The extended Lotka-Volterra lattice
and affine Jacobi varieties of spectral curves

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Abstract: Based on the work by Smirnov and Zeitlin, we study a simple realization of the matrix construction of the affine Jacobi varieties. We find that the realization is given by a classical integrable model, the extended Lotka-Volterra lattice. We investigate the integrable structure of the representative for the gauge equivalence class of matrices, which is isomorphic to the affine Jacobi variety, and make use it to discuss the solvability of the model.

1 Introduction

Consider an $N$ by $N$ matrix whose matrix elements are polynomials of $z$ of degree $M \in \mathbb{Z}_{>0}$. We write the characteristic equation of the matrix as

$$F(z, w) \equiv w^N - f_1(z)w^{N-1} + f_2(z)w^{N-2} - \cdots + (-1)^N f_N(z) = 0, \quad (1.1)$$

then each $f_i(z)$ satisfies $\deg f_i(z) \leq iM$. We assume that the algebraic curve $X$ defined by (1.1) is smooth. The genus of the curve $X$ is $g = \frac{1}{2}(N - 1)(MN - 2)$.

For the matrix and the curve $X$, Beauville introduced an isomorphism \[ \mathcal{M}_F \simeq X(g) - D. \]

Here the left hand side is the gauge equivalence class $\mathcal{M}_F$ defined as

$$\mathcal{M}_F = \{ \mathcal{M}(z) \mid \deg(\mathcal{M}(z))_{i,j} \leq M \text{ for all } i, j, \quad \text{Det} |wI - \mathcal{M}(z)| = F(z, w) \} / \text{GL}_N(\mathbb{C}),$$

and in the right hand side we have the set of nontrivial divisors $X(g) = X^g / \mathfrak{S}_g \subset \text{Div}(X)$ where $\mathfrak{S}_g$ is the symmetric group. The last term $D$ is a subset of $X(g)$, where by the Abel transformation $D$ is mapped to a $(g-1)$-dimensional subvariety of the Jacobi variety $J(X)$, which is called the theta divisor $\Theta$. The Abel transformation induces an isomorphism,

$$X(g) - D \simeq J(X) - \Theta,$$

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and we call \( J(X) - \Theta \) the affine Jacobi variety. In other words, the gauge equivalence class \( \mathcal{M}_F \) gives a matrix construction of the affine Jacobi variety. Before Beauville’s work, Mumford studied the case that the curve \( X \) is a hyperelliptic curve (the \( N = 2 \) case), and introduced a unique representative of the gauge equivalence class \( \mathcal{M}_F \).

As discussed in refs. \( 1 \), the above correspondence of matrices and Jacobi varieties closely relates to the study of finite dimensional integrable systems. The coefficients of the characteristic equation (1.1) correspond to commuting integrals of motion which generate \( g \) independent vector fields on the affine Jacobi variety. These fields determine the time-evolution of the divisors in \( X(g) - D \), which is linearized on the Jacobi variety \( J(X) \). Recently Nakayashiki and Smirnov studied Mumford’s representative of a 2 by 2 matrix from the view point of the affine ring for the affine Jacobi variety \( \mathcal{M}_F \). They investigated how the commuting integrals act on the polynomial ring generated by the matrix elements, by calculating the cohomology group. The extension of their work to that for integrable \( N \) by \( N \) matrices is studied by Smirnov and Zeitlin \( 5, 6 \). Starting with an \( N \) by \( N \) integrable monodromy matrix, they introduced a unique representative for the gauge equivalence class \( \mathcal{M}_F \) which is isomorphic to a divisor in \( X(g) - D \). The generalization of Mumford’s representative appears in refs. \( 7, 6 \). We should remark that the way of constructing the divisor from the monodromy matrix is nothing but the separation of variables (SoV) invented by Sklyanin \( 8 \).

The aim of this paper is to study a simple realization of the representative for \( \mathcal{M}_F \), based on ref. \( 5 \). For the characteristic equation (1.1) we have assumed

\[
\begin{align*}
f_k(z) &= f_k^{(0)} z^{kM} + f_k^{(1)} z^{kM-1} + \cdots + f_k^{(kM)}, \quad \text{for } k = 1, \ldots, N, \\
\text{and we add a condition}
\end{align*}
\]

\[
f_N^{(0)} = 0.
\]

The realization of the representative is given by the extended Lotka-Volterra lattice (In some papers we call it the Bogoyavlensky lattice). This is a classical integrable dynamical model defined by \((1 + 1)\)-dimensional differential-difference equation \( 9, 10, 11 \),

\[
\frac{dV_n}{dt} = 2 V_n \sum_{k=1}^{N-1} (V_{n+k} - V_{n-k}),
\]

where \( V_n \equiv V_n(t), n \in \mathbb{Z} \). In this article we denote this model using \( LV(N) \). The integrable structure of \( LV(N) \) is based on the Poisson algebra \( \mathcal{A}_{LV} \) generated by \( V_n \), and on the \( N \) by \( N \) Lax matrix given by \( 3, 4 \)

\[
\mathbf{\tilde{L}}_n(z) = (V_n)^{-\frac{N}{2}} (z^{\frac{N}{2}} E_{1,1} + (-1)^{N-1} V_n E_{1,N} + \sum_{k=1}^{N-1} E_{k+1,k}).
\]

Here \( z \in \mathbb{C} \) is a spectral parameter and \( (E_{i,j})_{m,n} = \delta_{m,i} \delta_{n,j} \). The Lax matrix composes the monodromy matrix which generates a family of commuting integrals of motion. This proves
the integrability of the model in Liouville’s sense [11]. It is remarkable that the model has an integrable quantization which can be applied to construct the vertex model linked on the crystal base theory [12]. In this paper, we study LV(N) of a periodic boundary condition, which gives a realization of the representative of \( \mathcal{M}_F \) introduced in ref. [5]. For the case of \( N = 2 \) our realization is essentially same as that introduced in ref. [13], and the previous article gives what corresponds to its generalization. Based on some special properties of the correspondence of LV(N) and the representative, we try to solve LV(N) by describing the dynamical variables \( V_n \) [1.3] in terms of the divisor in \( X(g) - D \).

