Smooth classifying spaces for differential $K$-theory

Eric Schlarmann

Abstract

We construct a version of differential $K$-theory based on smooth Banach manifold models for the homotopy types $BU \times \mathbb{Z}$ and $U$ that appear in the topological $K$-theory spectrum. These manifolds carry natural differential forms that refine the topological universal Chern character, together with natural addition and inversion operations that induce the respective structure on $\hat{K}$. Our models are norm completions of the usual stable Grassmannian and the stable unitary group. Their regularity allows us to work completely on the level of classifying spaces, and therefore we do not need a compactness assumption on our manifolds that is present in many other descriptions. The constructed groups $\hat{K}(M)$ are isomorphic to the unique differential extension of $K$-theory that admits an $S^1$-integration.

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1 Introduction

For a given cohomology theory $E$ restricted to the category of smooth manifolds, a differential refinement $\hat{E}$ provides a theory which makes use of the additional geometric information. In the case of topological $K$-theory, if a cycle is given by a vector bundle, then a lift to a class in $\hat{K}$ would be defined by the additional data of a connection. This connection will refine the Chern character of this bundle, normally only well-defined as a cohomology class, to a differential form. There is a set of axioms analogous to the Eilenberg–Steenrod axioms for cohomology that characterizes such extensions, given by Bunke and Schick [BS10, Def. 1.1].

For any smooth manifold $M$, we have a diagram of abelian groups

\[
\begin{array}{ccc}
\hat{K}^*(M) & \xrightarrow{I} & K^*(M) \\
\downarrow R & & \downarrow \text{ch} \\
\Omega^*_{d=0}(M) & \xrightarrow{\text{Rham}} & H^*(M),
\end{array}
\]

where $I$ and $R$ are forgetful functors that must come with any definition of $\hat{K}$, ch is the topological Chern character, and Rham is the de Rham map. Although this is not a cartesian
A construction of such functors (for any generalized cohomology theory $E$) was given by Hopkins and Singer [HS05, Def. 4.34], and from the modern viewpoint they can be described quite efficiently in a very general setting via sheaves of spectra [BNV16].

In order to understand and compute these abstractly defined refinements, it is however important to have concrete models. Differential $K$-theory is an especially prominent example of this, since it appears in mathematical as well as physics discussions, often in the form of a geometric model. In the case of $K$-theory, the differential version is $\mathbb{Z}_2$-graded and the even and odd part were developed independently. On the category of compact manifolds, a variety of descriptions are available. Simons and Sullivan [SS10, §3] show that even differential $K$-theory is defined by structured vector bundles, i.e. vector bundles with connection with a suitable equivalence relation. This picture was completed by Tradler, Wilson and Zeinalian [TWZ13 Thm. 5.7] by giving a geometric description of odd differential $K$-theory via operator theory, presented as maps into the stable unitary group, where the addition is induced by a blocksum operation. Later, via the Caloron correspondence, an interpretation of their model via $\Omega$-bundles was developed in [Hek+15, Thm. 3.17].

More recently, another approach has been implemented in [TWZ16, Thm. 4.25]. The authors discuss the question of representability of the $\hat{K}$-functor. As any cohomology theory, topological $K$-theory is represented by homotopy classes of maps into the corresponding spectrum, i.e.

$$K^0(M) \cong [M, BU \times \mathbb{Z}], \quad K^1(M) \cong [M, U],$$

where $U$ is the stable unitary group, i.e. the union over all $U(n)$. For compact manifolds, this agrees with the usual description as the Grothendieck group of the monoid of complex isomorphism classes of vector bundles. For non-compact manifolds, we can take this as a definition (the vector bundle definition would not yield a cohomology theory). Since only the homotopy type of these spaces is relevant in this description, one can find good models for $BU \times \mathbb{Z}$ and $U$, which carry the additional information needed to define a differential $K$-theory class from a map into it. In the end, the authors describe even and odd differential $K$-theory via smooth maps into explicit classifying spaces, equipped with differential forms that represent the universal Chern character. These universal forms are defined on filtrations of their spaces via compact smooth manifolds (the usual finite-dimensional Grassmannians and unitary groups). Therefore, again, this method relies heavily on the fact that a compact smooth manifold will always map to a finite stage in the filtration. The problem with working directly on the spaces $BU \times \mathbb{Z}$ and $U$ is of course their infinite-dimensional nature. As colimits of finite-dimensional smooth manifolds, they are Fréchet manifolds, and as such, it is harder to for example talk about differential forms on them.

The new approach employed in this paper is the use of operator theory to perform certain norm completions and slightly enlarge these spaces in order to improve their regularity. The result are well-behaved Banach manifolds, which we then equip with natural differential forms in the classical sense. These constructions are closely related to to the identification of $BU \times \mathbb{Z}$ with the space of Fredholm operators via a generalized index map, as shown by Atiyah and Jänich. While [TWZ16] ultimately work with Chen spaces as models for the spaces $BU \times \mathbb{Z}$ and $U$, our Banach manifolds allow us to do certain calculations directly in the universal example, without considering test manifolds. One immediate advantage of this approach is that our model also works in the case of non-compact manifolds. If we restrict to the compact case, we have explicit maps from the [TWZ16] classifying spaces into our completed versions, which induce isomorphisms in differential $K$-theory.

The addition map in $\hat{K}$ is implemented in both the odd and even case via an explicit blocksum operation on the classifying spaces, which we denote by $\boxplus$. Inverting an element corresponds to taking the operator adjoint in the odd case, and flipping the polarization on
the underlying polarized Hilbert space in the even case. Since the [TWZ16]-model operates always in a finite step in a filtration of the classifying spaces by compact manifolds, the authors have to implement certain finite-dimensional shifts which depend on choices of a concrete representative of a cocycle. Our model gets rid of the need for this shift by working directly in the infinite-dimensional setting. Same as in their model, a map to our geometrically enriched classifying spaces carries more data than just the isomorphism class of a bundle, and is actually all that is needed to define a differential $K$-theory class. In this sense, we take the classifying map approach seriously and do not need the additional data of a differential form that is present in other models. We also produce an obvious cycle map, that assigns a differential $K$-theory class to a vector bundle with connection. These are crucial differences to the abstract spectrum-based construction given by Hopkins and Singer. Our main result is therefore

**Theorem A.** Let $Gr_{res}$ be the restricted Grassmannian (Def. 3.1) and $U^1$ be the unitary group of operators which have a determinant (Def. 2.1). On the category of possibly non-compact smooth manifolds, the abelian group valued functors

\[
\hat{K}^0(M) = \text{Map}(M, Gr_{res})/\text{CS-homotopy + Stabilization}
\]

\[
\hat{K}^1(M) = \text{Map}(M, U^1)/\text{CS-homotopy + Stabilization}
\]

define differential $K$-theory.

The equivalence relation is a geometrically refined version of homotopy (see Def. 4.2) that makes it possible to extract a differential form representative for the Chern character out of an equivalence class, plus the additional stability requirement that blocksumming with a constant map to the basepoint does not change the equivalence class. It is a natural question whether the stability relation is already contained in the CS-homotopy equivalence class, i.e. whether $f \sim_{\text{CS}} f \oplus \text{const.}$ for any representative of a $\hat{K}$-class. In the compact case, we get an affirmative answer to this question and we prove

**Theorem B.** On the category of compact smooth manifolds, the abelian group valued functors

\[
\hat{K}^0(M) = \text{Map}(M, Gr_{res})/\text{CS-homotopy}
\]

\[
\hat{K}^1(M) = \text{Map}(M, U^1)/\text{CS-homotopy}
\]

define differential $K$-theory.

The idea that leads to this operator-theoretic approach can be described as follows: While $K$-theory is the study of stable vector bundles, it can also be interpreted as studying Hilbert space bundles with a reduction of the structure group to the stable general linear group $GL \subset GL(\mathcal{H})$, sitting in the (contractible) full general linear group of $\mathcal{H}$. By Palais' tame approximation theorem [Pal65, Thm. B] this group is homotopy equivalent via its natural inclusion to the group of operators which have a determinant, denoted by $GL^1$. Therefore we might as well study the space $BGL^1$. There happens to be a model of the universal smooth principal $GL^1$-fiber bundle, which has appeared in the study of loop groups [PS88, Sec. 7.5] and also in applications in physics in the form of fermionic second quantization (for a mathematical treatment see [War01, Sec. V.2]). This bundle carries a connection, which gives rise to a universal Chern character differential form via the usual Chern-Weil formula. The degree 2-part of this form is known in the physics literature as the Schwinger cocycle, where the discussion usually focusses on line bundles. We prove that we can get representatives also for the higher dimensional parts of the Chern character (as has already been observed by [Fres88, Thm. 3.9]), and along the way, we review some constructions in the world of restricted unitary groups, Grassmannians and Stiefel manifolds, which we could not find a good reference for.

Our proof proceeds in the following way. In Section 2 and 3, we review the construction of the restricted Grassmannian and the unitary group of operators which have a determinant,  

\footnote{Note that this is impossible for a homotopy class of maps.}
which will give the even resp. odd model for differential $K$-theory. The universal Chern class in the odd case is induced by the Maurer-Cartan form of $U^1$. In the even case, we review the construction of a certain universal bundles over $Gr_{res}$, the curvature of which gives rise to invariant representatives of the Chern character via Chern–Weil theory.

In Section 3, we equip these spaces with an $H$-space structure. The key difference to the purely homotopy theoretical approach is that we have to choose these structures in such a way that they are compatible with the Chern and Chern–Simons forms. For example, even though it induces addition in $K$-theory, operator multiplication on the unitary group will not work as an addition in $\hat{K}^1$, since it will not make the Chern character map into a monoid morphism on the level of differential forms.

Section 5 reviews geometric versions of the usual periodicity maps in the $K$-theory spectrum. The even to odd part is given by the holonomy map in the universal fibration, while the odd to even part is a certain multiplication operator map considered already by Pressley and Segal in their study of loop group representations [PS88, Sec. 6.3]. It is interesting, though not a key fact for us, that this map can be used to implement Bott periodicity as a smooth homomorphism of infinite-dimensional Lie groups. We also prove that the geometric spaces we use combine to an $\Omega$-spectrum representing $K$-theory, where the addition is implemented by our blocksum (Prop. 5.6).

In Section 6, we put together all the ingredients from the previous sections in order to prove that the previously discussed blocksum and inversion operations equip the Chern–Simons equivalence classes of maps into the classifying spaces with an abelian group structure. This is achieved by finding explicit homotopies directly on our classifying spaces, which need to have vanishing Chern–Simons forms. The discussion here is simplified considerably by the simple cohomological structure of the relevant spaces and the availability of a de Rham theorem for the Banach manifolds in question.

Having the abelian group structure on $\hat{K}^0$ and $\hat{K}^1$, what is left to do in Section 7 is to give the remaining structure maps for a differential extension and check the corresponding axioms. Here, the periodicity maps constructed in Section 5 play a key role.

In Section 8 we make a comparison to the [TWZ16]-model of differential $K$-theory and find isomorphisms induced by explicit maps on the classifying spaces. From this isomorphism we learn that the stability condition in our main theorem can actually be dropped if we restrict to compact manifolds. We close by discussing some examples of differential $K$-theory classes in Section 9.

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2 Universal representatives for the Chern character

Central to this work are the constructions of explicit smooth models for the classifying spaces of even and odd $K$-theory. Recall that the complex $K$-theory spectrum is two-periodic and consists of the spaces $BU \times \mathbb{Z}$ in the even degrees and $U$ in the odd degrees, where $U$ is the stable unitary group, i.e. the colimit along the inclusions $U(n) \hookrightarrow U(n+1)$. In order to build a differential extension of $K$-theory, we define smooth models for both of these spaces which carry natural invariant differential forms that represent the universal Chern character.

For the odd case, recall that on $U(n)$, we have the Maurer-Cartan form $\omega_n$. It is well known that the real cohomology of $U(n)$ is generated by the cohomology classes represented

\footnote{Since not all Banach manifolds admit a smooth partition of unity, this is not immediately obvious.}
by the invariant differential forms

\[
\left( \frac{i}{2\pi} \right)^k \frac{(-1)^{k-1}(k-1)!}{(2k-1)!} \text{tr} \left( \omega^{2k-1}_n \right) \in \Omega^{2k-1}(U(n)).
\]

(1)

The normalizations we have chosen here are in order to make this agree with the transgression of the Chern character in the universal fibration (see Section 5). We can stabilize using the usual inclusion \(U(n) \hookrightarrow U(n+1)\), but when one goes to limit, one has to deal with the intricacies of infinite-dimensional manifolds. Our preferred way of dealing with this is to work in the setting of Banach manifolds. The problem is that the Lie algebra of the stable unitary group \(U\) is supposed to consist of skew-adjoint finite rank operators of arbitrary dimension. Since this is not a closed subspace of the bounded operators, there are some complications if we want to consider \(U\) as a smooth manifold. A simple fix is to instead go one step further and complete with respect to the trace norm

\[
\|X\|_{L^1} = \text{tr}|X| = \text{tr}\sqrt{X^*X}.
\]

This leads to the ideal \(L^1\) of trace-class operators, and further to the Banach-Lie group \(U^1\), which we now define.

