‘Riemann Equations’
in Bidifferential Calculus

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Abstract

In the framework of bidifferential calculus, we consider equations that formally resemble a matrix Riemann (or Hopf) equation. Depending on the choice of first-order bidifferential calculus, besides a semi-discrete and a fully discrete version of the matrix Riemann equation, we also obtain quite different equations. They all share a simple universal solution-generating method, which in some examples turns out to be a (continuous or discrete) Cole-Hopf transformation. Furthermore, a recent (non-isospectral) binary Darboux transformation result in bidifferential calculus is specialized to generate solutions of the ‘Riemann equations’. If the bidifferential calculus extends to second order, solutions of a system of such equations are also solutions of an equation that arises, on the universal level of bidifferential calculus, as an integrability condition. Depending on the choice of bidifferential calculus, the latter can represent quite a number of prominent integrable equations, like self-dual Yang-Mills, as well as matrix versions of the two-dimensional Toda lattice, Hirota’s bilinear difference equation, (2+1)-dimensional NLS, KP and Davey-Stewartson equations. From matrix versions (of arbitrary size) of the associated ‘Riemann equations’, multi-soliton-type solutions of the latter equations can be generated. This includes (‘breaking’) multi-soliton-type solutions of the self-dual Yang-Mills and the (2+1)-dimensional NLS equation, which are parametrized by solutions of Riemann equations.

Keywords: bidifferential calculus, breaking soliton, Burgers equation, chiral model, Cole-Hopf transformation, Darboux transformation, Davey-Stewartson equation, Hirota bilinear difference equation, Hopf equation, hierarchy, integrable discretization, kink, KP, Riemann equation, self-dual Yang-Mills, soliton, Toda lattice.

1 Introduction

Given an associative algebra $\mathcal{A}$ and two derivations $d, \bar{d} : \mathcal{A} \to \Omega^1$ into an $\mathcal{A}$-bimodule, the two equations

$$\bar{d}\phi - (d\phi)\phi = 0$$

and

$$\bar{d}\phi + \phi d\phi = 0,$$

for $\phi \in \mathcal{A}$, resemble a Riemann equation (also known as Hopf, inviscid Burgers, dispersionless KdV, or nonlinear transport equation). For a simple choice of the first-order bidifferential calculus, given by $\mathcal{A}, \Omega^1, d, \bar{d}$, these are indeed matrix Riemann equations.\footnote{Matrix versions of the Riemann (or Hopf) equation appeared in many publications. In the context of the present work, the references $[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]$ are of particular interest.} For other choices, (1.1) and (1.2) turn out to be very different equations, however. Several examples will be presented in this
work. An important point is that (in an intermediate step) we have to go beyond solutions which are (matrices of) functions and allow $\phi$ to become an operator (e.g., a differential or difference operator). Accordingly, $A$ will then be an algebra involving operators.

In the examples considered in this work, (1.1) and (1.2) are related by a kind of transpose or adjoint operation. Therefore we will concentrate on (1.1).

Suppose there is an extension of the derivation $d$ to a map $A \overset{d}{\rightarrow} \Omega^1 \overset{d}{\rightarrow} \Omega^2$, with another $A$-bimodule $\Omega^2$, and correspondingly for $\bar{d}$, such that
\[
d^2 = \bar{d}^2 = d\bar{d} + \bar{d}d = 0. \tag{1.3}
\]
In this case we have a second-order bidifferential calculus, $(\Omega, d, \bar{d})$, with $\Omega = \bigoplus_{r=0}^2 \Omega^r$, $\Omega^0 := A$ \cite{11, 12}. Then, acting with $d$ on (1.1) or (1.2) yields
\[
d\bar{d}\phi + d\phi d\phi = 0 \tag{1.4}
\]
as an integrability condition. By choosing appropriate bidifferential calculi, this equation leads to various integrable partial differential and/or difference equations (PDDEs) \cite{12} and references therein). Equations like (1.1), (1.2) and (1.4) are of a universal nature and integrable PDDEs, derived from them, may be thought of as realizations.

A crucial point is that efficient solution generating methods can be easily derived for the universal equations. By choosing a bidifferential calculus in such a way that one of these equations becomes equivalent to some PDDE, the method applies to the latter, and in this way one typically recovers a known method for that specific equation. In particular, this shows that solution generating methods for various equations have a surprisingly simple origin and a universal proof.

For (1.1) and (1.2), there is a simple ‘linearization method’ (see Section 2), which in several cases is the origin of a Cole-Hopf-type transformation. This does not extend to (1.4), for which, however, there is another universal method. Indeed, in \cite{10} (also see Section 3), a solution-generating result representing an abstract version of binary Darboux transformations \cite{17, 18} has been derived for (1.4) and the ‘(Miura-) dual’ equation
\[
d[(\bar{d}g)g^{-1}] = 0, \tag{1.5}
\]
with (invertible) dependent variable $g \in A$. More precisely, this is a solution generating result for the ‘Miura equation’
\[
(\bar{d}g)g^{-1} = d\phi, \tag{1.6}
\]
which has both equations, (1.4) and (1.5), as integrability conditions, provided that (1.3) holds. This binary Darboux transformation method requires solutions of versions of (1.1) and (1.2) as inputs (cf. (3.1)), which is another motivation to explore these ‘Riemann equations’. In most cases, soliton families are obtained by choosing $d$- and $\bar{d}$-constant solutions of these equations. The non-autonomous chiral model equation that arises in integrable reductions of the vacuum Einstein (-Maxwell) equations is an important exception in this respect, see \cite{19, 16} and also Section 5.1.3.

More generally, this concerns equations possessing a non-isospectral linear problem (see \cite{20, 21} and also, e.g., \cite{22, 23}).

Furthermore, the present work partly originated from the simple observation that (1.6) becomes (1.1), respectively (1.2), if we set $g = \pm \phi$, respectively $g^{-1} = \pm \phi$. The solution-generating result in \cite{10} then still works and can indeed be applied to generate large classes of exact solutions of various incarnations of (1.1) (or (1.2)). If the first order bidifferential calculus extends to second order, this also provides us with a special class of solutions of the associated incarnations of (1.4).

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Footnotes:

\[\text{This step is crucial in the theory of integrable systems, where one considers a ‘zero curvature’ (Zakharov-Shabat, or Lax) equation in general for operator expressions.}\]

\[\text{Realizations of (1.1) (or (1.2)) are typically in the class of ‘C-integrable equations’ (see, e.g., [13, 14, 15]).}\]
and (1.5). Of course, more general classes of solutions of (1.4) and (1.5) can be generated using the binary Darboux transformation method (Theorem 3.1) and we will report on this in a separate work.

A Cole-Hopf-type transformation is certainly the simplest way to generate exact solutions. Darboux transformations for PDDEs resulting from (1.1) (or (1.2)) are in this respect not the first choice, but may be helpful, depending on the addressed problem. We are particularly interested in understanding how the two methods are related.

Another insight described in the present work concerns the way in which bidifferential calculi for many integrable equations are composed from those for ‘Riemann systems’.

Section 2 presents the aforementioned simple solution-generating method for (1.1) (or (1.2)), which in special cases becomes a Cole-Hopf transformation. In Section 3 we recall the binary Darboux transformation result from [16] and specialize it in a corollary in the way described above. Section 4 treats matrix Riemann equations, their integrable (semi- and full) discretizations, and corresponding hierarchies. In the semi-discrete case, this is the semi-discrete Burgers hierarchy first treated in [1]. The (fully) discrete Riemann hierarchy contains a matrix version of the discrete Burgers equation derived in [24]. For its first member, an integrable discrete Riemann equation, we are not aware of previous explorations. Sections 5-8 present a collection of important examples of integrable equations arising as integrability condition of a system of ‘Riemann equations’. Section 9 contains some concluding remarks.

2 A simple solution-generating method

Writing

$$\phi = \Phi \phi_0 \Phi^{-1},$$

(2.1)

with an invertible $\Phi$, (1.1) is equivalent to

$$\ddot{\phi} + (d\phi_0) \phi_0 + [\gamma, \phi_0] = 0,$$

(2.2)

where the 1-form (i.e., element of $\Omega^1$) $\gamma$ is defined by

$$\ddot{\phi} - (d\phi) \phi_0 - \Phi \gamma = 0.$$

(2.3)

Let us consider the last equation as an equation for $\Phi$, for a given 1-form $\gamma$. Then, if $\phi_0$ is a solution of (2.2), $\phi$ given by (2.1) solves (1.1). For fixed $\phi_0$, (2.1) and (2.3), written as linear equations in $\Phi$, can thus be regarded as a Lax pair for (1.1).

Here we only have to solve linear equations in order to construct new solutions. Obviously, (2.1) can only lead to a new solution if the solution $\Phi$ of the linear equation does not commute with $\phi_0$. This excludes the example of the scalar (continuous) Riemann equation (see Section 4.1), but non-trivial solutions of matrix (continuous) Riemann equations can be obtained (also see [8]). In the scalar case, (2.1) can be nontrivial if $\phi_0$ involves an operator. In fact, this includes the well-known Cole-Hopf transformation for the Burgers equation, see Section 7.

**Remark 2.1.** Replacing (2.3) by $\ddot{\phi} - (d\phi) \phi_0 + [\gamma, \phi] = 0$, then $\phi$ given by (2.1) satisfies (2.2). Note that the $\gamma$ term can be absorbed via a redefinition $\dddot{\phi} \rightarrow \dddot{\phi} - [\gamma, \cdot]$, which turns (2.2) into (1.1).

**Remark 2.2.** Instead of (1.1), we may consider the more general equation

$$\ddot{\phi} - (d\phi) \eta(\phi) = \rho(\phi),$$

(2.4)

4If we choose $\gamma = \frac{1}{2}d\phi_0$, then (2.2) takes the form $\ddot{\phi}_0 - d(\phi^2_0) = 0$.
where the function $\eta$ and $\rho \in \Omega^1$ are required to satisfy $\eta(\phi) = \Phi \eta(\phi_0) \Phi^{-1}$, $\rho(\phi) = \Phi \rho(\phi_0) \Phi^{-1}$. (2.4) is then equivalent to

$$
\bar{d}\phi_0 - (d\phi_0) \eta(\phi_0) + [\gamma, \phi_0] = \rho(\phi_0),
$$

where the 1-form (i.e., element of $\Omega^1$) $\gamma$ is defined by

$$
\bar{d}\Phi - (d\Phi) \eta(\phi_0) - \Phi \gamma = 0.
$$

Considering these as equations for a given $\gamma$, we have a solution generating method that works well as long as $\phi$ has values in an algebra of matrices of functions, which then is essentially the case treated in [8]. If the algebra is extended by operators and $\phi$ is an operator expression, it will typically be impossible to reduce (2.4) to a PDDE. There are exceptions when $\eta$ is homogeneous. In those cases we looked at, it turned out, however, that they can also be treated starting from (1.1), see Remarks 4.13 and 4.15.

### 3 Binary Darboux transformations in bidifferential calculus

In the following, let $\mathcal{A}$ be the algebra of all finite-dimensional matrices with entries in a unital associative algebra $\mathcal{B}$, where the product of two matrices is defined to be zero if the sizes of the two matrices do not match. We assume that there is an $\mathcal{A}$-bimodule $\Omega^1$ and derivations $d$ and $\bar{d}$ on $\mathcal{A}$ with values in $\Omega^1$, such that $d$ and $\bar{d}$ preserve the size of matrices. Mat$(m,n,B)$ denotes the set of $m \times n$ matrices over $B$. For fixed $m,n \in \mathbb{N}$, $I = I_m$ and $I = I_n$ denote the $m \times m$, respectively $n \times n$, identity matrix, and we assume that they are constant with respect to $d$ and $\bar{d}$. Let us recall a result from [16] in a slightly generalized form.

#### Theorem 3.1

Let $\phi_0, g_0 \in \text{Mat}(m,m,B)$ solve (1.6), and let $P, Q \in \text{Mat}(n,n,B)$ be solutions of

$$
\bar{d}P - (dP) P = -[\alpha, P], \quad \bar{d}Q - Q dQ = [\beta, Q],
$$

with any 1-forms $\alpha, \beta$.

Let $U \in \text{Mat}(m,n,B)$ and $V \in \text{Mat}(n,m,B)$ be solutions of the linear equations

$$
\bar{d}U = (dU) P + (d\phi_0) U + U \alpha, \quad \bar{d}V = Q dV - V d\phi_0 + \beta V.
$$

Furthermore, let $X \in \text{Mat}(n,n,B)$ be an invertible solution of the (inhomogeneous) linear equations

$$
XP - QX = VU, \quad \bar{d}X - (dX) P + (dQ) X + (dV) U = X \alpha + \beta X.
$$

Then

$$
\phi = \phi_0 + U X^{-1} V \quad \text{and} \quad g = (I + U (Q X)^{-1} V) g_0
$$

yields a new solution of the Miura equation (1.6). \hfill \square

---

5 In [8] also generalizations of (2.4) (in the continuous setting) to any number of independent variables are treated. In principle our formalism can incorporate this by extending $d$ to several commuting derivations $d_i : \mathcal{A} \to \mathcal{A}$, $i = 1, \ldots, N$. But we have not been able to find a PDDE arising in this way outside of the (continuous) framework of [8].

6 The possibility $\Omega^1 = \mathcal{A}$ is not excluded. Also see Section 4.

7 Here we do not require the conditions (1.3). All we need is that $d$ and $\bar{d}$ satisfy the Leibniz rule.

8 Of course, the requirement that these equations possess solutions imposes constraints on $\alpha$ and $\beta$. 
Remark 3.2. For $\alpha = \beta = 0$, the two equations in (3.1) are $n \times n$ matrix versions of (1.1) and (1.2), respectively. If the first-order bidifferential calculus extends to second order, $P$ and $-Q$ are solutions of the $n \times n$ matrix version of (1.4). The linear equations (3.2) then have (1.4), here for $\phi_0$, as an integrability condition. $P$ and $Q$ may be regarded as operator (in particular, matrix) versions of a spectral parameter. Unless $P$ and $Q$ are d- and $\bar{d}$-constant, the linear equations (3.2) constitute a non-isospectral problem (cf. [20] and also, e.g., [22] [23]).

Remark 3.3. Theorem 3.1 appeared in [16] with $\alpha = \beta = 0$. Let us start with this version. Correspondingly, in this remark a reference to an equation in Theorem 3.1 shall mean the respective equation with $\alpha = \beta = 0$. The introduction of the 1-forms $\alpha$ and $\beta$ is motivated by a freedom of transformations. Let us write

$$P = \Psi_1 \bar{P} \Psi_1^{-1}, \quad Q = \Psi_2^{-1} \bar{Q} \Psi_2,$$

(3.6)

with invertible $n \times n$ matrices $\Psi_1, \Psi_2$. (3.1) then takes the form

$$\bar{d} \bar{P} - (d \bar{P}) \bar{P} = -[\alpha, \bar{P}], \quad \bar{d} \bar{Q} - \bar{Q} d \bar{Q} = [\beta, \bar{Q}],$$

where the 1-forms $\alpha, \beta$ are now defined by

$$\bar{d} \Psi_1 - (d \Psi_1) \bar{P} = \Psi_1 \alpha, \quad \bar{d} \Psi_2 - \bar{Q} d \Psi_2 = \beta \Psi_2.$$

In terms of

$$\tilde{X} := \Psi_2 X \Psi_1, \quad \tilde{U} := U \Psi_1, \quad \tilde{V} := \Psi_2 V,$$

(3.3) and (3.4) are equivalent to

$$\tilde{X} \bar{P} - \bar{Q} \tilde{X} = \tilde{V} \tilde{U}, \quad \bar{d} \tilde{X} - (d \tilde{X}) \bar{P} + (d \bar{Q}) \tilde{X} + (d \tilde{V}) \tilde{U} = \tilde{X} \alpha + \beta \tilde{X},$$

where we used the linear equations for $\Psi_1$ and $\Psi_2$. The linear equations (3.2) are correspondingly transformed to

$$\bar{d} \tilde{U} = (d \tilde{U}) \bar{P} + (d \phi_0) \tilde{U} + \tilde{U} \alpha, \quad \bar{d} \tilde{V} = \bar{Q} d \tilde{V} - \tilde{V} d \phi_0 + \beta \tilde{V}.$$

The expressions (3.5) for the new solutions are invariant under $U \mapsto \tilde{U}, V \mapsto \tilde{V}$ and $X \mapsto \tilde{X}$. Abstracting $\alpha, \beta$ from their above origin, leads to a slightly generalized version of Theorem 2.1 in [16], which is our Theorem 3.1. The freedom in the choice of $\alpha$ and $\beta$ turns out to be very helpful in order to derive a convenient expression for the solution of (3.3) and (3.4) in concrete examples.

Remark 3.4. An important observation is the following. If the ‘Riemann equations’ (3.1) are completely solvable via the method in Section 2 with (3.6) and d- and $\bar{d}$-constant $\bar{P}$ and $\bar{Q}$, then the computation in Remark 3.3 shows that Theorem 3.1 is equivalent to its restriction, where $P$ and $Q$ are d- and $\bar{d}$-constant and commute with $\alpha$, respectively $\beta$. In this case, the apparent freedom expressed by (3.1) is fake and Theorem 3.1 reduces to a method that generates solutions of (1.6) from solutions of only linear equations. We meet this situation if the ‘Riemann equation’ (1.1) is solvable by a Cole-Hopf transformation. But it does not hold if (1.1) (hence (3.1)) involves a continuous Riemann equation.

The theorem includes a case, where solutions are generated from solutions of nonlinear ‘Riemann equations’, and the linear equations (3.2) are eliminated. According to Remark 3.4, this is mainly of relevance if the equations in (3.1) are not solvable by a Cole-Hopf-type transformation.
Corollary 3.5. Let \( \phi_0, g_0 \in \text{Mat}(m, m, \mathcal{B}) \) solve \( (1.6) \), and let \( P, Q \in \text{Mat}(n, n, \mathcal{B}) \) be solutions of
\[
\bar{d}P - (dP)P = -[\alpha, P], \quad \bar{d}Q - (dQ)Q = [\beta, Q],
\]
with 1-forms \( \alpha, \beta \). Let \( X \in \text{Mat}(n, n, \mathcal{B}) \) be an invertible solution of the linear equations
\[
XP - QX = V_0U_0, \quad \bar{d}X - (dX)P + (dQ)X = X\alpha + \beta X,
\]
where \( U_0 \in \text{Mat}(m, n, \mathcal{B}) \) and \( V_0 \in \text{Mat}(n, m, \mathcal{B}) \) are \( \bar{d} \)- and \( d \)-constant. Then
\[
\phi = \phi_0 + U_0X^{-1}V_0 \quad \text{and} \quad g = (I + U_0(QX)^{-1}V_0)g_0
\]
yields a new solution of the Miura equation \( (1.6) \), if
\[
(d\phi_0)U_0 + U_0\alpha = 0, \quad V_0d\phi_0 - \beta V_0 = 0. \tag{3.7}
\]

Proof. This is obtained by choosing \( U \) and \( V \) to be \( d \)- and \( \bar{d} \)-constant in Theorem 3.1 \( (3.2) \) is then satisfied iff \( (3.7) \) holds.

