{I} \left( \epsilon \gamma_1 \psi_I \partial_I W + \bar{\epsilon} \gamma_1 \bar{\psi}_I \bar{\partial}_I \bar{W} \right).$$

We need to introduce boundary interactions which compensate the above boundary terms. Now we restrict the supersymmetric transformation of the three dimensional $\mathcal{N} = 2$ multiplet to the boundary ($\vartheta = \frac{\pi}{2}$) in order to construct the BPS and gauge invariant boundary interaction. First of all, we will consider the vector multiplet (2.8) restricted on the boundary. Under the condition (2.10), associated supersymmetric transformations are given by

$$\delta A_2 = i \frac{1}{2} (\bar{\epsilon} \gamma_2 \lambda - \bar{\lambda} \gamma_2 \epsilon) = \epsilon' \lambda_1 - \bar{\lambda} \epsilon',
\delta A_3 = i \frac{1}{2} (\bar{\epsilon} \gamma_3 \lambda - \bar{\lambda} \gamma_3 \epsilon) = i (\epsilon' \lambda_1 - \bar{\lambda} \epsilon'),
\delta \lambda = -i F_{23} \epsilon - \hat{D} \epsilon,
\delta \bar{\lambda} = -i F_{23} \bar{\epsilon} + \hat{D} \bar{\epsilon},
\delta (-\hat{D} - i F_{23}) = 2 \epsilon' (D_2 + i D_3) \lambda_1 - \frac{2i}{r} \epsilon' \lambda_1,
\delta (-\hat{D} + i F_{23}) = -2 \epsilon' (D_2 + i D_3) \bar{\lambda}_1 - \frac{2i}{r} \epsilon' \bar{\lambda}_1.

Here we defined $\hat{D} := D - i D_1 \sigma$ and $\epsilon' := \frac{e^{\frac{i}{\sqrt{2}} \epsilon}}{\sqrt{2}} \epsilon$, $\bar{\epsilon}' := \frac{e^{-\frac{i}{\sqrt{2}} \bar{\epsilon}}}{\sqrt{2}} \bar{\epsilon}$. The commutation relations of these transformations are summarized in the appendix B. In the flat space limit $r \to \infty$, the set of the above transformations becomes that of a two-dimensional $\mathcal{N} = (0,2)$ vector multiplet. We shall call this multiplet a boundary $\mathcal{N} = (0,2)$ vector multiplet. A Lagrangian can be represented as $\delta_{\epsilon'}$-exact form and invariant under the supersymmetric transformations (3.2)

$$\bar{\epsilon}' \mathcal{L}_{\text{vec}}^{\mathcal{N} = (0,2)} = \delta_{\epsilon'} \text{Tr} (-\hat{D} + i F_{23}) \lambda_1 = \epsilon' \left( F_{23}^2 + \hat{D}^2 + 2 \lambda_1 (D_2 + i D_3) \lambda_1 - \frac{2i}{r} \lambda_1 \bar{\lambda}_1 \right).$$

This result implies the transformation can be regarded as the $\mathcal{N} = (0,2)$ supersymmetry on the boundary torus $\partial (S^1 \times D^2) = T^2$. But one wonders whether matter part has the same supersymmetry on the boundary. In fact, the supersymmetry on the boundary is
expressed for the chiral multiplet (2.16) under the boundary conditions (2.10) and (2.18)

\[
\begin{align*}
\delta \phi &= \bar{\epsilon} \psi = \epsilon' \psi', \\
\delta \bar{\phi} &= \epsilon \bar{\psi} = \epsilon' \bar{\psi}', \\
\delta \psi &= \gamma_3 \epsilon (D_2 + iD_3)\phi + \frac{i \Delta}{r} \gamma_3 \epsilon \phi, \\
\delta \bar{\psi} &= \gamma_3 \bar{\epsilon} (D_2 + iD_3)\bar{\phi} - \frac{i \Delta}{r} \bar{\phi} \gamma_3 \bar{\epsilon},
\end{align*}
\]

where we put \( \psi' := \psi_1 - \psi_2, \bar{\psi}' := \bar{\psi}_1 - \bar{\psi}_2 \). When we take the limit \( r \to \infty \), the above transformation becomes the \( \mathcal{N} = (0, 2) \) supersymmetric transformation of the boundary chiral multiplet \((\phi, \psi')\). The Lagrangian of this \( \mathcal{N} = (0, 2) \) multiplet is also represented as \( \delta_{\epsilon'} \)-exact form

\[
\mathcal{L}_{\chi}^{\mathcal{N}=(0,2)} = \delta_{\epsilon'} \left( \frac{1}{2} \bar{\phi} (D_2 - iD_3) \psi' + i \bar{\phi} \lambda_1 \phi \right)
= \epsilon' \left( \bar{\phi} (D_2 - iD_3) (D_2 + iD_3) \phi + \frac{1}{2} \bar{\psi}' (D_2 - iD_3) \psi' \\
+ \frac{i \Delta}{r} \bar{\phi} (D_2 - iD_3) \phi + i \bar{\phi} \lambda_1 \psi' + i \bar{\psi}_1 \lambda_1 \phi + \bar{\phi} (F_{23} - i \hat{D}) \phi \right).
\]

These \( \mathcal{N} = (0, 2) \) boundary (vector and chiral) multiplets are constructed through restriction of the bulk supersymmetry on the boundary. In addition, there are new multiplets for the \( \mathcal{N} = (0, 2) \) theory characterized by holomorphic functions \( E(\phi) \)'s, namely Fermi multiplets and we can construct the boundary interaction which cancel the variation of the superpotential term. We shall introduce the Fermi multiplet \((\Psi, G)\) on \( T^2 \) coupled to the boundary \( \mathcal{N} = (0, 2) \) (vector and chiral) multiplets in the supersymmetric way. The supersymmetric transformation of the boundary Fermi multiplet is given by

\[
\begin{align*}
\delta \Psi &= 2E \epsilon' + 2 \bar{\epsilon} G, \\
\delta \bar{\Psi} &= 2 \bar{E} \epsilon' + 2 \bar{\epsilon} G, \\
\delta G &= -\epsilon' \psi_E + \epsilon' (D_2 + iD_3) \Psi + \frac{i}{r} (\bar{\Delta} - 1) \epsilon' \Psi, \\
\delta \bar{G} &= -\epsilon' \bar{\psi}_E + \epsilon' (D_2 + iD_3) \bar{\Psi} + \frac{i}{r} (1 - \Delta) \epsilon' \bar{\Psi}.
\end{align*}
\]

Here \( \psi_E := \sum_I \frac{\partial E(\phi)}{\partial \phi_I} \psi_I, \bar{\psi}_E := \sum_I \frac{\partial E(\phi)}{\partial \bar{\phi_I}} \bar{\psi}_I \) and \((\phi_I, \psi'_I)\)'s are \( \mathcal{N} = (0, 2) \) boundary chiral multiplets. The subscript \( I \) represents the \( \mathcal{N} = (0, 2) \) chiral multiplet \( \phi_I \) contained in \( E(\phi) \) and we also require the relation \( \sum_I \Delta_I = \tilde{\Delta} \). One can show commutators of these supersymmetries turn to generate a translation, R-symmetry and gauge transformations.

So far, we discuss the Neumann type boundary condition (2.18), but we have another possibility, namely the Dirichlet type one. Here we make a remark: The supersymmetric
transformation (2.16) with the Dirichlet boundary condition on the torus leads to Eq. (3.9) with $E(\phi) = 0$ by the following redefinition:

$$
G := ie^\tau e^{i\phi} D_1 \phi + F, \quad \bar{G} := ie^{-\tau} e^{-i\phi} D_1 \bar{\phi} + \bar{F},
$$

$$
\Psi := \psi_1, \quad \bar{\Psi} := \bar{\psi}_1, \quad \bar{\Delta} = 2\Delta.
$$

(3.10)

It is the special case of the $\mathcal{N} = (0, 2)$ theory.

Now we return to the boundary Fermi multiplet. The Lagrangian of the boundary Fermi multiplet can be constructed as

$$
\epsilon' L_{\text{Fermi}}^{\mathcal{N}-(0,2)} = \delta_{\epsilon'}(\bar{\Psi}G + E\Psi) = \epsilon' \left( -\bar{\Psi}(D_2 + iD_3)\Psi + 2\bar{G}G + 2\bar{E}E - \bar{\psi}_E\Psi - \bar{\Psi}\psi_E + i\frac{1}{r}(1 - \bar{\Delta})\bar{\Psi}\Psi \right).
$$

(3.11)

We can also introduce the potential term for this multiplet

$$
\mathcal{L}_J = G_a J^a - \frac{1}{2}\bar{\psi}_a \psi_J^a + (\text{c.c}),
$$

(3.12)

which induces boundary terms on $T^2$ through the variation

$$
\int_{T^2} \sqrt{g} d^2 x \delta \mathcal{L}_J = \int_{T^2} \sqrt{g} d^2 x \sum_{I, a} \left( -\epsilon' \psi_I^J \frac{\partial (E_a J^a)}{\partial \phi_I} + (\text{c.c}) \right).
$$

(3.13)

Comparing this with the term (3.1), we find that cancellation between (3.1) and (3.13) is implemented when the relation $\sum_a E_a J^a = W$ is satisfied. We further require each monomial in $E_a J^a$ has R-charge 2 so that the relation $\sum_a E_a J^a = W$ can be regarded as the three dimensional matrix factorization.

### 3.2 Supersymmetric Chern-Simons term

In this subsection, we consider the $\mathcal{N} = 2$ supersymmetric Chern-Simons theory and investigate boundary terms induced by variations of the bulk action. The boundary effect is first studied for the $\mathcal{N} = 1$ Chern-Simons theory [11] and the $\mathcal{N} = 2$ Abelian case is also studied in [12].

We first treat the Chern-Simons term with the gauge group $G$ and construct consistent boundary interactions. Later we treat a quiver-type theory by gauging the flavor symmetry and discuss mixed terms between the dynamical gauge groups and the flavor groups.
The Chern-Simons theory has a vector boson $A_\mu$, bosonic fields $D, \sigma$ and fermions $\lambda, \bar{\lambda}$. They take their values in the Lie algebra of the gauge group $G$ and transform in the adjoint representation. This model has a parameter $\kappa$ called the level of the theory and its action is written by

$$S_{CS} = \frac{i\kappa}{4\pi} \int d^3 x \, \mathcal{L}_{CS},$$

$$\mathcal{L}_{CS} = \varepsilon^{\mu \nu \rho} \text{Tr} \left( \partial_\mu A_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho \right) + \sqrt{g} \text{Tr} (-\bar{\lambda} \lambda + 2\sigma D),$$

(3.14)

where $\varepsilon^{\mu \nu \rho}$ is an antisymmetric tensor density. The supersymmetric variation of this term is evaluated as

$$\delta \mathcal{L}_{CS} = \frac{i}{2} \partial_\mu \text{Tr} \varepsilon^{\mu \nu \rho} \left( \bar{\epsilon} \gamma_\nu \lambda A_\rho - \bar{\lambda} \gamma_\nu \epsilon A_\rho \right) - i \partial_\mu \text{Tr} \sqrt{g} \left( \bar{\epsilon} \gamma^\mu \lambda \sigma + \bar{\lambda} \gamma^\mu \epsilon \sigma \right).$$

(3.15)

On $S^1 \times D^2$ with the boundary condition (2.10), the second term in the above equation can be dropped out and the first term leads to the following boundary term

$$S_{CS} = -\frac{i\kappa}{8\pi} \int_{T^2} \sqrt{g} d^2 x \text{Tr} [A_\mu (\bar{\epsilon} \gamma^\mu \lambda - \bar{\lambda} \gamma^\mu \epsilon)],$$

(3.16)

where $\sqrt{g}$ is the measure of the two dimensional boundary and is related to the bulk three dimensional one $\sqrt{g}^{(3)} = r \sqrt{g}$. By introducing a boundary Chern-Simons term

$$S_{b,CS} = \frac{-\kappa}{8\pi} \int_{T^2} \sqrt{g} d^2 x \text{Tr} (A_\mu A^\mu),$$

(3.17)

we find that $S_{CS} + S_{b,CS}$ preserves the supersymmetry generated by $\delta \epsilon, \delta \bar{\epsilon}$. But $S_{CS} + S_{b,CS}$ still breaks gauge invariance in the presence of boundary and we should resolve this problem. There are two choices to restore this gauge symmetry:

(i) One introduces some boundary $\mathcal{N} = (0,2)$ multiplets to induce anomaly inflows.

(ii) One treats the gauge degrees of freedom as physical fields on the boundary and couples them with a chiral gauged WZW model.

We first explain the choice (i). The gauge non-invariant term of $S_{CS} + S_{b,CS}$ is compensated by gauge anomalies from the boundary $\mathcal{N} = (0,2)$ (chiral and Fermi) multiplets when the following condition is satisfied:

$$\kappa \text{Tr}_\square (T^a T^b) = \sum_{m: \text{chiral}} \text{Tr}_{\mathcal{R}_m} (T^a T^b) - \sum_{n: \text{Fermi}} \text{Tr}_{\mathcal{R}_n} (T^a T^b).$$

(3.18)

The left hand side comes from an infinitesimal gauge transformation of $S_{CS} + S_{b,CS}$. In the right hand side, $\text{Tr}_{\mathcal{R}_m} (T^a T^b), (\text{Tr}_{\mathcal{R}_n} (T^a T^b))$ comes respectively from the gauge anomaly
coefficient of the chiral (Fermi) multiplet with the representation \( R_m, (R_n) \). It means the classical gauge anomaly (the left hand side) and the one-loop gauge anomalies (the right hand side) cancel each other. This type of cancellation was considered in the context of an \( \mathcal{N} = (0, 2) \) gauged WZW model [20].

But there is a subtle point; in the case of \( \mathcal{N} = 2 \) theories in three dimensions, the bare Chern-Simons level is shifted by one-loop effect of dynamical fermions. In two dimensions, it is known that the gauge field acquires some mass term by the quantum effect [21] and the coefficient of the boundary Chern-Simons term is also shifted by effects of fermion loops. Then left hand side of (3.18) is expected to be replaced by some effective Chern-Simons level. We will revisit this problem in section 5.

Next we shall consider quiver Chern-Simons theory. Here gauge fields are not necessarily dynamical. In fact, there are possible mixing terms between the dynamical gauge fields and background flavor fields, namely, mixed Chern-Simons terms. For simplicity, we assume all the groups are Abelian, but it is straightforward to generalize to non-Abelian cases. We put the gauge groups \( U(1)_G^N \times U(1)_F^{N_f} \): the dynamical gauge group \( U(1)_G^N \), and the flavor symmetry group \( U(1)_F^{N_f} \). Further there are other possible mixings, for examples, the dynamical symmetry and the \( U(1)_R \) R-symmetry, or the flavor symmetry and the \( U(1)_R \) R-symmetry. We will discuss a mixed Chern-Simons term for each mixing in the later.

The \( \mathcal{N} = 2 \) Abelian quiver theory is described by the action

\[
S_{CS} = \frac{i \kappa_{st}}{4\pi} \int_{S^1 \times D^2} d^2x \left[ \varepsilon^{\mu\nu\rho} \partial_\mu A^{(s)}_\nu A^{(t)}_\rho + \sqrt{g} (-\bar{\chi}^{(s)} \chi^{(t)} + 2 \sigma^{(s)} D^{(t)}) \right].
\]

(3.19)

It has gauge, flavor and R-symmetries and there are mixed Chern-Simons terms characterized by mixed Chern-Simons levels. We will collect these levels into one symmetric \((N + N_f) \times (N + N_f)\) matrix \( \kappa_{st} \) that represents effective levels. \( A^{(s)}_\mu \) represents the collection of \( U(1)_G^N \) dynamical gauge fields and \( U(1)_F^{N_f} \) background gauge fields. As in the case of the non-Abelian theory (3.17), the supersymmetric boundary term is given by

\[
S_{b,CS} = -\frac{\kappa_{st}}{2\pi} \int_{T^2} \sqrt{g} d^2x A^{(s)}_z A^{(t)}_{\bar{z}}.
\]

(3.20)

The infinitesimal \( U(1)_G^N \times U(1)_F^{N_f} \) transformation of the mixed Chern-Simons term leads to non-invariant terms on the boundary:

\[
\delta(S_{CS} + S_{b,CS}) = \frac{\kappa_{st}}{2\pi} \int_{T^2} F^{(s)}_{zz} \alpha^{(t)}.
\]

(3.21)

The gauge-gauge and gauge-flavor mixed parts in (3.21) can be cancelled from corresponding mixed anomalies in the boundary \( \mathcal{N} = (0, 2) \) (chiral and Fermi) multiplets when the
following conditions are satisfied

\[ \kappa_{st} = \sum_{m:2d \text{ chiral}} Q^n_m Q^m_t - \sum_{n:2d \text{ Fermi}} \tilde{Q}^n_s \tilde{Q}^n_t. \]  

(3.22)

Here \( Q^n_m \) represents the \( U(1) \) charge for \( m \)-th \( \mathcal{N} = (0, 2) \) chiral multiplet associated to \( s \)-th gauge field and \( \tilde{Q}^n_s \) represents the \( U(1) \) charge for \( n \)-th \( \mathcal{N} = (0, 2) \) Fermi multiplet coupled to \( s \)-th gauge field. The summation of each term runs over the \( \mathcal{N} = (0, 2) \) chiral and Fermi multiplets respectively. If the bare Chern-Simons levels \( \kappa \)'s are replaced by the effective Chern-Simons levels, the equation (3.22) matches with the condition of Chern-Simons levels in the context of the holomorphic blocks [8](see also [19]).

