Abstract

In this Master of Science Thesis I introduce geometric algebra both from the traditional geometric setting of vector spaces, and also from a more combinatorial view which simplifies common relations and operations. This view enables us to define Clifford algebras with scalars in arbitrary rings and provides new suggestions for an infinite-dimensional approach.

Furthermore, I give a quick review of classic results regarding geometric algebras, such as their classification in terms of matrix algebras, the connection to orthogonal and Spin groups, and their representation theory. A number of lower-dimensional examples are worked out in a systematic way using so called norm functions, while general applications of representation theory include normed division algebras and vector fields on spheres.

I also consider examples in relativistic physics, where reformulations in terms of geometric algebra give rise to both computational and conceptual simplifications.
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1 Introduction

The foundations of geometric algebra, or what today is more commonly known as Clifford algebra, were put forward already in 1844 by Grassmann. He introduced vectors, scalar products and extensive quantities such as exterior products. His ideas were far ahead of his time and formulated in an abstract and rather philosophical form which was hard to follow for contemporary mathematicians. Because of this, his work was largely ignored until around 1876, when Clifford took up Grassmann’s ideas and formulated a natural algebra on vectors with combined interior and exterior products. He referred to this as an application of Grassmann’s geometric algebra.

Due to unfortunate historic events, such as Clifford’s early death in 1879, his ideas did not reach the wider part of the mathematics community. Hamilton had independently invented the quaternion algebra which was a special case of Grassmann’s constructions, a fact Hamilton quickly realized himself. Gibbs reformulated, largely due to a misinterpretation, the quaternion algebra to a system for calculating with vectors in three dimensions with scalar and cross products. This system, which today is taught at an elementary academic level, found immediate applications in physics, which at that time circled around Newton’s mechanics and Maxwell’s electrodynamics. Clifford’s algebra only continued to be employed within small mathematical circles, while physicists struggled to transfer the three-dimensional concepts in Gibbs’ formulation to special relativity and quantum mechanics. Contributions and independent inventions of Grassmann’s and Clifford’s constructions were made along the way by Cartan, Lipschitz, Chevalley, Riesz, Atiyah, Bott, Shapiro, and others.

Only in 1966 did Hestenes identify the Dirac algebra, which had been constructed for relativistic quantum mechanics, as the geometric algebra of spacetime. This spawned new interest in geometric algebra, and led, though with a certain reluctance in the scientific community, to applications and reformulations in a wide range of fields in mathematics and physics. More recent applications include image analysis, computer vision, robotic control and electromagnetic field simulations. Geometric algebra is even finding its way into the computer game industry.

There are a number of aims of this Master of Science Thesis. Firstly, I want to give a compact introduction to geometric algebra which sums up classic results regarding its basic structure and operations, the relations between different geometric algebras, and the important connection to orthogonal groups via Spin groups. I also clarify a number of statements which have been used in a rather sloppy, and sometimes incorrect, manner in the literature. All stated theorems are accompanied by proofs, or references to where a strict proof can be found.

Secondly, I want to show why I think that geometric and Clifford algebras are important, by giving examples of applications in mathematics and physics. The applications chosen cover a wide range of topics, some with no direct connection to geometry. The applications in physics serve to illustrate the computational and, most importantly, conceptual simplifications that the language of geometric algebra can provide.

Another aim of the thesis is to present some of the ideas of my supervisor Lars Svensson in the subject of generalizing Clifford algebra in the algebraic direction. I also present some of my own ideas regarding norm functions on geometric algebras.
The reader will be assumed to be familiar with basic algebraic concepts such as tensors, fields, rings and homomorphisms. Some basics in topology are also helpful. To really appreciate the examples in physics, the reader should be familiar with special relativity and preferably also relativistic electrodynamics and quantum mechanics. For some motivation and a picture of where we are heading, it could be helpful to have seen some examples of geometric algebras before. For a quick 10-page introduction with some applications in physics, see [16].

Throughout, we will use the name geometric algebra in the context of vector spaces, partly in honor of Grassmann’s contributions, but mainly for the direct and natural connection to geometry that this algebra admits. In a more general algebraic setting, where a combinatorial rather than geometric interpretation exists, we call the corresponding construction Clifford algebra.
2 Foundations

In this section we define geometric algebra and work out a number of its basic properties. We consider the definition that is most common in the mathematical literature, namely as a quotient space on the tensor algebra of a vector space with a quadratic form. We see that this leads, in the finite-dimensional case, to the equivalent definition as an algebra with generators \( \{ e_i \} \) satisfying \( e_i e_j + e_j e_i = 2g_{ij} \) for some metric \( g \). This is perhaps the most well-known definition.

We go on to consider an alternative definition of geometric algebra based on its algebraic and combinatorial features. The resulting algebra, here called Clifford algebra due to its higher generality but less direct connection to geometry, allows us to define common operations and prove fundamental identities in a remarkably simple way compared to traditional formulations.

Returning to the vector space setting, we go on to study some of the geometric features from which geometric algebra earns its name. We also consider parts of the extensive linear function theory which exists for geometric algebras. Finally, we note that the generalized Clifford algebra offers interesting views regarding the infinite-dimensional case.

2.1 Geometric algebra \( \mathcal{G}(\mathcal{V}, q) \)

The traditional definition of geometric algebra is carried out in the context of vector spaces with an inner product, or more generally a quadratic form. We consider here a vector space \( \mathcal{V} \) of arbitrary dimension over some field \( \mathbb{F} \).

**Definition 2.1.** A quadratic form \( q \) on a vector space \( \mathcal{V} \) is a map \( q: \mathcal{V} \to \mathbb{F} \) such that

\[
\begin{align*}
&i) \quad q(\alpha v) = \alpha^2 q(v) \quad \forall \ \alpha \in \mathbb{F}, v \in \mathcal{V} \\
&ii) \quad q(v + w) - q(v) - q(w) \quad \text{is linear in both } v \text{ and } w.
\end{align*}
\]

The bilinear form \( \beta_q(v, w) := \frac{1}{2} \left( q(v + w) - q(v) - q(w) \right) \) is called the polarization of \( q \).

**Example 2.1.** If \( \mathcal{V} \) has a bilinear form \( \langle \cdot, \cdot \rangle \) then \( q(v) := \langle v, v \rangle \) is a quadratic form and \( \beta_q \) is the symmetrization of \( \langle \cdot, \cdot \rangle \). This could be positive definite (an inner product), or indefinite (a metric of arbitrary signature).

**Example 2.2.** If \( \mathcal{V} \) is a normed vector space over \( \mathbb{R} \), with norm denoted by \( | \cdot | \), where the parallelogram identity \( |x + y|^2 + |x - y|^2 = 2|x|^2 + 2|y|^2 \) holds then \( q(v) := |v|^2 \) is a quadratic form and \( \beta_q \) is an inner product on \( \mathcal{V} \). This is a classic result, sometimes called the Jordan-von Neumann theorem.

Let \( \mathcal{T}(\mathcal{V}) := \bigoplus_{k=0}^{\infty} \otimes^k \mathcal{V} \) denote the tensor algebra on \( \mathcal{V} \), the elements of which are finite sums of tensors of different grades on \( \mathcal{V} \). Consider the ideal generated by all elements of the form\(^1 \) \( v \otimes v - q(v) \) for vectors \( v \),

\[
\mathcal{I}_q(\mathcal{V}) := \left\{ A \otimes (v \otimes v - q(v)) \otimes B : v \in \mathcal{V}, A, B \in \mathcal{T}(\mathcal{V}) \right\}.
\]

(2.1)

We define the geometric algebra over \( \mathcal{V} \) by quoting out this ideal from \( \mathcal{T}(\mathcal{V}) \).

---

\(^1\)Mathematicians often choose a different sign convention here, resulting in reversed signature in many of the following results. The convention used here seems more natural in my opinion, since e.g. squares of vectors in euclidean spaces become positive instead of negative.
Definition 2.2. The geometric algebra $\mathcal{G}(V, q)$ over the vector space $V$ with quadratic form $q$ is defined by

$$\mathcal{G}(V, q) := \mathcal{T}(V)/I_q(V).$$

When it is clear from the context what vector space or quadratic form we are working with, we will often denote $\mathcal{G}(V, q)$ by $\mathcal{G}(V)$, or just $\mathcal{G}$.

The product in $\mathcal{G}$, called the geometric or Clifford product, is inherited from the tensor product in $\mathcal{T}(V)$ and we denote it by juxtaposition (or · if absolutely necessary),

$$\mathcal{G} \times \mathcal{G} \to \mathcal{G}, \quad (A, B) \mapsto AB := [A \otimes B].$$

Note that this product is bilinear and associative. We immediately find the following identities on $\mathcal{G}$ for $v, w \in V$:

$$v^2 = q(v) \Rightarrow vw + wv = 2\beta_q(v, w). \quad (2.2)$$

One of the most important consequences of this definition of the geometric algebra is the following

Proposition 2.1 (Universality). Let $\mathcal{A}$ be an associative algebra over $F$ with a unit denoted by $1_\mathcal{A}$. If $f: V \to \mathcal{A}$ is linear and

$$f(v)^2 = q(v)1_\mathcal{A} \quad \forall \ v \in V \quad (2.3)$$

then $f$ extends uniquely to an $F$-algebra homomorphism $F: \mathcal{G}(V, q) \to \mathcal{A}$, i.e.

$$F(\alpha) = \alpha 1_\mathcal{A}, \quad \forall \ \alpha \in F,$$

$$F(v) = f(v), \quad \forall \ v \in V,$$

$$F(xy) = F(x)F(y),$$

$$F(x + y) = F(x) + F(y), \quad \forall \ x, y \in \mathcal{G}.$$  

Furthermore, $\mathcal{G}$ is the unique associative $F$-algebra with this property.

Proof. Any linear map $f: V \to \mathcal{A}$ extends to a unique algebra homomorphism $\bar{f}: \mathcal{T}(V) \to \mathcal{A}$ defined by $\bar{f}(u \otimes v) := f(u)f(v)$ etc. Property (2.3) implies that $\bar{f} = 0$ on the ideal $I_q(V)$ and so $\bar{f}$ descends to a well-defined map $F$ on $\mathcal{G}(V, q)$ which has the required properties. Suppose now that $\mathcal{C}$ is an associative $F$-algebra with unit and that $i: \mathcal{V} \hookrightarrow \mathcal{C}$ is an embedding with the property that any linear map $f: \mathcal{V} \to \mathcal{A}$ with property (2.3) extends uniquely to an algebra homomorphism $F: \mathcal{C} \to \mathcal{A}$. Then the isomorphism from $\mathcal{V} \subseteq \mathcal{G}$ to $i(\mathcal{V}) \subseteq \mathcal{C}$ clearly induces an algebra isomorphism $\mathcal{G} \to \mathcal{C}$. \qed

So far we have not made any assumptions on the dimension of $V$. We will come back to the infinite-dimensional case when discussing the more general Clifford algebra. Here we will familiarize ourselves with the properties of quadratic forms on finite-dimensional spaces. For the remainder of this subsection we will therefore assume that $\dim \mathcal{V} = n < \infty$.

Definition 2.3. A basis $\{e_1, \ldots, e_n\}$ of $(\mathcal{V}, q)$ is said to be orthogonal or canonical if $\beta_q(e_i, e_j) = 0$ for all $i \neq j$. The basis is called orthonormal if we also have that $q(e_i) \in \{-1, 0, 1\}$ for all $i$. 

4
We have a number of classical theorems regarding orthogonal bases. Proofs of these can be found e.g. in [23].

**Theorem 2.2.** If \( \dim V < \infty \) and \( \text{char } F \neq 2 \) then there exists an orthogonal basis of \((V, q)\).

Because this rather fundamental theorem breaks down for fields of characteristic two (such as \( \mathbb{Z}_2 \)), we will always assume that char \( F \neq 2 \) when talking about geometric algebra. General fields and rings will be treated by the general Clifford algebra, however.

**Theorem 2.3 (Sylvester’s Law of Inertia).** Assume that \( \dim V < \infty \) and \( F = \mathbb{R} \). If \( E_1 \) and \( E_2 \) are two orthogonal bases of \( V \) and

\[
E^+_i := \{ e \in E_i : q(e) > 0 \}, \\
E^-_i := \{ e \in E_i : q(e) < 0 \}, \\
E^0 := \{ e \in E_i : q(e) = 0 \}
\]

then

\[
|E^+_i| = |E^+_2|, \\
|E^-_i| = |E^-_2|, \\
\text{Span } E^0 = \text{Span } E^0.
\]

This means that there is a unique signature \((s, t, u) := (|E^+_1|, |E^-_1|, |E^0|)\) associated to \((V, q)\). For the complex case we have the following simpler result:

**Theorem 2.4.** If \( E_1 \) and \( E_2 \) are orthogonal bases of \( V \) with \( F = \mathbb{C} \) and

\[
E^\times_i := \{ e \in E_i : q(e) \neq 0 \}, \\
E^0 := \{ e \in E_i : q(e) = 0 \}
\]

then

\[
\text{Span } E^\times_i = \text{Span } E^\times_2, \\
\text{Span } E^0 = \text{Span } E^0.
\]

If \( E^0 = \emptyset \) (\( q \) is nondegenerate) then there exists a basis \( E \) with \( q(e) = 1 \) \( \forall e \in E \).

From the above follows that we can talk about the signature of a quadratic form or a metric without ambiguity. We use the short-hand notation \( \mathbb{R}^{s,t,u} \) to denote the \((s + t + u)\)-dimensional real vector space with a quadratic form of signature \((s, t, u)\), while \( \mathbb{C}^n \) is understood to be the complex \( n \)-dimensional space with a nondegenerate quadratic form. When \( u = 0 \) or \( t = u = 0 \) we may simply write \( \mathbb{R}^{s,t} \) or \( \mathbb{R}^s \). A space of type \( \mathbb{R}^{n,0} \) is called euclidean and \( \mathbb{R}^{0,n} \) anti-euclidean, while the spaces \( \mathbb{R}^{1,n} \) (\( \mathbb{R}^{n,1} \)) are called (anti-)lorentzian. Within real and complex spaces we can always find bases that are orthonormal.

**Remark.** The general condition for orthonormal bases to exist is that the field \( F \) is a so called spin field. This means that every \( \alpha \in F \) can be written as \( \alpha = \beta^2 \) or \(-\beta^2\) for some \( \beta \in \mathbb{F} \). The fields \( \mathbb{R}, \mathbb{C} \) and \( \mathbb{Z}_p \) for \( p \) a prime with \( p \equiv 3 \) (mod 4), are spin, but e.g. \( \mathbb{Q} \) is not.

Consider now the geometric algebra \( \mathcal{G} \) over a real or complex space \( V \). If we pick an orthonormal basis \( E = \{e_1, \ldots, e_n\} \) of \( V \) it follows from Definition [222] and [223] that \( \mathcal{G} \) is the free associative algebra generated by \( E \) modulo the relations

\[
e_i^2 = q(e_i) = \pm 1 \text{ or } 0 \quad \text{and} \quad e_i e_j = -e_j e_i, \ i \neq j. \quad (2.4)
\]
We also observe that \( \mathcal{G} \) is spanned by \( \{ E_{i_1 i_2 \ldots i_k} \}_{i_1 < i_2 < \ldots < i_k} \), where \( E_{i_1 i_2 \ldots i_k} := e_{i_1} e_{i_2} \ldots e_{i_k} \). Thus, one can view \( \mathcal{G} \) as vector space isomorphic to \( \wedge^* \mathcal{V} \), the exterior algebra of \( \mathcal{V} \). This is a description of geometric algebra (Clifford algebra) which may be more familiar to e.g. physicists.

**Remark.** If we take \( q = 0 \) we actually obtain an algebra isomorphism \( \mathcal{G} \cong \wedge^* \mathcal{V} \). In this case \( \mathcal{G} \) is called a Grassmann algebra.

One element in \( \mathcal{G} \) deserves special attention, namely the so called pseudoscalar

\[
I := e_1 e_2 \ldots e_n.
\]  

(2.5)

Note that this definition is basis independent up to orientation when \( q \) is non-degenerate. Indeed, let \( \{ Re_1, \ldots, Re_n \} \) be another orthonormal basis with the same orientation, where \( R \in O(\mathcal{V}, q) \), the group of linear transformations which leave \( q \) invariant\(^2\). Then \( Re_1 Re_2 \ldots Re_n = \sum_{\pi \in S_n} \text{sign}(\pi) R_{\pi(1)} \ldots R_{\pi(n)} e_1 e_2 \ldots e_n = I \) due to the anticommutativity of the \( e_i \)'s.

Note that, by selecting a certain pseudoscalar for \( \mathcal{G} \) we also impose a certain orientation on \( \mathcal{V} \). There is no such thing as an absolute orientation; instead all statements concerning orientation will be made relative to the chosen one.

The square of the pseudoscalar is given by (and gives information about) the signature and dimension of \( (\mathcal{V}, q) \). For \( \mathcal{G}(\mathbb{R}^{s,t,u}) \) we have that

\[
I^2 = (-1)^{\frac{n(s-1)}{2} + t} \delta_{s,0}, \quad \text{where } n = s + t + u.
\]  

(2.6)

We say that \( \mathcal{G} \) is degenerate if the quadratic form is degenerate, or equivalently if \( I^2 = 0 \). For odd \( n \), \( I \) commutes with all elements in \( \mathcal{G} \) and the center of \( \mathcal{G} \) is \( Z(\mathcal{G}) = \text{Span}_\mathbb{F} \{ 1, I \} \). For even \( n \), the center consists of the scalars \( \mathbb{F} \) only.

### 2.2 Combinatorial Clifford algebra \( Cl(X, R, r) \)

We now take a temporary step away from the comfort of fields and vector spaces and instead consider the purely algebraic features of geometric algebra that were uncovered in the previous subsection. Note that we could roughly write

\[
\mathcal{G}(\mathcal{V}) = \text{Span}_\mathbb{F} \{ E_A \}_{A \subseteq \{1,2,\ldots,n\}}
\]  

(2.7)

for an \( n \)-dimensional space \( \mathcal{V} \) over \( \mathbb{F} \), and that the geometric product of these basis elements behaves as

\[
E_A E_B = \tau(A, B) E_{A \triangle B}, \quad \text{where } \tau(A, B) = 1, -1 \text{ or } 0,
\]  

(2.8)

and \( A \triangle B := (A \cup B) \setminus (A \cap B) \) is the symmetric difference between the sets \( A \) and \( B \). Motivated by this we consider the following generalization.

**Definition 2.4.** Let \( X \) be a finite set and \( R \) a commutative ring with unit. Let \( r: X \to R \) be some function which is to be thought of as a signature on \( X \). The **Clifford algebra** over \( (X, R, r) \) is defined as the set

\[
Cl(X, R, r) := \bigoplus_{\mathcal{P}(X)} R,
\]

i.e. the free \( R \)-module generated by \( \mathcal{P}(X) \), the set of all subsets of \( X \). We may use the shorter notation \( Cl(X) \), or just \( Cl \), when the current choice of \( X, R \) and \( r \) is clear from the context. We call \( R \) the **scalars** of \( Cl \).

\(^2\)The details surrounding such transformations will be discussed in Section \( \[ \)
Example 2.3. A typical element of $\text{Cl}((\{x, y, z\}, \mathbb{Z}, r)$ could for example look like
\begin{equation}
5\emptyset + 3\{x\} + 2\{y\} - \{x, y\} + 12\{x, y, z\}.
\end{equation}

We have not yet defined a product on $\text{Cl}$. In addition to being $R$-bilinear and associative, we would like the product to satisfy $\{x\}^2 = r(x)\emptyset$ for $x \in X$, $\{x\}\{y\} = -\{y\}\{x\}$ for $x \neq y \in X$ and $\emptyset A = A\emptyset = A$ for all $A \in \mathcal{P}(X)$. In order to arrive at such a product we make use of the following

Lemma 2.5. There exists a map $\tau: \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow R$ such that
\begin{itemize}
  \item[i)] $\tau(\{x\}, \{x\}) = r(x)$ \quad $\forall \ x \in X$,
  \item[ii)] $\tau(\{x\}, \{y\}) = -\tau(\{y\}, \{x\}) \quad \forall \ x, y \in X : x \neq y$,
  \item[iii)] $\tau(\emptyset, A) = \tau(A, \emptyset) = 1 \quad \forall \ A \in \mathcal{P}(X)$,
  \item[iv)] $\tau(A, B)\tau(A\triangle B, C) = \tau(A, B\triangle C)\tau(B, C) \quad \forall \ A, B, C \in \mathcal{P}(X)$,
  \item[v)] $\tau(A, B) \in \{-1, 1\}$ if $A \cap B = \emptyset$.
\end{itemize}

Proof. We proceed by induction on the cardinality $|X|$ of $X$. For $X = \emptyset$ the lemma is trivial, so let $z \in X$ and assume the lemma holds for $Y := X \setminus \{z\}$. Hence, there is a $\tau': \mathcal{P}(Y) \times \mathcal{P}(Y) \rightarrow R$ which has the properties (i)-(v) above. If $A \subseteq Y$ we write $A' = A \cup \{z\}$ and, for $A, B \in \mathcal{P}(X)$ we extend $\tau'$ to $\tau: \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow R$ in the following way:
\begin{align*}
\tau(A, B) & := \tau'(A, B) \\
\tau(A', B) & := (-1)^{|B|}\tau'(A, B) \\
\tau(A, B') & := \tau'(A, B) \\
\tau(A', B') & := r(z)(-1)^{|B|}\tau'(A, B)
\end{align*}
Now it is straightforward to verify that (i)-(v) holds for $\tau$, which completes the proof.

Definition 2.5. Define the Clifford product
\begin{align*}
\text{Cl}(X) \times \text{Cl}(X) & \rightarrow \text{Cl}(X) \\
(A, B) & \mapsto AB
\end{align*}
by taking $AB := \tau(A, B)A\triangle B$ for $A, B \in \mathcal{P}(X)$ and extending linearly. We choose to use the $\tau$ which is constructed as in the proof of Lemma 2.5 by consecutively adding elements from the set $X$. A unique such $\tau$ may only be selected after imposing a certain order (orientation) on the set $X$.

Using Lemma 2.5 one easily verifies that this product has all the properties that we asked for above. For example, in order to verify associativity we note that
\begin{equation}
A(BC) = A(\tau(B, C)B\triangle C) = \tau(A, B\triangle C)\tau(B, C)A\triangle (B\triangle C),
\end{equation}
while
\begin{equation}
(AB)C = \tau(A, B)(A\triangle B)C = \tau(A, B)\tau(A\triangle B, C)(A\triangle B)\triangle C.
\end{equation}
Associativity now follows from (iv) and the associativity of the symmetric difference. As is expected from the analogy with $\mathcal{G}$, we also have the property that different basis elements of $\text{Cl}$ commute up to a sign.
We make this equivalence between $G$ and $\dim(\text{the unit } \emptyset)$.

Of special importance are the $f$ defined by $E = (\mod 2)$. By Proposition 2.1, $f$ and likewise for $X$ and likewise for $b_i$. If $A$ and $B$ have $m$ elements in common then $AB = (-1)^{(k-m)l + m(l-1)}BA = (-1)^{kl-m}BA$ by property (ii). But then we are done, since $\frac{1}{2}(-k(k-1) - l(l-1) + (k+l-2m)(k+l-2m-1)) \equiv kl + m \pmod{2}$.

We are now ready to make the formal connection between $G$ and $\mathcal{C}l$. Let $(\mathcal{V}, q)$ be a vector space over $\mathbb{F}$ with a quadratic form. Pick an orthogonal basis $E = \{e_1, \ldots, e_n\}$ of $\mathcal{V}$ and consider the Clifford algebra $\mathcal{C}l(E, F, q|E)$. Define $f: \mathcal{V} \to \mathcal{C}l$ by $f(e_i) := \{e_i\}$ for $i = 1, \ldots, n$ and extend linearly. We then have

\[ f(v)^2 = f(\sum_i v_i e_i)f(\sum_j v_j e_j) = \sum_{i,j} v_i v_j f(e_i)f(e_j) \]

\[ = \sum_{i,j} v_i v_j \{e_i\}\{e_j\} = \sum_i v_i^2 \{e_i\}^2 \]

\[ = \sum_i v_i^2 q|E(e_i)\emptyset = \sum_i v_i^2 q(e_i)\emptyset = q(\sum_i v_i e_i)\emptyset = q(v)\emptyset. \]

By Proposition 2.1 $f$ extends uniquely to a homomorphism $F: G \to \mathcal{C}l$. Since $\dim G = \dim \mathcal{C}l = 2^n$ and $F$ is easily seen to be surjective from the property (v), we arrive at an isomorphism

\[ G(\mathcal{V}, q) \cong \mathcal{C}l(E, F, q|E). \]

We make this equivalence between $G$ and $\mathcal{C}l$ even more transparent by suppressing the unit $\emptyset$ in expressions and writing simply $e$ instead of $\{e\}$ for singletons $e \in E$. For example, with an orthonormal basis $\{e_1, e_2, e_3\}$ in $\mathbb{R}^3$, both $G$ and $\mathcal{C}l$ are then spanned by

\[ \{1, e_1, e_2, e_3, e_1e_2, e_1e_3, e_2e_3, e_1e_2e_3\}. \]

There is a natural grade structure on $\mathcal{C}l$ given by the cardinality of the subsets of $X$. Consider the following

**Definition 2.6.** The **subspace of $k$-vectors** in $\mathcal{C}l$, or the **grade-$k$ part** of $\mathcal{C}l$, is defined by

\[ \mathcal{C}l^k(X, R, r) := \bigoplus_{A \in \mathcal{P}(X) : |A|=k} R. \]

Of special importance are the **even** and **odd subspaces**,

\[ \mathcal{C}l^\pm(X, R, r) := \bigoplus_{k \text{ is even/odd}} \mathcal{C}l^k(X, R, r). \]

This notation carries over to the corresponding subspaces of $G$ and we write $G^k$, $G^\pm$ etc., where for example $G^0 = \mathbb{F}$ and $G^1 = \mathcal{V}$. The elements of $G^2$ are also called **bivectors**, while arbitrary elements of $G$ are traditionally called **multivectors**.

\[ \text{Page 8} \]
We then have a split of $\mathcal{C}l$ into graded subspaces as

$$
\mathcal{C}l(X) = \mathcal{C}l^+ \oplus \mathcal{C}l^- = \mathcal{C}l^0 \oplus \mathcal{C}l^1 \oplus \mathcal{C}l^2 \oplus \ldots \oplus \mathcal{C}l^{|X|}.
$$

(2.15)

Note that, under the Clifford product, $\mathcal{C}l^+ \cdot \mathcal{C}l^+ \subseteq \mathcal{C}l^+$ and $\mathcal{C}l^- \cdot \mathcal{C}l^- \subseteq \mathcal{C}l^-$. Hence, the even-grade elements $\mathcal{C}l^+$ form a subalgebra of $\mathcal{C}l$.

