INTERNALLY 4-CONNECTED BINARY MATROIDS WITH EVERY ELEMENT IN THREE TRIANGLES

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Received November 12, 2016
Revised October 4, 2018
Online First July 9, 2019

Let \( M \) be an internally 4-connected binary matroid with every element in exactly three triangles. Then \( M \) has at least four elements \( e \) such that \( si(M/e) \) is internally 4-connected. This technical result is a crucial ingredient in Abdi and Guenin’s theorem determining the minimally non-ideal binary clutters that have a triangle.

1. Introduction

The terminology in this paper will follow [7]. A matroid is internally 4-connected if it is 3-connected and, for every 3-separation \((X,Y)\) of \( M \), either \( X \) or \( Y \) is a triangle or a triad of \( M \).

The purpose of this paper is to prove the following technical result.

Theorem 1.1. Let \( M \) be a binary internally 4-connected matroid in which every element is in exactly three triangles. Then \( M \) has at least four elements \( e \) such that \( si(M/e) \) is internally 4-connected.

We were motivated to prove this result because Abdi and Guenin [1] needed it to prove that the only minimally non-ideal binary clutters that have a triangle consist of the lines of the Fano matroid and the odd circuits of \( K_5 \). Indeed, the above result appears as Theorem 15 in [1].

Following [1,4,10,11,13], we now give the background relating to clutters needed to understand Abdi and Guenin’s theorem. A clutter \( \mathcal{A} \) on a finite set \( E(\mathcal{A}) \) is a family of subsets of \( E(\mathcal{A}) \) none of which is a proper subset of

Mathematics Subject Classification (2010): 05B35, 52B40
A clutter \( A \) is trivial if \( A = \emptyset \) or \( A = \{\emptyset\} \). The blocker \( b(A) \) of \( A \) is the family of minimal subsets of \( E(A) \) that have non-empty intersection with every member of \( A \). Edmonds and Fulkerson [5] showed that \( b(b(A)) = A \) for all clutters \( A \). A clutter \( A \) is binary if there is a binary matroid \( M \) having an element \( e \) such that \( E(A) = E(M) - \{e\} \) and \( A \) is the collection of sets of the form \( C - e \) where \( C \) is a circuit of \( M \) containing \( e \). It is well known that a clutter \( A \) is binary if and only if \( |S \cap R| \) is odd for all \( S \) in \( A \) and all \( R \) in \( b(A) \).

For a clutter \( A \), if \( C \) and \( D \) are disjoint subsets of \( E(A) \), then the minimal members of \( \{S - C : S \in A, S \cap D = \emptyset\} \) forms a clutter \( A/C\setminus D \) on \( E(A) - (C \cup D) \). Such a clutter is a minor of \( A \). This minor is proper if \( C \cup D \neq \emptyset \). Seymour [10] showed that if a clutter is binary, then so are its blocker and all of its minors.

Two important binary clutters are \( \mathbb{L}_7 \) and \( \mathbb{O}_5 \). The first consists of the seven lines in the Fano matroid while the second consists of the odd circuits in \( M(K_5) \).

Let \( A \) be a non-trivial clutter and \( A(A) \) be the matrix whose columns are indexed by the elements of \( E(A) \) and whose rows are the characteristic vectors of the members \( S \) of \( A \). If all of the extreme points of the real polyhedron \( \{x \geq 0 : A(A)x \geq 1\} \) have all of their coordinates in \( \{0, 1\} \), then the clutter \( A \) is ideal.

Seymour [11,13] proposed the following.

**Conjecture 1.2.** A binary clutter is ideal if and only if it has none of \( \mathbb{L}_7 \), \( \mathbb{O}_5 \), or \( b(\mathbb{O}_5) \) as a minor.

A member of a clutter with exactly three elements is called a triangle. Abdi and Guenin [1] proved the following partial result towards this conjecture. Theorem 1.1 is essential in their proof of this result.

**Theorem 1.3.** The only binary non-ideal clutters that have a triangle and have all of their proper minors ideal are \( \mathbb{L}_7 \) and \( \mathbb{O}_5 \).

The next section introduces some preliminaries. In Section 3, we prove the main theorem when \( M \) has at most thirteen elements, while Section 4 deals with when \( M \) has small cocircuits. Section 5 completes the proof of the main theorem. In Section 6, we show that the main theorem cannot be extended to ensure that \( si(M/e) \) is internally 4-connected for every element \( e \) of \( M \).

2. Preliminaries

This section introduces some basic material relating to matroids. Although most of this material can be found in [7], it is included here to make the
paper as self-contained as possible. Let $M$ be a matroid having ground set $E$ and rank function $r$. The simplification $\text{si}(M)$ of $M$ is the matroid that is obtained from $M$ by deleting all loops and deleting all but one element from each non-trivial parallel class. We note that the ground set of $\text{si}(M)$ is not uniquely defined since we are free to choose which elements are retained from the non-trivial parallel classes of $M$. The property that a circuit and a cocircuit in a matroid cannot meet in exactly one element is called orthogonality. In a binary matroid, the intersection of every circuit and every cocircuit must have even cardinality. We shall call this characterizing property of binary matroids [7, Theorem 9.1.2] binary orthogonality. We shall make frequent use of the fact that a matroid is binary if and only if the symmetric difference of every set of cocircuits is a disjoint union of cocircuits.

The binary projective and binary affine geometries of rank $r$ will be denoted by $PG(r-1,2)$ and $AG(r-1,2)$. In particular, the Fano matroid, $F_7$, is isomorphic to $PG(2,2)$.

The connectivity function $\lambda_M$ of $M$ is defined on all subsets $X$ of $E$ by $\lambda_M(X) = r(X) + r(E-X) - r(M)$. Equivalently, $\lambda_M(X) = r(X) + r^*(X) - |X|$. We will sometimes abbreviate $\lambda_M$ as $\lambda$. For a positive integer $k$, a subset $X$ or a partition $(X,E-X)$ of $E$ is $k$-separating if $\lambda_M(X) \leq k-1$. A $k$-separating partition $(X,E-X)$ of $E$ is a $k$-separation if $|X|,|E-X| \geq k$. A $k$-separation $(X,E-X)$ is exact if $\lambda_M(X) = k-1$. For $n \geq 2$, a matroid is $n$-connected if it has no $k$-separations for all $k<n$. Suppose $M$ is 3-connected. Then $M$ is both simple and cosimple provided $|E| \geq 4$. A subset $S$ of $E$ is a fan in $M$ if $|S| \geq 3$ and there is an ordering $(s_1,s_2,\ldots,s_k)$ of the elements of $S$ such that $\{s_1,s_2,s_3\},\{s_2,s_3,s_4\},\ldots,\{s_{k-2},s_{k-1},s_k\}$ alternate between triangles and triads, beginning with either. Let $(X,Y)$ be a 3-separation in a matroid $M$. If $|X|,|Y| \geq 4$, then we call $X,Y$, or $(X,Y)$ a $(4,3)$-violator since it certifies that $M$ is not internally 4-connected. For example, if $X$ is a 4-fan, that is, a 4-element fan, then $X$ is a $(4,3)$-violator provided $|Y| \geq 4$.

The next result is Seymour’s Splitter Theorem [12] (see also [7, Corollary 12.2.1]), a basic tool for dealing with 3-connected matroids.

**Theorem 2.1.** Let $M$ and $N$ be 3-connected matroids such that $N$ is a minor of $M$ having at least four elements and if $N$ is a wheel, then $M$ has no larger whirl as a minor. Then there is a sequence $M_0,M_1,\ldots,M_k$ of 3-connected matroids with $M_0 \cong N$ and $M_k = M$ such that $M_i$ is a single-element deletion or a single-element contraction of $M_{i+1}$ for all $i$ in $\{0,1,\ldots,k-1\}$.

In a matroid $M$, a set $U$ is fully closed if it is closed in both $M$ and $M^*$. The full closure $\text{fcl}(Z)$ of a set $Z$ in $M$ is the intersection of all fully closed
sets containing $Z$. The full closure of $Z$ may be obtained by alternating between taking the closure and the coclosure until both operations leave the set unchanged (see, for example, [8, p.262]). Let $(X,Y)$ be a partition of $E(M)$. If $(X,Y)$ is $k$-separating in $M$ for some positive integer $k$, and $y$ is an element of $Y$ that is also in $cl(X)$ or $cl^*(X)$, then it is well known and easily checked that $(X \cup y, Y - y)$ is $k$-separating, and we say that we have moved $y$ into $X$. More generally, $(fcl(X), Y - fcl(X))$ is $k$-separating in $M$.

