ENTIRE SIGN-CHANGING SOLUTIONS TO THE FRACTIONAL CRITICAL SCHRÖDINGER EQUATION

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ABSTRACT. We consider the fractional critical Schrödinger equation (FCSE)

\[- \Delta^s u - |u|^{2^*_s - 2} u = 0, \]

where \( u \in \dot{H}^s(\mathbb{R}^N) \), \( N \geq 2 \), \( 0 < s < 1 \) and \( 2^*_s = \frac{2N}{N - 2s} \). By virtue of the mini-max theory and the concentration compactness principle with the equivariant group action, we obtain the new type of non-radial, sign-changing solutions of (FCSE) in the energy space \( \dot{H}^s(\mathbb{R}^N) \). The key component is that we use the equivariant group to partition \( \dot{H}^s(\mathbb{R}^N) \) into several connected components, then combine the concentration compactness argument to show the compactness property of Palais-Smale sequences in each component and obtain many solutions of (FCSE) in \( \dot{H}^s(\mathbb{R}^N) \). The solutions and the argument here are different from those by Garrido, Musso in [19] and by Abreu, Barbosa and Ramirez in [1].

1. Introduction

This paper is concerned with the existence of sign-changing solutions to the following fractional critical Schrödinger equation

\[
\begin{cases}
(- \Delta)^s u - |u|^{2^*_s - 2} u = 0, & \text{in } \mathbb{R}^N, \\
u \in \dot{H}^s(\mathbb{R}^N),
\end{cases}
\]

where \( N \geq 2 \), \( 0 < s < 1 \), \( 2^*_s = \frac{2N}{N - 2s} \), \((- \Delta)^s\) denotes the usual fractional Laplace operator and \( \dot{H}^s(\mathbb{R}^N) \) denotes the homogenous Sobolev space of real-valued functions whose energy associated to \((- \Delta)^s\) is finite, i.e.

\[
\dot{H}^s(\mathbb{R}^N) = \{ u \in \mathcal{S}'(\mathbb{R}^N) \mid \|u\|_{\dot{H}^s} < +\infty \},
\]

with

\[
\|u\|_{\dot{H}^s}^2 = \int_{\mathbb{R}^N} \langle \xi \rangle^2 |(\mathcal{F}u)(\xi)|^2 d\xi,
\]

where \( \mathcal{F}u \) denotes the Fourier transform of \( u \):

\[
\mathcal{F}u(\xi) = \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} u(x) e^{-ix\xi} dx.
\]

Fractional Schrödinger equations (1.1) arise as models in the fractional quantum mechanics, including path integral over the Lévy flights paths (see for instance [24, 25, 23]), and as Euler-Lagrange equations for the Hardy-Littlewood-Sobolev inequalities (e.g., see [12, 18, 26]).

The problem about the positive solutions to (1.1) has attracted lots of attention. On the one hand, the existence of positive solutions to (1.1) is related to the existence of extremizers to
the Hardy-Littlewood-Sobolev inequalities. Lieb considered the following Hardy-Littlewood- Sobolev inequality in [26]
\[ \|u\|^{2}_{L^{\frac{2N}{N-2s}}} \leq S(N, s) \|u\|^{2}_{\dot{H}^{s}(\mathbb{R}^{N})}, \]  
and obtained that \( \omega_{\mu, \lambda, x_{0}} \) is the extremizer to (1.3) if and only if
\[ \omega_{\mu, \lambda, x_{0}}(x) = \frac{\mu}{\lambda^{\frac{N-2s}{2}}} \left( \frac{1}{1 + \frac{|x-x_{0}|^{2}}{\lambda^{2}}} \right)^{\frac{N-2s}{2}}, \quad \mu \neq 0, \quad \lambda > 0 \text{ and } x_{0} \in \mathbb{R}^{N}, \]  
by the layer cake representation technique. Note that (1.4) also solves (1.1) by taking suitable choices of \( \mu \). We can refer to [7, 12, 18], et.al., for more references. On the other hand, up to the symmetries of (1.1), Chen, Li and Ou made use of the moving plane method to show that (1.4) are the only positive (negative) solutions to (1.1) in \( L^{2N/(N-2s)}_{\text{loc}}(\mathbb{R}^{N}) \) in [9]. Moreover, Dávila, del Pino and Sire obtained the nondegeneracy of the extremizer (1.4) for the Hardy-Littlewood-Sobolev inequality (1.3) in [13].