**Definition 2.1.** Let \(\mathcal{H}\) be a complex separable infinite-dimensional Hilbert space. Then \(U^1\) is the subgroup of the unitaries of \(\mathcal{H}\) given by

\[
U^1 = \{ P \in U(\mathcal{H}) \mid P - 1 \in L^1 \}
\]

with topology induced by the inclusion

\[
U^1 \hookrightarrow L^1
\]

\[
P \mapsto P - 1.
\]

Palais [Pal65, Thm. B] showed that the inclusion of the stable unitary group \(U \hookrightarrow U^1\) is a homotopy equivalence, but \(U^1\) has better regularity, as it is actually a Banach-Lie group, locally modelled on the Banach space \(L^1\). It is well known that its cohomology is generated entirely by traces of odd powers of the Maurer-Cartan form \(\omega\), analogous to formula (1). It is therefore sensible to make the following definition.

**Definition 2.2.** The universal odd Chern character form \(ch_{\text{odd}} \in \Omega^\text{odd}(U^1)\) is

\[
ch_{\text{odd}} = \sum_{k \geq 1} ch_{2k-1} = \sum_{k \geq 1} \left( \frac{i}{2\pi} \right)^k \frac{(-1)^{k-1}(k-1)!}{(2k-1)!} \text{tr} \left( \omega^{2k-1}_n \right).
\]

In order to find a good model for the even case, we recall the construction of universal connections. We will first review the situation for the finite-dimensional Grassmannians, and then spend the next chapter to generalize to the infinite-dimensional setting. As one would hope, these universal connections will yield well suited differential form representatives for the universal Chern character on our Grassmannian model of \(BU \times \mathbb{Z}\).

The Stiefel bundle over the Grassmannian manifold is given by

\[
\text{St}_{k,N} = U(N)/I_k \times U(N-k) \to U(N)/U(k) \times U(N-k) = \text{Gr}_{k,N}.
\]

There is a map \(S: \text{St}_{k,N} \to M_{N \times k}\) which assigns to an element on the left a matrix \(A \in M_{N \times k}\) which satisfies \(A^*A = I_k\). The entries of \(A\) are just given by the first \(k\) columns of a representative of our left coset. Denote by \(S^*\) the map \(S\) followed by taking the adjoint matrix, and denote by \(dS\) the differential of \(S\), which is an \(M_{N \times k}\)-valued differential form. Then, there is a Lie algebra valued 1-form given by \(S^*dS\), and one can show that it takes values in the skew adjoint matrices and furthermore that it defines a connection for the given principal bundle. Narasimhan and Ramanan [NR61, Thm. 1] observed that the family of connections given by this construction for varying \(k\) and \(N\) have a universal property, meaning
that every smooth principal bundle for a unitary group with a given connection comes from pulling back such a bundle and its respective connection by a smooth classifying map.

By Chern–Weil theory, one can define representatives for the Chern character by choosing a connection and considering traces of powers of its curvature. The curvature of \( \omega = S^*dS \) can be calculated explicitly as follows. An element in the tangent space at \( (I_k,0) \) of \( \text{St}_{k,N} \) is given by an \( n \times n \) skew-hermitian block matrix \( \begin{pmatrix} P & -Q^* \\ Q & 0 \end{pmatrix} \) where \( P \) is a skew hermitian \( k \times k \) matrix and \( Q \) is an arbitrary \( (N \times (N-k)) \) - matrix. The horizontal subspace is given by the kernel of \( \omega \), which corresponds to matrices which have \( P = 0 \). Recall that the curvature according to [KN96, Thm. 5.2] is defined to be the covariant derivative of the connection, so we have \( \Omega = d\omega \circ h \), where \( h \) is the horizontal projection. We calculate

\[
\Omega \left( \begin{pmatrix} P_1 & -Q_1^* \\ Q_1 & 0 \end{pmatrix}, \begin{pmatrix} P_2 & -Q_2^* \\ Q_2 & 0 \end{pmatrix} \right) = d\omega \left( \begin{pmatrix} 0 & -Q_1^* \\ Q_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -Q_2^* \\ Q_2 & 0 \end{pmatrix} \right) \\
= -\omega \left[ \begin{pmatrix} 0 & -Q_1^* \\ Q_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -Q_2^* \\ Q_2 & 0 \end{pmatrix} \right] \\
= Q_1^*Q_2 - Q_2^*Q_1.
\]

(2)

Invariance under the transitive left \( U(N) \)-action allows us to extend this to at any point in \( \text{St}_{k,N} \). The usual Chern–Weil theory then gives explicit differential forms on the Grassmannian after we take traces.

As in the odd case, these invariant forms stabilize under the inclusions \( \text{Gr}_{k,N} \hookrightarrow \text{Gr}_{k,N+1} \), but again, when we want to work with a universal space, problems arise. The direct limit of the Grassmannians is not a Banach manifold, and so one needs more delicate tools to talk about connections and even differential forms on them. There is no obvious construction of a universal invariant connection for \( U \)-bundles in the stable case, and some of the problems that arise are discussed in in [Fre88, Prop. 2.3]. However there still exists an analog to the universal invariant connection for \( U \)-bundles in the stable case, and some of the problems about connections and even differential forms on them. There is no obvious construction of a universal invariant connection for \( U \)-bundles in the stable case, and some of the problems that arise are discussed in in [Fre88, Prop. 2.3]. However there still exists an analog to the

3 The restricted Stiefel manifold and Grassmannian

In the infinite-dimensional setting, for a Hilbert space \( \mathcal{H} \), the unitary group \( U(\mathcal{H}) \) becomes contractible, so one usually restricts to appropriate subgroups in order to generate non-trivial topology. Assume that our Hilbert space \( \mathcal{H} \) (complex, separable, infinite-dimensional) comes with a \( \mathbb{Z} \)-graded orthonormal basis \( \{e_i\}_{i \in \mathbb{Z}} \), thereby defining a grading (also sometimes called polarization) into two infinite-dimensional, complementary subspaces

\[
\mathcal{H} \cong \mathcal{H}_+ \oplus \mathcal{H}_- = \text{span} \{e_i \mid i \geq 0\} \oplus \text{span} \{e_i \mid i < 0\}.
\]

The grading can also be seen as given by the involution \( \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \). We define the Banach algebra of bounded operators

\[
\mathfrak{gl}_{\text{res}} = \left\{ \begin{pmatrix} X_{++} & X_{+-} \\ X_{-+} & X_{--} \end{pmatrix} \in \mathfrak{gl}(\mathcal{H}_+ \oplus \mathcal{H}_-) \mid X_{+-}, X_{--} \in L^2 \right\}
\]

with norm

\[
\| \begin{pmatrix} X_{++} & X_{+-} \\ X_{-+} & X_{--} \end{pmatrix} \| = \|X_{++}\| + \|X_{-+}\| + \|X_{+-}\|_{L^2} + \|X_{--}\|_{L^2}.
\]

Recall that \( L^2 \) denotes the ideal of Hilbert-Schmidt operators, i.e. operators that meet the summability condition \( \text{tr} X^*X < \infty \). One could equivalently define \( \mathfrak{gl}_{\text{res}} \) to be the subalgebra of bounded operators that commute with \( \varepsilon \) up to a Hilbert-Schmidt operator. The group of
units in this Banach algebra is the restricted general linear group $GL_{\text{res}}$ of [PSS88, Sec. 6.2]. It is easy to see that for $(X_{++} \, \, X_{+-}) \in GL_{\text{res}}$, the operators $X_{++}$ and $X_{+-}$ have to be Fredholm operators, since they are invertible up to compacts. Then, one can show that the projection

$$\psi: GL_{\text{res}} \to \text{Fred}$$

is a homotopy equivalence [Wur06, Cor. 3.1]. By polar decomposition, the restricted unitary group $U_{\text{res}} = GL_{\text{res}} \cap U(H)$ is homotopy equivalent to $GL_{\text{res}}$, and since by the Atiyah-Jänich theorem, $\text{Fred} \sim BU \times \mathbb{Z}$, this makes $U_{\text{res}}$ into a suitable Banach-Lie group model for $BU \times \mathbb{Z}$. We will now consider the associated Grassmanian to this situation.

**Definition 3.1.** The restricted Grassmannian $Gr_{\text{res}}$ is the set of all closed subspaces $W \subset H$ such that the orthogonal projection $\pi_+: W \to H_+$ is a Fredholm operator and $\pi_-: W \to H_-$ is a Hilbert-Schmidt operator.

Loosely speaking, we only consider subspaces here which are comparable in size with $H_+$, in the sense of a perturbation by a Hilbert-Schmidt operator. As in the finite-dimensional case, there are many equivalent descriptions of the Grassmannian.

**Proposition 3.2.** A point in $Gr_{\text{res}}$ can be thought of as

(i) A subspace $W \subset H$ such that $\pi_+|_W \in \text{Fred}$ and $\pi_-|_W \in L^2$.

(ii) A self-adjoint projection operator $\pi$ on $H$ such that $[\pi, \varepsilon] \in L^2$.

(iii) A self-adjoint involution $F$ on $H$ such that $F - \varepsilon \in L^1$.

(iv) An equivalence class $[X] \in U_{\text{res}}/U(H_+) \times U(H_-)$.

**Proof.** In [PSS88, Prop. 7.1.3], it is proved that $U_{\text{res}}$ acts transitively on $Gr_{\text{res}}$ with stabilizer $U(H_+) \times U(H_-)$ and thus we get the description (iv) as a homogeneous space. To a representative $X \in U_{\text{res}}$, we associate the subspace $W = X(H_+)$ to get back to (i). Furthermore, we can construct the self adjoint involution $F = X^*X$, which is $+\text{id}$ on the subspace $W$ and $-\text{id}$ on $W^\perp$, to get to (iii). Lastly, we can consider the projection operator $(F + \text{id})/2$, which gives (ii). \qed

It is often convenient to have multiple descriptions of $Gr_{\text{res}}$. Note that using (iv), we can endow $Gr_{\text{res}}$ with the structure of a Hilbert manifold modelled on

$$T_1Gr_{\text{res}} \cong u_{\text{res}}/u(H_+) \times u(H_-) \cong L^2(H_+, H_-).$$

By Kuiper’s theorem, $U(H_+)$ is contractible. Since the projection map $U_{\text{res}} \to Gr_{\text{res}}$ defines a locally trivial principal bundle, it is therefore actually a homotopy equivalence, in sharp contrast to the finite-dimensional case (for details, see [Wur06, Lemma 2.1]). It follows that the restricted Grassmannian has infinitely many diffeomorphic path components, indexed by $\mathbb{Z}$, which can be recovered from a given subspace $W$ by its virtual dimension

$$\text{virt.dim}(W) = \dim(\ker(\pi_+: W \to H_+)) - \dim(\coker(\pi_+: W \to H_+)).$$

If $W = X(H_+)$ for $X \in U_{\text{res}}$, then $\text{virt.dim}(W) = \text{ind}(X_{++})$. As in the finite-dimensional case, there is a corresponding Stiefel manifold.

**Definition 3.3.** The restricted Stiefel manifold is the set of operators

$$\text{St}_{\text{res}} = \left\{ w = \begin{pmatrix} w_+ \\ w_- \end{pmatrix} \in \mathfrak{g}(H_+, H) \mid w \text{ injective, } w_+ - 1 \in L^1, w_- \in L^2 \right\},$$

endowed with the topology and smooth structure coming from the inclusion as an open subset into the Banach space $L^1 \times L^2$. 

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An element in $\text{St}_{\text{res}}$ is sometimes called an admissible base. We have the following immediate observation.

**Proposition 3.4.** The restricted Stiefel manifold is contractible.

**Proof.** Consider the filtration of $\mathcal{H}_+$ by the finite-dimensional subspaces $V_N$ spanned by the first $N$ basis vectors. By Palais' tame approximation theorem [Pal65, Thm. A], $\text{St}_{\text{res}}$ is homotopy equivalent to the inductive limit of the subspaces

$$\text{St}_{\text{res}}^N = \{ w \in \text{St}_{\text{res}} \mid w_{\pi_{V_{N}}} = 0 \}.$$  

But those are just the usual $n$-dimensional Stiefel manifolds that serve as the total space of the universal $\text{GL}(n)$-bundle and as such are contractible. \qed

We have set up a situation very similar to the finite-dimensional one, where one has a principal $\text{U}(k)$-bundle $\text{St}_{k,N} \rightarrow \text{Gr}_{k,N}$. The correct structure group in our case is the group of invertible operators which have a determinant $\text{GL}_1 = \{ P \in \text{GL}(\mathcal{H}_+) \mid P - 1 \in L^1 \}$. This group acts on $\text{St}_{\text{res}}$ on the right via the usual change of basis $(w, Q) \mapsto wQ$. With this action, we have the following

**Proposition 3.5.** The map $q: \text{St}_{\text{res}} \rightarrow \text{Gr}^0_{\text{res}}$ defines a smooth principal $\text{GL}_1$-bundle over the path component of the basepoint $\mathcal{H}_+$ on the restricted Grassmannian.

**Proof.** The action is smooth since it is just multiplication of operators, and it is also clear that it is free. For fiberwise transitivity, we need to check that two admissible bases for the same subspace are related by right multiplication with elements in $\text{GL}_1$. Let $w, w'$ be two admissible bases for $W$. Then $w' = wQ$, where $Q = w^{-1}w' \in \text{GL}(\mathcal{H}_+)$ and we need to show that $Q \in \text{GL}_1$. We calculate

$$1 + L^1 = \pi_+ w' = \pi_+ wQ = Q + L^1.$$  

The only thing left to show is local triviality. As in the finite dimensional case, there exist graph coordinates for the restricted Grassmannian (cf. [PS88, Ch. 7]). Those are given by

$$L^2(W, W^\perp) \rightarrow U \subset \text{Gr}_{\text{res}} \quad T \mapsto \Gamma_T = \{ (v, Tv) \mid v \in W \}.$$  

Choose an $X \in U_{\text{res}}$ such that $W = X(\mathcal{H}_+)$. Then we define a local section by setting

$s(T) = X_{\mid \mathcal{H}_+} + TX_{\mid \mathcal{H}_+} \in \text{St}_{\text{res}}.$  

\qed

**Remark 3.6.** One would like to reduce the structure group of this bundle to the unitary group $\text{U}_1$. Interestingly, this is actually not possible, since it would determine a homogenous connection which would ultimately imply that the bundle is trivial. This is discussed after Prop. 3.15 in [Fre88].

