E8 LATTICE AND THE KODAIRA DIMENSION OF ORTHOGONAL MODULAR VARIETIES

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ABSTRACT. We prove that for any even lattice $L$ of signature $(2, n_0)$, the modular variety defined by the orthogonal group of the lattice $L \oplus mE_8$ is of general type when $m$ is sufficiently large.

1. INTRODUCTION

The purpose of this article is to show that a certain series of modular varieties of orthogonal type tend to be of general type in higher dimension. Let $L_0$ be an even lattice of signature $(2, n_0)$. Consider the orthogonal sum $L_m = L_0 \oplus mE_8$ with $m$ copies of the $E_8$-lattice. To the lattice $L_m$ we can associate a Hermitian symmetric domain of type IV, say $D_{L_m}$, as either of the two connected components of the space

$$\{\mathbb{C}\omega \in \mathbb{F}(L_m \otimes \mathbb{C}) \mid (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0\}.$$ 

Let $O^+(L_m)$ be the group of isometries of $L_m$ which preserve the component $D_{L_m}$. The quotient space

$$\mathcal{F}(L_m) = O^+(L_m) \backslash D_{L_m}$$

is a quasi-projective variety of dimension $n_0 + 8m$. Our main result is the following.

**Theorem 1.1.** The modular variety $\mathcal{F}(L_m)$ is of general type for sufficiently large $m$.

The birational type of orthogonal modular varieties in higher dimension was first studied by Gritsenko-Hulek-Sankaran [8]. They proved general-type results as above for $L_0 = 2U$ and also for a natural covering of $\mathcal{F}(L_m)$ for $L_0 = 2U \oplus (-2d)$, with explicit bounds of $m$. Our study was much inspired by their work. In general, given the lattice $L_0$ explicitly in Theorem 1.1, it would be possible (though cumbersome) to calculate a bound of $m$ explicitly. We have summarized in §4.3 the ingredients of such a computation.

Let us show an example of Theorem 1.1 which actually was our original motivation.

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Example 1.2. Let $D_i$ be the root lattice of type $D_i$ for $1 \leq i \leq 8$, where $D_1 = (-4)$ by convention. Since $2U \oplus D_i \oplus 8m \cong 2U \oplus D_i \oplus mE_8$, it follows that $\mathcal{F}(2U \oplus D_k)$ is of general type for sufficiently large $k$. For $k \equiv 1 \pmod{8}$, this is essentially proved in [8].

Let $I_{2,n}$ be the odd unimodular lattice $2\langle 1 \rangle \oplus n\langle -1 \rangle$. The maximal even sublattice of $I_{2,n}$ is isometric to $2U \oplus D_{n-2}$. This induces a natural inclusion $O^*(I_{2,n}) \subset O^*(2U \oplus D_{n-2})$. Therefore $\mathcal{F}(I_{2,n})$ is of general type when $n$ is sufficiently large.

The rest of the article is devoted to the proof of Theorem 1.1, which is a generalization of the argument in [8]. The outline is as follows. We first reduce the lattice $L_0$ to a simpler form. Then we take a nice toroidal compactification of $\mathcal{F}(L_m)$ as in [6], say $X_m$. Its canonical divisor is $\mathbb{Q}$-linearly equivalent to

$$K_{X_m} \sim \mathbb{Q} (n_0 + 8m) \mathcal{L} - \frac{1}{2} B - \Delta,$$

where $\mathcal{L}$ is the $\mathbb{Q}$-line bundle of modular forms of weight 1, $B$ the branch divisor of $D_{L_m} \to \mathcal{F}(L_m)$, and $\Delta$ the boundary divisor of the compactification. Since $X_m$ has canonical singularity, it is sufficient to show that the right side of (1.1) is big. We will find a division

$$(n_0 + 8m) \mathcal{L} - B/2 - \Delta = ((k_0 + 4m) \mathcal{L} - \Delta) + ((l_0 + 4m) \mathcal{L} - B/2),$$

where $k_0 + l_0 = n_0$, such that some multiple of $(k_0 + 4m) \mathcal{L} - \Delta$ is effective and that $(l_0 + 4m) \mathcal{L} - B/2$ is big. The first property means the existence of a modular form of weight $\delta(k_0 + 4m)$ for some $\delta > 0$ which vanishes of order $\geq \delta$ along the boundary. We construct such a cusp form using the generalized Maass lifting by Gritsenko [5] and an operation of average product. Our key observation is roughly that the upper bound $k_0 + 4m$ of the “slope” of our cusp form grows slower than the weight $n_0 + 8m$ of the canonical divisor, so that we come to be left with sufficient weight $l_0 + 4m$ for the remaining divisor to be big as $m$ grows. We then prove that $(l_0 + 4m) \mathcal{L} - B/2$ is big by a comparison of the Hirzebruch-Mumford volume ([7]) of $\mathcal{F}(L_m)$ with that of the branch divisors. We analyze those volumes as functions of $m$, using the formula of [7] and a sort of regularity of the branch divisors with respect to $m$.

Notation. Throughout the article $E_8$ will stand for the negative-definite even unimodular lattice of rank 8. $U$ stands for the even unimodular lattice of signature $(1, 1)$. For an even lattice $L$, its dual lattice is denoted by $L^\vee$. By $A_L$ we denote the discriminant form $L' / L$, whose quadratic form $A_L \to \mathbb{Q} / 2\mathbb{Z}$ is given by $\lambda + L \mapsto (\lambda, \lambda) + 2\mathbb{Z}$. The length of $A_L$ as a finite abelian group is written as $l(A_L)$. $mL$ denotes the orthogonal sum $L^{\oplus m}$, while $L(n)$ denotes the scaling of $L$ by $n$. 


2. Preliminaries

2.1. Reduction of the lattice. Before launching, let us make a simple reduction. This will be helpful in several places.

**Reduction 2.1.** In order to prove Theorem 1.1 it is sufficient to prove the assertion for even lattices of the form $L_0 = 2U \oplus M$ where $M$ is negative-definite with $\text{rk}(M) \geq l(A_M) + 2$.

**Proof.** Let $L_0$ be the arbitrarily given lattice. We just see that $L_1 = L_0 \oplus E_8$ has the desired properties. Since $\text{rk}(L_1) \geq l(A_{L_1}) + 8$, we can find an even lattice $M$ with $L_1 \cong 2U \oplus M$ by [11] Corollary 1.13.5. We then have

$$\text{rk}(M) = \text{rk}(L_0) + 4 \geq l(A_{L_0}) + 4 = l(A_M) + 4.$$ 

From now on we will prove Theorem 1.1 for lattices of the above form.

