Emergent electromagnetism in solids

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Abstract
The electromagnetic field (EMF) is the most fundamental field in condensed-matter physics. Interaction between electrons, electron–ion interaction and ion–ion interaction are all of electromagnetic origin, while the other three fundamental forces, i.e. the gravitational force and weak and strong interactions, are irrelevant in the energy/length scales of condensed-matter physics. Also the physical properties of condensed matter, such as transport, optical, magnetic and dielectric properties, are almost described as their electromagnetic responses. In addition to this EMF, it often happens that the gauge fields appear as the emergent phenomenon in the low-energy sector due to the projection of the electronic wavefunctions onto the curved manifold of the Hilbert sub-space. These emergent EMFs play important roles in many places in condensed-matter physics including the quantum Hall effect, strongly correlated electrons and also in non-interacting electron systems. In this paper, we describe the fundamental idea behind it and some of its applications studied recently.

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(Some figures may appear in colour only in the online journal)

1. Introduction and basic concepts

Electromagnetism is the first gauge theory which was recognized in physics, and essentially contains the principle of relativity. Most of the physical properties of solids are regarded as the responses of systems to the electromagnetic field (EMF). For example, the transport phenomena are described by the currents induced by the electric field, magnetism is the behavior of magnetization in response to the magnetic field and optical properties are described by the electric polarization driven by the EMF of light. On the side of the electronic system, the electron–electron and/or electron–ion interactions mediated by the EMF, i.e. mostly the Coulomb interaction, determine the quantum state even without the external EMF as a probe. This interaction often leads to the emergent electromagnetic fields (EEMFs) as will be described in more detail below. The other aspect of electromagnetism in solids is the various cross-correlation effects. Namely, various different combinations of inputs and responses have been explored. For example, an applied magnetic field can induce electric polarization, or magnetization emerges under the electric field in the magneto-electric (ME) effect. This effect was proposed a long time ago [1], but has been very weak. An explosion of research was triggered by the discovery of multiferroic materials showing both magnetic order and ferroelectricity, which are coupled strongly to each other (for recent reviews, see [2, 3]). These phenomena are driven by the internal degrees of freedom of electrons, i.e. the spin, charge and orbital, which are again described coherently by the EEMFs. This EEMF is analogous to the usual EMF, but is much richer because of the following aspects [4, 5]. (i) It is defined on the lattice because of the crystal structure and Bloch waves in solids, and the lattice gauge theories are relevant. Therefore, various topological defects play important roles. (ii) One can generalize the space where the EEMF is defined including the real space, the momentum (k-)space and other parameter spaces. (iii) The gauge group is not only \( U(1) \) as in the case of EMF, but can be more general such as the SU(2) corresponding to Kramer’s doublet in the time-reversal symmetric system and the \( U(N) \) corresponding to the \( N \)-band system. (iv) Topological terms such as the Chern–Simons term, the Wess–Zumion–Witter (WZW) term and the \( \theta \) term often appear in the effective action for EEMF. Therefore, solids can be an ideal arena to study various concepts and techniques in quantum field theory [6].
1.1. Berry phase of electrons in a perfect crystal

The Bragg diffraction of the electrons by the periodic electric field of the ions in a crystal leads to the Bloch states as described by

\[
\psi_{\alpha\kappa}(\mathbf{r}, s) = e^{ik \cdot \mathbf{r}} u_{\alpha\kappa}(\mathbf{r}) \chi_{\sigma}(s),
\]

where \(u_{\alpha\kappa}(\mathbf{r})\) is the periodic function with respect to the translation of lattice vectors, and \(\chi_{\sigma}(s)\) is the spin wavefunction for \(\sigma = \uparrow, \downarrow\) with the spin coordinate \(s = \pm\). The band index \(n\) and the crystal magnetic moment \(\mathbf{k}\) are good quantum numbers and the energy eigenvalues, \(\epsilon_n(\mathbf{k})\), with different \(n\)s are usually separated by band gaps. This means that the electrons in a solid are confined to each band in the lower energy sector than the band gaps, which corresponds to the projection of the wavefunctions to the sub-space of the Hilbert space. This sub-space is usually ‘curved’ and is represented by the gauge connection. (This is an application of the Berry connection to Bloch electrons [8].) It is given by the expression

\[
a_{\alpha\kappa}(\mathbf{k}) = -i \langle u_{\alpha\kappa} | \partial_{\mathbf{k}} | u_{\alpha\kappa} \rangle
\]

with \(a = x, y, z\). This quantity appears in the inner product of the two wavefunctions for the two neighboring \(\mathbf{k}\)-points as

\[
\langle u_{\alpha\kappa} | u_{\alpha\kappa + \Delta\mathbf{k}} \rangle = \exp(i\epsilon_{\alpha\kappa}(\mathbf{k}) \cdot \Delta\mathbf{k}).
\]

This Berry connection leads to the concept of Berry curvature corresponding to the magnetic field in the momentum space as

\[
b_{\alpha\kappa}(\mathbf{k}) = \epsilon_{\alpha\kappa\lambda} \partial_{\mathbf{k}} a_{\alpha\kappa}(\mathbf{k})
\]

with the totally antisymmetric tensor \(\epsilon_{\alpha\kappa\lambda}\).

What are the physical consequences of this Berry phase? To answer this question, it is useful to consider the motion of a wavepacket made of Bloch states, where the position and momentum of a particle are defined within the accuracy consistent with the uncertainty principle [9]. In the presence of the Berry connection, the position operator \(\mathbf{x}\) originally defined as \(x_\mu = \partial \epsilon_\mu / \partial k_\mu\) should be generalized to the gauge covariant form \(x_\mu = \partial / \partial x_\mu + \epsilon_{\mu\nu\lambda} a_{\nu\lambda}(\mathbf{k})\). This is dual to the momentum operator \(\pi_\mu\) in the presence of the magnetic field, i.e. \(\pi_\mu = p_\mu + eA_\mu(\mathbf{x}) = -i\partial / \partial x_\mu + e A_\mu(\mathbf{x})\), where \(A_\mu(\mathbf{x})\) is the vector potential of the EMF. Analogously to the commutator \([\pi_\mu, \pi_\nu] = i\epsilon_{\mu\nu\lambda} B_\lambda(\mathbf{x})\) with \(B_\lambda(\mathbf{x})\) being the \(\lambda\)-component of the magnetic field, one can derive \([x_\mu, y_\nu] = i\epsilon_{\mu\nu\lambda} b_\lambda(\mathbf{k})\). Namely, the real-space coordinates do not commute with each other. With this commutator, the equation of motion for the wavepacket reads [9, 10]

\[
\frac{d\mathbf{x}_\mu}{dt} = -i[x_\mu, \mathbf{H}],
\]

\[
\frac{d\pi_\mu}{dt} = -i[\pi_\mu, \mathbf{H}].
\]