In $\mathcal{C}l(X, R, r)$ we have the possibility of defining a unique pseudoscalar independently of the signature $r$, namely the set $X$ itself. Note, however, that it can only be normalized if $X^2 = \tau(X, X) \in R$ is invertible, which requires that $r$ is nondegenerate. We will almost always talk about pseudoscalars in the setting of nondegenerate vector spaces, so this will not be a problem.

2.3 Standard operations

A key feature of Clifford algebras is that they contain a surprisingly large amount of structure. In order to really be able to harness the power of this structure we need to introduce powerful notation. Most of the following definitions will be made on $\mathcal{C}l$ for simplicity, but because of the equivalence between $\mathcal{G}$ and $\mathcal{C}l$ they carry over to $\mathcal{G}$ in a straightforward manner.

We will find it convenient to introduce the notation that for any proposition $P$, $(P)$ will denote the number 1 if $P$ is true and 0 if $P$ is false.

**Definition 2.7.** For $A, B \in \mathcal{P}(X)$ define

- $A \wedge B := (A \cap B = \emptyset) AB$ outer product
- $A \lhd B := (A \subseteq B) AB$ left inner product
- $A \rhd B := (A \supseteq B) AB$ right inner product
- $A * B := (A = B) AB$ scalar product
- $\langle A \rangle_n := (|A| = n) A$ projection on grade n
- $A^* := (-1)^{|A|} A$ grade involution
- $A^\dagger := (-1)^{(|A|/2)} A$ reversion

and extend linearly to $\mathcal{C}l(X, R, r)$.

The grade involution is also called the (first) main involution. It has the property

$$(xy)^* = x^* y^*, \quad v^* = -v$$

(2.16)

for all $x, y \in \mathcal{C}l(X)$ and $v \in \mathcal{C}l^1(X)$, as is easily verified by expanding in linear combinations of elements in $\mathcal{P}(X)$ and using that $|A \triangle B| \equiv |A| + |B|$ (mod 2).

The reversion earns its name from the property

$$(xy)^\dagger = y^\dagger x^\dagger, \quad v^\dagger = v,$$

(2.17)

and it is sometimes called the second main involution or the principal antiautomorphism. This reversing behaviour follows directly from Proposition 2.6. We will find it convenient to have a name for the composition of these two involutions. Hence, we define the Clifford conjugate $x^\Box$ of $x \in \mathcal{C}l(X)$ by $x^\Box := x^* \dagger$ and observe the property

$$(xy)^\Box = y^\Box x^\Box, \quad v^\Box = -v.$$

(2.18)
Note that all the above involutions act by changing sign on some of the graded subspaces. We can define general involutions of that kind which will come in handy later.

**Definition 2.8.** For $A \in \mathcal{P}(X)$ define

$$[A] := (-1)^{(A \neq \emptyset)}A,$$

$$[A]_{p,q,\ldots,r} := (-1)^{|A|=p,q,\ldots,\text{or } r}A,$$

and extend linearly to $Cl(X, R, r)$.

We summarize the action of these involutions in Table 2.1. Note the periodicity.

|   | $Cl^0$ | $Cl^1$ | $Cl^2$ | $Cl^3$ | $Cl^4$ | $Cl^5$ | $Cl^6$ | $Cl^7$ |
|---|-------|-------|-------|-------|-------|-------|-------|-------|
| $*$ | +     | -     | +     | -     | +     | -     | +     | -     |
| $\dagger$ | +     | +     | -     | -     | +     | +     | -     | -     |
| $\box$ | +     | -     | -     | +     | -     | +     | -     | +     |
| $[ ]$ | +     | -     | -     | -     | -     | -     | -     | -     |

Table 2.1: The action of involutions on graded subspaces of $Cl$.

The scalar product has the symmetric property $x \ast y = y \ast x$ for all $x, y \in Cl$. Therefore, it forms a symmetric bilinear map $Cl \times Cl \to R$ which is degenerate if and only if $Cl$ (i.e. the signature $r$) is degenerate. This map coincides with the bilinear form $\beta_q$ when restricted to $V \subseteq G(V, q)$. Note also that subspaces of different grade are orthogonal with respect to the scalar product.

Another product that is often seen in the context of geometric algebra is the inner product, defined by $A \cdot B := (A \subseteq B \text{ or } A \supseteq B) \ AB = A \uplus B + A \downarrow B - A \ast B$. We will stick to the left and right inner products, however, because they admit a simpler handling of grades, something which is illustrated\(^3\) by the following

**Proposition 2.7.** For all $x, y, z \in Cl(X)$ we have

\[
\begin{align*}
x \land (y \land z) &= (x \land y) \land z, \\
x \uplus (y \downarrow z) &= (x \uplus y) \downarrow z, \\
x \uplus (y \uplus z) &= (x \land y) \uplus z, \\
x \ast (y \uplus z) &= (x \land y) \ast z.
\end{align*}
\]

**Proof.** This follows directly from Definition 2.7 and basic set logic. For example, taking $A, B, C \in \mathcal{P}(X)$ we have

\[
\begin{align*}
A \uplus (B \uplus C) &= (B \subseteq C)(A \subseteq B \triangle C)ABC \\
&= (B \subseteq C \text{ and } A \subseteq C \setminus B)ABC \\
&= (A \cap B = \emptyset \text{ and } A \cup B \subseteq C)ABC \\
&= (A \cap B = \emptyset)(A \triangle B \subseteq C)ABC \\
&= (A \land B) \downarrow C.
\end{align*}
\]

The other identities are proven in an equally simple way. \(\square\)

\(^3\)The corresponding identities with $\cdot$ instead of $\uplus, \downarrow$ need to be supplied with grade restrictions.
To work efficiently with geometric algebra it is crucial to understand how vectors behave under these operations.

**Proposition 2.8.** For all \( x, y \in \mathbb{C}(X) \) and \( v \in \mathbb{C}^1(X) \) we have
\[
\begin{align*}
v x &= v \perp x + v \wedge x, \\
v \perp x &= \frac{1}{2} (v x - x^* v) = -x^* \lrcorner v, \\
v \wedge x &= \frac{1}{2} (v x + x^* v) = x^* \lrcorner v, \\
v \lrcorner (xy) &= (v \lrcorner x) y + x^* (v \lrcorner y).
\end{align*}
\]
The first three identities are shown simply by using linearity and set relations, while the fourth follows immediately from the second. Note that for 1-vectors \( u, v \in \mathbb{C}^1 \) we have the basic relations
\[
\begin{align*}
u \lrcorner v &= v \lrcorner u = u \bullet v = \frac{1}{2} (uv + vu) \quad (2.20) \\
u \wedge v &= -v \wedge u = \frac{1}{2} (uv - vu). \quad (2.21)
\end{align*}
\]

It is often useful to expand the various products and involutions in terms of the grades involved. The following identities are left as exercises.

**Proposition 2.9.** For all \( x, y \in \mathbb{C}(X) \) we have
\[
\begin{align*}
x \wedge y &= \sum_{n,m \geq 0} \langle \langle x \rangle_n \langle y \rangle_m \rangle_{n+m}, \\
x \lrcorner y &= \sum_{0 \leq n \leq m} \langle \langle x \rangle_n \langle y \rangle_m \rangle_{m-n}, \\
x \bullet y &= \sum_{n \geq m \geq 0} \langle \langle x \rangle_n \langle y \rangle_m \rangle_{n-m}, \\
x \ast y &= \sum_{n,m \geq 0} \langle \langle x \rangle_n \langle y \rangle_m \rangle_{[n-m]}, \\
x^* &= \sum_{n \geq 0} (-1)^n \langle x \rangle_n, \\
x^\dagger &= \sum_{n \geq 0} (-1)^{\frac{n(n-1)}{2}} \langle x \rangle_n.
\end{align*}
\]

In the general setting of a Clifford algebra with scalars in a ring \( R \), we need to be careful about the notion of linear (in-)dependence. A subset \( \{x_1, x_2, \ldots, x_m\} \) of \( \mathbb{C} \) is called \textit{linearly dependent} iff there exist \( r_1, \ldots, r_m \in R \), not all zero, such that
\[
r_1 x_1 + r_2 x_2 + \ldots + r_m x_m = 0. \quad (2.22)
\]

Note that a single nonzero 1-vector could be linearly dependent in this context. We will prove an important theorem concerning linear dependence where we need the following

**Lemma 2.10.** If \( u_1, u_2, \ldots, u_k \) and \( v_1, v_2, \ldots, v_k \) are 1-vectors then
\[
(u_1 \wedge u_2 \wedge \cdots \wedge u_k) \ast (v_k \wedge v_{k-1} \wedge \cdots \wedge v_1) = \det [u_i \ast v_j]_{1 \leq i,j \leq k}.
\]

**Proof.** Since both sides of the expression are multilinear and alternating in both the \( u_i \)'s and the \( v_i \)'s, we need only consider ordered disjoint elements \( \{e_i\} \) in
the basis of singleton sets in $X$. Both sides are zero, except in the case
\[
(e_1 e_2 \ldots e_k) \ast (e_i e_{i+1} \ldots e_1) = \\
= r(e_1) r(e_2) \ldots r(e_k) = \det [r(e_i) \delta_{p,q}]_{1 \leq p, q \leq k} \tag{2.23}
\]
so we are done. \hfill \Box

**Theorem 2.11.** The 1-vectors $\{x_1, x_2, \ldots, x_m\}$ are linearly independent iff the $m$-vector $\{x_1 \wedge x_2 \wedge \cdots \wedge x_m\}$ is linearly independent.

**Proof.** Assume that $r_1 x_1 + \ldots + r_m x_m = 0$, where, say, $r_1 \neq 0$. Then
\[
(r_1 x_1 \wedge \cdots \wedge x_m) = (r_1 x_1) \wedge x_2 \wedge \cdots \wedge x_m = (r_1 x_1 + \ldots + r_m x_m) \wedge x_2 \wedge \cdots \wedge x_m = 0, \tag{2.24}
\]
since $x_i \wedge x_i = 0$.

Conversely, assume that $rX = 0$ for $r \neq 0$ in $R$ and $X = x_1 \wedge \cdots \wedge x_m$. We will use the basis minor theorem for arbitrary rings which can be found in the appendix. Assume that $x_j = x_{11} e_1 + \cdots + x_{1n} e_n$, where $x_{ij} \in R$ and $e_i \in X$ are basis elements such that $e_i^2 = 1$. This assumption on the signature is no loss in generality, since this theorem only concerns the exterior algebra associated to the outer product. It will only serve to simplify our reasoning below. Collect the coordinates in a matrix
\[
A := \begin{bmatrix}
  r x_{11} & x_{12} & \cdots & x_{1m} \\
  r x_{21} & x_{22} & \cdots & x_{2m} \\
  \vdots & \vdots & \ddots & \vdots \\
  r x_{n1} & x_{n2} & \cdots & x_{nm}
\end{bmatrix} \in R^{n \times m}, \ m \leq n \tag{2.25}
\]
and note that we can expand $rX$ in a grade-$m$ basis as
\[
rX = \sum_{E \subseteq X : |E| = m} (rX \ast E^\dagger) E = \sum_{E \subseteq X : |E| = m} \det (A_{E, \{1, \ldots, m\}}) E, \tag{2.26}
\]
where we used Lemma 2.10. We find that the determinant of each $m \times m$ minor of $A$ is zero.

Now, let $k$ be the rank of $A$. Then we must have $k < m$, and if $k = 0$ then $r x_{1i} = 0$ and $x_{ij} = 0$ for all $i = 1, \ldots, n$, $j > 1$. But that would mean that $\{x_1, \ldots, x_m\}$ are linearly dependent. Therefore we assume that $k > 0$ and, without loss of generality, that
\[
d := \det \begin{bmatrix}
  r x_{11} & x_{12} & \cdots & x_{1k} \\
  \vdots & \vdots & \ddots & \vdots \\
  r x_{k1} & x_{k2} & \cdots & x_{kk}
\end{bmatrix} \neq 0. \tag{2.27}
\]
By the basis minor theorem there exist $r_1, \ldots, r_k \in R$ such that
\[
r_1 x_1 + r_2 x_2 + \ldots + r_k x_k + d x_m = 0. \tag{2.28}
\]
Hence, $\{x_1, \ldots, x_m\}$ are linearly dependent. \hfill \Box
For our final set of operations, we will consider a nondegenerate geometric algebra \( \mathcal{G} \) with pseudoscalar \( I \). The nondegeneracy implies that there exists a natural duality between the inner and outer products.

**Definition 2.9.** We define the dual of \( x \in \mathcal{G} \) by \( x^c := xI^{-1} \). The dual outer product or meet \( \vee \) is defined such that the diagram

\[
\begin{array}{ccc}
\mathcal{G} \times \mathcal{G} & \xrightarrow{\vee} & \mathcal{G} \\
(\ )^c & \downarrow & (\ )^c \\
\mathcal{G} \times \mathcal{G} & \xrightarrow{\wedge} & \mathcal{G}
\end{array}
\]

commutes, i.e. \((x \vee y)^c := x^c \wedge y^c \Rightarrow x \vee y = ((xI^{-1}) \wedge (yI^{-1}))I\).

**Remark.** In \( \mathcal{C}(X) \), the corresponding dual of \( A \in \mathcal{P}(X) \) is \( A^c = AX^{-1} = \tau(X, X)^{-1} \tau(A, X)A \triangle X \times A^c \), the complement of the set \( A \). Hence, we really find that the dual is the linearization of a sign (or orientation) respecting complement. This motivates our choice of notation.

**Proposition 2.12.** For all \( x, y \in \mathcal{G} \) we have
\[
\begin{align*}
x \uplus y^c &= (x \wedge y)^c, \\
x \wedge y^c &= (x \uplus y)^c.
\end{align*}
\]  
(2.29)

**Proof.** Using Proposition 2.7 and the fact that \( xI = x \uplus I \), we obtain
\[
x \uplus (yI^{-1}) = x \uplus (y \uplus I^{-1}) = (x \wedge y) \uplus I = (x \wedge y)I^{-1},
\]  
(2.30)

and from this follows also the second identity
\[
(x \wedge y^c)I^{-1}I = (x \wedge y^c I) = (x \wedge y)I^{-2}I.
\]  
(2.31)

It is instructive to compare these results with those in the language of differential forms and Hodge duality, which are completely equivalent. In that setting one often starts with an outer product and then uses a metric to define a dual. The inner product is then defined from the outer product and dual according to (2.29).

### 2.4 Vector space geometry

We will now leave the general setting of Clifford algebra for a moment and instead focus on the geometric properties of \( \mathcal{G} \) and its newly defined operations.

**Definition 2.10.** A blade is an outer product of 1-vectors. We define the following:

\[
\begin{align*}
B_k &:= \{v_1 \wedge v_2 \wedge \cdots \wedge v_k \in \mathcal{G} : v_i \in \mathcal{V}\} \quad \text{the set of } k\text{-blades} \\
B &:= \bigcup_{k=0}^{\infty} B_k \quad \text{the set of all blades} \\
B^* &:= B \setminus \{0\} \quad \text{the nonzero blades} \\
B^\times &:= \{B \in B : B^2 \neq 0\} \quad \text{the invertible blades}
\end{align*}
\]

The basis blades associated to an orthogonal basis \( E = \{e_i\}_{i=1}^{\text{dim } \mathcal{V}} \) is the basis of \( \mathcal{G} \) generated by \( E \), i.e.

\[
B_E := \{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \in \mathcal{G} : i_1 < i_2 < \ldots < i_k\} \simeq \mathcal{P}(E) \text{ in } \mathcal{C}.
\]
We include the unit 1 among the blades and call it the 0-blade. Note that \( B_k \subseteq G^k \), and that by applying Proposition 2.8 recursively we can expand a blade as a sum of geometric products,
\[
a_1 \wedge a_2 \wedge \cdots \wedge a_k = \frac{1}{k!} \sum_{\pi \in S_k} \text{sign}(\pi) a_{\pi(1)} (a_{\pi(2)} \wedge \cdots \wedge a_{\pi(k)}).
\]

This expression is clearly similar to a determinant, except that this is a product of vectors instead of scalars.

The key property of blades is that they represent linear subspaces of \( V \). This is made precise by the following

**Proposition 2.13.** If \( A = a_1 \wedge a_2 \wedge \cdots \wedge a_k \neq 0 \) is a nonzero \( k \)-blade and \( a \in V \) then
\[
a \wedge A = 0 \iff a \in \text{Span}\{a_1, a_2, \ldots, a_k\}.
\]

**Proof.** This follows directly from Theorem 2.11 since \( \{a_i\} \) are linearly independent and \( a \wedge A = 0 \iff \{a, a_i\} \) are linearly dependent. \( \square \)

Hence, to every nonzero \( k \)-blade \( A = a_1 \wedge a_2 \wedge \cdots \wedge a_k \) there corresponds a unique \( k \)-dimensional subspace
\[
\bar{A} := \{a \in V : a \wedge A = 0\} = \text{Span}\{a_1, a_2, \ldots, a_k\}.
\]

Conversely, if \( \bar{A} \subseteq V \) is a \( k \)-dimensional subspace of \( V \), then we can find a nonzero \( k \)-blade \( A = a_1 \wedge a_2 \wedge \cdots \wedge a_k \) representing \( \bar{A} \) by simply taking a basis \( \{a_i\}_{i=1}^k \) of \( \bar{A} \) and forming
\[
A := a_1 \wedge a_2 \wedge \cdots \wedge a_k.
\]

We thus have the geometric interpretation of blades as subspaces with an associated orientation (sign) and magnitude. Since every element in \( G \) is a linear combination of basis blades, we can think of every element as representing a linear combination of subspaces. In the case of a nondegenerate algebra these basis subspaces are nondegenerate as well. On the other hand, any blade which represents a nondegenerate subspace can also be treated as a basis blade associated to an orthogonal basis. This will follow in the discussion below.

**Proposition 2.14.** Every \( k \)-blade can be written as a geometric product of \( k \) vectors.

**Proof.** Take a nonzero \( A = a_1 \wedge \cdots \wedge a_k \in B^* \). Pick an orthogonal basis \( \{e_i\}_{i=1}^k \) of the subspace \( (\bar{A}, q|_{\bar{A}}) \). Then we can write \( a_i = \sum_j \beta_{ij} e_j \) for some \( \beta_{ij} \in \mathbb{F} \), and \( A = \text{det} [\beta_{ij}] e_1 e_2 \cdots e_k \) by (2.32). \( \square \)

There are a number of useful consequences of this result.

**Corollary.** If \( A \in B \) then \( A^2 \) is a scalar.

**Proof.** Use the expansion of \( A \) above to obtain
\[
A^2 = (\text{det} [\beta_{ij}])^2 (-1)^{k(k-1)} q(e_1)q(e_2) \cdots q(e_k) \in \mathbb{F}.
\]

**Corollary.** If \( A \in B^\times \) then \( A \) has an inverse \( A^{-1} = \frac{1}{q(A)} A \).
Corollary. If $A \in \mathcal{B}^\perp$ then $q$ is nondegenerate on $\bar{A}$ and there exists an orthogonal basis $E$ of $V$ such that $A \in \mathcal{B}_E$.

Proof. The first statement follows directly from (2.35). For the second statement note that, since $q$ is nondegenerate on $\bar{A}$, we have $\bar{A} \cap \bar{A}^\perp = 0$. Take an orthogonal basis $\{e_i\}$ of $\bar{A}$. For any $v \in V$ we have that $v - \sum \beta q(v,e_i)q(e_i)^{-1}e_i \in \bar{A}^\perp$. Thus, $V = \bar{A} + \bar{A}^\perp$ and we can extend $\{e_i\}$ to an orthogonal basis of $V$ consisting of one part in $\bar{A}$ and one part in $\bar{A}^\perp$. By rescaling this basis we have $A = e_1 \wedge \cdots \wedge e_k$.

Remark. Note that if we have an orthogonal basis of a subspace of $V$ where $q$ is degenerate, then it may not be possible to extend this basis to an orthogonal basis for all of $V$. $\mathbb{R}^{1,1}$ for example has two null-spaces, but these are not orthogonal. If the space is euclidean or anti-euclidean, though, orthogonal bases can always be extended (e.g. using the Gram-Schmidt algorithm).

It is useful to be able to work efficiently with general bases of $V$ and $G$ which need not be orthogonal. Let $\{e_1, \ldots, e_n\}$ be any basis of $V$. Then $\{e_1\}$ is a basis of $G(V)$, where we use a multi-index notation

$$i = (i_1, i_2, \ldots, i_k), \quad i_1 < i_2 < \ldots < i_k, \quad 0 < k < n$$

and

$$e^{()} := 1, \quad e^{(i_1, i_2, \ldots, i_k)} := e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}. \quad (2.37)$$

Sums over $i$ are understood to be performed over all allowed such indices. If $G$ is nondegenerate then the scalar product $(A, B) \mapsto A \ast B$ is also nondegenerate and we can find a so-called reciprocal basis $\{e^1, \ldots, e^n\}$ of $V$ such that

$$e^i \ast e_j = \delta^i_j. \quad (2.38)$$

The reciprocal basis is easily verified to be given by

$$e^i = (-1)^{i-1}(e_1 \wedge \cdots \wedge e_i \wedge \cdots \wedge e_n)e_{1,\ldots,n}^{-1}, \quad (2.39)$$

where $\cdot$ denotes a deletion. Furthermore, we have that $\{e^i\}$ is a reciprocal basis of $G$, where $e^{(i_1, \ldots, i_k)} := e_i \wedge \cdots \wedge e_i$. This follows since by Lemma 2.10 and (2.38),

$$e^i \ast e_j = (e^1 \wedge \cdots \wedge e^1) \ast (e_{j_1} \wedge \cdots \wedge e_{j_1}) = \delta^k_i \det [e_p \ast e_q] e_p, e_q = \delta^j_i. \quad (2.40)$$

We now have the coordinate expansions

$$\begin{align*}
v &= \sum_i v \ast e^i e_i = \sum_i v \ast e_i e^i \quad \forall \ v \in V, \\
x &= \sum_i x \ast e^i e_i = \sum_i x \ast e_i e^i \quad \forall \ x \in G(V).
\end{align*} \quad (2.41)$$

In addition to being useful in coordinate expansions, the general and reciprocal bases also provide a geometric understanding of the dual operation because of the following

Theorem 2.15. Assume that $G$ is nondegenerate. If $A = a_1 \wedge \cdots \wedge a_k \in \mathcal{B}^*$ and we extend $\{a_i\}_{i=1}^k$ to a basis $\{a_i\}_{i=1}^n$ of $V$ then

$$A^\ast \propto a^{k+1} \wedge a^{k+2} \wedge \cdots \wedge a^n,$$

where $\{a_i\}$ is the reciprocal basis of $\{a_i\}$. 

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Proof. Using an expansion of the inner product into sub-blades (this will not be explained in detail here, see [9] or [23]) plus orthogonality (2.38), we obtain

\[
A^c = A \perp I^{-1} \propto (a_k \wedge \cdots \wedge a_1) \perp (a^1 \wedge \cdots \wedge a^k \wedge a^{k+1} \wedge \cdots \wedge a^n)
= (a_k \wedge \cdots \wedge a_1) \propto (a^1 \wedge \cdots \wedge a^k) a^{k+1} \wedge \cdots \wedge a^n \tag{2.42}
= a^{k+1} \wedge \cdots \wedge a^n.
\]

Corollary. If \(A\) and \(B\) are blades then \(A^c, A \wedge B, A \vee B\) and \(A \perp B\) are blades as well.

The blade-subspace correspondence then gives us a geometric interpretation of these operations.

Proposition 2.16. If \(A, B \in B^*\) are nonzero blades then \(A^c = A^\perp\) and

\[
A \wedge B \neq 0 \Rightarrow A \wedge B = A \perp B \text{ and } A \cap B = 0, \\
A \perp B = \mathcal{V} \Rightarrow A \vee B = A \cap B, \\
A \perp B \neq 0 \Rightarrow A \perp B = A^\perp \cap B, \\
A \subseteq B \Rightarrow A \perp B = AB, \\
A \cap B^\perp \neq 0 \Rightarrow A \perp B = 0.
\]

The proofs of the statements in the above corollary and proposition are left as exercises. Some of them can be found in [23] and [9].

2.5 Linear functions

Since \(G\) is itself a vector space which embeds \(V\), it is natural to consider the properties of linear functions on \(G\). There is a special class of such functions, called outermorphisms, which can be said to respect the structure of \(G\) in a natural way. We will see that, just as the geometric algebra \(G(V, q)\) is completely determined by the underlying vector space \((V, q)\), an outermorphism is completely determined by its behaviour on \(V\).

Definition 2.11. A linear map \(F : G \to G'\) is called an outermorphism or \(\wedge\)-morphism if

\[
i)\ F(1) = 1, \\
ii)\ F(G^m) \subseteq G'^m \ \forall \ m \geq 0, \ \text{(grade preserving)} \\
iii)\ F(x \wedge y) = F(x) \wedge F(y) \ \forall \ x, y \in G.
\]

A linear transformation \(F : G \to G\) is called a dual outermorphism or \(\vee\)-morphism if

\[
i)\ F(I) = I, \\
ii)\ F(G^m) \subseteq G'^m \ \forall \ m \geq 0, \\
iii)\ F(x \vee y) = F(x) \vee F(y) \ \forall \ x, y \in G.
\]

Theorem 2.17. For every linear map \(f : V \to W\) there exists a unique outermorphism \(f_\wedge : G(V) \to G(W)\) such that \(f_\wedge(v) = f(v) \ \forall \ v \in V\).
Proof. Take a general basis \( \{e_1, \ldots, e_n\} \) of \( \mathcal{V} \) and define, for \( 1 \leq i_1 < i_2 < \ldots < i_m \leq n, \)
\[
f_\wedge(e_{i_1} \wedge \cdots \wedge e_{i_m}) := f(e_{i_1}) \wedge \cdots \wedge f(e_{i_m}),
\]
(2.43)
and extend \( f_\wedge \) to the whole of \( \mathcal{G}(\mathcal{V}) \) by linearity. We also define \( f_\wedge(\alpha) := \alpha \) for \( \alpha \in \mathcal{F} \). Hence, (i) and (ii) are satisfied. (iii) is easily verified by expanding in the induced basis \( \{e_1\} \) of \( \mathcal{G}(\mathcal{V}) \). Unicity is obvious since our definition was necessary. \( \square \)

Uniqueness immediately implies the following.

