Topological superconductivity of spin-3/2 carriers in a three-dimensional doped Luttinger semimetal

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We investigate topological Cooper pairing, including gapless Weyl and fully gapped class DIII superconductivity, in a three-dimensional doped Luttinger semimetal. The latter describes effective spin-3/2 carriers near a quadratic band touching and captures the normal-state properties of the 227 pyrochlore iridates and half-Heusler alloys. Electron-electron interactions may favor non-s-wave pairing in such systems, including even-parity d-wave pairing. We argue that the lowest energy d-wave pairings are always of complex (e.g., $d + id$) type, with nodal Weyl quasiparticles. This implies $\rho(E) \sim |E|^2$ scaling of the density of states (DoS) at low energies in the clean limit, or $\rho(E) \sim |E|$ over a wide critical region in the presence of disorder. The latter is consistent with the $T$-dependence of the penetration depth in the half-Heusler compound YPtBi. We enumerate routes for experimental verification, including specific heat, thermal conductivity, NMR relaxation time, and topological Fermi arcs. Nucleation of any d-wave pairing also causes a small lattice distortion and induces an s-wave component; this gives a route to strain-engineer exotic $s + d$ pairings. We also consider odd-parity, fully gapped p-wave superconductivity. For hole doping, a gapless Majorana fluid with cubic dispersion appears at the surface. We invent a generalized surface model with $\nu$-fold dispersion to simulate a bulk with winding number $\nu$. Using exact diagonalization, we show that disorder drives the surface into a critically delocalized phase, with universal DoS and multifractal scaling consistent with the conformal field theory (CFT) $SO(n)_\nu$, where $n \to 0$ counts replicas. This is contrary to the naive expectation of a surface thermal metal, and implies that the topology tunes the surface renormalization group to the CFT in the presence of disorder.

Contents

I. Introduction 2
A. Even parity pairing: scenarios and main results 2
B. Odd parity pairing: surface states and topological protection 3
C. Outline 4

II. Pairing in the Luttinger semimetal 4
A. Luttinger Hamiltonian 4
B. Even-parity local pairings 5
C. Odd-parity, momentum-dependent pairings 7
D. Free energy and gap equation 7

III. Weyl superconductors 9
A. Topology of Weyl superconductors 9
B. $E_g$ pairing 10
1. Competition within $E_g$ 10
2. Nodal topology 11
3. Scaling of density of states 11
C. $T_{2g}$ pairing 12
1. Nodal topology 12
2. Alternative phase locking in $T_{2g}$ channel 13
D. Competition between $E_g$ and $T_{2g}$ pairings 14
E. Anomalous thermal and spin Hall conductivities 15

IV. External strain and $s + d$ pairing 17
A. Strain along the [1, 1, 1] direction 18
B. Strain along the [1, 1, 0] direction 19

V. Effects of impurities on BdG-Weyl quasiparticles 19

VI. Connection with experiments: Penetration depth in YPtBi 22
A. Source of $T^2$ dependence 23
B. Sources of $T$-linear dependence 23
C. Sources of the s-wave component 24
D. Future experiments and pairing symmetry 24

VII. $s + id$ pairing 25

VIII. Strong topological superconductivity: Odd-parity isotropic p-wave pairing 27
A. Bulk and surface theory 28
B. Quenched surface disorder, class DIII $SO(n)_\nu$ conformal field theory, and numerical results 30
C. Generalized surface: Higher winding numbers and numerical results 32

IX. Conclusions and outlook 34

Acknowledgments 35

A. Luttinger model components 35
I. INTRODUCTION

One of the most useful concepts of modern day condensed matter physics is the topological classification of quantum phases, which at the coarsest level divides into two categories: topological and trivial. A hallmark signature of a topologically non-trivial system is the existence of robust gapless states at an interface with the trivial vacuum, exposing the information about the bulk topological invariant to the external world. This classification encompasses insulators, semimetals and superconductors (both gapped and gapless). In this paper we establish that a doped three-dimensional Luttinger semimetal (LSM), describing a quadratic touching of Kramers degenerate valence and conduction bands of \( j = 3/2 \) (effective) fermions, can harbor myriad exotic gapless and gapped topological superconductors.

The LSM provides the low-energy normal-state description for a plethora of both strongly and weakly correlated compounds, such as the 227 pyrochlore iridates (\( \text{Ln}_2\text{Ir}_2\text{O}_7 \), with \( \text{Ln} \) being a lanthanide element), half-Heusler compounds (ternary alloys such as \( \text{LuPtBi} \), \( \text{LuPdBi} \)), \( \text{HgTe} \), and gray-tin. Among these materials, the 227 pyrochlore iridates might support only non-Fermi liquid or excitonic (particle-hole channel) orders (most likely magnetic such as the all-in all-out spin-ice orders), since the chemical potential lies extremely close to the band touching point. Nevertheless, it is possible at least in principle to move the chemical potential away from charge neutrality by chemical doping, for example, which in turn can be conducive for superconductivity. Even though the signature of Cooper pairing is yet to be found in \( \text{HgTe} \) or gray-tin, some half-Heusler compounds (such as \( \text{YPtBi} \), \( \text{LuPtBi} \), \( \text{LuPdBi} \)) become superconducting below a few Kelvin. This has led to a surge of theoretical works recently. Despite half-Heuslers standing as fertile ground for topological phases of matter, the nature of the actual pairing remains elusive so far and therefore demands comprehensive theoretical and experimental investigations.

In this paper, we study various experimental signatures of superconducting states that could arise in a three-dimensional LSM. Since the superconducting order parameter is formed from spin-\( 3/2 \) band electrons, the \( \text{SU}(2) \) angular momentum addition rule

\[
(3/2) \otimes (3/2) = 0 \oplus 1 \oplus 2 \oplus 3
\]

implies that simple paired states reside in two broad categories: (a) even-parity, such as local or intra-unit cell pairing (with order parameter spin \( j \in \{0, 2\} \)), and (b) odd-parity, momentum-dependent pairing (with order parameter spin \( j \in \{1, 3\} \)). We consider these two cases separately. We next provide a synopsis of our main findings for even- and odd-parity pairings.

A. Even parity pairing: scenarios and main results

Even-parity local pairings are represented by anomalous local bilinears of the spin-\( 3/2 \) fermion field. Local or intra-unit cell pairings can be mediated by short-range interactions, such as spin exchange scattering. For superconductivity at low densities in an LSM, momentum-dependent pairing interactions can be strongly suppressed relative to local ones. The mechanism for this is virtual renormalization from higher energies, as may also occur in bilayer graphene (a “two-dimensional LSM”) or structurally similar bilayer silicene. The local pairing amplitudes couple to \( j = 0 \) and \( j = 2 \) spin \( \text{SU}(2) \) tensor operators.

Although the even-parity pairings are local bilinears in the LSM, we focus on the situation where the superconductivity itself manifests mainly near the Fermi surface. The Fermi surface is assumed to reside at finite carrier density away from charge neutrality. The projection of the \( j = 0 \) \( (j = 2) \) pairing onto the conduction or valence bands give rise to momentum-independent (dependent) \( s \)-wave \((d \)-wave\) superconductivity on the Fermi surface. Both channels are \( \text{band-pseudospin singlets} \) due to Kramers degeneracy. In this paper we carefully catalog the bulk structure of the nodal loops, single or double Weyl nodes that arise via combinations of the \( d \)-wave pairings (at the cost of the time-reversal symmetry breaking). We show how strain can promote particular \( d \)-wave pairings, whilst simultaneously inducing a parasitic \( s \)-wave component. We also consider the effects of quenched disorder on the bulk quasiparticle density of states (DoS). In addition, we determine the anomalous spin and/or thermal Hall conductivities expected from possible time-reversal symmetry-breaking orders. We now highlight our main results.

1. We show that while pseudospin singlet \( s \)-wave pairing induces a full gap, each of the five simple \( d \)-wave pairings (belonging to \( T_{2g} \) and \( E_g \) representations of the \( O_h \) point group) produces two nodal loops on the Fermi surface (see Sec. II and Table. I). However, due to the underlying cubic symmetry it is always possible to find a specific phase locking among the \( d \)-wave components that eliminates the nodal loops from the spectra and yields
only few isolated simple Weyl nodes (characterized by monopole charges ±1). See Secs. HI B and HI C. The DoS around each Weyl node vanishes as \( \rho(E) \sim |E|^2 \). Within a weak coupling pairing picture, complex (e.g., \( d \to id \)) Weyl superconductors are therefore energetically favored over the simple \( d \)-wave nodal-loop pairings, since the former cause an additional power-law suppression of the DoS \( \rho(E) \sim |E| \) for nodal loop \( \sim |E|^2 \) for Weyl]. By contrast, double-Weyl nodes (with monopoles charges ±2) can only be found inside the \( d_{x^2−y^2}+id_{xy} \) phase, which results from a competition between \( E_g \) and \( T_{2g} \) pairings. This gives \( \rho(E) \sim |E| \) at low energies.

2. We show that despite possessing Weyl nodes the net anomalous pseudospin and thermal Hall conductivities inside the \( E_g \) paired state are precisely zero, while these are finite in any high symmetry plane inside the \( T_{2g} \) paired states. On the other hand, when pairing interactions in the \( E_g \) and \( T_{2g} \) channels are of comparable strength, only the \( d_{x^2−y^2}+id_{xy} \) paired state supports non-trivial anomalous pseudospin and thermal Hall conductivities (see Sec. III E). Such peculiar outcomes purely stem from the momentum-space distribution of the Abelian Berry curvature, shown in Figs. 3 and 4.

3. We demonstrate that the formation of any \( d \)-wave pairing breaks the cubic symmetry and causes a small lattice distortion or nematicity that in turn induces an even smaller \( s \)-wave component. Thus any \( d \)-wave paired state is always accompanied by a parasitic \( s \)-wave counterpart. Such non-trivial coupling between \( d \)-wave and \( s \)-wave superconductivity with the lattice distortion can be exploited to strain engineer various \( d+s \) paired states, by applying a weak external strain in specific directions (see Sec. IV).

4. Pursuing a renormalization group approach, controlled via an appropriate \( \epsilon \)-expansion, we find that Weyl superconductors, comprised of Weyl nodes with monopole charges ±1, remain stable for sufficiently weak disorder, while at stronger disorder the system can undergo a continuous quantum phase transition into a thermal metallic phase where \( \rho(0) \) is finite. The disorder-controlled quantum critical point is accompanied by a wide quantum critical regime, where \( \rho(E) \sim |E| \), as long as \( |E| \ll T_c \) (the superconducting transition temperature). By contrast, both double-Weyl and nodal-loop paired states enter into a diffusive thermal metallic phase for arbitrarily weak strength of disorder (see Sec. V).

5. In this work, we make an independent attempt to understand the peculiar power-law suppression of the penetration depth (\( \Delta \lambda \)) in YPtBi [16] by combining a power-law contribution (arising from gapless quasiparticles in the \( d \)-wave paired state, for example), with an exponential one, stemming from an \( s \)-wave component (due to its inevitable coexistence with any \( d \)-wave pairing). We find that even though both \( T \)-linear and \( T^2 \) fitting give qualitative agreement, the former one yields a better fit over a larger window of temperature (see Fig. 10). A \( T \)-linear dependence may arise from either double-Weyl nodes or nodal loop(s) in a clean system, but it might also represent BdG-Weyl quasiparticles in the presence of quenched disorder. The dirty BdG-Weyl system can exhibit \( \rho(E) \sim |E| \) scaling throughout a wide quantum critical fan. We propose future experiments to determine the scaling of specific heat, thermal conductivity, NMR relaxation time, STM measurements of surface Andreev bound states, and anomalous thermal Hall conductivity that can pin the nature of the pairing in this class of materials (see Sec. VI).

B. Odd parity pairing: surface states and topological protection

Odd-parity pairing can arise in the LSM via \( j = 1 \) or \( j = 3 \) spin SU(2) tensor operators, coupled to odd powers of momentum to satisfy the Pauli principle. The \( j = 3 \) operator [Eq. (A5) in Appendix A] plays the key role in proposals for \( p \)-wave, “septet” pairing that has been extensively discussed in the context of YPtBi [16] [50] [51]. It could also arise in an exotic, isotropic \( f \)-wave pairing scenario [38].

In this paper, we instead focus on simple isotropic \( p \)-wave pairing. This odd-parity pairing is the spin-3/2 generalization of the \( B \) phase of helium 3 [15], and gives rise to fully gapped, strong class DIII topological superconductivity [9] [47] [48] [51]. Unlike model spin-1/2 topological superconductors, the gapless surface Majorana fluid that arises from a higher-spin bulk can exhibit nonrelativistic dispersion [47] [48]. The robustness of “topological protection” for such a 2D surface fluid has not been generally established, and we have shown previously that interactions can destabilize such states [51]. In the same work, however, we demonstrated that topological protection can be enhanced by quenched surface disorder.

The motivation for studying strong topological superconductivity in the LSM is twofold. We seek to define topological protection for surface states of higher-spin superconductors, since this is an ingredient expected to arise in candidate materials with strong spin-orbit coupling. At the same time, the Eliashberg calculations in Ref. [54] suggest that isotropic \( p \)-wave pairing gives the dominant non-\( s \)-wave channel in a hole-doped LSM due to optical-phonon-mediated pairing interactions.

For isotropic \( p \)-wave pairing in the LSM, we show that the bulk winding number \( \nu = 3 \) describes superconductivity arising from either the \( |m_z| = 3/2 \) valence or \( |m_s| = 1/2 \) conduction band; here \( m_s \) denotes the spin projection. Unlike the winding number, the dispersion of the surface Majorana fluid does depend on \( |m_z| \). We investigate the effects of quenched disorder on the cubic-dispersing fluid that obtains in the hole-doped scenario. We compare numerics to conformal field theory predictions for the disorder-averaged surface density of states and wave function multifractal spectrum [51] [61] [62]. We also invent a generalized surface theory which allows the investigation of a Majorana surface fluid corresponding to a generic bulk winding number \( \nu \). We find excellent
agreement between the SO($n$)$_\nu$ conformal field theory (CFT) \cite{11,61} and numerics for higher winding numbers. Since the SO($n$)$_\nu$ theory is known \cite{61} to be stable against interactions, our results imply that surface states enjoy robust topological protection, with signatures such as a universal power-law divergence of the tunneling density of states that could be detected experimentally.

That we find critical delocalization for any $\nu$ is surprising, since the naive expectation would be a surface thermal metal phase. Indeed, the CFT is technically unstable towards flowing into the thermal metal, see Fig.\cite{12} Our results suggest that, in the presence of disorder, the topology fine-tunes the surface to the CFT.

C. Outline

This paper is organized as follows. The low-energy description of a LSM, possible pairings (both even- and odd-parity) and their classification are discussed in Sec.\textbf{II} In Sec.\textbf{III} we focus on the competition amongst various $d$-wave pairings belonging to different representations, and the emergence of Weyl superconductivity at low temperatures. We also compute the nodal topology of Weyl pairings and its manifestation through anomalous pseudospin and thermal Hall conductivities. Sec.\textbf{IV} is devoted to the effects of external strain, while the effects of impurities on BdG-Weyl quasiparticles are addressed in Sec.\textbf{V} Connections with a recent experiment in YPtBi and possible future experiments to pin the pairing symmetry are presented in Sec.\textbf{VI}. The competition between even parity $s$- and $d$-wave superconductivity is discussed in Sec.\textbf{VII} The bulk-boundary correspondence and the surface states of odd-parity $p$-wave pairing are discussed in Sec.\textbf{VIII} We conclude in Sec.\textbf{IX}. Additional technical details are relegated to the Appendices.

II. PAIRING IN THE LUTTINGER SEMIMETAL

We review the low-energy description of a LSM, followed by even- and odd-parity Cooper pairing scenarios. We enumerate the nodal-loop structure of all even-parity $d$-wave pairings. Finally, we compute the free energy, gap equation and transition temperature within BCS theory.

A. Luttinger Hamiltonian

Quadratic touching of Kramers degenerate valence and conduction bands at an isolated point [taken to be the $\Gamma = (0,0,0)$ point] in the Brillouin zone in three spatial dimensions can be captured by the $k\cdot p$ Hamiltonian

$$H_L = \int \frac{d^3k}{(2\pi)^3} \Psi_k^\dagger \hat{h}_L(k) \Psi_k,$$  

(2.1)

where the four-component spinor $\Psi_k$ is defined as

$$\Psi_k^\dagger = \left( c_{k,\frac{3}{2}}, c_{k,\frac{1}{2}}, c_{k,-\frac{1}{2}}, c_{k,-\frac{3}{2}} \right).$$  

(2.2)

Here $c_{k,m}$ is the band electron annihilation operator with spin projection $m_s \in \{3/2,1/2,-1/2,-3/2\}$. Such quadratic touching is protected by the cubic symmetry, which restricts the form of the Luttinger Hamiltonian \cite{16,17} operator to

$$\hat{h}_L(k) = \left( \frac{k^2}{2m_0} - \mu \right) \Gamma_0 - \frac{1}{2m_1} \sum_{a=1}^3 d_a(k) \Gamma_a$$

$$- \frac{1}{2m_2} \sum_{a=4}^5 d_a(k) \Gamma_a,$$  

(2.3)

where $\mu$ is the chemical potential measured from the band touching point. The $d$-vector appearing in the Luttinger Hamiltonian is given by $d(k) = k^2 \hat{d}(k)$, where $\hat{d}(k)$ is a five-dimensional unit vector that transforms in the $l = 2$ ("$d$-wave") representation under orbital SO(3) rotations. Its components can be constructed from the spherical harmonics $Y_l^m(\theta,\phi)$, see Appendix A. While $\Gamma_0$ is a four-dimensional unit matrix, the five mutually anticommuting matrices appearing in the Luttinger Hamiltonian are given by

$$\Gamma_1 = \kappa_3 \sigma_2, \quad \Gamma_2 = \kappa_3 \sigma_1, \quad \Gamma_3 = \kappa_2,$$

$$\Gamma_4 = \kappa_1, \quad \Gamma_5 = \kappa_3 \sigma_3.$$  

(2.4)

Two sets of Pauli matrices $\{\sigma_\alpha\}$ and $\{\sigma_\alpha\}$, with $\alpha = 0,1,2,3$ operate respectively on the sign $[\text{sgn}(m_s)]$ and the magnitude $[m_s] \in \{1/2, 3/2\}$ of the spin projection $m_s$. The $\Gamma$ matrices provide a basis for a symmetric traceless tensor operator formed from bilinear products of $j = 3/2$ matrices [Eqs. (A2) and (A3) in Appendix A], and transform in the $j = 2$ representation of the spin SU(2). Consequently, the Hamiltonian in Eq. (2.3), is an $A_1g$ quantity in a cubic environment. For $m_1 = m_2$, $\hat{h}_L(k)$ exhibits continuous SO(3) rotational invariance.

Besides five mutually anticommuting $\Gamma$ matrices and the identity matrix ($\Gamma_0$), we can define ten commutators as $\Gamma_{ab} = [\Gamma_a, \Gamma_b] / (2i) \equiv -i\Gamma_a \Gamma_b$ for $a, b = 1, \ldots , 5$ with $a \neq b$ that together close the basis for all four dimensional matrices. The ten commutators are the generators of a (fictitious) SO(5) symmetry. Since $d(k) = 0$ at the $\Gamma$ point of the Brillouin zone $k = 0$, the four degenerate bands possess an emergent SU(4) symmetry. Since $d(k) = 0$ at the $\Gamma$ point of the Brillouin zone $k = 0$, the four degenerate bands possess an emergent SU(4) symmetry. Since $d(k) = 0$ at the $\Gamma$ point of the Brillouin zone $k = 0$, the four degenerate bands possess an emergent SU(4) symmetry. Since $d(k) = 0$ at the $\Gamma$ point of the Brillouin zone $k = 0$, the four degenerate bands possess an emergent SU(4) symmetry. Since $d(k) = 0$ at the $\Gamma$ point of the Brillouin zone $k = 0$, the four degenerate bands possess an emergent SU(4) symmetry. Since $d(k) = 0$ at the $\Gamma$ point of the Brillouin zone $k = 0$, the four degenerate bands possess an emergent SU(4) symmetry. Since $d(k) = 0$ at the $\Gamma$ point of the Brillouin zone $k = 0$, the four degenerate bands possess an emergent SU(4) symmetry. Since $d(k) = 0$ at the $\Gamma$ point of the Brillouin zone $k = 0$, the four degenerate bands possess an emergent SU(4) symmetry. Since $d(k) = 0$ at the $\Gamma$ point of the Brillouin zone $k = 0$, the four degenerate bands possess an emergent SU(4) symmetry. Since $d(k) = 0$ at the $\Gamma$ point of the Brillouin zone $k = 0$, the four degenerate bands possess an emergent SU(4) symmetry. Since $d(k) = 0$ at the $\Gamma$ point of the Brillouin zone $k = 0$, the four degenerate bands possess an emergent SU(4) symmetry. Since $d(k) = 0$ at the $\Gamma$ point of the Brillouin zone $k = 0$, the four degenerate bands possess an emergent SU(4) symmetry. Since $d(k) = 0$ at the $\Gamma$ point of the Brillouin zone $k = 0$, the four degenerate bands possess an emergent SU(4) symmetry. Since $d(k) = 0$ at the $\Gamma$ point of the Brillouin zone $k = 0$, the four degenerate bands possess an emergent SU(4) symmetry. Since $d(k) = 0$ at the $\Gamma$ point of the Brillouin zone $k = 0$, the four degenerate bands possess an emergent SU(4) symmetry. Since $d(k) = 0$ at the $\Gamma$ point of the Brillouin zone $k = 0$, the four degenerate bands possess an emergent SU(4) symmetry. Since $d(k) = 0$ at the $\Gamma$ point of the Brillouin zone $k = 0$, the four degenerate bands possess an emergent SU(4) symmetry. Since $d(k) = 0$ at the $\Gamma$ point of the Brillouin zone $k = 0$, the four degenerate bands possess an emergent SU(4) symmetry. Since $d(k) = 0$ at the $\Gamma$ point of the Brillouin zone $k = 0$, the four degenerate bands possess an emergent SU(4) symmetry. Since $d(k) = 0$ at the $\Gamma$ point of the Brillouin zone $k = 0$, the four degenerate bands possess an emergent SU(4) symmetry. Since $d(k) = 0$ at the $\Gamma$ point of the Brillouin zone $k = 0$, the four degenerate bands possess an emergent SU(4) symmetry. Since $d(k) = 0$ at the $\Gamma$ point of the Brillouin zone $k = 0$, the four degenerate bands possess an emergent SU(4) symmetry. Since $d(k) = 0$ at the $\Gamma$ point of the Brillouin zone $k = 0$, the four degenerate bands possess an emergent SU(4) symmetry.
Without any loss of generality, but for the sake of technical simplicity, we work with the isotropic Luttinger model for which \( m_1 = m_2 = m \). The Luttinger Hamiltonian then has the alternative representation,

\[
\hat{h}_L(k) = \left[ \left( \lambda_1 + \frac{5}{2} \lambda_2 \right) k^2 - \mu \right] \Gamma_0 - 2 \lambda_2 (\mathbf{J} \cdot \mathbf{k})^2, \tag{2.5}
\]

with \( \mathbf{J} = (J^x, J^y, J^z) \) and \( \mathbf{k} = (k_x, k_y, k_z) \). Here \( J^{x,y,z} \) are SU(2) generators in the 3/2 representation. The correspondence between Eqs. (2.3) and (2.5) is \( \lambda_1 = (2m_0)^{-1}, \lambda_2 = (4m)^{-1} \). The Luttinger Hamiltonian can be diagonalized as \( D \hat{H}_L(k) D \), with the energy spectra

\[
\varepsilon_{\pm,\sigma}(k) = \left( k^2 / 2m_0 - \mu \right) \pm k^2 / 2m. \tag{2.6}
\]

Here \( + (-) \) corresponds to the \(|m_\sigma| = 3/2 \) conduction \((|m_\sigma| = 3/2 \) valence\) band. We have assumed that \( m_0 > m_1 \), so that these two bands bend oppositely. The “band pseudospin” index \( \sigma = \pm 1 \), and independence of \( \varepsilon_{\pm,\sigma}(k) \) on \( \sigma \) specifies the Kramers degenerate states in each band. For a given \( k \), one possible choice is \( \sigma = \text{sgn}(\mathbf{J} \cdot \mathbf{k}) \) (i.e. pseudospin-momentum locking), but we will not need to fix this basis. The diagonalizing matrix \( D \) is given by [17]

\[
D = \frac{1}{\sqrt{2 \left( 1 + \hat{d}_5 \right)}} \left[ \left( 1 + \hat{d}_5 \right) \Gamma_0 + i \sum_{a=1}^4 \Gamma_{a5} \right]. \tag{2.7}
\]

The pseudospin locking in the valence and conduction bands becomes transparent with a specific choice of the momentum \( \mathbf{k} = (0, 0, k_z) \) for which the Luttinger Hamiltonian from Eq. (2.3) readily assumes a diagonal form

\[
\hat{h}_L(k_z) = \text{Diag} \left[ -k_z^2 / 2m^+, k_z^2 / 2m^+, k_z^2 / 2m^+, -k_z^2 / 2m^+ \right] - \mu. \tag{2.8}
\]

in the spinor basis defined in Eq. (2.2), where \( m^\pm = m_0m/|m_0 \pm m| \). Therefore, for \( m_0 > m \) the first and fourth (second and third) entries yield Kramers degenerate spectra for the valence (conduction) band. Hence, the pseudospin projection on the valence (conduction) band is \(|m_\sigma| = 3/2 \) (1/2).