Next we explain the second choice (ii). Under the finite gauge transformation \( A^g_\mu = i\partial_\mu gg^{-1} + g A_\mu g^{-1}, (g \in G) \), the Chern-Simons term and the supersymmetric boundary term transform respectively

\[ \varepsilon^{\mu\nu\rho} \text{Tr} \left( \partial_\mu A_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho \right)^g = \varepsilon^{\mu\nu\rho} \text{Tr} \left( -\frac{1}{3} \partial_\mu gg^{-1} \partial_\nu gg^{-1} \partial_\rho gg^{-1} + i\partial_\mu (A_\nu g^{-1} \partial_\rho g) \right) 
+ \varepsilon^{\mu\nu\rho} \text{Tr} \left( \partial_\mu A_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho \right), \]  

(3.23)

\[ \text{Tr}(A_z A_\bar{z})^g = \text{Tr} \left( -\partial_z gg^{-1} \partial_{\bar{z}} gg^{-1} + ig^{-1} \partial_z g A_\bar{z} + iA_z g^{-1} \partial_{\bar{z}} g + A_z A_\bar{z} \right). \]  

(3.24)

Then the combined action transforms as

\[ (S_{CS} + S_{b,CS})^g = S_{CS} + S_{b,CS} - S_{c,GWZW}, \]  

(3.25)

with

\[ S_{c,GWZW}[g, A_\bar{z}] = -\frac{\kappa}{2\pi} \int_{T^2} \text{Tr}(\partial_\bar{z} gg^{-1} \partial_z gg^{-1}) + \frac{i\kappa}{12\pi} \int_{S^1 \times D^2} \varepsilon^{\mu\nu\rho} \text{Tr}(\partial_\mu gg^{-1} \partial_\nu gg^{-1} \partial_\rho gg^{-1}) 
+ \frac{i\kappa}{\pi} \int_{T^2} \text{Tr}(g^{-1} \partial_z g A_\bar{z}). \]  

(3.26)

The first line in (3.26) is the action of \( G \) Wess-Zumino-Witten (WZW) model and the second line is the chiral \( G/G' \) gauged term with a right \( G \)-action. The supersymmetric transformation trivially acts on the \( G \)-elements and (3.26) itself is supersymmetric invariant, since the fermionic superpartner of chiral \( G/G' \) gauged WZW model takes its value in the subspace orthogonal to \( \text{Lie}(G') \) in \( \text{Lie}(G) \). It is this combination of gauge fields \( A_\bar{z} \) that is invariant under supersymmetric transformations.

The combination \( S_{CS} + S_{b,CS} + S_{c,GWZW} \) is also gauge invariant because we have identities under the right \( g' \)-action with \( g \to gg', (g, g' \in G) \)

\[ (S_{CS} + S_{b,CS})^g = S_{CS} + S_{b,CS} - S_{c,GWZW}[g', A_\bar{z}], \]  

(3.27)

\[ S_{c,GWZW}[gg', A'_\bar{z}] = S_{c,GWZW}[g, A_\bar{z}] + S_{c,GWZW}[g', A_\bar{z}]. \]  

(3.28)
It is the chiral gauged WZW model. When we treat the gauge degrees of freedom as physical fields, we find that $S_{\text{CS}} + S_{b,\text{CS}} + S_{c,\text{GWZW}}$ is fully invariant under the $G$ gauge transformation. The partition function of this WZW model with the background gauge field is known as the holomorphic wave functional $\Psi_{\text{CS}}[A]$ in the pure Chern-Simons theory [22] or equivalently the conformal block of the $G$ WZW model. The quadratic term of the gauge field is the Kähler polarization which appears in the inner product in the Hilbert space of the Chern-Simons theory. In our calculation, the Kähler polarization appears from the half-BPS boundary condition, and it is interesting to study the relation between the Kähler polarization and the half-BPS boundary condition.

In the localization computation, the constant gauge field configuration "a" (see (4.5)) only appears in the $S_{b,\text{CS}} + S_{c,\text{GWZW}}$. Then the path integration over the $G$ elements is nothing but the holomorphic wave functional $\Psi_{\text{CS}}[a]$ with the constant gauge field which can be expressed in terms of the Weyl-Kac character formula [22].

It is possible to consider more general boundary interactions by gauging subgroup $H$ of $G$ and considering a left $H$-action: $g \rightarrow h^{-1}gg'$, $(g, g' \in G, h \in H)$. Although the chiral gauged WZW model is anomalous under the left $H$- and right $G$-actions, we can cancel the gauge anomaly for $H$ by introducing appropriate $\mathcal{N} = (0, 2)$ (chiral and Fermi) multiplets coupled to the Lie($H$)-valued gauged field. Namely, when these $\mathcal{N} = (0, 2)$ multiplets satisfy some suitable relations to cancel the anomaly, the theory becomes consistent. It is interesting to study these boundary interactions in detail and evaluate such 3d-2d coupled partition functions. But in the rest of this article, we mainly consider the case (i).

4 Index on $S^1 \times D^2$ and localization

In this section, we evaluate the partition function on $S^1 \times D^2$ via localization. On $S^1 \times S^2$, the supersymmetric variation parameter cannot be periodic along $S^1$ direction [7], so we impose the twisted periodic boundary condition along this $S^1$ direction:

$$\Phi(\tau + \beta r) = e^{(-J_3-R)\beta_1+J_3\beta_2+F\ell M}\Phi(\tau), \quad (\beta = \beta_1 + \beta_2). \quad (4.1)$$

Here $J_3$ is the generator of rotation along the $\varphi$-direction, $R$ is $U(1)$ R-charge and $F\ell$’s are Cartan generators of global symmetry groups. $\beta_1$, $\beta_2$ and $M\ell$ are fugacities for these charges. Under the twisted periodic boundary condition, the partition function on $S^1 \times S^2$ defines an superconformal index:

$$\mathcal{I}_{S^1 \times S^2} = \text{tr}_{H(S^2)} \left[ (-1)^F e^{-\beta_1(D-R-J_3)} e^{-\beta_2(D+J_3)} e^{-F\ell M\ell} \right]. \quad (4.2)$$
Here $D$ is the generator of the translation along the $S^1$-direction. The superconformal index counts the BPS operators which saturates the bound $D - R - J_3 \geq 0$ and does not depend on the fugacity $\beta_1$.

When we impose boundary conditions (2.10), (2.18) or (2.19) and the common twisted boundary condition along the $S^1$-direction (4.1), we will find all one-loop determinants on $S^1 \times D^2$ do not depend on $\beta_1$. Thus the partition function on $S^1 \times D^2$ can be interpreted as an index:

$$I_{S^1 \times D^2}^\alpha = \text{tr}_{\mathcal{H}(D^2;\alpha)} \left[ (\pm 1)^F e^{-\beta_1 (D-R-J_3)} e^{-\beta_2 (D+J_3)} e^{-F_\ell M_\ell} \right]$$

(4.3)

Here $\alpha$ labels the boundary conditions and a choice of boundary conditions is translated to a choice of contours of the integral in the localization procedure [17]. We often omit the suffix $\alpha$. In general, it is difficult to calculate directly the one-loop determinants with other boundary conditions generated by the vector-type $R$-rotation. But in the case of the two dimensional $\mathcal{N} = (2, 2)$ gauged linear sigma model (GLSM), we can use the argument of the holomorphy, that is, the hemisphere partition function only depends on the holomorphic (anti-holomorphic) combination of two real scalars $\sigma_1 + i\sigma_2$ ($\sigma_1 - i\sigma_2$). Then we can choose a more general Lagrangian submanifold in $\text{Lie}(G) \otimes \mathbb{C}$. The main difference from the $\mathcal{N} = (2, 2)$ GLSM is the presence of Chern-Simons terms which is making the use of the holomorphy difficult. Then we have to also directly evaluate one-loop determinants in the cases of other boundary conditions. This problem is left as our future work.

Recently the index on $S^1 \times S^0_6$ and $S^1 \times \mathbb{RP}^2$ are studied in [23]. They are some $S^1$-uplifts of the partition functions of the two dimensional squashed sphere [24] or the real projective space [25]. There the index on $S^1 \times S^0_6$ does not depend on the squashing parameter $b$ and corresponds with the ordinary superconformal index. At least, the (three dimensional) bulk part of the index on $S^1 \times D^2$ is also expected to be independent from the squashing deformation of the two dimensional hemisphere. Moreover the index on $S^1 \times D^2$ also does not explicitly depend on the radius $r$ of the two dimensional hemisphere, since the radius of the hemisphere enters in the index on $S^1 \times D^2$ only through the matrix integral variables when the localization method is applied. When the boundary interactions are absent, the index on $S^1 \times D^2$ is expected to be independent from both squashing and the radius of the hemisphere and be identical to an index on circle times two dimensional flat space, $I_{S^1 \times D^2} \simeq I_{S^1 \times \mathbb{R}^2}$

$$I_{S^1 \times D^2} = \text{tr}_{\mathcal{H}(\mathbb{R}^2)} \left[ (\pm 1)^F e^{-\beta_2 (R+2J_3)} e^{-F_\ell M_\ell} \right].$$

(4.4)
This is a BPS index of three dimensional $\mathcal{N} = 2$ supersymmetric theories. Our 3d-2d index should be thought of as a three dimensional BPS index with boundary interactions.

In the calculation of the localization of the super Yang-Mills Lagrangian (2.9), we find that the vector multiplet is given at the saddle point

$$A_3 = \text{constant} = a = \sum_c a^c T^c,$$

and the other fields are trivial. The boundary Chern-Simons term has a contribution at the saddle point and its value is given by

$$\exp \left( S_{CS. a} \right) = \exp \left( -\frac{\kappa}{4\beta} \Tr(i\beta r a)^2 \right).$$

As another possible term with $Q$-closed form, we have the FI-term

$$\mathcal{L}_{FI} = i\zeta \Tr \left( \frac{A_3}{r} - D \right).$$

In fact, the supersymmetric transformation of this FI-term does not have surface terms and becomes $Q$-closed under the boundary condition (2.10). This FI-term has a contribution at the saddle point;

$$-S_{FI} = \int_{S^1 \times D^2} \mathcal{L}_{FI} = 2\pi r \zeta Tr i\beta r a.$$

The invariance under the large gauge transformation requires a condition $2\pi r \zeta \in \mathbb{Z}$.

In the calculation of the localization, we deform the action $S$ by adding $Q$-exact term $tQ \cdot V$ and take the limit $t \to \infty$. Although, it is possible to consider boundary $\mathcal{N} = (0, 2)$ vector multiplets that are independent of the bulk $\mathcal{N} = 2$ vector multiplet, we only treat boundary $\mathcal{N} = (0, 2)$ vector multiplets which have originated from the $\mathcal{N} = 2$ bulk vector multiplets. Then the 3d-2d coupled index on $S^1 \times D^2$ can be evaluated by the one-loop calculation around the saddle point:

$$\mathcal{I}_{S^1 \times D^2} = \lim_{t_1, t_2 \to \infty} \int \mathcal{D}\Phi_{3d} \mathcal{D}\Phi_{2d} e^{-S[\Phi] - t_1 Q_{3d} V_{3d}[\Phi] - t_2 Q_{2d} V_{2d}[\Phi]}$$

$$= \frac{1}{|W_G|} \int \frac{d^N(\beta r a)}{(2\pi)^N} \left( \prod_{\alpha \neq 0} \sinh \frac{i\beta r \alpha(a)}{2} \right) e^{S_{3d} \mathcal{Z}_{1-\text{loop}}^3 \mathcal{Z}_{1-\text{loop}}^{2d}}.$$

Here $N$ is the rank of the gauge group $G$, $|W_G|$ is the order of the Weyl group $W_G$ and $\alpha$ runs over the roots. Also $S_{3d}$ is the $Q$-closed action evaluated at the saddle point. We perform the path integrals for the two dimensional boundary fields except for vector and chiral multiplets which are obtained by the restriction of the three dimensional bulk
vector and chiral multiplets. The factor "sinh" comes from the additional gauge fixing condition of $A_\tau$. For the closed manifolds $S^1 \times S^2$ and $S_6^3$, the Chern-Simons terms are $Q$-closed but not $Q$-exact and have contributions as classical terms. On the other hand, in our case $S^1 \times D^2$, the Chern-Simons term contributes to one-loop determinants of the boundary $\mathcal{N} = (0, 2)$ (chiral and Fermi) multiplets through anomaly inflows. We collect the resulting bulk and boundary one-loop determinants in the following lists and the derivation of these is summarized in the appendix D.

- The one-loop determinant of 3d $\mathcal{N} = 2$ vector multiplet

$$Z_{\text{1-loop}}^{3d, \text{vec}} = \prod_{\alpha \neq 0} e^{-\frac{(i\beta_\alpha a)^2}{8\beta_2}} (e^{i\beta_\alpha a}; q^2)_{\infty}. \quad (4.10)$$

Here we defined $q := e^{-\beta_2}$, $(a; q)_{\infty} := \prod_{n=0}^{\infty} (1 - aq^n)$ and included the "sinh" factor coming from the additional gauge fixing.

- The one-loop determinant of 3d $\mathcal{N} = 2$ chiral multiplet with Neumann boundary condition

$$Z_{\text{1-loop}}^{3d, \text{chi.N}} = \prod_{\rho} \prod_{l} e^\mathcal{E}(i\beta_\rho a + (2 - \Delta)\beta_2 + F_l M_l) (e^{i\beta_\rho a} - F_l M_l q^{\Delta}; q^2)_{\infty}^{-1}. \quad (4.11)$$

with

$$\mathcal{E}(x) := \frac{\beta_2}{12} - \frac{1}{4} x + \frac{1}{8\beta_2} x^2. \quad (4.12)$$

Here we defined $\rho$ as the fundamental weight and $l$ runs Cartan parts of the global symmetry groups. The factor $\mathcal{E}(x)$ comes from the regularization of zero point energies.

- The one-loop determinant of 3d $\mathcal{N} = 2$ chiral multiplet with Dirichlet boundary condition

$$Z_{\text{1-loop}}^{3d, \text{chi.D}} = \prod_{\rho} \prod_{l} e^{-\mathcal{E}(i\beta_\rho a + (2 - \Delta)\beta_2 + F_l M_l)} (e^{i\beta_\rho a} + F_l M_l q^{\Delta}; q^2)_{\infty}. \quad (4.13)$$

- The one-loop determinant of 2d $\mathcal{N} = (0, 2)$ vector multiplet

$$Z_{\text{1-loop}}^{2d, \text{vec}} = \prod_{\alpha \neq 0} e^{-\frac{(i\beta_\alpha a)^2}{4\beta_2}} \theta(q^2; e^{i\beta_\alpha a}). \quad (4.14)$$
• The one-loop determinant of 2d $\mathcal{N} = (0, 2)$ chiral multiplet
\[
Z_{2d.\chi}^{1\text{-loop}} = \prod_{\rho} \prod_{a} e^{2\xi(i\beta r_{\rho}(a) + \Delta_{2} + F_{l}M_{l})} \theta(e^{-i\beta r_{\rho}(a) - F_{l}M_{l}q^{\Delta}; q^{2}})_{\infty}^{-1}. \quad (4.15)
\]

• The one-loop determinant of 2d $\mathcal{N} = (0, 2)$ Fermi multiplet
\[
Z_{2d.\text{Fermi}}^{1\text{-loop}} = \prod_{\rho} \prod_{a} e^{2\xi(i\beta r_{\rho}(a) + \bar{\Delta}_{2} + F_{a}M_{a})} \theta(e^{-i\beta r_{\rho}(a) - F_{a}M_{a}q^{\bar{\Delta}}; q^{2}})_{\infty}. \quad (4.16)
\]

We find that all the one-loop determinants do not depend on the fugacity $\beta_{1}$.

