THE FAST SIGNAL DIFFUSION LIMIT IN NONLINEAR
CHEMOTAXIS SYSTEMS

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Abstract. For \( n \geq 2 \) let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with smooth boundary as well as some nonnegative functions \( 0 \not\equiv u_0 \in W^{1,\infty}(\Omega) \) and \( v_0 \in W^{1,\infty}(\Omega) \). With \( \varepsilon \in (0,1) \) we want to know in which sense (if any!) solutions to the parabolic-parabolic system

\[
\begin{align*}
    u_t &= \nabla \cdot (\chi (u+1)^{m-1} \nabla u) - \nabla \cdot (\nu \nabla v) \quad \text{in } \Omega \times (0,\infty), \\
    \varepsilon v_t &= \Delta v - v + \nu u \quad \text{in } \Omega \times (0,\infty), \\
    \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial \Omega \times (0,\infty), \\
    u(\cdot,0) &= u_0, \quad v(\cdot,0) = v_0 \quad \text{in } \Omega
\end{align*}
\]

converge to those of the system where \( \varepsilon = 0 \) and where the initial condition for \( v \) has been removed. We will see in our theorem that indeed the solutions of these systems converge in a meaningful way if \( m > 1 + \frac{n}{2} \) without the need for further conditions, e. g. on the size of \( \|u_0\|_{L^p(\Omega)} \) for some \( p \in [1,\infty] \).

1. Introduction and main result. In [6], Keller and Segel examined the systems

\[
\begin{align*}
    u_t &= d_1 \Delta u - a_1 \nabla \cdot (u \nabla v) \quad \text{in } \Omega \times (0,\infty), \\
    v_t &= d_2 \Delta v - a_2 v + a_3 u \quad \text{in } \Omega \times (0,\infty)
\end{align*}
\]

with positive numbers \( d_1, d_2, a_1, a_2 \) and \( a_3 \) in order to describe the phenomenon that is known as chemotaxis. Here, \( u \) denotes the cell density of a slime mold and \( v \) is the concentration of a chemical substance produced by the cells themselves, both depending on a spatial parameter \( x \) and the time \( t \). Chemotaxis is the name given to the movement of the cells which favours higher concentrations of that chemical.

With the substitutions

\[
\begin{align*}
    \frac{a_1}{d_1} &= \chi, \quad \frac{d_1}{d_2} = \varepsilon, \quad \frac{a_2}{d_2} = \gamma \quad \text{and} \quad \frac{a_3}{d_2} = \alpha
\end{align*}
\]

and transforming the second variable from \( t \) to \( \frac{t}{\varepsilon} \) we arrive at

\[
\begin{align*}
    u_t &= \Delta u - \nabla \cdot (u \chi \nabla v) \quad \text{in } \Omega \times (0,\infty), \\
    \varepsilon v_t &= \Delta v - \gamma v + \alpha u \quad \text{in } \Omega \times (0,\infty)
\end{align*}
\]

which brings us one step closer to our topic: If we take the limit \( \varepsilon \to 0 \), then the second equation in this system is formally turning into the inhomogeneous

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Helmholtz equation \(-\Delta v + \gamma v = \alpha u\) and for the arising parabolic-elliptic system results seem to be more easily obtained.

Comparatively early, several works ([5], [11], [12], [1]) have proven solutions to blow-up in the parabolic-elliptic setting. On the other hand, results for the case of positive \(\varepsilon\) have dealt only with one example ([4]) or followed at a later time after a significantly higher amount of effort ([24], [10]).

The picture is similar when one wants to extract quantitative results from the systems. While there are numerous works for \(\varepsilon = 0\) ([13], [14], [16], [19], [18], [17]), the findings for the fully-parabolic case are less abundant ([13], [9], [22]).

Accordingly, one might pose the following question: If we use the parabolic-elliptic system for the approximation of the non-simplified system, especially for \(\varepsilon\) small, i.e. situations where the signal diffusion is much faster than that of the cells, how close are we? Until quite recently, a first hint was only given by numerical results in [8], but with [21] we now also have a theoretical work linking the two systems: In a suitable sense the solutions of the fully parabolic system for decreasing \(\varepsilon\) do in fact converge to a solution of the parabolic-elliptic simplification.

Our work is concerned with a modification of these systems: instead of a linear diffusion, for some \(m > 1\) we replace \(\Delta u\) by \(\nabla \cdot ((u + 1)^{m-1} \nabla u)\) in the first equation.

In the fully parabolic system with \(\varepsilon = 1\), the behaviour changes drastically when in the first equation the diffusion is no longer linear. While for \(m = 1\) the importance of the initial data (or more specifically, the size thereof) cannot be stressed enough, superlinear diffusion removes the need for such conditions: In this case, demanding \(m > 1 + \frac{n-2}{\varepsilon}\) suffices to ensure global existence and boundedness of solutions ([2]).

As in the case of linear diffusion, with \(\varepsilon \in (0, 1)\) we want to know in which sense (if any!) and under which conditions solutions to the parabolic-parabolic system

\[
\begin{align*}
u_t &= \nabla \cdot (\nabla (u + 1)^{m-1} \nabla u) - \nabla \cdot (u \nabla v) \quad &\text{in} &\ 
\Omega \times (0, \infty), \\
v_t &= \Delta v - v + u \quad &\text{in} &\ 
\Omega \times (0, \infty), \\
\frac{\partial v}{\partial \nu} &= 0 \quad &\text{on} &\ 
\partial \Omega \times (0, \infty), \\
u(\cdot, 0) &= u_0, \ v(\cdot, 0) = v_0 \quad &\text{in} &\ 
\Omega
\end{align*}
\]

\[\text{(S}_\varepsilon\text{)}\]

converge to those of the system where \(\varepsilon = 0\), namely

\[
\begin{align*}
u_t &= \nabla \cdot (\nabla (u + 1)^{m-1} \nabla u) - \nabla \cdot (u \nabla v) \quad &\text{in} &\ 
\Omega \times (0, \infty), \\
0 &= \Delta v - v + u \quad &\text{in} &\ 
\Omega \times (0, \infty), \\
\frac{\partial v}{\partial \nu} &= 0 \quad &\text{on} &\ 
\partial \Omega \times (0, \infty), \\
u(\cdot, 0) &= u_0 \quad &\text{in} &\ 
\Omega
\end{align*}
\]

\[\text{(S)}\]

as \(\varepsilon \to 0\). Here, we demand that \(\Omega \subset \mathbb{R}^n\), \(n \geq 2\), be a bounded domain with smooth boundary and that \(m > 1 + \frac{n-2}{\varepsilon}\). Furthermore, the nonnegative initial data fulfill \(0 \not\equiv u_0 \in W^{1,\infty}(\Omega)\) and \(v_0 \in W^{1,\infty}(\Omega)\).

Once more we mention the virtually pioneering work [21], many results here follow in their footsteps without mentioning it every single time.

We will first translate the existence and boundedness results from [2] to versions of \((S)_\varepsilon\) where \(\varepsilon \in (0,1)\) instead of \(\varepsilon = 1\) before eventually discussing the limit of the corresponding solutions. Our main result reads as follows:

**Theorem 1.1.** Let \(\Omega \subset \mathbb{R}^n\) be a bounded domain for some \(n \geq 2\). Additionally let \(m > 1 + \frac{n-2}{\varepsilon}\) and some nonnegative function \(0 \not\equiv u_0 \in W^{1,\infty}(\Omega)\) be given as well as some zero sequence \((\varepsilon_j)_{j \in \mathbb{N}} \subset (0,1)\). Then, fixing some arbitrary nonnegative
function \(v_0\) belonging to \(W^{1,\infty}(\Omega)\), by \((u_\varepsilon, v_\varepsilon)\) we denote solutions to \((S_\varepsilon)\) for every \(\varepsilon \in (0, 1)\). Under these conditions there is a classical solution \((u, v)\) to \((S)\) such that for every \(T > 0\) there is a subsequence \((\varepsilon_k)_{k \in \mathbb{N}}\) with

\[
\begin{align*}
    u_{\varepsilon_k} &\to u \text{ in } C^0\left(\overline{\Omega} \times [0, T]\right), \\
v_{\varepsilon_k} &\to v \text{ in } L^2\left((0, T); W^{1,2}(\Omega)\right), \\
v_\varepsilon &\to v \text{ in } L^\infty\left((0, T); \mathcal{C}^0\left(\overline{\Omega}\right)\right) \cap L^2_{\text{loc}}\left((0, T); W^{1,2}(\Omega)\right), \\
    \nabla v_\varepsilon &\rightharpoonup \nabla v \text{ in } L^\infty\left((0, T); W^{1,\infty}(\Omega)\right).
\end{align*}
\]

2. Existence of global classical solutions to the fully parabolic system and some bounds. General remark: In the first two subsections of this section we follow the ideas of [20], but cite the modifications of [2] due to their more immediate applicability.

