Anomalous Drag in Double Bilayer Graphene Quantum-Hall Superfluids

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Introduction.—Because of kinetic energy quantization, states with broken symmetries, often referred to generically as quantum Hall ferromagnets, are common whenever several Landau levels are close to degeneracy and the total Landau level filling factor is an integer. In semiconductor double-layers, exciton condensate states are characterized by spontaneous interlayer phase coherence, counterflow superfluidity \cite{1-6}, and other unique anomalous transport properties, appear when the total Landau level filling factor is an odd integer and the states at the Fermi level have the same orbital character in both layers \cite{7}. In this Letter we address the interesting case in which one or both sides of the double layer are formed by bilayer graphene (BLG) systems with a low energy $N = 0$ Landau level. The 8-fold degenerate $N = 0$ Landau level multiplet of BLG is the direct product of real spin, valley pseudospin, and $n = 0, 1$ orbital pseudospin doublets. We show that in this case the exciton superfluid phase stiffness vanishes while the charged excitation gap remains finite whenever the orbital doublet in one or both layers approaches exact degeneracy. Because a large negative drag is expected whenever the resistivity for collective counterflow transport exceeds the resistivity for parallel quasiparticle charge flow, we argue that this property is behind recent discovery of strong anomalous drag near a $n = 0/1$ degeneracy point.

Bilayer Ground States.—The phase diagram of a bilayer system with a $n = 0$ orbital Landau level in the top layer and $n = 0, 1$ close to degeneracy in the bottom layer is illustrated in Fig. 1. To construct this phase diagram we have assumed that both real spin and valley pseudospin are polarized, allowing us to focus on orbital pseudospin, and compared the energies of translationally invariant single Slater determinant full Landau level states with arbitrary orbital content,

\[
|\Psi(z)| = \prod_X \left( \sum_{\alpha,n} z_{\alpha n} c_{\alpha n, X}^d \right)|\text{vac}\rangle. \quad (1)
\]

In Eq. 1 $|\text{vac}\rangle$ is the vacuum state in which the Landau levels of interest are empty, $X$ is the guiding center label of the individual quantum states within a Landau level, $c_{\alpha n, X}^d$ is the creation operator for a state with layer index $\alpha = t, b$ and orbital index $n = 0, 1$, and the $z_{\alpha n}$ are the components of the ground state layer/orbital spinor which satisfies the normalization constraint: $\sum_{\alpha,n} |z_{\alpha n}|^2 = 1$. Extrema of $E[z] = \langle \Psi[z]|H|\Psi[z]\rangle$ are eigenstates of the mean field Hamil-
tonian,
\[ \mathcal{H}_{\text{MF}}^{\alpha \beta} = \left[ \epsilon_{\alpha n} + \left( \sum_{n_1} |z_{\alpha n_1}|^2 - \frac{1}{2} \frac{\delta_{\alpha \beta}}{\ell} \right) \delta_{\alpha \beta} \delta_{nn'} \right] + \sum_{n_1, n_2} X_{\alpha \beta}^{n_1 n_2 n' n_2} z_{\alpha n_1} z_{\beta n_2}^*. \] (2)

When projected to the valence Landau levels of interest the many-body Hamiltonian \( \mathcal{H} \) includes only \( X \)-independent single-particle energies and Coulomb interactions. In Eq. 2 \( \epsilon_{\alpha n} \) is an orbital-dependent single-particle energy, which can be tuned by adjusting gate voltages [8] and includes a self-energy contribution [9] due to exchange interactions with occupied Dirac sea Landau levels. The second term in square brackets in Eq. 2 is the Hartree self-energy, the final term is the exchange self-energy, \( d \) is spacing between the two bilayers, and energies are in units of \( e^2/(\epsilon \ell) \), where \( \epsilon \) is the background dielectric constant. The exchange integrals in Eq. 2 are given by [10, 11]

\[ X_{\alpha \beta}^{n_1 n_2 n' n_2} = \int \frac{d^2 q}{(2\pi)^2} V_{\alpha \beta}(q) F_{n_1}(q) F_{n_2}(q) (-q) \] (3)

where \( V_{\alpha \beta}(q) = 2\pi e^2/\epsilon(q) \exp(-qd_{\alpha \beta}) \), \( F_{n_1}(q) \) is an orbital dependent form factor, and \( d_{\alpha \beta} = d(1-\delta_{\alpha \beta}) \). For many values of \( \epsilon_{b1} - \epsilon_{b0} \) and \( \epsilon_{b0} - \epsilon_{b0} \), there is more than one self-consistent solutions of these mean-field equations, and the ground state must be determined by comparing extremal energies.