Here it is worthwhile to make several remarks on features of these one-loop determinants. We shall recall there are two equivalent descriptions to express fluctuations along the normal directions to D-branes in the two dimensional hemisphere [16, 17]. One is to impose the Dirichlet boundary condition for chiral multiplets whose lowest components label coordinates along the normal directions of D-branes. The other is to impose the Neumann boundary condition and introduce a boundary interaction which effectively transmutes the one-loop determinant with the Neumann boundary condition into that with the Dirichlet boundary condition.

When we combine the three dimensional Neumann-type chiral multiplet and the boundary $\mathcal{N} = (0, 2)$ Fermi multiplet with common weight $\rho$ and $\bar{\Delta} = \Delta$, we can obtain a Dirichlet-type chiral multiplet in three dimensions:
\[
Z_{2d.\text{Fermi}}^{1\text{-loop}} Z_{3d.\chi.\text{N}}^{1\text{-loop}} = Z_{3d.\chi.\text{D}}^{1\text{-loop}}. \quad (4.17)
\]

When we consider the product of the Neumann-type and the Dirichlet-type chiral multiplets in three dimensions, we can construct a one-loop determinant of a chiral multiplet without magnetic charge $m = 0$ on $S^{1} \times S^{2}$:
\[
Z_{1\text{-loop}} Z_{1\text{-loop}}^{3d.\chi.\text{D}} = \frac{(e^{i\beta r_{\rho}(a) + F_{l}M_{l}q^{2}\Delta}; q^{2})_{\infty}}{(e^{-i\beta r_{\rho}(a) - F_{l}M_{l}q^{\Delta}}; q^{2})_{\infty}}. \quad (4.18)
\]

Now we comment on the two dimensional limit of these determinants. When the size of $S^{1}$ goes to zero ($\beta \sim \beta_{2}$) with rescaling $M_{l} \rightarrow \beta M_{l}$, the one-loop determinants of the three dimensional (vector and chiral) multiplets tend to the asymptotic forms up to divergent factors
\[
Z_{1\text{-loop}}^{3d.\text{vec}} \sim \prod_{\rho} \prod_{a} \prod_{j=0}^{\infty} (i\alpha_{a} + j + 1),
\]
\[
Z_{1\text{-loop}}^{3d.\chi.\text{N}} \sim \prod_{\rho} \prod_{a} \prod_{j=0}^{-1} (-i\beta r_{\rho}(a) - F_{l}M_{l} + \frac{\Delta}{2} + j), \quad (4.19)
\]
\[
Z_{1\text{-loop}}^{3d.\chi.\text{D}} \sim \prod_{\rho} \prod_{a} \prod_{j=0}^{\infty} (i\beta r_{\rho}(a) + F_{l}M_{l} - \frac{\Delta}{2} + 1 + j).
\]
Here we defined $a_{2d} := a\beta/2\beta_2$. These reproduce un-regularized one-loop determinants of the $\mathcal{N} = (2, 2)$ vector and chiral multiplets on the two dimensional hemisphere $D^2$. In this limit, fugacities of the flavor symmetries become the twisted masses of corresponding flavor symmetries in two dimensions.

The divergent factors in this limit come from the anomalous terms which are not invariant under the large gauge transformation. In section 5, we study the cancellation mechanism of the anomalous factors.

5 Relation to holomorphic blocks

In this section we study the relation between the 3d-2d coupled partition function and the holomorphic block in three dimensions [8]. We only consider the boundary vector multiplet which has originated in the bulk vector multiplet.

First we concentrate on the chiral multiplet. Its one-loop determinant with the Dirichlet boundary condition is expressed up to an anomalous contribution $\mathcal{E}$

$$Z_{1\text{-loop}}^{3d, \text{chir}, D} = (q^{2-\Delta}s^{-\rho}z_i^{-F_i}; q^2)_\infty,$$  \hspace{1cm} (5.1)

where we defined $s^\rho := e^{-i\beta \rho(a)}$. This Eq.(5.1) is precisely the same formula as the contribution of the chiral multiplet in the holomorphic block.

Next we discuss the condition to cancel anomalous terms. As we have seen in (4.6), there are anomalous contributions that break the single-valuedness under the large gauge transformation. First let us study anomalous terms of dynamical gauge fields. The suitable condition of canceling anomalies becomes

$$-\kappa + \frac{\beta}{2\beta_2} \left( -I_2(\text{Ad}) + \sum_{i:3d, \text{chir}} (-1)^{|i|} I_2(R_i) \right) = \frac{\beta}{\beta_2} \left( \sum_{n:2d, \text{Fermi}} I_2(R_n) - \sum_{m:2d, \text{chir}} I_2(R_m) \right),$$  \hspace{1cm} (5.2)

with

$$(-1)^{|i|} = \begin{cases} -1 & (i\text{-th chiral } = \text{Dirichlet}) \\ +1 & (i\text{-th chiral } = \text{Neumann}) \end{cases}.$$  \hspace{1cm} (5.3)

Here $I_2(R)$ is the quadratic index of the Lie algebra of the dynamical gauge group and defined by $I_2(R) := \text{Tr}_R(T^aT^b)/\text{Tr}_\square(T^aT^b)$. Each term in the left hand side (5.2) respectively comes from the boundary Chern-Simons term (4.6), the one-loop anomalous term of the vector multiplet (4.10) and the one-loop anomalous terms of the chiral multiplets (4.11), (4.13). The right hand side in (5.2) comes from anomalous terms $\mathcal{E}$’s in
the boundary multiplets (4.15), (4.16). If we set \( \beta = \beta_2 \), then anomalous contributions from three dimensional chiral multiplets induce corrected Chern-Simons levels through one-loop effects of three dimensional fermions, and the right hand side reflects the gauge anomaly coefficient of the boundary \( \mathcal{N} = (0,2) \) theory. Therefore (5.2) can be regards as an anomaly inflow condition for the effective Chern-Simons term shifted from classical one (3.18). In localization computation of four dimensional \( \mathcal{N} = 1 \) superconformal index, there also exist anomalous terms which are not invariant under large gauge transformations. In four dimensions, the anomalous terms are proportional to anomalies in four dimensions [26].

If there is a pair of a fundamental and an anti-fundamental chiral multiplets, we have an accidental cancellation. That is, we take a Neumann (Dirichlet) boundary condition for the fundamental (anti-fundamental) chiral multiplet respectively, then anomalous terms cancel each other. This means that the Chern-Simons level on \( S^1 \times D^2 \) cannot be shifted in \( \mathcal{N} \geq 3 \) supersymmetric cases, and the condition (5.2) for \( \mathcal{N} \geq 3 \) cases matches with Eq.(3.18). This observation is similar to level shifts on the flat space [27].

Next we will show a consistency relation for the mixed Chern-Simons terms of the gauge symmetry and the \( l \)-th flavor symmetry

\[
-k_{gl}\text{Tr}(T^a) + \frac{\beta}{2\beta_2} \sum_{i:3d.chi} (-1)^{|i|}\text{Tr}_{R_i}(T^a) F_i^l
\]

\[
= \frac{\beta}{\beta_2} \left( \sum_{n:2d.Fermi} \text{Tr}_{R_n}(T^a) F_{2d.Fermi,l}^n - \sum_{m:2d.chi} \text{Tr}_{R_m}(T^a) F_{2d.chi,l}^m \right). \tag{5.4}
\]

Here we used the fact that the gauge field for \( l \)-th flavor symmetry has an expectation value \( A_{l}(\mu) = (0,0,-iM_{l}/\beta r) \). \( F_{2d.Fermi,l} \) and \( F_{2d.chi,l} \) represent \( l \)-th \( U(1) \) flavor charge for two dimensional Fermi and chiral multiplet, respectively.

In addition to gauge and flavor symmetries, we have the R-symmetry which is mixed with the gauge and flavor symmetries. These have an effect on mixed Chern-Simons levels \( \kappa_{s,R} \) where the subscript ”s” runs over the bare gauge and bare flavor symmetries. Then the mixed Chern-Simons terms are given by

\[
S_{CS}^{s,R} = \frac{i\kappa_{s,R}}{4\pi} \int_{S^1 \times D^2} A^{(s)} dA^{(R)}. \tag{5.5}
\]

Here \( A_{l}^{(R)} \) is the background gauge field coupled to the R-symmetry current. As in the cases of the gauge-gauge or gauge-flavor mixed boundary Chern-Simons terms (3.20), these boundary terms are expected to contain quadratic terms of the gauge potentials

\[
\int_{T^2} A^{(s)}_{z} A^{(R)}_{z}. \tag{5.6}
\]
When we put \( \beta_2 = \beta \), the background gauge field \( A^{(R)}_\mu \) on \( S^1 \times D^2 \) or equivalently on \( S^1 \times S^2 \) has an expectation value \[ A^{(R)}_\mu = \left( 0, 0, -\frac{i}{r} \right). \] (5.7)

Then the cancellation condition for the gauge and R-symmetries become

\[-\kappa_{gR} \text{Tr}(T^a) + \sum_{i:3d.chi} \frac{(-1)^{|i|}}{2} \text{Tr}_{R_i}(T^a)(\Delta_i - 1) = \sum_{n:2d.Fermi} \text{Tr}_{R_n}(T^a)(\bar{\Delta}_n - 1) - \sum_{m:2d.chi} \text{Tr}_{R_m}(T^a)(\Delta_m - 1), \] (5.8)

and that for the bare flavor and R-symmetries is expressed as

\[-\kappa_{lR} + \sum_{i:3d.chi} \frac{(-1)^{|i|}}{2} F^i_1(\Delta_i - 1) = \sum_{n:2d.Fermi} F^{n}_{2d.Fermi,i}(\bar{\Delta}_n - 1) - \sum_{m:2d.chi} F^{m}_{2d.chi,i}(\Delta_m - 1). \] (5.9)

When we take an Abelian group for the dynamical gauge field and put \( \beta_2 = \beta \), the set of conditions (5.2), (5.4), (5.8), (5.9) matches with the decomposition rule for the effective mixed Chern-Simons levels proposed in the holomorphic blocks [8]. Let us study several examples of 3d-2d indices and compare holomorphic blocks with them in the next several subsections. We take mixed bare Chern-Simons levels and charges for gauge, flavor and R-symmetries as in [8]. There appear one-loop determinants in each model, which generally have anomalous factors. But it is useful to define one-loop determinants by omitting these anomalous factors. We use the term “one-loop determinant” in this sense in the following subsections.

5.1 Mirror to \( T_\Delta \)

As a first example, we take \( G = U(1)_G \) with a bare dynamical Chern-Simons level \( k_{GG} = +\frac{1}{2} \) and consider single chiral multiplet with a charge +1. This model is mirror to the tetrahedron \( T_\Delta \). The charge assignments and effective mixed Chern-Simons levels \(^1\) are listed in Table1. Then the one-loop determinant of the 3d bulk chiral multiplet is given by

\[ Z_{1\text{-loop}}^{3d.chi,D} = (q^2 s^{-1}; q^2)_{\infty}, \] (5.10)

\(^1\)The effective CS level is related to the bare CS level through an equation: \( \kappa_{\text{eff}}^{l} = \kappa_{\text{bare}}^{l} + \sum_{d} \frac{1}{2} \text{sign}(m_l) Q^2_l Q^1_l \). In [8], signatures of fermion masses are taken as \( \text{sign}(m_l) = +1 \).
Table 1: Left: charge assignment of the scalar in the 3d chiral multiplet. $U(1)_G$ is the dynamical gauge group and $U(1)_J$ is the topological flavor group. Right: the set of mixed Chern-Simons levels

$\kappa_{st}^\text{eff}$ | $U(1)_G$ | $U(1)_J$ | $U(1)_R$
---|---|---|---
$U(1)_G$ | 1 | 1 | -1
$U(1)_J$ | 1 | 0 | 0
$U(1)_R$ | -1 | 0 | 0

where $s := e^{-i\beta r a}$. The conditions (5.2), (5.4), (5.8), (5.9) are satisfied when we introduce a pair of boundary chiral multiplet and a Fermi multiplet, whose lowest components are respectively represented as a scalar $\phi'$ and a fermion $\Psi$. The charge assignments of these boundary multiplets are listed in table 2 and the one-loop contribution of the boundary multiplets is given by

$$Z^{2d}_{\text{1-loop}} = \frac{\theta(x; q^2)}{\theta(sx; q^2)}.$$  (5.11)

From (5.10) and (5.11), we obtain a 3d-2d index of this model $T'_\Delta$ mirror to the tetrahedron theory $T_\Delta$

$$I^{T'_\Delta}_{S^1 \times D^2} = \int \frac{ds}{2\pi i s} \frac{\theta(x; q^2)}{\theta(sx; q^2)}(q^2 s^{-1}; q^2)_{\infty}.  \quad (5.12)$$

We verify that this 3d-2d index on $S^1 \times D^2$ reproduces the holomorphic block of $T'_\Delta$. \footnote{The convention of the theta function is $\theta(x; q)(\text{here}) = \theta(q^{\frac{1}{2}} x; q)([8])$. The normalization of the fugacity $q$ is different from that of [8]. In addition, an extra sign difference comes from $(-1)^F$ which is used in the holomorphic blocks instead of $(-1)^F$. Then the identification becomes $q(\text{here}) = -q^2([8])$.}

That is to say, in our language, the contribution of the chiral multiplet in the holomorphic blocks [8] corresponds to the one-loop determinant of three dimensional chiral multiplets with the Dirichlet boundary condition. This result causes one question: Is it possible to reproduce the result of holomorphic blocks by imposing the Neumann boundary condition? At least when the gauge group is Abelian and the superpotential is absent, the answer
is always positive. To see this, we take the Neumann boundary condition for the chiral multiplet. The one-loop determinant of three dimensional chiral multiplet is given by

\[ Z_{3d,\text{chi.N}}^{1\text{-loop}} = (s; q^2)^{-1}_\infty. \] (5.13)

In this case, the sign of the level shift from the 3d anomalous factor is the opposite of the sign of the result for the Dirichlet boundary condition. For example, the dynamical Chern-Simons level is shifted by \(-\frac{1}{2}\). The effective Chern-Simons levels for the Neumann boundary condition are listed in table 3. In addition, we have to satisfy the conditions (5.2), (5.4), (5.8), (5.9) for the anomaly cancellation and introduce appropriate boundary multiplets. They are listed in table 4. Then the one-loop determinant of the boundary multiplets is give by

\[ Z_{2d}^{1\text{-loop}} = \frac{\theta(s; q^2)\theta(x; q^2)}{\theta(sx; q^2)}. \] (5.14)

The 3d-2d index with the Neumann boundary condition

\[ \mathcal{I}_{S^1 \times D^2}^{T \Delta, \text{Neu}} = \int \frac{ds}{2\pi i s} (s; q^2)^{-1}_\infty \theta(s; q^2)\theta(x; q^2) \theta(sx; q^2) \] (5.15)

gives the same result as the Dirichlet boundary condition (5.12).

| \( \kappa_{sl}^{\text{eff}} \) | \( U(1)_G \) | \( U(1)_J \) | \( U(1)_R \) |
|---|---|---|---|
| \( \bar{U}(1)_G \) | 0 | 1 | 0 |
| \( \bar{U}(1)_J \) | 1 | 0 | 0 |
| \( \bar{U}(1)_R \) | 0 | 0 | 0 |

Table 3: The set of mixed effective Chern-Simons levels for the Neumann boundary condition.

| \( \phi'' \) | \( \bar{U}(1)_G \) | \( \bar{U}(1)_J \) | \( \bar{U}(1)_R \) |
|---|---|---|---|
| 1 | 1 | 0 |
| \( \Psi_1 \) | 1 | 0 | -1 |
| \( \Psi_2 \) | 0 | 1 | -1 |

Table 4: For the Neumann boundary condition: Charge assignments of the scalar in the 2d boundary chiral multiplet and fermions in the Fermi multiplets.
Table 5: XYZ model. Left: charge assignments of scalars in the 3d chiral multiplets. Right: the set of mixed Chern-Simons levels

### 5.2 XYZ model

As a second example, we consider the XYZ model that consists of three chiral multiplets in bulk three dimensions (table 5). When we impose minimal boundary conditions on the bulk chiral multiplets, one-loop contributions are collected into a formula

\[
Z_{1\text{-loop}}^{3d,\text{chi}} = (q^2x^{-1}; q^2)_\infty (q^2y^{-1}; q^2)_\infty (xy; q^2)_\infty. \tag{5.16}
\]

Next we introduce a boundary chiral multiplet \( \phi \) to cancel the bulk-boundary anomalies (Table 6). Then this multiplet has the contribution at the one-loop level

\[
Z_{1\text{-loop}}^{2d,\text{chi}} = \theta(xy; q^2)^{-1}. \tag{5.17}
\]

From (5.16) and (5.17), the 3d-2d index on \( S^1 \times D^2 \) becomes

\[
\mathcal{I}_{XYZ}^{S^1 \times D^2} = \frac{(q^2x^{-1}; q^2)_\infty (q^2y^{-1}; q^2)_\infty (xy; q^2)_\infty}{\theta(xy; q^2)}. \tag{5.18}
\]

This matches with the holomorphic block of the XYZ model.

