A microstate for the 3-charge black ring

Stefano Giusto\textsuperscript{1}, Samir D. Mathur\textsuperscript{2} and Yogesh K. Srivastava\textsuperscript{3}

Department of Physics,
The Ohio State University,
Columbus, OH 43210, USA

Abstract

We start with a 2-charge D1-D5 BPS geometry that has the shape of a ring; this geometry is regular everywhere. In the dual CFT there exists a perturbation that creates one unit of excitation for left movers, and thus adds one unit of momentum $P$. This implies that there exists a corresponding normalizable perturbation on the near-ring D1-D5 geometry. We find this perturbation, and observe that it is smooth everywhere. We thus find an example of ‘hair’ for the black ring carrying three charges – D1, D5 and one unit of $P$. The near-ring geometry of the D1-D5 supertube can be dualized to a D6 brane carrying fluxes corresponding to the ‘true’ charges, while the quantum of $P$ dualizes to a D0 brane. We observe that the fluxes on the D6 brane are at the threshold between bound and unbound states of D0-D6, and our wavefunction helps us learn something about binding at this threshold.

\textsuperscript{1}giusto@mps.ohio-state.edu
\textsuperscript{2}mathur@mps.ohio-state.edu
\textsuperscript{3}yogesh@mps.ohio-state.edu
1 Introduction

The classical geometry of a black hole has ‘empty space’ near the horizon. Pair creation in this region leads to the information paradox. String theory suggests that the black hole interior is not the naive classical one; rather the information of the state of the hole is distributed throughout the interior of the horizon (for a review of some basic ideas see [1]). Such a picture holds for all 2-charge extremal states, and has been shown to also extend for a subset of 3-charge states.

In 4+1 dimensions we can have not only black holes but also black rings [2]. We would therefore like to construct microstates for the ring. The goal of this paper is to construct a simple 3-charge extremal state for the ring, where we start with a ring carrying two charges D1,D5 and add a wave carrying one unit of momentum P, the third charge.

There has been a lot of recent progress on black rings. The entropy for the ring can be obtained by computing it for a short straight segment of a ring and multiplying by the total length of the ring [3]. A subset of 3-charge rings can be obtained as supertubes made out of branes [4].

We are interested in the gravity description of microstates. In [5, 6] dual geometries were found for a discrete subset of CFT states. But even though these states have a large angular momentum, they do not look like ‘rings’, since we cannot find a sphere $S^2$ that will surround a ‘ring’ shaped interior. In [7, 8] a method was developed to find large families of 3-charge BPS states, in terms of the choice of locations of poles of certain harmonic functions appearing in the metric [9]. While the CFT states for these geometries are not known, it was argued that these geometries represented bound states because there is a nonzero flux on spheres $S^2$ linking the poles. Assuming that this argument is correct we can make geometries that are like rings, and that have no horizon and no singularity.

In the present paper we would like to construct a gravity description of microstates in a case where we also know the dual CFT state. Thus in gravity terms our construction will be more modest than the ones obtainable from [7, 8]; the third charge will be only a small perturbation on our 2-charge ring. On the other hand since we will know the microscopic origin of the state, we are assured that we have a bound state and we can also develop some intuition for how CFT operations act in the gravity picture.

In spirit our computation is similar to the computation in [3], where one unit of momentum was added to a D1-D5 extremal state. We will again take the same D1-D5 state, and add a unit of momentum using the same fields, but will be working in a very different limit from the one used in [5]. In [5] we had chosen our moduli so that the D1-D5 geometry had a large AdS type region, which went over to flat space at infinity. This geometry is depicted in Fig.1(a). The wavefunction of the quantum carrying the momentum is peaked in the AdS region, falling off at infinity in a normalizable way. By contrast in the present paper we will take a limit of the moduli so that the D1-D5 state looks like a thin ring, depicted in Fig.1(b). Consider a short segment of this ring, which looks like a straight tube (Fig.1(c)). The bound state wavefunction must now appear as a wavefunction localized in the vicinity of this tube, falling to zero away from this tube, and regular everywhere inside. We find this wavefunction, thus obtaining a simple but explicit example of ‘hair’ for the black ring. The fact that the wavefunction is regular everywhere suggests that no horizon or singularity should form even for a non-infinitesimal deformation, so the result supports a ‘fuzzball’ picture for the black ring.
2 The CFT state

It is important for us that the state we construct in the gravity description be known to be a BPS state in the dual CFT. In this section we review the discussion of [5] where this state was described.

2.1 The 2-charge geometry

We start with the 2-charge D1-D5 system. We compactify IIB string theory as $M_{9,1} \rightarrow M_{4,1} \times T^4 \times S^1$. We wrap $n_5$ D5 branes on $T^4 \times S^1$, and $n_1$ D1 branes on $S^1$. The system has a large class of BPS bound states, out of which we choose a simple one that was first noted in [10, 11]. If we reduce the metric on $T^4$ we find that the 6-d string metric is the same as the Einstein metric, so we will just call it the ‘metric’ below. The masses of the D1 and D5 branes are described by parameters $\bar{Q}_1, \bar{Q}_5$ which we will set to be equal

$$\bar{Q}_1 = \bar{Q}_5 = \bar{Q}$$

This will simplify our computations, but we expect that the state we construct will exist for all $\bar{Q}_1, \bar{Q}_5$. With the choice (2.1) the dilaton is constant, and the volume of the $T^4$ is also constant. The metric and gauge field for the solution are given by

$$ds^2 = -H^{-1}(dt^2 - dy^2) + Hf\left(\frac{d\bar{r}^2}{\bar{r}^2 + a^2} + d\bar{\theta}^2\right) - 2\frac{a\bar{Q}}{Hf}(\cos^2 \bar{\theta}dyd\bar{\psi} + \sin^2 \bar{\theta}dtd\bar{\phi})$$

$$+ H\left(\bar{r}^2 + \frac{a^2\bar{Q}^2 \cos^2 \bar{\theta}}{H^2f^2}\right) \cos^2 \bar{\theta}d\bar{\psi}^2 + H\left(\bar{r}^2 + a^2 - \frac{a^2\bar{Q}^2 \sin^2 \bar{\theta}}{H^2f^2}\right) \sin^2 \bar{\theta}d\bar{\phi}^2$$

(2.2)

$$C^{(2)} = -\frac{\bar{Q}}{Hf} dt \wedge dy - \frac{\bar{Q} \cos^2 \bar{\theta}}{Hf}(r^2 + a^2 + \bar{Q})d\bar{\psi} \wedge d\bar{\phi}$$

$$- \frac{Qa \cos^2 \bar{\theta}}{Hf} dt \wedge d\bar{\psi} - \frac{Qa \sin^2 \bar{\theta}}{Hf} dy \wedge d\bar{\phi}$$

(2.3)
where

\[ f = r^2 + a^2 \cos^2 \tilde{\theta}, \quad H = 1 + \frac{Q}{f} \] (2.4)

Here

\[ a = \frac{Q}{R_y} \] (2.5)

where \( R_y \) is the radius of the \( S^1 \).

### 2.2 The ‘inner’ region

Suppose we take

\[ \epsilon = \frac{a}{Q^{1/2}} \frac{Q^{1/2}}{R_y} << 1 \] (2.6)

(This can be achieved by taking \( R_y \) large holding all other moduli and the charges fixed.) We can then look at the ‘inner region’ of this geometry

\[ \bar{r} << \sqrt{Q} \] (2.7)

The metric here is

\[
\begin{align*}
    ds^2 &= -\left( \frac{r^2 + a^2 \cos^2 \tilde{\theta}}{Q} \right) (dt^2 - dy^2) + \frac{Q}{r^2 + a^2} \left( d\tilde{\theta}^2 + \cos^2 \tilde{\theta} d\bar{\psi}^2 + \sin^2 \tilde{\theta} d\bar{\phi}^2 \right) \\
    &- 2a (\cos^2 \tilde{\theta} dy d\bar{\psi} + \sin^2 \tilde{\theta} dt d\bar{\phi}) + \bar{Q} (\cos^2 \tilde{\theta} d\bar{\psi}^2 + \sin^2 \tilde{\theta} d\bar{\phi}^2)
\end{align*}
\] (2.8)

The change of coordinates

\[ \psi_{NS} = \bar{\psi} - \frac{a}{Q} y, \quad \phi_{NS} = \bar{\phi} - \frac{a}{Q} t \] (2.9)

brings (2.8) to the form \( AdS_3 \times S^3 \)

\[
\begin{align*}
    ds^2 &= -\left( \frac{\bar{r}^2 + a^2}{Q} \right) dt^2 + \bar{r}^2 dy^2 + Q \left( \frac{dr^2}{\bar{r}^2 + a^2} \right) + \bar{Q} (d\tilde{\theta}^2 + \cos^2 \tilde{\theta} d\psi_{NS}^2 + \sin^2 \tilde{\theta} d\phi_{NS}^2)
\end{align*}
\] (2.10)

This AdS geometry is dual to a 1+1 dimensional CFT. For this CFT we can construct chiral primaries, which are described in the gravity picture by certain BPS configurations. The CFT dual to the geometry (2.10) is in the Neveu-Schwarz (NS) sector. In the original form (2.8) the geometry described the CFT in the Ramond (R) sector, and the coordinate change (2.9) gives the gravity description of the ‘spectral flow’ between the NS and R sectors [10, 11].

The simplest chiral primaries can be obtained by finding normalizable solutions of the supergravity equations describing linear perturbations around \( AdS_3 \times S^3 \). The supergravity fields in the 6-d theory separate into different sets (with no coupling at the linear level between sets). One set described an antisymmetric 2-form \( B_{MN}^{(2)} \) and a scalar \( w \). We write

\[
\begin{align*}
    H_{MNP}^{(3)} &= \partial_M C_{NP}^{(2)} + \partial_N C_{PM}^{(2)} + \partial_P C_{MN}^{(2)}, \quad F_{MNP}^{(3)} = \partial_M B_{NP}^{(2)} + \partial_N B_{PM}^{(2)} + \partial_P B_{MN}^{(2)}
\end{align*}
\] (2.11)
Then the field equations for the perturbation \( B^{(2)}_{MN}, w \) are
\[
F^{(3)} + \ast_6 F^{(3)} + w H^{(3)} = 0 \\
\Delta_6 w - \frac{1}{3} H^{(3) MNP} F^{(3)}_{MNP} = 0
\]
(2.12)

Here the star operation \( \ast_6 \), the laplacian \( \Delta_6 \) and index contractions in (2.12) are defined with respect to the metric (2.2).

### 2.3 Constructing the chiral primary

We can solve the equations (2.12) in the ‘inner region’ geometry (2.10) and obtain normalizable solutions. The solution giving a chiral primary is [12, 5] (the coordinates on \( S^3 \) are \( a, b, \ldots \) and on the \( AdS_3 \) are \( \mu, \nu, \ldots \))

\[
w_{\text{inner}} = e^{-2i\frac{a}{Q} l \hat{\theta} - \frac{2l + 1}{2} \theta} \hat{Y}^{(l)}_{NS} \\
B^{(2)}_{ab} = B_{\epsilon abc} \partial^c \hat{Y}^{(l)}_{NS}, \quad B^{(2)}_{\mu\nu} = \left[ \frac{1}{\sqrt{Q}} \epsilon_{\mu\nu\lambda} \partial^\lambda B \right] \hat{Y}^{(l)}_{NS}
\]

where

\[
\hat{Y}^{(l)}_{NS} = (Y^{(l,l)}_{(l,l)})_{NS} = \sqrt{\frac{2l + 1}{2} e^{-2il\phi_{NS}} \pi} \sin^{2l} \hat{\theta}, \quad B = \frac{e^{-2i\frac{a}{Q} l \hat{\theta}}}{4l (r^2 + a^2)^l}
\]

(2.13)

(2.14)

In (2.13), the tensors \( \epsilon_{abc}, g^{ab} \) etc are defined using the metric on an \( S^3 \) with unit radius. The spherical harmonics \( Y \) are representations of the rotation group \( SO(4) \approx SU(2) \times SU(2) \) of the sphere \( S^3 \), and \( Y^{(l,m)}_{(l',m')} \) has quantum numbers \( (l, m) \) in the first \( SU(2) \) and \( (l', m') \) in the second \( SU(2) \). These two \( SU(2) \) groups become the R symmetries of the left and right movers respectively in the dual CFT. The perturbation (2.13-2.15) gives a state in the CFT with R charges and dimensions given by

\[
\hat{j}_{NS} = l, \quad \hat{h}_{NS} = l, \quad \bar{\hat{j}}_{NS} = l, \quad \bar{\hat{h}}_{NS} = l
\]

(2.16)

(Unbarred and barred quantities denote left and right movers respectively.) The quantities \( \hat{j}_{NS}, \bar{\hat{j}}_{NS} \) are the values of the azimuthal quantum numbers in the two \( SU(2) \) groups. The subscript \( NS \) denotes that we are in the Neveu-Schwarz sector of the CFT. If we spectral flow this perturbation to the Ramond sector then we will get a perturbation with

\[
h = \bar{h} = 0
\]

(2.17)

which is expected, since a chiral primary of the NS sector maps under spectral flow to a ground state of the R sector.\(^4\)

Let the CFT state dual to the perturbation (2.13-2.15) be called \( |\psi\rangle_{NS} \), and let \( |\psi\rangle_R \) be its image under spectral flow to the Ramond sector.