**Corollary 3.7.** The smooth fiber bundle of Banach manifolds

$$\text{GL}_1 \rightarrow \text{St}_{\text{res}} \rightarrow \text{Gr}^0_{\text{res}}$$  

is a model for the universal $\text{GL}_1$-fibration.

We will now construct a connection form for this principal bundle that is supposed to represent the limit of the finite dimensional connections on the bundles $\text{St}_{k,N} \rightarrow \text{Gr}_{k,N}$. It will in particular generate representatives for the Chern character which are compatible with the finite-dimensional versions. Consider the coordinate map

$$w: \text{St}_{\text{res}} \rightarrow L^1 \times L^2 \quad \left(\begin{array}{c} w_+ \\ w_- \end{array}\right) \mapsto \left(\begin{array}{c} w_+ - 1 \\ w_- \end{array}\right),$$  

and consider its differential $dw$ as an operator-valued differential form on $\text{St}_{\text{res}}$. Furthermore, we can associate to $w \in \text{St}_{\text{res}}$ the projection operator $\pi_W \in \text{Gr}^0_{\text{res}}$ onto $W = w(\mathcal{H}_+)$, which gives another operator-valued differential form $d\pi_W$ on $\text{St}_{\text{res}}$. 

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We calculate
\[
\Omega = d\Theta + \frac{1}{2}[\Theta, \Theta] = w^{-1}\pi_W d\pi_W d\pi_W w.
\]

Proof. We first check that \(\Theta\) is \(L^1\)-valued. We can write \(\Theta = w^{-1}\pi_W (\pi_+ + \pi_-) dw\), and since \(\pi_+ dw\) is trace class, it remains to show that the second summand is also trace class. A simple calculation shows that \(\pi_W \in \text{Gr}_{\text{res}}\) is equivalent to \(\pi_+ - \pi_- \in L^2\). Therefore, using that \(\pi_- dw = d(\pi_- w) \in L^2\), we have
\[
w^{-1}\pi_W \pi_- dw = w^{-1}(\pi_+ + L^2)\pi_- dw = 0 + L^2 \cdot L^2 = L^1.
\]

We now check the defining properties of a connection form. On the fundamental vector fields for \(X \in L^1\) of the form \(\bar{X}_w = \frac{d}{dt}_{|t=0} w \exp(tX)\), we clearly have \(\Theta(\bar{X}) = X\). On the other hand, we have
\[(R^*Q\Theta)_w = (wQ)^{-1}\pi_W (dw)Q = \text{Ad}_{Q^{-1}}\Theta_w,
\]
finishing the proof that \(\Theta\) is a connection form.

For the calculation of the curvature, we will need the identities
\[
dw = d(\pi_W w) = d\pi_W w + \pi_W dw
\]
\[
d\pi_W = d(w^{-1}\pi_W) = dw^{-1}\pi_W + wd(w^{-1}\pi_W).
\]
From the second identity, it follows that
\[
d(w^{-1}\pi_W) = w^{-1}\pi_W d\pi_W - w^{-1}\pi_W dw w^{-1}\pi_W.
\]
We calculate
\[
d\Theta = d(w^{-1}\pi_W dw) = d(w^{-1}\pi_W) dw
\]
\[
= (w^{-1}\pi_W d\pi_W - w^{-1}\pi_W dw w^{-1}\pi_W) dw
\]
\[
= w^{-1}\pi_W d\pi_W (d\pi_W w + \pi_W dw) - w^{-1}\pi_W dw w^{-1}\pi_W dw
\]
\[
= w^{-1}\pi_W d\pi_W d\pi_W w - \frac{1}{2}[\Theta, \Theta],
\]
since \(\pi_W d\pi_W \pi_W = 0\).

Proposition 3.8. The assignment \(\Theta = w^{-1}\pi_W dw\) defines a principal connection on \(GL^1\)-bundle \(\text{St}_{\text{res}} \to \text{Gr}_{\text{res}}\). The curvature of \(\Theta\) is given by the expression
\[
\Omega = d\Theta + \frac{1}{2}[\Theta, \Theta] = w^{-1}\pi_W d\pi_W d\pi_W w.
\]

Definition 3.9. The universal even Chern character form \(\text{ch}_{\text{even}} \in \Omega^{\text{even}}(\text{Gr}_{\text{res}})\) is
\[
\text{ch}_{\text{even}} = \sum_{k \geq 0} \text{ch}_{2k} = \text{ch}_0 + \sum_{k \geq 1} \left(\frac{i}{2\pi}\right)^k \frac{1}{k!} \text{tr} \left(\Omega^k\right),
\]
where \(\Omega = \pi_W d\pi_W d\pi_W\) is a trace class operator valued form. Here, \(\text{ch}_0 : \text{Gr}_{\text{res}} \to \mathbb{Z}\) is the map that assigns to \(W\) its virtual dimension.
The positive degree forms are actually invariant: Since the action of $\text{U}_{\text{res}}$ is by conjugation of both $\pi_W$ and $d\pi_W$ by a unitary, it leaves the trace invariant. Thus, it is useful to explicitly work out what happens at the tangent space of $H_+$. Recall that

$$T_1\text{Gr}_{\text{res}} \cong u_{\text{res}}(\mathcal{H}_+) \times u(\mathcal{H}_-) \cong \left\{ \begin{pmatrix} 0 & -c^* \\ c & 0 \end{pmatrix} \mid c \in L^2(\mathcal{H}_+, \mathcal{H}_-) \right\}.$$ 

Set $w_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \text{St}_{\text{res}}$. For $w = Xw_0 \in \text{St}_{\text{res}}$, we have that $\pi_W = \pi_{X(\mathcal{H}_+)} = X\pi_*X^*$ and therefore $(d\pi_W)\pi_* = [-, \pi_*]$, where the bracket indicates the commutator. Therefore, evaluation of $\Omega = \pi_W d\pi_W d\pi_W$ at the point $\pi_*$ yields

$$\Omega_{\pi_*} = \begin{pmatrix} 0 & -c_1^* \\ c_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -c_2^* \\ c_2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -c_1^* \\ c_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -c_2^* \\ c_2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \text{c}_1^2 \\ \text{c}_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \text{c}_2^2 \\ \text{c}_2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \text{c}_1^2 - \text{c}_2^2 \text{c}_1 \\ \text{c}_1 \text{c}_2^2 - \text{c}_2 \text{c}_1 \end{pmatrix} \begin{pmatrix} 0 & \text{c}_2^2 \\ \text{c}_2 & 0 \end{pmatrix},$$

and we recover the familiar formula from the finite-dimensional case.

There are natural smooth inclusions of the finite-dimensional Grassmannians into the restricted Grassmannian, given as follows: Pick a $\mathbb{Z}$-graded orthonormal basis $\{e_i\}$ for $\mathcal{H}$, where $\mathcal{H}_+ \cong \text{span} \{e_i \mid i \geq 0\}$. Considering for $N \in \mathbb{Z}$ the subspaces

$$\mathcal{H}_N = \text{span} \{e_i \mid i \geq N\},$$

one sees that the subsets

$$\text{Gr}_{\text{res}, N} = \{W \in \text{Gr}_{\text{res}} \mid \mathcal{H}_N \subset W \subset \mathcal{H}_-\}$$

are isomorphic to the full finite-dimensional Grassmannians $\text{Gr}(\mathbb{C}^{2N}) = \coprod_{k \leq 2N} \text{Gr}_{k,2N}$ by mapping $W$ to $W/\mathcal{H}_N \subset \mathcal{H}_-\mathcal{H}_N \cong \mathbb{C}^{2N}$. The inclusion of $\text{Gr}_{\text{res}, N}$ into $\text{Gr}_{\text{res}, N+1}$ corresponds to sending $V \in \text{Gr}(\mathbb{C}^{2N})$ to $\{0\} \oplus V \oplus \mathbb{C} \in \text{Gr}(\mathbb{C}^{2N+1})$. The union of these finite-dimensional Grassmannians, denoted by $\text{Gr}_{\text{res}, \infty}$, is dense in $\text{Gr}_{\text{res}}$, and the intersection $\text{Gr}_{\text{res}, N} \cap \text{Gr}_{\text{res}, \infty}$ is diffeomorphic to $\text{Gr}_{N+k,2N}$ (cf. [Wur01, Prop. III.5]). All in all, we have inclusion maps

$$i: \text{Gr}_{k,2N} = \text{Gr}_{N+(k-N),2N} \hookrightarrow \text{Gr}_{res, \infty} \subset \text{Gr}_{\text{res}},$$

$$W \mapsto W \oplus \mathcal{H}_N,$$

which are easily seen to be compatible with the chosen Chern character differential forms in the following sense:

**Proposition 3.10.** Under the natural inclusion $i: \text{Gr}_{k,2N} \hookrightarrow \text{Gr}_{\text{res}}$, the universal Chern character form $\text{ch}_{\text{ren}}$ pulls back to the corresponding forms on the finite-dimensional Grassmannian, which are given by the Chern–Weil forms of the universal connection (see [2]).

**Proof.** On the level of projections, with the above mentioned identification of $\mathbb{C}^{2N}$ with a subset of $\mathcal{H}$, we see that $\pi_W$ gets mapped by $i$ to $\pi_W + \pi_N$, where $\pi_N$ is the projection to $\mathcal{H}_N$. But that means that the pullback of the form can be written as

$$(\pi_W + \pi_N) d(\pi_W + \pi_N) d(\pi_W + \pi_N) = (\pi_W + \pi_N) d\pi_W d\pi_W$$

$$= \pi_W d\pi_W d\pi_W - d\pi_N \pi_W d\pi_W$$

$$= \pi_W d\pi_W d\pi_W,$$

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where we used that $\pi_N \pi_W = \pi_W \pi_N = 0$. We can now do the calculation \(^1\) again, but this time on the finite-dimensional Grassmannian, and see that (using additivity of the trace) the resulting Chern character forms $\text{tr}^* (\Omega^k)$ are precisely the ones coming from the Narasimhan–Ramanan curvature form. \(\blacksquare\)

**Remark 3.11.** We can use this calculation to cook up a “cycle map”: Given a connected manifold $M$ and a class in $\hat{K}_0(M)$ represented by a formal difference $[V, \nabla_V] - [W, \nabla_W]$ of smooth hermitean vector bundles with compatible connections of dimension $k$ and $k'$, we can use the Narasimhan–Ramanan theorem to get classifying maps $f_V: M \to \text{Gr}_{k,2N}$, $f_W: M \to \text{Gr}_{k',2N}$. Employing our above defined inclusions, we may as well assume that the target of these maps is actually $\text{Gr}_{\text{res}}$. Then, using the flip and blocksum map defined in Section \(^2\), we get a smooth map to the restricted Grassmannian, given by $f_V \oplus \text{flip}(f_W)$, which represents the differential $K$-theory class in our model. Note that $\text{ch}_0(f_V \oplus \text{flip}(f_W)) = (k - N) - (l - N) = k - l = \text{ch}_0(V) - \text{ch}_0(W)$, which justifies our definition of the degree zero part $\text{ch}_0$ of the Chern character.

4 Chern–Simons forms, the blocksum and the inversion operation

We begin this chapter by discussing the transgressions of the Chern character in the path loop fibration. The resulting Chern–Simons forms have first appeared in [CS74, Sec. 3] and they were one of the key ideas that lead to the development of differential cohomology theories.

Let us consider the universal situation of the smooth path-loop fibration over $U^1$ and $\text{Gr}_{\text{res}}$. There are some subtleties when one wants to consider path and loop spaces as smooth manifolds, but all we need is to have well-defined pullbacks to finite-dimensional manifolds. This situation can be made precise by Chen’s notion of diffeological spaces [Che77, Def. 1.2.1]. However, the identities that we want are provable via topological arguments, so this viewpoint is not too important for the present paper, and one might as well interpret the next paragraph as an informal motivation for the second part of Def. \(^4.1\).

By pulling back along the evaluation maps $P\text{Gr}_{\text{res}} \times I \to \text{Gr}_{\text{res}}$ and $P\text{U}^1 \times I \to U^1$ and then fiber integrating, we arrive at the universal Chern–Simons forms

\[
\begin{align*}
\text{cs}_{\text{odd}} &= \int_I \text{ev}_* (\text{ch}_{\text{even}}) \in \Omega^\text{odd}(P\text{U}^1) \\
\text{cs}_{\text{even}} &= \int_I \text{ev}_* (\text{ch}_{\text{odd}}) \in \Omega^\text{even}(P\text{Gr}_{\text{res}})
\end{align*}
\]

on the path spaces based at the identity. They famously fit into the equation

\[
d\text{cs} = \text{ev}_1^* \text{ch} - \text{ev}_0^* \text{ch}
\]

by an application of Stokes’ theorem. When we pull back the Chern–Simons forms to the based loop space in order to get a form $\text{cs}_\Omega$, this identity shows that $\text{cs}_\Omega$ is a transgression of $\text{ch}$ in the path-loop fibration. Using our universal representatives, we can now associate to a map into $U^1$ or $\text{Gr}_{\text{res}}$, i.e. to a representative for a $K$-theory class, certain differential forms.