Next we state conditions under which Theorem 3.1 generates solutions of the special cases \( (1.1) \) and \( (1.2) \) of the Miura equation \( (1.6) \).

Corollary 3.6. Let \( \phi_0 \) be a solution of \( (1.1) \), respectively \( (1.2) \). Let \( P, Q, U, V, X \) be solutions of \( (3.1) - (3.4) \) and
\[
QV = V\phi_0, \quad \text{respectively} \quad UP = -\phi_0 U. \tag{3.8}
\]
Let \( \phi \) be given by the expression in \( (3.5) \). Then \( \phi \) is a solution of \( (1.1) \), respectively \( (1.2) \). If the first order bidifferential calculus extends to second order, then \( \phi \) is also a solution of \( (1.4) \). Moreover, if \( \phi \) is invertible\(^9\), then \( g := \pm\phi \), respectively \( g := \pm\phi^{-1} \) solves \( (1.5) \).

Proof. Setting \( g = \pm\phi \) in \( (1.6) \), turns it into \( (1.1) \). The additional condition, the first of \( (3.8) \), originates from evaluating \( g = \pm\phi \) using the expressions \( (3.5) \) for \( \phi \) and \( g \). Correspondingly, setting \( g^{-1} = \pm\phi \) in \( (1.6) \), it becomes \( (1.2) \). Using \( g^{-1} = g_0^{-1}[I - U(XP)^{-1}V] \), we are led to the second of \( (3.8) \). If the first order bidifferential calculus extends to second order, the last statement in the corollary follows from the integrability condition of \( (1.6) \), using \( d^2 = 0 \).

Remark 3.7. As a consequence of the assumptions in Corollary 3.6 we obtain
\[
(XPX^{-1})V = V\phi, \quad \text{respectively} \quad U(X^{-1}QX) = -\phi U.
\]
These are counterparts of \( (3.8) \). If \( n = m \) and \( Q = \phi_0 = 0 \), so that the first of conditions \( (3.8) \) holds, and if \( U \) is invertible, then \( \phi = UP^{-1}U \) and the Corollary thus boils down to the method described in Section 2.

Remark 3.8. Let \( \mathcal{B} \) be an algebra of (real or complex) functions of independent variables. We assume the spectrum condition \( \sigma(P) \cap \sigma(Q) = \emptyset \), and \( n > m \). The first of \( (3.8) \) then implies that the \( n \times nm \) matrix \( (V|QV|\cdots|Q^{n-1}V) = (V|V\phi_0|\cdots|V\phi_0^{n-1}) \) has at most \( m \), hence less than \( n \) linearly independent columns. In this case the pair \( (Q, V) \) is said to be not controllable (see, e.g., [25]). A corresponding statement holds in case of the second of \( (3.8) \). Theorem 3 in [25] then says that \( (3.3) \) has no invertible solution, so that under the stated conditions the solution-generating method in Corollary 3.6 does not work. This concerns in particular the case of the (continuous) Riemann equation (see Section 4.1). This negative result should not come as a surprise since ‘soliton methods’ are known not to work in case of hydrodynamic-type systems of which the (continuous) Riemann equation is the prototype.

\(^9\)One way to satisfy these equations is to set \( \beta = -\alpha = V_0(d\phi_0)U_0 \), and \( U_0V_0 = I \) if \( d\phi_0 \neq 0 \).

\(^{10}\)Since \( \phi \) may involve operators, this can turn out to be a serious problem. More importantly, if we arrange that \( (1.4) \) is turned into a partial differential or difference equation, it may then be impossible to simultaneously turn also \( (1.5) \) into an equation not involving operator terms. This is so in the cases treated in Sections 7 and 8.
Remark 3.9. In (2.2) we met an apparently generalized version of (1.1). By a redefinition of the derivation δ, we can cast it into the form (1.1). In Theorem 3.1, we can perform the reverse step. The Miura equation then reads (δg + [γ, g]) g = dφ. We can preserve (3.1) and (3.4). (3.3) and the solution formulas (3.5) remain unchanged, only (3.2) is modified to δU = (dU) P + (dφ0) U + U α − γ U and dV = Q dV − V dφ0 + β V + V γ. The first part of Corollary 3.6 then holds correspondingly. For example, if the first of (3.8) is satisfied, and if φ0 solves (2.2), then φ̃ given by the formula in (3.5), solves the same equation, i.e., δφ̃ − (dφ̃) φ̃ + [γ, φ̃] = 0.

In the following sections we choose the graded algebra Ω to be of the form
\[ \Omega = A \otimes \bigwedge (C^K), \] (3.9)
where \( \bigwedge (C^K) \) is the exterior (Grassmann) algebra of the vector space \( C^K \). In this case it is sufficient to define d and δd on \( A \) and, if needed, to arrange that (1.3) holds. We denote by \( \xi_1, \ldots, \xi_K \) a basis of \( \bigwedge^1 (C^K) \). Furthermore, we will always assume that φ in (1.1) (or (1.2)) is an \( m \times m \) matrix (with entries in a unital associative algebra \( B \)).

4 Continuous and discrete Riemann equations

In this section, we consider the case \( \Omega^1 = A \). d and δd then have to be derivations of \( A \).

4.1 Riemann equation

Let \( A \) be the algebra of matrices of (real or complex) smooth functions of independent variables \( t \) and \( x^\mu, \mu = 1, \ldots, N \). For \( f \in A \), let
\[ df = \sum_{\mu=1}^N f_{x^\mu} \quad \text{and} \quad d\phi = \sum_{\mu=1}^N \phi_{x^\mu} \phi, \]
where a subscript means a partial derivative with respect to the corresponding independent variable. (1.1) is then the matrix Riemann equation
\[ \phi_t = \sum_{\mu=1}^N \phi_{x^\mu} \phi. \] (4.1)

As a consequence of (4.1), the eigenvalues of \( \phi \) satisfy the corresponding scalar version of this equation [2], hence a scalar Riemann equation. (2.1) only generates new solutions in the matrix case \( (m > 1) \). Let \( \phi_0 \) be a solution of (4.1) that commutes with its partial derivatives. By use of the method of characteristics, solutions of (2.3), with \( \gamma = 0 \), are then given by (also see [8])
\[ \Phi = A_0 + \sum_{i=1}^k A_i f_i(t \phi_0 + x^1 I, \ldots, t \phi_0 + x^N I), \]
with any analytic functions \( f_i \) and constant \( m \times m \) matrices \( A_0, A_i \). (2.1) then yields a new solution of (4.1). Choosing a constant \( \phi_0 \neq I \), this class contains regular and asymptotically constant solutions. According to Remark 3.8, Corollary 3.6 is not useful here. We will not consider the above matrix Riemann equation further, since a more general case has been elaborated in some detail in [8].

11 \( N > 1 \) is a somewhat trivial extension of the case \( N = 1 \), since there is a variable \( s \) such that \( \sum \partial/\partial x^\mu = \partial/\partial s \).
4.2 Semi-discrete Riemann equation

Let $\mathcal{A}_0$ be the algebra of matrices of functions on $\mathbb{R} \times \mathbb{Z}$, smooth in the first variable $t$, and $\mathcal{A} = \mathcal{A}_0[\mathbb{S}, \mathbb{S}^{-1}]$. For $f \in \mathcal{A}$, we set
\[
d f = [\mathbb{S}, f], \quad \bar{d} f = f_t, \tag{4.2}\]
where $\mathbb{S}$ is the shift operator in the discrete variable $k \in \mathbb{Z}$. Then, in terms of
\[
\varphi = \phi \mathbb{S}, \tag{4.3}\]
and using
\[
\varphi^+ := \mathbb{S}\varphi^{-1}, \quad \varphi^- := \mathbb{S}^{-1}\varphi\mathbb{S},
\]
(1.1) is the semi-discrete matrix Riemann equation\(^{12}\)
\[
\varphi_t = (\varphi^+ - \varphi) \varphi, \tag{4.4}\]
where $\varphi$ can now be restricted to be an $m \times m$ matrix of functions (not involving the shift operator explicitly)\(^{13}\). Such a matrix equation already appeared in \[1\]. The scalar version has been called (perhaps somewhat misleadingly\(^{14}\)) ‘lattice Burgers equation’ \[26, 17, 27\]. It also showed up as a symmetry of a discrete Burgers equation in \[24\] and, for example, in a model for socio-economical systems in \[28\].

A lattice spacing $h$ can be introduced via a rescaling $t \mapsto t/h$. If $\varphi$, $\varphi_t$ and $(\varphi^+ - \varphi)/h$ have limits as $h \to 0$, keeping $x := kh$ fixed, then $\varphi$ solves the continuous Riemann equation $\varphi_t = \varphi_x \varphi$.

**Remark 4.1.** The following presents another view of the integrability of (4.4). It also illustrates the ‘asymmetry’ arising from the presence of the forward difference in (4.4). At a fixed lattice point $k_0$ let us choose for $\varphi(k_0, t)$ any matrix-valued function of $t$. Writing (4.4) as $\varphi^+ = \varphi + \varphi_t \varphi^{-1}$, as long as the inverse of $\varphi$ exists, extends this to a solution for $k > k_0$. To the left of $k_0$ on the lattice, we obtain from (4.4) iteratively at each lattice point (with $k < k_0$) a matrix Riccati equation:
\[
\varphi_t(k_0 - n, t) = -\varphi(k_0 - n, t)^2 + \varphi(k_0 - n + 1, t) \varphi(k_0 - n, t) \quad n = 1, 2, \ldots.
\]

For example, $\varphi(k, t) = 0$ for all $k > k_0$, solves the equation on this part of the lattice and leaves us with
\[
\varphi_t(k_0, t) = -\varphi(k_0, t)^2, \quad \varphi_t(k_0 - 1, t) = -\varphi(k_0 - 1, t)^2 + \varphi(k_0, t) \varphi(k_0 - 1, t), \ldots.
\]

In the scalar case, corresponding solutions of (4.4) are given by
\[
\varphi(k_0 - n, t) = \Phi_t(k_0 - n, t) \Phi(k_0 - n, t),
\]
\[
\Phi(k_0, t) = t + a(k_0), \quad \Phi(k_0 - n, t) = \int_0^t \Phi(k_0 - n + 1, s) ds + a(k_0 - n) \quad n = 1, 2, \ldots.
\]

with an arbitrary function $a(k)$, $k \leq k_0$. Any real solution from this special class is singular, since $\varphi(k_0, t)$ has a singularity at $t = -a(k_0)$.

\(^{12}\)\[1\] takes the form $\varphi_t = -\varphi(\varphi - \varphi^-)$, which is obtained from (4.4) for the transpose of $\varphi$, if we reverse the shift direction.

\(^{13}\)There is an alternative semi-discretization of the Riemann equation, known as the (Lotka-) Volterra lattice equation, see Remark 9.1.

\(^{14}\)Unfortunately, some authors call the Riemann equation ‘Burgers equation’.
### 4.2.1 Cole-Hopf transformation

Choosing $\phi_0 = S^{-1}$, (2.1) becomes

$$
\varphi = \Phi (\Phi^-)^{-1},
$$

(4.5)

and (2.3) with $\gamma = A(t)$, where $A$ commutes with $S$, takes the form

$$
\Phi_t = \Phi^+ - \Phi + \Phi A,
$$

(4.6)

which for $A = 0$ is a semi-discrete version of the transport equation. We can eliminate $A$ by a redefinition of $\Phi$ that preserves (4.5). Equations (4.5) and (4.6) constitute a discrete Cole-Hopf transformation for the semi-discrete Riemann equation (4.4), also see [1, 26, 17]. This Cole-Hopf transformation also works in the scalar case, of course. If $\Phi$ has a continuum limit, then $\varphi$ tends to $I$.

**Example 4.2.** A set of solutions of (4.6), with $A = 0$, is given by

$$
\Phi = I + \sum_{i=1}^{N} A_i e^{\Theta_i} B_i , \quad \Theta_i := \Lambda_i k + (e^{\Lambda_i} - I) t ,
$$

where $A_i$ and $B_i$ are constant $m \times n$, respectively $n \times m$ matrices, and the $\Lambda_i$ are $n \times n$ matrices. In the scalar case ($m = 1$), we can set $N = 1$ without restriction of generality. Choosing $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$, we obtain

$$
\Phi = 1 + \sum_{i=1}^{n} e^{\theta_i} , \quad \theta_i := \lambda_i k + (e^{\lambda_i} - 1) t + \gamma_i ,
$$

(4.7)

with constants $\gamma_i$. If the constants are real, then (4.5) yields an $n$-kink solution of the scalar version of (4.4), cf. [26, 17] and also see Fig. 1. In the continuum limit, such solutions become constant. Thus, regarding (4.4) as a discretization of the Riemann equation, the kink solutions are simply artifacts of the discretization. Examples of matrix shock wave solutions already appeared in [1], derived via Bäcklund transformations. If $\Lambda$ has non-diagonal Jordan normal form, further solutions are obtained.

**Figure 1:** Plots (with interpolation) of a regular solution of the scalar semi-discrete Riemann equation (4.4), as given in Example 4.2, with $n = 2$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\gamma_1 = \gamma_2 = 0$. Evolution in time $t$ is from right to left. For negative time, this is a 2-kink solution (right curve), which turns into a single kink at $t = 0$ (middle curve) and then becomes steeper and steeper (shock wave, left curve).

The scalar semi-discrete Riemann equation (4.4) is perhaps the simplest ‘soliton equation’ (calling a kink a soliton). Although it is obtained via the simplest discretization from the (continuous) Riemann equation, which is not a soliton equation (though integrable by the method of characteristics, or hodograph method), it is of a rather different nature. The behavior of the multiple kink solutions is actually very similar to that of corresponding solutions of the scalar Burgers equation, see Section 7. This is explained by the fact that (4.4) is a member of a semi-discrete Burgers hierarchy [1], also see Remark 4.12 below.

\[\text{Since} \ (4.4) \text{ is autonomous, if } \varphi \text{ is a solution, then also } \varphi^+. \text{ So we can redefine } \varphi \text{ and replace (4.5) by } \varphi = \Phi^+ \Phi^{-1}.\]

\[\text{Recall that there is no ‘Cole-Hopf transformation’ for the scalar continuous Riemann equation.}\]
Remark 4.3. (4.5), written as $\Phi = \phi \Phi^-$, together with (4.6) is a linear system that has (4.4) as its compatibility condition. Choosing $A = (1 - \lambda) I$, with a parameter $\lambda$, (4.6) reads

$$\Phi_t - \Phi^+ + \lambda \Phi = 0.$$ 

If $\Phi$ is a solution, then also $\Phi^+ - \mu \Phi$, with any constant $\mu$. So we may replace (4.5) by

$$\Phi^+ - \mu \Phi = \varphi (\Phi - \mu \Phi^-)$$

(now with a different $\varphi$), which is

$$\Phi^+ = (\mu I + \varphi) \Phi - \mu \varphi \Phi^-.$$ 

Here we have a Lax pair for the semi-discrete Riemann equation, depending on two spectral parameters. Setting $\mu = 2 \lambda$, $\psi_1 := \Phi$ and $\psi_2 := \sqrt{2 \lambda} \Phi^-$, we recover the Lax pair considered in [27] in the scalar case.

4.2.2 Darboux transformations

We exploit Corollary 3.6 in the case where $\alpha, \beta, P, Q$ are constant. The restriction to $d$- and $\bar{d}$-constant $P$ and $Q$ is suggested by Remark 3.4. Setting $P = A S^{-1}$, $Q = B S^{-1}$, $X = S \tilde{X}$, eliminates explicit appearances of the shift operator in the equations in Theorem 3.1. Then (3.1) is satisfied if $[\alpha, A] = [\beta, B] = 0$. The remaining equations in Theorem 3.1 now take the form

$$U_t = (U^+ - U) A + (\varphi_0^+ - \varphi_0) U + U \alpha,$$

$$V_t = B (V - V^-) - V (\varphi_0^+ - \varphi_0) + \beta V,$$

and

$$\tilde{X}^+ A - B \tilde{X} - V U = 0,$$

$$\tilde{X}_t - (\tilde{X}^+ - \tilde{X}) A + (V - V^-) U - \tilde{X} \alpha - \beta \tilde{X} = 0,$$

which are compatible equations. By use of the first equation for $\tilde{X}$, we can replace the second by

$$\tilde{X}_t = V^- U + (\beta + B) \tilde{X} + \tilde{X} (\alpha - A).$$

In the case under consideration, the first condition of (3.8) in Corollary 3.6 has to be considered, which here takes the form

$$BV^- = V \varphi_0.$$ 

Using (4.3), the solution formula for $\phi$ in (3.5) reads

$$\varphi = \varphi_0 + U \tilde{X}^+ V^-.$$ 

(4.8)

In all these equations, we can now restrict $\varphi_0, U, V, \tilde{X}$ to $A_0$, and $A, B$ to be matrices over $\mathbb{C}$.

