Using the value of $\beta$ to help determine $\gamma$ from $B$ decays

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Abstract

It has been pointed out by Gronau and Rosner that the angle $\gamma$ of the unitarity triangle could be determined by combining future results on $B_s$ and $B_d$ decays to $K\pi$. Here we show that it is important to include in the analysis the information on the phase $\beta$ which will be determined in the near future. Omitting this information could lead to an error as large as $8^\circ$ in $\gamma$.

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A large number of experiments have been proposed to determine the phase $\gamma = \text{Arg}(V_{ub}^*)$ in the CKM matrix $[1–8]$. Before any of these experiments is completed it is likely that there will be a good measurement of $\sin 2\beta$. In many cases using the value of $\beta = \text{Arg}(V_{td}^*)$ derived from $\sin 2\beta$ can improve possible determinations of $\gamma$. We illustrate this for the case of a recent proposal by Gronau and Rosner $[9]$ to determine $\gamma$ by using U-spin symmetry (the exchange of $s$ and $d$ quarks) to relate the decays $B^0 \rightarrow K^+ \pi^-$ to $B_s \rightarrow K^- \pi^+$. Combining the rate of these decays with the rate for $B^+ \rightarrow K^0 \pi^+$ the value of $\gamma$ could be obtained. We assume throughout the constraints of the CKM model.

The tree amplitude for $B^0$ ($B_s$) decay is proportional to $V_{ub}^* V_{ud}$ ($V_{ub}^* V_{us}$). The penguin amplitude is dominated by the virtual $t$ quark and is proportional to $V_{tb}^* V_{ts}$ ($V_{tb}^* V_{td}$). Their approximation is to assume that the decay $B^+ \rightarrow K^0 \pi^+$ is purely penguin because only the penguin gives $b \rightarrow s \bar{d} d$. We then find for the decay amplitudes

\begin{align}
A(B^+ \rightarrow K^0 \pi^+) &= P, \quad (1a) \\
A(B^0 \rightarrow K^+ \pi^-) &= T e^{i(\delta + \gamma)} + P, \quad (1b) \\
A(B_s \rightarrow K^- \pi^+) &= \frac{1}{\lambda} T' e^{i(\delta' + \gamma)} - P' |V_{td}/V_{ts}| e^{-i\beta}, \quad (1c)
\end{align}

where $\lambda \equiv |V_{us}/V_{ud}| \simeq 0.226$. $|V_{td}/V_{ts}|$ is completely determined in terms of $\beta$, $\gamma$, and $\lambda$. The U-spin approximation is $P' = P$, $T' = T$, and $\delta' = \delta$.

In Ref. $[9]$ unitarity is used to set

$$V_{tb}^* V_{ts} = - (V_{cb}^* V_{ci} + V_{ub}^* V_{ui}), \quad (2)$$

for $i = d, s$. Thus part of what we have called the penguin is now in the $V_{ub}^* V_{ui}$ term and combined with the tree; therefore, they get

\begin{align}
A(B^+ \rightarrow K^0 \pi^+) &= \bar{P}, \quad (3a) \\
A(B^0 \rightarrow K^+ \pi^-) &= \bar{T} e^{i(\bar{\delta} + \gamma)} + \bar{P}, \quad (3b) \\
A(B_s \rightarrow K^- \pi^+) &= \frac{1}{\lambda} \bar{T}' e^{i(\bar{\delta}' + \gamma)} - \bar{\lambda} \bar{P}', \quad (3c)
\end{align}

where $\bar{\delta}$ and $\bar{\delta}'$ are in general different from $\delta$ and $\delta'$ in Eqs. (1) and the last term follows since $V_{cd}/V_{cs} = -\bar{\lambda}$. They thus obtain simple results independent of $\beta$. 

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FIG. 1. The dependence of $\gamma$ on $\beta$ for $R_d = 0.8$, $R_s = 0.78$ (solid line), $R_d = 0.85$, $R_s = 0.73$ (short dashed line), $R_d = 0.9$, $R_s = 0.68$ (dash-dot-dash line), $R_d = 0.95$, $R_s = 0.63$ (short-long dashed line), and $R_d = 1$, $R_s = 0.68$ (long dashed line), respectively. We assume $\cos\delta = 1$ and $r < 0$.