The following elementary result will be used repeatedly.

**Lemma 2.2.** Let $e$ be an element of an internally 4-connected binary matroid $M$. Then $si(M/e)$ is 3-connected.

**Proof.** The only 3-connected binary matroids with at most five elements are uniform of rank at most two [7, p.316] and the result is easily checked for such matroids. By Tutte’s Wheels-and-Whirls Theorem [14], the only 3-connected binary matroids with six or seven elements are $M(K_4), F_7,$ and $F_7^*$ and again the result is easily checked. Thus we may assume that $|E(M)| \geq 8.$

Now let $M' = si(M/e)$ and suppose that $M'$ has a 2-separation $(X,Y)$. We may assume that $|X| \geq |Y|$. Suppose $|Y| = 2$. Then $Y$ is a 2-cocircuit $\{y_1,y_2\}$ of $M'$. As $\{y_1,y_2\}$ is not a 2-cocircuit of $M/e$ and $M$ is binary, we see that, in $M/e$, either one or both of $y_1$ and $y_2$ is in a 2-element parallel class. Thus we may assume that $M/e$ has $\{y_1,y'_1\}$ as a circuit and $\{y_1,y'_1,y_2\}$ as a cocircuit, or $M/e$ has $\{y_1,y'_1\}$ and $\{y_2,y'_2\}$ as circuits and has $\{y_1,y'_1,y_2,y'_2\}$ as a cocircuit. Hence $M$ has $\{e,y_1,y'_1,y_2\}$ as a 4-fan or has $\{y_1,y'_1,y_2,y'_2\}$ as both a circuit and a cocircuit. Since $|E(M)| \geq 8$, each possibility contradicts the fact that $M$ is internally 4-connected. We conclude that $|Y| \geq 3$.

Let $(X',Y')$ be obtained from $(X,Y)$ by adjoining each element of $E(M/e) - E(M')$ to the side of $(X,Y)$ that contains an element parallel to it. Then $r_{M/e}(X') = r_{M'}(X)$ and $r_{M/e}(Y') = r_{M'}(Y)$, so $(X',Y')$ is a 2-separation of $M/e$. Hence $(X',Y' \cup e)$ and $(X' \cup e,Y')$ are 3-separations of $M$. As $|Y' \cup e| \geq 4$ and $|E(M)| \geq 8$, this gives a contradiction.

For $n \geq 2$ and an $n$-connected matroid $M$, an $n$-separation $(U,V)$ of $M$ is sequential if $fcl(U)$ or $fcl(V)$ is $E(M)$. In particular, when $fcl(U) = E(M)$, there is an ordering $(v_1,v_2,\ldots,v_m)$ of the elements of $V$ such that $U \cup \{v_m,v_{m-1},\ldots,v_i\}$ is $n$-separating for all $i$ in $\{1,2,\ldots,m\}$. When this occurs, the set $V$ is called sequential. The next lemma will be helpful in dealing with sequential and non-sequential 3-separations. The first part was proved in [3, Lemma 2.9]. For completeness, we include the well-known proof of the second part.
**Lemma 2.3.** Let \((X,Y)\) be a 3-separation of a 3-connected binary matroid \(M\).

(i) If \(X\) is sequential and \(|X| \leq 5\), then \(X\) is a fan. In particular, if \(M\) has no 4-fans, then \(M\) has no sequential 3-separations.

(ii) If \(X\) is not sequential, then \(|X - \text{fcl}(Y)| \geq 4\).

**Proof.** For (ii), suppose \(X\) is not sequential, let \(Z = X - \text{fcl}(Y)\), and assume that \(|Z| \leq 3\). Now \(|E(M)| = |X| + |Y| \geq 6\), so \(M\) is simple and cosimple. Hence \(|Z| \geq 3\). As the full closure of \(Y\) avoids \(Z\), it follows that \(Z\) is a triad of \(M\). By duality, \(Z\) must also be a triangle of \(M\) and we have a contradiction to binary orthogonality.

We conclude this section by noting two useful results.

**Lemma 2.4.** Let \(M\) be a matroid in which every element is in exactly three triangles. Then \(M\) has exactly \(|E(M)|\) triangles.

**Proof.** Consider the set of ordered pairs \((e,T)\) where \(e \in E(M)\) and \(T\) is a triangle of \(M\) containing \(e\). The number of such pairs is \(3|E(M)|\) since each element is in exactly three triangles. As each triangle contains exactly three elements, this number is also three times the number of triangles of \(M\).

**Lemma 2.5.** Let \(M\) be an internally 4-connected binary matroid in which every element is in exactly three triangles. Then \(M\) has no cocircuits of odd size.

**Proof.** For a cocircuit \(C^*\) of \(M\), we construct an auxiliary graph \(G\) as follows. Let \(V(G) = C^*\), and let \(c_1c_2\) be an edge of \(G\) exactly when \(c_1\) and \(c_2\) are members of \(C^*\) that are contained in a triangle of \(M\). Since every element is in three triangles of \(M\), every vertex in \(G\) has degree three by orthogonality and the fact that \(M\) is binary. Hence \(|C^*|\), which equals the number of vertices of \(G\) of odd degree, is even.

### 3. Small matroids

In this section, we prove the main theorem for matroids with at most thirteen elements. To prove this, we shall use the following theorem of Qin and Zhou [9].

**Theorem 3.1.** Let \(M\) be an internally 4-connected binary matroid with no minor isomorphic to any of \(M(K_{3,3})\), \(M^*(K_{3,3})\), \(M(K_5)\), or \(M^*(K_5)\). Then either \(M\) is isomorphic to the cycle matroid of a planar graph, or \(M\) is isomorphic to \(F_7\) or \(F_7^*\).
Geometrically, the matroid $M^*(K_{3,3})$ looks like a twisted $3 \times 3$ grid [7, p.652]. Its ground set can be partitioned into three disjoint triangles in exactly two distinct ways, $\{X_1, X_2, X_3\}$ and $\{Y_1, Y_2, Y_3\}$. These correspond to the two ways to partition the edge set of $K_{3,3}$ into disjoint bonds of size three. Note that $|X_i \cap Y_j| = 1$ for all $i$ and $j$. The following characterization of $M^*(K_{3,3})$ will be useful.

**Lemma 3.2.** If $M$ is a simple 9-element rank-4 binary matroid with three disjoint triangles, then $M \cong M^*(K_{3,3})$.

**Proof.** We view $M$ as a restriction of $PG(3,2)$. The latter has exactly 35 triangles with each element being in precisely seven of these. For two disjoint triangles, $T_1$ and $T_2$, of $M$, we see that $M|(T_1 \cup T_2) \cong U_{2,3} \oplus U_{2,3}$. Consider the triangles of $PG(3,2)$ meeting $T_1 \cup T_2$. Each point of $T_1 \cup T_2$ is in six such triangles other than $T_1$ or $T_2$. Nine of these triangles contain a point of $T_1$ and a point of $T_2$. Thus, there are $(6 \times 6) - 9$ triangles other than $T_1$ or $T_2$ that meet $T_1 \cup T_2$. Hence $PG(3,2)$ has 29 triangles meeting $T_1 \cup T_2$. This leaves six triangles of $PG(3,2)$ that avoid $T_1 \cup T_2$. Let $T_i = \{x_i, y_i, z_i\}$. Let $x', y'$, and $z'$ be the third elements on the triangles of $PG(3,2)$ through $\{x_1, x_2\}, \{y_1, y_2\}$, and $\{z_1, z_2\}$, respectively. By taking the symmetric difference of $T_1, T_2, \{x_1, x_2, x'\}, \{y_1, y_2, y'\}$, and $\{z_1, z_2, z'\}$, we see that $\{x', y', z'\}$ is also a triangle, $T_3$, of $PG(3,2)$. For each of the six permutations of the elements of $T_2$, there is such a triangle $T_3$ and it is straightforward to check that these six triangles are distinct. We deduce that the six possible matroids $M|(T_1 \cup T_2 \cup T_3)$ are isomorphic, so there is a unique simple 9-element rank-4 binary matroid with three disjoint triangles. As $M^*(K_{3,3})$ is such a matroid, we conclude that $M \cong M^*(K_{3,3})$.

**Lemma 3.3.** Let $M$ be an internally 4-connected binary matroid in which every element is in exactly three triangles and $|E(M)| \leq 13$. Then $M$ is isomorphic to $F_7$ or $M(K_5)$. Hence $\text{si}(M/e)$ is internally 4-connected for all elements $e$ of $M$.