### 5.3 SQED

In this subsection, we consider the SQED model. From table 7, the one-loop determinants of three dimensional chiral multiplets \( \phi_1, \phi_2 \) are collected into a formula

\[
Z_{1\text{-loop}}^{3d,\text{chi},D} = (s^{-1}q^2; q^2)_\infty (sq^2x^{-1}; q^2)_\infty. \tag{5.19}
\]

A pair of boundary multiplets should be introduced to cancel anomalous terms. It is a pair of a boundary \( \mathcal{N} = (0,2) \) chiral and a Fermi multiplets whose lowest components are
respectively a scalar $\phi'$ and a fermion $\Psi$. Their charge assignments are listed in Table 8. Then one-loop contributions of the boundary multiplets are given by

$$Z_{1\text{-loop}}^{2d,\text{chi}} Z_{1\text{-loop}}^{2d,\text{Fermi}} = \frac{\theta(y; q^2)}{\theta(sy; q^2)}.$$  

(5.20)

Table 8: SQED. Charge assignment of the scalar (fermion) in the boundary chiral (Fermi) multiplet.

Thus the 3d-2d index of this model becomes

$$I_{\text{SQED}} = \int \frac{ds}{2\pi is} (s^{-1}q^2; q^2)^\infty (sq^2x^{-1}; q^2)^\infty \frac{\theta(y; q^2)}{\theta(sy; q^2)}.$$  

(5.21)

This has the same expression as the result in the holomorphic block for the SQED. The holomorphic block for the SQED also matches with that in the XYZ model. Thus 3d-2d indices for these two models produce the identical result. We mention one comment here: The SQED and the XYZ model flow to the same IR fixed point and the set of them is the simplest example of the $\mathcal{N} = 2$ mirror pair in three dimensions [29]. The half BPS boundary conditions for the SQED and the XYZ model were studied in [30], where it is shown that the $\mathcal{N} = (0, 2)$-type BPS boundary condition in the SQED is mapped to the $\mathcal{N} = (0, 2)$-type supersymmetry in the XYZ model. Our result is consistent with their analysis of the boundary conditions because the 3d-2d index on $S^1 \times D^2$ preserve the boundary $\mathcal{N} = (0, 2)$ supersymmetry. This situation is different from the two dimensions. For a mirror pair in two dimensions, the A-type boundary supersymmetry is mapped to the B-type boundary supersymmetry [31, 32, 33].
5.4 Appetizer

All the above examples are Abelian theories for dynamical gauge fields. In such cases, vector multiplets do not contribute to the calculation. In this subsection, we consider the appetizer \[34\] which consists of a non-Abelian\((G = SU(2))\) dynamical gauge field with the bare Chern-Simons level \(1\) \(^3\) and single adjoint chiral multiplet. We assign charges for the 3d bulk field as in table 9. Here we assume the Chern-Simons level in this model is identical with the bare CS level in \([8]\).

| | SU(2) \(G\) | U(1) \(x\) | U(1) \(R\) |
|---|---|---|---|
| \(\phi\) | Ad | 1 | 0 |

\[\begin{array}{|c|c|c|c|}
\hline
K^\text{eff}_{\kappa} & SU(2) \(G\) & U(1) \(x\) & U(1) \(R\) \\
\hline
SU(2) \(G\) & 6 & 0 & 0 \\
U(1) \(x\) & 0 & 0 & 0 \\
U(1) \(R\) & 0 & 0 & 0 \\
\hline
\end{array}\]

Table 9: Left: charge assignment of the scalar in the 3d chiral multiplet. Right: the set of mixed Chern-Simons levels

Before we examine this model, we mention on the discrepancy of contributions in the sector of the vector multiplet between our model and holomorphic blocks for non-Abelian cases. In our calculation, the contribution of the vector multiplet is given by the one-loop determinant up to anomalous contributions

\[
\prod_{\alpha > 0} (s^\alpha ; q^2)_{\infty} (s^{-\alpha} ; q^2)_{\infty}. \tag{5.22}
\]

On the other hand, the contribution from the non-Abelian gauge field in the holomorphic block is read as

\[
\prod_{\alpha > 0} \frac{(qs^\alpha ; q^2)_{\infty}(q^{-1}s^{-\alpha} ; q^2)_{\infty}}{(q^2s^\alpha ; q^2)_{\infty}(q^{-2}s^{-\alpha} ; q^2)_{\infty}}. \tag{5.23}
\]

Moreover, in general, the result from the effective Chern-Simons terms in our model is different from that of the holomorphic block.

In spite of these differences, the result of the 3d-2d index of the appetizer is still consistent with the calculation in the holomorphic block. To see this, we write down the 3d-2d index of the appetizer. First of all, the one-loop determinant of the adjoint chiral multiplet is given by

\[
Z_{1-\text{loop}}^{3d,\chi} = (q^2s^2x^{-1} ; q^2)_{\infty}(q^2x^{-1} ; q^2)_{\infty}(q^2s^{-2}x^{-1} ; q^2)_{\infty}. \tag{5.24}
\]

\(^3\)Here we take the bare Chern-Simons level 2 as in \([8]\)
In order to satisfy the relations (5.2), (5.4), (5.8), (5.9), we have to introduce two $SU(2)_G$ fundamental chiral multiplets and an adjoint chiral multiplet whose lowest components are scalars $\phi'_1$, $\phi'_2$ and $\phi'_3$. Their properties are listed in table 10. Then one-loop determinants

$$Z^{2d,\text{chi}}_{1\text{-loop}} = \theta(qs; q^2)^{-2}\theta(s^2; q^2)^{-1}\theta(s^{-2}; q^2)^{-1}. \quad (5.25)$$

Then the 3d-2d index becomes

$$I_{\text{App}}^{S^1\times D^2} = \int \frac{ds}{2\pi is} \frac{\theta(qs; q^2)^2\theta(q^2s^2; q^2)^\infty\theta(q^2s^{-2}; q^2)^\infty}{(q^2s^{-1}; q^2)^\infty\theta(qs; q^2)^2(q^2s^2; q^2)^\infty(q^2s^{-2}; q^2)^\infty}. \quad (5.26)$$

Even though there are discrepancies in the sector of the vector multiplet, the result agrees with the holomorphic block of the appetizer.

## 6 Several models

In this section we study several models. We have chosen bare Chern-Simons levels to cancel the anomalous contributions coming from the one-loop determinants.

### 6.1 $S^1$-uplift of $\mathbb{C}P^N$ model, $q$-deformed Whittaker function and K-theoretic $J$-function

In this subsection we explain mathematical aspects of the index of $G = U(1)$ with $N + 1$ chiral multiplets with flavor charges $+1$’s:

$$I_{S^1\times D^2}^{\mathbb{C}P^N} = \int \frac{ds}{2\pi is} \frac{s^{-2\pi\zeta}}{\prod_{l=1}^{N+1} \prod_{j=0}^{\infty} (1 - sq^{2j}z_l)}. \quad (6.1)$$

Here $s = e^{-ib^a}$, $q = e^{-\beta_2}$, $z_l = e^{-M_l}$ and $M_l$’s ($l = 1, \cdots, N + 1$) represent the set of fugacities of $SU(N + 1)$ flavor symmetry.

We want to shed light on the meaning of this index. First let us recall the mathematical aspects of the hemisphere indices. In two dimensions, the partition functions on the
hemisphere $D^2$ are related to (equivariant) $J$-functions of the $\mathbb{C}P^N$ model in the large volume regime. In order to clarify the geometric data in our model, we evaluate the above integral explicitly. The integrand has poles and we consider residues at $s = q^{-2k}z_l^{-1}$, $(k = 0, 1, \cdots)$. Then Eq.(6.1) is rewritten:

$$
I_{S^1 \times D^2}^{\mathbb{C}P^N} = \oint_{|s-z_l^{-1}|=\epsilon} \frac{ds}{2\pi i s} z_l^{2\pi r\zeta} z^{-2\pi r\zeta} \left( \prod_{j=0}^{\infty} \frac{1}{\prod_{l=1}^{N+1} (1 - s z_l(q^{2j}))} \right)
\times \left( \sum_{k=0}^{\infty} \frac{Q^k}{\prod_{l=1}^{N+1} \prod_{j=1}^{k} (1 - s z_l(q^{-2j}))} \right).
$$

(6.2)

Here we defined $Q := q^{2\pi r\zeta}$ and assumed $\zeta > 0$. The region for $\zeta > 0$ corresponds to the Higgs branch in the two dimensional limit. The second line in the above equation

$$
J^{\mathbb{C}P^N}(Q, s, z, q) := \sum_{k=0}^{\infty} \frac{Q^k}{\prod_{l=1}^{N+1} \prod_{j=1}^{k} (1 - s z_l(q^{-2j}))}
$$

agrees with the equivariant K-theoretic $J$-function of $\mathbb{C}P^N$ [35] by rescaling parameters appropriately. On the other hand, the function $J^{\mathbb{C}P^N}$ reduces to the ordinary K-theoretic $J$-function in the un-equivariant limit $z_l \to 1$. In order to compare our model to two dimensional cases, we take the two dimensional limit (4.20). Then the index (6.1) reduces to the hemisphere partition function of the 2d model whose target space is $\mathbb{C}P^N$

$$
\lim_{\beta \to 0} I_{S^1 \times D^2}^{\mathbb{C}P^N} \sim \int \frac{dy}{2\pi i} z^{2\pi i \zeta_2 y} \prod_{l=1}^{N+1} \Gamma(\zeta - M_l).
$$

(6.4)

Namely, this has the same formula as the two dimensional hemisphere partition function for the $\mathcal{N} = (2, 2)$ $U(1)$ theory with $N + 1$ chiral multiplets with flavor charges $+1$'s and twisted masses $M_l$'s.

In [36, 37], an eigenfunction of the Hamiltonian of the $q$-deformed $\mathfrak{gl}_{N+1}$ Toda chain is constructed, that is, some kind of the (specialized) $q$-deformed Whittaker function. This specialized $q$-deformed Whittaker function $\Psi_{z_l}^{\mathfrak{gl}_{N+1}}(n, k)$ has a following contour integral representation

$$
\Psi_{z_l}^{\mathfrak{gl}_{N+1}}(n, k) = \left( \prod_{l=1}^{N} z_l^k \right) \oint \frac{ds}{2\pi i s} s^{-n} \prod_{l=1}^{N+1} (z_l s; q)_\infty^{-1}.
$$

(6.5)

When we set $n = 2\pi r\zeta$ and replace $q \to q^2$ in the above equation, the index on $S^1 \times D^2$ (6.1) agrees with this specialized $q$-deformed Whittaker function $\Psi_{z_l}^{\mathfrak{gl}_{N+1}}(n, k)$ up to the overall factor $\left( \prod_{l=1}^{N} z_l^k \right)$. Here the contour is chosen to take all the poles except for the
pole at the origin. We can also include the factor \( \left( \prod_{l=1}^{N} z_l^k \right) \) to the index on \( S^1 \times D^2 \) by turning on the FI term (4.7) for the flavor gauge field with the FI-parameter \( \frac{k}{2\pi i} \). As a geometrical interpretation of \( \Psi^{gl}_{M}(n, k) \), it is also conjectured in [37] \( \prod_{l=1}^{N} z_l^k \). As a geometrical interpretation of \( \Psi^{gl}_{M}(n, k) \), it is also conjectured in [37], it is also conjectured in [37]

\[
(6.6)
\]

Here \( QM_\infty(\mathbb{CP}^N) \) is the space of the degree-\( \infty \) quasi maps \( \mathbb{CP}^1 \rightarrow \mathbb{CP}^N \) and \( Ch_G \) and \( Td_G \) are \( G = U(1) \times GL(N+1) \)-equivariant Chern character and Todd class, respectively.

Remarkably, it was already pointed out that \( q \)-deformed Whittaker functions is related to the partition function of an equivariant A-type twisted model on \( S^1 \times D^2 \) in [38]. Three dimensional version of the A-type twisted Chern-Simon-Matter theory is also constructed in [39]. But the 3d-2d index considered in this paper is not topologically twisted counterpart and it seems that this index is not directly related to the topologically twisted theories. It is interesting to reveal the relation between the 3d-2d index and the A-type twisted theories in three dimensions.

### 6.2 Vortex partition function and surface operator

The vortex partition functions [40, 41, 42] are vortex counterparts of the Nekrasov instanton partition functions [43]. It is shown [44, 45, 46] that the partition function on \( S^3_b \) (\( S^1 \times S^2 \)) is respectively factorized into a pair of the vortex and anti-vortex partition functions. Looking at the Higgs branch calculation in three dimensions [47, 48], we can also construct some \( Q \)-exact term whose saddle points admit point-like vortices at the north pole of the hemisphere. So we expect the index on \( S^1 \times D^2 \) contains contributions from vortex partition functions.

Here we consider a non-Abelian model: The gauge group is \( G = U(N) \) and the flavor symmetry is \( SU(N_f) \times SU(\tilde{N}_f) \) with \( N_f \geq \tilde{N}_f \). We have \( N_f \) (flavor) fundamental chiral multiplets with Neumann boundary conditions and \( \tilde{N}_f \) (flavor) anti-fundamental chiral multiplets with Dirichlet boundary conditions.

The partition function of this matter content is given by

\[
\mathcal{I}_{S^1 \times D^2} = \frac{1}{N!} \int \frac{ds_a}{2\pi is_a} \left( \prod_{1 \leq a \neq b \leq N} (1 - s_a s_b^{-1} q^{2j}) \right) \prod_{a=1}^{N} \prod_{j=0}^{\infty} (1 - s_a q^{2j+2} z_l^{-1}) \prod_{l=1}^{\tilde{N}_f} (1 - s_a q^{2j} \tilde{z}_m).
\]

Here \( s_b = e^{-i\beta_b a} \), \( q = e^{-i\beta_2} \), \( z_l = e^{-M_l} \) and \( M_l \)'s \( (l = 1, \cdots, N_f) \) represent the set of fugacities of \( SU(N_f) \) flavor symmetry. Also we put \( \tilde{z}_m := e^{-\tilde{M}_m} \) and \( \tilde{M}_m \)'s \( (m = 1, \cdots, \tilde{N}_f) \) represent the set of fugacities of another flavor symmetry \( SU(\tilde{N}_f) \).
When we take residues at poles \( s_a = q^{-2j'_a} z_{l'_a}^{-1} \) \( (l'_a = 1, \ldots, N_f; a = 1, \ldots, N) \), then the partition function is written as a combination of classical terms, one-loop terms and vortex partition functions

\[
I_{S^1 \times D^2} = \sum_{\{l'_a\} \subset \{N_f\}} Z_{cl}^{(l')} Z_{1\text{-loop}}^{(l')} Z_{\text{vortex}, \{l'_a\}}^{(l')},
\]

with

\[
Z_{cl}^{(l')} = \prod_{a=1}^{N} (z_{l'_a}^{-1})^{2\pi r \zeta},
\]

\[
Z_{1\text{-loop}}^{(l')} = \prod_{j=0}^{\infty} \prod_{m=1}^{\tilde{N}_f} \prod_{a=1}^{N} \prod_{l \notin \{l'_a\}} \left( 1 - z_{l'_a}^{-1} z^{2j} \right)^{-1},
\]

\[
Z_{\text{vortex}, \{l'_a\}}^{(l')} = \prod_{k} Q^{\sum_{a=1}^{N} k_a} \prod_{j=1}^{k_a} \prod_{s=1}^{\tilde{N}_j} \left( 1 - q^{-2j + 2} z_{l'_a}^{-1} z^{2j} \right)^{-1}
\]

\[
\times \left( \prod_{1 \leq a,b \leq N} \prod_{j=0}^{k_a} \prod_{s=1}^{\tilde{N}_j} \left( 1 - z_{l'_a}^{-1} q^{2j-2k_a} \right) \prod_{a=1}^{N} \prod_{l \notin \{l'_a\}} \prod_{j=1}^{k_a} \left( 1 - z_{l'_a}^{-1} q^{-2j} \right) \right).
\]

Here we defined \( \{l'_a\} := \{l_1, l_2, \ldots, l_N\} \) with \( 1 \leq l_1 < l_2 < \cdots < l_N \leq N_f \) and \( \{N_f\} := \{1, 2, \cdots, N_f\} \). Also the sum is defined by \( \sum_{\{j'_a\}} := \sum_{i=1}^{N} \sum_{j_i=0}^{\infty} \).