### B. Even-parity local pairings

In this section we review even-parity local pairing operators that give rise to pseudospin-singlet \( s \)- and \( d \)-wave channels when projected to the Fermi surface [50, 54]. We enumerate the nodal loop states that arise from individual \( d \)-wave pairing (see Table [1]), and which form the basis for nodal \( d + id \) Weyl superconductors in the sequel. Odd-parity pairings are considered in Sec. 11C.

The effective single-particle pairing Hamiltonian in the presence of local or intra-unit cell superconductivity assumes the form

\[
H^{local}_{pp} = \Delta_M \int d^3 \mathbf{r} \Psi^T M \Psi + \text{H.c.}, \tag{2.9}
\]

where \( M \) is a \( 4 \times 4 \) matrix, \( \Delta_M \) is the pairing amplitude, \( T \) is the matrix transpose, and H.c. denotes the Hermitian conjugate. The Pauli principle mandates that \( M^T = -M \), implying that there are only six possible independent bilinears of the form \( \Psi^T M \Psi \), since the allowed matrices correspond to the generators of SO(4). Therefore, the effective single-particle Hamiltonian in the presence of all possible local pairings reads as

\[
H^{local}_{pp} = \int d^3 \mathbf{r} \Psi^T \left[ \Delta_0 \Gamma_{13} + \Delta_1 \Gamma_3 + \Delta_2 \Gamma_{45} + \Delta_3 \Gamma_1 + \Delta_4 \Gamma_{25} + \Delta_5 \Gamma_{24} \right] \Psi + \text{H.c.}, \tag{2.10}
\]

where we have used the product basis for the Clifford algebra [Eq. (2.4)] to express the antisymmetric matrices. We stress that the existence of the above six local pairing operators does not depend on the character of the normal state; it relies only on the fact that the low-energy description of this state is captured by a four-component spinor. Identical pairing operators arise for massive Dirac fermions describing either topological or normal insulators [33, 65]. Weyl semimetals (where the four-component representation accounts for a pair of Weyl nodes) [66], and \( T \)-preserving nodal-loop semimetals [67]. However, the physical meaning of these local pairings crucially depends on the band structure of the parent state.

To characterize local pairings, we now introduce an eight-component Nambu spinor

\[
\Psi_N = \begin{bmatrix} \Psi \\ i \Gamma_{13} (\Psi^\dagger)^T \end{bmatrix}, \tag{2.11}
\]

where \( \Psi \) is the four-component spinor defined in Eq. (2.2). We have absorbed the unitary part of the time-reversal operator \( T \) in the lower block of the Nambu spinor. This ensures that \( \Psi_N \) transforms the same way as \( \Psi \) under spin SU(2) rotations, because the \( j = 3/2 \) generators \( \{ J^\mu \} \) satisfy the pseudoreality condition

\[
-\Gamma_{13} (J^\mu)^T \Gamma_{13} = J^\mu, \quad \mu \in \{ x, y, z \}. \tag{2.12}
\]

In this basis the single-particle Hamiltonian operator in the presence of six local pairings [introduced in Eq. (2.10)] assumes a simple and instructive form

\[
H^{local}_{pp} = \int d^3 \mathbf{r} \Psi_N^\dagger \hat{h}^{local}_{pp} \Psi_N, \tag{2.13}
\]

\[
\hat{h}^{local}_{pp} = (\tau_1 \cos \phi + \tau_2 \sin \phi) \times \left[ \Delta_0 \Gamma_0 + \Delta_1 \Gamma_1 + \Delta_2 \Gamma_2 + \Delta_3 \Gamma_3 + \Delta_4 \Gamma_4 + \Delta_5 \Gamma_5 \right],
\]

where \( \phi = \text{U(1) superconducting phase} \). The Pauli matrices \( \{ \tau_\alpha \} \) act on the particle-hole (Nambu) space.
Pairing near Fermi surface

| Pairing in LSM | Pairing near Fermi surface | IREP (Nature) | Quasiparticle spectrum |
|----------------|---------------------------|---------------|-----------------------|
| $\Delta_0 \Gamma_0$ | $\Delta_0 \sigma_0$ | $A_{1g}$ (s-wave) | Fully gapped |
| $\Delta_1 \Gamma_1$ | $\Delta_1 \hat{d}_1 \sigma_0 \equiv \sqrt{3} \Delta_1 \left( \hat{k}_x \hat{k}_z \right) \sigma_0$ | $T_{2g}$ ($d_{yz}$) | Gapless: 2 Nodal loops, $\begin{cases} k_x^2 + k_z^2 = k_F^2, k_y = 0 \\ k_x^2 + k_z^2 = k_F^2, k_y = 0 \end{cases}$ |
| $\Delta_2 \Gamma_2$ | $\Delta_2 \hat{d}_2 \sigma_0 \equiv \sqrt{3} \Delta_2 \left( \hat{k}_x \hat{k}_y \right) \sigma_0$ | $T_{2g}$ ($d_{xz}$) | Gapless: 2 Nodal loops, $\begin{cases} k_y^2 + k_z^2 = k_F^2, k_x = 0 \\ k_y^2 + k_z^2 = k_F^2, k_x = 0 \end{cases}$ |
| $\Delta_3 \Gamma_3$ | $\Delta_3 \hat{d}_3 \sigma_0 \equiv \sqrt{3} \Delta_3 \left( \hat{k}_y \hat{k}_x \right) \sigma_0$ | $T_{2g}$ ($d_{xy}$) | Gapless: 2 Nodal loops, $\begin{cases} k_y^2 + k_z^2 = k_F^2, k_x = 0 \\ k_y^2 + k_z^2 = k_F^2, k_x = 0 \end{cases}$ |
| $\Delta_4 \Gamma_4$ | $\Delta_4 \hat{d}_4 \sigma_0 \equiv \sqrt{3} \Delta_4 \left( \hat{k}_x^2 - \hat{k}_y^2 \right) \sigma_0$ | $E_g$ ($d_{x^2-y^2}$) | Gapless: 2 Nodal loops, $\begin{cases} k_+^2 + k_-^2 = k_F^2, k_+ = +k_y \\ k_+^2 + k_-^2 = k_F^2, k_+ = -k_y \end{cases}$ |
| $\Delta_5 \Gamma_5$ | $\Delta_5 \hat{d}_5 \sigma_0 \equiv \Delta_5 \left( 2 \hat{k}_x^2 - \hat{k}_y^2 \right) \sigma_0$ | $E_g$ ($d_{3z^2-r^2}$) | Gapless: 2 Nodal loops, $\begin{cases} k_{\pm} = k_F \sqrt{2} / 3, k_+ = + k_F \\ k_{\pm} = k_F \sqrt{2} / 3, k_+ = - k_F \end{cases}$ |

The identity matrix $\Gamma_0$ represents s-wave pairing. By contrast, as the Clifford matrices transform irreducibly in the $j = 2$ representation of the spin SU(2), the corresponding pairing channels $\Delta_{1,...,5}$ are all (effectively) d-wave. In terms of cubic symmetry, the pairing proportional to $\Gamma_0$ belongs to the trivial $A_{1g}$ representation. The three pairings proportional to $\Gamma_1, \Gamma_2, \Gamma_3$ transform as a triplet under the $T_{2g}$ representation. By contrast, those proportional to $\Gamma_4$ and $\Gamma_5$ transform as a doublet under the $E_g$ representation in a cubic environment.

In the Nambu basis, a Bogoliubov-de Gennes Hamiltonian $\hat{h}(\mathbf{k})$ automatically satisfies the particle-hole symmetry

$$-M_P \hat{h}(-\mathbf{k}) M_P = \hat{h}(\mathbf{k}), \quad M_P = \tau_2 \Gamma_{13},$$

owing to the reality condition $\Psi_N^\dagger(\mathbf{k}) = \Psi_N(-\mathbf{k})$ $M_P$ and Pauli exclusion. For momentum-independent pairing operators, Eqs. (2.12) and (2.14) imply that only tensor operators composed from products with even numbers of spin generators (e.g., $\{J^\mu J^\nu\}$) are allowed. These are precisely the identity and the anticommuting Clifford matrices (see Appendix A). Therefore, all d-wave pairings can also be considered as quadrupolar pairings. Since Eq. (2.14) is automatic, we can combine it with the usual form of time-reversal symmetry to get the chiral condition

$$-M_S \hat{h}(\mathbf{k}) M_S = \hat{h}(\mathbf{k}), \quad M_S = \tau_2.$$  (2.15)

Thus pairings in Eq. (2.13) proportional to $\tau_1$ ($\tau_2$) are even (odd) under time-reversal (for a fixed phase $\phi$).

We focus on superconductivity in an LSM doped to finite electron or hole density, away from charge neutrality. The nature of these pairings becomes transparent after projecting onto the valence or conduction band. With local pairings the result is the same for both bands. The details of the projection are presented in Appendix B. If we assume that pairing occurs only in close proximity to the Fermi surface and does not mix the bands, the projected pairing Hamiltonian takes the form

$$\hat{h}_\text{band} = (\tau_1 \cos \phi + \tau_2 \sin \phi) \sigma_0 \left[ \Delta_0 + \sum_{j=1}^5 \Delta_j \hat{d}_j \right],$$

as shown in Table I (see also Appendix A). Here the Pauli matrices $\{\sigma_j\}$ act on the pseudospin (Kramers) degenerate states within the projected band. All even-parity pairing operators map to pseudospin singlets.

The band-projected kinetic energy term arising from Eq. (2.3) assumes the simple form in the Nambu basis

$$\hat{h}_0^\text{band}(\mathbf{k}) = \left[ \pm \left( \frac{k^2}{2m^*} \right) - \mu \right] \tau_3 \sigma_0,$$  (2.17)
where \( m_\pm = m_0/m_0 \pm m \) (and \( m_0 > m \)). Here \(+\) (−) denotes the \( |m_\pm| = 1/2 \) conduction (\( |m_\pm| = 3/2 \) valence) band. While both the kinetic and pairing terms involve the Clifford matrices before the projection, only the kinetic term of the Luttinger Hamiltonian depends on the five \( d \)-wave harmonics \( d(k) \). Post projection, the kinetic term is trivial and the five pairing operators become \( d(k) \) components. Therefore, the band structure in the normal state plays a paramount role in determining the projected form of the local pairing operators. The Nambu-doubled four-component spinor describing quasiparticle excitations around the Fermi surface is defined as \( \psi_k = [c_{\uparrow, k}, c_{\downarrow, k}, c_{\downarrow, -k}, -c_{\uparrow, -k}] \), and the time-reversal operator in the reduced space reads as \( \mathcal{T} = i\sigma_2 K \), so that \( \mathcal{T}^2 = -1 \) as usual.

As a counterexample, we note that the same six local pairings projected into the valence/conduction bands in a massive Dirac semiconductor give rise to two even-parity \( s \)-wave and four pseudospin-triplet odd-parity \( p \)-wave pairing operators (including those of the \( B \)-phase and three planar pairings of \( ^3 \)He). The details of the projection in this case are shown in Appendix C and the results are displayed in Table III of that appendix.

The quasiparticle spectra inside each of the six local pairing states of the LSM are as follows. The \( s \)-wave pairing gives a fully gapped spectrum everywhere on the Fermi surface. On the other hand, each component of the five \( d \)-wave pairings supports two nodal loops, along which the Fermi surface remains gapless. The equations determining the nodal loop for each \( d \)-wave pairing are reported in the rightmost column of Table I. As a result each \( d \)-wave pairing supports “topologically protected” flatband surface states that span the images of the bulk nodal loops on each surface.

The density of states (DoS) in the presence of isolated nodal loops vanishes as \( \varrho(E) \sim |E| \). In Sec. III we will show that underlying cubic symmetry causes specific phase locking among different components of the \( d \)-wave pairings belonging to either \( T_{2g} \) or \( E_g \) representation (see Table I), at the cost of the time-reversal symmetry. Consequently, the paired state only supports simple Weyl nodes at a few isolated points on the Fermi surface, around which the DoS vanishes as \( \varrho(E) \sim |E|^2 \). Such reconstruction of the quasiparticle spectra thus causes a power-law suppression of the DoS and that way optimizes the condensation energy gain. Thus we expect Weyl superconductors to be energetically favored over the nodal-loop pairings, at least within the framework of weak BCS superconductivity and for dominant \( d \)-wave pairing coupling strengths.

### C. Odd-parity, momentum-dependent pairings

In the isotropic case \((m_1 = m_2 = m)\), odd-parity pairings can be classified via angular momentum addition \( [54] \). A basis of 10 Hermitian, particle-hole-odd operators with well-defined SU(2) spin \( j \) is given by [c.f. Eq. (2.13)]

\[
\tau_{1,2} \otimes \{ J^\mu, \quad J^\nu \} = \{ J^\mu, \quad J^\nu \} \quad \text{(2.18)}
\]

Here \( T^\mu \) is a completely symmetric, traceless tensor operator formed from triple products of \( J^\mu \) generators, see Eq. (A5). Eq. (2.14) implies that particle-hole allowed pairing operators obtain by multiplying any of the matrices in Eq. (2.18) by odd powers of momentum. The resulting momentum-dependent pairing operator with particle-hole matrix \( \tau_1 (\tau_2) \) is even (odd) under time-reversal [Eq. (2.15)].

For orbital \( p \)-wave pairing \((l = 1)\), angular momentum addition gives \( l \otimes j \equiv j_{\text{tot}} = 0 \otimes 1 \) for \( j = 1 \) and \( j_{\text{tot}} = 2 \otimes 3 \) for \( j = 3 \). We highlight a few combinations. The \( j_{\text{tot}} = 0 \) corresponds to a fully gapped, isotropic \( p \)-wave superconductor. For weak pairing this represents strong topological superconductivity \( [9, 47, 48] \), which we study in detail in Sec. VIII below. The \( j_{\text{tot}} = 1 \) state corresponds to gapless “\( p_z \)-wave” pairing. The \( j_{\text{tot}} = 2 \) states arising from \( j = 1 \) and \( j = 3 \) spin can mix, since only the total angular momentum is well-defined \( [54] \).

The \( p \)-wave “septet” order considered in the context of \( \text{YPtBi} \) is also built from the \( j = 3 \) operator \( [40, 53, 52] \). Finally, we note that isotropic \( f \)-wave pairing of the form

\[
\tau_{1,2} \otimes T^{\mu \nu} k_{\mu} k_{\nu} k_{\gamma}, \quad j_{\text{tot}} = 0,
\]

turns out to give the same band-projected Bogoliubov-de Gennes Hamiltonians as the isotropic \( p \)-wave case, Eq. (8.3), except that the pairing potential is multiplied by an additional factor of \( k^2 \) in each case.

### D. Free energy and gap equation

We now discuss the free energy and resulting gap equation for the five \( d \)-wave pairings summarized in Table I. At zero temperature the free energy of a superconductor is given by [69]

\[
F_j = \frac{|\Delta_d|^2}{2g_d} - \frac{a^3}{3} \int_{|k| < \Omega_D} \frac{d^3k}{(2\pi)^3} \sqrt{\xi_k^2 + |\Delta_d|^2d^2(k)},
\]

where \( d_j(k), j \in \{1, \ldots, 5\} \) are the \( d \)-wave harmonics (see the second column of Table I). \( \xi_k \equiv \frac{k^4}{2m} - \mu \), \( a \) denotes the lattice spacing, \( g_d \) is the coupling strength, and we set \( \hbar = 1 \) throughout. The pairing is restricted to an energy window set by the (effective) Debye frequency \( \Omega_D \). We introduce dimensionless variables defined via \( f_j = F_j/|\mu|^2 \varrho(\mu) \), \( x = |k|/k_F, \lambda_d = g_d \varrho(\mu), \Delta_d = \Delta_d/\mu \), and \( \omega_D \equiv \Omega_D/\mu \), where \( \varrho(\mu) = 2a^3m^*\sqrt{2m\mu^2/(2\pi^2)} \) is the DoS at the Fermi level. Eq. (2.19) can be written as

\[
f_j = \frac{\hat{\Delta}_d^2}{2\lambda_d} - \int_{-\omega_D}^{\omega_D} dy \frac{\sqrt{1 + y^2}}{2} \int d\Omega \frac{4\pi}{4\pi} E_j(y, \Omega),
\]
where $y = x^2 - 1$, $\Omega$ denotes the angular variables, and $E_j(y, \Omega) \equiv \sqrt{y^2 + |\Delta_d|^2 j^2(y, \Omega)}$ is the (dimensionless) bulk quasiparticle energy.

The zero-temperature gap equation is obtained from $\frac{df_j}{d\Delta_d} = 0$, leading to

$$\frac{1}{\lambda_d} = \int_{-\omega_D}^{\omega_D} dy \frac{\sqrt{1 + y}}{2} \int d\Omega \frac{d^2 j(y, \Omega)}{4\pi E_j(y, \Omega)}. \quad (2.21)$$

The finite-temperature version is

$$\frac{1}{\lambda_d} = \int_{-\omega_D}^{\omega_D} dy \frac{\sqrt{1 + y}}{2} \int d\Omega \frac{d^2 j(y, \Omega)}{4\pi E_j(y, \Omega)} \tanh \left[ \frac{E_j(y, \Omega)}{2k_B T} \right]. \quad (2.22)$$

Unlike in the s-wave case, the complicated d-wave form factors do not allow an analytic solution to Eqs. (2.21) and (2.22), and one must seek it numerically. However, we can simplify the zero-temperature gap equation (2.21) in the weak-coupling limit, when pairing takes place within a thin shell around the Fermi momentum so that $\omega_D \ll 1$ (and thereby $y \ll 1$). We then obtain

$$\frac{1}{\lambda_d} = \int d\Omega \frac{d^2 \delta(\Omega)}{4\pi} \ln \left[ \frac{\omega_D + \sqrt{\omega_D^2 + |\Delta_d|^2 \cdot \delta^2(\Omega)}}{|\Delta_d| \cdot \delta(\Omega)} \right], \quad (2.23)$$

where $\delta(\Omega)$ is a purely angle dependent form-factor [e.g., for $d_{x^2-y^2}$ pairing, $\delta(\Omega) = \frac{\sqrt{2}}{\pi} \sin^2 \theta \cos(2\phi)$]. To solve this equation analytically, we make the standard weak-coupling approximation, $|\Delta_d| \ll \omega_D$, as is usually done in BCS theory. The quantity in the numerator under the logarithm can then be replaced by $2\omega_D$ and the integral over the solid angle can now be performed analytically, yielding

$$\Delta_{x^2-y^2}(0) = 3.355 \omega_D \exp(-5/\lambda_d); \quad \Delta_{3z^2-r^2}(0) = 3.501 \omega_D \exp(-5/\lambda_d). \quad (2.24)$$

The appearance of the factor of 5 in the exponent can be traced to the angle-average of the d-wave form-factors $\int d\Omega \frac{d^2 \delta(\Omega)}{(4\pi)} = 1/5$, resulting in an exponential suppression of the gap compared to the well known s-wave result: $\Delta_s(T = 0) \approx 2\omega_D \exp(-1/\lambda_s)$ [69].

It may appear surprising, at first glance, that the two $E_g$ d-wave harmonics appear to have different values of the superconducting gap in Eq. (2.24) at $T = 0$. This is not a mistake, and the numerically exact solution of the gap equation Eq. (2.22) leads to the identical conclusion, see Fig. 1 with about 4% difference between the zero-temperature values of the gap. The fundamental reason why $x^2 - y^2$ and $3z^2 - r^2$ harmonics have different solutions is because they feature different arrangements of the nodal loops, see Fig. 2. Despite belonging to the same irreducible representation of the cubic point group ($O_h$), it is obvious that there is no way to rotate these two harmonics into each other by any SO(3) rotation. By contrast, $d_{x^2-y^2}$ can be trivially rotated into any of the three $T_{2g}$ harmonics (namely $xy$, $yz$ and $zx$), which is why they must all have the same gap value at $T = 0$, even though they belong to different irreducible representations (assuming that the interaction strengths and Debye frequencies in $E_g$ and $T_{2g}$ channels are identical).

So how does one resolve the apparent contradiction with a commonly held belief that the cubic symmetry enforces the two $E_g$ solutions to have identical gap values? As a matter of fact, this belief, as phrased above, is incorrect; what is true is that the members of the two-dimensional irreducible representation $E_g$ must have identical transition temperature $T_c$, which can be reconciled from its expression from Eq. (2.22), leading to

$$\frac{1}{\lambda_d} = \int_{-\omega_D}^{\omega_D} dy \frac{(1+y)^\frac{3}{2}}{2y} \tanh \left( \frac{y}{2k_B T_c} \right) \int d\Omega \frac{d^2 j^2(\Omega)}{4\pi}, \quad (2.25)$$

Symmetry requires that $\int d\Omega \frac{d^2 j^2}{(4\pi)} = 1/5$ for all d-wave harmonics (belonging to $E_g$ as well as $T_{2g}$ representations), and thus all five d-wave pairings must have the identical $T_c$. In fact, one can calculate $T_c$ analytically in the weak-coupling approximation ($\omega_D \equiv \Omega_D/\mu \ll 1$), yielding

$$k_B T_c = \frac{2e^{\gamma}}{\pi} \omega_D e^{-\frac{\gamma}{2}} \approx 1.134 \omega_D e^{-\frac{\gamma}{2}}, \quad (2.26)$$

where $\gamma \approx 0.577$ is the Euler’s number. It follows from the above and from Eq. (2.24) that the ratio

$$\frac{\Delta_d(0)}{k_B T_c} = \begin{cases} 2.96 & \text{for } x^2 - y^2, \ xy, \ xz, \ yz \quad (2.27) \\ 3.09 & \text{for } 3z^2 - r^2, \end{cases}$$

FIG. 1: The temperature dependence of the superconducting gap for (a) $d_{x^2-y^2}$ and (b) $d_{3z^2-r^2}$ pairings, calculated with $\lambda_d = 1$ and $\omega_D = 0.02$. The two gaps have identical transition temperature ($T_c$) given by Eq. (2.26), but are generally different at any lower temperature $T < T_c$, with the zero-temperature values approximately given by Eq. (2.24).
We explicitly demonstrate below that pairing energy minimization and the underlying cubic symmetry cause specific phase locking amongst various components of the \( d \)-wave pairings in both the \( E_g \) and \( T_{2g} \) sectors. As a result, simple Weyl superconductors are expected to emerge at low temperature if the pairing strength in the \( d \)-wave channel dominates. We first review the nodal topology of such Weyl superconductors, since we will be interested in its manifestation in various measurable quantities (such as the anomalous thermal and pseudospin Hall conductivities, discussed in Sec. [III E]).

A. Topology of Weyl superconductors

Since the \( s \)-wave and all \( d \)-wave pairings are all band pseudospin-singlets we can further simplify the reduced BCS Hamiltonian \( \hat{h}_{\text{pair}} = \hat{h}_0 \text{band}(\mathbf{k}) + \hat{h}_{\text{pair}} \) [see Eqs. (2.16) and (2.17)] as a direct sum of two \( 2 \times 2 \) blocks (reflecting the pseudospin-degeneracy inside the paired state). To illustrate the nodal topology of such a system, it is now sufficient to consider one such block, which can schematically be written as

\[
\hat{h}_k = E_k \left[ \hat{u}_k \cdot \tau \right].
\]

For simple Weyl nodes \( E_k \) is a general linear function of all three momenta, but its specific form does not affect the following discussion. Here \( \hat{u}_k \) is a unit vector, a function of only polar (\( \theta \)) and azimuthal (\( \phi \)) angles, and the \( \tau \)s are three standard Pauli matrices operating on the particle-hole/Nambu index.

The monopole charge of a Weyl node \( (W_n) \) is then defined as

\[
W_n = \frac{1}{4\pi} \int_0^\pi d\phi_k \int_0^\pi d\theta_k \left[ \hat{u}_k \cdot \left( \frac{\partial \hat{u}_k}{\partial \theta_k} \times \frac{\partial \hat{u}_k}{\partial \phi_k} \right) \right],
\]

which for simple Weyl nodes \( W_n = \pm 1 \) (see also discussion in Refs. [66, 70–74]). The Weyl node with monopole charge +1 (−1) corresponds to a source (sink) of Abelian Berry curvature of unit strength; this is ultimately responsible for the anomalous thermal and spin-Hall conductivities, which we discuss in Sec. [III E].

The topological nature of the BdG-Weyl quasiparticles can also be assessed from the gauge invariant Abelian Berry curvature \( \Omega_{n,k} \), given by

\[
\Omega_{n,k,a} = \frac{(-1)^n}{4} \epsilon_{abc} \hat{u}_k \cdot \left( \frac{\partial \hat{u}_k}{\partial k_b} \times \frac{\partial \hat{u}_k}{\partial k_c} \right),
\]

with \( a, b, c = x, y, z \), and \( n = 1, 2 \) are the Bogoliubov band indices. The Berry curvature distribution in various Weyl superconducting phases will be displayed below.

Due to the bulk-boundary correspondence, Weyl superconductors (arising from time-reversal symmetry breaking combinations of \( d + id \) pairings, for example)
support topologically protected pseudospin-degenerate Fermi arc surface states, which connect the projections of the Weyl nodes on the surface in the reciprocal space. On the other hand, in the presence of a nodal-loop pairing the pseudospin degenerate surface states are completely flat and correspond to the images of the bulk loop (such as the ones shown in Fig. 2, 3). A detailed analysis of these topologically protected surface Andreev bound states is left as a subject for future investigation. In the absence of inversion symmetry such surface states lose the Kramer/pseudo-spin degeneracy, which could be directly observed in scanning tunneling microscopy (STM) measurements, for example.

