Synthesising a Database of Parameterised Linear and Non-Linear Invariants for Time-Series Constraints

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Abstract Many constraints restricting the result of some computations over an integer sequence can be compactly represented by register automata. We improve the propagation of the conjunction of such constraints on the same sequence by synthesising a database of linear and non-linear invariants using their register-automaton representation. The obtained invariants are formulae parameterised by a function of the sequence length and proven to be true for any long enough sequence. To assess the quality of such linear invariants, we developed a method to verify whether a generated linear invariant is a facet of the convex hull of the feasible points. This method, as well as the proof of non-linear invariants, are based on the systematic generation of constant-size deterministic finite automata that accept all integer sequences whose result verifies some simple condition. We apply such methodology to a set of 44 time-series constraints and obtain 1400 linear invariants from which 70% are facet defining, and 600 non-linear invariants, which were tested on short-term electricity production problems.

1 Introduction

We present a framework for synthesising necessary conditions for a conjunction of sequence constraints that are each represented by a register automaton [11], and are imposed on the same integer sequence of length \( n \). Our necessary conditions are in the form of linear inequalities, implications whose right-hand side is a linear inequality, and disjunctions of inequalities. In addition, they are parameterised by a function of \( n \) and instance-independent, i.e. they are true for any integer sequence of length \( n \) greater than some small constant.

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In order to synthesise linear inequalities and implications with linear inequalities we draw full benefit from register automata representing the constraints since they do not encode explicitly all potential values of registers as states, and allow a constant-size representation of many counting constraints imposed on a sequence of integer variables. Moreover their compositional nature permits representing a conjunction of sequence constraints as the intersection of the corresponding register automata [30,29], i.e. the intersection of the languages accepted by all register automata, without representing explicitly the Cartesian product of all register values. As a consequence, the size of such an intersection register automaton is often quite compact, even if maintaining domain consistency for such constraints is in general NP-hard [10]; for instance, the intersection of the 22 register automata for all nb_σ time-series constraints described in [3] has only 16 states.

To formally analyse the quality of the generated invariants we developed a method allowing us to verify whether a linear invariant is a facet of the convex hull or not. The method identifies two distinct points located on the line corresponding to the linear invariant, and shows that these points are always feasible provided the precondition associated with the invariant holds.

For synthesising disjunctions of inequalities, we use a slightly different approach, comprising three steps: data generation, mining of invariants, and proof of invariants. The proof part is based on the idea that, in order to prove that there is no sequence satisfying a conjunction of conditions, we can represent a set of sequences satisfying each condition by a constant-size automaton without registers. Then, a sequence satisfying all the conditions must be accepted by the intersection of such automata. If the intersection is empty, then such a sequence does not exist.

The contributions of this paper are:

- First, Section 4 provides the basis of a simple, systematic method to precompute linear inequalities and conditional linear inequalities for a conjunction of automaton constraints on the same sequence. We call such inequalities and implications linear invariants and conditional linear invariants, respectively. Each linear invariant and each conditional linear invariant involves the result variables of the different automaton constraints in a considered conjunction representing the fact that the result variables cannot vary independently. Such invariants may be parametrised by a function of the sequence length and are independent of the domains of the sequence variables. Finally, we describe a systematic method for verifying whether a linear invariant is a facet of the convex hull or not.

- Second, Section 5 shows how to obtain disjunctions of inequalities, possibly parameterised by a function of the sequence length. We call such disjunctions non-linear invariants.

- Third, to mechanise all proofs required in Section 4 for proving that a linear invariant is facet defining, and in Section 5 for proving non-linear invariants, Section 6 defines a special kind of constant-size automaton without registers, named conditional automata that recognises all (and only all) sequences satisfying some condition, e.g. all sequences maximising the number of peaks. It shows how to construct such conditional automata in a systematic way.

- Fourth, within the context of time-series constraints, Section 7 shows the impact of the database of 2000 synthesised invariants on the propagation of time-series constraints on short-term electricity production problems.

Note that all obtained parameterised invariants are formulae that are always true. Hence they are computed once and for all, put into a database of parameterised invariants, and consulted every time when required: there is no need to rerun our methods for synthesising invariants for every instance.

Adding redundant constraints to a constraint model has been recognised from the very beginning of Constraint Programming as a major source of improvement [20]. Attempts to generate such implied constraints in a systematic way were limited (1) by the difficulty to manually prove a large number of conjectures [26,8], (2) by the limitations of automatic proof systems [24,16], or (3) to special cases for very few constraints like alldifferent, cardinality, element [28,1,27]. Within the context of register automata, linear invari-
ants relating consecutive register values of the same constraint were obtained [21] using Farkas’s lemma [13] in a resource-intensive procedure.

2 Background

This section presents the necessary background and notation on regular expressions, register automata, and time-series constraints. Two complementary facets of time-series constraints will be presented: first, their declarative definition, second the transducers used to synthesise an implementation of time-series constraints. These transducers will be used in Section 6 to generate a constant-size automaton associated with an upper bound minus a constant shift of a time-series constraint.

2.1 Background on Regular Expressions and Register Automata

For a regular expression $\sigma$, its language [19] is denoted by $L_{\sigma}$. The size [5] of a regular expression $\sigma$, denoted by $|\sigma|$, is the number of letters in the shortest word of $L_{\sigma}$.

A register automaton [6] $M$ with $p > 0$ registers is a tuple $\langle Q, \Sigma, \delta, q_0, I, A, \alpha \rangle$, where $Q$ is the set of states, $\Sigma$ is the input alphabet, $\delta: (Q \times \mathbb{Z}^p) \times \Sigma \rightarrow Q \times \mathbb{Z}^p$ is the transition function, $q_0 \in Q$ is the initial state, $I$ is a sequence of length $p$ of the initial values of the $p$ registers, $A \subseteq Q$ is the set of accepting states, and $\alpha: \mathbb{Z}^p \rightarrow \mathbb{Z}$ is a function, called acceptance function, which maps the registers of an accepting state into an integer. If, by consuming the symbols of a word $w$ in $\Sigma^*$, the automaton $M$ triggers a sequence of transitions from $q_0$, its initial state, to some accepting state where $\langle d_1, d_2, \ldots, d_p \rangle$ are the values of the registers at this stage, then $M$ returns $\alpha(d_1, d_2, \ldots, d_p)$, otherwise it fails. In this paper, the input alphabet of the register automata is $\{<, =, >\}$.

Within all figures, the acceptance function is depicted by a box connected by dotted lines to each state. If a register is left unchanged while triggering a given transition, then we do not mention this register update on the corresponding transition.

2.2 Defining Time-Series Constraints

Given an integer sequence $X = \langle X_1, X_2, \ldots, X_n \rangle$, a time-series constraint $g \_f \_\sigma(X, R)$, introduced in [7], restricts $R$ to be the result of some computations over an integer sequence $X = \langle X_1, X_2, \ldots, X_n \rangle$, where:

- $\sigma$ is a regular expression [19] over the alphabet $\Sigma = \{<, =, >\}$ with which we associate two integer constants $b_\sigma$ and $a_\sigma$ whose role is explained below; the sequence $S = \langle S_1, S_2, \ldots, S_{n-1} \rangle$, called the signature and containing signature symbols, is linked to the sequence $X$ via the signature conditions $(X_i < X_{i+1} \Leftrightarrow S_i = '<')$ (with $S_i = '='$) (and $(X_i > X_{i+1} \Leftrightarrow S_i = '>')$ for all $i \in [1, n-1]$ [6, 33]. When $\langle S_i, S_{i+1}, \ldots, S_j \rangle$ (with $1 \leq i \leq j \leq n$) is a maximal word matching $\sigma$, the sequence $\langle X_{i+b_\sigma}, X_{i+b_\sigma+1}, \ldots, X_{j+a_\sigma-1} \rangle$ is called a $\sigma$-pattern;
- $f$ is a function over sequences, called feature, and is used for computing a value for each $\sigma$-pattern; the role of the two constants $b_\sigma$ and $a_\sigma$ is to trim the left and right borders of an occurrence of the regular expression $\sigma$ when computing the feature values;
- $g$ is a function over sequences, called aggregator, and is used for aggregating the feature values of the different $\sigma$-patterns.

The result value $R$ of a time-series constraints is restricted to be the result of aggregation, computed using $g$, of the list of values of feature $f$ for all $\sigma$-patterns in $X$. In this paper, we consider the following class of time-series constraints.
In this paper, we consider a simplified version of seed transducers of \[7,22\] that we now present. It was shown in \[22\] how to generate such seed transducer from a regular expression. For the purpose of the first (resp. second) maximal occurrence of a letter in the input alphabet, the result of a regular expression in \(\Sigma\) is a deterministic transducer where each transition is labelled with two letters: a letter in the input alphabet \(\Sigma\) and a letter in the output alphabet \(\Omega\). Every element of \(\Omega\) is called a phase letter and corresponds to a recognition phase of a new occurrence of \(\sigma\) in \(S\). Consider the regular expression introduced in Example 1, and its seed transducer given in Part (A) of Figure 9:

- A transition labelled by this output symbol corresponds to the discovery of a new occurrence of \(\sigma\) in \(S\).
- A transition labelled with \(\text{found}\) is called a found-transition. A found-path is any sequence of consecutive transitions of the transducer containing at least one found-transition.

Example 2 Consider the peak regular expression introduced in Example 1, and its seed transducer given in Part (A) of Figure 9:

- the transition from \(r\) to \(t\) is a single found-transition.
the sequence of transitions from $s$ to $r$, from $r$ to $t$ and from $t$ to $r$ is a found-path.

While consuming the signature $S = \langle <,=,>,=,>,=\rangle$ of the integer sequence $(0,1,2,2,0,0,4,1)$, the seed transducer produces the output sequence (not_found,not_found,not_found,found,not_found,not_found). As shown in Example 1, $S$ contains two maximal occurrences of peak, complying with the two found letters in $t$. △

3 Types of Synthesised Invariants

Consider a conjunction of two time-series constraints $\gamma_1(X,R_1)$ and $\gamma_2(X,R_2)$ imposed on the same sequence of integer variables $X = \langle X_1,X_2,\ldots,X_n \rangle$. In this section, we present a classification of different types of invariants that involves $R_1$, $R_2$ and $n$.

**Farkas Linear Invariants for a Single Constraint** The method for generating linear invariants based on the Farkas’s lemma was described in [21], and is used for generating linear invariants linking the registers of a register automaton representing a single constraint $\gamma_i$ with $i \in \{1,2\}$. Although, this method is fairly general, the generation of invariants can be time consuming and the set of generated invariants is too large. This requires an extra step for selecting the tightest generated invariants.

**Linear Invariants for a Conjunction of Constraints** A contribution of this paper is a systematic method for generating parameterised linear invariants linking the result variables $R_1$ and $R_2$ of two time-series constraints. This method applies for any conjunction of constraints, where each constraint can be represented by a register automaton, satisfying a certain property, named the incremental-automaton property, which will be introduced in Property 1 of Section 4. The class of automata satisfying the incremental-automaton property is smaller compared to the ones satisfying the conditions of the method of [21]. However, it still covers 35 constraints of the volume II of the Global Constraint Catalogue [3]. We further show in a systematic way that many of the generated invariants are facets of the convex hull of feasible combinations of $R_1$ and $R_2$.

**Conditional Linear Invariants for a Conjunction of Constraints** We also generate conditional parameterised linear invariants, where the condition may be a requirement on $n$, $R_1$ or $R_2$, e.g. $R_1 > 0 \land R_2 > 0$, $n > 3$. Such invariants are useful when, for example, a linear invariant is a facet of the convex hull and holds only for long enough sequences. The method for generating such invariants is based on the method for synthesising linear invariants, and the same conditions on register automata apply.

**Non-Linear Invariants** The non-linear invariants we synthesise are of the form $P_1 \lor P_2 \lor \cdots \lor P_k$, where every $P_k$ is a negation of an atomic relation. We define in Section 5 a set of 8 atomic relations, some of which are $R_i = c$, $R_i = \text{up}_{R_i}(n) - c$, where $c$ is a natural number, and $\text{up}_{R_i}(n)$ is the maximum value of $R_i$ among all time series of length $n$ [5]. Such invariants are required when the set of feasible combinations of $R_1$ and $R_2$ is non-convex and therefore linear invariants are not enough for fully describing it.

4 Synthesising Parameterised Linear Invariants

Consider $k$ register automata $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_k$ over the same alphabet $\Sigma$. Let $r_i$ denote the number of registers of $\mathcal{M}_i$, and let $R_i$ designate its returned value. In this section we show how to systematically generate linear invariants of the form
\[ e + e_0 \cdot n + \sum_{i=1}^{k} e_i \cdot R_i \geq 0 \quad \text{with} \quad e, e_0, e_1, \ldots, e_k \in \mathbb{Z}, \] (1)

which hold after the signature of the same input sequence \((X_1, X_2, \ldots, X_n)\) is completely consumed by the \(k\) register automata \(M_1, M_2, \ldots, M_k\). We call such linear invariant general since it holds regardless of any conditions on the result variables \(R_1, R_2, \ldots, R_k\). Stronger, but less general, invariants may be obtained when the initial values of the registers cannot be assigned to the result variables.

Our method for generating invariants is applicable to a restricted class of register automata that we now introduce.

**Property 1 (incremental-automaton property)** A register automaton \(M\) with \(r\) registers has the incremental-automaton property if the following four conditions are all satisfied:

1. For every register \(A_j\) of \(M\), its initial value \(a_j^0\) is a natural number.
2. For every register \(A_j\) of \(M\) and for every transition \(t\) of \(M\), the update of \(A_j\) upon triggering transition \(t\) is of the form \(A_j \leftarrow a_{j,0}^t + \sum_{i=1}^{r} a_{j,i}^t \cdot A_i\), with \(a_{j,0}^t, a_{j,1}^t, a_{j,2}^t, \ldots, a_{j,r}^t \in \{0, 1\}\).
3. The register \(A_r\) is called the main register and verifies all the following three conditions:
   (a) the value returned by \(M\) is the last value of its main register \(A_r\),
   (b) for every transition \(t\) of \(M\), \(a_{r,r}^t = 1\),
   (c) for a non-empty subset \(T\) of transitions of \(M\), \(\sum_{i=1}^{r-1} \alpha_{t,i}^r > 0, \forall t \in T\).
4. For all other registers \(A_j\) with \(j < r\), on every transition \(t\) of \(M\), we have \(\sum_{i=1, i \neq j}^{r} \alpha_{t,i}^j = 0\) and, if \(a_{r,j}^t > 0\), then \(\alpha_{j,j}^t = 0\).

The intuition behind the incremental-automaton property is that there is one register that we name the main register, whose last value is the final value, returned by the register automaton, (see 3a). At some transitions, the update of the main register is a linear combination of the other registers, while on the other transitions its value either does not change or is incremented by a non-negative constant, (see 3b and 3c). All other registers may only be incremented by a non-negative constant or assigned to some non-negative integer value, and they may contribute to the final value, (see 4). These registers are called potential registers. Both register automata in Parts (A) and (B) of Figure 1 have the incremental-automaton property, and their main registers are the main registers. Volumes I and II of the global constraint catalogue contain more than 50 such register automata. In particular, in Volume II, the register automata for all the constraints of the \(\text{NB}_{\cdot \sigma}\) and the \(\text{SUM}\_\text{WIDTH}_{\cdot \sigma}\) families have the incremental-automaton property. In the rest of this paper we assume that all register automata \(M_1, M_2, \ldots, M_k\) have the incremental-automaton property.

Our approach for systematically generating linear invariants of type \(e + e_0 \cdot n + \sum_{i=1}^{k} e_i \cdot R_i \geq 0\) considers each combination of signs of the coefficients \(e_i\) (with \(i \in [0, k]\)). It consists of three steps:

1. Construct a non-negative function \(v = e + e_0 \cdot n + \sum_{i=1}^{k} e_i \cdot R_i\), which represents the left-hand side of the sought linear invariant (see Section 4.1).
2. Select the coefficients \(e_0, e_1, \ldots, e_k\), called the relative coefficients of the linear invariant, so that there exists a constant \(C\) such that \(e_0 \cdot n + \sum_{i=1}^{k} e_i \cdot R_i \geq C\) (see Section 4.2).
3. Compute \(C\) and set the coefficient \(e\), called the constant term of the linear invariant, to \(-C\) (see Section 4.3).
Fig. 1: (A) Register automaton for nb_peak; (B) Register automaton for nb_valley; (C) Intersection of (A) and (B).

The three previous steps are performed as follows:

1. First, we assume a sign for each coefficient $e_i$ (with $i \in [0,k]$), which tells whether we have to consider or not the contribution of the potential registers; note that each combination of signs of the coefficients $e_i$ (with $i \in [0,k]$) will lead to a different linear invariant. Then, from the intersection $\mathcal{I}$ of $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_k$, we construct a digraph called the invariant digraph, where each transition $t$ of $\mathcal{I}$ is replaced by an arc whose weight represents the lower bound of the variation of the term $e_0 \cdot n + \sum_{i=1}^{k} e_i \cdot R_i$ while triggering $t$.

2. Second, we find the coefficients $e_i$ (with $i \in [0,k]$) so that the invariant digraph does not contain any negative cycles. When the invariant digraph has no negative cycles, the value of $e_0 \cdot n + \sum_{i=1}^{k} e_i \cdot R_i$ is bounded from below for any integer sequence.

3. Third, to obtain $C$ we compute the shortest path in the invariant digraph from the node of the invariant digraph corresponding to the initial state of $\mathcal{I}$, to all nodes corresponding to accepting states of $\mathcal{I}$.