2.2. Regularity of the branch divisors. Let $L_0$ be a lattice as in Reduction 2.1. We regard $L_0$ as a sublattice of $L_m = L_0 \oplus mE_8$ in the natural way. As a preparation for the proof of Theorem 1.1, we here show that the branch divisors of the projection $\pi: D_{L_m} \to \mathcal{F}(L_m)$ behave regularly with respect to $m$, in a sense. A vector $l \in L_m$ is called **reflective** if it is primitive, $(l, l) < 0$, and the reflection with respect to $l$ preserves $L_m$. By [6] Corollary 2.13, the ramification divisors of $\pi$ are precisely the hyperplane sections $l^\perp \cap D_{L_m}$ for reflective vectors $l \in L_m$.

**Lemma 2.2.** Any primitive vector $l \in L_m$ with $(l, l) \neq 0$ can be transformed by the action of $O^+(L_m)$ into $L_0 \subset L_m$.

**Proof.** Let $K = l^\perp \cap L_m$. The overlattice $L_m \supset \mathbb{Z}l \oplus K$ is obtained from the graph of an anti-isometry $G_1 \to G_2$ for some $G_1 \subset A_{\mathbb{Z}l}$ and $G_2 \subset A_K$ (see [11] Proposition 1.5.1). Since $G_1$ is cyclic, we have

$$l(A_K) \leq l(A_{\mathbb{Z}l \oplus K}) \leq l(A_{L_m}) + 2 \leq \text{rk}(L_m) - 4 - 8m,$$

and hence $\text{rk}(K) \geq l(A_K) + 3 + 8m$. By [11] Corollary 1.13.5, we have an isometry $\gamma: K' \oplus mE_8 \to K$ for some lattice $K'$. We put $L' = \gamma(mE_8) \perp L_m$. By the unimodularity of $E_8$ we have the splitting $L_m = L' \oplus \gamma(mE_8)$ with $l \in L'$. Since $L_0$ is unique in its genus by [11] Corollary 1.13.4, $L'$ is isometric to $L_0$. Then the component-wise isometry

$$L_m = L' \oplus \gamma(mE_8) \to L_0 \oplus mE_8 = L_m$$

of $L_m$ maps $l$ into $L_0$. We may arrange this isometry to be contained in $O^+(L_m)$, by using $-\text{id}_U$ of $U \subset L_0$ if necessary. \[\Box\]
Let
\[ l_1, \ldots, l_r \in L_0 \]
be representatives for the equivalence classes of reflective vectors in \( L_0 \) under the action of \( O^*(L_0) \). We set \( K_i = l_i^\perp \cap L_0 \) and
\[
K_{i,m} = l_i^\perp \cap L_m = K_i \oplus mE_8.
\]
Notice that we have \( \text{rk}(K_i) \geq l(A_{K_i}) + 3 \) as in the proof of the lemma, and so \( K_i \) contains \( U \) and is unique in its genus by \([11]\).

**Proposition 2.3.** The vectors \( l_1, \ldots, l_r \in L_0 \subset L_m \) are representatives for the equivalence classes of reflective vectors in \( L_m \) under the action of \( O^+(L_m) \).

**Proof.** Let \( l \) be any reflective vector of \( L_m \). By the lemma there exists \( \gamma \in O^+(L_m) \) such that \( \gamma(l) \in L_0 \subset L_m \). Then \( \gamma(l) \) is also reflective as a vector of \( L_0 \) and hence equivalent to some \( l_i \) under \( O^+(L_0) \). Thus \( l \) is equivalent to \( l_i \) under \( O^+(L_0) \).

Conversely, suppose we have \( \gamma \in O(L_m) \) with \( \gamma(l_i) = l_j \). Then we obtain the second splitting \( K_{j,m} = \gamma(K_i) \oplus \gamma(mE_8) \) of \( K_{j,m} \). In particular, \( K_i \simeq K_j \). By \([11]\) Corollary 1.9.6, we can find an isometry \( \gamma' : K_j \rightarrow \gamma(K_i) \) such that the isometry \( \gamma'' = \gamma' \oplus (\gamma|mE_8) \) of \( K_{j,m} \) acts trivially on \( A_{K_{j,m}} \). Then \( \gamma'' \) extends to an isometry of \( L_m \) fixing \( l_j \). The composition \( (\gamma'')^{-1} \circ \gamma \in O(L_m) \) maps \( l_i \) to \( l_j \) and preserves \( L_0 \). Thus \( l_i \) and \( l_j \) are \( O(L_0) \)-equivalent. As before, they are moreover \( O^+(L_0) \)-equivalent. \( \square \)

### 3. Comparison of the Hirzebruch-Mumford volumes

Let \( L_0 \) be an even lattice of signature \((2, n_0)\) as in Reduction 2.1. We study the Hirzebruch-Mumford volume of \( O^+(L_m) \) as a function of \( m \), and then compare its asymptotic behavior with that of \( O^+(K_{i,m}) \). Our conclusion in this section is Lemma 3.3, which will play a key role in §4.2.

**3.1. The Hirzebruch-Mumford volume.** Let \( L \) be a general even lattice of signature \((2, n)\). For a subgroup \( \Gamma \subset O^+(L) \) of finite-index, its Hirzebruch-Mumford volume \( \text{vol}_{HM}(\Gamma) \) was defined by Gritsenko-Hulek-Sankaran \([7]\) following the proportionality principle of Hirzebruch and Mumford. Let \( M_k(\Gamma) \) be the space of modular forms of weight \( k \in \mathbb{N} \) with respect to \( \Gamma \). Then \( \text{vol}_{HM}(\Gamma) \) appears in the leading term of the Hilbert polynomial of \( M_k(\Gamma) \) as
\[
\dim M_k(\Gamma) = \frac{2}{n!} \text{vol}_{HM}(\Gamma)k^n + O(k^{n-1}),
\]
where we restrict to even \( k \) if \(-1 \in \Gamma\). Although this is not the original definition of \( \text{vol}_{HM}(\Gamma) \), we may take it as like a definition in this article.
Gritsenko-Hulek-Sankaran calculated $\text{vol}_{HM}(O^+(L))$ out of various volume formulae concerning $O(L)$. When the lattice $L$ contains $U$, they obtained in [7] §3 that

\begin{equation}
\text{vol}_{HM}(O^+(L)) = 4 \cdot |A_L|^{(\alpha+3)/2} \cdot \prod_{k=1}^{n+2} \pi^{-k/2} \Gamma(k/2) \cdot \prod_p \alpha_p(L)^{-1},
\end{equation}

where $\alpha_p(L)$ is the local density of the $\mathbb{Z}_p$-lattice $L \otimes \mathbb{Z}_p$ that is also denoted as $\alpha_p(L, L)$ in some literature.