Here the Hamiltonian \(\mathbf{H}\) is given by \(\mathbf{H} = \epsilon_\mu(\mathbf{k}) + V(\mathbf{x})\) with \(\epsilon_\mu(\mathbf{k})\) being the energy dispersion of the band \(n\) of interest, and \(V(\mathbf{x})\) is the slowly varying external potential. Putting this into equations (5), we obtain

\[
\frac{dx_\mu}{dt} = \frac{\partial \epsilon_\mu(\mathbf{k})}{\partial k_\mu} - i[x_\mu, x_\nu] \frac{\partial V(\mathbf{x})}{\partial x_\nu} = \frac{\partial \epsilon_\mu(\mathbf{k})}{\partial k_\mu} + \epsilon_{\mu\nu\lambda} b_\lambda(\mathbf{k}) \frac{\partial V(\mathbf{x})}{\partial x_\nu} = \frac{\partial \epsilon_\mu(\mathbf{k})}{\partial k_\mu} + (\mathbf{b} \times \mathbf{F})_\mu,
\]

\[
\frac{dk_\mu}{dt} = -\frac{\partial V(\mathbf{x})}{\partial x_\mu} = F_\mu,
\]
where $F = -\nabla V(x) = -eE$ is the force acting on the electron. In equation (6), the real-space quantities $x_{\mu}$, $A_\mu(x)$, $V(x)$ and those in the momentum space $k_{\mu}$, $a_{\mu}(k)$, $\epsilon_{\mu}(k)$ enter the equations of motion in a symmetric way. Therefore one can translate the phenomena in real (momentum) space to those in the momentum (real) space. Note, however, that compared with the real-space magnetic field, which is divergence-free, i.e. $\nabla \cdot \mathbf{B}(x) = 0$, corresponding to the absence of the magnetic monopole in real space, $\nabla \cdot \mathbf{b}(k)$ can be non-zero in momentum space. Another important remark is that the symmetries give the following constraint. The time-reversal symmetry $T$ gives the relation $b_\mu(k) = -b_\mu(-k)$, while the inversion symmetry $I$ $b_\mu(k) = b_\mu(-k)$. Therefore, when both $T$ and $I$ symmetries are there, there is no Berry curvature $b_\mu(k)$. Also, in the non-centrosymmetric system where $I$-symmetry is absent, $b_\mu(k)$ can be non-zero although the contributions from $k$ and $-k$ cancel each other.

1.2. Emergent SU(2) gauge field from the Dirac equation

Starting from the Dirac equation, the projection of the wavefunctions onto the low-energy sub-space leads to gauge structure analogous to electromagnetism. This formalism is based on the real-space picture, and is complementary to the discussion in the last subsection where the momentum space geometry has been considered. When the SU(2) spin space is preserved, it leads to the SU(2) non-Abelian gauge field, which is coupled to the spin current, corresponding to the SOI.

We start with quantum electrodynamics (QED), where the Dirac relativistic electrons and their charge current are based on the real-space picture, and is complementary to the wavefunctions onto the low-energy sub-space leads to gauge covariant derivatives with $\mu = 0, 1, 2, 3$. The four-component charge current density is defined as

$$ j^\mu = -\frac{\partial L}{\partial A_\mu} = -e\bar{\psi}\gamma^\mu \psi, $$

whose zero-component is the charge density $\rho$, while the spatial components are the current density $\mathbf{j}$. From the gauge invariance, the conservation law of the charge is derived through Noether’s theorem as

$$ \partial_\mu j^\mu = 0. $$

By taking the variation, one can derive the Maxwell equation

$$ \partial_\mu F^{\mu\nu} = j^\nu $$

and the Dirac equation

$$ [i\gamma^\mu \hat{D}_\mu - m]\psi = 0. $$

As is well known, the solutions to the Dirac equation are classified into two classes, i.e. the positive energy and negative energy states separated by twice the rest mass energy of the electrons $2mc^2$. Since the energy $mc^2$ is of the order of MeV, for the low-energy phenomena typically of the order of $\sim eV$, the negative energy states are not relevant. Therefore, the non-relativistic Schrödinger equation is usually used, which describes the dynamics of the two-component spinor wavefunctions for positive energy states. However, one needs to take into account one important aspect of ‘projection’. Namely, the negligence of the negative energy states means the projection of the wavefunctions onto the positive energy states, i.e. the sub-Hilbert space. Usually the sub-space is not flat but curved, and associated geometrical structure is introduced. The derivation of the effective Lagrangian describing the low-energy physics is achieved by the expansion with respect to 1/$(mc^2)$, and the result reads [11, 12]

$$ L = \bar{\psi} i\gamma^\mu \hat{D}_\mu \psi + \frac{1}{2m} \left[ q^2 A^2 \right] + \frac{1}{4} F^2 \psi $$

where $\psi$ is now the two-component spinor and $D_\mu = \partial_\mu + ieA_\mu \gamma^\mu$ and $D_1 = \partial_1 - ieA_1 \gamma^1$ for $i = 1, 2, 3$ are the gauge covariant derivatives with $\mu$ being the quantity proportional to the Bohr magneton [11, 12]. $A_\mu$ is the vector potential for EMF, while the SU(2) gauge potentials are defined as $A^a_\mu = A_\mu^a = \epsilon_{abc} E_c$. The former is coupled to the charge current and the latter to the four-component spin current $j^a_\mu = \bar{\psi} \sigma^a \psi$, $j^μ = \frac{e}{2mc} \left[ \bar{\psi} \gamma^\mu D_\mu \psi - D_\mu \bar{\psi} \gamma^\mu \psi \right]$. Note that the spin current is the tensor quantity with one suffix for the direction of the spin polarization and the other for the direction of the flow. Note an important difference between the EMF and the SU(2) gauge field. The former has gauge symmetry, i.e. the freedom to choose the arbitrary gauge for the vector potential $A_\mu$, while the ‘vector potential’ $A^a_\mu$ for the latter is given by the physical field strengths $\mathbf{B}$ and $\mathbf{E}$. Actually, the relation $\partial_\mu A^a_\mu = 0$ holds. Therefore, the SU(2) gauge symmetry is absent. This is the basic reason why the spin is not conserved in the presence of the relativistic SOI. Instead, the spin current is ‘covariantly’ conserved and satisfies [11, 12]

$$ D_\mu J^a_\mu + \mathbf{D} \cdot \mathbf{J}^a = 0. $$

This means that in the co-moving frame the spin is conserved, while in the laboratory frame the spin source or sink appears when the electron forms a loop and comes back to the same position in space since the frame already changes [12]. (Note that the usual SU(2) gauge theory is a nonlinear theory and the gauge field is ‘charged’, and the sum of the spin current by the matter field and the gauge field is conserved in the non-Abelian gauge theory as Yang–Mills first showed [6]. However, the SU(2) gauge invariance is absent here and also the action $Tr[F_{\mu\nu}^a]$. There is no spin current from the non-Abelian gauge field when it is assumed to be the frozen background field.)

When magnetic ordering occurs, the wavefunctions are further projected onto the spin component at each site, which is described by the field operator $\psi_\sigma$ of electrons decomposed into

$$ \psi_\sigma(x) = \bar{z}_\sigma(x) f(x) $$

with $f$ being the spinless fermion corresponding to the charge degrees of freedom while $z_\sigma$ is the two-component
spinor corresponding to the direction of the magnetization at \( \mathbf{x} \). Putting equation (14) into equation (12), we obtain the effective Lagrangian for the \( f \)-field as

\[
L_{\text{eff}} = \psi^\dagger \left[ i\beta_0 + a_0^B + (\mathbf{v} \times \mathbf{A})_0 + \frac{(\mathbf{v} \times \mathbf{B}) + i\mathbf{A}}{2\mu} \right] \psi,
\]

where \( a_0^B = i (z|\partial_\mu |z) \) is the U(1) field originating from the Berry connection of the spin wavefunctions, and \( a_0^{SO} = A_0^g (z|x^\mu |z) \), \( a_0^{SO} = A_0^g (z|z^\mu |z) \) are U(1) fields originating from the SOI.