Proposition 4.4. Let $\varphi_0$ solve (4.4) and $U, V$ be solutions of

$$U_t = U^+ A + (\varphi_0^+ - \varphi_0) U,$$

$$V_t = -V \varphi_0^+,$$

$$BV^- = V \varphi_0,$$ 

(4.9)

with constant $n \times n$ matrices $A, B$. Then a new solution of (4.4) is given by (4.4) with

$$\tilde{X}_t = V^- U + (\beta + B) \tilde{X} + \tilde{X} (\alpha - A).$$

(4.10)

where $C$ does not depend on $t$ and satisfies the constraint

$$CA - BC^- = VU \bigg|_{t=0}.$$ 

(4.11)
Proof. We choose \( \alpha = A \) and \( \beta = -B \). Then \( \dot{X}_t = V^{-1} U \), which integrates to the stated expression. It remains to solve \( \dot{X}^* A - B \dot{X} = V U \). It is sufficient to do this at \( t = 0 \), where \( \dot{X} = C^- \), and this leads to (4.11). \hfill \square

Remark 4.5. In the scalar case \((m = 1)\), under the conditions specified in Proposition 4.4 we have

\[
\varphi = \varphi_0 + \text{tr}(V^{-1} U \dot{X}^{-1}) = \varphi_0 + \text{tr}(\dot{X}_t \dot{X}^{-1}) = \varphi_0 + (\ln \det \dot{X})_t.
\]

Alternatively, we can write

\[
\varphi = \varphi_0 + \text{tr}(V^{-1} U \dot{X}^{-1}) = \varphi_0 (1 + \text{tr}(B^{-1} V U \dot{X}^{-1})) = \varphi_0 (1 + U \dot{X}^{-1} B^{-1} V).
\]

Example 4.6. We consider the scalar case and set \( \varphi_0 = \mu \in \mathbb{C} \setminus \{0\} \), \( A = B = \mu I \). Writing \( U = (u_1, \ldots, u_n) \) and \( V = (v_1, \ldots, v_n)^\top \) (where \( ^\top \) denotes the transpose), the equations in (4.9) are solved by

\[
u_j = (e^{\lambda_j} - 1) e^{\lambda_j} k + \mu e^{\lambda_j} t + \gamma_j, \quad v_j = \mu e^{-\mu t}, \quad \lambda_j, \gamma_j \in \mathbb{C}, \quad j = 1, \ldots, n.\]

(4.10) and (4.11) are then solved by

\[
\dot{X} = (c_{ij} + e^{\theta_j})^{n}_{i,j=1},
\]

where \( c_{ij} \) are arbitrary constants, which we choose as \( \delta_{ij} \), and \( \theta_i := \lambda_i k + \mu (e^{\lambda_i} - 1) t + \gamma_i \). Using Sylvester’s determinant theorem, we obtain \( \det \dot{X} = 1 + \sum_{j=1}^{n} e^{\theta_j} \), which is (4.7) if we set \( \mu = 1 \). The latter can be achieved by an obvious scaling symmetry of (4.4).

4.3 Discrete Riemann equation

Let \( A_0 \) be the algebra of matrices of functions on \( \mathbb{Z}^2 \). Let \( S_0, S_1 \) be the corresponding commuting shift operators and \( A = A_0[S_0^{\pm 1}, S_1^{\pm 1}] \). We will use the notation

\[
f_{0,0} := S_0 f S_0^{-1}, \quad f_{1,0} := S_1 f S_1^{-1},
\]

and also \( f_{-0} := S_0^{-1} f S_0, \ f_{-1} := S_1^{-1} f S_1 \). Let

\[
df = \frac{1}{h_0} [S_1^{-1} S_0, f], \quad df = \frac{1}{h_1} [S_1^{-1} f],
\]

with constants \( h_0, h_1 \neq 0 \). In terms of

\[
\varphi = \phi S_0,
\]

(1.1) becomes

\[
\frac{1}{h_1} (\varphi_1 - \varphi) = \frac{1}{h_0} (\varphi_1 - \varphi_0) \varphi
\]

\[
= \frac{1}{h_0} (\varphi_1 - \varphi) - \frac{1}{h_0} (\varphi_0 - \varphi) \varphi.
\]

(4.12)
If \( h_0 = -1 \), this formally tends to the semi-discrete Riemann equation (4.4) as \( h_1 \to 0 \). In the following we set \( h_0 = h_1 = 1 \), hence we consider the discrete matrix Riemann equation

\[
\varphi_{,1} - \varphi = (\varphi_{,1} - \varphi_{,0}) \varphi, 
\]

which can be rewritten as \( \varphi_{,1} = (I - \varphi_{,0}) \varphi (I - \varphi)^{-1} \).

### 4.3.1 Cole-Hopf transformation

Choosing \( \phi_0 = S_{0}^{-1} \) in (2.1) and \( \gamma = -A S_{1}^{-1} \) in (2.3), with \( A_{,0} = A \), we obtain for (4.13) the discrete Cole-Hopf transformation

\[
\varphi = \Phi \Phi_{-,0}^{-1}, \quad \Phi_{,0} - \Phi = (\Phi A)_{,1}. 
\]

#### Example 4.7

A set of solutions of the linear equation in (4.14), with the choice \( A = -I \), is given by

\[
\Phi = \sum_{i=1}^{N} A_i A_i^{k_0} (I - A_i)^{k_1} B_i, 
\]

where \( A_i \) and \( B_i \) are constant \( m \times n \), respectively \( n \times m \) matrices, and the \( A_i \) are constant \( n \times n \) matrices. In the scalar case \( (m = 1) \), we can set \( N = 1 \) without restriction of generality. Choosing \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \), we obtain

\[
\Phi = \sum_{i=1}^{n+1} \gamma_i \lambda_i^{k_0} (1 - \lambda_i)^{k_1}, 
\]

with constants \( \gamma_i \). If the constants are real and \( \gamma_i > 0, 0 < \lambda_i < 1 \), then (4.14) yields an \( n \)-kink solution of the scalar version of (4.13), also see Fig. 2. We can write the last expression in the form

\[
\Phi = \mu^{k_0} (1 - \mu)^{k_1} \left( 1 + \sum_{i=1}^{n} \gamma_i \lambda_i^{k_0} \left( \frac{1 - \mu \lambda_i}{1 - \mu} \right)^{k_1} \right), 
\]

after the redefinitions \( \lambda_1 \mapsto \mu, \lambda_{i+1}/\lambda_1 \mapsto \lambda_i, \gamma_{i+1}/\gamma_1 \mapsto \gamma_i, i = 1, \ldots, n \), and a rescaling of \( \Phi \) that preserves \( \varphi \).

#### Figure 2: A regular 2-kink solution of the scalar discrete Riemann equation (4.13) obtained via the discrete Cole-Hopf transformation (4.14) with (4.15), where we chose \( \mu = 1/2, \lambda_1 = 1/4, \lambda_2 = 3/4, \gamma_1 = \gamma_2 = 1 \). The plots, from left to right, correspond to consecutive values \((-50, 0, 50)\) of \( k_1 \).

#### Remark 4.8

Choosing \( A = -\lambda I \), with a constant \( \lambda \), the second equation in (4.14) reads

\[
\Phi_{,0} - \Phi + \lambda \Phi_{,1} = 0.
\]

\textsuperscript{17} (1.2) has the form \( \varphi_{,0,1} - \varphi_{,0} = -\varphi_{,0,1} (\varphi_{,0} - \varphi_{,1}) \), which becomes (4.13) for the transpose of \( \varphi \), if we reverse the direction of the two shift operators.
Writing the first equation in (4.14) as a linear equation for \( \Phi \), we have a \textit{Lax pair} for the discrete Riemann equation, with spectral parameter \( \lambda \). If \( \Phi \) is a solution of the above equation, then also \( \Phi_{,0} - \mu \Phi \), with any constant \( \mu \). We may thus replace the first equation in (4.14) by

\[
\Phi_{,0} = (\mu I + \varphi) \Phi - \mu \varphi \Phi_{,-0}
\]

(with a different \( \varphi \)). The two equations for \( \Phi \) then constitute a Lax pair depending on two spectral parameters. This is the counterpart of the corresponding Lax pair for the semi-discrete Riemann equation, see Remark 4.3.

4.3.2 Darboux transformations

Let

\[
\alpha = \tilde{\alpha} S_1^{-1}, \quad \beta = \tilde{\beta} S_1^{-1}, \quad P = A S_0^{-1}, \quad Q = B S_0^{-1}, \quad X = \tilde{X} S_0,
\]

with constant matrices \( \tilde{\alpha}, \tilde{\beta}, A, B \). Then (3.1) is satisfied if \( [\tilde{\alpha}, A] = [\tilde{\beta}, B] = 0 \). The remaining equations we have to consider now take the form

\[
U - U_{,1} = (U_{,0} - U_{,1}) A + (\varphi_{0,0} - \varphi_{0,1}) U + U_{,1} \tilde{\alpha},
\]

\[
V_{,1} - V = B (V_{,0} - V_{,1}) - V_{,1} (\varphi_{0,1} - \varphi_{0,0}) - \tilde{\beta} V,
\]

and

\[
\tilde{X}_{,0} A - B \tilde{X} - V_{,0} U_{,0} = 0,
\]

\[
\tilde{X}_{,1} - \tilde{X} - (\tilde{X}_{,0} - \tilde{X}_{,0}) A + (V_{,1} - V_{,0}) U_{,0} + \tilde{X}_{,1} \tilde{\alpha} + \tilde{\beta} \tilde{X} = 0.
\]

By use of the first equation for \( \tilde{X} \), we can replace the second by

\[
\tilde{X}_{,1} - \tilde{X} + \tilde{X}_{,1} (\tilde{\alpha} - A) + (\tilde{\beta} + B) \tilde{X} + V_{,1} U_{,0} = 0.
\]

We have to consider the first condition of (3.8), which here takes the form

\[
BV = V_{,0} \varphi_{0,0}.
\]

The solution formula in (3.5) reads

\[
\varphi = \varphi_0 + U(\tilde{X}^{-1} V)_{,-0}.
\]

(4.16)

We can now restrict \( \varphi_0, U, V, \tilde{X} \) to \( \mathcal{A}_0 \).

**Proposition 4.9.** Let \( \varphi_0 \) solve (4.13), and let \( U, V \) be solutions of the linear equations

\[
U_{,1} - U + U_{,0} A = (\varphi_{0,1} - \varphi_{0}) U,
\]

\[
BV_{,0} = V \varphi_{0}, \quad V_{,1} - V = V_{,1} \varphi_{0,0},
\]

with constant matrices \( A, B \). Let\(^{18}\)

\[
\tilde{X}(k_0, k_1) = F(k_0) - \sum_{j=0}^{k_1-1} V(k_0, j + 1) U(k_0 + 1, j),
\]

where \( k_0 \) and \( k_1 \) are the discrete variables on which the shift operators \( S_0 \), respectively \( S_1 \) act, and \( F \) is an arbitrary \( n \times n \) matrix function satisfying

\[
F(k_0 + 1) A - BF(k_0) - V(k_0 + 1, 0) U(k_0 + 1, 0) = 0.
\]

Then a new solution of (4.13) is given by (4.16).

\(^{18}\)If the upper summation limit is smaller than the lower one, the two limits shall be exchanged.
Proof. We choose $\tilde{\alpha} = A$ and $\tilde{\beta} = -B$. Then the equations for $\tilde{X}$ read
\[
\begin{align*}
\tilde{X},_1 - \tilde{X} + V,_1 U,0 & = 0, \\
\tilde{X},_0 A - B \tilde{X} & = V,0 U,0.
\end{align*}
\]

The first equation is completely solved by the expression for $\tilde{X}$ in the proposition. The second equation then results in the stated constraint. □

Remark 4.10. In the scalar case, we have
\[
\varphi = \varphi_0 (1 + U \tilde{X},_0^{-1} B^{-1} V) = \varphi_0 \det(I + B^{-1} V U \tilde{X},_0^{-1})
\]
\[
= \varphi_0 \det(I + B^{-1} (\tilde{X} A - B \tilde{X},_0) \tilde{X},_0^{-1})
\]
\[
= \varphi_0 \det(A) \det(B)^{-1} \det(\tilde{X}) \det(\tilde{X},_0)^{-1},
\]
(4.17)

which makes contact with the Cole-Hopf transformation.

Example 4.11. Let $m = 1$ (scalar case) and $\varphi_0 = \mu \in \mathbb{C} \setminus \{0, 1\}$, $A = B = \mu I$. Writing $U = (u_1, \ldots, u_n)$ and $V = (v_1, \ldots, v_n)^T$, the linear equations for $U$ and $V$ are solved by
\[
u_j = a_j \lambda_j^{k_0 - 1} (1 - \lambda_j \mu)^{k_1}, \quad v_j = (1 - \mu)^{-k_1}, \quad a_j, \lambda_j \in \mathbb{C}, \quad j = 1, \ldots, n.
\]

$\tilde{X}$ is then given by
\[
\tilde{X} = \left(c_{ij} + \gamma_j \lambda_j^{k_0} \left(\frac{1 - \mu \lambda_j}{1 - \mu}\right)^{k_1}\right)_{i,j=1}^n, \quad \gamma_j := \frac{a_j}{\mu (\lambda_j - 1)},
\]
where $c_{ij}$ are arbitrary constants, which we set to $\delta_{ij}$. Then we obtain
\[
\det \tilde{X} = 1 + \sum_{i=1}^n \gamma_i \lambda_i^{k_0} \left(\frac{1 - \mu \lambda_i}{1 - \mu}\right)^{k_1},
\]
and $\varphi$ given by (4.17) coincides with $\varphi$ obtained from (4.15) via the Cole-Hopf transformation.

4.4 Hierarchies

4.4.1 Continuous Riemann hierarchy

Let $\mathcal{A}$ be the algebra of matrices of real (or complex) smooth functions of independent variables $t_k, k = 0, 1, 2, \ldots$, and $\lambda$ an arbitrary parameter (or an indeterminate). Let
\[
\begin{align*}
df & = \sum_{k=0}^{\infty} \lambda^k f_k, \\
\bar{d}f & = \sum_{k=0}^{\infty} \lambda^k f_{k+1},
\end{align*}
\]
where a subscript means a partial derivative with respect to the corresponding variable. (1.1), for all $\lambda$, is then equivalent to the matrix Riemann hierarchy
\[
\phi_{t_k} = \phi_x \phi^k,
\]
where $k = 0, 1, 2, \ldots$, and we set $x := t_0$. By taking linear combinations, we obtain equations of the form $\phi_t = \phi_x p(\phi)$, where $p(\phi)$ is any polynomial of $\phi$. 

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4.4.2 Semi-discrete Riemann hierarchy

Let \( \mathcal{A} \) be the algebra of matrices of functions of a discrete variable \( k \), and smoothly dependent on variables \( t_j, j = 1, 2, \ldots \). Let

\[
\mathcal{E}_\lambda := \exp \left( \sum_{j \geq 1} \frac{1}{j} \lambda^j \partial_{t_j} \right).
\]

We write \( f[\lambda] := \mathcal{E}_\lambda f \mathcal{E}_\lambda^{-1} \). In terms of \( \varphi = \phi \mathcal{S} \), (4.1) takes the form

\[
\lambda^{-1} (\varphi[\lambda] - \varphi) - (\varphi[\lambda]^+ - \varphi) \varphi[\lambda] = 0.
\]

Expanding in powers of \( \lambda \), to zero order we recover (4.4) with \( t = t_1 \). By use of it, the next hierarchy member can be written in the form

\[
\varphi_{t_2} = (\varphi^{++} - \varphi) \varphi^+ \varphi.
\]

Such a hierarchy apparently first appeared in [1] (where it has been called ‘discrete matrix Burgers hierarchy’). In the scalar case, Darboux transformations for this hierarchy have been studied in [27]. (2.3) with \( \gamma = 0 \) takes the form

\[
\lambda^{-1} (\Phi[\lambda] - \Phi) - (\Phi[\lambda]^+ - \Phi) \varphi[\lambda]_0 = 0.
\]

According to Section 2, together with (4.5) this determines a discrete Cole-Hopf transformation for the whole hierarchy (4.18). To zero order in \( \lambda \), we have

\[
\Phi_{t_1} = (\Phi^+ - \Phi) \varphi^+_0,
\]

which becomes (4.6) if \( \varphi_0 = I \). The next equation, which arises at first order in \( \lambda \), is

\[
\Phi_{t_2} = -\Phi_{t_1 t_1} + 2 (\Phi^+ - \Phi) \varphi^+_0, t_1 + 2 \Phi^+ t_1 \varphi^+_0 = (\Phi^{++} - \Phi^+) \varphi^{++}_0 \varphi^+_0 + (\Phi^+ - \Phi) [(\varphi^+_0)^2 + \varphi_{0,t_1}^+],
\]

by use of the first equation. For \( \varphi_0 = I \), this reduces to

\[
\Phi_{t_2} = \Phi^{++} - \Phi,
\]

which is the second member of the semi-discrete linear heat hierarchy.

**Remark 4.12.** The nonlinear hierarchy for \( \varphi \) contains a semi-discretization of a matrix Burgers equation and can thus be regarded as a semi-discrete version of a matrix Burgers hierarchy [1]. Indeed, a combination of the first two equations of the semi-discrete Riemann hierarchy is

\[
\varphi_s := \frac{1}{h^2}(\partial_{t_2} - 2\partial_{t_1})\varphi = \frac{1}{h^2} ((\varphi^{++} - \varphi) \varphi^+ \varphi - 2(\varphi^+ - \varphi) \varphi),
\]

introducing a lattice spacing \( h \) and a new variable \( s \). In terms of the new dependent variable

\[
\tilde{\varphi} := \frac{1}{h}(\varphi - I),
\]

this takes the form

\[
\tilde{\varphi}_s = \frac{1}{h^2} (\tilde{\varphi}^{++} - 2\tilde{\varphi}^+ + \tilde{\varphi})(I + h \tilde{\varphi}^+)(I + h \tilde{\varphi}) + \frac{2}{h} (\tilde{\varphi}^+ - \tilde{\varphi})\tilde{\varphi}^+(I + h \tilde{\varphi}),
\]
which formally tends to the Burgers equation $\tilde{\varphi}_t = \tilde{\varphi}_{xx} + 2\tilde{\varphi}_x \tilde{\varphi}$ as $h \to 0$\textsuperscript{19} The corresponding combination of the above first two equations of the semi-discrete linear heat hierarchy is

$$\Phi_s = \frac{1}{h^2} (\Phi^{++} - 2\Phi^+ + \Phi),$$

which tends to the heat equation as $h \to 0$. Correspondingly, the transformation (4.5) reads

$$\tilde{\varphi} = \frac{1}{h} (\Phi - \Phi^-) (\Phi^-)^{-1},$$

so we also recover the continuous Cole-Hopf transformation as $h \to 0$. This limit takes discrete kink solutions to kink solutions of the Burgers equation. The observation that the scalar semi-discrete Riemann equation possesses solutions of the type we meet in case of the scalar Burgers equation is explained by the fact that they extend to solutions of the whole hierarchy.