Phase stiffness of the Monolayer/Bilayer Exciton Condensate.—Corrections can be added to these mean-field states by deriving a quantum fluctuation Hamiltonian. For this purpose we expand the layer/orbital spinors in terms of the mean-field theory eigenspinors:

\[ |m, X\rangle = \sum_{\alpha, n} a_{\alpha n}^m |\alpha, n, X\rangle, \quad m = 0, 1, 2 \] (4)

where \( a_{\alpha n}^m \) is the eigenvector of the self-consistent mean-field Hamiltonian, the label \( m = 0 \) is reserved for the occupied lowest energy spinor, and \( m = 1, 2 \) for the two unoccupied excited state spinors. Note \( a_{\alpha n}^m \) is always zero because we assume that the \( m = 1 \) orbital energy is close to degeneracy only in the bottom layer.

The fluctuation energy functional is constructed by considering instantaneous Slater determinants

\[ |\Psi[z]\rangle = \prod_X \left( \sum_{m=0}^2 z_{m, X} c_{m, X}^\dagger |\text{vac}\rangle, \right) \] (5)

with guiding-center dependent orbitals that contain small admixtures of higher energy mean-field eigenspinors. To second order in the fluctuation amplitudes \( z_{1, X} \) and \( z_{2, X} \), the fluctuation energy is specified by the kernel

\[ \mathcal{K}_{ij}(X - X') = -\frac{\partial^2 \mathcal{E}[Z]}{\partial Z_{i, X} \partial Z_{j, X'}}. \] (6)

In Eq. 6 we have defined \( Z_{i, X} = \{ z_{1, X}, z_{1, X}^*, z_{2, X}, z_{2, X}^* \} \). Explicit forms for the total energy and for the fluctuation kernel are listed in the supplemental material. Because the elements of the fluctuation kernel depend only on the difference between guiding centers they can be Fourier transformed and are most conveniently expressed in terms of \( K_{ij}(q) = \sum_X \mathcal{E}_{ij}(X) \).

Our main interest here is in the green region of Fig. 1, where the mean field ground state is an exciton condensate with spontaneous phase coherence between \( n = 0 \) orbitals in the top and bottom layers, empty \( n = 1 \) orbitals, and counterflow superfluidity which we discuss at greater depth below. Concentrating first on the \( \epsilon_{b0} = 0 \) line, along which the top and bottom layers have equal weight, the bilayer exciton condensate state remains stable at the mean-field level even when \( \epsilon_{b1} < \epsilon_{b0} \) because the exchange energy between \( n = 0 \) orbitals is larger than the exchange energies that involve \( n = 1 \) orbitals. For \( \epsilon_{b1} - \epsilon_{b0} < -0.07e^2/\epsilon \ell \), the dark blue region marked as “mixed all” in Fig. 1, the mean field theory spinors have non-zero projections onto all three orbitals.

In Fig. 2 we plot eigenvalues of the fluctuation kernel \( K_{ij}(q) \) of the \( n = 0 \) bilayer exciton condensate state for \( \epsilon_{b0} = 0 \) and several values of \( \epsilon_{b1} \). (The relationship of these eigenvalues to the collective mode energies is explained in the supplemental material.) At quadratic order, fluctuations that influence the interlayer phase (solid lines) and density balance (dashed lines) decouple. If we choose the mean-field state to have the same phase in both layers, the basis functions for phase and density fluctuations are

\[ z_{1, X}^\pm = (z_{1, X} \pm z_{1, X}^*)/\sqrt{2}, \]
\[ z_{2, X}^\pm = (z_{2, X} \pm z_{2, X}^*)/\sqrt{2}. \] (7)
where the “+” sign corresponds to density and the “−” sign to phase modes. The \( i = 1 \) excited state is an antisymmetric \( n = 0 \) bilayer state, and the \( i = 2 \) excited state is a \( n = 1 \) orbital state localized in the bottom layer. The vanishing energy of the phase mode as \( q \to 0 \), reflects the continuously broken interlayer phase \( U(1) \) symmetry. The coefficient of \( q^2 \) in the phase mode energy is the superfluid phase stiffness, which is the key microscopic property of the exciton condensate \cite{12} because it controls the Kosterlitz-Thouless phase transition temperature and the relationship between counterflow superfluid currents and interlayer phase gradients. Our calculations show that the superfluid phase stiffness decreases as \( \epsilon_{b1} \to \epsilon_{b0} \) from above, vanishing along the \( \epsilon_{b1} = \epsilon_{b0} \) black dashed line in Fig. 1.