Sign-changing solutions to (1.1), in the case \( s = 1 \), has been intensively studied in [10, 11, 15, 16, 22, 29], et al. As far as the authors known, there are two different ways to study sign-changing solutions of (1.1). On the one hand, Ding obtained infinitely many sign-changing solutions by making use of variational methods restricted to the space of group invariant functions in [16]. Clapp showed the multiplicity of sign-changing solutions by making use of minimax argument restricted to the space of group equivariant functions in [10]. We can also refer to [11] for the application in critical Lane-Emden systems. On the other hand, del Pino, Musso, Pacard and Pistoia constructed sign-changing solutions by the Lyapunov-Schmidt reduction argument in [14, 15, 28]. Recently, Medina and Musso also constructed some kind of sign-changing solutions with maximal rank in [28].

Our main result is the following.

**Theorem 1.1.** Let \( N = 4n + m \) with \( n \geq 1 \) and \( m \in \{0, 1, 2, 3\} \). Then for any \( 0 < s < 1 \), the problem (1.1) has at least \( n \) non-radial sign-changing solutions.

Both the result and the argument in this paper are different from those in [1, 19]. Abreu, Barbosa and Ramirez obtained infinitely many sign-changing solutions of (1.1) by the Ljusternik-Schnirelman type mini-max method and group invariant technique in [1]. Garrido and Musso constructed the sign-changing solutions of (1.1) by the Lyapunov-Schmidt reduction argument in [19]. The key idea here is that we use the equivariant group to partion \( \dot{H}^{s}(\mathbb{R}^{N}) \) into several connected components, then combine the concentration compactness argument to show the compactness property of Palais-Smale sequences in each component and obtain many solutions of (1.1) in \( \dot{H}^{s}(\mathbb{R}^{N}) \), where the compactness property of the Palais-Smale sequences is nontrivial for the fractional case \( 0 < s < 1 \), please see Section 3 for more details.

The remainder of the paper is organized as follows: In Section 2 we introduce some well-known facts about the fractional Schrödinger equations (1.1), which will be used throughout the paper. In Section 3 we present some compactness property of the Palais-Smale sequences. In Section 4 we prove the main result Theorem 1.1.

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2. Preliminaries

In this section, we begin with some notation that will be useful throughout this paper.
2.1. Notation. There are two ways to define the fractional Laplacian \((-\triangle)^{s}\varphi\) for the real-valued functions \(\varphi \in \dot{H}^{s}(\mathbb{R}^{N})\) with \(0 < s < 1\). On the one hand, the fractional Laplacian of \(\varphi\) can be defined by the Fourier transform as

\[ \mathcal{F}((-\triangle)^{s}\varphi)(\xi) = |\xi|^{2s}(\mathcal{F}\varphi)(\xi). \]

On the other hand, for \(\varphi \in \dot{H}^{s}(\mathbb{R}^{N})\) with \(0 < s < 1\), one can obtain by the fractional heat kernel that

\[ (-\triangle)^{s}\varphi(x) = \frac{1}{C(N, s)} \int_{\mathbb{R}^{N}} \frac{\varphi(x) - \varphi(y)}{|x - y|^{N+2s}} \, dy, \quad (2.1) \]

where

\[ C(N, s) = \int_{\mathbb{R}^{N}} \frac{1 - \cos(\eta)}{|\eta|^{N+2s}} \, d\eta. \quad (2.2) \]

Let \(O(N)\) be the orthogonal group in \(\mathbb{R}^{N}\), and \(G\) be a closed subgroup of the group \(O(N)\). Let \(\mathbb{Z}_2 = \{1, -1\}\) be the group of 2nd roots of unity, and \(\sigma\) be a continuous group homomorphism from \(G\) to \(\mathbb{Z}_2\).