**Proof.** Assume that $M$ is not isomorphic to $F_7$ or $M(K_5)$. Suppose first that $M$ has none of $M(K_{3,3})$, $M^*(K_{3,3})$, $M(K_5)$, or $M^*(K_5)$ as a minor. As $F_7^*$ has no triangles, it follows by Theorem 3.1 that $M$ is isomorphic to the cycle matroid of a planar graph $G$. As every edge of $G$ is in exactly three triangles, and $M(G)$ is internally 4-connected, every vertex has degree at least four. Hence $|E(G)| \geq 2|V(G)|$. Moreover, by Lemma 2.5, every vertex of $G$ has even degree. Clearly, $|V(G)| \neq 4$. Moreover, $|V(G)| \neq 5$, otherwise $M \cong M(K_5)$; a contradiction. As $|E(G)| \leq 13$, it follows that $|V(G)| = 6$ and $|E(G)| = 12$. Then $G$ is obtained from $K_6$ by deleting the edges of a perfect matching. But no edge of this graph is in exactly three triangles.
We deduce that $M$ has an $N$-minor for some $N$ in
\[ \{M(K_{3,3}), M^*(K_{3,3}), M(K_5), M^*(K_5)\}. \]
By Theorem 2.1, there is a sequence $M_0, M_1, \ldots, M_k$ of 3-connected matroids such that $M_0 \cong N$ and $M_k = M$, while $M_i$ is a single-element deletion or contraction of $M_{i+1}$ for all $i$ in $\{0, 1, \ldots, k - 1\}$. Since $|E(N)| \geq 9$ and $|E(M)| \leq 13$, it follows that $k \in \{0, 1, 2, 3, 4\}$.

Suppose that some $M_i$ is obtained from its successor by contracting an element $e$. Then $M/e$ has an $N$-minor. But $\text{si}(M/e)$ has at most nine elements. Thus $|E(M)| = 13$ and $N$ is $M(K_{3,3})$ or $M^*(K_{3,3})$. Since $\text{si}(M/e)$ must contain triangles, $N$ is $M^*(K_{3,3})$. Now, by Lemma 2.5, every cocircuit of $M/e$ is exactly three 2-circuits. The union of these three 2-circuits cannot have rank two in $M/e$ otherwise $M$ has $F_7$ as a restriction and then the remaining six elements of $M$ cannot all be in exactly three triangles of $M$. Let $a, b$ and $c$ be the three elements of $M^*(K_{3,3})$ that are in 2-circuits in $M/e$. From the geometric description of $M^*(K_{3,3})$ given above, one can see that there are two intersecting triangles of $M^*(K_{3,3})$ whose union contains exactly two elements of $\{a, b, c\}$. The cocircuit of $M/e$ whose complement is the union of the closure of these two triangles is odd; a contradiction.

We now know that $M$ is an extension of $N$ by at most four elements. Let $M = PG(3,2) \backslash D_M$. Because binary matroids are uniquely representable, $M$ is uniquely determined by its complement $M|D_M$ in $PG(3,2)$ (see, for example, [7, p.554]). For each $e$ in $E(M)$, we must delete elements from exactly four of the seven triangles of $PG(3,2)$ containing $e$. Thus $|D_M| \geq 4$, so $|E(M)| \leq 11$. As $M \neq N$, we deduce that $|E(M)| \geq 10$. Let $N = M \backslash D$. Then $N$ has at least $|E(M)| - 3|D|$ triangles. Therefore $N$ cannot be $M(K_{3,3})$ or $M^*(K_5)$. Thus $N$ is $M^*(K_{3,3})$ or $M(K_5)$. Each element of $M(K_5)$ is in exactly three triangles. As $|E(M) - E(N)| \in \{1, 2\}$, we deduce that $N \neq M(K_5)$. Hence $N = M^*(K_{3,3})$. Now $M^*(K_{3,3})$ has exactly six triangles with each element being in precisely two of these. Adding one element to $M^*(K_{3,3})$ cannot create a matroid in which every element is in exactly three triangles. Thus $|E(M)| = 11$. As the complement of $M^*(K_{3,3})$ in $PG(3,2)$ is $U_{2,3} \oplus U_{2,3}$ [7, p.653], the complement of $M$ in $PG(3,2)$ is $U_{4,4}$ or $U_{2,3} \oplus U_{1,1}$. In the first case, $M$ has exactly thirteen triangles, while, in the second, an element of $M$ that lies in a triangle with two elements of $D$ is in four triangles in $M$. These contradictions complete the proof of the lemma.

\[ \Box \]

4. Small cocircuits

In this section, we move towards proving the main result by dealing with 4-cocircuits and certain special 6-cocircuits in $M$. Throughout the section,
we will assume that $M$ is an internally 4-connected binary matroid in which every element is in exactly three triangles, and $|E(M)| \geq 14$. By Lemma 2.5, $M$ has no odd cocircuits so, in particular, $M$ has no triads. The following elementary observation will be frequently used.

**Lemma 4.1.** In every partition $(X,Y)$ of the elements of $M(K_4)$ or $M^*(K_{3,3})$, either $X$ or $Y$ contains a basis of the matroid.

**Lemma 4.2.** If $C^*$ is a 4-cocircuit of $M$, then, for all $e$ in $C^*$, the matroid $\text{si}(M/e)$ is internally 4-connected having no triads.

**Proof.** Suppose that $C^* = \{e,f_1,f_2,f_3\}$ and $\text{si}(M/e)$ is not internally 4-connected. As $M$ is internally 4-connected, $r(C^*) = 4$. As $e$ is in three triangles of $M$, there are elements $\{g_1,g_2,g_3\}$ such that $\{e,f_i,g_i\}$ is a triangle for all $i$. As $f_i$ is in three triangles for all $i$, by binary orthogonality, there are elements $\{h_1,h_2,h_3\}$ such that $\{f_1,f_2,h_1\}, \{f_1,f_3,h_3\}$, and $\{f_2,f_3,h_2\}$ are triangles. The symmetric difference of $\{e,f_1,g_1\}, \{e,f_2,g_2\}$, and $\{f_1,f_2,h_1\}$ is $\{g_1,g_2,h_1\}$, so it is a triangle of $M$. By symmetry, so are $\{g_1,g_3,h_3\}$ and $\{g_2,g_3,h_2\}$. Thus, for each $g_i$, we have identified the three triangles containing it.

Let $M' = \text{si}(M/e) = M/e \setminus f_1,f_2,f_3$. Lemma 2.2 implies that $M'$ is 3-connected. The set $\{g_1,g_2,g_3,h_1,h_2,h_3\}$ forms an $M(K_4)$-restriction in $M'$. Suppose $M'$ has a non-sequential 3-separation $(X,Y)$. By Lemma 4.1, we may assume that $\{g_1,g_2,g_3,h_1,h_2,h_3\} \subseteq X$. As each $\{f_i,g_i\}$ is a circuit in $M/e$, we see that $(X \cup \{f_1,f_2,f_3\},Y - \{f_1,f_2,f_3\})$ is 3-separating in $M/e$. It follows, since $\{f_1,f_2,f_3,e\}$ is a cocircuit of $M$, that $(X \cup \{f_1,f_2,f_3\},Y - \{f_1,f_2,f_3\})$ is a 3-separation of $M$ and hence is a $(4,3)$-violation of $M$: a contradiction. We deduce that a $(4,3)$-violation of $\text{si}(M/e)$ is a sequential 3-separation of it.

We show next that

4.2.1. $M/e \setminus f_1,f_2,f_3$ has no triads.

Suppose $M/e \setminus f_1,f_2,f_3$ has a triad $\{\beta,\gamma,\delta\}$. Then $M \setminus f_1,f_2,f_3$ has $\{\beta,\gamma,\delta\}$ as a cocircuit. By Lemma 2.5, we may assume that $\{\beta,\gamma,\delta,f_1,f_2,f_3\}$ or $\{\beta,\gamma,\delta,f_1\}$ is a cocircuit of $M$. By orthogonality, in the first case, $\{\beta,\gamma,\delta\} = \{g_1,g_2,g_3\}$ while, in the second case, $g_1 \in \{\beta,\gamma,\delta\}$. In the first case, let $Z = \{e,f_1,f_2,f_3,g_1,g_2,g_3\}$. Then $r(Z) \leq 4$ while $Z$ contains at least two cocircuits of $M$. Hence $|Z| - r^*(Z) \geq 2$, so $\lambda(Z) \leq 2$; a contradiction as $|E(M)| \geq 14$.