### B. $E_g$ pairing

We first investigate the effect of underlying cubic symmetry in the $E_g$ channel. Since $E_g$ is a two-component representation, encompassing $d_{x^2-y^2}$ and $d_{3z^2-r^2}$ pairings, optimal minimization of the condensation energy then enforces nucleation of $d_{x^2-y^2} + id_{3z^2-r^2}$ pairing (within the framework of weak-coupling superconductivity). The matrix coefficients in the reduced BCS Hamiltonian

$$
\hat{h}_{\text{pair}}^{E_g} = \left( \frac{k^2}{2m} - |\mu| \right) \tau_3 + \frac{\sqrt{3} |\Delta_4|}{2k_F} (k_x^2 - k_y^2) \tau_1 \\
+ \frac{|\Delta_5|}{2k_F} (2k_x^2 - k_y^2 - k_z^2) \tau_2,
$$

then appear as sum of the squares in the expression for the Bogoliubov dispersion, given by

$$
E^E_g = \left[ \left( \frac{k^2}{2m} - |\mu| \right)^2 + \frac{|\Delta_{E_g}|^2}{4k_F^2} \left( 3 (k_x^2 - k_y^2)^2 + (2k_x^2 - k_y^2 - k_z^2)^2 \right) \right]^{1/2},
$$

where $k_F = \sqrt{2m|x|}$ is the Fermi momentum. Without any loss of generality, but for the sake of simplicity we set $\Delta_{E_g} = \Delta_4 = \Delta_5$ in the last expression. Here and in what follows, we assume that $\mu > 0$ for weak BCS superconductivity that arises from a spherical Fermi surface in the conduction band, and we use $m_*$ to represent the associated band mass $[m^*\mu$ in Eq. (2.17)]. The conclusions are identical for the hole-doped system. The time-reversal symmetry in such a paired state is spontaneously broken, and the quasiparticle spectra vanishes only at eight isolated points on the Fermi surface

$$
\pm k_x = \pm k_y = \pm k_z = k_F/\sqrt{3},
$$

precisely where the nodal loops for individual $d_{x^2-y^2}$ and $d_{3z^2-r^2}$ pairings cross each other (see Table). These isolated points are Weyl nodes and the phase can be considered a thermal Weyl semimetal, since the BdG-Weyl quasiparticles carry well-defined energy (but not well-defined electric charge). At the cost of shedding the time-reversal symmetry, the $d_{x^2-y^2} + id_{3z^2-r^2}$ paired state eliminates the line-nodes of its individual components (see the fifth and sixth rows of Table). The distribution of the Abelian Berry curvature for the $d_{x^2-y^2} + id_{3z^2-r^2}$ Weyl superconductor is shown in Fig. 3.

### I. Competition within $E_g$

Given the discussion in Sec. 3D, we know that the two $E_g$ components have different values of the superconducting gap below $T_c$. The basis of the $E_g$ representation obtained from two independent diagonal components of a symmetric, $3 \times 3$ traceless tensor [see Eq. (4.3)] is therefore not unique. Indeed, dropping the normalization factors for brevity, one can define the two basis functions as either

**Basis A:** $d_1(k) = k_x^2 - k_y^2, d_2(k) = 2k_z^2 - k_x^2 - k_y^2$. 

**Basis B:** $d_3(k) = k_x^2 - k_y^2, d_4(k) = k_z^2 - k_y^2$.

Note that in this subsection and in Appendix only, $d_1,...,d_8$ are the above-defined $E_g$ sector harmonics. This
is a different notation than that employed everywhere else in this paper, as exemplified by Table 1.

Notice that bases A, B, C are not independent of one another; for instance, \( d_4 - d_1 = d_1, d_2 + d_4 = d_2, d_5 + d_6 = -d_2 \) and so on. Nevertheless, these bases are distinct in the sense that no SO(3) rotation can convert one basis into another. As a result, the corresponding gaps will have different configurations of nodal loops that cannot be interconverted by rotations, and also different gap values! This raises a non-trivial question: which one of these three bases has the lowest energy, when we allow to form a time-reversal symmetry breaking \( (d_m + id_n) \) order parameter? The details of the analysis are provided in Appendix D. Here we quote only the final results.

It turns out that the lowest energy state is achieved from the Basis \( A \), with the zero-temperature value of the superconducting gap given by

\[
\Delta^{(A)}_{E_g}(T = 0) = 2.705 \omega_d \exp \left( -\frac{5}{2\lambda_d} \right),
\]

in the weak-coupling approximation. The numerical prefactor 2.705 is however non-universal (see Appendix D for derivation). Similarly, for the other two bases we find

\[
\Delta^{(B)}_{E_g}(T = 0) = 2.451 \omega_d \exp \left( -\frac{5}{2\lambda_d} \right),
\]

(3.10)

\[
\Delta^{(C)}_{E_g}(T = 0) = 2.145 \omega_d \exp \left( -\frac{5}{2\lambda_d} \right).
\]

(3.11)

The condensation energy gain in the \( (d_m + id_n) \) state is given, to the lowest order in \( |\Delta|^2 \), by

\[
f_{SC} - f_N \approx -\frac{\Delta_{E_g}}{4} \int \frac{d\Omega}{4\pi} \left[ \hat{d}_m^2(\Omega) + \hat{d}_n^2(\Omega) \right]
\]

\[
= -\frac{\Delta_{E_g}}{10} + \mathcal{O}(|\Delta_{dM}|^4),
\]

(3.12)

and thus the solution with the largest value of the zero-temperature gap, namely the \( (d_{x^2-y^2} + id_{3z^2-r^2}) \) paired state, has the lowest energy. However, the location of the nodal points in any \( d + id \) paired state is always given by Eq. (3.6), irrespective of the choice of basis.

2. Nodal topology

We now investigate the nodal topology of the eight isolated Weyl nodes inside the \( d_{x^2-y^2} + id_{3z^2-r^2} \) paired state. Since the Weyl nodes are placed along eight possible \( [1,1,1] \) directions, in order to comprehend the nodal topology of this state, we introduce a rotated co-ordinate frame according to

\[
q_x = \frac{k_x + k_y - 2k_z}{\sqrt{6}}, \quad q_y = \frac{k_x - k_y}{\sqrt{2}}, \quad q_z = \frac{k_x + k_y + k_z}{\sqrt{3}},
\]

(3.13)

keeping our focus on the Weyl nodes located at \( k = \pm (1,1,1)k_F/\sqrt{3} \). In this rotated co-ordinate system the Weyl nodes are located at \( q = (0,0,\pm 1)k_F \). The reduced BCS Hamiltonian [see Eq. (3.4)] for the \( d_{x^2-y^2} + id_{3z^2-r^2} \) state then becomes

\[
\hat{h}_{pair}^{E_g} = \left( \frac{q_t^2}{2m_s} - \mu \right) \tau_3 \pm \sqrt{2} |\Delta_{E_g}| k_F \left( \tau_1 q_x + \tau_2 q_y \right)
\]

\[
+ \frac{|\Delta_{E_g}|}{2k_F} \left( \tau_1 [q_x^2 - q_y^2] + \tau_2 [2q_x q_y] \right), \quad (3.14)
\]

after setting \( q_x = \pm k_F \) in the second term. Notice in the close proximity to the Weyl nodes \( q_x, q_y \ll k_F \), and the second term in the above expression becomes dominant over the last one, which therefore can be neglected. Upon defining \( \delta q_z = q_z \pm k_F \) (measuring the deviation from the Weyl nodes along the radial direction) and introducing a new set of parameters (bearing the dimension of Fermi velocity) according to

\[
v_x = v_y = \sqrt{2} |\Delta_{E_g}| / k_F, \quad v_z = k_F / m_s,
\]

(3.15)

the above reduced BCS Hamiltonian can be brought into a compact form

\[
\hat{h}_{pair}^{E_g} = \pm [v_x \tau_1 q_x + v_x \tau_2 q_y + v_z \tau_3 \delta q_z]. \quad (3.16)
\]

Eq. (3.2) then implies that the Weyl nodes located at \( k = \pm (1,1,1)k_F/\sqrt{3} \) are characterized by monopole charge \( W_n = \pm 1 \) [see Eq. (3.2)]. We also find that the Weyl nodes located at \( k = (-1,-1,1)k_F/\sqrt{3}, \quad (-1,1,-1)k_F/\sqrt{3} \) and \( (1,-1,-1)k_F/\sqrt{3} \) are characterized by monopole charge \( W_n = -1 \). On the other hand, the Weyl nodes located at \( k = (1,1,1)k_F/\sqrt{3}, \quad (-1,1,1)k_F/\sqrt{3} \) and \( (1,-1,-1)k_F/\sqrt{3} \) have monopole charge \( W_n = -1 \). See also Fig. 3.

3. Scaling of density of states

The identification of the nodal topology allows us to estimate the scaling of the DoS inside the \( d_{x^2-y^2} + id_{3z^2-r^2} \) paired state. Since the Weyl nodes bear monopole charge \( W_n = \pm 1 \), the DoS at low enough energy vanishes as \( \varrho(E) \sim |E|^2 / |W_n| \sim |E|^2 \). On the other hand, as we approach the high-energy regime the second term from Eq. (3.14) becomes relevant too. Note that if we completely neglect the first term (linear in \( q_x \) and \( q_y \)), the reduced BCS Hamiltonian assumes the form of a double Weyl node with monopole charge \( W_n = \pm 2 \), for which the DoS scales as \( \varrho(E) \sim |E| \). Therefore, as we scan from low to high energy the DoS for BdG quasiparticle shows a smooth crossover from \( \varrho \sim |E|^2 \to |E| \) behavior. Recall the DoS in the presence of a nodal line also scales as \( \varrho(E) \sim |E| \). Therefore, by sacrificing the time-reversal symmetry the system gains condensation energy through power-law suppression of the DoS at low energies.
C. $T_{2g}$ pairing

Since $T_{2g}$ is a three-component representation, we denote the phases of the complex superconducting pairing amplitudes associated with the $d_{xy}$, $d_{xz}$ and $d_{yz}$ pairings as $\phi_{xy}$, $\phi_{xz}$ and $\phi_{yz}$, respectively. The nodal loops associated to each of the three pairing channels in isolation can only be eliminated by the choice

$$ (\phi_{xy}, \phi_{xz}, \phi_{yz}) = P \left( 0, \frac{2\pi}{3}, \frac{4\pi}{3} \right), \quad (3.17) $$

where $P$ stands for eight possible permutations \[50\]. The resulting quasiparticle spectrum exhibits eight isolated gapless points on the Fermi surface. In particular, for the specific choice $(\phi_{xy}, \phi_{xz}, \phi_{yz}) = (0, 2\pi/3, 4\pi/3)$ the reduced BCS Hamiltonian reads as

$$ \hat{h}_{pair}^{T_{2g}} = \left( \frac{k^2}{2m_\ast} - |\mu| \right) \tau_0 + \frac{\Delta_{T_{2g}}}{k_F^2} \left[ \frac{\sqrt{3}}{4} k_z (k_x - k_y) \tau_2 + \frac{\sqrt{3}}{2} \left( k_x k_y - \frac{1}{2} k_z^2 k_x - \frac{1}{2} k_z k_y \right) \tau_1 \right], \quad (3.18) $$

and the energy spectrum

$$ E_k^{T_{2g}} = \left[ \left( \frac{k^2}{2m_\ast} - |\mu| \right) + \frac{3|\Delta_{T_{2g}}|^2}{4k_F^2} \left\{ k_y k_x - \frac{1}{2} k_z k_z \right. \right. 

$$

$$ \left. \left. - \frac{1}{2} k_y k_z \right\}^2 + \frac{3}{4} k_z^2 \left( k_x - k_y \right)^2 \right\}^{1/2}, \quad (3.19) $$

with $\Delta_{T_{2g}} = \Delta_1 = \Delta_2 = \Delta_3$, vanishes at

$$(a) = (0, 0, \pm 1) k_F, \quad (b) = (0, \pm 1, 0) k_F, \quad (c) = (\pm 1, 0, 0) k_F, \quad (d) = (1, 1, 1) k_F/\sqrt{3}. \quad (3.20)$$

Note that the pairs of Weyl nodes denoted by $(a)$, $(b)$ and $(c)$ are located on the three $C_{3v}$ axes, while the Weyl nodes $(d)$ are located on one of the four $C_{3v}$ axes. As we discuss below, any other phase locking amongst the three components of the $d$-wave pairing produces at least one nodal loop in the quasiparticle spectrum. Thus within the framework of a weak-coupling pairing mechanism the above phase locking is energetically most favored. The distribution of the Abelian Berry curvature in the presence of this pairing is shown in Fig. [4]. Next we discuss the nodal topology of the Weyl nodes reported in Eq. [3.20].

1. Nodal topology

The reduced BCS Hamiltonian in the close proximity to the Weyl nodes $(a)$ [see Eq. [3.20]], assumes the form

$$ \hat{h}_{pair,(a)}^{T_{2g}} = \pm [\tau_3 v_z \delta p_z - \tau_1 v_x p_x + \tau_2 v_y p_y], \quad (3.21) $$

where

$$ \delta p_z = k_z \pm k_F, \quad p_x = k_x + k_y, \quad p_y = k_x - k_y, $$

$$ v_z = k_F/m_\ast, \quad v_x = v_y = \sqrt{6} |\Delta_{T_{2g}}|/4k_F. \quad (3.22) $$

Therefore, the Weyl nodes located at $k = (0, 0, \pm k_F)$ have monopole charge $W_n = \mp 1$.

Next we focus near the Weyl nodes $(b)$ [see Eq. [3.20]], around which the reduced BCS Hamiltonian reads as

$$ \hat{h}_{pair,(b)}^{T_{2g}} = \pm \left[ \tau_3 v_y \delta p_y + \tau_1 v_x \left( \frac{3p_x + p_z}{2} \right) - \tau_2 v_z \sqrt{3} \left( -p_x + p_z \right) \right], \quad (3.23) $$

where

$$ \delta p_y = k_y \pm k_F, \quad p_x = k_x - k_y, \quad p_z = k_x + k_y, $$

$$ v_y = k_F/m_\ast, \quad v_x = v_z = \sqrt{3} |\Delta_{T_{2g}}|/2k_F. \quad (3.24) $$

Then following the arguments shown in Sec. [III A] we find that the Weyl nodes located at $k = (0, \pm k_F, 0)$ are accompanied by monopole charge $W_n = \pm 1$.

The reduced BCS Hamiltonian in the close proximity
to the Weyl nodes (c) [see Eq. (3.20)] assumes the form

\[
\hat{\mathcal{H}}_{\text{pair}, (c)} = \pm \left[ \tau_3 v_x \delta p_x + \tau_1 v_y \left( \frac{3p_y + p_z}{2} \right) + \tau_2 v_z \sqrt{3} \left( -\frac{p_y + p_z}{2} \right) \right],
\]

(3.25)

where

\[
\delta p_x = k_x \pm k_F, \quad p_y = \frac{k_y - k_x}{\sqrt{2}}, \quad p_z = \frac{k_y + k_z}{\sqrt{2}},
\]

\[
v_x = k_F/m_\star, \quad v_y = v_z = \sqrt{3} |\Delta T_{2g}|/2k_F.
\]

(3.26)

The Weyl nodes located at \( k = (\pm k_F, 0, 0) \) have monopole charge \( W_a = \mp 1 \).

Following the discussion presented in Sec. III B 2, we can immediately come to the conclusion that the Weyl nodes (d) [see Eq. (3.20)] are also simple, and the members \( k = (\pm 1, 1, 1)k_F/\sqrt{3} \) have monopole charge \( W_a = \pm 1 \). The scaling of the DoS due to this pair of Weyl nodes are identical to the one discussed in Sec. III B 3 while the DoS scales as \( g(E) \sim |E|^2 \) due to the pair of Weyl nodes tabulated as (a), (b) and (c) in Eq. (3.20). Therefore, all eight Weyl nodes arising due to the pairing in the \( T_{2g} \) sector are simple Weyl nodes with unit monopole charge, similar to the situation for \( E_g \) pairing. However, the arrangement of these Weyl nodes on the Fermi surface are completely different in these two sectors (see Fig. 4 and compare with Fig. 3), which bears important consequences for the anomalous thermal and pseudospin Hall conductivities. We will come back to this issue in Sec. III E.

2. Alternative phase locking in \( T_{2g} \) channel

We now briefly discuss a few other possible phase lockings among the three \( T_{2g} \) component pairings, the spectrum of BdG quasiparticles and emergent topology inside each corresponding ordered phase, as well as their competition with the paired state discussed above. Besides the \( T_{2g} \) phase locking in Eq. (3.17), other possible ones are the following: (i) \( \{\phi_{xy}, \phi_{xz}, \phi_{yz}\} = \mathcal{P} \{0, 0, 0\}, \quad \Delta_1 = \Delta_2 = 0 \},\)

(ii) \( \{\phi_{xy}, \phi_{xz}, \phi_{yz}\} = \mathcal{P} \{0, 0, \frac{\tau}{2}\}, \quad \Delta_3 = 0 \},\)

(iii) \( \{\phi_{xy}, \phi_{xz}, \phi_{yz}\} = (0, 0, 0) \). Here, \( \mathcal{P} \) encompasses all possible permutations.

The single-component paired state (i) supports two nodal loops. The equations of these two nodal loops, along which the gap on the Fermi surface vanishes, are given in the fourth row of Table I for example. The reduced BCS Hamiltonian with relative phase locking (ii) in the above list reads as

\[
\hat{\mathcal{H}}_{\text{pair}, (ii)} = \left( \frac{k^2}{2m_\star} - \mu \right) \tau_3 + \sqrt{3} |\Delta T_{2g}| k_z (k_x \tau_1 + k_y \tau_2),
\]

(3.27)

\[ \hat{\mathcal{H}}_{\text{pair}, (i)} = \pm \left[ v_z \delta k_z \tau_3 + v_x k_x \tau_1 + v_y k_y \tau_2 \right], \]

(3.28)
Note that while (i) and (iii) produce two nodal loops in the spectrum of the BdG quasiparticle excitations, (ii) yields only one nodal loop and a pair of simple Weyl points. Therefore, at least within the weak coupling scenario for pairing the relative phase locking following the pattern (ii) appears to be energetically more favored among these three possibilities. Recently, a similar pairing \([k_z(k_x + i k_y)]\) pairing has also been discussed in the context of URu\(_2\)Si\(_2\) [70], possessing tetragonal symmetry. However, in a cubic environment the \(k_z(k_x + i k_y)\) pairing can be energetically inferior to the one discussed in Sec. III C with \((\phi_{xy}, \phi_{xz}, \phi_{yz}) = (0, 2\pi/3, 4\pi/3)\) for example, since this pairing only produces eight isolated simple Weyl nodes on the Fermi surface, yielding \(g(E) \sim |E|^2\) and thereby causing power-law suppression of the DoS at low energies.

D. Competition between \(E_g\) and \(T_{2g}\) pairings

We now briefly discuss the competition among various \(d\)-wave pairings when the pairing interaction in the \(E_g\) and \(T_{2g}\) channels, respectively denoted by \(g_{E_g}\) and \(g_{T_{2g}}\) (say), are of comparable strength. Under this circumstance, two distinct possibilities can arise: (a) These two paired states are separated by a first-order transition with the pairings discussed in Sec. III B and Sec. III C residing on opposite sides of the discontinuous transition, respectively for \(g_{E_g} > g_{T_{2g}}\) and \(g_{T_{2g}} > g_{E_g}\), or (b) there can be a region, roughly when \(g_{E_g} \sim g_{T_{2g}}\), where pairings belonging to these two distinct representation can coexist. Leaving aside the possibility (a), we here further elaborate on the second scenario, by restricting ourselves to a weak coupling pairing picture.

When pairing from these two channels coexists, at the cost of the time-reversal symmetry, one can minimize the number of gapless points on the Fermi surface (thereby causing gain in the condensation energy). Since \(T_{2g}\) and \(E_g\) channels are respectively three- and two-component representations, all together we can find six possible time-reversal symmetry breaking paired phases (note these are simplest possibilities), shown in the first column of Table [III].

Following the discussion and methodology presented earlier in this section, we realize that only the \(d_{x^2−y^2} + id_{xy}\) paired state gives rise to double-Weyl points, with \(W_n = \pm 2\), on two poles of the Fermi surface. The DoS of low-energy BdG quasiparticles in the presence of double-Weyl nodes goes as \(g(E) \sim |E|\). More detailed discussion on the nodal topology of the \(d_{x^2−y^2} + id_{xy}\) paired state is presented in the next section. The rest of the pairings only support simple Weyl nodes with \(W_n = \pm 1\) [see Appendix F], and result in \(g(E) \sim |E|^2\) at low energies.

We also note that in the \(d_{xz} + id_{yz}\) and \(d_{yz} + id_{xz}\) paired states, besides the simple Weyl nodes in the \(k_x−k_y\) plane there also exist a pair of nodes at two opposite poles of the Fermi surface. With the former pairing the reduced BCS Hamiltonian around the poles reads as

\[
H_{d+id}^\text{pole} = \pm v_z \delta k_z \tau_3 \pm v_x k_x - \frac{|\Delta_{T_{2g}}|}{k_F^2} k_y^2 \tau_2,
\]

where \(v_z = k_F/m, v_x = |\Delta_{T_{2g}}|/k_F,\) and \(\delta k_z = k_z \pm k_F\). For such isolated nodes \(W_n = 0\), therefore, this pair of nodes are non-topological in nature and their exis-
tence is purely accidental. However, if such a node exists the DoS near the pole vanishes as \( g(E) \sim |E|^{3/2} \), and the low-energy thermodynamic responses of the \( d_{xz}/y_z + id_{x^2-y^2} \) states will be dominated by these accidental nodes. We postpone any further discussion on the competition among all six time-reversal symmetry breaking paired states and the nature of the ultimate ground state for a future work.

### E. Anomalous thermal and spin Hall conductivities

One hallmark signature of spin-singlet pairing is the separation of the spin and charge degrees of freedom. Electric charge is carried by the superconducting condensate, a macroscopic collection of charge 2e spinless bosonic Cooper pairs, while spin is fully carried by the fermionic excitations (BdG quasiparticles) that do not carry definite electric charge. In particular, such spin-charge separation bears important consequences for non-s-wave singlet pairing, such as \( d \)-wave superconductivity. For example, in a spin-singlet \( d \)-wave superconductor with broken time-reversal symmetry, the BdG quasiparticles can give rise to anomalous spin and thermal Hall conductivities.

One well-studied example is the \( d_{x^2-y^2} + id_{xy} \) state, which could be germane to cuprate high-\( T_c \) superconductors \( \text{[75, 80]} \). A state with this symmetry is also possible in the LSM (see Table \( \text{[1]} \)). Recently this pairing has also been discussed in the context of URu\(_2\)Si\(_2\) \( \text{[70]} \) and SrPtAs \( \text{[72]} \). Such a paired state bears close resemblance to the integer quantum Hall effect. In two dimensions (where it is fully gapped), the \( d_{x^2-y^2} + id_{xy} \) state supports quantized spin (since spin is a conserved quantity) and thermal (since energy is conserved) Hall conductivities\(^1\) \( \text{[81, 85]} \). We here do not discuss the experimental setup for the measurement of the anomalous spin or thermal Hall conductivities, which are readily available in the literature \( \text{[83, 85]} \). Instead we emphasize these two responses inside various Weyl superconductors that can directly probe the net Berry flux enclosed by the paired phase, while the lack of the time-reversal symmetry can directly be probed by Faraday and Kerr rotations \( \text{[80]} \). We also note that in the absence of inversion symmetry (which is the situation in half-Heusler compounds) the notion of (pseudo)spin Hall conductivity becomes moot, while thermal Hall conductivity remains well-defined.

Before turning our focus on the \( E_g \) and \( T_{2g} \) paired states, let us pick a specific example of a Weyl superconductor, \( d_{x^2-y^2} + id_{xy} \) pairing, which can be found (at least in principle) when pairing interactions in these two channels are of comparable strength. The Weyl nodes in this paired phase are located at \( \mathbf{k} = (0, 0, \pm k_F) \) (along

---

\(^1\) Note that there is no notion of a charge Hall conductivity inside a paired state since the global U(1) gauge symmetry gets broken by the Anderson-Higgs mechanism.
\( \hat{k}_z \) direction, and as we discussed above they are characterized by monopole charge \( W_n = \pm 2 \). The reduced BCS Hamiltonian for such a pairing in the \( k_z = 0 \) plane is

\begin{equation}
\hat{h}_{d+id}(k, k_z = 0) = \left\{ \left( \frac{k^2}{2m_*} - \mu \right) \tau_3 + \frac{\Delta T_{2g}}{k_F^2} (2k_z k_y) \tau_1 + \frac{\Delta E_g}{k_F^2} (k_x^2 - k_y^2) \tau_2 \right\} \otimes \sigma_0, \tag{3.31}
\end{equation}

where \( k^2 = k_x^2 + k_y^2 \) which describes a quantum anomalous thermal/spin Hall insulator, characterized by the Chern-Number \( C_n = 2 \) in the \((k_x, k_y)\) plane. Appearance of the Pauli matrix \( \sigma_0 \) reflects the singlet nature of this pairing and the fact that the band pseudospin is a good quantum number inside the paired state. Note that the pseudospin texture in the \((k_x, k_y)\) plane associated with the reduced BCS Hamiltonian in Eq. (3.31) assumes the form of a skyrmion, and the skyrmion number is the Chern number, which can be anchored in the following way. If we express the above Hamiltonian as \( \hat{h}_{d+id}(k, k_z = 0) = E_{k_{1z}} \left[ \hat{n}_{k_{1z}} \cdot \tau \right] \), the in-plane skyrmion or Chern number is given by

\[ C_n = \int \frac{d^2k_{1z}}{4\pi} \left[ \hat{n}_{k_{1z}} \cdot \left( \frac{\partial \hat{n}_{k_{1z}}}{\partial k_x} \times \frac{\partial \hat{n}_{k_{1z}}}{\partial k_y} \right) \right] \tag{3.32} \]

In this plane the above Hamiltonian describes a fully gapped two-dimensional \( d_{x^2−y^2} + id_{xy} \) paired state. At \( T = 0 \), such time-reversal symmetry breaking thermal insulator yields a quantized spin Hall conductivity

\[ \sigma_{xy}^0 = \sigma_{xy}(T = 0) = \frac{e}{4\pi} \times C_n = \frac{e}{4\pi}, \tag{3.33} \]

in the \( xy \)-plane, where \( e/2h \) is the spin-charge and \((e/2)^2/h = e/(8\pi)\) is the quantum of spin Hall conductance. The above thermal insulator also supports nonzero thermal Hall conductivity, which as \( T \to 0 \) is given by

\[ \kappa_{xy}^0 = \lim_{T \to 0} \kappa_{xy}(T) = 2 \times \frac{\pi^2 k^2 T}{6h} \times C_n = \frac{2\pi^2 k_F^2 T}{3h}. \tag{3.34} \]

In the above expression addition factor of 2 comes from the spin-degeneracy as two components of the spin projection carry the heat-current in the same direction. In two dimensions the unit of anomalous spin and thermal Hall conductivities are respective \( Js^{-1} \) and \( WK^{-1} \). Between the spin and thermal Hall conductivity as \( T \to 0 \) there exists a modified Wiedemann-Franz relation, given by

\[ \lim_{T \to 0} \frac{\kappa_{xy}(T)}{\sigma_{xy}^0} = \frac{4\pi^2}{3} \left( \frac{k_B}{\hbar} \right)^2 = L_m. \tag{3.35} \]

where \( L_m \approx 2.2731 \times 10^{23} \text{ K}^{-2} \text{s}^{-2} \) is the modified Lorentz number.