For each \(x \in \mathbb{R}^{N}\), let \(G \cdot x\) denote the \(G\)-orbit of the point \(x\), and \(G_{x}\) denote the stabilizer subgroup of the group \(G\) with respect to the point \(x\), i.e.

\[ G \cdot x = \{gx \mid g \in G\}, \quad \text{and} \quad G_{x} = \{g \in G \mid gx = x\}. \]

The domain \(\Omega\) in \(\mathbb{R}^{N}\) is said to be \(G\)-invariant, if for each \(x \in \Omega\), \(G \cdot x \subseteq \Omega\). For any \(G\)-invariant domain \(\Omega\), we denote

\[ \Omega^{G} := \{x \in \Omega \mid gx = x \quad \text{for all} \quad g \in G\}. \]

Any function \(u\) on the \(G\)-invariant domain \(\Omega\) is said to be \(\sigma\)-equivariant if

\[ u(gx) = \sigma(g)u(x), \quad \text{for all} \quad g \in G \quad \text{and} \quad x \in \Omega. \]

Now for each \(s\) with \(0 < s < 1\), we can obtain the representation for the norm on \(\dot{H}^{s}(\mathbb{R}^{N})\) by the fractional heat kernel (see [27, 30, 32] for instance):

\[ \|u\|^{2} := \iint_{\mathbb{R}^{N} \times \mathbb{R}^{N}} \frac{|u(x) - u(y)|^{2}}{|x - y|^{N+2s}} \, dx \, dy = 2C(N, s) \int_{\mathbb{R}^{N}} |\xi|^{2s}|(\mathcal{F}u)(\xi)|^{2} \, d\xi, \quad (2.3) \]

where \(C(N, s)\) is defined by (2.2). Moreover, for \(0 < s < \frac{N}{2}\) and any \(u \in \dot{H}^{s}(\mathbb{R}^{N})\), by the Sobolev embedding inequality in [12, 17, 18, 27], we have \(\dot{H}^{s}(\mathbb{R}^{N}) \hookrightarrow L^{2^{*}}(\mathbb{R}^{N})\). More precisely,

\[ \left( \int_{\mathbb{R}^{N}} |u(x)|^{2^{*}} \, dx \right)^{\frac{2}{2^{*}}} \leq S(N, s) \int \int_{\mathbb{R}^{N} \times \mathbb{R}^{N}} \frac{|u(x) - u(y)|^{2}}{|x - y|^{N+2s}} \, dx \, dy, \quad (2.4) \]

where

\[ S(N, s) = 2^{-2s} \pi^{-s} \frac{(\Gamma(N - 2s) \Gamma(N + 2s))^{\frac{2}{N}}}{\Gamma(\frac{N}{2})}. \quad (2.5) \]

For any domain \(\Omega \subset \mathbb{R}^{N}\) with smooth boundary, let \(\dot{H}^{s}(\Omega)\) denote the Sobolev space which is defined as the completion of \(C_{c}^{\infty}(\Omega)\) under the norm which is defined by (2.3). Since \(\partial \Omega\) is smooth, we have (see for instance [5, 21, 32])

\[ \dot{H}^{s}(\Omega) = \left\{ u \in \dot{H}^{s}(\mathbb{R}^{N}) \mid u(x) = 0 \quad \text{for a.e.} \quad x \in \mathbb{R}^{N} \setminus \Omega \right\}. \]
For any closed subgroup \( G \) of \( O(N) \) and any continuous group homomorphism \( \sigma : G \to \mathbb{Z}_2 \), we define the subspace of \( \dot{H}^s(\Omega) \) which coincides with all \( \sigma \)-equivariant functions under the group \( G \) as follows,

\[
\dot{H}^s(\Omega)_G^\sigma = \left\{ u \in \dot{H}^s(\Omega) \mid u(gx) = \sigma(g) u(x), \text{ for all } g \in G \text{ and } x \in \Omega \right\}.
\]

In what follows, we will always assume that the group homomorphism \( \sigma \) is surjective, and the group \( G \) satisfies

(G1) For every \( x \in \mathbb{R}^N \), either \( \dim(G \cdot x) > 0 \) or \( G \cdot x = x \).