**Definition 4.1.** Let $M$ be a smooth compact manifold. Then we define the maps

\[
\begin{align*}
\text{Ch}: \text{Map}(M, U^1) &\to \Omega^\text{odd}_{\cl}(M) \\
\text{Ch}: \text{Map}(M, \text{Gr}_{\text{res}}) &\to \Omega^\text{even}_{\cl}(M)
\end{align*}
\]

given by pullback of the universal Chern forms (Def. \(^2.2\) and Def. \(^3.9\)). Furthermore, we define the maps

\[
\begin{align*}
\text{CS}: \text{Map}(M \times I, U^1) &\to \Omega^\text{even}(M) \\
\text{CS}: \text{Map}(M \times I, \text{Gr}_{\text{res}}) &\to \Omega^\text{odd}(M)
\end{align*}
\]
given by “pullback of the universal Chern–Simons forms” via smooth homotopies, i.e. \( \text{CS}(H_t) = \int_I H_t^* \text{ch}. \)

We define a refined notion of homotopy by using these forms, following [TWZ16, Def. 3.4]. It is designed to retain more information in an equivalence class than just the isomorphism type of the corresponding bundle. One important feature is that we will have a well-defined map that assigns to a “CS-equivalence” class of maps the pullback of its universal Chern form, which is only possible up to exact forms for a homotopy class.

**Definition 4.2.** Let \( f, g : M \rightarrow U \) for \( U \in \{ \text{Gr}_{\text{res}}, U^1 \} \) be smooth maps. We say that \( f \) and \( g \) are Chern–Simons homotopic (CS-homotopic) if there is a smooth homotopy \( H_t \) connecting them such that the resulting Chern–Simons form given by integrating the universal Chern character

\[
\text{CS}_{\text{odd/even}}(H) = \int_I H_t^* (\text{ch}_{\text{even/odd}}) \in \Omega_{\text{odd/even}}(M)
\]

is exact.

We will also define the blocksum operation, which works in general for operators on an infinite-dimensional Hilbert space \( \mathcal{H} \). It will be used to implement addition in differential K-theory. In order to be explicit, we choose a specific isomorphism \( \rho : H \rightarrow H \oplus H \).

**Definition 4.3.** Let \( \rho : \mathcal{H} \rightarrow \mathcal{H} \oplus \mathcal{H} \) be the isometric isomorphism

\[
\rho : e_{2k} \mapsto (e_k, 0),
\]

\[
e_{2k+1} \mapsto (0, e_k),
\]
given on a \( \mathbb{N} \) or \( \mathbb{Z} \)-graded orthonormal basis \( \{ e_i \} \). We define the corresponding blocksum map

\[
\boxplus_\rho : \mathfrak{gl}(\mathcal{H}) \times \mathfrak{gl}(\mathcal{H}) \rightarrow \mathfrak{gl}(\mathcal{H})
\]

\[
(A, B) \mapsto \rho^*(A \oplus B)\rho.
\]

Note that various subgroups of operators which we consider are preserved by this construction, most importantly \( U_{\text{res}} \) and \( U^1 \). This also induces a well-defined operation on \( \text{Gr}_{\text{res}} \), where it corresponds to a direct sum of subspaces: If \( W = X(\mathcal{H}_+) \) and \( V = Y(\mathcal{H}_+) \) for \( X, Y \in U_{\text{res}} \), then

\[
W \boxplus_\rho V = (X \boxplus_\rho Y)(\mathcal{H}_+) = \rho^*(X \oplus Y)\rho(\mathcal{H}_+) = \rho^*(V \oplus W).
\]

Via pointwise application, we can now make sense of the blocksum of two maps \( f, g \) from a manifold into the bounded linear operators \( \mathfrak{gl}(\mathcal{H}) \). We write

\[
f \boxplus_\rho g = \rho^*(f \oplus g)\rho.
\]

Ultimately, one wants this blocksum operation on maps to not depend on the chosen unitary isomorphism \( \rho \) up to the right equivalence relation. This is easily seen to be true for homotopy classes of maps by using path-connectedness of the unitary group \( U \). The following technical lemma will show the corresponding statement for the more restricted class of CS-homotopies.

**Lemma 4.4.** Let \( f : M \rightarrow U^1, \ g : M \rightarrow U_{\text{res}} \) and \( h : M \rightarrow \text{Gr}_{\text{res}} \) be smooth maps and consider \( A \in U(\mathcal{H}_+) \) and \( B \in U(\mathcal{H}_+) \times U(\mathcal{H}_-) \subset U_{\text{res}} \). Then the pairs of maps

(i) \( AfA^* : M \rightarrow U^1 \) and \( f : M \rightarrow U^1 \)

(ii) \( BgB^* : M \rightarrow U_{\text{res}} \) and \( g : M \rightarrow U_{\text{res}} \)

(iii) \( Bh : M \rightarrow \text{Gr}_{\text{res}} \) and \( h : M \rightarrow \text{Gr}_{\text{res}} \)
are CS-homotopic\textsuperscript{3}, i.e. conjugation by a fixed such matrix does not change the Chern–Simons equivalence class. In particular, for any other unitary isomorphism $\rho' : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ (respecting the grading in the polarized case), we have that $f \boxplus \rho, g$ and $f \boxplus \rho', g$ are CS-homotopic.

**Proof.** For the first case, choose a smooth path $A_t$ from $A_0 = 1$ to $A_1 = A$. Then there is a smooth universal homotopy

$$H_t : U^1 \times I \to U^1$$

$$(X, t) \mapsto A_t X A_t^*,$$

which yields a homotopy as stated for any $f : X \to U^1$ by composition. We need to show that its CS-form is exact. We have

$$d\text{CS}_{2k}(H_t) = d \int_I H_t^* ch_{2k+1} = H_t^* ch_{2k+1} - H_0^* ch_{2k+1} =$$

$$- \left( \frac{1}{2\pi i} \right)^{k+1} \frac{(k)!}{(2k+1)!} (\text{tr}(AX^* dX A^*)^{2k+1} - \text{tr}(X^* dX)^{2k+1}) = 0.$$

Since the positive even cohomology of $U^1$ vanishes, this implies that the Chern–Simons forms for $k > 0$ are exact. For $k = 0$, we make a direct calculation. Note that the differential of $H_t$ splits according to the splitting of the tangent space of $U^1 \times I$ into a sum of a space part with a time derivative. Our notation for the space derivative is $dH_t$, while we denote the time derivative by $\dot{H}_t$. We have

$$\dot{H}_t = \dot{A}_t X A_t^* - A_t X A_t^* \dot{A}_t A_t^*$$

$$dH_t = A_t dX A_t^*.$$

We need to calculate

$$\text{CS}_0(H_t) = \int_I H_t^* ch_1 = \left( \frac{i}{2\pi} \right) \int_I t_{\partial t} (H_t^*(\text{tr}(\omega_{MC}))).$$

The integrand yields

$$t_{\partial t} (H_t^*(\text{tr}(\omega_{MC}))) = \text{tr}(A_t X^* A_t^*(\dot{A}_t X A_t^* - A_t X A_t^* \dot{A}_t A_t^*))$$

$$= \text{tr}(X^* A_t^* \dot{A}_t X - \dot{A}_t A_t^*) = 0$$

for all $t$ and therefore, $\text{CS}_0$ vanishes.

For the second case, we choose again a smooth path from 1 to $B$ in order to define a homotopy $H_t$ starting at $H_0 = C_B$ and ending at $H_1 = \text{id}_{U_{res}}$, where $C_B$ denotes conjugation by $B$. By the vanishing of $H^{\text{odd}}(\text{Gr}_{res})$, it is enough to show that the CS-form is closed, i.e. that $H_0$ and $H_1$ have the same Chern form. We argue as follows: The projection $U_{res} \to \text{Gr}_{res}$ sends a matrix $X$ to the projection $X \pi_+, X^*$. Conjugating $X$ by $B$ yields

$$BX B^* \pi_+ B X^* B^* = B X \pi_+ X^* B^*.$$

and therefore using the invariance of $\text{ch}$, the conjugation map $C_B : U_{res} \to U_{res}$ pulls back the universal Chern form to itself, i.e. $C_B^* \text{ch} = \text{ch} = \text{id}_{U_{res}}^* \text{ch}$. The third case follows by the same argument, using the invariance of $\text{ch}$ one more time.

The independence of the blocksum up to CS-equivalence is now easily deduced, since

$$\rho^* \rho(f \boxplus \rho, g) \rho^* \rho' = \rho'^* \rho \rho^* (f \boxplus g) \rho \rho^* \rho' = f \boxplus \rho' g$$

and therefore the two blocksums defined by $\rho$ and $\rho'$ just differ by a conjugation with the unitary matrix $\rho' \rho^*$ on $\mathcal{H}$, which in the polarized case respects the grading.\hfill $\Box$

\textsuperscript{3}On $U_{res}$, CS-equivalence is defined with respect to the universal Chern character that one gets from pulling back $\text{ch}_{even}$ via the projection $U_{res} \to \text{Gr}_{res}$.  

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Remark 4.5. By the preceding Lemma, it is now safe to suppress $\rho$ in our notation. For two elements in the restricted unitary group, which are by definition 2 by 2 block operators, we write

$$f \boxplus g = \begin{pmatrix} f_{++} & 0 & f_{+-} & 0 \\ 0 & g_{++} & 0 & g_{+-} \\ f_{-+} & 0 & f_{--} & 0 \\ 0 & g_{-+} & 0 & g_{--} \end{pmatrix}. \quad (5)$$

Proposition 4.6. Let $f,g,h: M \to U$ be smooth maps for $U \in \{ Gr_{\text{res}}, U^1 \}$. Then, the operation induced by blocksum is commutative and associative up to CS-homotopy, i.e. we have

$$f \boxplus g \sim_{\text{CS}} g \boxplus f \quad \text{and} \quad f \boxplus (g \boxplus h) \sim_{\text{CS}} (f \boxplus g) \boxplus h.$$  

Proof. This is just a consequence of Lemma 4.4, since the difference in each case is just a permutation of the basis. For commutativity in the case of $U^1$, one sees that for $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in U(\mathcal{H} \oplus \mathcal{H})$, one has

$$g \boxplus f = \rho^* U \rho (f \boxplus g) \rho^* U \rho.$$  

For the even case, acting by the same matrix $\rho^* U \rho$ on $\mathcal{H}_+$ and $\mathcal{H}_-$ separately does the job. For associativity, one has that

$$(f \boxplus g) \boxplus h = \rho^* (\rho^* \times \text{id}) \begin{pmatrix} f \\ g \\ h \end{pmatrix} (\rho \times \text{id}) \rho$$

$$= \rho^* (\rho^* \times \text{id})(\text{id} \times \rho)(f \boxplus (g \boxplus h)) \rho^* (\text{id} \times \rho^*)(\rho \times \text{id}) \rho$$

in the $U^1$ case and acting by the same matrix on $\mathcal{H}_+$ and $\mathcal{H}_-$ separately does the job in the $Gr_{\text{res}}$ case.

We will now discuss the involution on $U_{\text{res}}$ that will implement inversion in differential $K$-theory. Let $U$ be the unitary transformation that flips the role of $\mathcal{H}_+$ and $\mathcal{H}_-$, given by the matrix $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. In a $\mathbb{Z}$-basis $\{ e_i \}$ adapted to the polarization, the map $U$ sends $e_i$ to $e_{-i-1}$.

Definition 4.7. We define the polarization flip map $\text{flip}: U_{\text{res}} \to U_{\text{res}}$ to be conjugation with $U$. On the space of smooth maps from a manifold to $U_{\text{res}}$, this induces the operation

$$f \mapsto \text{flip}(f) = \text{flip} \circ f = U f U.$$  

Explicitly, we have

$$\text{flip}(f)(x) = \begin{pmatrix} f_{-+}(x) & f_{+-}(x) \\ f_{++}(x) & f_{-+}(x) \end{pmatrix}, \quad \text{when} \quad f(x) = \begin{pmatrix} f_{++}(x) & f_{-+}(x) \\ f_{++}(x) & f_{-+}(x) \end{pmatrix}.$$

Note that there is an induced flip map on the restricted Grassmannian, which corresponds to taking the orthogonal complement of a subspace and then changing the polarization. We have

$$W = X(\mathcal{H}_+) \mapsto \text{flip}(X)(\mathcal{H}_+) = UXU(\mathcal{H}_+) = UX(\mathcal{H}_-) = U(W^\perp) = \text{flip}(W),$$

which also extends to maps $M \to Gr_{\text{res}}$ via composition. One furthermore sees that pullback by flip preserves left invariance of forms: If $L_Y^* \eta = \eta$, we have that

$$L_Y^*(\text{flip}^* \eta) = (\text{flip} \circ L_Y)^*(L_Y^*(\text{flip}^{-1} \eta)) = \text{flip}^* \eta,$$

since flip is a group homomorphism on $U_{\text{res}}$. The following proposition shows compatibility of the inversion and addition operations on the classifying spaces with the Chern and Chern–Simons forms.
Proposition 4.8. Consider smooth maps $f, g : M \to \mathcal{U}$ and $\mathcal{U} \in \{\text{Gr}_{\text{res}}, U^1\}$. Then:

(i) The maps Ch and CS are monoid morphism, i.e. $\text{Ch}(f \oplus g) = \text{Ch}(f) + \text{Ch}(g)$ and $\text{CS}(H_t \boxplus G_t) = \text{CS}(H_t) + \text{CS}(G_t)$.

(ii) $\text{CS}(H_t \ast G_t) = \text{CS}(H_t) + \text{CS}(G_t)$ (composition of homotopies).