4.1 Constructing the Invariant Digraph for a Conjunction of Automaton Constraints wrt a Linear Function

First, Definition 2 introduces the notion of invariant digraph $G_\mathcal{I}^v$ of the register automaton $\mathcal{I} = \mathcal{M}_1 \cap \mathcal{M}_2 \cap \ldots \cap \mathcal{M}_k$ wrt a linear function $v$ involving the values returned by these register automata. Second, Definition 3 introduces the notion of weight of an accepting sequence $X$ wrt $\mathcal{I}$ in $G_\mathcal{I}^v$, which makes the link between a path in $G_\mathcal{I}^v$ and the vector of values returned by $\mathcal{I}$ after consuming the signature of $X$. Finally, Theorem 1 shows that the weight of $X$ in $G_\mathcal{I}^v$ is a lower bound on the linear function $v$.

**Definition 2 (invariant digraph)** Consider an accepting sequence $X = (X_1, X_2, \ldots, X_n)$ wrt the register automaton $\mathcal{I} = \mathcal{M}_1 \cap \mathcal{M}_2 \cap \ldots \cap \mathcal{M}_k$, and a linear function $v = c + e_0 \cdot n + \sum_{i=1}^{k} e_i \cdot R_i$, where $(R_1, R_2, \ldots, R_k)$ is the vector of values returned by $\mathcal{I}$ after consuming the signature of $X$. The invariant digraph of $\mathcal{I}$ wrt $v$, denoted by $G_\mathcal{I}^v$, is a weighted digraph defined in the following way:

- The set of nodes of $G_\mathcal{I}^v$ is the set of states of $\mathcal{I}$.
The set of arcs of $G^v_I$ is the set of transitions of $I$, where for every transition $t$, the corresponding symbol of the alphabet is replaced by an integer weight, which is $e_0 + \sum_{i=1}^{k} e_i \cdot \beta^r_i$, where $\beta^r_i$ is defined as follows:

$$\beta^r_i = \begin{cases} \alpha^r_{i,r_i,0} & \text{if } e_i \geq 0, \\ r_i \sum_{j=1}^{r_i} \alpha^r_{i,j,0} & \text{if } e_i < 0, \end{cases}$$

where $r_i$ denotes the number of registers of $M_i$, and $\alpha^r_{i,p,0}$ (with $p \in [1, r_i]$) is the constant in the update of the register $p$ of $M_i$.

**Definition 3 (walk and weight of an accepting sequence)** Consider an accepting sequence $X$ of length $n$ wrt the register automaton $I = M_1 \cap M_2 \cap \cdots \cap M_k$, and a linear function $v = e_0 \cdot n + \sum_{i=1}^{k} e_i \cdot R_i$, where $(R_1, R_2, \ldots, R_k)$ is the vector of values returned by $I$ after consuming the signature of $X$.

- The walk of $X$ in $G^v_I$ is a path in $G^v_I$ whose sequence of arcs is the sequence of the corresponding transitions of $I$ triggered upon consuming the signature of $X$.
- The weight of $X$ in $G^v_I$ is the weight of its path in $G^v_I$ plus a constant value, which is a lower bound on $v$ corresponding to the initial values of the registers and is called the initialisation weight in $G^v_I$. It equals $e + e_0 \cdot (p - 1) + \sum_{i=1}^{k} e_i \cdot \beta^0_i$, where $p$ is the arity of the signature, and where $\beta^0_i$ is defined as follows:

$$\beta^0_i = \begin{cases} \alpha^0_{i,r_i} & \text{if } e_i \geq 0, \\ r_i \sum_{j=1}^{r_i} \alpha^0_{i,j} & \text{if } e_i < 0, \end{cases}$$

where $r_i$ denotes the number of registers of $M_i$, and $\alpha^0_{i,p}$ (with $p \in [1, r_i]$) is the initial value of the register $p$ of $M_i$.

**Example 3** Consider the peak$(X, P)$ and the valley$(X, V)$ constraints introduced in Example 1 on the same sequence $X = (X_1, X_2, \ldots, X_n)$. Figure 1 gives the automata for peak, valley, and their intersection $I$. We aim to find inequalities of the form $e + e_0 \cdot n + e_1 \cdot P + e_2 \cdot V \geq 0$ for every integer sequence $X$. After consuming the signature of $X$, $I$ returns a pair of values $(P, V)$, which are the number of peaks (resp. valleys) in $X$. The invariant digraph of $I$ wrt $v = e + e_0 \cdot n + e_1 \cdot P + e_2 \cdot V$ is given in the figure on the right. As neither of the two automata has any potential registers, the weights of the arcs of $G^v_I$ do not depend on the signs of $e_1$ and $e_2$. Hence, for every integer sequence $X$, its weight in $G^v_I$ equals $e + e_0 \cdot n + e_1 \cdot P + e_2 \cdot V$.

**Theorem 1 (lower bound on the weight of an accepting sequence)** Consider an accepting sequence $X = (X_1, X_2, \ldots, X_n)$ wrt the register automaton $I = M_1 \cap M_2 \cap \cdots \cap M_k$, and a linear function $v = e + e_0 \cdot n + \sum_{i=1}^{k} e_i \cdot R_i$, where $(R_1, R_2, \ldots, R_k)$ is the vector of values returned by $I$. Then, the weight of $X$ in $G^v_I$ is less than or equal to $e + e_0 \cdot n + \sum_{i=1}^{k} e_i \cdot R_i$. 

\[ \sum \]
Proof Since, when doing the intersection of register automata we do not merge registers, the registers of $\mathcal{I}$ that come from different register automata do not interact, i.e. their updates are independent, hence their returned values are also independent. By definition of the invariant digraph, the weight of any of its arc is $e_0 + \sum_{i=1}^{k} e_i \cdot \beta_i^t$, where $\beta_i^t$ depends on the sign of $e_i$, and where $t$ is the corresponding transition in $\mathcal{I}$.

Then, the weight of $X$ in $G^v_{\mathcal{I}}$ is the constant $e + e_0 \cdot (p - 1) + \sum_{i=1}^{k} e_i \cdot \beta_i^0$ (see Definition 3) plus the weight of the walk of $X$, which is in total $e + e_0 \cdot (p - 1) + \sum_{i=1}^{k} e_i \cdot \beta_i^0 + e_0 \cdot (n - p + 1) + \sum_{j=1}^{n-p+1} \sum_{i=1}^{k} e_i \cdot \beta_i^t = e + e_0 \cdot n + \sum_{i=1}^{k} e_i \cdot \left( \beta_i^0 + \sum_{j=1}^{n-p+1} \beta_i^t \right)$, where $p$ is the arity of the considered signature, and $t_1, t_2, \ldots, t_{n-p+1}$ is the sequence of transitions of $\mathcal{I}$ triggered upon consuming the signature of $X$. We now show that the value $e_i \cdot \left( \beta_i^0 + \sum_{j=1}^{n-p+1} \beta_i^t \right)$ is not greater than $e_i \cdot R_i$. This will imply that the weight of the walk of $X$ in $G^v_{\mathcal{I}}$ is less than or equal to $v = e + e_0 \cdot n + \sum_{i=1}^{k} e_i \cdot R_i$.

Consider the $v_i = e_i \cdot R_i$ linear function. We show that the weight of $X$ in $G^c_{\mathcal{I}}$, which equals $e_i \cdot \left( \beta_i^0 + \sum_{j=1}^{n-p+1} \beta_i^t \right)$, is less than or equal to $e_i \cdot R_i$. Depending on the sign of $e_i$ we consider two cases.

Case 1: $e_i \geq 0$. In this case, the weight of every arc of $G^c_{\mathcal{I}}$ is $e_i$ multiplied by $\alpha_r^{t,0}$, where $t$ is the corresponding transition in $\mathcal{I}$, and $r_i$ is the main register of $M_i$ (see Case 2 of Definition 2). If, on transition $t$, some potential registers of $M_i$ are incremented by a positive constant, the real contribution of the register updates on this transition to $R_i$ is at least $\alpha_r^{t,0}$ since $e_i \geq 0$. The same reasoning applies to the contribution of the initial values of the potential registers to the final value $R_i$. Since this contribution is non-negative, it is ignored, and $\beta_i^0 = \alpha_r^0$ (see Case 2 of Definition 3). Hence $e_i \cdot \left( \beta_i^0 + \sum_{j=1}^{n-p+1} \beta_i^t \right) = e_i \cdot \left( \alpha_r + \sum_{j=1}^{n-p+1} \alpha_r^{t,0} \right) \leq e_i \cdot R_i$.

Case 2: $e_i < 0$. In this case, the weight of every arc of $G^c_{\mathcal{I}}$ is $e_i$ multiplied by the sum of the non-negative constants, which come from the updates of every register of $M_i$ (see Case 5 of Definition 2). The contribution of the potential registers is always taken into account, and since $e_i < 0$, it is always negative. The same reasoning applies to the contribution of the initial values of the potential registers to the returned value $R_i$. To obtain a lower bound on $v$, observe that the initial values of the potential registers are non-negative and that $e_i < 0$; therefore we assume that the initial values of the potential registers always contribute to $R_i$ (see Case 3 of Definition 3). Hence $e_i \cdot \left( \beta_i^0 + \sum_{j=1}^{n-p+1} \beta_i^t \right) \leq e_i \cdot R_i$. \hfill \Box

Note that, if all the considered register automata $M_1, M_2, \ldots, M_k$ do not have potential registers, then for every accepting sequence $X = (X_1, X_2, \ldots, X_n)$ wrt $\mathcal{I} = M_1 \cap M_2 \cap \cdots \cap M_k$ and for any linear function $v = e + e_0 \cdot n + \sum_{i=1}^{k} e_i \cdot R_i$, the weight of $X$ in $G^v_{\mathcal{I}}$ is equal to $v$. If there is at least one potential register for at least one register automaton $M_i$, then there may exist an accepting sequence $X = (X_1, X_2, \ldots, X_n)$ wrt $\mathcal{I} = M_1 \cap M_2 \cap \cdots \cap M_k$ whose weight in $G^v_{\mathcal{I}}$ is strictly less than $v$. 

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4.2 Finding the Relative Coefficients of the Linear Invariant

We now focus on finding the relative coefficients \( e_0, e_1, \ldots, e_k \) of the linear invariant \( v = e + e_0 \cdot n + \sum_{i=1}^{k} e_i \cdot R_i \geq 0 \) such that, after consuming the signature of any accepting sequence by the register automaton \( I = M_1 \cap M_2 \cap \cdots \cap M_k \), the value of \( v \) is non-negative.

For any accepting sequence \( X \) wrt \( I \), by Theorem 1, we have that the weight \( w \) of \( X \) in \( G^w_I \) is less than or equal to \( v \). Recall that \( w \) consists of a constant part, and of a part that depends on \( X \), which involves the coefficients \( e_0, e_1, \ldots, e_k \); thus, these coefficients must be chosen in a way that there exists a constant \( C \) such that, after consuming the signature of any accepting sequence by the register automaton \( I \), the value of \( v \) is non-negative.

Let \( \mathcal{C} \) denote the set of all simple circuits of \( G^w_I \), and let \( w_e \) denote the weight of an arc \( e \) of \( G^w_I \). In order to prevent negative cycles in \( G^w_I \), we solve the following minimisation problem, parameterised by \((s_0, s_1, \ldots, s_k)\), the signs of \( e_0, e_1, \ldots, e_k \):

\[
\begin{align*}
\text{minimise} & \quad \sum_{c \in \mathcal{C}} W_c + \sum_{i=1}^{k} |e_i| \\
\text{subject to} & \quad W_c \geq 0 \quad \forall c \in \mathcal{C} \\
& \quad W_c = \sum_{e \in c} w_e \quad \forall c \in \mathcal{C} \\
& \quad s_i = '-' \Rightarrow e_i \leq 0, \quad s_i = '+' \Rightarrow e_i \geq 0 \quad \forall i \in [0, k] \\
& \quad e_i \neq 0 \quad \forall i \in [1, k] 
\end{align*}
\]

(6) (7) (8) (9) (10)

In order to obtain the coefficients \( e_0, e_1, \ldots, e_k \), so that \( G^w_I \) does not contain any negative cycles, it is enough to find a solution to the satisfaction problem (7)-(10). Minimisation is required to obtain linear invariants that eliminate as many infeasible values of \((R_1, R_2, \ldots, R_k)\) as possible. Within the objective function (6), the term \( \sum_{c \in \mathcal{C}} W_c \) is for minimising the weight of every simple circuit, while the term \( \sum_{i=1}^{k} |e_i| \) is for obtaining the coefficients with the smallest absolute value. By changing the sign vector \((s_0, s_1, \ldots, s_k)\) we obtain different linear invariants.

**Example 4 (finding the relative coefficients)** Consider \( \text{nb\_peak}(X, P) \) and \( \text{nb\_valley}(X, V) \) with \( X \) being a time series of length \( n \). The invariant digraph of the intersection of the register automata for the \( \text{nb\_peak} \) and \( \text{nb\_valley} \) constraints wrt \( v = e + e_0 \cdot n + e_1 \cdot P + e_2 \cdot V \) was given in Example 3. This digraph has four simple circuits, namely \( s - s \), \( t - t \), \( r - r \), and \( r - t - r \), which are labelled by 1, 2, 3 and 4, respectively. Then, the minimisation problem for finding the relative coefficients of the linear invariant \( v \geq 0 \), parameterised by \((s_0, s_1, s_2)\), the signs of \( e_0, e_1 \) and \( e_2 \), is the following:

\[
\begin{align*}
\text{minimise} & \quad \sum_{j=1}^{4} W_j + \sum_{i=0}^{2} |e_i| \\
\text{subject to} & \quad W_j = e_0, \quad \forall j \in [1, 3] \\
& \quad W_4 = e_0 + e_1 + e_2 \quad \forall j \in [1, 4] \\
& \quad W_j \geq 0 \quad \forall j \in [1, 4] \\
& \quad s_i = '-' \Rightarrow e_i \leq 0, \quad s_i = '+' \Rightarrow e_i \geq 0 \quad \forall i \in [0, 2] \\
& \quad e_i \neq 0 \quad \forall i \in [1, 2] 
\end{align*}
\]

(11)
Fig. 2: (A) The invariant digraph of the register automata for the \texttt{nb\_peak} and the \texttt{nb\_valley} time-series constraints; (B) The set of feasible values of the result variables $P$ and $V$ of the \texttt{nb\_peak} and the \texttt{nb\_valley} time-series constraints, respectively, for sequences of length 11.

Note that the value of $e_0$ must be non-negative otherwise (11) cannot be satisfied for $j \in \{1, 2, 3\}$. Hence we consider only the combinations of signs of the form $(+, s_1, s_2)$ with $s_1$ and $s_2$ being either ‘$-$’ or ‘$+$’. The following table gives the optimal solution of the minimisation problem for the considered combinations of signs:

| $(s_0, s_1, s_2)$ | $(+, -, -)$ | $(+, +, -)$ | $(+, +, +)$ |
|-------------------|-------------|-------------|-------------|
| $(e_0, e_1, e_2)$ | $(1, -1, -1)$ | $(0, +, -)$ | $(0, 1, 1)$ |

4.3 Finding the Constant Term of the Linear Invariant

Finally, we focus on finding the constant term $e$ of the linear invariant $v = e + e_0 \cdot n + \sum_{i=1}^{k} e_i \cdot R_i \geq 0$, when the coefficients $e_0, e_1, \ldots, e_k$ are known, and when the digraph of the register automaton $I = M_1 \cap M_2 \cap \cdots \cap M_k$ wrt $v$ does not contain any negative cycles. By Theorem 1, the weight of any accepting sequence $X$ wrt $I$ in $G^v_I$ is less than or equal to $v$, then if the weight of $X$ is non-negative, it implies that $v$ is also non-negative. Since the invariant digraph $G^v_I$ does not contain any negative cycles, then the weight of $X$ cannot be smaller than some constant $C$. Hence it suffices to find this constant and set the constant term $e$ to $-C$. The value of $C$ is computed as the constant $e_0 \cdot (p - 1) - \sum_{i=1}^{k} \beta_i^0$ (see Definition 3) plus the shortest path length from the node of $G^v_I$ corresponding to the initial state of $I$ to all the nodes of $G^v_I$ corresponding to the accepting states of $I$.

Example 5 (obtaining invariants) Consider \texttt{nb\_peak}(X, P) and \texttt{nb\_valley}(X, V) with $X$ being a time series of length $n$ such that $n \geq 2$. In Example 4, we found four vectors for the relative coefficients $e_0, e_1, e_2$ of the linear invariant $v = e_0 \cdot n + e_1 \cdot P + e_2 \cdot V \geq 0$. For every found vector for the relative coefficients $(e_0, e_1, e_2)$, we obtain a weighted digraph, whose weights now are integer numbers. For example, for the vector $(e_0, e_1, e_2) = (0, -1, 1)$, the obtained digraph is given in Part (A) of Figure 2. We compute the length of the shortest path from the node $s$, which corresponds to the initial state of the register automaton in Part (C) of Figure 1 to every node corresponding to an accepting state of the register automaton in Part (C) of Figure 1. The length of the shortest path from $s$ to $s$ is 0, from $s$ to $t$ is 0, and from $s$ to $r$ is $-1$. The minimum of these values is $-1$, hence the constant term $e$ equals $-0 + (-1)) = 1$. The obtained linear invariant is $P \leq V + 1$. 

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In a similar way, we find the constant terms for the other found vectors of the relative coefficients $(e_0, e_1, e_2)$, and obtain three other linear invariants: $V \leq P + 1$, $V + P \leq n - 2$, $V + P \geq 0$.