\(^4\)The full spectral flow relations are \( h = h_{NS} - j_{NS} + \frac{c}{12}, \quad j = j_{NS} - \frac{c}{12} \). Spectral flow of the background \( |0\rangle_{NS} \) gives \( h^b = h^b_{NS} - \frac{c}{12}, \quad j^0 = j^0_{NS} - \frac{c}{12} \), so for the perturbation the spectral flow relations are just \( h = h_{NS} - j_{NS}, \quad j = j_{NS} \).
2.4 The state $J_0^- |\psi\rangle_{NS} \leftrightarrow J_{-1}^- |\psi\rangle_R$

Consider again the inner region in the NS sector coordinates (2.10). We now wish to make the perturbation dual to the NS sector state

$$J_0^- |\psi\rangle_{NS}$$

(2.18)

Since the operator $J_0^-$ in the NS sector is represented by just a simple rotation of the $S^3$, we can immediately write down the bulk wavefunction dual to the above CFT state

$$w_{inner} = e^{-\frac{2i a}{Q}lt}$$

(2.19)

$$B_{ab}^{(2)} = B_{abc} \partial^c Y_{NS}^{l(f)} , \quad B_{\mu\nu}^{(2)} = \left[ \frac{1}{\sqrt{Q}} \epsilon_{\mu\nu\lambda} \partial^\lambda \right] Y_{NS}^{l(f)}$$

(2.20)

$$Y_{NS}^{l(f)} = (Y_{(l,l)}^{(l,l)})_{NS} = -\frac{\sqrt{l(2l+1)}}{\pi} \sin^{2l-1} \bar{\theta} \cos \theta e^{i(-2l+1)\phi_{NS}+i\psi_{NS}} , \quad B = \frac{1}{4l} e^{-\frac{2i a}{Q}lt}$$

(2.21)

This perturbation has

$$j_{NS} = l - 1 , \quad \bar{j}_{NS} = l , \quad h_{NS} = l , \quad \bar{h}_{NS} = l$$

(2.22)

The spectral flow to the R sector coordinates should give

$$h = h_{NS} - j_{NS} = 1 , \quad \bar{h} = \bar{h}_{NS} - \bar{j}_{NS} = 0$$

(2.23)

This spectral flowed state can be written as

$$|\psi\rangle = J_{-1}^- |\psi\rangle_R$$

(2.24)

This is a state with nonzero $L_0 - \bar{L}_0$, which means that it is a state carrying momentum $P$ along $S^1$. It is a state in the R sector, which is the sector which we obtain for the CFT if we wrap D1,D5 branes around the compact directions in our original spacetime.

So far we have found the relevant fields only in the ‘inner’ region (2.8). But in [5] the perturbation equations were also solved in the ‘outer’ region $a \ll r < \infty$ and it was shown that the inner and outer region solutions agreed with each other to several orders in the small parameter $\epsilon$ given in (2.6). This agreement was very nontrivial, and indicated that there was an exact solution to the perturbation problem that was smooth in the inner region and normalizable at spatial infinity. This exact solution would be a state carrying three charges: D1, D5, and one unit of momentum $P$. Since it is regular everywhere, we learn that it is possible for 3-charge microstate to be nonsingular and horizon free, just like 2-charge microstates.

Even though the solution obtained in [5] was only found by matching inner and outer region solutions to some order in $\epsilon$, it was possible to guess, from the results, a closed form for the scalar $w$ which would conceivably hold for all orders in $\epsilon$:

$$w_{full} = e^{-\frac{i a}{Q} (t+y)} e^{-i(2l-1)\phi} e^{i\psi} \frac{\sin^{2l-1} \bar{\theta} \cos \bar{\theta}}{(\bar{r}^2 + a^2)^l (Q + f)}$$

(2.25)

We will see that this conjecture for $w$ will help us in obtaining the perturbation for the 3-charge ring.
3 The near ring limit

In the above section we took the limit (2.6) which sets \( R_y >> \sqrt{\bar{Q}} \); this gives the geometry of Fig.1(a) which has a large AdS type region. Now we will take the opposite limit

\[
R_y \ll \sqrt{\bar{Q}}
\]  

(3.26)

In this limit we get a geometry like that in Fig.1(b); we have flat space everywhere except around a thin ‘ring’. This ring has radius \( a = \bar{Q}/R_y \). Note that we have a large family of bound state D1-D5 geometries; these arise by duality from different vibration profiles of the NS1-P system [13]. In the limit (3.26) all these become thin tubes around the curves generated by the NS1-P profile. The near ring limit of any of these curves looks the same; thus for a local analysis of the perturbation equations we may start with any ring, and we have chosen to start with the ‘round’ ring because it is the simplest.

In the near ring limit the following coordinates are the natural ones: we take a coordinate \( z \) to measure length along the ring, and we introduce spherical polar coordinates \( r, \theta, \phi \) in the 3-dimensional space transverse to the ring. (We leave unchanged the coordinate along the compact directions \( S^1, T^4 \).) The coordinate change from \((\bar{r}, \bar{\theta}, \bar{\psi}, \bar{\phi})\) to \((r, \theta, \phi, z)\) is described in the Appendix. The result is

\[
\begin{align*}
\bar{r}^2 &= \frac{a^2 r (1 - \cos \theta)}{a + r \cos \theta} , \\
\sin^2 \bar{\theta} &= \frac{a - r}{a + r \cos \theta} , \\
\bar{\psi} &= \phi , \\
\bar{\phi} &= \frac{z}{a}
\end{align*}
\]  

(3.27)

The only length scale of the near ring geometry is the parameter characterizing the charge density along the ring

\[
Q = \frac{\bar{Q}}{2a}
\]  

(3.28)

The \( y \) radius is given in terms of \( Q \) by

\[
R_y = 2Q
\]  

(3.29)

The near ring region is described by

\[
r \ll a
\]  

(3.30)

From (3.27) one finds, in this limit

\[
\begin{align*}
\bar{r}^2 &\approx ar(1 - \cos \theta) , \\
\sin^2 \bar{\theta} &\approx 1 - \frac{r}{a} (1 + \cos \theta) , \\
f &\approx 2ar
\end{align*}
\]

\[
\frac{dr^2}{\bar{r}^2 + a^2} + d\bar{\theta}^2 = \frac{1}{2r(a + r \cos \theta)} \left[ \frac{a^2}{a^2 - r^2} dr^2 + r^2 d\theta^2 \right] \approx \frac{dr^2 + r^2 d\theta^2}{2ar}
\]  

(3.31)

In this limit the metric and RR field become

\[
\begin{align*}
ds^2 &= -H^{-1} \left( dt + \frac{Q}{r} dz \right)^2 + H \ dz^2 + ds^2_{TN} \\
ds^2_{TN} &= H^{-1} (dy - Q (1 + \cos \theta) d\phi)^2 + H (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) \\
C^{(2)} &= H^{-1} \frac{Q}{r} dy \wedge (dt - dz) + H^{-1} Q (1 + \cos \theta) d\phi \wedge (dt - dz)
\end{align*}
\]  

(3.32)

where

\[
H = 1 + \frac{Q}{r}
\]  

(3.33)
3.1 The Taub-NUT space

The part of the metric denoted as $ds_{TN}^2$ is Taub-NUT (TN) space: it is smooth due to the relation (3.29). The TN gauge field

$$A = -Q(1 + \cos \theta)d\phi$$

satisfies

$$dA = Q \sin \theta \, d\theta \wedge d\phi, \quad \star_3 dA = -dH$$

where $\star_3$ is the Hodge dual with respect to the flat $\mathbb{R}^3$ spanned by $r, \theta, \phi$. A convenient basis of 1-forms on TN is given by $\hat{\sigma}, dr, d\theta, d\phi$, with

$$\hat{\sigma} = dy - \frac{1 + \cos \theta}{2}d\phi, \quad \hat{y} = \frac{y}{2Q}$$

In terms of these forms the RR field strength can be written as

$$H^{(3)} = \Omega^{(2)} \wedge (dt - dz)$$

with

$$\Omega^{(2)} = -\frac{2Q^2}{H^2 r^2} \left[ d\tau \wedge \hat{\sigma} + \frac{Hr^2}{Q} d\hat{\sigma} \right] = -2Q \, d(H^{-1}\hat{\sigma})$$

We choose the orientations of the 6D and TN spaces so that

$$\epsilon_{ty\bar{r}\bar{\phi}\bar{\psi}} = \epsilon_{tz\bar{y}\bar{r}\bar{\phi}} = 1, \quad \epsilon_{r\theta\phi y} = 1$$

(and thus $\epsilon_{tz} = -1$). Then $H^{(3)}$ is self-dual with respect to the 6-D metric

$$\star_6 H^{(3)} = H^{(3)}$$

and $\Omega^{(2)}$ is self-dual with respect to the 4-dimensional TN metric

$$\star \Omega^{(2)} = \Omega^{(2)}$$

$\Omega^{(2)}$ is the unique closed and self-dual 2-form on TN.

3.2 The scalar $w$ in the near-ring limit

We had noted in section (2.4) that the computations of [5] had suggested an exact form for $w$, given in (2.25).

We would like to take the near ring limit of (2.25). Remember that in the geometry (2.2) the ring is spanned by the coordinate $\bar{\phi}$ and its length is $2\pi a$. From the $\bar{\phi}$ dependence in (2.25) we see that the the wavelength of the perturbation $w_{full}$ in the direction of the ring is

$$\lambda = \frac{2\pi a}{2t - 1} \equiv \frac{2\pi}{k}$$

We will be interested in the regime in which this wavelength is much shorter than the ring:

$$\lambda \ll a$$
This will enable us to take our limit in such a way that we see oscillations of the wavefunction along the z direction even when we take a near-ring limit and see only a short segment of the ring. Eqs. (3.42) and (3.43) imply

\[ l \gg 1 \]  

(3.44)

We can thus approximate \( 2l - 1 \approx 2l \) in the following. By applying the change of coordinates (3.27) and taking the limits (3.30) and (3.44), we find

\[
\cos \bar{\theta} = \sqrt{\frac{r(1 + \cos \theta)}{a + r \cos \theta}} \approx \sqrt{\frac{2}{a}} \sqrt{r} \cos \frac{\theta}{2}
\]  

(3.45)

and

\[
\frac{\sin^{2l-1} \bar{\theta}}{(r^2 + a^2)^l} \approx \left( \frac{\sin^2 \bar{\theta}}{r^2 + a^2} \right)^l = a^{-2l} \left( \frac{a - r}{a + r} \right)^l \approx a^{-2l} \left( 1 - \frac{kr}{l} \right)^l \approx a^{-2l} e^{-kr}
\]  

(3.46)

where we have used (3.42) and the identity \((1 + \epsilon \alpha)^{1/\epsilon} \approx e^\alpha\). Up to an overall normalization, the near ring limit of \( w_{\text{full}} \) is then

\[
w = e^{-i(pt + y)} e^{-i(y - \phi)} \cos \frac{\theta}{2} e^{-kr} \frac{\sqrt{r}}{Q + r}
\]  

(3.47)

where

\[
p = \frac{a}{Q} = \frac{1}{2Q}
\]  

(3.48)

4 The perturbation equations

The perturbation we seek carries one unit of momentum along y and is BPS: this fixes the t and y dependence to be of the form \( e^{-ip(t+y)} \). We also allow for a generic wave number \( k \) along the ring direction z; sometimes we will find it convenient to write this wave number as \( k = \kappa/(2Q) \). The perturbation fields then have the form

\[
B_{MN}^{(2)} = e^{-ip(t+y) - ikz} \tilde{B}_{MN}^{(2)}(r, \theta, \phi), \quad w = e^{-ip(t+y) - ikz} \tilde{w}(r, \theta, \phi)
\]  

