ON SELF-ORTHOGONAL DESIGNS AND CODES RELATED TO HELD’S SIMPLE GROUP

DEAN CRNKOVIĆ AND VEDRANA MIKULIĆ CRNKOVIĆ
Department of Mathematics, University of Rijeka
Radmile Matejić 2
51000 Rijeka, Croatia

BERNARDO G. RODRIGUES
School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal
University Road Private Bag X54001
Westville Campus
Durban 4041, South Africa

(Communicated by Mario Osvin Pavcevic)

Abstract. A construction of designs acted on by simple primitive groups is used to find some 1-designs and associated self-orthogonal, decomposable and irreducible codes that admit the simple group He of Held as an automorphism group. The properties of the codes are given and links with modular representation theory are established. Further, we introduce a method of constructing self-orthogonal binary codes from orbit matrices of weakly self-orthogonal designs. Furthermore, from the support designs of the obtained self-orthogonal codes we construct strongly regular graphs with parameters (21,10,3,6), (28,12,6,4), (49,12,5,2), (49,18,7,6), (56,10,0,2), (63,30,13,15), (105,32,4,12), (112,30,2,10) and (120,42,8,18).

1. Introduction

After having constructed Steiner systems from finite groups, Witt in the papers [32, 33] highlights the links between design theory and the theory of finite groups. Further results that are significant for the method of construction used in this paper were the constructions of a 2-(276, 100, 1458) on which Co3 acts doubly transitively on points and primitively on blocks in [12] and that of a 2-(126, 36, 14) design from a primitive group isomorphic to $U_3(5):2$ in [25]. In [23, 24] Moori and Rodrigues found binary codes of small dimension that are invariant under the Conway group Co2 and the Rudvalis group Ru in their primitive permutation representations of degrees 2300 and 4060, respectively.

The problem of finding the minimality of the $p$-ranks of designs or graphs under the action of a finite group bears a close relation to that of finding the dimension of the irreducible representations of the group. In many instances examined in the literature it was shown that the minimal $p$-rank of a design or a graph coincides with the dimension of the smallest irreducible representations of the group over $\mathbb{F}_p$, where

2010 Mathematics Subject Classification: Primary: 05E18, 94B05; Secondary: 20D08, 05B05, 05E30.

Key words and phrases: Design, orbit matrix, linear code, sporadic simple group, Held group.

This work has been supported by Croatian Science Foundation under the project 1637.

This work is based on the research supported by the National Research Foundation of South Africa (Grant Numbers 95725 and 106071).
p divides the order of the group ([6] and [20]). It is known (see [17]) that the smallest faithful irreducible representation of the Held group He over \( \mathbb{F}_2 \) has dimension 51. In line with the above it seems thus natural to ask which permutation representation of He contains this faithful irreducible representation. It turns out that the smallest permutation representation of He which contains an irreducible faithful module of dimension 51 is the representation of degree 2058.

We construct this irreducible representation as a subcode of codimension 1 in a decomposable code of dimension 52 invariant under the Held group of degree 2058. Moreover, we give the full weight distribution of these codes, describe the nature of the minimum weight codewords, and the structure of the stabilizer of a word in this weight class. It is a result of [19], (see also [10, 22, 21]) that if we form any union of orbits of the stabilizer of a point, including the orbit consisting of the single point, and orbit this under the full group, we will obtain a symmetric design with the group acting on.

For the representations of the Held group of degree 2058 and 8330 respectively, we take the images of the unions of the orbits of the point stabilizer including the singleton to form the designs and codes that we will be examining and which have the Held group acting as an automorphism group. In addition, using the representation of He of degree 8330 we construct symmetric 1-designs on 8330 points and non-symmetric 1-designs on 2058 points with 8330 blocks on which He acts point-primitively as an automorphism group. With help of tools from group theory we examine some of the relevant properties of these designs and respective binary codes. Further, we introduce a method of construction of self-orthogonal binary codes from orbit matrices of weakly self-orthogonal designs. Using this method we construct self-orthogonal codes corresponding to the respective weakly self-orthogonal designs. Furthermore, from the support designs of these codes we construct strongly regular graphs with parameters (21,10,3,6), (28,12,6,4), (49,12,5,2), (49,18,7,6), (56,10,0,2), (63,30,13,15), (105,32,4,12), (112,30,2,10) and (120,42,8,18).

The paper is organized as follows: after a brief description of our terminology and some background, Section 4 gives a brief account on the Held group, and in Sections 5, 6, 7, 8 and 9 we discuss our results.

2. Terminology and notation

Our notation will be standard, and it is as in [1] and ATLAS [9]. For the structure of groups and their maximal subgroups we follow the ATLAS notation. The groups \( G.H, G : H, \) and \( G \cdot H \) denote a general extension, a split extension and a non-split extension respectively. For a prime \( p \), the symbol \( p^\alpha \) denotes an elementary abelian group of that order. If \( p \) is an odd prime, \( p^{1+2n} \) and \( p^{1+2n} \) denote the extraspecial groups of order \( p^{1+2n} \) and exponent \( p^2 \) respectively. For \( G \) a finite group acting on a finite set \( \Omega \), the set \( \mathbb{F}_p \cdot \Omega \), that is, the vector space over \( \mathbb{F}_p \) with basis \( \Omega \) is called an \( \mathbb{F}_p G \) permutation module, if the action of \( G \) is extended linearly on \( \mathbb{F}_p \cdot \Omega \).

Our notation for designs and codes will follow [1]. An incidence structure \( D = (P, B, I) \), with point set \( P \), block set \( B \) and incidence \( I \) is a \( t-(v, k, \lambda) \) design, if \( |P| = v \), every block \( B \in B \) is incident with precisely \( k \) points, and every \( t \) distinct points are together incident with precisely \( \lambda \) blocks. The complement of \( D \) is the structure \( D = (P, B, I) \), where \( I = P \times B - I \). The dual structure of \( D \) is \( D^t = (B, P, I^t) \), where \( (B, P) \in I^t \) if and only if \( (P, B) \in I \). Thus the transpose of an incidence matrix for \( D \) is an incidence matrix for \( D^t \). The design is symmetric if it has the same number of points and blocks. A \( t-(v, k, \lambda) \) design is called weakly...
self-orthogonal if all the block intersection numbers have the same parity. A design is self-orthogonal if it is weakly self-orthogonal and if the block intersection numbers and the block size are even numbers. An isomorphism from one design to the other is a bijective mapping of points to points and blocks to blocks which preserves incidence. An isomorphism from a design $D$ onto itself is called an automorphism of $D$. The set of all automorphisms of $D$ forms its full automorphism group denoted by $\text{Aut}(D)$. Note the full automorphism group of a design $D$ is isomorphic to the full automorphism groups of its complementary design $\overline{D}$ and its dual design $D'$. A group $G$ acts flag-transitive on a design $D$ if it acts transitively on the set of flags of the design, i.e., on the set of incident point-block pairs. In that case, we will say that the design is flag-transitive. A design is transitive (primitive) if an automorphism group acts transitively (primitively) on the set of points and set of blocks of the design.

A strongly regular graph with parameters $(n, k, \lambda, \mu)$ is a $k$-regular graph with $n$ vertices such that any two adjacent vertices have $\lambda$ common neighbours and any two non-adjacent vertices have $\mu$ common neighbours.

Let $D$ be 1-design and $G$ be an automorphism group of the design acting on the set of points in the orbits $V_1, \ldots, V_n$ and on the set of blocks in orbits $B_1, \ldots, B_m$. Denote by $a_{i,j}$ the number of points of the orbit $V_j$ incident with a block of the orbit $B_i$. The orbit matrix of the design $D$ is the matrix

$$
\begin{bmatrix}
  a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\
  a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m,1} & a_{m,2} & \cdots & a_{m,n}
\end{bmatrix}
$$

The code $C_F(D)$ of the design $D$ over the finite field $F$ is the span of the incidence vectors of the blocks over $F$. The $E_F(D)$ denotes the code generated by the differences of the incidence vectors of the blocks over the $F$. Codes here will be linear codes, i.e., subspaces of the ambient vector space. A code $C$ over a field of order $q$, of length $n$, dimension $k$, and minimum weight $d$, is denoted by $[n, k, d]_q$. A generator matrix for the code is a $k \times n$ matrix made up of a basis for $C$. The weight enumerator of $C$ is defined as $W_C(x) = \sum_{i=0}^{n} A_i x^i$, where $A_i$ denotes the number of codewords of weight $i$ in $C$. The dual code $C^\perp$ is the orthogonal under the standard inner product $(,)$. i.e. $C^\perp = \{ v \in F^n | (v, c) = 0 \text{ for all } c \in C \}$. The hull of a code or a design, where $C = C_F(D)$, is $\text{Hull}(C) = C \cap C^\perp$. A code $C$ is self-orthogonal if $C \subseteq C^\perp$. Thus $C$ is self-orthogonal if and only if it is equal to its hull. If $D$ is a self-orthogonal design then the binary code of the design $D$ is self-orthogonal. The incidence matrix $M$ of a weakly self-orthogonal design such that $k$ is odd and the block intersection numbers are even can be extended to the generating matrix $(I_b, M)$ of the self-orthogonal code ([29]). The all-one vector will be denoted by $1$, and is the constant vector with all coordinate entries equal to 1. A binary code $C$ is self-complementary if it contains the all-one vector, and it is doubly-even if all codewords of $C$ have weight divisible by four. If $C_1$ is an $[n_1, k_1]$-code, and $C_2$ is an $[n_2, k_2]$-code, then we say that $C$ is the direct sum of $C_1$ and $C_2$ if (up to reordering of coordinates) $C = \{ (x, y) \mid x \in C_1, y \in C_2 \}$. We denote this by $C = C_1 \oplus C_2$. If moreover $C_1$ and $C_2$ are nonzero, then we say that $C$ decomposes into $C_1$ and $C_2$. A code $C$ is said to be decomposable if and only if it is equivalent to a code which is the direct sum of two or more linear codes. Otherwise it is called indecomposable. Two linear codes are isomorphic if they
can be obtained from one another by permuting the coordinate positions. And they are equivalent if they can be obtained from one another by multiplication of the coordinate positions by non-zero field elements or by permuting the coordinate positions. An automorphism of a code \( C \) is an isomorphism from \( C \) to \( C \). The full automorphism group will be denoted by \( \text{Aut}(C) \). If code \( C_F(D) \) is a linear code of a design \( D \) over a finite field \( F \), then the full automorphism group of \( D \) is contained in the full automorphism group of code \( C_F(D) \).

3. Preliminary results

The following construction of symmetric 1-designs and regular graphs was described in [19, Proposition 1] and later corrected in [22]. See also [21]:

**Theorem 1.** Let \( G \) be a finite primitive permutation group acting on the set \( \Omega \) of size \( n \). Further, let \( \alpha \in \Omega \), and let \( \Delta \neq \{\alpha\} \) be an orbit of the stabilizer \( G_\alpha \) of \( \alpha \). If 
\[ B = \{\Delta g : g \in G\} \]
then \( D = (\Omega, B) \) is a symmetric 1-(\( n, |\Delta|, |\Delta|\) design. \( G \) acts as an automorphism group of \( D \), primitive on points and blocks of the design. Furthermore, if \( \Delta \) is a self-paired orbit of \( G_\alpha \) then \( D \) is self-dual.