Part (B) of Figure 2 shows the polytope of feasible points $(P, V)$ when $n = 11$. Observe that three of the four linear invariants found are facets of the convex hull of this polytope. △

The next example illustrates how the method presented in this section can also be used for generating linear invariants for non-time-series constraints.

**Example 6 (generating invariants for non-time-series constraints)** Consider a sequence of integer variables $X = \langle X_1, X_2, \ldots, X_n \rangle$ with every $X_i$ ranging over $[0, 3]$, four among [9] constraints that restrict the variables $R_0$, $R_1$, $R_2$, $R_3$ to be the number of occurrences of values $0, 1, 2, 3$, respectively, in $X$, as well as the four corresponding stretch [31] constraints restricting the stretch length in $X$ to be respectively in $[1, 4]$, $[2, 5]$, $[3, 5]$, and $[1, 2]$. In addition assume that value $2$ (resp. $1$) cannot immediately follow a $3$ (resp. $2$). The intersection of the corresponding register automata has $17$ states and allows one to generate $16$ linear invariants, one of them being $2 + n + R_0 + R_1 - R_2 - 2 \cdot R_3 \geq 0$. Since the sum of all $R_i$ is $n$, this linear invariant can be simplified to $2 + 2 \cdot n - 2 \cdot R_2 - 3 \cdot R_3 \geq 0$, which is equivalent to $2 \cdot (R_2 + R_3 - n) \leq 2 - R_3$. This inequality means that if $X$ consists only of the values $2$ and $3$, i.e. $R_2 + R_3 - n = 0$, then $R_3 \leq 2$, which represents the conjunction of the conditions that the stretch length of $R_3 \in [1, 2]$ and $(X_1 = 3) \Rightarrow (X_{i+1} \neq 2)$.

4.4 Improving the Generated Linear Invariants

When at least one of the register automata $M_1, M_2, \ldots, M_k$ has at least one potential register, then there may exist an accepting sequence $X = \langle X_1, X_2, \ldots, X_n \rangle$ wrt $I = M_1 \cap M_2 \cap \cdots \cap M_k$ such that the weight of $X$ in the invariant graph $G^I_P$ is strictly less than $v = e + e_0 \cdot n + \sum_{i=1}^{k} e_i \cdot R_i$. This may lead to weaker invariants and Example 7 illustrates such a situation.

**Example 7 (weak invariant)** Given the proper plateau regular expression ‘$>\Rightarrow+<’$, consider a conjunction of \texttt{nb\_proper\_plateau}(X, R_1) and \texttt{sum\_width\_proper\_plateau}(X, R_2) imposed on the same time series $X$ of length $n$, and a linear function $v = e + e_0 \cdot n + e_1 \cdot R_1 + e_2 \cdot R_2$. The intersection of the register automata for these two constraints is given in Part (A) of Figure 3. By inspection we can derive the invariant $R_2 \geq 2 \cdot R_1$, which cannot be generated by the method described in Sections 4.1, 4.2 and 4.3, because of the following reason: when $e_0 = 0$, $e_1 = -2$, and $e_2 = 1$, the weights of the arcs from $a$ to $b$ and from $b$ to $c$ are both $e_0$, and the weight of the arcs from $c$ to $a$ is $e_0 + e_1 + e_2$, and thus the weight of the cycle $a - b - c - a$ is $3 \cdot e_0 + e_1 + e_2 = -1$.

Just before triggering the transition from $c$ to $a$, the value of the register $D_2$ is at least 1 since the register automaton had triggered the transition from $b$ to $c$ before, which incremented $D_2$. Let us modify the intersection $I$ so that the register $D_2$ is not updated on the transition from $b$ to $c$, and the register $R_2$ is updated as $R_2 + D_2 + 2$ on the transition from $c$ to $a$. The modified register automaton $I^*$ recognises the same set of signatures as $I$, and after consuming any accepting sequence wrt $I$, the register automaton $I^*$ returns the same tuple of final values as $I$. In addition, the weight of the cycle $a - b - c - a$ in $I^*$ is equal to $3 \cdot e_0 + e_1 + 2 \cdot e_2$, which is $0$ when $e_0 = 0$, $e_1 = -2$, and $e_2 = 1$. Hence, the invariant $R_2 \geq 2 \cdot R_1$ can be generated after some modifications of the intersection $I$.

△

To handle the issue presented in Example 7 we introduce a preprocessing technique of the intersection of register automata. The technique relies on the notion of delay of a potential register $A$ at a state $q$ of the intersection $I$, which is a lower bound on the value of $A$ when a sequence of triggered transitions of
Fig. 3: (A) Intersection of register automata for nb_proper_plateau and sum_width_proper_plateau, for which the method described in Sections 4.1, 4.2 and 4.3 does not generate facet-defining invariants; (B) Delayed intersection obtained from the intersection in (A); (C) Invariant digraph obtained from the delayed intersection in (B).

the register automaton ends up in state q. Intuitively, we can change the updates of some registers in a way that for any accepting sequence wrt I, the returned tuple of values does not change, but the arcs of the invariant digraph obtained from the modified intersection I* will have larger weights. The modified intersection that we obtain satisfies the three following conditions:

1. The set of accepting sequences wrt I coincides with the set of accepting sequences wrt I*.
2. For every accepting sequence X wrt I, the register automata I and I* return the same tuple of values.
3. For any accepting sequence X, the weight of X in G_2^v is greater than or equal to the weight of X in G_2^v, where v is e + c_0 \cdot n + \sum_{i=1}^{k} e_i \cdot R_i.

By Condition 3, since for every X, the weight of X in G_2^v is greater than or equal to the weight of X in G_2^v, the weight of every simple circuit in I may also increase, which may lead to stronger invariants.

To obtain such register automata I*, we first introduce in Definition 4 the notion of list of delays of a state q of the intersection I, denoted by d_q. An element i of d_q is an array whose values correspond to the potential registers of M_i. The value j of this array represents a lower bound on the value of the register I_j when the register automaton I arrives to the state q. Further, based on this notion, in Definition 5, we introduce the notion of delayed intersection. Finally, in Theorem 2 we show that the delayed intersection satisfies Conditions 1, 2, and 3.

**Definition 4 (list of delays of a state)** Consider a register automaton I = M_1 \cap M_2 \cap \cdots \cap M_k. The list of delays d_q of a state q is a list of arrays, where the size of the i-th array in d_q is the number of potential registers in the register automaton M_i. Let j be the index of a register of M_i, let \cal T_q denote the set of transitions entering q, and \cal T'_q denote a subset of transitions of \cal T_q starting from a state different from q, then the value d_q[i][j] is defined as

\[
d_q[i][j] = \begin{cases} 
0 & \exists t \in \cal T_q, \; \alpha^{t}_{i,j,j} = 0, \\
\min(\alpha^{0}_{i,j,j}, \min_{t \in \cal T'_q} \alpha^{t}_{i,j,j,0}) & q \text{ is the initial state of } I, \; \forall t \in \cal T_q, \; \alpha^{t}_{i,j,j} > 0, \\
\min_{t \in \cal T'_q} \alpha^{t}_{i,j,j,0} & \text{otherwise},
\end{cases}
\]

where \alpha^{t}_{i,j,j} (resp. \alpha^{t}_{i,j,j,0}) denotes the coefficient of the register A_j (resp. the free term) in the update of A_j in the automaton M_i.

**Example 8 (list of delays of a state)** Consider two register automata M_1 and M_2 such that their intersection I is given in Part (A) of Figure 3. The register automaton M_1 has one register R_1, and M_2 has two
It implies that, when the register automaton registers $D_2$ and $R_2$. Let us compute the list of delays of every state of $I$. Since only $M_1$ does not have any potential registers then for any state $q$ of $I$, the array $d_q[1]$ is empty. The following table gives the list of delays of every potential register of $I$.

| state | a  | b  | c  |
|-------|----|----|----|
| $d_q$ | [], [0] | [], [0] | [], [1] |

It implies that, when the register automaton $I$ is either in state $a$ or state $b$, we only know that its potential register $D_2$ is non-negative. However, when $I$ is in the state $c$, the value of its potential register is at least 1.

**Definition 5 (delayed intersection)** Consider the register automaton $I = M_1 \cap M_2 \cap \cdots \cap M_k$. The delayed intersection $I^*$ of $M_1, M_2, \ldots, M_k$ is obtained from $I$ using the following rules:

- The set of states and accepting states of $I^*$ coincide with those of $I$.
- The set of transitions of $I^*$ coincide with the one of $I$.
- The number of registers of $I^*$ is the same as for $I^*$, and is denoted by $r$.
- The initial values of main registers of $I^*$ are the same as for $I^*$. For every potential register $A_{i,j}$ of $I^*$, its initial value equals $\alpha_{i,j}^0 - d_q[i][j]$, where $q$ is the initial state of $I^*$ and $\alpha_{i,j}^0$ is the initial value of $A_{i,j}$ of $I$.
- For every transition $t$ from a state $q_1$ to a state $q_2$ and for any register $M_{i,j}$ of $I$, the update of $A_{i,j}$

\[
\text{on } t \text{ is equal to } \alpha_{i,j,0}^t + \sum_{k=1}^r \alpha_{i,j,k}^t \cdot A_{i,k}^t , \text{ while the update of the corresponding register } M_{i,j}^t \text{ on the corresponding transition of } I^* \text{ is equal to } \gamma_{i,j,0}^t + \sum_{k=1}^r \alpha_{i,j,k}^t \cdot A_{i,k}^t , \text{ where } \gamma_{i,j,0}^t \text{ is defined as follows:}
\]

- If $A_{i,j}$ is a main register of $I$, then $\gamma_{i,j,0}^t = \alpha_{i,j,0}^t + \sum_{k=1}^{r_i} \alpha_{i,j,k}^t \cdot d_q[i][k]$, where $r_i$ is the number of registers of the register automaton $M_{i,j}$.
- If $A_{i,j}$ is a potential register of $I$, then $\gamma_{i,j,0}^t = \alpha_{i,j,0}^t + d_q[i][j] - d_q[i][j]$.
- The acceptance function of $I^*$ is the same as for $I$.

**Example 9 (delayed intersection)** Consider two register automata $M_1$ and $M_2$ such that their intersection $I$ is given in Part (A) of Figure 3. The delayed intersection $I^*$ constructed according to Definition 5 is given in Part (B) of Figure 3. The main difference between $I^*$ and $I$ is that the register $D_2$ is no longer updated on the transition from $b$ to $c$, but its contribution is integrated directly to $R_2$ on the transition from state $c$ to state $a$.

**Theorem 2 (properties of delayed intersection)** Consider the register automaton $I = M_1 \cap M_2 \cap \cdots \cap M_k$ and the corresponding delayed intersection $I^*$. The three following conditions are satisfied:

1. The set of accepting sequence wrt $I$ coincides with the set of accepting sequence wrt $I^*$.
2. For every accepting sequence $X$ wrt $I$, the register automata $I$ and $I^*$ return the same tuple of values.
3. For any accepting sequence $X$, the weight of $X$ in $G^\nu_I$ is greater than or equal to the weight of $X$ in $G^\nu_{I^*}$,

where $\nu = e + e_0 \cdot n + \sum_{i=1}^k e_i \cdot R_i$.

**Proof** We prove each of the three statements separately.

[Proof of (1)] Since $I$ have the same sets of states, transitions and accepting states, and every $M_i$ has the incremental-automaton property, then the sets of accepting sequences of $I$ and $I^*$ are the same.
[Proof of (2)]. Since the acceptance function of both $\mathcal{I}$ and $\mathcal{I}^*$ returns a tuple of main registers, we will show that after consuming the signature $S$ of any accepting sequence, the main registers of $\mathcal{I}$ and $\mathcal{I}^*$ contain the same values. Let us prove this statement by induction on the length of $S$.

**Base case.** Let us consider a sequence $S = (S_1)$ consumed by $\mathcal{I}^*$. The register automaton $\mathcal{I}^*$ triggered one transition $t$ from its initial state $q$ to some other state $q'$. Then, let us consider a main register $A^*_{i,r}$. By definition, its value equals $\alpha_{i,j,0}^t + A^*_{i,r},_r + \sum_{k=1}^{r-1} \alpha_{i,j,k}^t (A^*_{i,k} + d_q[i][k])$. Since any potential register $A^*_{i,k}$ has not been updated, its contains the initial value, which equals $\alpha_{i,j}^0$. Furthermore, the value of $A^*_{i,r}$ after one transition is equal to $\alpha_{i,j,0}^t + \sum_{k=1}^{r-1} \alpha_{i,j,k}^t (A^*_{i,k} + d_q[i][k]) = \alpha_{i,j,0}^t + \sum_{k=1}^{r-1} \alpha_{i,j,k}^t (A^*_{i,k} + d_q[i][k])$, which coincides with the value of the corresponding register $A_{i,j}$ of $\mathcal{I}$.

**Induction step.** Assume that after having consumed a sequence $S = (S_1, S_2, \ldots, S_{m-1})$, the main registers of $\mathcal{I}^*$ contain the same values as the main register of $\mathcal{I}$ after having consumed the same sequence. Let us show that after consuming one another symbol $S_m$, which triggers a transition $t$, the main registers of $\mathcal{I}^*$ and $\mathcal{I}$ will have the same value. The update of $A^*_{i,r}$ on $t$ is equal to $\alpha_{i,j,0}^t + A^*_{i,r},_r + \sum_{k=1}^{r-1} \alpha_{i,j,k}^t (A^*_{i,k} + d_q[i][k])$. By assumption of induction the value of $A^*_{i,r}$ in $\mathcal{I}^*$ and $A_{i,r}$ in $\mathcal{I}$ are the same after consuming $S$. Hence, we only need to show that after having consumed $S$, that the value of the potential register $A_{i,k}$ of $\mathcal{I}$ equals $A_{i,k} + d_q[i][k]$. This can also be shown by induction, starting from a state that is a destination of a triggered transition $t'$ such that $\alpha_{i,k,k}^t = 0$.

[Proof of (3)]. We now prove the last statement. Let us consider the invariant digraphs $G^*_X$, and $G^v_X$, wrt $\mathcal{I}$, its weight in $G^*_X$. is greater than or equal to its weight in $G^v_X$. The weight of $X$ in $G^*_X$ is the constant $e + e_0 (p-1) + \sum_{i=1}^k e_i \cdot \beta_i^0$ (see Definition 3) plus the weight of the walk of $X$, which is in total $e + e_0 (p-1) + \sum_{i=1}^k e_i \cdot \beta_i^0 + e_0 (n-p+1) + \sum_{i=1}^k e_i \cdot \delta_i$. For simplicity, we assume $n = p$. By the property of the considered signature, and $t_1, t_2, \ldots, t_{n-p+1}$ is the sequence of transitions of $\mathcal{I}$ triggered upon consuming the signature of $X$. Similarly, the weight of $X$ in $G^v_X$ is equal to $e + e_0 \cdot n + \sum_{i=1}^k e_i \cdot \left( \delta_i^0 + \sum_{j=1}^{n-p+1} \delta_i^{t_j} \right)$, where $\delta_i^0$ is the initialisation weight in $\mathcal{I}^*$, and every $\delta_i^{t_j}$ is the weight of an arc $t_j$ in $G^v_X$.

We now show that the value $e_i \cdot \left( \delta_i^0 + \sum_{j=1}^{n-p+1} \delta_i^{t_j} \right)$ is not greater than $e_i \cdot \left( \delta_i^0 + \sum_{j=1}^{n-p+1} \delta_i^{t_j} \right)$. This will imply that the weight of the walk of $X$ in $G^v_X$ is less than or equal to the weight of the walk of $X$ in $G^*_X$.

By Definition 2, the weight of every arc of $G^*_X$ (resp. $G^v_X$), corresponding to a transition $t$ of $\mathcal{I}$, (resp. $\mathcal{I}^*$) is equal to $\sum_{i=1}^k e_i \cdot \beta_i^t$ (resp. $\sum_{i=1}^k e_i \cdot \delta_i^t$).

As in Theorem 1, we consider the function $v_t = e_i \cdot R_t$. Depending on the sign of $e_i$, we have two cases:

**Case (1):** $e_i \geq 0$. Then, the weight of $X$ in $G^*_X$ (resp. $G^v_X$) is equal to $e_i \cdot \alpha$ (resp. $e_i \cdot \gamma$), where $\alpha$ denotes $\beta_i^0 + \sum_{j=1}^{n-p+1} \beta_i^t = \sum_{k=1}^{r-1} \alpha_{i,k}^t + \sum_{k=1}^{r-1} \alpha_{i,k}^t (A^*_{i,r},_r + \sum_{k=1}^{r-1} \alpha_{i,k}^t (A^*_{i,k} + d_q[i][k]))$. Since
every \( \gamma^{t_i}_{i,r_i,0} = \alpha^{t_i}_{i,r_i,0} + \sum_{k=1}^{r_i-1} d_q[i][k] \), it implies that \( \gamma^{t_i}_{i,r_i,0} \geq \alpha^{t_i}_{i,r_i,0} \). Then, \( \alpha \leq \gamma \), and when \( \epsilon_i > 0 \), we have \( \epsilon_i \cdot \gamma \geq \epsilon_i \cdot \alpha \).

**Case (2):** \( \epsilon_i < 0 \). Then, the weight of \( X \) in \( G_X^{w_i} \) (resp. \( G_X^{w_i} \)) is equal to \( \epsilon_i \cdot \alpha \) (resp. \( \epsilon_i \cdot \gamma \)), where \( \alpha \) denotes \( \beta^0_i + \sum_{j=1}^{n-p+1} \beta^j_i = \sum_{k=1}^{r_i} \alpha^0_{i,k} + \sum_{r_i} \sum_{k=1}^{r_i} \alpha^j_{i,k,0} \) (resp. \( \gamma \) denotes \( \delta^0_i + \sum_{j=1}^{n-p+1} \delta^j_i = \sum_{k=1}^{r_i} \gamma^0_{i,k} + \sum_{r_i} \sum_{k=1}^{r_i} \gamma^j_{i,k,0} \)).