Here a remark about the geometrical meaning of the gauge field \( a_0^B \) is in order [13]. The magnetic flux made from \( a_0^B \), i.e. \( \mathbf{b} = \nabla \times a_0^B \), corresponds to the solid angle subtended by the spins. Namely, the scalar spin chirality given by \( \mathbf{b} \cdot (\mathbf{S}_i \times \mathbf{S}_j) \) is reduced to \( a_0^B \) along the direction normal to the plane made by the three sites \( i, j, k \) in the continuum limit (see figure 2). Therefore, the effective magnetic field is produced when the spin structure is non-coplanar. In summary, there are three 'electromagnetic fields' in magnetic systems, i.e. (i) \( a_0^B \) due to the Berry phase associated with the non-coplanar spin structure, (ii) \( a_0^{SO} \) from the SOI and (iii) the usual Maxwell EMF \( \mathbf{A} \).

### 1.3. Emergent electromagnetic field in correlated electronic systems

Up to now, we have considered the gauge fields in the band structure or the magnetically ordered state, i.e. the single-particle properties, where the gauge fields are the static background fields. The electron correlation effect corresponds to the fluctuating spin beyond the mean-field theory, and accordingly the gauge fields become dynamical [14]. This issue has been studied extensively in research on the low-energy physics of the quantum antiferromagnet [4, 15, 16]. Historically, motivated by studies on strongly correlated systems and quantum Hall systems from the gauge theoretical viewpoint during the 1990s, the conventional materials, which can be well described by band theory and/or the mean field theory, have been revisited from this new perspective. Below, we take some of the examples where new physical effects have been explored in conventional systems from the viewpoint of gauge fields.

### 2. Materials and phenomena of emergent electromagnetism

As described above, there are several sources of the EEMF, and correspondingly there are so many related phenomena in condensed matter. A common feature of these phenomena is that the topological currents play an essential role. In solids, there are many imperfections such as defects and impurities, which cause scattering of electrons and dissipation. This dissipation, i.e. Joule heating, is balanced with the energy supplied from the external electric field. This is the usual Ohmic current. In the superconducting state, the macroscopic quantum coherence and the associated 'rigidity' against the phase twist by the external magnetic field produce the superconducting current. This current neither decays nor causes any dissipation since the current-flowing state is in thermal equilibrium and is separated by the macroscopic energy barrier from the zero current state. To this list we would add here a third category of current, i.e. the topological current, induced by the gauge field or curvature of the Hilbert space, i.e. EEMF. This third category includes the quantum Hall current and the polarization current in ferroelectric materials, which are related to the Berry phase. This topological current does not require off-diagonal long-range order (ODLRO), and can exist in the normal states even at room temperature. Therefore, the topological currents will be of vital importance when applications to electronics are considered. We will describe below several novel phenomena driven by this topological current in solids.

#### 2.1. Multiferroics

The minimal coupling to the SU(2) gauge field \( A_\mu^a \) in equation (12) or in equation (15) for magnets means a close relation between the spin current \( j_\mu^a \) and the electric polarization \( \mathbf{P} \). More explicitly, \( \mathbf{P} \) is given by the derivative of the Lagrangian to \( \mathbf{E} \), i.e.

\[
P_i \propto \epsilon_{ia\sigma} j_\sigma^a,
\]
which means that the spin current produces the ferroelectric moment. Then the next problem is how one can produce the spin current. In the magnetically ordered state, the expectation value of the spin current \( \langle j_{\mu}^p \rangle \) in the ground state can be non-zero for the non-collinear spin configuration. Consider the expression for the spin current operator in the tight-binding model

\[
j_{\mu}^p = i \sum_i t_{ij} c_{i\alpha}^\dagger c_{j\beta} (\sigma^\nu)_{\alpha\beta} + \text{h.c.},
\]

(19)

where \( t_{ij} \) is the transfer integral, \( \sigma^\nu (\nu = x, y, z) \) the Pauli matrices and \( \alpha, \beta \) the spin indices. For simplicity, let us consider the case of the spin ordering within the \( xy \)-plane, and the corresponding wavefunction \( \chi_i = \left( \frac{1}{\sqrt{2}}, \frac{\sqrt{2}}{\sqrt{2}} \right) \). Taking the expectation value with this wavefunction assuming the half-filling, one obtains

\[
\langle j_{\mu}^p \rangle \propto \frac{t_{ij}^2}{U} \sin(\phi_i - \phi_{ij}),
\]

(20)

where we employed the perturbation theory in \( t/U \) (\( U \) is on-site Coulomb interaction or Hund’s coupling). This is analogous to the Josephson effect in a superconductor with \( \phi \) being the phase of the order parameter, and the spin supercurrent in the magnet is induced by the tilted spins. Putting this into equation (18), \( \langle P_i \rangle \propto \epsilon_{\mu\nu} \sin(\phi_i - \phi_{ij}) \). A more general expression is easily obtained for the polarization \( P_{ij} \) obtained by the two spins \( S_i \) and \( S_j \) as

\[
P = \eta \epsilon_{ij} \times (S_i \times S_j)
\]

(21)

with \( \eta \) being the coupling constant related to SOI, and \( \epsilon_{ij} \) is the unit vector connecting the two sites \( i \) and \( j \) [17, 18]. The vector product of the two spins \( \chi_{ij} = S_i \times S_j \) is called vector spin chirality. This quantity is even with respect to the time-reversal operation \( T \), similar to the spin current.

This generic argument is applied to the real materials as follows. Compared with the free electrons in vacuum, the strength of the relativistic SOI can be enhanced by a factor of \( \sim 10^5 \), which is the ratio of the rest mass of the electrons \( mc^2 \) to the band gap. For 3d electrons in transition metal atoms, the SOI \( \lambda \) is typically of the order of \( \sim 20–50 \) meV, while it becomes \( \sim 0.5 \) eV for 5d electrons. The electron correlation energy, on the other hand, decreases from 3d to 5d since the wavefunction is more and more expanded. In the cubic crystal field in transition metal oxides, the fivefold degeneracy of \( d \)-orbitals is lifted due to the ligand field of oxygens. As a result, threefold degenerate \( t_{2g} \)-orbitals (\( xy-,yz-,zx \)-orbitals) with lower energy, and doubly degenerate \( e_g \)-orbitals (\( x^2−y^2, 3z^2−r^2 \)-orbitals) with higher energy are formed. The matrix elements of the orbital angular momentum \( \ell \) are zero within the \( e_g \)-orbitals. On the other hand, they are non-zero among \( t_{2g} \)-orbitals and also between the \( e_g \) and \( t_{2g} \)-orbitals. This SOI is the origin of the relativistic coupling between magnetism and electric polarization. To give a more explicit prediction for transition metal oxides, the cluster model of magnetic ions sandwiching an oxygen ion has been studied theoretically by taking into account the SOI when deriving the super-exchange interaction [17] (see figure 3). As mentioned above, the spin current flows between the two non-collinear spins \( S_i \) and \( S_j \), which produces the electric polarization \( P \) as given by

\[
P = -\frac{4eV}{9} \Delta \epsilon_{ij} \times (S_i \times S_j),
\]

(22)

where \( \Delta = (p_i | d_{xz} | p_j) \), and \( \Delta(V) \) is the energy difference (hybridization) between the \( p \)-orbitals and the \( d \)-orbitals. The SOI interaction is implicitly included in this model by picking up one doublet after splitting by the SOI. Applying this result to various magnetic structures, one can easily predict the presence or absence, and the direction of the polarization. This theory does not contradict the symmetry argument developed for magnets [19, 20], but stresses the physical mechanism of the spin current-induced polarization.