2.1. Auxiliary results. Since the second equation in \((S_\varepsilon)\) basically retains the shape of the heat equation, we can use the well-known \(L^p-L^q\)-estimates to translate information on \(u\) into bounds for \(v\). More precisely, from lemma 1.3 in [23] we derive

**Lemma 2.1.** Let \(p \geq 1, s \in [1, \infty]\) with \(s < \frac{np}{n-p}\) in the case that \(p < n\) or \(s < \infty\) for \(p = n\), and \(\bar{v}_0 \in W^{1,s}(\Omega)\), then there is \(C > 0\) with the following property: Whenever \(\bar{v}\) is a solution to

\[
\begin{align*}
    \bar{v}_t &= \Delta \bar{v} - \bar{v} + f \quad \text{in } \Omega \times (0, T), \\
    \frac{\partial \bar{v}}{\partial \nu} &= 0 \quad \text{on } \partial \Omega \times (0, T), \\
    \bar{v}(\cdot, 0) &= \bar{v}_0 \quad \text{in } \Omega
\end{align*}
\]

for some \(T \in (0, \infty]\) and a function \(f\) for which we have \(f \in L^\infty((0, T); L^p(\Omega))\), then we already know that

\[
\|\bar{v}(\cdot, t)\|_{W^{1,s}(\Omega)} \leq C \left(1 + \sup_{t \in (0, T)} \|f(\cdot, t)\|_{L^p(\Omega)}\right)
\]

holds for every \(t \in (0, T)\).

From this we can derive the following corollary:

**Corollary 2.1.** Let \(p \geq 1\) and \(s \in [1, \infty]\) with \(s < \frac{np}{n-p}\) in the case that \(p < n\) or \(s < \infty\) for \(p = n\), then there is \(C > 0\) with the following property: Assuming \(u_\varepsilon\) to belong to \(L^\infty((0, T); L^p(\Omega))\) for some \(\varepsilon \in (0, 1)\) and some \(T \in (0, \infty]\) we know that

\[
\|v_\varepsilon(\cdot, t)\|_{W^{1,s}(\Omega)} \leq C \left(1 + \sup_{t \in (0, T)} \|u_\varepsilon(\cdot, t)\|_{L^p(\Omega)}\right)
\]

holds for every \(t \in (0, T)\). In particular, for any \(s \in \left[1, \frac{np}{n-p}\right]\) we can find \(C(s) > 0\) such that for every \(\varepsilon \in (0, 1)\) and \(t \in (0, T)\) (where \(T > 0\) is chosen in such a fashion that the functions \(u_\varepsilon(x, \cdot)\) and \(v_\varepsilon(x, \cdot)\) are defined on \((0, T)\) for every \(x \in \Omega\))

\[
\|v_\varepsilon(\cdot, t)\|_{W^{1,s}(\Omega)} \leq C(s)
\]

holds.
Proof. For \( t \in \left( 0, \frac{T}{\varepsilon} \right) \) we define \( \tilde{u}_\varepsilon(\cdot, t) := u_\varepsilon(\cdot, \varepsilon t) \) and \( \tilde{v}_\varepsilon(\cdot, t) := v(\cdot, \varepsilon t) \). A straightforward computation proves
\[
\partial_t \tilde{v}_\varepsilon(\cdot, t) = \mathcal{A} \tilde{v}_\varepsilon(\cdot, t) - \tilde{v}_\varepsilon(\cdot, t) + \bar{u}_\varepsilon(\cdot, t)
\]
for every \( t \in \left( 0, \frac{T}{\varepsilon} \right) \) so that the previous lemma can be applied to \( \tilde{v}_\varepsilon \). This results in our claim upon re-substituting the time variable and we stress that \( \varepsilon \) did not influence the arising constant. For the second half of the claim, we choose \( p = 1 \) and observing
\[
\frac{d}{dt} \int_\Omega u_\varepsilon = 0
\]
we find \( \| u_\varepsilon \|_{L^1(\Omega)} \equiv \| u_0 \|_{L^1(\Omega)} \in [0, T). \)
\( \square \)

Using this and a lemma from the appendix of [20], we can find that \( u_\varepsilon \) is uniformly (and independently from \( \varepsilon \)) bounded if it belongs to \( L^\infty ((0, T); L^p(\Omega)) \) for some large \( p > 1 \).

**Lemma 2.2.** Let \( p > n + 2 \) and \( C_p > 0 \). Then there is \( C > 0 \) with the following property: Whenever for some \( \varepsilon \in (0, 1) \) and \( T \in (0, \infty) \) we have
\[
\| u_\varepsilon(\cdot, t) \|_{L^p(\Omega)} \leq C_p
\]
for every \( t \in (0, T) \), then we also have
\[
\| u_\varepsilon(\cdot, t) \|_{L^\infty(\Omega)} \leq C
\]
for every \( t \in (0, T) \).

**Proof.** Firstly, for any such \( p \) from corollary 2.1 we have a constant \( C_1 > 0 \) which does not depend on \( \varepsilon \) such that
\[
\| v_\varepsilon(\cdot, t) \|_{W^{1,\infty}(\Omega)} \leq C_1
\]
holds for every \( t \in (0, T) \). Accordingly, we may employ lemma A.1 from [20] to find that our condition on \( p \) suffices to complete the proof if we choose the functions therein as \( D(x, t, u_\varepsilon) = (u_\varepsilon(x, t) + 1)^{n-1} \), \( f(x, t) = u_\varepsilon(x, t) \nabla v_\varepsilon(x, t) \) and \( g \equiv 0 \) for \( x \in \Omega \) and \( t \in (0, T) \). \( \square \)

2.2. **Local solutions and their extension.** With these results we are not only able to show that the system \((S_\varepsilon)\) possesses a global classical solution for every \( \varepsilon \in (0, 1) \) but that we also find \( \varepsilon \)-independent bounds for them.

**Lemma 2.3.** For every \( \varepsilon \in (0, 1) \) there is a global classical solution \( (u_\varepsilon, v_\varepsilon) \) to \((S_\varepsilon)\). Furthermore there exists some \( C > 0 \) such that
\[
\| u_\varepsilon(\cdot, t) \|_{L^\infty(\Omega)} + \| v_\varepsilon(\cdot, t) \|_{W^{1,\infty}(\Omega)} \leq C
\]
holds for every \( \varepsilon \in (0, 1) \) and every \( t \in (0, \infty) \).

**Proof.** We begin with some fixed \( \varepsilon \in (0, 1) \) but it is to be noted that none of the arising constants depend on this choice - whenever there is a contribution of \( \varepsilon \) to the estimates, it will be marked explicitly.

Once more, [7] provides us with a local solution and a maximum existence time \( T_{\text{max}} \in (0, \infty] \) such that either \( T_{\text{max}} = \infty \) or
\[
\limsup_{t \searrow T_{\text{max}}} \left( \| u_\varepsilon(\cdot, t) \|_{L^\infty(\Omega)} + \| v_\varepsilon(\cdot, t) \|_{W^{1,\infty}(\Omega)} \right) = \infty.
\]
Thus, in light of lemma 2.2 and corollary 2.1, \( T_{\text{max}} = \infty \) follows from the boundedness of \( u_\varepsilon \) in \( L^\infty(0, T_{\text{max}}); L^p(\Omega) \) for some \( p > n + 2 \). From corollary 2.1 we know that for any \( s \in [1, \frac{n}{n-1}] \) there is some \( C(s) > 0 \) such that

\[
\|\nabla u_\varepsilon(\cdot, t)\|_{L^s(\Omega)} \leq C(s)
\]

holds for every \( t \in (0, T_{\text{max}}) \). Next, for some \( p > \max\{n+2, m-1\} \) (as well as some \( q > 1 \) the exact size of which is entirely inconsequential here) and similarly to lemma 3.15 in [2] for the quantity

\[
y_\varepsilon(t) \coloneqq \frac{1}{p} \int_\Omega u_\varepsilon^p(\cdot, t) + \frac{\varepsilon}{q} \int_\Omega |\nabla u_\varepsilon(\cdot, t)|^{2q}
\]

with \( t \in (0, T_{\text{max}}) \) we find some constant \( C(p, q) > 0 \) which does not depend on \( \varepsilon \) and for which we have

\[
y_\varepsilon(t) + C(p, q) \left( \frac{1}{p} \int_\Omega u_\varepsilon^p(\cdot, t) + \frac{1}{q} \int_\Omega |\nabla u_\varepsilon(\cdot, t)|^{2q} \right) \leq \frac{1}{C(p, q)}
\]

for every \( t \in (0, T_{\text{max}}) \). Since this contains only very few differences compared to [2], this lengthy but barely captivating estimate has been moved to the appendix in section 4.

Here it is to be noted that the added \( \varepsilon \) in the second summand of \( y_\varepsilon \) disappears while computing the derivative so that the resulting term cannot immediately be written in terms of \( y_\varepsilon \). However, since \( \varepsilon \in (0, 1) \), we can estimate from below and find via a comparison argument that

\[
\|u_\varepsilon(\cdot, t)\|_{L^p(\Omega)} \leq y_\varepsilon(t) \leq \max \left\{ y_1(0), C(p, q)^{-2} \right\}
\]

holds for every \( \varepsilon \in (0, 1) \) and \( t \in (0, T_{\text{max}}) \). The proof is then complete since also the bound on \( \|v_\varepsilon\|_{L^\infty(\Omega \times [0, T_{\text{max}}])} \) given by corollary 2.1 is not connected to \( \varepsilon \).

2.3. Further estimates for the solutions to the fully parabolic system.

Before discussing the nature of the convergence of the solutions to (S\(_\varepsilon\)), we collect a number of estimates. Here, \( T \in (0, \infty) \) is at all times some arbitrarily large number and we assume that we are given some zero sequence \( E \coloneqq (\varepsilon_j)_{j \in \mathbb{N}} \subset (0, 1) \).

We begin this subsection by showing the Hölder continuity of every \( u_\varepsilon \) in both components.