Because of kinetic-energy quenching in a strong magnetic field, the superfluid phase stiffness of quantum Hall bilayer exciton condensates is entirely due to electron-electron interactions, with no contribution from electron-hole pair kinetic energy. To explain why the superfluid phase stiffness vanishes when \( \epsilon_{b1} \to \epsilon_{b0} \), we perform an analytic Taylor series expansion of the phase subspace of the fluctuation kernel in powers of \( q = |q| \) for \( \epsilon_{b0} = \epsilon_{b} \) which yields

\[
\mathcal{K}^-(q) = \left( \frac{f(d)}{f(d)q^2} - \frac{i f(d)q^\ell}{f(d)q^\ell (\epsilon_{b1} - \epsilon_{b0}) + f(d) + \Delta(d)q^2\ell^2} \right).
\]

Here \( f(d) = -\frac{d}{2} + \frac{1}{2} \sqrt{2(1 + \frac{d}{2})} e^{\frac{\sigma^2}{2\ell^2}} \text{Erfc}(\frac{d}{\sqrt{2\ell^2}}) \), and \( \Delta(d) \) are layer-separation dependent positive constants. To second order in \( q \), the lower eigenenergy of \( \mathcal{K}^{-}(q) \) is

\[
E^{-}(q) = \frac{f(d)(\epsilon_{b1} - \epsilon_{b0})}{f(d) + (\epsilon_{b1} - \epsilon_{b0})} q^2\ell^2,
\]

which vanishes as \( \epsilon_{b1} \to \epsilon_{b0} \) from above as illustrated in Fig. 2. For \( \epsilon_{b1} - \epsilon_{b0} \gg f(d) \) the coefficient of \( q^2\ell^2 \) in Eq. 8, which is the superfluid phase stiffness, approaches \( f(d) \), its standard bilayer limit.

When the interlayer phase has a spatial gradient \cite{13} nearby guiding centers are in different bilayer states, reducing the magnitude of the attractive interaction exchange. Because the transverse orbital of the \( n = 1 \) guiding center state is the derivative of the \( n = 0 \) orbital, mixing \( n = 1 \) orbitals relaxes the constraint that locks the Landau gauge guiding center label to the wavefunction-maximum. The structure of Eq. 8, in which the same quantity \( f(d) \) appears in both the \((1,1)\) and \((2,2)\) diagonal and the \((1,2)\) and \((2,1)\) off-diagonal matrix elements, reflects the property that the exchange energy cost of small phase gradients can be completely eliminated. The superfluid phase stiffness is therefore finite only if there is a single-particle energy cost of mixing \( n = 1 \) orbitals into the ground state.

**Negative Drag.**—Even though the superfluid phase stiffness approaches zero as \( \epsilon_{b1} \to \epsilon_{b0} \), the condensation energy of the excitonic state and its charge excitation gap remain large. In this case we expect that the excitonic character of the many-particle ground state will remain intact in this region of the phase diagram. The small superfluid phase stiffness then implies a broad region of temperature in which charged excitations are dilute and the Kosterlitz-Thouless transition temperature of the exciton fluid, which is bounded by the zero temperature superfluid density \( \rho_0 \), is substantially exceeded:

\[
k_B T_{KT} < \frac{\pi}{2} \rho_0 = \frac{\pi}{2} \frac{f(d)(\epsilon_{b1} - \epsilon_{b0})}{f(d) + (\epsilon_{b1} - \epsilon_{b0})}.
\]

\( f(d) \approx 0.07 \, e^2/\ell \) for \( d/\ell = 1 \) and decreases monotonically as interlayer spacing \( d \) is increased. Under these circumstances exciton currents are not supercurrents, but can still contribute to transport.