We refer to [9] §5.6 (as in [7]) for the following formula of $\alpha_p(L)$. Let $L \otimes \mathbb{Z}_p = \oplus \mathbb{Z}_p N_{p,j}(p^l)$ be a Jordan decomposition where $N_{p,j}$ is unimodular of rank $n_{p,j} \geq 0$. Let $s_p$ be the number of indices $j$ with $N_{p,j} \neq 0$, and set

$$w_p = \sum_j j n_{p,j}\left(\frac{n_{p,j}+1}{2} + \sum_{k>j} n_{p,k}\right).$$

For an even unimodular $\mathbb{Z}_p$-lattice $N$ of rank $r \geq 0$, we define $\chi(N)$ by $\chi(N) = 0$ if $r$ is odd, $\chi(N) = 1$ if $N \simeq (r/2)U \otimes \mathbb{Z}_p$, and $\chi(N) = -1$ otherwise. Moreover, for a natural number $l$ we put

$$P_p(l) = \prod_{k=1}^{l} (1 - p^{-2k}),$$

and $P_p(0) = 1$. Then for $p \neq 2$, we have

$$\alpha_p(L) = 2^{w_p-1} \cdot P_p^{w_p} \cdot \prod_j P_p([n_{p,j}/2]) \cdot \prod_j (1 + \chi(N_{p,j})p^{-n_{p,j}/2})^{-1},$$

where $j$ ranges over indices with $N_{p,j} \neq 0$.

The 2-adic density $\alpha_2(L)$ is much more complicated. Consider a decomposition $N_{2,j} = N_{2,j}^+ \oplus N_{2,j}^-$ such that $N_{2,j}^+$ is even and $N_{2,j}^-$ is either 0 or odd of rank $\leq 2$. Put $n_{2,j}^\pm = \text{rk}(N_{2,j}^\pm)$. We also set $q = \sum_j q_j$, where $q_j = 0$ if $N_{2,j}$ is even, $q_j = n_{2,j}$ if $N_{2,j}$ is odd and $N_{2,j+1}$ is even, and $q_j = n_{2,j} + 1$ if both $N_{2,j}$ and $N_{2,j+1}$ are odd. Here zero-lattice is counted as an even lattice. For those $j$ with $N_{2,j} \neq 0$, we define $E_{2,j}(L)$ by $E_{2,j}(L) = 1 + \chi(N_{2,j}^+)2^{-n_{2,j}^+/2}$ if both $N_{2,j-1}$ and $N_{2,j+1}$ are even and $N_{2,j}^+ \neq \langle e_1, e_2 \rangle$ with $e_1 \equiv e_2 \text{ mod } 4$, and $E_{2,j}(L) = 1$ otherwise. We also let $s_j^*$ be the number of indices $j$ such that $N_{2,j} = 0$ and either $N_{2,j-1}$ or $N_{2,j+1}$ is odd. Then we have

$$\alpha_2(L) = 2^{n_{2,j}+w_2-q+s_2+s_2^*} \cdot \prod_j P_2(n_{2,j}^*/2) \cdot \prod_j E_{2,j}(L)^{-1},$$

where $j$ runs over indices with $N_{2,j} \neq 0$. 
3.2. Dependence on \( m \).

Now we substitute the lattice \( L_m = L_0 \otimes mE_8 \) into the formula (3.2) and express \( \text{vol}_{HM}(O^+(L_m)) \) as a function of \( m \). The final form will be presented in Lemma [3.1].

Our calculation, which is a generalization of the examples in [7] §3, is built upon the following observations.

- Since \( E_8 \) is unimodular, the discriminant form \( A_{L_m} \) does not change under \( m \).
- Since \( E_8 \otimes \mathbb{Z}_p \cong 4U \otimes \mathbb{Z}_p \) at each \( p \), we have \( L_m \otimes \mathbb{Z}_p \cong (L_0 \otimes \mathbb{Z}_p) \oplus 4m(U \otimes \mathbb{Z}_p) \).
- The lattices \( N_{p,j} \) for \( j > 0 \) do not change under \( m \), and \( N_{p,0} \) is added by \( 4mU \otimes \mathbb{Z}_p \).
- For \( p = 2 \), the unimodular component \( N_{2,0} \) is always even. Since the lattices \( N_{2,j} \) for \( j > 0 \) do not change under \( m \), the numbers \( q, s'_2, n^2_j \) for \( j > 0 \), and \( E_{2,j} \) for \( j > 0 \) are independent of \( m \) too.

Below let us rewrite the unimodular component of \( L_0 \otimes \mathbb{Z}_p \) as \( N_p \), which is non-zero because \( L_0 \) contains \( U \). We put \( n_p = \text{rk}(N_p) \). Denote by \( d \) the discriminant of \( L_0 \), whose absolute value is \( |A_{L_0}| \).

Putting the above observations together, we obtain from (3.2) the following tentative form:

\[
\text{vol}_{HM}(O^+(L_m)) = C \cdot |d|^{4m} \cdot 2^{-8m} \cdot \prod_{k=1}^{n_0+2+8m} \pi^{-k/2} \Gamma(k/2) \\
\times \prod_p P_p([n_p/2] + 4m)^{-1} \\
\times \prod_{p>2} (1 + \chi(N_p)p^{-n_p/2-4m}) \cdot E_{2,0}(L_m).
\]

(3.3)

Here \( C \) is some constant that does not depend on \( m \). We are going to simplify this expression.

