LOWER BOUNDS ON THE BLOW-UP RATE OF THE 3D NAVIER-STOKES EQUATIONS IN $H^{-\{5/2\}}$ \\
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Abstract. Under assumption that $T^*$ is the maximal time of existence of smooth solution of the 3D Navier-Stokes equations in the Sobolev space $H^s$, we establish lower bounds for the blow-up rate of the type $(T^* - t)^{-\varphi(n)}$, where $n$ is a natural number independent of $s$ and $\varphi$ is a linear function. Using this new type in the 3D Navier-Stokes equations in the $H^{5/2}$, both on the whole space and in the periodic case, we give an answer to a question left open by James et al (2012, J. Math. Phys.). We also prove optimal lower bounds for the blow-up rate in $\dot{H}^{3/2}$ and in $H^1$.

1. Introduction

We consider, in this paper, the 3D incompressible Navier-Stokes equations

$$ \frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p + \nu \Delta u, \quad \text{in } \Omega \times (0, \infty) $$

$$ \text{div } u = 0, \quad \text{in } \Omega \times (0, T) \quad \text{and} \quad u(x, 0) = u_0, \quad \text{in } \Omega,$$  

(1.1)

where $u = u(x, t)$ is the velocity vector field, $p$ is the pressure and $\nu$ is the viscosity of the fluid. The domain $\Omega$ may have periodic boundary conditions or $\Omega = \mathbb{R}^3$.

For small data $\|\nabla u\|_{L^2} \leq c(\nu)$ the global existence of strong solutions for the 3D Navier-Stokes equations it is well known, see Constantin [2, Theorem 9.3 P 80]. But for the 3D Navier-Stokes equations with large data, we don’t have a result of global existence. Under the assumption that the solution of the three-dimensional Navier-Stokes equations becomes irregular at finite time $T^*$ Leray 1934 [4, P 224] proved that there exists a constant $c(\nu) > 0$ such that

$$ \|\nabla u(., t)\|_{\dot{H}^1(\mathbb{R}^3)} \geq \frac{c(\nu)}{(T^* - t)^2}.$$  

(1.2)

In 2010, Benameur [11, Theorem 1.3.] showed in the whole space

$$ \|u(., t)\|_{H^{-s}(\mathbb{R}^3)} \geq c(s) \frac{\|u(., t)\|_{L^2(\mathbb{R}^3)}}{(T^* - t)^{\frac{s}{2}}} \quad \text{with } s > \frac{5}{2}.$$  

(1.3)

The result above was improved by Robinson, Sadowski, and Silva in [5] to

$$ \|u(., t)\|_{H^{-s}(\Omega)} \geq c(s) \frac{\|u_0\|_{L^2(\Omega)}}{(T^* - t)^{\frac{s}{2}}} \quad \text{with } \Omega = [0, 1]^3 \text{ or } \mathbb{R}^3.$$  

(1.4)

2010 Mathematics Subject Classification. 35Q30, 35B44.

Key words and phrases. Blow-up rate; Lower bounds; Navier-Stokes equations.
In the homogeneous Sobolev space \( \dot{H}^{5/2}(\mathbb{T}^3) \) of real valued periodic functions, Cortissoz, Montero, & Pinilla 2014 [3, Theorem 1.3, Page 2] proved lower bounds on the blow up with logarithmic corrections,

\[
\|u(.,t)\|_{\dot{H}^{5/2}(\mathbb{T}^3)} \geq \frac{c}{(T^*-t)\log (T^*-t)} \quad \text{with} \quad T^*-t \neq 1.
\]  

(1.5)

In the case of the \( \dot{H}^{3/2}(Q) \)-norm, (see also the work of Robinson et al [5]) it is known that for every \( \epsilon > 0 \) there exists a \( c_{\frac{3}{2},\epsilon} > 0 \) such that

\[
\|u(.,t)\|_{\dot{H}^{3/2}(\mathbb{T}^3)}^{\frac{4+\epsilon}{2}} \geq \frac{c_{\frac{3}{2},\epsilon}}{(T^*-t)^{\frac{3}{2}+\epsilon}}.
\]