**Corollary.** If \( f: \mathcal{V} \to \mathcal{V}' \) and \( g: \mathcal{V}' \to \mathcal{V}'' \) are linear then \( (g \circ f)_\wedge = g_\wedge \circ f_\wedge \).

**Corollary.** If \( F: \mathcal{G}(\mathcal{V}) \to \mathcal{G}(\mathcal{W}) \) is an outermorphism then \( F = (F|_{\mathcal{V}})_\wedge \).

In the setting of \( \mathcal{C}l \) this means that an outermorphism \( F: \mathcal{C}l(X, R, r) \to \mathcal{C}l(X', R, r') \) is completely determined by its values on \( X \).

We have noted that a nondegenerate \( \mathcal{G} \) results in a nondegenerate bilinear form \( x \wedge y \). This gives us a canonical isomorphism \( \theta: \mathcal{G} \to \mathcal{G}^* = \text{Lin}(\mathcal{G}, \mathcal{F}) \) between the elements of \( \mathcal{G} \) and the linear functionals on \( \mathcal{G} \) as follows. For every \( x \in \mathcal{G} \) we define a linear functional \( \theta(x) \) by \( \theta(x)(y) := x \wedge y \). Taking a general basis \( \{e_1\} \) of \( \mathcal{G} \) and using (2.40) we obtain a dual basis \( \{\theta(e^i)\} \) such that \( \theta(e^i)(e_j) = \delta^i_j \).

Now that we have a canonical way of moving between \( \mathcal{G} \) and its dual space \( \mathcal{G}^* \), we can for every linear map \( F: \mathcal{G} \to \mathcal{G} \) define an adjoint map \( F^*: \mathcal{G} \to \mathcal{G} \) by
\[
F^*(x) := \theta^{-1}(\theta(x) \circ F).
\]
(2.44)
Per definition, this has the expected and unique property
\[
F^*(x) \wedge y = x \wedge F(y)
\]
(2.45)
for all \( x, y \in \mathcal{G} \). Note that if we restrict our attention to \( \mathcal{V} \) this construction results in the usual adjoint \( f^* \) of a linear map \( f: \mathcal{V} \to \mathcal{V} \).

**Theorem 2.18 (Hestenes’ Theorem).** Assume that \( \mathcal{G} \) is nondegenerate and let \( F: \mathcal{G} \to \mathcal{G} \) be an outermorphism. Then the adjoint \( F^* \) is also an outermorphism and
\[
F(x) \wedge y = F(x \wedge F^*(y)),
\]
for all \( x, y \in \mathcal{G} \).

**Proof.** We first prove that \( F^* \) is an outermorphism. The fact that \( F^* \) is grade preserving follows from (2.43) and the grade preserving property of \( F \). Now take basis blades \( x = x_1 \wedge \cdots \wedge x_m \) and \( y = y_m \wedge \cdots \wedge y_1 \) with \( x_i, y_j \in \mathcal{V} \). Then
\[
F^*(x_1 \wedge \cdots \wedge x_m) \wedge y = (x_1 \wedge \cdots \wedge x_m) \wedge (y_1 \wedge \cdots \wedge y_m)
\]
\[
= (x_1 \wedge \cdots \wedge x_m) \wedge (F(y_m) \wedge \cdots \wedge F(y_1))
\]
\[
= \det [x_i \wedge F(y_j)]_{i,j} = \det [F^*(x_i) \wedge y_j]_{i,j}
\]
\[
= (F^*(x_1) \wedge \cdots \wedge F^*(x_m)) \wedge (y_1 \wedge \cdots \wedge y_m)
\]
\[
= (F^*(x_1) \wedge \cdots \wedge F^*(x_m)) \wedge y,
\]
where we have used Lemma 2.10. By linearity and nondegeneracy it follows that $F^*$ is an outerorphism. The first identity stated in the theorem now follows quite easily from Proposition 2.7. For any $z \in G$ we have

$$z \ast (x \uplus F(y)) = (z \wedge x) \ast F(y) = F^*(z \wedge x) \ast y$$

$$= (F^*(z) \wedge F^*(x)) \ast y = F^*(z) \ast (F^*(x) \uplus y)$$

$$= z \ast F(F^*(x) \uplus y).$$

The nondegeneracy of the scalar product then gives the first identity. The second identity is proven similarly, using that $(x \uplus y) \ast z = x \ast (y \wedge z)$. □

From uniqueness of outermorphisms we also obtain the following

**Corollary.** If $f : \mathcal{V} \to \mathcal{V}$ is a linear transformation then $(f^*) \lambda = (f_\lambda)^*$. This means that we can simply write $f^*_\lambda$ for the adjoint outermorphism of $f$.

A powerful concept in geometric algebra (or exterior algebra) is the generalization of eigenvectors to so called eigenblades. For a function $f : \mathcal{V} \to \mathcal{V}$, a $k$-eigenblade with eigenvalue $\lambda \in \mathbb{F}$ is a blade $A \in \mathcal{B}_k$ such that

$$f_\lambda(A) = \lambda A. \quad (2.46)$$

Just as eigenvectors can be said to represent invariant 1-dimensional subspaces of a function, a $k$-blade with nonzero eigenvalue represents an invariant $k$-dimensional subspace. One important example of an eigenblade is the pseudoscalar $I$, which represents the whole invariant vector space $\mathcal{V}$. Since $f_\lambda$ is grade preserving, we must have $f_\lambda(I) = \lambda I$ for some $\lambda \in \mathbb{F}$ which we call the determinant of $f$, i.e.

$$f_\lambda(I) = \det f\ I. \quad (2.47)$$

Expanding $\det f = f(I) \ast I^{-1}$ in a basis using Lemma 2.10 one finds that this agrees with the usual definition of the determinant of a linear function.

In the following we assume that $\mathcal{G}$ is nondegenerate, so that $I^2 \neq 0$.

**Definition 2.12.** For linear $F : \mathcal{G} \to \mathcal{G}$ we define the dual map $F^c : \mathcal{G} \to \mathcal{G}$ by $F^c(x) := F(xI)I^{-1}$, so that the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{G} & \overset{F}{\rightarrow} & \mathcal{G} \\
(\ )^c & \downarrow & (\ )^c \\
\mathcal{G} & \overset{F^c}{\rightarrow} & \mathcal{G}
\end{array}$$

**Proposition 2.19.** We have the following properties of the dual map:

i) \quad $F^{cc} = F$,

ii) \quad $(F \circ G)^c = F^c \circ G^c$,

iii) \quad $id^c = id$,

iv) \quad $F(G^*) \subseteq \mathcal{G}^t \Rightarrow F^c(G^{dim \vee -s}) \subseteq \mathcal{G}^{dim \vee -t}$,

v) \quad $F \wedge$-morphism $\Rightarrow F^c \vee$-morphism,

for all linear $F, G : \mathcal{G} \to \mathcal{G}$.
The proofs are simple and left as exercises to the reader. As a special case of Theorem 2.18 we obtain, with \( y = I \) and a linear map \( f : V \to V \),

\[
\det f \ xI = f_\wedge(f_\wedge(x)I),
\]

(2.48)

so that

\[
\det f \ id = f_\wedge \circ f_\wedge = f_\wedge \circ f_\wedge^e.
\]

(2.49)

If \( \det f \neq 0 \) we then have a simple expression for the inverse:

\[
f_\wedge^{-1} = (\det f)^{-1} f_\wedge^e,
\]

(2.50)

which is essentially the dual of the adjoint. \( f^{-1} \) is obtained by restricting to \( V \).

An orthogonal transformation \( f \) has \( f^{-1} = f^* \) and \( \det f = 1 \), so in that case \( f_\wedge = f_\wedge^e \).

### 2.6 Infinite-dimensional Clifford algebra

This far we have only defined the Clifford algebra \( Cl(X, R, r) \) of a finite set \( X \), resulting in a finite-dimensional algebra \( G(V) \) whenever \( R \) is a field. In order for this combinatorial construction to qualify as a complete generalization of \( G \), we would at least like to be able to define the corresponding Clifford algebra of an infinite-dimensional vector space, something which was possible for \( G \) in Definition 2.2.

The treatment of \( Cl \) in the previous subsections has been deliberately put in a form which eases the generalization to an infinite \( X \). Reconsidering Definition 2.4 we now have two possibilities; either we consider the set \( \mathcal{P}(X) \) of all subsets of \( X \), or the set \( \mathcal{F}(X) \) of all finite subsets. We therefore define, for an arbitrary set \( X \), ring \( R \), and signature \( r : X \to R \),

\[
Cl(X, R, r) := \bigoplus_{\mathcal{P}(X)} R \quad \text{and} \quad Cl_\mathcal{F}(X, R, r) := \bigoplus_{\mathcal{F}(X)} R.
\]

(2.51)

Elements in \( Cl \) (\( Cl_\mathcal{F} \)) are finite linear combinations of (finite) subsets of \( X \).

Our problem now is to define a Clifford product for \( Cl \) and \( Cl_\mathcal{F} \). This can be achieved just as in the finite case if only we can find a map \( \tau : \mathcal{P}(X) \times \mathcal{P}(X) \to R \) satisfying the conditions in Lemma 2.6. This is certainly not a trivial task. Starting with the case \( Cl_\mathcal{F} \) it is sufficient to construct such a map on \( \mathcal{F}(X) \).

We call a map \( \tau : \mathcal{F}(X) \times \mathcal{F}(X) \to R \) grassmannian on \( X \) if it satisfies (i)-(v) in Lemma 2.6 with \( \mathcal{P}(X) \) replaced by \( \mathcal{F}(X) \).

**Theorem 2.20.** For any \( X, R, r \) there exists a grassmannian map on \( \mathcal{F}(X) \).

**Proof.** We know that there exists such a map for any finite \( X \). Let \( Y \subseteq X \) and assume \( \tau : \mathcal{F}(Y) \times \mathcal{F}(Y) \to R \) is grassmannian on \( Y \). If there exists \( z \in X \setminus Y \) we can, by proceeding as in the proof of Lemma 2.6, extend \( \tau \) to a grassmannian map \( \tau : \mathcal{F}(Y \cup \{z\}) \times \mathcal{F}(Y \cup \{z\}) \to R \) on \( Y \cup \{z\} \) such that \( \tau_{|\mathcal{F}(Y) \times \mathcal{F}(Y)} = \tau' \).

We will now use transfinite induction, or the Hausdorff maximality theorem\(^4\), to prove that \( \tau \) can be extended to all of \( \mathcal{F}(X) \subseteq \mathcal{P}(X) \). Note that if \( \tau \) is grassmannian on \( Y \subseteq X \) then \( \tau \) is also a relation \( \tau \subseteq \mathcal{P}(X) \times \mathcal{P}(X) \times R \). Let

\[
\mathcal{H} := \left\{(Y, \tau) \in \mathcal{P}(X) \times \mathcal{P}(X) \times \mathcal{P}(X) \times R : \tau \text{ is grassmannian on } Y \right\}.
\]

(2.52)

\(^4\)This theorem should actually be regarded as an axiom of set theory since it is equivalent to the Axiom of Choice.
Then $\mathcal{H}$ is partially ordered by

$$(Y, \tau) \leq (Y', \tau') \quad \text{iff} \quad Y \subseteq Y' \quad \text{and} \quad \tau'|_{\mathcal{F}(Y) \times \mathcal{F}(Y)} = \tau.$$  \hfill (2.53)

By the Hausdorff maximality theorem, there exists a maximal totally ordered “chain” $\mathcal{K} \subseteq \mathcal{H}$. Put $Y^* := \bigcup_{(Y, \tau) \in \mathcal{K}} Y$. We want to define a grassmannian map $\tau^*$ on $Y^*$, for if we succeed in that, we find $(Y^*, \tau^*) \in \mathcal{H} \cap \mathcal{K}$ and can conclude that $Y^* = X$ by maximality and the former result.

Take finite subsets $A$ and $B$ of $Y^*$. Each of the finite elements in $A \cup B$ lies in some $Y$ such that $(Y, \tau) \in \mathcal{K}$. Therefore, by the total ordering of $\mathcal{K}$, there exists one such $Y$ containing $A \cup B$. Put $\tau^*(A, B) := \tau(A, B)$, where $(Y, \tau)$ is this chosen element in $\mathcal{K}$. $\tau^*$ is well-defined since if $A \cup B \subseteq Y$ and $A \cup B \subseteq Y'$ where $(Y, \tau), (Y', \tau') \in \mathcal{K}$ then $Y \subseteq Y'$ or $Y' \subseteq Y$ and $\tau, \tau'$ agree on $(A, B)$. It is easy to verify that this $\tau^*$ is grassmannian on $Y^*$, since for each $A, B, C \in \mathcal{F}(X)$ there exists $(Y, \tau) \in \mathcal{K}$ such that $A \cup B \cup C \subseteq Y$.

We have shown that there exists a map $\tau : \mathcal{F}(X) \times \mathcal{F}(X) \to R$ with the properties in Lemma 2.25. We can then define the Clifford product on $\text{Cl}_\mathcal{F}(X)$ as usual by $AB := \tau(A, B)A \Delta B$ for $A, B \in \mathcal{F}(X)$ and linear extension. Since only finite subsets are included, most of the previous constructions for finite-dimensional $\text{Cl}$ carry over to $\text{Cl}_\mathcal{F}$. For example, the decomposition into graded subspaces remains but now goes up towards infinity,

$$\text{Cl}_\mathcal{F} = \bigoplus_{k=0}^{\infty} \text{Cl}_\mathcal{F}^k.$$  \hfill (2.54)

Furthermore, Proposition 2.6 still holds, so the reverse and all other involutions behave as expected.

The following theorem shows that it is possible to extend $\tau$ all the way to $\mathcal{P}(X)$ even in the infinite case. We therefore have a Clifford product also on $\text{Cl}(X)$.

**Theorem 2.21.** For any set $X$ there exists a map $| \cdot |_2 : \mathcal{P}(\mathcal{P}(X)) \to \mathbb{Z}_2$ such that

1. $|A|_2 \equiv |A| \pmod{2}$ for finite $A \subseteq \mathcal{P}(X),$

2. $|A \cup B|_2 = |A|_2 + |B|_2 \pmod{2}$ if $A \cap B = \emptyset$.

Furthermore, for any commutative ring $R$ and signature $r : X \to R$ such that $r(X)$ is contained in a finite and multiplicatively closed subset of $R$, there exists a map $\tau : \mathcal{P}(X) \times \mathcal{P}(X) \to R$ such that properties (i)-(v) in Lemma 2.25 hold, plus

3. $\tau(A, B) = (-1)^{|A - B|_2} |A|_2 + |B|_2 + |A \Delta B|_2$ $\forall A, B \in \mathcal{P}(X).$

Here, $\binom{A}{n}$ denotes the set of all subsets of $A$ with $n$ elements. Note that for a finite set $A$, $|\binom{A}{n}| = \binom{|A|}{n}$ so that for example $|\binom{A}{1}| = |A|$ (in general, card $\binom{A}{n} = \text{card } A$ and $\binom{n}{2} = \frac{1}{2}|A|(|A| - 1)$). This enables us to extend the basic involutions $\star$, $\dagger$ and $\circ$ to infinite sets as

$$A^\star := (-1)^{|\binom{A}{1}}_2 A,$$

$$A^\dagger := (-1)^{|\binom{A}{2}}_2 A,$$
and because $|(A \wedge B)|_2 = |(A)|_2 + |(B)|_2 \pmod{2}$ still holds, we find that they satisfy the fundamental properties (2.11)-(2.13) for all elements of $\mathcal{C}l(X)$. The extra requirement (vi) on $\tau$ was necessary here since we cannot use Proposition 2.16 for infinite sets. Moreover, we can no longer write the decomposition (2.54) defined by extra requirement (vi) on $\tau$. We will not consider this here.

Let us now see how $\mathcal{C}l_\tau$ and $\mathcal{C}l$ can be applied to the setting of an infinite-dimensional vector space $\mathcal{V}$ over a field $\mathbb{F}$ and with a quadratic form $q$. By the Hausdorff maximality theorem one can actually find a (necessarily infinite) orthogonal basis $E$ for this space in the sense that any vector in $\mathcal{V}$ can be written as a finite linear combination of elements in $E$ and that $\beta_q(e, e') = 0$ for any pair of disjoint elements $e, e' \in E$. We then have

$$\mathcal{G}(\mathcal{V}, q) \cong \mathcal{C}l_\tau(\mathcal{V}, \mathbb{F}, q|_{\mathcal{V}}).$$

which is proved just like in the finite-dimensional case, using Proposition 2.21. The only difference is that one needs to check that the homomorphism $F: \mathcal{G} \to \mathcal{C}l_\tau$ is also injective.

The $k$-blades of $\mathcal{C}l_\tau$ represent $k$-dimensional subspaces of $\mathcal{V}$ even in the infinite case. Due to the intuitive and powerful handling of finite-dimensional geometry which was possible in a finite-dimensional $\mathcal{G}$, it would be extremely satisfying to be able to generalize the blade concept to e.g. closed subspaces of an infinite-dimensional Hilbert space. One could hope that the infinite basis subsets in $\mathcal{C}l(E)$ provide this generalization. Unfortunately, this is not so easy since $\mathcal{C}l(E)$ depends heavily on the choice of basis $E$. Let us sketch an intuitive picture of why this is so.

With a countable basis $E = \{e_i\}_{i=1}^\infty$, an infinite basis blade in $\mathcal{C}l$ could be thought of as an infinite product $A = e_{i_1}e_{i_2}e_{i_3} \ldots = \prod_{k=1}^{\infty} e_{i_k}$. A change of basis to $E'$ would turn each $e \in E$ into a finite linear combination of elements in $E'$, e.g. $e_j = \sum_{k}^{} \beta_{jk}e'_k$. However, this would require $A$ to be an infinite sum of basis blades in $E'$, which is not allowed. Note that this is no problem in $\mathcal{C}l_\tau$ since a basis blade $A = \prod_{k=1}^{\infty} e_{i_k}$ is a finite product and the change of basis therefore results in a finite sum. It may be possible to treat infinite sums in $\mathcal{C}l$ by taking the topology of $\mathcal{V}$ into account, but at present this issue is not clear.

Finally, we consider a nice application of the infinite-dimensional Clifford algebra $\mathcal{C}l_\tau$. For a vector space $\mathcal{V}$, define the simplicial complex algebra

$$\mathcal{C}(\mathcal{V}) := \mathcal{C}l_\tau(\mathcal{V}, \mathbb{R}, 1),$$

where we forget about the vector space structure of $\mathcal{V}$ and treat individual points $\hat{v} \in \mathcal{V}$ as orthogonal basis 1-vectors in $\mathcal{C}l_\tau^1$ with $\hat{v}^2 = 1$. The dot indicates that we think of $v$ as a point rather than a vector. A basis $(k+1)$-blade in $\mathcal{C}(\mathcal{V})$ consists of a product $\hat{v}_0\hat{v}_1 \ldots \hat{v}_k$ of individual points and represents a (possibly
degenerate) oriented $k$-simplex in $V$. This simplex is given by the convex hull

\[
\text{Conv}\{v_0, v_1, \ldots, v_k\} := \left\{ \sum_{i=0}^{k} \alpha_i v_i \in V : \alpha_i \geq 0, \sum_{i=0}^{k} \alpha_i = 1 \right\}.
\] (2.58)

Hence, an arbitrary element in $C(V)$ is a linear combination of simplices and can therefore represent a simplicial complex in $V$. The restriction of $C(V)$ to the $k$-simplices of a simplicial complex $K$ is usually called the $k$-chain group $C_k(K)$. Here the generality of the ring $R$ comes in handy because one often works with $R = \mathbb{Z}$ in this context.

The Clifford algebra structure of $C(V)$ handles the orientation of simplices, so that e.g. the line from the point $\dot{v}_0$ to $\dot{v}_1$ is $\dot{v}_0 \dot{v}_1 = -\dot{v}_1 \dot{v}_0$. Furthermore, it allows us to define the boundary operator

\[
\partial : C(V) \rightarrow C(V),
\]

\[
\partial(x) := \sum_{\dot{v} \in V} \dot{v} \cup x.
\]

Note that this is well-defined since only a finite number of points $\dot{v}$ can be present in any fixed $x$. For a $k$-simplex, we have

\[
\partial(\dot{v}_0 \dot{v}_1 \ldots \dot{v}_k) = \sum_{i=0}^{k} \dot{v}_i \cup (\dot{v}_0 \dot{v}_1 \ldots \dot{v}_k) = \sum_{i=0}^{k} (-1)^i (\dot{v}_0 \dot{v}_1 \ldots \dot{v}_i \ldots \dot{v}_k).
\] (2.59)

This shows that $\partial$ really is the traditional boundary operator on simplices. Proposition 2.7 now makes the proof of $\partial^2 = 0$ a triviality,

\[
\partial^2(x) = \sum_{\dot{u} \in V} \dot{u} \cup \left( \sum_{\dot{v} \in V} \dot{v} \cup x \right) = \sum_{\dot{u}, \dot{v} \in V} \dot{u} \cup (\dot{v} \cup x) = \sum_{\dot{u}, \dot{v} \in V} (\dot{u} \wedge \dot{v}) \cup x = 0.
\] (2.60)

We can also assign a geometric measure $\sigma$ to simplices, by mapping a $k$-simplex to a corresponding $k$-blade in $G(V)$ representing the directed volume of the simplex. Define $\sigma : C(V) \rightarrow G(V)$ by

\[
\sigma(1) := 0,
\]

\[
\sigma(\dot{v}) := 1,
\]

\[
\sigma(\dot{v}_0 \dot{v}_1 \ldots \dot{v}_k) := \frac{1}{k!} (v_1 - v_0) \wedge (v_2 - v_0) \wedge \cdots \wedge (v_k - v_0),
\]

and extending linearly. One can verify that this is well-defined and that the geometric measure of a boundary is zero, i.e. $\sigma \circ \partial = 0$. One can take this construction even further and arrive at “discrete” equivalents of differentials, integrals and Stokes’ theorem. See [23] or [17] for more on this.

This completes our excursion to infinite-dimensional Clifford algebras. In the following sections we will always assume that $X$ is finite and $V$ finite-dimensional.
3 Isomorphisms

In this section we establish an extensive set of relations between real and complex geometric algebras of varying signature. This eventually leads to an identification of these algebras as matrix algebras over \( \mathbb{R} \), \( \mathbb{C} \), or the quaternions \( \mathbb{H} \). The complete listing of such identifications is usually called the classification of geometric algebras.

We have seen that the even subspace \( \mathcal{G}^+ \) of \( \mathcal{G} \) constitutes a subalgebra. The following proposition tells us that this subalgebra actually is the geometric algebra of a space of one dimension lower.

**Proposition 3.1.** We have the algebra isomorphisms

\[
\mathcal{G}^+(\mathbb{R}^{s,t}) \cong \mathcal{G}(\mathbb{R}^{s,t-1}), \\
\mathcal{G}^+(\mathbb{R}^{s,t}) \cong \mathcal{G}(\mathbb{R}^{t,s-1}),
\]

for all \( s, t \) for which the expressions make sense.

**Proof.** Take an orthonormal basis \( \{e_1, \ldots, e_s, \epsilon_1, \ldots, \epsilon_t\} \) of \( \mathbb{R}^{s,t} \) such that \( e_i^2 = 1 \), \( \epsilon_i^2 = -1 \), and a corresponding basis \( \{\underline{e}_1, \ldots, \underline{e}_t, \underline{\epsilon}_1, \ldots, \underline{\epsilon}_{s-1}\} \) of \( \mathbb{R}^{s,t-1} \). Define \( f: \mathbb{R}^{s,t-1} \to \mathcal{G}^+(\mathbb{R}^{s,t}) \) by mapping

\[
e_i \mapsto e_i \epsilon_t, \quad i = 1, \ldots, s, \\
\epsilon_i \mapsto \epsilon_i e_t, \quad i = 1, \ldots, t - 1,
\]

and extending linearly. We then have

\[
f(\underline{e}_i)f(\underline{e}_j) = -f(\underline{e}_j)f(\underline{e}_i), \\
f(\underline{e}_i)f(\underline{\epsilon}_j) = -f(\underline{\epsilon}_j)f(\underline{e}_i)
\]

for \( i \neq j \), and

\[
f(\underline{e}_i)f(\underline{\epsilon}_j) = -f(\underline{\epsilon}_j)f(\underline{e}_i), \\
f(\underline{e}_i)^2 = 1, \quad f(\underline{\epsilon}_i)^2 = -1
\]

for all reasonable \( i, j \). By Proposition 2.1 (universality) we can extend \( f \) to a homomorphism \( F: \mathcal{G}(\mathbb{R}^{s,t-1}) \to \mathcal{G}^+(\mathbb{R}^{s,t}) \). Since \( \dim \mathcal{G}(\mathbb{R}^{s,t-1}) = 2^{s+t-1} = 2^{s+t-1}/2 = \dim \mathcal{G}^+(\mathbb{R}^{s,t}) \) and \( F \) is easily seen to be surjective, we have that \( F \) is an isomorphism.