Note that the three-dimensional \( d_{x^2−y^2} + id_{xy} \) Weyl superconductor can be envisioned as stacking (in the momentum space) of corresponding two-dimensional class C spin quantum Hall Chern insulators [described by Eq. (3.31)] along the \( k_z \) direction within the range \( -k_F \leq k_z \leq k_F \). The interlayer tunneling is captured by \((k_F^2/2m)\tau_3\sigma_0 \). Concomitantly, the contribution to the anomalous spin and thermal Hall conductivity from each such layer is respectively given by Eq. (3.33) and Eq. (3.34). Therefore, the anomalous spin Hall conductivity at \( T = 0 \) of a three dimensional \( d_{x^2−y^2} + id_{xy} \) paired state is given by

\[ \sigma_{xy}^0 = \sigma_{xy}(T = 0) = \frac{e}{4\pi} \times \left( \frac{k_F}{\pi} \right). \tag{3.36} \]

The anomalous thermal Hall conductivity of such Weyl superconductor is

\[ \kappa_{xy}^0 = \kappa_{xy}(T = 0) = \frac{2\pi^2 k^2 T}{3h} \times \left( \frac{k_F}{\pi} \right), \tag{3.37} \]

as \( T \to 0 \). In three dimensions the unit of anomalous spin and thermal Hall conductivities are respectively \( Js^{-1} \) and \( WK^{-1} \). Also note that the two double-Weyl nodes located at \( k = (0, 0, \pm k_F) \) acts as source and sink of Abelian Berry curvature in the reciprocal space, and the \((k_x, k_y)\) plane encloses quantized Berry flux. The anomalous spin Hall conductivity [and thus also the anomalous thermal Hall conductivity, tied with the spin Hall conductivity via the modified Wiedemann-Franz relation, see Eq. (3.35)] is directly proportional to the enclosed Berry flux since

\[ \sigma_{xy}^0 = \frac{e}{8\pi} \times \left( \frac{k_F}{\pi} \right) \times \left( \frac{k_F}{\pi} \right) \times \left( \frac{k_F}{\pi} \right). \tag{3.38} \]

Upon unveiling the topological source of anomalous spin and thermal Hall conductivities in a Weyl superconductor, we can now proceed with the estimation of these two quantities in the \( E_g \) and \( T_{2g} \) paired states.

We first focus on the \( E_g \) channel. Recall that the \( d_{x^2−y^2} + id_{z^2−r^2} \) paired state supports eight simple Weyl nodes [see Eq. (3.6) for their location] with \( W_n = \pm 1 \). From the arrangement of the source and the sink of the Abelian Berry curvature discussed in Sec. III B 2 we immediately come to the conclusion that the net Berry flux passing through any high symmetry plane (such as the \( xy \) or planes perpendicular to any \([1,1,1]\) directions, for example) is precisely zero (see Fig. 3). Therefore, the \( d_{x^2−y^2} + id_{z^2−r^2} \) paired state, despite possessing Weyl nodes, gives rise to net zero anomalous spin or thermal Hall conductivity. Qualitatively, this situation is similar to the all-in-all out-ordered phase in the presence of sufficiently strong repulsive electronic interactions [23].

In the \( T_{2g} \) paired state, with a specific phase locking \((\phi_x, \phi_y, \phi_z) = (0, 2\pi/3, 4\pi/3)\), shown in Sec. III C the low-temperature phase also supports eight simple Weyl nodes [see Eq. (3.20) for their locations] with monopole
charge \( W_n = \pm 1 \). The topology of each such nodal point has been discussed in details in Sec.

Notice that even through time-reversal symmetry is broken at low temperature \( T < T_c \) in both \( E_g \) and \( T_{2g} \) channels and the ground state supports eight simple Weyl nodes, their location and distribution of the Berry flux in various high symmetry planes are completely different (compare Figs. 2 and 4). Consequently, the anomalous spin and thermal Hall conductivity in the \( T_{2g} \) paired state are distinct from its counterpart in the \( E_g \) channel. For concreteness, we here focus on these two responses in the \( xy \) plane and a plane perpendicular to a \([1,1,1]\) direction.

For anomalous spin and thermal Hall conductivity in the \( xy \) plane the Weyl nodes denoted as (b) and (c) do not contribute and contributions come only from the two pairs of Weyl nodes identified as (a) and (d) in Eq. (3.20). Recall that all the Weyl nodes are characterized by monopole charges \( W_n = \pm 1 \). Thus any such pair can be considered as stacking of two dimensional thermal-spin Hall insulators, but with \( C_n = 1 \), along the separation of two Weyl nodes in the reciprocal space. After carefully accounting for the enclosed Berry flux we find the anomalous spin and thermal Hall conductivities in the \( xy \) plane are respectively given by

\[
\sigma^{xy,0}\ = \frac{h}{8\pi} \left[ k_F \left( 1 - \frac{1}{\sqrt{3}} \right) \right],
\]

\[
\kappa^{0}_{xy,3D} = \frac{\pi^2 k_B^2 T}{3h} \left[ 2 \frac{k_F}{\pi} \right].
\]

Following the same spirit, we find that these two quantities in the \( yz \) plane are identical to the above expressions, while those in the \( xz \) plane is obtained by replacing \( (1 - 1/\sqrt{3}) \) in the above expressions by \( (1 + 1/\sqrt{3}) \).

On the other hand, in a plane perpendicular to the \([1,1,1]\) direction all four pairs of Weyl nodes contribute to anomalous spin and thermal Hall conductivities, and we obtain

\[
\sigma^{[1,1,1],0}_{s,3D} = \frac{h}{8\pi} \left[ 2 \frac{k_F}{3} \right], \quad \kappa^{0}_{[1,1,1],3D} = \frac{\pi^2 k_B^2 T}{3h} \left[ 2 \frac{k_F}{3} \right].
\]

Note that anomalous spin and thermal Hall conductivities are also finite along three other body diagonals.

Recall that the \( k_z, k_y \) paired state supports only a single pair of simple Weyl nodes at two poles of the Fermi surface. Consequently, a net non-zero quantized Berry flux is enclosed by the \( k_z - k_y \) plane and the paired state yields

\[
\sigma^{xy,0}_{s,3D} = \frac{h}{8\pi} \left[ \frac{k_F}{\pi} \right], \quad \kappa^{0}_{xy,3D} = \frac{\pi^2 k_B^2 T}{3h} \left[ \frac{k_F}{\pi} \right].
\]

Similarly, the two other degenerate paired states \( k_z \pm ik_y \) and \( k_y \pm ik_z \) support anomalous spin and thermal Hall conductivities of equal magnitude, but respectively in the \( k_z - k_y \) and \( k_z - k_x \) planes.

Following the same set of arguments we find that all five time-reversal odd paired states, resulting from the competition between \( T_{2g} \) and \( E_g \), yield net zero anomalous spin and thermal Hall conductivities, apart from the \( d_{xz} + id_{xy} \) phase, as shown in Table II.

IV. EXTERNAL STRAIN AND \( s + d \) PAIRING

We now discuss the effects of external strain on the paired states. Generic external strain in a Luttinger semimetal can be captured by the Hamiltonian

\[
\hat{h}_{str} = \Phi_1 \Gamma_1 + \Phi_2 \Gamma_2 + \Phi_3 \Gamma_3 + \Phi_4 \Gamma_4 + \Phi_5 \Gamma_5,
\]

where \( \Phi_j \) (for \( j = 1, \ldots, 5 \)) represents the strength of the strain. Since we are interested in the effects of external strain on the paired state that only exists in the close proximity to the Fermi surface, we also project the above five strain operators onto the Fermi surface. We assume that the external strain is too weak to significantly mix the valence and conduction bands. In the proximity to the Fermi surface the effects of generic strain are then encoded in

\[
\hat{h}_{str}^{FS} = \sum_{j=1}^{5} \Phi_j \hat{d}_j \left( \tau_3 \sigma_0 \right),
\]

where \( \sigma_0 \) is the diagonal particle-hole (Nambu space) matrix and the \( \hat{d}_j \)s are defined in Appendix A [Eq. (A1)]. Note that external strain does not couple with the spin degrees of freedom and preserves time-reversal and inversion symmetries, but breaks the cubic symmetry.
Since each component of d-wave pairing breaks the cubic symmetry, nucleation of any such pairing causes a small lattice distortion or electronic nematicity. In experiment, the onset of such nematicity can be probed from the measurement of the divergent nematic susceptibility around the transition temperature \( T_c \) (see for example Refs. [77, 88]). Externally applied strain can directly couple with the appropriate d-wave pairing (depending on the direction of the applied strain), and in that way can be conducive for the nucleation of a specific component of this pairing. In other words, strain couples with d-wave pairing as an external field. In particular, an externally applied strain induces nontrivial coupling between s-wave and d-wave pairings and such coupling enters the expression for Landau free energy as \( f_{str} \sim \Phi_j \Delta_j \Delta_0 \). Here the index \( j \) corresponds to a particular component of external strain/d-wave pairing, bearing the same symmetry, and \( \Delta_0 \) is the order parameter for s-wave pairing. To gain quantitative estimation of such non-trivial coupling we compute the triangle diagram shown in Fig. [7(a)]

The contribution of the triangle diagram to the Landau free energy is

\[
f_{str} = -\Phi_j \Delta^\mu \Delta_0^\nu \frac{1}{\beta} \sum_{\tau} \int \frac{d^3k}{(2\pi)^3} \text{Tr} \left[ (\tau_3 \hat{d}_j) G(i\omega_n, \mathbf{k}) (\tau_3 \hat{d}_j) G(i\omega_n, \mathbf{k}) \right],
\]

(4.3)

where \( \beta = (k_BT)^{-1} \) is the inverse temperature and \( \omega_n = \beta^{-1}(2n+1)\pi \) is the fermionic Matsubara frequency. In the above expression \( \mu, \nu = 1, 2 \). To test whether the coexistence of s and d wave pairing breaks time-reversal symmetry or not we have introduced the superscript \( \mu, \nu \) to the pairing amplitudes \( \Delta_\tau \) and \( \Delta_0 \). Specifically, non-zero \( \text{Tr} \) for (i) \( \mu = \nu \) corresponds to time-reversal symmetry preserving s + d pairing, (ii) \( \mu \neq \nu \) implies onset of time-reversal symmetry breaking s + id pairing, due to an external strain. The minus (−) sign in the above expression comes from the fermion bubble. We here assume that all bosonic fields [see Fig. [7(a)] are carrying zero external momentum and frequency, yielding the leading order contribution to the Landau potential. The fermionic Greens function is

\[
G(i\omega_n, \mathbf{k}) = \frac{i\omega_n + \tau_3 \xi_k}{\omega_n^2 + \xi_k^2}, \quad \xi_k = \left[ \frac{|\mathbf{k}|^2}{2m} - \mu \right].
\]

(4.4)

We find that \( f_{str} \propto \text{Tr} \left[ \tau_3 \tau_3 \right] = 2\delta_{\mu\nu} \). Hence, external strain supports a time-reversal-symmetry preserving combination of s-wave and d-wave pairings. For strain assisted time-reversal symmetry breaking superconductivity, see Ref. [39]. After the Tr algebra we arrive at the following expression

\[
f_{str} = 2\delta_{\mu\nu} \Phi_j \Delta^\mu \Delta_0^\nu \frac{1}{\beta} \sum_{\tau} \int \frac{d^3k}{(2\pi)^3} \text{Tr} \left[ (\tau_3 \hat{d}_j) G(i\omega_n, \mathbf{k}) (\tau_3 \hat{d}_j) G(i\omega_n, \mathbf{k}) \right],
\]

(4.3)

\[
= \frac{1}{2} \delta_{\mu\nu} \delta_{j,l} \Phi_j \Delta^\mu \Delta_0^\nu \frac{40\pi^2}{\Omega D} \int dk \frac{k^2}{\xi_k^2} \times \text{sech}^2 \left( \frac{\xi_k}{2kB T} \right) \left[ \sinh \left( \frac{\xi_k}{k_BT} \right) - \frac{\xi_k}{k_BT} \right].
\]

(4.5)

In the final expression the Kronecker delta \( \delta_{j,l} \) arises from the integral over the solid angle in three dimensions. This delta function indicates that the external strain and the d-wave pairing must break the cubic symmetry in the exact same way, such that \( f_{str} \) is ultimately an \( A_{1g} \) quantity. The final integral over momentum will be performed in the close proximity to the Fermi surface. We then arrive at the final expression

\[
f_{str} = [\delta_{\mu\nu} \delta_{j,l} \Phi_j \Delta^\mu \Delta_0^\nu] \frac{g(E_F)}{40\pi^2\Omega D} \frac{\sqrt{1 + \frac{\xi_k^2}{2\pi}\Omega D}}{\sqrt{1 + \frac{\xi_k^2}{2\pi}\Omega D}} \times \frac{x^2}{(x^2 - 1)^2} \times \text{sech}^2 \left( \frac{\xi_k}{2kB T} \right) \left[ \sinh \left( \frac{x - 1}{t} \right) - \frac{x - 1}{t} \right] dx,
\]

(4.6)

where \( g(E_F) \) is the DoS at Fermi energy \( E_F \), \( \Omega_D \) is the Debye frequency, \( \omega_D = \Omega_D/E_F \), and \( t = k_BT/E_F \). The functional dependence of \( F(x, y) \) is displayed in Fig. [7(b)].

Next we discuss some specific examples when external strain is applied along certain high symmetry directions.

### A. Strain along the \([0,0,1] \) direction

First, we consider a situation when the external strain is applied along one of the \( C_{4v} \) axes. For the sake of simplicity we consider the external strain to be applied along the \( \hat{z} \) direction. Such strain can only couple with \( d_{3z^2-r^2} \) pairing. Thus, a strain along \( \hat{z} \) direction results in an \( s + d_{3z^2-r^2} \) paired state, a time-reversal symmetry preserving combination of s-wave and \( E_g \) pairings.

### B. Strain along the \([1,1,1] \) direction

Next we consider a situation when the external strain is applied along one of the body diagonal or \([1,1,1] \) directions (one of the \( C_{3v} \) axes). The coupling between such strain and the d-wave pairings can be appreciated most conveniently if we rotate the reference coordinate according to Eq. (3.13). In the rotated basis strain is applied along the \( \hat{z} \) direction (now aligned along the body diagonal). After performing the same transformation for all d-wave pairings, augmented by the argument we presented

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2 For discussion on the coupling amongst various magnetic, namely the all-in all-out and itinerant spin-ice, orders with an external strain, see Ref. [23].
above, we realize that only \( d_{xy} + d_{yz} + d_{xz} \) pairing directly couples with the \([1, 1, 1]\) strain. Thus, strain along \([1, 1, 1]\) direction results in an \( s+ d_{xy} + d_{yz} + d_{xz} \) paired state, a time-reversal symmetry preserving combination of \( s\)-wave and \( T_{2g} \) pairings.

C. Strain along the \([1, 1, 0]\) direction

Finally, we discuss the effect of an in-plane external strain, applied along \([1, 1, 0]\) direction. Following the same set of arguments we conclude that when the strain is applied along the \([1, 1, 0]\), it directly couples with \( d_{xy} + d_{3z^2-r^2} \) pairing. Thus, an external strain along \([1, 1, 0]\) direction is conducive to the formation of an \( s+ d_{xy} + d_{3z^2-r^2} \) paired state, a time-reversal symmetry preserving combination of \( s\)-wave, \( T_{2g} \) and \( E_g \) pairings.

We conclude that by applied strain along different directions one can engineer various time-reversal symmetry preserving combinations of \( s\)-wave and \( d\)-wave pairings. This mechanism can in particular be useful to induce exotic paired states in weakly correlated materials, such as HgTe and gray tin, which possibly can only accommodate phonon-driven \( s\)-wave pairing in the absence of strain. The above outcome can also be stated in a slightly different words as follows. Anytime a \( d\)-wave pairing nucleates in a doped Luttinger semimetal, it immediately causes a lattice distortion or nematicity. Consequently, any \( d\)-wave pairing will always be accompanied by an induced \( s\)-wave component, which, as discussed in Sec. VII may bear important consequences in experiments. Notice existence of a small \( s\)-wave component does not break any additional symmetry deep inside the \( d\)-wave (or \( d+i\)d-type Weyl) paired state. Thus, a non-trivial coupling between \( d\)-wave and \( s\)-wave pairings and the lattice distortion does not affect flat-band (for pure \( s\)-wave) paired state, a time-reversal symmetry preserving combination of \( s\)-wave, \( T_{2g} \) and \( E_g \) pairings.

V. EFFECTS OF IMPURITIES ON BdG-WEYL QUASIPARTICLES

We now discuss the effects of quenched disorder (static impurities) on BdG-Weyl quasiparticles. Understanding the effects of impurities on regular Weyl and Dirac fermions has attracted ample attention in recent times \([90, 109]\). However, the role of randomness on BdG-Weyl/Dirac quasiparticles is still at an early stage of exploration (see however Refs. \([104, 105]\)).

In the context of superconductivity in the Luttinger semimetal (LSM), the band projection (Sec. \([13]\) and Appendix \([8]\) modifies the form of the particle-hole and chiral time-reversal symmetries in Eqs. \([2.14]\) and \([2.15]\), respectively. For the conduction band (say), we can express particle-hole (\( P \)), time-reversal (\( T \)), and chiral time-reversal (\( S \equiv P \otimes T \)) symmetry conditions in terms of the \( 4 \times 4 \) band-projected Bogoliubov-de Gennes (BdG) Hamiltonian \( \hat{h}(k) \) as follows,

\[
-M_p^{(s)} \hat{h}^T (-\mathbf{k}) M_p^{(s)} = \hat{h}(\mathbf{k}), \quad M_p^{(s)} = \tau_2 \sigma_2 : P,
\]

\[
-M_T^{(s)} \hat{h}^* (-\mathbf{k}) M_T^{(s)} = \hat{h}(\mathbf{k}), \quad M_T^{(s)} = \sigma_2 : T, \quad (5.1)
\]

\[
-M_s^{(s)} \hat{h}(\mathbf{k}) M_s^{(s)} = \hat{h}(\mathbf{k}), \quad M_s^{(s)} = \tau_2 : S.
\]

The band Hamiltonian has indices in Nambu (\( \tau \)) and band pseudospin (\( \sigma \)) spaces. Eq. \([5.1]\) obtains from the corresponding conditions in the \( 8 \times 8 \) LSM-BdG Hamiltonian by replacing \( \Gamma_{13} \to \sigma_2 \), which is the band projection of the (unitary part) of the time-reversal operator, see Eq. \([2.14]\).

Even though all candidates for Weyl superconductors have multiple Weyl nodes (\( > 2 \)), for the sake of the simplicity of the discussion, we consider its simplest realization with only two Weyl nodes, with opposite chiralities (left and right), located at \( \pm \mathbf{K} \). Linearizing the band Hamiltonian \( \hat{h}(k) \) in the vicinity of the pair, we get

\[
\hat{h}_0^{(w)} = -i \eta_3 \sigma_0 \sum_{j=1}^{3} v_j \tau_j \partial_j. \quad (5.2)
\]
Here $\eta_3 = +1$ (-1) for the Weyl node with $W_n = +1$ (-1) [Eq. (5.2)]. The Weyl Hamiltonian is $8 \times 8$, with Pauli matrices $\{\eta_\mu\}$ acting on the chirality.

The symmetry condition in Eq. (5.1) become

$$-M_\rho^{(W)} \left( \hat{h}^{(W)} \right)^T (-k) M_\rho^{(W)} = \hat{h}^{(W)}(k), \quad M_\rho^{(W)} = \eta_1 \tau_2 \sigma_2, \quad M_T^{(W)} \left( \hat{h}^{(W)} \right)^* (-k) M_T^{(W)} = \hat{h}^{(W)}(k), \quad M_T^{(W)} = \eta_1 \sigma_2, \quad -M_S^{(W)} \hat{h}^{(W)}(k) M_S^{(W)} = \hat{h}^{(W)}(k), \quad M_S^{(W)} = \tau_2.$$

(5.3)

The particle-hole and time-reversal matrices $M_\rho^{(W)}$ and $M_T^{(W)}$ both flip the node chirality. By contrast, the “chiral” version of time-reversal is the same in all cases $M_S^{(W)} = \tau_2$ in Eqs. (2.15), (5.1), and (5.3). Note also that in all cases, the time-reversal matrix is antisymmetric [e.g., $M_T^{(W)} = -(M_T^{(W)})^T$], while the particle-hole matrix is symmetric [e.g., $M_\rho^{(W)} = (M_\rho^{(W)})^T$]. These conditions imply that $T^2 = -1$ and $P^2 = +1$ [9].

It is easy to check that Eq. (5.2) satisfies particle-hole symmetry with $M_\rho^{(W)}$, as defined above. In the LSM, such a Weyl pair arises from $d + id$ pairing. It means that one component of the Hamiltonian $\hat{h}_0^{(W)}$ is time-reversal-odd. Here the component proportional to $\tau_2$ breaks time-reversal invariance. The symmetry can be restored by setting $\nu_0 = 0$, although this flattens the band along the parent nodal loop.

Since Weyl superconductors in our analysis arise from singlet pairings, we will assume that the disorder preserves band pseudospin ($\sigma$) SU(2) symmetry. Band pseudospin is not identical to the $m_s = \pm 1/2$ (conduction band) or $m_s = \pm 3/2$ (valence band) physical spin-3/2 index, since the band projection of the spin generator $\left[ J_{\mu,\nu}^\mu(k) \right]$ is momentum-dependent, and moreover dependent upon the explicit gauge choice of the band diagonalizer. Nevertheless, one can check that

$$-\sigma_2 \left[ J_{\mu,\nu}^\mu(-k) \right]^T \sigma_2 = J_{\mu,\nu}^\mu(k),$$

so that the projected spin operators are odd under time-reversal, just like the band pseudospin operators $\{\sigma_{1,2,3}\}$. Magnetic impurities would therefore effectively couple to the band pseudospin. Although time-reversal is broken in any of the $d + id$ scenarios outlined in Sec. III, that give rise to isolated Weyl nodes, we assume that there is no magnetic impurity that breaks pseudospin symmetry.

Under this assumption, the BdG-Weyl Hamiltonian $\hat{h}^{(W)}$ commutes with all $\{\sigma_\mu\}$. Then we can replace the $P^2 = +1$ physical particle-hole condition in Eq. (5.3) with an effective $P^2 = -1$ one,

$$-M_\rho^{(EP)} \left( \hat{h}^{(W)} \right)^T M_\rho^{(EP)} = \hat{h}^{(W)}, \quad M_\rho^{(EP)} = \eta_1 \tau_2.$$

(5.4)

Since time-reversal symmetry is already broken in the absence of disorder, Eq. (5.4) is the only effective symmetry expected to hold even in the presence of disorder. Since $P^2 = -1$, the system belongs to class C. By contrast, the $P^2 = +1$ condition in Eq. (5.3) would give class D. As often occurs, a continuous symmetry [here band pseudospin SU(2)] changes the random matrix classification of a Hamiltonian with a given “microscopic” specification of $P$, $T$, and $S$ [9].

Incorporating generic quenched disorder in class C, we get the BdG-Weyl Hamiltonian

$$\hat{h}^{(W)} = \hat{h}_0^{(W)} + A_0(r) \cdot \tau + B_1(r) \cdot \tau \eta_1 + B_2(r) \cdot \tau \eta_2 + v_3(r) \eta_3.$$

(5.5)

In this equation, $\tau = \tau_1 \hat{x} + \tau_2 \hat{y} + \tau_3 \hat{z}$ is the vector of Nambu matrices. There are 10 allowed perturbations, which take the form of (i) an axial [since it is missing the $\eta_3$, c.f. Eq. (5.2)] vector potential $A_0(r)$, (ii) two vector components $B_{1,2}(r)$ of a tensor disorder potential,3 and (iii) a Weyl node-graded (axial) scalar potential $v_3(r)$.