(1.6)

Recently, Cortissoz et al [3, Theorem 1.1, Page 2] showed that

\[
\|u(.,t)\|_{\dot{H}^{3/2}(\mathbb{T}^3)} \geq \frac{c}{\sqrt{(T^*-t)\log (T^*-t)}}.
\]

(1.7)

In this paper, we concentrate on the case of estimating lower bounds for the blow-up rate in \( \dot{H}^{5/2}(\mathbb{T}^3) \) and \( \dot{H}^{3/2}(Q) \) of a possible blow-up solution to 3D Navier-Stokes equations. First we prove a new lower bound on blowup solutions in the \( \dot{H}^{5/2} \)-norm both on the whole space and in the periodic case. This result gives a response to the question left open in [3] and improves previous known lower bounds. Therefore it is possible to prove the correct rate of blow up (1.0) in \( \dot{H}^{3/2}(Q) \), which is, in this case, \((T^*-t)^{-\frac{1}{2}}\), see [3, Sec.V.A P11]). Finally, we improve the order \( \frac{1}{4} \) in the result (1.2) to get a rate of the order \(-1\). We prove that is possible to get a rate of blowup of the type \((T^*-t)^{-1}\) in several spaces \( \dot{H}^{5/2}, \dot{H}^{3/2} \) and \( \dot{H}^{1} \) for \( t \leq T_* \) with \( T_* < T^* \). Those estimates are particularly useful for obtaining a control of degree \(-1\) for the comportment of strong solutions before the moment of the blow up.

The technique used in this paper are fairly standard based on the properties of trigonometric functions. This method can be used to estimate lower bounds on solutions that blowup at some finite time \( T^*>0 \) for the strict positive solution of ordinary differential inequality of the type \( y' \leq y^n \) with \( 0 < n \leq 3 \).

2. Main Result

Let \( Q = [0,2\pi]^3 \), we write \( \mathbb{Z}^3 = \mathbb{Z}^3/\{0,0,0\} \), let \( \dot{H}^s(Q) \) be the subspace of the Sobolev space \( H^s \) consisting of divergence-free, zero-average, periodic real functions,

\[
\dot{H}^s(Q) = \left\{ u = \sum_{\xi \in \mathbb{Z}^3} \hat{u}_\xi e^{-i\xi \cdot x} : \quad \overline{\hat{u}_\xi} = \hat{u}_{-\xi}, \sum_\xi |\xi|^{2s}|\hat{u}_\xi|^2 < \infty \text{ and } \xi, \hat{u}_\xi = 0 \right\}
\]

(2.1)

and equip \( \dot{H}^s(Q) \) with the norm

\[
\|u\|^2_{\dot{H}^s} = \|u\|^2_{\dot{H}^s(Q)} = \sum_\xi |\xi|^{2s}|\hat{u}_\xi|^2.
\]

(2.2)

On the whole space the corresponding definition of the \( \dot{H}^s(\mathbb{R}^3) \) norm is

\[
\|u\|^2_{\dot{H}^s(\mathbb{R}^3)} := \int_{\mathbb{R}^3} |\xi|^{2s}|\hat{u}(\xi)|^2 \, d\xi < \infty,
\]

(2.3)

where \( F[u](\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^3} e^{-i\xi \cdot x} |u(x)| \, dx \) is the Fourier transform of \( u \), for more details see [5]. We prove our estimate in the periodic case, but it also holds in the
Proof. We start our proof from the fourth differential inequality \[ (T^* - t) \leq m^n (T^* - t)^{n+1} \] (2.4) for all \( t \in \left[0, T^* - \frac{1}{m}\right] \) and \( n \in \mathbb{N}^* \).