**Remark 4.13.** The equations of the semi-discrete Riemann hierarchy can also be obtained in a different way. Let us consider the following generalizations of (1.1),

$$\overline{d}\varphi - (d\varphi) \varphi^r = 0 \quad r = 1, 2, \ldots, \quad (4.21)$$

and a corresponding generalization of the calculus determined by (4.2),

$$df = [S^r, f], \quad \overline{d}f = f_t.$$

Setting $\varphi = \varphi S^{-1}$, where $\varphi$ is a matrix of functions, all the above equations turn out to be PDDEs for $\varphi$, namely

$$\varphi_t = (\varphi^{(r)} - \varphi) (\varphi^{(r-1)} S^{-1}) S^r \quad r = 1, 2, \ldots,$$

where $\varphi^{(r)} := S^r \varphi S^{-r}$. For $r = 1, 2$, we recover (4.4) and (4.19), respectively.

### 4.4.3 Discrete Riemann hierarchy

Let $S_0, S_1, S_2, \ldots$ be commuting shift operators and

$$df = \sum_{i=1}^{\infty} \lambda^i \alpha_i [S_i^{-1} \prod_{j=0}^{i-1} S_j, f], \quad \overline{d}f = \sum_{i=1}^{\infty} \lambda^i \beta_i [S_i^{-1} \prod_{j=1}^{i-1} S_j, f],$$

with constants $\alpha_i, \beta_i$. At first and second order in $\lambda$, we obtain from (1.1), with $\varphi = \varphi S_0$, the following equations,

$$\beta_1 (\varphi_1 - \varphi) - \alpha_1 (\varphi_1 - \varphi_0) \varphi = 0, \quad \beta_2 (\varphi_{2} - \varphi_{,1}) - \alpha_2 (\varphi_{2} - \varphi_{,0,1}) \varphi_{,1} = 0.$$

The first coincides with (4.13) if $\alpha_1 = \beta_1 = 1$. Solving these equations for $\varphi_{,1}$, respectively $\varphi_{,2}$, and assuming the necessary invertibility conditions, we have

$$\varphi_{,1} = (\beta_1 - \alpha_1 \varphi_0) \varphi (\beta_1 - \alpha_1 \varphi)^{-1}, \quad \varphi_{,2} = (\beta_1 \beta_2 - (\alpha_1 \beta_2 + \alpha_2 \beta_1) \varphi + \alpha_1 \alpha_2 \varphi_0 \varphi)_0 \varphi (\beta_1 \beta_2 - (\alpha_1 \beta_2 + \alpha_2 \beta_1) \varphi + \alpha_1 \alpha_2 \varphi_0 \varphi)^{-1}. \quad (4.22)$$

If $\alpha_i = \beta_i = 1$, then we have $\varphi_{,2} = \varphi_{,1,1}$, and a corresponding relation holds for all equations of the hierarchy. The $(n + 1)$-th flow is then simply the $n$-th shift of the first hierarchy equation with respect to its ‘evolution variable’. In this respect the hierarchy is ‘trivial’.

\textsuperscript{19}In [30] an additional transformation of independent variables is made to turn the scalar version of (4.19) into a different form which then leads to the continuum Burgers equation as $h \to 0$. 

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Choosing $\phi_0 = S_0^{-1}$ and $\gamma = \sum_{i=1}^{\infty} \lambda^i (1 + \alpha_i - \beta_i) I S_i^{-1} \prod_{j=1}^{i-1} S_j$, according to Section 2 a Cole-Hopf transformation is given by

$$\varphi := \phi S_0 = \Phi \Phi^{-1}_0, \quad \Phi_i = (\beta_i \Phi - \alpha_i \Phi_0)_{1, \ldots, i-1} \quad i = 1, 2, \ldots,$$

where the linear equations are compatible. Using the first in the second equation, the latter becomes

$$\Phi_{i,2} = \beta_1 \beta_2 \Phi - (\alpha_1 \beta_2 + \alpha_2 \beta_1) \Phi_{0,0} + \alpha_1 \alpha_2 \Phi_{0,0}. \quad (4.23)$$

**Remark 4.14.** Let $\alpha_1 = \alpha_2 = \sqrt{t_2}, \beta_1 = \sqrt{t_2} + h_0, \beta_2 = \sqrt{t_2} - h_0$. In terms of the new dependent variable $\tilde{\varphi}$ given by $\varphi = I + h_0 \tilde{\varphi}$ (cf. (4.20)), the second hierarchy equation takes the form

$$- \Delta_2 \tilde{\varphi} = \left( h_0 (\Delta_0 \Theta) \tilde{\varphi} + \Delta_0 \Theta + [\Theta, \tilde{\varphi}] \right) (I - h_2 \Theta)^{-1}, \quad \Theta := \tilde{\varphi}_0 \tilde{\varphi} + \Delta_0 \tilde{\varphi}, \quad (4.24)$$

where $\Delta_2 \tilde{\varphi} := (S_2 \tilde{\varphi} S_0^{-1} - \tilde{\varphi})/h_2, \Delta_0 \tilde{\varphi} := (S_0 \tilde{\varphi} S_0^{-1} - \tilde{\varphi})/h_0$, and correspondingly for $\Delta_0$ acting on $\Theta$. In the scalar case, and after replacing $h_2$ by $-h_2$, the latter equation coincides with the discrete Burgers equation in [24, 31] (up to differences in notation). (4.24) is thus a matrix version of the latter. Here we interpret $h_0$ and $h_2$ as lattice spacings.

**Remark 4.15.** Let us consider $d \phi - (d \phi) \phi' = 0$ (which is (4.21)), with

$$df = [S^{-1}_r S^0_r, f], \quad d\bar{f} = [S^{-1}_r, f].$$

Setting $\phi = \varphi S_0^{-1}$, where $\varphi$ is a matrix of functions, (4.21) becomes a partial difference equation for $\varphi$, namely

$$\varphi_r - \varphi = (\varphi_r - S_0 \varphi S_0^{-r}) (\varphi S_0^{-1})^r S_0^r.$$

For $r = 1$, this is (4.13), i.e., the first of (4.22) if $\alpha_1 = \beta_1 = 1$. For $r = 2$, this is the second of (4.22) if $\alpha_1 = \beta_1 = \beta_2 = 1$ and $\alpha_2 = -1$.

### 5 Some integrable equations associated with Riemann equations

#### 5.1 Self-dual Yang-Mills equation

Let us choose (3.9) with $K = 2$ and combine two bidifferential calculi of the kind considered in Section 4.1 with $N = 1$, to

$$df = -f_\xi \xi_1 + f_y \xi_2, \quad d\bar{f} = f_y \xi_1 + f_\bar{z} \xi_2, \quad (5.1)$$

where $B$ is the space of smooth complex functions of four (real or complex) variables $y, \bar{y}, z, \bar{z}$. (1.3) holds and (1.4) takes the form

$$\phi_{2z} + \phi_{2\bar{z}} + [\phi_z, \phi_{\bar{z}}] = 0 \quad (5.2)$$

(also see [12]), which is a well-known potential form of the self-dual Yang-Mills (sdYM) equation (cf. [32]). Another well-known potential form of the sdYM equation is obtained from (1.5):

$$(g_{\bar{z}} g^{-1})_z + (g_y g^{-1})_{\bar{y}} = 0. \quad (5.3)$$

In the following subsections, we consider the Riemann system associated with these versions of the sdYM equation. Then we derive a method to construct breaking [21] multi-soliton-type solutions from Corollary 3.5. Finally, we consider a non-autonomous chiral model as an example of a reduction of the sdYM equation, making contact with the work in [19, 16].

---

20Replacing $h_2$ by $-h_2$ means that $\alpha_i$ and $\beta_i, i = 1, 2$, are complex. But the coefficients in the associated discrete heat hierarchy equation (4.23) are real if $h_0, h_2$ are real.

21Such solutions may be regarded as ‘breaking waves’.
5.1.1 The sdYM Riemann system

Using (5.1), (1.1) is equivalent to the system

\[ \begin{align*}
\phi_y &= -\phi_z \phi, \\
\phi_z &= \phi_y \phi
\end{align*} \]  

of Riemann equations. Any solution of this system also solves (5.2) and, if it is invertible, also (5.3). Solutions are implicitly given by

\[ \phi = f(y \phi - z I, \bar{z} \phi + \bar{y} I), \]

where \( f \) is any analytic function. More generally, \( f \) may depend in addition on constant \( m \times m \) matrices, but we have to ensure that \( \phi \) commutes with them.

Let us turn to the linearization method of Section 2. Setting \( \gamma = 0 \) and assuming that \( \phi_0 \) solves (5.4) and commutes with its partial derivatives, then (2.3), elaborated with (5.1), is solved by

\[ \Phi = A_0 + \sum_{i=1}^k A_i f_i(y \phi_0 - z I, \bar{z} \phi_0 + \bar{y} I) \]

(cf. [10]), with any analytic functions \( f_i \) and constant \( m \times m \) matrices \( A_0, A_i \). \( \phi = \Phi \phi_0 \Phi^{-1} \) is then a new solution of (5.4), and thus of (5.2), also see [10] 22 Choosing a constant \( \phi_0 \neq I \), this class contains regular and asymptotically constant solutions, cf. Section 4.1.

5.1.2 Breaking multi-soliton-type solutions of the sdYM equation

From Corollary 3.5, setting \( \alpha = \beta = d\phi_0 = 0 \), we obtain the following result.

**Proposition 5.1.** Let \( \phi_0 \) and \( g_0 \) be constant \( m \times m \) matrices, \( g_0 \) invertible. Let \( P \) and \( Q \) be solutions of the \( n \times n \) matrix Riemann equations

\[ \begin{align*}
P_y &= -P_z P, \\
P_z &= P_y P, \\
Q_y &= -Q_z Q, \\
Q_z &= Q Q_y,
\end{align*} \]

where \( Q \) is invertible and such that \( \sigma(P) \cap \sigma(Q) = \emptyset \). Let \( X \) be the unique solution of the Sylvester equation

\[ X P - Q X = V_0 U_0, \]

where \( U_0 \) is a constant \( m \times n \) and \( V_0 \) a constant \( m \times n \) matrix. Then

\[ \phi = \phi_0 + U_0 X^{-1} V_0, \]

respectively

\[ g = (I + U_0 (Q X)^{-1} V_0) g_0, \]

(\text{except at singular points}) solves the respective potential form of the sdYM equation.

Here we used the fact that the differential equation for \( X \) in Corollary 3.5 is a consequence of the Sylvester equation if \( \sigma(P) \cap \sigma(Q) = \emptyset \) holds (see the proof of Theorem 2.1 in [16]). The above equations for \( P \) form an \( n \times n \) version of the sdYM Riemann system considered and solved above. The equations for \( Q \) are obtained by transposition. Proposition 5.1 expresses a nonlinear superposition rule for ‘breaking wave’ solutions of the sdYM equation. Also see [10].

---

22 Equation (7) in [10] is a special case of the linear systems (3.2), where the matrix \( P \), respectively \( Q \), is given by \( \lambda I \), with a spectral parameter \( \lambda \) (allowed to be a function). Further results in [10] are closely related to Theorem 3.1 specialized to the sdYM case.
5.1.3 A non-autonomous chiral model in three dimensions

It is well-known that many integrable equations are reductions of the sdYM equation. As an example, the reduction condition $\partial_{\bar{z}} = \epsilon \partial_z$, where $\epsilon = \pm 1$, reduces (5.1) to

$$
\text{d} f = -f_z \xi_1 + f_y \xi_2, \quad \bar{\text{d}} f = f_y \xi_1 + \epsilon f_z \xi_2,
$$

Performing a change of variables $(y, \bar{y}) \mapsto (\rho, \theta)$ via $y = \frac{1}{2} \rho e^\theta$, $\bar{y} = \frac{1}{2} \rho e^{-\theta}$, we obtain

$$
\text{d} f = -f_z \xi_1 + e^\theta (f_\rho - \rho^{-1} f_\theta) \xi_2, \quad \bar{\text{d}} f = e^{-\theta} (f_\rho + \rho^{-1} f_\theta) \xi_1 + \epsilon f_z \xi_2.
$$

This is the bidifferential calculus exploited in [19, 16]. In terms of $\phi := e^\theta \phi$, (5.5) reads

$$
\epsilon \phi_{zz} + \Phi_\rho + \rho^{-1} \Phi_\theta = [\Psi, \phi_z], \quad \text{where} \quad \Psi := \phi_\rho + \rho^{-1}(\phi - \phi_\theta),
$$

and (1.5) takes the form

$$
(\rho g_z g^{-1})_z + \epsilon (\rho g_\rho g^{-1})_\rho - [(g_\rho + \rho^{-1} g_\theta) g^{-1}]_\theta + (g_\theta g^{-1})_\rho = 0.
$$

This is a three-dimensional generalization of the non-autonomous chiral model that underlies integrable reductions of the vacuum Einstein (-Maxwell) equations and to which it reduces if $g$ does not depend on $\theta$ (see [19, 16] and references cited there).

Now we apply the reduction condition and the change of variables to the sdYM Riemann system (5.4). Imposing $\partial_{\bar{z}} = \epsilon \partial_z$, (5.4) becomes

$$
\phi_y = -\phi_z \phi, \quad \epsilon \phi_z = \phi_y \phi,
$$

which is implicitly solved by

$$
\phi = f(\epsilon \bar{y} I + z \phi - y \phi^2).
$$

In terms of the independent variables $\rho, \theta, z$, the above system contains explicit factors $e^\theta$ (and is thus non-autonomous not only in $\rho$, but also in $\theta$). But they are eliminated by passing over to $\phi$ given by (5.5). Now the Riemann system reads

$$
\phi_\rho + \rho^{-1}(\phi_\theta - \phi) + \phi_z \phi = 0, \quad \epsilon \phi_z - [\phi_\rho - \rho^{-1}(\phi_\theta - \phi)] \phi = 0,
$$

and is implicitly solved by

$$
\phi = e^\theta f\left(e^{-\theta} \rho \frac{\rho}{2} (\epsilon I + 2 \rho^{-1} z \phi - \phi^2)\right).
$$

If we require $\phi$ (assumed to be invertible) to be independent of $\theta$, this fixes the function $f$ to $f(x) = A^{-1} x$, with a constant matrix $A$ (subject to conditions that ensure that $[A, \phi] = 0$). In this case we have

$$
\phi^2 - 2 \rho^{-1}(z I - A) \phi - \epsilon I = 0,
$$

which is a matrix version [19, 16] of the ‘pole trajectories’ in the Belinski-Zakharov approach to solutions of the integrable reductions of the Einstein vacuum equations [33]. The non-autonomous chiral model is an example, where non-constant solutions of the ‘Riemann equations’ (3.1) – here given by solutions of the above quadratic matrix equation – in the binary Darboux transformation theorem are crucial in order to recover relevant solutions of integrable reductions of Einstein’s equations, see [19, 16].

\[23\] Here they are indeed Riemann equations.
5.2 Two-dimensional Toda lattice

Now we compose a bidifferential calculus from two calculi of the kind considered in Section 4.2,

\[
df = [S, f] \xi_1 + [\partial_x, f] \xi_2, \quad df = [\partial_t, f] \xi_1 - [S^{-1}, f] \xi_2,
\]

where \(\partial_x := \partial/\partial x\) and \(\partial_t := \partial/\partial t\). Setting

\[
\varphi = \phi S,
\]

(1.1) takes the form

\[
\varphi_t = (\varphi^+ - \varphi) \varphi, \quad \varphi - \varphi^- = \varphi_x \varphi^-. \quad (5.6)
\]

The integrability condition (1.4) is

\[
\varphi_{tx} = (\varphi^+ - \varphi)(I + \varphi_x) - (I + \varphi_x)(\varphi - \varphi^-) \quad (5.7)
\]

(also see [12]). This is a matrix version of the two-dimensional Toda lattice equation

The Miura-dual equation (1.5) takes the form

\[
(g_x g^{-1}) x - g^+ g^{-1} + g (g^{-1})^- = 0.
\]

5.2.1 Cole-Hopf transformation for the Riemann system

Choosing \(\phi_0 = S^{-1}\) in (2.1), and \(\gamma = A \xi_1 + B S^{-1} \xi_2\) in (2.3), where \(A\) and \(B\) are constant, we obtain the following Cole-Hopf transformation for (5.6),

\[
\varphi = \Phi (\Phi^-)^{-1}, \quad \Phi_t = \Phi^+ - \Phi + \Phi A, \quad \Phi_x = \Phi - \Phi^- - \Phi B.
\]

The second equation is (4.6). The last two equations are compatible. Without restriction of generality we can set \(A = B = 0\), since \(A\) and \(B\) can be eliminated by a transformation of \(\Phi\) that preserves the expression for \(\varphi\).

**Example 5.2.** A class of solutions of (5.6) is determined by

\[
\Phi = I + \sum_{i=1}^{n} A_i e^{\Theta_i} B_i, \quad \Theta_i = \Lambda_i k + (I - e^{-\Lambda_i}) x + (e^{\Lambda_i} - I) t,
\]

where \(A_i, B_i\) are constant \(m \times n\), respectively \(n \times m\) matrices, and \(\Lambda_i\) are constant \(n \times n\) matrices. In the scalar case \((m = 1)\), and if \(\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)\), this takes the form

\[
\Phi = 1 + \sum_{i=1}^{n} e^{\lambda_i k + (1 - e^{-\lambda_i}) x + (e^{\lambda_i} - 1) t + \gamma_i},
\]

with constants \(\gamma_i\). If all constants are real, it determines an \(n\)-kink solution of the two-dimensional Toda lattice equation.