In the second case, $M$ has a 4-cocircuit $D^*$ such that $C^* \cap D^* = \{f_1\}$ and $g_1 \in D^*$. Apart from $\{f_1,e,g_1\}$, the other triangles containing $f_1$ must
meet \( C^* - \{ f_1, e \} \) in distinct elements and must meet \( D^* - \{ f_1, g_1 \} \) in distinct elements. Thus \( r(C^* \cup D^*) \leq 4 \) and \( |C^* \cup D^*| - r^*(C^* \cup D^*) \geq 2 \), so \( \lambda(C^* \cup D^*) \leq 2 \); a contradiction since \( |E(M)| \geq 14 \). Thus 4.2.1 holds.

By 4.2.1, \( M/e \setminus f_1, f_2, f_3 \) has no 4-fans and so has no sequential 3-separation that is a (4,3)-violator. This contradiction completes the proof. 

**Lemma 4.3.** Take \( e \in E(M) \) and the three triangles \( T_1, T_2, \) and \( T_3 \) containing \( e \). If \( (T_1 \cup T_2 \cup T_3) - e \) is a cocircuit \( C^* \), then \( \text{si}(M/x) \) is internally 4-connected for every element \( x \) of \( C^* \).

**Proof.** Let \( T_i = \{ e, f_i, g_i \} \) for each \( i \in \{ 1, 2, 3 \} \). Note that \( T_1, T_2, \) and \( T_3 \) are not coplanar, otherwise their union forms an \( F_7 \)-restriction, and \( C^* \) contains a triangle; a contradiction to the fact that \( M \) is binary. Suppose the lemma fails. Then we may assume that \( \text{si}(M/f_3) \) is not internally 4-connected.

As \( f_1 \) is in two triangles other than \( T_1 \), orthogonality and the fact that \( M \) is binary imply that each of these triangles contains an element of \( \{ f_2, g_2, f_3, g_3 \} \). If \( \{ f_1, f_2 \} \) and \( \{ f_1, g_2 \} \) are each contained in a triangle, then the plane containing \( T_1 \) and \( T_2 \) is an \( F_7 \)-restriction, so \( e \) is in a fourth triangle; a contradiction. Hence \( f_1 \) is in a single triangle with an element of \( \{ f_2, g_2 \} \) and a single triangle with an element of \( \{ f_3, g_3 \} \). Without loss of generality, \( \{ f_1, g_2, x_1 \} \) and \( \{ f_1, g_3, x_2 \} \) are triangles. By taking the symmetric difference of these triangles with the circuits \( \{ f_1, g_1, f_2, g_2 \} \) and \( \{ f_1, g_1, f_3, g_3 \} \), respectively, we see that \( \{ g_1, f_2, x_1 \} \) and \( \{ g_1, f_3, x_2 \} \) are also triangles. We have now identified all three of the triangles containing each element in \( \{ f_1, g_1 \} \).

But, for each element in \( \{ f_2, g_2, f_3, g_3 \} \), one of the triangles containing the element remains undetermined.

By binary orthogonality, either \( \{ f_2, g_3, x_3 \} \) and \( \{ g_2, f_3, x_3 \} \) are triangles for some element \( x_3 \), or \( \{ f_2, f_3, y_3 \} \) and \( \{ g_2, g_3, y_3 \} \) are triangles for some element \( y_3 \). In each of these cases, we will obtain the contradiction that \( \text{si}(M/f_3) \) is internally 4-connected. By Lemma 2.2, \( M' = \text{si}(M/f_3) \) is 3-connected. Take \( (U, V) \) to be a (4,3)-violator in \( M' \).

Let \( X = \{ e, f_1, f_2, g_1, g_2, x_1 \} \). Clearly, the restriction of \( M/f_3 \) to \( X \) is isomorphic to \( M(K_4) \). We may assume that \( M' = M/f_3 \setminus Y \) where \( Y \) is \( \{ g_3, x_2, x_3 \} \) or \( \{ g_3, x_2, y_3 \} \) depending on whether \( \{ g_2, f_3, x_3 \} \) or \( \{ f_2, f_3, y_3 \} \) is a triangle of \( M \). By Lemma 4.1, we may also assume that \( U \) spans \( X \) in \( M' \). Then \( (U \cup X, V - X) \) is 3-separating in \( M' \). In \( M/f_3 \), the elements of \( Y \) are parallel to elements of \( X \), so \( (U \cup X \cup Y, V - X) \) is 3-separating. Because \( U \cup X \cup Y \) contains all but the element \( f_3 \) of the cocircuit \( C^* \) of \( M \), it follows that \( (U \cup X \cup Y \cup f_3, V - X) \) is 3-separating in \( M \). Since \( M \) is internally 4-connected, \( |V - X| \leq 3 \). Thus, by Lemma 2.3(ii), \( V \) is a sequential 3-separating set in \( M' \). Thus, by Lemma 2.3(i), \( M' \) has a triad \( \{ \beta, \gamma, \delta \} \). By
Lemma 2.5, $M$ has a cocircuit $D^*$ where $D^*$ is $\{\beta, \gamma, \delta\} \cup Y$ or $\{\beta, \gamma, \delta\} \cup y$ for some $y$ in $Y$. Then $D^*$ is a cocircuit of $M/f_3$. If $D^* = \{\beta, \gamma, \delta\} \cup Y$, then, by orthogonality, $\{\beta, \gamma, \delta\}$ must consist of the elements of $X$ that are parallel in $M/f_3$ to the elements of $Y$. Hence $\{\beta, \gamma, \delta\} \subseteq X$. The last inclusion also follows by orthogonality when $D^* = \{\beta, \gamma, \delta\} \cup y$ since $\{\beta, \gamma, \delta\}$ must meet $X$ and $M|X \cong M(K_4)$. Hence $X \cup Y \cup f_3$ contains at least two cocircuits, so $|X \cup Y \cup f_3| - r^*(X \cup Y \cup f_3) \geq 2$. Since $r(X \cup Y \cup f_3) = 4$, it follows that $\lambda(X \cup Y \cup f_3) \leq 2$; a contradiction as $|E(M)| \geq 14$.

Lemma 4.4. Let $(X, Y)$ be an exact 4-separation in $M$ with $X \subseteq \text{fcl}(Y)$. If $M$ has no 4-cocircuits, then $X$ is coindependent, $r(X) = 3$, and $X \subseteq \text{cl}(Y)$.

Proof. If $X \subseteq \text{cl}(Y)$, then $Y$ contains a basis of $M$, and $X$ is coindependent. As $r(X) + r^*(X) - |X| = 3$, we see that $r(X) = 3$, and the result holds. If $X \subseteq \text{cl}^*(Y)$, then $X$ is independent, so $r^*(X) = 3$. As $|X| \geq 4$, it follows that $X$ contains a 4-cocircuit; a contradiction.

Beginning with $Y$, look at $\text{cl}(Y), \text{cl}^*(\text{cl}(Y)), \text{cl}(\text{cl}^*(\text{cl}(Y))), \ldots$ until the first time we get $E(M)$. Consider the set $Y'$ that occurs before $E(M)$ in this sequence, let $X' = E(M) - Y'$, and let $e$ be the last element that was added in taking the closure or coclosure that equals $Y'$. Then either $Y'$ is a hyperplane and $X'$ is a cocircuit, or $Y'$ is a cohyperplane and $X'$ is a circuit.

Suppose $X'$ is a circuit. As $r(X') + r^*(X') - |X'| \leq 3$, we see that $r^*(X') \leq 4$. Thus, as $X'$ does not contain a 4-cocircuit, it is coindependent, so it has size at most four. We may assume that $X' \subseteq X$, otherwise the lemma holds. Suppose $|X'| = 4$. Then both $(X' \cup e, Y' - e)$ and $(X', Y')$ are exact 4-separations. Thus $e \in \text{cl}^*(X') \cap \text{cl}^*(Y' - e)$ or $e \in \text{cl}(X') \cap \text{cl}(Y' - e)$. The latter holds otherwise $M$ has a 4-cocircuit; a contradiction. But $Y'$ is coclosed, so $e$ was added by coclosure; that is, $e \in \text{cl}^*(Y - e)$ and we have a contradiction to orthogonality since $e \in \text{cl}(X)$. It remains to consider the case when $|X'| = 3$. Then $|X' \cup e| = 4$. The lemma holds if $X' \cup e = X$, so there is an element $f$ of $Y' - e$ that was added immediately before $e$ in the construction of $Y'$. Now if $f$ is added via closure, then we can also add $e$ and $X'$ via closure, so we violate our choice of $Y'$. Thus $f$ is added via coclosure so $f \in \text{cl}^*(Y' - e - f) \cap \text{cl}^*(X' \cup e)$. Hence $M$ has a 4-cocircuit; a contradiction.