Further, by construction of \( I^* \), every \( \gamma^{t_i}_{i,k,0} \) (with \( i \in [1, r_i] \)) is equal to \( \alpha^{t_i}_{i,k,0} + d_{q_i}[i][k] - d_{q_2}[i][k] \), where \( q_i \) and \( q_2 \) are the source and the destination of the transition \( t_i \), respectively. In addition, \( \gamma^{t_i}_{i,r_i,0} = \alpha^{t_i}_{i,r_i,0} \). By replacing every \( \gamma^{t_i}_{i,k,0} \) with its expression, and simplifying the sum, we obtain

\[
\sum_{k=1}^{r_i} \alpha^0_{i,k} + \sum_{r_i} \sum_{k=1}^{r_i} (\alpha^j_{i,k,0} - d_{q}[i][k]),
\]

where \( q ' \) is the last state visited by \( I \) upon consuming \( X \). Since every \( d_{q}[i][k] \) is non-negative, \( \alpha^j_{i,k,0} - d_{q}[i][k] \leq \alpha^j_{i,k,0} \). This implies that \( \gamma \leq \alpha \), and when \( \epsilon_i < 0 \), \( \epsilon_i \cdot \gamma \geq \epsilon_i \cdot \alpha \). \( \square \)

Note that in the register automaton \( I^* \), all the constants \( \gamma_{i,j,0} \) introduced in Definition 5 are non-negative by definition of the delay (see Definition 4). It means that the reasoning used in the proof of Theorem 1 requiring the non-negativity of these constants remains valid for the invariant digraph \( G^v_{I^*} \).

**Example 10 (generating stronger invariants)** Consider two register automata \( M_1 \) and \( M_2 \) such that their intersection \( I \), and their delayed intersection \( I^* \) are respectively given in Parts (A) and (B) of Figure 3. The invariant digraph \( G^v_{I^*} \) is given in Part (C) of Figure 3 when \( e_0 > 0 \), \( e_1 > 0 \), and \( e_2 < 0 \). By stating the minimisation problem from Section 4.2, we obtain the following coefficients: \( e_0 = 0 \), \( e_1 = -2 \), and \( e_2 = 1 \). The constant \( \epsilon \) is found to be 0, and we obtain the invariant \( 2 \cdot R_1 \geq R_2 \), which could not be found with the invariant digraph \( G^v_{I} \).

\[ \triangle \]

4.5 Generating Conditional Linear Invariants with the Non-Default Value Condition

Quite often a register automaton \( M_i \) (with \( i \in [1, k] \)) returns the initial value of one of its registers only when the signature of \( X \) does not contain any occurrence of some regular expression \( \sigma_i \). This may lead to a convex hull of points of coordinates \((R_1, R_2, \ldots, R_k)\) returned by \( I \) containing infeasible points, e.g. see Part (A) of Figure 4. Some of these infeasible points can be eliminated by stronger invariants subject to a condition, called the *non-default value condition*, that no variable of the returned vector is assigned to the initial value of the corresponding register. We first illustrate the motivation for such conditional linear invariants.

**Example 11 (motivation for conditional invariants)** Consider the \texttt{nb_decreasing_terrace}(X, R_1) and the \texttt{sum_width_increasing_terrace}(X, R_2) constraints, where \( X \) is a time series of length \( n \), \( R_1 \) is restricted to be the number of maximal occurrences of \texttt{decreasing_terrace = '>=='} in the signature of \( X \), and \( R_2 \) is restricted to be the sum of the number of elements in subsseries of \( X \) whose signatures correspond to words of the language of \texttt{increasing_terrace = '<=<+'}. In Figure 4, for \( n = 12 \), the squared points represent feasible pairs \((R_1, R_2)\), while the circled points stand for infeasible pairs \((R_1, R_2)\) inside the convex hull. The linear invariant \( 2 \cdot R_1 + R_2 \leq n - 2 \) is a facet of the polytope, which does not eliminate the points \((1, 8), (2, 6), (3, 4), (4, 2)\). However, if we assume that both \( R_1 > 0 \) and \( R_2 > 0 \), then we can add a linear invariant eliminating these four infeasible points, namely \( 2 \cdot R_1 + R_2 \leq n - 3 \), shown in Part (B) of Figure 4. In addition, the infeasible points on the straight line \( R_2 = 1 \) will also be eliminated by the restriction \( R_2 = 0 \lor R_2 \geq 2 \) given in \cite[p. 2962]{3}.

\[ \triangle \]
Consider two time-series constraints hold when the non-default value condition is satisfied.

Sections 4.1, 4.2 and 4.3 we generate the linear invariants for words of the language some regular expression and (B) with the Non-Default Value condition. sum_width_increasing_terrace for a sequence length of signature of an accepting sequence R

Fig. 4: Invariants on the result values $R_1$ and $R_2$ of NB_DECREASING_TERRACE and SUM_WIDTH_INCREASING_TERRACE for a sequence length of 12 (A) with the general linear invariants, and (B) with the Non-Default Value condition.

Consider that each register automaton $M_i$ (with $i \in [1, k]$) returns its initial value after consuming the signature of an accepting sequence $X$ wrt $M_i$ iff the signature of $X$ does not contain any occurrence of some regular expression $\sigma_i$ over the alphabet $\Sigma$. Let $M'_i$ denote the register automaton which accepts the words of the language $\Sigma^* \sigma_i \Sigma^*$, where $\Sigma^*$ denotes any word over $\Sigma$. Then, using the method described in Sections 4.1, 4.2 and 4.3 we generate the linear invariants for $M'_1 \cap M'_2 \cap \cdots \cap M'_k$. These linear invariants hold when the non-default value condition is satisfied.

4.6 Facet Analysis of Linear Invariants

Consider two time-series constraints $\gamma_1(X, R_1)$ and $\gamma_2(X, R_2)$ imposed on the same sequence $X$ of length $n$. After having generated linear and conditional linear invariants linking $R_1$, $R_2$ and $n$, an essential question is whether these invariants are facets of the convex hull of feasible combinations $R_1$ and $R_2$, or not. Given a linear invariant $f = c + e_0 \cdot n + e_1 \cdot R_1 + e_2 \cdot R_2 \geq 0$, this section presents a three-step method for answering this question:

1. Assume an infinite set $A$ of values of $n$ such that the set of sequences whose length is in $A$ can be represented by a constant-size automaton, e.g. $n \geq 5, n \mod 2 = 1, n \in \mathbb{N}$.
2. Find two distinct points $P_1$ and $P_2$, possibly parameterised by $n \in A$, laying on the straight line $f = 0$.
3. Prove that $P_1$ and $P_2$ are feasible for any $n \in A$.

The challenge here is the third step, which requires to prove the feasibility of $P_1$ and $P_2$ for an infinite set of values of $n$. Let $\text{up}_{R_i}(n)$ denote the maximum value of $R_i$ among all time series of length $n$, let $a_x, a_y$ be in $\{0, 1\}$ and let $b_x$ and $b_y$ be natural numbers. It turns out that for points of the form $\left(\frac{hx}{hy}\right)$ = $\left(\frac{a_x \cdot \text{up}_{R_1}(n) + (1 - 2 \cdot a_x) \cdot b_x}{a_y \cdot \text{up}_{R_2}(n) + (1 - 2 \cdot a_y) \cdot b_y}\right)$ we can represent the set of time series corresponding to such a point as the intersection of three constant-size automata, namely (i) the automaton representing the assumed condition on $n$, (ii) the automaton that accepts only and only all time series yielding $h_x$ as the value of $R_1$, and (iii) the automaton that accepts only and only all time series yielding $h_y$ as the value of $R_2$. The constant-size automata representing a condition on $R_1$ and $R_2$ can be synthesised from the seed transducers for the regular expressions associated with $\gamma_1$ and $\gamma_2$, as shown in Section 6. We now give in Sections 4.6.1, 4.6.2 and 4.6.3 more details for each of the three steps.
4.6.1 Step One: Assuming a Condition on the Sequence Length

Some of the invariants we generate are facets of the convex hull only for a subset of values of $n$, e.g., only even-length sequences. This requires to assume a condition on $n$ that can be represented by a constant-size automaton. We start with the less restrictive condition and try to prove that an invariant is a facet, and then gradually restrict the condition if we cannot prove it in full generality.

4.6.2 Step Two: Finding Two Integer Points on a Straight Line

To find two distinct points on the straight line $f = 0$, we assume a value of $R_1$ as $a_x \cdot \text{up}_{R_1}(n) + (1 - 2 \cdot a_x) \cdot b_x$, which by [5] is equal to $a_x \cdot \frac{n - c_1}{d_1} + (1 - 2 \cdot a_x) \cdot b_x$, with $c_1$ and $d_1$ being integer constants depending on the regular expression associated with $\gamma_1$. If the coefficient of $R_2$ in $f$ is 0, then the value of $R_2$ is not relevant and we can take, for example, 0 or 1 as the value of $R_2$. Otherwise, by isolating $R_2$ from the equation $f = 0$ we obtain:

$$R_2 = \frac{(-c_0 \cdot d_1 - e_1 \cdot a_x \cdot n + (-e \cdot d_1 + e_1 \cdot a_x \cdot c_1 - e_1 \cdot (1 - 2 \cdot a_x) \cdot b_x \cdot d_1) + e_1 \cdot a_x \cdot (n - c_1) \mod d_1}{d_1}$$

Then we verify that the right-hand side of (12) is of the form $a_y \cdot \frac{n - c_2}{d_2} + (1 - 2 \cdot a_y) \cdot b_y$, with $c_2$ and $d_2$ being integer constants depending on the regular expression associated with $\gamma_2$, with $a_y$ being in $\{0, 1\}$, and with $b_y$ being a natural number. This is done by solving a system of constraints assuming that $n$ belongs to $A$. The solutions of such system are the candidate points of the next step.

4.6.3 Step Three: Proving Feasibility of an Integer Point

Once we found two distinct integer points lying on the straight line $f = 0$, we show that both points are feasible for any $n$ in $A$.

For a point of coordinates $(h_x, h_y)$ we construct two constant-size automata $M_1$ and $M_2$, where $M_1$ (resp. $M_2$) is an automaton recognising the signatures of all and only time series yielding $h_x$ (resp. $h_y$) as the value of $R_1$ (resp. $R_2$). Let $M_n$ be a constant-size automaton representing the $n \in A$ condition, and $d$ denote the smallest difference between two values in $A$. If, in the intersection $M$ of $M_1$, $M_2$, $\ldots$, $M_n$ there are cycles of length $d$, then the point $(h_x, h_y)$ is feasible for any sequence whose length is in $A$. From this intersection we also compute the smallest value of $n$, for which these two points are feasible. This is the length of the shortest path from the initial state of $M$ to an accepting state of $M$ that goes through a state belonging to a cycle of length $d$.

If we cannot prove the feasibility of our two current points, then we try a different combination of $a_x$ and $b_x$, and obtain two other distinct points. Since the set of values of $b_x$ is, potentially, unbounded we limit ourselves only to the values of $b_x$ belonging to the set $\{0, 1, 2, 3\}$.

Example 12 Consider the conjunction of the \textsc{nb\_peak}(X, P) and the \textsc{nb\_valley}(X, V) time-series constraints imposed on the same time series $X = \langle X_1, X_2, \ldots, X_n \rangle$, and the linear invariant $P + V \leq n - 2$. Let us now analyse whether this invariant is facet defining or not. By [5], both $\text{up}_P(n)$ and $\text{up}_V(n)$ are equal to $\frac{n - 1 - (n - 1) \mod 2}{2}$.

- When $P$ is equal to $\text{up}_P(n)$, then by (12), $V$ is equal to $\frac{n - 3 + (n - 1) \mod 2}{2}$; we consider two cases:
  i. If $(n - 1) \mod 2 = 0$, then $\frac{n - 3 + (n - 1) \mod 2}{2} = \frac{(n - 1) - 2}{2} = \text{up}_V(n) - 1$.
  ii. If $(n - 1) \mod 2 = 1$, then $\frac{n - 3 + (n - 1) \mod 2}{2} = \frac{(n - 2) - 2}{2} = \text{up}_V(n) - 1$.

In both cases, we obtain the candidate point $P_1 = (\text{up}_P(n), \text{up}_V(n) - 1)$. 

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When $P$ is equal to $\text{up}_P(n) - 1$, then by (12), $V$ is $\frac{n-1+(n-1) \mod 2}{2}$; we consider two cases:

1. If $(n-1) \mod 2 = 0$, then $\frac{n-1+(n-1) \mod 2}{2} = \frac{n-1}{2} = \text{up}_V(n)$ and we obtain the candidate point $P_2 = (\text{up}_P(n) - 1, \text{up}_V(n))$.

2. If $(n-1) \mod 2 = 1$, then $\frac{n-1+(n-1) \mod 2}{2} = \frac{(n-2)+2}{2} = \text{up}_V(n) + 1$ and we obtain the candidate $(\text{up}_P(n) - 1, \text{up}_V(n) + 1)$. This candidate is not feasible since its second coordinate is strictly greater than the maximum value of the second coordinate of any feasible point.

Hence, for the case $(n-1) \mod 2 = 0$, we obtain two distinct candidate points $P_1$ and $P_2$ located on the straight line $P + V = n - 2$. To prove that $P_2 = (\text{up}_P(n) - 1, \text{up}_V(n))$ is feasible, we construct and intersect the automata for the $R_1 = \text{up}_P(n)$, $R_2 = \text{up}_V(n) - 1$, and $(n-1) \mod 2 = 0$ conditions, and observe that the intersection has a cycle of length 2, which implied the feasibility of $P_2$ for any odd sequence size. The same procedure is used for proving the feasibility of $P_1$ for any odd sequence size.

Since both $P_1$ and $P_2$ lay on the straight line $R_1 + R_2 = n - 2$, and are feasible for any odd length, then the straight line $R_1 + R_2 = n - 2$ is a facet of the convex hull of feasible points, when $n$ is odd. 

5 Synthesising Parameterised Non-Linear Invariants

The contribution of this section is a methodology for two families of time-series constraints, namely the \texttt{NB} \_ \texttt{σ} and the \texttt{SUM} \_ \texttt{WIDTH} \_ \texttt{σ} families, which both proposes conjectures and proves them automatically by using constant-size automata, i.e. automata whose number of states, and whose input alphabet size are independent both from an input time-series length and from the values in an input time series. For a conjunction of two time-series constraints $\gamma_1(X, R_1)$ and $\gamma_2(X, R_2)$ imposed on the same time series $X = (X_1, X_2, \ldots, X_n)$, our method describes sets of infeasible result-value pairs for $(R_1, R_2)$. We assume that every time-series constraint mentioned in this section belongs either to the \texttt{NB} \_ \texttt{σ} or to the \texttt{SUM} \_ \texttt{WIDTH} \_ \texttt{σ} family. Each set of infeasible pairs is described by a formula $f_i(R_1, R_2, n)$ expressed as a conjunction $C_i^1 \land C_i^2 \land \cdots \land C_i^{f_i}$ of elementary conditions $C_i^j$ between $R_1$, $R_2$ and $n$. The learned Boolean function $f_1 \lor f_2 \lor \cdots \lor f_m$ represents the union of sets of infeasible pairs $(R_1, R_2)$, while its negation $\neg f_1 \land \neg f_2 \land \cdots \land \neg f_m$ corresponds to an implied constraint, which is a universally true Boolean formula, namely

$$\forall X, \gamma_1(X, R_1) \land \gamma_2(X, R_2) \Rightarrow \bigwedge_{i=1}^{m} \neg f_i(R_1, R_2, n) \tag{13}$$

In order to prove that (13) is universally true we need to show that for every $f_i(R_1, R_2, n)$, there does not exist a time series of length $n$ yielding $R_1$ (resp. $R_2$) as the result value of $\gamma_1$ (resp. $\gamma_2$) and satisfying $f_i(R_1, R_2, n)$. The key idea of our proof scheme is to represent the infinite set of time series satisfying each elementary condition $C_i^j$ of $f_i(R_1, R_2, n)$ as a constant-size automaton $M_i,j$. Then checking that the intersection of all automata $M_{i,1}, M_{i,2}, \ldots, M_{i,k}$ is empty implies that $f_i(R_1, R_2, n)$ is indeed infeasible. Note that such proof scheme is independent of the time-series length $n$; moreover, it does not explore any search space.

As for the linear invariants, the generation process of non-linear invariants is offline: it is done once and for all to build a reusable database of generic invariants. This section is organised as follows:

- Section 5.1 motivates this work with a running example, which illustrates the need for deriving non-linear invariants.
- Section 5.2 presents our method for deriving non-linear invariants for a conjunction of time-series constraints. It starts with an overview of the three phases of our method, and then details each phase:

1. A generating data phase is detailed in the introduction of Section 5.2. Its goal is to generate a dataset, from which we will extract non-linear invariants.
Fig. 5: Feasible points, shown as blue squares, for the result variables $R_1, R_2$ of the conjunction of `sum_width_decreasing_sequence(X, R_1)` and `sum_width_zigzag(X, R_2)` on the same time series $X = \langle X_1, X_2, \ldots, X_n \rangle$ for the values of $n$ in $\{9, 10, 11, 12\}$; red circles represent infeasible points inside the convex hull of feasible points.

2. A mining phase is detailed in Section 5.2.1. It extracts, from the data generated in the mining phase, a hypothesis $H$ consisting of Boolean functions of the form $f_1 \lor f_2 \lor \cdots \lor f_m$.