We first rewrite the second line \( \prod_p P_p^{-1} \). When \( p \nmid d \), \( L_0 \otimes \mathbb{Z}_p \) is unimodular and in particular \( n_p = n_0 + 2 \). As a correction term for \( p|d \) we consider the finite product

\[
F_m(L_0) = \prod_p (\prod_k (1 - p^{-2k})),
\]

(3.4)

where \( p \) runs over primes with \( p|d \) and \([n_p/2] \leq [n_0/2]\), and the range of \( k \) is \([n_p/2] + 4m + 1 \leq k \leq [n_0/2] + 4m + 1 \). We then have

\[
\prod_p P_p([n_p/2] + 4m)^{-1} = F_m(L_0) \cdot \prod_p P_p([n_0/2] + 4m + 1)^{-1} \\
= F_m(L_0) \cdot \prod_{k=1}^{[n_0/2]+4m+1} \zeta(2k),
\]

where \( \zeta(s) = \prod_p (1 - p^{-s})^{-1} \) is the Riemann zeta function.
Next we rewrite the third line

\[(3.5) \prod_{p \geq 2} \left(1 + \chi(N_p)p^{-n_p/2-4m}\right) \cdot E_{2,0}(L_m)\]

according to the parity of \(n_0\).

(A) Let \(n_0\) be odd. When \(p\) is odd with \(p \nmid d\), the unimodular lattice \(N_p = L_0 \otimes \mathbb{Z}_p\) has odd rank so that \(\chi(N_p) = 0\). Therefore (3.5) reduces to the finite product

\[(3.6) G_m(L_0) = \prod_{p \mid d, p > 2} \left(1 + \chi(N_p)p^{-n_p/2-4m}\right) \cdot E_{2,0}(L_m).\]

Notice that \(d\) must be even whenever \(n_0\) is odd.

(B) Let \(n_0\) be even. When \(p\) is odd with \(p \nmid d\), the unimodular \(\mathbb{Z}_p\)-lattice \(N_p = L_0 \otimes \mathbb{Z}_p\) is isometric to \((n_0/2 + 1)U \otimes \mathbb{Z}_p\) if and only if they have the same discriminant, namely \(d \equiv (-1)^{n_0/2+1} \mod \mathbb{Z}_p^\times(\mathbb{Z}_p^\times)^2\). If we put

\[d' = (-1)^{n_0/2+1}d,\]

this is equivalent to \([d'] \in \mathbb{F}_p^\times\) being square. Hence \(\chi(N_p)\) is given by the Legendre symbol \(\left(\frac{d'}{p}\right)\). Similarly, if \(p = 2\) with \(2 \nmid d\), \(\chi(N_2)\) is equal to the Kronecker symbol \(\left(\frac{d}{2}\right)\).

Since \(d' \equiv 0, 1 \mod 4\), we can factorize \(d'\) as

\[(3.7) d' = \ell^2D\]

with \(D\) a fundamental discriminant. If we denote by \(\chi_D\) the Kronecker symbol \(\left(\frac{D}{\ell}\right)\), which is the quadratic character for the field \(\mathbb{Q}(\sqrt{D})\), we thus obtain

\[\chi(N_p) = \chi_D(p) \quad \text{when} \quad p \nmid d.\]

We also notice that when \(d\) is odd, \(E_{2,0}(L_m)\) is given by \(1 + \chi_D(2)2^{-n_0/2-1-4m}\). Therefore, if we consider the finite product

\[H'_m(L_0) = \prod_{p \mid d, p > 2} \left(1 + \chi(N_p)p^{-n_p/2-4m}\right) \cdot \begin{cases} 1 & \text{d : odd,} \\ E_{2,0}(L_m) & \text{d : even,} \end{cases}\]

the expression (3.5) is equal to

\[H'_m(L_0) \cdot \prod_{p \mid d} \left(1 + \chi_D(p)p^{-n_0/2-1-4m}\right).\]

Since \(\chi_D(p) = \pm 1\) for \(p \nmid d\), this can be written as

\[H'_m(L_0) \cdot \prod_{p \mid d} \frac{1 - p^{-n_0-2-8m}}{1 - \chi_D(p)p^{-n_0/2-1-4m}}.\]
If we put

(3.8) \[ H_m(L_0) = H'_m(L_0) \cdot \prod_{p \mid d} \frac{1 - \chi_D(p)p^{-n_0/2-1-4m}}{1 - p^{-n_0-2-8m}} \]

and consider the Dirichlet \( L \)-function \( L(s, \chi_D) = \prod \rho(1 - \chi_D(p)p^{-s})^{-1} \), we can then rewrite (3.5) as

\[ H_m(L_0) \cdot \zeta(n_0 + 2 + 8m)^{-1} \cdot L(n_0/2 + 1 + 4m, \chi_D). \]

Now we combine the above calculations and use Euler’s formula

\[ \zeta(2k) \cdot \Gamma(k) \cdot \pi^{-k} \cdot \Gamma(k + 1/2) \cdot \pi^{-k-1/2} = \frac{|B_{2k}|}{2k}, \]

where \( B_{2k} \) is the Bernoulli number. This simplifies (3.3) to the following form.

**Lemma 3.1.** Let \( F_m(L_0), G_m(L_0) \) and \( H_m(L_0) \) be the finite products defined in (3.4), (3.6) and (3.8) respectively. Let \( \chi_D \) be the quadratic character associated to \( \mathbb{Q}(\sqrt{D}) \) where \( D \) is as defined in (3.7). Then we can express \( \text{vol}_{HM}(O^+(L_m)) \) as follows.

(A) When \( n_0 \) is odd,

\[ \text{vol}_{HM}(O^+(L_m)) = C \cdot F_m(L_0) \cdot G_m(L_0) \cdot |d/4|^{4m} \cdot \prod_{k=1}^{(n_0+1)/2+4m} \frac{|B_{2k}|}{2k}. \]

(B) When \( n_0 \) is even,

\[ \text{vol}_{HM}(O^+(L_m)) = C \cdot F_m(L_0) \cdot H_m(L_0) \cdot \left| \frac{d}{4\pi} \right|^{4m} \cdot \prod_{k=1}^{n_0/2+4m} \frac{|B_{2k}|}{2k} \times (n_0/2 + 4m)! \cdot L(n_0/2 + 1 + 4m, \chi_D). \]

Here \( C \) denote some constants that do not depend on \( m \).