In three dimensions, BPS vortices are point-like objects (particles) and the K-theoretic vortex partition functions contribute to the BPS index. Thus appearance of the K-theoretic vortex partition functions in (6.8) is also consistent with the observation that the index on \( S^1 \times D^2 \) is related to the 3d \( \mathcal{N} = 2 \) BPS index on \( S^1 \times \mathbb{R}^2 \). This is quite analogous to the fact that instantons on \( S^1 \times \mathbb{R}^4 \) are particle-like objects and the K-theoretic instanton partition functions [49] contribute to the BPS index in five dimensions.

From the point of view of the geometric engineering, instanton counting with surface operators in five-dimensions is encoded into partition functions of open-closed (refined) topological strings. An appropriate decoupling limit of the bulk five dimensional part leads us to realize open topological strings. It is widely believed the existence of relations among theses three functions; the BPS index in three dimensions, the vortex partition function, and the open topological string partition functions [41]. These relations have been intensively investigated. For example, in non-Abelian cases, the correspondence between the vortex partition function and the open topological strings has been already studied in [46].
Next, we will study the instanton counting with surface operators in the sector of vanishing instanton number. The result leads to the vortex counting for the non-Abelian $U(N)$ gauge theory with $N_f$-flavors fundamental chiral multiplets.

We consider a five dimensional $\mathcal{N} = 1$ pure $SU(N_f)$ gauge theory on $S^1 \times \mathbb{C}^2$. We further take the surface operator specified by a Levi subgroup $L = S(U(N) \times U(N_f - N)) \subset SU(N_f)$. Then the instanton counting with the surface operator is replaced by the instanton counting on the orbifold $\mathbb{C} \times \mathbb{C}/\mathbb{Z}_2$ [50, 51]. By the orbifold action $(z, \omega) \rightarrow (z, -\omega)$, the set of the ADHM data $(B_1, B_2, I, J)$;

$$B_1, B_2 \in \text{End}(V), \quad I \in \text{Hom}(W, V), \quad I \in \text{Hom}(V, W)$$

is divided into two groups with $\mathbb{Z}_2$-grading indices "$i$" ($i = 1, 2$)

$$V = V_1 \oplus V_2, \quad W = W_1 \oplus W_2,$$

and

$$A_i = B_1|_{V_i} \in \text{End}(V_i), \quad B_i = B_2|_{V_i} \in \text{Hom}(V_i, V_{i+1}),$$

$$I_i = I|_{W_i} \in \text{Hom}(W_i, V_i), \quad J_i = J|_{V_i} \in \text{Hom}(V_i, W_{i+1}).$$

These spaces $V_i, W_i$ have some dimensions: $\dim V_i = k_i$, $\dim W_1 = N$, $\dim W_2 = N_f - N$. Also the instanton number $k$ and the vortex number $k_1$ are related by a relation $k_1 - k_2 = k$.

Then the moduli space is represented as a quotient

$$\mathcal{M}_{L,k_1,k_2} = \left\{ (A_i, B_i, I_i, J_i) \big| A_{i+1}B_i - B_i A_i + I_{i+1}J_i = 0 \right\} / \prod_i GL(k_i, \mathbb{C}),$$

with some stability condition. The equivariant character for the tangent space of $\mathcal{M}_{L,k_1,k_2}$ is evaluated as

$$\chi(T_p\mathcal{M}_{L,k_1,k_2}) = \sum_{i,j=1}^2 \left[ e^{\varepsilon_1 + \frac{\varepsilon_2}{2}} \chi(W_i)\chi(V_j^*) + \chi(V_i)\chi(W_j^*) \right.$$

$$\left. - (1 - e^{\varepsilon_1})(1 - e^{\frac{\varepsilon_2}{2}})\chi(V_i)\chi(V_j^*) \right] \bigg|_{\mathbb{Z}_2-\text{even}},$$

with

$$\chi(W_1) = e^{\frac{\varepsilon_2}{2}} \sum_{a=1}^N e^{M_a}, \quad \chi(W_2) = \sum_{b=1}^{N_f-N} e^{-M'_b},$$

$$\chi(V_1) = e^{\frac{\varepsilon_2}{2}} \sum_{a=1}^N e^{M_a} \sum_{(i,2j) \in Y_a} e^{-(i-1)\varepsilon_1 -(j-1)\varepsilon_2} + \sum_{b=1}^{N_f-N} e^{M'_b} \sum_{(i,2j) \in X_b} e^{-(i-1)\varepsilon_1 -(j-\frac{1}{2})\varepsilon_2},$$

$$\chi(V_2) = e^{\frac{\varepsilon_2}{2}} \sum_{a=1}^N e^{M_a} \sum_{(i,2j) \in Y_a} e^{-(i-1)\varepsilon_1 -(j-1)\varepsilon_2} + \sum_{b=1}^{N_f-N} e^{M'_b} \sum_{(i,2j-1) \in X_b} e^{-(i-1)\varepsilon_1 -(j-\frac{1}{2})\varepsilon_2}.$$
Here $Y_a$’s, $X_b$’s are Young diagrams. The collection $(M_1 \cdots M_N, M_1', \cdots, M_{N_f-N})$ is the set of Coulomb moduli parameters and $\varepsilon_i$’s are $\Omega$-background parameters. In the above equation for the equivariant character, the symbol “$\mathbb{Z}_2$-even” means that we remove terms expressed as $e^{r \varepsilon_2}$ ($r = \frac{1}{2}, \frac{3}{2}, \cdots$). The non-negative integers $k_i$’s are related to the number of boxes by the following equations

\[
k_1 = \sum_{a=1}^{N} \#\{(i, 2j-1)|(i, 2j-1) \in Y_a\} + \sum_{b=1}^{N_f-N} \#\{(i, 2j)|(i, 2j) \in X_b\}
\]

\[
k_2 = \sum_{a=1}^{N} \#\{(i, 2j)|(i, 2j) \in Y_a\} + \sum_{b=1}^{N_f-N} \#\{(i, 2j-1)|(i, 2j-1) \in X_b\},
\]

with $i, j \in \mathbb{N}$. Here $\#\{\cdots\}$ expresses the cardinality of the set.

Now we put the instanton number $k_2 = 0$. Then the ADHM data on $\mathbb{C} \times \mathbb{C}/\mathbb{Z}_2$ reduces to $A_1, I_1$ and $J_1$:

\[
A_1 \in \text{End}(V_1), \quad I_1 \in \text{Hom}(W_1, V_1), \quad J_1 \in \text{Hom}(V_1, W_2).
\]

This $(A_1, I_1, J_1)$ is precisely the same as the data of the moduli space of the $k_1$-vortex of the $G = U(N)$ gauge theory with $N_f$-flavor symmetry. Moreover, from the relation (6.19), we find that $Y_a = \{(i, 1)|i = 1, \cdots, k_1^a\}$ with $\sum_{a=1}^{N} k_1^a = k_1$, but $X_b$’s are absent. Thus the equivariant character $\chi(T_p\mathcal{M}_{L,k_1,k_2})$ reduces to

\[
\chi(T_p\mathcal{M}_{L,k_1,k_2=0}) = e^{\varepsilon_1+\frac{N_f}{2}} \chi(W_2) \chi(V_1^*) + \chi(V_1) \chi(W_1^*) - (1 - e^{\varepsilon_1}) \chi(V_1) \chi(V_1^*)
\]

\[
= \sum_{a,a'=1}^{N} e^{M_a-M_{a'}} \sum_{i=1}^{k_1^a} e^{(k_1^a+1-i)\varepsilon_1} + \sum_{b=1}^{N_f-N} \sum_{a=1}^{N} e^{M_b-M_a} \sum_{i=1}^{k_1^a} e^{i\varepsilon_1},
\]

with

\[
\chi(W_1) = e^{\frac{N_f}{2}} \sum_{a=1}^{N} e^{M_a}, \quad \chi(W_2) = \sum_{b=1}^{N_f-N} e^{M_b}, \quad \chi(V_1) = e^{\frac{N_f}{2}} \sum_{a=1}^{N} e^{M_a} \sum_{i=1}^{k_1^a} e^{-(i-1)\varepsilon_1}.
\]

Explicitly $\chi(T_p\mathcal{M}_{L,k_1,k_2=0})$ does not depend on the equivariant parameter $\varepsilon_2$ and is precisely the same as the equivariant character of the tangent space of the $k_1$-vortex moduli space of the $G = U(N)$ gauge theory with $N_f$-flavor fundamental chiral multiplets. The latter has the fixed point $p$ labeled by $(k_1^1, \cdots, k_1^N)$ under the equivariant action. In the context of the three dimensional theory on $S^1 \times \mathbb{C}$, the Coulomb moduli parameters are regarded as real masses or fugacities associated to the $SU(N_f)$-flavor symmetry.
The K-theoretic vortex partition function can be obtained from the equivariant character $\sum_i \pm e^{\omega_i,p}$ by the replacement $\sum_i \pm e^{\omega_i,p} \to \sum_p \prod_i (1 - e^{\omega_i,p})^{\pm 1}$ as in the case of the K-theoretic instanton partition function. Here $p = (k_1^1, \cdots, k_N^1)$ denotes a fixed point under $U(1)^{N_f - 1} \times U(1)^{\varepsilon_1}$-equivariant action and $\omega_{i,p}$'s denote the equivariant weights at the point $p$. Then, from the equivariant character (6.21), the restricted vortex partition function with vortex number $k_1$ is written down

$$Z_{k_1\text{-vortex}} = \sum_{\sum_n k_i^n = k_1} \prod_{a,a'=1}^N (1 - e^{M_a - M_{a'} + (k_{1}^{a'} + 1 - i)\varepsilon_1})^{-1} \prod_{b=1}^{N_f - N} \prod_{a=1}^N \prod_{i=1}^{k_i^b} (1 - e^{M'_b - M_a + i\varepsilon_1})^{-1}. \quad (6.23)$$

When we identify the parameters as $\varepsilon_1 = q^{-2}$, $e^{M'_b} = z_l (b + N = l)$ and $e^{M_a} = z_{l'} (a = l')$, (6.23) agrees with $k_1$-vortex number sector of (6.11) with $\tilde{N}_f = 0$.

### 6.3 Calabi-Yau model and 3d matrix factorization

So far all the models we considered do not have surface terms of superpotentials. In this section, we will consider a simple model which has non-trivial three dimensional analogue of the matrix factorization and study its two-dimensional limit.

In this section we consider an $S^1$-uplift of an $\mathcal{N} = (2, 2)$ GLSM which flows in the IR limit to some non-linear sigma model that describes a hypersurface defined by a degree $N$ homogeneous polynomial $f(x_1, \cdots, x_N) = 0$ in $\mathbb{C}P^{N-1}$ in the large volume regime. As a set-up, we consider an Abelian model with $G = U(1)$ and take a set of chiral multiplets $P, \phi_I, (I = 1, \cdots, N)$ with a superpotential

$$W(P, \phi_I) = P \cdot f(\phi_I). \quad (6.24)$$

Here $f(\phi_I)$ is the homogeneous polynomial of a degree $N$. The charge assignments of the chiral multiplets are listed in table 11. We impose Neumann boundary conditions for these chiral multiplets. Then the one-loop contribution of the bulk three dimensional

| | $U(1)_G$ | $U(1)_R$ |
|---|---|---|
| $P$ | $-N$ | +2 |
| $\phi_I$ | +1 | 0 |

Table 11: Charge assignments of scalars of three dimensional chiral multiplets.

The partition function of chiral multiplets is given by

$$Z_{3d.chi.} = (e^{-i\beta r_a}; q^2)^{-N} (e^{iN\beta r_a q^2}; q^2)^{-1}. \quad (6.25)$$
In addition, we have another contribution from the two dimensional boundary and the corresponding boundary theory is characterised by funtuns $E_a$ and $J^a$. This boundary effect is correlated to the bulk 3d theory through some kind of a three dimensional matrix factorization. We choose $E_a$ and $J^a$ with $E_a J^a = W$ in order to realise this factorization

$$E(P, \phi_I) = P, \quad J(P, \phi_I) = f(\phi_I). \quad (6.26)$$

Then the boundary contribution in the partition function comes from the one-loop determinant of the Fermi multiplet coupled to $E(P, \phi_I)$ and is given by

$$Z_{1\text{-loop}}^{\text{2d,Fermi}} = \theta(e^{N_i \beta_1 a} q_1; q_1). \quad (6.27)$$

We can also include the FI-term and write down the expression of the partition function of the model

$$I_{S^1 \times D^2}^{\text{CY}_{N-2}} = \int \frac{d(1/2\pi)}{2\pi} e^{-S_{\text{FI}}} Z_{1\text{-loop}}^{\text{3d,chi.N}} Z_{1\text{-loop}}^{\text{2d,Fermi}} = \int \frac{d(1/2\pi)}{2\pi} e^{2\pi \tau \xi (1/2\pi)} \frac{\theta(e^{N_i \beta_1 a} q_1; q_1)}{(e^{-i \beta_1 a}; q_2)^2_N(e^{i N \beta_1 a} q_2; q_2)^2_\infty}. \quad (6.28)$$

In the two dimensional limit $\beta \to 0$, Eq.(6.28) reduces to

$$\lim_{\beta \to 0} I_{S^1 \times D^2}^{\text{CY}_{N-2}} \sim \int \frac{dy}{2\pi} e^{-2\pi \zeta (2x - 2\eta)} \Gamma(1 - Ny) \Gamma(y) N \left(e^{-\pi i Ny} - e^{\pi i Ny}\right). \quad (6.29)$$

Here we set $y = \nu a$ and $\beta \zeta = \zeta_{2d} + \frac{\nu}{2\pi}$. This agrees with the hemisphere partition function of the line bundle $O(n)$ over $\text{CY}_{N-2}$.

Next we want to relate this formula to some geometric data. We pick up poles $e^{-i \beta_1 a} q_2 k = 1, (k = 0, 1, 2, \cdots)$ of the integrand and evaluate Eq.(6.28).