**Lemma 2.4.** For certain \( \theta \in (0, 1) \) and \( C > 0 \)

\[
\sup_{\varepsilon \in E} \|u_\varepsilon\|_{C^{\alpha, \frac{\alpha}{2}}(\overline{\Omega} \times [0, T])} \leq C
\]

holds.

**Proof.** We can write \( \partial_t u_\varepsilon = \nabla \cdot a(x, t, u_\varepsilon, \nabla u_\varepsilon) \) for

\[
a(x, t, \alpha, \beta) = (\alpha + 1)^{m-1} \beta - a \nabla u_\varepsilon(x, t).
\]

Fixing \( C \coloneqq \max \left\{ 1, \sup_{\varepsilon \in E} \|\nabla u_\varepsilon\|_{L^\infty(\Omega \times (0, T))} \right\} \) which is finite due to lemma 2.3, we estimate

\[
a(x, t, \alpha, \beta) \cdot \beta \geq \frac{1}{2} (\alpha + 1)^{m-1} \|\beta\|^2 - \frac{C}{2} \alpha^2
\]

and

\[
|a(x, t, \alpha, \beta)| \leq |\alpha + 1|^{m-1} \|\beta\| + C |\alpha|.
\]

From theorem 1.3 and remark 1.4 in [15] we therefore have the desired regularity result. \( \square \)
We also have an estimate for the gradient of $u_\varepsilon$:

**Lemma 2.5.** There is $C > 0$ such that

$$\int_0^T \int_{\Omega} |\nabla u_\varepsilon|^2 \leq C$$

holds for every $\varepsilon \in E$.

**Proof.** Using lemma 2.3 we can fix $C := \int_{\Omega} u_0^2 + T \cdot \sup_{\varepsilon \in E} \int_{\Omega} u_\varepsilon^2 |\nabla v_\varepsilon|^2$ while direct computation and Young’s inequality yield

$$\frac{d}{dt} \int_{\Omega} u_\varepsilon^2 = -2 \int_{\Omega} (u_\varepsilon + 1)^{m-1} |\nabla u_\varepsilon|^2 + 2 \int_{\Omega} u_\varepsilon \nabla u_\varepsilon \cdot \nabla v_\varepsilon$$

$$\leq - \int_{\Omega} (u_\varepsilon + 1)^{m-1} |\nabla u_\varepsilon|^2 + \int_{\Omega} u_\varepsilon^2 |\nabla v_\varepsilon|^2$$

in $(0, T)$. Integration over $(0, T)$ and a trivial estimate then show

$$\int_0^T \int_{\Omega} |\nabla u_\varepsilon|^2 \leq \int_{\Omega} u_0^2 + \int_0^T \int_{\Omega} u_\varepsilon^2 |\nabla v_\varepsilon|^2 \leq C. \quad \blacksquare$$

As the final step in this section we define $W^{2,2}_N(\Omega) := \{ \psi \in W^{2,2}(\Omega) : \frac{\partial \psi}{\partial \nu} |_{\partial \Omega} = 0 \}$

and prove

**Lemma 2.6.** There is $C > 0$ such that

$$\int_0^T \left\| \frac{\partial}{\partial t} u_\varepsilon(\cdot, t) \right\|^2_{(W^{2,2}_N(\Omega))} \ dt \leq C$$

holds for every $\varepsilon \in E$.

**Proof.** Fixing some $\psi \in W^{2,2}_N(\Omega)$, from Hölder’s inequality we see for

$$C := \max \left\{ \frac{1}{m} \sup_{\varepsilon \in E} \left\| (u_\varepsilon + 1)^m \right\|_{L^2(\Omega)}, \sup_{\varepsilon \in E} \left\| u_\varepsilon \nabla v_\varepsilon \right\|_{L^2(\Omega)} \right\}$$

that

$$\left\| \int_{\Omega} \partial_t u_\varepsilon \psi \right\| = \left\| \int_{\Omega} (u_\varepsilon + 1)^m \Delta \psi + \int_{\Omega} u_\varepsilon \nabla v_\varepsilon \nabla \psi \right\| \leq C \left( \left\| \nabla \psi \right\|_{L^2(\Omega)} + \|\Delta \psi\|_{L^2(\Omega)} \right)$$

holds in $(0, T)$ so that upon integration the proof can be completed by using lemma 2.3 to ensure that $C$ is finite. \hfill $\blacksquare$

3. **Convergence to solutions of the parabolic-elliptic system.** Even without explicitly mentioning [21] in the preamble of every concerned lemma, it is to be understood that many ideas in this section originate there and are mostly mere adaptations for our purposes.

As in the previous subsection we assume $T \in (0, \infty)$ to be some arbitrarily large number.

To start this section off, we define a candidate $(u, v)$ that is to become our solution to $(S)$. It is found by combining estimates from the previous section and consecutively picking subsequences.
Lemma 3.1. Let \((\epsilon_j)_{j \in \mathbb{N}} \subset (0, 1)\) be a zero sequence. Then there exist a subsequence thereof and some \(\theta \in (0, 1)\) as well as functions \(u \in C^0(\mathbb{R} \times [0, T])\) and \(v \in L^2((0, T); W^{1,2}(\Omega))\) such that

\[
\begin{align*}
    u_{\epsilon} &\to u \text{ in } C^0(\mathbb{R} \times [0, T]), \\
    u_{\epsilon} &\to u \text{ in } L^2((0, T); W^{1,2}(\Omega)), \\
    v_{\epsilon} &\to v \text{ in } L^2((0, T); W^{1,2}(\Omega)), \\
    \partial_t u_{\epsilon} &\to u_t \text{ in } L^2((0, T); (W^{2,2}_N(\Omega))^*)
\end{align*}
\]

as \(\epsilon \to 0\) along that subsequence.

Proof. This follows lemma 2.3, lemma 2.4, lemma 2.5 and lemma 2.6 together with the Arzelà-Ascoli theorem. \(\square\)

In [21], namely in lemma 5.1, we find an almost directly applicable result concerning a first solution property of \(u\) and \(v\). The only real difference is that our system consists only of two components and therefore the proof is essentially the same but not an exact copy, which is why the result is not merely cited.

Lemma 3.2. There is a null set \(\mathcal{N} \subset (0, T)\) such that for every \(t \in (0, T) \setminus \mathcal{N}\) and \(v\) as in lemma 3.1 we have \(v(\cdot, t) \in W^{1,2}(\Omega)\) and

\[
\int_{\Omega} \nabla v \cdot \nabla \psi + \int_{\Omega} v \psi = \int_{\Omega} u \psi
\]

for every \(\psi \in W^{1,2}(\Omega)\).

Proof. Let \(E\) be the zero sequence provided by lemma 3.1. Then for every \(\epsilon \in E\) and every \(\varphi \in C^0(\mathbb{R} \times (0, T))\) we have

\[
-\epsilon \int_0^T \int_{\Omega} v_{\epsilon} \varphi_t + \int_0^T \int_{\Omega} \nabla v_{\epsilon} \cdot \nabla \varphi + \int_0^T \int_{\Omega} v_{\epsilon} \varphi = \int_0^T \int_{\Omega} u_{\epsilon} \varphi
\]

and the convergence in the previous lemma 3.1 shows

\[
-\epsilon \int_0^T \int_{\Omega} v_{\epsilon} \varphi_t \to 0,
\]

\[
\int_0^T \int_{\Omega} \nabla v_{\epsilon} \cdot \nabla \varphi \to \int_0^T \int_{\Omega} \nabla v \cdot \nabla \varphi
\]

and

\[
\int_0^T \int_{\Omega} v_{\epsilon} \varphi \to \int_0^T \int_{\Omega} v \varphi
\]

as well as

\[
\int_0^T \int_{\Omega} u_{\epsilon} \varphi \to \int_0^T \int_{\Omega} u \varphi
\]

as \(\epsilon \to 0\). As in [21], from the separability of \(W^{1,2}(\Omega)\) and a mollification argument we get \((\psi_j)_{j \in \mathbb{N}} \subset C^0(\mathbb{R})\) such that \(X_0 := \{|\psi_j| j \in \mathbb{N}\}\) is dense in \(W^{1,2}(\Omega)\). For \(j \in \mathbb{N}\) and \(t \in (0, T)\) we define

\[
\xi_j(t) := \int_{\Omega} \nabla v(\cdot, t) \cdot \nabla \psi_j \text{ and } \xi_j(t) := \int_{\Omega} v(\cdot, t) \psi_j
\]
as elements of $L^1((0,T))$. Given any $j \in \mathbb{N}$ we therefore find such a null set $\mathcal{N}_j \subset (0,T)$ that every $t \in (0,T) \setminus \mathcal{N}_j$ is a Lebesgue point of $\xi_j$ and $\xi_j$. We define the null set

$$\mathcal{N} := \left( \bigcup_{j \in \mathbb{N}} \mathcal{N}_j \right) \cup \{ t \in (0,T) \mid v(\cdot,t) \not\in W^{1,2}(\Omega) \},$$

which only contains mutual Lebesgue points of every $\xi_j$ and $\xi_j$ and within which $v$ belongs to $W^{1,2}(\Omega)$. Fixing $t_0 \in (0,T) \setminus \mathcal{N}$ as well as $h \in (0,T-t_0)$ and a sequence $(\chi_k)_{k \in \mathbb{N}} \subset C_0^\infty((0,T))$ with

$$\chi_k \xrightarrow{k \to \infty} \chi_{(t_0,t_0+h)} \in L^\infty((0,T))$$

where $\chi_M$ is the characteristic function on some set $M$, in $\Omega \times (0,T)$ we apply the identity from before to the function

$$\varphi(x,t) := \chi_k(t) \cdot \psi(x)$$

with fixed $k \in \mathbb{N}$ and $\psi \in X_0$. Accordingly, for any $k \in \mathbb{N}$ we see

$$\int_0^T \int_\Omega \chi_k \nabla v \cdot \nabla \psi + \int_0^T \int_\Omega \chi_k v \psi = \int_0^T \int_\Omega \chi_k u \psi$$

and that means for arbitrary $h \in (0,T-t_0)$

$$\frac{1}{h} \int_{t_0}^{t_0+h} \int_\Omega \nabla v \cdot \nabla \psi + \frac{1}{h} \int_{t_0}^{t_0+h} \int_\Omega \psi = \frac{1}{h} \int_{t_0}^{t_0+h} \int_\Omega \psi.$$