A recent experimental breakthrough is the discovery of multiferroic behavior in \( R_{MnO_3} \) (\( R = \text{Gd, Tb, Dy} \)) [2, 3]. In these materials, spontaneous electric polarization \( P_s \) appears accompanied by magnetic order, and they are necessarily strongly coupled. It has also been revealed that the magnetic structure that induces the electric polarization is the cycloidal spiral in good accordance with the theoretical prediction above [21–23]. Figure 3 shows the representative multiferroic behavior of DyMnO\(_3\). In this material, the spin rotation plane flop from the \( bc \)- to \( ab \)-planes and accordingly the direction of the electric polarization change from the \( c \)- to \( a \)-axes [3].

Now, extensive experimental studies have been done to explore the multiferroic materials, e.g. Ni\(_3\)V\(_2\)O\(_8\) [24], Ba\(_3\)Sr\(_{1.5}\)Zn\(_2\)Fe\(_{12}\)O\(_{22}\) [25], CoCr\(_2\)O\(_4\) [26], MnWO\(_4\) [27], CuFeO\(_2\) [28], LiCuV\(_2\)O\(_4\) [29] and LiCuO\(_2\) [30] have been found to be multiferroics. Namely, multiferroicity is not a special phenomenon but is a rather ubiquitous phenomenon in Mott insulators. These experimental findings urged systematic theoretical studies on the microscopic mechanisms of the spin-related electric polarization [31, 32]. The perturbative approach in both \( V/\Delta \) and \( \lambda/\Delta \) is employed, where \( V \) and \( \Delta \) represent the transfer integral and the charge transfer energy between the transition-metal (TM) \( d \)- and ligand (L) \( p \)-orbitals. This analysis concludes that the polarization \( P_{r \perp z} \) appearing on the bond between the sites \( r \) and \( r + e \) is given by

\[
P_{r \perp z} = \frac{P^\text{m} \alpha \cdot \mu_r \cdot \mu_{r\perp z} + P^\text{q} \epsilon \cdot (\mu_r \times \mu_{r\perp z})}{(V/\Delta)^3},
\]

(23)

where \( \mu_r \) is the spin direction at \( r \). The first term \( P^\text{m} \propto (V/\Delta)^3 \) is the polarization due to the exchange striction,
By a careful analysis of the super-exchange processes in Mott insulators taking into account the quantum dynamics of the spins in the intermediate states, we have derived the following effective Lagrangian [33]:

$$L_E = - \int d^2r T_{ab} E_a e_b,$$

(24)

where $E$ is the electric field and

$$e = \frac{1}{2} \sin \theta (\partial_\theta \nabla \phi - \partial_\phi \nabla \theta)$$

(25)

is the electric field component of EEMF associated with the spin structure derived from the $U(1)$ Berry connection. Here $n = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta)$ is the unit vector of the magnetization in the continuum approximation, and $T_{ab}$ is the tensor which depends on the details of the atomic configurations. One important remark here is that $T_{ab}$ vanishes in the single-orbital case since the spins form singlet and has no quantum dynamics in the intermediate states in the processes of the super-exchange interaction. However, in most of the cases, the orbital degrees of freedom are active and $T_{ab}$ is non-zero. This coupling is analogous to the spin motive force in the metallic ferromagnetic systems by which the domain wall or vortex motions produce the voltage drop [34–36].

2.2. Topological Hall effects

As explained in equation (6), the most natural phenomena expected from the gauge fields in momentum space are related to the transverse motion of the electrons to the external electric field, i.e. the Hall effects. Contrary to the usual Hall effect driven by the Lorentz force, we call the Hall effects originating from the gauge fields as topological Hall effects, and we describe below some of the examples.

2.2.1. Spin–orbit interaction and the anomalous Hall effect. Hall discovered the Hall effect in metallic ferromagnetic systems due to the spontaneous magnetization instead of the external magnetic field [37]. This effect is called the anomalous Hall effect (AHE) and its mechanism has been a controversial issue for more than a century. It is agreed that the effect is due to the SOI combined with the spontaneous magnetization, but the issue was the role of impurity scatterings. The intrinsic mechanism was first proposed by Karplus–Luttinger (KL) [38], who considered the anomalous velocity for the first time discussed in section 1. However, their theory has been criticized by Smit [39] saying that impurity scattering, which is inevitable and also indispensable for reaching the steady state under electric field, invalidates the KL theory. This posed the question as to whether the dissipationless topological current can survive even in the presence of the dissipative Ohmic current. Smit [39] proposed the skew scattering mechanism, where the impurity scattering in the presence of SOI gives the asymmetry of the transition rates between $W_{k-k'}$ and $W_{k'-k}$. This breakdown of the detailed balance leads to the net current perpendicular to the applied electric field, i.e. the Ohmic transport current is slightly distorted in the perpendicular direction. Later, Berger [40] proposed another mechanism called side-jump,
where the transverse shift of the electron trajectory occurs at the scattering in the presence of the SOI. These two mechanisms are called extrinsic as opposed to the intrinsic one by KL. Recent advances in this problem are twofold [37]. One is the recognition that the band structures of the ferromagnetic materials can be topologically non-trivial characterized by Chern numbers. Haldane [41] was the first to note that the quantum Hall effect can be realized without the Landau level formation in a tight-binding model on a lattice with the complex transfer integrals. What has been recognized recently is that this scenario can be realized in the ferromagnetic materials with the SOI [42]. In this case, even though the overlaps of the band dispersions make the system metallic, one can define the Chern number of each band when the gap opens at each \( k \)-point. Therefore, the ferromagnetic metal could be an implicit quantum Hall system with \( k \)-dependent Chern numbers. The other is the state-of-art first-principles calculations of electronic states and the Hall response as well as the accurate experimental measurements of the AHE in various materials [37]. In short, the transverse conductivity \( \sigma_{xy} \) due to the intrinsic mechanism can be written in terms of the Berry phase curvature, i.e. the magnetic flux \( b_{nz}(k) \) for the Bloch wave state with crystal momentum \( k \) and the band index \( n \) [43]:

\[
\sigma_{xy} = \frac{e^2}{2\pi h} \sum_{nk} f(\varepsilon_n(k)) b_{nz}(k),
\]

where \( f(\varepsilon_n(k)) \) is the Fermi distribution function for the energy \( \varepsilon_n(k) \) of the Bloch state, and the vector potential \( a_{n\mu}(k) \) is defined in equation (2), and \( b_{n\mu}(k) = \nabla_k \times a_{n\mu}(k) \). The meaning of this expression is that the Hall current is the sum of the anomalous velocities of Bloch states occupied by electrons in the equilibrium distribution. Therefore the transverse conductivity \( \sigma_{xy} \) represents the gauge field distribution in the momentum space. In particular, when the chemical potential is in the gap, the \( K \)-integral in equation (26) is over the whole first Brillouin zone (first BZ), and one might think it is zero due to the periodicity with respect to \( k \rightarrow k + G \) with \( G \) being the reciprocal lattice vector. Namely, the contour integral over the boundary of the first BZ appears to cancel. However, this is not always the case, and the integral is \((e^2/h) \times \text{integer}\), especially in two dimensions. This integer is the topological number called the Chern number for the U(1) fiber bundle of the Bloch wavefunction [43]. The finite Chern number means that the phase of the Bloch wavefunction cannot be defined continuously over the whole first BZ analogous to the Yang–Wu construction of Dirac monopole [4, 5, 44]. Here, plural patches need to be introduced to cover the first BZ. The relevance of the Dirac monopole to the Hall effect is understood more explicitly in the following three-dimensional (3D) model:

\[
H(k) = \sum_{a=x,y,z} k_a \sigma_a.
\]

The corresponding magnetic field associated with the Berry phase reads

\[
b_{\pm}(k) = \pm \frac{k}{2|k|^3},
\]

where \( \pm \) corresponds to the two bands with energy \( \varepsilon_{\pm}(k) = \pm |k| \). This means that the magnetic monopole exists at the band crossing point \( k = 0 \), and \( \nabla_k \cdot b_{\pm}(k) = \pm 2\pi \delta(k) \) has the delta-functional singularity. The contribution from the fixed \( k_c \) is given by

\[
\sigma_{xy}(k_c) = \pm \frac{e^2}{2h} \text{sgn} k_c.
\]

For the 2D system where \( k_c \) is regarded as a parameter \( m \) characterizing the system, it describes the quantum Hall system with \( \sigma_{xy} = \pm \frac{e^2}{2h} \). The factor \( 1/2 \) is not realized in a real system because the Dirac fermion appears always in the pair, i.e. species doubling, in the first BZ and hence the Chern number is an integer.

