A GEOMETRIC REALIZATION OF SYMMETRIC PAIRS OF TYPE AIII

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ABSTRACT. In this paper, we study the fixed loci of Nakajima quiver variety \( L \) under an involution, which is related to the \( \sigma \)-quiver variety introduced in \[8\]. We give a geometric realization of symmetric pair of type AIII over \( \sigma \)-quiver varieties as an analogy of the construction over Nakajima quiver varieties in \[17\]. It partially recovers the construction of sympletic partial flag varieties studied in \[1\]. This gives an affirmative answer to a variation of a conjecture raised in \[8\] for type AIII. To achieve this, we define the \( \iota \)-analogy of Hecke correspondences and study the properties of their fibres and products.

1. INTRODUCTION

Let \( U(\mathfrak{g}) \) be the universal enveloping algebra of a simple Lie algebra \( \mathfrak{g} \). Let \( U_q(\mathfrak{g}) \) be its quantization. In \[2\], the authors constructed an algebra homomorphism

\[
U_q(\mathfrak{sl}_n) \longrightarrow \text{End}(\text{Fun}(\mathcal{F}))
\]

where \( \text{Fun}(\mathcal{F}) \) is the space of functions over partial flag varieties \( \mathcal{F} \) of length \( n \) of a vector space over finite fields. This action is very instructive due to the following reasons

- The weight decomposition of \( \text{Fun}(\mathcal{F}) \) under this action corresponds to the decomposition of \( \mathcal{F} \) into connected components, i.e. the dimension vectors of the subspaces in the flags.
- The Chevalley generators of positive/negative part of \( U_q(\mathfrak{sl}_n) \), up to some power of \( q \), are given by “Hall-algebra-like” correspondences, i.e. “adding/lowering the flag by one dimension”.
- It is essentially a geometric realization of the representation of \( N \)-th symmetric power of natural representation where \( N \) is the dimension of the ambient vector space.

Their argument works over \( \mathbb{C} \) either, where it gives a Lie algebra homomorphism

\[
\mathfrak{sl}_n \longrightarrow \text{End}(\text{Fun}(\mathcal{F}))
\]

where \( \text{Fun}(\mathcal{F}) \) is the space of constructible functions over partial flag varieties \( \mathcal{F} \) of length \( n \) of a vector space over \( \mathbb{C} \).

Later, Nakajima \[17\] gave a much more general construction for all symmetric Kac–Moody algebra and dominant weights \( \mathbf{w} \). To be exact, Nakajima constructed a Lie-algebra homomorphism

\[
\mathfrak{g} \longrightarrow \text{End}(\text{Fun}(L(\mathbf{w})))
\]

where \( \text{Fun}(L(\mathbf{w})) \) is the space of constructible functions over Nakajima quiver variety \( L(\mathbf{w}) \). Similarly to the construction of \[2\], the weight spaces correspond to

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the connected components, and the action is “Hall algebra-like”, see Theorem 9. It is a geometric realization of the representation of highest weight \( w \). Moreover, this construction recovers the construction of \( [2] \) for \( \mathfrak{sl}_n \) at \( q = 1 \) and \( w \) is the highest weight for the symmetric power of natural representation, see Example 3.

The correspondence is known as Hecke correspondences (after Nakajima [17] and [18]).

From the theory of real simple Lie algebras, it is natural to consider symmetric pairs \((g, \mathfrak{k})\) where \( \mathfrak{k} \) is the fixed subalgebra of \( g \) under some involution. It has a quantization \((U_q(g), U_q^t(\mathfrak{k}))\), known as the quantum symmetric pair [7] where \( U_q^t(\mathfrak{k}) \) is also known as the \( t \)-quantum group. The theory is recently strongly developed by works of Wang and his colleagues, see [23] for a survey.

Consider the symmetric pair \((g, \mathfrak{k})\) of type AIII induced by the longest word of the Weyl group of type \( A \) odd. It can be illustrated by the following diagram

In [1], the authors constructed an algebra homomorphism

\[
U_q^t(\mathfrak{k}) \longrightarrow \text{End}(\text{Fun}(\mathcal{F}^i))
\]

where \( \mathcal{F}^i \) is the space of functions over partial flag varieties of classical types over finite fields.

The main result of this paper is to construct an \( t \)-analogy of the Nakajima picture. To be exact, we will construct a Lie-algebra homomorphism

\[
\mathfrak{k} \longrightarrow \text{End}(\text{Fun}(\mathcal{R}(w)))
\]

for symmetric pair of type AIII, where \( \text{Fun}(\mathcal{R}(w)) \) is the space of constructible functions over the \( \sigma \)-quiver variety \( \mathcal{R}(w) \). See Theorem 26 for the detailed description.

Our construction has the following features

- We have a decomposition \( \mathcal{R}(w) = \bigsqcup \mathcal{R}(v, w) \) which corresponds to the weight decomposition.
- The Chevalley generators act by “Hall-algebra-like” correspondences, which we will refer them as \( t \)-Hecke correspondences.
- The partial flag varieties \( \mathcal{F}^i \) of type \( C \) appears as a special case of \( \mathcal{R}(w) \), see Example 17.

Since the representation theory for symmetric pair is not clear at the present stage, we cannot describe which representation our geometric realization gives.

In [8], the \( \sigma \)-quiver variety \( \mathcal{G}(w) \) was introduced to give an \( \iota \)-analogy of Nakajima’s picture. Our \( \sigma \)-quiver variety \( \mathcal{R}(w) \) is a closed subvariety of the \( \sigma \)-quiver \( \mathcal{G}(w) \), which is parallel to the Nakajima quiver variety \( \mathcal{L}(w) \) and \( \mathcal{R}(w) \), see Remark 15 and Theorem 16. It was conjectured that there is a Lie algebra homomorphism

\[
\mathfrak{k} \longrightarrow H_{\text{top}}^{BM}(\mathcal{Z}(w))
\]

where \( H_{\text{top}}^{BM}(\mathcal{Z}(w)) \) is the algebra of top degree part of Borel–Moore homology of certain Steinberg variety under convolution product. This is the \( \iota \)-analogy to Borel–Moore homology construction in [18]. The special cases of partial flag variety of classical types are checked recently in [9] which is parallel to the Springer theory of \( U(\mathfrak{sl}_n) \), see [3, Chapter 4]. Our result can be viewed as a variation of this
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conjecture. The existence of such homomorphism can also be viewed as evidence of this conjecture.

With the development of geometric representation theory, it is noted that geometric realization usually helps us understand the algebra itself. But the problem of geometric realization of (quantum) symmetric pair is still very open, see [23, Section 9 (2)]. From our proof, we see that similar construction for $\sigma$-quiver variety might be difficult for other types. Due to the private discussion with Wang, the reason would be that the $\sigma$-quiver varieties are defined by Vogan diagrams rather than Satake diagrams. It is worth mentioning that there are other geometric ways to realize quantum symmetric pairs, for example [11]. It is closely related to the Hall algebra construction [10]. Though there is no evidence that it is related to the partial flag varieties of type B or C of [11] in a geometric way.

Last but no mean least, to achieve an action of (affine) $\iota$-quantum group, it is plausible to use equivariant K-theory as the $\iota$-analogy of [20]. Note that the Faddeev–Reshetikhin–Takhtajan construction of symmetric pair of Yangian was achieved in [8], which is parallel to the construction of [10]. But it cannot be moved to equivariant K-theory directly, since we do not know whether a K-theoretic stable envelope exists over $\sigma$-quiver variety.

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2. NAKAJIMA QUIVER VARIETIES

Let $Q$ be the Dynkin diagram of type $A_{2d-1}$ labelled as follows

$$
\circ \rightarrow \cdots \rightarrow \circ \\
1 \quad 2 \quad \cdots \quad 2d-2 \quad 2d-1
$$

Denote $\mathbb{I} = \{1, \ldots, 2d-1\}$ the index set of vertices. Let $(c_{ij})$ be the Cartan matrix. That is,

$$c_{ij} = \begin{cases} 
2, & i = j, \\
-1, & |i - j| = 1, \\
0, & \text{otherwise}. 
\end{cases}$$

We construct a new quiver $Q^\triangleright$ as follows. The vertices of $Q^\triangleright$ is $\mathbb{I} \cup \mathbb{I}_r$ with $\mathbb{I}_r$ a copy of $\mathbb{I} = \{i \uparrow : i \in \mathbb{I}\}$. For unframed vertices, we have an arrow $i \rightarrow j$ if there is an edge in $Q$. Besides, we have an arrow from $i \rightarrow i \uparrow$ for each $i \in \mathbb{I}$. Here is an example of $A_3$.

$$
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\end{array} \\
\begin{array}{c}
\uparrow \\
\downarrow \\
\downarrow \\
\end{array}$$
For any arrow \( h : i \to j \), denote \( \bar{h} : j \to i \) the reversed arrow. Let \( \mathcal{L} \) be the \( \mathbb{C} \)-linear category generated by the quiver \( Q^h \) with commutative relations for each \( i \in I \),

\[
\sum_{h : i \to j} \pm \bar{h} h = 0, \quad \text{with } \pm = \begin{cases} 1, & j = i + 1, \\ -1, & j = i - 1. \end{cases}
\]

This is in principle the category introduced in [4]. Denote \( \mathcal{Q} \) (resp. \( \mathcal{S} \)) be the full subcategory generated by unframed vertices (resp. framed vertices). Note that \( \mathcal{S} \) is discrete.