For the second statement, we take a corresponding basis \( \{\underline{e}_1, \ldots, \underline{e}_t, \underline{\epsilon}_1, \ldots, \underline{\epsilon}_{s-1}\} \) of \( \mathbb{R}^{t,s-1} \) and define \( f: \mathbb{R}^{t,s-1} \to \mathcal{G}^+(\mathbb{R}^{s,t}) \) by

\[
e_i \mapsto \epsilon_i e_s, \quad i = 1, \ldots, t, \\
\epsilon_i \mapsto e_i \epsilon_s, \quad i = 1, \ldots, s - 1.
\]

Proceeding as above, we obtain the isomorphism. \( \square \)

**Corollary.** It follows immediately that

\[
\mathcal{G}(\mathbb{R}^{s,t}) \cong \mathcal{G}(\mathbb{R}^{t+1,s-1}), \\
\mathcal{G}^+(\mathbb{R}^{s,t}) \cong \mathcal{G}^+(\mathbb{R}^{t,s}).
\]
In the above and further on we use the notation $\mathcal{G}(\mathbb{F}^{0,0}) := Cl(\otimes, \mathbb{F}, \otimes) = \mathbb{F}$ for completeness.

The property of geometric algebras that leads us to their eventual classification as matrix algebras is that they can be split up into tensor products of geometric algebras of lower dimension.

**Proposition 3.2.** We have the algebra isomorphisms

$$
\mathcal{G}(\mathbb{R}^{n+2,0}) \cong \mathcal{G}(\mathbb{R}^{0,n}) \otimes \mathcal{G}(\mathbb{R}^{2,0}),
$$

$$
\mathcal{G}(\mathbb{R}^{0,n+2}) \cong \mathcal{G}(\mathbb{R}^{n,0}) \otimes \mathcal{G}(\mathbb{R}^{0,2}),
$$

$$
\mathcal{G}(\mathbb{R}^{s+1,t+1}) \cong \mathcal{G}(\mathbb{R}^{s,t}) \otimes \mathcal{G}(\mathbb{R}^{1,1}),
$$

for all $n, s$ and $t$ for which the expressions make sense.

**Proof.** For the first expression, take orthonormal bases $\{e_i\}$ of $\mathbb{R}^{n+2}$, $\{e_i\}$ of $\mathbb{R}^{0,n}$ and $\{e_i\}$ of $\mathbb{R}^{2}$.

Define a mapping $f : \mathbb{R}^{n+2} \rightarrow \mathcal{G}(\mathbb{R}^{0,n}) \otimes \mathcal{G}(\mathbb{R}^{2})$ by

$$
e_j \mapsto e_j \otimes e_2, \quad j = 1, \ldots, n,
$$

$$
e_j \mapsto 1 \otimes e_j - n, \quad j = n + 1, n + 2,
$$

and extend to an algebra homomorphism $F$ using the universal property. Since $F$ maps onto a set of generators for $\mathcal{G}(\mathbb{R}^{0,n}) \otimes \mathcal{G}(\mathbb{R}^{2})$ it is clearly surjective. Furthermore, $\dim \mathcal{G}(\mathbb{R}^{n+2}) = 2^{n+2} = \dim \mathcal{G}(\mathbb{R}^{0,n}) \otimes \mathcal{G}(\mathbb{R}^{2})$, so $F$ is an isomorphism.

The second expression is proved similarly. For the third expression, take orthonormal bases $\{e_i\}, \{e_i\}$ of $\mathbb{R}^{s+1,t+1}$, $\{e_i\}$ of $\mathbb{R}^{s+1,t+1}$ and $\{e_i\}$ of $\mathbb{R}^{1,1}$, where $e_i^2 = 1, e_i^2 = -1$ etc. Define $f : \mathbb{R}^{s+1,t+1} \rightarrow \mathcal{G}(\mathbb{R}^{s,t}) \otimes \mathcal{G}(\mathbb{R}^{1,1})$ by

$$
e_j \mapsto e_j \otimes e_2, \quad j = 1, \ldots, s,
$$

$$
e_j \mapsto e_j \otimes e_1, \quad j = 1, \ldots, t,
$$

$$
e_{s+1} \mapsto 1 \otimes e_1,
$$

$$
e_{t+1} \mapsto 1 \otimes e_2.
$$

Proceeding as above, we can extend $f$ to an algebra isomorphism.

We can also relate certain real geometric algebras to complex equivalents.

**Proposition 3.3.** If $s + t$ is odd and $l^2 = -1$ then

$$
\mathcal{G}(\mathbb{R}^{s,t}) \cong \mathcal{G}^+(\mathbb{R}^{s,t}) \otimes \mathbb{C} \cong \mathcal{G}(\mathbb{C}^{s+t-1}).
$$

**Proof.** Since $s + t$ is odd, the pseudoscalar $I$ commutes with all other elements. This, together with the property $I^2 = -1$, makes it a good candidate for a scalar imaginary. Define $F : \mathcal{G}^+(\mathbb{R}^{s,t}) \otimes \mathbb{C} \rightarrow \mathcal{G}(\mathbb{R}^{s,t})$ by linear extension of

$$
E \otimes 1 \mapsto E \in \mathcal{G}^+,
$$

$$
E \otimes i \mapsto EI \in \mathcal{G}^-,
$$

for even basis blades $E$. $F$ is easily seen to be an injective algebra homomorphism. Using that the dimensions of these algebras are equal, we have an isomorphism.

For the second isomorphism, note that Proposition 3.1 gives us $\mathcal{G}^+(\mathbb{R}^{s,t}) \otimes \mathbb{C} \cong \mathcal{G}(\mathbb{R}^{s-1,t-1}) \otimes \mathbb{C}$. Finally, the order of complexification is unimportant since all nondegenerate complex quadratic forms are equivalent.
Corollary. It follows immediately that, for these conditions,
\[ \mathcal{G}(\mathbb{R}^{s,t}) \cong \mathcal{G}(\mathbb{R}^{p,q-1}) \otimes \mathbb{C} \]
for any \( p \geq 0, q \geq 1 \) such that \( p + q = s + t \).

One important consequence of the tensor algebra isomorphisms in Proposition 3.2 is that geometric algebras experience a kind of periodicity over 8 real dimensions in the underlying vector space.

Proposition 3.4. For all \( n \geq 0 \), there are periodicity isomorphisms
\[ \mathcal{G}(\mathbb{R}^{n+8,0}) \cong \mathcal{G}(\mathbb{R}^{n,0}) \otimes \mathcal{G}(\mathbb{R}^{8,0}), \]
\[ \mathcal{G}(\mathbb{R}^{0,n+8}) \cong \mathcal{G}(\mathbb{R}^{0,n}) \otimes \mathcal{G}(\mathbb{R}^{0,8}), \]
\[ \mathcal{G}(\mathbb{C}^{n+2}) \cong \mathcal{G}(\mathbb{C}^{n}) \otimes_{\mathbb{C}} \mathcal{G}(\mathbb{C}^{2}). \]

Proof. Using Proposition 3.2 repeatedly, we obtain
\[ \mathcal{G}(\mathbb{R}^{n+8,0}) \cong \mathcal{G}(\mathbb{R}^{0,n+6}) \otimes \mathcal{G}(\mathbb{R}^{2,0}) \]
\[ \cong \mathcal{G}(\mathbb{R}^{n,0}) \otimes \mathcal{G}(\mathbb{R}^{0,2}) \otimes \mathcal{G}(\mathbb{R}^{2,0}) \otimes \mathcal{G}(\mathbb{R}^{2,0}) \]
\[ \cong \mathcal{G}(\mathbb{R}^{n,0}) \otimes \mathcal{G}(\mathbb{R}^{8,0}), \]
and analogously for the second statement.

For the last statement we take orthonormal bases \( \{e_i\} \) of \( \mathbb{C}^{n+2} \), \( \{e_i\} \) of \( \mathbb{C}^{n} \) and \( \{\bar{e}_i\} \) of \( \mathbb{C}^{2} \). Define a mapping \( f: \mathbb{C}^{n+2} \rightarrow \mathcal{G}(\mathbb{C}^{n}) \otimes_{\mathbb{C}} \mathcal{G}(\mathbb{C}^{2}) \) by
\[ e_j \mapsto i \bar{e}_j \otimes \bar{e}_1 \bar{e}_2, \quad j = 1, \ldots, n, \]
\[ e_j \mapsto 1 \otimes \bar{e}_{j-n}, \quad j = n+1, n+2, \]
and extend to an algebra isomorphism as usual. \( \square \)

Theorem 3.5. We obtain the classification of real geometric algebras as matrix algebras, given by Table 3.1 together with the periodicity
\[ \mathcal{G}(\mathbb{R}^{s+8,t}) \cong \mathcal{G}(\mathbb{R}^{s,t+8}) \cong \mathcal{G}(\mathbb{R}^{s,t}) \otimes \mathbb{R}^{16 \times 16}. \]

Proof. We have the following easily verified isomorphisms:
\[ \mathcal{G}(\mathbb{R}^{1,0}) \cong \mathbb{R} \oplus \mathbb{R}, \]
\[ \mathcal{G}(\mathbb{R}^{0,1}) \cong \mathbb{C}, \]
\[ \mathcal{G}(\mathbb{R}^{2,0}) \cong \mathbb{R}^{2 \times 2}, \]
\[ \mathcal{G}(\mathbb{R}^{0,2}) \cong \mathbb{H}. \]

Some of these will be explained in detail in Section 5. We can now work out the cases \( (n, 0) \) and \( (0, n) \) for \( n = 0, 1, \ldots, 7 \) in a criss-cross fashion using Proposition 3.2 and the tensor algebra isomorphisms
\[ \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}, \]
\[ \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{C}^{2 \times 2}, \]
\[ \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{R}^{4 \times 4}. \]

For proofs of these, see e.g. [14]. With \( \mathcal{G}(\mathbb{R}^{1,1}) \cong \mathcal{G}(\mathbb{R}^{2,0}) \) and Proposition 3.2 we can then work our way through the whole table diagonally. The periodicity follows from Proposition 3.4 and \( \mathcal{G}(\mathbb{R}^{8,0}) \cong \mathbb{H} \otimes \mathbb{R}^{2 \times 2} \otimes \mathbb{H} \otimes \mathbb{R}^{2 \times 2} \cong \mathbb{R}^{16 \times 16}. \) \( \square \)
Table 3.1: The algebra $G(R^{s,t})$ in the box $(s,t)$, where $F[N] = F^{N \times N}$.
Because all nondegenerate complex quadratic forms on $\mathbb{C}^n$ are equivalent, the complex version of this theorem turns out to be much simpler.

**Theorem 3.6.** We obtain the classification of complex geometric algebras as matrix algebras, given by

$$
\begin{align*}
G(\mathbb{C}^0) &\cong \mathbb{C}, \\
G(\mathbb{C}^1) &\cong \mathbb{C} \oplus \mathbb{C},
\end{align*}
$$

**Proof.** The isomorphism $G(\mathbb{C}^n) \cong G(\mathbb{R}^n) \otimes \mathbb{C}$ gives us

$$
\begin{align*}
G(\mathbb{C}^0) &\cong \mathbb{C} \\
G(\mathbb{C}^1) &\cong (\mathbb{R} \oplus \mathbb{R}) \otimes \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C} \\
G(\mathbb{C}^2) &\cong \mathbb{R}^{2 \times 2} \otimes \mathbb{C} \cong \mathbb{C}^{2 \times 2}.
\end{align*}
$$

Then use Proposition 8.4 for periodicity.

The periodicity of geometric algebras actually has a number of far-reaching consequences. One example is Bott periodicity, which simply put gives a periodicity in the homotopy groups $\pi_k$ of the unitary, orthogonal and symplectic groups. See [14] for proofs using K-theory or [18] for examples.
4 Groups

One of the foremost reasons that geometric algebras appear naturally in so many areas of mathematics and physics is the fact that they contain a number of important groups. These are groups under the geometric product and thus lie embedded within the group of invertible elements in $\mathcal{G}$. In this section we will discuss the properties of various embedded groups and their relation to other familiar transformation groups such as the orthogonal and Lorentz groups. We will also introduce a generalized concept of spinor and see how such objects are related to the embedded groups.

**Definition 4.1.** We identify the following groups embedded in $\mathcal{G}$:

- $\mathcal{G}^\times := \{ x \in \mathcal{G} : \exists y \in \mathcal{G} : xy = yx = 1 \}$ \textit{the group of invertible elements}
- $\bar{\Gamma} := \{ x \in \mathcal{G}^\times : x^* \mathcal{V} x^{-1} \subseteq \mathcal{V} \}$ \textit{the Lipschitz group}
- $\Gamma := \{ v_1 v_2 \ldots v_k \in \mathcal{G} : v_i \in \mathcal{V}^\times \}$ \textit{the versor group}
- $\text{Pin} := \{ x \in \Gamma : xx^\dagger = \pm 1 \}$ \textit{the group of unit versors}
- $\text{Spin} := \text{Pin} \cap \mathcal{G}^+$ \textit{the group of even unit versors}
- $\text{Spin}^+ := \{ x \in \text{Spin} : xx^\dagger = 1 \}$ \textit{the rotor group}

where $\mathcal{V}^\times := \{ v \in \mathcal{V} : v^2 \neq 0 \}$ is the set of invertible vectors.

The versor group $\Gamma$ is the smallest group which contains $\mathcal{V}^\times$. Its elements are finite products of invertible vectors called \textit{versors}. As is hinted in Definition 4.1, many important groups are subgroups of this group. One of the central, and highly non-trivial, results of this section is that the versor and Lipschitz groups actually are equal. Therefore, $\Gamma$ is also called the Lipschitz group in honor of its creator. Sometimes it is also given the name \textit{Clifford group}, but we will, in accordance with other conventions, use that name to denote the finite group generated by an orthonormal basis.

The Pin and Spin groups are both generated by unit vectors, and in the case of Spin, only an even number of such vector factors can be present. The elements of $\text{Spin}^+$ are called \textit{rotors}. As we will see, these groups are intimately connected to orthogonal groups and rotations.

Throughout this section we will always assume that our scalars are real numbers unless otherwise stated. This is reasonable both from a geometric viewpoint and from the fact that e.g. many complex groups can be represented by groups embedded in real geometric algebras. Furthermore, we assume that $\mathcal{G}$ is nondegenerate so that we are working with a vector space of type $\mathbb{R}^{s,t}$. The corresponding groups associated to this space will be denoted Spin($s,t$) etc.

4.1 Group actions on $\mathcal{G}$

In order to understand how groups embedded in a geometric algebra are related to more familiar groups of linear transformations, it is necessary to study how groups in $\mathcal{G}$ can act on the vector space $\mathcal{G}$ itself and on the embedded underlying vector space $\mathcal{V}$. The following are natural candidates for such actions.
Definition 4.2. Using the geometric product, we have the following natural actions:

- **$L: \mathcal{G} \to \text{End } \mathcal{G}$** (left action)
  \[
  x \mapsto L_x : y \mapsto xy
  \]

- **$R: \mathcal{G} \to \text{End } \mathcal{G}$** (right action)
  \[
  x \mapsto R_x : y \mapsto yx
  \]

- **$\text{Ad}: \mathcal{G}^\times \to \text{End } \mathcal{G}$** (adjoint action)
  \[
  x \mapsto \text{Ad}_x : y \mapsto xyx^{-1}
  \]

- **$\tilde{\text{Ad}}: \mathcal{G}^\times \to \text{End } \mathcal{G}$** (twisted adjoint action)
  \[
  x \mapsto \tilde{\text{Ad}}_x : y \mapsto x^v y x^{-1}
  \]

where $\text{End } \mathcal{G}$ are the (vector space) endomorphisms of $\mathcal{G}$.

Note that $L$ and $R$ are algebra homomorphisms while $\text{Ad}$ and $\tilde{\text{Ad}}$ are group homomorphisms. These actions give rise to canonical representations of the groups embedded in $\mathcal{G}$. The twisted adjoint action takes the graded structure of $\mathcal{G}$ into account and will be seen to play a more important role than the normal adjoint action in geometric algebra. Using the expansion (2.3) one can verify that $\text{Ad}_x$ is always an outermorphism, while in general $\tilde{\text{Ad}}_x$ is not. Note, however, that these actions agree on the subgroup of even elements $\mathcal{G}^\times \cap \mathcal{G}^+$.

**Remark.** We note that, because the algebra $\mathcal{G}$ is assumed to be finite-dimensional, left inverses are always right inverses and vice versa. This can be seen as follows. First note that the left and right actions are injective. Namely, assume that $L_x = 0$. Then $L_x(y) = 0 \forall y$ and in particular $L_x(1) = x = 0$. Suppose now that $xy = 1$ for some $x, y \in \mathcal{G}$. But then $L_x L_y = id$, so that $L_y$ is a right inverse to $L_x$. Now, using the dimension theorem

$$\dim \ker L_y + \dim \text{im } L_y = \dim \mathcal{G}$$

with $\ker L_y = 0$, we can conclude that $L_y$ is also a left inverse to $L_x$. Hence, $L_y L_x = id \Rightarrow L_y x^{-1} = 0$, and $yx = 1$.

Let us study the properties of the twisted adjoint action. For $v \in \mathcal{V}^\times$ we obtain

$$\tilde{\text{Ad}}_v(v) = v^* v v^{-1} = -v, \quad (4.1)$$

and if $w \in \mathcal{V}$ is orthogonal to $v$,

$$\tilde{\text{Ad}}_v(w) = v^* w w^{-1} = -w v v^{-1} = w v v^{-1} = w. \quad (4.2)$$

Hence, $\tilde{\text{Ad}}_v$ is a reflection in the hyperplane orthogonal to $v$. For a general versor $x = u_1 u_2 \ldots u_k \in \Gamma$ we have

$$\tilde{\text{Ad}}_x(v) = (u_1 \ldots u_k)^* v (u_1 \ldots u_k)^{-1} = u_1^* \ldots u_k^* v u_k^{-1} \ldots u_1^{-1} = \tilde{\text{Ad}}_{u_1} \circ \ldots \circ \tilde{\text{Ad}}_{u_k}(v), \quad (4.3)$$

i.e. the twisted adjoint representation (restricted to act only on $\mathcal{V}$ which is clearly invariant) gives a homomorphism from the versor group into the group of orthogonal transformations,

$$\text{O}(\mathcal{V}, q) := \{ f : \mathcal{V} \to \mathcal{V} : f \text{ linear bijection s.t. } q \circ f = q \}.$$  

We have the following fundamental theorem regarding the orthogonal group.

We have the following fundamental theorem regarding the orthogonal group.
Theorem 4.1 (Cartan-Dieudonné). Every orthogonal map on a non-degenerate space \((V, q)\) is a product of reflections. The number of reflections required is at most equal to the dimension of \(V\).

For a constructive proof which works well for arbitrary signatures, see [23].

Corollary. \(\tilde{\text{Ad}}: \Gamma \to O(V, q)\) is surjective.

Proof. We know that any \(R \in O(V, q)\) can be written \(R = \tilde{\text{Ad}}_{v_1} \circ \ldots \circ \tilde{\text{Ad}}_{v_k}\) for some invertible vectors \(v_1, \ldots, v_k, k \leq n\). But then \(R = \tilde{\text{Ad}}_{v_1 \ldots v_k}\), where \(v_1 v_2 \ldots v_k \in \Gamma\).

4.2 The Lipschitz group

We saw above that the twisted adjoint representation maps the versor group onto the group of orthogonal transformations of \(V\). The largest group in \(G\) for which \(\tilde{\text{Ad}}\) forms a representation on \(V\), i.e. leaves \(V\) invariant, is per definition the Lipschitz group \(\tilde{\Gamma}\). We saw from (4.3) that \(\Gamma \subseteq \tilde{\Gamma}\).

We will now introduce an important function on \(G\), traditionally called the norm function,

\[
N: G \to G, \\
N(x) := x \square x. \tag{4.4}
\]

The name is a bit misleading since \(N\) is not even guaranteed to take values in \(\mathbb{R}\). For some special cases of algebras, however, it does act as a natural norm and we will see that it can be extended in many lower-dimensional algebras where it will act as a kind of determinant. Our first main result for this function is that it acts as a determinant on \(\tilde{\Gamma}\). This will help us prove that \(\Gamma = \tilde{\Gamma}\).

Lemma 4.2. Assume that \(G\) is nondegenerate. If \(x \in G\) and \(x^* v = vx\) for all \(v \in V\) then \(x\) must be a scalar.

Proof. Using Proposition 2.8 we have that \(v \sqcup x = 0\) for all \(v \in V\). This means that, for a \(k\)-blade, \((v_1 \wedge \cdots \wedge v_{k-1} \wedge v_k) \ast x = (v_1 \wedge \cdots \wedge v_{k-1}) \ast (v_k \sqcup x) = 0\) whenever \(k \geq 1\). The nondegeneracy of the scalar product implies that \(x\) must have grade 0.

Theorem 4.3. The norm function is a group homomorphism \(N: \tilde{\Gamma} \to \mathbb{R}^\times\).

Proof. First note that if \(xx^{-1} = 1\) then also \(x^*(x^{-1})^* = 1\) and \((x^{-1})^\dagger x^\dagger = 1\), hence \((x^*)^{-1} = (x^{-1})^*\) and \((x^\dagger)^{-1} = (x^{-1})^\dagger\).

Now take \(x \in \tilde{\Gamma}\). Then \(x^\dagger vx^{-1} \in V\) for all \(v \in V\) and therefore

\[
x^\dagger vx^{-1} = (x^* vx^{-1})^\dagger = (x^{-1})^\dagger v x \square.
\tag{4.5}
\]

This means that \(x^\dagger x^* v = vx \square x\), or \(N(x)^* v = vN(x)\). By Lemma 4.2 we find that \(N(x) \in \mathbb{R}\). The homomorphism property now follows easily, since for \(x, y \in \tilde{\Gamma}\),

\[
N(xy) = (xy)^\square xy = y^\square x^\square xy = y^\square N(x)y = N(x)N(y). \tag{4.6}
\]

Finally, because \(1 = N(1) = N(xx^{-1}) = N(x)N(x^{-1})\), we must have that \(N(x)\) is nonzero.

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Lemma 4.4. The homomorphism $\tilde{\text{Ad}} : \tilde{\Gamma} \to O(\mathcal{V}, q)$ has kernel $\mathbb{R}^\times$.

Proof. We first prove that $\tilde{\text{Ad}}_x$ is orthogonal for $x \in \tilde{\Gamma}$. Note that, for $v \in \mathcal{V}$,

$$N(\tilde{\text{Ad}}_x(v)) = N(x^*vx^{-1}) = (x^*vx^{-1})^2x^*vx^{-1} = (x^*)^2v^2x^*vx^{-1} = (x^*vx^{-1})N(x)^*v^2x^{-1} = (x^*)^2N(x)^*v^2x^{-1} = N(x^*)N(x)N(v) = N(v).$$

Then, since $N(v) = v^2 = -v^2$, we have that $\tilde{\text{Ad}}_x(v)^2 = v^2$.

Now, if $\tilde{\text{Ad}}_x = \text{id}$ then $x^*v = vx$ for all $v \in \mathcal{V}$ and by Lemma 4.2 we must have $x \in \mathbb{R} \cap \tilde{\Gamma} = \mathbb{R}^\times$.

We finally obtain

**Theorem 4.5.** We have that $\Gamma = \tilde{\Gamma}$.

Proof. We saw earlier that $\Gamma \subseteq \tilde{\Gamma}$. Take $x \in \tilde{\Gamma}$. By the above lemma we have $\tilde{\text{Ad}}_x \in O(\mathcal{V}, q)$. Using the corollary to Theorem 4.1 we then find that $\tilde{\text{Ad}}_x = \tilde{\text{Ad}}_y$ for some $y \in \Gamma$. Then $\tilde{\text{Ad}}_{xy^{-1}} = \text{id}$ and $xy^{-1} = \lambda \in \mathbb{R}^\times$. Hence, $x = \lambda y \in \Gamma$.

### 4.3 Properties of Pin and Spin groups

From the discussion above followed that $\tilde{\text{Ad}}$ gives a surjective homomorphism from the versor, or Lipschitz, group $\Gamma$ to the orthogonal group. The kernel of this homomorphism is the set of invertible scalars. Because the Pin and Spin groups consist of normalized versors ($N(x) = \pm 1$) we find the following

**Theorem 4.6.** The homomorphisms

$$\begin{align*}
\tilde{\text{Ad}} &: \text{Pin}(s,t) \to O(s,t) \\
\tilde{\text{Ad}} &: \text{Spin}(s,t) \to \text{SO}(s,t) \\
\text{Ad} &: \text{Spin}^+(s,t) \to \text{SO}^+(s,t)
\end{align*}$$

are surjective with kernel $\{\pm 1\}$.

The homomorphism onto the special orthogonal group,

$$\text{SO}(\mathcal{V}, q) := \{f \in O(\mathcal{V}, q) : \det f = 1\}$$

follows since it is generated by an even number of reflections. $\text{SO}^+$ denotes the connected component of $\text{SO}$ containing the identity. This will soon be explained.

In other words, the Pin and Spin groups are two-sheeted coverings of the orthogonal groups. Furthermore, we have the following relations between these groups.

Take a unit versor $\psi = u_1u_2 \ldots u_k \in \text{Pin}(s,t)$. If $\psi$ is odd we can always multiply by a unit vector $e$ so that $\psi = \pm \psi e e$ and $\pm \psi e \in \text{Spin}(s,t)$. Furthermore, when the signature is euclidean we have $\psi \psi^\dagger = 1$ for all unit versors. The same holds for even unit versors in anti-euclidean spaces since the signs cancel out. Hence, $\text{Spin} = \text{Spin}^+$ unless there is mixed signature. But in that case we can find two orthogonal unit vectors $e_+, e_-$ such that $e_+^2 = 1$ and

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\(e^2 = -1\). Since \(e_+ e_- (e_+ e_-)^\dagger = -1\) we then have that \(\psi = \psi(e_+ e_-)^2\), where \(\psi e_+ e_- (\psi e_+ e_-)^\dagger = 1\) if \(\psi \neq -1\).