**Lemma 2.1.** For any real numbers \( T^* > 0 \) there exists \( m \in \mathbb{N}^* \) such that we have

\[ (T^* - t) \leq m^n (T^* - t)^{n+1} \] (2.4)

Proof. We will prove (2.4) by induction. We show that the statement (2.4) holds for \( n = 1 \). Since \( \mathbb{R}_1^* \) is archimedean then for any finite real number \( T^* \) strictly positive there exists \( m \in \mathbb{N}^* \) such that \( T^* \geq \frac{1}{m} \). For any real \( t \) such that \( 0 \leq t \leq T^* - \frac{1}{m} \), the following inequality holds

\[ \frac{1}{m} \leq (T^* - t). \] (2.5)

Since \( (T^* - t) \geq 0 \), multiplying (2.5) by \( (T^* - t) \) we find

\[ (T^* - t) \leq m (T^* - t)^2, \] (2.6)

thus (2.5) is true for \( n = 1 \). Let \( k \in \mathbb{N}^* \) be given and suppose (2.4) is true for \( n = k \). Then

\[ (T^* - t) \leq m^k (T^* - t)^{k+1}. \] (2.7)

Multiplying the induction hypothesis (2.7) by \( (T^* - t) \) we find

\[ (T^* - t)^2 \leq m^k (T^* - t)^{k+2}. \] (2.8)

Using inequality (2.6), we can rewrite (2.8) as

\[ (T^* - t) \leq m^{k+1} (T^* - t)^{k+2}. \] (2.9)

Thus, (2.4) holds for \( n = k + 1 \), and the proof of the induction step is complete. Wich prove that (2.4) is true for all \( n \in \mathbb{N}^* \). \( \square \)

Note that the lemma above holds if we replace \( m \) with \( r \in \mathbb{R}_1^* \) and \( r \geq 1 \).

**Theorem 2.2.** Let \( u(., t) \in \mathring{H}^{5/2} (Q) \) be a smooth Leray-Hopf solution of the 3D Navier-Stokes equations (1.1) with non zero \( \|u_0\|_{H^{5/2}} \) and with maximal interval of existence \((0, T_*)\), \( T_* < \infty \). Then there exists a positive time \( T_* < T^* \) and a positive constant \( \eta_1 > 0 \) such that

\[ \|u(., t)\|_{\mathring{H}^{5/2}} \geq \frac{\eta_1}{(T_* - t)^{\frac{m+1}{m+2}}} \text{ for } t \leq T_. \] (2.10)

Proof. We start our proof from the fourth differential inequality \[ \frac{1}{4} \text{ (V.A P11)} \]

\[ \frac{d}{dt} \|u(., t)\|_{\mathring{H}^{5/2}}^2 \leq c_1 \|u(., t)\|_{L^2} \|u(., t)\|_{\mathring{H}^{5/2}} \text{ with } \xi = \frac{\epsilon}{5 (4 - \epsilon)}, \] (2.11)

for \( 0 \leq \epsilon \leq 1 \) yields \( 3 \leq 3 + \xi \leq \frac{45}{15} \). Since \( \|u(., t)\|_{L^2} \) is bounded it follows that

\[ \frac{d}{dt} \|u(., t)\|_{\mathring{H}^{5/2}}^2 \leq c_2 \|u(., t)\|_{\mathring{H}^{5/2}} \] (2.12)

Setting \( z(t) = \|u(., t)\|_{\mathring{H}^{5/2}}^2 \) in (2.12), we can obtain

\[ \frac{d}{dt} (z(t) + 1) \leq c_2 z^{\frac{m+1}{m+2}} (t). \] (2.13)
Multiplying (2.13) by cos \( \left( \frac{1}{z(t)+1} \right) \), and using \( \cos(1) \leq \cos \left( \frac{1}{z(t)+1} \right) \leq 1 \), we get

\[
\cos \left( \frac{1}{z(t)+1} \right) \frac{d(z(t)+1)}{dt} \leq c_2 z^{-\frac{3+\epsilon}{\epsilon}}(t). \tag{2.14}
\]

Dividing (2.14) by \((z+1)^2\), and using \(\frac{z^n}{(z+1)^2} \leq 1\) for \(0 \leq n \leq 2\), we obtain

\[
\frac{(z(t)+1)t}{(z(t)+1)^2} \cos \left( \frac{1}{z(t)+1} \right) \leq c_2. \tag{2.15}
\]