Essential to AHE is the fact that the non-trivial topological structure described above is not special as in the case of a quantum Hall system, but is generic for magnets in the presence of the SOI. A simple 2D three-band model for \( t_{2g} \)-orbitals was constructed, and the uniform magnetization produces the non-zero Chern number for each band [42]. In the absence of the SOI, the up and down spin bands are independent of each other except the exchange energy splitting, and there occur several band crossing points but no finite Chern numbers. Then the SOI lifts the degeneracies to generate the mass term \( m \) as discussed above and produce the finite Chern numbers. This is the reason why the SOI cannot be treated perturbatively in sharp contrast to the previous theoretical treatments. This non-trivial behavior has been found both in the first-principles band calculation and in the experiment in SrRuO\(_3\), a metallic ferromagnet with \( T_c = 130 \text{K} \) [45]. In figure 5 are shown the transport properties of SrRuO\(_3\). Figure 5(c) shows the temperature dependence of the Hall resistivity, which is dominated by the anomalous contribution. Just below \( T_c \), it is negative and turns into positive and again has the maximum. This characteristic behavior strongly suggests that the perturbative expansion in \( \lambda M \) (\( \lambda \) is SOI and \( M \) is magnetization) is not allowed and the AHE is a fingerprint of the Berry phase of the Bloch wavefunction. We plot in figure 5(d) the transverse conductivity \( \sigma_{xy} \) as a function of the spontaneous magnetization compared with the first-principles band structure calculation. This non-monotonic temperature dependence including the sign change is due to the band crossings acting as the magnetic monopoles in momentum space. Accordingly, figure 6 shows the distribution of the Berry curvature \( b_{nz}(k) \) as a function of \( (k_x, k_y) \) at fixed \( k_z = 0 \). It shows a sharp peak around \( k_c = k_y = 0 \) and the ridges along \( k_x = \pm k_y \). This sharp peak represents the monopole corresponding to the band crossing. When the Fermi energy \( \varepsilon_F \) is near the magnetic monopoles, the electrons are subject to the strong gauge field and hence the contribution to the Hall constant is resonantly enhanced. The integral over \( k_c \) leads to the partial cancellation of the positive and negative contributions, but still a rapid change of \( \sigma_{xy} \) as a function of \( \varepsilon_F \) results. This explains both the first-principle band calculation and experimental results, which are shown in figures 5(c) and (d). A similar conclusion is obtained also for the AHE in Fe [46], where sharp spiky structures of the Berry phase distribution have been found.

An important recent development is the experimental measurements of dynamical AHE, i.e. \( \sigma_{xy}(\omega) \) in
AHE in SrRuO₃. (a) Magnetization, (b) resistivity and (c) transverse resistivity as functions of temperature. Panel (d) shows the transverse conductivity as a function of spontaneous magnetization compared with the first-principles band structure calculations. The non-monotonic dependence is due to the magnetic monopoles in momentum space. Reproduced with permission from [45].

The low-frequency structures of the dynamical AHE are being revealed. In particular, the dominance of the contributions from the band crossings is expected to appear in the \( \omega \) dependence of \( \sigma_{xy}(\omega) \) as well as the non-monotonic temperature dependence of \( \sigma_{xy}(T) \). Figure 7 shows an example of the low-frequency data for AHE in SrRuO₃. It is clearly seen that a non-trivial structure exists at around 4 meV, which is well fitted by a model assuming the band crossing near the Fermi energy.

Until now, it is argued that in some materials, the AHE can be explained by the intrinsic mechanism based on the Berry phase concept, but the controversy between the intrinsic and the extrinsic needs to be resolved. As to this problem, again the roles of the enhanced Berry curvature near the band crossings and the topological nature are crucial. In [49], a model with a band anti-crossing is considered and its Hall response is calculated taking into account the disorder scatterings. The Hall conductivity \( \sigma_{xy} \) is obtained as a function of \( \sigma_{xx} \) representing the strength of the disorder. Figure 8 summarizes the results including the various experimental data plotted. (Note that the calculation has been performed for a 2D model, and \( \sigma_{xx} \) are scaled by the lattice constant along the \( z \)-direction, which is assumed to be \( \approx 0.4 \) nm.) This result captures the respective role of the transport and topological currents. In the very clean region where \( \sigma_{xx} > 10^6 \, \text{Ω}^{-1} \, \text{cm}^{-1} \), the skew scattering mechanism (extrinsic mechanism) is dominant and hence \( \sigma_{xy} \propto \sigma_{xx} \). When the disorder strength becomes larger, the transport current is suppressed much more rapidly compared with the topological current, the latter...
of which is ‘protected’ topologically, leading to a plateau region in the region $10^4 \Omega^{-1} \text{cm}^{-1} < \sigma_{xy} < 10^6 \Omega^{-1} \text{cm}^{-1}$. In this intermediate region, to which many of the metallic ferromagnets belong, the intrinsic mechanism is dominant and hence the first-principles calculations can predict the AHE semi-quantitatively. When the disorder becomes even stronger, $\sigma_{xy}$ begins to decrease obeying the approximate scaling law $\sigma_{xy} \propto \alpha^{1/6}$. This scaling is in good agreement with the experimental results as seen in figure 8, but its understanding is still left for future studies.

To conclude this subsection, AHE offers an interesting laboratory where both the Ohmic transport and topological currents coexist, and it has been established that the latter can contribute to the Hall effect even in metallic systems. The quantized version of the AHE, where most of the bulk states are localized [50] or the chemical potential is in the gap [51], has also been discussed. In this case, the current is carried by the edge channels as in the case of a quantum Hall system.

2.2.2. Scalar spin chirality and the anomalous Hall effect. There are a wide variety of non-collinear spin structures found experimentally. These systems are the ideal laboratory to test the idea of U(1) EEMF discussed in section 1. Of particular interest is the idea of ‘scalar spin chirality’ $\chi_{ijk}$ defined for the three spins as

$$\chi_{ijk} = S_i \cdot (S_j \times S_k),$$

which is $T$-odd. As discussed in section 1.2 and figure 2, the scalar spin chirality acts as an effective magnetic field and leads to the Hall response [52, 53]. First, we consider the case where the magnetic unit cell is small, and the Bloch wavefunctions are defined in the first BZ. The total gauge flux penetrating the unit cell is zero or an integer multiple of $2\pi$ due to the periodicity since the contour integral of $\mathbf{a}(r)$ along its boundary vanishes (mod $2\pi$) due to the periodicity. On the other hand, the gauge field corresponding to the Berry phase becomes non-zero for the Bloch wavefunctions in the momentum space with a possible finite Chern number of each band. Roughly speaking, this happens when the unit cell contains more than two different types of loop, and each band feels the fluxes with different weights to obtain the Chern number. Ohgushi et al [53] showed that this idea can be realized in a ferromagnet on a Kagome lattice with the non-coplanar spin configuration.