**Remark 3.2.** It is not difficult to trace back the way to see an explicit form of the constants \( C \) in the lemma. For those \( p \) dividing \( d \), we put

\[
C_p(L_0) = \begin{cases} 
2^{1-\epsilon_p} p^{-w_p} \prod_{p < 0} P_p([n_{p,j}/2])^{-1} \cdot \prod_{p > 0} (1 + \chi(N_{p,j})) p^{-n_{p,j}/2} & \text{if } p > 2, \\
2^{1-\epsilon_2} \epsilon_2^{-w_2+q} \prod_{p < 0} P_2([n_{2,j}/2])^{-1} \cdot \prod_{p > 0} E_2,j(L_0) & \text{if } p = 2.
\end{cases}
\]

Then we have

\[
C = \begin{cases} 
8 \cdot \frac{|d/4|^{(n_0+3)/2}}{|4\sqrt{\pi}|} \cdot \prod_{p \mid d} C_p(L_0) & n_0: \text{odd}, \\
8 \sqrt{\pi} \cdot \frac{|d/4\pi|^{(n_0+3)/2}}{|d/4\pi|^{(n_0+3)/2}} \cdot \prod_{p \mid d} C_p(L_0) & n_0: \text{even}.
\end{cases}
\]

We do not need this information in the proof of Theorem 1.1.

Note that in the calculation we used only the fact that \( L_0 \) contains \( U \). So Lemma 3.1 and Remark 3.2 actually hold for any such \( L_0 \).
3.3. **Comparison with \( K_{i,m} \).** We compare the Hirzebruch-Mumford volume of \( O^+(L_m) \) with that of \( O^+(K_{i,m}) \). See (2.1) for the definition of \( K_{i,m} \), but actually we need only the inequality \( \text{rk}(K_i) < \text{rk}(L_0) \). A formula for \( \text{vol}_{HM}(O^+(K_{i,m})) \) can be obtained by replacing \( L_0 \) with \( K_i \), \( n_0 \) with \( n_0 - 1 \), and \( d \) with the discriminant \( d_i \) of \( K_i \) in Lemma 3.1.

**Lemma 3.3.** For each \( 1 \leq i \leq r \), the ratio

\[
\frac{\text{vol}_{HM}(O^+(K_{i,m}))}{\text{vol}_{HM}(O^+(L_m))}
\]

converges to 0 as \( m \to \infty \).

**Proof.** Below \( C \) stand for some constants that are independent of \( m \). We first consider the case when \( n_0 \) is even. By Lemma 3.1, the ratio (3.9) equals

\[
C \cdot \left| \frac{d}{d} \right|^{4m} \cdot \frac{F_m(K_i)G_m(K_i)}{F_m(L_0)H_m(L_0)} \cdot (n_0/2 + 4m)!^{-1} \cdot L(n_0/2 + 4m + 1, \chi_D)^{-1}.
\]

It is clear that \( F_m(K_i), G_m(K_i), F_m(L_0) \) and \( H_m(L_0) \) converge to 1 as \( m \to \infty \). We also have \( \lim_{n \to \infty} L(n, \chi_D) = 1 \). Then (3.9) converges to 0 by Stirling’s formula

\[
n! \sim \sqrt{2\pi n}^{n+1/2}e^{-n}.
\]

Next we consider the case when \( n_0 \) is odd. In this case, abbreviating \( n = (n_0 + 1)/2 + 4m \), the ratio (3.9) is written as

\[
C \cdot \left| \frac{d}{d} \right|^{4m} \cdot \frac{F_m(K_i)H_m(K_i)}{F_m(L_0)G_m(L_0)} \cdot (n - 1)! \cdot L(n, \chi_D) \cdot \frac{2n}{|B_{2n}|},
\]

where \( D_i \) is the fundamental discriminant for \( (-1)^{(n_0+1)/2}d_i \). As before, the terms \( F_m(\cdot), G_m(\cdot), H_m(\cdot) \) and \( L(n, \chi_D) \) converge to 1. The remaining term is of the form

\[
C \cdot |d_i/\pi d|^{n} \cdot n! \cdot |B_{2n}|^{-1}.
\]

This converges to 0 in \( n \to \infty \) because of the asymptotic behavior (cf. [4])

\[
|B_{2n}| \sim 2(2\pi)^{-n}(2n)!.
\]

\( \square \)

4. **Proof of the theorem**

In this section we assume throughout that \( L_0 = 2U \oplus M \) is an even lattice of signature \((2, n_0)\) as in Reduction 2.1. We are going to prove Theorem 1.1 for such a lattice.

By [6] Theorem 2.1, when \( n_0 + 8m \geq 9 \), we can take a projective toroidal compactification of \( \mathcal{F}(L_m) \) that has only canonical quotient singularities and that has no branch divisor in the boundary. Moreover, the branch divisors
are defined by reflective vectors in $L_m$. We shall fix one such compactification and denote it by $X_m$. Let $\Delta \subset X_m$ be the boundary divisor and $B \subset X_m$ the branch divisor. Let $\mathcal{L}$ be the ($\mathbb{Q}$-)line bundle over $X_m$ of modular forms of weight 1. Then over the regular locus $(X_m)_{\text{reg}}$ the canonical divisor $K(X_m)_{\text{reg}}$ is $\mathbb{Q}$-linearly equivalent to the $\mathbb{Q}$-Cartier divisor

\[(n_0 + 8m)\mathcal{L} - B/2 - \Delta\]  

(see, e.g., [6] §1). Since $X_m$ has canonical singularity, in order to show that (a desingularization of) $X_m$ is of general type it is sufficient to prove that the $\mathbb{Q}$-Cartier divisor (4.1) of $X_m$ is big. In §4.1 we construct for each $m$ a modular form of weight $\delta(k_0 + 4m)$ with respect to $O^+(L_m)$ which vanishes of order $\geq \delta$ along $\Delta$, where $\delta$ and $k_0$ are some natural numbers independent of $m$. If we set $l_0 = n_0 - k_0$, then (4.1) is divided as

\[((k_0 + 4m)\mathcal{L} - \Delta) + ((l_0 + 4m)\mathcal{L} - B/2)\]

such that $\delta((k_0 + 4m)\mathcal{L} - \Delta)$ is effective. Hence Theorem 1.1 follows if we could show that the remaining divisor $(l_0 + 4m)\mathcal{L} - B/2$ is big when $m$ is sufficiently large. We do this in §4.2, of which one key point has been prepared in Lemma 3.3. After finishing the proof of Theorem 1.1 we supplement in §4.3 a few words on the calculation of an explicit range of $m$ where $\mathcal{F}(L_m)$ is of general type.

