Entropy and typical properties of Nash equilibria in two-player games

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We use techniques from the statistical mechanics of disordered systems to analyse the properties of Nash equilibria of bimatrix games with large random payoff matrices. By means of an annealed bound, we calculate their number and analyse the properties of typical Nash equilibria, which are exponentially dominant in number. We find that a randomly chosen equilibrium realizes almost always equal payoffs to either player. This value and the fraction of strategies played at an equilibrium point are calculated as a function of the correlation between the two payoff matrices. The picture is complemented by the calculation of the properties of Nash equilibria in pure strategies.

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Game theory seeks to model problems of strategic decision-making arising in economics, sociology, or international relations. In the generic set-up, a number of players choose between different strategies, the combination of which determines the outcome of the game specified by the payoff to each player. Contrary to the situation in ordinary optimization problems, these payoffs are in general different for different players leading to a competitive situation where each player tries to maximize his individual payoff. One of the cornerstones of modern economics and game theory is therefore the concept of a Nash equilibrium [3], see also [4], which describes a situation where no player can unilaterally improve his payoff by changing his individual strategy given the other players all stick to their strategies. However this concept is thought to suffer from the serious drawback that in a typical game-theoretical situation there is a large number of Nash equilibria with different characteristics but no means of telling which one will be chosen by the players, as would be required of a predictive theory.

This conceptual problem already shows up in the paradigmatic model of a bimatrix game between two players X and Y where player X chooses strategy \(i \in (1 \ldots N)\) with probability \(x_i \geq 0\) and player Y chooses strategy \(j \in (1 \ldots N)\) with probability \(y_j \geq 0\). The vectors \(\mathbf{x} = (x_1, \ldots, x_N)\), \(\mathbf{y} = (y_1, \ldots, y_N)\) are called mixed strategies and are constrained to the \((N - 1)\)-dimensional simplex by normalization. For a pair of pure strategies \((i, j)\) the payoff to player X is given by the corresponding entry in his payoff matrix \(a_{ij}\) whereas the payoff to player Y is given by \(b_{ij}\).

The expected payoff to player X is thus given by \(\nu_x(\mathbf{x}, \mathbf{y}) = \sum_{i,j} x_i a_{ij} y_j\) and analogously for player Y. Every player has the intention to maximize his own payoff. A Nash equilibrium (NE) \((\mathbf{x}, \mathbf{y})\) is defined by

\[
\nu_x(\mathbf{x}, \mathbf{y}) = \max_{\mathbf{x}'}, \nu_x(\mathbf{x}', \mathbf{y})
\]

\[
\nu_y(\mathbf{x}, \mathbf{y}) = \max_{\mathbf{y}'}, \nu_y(\mathbf{x}, \mathbf{y}')
\]

(1)
since in this situation there exists no mixed strategy \(\mathbf{x}\) which will increase the expected payoff to X given Y does not alter his strategy and vice versa for Y. Thus no player has an incentive to unilaterally change his strategy.

Apart from applications in economics, politics, sociology, and mathematical biology, there exists a wide body of mathematical literature on bimatrix games concerned with fundamental topics such as exact bounds for e.g. the number of NE [5] and efficient algorithms for locating them [6]. For games even of moderate size a large number of NE are found, forming a set of disconnected points. In general the different NE all correspond to different payoffs to the players.

However many situations of interest are characterized by a large number of possible strategies and complicated relations between the strategic choices of the players and the resulting payoffs. In such cases it is tempting to model the payoffs by random matrices in order to calculate typical properties of the game. This idea is frequently used in the statistical mechanics approach to complex systems such as spin glasses [8], neural networks [9], evolutionary models [10], or hard optimization problems [11]. Recently this approach has been used to investigate the typical properties of so-called zero-sum games [12].

In this Letter we investigate the properties of Nash equilibria in large bimatrix games with random payoffs, i.e. with random entries of the payoff matrices. Using techniques from the statistical mechanics of disordered systems we estimate the number of NE with a given payoff. We find that the NE are exponentially dominated in number by equilibria with a certain payoff.