The plan of this paper is as follows; in §2, starting with LV(N) we construct a monodromy matrix \( \mathbf{T}(z) \) whose matrix elements have a special form of polynomials of \( z \), and whose characteristic equation is (1.1) with (1.2). By applying the method SoV we obtain a divisor in \( X(g) - D \). In §3, following ref. [5] we review the gauge transformation which derives the representative of \( \mathcal{M}_F \), \( \mathcal{M}_F(z) \), from \( \mathbf{T}(z) \). We let \( \mathcal{A}_{M_F} \) be the Poisson algebra for the polynomial ring generated by the coefficients of the matrix elements of the representative \( \mathcal{M}_F(z) \). Next we investigate some nice properties of this gauge transformation; the transformation does not change the zeros of the separating equation (2.16), and erases the zero mode \( B_0 \). These assure the injection from a divisor to the representative. After eliminating the center of \( \mathcal{A}_{M_F} \), a family of non-trivial integrals for \( \mathcal{A}_{M_F} \) is composed of \( g \) independent variables. These integrals govern the evolution of the divisor, which is linearized on \( J(X) \). In §4, we investigate LV(N) by making use of the results in §3. We show that the center of \( \mathcal{A}_{M_F} \) is a subset of the center for \( \mathcal{A}_{LV} \), and the non-trivial integrals of motion of both algebras coincide. Our claims are that the representative \( \mathcal{M}_F(z) \) can be written in terms of the dynamical variable \( V_n \), and that the structure of \( \mathcal{M}_F(z) \) has a close relationship with the solvability of LV(N). The last section §5 is devoted to summary and remarks. We mention the quantization of LV(N) and propose some future problems.

2 Spectral curve of LV(N) and divisor

2.1 Derivation of a proper monodromy matrix

We study the integrable structure of LV(N), and derive a monodromy matrix of a special form which fits to the construction in ref. [5].

We consider the dynamical system (1.3) with a periodic boundary condition \( V_{n+L} = V_n \), and set \( L = N(N-1)M \), \( M \in \mathbb{Z}_{>0} \) for later convenience. The Hamiltonian structure of LV(N) is defined by the Poisson brackets

\[
\{V_n, V_m\} = 2V_mV_n \sum_{k=1}^{N-1} (\delta_{m,n+k} - \delta_{m,n-k}),
\]

and the Hamiltonian \( H_1 = \sum_{n=1}^L V_n \) [5]. We let \( \mathcal{A}_{LV} \) be the Poisson algebra for \( \mathbb{C}[V_n, V_n^{-1}; n \in \mathbb{Z}] \).
\( \text{Z], whose defining relations are given by (2.1). After a variable transformation} \) 
\[
V_n = (P_n P_{n+1} \cdots P_{n+N-1})^{-1} Q_n^{-1} Q_{n+N-1}, \tag{2.2}
\]
\( (2.1) \) is transformed into the Poisson brackets
\[
\{P_n, Q_m\} = \delta_{n,m} P_n Q_n, \quad \{P_n, P_m\} = \{Q_n, Q_m\} = 0, \tag{2.3}
\]
where \( P_n \) and \( Q_n \) are canonical variables. Using these variables we apply a gauge transformation to the Lax matrix \((1.4)\), 
\[
L_n(z) = \Omega_{n+1}(z) \tilde{L}_n(z) \Omega_n(z)^{-1},
\]
and obtain the following local Lax matrix;
\[
L_n(z) = z^{\frac{N}{2}} \left( P_n E_{1,1} + Q_n E_{1,2} + \frac{1}{z}(-1)^{N-1} Q_n^{-1} E_{N,1} + \sum_{k=2}^{N-1} E_{k,k+1} \right). \tag{2.4}
\]
See Appendix A for the concrete description of the gauge matrix \( \Omega_n(z) \). We introduce another Lax matrix
\[
\mathbf{L}_n(z) = (L_n^{-1}(z))^T = \frac{1}{z^{\frac{N}{2}}} \left( Q_n^{-1} E_{1,2} + \sum_{k=2}^{N-1} E_{k,k+1} + z(-1)^{N-1} Q_n E_{N,1} + z(-1)^{N-2} P_n E_{N,2} \right), \tag{2.5}
\]
where an superscript \( T \) denotes a transposition of the matrices. Note that \( \text{Det} L_n(z) = \text{Det} \mathbf{L}_n(z) = 1 \). These Lax matrices satisfy the Poisson relations as
\[
\{L_n(z) \otimes L_m(z')\} = \delta_{n,m} \left[ r(z/z') , L_n(z) \otimes L_n(z') \right],
\]
\[
\{\mathbf{L}_n(z) \otimes \mathbf{L}_m(z')\} = \delta_{n,m} \left[ -r(z'/z) , \mathbf{L}_n(z) \otimes \mathbf{L}_n(z') \right],
\]
\[
\{L_n(z) \otimes \mathbf{L}_m(z')\} = \delta_{n,m} \left[ -r^{T_2}(z/z') , L_n(z) \otimes \mathbf{L}_n(z') \right], \tag{2.6}
\]
where \( r(z) \) is a classical r-matrix;
\[
r(z) = \frac{z+1}{z-1} \sum_{k=1}^{N} E_{k,k} \otimes E_{k,k} + \frac{2}{z-1} \sum_{1 \leq j<k \leq N} \left( E_{k,j} \otimes E_{j,k} + z E_{j,k} \otimes E_{k,j} \right).
\]
In deriving the second Poisson relation in \( (2.6) \), we have used \( r(z/z')^{T_1 T_2} = -r(z'/z) \), where \( T_i \) denotes a transposition in the \( i \)-th space. Now the meaning of the local Lax matrix becomes clear that the Lax matrices \( (2.4) \) and \( (2.5) \) satisfy the Poisson relations with \( \delta_{n,m} \).