Integrating the differential inequality (2.15) from time \( t \) to blow-up time \( T^* \) and using the fact that \( \lim_{t \to T^*} \|u(.,t)\|_{\dot{H}^{5/2}} = \infty \), yields

\[
\sin \left( \frac{1}{z(t)+1} \right) \leq c_2 (T^*-t). \tag{2.16}
\]

Using (2.4) in (2.16), we obtain the following estimate

\[
\sin \left( \frac{1}{z(t)+1} \right) \leq c_2 m^n (T^*-t)^{n+1}, \tag{2.17}
\]

with \( m \in \mathbb{N}^* \) for all \( t \leq T^* - \frac{1}{m} = T_*. \) Multiplying (2.17) by \( \|u(.,t)\|_{\dot{H}^{5/2}}^2 \), we get

\[
\|u(.,t)\|_{\dot{H}^{5/2}}^2 \sin \left( \frac{1}{\|u(.,t)\|_{\dot{H}^{5/2}}^2 + 1} \right) \leq c_2 m^n (T^*-t)^{n+1} \|u(.,t)\|_{\dot{H}^{5/2}}^2. \tag{2.18}
\]

Equation (2.12) is a differential equation of Bernoulli type

\[
y' = g^n \text{ with } n > 1. \tag{2.19}
\]

To get in the left-hand side of (2.18) a minimum different to zero, we must assume that \( \|u\|_{\dot{H}^{5/2}} \neq 0 \). For \( n > 1, y = 0 \) must be the only solution to (2.19) satisfying \( y(t_0) = 0 \). Then yields that \( \|u\|_{\dot{H}^{5/2}} \) is non-zero, for \( \|u_0\|_{\dot{H}^{5/2}} \neq 0 \). Thus, there exist a positive constant \( \beta_1 = \min_{t \geq 0} \|u(.,t)\|_{\dot{H}^{5/2}}^2 \) such that \( \|u(.,t)\|_{\dot{H}^{5/2}}^2 \geq \beta_1 \) for all \( t \geq 0 \). Note that for \( \theta \geq 0 \), the function \( f(\theta) = \theta \sin \left( \frac{1}{\theta+1} \right) \) is an increasing positive function, this gives

\[
\alpha_1 = \beta_1 \sin \left( \frac{1}{\beta_1+1} \right) \leq \|u\|_{\dot{H}^{5/2}}^2 \sin \left( \frac{1}{\|u\|_{\dot{H}^{5/2}}^2 + 1} \right). \tag{2.20}
\]

Using this estimate in (2.18) yields the bound

\[
\alpha_1 \leq c_2 m^n (T^*-t)^{n+1} \|u(.,t)\|_{\dot{H}^{5/2}}^2. \tag{2.21}
\]

Then we can deduce (2.10) with \( \eta_1 = \sqrt{\frac{\alpha_1}{c_2 m^n}} \), which completes the proof. \( \square \)

Theorem 2.1 is valid when we consider the case of the whole space, i.e., for solutions \( u(t) \in \dot{H}^{5/2}(\mathbb{R}^3) \), this because the equation (2.11) in the proof valid in the whole space and for periodic boundary conditions see [5 Sec.V.A P11] and all the calculations leading to its proof are valid on \( \mathbb{R}^3 \) if we change Fourier series by Fourier integrals.

Therefore is possible to prove the improve the rate of blow up (1.10) in \( \dot{H}^{3/2}(Q) \) obtained in [5].
Theorem 2.3. Let $u(\cdot, t) \in \dot{H}^{3/2}(Q)$ be a smooth Leray-Hopf solution of the 3D Navier-Stokes equations (1.1) with non zero $\|u_0\|_{\dot{H}^{3/2}}$ and with maximal interval of existence $(0, T^*)$, $T^* < \infty$. Then there exists a positive time $T_* < T^*$ and a constant $\eta_2 > 0$ such that

$$\|u(\cdot, t)\|_{\dot{H}^{3/2}} \geq \frac{\eta_2}{(T^* - t)^{\frac{3}{2} + \delta}} \quad \text{for} \ t \leq T_*.$$  \hspace{1cm} (2.22)