Exciton chemical potential gradients, i.e. differences between the electrochemical potential gradients in the top and bottom layers, will drive exciton currents. We choose to characterize the counterflow current response by an exciton conductivity defined by:

\[
j_t - j_b = \sigma_{ex} \frac{(E_t - E_b)}{2}
\]

were \( E \) and \( j \) are the electrochemical potential gradients and currents, and the subscripts \( t \) and \( b \) refer to the top and bottom layers. Similarly, charged quasiparticle currents are carried in parallel by the two layers, sensitive only to the average of the electrochemical potential gradients in the two layers, and characterized by a quasiparticle conductivity:

\[
j_t + j_b = \sigma_{QP} \frac{(E_t + E_b)}{2}
\]

It follows that when current flows only in the top layer

\[
E_t = (\rho_{QP} + \rho_{ex}) \, j \equiv \rho_{\text{Drive}} \, j
\]

where \( \rho_{ex} = (\sigma_{ex})^{-1} \) and \( \rho_{QP} = (\sigma_{QP})^{-1} \) are the exciton and quasiparticle resistivities. The drag voltage measured in the bottom layers is then

\[
E_b = (\rho_{QP} - \rho_{ex}) \, j \equiv \rho_{\text{Drag}} \, j.
\]

When the exciton fluid condenses into a two-dimensional counterflow superfluid, \( \rho_{ex} \) vanishes for small currents and the longitudinal and drag resistivities are identical. This limit is often closely approached experimentally in bilayer exciton condensates. For this reason large positive drag voltages are routinely used as a fingerprint of exciton condensates. Since the charge gap does not change appreciably, we do not expect that \( \rho_{QP} \) will vary strongly as the \( \epsilon_{b1} = \epsilon_{b0} \) boundary of the green region of Fig. 1 is approached. On the other hand, \( \rho_{ex} \) becomes finite when the ambient temperature exceeds \( T_{KT} \), and we believe that it can become large as we explain below.
The ground state of density-balanced quantum Hall excitonic superfluids in the quantum Hall regime can be viewed as a fluid of excitons that interact weakly\cite{30} in the limit of small layer separations ($d/\ell < 1$) appropriate to graphene-based double layer systems. In this picture both electrons and holes have density $n_{ex} = (\pi \ell^2)^{-1}$. For a two-dimensional system of interacting bosons the superfluid phase stiffness

$$\rho = \frac{\hbar^2 n_s}{2 m^*}, \quad (15)$$

where $n_s$ is the boson density and $m^*$ is the particle mass. Since the exciton density is constant as a function of band parameters, it follows that the phase stiffness in double bilayer systems vanishes as $\epsilon_{61} \to \epsilon_{60}$ from above not because the exciton density $n_{ex}$ vanishes, but because the exciton mass ($m^* \to m_{ex}$) diverges. In a simple Drude picture the exciton conductivity

$$\sigma_{ex} = \frac{n_{ex} e^2 v_{ex}^2}{m_{ex}} = \frac{e^2}{\hbar} \frac{4 \pi \rho \tau_{ex}}{\hbar} = \frac{\sigma_0^{QP} \tau_{ex} m_{QP}}{2 \tau_{QP} m_{ex}}, \quad (16)$$

where $\sigma_0^{QP}$, $m_{QP}$, and $\tau_{QP}$ are the values appropriate for the quasiparticle system in the absence of a magnetic field. We conclude that at any fixed temperature $\sigma_{ex}$ should become smaller than $\sigma^{QP}$ when $m_{ex}$ diverges as $\epsilon_{61} \to \epsilon_{60}$, and the drag resistance should become large and negative.

**Stripe phase instability.**—When $\epsilon_{61} < \epsilon_{60}$ the phase energy kernel has the form $-4q^2 + Bq^4$, becoming negative over a finite range of small q values as shown in the inset of Fig. 2. To identify the nature of the states implied by this instability, we have performed self-consistent mean field calculations that allow translational symmetry to be broken along one direction, taken be the $x$ direction. We find that the resulting stripe states have their lowest energies when their periods in $x$ are close to $2\pi/q^*$ where $q^*$ is the value of $q$ at which the phase kernel eigenvalue reaches its minimum, Fig. 3 illustrates the variation in the guiding center spinors, whose orbital content corresponds to the eigenvector of the negative eigenvalue phase mode. As one passes through the stripe state red region of Fig. 1 from right to left, the $n = 1$ orbital content of the wave function increases.