(4.49)

4.1 Reducing to equations on TN

In this subsection we reduce the equations (2.12) into a system of equations for a set of p-forms on TN. Indices on TN are denoted by \( i, j, \ldots \). Here and in the following \( d \), \( \Delta \) and \( \star \) are the differential, scalar laplacian and Hodge dual on TN. The 2-form \( B^{(2)} \) reduces to a 2-form on TN denoted by \( B \), two 1-forms \( a \) and \( b \) and a scalar \( \Phi \):

\[
B_{ij}^{(2)} = B_{ij}, \quad B_{it}^{(2)} = a_i, \quad B_{iz}^{(2)} = b_i, \quad B_{tiz}^{(2)} = \Phi
\]  

(4.50)

Let \( f^{(a)} \) and \( f^{(b)} \) be the field strengths of \( a \) and \( b \):

\[
f^{(a)}_{ij} = \partial_i a_j - \partial_j a_i, \quad f^{(b)}_{ij} = \partial_i b_j - \partial_j b_i
\]  

(4.51)
One has the identities
\[ F_{ij}^{(3)} = f^{(a)}_{ij} - ipB_{ij}, \quad F_{ij}^{(3)} = f^{(b)}_{ij} - ikB_{ij}, \quad F_{ij}^{(3)} = \partial_i \Phi - ika_i + ipb_i \] (4.52)

We will need the following relations
\[
\begin{align*}
g_{tt} &= -H + \frac{Q^2}{Hr^2}, \quad g_{tz} = -\frac{Q}{Hr}, \quad g_{zz} = \frac{1}{H} \\
g_{zz} - g_{tz} &= 1, \quad g_{tt} - g_{tz} = -1
\end{align*}
\] (4.53)

By virtue of these relations we can rewrite the 6D laplacian, acting on \( w \), as
\[
\Delta_6 w = \Delta w - \left( p^2 g_{tt} + k^2 g_{zz} + 2pk g_{tz} \right) w = \Delta w - (p + k)(pg_{tt} + kg_{zz})w
\] (4.54)

We also find
\[
H^{(3)}_{ij} = H^{(3)}_{ijz} = -\Omega^{(2)}_{ij}
\] (4.55)

Using (4.52), (4.54) and (4.55), it is easy to see that the equations (2.12) reduce to the following system of equations:
\[
\begin{align*}
\Delta w - (p + k)(pg_{tt} + kg_{zz})w + \Omega^{(2)}_{ij}(f^{(a)}_{ij} + f^{(b)}_{ij} - i(p + k)B_{ij}) &= 0 \\
f^{(a)} - ipB - g_{zz} * (f^{(b)} - ikB) - g_{tz} * (f^{(a)} - ipB) + w\Omega^{(2)} &= 0 \\
f^{(b)} - ikB + g_{tt} * (f^{(a)} - ipB) + g_{tz} * (f^{(b)} - ikB) - w\Omega^{(2)} &= 0 \\
d\Phi - ika + ipb - *dB &= 0
\end{align*}
\] (4.56)

If we take the sum of eq. (4.57) and eq. (4.58), use (4.53), and define
\[
K = f^{(a)} + f^{(b)} - i(p + k)B
\] (4.60)

we find
\[
K = *K
\] (4.61)
i.e. \( K \) is a self-dual 2-form on \( TN \). Applying \( d \) to eq. (4.59) leads to
\[
pf^{(b)} - kf^{(a)} = -id \star dB = \frac{d \star dK}{p + k}
\] (4.62)

Taking \( p \) times eq. (4.58) minus \( k \) times eq. (4.57), and using again (4.53), gives
\[
pf^{(b)} - kf^{(a)} + *(pf^{(b)} - kf^{(a)}) + (pg_{tt} + kg_{zz})K - (p + k)w\Omega^{(2)} = 0
\] (4.63)

and thus, by virtue of (4.62) and (4.61),
\[
\Delta K + (p + k)(pg_{tt} + kg_{zz})K - (p + k)w\Omega^{(2)} = 0
\] (4.64)

where
\[
\Delta K = d \star d \star K + \star d \star dK = d \star dK + \star d \star dK
\] (4.65)
is the TN laplacian acting on the 2-form \( K \). Eq. (4.56) can also be rewritten in form language via the identity
\[
\Omega^{(2)}_{ij}K_{ij} = 2 * (\star \Omega^{(2)} \wedge K) = 2 * (\Omega^{(2)} \wedge K)
\] (4.66)
4.2 The equations to be solved

With all this, we have reduced the system (4.56-4.59) to a coupled system of equations for a self-dual 2-form \(K\) and scalar \(w\):

\[
\Delta w - (p + k)(pg^{tt} + kg^{zz})w + 2 \ast (\Omega^{(2)} \wedge K) = 0 \tag{4.67}
\]

\[
\Delta K + (p + k)(pg^{tt} + kg^{zz}) K - (p + k)^2 w \Omega^{(2)} = 0 \tag{4.68}
\]

Moreover, the definition of \(K\) (4.60) and eq. (4.59) imply the relations

\[
K = f^{(a)} + f^{(b)} - i(p + k)B, \quad \frac{i}{(p + k)} \ast dK = d\Phi - ika + ipb \tag{4.69}
\]

If \(K\) is known, these relations determine \(B, a, b\) and \(\Phi\), up to gauge transformations.

5 Harmonics on Taub-NUT

We would like to solve the above equations by expanding functions on the Taub-NUT space in harmonics on the \((\theta, \phi, \tilde{y})\) space. At the core of the Taub-NUT (i.e. at \(r \approx 0\)) this angular space has the geometry of a ‘round’ \(S^3\), but for larger \(r\) we get a ‘squashed sphere’. Forms on the squashed sphere can be expanded in ‘monopole spherical harmonics’, which have been widely studied; see for example [14]. We will however find it more convenient to develop this expansion in our own notation, in a way that relates it closely to the expansion used for the round sphere in [5].

5.1 Symmetries of Taub-NUT

Let us start with the metric on the round sphere

\[
d s^2_{S^3} = d\tilde{\theta}^2 + \cos^2 \tilde{\theta} d\tilde{\psi}^2 + \sin^2 \tilde{\theta} d\tilde{\phi}^2 \tag{5.70}
\]

The symmetry group is \(SO(4) \approx SU(2) \times SU(2)\). We write the elements of \(SU(2)\) as \(e^{\alpha a J_a}\), with the antihermitian generators \(J_a\) satisfying \([J_a, J_b] = -\epsilon_{abc} J_c\). Writing \(J_\pm = J_1 \pm iJ_2\), we get \([J_3, J_\pm] = \pm iJ_\pm, [J_\pm, J_-] = 2iJ_3\). For the first \(SU(2)\) the generators are

\[
J_+ = \frac{1}{2} e^{-i(\tilde{\psi} + \tilde{\phi})} [\partial_{\tilde{\phi}} - i \cot \tilde{\theta} \partial_{\tilde{\phi}} + i \tan \tilde{\theta} \partial_{\tilde{\psi}}] \\
J_- = \frac{1}{2} e^{i(\tilde{\psi} + \tilde{\phi})} [\partial_{\tilde{\phi}} + i \cot \tilde{\theta} \partial_{\tilde{\phi}} - i \tan \tilde{\theta} \partial_{\tilde{\psi}}] \\
J_3 = \frac{-i}{2} [\partial_{\tilde{\psi}} + \partial_{\tilde{\phi}}] \tag{5.71}
\]

and for the second \(SU(2)\) they are

\[
\tilde{J}_+ = \frac{1}{2} e^{i(\tilde{\psi} - \tilde{\phi})} [\partial_{\tilde{\phi}} - i \cot \tilde{\theta} \partial_{\tilde{\phi}} - i \tan \tilde{\theta} \partial_{\tilde{\psi}}] \\
\tilde{J}_- = \frac{1}{2} e^{-i(\tilde{\psi} - \tilde{\phi})} [\partial_{\tilde{\phi}} + i \cot \tilde{\theta} \partial_{\tilde{\phi}} + i \tan \tilde{\theta} \partial_{\tilde{\psi}}] \\
\tilde{J}_3 = \frac{i}{2} [\partial_{\tilde{\psi}} - \partial_{\tilde{\phi}}] \tag{5.72}
\]
To relate these generators to Taub-NUT we write the metric (5.70) for the round $S^3$ in different coordinates. Thus define
\[ \theta = 2\tilde{\theta}, \quad \hat{y} \equiv \tilde{\phi}, \quad \phi = \tilde{\phi} - \tilde{\psi} \] (5.73)
This gives
\[ ds^2_{S^3} = \frac{1}{4}d\theta^2 + \frac{1}{4}\sin^2\theta d\phi^2 + \left[ d\hat{y} - \frac{1}{2}(1 + \cos\theta) d\phi \right]^2 \] (5.74)
The generators (5.71), (5.72) become
\[ J_+ = \frac{1}{2}e^{-i(2\hat{y} - \phi)}[2\partial_\theta - i(\cot\frac{\theta}{2} + \tan\frac{\theta}{2})\partial_\phi - i\cot\frac{\theta}{2}\partial_y] \]
\[ J_- = \frac{1}{2}e^{i(2\hat{y} - \phi)}[2\partial_\theta + i(\cot\frac{\theta}{2} + \tan\frac{\theta}{2})\partial_\phi + i\cot\frac{\theta}{2}\partial_y] \]
\[ J_3 = -\frac{1}{2}\partial_y \] (5.75)
\[ \bar{J}_+ = \frac{1}{2}e^{-i\phi}[2\partial_\theta - i(\cot\frac{\theta}{2} - \tan\frac{\theta}{2})\partial_\phi - i\cot\frac{\theta}{2}\partial_y] \]
\[ \bar{J}_- = \frac{1}{2}e^{i\phi}[2\partial_\theta + i(\cot\frac{\theta}{2} - \tan\frac{\theta}{2})\partial_\phi + i\cot\frac{\theta}{2}\partial_y] \]
\[ \bar{J}_3 = -\frac{1}{2}[\partial_y + 2\partial_\phi] \] (5.76)
In the Taub-NUT metric if we fix $r$ then we get a 3-dimensional surface with metric of the form
\[ ds^2 = A(d\theta^2 + \sin^2\theta d\phi^2) + 4B \left[ d\hat{y} - \frac{1}{2}(1 + \cos\theta) d\phi \right]^2 \] (5.77)
At the center of Taub-NUT we get $A = B$, and the metric becomes that of a round $S^3$. For larger $r$ we have $A \neq B$, and this gives the squashed sphere. We can now check that the vector fields (5.71), (5.72) are Killing vectors of (5.77), for all $A, B$. But out of the vector fields (5.75) only $J_3$ is a Killing vector if $A \neq B$. Thus the $SU(2) \times SU(2)$ symmetry of the round sphere is broken to $U(1) \times SU(2)$.

### 5.2 Harmonics on the squashed sphere

On the round sphere we can expand any form in spherical harmonics, which are characterized by quantum numbers $(j, m), (j', m')$ in the two $SU(2)$ factors. On the squashed sphere, we can use the same functions, in the following sense. We take the map from the squashed $S^3$ to the round $S^3$ which sends each point $(\theta, \phi, \hat{y})$ on the former to the point with the same coordinates on the latter. The harmonics on the round $S^3$ then give harmonics on the squashed $S^3$ via the pullback under this map. These pulled back harmonics can be used to expand any form on the squashed sphere, though the harmonics are not orthogonal to each other as they were on the round sphere.

The quantum numbers $m$ and $(j', m')$ correspond to symmetries of the squashed sphere, and so are ‘good’ quantum numbers in the sense that we can restrict all terms in an equation to have the same values of these numbers. On the other hand a form of order $p$ on the round
$S^3$ was characterized by four quantum numbers, $(j, m)$, $(j', m')$. In the latter case the quantum numbers uniquely specify the form. A form on the squashed sphere will therefore be a sum

$$\omega_{(m,j',m')} = \sum_j C_j \omega_{(j,m),(j',m')} \tag{5.78}$$

It turns out however that if $\omega$ is a p-form then for its harmonics on the round sphere we must have $|j - j'| \leq p$. This tells us that the sum in (5.78) will be a finite one, and this makes the expansion in harmonics useful for the squashed sphere.