Summing up, we have that, for \(s, t \geq 1\) and any pair of orthogonal vectors \(e_+, e_-\) such that \(e_+^2 = 1, e_-^2 = -1\),

\[
\text{Pin}(s, t) = \text{Spin}(s, t) \cdot \{1, e_+, e_-, e_+ e_-\},
\]

\[
\text{Spin}(s, t) = \text{Spin}(s, t) \cdot \{1, e_+\}.
\]

Otherwise,

\[
\text{Pin}(s, t) = \text{Spin}(s, t) \cdot \{1, e\}
\]

for any \(e \in \mathcal{V}\) such that \(e^2 = \pm 1\). From the isomorphism \(\mathcal{G}^+(\mathbb{R}^{n, t}) \cong \mathcal{G}^+(\mathbb{R}^{t, s})\) we also have the signature symmetry \(\text{Spin}(s, t) \cong \text{Spin}(t, s)\). In all cases,

\[
\Gamma(s, t) = \mathbb{R}^\times \cdot \text{Pin}(s, t).
\]

From these considerations it is sufficient to study the properties of the rotor groups in order to understand the Pin, Spin and orthogonal groups. Fortunately, it turns out that the rotor groups have very convenient topological features.

**Theorem 4.7.** The groups \(\text{Spin}(s, t)\) are pathwise connected for \(s \geq 2\) or \(t \geq 2\).

**Proof.** Pick a rotor \(R \in \text{Spin}(s, t)\), where \(s\) or \(t\) is greater than one. Then \(R = v_1 v_2 \ldots v_{2k}\) with an even number of \(v_i \in \mathcal{V}\) such that \(v_i^2 = 1\) and an even number such that \(v_i^2 = -1\). Note that for any two invertible vectors \(a, b\) we have \(ab = aba^{-1}a = b'a\), where \(b'^2 = b^2\). Hence, we can rearrange the vectors so that those with positive square come first, i.e.

\[
R = a_1 b_1 \ldots a_p b_p a'_1 b'_1 \ldots a'_q b'_q = R_1 \ldots R_p R'_1 \ldots R'_q,
\]

where \(a_i^2 = b_i^2 = 1\) and \(R_i = a_i b_i = a_i \wedge b_i\) are so called simple rotors which are connected to either 1 or -1. This holds because \(1 = R_i R_i^\dagger = (a_i \wedge b_i)^2 - (a_i \wedge b_i)^2\), so we can, as is easily verified, write \(R_i = \pm e^{\phi_i a_i \wedge b_i}\) for some \(\phi_i \in \mathbb{R}\) (exponentials of bivectors will be treated shortly). Depending on the signature of the plane associated to \(a_i \wedge b_i\), i.e. on the sign of \((a_i \wedge b_i)^2 \in \mathbb{R}\), the set \(e^{\phi a_i \wedge b_i} \subseteq \text{Spin}(s, t)\) forms either a circle, a line or a hyperbola. In any case, it goes through the unit element. Finally, since \(s > 1\) or \(t > 1\) we can connect -1 to 1 with for example the circle \(e^{R_{e_1 e_2}}\), where \(e_1, e_2\) are two orthonormal basis elements with the same signature.

Continuity of \(\tilde{\text{Ad}}\) now implies that the set of rotations represented by rotors, i.e. \(\text{SO}^+\), forms a continuous subgroup containing the identity. For euclidean and lorentzian signatures, we have an even simpler situation.

**Theorem 4.8.** The groups \(\text{Spin}(s, t)\) are simply connected for \(s, t = (n, 0), (0, n), (1, n)\) or \((n, 1)\), where \(n \geq 3\). Hence, these are the universal covering groups of \(\text{SO}^+(s, t)\).

This follows because \(\pi_1(\text{SO}^+(s, t)) = \mathbb{Z}_2\) for these signatures. See e.g. [13] for details.

This sums up the situation nicely for higher-dimensional euclidean and lorentzian spaces: The Pin group, which is a double-cover of the orthogonal
group, consists of two or four simply connected components. These components are copies of the rotor group. In physics-terminology these components correspond to time and parity reflections.

It is also interesting to relate the rotor group to its Lie algebra, which is actually the bivector space \( \mathcal{G}^2 \) with the usual commutator \([\cdot, \cdot]\). This follows because there is a Lie algebra isomorphism between \( \mathcal{G}^2 \) and the algebra of antisymmetric transformations of \( \mathcal{V} \), given by

\[
f = [B, \cdot] \Leftrightarrow B = \frac{1}{2} \sum_{i,j} e^i \cdot f(e^j) \, e_i \wedge e_j,
\]

where \( \{e_i\} \) is some general basis of \( \mathcal{V} \). One verifies, by expanding in the geometric product, that

\[
\frac{1}{2} [e_i \wedge e_j, e_k \wedge e_l] = e_j \wedge e_k e_l e_i - e_j \wedge e_k e_l e_i + e_i \wedge e_k e_j - e_i \wedge e_k e_j e_l.
\]  

Actually, this bivector Lie algebra is more general than it might first seem. Doran and Lasenby (see [3] or [4]) have shown that the Lie algebra \( \mathfrak{gl} \) of the general linear group can be represented as a bivector algebra. From the fact that any finite-dimensional Lie algebra has a faithful finite-dimensional representation (Ado’s Theorem for characteristic zero, Iwasawa’s Theorem for nonzero characteristic, see e.g. [11]) we have that any finite-dimensional real or complex Lie algebra can be represented as a bivector algebra.

We define the exponential of a multivector \( x \in \mathcal{G} \) as the usual power series

\[
e^x := \sum_{k=0}^{\infty} \frac{x^k}{k!}.
\]  

Since \( \mathcal{G} \) is finite-dimensional we have the following for any choice of norm on \( \mathcal{G} \).

**Proposition 4.9.** The sum in (4.11) converges for all \( x \in \mathcal{G} \) and

\[
e^x e^{-x} = e^{-x} e^x = 1.
\]

The following now holds for any signature.

**Theorem 4.10.** For any bivector \( B \in \mathcal{G}^2 \) we have that \( \pm e^B \in \text{Spin}^+ \).

**Proof.** It is obvious that \( \pm e^B \) is an even multivector and that \( e^B (e^B)^t = e^B e^{-B} = 1 \). Hence, it is sufficient to prove that \( \pm e^B \in \Gamma \), or by Theorem 4.5 that \( e^B \mathcal{V} e^{-B} \subseteq \mathcal{V} \). This can be done by considering derivatives of the function \( f(t) := e^{tB} v e^{-tB} \) for \( v \in \mathcal{V} \). See e.g. [23] or [19] for details. \( \square \)

The converse is true for (anti-) euclidean and lorentzian spaces.

**Theorem 4.11.** For \((s, t) = (n, 0), (0, n), (1, n) \) or \((n, 1) \), we have

\[
\text{Spin}^+ (s, t) = \pm e^{\mathcal{G}^2(\mathbb{R}^{s+t})},
\]

i.e. any rotor can be written as (minus) the exponential of a bivector. The minus sign is only required in the cases (0, 0), (1, 0), (0, 1), (1, 1), (1, 2), (2, 1), (1, 3) and (3, 1).

The proof can be found in [19]. Essentially, it relies on the fact that any isometry of an euclidean or lorentzian space can be generated by a single infinitesimal transformation. This holds for these spaces only, so that for example \( \text{Spin}^+ (2, 2) \neq \pm e^{\mathcal{G}^2(\mathbb{R}^{3})} \), where for instance \( \pm e_1 e_2 e_3 e_4 e^0 e_1 e_2 + 2 e_1 e_4 + e_3 e_4 \), \( \beta > 0 \), cannot be reduced to a single exponential.
4.4 Spinors

Spinors are objects which originally appeared in physics in the early days of quantum mechanics, but have by now made their way into other fields as well, such as differential geometry and topology. They are traditionally represented as elements of a complex vector space since it was natural to add and subtract them and scale them by complex amplitudes in their original physical applications. However, what really characterizes them as spinors is the fact that they can be acted upon by rotations, together with their rather special transformation properties under such rotations. While a rotation needs just one revolution to get back to the identity in a vector representation, it takes two revolutions to come back to the identity in a spinor representation. This is exactly the behaviour experienced by rotors, since they transform vectors double-sidedly with the (twisted) adjoint action as

\[ v \mapsto \psi v \psi^\dagger \mapsto \phi \psi v \psi^\dagger = \phi \psi^\dagger v (\phi \psi)^\dagger, \tag{4.12} \]

where we applied consecutive rotations represented by rotors \( \psi \) and \( \phi \). Note that the rotors themselves transform according to

\[ 1 \mapsto \psi \mapsto \phi \psi, \tag{4.13} \]

that is single-sidedly with the left action.

This hints that rotors could represent some form of spinors in geometric algebra. However, the rotors form a group and not a vector space, so we need to consider a possible enclosing vector space with similar properties. Some authors (see e.g. Hestenes \[9\]) have considered so called operator spinors, which are general even multivectors that leave \( \mathcal{V} \) invariant under the double-sided action (4.12), i.e. elements of

\[ \Sigma := \{ \Psi \in \mathcal{G}^+ : \mathcal{V} \Psi \Psi^\dagger \subseteq \mathcal{V} \}. \tag{4.14} \]

Since \( \Psi \Psi^\dagger \) is both odd and self-reversing, it is an element of \( \bigoplus_{k=0}^{\infty} \mathcal{G}^{4k+1} \). Therefore \( \Sigma \) is only guaranteed to be a vector space for \( \dim \mathcal{V} \leq 4 \), where it coincides with \( \mathcal{G}^+ \).

For a general spinor space embedded in \( \mathcal{G} \), we seek a subspace \( \Sigma \subseteq \mathcal{G} \) that is invariant under left action by rotors, i.e. such that

\[ \Psi \in \Sigma, \quad \psi \in \text{Spin}^+(s,t) \quad \Rightarrow \quad \psi \Psi \in \Sigma. \tag{4.15} \]

One obvious and most general choice of such a spinor space is the whole space \( \mathcal{G} \). However, we will soon see from examples that spinors in lower dimensions are best represented by another natural suggestion, namely the set of even multivectors. Hence, we follow Francis and Kosowsky \[5\] and define the space of spinors \( \Sigma \) for arbitrary dimensions as the even subalgebra \( \mathcal{G}^+ \).

Note that action by rotors, i.e. \( \text{Spin}^+ \) instead of \( \text{Spin} \), ensures that \( \Psi \Psi^\dagger \) remains invariant under right action and \( \Psi^\dagger \Psi \) remains invariant under left action on \( \Psi \). In lower dimensions these are invariant under both actions, so they are good candidates for invariant or observable quantities in physics. Also note that if \( \dim \mathcal{V} \leq 4 \) then the set of unit spinors and the set of rotors coincide, i.e.

\[ \text{Spin}^+(s,t) = \{ \psi \in \mathcal{G}^+ : \psi \psi^\dagger = 1 \} \quad \text{for } s + t \leq 4. \tag{4.16} \]
This follows because then \( \psi \) is invertible and \( \psi^* v \psi^{-1} = \psi v \psi^\dagger \) is both odd and self-reversing, hence a vector. Thus, \( \psi \) lies in \( \hat{\Gamma} \) and is therefore an even unit versor.

A popular alternative to the above is to consider spinors as elements in left-sided ideals of (mostly complex) geometric algebras. This is the view which is closest related to the original complex vector space picture. We will see an example of how these views are related in the case of the Dirac algebra. A motivation for this definition of spinor is that it admits a straightforward transition to basis-independent spinors, so called covariant spinors. This is required for a treatment of spinor fields on curved manifolds, i.e. in the gravitational setting. However, covariant spinors lack the clearer geometrical picture provided by operator and even subalgebra spinors. Furthermore, by reconsidering the definition and interpretation of these geometric spinors, it is possible to deal with basis-independence also in this case. These and other properties of spinors related to physics will be discussed in Section 7.
5 A study of lower-dimensional algebras

We have studied the structure of geometric algebras in general and saw that they are related to many other familiar algebras and groups. We will now go through a number of lower-dimensional examples in detail to see just how structure-rich these algebras are. Although we go no higher than to a five-dimensional base vector space, we manage to find a variety of structures related to physics.

We choose to focus around the groups and spinors of these example algebras. It is highly recommended that the reader acquires a more complete understanding of at least the plane, space and spacetime algebras from other sources. Good introductions aimed at physicists can be found in [4] and [7]. A more mathematical treatment is given in [15].

5.1 \( \mathcal{G}(\mathbb{R}^1) \)

Since \( \mathcal{G}(\mathbb{R}^0,0) = \mathbb{R} \) is just the field of real numbers, which should be familiar, we start instead with \( \mathcal{G}(\mathbb{R}^1) \), the geometric algebra of the real line. Let \( \mathbb{R}^1 \) be spanned by one basis element \( e \) such that \( e^2 = 1 \). Then

\[
\mathcal{G}(\mathbb{R}^1) = \text{Span}_\mathbb{R}\{1, e\}. \tag{5.1}
\]

This is a commutative algebra with pseudoscalar \( e \). One easily finds the invertible elements \( \mathcal{G}^\times \) by considering the norm function, which with a one-dimensional \( V \) is given by

\[
N_1(x) := x \overline{x} = x^* x. \tag{5.2}
\]

For an arbitrary element \( x = \alpha + \beta e \) then

\[
N_1(\alpha + \beta e) = (\alpha - \beta e)(\alpha + \beta e) = \alpha^2 - \beta^2 \in \mathbb{R}. \tag{5.3}
\]

When \( N_1(x) \neq 0 \) we find that \( x \) has an inverse \( x^{-1} = \frac{1}{N_1(x)} x^* = \frac{1}{\alpha^2 - \beta^2}(\alpha - \beta e) \).

Hence,

\[
\mathcal{G}^\times(\mathbb{R}^1) = \{ x \in \mathcal{G} : N_1(x) \neq 0 \} = \{ \alpha + \beta e \in \mathcal{G} : \alpha^2 \neq \beta^2 \}. \tag{5.4}
\]

Note also that \( N_1(xy) = x^* y^* x y = N_1(x)N_1(y) \) for all \( x, y \in \mathcal{G} \) since the algebra is commutative.

The other groups are rather trivial in this space. Because \( \mathcal{G}^+ = \mathbb{R} \), we have

\[
\text{Spin}^+(1,0) = \{ 1, -1 \},
\]

\[
\text{Pin}(1,0) = \{ 1, -1, e, -e \},
\]

\[
\Gamma(1,0) = \mathbb{R}^\times \bigcup \mathbb{R}^\times e,
\]

where we write \( \bigcup \) to emphasize a disjoint union. The spinors in this algebra are just real scalars.

5.2 \( \mathcal{G}(\mathbb{R}^{0,1}) \cong \mathbb{C} - \text{The complex numbers} \)

As one might have noticed from previous discussions on isomorphisms, the complex numbers are in fact a real geometric algebra. Let \( i \) span a one-dimensional anti-euclidean space and be normalized to \( i^2 = -1 \). Then

\[
\mathcal{G}(\mathbb{R}^{0,1}) = \text{Span}_\mathbb{R}\{1, i\} \cong \mathbb{C}. \tag{5.5}
\]

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This is also a commutative algebra, but, unlike the previous example, this is a field since every nonzero element is invertible. The norm function is an actual norm (squared) in this case,

\[ N_1(\alpha + \beta i) = (\alpha - \beta i)(\alpha + \beta i) = \alpha^2 + \beta^2 \in \mathbb{R}^+, \] (5.6)

namely the modulus of the complex number. Note that the grade involution represents the complex conjugate and \( x^{-1} = \frac{1}{N_1(x)}x^* \) as above. The relevant groups are

\[
\begin{align*}
\mathcal{G}^\times(\mathbb{R}^{0,1}) &= \mathcal{G} \setminus \{0\}, \\
\text{Spin}^+(0,1) &= \{1, -1\}, \\
\text{Pin}(0,1) &= \{1, -1, i, -i\}, \\
\Gamma(0,1) &= \mathbb{R}^x \cup \mathbb{R}^x i.
\end{align*}
\]

The spinor space is still given by \( \mathbb{R} \).

### 5.3 \( \mathcal{G}(\mathbb{R}^{0,0,1}) \)

We include this as our only example of a degenerate algebra, just to see what such a situation might look like. Let \( n \) span a one-dimensional space with quadratic form \( q = 0 \). Then

\[ \mathcal{G}(\mathbb{R}^{0,0,1}) = \text{Span}_\mathbb{R}\{1, n\} \] (5.7)

and \( n^2 = 0 \). The norm function depends only on the scalar part,

\[ N_1(\alpha + \beta n) = (\alpha - \beta n)(\alpha + \beta n) = \alpha^2 \in \mathbb{R}^+. \] (5.8)

An element is invertible if and only if the scalar part is nonzero. Since no vectors are invertible, we are left with only the empty product in the versor group. This gives

\[
\begin{align*}
\mathcal{G}^\times(\mathbb{R}^{0,0,1}) &= \{\alpha + \beta n : \alpha \neq 0\}, \\
\Gamma &= \{1\}.
\end{align*}
\]

Note, however, that for \( \alpha \neq 0 \)

\[ (\alpha + \beta n)^* n (\alpha + \beta n)^{-1} = (\alpha - \beta n)n \frac{1}{N_1(x)}(\alpha - \beta n) = n, \] (5.9)

so the Lipschitz group is

\[
\tilde{\Gamma} = \mathcal{G}^\times \neq \Gamma. \] (5.10)

This shows that the assumption on nondegeneracy was necessary in the discussion about the Lipschitz group in Section 4.

### 5.4 \( \mathcal{G}(\mathbb{R}^2) \) - The plane algebra

Our previous examples were rather trivial, but we now come to our first really interesting case, namely the geometric algebra of the euclidean plane. Let \( \{e_1, e_2\} \) be an orthonormal basis of \( \mathbb{R}^2 \) and consider

\[ \mathcal{G}(\mathbb{R}^2) = \text{Span}_\mathbb{R}\{1, e_1, e_2, e_1 e_2\}. \] (5.11)
An important feature of this algebra is that the pseudoscalar \( I := e_1 e_2 \) squares to \(-1\). This makes the even subalgebra isomorphic to the complex numbers, in correspondence with the relation \( G^+(\mathbb{R}^2) \cong G(\mathbb{R}^{0,1}) \).

Let us find the invertible elements of the plane algebra. For two-dimensional algebras we use the original norm function

\[
N_2(x) := x \circ x
\]

since it satisfies \( N_2(x) \circ x = N_2(x) \) for all \( x \in \mathcal{G} \). The sign relations for involutions in Table 2.1 then require this to be a scalar, so we have a map

\[
N_2 : \mathcal{G} \rightarrow \mathbb{G}^0 = \mathbb{R}.
\]

For an arbitrary element \( x = \alpha + a_1 e_1 + a_2 e_2 + \beta I \in \mathcal{G} \) we have

\[
N_2(x) = (\alpha - a_1 e_1 - a_2 e_2 - \beta I)(\alpha + a_1 e_1 + a_2 e_2 + \beta I) = \alpha^2 - a_1^2 - a_2^2 + \beta^2.
\]

Furthermore, \( N_2(x \circ x) = N_2(x) \) and \( N_2(xy) = N_2(x)N_2(y) \) for all \( x, y \). Proceeding as in the one-dimensional case, we find that \( x \) has an inverse \( x^{-1} = \frac{1}{N_2(x)} x \circ x \) if and only if \( N_2(x) \neq 0 \), i.e.

\[
\mathcal{G}^\times(\mathbb{R}^2) = \{ x \in \mathcal{G} : N_2(x) \neq 0 \} = \{ \alpha + a_1 e_1 + a_2 e_2 + \beta I \in \mathcal{G} : \alpha^2 - a_1^2 - a_2^2 + \beta^2 \neq 0 \}.
\]

For \( x = \alpha + \beta I \) in the even subspace we have \( x \circ x = x^\dagger = \alpha - \beta I \), so the Clifford conjugate acts as complex conjugate in this case. Again, the norm function (here \( N_2 \)) acts as modulus squared. We find that the rotor group, i.e. the group of even unit versors, corresponds to the group of unit complex numbers,

\[
\text{Spin}^+(2,0) = e^{\mathbb{R}I} \cong \text{U}(1).
\]

Note that, because \( e_1 \) and \( I \) anticommute,

\[
e^{\varphi I} e_1 e^{-\varphi I} = e_1 e^{-2\varphi I} = e_1 (\cos 2\varphi - I \sin 2\varphi) = e_1 \cos 2\varphi - e_2 \sin 2\varphi,
\]

so a rotor \( \pm e^{-\varphi I/2} \) represents a counter-clockwise rotation in the plane by an angle \( \varphi \). The Pin group is found by picking for example \( e_1 \);

\[
\text{Pin}(2,0) = e^{\mathbb{R}I} \bigcup e^{\mathbb{R}I} e_1, \quad \Gamma(2,0) = \mathbb{R} \times e^{\mathbb{R}I} \bigcup \mathbb{R} \times e^{\mathbb{R}I} e_1.
\]

As we saw above, the spinors of \( \mathbb{R}^2 \) are nothing but complex numbers. We can write any spinor or complex number \( \Psi \in \mathcal{G}^+ \) in the polar form \( \Psi = \rho e^{\varphi I} \), which is just a rescaled rotor. The spinor action

\[
a \mapsto \Psi a \Psi^\dagger = \rho^2 e^{\varphi I} a e^{-\varphi I}
\]

then gives a geometric interpretation of the spinor \( \Psi \) as an operation to rotate by an angle \(-2\varphi\) and scale by \( \rho^2 \).

---

5 Assuming, of course, that \( e_1 \) points at 3 o’clock and \( e_2 \) at 12 o’clock.
5.5 $\mathcal{G}(\mathbb{R}^{0,2}) \cong \mathbb{H}$ - The quaternions

The geometric algebra of the anti-euclidean plane is isomorphic to Hamilton’s quaternion algebra $\mathbb{H}$. This follows by taking an orthonormal basis $\{i, j\}$ of $\mathbb{R}^{0,2}$ and considering
\[ \mathcal{G}(\mathbb{R}^{0,2}) = \text{Span}_\mathbb{R}\{1, i, j, k\} , \]
where $k := ij$ is the pseudoscalar. We then have the classic identities defining quaternions,
\[ i^2 = j^2 = k^2 = ijk = -1. \]

We write an arbitrary quaternion as $x = \alpha + a_1i + a_2j + \beta k$. The Clifford conjugate acts as the quaternion conjugate, $\overline{x} \equiv x^\dagger = x$. The norm function $N_2$ has the same properties as in the euclidean algebra, but in this case it once again represents the square of an actual norm, namely the quaternion norm,
\[ N_2(x) = \alpha^2 + a_1^2 + a_2^2 + \beta^2. \]

Just as in the complex case then, all nonzero elements are invertible,
\[ \mathcal{G}^\times(\mathbb{R}^{0,2}) = \{ x \in \mathcal{G} : N_2(x) \neq 0 \} = \mathcal{G} \setminus \{ 0 \}. \]

The even subalgebra is also in this case isomorphic to the complex numbers, so the spinors and groups $\Gamma, \text{Pin}$ and $\text{Spin}$ are no different than in the euclidean case.

5.6 $\mathcal{G}(\mathbb{R}^{1,1})$

This is our simplest example of a lorentzian algebra. An orthonormal basis $\{e_+, e_-\}$ of $\mathbb{R}^{1,1}$ consists of a timelike vector, $e_+^2 = 1$, and a spacelike vector, $e_-^2 = -1$. In general, a vector (or blade) $v$ is called timelike if $v^2 > 0$, spacelike if $v^2 < 0$, and lightlike or null if $v^2 = 0$. This terminology is taken from relativistic physics. The two-dimensional lorentzian algebra is given by
\[ \mathcal{G}(\mathbb{R}^{1,1}) = \text{Span}_\mathbb{R}\{1, e_+, e_-, e_+e_-\}. \]

The group of invertible elements is as usual given by
\[ \mathcal{G}^\times(\mathbb{R}^2) = \{ x \in \mathcal{G} : N_2(x) \neq 0 \} = \{ \alpha + a_+e_+ + a_-e_- + \beta I \in \mathcal{G} : \alpha^2 - a_+^2 + a_-^2 - \beta^2 \neq 0 \}. \]

The pseudoscalar $I := e_+e_-$ squares to the identity in this case and the even subalgebra is therefore $\mathcal{G}^\times(\mathbb{R}^{1,1}) \cong \mathcal{G}(\mathbb{R}^1)$. This has as an important consequence that the rotor group is fundamentally different from the euclidean case,
\[ \text{Spin}^+(1,1) = \{ \psi = \alpha + \beta I \in \mathcal{G}^+ : \psi^\dagger \psi = \alpha^2 - \beta^2 = 1 \} = \pm e^{RI}. \]

This is a pair of disjoint hyperbolas passing through the points 1 and $-1$, respectively. The Spin group consists of four such hyperbolas and the Pin group of eight,
\[
\begin{align*}
\text{Spin}(1,1) &= \pm e^{RI} \sqcup \pm e^{RI}I, \\
\text{Pin}(1,1) &= \pm e^{RI} \sqcup \pm e^{RI}e_+ \sqcup \pm e^{RI}e_- \sqcup \pm e^{RI}I, \\
\Gamma(1,1) &= \mathbb{R} \times e^{RI} \sqcup \mathbb{R} \times e^{RI}e_+ \sqcup \mathbb{R} \times e^{RI}e_- \sqcup \mathbb{R} \times e^{RI}I.
\end{align*}
\]
The rotations that are represented by rotors of this kind are called Lorentz boosts. We will return to the Lorentz group in the 4-dimensional spacetime, but for now note the hyperbolic nature of these rotations,

\[ e^{\alpha I} e^{-\alpha I} = e_+ e^{-2\alpha I} = e_+ (\cosh 2\alpha - I \sinh 2\alpha) \]

\[ = e_+ \cosh 2\alpha - e_- \sinh 2\alpha. \]  

(5.26)

Hence, a rotor \( \pm e^{\alpha I/2} \) transforms (or boosts) timelike vectors by a hyperbolic angle \( \alpha \) away from the positive spacelike direction.

The spinor space \( G^+ \) consists partly of scaled rotors, \( \rho e^{\alpha I} \), but there are also two subspaces \( \rho (1 \pm I) \) of null spinors which cannot be represented in this way. Note that such a spinor \( \Psi \) acts on vectors as

\( \Psi e^{+} \Psi^d = 2 \rho^2 (e_+ \mp e_-) \),

\( \Psi e^{-} \Psi^d = \mp 2 \rho^2 (e_+ \mp e_-) \),

(5.27)

so it maps the whole space into one of the two null-spaces. The action of a non-null spinor has a nice interpretation as a boost plus scaling.