A pyrochlore lattice can be regarded as the 3D generalization of the Kagome lattice, i.e. the pyrochlore lattice contains the Kagome lattice normal to the [111]-direction. In the material Nd$_2$Mo$_2$O$_7$ (hereafter we denote it as NMO), there are two interpenetrating sublattices composed of tetrahedrons of Nd and Mo atoms shifted along the c-axis as shown in figure 9(a) [54]. The conduction electrons are on the Mo sublattice, while the localized spins on the Nd sublattice are subject to strong anisotropy to form non-coplanar spin configurations. The localized spins and conduction electron spins are coupled antiferromagnetically, which transmit the scalar spin chirality of Nd to conduction electrons (figure 9(b)). Figure 9(c) shows the temperature dependence of the Hall resistivity, which shows a steep increase as the temperature is lowered, corresponding to the increase of exchange field from Nd spins and associated scalar spin chirality. The absolute value of the low temperature value is consistent with the theoretical calculation taking into account the tilting angle of the Mo spins estimated from the neutron scattering experiment [54]. Furthermore, the observation of the sign change of $\sigma_{xy}$ under the external magnetic field along the [111]-direction [55] is consistent with the sign change of the spin chirality.

The analysis in $k$-space above is justified when the magnetic unit cell is small compared with the mean free path $\ell$. In this case, the EEMF is defined by the Berry phase in $k$-space. In the other limit, i.e. when $\ell$ is shorter than the size of the spin texture $\xi$, the EEMF is defined more appropriately in $r$-space, which acts as the effective EMF with $\alpha_2 \tau \ll 1$ ($\alpha_2$ is cyclotron frequency and $\tau$ is mean free time). This situation is realized in the recently discovered Skyrmion crystals in non-centrosymmetric magnets with B20 structure [56, 57]. MnSi, (Fe, Co)Si and MnGe are examples of this class of materials, which has Dzyaloshinskii–Moriya (DM) interaction as described by the following Hamiltonian in the continuum approximation:

$$H = \int \mathbf{r} \left[ \frac{J}{2} (\nabla \mathbf{M}(\mathbf{r}))^2 + \alpha \mathbf{M}(\mathbf{r}) \cdot (\nabla \times \mathbf{M}(\mathbf{r})) - \mathbf{H} \cdot \mathbf{M}(\mathbf{r}) \right],$$

(31)

where $\mathbf{M}(\mathbf{r})$ is the magnetization, $\mathbf{H}$ the magnetic field, $J$ the ferromagnetic interaction and $\alpha$ the DM interaction constant. The lattice constant is taken to be unity. The ground state under zero magnetic field is the single spiral with the spins rotating in the plane perpendicular to the wavevector $\mathbf{q}$. The length of $q = |\mathbf{q}|$ is determined to be the ratio $\alpha/J$, while its direction is determined by the weak spin anisotropies not included in the Hamiltonian equation (31).

Recently, a neutron scattering experiment identified the mysterious A-phase in MnSi as the Skyrmion crystal state stabilized by the external magnetic field and thermal fluctuations [56]. The conical spin structure is the most stable state under the magnetic field in the major part of the phase diagram in a 3D crystal. On the other hand, when one reduces
Figure 9. AHE in Nd$_2$Mo$_2$O$_7$. (a) The crystal structure contains two inter-penetrating networks of tetrahedrons of Nd and Mo atoms, which are coupled antiferromagnetically. Due to the strong spin anisotropy Nd spins form a non-coplanar configuration with scalar spin chirality at low temperature, which is transferred to conduction electron spins on Mo atoms as shown in (b). (c) The magnetic field dependence of the Hall resistivity for several temperatures. Due to the scalar spin chirality, it shows a rapid increase as the temperature is lowered (inset). Reproduced with permission from [54].

Figure 10. Skyrmions and Skyrmion crystal in (Fe, Co)Si. Left panel: experimental observation by a Lorentz microscope (a)–(c) at each point shown by the star in the phase diagram (g). Right panel: the corresponding figures obtained by the Monte Carlo simulation for a DM magnet in 2D. Reproduced with permission from [57].

As shown schematically in figure 11, each Skyrmion wraps the sphere once in the order parameter space of $\mathbf{M}$, which means that the integral of the magnetic flux $b \parallel \hat{z}$ over one Skyrmion is $2\pi$. From this one can estimate the effective magnetic field induced by the Skyrmion. It is typically 4000 T when the size of the Skyrmion $\xi = 1$ nm, and inversely proportional to $\xi^2$. Since MnSi, (Fe, Co)Si and MnGe are metallic systems, the conduction electrons are coupled to the spins and spin chirality. Therefore, the Hall effect is expected due to the Lorentz force driven by $b$ replacing the...
external magnetic field $\mathbf{B}$. Recently, the Hall measurement on MnGe was carried out and the contribution from the Skyrmion was analyzed [59]. The Hall resistivity $\Delta\rho_{yx} \equiv 200$ n$\Omega$cm is due to this topological contribution, which corresponds to the effective magnetic field $b \approx 1100$ T for $\xi = 3$ nm. These values are compared with the case of MnSi, where $\xi = 18$ nm, $b \approx 28$ T and $\Delta\rho_{yx} \approx 5$ n$\Omega$cm.

The dynamics of Skyrmions and Skyrmion crystal is an intriguing issue to be studied. A recent experiment [60] shows the Skyrmion crystal motion driven by a current with the threshold value $j_c \approx 10^2$ $\text{A cm}^{-2}$ m, which is orders of magnitude smaller than that for the domain wall motion in ferromagnets [61]. Theoretically, the motion of the Skyrmions will induce the electric field $\mathbf{e}$ as an electromotive force due to $d\mathbf{b}/dt$. This leads to several physical consequences such as a new mechanism for the damping of the Skyrmion motion, and the motion of the Skyrmion transverse to the current (the Skyrmion Hall effect) [62].

2.2.3. The Hall effect of light. The topological Hall effect does not require that the particles are charged, and hence is possible even for uncharged particles. A representative example of uncharged particles is the photon, and one can ask if the Hall effect of light can occur or not. The constraint which induces the Berry curvature in this case is that the electric and magnetic fields of light are always perpendicular to the direction of the propagation, i.e. that of the wavevector $\mathbf{k}$. This situation corresponds to the strong coupling limit of SOI in electrons. Actually, the effect of the Berry phase in the propagation of light in the optical fiber has been studied both experimentally and theoretically, and the rotation of the polarization was confirmed for the light with the $\mathbf{k}$-vector slowly changing and enclosing an area on the sphere [63].