A framed representation is a \( \mathbb{C} \)-linear functor from \( \mathcal{L} \) to \( \mathbb{C} \text{-Vec} \) the category of \( \mathbb{C} \)-vector spaces. Morphisms of framed representations are defined to be natural transforms. We say a representation \( F \) is stable if for any representation \( G \)

\[
G|_{\mathcal{S}} = 0 \implies \text{Hom}(G, F) = 0.
\]

Let \( w = (w_i)_{i \in I} \) and \( v = (v_i)_{i \in I} \) be two dimension vectors. The Nakajima quiver variety \( \mathcal{L}(v, w) \) is the moduli space of stable framed representations \( F \) with

\[
\dim F|_{\mathcal{Q}} = v, \quad \text{and } \dim F|_{\mathcal{S}} = w.
\]

where \( \dim X \in \mathbb{N}^\mathbb{Z} \) is the dimension vector for any \( \mathbb{Z} \)-graded vector space \( X \). This coincides with Nakajima’s original definition [17], see [4] for the proof. We will write \( \mathcal{L}(w) = \bigsqcup \mathcal{L}(v, w) \).

Assume we are given a functor \( W \) from \( \mathcal{S} \) to \( \mathbb{C} \text{-Vec} \), i.e. an \( \mathbb{Z} \)-graded vector space \( (W(i))_{i \in I} \). We can define a stable framed representation \( KRW \) with \( KRW|_{\mathcal{S}} = W \) as follows. For each unframed vertex \( i \),

\[
KRW(i) = \bigoplus_{j \in \mathbb{Z}} \text{Hom}_\mathbb{C}(\mathcal{Q}(i, j), W(j)).
\]

For each arrow \( h : i \to j \) in \( \mathcal{Q} \),

\[
KRW(h) = \left[ (f_j) \mapsto (f_j \circ \mathcal{Q}(h, j)) \right] : KRW(i) \to KRW(j).
\]

For each arrow \( \alpha : i \to j \),

\[
KRW(\alpha) = \left[ (f_j) \mapsto f_i(id_j) \right] : KRW(i) \to W(i).
\]

For any representation \( F \) with \( F|_{\mathcal{S}} = W \), there is morphism of framed representation \( \epsilon : F \to KRW \) given by evaluation, say

\[
\epsilon(i)(v) = (f_j) \quad \text{with } f_j(p) = \alpha_j(p(v)),
\]

where \( \alpha_j : F(j) \to W(j) \) the morphism corresponding to \( j \to j \).

Actually, \( KRW \) is the right adjoint functor (so-called right Kan extension, see [4]) of restriction functor from \( \mathcal{L} \) to \( \mathcal{S} \).

**Proposition 1** (Keller and Scherotzke [4]). The representation \( F \) is stable if and only if the natural map \( \epsilon : F \to KRW \) is injective. In particular, \( \mathcal{L}(v, w) \) is the quiver Grassmannian of \( KRW|_{\mathcal{Q}} \) for any \( W \) such that \( \dim W = w \).

**Proposition 2.** Let \( v' = \dim KRW|_{\mathcal{Q}} \). Then \( w_0w = w - Cv' \) for the longest element \( w_0 \) of Weyl group.

**Proof.** By [18], \( \mathcal{L}(v', w) \) corresponds to the space of lowest weight \( w - Cv' \) of the representation of highest weight \( w \). \( \square \)
Example 3. Consider the following diagram

That is, \( w \) has only one nonzero component \( n \) at the left end of Dynkin diagram of type \( A \). Let \( W \) be a vector space of dimension \( n \). Then \( K_RW \) is the representation

The subrepresentations of \( K_RW|Q \) is simply the partial flags of \( W \). In particular, \( \mathcal{L}(w) \) is the variety of partial flags of \( W \).

Example 4. Consider the following diagram

That is, \( w \) has only one nonzero component \( n \) at the middle vertex of Dynkin diagram \( A_3 \). Let \( W \) be a vector space of dimension \( n \). Now \( K_RW \) is exactly

where all illustrated arrows are identities, and other arrows (not figured out) are zero. To be exact, \( (K_RW)(i) = W \oplus W \) for the middle vertex \( i \) and \( (K_RW)(i) = W \) for the left or right vertices \( i \). Actually, this representation achieves the dimension predicted in Proposition 2, and it is stable, thus it is exactly \( K_RW \). In particular, when \( n = 1 \), each nonempty \( \mathcal{L}(v,w) \) is a point.

Example 5. Consider the following diagram

Let \( N \) (resp. \( M \)) be a vector space of dimension \( n \) (resp. \( m \)). Then \( K_RW \) can be represented by the following diagram
Actually, this representation achieve the dimension predicted in Proposition 3 and it is stable, thus it is exactly $KRW$. In particular, when $n = 1$, and $v = (1, 1, 1)$, $\Sigma(v, w)$ is a union of three copies of $\mathbb{P}^1$ with intersection diagram $A_3$, see [19].

3. Geometric realization of $U$

Let $g$ be the complex simple Lie algebra correspondent to $Q$. By a theorem of Serre, for example [6], $g$ is the Lie algebra generated by $E_i$, $F_i$ and $H_i$ for $i \in I$ with relations

\begin{align*}
[H_i, H_j] &= 0, \ [H_i, E_j] = c_{ij}E_j, \ [H_i, F_j] = -c_{ij}F_j \\
[E_i, F_j] &= \delta_{ij}H_i \\
[E_i, [E_i, E_j]] &= [F_i, [F_i, F_j]] = 0 \quad \text{if } c_{ij} = 0 \\
[E_i, [E_i, F_j]] &= [F_i, [F_i, F_j]] = 0 \quad \text{if } c_{ij} = -1
\end{align*}

Here $\delta_{ij}$ stands the Kronecker’s delta. Denote $U$ the universal enveloping algebra of $g$, Nakajima [17] constructed a $U$-action using the constructible functions $\text{Fun}(L(w))$ over $L(w)$. We will shortly review the theory of the basic properties of such space.

Let $X$ be a variety over $\mathbb{C}$. Denote $\chi(X)$ the Euler characteristic of $X$, that is

\begin{equation}
\chi(X) = \sum_n (-1)^n \dim H^n_c(X; \mathbb{Q}),
\end{equation}

where $H^n_c(X; \mathbb{Q})$ is the cohomology of compact support. The following properties are well-known.

**Proposition 6.** We have the following properties of $\chi$.

- For a closed subvariety $F \subseteq X$, with complement $U = X \setminus F$

\begin{equation}
\chi(F) + \chi(U) = \chi(X).
\end{equation}

- For a fibre bundle $X \rightarrow B$ with fibre $F$

\begin{equation}
\chi(F)\chi(B) = \chi(X).
\end{equation}

Let $\text{Fun}(X)$ be the space of constructible functions with value in $\mathbb{Z}$ over algebraic variety $X$. For any morphism $f : X \rightarrow Y$, we have a natural pull back $f^* : \text{Fun}(Y) \rightarrow \text{Fun}(X)$ by

\begin{equation}
f^*\psi = \psi \circ f.
\end{equation}

We can define push forward $f_* : \text{Fun}(X) \rightarrow \text{Fun}(Y)$ by

\begin{equation}
(f_*\varphi)(y) = \sum_n \chi \left( \{ x \in X : f(x) = y, \varphi(x) = n \} \right) n.
\end{equation}

with $n$ going through all values of image $\psi$.

The following proposition is standard.

**Proposition 7.** These functors have the following properties

- For any morphism $f : X \rightarrow Y$ and $\psi_1, \psi_2 \in \text{Fun}(Y)$ for $\bullet = 1, 2$

\begin{equation}
f^*\psi_1 \cdot f^*\psi_2 = f^*(\psi_1 \cdot \psi_2).
\end{equation}

- For any morphism $f : X \rightarrow Y$, $\varphi \in \text{Fun}(X)$ and $\psi \in \text{Fun}(Y)$,

\begin{equation}
f_*(\varphi \cdot f^*\psi) = f_*\varphi \cdot \psi.
\end{equation}
Let \( i : X \to Y \) be an inclusion of subvariety. For any \( \psi \in \text{Fun}(Y) \),

\[
i_*\psi = 1_X \cdot \psi
\]

where \( 1_X \) is the characteristic function of \( X \).