The method that allows a construction of all 1-designs such that \( G \) acts primitively on the point set and the block set of the design, including designs that have a point stabilizer and a block stabilizer that are not conjugate, is given in [10].

**Theorem 2.** Let \( G \) be a finite permutation group acting primitively on the sets \( \Omega_1 \) and \( \Omega_2 \) of size \( m \) and \( n \), respectively. Let \( \alpha \in \Omega_1 \), \( \delta \in \Omega_2 \), and let \( \Delta_2 = \delta G_\alpha \) be the \( G_\alpha \)-orbit of \( \delta \in \Omega_2 \) and \( \Delta_1 = \alpha G_\delta \) be the \( G_\delta \)-orbit of \( \alpha \in \Omega_1 \). If \( \Delta_2 \neq \Omega_2 \) and 
\[ B = \{\Delta_2 g : g \in G\}, \]
then \( D(G, \alpha, \delta) = (\Omega_2, B) \) is a 1-(\( n, |\Delta_2|, |\Delta_1|\) design with \( m \) blocks, and \( G \) acts as an automorphism group, primitive on points and blocks of the design.

If we form any union of orbits of the stabilizer of a point in Result 1 or Result 2, including the orbit consisting of the single point, and orbit this under the full group, we will still get a 1-design with the group acting on. Note that if a design \( D \) has a base block \( \Delta_2 \) then its complement has a base block \( \Omega_2 \setminus \Delta_2 \) and we will construct only designs such that \( k \leq \frac{n}{2} \).

**Corollary 1.** If a group \( G \) acts primitively on the points and the blocks of a 1-design \( D \) then the design can be obtained by orbiting a union of orbits of a maximal subgroup of \( G \).

Some further results dealing with the automorphism groups of the designs and codes constructed from a finite primitive permutation group can be found in [27].

4. The Held group \( He \)

In this section we give a brief overview of the Held group \( He \). For an account on Held group the reader is referred to [9, p.104], [31, Section 5.8.9] or [26, 4.18.4]. The simple group \( He \) is the automorphism group of a directed graph, (i.e. edges are ordered pairs of vertices called arcs), of rank 5 with 2058 vertices having in-degree (i.e. the number of arcs ending in the vertex) and out-degree (i.e. the number of arcs beginning in the vertex) 136. The stabilizer of a point \( \omega \) in this representation is a maximal subgroup isomorphic to the symplectic group \( S_4(4):2 \), the split extension
of the symplectic group $S_4(4)$ by its field automorphism, with orbits $\Omega_1 = \{\omega\}, \Omega_2, \Omega_3, \Omega_4, \Omega_5$ and subdegrees 1, 136, 136, 425, and 1360 respectively. The two suborbits of length 136 are interchanged by the outer automorphism of He thus extending $S_4(4):2$ to $S_4(4):4$ in $He:2$. The Held group was first discovered by Dieter Held, [14, 15], during an investigation of simple groups containing an involution whose centralizer is isomorphic to that of an involution in the Mathieu group $M_{24}$ (see [14]). The centralizer of a $2B$-involution is isomorphic to the centralizer of suitable involutions in $L_5(2)$ and $M_{24}$ (see [9]). The Held group $He$ centralises an element of order 7 in the Monster, and an element of order 3 which normalises the given element of order 7. The Held group is also a subgroup of the simple Fischer group $Fi'_{24}$. In fact, the full normaliser of the group of order 7 in the Monster is $(7 : 3 \times He):2$, containing $\text{Aut}(He) = He:2$, and there are two conjugacy classes of $He$ in $Fi'_{24}$, interchanged by the outer automorphism. The maximal subgroups of $He$ and $He:2$ are determined by Buttler [7] and Wilson [30]. In Result 3 we present a summary of the maximal subgroups of $He$ as given in [7].

**Theorem 3.** (Butler [7]) The sporadic simple group $He$ of order $2^{10} \times 3^3 \times 5^2 \times 7^3 \times 17$ has 11 conjugacy classes of maximal subgroups as follows:

(A) Four classes of 2-local subgroups:
- $2^2 : L_3(4) : S_3$, $2^6 : 3 \cdot S_6$ (two classes), $2^{1+6} : L_3(2)$.

(B) Five classes of odd-local subgroups:
- $3 \cdot S_7$, $5^2 : 4A_4$, $7^2 : L_2(7)$, $7^{1+2} : (3 \times S_3)$, $7 : 3 \times L_3(2)$.

(C) Two classes of non-local subgroups:
- $S_4(4) : 2$, $S_4 \times L_3(2)$.

The primitive representations referred to in Result 3 are listed in Table 1. The first column gives the ordering of the primitive representations as given by the ATLAS [9]; the second gives the structures maximal subgroups; the third gives the degree (the number of cosets of the point stabilizer).

| No. | Max. sub.         | Deg. | No. | Max. sub.         | Deg. |
|-----|-------------------|------|-----|-------------------|------|
| 1   | $S_4(4) : 2$      | 2058 | 7   | $3 \cdot S_7$    | 266560 |
| 2   | $2^2 : L_3(4) : S_3$ | 8330 | 8   | $7^{1+2} : (3 \times S_3)$ | 652800 |
| 3   | $2^6 : 3 \cdot S_6$ | 29155 | 9   | $S_4 \times L_3(2)$ | 999600 |
| 4   | $2^6 : 3 \cdot S_6$ | 29155 | 10  | $7 : 3 \times L_3(2)$ | 1142400 |
| 5   | $2^{1+6} : L_3(2)$ | 187425 | 11  | $5^3 : 5A_4$     | 3358656 |
| 6   | $7^2 : L_2(7)$    | 244800 |      |                   |      |

**Table 1.** Maximal subgroups of $He$, up to conjugation.

5. **The designs**

5.1. **Symmetric designs on 2058 points.** Let $G$ be a finite primitive permutation group acting on the set $\Omega$. It follows from Result 1 that if we form any union of orbits of the stabilizer of a point, and take their images under the action of the full group, we will obtain the blocks of a symmetric design with the group $G$ acting as an automorphism group.

Considering $G$ to be the simple group $He$, in the sequel we examine some designs which are invariant under $G$, in particular designs constructed from unions of the suborbits of its rank-5 permutation representation of degree 2058. Let $\Omega$ be the
primitive $G$-set of degree 2058, and $\Omega_i = \{\omega\}, \Omega_2, \Omega_3, \Omega_4, \Omega_5$ with subdegrees 1, 136, 136, 425, and 1360 respectively, denote the suborbits of $G$ on $\Omega$ with respect to the point stabilizer $S_i(2):2$. We consider the $l$-element subsets $\{i_1, \ldots, i_l\}$ of the set $\{1, 2, 3, 4, 5\}$ to form $\binom{5}{l}$ (to avoid trivial cases, we exclude $l = 0, 5$) distinct unions of $l$ suborbits $\Omega_i$. Subsequently, we form the multiset $R$, which represents the possible cardinalities $k$ of the block, where $k \leq \frac{2058}{2} = 1029$. Thus

$$R = \{136, 136, 137, 137, 272, 273, 425, 426, 561, 561, 562, 562, 697, 698\}.$$ 

For $k \in R$, using Result 1, we take the images of these unions under the action of $He$ and form the blocks of the self-dual symmetric 1-design $D_k$ whose properties we will be examining in Lemma 1. Observe that $k = |\bigcup_{j=1}^5 \Omega_{i_j}|$, where $1 \leq l \leq 4$ and $1 < k \leq 1029$.

**Lemma 1.** Let $G$ be the sporadic simple group $He$, and $\Omega$ the primitive $G$-set of size 2058 defined by the action on the cosets of $S_4(4):2$. Let $\omega \in \Omega$, and $\Delta = \bigcup_{j=1}^5 \Omega_{i_j}$, $1 \leq l \leq 4$, be a union of $(S_4(4):2)$-orbits. Let

$$B = \{\Delta^g : g \in G\},$$

and $D_k = (\Omega, B)$ with $k = |\Delta|$. Define the sets $M$ and $N$ such that $M = \{136, 137, 561, 562\}$ and $N = \{272, 273, 425, 426, 697, 698\}$. Then the following hold:

(i) $D_k$ is a primitive symmetric 1-(2058, $|\Delta|$, $|\Delta|$) design.

(ii) If $k \in M$ then $\text{Aut}(D_k) \cong He$, and if $k \in N$ then $\text{Aut}(D_k) \cong He:2$.

(iii) If $k = 272$ then $\text{Aut}(G)$ acts flag-transitively on the design $D_k$.

(iv) If $k$ is even then $D_k$ and $\bar{D}_k$ are self-orthogonal designs.

**Proof.** We will use the fact that $He$ and $\text{Aut}(He) = He:2$ are the only primitive groups on 2058 points except $A_{2058}$ and $S_{2058}$ ([28, Table 9, p. 179] and [26, 4.18.4]). For the benefit of the reader we point out that there is typing error in [28, Table 9, p. 179]: four orbits of $S_4(4):2$ should be corrected to five.

(i) The definition of $\Omega$ and $B$ is inferred from Result 1, and from this it is clear that $G$ acts as an automorphism group, primitive on points and on blocks of the design and so $G \subseteq \text{Aut}(D_k)$.

(ii) First, we consider the case when $k = 136$. Since $D_{136}$ is a primitive symmetric 1-design, we need only show that $G = \text{Aut}(D_{136})$. Now $G \subseteq \text{Aut}(D_{136}) \subseteq S_{2058}$, so $\text{Aut}(D_{136})$ is a primitive permutation group on $\Omega$ of degree 2058. Moreover, $\text{Aut}(D_{136})_\omega$ must fix $\Omega_2$ setwise, and hence $\text{Aut}(D_{136})_\omega$ also has an orbit of length 136 in $\Omega$. The only primitive group of degree 2058 that has a suborbit of length 136 is $He$. Hence $G = \text{Aut}(D_{136})$.

Now, consider the case when $k = 272$. Again, since $D_{272}$ is a primitive symmetric 1-design, it suffices to show that $He:2 \cong \text{Aut}(D_{272})$. Obviously $G \subseteq \text{Aut}(D_{272}) \subseteq S_{2058}$, so $\text{Aut}(D_{272})$ is a primitive permutation group on $\Omega$ of degree 2058. Also $\text{Aut}(D_{272})_\omega$ must fix $\Theta = \Omega_2 \cup \Omega_3$ setwise, and thus $\text{Aut}(D_{272})_\omega$ has an orbit of length 272 in $\Omega$. The only primitive group of degree 2058 that has a suborbit of length 272 is $He_2$, and $\text{Aut}(D_{272})_\omega \cong S_4(4):4$.