We will further simplify our treatment by neglecting quenched random fluctuations of the Weyl $(d + id)$ pairing amplitudes, so that we drop disorder terms in $B_{1,2}$ that couple to $\tau_{1,2}$. In our renormalization group (RG) scheme explained below, the remaining subset still closes under the one-loop RG. Thus we retain six random potentials, $\{A_0^{1,2}, B_1^{1,2}, B_3^{1,2}, v_3\}$. Physically, $A_0^{1,2}(r)$ corresponds to the electric charge density, i.e. encodes scattering off of Coulomb impurities (despite the fact that it appears as a vector potential component to the BdG-Weyl quasiparticles). Real and imaginary quenched fluctuations of the s-wave pairing are encoded in $\{A_0^{1,2}\}$. The potential $v_3(r)$ is a node-staggered chemical potential or “random Doppler” shift. $A_0$ and $v_3$ scatter only within a given node. The remaining potentials $\{B_{1,2}^{1,2}, B_3^{1,2}\}$ describe internode backscattering due to short-ranged impurities.4

3 In relativistic notation, the 6 independent components of $B_1$ and $B_2$ couple to the independent elements of $\sigma^{\mu\nu} = \gamma^\mu \gamma^\nu$, where $\{\gamma^\mu\}$ are the four $4 \times 4$ $\gamma$-matrices acting on the spinor field formed from the sum of left- and right-handed Weyl components. See for example [110].

4 Note that random charge impurities couple as the third component of the axial vector potential, while the two planar components stem from the real and imaginary components of the random singlet s-wave pairing. Therefore, the strength of these two types of disorder coupling (respectively described by $\Delta_0$ and $\Delta_1$ below) at the microscopic level are different. In the presence of generic disorder the Fermi velocities along the $z$ direction (along which the quasiparticle spectrum supports Weyl nodes in clean system), denoted by $v_3$ and in the $x - y$ plane, denoted by $v_1 = v_2 = v_{\perp,1}$ receive different renormalizations from the disorder. Even if we impose isotropy at the bare level (assuming $v_1 = v_2 = v_3 = v$), such symmetry is no longer respected at intermediate scale as we coarse-grain the theory. Nonetheless, we are allowed to perform the perturbative RG analysis with one Fermi velocity, but we need to treat the anisotropy parameter, defined here as $\alpha = v_{\perp,1}/v_3$, as a running coupling. But, as we demonstrate below that at the clean Weyl fixed point as well as the thermal Weyl semimetal-thermal metal quantum critical point, $\alpha$ is a marginal variable and does not affect the disorder-driven
In Eqs. (5.7) and (5.8) above, the disorder potentials \( \varphi_j \in \{A^{1,2}_0, B^1_2, B^2_3, v_3\} \). However, for these six we assign only four variances,

\[
\begin{align*}
A^0_3 & : \Delta_0, \quad \text{(Electric potential)}, \\
A^{0,2}_1 & : \Delta_1, \quad \text{(Real and imaginary s-wave pairing)}, \\
B^3_2 & : \Delta_2, \quad \text{(Internode backscattering)}, \\
v_3 & : \Delta_3, \quad \text{(Random Doppler [axial potential])}.
\end{align*}
\]

We here control the renormalization group (RG) calculation in the following way. Each disorder field is assumed to obey the following distribution

\[
\langle \varphi_j(\mathbf{x}) \varphi_k(\mathbf{y}) \rangle = \delta_{jk} \frac{\Delta_j}{|x-y|^{d-m}}
\]

in position space or

\[
\langle \varphi_j(\mathbf{q}) \varphi_k(0) \rangle = \delta_{jk} \frac{\Delta_j}{|q|^m},
\]

in momentum space and the limit \( m \rightarrow 0 \) corresponds to the Gaussian white noise distribution, which we are ultimately interested in. This form of the white noise distribution stems from the following representation of the \( d \)-dimensional \( \delta \)-function

\[
\delta^{(d)}(\mathbf{x} - \mathbf{y}) = \lim_{m \rightarrow 0} \frac{\Gamma \left( \frac{d-m}{2} \right)}{2^{m-1} \pi^{d/2} \Gamma(m/2)} \frac{1}{|x-y|^{d-m}}.
\]

For additional details of this methodology readers should consult Refs. [100, 103]. An \( \epsilon \)-expansion can be performed with the construction \( m = 1 - \epsilon \), and ultimately for Gaussian white noise disorder we set \( \epsilon = 1 \) at the end of the calculation.

The RG flow equations to the leading order in the \( \epsilon \)-expansion read as

\[
\begin{align*}
\beta_{v_3} &= -\frac{4v_3}{3} \left[ \Delta_0 + \Delta_1 + \Delta_2 + \Delta_3 \right] = v_3(1 - z), \\
\beta_{\alpha} &= \alpha \frac{2}{3} \left[ \Delta_0 - \Delta_1 \right], \\
\beta_{\Delta_1} &= \Delta_1 \left[ -\epsilon + \frac{4}{3} \left( \Delta_0 - \Delta_1 \right) \right], \\
\beta_{\Delta_2} &= \Delta_2 \left[ -\epsilon + \frac{4}{3} \left( \Delta_0 - 2\Delta_1 - \Delta_2 \right) \right], \\
\beta_{\Delta_3} &= \Delta_3 \left[ -\epsilon + \frac{8}{3} \left( \Delta_0 + \Delta_1 - 2\Delta_2 + \Delta_3 \right) \right],
\end{align*}
\]

in terms of dimensionless disorder couplings \( \hat{\Delta}_j = \Delta_j \Lambda^*/(2\pi^2v_3^2) \). But for brevity we drop the ‘hat’ notation in the above flow equations. Here, \( \Lambda \) is the ultra-

quantum critical behavior in a dirty thermal Weyl semimetal (at least to the one-loop order).
and BdG quasiparticles possess finite elastic impurity lifetime and mean-free path (in the plane-wave basis).

It is worth pointing out that even though we can tune the strength of any one of the four disorder couplings, the thermal Weyl semimetal-thermal metal QPT is ultimately driven by the random Doppler shift, which couples to the Weyl fermions as the axial potential. Also note that the anisotropy parameter $\alpha$ is a marginal parameter at both fixed points. Therefore both fixed points are multicritical in the five-dimensional space $\{\alpha, \Delta_{1,2,3}\}$. For now we neglect the effect of $\alpha$ and focus only on the four-dimensional subspace spanned by the disorder.

At the thermal Weyl semimetal-metal QCP the dynamic scaling exponent [see Eq. (5.11)] is given by

$$z = 1 + \epsilon/2 \Rightarrow z = 3/2, \quad (5.12)$$

for the Gaussian white noise distribution ($\epsilon = 1$). The correlation length exponent at the disorder controlled QCP is given by

$$\nu^{-1} = \epsilon \Rightarrow \nu = 1, \quad (5.13)$$

for $\epsilon = 1$. Eq. (5.12) implies that the average density of states at the QCP scales as $g(E) \sim |E|^{(d-2)/2} = |E|$. The product $\nu \zeta = 3/2$. Consequently, the crossover boundaries at finite temperature or energy are determined by $T^*$ or $E^* \sim \delta^{|\nu_z|}$ and are concave upward, where $\delta = (\Delta - \Delta_s)/\Delta_s$ is the reduced distance from the disorder-controlled QCP located at $\Delta = \Delta_s$. As a result a wide quantum critical regime occupies the largest portion of the phase diagram of a dirty thermal Weyl semimetal at finite temperature, as shown in Fig. 8.

On the other hand, the thermal double-Weyl semimetal as well as the thermal nodal-loop semimetal become unstable towards the formation of a thermal metal for arbitrary weak strength of disorder due to the clean linearly vanishing density of states $g(E) \sim |E|$ \cite{71, 97, 109}. This can be substantiated from the computation of the scattering lifetime ($\tau$) from a self-consistent Born approximation, leading to

$$W \int_0^{E_\Lambda} dE \frac{g(E)}{\hbar \tau^{-2} + E^2} = 1 \Rightarrow \frac{\hbar}{\tau} = E_\Lambda \exp \left(-\frac{A}{W}\right), \quad (5.14)$$

where $W$ is the strength of disorder, $E_\Lambda$ is the ultraviolet energy cut-off, and $A$ is a non-universal (material dependent) constant. Thus, the self-consistent solution of $\tau$ indicates that BdG-Weyl fermions in thermal double Weyl and nodal-loop semimetals acquire finite lifetime and the system immediately becomes a diffusive thermal metal. Note that the DoS at zero energy also follows the profile of $1/\tau$, and the phase boundary in Fig. 9 follows the functional form in the above equation.\footnote{We note that the accidental nodes found at two opposite poles in the presence of $d_{xz}/y^2 + id_{yz}x^2$ pairing (see Table II) are, however, stable against sufficiently weak randomness \cite{105}.}

FIG. 10: Various possible fits for the experimentally measured data of the penetration depth ($\Delta \lambda$) in YPtBi as a function of the reduced temperature $t = T/T_c$ (replotted from Ref. 46 with permission). Here $T_c = 0.78K$ is the superconducting transition temperature in YPtBi. For details of these fittings and the corresponding physical picture, see Sec. VI

VI. CONNECTION WITH EXPERIMENTS: PENETRATION DEPTH IN YPtBi

Even though there exists experimental evidence suggestive of superconductivity in some half-Heusler materials, the actual nature of the pairing in these compounds is not very clear at this stage \cite{38, 40}. In this respect, a recent experiment revealed a very interesting feature through the measurement of the penetration depth \cite{40}. This experiment \cite{40} suggests that the change in the penetration depth ($\Delta \lambda$) vanishes in a power-law fashion with
temperature $\Delta \lambda \sim T^n$, which is suggestive of the existence of gapless BdG quasiparticles inside the paired state. However, the precise value of $n$ remains a subject of debate. In Ref. [46], a reasonably good fit was found with $n = 1.20 \pm 0.02$, but only over a limited window of temperature $0.1 \leq T/T_c \leq 0.2$ (approximately), where $T_c \approx 0.78$K is the superconducting transition temperature in YPtBi. Such power-law dependence was then interpreted as the signature of a paired state with nodal loops, for which the DoS vanishes as $g(E) \sim |E|^{n}$ in a clean system. Since $\Delta$ follows the power-law of the DoS, the deviation from a pure $T$-linear dependence was attributed to impurities.

We here make an independent attempt to understand the dependence of the penetration depth $(\Delta \lambda)$ on the reduced temperature $t = T/T_c$ and fit the available data with the following functional form

$$\Delta \lambda_{s}(t) = \lambda(0) \sqrt{\frac{\pi \Delta_{0}}{2k_{B}T_{c}}} \cdot \exp \left( -\frac{\Delta_{0}}{k_{B}T_{c}} t \right) + c_{n} t^{n}, \quad (6.1)$$

and provide alternative explanation for the experimental observation in Ref. [46]. The first term on the right-hand side [denoted by $\Delta \lambda_{s}(t)$] is the canonical penetration depth dependence in an $s$-wave superconductor, whereas the power law terms correspond to the presence of gapless BdG quasiparticles for which the DoS vanishes as $g(E) \sim |E|^n$. The resulting fits are displayed in Fig. 10.

The solid line in Fig. 10(a) shows the best fit obtained by keeping only the linear in temperature term in Eq. (6.1) in addition to the $s$-wave contribution. We find the best fit with the zero-temperature value of the $s$-wave gap $\Delta_{0} = 2.08k_{B}T_{c}$, which is close to the BCS value $\Delta_{BCS} = 1.76k_{B}T_{c}$, with the amplitude of the linear term $c_{1} = 1.04\mu$m. Interestingly, the fit results in the zero-temperature value of the penetration depth $\lambda(0) \approx 23\mu$m, which is three orders of magnitude larger than found in conventional superconductors such as aluminum, as remarked by the authors of Ref. [46] who also measured a high value of $\lambda(0)_{exp} = 2\mu$m in YPtBi. As Fig. 10(a) illustrates, a pure $s$-wave dependence (dotted line) is not a good fit to the data, even though it would result in a value of $\lambda(0) \approx 3\mu$m that is closer to the experiment [46].

On the other hand, including the $s$-wave component is crucial to accurately fit the data. Indeed, attempts to fit to a pure power-law, see Fig. 10(b), are unsatisfactory as the fits only work in the low-temperature regime $t \lesssim 0.2$. Moreover, the extracted power-law exponent depends sensitively on the width of the fitted temperature region and given the narrow temperature range, it is difficult to distinguish between a $T$-linear fit [solid red line in Fig. 10(b)], the $T^2$ fit [dash-dotted blue line in Fig. 10(b)] or, say, the $T^{1.2}$ fit adopted by the authors in Ref. [46] [dashed line in panels (a) and (b) of Fig. 10]. Of course the three curves deviate from each other at higher temperatures $t > 0.25$, but by that time neither one of them fits the experimental data even remotely.

Based on the above analysis and the comparison between panels (a) and (b) in Fig. 10, we conclude that it is necessary to include the $s$-wave component to properly fit the penetration depth data. The pure $s$-wave fit is unsatisfactory, as remarked earlier, and a power-law contribution must be considered. We now turn to the comparison between different such power-law contributions.

The red and blue lines in Fig. 10(c) show the $\Delta \lambda_{s}(t) + c_{1}t$ and $\Delta \lambda_{s}(t) + c_{2}t^{2}$ fits to the data, respectively. The red curve incorporating the $T$-linear component fits the data better [the same fit as the solid line in panel (a)]. By contrast, attempts to fit the data with $\Delta \lambda_{s}(t) + c_{2}t^{2}$ form not only fall below the target in the range $0.05 < t < 0.25$, but also requires in an unphysically large fitting parameter $\lambda(0) \approx 10^{3}\mu$m. We cannot however exclude possible presence of a small $T^2$ component in addition to the dominant $s$-wave and $T$-linear terms.

Based on the above analysis, we conclude that superconductivity in YPtBi is best described by a combination of a fully gapped $s$-wave component and a gapless BdG quasiparticles with linear in energy density of states. The latter is commonly attributed to the nodal lines in the gap, such as a $d$-wave or $p$-wave. However we stress that this is not a unique explanation and we list several possible sources of $T$-linear dependence in section III below.

### A. Source of $T^2$ dependence

Within a simple picture for pairing in the Luttinger system, there is only one possible source for $T^2$ dependence of the penetration depth: namely, the existence of gapless quasiparticles at isolated points on the Fermi surface where the DoS vanishes as $g(E) \sim |E|^2$ at low energy. Only a Weyl superconductor, constituted by Weyl nodes with monopole charge $W_{n} = \pm 1$ yields such DoS, see Secs. III B and III C. In this work, we have presented several examples of Weyl superconductors; any such candidate is capable of producing a $T^2$ dependence of the penetration depth at low temperatures.

### B. Sources of $T$-linear dependence

Unlike the aforementioned case of $T^2$ dependence, the origin of a $T$ linear contribution to $\Delta \lambda$ is not unique, with several possible sources resulting in the gapless BdG fermions displaying the linear scaling $g(E) \sim |E|$ of the DoS. Once again within a simple picture of pairing in the Luttinger system, we can identify three possible origins of such $E$-linear scaling of the DoS. They are the following.

1. A Weyl superconductor, with isolated Weyl nodes characterized by the monopole charge $W_{n} = \pm 2$. Such Weyl nodes are also referred to as double-Weyl nodes, and yield $g(E) \sim |E|$ at low energies (below the superconducting gap). But, as we have seen above [see Table II and Sec. III D], examples of such a double-Weyl superconductor are sparse and so far we can only identify one candidate, namely, the $d_{x^2-y^2} + id_{xy}$ paired state, that
can support a linear scaling of the DoS. Thus, based on various examples of Weyl superconductors discussed in this work, one may conclude that this possibility is the least likely one. Nucleation of such a phase will, however, be associated with a two-stage transition.

2. A tempting source of $\rho(E) \sim |E|$ is existence of at least one nodal loop in the spectrum of BdG quasiparticles. So far we have found ample examples in the presence of local or intra-unit cell pairings that support nodal loops in the ordered phase [see Table I, Sec. III C 2]. However, for each such possibility there is always a competing ordered phase of the $d + id$ type that accommodates simple and isolated Weyl nodes (with $W_n = \pm 1$) on the Fermi surface. Since $\rho(E) \sim |E|^2$ inside the simple Weyl superconductors, nucleation of such paired state will cause power-law suppression of the DoS, thus optimizing the gain in the condensation energy. We therefore conclude that Weyl nodal-point superconductors are energetically superior to the ones with nodal loops, at least within the framework of the weak coupling BCS pairing.

3. The last, but most likely, possibility is the following. The underlying paired state supports simple BdG-Weyl quasiparticles with the Weyl nodes that are characterized by the monopole charge $W_n = \pm 1$ in Eq. (3.2). But, due to the presence of impurities in the system, the thermodynamic responses measuring the DoS (for instance, the penetration depth $\Delta \lambda$) are determined by the disorder-driven quantum critical regime associated with the Weyl superconductor-thermal metal QPT, see Fig. 8. At such a transition, the dynamical critical exponent $z = 3/2$ (almost exact in light of recent field-theoretic and numerical works, see Sec. IV) yields the density of states $\rho(E) \sim |E|^{1+4/d_z} \sim |E|$ linear in energy. Such an origin of a $T$-linear contribution to the penetration depth is quite natural in YPtBi, since the carrier density is extremely low, making the system prone to impurities.

We believe that the last possibility is the most likely scenario in YPtBi and other half-Heusler compounds for the following reasons. Nucleation of simple-Weyl nodes in the gap is possibly energetically the best option among various available candidates for nodal pairings as it provides the optimal power-law suppression of DoS at low-energy (leaving aside fully gapped paired states that require non-local pairing, see Sec. II C). But, due to the presence of randomness/impurities, the thermodynamic responses are dominated by the wide quantum critical regime associated with disorder-controlled Weyl-to-thermal metal QPT. Note that this critical regime occupies the largest portion of the phase diagram of the disordered Weyl superconductor at finite temperature, as shown in Fig. 8.

C. Sources of the $s$-wave component

Perhaps the most enigmatic aspect of our analysis of the recent experimental data on the penetration depth [46] is the unambiguous presence of an $s$-wave component, which is however quite natural in light of the discussion presented in Sec. IV. Recall that nucleation of any $d$-wave pairing (or any combination of multiple $d$-wave pairing) breaks the cubic symmetry, which naturally introduces a lattice distortion or electronic nematicity in the system. In turn, the cooperative effect of the $d$-wave pairing and such lattice distortion introduces a non-trivial $s$-wave pairing in the system. Therefore, incorporating the contribution of $s$-wave pairing is fully consistent with the symmetry of the problem: indeed, the $s$-wave $A_{1g}$ pairing appears on equal footing with the nodal $d$-wave channels in the vicinity of the Fermi surface, see Table I. Additionally, one could of course imagine a more trivial origin of the $s$-wave pairing, such as due to the electron-phonon coupling, considered in a recent theoretical work [51].

Since the chemical potential in superconducting half-Heusler compounds, such as YPtBi, lies in close proximity to the quadratic band touching points, it is quite natural to anticipate that intra-unit cell pairings or local pairings ($s$-wave and five $d$-waves listed in Table I) stand as prominent candidates. This is the reason why so far we have focused on these pairings, leaving aside non-local or longer-range pairings, which will be the subject of discussion in Sec. VII. Since some of the half-Heusler compounds display magnetic order, we also believe that non-$s$-wave pairing is possibly the dominant intra-unit cell pairing, which has received some support from simple microscopic calculations [52, 53]. In light of the above discussion, the $s$-wave component, when it is manifest, is not interaction driven but rather an induced one. Our whole discussion on experimental aspects of pairing in Luttinger system thus evolves around various $d$-wave pairings that can ultimately lead to simple Weyl superconductors via the formation of the $d + id$ state (see section III). Finally, we briefly comment on possible future experiments that can possibly pin down the pairing symmetry in these compounds.

D. Future experiments and pairing symmetry

The existence of simple Weyl superconductors can be pinned down, at least in principle, by systematically controlling the impurity concentration in the system, for example. Note that with the decreasing strength of impurity scattering (as the material gets cleaner), the $T$-linear dependence of the penetration depth is expected to get suppressed and a $T^2$ dependence should become dominant. This feature, for example, can be probed by the following measurements:

- Specific heat ($C_v$) measurements inside the superconducting phase, although difficult given the low critical temperatures (0.78 K in YPtBi) in half-Heuslers, can be instrumental in unveiling the pairing symmetry. With an underlying simple Weyl superconductor, $C_v$ should display a gradual onset of $T^3$ dependence at low temperature ($T \ll T_c$) and disappearance of $T^2$ scaling as the
concentration of impurities is reduced (the system then falls on the Weyl side of the phase diagram, escaping the critical regime), see Fig. 8. By contrast, with increasing impurity scattering, the \( T \) dependence of specific heat is expected to gradually get replaced by the Fermi-liquid-like \( T \)-linear dependence as the system moves into the thermal metallic side of the transition.

- Measurements of the anomalous thermal Hall conductivity [see Sec. IIIE 6, as well as probing the surface Andreev bound Fermi arc states with the STM quasiparticle interference (QPI) techniques are expected to distinguish among various candidates of Weyl pairing listed in Table I.]

- The nuclear magnetic resonance (NMR) relaxation time \( (T_1) \) can also be a good probe to elucidate the scaling of low-energy DoS since \( 1/T_1 \sim [\rho(E \rightarrow T)]^2 \) (Korringa’s relation). Therefore, as the impurity concentration is gradually increased, the inverse of the NMR relaxation time should display a crossover from \( T^{-1} \sim T^5 \) (dominated by BdG-Weyl quasiparticles) to \( T_{1\perp}^{-1} \sim T^3 \) (inside the quantum critical regime), to \( T \)-linear behavior (in the thermal metallic phase) scaling, see Fig. 8.

- The longitudinal thermal conductivity \( (\kappa_{jj}) \) can also be a good probe to expose various regimes of the phase diagram. Specifically, \( \kappa_{jj}/T \) scales as (a) \( T \) when the BdG-Weyl quasiparticles dominate the transport (clean limit), (b) \( T^{2/3} \) in the entire quantum critical regime, (c) approaches a constant value in the thermal metallic side as \( T \rightarrow 0 \), see Fig. 8. Note that the magnitude of thermal conductivity will in general be different along various crystallographic directions due to the natural anisotropy in the Weyl paired state.

Above, we have discussed the experimental implications of the Weyl-paired superconducting state and how it evolves as a function of impurity scattering strength. If, on the other hand, the \( T \)-linear dependence of the penetration depth in Fig. 11 arises from an underlying line node, then the measurements of the specific heat can be a good tool to pin down such a nodal structure. In the presence of a nodal loop (or double-Weyl node), \( T^2 \)-dependence of the specific heat is expected to occur a progressively wider window of temperature with a gradual decrease of impurity scattering, see Fig. 9. By contrast, with increasing strength of impurity scattering, the \( T^2 \) dependence will be gradually replaced by the \( T \)-linear dependence in \( C_v \) (dominated by a thermal metal). The inverse of the NMR relaxation time in the presence of a nodal loop in the quasiparticle spectra is then expected to display a smooth crossover from \( T_{1\perp}^{-1} \sim T^3 \) (dominated by pseudoballistic quasiparticles) to \( T \)-linear dependence (governed by thermal metallic phase), as the disorder in the system increases. Notice \( \kappa_{jj}/T \) is expected to display a \( T \)-linear scaling, but only when the heat-current flows in the basal plane containing the nodal loop. Otherwise, with increasing \( \kappa_{jj}/T \) as \( T \rightarrow 0 \) should increase (decrease) \[112\]. By contrast, in the presence of double-Weyl nodes (separated along \( \hat{z} \) direction, for example), \( \kappa_{zz}/T \sim \text{constant}, \) while \( \kappa_{jj}/T \sim T \) at low temperature, where \( j = x, y \).

We realize that it may be very difficult to tune the concentration of impurities experimentally, the task further complicated by the fact that nodal superconductivity is easily destroyed by (non-magnetic) impurities. Nevertheless, one could still make meaningful conclusions about the pairing by exploring the phase diagram at a fixed impurity concentration, as a function of temperature. For instance, descending in temperature to the left of the disorder controlled diffusive QCP [the red dot in Fig. 8], one would expect to see the crossover from the critical regime (where \( C_v \sim T^2, T_1^{-1} \sim T^3 \)) to the pure Weyl regime (with \( C_v \sim T^3, T_1^{-1} \sim T^5 \) and \( \kappa_{jj}/T \sim T^1 \)). Therefore, we believe that it is still conceivable to pin the actual nature of the pairing symmetry in half-Heusler compounds with the presently available experimental tools. We hope our detailed discussion will motivate future experiments in this class of materials.

### VII. \( s + id \) Pairing

In Section IIII we have investigated the \( d + id \) pairing in detail, which results in the topological nodal Weyl superconductor. Recall however that in addition to the five \( d \)-wave pairings, the even-parity local pairing also contains an \( s \)-wave component that transforms under the \( A_{1g} \) representation of the cubic point group (first row in Table I). Such solution is generically fully gapped, with the exception of accidental nodes in an extended \( s \)-wave, which occur if the Fermi surface happens to cross the lines of nodes (for instance, \( \Delta(k) = \cos(k_x) \cos(k_y) \) has nodes at \( k_{x,y} = \pm \pi/2 \)). We shall exclude this latter possibility based on the fact that (a) this would require a very large doping of the Luttinger semimetal in order to achieve the necessarily large \( k_F \), and (b) such \( k \)-dependent gap structure comes at a cost of spatial derivatives in the Landau–Ginzburg functional, thus making it less energetically favorable than featureless \( s \)-wave.