Now, since $t_0$ is a Lebesgue point and due to the continuity of $u$ in $\overline{\Omega} \times (0,T)$ (cf. lemma 3.1) we can take the limit $h \to 0$ and see

$$\int_\Omega \nabla v(\cdot,t_0) \cdot \nabla \psi + \int_\Omega v(\cdot,t_0) \psi = \int_\Omega u(\cdot,t_0) \psi$$

for every $\psi \in X_0$. Due to the density property of $X_0$ in $W^{1,2}(\Omega)$ the claim follows upon another approximation. $\square$

Within this null set we can find two more results regarding boundedness and continuity of $v$:

**Lemma 3.3.** For $v$ from lemma 3.1 and $\mathcal{N}$ from lemma 3.2 there are $\theta \in (0,1)$ and $C > 0$ such that

$$\left\| v(\cdot,t) \right\|_{W^{1,2}(\Omega)} \leq C \text{ for every } t \in (0,T) \setminus \mathcal{N}$$

and

$$\left\| v(\cdot,t) - v(\cdot,s) \right\|_{W^{1,2}(\Omega)} \leq C \left| t - s \right|^\theta \text{ for every } t, s \in (0,T) \setminus \mathcal{N}$$

hold.

**Proof.** For $t \in (0,T) \setminus \mathcal{N}$ we may pick $\psi = v$ in lemma 3.2 and together with Young's inequality this directly shows the first statement. Fixing an additional $s \in (0,T) \setminus \mathcal{N}$ for $x \in \Omega$ we define

$$z(x) := v(x,t) - v(x,s)$$

which gives us a function belonging to $W^{1,2}(\Omega)$ so that it too may be inserted into the identity in lemma 3.2. Evaluation in $s$ and $t$ and subtraction of the two identities give us

$$\int_\Omega |\nabla z|^2 + \int_\Omega z^2 = \int_\Omega (u(\cdot,t) - u(\cdot,s)) z.$$
Accordingly, lemma 3.1 and Young’s inequality allow for the estimate
\[
\int_{\mathcal{Q}} |\nabla v| + \int_{\Omega} z^2 \leq \int_{\mathcal{Q}} C|v-s|^2 |z| \leq \frac{1}{2} \int_{\Omega} z^2 + \frac{C^2 |\mathcal{Q}|}{2} |v-s|^\theta
\]
with some \( C > 0 \) and \( \theta \in (0, 1) \).

We will now see that \( v \) has further helpful properties and that at least the second equation in \((S)\) holds.

**Lemma 3.4.** For \( u \) and \( v \) as in lemma 3.1 we have some \( C > 0 \) and \( \theta \in (0, 1) \) such that \( \|v(\cdot,t)\|_{C^{2+\theta}(\mathcal{Q})} \leq C \) holds for every \( t \in (0,T) \). Furthermore, we have
\[
\begin{cases}
-\Delta v + v = u & \text{in } \Omega \times (0,T), \\
\frac{d}{dt} = 0 & \text{on } \partial\Omega \times (0,T).
\end{cases}
\]

**Proof.** We start by proving the existence of some \( q > n \) and some \( C(q) > 0 \) such that
\[
\|v(\cdot,t)\|_{W^{2,q}(\Omega)} \leq C(q)
\]
holds for every \( t \in (0,T) \setminus \mathcal{N} \) where \( \mathcal{N} \) is as in lemma 3.2. Since lemma 3.1 provides us with some \( \theta_1 \in (0,1) \) and \( C_1 > 0 \) such that
\[
\|u(\cdot,t)\|_{C^{\theta_1}(\mathcal{Q})} \leq C_1
\]
holds for every \( t \in (0,T) \), for arbitrary \( q > n \) we find a positive constant \( C_2 \) with
\[
\|u(\cdot,t)\|_{L^q(\Omega)} \leq C_2
\]
for every \( t \in (0,T) \). According to lemma 3.2, for every \( t \in (0,T) \setminus \mathcal{N} \) the function \( v \) belongs to \( W^{1,2}(\Omega) \) and it is a weak solution to the Neumann boundary value problem to \(-\Delta v(\cdot,t) + v(\cdot,t) = u(\cdot,t) \) in \( \Omega \). Elliptic estimates \((3)\) show
\[
\|v(\cdot,t)\|_{W^{2,q}(\Omega)} \leq C_3 \|u(\cdot,t)\|_{L^q(\Omega)} \leq C_2 C_3
\]
with some \( C_3 > 0 \) and for every \( t \in (0,T) \setminus \mathcal{N} \). For any \( \theta_2 \in (0,1-\frac{2}{q}) \) from the embedding \( W^{2,q}(\Omega) \hookrightarrow C^{1+\theta_2}(\Omega) \) we can therefore conclude the boundedness of the family \( (\nabla v(\cdot,t))_{t \in (0,T) \setminus \mathcal{N}} \subset C^{\theta_2}(\Omega) \). By the same source as before (“elliptic Schauder theory”), together with the boundedness of \( (u(\cdot,t))_{t \in (0,T) \setminus \mathcal{N}} \subset C^{\theta_1}(\Omega) \), we are provided with some \( C_4 > 0 \) such that
\[
\|v(\cdot,t)\|_{C^{2+\theta_1}(\Omega)} \leq C_4
\]
holds for every \( t \in (0,T) \setminus \mathcal{N} \). Since \( v \in C^0\left([0,T]; W^{1,2}(\Omega)\right) \) for some \( \theta \in (0,1) \) by lemma 3.3 implies continuity of \( v \) with respect to time, the statement also holds for \( t \in \mathcal{N} \). The rest follows from the identity in lemma 3.2.

Furthermore, we can also prove Hölder continuity of \( v \) with respect to the second variable \( t \in (0,T) \).

**Lemma 3.5.** For \( v \) from lemma 3.1 there are \( \theta \in (0,1) \) and \( C > 0 \) with
\[
\|v(\cdot,s) - v(\cdot,t)\|_{C^{2+\theta}(\mathcal{Q})} \leq C|t-s|^\theta
\]
for every \( s \in (0,T) \) and \( t \in (0,T) \).
Proof. Lemma 3.3 gives us some $\theta_1 \in (0, 1)$ such that $v \in C^0 \left([0, T]; W^{1,2}(\Omega)\right)$. After fixing some $\theta_2 \in (0, \theta_1)$ via “straight-forward interpolation” we find $a \in (0, 1)$ and $C_1 > 0$ such that
\[
\|\psi\|_{C^{2+\theta_2}(\Omega)} \leq C_1 \|\psi\|_{C^{2+\theta_1}(\Omega)}^a \|\psi\|_{W^{1,2}(\Omega)}^{1-a}
\]
holds for any $\psi \in C^{2+\theta_1}(\Omega)$. Accordingly, for any $s \in (0, T)$ and $t \in (0, T)$, upon inserting $v(\cdot, t) - v(\cdot, s)$ we see
\[
\|v(\cdot, t) - v(\cdot, s)\|_{C^{2+\theta_2}(\Omega)} \leq C_1 \left(\|v(\cdot, t)\|_{C^{2+\theta_1}(\Omega)} + \|v(\cdot, s)\|_{C^{2+\theta_1}(\Omega)}\right)^a \|v(\cdot, t) - v(\cdot, s)\|_{W^{1,2}(\Omega)}^{1-a}
\]
and for suitably large $C > 0$ as well as $\theta := (1 - a)\theta_1$ lemma 3.3 and lemma 3.4 complete the proof. \qed

Next we see that a result from [21] regarding the regularity of $v_t$ holds in our system as well.