In general, when the base block $\Delta$ contains only one $(S_4(4):2)$-orbit of length 136, then the full automorphism group of the design is isomorphic to $He$. However, if $\Delta$ contains none or two $(S_4(4):2)$-orbits of length 136 then the full automorphism group of the design is isomorphic to $He:2$. This is a consequence of the fact that the two orbits of length 136 are interchanged by the involutory outer automorphism.

(iii) The incidence matrix of $D_k$ is the adjacency matrix of the graph on 2058 vertices.
described in [3] and [31]. It is proved in [3] that $\text{Aut}(G)$ acts edge-transitively on the graph which implies that $\text{Aut}(G)$ acts flag-transitively on the design.

(iv) Notice that the if block size of $D_k$ is even so is that of its complementary design $\overline{D}_k = D_{2058-k}$. One can check, using Magma ([4]), that if $\Delta = \bigcup_{l=1}^{4} \Omega_{i,l}$, $1 \leq l \leq 4$, a union of $S_4(4):2$-orbits is of even size then the intersection $\Delta \cap \Delta^g, \forall g \in G \setminus G_{\Delta}$ is of even size. So, for $i \neq j$ consider two distinct blocks $B_i$ and $B_j$ in $D_k$ (respectively $\overline{D}_k$). Since both $|B_i \cap B_j| = \lambda_{i,j}$ and $k$ are even we have that $D_k$ (respectively $\overline{D}_k$) is a self-orthogonal design. The details of the block intersection numbers can be verified from Table 2.

In Table 2 we give the numbers $l_i$ of blocks intersecting a given block in precisely $i$ points.

| $k$ | $D_k$ | $i$ | $l_i$ | $\overline{D}_k$ | $i$ | $l_i$ |
|-----|-------|-----|------|-----------------|-----|------|
| 136 | 1-(2058, 136, 136) | 272 | 1-(2058, 272, 272) | 1786 | 1-(2058, 1786, 1786) | 1546 | 1360 |
| 272 | 1-(2058, 272, 272) | 1786 | 1-(2058, 1786, 1786) | 1546 | 1360 |
| 426 | 1-(2058, 426, 426) | 1632 | 1-(2058, 1632, 1632) | 1562 | 1360 |
| 562 | 1-(2058, 562, 562) | 1496 | 1-(2058, 1496, 1496) | 1344 | 1360 |
| 698 | 1-(2058, 698, 698) | 1360 | 1-(2058, 1360, 1360) | 1128 | 1360 |

Table 2. Block intersection numbers of $D_k$ and $\overline{D}_k = D_{2058-k}$

Since the designs obtained are transitive, we have that all block intersection numbers are even.

In Table 3 we give all 1-designs constructed on 2058 points ($k \leq \frac{n}{2}$) having He acting as an automorphism group.

| Orbits | Parameters | Full Automorphism Group |
|--------|------------|-------------------------|
| $\Omega_1, \Omega_4$ | 1-(2058, 2058, 2058) | He |
| $\Omega_1, \Omega_4, \Omega_3$ | 1-(2058, 562, 562) | He |
| $\Omega_2, \Omega_3, \Omega_4$ | 1-(2058, 1360, 1360) | He |
| $\Omega_1, \Omega_3, \Omega_4$ | 1-(2058, 137, 137) | He |
| $\Omega_3, \Omega_4$ | 1-(2058, 425, 425) | He |
| $\Omega_2, \Omega_3$ | 1-(2058, 1360, 1360) | He |
| $\Omega_1, \Omega_3$ | 1-(2058, 1360, 1360) | He |

Table 3. Symmetric 1-designs on 2058 points

The fact that the two orbits of length 136 are interchanged by the outer automorphism, and the other ($S_4(4):2$)-orbits are stabilized by the outer automorphism, implies that the designs with equal block sizes are isomorphic.
5.2. Symmetric designs on 8330 points. In a manner similar to that in Section 5.1, here we construct symmetric 1-designs from the rank-7 permutation representation of the Held group on 8330 points. In Table 4 we give all 1-designs on 8330 points that can be constructed in the described way.

| Orbits | Parameters | Full Automorphism Group |
|--------|------------|-------------------------|
| \(\Delta_1, \Delta_2\) | \(1\)-\((8330, 105, 105)\) | He:2 |
| \(\Delta_1, \Delta_2, \Delta_3\) | \(1\)-\((8330, 1450, 1450)\) | He:2 |
| \(\Delta_1, \Delta_2, \Delta_3, \Delta_4\) | \(1\)-\((8330, 2290, 2290)\) | He |
| \(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5\) | \(1\)-\((8330, 2010, 2010)\) | He |
| \(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6\) | \(1\)-\((8330, 3850, 3850)\) | He:2 |
| \(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6, \Delta_7\) | \(1\)-\((8330, 3130, 3130)\) | He:2 |
| \(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6, \Delta_7, \Delta_8\) | \(1\)-\((8330, 2710, 2710)\) | He:2 |
| \(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6, \Delta_7, \Delta_8, \Delta_9\) | \(1\)-\((8330, 3010, 3010)\) | He |
| \(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6, \Delta_7, \Delta_8, \Delta_9, \Delta_{10}\) | \(1\)-\((8330, 2290, 2290)\) | He |
| \(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6, \Delta_7, \Delta_8, \Delta_9, \Delta_{10}, \Delta_{11}\) | \(1\)-\((8330, 945, 945)\) | He |
| \(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6, \Delta_7, \Delta_8, \Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12}\) | \(1\)-\((8330, 825, 825)\) | He |
| \(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6, \Delta_7, \Delta_8, \Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12}, \Delta_{13}\) | \(1\)-\((8330, 840, 840)\) | He |
| \(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6, \Delta_7, \Delta_8, \Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{14}\) | \(1\)-\((8330, 2400, 2400)\) | He |
| \(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6, \Delta_7, \Delta_8, \Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{14}, \Delta_{15}\) | \(1\)-\((8330, 2040, 2040)\) | He |
| \(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6, \Delta_7, \Delta_8, \Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{14}, \Delta_{15}, \Delta_{16}\) | \(1\)-\((8330, 2040, 2040)\) | He |
| \(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6, \Delta_7, \Delta_8, \Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{14}, \Delta_{15}, \Delta_{16}, \Delta_{17}\) | \(1\)-\((8330, 1680, 1680)\) | He |
| \(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6, \Delta_7, \Delta_8, \Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{14}, \Delta_{15}, \Delta_{16}, \Delta_{17}, \Delta_{18}\) | \(1\)-\((8330, 1680, 1680)\) | He |
| \(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6, \Delta_7, \Delta_8, \Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{14}, \Delta_{15}, \Delta_{16}, \Delta_{17}, \Delta_{18}, \Delta_{19}\) | \(1\)-\((8330, 1680, 1680)\) | He |
| \(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6, \Delta_7, \Delta_8, \Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{14}, \Delta_{15}, \Delta_{16}, \Delta_{17}, \Delta_{18}, \Delta_{19}, \Delta_{20}\) | \(1\)-\((8330, 1680, 1680)\) | He |
| \(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6, \Delta_7, \Delta_8, \Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{14}, \Delta_{15}, \Delta_{16}, \Delta_{17}, \Delta_{18}, \Delta_{19}, \Delta_{20}, \Delta_{21}\) | \(1\)-\((8330, 1680, 1680)\) | He |
| \(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6, \Delta_7, \Delta_8, \Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{14}, \Delta_{15}, \Delta_{16}, \Delta_{17}, \Delta_{18}, \Delta_{19}, \Delta_{20}, \Delta_{21}, \Delta_{22}\) | \(1\)-\((8330, 1680, 1680)\) | He |
| \(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6, \Delta_7, \Delta_8, \Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{14}, \Delta_{15}, \Delta_{16}, \Delta_{17}, \Delta_{18}, \Delta_{19}, \Delta_{20}, \Delta_{21}, \Delta_{22}, \Delta_{23}\) | \(1\)-\((8330, 1680, 1680)\) | He |
| \(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6, \Delta_7, \Delta_8, \Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{14}, \Delta_{15}, \Delta_{16}, \Delta_{17}, \Delta_{18}, \Delta_{19}, \Delta_{20}, \Delta_{21}, \Delta_{22}, \Delta_{23}, \Delta_{24}\) | \(1\)-\((8330, 1680, 1680)\) | He |
| \(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6, \Delta_7, \Delta_8, \Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{14}, \Delta_{15}, \Delta_{16}, \Delta_{17}, \Delta_{18}, \Delta_{19}, \Delta_{20}, \Delta_{21}, \Delta_{22}, \Delta_{23}, \Delta_{24}, \Delta_{25}\) | \(1\)-\((8330, 1680, 1680)\) | He |

Table 4. Symmetric 1-designs on 8330 points

Notice that the maximal subgroup isomorphic to \(2^2 L_3(4)\):S_3 acts on the set \(\{1, 2, \ldots, 8330\}\) with 7 orbits \(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6, \text{ and } \Delta_7\) and subdegrees 1, 105, 1344, 840, 720, 840 and 4480, respectively. The two orbits of length 840 are interchanged by the outer automorphism of the group He, and the other \(2^2 L_3(4)\):S_3-orbits are stabilized by the group He:2, which implies that the designs with equal
block sizes are isomorphic and have He acting as the full automorphism group. All other designs in Table 4 are pairwise non-isomorphic and have He:2 acting as the full automorphism group. From these considerations, we deduce the following lemma.

**Lemma 2.** Let $\Delta$ be a union of some $2^2 \cdot L_3(4); S_4$-orbits of the set $\{1, \ldots, 8330\}$ such that $1 < |\Delta| \leq 4165$. Then $B = \{ \Delta^g : g \in G \}$ is a set of blocks of a symmetric design $D'_{|\Delta|}$ with parameters $1-(8330, |\Delta|, |\Delta|)$ having He acting primitively as the automorphism group on the points and the blocks of the design. Up to isomorphism, there are 46 symmetric 1-designs on 8330 points admitting He as a primitive automorphism group. Of these, 30 have He:2 acting as the full automorphism group and 16 have He acting as the full automorphism group.

**Remark 1.** On computing the block intersection sizes we conclude that 5 designs in Table 4, namely the designs with parameters: $1-(8330, 1450, 1450)$, $1-(8330, 3130, 3130), 1-(8330, 1666, 1666), 1-(8330, 2904, 2904), 1-(8330, 1680, 1680)$ are self-orthogonal designs, and the 3 designs with parameters: $1-(8330, 1681, 1681), 1-(8330, 1449, 1449), 1-(8330, 3129, 3129)$ are weakly self-orthogonal with $k$ odd and even block intersection numbers.

### 5.3. Non-Symmetric Designs

Let $G_1$ and $G_2$ be permutation representations of the group He on 2058 and 8330 points, respectively. Let $S_3$ be stabilizer of a point (i.e. maximal subgroup) in the permutation group $G_2$ and $S_1$ be a maximal subgroup of $G$ isomorphic to the group $S_2$. By using [4] we obtained that $S_1$ acts on the set $\{1, 2, \ldots, 2058\}$ with five orbits $\Delta_1, \Delta_2, \Delta_3, \Delta_4$, and $\Delta_5$ with subdegrees $21, 21, 336, 840$, and 840 respectively.

