Strange Integrals Derived By Elementary Complex Analysis

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Abstract

In this short note, we provide an elementary complex analytic method for converting known real integrals into numerous strange and interesting looking real integrals.

Consider the odd looking integral

\[
\lim_{R \to \infty} \int_0^{\pi} R e^{-R^2 \cos^2 \theta} \sin \left( R^2 \sin 2\theta - \theta \right) d\theta,
\]

which typically invites responses like "what the...!?" Despite the appearance of such an integral, however, we provide a method to compute integrals like equation (1) using certain well known real integrals by simply modifying the typical complex analytic method for computing real integrals. Moreover, the method we describe provides an easy way to motivate the beauty and simplicity of complex analysis early on in a complex analysis course.

The typical complex analytic method for solving real integrals (which is found in nearly every complex analysis book, for example [1] or [2]) is to consider the integral in the complex plane where we start with a line segment on the real line, appropriately close the line segment, integrate over the resulting closed curve, and finally use this complex integral to obtain
the value of the original real integral. Here is an example taken from [1].
If we want to compute $\int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx$, then we can equivalently compute the imaginary part of the principle value of $\int_{-\infty}^{\infty} \frac{e^{iz}}{z} \, dx$.

Figure 1: Contour for the current example.

To do this, consider $\int_{\Gamma} \frac{e^{iz}}{z} \, dz$ so that basic complex analytic results gives us that

$$\lim_{R \to \infty} \int_{\Gamma_R} \frac{e^{iz}}{z} \, dz = 0$$

and

$$\lim_{\epsilon \to 0} \int_{\Gamma_\epsilon} \frac{e^{iz}}{z} \, dz = -i\pi$$

where $\Gamma_R = \{z : |z| = R \text{ and } Im(z) > 0\}$, $\Gamma_\epsilon = \{z : |z| = \epsilon \text{ and } Im(z) > 0\}$, and $\Gamma = \Gamma_\epsilon \cup \Gamma_R \cup [-R, -\epsilon] \cup [\epsilon, R]$, so that

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx = \pi.$$ 

**Main Examples**  Now we show how integrals like equation (1) are derived from certain well known real integrals. We know that $\int_{-\infty}^{\infty} \frac{\sin ax}{x} \, dx = \pi$
and that \( f(z) = \frac{\sin z}{z} \) is entire (more precisely, the function \( f(z) = \frac{\sin z}{z} \) on \( z \in \mathbb{C}\setminus\{0\} \) and \( f(z) = 1 \) for \( z = 0 \) is entire.) Therefore, we have

\[
\lim_{R \to \infty} \int_{\Gamma_R} \frac{\sin z}{z} \, dz = -\int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx = -\pi \tag{2}
\]

where again, \( \Gamma_R = \{z : |z| = R \text{ and } Im(z) > 0\} \).

However, if we put the first integral of equation (2) in terms of real quantities, then we obtain a very odd looking result. First notice that \( z = Re^{i\theta} \) on \( \Gamma_R \) so that \( dz = iRe^{i\theta} \, d\theta \). Therefore, directly plugging in for \( z \) and \( dz \) into the first integral of equation (2) and plowing through some tedious but completely intuitive and elementary calculations involving multiplication, the \( \sin \) addition formula, the \( \sin/\sinh \) and \( \cos/cosh \) identities (and only considering the real part of the integral, since we know that the imaginary part must be zero) gives us that

\[
-\lim_{R \to \infty} \int_{\Gamma_R} \frac{\sin z}{z} \, dz = \lim_{R \to \infty} \int_{0}^{\pi} \cos (R \cos \theta) \sinh (R \sin \theta) \, d\theta = \pi, \tag{3}
\]

which is indeed a very odd looking integral. Now we will show how to use this method to derive the integral in equation (1).

Consider the entire function \( f(z) = e^{-z^2} \). Using the same contours and parametrization as in the previous example, we obtain

\[
-\lim_{R \to \infty} \int_{\Gamma_R} e^{-z^2} \, dz = \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}.
\]

Therefore, substituting in for \( z \) and \( dz \) and again plowing through some tedious but completely intuitive and elementary calculations involving multiple uses of Euler’s identity, multiplication, the \( \sin \) addition formula (where again disregarding the imaginary part of the integral) gives us that
\[-\lim_{R \to \infty} \int_0^\pi R e^{-R^2 \cos 2\theta} \sin (R^2 \sin 2\theta - \theta) \, d\theta = \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi},\]

which is of course the integral in equation (1).

Finally, we use the same integral \( \int_{-\infty}^{\infty} e^{-x^2} \, dx \), but use a different curve to close the line segment \([-R, R]\). This time, let

\[ \Gamma_R = \{ t + i (t^2 - R^2) : t \in [-R, R] \} \]

so that again,

\[-\lim_{R \to \infty} \int_{\Gamma_R} \, e^z^2 \, dz = \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}.\]

![Figure 2: Contour for the current example.](image)

Thus, plugging in for \( z = t + i(t^2 - R^2) \) and \( dz = (1 + 2it) \, dt \), we obtain
the following:

\[- \lim_{R \to \infty} \int_{-R}^{R} e^{(t^2-R^2)^2-t^2} \left( \cos (2t^3 - tp^2) + 2t \sin (2t^3 - tp^2) \right) \, dt = \sqrt{\pi}, \]

which is yet another strange looking integral.

**Conclusion** At this point, we could evaluate numerous other integrals by merely changing the integrands or using various paths to close the line segment \([-R, R]\). For example, we could have simply used any polynomial with its only real roots as \([-R, R]\) or ellipses to close the line segment \([-R, R]\). However, doing so involves nothing more than reapplying our method to whatever integrand or path to close the line segment one chooses, and so we leave this for the interested reader to do. After all, what’s more fun than deriving a formula that is quite possibly more weird looking than

\[- \lim_{R \to \infty} \int_{-R}^{R} e^{(t^2-R^2)^2-t^2} \left( \cos (2t^3 - tp^2) + 2t \sin (2t^3 - tp^2) \right) \, dt? \]

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**References**

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2. James Brown, Reul Churchill, and Roger Verhey, Complex Variables and Applications, McGraw-Hill, New York, 1996.