- Let \( f : X \to Y \) be any morphism. We have

\[
f_*1_X(y) = \chi(f^{-1}(y)).
\]

- Assume we have a Cartesian square

\[
\begin{array}{ccc}
X' & \xrightarrow{q} & X \\
\downarrow{f'} & & \downarrow{f} \\
Y' & \xrightarrow{p} & Y
\end{array}
\]

Then, for any \( \varphi \in \text{Fun}(X) \),

\[
f'_*q^*\varphi = p^*f_*\varphi
\]

Now, let us turn to Nakajima quiver variety \( \mathfrak{L}(w) \). Denote \( e_j = (\delta_{ij})_{j \in \mathbb{I}} \) with \( \delta_{ij} \) the Kronecker’s delta. Let \( \mathfrak{P}_i \) be the variety of pairs \( (F_1, F_2) \) of representations in \( \mathfrak{L}(w) \) such that \( F_1 \subseteq F_2 \) with \( \text{dim } F_2/F_1 = e_i \). The variety \( \mathfrak{P}_i \) is called the Hecke correspondence (after [17] and [18]). We have two projections \( \rho_1, \rho_2 : \mathfrak{P}_i \to \mathfrak{L}(w) \) for \( i = 1, 2 \).

\[
\begin{array}{ccc}
\mathfrak{P}_i & \xrightarrow{\rho_1} & \mathfrak{L}(w) \\
& \xrightarrow{\rho_2} & \\
\mathfrak{L}(w)
\end{array}
\]

For \( F \in \mathfrak{L}(v, w) \) and \( i \in \mathbb{I} \), denote \( C_i(F) \) the complex of vector spaces,

\[
\begin{array}{ccc}
-1 & 0 & 1 \\
0 & \rightarrow & F(i) \rightarrow W(i) \oplus \bigoplus_{h:i \to j} F(j) \rightarrow F(i) \rightarrow 0.
\end{array}
\]

**Proposition 8** (Nakajima [17]). We have

\[
H^{-1}(C_i(F)) = 0,
\]

\[
\mathfrak{P}(H^0(C_i(F))) \cong \rho_{i1}^{-1}(F),
\]

\[
\mathfrak{P}(H^1(C_i(F))^*) \cong \rho_{i2}^{-1}(F),
\]

where \( \mathfrak{P}(-) \) is the associated projective space of a vector space.

Note that \( \rho_{i1}^{-1}(F) \) (resp. \( \rho_{i2}^{-1}(F) \)) is the variety of sup-representation (resp. subrepresentation) \( F' \) of \( F \) with dimension difference \( e_i \).

Let

\[
\phi_i(F) = \dim H^0(C_i(F)) = \chi(\rho_{i1}^{-1}(F)),
\]

\[
\epsilon_i(F) = \dim H^1(C_i(F)) = \chi(\rho_{i2}^{-1}(F)).
\]

It is clear that \( \phi_i(F) - \epsilon_i(F) = (w - Cv)_i \) for \( v = \text{dim } F \).

We know \( \mathfrak{L}(w) = \bigsqcup \mathfrak{L}(v, w) \), thus \( \text{Fun}(\mathfrak{L}(w)) = \bigoplus_v \text{Fun}(\mathfrak{L}(v, w)) \). For any function \( f \) of \( v \), it defines a map \( f : \text{Fun}(\mathfrak{L}(w)) \to \text{Fun}(\mathfrak{L}(w)) \) by

\[
f(\varphi_v) = (f(\varphi)) \varphi_v.
\]
Theorem 9 (Nakajima [17]). There is a well-defined action of $U$-action over $\text{Fun}(\mathcal{L}(w))$ by

\begin{align}
H_i \mapsto (w - Cv)_i \\
E_i \mapsto \rho_{1i} \rho^*_{2i} \\
F_i \mapsto \rho_{2i} \rho^*_{1i}
\end{align}

Example 10. Especially, the Example 3 recovers the Beilinson–Lusztig–MacPherson construction of $U_q(\mathfrak{sl}_n)$ in [2] at $q = 1$.

4. $\sigma$-quiver varieties

Denote $\sigma$ the involution over $Q$ induced by the longest element $w_0$ in the Weyl group, that is, $\sigma(i) = 2d - i$.

This is known as type AIII.

Lemma 11. There is a natural non-degenerate pairing

\begin{align}
B_{ij} : Q(i,j) \otimes Q(\sigma(i),j) \to \mathbb{C},
\end{align}

such that

\begin{align}
B_{ij}(p,q) &= B_{\sigma(i),j}(q,p), \\
B_{i',j}(p \circ h, q) &= B_{\sigma(i'),j}(p, q \circ \sigma(h))
\end{align}

for any $p \in Q(i,j)$, $q \in Q(\sigma(i),j)$ and $h \in Q(i,i')$.

Proof. By a simple combinatorial argument, there is a unique path $[d_i]$ of length $|I| - 1 = 2d - 2$ from $i$ to $\sigma(i)$ up to commutative relations (6). Let us denote $B_{ij}(p,q)$ to be the coefficient of $[d_i]$ in $\sigma(q)p$. It is clear that this pairing satisfies the properties in the assertion.

Remark 12. This pairing is in principle the Auslander–Reiten formulae

\[ \text{Hom}_Q(U,V)^* \cong \text{Ext}_Q(V,\tau U), \]

for an orientation $\check{Q}$ of $Q$, where $\tau$ is the Auslander–Reiten translation, see for example [5]. But for other Dynkin types, the pairing is not explicit as that for type $A$, and the signs would not be complicated.

Let $W = (W_i)_{i \in I}$ be an $I$-graded symplectic vector space.

Corollary 13. There is a symplectic form over $\bigoplus_{i \in I} KRW(i)$ induced by

\begin{align}
\omega : KRW(i) \otimes KRW(\sigma(i)) \to \mathbb{C}
\end{align}

such that

\begin{align}
\omega(x,y) &= -\omega(y,x), \\
\omega(h(x),y) &= \omega(x,\sigma(h)y)
\end{align}

for any arrows $h$ in $KRW$. 

Proof. By identifying $\text{Hom}(Q(i,j), W(j)) = Q(i,j)^* \otimes W(j)$, the form is given by
\[
\omega(p \otimes x, q \otimes y) = B_{ij}^1(p, q)\omega(x, y) \quad p \in Q(i,j)^*, x \in W(j)
\]
where $B_{ij}^1$ is the nondegenerated pairing $Q(i,j)^* \otimes Q(\sigma(i),j)^* \to \mathbb{C}$ induced by $B_{ij}$. □

Now we will define an involution over $\mathcal{L}(w)$. To ensure the well-definedness, we need the following lemma.

Lemma 14. For any subrepresentation $F$ of $K_RW|_{Q}$, the annihilator $F^\perp$ is also a subrepresentation of $K_RW|_{Q}$.

Proof. Note that $h(F(i)) \subseteq F(j)$ if and only if $\sigma(h)(F(j)^\perp) \subseteq F(i)^\perp$. □

Thanks to the lemma above, it is safe to define
\[
\sigma : \mathcal{L}(w) \to \mathcal{L}(w)
\]
the involution sending $F$ to $F^\perp$. Define the $\sigma$-quiver variety $\mathcal{R}(w)$ to be the fixed loci of fixed points of $\mathcal{L}(w)$ under this involution.

Remark 15. Actually, the Nakajima quiver variety $\mathcal{L}(w)$ is a closed subvariety of the Nakajima quiver variety $\mathcal{M}(w)$. To be exact, it is the fibre of $0 \in \mathcal{M}_0(w)$ under a proper morphism $\mathcal{M}(w) \to \mathcal{M}_0(w)$. Our definition of $\mathcal{R}(w)$ is inspired by [8]. Actually, in our case (type AIII), the variety $\mathcal{R}(w)$ is the intersection of $\mathcal{L}(w)$ and the $\sigma$-quiver variety $\mathcal{R}(w)$ in $\mathcal{M}(w)$. It follows from the next theorem 16. In particular, $\mathcal{R}(w)$ gives a family of Spaltenstein varieties, see [8].

We will not use this fact in the rest of this paper.

Theorem 16. In type AIII, our involution $\sigma$ defined in (43) coincides with the involution defined in [8] after restricting to $\mathcal{L}(w)$.

Proof. We will write $\mathcal{L}(W)$ rather than $\mathcal{L}(w)$ for an $\mathbb{I}$-graded vector space $W$ to emphasize the underlying frame space.

Firstly, the involution defined in [8] is based on reflection functors introduced in [14] and [15] independent, see also [21]. We will only use the reflection functor corresponding to the longest element of the Weyl group. It is usually complicated to describe, but over $\mathcal{L}(w)$, it has some explicit description.

Recall the Lusztig’s new symmetry introduced in [14], it is an isomorphism of varieties given by
\[
\mathcal{L}(W) \to \mathcal{L}(W^*)
\]
It is given by taking the annihilator of $K_RW$. In [21], Nakajima proved this coincides with the composition of the reflection functor corresponding to the longest element of the Weyl group and the isomorphism of taking the dual representation. Actually, the involution defined in [8] is the composition of the longest element of the Weyl group and the isomorphism of taking the annihilator. In particular, it is essentially (44) by identifying $W^*$ and $W$ by a bilinear form.