3. A proof phase is detailed in Section 5.2.2. For every Boolean function $f_i$ (with $i \in [1, m]$) in the extracted hypothesis $H$, the proof phase either proves its validity for every time-series length, or refute it by generating a counter example. The counter example is used to modify the current hypothesis and the process is repeated.

Note that our generated data is noise-free, and that our goal is not to discover statistical properties of time-series constraints, but rather to extract non-linear invariants, which are always true.

5.1 Motivation and Running Example

Consider a conjunction of time-series constraints $\gamma_1(X, R_1) \land \gamma_2(X, R_2)$ imposed on the same time series $X$. In Section 4, using the representation of $\gamma_1$ and $\gamma_2$ as register automata, we presented a method for deriving parameterised linear invariants linking the values of $R_1, R_2$. Although, in most cases the derived inequalities were proven to be facet-defining, we observe that in some cases, even when using these invariants, the solver could still take a lot of time to find a feasible solution or to prove infeasibility. This happens because of some infeasible combinations of values of the result variables that were located inside the convex hull of all feasible combinations. The following example illustrates such a situation.

**Example 13 (running example)** Consider the conjunction of `sum_width_decreasing_sequence(X, R_1)` and `sum_width_zigzag(X, R_2)` time-series constraints imposed on the same time series $X$ of length $n$, where a decreasing sequence and a zigzag respectively correspond to `‘(> (> | =)*)’` and `‘(><)’`. For the values of $n$ in the interval $[9, 12]$, Figure 5 represents feasible pairs of $(R_1, R_2)$ as blue squares, and infeasible pairs lying inside the convex hull of feasible (blue) points as red circles. The convex hull contains a significant number of infeasible (red) points, which we want to characterise automatically.

Next section develops a systematic approach for generating non-linear invariants characterising infeasible combinations of $R_1$ and $R_2$ located within the convex hull of feasible combinations.
5.2 Discovering and Proving Invariants

Consider a conjunction of time-series constraints $\gamma_1(X, R_1)$ and $\gamma_2(X, R_2)$ imposed on the same time series $X$. This work focuses on automatically extracting and proving invariants that characterise some subsets of infeasible combinations of $R_1$ and $R_2$ that are all located inside the convex hull of feasible combinations of $R_1$ and $R_2$. Our approach uses three sequential phases.

- [Generating Data Phase] The first phase is a preparatory work, namely generating data. For each time-series length $n$ in [7, 12], we generate all feasible combinations of the values of $R_1$ and $R_2$. For each of the 6 lengths, (i) we compute the convex hull of feasible points of $R_1$ and $R_2$ using Graham’s scan [26], and (ii) we detect the set $I$ of infeasible combinations of $R_1$ and $R_2$ in this convex hull.

- [Mining Phase] The second phase, called the mining phase, consists of extracting a hypothesis $H$ describing the set $I$ of infeasible combinations of $R_1$ and $R_2$ from the generated data. We represent this hypothesis as a disjunction of Boolean functions $f_i(R_1, R_2, n)$.

- [Proof Phase] The third phase, called the proof phase, consists in refining the discovered hypothesis $H$ by validating some Boolean functions $f_i$ and by refuting and eliminating others using constant-size automata. A refined hypothesis, which is proved to be correct in the general case, i.e. for any time-series length, is called a description of the set $I$.

5.2.1 Mining Phase

Consider a conjunction of two time-series constraints $\gamma_1(X, R_1)$ and $\gamma_2(X, R_2)$, imposed on the same time series $X$. This section shows how to extract a hypothesis in the form of a disjunction of Boolean functions, describing the infeasible combinations of values of $R_1$ and $R_2$ that are located within the convex hull of feasible combinations.

There exist a number of works on learning a disjunction of predicates [14], and some special case, where disjunction corresponds to a geometric concept [15, 17]. Usually, the learner interacts with an oracle through various types of queries or with the user by receiving positive and negative examples; the learner tries to minimise the number of such interactions to speed up convergence.

In our case, the input data consists of the set of positive, called infeasible, and negative, called feasible, examples, which is finite and which is completely produced by our generating phase. This allows exploring all possible inputs without any interaction.

We now present the components of our mining phase:

- First, we describe our dataset, which consists of feasible and infeasible pairs of the result values $R_1, R_2$.
- Second, we define the space of concepts, hypotheses, we can potentially extract from our dataset.
- Third, we outline the target hypothesis for time-series constraints, i.e. what we are searching for.
- Finally, we briefly describe the algorithm used for finding the target hypothesis.

**Input Dataset** We represent our generated data as the union of two sets of triples $D^+$ (resp. $D^-$) called the set of feasible (resp. infeasible) examples, such that:

- For every $(k, p_1, p_2)$ (with $k \in [7, 12]$) in $D^+$, there exists at least one time series of length $k$ that yields $p_1$ and $p_2$ as the values of $R_1$ and $R_2$, respectively.
- For every $(k, p_1, p_2)$ (with $k \in [7, 12]$) in $D^-$,
  1. there does not exist any time series of length $k$ that would yield $p_1$ and $p_2$ as the values of $R_1$ and $R_2$, respectively.
  2. $(p_1, p_2)$ is located within the convex hull of feasible combinations of $R_1$ and $R_2$. 

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**Space of Hypotheses** Every element of our hypothesis space is a disjunction of Boolean functions from a finite predefined set $H$. Each element of $H$ is a conjunction $C_1 \land C_2 \land \cdots \land C_p$ with every $C_i$ being a predicate, called an atomic relation, where the main atomic relations are:

\[
\begin{align*}
(i) \quad & n \geq c, \\
(ii) \quad & n \mod c = d, \\
(iii) \quad & R_j \mod c = d, \\
(iv) \quad & R_j \geq d, \\
(v) \quad & R_j \leq d, \\
(vi) \quad & R_j = c, \\
(vii) \quad & R_j = R_k + d,
\end{align*}
\]

with $c$ and $d$ being natural numbers, and $\uparrow R_k(n)$ being the maximum possible value of $R_k$ given the constraint $\gamma_k(X_1, X_2, \ldots, X_n, R_k)$. The intuition of these atomic relations is now explained:

- (i) stands from the fact that many invariants are only valid for long enough time series.
- (ii) is motivated by the fact that the parity of the length of a time series is sometimes relevant.
- (iii) is justified by the fact that the parity of $R_1$ or $R_2$ can come into play.
- (iv) and (v) are related to the fact that infeasible combinations of $R_1$ and $R_2$ can be located on a ray or an interval.
- (vi) and (vii) are respectively linked to the fact that quite often infeasible combinations of $R_1$ and $R_2$ within the convex hull are very close to the minimum or the maximum values $[5]$ of $R_k$ (with $k \in [1, 2]$), i.e. $c$ is a very small constant, typically 0 or 1.
- (viii) denotes the fact that some invariants correspond to a linear combination of $R_1$ and $R_2$.

**Target Hypothesis**

**Definition 6 (Boolean function consistent wrt a dataset)** A Boolean function of $H$ is consistent wrt a dataset $D$ if it is true for at least one infeasible example of $D$, and false for every feasible example of $D$.

For example, $R_1 = R_2 \land R_1 \mod 2 = 1$ is consistent with the dataset of Figure 5, but the two Boolean functions $R_1 = 13$ and $R_1 = R_2$ are not.

**Definition 7 (universally true Boolean function)** A Boolean function of $H$ is universally true if it is true for any time series of any length.

**Definition 8 (target hypothesis)** The target hypothesis $H$ is the disjunction of all Boolean functions of $H$ consistent with $D$.

Note that in the target hypothesis some Boolean functions can be subsumed by other Boolean functions. We cannot do the subsumption analysis at this point since we do not yet know which Boolean functions are true or not.

**Mining Algorithm** Our mining algorithm filters out all the Boolean functions not consistent with our dataset and returns the disjunction of the remaining Boolean functions. Note that the mining algorithm ignores Boolean functions involving the atomic relation (i) $n > c$, which is handled in the proof phase. Remember that we run the algorithm only on the limited dataset $D_{[7, 12]}$, i.e. the dataset generated from time series of length in $[7, 12]$. This is because sizes that are too small lead to degenerate polytopes, while sizes that are too large are too expensive in terms of computation.
5.2.2 Proof Phase

After extracting from $\mathcal{D}_{1,12}$ the target hypothesis $H = f_1 \lor f_2 \lor \cdots \lor f_m$ characterising subsets of infeasible combinations of $R_1$ and $R_2$ that are all located within the convex hull of feasible combinations of $R_1$ and $R_2$, we refine this hypothesis, by keeping only universally true Boolean functions $f_i$.

Before presenting our proof technique, we look at the structure of the hypothesis $H$. Every Boolean function $f$ in $H$ is of the form $f = C_1 \land C_2 \land \cdots \land C_p$ and can be classified into one of the two following categories:

- **Independent Boolean Function** means that every $C_i$ is an *independent atomic relation*, i.e. depends either on $R_1$ or $R_2$, but not on both. For instance, $R_1 = \text{up}_{R_1}(n) \land R_2 \mod 2 = 1$ is an independent Boolean function.

- **Dependent Boolean Function** means that there exists at least one $C_i$ that is a *dependent atomic relation*, i.e. mentions both $R_1$ and $R_2$. For instance, $R_1 \mod 2 = 1 \land R_1 = R_2 + 1$ is a dependent Boolean function.

The proof of an invariant depends on its category. We now show how to prove that an independent (resp. dependent) Boolean function is universally true.

**Proof of Independent Boolean Functions** Since most atomic relations are independent, i.e. cases (i) to (vii), we first focus on a necessary and sufficient condition for proving that an independent Boolean function is universally true. Such necessary and sufficient condition is given in the main result of this section, namely Theorem 3, provided that there exists constant-size automata associated with the atomic relations in $f$.

**Definition 9 (set of supporting signatures for an atomic relation)** For an atomic relation $C$, the *set of supporting signatures* $T_C$ is the set of words in $\Sigma^*$ such that, for every word in $T_C$ there exists a time series satisfying $C$, whose signature is this word.

**Definition 10 (set of supporting signatures for a Boolean function)** For an independent Boolean function $f = C_1 \land C_2 \land \cdots \land C_p$, we define the *set of supporting signatures* $T_f$ as $\bigcap_{i=1}^{p} T_{C_i}$.

A Boolean function $f$ is universally true iff it describes infeasible combinations of $R_1$ and $R_2$ for any time-series length, and thus the set $T_f$ is empty.

For any atomic relation $C$ from (i) to (vii), i.e. an independent atomic relation, the corresponding set of supporting signatures is represented as the language of a constant-size automaton $M_C$. Constant size means that the number of states of this automaton does not depend on the length of the input time series. For a Boolean function $f = C_1 \land C_2 \land \cdots \land C_p$, $T_f$ is simply the set of signatures recognised by the automaton obtained after intersecting all $M_{C_i}$ (with $i \in [1, p]$). This provides a necessary and sufficient condition for proving that a Boolean function $f$ is universally true.

**Theorem 3 (necessary and sufficient condition for an independent Boolean function to be universally true)** Consider two time-series constraints $\gamma_1(X, R_1)$ and $\gamma_2(X, R_2)$ on the same time series $X$, and a Boolean function $f(R_1, R_2, n) = C_1 \land C_2 \land \cdots \land C_p$ such that, for every $C_i$ there exists a constant-size automaton $M_{C_i}$. The function $f$ is universally true iff the intersection of all automata for $M_{C_i}$ (with $i \in [1, p]$) is empty.
The proof of Theorem 3 follows from Definitions 9 and 10.

For some Boolean function \( f = C_1 \land C_2 \land \cdots \land C_p \), the set \( T_f = \bigcap_{i=1}^{p} T_{C_i} \) may not be empty, but finite. In this case, we compute the length \( c \) of the longest signature in \( T_f \), and obtain a new Boolean function \( f' = f \land n \geq c + 1 \). By construction, the set \( T_{f'} \) is empty, thus \( f' \) is universally true.

Section 6 will further show how to generate automata for independent atomic relations. Every such automaton is called a conditional automaton.

**Proof of Dependent Boolean Functions** Some dependent Boolean functions, i.e. case (viii), can be handled by adapting the technique for generating linear invariants described in Section 4.

Consider two time-series constraints \( \gamma_1(X, R_1) \) and \( \gamma_2(X, R_2) \) on the same time series \( X \). We present here a method for verifying that the dependent Boolean function \( R_1 \lor d \cdot R_2 = 1 \), with \( d \) being either 1 or 2, is universally true. Note that such Boolean function was extracted during the mining phase for 17 pairs of time-series constraints.

We prove by contradiction that the corresponding Boolean function is universally true. Our proof consists of the following steps:

1. **Assumption.** Assume that there exists a time series \( X \) such that \( R_1 \lor d \cdot R_2 = 1 \).
2. **Implication for the parity of \( R_1 \) and \( d \cdot R_2 \).** When \( R_1 \lor d \cdot R_2 = 1 \), then \( R_1 \) and \( d \cdot R_2 \) have different parity.
3. **Obtaining a contradiction.** Since \( R_1 \) and \( d \cdot R_2 \) must have different parity, there exists a value of \( b \) that is either 0 or 1 such that the conjunction \( R_1 \lor d \cdot R_2 = 1 \land R_1 \mod 2 = b \land d \cdot R_2 \mod 2 = 1 - b \) holds. In order to prove that \( R_1 \lor d \cdot R_2 = 1 \) is infeasible, for either value of parameter \( b \), we need to show that, either the obtained conjunction is infeasible, e.g. when \( d = 2 \) and \( b = 0 \), or the method of Section 4 produces a linear invariant \( R_1 \lor d \cdot R_2 \geq c \), with \( c \) being strictly greater than 1.

If at this third step of our proof method the considered conjunction is feasible, and the desired invariant \( R_1 \lor d \cdot R_2 \geq c \) was not obtained, then we cannot draw any conclusion about the infeasibility of \( R_1 \lor d \cdot R_2 = 1 \).

In practice, for the 17 pairs of time-series constraints, for which we extracted the Boolean function \( R_1 \lor d \cdot R_2 = 1 \), the method of Section 4 did indeed generate the desired linear invariant, which proved that the considered Boolean function is universally true.

**Example 14 (mining, proving and filtering non-linear invariants for the running example)** Consider the conjunction of the \texttt{sum\_width\_decreasing\_sequence}(\( X, R_1 \)) and \texttt{sum\_width\_zigzag}(\( X, R_2 \)) time-series constraints on the same time series \( X \), introduced in Example 13. For this conjunction, we now describe the result of the mining and the proving phases of our method, as well as the dominance filtering, i.e. discarding Boolean functions subsumed by some other Boolean function.

- During the mining phase we extracted a disjunction of 156 Boolean functions. Most Boolean functions, even if they are true, are redundant. For example, the Boolean function \( R_1 = 1 \land R_2 = 1 \) is subsumed by \( R_1 = 1 \) and thus can be discarded. However, at this point we cannot do the dominance filtering since we do not yet know which Boolean functions are universally true.
- During the proof phase we proved that 95 out of the extracted 156 Boolean functions are universally true.
- Finally, after the dominance filtering of the 95 proved Boolean functions we obtain the disjunction of the following seven Boolean functions:
  \begin{align*}
  &1 \quad R_1 = 1, \\
  &2 \quad R_2 = 1, \\
  &3 \quad R_1 = 3 \land R_2 \geq 1, \\
  &4 \quad R_1 = \text{up}_{R_1}(n) \land R_2 \mod 2 = 1, \\
  &5 \quad R_1 \mod 2 = 1 \land R_1 = R_2, \\
  &6 \quad n \mod 2 = 0 \land R_1 = \text{up}_{R_1}(n) - 1 \land R_2 = \text{up}_{R_2}(n).
  \end{align*}
Fig. 6: Seven groups of infeasible combinations of $R_1$ and $R_2$, where $R_1$ and $R_2$ are, respectively, constrained by $\text{sum\_width\_decreasing\_sequence}(X, R_1)$ and $\text{sum\_width\_zigzag}(X, R_2)$ on the same sequence $X$ of length 9 (all plots except the two plots at the bottom right) and of lengths 10 and 12 (the two plots at the bottom right).

All four upper plots and the two lower plots on the left of Figure 6 contain the groups of infeasible combinations of $R_1$ and $R_2$, corresponding to the Boolean functions from $\text{x}$ to $\text{|}$ for $n$ being 9. The two lower plots on the right of Figure 6 contain the infeasible combinations of $R_1$ and $R_2$, corresponding to the $\text{~}$ Boolean function for $n$ being 10 and 12, respectively.

The Boolean functions from $\text{x}$ to $\text{|}$ and $\text{~}$ were proved by intersecting the automata for the atomic relations in these Boolean functions, and check that it was empty.

In order to prove the dependent Boolean function $\text{~}$, we consider the conjunction of three constraints, namely $R_1 \equiv 1$, $\text{sum\_width\_decreasing\_sequence}$, and $\text{sum\_width\_zigzag}$. Each of the three constraints can be represented by an automaton or by a register automaton satisfying the required properties of the method of Section 4, which generates for this conjunction the invariant $R_1 \geq R_2 + 2$. This proves that $\text{~}$ is a universally true Boolean function.

We now give an interpretation of five of those Boolean functions:

- $\text{x}$ and $\text{y}$ means that, in the languages of $\text{decreasing\_sequence}$ and $\text{zigzag}$, respectively, there is no word consisting of one letter.
- $\text{|}$ means that, when a time series yields up$_{R_1}(n)$ as the value of $R_1$, every occurrence of zigzag in its signature must start and end with ‘>’, and the length of every word in the language of zigzag starting and ending with the same letter is even.
- $\text{~}$ is related to the fact that every word in the language of zigzag contains at least one word of the language of decreasing_sequence as a factor, and every such factor is of even length.
meaning, when a time series yields \( u_{R_i}(n) \) as the value of \( R_2 \), then its signature is a word in the language of zigzag, and every occurrence of decreasing_sequence is of even length, and thus \( R_1 \) must be even. At the same time, \( u_{R_i}(n) - 1 = n - 1 \) is odd, when \( n \) is even.