(iii) $\text{Ch}_{\text{even}}(\text{flip}(f)) = -\text{Ch}_{\text{even}}(f)$, and $\text{Ch}_{\text{odd}}(f^*) = -\text{Ch}_{\text{odd}}(f)$.

(iv) $\text{CS}_{\text{odd}}(\text{flip}(H_t)) = -\text{CS}_{\text{odd}}(H_t)$, and $\text{CS}_{\text{even}}(H_t^*) = -\text{CS}_{\text{even}}(H_t)$.

Proof. The monoid morphism property follows directly from the additivity of the trace under block sum and linearity of the integral, and the additivity under composition follows from additivity of the integral under partition of the interval.

We check the third identity directly on $\text{Gr}_{\text{res}}$ and $U^1$. For $\text{Gr}_{\text{res}}$, we need to compute the pullback of the curvature $\text{flip}^*\Omega$, and it suffices to do this in the tangent space at $\mathcal{H}_+$ by left invariance. Take $X = \begin{pmatrix} 0 & X_{+} \ 1 - & X_{-} \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & Y_{+} \ 1 - & Y_{-} \end{pmatrix} \in T\mathcal{H}_+\text{Gr}_{\text{res}}$. We have

$$(\text{flip}^*\Omega)_{\mathcal{H}_+}(X, Y) = \Omega \left( \begin{pmatrix} 0 & X_{+} \ 1 - & X_{-} \end{pmatrix}, \begin{pmatrix} 0 & Y_{+} \ 1 - & Y_{-} \end{pmatrix} \right) = Y_{+}X_{+-} - X_{+}Y_{+-}$$

Therefore, we see that $(\text{flip}^*(\text{tr}\Omega^k))_{\mathcal{H}_+}(X^1, \ldots, X^{2k})$ equals

$$\frac{1}{2^k} \sum_{\sigma \in S_{2k}} \text{sgn}(\sigma) \text{tr}(X^{\sigma(1)} X^{\sigma(2)} \cdots X^{\sigma(2k-1)} X^{\sigma(2)})$$

This is just a big sum of products of $2k$-operators with many redundant terms. One sees that it is equal to

$$\sum_{\sigma \in S_{2k}} \text{sgn}(\sigma) \text{tr} X^{\sigma(2)} X^{\sigma(1)} X^{\sigma(4)} X^{\sigma(3)} \cdots X^{\sigma(2k)} X^{\sigma(2k-1)}$$

The first equality is cyclic invariance of the trace, the second one comes applying a cyclic permutation with $2k$-elements. Finally, the last equality comes from going through the same calculation without applying the flip map. This proves (iii) for the odd case.

For even case, notice that from $0 = df^* f = df^* + df^*$ it follows that $\text{tr}(df^* f)^{2k-1} = -\text{tr}(df^* f)^{2k-1}$. Therefore pulling back $\text{ch}_{\text{odd}}$ via the adjoint-operator map $U^1 \to U^1$ gives the desired minus sign. Part (iv) easily follows from part (iii) by definition of the Chern–Simons form.

5 Geometric structure maps in the $K$-theory spectrum

The goal of this section is to recall explicit homotopy equivalences $\Omega\text{Gr}_{\text{res}} \to U^1$ and $\Omega U^1 \to \text{Gr}_{\text{res}}$ which are compatible with our Chern and Chern–Simons forms. We will consider the even case first.

In Section 3 we constructed a smooth model for the universal $\text{GL}_1^1$ principal bundle as $\text{St}_{\text{res}} \to \text{Gr}_{\text{res}}^0$, together with an explicit connection form $\Theta$. Consider the loop space $\Omega\text{Gr}_{\text{res}}$, based at $\mathcal{H}_+ \in \text{Gr}_{\text{res}}^0$. Parallel transport via the connection $\Theta$ gives rise to the holonomy
map, which assigns to such a loop the fiber coordinate of the endpoint of the horizontal lift of this loop, starting at \(w_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in St_{res}\). After applying the homotopy equivalence

\[
\begin{align*}
GL^1 & \to U^1 \\
T & \mapsto T|T|^{-1},
\end{align*}
\]

we have constructed a map \(h_{even} : \Omega Gr_{res} \to U^1\). Since it implements holonomy in the fibration \(U \to EU \to BU\), it is clear that this is a homotopy equivalence. It remains to check compatibility with the Chern and Chern–Simons forms.

It is well known that the transgression of \([ch_{even}] \in H^{even}(BU; \mathbb{R})\) in the universal fibration \(U \to EU \to BU\) is the class \([ch_{odd}] \in H^{odd}(U; \mathbb{R})\). We can actually recover this fact from the universal bundle which we constructed in Section 3 by a direct calculation.

**Lemma 5.1.** The transgression map \(T : H^k(Gr_{res}; \mathbb{R}) \to H^{k-1}(St_{res}; \mathbb{R})\) in the universal \(GL^1\)-fibration \(St_{res} \to Gr_{res}^0\) maps the even Chern character to the odd one, i.e. \(T([ch_{2k}]) = [ch_{2k-1}]\).

**Proof.** We use the connection \(\Theta\) as constructed in Prop. 3.8. Then the Chern character is given by an invariant polynomial, evaluated at the curvature \(\Omega\). For this situation, Chern and Simons [CS74, Sec. 3] gave a formula for the transgression. Define \(\phi_t = \frac{t}{2\pi} \varphi_{t-1}^{k-1}\).

Then one easily checks that the form

\[
\eta = \left(\frac{i}{2\pi}\right)^k \frac{1}{k!} \int_0^1 ktr(\Theta \wedge \varphi_{t-1}^k) dt
\]

satisfies the two identities \(d\eta = \pi^* ch_{2k}\) and \(i^* \eta = ch_{2k-1}\). Therefore \(ch_{2k-1}\) represents a transgression of \(ch_{2k}\).

From this result we can deduce the needed compatibility between the Chern and Chern–Simons forms.

**Proposition 5.2.** Let \(H : M \times I \to Gr_{res}^0\) be a smooth homotopy, starting and ending with \(H_0 = H_1 = \text{const}_{Gr}\) with adjoint map \(\hat{H} : M \to \Omega Gr_{res}^0\). Let \(h_{even} : \Omega Gr_{res}^0 \to GL^1 \to U^1\) be the holonomy map composed with the homotopy equivalence \(GL^1 \to U^1\) given by \(X \mapsto X|X|^{-1}\). Then, on the level of differential forms, we have the congruence modulo exact forms

\[
\int_I H^*_t ch_{even} = (h_{even} \circ \hat{H})^* ch_{odd} + \text{exact}.
\]

**Proof.** We have a diagram of fibrations

\[
\begin{array}{cccc}
GL^1 & \to & St_{res} & \to & Gr_{res}^0 \\
\text{hol} & & \text{hol} & & \\
\Omega Gr_{res}^0 & \to & PGr_{res}^0 & \to & Gr_{res}^0,
\end{array}
\]

where the vertical maps are induced by holonomy. Now since transgression commutes with maps of fibrations, we see that \(\int_I ev_t^*[ch_{even}] = \text{hol}_{Gr}^*[ch_{odd}] = h_{even}^*[ch_{odd}]\) as cohomology classes on \(\Omega Gr_{res}^0\).

We now pull back the equation via \(\hat{H}\) and get

\[
(h_{even} \circ \hat{H})^*[ch_{odd}] = \hat{H}^* \int_I ev_t^*[ch_{even}] = \int_I (\hat{H} \times \text{id}_I)^* ev_t^*[ch_{even}] = \int_I H^*_t [ch_{even}]
\]

Since the domain now is a finite-dimensional manifold, we can smoothly approximate the maps up to homotopy and see that the claimed equality is true on the level of differential forms, up to exact forms. \(\square\)
Remark 5.3. This argument (and also the one in the odd case given below) avoids the discussion of \( h_{\text{even}} : \Omega \text{Gr}^0_{\text{res}} \to U^1 \) as a smooth map between infinite-dimensional manifolds. It would be interesting to compute directly the derivative of \( h_{\text{even}} \) and try to pull back \( ch_{\text{odd}} \) as a differential form.

The construction of the map for the odd case has appeared in [CM00 Appendix 2], based on the ideas of [PSS88 Sec. 6.3]. We choose a concrete model for the generic polarized Hilbert space that has been used before. Let \( \mathcal{H}^\infty = L^2(S^1, \mathcal{H}) = L^2(S^1) \otimes \mathcal{H} \) be the space of \( L^2 \)-functions on the circle to an infinite-dimensional separable complex Hilbert space \( \mathcal{H} \) with fixed basis \( \{e_i\}_{i \geq 0} \). There is a natural \( \mathbb{Z}/2 \)-grading given by the positive resp. negative exponent part of the Fourier decomposition, i.e.

\[
\begin{align*}
\mathcal{H}^\infty_+ &= \left\{ f \in \mathcal{H}^\infty \mid f = \sum_{k \geq 0} f_k z^k, f_k \in \mathcal{H} \right\}, \\
\mathcal{H}^\infty_- &= \left\{ f \in \mathcal{H}^\infty \mid f = \sum_{k < 0} f_k z^k, f_k \in \mathcal{H} \right\},
\end{align*}
\]

where \( z = \exp(i\theta) \). We consider the multiplication operator map

\[
h_{\text{odd}} : \Omega U^1(\mathcal{H}) \to U(\mathcal{H}^\infty)
\]

\[
\gamma \mapsto M_\gamma,
\]

where \( (M, f)(\theta) = \gamma(\theta) f(\theta) \). We have the following

Lemma 5.4. The map \( h_{\text{odd}} \) has image in the restricted unitary group. Furthermore, as a map to \( U_{\text{res}}, h_{\text{odd}} \) is a homotopy equivalence.

Proof. We will rely on the corresponding statements for the finite-dimensional version of said map which were proved by Pressley and Segal in [PSS88 Ch. 8].

First, we have that the analogously defined map \( h^0_{\text{odd}} : \Omega U(n) \to U(\mathcal{H}^{(n)}) \) has image contained in \( U_{\text{res}}(\mathcal{H}^{(n)}) \subset U_{\text{res}}(\mathcal{H}^\infty) \), where \( \mathcal{H}^{(n)} \) is the finite-dimensional version of our space \( \mathcal{H}^\infty \), i.e. \( \mathcal{H}^{(n)} = L^2(S^1, \mathbb{C}^n) \), and the inclusion \( \mathbb{C}^n \hookrightarrow \mathcal{H} \) is via the first \( n \) basis vectors \( \{e_i\}_{i \in \mathbb{N}} \). This is a consequence of the decay condition on the Fourier coefficients of the loop \( \gamma \), using the boundedness of its first derivative. Furthermore, we have that this map is \((2n-2)\)-connected.

In order to conclude the corresponding statements for our stabilized version of the map, we note that the restriction of \( h_{\text{odd}} \) to \( \Omega U(n) \) has image contained in \( U_{\text{res}} \) by the previous paragraph. The union of these loop spaces is the loop space \( \Omega U \) of the stable unitary group, and this still maps to \( U_{\text{res}} \). But \( \Omega U \) is a dense subspace of \( \Omega U^1 \) and therefore the first claim follows, since \( U_{\text{res}} \) is a complete Riemannian manifold.

Since the connectivity of these maps increases, we can use a similar argument for \( h_{\text{odd}} \) being a homotopy equivalence. It is enough to show that it induces an isomorphism on all homotopy groups. We have the commutative diagram

\[
\begin{array}{ccc}
\Omega U(n) & \xrightarrow{h^0_{\text{odd}}} & U_{\text{res}}(\mathcal{H}^{(n)}) \\
\downarrow i & & \downarrow j \\
\Omega U^1(\mathcal{H}) & \xrightarrow{h_{\text{odd}}} & U_{\text{res}}(\mathcal{H}^\infty),
\end{array}
\]

where both horizontal maps come from the inclusion of \( \mathbb{C}^n \) into \( \mathcal{H} \) and then filling up with the identity matrix. The maps \( h^0_{\text{odd}} \) and \( i \) are \( 2n-2 \)-connected, while the map \( j \) is a homotopy equivalence and therefore \( h_{\text{odd}} \) is also \( 2n-2 \)-connected, for any \( n \).

The map \( h_{\text{odd}} \) realizes the inverse of the Bott periodicity map as a homomorphism of infinite-dimensional Lie groups. We can append the projection \( U_{\text{res}} \to \text{Gr}_{\text{res}} \) in order to get our desired periodicity map for our Grassmannian model, and we will also denote this map by \( h_{\text{odd}} \).
Proposition 5.5. Let \( H : M \times I \to U^1 \) be a smooth homotopy, starting and ending with \( H_0 = H_1 = \text{const}_{id} \) with adjoint map \( \hat{H} : M \to \Omega U^1 \). Let \( h_{\text{odd}} : \Omega U^1 \to U_{\text{res}} \to \text{Gr}_{\text{res}} \) be the assignment of the corresponding multiplication operator, composed with the homotopy equivalence given by the projection. Then on the level of differential forms, we have the congruence modulo exact forms

\[
\int_I H^*_t \text{ch}_{\text{odd}} = (h_{\text{odd}} \circ \hat{H})^* \text{ch}_{\text{even}} + \text{exact}.
\]

Proof. In the proof of Prop. 5.2 we have seen that the even Chern character transgresses to the odd Chern character as cohomology classes in the path loop fibration over \( \text{Gr}_{\text{res}}^0 \). Now since the Chern character is compatible with Bott periodicity, if we apply transgression again in the path loop fibration over \( U^1 \), we must get \( T^2([\text{ch}_{2k}]) = [\text{ch}_{2k-2}] \) after identifying \( \Omega^2 \text{Gr}_{\text{res}}^0 \) with \( \text{Gr}_{\text{res}} \) explicitly via \( h_{\text{odd}} \circ (\Omega h_{\text{even}}) \). This allows us to make the calculation

\[
(h_{\text{odd}} \circ \Omega h_{\text{even}})^* [\text{ch}_{\text{even}}] = T(T([\text{ch}_{\text{even}}])) = T(h_{\text{even}}^* [\text{ch}_{\text{odd}}]) = (\Omega h_{\text{even}})^* T([\text{ch}_{\text{odd}}]),
\]

where the first equality is Bott periodicity, the second one is from (the proof of) Prop. 5.2, and the last one follows from the naturality of transgression in the diagram of fibrations

\[
\begin{array}{ccc}
\Omega U^1 & \xrightarrow{PH^1} & U^1 \\
\Omega h_{\text{even}} & & h_{\text{even}} \\
\Omega^2 \text{Gr}_{\text{res}}^0 & \xrightarrow{P \Omega \text{Gr}_{\text{res}}^0} & \Omega \text{Gr}_{\text{res}}^0.
\end{array}
\]

Since \( \Omega h_{\text{even}} \) is a homotopy equivalence, we get that \( T[\text{ch}_{\text{odd}}] = h_{\text{odd}}^* [\text{ch}_{\text{even}}] \). Pulling back by \( H \) and using the same argument as in the even case now gives the claim.  