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In this approach characteristics of the game are encoded in the distribution of payoff matrices – with only a few parameters – instead of the payoff matrices themselves. Clearly the choice of the probability distribution to be averaged over can influence the results. However a number of simplifying observations may be made: the two payoff matrices may be multiplied by any constant or have any constant added to them without changing the properties of the game in any material way. Thus there is no loss of generality involved in considering payoffs of order $N^{-1/2}$, and of zero mean. We assume that the entries of the payoff matrices at different sites are identically and independently distributed. In the thermodynamic limit one finds that only the first two moments of the payoff distribution are relevant so the entries of the payoff matrices may be considered to be Gaussian distributed. The only property of the distribution of payoffs which is not fixed by these specifications is the correlation $\kappa$ between entries of the same site of the two payoff matrices. We thus choose the entries of the payoff matrices to be drawn randomly according to the probability distribution

$$p(\{a_{ij}\}, \{b_{ij}\}) = \prod_{ij} \frac{N}{2\pi \sqrt{1-\kappa^2}} \exp \left( -\frac{N(a_{ij}^2 - 2\kappa a_{ij} b_{ij} + b_{ij}^2)}{2(1-\kappa^2)} \right),$$

\(i.e.\) a Gaussian distribution with zero mean, variance $1/N$ and correlation $a_{ij} b_{kl} = \kappa \delta_{ik} \delta_{jl}/N$ for all pairs $(i,j)$ and $(k,l)$. Here and in the following, the overbar $\overline{\cdot}$ denotes the average over the payoff distribution $\mathbb{E}$.

Thus for $\kappa = -1$ one finds a Dirac-delta factor $\delta(a_{ij} + b_{ij})$ in \((3)\) corresponding to a zero-sum game. $\kappa = 0$ corresponds to uncorrelated payoff matrices and $\kappa = 1$ gives the so-called symmetric case $a_{ij} = b_{ij}$ where the two players always receive identical payoffs. The parameter $\kappa$ consequently describes the degree of similarity between the payoffs to either player and may be used to continuously tune the game from a zero-sum game to a purely symmetric one. The former admits no cooperation at all between the players. Negative $\kappa$ correspond to an on average competitive situation, whereas for increasing $\kappa$ there are more and more pairs of strategies which are beneficial to both players.

As a starting point for the statistical mechanics calculations, we remark that \((1)\) is equivalent to the set of inequalities

$$\begin{align*}
\sum_j a_{ij} y_j - \nu_x(x, y) &\leq 0 \quad \forall i \\
\sum_i x_i b_{ij} - \nu_y(x, y) &\leq 0 \quad \forall j .
\end{align*}$$

\(3\)

Due to the non-negativity of the probabilities $x_i$ and $y_j$ this directly results in the local equations specifying a NE with payoffs $\nu_x$ and $\nu_y$

$$\begin{align*}
x_i \left( \sum_j a_{ij} y_j - \nu_x \right) &= 0 \quad \left( \sum_j a_{ij} y_j - \nu_x \right) \leq 0 \quad \forall i \\
y_j \left( \sum_i x_i b_{ij} - \nu_y \right) &= 0 \quad \left( \sum_i x_i b_{ij} - \nu_y \right) \leq 0 \quad \forall j .
\end{align*}$$

\(4\)

We introduce real-valued variables $\tilde{x}_i$ and $\tilde{y}_j$ with $x_i = \tilde{x}_i \Theta(\tilde{x}_i)$ (where $\Theta(x)$ is the Heaviside step-function) and likewise for $y_j$. Then the last conditions \((4)\) may be written as

$$\begin{align*}
I_x^\nu(\tilde{x}, \tilde{y}) &= \tilde{x}_i \Theta(-\tilde{x}_i) - (\sum_j a_{ij} \tilde{y}_j \Theta(\tilde{y}_j) - \nu_x) = 0 \\
I_y^\nu(\tilde{x}, \tilde{y}) &= \tilde{y}_j \Theta(-\tilde{y}_j) - (\sum_i \tilde{x}_i \Theta(\tilde{x}_i) b_{ij} - \nu_y) = 0 ,
\end{align*}$$

\(5\)

\(i.e.\) we have constructed indicator functions $I_x^\nu$ and $I_y^\nu$ which are zero at a NE $(x, y)$ and non-zero elsewhere \([3]\).