5.7 \( G(\mathbb{R}^3) \cong G(\mathbb{C}^2) \) - The space algebra / Pauli algebra

Since the 3-dimensional euclidean space is the space that is most familiar to us humans, one could expect its geometric algebra, the space algebra, to be familiar as well. Unfortunately, this is generally not the case. Most of its features, however, are commonly known but under different names and in separate contexts. For example, using the isomorphism \( G(\mathbb{R}^3) \cong G(\mathbb{C}^2) \cong \mathbb{C}^{2 \times 2} \) from Proposition 3.3, we find that this algebra also appears in quantum mechanics in the form of the complex Pauli algebra.

We take an orthonormal basis \( \{e_1, e_2, e_3\} \) in \( \mathbb{R}^3 \) and obtain

\[ G(\mathbb{R}^3) = \text{Span}_\mathbb{R} \{1, e_1, e_2, e_3, e_1 I, e_2 I, e_3 I, I\}, \]

(5.28)

where \( I := e_1 e_2 e_3 \) is the pseudoscalar. We write in this way to emphasize the duality between the vector and bivector spaces in this case. This duality can be used to define cross products and rotation axes etc. However, since the use of such concepts is limited to three dimensions only, it is better to work with their natural counterparts within geometric algebra.

To begin with, we would like to find the invertible elements of the space algebra. An arbitrary element \( x \in G \) can be written as

\[ x = \alpha + a + bI + \beta I, \]

(5.29)

where \( \alpha, \beta \in \mathbb{R} \) and \( a, b \in \mathbb{R}^3 \). Note that, since the algebra is odd, the pseudoscalar commutes with everything and furthermore \( I^2 = -1 \). The norm function \( N_2 \) does not take values in \( \mathbb{R} \) in this algebra, but due to the properties of the Clifford conjugate we have \( N_2(x) = N_2(x) \square \in G^0 \oplus G^3 \). This subspace is, from our observation, isomorphic to \( \mathbb{C} \) and its corresponding complex conjugate is given by \( [x]_3 \). Using this, we can construct a real-valued map \( N_3 : G \rightarrow \mathbb{R}^+ \) by taking the complex modulus,

\[ N_3(x) := [N_2(x)]_3 N_2(x) = [x \square x] x \square x. \]

(5.30)
Plugging in (5.29) we obtain
\[
N_3(x) = (\alpha^2 - \alpha^2 + b^2 - \beta^2)^2 + 4(\alpha\beta - a * b)^2.
\] (5.31)

Although \(N_3\) takes values in \(\mathbb{R}^+\), it is not a real norm\(^6\) on \(\mathcal{G}\) since there are nonzero elements with \(N_3(x) = 0\). It does however have the multiplicative property
\[
N_2(xy) = (xy)^\gamma xy = y^\gamma N_2(x)y = N_2(x)N_2(y)
\]
\[
\Rightarrow N_3(xy) = [N_2(xy)]_3N_2(xy) = [N_2(x)]_3[N_2(y)]_3N_2(x)N_2(y)
\] (5.32)
\[
= N_3(x)N_3(y),
\]
for all \(x,y\), since \(N_3(x^\gamma) = N_3(x)\). The expression (5.30) singles out the invertible elements as those elements \((5.29)\) for which \(N_3(x) \neq 0\), i.e.
\[
\mathcal{G}^\times(\mathbb{R}^3) = \{x \in \mathcal{G} : N_3(x) \neq 0\}
\]
\[
= \{x \in \mathcal{G} : (\alpha^2 - \alpha^2 + b^2 - \beta^2)^2 + 4(\alpha\beta - a * b)^2 \neq 0\}.\] (5.33)

The even subalgebra of \(\mathcal{G}(\mathbb{R}^3)\) is the quaternion algebra, as follows from the isomorphism \(\mathcal{G}^+(\mathbb{R}^3,0) \cong \mathcal{G}(\mathbb{R}^0,2) \cong \mathbb{H}\). The rotor group is then, according to the observation (4.16), the group of unit quaternions (note that the reverse here acts as the quaternion conjugate),
\[
\text{Spin}^+(3,0) = \{\alpha + bI \in \mathcal{G}^+ : \alpha^2 + b^2 = 1\} = e^{\mathcal{G}^2(\mathbb{R}^3)} \cong \text{SU}(2),\] (5.34)
where the exponentiation of the bivector algebra followed from Theorem 4.11. The last isomorphism shows the relation to the Pauli algebra and is perhaps the most famous representation of a Spin group. An arbitrary rotor \(\psi\) can according to (5.34) be written in the polar form \(\psi = e^{\hat{n}I}\), where \(\hat{n}\) is a unit vector, and represents a rotation by an angle \(-2\varphi\) in the plane \(\hat{n}I = -\hat{n}e\) (i.e. \(2\varphi\) counter-clockwise around the axis \(\hat{n}\)).

The Pin group consists of two copies of the rotor group,
\[
\text{Pin}(3,0) = e^{\mathcal{G}^2(\mathbb{R}^3)} \cup e^{\mathcal{G}^2(\mathbb{R}^3)}\hat{n},
\]
\[
\Gamma(3,0) = \mathbb{R}^x e^{\mathcal{G}^2(\mathbb{R}^3)} \cup \mathbb{R}^x e^{\mathcal{G}^2(\mathbb{R}^3)}\hat{n},\] (5.35)
for some unit vector \(\hat{n}\). The Pin group can be visualized as two unit 3-spheres \(S^3\) lying in the even and odd subspaces, respectively. The odd one includes a reflection and represents the non-orientation-preserving part of \(O(3)\).

As we saw above, the spinors in the space algebra are quaternions. An arbitrary spinor can be written \(\Psi = \rho e^{\hat{n}\varphi I/2}\) and acts on vectors by rotating in the plane \(\hat{n}e\) with the angle \(\varphi\) and scaling with \(\rho^2\). We will continue our discussion on these spinors in Section 6.

5.8 \(\mathcal{G}(\mathbb{R}^{1,3})\) - The spacetime algebra

We take as a four-dimensional example the \textit{spacetime algebra} (STA), which is the geometric algebra of Minkowski spacetime, \(\mathbb{R}^{1,3}\). This is the stage for special

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\(^6\)This will also be seen to be required from dimensional considerations and the remark to Hurwitz' Theorem in Section 4.
relativistic physics and what is fascinating with the STA is that it embeds a lot of important physical objects in a natural way.

By convention, we denote an orthonormal basis of the Minkowski space by \( \{ \gamma_0, \gamma_1, \gamma_2, \gamma_3 \} \), where \( \gamma_0 \) is timelike and the other \( \gamma_i \) are spacelike. This choice of notation is motivated by the Dirac representation of the STA in terms of so-called gamma matrices which will be explained in more detail later. The STA expressed in this basis is

\[
\mathcal{G}(\mathbb{R}^{1,3}) = \text{Span}_\mathbb{R} \{ 1, \gamma_0, \gamma_1, \gamma_2, \gamma_3, \ e_1, e_2, e_3, e_1 I, e_2 I, e_3 I, \ \gamma_0 I, \gamma_1 I, \gamma_2 I, \gamma_3 I, I \},
\]

where the pseudoscalar is \( I := \gamma_0 \gamma_1 \gamma_2 \gamma_3 \) and we set \( e_i := \gamma_i \gamma_0, \ i = 1, 2, 3 \). The form of the STA basis chosen above emphasizes the duality which exists between the graded subspaces. It also hints that the even subalgebra of the STA is the space algebra \( \mathbb{R}^{2,1} \). This is true from the isomorphism \( \mathcal{G}^+(\mathbb{R}^{1,3}) \cong \mathcal{G}(\mathbb{R}^{3,0}) \), but we can also verify this explicitly by noting that \( e_i^2 = 1 \) and \( e_1 e_2 e_3 = I \). Hence, the timelike (positive square) blades \( \{ e_i \} \) form a basis of a 3-dimensional euclidean space called the "relative space" to \( \gamma_0 \). For any timelike vector \( a \) we can find a similar relative space spanned by the bivectors \( \{ b \wedge a \} \) for \( b \in \mathbb{R}^{1,3} \). These spaces all generate the relative space algebra \( \mathcal{G}^+ \). This is a very powerful concept which helps us visualize and work efficiently in Minkowski spacetime and the STA.

Using boldface to denote relative space elements, an arbitrary multivector \( x \in \mathcal{G} \) can be written

\[
x = \alpha + a + a^b I + b I + \beta I, \quad (5.36)
\]

where \( \alpha, \beta \in \mathbb{R} \), \( a, b \in \mathbb{R}^{1,3} \) and \( a, b \) in relative space \( \mathbb{R}^3 \). As usual, we would like to find the invertible elements. Looking at the norm function \( N_2 : \mathcal{G} \to \mathcal{G}^0 \oplus \mathcal{G}^3 \oplus \mathcal{G}^4 \), it is not obvious that we can extend this to a real-valued function on \( \mathcal{G} \). Fortunately, we have for \( X = \alpha + b I + \beta I \in \mathcal{G}^0 \oplus \mathcal{G}^3 \oplus \mathcal{G}^4 \) that

\[
[X]_{3,4} X = (\alpha - b I - \beta I)(\alpha + b I + \beta I) = \alpha^2 - b^2 + \beta^2 \in \mathbb{R}. \quad (5.37)
\]

Hence, we can define a map \( N_4 : \mathcal{G} \to \mathbb{R} \) by

\[
N_4(x) := [N_2(x)]_{3,4} N_2(x) = [x \Box x] \Box x. \quad (5.38)
\]

Plugging in (5.36) into \( N_2 \), we obtain after a tedious calculation

\[
N_2(x) = \alpha^2 - a^2 - a^2 + b^2 + b^2 - \beta^2
+ 2(\alpha b - \beta a - a \wedge b + b \wedge a - a \wedge a^e - b \wedge b^e) I
+ 2(\alpha \beta - a \wedge b - a \wedge b) I
\quad (5.39)
\]

and, by (5.37),

\[
N_4(x) = (\alpha^2 - a^2 - a^2 + b^2 + b^2 - \beta^2)^2
- 4(\alpha b - \beta a - a \wedge b + b \wedge a - a \wedge a^e - b \wedge b^e)^2
+ 4(\alpha \beta - a \wedge b - a \wedge b)^2. \quad (5.40)
\]

We will prove some rather non-trivial statements about this norm function where we need that \( [xy] x = x [yx] \) for all \( x, y \in \mathcal{G} \). This is a quite general property of this involution.
Lemma 5.1. In any Clifford algebra $\mathrm{Cl}(X, R, r)$ (even when $X$ is infinite), we have
\[ [xy]x = x[yx] \quad \forall x, y \in \mathrm{Cl}. \]

Proof. Using linearity, we can set $y = A \in \mathcal{P}(X)$ and expand $x$ in coordinates $x_B \in R$ as $x = \sum_{B \in \mathcal{P}(X)} x_B B$. We obtain
\[
x[Ax] = \sum_{B, C} x_B x_C \ B[AC] \\
= \sum_{B, C} x_B x_C \ ((A \triangle C = \emptyset) - (A \triangle C \neq \emptyset)) \ BAC \\
= \sum_{B, C} x_B x_C \ ((A = C) - (A \neq C)) \ BAC \\
= \sum_{B} x_B x_A \ BAA - \sum_{C \neq A} \sum_{B} x_B x_C \ BAC
\]
and
\[
x[A]x = \sum_{B, C} x_B x_C \ [BA]C \\
= \sum_{B, C} x_B x_C \ ((B = A) - (B \neq A)) \ BAC \\
= \sum_{C} x_A x_C \ AAC - \sum_{B \neq A} \sum_{C} x_B x_C \ BAC \\
= x_A^3 \ AAA + \sum_{C \neq A} x_A x_C \ AAC - \sum_{B \neq A} x_B x_A \ [BAA]_{AAB} \\
\quad - \sum_{B \neq A} \sum_{C \neq A} x_B x_C \ BAC \\
= x_A^3 \ AAA - \sum_{B \neq A} \sum_{C \neq A} x_B x_C \ BAC \\
= x[Ax].
\]

We now have the following

Theorem 5.2. $N_4(x^\square) = N_4(x)$ for all $x \in \mathcal{G}(\mathbb{R}^{1,3})$.

Remark. Note that this is not at all obvious from the expression (5.40).

Proof. Using Lemma 5.1, we have that
\[ N_4(x^\square) = [xx^\square]xx^\square = x[x^\square x]x^\square. \quad (5.41) \]
Since $N_4$ takes values in $\mathbb{R}$, this must be a scalar, so that
\[ N_4(x^\square) = (x[x^\square x]x^\square)_0 = ([x^\square x]x^\square)_0 = (N_4(x))_0 = N_4(x), \quad (5.42) \]
where we used the symmetry of the scalar product.

Lemma 5.3. For all $X, Y \in \mathcal{G^0} \oplus \mathcal{G^3} \oplus \mathcal{G^4}$ we have
\[ [XY] = [Y][X]. \]

Proof. Take arbitrary elements $X = \alpha + bI + \beta I$ and $Y = \alpha' + b'I + \beta' I$. Then
\[
[XY] = [(\alpha + bI + \beta I)(\alpha' + b'I + \beta' I)] \\
= \alpha\alpha' - \alpha b'I - \alpha' \beta I - bI\alpha' + b * b' - b \wedge b' + b\beta' - \beta I\alpha' - \beta b' - \beta' \beta
\]
and
\[
[Y][X] = (\alpha' - b'I - \beta' I)(\alpha - bI - \beta I) \\
= \alpha'\alpha - \alpha' bI - \alpha' \beta I - b'I\alpha + b * b' + b' \wedge b - b/\beta - \beta'I\alpha + \beta'b - \beta' \beta.
\]
Comparing these expressions we find that they are equal.
We can now prove that $N_4$ really acts as a determinant on the STA.

**Theorem 5.4.** The norm function $N_4$ satisfies the product property

$$N_4(xy) = N_4(x)N_4(y) \quad \forall x, y \in G(\mathbb{R}^{1,3}).$$

**Proof.** Using that $N_4(xy)$ is a scalar and that $N_2$ takes values in $G^0 \oplus G^3 \oplus G^4$, we obtain

$$N_4(xy) = [(xy)\mathcal{D}(xy)(xy)\mathcal{D}xy = [y^\mathcal{D}x^\mathcal{D}xy]y^\mathcal{D}x^\mathcal{D}xy
= \langle x^\mathcal{D}y[y^\mathcal{D}x^\mathcal{D}xy]y^\mathcal{D}\rangle_0
= \langle N_2(x)[N_2(y^\mathcal{D})N_2(x)]N_2(y^\mathcal{D})\rangle_0
= \langle N_2(x)[N_2(x)][N_2(y^\mathcal{D})]N_2(y^\mathcal{D})\rangle_0
= \langle N_4(x)N_4(y^\mathcal{D})\rangle_0
= N_4(x)N_4(y^\mathcal{D}),$$

where we applied Lemma 5.1 and then Lemma 5.3. Theorem 5.2 now gives the claimed identity.

From (5.38) we find that the group of invertible elements is given by

$$G^\times(\mathbb{R}^{1,3}) = \{ x \in G : N_4(x) \neq 0 \}$$

and the inverse of $x \in G^\times$ is

$$x^{-1} = \frac{1}{N_4(x)}[x^\mathcal{D}x]^\mathcal{D}. \quad (5.44)$$

Note that the above theorems regarding $N_4$ only rely on the commutation properties of the different graded subspaces and not on the actual signature and field of the vector space. Therefore, these hold for all $\mathcal{C}(X,R,r)$ such that $|X| = 4$, and

$$\mathcal{C}^\times = \{ x \in \mathcal{C} : [x^\mathcal{D}x][x^\mathcal{D}x] \in R \text{ is invertible} \}. \quad (5.45)$$

Let us now turn our attention to the rotor group of the STA. The reverse equals the Clifford conjugate on the even subalgebra (it also corresponds to the Clifford conjugate defined on the relative space), so we find from (5.39) that the rotor group is

$$\text{Spin}^+(1,3) = \{ x \in \mathcal{G}^+ : N_2(x) = x^\mathcal{D}x = 1 \}
= \{ \alpha + a + bI + \beta I \in \mathcal{G}^+ : \alpha^2 - a^2 + b^2 - \beta^2 = 1, \ \alpha \beta = a \star b \} \quad (5.46)
= \pm e^{\sigma^2(\mathbb{R}^{1,3})} \cong \text{SL}(2,\mathbb{C}).$$

The last isomorphism is related to the Dirac representation of the STA, while the exponentiation identity was obtained from Theorem 4.11 and gives a better picture of what the rotor group looks like. Namely, any rotor $\psi$ can be written $\psi = \pm e^{a + bI}$ for some relative vectors $a, b$. A pure $\pm e^{bI}$ corresponds to a rotation in the spacelike plane $b^\mathcal{D}$ with angle $2|b|$ (which is a corresponding rotation also in relative space), while $\pm e^a$ corresponds to a “rotation” in the timelike plane $a$, i.e. a boost in the relative space direction $a$ with velocity $\arctanh(2a)$ times the speed of light.
Picking for example $\gamma_0$ and $\gamma_1$, we obtain the Spin and Pin groups,

\[
\begin{align*}
\text{Spin}(1, 3) &= \pm e^{G^2(\mathbb{R}^{1, 3})}_0 \gamma_0 \gamma_1, \\
\text{Pin}(1, 3) &= \pm e^{G^2(\mathbb{R}^{1, 3})}_0 \gamma_0 \pm e^{G^2(\mathbb{R}^{1, 3})}_1 \gamma_1 \pm e^{G^2(\mathbb{R}^{1, 3})}_5 \gamma_0 \gamma_1, \\
\Gamma(1, 3) &= \mathbb{R}^5 \times e^{G^2(\mathbb{R}^{1, 3})}_0 \gamma_0 \mathbb{R}^5 \times e^{G^2(\mathbb{R}^{1, 3})}_1 \gamma_1 \mathbb{R}^5 \times e^{G^2(\mathbb{R}^{1, 3})}_5 \gamma_0 \gamma_1.
\end{align*}
\]

The Pin group forms a double-cover of the so called Lorentz group $O(1, 3)$. Since the rotor group is connected, we find that $O(1, 3)$ has four connected components. The Spin group covers the subgroup of proper Lorentz transformations preserving orientation, while the rotor group covers the connected proper orthochronous Lorentz group which also preserves the direction of time.

The spinor space of the STA is the relative space algebra. We will discuss these spinors in more detail later, but for now note that an invertible spinor

\[
\Psi \Psi^\dagger = \rho e^{\frac{i}{2}} \in \mathbb{C}^4 \rightarrow \Psi = \rho^{1/2} e^{i \frac{\varphi}{2}} \psi
\]

is the product of a rotor $\psi$, a duality rotor $e^{i \varphi/2}$ and a scale factor $\rho^{1/2}$.

### 5.9 $\mathcal{G}(\mathbb{R}^{4, 1}) \cong \mathcal{G}(\mathbb{C}^4)$ - The Dirac algebra

The Dirac algebra is the representation of the STA which is most commonly used in physics. This is due to historic reasons, since the geometric nature of this algebra from its relation to the spacetime algebra was not uncovered until the 1960s. The relation between these algebras is observed by noting that the pseudoscalar in $\mathcal{G}(\mathbb{R}^{4, 1})$ commutes with all elements and squares to minus the identity. By Proposition 3.3 we have that the Dirac algebra is the complexification of the STA,

\[
\mathcal{G}(\mathbb{R}^{4, 1}) \cong \mathbb{C} \otimes \mathcal{G}(\mathbb{R}^{1, 3}) \cong \mathcal{G}(\mathbb{C}^4) \cong \mathbb{C}^{4 \times 4}.
\]

We construct this isomorphism explicitly by taking bases $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ of $\mathbb{R}^{1, 3}$ as usual and $\{e_0, \ldots, e_4\}$ of $\mathbb{R}^{4, 1}$ such that $e_0^2 = -1$ and the other $e_j^2 = 1$. We write $\mathcal{G}_5 := \mathcal{G}(\mathbb{R}^{4, 1})$ and $\mathcal{G}_5^C := \mathcal{G}(\mathbb{R}^{1, 3}) \otimes \mathbb{C}$, and use the convention that Greek indices run from 0 to 3. The isomorphism $F: \mathcal{G}_5 \rightarrow \mathcal{G}_5^C$ is given by the following correspondence of basis elements:

\[
\begin{align*}
\mathcal{G}_5^C: & \quad [1 \otimes 1, \gamma_\mu \otimes 1, \gamma_\mu \gamma_\nu \otimes 1, \gamma_\mu \gamma_\nu \gamma_\lambda \otimes 1, I_4 \otimes 1, 1 \otimes i] \\
\mathcal{G}_5: & \quad [e_\mu e_4, -e_\mu \otimes e_\nu, -e_\mu \otimes e_\nu \otimes e_\lambda e_4, e_0 e_1 e_2 e_3, I_5] \\
x^\square & \text{ in } \mathcal{G}_5^C: \quad + \quad + \quad + \\
[x] & \text{ in } \mathcal{G}_5^C: \quad + \quad + \quad + + \\
\overline{x} & \text{ in } \mathcal{G}_5^C: \quad + \quad + \quad + + \\
\end{align*}
\]

The respective pseudoscalars are $I_4 := \gamma_0 \gamma_1 \gamma_2 \gamma_3$ and $I_5 := e_0 e_1 e_2 e_3 e_4$. We have also noted the correspondence between involutions in the different algebras. Clifford conjugate in $\mathcal{G}_5^C$ corresponds to reversion in $\mathcal{G}_5$, the $[\ ]$-involution becomes the $[\ ]_{1, 2, 3, 4}$-involution, while complex conjugation in $\mathcal{G}_5^C$ corresponds to grade involution in $\mathcal{G}_5$. In other words,

\[
F(x^\square) = F(x)^\square, \quad F([x]_{1, 2, 3, 4}) = [F(x)], \quad F(x^*) = \overline{F(x)}.
\]
We can use the correspondence above to find a norm function on $G_5$. Since $N_4 : G(\mathbb{R}^{1,3}) \to \mathbb{R}$ was independent of the choice of field, we have that the complexification of $N_4$ satisfies
\[
N_4^C : G(\mathbb{C}^4) \to \mathbb{C},
\]
\[
x \mapsto |x^\Box x|^2.
\]
Taking the modulus of this complex number, we arrive at a real-valued map $N_5 : G(\mathbb{R}^{4,1}) \to \mathbb{R}$ with
\[
N_5(x) := \frac{N_4^C(F(x)) N_4^C(F(x))}{|F(x)^\Box F(x)| |
\]
\[
\begin{align*}
&= \frac{[F(x)^\Box F(x)] F(x)^\Box F(x)}{|F(x)^\Box F(x)|} |F(x)^\Box F(x)|
\end{align*}
\]
\[
= \frac{[x^\dagger x]_{1,2,3,4} x^\dagger x}{x^\dagger x}.
\]
In the final steps we noted that $x^\dagger x \in G^0 \oplus G^3 \oplus G^4 \oplus G^5$ and that $\mathbb{C} \subseteq G^C_4$ corresponds to $G^0 \oplus G^5 \subseteq G_5$. Furthermore, since $N_4^C(xy) = N_4^C(x)N_4^C(y)$, we have
\[
N_5(xy) = \frac{N_4^C(F(x) F(y))}{N_4^C(F(x)) N_4^C(F(y))}.
\]
for all $x, y \in G$. The invertible elements of the Dirac algebra are then as usual
\[
G^\times(\mathbb{R}^{4,1}) = \{ x \in G : N_5(x) \neq 0 \}
\]
and the inverse of $x \in G^\times$ is
\[
x^{-1} = \frac{1}{N_5(x)} [x^\dagger x]_{1,2,3,4} x^\dagger x.
\]
The above strategy could also have been used to obtain the expected result for $N_3$ on $G(\mathbb{R}^{5,0}) \cong G(\mathbb{C}^2)$ (with a corresponding isomorphism $F$):
\[
N_3(x) := \frac{N_2^C(F(x)) N_2^C(F(x))}{|F(x)^\Box F(x)|} |F(x)^\Box F(x)|.
\]
We briefly describe how spinors are dealt with in this representation. This will not be the same as the even subspace spinors which we usually consider. For the selected basis of $G^C_5$, we form the idempotent element
\[
f := \frac{1}{2}(1 + \gamma_0) \frac{1}{2}(1 + i\gamma_1 \gamma_2).
\]
Spinors are now defined as elements of the ideal $G_5^C f$ and one can show that every such element can be written as
\[
\Psi = \sum_{i=1}^4 \psi_i f_i, \quad \psi_i \in \mathbb{C},
\]
where
\[
f_1 := f, \quad f_2 := -\gamma_1 \gamma_3 f, \quad f_3 := \gamma_3 \gamma_0 f, \quad f_4 := \gamma_1 \gamma_0 f.
\]
With the standard representation of $\gamma_\mu$ as generators of $\mathbb{C}^{4\times 4}$,

$$\gamma_0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad \gamma_i = \begin{bmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{bmatrix}, \quad i = 1, 2, 3,$$

(5.56)

where $\sigma_i$ are the Pauli matrices which generate a representation of the Pauli algebra, one finds that

$$\Psi = \begin{bmatrix} \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \\ \psi_4 & 0 & 0 & 0 \end{bmatrix}.$$

(5.57)

Hence, these spinors can be thought of as complex column vectors. Upon a transformation of the basis $\{\gamma_\mu\}$, the components $\psi_i$ will transform according to the representation $D^{(1/2,0)} \oplus D^{(0,1/2)}$ of $\text{SL}(2,\mathbb{C})$. We will come back to discuss these spinors in Section 7. See [20] for more details on this correspondence between spinors and ideals.

### 5.10 Summary of norm functions

The norm functions

$$N_0(x) := x,$$

$$N_1(x) = x^\square x,$$

$$N_2(x) = x^\square x,$$

$$N_3(x) = [x^\square x]x^\square x,$$

$$N_4(x) = [x^\square x]x^\square x,$$

$$N_5(x) = [[x^\dagger x]_{1,4} x^\dagger x][x^\dagger x]_{1,4} x^\dagger x$$

constructed above (where we added $N_0$ for completeness) all have the product property

$$N_k(xy) = N_k(x)N_k(y)$$

(5.58)

for all $x, y \in \mathcal{G}(\mathcal{V})$ when $\mathcal{V}$ is $k$-dimensional. Because these functions only involve products and involutions, and the proofs of the above identities only rely on commutation properties in the respective algebras, they even hold for any Clifford algebra $\text{Cl}(X,R,r)$ with $|X| = 0, 1, \ldots, 5$, respectively.