6 Synthesising Conditional Automata

For the time-series constraints considered in this work we need to generate constant-size finite automata representing a certain condition, e.g. an automaton recognising the signatures of all and only all time series with the maximum number of peaks. Such automata are required for proving non-linear invariants parameterised by the time-series length, described in Section 5, and also for the facet analysis of linear invariants, described in Section 4.6. This section shows how to synthesise a constant-size automaton, i.e. an automaton whose number of states is independent, both from the input time-series length and from the values in an input time series, accepting the signatures of all, and only all, time series satisfying atomic relations of Section 5.2.1. For brevity, we only consider the atomic relation (vii) \( R = u_{R_i}(n) - d \), where \( R \) is constrained by some time-series constraint \( \gamma((X_1, X_2, \ldots, X_n), R) \), with \( \gamma \) being nb_\( \sigma \) or sum_width_\( \sigma \), and where \( u_{R_i}(n) \) is the maximum possible value of \( R \) yielded by a time series of length \( n \). This atomic relation is indeed the most difficult case for generating a constant-size automaton. The construction associated with other atomic relations are described in [2]. We start with an illustrative example.

Example 15 (automaton for a gap atomic relation) Consider the nb_peak\((X_1, X_2, \ldots, X_n), R\) time-series constraint and a gap atomic relation \( C \) defined by \( R = u_{R_i}(n) \). We showed in [5] that the maximum value of \( R \) for a given time-series length \( n \) is \( \max(0, \lfloor \frac{n+1}{2} \rfloor) \). Hence, the automaton for \( C \) must recognise the signatures of all and only time series yielding \( \max(0, \lfloor \frac{n+1}{2} \rfloor) \) as the value of \( R \).

Part (A) of Figure 7 gives the minimal automaton accepting the set of signatures reaching this upper bound, while Part (B) lists all words of length 4 and 5 over the alphabet \{‘<’, ‘=’, ‘>’\} having the maximum number of peaks, 2 in this case, that can be obtained from the corresponding automaton.

Fig. 7: (A) Automaton achieving the maximum number of peaks in a time series of length \( n \), i.e. \( \max(0, \lfloor \frac{n+1}{2} \rfloor) \), and (B) all corresponding accepted words for \( n - 1 \in \{4, 5\} \), where each peak is surrounded by two vertical bars, and is highlighted in yellow. (C) The signatures of time series with gap 1 and 2, and with loss 3 and 5.

The rest of this section is organised as follows:

- **[Gap Automaton]** In the context of time-series constraints of the form nb_\( \sigma \) or sum_width_\( \sigma \), Section 6.1 first introduces the notion of gap of a time series \( X \), which indicates how far apart the result value of a time-series constraint yielded by \( X \) is from the given upper bound; it then presents the main contribution of this section, namely, the notion of \( \delta \)-gap automaton for a time-series constraint, i.e. a constant-size automaton that only accepts integer sequences whose gap is \( \delta \). Second, it gives
a sufficient condition on the time-series constraint for the existence of such an automaton. Third, it describes how to synthesise such δ-gap automaton.

1. Section 6.1.1 introduces an intermediate notion, the *loss of a time series* wrt a time-series constraint, which is the maximum difference between the length of this time series and the length of the shortest time series yielding the same result value of a time-series constraint. For example, all words of length 4 (resp. 5) in Part (B) of Figure 7 are the signatures of time series whose gap is 0 and whose loss is 0 (resp. 1). Part (C) of Figure 7 gives two signatures of time series with gap (resp. loss) 1 and 2 (resp. 3 and 5).

Finally, it introduces the notion of *loss automaton*, i.e. a register automaton used to compute the loss. How to synthesise a loss automaton will be explained in Section 6.2.

2. Section 6.1.2 introduces a sufficient condition in the form of a conjunction of four conditions on a time-series constraint, called *principal conditions* that, when satisfied, guarantee the existence of the δ-gap automaton.

   - When the first three principal conditions hold, describing the set of time series whose gap is δ is equivalent to describing the set of time series whose loss belongs to a certain interval, depending on δ.
   - When the fourth principal condition holds, there exists a loss automaton whose registers can either be monotonously increased or reset to a natural number.

3. For a given time-series constraint satisfying the four principal conditions and for any non-negative integer δ, Section 6.1.3 constructively proves the existence of the δ-gap automaton, i.e. assuming the loss automaton is known it shows how to construct the δ-gap automaton.

   - [Loss Automaton] For space reason Section 6.2 focuses only on the construction of the loss automaton for the *nb_*σ family, the construction for the *sum_width_*σ family being described in [2].

   It introduces a sufficient condition on a regular expression σ such that, when σ satisfies this condition, the *nb_*σ family satisfies the principal conditions of Section 6.1.2. It also shows how to obtain a loss automaton for a *nb_*σ time-series constraint from the seed transducer [7] for σ. The main idea is to compute the regret of every transition of the seed transducer as a special case of minimax regret [23, 32] from decision theory, which gives the minimum additional cost to pay when one action is chosen instead of another. In CP, the minimax regret has been used for assessing an extra cost when a variable is assigned to a given value [12].

6.1 Synthesising a δ-gap Automaton for a Time-Series Constraint

We present the main contribution of this section namely a systematic method for deriving a δ-gap automaton for a time-series constraint, see Definition 12, satisfying certain conditions that will be given in Definition 16. We first introduce the *gap of a ground time series* in Definition 11, and the *δ-gap automaton for a time-series constraint* in Definition 12. Let \( S \) denote the set of time-series constraints of the *nb_*σ and *sum_width_*σ families.

**Definition 11 (gap of a ground time series)** Consider a time-series constraint \( \gamma \) and a ground time series \( X \) of length \( n \). The *gap* of \( X \) wrt \( \gamma \), denoted by \( \text{gap}_\gamma(X) \), is a function that maps an element of \( S \times \mathbb{Z}^+ \) to \( \mathbb{N} \). It is the difference between the maximum value of \( R \) that could be yielded by a time series of length \( n \), and the value of \( R \) yielded by \( X \).

Example 17 will illustrate the notion of gap for different time series.

**Definition 12 (δ-gap automaton)** Consider a time-series constraint \( \gamma \) and a natural number \( \delta \). The *δ-gap automaton for \( \gamma \)* is a minimal automaton that accepts the signatures of all, and only all, ground time series whose gap wrt \( \gamma \) is \( \delta \).


Definition 16 will further give a sufficient condition on a time-series constraint \( \gamma \) for the existence of a \( \delta \)-gap automaton for \( \gamma \).

**Example 16 (0-gap automaton)** The 0-gap automaton for \( \text{nb}_\text{peak} \) was given in Part (A) of Figure 7. It only recognises the signatures of ground time series containing the maximum number of peaks. \( \triangle \)

To construct the \( \delta \)-gap automaton for a time-series constraint \( \gamma \) we introduce the notion of *loss of a time series*. For a time series of length \( n \), its loss is the difference between \( n \) and the length of a shortest time series yielding the same result value of \( \gamma \). The main idea of our method for generating \( \delta \)-gap automata is that by knowing the loss of a time series, and whether it contains at least one \( \sigma \)-pattern or not, we can determine its gap.

We now describe how to derive the \( \delta \)-gap automaton for a time-series constraint \( \gamma \).

### 6.1.1 Defining the Loss and the Loss Automaton

Consider a time-series constraint \( \gamma \) and a natural number \( \delta \). Definition 13 introduces the *loss of a time series* wrt \( \gamma \), and Definition 14 presents the notion of *loss automaton* for \( \gamma \).

**Definition 13 (loss of a time series)** Consider a time-series constraint \( \gamma \) and a ground time series \( X \) of length \( n \). The *loss of \( X \) wrt \( \gamma \)*, denoted by \( \text{loss}_\gamma(X) \), is a function that maps an element of \( S \times Z^* \) to \( \mathbb{N} \). It is the difference between \( n \) and the length of a shortest time series that yields the same result value of \( \gamma \) as \( X \).

**Example 17 (gap and loss of a time series)** Now we illustrate the computation of the gap and the loss. Consider the \( \text{nb}_\text{peak} \) time-series constraint. From [5], the maximum number of peaks in a time series of length \( n \) is \( \max(0, \lfloor \frac{n-1}{2} \rfloor) \).

- The time series \( X^1 = (1, 2, 1, 2, 1, 2, 1) \) has a gap of 0 since it contains three peaks, which is maximum, and a loss of 0 since any shorter time series has a smaller number of peaks.
- The time series \( X^2 = (1, 2, 1, 2, 1, 1, 1, 1) \) has a gap of 1 since it has only two peaks, when three is the maximum, and a loss of 3 since a shortest time series with 2 peaks is of length 5.
- The time series \( X^3 = (1, 1, 1, 0, 0, 1, 1, 1, 1) \) has a gap of 4 since it has no peaks, when the maximum is 4, and a loss of 8 since a shortest time series without any peaks is of length 1. \( \triangle \)

**Definition 14 (loss automaton for a time-series constraint)** Consider a time-series constraint \( \gamma \). A *loss automaton* for \( \gamma \) is a register automaton over the alphabet \( \{<,=,>\} \) with a constant number of registers such that, for any ground time series \( X \), it returns \( \text{loss}_\gamma(X) \) after having consumed the signature of \( X \).

For the \( \text{nb}_\sigma \) and \( \text{sum}_\text{width}_\sigma \) families, a loss automaton can be synthesised from the seed transducer of the regular expression \( \sigma \). For the \( \text{nb}_\sigma \) family, this will be explained in Section 6.2.

### 6.1.2 Principal Conditions for Deriving a \( \delta \)-Gap Automaton

Consider a \( g\_f\_\sigma \) time-series constraint, denoted by \( \gamma \), and a natural number \( \delta \). Definition 16 formulates a sufficient condition, consisting of a conjunction of four conditions, named *principal conditions*, for the existence of the \( \delta \)-gap automaton for \( \gamma \). The first three principal conditions express the idea that, knowing the loss of a time series and, whether it has at least one \( \sigma \)-pattern or not, fully determines the gap of this time series. The fourth condition requires the existence of a loss automaton \( M \) for \( \gamma \), whose registers may either monotonously increase, or be reset to a natural number, and each accepting state of \( M \) either accepts only signatures with at least one occurrence of \( \sigma \), or accepts only signatures without any occurrence of \( \sigma \).
Before formulating the principal conditions, Definition 15 introduces the notions of before-found and after-found state of a loss automaton.

**Definition 15 (before-found and after-found states)** Consider a loss automaton $\mathcal{M}$ for the $g \_ f \_ \sigma$ time-series constraint. An accepting state $q$ of $\mathcal{M}$ is a before-found (resp. after-found) state, if there exists a time series $X$ without any $\sigma$-patterns (resp. with at least one $\sigma$-pattern) such that, after having consumed the signature of $X$, $q$ is the final state of $\mathcal{M}$.

Note that an accepting state of a loss automaton can have both statuses.

**Definition 16 (principal conditions)** Consider a $\gamma(X, R)$ time-series constraint. The **four principal conditions on $\gamma$** are defined as follows:

1. **Gap-to-loss condition.** There exists a function $h_\gamma: \mathbb{S} \times \mathbb{N} \times \{0,1\} \times \mathbb{N} \rightarrow \mathbb{N}$, called the gap-to-loss function, such that for any ground time series $X = \langle X_1, X_2, \ldots, X_n \rangle$, we have $\text{loss}\_\gamma(X)$ being equal to $h_\gamma(\text{gap}\_\gamma(X), \text{sgn}(R), n)$, where $\text{sgn}$ is the signum function. Hence, in order to compute the loss of a ground time series it is enough to know (i) its gap, (ii) whether it has at least one $\sigma$-pattern or not, and (iii) the length of this time series.

2. **Boundedness condition.** For given values of $\text{gap}\_\gamma(X)$ and $\text{sgn}(R)$, and for any $n$ in $\mathbb{N}$, the value of the gap-to-loss function $h_\gamma(\text{gap}\_\gamma(X), \text{sgn}(R), n)$ belongs to a bounded integer interval, called the loss interval $\sigma$ of $(\text{gap}\_\gamma(X), \text{sgn}(R))$.

3. **Disjointedness condition.** For a given value of $\text{sgn}(R)$, and two different values of gap, $\delta_1$ and $\delta_2$, the loss intervals $\sigma$ (with $\text{sgn}(R)$) and $\sigma$ (with $\text{sgn}(R)$) are disjoint.

4. **Loss-automaton condition.** There exists a loss automaton $\mathcal{M}$ for $\gamma$ satisfying all the following conditions:
   (a) Every register update of $\mathcal{M}$ has one of the following forms:
      i. The register is incremented by a natural number, or by the value of another register.
      ii. The value of the register is reset to a natural number.
   (b) The initial values of the registers of $\mathcal{M}$ are natural numbers.
   (c) The acceptance function of $\mathcal{M}$ is a weighted sum with natural number coefficients of the last values of the registers of $\mathcal{M}$ after having consumed an input signature.
   (d) The sets of before-found states and after-found states of $\mathcal{M}$ are disjoint. It means that, by knowing the final state of $\mathcal{M}$ after having consumed the signature of any ground time series $X$, we also know the value of $\text{sgn}(R)$ yielded by $X$.

Conditions 1., 2., 3. are called the gap-loss-relation conditions. Conditions 4a, 4b, 4c are called the non-negativity conditions, while Condition 4d is called the separation condition on $\mathcal{M}$.

**Example 18 (principal conditions)** Consider a $\gamma(X, R)$ time-series constraint. For the time series $X^1$, $X^2$, and $X^3$ of Example 17, Figure 8 shows the relation between the gap, the loss, the time-series lengths, and $R$ when $\gamma$ is NB\_peak. For any time series $X^i$ (with $i \in [1,3]$) of length $n_i$ yielding $R_i$ as the value of $R$, its gap (resp. loss) is equal to the length of the violet (resp. blue) dotted line segment starting from the point $X^i$ of coordinates $(n_i, R_i)$. Note that the boundedness and the disjointedness conditions are satisfied for NB\_peak. \hfill $\triangle$

### 6.1.3 Synthesising the $\delta$-Gap Automaton

Consider a $\gamma$ time-series constraint satisfying all four principal conditions of Section 6.1.2, and a natural number $\delta$. We prove that the $\delta$-gap automaton for $\gamma$ exists. First, Lemma 1 states a necessary and sufficient
Theorem 4 (existence of the \( \gamma \text{-gap relation} \)) Let us denote by \( \delta \)-gap automaton for \( \gamma \) exists.

Lemma 1 (relation between gap and loss) Consider a \( \gamma(X, R) \) time-series constraint such that the gap-loss-relation conditions, see Definition 16, are all satisfied, and a natural number \( \delta \). Then, for a time series \( X \), \( \gamma(X) \) is \( \delta \)-gap lip \( \gamma(X) \) belongs to the loss interval \( \delta \text{-gap automaton} \) for \( \gamma \) exists.

Proof Let us denote by \( \mathcal{M} \) the loss automaton for \( \gamma \), satisfying the non-negativity and the separation conditions. Note that such automaton necessarily exists since the loss-automaton condition, see Condition 4 of Definition 16, is satisfied. We prove the theorem by explicitly constructing a constant-size automaton \( A_M \) using \( \mathcal{M} \); after minimising \( A_M \) we obtain the sought \( \delta \)-gap automaton.

[Construction of \( A_M \)] By Lemma 1, there exist a loss interval \( \mathcal{L}_{\delta, 0} \) wrt \( \langle \delta, 0 \rangle \) and a loss interval \( \mathcal{L}_{\delta, 1} \) wrt \( \langle \delta, 1 \rangle \) such that any ground time series \( X \), whose gap is \( \delta \), belongs to one of the following types:

- **Type 1.** The time series \( X \) has no \( \sigma \)-patterns and the value of \( \text{loss}_\gamma(X) \) is in \( \mathcal{L}_{\delta, 0} \).
- **Type 2.** The time series \( X \) has at least one \( \sigma \)-pattern and the value of \( \text{loss}_\gamma(X) \) is in \( \mathcal{L}_{\delta, 1} \).

Hence, our goal is to construct a constant-size automaton \( A_M \) that recognises the signatures of all, and only all, ground time series that belongs either to Type 1 or to Type 2.

Let \( (A_1, A_2, \ldots, A_p) \) denote the \( p \) registers of the loss automaton \( \mathcal{M} \), whose initial values are \( (v_1, v_2, \ldots, v_p) \), let \( \alpha(A_1, A_2, \ldots, A_p) \) denote the acceptance function of \( \mathcal{M} \), let \( \delta \) be the transition function of \( \mathcal{M} \), and let \( \phi \) be the maximum element in \( \mathcal{L}_{\delta, 0} \cup \mathcal{L}_{\delta, 1} \). Then, the states, the initial state, the accepting states, and the transitions of \( A_M \) are defined as follows:

- **States.** For every state \( q \) of \( \mathcal{M} \), there are \( (\phi + 2)^p \) states in \( A_M \), each of which is labelled with \( q_{i_1, i_2, \ldots, i_p} \), with every \( i_j \) (with \( j \in [1, p] \)) being in \( [0, \phi + 1] \).

- **Initial state.** If \( q_0 \) is the initial state of \( \mathcal{M} \), then \( q_0^{v_1, v_2, \ldots, v_p} \) is the initial state of \( A_M \).