We define two monodromy matrices,
\[
T(z) = \prod_{k=1}^{L} L_k(z), \quad \mathbf{T}(z) = \prod_{k=1}^{L} \mathbf{L}_k(z).
\]
Due to (2.6) the monodromy matrices satisfy the following Poisson relations

\[ \{ T(z) \otimes T(z') \} = [ r(z/z') , T(z) \otimes T(z') ] , \quad (2.7) \]
\[ \{ \bar{T}(z) \otimes \bar{T}(z') \} = [ -r(z'/z) , \bar{T}(z) \otimes \bar{T}(z') ] , \quad (2.8) \]
\[ \{ T(z) \otimes \bar{T}(z') \} = [ -rT(z/z') , T(z) \otimes \bar{T}(z') ] . \quad (2.9) \]

The first relation denotes that the commuting integrals of motion for \( \text{LV}(N) \) are generated by \( \text{Tr} T(z) \), since the Hamiltonian \( H_1 \) is obtained by expanding \( \text{Tr} T(z) \) by \( z \). Latter two denote that the matrix \( \bar{T}(z) \) also generate the commuting integrals of motion for \( \text{LV}(N) \).

The matrix elements of \( T(z) \) and \( \bar{T}(z) \) turn out to be polynomials of \( z \), and these matrices have forms as

\[ T(z) = T_-(z) + T_0(z) + zT_+(z) , \quad (2.10) \]
\[ \bar{T}(z) = z\bar{T}_-(z) + \bar{T}_0(z) + \bar{T}_+(z) . \quad (2.11) \]

Here \( T_\pm(z), \bar{T}_\pm(z) \) are an upper/lower triangular matrices without diagonal terms, and \( T_0(z), \bar{T}_0(z) \) are diagonal matrices. All matrix elements of \( T_\pm(z) \) and \( T_0(z) \) are polynomials of degree \( N(N-1) - 1 \) but \((T_0(z))_{1,1}\) which has a polynomial of degree \( N(N-1) \). On the other hand, elements of \( T_0(z) \) are degree \( M \) except for \((\bar{T}_0(z))_{1,1}\) which is degree \( M-1 \), and \( \bar{T}_\pm(z) \) has polynomials of degree \( M-1 \).

We find that the matrix \( \bar{T}(z) \) has the characteristic equation, \( \text{Det}|wI - \bar{T}(z)| = 0 \), which coincides with (1.1) of (1.2) and satisfies \( f_N(z) = 1 \). Moreover, the construction of the matrix \( \bar{T}(z) \) (2.11) and its Poisson relation (2.8) are exactly same as those discussed in ref. 5 where the Poisson relation (2.8) defines what is called the classical algebra of observables generated by the coefficients of polynomials which compose the matrix. Therefore we conclude that \( \text{LV}(N) \) gives a realization of the algebra of observables. In the following, unless we give a notification, we let \( \bar{T}(z) \) be a matrix of a special form (2.11) whose Poisson structure is given by (2.8) and forget about the model \( \text{LV}(N) \).

### 2.2 Separation of variables and divisor

We apply SoV method to obtain the eigenvalues of the monodromy matrix \( \bar{T}(z) \) algebraically, following refs. [14][15][6]. This method gives a surjective map from the monodromy matrix \( \bar{T}(z) \) to a divisor on the curve \( X \).

Divide the matrix \( \bar{T}(z) \) into parts as

\[ \bar{T}(z) = \begin{pmatrix} a(z) & \bar{b}(z) \\ \bar{c}(z)^T & d(z) \end{pmatrix} , \quad (2.12) \]

where \( a(z) = (\bar{T}(z))_{1,1}, \bar{b}(z) \) and \( \bar{c}(z) \) are low vectors of \( N-1 \) entries, and \( d(z) \) is an \( N-1 \)
by $N - 1$ matrix. We transform $\mathbf{T}(z)$ as

$$U(z) = K \mathbf{T}(z) K^{-1}, \quad K = \mathbb{1} + \sum_{j=1}^{N-2} k_i E_{i+1,N},$$

where $k_i \in \mathbb{C}$. On the matrix $U(z)$ we impose some conditions;

$$(U(z))_{i,N} = 0, \quad \text{for } i = 1, \cdots, N - 1. \quad (2.13)$$

One sees that these conditions reduce to

$$\vec{b}(z) \cdot \vec{x}^T = 0, \quad \vec{x}_i \cdot \vec{d}(z) \cdot \vec{x}^T = 0, \quad \text{for } i = 1, \cdots, N - 2, \quad (2.14)$$

where $\vec{x}, \vec{x}_i \in \mathbb{C}^{N-1}$ are low vectors,

$$\vec{x}_i = (0, \cdots, 0, 1, 0, \cdots, 0, k_i), \quad \vec{x} = (-k_1, -k_2, \cdots, k_{N-2}, 1).$$

These vectors satisfy $\vec{x}_i \perp \vec{x}$ for all $i$, then the vectors $\vec{x}_i$ compose basis of the plane normal to $\vec{x}$. Since the vector $\vec{b}(z)$ is also orthogonal to $\vec{x}$, it can be uniquely written as

$$\vec{b}(z) = \sum_{i=1}^{N-2} \lambda_i \vec{x}_i, \quad \lambda_i \in \mathbb{C}.$$

By using (2.14), we have $\vec{b}(z) \cdot \vec{d}(z) \cdot \vec{x}^T = 0$ which enables to write $\vec{b}(z) \cdot \vec{d}(z)$ as a linear combination of $\vec{x}_i$ again. By repeating this procedure, we obtain $\vec{b}(z) \cdot \vec{d}^k \cdot \vec{x}^T = 0$ for $k \in \mathbb{Z}_{\geq 0}$. Since $\vec{x}$ is not a zero vector, the condition (2.14) finally reduces to

$$B(z) \equiv \text{Det} \begin{pmatrix}
\vec{b}(z) \\
\vec{b}(z) \cdot \vec{d}(z) \\
\vec{b}(z) \cdot \vec{d}(z)^2 \\
\vdots \\
\vec{b}(z) \cdot \vec{d}(z)^{N-2}
\end{pmatrix} = 0. \quad (2.15)$$