As an application of this approach consider the scalar $w$ in (3.47): its angular dependence is captured by the function

$$\omega_0 = e^{-i(\hat{y} - \phi)} \cos \frac{\theta}{2} = e^{-i\tilde{\psi}} \cos \tilde{\theta} \tag{5.79}$$

All scalars on $S^3$ have quantum numbers $(j, m)$, $(j, m')$; i.e. $j = j'$. From the $\tilde{\psi}$ dependence of (5.79) we find that $m = \frac{1}{2}, m' = -\frac{1}{2}$. The lowest $j$ this can come from is $j = \frac{1}{2}$, so we look at the scalar spherical harmonic on the round $S^3$ given by the quantum numbers $(\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2})$. Such harmonics were given explicitly in [5], and we find that indeed the function (5.79) is proportional to the required scalar harmonic.

Now consider 1-forms. On the round $S^3$, there are two kinds of 1-forms. The first kind are obtained by just applying $d$ to the scalar harmonics, so these have quantum numbers $(j, m)$, $(j, m')$. The second kind have $j - j' = \pm 1$, so they come in two varieties: with quantum numbers $(j+1, m)$, $(j, m')$, and $(j, m)$, $(j+1, m')$. Let us examine these 1-forms for our problem.

Since $m, j', m'$ are good quantum numbers for the problem these must be the same for the 1-forms as for the scalar $w$. Thus the first kind of 1-form must be

$$d\omega_0 = -e^{-i(\hat{y} - \phi)} \left[ \frac{1}{2} \sin \frac{\theta}{2} d\theta + i \cos \frac{\theta}{2} (d\hat{y} - d\phi) \right] \tag{5.80}$$

For the second kind of 1-form we find only one set of quantum numbers that are consistent with the given $m, j', m'$: the set $(\frac{3}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{3}{2})$. The corresponding harmonic was constructed in [5]

$$\tilde{\omega}_1 = e^{-i\tilde{\psi}} [\sin \tilde{\theta} d\tilde{\theta} - i \cos \tilde{\theta} (3 \cos^2 \tilde{\theta} - 1)d\tilde{\psi} - 3i \cos \tilde{\theta} \sin^2 \tilde{\theta} d\tilde{\phi}] \tag{5.81}$$

In the coordinates $(\theta, \hat{y}, \phi)$ this is

$$\tilde{\omega}_1 = e^{-i(\hat{y} - \phi)} \left[ \frac{1}{2} \sin \frac{\theta}{2} d\theta + i \cos \frac{\theta}{2} (3 \cos^2 \frac{\theta}{2} - 1)d\hat{y} - 2i \cos \frac{\theta}{2} d\phi \right] \tag{5.82}$$

### 5.3 Decomposing along base and fiber

At this stage we may think of expanding the angular components of our 1-forms using (5.80) and (5.82), and the $dr$ component using the scalar harmonic $\omega_0$. But actually we can do better, by exploiting the spherical symmetry of the background in the $r, \theta, \phi$ space. The Taub-NUT has such a spherical symmetry, though any choice of coordinates prevents this symmetry from being manifest.

Consider the squashed sphere at any $r$. This space can be regarded as a $S^1$ fiber (parameterized by $\hat{y}$) over a $S^2$ base (parametrized by $\theta, \phi$). We can geometrically identify the fiber
direction over any point, and thus also the 2-plane orthogonal to the fiber. We can thus decompose any 1-form into two parts: \( \omega_1 = \alpha + \beta \). The part \( \alpha \) will have no component along the base; thus \( \langle v, \alpha \rangle = 0 \) for all \( v \) perpendicular to \( \partial \hat{y} \). The part \( \beta \) will have no component along the fiber; thus \( \langle \partial \hat{y}, \beta \rangle = 0 \). We find that \( \alpha \) must be proportional to 
\[
\hat{\sigma} = d\hat{y} - \frac{1 + \cos \theta}{2} d\phi \tag{5.83}
\]

while \( \beta \) is just characterized by having no term proportional to \( d\hat{y} \).

Let us now apply this decomposition to our 1-form. The part \( \alpha \) can be written as \( \alpha = f \hat{\sigma} \) where \( f \) is a function on the squashed sphere. This function must carry the quantum numbers \( (m', j, m) = \left( \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right) \). Since \( f \) is a scalar it must have \( j = j' \) and so it must actually be proportional to the scalar harmonic that is the pullback of \( \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right) \), which is just \( \omega_0 \). So we find that \( \omega_0 \) is just characterized by having no term proportional to \( \omega_0 \).

To summarize, our 1-form must have the form \( f_0(r) \omega_0 \hat{\sigma} + f_1(r) \omega_1 \).

### 5.4 Some relations on forms

A decomposition of the type \( \omega_1 = \alpha + \beta \) which we did for 1-forms can be done for any p-form \( \omega \) on the squashed 3-sphere. The \( \hat{y} \) dependence of our forms is \( e^{-i\hat{y}} \). Using this fact we find

\[
d\omega = -i\hat{\sigma} \wedge \omega + D\omega \tag{5.85}
\]

where

\[
D\omega = d_2\omega - i \frac{1 + \cos \theta}{2} d\phi \wedge \omega \tag{5.86}
\]

is the covariant derivative of \( \omega \) and \( d_2 \) denotes the differential with respect to \( \theta, \phi \). The square of \( D \) is proportional to the monopole field strength

\[
D^2\omega = \left( \frac{\sin \theta}{2} d\theta \wedge d\phi \right) \wedge \omega \tag{5.87}
\]

If we denote by \( *_2 \) the Hodge dual with respect to the \( S^2 \) metric,\(^5\) the monopole harmonics \( \omega_0 \) and \( \omega_1 \) satisfy

\[
*_2 \omega_1 = -i \omega_1 \\
D_2 \omega_0 = -\frac{1}{2} \omega_1 \\
D^2 \omega_0 = i \frac{\sin \theta}{2} \omega_0 d\theta \wedge d\phi \\
D_2 \omega_1 = -2D^2 \omega_0 = -i \sin \theta \omega_0 d\theta \wedge d\phi \tag{5.88}
\]

\(^5\)Note that \((*_2)^2 = -1\) on 1-forms. We have \( \epsilon_{\theta\phi} = 1 \).
5.5 The 2-form $K$

We can use the structure above to write a general ansatz for the 2-form $K$. Any self-dual 2-form on $TN$ can be written as

$$dr \wedge \tilde{\omega} + \star (dr \wedge \tilde{\omega})$$  \hspace{1cm} (5.89)

where $\tilde{\omega}$ is a 1-form on $TN$. The form $dr$ has all angular quantum numbers zero, so the quantum numbers of the perturbation must be carried by $\tilde{\omega}$. But we have seen in section (5.3) that any such 1-form, with quantum numbers $(m', j, m) = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$, is of the form

$$\tilde{\omega} = f_0(r) \omega_0 \hat{\sigma} + f_1(r) \omega_1$$  \hspace{1cm} (5.90)

The 2-form $K$ is self-dual, as a form on $TN$, and depends on $t$ and $z$ as in (4.49): it can thus be written as

$$K = e^{-i(t+\kappa z)/(2Q)} (K_0 + K_1)$$  \hspace{1cm} (5.91)

where the $K_0, K_1$ parts correspond to the first and second parts on the RHS of (5.90)

$$K_0 = f_0(r) \omega_0 [dr \wedge \hat{\sigma} + \star (dr \wedge \hat{\sigma})] = f_0(r) \omega_0 \left[ dr \wedge \hat{\sigma} + \frac{Hr^2}{Q} d\hat{\sigma} \right]$$

$$K_1 = f_1(r) [dr \wedge \omega_1 + \star (dr \wedge \omega_1)] = f_1(r) \left[ dr \wedge \omega_1 - \frac{2Q}{H} \hat{\sigma} \wedge \omega_1 \right]$$  \hspace{1cm} (5.92)

Note that we have used only the scalar $\omega_0$ and the 1-form $\omega_1$ in our expansion, and avoided a separate coefficient function for the 1-form $d\omega_0$.

6 The radial equations and their solution

The ansatz (5.92) reduces the unknowns to two functions of $r$: $f_0(r)$ and $f_1(r)$. In this section we will derive the system of differential equations these functions have to satisfy, and then see how they are solved.

6.1 Obtaining the radial equations

Let us start from eq. (4.67). We note that

$$\Omega^{(2)} \wedge K_1 = 0$$  \hspace{1cm} (6.93)

and that, by comparing (6.92) with (5.38),

$$K_0 = -f_0 \omega_0 \left( \frac{rH^2}{2Q^2} \Omega^{(2)} \right)$$  \hspace{1cm} (6.94)

Thus

$$\star (\Omega^{(2)} \wedge K) = -e^{-i(t+\kappa z)/(2Q)} f_0 \omega_0 \left( \frac{rH^2}{2Q^2} \star (\Omega^{(2)} \wedge \Omega^{(2)}) \right)$$  \hspace{1cm} (6.95)

An easy computation gives

$$\Omega^{(2)} \wedge \Omega^{(2)} = \frac{4Q^3}{H^3 r^2} \sin \theta \ dr \wedge d\theta \wedge d\phi \wedge \hat{\sigma}$$  \hspace{1cm} (6.96)
and

\[ * (\Omega^{(2)} \wedge \Omega^{(2)}) = \frac{2Q^2}{(Hr)^4} \] (6.97)

Using this in (6.95) we find

\[ * (\Omega^{(2)} \wedge K) = -e^{-i(t+\kappa z)/(2Q)} f_0 \omega_0 \frac{1}{(Hr)^2} \] (6.98)

Using the expression for \( w \) in (6.97), it is also straightforward to compute (for example with the help of Mathematica)

\[ \Delta w - (p + k)(pg_{tt} + kg_{zz})w = -e^{-i(t+\kappa z)/(2Q)} e^{-\kappa r/(2Q)} \frac{i}{(Hr)^2} \frac{\sqrt{T}}{Hr^4} [(3 + \kappa)Q + (1 + \kappa)r] \omega_0 \] (6.99)

Using (6.98) and (6.99), we see that equation (4.67) becomes

\[ e^{-\kappa r/(2Q)} \frac{i}{(Hr)^2} [(3 + \kappa)Q + (1 + \kappa)r] + 2f_0 = 0 \] (6.100)

Let us now turn to eq. (4.68). We first need to compute

\[ \Delta K_a = *d * dK_a + d * dK_a \]

\[ = - \left( \nabla^k \nabla_k K_{a ij} + [\nabla^k, \nabla_i] K_{a jk} - [\nabla^k, \nabla_j] K_{a ik} \right) dx^i \wedge dx^j \] (6.101)

for \( a = 0, 1 \). The covariant derivatives and index contractions in the second line of (6.101) are done with the TN metric. A lengthy but straightforward computation, that makes use of identities (5.88), leads to

\[ \Delta K_0 = -\omega_0 [dr \wedge \sigma + * (dr \wedge \sigma)] \left[ \frac{f''}{H} + 2 \frac{f'}{(Hr)} \right] \frac{r^4 + 4Qr^3 + 16Q^2r^2 - 8Q^3r + 3Q^4}{4Q^2r(Q + r)^3} f_0 \]

\[ - \frac{i}{2Q} [dr \wedge \omega_1 + *(dr \wedge \omega_1)] \frac{f_0}{Hr} \]

\[ \Delta K_1 = -[dr \wedge \omega_1 + *(dr \wedge \omega_1)] \left[ \frac{f''}{H} + 2 \frac{f'}{(Hr)^2} \right] \frac{r^4 + 4Qr^3 + 8Q^2r^2 + 16Q^3r + 3Q^4}{4Q^2r(Q + r)^3} f_1 \]

\[ + i4Q [dr \wedge \sigma + *(dr \wedge \sigma)] \frac{f_1}{(Hr)^3} \] (6.102)