We may now assume that $X'$ is a cocircuit. Then $X'$ has at least six elements. As $X'$ is 4-separating, $3 = r(X') + r^*(X') - |X'| = r(X') - 1$. Hence $r(X') = 4$, so $M|X'$ is a restriction of $PG(3, 2)$. As $M$ is binary, $X'$ contains no triangle and no 5-circuits, so $M|X'$ is a restriction of $AG(3, 2)$. As $X'$ has six or eight elements, it follows that $X'$ is a union of 4-circuits so $\text{cl}^*(Y')$ cannot contain $X'$; a contradiction.
Lemma 4.5. Assume $M$ has no 4-cocircuits. If every exact 4-separation in $M$ is sequential, then, for every element $e \in E(M)$, the matroid $\text{si}(M/e)$ is internally 4-connected with no triads.

Proof. Let \{\emph{e}, f\emph{i}, g\emph{i}\} be a triangle for all $i \in \{1, 2, 3\}$. The matroid $M' = \text{si}(M/e) = M/e \setminus f_1, f_2, f_3$ is 3-connected by Lemma 2.2. We show first that $M'$ is internally 4-connected. Assume it is not, letting $(U, V)$ be a $(4, 3)$-violator of it. Then $|U|, |V| \geq 4$. Add $f_i$ to the side of the 3-separation containing $g_i$ for all $i \in \{1, 2, 3\}$ to obtain $(U', V')$, a 3-separation in $M/e$. Neither $(U' \cup e, V')$ nor $(U', V' \cup e)$ is a 3-separation in $M$. Hence both are 4-separations in $M$. Thus, by hypothesis, each is a sequential 4-separation in $M$. Lemma 4.4 implies that, without loss of generality, either $U' \cup e$ is coindependent and has rank at most three in $M$; or both $U'$ and $V'$ have rank at most three and are contained in $\text{cl}(V' \cup e)$ and $\text{cl}(U' \cup e)$, respectively. In the first case, as $U' \cup e$ is contained in a plane, $U$ is contained in a triangle in $\text{si}(M/e)$; a contradiction. In the second case, $r(M) = 4$, so $U'$ and $V'$ span planes in $PG(3, 2)$. These planes meet in a line, so $|U' \cup V'| \leq 7 + 7 - 3 = 11$. Hence $E(M) \leq 12$; a contradiction. We conclude that $\text{si}(M/e)$ is internally 4-connected.

Suppose $M/e \setminus f_1, f_2, f_3$ has a triad \{\emph{a}, \emph{b}, \emph{c}\}. Then, by Lemma 2.5, since $M$ has no 4-cocircuits, $M$ has \{\emph{a}, \emph{b}, \emph{c}, \emph{f}1, \emph{f}2, \emph{f}3\} as a cocircuit, so we may assume that $(a, b, c) = (g_1, g_2, g_3)$. Now $M$ has a triangle containing $f_1$ and exactly one of $f_2, g_2, f_3$, or $g_3$. It follows that $\text{si}(M/e)$ has a triangle meeting \{\emph{g}1, \emph{g}2, \emph{g}3\}, so the internally 4-connected matroid $\text{si}(M/e)$ has a 4-fan; a contradiction.

Lemma 4.7 deals with a structure that arises in the proof of Lemma 5.1. This structure consists of a plane $P$ and a line $L$ in $M$. Clearly, $M|P$ and $M|L$ are restrictions of $F_7$ and $U_{2,3}$, respectively, although some elements of $F_7$ and $U_{2,3}$ may not be present in $M$. The next lemma is a preliminary result for Lemma 4.7 and treats the case when $P$ and $L$ are skew, that is, $r(P \cup L) = r(P) + r(L)$.
Lemma 4.6. Suppose $M$ contains a plane $P$ and a line $L$ that are skew and are labelled as in Figure 1 where not every element in the figure must be in $M$. If $a,b,c,d,e,f,x,y,$ and $z$ are in $M$, and $\{x,y,a,b,d,e\}$ and $\{y,z,b,c,e,f\}$ are cocircuits in $M$, then $\text{si}(M/w)$ is internally 4-connected for all $w$ in $\{a,b,c,d,e,f\}$.

Proof. By symmetric difference, $\{x,z,a,c,d,f\}$ is a cocircuit. As $z$ is in three triangles of $M$, orthogonality implies that $z$ is in a triangle with $c$, say $\{z,c,c'\}$, and a triangle with $f$, say $\{z,f,f'\}$. Likewise, $x$ is in triangles $\{x,a,a'\}$ and $\{x,d,d'\}$, while $y$ is in triangles $\{y,b,b'\}$ and $\{y,e,e'\}$, for some elements $a',d',b',e'$. As $P$ and $L$ are skew, all of $a',b',c',d',e',f'$ are distinct and none is in $P$ or $L$.

By symmetry, it suffices to show that $\text{si}(M/a)$ is internally 4-connected. Let $M' = \text{si}(M/a) = M/a \setminus a',b,f$. Let $Z = \{c,d,e,x,y,z,d',b',f'\}$. Then $M'|Z$ is a simple 9-element matroid of rank 4 having three disjoint triangles. Thus, by Lemma 3.2, $M'|Z \cong M^*(K_{3,3})$. Suppose $(U,V)$ is a $(4,3)$-violator of $M'$. By Lemma 4.1, we may assume that $U$ spans $Z$ in $M'$, thus $U$ spans $\{e',e''\}$. Hence $(U\cup Z\cup \{e',e''\}\cup \{a',b,f\},V-Z-\{e',e''\})$ is 3-separating in $M/a$, so $(U\cup Z\cup \{e',e''\}\cup \{a',b,f\}\cup a, V-Z-\{e',e''\})$ is 3-separating in $M$. Thus $V$ is a sequential 3-separating set in $M'$, so $V$ contains a triad $\{\beta, \gamma, \delta\}$. Thus either $\{x,c,e,a',b,f\}$ or $\{\beta,\gamma,\delta\} \cup t$ is a cocircuit of $M$ for some $t$ in $\{a',b,f\}$. The first possibility gives a contradiction to orthogonality with $\{y,b,b'\}$. Thus $\{\beta,\gamma,\delta,b\}$, $\{\beta,\gamma,\delta,f\}$, or $\{\beta,\gamma,\delta,a'\}$ is a cocircuit. Suppose $\{\beta,\gamma,\delta,b\}$ or $\{\beta,\gamma,\delta,f\}$ is a cocircuit. Then orthogonality implies that $\{\beta,\gamma,\delta\}$ contains $\{b,c,d\}$ or $\{f,e,d\}$ and so we get a contradiction to orthogonality with at least one of $\{x,d,d'\}$, $\{z,c,c'\}$, $\{z,f,f'\}$, $\{y,b,b'\}$ and $\{y,e,e'\}$. Thus $\{\beta,\gamma,\delta,a'\}$ is a cocircuit. This cocircuit also contains $x$ so either contains $y$ and elements from each of $\{b,b'\}$ and $\{e,e'\}$, or contains $z$ and elements from each of $\{f,f'\}$ and $\{c,c'\}$. Each case gives a contradiction to orthogonality. We conclude that $\text{si}(M/a)$ is internally 4-connected, so the lemma holds.

Lemma 4.7. Assume $M$ has no 4-cocircuits. Let $(U,V)$ be a non-sequential 4-separation of $M$ where $U$ is closed and $V$ is contained in the union of a plane $P$ and a line $L$ of $M$. Then either $V$ is a 6-cocircuit, or $|V|=9$ and $|P|=6$. Moreover, $\text{si}(M/v)$ is internally 4-connected for at least six elements $v$ of $V$.

Proof. By Lemma 2.5, each cocircuit contained in $V$ has exactly six elements otherwise it contains a triangle. Suppose $r(V) = 3$. As $r(V)+r^*(V)-|V|=3$, we know that $V$ is coindependent. Hence it is contained in $\text{cl}(U)$; a contradiction. Evidently $r(V) \geq 4$. We use Figure 1 as a guide for the points that
may exist in $V$. We consider which positions are filled, keeping in mind that $V$ is the union of circuits and the union of cocircuits.