In Table 5 we give all 1-designs on 2058 points with 8330 blocks ($k \leq \frac{v}{2}$) and invariant under He.

| Design | Orbits | Parameters | Full Automorphism Group |
|--------|--------|------------|-------------------------|
| $D''_1$ | $\Delta_1$ | 1-(2058, 840, 3400) | He |
| $D''_2$ | $\Delta_1, \Delta_2$ | 1-(2058, 861, 3485) | He |
| $D''_3$ | $\Delta_1, \Delta_3$ | 1-(2058, 861, 3485) | He |
| $D''_4$ | $\Delta_1, \Delta_4$ | 1-(2058, 861, 3485) | He |
| $D''_5$ | $\Delta_1, \Delta_5$ | 1-(2058, 861, 3485) | He |
| $D''_6$ | $\Delta_1, \Delta_6$ | 1-(2058, 861, 3485) | He |
| $D''_7$ | $\Delta_1, \Delta_7$ | 1-(2058, 861, 3485) | He |
| $D''_8$ | $\Delta_1, \Delta_8$ | 1-(2058, 861, 3485) | He |
| $D''_9$ | $\Delta_1, \Delta_9$ | 1-(2058, 861, 3485) | He |
| $D''_{10}$ | $\Delta_1, \Delta_{10}$ | 1-(2058, 861, 3485) | He |
| $D''_{11}$ | $\Delta_1, \Delta_{11}$ | 1-(2058, 861, 3485) | He |
| $D''_{12}$ | $\Delta_1, \Delta_{12}$ | 1-(2058, 861, 3485) | He |
| $D''_{13}$ | $\Delta_1, \Delta_{13}$ | 1-(2058, 861, 3485) | He |
| $D''_{14}$ | $\Delta_1, \Delta_{14}$ | 1-(2058, 861, 3485) | He |
| $D''_{15}$ | $\Delta_1, \Delta_{15}$ | 1-(2058, 861, 3485) | He |

**Table 5.** 1-designs on 2058 points and 8330 blocks

Notice that the outer automorphism of the group He interchanges the two orbits of the length 21 among themselves and, similarly, the orbits of the length 840 and therefore the following pair of designs are isomorphic: $D''_1 \cong D''_5, D''_2 \cong D''_7, D''_3 \cong D''_6, D''_4 \cong D''_6, D''_6 \cong D''_7, D''_7 \cong D''_6, D''_8 \cong D''_9, D''_9 \cong D''_{10}$, and $D''_{13} \cong D''_{15}$.

Based on the above observations, we state the following:

**Lemma 3.** Let $\Delta$ be a union of some $S_1$-orbits on the set $\{1, \ldots, 2058\}$ such that $1 < |\Delta| \leq 1028$ and $S_1 \cong 2^2 \cdot L_3(4); S_3$. Then $B = \{ \Delta^g : g \in G \}$ is a set of blocks of a 1-design $D''_{|\Delta|}$ on 2058 points with $b = 8330$ admitting He as a primitive automorphism group. Up to isomorphism, there are nine 1-designs on 2058 points.
with 8330 blocks having He acting primitively as automorphism group. Three of these have He:2 acting as the full automorphism group and six have He acting as the full automorphism group.

Remark 2. The duals of the 1-designs in Table 5 have 8330 points and 2058 blocks. The definition of 1-design allows the number of blocks to be less than the number of points. So, the above dual designs could be constructed from a union of some \( S_2 \)-orbits. If \( S_2 \cong S_4(4):2 \), for the set \{1, \ldots, 8330\} as a base block.

Remark 3. If \( \Delta \) is a union of some \( S_1 \)-orbits, where \( S_1 \cong 2^2 \cdot L_3(4) : S_3 \) acts on the set \{1, \ldots, 2058\} of even size, then the intersection \( \Delta \cap \Delta^g \), \( \forall g \in G \setminus G_\Delta \), is of even size, and vice-versa. The five designs with even block sizes in Table 5 are self-orthogonal designs. Among the nine dual designs described in Lemma 3, there are seven weakly self-orthogonal designs. Of these, five are self-orthogonal designs and two have odd \( k \), and even block intersection numbers.

6. The binary codes

Notice that the binary row span of the incidence matrices of each design \( D_k \) (see Table 3) (respectively \( D_k = D_{2058-k} \)) yield the code denoted \( C_k \) (respectively \( \bar{C}_k = C_{2058-k} \)), while the difference of the incidence vectors of the blocks of the design \( D_k \) span over \( \mathbb{F}_2 \) yield the code denoted \( E_k \). In the following lemma, we examine the properties of each of the codes \( C_k \) (respectively \( \bar{C}_k \)) and show that if \( C_k \) and \( \bar{C}_k \) are different from \( V_{2058}(\mathbb{F}_2) \), then \( \text{Aut}(C_k) \cong \text{He} \) or \( \text{Aut}(\bar{C}_k) \cong \text{He}:2 \), (respectively \( \text{Aut}(C_k) \cong \text{He} \) or \( \text{Aut}(\bar{C}_k) \cong \text{He}:2 \)).

Remark 4. If \( v \) and \( k \) are even numbers, and \( D \) is a self-orthogonal 1-(v, k, r) design with an incidence matrix \( M \), then all entries of the matrix \( MM^t \) are even numbers and \( \det(MM^t) \equiv 0 \pmod{2} \). Thus, \( \det(M) \equiv 0 \pmod{2} \) and the binary code of the design \( D \) is non-trivial.

In the following remark we state the results from [1] that we will use from this point onwards.

Remark 5. Let \( E_\mathbb{F}(D) \) be the code generated by the differences of the incidence vectors of the blocks of the structure \( D \) over \( \mathbb{F} \). Then \( E_\mathbb{F}(D) \) is of codimension at most 1 in \( C_\mathbb{F}(D) \) and, for the binary field, \( 1 \in E_\mathbb{F}(D) \) if and only if \( C_\mathbb{F}(D) = E_\mathbb{F}(D) \).

Lemma 4. Let \( C_k \) and \( \bar{C}_k \) denote the binary codes defined by the row span of the incidence matrices of \( D_k \) (respectively \( \bar{D}_k \)). Then the following hold:

(i) If \( k \) is odd then \( \bar{C}_k = C_k = V_{2058}(\mathbb{F}_2) \).
(ii) If \( k \in \{136, 272, 1360, 1496, 1632\} \) then \( \bar{C}_k = C_{2058-k} = \langle C_k, 1 \rangle = C_k \oplus \langle 1 \rangle \) is a decomposable self-orthogonal code. \( 1 \notin E_k \) and \( |\dim \bar{C}_k - \dim C_k| = 1 \). Moreover, \( C_k \) is indecomposable and doubly-even, \( 1 \in C_k^\perp \) (respectively \( 1 \in \bar{C}_k^\perp \)) and \( 1 \in \bar{C}_k \). \( \text{Aut}(C_k) \cong \text{He} \) or \( \text{Aut}(\bar{C}_k) \cong \text{He}:2 \). (Note that \( C_{1360} \) is the code \( C_{698}, C_{1496} \) is the code \( C_{562} \), and \( C_{4632} \) is the code \( C_{426} \) in Table 6).
(iii) If \( k = 1360 \) or \( k = 1496 \), then \( C_k \) is an irreducible and faithful He-invariant \( \mathbb{F}_2 \)-module of dimension 680 or 51. Further, each of \( C_{1360} \) and \( C_{1496} \) are respectively the unique and smallest He-invariant \( \mathbb{F}_2 \)-submodules.

Proof. (i) The result is obtained by computations with Magma ([4], [8]). Since \( \text{Aut}(V_{2058}(\mathbb{F}_2)) = S_{2058} \), it follows that \( \text{Aut}(C_k) = \text{Aut}(\bar{C}_k) = S_{2058} \).
(ii) Since \( k \) is even, by Lemma 1(iv) it follows that \( D_k \) is self-orthogonal designs. Thus, the row span over \( \mathbb{F}_2 \) of the block-point incidence matrix.
of the design generates a binary self-orthogonal code of length 2058, namely $C_k$ (respectively $\bar{C}_k$). Further, it can be observed that the block size of $D_k$ is divisible by 4. Since the incidence vectors of the blocks of the design span the code, and these vectors have weight divisible by 4, we deduce that $C_k$ is doubly-even. The reader should notice that $1 \in C_k^\perp$ (respectively $\bar{C}_k^\perp$) as the blocks of $D_k$ (respectively $\bar{D}_k$) are of even size, hence 1 meets evenly every vector of $C_k$ (respectively $\bar{C}_k$).

As $D_k$ is the complement of $\bar{D}_k$, the difference of any two codewords in $C_k$ is in $C_k$. By using Magma ([4], [8]) we verified that these differences span a subcode of co-dimension 1 in $C_k$. Hence, this subcode must be $C_k$. Now, 1 can be written, in many ways, as the sum of a codeword in $C_k$ and a codeword in $\bar{C}_k$, and so it is in $C_k$. From this we infer that $C_k = (C_k, 1) = C_k \oplus \langle 1 \rangle$. In particular, since $C_k$ and $\langle 1 \rangle$ are He-invariant (respectively He:2-invariant) subspaces of $C_k$ we deduce that $C_k$ is a decomposable $\mathbb{F}_2$-module of He (respectively of He:2) containing the $\mathbb{F}_2$-module $C_k$.

In order to determine the full automorphism groups of the codes we argue as follows. Suppose He acts primitively on a design $D$. Then He acts primitively on the set of coordinate positions of the code of the design $D$, which means that the full automorphism group of the code is a primitive group of degree 2058 which contains the group He. Since He and He:2 are the only primitive groups of degree 2058 (except for $A_{2058}$ and $S_{2058}$), we deduce that either He or He:2 acts primitively on the non-trivial code as the full automorphism group. Moreover, if $D$ is a design with full automorphism group isomorphic to He:2 and the code of $D$ is non-trivial, then the code has for full automorphism group a group isomorphic to He:2.

Further, suppose that $D'$ is a non-trivial design with the full automorphism group isomorphic to He and that the code of the design is non-trivial. Then, by Lemma 1, there exists a design $D_1$ (see the designs constructed and given in Table 5) isomorphic but not equal to the design $D'$. If the code of the design $D'$ and the code of the design $D_1$ are equal then their full automorphism groups are isomorphic to He:2. Otherwise, the full automorphism groups of the codes are isomorphic to the group He.