**Discussion.**—In systems of fermions with attractive effective interactions, it is known that the crossover from the BCS-theory weak interaction limit, to the BEC limit of weakly interacting boson limit is smooth. The microscopic physics of condensed electron-hole pairs in the quantum Hall regime is distinct from the familiar BCS-BEC crossover paradigm because of the way in which Landau quantization cuts off the many-particle Hilbert space. The elementary excitation spectrum at long wavelengths consists only of bosonic collective modes formed by electron-hole pairs that are more and more weakly bound as wavelengths shorten. Although strong-positive drag signals suggesting excitonic superfluidity have been observed regularly, there have been few observations that signal transport contributions from uncondensed bosonic excitations. In this Letter we have shown that the superfluid phase stiffness of double bilayer quantum Hall exciton condensates vanishes as $n = 0$ and $n = 1$ orbitals approach degeneracy in one of the layers. We associate the increase in stiffness with a diverging exciton mass which, we argue, will also lead to a diverging excitonic counterflow resistivity and to large negative drag resistivities. Indeed large negative drag signals do\cite{31} appear in double bilayer graphene near narrow regions of gate voltage settings close to orbital degeneracy conditions. If our interpretation of the drag anomalies is correct, the sign of the drag is determined by a competition between quasiparticle resistivities that diverge as $T \to 0$ for any gate setting, and exciton resistivities that diverge at any temperature as gate settings are tuned to orbital degeneracy.

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[1] H. A. Fertig, Phys. Rev. B 40, 1087 (1989).
[2] X. G. Wen and A. Zee, Phys. Rev. Lett. 69, 1811 (1992).
[3] A. H. MacDonald, P. M. Platzman, and G. S. Boebinger, Phys. Rev. Lett. 65, 775 (1990).
[4] A. H. MacDonald and E. H. Rezayi, Phys. Rev. B 42, 3224(R) (1990).
[5] J. P. Eisenstein and A. H. MacDonald, Nature 432, 691 (2004).
[6] E. Tutuc, M. Shayegan, and D. A. Huse, Phys. Rev. Lett. 93, 036802 (2004).
[7] T. Jungwirth and A. H. MacDonald, Phys. Rev. B 63, 035305 (2000).
[8] E. McCann and V. I. Fal’ko, Phys. Rev. Lett. 96, 086805 (2006).
[9] K. Shizuya, Phys. Rev. B 81, 075407 (2010).
[10] Y. Barlas, R. Côté, J. Lambert, and A. H. MacDonald, Phys. Rev. Lett. 104, 096802 (2010).
[11] R. Côté, J. Lambert, Y. Barlas, and A. H. MacDonald,
Phys. Rev. B 82, 035445 (2010).
[12] K. Yang et al., Phys. Rev. B 54, 11644 (1996).
[13] L. Radzihovsky, Phys. Rev. Lett. 87, 236802 (2001).
[14] M. Abolfath, L. Radzihovsky, and A. H. MacDonald, Phys. Rev. B 65, 233306 (2002).
[15] M. Abolfath, A. H. MacDonald, and L. Radzihovsky, Phys. Rev. B 68, 155318 (2003).
[16] R. Côté and A. H. MacDonald, Phys. Rev. B 44, 8759 (1991).
[17] K. Nomura, S. Ryu, and D.-H. Lee, Phys. Rev. Lett. 103, 216801 (2009).
[18] X. Liu, K. Watanabe, T. Taniguchi, B. I. Halperin and P. Kim, Nat. Phys. 13, 746 (2017).
[19] J. I. A. Li, T. Taniguchi, K. Watanabe, J. Hone and C. R. Dean, Nat. Phys. 13, 751 (2017).
[20] B. M. Hunt et al., Nat. Comm. 8, 948 (2017).
[21] J. I. A. Li et al., Phys. Rev. Lett. 117, 046802 (2016).
[22] Y. Zhao, P. Cadden-Zimansky, Z. Jiang, and P. Kim, Phys. Rev. Lett. 104, 066801 (2010).
[23] J.-J. Su and A. H. MacDonald, Phys. Rev. B 95, 045416 (2017).
[24] M. Kellogg, J. P. Eisenstein, L. N. Pfeiffer, and K. W. West, Phys. Rev. Lett. 93, 036801 (2004).
[25] G. M. Rutter et al., Nat. Phys. 7, 649 (2011).
[26] D. Nandi, A. D. K. Finck, J. P. Eisenstein, L. N. Pfeiffer, and K. W. West, Nature 488, 481 (2012).
[27] S. A. Brazovskii, Zh. Eksp. Teor. Fiz. 68, 175 (1975).
[28] K. Lee et al., Science 345, 58 (2014).
[29] A. F. Young et al., Nat. Phys. 8, 550 (2012).
[30] I.V. Lerner and I.E. Lozovik, Zh. Eksp. Teor. Fiz. 80, 1488 (1981).
[31] J.I.A. Li, Q. Shi, T. Cheng, K. Watanabe, T. Taniguchi, J. Hone, C.R. Dean, to be published.
Supplemental material