Assume we have an isomorphism $W^* \to W$ by a bilinear form, then it induces an isomorphism,
\[
K_RW^* \to K_RW
\]
since the construction of $K_R$ is functorial. By the construction of Lusztig’s new symmetry, after the above identification, the involution $\sigma$ introduced in [8] coincides with ours.

**Example 17.** In the Example [8] assume $n$ is an even number, and $W$ is equipped with a symplectic form $\omega$. The pairing over $K_R(w)$ is given by

$$\omega((v_i), (v'_i)) = \sum_i \omega(v_i, v'_{\sigma(i)}).$$

Thus in particular, for a partial flag

$$0 = V_0 \subseteq \cdots \subseteq V_i \subseteq \cdots \subseteq V_n = W$$

of $W$, its image under $\sigma$ is exactly

$$0 = V_n^\perp \subseteq \cdots \subseteq V_{n-i}^\perp \subseteq \cdots \subseteq V_0^\perp = W.$$

In particular, the $\sigma$-quiver variety is exactly the partial flags of type C studied in [1].

**Example 18.** Let us analyse the Example 4 when $n = 2$. Let us equip $W$ with a symplectic form $\omega$. In this case,

$$\omega(v_1^+, v_2^+, v_3, v'_1^+, v'_2^+, v'_3) = \omega(v_1, v'_3) + \omega(v_3, v'_1) + \omega(v'_2, v'_2) + \omega(v_2^+, v'_2^+).$$

As a result, we are finding subspaces $V_1, V_3$ in $W$ with $V_1^\perp = V_3$ and $V_2$ a Lagrangian subspace of $W \oplus W$ under above symplectic form such that

$$\text{pr}_2(V_2) \subseteq V_1 \cap V_3, \quad (V_1 + V_3) \oplus 0 \subseteq V_2,$$

where $\text{pr}_2 : W \oplus W \to W$ is the second projection.

- When $v = 022$ or $220$, $\mathcal{R}(v, w)$ is a point, given by

$$\begin{array}{c}
0 \quad W \quad \text{or} \quad W \quad 0
\end{array}$$

Actually, the choice of $V_2$ can only be $\text{ker} \text{pr}_2$.

- When $v = 121$, it forces $V_1 = V_3$ since $V_1^\perp = V_1$. As a result, $\mathcal{R}(v, w)$ is the variety of one dimensional vector spaces $V_1$ and $V_2$ with $V_1 \subseteq W$ and $V_2 \subseteq (W \oplus V_1)/(V_1 \oplus 0)$. Thus it is the associated projective bundle of $\mathcal{T} \oplus \mathcal{Q}$ over $\mathbb{P}^1$, with $\mathcal{T}$ and $\mathcal{Q}$ to be the tautological and quotient bundle respectively.

**Example 19.** Let us deal with Example 5 when $n = m = 2$. We take $N = M = W$ and equip with the same symplectic form $\omega$. Now, the symplectic form is given by

$$\omega(v_1^+, v_2^+, v_3, v'_1^+, v'_2^+, v'_3) = \sum_{\pm \in \{+,-\}} \sum_{\bullet=1,2,3} \omega(v^\pm_\bullet, v'^\pm_\bullet).$$
We take the following notations

\[ W \xrightarrow{V_1} V_2 \xrightarrow{V_3} \]

We have

- For \( \mathbf{v} = 024 \) and \( \mathbf{v} = 402 \), the sigma quiver variety is empty. This is because \( \text{pr}_2(V_2) \) is always nonzero for \( \bullet = 1, 2 \), since \( \ker \text{pr}_2 \) is not Lagrangian.
- For \( \mathbf{v} = 123 \) or \( \mathbf{v} = 321 \), the representation in \( \mathfrak{R}(\mathbf{v}, \mathbf{w}) \) takes the form

\[ W \xrightarrow{0} 1 \xrightarrow{1} 2, \quad \text{or} \quad W \xrightarrow{1} 1 \xrightarrow{1} 0. \]

Here the numbers indicate the dimension. As a result, \( \mathfrak{R}(\mathbf{v}, \mathbf{w}) \) are both isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \).
- When \( \mathbf{v} = 222 \), it is slightly complicated.
  - If \( \text{pr}_2(V_2) \) has dimension 2 for \( \bullet = 1 \) or \( \bullet = 2 \), then \( V_1 = 0 \oplus W \) and \( V_3 = W \oplus 0 \). In this case the choice of \( V_2 \) is arbitrary.
  - If \( \text{pr}_2(V_2) \) are both of dimension 1 for \( \bullet = 1, 2 \), then \( V_2 = V_2^+ \oplus V_2^- \) for \( V_2^+ \subseteq W \oplus 0 \) and \( V_2^- \subseteq 0 \oplus W \). We need to require \( 0 \oplus V_2^- \subseteq V_1 \subseteq V_2^+ \oplus W \).

As a result, \( \mathfrak{R}(\mathbf{v}, \mathbf{w}) \) is a union of two irreducible components \( \Sigma_1 \) and \( \Sigma_2 \) with \( \Sigma_1 \) isomorphic to the Lagrangian Grassmannian of \( W \oplus W \) and \( \Sigma_2 \) isomorphic to the associated Grassmannian of \( Q_+ \oplus T_+ \) over \( \mathbb{P}^1 \times \mathbb{P}^1 \) with \( T^+ \) the tautological bundle of the first factor, and \( Q^- \) the quotient bundle of the second factor. Their intersection is \( \mathbb{P}^1 \times \mathbb{P}^1 \).

Proposition 20. We have

\[ \epsilon_i(F) = \phi_{\sigma(i)}(F^\perp), \quad \phi_i(F) = \epsilon_{\sigma(i)}(F^\perp). \]

Proof. Note that \( \sigma \) sets an isomorphism between \( \rho_{\sigma(i)}^{-1}(F) \) and \( \rho_{\sigma(i)}^{-1}(F^\perp) \). Thus the results follows from Proposition 8. \( \square \)

Corollary 21. Assume \( \dim F = \mathbf{v} \), and \( \dim F^\perp = \mathbf{v}' \). Then \( -\sigma(\mathbf{w} - C\mathbf{v}) = \mathbf{w} - C\mathbf{v}' \). In particular, the variety \( \mathfrak{R}(\mathbf{v}, \mathbf{w}) \) is empty unless \( \sigma(\mathbf{w} - C\mathbf{v}) = -(\mathbf{w} - C\mathbf{v}') \).

Proof. This can be shown simply by taking difference of \[20\] but let us prove by direct computation. Note that by our construction, \( \dim F + \sigma \dim F^\perp = \mathbf{v}' + \sigma \mathbf{v} = \sigma \mathbf{v}' + \mathbf{v} = \dim K_R \mathcal{W} =: \mathbf{v}_0 \). By proposition 2, we have \( C\mathbf{v}_0 = \mathbf{w} + \sigma \mathbf{w} \). Hence we have \( \mathbf{w} + \sigma \mathbf{w} - C(\sigma \mathbf{v} + \mathbf{v}') = 0 \). \( \square \)

5. Geometric realization of \( U^\epsilon \)

Given an involution \( \sigma \) over \( Q \), it induces an involution over \( \mathfrak{g} \)

\[ E_i \mapsto F_{\sigma(i)}, \quad F_i \mapsto E_{\sigma(i)}, \quad H_i \mapsto -H_{\sigma(i)}. \]

We have a Cantan decomposition

\[ \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p} \]
with \( \mathfrak{t} \) (resp. \( \mathfrak{p} \)) the subspace of \( \mathfrak{g} \) the eigenspace of \( \sigma \) to 1 (resp. \(-1\)). The pair \((\mathfrak{g}, \mathfrak{t})\) forms a symmetric pair, see [22] for more background. Denote

\[
B_i = E_i + F_{\sigma(i)}, \quad h_i = H_1 - H_{\sigma(i)}.
\]

It is easy to show that \( \mathfrak{t} \) is the Lie algebra generated by \( B_i \) and \( h_i \) with relations

\[
\begin{align*}
(50) & \quad h_i + h_{\sigma(i)} = 0, \\
(51) & \quad [h_i, h_j] = 0, \quad [h_i, B_j] = (c_{ij} - c_{\sigma(i)j})B_j \\
(52) & \quad [B_i, B_{\sigma(i)}] = h_i \\
(53) & \quad [B_i, B_j] = 0 \quad \text{for} \quad \sigma(i) \neq j \quad \text{and} \quad c_{ij} = 0 \\
(54) & \quad [B_i, [B_i, B_j]] = 0 \quad \text{for} \quad \sigma(i) \neq i \quad \text{and} \quad c_{ij} = -1 \\
(55) & \quad [B_i, [B_i, B_j]] = B_j \quad \text{for} \quad \sigma(i) = i \quad \text{with} \quad c_{ij} = -1
\end{align*}
\]

Let \( \text{dim} \) be the universal enveloping algebra of \( \mathfrak{t} \). Actually, this is the classical limit of the quasi-split quantum symmetric pair of type AIII introduced in [7].