However, terms of $O(\tilde{\lambda}^2)$ and with dependence on both $\beta$ and $\gamma$ have been omitted from the first equation in (3). Since $\beta$ will be known when this analysis can be used there is no purpose in eliminating $\beta$. We instead use Eqs. (1) to determine $\gamma$ and the ratio $r \equiv P/T$ from the quantities $R_d$ and $R_s$ defined in Ref. [9] for any value of $\beta$. Typical results are shown in Figs. 1 and 2 where we fix the sum of $R_d$ and $R_s$ and consider the limiting case $\delta = \delta' = 0$. The results of Ref. [9] are reproduced in the limit $\beta = 0$.

It is seen that for values of $\gamma$ in the neighborhood of 50$^\circ$ and for $\beta = 30^\circ$ ($\sin 2\beta = 0.87$) the values of $\gamma$ is shifted from $\sim 45^\circ$ to $\sim 53^\circ$ from the $\beta = 0$ approximation. For values of $\gamma$ in the neighborhood of 130$^\circ$ and for $\beta = 20^\circ$ the shift is from $\sim 134^\circ$ to $\sim 127^\circ$. We assume $r < 0$ in accordance with the factorization assumption.

It is instructive to analyze the difference in the two approximations. The effective inter-

$R_d$ is the ratio of the sum of $B^0$ and $\bar{B}^0$ decays to that of $B^+$ and $B^-$ decays. $R_s$ is the ratio of the sum of $B_s$ and $\bar{B}_s$ decays to that of $B^+$ and $B^-$ decays.
FIG. 2. The dependence of $\gamma$ on $\beta$ for $R_d = 1.05$, $R_s = 0.53$ (long dashed line), $R_d = 1.1$, $R_s = 0.48$ (short dashed line), $R_d = 1.15$, $R_s = 0.43$ (dash-dot-dash line), $R_d = 1.2$, $R_s = 0.38$ (short-long dashed line), and $R_d = 1.25$, $R_s = 0.33$ (solid line), respectively. We assume $\cos \delta = 1$ and $r < 0$.

Action can be written as

$$H_{\text{eff}} = V_{tb}^* V_{tq} \sum_{i=3}^6 Q_i + V_{ub}^* V_{uq} \sum_{i=1}^2 Q_i^{(u)} + V_{cb}^* V_{cq} \sum_{i=1}^2 Q_i^{(c)},$$

(4)

where $q = d$ or $s$ and $Q_i$ are the standard operators including the Wilson coefficients. We use the approximation that annihilation diagrams can be neglected so that $B^+ \to K^0 \pi^+$ is due to the penguin operators $Q_3 \sim Q_6$. Thus, as assumed in deriving Eqs. (1) the terms $P (P')$ are proportional to $V_{tb}^* V_{ts}$ ($V_{tb}^* V_{td}$). In Ref. [9] unitarity is used to set

$$- V_{tb}^* V_{ts} = V_{cb}^* V_{cs} + V_{ub}^* V_{us},$$

(5)

and then the term proportional to $V_{ub}^* V_{us}$ is just omitted on the ground that it is smaller by a factor $\lambda^2$. In the limit that we neglect the strong phases we can include this term by replacing Eq. (3a) by

$$A(B^+ \to K^0 \pi^+) = \bar{P} \left[ 1 + \tilde{\lambda}^2 \frac{\sin \beta}{\sin (\beta + \gamma)} e^{i\gamma} \right].$$

(6)

Formally our results reduce to theirs in the limit $\beta = 0$. It is the amplification of this factor $\tilde{\lambda}^2$ that is responsible for the difference.
The equations of Ref. [9] for $R_d$ and $R_s$ become equations for $K R_d$ and $K R_s$, where

$$K = 1 + 2 \tilde{\lambda}^2 \frac{\sin \beta \cos \psi}{\sin (\beta + \gamma)} + \tilde{\lambda}^4 \left( \frac{\sin \beta}{\sin (\beta + \gamma)} \right)^2. \quad (7)$$

The same factor $K$ enters for $R_d$ and $R_s$ because both are defined as ratios to the $B^+$ decay. Then

$$K R_d = 1 + r^2 + 2r \cos \delta \cos \gamma,$$
$$K R_s = \tilde{\lambda}^2 + \left( \frac{r}{\tilde{\lambda}} \right)^2 - 2r \cos \delta \cos \gamma. \quad (8)$$