The number $\mathcal{N}(\nu_x, \nu_y)$ of NE with payoffs $\nu_x$ and $\nu_y$ may thus be calculated from

$$\mathcal{N}(\nu_x, \nu_y) = \int d\tilde{x} \ d\tilde{y} \ \delta \left( \sum_i \tilde{x}_i \Theta(\tilde{x}_i) - N \right) \delta \left( \sum_j \tilde{y}_j \Theta(\tilde{y}_j) - N \right) \prod_i \delta (I_x^\nu(\tilde{x}, \tilde{y})) \prod_j \delta (I_y^\nu(\tilde{x}, \tilde{y})) \left\| \frac{\partial (\mathbf{T}^x, \mathbf{Y}^y)}{\partial (\tilde{x}, \tilde{y})} \right\| ,$$

\(6\)

where the mixed strategies have been rescaled to $\sum_i x_i = \sum_j y_j = N$ for convenience. Although we expect an exponential number of NE, so $\ln \mathcal{N}$ is an extensive quantity, we calculate the so-called annealed average $\ln \overline{\mathcal{N}}$ instead
of the quenched one $\ln N$. This gives an exact upper bound to the typical number of NE at given payoffs $^1$. Using integral representations of the delta functions in (4), averaging over the disorder (3), and introducing the order parameters $q^x = N^{-1} \sum_i \hat{x}_i^2 \Theta(\hat{x}_i)$, $R^x = N^{-1} \sum_i \hat{x}_i \hat{x}_i \Theta(\hat{x}_i)$, and similarly for player $Y$ we obtain

$$N(v_x, v_y) = \int \frac{dq^x \cdot dq^y \cdot dp^x \cdot dp^y}{(2\pi N)^2} \int \frac{dE^x \cdot dE^y}{(2\pi N)^2} \int \frac{dR^x \cdot dR^y}{(2\pi N)^2} \exp\{iN(q^x \hat{q}^x + q^y \hat{q}^y - \kappa R^x R^y + E^x + E^y)\}$$

$$\int \prod_i d\hat{x}_i \int \prod_j d\hat{y}_j \exp\left\{-\frac{i\hat{q}^x}{2} \sum_i \hat{x}_i^2 \Theta(\hat{x}_i) + i\kappa R^y \sum_i \hat{x}_i \hat{x}_i \Theta(\hat{x}_i) - \frac{q^y}{2} \sum_i \hat{x}_i^2\right\}$$

$$-i \sum_i \hat{x}_i \hat{x}_i \Theta(-\hat{x}_i) - i\nu_x \sum_i \hat{x}_i - iE^x \sum_i \hat{x}_i \Theta(\hat{x}_i) + (x \leftrightarrow y \text{ with } iR^y \to R^x) \} \| \det(B) \|,$$

(7)

with $H(x) = \int_0^\infty dx x \exp(-x^2/2)/\sqrt{2\pi}$. Here we have used the assumption that the normalizing determinant $\| \det(B) \| = \sqrt{\frac{\partial^2 F}{\partial x^2 \partial y^2}}$ is self-averaging and is effectively uncorrelated with the rest of expression (3) and may therefore be averaged over the disorder independently of the rest of the expression $^3$. We obtain

$$\| \det(B) \| = \exp\left\{ \frac{N}{2} (p_x \ln p_x - p_x + p_y \ln p_y - p_y) \right\},$$

(8)

where the fraction of strategies played with non-zero probability $p_x = N^{-1} \sum_i \Theta(\hat{x}_i)$ is determined self-consistently by introducing $p_x$, $p_y$ and their conjugates $\hat{p}_x$, $\hat{p}_y$ in (6). After the transformation $i\hat{q}^x \to \hat{q}^x$ and similarly for $E^x$, $\hat{p}^x$, $R^y$, the integrals over $\hat{x}, \hat{y}$ may be performed. In the limit of large payoff matrices $N \to \infty$, the integrals over the order parameters are dominated by their saddle point. Furthermore (7) is now symmetric under an interchange of the players, and the maximum (and hence exponentially in $N$ dominating) number of NE is found at equal payoffs to each player. Thus it makes sense to restrict the analysis to the case $\nu_x = \nu_y = \nu$. We obtain

$$S_\nu(\nu) = \frac{1}{N} \ln N(\nu, \nu) = 2 \exp_{q, \hat{q}, E, R} \left[ \frac{q \hat{q}}{2} - \frac{\kappa R^2}{2} - \frac{p}{2} \right]$$

$$+ \ln \left( H \left( -\frac{\nu}{\sqrt{\hat{q}}} \right) + \sqrt{\frac{p}{\hat{q} + \kappa^2 R^2}} \exp \left\{ -\frac{\nu^2}{2\hat{q}} + \frac{(E - \kappa R\nu/q)^2}{2\hat{q} + 2\kappa^2 R^2 / q} \right\} H \left( \frac{E - \kappa R\nu/q}{\sqrt{\hat{q} + \kappa^2 R^2 / q}} \right) \right)$$

(9)