Note that the quantum nature of the photon is not required for this Berry phase effect, which is solely due to the wave nature of light. Therefore, this offers a unique opportunity to study the Berry phase effect in classical systems. Extending this idea, we have derived the semiclassical equation of motion for a wavepacket of light including the effect of finite wavelength [64]. (Note that the wave optics is reduced to geometric optics in the short-wavelength limit, the former of which corresponds to quantum mechanics while the latter to classical Newtonian mechanics.) These semiclassical equations of motion for the position $\mathbf{r}$, the wavevector (momentum) $\mathbf{k}$ and the spin state $|z\rangle$ (two-component spinor describing the polarization of light) read

$$
\frac{d\mathbf{r}}{dt} = \mathbf{v}(\mathbf{r}) \frac{\mathbf{k}}{|\mathbf{k}|} + \frac{d\mathbf{k}}{dt} \times (z|\Omega_{\mathbf{k}}|z),
$$

$$
\frac{d\mathbf{k}}{dt} = -[\nabla \mathbf{v}(\mathbf{r})] \mathbf{k},
$$

$$
\frac{d|z\rangle}{dt} = -\frac{i}{\hbar} \frac{d\mathbf{k}}{dt} \cdot \Lambda_{\mathbf{k}} |z\rangle,
$$

where $\mathbf{v}(\mathbf{r}) = c/n(\mathbf{r})$ is the velocity of light in the medium, $[\Lambda_{\mathbf{k}}]_{\lambda\lambda} = -i\mathbf{e}_\lambda \nabla_k \mathbf{e}_\lambda$ is the connection of the $2 \times 2$ matrix form, i.e. SU(2) gauge connection, and $\Omega_{\mathbf{k}} = \nabla_k \times \mathbf{A}_k$ is the field strength. It is diagonal in the basis of right and left-circular polarization, i.e. $\Omega_{\mathbf{k}} = \sigma^\lambda_k |k|$. This equation indicates that the Berry curvature has the opposite sign for opposite circularly polarized light, which gives the polarization-dependent anomalous velocity adding to the group velocity of the wavepacket. More explicitly, one can consider the reflection and transmission of light at the interface of two media with different dielectric constants [64]. The anomalous velocity is finite when the particle is subject to the gradient of the refractive index $\nabla n(\mathbf{r})$, and results in finite transverse shifts of the transmitted and reflected light at this reflection/transmission. These shifts are in the opposite direction for opposite circular polarization. This effect has recently been observed experimentally using the ‘weak measurement’ [65].

The idea of the Berry phase for light can be generalized to the photonic crystal, where the Bloch waves are formed. In particular, one can consider the crystal with distortion, where the Berry phase is defined in the 6D space $(\mathbf{k}, \mathbf{r})$. In particular, the curvature is enhanced near the gap edge when the gap is small. This situation is realized for x-rays propagating in crystals since the deviation of the dielectric function from unity is typically of the order of $10^{-6}$, and hence the ‘periodic potential’ for the Bloch wave is very weak. Sawada and Nagaosa [66] considered the role of this enhanced Berry curvature in the propagation of x-rays in a deformed crystal. The prediction is that the shift in the trajectory $\Delta \mathbf{r}$ of the x-ray is given by

$$
\Delta \mathbf{r} \approx \mathbf{G}(\mathbf{G} \cdot \mathbf{u}) \frac{\omega}{\Delta\omega} \frac{1}{|\mathbf{k}|^2},
$$

Figure 11. A single Skyrmion viewed as the mapping from the 2D real space to the unit sphere. This mapping is characterized by a topological number called the Skyrmion number, i.e. how many times the mapping wraps the unit sphere.

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where \( \mathbf{G} \) is the reciprocal lattice vector satisfying the Bragg condition for the wavevector \( \mathbf{k} \) of the x-ray, \( \mathbf{u} \) the displacement of the crystal, \( \omega \) the frequency of x-ray and \( \Delta \omega \) the gap in the Bloch wave of x-rays. As mentioned above, \( \omega / \Delta \omega \) can be as large as \( 10^8 \), which determines the magnification of \( \Delta \mathbf{r} \) compared with \( \mathbf{u} \) considering that \( |\mathbf{G}| \sim |\mathbf{k}| \). This theoretical prediction has recently been confirmed experimentally using the single crystal of Si [67], and offers a new principle for the x-ray microscope.

2.2.4. Hall effect of magnons. Another example of the uncharged particle is the magnons in insulating magnets. Although the charge current is not available in this case, the energy current can be carried by the magnons and hence the thermal Hall effect without resorting to the Lorentz force is expected [68]. When one considers the propagation of the magnon wave in a ferromagnet, it behaves as a Bloch wave and the band structure is formed. When more than two atoms are in the unit cell, the Berry phase is generally expected for this Bloch wavefunction. We consider the Hamiltonian with the DM SOI as

\[
H = -\sum_{ij} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j + \mathbf{D}_{ij} \cdot (\mathbf{S}_i \times \mathbf{S}_j) - \sum_i \mathbf{H} \cdot \mathbf{S}_i. \tag{34}
\]

Suppose the external magnetic field \( \mathbf{H} \) and ferromagnetic moments align along the +z-direction. Then, the spin wave Hamiltonian can be written as

\[
H_{SW} = -\sum_{ij} \tilde{J}_{ij} S(e^{-i\phi_j}b_j^\dagger b_j + h.c.) + \sum_i H^z b_i^\dagger b_i, \tag{35}
\]

where \( \tilde{J}_{ij} e^{i\phi_j} = J_{ij} + iD_{ij} \). This indicates that the DM interaction acts as a vector potential and effective magnetic flux for the propagating magnons [69]. Similar to the case of AHE, the distribution is such that the total effective magnetic flux is zero (or equivalently an integer multiple of \( 2\pi \)) when integrated over the unit cell. Therefore, the finite effect survives when the equivalent loops exist in the unit cell. The Kagome lattice and its 3D generalization pyrochlore lattice are representative crystal structures satisfying this condition. And actually the thermal Hall effect has recently been observed in an insulating ferromagnet Lu2V3O7 with pyrochlore structure [69]. In the pyrochlore structure, the midpoint between any two corners of a tetrahedron is not an inversion symmetry center, and hence the non-zero DM interaction is expected. Symmetry further determines the direction of the DM vectors \( \mathbf{D}_{ij} \) as shown in figure 12(a).

In this material, spontaneous magnetization \( M \) emerges below Curie temperature \( T_c = 70 \text{ K} \), which is isotropic and almost coincides with 1 Bohr magneton (\( \mu_B \)) at low temperatures, indicating the collinear ferromagnetic state with spin \( S = 1/2 \). Below \( T_c \), the thermal Hall conductivity is discernible, whereas it is very small above 80 K. The magnitude of the thermal Hall conductivity has a maximum at about 50 K. Similar to the magnetization, the thermal Hall conductivity steeply increases and saturates in the low magnetic field region. Therefore, the observed thermal Hall effect is due to the spontaneous magnetization \( M \) as in the case of AHE. \( \kappa_{th} \) gradually decreases with increasing magnetic field furthermore due to the opening of the gap in the magnon spectrum induced by the magnetic field. All these effects are contained in the following theoretical expression for \( \kappa_{th} \) [69]:

\[
\kappa_{th} = \Phi_{ag} \frac{k_B^2 T}{pi^{3/2} a} \left[ 2 + \frac{g \mu_B H}{2JT} \right]^2 \times \frac{k_B T}{2JT} \left[ \exp \left( -\frac{g \mu_B H}{k_B T} \right) \right]. \tag{36}
\]

where \( \Phi_{ag} = -e_{ag} n_d D/(8\sqrt{2}J) \) (\( e_{ag} \) is the totally antisymmetric tensor, \( n \) the direction of magnetization, \( a \) the lattice constant and \( D = |\mathbf{D}_{ij}| \), and \( \exp(z) = \sum_{k=1,\infty} z^k/k^k \). Figure 12(b) shows a comparison of the experimental result and equation (36). The only adjustable parameter is the ratio \( D/J = 0.32 \), which is a reasonable value for transition metal oxides. From these facts, it is convincing that the thermal Hall effect observed in Lu2V3O7 is due to the magnons affected by the Berry curvature due to the DM interaction.