By the construction of $\mathbf{T}(z)$, $B(z)$ becomes a polynomial of $z$ of degree $g$,

$$B(z) = B_0 \prod_{i=1}^{g}(z - z_i). \quad (2.16)$$

The Poisson relation (2.8) ensures that all $z_i$ and $B_0$ are Poisson commutative each other. For each $z_i$ the eigenvalue of the matrix $U(z)$, $w_i \equiv (U(z_i))_{N,N}$, is obtained as

$$w_i = \text{Det} \begin{pmatrix}
\vec{b}(z_i) \\
\vec{b}(z_i) \cdot \vec{d}(z_i) \\
\vdots \\
\vec{b}(z_i) \cdot \vec{d}(z_i)^{N-3} \\
\vec{d}(z_i)
\end{pmatrix} \quad \text{Det} \begin{pmatrix}
\vec{b}(z_i) \\
\vec{b}(z_i) \cdot \vec{d}(z_i) \\
\vdots \\
\vec{b}(z_i) \cdot \vec{d}(z_i)^{N-3} \\
\vec{d}(z_i)
\end{pmatrix}^{-1}. \quad (2.17)$$

By the construction of $\mathbf{T}(z)$, $B(z)$ becomes a polynomial of $z$ of degree $g$,

$$B(z) = B_0 \prod_{i=1}^{g}(z - z_i). \quad (2.16)$$
where $\vec{\xi}$ is a low vector of $N - 1$ entries, $\vec{\xi} = (0, \cdots, 0, 1)$. The Poisson relation (2.8) shows that the separated variables, $w_i$ and $z_i$ ($i = 1, \cdots, g$), satisfy the canonical Poisson brackets;

$$\{ z_i, z_j \} = \{ w_i, w_j \} = 0, \quad \{ z_i, w_j \} = 2 \delta_{i,j} z_i w_i,$$

and $B_0$ is a zero mode,

$$\{ B_0, z_i \} = 0, \quad \{ B_0, w_i \} = -B_0 w_i.$$

We conclude that via SoV we get the map from the matrix $\bar{T}(z)$ to a divisor over $X$, $P = \sum_{i=1}^{g} [(w_i, z_i)]$, as each pair of separated variables $(w_i, z_i)$ is a point on the curve $X$. We assume that (2.16) has different zeros, $z_i \neq z_j$ for all $i \neq j$, and that no point $(w_i, z_i)$ coincide with the ramification points of the map from $X$ to $\mathbb{P}^1$. These assumptions assure $P \in X(g) - D$.

3 Integrable monodromy matrix and affine Jacobi variety

3.1 Representative of $M_F$

Let $\{ \bar{T}(z) \}_F$ be a set of matrices with a form (2.11) and whose characteristic equations coincide with (1.1). In the previous section SoV define an surjective map from the set $\{ \bar{T}(z) \}_F$ to a certain set of divisors $P \in X(g) - D$, but it is not a injective map. One easily sees the reason by comparing their dimensions, namely $\{ \bar{T}(z) \}_F$ and $X(g) - D$ respectively have $(g+N-1)$-dimension and $g$-dimension as affine spaces. Smirnov and Zeitlin introduced a representative of $M_F$ by setting a gauge transformation which eliminate the excessive dimension $N - 1$ of $\{ \bar{T}(z) \}_F$. Following ref. 5, we review the way to introduce the representative of $M_F$.

For the matrix $\bar{T}(z)$ we set

$$\bar{T}(z) = \mu_0 z^M + \mu_1 z^{M-1} + \cdots + \mu_M,$$

and define $\vec{\nu} = \vec{\epsilon}_1 \cdot \mu_1$ where $\vec{\epsilon}_1$ is a $N$-dimensional low vector whose entries are zero but $i$-th is 1. The gauge transformation from the monodromy matrix $\bar{T}(z)$ to the representative of $M_F$, $M_F(z)$, is

$$M_F(z) = S \bar{T}(z) S^{-1}, \quad \text{where} \quad S = \begin{pmatrix} \vec{\epsilon}_1 & \vec{\nu} \mu_0^{N-2} \\ \vdots & \vdots \\ \vec{\nu} \mu_0 & \vec{\nu} \end{pmatrix}. \tag{3.2}$$
Then we obtain \( M_F(z) \) as

\[
M_F(z) = U z^M + O(z^{M-1}), \quad U = \sum_{k=1}^{N} m^{(k)} E_{2,k} + \sum_{k=3}^{N} E_{k,k-1},
\]  

(3.3)

where \( m^{(k)} \) are given by

\[
m^{(1)} = (-1)^N \text{Det} \begin{pmatrix}
\vec{\nu} \\
\vec{e}_2 \mu_0 \\
\vdots \\
\vec{e}_N \mu_0
\end{pmatrix}
\]

\[
z^{N-1} - \sum_{k=2}^{N} m^{(k)} z^{N-k} = \prod_{k=2}^{N} (z - (\mu_0)_{k,k}).
\]

Especially we have

\[
(M_F(z))_{1,N} = z^{N-1} + O(z^{N-2}), \quad (M_F(z))_{1,i} = O(z^{N-2}), \text{ for } i = 1, \cdots, N - 1.
\]

The set \( \{ \mathbf{T}(z) \}_F \) is transformed to \( \{ M_F(z) \} \), and one sees that \( \{ M_F(z) \} \) is a \( g \)-dimensional affine space.

Under the gauge transformation (3.2), the zeros of \( B(z) \) (2.16) are invariant and the zero mode \( B_0 \) is canceled (see Appendix B for the proof);

\[
B(z) \mapsto B_F(z) = ( -1 )^\frac{1}{2} (N-1)(N-2) \prod_{k=1}^{g} (z - z_k).
\]

Therefore a divisor \( P = \sum_{i=1}^{g} [(w_i, z_i)] \) determines \( B_F(z) \) uniquely. In conclusion, we get the isomorphism, \( M_F \simeq X(g) - D \), where the representative \( M_F(z) \) concretely gives the matrix construction of the affine Jacobi variety.

### 3.2 Integrable system on the Jacobi variety

Let us see how the integrable structure of the monodromy matrix \( \mathbf{T}(z) \) is translated to that of the matrix \( M_F(z) \). Via (3.2), the Poisson structure of the matrix elements of \( \mathbf{T}(z) \) (2.8) induces the Poisson algebra \( \mathcal{A}_{M_F} \) generated by the matrix elements of \( M_F(z) \). For the defining relation of \( \mathcal{A}_{M_F} \), see the last part of §2 in ref. 3 and take its classical limit. We study a commuting family of integrals of motion for \( \mathcal{A}_{M_F} \) without referring the defining relation of \( \mathcal{A}_{M_F} \).