From this expression the statistical properties of the entire spectrum of player-symmetric NE may be deduced. In the thermodynamic limit, a randomly chosen NE will give the payoff $\nu = \nu_x = \nu_y = \arg\max S_\nu(\nu)$ with probability one, because the number of NE with this payoff $N(\nu, \nu) = \exp\{N \max S_\nu(\nu)\}$ is exponentially larger than the number of all other NE. Similarly, the self-overlap $q = \sum_i \hat{x}_i^2 = \sum_j \hat{x}_j^2$ and the fraction of strategies played with non-zero probability $p = p_x = p_y$ will take on their saddle-point values of (6) evaluated at the maximum of $S_\nu(\nu)$ with probability one.

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$^1$The annealed approximation does not differ by more than 3% in the entropy and by 10% in the order parameters from the results of the more complicated replica calculation of the quenched average, which will be presented elsewhere.

$^2$This assumption has been verified by explicitly including the calculation of the effective normalizing determinant into the disorder average.
FIG. 1. The entropy $S_\kappa$ of the NE which exponentially dominate the spectrum $S_\kappa(\nu)$. The analytic results are compared with numerical simulations for $N = 18$ averaged over 100 samples.

Figure 1 shows the entropy $S_\kappa$ at the maximum of $S_\kappa(\nu)$ as a function of $\kappa$. The numerical results were obtained by enumerating all NE for $N=18$ [6,14]. Given that enumeration is only possible for small sample sizes, the agreement between the annealed approximation and the numerical results is quite good.

Figure 2 shows the payoff $\nu$ and the fraction $p$ of strategies played with non-zero probability again as a function of $\kappa$. The numerical results were obtained for $N=50$ by using the iterated Lemke-Howson algorithm [15,6], which locates a single but typical NE. The deviation between analytic and numerical results may be accounted for by the use of the annealed calculation and finite-size effects.

For zero-sum games ($\kappa = -1$) there is only a single NE [12] so $S_{\kappa=0} = 0$, and the payoff is zero due to the symmetry between the players. As $\kappa$ is increased the number of NE rises. The maximum of the typical number of NE is reached for the case of symmetric games, where $S_{\kappa=1} \sim 0.362$. This result may be compared with a rigorous upper bound derived using geometric methods [3,8], which states that for any non-degenerate $N$-by-$N$ bimatrix game with large $N$ there are at most $e^{0.955N}$ equilibrium points. Thus the typical-case scenario investigated here does not saturate this bound.

The increase of $\nu$ with $\kappa$ may be understood as follows: At increasing $\kappa$ the outcome of a pair of strategies $(i,j)$ which is beneficial to player $X$ say, tends to become more beneficial to player $Y$. As a result players focus on these strategies and the payoff to both players rises. By the same token the fraction $p$ of strategies which are played with non-zero probability at a NE decreases with $\kappa$ and the self-overlap of the mixed strategies $q$ increases.

Even though the properties of the NE at a given $\kappa$ are dominated by the peak of $S_\kappa(\nu)$, there is an important qualitative difference between the curves of different values of $\kappa$. For $-1 < \kappa < \kappa_c \sim -.59$, $S_\kappa(\nu)$ reaches its
maximum and then decreases crossing the $S = 0$ axis at some point. For $\kappa > \kappa_c$ however, $S_\kappa(\nu \to \infty) \to 0^+$ and $p \to 0^+$. Hence there is an exponential number of NE offering an arbitrarily large (but finite) payoff to either player.

This observation can be corroborated by considering the extreme case of pure strategy Nash equilibria (PSNE) $(i,j)$ where each player only plays a single (pure) strategy [10], i.e. $x_i = N\delta_{i'\,i} \;\text{and} \; y_j = N\delta_{j'\,j} \;\forall i',j' = 1, ..., N$. According to [3], a PSNE corresponds to a site $(i,j)$ which is simultaneously a maximum of the column $a_{i'\,j}$ and a maximum of the corresponding row of $b_{ij'}$. The average of the number $\mathcal{M}$ of PSNE is thus given by

$$\overline{\mathcal{M}} = \sum_{ij} \prod_{k \neq i} \Theta(a_{ij} - a_{kj}) \prod_{l \neq j} \Theta(b_{ij} - b_{il})$$

$$= N^2 \int da \, db \, p(a, b) H^{N-1}(-\sqrt{Na}) H^{N-1}(-\sqrt{Nb})$$

where $p(a, b)$ denotes one of the Gaussian factors in [3].