Suppose $V$ has rank four and view $V$ as a restriction of $Q = PG(3,2)$. Then $\text{cl}_Q(P) \cap \text{cl}_Q(L)$ is a point of $Q$, so we may suppose $e = z$. Furthermore, as $r(V) + r^*(V) - |V| = 3$, we know that $V$ contains a single cocircuit. As $U$ is closed, $V$ is a cocircuit. Thus $|V| = 6$. As $V$ contains no triangles, $|(P \cup L) \cap \text{cl}_Q(P)| \leq 4$, and $|(P \cup L) \cap \text{cl}_Q(L)| \leq 2$. Thus $e \notin P \cup L$. Without loss of generality, the points in $V$ are $a,b,f,g,x$, and $y$, and the result follows by Lemma 4.3 provided $e \in E(M)$.

We may assume therefore that $e \notin E(M)$. We know that $V = \{x,y,a,b,f,g\}$. By orthogonality, without loss of generality, the three triangles of $M$ containing $x$ are $\{x,a,a'\}$, $\{x,f,f'\}$, and $\{x,b,b'\}$. Also $M$ has $\{x,y,a,f\}$ and $\{x,y,b,g\}$ as circuits. Thus $M$ has as triangles each of $\{y,a',f\}$, $\{y,a,f'\}$, and $\{y,b',g\}$. Hence $M$ has no other triangles containing $x$ or $y$. By orthogonality with the cocircuit $V$, the remaining triangles containing $g$ must be in $P$, and so contain $c$ and $d$. But then $\{a,b,c\}$ and $\{a,g,d\}$ are triangles of $M$, so $a$ is in four triangles; a contradiction.

Suppose that $r(V) = 5$. Then $P$ and $L$ are skew, and $V$ is the union of two 6-cocircuits, $C^*$ and $D^*$. By orthogonality, each of $C^*$ and $D^*$ contains at most four elements of $P$. Thus, by orthogonality, $|P| \leq 6$ so $|C^* \cup D^*| \leq 9$. Hence $|C^* \triangle D^*| = 6$ and $|V| = 9$. Then, without loss of generality, each of $C^*$ and $D^*$ meets $P$ in four elements and $L$ in two elements. The result now follows by Lemma 4.6.

The following lemma will be used in the proof of its successor.

**Lemma 4.8.** If $M$ has a 6-element cocircuit $C^* = \{a,b,c,d,e,f\}$ where $\{a,b,c,d\}$ and $\{a,b,e,f\}$ are circuits, then $\text{si}(M/x)$ is internally 4-connected for all $x$ in $C^*$.

**Proof.** By symmetric difference, $\{c,d,e,f\}$ is also a circuit. Thus $C^*$ is the union of three disjoint pairs, $\{a,b\}$, $\{c,d\}$, and $\{e,f\}$ such that the union of any two of these pairs is a circuit. If one of these pairs is in a triangle with some element $x$, then each of the pairs is in a triangle with $x$ and the lemma follows by Lemma 4.3. Thus we may assume that each of $\{a,c\}$ and $\{a,d\}$ is in a triangle. Hence so are $\{b,c\}$ and $\{b,d\}$. Thus each of $a,b,c$ and $d$ is in exactly one triangle with an element of $\{e,f\}$. Hence $e$ and $f$ cannot both be in exactly three triangles; a contradiction.

**Lemma 4.9.** Assume $M$ has no 4-cocircuits. Let $(J,K)$ be an exact 4-separation of $M$ with $J$ closed. If $|K| \leq 6$, then $K$ is a 6-cocircuit and $\text{si}(M/k)$ is internally 4-connected for all $k$ in $K$. 
Proof. We have $r(K) + r^*(K) - |K| = 3$ and $|K| \geq 4$. If $|K| = 4$, then $K$ is a cocircuit; a contradiction. Thus $|K| \geq 5$. Since $K$ is a union of cocircuits each of which has even cardinality, it follows that $|K| \geq 6$. Hence $K$ is a 6-cocircuit. Thus $r(K) = 4$ so $K$ contains two circuits such that they and their symmetric difference have even cardinality. Hence $K$ is the union of two 4-circuits that meet in exactly two elements and the result follows by Lemma 4.8.

5. The proof of the main result

The next lemma essentially completes the proof of Theorem 1.1.

Lemma 5.1. Let $M$ be an internally 4-connected binary matroid in which every element is in exactly three triangles. Suppose $M$ has no 4-cocircuits. Then $M$ has at least six elements $e$ such that $si(M/e)$ is internally 4-connected.

Proof. By Lemma 3.3, we know that $|E(M)| \geq 14$ otherwise $M$ is isomorphic to $F_7$ or $M(K_5)$ and so has a 4-cocircuit; a contradiction. Assume that the lemma fails. By Lemma 4.5, $M$ has a non-sequential 4-separation $(X,Y)$ where $|X|$ is minimal. Then $Y$ is fully closed and so is closed. By Lemma 4.9, $|X| \geq 7$. In fact,

5.1.1. $|X| \geq 9$.

To see this, note that, as $Y$ is closed, $X$ is a union of cocircuits of $M$. Since the symmetric difference of cocircuits also contains a cocircuit, 5.1.1 holds unless $X$ is an 8-cocircuit of $M$. In the exceptional case, since $r(X) + r^*(X) - |X| = 3$, we deduce that $r(X) = 4$. By binary orthogonality, every circuit of $M|X$ has even cardinality. Thus, by a result of Brylawski [2] and Heron [6] (see also [7, Proposition 9.4.1]), $M|X$ is a restriction of $AG(3, 2)$. As the last matroid has eight elements, $M|X \cong AG(3, 2)$. Consider a triangle $T$ of $M$ meeting $X$. Binary orthogonality means that $T$ contains exactly two elements of $X$. Let $T - X = \{y\}$. Then $r(M|(X \cup y)) = 4$. Because the complement of $AG(3, 2)$ in $PG(3, 2)$ is $F_7$, the matroid $M|(X \cup y)$ is uniquely determined and has four triangles containing $y$. This contradiction implies that 5.1.1 holds.

Because $|X| \geq 9$ and the lemma fails, $X$ contains an element $\alpha$ such that $si(M/\alpha)$ is not internally 4-connected. Let $\{\alpha, f_i, g_i\}$ be a triangle for all $i \in \{1, 2, 3\}$. Now $M' = si(M/\alpha) = M/\alpha \setminus f_1, f_2, f_3$ is not internally 4-connected. By Lemma 2.2, it is 3-connected. Take a $(4, 3)$-violator $(U', V')$ in $M'$. Then
\[|U'|,|V'| \geq 4. \text{ Hence } r_{M/\alpha}(U') \text{ and } r_{M/\alpha}(V') \text{ exceed two. Add } f_i \text{ to the side containing } g_i \text{ for all } i \in \{1,2,3\} \text{ to obtain } (U'',V''). \text{ Then both } (U'' \cup \alpha, V'') \text{ and } (U'',V'' \cup \alpha) \text{ are exact 4-separations of } M. \text{ Since } \alpha \in \text{cl}(U'') \text{ and } \alpha \in \text{cl}(V''), \text{ we deduce that } r_M(U'') \geq 4 \text{ and } r_M(V'') \geq 4. \text{ It follows by Lemma 4.4 that neither of the 4-separations } (U'' \cup \alpha, V'') \text{ and } (U'',V'' \cup \alpha) \text{ is sequential. Without loss of generality, we may assume that } r_M(U'' \cap X) \geq r_M(V'' \cap X) \text{ and, when equality holds, } |U'' \cap X| \geq |V'' \cap X|. \text{ Let } (U,V) = (\text{cl}(U''),V''-\text{cl}(U'')). \text{ Then}

5.1.2. \[r_M(U \cap X) \geq r_M(V \cap X), \text{ and, when equality holds, } |U \cap X| > |V \cap X|.

We show next that

5.1.3. \[X \cap U,X \cap V,Y \cap U, \text{ and } Y \cap V \text{ are all non-empty.}

As } \alpha \in X \cap U, \text{ the first set is not empty. If the second is empty, then, as } \alpha \text{ is in the closure of } V = V \cap Y, \text{ we can move } \alpha \text{ to } Y \text{ to get } (X - \alpha, Y \cup \alpha) \text{ as a non-sequential 4-separation of } M; \text{ a contradiction to our choice of } (X,Y). \text{ If the third is empty, then } U = X \cap U, \text{ and } (X \cap U, Y \cup V) \text{ contradicts our choice of } (X,Y) \text{ because } X \cap V \neq \emptyset. \text{ Likewise, if the fourth set is empty, then } V = X \cap V, \text{ and } (X \cap V, Y \cup U) \text{ violates our choice of } (X,Y). \text{ This completes our proof of 5.1.3.}

By submodularity of the connectivity function, \[\lambda_M(X \cup U) + \lambda_M(X \cap U) \leq \lambda_M(X) + \lambda_M(U) = 3 + 3. \text{ We now break the rest of the argument into the following two cases, which we shall then consecutively eliminate.}

(A) \[\lambda(X \cap U) \geq 4 \text{ and } \lambda(X \cup U) = \lambda(Y \cap V) \leq 2; \text{ or}

(B) \[\lambda(X \cap U) \leq 3.