From (2.8), one obtains

\[
\{ \text{Det} \mathbf{T}(z) \cap \mathbf{T}(z') \} = 0, \quad (3.4)
\]

\[
\{ \text{Det}(w \cdot 1 - \mathbf{T}(z)), \text{Det}(w' \cdot 1 - \mathbf{T}(z')) \} = 0. \quad (3.5)
\]
Eq. (3.4) denotes that Det $\overline{T}(z)$ is Poisson commutative with all elements of $\overline{T}(z')$, namely Det $\overline{T}(z)$ belongs to the center of $A_{M_F}$, $A^0_{M_F}$. Eq. (3.3) assures that the variables $f^{(j)}_k$ compose a commutative subalgebra of $A_{M_F}$, \( \{ f^{(j)}_k, f^{(j')}_k \} = 0 \). Therefore, the dynamical system in $A_{M_F}$ has a family of integrals of motion, \( \{ f^{(j)}_k \mid k = 1, \cdots N - 1, j = 0, \cdots kM \} \), whose number is \( g + 2(N - 1) \). In the following we show that \( 2(N - 1) \) integrals, $f^{(0)}_k$ and $f^{(kM)}_k$, $k = 1, \cdots N - 1$, belong to $A^0_{M_F}$, namely the number of non-trivial integrals of motion is $g$. What we should show is

\[
\{ f^{(j)}_k \odot M_F(z) \} = 0, \quad \text{for } k = 1, \cdots, N - 1 \text{ and } j = 0, kM. \tag{3.6}
\]

One sees that since $f_k(z)$ can be written in terms of $t_k(z) \equiv \text{Tr}(\overline{T}(z))^k$;

\[
f_1(z) = t_1(z), \quad f_2(z) = \frac{1}{2}(t_1(z)^2 - t_2(z)), \quad \cdots
\]

(3.6) are reduced to

\[
\{ t^{(j)}_k \odot M_F(z) \} = 0, \quad \text{for } j = 0, kM. \tag{3.7}
\]

Here we denote the dominant terms of $\text{Tr}(\overline{T}(z))^k$ in the $z \rightarrow 0, \infty$ limits using $t^{(kM)}_k$, $t^{(0)}_k$ respectively. The Poisson relation (2.8) reduces to

\[
\{ t_k(z) \odot \overline{T}(z') \} = \text{Tr}_1\{ (\overline{T}(z))^k \odot \overline{T}(z') \} = k \text{ Tr}_1 [ r(z/z'), (\overline{T}(z))^k \odot \overline{T}(z') ],
\]

which derives

\[
\begin{align*}
\{ t^{(j)}_k \odot \overline{T}(z') \} &= k [(K^{(j)})^k, \overline{T}(z')], \\
\{ t^{(j)}_k \odot \overline{v} \} &= -k \overline{v} (K^{(j)})^k, \\
\{ t^{(j)}_k \odot \mu_0 \} &= k [(K^{(j)})^k, \mu_0],
\end{align*}
\tag{3.8}
\]

for $j = 0, kM$. Here we use the matrices $\mu_k$ (3.1) and

\[
K^{(0)} = \text{diag}[0, (\mu_0)_{2,2}, \cdots, (\mu_0)_{N,N}], \\
K^{(kM)} = -\text{diag}[0, (\mu_M)_{2,2}, \cdots, (\mu_M)_{N,N}].
\]

Due to the relations (3.8) we get

\[
\{ t^{(i)}_k \odot S \} = -k (S - E_{1,1}) (K^{(i)})^k, \quad \text{for } i = 0, kM, \tag{3.9}
\]

and (3.7) is proved.

We arrange the non-trivial integrals as

\[
f^{(1)}_1, \cdots, f^{(M-1)}_1, f^{(1)}_2, \cdots, f^{(2M-1)}_2, \cdots, f^{(1)}_{N-1}, \cdots, f^{(M(N-1)-1)}_{N-1},
\]

and number them in order,

\[
\mathcal{H}_1, \mathcal{H}_2, \cdots, \mathcal{H}_g. \tag{3.10}
\]
In conclusion, we obtain the integrable structure of $\mathcal{A}_M$ that the $g$ commuting integrals $\mathcal{H}_i$ describe the time evolution for $\mathcal{O} \in \mathcal{A}_M$;

$$\frac{\partial \mathcal{O}}{\partial \tau_i} \equiv \{\mathcal{H}_i, \mathcal{O}\}, \quad \text{for } i = 1, \cdots, g.$$ (3.11)

On the Jacobi variety $J(X)$, $\mathcal{H}_i$ generate the invariant vector field where the time evolution of the image of the divisor $P$ is linearized. By the inverse map of the Abel transformation, we get $z_i$ as functions of times $\tau_i, z_i = z_i(\tau_1, \cdots, \tau_g)$ once the initial values $z_i(0, \cdots, 0) = z_i^0$ is given. Note that the curve $X$ is determined by the eigenvalues of $\mathcal{H}_i$.

4 Description of the $\text{LV}(N)$

Now we study the realization of $\mathcal{M}_F(z)$ given by $\text{LV}(N)$. Since the characteristic equation (1.1) is invariant under the gauge transformation, the commuting family can be written in terms of the dynamical variables of $\text{LV}(N)$ by making use of the Lax matrix $\tilde{L}_n(z)$ (1.4).

We introduce variables $\mathcal{P}_k$ and $\mathcal{P}'_k$,

$$\mathcal{P}_0 = \prod_{n=1}^{L} (V_n)^{-\frac{1}{2}},$$

$$\mathcal{P}_k = \prod_{n=1}^{NM} (V_{(N-1)n+k}), \quad \text{for } k = 1, \cdots, N-1,$$

$$\mathcal{P}'_k = \prod_{n=1}^{(N-1)M} (V_{Nn+k}), \quad \text{for } k = 1, \cdots, N.$$