---

\(^{24}\)In the commutative case, setting \(V := \varphi_x\) and differentiating once with respect to \(t\), (5.7) becomes (cf. [12]) the well-known equation \((\log(1 + V))_t = V^+ - 2V + V^- [34] [55].\)
5.2.2 Darboux Transformations for the Riemann system

Let
\[ \alpha = A \xi_1 + S^{-1} \xi_2, \quad \beta = -B \xi_1 - S^{-1} \xi_2, \quad P = AS^{-1}, \quad Q = BS^{-1}, \quad X = S \dot{X}, \]
with invertible constant matrices \( A, B \). Then Corollary [3.6] yields the following system of equations,
\[ \begin{align*}
U_t &= U^+ A + (\varphi_0^+ - \varphi_0) U, \quad U_x = -(I + \varphi_{0,x}) U^{-1} A^{-1}, \\
V_t &= -V \varphi_0^+, \quad V_x = B^{-1} V^+ (I + \varphi_{0,x}), \quad V^+ = B^{-1} V \varphi_0,
\end{align*} \]
and
\[ \dot{X}^+ = (V U + B \dot{X}) A^{-1}, \quad \dot{X}_t = V^- U, \quad \dot{X}_x = V_x U A^{-1}. \]

The latter system is compatible, which allows us to write
\[ \dot{X} = C^- + \int_{(0,0)}^{(t,x)} \left( V^- U dt + V_x^- U A^{-1} dx \right), \]
where the integral does not depend on the path of integration. Here \( C \) does not depend on \( x \) and \( t \) and has to satisfy the constraint \( CA - BC^- = V U \|_{t=0,x=0} \). If \( \varphi_0 \) solves the Riemann system [5.6], and if \( U, V \) satisfy the above linear equations, a new solution of (5.6) is given by
\[ \varphi = \varphi_0 + U \dot{X}^{-1} V^-. \]

5.3 A matrix version of Hirota’s bilinear difference equation

Here we compose calculi of the kind considered in Section 4.3. In the following, \( c_i \) are constants and \( S_0, S_i : \mathcal{A}_0 \to \mathcal{A}_0 \) commuting shift operators, where \( \mathcal{A}_0 \) is the algebra of matrices of functions of corresponding discrete variables. Let \( \mathcal{A} = \mathcal{A}_0[S_0^\pm, S_1^\pm, \ldots, S_K^\pm] \). We use the notation introduced in Section 4.3: \( f_0 := S_0 f S_0^{-1} \) and \( f_i := S_i f S_i^{-1} \).

5.3.1 First version

Let
\[ df = \sum_{i=1}^{K} [S_i^{-1} S_0, f] \xi_i, \quad \delta f = \sum_{i=1}^{K} c_i^{-1} [S_i^{-1}, f] \xi_i. \]

Then, in terms of
\[ \varphi = \phi S_0, \]
(1.1) reads
\[ \Delta_i \varphi = (\varphi_i - \varphi_0) \varphi, \quad i = 1, \ldots, K, \tag{5.8} \]
where
\[ \Delta_i \varphi := c_i^{-1} (\varphi_i - \varphi). \]
(1.4) takes the form
\[ (\Delta_i \varphi)_0 - (\Delta_i \varphi)_j + (\varphi_i - \varphi_0)_j (\varphi_j - \varphi_0) = (\Delta_j \varphi)_0 - (\Delta_j \varphi)_i + (\varphi_j - \varphi_0)_i (\varphi_i - \varphi_0), \tag{5.9} \]
where \( i, j = 1, \ldots, K, i \neq j \). The Miura-dual (1.5) is
\[ (\Delta_i g)_0 g_0^{-1} - (\Delta_i g)_j g_j^{-1} = (\Delta_j g)_0 g_0^{-1} - (\Delta_j g)_i g_i^{-1} \]
\[ i, j = 1, \ldots, K, \quad i \neq j, \]
which is
\[ c_i^{-1} [(g_i g_j^{-1})_0 - (g_i g_j^{-1})_j] = c_j^{-1} [(g_j g_i^{-1})_0 - (g_j g_i^{-1})_i]. \tag{5.10} \]

\[ \text{More generally, we may consider any commuting invertible operators.} \]
Cole-Hopf transformation. Choosing $\phi_0 = S_0^{-1}$ in (2.1), and $\gamma = - \sum_i A_i S_i^{-1} \xi_i$, with constant matrices $A_i$, in (2.3), we obtain the following Cole-Hopf transformation for (5.8),

$$
\varphi = \Phi \Phi^{-1}_{\gamma_0}, \quad \Phi_0 - c_i^{-1} \Phi = \Phi, i (A_i - (c_i^{-1} - 1) I) \quad i = 1, \ldots, K.
$$

If $c_i = 1$, these equations are of the form (4.14). Choosing $A_i = (c_i^2 - 2) I$, a set of solutions of the linear equations is given by

$$
\Phi = \sum_{i=1}^N A_i \Lambda_i^{k_0} \prod_{j=1}^K (c_i^{-1} I - \Lambda_i^{k_j}) B_i,
$$

where $A_i$ and $B_i$ are constant $m \times n$, respectively $n \times m$ matrices, and the $\Lambda_i$ are constant $n \times n$ matrices.

Darboux transformations for the Riemann system. Let

$$
P = AS_0^{-1}, \quad Q = BS_0^{-1}, \quad \alpha = \sum_{i=1}^K A S_i^{-1} \xi_i, \quad \beta = \sum_{i=1}^K B S_i^{-1} \xi_i, \quad \bar{X} = X S_0,
$$

with constant $n \times n$ matrices $A, B$. Then (3.1)-(3.4) result in

$$
\Delta_i U + U,0 A = c_i \Delta_i \varphi_0 U, \quad \Delta_i V = V,0 \varphi_0, \quad B V,0 = V \varphi_0,
$$

and

$$
\Delta_i \bar{X} + V,0 U,0 = 0, \quad \bar{X} A - B \bar{X},0 = V U.
$$

The latter equations are compatible, hence

$$
\bar{X}(k_0, k_1, \ldots, k_K) = F(k_0) - \sum_{i=1}^K \sum_{j_i=0}^{k_i-1} V(k_0, k_1, \ldots, k_{i-1}, j_i + 1, 0, \ldots, 0) \times U(k_0 + 1, k_1, \ldots, k_{i-1}, j_i, 0, \ldots, 0),
$$

where $F(k_0)$ satisfies

$$
F(k_0) A - BF(k_0 - 1) - V(k_0, 0, \ldots, 0) U(k_0, 0, \ldots, 0) = 0.
$$

If $\varphi_0$ solves (5.8), then also

$$
\varphi = \varphi_0 + U(\bar{X}^{-1})V,0.
$$

5.3.2 Second version

Here we exchange $d$ and $\bar{d}$ in the first version:

$$
df = \sum_{i=1}^K c_i^{-1} [S_i^{-1}, f] \xi_i, \quad \bar{d} f = \sum_{i=1}^K [S_i^{-1} S_0, f] \xi_i,
$$

In terms of

$$
\varphi = \phi S_0^{-1},
$$

(1.1) becomes

$$
\varphi, i - \varphi, 0 = (\Delta_i \varphi) \varphi, 0 \quad i = 1, \ldots, K.
$$
This system is simply obtained from (5.8) with \( \varphi \) replaced by \( \varphi^{-1} \), which is a special case of (9.2). The integrability condition (1.4) takes the form

\[
(I + \Delta_i \varphi)_j (I + \Delta_j \varphi)_i = (I + \Delta_j \varphi)_i (I + \Delta_i \varphi)_j, \quad i, j = 1, \ldots, K, \quad i \neq j. \tag{5.11}
\]

The Miura-dual is

\[
\Delta_j (g, g^{-1}) = \Delta_i (g, g^{-1}) \quad i, j = 1, \ldots, K, \quad i \neq j. \tag{5.12}
\]

Via \( g \mapsto g^{-1} \), (5.12) becomes (5.10), also see Section 9. There is no such relation between (5.9) and (5.11).

For \( K = 2 \), (5.12) (or (5.10)) can be regarded as a matrix version of Hirota’s bilinear difference equation [37, 38]. This is explained in the following remark.

**Remark 5.3.** In the scalar case \( (m = 1) \), setting \( g = \tau_{-0} \), (5.12) becomes

\[
c_i \left( \frac{\tau_{i,j}, -0 \tau_{j}, 0}{\tau_{i,j} \tau_j} - \frac{\tau_{i,j}, -0 \tau_{i}, 0}{\tau_{i} \tau_j} \right) = c_j \left( \frac{\tau_{i,j}, -0 \tau_{i}, 0}{\tau_{i,j} \tau_i} - \frac{\tau_{j}, -0 \tau_{0}}{\tau_{j} \tau_i} \right),
\]

which is

\[
\left( \frac{c_i \tau_{i,j} \tau_{j}, 0 - c_j \tau_{i,j} \tau_{i}, 0}{\tau_{i,j} \tau_0} \right) = \left( \frac{c_i \tau_{i,j} \tau_{j}, 0 - c_j \tau_{i,j} \tau_{i}, 0}{\tau_{i,j} \tau_0} \right).
\]

Hence, we obtain

\[
c_i \tau_{i,j} \tau_{j}, 0 - c_j \tau_{i,j} \tau_{i}, 0 = c_{i,j} \tau_{i,j} \tau_0,
\]

with new arbitrary constants \( c_{i,j} = -c_{j,i} \). For \( K = 2 \), this is Hirota’s bilinear difference (or Hirota-Miwa) equation [37, 38].

### 6 (2+1)-dimensional Nonlinear Schrödinger equation

The example worked out in this section can also be treated as a reduction of the sdYM equation, but here we will take a more direct approach. Let \( \mathcal{B} \) be the space of smooth complex functions of independent variables \( x, y, t \). Let \( J \neq I \) be an invertible constant \( m \times m \) matrix. We consider two calculi on \( \text{Mat}(m, m, \mathcal{B}) \).

1. Let

\[
\text{df} = f_y, \quad \bar{\text{df}} = -i f_t.
\]

Then (1.1) is the matrix Riemann equation

\[
i \phi_t + \phi_y \phi = 0. \tag{6.1}
\]

2. Let

\[
\text{df} = \frac{1}{2} [J, f], \quad \bar{\text{df}} = f_x.
\]
In this case, (1.1) is the ordinary differential equation
\[ \phi_x - \frac{1}{2} [J, \phi] \phi = 0 . \] (6.2)

Now we combine the two calculi to
\[ df = f_y \xi_1 + \frac{1}{2} [J, f] \xi_2 , \quad \bar{df} = -i f_t \xi_1 + f_x \xi_2 \] (6.3)
(also see [39]). Then (1.1) consists of the pair of equations given above. The integrability condition (1.4) takes the form
\[ -\frac{i}{2} [J, \phi_t] = \phi_{xy} + \frac{1}{2} [\phi_y, [J, \phi]] . \]

Writing
\[ \phi = J \varphi , \]
it becomes
\[ -\frac{i}{2} [J, \varphi_t] = \varphi_{xy} + \frac{1}{2} \left( \varphi_y J [J, \varphi] - [J, \varphi] J \varphi_y \right) . \] (6.4)

From now on we set
\[ J := J_{(m_1, m_2)} := \begin{pmatrix} I \cdot m_1 & 0 \\ 0 & -I \cdot m_2 \end{pmatrix} , \quad \varphi := \begin{pmatrix} u \\ q \\ r \\ v \end{pmatrix} , \] (6.5)
with \( m = m_1 + m_2 \). Then (6.4) splits into
\[ -i q_t = q_{xy} + q v_y + u_y q , \quad i r_t = r_{xy} + r u_y + v_y r , \quad u_x = -q r , \quad v_x = -r q . \] (6.6)

Correspondingly, the associated ‘Riemann system’ becomes
\[ i \varphi_t + \varphi_y J \varphi = 0 , \quad \varphi_x - \frac{1}{2} (J \varphi J - \varphi) \varphi = 0 , \] (6.7)
which decomposes into
\[ \begin{align*}
    i \varphi_t + u_y u - q_y r = 0 , & \quad i v_t - v_y v + r_y q = 0 , \quad u_x + q r = 0 , \quad v_x + r q = 0 , \\
    i q_t + u_y q - q_y v = 0 , & \quad i r_t - r_y u - v_y r = 0 , \quad q_x + q v = 0 , \quad r_x + r u = 0 .
\end{align*} \] (6.8)

The first two equations can be dropped, they are a consequence of the others if \( q, r \neq 0 \). The third and the fourth equation are just the last two equations in (6.6). The last four equations imply the first two equations in (6.6).

Now we consider the reduction conditions [29]
\[ \varphi^\dagger = \begin{cases} \varphi + C \quad \text{defocusing case} \\ J \varphi J + C \quad \text{focusing case} \end{cases} \] (6.9)
where \( C \) is an anti-Hermitian matrix commuting with \( J \). Writing \( C = \text{block-diag}(c_1, c_2) \) and using (6.5), these conditions result in
\[ r = \pm q^\dagger , \quad u^\dagger = u + c_1 , \quad v^\dagger = v + c_2 , \] (6.10)

[28] Imposing \( \partial_x = \partial_y \), this reduces to the matrix NLS system [40, 41], also see the references in [42].

[29] This generalizes the ‘naive reductions’ where \( C = 0 \). The latter would exclude the solutions presented in Example 6.1 below.
where \( c_i^\dagger = -c_i \), \( i = 1, 2 \). If
\[
c_{1,x} = c_{2,x} = 0, \quad c_{1,y} q + q c_{2,y} = 0, \tag{6.11}
\]
this reduces \( (6.6) \) to a matrix version of the \((2+1)- dimensional NLS equation\) \[43, 44\]
\[
\dot{q} t + q_{xy} + 2 q \left( \int_0^x |q|^2 \, dx \right)_y = 0, \tag{6.12}
\]
which we obtain from \( (6.6) \) in the scalar case \( (m_1 = m_2 = 1) \). The upper (lower) choice of sign corresponds to the defocusing (focusing) NLS case. The matrix version probably first appeared in \[45\].

### 6.1 Reduction of the associated ‘Riemann system’

The reduction conditions \( (6.9) \), respectively \( (6.10) \), imposed on the ‘Riemann system’ \( (6.8) \), reduce it to
\[
i q_t + u_y q - q_y v = 0, \quad i u_t + u_y u \mp q_y q^\dagger = 0, \quad i v_t - v_y v \pm q_y^\dagger q = 0,
\]
and
\[
q v = (u + c_1) q, \quad q_x = -q v, \quad u_x = \mp q q^\dagger, \quad v_x = \mp q^\dagger q, \quad c_{1,y} q + q c_{2,y} = 0.
\]

We already met the last equation in \( (6.11) \). Recall that \( u^\dagger = u + c_1 \) and \( v^\dagger = v + c_2 \). If \( m_1 = m_2 \) \((q, u, v \text{ are then square matrices of same size})\), and assuming invertibility of \( q \), the above system reduces to
\[
u = -q_x q^{-1} - c_1, \quad v = -q^{-1} q_x, \quad (q_x q^{-1})^\dagger = q_x q^{-1} + c_1, \quad (q^{-1} q_x)^\dagger = q^{-1} q_x - c_2,
\]
and
\[
i q_t - q_{xy} + q_x q^{-1} q_y + q_y q^{-1} q_x - c_{1,y} q = 0, \quad (q_x q^{-1})_x \mp q q^\dagger = 0, \quad (q^{-1} q_x)_x \mp q^\dagger q = 0. \tag{6.13}
\]

**Example 6.1.** In the scalar case, i.e., \( m_1 = m_2 = 1 \), the above equations require \( c_2 = -c_1 \). Writing \( q = \sqrt{\rho} e^{i S} \), with real functions \( \rho \) and \( S \), the second equation in \( (6.13) \) becomes \( \log(\rho)_{xx} \mp 2 \rho = 0 \) and \( S_{xx} = 0 \), hence \( S = \lambda x + \beta \) with real functions \( \lambda, \beta \) not depending on \( x \). We obtain
\[
q = \begin{cases} 
  a \sec(a x + b) e^{i(\lambda x + \beta)} & \text{defocusing case (upper sign)} \\
  a \sech(a x + b) e^{i(\lambda x + \beta)} & \text{focusing case (lower sign)}
\end{cases}
\]
where \( a, b \) are another two real functions that do not depend on \( x \). We find \( c_1 = -2i \lambda \). Inserting the expression for \( q \) in the first of \( (6.13) \), leads to
\[
\lambda_t + \lambda \lambda_y \pm a a_y = 0, \quad a_t + (\lambda a)_y = 0, \tag{6.14}
\]
and the linear system
\[
b_t + \lambda b_y + a \beta_y = 0, \quad \beta_t + \lambda \beta y \pm a b_y = 0. \tag{6.15}
\]

In the defocusing case (upper sign), the first pair of equations decouples into the two Riemann equations
\[
w_i t + w_i w_j^i = 0 \quad i = 1, 2, \quad \text{where} \quad w^1 := \lambda + a, \quad w^2 := \lambda - a.
\]

With the reduction \( a = \lambda \) (i.e., \( w^2 = 0 \)), the function \( \lambda \) satisfies the Riemann equation \( \lambda_t + 2 \lambda \lambda_y = 0 \) and we recover (with \( b = \beta \)) a case treated in \[46\] (see §5 therein), where ‘breaking soliton’ solutions
of (6.12) were searched for. We thus showed that this case is a subcase of what is covered by the ‘Riemann system’ associated with the (2+1)-dimensional NLS equation.

In the focusing case, (6.14) can be expressed as the complex Riemann equation

\[ w_t + w w_y = 0, \]

where \( w := \lambda + i a \). In terms of \( \zeta := b - i \beta \), (6.15) (with the lower sign) reads \( \zeta_t + w \zeta_y = 0 \), which is solved by \( \zeta = f(w) \), with an arbitrary differentiable function \( f \), as a consequence of the Riemann equation for \( w \). A hodograph transformation (with \( x(\lambda, a), t(\lambda, a) \)) turns (6.14) into the linear equations

\[
\begin{align*}
y_\lambda + a t_a - \lambda t_\lambda &= 0, \\
y_a - \lambda t_a - a t_\lambda &= 0,
\end{align*}
\]

if \( y_\lambda t_a - y_a t_\lambda \neq 0 \) (also see, e.g., [47]). If \( t_\lambda = 0 \), then \( y = y_0 + \lambda a_0/a \) and \( t = t_0 + a_0/a \), with arbitrary constants \( a_0, t_0, y_0 \in \mathbb{R} \), hence

\[ w = \frac{y - y_0}{t - t_0} + i \frac{a_0}{t - t_0}. \]

With this special explicit solution of the complex Riemann equation, we recover the singular solitary wave solution, of the scalar focusing (2+1)-dimensional NLS equation, that appeared in [48]. If we choose a ‘breaking wave’ solution of the complex Riemann equation, the function \( a \) inherits the singularity of \( \lambda_y \) and thus the solution \( q \) of the (2+1)-dimensional NLS equation itself is singular (and not just a partial derivative of it). Therefore, such solutions are not breaking waves.