Proposition 5.6. The sequence of pointed spaces and pointed maps \((E_n, h_n), n \in \mathbb{Z}\) given by

\[
E_{2n} = \text{Gr}_{\text{res}} \quad \text{and} \quad E_{2n+1} = U^1 \\
h_{2n} = h_{\text{even}} \quad \text{and} \quad h_{2n+1} = h_{\text{odd}}
\]
defines an \( \Omega \)–spectrum that represents complex topological \( K \)-theory. Furthermore, addition in \( K \)-theory is implemented by the blocksum operation on both \( \text{Gr}_{\text{res}} \) and \( U^1 \).

Proof. Since the structure maps are homotopy equivalences, the first part of the theorem follows. For the second part, we have to prove that blocksum is homotopic to composition of loops, i.e. that the squares

\[
\begin{array}{ccc}
\Omega U^1 \times \Omega U^1 & \xrightarrow{\ast} & \Omega U^1 \\
\downarrow h_{\text{odd}} \times h_{\text{odd}} & & \downarrow h_{\text{odd}} \\
\text{Gr}_{\text{res}} \times \text{Gr}_{\text{res}} & \xrightarrow{\oplus} & \text{Gr}_{\text{res}}
\end{array}
\]

\[
\begin{array}{ccc}
\Omega \text{Gr}_{\text{res}} \times \Omega \text{Gr}_{\text{res}} & \xrightarrow{\ast} & \Omega \text{Gr}_{\text{res}} \\
\downarrow h_{\text{even}} \times h_{\text{even}} & & \downarrow h_{\text{even}} \\
U^1 \times U^1 & \xrightarrow{\oplus} & U^1.
\end{array}
\]

commute up to homotopy, where the star denotes loop composition.

For the left one, recall that \( h_{\text{odd}} \) assigns to a loop the corresponding multiplication operator in \( U_{\text{res}} \) and then projects to \( \text{Gr}_{\text{res}} \). Since the projection \( U_{\text{res}} \to \text{Gr}_{\text{res}} \) is a map of \( H \)-spaces for the blocksum, we can work on \( U_{\text{res}} \). It is clear that \( h_{\text{odd}} \) respects the alternative \( H \)-space structures on target and domain: the pointwise multiplication of two loops maps to the product of their operators. Now by the usual Eckmann–Hilton argument, pointwise multiplication of loops is homotopic to loop concatenation. On the other hand, if \( C_t \) denotes the grading preserving rotation

\[
C_t = \begin{pmatrix}
\cos(t) & \sin(t) & 0 & 0 \\
-\sin(t) & \cos(t) & 0 & 0 \\
0 & 0 & \cos(t) & \sin(t) \\
0 & 0 & -\sin(t) & \cos(t)
\end{pmatrix},
\]

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we have the homotopy
\[ U_{\text{res}} \times U_{\text{res}} \times I \to U_{\text{res}} \]

\[(A, B, t) \mapsto (A \oplus 1)G_t(1 \oplus B)G_t^*, \]

which shows that \( A \oplus B \sim AB \oplus 1 \). We will finish the proof by showing that the map
\[ f: A \to A \oplus 1 \] is homotopic to the identity. It is enough to show this over each path component of \( U_{\text{res}} \), so let \( U_{\text{res}}^0 \) denote the identity component. We have homotopy equivalences \( BU \xrightarrow{\sim} U_{\text{res}}^0 \xrightarrow{\psi} \text{Fred}^0 \) and the second map is given explicitly by the projection to the \( ++ \)-component (recall (3)). Notice that this respects taking the blocksums of operators, i.e. \( f \circ \psi = \psi \circ f \).

We have \( BU = \text{colim} BU(n) \). Since each of the inclusions in this colimit induces a surjection in \( K \)-theory, there is no \( \lim^1 \)-term, and we have
\[
\begin{align*}
[U_{\text{res}}^0 \times U_{\text{res}}^0] \xrightarrow{\sim} [BU, \text{Fred}^0] \xrightarrow{\sim} [\text{colim} BU(n), \text{Fred}^0] \xrightarrow{\sim} \text{lim}[BU(n), \text{Fred}^0] \\
\psi \quad \psi \quad \psi \\
[f] \xrightarrow{[\psi \circ f \circ \varphi]} [f \circ \psi \circ \varphi] \xrightarrow{[\varphi]} (f \circ \psi \circ \varphi)_{BU(n)}
\end{align*}
\]

But \( BU(n) \) is compact, and by the Atiyah–Jänich index map, \([BU(n), \text{Fred}^0] \cong \tilde{K}(BU(n)) \). Since blocksum with the identity does not change the kernel or cokernel of an operator, we have that \((f \circ \psi \circ \varphi)_{BU(n)} = (\psi \circ \varphi)_{BU(n)} \), and tracing back through the isomorphisms, we see that \([f] = [\text{id}] \).

The argument for the second square in the above diagram is similar and follows from the fact that the holonomy map \( \psi_{\text{even}} \) takes compositions of loops to products of operators, and that \( U^1 \simeq U \) is filtered by the compact manifolds \( U(n) \) \( \square \)

### 6 The Differential \( K \)-theory groups

Let us recall the axiomatic definition of differential extensions according to [BS10, Def. 1.1], adapted to the special case of \( K \)-theory. Let \( V = K^*(*) \otimes Z \otimes R = \mathbb{R}[u, u^{-1}] \) be the coefficients of complex \( K \)-theory with the Bott element of degree 2, and let \( \Omega^*(M; V) = \mathcal{C}^\infty(M, \Lambda^*T^*M \otimes_B V) \) denote the \( V \)-valued differential forms on \( M \), where the degree is induced by the sum of the degrees as a differential form and as an element of \( V \). There is also a version of cohomology with coefficients in the graded vector space \( V \), which we will denote by \( H^*(M; V) \). In the following, we will consider all \( \mathbb{Z} \)-graded vector spaces that arise as \( \mathbb{Z}_2 \)-graded by restricting to even or odd degrees. Denote by \( \text{ch}: K^*(M) \to H^*(M; V) \) the topological Chern character.

**Definition 6.1.** A differential extension of \( K \)-theory is a contravariant functor from the category of smooth manifolds to \( \mathbb{Z}_2 \)-graded abelian groups, together with natural transformations

1. **R:** \( \hat{K}^*(M) \to \Omega^*_d(M; V) \), called the curvature,
2. **I:** \( \hat{K}^*(M) \to K^*(M) \), called the underlying class,
3. **a:** \( \Omega^*-1(M; V) / \text{im}(d) \to \hat{K}^*(M) \), called the action of forms,

such that

(i) the diagram
\[
\begin{array}{ccc}
\hat{K}^*(M) & \xrightarrow{I} & K^*(M) \\
\downarrow R & & \downarrow \text{ch} \\
\Omega^*_d(M; V) & \xrightarrow{\text{Rham}} & H^*(M; V).
\end{array}
\]

commutes,
(ii) \( R \circ a = d \), so the action map is a lift of the exterior derivative, and
(iii) we have the exact sequence
\[
K^{*-1}(M) \xrightarrow{\text{Ch}} \Omega^{*-1}(M; V)/\text{im}(d) \xrightarrow{a} \hat{K}^*(M) \xrightarrow{f} K^*(M) \xrightarrow{} 0.
\]

We will now give a concrete implementation of such a differential refinement via smooth
classifying spaces. The goal of this section is to define the underlying group-valued functors \( \hat{K}^* \).

**Definition 6.2.** Let \( M \) be a smooth manifold and \( \mathcal{H} \cong \mathcal{H}_+ \oplus \mathcal{H}_- \) be an infinite-dimensional
\( \mathbb{Z}_2 \)-graded separable complex Hilbert space with both \( \mathcal{H}_+ \) and \( \mathcal{H}_- \) infinite-dimensional.
Define the underlying sets of the odd and even differential \( K \)-theory groups via
\[
\hat{K}^0(M) = \text{Map}(M, \text{Gr}_{\text{res}}(\mathcal{H}))/\sim
\]
\[
\hat{K}^1(M) = \text{Map}(M, U^1(\mathcal{H}_+))/\sim.
\]

The equivalence relation is induced by Chern–Simons homotopy equivalence (see Def. 4.2),
together with a stabilization step that identifies any map \( f \) with \( f \oplus \text{const.}, \) where \( * \) is
the basepoint, i.e. the subspace \( \mathcal{H}_+ \) or the identity.

**Lemma 6.3.** The operation \( \boxplus \) induces an abelian group structure on \( \hat{K} \). The neutral elements
are given by the equivalence class of the constant map to the basepoint, and inversion is given
by \( f \mapsto f^* \) and \( f \mapsto \text{flip}(f) \) in the odd/even case respectively.

**Proof.** We need to check well-definedness. If \( f_0 \sim_{\text{CS}} f_1 \) and \( g_0 \sim_{\text{CS}} g_1 \), then we need to
show that \( f_0 \boxplus g_0 \sim_{\text{CS}} f_1 \boxplus g_1 \). This is achieved by the homotopy \( f_1 \boxplus g_1 \), which is again a
CS-homotopy by Prop. 4.6. Furthermore, the matrices
\[
(f \boxplus 1) \boxplus (g \boxplus 1) \quad \text{and} \quad (f \boxplus g) \boxplus 1 = (f \boxplus g) \boxplus (1 \boxplus 1)
\]
are CS-equivalent by Prop. 4.6 and so stabilization is also fine. Commutativity and
associativity are also proven in Prop. 4.6. That blocksumming with const, is the identity is
built into the definition of our equivalence relation. It remains to show that inversion is given
by the proposed operations.

Start with the even case and consider the rotation matrix in the 2-3-plane, given by
\[
C_t = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos(t) & -\sin(t) & 0 \\
0 & \sin(t) & \cos(t) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

We define the universal homotopy \( H_t : U_{\text{res}} \times [0, \pi/2] \to U_{\text{res}} \), which at \( t = 0 \) is
\[
H_t(x) = \rho^* C_t^*(X \oplus \text{flip}(X))C_t\rho
= \begin{pmatrix}
X_{++} & \sin(t)X_{+-} & \cos(t)X_{+-} & 0 \\
\sin(t)X_{+-} & X_{+-} & 0 & \cos(t)X_{+-} \\
\cos(t)X_{+-} & 0 & X_{+-} & -\sin(t)X_{+-} \\
0 & \cos(t)X_{+-} & -\sin(t)X_{+-} & X_{++}
\end{pmatrix}.
\]

It is a unitary matrix whose \(+−\) and \(−±\) components are \( L^2 \)-operators, and so it lies in
\( U_{\text{res}} \). The map \( H_t \) has values always in the subgroup \( U_+ \times U_- \subset U_{\text{res}} \), while \( H_0 \) is just
\( X \mapsto X \boxplus \text{flip}(X) \). This homotopy furthermore induces a well-defined homotopy on the
quotient \( \text{Gr}_{\text{res}} \): If we consider another representative \( XV = X \begin{pmatrix} v_+ & 0 \\ 0 & v_- \end{pmatrix} \) for unitary matrices
\( v_\pm \), it is easy to see that the operator \( V \oplus \text{flip}(V) \) commutes with \( C_t \). Therefore, we have
\[
\rho^* C_t^*(XV \oplus \text{flip}(XV))C_t\rho = \rho^* C_t^*(X \oplus \text{flip}(X))(V \oplus \text{flip}(V))C_t\rho
= \rho^* C_t^*(X \oplus \text{flip}(X)) C_t\rho \rho^* (V \oplus \text{flip}(V)) \rho
\]
and since \( \rho^*(V \oplus \text{flip}(V))_p \in U_+ \times U_- \), the homotopy is well-defined as a map \( \text{Gr}_{\text{res}} \times [0, \pi/2] \to \text{Gr}_{\text{res}} \). As it goes from \( X \oplus \text{flip}(X) \) to the constant map on the basepoint, we are reduced to showing that \( H \) is a CS-homotopy. The Chern–Simons form is certainly closed, since

\[
d \int_I H_i^* \chi = H_i^* \chi - H_0^* \chi = (X \oplus \text{flip} X)^* \chi - \text{const}^* \pi_+ \chi = 0
\]

by Prop. 4.8. Since \( H^{\text{odd}}(\text{Gr}_{\text{res}}) = 0 \), it follows that it must be exact, and we are done.