Analogy to Hecke correspondence \( \mathfrak{B}_i \), we define the \textit{iHecke correspondence} \( \mathfrak{B}_i \) to be the variety of pairs \((F_1, F_2) \in \mathfrak{B}(w) \times \mathfrak{B}(w)\) such that

\[
\text{dim} F_1/F_1 \cap F_2 = \mathfrak{e}_{\sigma(i)} \quad \text{and} \quad \text{dim} F_2/F_1 \cap F_2 = \mathfrak{e}_i.
\]

Now we have two projections

\[
\mathfrak{B}_i \xrightarrow{\pi_{i2}} \mathfrak{B} \xleftarrow{\pi_{i2}} \mathfrak{B}(w).
\]

For \( i \in \mathbb{I} \), assume \( \sigma(i) \neq i \). Let \( O(i) \) be the neighborhood of \( i \), i.e. it contains all edges incident to \( i \). Assume we are given two representations \( F \) and \( G \) such that \( F \) and \( G \) agree over the full quiver of vertices \( \mathbb{I} \setminus \{i, \sigma(i)\} \). Then we can define \( F_{\iota_i} G \) to be the representation \( H \) such that

\[
H|_{O(i)} = F|_{O(i)} \cap G|_{O(i)}, \quad \text{and} \quad H|_{O(\sigma(i))} = F|_{O(\sigma(i))} \cap G|_{O(\sigma(i))}.
\]

This is well-defined since there is no common edge in \( O(i) \) and \( O(\sigma(i)) \) by our assumption. Define \( F_{\iota_i} G = F_{\iota_i} G \). We have the following equality of dimension vector

\[
\text{dim} (F_{\iota_i} G) + \text{dim} (F_{\iota_i} G) = \text{dim}(F) + \text{dim}(G).
\]

Moreover, if \( F, G \in \mathfrak{B}(w) \), then so are \( F_{\iota_i} G \) and \( F_{\iota_i} G \).

**Lemma 22.** For \( \sigma(i) \neq i \), consider the following commutative diagram

\[
\begin{array}{ccc}
\mathfrak{B}_i & \xrightarrow{g} & \mathfrak{B}_i \\
\downarrow{\pi_{i2}} & & \downarrow{\rho_{i2}} \\
\mathfrak{B}(w) & \xleftarrow{\rho_{i2}} & \mathfrak{B}(w).
\end{array}
\]

where \( g(F_1, F_2) = (F_1 \cap F_2, F_2) \). Then \( g \) is an isomorphism onto \( \rho_{i2}^{-1}(\mathfrak{B}(w)) \).

**Proof.** The inverse is given by \( (F_0, F_2) \mapsto (F_0 \cap F_2, F_2) \). \( \square \)
Corollary 23. For $F \in \mathcal{R}(v, w)$, if $\sigma(i) \neq i$,
\begin{align}
\chi(p_{11}^{-1}(F)) &= \phi_i(F), \\
\chi(p_{12}^{-1}(F)) &= \epsilon_i(F).
\end{align}

Now turn to the case $\sigma(i) = i$. Note that $(w - Cv)_i = 0$ by Proposition 21. In particular $\epsilon_i(F) = \phi_i(F)$ for $F \in \mathcal{R}(v, w)$. We need the following linear algebra.

Lemma 24. Let $V$ be a symplectic vector space of dimension $2n$. For a given Lagrangian subspace $L_0$, the variety of Lagrangian subspace $L$ such that $\dim L \cap L_0 = n - 1$ has Euler characteristic $n$.

Proof. Let this variety be $\Sigma$. Then we have a morphism $\Sigma = \mathbb{P}(L^*)$ by sending $L$ to $L \cap L_0$. This is a fibre bundle with fibre at $L'$ the variety of Lagrangian subspaces between $L'$ and $L'^\perp$ other than $L_0$. The fibre is just $\mathbb{C}P^1 \setminus \text{pt}$ whose Euler characteristic is 1.

Corollary 25. For $F \in \mathcal{R}(v, w)$, if $\sigma(i) = i$,
\begin{equation}
\chi(p_{11}^{-1}(F)) = \phi_i(F) = \chi(p_{12}^{-1}(F)) = \epsilon_i(F).
\end{equation}

Proof. It is clear that $\phi_i(F)$ is the variety of Lagrangian subspaces of $K_iW(i)$ containing $\sum_{h:j \rightarrow i} h(F(j))$ other than $F(i)$. By the Lemma 24 above, it is exactly $\epsilon_i(F)$.

The following theorem is the main theorem of this paper.

Theorem 26. For type $A_{III}$, there is a well-defined action of $U'$-action over $\text{Fun}(\mathcal{R}(w))$ by
\begin{align}
h_i &\mapsto (w - Cv)_i \\
B_i &\mapsto p_{11}^{-1}p_{12}^{-1}.
\end{align}

We will prove this theorem in the rest of paper.

Example 27. In the case of the Example 7, this is the result of [1] at $q = 1$.

Remark 28. Let us denote $\mathfrak{B}_i$ be the variety of pairs $(F_1, F_2) \in \mathcal{R}(w) \times \mathcal{R}(w)$ such that there exists an $F_0 \in \Omega(w)$ with
\begin{equation}
F_0 \subseteq F_1 \quad \text{dim} F_1/F_0 = e_{\sigma(i)} \quad \text{and} \quad F_0 \subseteq F_2 \quad \text{dim} F_2/F_0 = e_i.
\end{equation}

Note that $\mathfrak{B}_i = \mathfrak{B}_i$ if $\sigma(i) \neq i$, $\mathfrak{B}_i = \mathfrak{B}_i \cup \Delta \mathcal{R}(w)$ if $\sigma(i) = i$ where $\Delta$ is the diagonal. This also defines a well-defined action of $U'$ in a similar manner. This corresponds to the correspondence used in [1]. This is because $B_i \mapsto B_i + \delta_{i=\sigma(i)}$ is an automorphism of $U'$.

Note that $\mathfrak{B}_i$ is closed, since $\mathfrak{B}_i$ is proper by the proof of Lemma 24 and Corollary 25. Here we take this convention since it is parallel to the geometric realization of Hecke algebras where the correspondences are not closed.

6. Relations (1)

In this section, we will prove the relations (50), (52) and (53). They are parallel to what was done in [17]. Especially, the proof of the last relations (53), known as Serre relations, was originally due to Lusztig [12].
Proposition 29. The relations \(50\) and \(51\) hold.

Proof. The relation \(50\) follows from Proposition 21 that \(w = \dim F_1 + \dim S_q \leq 0\). For \((F_1, F_2) \in \mathfrak{B}_i\), we have
\[
(w - C \dim F_1)_j - (w - C \dim F_2)_j = (C(e_i - e_{\sigma(i)}))_j = c_{ij} - c_{\sigma(i)j}.
\]
Thus the relation \(51\) holds.

\[\square\]

Proposition 30. The relation \(52\) holds.

Proof. It suffices to show when \(\sigma(i) \neq i\). Let \(\mathfrak{B}_i \mathfrak{B}_{\sigma(i)} = \mathfrak{B}_i \times_{\mathfrak{R}(w)} \mathfrak{B}_{\sigma(i)}\) be the fibre product. Consider the following diagram
\[
\begin{array}{ccc}
\mathfrak{B}_i \mathfrak{B}_{\sigma(i)} & \xrightarrow{f} & \mathfrak{R}(w) \\
\downarrow{p_1} & & \downarrow{q_1} \\
\mathfrak{R}(w) & \xrightarrow{p_3} & \mathfrak{R}(w) \\
\end{array}
\]
Let \(\mathfrak{S}_i\) be the preimage of the diagonal \(\Delta \mathfrak{R}(w) \subseteq \mathfrak{R}(w) \times \mathfrak{R}(w)\) under \(f\) and \(\mathfrak{A}_i\) its complement. By \(27\) and \(13\) for any \(\varphi \in \Fun(\mathfrak{R}(w))\),
\[
B_i B_{\sigma(i)} \varphi = p_{1*} p_{3*} \varphi = q_{1*}(f_1 \mathbf{1}_{\mathfrak{A}_i} \cdot q_{3*} \varphi) = q_{1*}(f_{\sigma(i)} \mathbf{1}_{\mathfrak{A}_{\sigma(i)}} \cdot q_{3*} \varphi).
\]
Note that \(\mathfrak{S}_i\) is isomorphic to \(\mathfrak{B}_i\), and thus \(q_{1*}(f_{\sigma(i)} \mathbf{1}_{\mathfrak{A}_{\sigma(i)}} \cdot q_{3*} \varphi) = \pi_{i1*} \pi_{11*} \varphi\). We will use the same notation \(f\) to denote the counterpart of \(\sigma(i)\) and we have similarly \(q_{1*}(f_1 \mathbf{1}_{\mathfrak{A}_i} \cdot q_{3*} \varphi) = \pi_{i2*} \pi_{22*} \varphi\). By Corollary 23 we have
\[
\pi_{i1*} \pi_{11*} \varphi - \pi_{i2*} \pi_{22*} \varphi = (w - C \mathbf{v})_i \varphi = h_i \varphi.
\]
It rests to show \(f_{\sigma(i)} \mathbf{1}_{\mathfrak{A}_i} = f_{\sigma(i)} \mathbf{1}_{\mathfrak{A}_{\sigma(i)}}\). By definition, the fibre of \(f\) at \((F_1, F_2)\) when \(F_1 \neq F_2\) in \(\mathfrak{A}_i\) (resp \(\mathfrak{A}_{\sigma(i)}\)) is the point \(F_1 F_2\) \(F_1 \triangleright F_2\) (resp \(F_1 \triangleright F_2\)) if it has correct dimension. This shows \(f_{\sigma(i)} \mathbf{1}_{\mathfrak{A}_i} = f_{\sigma(i)} \mathbf{1}_{\mathfrak{A}_{\sigma(i)}}\).

\[\square\]

Proposition 31. The relation \(53\) holds.