If \( \lambda \) is constant, the above system admits (regular) solitary wave solutions. From the above, we can conclude that the slightest generic perturbation will lead to a solution that breaks or blows up in finite (positive or negative) time \( t \). This feature is absent in the (1+1)-dimensional NLS equation.

### 6.1.1 The linearization method

Let \( \phi_0 = i \Lambda \) and \( \gamma = i \frac{1}{2} J \Lambda \xi_2 \). We assume that \( \Lambda \) does not depend on \( x \) and \([\Lambda, J] = 0\). Then (2.3) with (6.3) reads

\[ \Phi_t + \Phi_y \Lambda = 0, \quad \Phi_x = i \frac{1}{2} J \Phi \Lambda, \]

and (2.2) is the matrix Riemann equation

\[ \Lambda_t + \Lambda_y \Lambda = 0. \]

For any solution \( \Phi \) of (6.16),

\[ \varphi = i J \Phi \Lambda \Phi^{-1} \]

is a solution of the matrix (2+1)-dimensional NLS system (6.7). The general solution of the second of (6.16) is

\[ \Phi = A_1 e^{\frac{i}{2} i J \Lambda x} + A_2 e^{-\frac{i}{2} i J \Lambda x}, \]

where the matrices \( A_i \) do not depend on \( x \) and satisfy \( J A_1 = A_1 J \) and \( J A_2 = -A_2 J \). The first of (6.16) now results in

\[ A_i, t + A_i, y \Lambda = 0 \quad i = 1, 2. \]

The reduction conditions (6.9) are translated as follows,

\[ \varphi^\dagger = \varphi + C \quad \iff \quad \Lambda^\dagger \Phi^\dagger J \Phi + \Phi^\dagger J \Phi \Lambda = i \Phi^\dagger C \Phi, \]

\[ \varphi^\dagger = J \varphi J + C \quad \iff \quad \Lambda^\dagger \Phi^\dagger \Phi + \Phi^\dagger \Phi \Lambda = i \Phi^\dagger C J \Phi. \]
Inserting the formula for $\Phi$, we obtain
\[
\Gamma_2 = e^{-i\Lambda_1^\dagger x} \Gamma_1 e^{i\Lambda_1 x} = 0, \quad \Xi^\dagger = e^{-i\Lambda_1^\dagger x} \Xi e^{-i\Lambda_1 x} = 0,
\]
where the upper sign refers to the defocusing case (6.20), the lower sign to the focusing case (6.21), and we introduced the $x$-independent expressions
\[
\Gamma_j := \Lambda_j^\dagger A_j^\dagger A_j + A_j^\dagger A_j \Lambda - i A_j^\dagger C J A_j \quad j = 1, 2, \quad \Xi := \Lambda_j^\dagger A_2 + A_j^\dagger A_2 \Lambda - i A_j^\dagger C J A_2.
\]
$\Gamma_j$ is Hermitian and commutes with $J$. $\Xi$ anti-commutes with $J$. Since $\Gamma_j, \Xi$ are independent of $x$, differentiation of the above equations with respect to $x$ yields
\[
\Lambda^\dagger \Gamma_1 = \Gamma_1 \Lambda, \quad \Lambda^\dagger \Xi = \Xi \Lambda.
\] (6.22)
If these relations hold, the above conditions are reduced to
\[
\Gamma_2 = \pm \Gamma_1, \quad \Xi^\dagger = \pm \Xi.
\] (6.23)
The reduction condition (6.20), respectively (6.21), is thus equivalent to (6.22) and (6.23), with the respective choice of sign.

**Example 6.2.** If we set $\Gamma_j = 0$, $j = 1, 2$, those equations in (6.22) and (6.23) that involve $\Gamma_j$ are satisfied and, assuming that $A_1$ is invertible, from their definitions we obtain
\[
C = -i (A_1 A_1^{-1} + \text{h.c.}) J, \quad A_2^\dagger \Theta = -\Theta^\dagger A_2,
\]
where $\Theta := A_2 \Lambda - A_1 A_1^{-1} A_2$ and ‘h.c.’ means Hermitian conjugate. Furthermore, we have
\[
\Xi = A_1^\dagger \Theta + A_1^\dagger A_2 \Lambda - \Lambda^\dagger A_1^\dagger A_2,
\]
which still has to satisfy the corresponding equations in (6.22) and (6.23). Also (6.17) and (6.19) have to be satisfied. We were unable to solve the algebraic equations in general. A special solution of (6.16) and the reduction conditions is given by
\[
\Phi = \begin{pmatrix}
A e^{\frac{1}{2}i\Lambda_1 x} & e^{\frac{1}{2}i\Lambda_1^\dagger x} \\
\pm A^\dagger e^{\frac{1}{2}i\Lambda_1^\dagger x} & e^{-\frac{1}{2}i\Lambda_1 x}
\end{pmatrix},
\]
where $\Lambda_1$ (originating from $\Lambda = \text{block-diag}(\Lambda_1, \Lambda_1^\dagger)$) has to be normal (i.e., $[\Lambda_1, \Lambda_1^\dagger] = 0$) and to commute with $A$ and $A^\dagger$, and
\[
\Lambda_{1,t} + \Lambda_{1,y} \Lambda_1 = 0, \quad A_t + A_y \Lambda_1 = 0.
\]
If $m_1 = m_2 = 1$ (scalar case), setting $A = \pm e^{i\beta-b}$, $\Lambda_1 = \lambda + i a$, we obtain
\[
\varphi = \begin{cases}
\begin{pmatrix} i \lambda + a \coth(a x + b) & -a \coth(a x + b) e^{i(\lambda x + \beta)} \\
-\lambda + a \coth(a x + b) & i \lambda + a \coth(a x + b)
\end{pmatrix} & \text{defocusing case (upper sign)} \\
\begin{pmatrix} i \lambda + a \tanh(a x + b) & a \tanh(a x + b) e^{i(\lambda x + \beta)} \\
-\lambda + a \tanh(a x + b) & i \lambda + a \tanh(a x + b)
\end{pmatrix} & \text{focusing case (lower sign)}
\end{cases}
\]
where $a, b, \beta$ and $\lambda$ have to satisfy the system (6.14), (6.15). In this way we recover the focusing case treated in Example 6.1. The above $\Phi$ thus leads, via (6.18), to matrix versions of the solutions treated in Example 6.1. They solve the matrix version of the (2+1)-dimensional NLS equation.

---

\[30\] We omit the left lower component which is ± the Hermitian conjugate of the right upper component: $r = \pm q^\dagger$. 

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28
Example 6.3. Now we set $\Xi = 0$. Assuming that $A_1$ and $A_2$ are invertible, this yields $C = -i ((A_1^{-1})^* A_1 + A_2 A A_2^{-1})$. Eliminating $C$ from the definitions of $\Gamma_1, \Gamma_2$, we are then left with the algebraic equations for $\Gamma_i$ in (6.22) and (6.23), besides the differential equations (6.17) and (6.19). A special solution of (6.16) and the defocusing reduction condition is given by

$$
\Phi = \begin{pmatrix}
    e^{\frac{1}{2} i \Lambda_1 x} & e^{\frac{1}{2} i \Lambda_2 x} \\
    B e^{-\frac{1}{2} i \Lambda_1 x} & A e^{-\frac{1}{2} i \Lambda_2 x}
\end{pmatrix},
$$

with Hermitian $\Lambda = \text{block-diag}(\Lambda_1, \Lambda_2)$, provided that $[\Lambda_1, \Lambda_2] = 0$, and $A, B$ are unitary matrices with $[A^* B, A_i] = 0$, $i = 1, 2$. (6.17) and (6.19) then result in

$$
\Lambda_{i,t} + \Lambda_{i,y} \Lambda_i = 0, \quad A_t + A_y \Lambda_2 = 0, \quad B_t + B_y \Lambda_1 = 0.
$$

In the scalar case, setting $A = i e^{-i (b+\beta)}$, $B = -i e^{i (b-\beta)}$, $\Lambda = (\lambda - a, \lambda + a)$, we obtain

$$
\varphi = \begin{pmatrix}
    i \lambda - a \tan(a x + b) & a \sec(a x + b) e^{i (\lambda x + \beta)} \\
    -i \lambda - a \tan(a x + b)
\end{pmatrix},
$$

from which we recover a case treated in Example 6.1.

According to Remark 3.8 addressing solutions of the above Riemann system, we cannot expect more from Corollary 3.6 than what we obtain from the method in Section 2, which we elaborated above.

6.2 Multi-soliton solutions of the (2+1)-dimensional NLS equation, parametrized by solutions of Riemann equations

Breaking multi-solitons of the scalar $(2+1)$-dimensional NLS equation have been mentioned in [46]. A special class of such solutions in the focusing case was obtained a few years later in [48, 51]. More general solutions, moreover for the matrix $(2+1)$-dimensional NLS system, are immediately obtained via Corollary 3.5. As in Proposition 5.1, we can drop the differential equation for $X$ if we impose the spectrum condition $\sigma(P) \cap \sigma(Q) = \emptyset$. In the following, $J$ is an $n \times n$ counterpart of $J$. The latter is given in (6.5).

Proposition 6.4. Let $\varphi_0$ be a constant $m \times m$ matrix with $[J, \varphi_0] = 0$. Let $P$ and $Q$ be solutions of the $n \times n$ matrix equations

$$
i P_t + P_y J P = 0, \quad P_x = \frac{1}{2} (J P J - P) P, \quad i Q_t + Q J Q_y = 0, \quad Q_x = \frac{1}{2} Q (Q - J Q J).
$$

We further assume that $\sigma(JP) \cap \sigma(QJ) = \emptyset$. Let $X$ be the unique solution of the Sylvester equation

$$
X (J P) - (Q J) X = V_0 U_0,
$$

where $U_0$ and $V_0$ are constant matrices of size $m \times n$, respectively $n \times m$, and have to satisfy $J U_0 = U_0 J$ and $J V_0 = V_0 J$. Then

$$
\varphi = \varphi_0 + J U_0 X^{-1} V_0
$$

(exception at singular points) solves (6.4). \qed

31 Apparently, the authors of [48] were not aware of Bogoyavlenskii’s related work (in particular, [46]). They used the (AKNS) inverse scattering method and made an ansatz for solving the equations for the scattering data to generate multi-soliton-type solutions. That, more generally, the solutions can be parametrized by solutions of a Riemann equation, is not deducible from their work.

32 See [22] for the way in which the bidifferential calculus (6.3) is extended to $n \times n, m \times n$ and $n \times m$ matrices.

33 We made redefinitions $P \mapsto J P$ and $Q \mapsto Q J$ in Corollary 3.5.
It remains to implement the reduction conditions, so that we obtain solutions of the (focusing, respectively defocusing) matrix (2+1)-dimensional NLS system, which is given by (6.6) and (6.10), where the \(c_i\) have to be anti-Hermitian and satisfy (6.11).

**Corollary 6.5.** Let \(\varphi_0\) be a constant \(m \times m\) matrix, with \([J, \varphi_0] = 0\) and satisfying one of the reduction conditions (6.9). Let \(P\) be a solution of the \(n \times n\) matrix equations

\[
i P_t + P_y J P = 0, \quad P_x = \frac{1}{2}(J P J - P) P,
\]

and such that \(\sigma(JP) \cap \sigma(-P^\dagger J) = \emptyset\). Let \(X\) be the unique solution of the Lyapunov equation

\[
X (JP) + (JP)^\dagger X = V_0 U_0,
\]

where \(U_0\) is a constant \(m \times n\) matrix with \(J U_0 = U_0 J\), and

\[
V_0 = \begin{cases} 
U_0^\dagger J & \text{defocusing case} \\
U_0^\dagger & \text{focusing case}.
\end{cases}
\]

Then the blocks \(q, u, v\), read off from \(\varphi = \varphi_0 + J U_0 X^{-1} V_0\), solve the matrix (2+1)-dimensional NLS system. If \(m_1 = m_2 = 1\), then \(q\) solves the (2+1)-dimensional NLS equation (6.12).

**Proof.** Setting \(Q = -P^\dagger\) in the preceding proposition reduces the differential equations for \(Q\) to those for \(P\). The relation between \(V_0\) and \(U_0\) then ensures that \(\varphi\) given by (6.24) satisfies the same reduction condition as \(\varphi_0\). Note that \(X^\dagger = X\), since the solution of the Lyapunov equation is unique if the stated spectrum condition holds.

By choosing

\[
P = \text{block-diag}(\varphi_1, \ldots, \varphi_N), \quad J = \text{block-diag}(J, \ldots, J),
\]

where, for \(i = 1, \ldots, N\), \(\varphi_i\) solves (6.1) and (6.2), we obtain a nonlinear superposition of \(N\) singular solitons. Still more general solutions can be generated via Theorem 3.1. This includes nonlinear superpositions of regular and singular solitons. We should stress again that the singular soliton-type solutions cannot be called ‘breaking solitons’. Although we cannot exclude presently that the (2+1)-dimensional NLS equation possesses hitherto unknown solutions representing breaking waves, the soliton-like solutions obtained here are not quite of this type and thus do not justify to call this equation a ‘breaking soliton equation’, as sometimes done in the literature.

7 Burgers and KP

7.1 Burgers equation

Let \(\Omega^1 = A := A_0[\partial_x]\), where \(A_0\) is the algebra of matrices of smooth functions of variables \(x, y\), and \(\partial_x = \partial/\partial x\). Let

\[
df = f_x, \quad \bar{df} = \frac{1}{2} [\alpha \partial_y - \partial_x^2, f],
\]

with a constant \(\alpha\). (1.1) then reads

\[
\alpha \phi_y - \phi_{xx} - 2 \phi_x (\partial_x + \phi) = 0.
\]

A solution \(\phi\) has to be an operator. Writing

\[
\phi = -\partial_x + \varphi,
\]

turns (1.1) into the matrix Burgers equation

\[
\alpha \varphi_y - \varphi_{xx} - 2 \varphi_x \varphi = 0,
\]

and \(\varphi\) can now be taken to be a matrix of functions.
7.1.1 Cole-Hopf transformation

Choosing \( \phi_0 = -\partial_x \), (2.1) becomes the Cole-Hopf transformation

\[
\varphi = \Phi_x \Phi^{-1},
\]

and (2.3), with \( \gamma = 0 \), is the linear heat equation

\[
\alpha \Phi_y - \Phi_{xx} = 0. \tag{7.2}
\]

**Example 7.1.** A class of solutions of (7.1) is determined by the following solutions of (7.2),

\[
\Phi = I + \sum_{i=1}^n A_i e^{\Theta_i} B_i, \quad \Theta_i = \Lambda_i x + \frac{1}{\alpha} \Lambda_i^2 y,
\]

where \( A_i, B_i \) are constant \( m \times n \), respectively \( n \times m \) matrices, and \( \Lambda_i \) are constant \( n \times n \) matrices. In the scalar case (\( m = 1 \)), and if \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \), this takes the form

\[
\Phi = 1 + \sum_{i=1}^n e^{\lambda_i x + \lambda_i^2 y/\alpha + \gamma_i}, \tag{7.3}
\]

with constants \( \gamma_i \). It determines an \( n \)-kink solution of the scalar Burgers equation merging (at some value of \( y \)) into a single kink, which then develops into a shock wave (in the sense that it becomes arbitrarily steep), see Fig. 3.

![Figure 3: Plots of a solution of the scalar Burgers equation (7.1), with \( \alpha = 1 \), at four different values \((-20, -10, -4, 10)\) of \( y \). It is obtained from (7.3) with \( n = 3, \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3, \gamma_1 = -10, \gamma_2 = 0, \gamma_3 = 10 \). Evolution in the variable \( y \) is from right to left. For large negative \( y \), this is a 3-kink solution (right curve). It then turns into a single kink and finally evolves into a shock wave (left curve). The second figure shows a contour plot of \( \varphi_x \), which has the form of a rooted binary tree.](image)

7.1.2 Darboux transformations

The following is obtained from Corollary 3.6. Setting \( \alpha = \beta = 0 \) and \( P = Q = -\partial_x \), we have to consider the following equations,

\[
\alpha U_y - U_{xx} = \varphi_{0,x} U, \quad V_x = -V \varphi_0, \quad \alpha V_y = -V \varphi_0^2, \quad X_x = VU, \quad \alpha X_y = VU_x - V_x U.
\]

The last two equations are solved by

\[
X = C + \int \left( VU \, dx + \frac{1}{\alpha} (VU_x - V_x U) \, dy \right), \tag{7.4}
\]

where \( C \) is constant and the integration is along any path from a fixed point \((x_0, y_0)\) to \((x, y)\). Then \( \varphi = \varphi_0 + UX^{-1}V \) solves the Burgers equation (7.1).
Remark 7.2. In the scalar case (i.e., \( m = 1 \)), we have
\[
\varphi - \varphi_0 = U X^{-1} V = \text{tr}(V U X^{-1}) = \text{tr}(X_x X^{-1}) = (\ln \det X)_x.
\]
Expressing the seed solution as \( \varphi_0 = (\ln \Phi_0)_x \) and setting \( \Phi = \Phi_0 \det X \), we find
\[
\alpha \Phi_y - \Phi_{xx} = (\alpha \Phi_{0,y} - \Phi_{0,xx}) \Phi - 2 \Phi_0 \left( \text{tr}(U X^{-1} V) \varphi_0 + \text{tr}(U X^{-1} V x) \right),
\]
which vanishes by use of \( V_x = -V \varphi_0 \) and if \( \Phi_0 \) satisfies (7.2). Hence \( \Phi \) solves (7.2) and we expressed the Darboux transformation as a Cole-Hopf transformation.

Example 7.3. If the seed solution \( \varphi_0 \) vanishes, \( V \) has to be constant and it only remains to solve \( \alpha U_y - U_{xx} = 0 \). In the scalar case, writing \( U = (u_1, \ldots, u_n) \) and \( V = (v_1, \ldots, v_n)^T \), for vanishing seed we find \( u_i = a_i e^{\lambda_i x + \lambda_i^2 y / \alpha} \), with constants \( a_i, \lambda_i \), and \( X_{ij} = C_{ij} + v_i u_j / \lambda_i \). If \( C \) is invertible, we obtain the same solution \( \varphi \) with \( C = I \) and a redefined \( V \). Computing \( \det X \) via Sylvester’s determinant theorem and setting \( a_i v_i / \lambda_i =: e^n \), yields (7.3).