I. Total energy functional and fluctuation kernel matrix

We derive the total energy functional of fluctuations around mean field ground states and the fluctuation kernel matrix elements. For given band parameters \( \epsilon_{\alpha,n} \), self-consistent mean field theory predicts eigenstate wavefunctions \( \{ \phi_{n}^{m} \} \) for both the ground state \( (m = 0) \) and the excited states \( (m = 1, 2) \). The many-body wavefunction of collective fluctuations are given by Eq. 5 where the excited states are mixed into the ground state wavefunction leading to the instantaneous Slater determinant states. The total energy functional can be written as

\[
\mathcal{E}[\{z,z^{*}\}] = \mathcal{E}_{0}[\{z,z^{*}\}] + \mathcal{E}_{H}[\{z,z^{*}\}] + \mathcal{E}_{F}[\{z,z^{*}\}]
\]

where

\[
\mathcal{E}_{0}[\{z,z^{*}\}] = \sum_{m',m,X} \sum_{\alpha,n} \epsilon_{\alpha,n} a_{\alpha,n}^{m'} a_{\alpha,n}^{m} z_{m',X}^{*} z_{m,X}
\]

is the single-particle energy and the sign function.

The fluctuation kernel is the coefficient of second order expansion of the total energy around the mean field ground state. For small amplitude fluctuations, i.e. \( |z_{m=1,2}| \ll 1 \) and \( |z_{m=0}| \approx 1 \), the normalization condition up to second order in the amplitudes can be expressed as

\[
z_{0,X} \approx 1 - \frac{1}{2} |z_{1,X}|^2 - \frac{1}{2} |z_{2,X}|^2
\]

where we have chosen the phase of \( z_{0,X} \) to be zero anywhere. By substituting Eq. 23, we can rewrite the total energy functional which then depends only on \( Z_{i,X} = \{ z_{1,X}, z^{*}_{1,X}, z_{2,X}, z^{*}_{2,X} \} \) where \( i = 1, \ldots A \). By taking second order derivative with respect to \( Z_{i,X} \) and Fourier transformation in guiding center coordinate \( X \), we obtain the following independent fluctuation kernel matrix elements.

\[
\begin{align*}
\mathcal{F}_{mm'}(q) &= \sum_{n,n'} a_{\alpha,n}^{m} a_{\alpha,n'}^{m'} F_{nn'}(q) \\
F_{nn'}(q) &= \left( \frac{n_<}{n_>} \right)^{1/2} \left[ \frac{\{\text{sgn}(n-n')q_{y} + iq_{z}\}}{\sqrt{2}} \right]^{n_<^{*} - n_<} \\
&\quad \times L_{n_<^{*} - n_<}^{n_<} \left( \frac{q_{y}^{2} + q_{z}^{2}}{2} \right) \exp \left( -\frac{q_{y}^{2} + q_{z}^{2}}{4} \right)
\end{align*}
\]

where \( n_>^{*} \) is the greater (less) one of \( n \) and \( n' \). \( L_{n}^{m}(x) \) is the generalized Laguerre polynomial and \( \text{sgn}(x) \)

For \( i, j = 2, 4 \) and \( j = 1, 3 \),

\[
\mathcal{K}_{ij}(q) = \sum_{\alpha,n} \epsilon_{\alpha,n} (a_{\alpha,n}^{m} a_{\alpha,n}^{m'} - a_{\alpha,n}^{0} a_{\alpha,n}^{0} \delta_{mm'})
\]

\[
+ \frac{1}{L_{y}} \sum_{n_{1}^{*}n_{2}^{*}} \sum_{\alpha} H_{n_{1}^{*}n_{2}^{*}}^{\alpha \beta} (0) (a_{\alpha,n_{1}^{*}}^{m} a_{\alpha,n_{2}^{*}}^{m'} - a_{\alpha,n_{1}^{*}}^{0} a_{\alpha,n_{2}^{*}}^{0} \delta_{mm'}) a_{\alpha,n_{1}^{*}}^{0} a_{\alpha,n_{2}^{*}}^{0} = H_{n_{1}^{*}n_{2}^{*}}^{\alpha \beta} (q) a_{\alpha,n_{1}}^{m} a_{\alpha,n_{2}}^{0} a_{\alpha,n_{2}}^{0}
\]