In this section, we shall instead investigate the possibility of a time-reversal symmetry broken solution of \( s + id \) type. Such a solution necessarily involves a combination of two different irreducible representations: the \( A_{1g} \) for the \( s \)-wave component and either \( E_g \) or \( T_{2g} \) for the \( d \)-wave component. This implies that there is no symmetry requirement for the amplitudes of the two components to be equal, and one will generically find \( \Delta_s \neq \Delta_d \) in the Bogoliubov quasiparticle dispersion:

\[
E_{s+id}(k) = \left[ \xi_k^2 + |\Delta_s|^2 + |\Delta_d|^2 d^2(k) \right]^{1/2},
\]  

(7.1)
where the five representative $d$-wave form-factors $d(k)$ are listed in Appendix A. It is intuitively clear that for such an $s + id$ solution to be realized, pairing strengths $\lambda_s$ and $\lambda_d$ need to be comparable: otherwise, a pure $s$-wave or a pure $d$-wave (more precisely, $d + id$) will dominate. One can therefore imagine that by tuning the ratio $r = \lambda_d/\lambda_s$, the $s + id$ solution might be realized in an intermediate parameter range. To see whether this is indeed the case, we must solve a gap equation similar to Eq. (2.21) in Sec. IID for each of the two gap components

$$
\frac{\Delta_s}{\lambda_s} = \frac{\int dy}{2} \int \frac{d\Omega}{4\pi} \frac{\sqrt{1+y} \Delta_s}{\sqrt{y^2 + |\Delta_s|^2 + |\Delta_d|^2} d^2(y, \Omega)},
$$

(7.2)

$$
\frac{\Delta_d}{\lambda_d} = \frac{\int dy}{2} \int \frac{d\Omega}{4\pi} \frac{\sqrt{1+y} \Delta_d d^2(y, \Omega)}{\sqrt{y^2 + |\Delta_s|^2 + |\Delta_d|^2} d^2(y, \Omega)},
$$

(7.3)

where $\Delta_j = \Delta_j/\mu$ and $y = (k/k_F)^2 - 1$ are the dimensionless parameters, introduced in Sec. IID.

The coupled system of equations (7.2)–(7.3) does not lend itself to an analytical solution, nevertheless the solution can be obtained numerically, with the result shown in Fig. 11(a). At first, for low values of $r = \lambda_d/\lambda_s$, the only solution to Eq. (7.3) is a trivial one: $\hat{\Delta}_d = 0$, resulting in a pure $s$-wave solution. As the strength of $d$-wave pairing grows, a non-zero value of $\Delta_d$ (blue diamonds) starts developing above a certain value $r_{c1} \lesssim 5$, and an $s + id$ solution appears in a finite region of the phase diagram $r_{c1} < r < r_{c2}$ (magenta shading in Fig. 11). Above the second critical point, $r > r_{c2} \approx 7.75$, only a trivial solution $\Delta_s = 0$ is possible, resulting in a pure $d$-wave for large coupling strength $\lambda_d$.

So far, it appears that the initial intuition was correct and that the $s + id$ solution exists in an intermediate regime of coupling strength $r_{c1} < r < r_{c2}$. However, one must be carefully consider other competing orders: in particular, since we are entertaining the possibility of time-reversal symmetry broken phases, we must also include $d + id$ order into the consideration. Allowing for the $d + id$ solution (specifically, the $d_s z - y^2 + id_{3z} r^2$ pairing, as it was shown to be energetically the most favorable state in the $E^d$ sector, see Sec. IID and Appendix D, we find that $\Delta_{d+id}$ rises precipitously with increasing $\lambda_d$ (black squares in Fig. 11a). It is clear that the $d + id$ order parameter grows parametrically faster than that of the pure $d$-wave and that it should dominate for sufficiently large $\lambda_d$. This is intuitively clear since the $d + id$ solution only has point nodes, as discussed already in Sec. IID and is therefore energetically more favorable than the pure $d$-wave with its line nodes. This argument can be made rigorous by comparing the behavior of the two solutions from Eqs. (2.24) and (3.9):

$$
\hat{\Delta}_d = \omega_D C_d \exp \left( -\frac{5}{\lambda_d} \right),
$$

$$
\hat{\Delta}_{d+id} = \omega_D C_{d+id} \exp \left( -\frac{5}{2\lambda_d} \right),
$$

(7.4)

with $C_{d+id} > C_d$, confirming that the $d + id$ order parameter is parametrically larger than the pure $d$-wave one (the exact values of the constants $C_d$ and $C_{d+id}$ are non-essential). Therefore the question is – can the $s + id$ phase survive the competition against its $d + id$ rival?

To answer this question, we plot the energies of the two solutions in Fig. 11(b), from which it becomes evident that $d + id$ has lower energy than pure $s$-wave or $s + id$, provided $r > r_0 \approx 2.9$ (to the right of the vertical dashed line in Fig. 11). The entire region of existence of the putative $s + id$ phase lies at coupling strength $r > r_0$, and we conclude that the $s + id$ phase is therefore energetically unstable. Instead, there is a first-order phase transition (i.e. an energy level crossing) at $r = r_0$ from pure $s$-wave directly into the $d + id$ phase.

We can shed more light on the reason why the $s + id$ solution is less energetically stable by considering the gap equations (7.2)–(7.3) in the weak-coupling approximation $\Delta \ll \omega_D \ll 1$. The gap equations [see Eqs. (7.2)
FIG. 12: Schematic phase diagram for the noninteracting 2D surface states of a class DIII bulk topological superconductor. The fixed point representing the clean surface band structure (red dot) is unstable in the presence of time-reversal preserving quenched disorder for any $\nu \geq 3$, where $\nu$ is the integer bulk winding number. The precise form of the clean limit depends on details. For a spin-1/2 bulk, one can have $\nu$ species of massless relativistic Majorana fermions, with disorder that enters as a nonabelian gauge potential scattering between these [61]. For isotropic $p$-wave pairing in the LSM studied here with winding number $\nu = 3$, the surface states in the hole-doped case consist of a single two-component surface fermion with cubic dispersion, see Fig. 13(a) [17], c.f. Refs. [48, 51]. Our generalized surface theory in Eq. (8.23) has $\nu$-fold dispersion for the corresponding winding number. The disordered system should be described by a class DIII non-linear sigma model with a Wess-Zumino-Novikov-Witten (WZNW) term. The WZNW term prevents Anderson localization [61, 116]. This theory has a stable thermal metal (green dot) and an unstable, critically delocalized fixed point. The latter (yellow dot) is governed by the $SO(n)\nu$ CFT [61][117]. Our numerical results are generally consistent with the $SO(n)\nu$ theory, see Figs. 14–16 implying that the renormalization group trajectory away from the clean limit is fine-tuned by the topology to flow into the CFT (solid vertical flow), instead of flowing into the thermal metal (dashed flow). The same conclusion was reached for a model with $\nu = 4$ in Ref. [51].

and (7.3)] can further be simplified in the vicinity of the second-order phase transition at $T = r_{c1}$, where the magnitude of $\Delta_d$ can be made arbitrarily small. In this limit we can approximate $\Delta_s \approx 2\omega_D \exp(-1/\lambda_s)$ and the $d$-wave pairing amplitude can then be obtained from [follows from Eqs. (E1)- (E5) in Appendix E

$$\left(\frac{\Delta_d}{\Delta_s}\right)^2 \approx \frac{280}{27} \left[ \frac{1}{5\lambda_s} - \frac{1}{\lambda_d} \right].$$

(7.5)

The non-trivial solution for $\hat{\Delta}_d$ is only possible provided $\lambda_d/\lambda_s > 5$, explaining the value $r_{c1} \approx 5$ obtained numerically, see Fig. 11(a). We now must compare the free energy of this solution (which is essentially a pure $s$-wave in the vicinity of $r_{c1}$) to that of $d + id$ from Eq. (3.12). The technical details are provided in the Appendix E and we quote the final result here: in order for $s + id$ phase to have lower energy than the $d + id$, in the vicinity of $r_{c1}$, we require that $\lambda_s^{-1} < -\ln(C_{d+id}^{-1}/10) \approx 0.309$, where $C_{d+id} = 2.705$, see Eq. (3.12). Substituting this into Eq. (7.5), we see that for a non-trivial $s + id$ solution to exist, one must ensure the lower bound on the pairing strength $\lambda_d > 5\lambda_s \gtrapprox 16.2$. Such a huge value of $\lambda_d$ is unphysical, and moreover, the assumption of the weak coupling would break down in this case. Even allowing for different Debye frequencies for the $s$- and $d$-wave components – justified if the origin of $s$-wave pairing is not the same, for instance due to conventional electron-phonon mechanism – does not alter the outcome, with the lower bound on $\lambda_d \gtrapprox 16.2$ remaining roughly same as before (see Appendix E). This conclusion is corroborated by numerics, where we certainly have not been able to find an energetically viable $s + id$ solution for any $\lambda_d \lesssim 8$.

We therefore conclude that the $s + id$ solution, although possible as a matter of principle, turns out to be energetically inferior to a rival $d + id$ phase in a physical parameter regime. The phase diagram is summarized in Figure 11(b), with a first-order phase transition from a pure $s$-wave to a $(d + id)$ phase. Such outcome is rooted in the underlying cubic symmetry of the system, for which $E_g$ is a two-component representation, permitting a $d_{x^2-y^2} + i d_{3z^2-r^2}$ pairing to compete with (and finally win over) the $s + id_{x^2-y^2}/3z^2-r^2$ pairing. By contrast, in a tetragonal environment, the $d_{x^2-y^2}$ pairing belongs to a single-component $E_g$ representation, and consequently a $s + id_{x^2-y^2}$ pairing can easily be found for comparably strong $\lambda_s$ and $\lambda_d$ (for example, see the magenta shaded region in Fig. 11). Therefore, our formalism is specifically tailored to address the competition among the pairings (including the local as well as the non-local ones), belonging to different representations, in a cubic environment; even though we here explicitly study only the competition between the simplest $A_{1g}$ pairing and the $E_g$ pairings.

VIII. STRONG TOPOLOGICAL SUPERCONDUCTIVITY: ODD-PARITY ISOTROPIC P-WAVE PAIRING

In this section we study superconductivity in the Luttinger semimetal (LSM) with isotropic $p$-wave pairing. This odd-parity pairing is the spin-3/2 generalization of the $B$ phase of $^3$He [13]. The paired state represents a time-reversal invariant, class DIII (strong) topological superconductor (TSC) [9]. The topology induces a two-dimensional (2D) gapless Majorana fluid to appear at the material surface. Our goal is to investigate the stability (“topological protection”) of this 2D Majorana fluid to perturbations that are inevitable at the surface of a real material: quenched impurities and residual interparticle interactions.

In a previous work [51], we investigated exactly these questions in the Luttinger Hamiltonian [Eqs. (2.1) and (2.5)] with isotropic $p$-wave pairing, but with one crucial difference. In Ref. [51], we assumed that $m > m_0$, i.e. that both bands in Eq. (2.1) “bend together”, as in the
light and heavy hole bands of GaAs [17]. Assuming that both bands participate in superconductivity, the bulk winding number \( \nu = 4 \) in that case, and the surface Majorana fluid exhibits coexisting linear and cubic dispersing branches [25]. We showed that interactions can destabilize the clean fluid [31], inducing spontaneous time-reversal symmetry breaking and surface thermal quantum Hall order [9, 84, 113, 114]. By contrast, we demonstrated that quenched surface disorder is a strong perturbation that induces critical Anderson delocalization, with multifractal surface wave functions and a power-law divergence of the disorder-averaged density of states. These results were obtained numerically via exact diagonalization, and were found to agree very well with the predictions of a certain 2D conformal field theory (CFT). The CFT is the current algebra \( \text{SO}(n)_\nu \) (with \( n = 4 \)), where \( n \to 0 \) is a replica index [61]. We concluded that the surface states are governed by this CFT in the presence of arbitrarily weak disorder. Moreover, in a separate work we established that the class DIII SO(\( n \)) theory is stable against the effects of residual quasiparticle-quasiparticle interactions [61]. The main takeaway of Ref. [51] was that disorder can enhance topological protection at the surface of a higher-spin TSC.

The SO(\( n \)) CFT can be “derived” via certain conformal embedding rules for surface states of model spin-1/2 TSCs [61]. In the case of the LSM with \( p \)-wave pairing studied here and for the closely related model in [51], these rules do not obviously apply. In particular, the conformal embedding argument assumes that the clean limit is also a CFT, i.e. free relativistic fermions (in 2+0 dimensions; in the absence of interactions, we can study the problem at a fixed single particle energy [61]). By contrast, the clean surface states of higher-spin TSCs typically have higher (e.g. cubic) dispersion [47, 48], and are not conformally invariant.

Here we consider the problem in the LSM, where electron and hole bands bend oppositely. This gives rise to a different winding number (\( \nu = 3 \)) and different surface states, depending on the doping. In fact, we invent here a generalized surface model (see Sec. VIIIC) that allows us to efficiently simulate noninteracting surface states corresponding to a bulk TSC in class DIII with arbitrary integer winding number \( \nu \). The model has \( \nu \)-fold dispersion, such that the large-\( \nu \) limit corresponds to a highly flattened surface band with a strongly diverging clean DoS.

On physical grounds, the most general expectation for class DIII in this case would be that disorder induces a surface thermal metal [116]. In two spatial dimensions, the thermal metal phase in class DIII is stable due to weak antilocalization. Moreover, the SO(\( n \)) CFT fixed point, while stable against interactions, is technically unstable towards flowing into the thermal metal [51]; see Fig. 12. Despite this, in Ref. [51] for winding number \( \nu = 4 \) and here for generic \( \nu \geq 3 \), we provide strong numerical evidence that any disorder induces the quantum critical scaling associated to SO(\( n \)), with universal predictions for experiment that depend only on \( \nu \). These include power-law scaling for the tunneling density of states, a quantized thermal conductivity divided by temperature [51, 117], and a universal multifractal spectrum of local DoS fluctuations. These states are also robust against interactions for any \( \nu [61].

Our results suggest a deep connection between the bulk topology of three-dimensional TSC and the universal physics of the quench-disordered two-dimensional Majorana surface fluid, despite the fact that key attributes of the clean surface depend on details of the bulk. In particular, it suggests a topological generalization of the conformal embedding rule [SO(\( n \nu \)) \( \supset \) SO(\( n \nu \)) \( \oplus \) SO(\( n \nu \))] used to link \( \nu \) clean relativistic Majorana fermions to the SO(\( n \nu \)) CFT in the presence of disorder [61]. This topological generalization should apply in the replica limit \( n \to 0 \) to any surface band structure for any strong class DIII TSC with winding number \( \nu \geq 3 \), subject to time-reversal invariant quenched disorder.

Beyond fundamental interest, the Eliashberg calculations in Ref. [54] suggest that isotropic \( p \)-wave pairing gives the dominant non-\( s \)-wave channel in a hole-doped LSM due to optical-phonon-mediated pairing. For this reason we focus mainly on the hole-doped model in the following, which has \( \nu = 3 \) (see below).

### A. Bulk and surface theory

We write the Luttinger Hamiltonian in terms of the Nambu spinor defined by Eq. (2.11).

\[
H = \frac{1}{2} \int \frac{d^2 k}{(2\pi)^2} \Psi^\dagger_N(k) \hat{h}(k) \Psi_N(k),
\]

where the 8 \( \times \) 8 Bogoliubov-de Gennes (BdG) Hamiltonian is

\[
\hat{h}(k) = \hat{h}_L(k) \tau_3 + \Delta_p (J \cdot k) \tau_1.
\]

Here \( \hat{h}_L \) is the Luttinger operator from Eq. (2.9) and \( J \) denotes the vector of spin-3/2 generators [see Eq. (A3)]. The Pauli matrices \( \{\tau_m\} \) act on the particle-hole (Nambu) space. The parameter \( \Delta_p \) is the real \( p \)-wave pairing amplitude; with this choice, Eq. (8.2) is time-reversal invariant [see Eq. (2.15)]. It also satisfies the particle-hole condition in Eq. (2.14), using Eq. (2.12).

We assume weak BCS pairing so that \( \mu > 0 \) (\( \mu < 0 \)) describes superconductivity in the \( |m_s| = 3/2 \) valence band of the Luttinger Hamiltonian. The physical bulk quasiparticle energy spectrum

---

7 We note that such surface order can also arise in the presence of \( p + is \) pairing in the bulk [175].
of Eq. (8.2) is fully gapped,

\[ E_{\pm}(k) = \sqrt{(|\lambda_1 + 2\lambda_2| k^2 - |\mu|)^2 + \left(\frac{2\lambda_1}{k^2}\right)^2}, \]

where \(2\lambda_2 > \lambda_1\) is required so that conduction and valence bands bend oppositely [or \(m_0 > m\) in Eq. (2.6)], and \(E_{+}(E_{-})\) corresponds to superconductivity in the conduction (valence) band. The assumption of weak BCS pairing around a finite Fermi surface means that we can project the BdG Hamiltonian into the \(|m_s| = 1/2\) conduction or \(|m_s| = 3/2\) valence band. The results are

\[ \hat{h}_{1/2}(k) = \left[ (\lambda_1 + 2\lambda_2) k^2 - \mu \right] \tau_3 + \frac{\Delta_p}{2} \begin{pmatrix} -k_z & -\frac{\epsilon^2}{k} \\ k_z & k_z \end{pmatrix} \tau_1, \]

\[ \hat{h}_{3/2}(k) = \left[ (\lambda_1 - 2\lambda_2) k^2 - \mu \right] \tau_3 + \frac{\Delta_p}{\alpha(k)} \begin{pmatrix} -k_z \beta(k) \\ k_z \end{pmatrix} \tau_1, \]

where \(k \equiv k_x - ik_y, \bar{k} = k^*\), and

\[ \alpha(k) = \frac{2}{3} \left(4k_z^2 + |k|^2\right), \quad \beta(k) = \left(4k_z^2 + 3|k|^2\right). \]

Here we have diagonalized \((\mathbf{J} \cdot \mathbf{k})^2\) but not \((\mathbf{J} \cdot \mathbf{k})\), so that the matrix elements are rational functions of the momentum components. This is essential for obtaining a local surface theory, derived below.

We employ the winding number defined by Schnyder et al. \[118\] to characterize the topology of the bulk. After rotating \(\tau_3 \rightarrow \tau_2\), one introduces the matrix \(4 \times 4\) matrix \(Q(k) = U^{-1}(k) \Lambda U(k)\), where \(U(k)\) diagonalizes \(\hat{h}_{|m_s|}(k)\), and \(\Lambda = \text{diag}(1, 1, -1, -1)\) is the flattened matrix of energy eigenvalues. Then \(Q\) is off-diagonal,

\[ Q = \begin{pmatrix} 0 & q^{-1} \\ q^{-1} & 0 \end{pmatrix}, \]

and the winding number is given by

\[ \nu = \int \frac{d^3k}{24\pi^2} \epsilon^{ijk} \text{Tr} \left[ (q^{-1} \partial_i q)(q^{-1} \partial_j q)(q^{-1} \partial_k q) \right], \]

with repeated indices summed. We find that \(|\nu| = 3\) for the valence and conduction bands. From here on we ignore \(\text{sgn}(\nu)\), which can only be important at an interface (e.g., a physical surface) where this sign flips. Our winding number is in agreement with Ref. \[47\] for the \(|m_s| = 3/2\) band, but differs from that obtained for the \(|m_s| = 1/2\) band, in which \(\nu = 1\) was claimed. We show below that surface state calculations support our results. We believe that the discrepancy comes from the fact the authors of Ref. \[47\] used a Fermi surface winding number method \[118\], which gives the correct winding number only if the system is non-degenerate.

To obtain the effective surface Hamiltonian we follow the conventional approach of terminating in the \(z\) direction and diagonalizing \(\hat{h}_{|m_s|}(k, k_z \rightarrow -i\partial_z)\) where \(k = k_x, k_y\) denotes momentum parallel to the surface. For the \(|m_s| = 3/2\) valence band, applying hard-wall boundary conditions we obtain zero energy surface states at \(k = 0\) of the form

\[ |\psi_{0,m_s}\rangle = |\tau_2 = \text{sgn}(m_s)\rangle \otimes |m_s\rangle \otimes |f_{m_s}\rangle. \]

The particle-hole spin locks along the \(+\tau_2 (-\tau_2)\) direction for positive (negative) \(m_s\). In Eq. (8.8), \(\langle z | f_{m_s}\rangle = f_{m_s}(z)\) denotes the bound state envelope function.

Using first-order \(k \cdot p\) perturbation theory we obtain the surface effective Hamiltonian,

\[ \hat{h}^{(s)}_{3/2}(k) \propto \frac{\Delta_p}{k_F} \begin{pmatrix} 0 & i k^3 \\ -i k^3 & 0 \end{pmatrix}. \]
Eq. (8.9) satisfies the projected version of the particle-hole symmetry in Eq. (2.14),
\[-\hat{M}^{(5)} \left[ \hat{h}^{(5)}_{3/2} \right]^\dagger (-k) \hat{M}^{(5)} = \hat{h}^{(5)}_{3/2}(k), \quad \hat{M}^{(5)} = \sigma_1, \quad (8.10)\]
and the projected time-reversal symmetry [Eq. (2.15)]
\[-\hat{M}^{(5)} \hat{h}^{(5)}_{3/2}(k) \hat{M}^{(5)} = \hat{h}^{(5)}_{3/2}(k), \quad \hat{M}^{(5)} = \sigma_3. \quad (8.11)\]
Here the matrices \(\{\sigma_\mu\}\) act on the components \(m_\nu = \pm 3/2\).

Fig. 13 shows the clean Majorana surface bands obtained numerically from a lattice regularization of Eq. (8.2) for (a) \(|m| = 3/2\) valence-band–hole and (b) \(|m| = 1/2\) conduction-band–electron superconductivity. Below we focus on the hole-doped case in which the surface fluid has cubic dispersion [Eq. (8.9)]. The surface fluid in the electron-doped case is depicted in Fig. 13(b), and exhibits a linear Majorana cone around \(k = 0\) and a zero-mode ring at finite surface momentum; the latter structure is inconsistent with \(\nu = 1\) [47] [48].

B. Quenched surface disorder, class DIII SO(\(n_\nu\)) conformal field theory, and numerical results

We now turn to perturbations of the surface theory, focusing on the cubic-dispersing Majorana fluid that arises from hole-doped superconductivity. We can write the surface Hamiltonian as
\[
H_0^{(5)} = \frac{1}{2} \int d^2 \mathbf{r} \eta^\dagger \hat{M}^{(5)} (\sigma_- \partial^3 - \sigma_+ \bar{\partial}^3) \eta, \quad (8.12)
\]
where \(\eta \to \eta_{m_\nu}\) is a two-component Majorana spinor and \(\mathbf{r}\) is the position vector. The chiral derivative operators are \(\{\partial, \bar{\partial}\} \equiv (1/2)\{\partial_x + i\partial_y\}\), while \(\sigma_\pm \equiv \sigma_1 \pm i\sigma_2\). Here we have set the prefactor of Eq. (8.9) equal to one.

The simplest class of surface perturbations are constant bilinears. Such an operator can be written as \(\eta^\dagger \hat{M}^{(5)} \Lambda \eta\), with \(\Lambda\) a \(2 \times 2\) Hermitian matrix. The only bilinear that satisfies particle-hole in Eq. (8.10) (i.e., which does not vanish under Pauli exclusion) is the mass term \(\Lambda = \sigma_3 \simeq J^2\). This is the projection of the spin operator perpendicular to the surface. The nonzero expectation value of this term (due e.g. to a coupling with an external Zeeman field) would open a surface energy gap and signal time-reversal symmetry breaking. The time-reversal broken state would reside in a plateau of a surface thermal quantum Hall effect [9] [84] [113] [114].

These considerations are almost identical to \(^3\)He-B [9] [15], which has spin-1/2 and \(\nu = 1\). The only difference is that the derivatives in Eq. (8.12) appear to the first power for \(\nu = 1\), whereas here we get the \(\nu = 3\) power for the spin-3/2 bulk.

Residual quasiparticle-quasiparticle interactions should be short-ranged (due to screening by the bulk superfluid). Since \(\eta\) is a two-component Majorana field, the most relevant interaction that we can write down is
\[
H_I^{(5)} = u \int d^2 \mathbf{r} \eta_1 \nabla \eta_1 \cdot \eta_2 \nabla \eta_2. \quad (8.13)
\]
The coupling strength \(u\) has dimensions of length for cubic dispersion and is therefore irrelevant in the sense of the renormalization group (RG) [47].

Finally we turn to quenched disorder, which is always present at the surface of a real sample. We assume the disorder is non-magnetic, but may arise due to neutral adatoms, charged impurities, grain boundaries, etc. In other words, any time-reversal invariant surface potential perturbation is allowed. Since the only bilinear without derivatives is the massive, time-reversal odd \(J^2\) operator discussed above, we must broaden the search to include bilinears with derivatives. The most relevant possible potential can be encoded in the Hamiltonian
\[
H_D^{(5)} = -\frac{i}{2} \int d^2 \mathbf{r} \left[ \eta^\dagger (\mathbf{r}) \hat{M}^{(5)} \sigma_\alpha \frac{\partial}{\partial x^\alpha} \eta(\mathbf{r}) \right] P_{\alpha \beta}(\mathbf{r}). \quad (8.14)
\]
In this equation repeated indices are summed, \(\alpha, \beta \in \{x, y\}\). We assume that \(P_{\alpha \beta}(\mathbf{r})\) is a white-noise-correlated random potential with variance \(\lambda\). Then \(\lambda\) has dimensions of \(1/(\text{length})^2\) and is a relevant perturbation to the clean cubic band structure.

The effects of disorder cannot be treated perturbatively. The standard procedure would produce a disorder-averaged nonlinear sigma model in class DIII, which possesses a stable thermal metal phase [116] [119]. Although the thermal metal is perturbatively accessible in the sigma model with the WZNW term, the critical SO(\(n_\nu\)) CFT fixed point is not, except for the limit of large \(\nu\). Therefore we resort to numerics in the remainder of this section. The question we want to answer is whether disorder flows into the SO(\(n_\nu\)) CFT or the thermal metal, see Fig. 12.