When we introduce $b$ as $e^{i \beta_1 b} = e^{-i \beta_1 a} q_2^k$, then Eq.(6.28) is expressed as

$$I_{S^1 \times D^2}^{\text{CY}_{N-2}} = -\sum_{k=0}^{\infty} \int \frac{d(1/2\pi)}{2\pi} e^{2\pi \xi (1/2\pi)(-2k)} \frac{\theta(e^{N_i \beta_1 b} q_2^{2+2Nk}; q_2^2)}{(e^{-i \beta_1 b} q_2^{-2k}; q_2^2)^2_N(e^{-i N \beta_1 b} q_2^{2+2Nk}; q_2^2)^2_\infty} \times \frac{(e^{-i \beta_1 b} q_2^{2+2k}; q_2^2)^2_N}{(e^{-i N \beta_1 b} q_2^{2+2Nk}; q_2^2)^2_\infty} \frac{\theta(e^{N_i \beta_1 b}; q_2^2)}{\theta(e^{-i \beta_1 b}; q_2^2)^N}. \quad (6.30)$$

This formula is thought of a kind of the $S^1$-uplift of the two dimensional model and it should be related to K-theoretic Gromov-Witten invariants. Also the integrand should be interpreted as some kind of an $S^1$-uplift of the $\Gamma$-class when we put $k = 0$ in the integrand

$$\Gamma(x; q) = \frac{(e^{-x} q^2; q^2)^N_\infty \theta(e^{-N_x} q^2)}{(e^{N_x} q^2; q^2)^2_\infty} \theta(e^x; q^2)^N. \quad (6.31)$$
6.4 3d $\mathcal{N} \geq 3$ models and gauge/Bethe correspondence

In this subsection, we study three dimensional $\mathcal{N} \geq 3$ supersymmetric theories and Bethe ansatz for quantum integrable models. We consider the $G = U(N)$ theory with $N_f$ (flavor) fundamental hypermultiplets. In the language of the $\mathcal{N} = 2$ multiplets, an $\mathcal{N} = 4$ vector multiplet consists of an $\mathcal{N} = 2$ vector multiplet $(A_\mu, \sigma, D, \lambda, \bar{\lambda})$ and an $\mathcal{N} = 2$ adjoint chiral multiplet $(\sigma', \chi', D')$. On the other hand, an $\mathcal{N} = 4$ hyper multiplet consists of an $\mathcal{N} = 2$ fundamental chiral multiplet $(\phi, \psi, F)$ and an anti-fundamental chiral multiplet $(\bar{\phi}, \bar{\psi}, \bar{F})$.

| $U(1)_R$ | $A_\mu$ | $\lambda$ | $\sigma$ | $D$ | $\sigma'$ | $\chi'$ | $D'$ |
|-----------|---------|-----------|---------|-----|---------|--------|-----|
| $U(1)_R$ | 0       | 1         | 0       | 0   | 1       | 0      | -1  |
| $U(1)_F$ | 0       | 0         | 0       | 0   | 1       | 1      | 1   |

Table 12: $U(1)_R \times U(1)_F$ R-charge assignments for the $\mathcal{N} = 4$ vector multiplet

| $U(1)_R$ | $\phi$ | $\psi$ | $\bar{\phi}$ | $\bar{\psi}$ |
|-----------|--------|--------|--------------|--------------|
| $U(1)_R$ | $\frac{3}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $U(1)_F$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ |

Table 13: $U(1)_R \times U(1)_F$ R-charge assignments for the $\mathcal{N} = 4$ hyper multiplet

The R-symmetry group of the $\mathcal{N} = 4$ supersymmetric theory is $SU(2)_L \times SU(2)_R$ and the Cartan generator of $SU(2)_L(SU(2)_R)$ is respectively $J_L(J_R)$. The $U(1)_R$ R-symmetry in the $\mathcal{N} = 2$ susy is generated by a generator $R$, which is related to the $SU(2)_L \times SU(2)_R$ generators by $R = J_L - J_R$. On the other hand, the combination $F = J_L + J_R$ defines another global charge which commutes with the $U(1)_R$ and we introduce the fugacity $t$ of the $U(1)_F$. The R-charge assignments for the $\mathcal{N} = 4$ multiplets are given in the tables 12, 13. Now we impose the Neumann boundary conditions for the adjoint chiral multiplet $(\sigma', \chi', D')$ and the fundamental chiral multiplet $(\phi, \psi, F)$ and the Dirichlet boundary conditions for the anti-fundamental chiral multiplet $(\bar{\phi}, \bar{\psi}, \bar{F})$. In this choice, all the anomalous terms from the three dimensional multiplets cancel each other, thus Chern-Simons levels are not shifted from the bare ones. If we introduce the Chern-Simons term, the supersymmetry is broken to $\mathcal{N} = 3$. When we take bare Chern-Simons levels as zero, the 3d index is well-defined without introducing any boundary multiplets. Then the
$\mathcal{N} = 4$ index on $S^1 \times D^2$ is given by

$$\mathcal{I}_{S^1 \times D^2}^{\mathcal{N}=4} = \frac{(t; q^2)^{-N}}{N!} \int \prod_{a=1}^{N} \frac{ds_a}{2\pi is_a} \prod_{1 \leq a \neq b \leq N} (s_a s_b^{-1}; q^2)_{\infty} \prod_{l=1}^{N_f} \prod_{a=1}^{N} (s_a q^2 t^{-\frac{1}{2} z_l^{-1}}; q^2)_{\infty}. $$

(6.32)

Here it is an interesting point that the one-loop determinant of the $\mathcal{N} = 4$ vector multiplet are expressed by the Macdonald measure.

Let us study the relation to integrable spin models. First we examine the relation between the $\mathcal{N} = 4$ SQCD and the spin-$\frac{1}{2}$ XXZ quantum spin chain. It is shown in [53] that the saddle point equation of the twisted effective superpotential for the mass deformed $\mathcal{N} = 4$ SQCD corresponds to the Bethe ansatz equation for the spin-$\frac{1}{2}$ XXZ model. We recall that the partition functions on the three dimensional ellipsoid $S^3_b$ depend on the squashing parameter $b$. In the limit $b \to 0$, the effective twisted superpotential appears. Similarly, in the semi classical limit, an effective twisted superpotential appears. Here the semiclassical limit means that $q \to 1$ with keeping the other parameters finite. It is different from the two dimensional limit. In the Abelian case (the rank of the gauge group $N = 1$), semiclassical behaviors of holomorphic blocks correctly reproduce the Bethe ansatz equation for the XXZ spin chain [54](see also [55]). Therefore we expect the Bethe ansatz also appears in the semiclassical limit of the 3d index (6.32) for generic rank $N$. To see this, we take the semiclassical limit and look at the behavior of the index;

$$\mathcal{I}_{S^1 \times D^2}^{\mathcal{N}=4} \sim \frac{1}{N!} \int \prod_{a=1}^{N} \frac{ds_a}{2\pi is_a} \exp \left( \frac{1}{2\beta} \mathcal{W}_{\text{eff}}^{\mathcal{N}=4} \right), \quad (q \to 1), $$

(6.33)

with

$$\mathcal{W}_{\text{eff}}^{\mathcal{N}=4} = \sum_{a \neq b} \left( \text{Li}_2(s_a s_b^{-1}) - \text{Li}_2(t s_a s_b^{-1}) \right) + \sum_{l=1}^{N_f} \sum_{a=1}^{N} \left( \text{Li}_2(s_a t^{-\frac{1}{2} z_l^{-1}}) - \text{Li}_2(s_a t^{\frac{1}{2} z_l}) \right). $$

(6.34)

We can evaluate the saddle point of this system and obtain a set of equations \( \text{exp} \left( s_a \partial_{s_a} \mathcal{W}_{\text{eff}}^{\mathcal{N}=4} \right) = 1 \), which is written as

$$\prod_{b=1}^{N} \frac{\sinh(y_b - y_a - c)}{\sinh(y_b - y_a + c)} = \prod_{l=1}^{N_f} \frac{\sinh(y_a - \frac{5}{2} - M_l)}{\sinh(y_a + \frac{5}{2} - M_l)}. $$

(6.35)

Here we defined $y_a = 2 \log s_a$, $c := 2 \log t$. By redefining parameters, this is equivalent to the Bethe ansatz equation for the $\mathfrak{sl}(2)$ spin-$\frac{1}{2}$ inhomogeneous XXZ quantum spin chain.
The parameters in the gauge theory are related to those of the spin chain: the rank of the color $N$ corresponds to the number of excitations and the number of flavor $N_f$ corresponds to the number of sites of the spin chain. Flavor fugacities for the hyper multiplets correspond to impurity parameters. So far the periodic boundary condition is imposed in the spin chain. But twisted periodic boundary condition in the spin chain is reproduced by introducing the FI-term. Also the generalization to the $\mathfrak{sl}(K)$ case is straightforward by considering quiver gauge theory.

The correspondence between the $\mathcal{N} = 4$ SQCD and the XXZ model is already known in the context of the gauge/Bethe correspondence. But we will propose a new example for the gauge/Bethe correspondence: The ”gauge” side is the pure $\mathcal{N} = 3$ Chern-Simons-Matter theory, namely, we take $N_f = 0$ in (6.32) and introduce a dynamical Chern-Simons term with level $\kappa$. The ”Bethe” side (quantum integrable model) is the $q$-boson hopping model [56]. The existence of the gauge/Bethe correspondence for Chern-Simons(-Matter) theories was first pointed out in [57, 58]. In this case, we have to introduce matters on the boundary to cancel the anomalous Chern-Simons terms. But in the semiclassical limit, it is not necessary to know precisely matter contents of the boundary multiplets because one-loop determinants of the boundary multiplets without anomalous terms $Z_{1\text{-loop}}^{2d}$ behave

$$Z_{1\text{-loop}}^{2d} \sim \sum_{a=1}^{N} \frac{\kappa}{2} (\log s_a)^2.$$

Thus we can read the effective twisted superpotential

$$W_{\text{eff}}^{\mathcal{N} = 3} = \frac{\kappa}{2} \sum_{a=1}^{N} (\log s_a)^2 + \sum_{a \neq b} \left( \text{Li}_2(s_a s_b^{-1}) - \text{Li}_2(t s_a s_b^{-1}) \right).$$

and obtain the set of saddle point equations

$$s_a^\kappa = \prod_{\substack{b=1 \atop b \neq a}}^{N} \frac{s_a t - s_b}{s_a - t s_b}.$$  

This is the Bethe ansatz equation for the $q$-boson hopping model for the $\kappa$-particles sector with the periodic boundary condition. The number of sites corresponds to the rank of the gauge group $N$.

6.5 Domain wall index on $S^1 \times S^2$

In four dimensions, 4d-3d coupled partition functions or superconformal indices have been introduced in [59], [60]. In this section we briefly mention on 3d-2d domain wall index on $S^1 \times S^2$.  

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The domain wall $\mathcal{N} = (0, 2)$ theory lives on $T^2 = S^1 \times S^1_{\vartheta = \frac{\pi}{2}} \subset S^1 \times S^2$ and couples with two different theories, each of which lives on a northern part $S^1 \times D^2_N(\vartheta \leq \frac{\pi}{2})$ or a southern part $S^1 \times D^2_S(\vartheta \geq \frac{\pi}{2})$. Let $G_N$ and $G_S$ be gauge groups of the $\mathcal{N} = 2$ theory on $S^1 \times D^2_N$ and $S^1 \times D^2_S$, respectively. Their partition functions $Z_{S^1 \times D^2_N}^{1\text{-loop}}(s)$, $Z_{S^1 \times D^2_S}^{1\text{-loop}}(\tilde{s})$ are expressed by one-loop determinants of the $\mathcal{N} = 2$ theory on northern $S^1 \times D^2_N$ and southern $S^1 \times D^2_S$. Here $s_a$'s and $\tilde{s}_b$'s label Cartan parts of exponentiated holonomies of $G_N$ and $G_S$ along the $S^1$-direction. On the boundary torus, the multiplets of the $\mathcal{N} = (0, 2)$ theory have charges associated with the gauge groups $G_N \times G_S$ and its partition function $Z_{T^2}^{1\text{-loop}}(s, \tilde{s})$ is represented by the one-loop determinant of the $\mathcal{N} = (0, 2)$ theory. By collecting these functions, we can write down the domain wall index:

$$I_{DW}^{S^1 \times S^2} = \int \int \prod_{a=1}^{\text{rk}(G_N)} \frac{ds_a}{2\pi s_a} \prod_{b=1}^{\text{rk}(G_S)} \frac{d\tilde{s}_b}{2\pi \tilde{s}_b} Z_{S^1 \times D^2_N}^{1\text{-loop}}(s) Z_{S^1 \times D^2_S}^{1\text{-loop}}(\tilde{s}) Z_{T^2}^{1\text{-loop}}(s, \tilde{s}).$$

(6.39)

The matter contents and two bare Chern-Simons levels have to be chosen to cancel the total bulk-boundary anomalies.

### 6.6 Wilson-Vortex loop and $q$-shifting operator

In this section we study properties of flavor Wilson loops and vortex loops on the 3d index $I_{S^1 \times D^2}(q^2, z)$. When we switch on the $l$-th flavor background gauge field $A_{\mu}^{(l)} = (0, 0, -iM_l/\beta r)$, an associated flavor Wilson loop with a charge $Q$ is defined by

$$W_F^{(l)} = \exp \left( iQ \int A_{\tau}^{(l)} d\tau \right).$$

(6.40)

Then the expectation value of the flavor Wilson loop is given by using the index $\langle W_F^{(l)} \rangle = z_l^{-Q} I_{S^1 \times D^2}$. The localization computation of vortex loops was also studied in [61, 62]. Vortex loops are defined as some defect operators specified by line singularities similar to ’t Hooft loops. This means the boundary conditions for the component fields in three dimensional theories are modified near the vortex loops. Since the appropriate equivariant index theorem for the manifold with the boundary has not been known yet, it is difficult to directly evaluate the effects of the vortex loops on the manifold with the boundary, for example, $S^1 \times D^2$. In stead of the direct computation, we can apply the method studied in [61] to our model. That is, the vortex loop is obtained by acting $S \in SL(2, \mathbb{Z})$ transformation on the flavor Wilson loop

$$V_F^{(l)} = S^{-1} W_F^{(l)} S.$$

(6.41)
Here the $S$-transformation is defined by adding an FI-term for the $l$-th flavor background gauge field and gauging this background field

\[
(S \cdot I_{S^1 \times D^2})(q^2, \zeta_l) = \int dM_l \ I_{S^1 \times D^2} \ e^{2\pi i \zeta_l M_l}.
\]  

(6.42)

The transformation $S^{-1}$ is also given as the inverse transformation of $S$ and the vortex loop acts on the 3d index as

\[
(S^{-1} W_F^{(l)} S) \cdot I_{S^1 \times D^2}(q, z'_l) = \int d\zeta_l e^{-2\pi i M'_l \zeta_l} e^{2\pi i Q \zeta_l} \int dM_l \ I_{S^1 \times D^2} \ e^{2\pi i \zeta_l M_l}
\]

\[
= I_{S^1 \times D^2}(q^2, z'_l q^Q).
\]  

(6.43)

Thus we obtain the expectation value of the vortex loop for $l$-th flavor gauge field by shifting the $l$-th flavor fugacity $\langle V_F^{(l)} \rangle = I_{S^1 \times D^2}(q^2, z_l q^Q)$. As a result, flavor vortex loops are regarded as $q$-shifting operators. The successive actions of Wilson-vortex loops on the 3d index do not commute, but they satisfy the following commutation relations

\[
\langle V_F^{(l)} W_F^{(k)} \rangle = q^{-Q} \delta_{lk} \langle W_F^{(k)} V_F^{(l)} \rangle.
\]  

(6.44)

7 Summary and discussion

In this paper, we have evaluated the partition functions of the $\mathcal{N} = 2$ supersymmetric Chern-Simons-Matter theories on $S^1 \times D^2$ in terms of localization techniques. In the particular choice of the fugacity $\beta_2 = \beta$, we find the conditions to cancel anomalous terms are reduced to the decomposition rule for effective mixed Chern-Simons levels in the holomorphic blocks for the Abelian gauge theories. In these cases, 3d-2d indices we have evaluated reproduce the holomorphic blocks. On the other hand, there might be a mismatch in the sector of the vector multiplet and effective Chern-Simons levels. One possibility of the mismatch seems to come from the difference in the metrics of $S^1 \times D^2$ and Melvin cigar. Both spaces have the same topology as the solid torus, but have different metrics. It is desirable to study further to reveal the origin of this discrepancy. We postpone this problem in our future work.

We also study the connection between our indices on $S^1 \times D^2$ and several topics; the K-theoretic $J$-function for the $\mathbb{CP}^N$ model, vortex partition functions and surface operators, the 3d analogue of the matrix factorization, gauge/Bethe correspondence and loop operators.

We have constructed boundary interactions which can be regarded as three dimensional uplifts of the matrix factorizations. Although the boundary interactions in three
dimensions (the $\mathcal{N} = (0, 2)$ superpotential term (3.13)) have quite different expressions from two dimensional ones (roughly speaking, Wilson loops for superconnections). After the localization computation is performed, the partition functions can reproduce the two dimensional partition functions on the hemisphere by the dimensional reduction.

In this paper we did not study the boundary interaction described by the $G/G$ chiral gauged WZW model in detail. It is interesting to study the 3d-2d index with this boundary interactions.

In [19], the 2d-4d correspondence was proposed. The 2d side describes the $\mathcal{N} = (0, 2)$ flavored elliptic genus [63, 64] and the 4d side is related to the Vafa-Witten partition functions [65] on four-manifolds. In this paper, we have realized $\mathcal{N} = (0, 2)$ indices as the boundary interactions of 3d $\mathcal{N} = 2$ supersymmetric theories. The 3d-2d coupled index is expected to be related to some Vafa-Witten partition function with degrees of freedom on the three dimensional boundary, which is realized as the asymptotic boundary of the four-manifold. It is interesting to explore the connection between the 3d-2d indices and the partition functions of the 3d-4d coupled systems.