5.1.4. \{(A) \text{ does not hold.}

Suppose that (A) holds. As } M \text{ is internally 4-connected, } Y \cap V \text{ is a triangle or a triad, or it contains at most two elements. By Lemma 2.5, this set is not a triad. Thus } r(Y \cap V) \leq 2. \text{ Suppose } \lambda(X \cap V) \geq 4. \text{ Then, by submodularity again, } \lambda(Y \cap U) \leq 2, \text{ so } |Y \cap U| \leq 3. \text{ Then } |Y| \leq 6, \text{ so } (Y,X) \text{ contradicts the choice of } (X,Y). \text{ Thus } \lambda(X \cap V) \leq 3. \text{ If } \lambda(X \cap V) \leq 2, \text{ then } X \cap V \text{ is contained in a triangle, so } V \text{ is contained in the union of two lines; a contradiction as } V \text{ contains a cocircuit that must have six elements and so contain a triangle. We deduce that } \lambda(X \cap V) = 3. \text{ Hence } X \cap V \subseteq \text{cl}(Y \cup U) \text{ by the choice of } (X,Y). \text{ Lemma 4.4 implies that } r(X \cap V) \leq 3. \text{ Thus } V \text{ is contained in the union of a line } L \text{ and a plane } P. \text{ By Lemma 4.7 we get a contradiction. Thus 5.1.4 holds.}

Next we show that
5.1.5. *(B) does not hold.*

Assume that *(B) holds.* Since \( \lambda(X \cap U) \leq 3 \) and \( X \cap U \) is properly contained in \( X \), either \( X \cap U \subseteq \text{fcl}(Y \cup V) \), or \( \lambda(X \cap U) \leq 2 \). In the latter case, as \( M \) is internally 4-connected, \( r(X \cap U) \leq 3 \). But, by Lemma 4.3, this inequality also holds in the former case. Thus, by 5.1.2, \( r(X \cap V) \leq 3 \). If \( r(X \cap V) \leq 2 \), then \( X \) is contained in the union of a plane and a line. Then, by Lemma 4.7, \( |X| \geq 6 \) and \( \text{si}(M/x) \) is internally 4-connected for all \( x \in X \); a contradiction. Thus, by 5.1.2,

\[
(1) \quad r(X \cap V) = 3 = r(X \cap U)
\]

and \( |X \cap V| < |X \cap U| \leq 7 \). Hence \( 3 \leq r(X) \leq 6 \). But, if \( r(X) = 3 \), then we get a contradiction by Lemma 4.7. Thus \( 4 \leq r(X) \leq 6 \).

Now view \( M \) as a restriction of \( Q = PG(r-1,2) \), where \( r = r(M) \). As \((X,Y)\) is an exact 4-separation, \( \text{cl}_Q(X) \cap \text{cl}_Q(Y) \) is a plane \( P \) of \( Q \). Because \( Y \) is fully closed, no element of \( X \) is in \( P \). It follows by orthogonality, since \( X \) is a union of cocircuits of \( M \), that each triangle that meets an element of \( X \) is either contained in \( X \) or contains exactly two elements of \( X \) with the third element being in \( P \).

5.1.6. \( r(X) \neq 4 \).

Suppose, instead, that \( r(X) = 4 \). Then \( X \subseteq \text{cl}_Q(X) - P \). So \( X \) is contained in an \( AG(3,2) \)-restriction of \( M \). Hence \( |X| \leq 8 \); a contradiction to 5.1.1. Thus 5.1.6 holds.

We show next that

5.1.7. \( r(X) = 5 \).

Suppose not. Then \( r(X) = 6 \). As \( r(X \cap U) = r(X \cap V) = 3 \), we deduce that \( \text{cl}_Q(X \cap U) \cap \text{cl}_Q(X \cap V) = \emptyset \), where we recall that \( Q = PG(r-1,2) \) and \( P = \text{cl}_Q(X) \cap \text{cl}_Q(Y) \).

Suppose \( \text{cl}_Q(X \cap V) \) meets \( P \). As \( 3 = \lambda(X) = r(X) + r(Y) - r(M) \), we know that \( r(Y) = r(M) - 3 \). Then \( \text{cl}_M(Y \cup (X \cap V)) \) is a flat with rank at most \( r(M) - 1 \). Hence its complement, which is contained in \( X \cap U \), contains a cocircuit. But this cocircuit contains at least six elements by Lemma 2.5. Since \( M|_{(X \cap U)} \) is a restriction of \( F_7 \) having at least six elements, it contains a triangle in \( X \cap U \). This contradiction to binary orthogonality implies that \( \text{cl}_Q(X \cap V) \) avoids \( P \). By symmetry, so does \( \text{cl}_Q(X \cap U) \). It follows that each triangle that meets \( X \) is either contained in \( X \cap U \) or \( X \cap V \), or contains an element of each of \( X \cap U, X \cap V \), and \( P \). If \( |X \cap U| = 7 \), then \( M|_{(X \cap U)} \cong F_7 \), so each element in \( X \cap U \) is in three triangles contained in \( X \cap U \). Then each
element in \( X \cap V \) is contained in three triangles in \( X \cap V \), so \( M | (X \cap V) \cong F_7 \),
and \( |X \cap U| = |X \cap V| \); a contradiction to 5.1.2. Thus \( |X \cap U| \leq 6 \) and 5.1.2
implies that \( |X \cap V| \leq 5 \). Thus \( X \cap V \) contains an element \( v \) that is in at most
one triangle in \( X \cap V \). Hence \( v \) is in triangles \( \{v, u_1, p_1\} \) and \( \{v, u_2, p_2\} \) for
some \( u_1 \) and \( u_2 \) in \( X \cap U \), and \( p_1 \) and \( p_2 \) in \( P \). Take \( u_3 \) in \( X \cap U \) such that
\( \{u_1, u_2, u_3\} \) is a basis for \( X \cap U \). Then \( \text{cl}(Y \cup \{v, u_3\}) \) is a flat of rank at most
\( r(M) - 1 \) whose complement, which is contained in \( X \cap V \), contains a cocircuit. This
 cocircuit has at most five elements; a contradiction. Hence 5.1.7 holds.

We now know that \( r(X) = 5 \). It follows, since \( r(X \cap U) = r(X \cap V) = 3 \),
that \( \text{cl}_Q(X \cap U) \cap \text{cl}_Q(X \cap V) \) is a point \( p \) of \( Q \). Moreover, \( r(Y) = r(M) - 2 \),
so \( r(\text{cl}_Q(Y) \cap \text{cl}_Q(X \cap U)) = 1 \) since \( r(\text{cl}_Q(Y \cup (X \cap U))) = r(M) \), otherwise
\( X \cap V \) contains a cocircuit of \( M \) that either has fewer than six elements or
contains a triangle. Similarly, \( r(\text{cl}_Q(Y) \cap \text{cl}_Q(X \cap V)) = 1 \).

The following is an immediate consequence of the fact that \( U \) is closed.

5.1.8. If \( p \in X \), then \( p \in X \cap U \).

Let \( \text{cl}_Q(Y) \cap \text{cl}_Q(X \cap U) = \{s\} \) and \( \text{cl}_Q(Y) \cap \text{cl}_Q(X \cap V) = \{t\} \). Neither \( s \)
 nor \( t \) is in \( X \) because \( Y \) is fully closed. Thus
\[
|X \cap U| \leq 6.
\]
Hence \( |X \cap V| \leq |X \cap U| - 1 \leq 5 \). By 5.1.1, \( |X| \geq 9 \). As \( |X \cap U| \geq |X \cap V| \),
it follows that \( |X \cap U| \geq 5 \). Hence

5.1.9. \( |X \cap U| \in \{5, 6\} \).

Call a triangle of \( M \) special if it contains an element of \( X \cap U \), an element
of \( X \cap V \), and an element of \( P \). Construct a bipartite graph \( H \) with vertex
classes \( X \cap U \) and \( X \cap V \) with \( uv \) being an edge, where \( u \in X \cap U \) and \( v \in X \cap V \),
precisely when \( \{u, v\} \) is contained in a special triangle. Clearly
\[
(2) \quad \sum_{u \in X \cap U} d_H(u) = \sum_{v \in X \cap V} d_H(v).
\]

Next we show the following.