Proof. We start with the following inequality (inequality (7) in [5, P7])

$$\frac{d}{dt} \|u(\cdot, t)\|_{\dot{H}^{3/2}}^2 \leq c_3 \|u(\cdot, t)\|_{L^2}^{2^n} \|u(\cdot, t)\|_{\dot{H}^{3/2}}^{4 + \gamma} \quad \text{with} \ \gamma = \frac{2\delta}{(2 - \delta)},$$  \hspace{1cm} (2.23)

for $\delta > 0$ small. Since $\|u(\cdot, t)\|_{L^2}$ is bounded, we obtain

$$\frac{d}{dt} \|u(\cdot, t)\|_{\dot{H}^{3/2}}^2 \leq c_4 \|u(\cdot, t)\|_{\dot{H}^{3/2}}^{4 + \gamma}.$$  \hspace{1cm} (2.24)

Setting $y(t) = \|u(\cdot, t)\|_{\dot{H}^{3/2}}^2$, the inequality (2.24) can be written in the form

$$\frac{d(y(t) + 1)}{dt} \leq c_4 y^{1 + \frac{\gamma}{2}}(t).$$  \hspace{1cm} (2.25)

Multiplying (2.25) by $\sin \left(\frac{1}{y(t) + 1}\right)$, and using $y \sin \left(\frac{1}{y(t) + 1}\right) \leq 1$, we find

$$\sin \left(\frac{1}{y(t) + 1}\right) \frac{d(y(t) + 1)}{dt} \leq c_4 y^{1 + \frac{\gamma}{2}}(t).$$  \hspace{1cm} (2.26)

Dividing (2.26) by $(y + 1)^2$, and using $\frac{y^n}{(y + 1)^2} \leq 1$ for $0 \leq n \leq 2$, we obtain

$$\frac{(y(t) + 1)t}{(y(t) + 1)^2} \sin \left(\frac{1}{y(t) + 1}\right) \leq c_4.$$  \hspace{1cm} (2.27)

Integrating the differential inequality (2.27) from time $t$ to blow-up time $T^*$ and using the fact that $\lim_{t \to T^*} \|u(\cdot, t)\|_{\dot{H}^{3/2}}^2 = \infty$, we find that

$$1 \leq \cos \left(\frac{1}{z(t) + 1}\right) + c_4 (T^* - t).$$  \hspace{1cm} (2.28)

Using (2.17) in (2.28), it follows that

$$1 - \cos \left(\frac{1}{z(t) + 1}\right) \leq c_4 m^n (T^* - t)^{n+1} \quad \text{with} \ m \in \mathbb{N}^*, \hspace{1cm} (2.29)$$

for all $t \leq T_*$ (see proof of Theorem 2.1). Using the trigonometric formula $\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$, we obtain

$$\sin \left(\frac{1}{2z(t) + 2}\right)^2 \leq c_4 m^n (T^* - t)^{n+1}.$$  \hspace{1cm} (2.30)

Multiplying (2.30) by $\|u(\cdot, t)\|_{\dot{H}^{3/2}}^4$, we get

$$\|u(\cdot, t)\|_{\dot{H}^{3/2}}^4 \sin \left(\frac{1}{2\|u(\cdot, t)\|_{\dot{H}^{3/2}}^2 + 2}\right)^2 \leq c_4 m^n (T^* - t)^{n+1} \|u(\cdot, t)\|_{\dot{H}^{3/2}}^4.$$  \hspace{1cm} (2.31)
For \( \|u_0\|_{\dot{H}^{3/2}} \neq 0 \), there exists a positive constant \( \alpha_2 \) (see proof of Theorem 2.1) such that
\[
\alpha_2 = \left( \beta_2 \sin \left( \frac{1}{2\beta_2 + 2} \right) \right)^2 \leq \|u(.,t)\|_{\dot{H}^{3/2}}^4 \sin \left( \frac{1}{2 \|u(.,t)\|_{\dot{H}^{3/2}}^2 + 2} \right)^2. \tag{2.32}
\]
Using this estimate in (2.32), we find that
\[
\alpha_2 \leq c_4m^n (T^* - t)^{n+1} \|u(.,t)\|_{\dot{H}^{3/2}}^4 \tag{2.33}
\]
Then we can deduce that \( \eta_2 = \left( \frac{\alpha_2}{c_4m^n} \right)^{1/2} \), which completes the proof. \( \square \)

We now present another application of this trigonometric method, leading to improve the lower bound (1.2).