**Lemma 3.6.** For $v$ from lemma 3.1 we have $v_t \in L^2_{\text{loc}}(\Omega \times (0, T))$.

**Proof.** We fix some $\tau \in (0, T)$ as well as some $h_0 \in (0, T - \tau)$ and for $h \in (0, h_0)$ we define
\[
z_h(x, t) := \frac{v(x, t + h) - v(x, t)}{h}
\]
for $x \in \Omega$ and $t \in (\tau, T - h_0)$. From lemma 3.3 we know that for any $t \in (\tau, T - h_0)$ by $z_h(\cdot, t) \in C^{2}(\Omega)$ we have found a classical solution for the Neumann boundary problem to
\[
-Az_h(\cdot, t) + z_h(\cdot, t) = \frac{u(\cdot, t + h) - u(\cdot, t)}{h}
\]
in $\Omega$. If $B$ is the realisation of $-A + 1$ in $W^{2,2}_N(\Omega)$, then we have
\[
\|z_h(\cdot, t)\|_{L^2(\Omega)} = \left\|B^{-1}u(\cdot, t + h) - u(\cdot, t)\right\|_{L^2(\Omega)} = \left\|\frac{1}{h} \int_t^{t+h} B^{-1} u_t(\cdot, s) \, ds\right\|_{L^2(\Omega)}
\]
for every $t \in (\tau, T - h_0)$. From the Cauchy-Schwarz inequality we thus know that
\[
\int_{\tau}^{T-h_0} \|z_h(\cdot, t)\|_{L^2(\Omega)}^2 \leq \frac{1}{h} \int_\tau^{T-h_0} \int_t^{t+h} \|B^{-1} u_t(\cdot, s)\|_{L^2(\Omega)}^2 \, ds \, dt
\]
holds for every $h \in (0, T - h_0 - \tau)$ and using the Fubini theorem this can be rearranged. To this end we consider
\[
\{(s, t) \mid t \in (\tau, T - h_0), \, s \in (t + h, T)\} = X_1 \cup X_2 \cup X_3
\]
where
\[
X_1 = \{(s, t) \mid s \in (\tau, \tau + h), \, t \in (\tau, s)\},
X_2 = \{(s, t) \mid s \in (\tau + h, T - h_0), \, t \in (s - h, s)\} \quad \text{and} \quad X_3 = \{(s, t) \mid s \in (T - h_0, T - h_0 + h), \, t \in (s - h, T - h_0)\}.
\]
Accordingly we have
\[
\frac{1}{h} \int_{T-h}^{T-h_0} \int_{t-h}^{t+h} \left\| B^{-1} u_t(\cdot, s) \right\|_{L^2(\Omega)}^2 \, dt \, ds = \frac{1}{h} \int_{T-h}^{T-h_0} \int_{t-h}^{t+h} \left\| B^{-1} u_t(\cdot, s) \right\|_{L^2(\Omega)}^2 \, dt \, ds \\
+ \frac{1}{h} \int_{T-h}^{T-h_0} \int_{s-h}^{s+h} \left\| B^{-1} u_t(\cdot, s) \right\|_{L^2(\Omega)}^2 \, dt \, ds \\
+ \frac{1}{h} \int_{T-h}^{T-h_0} \int_{s-h}^{T-h_0+h} \left\| B^{-1} u_t(\cdot, s) \right\|_{L^2(\Omega)}^2 \, dt \, ds
\]
wherein the summands together can be elementarily estimated from above by the integral \( \int_T \left\| B^{-1} u_t(\cdot, s) \right\|_{L^2(\Omega)}^2 \, ds \). Standard regularity theory in [25] shows that the mapping \( B^{-1} : L^2(\Omega) \to W^{2,2}_N(\Omega) \) is continuous so that there is some \( C_1 > 0 \) such that
\[
\left\| B^{-1} \psi \right\|_{L^2(\Omega)} \leq C_1 \left\| \psi \right\|_{W^{2,2}_N(\Omega)},
\]
holds for every \( \psi \in W^{2,2}_N(\Omega) \). Together with lemma 2.6 we therefore have some \( C_2 > 0 \) such that for every \( h \in (0, T-h_0-\tau) \)
\[
\int_{T-h}^{T-h_0} \left\| z_h(\cdot, t) \right\|_{L^2(\Omega)}^2 \leq C_2
\]
holds and so there are some zero sequence \( (h_k)_{k \in \mathbb{N}} \subset (0, T-h_0-\tau) \) and some function \( z \in L^2(\Omega \times (\tau, T-h_0)) \) with \( z_{h_k} \to z \) in \( L^2(\Omega \times (\tau, T-h_0)) \) as \( k \to \infty \). As per the definition of distributional derivatives, this \( z \) coincides with \( v_t \) almost everywhere in \( \Omega \times (\tau, T-h_0) \) for any \( \tau > 0 \) and \( h_0 > 0 \) which completes the proof. \( \Box \)

There is one more convergence result for \( v_e \) and \( v \) which we want to tackle:

**Lemma 3.7.** For \( v \) and \( (v_{e_k})_{k \in \mathbb{N}} \) found in lemma 3.1 we have
\[
\begin{align*}
\frac{v_{e_k}}{v} & \to v \quad \text{in } L^\infty_{\text{loc}}((0,T]; L^2(\Omega)) \\
\nabla v_{e_k} & \to \nabla v \quad \text{in } L^2_{\text{loc}}((\Omega \times (0,T))
\end{align*}
\]
as \( e \to 0 \).

**Proof.** We begin with the definitions
\[
z_e(x,t) := v_e(x,t) - v(x,t)
\]
for \( (x,t) \in \Omega \times (0,T) \) as well as
\[
y_e(t) := \int_\Omega \nabla z_e^2(\cdot,t)
\]
for \( t \in (0,T) \) and we see \( z_e \in L^\infty(\Omega \times (0,T)) \), \( \partial_t z_e = \partial_t v_e - v_t \in L^2_{\text{loc}}((0,T]; L^2(\Omega)) \).
From lemma 3.6 and a standard argument we gather \( y_e \in W^{1,2}_{\text{loc}}((0,T]) \), therefore \( y_e \) is locally absolutely continuous in \( (0,T] \) with
\[
y_e(t) = 2 \int_\Omega \nabla z_e^2(\cdot,t) \partial_t z_e(\cdot,t)
\]
for almost every \( t \in (0,T) \). From lemma 3.4 we know
\[
\epsilon \partial_t z_e = \Delta v_e - v_e + u_e - \epsilon v_t \\
= \Delta z_e + \Delta v - z_e - v + u_e - \epsilon v_t \\
= \Delta z_e - z_e + u_e - u - \epsilon v_t
\]
almost everywhere in $\Omega \times (0, T)$. We see

$$\frac{\epsilon}{2} \dot{y}_\epsilon(t) = \int_{\Omega} z_\epsilon(\cdot, t) \, d\dot{z}_\epsilon(\cdot, t) - \int_{\Omega} z_\epsilon^2(\cdot, t) + \int_{\Omega} z_\epsilon(\cdot, t) \left( u_\epsilon(\cdot, t) - u(\cdot, t) \right) - \epsilon \int_{\Omega} z_\epsilon(\cdot, t) v_\epsilon(\cdot, t)$$

for almost every $t \in (0, T)$ and herein the first term (using integration by parts) is transformed into $-\int_{\Omega} |\nabla z_\epsilon(\cdot, t)|^2$ while the last two terms are estimated via Young’s inequality. Together this leads us to

$$\epsilon \dot{y}_\epsilon(t) + 2 \int_{\Omega} |\nabla z_\epsilon(\cdot, t)|^2 + \int_{\Omega} z_\epsilon^2(\cdot, t) \leq 2 \int_{\Omega} |u_\epsilon(\cdot, t) - u(\cdot, t)|^2 + 2\epsilon^2 \int_{\Omega} v_\epsilon^2(\cdot, t) \quad (*)$$

for almost every $t \in (0, T)$ and we already know from lemma 3.1 that the first term on the right vanishes as $\epsilon \to 0$. We now fix some $\tau \in (0, T)$ and some $\eta > 0$. Using $u_\epsilon \to u$ in $L^\infty(\Omega \times (0, T))$, lemma 3.6 and the boundedness of $(y_\epsilon)_{\epsilon \in (\epsilon_h)_{k \in \mathbb{N}}} \subset L^\infty(\Omega \times (0, \infty))$ provided by lemma 3.4, we can fix $\epsilon_0 > 0$ with the following property: Whenever $\epsilon \in (\epsilon_h)_{k \in \mathbb{N}}$ is smaller than $\epsilon_0$, we have

$$4 |\Omega| \cdot \|u_\epsilon - u\|_{L^\infty(\Omega \times (0, T))}^2 \leq \frac{\eta}{3},$$

$$2\epsilon \int_{\frac{T}{2}}^T \int_{\Omega} v_\epsilon^2 \leq \frac{\eta}{3} \text{ and } y_\epsilon \left( \frac{T}{2} \right) \cdot e^{-\frac{\epsilon}{2}} \leq \frac{\eta}{3}.$$  

For the absolutely continuous function $[\frac{T}{2}, T] \ni t \mapsto e^{\frac{\epsilon}{2}(-t - \frac{T}{2})} y_\epsilon(t)$ we see

$$\frac{d}{dt} \left( e^{\frac{\epsilon}{2}(-t - \frac{T}{2})} y_\epsilon(t) \right) = e^{\frac{\epsilon}{2}(-t - \frac{T}{2})} \left( y_\epsilon(t) + \frac{1}{2 \epsilon} y_\epsilon(t) \right) \leq e^{\frac{\epsilon}{2}(-t - \frac{T}{2})} \left( -y_\epsilon(t) + 2 |\Omega| \cdot \|u_\epsilon - u\|_{L^\infty(\Omega \times (0, T))}^2 + 2\epsilon \int_{\Omega} v_\epsilon^2 + \frac{1}{2} y_\epsilon(t) \right)$$

for every $t \in \left( \frac{T}{2}, T \right)$ and upon integration this shows

$$y_\epsilon(t) \leq y_\epsilon \left( \frac{T}{2} \right) \cdot e^{-\frac{\epsilon}{2}(-t - \frac{T}{2})} - \frac{1}{2\epsilon} \int_{\frac{T}{2}}^T e^{\frac{\epsilon}{2}(s-t)} y_\epsilon(s) \, ds + \frac{2 |\Omega| \cdot \|u_\epsilon - u\|_{L^\infty(\Omega \times (0, T))}^2}{\epsilon} \int_{\frac{T}{2}}^T e^{\frac{\epsilon}{2}(s-t)} \, ds + 2\epsilon \int_{\frac{T}{2}}^T e^{\frac{\epsilon}{2}(s-t)} \int_{\Omega} v_\epsilon^2(\cdot, s) \, ds$$