For the odd case, we use the homotopy from Lem. [TWZ13, Lemma 3.7] in the universal case:

\[
H_i : U^1 \times I \to U^1
\]

\[
(A, t) \mapsto (A \oplus 1)C_t(1 \oplus A^*)C_t^*,
\]

where \( C_t = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \) is again a rotation matrix. This is a homotopy from \( H_0(A) = A \oplus A^* \) to \( H_2^*(A) = 1 \). The calculation

\[
d \int_I H_i^* \chi = H_i^* \chi - H_0^* \chi = \text{const}^* \chi - (A \oplus A^*)^* \chi = 0
\]

shows that it has a closed CS-form and the vanishing of \( H^{\text{even},>0}(U^1) = 0 \) shows that it has to be exact in positive degree. In the degree 0 case, we again make an explicit calculation:

\[
\text{CS}_0(H_i) = \int_I H_i^* \chi_1 = \left( \frac{i}{2\pi} \right) \int_I \iota_{\partial_1}(H_i^*(\text{tr}(\omega_{\text{MC}})))
\]

The integrand resolves to

\[
\iota_{\partial_1}(H_i^*(\text{tr}(\omega_{\text{MC}}))) = \text{tr}(C_t(1 \oplus A)^*C_t^*(A \oplus 1)^*)(1 \oplus 1)C_t(1 \oplus A)X_t^* - (A \oplus 1)C_t(1 \oplus A)C_t^*(\dot{C}_t C_t^*)
\]

\[
= \text{tr} C_t^* \dot{C}_t - \text{tr} \dot{C}_t C_t^* = 0,
\]

thus proving the exactness of the Chern–Simons form.

\[\square\]

7 Natural transformations and exact sequences

It remains to define the curvature map \( R \), the integration map \( I \) and the action map \( a \) from the definition of a differential extension and see that they have the required properties. Two of those are easy: The underlying class \( I([f]) \) of a CS-class \( [f] \in K^*(M) \) is given by dropping to the same equivalence class under the weaker equivalence relation of homotopy. Note that the stability equivalence relation is also compatible with homotopy classes by Prop. 5.6. The curvature map \( R \) is just the map \( \text{Ch} \) from Def. 4.1, which pulls back the universal Chern form, i.e.

\[
R([f]) = f^* \chi,
\]

where we interpret \( \chi \) as a differential form with values in \( V = \mathbb{R}[u, u^{-1}] \), i.e.

\[
\text{ch}_{\text{even}} = \sum_{k \geq 0} \text{ch}_{2k} u^{-k} \in \Omega^0(\text{Gr}_{\text{res}}; V), \quad \text{ch}_{\text{odd}} = \sum_{k \geq 0} \text{ch}_{2k+1} u^{-k} \in \Omega^1(U^1; V).
\]

This is well defined on the CS-equivalence class, since Stokes’ theorem implies that

\[
f_i^* \chi - f_0^* \chi = d \int_I f_i^* \chi = d\text{CS}(f_i) = 0,
\]

if \( f_i \) is a CS-homotopy. Stabilization is also fine, since \( \text{Ch}(f \oplus \text{const.}) = \text{Ch}(f) \). It is easy to see that these maps are homomorphisms. For \( R \), this follows from additivity of Chern forms.
(Prop. 4.8), while for \( I \), we use that addition in \( K \)-theory can also be implemented by the blocksum (Prop. 5.6). It remains to define the action map

\[
a : \Omega^{*-1}(M; V)/\text{im}(d) \to \hat{K}^*(M).
\]

Our construction is adapted from [TWZ16, Def. 3.26] [TWZ13, Prop. 5.3]. The map has to fit into the exact sequence

\[
K^{*-1}(M) \xrightarrow{\text{Ch}} \Omega^{*-1}(M; V)/\text{im}(d) \xrightarrow{a} \hat{K}^*(M) \xrightarrow{f} K^*(M) \xrightarrow{} 0.
\]

Finding such a map is algebraically equivalent to constructing an isomorphism

\[
\hat{CS} : \ker(I) \to \Omega^{*-1}(M; V)/\text{im}(d)/\text{im}(\text{Ch}).
\]

We define \( \hat{CS} \) as follows: For a map \( f_1 \) to lie in the kernel of \( I \), it means that there is a homotopy \( f_t \) such that \( f_0 = \text{const.} \). We define \( \hat{CS}(f_1) = CS(f_t) \) for a choice of such a nullhomotopy.

**Lemma 7.1.** The map \( \hat{CS} \) is a well-defined isomorphism of groups, and the resulting map

\[
a : \Omega^{*-1}(M; V)/\text{im}(d) \longrightarrow \Omega^{*-1}(M; V)/\text{im}(\text{Ch}) \xrightarrow{\hat{CS}^{-1}} \ker(I) \longrightarrow \hat{K}^*(M)
\]

meets the axiomatic requirements for the action map in differential \( K \)-theory.

**Proof.** Let \( g_t \) be another nullhomotopy of \( f_t \). We need to show that the resulting CS-forms only differ by a Chern form up to exact forms. Since both homotopies start and end at the same point, we can construct a loop \( f_t \circ g_{1-t} \) at the base point and calculate

\[
CS(f_t) - CS(g_t) = CS(f_t) + CS(g_{1-t}) = CS(f_t \circ g_{1-t}).
\]

Recall that we have the homotopy equivalences given by explicit periodicity maps

\[
\Omega\text{Gr}_{\text{res}} \to U^1 \quad \text{and} \quad \Omega U^1 \to \text{Gr}_{\text{res}},
\]

defined in Sec. 3. We will denote them both by the letter \( h \). It was shown in Prop. 5.2 and 5.5 that we have

\[
CS(f_t \circ g_{1-t}) = \text{Ch}(h \circ (f_t \circ g_{1-t})) + \text{exact}.
\]

Therefore, the map \( \hat{CS} \) is well-defined. Furthermore, if we have \( f_0 \oplus g_0 \) with nullhomotopies \( f_t \) and \( g_t \), then

\[
CS(f_t \oplus g_t) = CS(f_t) + CS(g_t)
\]

by Prop. 4.8 and so \( \hat{CS} \) is a homomorphism.

For injectivity, suppose that \( \hat{CS}(f_0) = 0 \), so \( CS(f_t) = \text{Ch}(g) + \text{exact} \) for some \( g : M \to U \) for \( U \in \{ \text{Gr}_{\text{res}}, U^1 \} \). Using any homotopy inverse of the periodicity map \( h \), we can construct a based loop \( H := h^{-1} \circ g : M \to \Omega U \). Since the domain is just a finite-dimensional smooth manifold, we can up to homotopy assume that this map is smooth. As an immediate corollary of Prop. 5.2 and 5.5, we have that (modulo exact forms)

\[
CS(H_t) = \text{Ch}(h \circ H) = \text{Ch}(h \circ h^{-1} \circ g) = \text{Ch}(g).
\]

The composition \( H_{1-t} \circ f_t \) is now a nullhomotopy of \( f_0 \) with exact Chern–Simons form, since

\[
CS(H_{1-t} \circ f_t) = -CS(H_t) + CS(f_t) = -\text{Ch}(g) + \text{Ch}(g) + \text{exact} = \text{exact}.
\]

We still have to prove surjectivity. The key result here is a surjectivity statement for the Chern character on the level of differential forms. It is proved in [PT14, Prop. 2.1 and Rem.
that any exact even form is the Chern form of a trivial hermitean bundle with compatible
connection. On the other hand, in [TWZ13, Cor. 2.7] it is shown that every exact odd
form is the Chern form of a nullhomotopic map $M \to U^4$. Since our odd Chern character is
compatible with the [TWZ16]-Chern character (cf. Sec. 8) and the even Chern character is
compatible with classifying maps of connections, we can conclude that the maps
$$\text{Ch}: \text{Map}^0(M, \text{Gr}_{\text{res}}) \to \Omega^0(M; V)$$
$$\text{Ch}: \text{Map}^0(M, U^1) \to \Omega^1(M; V)$$
given by pullback of universal forms hit all exact forms. The decoration here means that we
only consider maps which are smoothly homotopic to the constant map to the basepoint.
The surjectivity of CS can be deduced from this in the following way. Let $\omega$ be an even or
odd form on $M$. Then, we can construct a form $\tilde{\omega} \in \Omega(M \times I)$ such that under the inclusions
at the endpoints, one has $i_0^* \tilde{\omega} = 0$ and $i_1^* \tilde{\omega} = \omega$. The previous result allows us to write
$$d \tilde{\omega} = \text{Ch}(g_t)$$
for some map $g_t$. Via Stokes’ theorem, this yields
$$\text{CS}(g_t) = \int_I \text{Ch}(g_t) = \int_I d \tilde{\omega} = \omega + \text{exact},$$
and we are done.

The only thing left to show is that we have $R \circ a = d$, but this follows from the calculation
$$d \int_I f_t^* \text{ch} = f_1^* \text{ch} - f_0^* \text{ch} = f_1^* \text{ch} - \text{const}_d \text{ch} = f_1^* \text{ch} = R(\int_I f_t^* \text{ch})).$$

With the definitions of $R$, $a$ and $I$ as above and the abelian group structure given by blocksum, we have proved our main theorem.

**Theorem A.** On the category of possibly non-compact smooth manifolds, the abelian group valued functors
$$\hat{K}^0(M) = \text{Map}(M, \text{Gr}_{\text{res}})/\text{CS-homotopy + Stabilization}$$
$$\hat{K}^1(M) = \text{Map}(M, U^1)/\text{CS-homotopy + Stabilization}$$
define differential K-theory.

**Remark 7.2.** Differential K-theory is functorial for CS-equivalence classes of maps: if two maps $f_0, f_1: M \to N$ are homotopic and the homotopy $f_t$ satisfies the additional condition that for any $g: Y \to \text{Gr}_{\text{res}}$ or $g: Y \to U^1$, the Chern–Simons form $\text{CS}(f_t \circ g)$ is exact, then $f_0$ and $f_1$ induce the same map on $\hat{K}^0$ resp. $\hat{K}^1$. This feature of a descent to a quotient category of smooth manifolds is a general property of differential cohomology theories and is discussed in [TWZ16, Cor. 2.5].

### 8 Comparison to the Tradler-Wilson-Zeinalian model

Though all versions of even differential K-theory that meet the Bunke-Schick axioms are
uniquely isomorphic, there are infinitely many inequivalent versions of odd differential K-
theory. There is however, only one unique isomorphism type that is compatible with the
additional structure of an $S^1$-integration map. One such model for compact manifolds is the
one proposed in [TWZ16, Thm. 4.25]. We will establish an explicit isomorphism of both the
even and odd part of the restriction of our model to compact manifolds to the [TWZ16]-model
in this section.

---

4The proof given there applies without change to the case of a noncompact manifold $M$.  

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The $\text{[TWZ16]}$-model is also based on smooth classifying spaces. For the odd part, they use the stable unitary group $U$ and define

$$\hat{K}^1_+(M) = \text{Map}(M, U)/\text{CS-equivalence}.$$ 

Since $U$ does not admit a Banach manifold structure, the authors work with universal cocycles given by the finite-dimensional differential forms $[1]$ on the filtration defined by the inclusions of $U(n)$ for $n \in \mathbb{N}$. It is immediately clear that the forms $\text{ch}_{\text{odd}} \in \Omega^{\text{odd}}(U^1)$ that we use pull back to give the same forms under the natural inclusions

$$U(n) \hookrightarrow U^1.$$ 

The second map also preserves the blocksum on the nose. Since CS-homotopies go to CS-homotopies, it induces a well-defined homomorphism $i_*: \hat{K}^1_+(M) \to \hat{K}^1_+(M)$.

**Proposition 8.1.** The homomorphism $i_*: \hat{K}^1_+ \to \hat{K}^1_+$ preserves all the structure of a differential extension, i.e. $I \circ i_* = I_T$, $i_* \circ a_T = a$ and $R \circ i_* = R_T$. Furthermore, $i_*$ is an isomorphism.