2.2.5. The spin Hall effect. The concept of the anomalous velocity can be generalized to the time-reversal \( T \)-symmetric systems. As already discussed in equation (32), the SU(2) Berry curvature is non-zero even in the \( T \)-invariant system, and an analogous effect is expected for the electrons also. This idea has been explored for the band structure of semiconductors [70]. In GaAs and Ge, the valence bands...
The case of the p-doped GaAs/Ge the spin current, $j$ of the extrinsic spin Hall effect. This is called the (intrinsic) spin Hall effect, and in the external electric field can induce the transverse spin current. As discussed above, the degeneracy acts as the magnetic monopole for the Berry curvature, and the $\Gamma$-point is especially interesting since the two doubly degenerate bands touch there. Since there is Kramers degeneracy at each $\mathbf{k}$-point, the Berry connection is the $2 \times 2$ matrix ($SU(2)$). There are two pseudo-spin states, i.e. parallel and anti-parallel pseudo-spin to $\mathbf{k} = \mathbf{k}/|\mathbf{k}|$, which is called helicity. The gauge field and hence the anomalous velocity depend on the helicity of the state, i.e. it has the opposite sign for different helicity states. Therefore, even though there is no net charge current, the external electric field can induce the transverse spin current. This is called the (intrinsic) spin Hall effect, and in the case of the p-doped GaAs/Ge the spin current, $j^i = J^i \sigma^i_H \epsilon^i$ was predicted to be [70]

$$J^i = \sigma^i_H \epsilon^{ijk} E_j$$

with

$$\sigma^i_H = \frac{1}{6\pi} (k^F - k^J)$$

where $k^F$ ($k^J$) is the Fermi wavenumber for the heavy (light) holes. This intrinsic spin Hall effect is driven by the band structure and its Berry phase connection, in sharp contrast to the extrinsic mechanisms previously proposed by Dyakonov and Perel in 1971 [71] and by others [72, 73]. Sinova et al. [74] independently proposed the intrinsic spin Hall effect for n-GaAs using the Rashba model. Note that equation (38) looks natural when one remember that the SU(2) gauge potential $A^i = \epsilon^{ijk} E_j$ is coupled to the spin current as discussed in section 1. However, the topological structure of the Bloch wavefunction is needed to realize the intrinsic spin Hall effect.

There are several experimental reports on the spin Hall effect. In semiconductors, the spin accumulation near the edge of the sample is measured to confirm the spin Hall effect. Kato et al. [75] observed the spin accumulation at the edge of the n-type GaAs in terms of the Kerr rotation. Wunderlich et al. [76] observed the circular polarized LED from the recombination process of the holes in the p-type GaAs. Both experiments seem to be relevant to the spin Hall effect, but its origin, i.e. intrinsic or extrinsic, is still controversial [77]. The experiment on metals usually measures the inverse spin Hall effect, i.e. the voltage induced by the injection of spin current from the ferromagnetic metals [78]. For some of the metals, the first-principles band structure calculations are applied to predict the intrinsic spin Hall effect in e.g. Pt, which agrees reasonably well with the experimental observations [79]. See the review [77] for more details.

3. Other interesting systems and conclusions

There are several important and intriguing systems from the viewpoint of EEMF which are not covered in section 2. One natural question to ask is, what are the global aspect of the sub-Hilbert space and gauge fields, and their implications on the physical properties? Of particular importance from this viewpoint are the topological insulators [80]. One can imagine the two copies of quantum Hall systems with opposite chiralities for up and down spins. These systems will show the quantized spin Hall conductance due to the non-zero Chern numbers $C_\uparrow = -C_\downarrow$. The spin Chern number is defined as $C_\uparrow - C_\downarrow$, which can be finite even for $T$-symmetric systems. However, in the presence of SOI, usually all the components of the spin are not conserved, and hence the spin current and Chern number are not well defined. Actually, the spin Hall conductance in a spin Hall insulator, where the chemical potential is inside the band gap but still the spin Hall conductance is finite due to the band inversion, is not quantized [81]. Kane–Mele discovered a $T$-symmetric model with SOI, which shows a stable non-trivial topological phase with the helical edge channels even with the mixing of the spins [80]. This phase is characterized by the $Z_2$ topological number instead of the Chern number. This idea is now generalized to three dimensions and also to arbitrary dimensions to classify all the possible topologically non-trivial states as far as the band structure can be defined [82]. The basic concept is that the Bloch wavefunctions as the fiber bundle can have a globally non-trivial structure characterized by the index defined as the integral of the Berry curvature or EEMF over the first BZ. Combining the mapping between the different dimensions and symmetry class enables the classification of the topological classes including both the insulators and superconductors with gaps. Recently, generalization to include the $\mathbf{k}$- and $\mathbf{r}$-spaces was also achieved [83]. See the recent reviews [80] for more details.

Noncentrosymmetric systems with SOI are also an interesting laboratory to study the topological nature and EEMF: The Rashba system is the representative one described by the Hamiltonian

$$H_R = \frac{p^2}{2m} + e \mathbf{r} \cdot (\mathbf{s} \times \mathbf{p})$$

where $e$ is the direction of the potential gradient which breaks $I$-symmetry and $\mathbf{s}$ and $\mathbf{p}$ are the spin and momentum operators, respectively. This interaction has been experimentally demonstrated for the 2D electrons at the interface of the GaAs system [84] or at the interface electrons in the oxide system [85, 86], and also the bulk 3D material BiTeI [87]. The spin splitting at each $\mathbf{k}$-point means that the spin and the velocity are tightly coupled, and hence the transport and magnetic properties. Therefore, various cross correlations between the magnetic and transport degrees of freedom are expected and actually observed experimentally in the Rashba systems, which constitute an important part of the spintronics. The spin Hall effect, the spin galvanic effect and the dynamical magneto-electric effect.
are examples of this cross-correlation driven by the Rashba interaction [88, 89]. Combined with the superconducting order parameter, the Rashba-type SOI leads to even more interesting phenomena [90]. Because of the absence of the inversion symmetry, the classification of even and odd parity does not work there, and hence the singlet and triplet pairings are mixed. This fact leads to several unusual features such as the $H_\Omega$ beyond the Pauli limit [90]. Furthermore, it is shown that the topological superconductor with helical Majorana edge channels as the Andreev bound states can be realized when the p-wave pairing is dominant over the s-wave pairing component [91]. When the magnetic field or exchange field is applied to open the gap at the band crossing and the Fermi energy is within that gap, the spinless pairing superconductor is realized, and the Majorana fermions are expected to appear there [92]. These are consistent with the more general classification scheme of the topological superconductors [82].

In this paper, we have discussed the physics of EEMF in condensed-matter systems. The gauge connections and fields are naturally introduced in various situations when the wavefunctions are constrained on some manifold in the Hilbert space. In the band structure and/or in the mean field theory, this gauge fields are not dynamical but frozen ones or are controlled as parameters by the EMF. However, the physical phenomena driven by EEMF are even richer than those of EMF because the lattice gauge theory, non-Abelian gauge fields, higher dimensions and topological terms are often relevant to EEMF. In this sense, the gauge fields can be a guiding and unifying principle in condensed-matter physics. Needless to say, the dynamical nature of EEMF in correlated systems and spin liquids, which was not covered in this paper, also remains an important subject and continues to be a central issue in condensed-matter theory [4, 5, 16].

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