The full wave operator is

\[ \Delta_6 K_a \equiv \Delta K_a + (p + k)(pg_{tt} + kg_{zz}) K_a, \ a = 0, 1 \] (6.103)

and we find

\[ \Delta_6 K_0 = -\omega_0 [dr \wedge \sigma + *(dr \wedge \sigma)] \left[ \frac{f''}{H} + 2 \frac{f'}{(Hr)} \right] \frac{r^4 + 4Qr^3 + 16Q^2r^2 - 8Q^3r + 3Q^4}{r(Hr)^3} \]

\[ - (1 + \kappa) \frac{2Q + (1 - \kappa)r}{Hr} \frac{f_0}{(2Q)^2} - \frac{i}{2Q} [dr \wedge \omega_1 + *(dr \wedge \omega_1)] \frac{f_0}{Hr} \]

\[ \Delta_6 K_1 = -[dr \wedge \omega_1 + *(dr \wedge \omega_1)] \left[ \frac{f''}{H} + 2 \frac{f'}{(Hr)^2} \right] \frac{r^4 + 4Qr^3 + 8Q^2r^2 + 16Q^3r + 3Q^4}{r(Hr)^3} \]

\[ - (1 + \kappa) \frac{2Q + (1 - \kappa)r}{Hr} \frac{f_1}{(2Q)^2} + i4Q [dr \wedge \sigma + *(dr \wedge \sigma)] \frac{f_1}{(Hr)^3} \] (6.104)
In (4.68) the first two terms constitute $\Delta_6$; thus the last term will act as a ‘source term’ for this Laplacian. The source term is

$$-(p + k)^2 wQ^{(2)} = \frac{(1 + \kappa)^2}{2} e^{-i(t+\kappa z)/(2Q)} e^{-\kappa r/(2Q)} \frac{\sqrt{r}}{(Hr)^3} \omega_0 [dr \wedge \hat{s} + \star(dr \wedge \hat{s})] \quad (6.105)$$

Collecting the terms proportional to $\omega_0 [dr \wedge \hat{s} + \star(dr \wedge \hat{s})]$ and to $[dr \wedge \omega_1 + \star(dr \wedge \omega_1)]$ in eq. (4.68), we find the following system of equations for $f_0$ and $f_1$:

$$\frac{f_0''}{H} + 2 \frac{f_0'}{Hr} - \left( \frac{r^4 + 4Qr^3 + 16Q^2r^2 - 8Q^3r + 3Q^4}{r(Hr)^3} - (1 + \kappa) \frac{2Q + (1 - \kappa)r}{Hr} \right) f_0 \frac{f_0}{(2Q)^2}$$

$$-i 4Q \frac{f_1}{(Hr)^3} - \frac{(1 + \kappa)^2}{2} e^{-\kappa r/(2Q)} \frac{\sqrt{r}}{(Hr)^3} = 0 \quad (6.106)$$

$$\frac{f_1''}{H} + 2 \frac{f_1'}{(Hr)^2} - \left( \frac{r^4 + 4Qr^3 + 8Q^2r^2 + 16Q^3r + 3Q^4}{r(Hr)^3} - (1 + \kappa) \frac{2Q + (1 - \kappa)r}{Hr} \right) f_1 \frac{f_1}{(2Q)^2}$$

$$+ \frac{i}{2Q} \frac{f_0}{(Hr)^3} = 0 \quad (6.107)$$

### 6.2 Solving the radial equations

Eq. (6.100) can be readily solved for $f_0$, giving:

$$f_0 = -e^{-kr} \frac{3 + \kappa)Q + (1 + \kappa)r \sqrt{r}}{2} \frac{(Hr)^2}{(2Q)^2} \quad (6.108)$$

By substituting $f_0$ into (6.106) we can derive $f_1$. With the help of Mathematica we compute:

$$\frac{f_0''}{H} + 2 \frac{f_0'}{Hr} - \left( \frac{r^4 + 4Qr^3 + 16Q^2r^2 - 8Q^3r + 3Q^4}{r(Hr)^3} - (1 + \kappa) \frac{2Q + (1 - \kappa)r}{Hr} \right) f_0 \frac{f_0}{(2Q)^2}$$

$$= e^{-kr} \frac{2 + (1 + \kappa)^2}{2} \frac{\sqrt{r}}{(Hr)^3} \quad (6.109)$$

and thus, from (6.106), we obtain

$$f_1 = -\frac{i}{4Q} e^{-kr} \sqrt{r} \quad (6.110)$$

Eq. (6.107) is a consistency condition for our previously determined values of $f_0$ and $f_1$. By Mathematica we compute

$$\frac{f_1''}{H} + 2 \frac{f_1'}{(Hr)^2} - \left( \frac{r^4 + 4Qr^3 + 8Q^2r^2 + 16Q^3r + 3Q^4}{r(Hr)^3} - (1 + \kappa) \frac{2Q + (1 - \kappa)r}{Hr} \right) f_1 \frac{f_1}{(2Q)^2}$$

$$= \frac{i}{4Q} e^{-kr} [(3 + \kappa)Q + (1 + \kappa)r \frac{\sqrt{r}}{(Hr)^3}] \quad (6.111)$$

Substituting this in (6.107) and using (6.108), we see that (6.107) is satisfied.
To summarize, we have found the solution

$$\begin{align*}
w &= e^{-i(pt+kz)} e^{-kr} \frac{\sqrt{r}}{Q + r} \omega_0 \\
K &= e^{-i(pt+kz)} (K_0 + K_1) \\
K_0 &= -e^{-kr} \frac{\sqrt{r}}{2(3Q + r)^2} \left[ (3Q + r) + \kappa (Q + r) \right] \omega_0 \left[ dr \wedge \hat{\sigma} + \frac{H r^2}{2Q} \sin \theta d\theta \wedge d\phi \right] \\
K_1 &= -ie^{-kr} \frac{\sqrt{r}}{4Q} \left[ dr \wedge \omega_1 - i \frac{2Q}{H} \hat{\sigma} \wedge \omega_1 \right]
\end{align*}$$

(6.112)

Knowing $K$ we can derive the values of $a$, $b$, $B$ and $\Phi$. Before we do this we discuss the gauge invariance of our problem.

6.2.1 Gauge invariance

One can easily check, using the identities (4.52), that the following two gauge transformations leave all the components of $F^{(3)}$ invariant:

$$\begin{align*}
B &\rightarrow B + d\lambda^{(1)}, \quad a \rightarrow a + ip \lambda^{(1)}, \quad b \rightarrow b + ik \lambda^{(1)} \\
\Phi &\rightarrow \Phi + \lambda^{(0)}, \quad a \rightarrow a + d\lambda_a^{(0)}, \quad b \rightarrow b + d\lambda_b^{(0)} \quad \text{with} \quad \lambda^{(0)} - ik \lambda_a^{(0)} + ip \lambda_b^{(0)} = 0
\end{align*}$$

(6.114)

Here $\lambda^{(1)}$ is a 1-form and $\lambda^{(0)}$, $\lambda_a^{(0)}$ and $\lambda_b^{(0)}$ are 0-forms on $TN$. The 2-form $K$ is gauge invariant.

6.2.2 Deriving the gauge fields

By making use of the transformation (6.115), we can set $\Phi = 0$. Then the second equation in (4.68) implies

$$b - \kappa a = (2Q)^2 \frac{\ast dK}{1 + \kappa}$$

(6.116)

One can compute

$$\frac{\ast dK}{1 + \kappa} = e^{-i(pt+kz)} e^{-kr} \sqrt{r} \left[ -i \frac{1}{8Q^2} \omega_1 + i \frac{1}{8Q^2} \omega_0 dr + \frac{1}{4(Q + r)^2} \left( 1 - \kappa \frac{r}{Q} \right) \omega_0 \hat{\sigma} \right]$$

(6.117)

Then a solution of (6.116) for $a$ and $b$ is

$$\begin{align*}
a &= e^{-i(pt+kz)} e^{-kr} \frac{Qr}{(Q + r)^2} \omega_0 \hat{\sigma} \\
b &= e^{-i(pt+kz)} e^{-kr} \left[ \frac{i}{2r} \omega_0 dr - \frac{i}{2} \omega_1 + \frac{Q^2}{(Q + r)^2} \omega_0 \hat{\sigma} \right]
\end{align*}$$

(6.118)

By picking this solution we have fixed the gauge freedom implied by the transformation (6.114).

Substituting these values of $a$ and $b$ into the first equation in (4.68) we derive $B$

$$B = e^{-i(pt+kz)} e^{-kr} \sqrt{r} \left[ \frac{1}{2} dr \wedge \omega_1 - \frac{ir}{2} \omega_0 \sin \theta d\theta \wedge d\phi \right]$$

(6.119)
7 Regularity of the solution

We show that the solution given in (6.112,6.118,6.119) is both regular and normalizable.

7.1 Normalizability

For $k > 0$ normalizability is guaranteed by the exponential fall off $e^{-kr}$. Note however that waves with $k \leq 0$ give rise to non-normalizable perturbations. This is obvious for $k < 0$. For $k = 0$ let us look, for example, at the scalar $w$: its large $r$ behavior is

$$w \approx e^{-ipt} \frac{1}{\sqrt{r}} \omega_0$$

and thus

$$|w|^2 \sim 1/r$$

Since the volume element of the space transverse to the ring grows as $\sim r^2 dr$ for large $r$, the norm of $w$ is quadratically divergent at $r \to \infty$. This shows that for $k = 0$ the perturbation leaks out to the center of the ring ($r \sim a$) and does not stay confined to the vicinity of the tube.\textsuperscript{6} For $k > 0$ the wavefunction becomes confined closer to the ring, and in the limit (3.30) we find a normalizable solution in the near ring limit.

The fact that positive and negative $k$ behave differently is to be expected; the 2-charge background does not have the symmetry $z \leftrightarrow -z$. In the NS1-P picture the geometry is created by a string carrying a wave, and the strands of the string carry momentum along the ring, thus breaking the $z \leftrightarrow -z$ symmetry. In \cite{15} it was found that there are ‘left-moving’ non-BPS perturbations that move in one direction along the ring, while ‘right-moving’ perturbations create time independent distortions of the 2-charge geometry. For our present problem note that in $w_{full}$ (eq. (2.25)) we have $l \geq \frac{1}{2}$, so we must have $k \geq 0$, and negative values of $k$ do not appear.\textsuperscript{7} We need to take large $|k|$ to be able to use the ‘straight segment’ limit of the ring so the case $k = 0$ is not relevant for our discussion, and we naturally find ourselves at large positive $k$.

7.2 Regularity at $\theta = 0, \pi$

Let us now consider the regularity of the solution. The fields $w, a, b$ and $B$ are manifestly regular away from the points where our system of coordinates degenerates. This degeneration happens at $\theta = 0$ or $\pi$ and at $r = 0$. Around $\theta = 0, \pi$ it is convenient to change to $S^3$ coordinates (5.63). From the expression of $\omega_0$ in (5.73) it is apparent that $\omega_0$ is regular: indeed for $\tilde{\theta} = \pi/2$, where the $\tilde{\psi}$ coordinate degenerates, the coefficient of $e^{-i\tilde{\psi}}$ vanishes. The second identity in (5.88) expresses $\omega_1$ as the covariant derivative of $\omega_0$, and thus $\omega_1$ is regular too. The 1-form along the fiber $\hat{\sigma}$ can be expressed in $S^3$ coordinates as

$$\hat{\sigma} = \sin^2 \tilde{\theta} d\tilde{\phi} + \cos^2 \tilde{\theta} d\tilde{\psi}$$

which is also regular. Since the angular dependence of $w, a, b$ and $B$ is entirely expressed in terms of $\omega_0, \omega_1$ and $\hat{\sigma}$, this proves that our solution is regular at $\theta = 0, \pi$.\textsuperscript{8}

\textsuperscript{6}If we construct the exact wavefunction for the ring (instead of just constructing it for the near ring limit) then we expect to have a solution normalizable at spatial infinity, since the state exists in the dual CFT.