**Proof.** The compatibilities are easy to check and follow from $i$ being a homotopy equivalence and pulling back $\text{ch}$ to $\text{ch}_T$. The isomorphism property of $i_*$ follows from an application of the five lemma to the diagram

$$
\begin{array}{ccccccc}
K^0(M) & \xrightarrow{\text{Ch}} & \Omega^{\text{even}}(M; \mathbb{R})/\text{im}(d) & \xrightarrow{a} & \hat{K}^1_+(M) & \xrightarrow{I} & K^1(M) & \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
K^0(M) & \xrightarrow{\text{Ch}} & \Omega^{\text{even}}(M; \mathbb{R})/\text{im}(d) & \xrightarrow{a} & \hat{K}^1_+(M) & \xrightarrow{I} & K^1(M) & \to 0.
\end{array}
$$

\[\Box\]

The even part of the $\text{[TWZ16]}$-theory is given by maps into the space of finite rank projections on $\mathbb{C}^\infty = \bigoplus_{\mathbb{Z}} \mathbb{C} \subset \mathcal{H}$, defined as

$$\text{Proj} = \left\{ \pi \in \text{End}(\mathbb{C}^\infty) \mid \pi^* = \pi, \text{Spec}(\pi) \subset \{0, 1\}, \text{rank}(\pi - \pi_{\mathbb{C}^\infty}) < \infty \right\} \approx \left\{ V \subset \mathbb{C}^\infty \mid \mathbb{C}^p \subset V \subset \mathbb{C}^q \text{ for some } p, q \in \mathbb{Z} \right\}.$$ 

Their basepoint is the space $\mathbb{C}^\infty_{-\infty}$. Apart from a change of basis, we can identify $\text{Proj}$ with the colimit of the finite-dimensional Grassmannians, which we denoted by $\text{Gr}_{\text{res,\infty}}$ in Section 3 as follows: Denote by $A: \mathcal{H} \to \mathcal{H}$ the change of basis which maps $e_i$ to $e_{-i}$ for all $i$. Then, we have a natural map

$$i: \text{Proj} \to \text{Gr}_{\text{res,\infty}} \xrightarrow{\sim} \text{Gr}_{\text{res}} \quad \pi \mapsto A\pi A.$$ 

This is well-defined, since $A\pi A - \pi_+ = A(\pi - \pi_{\mathbb{C}^\infty})A$ has image contained in some $\mathcal{H}_N \subset \text{im}(A\pi A) \subset \mathcal{H}_{-N}$. We check that it is a homomorphism for the blocksum. We have that $\pi_1 \oplus \pi_2 = \rho^* (\pi_1 \oplus \pi_2) \rho$ gets mapped to $A(\pi_1 \oplus \pi_2) A$. On the other hand, the blocksum of the images is $\rho^*(A \oplus A)(\pi_1 \oplus \pi_2)(A \oplus A)$. Comparing $(A \oplus A)\rho$ and $\rho A$ as operators from $\mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$ (see Definition 4.3), we see that they both map basis vectors $e_{2i}$ to $(e_{-i}, 0)$. On odd basis vectors, we have

$$\begin{align*}
(A \oplus A)\rho (e_{2i+1}) &= (0, e_i), \\
\rho A (e_{2i+1}) &= \rho (e_{-2(i+1)+1}) = (0, e_{i+1}).
\end{align*}$$

Therefore, if we have $f, g: M \to \text{Proj}$, then $i \circ (f \oplus g)$ and $(i \circ f) \oplus (i \circ g)$ differ only by conjugation with a fixed unitary matrix $B \in U_+ \times U_-$. This shifts odd basis vectors by one. By Lem. 4.7 these are therefore CS-equivalent. We conclude that $i$ induces a homomorphism of differential $K$-theory groups.
**Proposition 8.2.** The homomorphism \( i_* : \tilde{K}_T^0 \to \tilde{K}_T^0 \) preserves all the structure of a differential extension, i.e. \( I \circ i_* = I_T \), \( i_* \circ a_T = a \) and \( R \circ i_* = R_T \). Furthermore, \( i_* \) is an isomorphism.

*Proof.* As in the even case, we have that \( i \) is a homotopy equivalence. We need to check that \( i^* \text{ch} = \text{ch}_T \). The path components of Proj are given by the rank map, where by definition \( \text{rank}(V) = \dim(V/\mathbb{C}_{-N}) - N \), if we and assume that \( \mathbb{C}_{-N} \subset V \subset \mathbb{C}_{-N} \). This agrees with the path component of the image, which is indexed by \( \text{virt} \dim(A(V)) \). In order to check that also the positive degree parts of \( \text{ch} \) are compatible, we note that for the \( \text{C} \)-model, the Chern character is calculated in terms of traces of powers of the differential forms \( \pi d \pi^* \). Pulling back along the composition

\[
\text{Gr}_{k,2N} \to \text{Proj} \to \text{Gr}_{\text{res},\infty}
\]

on the other hand gives the forms \( \pi d \pi^* A \) whose powers of traces agree with those. Since any map from a compact manifold factors through one of these Grassmannians, we are done. \( \Box \)

We can use the isomorphism \( i_* \) in order to slightly improve our model for the case of a compact manifold: The stabilization condition that \( f \boxplus \text{const}_* \) is equivalent to just \( f \) is actually not needed. We can now prove our second theorem.

**Theorem B.** On the category of compact smooth manifolds, the abelian group valued functors

\[
\tilde{K}^0(M) = \text{Map}(M, \text{Gr}_{\text{res}})/\text{CS-homotopy}
\]

\[
\tilde{K}^1(M) = \text{Map}(M, U^1)/\text{CS-homotopy}
\]

define differential \( K \)-theory.

*Proof.* The only thing left to show is that \( f \boxplus \text{const}_* \sim \text{CS} \ f \) for any map \( f \). Using the unique isomorphism \( (i_*)^{-1} : \tilde{K}^* \to \tilde{K}_T^* \) to the \( \text{TWZ16} \)-model, we see that

\[
[f \boxplus \text{const}_*] = i_* ((i_*)^{-1} [f] \boxplus_T [\text{const}_*]_T) = i_* (i_*^{-1} [f]) = [f].
\]

In the second equality we used that in the \( \text{TWZ16} \)-model, we are always on a finite stage in the filtration of \( U \) or Proj. In that case, the shuffle blocksum operation as we defined it is CS-equivalent to the naive blocksum of finite-dimensional matrices, for which the constant map to the identity is easily seen to be a unit (compare \( \text{TWZ16} \) Lem. 3.9 and Lem 3.24). \( \Box \)

### 9 Examples

Already the point is an interesting example, since it illustrates the role that is played by Chern-Simons homotopies.

**Proposition 9.1.** We have isomorphisms \( \tilde{K}^0(*) \cong K^0(*) \cong \mathbb{Z} \) and \( \tilde{K}^1(*) = \mathbb{R}/\mathbb{Z} \), given by the underlying class map \( I \) and the determinant map.

*Proof.* Since there are no odd forms on the point, every homotopy is a CS-homotopy, and the first part follows. In the odd case, we need to check that the homotopies \( f_t \) we use are CS-homotopies, i.e. that the CS-form \( \text{CS}(f_t) = \int f_t^* f_t^*(\text{ch}_1) = \frac{1}{2\pi} \int f_t^* \text{tr} f_t^{-1} f_t \) is exact. We have a splitting induced by the (Fredholm)-determinant map

\[
SU^1 \times U(1) \cong U^1.
\] (6)

Under the isomorphism, the semi-direct group structure is given by

\[
(n_1, h_1) \cdot (n_2, h_2) = (n_1 h_1 n_2 h_1^{-1}, h_1 h_2), \quad (n, h)^{-1} = (h^{-1} n^{-1} h, h^{-1}),
\]
which yields that for \( f_t = (nt, h_t) \), we have
\[
f_t^{-1} \hat{f}_t = h_t^{-1} n_t^{-1} n_t + h_t^{-1} h_t,
\]
where the first term is in \( su(n) \) and the second one in \( u(1) \). Since \( su(n) \) consists of matrices with trace zero, every homotopy that leaves the second factor in \([0]\) alone will be fine. Therefore, all the information is in the second factor, and since the isomorphism above is induced by the determinant map, we are done.

Next, we will study the circle \( S^1 \). One can deduce from the exact sequence
\[
0 \to \Omega^{-1}(S^1; \mathbb{R})/\text{im}(d)/\text{im}(\text{Ch}) \xrightarrow{f} \hat{K}^*(S^1) \xrightarrow{I} K^*(S^1) \to 0
\]
coming from the axioms of \( \hat{K} \) that there are exact sequences
\[
0 \to H^1(S^1; \mathbb{R})/\mathbb{Z} \to \hat{K}^0(S^1) \to K^0(S^1) \to 0 \tag{7}
\]
\[
0 \to C^\infty(S^1) \to \hat{K}^1(S^1) \to K^1(S^1) \to 0.
\]
We will give a description of the kernel of \( I \) in both cases. The following Lemma has already been computed in a different geometric model in \[BS09\] Lem. 5.3.

**Lemma 9.2.** Let \((E_\pm, \nabla_\pm)\) be a pair of vector bundles with connection over \( S^1 \) with \( \dim(E_+) = \dim(E_-) \). Then, the corresponding element in \( \hat{K}^0(S^1) \) is
\[
[(E_+, \nabla_+)] - [(E_-, \nabla_-)] = a \left( \frac{1}{2\pi i} \log \frac{\det \text{hol}(E_+, \nabla_+)}{\det \text{hol}(E_-, \nabla_-)} dz \right),
\]
where \( dz \) is the volume form on \( S^1 \) and \( \text{hol}(E_\pm, \nabla_\pm) \in U(n)/\text{conjugation} \) is the holonomy of the bundle.

**Proof.** Let \( f^\pm \) be the classifying maps of the bundles with connections with corresponding nullhomotopies \( f^\pm_t \) with \( f^\pm_0 \) some constant map. We get a corresponding homotopy of the classifying map of the virtual bundle (cf. Rem. 3.11)
\[
f_t = f_t^+ \text{flip} f_t^- : S^1 \to \text{Gr}^0_{\text{res}}.
\]
By the construction of the map \( a \) (Lem. 7.1), the differential form corresponding to \( f_1 = I \) is the Chern-Simons form of a nullhomotopy to \( \text{const} \). It is easy to see that \( f_0 \), which is a constant map to some subspace in \( \text{Gr}^0_{\text{res}} \), can be connected by a CS-homotopy to \( \text{const} \). We are therefore reduced to computing the CS-form of \( f_t \). Notice that \( f_t \) has image contained in some finite-dimensional \( \text{Gr}_{k,2} \subset \text{Gr}^0_{\text{res}} \), which means that we really have a path of actual finite-dimensional bundles with connections. We have
\[
\omega_f = \int_I f_t^* \text{ch}_2 = \int_I \text{ch}_2(f_t^+) - \text{ch}_2(f_t^-) = \frac{i}{2\pi} \int I \Omega_{\text{det}(f_t^+)} - \Omega_{\text{det}(f_t^-)}.
\]
In the last step, we used that \( \text{ch}_2 \) of a bundle is the same as the first Chern class of its determinant line bundle, i.e. the integral over \( \frac{1}{2\pi} \) times its curvature. In order to see what is the cohomology class of \( \omega_f \in \Omega^1(S^1) \), we can just integrate over \( S^1 \). This yields
\[
\int_{S^1} \omega_f = \frac{i}{2\pi} \int_{S^1} \int_I \Omega_{\text{det}(f_t^+)} - \Omega_{\text{det}(f_t^-)} = \frac{1}{2\pi i} \left( - \int_{D^2} \Omega_{\text{det}(f^+)} + \int_{D^2} \Omega_{\text{det}(f^-)} \right)
\]
\[
= \frac{1}{2\pi i} \left( \log \text{hol}(\det(f^+)) - \log \text{hol}(\det(f^-)) \right),
\]
which shows that the function \( f \) corresponds to the logarithm of the determinant of the holonomy of its induced bundles, as claimed. 

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Proposition 9.3. A cohomology class on $S^1$ represented by a one-form $\alpha$ gets mapped in the exact sequence to $a([\alpha] + \mathbb{Z}[dz]) = [f_\alpha]$, where $f_\alpha$ is the classifying map of the trivial line bundle with local connection form $i\alpha \in \Omega^1(S^1; u(1))$. In the odd case, a function $\varphi \in \mathbb{C}_c^\infty(S^1)$ gives rise to an element in $\check{K}^1(S^1)$ via the exponential map, i.e. $a(\varphi + \mathbb{Z}) = [exp(\frac{1}{2\pi i} \varphi)]$.

Proof. Let $i\alpha \in \Omega^1(S^1; u(1))$ be a local connection form for the trivial $U(1)$-bundle $E$. Furthermore, let $z$ be a horizontal lift of the fundamental loop on $S^1$, where $z = \exp(2\pi it)$. If we write $\alpha_z = \alpha(z)dz$, it is determined by the equation $i\alpha(z)dz = H(z)^{-1}H'(z)dz$.

which we can integrate over the interval and exponentiate in order to get $\exp \left( i \int_{S^1} \alpha(z)dz \right) = \exp \left( \int_I H(z)^{-1}H'(z)dz \right) = H(1)$.

In the last step, we used that for any path $h: I \to U(1)$ starting at $h(0) = 1$, we have $h(s) = \exp \left( \int_0^s h^{-1}_t dt \right)$, which can be seen by noting that $k(s) = h^{-1}_s \exp(\int_0^s h^{-1}_t dt)$ satisfies $k(0) = 1$ and $\dot{k}(s) = 0$ for all $s$. Note that $H(1)$ is precisely the holonomy of the connection $i\alpha$. By Lemma 9.2 we have $[E, i\alpha] = a \left( \frac{1}{2\pi i} \log H(1) \right) dz$

$= a \left( \frac{1}{2\pi} \int_{S^1} \alpha(z)dz \right) dz$,

and the form in the brackets is cohomologous to $\alpha$, which proves the claim.

For the odd part, let $f: S^1 \to U^1$ be a representative of a class in $\check{K}^1(S^1)$ that is in the kernel of $I$. Choose a nullhomotopy $f_t$. The corresponding 0-form $\omega_f$ is the Chern-Simons form of $f_t$, and we calculate for $z \in S^1$:

$$\omega_f(z) = \int_I (f_t^* ch_1)(z) = \frac{i}{2\pi} \int_I \text{tr} f_t^* (z) f_t(z)$$

By the splitting in [6], we can assume that $f_t(z)$ takes values in $U(1)$. Then, for fixed $z$, the integral on the right is over a path that starts at $1 \in S^1$, and we use the same argument as in the even case to conclude

$$\exp \left( \frac{2\pi}{i} \omega_f(z) \right) = \exp \left( \int_I f_t^* (z) f_t(z) \right) = f(z). \quad (8)$$

Therefore, by defining $f: S^1 \to U^1$ to be the left hand side of (8), followed by the inclusion $U(1) \hookrightarrow U^1$, we have successfully recovered the function $f$ from the given $\omega_f$. $\square$

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Eric Schlarmann  
Institut für Mathematik, Universität Augsburg  
EMAIL: eric.schlarmann@math.uni-augsburg.de