(G2) There exists at least one point \( \xi \in \mathbb{R}^N \) such that \( \sigma(G\xi) = \{1\} \).

Lemma 2.1 ([4 page 195]). Let \( G \) be closed subgroup of the group \( O(N) \), \( \Omega \) be a \( G \)-invariant domain in \( \mathbb{R}^N \), and \( \sigma : G \to \mathbb{Z}_2 \) be a continuous group homomorphism. If the group \( G \) satisfies (G2), then the space \( \dot{H}^s(\Omega)_G^\sigma \) is infinite dimensional.

2.2. The \( \sigma \)-equivariant solutions of (1.1) vanishing outside a \( G \)-invariant domain.

Let \( G \) be closed subgroup of the group \( O(N) \), \( \Omega \) be a \( G \)-invariant domain in \( \mathbb{R}^N \), and \( \sigma : G \to \mathbb{Z}_2 \) be a continuous group homomorphism. Now, we consider the fractional critical Schrödinger equation,

\[
\begin{cases}
(-\triangle)^s u - |u|^{2^*_s - 2} u = 0, & \text{in } \mathbb{R}^N, \\
u \in \dot{H}^s(\Omega)_G^\sigma.
\end{cases}
\] (2.6)

By the critical point theory (for instance, see [8 31 33]), the function \( u \) satisfies (2.6) if and only if \( u \in \dot{H}^s(\Omega)_G^\sigma \) is a critical point of the Lagrange functional as follows:

\[
\mathcal{E}(u; \Omega) = \frac{1}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2s}} \, dx \, dy - \frac{1}{2^*_s} \int_{\Omega} |u(x)|^{2^*_s} \, dx.
\] (2.7)

Moreover, if \( u \) is a nontrivial solution to (2.6), then \( u \) also belongs to the Nehari manifold \( \mathcal{N}(\Omega)_G^\sigma \),

\[
\mathcal{N}(\Omega)_G^\sigma = \left\{ u \mid u \in \dot{H}^s(\Omega)_G^\sigma \setminus \{0\}, \mathcal{N}(u; \Omega) = 0 \right\},
\] (2.8)

where the Nehari functional \( \mathcal{N}(u; \Omega) \) is defined by

\[
\mathcal{N}(u; \Omega) = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2s}} \, dx \, dy - \int_{\Omega} |u(x)|^{2^*_s} \, dx.
\] (2.9)

Now, we can reduce the variational problem for (2.6) to seek the critical points of \( \mathcal{E}(\cdot; \Omega) \) restricted to the subspace \( \dot{H}^s(\Omega)_G^\sigma \) by the following lemma.

Lemma 2.2. Under the assumptions of Lemma 2.1 if \( u \in \dot{H}^s(\Omega)_G^\sigma \) satisfies

\[
\langle \nabla \mathcal{E}(u; \Omega), \varphi \rangle = 0, \quad \text{for all } \varphi \in C_c^\infty(\Omega)_G^\sigma,
\]

where

\[
C_c^\infty(\Omega)_G^\sigma = \left\{ u \in C_c^\infty(\Omega) \mid u(gx) = \sigma(g) u(x), \text{ for all } g \in G \text{ and } x \in \Omega \right\},
\] (2.10)

then

\[
\langle \nabla \mathcal{E}(u; \Omega), \tilde{\varphi} \rangle = 0, \quad \text{for all } \tilde{\varphi} \in \dot{H}^s(\Omega).
\]

