A SEMILINEAR PARTIAL DIFFERENTIAL EQUATION INDUCED BY HERMITIAN YANG-MILLS METRICS

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ABSTRACT. This paper will discuss a class of semilinear partial differential equations induced by studying the limiting behaviour of Hermitian Yang-Mills metrics. We will study the radial symmetry of the $C^2$ global solution of this equation in $\mathbb{R}^2$ and the existence of $C^{2,\alpha}$ solution of the Dirichlet boundary value problem in any bounded domain.

KEYWORDS. Hermitian Yang-Mills metric, $C^k$-estimates, Boundary value problems

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1. Introduction

Let $X$ be a Kähler manifold with a family of Kähler metrics $\omega_\varepsilon$, and let $V$ be a slope stable holomorphic vector bundle over $X$. According to the Donaldson-Uhlenbeck-Yau theorem [5], $V$ admits unique fully irreducible Hermitian-Yang-Mills metrics $H_\varepsilon$ associated to each $\omega_\varepsilon$. Similar to study the limiting behaviour Ricci flat metrics, Professor Jixiang Fu [2] studied the limiting behaviour of Hermitian Yang-Mills metrics $H_\varepsilon$ when $\omega_\varepsilon$ goes to a large Kähler metric limit. A critical step in [2] is to explicitly construct a family of Hermitian-Yang-Mills metrics by solving the following semilinear partial differential equation on unit ball $B_1(0)$ of $\mathbb{R}^2$

\begin{equation}
\begin{aligned}
\Delta u & = \varepsilon^{-2} \left( e^u - (x^2 + y^2) e^{-u} \right) \quad \text{in } B_1(0), \\
\quad \quad u & = 0 \quad \text{on } \partial B_1(0).
\end{aligned}
\end{equation}

Here $\varepsilon$ is a constant and $(x, y)$ is the coordinate of $\mathbb{R}^2$. By the symmetry of the domain $B_1(0)$ and using reference [3], Jixiang Fu proved the equation (1) has a unique radially symmetric solution. However, this method can not be applied to non-symmetric domain $\Omega$ in $\mathbb{R}^2$.

In this paper, we first study the following equation defined a bounded connected domain $\Omega \subset \mathbb{R}^2$ with Dirichlet boundary value

\begin{equation}
\begin{aligned}
\Delta u & = \varepsilon^{-2} \left( e^u - (x^2 + y^2) e^{-u} \right) \quad \text{in } \Omega, \\
\quad \quad u & = g \quad \text{on } \partial \Omega.
\end{aligned}
\end{equation}

For existence and unique of the solution of (2), we have the following theorem
Theorem 1.1. If \( \partial \Omega \) is \( C^{2,\alpha} \) and \( g \in C^{2,\alpha}(\partial \Omega) \), there is a unique solution \( u \in C^{2,\alpha}(\Omega) \) to equation (2). Especially if \( \partial \Omega \) and \( g \) is smooth, the solution \( u \) is smooth.

This theorem will give a Hermitian Yang-Mills metrics on a certain Kähler manifold given by [2]. On the other hand, the equation (1) can be defined on whole space \( \mathbb{R}^2 \). It is natural to explore whether the global solution of (1) is radially symmetric. The symmetry of global solutions of some semilinear equations has been investigated in [1] and [3] under the assumption \( u(x, y) \) decays to zero at a certain rate as \( r^2 = x^2 + y^2 \to +\infty \). But they do not fit the equation (1) since one can see the global solution \( u \) is not bounded. Similar to [1, 3], by using moving plane method and maximum principle, we get the following theorem

Theorem 1.2. For any given constant \( c \), if the global \( C^2 \) solution \( u \) of

\[
\Delta u = \varepsilon^{-2} \left( e^u - (x^2 + y^2) e^{-u} \right)
\]

in \( \mathbb{R}^2 \) satisfies

\[
u(s) - u(t) \to 0 \quad \text{as} \quad |s|, |t| \to \infty \quad \text{and} \quad |s| - |t| = c,
\]

then \( u \) is radially symmetric and \( \frac{\partial u}{\partial r} \geq 0 \). Here \( s, t \in \mathbb{R}^2 \).

One may observe \( \frac{1}{2} \log(x^2 + y^2) \) is a singular solution to the equation (3) and also satisfies \( \log(|s|) - \log(|t|) \to 0 \) as \( |t| - |s| = c \) and \( |s|, |t| \to \infty \), so the assumption of Theorem 1.2 is natural and reasonable.

The next part of this paper will give the detailed proof of Theorem 1.1 and Theorem 1.2.

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2. Existence of solution of the Dirichlet boundary value problem

In this section we will prove Theorem 1.1. One can use Chapter 14 in [6] to show the existence of the equation 2 by using variational method. Here we take Leray-Schauder existence theorem to prove it.