\[
- \frac{1}{L_{y}} \sum_{n_{1}^{*}n_{2}^{*}} \sum_{\alpha} J_{n_{1}^{*}n_{2}^{*}}^{\alpha \beta} (0) (a_{\alpha,n_{1}}^{m} a_{\alpha,n_{2}}^{m'} - a_{\alpha,n_{1}}^{0} a_{\alpha,n_{2}}^{0} \delta_{mm'}) a_{\alpha,n_{1}}^{0} a_{\alpha,n_{2}}^{0} = J_{n_{1}^{*}n_{2}^{*}}^{\alpha \beta} (q) a_{\alpha,n_{1}}^{m} a_{\alpha,n_{2}}^{0} a_{\alpha,n_{2}}^{0},
\]
and for \(i,j = 1,3\), we have

\[
\mathcal{K}_{ij}(q) = \frac{1}{L_y} \sum_{n_1' n_2'} \sum_{\alpha \beta} H^{\alpha \beta}_{n_1' n_2'}(q) a_{\alpha n_1'} a_{\beta n_2'} a_{\alpha n_1} a_{\beta n_2} - \frac{1}{L_y} \sum_{n_1' n_2'} \sum_{\alpha \beta} I^{\alpha \beta}_{n_1' n_2'}(q) a_{\alpha n_1} a_{\alpha n_1'} a_{\beta n_2} a_{\beta n_2'},
\]

(25)

where \(H\) and \(I\) are interaction integrals defined as

\[
H^{\alpha \beta}_{n_1' n_2'}(q) = \frac{1}{2\pi \ell^2} V_{\alpha \beta}(q) F_{n_1'}(q) F_{n_2'}(-q)
\]

(26)

\[
I^{\alpha \beta}_{n_1' n_2'}(q) = \int \frac{dp}{(2\pi)^2} e^{ipq\sqrt{\ell^2}} V_{\alpha \beta}(p) F_{n_1'}(p) F_{n_2'}(-p)
\]

(27)

The first line of Eq. 24 is the single-particle term whereas the second and third lines correspond to Hartree and Fock interaction contributions, respectively. In contrast, Eq. 25 has only interaction terms, the first and the second of which correspond to Hartree and Fock terms, respectively. The rest of the matrix elements can be found by Hermitian conjugation relation \(\mathcal{K}_{ij}(q) = \mathcal{K}_{ji}(q)\).

II. Eigenenergies of the Fluctuation Kernel \(\mathcal{K}(q)\)

In this section, we present examples of the quadratic fluctuation and collective mode energies for several different regions of the double-bilayer phase diagram illustrated in Fig. 1.

II. A. Fully Polarized in Top Layer

When the top layer Landau \(n = 0\) is lowest in energy by a sufficiently large margin, the ground state is both layer and orbital polarized to the \(n = 0\) orbital of the top layer. For example for \(\epsilon_{b0} = 0.9\) and \(\epsilon_{b1} = 1.1\), the mean field eigenstates are (ascending order in eigenenergy):

\[
\begin{align*}
|m = 0\rangle &= |t, n = 0\rangle, \\
|m = 1\rangle &= |b, n = 0\rangle, \\
|m = 2\rangle &= |b, n = 1\rangle.
\end{align*}
\]

(28)

The fluctuation kernel which specifies the energies of transitions from the \(|m = 0\rangle\) level to the \(|m = 1\rangle\) and \(|m = 2\rangle\) levels is plotted in Fig. 4.