For matrix algebras, a similar product property is satisfied by the determinant. On the other hand, we have the following theorem for matrices.

**Theorem 5.5.** Assume that $d: \mathbb{R}^{n\times n} \to \mathbb{R}$ is continuous and satisfies

$$d(AB) = d(A)d(B)$$

(5.59)

for all $A, B \in \mathbb{R}^{n\times n}$. Then $d$ must be either $0$, $1$, $|\det|^\alpha$ or $(\text{sign} \circ \det)|\det|^\alpha$ for some $\alpha > 0$.

In other words, we must have that $d = d_1 \circ \det$, where $d_1: \mathbb{R} \to \mathbb{R}$ is continuous and $d_1(\lambda u) = d_1(\lambda)d_1(u)$. This $d_1$ is uniquely determined e.g. by whether $d$ takes negative values, together with the value of $d(\lambda I)$ for any $\lambda > 1$. This
means that the determinant is the *unique* real-valued function on real matrices with the product property \( (5.59) \). The proof of this theorem can be found in the appendix.

Looking at Table 3.1, we see that \( \mathcal{G}(\mathbb{R}^{k,k}) \cong \mathbb{R}^{2^k \times 2^k} \) for \( k = 0, 1, 2, \ldots \). From the above theorem we then know that there are *unique* \( 7 \) continuous functions \( N_{2k} : \mathcal{G}(\mathbb{R}^{k,k}) \to \mathbb{R} \) such that \( N_{2k}(xy) = N_{2k}(x)N_{2k}(y) \) and \( N_{2k}(\lambda) = \lambda^{2^k} \). These are given by the determinant on the corresponding matrix algebra. What we do not know, however, is if every one of these can be expressed in the same simple form as \( N_0, N_2 \) and \( N_4 \), i.e. as a composition of products and grade-based involutions. Due to the complexity of higher-dimensional algebras, it is not obvious whether a continuation of the strategy employed so far can be successful or not. It is even difficult\(^8\) to test out suggestions of norm functions on a computer, since the number of operations involved grows as \( 2^{2k} \cdot 2^k \). We therefore leave this question as a suggestion for further investigation.

Because of the product property \( (5.58) \), the norm functions also lead to interesting factorization identities on rings. An example is \( N_2 \) for quaternions,

\[
\begin{align*}
(x_1^2 + x_2^2 + x_3^2 + x_4^2)(y_1^2 + y_2^2 + y_3^2 + y_4^2) &= (x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4)^2 + (x_1y_2 + x_2y_1 + x_3y_4 - x_4y_3)^2 \\
&+ (x_1y_3 - x_2y_4 + x_3y_1 + x_4y_2)^2 + (x_1y_4 + x_2y_3 - x_3y_2 + x_4y_1)^2.
\end{align*}
\]

This is called the *Lagrange identity*. These types of identities can be used to prove theorems in number theory. Using \( (5.60) \), one can for example prove that every integer can be written as a sum of four squares of integers. Or, in other words, every integer is the norm (squared) of an integral quaternion. See e.g. \[^8\] for the proof.

Another possible application of norm functions could be in public key cryptography and one-way trapdoor functions. We have not investigated this idea further, however.

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7Actually, the functions are either \( \det \) or \( |\det| \). \( N_2 \) and \( N_4 \) constructed previously are smooth, however, so they must be equal to \( \det \).

8The first couple of \( N_4 \) can be verified directly using a geometric algebra package in Maple, but already for \( N_4 \) this becomes impossible to do straight-away on a standard desktop computer.
6 Representation theory

In this section we will use the classification of geometric algebras as matrix algebras, which was developed in Section 3, to work out the representation theory of these algebras. Since one can find representations of geometric algebras in many areas of mathematics and physics, this leads to a number of interesting applications. We will consider two main examples in detail, namely normed division algebras and vector fields on higher-dimensional spheres.

**Definition 6.1.** For $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, we define a $\mathbb{K}$-representation of $\mathcal{G}(V, q)$ as an $\mathbb{R}$-algebra homomorphism

$$\rho: \mathcal{G}(V, q) \to \text{End}_{\mathbb{K}}(W),$$

where $W$ is a finite-dimensional vector space over $\mathbb{K}$. $W$ is called a $\mathcal{G}(V, q)$-module over $\mathbb{K}$.

Note that a vector space over $\mathbb{C}$ or $\mathbb{H}$ can be considered as a real vector space together with operators $J$ or $I, J, K$ in $\text{End}_{\mathbb{R}}(W)$ that anticommute and square to minus the identity. In the definition above we assume that these operators commute with $\rho(x)$ for all $x \in \mathcal{G}$, so that $\rho$ can be said to respect the $\mathbb{K}$-structure of the space $W$. When talking about the dimension of the module $W$ we will always refer to its dimension as a real vector space.

The standard strategy when studying representation theory is to look for irreducible representations.

**Definition 6.2.** A representation $\rho$ is called reducible if $W$ can be written as a sum of proper (not equal to 0 or $W$) invariant subspaces, i.e.

$$W = W_1 \oplus W_2 \quad \text{and} \quad \rho(x)(W_j) \subseteq W_j \quad \forall \ x \in \mathcal{G}.$$ 

In this case we can write $\rho = \rho_1 \oplus \rho_2$, where $\rho_j(x) := \rho(x)|_{W_j}$. A representation is called irreducible if it is not reducible.

The traditional definition of an irreducible representation is that it does not have any proper invariant subspaces. However, because $\mathcal{G}$ is generated by a finite group (the Clifford group) one can verify that these two definitions are equivalent in this case.

**Proposition 6.1.** Every $\mathbb{K}$-representation $\rho$ of a geometric algebra $\mathcal{G}(V, q)$ can be split up into a direct sum $\rho = \rho_1 \oplus \ldots \oplus \rho_m$ of irreducible representations.

**Proof.** This follows directly from the definitions and the fact that $W$ is finite-dimensional.

**Definition 6.3.** Two $\mathbb{K}$-representations $\rho_j: \mathcal{G}(V, q) \to \text{End}_{\mathbb{K}}(W_j)$, $j = 1, 2$, are said to be equivalent if there exists a $\mathbb{K}$-linear isomorphism $F: W_1 \to W_2$ such that

$$F \circ \rho_1(x) \circ F^{-1} = \rho_2(x) \quad \forall \ x \in \mathcal{G}.$$ 

**Theorem 6.2.** Up to equivalence, the only irreducible representations of the matrix algebras $\mathbb{K}^{n \times n}$ and $\mathbb{K}^{n \times n} \oplus \mathbb{K}^{n \times n}$ are

$$\rho: \mathbb{K}^{n \times n} \to \text{End}_{\mathbb{K}}(\mathbb{K}^n)$$

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and

\[ \rho_{1,2}: K^{n \times n} \oplus K^{n \times n} \to \text{End}_K(K^n) \]

respectively, where \( \rho \) is the defining representation and

\[ \rho_1(x, y) := \rho(x), \]
\[ \rho_2(x, y) := \rho(y). \]

**Proof.** This follows from the classical fact that the algebras \( K^{n \times n} \) are simple and that simple algebras have only one irreducible representation up to equivalence. See e.g. [12] for details.

**Theorem 6.3.** From the above, together with the classification of real geometric algebras, follows the table of representations in Table 6.1, where \( \nu_{s,t} \) is the number of inequivalent irreducible representations and \( d_{s,t} \) is the dimension of an irreducible representation for \( \mathcal{G}(R^{s,t}) \). The cases for \( n > 8 \) are obtained using the periodicity

\[ \nu_{m+8k} = \nu_m, \]
\[ d_{m+8k} = 16^k d_m. \]

(6.1)

| \( n \) | \( \mathcal{G}(\mathbb{R}^{n,0}) \) | \( \nu_{n,0} \) | \( d_{n,0} \) | \( \mathcal{G}(\mathbb{R}^{0,n}) \) | \( \nu_{0,n} \) | \( d_{0,n} \) |
|---|---|---|---|---|---|---|
| 0 | \( \mathbb{R} \) | 1 | 1 | \( \mathbb{R} \) | 1 | 1 |
| 1 | \( \mathbb{R} \oplus \mathbb{R} \) | 2 | 1 | \( \mathbb{C} \) | 1 | 2 |
| 2 | \( \mathbb{R}^{2 \times 2} \) | 1 | 2 | \( \mathbb{H} \) | 1 | 4 |
| 3 | \( \mathbb{C}^{2 \times 2} \) | 1 | 4 | \( \mathbb{H} \oplus \mathbb{H} \) | 2 | 4 |
| 4 | \( \mathbb{H}^{2 \times 2} \) | 1 | 8 | \( \mathbb{H}^{2 \times 2} \) | 1 | 8 |
| 5 | \( \mathbb{H}^{2 \times 2} \oplus \mathbb{H}^{2 \times 2} \) | 2 | 8 | \( \mathbb{C}^{4 \times 4} \) | 1 | 8 |
| 6 | \( \mathbb{H}^{4 \times 4} \) | 1 | 16 | \( \mathbb{R}^{8 \times 8} \) | 1 | 8 |
| 7 | \( \mathbb{C}^{8 \times 8} \) | 1 | 16 | \( \mathbb{R}^{8 \times 8} \oplus \mathbb{R}^{8 \times 8} \) | 2 | 8 |
| 8 | \( \mathbb{R}^{16 \times 16} \) | 1 | 16 | \( \mathbb{R}^{16 \times 16} \) | 1 | 16 |

Table 6.1: Number and dimension of irreducible representations of euclidean and anti-euclidean geometric algebras.

We will now consider the situation when the representation space \( W \) is endowed with an inner product. Note that if \( W \) is a vector space over \( \mathbb{K} \) with an inner product, we can always find a \( \mathbb{K} \)-invariant inner product on \( W \), i.e. such that the operators \( J \) or \( I, J, K \) are orthogonal. Namely, let \( \langle \cdot, \cdot \rangle_\mathbb{R} \) be an inner product on \( W \) and put

\[ \langle x, y \rangle_\mathbb{C} := \sum_{\Gamma \in \{id, J\}} \langle \Gamma x, \Gamma y \rangle_\mathbb{R}, \quad \langle x, y \rangle_\mathbb{H} := \sum_{\Gamma \in \{id, I, J, K\}} \langle \Gamma x, \Gamma y \rangle_\mathbb{R}. \]

(6.2)

Then \( \langle Jx, Jy \rangle_\mathbb{K} = \langle x, y \rangle_\mathbb{K} \) and \( \langle Jx, Jy \rangle_\mathbb{K} = -\langle x, Jy \rangle_\mathbb{K} \), etc.

In the same way, when \( \mathcal{V} \) is euclidean or anti-euclidean, we can for a representation \( \rho: \mathcal{G}(\mathcal{V}) \to \text{End}_K(W) \) find an inner product such that \( \rho \) acts orthogonally with unit vectors, i.e. such that \( \langle \rho(e)x, \rho(e)y \rangle = \langle x, y \rangle \) for all \( x, y \in W \) and \( e \in \mathcal{V} \) with \( e^2 = \pm 1 \). We construct such an inner product by averaging
a, possibly $K$-invariant, inner product $\langle \cdot, \cdot \rangle_K$ over the Clifford group. Take an orthonormal basis $E$ of $V$ and put
\[
\langle x, y \rangle := \sum_{\Gamma \in B_E} \langle \rho(\Gamma)x, \rho(\Gamma)y \rangle_K.
\]
(6.3)

We then have that
\[
\langle \rho(e_i)x, \rho(e_i)y \rangle = \langle x, y \rangle
\]
(6.4) and
\[
\langle \rho(e_i)x, \rho(e_j)y \rangle = \langle \rho(e_j)x, \rho(e_j)y \rangle = \pm \langle \rho(e_j)x, \rho(e_i)y \rangle
\]
(6.5)
for $e_i \neq e_j$ in $E$. Thus, if $e = \sum_i a_i e_i$ and $\sum_i a_i^2 = 1$, we obtain
\[
\langle \rho(e)x, \rho(e)y \rangle = \sum_{i,j} a_i a_j \langle \rho(e_i)x, \rho(e_j)y \rangle = \langle x, y \rangle.
\]
(6.6)

Hence, this inner product has the desired property. Also note that, for $v \in V = \mathbb{R}^{n,0}$, we have
\[
\langle \rho(v)x, y \rangle = \langle x, \rho(v)y \rangle,
\]
(6.7)
while for $V = \mathbb{R}^{0,n}$,
\[
\langle \rho(v)x, y \rangle = -\langle x, \rho(v)y \rangle,
\]
(6.8)
i.e. $\rho(v)$ is symmetric for euclidean spaces and antisymmetric for anti-euclidean spaces.

We are now ready for some examples which illustrate how representations of geometric algebras can appear in various contexts and how their representation theory can be used to prove important theorems.

6.1 Example I: Normed division algebras

Our first example concerns the possible dimensions of normed division algebras. A normed division algebra is an algebra $\mathcal{A}$ over $\mathbb{R}$ (not necessarily associative) with a norm $|\cdot|$ such that
\[
|xy| = |x||y|
\]
(6.9)
for all $x, y \in \mathcal{A}$ and such that every nonzero element is invertible. We will prove the following

Theorem 6.4 (Hurwitz’ Theorem). If $\mathcal{A}$ is a finite-dimensional normed division algebra over $\mathbb{R}$, then its dimension is either 1, 2, 4 or 8.

Remark. This corresponds uniquely to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and the octonions $\mathbb{O}$, respectively. The proof of unicity requires some additional steps, see e.g. [2].

Let us first consider the restrictions that the requirement [K3] puts on the norm. Assume that $\mathcal{A}$ has dimension $n$. For every $a \in \mathcal{A}$ we have a linear transformation
\[
L_a : \mathcal{A} \to \mathcal{A},
\]
\[
x \mapsto a x
\]
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given by left multiplication by \(a\). When \(|a| = 1\) we then have
\[
|L_ax| = |ax| = |a||x| = |x|,
\]
i.e. \(L_a\) preserves the norm. Hence, it maps the unit sphere \(S := \{x \in \mathcal{A} : |x| = 1\}\) in \(\mathcal{A}\) into itself. Furthermore, since every element in \(\mathcal{A}\) is invertible, we can for each pair \(x, y \in S\) find an \(a \in S\) such that \(L_ax = ax = y\). Now, these facts imply a large amount of symmetry of \(S\). In fact, we have the following

**Lemma 6.5.** Assume that \(V\) is a finite-dimensional normed vector space. Let \(S_V\) denote the unit sphere in \(V\). If, for every \(x, y \in S_V\), there exists an operator \(L \in \text{End}(V)\) such that \(L(S_V) \subseteq S_V\) and \(L(x) = y\), then \(V\) is an inner product space.

**Proof.** We will need the following fact: Every compact subgroup \(G\) of \(\text{GL}(n)\) preserves some inner product on \(\mathbb{R}^n\). This can be proven by picking a Haar-measure \(\mu\) on \(G\) and averaging any inner product \(\langle \cdot, \cdot \rangle\) on \(\mathbb{R}^n\) over \(G\) using this measure,
\[
\langle x, y \rangle_G := \int_G \langle gx, gy \rangle \, d\mu(g).
\]

Now, let \(G\) be the group of linear transformations on \(V \cong \mathbb{R}^n\) which preserve its norm \(|\cdot|\). \(G\) is compact in the finite-dimensional operator norm topology, since \(G = \bigcap_{x \in V} \{L \in \text{End}(\mathbb{R}^n) : |Lx| = |x|\}\) is closed and bounded by 1. Furthermore, \(L \in G\) is injective and therefore an isomorphism. The group structure is obvious. Hence, \(G\) is a compact subgroup of \(\text{GL}(n)\).

From the above we know that there exists an inner product \(\langle \cdot, \cdot \rangle\) on \(\mathbb{R}^n\) which is preserved by \(G\). Let \(|\cdot|_o\) denote the norm associated to this inner product, i.e. \(|x|_o^2 = \langle x, x \rangle\). Take a point \(x \in \mathbb{R}^n\) with \(|x| = 1\) and rescale the inner product so that also \(|x|_o = 1\). Let \(S\) and \(S_o\) denote the unit spheres associated to \(|\cdot|\) and \(|\cdot|_o\), respectively. By the conditions in the lemma, there is for every \(y \in S\) an \(L \in G\) such that \(L(x) = y\). But \(G\) also preserves the norm \(|\cdot|_o\), so \(y\) must also lie in \(S_o\). Hence, \(S\) is a subset of \(S_o\). However, being unit spheres associated to norms, \(S\) and \(S_o\) are both homeomorphic to the standard sphere \(S^{n-1}\), so we must have that they are equal. Therefore, the norms must be equal.

We now know that our normed division algebra \(\mathcal{A}\) has some inner product \(\langle \cdot, \cdot \rangle\) such that \(\langle x, x \rangle = |x|^2\). We call an element \(a \in \mathcal{A}\) imaginary if \(a\) is orthogonal to the unit element, i.e. if \(\langle a, 1_A \rangle = 0\). Let \(\text{Im} \mathcal{A}\) denote the \((n-1)\)-dimensional subspace of imaginary elements. We will observe that \(\text{Im} \mathcal{A}\) acts on \(\mathcal{A}\) in a special way.

Take a curve \(\gamma : (-\epsilon, \epsilon) \to S\) on the unit sphere such that \(\gamma(0) = 1_A\) and \(\gamma'(0) = a \in \text{Im} \mathcal{A}\). (Note that \(\text{Im} \mathcal{A}\) is the tangent space to \(S\) at the unit element.) Then, because the product in \(\mathcal{A}\) is continuous,
\[
\left.\frac{d}{dt}\right|_{t=0} L_{\gamma(t)}x = \lim_{h \to 0} \frac{1}{h} (L_{\gamma(h)}x - L_{\gamma(0)}x) = \lim_{h \to 0} \frac{1}{h} (\gamma(h) - \gamma(0))x = \gamma'(0)x = ax = L_ax
\]
and
\[
0 = \left.\frac{d}{dt}\right|_{t=0} \langle x, y \rangle = \left.\frac{d}{dt}\right|_{t=0} \langle L_{\gamma(t)}x, L_{\gamma(t)}y \rangle = \langle \left.\frac{d}{dt}\right|_{t=0} L_{\gamma(t)}x, L_{\gamma(0)}y \rangle + \langle L_{\gamma(0)}x, \left.\frac{d}{dt}\right|_{t=0} L_{\gamma(t)}y \rangle = \langle L_ax, y \rangle + \langle x, L_ay \rangle.
\]
Hence, $L_a^* = -L_a$ for $a \in \operatorname{Im} \mathcal{A}$. If, in addition, $|a| = 1$ we have that $L_a \in O(\mathcal{A}, |\cdot|^2)$, so $L_a^2 = -L_a L_a^* = -id$. For an arbitrary imaginary element $a$ we obtain by rescaling

$$L_a^2 = -|a|^2. \quad (6.14)$$

This motivates us to consider the geometric algebra $\mathcal{G}(\operatorname{Im} \mathcal{A}, q)$ with quadratic form $q(a) := -|a|^2$. By (6.14) and the universal property of geometric algebras (Proposition 2.1) we find that $L$ extends to a representation of $\mathcal{G}(\operatorname{Im} \mathcal{A}, q)$ on $\mathcal{A}$,

$$\hat{L} : \mathcal{G}(\operatorname{Im} \mathcal{A}, q) \to \operatorname{End}(\mathcal{A}), \quad (6.15)$$

i.e. a representation of $\mathcal{G}(\mathbb{R}^{0,n-1})$ on $\mathbb{R}^n$. The representation theory now demands that $n$ is a multiple of $d_0(n-1)$. By studying Table 6.1 and taking periodicity (6.1) into account we find that this is only possible for $n = 1, 2, 4, 8$.

### 6.2 Example II: Vector fields on spheres

In our next example we consider the $N$-dimensional unit spheres $S^N$ and use representations of geometric algebras to construct vector fields on them. The number of such vector fields that can be found gives us information about the topological features of these spheres.

**Theorem 6.6 (Radon-Hurwitz).** On $S^N$ there exist $n_N$ pointwise linearly independent vector fields, where, if we write $N$ uniquely as

$$N + 1 = (2t + 1)2^{4a+b}, \quad t, a \in \mathbb{N}, \ b \in \{0, 1, 2, 3\}, \quad (6.16)$$

then

$$n_N = 8a + 2^b - 1. \quad (6.17)$$

For example,

| $N$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|-----|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|
| $n_N$ | 0 | 1 | 0 | 3 | 0 | 1 | 0 | 7 | 0 | 1 | 0 | 3 | 0 | 1 | 0 | 8 | 0 |

**Corollary.** $S^1$, $S^3$ and $S^7$ are parallelizable.

**Remark.** The number of vector fields constructed in this way is actually the maximum number of possible such fields on $S^N$. This is a much deeper result proven by Adams [1] using algebraic topology.

Our main observation is that if $\mathbb{R}^{N+1}$ is a $\mathcal{G}(\mathbb{R}^{0,n})$-module then we can construct $n$ pointwise linearly independent vector fields on $S^N = \{x \in \mathbb{R}^{N+1} : \langle x, x \rangle = 1 \}$. Namely, suppose we have a representation $\rho$ of $\mathcal{G}(\mathbb{R}^{0,n})$ on $\mathbb{R}^{N+1}$. Take an inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{N+1}$ such that the action of $\rho$ is orthogonal and pick any basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^{0,n}$. We can now define a collection of smooth vector fields $\{V_1, \ldots, V_n\}$ on $\mathbb{R}^{N+1}$ by

$$V_i(x) := \rho(e_i)x, \quad i = 1, \ldots, n. \quad (6.18)$$

According to the observation (6.18) this action is antisymmetric, so that

$$\langle V_i(x), x \rangle = \langle \rho(e_i)x, x \rangle = 0. \quad (6.19)$$
Hence, \( V_i(x) \in T_xS^N \) for \( x \in S^N \). By restricting to \( S^N \) we therefore have \( n \) tangent vector fields. It remains to show that these are pointwise linearly independent. Take \( x \in S^N \) and consider the linear map

\[
i_x: \mathbb{R}^{0,n} \to T_xS^N
\]

\[
v \mapsto i_x(v) := \rho(v)x
\]

Since the image of \( i_x \) is \( \text{Span}_\mathbb{R}\{V_i(x)\} \) it is sufficient to prove that \( i_x \) is injective. But if \( i_x(v) = \rho(v)x = 0 \) then also \( v^2x = \rho(v)^2x = 0 \), so we must have \( v = 0 \).

Now, for a fixed \( N \) we want to find as many vector fields as possible, so we seek the highest \( n \) such that \( \mathbb{R}^{N+1} \) is a \( G(\mathbb{R}^{0,n}) \)-module. From the representation theory we know that this requires that \( N + 1 \) is a multiple of \( d_{0,n} \). Furthermore, since \( d_{0,n} \) is a power of 2 we obtain the maximal such \( n \) when \( N + 1 = p2^m \), where \( p \) is odd and \( d_{0,n} = 2^m \). Using Table 6.1 and the periodicity (6.1) we find that if we write \( N + 1 = p2^{a+b} \), with \( 0 \leq b \leq 3 \), then \( n = 8a + 2^b - 1 \). This proves the theorem.
7 Spinors in physics

In this final section we discuss how the view of spinors as even multivectors can be used to reformulate physical theories in a way which clearly expresses the geometry of these theories, and therefore leads to conceptual simplifications.

In the geometric picture provided by geometric algebra we consider spinor fields as (smooth) maps from the space or spacetime $V$ into the spinor space of $G(V)$,

$$\Psi: V \rightarrow G^+(V).$$

(7.1)

The field could also take values in the spinor space of a subalgebra of $G$. For example, a relativistic complex scalar field living on Minkowski spacetime could be considered as a spinor field taking values in a plane subalgebra $G^+(\mathbb{R}^{0,2}) \subset G(\mathbb{R}^{1,3})$.

In the following we will use the summation convention that matching upper and lower Greek indices implies summation over 0,1,2,3. We will not write out physical constants such as $c, e, \hbar$.

7.1 Pauli spinors

Pauli spinors describe the spin state of a non-relativistic fermionic particle such as the non-relativistic electron. Since this is the non-relativistic limit of the Dirac theory discussed below, we will here just state the corresponding representation of Pauli spinors as even multivectors of the space algebra. We saw that such an element can be written as $\Psi = \rho e^{\varphi nI/2} e^{\phi e^3}$, i.e. a scaled rotor. For this spinor field, the physical state is expressed by the observable vector (field)

$$s := \Psi e^3 \Psi^\dagger = \rho e^{\varphi nI/2} e^3 e^{-\varphi nI/2},$$

(7.2)

which is interpreted as the expectation value of the particle’s spin, scaled by the spatial probability amplitude $\rho$. The vector $e_3$ acts as a reference axis for the spin. The up and down spin basis states in the ordinary complex representation correspond to the rotors which leave $e_3$ invariant, respectively the rotors which rotate $e_3$ into $-e_3$. Observe the invariance of $s$ under right-multiplication of $\Psi$ by $e^{\varphi e^3}$. This corresponds to the complex phase invariance in the conventional formulation.