The noninteracting BdG Hamiltonian implied by Eqs. (8.12) and (8.14) has momentum space matrix elements
\[
\left[ \hat{h}_z \right]_{\mathbf{k}, \mathbf{k}'} = \begin{bmatrix} 0 & \frac{i k^3}{2} & \delta_{\mathbf{k}, \mathbf{k}'} + (k_x + k'_x) \delta_{\mathbf{k}, \mathbf{k}'} & 0 & 1 & 0 \\ -\frac{i k^3}{2} & 0 & 1 & 0 & \delta_{\mathbf{k}, \mathbf{k}'} + (k_y + k'_y) \delta_{\mathbf{k}, \mathbf{k}'} & 0 \\ \delta_{\mathbf{k}, \mathbf{k}'} + (k_x + k'_x) & 1 & 0 & \delta_{\mathbf{k}, \mathbf{k}'} + (k_y + k'_y) & 0 \\ 0 & 1 & 0 & \delta_{\mathbf{k}, \mathbf{k}'} + (k_x + k'_x) & 0 \\ 0 & 0 & \delta_{\mathbf{k}, \mathbf{k}'} + (k_y + k'_y) & 0 & \delta_{\mathbf{k}, \mathbf{k}'} + (k_x + k'_x) & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta_{\mathbf{k}, \mathbf{k}'} + (k_x + k'_x) \end{bmatrix} P_z (\mathbf{k} - \mathbf{k}') + (k_y + k'_y) \delta_{\mathbf{k}, \mathbf{k}'} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & \delta_{\mathbf{k}, \mathbf{k}'} + (k_y + k'_y) \end{bmatrix} P_y (\mathbf{k} - \mathbf{k}'), \quad (8.15)
\]
where we have taken \(P_{\alpha \beta}(\mathbf{r})\) to be diagonal in its lower indices. Gaussian white noise disorder can be efficiently simulated in momentum space using a random phase method [62],
\[
P_{\alpha}(\mathbf{k}) = \frac{\sqrt{\lambda}}{L} e^{i \theta_\alpha(\mathbf{k})} \exp \left( -\frac{k^2 x^2}{4} \right), \quad (8.16)
\]
where \(\theta_\alpha(\mathbf{k}) = -\theta_\alpha(\mathbf{k})\), but these are otherwise inde-
dependent, uniformly distributed random phases. The parameters $L$, $\xi$ and $\lambda$ denote the system size, correlation length, and disorder strength respectively. For exact diagonalization, we choose periodic boundary conditions so that $k = (2\pi/L)n$, and the components of $n \in \{\mathbb{Z}, \mathbb{Z}\}$ run over a square with $-N_k \leq n_i \leq N_k$, for $i = 1, 2$. Here $N_k$ determines the size of the vector space in which we diagonalize, which is $2(2N_k + 1)^2$. While the choice of $L$ is arbitrary, we use it to fix the ultraviolet momentum cutoff $\Lambda = 2\pi N_k/L$. The correlation length $\xi$ and the dimensionful disorder strength $\lambda$ are then measured in terms of powers of $\Lambda$. The random-phase approach is equivalent to the disorder-average up to finite-size corrections [62]. We perform the calculations in momentum space in terms of powers of $\Lambda$. The random-phase approach is equivalent to the disorder-average up to finite-size corrections [62].

To characterize the disordered surface theory, we study the scaling of the disorder-averaged DoS $\rho_S(\varepsilon)$ and wave function multifractality, measures that are expected to show universal behavior at the $SO(n)_3$ fixed point. The clean surface has $\rho_S(\varepsilon) \propto |\varepsilon|^{-1/3}$ due to the cubic dispersion. For winding number $\nu$, in the presence of time-reversal preserving disorder the $SO(n)_\nu$ theory predicts the scaling behavior of the disorder-averaged DoS [61, 120] to be $\rho_S(\varepsilon) \propto |\varepsilon|^{-1/(2\nu-3)}$. In the case of the hole-doped LSM with $\nu = 3$, the clean and dirty predictions coincide. For the generalized surface theory introduced below [defined via Eq. (8.23)], or the $\nu = 4$ model studied in Ref. [51], the clean and dirty predictions differ, so that the DoS provides a useful diagnostic. We will plot the integrated density of states (IDoS) $N(\varepsilon)$. For the $SO(n)_\nu$ theory

$$N(\varepsilon) \equiv \int_{\varepsilon_0}^{\varepsilon} d\varepsilon' \rho_S(\varepsilon') \sim |\varepsilon|^{(2\nu-4)/(2\nu-3)}. \quad (8.17)$$

The other measure that we will employ here as a numerical test for the $SO(n)_\nu$ CFT is wave function multifractality. The disorder-induced spatial fluctuations of the local DoS $\rho_S(\varepsilon, \mathbf{r})$ are encoded in the multifractal spectrum $\tau(q)$. The $\tau(q)$ spectrum measures the sensitivity of extended wave functions to the sample boundary. By partitioning a large area $L \times L$ of the surface with boxes of small size $b \ll L$, one can define the box probability $\mu_n$ and inverse participation ratio (IPR) $\mathcal{P}_q$ in terms of a particular wave function $\psi(\mathbf{r})$ via

$$\mu_n = \frac{\int_{A_n} d^2 \mathbf{r} |\psi(\mathbf{r})|^2}{\int_{L^2} d^2 \mathbf{r} |\psi(\mathbf{r})|^2}, \quad \mathcal{P}_q \equiv \sum_n \mu_n^q, \quad (8.18)$$

where $A_n$ denotes the $n$th box. In the case of a typical critically delocalized wave function, one expects that

$$\mathcal{P}_q \sim (b/L)^{\tau(q)}, \quad (8.19)$$

where $\tau(q)$ is self-averaging and universal [116]. For TSC surface states, the multifractal spectrum $\tau(q)$ is expected to be $\nu$-dependent.
to have the form \[61, 62, 121\]

\[
\tau(q) = \begin{cases}
(q-1)(2-\theta(q), & q < |q_c|, \\
(\sqrt{2}-\sqrt{\theta(q)})^2 q, & q > q_c, \\
(\sqrt{2}+\sqrt{\theta(q)})^2 q, & q < -q_c,
\end{cases}
\tag{8.20}
\]

where

\[
q_c = \sqrt{2/\theta(q)}.
\tag{8.21}
\]

The spectrum is quadratic below the termination threshold \(q = \pm q_c\), beyond which it is linear \[116, 121, 122\].

For disordered class DIII surface states and winding number \(\nu\), the SO(\(n\))\(_\nu\) theory predicts \[61\]

\[
\theta(q) = \frac{1}{\nu-2}, \quad \nu \geq 3.
\tag{8.22}
\]

Eqs. 8.17 and 8.22 are exact results that obtain from the primary field spectrum of SO(\(n\))\(_\nu\) in the replica \(n \to 0\) limit. Isotropic \(p\)-wave pairing in the Luttinger semimetal gives \(\nu = 3\), so that \(\theta(q) = 1\) and \(q_c = \sqrt{2} \approx 1.4\). This corresponds to quite strong multifractality, which presents some difficulties as we will see. By contrast, large \(\nu\) gives \(\theta(q) \ll 1\) and \(q_c \gg 1\), corresponding to weakly multifractal (nearly plane-wave) states.

Fig. 14 depicts our numerical results for the \(\tau(q)\) spectrum for two different disorder strengths, and the IDoS \(N(\varepsilon)\) for one disorder strength. As mentioned above, the IDoS is not particularly useful for \(\nu = 3\) because both clean and dirty CFT predictions coincide. Moreover, the strong divergence in the corresponding DoS \(\rho(\varepsilon)\) makes it difficult to get sufficient resolution in the peak itself.

We find that the multifractal spectrum becomes disorder-independent for sufficiently large disorder strengths. This is important, because the thermal metal phase should exhibit weak multifractality and a weak DoS divergence, but both features would be disorder- and scale-dependent due to weak antilocalization \[119\]. We observe rough agreement between the analytical SO(\(n\))\(_3\) CFT prediction [Eqs. 8.20–8.22] and the numerics. This should be compared to the \(\nu = 4\) model studied in Ref. 51, wherein quite good agreement was obtained. Even better results are found for the higher-\(\nu\) model explained in the next section, see Figs. 15 and 16.

We attribute the relatively poor fit for \(\tau(q)\) in Fig. 14 to the strong multifractality predicted by SO(\(n\))\(_3\). This is indicated by the solid blue curve in the top panel of Fig. 14. The analytical \(\tau(q)\) is almost “frozen.” A frozen state has \(\tau(q) = 0\) for \(q > q_c \geq 1\) \[121\]. A critically delocalized, but frozen state consists of a few rare probability peaks, with arbitrarily large separation between these \[62, 121, 123–125\]. The peaks are sufficiently rare that their heights do not scale with a power of the system size \(L\), similar to an Anderson localized state. (The term “frozen” originates via a mapping to the classical glass transition in the random energy model \[121, 123, 125\].) Frozen states also resemble the “random singlet” wave functions of the Jordan-Wignerized random bond XY model in 1D, which have the quality of random telegraph signals \[126\]. In a previous study \[62\], we found that the momentum space method does not scale well for frozen states, and we believe this is the source of the relatively poor fit in Fig. 14.

C. Generalized surface: Higher winding numbers and numerical results

If we believe that finite size effects are responsible for the relatively poor fit between numerics and the SO(\(n\))\(_3\) CFT in Fig. 14, the obvious way to improve is to increase the system size. Instead of doing this (which requires more computer memory), we take another approach.

We conjecture that the Majorana surface fluid of a class DIII TSC with bulk winding number \(\nu \in 2\mathbb{Z} + 1\) (odd) can be captured by the generalized \(2 \times 2\) surface
However, if the “topological tuning” scenario articulated in Fig. [12] is correct, then any clean starting point should lead to the same disordered fixed point, the SO(n)ν CFT [61].

We can infer the bulk winding number ν by computing the surface winding number W5. This obtains by adding the homogeneous time-reversal symmetry-breaking mass term mσ3 to the clean band structure in Eq. (8.23), and computing [15]

$$W_5(m) = \frac{\epsilon_{\alpha \beta \gamma}}{3! (2\pi)^2} \int_{-\infty}^{\infty} d\omega \int d^2k \text{Tr} \left[ (\hat{G}^{-1} \partial_\omega \hat{G}) \times (\hat{G}^{-1} \partial_\beta \hat{G}) (\hat{G}^{-1} \partial_\gamma \hat{G}) \right], \quad (8.25)$$

where Tr denotes the trace over the two spinor components, and α, β, γ ∈ {ω, kx, ky} (repeated indices are summed). The surface state Green function’s \( \hat{G}(\omega, \mathbf{k}, m) \) is given by

$$\hat{G}(\omega, \mathbf{k}, m) \equiv \left[ -i \omega 1 + \hat{h}_m(\mathbf{k}) \right]^{-1}, \quad (8.26)$$

where \( \hat{h}_m = \hat{h}_0^{(\nu)} |_{p_\nu = 0} + m \sigma_3 \) is the clean, gapped surface Hamiltonian. The surface winding number determines the thermal Hall conductivity [84, 113, 114, 127, 129]

$$\kappa_{xy} = W_5 \kappa_0, \quad (8.27a)$$

$$\kappa_0 = \pi^2 k_B^2 T / 6 h. \quad (8.27b)$$

Here h is Planck’s constant.

It is easy to check that

$$W_5(m) = (\nu/2) \text{sgn}(m). \quad (8.28)$$

For ν = 1 this is the standard “half-integer” (shifted) surface quantum Hall effect familiar from 3He-B and topological insulators [9, 15]. For a relativistic Majorana surface fluid, it can be shown that the maximum possible value of W5 is the bulk winding number divided by two [61]. We conclude that the surface Hamiltonian in Eq. (8.23) is a representative surface band structure for a class DIII TSC with winding number ν.

Following the same logic of the previous section, we compare the numerical diagonalization of Eq. (8.23) in momentum space to the predictions of the SO(n)ν CFT. Results for ν = 5 and ν = 7 are shown in Figs. [15] and [16] respectively. In these cases, the multifractal spectrum τ(q) and the IDoS N(ε) match very well the corresponding CFT predictions in Eqs. (8.20)–(8.22) and (8.17), respectively. The reason for the better matching is the weaker multifractality of the critical wave functions with increasing ν, as predicted by the CFT.

The SO(n)ν fixed point is stable against residual quasiparticle-quasiparticle interactions [61]. In addition to universal energy scaling of the DoS and wave function multifractality (both which could be detected via STM), the ratio of the thermal conductivity to temperature T
is predicted to be quantized in the $T \to 0$ limit: \[ \lim_{T \to 0} \frac{\kappa_{xx}}{T} = \frac{|\nu|}{\pi} \frac{\kappa_0}{T}, \]
where $\kappa_0$ was defined by Eq. (8.27b).

**IX. CONCLUSIONS AND OUTLOOK**

To summarize, we have presented possible topological superconducting phases, including both gapless and gapped, in a doped Luttinger system (see Sec. II). We showed that while pseudo-spin singlet $s$-wave pairing yields a trivial fully gapped state, the $d$-wave counterparts (belonging to either $T_{2g}$ or $E_g$ representations) often (if not always) lead to Weyl superconductors at low temperature at the cost of the time-reversal symmetry (see Sec. III). We argued that the simple Weyl nodes (sources and sinks of Abelian Berry curvature, see Figs. 3E] that arise from complex combinations of simple $d$-wave nodal loops cause a power-law suppression of the density of states $\varrho(E) \sim |E|$ (for nodal-loop) $\to |E|^2$ (for simple Weyl nodes) at low energies. Therefore, Weyl paired states are generically expected, at least within the framework of weak coupling pairing.

While any Weyl pairing supports one-dimensional (pseudospin degenerate) Fermi arcs as surface Andreev bound states, only the $T_{2g}$ paired state can lead to nontrivial anomalous pseudospin and thermal Hall conductivities at low temperature (Sec. III E). The simple Weyl BdG quasiparticles remain sharp in the presence of weak randomness in the system (in contrast to double-Weyl fermions in the $d_{xy} + id_{x^2−y^2}$ phase or nodal-loop states, see Fig. 9). Stronger bulk disorder in the BdG-Weyl system induces a continuous quantum phase transition into a thermal metallic phase (Sec. V and Fig. 8). The critical regime occupies a large portion of the phase diagram, where $\varrho(E) \sim |E|$, as shown in Fig. 8, which induces the corresponding scaling of physical observables such as specific heat, thermal conductivity, etc.

We demonstrated that nucleation of any $d$-wave pairing always causes a small lattice distortion or nematicity that in turn gives rise to a non-trivial $s$-wave component in the paired state (see Fig. 7). Such symmetry-guaranteed coupling between $d$- and $s$-wave pairing with a lattice distortion may allow one to strain-engineer various exotic $s + d$ pairings, specifically in weakly correlated materials (Sec. IV). We found that, within a simple picture of pairing, time-reversal symmetry breaking $s + id$ order seems to be extremely unlikely (with $s$-wave and $d + id$-type pairings being separated by a first order transition, see Fig. 11). This interesting possibility cannot be completely ruled out (Sec. VII). We also showed that when the pairing interactions in the $T_{2g}$ and $E_g$ channels are of comparable strength, a myriad of gapless topological superconductors can be realized in the system (see Sec. IIII Table II and Fig. 6), while only the $d_{xy} + id_{x^2−y^2}$ paired state, supporting double-Weyl fermions, would exhibit non-trivial anomalous thermal and spin Hall conductivities (Sec. IIII E). However, in the presence of inversion symmetry breaking (the situation in half-Heusler compounds) only thermal Hall conductivity remains sharply defined.

In terms of these nodal pairings we also attempted to understand the recent experimental data for the penetration depth in YPtBi [46], suggestive of the existence of gapless quasiparticles inside the paired state. We showed that $T$-linear fit, when augmented by a contribution from an ordinary $s$-wave component (always present with any $d$-wave pairing via the aforementioned coupling to the strain), matches extremely well with the experimental penetration depth data in YPtBi [46], see Fig. 10. We argued that this $T$-linear contribution may originate from either nodal loops in simple $d$-wave pairing for example (see Fig. 2), or from the effect of quenched disorder (such as half-Heuslers and 227 pyrochlore iridates also disperse). Perhaps the most urgent issue in the context of superconductivity in a doped Luttinger semimetal is that of pairing mechanisms. Recently, it has been argued that such pairing can in principle be mediated by electron-phonon interactions, specifically due to optical phonons [54]. However, given that promising candidates such as half-Heuslers and 227 pyrochlore iridates also display magnetic orders, pairing in these materials may also arise from strong electronic interactions. Understanding the effects of magnetic fluctuations on various pairing scenarios is a challenging, but crucial question that we leave for a future investigations.
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Appendix A: Luttinger model components

In this Appendix we present some essential details of the Luttinger Hamiltonian. The \( \mathbf{d} \)-vector appearing in Eq. (2.3) is a quadratic function of momentum measured from the \( \Gamma = (0, 0, 0) \) point of the Brillouin zone, where Kramers degenerate valence and conduction bands touch each other, namely \( \mathbf{d} = k^2 \hat{\mathbf{d}} \). The quantity \( \hat{\mathbf{d}} \) is a five-component unit vector and its components are given by

\[
\hat{d}_1 = \frac{i}{\sqrt{2}} \left[ Y^1_2 + Y^{-1}_2 \right] = \frac{\sqrt{3}}{2} \sin 2\theta \sin \phi = \frac{\sqrt{3}}{2} k_y k_z,
\]

\[
\hat{d}_2 = \frac{i}{\sqrt{2}} \left[ Y^{-1}_2 + Y^1_2 \right] = \frac{\sqrt{3}}{2} \sin 2\theta \cos \phi = \frac{\sqrt{3}}{2} k_y k_z,
\]

\[
\hat{d}_3 = \frac{i}{\sqrt{2}} \left[ Y^{-2}_2 + Y^2_2 \right] = \frac{\sqrt{3}}{2} \sin^2 \theta \sin 2\phi = \frac{\sqrt{3}}{2} k_y k_z,
\]

\[
\hat{d}_4 = \frac{i}{\sqrt{2}} \left[ Y^{-2}_2 + Y^2_2 \right] = \frac{\sqrt{3}}{2} \sin^2 \theta \cos 2\phi = \frac{\sqrt{3}}{2} k_y k_z,
\]

\[
\hat{d}_5 = Y^0_2 = \frac{1}{2} \left( 3 \cos^2 \theta - 1 \right) = \frac{1}{2} \left[ 2k_x^2 - \hat{k}_x^2 - \hat{k}_y^2 \right]. \quad (A1)
\]

Note that \( \sum_{j=1}^{5} \left( \hat{d}_j \right)^2 = 1 \). In the above expression \( Y^m_l = Y^m_l(\theta, \phi) \) are the spherical harmonics with angular momentum \( l = 2 \).

Five mutually anticommuting \( \Gamma \) matrices appearing in Eq. (2.3) are constructed from \( J = 3/2 \) matrices according to

\[
\Gamma_1 = \frac{1}{\sqrt{3}} \{ J^y, J^z \}, \quad \Gamma_2 = \frac{1}{\sqrt{3}} \{ J^x, J^z \},
\]

\[
\Gamma_3 = \frac{1}{\sqrt{3}} \{ J^x, J^y \}, \quad \Gamma_4 = \frac{1}{\sqrt{3}} \{ (J^x)^2 - (J^y)^2 \},
\]

\[
\Gamma_5 = \frac{1}{3} \left[ 2(J^x)^2 - (J^y)^2 - (J^z)^2 \right], \quad (A2)
\]

while \( \Gamma_0 \) denotes the four dimensional identity matrix. Here \( \{ A, B \} = AB + BA \) is the anticommutator. The \( \{ \Gamma_1, \ldots, \Gamma_5 \} \) are components of the rank-two symmetric traceless tensor operator

\[
T^{\mu\nu} = \frac{1}{\sqrt{3}} \left[ \{ \mu, \nu \} - \frac{2}{3} \delta^{\mu\nu} J^2 \right], \quad (A3)
\]

which transforms in the \( j = 2 \) representation of SU(2) under spin rotations. In the basis specified by Eq. (2.2), the spin-3/2 matrices are defined as

\[
J^x = \frac{1}{2} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad J^y = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad (A4)
\]

A full basis for \( 4 \times 4 \) matrices can be formed by adding the identity and the 10 commutators \( \{ \Gamma_{ab} = -i\Gamma_a \Gamma_b \} \) to the 5 \( \{ \Gamma_a \} \) matrices. The matrices in the product basis \( \{ \Gamma_{ab} \} \) do not transform irreducibly under the spin SU(2).

An irreducibly decomposed basis of tensor operators instead uses the three \( J^x,y,z \) generators \( (j = 1) \) and seven components of a rank-three, traceless symmetric tensor formed from products of these,

\[
T^{\mu\nu\rho} = J^{( \mu} J^{\nu} J^{\rho) - (\text{traces})}, \quad \sum_{\mu=1}^{3} T^{\mu\mu\nu} = 0, \quad (A5)
\]

where \( (\mu\nu\rho) \) means complete symmetrization of these indices. The latter transforms as \( j = 3 \) under spin SU(2) rotations. The tensor operator \( T^{\mu\nu\rho} \) plays the key role in the proposal for odd-parity, orbital \( p \)-wave “septet” superconductivity in [46, 50, 52].

Appendix B: Band Projection in Luttinger system

In this Appendix we present the band projection method in the Luttinger system. We first focus on the non-interacting part of the theory. In the eight-component spinor basis, defined in Eq. (2.11), the isotropic Luttinger Hamiltonian reads as

\[
\hat{h}_L^N(\mathbf{k}) = \tau_3 \hat{h}_L(\mathbf{k}), \quad (B1)
\]

where \( \hat{h}_L(\mathbf{k}) \) is the four-dimensional Luttinger Hamiltonian defined in Eq. (2.3), with \( m_1 = m_2 = m \). The Luttinger Hamiltonian can be brought into diagonal form under the following unitary transformation \( \mathcal{D}_N^J \hat{h}_L^N \mathcal{D}_N^{-1} \), where \( \mathcal{D}_N = U_1 U_2 \) is the diagonalizing matrix, with

\[
U_1 = \tau_0 \otimes \mathcal{D}, \quad U_2 = \tau_0 \otimes \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad (B2)
\]

with \( \mathcal{D} \) is already defined in Eq. (2.7). Upon performing the unitary rotation with the diagonalizing matrix the
Luttinger Hamiltonian takes the form

\[
\hat{D}_N \hat{H} \hat{D}_N^{-1} = \left( \frac{k^2}{2m_0} - \mu \right) \text{diag.}(\sigma_0, \sigma_0, -\sigma_0, -\sigma_0) + \frac{k^2}{2m} (\sigma_0, -\sigma_0, -\sigma_0, \sigma_0). \tag{B3}
\]

In the last expression the bold (normal) entries correspond to conduction (valence) band, which then leads to the kinetic energy in the conduction band reported in Eq. (2.17). The kinetic energy in the valence band also assumes the form of Eq. (2.17), but with the modification that effective mass parameter reads as \(m_e = mn_0/(m - m_0)\). Therefore, valence and conduction bands bend in opposite directions when the effective mass parameters in the Luttinger model satisfy \(m > m_0\).

The diagonalization process on pairing operators yields

\[
\hat{D}_N [\tau_1 \cos \phi_{sc} + \tau_2 \sin \phi_{sc}] \hat{M}_{4 \times 4} \hat{D}_N^{-1} = \left[ \hat{a}_{2 \times 2} \hat{b}_{2 \times 2} \right] \left[ \hat{d}_{2 \times 2} \right]. \tag{B4}
\]

In the final expression \(\hat{a}_{2 \times 2}, \hat{b}_{2 \times 2}, \hat{d}_{2 \times 2}\) captures the form of a given pairing \(\hat{M}_{4 \times 4}\) on conduction (valence) bands, while two other entries namely \(\hat{\bar{b}}_{2 \times 2}, \hat{\bar{d}}_{2 \times 2}\) capture the coupling between the valence and conduction bands. Throughout this work we neglect such coupling assuming that the pairing only takes place in the close proximity to the Fermi surface.

### Appendix C: Band Projection in massive Dirac system

In this Appendix we present the band diagonalization procedure and transformation of six local pairings in a Dirac semiconductor. In the basis of a four-component Dirac-spinor defined as \(\Psi^T = (c^+_\uparrow, c^+_{\downarrow}, c^-_{\uparrow}, c^-_{\downarrow})\), where \(c^\alpha_{\sigma}\) is the fermionic annihilation operator with parity \(\kappa = \pm\) and spin-projection \(\sigma = \uparrow, \downarrow\), the massive Dirac Hamiltonian reads as

\[
H_D = v \gamma_0 \gamma_j k_j + m \gamma_0 - \mu, \tag{C1}
\]

where \(j = 1, 2, 3\), and summation over repeated indices are assumed. Here, \(k_j\)s are three spatial component of momentum, measured from the \(\Gamma = (0, 0, 0)\) point of the Brillouin zone, \(m\) is the Dirac mass and \(\mu\) is the chemical potential. Fermi velocity \(v\) is assumed to be isotropic for the sake of simplicity, which from now onward we set to be unity. Mutually anti-commuting four-component Hermitian \(\gamma\) matrices are defined as

\[
\gamma_0 = \kappa_3 \sigma_0, \quad \gamma_1 = \kappa_2 \sigma_1, \quad \gamma_2 = \kappa_2 \sigma_2, \quad \gamma_3 = \kappa_2 \sigma_3.
\]

Two sets of Pauli matrices \(\{\kappa_\mu\}\) and \(\{\sigma_\mu\}\) with \(\mu =
0, 1, 2, 3 respectively operate on parity and spin index. The above Hamiltonian is invariant under the following discrete symmetry operations: (a) reversal of time \((T)\), generated by \(T = i\gamma_1\gamma_3K\), where \(K\) is the complex conjugation, (b) parity or inversion \((P)\), under which \(r \rightarrow -r\) and \(P = \gamma_0\), (c) charge conjugation \((C)\), under which \(C\Psi C^{-1} = \gamma_2\Psi^*\). In the massless limit \((m \rightarrow 0)\), describing a quantum critical point between two topologically distinct insulating phases, the Dirac Hamiltonian also enjoys an emergent continuous \(U(1)\) chiral symmetry, generated by \(\gamma_5\).