Let \( \Omega \) be a \( C^{2,\alpha} \) bounded domain in \( \mathbb{R}^2 \) and \( g \in C^{2,\alpha}(\partial \Omega) \) with \( \alpha \in (0, 1) \). We first have a \( C^0(\Omega) \) estimate.

Lemma 2.1. Let \( \Phi \) be the \( C^{2,\alpha} \) solution of Dirichlet boundary value problem

\[
\begin{aligned}
\Delta \Phi &= \varepsilon^{-2}(1 - (x^2 + y^2)) \quad \text{in} \ \Omega, \\
\Phi &= g \quad \text{on} \ \partial \Omega.
\end{aligned}
\]

Then a solution \( u \) to (3) satisfies

\[
\sup_{\Omega} |u| \leq \sup_{\Omega} 2|\Phi|.
\]
Proof. The existence of $\Phi$ is from Green formula (one can see [4]). Consider $u - \Phi$ for $u > 0$. Note that on $\mathcal{O} = \{ x \in \Omega : u(x) > 0 \}$
\[
\Delta (u - \Phi) = \varepsilon^{-2} (e^{u} - 1 + (x^2 + y^2) (1 - e^{-u})) > 0.
\]
By maximum principle, we have
\[
\sup_{\mathcal{O}} (u - \Phi) = \sup_{\partial \mathcal{O}} (u - \Phi) \leq \sup_{\Omega} (-\Phi, 0) \leq \sup_{\Omega} |\Phi|.
\]
It follows
\[
(6) \quad \sup_{\Omega} u \leq \sup_{\Omega} 2|\Phi|.
\]
Similarly, if $u < 0$, \[
\Delta (u - \Phi) = \varepsilon^{-2} (e^{u} - 1 + (x^2 + y^2) (1 - e^{-u})) < 0.
\]
Hence we obtain on $\mathcal{O}^- = \{ x \in \Omega : u(x) < 0 \}$
\[
\sup_{\mathcal{O}^-} (\Phi - u) = \sup_{\partial \mathcal{O}^-} (\Phi - u) \leq \sup_{\Omega} (\Phi, 0) \leq \sup_{\Omega} |\Phi|
\]
which implies
\[
(7) \quad \sup_{\Omega} -u \leq \sup_{\Omega} 2|\Phi|.
\]
Therefore from (6) and (7) one can get the estimate (5).

Second, we give the gradient estimate of $u$.

Lemma 2.2. Suppose $u \in C^2(\Omega)$ satisfies the equation (2) in $\Omega$, then there is positive constant $C$ depend only on $\Omega$ and $g$ such that
\[
(8) \quad \sup_{\Omega} |\nabla u| \leq C.
\]

Proof. From the equation (2) and by standard regularity, one can see $u$ is $C^4$ since $u$ and $g$ are $C^2$. Then we have
\[
(9) \quad \Delta |\nabla u|^2 = \langle \nabla \Delta u, \nabla u \rangle + |\nabla^2 u|^2 = \varepsilon^{-2} (e^u + r^2 e^{-u}) |\nabla u|^2 - \varepsilon^{-2} e^{-u} < \nabla r^2, \nabla u > + |\nabla^2 u|^2.
\]
If $|\nabla u|^2$ attains its maximum on the boundary $\partial \Omega$, we have \[
\sup_{\partial \Omega} |\nabla u| = \sup |\nabla g| \quad \text{which leads to (8)}. \]
Now assume $|\nabla u|^2$ attains its maximum at $z_0 \in \Omega$. Then from (9), at the point $z_0$ we have
\[
\varepsilon^{-2} (e^u + r^2 e^{-u}) |\nabla u|^2 - \varepsilon^{-2} e^{-u} < \nabla r^2, \nabla u > \leq 0
\]
or
\[
(e^u + r^2 e^{-u}) |\nabla u|^2 \leq e^{-u} |\nabla r^2| |\nabla u|.
\]
Since $|u|$ is bounded from Lemma 2.1 there is a constant $C$ dependent on $\Omega$ and $g$ such that
\[
|\nabla u|(z_0) \leq C.
\]
Then we finish the proof.
Now we give the proof of Theorem 1.1.

Proof. Let $\sigma \in [0,1]$, we claim if $u_{\sigma} \in C^{2,\alpha}(\Omega)$ is the solution of boundary value problem

$$\begin{cases} 
\Delta u = \sigma \varepsilon^{-2} (e^u - (x^2 + y^2) e^{-u}) & \text{in } \Omega, \\
u = \sigma g & \text{on } \partial \Omega, 
\end{cases}$$

(10)

then there is a constant $M$ independent of $u_{\sigma}$ and $\sigma$ such that

$$||u_{\sigma}||_{C^{1,\alpha}(\bar{\Omega})} \leq M.$$  

(11)

Then one can use the Leray-Schauder existence theorem (see Theorem 6.23 in [4]) to show the Dirichlet problem (2) is solvable in $C^{2,\alpha}(\bar{\Omega})$.