Proof. Let \( \tilde{\varphi} \in \dot{H}^s(\Omega) \). Define

\[
\varphi(x) = \frac{1}{\mu(G)} \int_G \sigma(g) \tilde{\varphi}(gx) \, d\mu,
\]
where $\mu$ is the Haar measure on $G$. Then $\varphi \in C^\infty_c(\Omega)_G^\sigma$, and note that $\langle \nabla \mathcal{E}(u; \Omega) \; , \; \varphi \rangle = 0$, therefore, by Fubini’s theorem and a change of variable, we obtain

\[
0 = \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy - \int_{\Omega} |u(x)|^{2s-2} u(x)\varphi(x) \, dx
\]

\[
= \frac{1}{\mu(G)} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{\Omega} \frac{\sigma(g)(u(x) - u(y))(\varphi(g x) - \varphi(g y))}{|x - y|^{N+2s}} \, d\mu \, dx \, dy
\]

\[
- \frac{1}{\mu(G)} \int_{\Omega} \int_{G} |u(x)|^{2s-2} u(x)\sigma(g)\varphi(g) \, d\mu \, dx
\]

\[
= \frac{1}{\mu(G)} \int_{\Omega} \int_{G} \frac{u(g x) - u(g y))(\varphi(g x) - \varphi(g y))}{|x - y|^{N+2s}} \, dx \, dy \mu(d\mu)
\]

\[
- \frac{1}{\mu(G)} \int_{\Omega} \int_{G} |u(g x)|^{2s-2} u(g x)\varphi(g x) \, d\mu \, dx \mu(d\mu)
\]

\[
= \frac{1}{\mu(G)} \int_{\Omega} \int_{G} (u(x) - u(y))(\varphi(x) - \varphi(y)) \, dx \, dy \mu(d\mu)
\]

\[
- \frac{1}{\mu(G)} \int_{\Omega} \int_{G} |u(x)|^{2s-2} u(x)\varphi(x) \, dx \mu(d\mu)
\]

\[
= \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy - \int_{\Omega} |u(x)|^{2s-2} u(x)\varphi(x) \, dx
\]

\[
= \langle \nabla \mathcal{E}(u; \Omega) \; , \; \varphi \rangle .
\]

This completes the proof. \hfill \Box

**Lemma 2.3.** Let the functionals $\mathcal{E}(\cdot; \Omega), N(\cdot; \Omega)$ be defined by (2.7), (2.9) respectively, and $S(N, s)$ be given by (2.5). Under the assumptions of Lemma 2.1, the following statements hold:

(a) for any $u \in \dot{H}^s(\Omega)_G^\sigma \setminus \{0\}$ satisfying $\|u\| < S(N, s)^{-\frac{N}{2s}}$, we have $N(u; \Omega) > 0$;

(b) for any $u \in \dot{H}^s(\Omega)_G^\sigma \setminus \{0\}$ satisfying $\|u\| < \left(\frac{2^s}{2} S(N, s)^{-\frac{2}{2s}}\right)^{\frac{1}{2^s-2}}$, we have $\mathcal{E}(u; \Omega) > 0$.

(c) for any $u \in \dot{H}^s(\Omega)_G^\sigma \setminus \{0\}$ satisfying $\mathcal{E}(u; \Omega) \leq 0$, we have $N(u; \Omega) < 0$.

**Proof.** (a) By (2.4), we have

\[
\int_{\Omega} |u(x)|^{2s} \, dx \leq S(N, s)^{\frac{2}{2s}}\|u\|^{2s} < \|u\|^2.
\]

Inserting (2.11) into (2.9), we obtain that $N(u; \Omega) > 0$.

\[\text{(b) By (2.4), we get}
\]

\[
\frac{1}{2^s} \int_{\Omega} |u(x)|^{2s} \, dx \leq \frac{1}{2^s} S(N, s)^{\frac{2}{2s}}\|u\|^{2^s} < \frac{1}{2} \|u\|^2,
\]

which implies that $\mathcal{E}(u; \Omega) > 0$. 

\[\text{(c) By (2.4), we have}
\]

\[
\frac{1}{2^s} \int_{\Omega} |u(x)|^{2s} \, dx \leq \frac{1}{2^s} S(N, s)^{\frac{2}{2s}}\|u\|^{2^s} < \frac{1}{2} \|u\|^2,
\]

which implies that $\mathcal{E}(u; \Omega) < 0$. 