\textsuperscript{7}Spherical harmonics for the scalar have $l = 0, \frac{1}{2}, 1, \ldots$, but for $l = 0$ we get zero if we apply $J_{\tilde{\phi}}$, and so we cannot construct the required perturbation of section (2.4).
7.3 Regularity at $r = 0$

At $r \to 0$ the TN space becomes flat $\mathbb{R}^4$: the change of coordinates that brings the TN metric into explicitly flat form is (5.73) for the angular variables and

$$r = \frac{\rho^2}{4Q} \quad (7.123)$$

for the radial coordinate. In these coordinates the $r \to 0$ limit of $w$ is

$$w \sim \rho e^{-i\tilde{\psi}} \cos \tilde{\theta} = x_1 - ix_2 \quad (7.124)$$

where $x_i$, with $i = 1, \ldots, 4$ are Cartesian coordinates in $\mathbb{R}^4$. This shows that $w$ is regular at $r \to 0$. Similarly, the gauge fields $a$ and $B$ behave like

$$a \sim \rho^3 e^{-i\tilde{\psi}} \cos \tilde{\theta} (\sin^2 \tilde{\theta} \, d\tilde{\phi} + \cos^2 \tilde{\theta} \, d\tilde{d}) = (x_1 - ix_2) \left[ x_1 dx_2 - x_2 dx_1 + x_3 dx_4 - x_4 dx_3 \right]$$

$$B \sim -i\rho^2 e^{-i\tilde{\psi}} \sin \tilde{\theta} \left[ id\rho \wedge d\tilde{\phi} + \sin \tilde{\theta} dp + \rho \cos \tilde{\theta} d\tilde{\theta} \wedge (d\tilde{\phi} - d\tilde{d}) \right]$$

$$= -i[(x_1 - ix_2)(dx_1 \wedge dx_2 + dx_3 \wedge dx_4) - i \sum_i x_i dx_i \wedge (dx_1 - idx_2)] \quad (7.126)$$

and are hence regular. Regularity of $b$ is not manifest in the form in which it appears in (6.118). This form was derived after making the arbitrary gauge choice $\Phi = 0$. By using the transformation (6.115) we can change $\Phi$ and $b$ and write them in an explicitly smooth form; since $a$ was already shown to be smooth, we can take $\lambda_a^{(0)} = 0$ in (6.115) and leave it unchanged. If we choose

$$\lambda^{(0)} = -e^{-i(pt+kz)} e^{-kr} \frac{\sqrt{r}}{2Q} \omega_0, \quad \lambda_b^{(0)} = \frac{i}{p} \lambda^{(0)} \quad (7.127)$$

in (6.115), then the fields $\Phi$ and $b$ are changed into

$$\Phi = -e^{-i(pt+kz)} e^{-kr} \frac{\sqrt{r}}{2Q} \omega_0$$

$$b = e^{-i(pt+kz)} e^{-kr} \frac{\sqrt{r}}{2Q} \left[ ik \omega_0 dr + \left( \frac{Q^2}{(Q+r)^2} - 1 \right) \omega_0 \tilde{\sigma} \right] \quad (7.128)$$

At $r \to 0$ both $\Phi$ and $b$ are now explicitly regular:

$$\Phi \sim \rho e^{-i\tilde{\psi}} \cos \tilde{\theta} = x_1 - ix_2$$

$$b \sim \rho e^{-i\tilde{\psi}} \cos \tilde{\theta} \left[ \frac{ik}{2Q} \rho d\rho - \frac{\rho^2}{2Q^2} (\sin^2 \tilde{\theta} \, d\tilde{\phi} + \cos^2 \tilde{\theta} \, d\tilde{d}) \right]$$

$$= (x_1 - ix_2) \left[ \frac{ik}{2Q} \sum_i x_i dx_i - \frac{x_1 dx_2 - x_2 dx_1 + x_3 dx_4 - x_4 dx_3}{2Q^2} \right] \quad (7.129)$$

---

8Explicitly,

$$x_1 + ix_2 = \rho e^{i\tilde{\psi}} \cos \tilde{\theta}, \quad x_3 + ix_4 = \rho e^{i\tilde{\phi}} \sin \tilde{\theta} \quad (7.125)$$
8 Summary of the solution

We summarize here the full solution for $w$ and $B^{(2)M\bar{N}}$, in the gauge of section [3] where all fields are regular:

$$w = e^{-\frac{1}{2Q}(t+y)} e^{i(\phi-kz)} \cos \frac{\theta}{2} e^{-kr} \frac{r^{1/2}}{Q + r}$$

$$B^{(2)}_{i\bar{z}} = -e^{-\frac{1}{2Q}(t+y)} e^{i(\phi-kz)} \cos \frac{\theta}{2} e^{-kr} \frac{r^{1/2}}{2Q}$$

$$B^{(2)}_{\bar{y}t} = e^{-\frac{1}{2Q}(t+y)} e^{i(\phi-kz)} \cos \frac{\theta}{2} e^{-kr} \frac{r^{3/2}}{2(Q + r)^2}$$

$$B^{(2)}_{\phi\bar{t}} = -e^{-\frac{1}{2Q}(t+y)} e^{i(\phi-kz)} \cos^3 \frac{\theta}{2} e^{-kr} \frac{Q r^{3/2}}{(Q + r)^2}$$

$$B^{(2)}_{i\bar{z}} = e^{-\frac{1}{2Q}(t+y)} e^{i(\phi-kz)} \cos \frac{\theta}{2} e^{-kr} \frac{r^{1/2}}{2} \left[ \frac{Q^2}{(Q + r)^2} - 1 \right]$$

$$B^{(2)}_{\phi\bar{z}} = -e^{-\frac{1}{2Q}(t+y)} e^{i(\phi-kz)} \cos^3 \frac{\theta}{2} e^{-kr} r^{1/2} \left[ \frac{Q^2}{(Q + r)^2} - 1 \right]$$

$$B^{(2)}_{i\bar{r}} = i e^{-\frac{1}{2Q}(t+y)} e^{i(\phi-kz)} \cos \frac{\theta}{2} e^{-kr} r^{1/2}$$

$$B^{(2)}_{\bar{r}\phi} = -i e^{-\frac{1}{2Q}(t+y)} e^{i(\phi-kz)} \sin \frac{\theta}{2} e^{-kr} r^{1/2}$$

$$B^{(2)}_{\theta\phi} = -i e^{-\frac{1}{2Q}(t+y)} e^{i(\phi-kz)} \cos \frac{\theta}{2} \sin \theta e^{-kr} r^{3/2}$$

We have added one unit of $P$ to a $D1$-$D5$ bound state. For the moduli we have used this is a `threshold bound' state; i.e., the mass of the bound state is the mass of the $D1$-$D5$ plus the mass of the $P$.

In [16] it was noted that a $D0$ brane is repelled by a $D6$ brane, but if a suitable flux $F$ was turned on in the $D6$ worldvolume then the $D0$ and $D6$ will form a bound state. In this section we find a relation between our threshold bound state and the condition in [10] which separates the domain of bound states from unbound states in the $D0$-$D6$ system.

9 Relation to D0-D6 bound states

We have added one unit of $P$ to a $D1$-$D5$ bound state. For the moduli we have used this is a `threshold bound' state; i.e., the mass of the bound state is the mass of the $D1$-$D5$ plus the mass of the $P$.

In [16] it was noted that a $D0$ brane is repelled by a $D6$ brane, but if a suitable flux $F$ was turned on in the $D6$ worldvolume then the $D0$ and $D6$ will form a bound state. In this section we find a relation between our threshold bound state and the condition in [10] which separates the domain of bound states from unbound states in the $D0$-$D6$ system.

---

9A similar system was also studied in [17].
9.1 The near ring limit

Strictly speaking, we have a threshold bound state between the entire ring shaped D1-D5 and the entire P wavefunction. The threshold nature of this state follows from the general supersymmetry relation between D1,D5,P charges. But we have seen that for large $k$ the P wavefunction is confined to the vicinity of the ring, so we expect to get threshold binding between a short straight segment of the ring (like that in Fig.1(c)) and the wavefunction carrying P that we found in this short segment approximation. We will take the further step of identifying the two ends of our ring segment; this will enable us to perform a T-duality in the $z$ direction.

Let us review the charges carried by this segment of the ring:

(a) We have the KK monopole charge that can be measured by a $S^2$ surrounding the tube; the nontrivially fibered circle of this KK is the $y$ circle, and the directions $T^4, z$ are ‘homogeneous directions’, so they behave like directions along the ‘KK-brane’.

(b) We have the ‘true charge’ D5 along $T^4 \times S^1$.

(c) We have the ‘true charge’ D1 along the $S^1$.

(d) We have ‘dipole momentum’ along the ring direction $z$; we call this $P_z$.

(e) The ‘test quantum’ that we seek to bind to this background is a unit of $P_y$ (momentum along $S^1$).

The dipole charges are created automatically by the binding of the true charges, and so there are relations between the values of the true and dipole charges. To find these relations it is convenient to dualize the D5-D1 charges to NS1-P. The near ring limit of NS1-P was discussed in some detail in [15]; we reproduce some relevant details here.

The NS1 string carries the momentum $P$ through transverse oscillations, described by a profile $\vec{F}(t - y)$. In Fig.2(a) we open up this multiwound NS1 to exhibit this vibration profile. Since the NS1 is wrapped many times on the $S^1$ we find that a short segment of this oscillating string looks like Fig.2(b). We can see that the winding along the direction $y$ is due to the ‘winding charge’ of the NS1, while the slant along the $z$ direction is due to the ‘derivative of the transverse displacement’ $\frac{d\vec{F}}{dy}$ which we take to point along the $z$ direction.

Figure 2: (a) The NS1 carrying a transverse oscillation profile in the covering space of $S^1$. (b) The strands of the NS1 as they appear in the actual space.
Figure 3: The winding and momentum charges of a segment of the NS1; we have used a multiple cover of the $S^1$ so that the NS1 looks like a diagonal line.

For our present discussion we have compactified the $z$ direction, so that we can assign well defined ‘charges’ to all elements. We find the following charges in the NS1-P frame:

(a) We have one unit of $NS_{1z}$, winding charge of the NS1 in the $z$ direction. This is the dual of the KK dipole charge in the D1-D5 frame. Thus we write $n_1^{dipole} = 1$.

(b) The ‘true’ D5 charge becomes winding along $S^1$. We write this as $n_1^{true}$ units of $NS_{1y}$.

(c) The ‘true’ D1 charge becomes momentum along $S^1$. We write this as $n_p^{true}$ units of $P_y$.

(d) We have momentum $P_z$ along the dipole direction (this has been unchanged in the duality from D1-D5). The number of units of this momentum we call $n_p^{dipole}$.

(e) The quantum that we wish to bind to the background changes from $P_y$ to an NS5 along $S^1 \times T^4$.

Note that we must choose the compactification lengths of the $y, z$ directions judiciously so that we get integer values for all charges. This can be done by choosing $L_z, L_y$ so that $n_1^{true}, n_p^{true}$ are integers. We will see below (eq. (9.133)) that this will set $n_p^{dipole}$ to be integral. Note that $n_1^{dipole} = 1$ so it is already integral.

### 9.2 Relations between true and dipole charges

**Note on notation:** In this section we will encounter three different duality related systems: D1-D5, NS1-P, and a system where these true charges become D4 branes. We will not need to compute in the D1-D5 frame. For the NS1-P frame we use unprimed symbols for all quantities (for example lengths are $L_y, L_z$ etc.). These should not be confused with unprimed symbols used in earlier sections of this paper; the computations here will not use results from those sections. For the frame using D4 branes we use primes on all symbols (e.g. $L_y', L_z'$).

Let us ignore for now the charge (e) in the above list and look at the other charges which together give the background geometry of the ring. These charges are depicted in Fig.3. We have a NS1 moving in a direction perpendicular to itself with some velocity $v_\perp$; this gives all the four charges (a)–(d) above. We have denoted the lengths of the $y, z$ directions by $L_y, L_z$. 

22
We will now derive the relations between the true charges and the dipole charges. The first relation comes from the fact that the momentum carried by the NS1 is in a direction perpendicular to the NS1. Indeed if the momentum had a component along along the NS1 then there would be oscillations along the NS1 and a corresponding entropy. The entire NS1-P bound state does have an entropy, which is manifested in different possible shapes for the entire ring. But we are now zooming in on a short segment of the ring, and so by definition should have no entropy visible in oscillations of this segment.