for every $t \in (\tau, T)$. For any such $t$ and for $\epsilon < \epsilon_0$ the right-hand side is therefore bounded by $\eta$ which proves

$$z_\epsilon \to 0 \text{ in } L^\infty((\tau, T); L^2(\Omega))$$

as $\epsilon \to 0$. Integrating $(*)$, for any $\tau \in (0, T)$ we have

$$2 \int_{\tau}^T \int_{\Omega} |\nabla z_\epsilon|^2 \leq \epsilon \dot{y}_\epsilon(\tau) + 2 |\Omega| \cdot (T - \tau) \cdot \|u_\epsilon - u\|_{L^\infty(\Omega \times (0, T))}^2 + 2\epsilon \int_{\tau}^T \int_{\Omega} v_\epsilon^2$$

for every $\epsilon \in (\epsilon_h)_{k \in \mathbb{N}}$ and the terms on the right-hand side vanish as $\epsilon \to 0$ due to the boundedness of $y_\epsilon$ in $L^\infty((0, T))$, the uniform convergence of $u_\epsilon$ as before and lemma 3.6. For any $\tau \in (0, T)$ we therefore have

$$z_\epsilon \to 0 \text{ in } L^2((\tau, T); W^{1,2}(\Omega))$$

as $\epsilon \to 0$ which completes the proof. □
Once more we cite a result we can apply without modification, it can be found as lemma 5.7 in [21].

**Lemma 3.8.** For any \( q \geq 2 \) and \( \varphi \in C^2(\bar{\Omega}) \) with \( \partial_{\nu} \varphi \big|_{\partial\Omega} = 0 \) we have

\[
\int_\Omega \nabla \varphi^q \leq \left( \sqrt{n} + q - 2 \right)^{\frac{2}{q-2}} \left( \int_\Omega |\nabla \varphi|^{q-2} |D^2 \varphi|^2 \right)^{\frac{q}{2}} \left( \int_\Omega |\varphi|^{q} \right)^{\frac{2}{q}}.
\]

As the final ingredient, we show that the first equation in (5) holds together with the respective boundary condition:

**Lemma 3.9.** For every \( x \in \Omega \) and \( t \in (0, T) \) we have

\[
u_t = \nabla \cdot \left( (u + 1)^{m-1} \nabla u \right) - \nabla \cdot (u \nabla v)
\]

and for \( x \in \partial \Omega \) as well as \( t \in (0, T) \) we have

\[
\frac{\partial v}{\partial n} = 0.
\]

**Proof.** For any \( \varphi \in C^0_0(\bar{\Omega} \times [0, T)) \) and \( \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}} \) we have

\[
- \int_0^T \int_\Omega u_\varepsilon \varphi_t - \int_0^T \int_\Omega u_0 \varphi(\cdot, 0) = - \int_0^T \int_\Omega (u_\varepsilon + 1)^{m-1} \nabla u_\varepsilon \cdot \nabla \varphi + \int_0^T \int_\Omega u_\varepsilon \nabla u_\varepsilon \nabla \varphi
\]

wherein lemma 3.1 and lemma 3.7 show the convergence of all occurring integrals; the identity holds “without \( \varepsilon \)” which means that \( u \in L^2 \left( (0, T); W^{1,2}(\Omega) \right) \) (in the standard generalised sense from e. g. [7]) defines a weak solution for the initial-boundary value problem given by the two statements in this lemma (and \( u(\cdot, 0) = u_0 \)). Classical results of parabolic regularity theory and Hölder continuity of \( u, v, \nabla v \) and \( D^2 v \) in \( \bar{\Omega} \times [0, T] \) as given by lemma 3.1 and lemma 3.5 prove \( u \in C^{1+\theta_1, \frac{1+\theta_1}{2}}(\bar{\Omega} \times [0, T]) \) for some \( \theta_1 > 0 \) and even \( u \in C^{2+\theta_2, 1+\frac{\theta_2}{2}}(\bar{\Omega} \times [0, T]) \) for some \( \theta_2 \in (0, 1) \). Using a standard variational argument, this combined with the integral identity yields the desired result.

By collecting the results from this section we can now return to the theorem we originally wanted to prove:

**Proof of theorem 1.1.** From lemma 3.1 and lemma 3.7 we already have the first two claims as well as

\[
v_\varepsilon \rightarrow v \text{ in } L^\infty_{\text{loc}} \left( (0, T]; L^2(\Omega) \right) \cap L^2 \left( (0, T]; W^{1,2}(\Omega) \right)
\]

as \( \varepsilon \rightarrow 0 \) along some suitable subsequence. On the other hand, lemma 2.3 provides us with some \( C_1 > 0 \) such that

\[
\|v_\varepsilon(\cdot, t)\|_{W^{1,2}(\Omega)} \leq C_1
\]

holds for every \( t \in (0, T) \) and every \( \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}} \) which immediately gives us the fourth of the claimed convergences upon another suitable restriction of \((\varepsilon_j)_{j \in \mathbb{N}}\).

Using the Gagliardo-Nirenberg inequality we can use this a second time to see that for some \( a \in (0, 1) \) and some \( C_2 > 0 \) we have

\[
\|v_\varepsilon(\cdot, t) - v(\cdot, t)\|_{C^0(\bar{\Omega})} \leq C_2 \|v_\varepsilon(\cdot, t) - v(\cdot, t)\|_{W^{1,2}(\Omega)}^a \cdot \|v_\varepsilon(\cdot, t) - v(\cdot, t)\|_{L^2(\Omega)}^{1-a}
\]
for every \( t \in (0, T) \) and \( \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}} \) wherein the first term on the right-hand side is bounded via
\[
\|v_\varepsilon(\cdot, t) - v(\cdot, t)\|_{W^{1,\infty}(\Omega)}^a \leq \left( \|v_\varepsilon(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \right)^a.
\]

Therefore, (*) and (**) combine to prove
\[
v_\varepsilon \to v \text{ in } L^{\infty}_{\text{loc}}(0, T; C^0(\overline{\Omega}))
\]
while lemma 3.9 and lemma 3.4 complete our proof by showing that the differential equations are actually solved. \( \square \)

4. **Appendix: Extended proof of Lemma 2.4.** In the proof of lemma 2.3 we referred to [2] in order to show that for suitable \( p > 1 \), \( q > 1 \) and \( T \in (0, \infty] \) as well as
\[
y_\varepsilon(t) = \frac{1}{p} \int_{\Omega} u_\varepsilon^p(\cdot, t) + \frac{\varepsilon}{q} \int_{\Omega} |\nabla v_\varepsilon(\cdot, t)|^{2q}
\]
we can find \( C > 0 \) such that for any \( \varepsilon > 0 \) and every \( t \in (0, T) \)
\[
\dot{y}_\varepsilon(t) + C \left( \frac{1}{p} \int_{\Omega} u_\varepsilon^p(\cdot, t) + \frac{1}{q} \int_{\Omega} |\nabla v_\varepsilon(\cdot, t)|^{2q} \right) \leq \frac{1}{C}
\]
holds. In this section we want to elaborate on the steps inbetween. We begin by fixing several parameters, among which only \( p \) is of any actual interest to us.

**Lemma 4.1.** Given any lower bound \( p^* \geq 1 \) we can find
\[
p > \max\{p^*, 2n - m - 1\}, \tag{p}
\]
\( \theta > 1 \) and \( \mu > 1 \) as well as a nonempty interval \( Q := (n, \frac{n}{n-1} - \frac{m+p-1}{2}) \) such that together with \( s_+ = \frac{n}{\theta-1} \) and the conjugate exponents \( \theta' := \frac{\theta}{\theta-1} \) and \( \mu' := \frac{\mu}{\mu-1} \) we have
\[
\theta \leq \frac{n}{n-2}, \tag{\theta 1}
\]
\[
\theta > \frac{1}{-m+p+1} \tag{\theta 2}
\]
and
\[
\theta \geq \frac{ng}{nq - n + 2} \text{ for every } q \in Q \tag{\theta 3}
\]
as well as
\[
\mu \leq \frac{n}{n-2} \frac{m+p-1}{2} \tag{\mu 1}
\]
and
\[
\mu \geq \frac{n}{2}. \tag{\mu 2}
\]
Lastly, the inequalities
\[
s_+ < 2(\theta - 1)\mu' \tag{s 1}
\]
and
\[
s_+ < 2q \text{ for every } q \in Q \tag{s 2}
\]
hold, too.
Proof. Due to (p), the interval $Q$ is well-defined. Since $\frac{1}{n+m+p+1} < 1$, we only need to prove
\[
\frac{nq}{nq - n + 1} \leq \frac{n}{n-2}
\]
for every $q \in Q$ and then $(\theta_1)$, $(\theta_2)$ and $(\theta_3)$ can be achieved simultaneously. This however is a trivial consequence of $q > n$ so that we may fix an appropriate $\theta > 1$. For the compatibility of $(\mu_1)$ and $(\mu_2)$ on the other hand we need
\[
\frac{n}{2} = \frac{n - m + p - 1}{2}
\]
which is clearly guaranteed by (p). Lastly, after fixing $\mu > 1$ according to these constraints and due to $\mu' > 1$ as well as $q > 2$ for every $q \in Q$, the properties (s1) and (s2) are equally self-evident. 