\[\text{ }
\]
Since for any \( u \in \dot{H}^s(\Omega)_G \setminus \{0\} \) with \( E(u; \Omega) \leq 0 \), we have \( \frac{2s}{2} \|u\|^2 \leq \int_{\Omega} |u(x)|^{2^*} \, dx \). Therefore, we obtain

\[
N(u; \Omega) \leq \left( 1 - \frac{2s}{2} \right) \|u\|^2 < 0.
\]

This ends the proof of Lemma 2.3. \( \square \)

As a corollary of Lemma 2.3 that under the assumptions of Lemma 2.1, for any \( u \in \dot{H}^s(\Omega)_G \setminus \{0\} \) satisfying \( N(u; \Omega) = 0 \), we have \( E(u; \Omega) > 0 \), which enables us to minimize the functional \( E(\cdot; \Omega) \) constrained on the Nehari manifold \( \mathcal{N}(\Omega)_G^\sigma \). More precisely, let us define

\[
m(\Omega)_{\mathcal{N},G}^\sigma = \inf \{ E(u; \Omega) \mid u \in \mathcal{N}(\Omega)_G^\sigma \},
\]

(2.12) then \( m(\Omega)_{\mathcal{N},G}^\sigma \geq 0 \). In fact, the Nehari manifold \( \mathcal{N}(\Omega)_G^\sigma \) is a natural constraint for the minimizer to the functional \( E(\cdot; \Omega) \). More precisely, we have:

**Lemma 2.4.** Let \( m(\Omega)_{\mathcal{N},G}^\sigma \) be defined by (2.12). Under the assumptions of Lemma 2.1, if \( \varphi \in \mathcal{N}(\Omega)_G^\sigma \) satisfies \( E(\varphi; \Omega) = m(\Omega)_{\mathcal{N},G}^\sigma \), then

\[
\langle \nabla E(\varphi; \Omega) , u \rangle = 0, \quad \text{for all} \quad u \in \dot{H}^s(\Omega)_G^\sigma.
\]

**Proof.** Since \( \varphi \in \mathcal{N}(\Omega)_G^\sigma \) is a minimizer of the functional \( E(u; \Omega) \) subject to \( N(\varphi; \Omega) = 0 \), there exists a Lagrange multiplier \( \lambda \in \mathbb{R} \) such that

\[
\langle \nabla E(\varphi; \Omega) , u \rangle = \lambda \langle \nabla N(\varphi; \Omega) , u \rangle, \quad \text{for all} \quad u \in \dot{H}^s(\Omega)_G^\sigma.
\]

(2.13)

By choosing \( u = \varphi \) in (2.13), and using the fact that

\[
\langle \nabla E(\varphi; \Omega) , \varphi \rangle = N(\varphi; \Omega),
\]

we have

\[
0 = \langle \nabla E(\varphi; \Omega) , \varphi \rangle
= \lambda \langle \nabla N(\varphi; \Omega) , \varphi \rangle
= \lambda \left( 2\|\varphi\|^2 - 2^* \int_{\Omega} |\varphi(x)|^{2^*} \, dx \right),
\]

(2.14)

which, together with \( N(\varphi; \Omega) = 0 \) and \( \varphi \neq 0 \), implies that \( \lambda = 0 \). This completes the proof. \( \square \)

The following lemma shows that \( m(\Omega)_{\mathcal{N},G}^\sigma \) defined by (2.12) coincides with the critical value which is characterized via the well-known mountain pass theorem.

**Lemma 2.5.** Suppose that the assumptions of Lemma 2.1 hold, and let

\[
m(\Omega)_{\text{MP},G}^\sigma = \inf_{\gamma \in \Theta} \max_{t \in [0,1]} E(\gamma(t); \Omega).
\]

(2.15)

where

\[
\Theta = \left\{ \gamma \in \mathcal{E}([0,1] , \dot{H}^s(\Omega)_G^\sigma) \mid \gamma(0) = 0, \gamma(1) \neq 0, E(\gamma(1); \Omega) \leq 0 \right\},
\]

then we have

\[
m(\Omega)_{\text{MP},G}^\sigma = m(\Omega)_{\mathcal{N},G}^\sigma.
\]
Proof. Firstly, we show that
\[ m(\Omega)^\sigma_{\mathcal{V},G} \geq m(\Omega)^\sigma_{MP,G}. \]
For each \( u \in \mathcal{N}(\Omega)^\sigma_G \), we define \( \gamma_{\mathcal{V},u}(t) \doteq \left( \frac{2^{s^2}}{2^{s^2}} \right)^{\frac{1}{2^{s^2}-2}} t \cdot u \), and \( \Theta_{\mathcal{V}} \doteq \left\{ \gamma_{\mathcal{V},u} \mid u \in \mathcal{N}(\Omega)^\sigma_G \right\} \).