**Theorem 2.4.** Let \( u(.,t) \in \dot{H}^1(Q) \) be a smooth Leray-Hopf solution of the 3D Navier-Stokes equations (1.1) with non zero \( \|u_0\|_{\dot{H}^1} \) and with maximal interval of existence \((0,T^*), T^* < \infty \). Then there exists a positive time \( T_* < T^* \) and a constant \( \eta_3 > 0 \) such that
\[
\|u(.,t)\|_{\dot{H}^1} \geq \eta_3 (T^* - t)^{-\frac{n+1}{2}} \text{ for } t \leq T_*.
\]

**Proof.** We consider the Navier–Stokes equations (1.1) in periodic domain. Multiplying (1.1) by \( \Delta u \) and integrate, we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\nabla u(.,t)\|_{L^2}^2 + \nu \|\Delta u\|_{L^2}^2 = ((u,\nabla u) \cdot \Delta u). \tag{2.35}
\]
Using the Holder inequality and the Sobolev theorem, we get
\[
\|((u,\nabla u) \cdot \Delta u)\| \leq c_5 \|\nabla u\|_{L^p}^{\frac{2}{p}} \|\Delta u\|_{L^q}^{\frac{2}{q}}, \tag{2.36}
\]
see [3] P. 79 (2.22). Combining (2.35) and (2.36), we obtain
\[
\frac{d}{dt} \|\nabla u(.,t)\|_{L^2}^2 + \nu \|\Delta u\|_{L^2}^2 \leq c_5 \|\nabla u\|_{L^2}^{\frac{2}{p}} \|\Delta u\|_{L^2}^{\frac{2}{q}}. \tag{2.37}
\]
However, an application of Young’s inequality to the right-hand side of (2.37) yields
\[
\frac{d}{dt} \|\nabla u(.,t)\|_{L^2}^2 + \nu \|\Delta u\|_{L^2}^2 \leq c_6 \|\nabla u\|_{L^2}^6 + \frac{\nu}{2} \|\Delta u\|_{L^2}^2. \tag{2.38}
\]
We obtain
\[
\frac{d}{dt} \|\nabla u(.,t)\|_{L^2}^2 + \nu \|\Delta u\|_{L^2}^2 \leq c_6 \|\nabla u\|_{L^2}^6. \tag{2.39}
\]
If we drop the \( \nu \|\Delta u\|_{L^2}^2 \) term in (2.39) then we have
\[
\frac{d}{dt} \|\nabla u(.,t)\|_{L^2}^2 \leq c_6 \|\nabla u\|_{L^2}^6. \tag{2.40}
\]
Setting \( y(t) = \|\nabla u(.,t)\|_{L^2}^2 \) in (2.40), this gives
\[
\frac{d}{dt} y \leq c_6 y^3. \tag{2.41}
\]
Multiplying (2.41) by \( \sin \left( \frac{1}{y(t)+1} \right) \), and using \( y \sin \left( \frac{1}{y(t)+1} \right) \leq 1 \), we get
\[
\sin \left( \frac{1}{y(t)+1} \right) \frac{d(y(t)+1)}{dt} \leq c_6 y^2 (t). \tag{2.42}
\]
Dividing (2.42) by \((y + 1)^2\), and using \(\frac{y^n}{(y+1)^2} \leq 1\) for \(0 \leq n \leq 2\), we obtain

\[
\frac{(y(t) + 1)^t}{(y(t) + 1)^2} \sin \left(\frac{1}{y(t) + 1}\right) \leq c_6. \tag{2.43}
\]