Proof. Let \(\mathfrak{R}\) be the variety of pair \((F_1, F_2) \in \mathfrak{R}(w) \times \mathfrak{R}(w)\) such that
\[
\dim F_1 F_1 \cap F_2 = e_{\sigma(i)} + e_{\sigma(j)}\quad \text{and} \quad \dim F_2 F_1 \cap F_2 = e_i + e_j.
\]
We have two morphisms
\[-\begin{array}{c}
\mathfrak{B}_i \times_{\mathfrak{R}(w)} \mathfrak{B}_j \rightarrow \mathfrak{R} \leftarrow \mathfrak{B}_j \times_{\mathfrak{R}(w)} \mathfrak{B}_i
\end{array}\]
We will show both of them are isomorphisms for \(\sigma(i) \neq j\). If \(e_{\sigma(i)j} = 0\), then \(O(i) \cup O(\sigma(i))\) and \(O(j) \cup O(\sigma(j))\) has no common edge, thus the above two morphisms are both isomorphisms. If \(e_{\sigma(i)j} = -1\) and \(\sigma(j) \neq i\) and \(\sigma(i) \neq i\). The relation of \((F_1, F_2) \in \mathfrak{R}\) can be illustrated as follows,
\[
\begin{array}{c}
\cdots \rightarrow F_1(i) \rightarrow F_1(\sigma(j)) \cdots \rightarrow F_1(j) \rightarrow F_1(\sigma(i)) \rightarrow \cdots \\
\uparrow{f_i} \quad \uparrow{f_j} \\
\cdots \rightarrow F_2(i) \rightarrow F_2(\sigma(j)) \cdots \rightarrow F_2(j) \rightarrow F_2(\sigma(i)) \rightarrow \cdots
\end{array}
\]
It is clear that \(h(F_1(i)) \subseteq F_2(\sigma(j))\) for \(h : i \rightarrow \sigma(j)\) and so on. In particular,
\[
\cdots \oplus F_1(i) \oplus F_2(\sigma(j)) \oplus \cdots \oplus F_2(j) \oplus F_1(\sigma(i)) \oplus \cdots
\]
is a well-defined representation which is the unique representation in the fibre of the left morphism. The same argument for the right morphism shows the equality.

\[\square\]
Proposition 32. The relation (74) holds.

Proof. We will show the case \( \sigma(j) = j \), and the case \( \sigma(j) \neq j \) will be left to readers. Construct the following diagram

\[
\begin{array}{c}
\mathcal{B}_i \mathcal{B}_j \\
\mathcal{R}(w) \quad \quad \quad \mathcal{R}(w) \times \mathcal{R}(w) \quad \quad \quad \mathcal{R}(w),
\end{array}
\]

where

\[ \mathcal{B}_i \mathcal{B}_j = \mathcal{B}_i \times_{\mathcal{R}(w)} \mathcal{B}_i \times_{\mathcal{R}(w)} \mathcal{B}_j. \]

By (27), for any \( \varphi \in \text{Fun}(\mathcal{R}(w)) \),

\[ B_iB_jB_j = q_1\{f_1 \mathcal{B}_i, \mathcal{B}_j, \mathcal{B}_j, q_2^* \varphi \}. \]

We will abuse of notation to denote \( f \) the counterpart of \( \mathcal{B}_i \mathcal{B}_j \mathcal{B}_j \) or \( \mathcal{B}_j \mathcal{B}_j \mathcal{B}_i \). We will show that

\[ f_1 \mathcal{B}_i, \mathcal{B}_j, \mathcal{B}_j, -2f_1 \mathcal{B}_i, \mathcal{B}_j, \mathcal{B}_j, + f_1 \mathcal{B}_i, \mathcal{B}_j, \mathcal{B}_j, = 0. \]

Let \( h \) the arrow from \( i \) to \( j \). For \( (F_1, F_2) \in \mathcal{R}(w) \times \mathcal{R}(w) \) with

\[ \dim F_1/F_1 \cap F_2 = 2e_{\sigma(i)} + e_j, \quad \text{and} \quad \dim F_2/F_1 \cap F_2 = 2e_i + e_j, \]

we will compute (63). We can illustrate the relation of \( (F_1, F_2) \) as follows

\[
\begin{array}{c}
\cdots \quad F_1(i) \xrightarrow{h} F_1(j) \xrightarrow{h} F_1(\sigma(i)) \xrightarrow{h} \cdots \\
\cdots \quad F_2(i) \xrightarrow{h} F_2(j) \xrightarrow{h} F_2(\sigma(i)) \xrightarrow{h} \cdots 
\end{array}
\]

We have

- The fibre of \( f \) in \( \mathcal{B}_j \mathcal{B}_i \mathcal{B}_i \) is exactly the variety of subspaces \( X \) of correct dimension between \( F_1(i) \) and \( F_2(i) \) if \( \bar{h}(F_2(j)) \subseteq F_1(i) \), otherwise, it is empty.
- The fibre of \( f \) in \( \mathcal{B}_i \mathcal{B}_j \mathcal{B}_i \) is exactly the variety of subspaces \( X \) of correct dimension between \( F_1(i) \) and \( F_2(i) \) of correct dimension such that

\[ h(X) \subseteq F_1(j) \cap F_2(j), \quad \bar{h}(F_1(j) + F_2(j)) \subseteq X. \]

- The fibre of \( f \) in \( \mathcal{B}_i \mathcal{B}_i \mathcal{B}_i \) is exactly the variety of subspaces \( X \) of correct dimension between \( F_1(i) \) and \( F_2(i) \) if \( h(F_2(i)) \subseteq F_1(j) \), otherwise, it is empty.

There are four cases.

- If \( h(F_2(i)) \subseteq F_1(j) \) and \( \bar{h}(F_2(j)) \subseteq F_1(i) \), the three fibres are all \( \mathbb{C}P^1 \). In particular, (63) holds.
- If \( h(F_2(i)) \not\subseteq F_1(j) \) and \( \bar{h}(F_2(j)) \subseteq F_1(i) \). Then the fibre of \( \mathcal{B}_i \mathcal{B}_j \mathcal{B}_i \), has only one point, i.e. the subspace

\[ h^{-1}(F_1(j) \cap F_2(j)) \cap F_2(i). \]
To prove this, it suffices to show it has a correct dimension. Otherwise, \( h^{-1}(F_1(j)) \cap F_2(i) = F_1(i) \). Actually, we have an injection induced by \( h \)

\[
\frac{F_2(i)}{h^{-1}(F_1(j) \cap F_2(j)) \cap F_2(i)} \longrightarrow \frac{F_2(j)}{F_1(j) \cap F_2(j)}.
\]

This is a contradiction. As a result, the fibres are \( \mathbb{C}P^1 \), pt and empty respectively.

- If \( h(F_2(i)) \subseteq F_1(j) \) and \( h(F_2(j)) \not\subseteq F_1(i) \). Then the fibre of \( \mathcal{B}_1 \mathcal{B}_2 \mathcal{B}_1 \), has only one point, i.e. the subspace

\[
\tilde{h}(F_1(j) + F_2(j)) + F_1(i).
\]

It suffices to show it has a correct dimension. Otherwise, \( \tilde{h}(F_1(j) + F_2(j)) + F_1(i) = F_2(i) \). Actually, we have a surjection induced by \( \tilde{h} \)

\[
\frac{F_2(j)}{F_1(j) \cap F_2(j)} = \frac{F_1(j) + F_2(j)}{F_1(j)} \longrightarrow \frac{\tilde{h}(F_1(j) + F_2(j)) + F_1(i)}{F_1(i)}.
\]

The is a contradiction. As a result, the fibres are empty, pt and \( \mathbb{C}P^1 \) respectively.