Figure 3 shows the average number of PSNE as well as its variance as a function of $\kappa$. For $N \to \infty$ one finds from [10] that $\overline{\mathcal{M}}$ scales with $N^{2\kappa/(1+\kappa)}$, so the number of PSNE vanishes for $\kappa < 0$, at $\kappa = 0$ there is on average 1 PSNE, and for $\kappa > 0$ the number of PSNE diverges. The increase of the number of PSNE with $\kappa$ may be understood as follows. At low $\kappa$, $a$ and $b$ are anticorrelated, so in the extreme case $\kappa = -1$ a PSNE corresponds to a site $(i,j)$ which is simultaneously a maximum of the column $a_{i'\,j}$ and a minimum of the row $a_{ij'} = -b_{ij'}$. In the thermodynamic limit, such points are exponentially rare. For large $\kappa$, the condition for a PSNE is much easier to fulfill and in the case $\kappa = +1$, there is always at least one PSNE, namely the maximum entry of $a_{ij} = b_{ij}$.

Note that apart from the point $\kappa = 0$, the number of PSNE is also self-averaging as the relative sample-to-sample fluctuations of $\mathcal{M}$ are found to vanish in the large $N$-limit. Furthermore for all $\kappa > -1$ the typical PSNE payoffs to leading orders of $N$ are found to be independent of $\kappa$, they are $\sqrt{N}(2 \ln N - \ln \ln N + O(1))$.

![Figure 3](image.png)

**FIG. 3.** Average number $\overline{\mathcal{M}}$ of PSNE for $N = 25, 50, 100$ (full, dashed, dotted lines). The lines show analytical result [10], the symbols numerical results, averaged over 1000 samples. Inset: Numerical results for the logarithm of $\overline{\mathcal{M}}$ (circles) and of $\overline{\mathcal{M}}^2 - \overline{\mathcal{M}}^2$ with $\kappa = 0.25, 0.5, 0.75$ (bottom to top) against $\ln N$. The dashed lines show the asymptotic slope $2\kappa/(1+\kappa)$ of $\overline{\mathcal{M}}$.

In conclusion we have used methods from the statistical mechanics of disordered systems to describe the typical properties of Nash equilibria in bimatrix games with large random payoff matrices. We find that in the thermodynamic limit for a randomly chosen Nash equilibrium quantities such as the fraction of strategies played with non-zero probability, the self-overlap of the mixed strategies and most importantly the payoff to either player take on a specific value with probability 1. We have analytically calculated these quantities in the annealed approximation as a function of the correlation $\kappa$ between the payoff matrices and found good agreement with numerical simulations. Furthermore the properties of Nash equilibria in pure strategies were calculated as a limiting case. A number of extensions of this work may be considered, including the case where the players have different numbers of strategies at their disposal and the generalisation to games of several players.

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[2] weigt@physique.ens.fr
[3] J. F. Nash, *Annals of Mathematics* **54**(2), 286 (1951)
[4] For an introductory text see: Wang Jianhua, *The Theory of Games* (Oxford University Press, 1988)
[5] H. Keiding, *Games and Economic Behaviour* **21**, 148 (1997)
[6] B. v. Stengel, to appear in *Handbook of Game Theory*.
[7] M. Mézard, G. Parisi, and M. A. Virasoro, *Spin Glass Theory and Beyond* (World Scientific, Singapore 1987)
[8] A. P. Young, *Spin Glasses and Random Fields* (World Scientific, Singapore 1998)
[9] J. Hertz, A. Krogh, and R. G. Palmer, *Introduction to the Theory of Neural Computation* (Addison Wesley, Redwood City 1991)
[10] S. Diederich and M. Opper, Phys. Rev. **A39**, 4333 (1989), M. Opper and S. Diederich, Phys. Rev. Lett. **69**, 1616 (1992)
[11] R. Monasson and R. Zecchina, *Phys. Rev. Lett.* **76**, 3881 (1996)
[12] M. Mézard and G. Parisi, *J. Phys. (France)* **47**, 1285 (1986)
[13] J. Berg and A. Engel, *Phys. Rev. Lett.* **81**, 4999 (1998)
[14] D. Avis and K. Fukuda, *Discrete and Computational Geometry* **8**, 295 (1992)
[15] C.E. Lemke and J.T. Howson, *J. Soc. Indust. Appl. Math.* **12**, 413 (1964)
[16] A. J. Goldman, *American Mathematical Monthly* **64**, 729 (1957)