Since all four dimensional representation of five mutually anti-commuting matrices are unitarily equivalent, we can express the above five matrices from Eq. (C2) in term of \(\Gamma_j.s\) and \(\Gamma_j.k.s\) introduced in Eq. (2.4) according to

\[
\gamma_0 = \Gamma_{43}, \quad \gamma_1 = \Gamma_{24}, \quad \gamma_2 = \Gamma_{14}, \quad \gamma_3 = \Gamma_5, \quad \gamma_5 = \Gamma_4, \quad (C3)
\]

or equivalently

\[
\Gamma_1 = \gamma_{25}, \quad \Gamma_2 = \gamma_{15}, \quad \Gamma_3 = \gamma_{05}, \quad \Gamma_4 = \gamma_5, \quad \Gamma_5 = \gamma_3. \quad (C4)
\]

where \(\gamma_{lm} = i\gamma_l\gamma_m\). Therefore, all possible, namely six, local-pairings in this system are also captured by the effective single-particle Hamiltonian \(H_{pp}^{\text{local}}\) in Eq. (2.10), which in terms of the \(\gamma\) matrices can be expressed as

\[
H_{pp}^{\text{local}} = \int d\mathbf{r} \left\{ \Delta_s \Psi_{\gamma_{13}} \Psi + \Delta_p \Psi_{\gamma_{02}} \Psi + \Delta_1 \Psi_{\gamma_{3}} \Psi + \Delta_2 \Psi_{\gamma_{05}} \Psi + \Delta_3 \Psi_{\gamma_{25}} \Psi \right\}, \quad (C5)
\]

where \(\Psi\) is the four-component Dirac-spinor, introduced earlier. Also notice that we have changed the notations for the pairing amplitudes \(\Delta_s,\Delta_p\) from Eq. (2.10). The purpose will be clear in a moment.

We now conveniently define an eight-component Nambu spinor to cast all local pairing in a massive Dirac system in a compact form

\[
\Psi_N = \begin{bmatrix} \Psi \\ \gamma_{13} \Psi^\dagger \end{bmatrix}. \quad (C6)
\]

Note that \(\gamma_{13}\) is the unitary part of the time-reversal operator. In this eight-component basis, the Nambu-doubled massive Dirac Hamiltonian from Eq. (C1) takes the form

\[
H_D^{\text{Namb}} = \tau_3 \otimes \left[ \gamma_{01} k_x + \gamma_{02} k_y + \gamma_{03} k_z + \gamma_0 m - \mu \right]. \quad (C7)
\]

The newly introduced set of Pauli matrices \(\{\tau_3\}\) operate on the Nambu index. In the same basis the single-particle Hamiltonian in the presence of all local pairings is given by

\[
H_{pp}^{\text{local}} = (\tau_1 \cos \phi + \tau_2 \sin \phi) \left[ \Delta_s + \Delta_p \gamma_5 \right] + \sum_{\mu=0}^3 \Delta_\mu \gamma_\mu, \quad (C8)
\]

where \(\phi\) is the superconducting phase. Now we can take the advantage of the spinor basis introduced in Eq. (C6), to classify all six local pairings according to their transformation under the Lorentz transformation \((LT)\). Respectively the pairing proportional to \(\Delta_s\) and \(\Delta_p\) transforms as scalar and pseudo-scalar Majorana mass under the LT. The set of pairings \(\{\Delta_\mu\}\), with \(\mu = 0, 1, 2, 3\), transforms as a “four”-vector under the LT. While the pseudo-scalar and the spatial-components \(\{\mu = 1, 2, 3\}\) of the vector pairings are odd under the parity \((P)\), the scalar and the temporal-component of the vector pairing are even under parity [see Table 3].

The unitary operator that diagonalizes the massive Hamiltonian [see Eq. (C7)] is given by \(\tau_0 \tilde{D}(k, m)\), where \(k = (k_x, k_y, k_z)\),

\[
\tilde{D}(k, m) = \begin{bmatrix} k_x - ik_y & k_y & k_x + ik_y \\ \sqrt{2\lambda(\lambda - m)} & \sqrt{2\lambda(\lambda + m)} & -\sqrt{2\lambda(\lambda - m)} \\ -\sqrt{2\lambda(\lambda + m)} & \sqrt{2\lambda(\lambda - m)} & k_x - ik_y \\ \sqrt{2\lambda(\lambda - m)} & 0 & \sqrt{2\lambda(\lambda + m)} \\ 0 & -\sqrt{2\lambda(\lambda - m)} & \sqrt{2\lambda(\lambda + m)} \\ \sqrt{2\lambda(\lambda + m)} & 0 & 0 \\ \sqrt{2\lambda(\lambda - m)} & 0 & 0 \\ 0 & \sqrt{2\lambda(\lambda + m)} & 0 \end{bmatrix}, \quad (C9)
\]

and \(\lambda = [k^2 + m^2]^{1/2}\). After performing a unitary rotation with diagonalizing matrix, the Nambu-doubled Massive Dirac Hamiltonian from Eq. (C7) becomes

\[
\tau_3 \tilde{D}^\dagger H_D^{\text{Namb}} \tau_3 \tilde{D} = -\mu \text{ diag.} (\sigma_0, \sigma_0, -\sigma_0, -\sigma_0) + \sqrt{k^2 + m^2} \text{ diag.} (\sigma_0, -\sigma_0, -\sigma_0, \sigma_0). \quad (C10)
\]

In the above expression, the quantities in the bold (normal) font correspond to the conduction (valence) band. We here restrict ourselves with the situation when the chemical potential is placed in the conduction band, i.e., when \(\mu > 0\). We now make a large mass expansion of the kinetic energy term, yielding

\[
\sqrt{k^2 + m^2} - \mu = \frac{k^2}{2m} - (\mu - m) \equiv \frac{k^2}{2m} - \tilde{\mu}, \quad (C11)
\]

where \(\tilde{\mu} = \mu - m\) is the renormalized chemical potential, measured from the bottom of the conduction band, and \(k_F = \sqrt{2m\tilde{\mu}}\) is the Fermi vector. Hence, the kinetic energy for massive Dirac fermions in the close proximity to the Fermi surface assumes the form

\[
H_0 = \left(\frac{k^2}{2m} - \tilde{\mu}\right) \tau_3 \sigma_0. \quad (C12)
\]

Next we perform the same band projection on six local pairings shown in Eq. (C8). After this transformation the pairing assumes the schematic form shown in
Eq. [B4]. Since, we have assumed that chemical potential lies within the conduction band, we are only interested in the two-component representation of the pairings, namely $\hat{a}_{2\times2}$. The band-projected version of all six local pairings, quasiparticle spectra etc. are displayed in Table [III]

Appendix D: $d + id$ pairing: energetics and competition within $E_g$ channel

In Sec. [III.B.1] we have established that there are three inequivalent ways of choosing the basis functions for the two-dimensional representation $E_g$, namely:

A: $d_1(k) = \frac{\sqrt{3}}{2}(k_x^2 - k_z^2)$, $d_2(k) = \frac{1}{2}(2k_x^2 - k_z^2)$;

B: $d_3(k) = \frac{\sqrt{3}}{2}(k_x^2 - k_z^2)$, $d_4(k) = \frac{\sqrt{3}}{2}(k_y^2 - k_z^2)$;

C: $d_5(k) = \frac{1}{2}(2k_y^2 - k_y^2 - k_z^2)$, $d_6(k) = \frac{1}{2}(2k_y^2 - k_y^2 - k_z^2)$.

In this Appendix, we provide the details of the derivation and solution of the gap equation for these cases.

The zero-temperature gap equation, Eq. (2.24), acquires the following form in the case of $d + id$ pairing:

$$\frac{1}{\lambda_d} = \frac{1}{2} \int \frac{d\Omega}{4\pi} \int d\omega_D \frac{|d_m|^2 + |d_n|^2}{\omega_D} \sqrt{1 + y} \int d\omega_D \frac{|d_m|^2 + |d_n|^2}{\omega_D},$$

where $y = (k/k_F)^2 - 1$. The general structure of the form-factors is as follows:

$$|d_m(k)|^2 + |d_n(k)|^2 = \left(\frac{k}{k_F}\right)^4 \left[a^2(\theta) + b^2(\theta) \cos^2(2\phi)\right] = (1 + y)^2 \left[a^2(\theta) + b^2(\theta) \cos^2(2\phi)\right].$$

In order to make progress analytically, we shall adopt the weak-coupling approximation $\omega_D \ll 1$, leading to $y \ll 1$ and allow us to simplify $1 + y \approx 1$. The integral over $y$ can then be computed analytically, resulting in

$$\frac{1}{\lambda_d} = \frac{\omega_D}{\pi} \left[ a^2(\theta) + b^2(\theta) \cos^2(2\phi) \right] \times$$

$$\times \ln \left( \frac{\omega_D + \sqrt{\omega_D^2 + |\Delta_d|^2(a^2 + b^2 \cos^2(2\phi))}}{\Delta_d \sqrt{a^2 + b^2 \cos^2(2\phi)}} \right).$$

Under the assumption of weak coupling, $|\Delta_d| \ll \omega_D$, the numerator under the logarithm can be approximated by $2\omega_D$, allowing one to make further progress analytically.

The integration over $\phi$ can then readily be performed using the following two identities

$$I_0(a, b) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \ln (\sqrt{a^2 + b^2 \cos^2(2\phi)}),$$

$$I_1(a, b) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \cos^2(2\phi) \ln (\sqrt{a^2 + b^2 \cos^2(2\phi)})$$

where $F = \frac{1}{2\pi} \int_0^{2\pi} d\phi \cos^2(2\phi)$,

$$I_1(a, b) = \frac{1}{4} + \frac{1}{2} \ln \left( a + \sqrt{a^2 + b^2} \right) - a \frac{\sqrt{a^2 + b^2} - a}{2b^2}.$$

Further denoting $f = \frac{1}{2} \int_0^1 d\phi \cos^2(2\phi)$, we can write down the gap equation as follows

$$\frac{1}{\lambda_d} = \left( a^2 + \frac{b^2}{2} \right) \ln \left( \frac{2\omega_D}{\Delta_d} \right) - \langle a^2 I_0(a, b) \rangle - \langle b^2 I_1(a, b) \rangle$$

Note that the angle average

$$\int \frac{d\omega}{4\pi} \left[ d_m(\theta, \phi) \right] \langle d_m(\theta, \phi) \rangle^2 = \langle a^2 \rangle + \langle b^2 \rangle = \frac{2}{5},$$

since each of the 5 $d$-wave harmonics averages to 1/5. This allows us to finally write down the solution for the gap in a closed form:

$$\Delta_d = 2\omega_D \exp \left[ \frac{5}{2}(P_0 + P_1) \right] \exp \left[ -\frac{5}{2}\lambda_d \right],$$

where $P_0 = \langle a^2 I_0(a, b) \rangle$ and $P_1 = \langle b^2 I_1(a, b) \rangle$ are simple $c$-numbers that can be computed explicitly for each of the three choices of the basis, shown in Eq. (D1), yielding

A: $P_0 = -0.1008$, $P_1 = -0.0199,$

B: $P_0 = -0.0487$, $P_1 = -0.0326,$

C: $P_0 = -0.0099$, $P_1 = -0.0181.$

We then arrive at the final expression for the gap amplitudes for each choice of the basis

$$\Delta^{(A)}(T = 0) = 2.705 \omega_D \exp \left( -\frac{5}{2}\lambda_d \right), \quad \Delta^{(B)}(T = 0) = 2.451 \omega_D \exp \left( -\frac{5}{2}\lambda_d \right), \quad \Delta^{(C)}(T = 0) = 2.145 \omega_D \exp \left( -\frac{5}{2}\lambda_d \right).$$

As anticipated, different choices of bases result in different zero-temperature values of the $(d + id)$ gap, with the basis A ($d_{x^2+y^2} + id_{3z^2-r^2}$) having the largest gap value and therefore the lowest energy. But any $d + id$ paired state from same basis produces identical nodal structure in the ordered phase.

We note that the choice of the basis does not however affect the value of the superconducting transition temperature, with all three choices resulting in the same value of $T_c$ given by the solution of the equation similar
to Eq. (2.25):
\[
\frac{1}{\lambda_d} = \frac{1}{2} \int_{-\omega_D}^{\omega_D} dy \frac{(1+y)^{\frac{3}{2}}}{y} \tanh \left( \frac{y}{2k_BT_c} \right) \times \int \frac{d\Omega}{4\pi} \left[ \tilde{d}^2_{n} (\Omega) + \tilde{d}^2_{n} (\Omega) \right].
\] (D12)

Using the fact that \(1+y \approx 1\) in the weak-coupling approximation, we obtain
\[
k_BT_c (d+id) = \frac{2e^\gamma}{\pi} \omega_D e^{-\frac{\lambda_s}{\lambda_d}} \approx 1.134 \omega_D e^{-\frac{\lambda_s}{\lambda_d}},
\] (D13)

where \(\gamma \approx 0.577\) is the Euler’s number. Therefore, despite possessing the same transition temperature, \(d_{s}-d_{-}+id_{32}\) has the lowest energy among three time-reversal broken candidates for the \(d+id\) paired states in the \(E_g\) channel, and thus always wins at low temperature.

**Appendix E: \(s+id\) pairing and competition with \(d+id\) phase**

In this Appendix, we provide technical details of the derivation for the \(s+id\) phase, which we studied in Sec. VII Our starting point is the system of coupled gap equations Eqs. (7.2), valid in the weak-coupling approximation \(\Delta_d \ll \omega_D \ll 1\). We stress that the weak coupling approximation is justified here, since we are concerned with solving the coupled gap equations in the vicinity of the second-order phase transition at \(r = r_{c1}\) \((r \equiv \lambda_d/\lambda_s)\), where \(\Delta_d\) vanishes. In the following, we consider the most general case when the Debye frequencies for the \(s\)-wave and \(d\)-wave components, \(\omega_D^{(s)}\) and \(\omega_D^{(d)}\) respectively, are not necessarily the same. This would be the case if, for instance, the origin of the \(s\)-wave component is due to electron-phonon coupling, whereas the \(d\)-wave pairing is mediated by electron interactions.

To avoid the unnecessary complications associated with the integration over the angle \(\phi\), we hereby consider the case of \(s+id_{32}\) pairing, since the form-factor only depends on the polar angle \(\theta\). We then have the following coupled gap equations

\[
\frac{1}{\lambda_s} = \ln (2\omega_D^{(s)}) - \frac{1}{2} \int_{-1}^{1} d(\cos \theta) \ln \left( \sqrt{\Delta^2_s + \Delta_d^2\hat{d}^2(\theta)} \right),
\] (E1)

\[
\frac{1}{\lambda_d} = \ln (2\omega_D^{(d)}) - \frac{1}{2} \int_{-1}^{1} d(\cos \theta) \hat{d}^2(\theta) \ln \left( \sqrt{\Delta^2_s + \Delta_d^2\hat{d}^2(\theta)} \right).
\] (E2)

The integration cannot be completed in the closed form, however it is possible to obtain an approximate solution in the vicinity of the transition \(r_{c1}\) by expanding in the powers of the small parameter \(\Delta_d/\Delta_s\). We then obtain

\[
\frac{1}{\lambda_s} = \ln \left( \frac{2\omega_D^{(s)}}{\Delta_s} \right) - \frac{3}{20} \left( \frac{\Delta_d}{\Delta_s} \right)^2 + O \left( \frac{\Delta_d}{\Delta_s} \right)^4,
\] (E3)

\[
\frac{1}{\lambda_d} = \frac{1}{5} \ln \left( \frac{2\omega_D^{(d)}}{\Delta_s} \right) - \frac{27}{280} \left( \frac{\Delta_d}{\Delta_s} \right)^2 + O \left( \frac{\Delta_d}{\Delta_s} \right)^4,
\] (E4)

from the first equation, we get \(\Delta_s \approx 2\omega_D^{(s)} \exp (-1/\lambda_s)\), since the vanishing \(d\)-wave component does not alter the pure \(s\)-wave solution at \(r_{c1}\) to the leading order. Substituting \(\Delta_s\) into the second equation, we find

\[
\left( \frac{\Delta_d}{\Delta_s} \right)^2 = \frac{56}{27} \left[ \frac{1}{\lambda_s} - \ln \left( \frac{\omega_D^{(s)}}{\omega_D^{(d)}} \right) - \frac{5}{\lambda_d} \right],
\] (E5)

from which it follows that for \(\Delta_d\) to have a non-trivial solution, \(\lambda_d\) must have a lower bound:

\[
\frac{\lambda_d}{\lambda_s} > \frac{5}{1 - \lambda_s \ln \left( \frac{\omega_D^{(s)}}{\omega_D^{(d)}} \right)},
\] (E6)

and the results in Sec. VII are quoted for \(\omega_D^{(s)} = \omega_D^{(d)}\). In particular, if the two Debye frequencies are the same, we have \(r_{c1} = 5\), as verified by direct numerical calculation [see Fig. VII(a) and Sec. VII]. Eq. (E6) also tells us that the ratio of the Debye frequencies cannot be chosen arbitrarily and must satisfy \(\omega_D^{(s)}/\omega_D^{(d)} < e^{-\frac{1}{\lambda_s}}\) in order for the \((s+id)\) solution to exist. Put alternatively, it requires that the \(s\)-wave coupling constant is not too large

\[
\lambda_s < 1/\ln \left( \frac{\omega_D^{(s)}}{\omega_D^{(d)}} \right),
\] (E7)

otherwise the pure \(s\)-wave will dominate and the \(d\)-wave component will never have a chance to develop.

As noted in Sec. VII, the mere existence of the \(s+id\) solution does not guarantee that it will be realized in nature, unless it is lower in energy than the competing \(d+id\) phase. We must therefore compare the free energy of the \(s+id\) solution (which is essentially a pure \(s\)-wave in the vicinity of \(r_{c1}\)) to that of the \(d+id\) phase from Eq. (3.12), leading to

\[
f_s - f_N = -\frac{(\omega_D^{(s)})^2}{2} \int \frac{d\Omega}{4\pi} \left[ \sqrt{1 + \left( \frac{\Delta_d}{\omega_D^{(s)}} \right)^2} - 1 \right] = -\frac{\lambda_s^2}{4} + O \left( \Delta_d^4 \right),
\] (E8)

\[
f_{d+id} - f_N = -\frac{\Delta_d^{+id}}{10} + O \left( \Delta_d^{+id} \right).
\] (E9)

Substituting the zero-temperature gap values \(\Delta_s = 2\omega_D^{(s)} \exp (-1/\lambda_s)\) and \(\Delta_d^{+id} = 2.705 \omega_D^{(d)} \exp (-2.5/\lambda_d)\)
from Eq. (3.9) into the free energies, we find

\[ f_s - f_{d+id} = -\left(\omega_D^{(s)}\right)^2 e^{-\frac{x^2}{2\lambda_s^2}} + 0.734 \left(\omega_D^{(d)}\right)^2 e^{-\frac{x^2}{2\lambda_s^2}}. \] (E10)

For a non-trivial \( s + id \) wave solution to exist, \( \lambda_s \) must exceed the minimal value given in Eq. (E6). Substituting it into the above expression for the free energy, we find

\[ f_s - f_{d+id} > \left(\omega_D^{(s)}\right)^2 e^{-\frac{x^2}{2\lambda_s^2}} \left[0.734 e^{\frac{x^2}{2\lambda_s^2}} \left(\frac{\omega_D^{(d)}}{\omega_D^{(s)}}\right) - 1\right]. \] (E11)

Therefore, for \( s \)-wave to be more stable than \( d + id \) at \( r_{c1} \), the necessary condition is \( \exp(1/\lambda_s) < 1.362 (\omega_D^{(s)}/\omega_D^{(d)}) \), or equivalently

\[ \frac{1}{\lambda_s} < 0.309 + \ln \left(\frac{\omega_D^{(s)}}{\omega_D^{(d)}}\right). \] (E12)

Combining this expression with Eq. (E7), we see that \( \lambda_s^{-1} \) must belong to a rather narrow interval, given by

\[ \ln \left(\frac{\omega_D^{(s)}}{\omega_D^{(d)}}\right) < \frac{1}{\lambda_s} < 0.309 + \ln \left(\frac{\omega_D^{(s)}}{\omega_D^{(d)}}\right), \] (E13)

for the \((s+id)\) phase to have a chance to exist. Substituting Eq. (E12) into the inequality Eq. (E6), we conclude that for \( s + id \) to be energetically stable, the following lower bound on \( \lambda_d \) is necessary

\[ \lambda_d > \frac{5}{0.309} \approx 16.2. \] (E14)

Note that this bound is universal, independent of the ratio of the Debye frequencies. We reiterate that the above derivation is rigorous since the weak-coupling approximation is always justified near the \( r = r_{c1} \) transition (since \( \hat{\Delta}_q \) vanishes at that point).

As remarked in Sec. VII, the condition in Eq. (E14) is extremely unlikely to be realized in nature, and if true, it would certainly lie outside the realm of weak-coupling approximation at the apogee of \( s + id \) phase (it would imply that \( \Delta_{s+id}^{max} > \omega_D^{(d)} \)). Therefore for all practical applications, we can safely conclude that \((s + id)\) order is always energetically inferior to its rival \((d + id)\) phase. This conclusion is corroborated by the direct numerical evaluation of the free energies, an example of which is shown in Fig. 11b. Invariably, we find a direct first-order phase transition from a pure \( s \)-wave into the \( d + id \) phase as the ratio of the coupling constants \( r = \lambda_d/\lambda_s \) is increased.

**Appendix F: Nodal topology of \( d + id \) pairing: competing \( T_{2g} \) and \( E_g \) channels**

In Sec. III D we have highlighted six possible time-reversal symmetry breaking paired states when the pairing interaction in the \( T_{2g} \) and \( E_g \) channels are of comparable strength. Nodal topology of these paired states were announced in Table III. In Sec. III E we have discussed the nodal topology of \( d_{x^2-y^2} + id_{1g} \) paired state in details. We here present the computation of nodal topology of other five paired states, shown in Table III

- \( d_{xy} + id_{3z^2-r^2} \) pairing: The location of the nodal points for this paired state are given in the second row of Table III. To extract the nodal topology of these isolated points on the Fermi surface, we first focus on the Weyl nodes located at \( k_x = 0, k_y = k_F \sqrt{2}/3, k_z = k_F \sqrt{1/3} \), and conveniently define a set of new momentum variables

\[ p_z = \frac{k_z}{\sqrt{3}} + \frac{\sqrt{2}k_y}{\sqrt{3}}, \quad p_y = \frac{\sqrt{2}k_z}{\sqrt{3}} - \frac{k_y}{\sqrt{3}}, \quad p_x = k_x. \] (F1)

The Weyl nodes are now located at \( p = (0, 0, k_F) \). Expanding the kinetic energy around the Weyl node we obtain the following reduced BCS Hamiltonian

\[ \hat{h}_{d_{xy}+id_{3z^2-r^2}} = \tau_3 v_3 \delta p_z + v_x \tau_1 p_x + v_y \tau_2 p_y, \] (F2)

where \( \delta p_z = p_z - k_F, \quad v_x = \sqrt{2} \Delta_T / |k_F|, \quad v_y = |\Delta_E| / (\sqrt{3} |k_F|), \) and \( v_z = k_F/m \). The above nodal point, as well as the one located at \( k = -\left(0, \sqrt{2/3}, -\sqrt{1/3}\right) k_F \) are characterized by monopole charge \( W_n = +1 \) [follows from Eq. (3.2)]. By contrast, \( W_n = -1 \) for the Weyl nodes at \( k = \pm \left(0, \sqrt{2/3}, -\sqrt{1/3}\right) k_F \). We denote the above four Weyl nodes as “(a)”.

- \( d_{xz} + id_{2g} \) pairing: Let us first focus on the Weyl nodes located at \( k = \pm (k_F \sqrt{2/3}, 0, k_F \sqrt{1/3}) \) the monopole charge is \( W_n = +1 \), and Weyl nodes located at \( k = \pm \left(\sqrt{2/3}, 0, -\sqrt{1/3}\right) k_F \) have \( W_n = -1 \). The last four Weyl nodes are denoted as “(b)”.

- \( d_{yy} + id_{1g} \) pairing: Analysis of the nodal topology for this paired state is identical to the previous one. We find that Weyl nodes at \( k = \pm (1, 1, 0) k_F / \sqrt{2} \)
have monopole charge $W_n = -1$ and Weyl nodes at $k = \pm (1, -1, 0) k_F / \sqrt{2}$ have monopole charge $W_n = -1$.

- $d_{xz} + id_{3z^2-r^2}$ pairing: Following the analysis of nodal topology for $d_{xy} + id_{3z^2-r^2}$ pairing, denoted as “(a)”, we find that Weyl nodes at $k = \pm \left( 0, \sqrt{2/3}, \sqrt{1/3} \right) k_F$ have monopole charge $W_n = +1$, while $W_n = -1$ for the ones located at $k = \pm \left( 0, -\sqrt{2/3}, \sqrt{1/3} \right) k_F$.

\begin{itemize}
  \item $d_{yz} + id_{3z^2-r^2}$ pairing: From the analysis of nodal topology for $d_{xy} + id_{3z^2-r^2}$ pairing, denoted as “(b)”, we find that Weyl nodes at $k = \pm \left( \sqrt{2/3}, 0, \sqrt{1/3} \right) k_F$ have monopole charge $W_n = +1$, while $W_n = -1$ for the ones located at $k = \pm \left( -\sqrt{2/3}, 0, \sqrt{1/3} \right) k_F$.
\end{itemize}

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