(iii) If $k = 1360$, it follows from [18] that the submodule of dimension 680, or equivalently the code $C_{1360} = [2058, 680, d]$ where $d \geq 1360$ is an irreducible $\mathbb{F}_2$-submodule invariant under He. For $k = 1496$, the submodule $C_{1496}$ has dimension 51 and is irreducible. Now, the reduction modulo 2 of the ordinary character of He of degree 51 gives rise to a faithful 2-modular character of He. Moreover, since the smallest faithful representation of He in characteristic 2 has degree 51, see Jansen [17, Table 1], the minimality and hence irreducibility of $C_{1496}$ is established. Since its Brauer character is not real the 51-dimensional module is not unique ([16]). The lattice of submodules of the permutation module of dimension 2058 possesses two non-isomorphic absolutely irreducible submodules of dimension 51 (see Table 7). As codes these submodules are isomorphic. They are in fact isomorphic as codes to $C_{1496}$. That the 680-dimensional code $C_{1360}$ is unique follows from [18]. Moreover, we note that the permutation representation on the point set of $D_k$ has character $1_a + 51_a + 51_b + 680_a + 1275_a$. \qed

The codes of the same dimension in Table 6 are equal, and all codes are subcodes of the code with parameters [2058, 783]. In total, we constructed 10 non-trivial pairwise non-isomorphic codes of length 2058 all of which are self-orthogonal.

Remark 6. Note that the constructed codes are of large dimension and we were not able to compute the minimum distance. In the following results, whenever the
Table 6. Non-trivial binary codes of the pairwise non-isomorphic symmetric 1-designs on 2058 points

| $k'$ | $C_k$ | $\text{Aut}(C_k)$ | $\bar{C}_k$ | $\text{Aut}(\bar{C}_k)$ | $E_k$ |
|------|-------|-------------------|--------------|------------------------|-------|
| 126  | [2058, 782] | He:2 | [2058, 782] | He:2 | [2058, 782] |
| 562  | [2058, 52] | He | [2058, 51] | He | [2058, 51] |
| 698  | [2058, 631] | He:2 | [2058, 630] | He:2 | [2058, 630] |
| 116  | [2058, 731] | He | [2058, 732] | He | [2058, 731] |
| 272  | [2058, 102] | He:2 | [2058, 102] | He:2 | [2058, 102] |

Table 7. Number of submodules of length 2058 invariant under He

| $k'$ | $\# C_k$ | $k'$ | $\# C_k$ |
|------|---------|------|---------|
| 0    | 1       | 731  | 2       |
| 1    | 1       | 732  | 2       |
| 51   | 2       | 782  | 1       |
| 52   | 2       | 783  | 3       |
| 102  | 1       | 784  | 1       |
| 103  | 3       | 977  | 2       |
| 104  | 1       | 978  | 2       |
| 680  | 1       | 1028 | 2       |
| 681  | 1       | 1029 | 10      |

Observe that in Table 7, the number of submodules of dimension 2058 − $k'$ equals the number of those of dimension $k'$.

(ii) The 66 codes (i.e., submodules), including their duals, described in Table 7 are only claimed to be distinct and not necessarily inequivalent. In general distinct G-invariant codes are inequivalent. In particular, if $C$ and $C'$ are two equivalent codes (i.e., there exists $\sigma \in S_{2058}$ with $\sigma(C) = C$) for which $\text{Aut}(C) = \text{Aut}(C') = G$, then it follows that $C = C'$. In fact, $\text{Aut}(C) = \sigma \text{Aut}(C') \sigma^{-1} = \text{Aut}(C') = G$, which implies that $\sigma \in N_{S_{2058}}(G)$. But, since $G$ is equal to its normalizer we deduce that $\sigma \in G$, and so $C = C'$.

The discussion in Section 6 and a careful examination of Remark 7 affords us results on non-existence of codes with certain properties. In particular, we deduce the following proposition.

**Proposition 1.** Let $G$ denote the sporadic simple group He, and $\Omega$ the primitive $G$-set of size 2058 defined by the action on the cosets of $S_4(4):2$. Let $C$ be a linear code over $\mathbb{F}_2$ obtained from the 2-modular primitive representation as an $\mathbb{F}_2G$-submodule of the permutation module $\mathbb{F}_2\Omega$ of dimension 2058 admitting $G$ as an automorphism group. The following occurs:

(i) Up to isomorphism there are exactly 62 non-trivial codes of length 2058 (including their duals) invariant under $G$.

(ii) There is no $G$-invariant self-dual code $C$ of length 2058.

From Lemma 2 and Lemma 3 and by using Magma ([4], [8]) we deduce the following results.

**Lemma 5.** Let $C'_k$ and $\bar{C}'_k$ denote the binary codes of length 8330 defined by the row span of the incidence matrices of $D'_k$ (respectively $\bar{D}'_k$). Then the following hold:
(i) If \( k \in \{1450, 1666, 1680, 2904, 3130\} \), then \( C'_k \) and \( C'_1 \) are self-orthogonal codes and \( 1 \in C_k \) and \( 1 \in C'_k \).

(ii) If \( k \in \{1449, 1681, 3129\} \) and \( M_k \) is the \( b \times v \) incidence matrix of the design \( D'_k \), then the matrices \((J_{8330}, M_k)\) and \((J_{8330}, J_{8330} - M)\), where \( J_{8330} \) is the all-one matrix, generate self-orthogonal codes of length 16660.

(iii) If \( k \) is even then \( 1 \in E'_k \) and \( \dim C'_k - \dim C'_1 = 1 \) (\( C'_k \) is even when \( k \) is even). If \( k \) is odd then \( 1 \in E'_k \) and \( \dim C'_k = \dim C'_1 \) (\( C'_k = C'_1 \)).

(iv) All constructed codes are invariant under the action of the group \( \text{He} \).

| \( k \) | \( C'_k \) | \( C'_1 \) | \( E'_k \) |
|------|------|------|------|
| 106  | [8330, 7055] | [8330, 7055] | [8330, 7055] |
| 1150 | [8330, 782]  | [8330, 782]  | [8330, 782]  |
| 2290 | [8330, 1972]* | [8330, 1971]* | [8330, 1971]* |
| 3010 | [8330, 7004]* | [8330, 7003]* | [8330, 7003]* |
| 3850 | [8330, 4352]* | [8330, 4352]* | [8330, 4352]* |
| 3130 | [8330, 680]  | [8330, 680]  | [8330, 680]  |
| 2170 | [8330, 4445] | [8330, 4445] | [8330, 4445] |
| 946  | [8330, 4404]* | [8330, 4403]* | [8330, 4403]* |
| 1666 | [8330, 732]  | [8330, 731]  | [8330, 731]  |
| 2505 | [8330, 192]  | [8330, 192]  | [8330, 192]  |
| 1792 | [8330, 6955] | [8330, 6952] | [8330, 6952] |
| 826  | [8330, 2023] | [8330, 2022] | [8330, 2022] |
| 1345 | [8330, 2058] | [8330, 2058] | [8330, 2058] |
| 2165 | [8330, 3978] | [8330, 3978] | [8330, 3978] |
| 3745 | [8330, 6410] | [8330, 6410] | [8330, 6410] |
| 3025 | [8330, 6410] | [8330, 6410] | [8330, 6410] |
| 2065 | [8330, 6410] | [8330, 6410] | [8330, 6410] |
| 841  | [8330, 6410] | [8330, 6410] | [8330, 6410] |
| 1561 | [8330, 2058] | [8330, 2058] | [8330, 2058] |
| 2401 | [8330, 3978] | [8330, 3978] | [8330, 3978] |
| 721  | [8330, 3978] | [8330, 3978] | [8330, 3978] |
| 105  | [8330, 2058] | [8330, 2058] | [8330, 2058] |
| 2289 | [8330, 6410] | [8330, 6410] | [8330, 6410] |
| 3009 | [8330, 2058] | [8330, 2058] | [8330, 2058] |
| 3849 | [8330, 3978] | [8330, 3978] | [8330, 3978] |
| 2169 | [8330, 3978] | [8330, 3978] | [8330, 3978] |
| 945  | [8330, 3978] | [8330, 3978] | [8330, 3978] |
| 2505 | [8330, 3978] | [8330, 3978] | [8330, 3978] |
| 1785 | [8330, 2058] | [8330, 2058] | [8330, 2058] |
| 825  | [8330, 6410] | [8330, 6410] | [8330, 6410] |
| 1344 | [8330, 6272] | [8330, 6273] | [8330, 6273] |
| 2184 | [8330, 5081]* | [8330, 5084]* | [8330, 5084]* |
| 2904 | [8330, 51]*  | [8330, 52]*  | [8330, 51]  |
| 3744 | [8330, 2762] | [8330, 2763] | [8330, 2762] |
| 3024 | [8330, 6374] | [8330, 6375] | [8330, 6374] |
| 2064 | [8330, 2600] | [8330, 2601] | [8330, 2600] |
| 840  | [8330, 2651]* | [8330, 2652]* | [8330, 2651]* |
| 1560 | [8330, 6323]* | [8330, 6324]* | [8330, 6323]* |
| 2400 | [8330, 5134] | [8330, 5135] | [8330, 5134] |
| 1600 | [8330, 102]  | [8330, 103]  | [8330, 102]  |
| 720  | [8330, 5032] | [8330, 5033] | [8330, 5032] |

Table 8. Non-trivial binary codes of the pairwise non-isomorphic symmetric 1-designs on 8330 points

Note that the codes of the same dimension in Table 8 are equivalent. In total, we constructed 52 non-trivial pairwise non-isomorphic codes of length 8330 of which 10 are self-orthogonal. By an argument similar to that used in the proof of Lemma 4, we conclude that the full automorphism group of a constructed code of length 8330 is a primitive group of degree 8330 which contains a subgroup \( P \) isomorphic to \( \text{He} \), but does not necessarily contain \( \text{Aut}(P) \) as a subgroup. Hence, the full automorphism group of the codes denoted with * in Table 8 is a primitive group of degree 8330 which does not contain \( \text{Aut}(\text{He}) \).

**Lemma 6.** Let \( C''_k \) and \( C''_{k'} \) denote the binary codes of length 2058 defined by the row span of the incidence matrices of \( D''_k \) with blocks of size \( k \) (respectively \( D''_{k'} \)). Then the following hold:
(i) If $k$ is odd, then $C''_k = \bar{C}''_k = V_{2058}(F_2)$.

(ii) If $k$ is even, then $C''_k$ and $C'''_k$ are non-trivial self-orthogonal codes. $1 \not\in E''_k$ and, $|\dim C''_k - \dim C'''_k| = 1$ ($C''_k \leq C_k$ or $C'''_k \leq C_k$). Moreover, $1 \in (C''_k)^\perp$ and $1 \in (C'''_k)^\perp$.

(iii) If $k$ is even, then $C''_k$ and $\bar{C}''_k$ (codes of the dual designs and their complementary designs) are self-orthogonal codes of length 8330 and $1 \in (C''_k)^\perp$ and $1 \in (\bar{C}''_k)^\perp$. Moreover, $C''_k \leq C''_k$ or $\bar{C}''_k \leq \bar{C}''_k$.

(iv) If $k \in \{21, 357\}$ and $M_k$ is the incidence matrix of the design $D'_k$ then the matrices $(I_{2058}, M_k)$ and $(I_{2058}, \bar{M}_k)$ generate self-orthogonal codes of length 10388 invariant under the action of $\text{He}$.