4.1. **Construction of cusp form.** We will construct a modular form with respect to $O^+(L_m)$ (Lemma 4.1). As the first step, for the lattice $L_0 = 2U \oplus M$ we choose an even overlattice $M'$ of $M$ that is maximal. Then the lattice

\[L'_m = 2U \oplus M' \oplus mE_8\]

is a maximal even overlattice of $L_m$. Let $\widetilde{O}^+(L_m)$, $\widetilde{O}^+(L'_m)$ be the subgroups of $O^+(L_m)$, $O^+(L'_m)$ that act trivially on the discriminant groups $A_{L_m}$, $A_{L'_m}$ respectively. Since any element of $\widetilde{O}^+(L_m)$ preserves the overlattice $L'_m$ with trivial action on $A_{L'_m}$, we have a natural inclusion

\[\widetilde{O}^+(L_m) \subset \widetilde{O}^+(L'_m)\]  

(4.2)

This inclusion is compatible with the canonical identification $\mathcal{D}_{L_m} = \mathcal{D}_{L'_m}$ of the symmetric domains. We first construct a cusp form with respect to $\widetilde{O}^+(L'_m)$ using the Jacobi lifting by Gritsenko [5], and then produce the desired modular form with respect to $O^+(L_m)$ by some general constructions.

We shall begin with recollection of Jacobi forms following [5]. Let

\[K = M'(-1),\]

which is a maximal even positive-definite lattice of rank $n_0 - 2$. A Jacobi form of weight $k \in \mathbb{N}$ and index 1 for $K$ is a holomorphic function $\phi(\tau, Z)$
on $\mathbb{H} \times (K \otimes \mathbb{C})$ which satisfies the transformation laws

$$\phi\left(\gamma \tau, \frac{Z}{c \tau + d}\right) = (c \tau + d)^k \exp\left(\frac{\pi i c(Z, Z)}{c \tau + d}\right) \phi(\tau, Z), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}),$$

$$\phi(\tau, Z + l \tau + m) = q^{-(l(l)/2)} \zeta^{-l} \phi(\tau, Z), \quad l, m \in K,$$

where $q = e^{2\pi i c}$ and $\zeta^l = e^{2\pi i (lZ)}$ for $l \in K$, and which has a Fourier expansion of the form

$$\phi(\tau, Z) = \sum_{n \in \mathbb{N}, \lambda \in K'} c(n, \lambda) q^n \zeta^l,$$

where $c(n, \lambda) \neq 0$ only when $(l, l) \leq 2n$. If $c(n, \lambda) = 0$ for any $(n, \lambda)$ with $(l, l) = 2n$, $\phi$ is called a cusp form. We denote by $J_{k,1}(K)$ the space of Jacobi forms of weight $k$ and index 1 for $K$. For $\lambda \in A_K$ consider the theta function

$$\theta^1_k(\tau, Z) = \sum_{l \in K + \lambda} q^{(l(l)/2)} \zeta^l.$$

Then a Jacobi form $\phi \in J_{k,1}(K)$ can be uniquely expanded as

$$\phi(\tau, Z) = \sum_{\lambda \in A_k} \phi(\lambda) \theta^1_k(\tau, Z)$$

for some $\mathbb{C}[A_k]$-valued holomorphic function $\Phi(\tau) = (\phi(\lambda))_{\lambda \in A_k}$ on $\mathbb{H}$. Let $\text{Mp}_2(\mathbb{Z})$ be the metaplectic double cover of $\text{SL}_2(\mathbb{Z})$ and

$$\rho_K : \text{Mp}_2(\mathbb{Z}) \rightarrow \text{U}(\mathbb{C}[A_k])$$

be the Weil representation attached to $K$, for which we follow the same convention as [2]. Comparing the transformation rule of $\phi(\tau, Z)$ under $\text{SL}_2(\mathbb{Z})$ with that of $(\theta^1_k)_{\lambda \in A_k}$ under $\text{Mp}_2(\mathbb{Z})$, we see that $\Phi(\tau)$ is a modular form of weight $k - (n_0 - 2)/2$ and type $\rho^\vee_k$ for $\text{Mp}_2(\mathbb{Z})$. Furthermore, $\phi$ is a Jacobi cusp form if and only if $\Phi$ is a cusp form. Denote by $M_{K'}(\rho^\vee_K)$ for $K' \in \frac{1}{2}\mathbb{Z}$ the space of modular forms of weight $k'$ and type $\rho^\vee_K$. Then this correspondence establishes the isomorphism

$$J_{k,1}(K) \rightarrow M_{k+1-n_0/2}(\rho^\vee_K), \quad \phi = \sum_{\lambda} \phi(\lambda) \theta^1_k(\tau, Z) = \Phi = (\phi(\lambda)),$$

which preserves the cusp forms. Notice that since $\rho^\vee_K = \rho_{M'}$ for $M' = K(-1)$, we may write $M_{K'}(\rho_{M'})$ in place of $M_{K'}(\rho^\vee_K)$.

Now we replace $K$ with $K \oplus mE_8(-1)$. Since $K \oplus mE_8(-1)$ has the same discriminant form as $K$, from (4.3) we obtain for each $m$ an isomorphism

$$M_{k+1-n_0/2}(\rho^\vee_K) \rightarrow J_{k+4m,1}(K \oplus mE_8(-1))$$

preserving the cusp forms. Note that the source space is independent of $m$. A dimension formula for $M_{K'}(\rho^\vee_K)$ is given in [2] p. 228 (see also [3] §2 in
case \( k' \equiv 1 - n_0/2 \mod 2 \). Looking it with the fact that the subspace of cusp forms in \( M_{k'}(\rho_K^{\vee}) \) has codimension at most
\[
\#(\lambda \in A_K(\lambda, \lambda) \in 2\mathbb{Z})/ \pm 1,
\]
we can find a cusp form \( \Phi_0 \) of weight \( k_0 + 1 - n_0/2 \) and type \( \rho_K^{\vee} \) for some natural number \( k_0 \). Then for each \( m \) we obtain a Jacobi cusp form \( \phi_m \) of weight \( k_0 + 4m \) and index 1 for \( K \oplus mE_8(-1) \), as the image of \( \Phi_0 \) by the isomorphism (4.4).