- **Accepting states.** A state \( q_{i_1, i_2, \ldots, i_p} \) of \( A_M \) is accepting if and only if

  1. \( q \) is a before-found state of \( \mathcal{M} \) and the value of \( \alpha(i_1, i_2, \ldots, i_p) \) is within \( \mathcal{L}_{\delta, 0} \), or

  2. \( q \) is an after-found state of \( \mathcal{M} \) and the value of \( \alpha(i_1, i_2, \ldots, i_p) \) is within \( \mathcal{L}_{\delta, 1} \).

- **Transitions.** There is a transition from state \( q_{i_1, i_2, \ldots, i_p} \) (with \( i_1, i_2, \ldots, i_p \in [0, \phi + 1] \)) to state \( q_{k_1, k_2, \ldots, k_p} \) labelled with \( s \) in \( \{', '<', '==', '>', '\} \), if the value of the transition function \( \delta(q, i_1, i_2, \ldots, i_p) = s \) is equal to \( q^* \), \( (i_1^*, i_2^*, \ldots, i_p^*) \), where every \( k_j \) is equal to \( \min(\phi + 1, i_j^*) \), with \( j \) in \( [1, p] \).
[Interpretation of the states of $A_M$] If after consuming the signature of some ground time series, the automaton $A_M$ arrives in a state $q_{i_1,i_2,...,i_p}$, then after consuming the same signature, the loss automaton $M$ arrives in state $q$; for every $j \in [1,p]$, when $i_j \leq \phi$ (resp. $i_j = \phi + 1$), the register $A_j$ has value $i_j$ (resp. is strictly greater than $\phi$). Hence, the states of $A_M$ encode the register values of $M$ when consuming the same input signature.

[Size of $A_M$] By construction, the automaton $A_M$ has a constant size, i.e. its number of states is $m \cdot (\phi+2)^p$, where $m$, $p$ and $\phi$ are parameters, i.e. independent from the time-series length, respectively defined as:

- the number of states of $M$,
- the number of registers of $M$,
- the maximum value of $L_{\delta,0} \cup L_{\delta,1}$, where $L_{\delta,0}$ and $L_{\delta,1}$ are bounded intervals depending only on the constraint $\gamma$ and the gap $\delta$.

We explain why $A_M$ needs only $m \cdot (\phi+2)^p$ states to recognise the signatures of all, and only all, ground time series of either Type 1 or Type 2. By the boundedness condition (Condition 2 of Definition 16) and by definition of $\phi$, for any ground time series whose gap is $\delta$, its loss cannot exceed $\phi$. We show that if, when consuming the signature of some ground time series, the value of some register of $M$ becomes greater than $\phi$, then we no longer need to know its exact value.

Recall that the acceptance function $\alpha$ of $M$ is a weighted sum with natural coefficients of the last values of the registers of $M$. If, for a register $A_j$, the corresponding coefficient in $\alpha$ is zero, then it does not affect the value of $\alpha$, and the exact value of $A_j$ is irrelevant. Otherwise, once the value of $A_j$ exceeds $\phi$, the value of $\alpha$ also exceeds $\phi$, and the loss of such a time series is greater than $\phi$. By the non-negativity conditions, the value of $A_j$ exceeds $\phi$ if it can either increase even more, or it can be reset to a natural constant. In either case, the exact value of $A_j$ is irrelevant, and it is enough to know a lower bound, $\phi + 1$ of its value.

[Correctness of $A_M$] We now prove that the constructed automaton $A_M$ is sound, i.e. it recognises the signatures of all ground time series of either Type 1 or Type 2, and complete i.e. it recognises the signatures of all ground time series of either Type 1 or Type 2.

- **Soundness of $A_M$.** We prove the soundness of $A_M$ by contradiction. Assume there exists a ground time series $X$ recognised by $A_M$ and whose gap is not $\delta$. Let $q_{i_1,i_2,...,i_p}$ be the final state of $A_M$ after consuming the signature $S$ of $X$. Due to the non-negativity conditions, by construction of $A_M$ this means that, after consuming $S$, the register automaton $M$ finishes in the state $q$ of $M$, and for every $j \in [1,p]$, if $i_j \leq \phi$ (resp. $i_j = \phi + 1$), then the register $A_j$ has value $i_j$ (resp. is strictly greater than $\phi$). By the separation condition on $M$, the state $q$ of $M$ is either a before-found or an after-found state. Since $q_{i_1,i_2,...,i_p}$ is an accepting state of $A_M$, then either $q$ is a before-found state and $\alpha(i_1,i_2,...,i_p) \in L_{\delta,0}$, or $q$ is an after-found state and $\alpha(i_1,i_2,...,i_p) \in L_{\delta,1}$. In the former (resp. latter) case, $X$ belongs to Type 1 (resp. Type 2), and by Lemma 1, the gap of $X$ is $\delta$, a contradiction.

- **Completeness of $A_M$.** We prove the completeness of $A_M$ also by contradiction. Assume there exists a ground time series $X$ whose gap is $\delta$, i.e. it belongs either to Type 1 or to Type 2, but its signature $S$ is not recognised by $A_M$. Then,

1. either the final state $q_{i_1,i_2,...,i_p}$ of $A_M$ after consuming $S$ is not accepting,
2. or the automaton $A_M$ cannot consume the full signature $S$.

We show that both situations are impossible.

- **Impossibility of Situation 1.** Due to the non-negativity conditions, and by construction of $A_M$, after having consumed the signature of $X$, the automaton $M$ ends in state $q$ of $M$, and the value of the acceptance function is equal to $\alpha(i_1,i_2,...,i_p)$. Since the gap of $X$ is $\delta$, by Lemma 1 and by the separation condition, either $q$ is a before-found state of $M$ and $\alpha(i_1,i_2,...,i_p)$ belongs to $L_{\delta,0}$ or $q$ is an after-found state of $M$ and $\alpha(i_1,i_2,...,i_p)$ belongs to $L_{\delta,1}$. In either case, the state $q_{i_1,i_2,...,i_p}$ of $A_M$ must be accepting by construction, thus Situation 1 is impossible.
– **Impossibility of Situation 2.** Assume that (1) at a state \( q_{i_1,i_2,...,i_p} \) of \( A_M \), there does not exist a transition labelled with some input symbol \( s \), and that (2) \( A_M \) needs to trigger this transition when consuming the signature of \( X \). Then, at state \( q \) of \( M \), there does not exist a transition labelled with \( s \). This contradicts the nature of the loss automaton \( M \) since it must compute the loss of any ground time series, and thus accept any time series. Hence, Situation 2 is also impossible.

Therefore, both situations are impossible, which implies that the time series \( X \) does not exist, and thus the automaton \( A_M \) is complete.

Since \( A_M \) is sound and complete, the minimisation of \( A_M \) gives the sought \( \delta \)-gap automaton. \( \square \)

6.2 Synthesising the Loss Automaton for the \texttt{nb\_} Family

First, for the \texttt{nb\_} family, we show that, when \( \sigma \) has a property, named the homogeneity property, the first three principal conditions of Definition 16 are satisfied. Second, based on the homogeneity property we show how to satisfy the fourth principal condition by constructing from the seed transducer for \( \sigma \) a loss automaton satisfying the loss-automaton condition. Consequently, the constructive proof of Theorem 4 can be used to derive the \( \delta \)-gap automaton.

1. Section 6.2.1 introduces the homogeneity property. Sections 6.2.2 and 6.2.3 both assume the homogeneity property.
2. Section 6.2.2 proves three theorems stating that, the gap-to-loss, the boundedness, and the disjointedness conditions are satisfied for \texttt{nb\_}.
3. Section 6.2.3 gives a systematic method for constructing a loss automaton \( M \) satisfying the non-negativity and the separation conditions.

6.2.1 The homogeneity Property

**Property 2 (homogeneity property)** A regular expression \( \sigma \) has the homogeneity property if the following conditions are both satisfied:

1. The pair \( \langle \sigma, b_\sigma \rangle \) is a recognisable pattern [22]. This implies that the seed transducer \( T_\sigma \) for \( \sigma \) exists and can be constructed by the method of [22].
2. For any state \( q \) of \( T_\sigma \) that is the destination state of a found-transition, the number of transitions in the shortest found-path starting from \( q \) is a constant that does not depend on \( q \).

For a regular expression \( \sigma \) with the homogeneity property, the following lemma gives the maximum number of \( \sigma \)-patterns in a time series of length \( n \).

**Lemma 2 (maximum of the result value)** Consider a time-series constraint \texttt{nb\_} such that \( \sigma \) has the homogeneity property, and \( T_\sigma \) denotes the seed transducer for \( \sigma \). Let \( d_\sigma \) denote the length of shortest found-path in \( T_\sigma \) starting from any state that is the destination of a found-transition, and let \( c_\sigma \) denote the difference between \( d_\sigma \) and the length of shortest found-path in \( T_\sigma \) starting from the initial state of \( T_\sigma \). Then, the maximum number of \( \sigma \)-patterns in a time series of length \( n \) is computed as

\[
\left\lfloor \frac{n - c_\sigma}{d_\sigma} \right\rfloor.
\]

**Proof** For any time series \( X = \langle X_1, X_2, \ldots, X_n \rangle \), there is a bijection between its set of \( \sigma \)-patterns and the found symbols in the output sequence of \( T_\sigma \) after consuming the signature of \( X \). Hence, we need to show that \( \left\lfloor \frac{n - c_\sigma}{d_\sigma} \right\rfloor \) is the maximum number of the found symbols in the output sequence \( T \) of \( T_\sigma \) after
having consumed the signature of any time series of length $n$. The first found symbol in $T$ cannot occur before the position $\ell$, where $\ell$ is the length of the shortest found-path starting from the initial state. Since $T_\sigma$ has the homogeneity property then every other found symbol can occur in $T$ with the interval of $d_\sigma$. Such an $T$ output sequence has the number of found symbols being equal to \( \left\lfloor \frac{n-(\ell-d_\sigma)}{d_\sigma} \right\rfloor \). We replace $\ell - d_\sigma$ with $c_\sigma$ and obtain Formula (14).

\[ \square \]

6.2.2 Verifying the Gap-Loss-Relation Conditions

This section shows that the gap-loss-relation conditions, see Definition 16, for a $\text{nb}_\sigma$ time-series constraint are satisfied, assuming $\sigma$ has the homogeneity property. Theorem 5 proves the gap-to-loss condition and derives the formula for the gap-to-loss function; Theorem 6 proves the boundedness condition and derives the formula of loss interval for a given gap and sign of the result value, and, finally, Theorem 7 proves the disjointedness condition.

**Theorem 5 (gap-to-loss condition)** Consider a $\gamma(X, R)$ time-series constraint that belongs to the $\text{nb}_\sigma$ family with $\sigma$ having the homogeneity property. First, the gap-to-loss condition is satisfied for $\gamma$. Second, for any ground time series $X$ of length $n$, the gap-to-loss function is defined by:

\[
\text{loss}_\gamma(X) = \text{gap}_\gamma(X) \cdot d_\sigma + (1 - \text{sgn}(R)) \cdot (\min(n, c_\sigma) - 1) + \max(0, n - c_\sigma) \mod d_\sigma,
\]

where $\text{sgn}$ is the signum function, and $c_\sigma$ and $d_\sigma$ are the constants from the maximum value of $R$ given in Lemma 2.

**Proof** We successively consider two disjoint cases wrt $\text{sgn}(R)$.

**[sgn($R$) is zero]** We need to prove that loss$_\gamma(X)$ is equal to gap$_\gamma(X) \cdot d_\sigma + \min(n, c_\sigma) - 1 + \max(0, n - c_\sigma) \mod d_\sigma$. When $R$ is zero, the loss of $X$ is $n - 1$ since a shortest time series without any $\sigma$-patterns is of length 1. Thus, we need to show that gap$_\gamma(X) \cdot d_\sigma + \min(n, c_\sigma) - 1 + \max(0, n - c_\sigma) \mod d_\sigma$ is equal to $n - 1$. From the maximum value of $R$, given by the homogeneity property, we have the following equality:

\[
\text{gap}_\gamma(X) = \max(0, \left\lfloor \frac{n-c_\sigma}{d_\sigma} \right\rfloor) - R = \max(0, \left\lfloor \frac{n-c_\sigma}{d_\sigma} \right\rfloor).
\]

Let us consider two cases wrt the value of gap$_\gamma(X)$, namely:

- gap$_\gamma(X)$ is zero. By (16), $n < c_\sigma + d_\sigma$, and the value of the right-hand side of (15) is equal to $\min(n, c_\sigma) - 1 + \max(0, n - c_\sigma)$, which is $n - 1$.

- gap$_\gamma(X)$ is positive. Then, by (16), $n \geq c_\sigma + d_\sigma$, and we have the following equality:

\[
\text{gap}_\gamma(X) = \left\lfloor \frac{n-c_\sigma}{d_\sigma} \right\rfloor = \frac{n-c_\sigma - (n-c_\sigma) \mod d_\sigma}{d_\sigma}.
\]

From (17) we obtain the expression for $n - 1$, which is gap$_\gamma(X) \cdot d_\sigma + c_\sigma - 1 + (n-c_\sigma) \mod d_\sigma$.

**[sgn($R$) is one]** We need to prove that loss$_\gamma(X)$ is equal to gap$_\gamma(X) \cdot d_\sigma + \max(0, n-c_\sigma) \mod d_\sigma$. Since $R$ is positive, $n$ is strictly greater than $c_\sigma$, and thus $\max(0, n-c_\sigma)$ is equal to $n-c_\sigma$. Further, by definitions of gap and loss, we have:

\[
\text{gap}_\gamma(X) = \left\lfloor \frac{n-c_\sigma}{d_\sigma} \right\rfloor - R = \frac{n-c_\sigma - (n-c_\sigma) \mod d_\sigma}{d_\sigma} - \frac{(n - \text{loss}_\gamma(X)) - c_\sigma}{d_\sigma}.
\]
Since on the right-hand side of (18), both divisions are integer divisions we obtain:

$$\text{gap}_{\gamma}(X) = \frac{\text{loss}_{\gamma}(X) - (n - c_{\sigma})}{d_{\sigma}} \mod d_{\sigma}. \quad (19)$$

By isolating $\text{loss}_{\gamma}(X)$ from (19) we obtain the formula of the theorem. \(\square\)

**Example 19 (gap-to-loss condition)** Consider a $\text{nb}_{\sigma}((X_1, X_2, \ldots, X_n), R)$ time-series constraint with $\sigma$ being the peak regular expression, which has the homogeneity property. Hence, we can apply Theorem 5 for computing the gap-to-loss function for $\text{nb}_{\sigma}$. By Lemma 2, the maximum value of $R$ is $\max \left(0, \left\lfloor \frac{n - 1}{2} \right\rfloor \right)$, and thus $c_{\sigma}$ and $d_{\sigma}$, are 1 and 2, respectively. Then the gap-to-loss function for $\text{nb}_{\sigma}$ is $\text{loss}_{\gamma}(X) = 2 \cdot \text{gap}_{\gamma}(X) + \max(0, n - 1) \mod 2$. \(\triangle\)

**Theorem 6 (boundedness condition)** Consider a $\gamma(X, R)$ time-series constraint that belongs to the $\text{nb}_{\sigma}$ family with $\sigma$ having the homogeneity property. First, the boundedness condition is satisfied for $\gamma$; second, for any given gap $\delta$ and any value of $\text{sgn}(R)$, the loss interval $[\ell_{\min}, \ell_{\max}]$ wrt $\langle \delta, \text{sgn}(R) \rangle$ is defined by:

(i) $\ell_{\min} = \delta \cdot d_{\sigma} + (1 - \text{sgn}(R)) \cdot \text{sgn}(\delta) \cdot (c_{\sigma} - 1)$,

(ii) $\ell_{\max} = d_{\sigma} \cdot (\delta + 1) - 1 + (1 - \text{sgn}(R)) \cdot (c_{\sigma} - 1)$.

**Proof** Let $X$ be a ground time series of length $n$ whose gap is $\delta$. From Theorem 5, we have that $\text{loss}_{\gamma}(X)$ is $\delta \cdot d_{\sigma} + (1 - \text{sgn}(R)) \cdot \text{sgn}(\delta) \cdot (c_{\sigma} - 1) + \max(0, n - c_{\sigma}) \mod d_{\sigma}$. By case analysis wrt the value of $\text{sgn}(R)$, i.e. either 0 or 1, we now show that $\ell_{\min} \leq \text{loss}_{\gamma}(X) \leq \ell_{\max}$.

**[sgn](R) is zero** In this case, $\text{loss}_{\gamma}(X)$ simplifies to $\delta \cdot d_{\sigma} + \min(n, c_{\sigma}) - 1 + \max(0, n - c_{\sigma}) \mod d_{\sigma}$. Since $\delta \cdot d_{\sigma} - 1$ is a constant, in order to prove that $\ell_{\min}$ (resp. $\ell_{\max}$) is a lower (resp. upper) bound on $\text{loss}_{\gamma}(X)$, we need to find the minimum (resp. maximum) of the function $z(n) = \min(n, c_{\sigma}) + \max(0, n - c_{\sigma}) \mod d_{\sigma}$.

(i) $\ell_{\min} \leq \text{loss}_{\gamma}(X)$. We prove that $\text{loss}_{\gamma}(X) = \delta \cdot d_{\sigma} + z(n) \geq \ell_{\min}$ by case analysis on $\delta$:

(a) **[sgn](\delta) is zero** As shown in the proof of Theorem 5, $n < c_{\sigma} + d_{\sigma}$ and the minimum value of the function $z(n) = 1$, and is reached for $n$ being 1.