(v) All codes constructed in this way are invariant under the action of the group $\text{He}$.

| $k$ | $C''_k$ | $\text{Aut}(C''_k)$ | $\bar{C}''_k$ | $\text{Aut}(\bar{C}''_k)$ | $E''_k$ |
|-----|---------|----------------------|---------------|----------------------|---------|
| 840 | [2058, 731] | He | [2058, 731] | He | [2058, 731] |
| 882 | [2058, 52] | He | [2058, 51] | He | [2058, 51] |
| 336 | [2058, 680] | He:2 | [2058, 681] | He:2 | [2058, 680] |
| 378 | [2058, 103] | He:2 | [2058, 102] | He:2 | [2058, 102] |
| 42 | [2058, 783] | He:2 | [2058, 782] | He:2 | [2058, 782] |

Table 9. Non-trivial binary codes of the pairwise non-isomorphic 1-designs on 2058 points and 8330 blocks

Notice that the codes of the same dimension in Table 9 are isomorphic. Furthermore, they are isomorphic to the codes with the same dimension in Table 6.

| $k$ | $C'''_k$ | $\text{Aut}(C'''_k)$ | $\bar{C}'''_k$ | $\text{Aut}(\bar{C}'''_k)$ |
|-----|---------|----------------------|---------------|----------------------|
| 840 | [8330, 731] | He | [8330, 731] | He |
| 882 | [8330, 52] | He | [8330, 51] | He |
| 336 | [8330, 680] | He:2 | [8330, 681] | He:2 |
| 378 | [8330, 103] | He:2 | [8330, 102] | He:2 |
| 42 | [8330, 783] | He:2 | [8330, 782] | He:2 |

Table 10. Non-trivial binary codes of the pairwise non-isomorphic 1-designs on 8330 points and 2058 blocks

7. Codes from orbit matrices of weakly self-orthogonal designs

In this section we will describe self-orthogonal codes obtained from the orbit matrices of the weakly self-orthogonal 1-designs constructed in Section 5.1. Notice that a method of constructing self-orthogonal codes from orbit matrices of a 2-design was introduced in [13] and generalized in [11]. In this paper we generalize that method further by constructing the codes from weakly self-orthogonal designs.

**Remark 8.** Let $D$ be a self-orthogonal 1-$(v, k, r)$ design and $G$ be an automorphism group of the design. Let $V = |V_1|, \ldots, v_n = |V_n|$ be the sizes of point orbits and $b_1 = |B_1|, \ldots, b_m = |B_m|$ be the sizes of block orbits under the action of the group $G$. Moreover, let $a_{i,j}$ be the number of points of the orbit $V_j$ incident with a block of the orbit $B_i$ and let $c_{i,j}$ be the number of blocks of the orbit $B_i$ incident with a point of the orbit $V_j$. It is easy to see that $\sum_{j=1}^{n} a_{i,j} = k$ and $\sum_{i=1}^{m} c_{i,j} = \sum_{j=1}^{n} \frac{b_j}{v_j} a_{i,j} = r$.

For $x \in B_i$, by counting the incidence pairs $(P, x')$ such that $x' \in B_i$ and $P$ is incident with the block $x$, we obtain

$$\sum_{x' \in B_i} |x \cap x'| = \sum_{j=1}^{n} a_{s,j} c_{t,j} = \sum_{j=1}^{n} \frac{b_j}{v_j} a_{s,j} a_{t,j}.$$
7.1. Codes from orbit matrices of self-orthogonal 1-designs.

**Theorem 4.** Let \( D \) be a self-orthogonal \( 1-(v, k, r) \) design and \( G \) be an automorphism group of the design acting on \( D \) with \( n \) point orbits of length \( w \) and block orbits of lengths \( b_1, \ldots, b_m \), such that \( \frac{w}{w} \) is an odd number, for \( i \in \{1, \ldots, m\} \). Then the binary linear code spanned by the rows of the orbit matrix of the design \( D \) (with respect to the action of the group \( G \)) is a self-orthogonal code of length \( \frac{w}{w} \).

**Proof.** Since \( D \) is a self-orthogonal 1-design and \( w = v_1 = v_2 = \ldots = v_n \), it follows from Remark 8 that \( \sum_{j=1}^{n} a_{x,j}a_{t,j} \equiv 0 \mod 2 \). Thus, we conclude that the matrix

\[
\begin{bmatrix}
a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\
a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m,1} & a_{m,2} & \cdots & a_{m,n}
\end{bmatrix}
\]

spans a binary self-orthogonal code of length \( \frac{w}{w} \).

By using the above method we constructed self-orthogonal codes from the orbit matrices of the self-orthogonal 1-designs on 2058 and 8330 points given in Section 5.1. In order to construct the orbit matrices we determined orbits of some cyclic subgroups of prime order of the group \( H_e \) acting with orbits of the same length. Precisely,

- cyclic subgroup of order 3 of the group \( H_e \) acting on the set \( \{1, 2, \ldots, 2058\} \) with orbits of length 3 (Table 11 and Table 12),
- cyclic subgroup of order 7 of the group \( H_e \) acting on the set \( \{1, 2, \ldots, 2058\} \) with orbits of length 7 (Table 11 and Table 12),
- cyclic subgroup of order 7 of the group \( H_e \) acting on the set \( \{1, 2, \ldots, 8330\} \) with orbits of length 7 (Table 13 and Table 14),
- cyclic subgroup of order 17 of the group \( H_e \) acting on the set \( \{1, 2, \ldots, 8330\} \) with orbits of length 17 (Table 13 and Table 14).

| [294, 111] | [294, 110] | [686, 261] | [686, 260] |
|-----------|-----------|-----------|-----------|
| [294, 10, 126] | [294, 6, 120] | [686, 18, 224] | [686, 17, 224] |
| [294, 93] | [294, 98] | [686, 227] | [686, 226] |
| [294, 101] | [294, 105] | [686, 243] | [686, 244] |
| [294, 18, 96] | [294, 13, 112] | [686, 34, 192] | [686, 35, 126] |

**Table 11.** Non-trivial binary pairwise non-equivalent codes of the orbit matrices of the symmetric self-orthogonal 1-designs on 2058 points

| [294, 104] | [294, 105] | [686, 244] | [686, 244] |
|-----------|-----------|-----------|-----------|
| [294, 7, 120] | [294, 6, 120] | [686, 18, 224] | [686, 17, 224] |
| [294, 98] | [294, 99] | [686, 226] | [686, 227] |
| [294, 13, 112] | [294, 12, 112] | [686, 35, 126] | [686, 34, 192] |
| [294, 111] | [294, 110] | [686, 261] | [686, 260] |

**Table 12.** Non-trivial binary pairwise non-equivalent codes of the orbit matrices of the self-orthogonal 1-designs on 2058 points and 8330 blocks

**Theorem 5.** Let \( D \) be a self-orthogonal \( 1-(v, k, r) \) design and \( G \) be an automorphism group of \( D \) acting with \( f_1 \) fixed points and \( n \) point orbits of length 2, and with \( f_2 \) fixed blocks and \( m \) block orbits of length 2. Then
Proof.

1. the binary linear code spanned by the matrix

\[
OM_1 = \begin{bmatrix}
a_{1,1} & a_{1,2} & \ldots & a_{1,f_1} \\
a_{2,1} & a_{2,2} & \ldots & a_{2,f_1} \\
\vdots & \vdots & \ddots & \vdots \\
a_{f_2,1} & a_{f_2,2} & \ldots & a_{f_2,f_1}
\end{bmatrix},
\]

where the columns 1, \ldots, \( f_1 \) correspond to the fixed points and the rows 1, \ldots, \( f_2 \) correspond to the fixed blocks, is a self-orthogonal code of length \( f_1 \);

2. the binary linear code spanned by the matrix

\[
OM_2 = \begin{bmatrix}
a_{f_2+1,f_1+1} & a_{f_2+1,f_1+2} & \ldots & a_{f_2+1,f_1+n} \\
a_{f_2+2,f_1+1} & a_{f_2+2,f_1+2} & \ldots & a_{f_2+2,f_1+n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{f_2+m,f_1+1} & a_{f_2+m,f_1+2} & \ldots & a_{f_2+m,f_1+n}
\end{bmatrix}
\]

is a self-orthogonal code of length \( n \).

Table 13. Non-trivial binary pairwise non-equivalent codes of the orbit matrices of the self-orthogonal 1-designs on 8330 points and 2058 blocks

| [490, 4] | [490, 4] | [1190, 9] | [1190, 9] |
| [490, 4, 210] | [490, 3, 240] | [1190, 10, 5] | [1190, 9, 5] |
| [490, 40] | [490, 41] | [1190, 92] | [1190, 93] |
| [490, 7, 90] | [490, 6, 224] | [1190, 19, 360] | [1190, 18, 360] |
| [490, 47] | [490, 46] | [1190, 111] | [1190, 110] |

Table 14. Non-trivial binary pairwise non-equivalent codes of the orbit matrices of the self-orthogonal symmetric 1-designs on 8330 points

| [490, 47] | [490, 46] | [1190, 111] | [1190, 110] |
| [490, 41] | [490, 40] | [1190, 99] | [1190, 98] |
| [490, 44] | [490, 43] | [1190, 105] | [1190, 104] |
| [490, 3, 240] | [490, 4, 210] | [1190, 6, 568] | [1190, 7, 568] |
| [490, 6, 224] | [490, 7, 90] | [1190, 12, 512] | [1190, 13, 512] |

1. If \( B_1 \) and \( B_2 \) are blocks fixed under the action of the group \( G \) on the design, then \( B_1, B_2 \) and \( B_1 \cap B_2 \) are unions of some \( G \)-orbits of the point set. Both \( k \) and block intersection sizes are even and, therefore, each fixed block and intersection of any two fixed blocks contains even number of fixed points. It follows that the matrix

\[
\begin{bmatrix}
a_{1,1} & a_{1,2} & \ldots & a_{1,f_1} \\
a_{2,1} & a_{2,2} & \ldots & a_{2,f_1} \\
\vdots & \vdots & \ddots & \vdots \\
a_{f_2,1} & a_{f_2,2} & \ldots & a_{f_2,f_1}
\end{bmatrix}
\]

spans a binary self-orthogonal code.

2. Let \( B_3 \) and \( B_4 \) be block orbits of size 2 under the action of the group \( G \) on the design. It follows from the Remark 8 that

\[
\sum_{x' \in B_3} |x \cap x'| = \sum_{j=1}^{f_1} \frac{b_i}{v_j} a_{s,j} a_{t,j} + \sum_{j=f_1+1}^{f_1+n} \frac{b_i}{v_j} a_{s,j} a_{t,j} = 2 \sum_{j=1}^{f_1} a_{s,j} a_{t,j} + \sum_{j=f_1+1}^{f_1+n} a_{s,j} a_{t,j}.
\]
The design is self-orthogonal and we conclude that \( \sum_{j=1}^{f_1+n} a_{s,j} a_{t,j} \) is even and that the binary code spanned by the matrix

\[
\begin{bmatrix}
a_{f_2+1,f_1+1} & a_{f_2+1,f_1+2} & \cdots & a_{f_2+1,f_1+n} \\
a_{f_2+2,f_1+1} & a_{f_2+2,f_1+2} & \cdots & a_{f_2+2,f_1+n} \\
\vdots & \vdots & \ddots & \vdots \\
\end{bmatrix}
\]

is self-orthogonal.