The NS1 winds in a direction given by the vector
\[
\vec{W} = L_z n_1^{\text{dipole}} \hat{z} + L_y n_1^{\text{true}} \hat{y} = L_z \hat{z} + L_y n_1^{\text{true}} \hat{y}
\]  
(9.131)

The momentum vector is
\[
\vec{P} = \frac{2\pi n_p^{\text{dipole}}}{L_z} \hat{z} + \frac{2\pi n_p^{\text{true}}}{L_y} \hat{y}
\]  
(9.132)

Requiring \(\vec{W} \cdot \vec{P} = 0\) gives
\[
n_1^{\text{dipole}} n_p^{\text{dipole}} + n_1^{\text{true}} n_p^{\text{true}} = 0, \quad \Rightarrow \quad n_p^{\text{dipole}} = -n_1^{\text{true}} n_p^{\text{true}}
\]  
(9.133)

The second condition comes from the fact that the waveform \(\vec{F}(t - y)\) moves along the \(y\) direction at the speed of light \(v = 1\). This implies that the velocity of the NS1 in the direction normal to itself is
\[
v_\perp = \cos \alpha = \frac{L_z}{\sqrt{(L_z)^2 + (n_1^{\text{true}} L_y)^2}}
\]  
(9.134)

The mass of the NS1 is
\[
M = T \sqrt{(L_z)^2 + (n_1^{\text{true}} L_y)^2}
\]  
(9.135)

where \(T = 1/2\pi \alpha'\) is the tension of the NS1. The momentum of the NS1 is in the direction normal to itself, and has magnitude
\[
|\vec{P}| = \sqrt{\left(\frac{2\pi n_p^{\text{dipole}}}{L_z}\right)^2 + \left(\frac{2\pi n_p^{\text{true}}}{L_y}\right)^2}
\]  
(9.136)

Setting \(|\vec{P}| = \frac{M \nu_\perp}{\sqrt{1-v_\perp^2}}\) we get
\[
\sqrt{\left(\frac{2\pi n_p^{\text{dipole}}}{L_z}\right)^2 + \left(\frac{2\pi n_p^{\text{true}}}{L_y}\right)^2} = T \sqrt{(L_z)^2 + (n_1^{\text{true}} L_y)^2} \frac{L_z}{n_1^{\text{true}} L_y}
\]  
(9.137)

Using (9.133) gives
\[
\sqrt{\left(\frac{2\pi n_p^{\text{dipole}}}{L_z}\right)^2 + \left(\frac{2\pi n_p^{\text{true}}}{L_y}\right)^2} = \frac{(2\pi)n_p^{\text{true}}}{L_y L_z} \sqrt{(L_z)^2 + (n_1^{\text{true}} L_y)^2}
\]  
(9.138)

We thus find that that (9.137) is equivalent to
\[
[T n_1^{\text{true}} L_y] \left[\frac{2\pi n_p^{\text{true}}}{L_y}\right] = [T L_z]^2
\]  
(9.139)
which tells us that

\[ \text{[Mass of true NS1 charge]} \times \text{[Mass of true P charge]} = \text{[Mass of NS1 dipole charge]}^2 \] (9.140)

In this form the condition is valid in all duality frames, with only the names of the charges changing under the dualities.

To summarize we have two relations between the true and dipole charges. The relation (9.133) comes from requiring that there be no entropy in the ring segment after we have zoomed into a sufficiently small region of the ring. The other condition (9.140) is related to the supersymmetry of the charges distributed along the ring. The supersymmetry is assured by the fact that the entire waveform moves in one direction with the speed of light. Different parts of the NS1 have different slopes and different velocities \( v_\perp \), but for a profile of the form \( \vec{F}(t - y) \) the slope and velocity are always correlated in such a way that the different parts are mutually BPS.

### 9.3 Dualizing to D6-D0

We now wish to perform dualities that will map the dipole charge of the ring (KK in the case of D1-D5, NS1 in the case of NS1-P) to a D6 brane charge. The quantum carrying one unit of \( P_y \) will be converted to a D0. Since we have found the relations between true and dipole charges in the NS1-P frame let us start with NS1-P and perform the required dualities:

\[
\begin{pmatrix}
NS_1z \\
NS_1y \\
P_y \\
P_z \\
NS_5y_{1234}
\end{pmatrix}
\mathcal{S}
\begin{pmatrix}
D1z \\
D1y \\
P_y \\
P_z \\
D5y_{1234}
\end{pmatrix}
\mathcal{T}_{yz_{12}}
\begin{pmatrix}
\overline{D3}_y_{12} \\
\overline{D3}_z_{12} \\
NS_1y \\
NS_1z \\
D3z_{34}
\end{pmatrix}
\mathcal{S}
\begin{pmatrix}
D6_{yz_{1234}} \\
D4_{1234} \\
D4_{yz_{34}} \\
D2_{34} \\
D0
\end{pmatrix}
\] (9.141)

The true charges \( n_{1}^{\text{true}}, n_{p}^{\text{true}} \) have become D4 branes which can be described by fluxes in the D6:

\[
n_{1}^{\text{true}} = n_{4}^{(1234)} = \frac{1}{2\pi} \int_{zy} F = \frac{L'_z L'_y}{2\pi} F_{zy}
\]

\[n_{p}^{\text{true}} = n_{4}^{(yz34)} = -\frac{1}{2\pi} \int_{12} F = -\frac{L'_z L'_y}{2\pi} F_{12} \] (9.142)

where \( L'_i \) are the lengths of cycles after the dualities. The minus sign in the expression for \( n_{p}^{\text{true}} \) arises from the orientation of the D6: the positive orientation is \( (zy1234) \) while the \( n_4 \) is oriented as \( (yz34) \). The presence of the above components of \( F \) also induces a D2 charge

\[
n_{2}^{(34)} = \frac{1}{2} \frac{1}{(2\pi)^2} \int_{zy12} F \wedge F = \frac{L'_z L'_y L'_y L'_z}{(2\pi)^2} F_{zy} F_{12} = -n_{4}^{(1234)} n_{4}^{(yz34)} \] (9.143)

Since under the dualities \( n_{p}^{\text{dipole}} = n_{2}^{(34)} \), we observe that (9.133) is equivalent to (9.143). In other words the relation (9.133) translates in the D6 duality frame to the statement that the D2 charge comes entirely from the fluxes needed to induce the required D4 charges; there is no ‘additional’ D2 charge.
9.4 The condition of \[16\]

Consider a D6 brane along the directions \((zy1234)\). Suppose that there is a background NS-NS 2-form turned on; by a suitable change of coordinates we can bring this to a form where the nonzero components are \(b_1 = B_{zy}, b_2 = B_{12}, b_3 = B_{34}\). Write

\[
e^{2\pi i a} = \frac{1 + i b_a}{1 - i b_a} \quad a = 1, 2, 3
\]

The threshold value of \(B\), beyond which a D0 will bind to the D6, is given by \[16\]

\[
v_1 + v_2 + v_3 = \frac{1}{2}
\]

In terms of the \(b_a\) this condition becomes

\[
b_1b_2 + b_1b_3 + b_2b_3 = 1
\]

We can replace the \(B\) field with a field strength on the D6:

\[
b_a \rightarrow 2\pi \alpha' F_a
\]

We take \(\alpha' = 1\) in the following. In our case we have \(b_3 = 0\) and \(b_1 = \frac{2\pi}{L_z} L'_y, \quad b_2 = \frac{2\pi}{L_z} L'_1 L'_2\). One has the freedom to change the orientation in each of the 2-planes \((z, y), (1, 2), (3, 4)\): this flips the sign of \(v_a\) and \(b_a\), and thus the sign of each of the terms in \((9.146)\) is arbitrary. Taking this into account the threshold condition of \[16\] for our case is

\[
(2\pi)^2 |F_{zy} F_{12}| = 1
\]

9.5 Checking the threshold condition

Let us now see if the condition \((9.148)\) is satisfied by our ring segment. From \((9.142)\) we find that

\[
(2\pi)^2 F_{zy} F_{12} = - \frac{(2\pi)^4}{L'_z L'_y L'_1 L'_2} n_1^{true} n_p^{true}
\]

Under the dualities \(9.141\) the moduli change as follows (primed quantities refer to D6 frame, unprimed to the NS1-P frame, and \(L_i = 2\pi R_i\))

\[
\begin{align*}
g' &= g \sqrt{\frac{R_z}{R_y R_1 R_2 R_3 R_4}}, & R'_3 &= \frac{g}{R_4 \sqrt{R_z R_y R_1 R_2}}, & R'_4 &= \frac{g}{R_3 \sqrt{R_z R_y R_1 R_2}} \\
R'_{zy} &= \sqrt{\frac{R_z R_1 R_2}{R_y}}, & R'_z &= \sqrt{\frac{R_z}{R_y R_1 R_2}}, & R'_1 &= \sqrt{\frac{R_z R_y R_2}{R_1}}, & R'_2 &= \sqrt{\frac{R_z R_y R_1}{R_2}}
\end{align*}
\]

Thus

\[
(2\pi)^2 F_{zy} F_{12} = - \frac{(2\pi)^4 n_1^{true} n_p^{true}}{L'_z L'_y L'_1 L'_2} = - \frac{(2\pi)^2 n_1^{true} n_p^{true}}{L'_z^2}
\]

If we now use the relation \(9.139\) we find

\[
(2\pi)^2 F_{zy} F_{12} = - 1
\]

We thus see that the charges carried by our ring satisfy the condition \((9.148)\) noted in \[16\].
9.6 Depth of the tachyon potential

Let us see what we have learned. The 2-charge system has true charges and dipole charges, and these satisfy the relations (9.133), (9.140). The system can be mapped to a D6 brane carrying fluxes, and the fluxes have a value which puts the system at the boundary of the domain where a D0 brane will bind to the D6.

In the D1-D5 picture the analogue of the D0 is the P charge. In section (9.1) we listed the charges carried by the 2-charge D1-D5 system and the charge P carried by the wavefunction we are trying to construct. But there is one more charge carried by the wavefunction, which comes from the momentum of this wavefunction along the $z$ direction. In the wavefunction this momentum arises from the factor $e^{-ikz}$. So we would label this charge as an additional amount of $P_z$, carried by the quantum that we are trying to bind to the 2-charge D1-D5 ring segment.

In the D6 duality frame this additional $P_z$ becomes a $D_{234}$. Thus the quantum that we are trying to bind to the D6 is not just a D0, but a ‘D0 plus some $D_{234}$’. We now draw some conclusions about the D0-D6 bound state from our construction of the wavefunction (8.130).

9.6.1 The case $k = 0$

Since $k$ is a free parameter, we can try to set $k = 0$. This would correspond to letting the test quantum be just the D0 (not bound to any $D_{234}$), and asking if at the threshold value of fluxes (9.148) we get a good bound state with the D6. But from the discussion of section (7.1) we see that the wavefunction is not normalizable for the case $k = 0$, so there is no bound state in this case. We therefore conclude that for a D6 wrapped on a torus $T^6$ carrying fluxes at the threshold value (9.148) we do not get a bound state with the D0. As argued in [16] we would of course get a bound state for larger values of $F$ and no bound state for smaller $F$, but our explicit construction of the wavefunction (in the dual D1-D5 case) tells us the situation at the threshold value of $F$.

9.6.2 The case $k > 0$

In this case the test quantum to be bound has some $D_{234}$ branes along with the D0. The mass of a ‘D0 plus some $D_{234}$’ is obviously more than the mass of just the D0. But after we bind the ‘D0 plus some $D_{234}$’ to the D6 carrying fluxes, the final mass of the bound state is independent of the mass of the $D_{234}$ branes coming with the D0, since in the D1-D5 frame the energy of the wavefunction is given by $e^{-i\frac{2\pi}{\tau}Q}$ for all values of $k$.

This observation tells us the binding energy of the $D_{234}$ branes in the situation where we have a D6 carrying fluxes equal to their ‘threshold’ value (9.148). The binding energy $\Delta E$ must be equal to the mass $M_2$ of the $D_{234}$ in order that these branes do not show up in the final result for the mass of the composite:

$$\Delta E = M_2$$

(9.153)

Note that the D0 is repelled by a D6, is neutral with respect to the D4’s in the D6, and is attracted by the D2 charge in the D6. At the ‘threshold’ value of fluxes it becomes neutral with respect to the ‘D6-D4-D4-D2’ bound state created by the D6 with fluxes. By contrast the $D_{234}$ is neutral with respect to the D6, is attracted to the D4’s in the D6, and is neutral with respect to the D2 in the D6. Thus we expect a binding energy $\Delta E$ for the $D_{234}$, and our construction of the wavefunction tells us that this energy is (9.153). In CFT terms we get a tachyon in the
open string spectrum between the $D_{234}$ and the D6 with fluxes. For the threshold value of these fluxes the depth of the tachyon potential must equal the mass of the $D_{234}$.

10 Discussion

We have constructed a simple case of ‘3-charge hair’ for the BPS black ring, by starting with a D1-D5 ring and adding a perturbation carrying one unit of $P$. A normalizable perturbation carrying this $P$ was expected to exist because there was a corresponding state in the dual CFT. After constructing the perturbation we observe that it is smooth everywhere, so the result supports a ‘fuzzball’ picture for black ring microstates.

A similar perturbation was constructed (up to several orders in a small parameter) for the 3-charge black hole in [5], and solutions dual to specific CFT states carrying nonperturbative amounts of $P$ were found in [6]. But these solutions carried a large amount of angular momentum. Thus it may be said that they did not give generic microstates for the 3-charge hole. By contrast, the black ring is supposed to carry a sizable amount of angular momentum, which gives it the ‘ring shape’. Thus even though we have only one unit of $P$ in our present construction, the hair we have constructed might be considered a good indicator of the nature of generic states of the ring.