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A Conventions of 3d $\mathcal{N} = 2$ theory on $S^1 \times D^2$

We use gamma matrices $\gamma_a$ in the local Lorentz frame:

$$
\gamma_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 -i \\ i & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

(A.1)

In curved spaces, one can define $\gamma_\mu = e_\mu^a \gamma_a$ by using dreivein $e_\mu^a$. The charge conjugation matrix is expressed by

$$
C_{\alpha\beta} = -i\gamma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
$$

(A.2)

which satisfies $C\gamma^\mu C^{-1} = - (\gamma^\mu)^T$. With this $C$, the spinor product is defined by

$$
\epsilon \psi := \epsilon^\alpha \psi_\alpha = \epsilon^\alpha C_{\alpha\beta} \psi_\beta = \epsilon^T C \psi.
$$

(A.3)
The supersymmetric transformation of the $\mathcal{N} = 2$ vector multiplet is given by

$$\delta A_{\mu} = i\frac{1}{2}(\bar{\epsilon}\gamma_{\mu}\lambda - \bar{\lambda}\gamma_{\mu}\epsilon),$$

$$\delta \sigma = \frac{1}{2}(\bar{\epsilon}\lambda - \bar{\lambda}\epsilon),$$

$$\delta \lambda = -\frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}\epsilon + D\epsilon + i\gamma^{\mu}\sigma\sigma\epsilon + \frac{2i}{3}\sigma\gamma^{\mu}D_{\mu}\epsilon,$$

$$\delta \bar{\lambda} = -\frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}\bar{\epsilon} + D\bar{\epsilon} - i\gamma^{\mu}\sigma\sigma\bar{\epsilon} - \frac{2i}{3}\sigma\gamma^{\mu}D_{\mu}\bar{\epsilon},$$

$$\delta D = -\frac{i}{2}\bar{\epsilon}\gamma^{\mu}D_{\mu}\lambda - \frac{i}{2}D_{\mu}\bar{\lambda}\gamma^{\mu}\epsilon + \frac{i}{2}[\bar{\epsilon}\lambda, \sigma] + \frac{i}{2}[\bar{\lambda}\epsilon, \sigma] - \frac{i}{6}(D_{\mu}\bar{\epsilon}\gamma^{\mu}\lambda + \bar{\lambda}\gamma^{\mu}D_{\mu}\epsilon).$$

(A.4)

For the $\mathcal{N} = 2$ chiral multiplet, the supersymmetric transformation is expressed as

$$\delta \phi = \bar{\epsilon}\psi,$$

$$\delta \bar{\phi} = \epsilon\bar{\psi},$$

$$\delta \psi = i\gamma^{\mu}\sigma\epsilon D_{\mu}\phi + i\sigma\epsilon D_{\mu}\phi + \frac{2i\Delta}{3}\gamma^{\mu}D_{\mu}\epsilon\phi + \bar{\epsilon}F,$$

$$\delta \bar{\psi} = i\gamma^{\mu}\bar{\sigma}\bar{\epsilon} D_{\mu}\bar{\phi} + i\bar{\sigma}\bar{\epsilon} D_{\mu}\bar{\phi} + \frac{2i\Delta}{3}\bar{\sigma}\bar{\gamma}^{\mu}D_{\mu}\bar{\epsilon} + F\epsilon,$$

$$\delta F = \epsilon(\epsilon\gamma^{\mu}D_{\mu}\bar{\psi} + i\bar{\sigma}\psi - i\lambda\phi) + \frac{i}{3}(2\Delta - 1)D_{\mu}\bar{\epsilon}\gamma^{\mu}\psi,$$

$$\delta \bar{F} = \bar{\epsilon}(\bar{\epsilon}\gamma^{\mu}D_{\mu}\bar{\psi} - i\bar{\sigma}\bar{\psi} - i\bar{\phi}\lambda) + \frac{i}{3}(2\Delta - 1)D_{\mu}\bar{\epsilon}\gamma^{\mu}\bar{\psi}.$$

(A.5)

For the $\mathcal{N} = 2$ chiral multiplet, the supersymmetric transformation is expressed as

Here covariant derivative is defined by $D_{\mu} = \nabla_{\mu} + iA_{\mu}^{a}T_{R}^{a}$ with the vector boson $A_{\mu} = A_{\mu}^{a}T_{R}^{a}$ in the representation $\mathcal{R}$.

**B 2d $\mathcal{N} = (0, 2)$ supersymmetry on the boundary torus**

The generators of the supersymmetric transformations are defined by the restriction of the three dimensional Killing spinors $\epsilon'$ and $\bar{\epsilon}'$ on the boundary torus:

$$(D_{2} + iD_{3})\epsilon' = \frac{i}{r}\epsilon', \quad (D_{2} - iD_{3})\epsilon' = 0,$$

$$(D_{2} + iD_{3})\bar{\epsilon}' = \frac{-i}{r}\bar{\epsilon}', \quad (D_{2} - iD_{3})\bar{\epsilon}' = 0.$$  

(B.1)
The set of commutators of the supersymmetric transformations of the vector multiplet is given by

\[
\begin{align*}
\lbrack \delta_1, \delta_2 \rbrack (A_2 - iA_3) &= \alpha(-2i)F_{2\bar{3}}, \\
\lbrack \delta_1, \delta_2 \rbrack \lambda_1 &= \alpha \left[ -2(D_2 + iD_3)\lambda_1 + \frac{2i}{r}\lambda_1 \right], \\
\lbrack \delta_1, \delta_2 \rbrack \bar{\lambda}_1 &= \alpha \left[ -2(D_2 + iD_3)\bar{\lambda}_1 - \frac{2i}{r}\bar{\lambda}_1 \right], \\
\lbrack \delta_1, \delta_2 \rbrack \hat{D} &= \alpha 2(D_2 + iD_3)\hat{D}.
\end{align*}
\] (B.2)

Next the set of commutators of the supersymmetric transformations of the chiral multiplet is expressed as

\[
\begin{align*}
\lbrack \delta_1, \delta_2 \rbrack \phi &= \alpha \left[ 2(D_2 + iD_3)\phi + \frac{2i}{r}\phi \right], \\
\lbrack \delta_1, \delta_2 \rbrack \psi' &= \alpha \left[ -2(D_2 + iD_3)\psi' \frac{2i}{r}(\Delta - 1)\psi' \right], \\
\lbrack \delta_1, \delta_2 \rbrack \bar{\psi}' &= \alpha \left[ 2(D_2 + iD_3)\bar{\psi}' - \frac{2i}{r}\bar{\psi}' \right], \\
\lbrack \delta_1, \delta_2 \rbrack \bar{\phi} &= \alpha \left[ -2(D_2 + iD_3)\bar{\phi} + \frac{2i}{r}(\Delta - 1)\bar{\phi} \right].
\end{align*}
\] (B.3)

At last the set of commutators of the supersymmetric transformations of the Fermi multiplet is given by

\[
\begin{align*}
\lbrack \delta_1, \delta_2 \rbrack \Psi &= \alpha \left[ 2(D_2 + iD_3)\Psi + \frac{2i}{r}(\bar{\Delta} - 1)\Psi \right], \\
\lbrack \delta_1, \delta_2 \rbrack G &= \alpha \left[ 2(D_2 + iD_3)G + \frac{2i}{r}(\bar{\Delta} - 2)G \right], \\
\lbrack \delta_1, \delta_2 \rbrack \bar{\Psi} &= \alpha \left[ 2(D_2 + iD_3)\bar{\Psi} + \frac{2i}{r}(1 - \bar{\Delta})\bar{\Psi} \right], \\
\lbrack \delta_1, \delta_2 \rbrack \bar{G} &= \alpha \left[ 2(D_2 + iD_3)\bar{G} + \frac{2i}{r}(2 - \bar{\Delta})\bar{G} \right].
\end{align*}
\] (B.4)

Here we defined \( \alpha = \bar{\epsilon}_2 \epsilon'_1 - \bar{\epsilon}_1 \epsilon'_2 \).

**C Definitions of functions**

The dilogarithm function is defined by

\[
\text{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2},
\] (C.1)

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and an integral representation of this dilogarithm function is given by

$$\text{Li}_2(x) = -\int_0^x dt \frac{\log(1-t)}{t}. \quad (C.2)$$

Next the q-Pochhammer symbol is defined by

$$(a; q)_n = \prod_{i=0}^{n-1} (1 - aq^i), \quad (C.3)$$

and the q-theta function is defined for $|q| < 1$

$$\theta(y; q) = \prod_{n=0}^{\infty} (1 - yq^n)(1 - y^{-1}q^{n+1}), \quad y \in \mathbb{C}^*, |q| < 1. \quad (C.4)$$

The quantum dilogarithm function is defined by

$$\text{Li}_2(x; q) = \sum_{n=1}^{\infty} \frac{x^n}{n(1 - q^n)}, \quad |x|, |q| < 1. \quad (C.5)$$

The q-Pochhammer symbol is expressed by the quantum dilogarithm as

$$(x; q)_\infty = \exp(-\text{Li}_2(x; q)), \quad (C.6)$$

and the semiclassical limit is given by

$$\text{Li}_2(x; e^{2\hbar}) \sim -\frac{1}{2\hbar} \text{Li}_2(x), \quad \hbar \to 0 \quad (C.7)$$

### D Derivation of one-loop determinants

#### D.1 3d vector multiplet

In this subsection, we evaluate one-loop determinants of the super Yang-Mills theory (2.9). The evaluation of one-loop determinant on $S^1 \times D^2$ can be performed in the similar manner to [16, 17]. Because we treat bosonic fields on $S^1 \times D^2$, we shall introduce scalar harmonics $Y_{jm}$ and vector harmonics $(C^\lambda_{jm})_i$ on $S^2$ labeled by sets of $(j, m)$’s with $j \leq |m|$ $(j = 1, 2, 3, \ldots)$. Each field in the multiplet is expanded by these harmonics and generators $E_\alpha$’s associated with roots $\alpha$’s in the Cartan Weyl basis: (Cartan parts for the fluctuations are omitted)
\[ \sigma = \sum_{j} \sum_{m} \sum_{\alpha>0} \sigma^\alpha Y_{jm} E_\alpha + (\text{h.c}), \quad (D.1) \]

\[ A_i = \sum_{\lambda=1}^{2} \sum_{j} \sum_{m=-j}^{\infty} A_{\lambda jm}^\alpha (C_{jm}^\lambda), E_\alpha + (\text{h.c}), \quad (i = 1, 2) \quad (D.2) \]

\[ A_3 = \sum_{\alpha>0} \sum_{j=1}^{\infty} \sum_{j-m=\text{even}} j \sum_{m} A_{\alpha jm}^{3} Y_{jm} E_\alpha + (\text{h.c}). \quad (D.3) \]

Here symbol "(h.c)" means the Hermitian conjugate and the sum \( \sum' \) represents surviving modes under the boundary condition (2.10)

\[ Y_{jm} : j - m = \text{even}, \quad \text{(Neumann)}, \quad (D.4) \]

\[ Y_{jm} : j - m = \text{odd}, \quad \text{(Dirichlet)}, \quad (D.5) \]

\[ C_{jm}^1 : j - m = \text{even}, \quad C_{jm}^2 : j - m = \text{odd}. \quad (D.6) \]

Next we turn to ghost fields \((c, \bar{c})\). The theory has gauge symmetry and we need to introduce ghost terms for fixing the symmetry

\[ \mathcal{L}_{\text{ghost+g.f.}} = -\bar{c} \nabla^i D_i c + \frac{1}{2\xi} (\nabla^i A_i)^2. \quad (D.7) \]

We take the Neumann boundary condition for ghost fields \((c, \bar{c})\) and expand them by the harmonics

\[ c = \sum_{j} \sum_{m} \sum_{\alpha \neq 0} c_{jm}^\alpha Y_{jm} E_\alpha. \quad (D.8) \]

One can represent the bosonic part \(\mathcal{L}_{\text{vec.b}}\) of the super Yang-Mills and the ghost part \(\mathcal{L}_{\text{ghost}}\) at the quadratic order as

\[ \int_{D^2} \mathcal{L}_{\text{ghost}} = \frac{1}{2r^2} \sum_{\alpha} \sum_{j,m} \text{Tr}(E_\alpha E_{-\alpha}) \cdot (-1)^m j(j+1) c_{jm}^\alpha c_{-jm,-\alpha} \]

\[ \int_{D^2} \mathcal{L}_{\text{vec.b}} = \frac{1}{2r^2} \sum_{\alpha>0} \sum_{j,m} \text{Tr}(E_{-\alpha} E_\alpha) \cdot \mathcal{V}_{jm}^\alpha \cdot \mathcal{M} \cdot \mathcal{V}_{jm}^\alpha, \]

\[ \mathcal{V}_{jm}^\alpha = (A_{jm}^{\alpha 1} A_{jm}^{\alpha 2} A_{jm}^{\alpha 3} \sigma_{jm}^\alpha)^T, \]

\[ \mathcal{M} = \begin{pmatrix} \xi^{-1} \cdot j(j+1) - r^2 D^3 D_3 & 0 & \sqrt{j(j+1)} \cdot rD_3 & 0 \\ 0 & j(j+1) - r^2 D^3 D_3 & 0 & -\sqrt{j(j+1)} \\ -\sqrt{j(j+1)} \cdot rD_3 & 0 & j(j+1) & 0 \\ 0 & -\sqrt{j(j+1)} & 0 & j(j+1) + 1 - r^2 D^3 D_3 \end{pmatrix}. \quad (D.9) \]
Then one-loop determinants of bosonic and ghost parts respectively become products of modes \((j, m)\)

\[
Z_{\text{vec.b}} = \prod_{\alpha \neq 0} \prod_{j-m=\text{even}} [j(j + 1)]^{-1} \times \prod_{\alpha \neq 0} \prod_{j-m=\text{odd}} [(j + 1 + rD_3)(j - rD_3)(j + 1 - rD_3)(j + rD_3)]^{-\frac{1}{2}}, \tag{D.10}
\]

\[Z_{\text{ghost}} = \prod_{\alpha \neq 0} \prod_{j-m=\text{even}} [j(j + 1)]. \tag{D.11}\]

Here \(Z_{\text{ghost}}\) is cancelled by the first line in the right hand side of (D.10). The twisted boundary condition along the \(S^1\) requires that \(D_3\) acts on an operator \(O\) with an R-charge \(R\) and a set of modes \((n, m)\)

\[
\beta rD_3O = [2\pi in + i\beta r\rho(a) - (R + m)\beta_1 + m\beta_2 + F_iM_i]\cdot O. \tag{D.12}
\]

For examples, the \(R\)-charge for the gauge field is 0 and that for the fermion \(\lambda(\bar{\lambda})\) is \(-1(+1)\). Then the unregularized one-loop determinant of the bosonic part is given by

\[
\prod_{\alpha \neq 0} \prod_{n \in \mathbb{Z}} \prod_{j=1}^{j-1} \prod_{j-m=\text{odd}} (2\pi in + i\beta r\alpha(a) + (j - m + 1)\beta_1 + (j + m + 1)\beta_2)^{-1} \times (2\pi in + i\beta r\alpha(a) + (j + m)\beta_1 + (j - m)\beta_2)^{-1}.
\]

We make a remark here: We have fixed the gauge symmetry with \(\nabla_i A_i = 0\), but there is a residual gauge symmetry \(\delta A_\tau = D_\tau \kappa\) with a parameter \(\kappa(\tau)\). In order to fix this symmetry, we impose a condition \(\partial_\tau \omega = 0\) with \(\omega := \frac{1}{\text{vol}(D^2)} \int_{D^2} A_\tau\) and introduce a set of a ghost and an anti-ghost. When we integrate these ghost fields, they induce a contribution \(\det D_\tau\) to the partition function. It is evaluated up to an overall constant

\[
\det D_\tau = \prod_{n \neq 0} \prod_{\alpha} \left(2\pi i \frac{n}{\beta r} + i\alpha(\lambda)\right) \approx \prod_{\alpha > 0} \frac{1}{\alpha(\lambda)^2} \sin^2 \frac{\beta r\alpha(\lambda)}{2} \tag{D.13}
\]

where \(\lambda_i (i = 1, 2, \cdots, N)\) is the set of eigenvalues of the matrix \(\omega\) and \(\alpha\) is the root. The measure of the matrix integral is represented by \(d\omega = \prod_i d\lambda_i \prod_{\alpha > 0} \alpha(\lambda)^2\) and \(d\omega\cdot\det D_\tau = \prod_i d\lambda_i \prod_{\alpha > 0} \sin^2 \frac{\beta r\alpha(\lambda)}{2}\).