There are, up to conjugation, 2 cyclic subgroups of order 2 of the group \( H_e \). These subgroups act on the set \( \{1, \ldots, 2058\} \) with 42 and 154 fixed points, respectively, and on the set \( \{1, \ldots, 8330\} \) they act with 42 and 346 fixed points, respectively. By applying Theorem 5 we constructed self-orthogonal linear codes from the orbit matrices (with respect to the action of the cyclic groups of order 2) of the constructed self-orthogonal designs with 2058 or 8330 points. In that way we constructed 160 codes. The results are presented in Table 15, Table 16, Table 17, and Table 18.

**Table 15.** Parameters of non-trivial binary codes of the orbit matrices of the self-orthogonal symmetric 1-designs with 2058 points (Theorem 5, item 1, fixed points and blocks)

| Parameters | Code length |
|------------|-------------|
| [154, 75, 8] | [154, 74, 8] | [42, 15, 8] | [42, 14, 8] |
| [154, 12, 42] | [154, 11, 48] | [42, 4, 18] | [42, 3, 24] |
| [154, 57, 12] | [154, 56, 12] | [42, 9, 12] | [42, 8, 12] |
| [154, 65, 8] | [154, 66, 8] | [42, 11, 8] | [42, 12, 8] |
| [154, 20, 32] | [154, 21, 32] | [42, 6, 16] | [42, 7, 16] |

**Table 16.** Parameters of non-trivial binary codes of the orbit matrices of the self-orthogonal symmetric 1-designs with 2058 points (Theorem 5, item 2, non-fixed part of orbit matrices)

| Parameters | Code length |
|------------|-------------|
| [952, 352] | [1008, 384] |
| [952, 20, 368] | [1008, 24, 336] |
| [952, 312] | [1008, 336] |
| [952, 332] | [1008, 360] |
| [952, 40, 224] | [1008, 48] |

**Table 17.** Parameters of non-trivial binary codes of the orbit matrices of the self-orthogonal symmetric 1-designs with 8330 points (Theorem 5, item 1, fixed points and blocks)

| Parameters | Code length |
|------------|-------------|
| [346, 75, 26] | [346, 74, 32] | [42, 15, 6] | [42, 14, 8] |
| [346, 57, 32] | [346, 56, 32] | [42, 9, 12] | [42, 8, 12] |
| [346, 66, 26] | [346, 65, 32] | [42, 12, 8] | [42, 11, 8] |
| [346, 11, 152] | [346, 12, 106] | [42, 3, 24] | [42, 4, 18] |
| [346, 20, 96] | [346, 21, 96] | [42, 6, 16] | [42, 7, 16] |

**Remark 9.**
1. The code with parameters [42, 3, 24] is optimal.
2. Codes constructed (Theorem 5, item 2) from the self-orthogonal symmetric 1-designs on 2058 points with \( k > 1029 \) have the same parameters as the ones listed in Table 16.
3. Codes constructed (Theorem 5) from non-symmetric self-orthogonal designs with 2058 points and 8330 blocks have the same parameters as the codes in Tables 15 and 16.
Table 18. Parameters of non-trivial binary codes of the orbit matrices of the self-orthogonal symmetric 1-designs with 8330 points (Theorem 5, item 2, non-fixed part of orbit matrices)

| Parameters | Length |
|------------|--------|
| [3992, 352] | 4144, 384 |
| [3992, 332] | 4144, 360 |
| [3992, 20, 1776] | 4144, 24, 1776 |
| [3992, 40] | 4144, 48 |

4. Codes constructed (Theorem 5, item 2) from the self-orthogonal symmetric 1-designs on 8330 points with \( k > 4165 \) have the same parameters as the ones listed in Table 18.

5. Codes constructed (Theorem 5) from non-symmetric self-orthogonal designs with 8330 points and 2058 blocks have the same parameters as the codes in Tables 17 and 18.

7.2. Codes from extended orbit matrices of weakly self-orthogonal 1-designs with \( k \) odd and even block intersection sizes.

Theorem 6. Let \( D \) be a weakly self-orthogonal 1-design with \( k \) is odd and even block intersection numbers. Let \( G \) be an automorphism group of the design which acts on \( D \) with \( n \) point orbits of length \( w \) and block orbits of lengths \( b_1, \ldots, b_m \) such that \( \frac{b_i}{w} \) is an odd number for \( i \in \{1, \ldots, m\} \). Let the matrix

\[
\begin{bmatrix}
    a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\
    a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m,1} & a_{m,2} & \cdots & a_{m,n}
\end{bmatrix}
\]

be the orbit matrix of the design of that action. Then the matrix

\[
\begin{bmatrix}
    1 & 0 & \cdots & 0 & a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\
    0 & 1 & \cdots & 0 & a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & 1 & a_{m,1} & a_{m,2} & \cdots & a_{m,n}
\end{bmatrix}
\]

generates a binary self-orthogonal linear code of length \( m + \frac{v}{w} \).

Proof. Recall that \( D \) is a weakly self-orthogonal 1-design such that \( k \) is odd and the block intersection numbers are even. Now since \( w = v_1 = v_2 = \ldots = v_n \) and, since by Remark 8 we have \( \sum_{j=1}^{n} a_{s,j}a_{t,j} \equiv 0 \) mod 2 for \( s \neq t \) and \( \sum_{j=1}^{n} a_{s,j}a_{s,j} \equiv 1 \) mod 2, we conclude that the matrix

\[
\begin{bmatrix}
    1 & 0 & \cdots & 0 & a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\
    0 & 1 & \cdots & 0 & a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & 1 & a_{m,1} & a_{m,2} & \cdots & a_{m,n}
\end{bmatrix}
\]

generates a binary self-orthogonal code of the length \( m + \frac{v}{w} \). \( \square \)

Using this method we construct 6 self-orthogonal binary codes with parameters [980, 490] (from the action of the cyclic subgroup of order 17) and 6 self-orthogonal binary codes with parameters [2380, 1190] (from the action of the cyclic subgroup of order 7) from the obtained weakly self-orthogonal 1-design on 8330 points with odd
k and even block intersection numbers. Moreover, we construct 4 self-orthogonal binary codes with parameters [1484, 294] and 4 self-orthogonal binary codes with parameters [612, 122] from the weakly self-orthogonal 1-designs, with odd k and even block intersection numbers, on 8330 points and with 2058 blocks.

8. STRONGLY REGULAR GRAPHS CONSTRUCTED FROM THE SUPPORT DESIGNS OF CODES

In this section we construct strongly regular graphs from the support designs of the binary codes obtained in Section 7.

Let C be a linear code of length n and let D be a support design, i.e. a design with n points and block set B such that the incidence vectors of blocks are the codewords of the code C of weight w. Denote by \( S = \{|x \cap y|, x, y \in B\} \) and choose a subset \( A \subset S \). One can define a graph whose vertices are the elements of the set B and two vertices are adjacent if the size of the corresponding blocks intersection is in A.

We analyse the constructed codes of small dimension (Table 11 - Table 18) and obtained the following strongly regular graphs.

| SRG   | Automorphism group | Code       | w  |
|-------|-------------------|------------|----|
| (21, 10, 3, 6) | \( S_7 \) | [42, 11]  | 8  |
|        |                   | [42, 12]  | 8  |
|        |                   | [42, 12]  | 34 |
|        |                   | [1190, 12]| 512|
|        |                   | [1190, 13]| 512|
|        |                   | [1190, 13]| 678|
| (28, 12, 5, 4) | \( S_8 \) | [42, 8]   | 12 |
|        |                   | [42, 9]   | 12 |
|        |                   | [42, 9]   | 30 |
|        |                   | [1008, 24]| 660|
| (49, 12, 5, 2) | \( (S_7 \times S_7)/2 \) | [294, 10] | 138|
|        |                   | [294, 10] | 156|
|        |                   | [294, 10] | 636|
|        |                   | [1190, 9] | 636|
|        |                   | [1190, 10]| 554|
|        |                   | [1190, 10]| 636|
|        |                   | [1190, 18]| 360|
|        |                   | [1190, 18]| 360|
|        |                   | [1190, 19]| 360|
|        |                   | [1190, 19]| 830|
| (49, 18, 7, 6) | \( ((7 \times 7 : 3) / 2) / 2 \) | [294, 10] | 138|
|        |                   | [294, 10] | 156|
|        |                   | [294, 10] | 96 |
|        |                   | [1190, 9] | 636|
|        |                   | [1190, 10]| 554|
|        |                   | [1190, 10]| 636|
|        |                   | [1190, 18]| 360|
|        |                   | [1190, 18]| 360|
|        |                   | [1190, 19]| 830|
| (66, 10, 3, 2) | \( (L_3(4) / 2) / 2 \) | [154, 20] | 32 |
|        |                   | [154, 21] | 32 |
|        |                   | [154, 21] | 122|
| (63, 30, 14, 15) | \( 2^5 \cdot L_3(2) \) | [1008, 24]| 336|
|        |                   | [1008, 24]| 2512|
| (105, 32, 14, 12) | \( ((L_3(4) / 3) / 2) / 2 \) | [346, 20] | 96 |
|        |                   | [346, 21] | 96 |
|        |                   | [346, 21] | 250|
| (112, 30, 2, 10) | \( (L_3(4) / 3) / 2 \) | [952, 20] | 416 |
|        |                   | [952, 20] | 2080|
| (120, 42, 8, 18) | \( (L_3(4) / 2) / 3 \) | [346, 20] | 112|
|        |                   | [346, 21] | 112|
|        |                   | [346, 21] | 234|
|        |                   | [952, 20] | 588|
|        |                   | [952, 20] | 2284|

Table 19. Strongly regular graphs constructed from the support designs of the codes of small dimension

Up to isomorphism, we constructed 9 strongly regular graphs and their complements. The strongly regular graphs with parameters \((21, 10, 3, 6), (49, 12, 5, 2)\),
(56, 10, 0, 2), (105, 32, 4, 12), (112, 30, 2, 10) and (120, 42, 8, 18) are unique, up to isomorphism (see [5]). The graph with parameters (28, 12, 6, 4) is the triangular graph \( T(8) \). The graph with parameters (49, 18, 7, 9) was constructed within the classification of all strongly regular graphs admitting an automorphism group of order 5 or order 7 (see [2]).

9. The minimal 2-modular irreducible example

In what follows, as an illustration of our discussion we give an account on the codes \( C_{562} = [2058, 52, 562]_2 \) and \( C_{1496} = [2058, 51, 672]_2 \). In particular, we show that \( C_{1496} \) is the smallest non-trivial faithfully \( \mathbb{F}_2 \)-submodule on which the Held group acts irreducibly.