Next, using the generalized Maass lifting ([5] Theorem 3.1), from the Jacobi cusp form \( \phi_m \) we obtain a modular form \( f_m \) of weight \( k_0 + 4m \) with respect to \( \mathcal{O}^+(L_m) \). Since \( K \oplus mE_8(-1) \) is maximal, \( f_m \) is actually a cusp form. By the inclusion (4.2), we may regard \( f_m \) as a cusp form with respect to \( \mathcal{O}^+(L_m) \) of the same weight.

To obtain a modular form with respect to \( \mathcal{O}^+(L_m) \), we choose representatives \( \gamma_1, \cdots, \gamma_\delta \in \mathcal{O}^+(L_m) \) of the quotient group \( \mathcal{O}^+(L_m)/\mathcal{O}^+(L_m) \) where
\[
\delta := [\mathcal{O}^+(L_m) : \mathcal{O}^+(L_m)] = |\mathcal{O}(A_{L_m})|,
\]
the last equality being a consequence of the surjectivity of \( \mathcal{O}^+(L_m) \to \mathcal{O}(A_{L_m}) \) by [11]. We can consider the pullback \( f_m|_{\gamma_i} \) of the modular form \( f_m \) by \( \gamma_i \) as usual. It depends only on the class of \( \gamma_i \) modulo \( \mathcal{O}^+(L_m) \), and is a cusp form with respect to \( \gamma_i^{-1}\mathcal{O}^+(L_m)\gamma_i = \mathcal{O}^+(L_m) \). Then take the product
\[
(4.5)
F_m = \prod_{i=1}^{\delta} (f_m|_{\gamma_i}).
\]
This is a non-zero modular form of weight \( \delta k_0 + 4m \) with respect to \( \mathcal{O}^+(L_m) \). Moreover, since each \( f_m|_{\gamma_i} \) is a cusp form, \( F_m \) vanishes of order \( \geq \delta \) along each component of the boundary divisor \( \Delta \) (cf. Theorem 1.1 in [1] Chapter IV). Summing up, we have obtained

**Lemma 4.1.** Let \( \delta = |\mathcal{O}(A_{L_m})| \). We have a natural number \( k_0 \) such that for each \( m \) there exists a modular form of weight \( \delta k_0 + 4m \) with respect to \( \mathcal{O}^+(L_m) \) which vanishes of order \( \geq \delta \) along the boundary.

The construction (4.5), giving equality of “slopes” of cusp forms between \( \mathcal{F}(L_m) \) and \( \mathcal{O}^+(L_m)\mathcal{D}_{L_m} \), might be also useful for some similar problems of Kodaira dimension.

4.2. **Completion of the proof.** By Lemma 4.1, the proof of Theorem 1.1 is reduced to showing that the \( \mathbb{Q} \)-divisor \( (l_0 + 4m)\mathcal{L} - B/2 \) of \( X_m \) is big, where \( l_0 = n_0 - k_0 \). Let \( B = \sum_{i=1}^{r} B_i \) be the irreducible decomposition such that \( B_i \) is defined by the reflective vector \( l_i \in L_m \) as in Proposition 2.3. Then the hyperplane section \( \mathcal{D}_{L_m} = l_i^+ \cap \mathcal{D}_{L_m} \) is one of the ramification divisors over \( B_i \). Let \( \Gamma_{i,m} \subset \mathcal{O}^+(L_m) \) be the stabilizer of \( l_i \), and \( \Gamma_{i,m} \subset \mathcal{O}^+(K_{i,m}) \) be
its natural image. For \( k, j \geq 0 \), the space \( H^0(k\mathcal{L} - jB/2) \) is identified with
the subspace of \( M_k(O^+(L_m)) \) consisting of modular forms of weight \( k \) that
vanish of order \( \geq j \) along each \( \mathcal{D}_{i,m}, 1 \leq i \leq r \). Notice that for \( k \) even, any
modular form in \( M_k(O^+(L_m)) \) must have zero of even order (\( \geq 0 \)) along \( \mathcal{D}_{i,m} \)
(cf. [8]). Indeed, consider the quasi-pullback
\[
H^0(k\mathcal{L} - jB/2) \rightarrow M_{k+j}(\Gamma_{i,m}), \quad F \mapsto (F/(l_i, \cdot)^j)_{|\mathcal{D}_{i,m}}.
\]
Its kernel is \( H^0(k\mathcal{L} - (j+1)B/2) \), and we have \( M_{k+j}(\Gamma_{i,m}) = 0 \) when \( k + j \) is
odd because \(-1 \in \Gamma_{i,m}\).

Now what we want to show is that when \( m \) is large enough (and fixed), we have the growth estimate
\[(4.6) \quad h^0(k(l_0 + 4m)\mathcal{L} - k/2B) = O(k^{n_0+8m})
\]
with respect to even \( k \). We assume first of all that \( l_0 + 4m > 0 \). By iteration
of quasi-pullback for \( j = 0, 2, 4, \ldots, k-2 \) as in [8] Proposition 4.1 (see also
[10] §9), we obtain the estimate
\[
h^0(k(l_0 + 4m)\mathcal{L} - k/2B) \geq \dim M_{k(l_0+4m)}(O^+(L_m)) - \sum_{i=1}^r \sum_{j=0}^r \dim M_{k(l_0+4m)+2j}(\Gamma_{i,m}).
\]
By the property (3.1) of the Hirzebruch-Mumford volume, the first term has the asymptotic estimate
\[
\dim M_{k(l_0+4m)}(O^+(L_m)) = \frac{2 \cdot \text{vol}_{HM}(O^+(L_m))}{(n_0 + 8m)!} \cdot (l_0 + 4m)^{n_0+8m} \cdot k^{n_0+8m} + O(k^{n_0+8m-1}).
\]

On the other hand, for each \( 1 \leq i \leq r \), the second term is estimated as
\[
\sum_{j=0}^{k/2-1} \dim M_{k(l_0+4m)+2j}(\Gamma_{i,m})
\leq \sum_{j=0}^{k/2-1} \left\{ \frac{2 \cdot \text{vol}_{HM}(\Gamma_{i,m})}{(n_0 + 8m - 1)!} \cdot (l_0 + 4m + 1)^{n_0+8m-1} \cdot k^{n_0+8m-1} + O(k^{n_0+8m-2}) \right\}
\leq \sum_{j=0}^{k/2-1} \left\{ \frac{2 \cdot \text{vol}_{HM}(\Gamma_{i,m})}{(n_0 + 8m - 1)!} \cdot (l_0 + 4m + 1)^{n_0+8m-1} \cdot k^{n_0+8m-1} + O(k^{n_0+8m-2}) \right\}
= \frac{\text{vol}_{HM}(\Gamma_{i,m})}{(n_0 + 8m - 1)!} \cdot (l_0 + 4m + 1)^{n_0+8m-1} \cdot k^{n_0+8m} + O(k^{n_0+8m-1}).
\]