In fact, one can see $\sigma \Phi$ solves

$$\begin{cases} 
\Delta \Phi = \sigma \varepsilon^{-2} (1 - (x^2 + y^2)) & \text{in } \Omega, \\
\Phi = \sigma g & \text{on } \partial \Omega. 
\end{cases}$$

(12)

Then from Lemma 2.1, we have

$$||u_{\sigma}||_{C^0(\bar{\Omega})} \leq \sup_\Omega 2|\sigma \Phi| \leq \sup_\Omega 2|\Phi|.$$  

(13)

Therefore, from (10), there is a constant $C$ independent on $\sigma$ and $u_{\sigma}$, such that

$$|\Delta u| \leq C.$$  

This means $|\nabla^2 u|$ is also bounded. From Lemma 2.2 and using interpolation inequality in Hölder space, there is a constant $M$ independent on $u$ and $\sigma$ such that (11) is satisfied.

In the end, by standard bootstrap argument of the regularity we have $u$ is smooth if $\Omega$ and $g$ are smooth. This finishes the proof.

3. Radial symmetry of the global $C^2$ solution of the equation in $\mathbb{R}^2$

In this section we will prove Theorem 1.2. In [3] and [1], the radial symmetry of the $C^2$ positive solutions of the following second order elliptic equation is studied

$$\Delta u + f(u) = 0 \text{ in } \mathbb{R}^n$$

under the assumption on $f$ and $u$. For example, they assumed $u(x) \to 0$ as $x \to \infty$. Obviously, our equation (2) is different from this type since $e^u - r^2 e^{-u}$ has the term $r^2$. Also we cannot assume $|u| \to 0$ as $r \to +\infty$. In fact, it will lead $\Delta u \to -\infty$ and then $u$ is unbounded. It contradicts the hypothesis. In this paper, we assume for any finite constant $c$

$$u(s) - u(t) \to 0 \quad \text{and} \quad |s| - |t| = c, \quad \text{as} \quad |s|, |t| \to \infty$$

(14)

where $s, t \in \mathbb{R}^2$. 


Proof. Proof of Theorem 1.2 Since the partial differential equation (3) is rotationally symmetric, we only have to prove the symmetry about a line across origin. Here we choose the line $y$ axis. Define

$$
\Sigma(\lambda) = \{(x, y) \in \mathbb{R}^2 \mid x < \lambda\}
$$

and let

$$
v = u(2\lambda - x, y), \quad x^\lambda = 2\lambda - x.
$$

In $\Sigma(\lambda)$ we define

$$
w = v(x) - u(x).
$$

When $\lambda = 0$ and $x \in \Sigma(\lambda)$, we have $x + x^\lambda = 0$ and $x < x^\lambda$. Then

$$
x^2 = (x^\lambda)^2
$$

and

(15)$$
\Delta v - \varepsilon^{-2} \left( e^v - \left( (x^\lambda)^2 + y^2 \right) e^{-v} \right) = \Delta v - \varepsilon^{-2} \left( e^v - \left( x^2 + y^2 \right) e^{-v} \right) = 0.
$$

By the mean value theorem, we have

$$
\Delta w + \bar{c}w = 0
$$

where

$$
\bar{c} = -\int_0^1 \varepsilon^{-2} (e^{u+tw} + t^2 e^{-u-wt}) dt < 0.
$$

Then from the assumption (14) and $w(0, 0) = 0$ on the $y$ axis, we have by maximum principle and minimum principle

$$
w = 0
$$

in $\Sigma(0)$. That’s to say the global solution of (3) in $\mathbb{R}^2$ is symmetric about $y$ axis.

In the end, assuming $\lambda > 0$ and $x \in \Sigma(\lambda)$, then we have $x + x^\lambda > 0$ and $x < x^\lambda$. It follows

$$
x^2 < (x^\lambda)^2
$$

which implies

(16)$$
\Delta v - \varepsilon^{-2} \left( e^v - \left( x^2 + y^2 \right) e^{-v} \right) < 0.
$$

Then by mean value theorem, in $\Sigma(\lambda)$

$$
\Delta w + \bar{c}w < 0
$$

with $c < 0$. Using the infinite boundary condition (14) and $w(\lambda, \lambda) = 0$, we have by maximum principle, in $\Sigma(\lambda)$

$$
w \geq 0.
$$

Then if $x > 0$ and let $x_\lambda \to x$, we have $\frac{\partial u}{\partial x} \geq 0$. Since $u$ is radially symmetric and from

$$
\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} = \frac{\partial u}{\partial r} \frac{x}{r}
$$

it follows $\frac{\partial u}{\partial r} \geq 0$ and we finish the proof.
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