These variables constitute the center of $\mathcal{A}_{\text{LV}}, \mathcal{A}^0_{\text{LV}}$. Note that not all of them are independent, and the generators of $\mathcal{A}^0_{\text{LV}}$ are obtained by choosing any $2(N-1)$ variables from $\{\mathcal{P}_{k \geq 1}, \mathcal{P}'_k\}$. The direct calculations show that the elements of $\mathcal{A}^0_M$ are written in terms of these variables as

$$f_N(z) = \text{Det} \overline{T}(z) = 1,$$

$$f^{(0)}_{N-1} = \mathcal{P}_0,$$

$$f^{(0)}_1 = \mathcal{P}^{-1}_0 (\mathcal{P}^{-1}_1 + \cdots + \mathcal{P}^{-1}_{N-1}),$$

$$f^{(N-1)M}_{N-1} = \mathcal{P}_0 (\mathcal{P}'_1 + \cdots + \mathcal{P}'_N),$$

$$f^{(N-1)M}_1 = \mathcal{P}^{-1}_0 (\mathcal{P}'^{-1}_1 + \cdots + \mathcal{P}'^{-1}_N),$$

and that other elements, $f^{(0)}_k, f^{(kM)}_k$ for $k = 2, \cdots, N-2$, are obtained from the above. Therefore we see $\mathcal{A}^0_M \subset \mathcal{A}^0_{\text{LV}}$, and the non-trivial integrals of motion for $\text{LV}(N)$ have one-to-one correspondence to $\mathcal{H}_i$. 

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Based on the above observation, we conjecture that

(i) the matrix $\mathbf{M}_F(z)$ can be written in terms of $V_n$,

(ii) then all zeros of $B_F(z)$ are given by $V_n$, $z_i = z_i(\{V_n\})$, and we can solve $\text{LV}(N)$ as $V_n = V_n(\mathcal{P}_k, \mathcal{P}'_k, \mathcal{H}_i; z_i)$.

To discuss the conjecture, using (A.1) and (3.2) we rewrite $\mathbf{M}_F(z)$ as

$$\mathbf{M}_F(z) = \mathbf{S} \left( \Omega_1(z)^{-1} \right)^T \left( (\tilde{\mathbf{L}}_L(z) \cdots \tilde{\mathbf{L}}_1(z))^{-1} \right)^T \Omega_1(z)^T \mathbf{S}^{-1}. \quad (4.1)$$

Due to the construction of the gauge matrix $\Omega_1(z)$ (A.2), we reduce (4.1) to

$$\mathbf{M}_F(z) = \tilde{\mathbf{S}} \mathbf{X}(z)^{-1} \left( (\tilde{\mathbf{L}}_L(z) \cdots \tilde{\mathbf{L}}_1(z))^{-1} \right)^T \mathbf{X}(z) \tilde{\mathbf{S}}^{-1}. \quad (4.2)$$

We have conjectured by (i) that the matrix $\tilde{\mathbf{S}}$ is written in terms of $V_n$. Remember that $\text{LV}(N)$ has $N(N - 1)M$ dynamical variables $V_n$. Once we accept (i), (ii) follows (i) since we have enough number of relations to describe $V_n$ in terms of $\mathbf{P}_k, \mathbf{P}'_k, \mathbf{H}_i$ and $z_i$. Actually we have $g$ relation equations between $z_i$ and $V_n$, $g$ non-trivial integrals of motion $\mathcal{H}_i$, and $2(N - 1)$ independent generators of $\mathcal{A}^{0}_{LV}$, whose summation coincides with $N(N - 1)M$.

It should be remarked that due to the periodic boundary condition of the system, we essentially have the translation invariant such as $\mathbf{P}_1 = \cdots = \mathbf{P}_{N-1}$ and $\mathbf{P}'_1 = \cdots = \mathbf{P}'_N$.

In the following, we study the cases of $N = 2, 3$ which illustrate the correspondence of $\text{LV}(N)$ and the integrable structure on the affine Jacobi variety. We prove the conjecture in the $N = 2$ case, and the simplest case of $N = 3$. For general $N$, it seems to be very complicated even to show (i).

- $N = 2$ case

  We have $L = 2M$, $g = M - 1$, the integrals of motion $\mathcal{H}_i$, $i = 1, \cdots M - 1$, and the elements of $\mathcal{A}^{0}_{LV}$,

$$\mathcal{P}_1 = (\mathcal{P}_0)^{-2} = \prod_{k=1}^{2M} V_k, \quad \mathcal{P}'_1 = \prod_{k=1}^{M} V_{2k-1}, \quad \mathcal{P}'_2 = \prod_{k=1}^{M} V_{2k}.$$ 

By definition, we have

$$\mathbf{S} = \begin{pmatrix} 1 & 0 \\ -P_2 \cdots P_{L-1} Q_1 Q^{-1}_L & P_1 \cdots P_{L-1} Q^{-1}_L \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\mathcal{P}_0 V_L & \mathcal{P}_0 P^{-1}_L Q^{-1}_L \end{pmatrix},$$

$$\mathbf{B}_1^{-1} = \left( \frac{1}{P_1} Q_1^{-\frac{1}{2}} \right)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & P^{-1}_1 Q_1 \end{pmatrix},$$

where $P_n, Q_n$ are canonical variables (2.3), and $V_n = (\mathcal{P}_n \mathcal{P}_{n+1})^{-1} Q^{-1}_n Q_{n+1}$ (2.2). Then (4.1) reduces to (1.2) where the matrix $\mathbf{S}$ is

$$\tilde{\mathbf{S}} = \begin{pmatrix} 1 & 1 \\ 0 & -\mathcal{P}_0 V_L \end{pmatrix}. $$
which justifies (i).

Let us consider the case of $M = 2$, $L = 4$ and $g = 1$. Now (1.1) becomes
\[ w^2 - \mathcal{P}_0(z^2 - H_1 z + P_1' + P_2')w + 1 = 0, \]
where we have three independent integrals of motion;
\[ H_1 = \sum_{k=1}^{4} V_n, \quad P_1' = V_1V_5, \quad P_2' = V_2V_4. \] (4.3)
Due to the translation invariant we set $\mathcal{P} \equiv \mathcal{P}_1' = \mathcal{P}_2'$ which yields $\mathcal{P}_0^2 = \mathcal{P}^{-2}$. The polynomial $B_F(z)$ has a zero $z_1$,
\[ z_1 = V_1 + V_2. \] (4.4)
Here $z_1$ is a function of $\tau_1$ defined by (3.11) with $H_1 = P_0 H_1$. Finally, the dynamical variables $V_n = V_n(\mathcal{P}, H_1; z_1(\tau_1))$ for $n = 1, 2, 3, 4$ are obtained from (4.3) (4.4).