### 7.2 The second equation of the Burgers hierarchy

Now we extend \( A_0 \) to the algebra of matrices of smooth functions of variables \( x, y \) and \( t \), and we set
\[
df = \frac{1}{2} [\alpha \partial_y + \partial_x^2, f], \quad \bar{d} f = \frac{1}{3} [\partial_t - \partial_y^3, f] .
\]
Writing again \( \phi = -\partial_x + \varphi \), and assuming that \( \varphi \) is a matrix of functions, (1.1) splits into (7.1) and
\[
\frac{1}{3} (\varphi_t - \varphi_{xxx}) - \frac{1}{2} \varphi_{xx} \varphi - (\varphi_x)^2 - \frac{\alpha}{2} \varphi y \varphi = 0 .
\]
Using (7.1) in the last equation, converts the latter into the second member of a matrix Burgers hierarchy,
\[
\varphi_t - \varphi_{xxx} - 3 (\varphi_x \varphi)_x - 3 \varphi_x \varphi^2 = 0 . \tag{7.5}
\]
We should stress that, though (1.1) is only a single equation for \( \phi \), in terms of \( \varphi \) it consists of two equations (which arise as the coefficients of \( \partial_y^3 \) and \( \partial_x^3 \) ), and these are equivalent to the first two equations of the Burgers hierarchy.

Setting \( \phi_0 = -\partial_x \) and \( \gamma = 0 \), (2.1) is the Cole-Hopf transformation and (2.3) is equivalent to (7.2) and the second member of a matrix heat hierarchy, \( \Phi_t - \Phi_{xxx} = 0 \).

### 7.3 KP

Let us now choose \( \Omega \) according to (3.9) with \( K = 2 \). We combine the above two bidifferential calculi, associated with the first two members of a matrix Burgers hierarchy, to
\[
df = [\partial_x, f] \xi_1 + \frac{1}{2} [\alpha \partial_y + \partial_x^2, f] \xi_2 , \quad \bar{d} f = \frac{1}{2} [\alpha \partial_y - \partial_x^2, f] \xi_1 + \frac{1}{3} [\partial_t - \partial_x^2, f] \xi_2 . \tag{7.6}
\]
With \( \phi = -\partial_x + \varphi \), (1.1) is then equivalent to (7.1) and (7.5). Now (1.3) holds, and the integrability condition (1.4) turns out to be the matrix potential KP equation
\[
(-4 \varphi_t + \varphi_{xxx} + 6 (\varphi_x)^2)_x + 3 \alpha^2 \varphi_{yy} - 6 \alpha [\varphi_x, \varphi_y] = 0 \tag{7.7}
\]
(also see [12]). We recover the well-known fact that any solution of the first two members of the (matrix) Burgers hierarchy solves the (matrix) potential KP equation (see [49] and the references cited there). This is simply a special case of the implication (1.1) \( \Rightarrow \) (1.4).
The resulting class of solutions of the scalar KP-II equation (with $\alpha = 1$) for the variable $u = \varphi_x$, corresponding to the above class of multi-kink solutions of the Burgers equation, consists of KP line soliton solutions that form, at any time $t$, a rooted (and generically binary) tree-like structure in the $xy$-plane (see Fig. [3] for an example). An analysis (in a ‘tropical limit’) of this class of KP-II solutions has been carried out in [50, 51]. More complicated line soliton solutions of the scalar KP-II equation are obtained from $n \times n$ matrix Burgers equations via Corollary 3.5. But according to Remark 3.4, these solutions are obtained in a simpler way from Theorem 3.1 with constant $P$ and $Q$. The resulting class of line-soliton solutions is well-known.

**Proposition 7.4.** Let $\varphi_0$ be a constant $m \times m$ matrix, and $\tilde{P}$ and $\tilde{Q}$ solutions of the following $n \times n$ matrix equations

\[
\begin{align*}
\alpha \tilde{P}_y &= \tilde{P}_{xx} + 2 \tilde{P}_x \tilde{P}, \\
\alpha \tilde{Q}_y &= -\tilde{Q}_{xx} + 2 \tilde{Q} \tilde{Q}_x,
\end{align*}
\]

which are the first two members of two versions of a matrix Burgers hierarchy. Let $X$ be an invertible solution of the system of linear ordinary differential equations

\[
\begin{align*}
X_x &= V_0 U_0 + \tilde{Q} X - X \tilde{P}, \\
\alpha X_y &= \tilde{Q} V_0 U_0 + V_0 U_0 \tilde{P} - (\tilde{Q}_x - \tilde{Q}) X - X (\tilde{P}_x + \tilde{P}^2), \\
X_t &= \tilde{Q} V_0 U_0 \tilde{P} - (\tilde{Q}_x - \tilde{Q}) V_0 U_0 + V_0 U_0 (\tilde{P}_x + \tilde{P}^2) + (\tilde{Q}_{xx} - \tilde{Q}_x \tilde{Q} - 2 \tilde{Q} \tilde{Q}_x + \tilde{Q}^2) X - X (\tilde{P}_{xx} + \tilde{P} \tilde{P}_x + 2 \tilde{P}_x \tilde{P} + \tilde{P}^3),
\end{align*}
\]

with any constant matrices $U_0$, $V_0$, of size $m \times n$, respectively $n \times m$. Then $\varphi = \varphi_0 + U_0 X^{-1} V_0$ solves the $m \times m$ matrix potential KP equation (7.7).

**Proof.** This is obtained from Corollary 3.5 using (7.6) and setting $P = \tilde{P} - \partial_x$, $Q = \tilde{Q} - \partial_x$, $\alpha = \beta = 0$. \hfill $\square$

**Remark 7.5.** Proposition 7.4 is not really of practical use. But reducing the matrix Burgers hierarchy equations to heat hierarchy equations via Cole-Hopf transformations,

\[
\begin{align*}
\tilde{P} &= \Phi_x \Phi^{-1}, \\
\alpha \Phi_y &= \Phi_{xx}, \\
\tilde{Q} &= -\Psi^{-1} \Psi_x, \\
\alpha \Psi_y &= -\Psi_{xx},
\end{align*}
\]

in terms of $U := U_0 \Phi$ and $V := \Psi V_0$, the solution of the equations for $X$ can be expressed as

\[
X = \Psi^{-1} \left( C + \int_{(x_0,y_0,t_0)}^{(x,y,t_0)} VU \, dx + \alpha^{-1} \int_{(x_0,y_0,t_0)}^{(x,y,t_0)} (V U_x - U_x V) \, dy \\
+ \int_{(x_0,y_0,t_0)}^{(x,y,t)} (V_{xx} U - U_x V_x + VU_{xx}) \, dt \right) \Phi^{-1},
\]

with a constant matrix $C$. This expression generalizes (7.4). As expected from well-known results, now one only has to solve linear (matrix) heat hierarchy equations in order to construct solutions of the KP equation.

Of course, all this can be extended to the whole matrix KP hierarchy, see [12] for a corresponding bidifferential calculus.
8 Davey-Stewartson equation

8.1 Another ‘Riemann equation’

Let $\mathcal{B}$ be the space of smooth complex functions on $\mathbb{R}^2$. We extend it to $\mathcal{B} = \mathcal{B}_0[\partial_x]$, where $\partial_x = \partial/\partial x$. On $\text{Mat}(m, m, \mathcal{B})$ we define

$$df = [J, f], \quad \bar{df} = [\partial_y - J\partial_x, f],$$

where $J \neq I$ is a constant $m \times m$ matrix. Then (1.1) becomes

$$\phi_y = [J\partial_x, \phi] + [J, \phi] \phi = J \phi_x + [J, \phi] (\phi + \partial_x).$$

This suggests to introduce a new dependent variable $\varphi$ via

$$\varphi = J \varphi - \partial_x,$$

in terms of which (8.2) reads

$$\varphi_y = J \varphi_x + [J, \varphi] J \varphi,$$

assuming $J^2 = I$. We choose $J$ and decompose $\varphi$ as in (6.5). The last equation then splits into

$$u_y - u_x + 2qr = 0, \quad v_y + v_x + 2rq = 0, \quad q_y - q_x + 2qv = 0, \quad r_y + r_x + 2ru = 0. \quad (8.5)$$

**Remark 8.1.** Via a change of variables corresponding to $\partial \tilde{y} = \partial y - \partial x$, $\partial \tilde{x} = \partial y + \partial x$, the system (8.5) can be expressed as

$$u = -2 \int q r d\tilde{y}, \quad v = -2 \int r q d\tilde{x}, \quad q_{\tilde{y}} = 4q \int r q d\tilde{x}, \quad r_{\tilde{x}} = 4r \int q r d\tilde{y}. \quad (8.6)$$

8.1.1 Hermitian conjugation reduction

Let us decompose $\varphi$ as follows,

$$\varphi = D + U, \quad D = \frac{1}{2}(\varphi + J \varphi J) = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}, \quad U = \frac{1}{2}(\varphi - J \varphi J) = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}. \quad (8.7)$$

The Hermitian conjugation reduction conditions\(^{34}\)

$$U = \mu U^\dagger, \quad D_x = D_x^\dagger, \quad D_y = D_y^\dagger, \quad (8.6)$$

with a real constant $\mu \neq 0$, are equivalent to

$$r = \mu q^\dagger, \quad u_x = u_x^\dagger, \quad u_y = u_y^\dagger, \quad v_x = v_x^\dagger, \quad v_y = v_y^\dagger. \quad (8.6)$$

These conditions reduce (8.5) to

$$u_y - u_x + 2\mu q q^\dagger = 0, \quad v_y + v_x + 2\mu q^\dagger q = 0, \quad q_y - q_x + 2qv = 0, \quad q_y + q_x + 2v^\dagger q = 0. \quad (8.7)$$

**Remark 8.2.** In the scalar case, the system (8.7) has the following consequences,

$$(\ln q)_{yy} - (\ln q)_{xx} = 4\mu |q|^2,$$

$$u_{yy} - u_{xx} = 2 \text{Re}(u^2)_x - 2 \text{Re}(u^2)_y, \quad v_{yy} - v_{xx} = -2 \text{Re}(v^2)_x - 2 \text{Re}(v^2)_y. \quad (8.7)$$

In terms of $w := \ln q$, the first becomes a Liouville equation. The last two equations can be integrated to first order equations, using the variables introduced in Remark 8.1.\(^{34}\)

\(^{34}\)It should be noticed that the conditions for $D$ are weaker than $D^\dagger = D$. This is important in order to recover relevant solutions of the DS equation, see Section 8.2.
Remark 8.3. In the scalar case \((m_1 = m_2 = 1)\), the equations (8.7) are solved by

\[
q = F^{-1} e^{i\phi(x+y) - i\phi'(y-x)}, \quad F(x,y) := H_1(x-y) F_1(x+y) + H_2(x-y) F_2(x+y),
\]

\[
u = \frac{1}{2} \left( (\ln F)_x + (\ln F)_y \right) + ic, \quad \psi = \frac{1}{2} \left( (\ln F)_y - (\ln F)_x \right) + i\tilde{c},
\]

with real constants \(c, \tilde{c}\), and any real functions \(F_i, H_i\), subject to the determinant conditions

\[
\begin{vmatrix}
F_1 & F_2 \\
2F'_1 & 2F'_2
\end{vmatrix} = \alpha, \quad \begin{vmatrix}
H_1 & H_2 \\
2H'_1 & 2H'_2
\end{vmatrix} = \frac{4\mu}{\alpha},
\]

(8.8) where \(\alpha \in \mathbb{R} \setminus \{0\}\), and a prime denotes a derivative with respect to the argument.

8.1.2 Cole-Hopf transformation

Setting \(\phi_0 = -\partial_x\), and using (8.1) and (8.3), (2.1) becomes

\[
\varphi = J(\phi + \partial_x) = -J\Phi \partial_x \Phi^{-1} + J\partial_x = J\Phi \Phi^{-1},
\]

(8.9) and (2.3), with \(\gamma = 0\), reads

\[
\Phi_y - J\Phi_x = 0.
\]

(8.10) All solutions of (8.4) can be reached in this way. With \(J\) given by (6.5), and writing

\[
\Phi = \begin{pmatrix}
f_1 & f_2 \\
g_1 & g_2
\end{pmatrix},
\]

(8.10) means that the dependence of the blocks on the independent variables is restricted to

\[
f_i = f_i(x+y), \quad g_i = g_i(x-y), \quad i = 1, 2.
\]

It seems to be difficult, however, to implement the Hermitian conjugation reduction conditions (8.6) via conditions on \(\Phi\), in general.

Remark 8.4. The solutions obtained in Remark 8.3 in the scalar case can also be obtained via the Cole-Hopf transformation by writing \(f_i(x+y) = F_i(x+y) e^{i\phi(x+y)}\) and \(g_1(x-y) = -\frac{\alpha}{2} H_2(x-y) e^{i\phi'(y-x)}\), \(g_2(x-y) = \frac{\alpha}{2} H_1(x-y) e^{i\phi'(y-x)}\), and imposing (8.8) on \(F_i, H_i\).

Example 8.5. In the scalar case, let us set

\[
f_i = a_i e^{\theta} + b_i e^{-\theta^*}, \quad g_i = c_i e^{\tilde{\theta}} + d_i e^{-\tilde{\theta}^*}, \quad i = 1, 2,
\]

\[
\theta = \lambda(x+y), \quad \tilde{\theta} = \tilde{\lambda}(x-y),
\]

with complex constants \(\lambda, \tilde{\lambda}\), and real constants \(a_i, b_i, c_i\) and \(d_i\). An asterisk denotes complex conjugation. From (8.9) we obtain

\[
q = -\frac{2}{\Delta} \text{Re}(\lambda) (a_1 b_2 - a_2 b_1) e^{i\text{Im}(\theta - \tilde{\theta})}, \quad r = -\frac{2}{\Delta} \text{Re}(\tilde{\lambda}) (c_1 d_2 - c_2 d_1) e^{i\text{Im}(\tilde{\theta} - \theta)},
\]

\[
u = i \text{Im}(\lambda) + \text{Re}(\lambda) \frac{\Delta_1 + \Delta_2}{\Delta}, \quad \psi = -i \text{Im}(\tilde{\lambda}) + \text{Re}(\tilde{\lambda}) \frac{\Delta_1 - \Delta_2}{\Delta},
\]

where

\[
\Delta_1 := (a_1 d_2 - a_2 d_1) e^{\text{Re}(\theta - \tilde{\theta})} - (b_1 c_2 - b_2 c_1) e^{\text{Re}(\theta^* - \tilde{\theta}^*)},
\]

\[
\Delta_2 := (a_1 c_2 - a_2 c_1) e^{\text{Re}(\theta^* + \tilde{\theta})} - (b_1 d_2 - b_2 d_1) e^{\text{Re}(\theta + \tilde{\theta})},
\]

\[
\Delta := (a_1 c_2 - a_2 c_1) e^{\text{Re}(\theta + \tilde{\theta})} + (b_1 d_2 - b_2 d_1) e^{\text{Re}(\theta + \tilde{\theta})} + (b_1 c_2 - b_2 c_1) e^{\text{Re}(\theta - \tilde{\theta})} + (a_1 d_2 - a_2 d_1) e^{\text{Re}(\theta - \tilde{\theta})}.
\]
This provides us with solutions of the reduction conditions (8.7) if

$$\frac{\text{Re}(\lambda)}{\text{Re}(\lambda)} = \mu \frac{c_1 d_2 - c_2 d_1}{a_1 b_2 - a_2 b_1}.$$  \hspace{1cm} (8.11)

Excluding the trivial case where \( q = r = 0 \), we need \( a_1 b_2 \neq a_2 b_1 \) and \( c_1 d_2 \neq c_2 d_1 \). A solution from the above family is regular iff \( \Delta \) is everywhere different from zero. This is so if one of the coefficients of the sum of exponentials is positive and all others greater or equal to zero.

### 8.2 A ‘Riemann system’ associated with the Davey-Stewartson equation

Let now \( B \) be the space of smooth complex functions on \( \mathbb{R}^3 \). We extend \(^{35}\) (8.1) as follows,

$$df = [J, f] \xi_1 + [\partial_y + J \partial_x, f] \xi_2, \quad \bar{df} = [\partial_y - J \partial_x, f] \xi_1 - [i \partial_t + J \partial_x^2, f] \xi_2.$$  \hspace{1cm} (8.12)

Then (1.1) becomes (8.2) together with

$$i \phi_t = -[J \partial_x^2, \phi] - \phi_y \phi - [J \partial_x, \phi] \phi = -J \partial_{xx} - 2 J \partial_x \partial_y - [J, \phi] \partial_x^2 - \phi_y \phi - J \partial_x \phi - [J, \phi] (\phi_x + \phi \partial_x).$$

In terms of \( \varphi \) given by (8.3), (1.1) is equivalent to (8.4) and

$$i \varphi_t = -J \varphi_{xx} - \varphi_y J \varphi - J \varphi_x J \varphi - [J, \varphi] J \varphi_x.$$ \hspace{1cm} (8.13)

The integrability condition (1.4) takes the form

$$-i [J, \varphi_t] = \phi_{yy} - J \phi_{x} J - \phi_y J + [\phi_y + J \phi_x, J, \phi] + [J, \phi] J, \phi_x.$$  \hspace{1cm} 

Using the decomposition (6.5) of \( \varphi = J (\phi + \partial_x) \), and introducing new variables via

$$\tilde{u} := u_y + u_x, \quad \tilde{v} := v_y - v_x,$$

this splits into

$$2i q_t + q_{xx} + q_{yy} + 2 (q \tilde{v} + \tilde{u} q) = 0, \quad -2i r_t + r_{xx} + r_{yy} + 2 (r \tilde{v} + \tilde{v} r) = 0,$$

$$\tilde{u}_x - \tilde{u}_y = 2 ((q r)_x + (q r)_y), \quad \tilde{v}_x + \tilde{v}_y = 2 ((r q)_x + (r q)_y).$$

The conditions

$$r = \mu q^\dagger, \quad \tilde{u} = \tilde{u}^\dagger, \quad \tilde{v} = \tilde{v}^\dagger,$$

reduce this to

$$2i q_t + q_{xx} + q_{yy} + 2 (q \tilde{v} + \tilde{u} q) = 0,$$

$$\tilde{u}_x - \tilde{u}_y = 2 \mu ( (q q^\dagger)_x + (q q^\dagger)_y), \quad \tilde{v}_x + \tilde{v}_y = 2 \mu ( (q^\dagger q)_x + (q^\dagger q)_y).$$ \hspace{1cm} (8.14)

which is a matrix version \(^{53, 52, 51, 49, 55, 56}\) of the Davey-Stewartson (DS) equation \(^{57}\).