These choices allow us to pick $q \in Q$ and $s \in [1, s^+]$ such that arising parameters in upcoming lemmata behave helpfully:

**Lemma 4.2.** Taking $p$, $Q$, $s^+$, $\theta$, $\theta'$, $\mu$ and $\mu'$ from lemma 4.1 we find $q \in Q$ and $s \in [1, s^+]$ with $\frac{q}{s} < 2$ such that for the positive quantities
\[
\beta_1 := \frac{n}{2} \left( \frac{-m + p + 1 - \frac{1}{2}}{1 - \frac{n}{2} + \frac{n(m + p + 1)}{2}} \right),
\]
\[
\gamma_1 := \frac{n}{2} \left( \frac{2}{3} - \frac{1}{s} \right) \frac{1 - \frac{q}{s} + \frac{m q}{s}}{1 - \frac{n}{2} + \frac{n q}{s}},
\]
\[
\beta_2 := \frac{n}{2} \left( 2 - \frac{1}{\mu} \right) \frac{1 - \frac{n}{2} + \frac{n(m + p + 1)}{2}}{1 - \frac{n}{2} + \frac{n q}{s}},
\]
\[
\gamma_2 := \frac{n}{2} \left( \frac{2(q-1)}{s} - \frac{1}{\mu} \right) \frac{1 - \frac{n}{2} + \frac{n q}{s}}{1 - \frac{n}{2} + \frac{n q}{s}}.
\]
we have
\[
\beta_1 + \gamma_1 < 1
\]
and
\[
\beta_2 + \gamma_2 < 1
\]

Proof. The positivity follows from lemma 4.1. For $q \in Q$ and $s \in [1, s^+]$ we consider the functions
\[
f(q, s) := \beta_1 + \frac{n}{2} \left( \frac{2}{3} - \frac{1}{s} \right) \frac{1 - \frac{n}{2} + \frac{n q}{s}}{1 - \frac{n}{2} + \frac{n q}{s}},
\]
as well as
\[
g(q, s) := \beta_2 + \frac{n}{2} \left( \frac{2(q-1)}{s} - \frac{1}{\mu} \right) \frac{1 - \frac{n}{2} + \frac{n q}{s}}{1 - \frac{n}{2} + \frac{n q}{s}}.
\]
and we claim that for large enough $q$ and $s$ these functions are smaller than 1. For $q^+ := \frac{n-1}{n-2}$ we see $g(q^+, s^+) = 1$ and equally easily one can compute
\[
\frac{2}{n} \left( 1 - \frac{n}{2} + q(n-1) \right)^2 \frac{\partial g}{\partial q}(q, s^+) = (n-1) \left( \frac{1}{n} + \frac{1}{\mu'} \right) > 0.
\]
On the other hand we find that

\[ f(q^+, s^+) = \frac{-m + p + 2 - \frac{2}{n}}{m + p - 2 + \frac{2}{n}} \]

which is less than 1 because of our restriction on \( m \). Therefore our claim holds true if we marginally decrease both variables. \( \square \)

Since we dropped the condition that \( \Omega \) be convex, we cannot rely on an estimate like \( \frac{\partial |\nabla v|^2}{\partial v} \leq 0 \) on \( \partial \Omega \). Instead we use the observation of Mizoguchi and Souplet in [9], that instead \( \frac{\partial |\nabla v|^2}{\partial v} \leq C |\nabla v|^2 \) on \( \partial \Omega \). This gives us (cf. lemma 3.11 in [2])

**Lemma 4.3.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain. Additionally, if there are \( s \geq 1 \) and \( q \geq 1 \) such that \( \frac{s}{q} < 2 \) as well as some \( C(s) > 0 \), then there is \( C > 0 \) with the following property: For every \( w \in C^2(\Omega) \) with \( \frac{\partial w}{\partial v} = 0 \) on \( \partial \Omega \) and \( \|\nabla w\|_{L^q(\Omega)} \leq C \) we have

\[ \int_{\Omega} |\nabla w|^{2q-2} A |\nabla w|^2 \leq -\frac{q-1}{q^2} \int_{\Omega} |\nabla w|^q + C. \]

With this final ingredient we can prove the central

**Lemma 4.4.** Given some \( p \geq 1 \), \( q \geq 1 \), \( \epsilon \in (0,1) \) and \( T_{\text{max}} \in (0,\infty] \) from the beginning of the proof of lemma 2.3 we define

\[ \gamma_\epsilon(t) := \frac{1}{p} \int_{\Omega} u_\epsilon^p(\cdot,t) + \frac{\epsilon}{q} \int_{\Omega} |\nabla u_\epsilon(\cdot,t)|^{2q} \]

for \( t \in (0,T_{\text{max}}) \). For any choice of \( p^* \geq 1 \) we can find \( p \geq p^* \), \( q \geq 1 \) and \( C > 0 \) such that

\[ \gamma_\epsilon(t) + \frac{1}{p} \int_{\Omega} u_\epsilon^p(\cdot,t) + \frac{\epsilon}{q} \int_{\Omega} |\nabla u_\epsilon(\cdot,t)|^{2q} \leq \frac{1}{C} \]

holds for every \( \epsilon \in (0,1) \) in the respective maximum interval of existence.

**Proof.** For the sake of clarity in this proof we write \( u \) instead of \( u_\epsilon \) and \( v \) instead of \( v_\epsilon \). We fix \( p \geq p^* \), \( q \), \( s \), \( \theta \), \( \theta^* \), \( \mu \) and \( \mu^* \) as in lemma 4.1 and lemma 4.2 and firstly observe that the standard procedure via integration by parts and Young’s inequality gives us

\[ \frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p = -\int_{\Omega} u^{p-2} (u + 1)^{m-1} |\nabla u|^2 + \int_{\Omega} u^{p-1} \nabla u \cdot \nabla v \leq \frac{1}{2} \int_{\Omega} u^{p-2} (u + 1)^{m-1} |\nabla u|^2 + \frac{1}{2} \int_{\Omega} u^p |\nabla v|^2 \]

in \( (0,T_{\text{max}}) \) since obviously \( (u + 1)^{1-m} \leq 1 \). Accordingly, we have

\[ \frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p + \frac{2}{(m + p - 1)^2} \int_{\Omega} \left| \nabla u^{\frac{p-1}{m}} \right|^2 \leq \frac{1}{2} \int_{\Omega} u^p |\nabla v|^2 \]

in \( (0,T_{\text{max}}) \). For the second summand in \( \gamma_\epsilon \) we need the identities

\[ \frac{\partial}{\partial t} |\nabla v|^2 = 2 \nabla v \cdot \Delta v = 2 \nabla v \cdot \nabla \Delta v - 2 |\nabla v|^2 + 2 \nabla u \cdot \nabla v \]

and

\[ A|\nabla v|^2 = 2 \nabla \cdot (D^2 v \nabla v) = 2 |D^2 v|^2 + 2 \nabla v \cdot \Delta v. \]
With these computations and lemma 4.3 we find some $C_1 > 0$ such that
\[
\frac{e}{q} \frac{d}{dt} \int_{\Omega} |\nabla v|^{2q} = \int_{\Omega} |\nabla v|^{2q-2} \left(\partial_t |\nabla v|^2 - 2|D^2 v|^2 - 2|\nabla v|^2 + 2\nabla u \cdot \nabla v\right)
\leq \frac{q-1}{q^2} \int_{\Omega} |\nabla |\nabla v|^q|^2 - 2 \int_{\Omega} |\nabla v|^{2q-2}|D^2 v|^2 + C_1
\]
\[+ 2 \int_{\Omega} |\nabla v|^{2q-2} \nabla u \cdot \nabla v
\]
in $(0, T_{\text{max}})$. Using integration by parts, Young’s inequality and the pointwise estimate $|Dv|^2 \leq n|D^2 v|^2$ we see therein
\[
2 \int_{\Omega} |\nabla v|^{2q-2} \nabla u \cdot \nabla v = -2 \int_{\Omega} u \nabla \cdot (|\nabla v|^{2q-2} \nabla v)
\]
\[= -2(q-1) \int_{\Omega} u \nabla v^{2q-4} \nabla v \cdot \nabla |\nabla v|^2 - 2 \int_{\Omega} u |\nabla v|^{2q-2} \partial_v
\]
\[\leq \frac{q-1}{8} \int_{\Omega} \nabla v^{2q-4} |\nabla v|^2 + 8(q-1) \int_{\Omega} u^2 |\nabla v|^{2q-2}
\]
\[+ \frac{2}{n} \int_{\Omega} |\nabla v|^{2q-2}|Dv|^2 + \frac{n}{2} \int_{\Omega} u^2 |\nabla v|^{2q-2}
\]
\[\leq \frac{q-1}{2q^2} \int_{\Omega} \nabla |\nabla v|^q + \left[8(q-1) + \frac{n}{2}\right] \int_{\Omega} (u+1)^2 |\nabla v|^{2q-2}
\]
\[+ 2 \int_{\Omega} |\nabla v|^{2q-2}|D^2 v|^2
\]
in $(0, T_{\text{max}})$ so that for $C_2 := 8(q-1) + \frac{n}{2} > \frac{1}{2}$
\[
\dot{y}_e + \frac{2}{(m+p-1)^2} \int_{\Omega} \nabla u^{\frac{m+p-1}{2}} + \frac{q-1}{2q^2} \int_{\Omega} |\nabla v|^{2q-2}
\]
\[\leq C_2 \int_{\Omega} u^p |\nabla v|^2 + C_2 \int_{\Omega} (u+1)^2 |\nabla v|^{2q-2} + C_1
\]
follows in $(0, T_{\text{max}})$. In the next step we combine both Hölder’s inequality and the Gagliardo-Nirenberg inequality to show that the quantities on the right can be absorbed by the terms on the left and an additional constant. More precisely, we intend to find some $C > 0$ such that
\[
\int_{\Omega} u^{m+p+1}|\nabla v|^2 \leq C \left[1 + \left(\int_{\Omega} |\nabla u^{\frac{m+p-1}{2}}|^{2}\right)^{\frac{q}{2}}\left(\int_{\Omega} |\nabla v|^{2q}|\right)^{\frac{1}{2}}\right]
\]
and
\[
\int_{\Omega} (u+1)^2 |\nabla v|^{2q-2} \leq C \left[1 + \left(\int_{\Omega} |\nabla u^{\frac{m+p-1}{2}}|^{2}\right)^{\frac{q}{2}}\left(\int_{\Omega} |\nabla v|^{2q}|\right)^{\frac{1}{2}}\right]
\]
hold in $(0, T_{\text{max}})$. The first step is simple, Hölder’s inequality lets us estimate
\[
\int_{\Omega} u^{m+p+1}|\nabla v|^2 \leq \left(\int_{\Omega} u^{(m+p+1)}\right)^{\frac{1}{p}} \left(\int_{\Omega} |\nabla v|^{2q}|\right)^{\frac{1}{2}}
\]
and
\[
\int_{\Omega} u^2 |\nabla v|^{2q-2} \leq \left(\int_{\Omega} u^2 u^{(q-1)}\right)^{\frac{1}{p}} \left(\int_{\Omega} |\nabla v|^{2(q-1)}|\right)^{\frac{1}{2}}.
\]
both in \((0,T_{\text{max}})\). For each of the four resulting integrals the Gagliardo-Nirenberg inequality is now helpful since both