- **$N = 3$ case**

We consider the $L = 6$, $M = 1$ and $g = 1$ case. Now the characteristic equation is
\[ w^3 + \mathcal{P}_0^2(z(P_1 + P_2) - (P_1'P_2' + P_1'P_3' + P_2'P_3'))w^2 + \mathcal{P}_0(z^2 + zH_1 + P_1' + P_2' + P_3')w - 1 = 0, \]
where
\[ P_i = V_iV_{i+2}V_{i+4}, \quad \text{for} \quad i = 1, 2, \quad P_i' = V_iV_{i+3}, \quad \text{for} \quad i = 1, 2, 3, \quad H_1 = \sum_{k=1}^{6} V_n. \] (4.5)
We set $\mathcal{P}_i' \equiv \mathcal{P}$, and for simplicity consider the case of $\mathcal{P}_i \equiv \mathcal{P}_i^2$ and $\mathcal{P}_0 \equiv \mathcal{P}^{-1}$. Then the matrix $\mathbf{M}(z)$ reduces to (4.2) where the gauge matrix $\tilde{\mathbf{S}}$ is written in terms of $V_n$,
\[ \tilde{\mathbf{S}} = \begin{pmatrix} 0 & 1 & 1 \\ \mathcal{P}^{-1}(V_5 + V_6) + 2\mathcal{P}^{-\frac{1}{2}} & -\mathcal{P}^{-1}V_6 & 1 \\ -1 & -\mathcal{P}^{-\frac{1}{2}}V_6 & 1 \end{pmatrix}. \]
The polynomial $B_F(z)$ has a zero $z_1$,
\[ z_1 = -(V_1 + V_2)\left(\frac{V_3V_4}{\mathcal{P}^2(V_3 + V_4)} + 1\right). \] (4.6)
As same as the $N = 2$ case, we obtain $V_n = V_n(\mathcal{P}, H_1; z_1(\tau_1))$ by using (4.3) and (4.6).

For the general $N$ cases we support (i) and (ii), and the dynamical variables of $LV(N)$ should be solved as
\[ V_n = V_n(\mathcal{P}, H_1, \ldots, H_g; z_1, \ldots, z_g), \]
where $z_i = z_i(\tau_1, \ldots, \tau_g)$. 

5 Summary and remarks

In this paper, we have studied the realization of the representative of the gauge equivalence class \( \mathcal{M}_F \), which is given by the classical integrable model, the extended Lotka-Volterra lattice.

The gauge equivalence class \( \mathcal{M}_F \) have the elements whose characteristic equation (1.1) is common, and the coefficients of (1.1) correspond to a set of commuting integrals of motion. There is the isomorphism from \( \mathcal{M}_F \) to a set of divisors \( X(g) - D \), and the time-evolution of the divisor is linearized on the Jacobi variety \( J(X) \). In ref. 5, it was introduced that the way to construct the representative of \( \mathcal{M}_F \) by starting with the integrable monodromy matrix \( \mathbf{T}(z) \) (2.11). Based on the integrable Poisson structure of the monodr omy matrix, the divisor is determined via SoV.

We have found that \( \text{LV}(N) \) gives the realizations not only for the monodromy matrix \( \mathbf{T}(z) \) but also for the representative \( \mathcal{M}_F(z) \). We have studied the correspondence of \( \text{LV}(N) \) and the representative and their Poisson algebras in detail. Then we have shown that the family of non-trivial integrals of motion for the representative coincides with that of \( \text{LV}(N) \), and that the number of these integrals are necessary and sufficient to describe the model. Especially our claim is that \( \mathcal{M}_F(z) \) can be written in terms of the dynamical variables of \( \text{LV}(N) \). These make possible to solve the model, and the time evolutions of the dynamical variables \( V_n \) are obtained as

\[ V_n = V_n(\mathcal{P}, \mathcal{H}_1, \ldots, \mathcal{H}_g; z_1(\{\tau_i\}), \ldots, z_g(\{\tau_i\})) \]

Here \( \mathcal{P} \) and \( \mathcal{H}_i \) are the integrals of motion, where each \( \mathcal{H}_i \) generates the independent time \( \tau_i \).

In closing, we would like to mention the quantization of \( \text{LV}(N) \). By replacing the canonical variables (2.3) in the Lax matrices \( \mathbf{L}_n(z) \) (2.4) and \( \mathbf{L}_n(z') \) (2.5) with the Weyl operators,

\[ [\hat{P}_n, \hat{Q}_m] = \delta_{n,m} \hat{P}_n \hat{Q}_n, \quad [\hat{P}_n, \hat{P}_m] = [\hat{Q}_n, \hat{Q}_m] = 0, \]

we get the quantum integrable model [11, 12]. As same as the classical case we note the Lax matrix \( \mathbf{L}_n(z) \) (2.5). Now this matrix with Weyl operators satisfies the fundamental commuting relation

\[ R(z/z'; q) (\mathbf{L}_n(z) \otimes \mathbb{1}) (\mathbb{1} \otimes \mathbf{L}_n(z')) = (\mathbb{1} \otimes \mathbf{L}_n(z')) (\mathbf{L}_n(z) \otimes \mathbb{1}) R(z/z'; q), \]

where \( R \)-matrix is

\[ R(z; q) = \sum_{k=1}^{N} (z - q^2) \mathbf{E}_{k,k} \otimes \mathbf{E}_{k,k} + \sum_{j=1}^{N} \sum_{k=1}^{N-1} q(z - 1) \mathbf{E}_{j,j} \otimes \mathbf{E}_{j+k,j+k} + \sum_{1 \leq j < k \leq N} (1 - q^2) \left( \mathbf{E}_{j,k} \otimes \mathbf{E}_{k,j} + z \mathbf{E}_{k,j} \otimes \mathbf{E}_{j,k} \right). \]
Especially in the $N = 2$ case the Lax matrix becomes

$$\mathbf{L}_n(z) = \frac{1}{z^2} \begin{pmatrix} 0 & \hat{Q}_n^{-1} \\ -z\hat{Q}_n & z\hat{P}_n \end{pmatrix}. \tag{5.1}$$