**Lemma 7.** \( \text{He} \) is the automorphism group of the code \( C_{562} = [2058, 52, 562]_2 \) obtained from \( D_{1496} = D_{562} \). The code \( C_{562} \) is self-orthogonal and singly-even. Its dual is a \( [2058, 2006, 4]_2 \) code. \( C_{1496} = [2058, 51, 672]_2 \) is the smallest non-trivial and faithfully irreducible \( \mathbb{F}_2 \)-submodule on which the Held group acts. If \( w \in \mathbb{C}_{562} \) or \( w \in \mathbb{C}_{1496} \) is a codeword of minimum weight, then \( (\text{He})_w \) is a maximal subgroup.

**Proof.** From Lemma 4 we deduce that \( \text{Aut}(C_{562}) \cong \text{He} \). The dimension is at most 2058/2 = 1029, but it turns out to be as small as 52. The dimension and minimum weight were obtained through computations with Magma ([4]). In Table 20, \( m \) represents the weight of a codeword and \( A_m \) denotes the number of codewords in \( C_{562} \) of weight \( m \). Note that \( w_{2058} = 1 \in C_{562} \) and that \( \langle 1 \rangle \) is an 1-dimensional \( \text{He} \)-invariant subspace of \( C_{562} \). Also \( A_{2058-m} = |\{w_m + 1 : w_m \in C_{562}\}| = |\{w_m : w_m \in C_{562}\}| = A_m \).

Notice that there are 2058 codewords of minimum weight 562 in \( C_{562} \). Thus, the incidence matrix of \( D_{562} \) is determined by the set of all minimum weight codewords up to a column permutation. That \( C_{562}^+ \) has minimum weight 4 was found using a programme of Peter Vandendriessche. It follows from Lemma 4(ii) that \( C_{562} \supseteq C_{1496} \), and that \( C_{562} \) is in fact \( C_{1496} \) adjoined by the all-one vector, and hence \( C_{562} \) is a \( [2058, 52, 562]_2 \) code. Furthermore, \( C_{1496} \) is a \( [2058, 51, 672]_2 \) code with 29155 codewords of minimum weight 672. Now, part (iii) of Lemma 4 shows that \( C_{1496} \) is the smallest non-trivial and faithfully irreducible \( \mathbb{F}_2 \)-submodule invariant under the Held group. The dual \( C_{1496}^⊥ \) of \( C_{1496} \) is a \( [2058, 2007, 6]_2 \) code with 6676495 words of minimum weight 6. For \( w \) a minimum weight codeword in \( C_{562} \) or \( C_{1496} \), we can show using Table 1, Table 20 and the orbit-stabilizer theorem that the stabilizer \( \text{Aut}(C_{562})_w \cong S_4(4):2 \), and similarly the stabilizer \( \text{Aut}(C_{1496})_w \cong 2^{16}3^3 S_6 \). In both cases these are maximal subgroups of \( \text{He} \) and thus the designs constructed by taking the images of the supports of the codewords of minimum weight under \( \text{He} \) are primitive.

\( \square \)

**Acknowledgments**

The authors would like to thank Peter Vandendriessche from Ghent University, for the valuable help provided with the computations of the weight distributions of the codes \( C_{562} \) and \( C_{1496} \) respectively. We would also like to thank the referees very much for their valuable comments and suggestions.
### Table 20. The weight distribution of $C_{562}$ ($\tilde{m} = 2058 - m$)

| $m$ | $A_m$ | $m$ | $A_m$ | $m$ | $A_m$ | $m$ | $A_m$ |
|-----|--------|-----|--------|-----|--------|-----|--------|
| 0   | 1582   | 62  | 195459 | 121 | 2449423 | 184 | 763956 |
| 562 | 2058   | 64  | 276059 | 123 | 1755804 | 186 | 767536 |
| 672 | 29155  | 66  | 355869 | 125 | 7755404 | 188 | 770912 |
| 736 | 43732  | 68  | 455369 | 127 | 8364004 | 190 | 774288 |
| 798 | 291550  | 70  | 564179 | 129 | 3628004 | 192 | 777664 |
| 822 | 932960  | 72  | 682689 | 131 | 1814004 | 194 | 781040 |
| 824 | 1399440 | 74  | 801199 | 133 | 7060004 | 196 | 784416 |
| 828 | 30513137860 | 76  | 922509 | 135 | 2120004 | 198 | 787792 |
| 832 | 1399440 | 78  | 1047109 | 137 | 5160004 | 200 | 791168 |
| 834 | 2826320 | 80  | 1171709 | 139 | 10320004 | 202 | 794544 |
| 840 | 1049580 | 82  | 1296309 | 141 | 20640004 | 204 | 797920 |
| 842 | 170955200 | 84  | 1420909 | 143 | 41280004 | 206 | 801304 |
| 844 | 574703360 | 86  | 1545509 | 145 | 82560004 | 208 | 804680 |
| 848 | 636745200 | 88  | 1670109 | 147 | 165120004 | 210 | 808056 |
| 850 | 13994400 | 90  | 1794709 | 149 | 330240004 | 212 | 811432 |

### References

[1] E. F. Assmus, Jr and J. D. Key, *Designs and Their Codes*, Cambridge: Cambridge University Press, 1992. Cambridge Tracts in Mathematics, Vol. 103 (Second printing with corrections, 1993).

[2] M. Behbahani and C. Lam, Strongly regular graphs with non-trivial automorphisms, *Discrete Math.*, 311 (2011), 132–144.

[3] J. v. Bon, A. M. Cohen and H. Cuypers, Graphs related to Held’s simple group, *J. Algebra*, 123 (1989), 6–26.

[4] W. Bosma, J. Cannon and C. Playoust, The Magma algebra system I: The user language, *J. Symb. Comp.*, 24 (1997), 235–265.

[5] A. E. Brouwer, Strongly regular graphs, C.J. Colbourn, J.H. Dinitz (Eds.), *Handbook of Combinatorial Designs (second ed.)*, Chapman & Hall/CRC, Boca Raton (2007), 852–868.

[6] A. E. Brouwer and C. J. van Eijl, On the $p$-rank of the adjacency matrices of strongly regular graphs, *J. Algebraic Combin.*, 1 (1992), 329–346.

[7] G. Butler, The maximal subgroups of the sporadic simple group of Held, *J. Algebra*, 69 (1981), 67–81.

[8] J. Cannon, A. Steel and G. White, Linear codes over finite fields, In J. Cannon and W. Bosma, editors, *Handbook of Magma Functions*, pages 3951–4023. Computational Algebra Group, Department of Mathematics, University of Sydney, 2006. V2.13, http://magma.maths.usyd.edu.au/magma.

[9] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *An Atlas of Finite Groups*, Oxford: Oxford University Press, 1985.

[10] D. Crnković and V. Mikulić, Unitals, projective planes and other combinatorial structures constructed from the unitary groups $U_3(q)$, *Ars Combin.*, 110 (2013), 3–13.

[11] D. Crnković, B. G. Rodrigues, L. Simić and S. Rukavina, Self-orthogonal codes from orbit matrices of 2-designs, *Adv. Math. Commun.*, 7 (2013), 161–174.

[12] W. Haemers, C. Parker, V. Pless and V. Tonchev, A design and a code invariant under the simple group $Co_3$, *J. Combin. Theory, Ser. A.*, 62 (1993), 225–233.

[13] M. Harada and V. D. Tonchev, Self-orthogonal codes from symmetric designs with fixed-point-free automorphisms, *Discrete Math.*, 264 (2003), 81–90.

[14] D. Held, The simple groups related to $M_{24}$, *Journal of Algebra*, 13 (1969), 253–296.

[15] D. Held, The simple groups related to $M_{24}$, *Journal of Austral. Math. Soc.*, 16 (1973), 24–28.
[16] J. Hrabě de Angelis, A Presentation and a Representation of the Held Group, *Acta Appl. Math.*, 52 (1998), 285–290.
[17] C. Jansen, The minimal degrees of faithful representations of the sporadic simple groups and their covering groups, *LMS J. Comput. Math.*, 8 (2005), 122–144.
[18] Decomposition Matrices, The Modular Atlas homepage, 2014. http://www.math.rwth-aachen.de/~MOC/decomposition/tex/He/Hemod2.pdf, 2014.
[19] J. D. Key and J. Moori, Designs, codes and graphs from the Janko groups $J_1$ and $J_2$, *J. Combin. Math. and Combin. Comput.*, 40 (2002), 143–159.
[20] J. D. Key and J. Moori, Some irreducible codes invariant under the Janko group, $J_1$ or $J_2$, *J. Combin. Math. Combin. Comput.*, 81 (2012), 165–189.
[21] J. D. Key, J. Moori and B. G. Rodrigues, On some designs and codes from primitive representations of some finite simple groups, *J. Combin. Math. and Combin. Comput.*, 45 (2003), 3–19.
[22] J. D. Key and J. Moori, Correction to: “Codes, designs and graphs from the Janko groups $J_1$ and $J_2$”, *J. Combin. Math. Combin. Comput.*, 40 (2002), 143–159, *J. Combin. Math. Combin. Comput.*, 64 (2008), 153.
[23] J. Moori and B. G. Rodrigues, Some designs and codes invariant under the simple group $Co_2$, *J. Algebra*, 316 (2007), 649–661.
[24] J. Moori and B. G. Rodrigues, Some designs and binary codes preserved by the simple group $Ru$ of Rudvalis, *J. Algebra*, 372 (2012), 702–710.
[25] C. Parker and V. D. Tonchev, Linear Codes and Double Transitive Symmetric Design, *Linear Algebra Appl.*, 226/228 (1995), 237–246.
[26] C. E. Praeger and L. H. Soicher, Low Rank Representations and Graphs for Sporadic Groups, Cambridge: Cambridge University Press. 1997. Australian Mathematical Society Lecture Series, Vol. 8.
[27] B. G. Rodrigues, Codes of Designs and Graphs from Finite Simple Groups, Ph.D. thesis, University of Natal, Pietermaritzburg, 2002.
[28] C. M. Roney-Dougal, The primitive permutation groups of degree less than 2500, *J. Algebra*, 292 (2005), 154–183.
[29] V. D. Tonchev, Self-orthogonal designs and extremal doubly even codes, *J. Combin. Theory, A*, 52 (1989), 197–205.
[30] R. A. Wilson, Maximal subgroups of automorphism groups of simple groups, *J. London Math. Soc.*, 32 (1985), 406–466.
[31] R. A. Wilson, The Finite Simple Groups, London: Springer-Verlag London Ltd., 2009. Graduate Texts in Mathematics, Vol. 251.
[32] E. Witt, Die 5-Fach transitiven Gruppen von Mathieu, *Abh. Math. Sem. Univ. Hamburg*, 12 (1937), 256–264.
[33] E. Witt, Über Steinersche Systeme, *Abh. Math. Sem. Univ. Hamburg*, 12 (1937), 265–275.

Received for publication November 2017.

E-mail address: deanc@math.uniri.hr
E-mail address: vmikulic@math.uniri.hr
E-mail address: Rodrigues@ukzn.ac.za