Obviously, we have
\[ \gamma_{\mathcal{V},u} \in C\left([0,1], \dot{H}^{s}(\Omega)^\sigma_G \right) \quad \text{with} \quad \gamma_{\mathcal{V},u}(0) = 0, \quad \text{and} \quad \gamma_{\mathcal{V},u}(1) \neq 0. \]
On the one hand, a direct computation shows that
\[ E(\gamma_{\mathcal{V},u}(1) ; \Omega) = \left( \frac{2^{s^2}}{2^{s^2}} \right)^{\frac{1}{2^{s^2}-2}} N(u) = 0, \]
which implies that, \( \Theta_{\mathcal{V}} \subseteq \Theta \). On the other hand, for all \( t \in [0,1] \), we have
\[ E(\gamma_{\mathcal{V},u}(t) ; \Omega) = \left( \frac{2^{s^2}}{2^{s^2}} \right)^{\frac{1}{2^{s^2}-2}} \|u\|^2 \left( t^2 - t^{2^s} \right). \]
By the elementary fact that
\[ t^2 - t^{2^s} \leq \left( \frac{2^{s^2}}{2^{s^2}} \right)^{\frac{1}{2^{s^2}-2}} - \left( \frac{2^{s^2}}{2^{s^2}} \right)^{\frac{1}{2^{s^2}-2}} 2^{s^2}, \]
we have
\[ E(\gamma_{\mathcal{V},u}(t) ; \Omega) \leq E\left( \gamma_{\mathcal{V},u} \left( \frac{2^{s^2}}{2^{s^2}} \right)^{\frac{1}{2^{s^2}-2}} ; \Omega \right) = E(u ; \Omega). \]
Hence,
\[ m(\Omega)^\sigma_{\mathcal{V},G} = \inf \left\{ E(u ; \Omega) \mid u \in \mathcal{N}(\Omega)^\sigma_G \right\} = \inf_{\gamma_{\mathcal{V},u} \in \Theta_{\mathcal{V}}} \max_{t \in [0,1]} E(\gamma_{\mathcal{V},u}(t) ; \Omega) \geq \inf_{\gamma \in \Theta} \max_{t \in [0,1]} E(\gamma(t) ; \Omega) = m(\Omega)^\sigma_{MP,G}. \quad (2.16) \]

Next, we show that
\[ m(\Omega)^\sigma_{MP,G} \geq m(\Omega)^\sigma_{\mathcal{V},G}. \]
Indeed, for each \( \gamma \in \Theta \), one has \( \gamma(0) = 0 \), by (a) of Lemma 2.3 there exists \( s_u \in (0,1) \) such that for all \( t \in (0, s_u) \),
\[ N(\gamma(t) ; \Omega) > 0. \quad (2.17) \]
However \( E(\gamma(1) ; \Omega) \leq 0 \), by (e) of Lemma 2.3 we have
\[ N(\gamma(1) ; \Omega) < 0. \quad (2.18) \]
By (2.17) and (2.18), there exists $t_{u,\text{max}} \in (s_u,1) \subset [0,1]$ such that $N(\gamma(t_{u,\text{max}}) ; \Omega) = 0$, which means that $\gamma(t_{u,\text{max}}) \in \mathcal{N}(\Omega)^{\sigma}_G$. Therefore,

$$m(\Omega)^{\sigma}_{MP,G} = \inf_{\gamma \in \Theta} \max_{t \in [0,1]} \mathcal{E}(\gamma(t); \Omega) \geq \inf_{\gamma \in \Theta} \mathcal{E}(\gamma(t_{u,\text{max}}); \Omega) \geq \inf \{ \mathcal{E}(u; \Omega) \mid u \in \mathcal{N}(\Omega)^{\sigma}_G \} = m(\Omega)^{\sigma}_{\mathcal{N},G}.$$