- If \( h(F_2(i)) \not\subseteq F_1(j) \) and \( \tilde{h}(F_2(j)) \not\subseteq F_1(i) \). From the discussion above, if the fibre of \( \mathcal{B}_1 \mathcal{B}_2 \mathcal{B}_1 \mathcal{B}_j \) is non-empty, then it is

\[
h^{-1}(F_1(j) \cap F_2(j)) \cap F_2(i) = \tilde{h}(F_1(j) + F_2(j)) + F_1(i).
\]

Note that we have the following complex (see [12])

\[
\frac{F_2(i)}{h^{-1}(F_1(j) \cap F_2(j)) \cap F_2(i)} \longrightarrow \frac{F_2(j)}{F_1(j) \cap F_2(j)} \longrightarrow \frac{\tilde{h}(F_1(j) + F_2(j)) + F_1(i)}{F_1(i)}
\]

which is injective on the left and surjective on the right. But by dimension reason, it is impossible. As a result, the fibres are all empty.

In particular, the relation (63) holds. \( \square \)

7. Relations (II)

In this section, we will show the relation (55) which is known as Serre relations. Assume \( \sigma(i) = i \) and \( c_{ij} = -1 \) in this section. For any \( F \in \mathfrak{L}(v, w) \), denote

\[
\mathcal{I}(F) = \sum_{h : j' \rightarrow i} h'(F(j')).
\]

Note that \( \dim \mathcal{I}(F) = v_i - \epsilon_i(F) \) by definition.

**Lemma 33.** If \( (F_1, F_2) \in \mathcal{B}_j \), then

\[
\dim \mathcal{I}(F_1 + F_2) - \dim \mathcal{I}(F_1 \cap F_2) = 1.
\]

In particular \( \mathcal{I}(F_1) \subseteq \mathcal{I}(F_2) \) or \( \mathcal{I}(F_2) \subseteq \mathcal{I}(F_1) \).

**Proof.** By Proposition [20]

\[
\dim \mathcal{I}(F_1 + F_2) - \dim \mathcal{I}(F_1 \cap F_2) = \epsilon_i(F_1 \cap F_2) - \epsilon_i(F_1 + F_2) = \phi_i(F_1 + F_2) - \epsilon_i(F_1 + F_2) = (w - C \dim(F_1 + F_2))_i = -Ce_j = 1.
\]

The equality of the last row is due to Corollary [21] \( \square \)

We can say more about the relations of \( \mathcal{I}(F_2) \) and \( \mathcal{I}(F_1) \) where we essentially use the symplectic form in Corollary [13]
Lemma 34. If \((F_1, F_2) \in \mathcal{B}_j\), then \(\mathcal{I}(F_1) \neq \mathcal{I}(F_2)\) and \(\mathcal{I}(F_1) \subset \mathcal{I}(F_2)\) or \(\mathcal{I}(F_2) \subset \mathcal{I}(F_1)\). In particular,
\[
\mathcal{I}(F_1) \cap \mathcal{I}(F_2) = \mathcal{I}(F_1 \cap F_2).
\]

Proof. Otherwise \(\mathcal{I}(F_1) = \mathcal{I}(F_2) = \mathcal{I}(F_1 + F_2)\). Let \(A = \mathcal{I}(F_1 + F_2)\), and \(B = \mathcal{I}(F_1 \cap F_2)\). Let \(V_\bullet = F_\bullet(j) \oplus F_\bullet(\sigma(j))\) for \(\bullet = 1, 2\). Let \(h\) be the arrow \(i \to j\).

Denote \(g : V_\bullet \to F_\bullet(i)\) the sum of \(h\) and \(\sigma(h)\). By definition,
\[
g(V_1 + V_2) \subset A, \quad g(V_\bullet) \not\subset B, \quad g(V_1 \cap V_2) \subset B.
\]

Therefore,
\[
\bar{g}(A^\perp) \subset V_1 \cap V_2, \quad \bar{g}(B^\perp) \not\subset V_\bullet, \quad \bar{g}(B^\perp) \subset V_1 + V_2.
\]

Pick any \(x \in B^\perp \setminus A^\perp\). By definition
\[
\bar{g}(x) \in V_1 + V_2, \quad \bar{g}(x) \not\in V_\bullet.
\]

Note that \(\bar{g}(x) = (h(x), \sigma(h)(x)) \in K_RW(j) \oplus K_RW(\sigma(j))\). Denote \((u, v) = \bar{g}(x) = (h(x), \sigma(h)(x)) \in K_RW(j) \oplus K_RW(\sigma(j))\). The condition is equivalent to say
\[
u \in F_2(j), v \in F_2(\sigma(j)) \text{ and } u \notin F_1(j), v \notin F_2(j).
\]

In particular \(F_1(j) + Cu = F_2(j)\) and \(F_2(\sigma(j)) + Cv = F_1(j)\). In particular, \(\omega(u, v) \neq 0\). But
\[
\omega(u, v) = \omega(h(x), \sigma(h)(x)) = \omega(x, \sigma(\bar{h}h)x) = \omega(x, h\bar{h}x) = \omega(\sigma(h)(x), h(x)) = \omega(v, u)
\]
a contradiction. \(\square\)

Example 35. We can examine the examples before.

- In Example 17 \(\mathcal{I}\) is exactly \(V_{n/2-1}\).
- In Example 18 consider the cases of 121 and 022, we see that \(\mathcal{I} = V_1\) and \(\mathcal{I} = W\) respectively.
- In Example 19 we see \(\dim \mathcal{I} = 1\) for \(v = 321\). For \(v = 222\), if \(F\) corresponds to a point over Langrangian Grassmannian, then \(\mathcal{I} = 0\). If \(F\) corresponds a point out of it, then \(\text{pr}_1(V_1) \neq 0\) and similarly \(\text{pr}_2(V_1^\perp) \neq 0\). So \(\dim \mathcal{I} = 2\).

As a result, we cannot predict which \(\mathcal{I}(F_\bullet)\) is bigger for \((F_1, F_2) \in \mathcal{B}_j\) from the dimension vector.

Construct the following diagram

\[
\begin{array}{ccc}
\mathcal{B}_i \otimes \mathcal{B}_j & & \mathcal{B}_i \otimes \mathcal{B}_j \\
\downarrow f & & \downarrow f \\
\mathcal{R}(w) & \xrightarrow{q_1} & \mathcal{R}(w) \times \mathcal{R}(w) & \xrightarrow{q_2} & \mathcal{R}(w),
\end{array}
\]

where
\[
(65) \quad \mathcal{B}_i \otimes \mathcal{B}_j \otimes \mathcal{B}_i = \mathcal{B}_i \times_{\mathcal{R}(w)} \mathcal{B}_i \times_{\mathcal{R}(w)} \mathcal{B}_j.
\]

We will abuse of notation to denote \(f\) the counterpart of \(\mathcal{B}_i \otimes \mathcal{B}_j \otimes \mathcal{B}_i, \mathcal{B}_j \otimes \mathcal{B}_i \otimes \mathcal{B}_i\). By (27), it suffices to show
\[
(66) \quad f_*1_{\mathcal{B}_i \otimes \mathcal{B}_j \otimes \mathcal{B}_i} - 2f_*1_{\mathcal{B}_i \otimes \mathcal{B}_j \otimes \mathcal{B}_i} + f_*1_{\mathcal{B}_j \otimes \mathcal{B}_i \otimes \mathcal{B}_i} = 1_{\mathcal{B}_j}.
\]

From the next lemma, we see that it suffices to check over \((F_1, F_2) \in \mathcal{R}(w) \times \mathcal{R}(w)\) with \(\dim(F_1(i) \cap F_2(i)) \leq \dim F_1(i) - 2\).
Lemma 36. Let $V$ be a symplectic vector space of dimension $2n$. For three Lagrangian subspaces $L_1, L_2, L_3$, denote $L_{ij} = L_i \cap L_j$ for $i, j \in \{1, 2, 3\}$, and $L_{123} = \cap L_i$. Assume $\dim L_{12} = \dim L_{23} = n - 1$, then

$$
\dim L_{13} = n - 2 \quad \text{or} \quad \dim L_{13} = n - 1
$$

or $L_1 = L_3$.

Proof. Assume $L_1 \neq L_3$. Since $\dim L_{123} \geq \dim L_{12} + \dim L_{23} - \dim(L_{12} + L_{23}) \geq \dim L_{12} + \dim L_{23} - \dim L_2 = n - 2$, $\dim L_{123} \geq n - 2$. Assume $L_{12} \neq L_{23}$, then $\dim L_{123} = n - 2$ and $L_{12} + L_{23} = L_2$. Without loss of generality, we can assume $L_{123} = 0$ i.e. $n = 2$. Assume $L_{12} = \mathbb{C}x$, and $L_{23} = \mathbb{C}y$. Then $L_2 = L_{12} + \mathbb{C}y$ and $L_1 = L_{23} + \mathbb{C}x$. Assume further that $L_1 = L_{12} + \mathbb{C}z$ and $L_3 = L_{23} + \mathbb{C}w$. We will show $x, y, z, w$ are linearly independent, then $L_1 \cap L_3 = L_{123} = 0$.