(b) **[sgn](\delta) is one** We have $n \geq c_{\sigma} + d_{\sigma}$, and thus $\min(n, c_{\sigma})$ is equal to $c_{\sigma}$, and the minimum value of the function $z(n)$ is 1. Hence, $\delta \cdot d_{\sigma} + \text{sgn}(\delta) \cdot (c_{\sigma} - 1)$ is indeed a lower bound on $\text{loss}_{\gamma}(X)$ when $\text{sgn}(R)$ is zero.

(ii) $\ell_{\max} \geq \text{loss}_{\gamma}(X)$. We prove that $\text{loss}_{\gamma}(X) \leq \ell_{\max}$. The maximum value of $z(n)$ is $c_{\sigma} + d_{\sigma} - 1$. Hence, $d_{\sigma} \cdot (\delta + 1) - 1 + c_{\sigma} - 1$ is indeed an upper bound on $\text{loss}_{\gamma}(X)$.

**[sgn](R) is one** In this case, $\text{loss}_{\gamma}(X)$ simplifies to $\delta \cdot d_{\sigma} + \max(0, n - c_{\sigma}) \mod d_{\sigma}$. A lower (resp. upper) bound on $(n - c_{\sigma}) \mod d_{\sigma}$ is zero (resp. $d_{\sigma} - 1$). Hence, $\ell_{\min}$ and $\ell_{\max}$ are, respectively, a lower and an upper bound on $\text{loss}_{\gamma}(X)$. \(\square\)

**Example 20 (boundedness condition)** Consider a $\text{nb}_{\sigma}(X, R)$ time-series constraint with $\sigma$ being the peak regular expression. Since $\sigma$ has the homogeneity property we can apply Theorem 6 for computing the loss interval for $\text{nb}_{\sigma}$. Recall that the values of $c_{\sigma}$ and $d_{\sigma}$, are, respectively, 1 and 2. Then, for any value $\delta$ of gap and any value of $\text{sgn}(R)$, the loss interval wrt $\langle \delta, \text{sgn}(R) \rangle$ is $[2 \cdot \delta, 2 \cdot \delta + 1]$. \(\triangle\)

**Theorem 7 (disjointedness condition)** Consider a $\text{nb}_{\sigma}((X_1, X_2, \ldots, X_n), R)$ time-series constraint such that $\sigma$ has the homogeneity property. Then the disjointedness condition is satisfied for $\text{nb}_{\sigma}$. \(\triangle\)

**Proof** The disjointedness condition can be proved using the formula of the loss interval of Theorem 6. For each value of $\text{sgn}(R)$, i.e. either 0 or 1, we take two different values of gap, w.l.o.g. $\delta$ and $\delta + t$ with a non-negative integer $t$, and show that the upper limit of the loss interval wrt $\langle \delta, \text{sgn}(R) \rangle$ is strictly less than the lower limit of the loss interval wrt $\langle \delta + t, \text{sgn}(R) \rangle$. This implies the disjointedness condition. \(\square\)
6.2.3 Verifying the Loss-Automaton Condition

We focus on the loss-automaton condition for the \( \text{nb}_\sigma \) time-series constraints, i.e. we construct a loss automaton \( M \) for \( \text{nb}_\sigma \) satisfying the non-negativity and the separation conditions. This is done by deriving \( M \) from a seed transducer for \( \sigma \), which exists assuming \( \sigma \) has the homogeneity property \[22\]. In order to satisfy the separation condition for the loss automaton for \( \text{nb}_\sigma \), we require the seed transducer for \( \sigma \) to have a specific form that we now introduce in Definition 17.

**Definition 17 (separated seed transducer)** Given a regular expression \( \sigma \), a seed transducer \( T_\sigma \) for \( \sigma \) is separated iff for any state \( q \) of \( T_\sigma \), one of the two following conditions holds:

1. Any path from the initial state of \( T_\sigma \) to \( q \) is a found-path.
2. There are no found-paths from the initial state of \( T_\sigma \) to \( q \).

![Fig. 9: (A) Seed transducer and (B) separated seed transducer for the PEAK regular expression.](image)

**Example 21 (separated seed transducer)** Part (B) of Figure 9 gives the separated seed transducer for \( \text{PEAK} \) obtained from the seed transducer in Part (A).

Note that, even if the seed transducer for \( \sigma \) constructed by the method of \[22\] is not separated, it can be easily made so by duplicating some of its states. Subsequently we assume that the seed transducer for \( \sigma \) is separated, and we derive the loss automaton \( M \) in the same way as we generate register automata for time-series constraints \[7\], namely:

1. First, we identify the required registers of \( M \) and their role.
2. Second, to each phase letter of the output alphabet of the seed transducer for \( \sigma \), we associate a set of instructions, i.e. register updates. The loss automaton \( M \) is obtained by replacing every phase letter of the seed transducer for \( \sigma \) by the corresponding set of instructions.

**Identifying the Required Registers of the Loss Automaton** Consider a \( \text{nb}_\sigma \) time-series constraint. Intuitively, when consuming the signature of a ground time series, every transition triggered by the seed transducer \( T_\sigma \) for \( \sigma \) has a certain impact on the loss of this time series. To quantify this impact for the case of \( \text{nb}_\sigma \) time-series constraints, Definition 18 introduces the notion of *regret of a transition* of a seed transducer for \( \sigma \). The regret of a transition \( t \) gives how many additional transitions \( T_\sigma \) has to trigger, before it can trigger the next found-transition, if it triggers \( t \) rather than the transition on a shortest found-path.
Definition 18 (regret of a transition) Consider a regular expression $\sigma$ and its seed transducer $T_\sigma$. For any transition $t$ of $T_\sigma$ from state $q_1$ to state $q_2$, the regret of $t$ equals one plus the difference between the lengths of the shortest found-paths from $q_2$, respectively $q_1$.

Example 22 (regret of a transition) Consider the peak regular expression, whose separated seed transducer is given in Part (B) of Figure 9. We denote by $q_1 \xrightarrow{s} q_2$ a transition of the seed transducer from state $q_1$ to state $q_2$ whose input symbol is $a$. All transitions in $\{ s \xrightarrow{r} r \xrightarrow{r} t, t \xrightarrow{r'} r' \xrightarrow{r} t \}$ between two distinct states have a regret of 0, while all transitions in $\{ s \xrightarrow{s}, s \xrightarrow{r}, r \xrightarrow{r}, t \xrightarrow{r} t, t \xrightarrow{r} t, r' \xrightarrow{r} r' \xrightarrow{r} r' \}$ have a regret of 1.

Lemma 3 shows the connection between the loss of a ground time series $X$ and the regret of the transitions triggered by the seed transducer for $\sigma$ when consuming the signature of $X$.

Lemma 3 (regret-loss relation) Consider a $\gamma(X,R)$ time-series constraint with $\gamma$ being $\text{NB } \sigma$ such that $\sigma$ has the homogeneity property. Let $t = \langle t_1, t_2, \ldots, t_{n-1} \rangle$ denote the sequence of transitions triggered by the seed transducer $T_\sigma$ for $\sigma$ upon consuming the signature of $X = \langle X_1, X_2, \ldots, X_n \rangle$, and let $t^*$ denote the index of the last found-transition in $t$, if no such transition exists, $t^*$ is zero. The following equality holds:

$$\text{loss}_\gamma(X) = n - 1 - t^* + \sum_{i=1}^{t^*} \rho(t_i), \text{ where } \rho(t_i) \text{ denotes the regret of transition } t_i.$$  

Proof Since $\langle t_{i+1}, t_{i+2}, \ldots, t_{n-1} \rangle$ does not contain any found-transition, it implies that the loss of $X$ is at least $n - 1 - t^*$. Then, the sum $\sum_{i=1}^{t^*} \rho(t_i)$ shows how many additional transitions were triggered to achieve the same number of found-transitions in the output sequence. Hence, the loss of $X$ is the sum of $n - 1 - t^*$ and $\sum_{i=1}^{t^*} \rho(t_i)$.

Example 23 (regret-loss transition) Consider the peak regular expression, whose separated seed transducer $T_{\text{peak}}$ is given in Part (B) of Figure 9. Upon consuming the signature of the time series $X = \langle 1, 1, 2, 1, 2, 1, 2, 1, 2 \rangle$, the seed transducer $T_{\text{peak}}$ triggers the following sequence of transitions $\langle s \xrightarrow{s}, s \xrightarrow{r}, r \xrightarrow{t}, t \xrightarrow{r'}, r' \xrightarrow{t}, t \xrightarrow{r'}, r' \xrightarrow{t}, t \xrightarrow{r'} \rangle$. The index of the last triggered found-transition is 8. From Lemma 3, we obtain $\text{loss}_\gamma(X) = 10 - 1 - 8 + (1 + 0 + 0 + 0 + 1 + 0 + 0 + 0) = 3$.

From Lemma 3, three registers are needed for the loss automaton. Given a prefix of a signature consumed by the seed transducer, let $t^*$ denote the last triggered found-transition:

- Register $R$ gives the sum of the regrets of the transitions triggered before $t^*$. Note that the regret of $t^*$ is zero.
- Register $D$ gives the sum of the regrets of the transitions triggered after $t^*$.
- Register $C$ gives the number of transitions triggered after $t^*$.

The initial value of these three registers is zero. The decoration table, given in the next section, follows from Lemma 3.

Decoration Table of a Loss Automaton As stated before, a loss automaton for $\text{NB } \sigma$ has three registers $C$, $D$ and $R$. Given a prefix of some signature consumed by the seed transducer $T_\sigma$, let $t^*$ denote the last triggered found-transition. When $T_\sigma$ triggers the transition $t$, we have one of the two following cases:

1. If it is not a found-transition] Then $t^*$ is still the last triggered found-transition. There is one more transition triggered after $t^*$, and the register $C$ must be increased by 1. Further, the value of $D$ should be increased by the regret of $t$. Finally, register $R$ remains unchanged.
2. [i is a found-transition] Then i becomes the last triggered found-transition. Since there is no transition triggered after i, registers C and D must both be reset to 0. Register R must be increased by the sum of the regrets of all the transitions triggered after i and before i, i.e. the value of D.

By Lemma 3, the loss of a time series is the sum between the sum of the regrets of all the triggered transitions before the last found-transition and the number of transitions triggered after the last found-transition. This is the sum of the last values of C and R. Part (A) of Figure 10 summarises how registers are updated.

| initial values | acceptance function | phase letters |
|----------------|---------------------|---------------|
| C ← 0          | R + C               | update of C   |
| D ← 0          |                     | update of D   |
| R ← 0          |                     | update of R   |

Part (A) of Figure 10 summarises how registers are updated.

Fig. 10: (A) Decoration table for the loss automaton for nb_group time-series constraints, where ρ(t) denotes the regret of a transition t of the seed transducer for σ; (B) Loss automaton for nb_group; the initial value of the registers C, D, and R is zero; as the regret of the not found transitions s → r and t → r′ of the seed transducer for σ is zero, the register D remains unchanged while triggering these two transitions.

To obtain the loss automaton for a nb_group time-series constraint, we replace every output letter in the separated seed transducer for σ with the corresponding set of register updates according to the decoration table shown in Part (A) of Figure 10. The initial value of all three registers is zero, and the acceptance function is C + R.

Example 24 (loss automaton) The loss automaton for nb_group, obtained from the seed transducer in Part (B) of Figure 9 and from the decoration table in Part (A) of Figure 10, is given in Part (B) of Figure 10.

6.3 Summary

We presented a systematic approach for generating δ-gap automata for time-series constraints, and demonstrated its applicability for the nb_group family. We used the obtained automata both (i) for proving that 70% of our synthesised linear invariants were facet defining, and (ii) for proving the correctness of all non-linear invariants of a database of invariants on conjunctions of time-series constraints.

Although, we did this work in the context of time series, the same method can be used for generating δ-gap automata for any constraint satisfying the four principal conditions. As an example, consider the nb_group(X, R, P) constraint [18,8], where X is a sequence of n integer variables, R is an integer variable, and P is a non-empty finite set of integer numbers. This constraint restricts R to be the number of maximal subsequences of X whose elements are in P. For example, nb_group((1, 3, 4, 1, 0, 9, 0), 3, (0, 1)) holds. Then a sharp upper bound on R is \( \lceil \frac{n}{2} \rceil \), and it can be shown that all the four principal conditions are satisfied for nb_group. Hence by Theorem 4 for any natural δ, the δ-gap automaton for nb_group exists and can be constructed by the method given in the proof of Theorem 4.
7 Evaluation

To test the generated invariants, we use real-world electricity demand data from an industrial partner. The dataset contains time series of length 96 (2 days in half-hour resolution) for multiple years. We use fixed size prefixes of the data to show scaleability of our methods.

In a first experiment we consider prefixes of length 25 and test all binary combinations of the considered constraints both with our baseline implementation of the individual constraints (version pure) and with the added, generated invariants applied to each suffix (version incremental). From the dataset, we extract as features the observed values for a pair of constraints for a time-series instance, and then try to find an assignment that achieves these values. Each problem is feasible, as it is based on an existing assignment. Any improvement of the propagation is due to detecting failures in partial assignments more quickly by applying the invariants to suffixes of the complete series. Our default search strategy labels the signature variables first, followed by the decision variables, always starting with the smallest values. As all constraints used here operate on the signature variables only, we can always find an assignment of the decision variables once a feasible assignment of the signatures is found.

Figure 11 shows the results, with the pure baseline above the main diagonal, and the results with the added invariants (incremental) below the main diagonal. Each box represents the results for 100 time series. The number in the box, if present, shows how many of the 100 experiments timed out (limit 2 seconds) with the default search strategy. The colour of the cell indicates the average number of backtracks required for the solved instances, based on the legend below the matrix. All experiments were run using SICStus Prolog 4.3.5 on a Windows 10 laptop with 64 GB of memory, using a single core of the Intel i7 processor running at 2.9 GHz base speed.

Adding the invariants decreases both the number of timeouts and the number of backtracks for most, but not all, constraint combinations. While some constraint combinations are easily solved even without the invariants, there are many cases where the baseline constraints are not able to find a solution quickly, but the added invariants reduce the backtrack count close to zero. It is interesting to note that all combinations of the nb_ constraints are solved with less than 20 backtracks when the invariants are added, while the baseline constraint do not find any solutions for several combinations of such constraints.

We repeat the experiments, but now for time-series length increasing from 20 to 90, to investigate scaleability of the approach. Figure 12 shows the baseline results on the left, the results with added invariants on the right. We plot the percentage of instances solved as a function of execution time. For the baseline, we see that with increasing problem size the percentage of problems solved steadily drops from 93.9% for size 20 to 75.9% for size 90 with a timeout of 2 seconds. Adding the invariants improves the percentage to 99.3% for size 20, while still achieving 97.9% for size 90.

To test the method in a realistic setting, we consider the conjunction of all 35 considered time-series constraints on the dataset. To capture the shape of the time series more accurately, we split the series into overlapping segments from 00-12, 06-18, and 12-24 hours, each segment containing 24 data points, overlapping in 12 data points with the previous segment. We then set up the conjunction of the 35 time-series constraints for each segment, using the pure and incremental variants described above. This leads to $3 \times 35 \times 2 = 210$ automaton constraints with decision variables. The invariants are created for every pair of constraints, and every suffix, leading to a large number of inequalities. The search routine assigns all signature variables from left to right, and then assigns the decision variables, with a timeout of 120 seconds.

In order to understand the scaleability of the method, we also consider time series of 44 resp. 50 data points (three segments of length 22 and 25), extracted from the daily data stream covering a four-year period (1448 samples). In Figure 14 we show the time and backtrack profiles for finding a first solution. The top row shows the percentage of instances solved within a given time budget, the bottom row shows the percentage of problems solved within a backtrack budget. For easy problems, the pure variant finds solutions more quickly, but the incremental version pays off for more complex problems, as it reduces the number
Fig. 11: Comparing baseline (top-left) and added invariants (bottom-right) models on all binary combinations of considered constraints: Length 25 variables, 100 feasible samples. Number of timeouts as numbers, average number of backtracks as cell colour.

The results show that adding the generated invariants drastically improves the propagation, even for feasible problems. The improvement is due to detecting infeasibility of a generated sub-problem. The problems for segment length 20 (not shown) can be solved without timeout for both variants, as the segment length increases, the number of timeouts increases much more rapidly for the pure variant.

The problems for segment length 20 (not shown) can be solved without timeout for both variants, as the segment length increases, the number of timeouts increases much more rapidly for the pure variant.
Fig. 12: Comparing baseline (left) and added invariants (right) models on time series of sizes 20-90; 100 feasible sample; Showing cumulative percentage of problems solved as a function of execution time, timeout 2 seconds.

Fig. 14: Percentage of Problems Solved for 3 Overlapping Segments of Lengths 22, 24, and 25; Execution time in top row, backtracks required in bottom row.
remaining suffix of the unassigned variables more rapidly, and therefore avoiding having to explore this infeasible subtree in the overall search.

8 Conclusion

Using the operational view of time-series constraints, i.e. the seed transducers for each regular expression and register automata, we presented systematic methods for synthesising 1) linear and 2) non-linear invariants linking the result values of several time-series constraints and parameterised by a function of the time-series length, and 3) conditional automata representing a condition on the result value of a time-series constraint. Since all these conditional automata have a number of states and an input alphabet that do not depend on the length of an input sequence, these automata allow us to prove both the fact that linear invariants are facet defining or not, and the validity of non-linear invariants, for any long enough sequence length. All the 2000 synthesised parametrised invariants were put in a publicly available database of invariants [3] linked to the time-series catalogue that was used to automatically enhance short-term electricity production models that were acquired from real production data.

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