In [18] ‘ring-like’ 2-charge states were considered, and it was observed that the area of a ‘horizon’ drawn around such states has an area satisfying a Bekenstein type relation $A/4G \sim \sqrt{n_1 n_5 - J} \sim S$ where $S$ is the entropy of these states and $J$ is the angular momentum of the ring. (Such 2-charge systems have been further studied recently [19, 20, 21, 22].) In the present paper we have taken the simplest of the 2-charge ring states and added one unit of $P$. We have made the wavefunction only in the near ring limit, where the segment of the ring looked like a straight line. But we will get a similar near ring limit from any sufficiently smooth microstate out of the collection used in [18], so our wavefunction adding $P$ should describe the nature of $P$ excitations for any of these 2-charge microstates.

A large class of 3-charge BPS solutions for the black hole and black ring were found in [7, 8]. While the explicit examples studied there had axial symmetry (and thus a nontrivial amount of rotation) one may be able to construct nonrotating solutions by extending such techniques. Thus this approach may lead to generic nonperturbative hair for the black hole as well as for the black ring. It would therefore be very interesting to identify microstates in this approach. In the perturbative construction of the present paper we have excited the NS-NS 2-form gauge field, which was not excited in the solutions of [7, 8]. It would be interesting to find an extension of the solutions of [7, 8] which give nonperturbative hair involving this gauge field.

Acknowledgements

This work is supported in part by DOE grant DE-FG02-91ER-40690. We thank Ashish Saxena for discussions.
A Coordinates for the ring

In this appendix we explain the geometric meaning of the coordinates (A.1) useful in describing the ring, and also obtain the near ring limit used in our analysis. The coordinates we define are constructed on the lines of the coordinates used in [2], and are related to them by a simple transformation.

The D1-D5 geometry (2.2) can be generated by starting with an NS1-P system where the NS1 describes one turn of a uniform helix. Let this helix lie in the $x_1 - x_2$ plane of the noncompact 4-dimensional space $x_1, x_2, x_3, x_4$. We introduce polar coordinates in this space

$$
x_1 = \tilde{r} \sin \tilde{\theta} \cos \tilde{\phi}, \quad x_2 = \tilde{r} \sin \tilde{\theta} \sin \tilde{\phi}, \quad x_3 = \tilde{r} \cos \tilde{\theta} \cos \tilde{\psi}, \quad x_4 = \tilde{r} \cos \tilde{\theta} \sin \tilde{\psi}
$$

Then the coordinates $\bar{r}, \bar{\theta}$ appearing in (2.2) are related to $\tilde{r}, \tilde{\theta}$ by

$$
\tilde{r} = \sqrt{\bar{r}^2 + a^2 \sin^2 \theta}, \quad \cos \tilde{\theta} = \frac{\bar{r} \cos \bar{\theta}}{\sqrt{\bar{r}^2 + a^2 \sin^2 \theta}}, \quad \tilde{\phi} = \bar{\phi}, \quad \tilde{\psi} = \bar{\psi}
$$

In these coordinates the ring is easy to see; the center of the ‘tube’ runs along the circle at $\tilde{r} = a, \tilde{\theta} = \pi/2$. We will start by defining our ring coordinates with the help of these variables, and later convert to the coordinates ($\bar{r}, \bar{\theta}, \bar{\phi}, \bar{\psi}$).

We want to define coordinates near the ring such that the direction along the ring becomes a linear coordinate

$$
z = a \bar{\phi}
$$

We now wish to choose coordinates in the 3-dimensional space perpendicular to the ring. Choose a point $P = (a \cos \bar{\phi}, a \sin \bar{\phi}, 0, 0)$ on the ring. Close to the ring we would like these to be spherical polar coordinates $r, \theta, \phi$ centered at $P$, with the direction $\theta = 0$ pointing towards the center of the ring. Close to the ring the coordinate $r$ should measure distance from the ring, but when $r \sim a$ we will see the diametrically opposite point $P' = (-a \cos \bar{\phi}, -a \sin \bar{\phi}, 0, 0)$ on the ring, and should use a radial coordinate that vanishes at $P'$.

Consider all points that have azimuthal coordinate $\bar{\phi}$ and for these points define

$$
\frac{1}{r} = \frac{1}{2a} \left( \frac{r_p}{r_{p'}} + \frac{r_{p'}}{r_p} \right)
$$

where $r_p, r_{p'}$ measure distances from the points $P, P'$ respectively

$$
r_p = \sqrt{\bar{r}^2 + a^2 - 2a\bar{r} \sin \bar{\theta}}, \quad r_{p'} = \sqrt{\bar{r}^2 + a^2 + 2a\bar{r} \sin \bar{\theta}}
$$

If we approach the point $P$ we have $r_p \to 0$, and

$$
\frac{1}{r} \approx \frac{1}{2a \bar{r}} \approx \frac{1}{r_p}
$$

So we see that $r \approx r_p$ near $P$, and similarly $r \approx r_{p'}$ near $P'$.

Note that

$$r^2 r'^2 = (a^2 + \bar{r}^2)^2 - 4\bar{r}^2 a^2 \sin^2 \bar{\theta} = (a^2 - \bar{r}^2)^2 + 4\bar{r}^2 a^2 \cos^2 \bar{\theta}
$$
Thus
\[ |a^2 - \tilde{r}^2| \leq r_P r_{P'} \]  \hspace{1cm} (A.8)
with equality only for points on the ring diameter passing through \( P, P' \). Thus we can define
\[ \cos \theta = \frac{(a^2 - \tilde{r}^2)}{r_P r_{P'}} \]  \hspace{1cm} (A.9)

Near \( P \) we have
\[ r_P \approx r, \quad r_{P'} \approx 2a, \quad a^2 - \tilde{r}^2 = (a + \tilde{r})(a - \tilde{r}) \approx 2a(a - \tilde{r}) \]  \hspace{1cm} (A.10)

Close to \( P \) we have
\[ \tilde{r} = \sqrt{(x_1^2 + x_2^2) + (x_3^2 + x_4^2)} \approx \sqrt{x_1^2 + x_2^2} \]  \hspace{1cm} (A.11)

where we have kept terms up to linear order in the displacement from \( P \). Thus \( a - \tilde{r} \) measures the distance \( d \) from the \( P \) along the diameter through \( P \) (with \( d \) positive for points inside the ring). We then see that
\[ \cos \theta \approx \frac{(2a)d}{(2a)^r} = \frac{d}{r} \]  \hspace{1cm} (A.12)

and thus \( \theta \) is the desired polar coordinate near \( P \). Finally note that the \( x_3 - x_4 \) plane is perpendicular to the ring and also to the diameter through \( P \), so we define the azimuthal angle
\[ \phi = \tan^{-1} \frac{x_4}{x_3} = \tilde{\psi} \]  \hspace{1cm} (A.13)

Using (A.1) we write the ring coordinates in terms of the coordinates \((\tilde{r}, \tilde{\theta}, \tilde{\phi}, \tilde{\psi})\)
\[ r = a\left(\frac{\tilde{r}^2 + a^2 \cos^2 \tilde{\theta}}{\tilde{r}^2 + a^2 + a^2 \sin^2 \tilde{\theta}}\right), \quad \cos \theta = \frac{a^2 \cos^2 \tilde{\theta} - \tilde{r}^2}{a^2 \cos^2 \tilde{\theta} + \tilde{r}^2}, \quad z = a\tilde{\phi}, \quad \phi = \tilde{\psi} \]  \hspace{1cm} (A.14)

The inverse of these relations gives
\[ \tilde{r}^2 = \frac{a^2 r (1 - \cos \theta)}{a + r \cos \theta}, \quad \sin^2 \tilde{\theta} = \frac{a-r}{a + r \cos \theta}, \quad \tilde{\psi} = \phi, \quad \tilde{\phi} = \frac{z}{a} \]  \hspace{1cm} (A.15)

References

[1] S. D. Mathur, Fortsch. Phys. 53, 793 (2005) [arXiv:hep-th/0502050]; S. D. Mathur, arXiv:hep-th/0510180.

[2] H. Elvang, R. Emparan, D. Mateos and H. S. Reall, Phys. Rev. Lett. 93, 211302 (2004) [arXiv:hep-th/0407065]; H. Elvang, R. Emparan, D. Mateos and H. S. Reall, Phys. Rev. D 71, 024033 (2005) [arXiv:hep-th/0408120].

[3] I. Bena and P. Kraus, JHEP 0412, 070 (2004) [arXiv:hep-th/0408186]; M. Cyrier, M. Guica, D. Mateos and A. Strominger, Phys. Rev. Lett. 94, 191601 (2005) [arXiv:hep-th/0411187]; R. Emparan and D. Mateos, Class. Quant. Grav. 22, 3575 (2005) [arXiv:hep-th/0506110].
[4] I. Bena and P. Kraus, Phys. Rev. D 70, 046003 (2004) [arXiv:hep-th/0402144]; I. Bena, Phys. Rev. D 70, 105018 (2004) [arXiv:hep-th/0404073]; D. Bak, K. Kim and N. Ohta, arXiv:hep-th/0511051.

[5] S. D. Mathur, A. Saxena and Y. K. Srivastava, Nucl. Phys. B 680, 415 (2004) [arXiv:hep-th/0311092].

[6] S. Giusto, S. D. Mathur and A. Saxena, Nucl. Phys. B 701, 357 (2004) [arXiv:hep-th/0405017]; S. Giusto, S. D. Mathur and A. Saxena, arXiv:hep-th/0406103; O. Lunin, JHEP 0404, 054 (2004) [arXiv:hep-th/0404006].

[7] I. Bena and N. P. Warner, arXiv:hep-th/0505166.

[8] P. Berglund, E. G. Gimon and T. S. Levi, arXiv:hep-th/0505167.

[9] J. P. Gauntlett, J. B. Gutowski, C. M. Hull, S. Pakis and H. S. Reall, Class. Quant. Grav. 20, 4587 (2003) [arXiv:hep-th/0209114]; J. B. Gutowski, D. Martelli and H. S. Reall, Class. Quant. Grav. 20, 5049 (2003) [arXiv:hep-th/0306235]; I. Bena and N. P. Warner, arXiv:hep-th/0408106; I. Bena, C. W. Wang and N. P. Warner, arXiv:hep-th/0411072.

[10] V. Balasubramanian, J. de Boer, E. Keski-Vakkuri and S. F. Ross, Phys. Rev. D 64, 064011 (2001) [arXiv:hep-th/0011217].

[11] J. M. Maldacena and L. Maoz, JHEP 0212, 055 (2002) [arXiv:hep-th/012025].

[12] J. M. Maldacena and A. Strominger, JHEP 9812, 005 (1998) [arXiv:hep-th/9804085].

[13] O. Lunin and S. D. Mathur, Nucl. Phys. B 623, 342 (2002) [arXiv:hep-th/0109154].

[14] “Monopole harmonics” T. T. Wu and C. N. Yang, Nucl. Phys. B 107, 365 (1976); H. A. Olsen, P. Osland and T. T. Wu, Phys. Rev. D 42, 665 (1990); E. J. Weinberg, Phys. Rev. D 49, 1086 (1994) [arXiv:hep-th/9308054].

[15] S. Giusto, S. D. Mathur and Y. K. Srivastava, arXiv:hep-th/0510235.

[16] E. Witten, JHEP 0204, 012 (2002) [arXiv:hep-th/0012054].

[17] M. Mihailescu, I. Y. Park and T. A. Tran, Phys. Rev. D 64, 046006 (2001) [arXiv:hep-th/0011107].

[18] O. Lunin and S. D. Mathur, Phys. Rev. Lett. 88, 211303 (2002) [arXiv:hep-th/0202072].

[19] N. Iizuka and M. Shigemori, JHEP 0508, 100 (2005) [arXiv:hep-th/0506215].

[20] V. Balasubramanian, P. Kraus and M. Shigemori, Class. Quant. Grav. 22, 4803 (2005) [arXiv:hep-th/0508110].

[21] A. Dabholkar, N. Iizuka, A. Iqubal and M. Shigemori, arXiv:hep-th/0511120.

[22] L. F. Alday, J. de Boer and I. Messamah, arXiv:hep-th/0511246.

[23] O. Lunin and S. D. Mathur, Nucl. Phys. B 610, 49 (2001) [arXiv:hep-th/0105136].

30