Then compare these two asymptotic estimates. Notice that \( r \) is independent
of \( m \) by Proposition 2.3. Hence the property (4.6) is satisfied if we can show
that for each $1 \leq i \leq r$, the ratio
\[
\frac{n_0 + 8m}{2l_0 + 8m} \cdot \left( \frac{l_0 + 4m + 1}{l_0 + 4m} \right)^{n_0 + 8m - 1} \cdot \frac{\text{vol}_{HM}(\Gamma_{i,m})}{\text{vol}_{HM}(O^+(L_m))}
\]
of the two leading coefficients above converges to 0 in $m \to \infty$. We first see that
\[
\lim_{m \to \infty} \frac{n_0 + 8m}{2l_0 + 8m} \left( \frac{l_0 + 4m + 1}{l_0 + 4m} \right)^{n_0 + 8m - 1} = 1 \cdot e^2.
\]
By the very definition of Hirzebruch-Mumford volume (see [7]), we have
\[
\text{vol}_{HM}(\Gamma_{i,m}) = [O^+(K_{i,m}) : \Gamma_{i,m}] \cdot \text{vol}_{HM}(O^+(K_{i,m})).
\]
Since $\Gamma_{i,m} \supset \widetilde{O}^+(K_{i,m})$, we have the estimate
\[
[O^+(K_{i,m}) : \Gamma_{i,m}] \leq [O^+(K_{i,m}) : \widetilde{O}^+(K_{i,m})] = |O(A_{K})|,
\]
where the last equality follows from the surjectivity of $O^+(K_{i,m}) \to O(A_{K_{i,m}})$ by [11]. Therefore we deduce from Lemma 3.3 that
\[
\lim_{m \to \infty} \frac{\text{vol}_{HM}(\Gamma_{i,m})}{\text{vol}_{HM}(O^+(L_m))} = 0.
\]
This proves the asymptotic behavior (4.6) when $m$ is sufficiently large, and the proof of Theorem 1.1 is completed.

4.3. **Computation of a bound.** In our argument in §4.2, the modular variety $\mathcal{F}(L_m)$ is of general type when
\[
n_0 + 8m \geq 9, \quad l_0 + 4m > 0,
\]
and the sum of the ratios (4.7) over $1 \leq i \leq r$ is smaller than 1:
\[
\sum_{i=1}^{r} \frac{\text{vol}_{HM}(\Gamma_{i,m})}{\text{vol}_{HM}(O^+(L_m))} < \frac{2l_0 + 8m}{n_0 + 8m} \cdot \left( \frac{l_0 + 4m}{l_0 + 4m + 1} \right)^{n_0 + 8m - 1}.
\]
Let us conclude this article with remarks concerning how to compute an explicit range of $m$ for these inequalities. The inputs required in the calculation are the following:

(I) the weight $k_0$ of which there exists a Jacobi cusp form of index 1 for $M'(-1)$, where $M'$ is a maximal even overlattice of $M$ for $L_0 = 2U \oplus M$ (actually, in view of [6] Theorem 4.2, we only need to choose $M'$ so that any isotropic subgroup of $A_{M'}$ is cyclic),

(II) classification of the reflective vectors $l_1, \cdots, l_r \in L_0$ up to $O^+(L_0)$,

(III) the index $[O^+(K_{i,m}) : \Gamma_{i,m}]$, and

(IV) precise forms of $\text{vol}_{HM}(O^+(L_m))$ and $\text{vol}_{HM}(O^+(K_{i,m}))$. 
At least in principle, these datum could be calculated or estimated explicitly as follows. See [8] for the model cases $L_0 = 2U, 2U \oplus \langle -2d \rangle$.

(I) This is equivalent to the weight $k_0 + 1 - n_0/2$ of which there exists a cusp form of type $\rho_M$ for $Mp_2(\mathbb{Z})$. One can find such a weight by looking the dimension formula for $M'_k(\rho_M')$ presented in [2] p. 228 (which is worked out in [3] §2 in case $k' \equiv 1 - n_0/2 \mod 2$).

(II) This could be done, e.g., in the following steps:

(a) enumerate possible norms $-2d$ of reflective vectors $l$, which are either $-\text{div}(l)$ or $-2\text{div}(l)$;

(b) enumerate finite quadratic forms $A$ of signature $[3-n_0] \in \mathbb{Z}/8\mathbb{Z}$ with gluings between $A$ and $A_{(-2d)}$ that give rise to $A_{L_0}$;

(c) construct even lattices $K$ of signature $(2,n_0-1)$ with $A_K \cong A$, and the embeddings $K \oplus \langle -2d \rangle \subset L_0$ given by the gluings in (b); and

(d) exclude the cases where $\langle -2d \rangle$ is not reflective in $L_0$.

(III) Perhaps the estimate (4.8) might be sufficient.

(IV) The formula of $\text{vol}_{HM}(O^+(L_m))$ given in Lemma 3.1 consisted of the following terms:

(i) the constant $C$, whose explicit value is shown in Remark 3.2,

(ii) the elementary functions $F_m(L_0), G_m(L_0), H_m(L_0)$ and $(|A_{L_0}|/4)^{4m}$ or $(|A_{L_0}|/4\pi)^{4m}$;

(iii) the product $\prod_k (B_{2k}/2k)$; and

(iv) a special value of the $L$-function $L(s,\chi_D)$ with a factorial $n!$.

One can evaluate or estimate (iv) by referring to, e.g., [4] §10.2. The terms in (i) and (ii) could be worked out from the information of $A_{L_0}$, because the $\mathbb{Z}_p$-lattices $L_0 \otimes \mathbb{Z}_p$ are encoded in $A_{L_0}$ and $n_0$. The Bernoulli numbers can be estimated by Stirling’s formula (see, e.g., [4] Chapter 9). Note that when comparing $\text{vol}_{HM}$ between $O^+(L_m)$ and $O^+(K_{i,m})$, all but at most one Bernoulli numbers are canceled out.

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