Firstly, since $L_1, L_2, L_3$ are Lagrangian, $\omega(x, y) = \omega(x, z) = \omega(y, w) = 0$. Secondly, since Langriang is maximal isotropic subspaces, $\omega(z, y) \neq 0 \neq \omega(x, w)$. Thus

$$
\begin{pmatrix}
\omega(x, x) & \omega(x, y) & \omega(x, z) & \omega(x, w) \\
\omega(y, x) & \omega(y, y) & \omega(y, z) & \omega(y, w) \\
\omega(z, x) & \omega(z, y) & \omega(z, z) & \omega(z, w) \\
\omega(w, x) & \omega(w, y) & \omega(w, z) & \omega(w, w)
\end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \mathbb{C}^x \\
0 & 0 & \mathbb{C}^x & 0 \\
0 & \mathbb{C}^x & 0 & \omega(z, w) \\
\mathbb{C}^x & 0 & \omega(w, z) & 0
\end{pmatrix}
$$

This shows the linear independence. \qed

Proposition 37. At the point $(F_1, F_2)$ with $F_1(i) = F_2(i)$, the relation (66) holds.

Proof. The conditions of the first three fibres can be simplified as follows.

$$
\begin{align*}
F_1(i) & \sim F_1(i) \sim F_1(\sigma(i)) \\
F_2(i) & \sim F_2(i) \sim F_2(\sigma(i)) \\
\end{align*}
$$

Thus when $(F_1, F_2) \notin \mathcal{B}_j$, the fibres are all empty. So it suffices to consider the case $(F_1, F_2) \in \mathcal{B}_j$. The fibres are the variety of Lagrangian subspaces $X$ of $K_R W(i)$ such that

$$
\dim(X \cap F_1(i)) = \dim F_1(i) - 1,
$$

containing $\mathcal{I}(F_1)$, $\mathcal{I}(F_1 + F_2)$ and $\mathcal{I}(F_2)$ respectively. By Lemma 24 the Euler characteristic is exactly the codimension of $\mathcal{I}$ in $X$. Thus the left-hand-side of (66) is $\dim \mathcal{I}(F_1 + F_2) - \dim \mathcal{I}(F_1 \cap F_2)$ by Lemma 34. The result follows from 33. \qed

Proposition 38. At the point $(F_1, F_2)$ with $\dim(F_1(i) \cap F_2(i)) = \dim F_1(i) - 1$, the relation (66) holds.

Proof. By Lemma 30, the fibre is empty or $\mathbb{CP}^1$ deleting two points, whose Euler characteristics are both 0. \qed

Lemma 39. Let $V$ be a symplectic vector space of dimension $2n$. For two Lagrangian subspaces $L_1$ and $L_2$ with $\dim(L_1 \cap L_2) = n - 2$. Then the choice of Lagrangian subspaces $L$ such that $\dim(L_1 \cap V) = \dim(L_2 \cap V) = n - 1$ are in one-to-one correspondence to the choice of subspaces $U$ between $L_1$ and $L_1 \cap L_2$ with $\dim U = n - 1$.

Proof. The correspondence is given by $U = L \cap L_1$ and $L_1 = U + (U^\perp \cap L_2)$. It suffices to prove when $n = 2$ and $L_1 \cap L_2 = 0$. Assume $U = \mathbb{C}u$. Then we can find a nonzero element $v \in L_2$ such that $\omega(u, v) = 0$, since $L_2$ has dimension 2. This choice is unique up to scalar, otherwise $\omega(u, L_2) = 0$, which implies $u \in L_2$. \qed
Proposition 40. At the point $(F_1, F_2)$ with \( \dim(F_1(i) \cap F_2(i)) = \dim F_1(i) - 2 \), the relation (66) holds.

Proof. By lemma 39 the three fibres are the variety of \( X \) such that

\[
F_1(i) \cap F_2(i) \subseteq X \not\subseteq F_1(i)
\]

such that

\[
I(F_1) \subseteq X, \quad I(F_1) \subseteq X, \quad I(F_2) \subseteq X, \quad I(F_1) \subseteq Y; \quad I(F_2) \subseteq Y; \quad I(F_2) \subseteq Y;
\]

respectively, where \( Y = X \perp \cap F_2(i) \). Note that \( X \cap Y = F_1(i) \cap F_2(i) \). There are four cases.

- The case \( I(F_1 + F_2) \subseteq F_1(i) \cap F_2(i) \). Now, three fibres are the same.
- The case \( I(F_1) \not\subseteq F_1(i) \cap F_2(i) \supseteq I(F_2) \). The first fibre is empty. The second fibre is the point of the subspace \( X = I(F_1) + (F_1(j) \cap F_2(i)) \).

Actually, \( X \subseteq F_1(i) \) since

\[
\dim \frac{I(F_1) + (F_1(j) \cap F_2(i))}{F_1(j) \cap F_2(i)} = \dim \frac{I(F_1) + (F_1(j) \cap F_2(i))}{I(F_2) + (F_1(j) \cap F_2(i))} \leq 1.
\]

The last fibre is \( \mathbb{C}P^1 \) by Lemma 39.
- The case \( I(F_2) \not\subseteq F_1(i) \cap F_2(i) \supseteq I(F_1) \) is similar.
- The case \( I(F_2) \not\subseteq F_1(i) \cap F_2(i) \not\supseteq I(F_1) \). The fibres are all empty. The first and the last one are easy to show. For the second fibre, if it is not empty, then by the discussion above, the only choice is \( X = I(F_1) + (F_1(i) \cap F_2(j)) \) with \( Y = I(F_2) + (F_1(i) \cap F_2(j)) \). Note that \( X \) and \( Y \) do not have any inclusion relation between them, which contradicts to Lemma 33.

In particular, the relation (66) holds. \( \square \)

In all, we have the following proposition which finishes the proof of Theorem 26.

Proposition 41. The relation (55) holds.

Conflict of Interest. The authors declare no competing interests.

References

[1] Huanchen Bao, Jonathan Kujawa, Yiqiang Li, and Weiqiang Wang, Geometric Schur duality of classical type, Transformation Groups 23 (201806).
[2] A. Beilinson, G. Lusztig, and R. MacPherson, A geometric setting for the quantum deformation of \( \text{GL}_n \), Duke Mathematical Journal 61 (199010).
[3] N. Chriss and Victor Ginzburg, Representation theory and complex geometry (199701).
[4] Bernhard Keller and Sarah Scherotzke, Desingularizations of quiver Grassmannians via graded quiver varieties, Advances in Mathematics 256 (201305).
[5] Alexander Kirillov Jr, Quiver representations and quiver varieties, Vol. 174, American Mathematical Soc., 2016.
[6] Anthony Knapp, Lie groups beyond an introduction, second edition, Vol. 140, 2002.
[7] Gail Letzter, Symmetric pairs for quantized enveloping algebras, Journal of Algebra 220 (199910), 729–767.
[8] Yiqiang Li, Quiver varieties and symmetric pairs, Representation Theory of the American Mathematical Society 23 (201801).
[9] ________, Quasi-split symmetric pairs of \( U_q(\mathfrak{sl}_n) \) Steinberg varieties of classical type, Representation Theory of the American Mathematical Society 25 (202110), 903–934.
[10] Ming Lu and Weiqiang Wang, Hall algebras and quantum symmetric pairs I: Foundations, Proceedings of the London Mathematical Society 124, no. 1, 1–82.
[11] ______, *Hall algebras and quantum symmetric pairs III: Quiver varieties*, Advances in Mathematics 393 (202112), 108071.
[12] G. Lusztig, *Quivers, perverse sheaves, and quantized enveloping algebras*, J. Am. Math. Soc 4 (199104), 365–421.
[13] George Lusztig, *Quiver varieties and weyl group actions*, Annales de l’Institut Fourier 50 (200001).
[14] ______, *Remarks on quiver varieties*, Duke Mathematical Journal 105 (200011).
[15] Andrea Maffei, *A remark on quiver varieties and weyl groups*, arXiv, 2000.
[16] Davesh Maulik and Andrei Okounkov, *Quantum groups and quantum cohomology*, Astérisque 408 (201211).
[17] Hiraku Nakajima, *Instantons on ALE spaces, quiver varieties, and Kac–Moody algebras*, Duke Mathematical Journal 76 (199411).
[18] ______, *Quiver varieties and Kac–Moody algebras*, Duke Mathematical Journal 91 (199802).
[19] ______, *Lectures on Hilbert schemes of points on surfaces*, American Mathematical Society, 1999.
[20] ______, *Quiver varieties and finite dimensional representations of quantum affine algebras*, Journal of the American Mathematical Society 14 (200001).
[21] ______, *Reflection functors for quiver varieties and Weyl group actions*, Mathematische Annalen 327 (200008).
[22] Arkadij L Onishchik and Ernest B Vinberg, *Lie groups and algebraic groups*, Springer Science & Business Media, 2012.
[23] Weiqiang Wang, *Quantum symmetric pairs*, 2021.

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