II. B. Ising Quantum Hall Ferromagnet

In this region mean-field predicts ground state that are mixtures of \(|t, n = 0\rangle\) and \(|b, n = 1\rangle\). We take, for example, the single particle energies \(\epsilon_{b0} = 1\) and \(\epsilon_{b1} = 0.1\). The mean field eigenstates are

\[
\begin{align*}
|m = 0\rangle &= 0.90|t, n = 0\rangle + 0.43|b, n = 1\rangle, \\
|m = 1\rangle &= 0.43|t, n = 0\rangle - 0.90|b, n = 1\rangle, \\
|m = 2\rangle &= |b, n = 0\rangle.
\end{align*}
\]

(29)

The eigenenergy of fluctuations to \(|m = 1\rangle\) and \(|m = 2\rangle\) states is plotted in Fig. 5. The negative eigenenergy shows that the mean-field ground state in this region is not stable.
II. C. $n = 1$ Layer and Orbitally Polarized state

In the region where ground state is both layer and orbital polarized to $|b, n = 1\rangle$. We set, for example, $\epsilon_{b0} = 0.5$ and $\epsilon_{b1} = -1$. The mean field eigenstates are

$$
|m = 0\rangle = |b, n = 1\rangle, \\
|m = 1\rangle = |t, n = 0\rangle, \\
|m = 2\rangle = |b, n = 0\rangle.
$$

(30)

The eigenenergy of fluctuations to $|m = 1\rangle$ and $|m = 2\rangle$ states is plotted in Fig.6.

![FIG. 6. Eigenenergy of fluctuation kernel](image)

FIG. 6. Eigenenergy of fluctuation kernel $K(\ell)$ at $\epsilon_{b0} = 0.5$ and $\epsilon_{b1} = -1$. Red line is two fold degenerate and correspond to degenerate modes of $z_{1, q}$, $z_{1, -q}$ and blue lines to modes $(z_{2, q} \pm z_{2, -q})/\sqrt{2}$.

II. D. Layer Polarized Mixed Orbital State

We consider the region where mean-field ground state is mixture of $|b, n = 0\rangle$ and $|b, n = 1\rangle$. As an example, we set $\epsilon_{b0} = -1$ and $\epsilon_{b1} = -1.3$. The mean field eigenstates are

$$
|m = 0\rangle = 0.21|b, n = 0\rangle + 0.98|b, n = 1\rangle, \\
|m = 1\rangle = 0.98|b, n = 0\rangle - 0.21|b, n = 1\rangle, \\
|m = 2\rangle = |t, n = 0\rangle.
$$

(31)

The eigenenergy of fluctuations to $|m = 1\rangle$ and $|m = 2\rangle$ states is plotted in Fig.7.

![FIG. 7. Eigenenergy of fluctuation kernel $K(\ell)$](image)

FIG. 7. Eigenenergy of fluctuation kernel $K(\ell)$ at $\epsilon_{b0} = -1$ and $\epsilon_{b1} = -1.3$. Red lines correspond to linear superposition of modes $z_{1, q}$, $z_{1, -q}$ and blue lines are two fold degenerate and correspond to degenerate modes of $z_{2, q}$, $z_{2, -q}$.

II. E. Layer and orbital polarized state

Setting $\epsilon_{b0} = -1$ and $\epsilon_{b1} = 0$ leads to the mean field eigenstates:

$$
|m = 0\rangle = |b, n = 0\rangle, \\
|m = 1\rangle = |t, n = 0\rangle, \\
|m = 2\rangle = |b, n = 1\rangle.
$$

(32)

The eigenenergy of fluctuations to $|m = 1\rangle$ and $|m = 2\rangle$ states is plotted in Fig.8.

![FIG. 8. Eigenenergy of fluctuation kernel $K(\ell)$](image)

FIG. 8. Eigenenergy of fluctuation kernel $K(\ell)$ at $\epsilon_{b0} = -1$ and $\epsilon_{b1} = 0$. Red line is two fold degenerate and correspond to degenerate modes of $z_{1, q}$, $z_{1, -q}$ and blue lines to modes $(z_{2, q} \pm z_{2, -q})/\sqrt{2}$.

II. F. “Mixing all” region

In this region, mean field calculation predicts ground state to be a mixture of $|t, n = 0\rangle$, $|b, n = 0\rangle$ and $|b, n = 1\rangle$. As an example, we set $\epsilon_{b0} = 0$ and $\epsilon_{b1} = -0.2$.

The eigenenergy of fluctuations to $|m = 1\rangle$ and $|m = 2\rangle$ states is plotted in Fig.9. As expected, this mean field ground state is not stable.
FIG. 9. Eigenenergy of fluctuation kernel $K(q)$ at $\epsilon_{00} = 0$ and $\epsilon_{01} = -0.2$. Each eigenstate branch is a mixture of all four modes $z_{1,q}$, $z_{1,-q}^*$, $z_{2,q}$, $z_{2,-q}^*$. 