7.2 Dirac-Hestenes spinors

Dirac spinors describe the state of a relativistic Dirac particle, such as an electron or neutrino. Conventionally, Dirac spinors are represented by four-component complex column vectors, $\psi = [\psi_1, \psi_2, \psi_3, \psi_4]^T \in \mathbb{C}^4$. For a spinor field the components will be complex-valued functions on spacetime. Acting on these spinors are the complex $4 \times 4$-matrices $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ given in (5.56), which generate a matrix representation of the Dirac algebra. The Dirac adjoint of a column spinor is a row matrix

$$\psi^\dagger \gamma_0 = [\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*],$$

(7.3)

where, in this context, complex conjugation is denoted by $^*$ and hermitian conjugation by $^\dagger$. The physical state of a Dirac particle is determined by the
following 16 so called \textit{bilinear covariants}:

\[
\begin{align*}
\alpha & := \psi^\dagger \gamma_0 \psi \\
J^\mu & := \psi^\dagger \gamma_\mu \psi \\
S^{\mu\nu} & := \psi^\dagger \gamma_0 i\gamma^\mu \gamma^\nu \psi \\
K^\mu & := \psi^\dagger \gamma_0 I^{-1} \gamma_\mu \psi \\
\beta & := \psi^\dagger \gamma_0 I^{-1} \psi \\
\end{align*}
\]

(7.4)

Their integrals over space give expectation values of the physical observables. For example, \(J^0\) integrated over a spacelike domain gives the probability\(^9\) of finding the particle in that domain, and \(J^k, k = 1, 2, 3\), give the current of probability. These are components of a spacetime current vector \(J\). The quantities \(S^{\mu\nu}\) describe the probability density of the particle’s electromagnetic moment, while \(K^\mu\) gives the direction of the particle’s spin\(^{10}\).

In Hestenes’ reformulation of the Dirac theory, we represent spinors by even multivectors \(\Psi \in \mathcal{G}^+\) in the real spacetime algebra \(\mathcal{G}(\mathbb{R}^{1,3})\). Note that both \(\Psi\) and \(\psi\) have eight real components, so this is no limitation. In this representation, the gamma matrices are considered as orthonormal basis vectors of the Minkowski spacetime and the bilinear covariants are given by

\[
\begin{align*}
\alpha + \beta I & = \Psi \Psi^\dagger \\
J & = \Psi \gamma_0 \Psi^\dagger \\
S & = \Psi \gamma_1 \gamma_2 \Psi^\dagger \\
K & = \Psi \gamma_3 \Psi^\dagger \\
\end{align*}
\]

(7.5)

where \(J = J^\mu \gamma_\mu, K = K^\mu \gamma_\mu\) are spacetime vectors and \(S = \frac{1}{2} S^{\mu\nu} \gamma_\mu \wedge \gamma_\nu\) a bivector. This reformulation allows for a nice geometric interpretation of the Dirac theory. Here, spinors are objects that transform the reference basis \(\{\gamma_\mu\}\) into the observable quantities.

Since a spinor only has eight components, the bilinear covariants cannot be independent. From (7.5) we easily find a number of relations called the \textit{Fierz identities},

\[
\begin{align*}
J^2 &= -K^2 = \alpha^2 + \beta^2, & J \ast K &= 0, & J \wedge K &= -(\alpha I + \beta)S.
\end{align*}
\]

(7.6)

The Fierz identities also include a bunch of relations which in the case \(\alpha^2 + \beta^2 \neq 0\) can be derived directly from these three. In total, there are seven degrees of freedom, given for example by the spacetime current \(J\), the relative space direction of the spin vector \(K\) (two angles) and the so called Yvon-Takabayasi angle \(\chi := \arctan(\beta/\alpha)\). The eighth degree of freedom present in a spinor is the phase-invariance, which in the original Dirac theory corresponds to the overall complex phase of \(\psi\), while in the Dirac-Hestenes picture corresponds to a rotational freedom in the \(\gamma_1 \gamma_2\)-plane, or equivalently around the spin axis in relative space. This is explained by the invariance of (7.5) under a transformation \(\Psi \mapsto \Psi e^{x_3 i}\).

In the null case, i.e. when \(\alpha = \beta = 0\), we have the additional identities

\[
\begin{align*}
S^2 = 0, & \quad JS = SJ = 0, & \quad KS = SK = 0.
\end{align*}
\]

(7.7)

\(^9\)Or rather the probability multiplied with the charge of the particle. For a large number of particles this can be interpreted as a charge density.

\(^{10}\)In the formulation below, we obtain the relative space spin vector as \(K = \frac{1}{2} K \wedge \gamma_0 / |K \wedge \gamma_0|\).
The geometric interpretation is that $J \propto K$ are both null vectors and $S$ is a null bivector blade with $J$ and $K$ in its null subspace. Hence, the remaining five degrees of freedom are given by the direction of the plane represented by $S$, which must be tangent to the light-cone (two angles), plus the magnitudes of $S, J$ and $K$.

The equation which describes the evolution of a Dirac spinor in spacetime is the Dirac equation, which in this representation is given by the Dirac-Hestenes equation,
\[ \nabla \Psi \gamma_1 \gamma_2 - A \Psi = m \Psi \gamma_0, \] (7.8)
where $\nabla := \gamma^\mu \frac{\partial}{\partial x^\mu}$ (we use the spacetime coordinate expansion $x = x^\mu \gamma_\mu$) and $A$ is the electromagnetic potential vector field. Here, $\gamma_1 \gamma_2$ again plays the role of the complex imaginary unit $i$.

Another interesting property of the STA is that the electromagnetic field is most naturally represented as a bivector field in $G(R^{1,3})$. We can write any such bivector field as $F = E + I B$, where $E$ and $B$ are relative space vector fields. In the context of relativistic electrodynamics, these are naturally interpreted as the electric and magnetic fields, respectively. Maxwell’s equations are compactly written as
\[ \nabla F = J \] (7.9)
in this formalism, where $J$ is the source current. The physical quantity describing the energy and momentum present in an electromagnetic field is the Maxwell stress-energy tensor which in the STA formulation can be interpreted as a map $T: R^{1,3} \rightarrow R^{1,3}$, given by
\[ T(x) := -\frac{1}{2} F_x F = \frac{1}{2} F_x F^\dagger. \] (7.10)
For example, the energy of the field $F$ relative to the $\gamma_0$-direction is $\gamma_0 \ast T(\gamma_0) = \frac{1}{2}(E^2 + B^2)$.

Rodrigues and Vaz [25], [26] have studied an interesting correspondence between the Dirac and Maxwell equations. With the help of the following theorem, they have proved that the electromagnetic field can be obtained from a spinor field satisfying an equation similar to the Dirac equation. This theorem also serves to illustrate how efficiently computations can be performed in the STA framework.

**Theorem 7.1.** Any bivector $F \in G^2(R^{1,3})$ can be written as
\[ F = \Psi \gamma_0 \gamma_1 \Psi^\dagger, \]
for some (nonzero) spinor $\Psi \in G^+(R^{1,3})$.

**Proof.** Take any bivector $F = E + I B \in G^2$. Note that
\[ F^2 = (E^2 - B^2) + 2 E \ast B I = \rho e^{\phi I} \] (7.11)
for some $\rho \geq 0$ and $0 \leq \phi < 2\pi$. We consider the cases $F^2 \neq 0$ and $F^2 = 0$ separately.

If $F^2 \neq 0$ then $E^2 - B^2$ and $E \ast B$ are not both zero and we can apply a duality rotation of $F$ into
\[ F' = E' + I B' := e^{-\phi I / 4} F e^{-\phi I / 4} \Rightarrow F'^2 = \rho, \] (7.12)
i.e. such that \( E'^2 - B'^2 > 0 \) and \( E' \neq B' = 0 \). Hence, we can select an orthonormal basis \( \{ e_i \} \) of the relative space, aligned so that \( E' = E'e_1 \) and \( B' = B'e_2 \), where \( E' = |E'| \) etc. Consider now a boost \( a = \alpha e_3 \) of angle \( \alpha \) in the direction orthogonal to both \( E' \) and \( B' \). Using that
\[
e^{-\frac{\alpha}{2} e_1} e_1 e^{\frac{\alpha}{2} e_3} = e_1 e^{\alpha e_3} = e_1 (\cosh \alpha + \sinh \alpha \ e_3)
\]
and likewise for \( e_2 \), we obtain
\[
F'' := e^{-a/2} F' e^{a/2} = E' e^{-\frac{\alpha}{2} e_3} e_1 e^{\frac{\alpha}{2} e_3} + IB' e^{-\frac{\alpha}{2} e_3} e_2 e^{\frac{\alpha}{2} e_3}
\]
\[
= E' e_1 (\cosh \alpha + \sinh \alpha \ e_3) + IB' e_2 (\cosh \alpha + \sinh \alpha \ e_3)
\]
\[
= (E' \cosh \alpha - B' \sinh \alpha) e_1 + I (B' \cosh \alpha - E' \sinh \alpha) e_2
\]
\[
= \cosh \alpha ((E' - B' \tanh \alpha) e_1 + I (B' - E' \tanh \alpha) e_2),
\]
where we also noted that \( e_1 e_3 = -I e_2 \) and \( I e_2 e_3 = -e_1 \). Since \( E'^2 - B'^2 > 0 \) we can choose \( \alpha := \arctanh \left( \frac{B'}{E'} \right) \) and obtain \( F'' = \sqrt{1 - (\frac{B'}{E'})^2} E' = E'' e_1 \), where \( E'' > 0 \). Finally, some relative space rotor \( e^{ib/2} \) takes \( e_1 \) to our timelike target blade (relative space vector) \( \gamma_0 \gamma_1 \), i.e.
\[
F'' = E'' e^{ib/2} \gamma_0 \gamma_1 e^{-ib/2}.
\]
Summing up, we have that \( F = \Psi \gamma_0 \gamma_1 \Psi^+ \), where
\[
\Psi = \sqrt{E''} e^{ia/2} e^{ib/2} \in \mathbb{G}^+.
\]

When \( F^2 = 0 \) we have that both \( E^2 = B^2 \) and \( E \neq B = 0 \). Again, we select an orthonormal basis \( \{ e_i \} \) of the relative space so that \( E = E e_1 \) and \( B = B e_2 = E e_2 \). Note that
\[
(1 - I e_1 e_2) e_1 (1 + I e_1 e_2) = e_1 - I e_1 e_2 e_1 + I e_2 - I e_1 e_2 I e_2
\]
\[
= 2(e_1 + I e_2).
\]
Thus, \( \frac{1}{\sqrt{2}} (1 - I e_1 e_2) E \frac{1}{\sqrt{2}} (1 + I e_1 e_2) = E + I B \). Using that \( e_1 \) can be obtained from \( \gamma_0 \gamma_1 \) with some relative space rotor \( e^{ib/2} \), we have that \( F = \Psi \gamma_0 \gamma_1 \Psi^+ \), where
\[
\Psi = (\frac{E}{\gamma})^{1/2} (1 - \frac{1}{\gamma} I E B) e^{ib/2} \in \mathbb{G}^+.
\]
The case \( F = 0 \) can be achieved not only using \( \Psi = 0 \), but also with e.g. \( \Psi = (1 + \gamma_0 \gamma_1) \).

Note that we can switch \( \gamma_0 \gamma_1 \) for any other non-null reference blade, e.g. \( \gamma_1 \gamma_2 \).

**Remark.** In the setting of electrodynamics, where \( F = E + I B \) is an electromagnetic field, we obtain as a consequence of this theorem and proof the following result due to Rainich, Misner and Wheeler. If we define an *extremal field* as a field for which the magnetic (electric) field is zero and the electric (magnetic) field is parallel to one coordinate axis, the theorem of Rainich-Misner-Wheeler says that: “At any point of Minkowski spacetime any nonnull electromagnetic field can be reduced to an extremal field by a Lorentz transformation and a duality rotation.”

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The reformulation of the Pauli and Dirac theory observables above depended on the choice of fixed reference bases \( \{e_1, e_2, e_3\} \) and \( \{\gamma_0, \gamma_1, \gamma_2, \gamma_3\} \). When a different basis \( \{e'_1, e'_2, e'_3\} \) is selected, we cannot apply the same spinor \( \Psi \) in (7.2) since this would in general yield an \( s' = \Psi e'_3 \Psi^+ \neq s \). This is not a problem in flat space since we can set up a globally defined field of such reference frames without ambiguity. However, in the covariant setting of a curved manifold, i.e. when gravitation is involved, we cannot fix a certain field of reference frames, but must allow a variation in these and equations which transform covariantly under such variations. In fibre bundle theory this corresponds to picking different sections of an orthonormal frame bundle. We therefore seek a formulation of spinor that takes care of this required covariance. Rodrigues, de Souza, Vaz and Lounesto [20] have considered the following definition.

**Definition 7.1.** A Dirac-Hestenes spinor (DHS) is an equivalence class of triplets \((\Sigma, \psi, \Psi)\), where \( \Sigma \) is an oriented orthonormal basis of \( \mathbb{R}^{1,3} \), \( \psi \) is an element in \( \text{Spin}^+(1,3) \), and \( \Psi \in G^+(\mathbb{R}^{1,3}) \) is the representative of the spinor in the basis \( \Sigma \). We define the equivalence relation by \( (\Sigma, \psi, \Psi) \sim (\Sigma_0, \psi_0, \Psi_0) \) if and only if \( \Sigma = \tilde{\text{Ad}}_{\psi_0}^{-1} \Sigma_0 \) and \( \Psi = \Psi_0 \psi_0 \psi^{-1} \). The basis \( \Sigma_0 \) should be thought of as a fixed reference basis and the choice of \( \psi_0 \) is arbitrary but fixed for this basis. We suppress this choice and write just \( \Psi_\Sigma \) for the spinor \((\Sigma, \psi, \Psi)\).

Note that when for example \( J = \Psi_E e_0 \Psi_E^+ \) for some basis \( E = \{e_i\} \) we now have the desired invariance property \( J = \Psi_{E'} e'_0 \Psi_{E'}^+ \) for some other basis \( E' = \{e'_i\} \). Hence, \( J \) is now a completely basis independent object which in the Dirac theory represents the physical and observable local current produced by a Dirac particle.

The definition above allows for the construction of a covariant Dirac-Hestenes spinor field. The possibility of defining such a field on a certain manifold depends on the existence of a so called spin structure on it. Geroch [4] has shown that in the spacetime case, i.e. when the tangent space is \( \mathbb{R}^{1,3} \), this is equivalent to the existence of a globally defined field of time-oriented orthonormal reference frames. In other words, the principal \( \text{SO}^+ \)-bundle of the manifold must be trivial. We direct the reader to [20] for a continued discussion.

We end by mentioning that other types of spinors can be represented in the STA as well. See e.g. [5] for a discussion on Lorentz, Majorana and Weyl spinors.
8 Summary and discussion

We have seen that a vector space endowed with a quadratic form naturally embeds in an associated geometric algebra. This algebra depends on the signature and dimension of the underlying vector space, and expresses the geometry of the space through the properties of its multivectors. By introducing a set of products, involutions and other operations, we got access to the rich structure of this algebra and could identify certain significant types of multivectors, such as blades, rotors, and spinors. Blades were found to represent subspaces of the underlying vector space and gave a geometric interpretation to multivectors and the various algebraic operations, while rotors connected the groups of structure-respecting transformations to corresponding groups embedded in the algebra. This enabled a powerful encoding of rotations using geometric products and allowed us to identify candidates for spinors in arbitrary dimensions.

The introduced concepts were put to practice when we worked out a number of lower-dimensional examples. These had obvious applications in mathematics and physics. Norm functions were found to act as determinants on the respective algebras and helped us find the corresponding groups of invertible elements. We noted that the properties of such norm functions also lead to totally non-geometric applications in number theory.

We also studied the relation between geometric algebras and matrix algebras, and used the well-known representation theory of such algebras to work out the corresponding representations of geometric algebras. The dimensional restrictions of such representations led to proofs of classic theorems regarding normed division algebras and vector fields on spheres.

Throughout our examples, we saw that complex structures appear naturally within real geometric algebras and that many formulations in physics which involve complex numbers can be identified as structures within real geometric algebras. Such identifications also resulted in various geometric interpretations of complex numbers. This suggests that, whenever complex numbers appear in an otherwise real or geometric context, one should ask oneself if not a real geometric interpretation can be given to them.

The combinatorial construction of Clifford algebra which we introduced mainly served as a tool for understanding the structure of geometric or Clifford algebras and the behaviour of, and relations between, the different products. This construction also expresses the generality of Clifford algebras in that they can be defined and find applications in general algebraic contexts. Furthermore, it gives new suggestions for how to proceed with the infinite-dimensional case. Combinatorial Clifford algebra has previously been applied to simplify proofs in graph theory [24].

Finally, we considered examples in physics and in particular relativistic quantum mechanics, where the representation of spinors as even multivectors in the geometric algebra of spacetime led to conceptual simplifications. The resulting picture is a rather classical one, with particles as fields of operations which rotate and scale elements of a reference basis into the observable expectation values. Although this is a geometric and conceptually powerful view, it is unfortunately not that enlightening with respect to the quantum mechanical aspects of states and measurement. This requires an operator-eigenvalue formalism which of course can be formulated in geometric algebra, but sort of breaks the geometric picture. The geometric view of spinors does fit in the context of quantum field
theory, however, since spinor fields there already assume a classical character. It is not clear what conceptual simplifications that geometric algebras can bring to other quantum mechanical theories than the Pauli and Dirac ones, since most realistic particle theories are formulated in infinite-dimensional spaces. Doran and Lasenby \cite{4} have presented suggestions for a multi-particle formulation in geometric algebra, but it still involves a fixed and finite number of particles.

Motivated by the conceptual simplifications of Dirac theory brought by the spacetime algebra, one can argue about the geometric significance of all particles. The traditional classification of particles in terms of spin quantum numbers relies on the complex representation theory of the (inhomogeneous) Lorentz group. There are complex (or rather complexified) representations of the STA-embedded scalar, spinor (through the Dirac algebra), and vector fields, corresponding to spin 0, $\frac{1}{2}$, and 1, respectively. Coincidentally, the fundamental particles that have been experimentally verified all have spin quantum numbers $\frac{1}{2}$ or 1, corresponding to spinor fields and vector fields. Furthermore, the proposed Higgs particle is a scalar field with spin 0. Since all these types of fields are naturally represented within the STA it then seems natural to me that exactly these spins have turned up.

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Appendix: Matrix theorems

In order to avoid long digressions in the text, we have placed proofs to some, perhaps not so familiar, theorems in this appendix.

In the following theorem we assume that $R$ is an arbitrary commutative ring and

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \in R^{n \times m}, \quad a_j = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{bmatrix},$$

i.e. $a_j$ denotes the $j$:th column in $A$. If $I \subseteq \{1, \ldots, n\}$ and $J \subseteq \{1, \ldots, m\}$ we let $A_{I,J}$ denote the $|I| \times |J|$-matrix minor obtained from $A$ by deleting the rows and columns not in $I$ and $J$. Further, let $k$ denote the rank of $A$, i.e. the highest integer $k$ such that there exists $I, J$ as above with $|I| = |J| = k$ and $\det A_{I,J} \neq 0$. By renumbering the $a_{ij}$:s we can without loss of generality assume that $I = J = \{1, 2, \ldots, k\}$.

**Theorem A.1 (Basis minor).** If the rank of $A$ is $k$, and 

$$d := \det \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{bmatrix} \neq 0,$$

then every $d \cdot a_j$ is a linear combination of $a_1, \ldots, a_k$.

**Proof.** Pick $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$ and consider the $(k+1) \times (k+1)$-matrix 

$$B_{i,j} := \begin{bmatrix} a_{11} & \cdots & a_{1k} & a_{1j} \\ \vdots & & \vdots & \vdots \\ a_{k1} & \cdots & a_{kk} & a_{kj} \\ a_{i1} & \cdots & a_{ik} & a_{ij} \end{bmatrix}.$$ 

Then $\det B_{i,j} = 0$. Expanding $\det B_{i,j}$ along the bottom row for fixed $i$ we obtain

$$a_{11}C_1 + \ldots + a_{ik}C_k + a_{ij}d = 0, \quad (1)$$

where the $C_l$ are independent of the choice of $i$ (but of course dependent on $j$). Hence,

$$C_1a_1 + \ldots + C_ka_k + da_j = 0, \quad (2)$$

and similarly for all $j$. \qed

The following shows that the factorization $\det(AB) = \det(A)\det(B)$ is a unique property of the determinant.

**Theorem A.2 (Uniqueness of determinant).** Assume that $d : \mathbb{R}^{n \times n} \to \mathbb{R}$ is continuous and satisfies

$$d(AB) = d(A)d(B) \quad (3)$$

for all $A, B \in \mathbb{R}^{n \times n}$. Then $d$ must be either $0$, $1$, $|\det|^\alpha$ or $(\text{sign} \circ \det)|\det|^\alpha$ for some $\alpha > 0$. 
Proof. First, we have that
\[ d(0) = d(0^2) = d(0)^2, \]
\[ d(I) = d(I^2) = d(I)^2, \] (4)
so \( d(0) \) and \( d(I) \) must be either 0 or 1. Furthermore,
\[ d(0) = d(0A) = d(0)d(A), \]
\[ d(A) = d(AA) = d(I)d(A), \] (5)
for all \( A \in \mathbb{R}^{n \times n} \), which implies that \( d = 1 \) if \( d(0) = 1 \) and \( d = 0 \) if \( d(I) = 0 \). We can therefore assume that \( d(0) = 0 \) and \( d(I) = 1 \).

Now, an arbitrary matrix \( A \) can be written as
\[ A = E_1E_2 \ldots E_kR, \] (6)
where \( R \) is on reduced row-echelon form (as close to the identity matrix as possible) and \( E_i \) are elementary row operations of the form
\[ R_{ij} := (\text{swap rows } i \text{ and } j), \]
\[ E_i(\lambda) := (\text{scale row } i \text{ by } \lambda), \] (7)
\[ E_{ij}(c) := (\text{add } c \times \text{row } j \text{ to row } i). \]
Because \( R_{ij}^2 = I \), we must have \( d(R_{ij}) = \pm 1 \). This gives, since
\[ E_i(\lambda) = R_{1i}E_1(\lambda)R_{1j}, \] (8)
that \( d(E_i(\lambda)) = d(E_1(\lambda)) \) and
\[ d(\lambda I) = d(E_1(\lambda) \ldots E_n(\lambda)) = d(E_1(\lambda)) \ldots d(E_n(\lambda)) = d(E_1(\lambda))^n. \] (9)
In particular, we have \( d(E_1(0)) = 0 \) and of course \( d(E_1(1)) = d(I) = 1 \).

If \( A \) is invertible, then \( R = I \). Otherwise, \( R \) must contain a row of zeros so that \( R = E_i(0)R \) for some \( i \). But then \( d(R) = 0 \) and \( d(A) = 0 \). When \( A \) is invertible we have \( I = AA^{-1} \) and \( 1 = d(I) = d(A)d(A^{-1}) \), i.e. \( d(A) \neq 0 \) and \( d(A^{-1}) = d(A)^{-1} \). Hence,
\[ A \in \text{GL}(n) \iff d(A) \neq 0. \] (10)
We thus have that \( d \) is completely determined by its values on \( R_{ij}, E_1(\lambda) \) and \( E_{ij}(c) \). Note that we have not yet used the continuity of \( d \), but it is time for that now. We can split \( \mathbb{R}^{n \times n} \) into three connected components, namely \( \text{GL}^{-}(n), \det^{-1}(0) \) and \( \text{GL}^{+}(n) \), where the determinant is less than, equal to, and greater than zero, respectively. Since \( E_1(1), E_{ij}(c) \in \text{GL}^{+}(n) \) and \( E_1(-1), R_{ij} \in \text{GL}^{-}(n) \), we have by continuity of \( d \) that
\[ d(R_{ij}) = +1 \implies d \text{ is } > 0, = 0, \text{ resp. } > 0 \]
\[ d(R_{ij}) = -1 \implies d \text{ is } < 0, = 0, \text{ resp. } > 0 \] (11)
on these parts. Using that \( d(E_1(-1))^2 = d(E_1(-1)^2) = d(I) = 1 \), we have
\[ d(E_1(-1)) = \pm 1 \text{ and } d(E_1(-\lambda)) = d(E_1(-1))d(E_1(\lambda)) = \pm d(E_1(\lambda)) \] where
the sign depends on $11$. On $\mathbb{R}^{++} := \{ \lambda \in \mathbb{R} : \lambda > 0 \}$ we have a continuous map $d \circ E_1 : \mathbb{R}^{++} \to \mathbb{R}^{++}$ such that

$$d \circ E_1(\lambda \mu) = d \circ E_1(\lambda) \cdot d \circ E_1(\mu) \quad \forall \lambda, \mu \in \mathbb{R}^{++}. \quad (12)$$

Forming $f := \ln \circ d \circ E_1 \circ \exp$, we then have a continuous map $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(\lambda + \mu) = f(\lambda) + f(\mu). \quad (13)$$

By extending linearity from $\mathbb{Z}$ to $\mathbb{Q}$ and $\mathbb{R}$ by continuity, we must have that $f(\lambda) = \alpha \lambda$ for some $\alpha \in \mathbb{R}$. Hence, $d \circ E_1(\lambda) = \lambda^\alpha$. Continuity also demands that $\alpha > 0$.

It only remains to consider $d \circ E_{ij} : \mathbb{R} \to \mathbb{R}^{++}$. We have $d \circ E_{ij}(0) = d(I) = 1$ and $E_{ij}(c)E_{ij}(\gamma) = E_{ij}(c + \gamma)$, i.e.

$$d \circ E_{ij}(c + \gamma) = d \circ E_{ij}(c) \cdot d \circ E_{ij}(\gamma) \quad \forall c, \gamma \in \mathbb{R}. \quad (14)$$

Proceeding as above, $g := \ln \circ d \circ E_{ij} : \mathbb{R} \to \mathbb{R}$ is linear, so that $g(c) = \alpha_{ij} c$ for some $\alpha_{ij} \in \mathbb{R}$, hence $d \circ E_{ij}(c) = e^{\alpha_{ij} c}$. One can verify that the following identity holds for all $i, j$:

$$E_{ji}(-1) = E_i(-1)R_{ij}E_{ji}(1)E_{ij}(-1). \quad (15)$$

This gives $d(E_{ji}(-1)) = (\pm 1)(\pm 1)d(E_{ji}(1))d(E_{ij}(-1))$ and, using $(14)$,

$$d(E_{ij}(1)) = d(E_{ji}(2)) = d(E_{ji}(1 + 1)) = d(E_{ij}(1))d(E_{ij}(1))$$

$$= d(E_{ij}(2))d(E_{ij}(2)) = d(E_{ij}(4)), \quad (16)$$

which requires $\alpha_{ij} = 0$.

We conclude that $d$ is completely determined by $\alpha > 0$, where $d \circ E_1(\lambda) = \lambda^\alpha$ and $\lambda \geq 0$, plus whether $d$ takes negative values or not. This proves the theorem. \qed
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