---

\(^{35}\) We can extend \(^{8.1}\) in a different way: \( df = [J, f] \xi_1 + [\bar{J}, f] \xi_2, \) \( \bar{df} = [\partial_t - J \partial_y, f] \xi_1 + [\partial_t - \bar{J} \partial_y, f] \xi_2, \) where the constant matrices \( J, \bar{J} \) have to commute. Then, in terms of \( \varphi \), (1.1) is a system of two equations of the form (8.4), and (1.4) reads \([J, \phi_t] - [J, \phi_x] + \bar{J} \phi_y J - \bar{J} \phi_y J = [\bar{J}, \phi] [J, \phi].\) Imposing a reduction condition \( \phi^\dagger = B \phi B, \) where \( B = \text{diag}(b_1, \ldots, b_n) \) with \( b_i \in \{\pm 1\} \), and choosing \( J \) and \( \bar{J} \) diagonal with positive diagonal entries, this is the generalized nonlinear optics equation (see, e.g., \(^{52}\)).
8.2.1 Solutions of the scalar ‘Riemann system’

From our general framework we know that any solution of the system (8.5), (8.13) is also a solution of the DS system. From the solution generating method of Section 2 (with \(d\) and \(\bar{d}\) determined by (8.12)), we can conclude that \(\varphi\) given by (8.9), with \(\Phi\) satisfying (8.10) and \(i\Phi_t + J\Phi_{xx} = 0\), solves (8.5) and (8.13).

The Cole-Hopf transformation provides us with solutions of the ‘Riemann system’, if \(f_j\) and \(g_j\), now also depending on \(t\), satisfy \(if_{j,t} + f_{j,xx} = 0\) and \(ig_{j,t} - g_{j,xx} = 0\). In Example 8.5, we simply have to make the substitutions
\[
\theta \mapsto \lambda (x + y) + i\lambda^2 t, \quad \tilde{\theta} \mapsto \tilde{\lambda} (x - y) - i\tilde{\lambda}^2 t.
\]

In this way we recover solutions of the scalar DS system (8.14), with \(\mu = -1\), in a similar form as presented in [17]. We have a single dromion solution if \(a_1c_2 - a_2c_1 > 0\), \(b_1d_2 - b_2d_1 > 0\), \(b_1c_2 - b_2c_1 > 0\), and \(a_1d_2 - a_2d_1 > 0\). This degenerates to a single solitoff solution if \(a_1c_2 - a_2c_1 = 0\), which in turn degenerates to a single soliton solution if \(b_1d_2 - b_2d_1 = 0\). In all these cases, also (8.11), with \(\mu = -1\), has to hold.

9 Concluding and further remarks

In this work we showed that some prominent nonlinear equations can be regarded as realizations of the ‘Riemann equation’ (1.1) (or (1.2)) in bidifferential calculus.

The most basic examples are the matrix Riemann equation, and a semi- and a full discretization of it. These integrable discretizations are easily obtained in bidifferential calculus from the continuous model, essentially by replacing a partial derivative by a commutator with a shift operator.

The semi-discrete Riemann equation (4.4) appeared in [24] (see (3.23) therein) as an ‘infinitesimal symmetry’ (analog of infinitesimal Lie point symmetry) of the discrete Burgers equation (4.24), in the scalar case. The (fully) discrete Riemann equation (4.12) is a corresponding ‘finite symmetry’ of the discrete Burgers equation.

Furthermore, it can be easily verified that the semi-discrete Riemann equation (4.4) and the discrete Riemann equation (4.13) are compatible. More generally, the semi-discrete Riemann hierarchy and the (fully) discrete Riemann hierarchy form a common hierarchy. This can be concluded from the fact that they share the same Cole-Hopf formula (4.5) (cf. (4.14)), and the corresponding linear equations are compatible.

**Remark 9.1.** The alternative integrable semi-discretization
\[
u_t = u^+ u - uu^- \quad (9.1)
\]
of the Riemann equation is known as the (Lotka-) Volterra lattice equation (see, e.g., [59]). The above is a matrix version. Though not as a realization of (1.1), a kind of potential version of it can be obtained as a realization of (1.4). Let us consider the bidifferential calculus given by
\[
df = [S^+, f] \xi_1 + [S, f] \xi_2, \quad \bar{d}f = f_t \xi_1 + [S^{1-r}, f] \xi_2,
\]
with some integer \(r > 1\). Setting \(\phi = \varphi S^{-r}\) in (1.4) yields
\[
\varphi^{(1)}_t - \varphi_t = (\varphi^{(r)} - \varphi)(\varphi^{(1)} - \varphi - I) - (\varphi^{(1)} - \varphi - I)(\varphi^{(r)} - \varphi)^{(1-r)},
\]

---

36 Also see [42] for an analogous treatment of continuous, semi- and fully discrete matrix NLS equations.

37 In the scalar case, for positive solutions, writing \(u = a^2\), it becomes the (integrable) Kac-VanMoerbeke lattice equation \(a_t = a [(a^+)^2 - (a^-)^2]\). See [58], for example.
where \( \varphi^{(r)} = S^r \varphi S^{-r} \). In terms of \( u := \varphi^{(1)} - \varphi - I \), it takes the form

\[
u_t = \sum_{i=1}^{r-1} u^{(i)} u - u \sum_{i=1}^{-1} u^{(i)},
\]

which for \( r = 2 \) is \((9.1)\) (cf. \[12\]). The associated ‘Riemann system’, obtained from \((1.1)\), is

\[
\varphi_t = (\varphi^{(r)} - \varphi) \varphi, \quad \varphi^{(r)} = I + \varphi^{(r-1)} (I - \varphi^{-1}).
\]

The first is the semi-discrete Riemann equation \((4.2)\) (where \( S \) is replaced by \( S^r \)), the second a recursion relation.

Another point we concentrated on in this work is the implication

\[
d \bar{\phi} - (d \phi) \phi = 0 = \Leftrightarrow d \bar{\phi} + (d \phi) d \phi = 0,
\]

and there is a corresponding implication with the right hand side replaced by \((1.5)\). Special cases are the relation between the Burgers and the KP hierarchy, as well as an observation made in case of the sdYM equation in \[10\]. From the above implication it is clear that this is in fact a general feature, so there are counterparts in case of other integrable equations, and we demonstrated this for (matrix versions of) the two-dimensional Toda lattice, a variant of Hirota’s bilinear difference equation, \((2+1)\)-dimensional NLS and DS equations.

If a ‘Riemann system’ involves an ordinary (continuous) Riemann equation, perhaps via a reduction, we may expect to obtain a ‘breaking soliton’ case in the sense, e.g., of Bogoyavlenskii’s work (see \[22, 46\], in particular). Here the sdYM equation is the prime example. In the case of the \((2+1)\)-dimensional NLS equation, the resulting solutions are singular, rather than just ‘breaking’, however. Here we had to weaken the commonly used Hermitian conjugation reduction condition.

In contrast to the (continuous) Riemann equation, the integrable discrete versions possess infinite families of regular solutions that describe multi-kinks. The derivative, respectively discrete difference of the dependent variable, with respect to the ‘spatial’ independent variable, then has a multi-soliton character, with ‘solitons’ in the sense of localized objects. The appearance of very much the same families of solutions of seemingly quite different, integrable equations, like discrete Riemann equations, Toda, Hirota-Miwa, Burgers and KP equations, is traced back to simple relations between the associated ‘Riemann equations’.

In the examples presented in this work, we may think of exchanging \( d \) and \( \bar{d} \) in \((1.1)\) to get further integrable equations \[95\]. However, a simple computation, using the Leibniz rule, shows that

\[
d \bar{\phi} - (d \phi) \phi = 0 \quad \Leftrightarrow \quad d \bar{\phi} - (d \phi) \phi = 0,
\]

assuming that \( \phi \) has an inverse. A corresponding statement also holds for \((1.5)\):

\[
d [((d g) g^{-1})] = 0 \quad \Leftrightarrow \quad d [((d g) g^{-1})] = 0.
\]

In contrast, in case of \((1.4)\) an exchange of \( d \) and \( \bar{d} \) can lead to an inequivalent equation (cf. Section \[5.3\]). In particular, it may relate a member of a hierarchy with a member of a corresponding ‘negative’ or ‘reciprocal’ hierarchy, see \[60\]. \((9.2)\) shows that the corresponding ‘Riemann systems’ are simply related via \( \phi \mapsto \phi^{-1} \).

The linearization method of Section \[2\] does not extend to realizations of \((1.4)\) or \((1.5)\), but the binary Darboux transformation theorem (see Section \[3\]) applies to them and quickly leads to infinite (soliton-type) families of explicit solutions. This will be the subject of a follow-up work. In the present work we demonstrated that the binary Darboux transformation method also makes sense for a ‘Riemann system’. But a Cole-Hopf-type transformation is certainly the simpler way

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38This does not always lead to a meaningful equation.
to access the solutions. Needless to say, the list of examples presented in this work can easily be extended.

We should also mention the special case of Theorem 3.1 formulated in Corollary 3.5. If the seed solution satisfies \( d \phi_0 = 0 \), and if the bidifferential calculus extends to second order, this expresses a way to generate solutions of (1.4) or (1.5) from solutions of \( n \times n \) versions of the associated ‘Riemann system’, for arbitrary \( n \in \mathbb{N} \). This includes a construction of breaking multi-soliton-type solutions of the sdYM (also see [10]) and similar solutions of the (matrix) (2+1)-dimensional NLS equation.

It was not our aim in this work to obtain new integrable equations, but according to our knowledge the matrix version (4.24) of the fully discrete Burgers equation is new, and possibly also the integrable full discretization (4.13) of the Riemann equation (although this would be surprising). It is not difficult to construct further examples in the framework of bidifferential calculus, which have a chance to be new and to be of certain interest. The present work shows some universal structures and solution-generating methods with which they come along.

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**References**

[1] Levi D, Ragnisco O and Bruschi M 1983 Continuous and discrete matrix Burgers’ hierarchies *Nuovo Cimento B* **74** 33–51

[2] Bogoyavlenski˘ı O 1992 Breaking solitons. V. Systems of hydrodynamic type *Math. USSR Izvestiya* **38** 439–454

[3] Ferapontov E 1993 On the matrix Hopf equation and integrable Hamiltonian systems of hydrodynamic type, which do not possess Riemann invariants *Phys. Lett. A* **179** 391–397

[4] Ferapontov E 1994 Several conjectures and results in the theory of integrable Hamiltonian systems of hydrodynamic type, which do not possess Riemann invariants *Theor. Math. Phys.* **99** 567–570

[5] Guil F, Mañas M and Álvarez G 1994 The Hopf-Cole transformation and the KP equation *Phys. Lett. A* **190** 49–52

[6] Joseph K and Vasudeva Murthy A 2001 Hopf-Cole transformation to some systems of partial differential equations *Nonlinear Diff. Equ. Appl.* **8** 173–193

[7] Arrigo D and Hickling F 2002 Darboux transformations and linear parabolic partial differential equations *J. Phys. A: Math. Gen.* **35** L389–L399

[8] Santini P and Zenchuk A 2007 The general solution of the matrix equation \( w_t + \sum_{k=1}^{n} w_{x^k} \rho^{(k)}(w) = \rho(w) + [w,T \rho] \) *Phys. Lett. A* **368** 48–52

[9] Zenchuk A 2007 Matrix equations of hydrodynamic type as lower-dimensional reductions of self-dual type s-integrable systems [arXiv:0708.2050](http://arxiv.org/abs/0708.2050)

[10] Zenchuk A 2008 Lower-dimensional reductions of \( GL(M, \mathbb{C}) \) self-dual Yang-Mills equation: Solutions with break of wave profiles *J. Math. Phys.* **49** 063502

[11] Dimakis A and Müller-Hoissen F 2000 Bi-differential calculi and integrable models *J. Phys. A: Math. Gen.* **33** 957–974

[12] Dimakis A and Müller-Hoissen F 2009 Bidifferential graded algebras and integrable systems *Discr. Cont. Dyn. Systems Suppl.* **2009** 208–219
[13] Calogero F and Eckhaus W 1987 Nonlinear evolution equations, rescalings, model PDEs and their integrability. I Inv. Problems 3 229–262
[14] Calogero F 1991 C-Integrable nonlinear partial differential equations, I J. Math. Phys. 32 375–887
[15] Calogero F 1992 C-Integrable nonlinear partial differential equations in N + 1 dimensions J. Math. Phys. 33 1257–1271
[16] Dimakis A and Müller-Hoissen F 2013 Binary Darboux transformations in bidifferential calculus and integrable reductions of vacuum Einstein equations SIGMA 9 009 (31 pages)
[17] Matveev V and Salle M 1991 Darboux Transformations and Solitons Springer Series in Nonlinear Dynamics (Berlin: Springer)
[18] Dimakis A, Kanning N and Müller-Hoissen F 2011 The non-autonomous chiral model and the Ernst equation of general relativity in the bidifferential calculus framework SIGMA 7 118
[19] Maison D 1978 Are the stationary, axially symmetric Einstein equations completely integrable? Phys. Rev. Lett. 41 521–522
[20] Burtsev S, Zakharov V and Mikhailov A 1987 Inverse scattering method with variable spectral parameter Theor. Math. Phys. 70 227–240
[21] Bogoyavlenskii O 1990 Breaking solitons in 2 + 1-dimensional integrable equations Russ. Math. Surveys 45 1–86
[22] Clarkson P, Gordoa P and Pickering A 1997 Multicomponent equations associated to non-isospectral scattering problems Inv. Problems 13 1463–1476
[23] Hernández Heredero R, Levi D and Winternitz P 1999 Symmetries of the discrete Burgers equation J. Phys. A: Math. Gen. 32 2685–2695
[24] Hearan J 1977 Nonsingular solutions of $TA−BT=C$ Lin. Alg. Appl. 16 57–63
[25] Mason L and Woodhouse N 1996 Integrability, Self-Duality, and Twistor Theory (Oxford: Clarendon Press)
[26] Belinski V and Zakharov V 1978 Integration of the Einstein equations by means of the inverse scattering problem technique and construction of exact soliton solutions Sov. Phys. JETP 48 985–994
[27] Hirota R, Ito M and Kako F 1988 Two-dimensional Toda lattice equations Prog. Theor. Phys. Suppl. 94 42–58
[28] Nimmo J 2006 On a non-Abelian Hirota-Miwa equation J. Phys. A: Math. Gen. 39 5053–5065
[37] Hirota R 1981 Discrete analogue of a generalized Toda equation J. Phys. Soc. Japan 50 3785–3791
[38] Miwa T 1982 On Hirota’s difference equation Proc. Japan Acad. A 58 9–12
[39] Dimakis A and Müller-Hoissen F 2001 Bicomplex formulation and Moyal deformation of (2+1)-
dimensional Fordy-Kulish systems J. Phys. A: Math. Gen. 34 2571–2581
[40] Fordy A and Kulish P 1983 Nonlinear Schrödinger equations and simple Lie algebras Commun. Math. Phys. 89 427–443
[41] Ablowitz M, Prinari B and Trubatch A 2004 Discrete and Continuous Nonlinear Schrödinger Systems
(London Mathematical Society Lecture Note Series vol 302) (Cambridge: Cambridge University Press)
[42] Dimakis A and Müller-Hoissen F 2010 Solutions of matrix NLS systems and their discretizations: a
unified treatment Inverse Problems 26 095007
[43] Calogero F and Degasperis A 1976 Nonlinear evolution equations solvable by the inverse spectral trans-
form Il Nuovo Cim. B 32 201–242
[44] Zakharov V 1980 Solitons (Topics in Current Physics vol 17) ed Bullough R and Caudrey P (Berlin: Springer) pp 243–285
[45] Strachan I 1992 A new family of integrable models in 2 + 1 dimensions associated with Hermitian
symmetric spaces J. Math. Phys. 33 2477–2482
[46] Bogoyavlenskii O 1991 Breaking solitons. III Math. USSR Izvestiya 36 129–137
[47] Chae D, Córdoba A, Córdoba D and Fontelos M 2005 Finite time singularities in a 1D model of the
quasi-geostrophic equation Adv. Math. 194 203–223
[48] Jiang Z and Bullough R 1994 Integrability and a new breed of solutions of an NLS type equation in
2 + 1 dimensions Phys. Lett. A 190 249–254
[49] Dimakis A and Müller-Hoissen F 2009 Multicomponent Burgers and KP hierarchies, and solutions from
a matrix linear system SIGMA 5 002
[50] Dimakis A and Müller-Hoissen F 2011 KP line solitons and Tamari lattices J. Phys. A: Math. Theor. 44 025203
[51] Dimakis A and Müller-Hoissen F 2012 Associahedra, Tamari Lattices and Related Structures (Progress
in Mathematics vol 299) ed Müller-Hoissen F, Pullo J and Stasheff J (Basel: Birkhäuser) pp 391–423
[52] Sakhnovich A 1994 Dressing procedure for solutions of non-linear equations and the method of operator
identities Inv. Problems 10 699–710
[53] Marchenko V 1988 Nonlinear Equations and Operator Algebras Mathematics and Its Applications (Dor-
drecht: Reidel)
[54] Leznov A and Yuzbashyan E 1997 Multi-soliton solutions of the two-dimensional matrix Davey-
Stewartson equation Nucl. Phys. B 496 643–653
[55] Gilson C and Macfarlane S 2009 Dromion solutions of noncommutative Davey-Stewartson equations J. Phys. A: Math. Theor. 42 235202
[56] Macfarlane S 2010 Quasideterminant solutions of noncommutative integrable systems PhD thesis University of Glasgow, UK
[57] Davey A and Stewartson K 1974 On three-dimensional packets of surface waves Proc. R. Soc. London A 338 101–110
[58] Liu H 2008 On discreteness of the Hopf equation Acta Math. Appl. Sinica 24 423–440
[59] Suris Y 2003 The Problem of Integrable Discretization: Hamiltonian Approach (Progress in Mathematics vol 219) (Basel: Birkhäuser)
[60] Dimakis A and Müller-Hoissen F 2010 Bidifferential calculus approach to AKNS hierarchies and their
solutions SIGMA 6 055