The amplification arises from the fact that $r \cos \gamma$ is proportional to $(K R_d - 1 - r^2)$. Thus, for example, with values of $R_d = 0.8$ and $R_s = 0.78$ (corresponding to $\gamma \sim 50^\circ$) a change of $K$ from 1 to 1.03 decreases $|r \cos \gamma|$ by about 10%. The value of $r^2$ is proportional to $[K (R_d + R_s) - (1 + \tilde{\lambda}^2)]$. For our example with $R_d + R_s = 1.58$ a change of $K$ from 1 to 1.03 increases $r$ by about 5%. Thus, a change of $K$ from 1 to 1.03 can decrease $\cos \gamma$ by about 15%.

Unfortunately, the difficulty of using this method arises from the same sensitivity; small errors on $R_d$ and $R_s$ can cause a significant error on the determined $\gamma$. As an example, let the experimental errors be

$$\frac{\Delta R_s}{R_s} = 2 \frac{\Delta R_d}{R_d} \equiv 2\epsilon. \quad (9)$$

For the case shown in Fig. 1 with $\beta = 30^\circ$ and $\gamma = 53^\circ$, a value of $\epsilon = 6\%$ corresponds to an uncertainty of about 24\% in $\cos \gamma$, yielding a value $\gamma = 53^\circ \pm 10^\circ$. For another case in Fig. 2 with $\beta = 18^\circ$ and $\gamma = 128^\circ$ and assuming instead $\Delta R_s/R_s = 4\Delta R_d/R_d \equiv 4\epsilon$, the same value of $\epsilon$ would correspond to an error of about 23\% in $\cos \gamma$ and $\gamma = 128^\circ \pm 10^\circ$.

The accuracy of this method requires including the strong phase $\delta$. In principle this can be determined by measuring the asymmetry between the rates for $B^0$ and $\bar{B}^0$, which is proportional to $\sin \gamma \sin \delta$. To a first approximation, the quantity that is determined in the method discussed here is $\cos \gamma \cos \delta$. Assuming $\delta$ is small probably only a limit on $\sin \gamma \sin \delta$ can be achieved. If $\cos^2 \gamma < 1/2$ and $\sin \gamma \sin \delta < X$, then the uncertainty in $\delta$ leads to an
error of no more than 0.35 $X^2$ in $\cos \gamma$. It should be emphasized that this method depends upon the assumption that the sign of $r$ is as given by factorization.

The approximation of neglecting contributions from $Q_1$ and $Q_2$ needs to be considered. The contribution of $Q_i^{(c)}$ can be included in $\bar{P}$ since in going from Eqs. (3a) to (3b) all that is required is that $\bar{P}$ corresponds to no change in isospin. As a result the only effect is a correction to the term proportional to $\tilde{\lambda}^2$ in Eq. (8). The contributions to $Q_1$ and $Q_2$ are long-distance effects due to rescattering which mixes processes of different topologies; calculations of these effects are very model dependent [10–12]. If we call $P_u(P_c)$ the amplitudes due to $Q_1^{(u)} + Q_2^{(u)}(Q_1^{(c)} + Q_2^{(c)})$ then the $\tilde{\lambda}^2$ terms in Eq. (8) must be multiplied by $1 + (P_u - P_c)/\bar{P}$. Ciuchini et. al. [11], who call $P_c$ the “charming penguin”, suggest that $P_c/\bar{P}$ could be of order unity and Falk et. al. [12] suggest that $P_u/\bar{P}$ could be large. However, a recent analysis by Kamal [13] suggests that $(P_u - P_c)/\bar{P}$ is probably of order 0.1. As pointed out in these papers, it should be possible in the future to limit the values of $P_u$ and $P_c$ by detecting decays where they would make a major contribution.

In conclusion we emphasize that in determining $\gamma$ from future experiments, optimum use should take into account the value of $\beta$ which will be measured via $\sin 2\beta$ in the near future. In the examples we have discussed of $B_d$ ($B_s$) decays to $K\pi$, the omission of the $\beta$ dependence could lead to an error as large as $8^\circ$ in special cases. In the longer run it would be valuable to determine the phase of the penguin amplitude and the phase $2\beta$ of the mixing independently so as to detect new physics contributions. Here we have limited the discussion to the standard CKM model.

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