5.1.10. Every vertex \( x \) of \( V(H) - \{p\} \) has its degree in \( \{1, 2\} \).

Let \( \{X', X''\} = \{X \cap U, X \cap V\} \) and take \( x \in X' \) such that \( x \neq p \). Let \( x'' \)
be the element of \( \text{cl}_Q(X'') \cap P \). Thus \( x'' \in \{s, t\} \). Clearly \( d_H(x) \leq 3 \). Assume
\( d_H(x) = 3 \). Then \( \text{cl}_Q(Y \cup x) \) contains \( x \), at least three distinct elements of
\( X'' \), and \( x'' \). Thus \( \text{cl}_Q(Y \cup x) \) contains \( X'' \). Hence \( E(M) - \text{cl}_M(Y \cup x) \) contains
at most five elements of $M$; a contradiction to the fact that every cocircuit of $M$ has at least six elements. Thus $d_H(x) < 3$.

Next suppose that $d_H(x) = 0$. Then all three triangles containing $x$ are contained in $cl_M(X')$. Thus $M|cl_M(X') \cong F_7$. Hence, for $z \in X'' - cl_M(X')$, the three triangles containing $z$ are contained in $cl_M(X'')$. Thus $M|cl_M(X'') \cong F_7$. Hence $cl_M(X') \cap cl_M(X'')$ contains a point of $M$ that is in six triangles; a contradiction. Thus 5.1.10 holds.

Now either

(i) $s = t = p$; or
(ii) $s, t$, and $p$ are distinct.

Suppose that (i) holds. Assume first that $p \notin Y$. By 5.1.10, for $W \in \{U, V\}$, every element of $M|(X \cap W)$ is in a triangle contained in $X \cap W$. Thus either $M|(X \cap W) \cong M(K_4)$ and $\sum_{w \in X \cap W} d_H(w) = 6$; or $M|(X \cap W) \cong M(K_4 \setminus e)$ and $\sum_{w \in X \cap W} d_H(w) = 9$. Since $|X \cap U| > |X \cap V|$, we obtain a contradiction using (2). Thus $p \in Y$.

As $|X \cap U| \in \{5, 6\}$ by 5.1.9, we see that $|X \cap U| = 5$, otherwise $M|((X \cap U) \cup p) \cong F_7$, and $d_H(x) = 0$ for every $x \in X \cap V$; a contradiction to 5.1.10. We deduce that $M|((X \cap U) \cup p) \cong M(K_4)$, and $5 = \sum_{u \in X \cap U} d_H(u)$. Now $p$ is in two triangles in $(X \cap U) \cup p$. Thus, of the three triangles in $cl_Q(X \cap V)$ containing $p$, at most one contains two elements of $X \cap V$. Hence, using 5.1.10, we see that $M|cl_M(X \cap V)$ comprises two triangles with a single element, not $p$, in common. Thus $\sum_{v \in X \cap V} d_H(v) = 7$; a contradiction to (2). Therefore (i) does not hold.

We now know that $s, t$, and $p$ are distinct. We show next that

5.1.11. $p \in X$.

Suppose $p \notin X$. Then $|X \cap U| = 5$ so $|X \cap V| = 4$. Thus $\sum_{u \in X \cap U} d_H(u)$ is five when $s \in Y$ and is nine otherwise. By 5.1.10, $d_H(v) < 3$ for each $v \in X \cap V$, so $t \in Y$. Since $r(X \cap V) = 3$, we deduce that $M|(X \cap V)$ is $U_{3,4}$ or $U_{2,3} \oplus U_{1,1}$. In these two cases, $\sum_{v \in X \cap V} d_H(v)$ is eight or seven, respectively. Thus, by (2), we have a contradiction. Hence 5.1.11 holds.

Suppose $|X \cap U| = 6$. Then $s \notin Y$, otherwise there is an element of $(X \cap U) - p$ with degree zero in $H$; a contradiction to 5.1.10. Thus $\sum_{u \in X \cap U} d_H(u) = 6$. Suppose $t \in Y$. If the line through $\{p, t\}$ contains a third point of $M$, say $q$, then each of the other two lines through $p$ in $cl_Q(X \cap V)$ contains at most one point of $M$ otherwise $p$ is in more than three triangles of $M$. Thus $|X \cap V| = 3$ and, as $r(X \cap V) = 3$, we see that $\{p, q, t\}$ is the unique triangle in $M|cl_M(X \cap V)$ containing $q$. As this triangle is special, it follows that $d_H(q) = 3$; a contradiction to 5.1.10. Evidently the line through $\{p, t\}$ does not contain
a third point of $M$. We deduce that $M|_{\text{cl}(X \cap V)}$ comprises two triangles that have one element, not $p$ or $t$, in common. Then $\sum_{v \in X \cap V} d_H(v) = 5$; a contradiction. We deduce that $t \notin Y$. Then exactly one of the lines in $\text{cl}(X \cap V)$ through $p$ contains exactly three points of $M$. As no point of $X \cap V$ has degree three in $H$, it follows that $M|_{\text{cl}(X \cap V)}$ comprises two triangles with a point, not $p$, in common. As $p \notin X \cap V$, it follows that $\sum_{v \in X \cap V} d_H(v) = 7$; a contradiction. Hence $|X \cap U| \neq 6$.

It remains to consider the case that $|X \cap U| = 5$ and $|X \cap V| = 4$. Then $\sum_{u \in X \cap U} d_H(u)$ is five or nine depending on whether or not $s$ is in $Y$. From 5.1.11, $p \in X$. Thus $M|[(X \cap V) \cup p]$ consists of two three-point lines meeting in a point $z$. If $z = p$, then $\sum_{v \in X \cap V} d_H(v)$ is four or eight, depending on whether or not $t$ is in $Y$; a contradiction. Hence $z \neq p$. Thus the third element on the line containing $\{p, t\}$ is in $X$. Again $\sum_{v \in X \cap V} d_H(v)$ is seven, if $t \notin Y$, or four, if $t \in Y$; a contradiction to (2). We conclude that 5.1.5 holds and the lemma follows.

It is now straightforward to complete the proof of our main result.

**Proof of Theorem 1.1.** If $M$ has a 4-cocircuit, then the result follows by Lemma 4.2. If $M$ has no 4-cocircuits, then the theorem follows by Lemma 5.1.

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6. A (non)-extension

It is natural to ask whether, for an internally 4-connected binary matroid $M$ with every element in exactly three triangles, $\text{si}(M/e)$ is internally 4-connected for every element $e$. We now describe an example where this is not the case.

Let $G$ be a copy of $K_{3,3,3}$ having vertex classes $\{a_1, a_2, a_3\}$, $\{b_1, b_2, b_3\}$, and $\{u, v, w\}$. The vertex-edge incidence matrix of $G$ is the matrix $A$ shown below.

\[
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
\]
Then $M(G)$ is an internally 4-connected matroid in which every element is in exactly three triangles. Now adjoin the matrix $B$ to $A$ where $B$ is shown below.

\[
\begin{pmatrix}
a & b & c & d & e & f \\
1 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 \\
\end{pmatrix}
\]

The matroid $N$ that is represented by $[A|B]$ has each element in $M(G)$ in exactly three triangles, and each element of $\{a,b,c,d,e,f\}$ is in exactly two triangles. To see this, observe that $N|\{a,b,c,d,e,f\} \cong M(K_4)$. Moreover, no element of $M(G)$ lies on a line with two elements of $\{a,b,c,d,e,f\}$ and it is straightforward to check that no element of $\{a,b,c,d,e,f\}$ is in a triangle with two elements of $M(G)$.

Within $M(K_5)$, take a copy of $M(K_4)$ labelled as $N|\{a,b,c,d,e,f\}$. We can join $M(K_5)$ and $N$ across this common restriction using the operation of generalized parallel connection [7, p.441]. The ground set of the resulting matroid $M$ is the union of the ground sets of $M(K_5)$ and $N$ and its flats are the sets $F$ that meet both $M(K_5)$ and $N$ in flats of these two matroids. The matroid $M$ is binary and internally 4-connected and has every element in exactly three triangles. Evidently $\text{si}(M/z)$ is not internally 4-connected for all $z$ in $\{a,b,c,d,e,f\}$.

Acknowledgements. The authors thank the referees for numerous suggestions that both corrected errors and improved the exposition.

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