Next we shall consider the contribution of fermions to the one-loop determinant. We consider the fermionic part of the Yang-Mills Lagrangian and evaluate the fluctuations around the saddle point at the quadratic order. We expand the gauging in terms of the spinor harmonics \(\chi_{j,m}^\pm(\theta, \varphi)\)

\[
\lambda = \sum_{\alpha \neq 0} \sum_{s=\pm} \sum_j \sum_m \chi_{j,m}^\pm \lambda_{j,m} E_\alpha. \tag{D.14}
\]
Here the sum $\sum'$ expresses the surviving modes under the boundary condition (2.10)
\[ r\gamma^i D_i \chi_{jm}^\pm = \pm \left( j + \frac{1}{2} \right) \chi_{jm}^\pm, \]
\[ \chi_{jm}^+: j - m = \text{even}, \quad \chi_{jm}^-: j - m = \text{odd}. \] (D.15)

By using this spinor harmonics, we can write down the fermion part of the Yang-Mills Lagrangian
\[
S^{(2)}_{\text{vec}} \big|_{\text{fer}} = \int S_1 \sum_{\alpha \neq 0} \sum_{j = \frac{1}{2}}^{\infty} \frac{i}{4r} \left[ \sum_{m: j - m = \text{even}} (-1)^{-m + \frac{1}{2}} \chi_{j,-m}^\alpha (j + rD_3) \lambda_{j,m}^{-\alpha,+} \right.
\]
\[ + \sum_{m: j - m = \text{odd}} (-1)^{-m + \frac{1}{2}} \chi_{j,-m}^\alpha (j - rD_3 + 1) \lambda_{j,m}^{\alpha,-} \big] \text{Tr}(E_\alpha E_{-\alpha}), \]
and calculate the one-loop determinant
\[
\prod_{j = \frac{1}{2}}^{\infty} \prod_{m: j - m = \text{even}} \text{Det}(j + rD_3) \prod_{m: j - m = \text{odd}} \text{Det}(j - rD_3 + 1). \] (D.16)

First we can evaluate the factor $\text{Det}(j - rD_3 + 1)$ with $j = j' + \frac{1}{2}$, $m = m' + \frac{1}{2}$
\[
\text{Det}(\beta(j + 1) - \beta rD_3) = \prod_{n \in \mathbb{Z}} \prod_{\alpha \neq 0} \prod_{j' = 0}^{j' - 1} \prod_{m' = -j' - 1}^{j' - m' = \text{odd}} \left( 2\pi in + i\beta r\alpha(a) + (j' + m' + 1)\beta_1 + (j' + 1 - m')\beta_2 \right). \]
Similarly the other factor $\text{Det}(j + rD_3)$ is evaluated with $j = j' - \frac{1}{2}$, $m = m' + \frac{1}{2}$
\[
\text{Det}(\beta j + \beta rD_3) = \prod_{n \in \mathbb{Z}} \prod_{\alpha \neq 0} \prod_{j' = 1}^{j' - 1} \prod_{m' = -j' + 1}^{j' - m' = \text{odd}} \left( 2\pi in + i\beta r\alpha(a) + (j' - m')\beta_1 + (j' + m')\beta_2 \right). \]

Then the one-loop determinant of the vector multiplet results in the product formula
\[
Z_{1\text{-loop}}^{3d,\text{vec}} = \prod_{\alpha \neq 0} e^{-\frac{(i\beta r\alpha(a))^2}{4\beta_2}} (q^2 e^{-i\beta r\alpha(a)}; q^2). \] (D.17)

Here we adopted the zeta function regularization used in [23]. In the evaluation of the one-loop determinant in the following subsections, we use the same regularization scheme. As expected, the one-loop determinant of the vector multiplet does not depend on the fugacity $\beta_1$. 

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D.2 3d chiral multiplet

D.2.1 Neumann boundary condition

We first evaluate the one-loop bosonic determinant for the chiral multiplet. When the Neumann boundary condition (2.18) is imposed, \( \phi \) and \( \bar{\phi} \) can be expanded as follows:

\[
\phi = \sum_{\rho} \sum_{j=0}^{\infty} \sum_{m=-j}^{j} \phi_{jm}^\rho Y_{jm}(\theta, \varphi) E_\rho, \tag{D.18}
\]

Here \( \rho \) runs over the weight of the representation \( R \) of the Lie algebra \( \text{Lie}(G) \). At the quadratic order of fluctuation, the action of the chiral multiplet is expanded in terms of scalar harmonics

\[
S^{(2)}_{\text{chi}}|_{\text{bos}} = \frac{1}{2 r^2} \int_{S^1} \sum_{j=0}^{\infty} \sum_{m=-j}^{j} \phi_{jm}^\rho \phi_{jm} (j + \Delta + r D_3) (j + 1 - \Delta - r D_3) \phi_{jm}^\rho. \tag{D.19}
\]

Under the twisted periodic condition (4.1), the factor \((j + \Delta + r D_3)\) in (D.19) contributes to the unregularized one-loop determinant of the bosonic fields

\[
\prod_{j=0}^{\infty} \prod_{m=-j}^{j} \prod_{n \in \mathbb{Z}} \prod_{j=0}^{\infty} \prod_{m=-j}^{j} (2\pi n + i \beta r \rho(a) + (j - m) \beta_1 + (j + \Delta + m) \beta_2 + F_l M_l)^{-1}. \tag{D.20}
\]

Similarly the other factor \((j + 1 - \Delta - r D_3)\) contributes to the unregularized one-loop determinant of the bosonic fields

\[
\prod_{j=0}^{\infty} \prod_{m=-j}^{j} \prod_{n \in \mathbb{Z}} \prod_{j=0}^{\infty} \prod_{m=-j}^{j} (-2\pi n - i \beta r \rho(a) + (j + m + 1) \beta_1 + (j + 1 - \Delta - m) \beta_2 - F_l M_l)^{-1}. \tag{D.21}
\]

Next we evaluate the one-loop determinant of fermions. We expand the \( \psi, \bar{\psi} \) by the spinor harmonics \( \chi_{jm}^s \) as

\[
\psi = \sum_{\rho} \sum_{s = \pm} \sum_{j=0}^{\infty} \sum_{m=-j}^{j} \psi_{jm}^{\rho s}(\tau) \chi_{jm}^s(\theta, \varphi) E_\rho. \tag{D.22}
\]
Here the sum $\sum'$ expresses the surviving modes under the boundary condition (2.18)

$$\chi_{jm}^+: j - m = \text{odd}, \quad \chi_{jm}^-: j - m = \text{even}. \quad (D.23)$$

Then the action of the chiral multiplet (2.17) can be expanded at the quadratic order

$$S_{\chi}^{(2)}_{\text{fer}} = \frac{i}{2r} \int_{S^1} \sum_\rho \sum_j \sum_{m: j - m = \text{even}} \bar{\psi}_{j,-m}^{\rho,+} (j - rD_3 + 1 - \Delta) \psi_{j,m}^{\rho,-} \cdot (-1)^{m-1/2}$$

$$+ \frac{i}{2r} \int_{S^1} \sum_\rho \sum_j \sum_{m: j - m = \text{odd}} \bar{\psi}_{j,-m}^{\rho,-} (j + rD_3 + \Delta) \psi_{j,m}^{\rho,+} \cdot (-1)^{m-1/2}. \quad (D.24)$$

The factor $(j - rD_3 + 1 - \Delta)$ contributes to the one-loop determinant of the chiral multiplet with $j = j' + \frac{1}{2}$, $m = m' + \frac{1}{2}$

$$\prod_{j=\frac{1}{2}}^\infty \text{Det} \left( (j - \Delta + 1)\beta - \beta rD_3 \right)$$

$$= \prod_{\rho \in \mathbb{R}} \prod_{n \in \mathbb{Z}} \prod_{j' = 0}^{j'} \prod_{m' = j' - \Delta} \prod_{l} \left( -2\pi i n - i\beta r (a) + (j' + m' + 1)\beta_1 + (j' + 1 - \Delta - m')\beta_2 - F_l M_l \right). \quad (D.25)$$

The other factor $(j + rD_3 + \Delta)$ contributes to the one-loop determinant of the chiral multiplet with $j = j' - \frac{1}{2}$, $m = m' + \frac{1}{2}$

$$\prod_{j=\frac{1}{2}}^\infty \text{Det} \left( (j + \Delta)\beta + \beta rD_3 \right)$$

$$= \prod_{\rho \in \mathbb{R}} \prod_{n \in \mathbb{Z}} \prod_{j' = 1}^{j'-2} \prod_{m' = j' - \Delta} \prod_{l} \left( 2\pi i n + i\beta r (a) + (j' - m')\beta_1 + (j' + \Delta + m')\beta_2 + F_l M_l \right).$$

Therefore the one-loop determinant of the chiral multiplet with the Neumann boundary condition is given by

$$Z_{3d, \chi}^{1\text{-loop}, N} = \prod_{\rho \in \mathbb{R}} \prod_{n \in \mathbb{Z}} \prod_{j=0}^{\infty} (2\pi i n + i\beta r (a) + (2j + \Delta)\beta_2 + F_l M_l)^{-1}$$

$$= \prod_{\rho \in \mathbb{R}} \prod_{l} e^{\mathcal{E}(i\beta r (a) + \Delta \beta_2 + F_l M_l) (e^{-i\beta r (a) - F_l M_l q^2} - q^2)^{-1}}. \quad (D.26)$$

The one-loop determinant of the chiral multiplet also does not depend on the parameter $\beta_1$. 

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D.2.2 Dirichlet boundary condition

Next we evaluate the bosonic one-loop determinant for the chiral multiplet with Dirichlet boundary condition (2.19). At quadratic order, bosonic part of the action is expanded as

$$ S_{\text{chi}}^{(2)} = \frac{1}{2r^2} \int_{S^1} \sum_{j=0}^{\infty} \sum_{m=-j}^{rD_3} \sum_{\rho} \bar{\phi}_{j,m}^\rho (j + \Delta + rD_3)(j + 1 - \Delta - rD_3) \phi_{j,m}^\rho. $$

(D.27)

Thus, the factor \((j + \Delta + rD_3)\) in (D.27) contributes to the unregularized one-loop determinant of the bosonic fields

$$ \prod_{j=1}^{\infty} \prod_{m=-j+1}^{j-1} \prod_{n \in \mathbb{Z}} (2\pi in + i\beta r\rho(a) + (j - m)\beta_1 + (j + \Delta + m)\beta_2 + F_1 M_1)^{-1}. $$

The other factor \((j + 1 - \Delta - rD_3)\) contributes to the unregularized one-loop determinant of the bosonic fields

$$ \prod_{j=1}^{\infty} \prod_{m=-j+1}^{j-1} \prod_{n \in \mathbb{Z}} (2\pi in - i\beta r\rho(a) + (j + m + 1)\beta_1 + (j + 1 - \Delta - m)\beta_2 - F_1 M_1)^{-1}. $$

Next we evaluate the one-loop determinant of fermions. In this case, the sum \(\sum'\) in (D.22) expresses the surviving modes under the boundary condition (2.19)

$$ \chi^+_jm : j - m = \text{even}, \quad \chi^-jm : j - m = \text{odd}. $$

(D.28)

Then the action of the chiral multiplet (2.17) can be expanded at the quadratic order

$$ S_{\text{chi}}^{(2)} \bigg|_{\text{fer}} = \frac{i}{2r} \int_{S^1} \sum_{\rho} \sum_{j} \sum_{m;j-m=\text{odd}} \bar{\psi}_{j,-m}^{\rho+} (j - rD_3 + 1 - \Delta) \psi_{j,m}^{\rho+} \cdot (-1)^{m-1/2} $$

$$ + \frac{i}{2r} \int_{S^1} \sum_{\rho} \sum_{j} \sum_{m;j-m=\text{even}} \bar{\psi}_{j,-m}^{\rho-} (j + rD_3 + \Delta) \psi_{j,m}^{\rho+} \cdot (-1)^{m-1/2}. $$
Then, the factor \((j - rD_3 + 1 - \Delta)\) contributes to the one-loop determinant of the chiral multiplet

\[
\prod_{j=\frac{1}{2}}^\infty \det ((j - \Delta + 1)\beta - \beta rD_3)
\]

\[
= \prod_{j'=0}^{\infty} \prod_{j'=1}^{j'-1} \prod_{\substack{n \in \mathbb{Z} \backslash j' \rightarrow 1 \, \text{odd}}} (j' - rD_3 + 1 - \Delta)
\]

where we defined \(j' = j - \frac{1}{2}, \, m' = m - \frac{1}{2}\). The other factor \((j + rD_3 + \Delta)\) contributes to the one-loop determinant of the chiral multiplet

\[
\prod_{j=\frac{1}{2}}^\infty \det ((j + \Delta)\beta + \beta rD_3)
\]

\[
= \prod_{j'=1}^{\infty} \prod_{j'=1}^{j'-1} \prod_{\substack{n \in \mathbb{Z} \backslash j' \rightarrow 1 \, \text{odd}}} (2\pi in - \beta r\rho(a) + (j' - m')\beta_1 + (j' + \Delta + m')\beta_2 + F_1 M_l),
\]

Here we defined \(j' = j + \frac{1}{2}, \, m' = m - \frac{1}{2}\). Therefore the one-loop determinant of the chiral multiplet with the Dirichlet boundary condition is given by

\[
Z_{3d,\chi}^{D,1\text{-loop}} = \prod_{\rho \in \mathbb{R}} \prod_{n \in \mathbb{Z}} \prod_{j=0}^{\infty} \prod_{\substack{\rho \in \mathbb{R} \backslash j \rightarrow 1 \, \text{odd}}} (2\pi in - \beta r\rho(a) + (2j + 2 - \Delta)\beta_2 - F_1 M_l)
\]

\[
= \prod_{\rho \in \mathbb{R}} \prod_{l \in \mathbb{Z}} e^{-\xi(i\beta r\rho(a) + (2-\Delta)\beta_2 + F_1 M_l)} \times (\text{Det}^{\infty}(\text{Det}^{\infty} - \Delta + 2m)\beta_1 + F_1 M_l)) \quad (D.29)
\]

### D.3 2d \(\mathcal{N} = (0, 2)\) chiral multiplet

We have a boundary theory with \(\mathcal{N} = (0, 2)\) supersymmetry. The Lagrangian for the \(\mathcal{N} = (0, 2)\) chiral multiplet is given by

\[
\mathcal{L}^{\mathcal{N} = (0,2)}_{\chi} = \bar{\phi}(D_2 - iD_3)(D_2 + iD_3)\phi + \frac{1}{2}\bar{\psi}'(D_2 - iD_3)\psi'
\]

\[
+ \frac{i\Delta}{r} \bar{\phi}(D_2 - iD_3)\phi + i\bar{\phi}_1 \phi' + i\bar{\psi}_1 \lambda_1 \phi + \bar{\phi}(F^2_{23} - i\bar{D})\phi. \quad (D.30)
\]

The one-loop determinant of the bosonic part is given by

\[
\text{Det}(D_2 + iD_3 + \frac{i\Delta}{r})(D_2 - iD_3) 
\]

\[
\approx \prod_{\rho} \prod_{l} \prod_{m \in \mathbb{Z}} \prod_{n \in \mathbb{Z}} (2\pi in + i\beta r\rho(a) + (2m + \Delta)\beta_2 + F_1 M_l)
\]

\[
\times \prod_{\rho} \prod_{l} \prod_{m \in \mathbb{Z}} \prod_{n \in \mathbb{Z}} (2\pi in + i\beta r\rho(a) - (\Delta + 2m)\beta_1 + F_1 M_l) \quad (D.31)
\]
The one-loop determinant of the fermionic part is given by

\[ \text{Det}(D_2 - iD_3) \approx \prod_{\rho} \prod_{m' \in \mathbb{Z}} \prod_{n \in \mathbb{Z}} (2\pi in + i\beta \rho(a) - (\Delta + 2m')\beta + F_lM_l), \]  

(D.32)

with \( m = m' + \frac{1}{2} \). So the one-loop determinant of the chiral multiplet can be written by

\[ Z_{2\text{d.chi}}^{1\text{-loop}} = \prod_{\rho} \prod_{l} e^{2\xi (i\beta \rho(a) + \Delta \beta + F_lM_l)} \theta(\theta e^{-i\beta \rho(a)} - F_lM_lq^{\Delta} q^2)^{-1}. \]  

(D.33)

**D.4 2d \( \mathcal{N} = (0, 2) \) Fermi multiplet**

Next we evaluate the one-loop determinant of the Fermi multiplet on torus \( T^2 \):

\[ \mathcal{L}_{\text{Fermi}}^{\mathcal{N} = (0, 2)} = -\bar{\Psi}(D_2 + iD_3)\Psi + 2\bar{G}G + 2\bar{E}E - \bar{\Psi}E\Psi - \bar{\Psi}\psi_E + \frac{i}{r}(1 - \tilde{\Delta})\bar{\Psi}\Psi. \]

At the quadratic order, the one-loop contribution comes from the determinant of the fermion

\[ Z_{2\text{d.Fermi}}^{1\text{-loop}} \text{Det}[-(D_2 + iD_3) + \frac{i}{r}(1 - \tilde{\Delta})] \]

\[ \approx \prod_{a} \prod_{\rho} \prod_{m' \in \mathbb{Z}} \prod_{n \in \mathbb{Z}} (2\pi in + (2m' + \tilde{\Delta})\beta + i\beta \rho(a) + F_aM_a) \]

\[ = \prod_{a} \prod_{\rho} e^{2\xi (i\beta \rho(a) + \tilde{\Delta} \beta + F_aM_a)} \theta(\theta e^{-i\beta \rho(a)} - F_aM_aq^{\tilde{\Delta}} q^2), \]  

(D.34)

with \( m = m' + \frac{1}{2} \).

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