Taking into account the canonical transformation, this is essentially same as what discussed in ref. [13]. In general $N$ case, the monodromy matrix $\mathbf{T}(z)$ is written as

$$\mathbf{T}(z) = \prod_{k=1}^{MN} \mathbb{L}_k(z),$$

where $\mathbb{L}_k(z) = \left( \mathbf{L}_{k(N-1)}(z) \mathbf{L}_{k(N-1)-1}(z) \cdots \mathbf{L}_{(k-1)(N-1)+1}(z) \right)$ is

$$\frac{1}{z^{N-1}} \begin{pmatrix} 0 & 0 & \cdots & 0 & \hat{Q}_n^{-1} \\ z(-)^{N-1}\hat{Q}_1 & z(-)^{N-2}\hat{P}_1 & 0 & \cdots & 0 \\ 0 & z(-)^{N-1}\hat{Q}_2\hat{Q}_1^{-1} & z(-)^{N-2}\hat{P}_2 & 0 & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & z(-)^{N-1}\hat{Q}_{N-1}\hat{Q}_{N-2}^{-1} & z(-)^{N-2}\hat{P}_{N-1} \end{pmatrix}_{[k-1]}.$$

Here in the matrix with the subscript $[k-1]$ the operators $\hat{P}_i$ and $\hat{Q}_i$ are regarded as $\hat{P}_{i+(k-1)(N-1)}$ and $\hat{Q}_{i+(k-1)(N-1)}$ respectively. We expect that the matrix $\mathbb{L}_n(z)$ gives a key to generalize the Baxter equations and their dual structure in the sense of ref. [13], which is a future problem. The matrix $\mathbb{L}_k(z)$ may link to the relatives or the extension to $\mathfrak{sl}_N$ of the Toda lattice. It seems to be interesting to study this matrix in both of classical and quantum cases from this point of view.

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Appendix A  Local Lax matrix for $LV(N)$

We define a gauge transformation of $\tilde{L}(x)$ (1.4) as follows;

$$L_n(z) = \Omega_{n+1}(z)\tilde{L}_n(z)\Omega_n(z)^{-1},$$  \hspace{1cm} (A.1)

Here the gauge matrix $\Omega_n(z)$ is

$$\Omega_n(z) = B_n A X(z),$$  \hspace{1cm} (A.2)

where

$$A = \left( \sum_{k=1}^{N} E_{k,N+1-k} \right) \left( I - \sum_{k=1}^{N-1} E_{k,k+1} \right),$$

$$B_n = \prod_{k=0}^{N-2} \left( P_{n+k} \right)^{\frac{1}{N}} - \sum_{j=k+2}^{N} D^{(j)} \left( Q_{n+k} \right) D^{(k+2)},$$

$$X(z) = \sum_{k=1}^{N} z^{N-2} E_{k,k}$$

and we use

$$D^{(j)} = \frac{1}{N} I - E_{j,j}, \hspace{1cm} \left( P_n \right)^{D^{(j)}} = \sum_{k=1}^{N} P_n^{D^{(j)k,k}} E_{k,k}.$$

In the above we have used a notation,

$$P_n^{D} = \text{diag}[P_n^{d_1}, P_n^{d_2}, \ldots, P_n^{d_N}], \text{ where } D = \text{diag}[d_1, \ldots, d_N].$$

Finally we obtain the local Lax matrix (2.5),

$$L_n(z) = z^{\frac{N}{2}} \left( P_n E_{1,1} + Q_n E_{1,2} + \frac{1}{z} (-1)^{N-1}Q_n^{-1} E_{N,1} + \sum_{k=2}^{N-1} E_{k,k+1} \right).$$

Note that the gauge matrix $\Omega_n(z)$ is different from that introduced in ref. [12].

Appendix B  Gauge transformation of $B(z)$

We divide the matrix $S$ (3.2) in the same way as (2.12),

$$S = \begin{pmatrix} 1 & \tilde{0} \\ s_1^T & s_2 \end{pmatrix},$$

$$15$$
where we use
\[ \vec{s}_1 = \vec{c}_0 \left( \vec{d}_0^{N-1-i} \right)^T, \quad \vec{s}_2 = \begin{pmatrix} \vec{b}_0 \vec{d}_0^{N-2} \\ \vdots \\ \vec{b}_0 \vec{d}_0 \\ \vec{b}_0 \end{pmatrix}, \]

and $\vec{c}_0, \vec{b}_0$ and $\vec{d}_0$ are dominant parts of $\vec{c}(z), \vec{b}(z)$ and $\vec{d}(z)$ in $z \to \infty$. The matrix $S$ transforms the monodromy matrix $\overline{T}(z)$ to
\[ \mathbf{M}_F(z) = S \overline{T}(z) S^{-1} = \begin{pmatrix} \vec{b}(z) \vec{s}_2^{-1} \\ \vec{s}_2^{-1} \vec{b}(z) \vec{d}(z) \vec{s}_2^{-1} \end{pmatrix}, \]

where the parts indicated by $\#_1$ and $\#_2$ are not important now. Following this transformation, the polynomial $B(z)$ \[2.16\] becomes $B_F(z)$ as
\[ B_F(z) = \text{Det} \left( \begin{array}{c} \vec{b}(z) \vec{s}_2^{-1} \\ \vec{s}_2^{-1} \vec{b}(z) \vec{d}(z) \vec{s}_2^{-1} + \vec{d}(z) \vec{s}_2^{-1} \end{array} \right) \]
\[ = \text{Det} \left( \begin{array}{c} \vec{b}(z) \\ \vec{b}(z) \vec{d}(z) \\ \vdots \\ \vec{b}(z) \vec{d}(z)^{N-2} \end{array} \right) \text{Det}(\vec{s}_2^{-1}) \]
\[ = \left( -\frac{1}{2} \right)^{(N-1)(N-2)} B(z) B_0^{-1} \]
The second equality is due to
\[ \vec{b}(z) \vec{s}_2^{-1} \vec{s}_1^T \vec{b}(z) \propto \vec{b}(z), \]
and the third one follows \[2.15\] and
\[ \text{Det} \vec{s}_2 = \left( -\frac{1}{2} \right)^{(N-1)(N-2)} B_0 \]
where $B_0$ is the zero mode of $B(z)$. Finally we obtain $B_F(z)$ which do not have the zero mode.

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