\[
C_{m_0} := \int_{\Omega} u_0 \equiv \int_{\Omega} u_E
\]
and due to corollary 2.1 also

\[
C_{v_0} := \sup_{\varepsilon \in (0,1)} \int_{\Omega} |\nabla v|^{r'}
\]
are independent of \(\varepsilon\) and give us upper bounds for many arising terms. According to the Gagliardo-Nirenberg inequality, for arbitrary \(\bar{p} \in [1,\infty)\) (for \(n = 2\)) or \(\bar{p} \in [1,\frac{2n}{n-2}]\) (in higher dimensions) and \(\bar{r} \in (0,\bar{p})\) there are two constants \(C(\bar{p},\bar{r}) > 0\) and \(a(\bar{p},\bar{r}) \in (0,1)\) given by

\[
-\frac{n}{\bar{p}} = \left(1 - \frac{n}{2}\right)a(\bar{p},\bar{r}) - \frac{n}{\bar{p}} (1 - a(\bar{p},\bar{r}))
\]
such that

\[
\|\varphi\|_{L^{\bar{p}}(\Omega)} \leq C(\bar{p},\bar{r}) \|\nabla\varphi\|^{a(\bar{p},\bar{r})}_{L^{\bar{r}}(\Omega)} \|\varphi\|^{1-a(\bar{p},\bar{r})}_{L^{\bar{r}}(\Omega)} + C(\bar{p},\bar{r}) \|\varphi\|_{L^{\bar{r}}(\Omega)}
\]
holds for every \(\varphi \in C^1(\Omega)\). Apart from the computation we must be careful to move within the prescribed ranges which is where lemma 4.1 comes in.

Our first choices are \(\bar{p} = \frac{2(m+p+1)}{m+p-1}\) \(\bar{r} = \frac{2}{m+p-1}\) and \(\varphi = u^{\frac{m+p-1}{2}}\) which are admissible due to (\(\theta 1\)) and (\(\theta 2\)). For some suitable \(C_3 > 0\) we therefore have

\[
\left(\int_{\Omega} u^{-(m+p+1)/2}\right)^\frac{1}{2} \leq \left\|u^{\frac{m+p-1}{2}}\right\|_{L^{\frac{2(m+p+1)}{m+p-1}}(\Omega)} \leq C_3 + C_3 \left(\int_{\Omega} |\nabla u|^{2} \right)^{\frac{1}{2}}
\]
in \((0,T_{\text{max}})\). Similarly for \(\bar{r} = \frac{2\nu}{q}\) and \(\bar{r} = \frac{\gamma}{q}\) (cf. condition (\(\theta 3\)) coupled with the trivial observations \(s < 2\) and \(\theta' > 1\)) as well as \(\varphi = |\nabla v|^{q}\) we find some constant \(C_4 > 0\) with

\[
\left(\int_{\Omega} |\nabla v|^{2\nu}\right)^{\frac{1}{2\nu}} \leq \left\|\nabla v\right\|_{L^{2\nu}(\Omega)}^{\frac{1}{2\nu}} \leq C_4 + C_4 \left(\int_{\Omega} |\nabla |\nabla v|^{q}|^{2} \right)^{\frac{1}{2}}
\]
in \((0,T_{\text{max}})\). Picking \(\bar{p} = \frac{4\nu}{m+p-1}\) (which is a valid choice due to (\(\mu 1\)), \(\bar{r} = \frac{2}{m+p-1}\) and again \(\varphi = u^{\frac{m+p-1}{2}}\) we find another positive constant \(C_5\) such that

\[
\left(\int_{\Omega} u^{2\nu}\right)^{\frac{1}{2\nu}} = \left\|u^{\frac{4}{m+p-1}}\right\|_{L^{\frac{m+p}{4\nu}}(\Omega)} \leq C_5 + C_5 \left(\int_{\Omega} |\nabla u|^{2\nu}\right)^{\frac{1}{2}}
\]
holds in \((0,T_{\text{max}})\) while the final choices \(\bar{p} = \frac{2(q-1)}{2q}\), \(\bar{r} = \frac{\gamma}{q}\) and once more \(\varphi = |\nabla v|^{q}\) are admissible due to (\(\mu 2\)) as well as (\(s 1\)) and they give us some \(C_6 > 0\) with

\[
\left(\int_{\Omega} |\nabla v|^{2(q-1)\gamma}\right)^{\frac{1}{2}} \leq \left\|\nabla v\right\|_{L^{\frac{2(q-1)\gamma}{2q}}(\Omega)}^{\frac{2(q-1)\gamma}{2q}} \leq C_6 + C_6 \left(\int_{\Omega} |\nabla |\nabla v|^{q}|^{2} \right)^{\frac{1}{2}}
\]
in \((0,T_{\text{max}})\). Applying Young’s inequality twice shows that for any given combination of \(\beta, \gamma \in (0,1)\) such that \(\beta + \gamma < 1\) and for arbitrary \(\eta > 0\) there is \(C(\beta, \gamma, \eta) > 0\) with

\[
(1 + \varepsilon^\beta)(1 + b^\gamma) \leq \eta(a + b) + C(\beta, \gamma, \eta)
\]
for any choice of positive numbers $a$ and $b$. Together with the previous computations this proves the existence of some $C_7 > 0$ such that

$$ C_2 \int_{\Omega} u^\rho |\nabla v|^2 + C_2 \int_{\Omega} (u + 1)^2 |\nabla v|^{2q} - 2 
\leq C_7 + \frac{1}{(m + p - 1)^2} \int_{\Omega} |\nabla u|^{\frac{m + p - 1}{2}}^2 + \frac{q - 1}{4q^2} \int_{\Omega} |\nabla |\nabla v||^2 
$$

holds in $(0, T_{\text{max}})$. Accordingly and penultimately we have

$$ \dot{\gamma}_c + \frac{1}{(m + p - 1)^2} \int_{\Omega} |\nabla u|^{\frac{m + p - 1}{2}}^2 + \frac{q - 1}{4q^2} \int_{\Omega} |\nabla |\nabla v||^2 \leq C_1 + C_7 
$$
in $(0, T_{\text{max}})$ and for the final step we use the Gagliardo-Nirenberg inequality twice more: First we take $\tilde{\rho} = \frac{2p}{m + p - 1}$, $\tilde{r} = \frac{2}{m + p - 1}$ and $\varphi = u^{\frac{m + p - 1}{2}}$ to find $C_8 > 0$ with

$$ \int_{\Omega} u^{\rho} = \left\| u^{\frac{m + p - 1}{2}} \right\|_{L^{\tilde{r}}(\Omega)}^{\frac{2p}{m + p - 1}} \leq C_8 + C_8 \int_{\Omega} |\nabla u|^{\frac{m + p - 1}{2}}^2 
$$
in $(0, T_{\text{max}})$. Lastly, choosing $\tilde{\rho} = 2$, $\tilde{r} = \frac{q}{2}$ and $\varphi = |\nabla v|^q$, we find some positive constant $C_9$ such that

$$ \int_{\Omega} |\nabla v|^{2q} = \left\| |\nabla v||^q \right\|_{L^2(\Omega)} \leq C_9 + C_9 \int_{\Omega} |\nabla |\nabla v||^2 
$$
holds in $(0, T_{\text{max}})$ and in both cases we easily confirm the choices as admissible (the only not entirely trivial observation being (s2)). After another elemental estimate we therefore arrive at our claim for a suitable choice of $C$.

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