Integrating the differential inequality (2.43) from \(t = t_{\ast}\) to blow-up time \(T_{\ast}\) and using the fact that \(\lim_{t \to T_{\ast}} \|u(\cdot, t)\|_{H^1}^2 = \infty\), yields

\[
1 \leq \cos \left(\frac{1}{y(t) + 1}\right) + c_6 (T_{\ast} - t). \tag{2.44}
\]

Using (2.43) in (2.44), we obtain

\[
1 - \cos \left(\frac{1}{y(t) + 1}\right) \leq c_6 \|u(\cdot, t)^n\|_{(T_{\ast} - t)^n+1} \mbox{ with } m \in \mathbb{N}^\ast, \tag{2.45}
\]

for all \(t \leq T_{\ast}\) (see proof of Theorem 2.2). This gives

\[
\sin \left(\frac{1}{2y(t) + 2}\right) \leq c_6 \|u(\cdot, t)^n\|_{(T_{\ast} - t)^n+1}. \tag{2.46}
\]

Multiplying (2.46) by \(\|u(\cdot, t)^n\|_{H^1}^2\), we get

\[
\|u(\cdot, t)^n\|_{H^1}^2 \sin \left(\frac{1}{\|u(\cdot, t)^n\|_{H^1} + 1}\right) \leq c_6 \|u(\cdot, t)^n\|_{(T_{\ast} - t)^n+1} \|u(\cdot, t)^n\|_{H^1}. \tag{2.47}
\]

Thus, there exist a positive constant \(\beta_3 = \min_{t \geq 0} \|u(\cdot, t)^n\|_{H^1}^2\) such that \(\|u(\cdot, t)^n\|_{H^1}^2 \geq \beta_3\) for all \(t \geq 0\), this gives

\[
\alpha_3 = \left(\beta_3 \sin \left(\frac{1}{\beta_3 + 1}\right)\right)^2 \leq \|u(\cdot, t)^n\|_{H^1}^2 \sin \left(\frac{1}{\|u(\cdot, t)^n\|_{H^1} + 1}\right)^2. \tag{2.48}
\]

Using this estimate in (2.48) yields the bound

\[
\alpha_3 \leq c_6 \|u(\cdot, t)^n\|_{(T_{\ast} - t)^n+1} \|u(\cdot, t)^n\|_{H^1}. \tag{2.49}
\]

This completes the proof of Theorem 2.4. \(\square\)

In the classical method there is relation between \(s\) the order of the Sobolev spaces \(H^s\) and the lower bound \((T_{\ast} - t)^{-\varphi(s)}\). \(\varphi\) is a nonlinear function. In our previous estimates, the bounds on the blow up are independent of \(s\). Since \(n \in \mathbb{N}^\ast\), we can recover more case, we can get correct rate of blow up in the form \((T_{\ast} - t)^{-1}\) in several spaces, in \(\dot{H}^{3/2}\) for \(n = 3\) and also in \(\dot{H}^1\) for \(n = 3\). Theorem 2.2 includes the optimal lower bound for blow-up rate in \(\dot{H}^{5/2}\). This particular case was not achieved [5], for \(n = 1\) in (2.10) we get a positive answer to this question

\[
\|u(\cdot, t)\|_{\dot{H}^{5/2}} \geq \frac{\eta_1}{(T_{\ast} - t)} \mbox{ for } t \leq T_{\ast}. \tag{2.50}
\]

Setting \(n = 1\) in (2.22), Theorem 2.3 gives an improvement of the rate of blow up (1.6) in \(\dot{H}^{3/2}(Q)\) obtained in [6]

\[
\|u(\cdot, t)\|_{\dot{H}^{3/2}} \geq \frac{\eta_2}{(T_{\ast} - t)^2} \mbox{ for } t \leq T_{\ast}. \tag{2.51}
\]

Since the results above are valid for all real \(m \geq 1\), we can control the distance between \(T_{\ast}\) and \(T_{\ast}\), where \(T_{\ast} - T_{\ast} = \frac{1}{m}\). This distance can be minimized by
choosing a large value of $m$, which enhances the study of the behavior of strong solutions in the neighborhood of the blow up.

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