(2.19)

Combining (2.16) with (2.19), we hence complete the proof. □

In view of the above result, we will denote for short that

$$m(\Omega)^{\sigma}_G := m(\Omega)^{\sigma}_{MP,G} = m(\Omega)^{\sigma}_{\mathcal{N},G}.$$  

(2.20)

**Lemma 2.6.** If $\Omega$ is a $G$-invariant domain in $\mathbb{R}^N$ and $\Omega^G \neq \emptyset$, then

$$m(\Omega)^{\sigma}_G = m(\mathbb{R}^N)^{\sigma}_G.$$  

Proof. On one hand, by the embedding that $\Omega \subset \mathbb{R}^N$, we have

$$m(\Omega)^{\sigma}_G \geq m(\mathbb{R}^N)^{\sigma}_G.$$  

On the other hand, we fix $x_0 \in \Omega^G$ and choose a sequence $\{\varphi_n\}$ in $\mathcal{N}(\mathbb{R}^N)^{\sigma}_G \cap C_0^\infty(\mathbb{R}^N)$ such that $\mathcal{E}(\varphi_n; \mathbb{R}^N) \to m(\mathbb{R}^N)^{\sigma}_G$. Since $\varphi_n$ has compact support, we may choose $\lambda_n > 0$ such that

$$\text{supp} \tilde{\varphi}_n = \text{supp} \lambda_n^{-\frac{(N-2s)}{2}} \varphi_n(\frac{x-x_0}{\lambda_n}) \subset \Omega.$$  

As $x_0$ is a $G$-fixed point, $\tilde{\varphi}_n$ is $\sigma$-equivariant. Using the fact that

$$\| \varphi_n \|^2 = \| \tilde{\varphi}_n \|^2$$

and

$$\int_{\mathbb{R}^N} |\varphi_n|^2^* \, dx = \int_{\Omega} |\tilde{\varphi}_n|^2^* \, dx.$$  

We have $\tilde{\varphi}_n \in \mathcal{N}(\Omega)^{\sigma}_G$, hence

$$m(\Omega)^{\sigma}_G \leq \mathcal{E}(\tilde{\varphi}_n; \Omega) = \mathcal{E}(\varphi_n; \mathbb{R}^N) \to m(\mathbb{R}^N)^{\sigma}_G,$$

hence $m(\Omega)^{\sigma}_G \leq m(\mathbb{R}^N)^{\sigma}_G$, which implies the result. □

2.3. Some useful estimates.

**Lemma 2.7 ([6, Proposition 2.9]).** Let $f \in \left( \dot{H}^s(\Omega) \right)'$, and if $u \in \dot{H}^s(\Omega)$ satisfies that

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y)) (\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \, dx \, dy = (f, \varphi), \quad \text{for any } \varphi \in \dot{H}^s(\Omega),$$
then for any open subset \( \tilde{\Omega} \) of \( \mathbb{R}^N \) with \( \tilde{\Omega} \cap \Omega \neq \emptyset \) and any nonnegative function \( \phi \in C^\infty_0 (\tilde{\Omega}) \), we have

\[
\int\int_{\tilde{\Omega} \times \tilde{\Omega}} \frac{|u(x) \phi(x) - u(y) \phi(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \\
\leq C_0 \int\int_{\tilde{\Omega} \times \tilde{\Omega}} \frac{|\phi(x) - \phi(y)|}{|x - y|^{N+2s}} \left(|u(x)|^2 + |u(y)|^2\right) \, dx \, dy \\
+ C_0 \left( \sup_{y \in \text{supp}(\phi)} \int_{\mathbb{R}^N \setminus \tilde{\Omega}} \frac{|u(x)|}{|x - y|^{N+2s}} \, dx \right) \int_{\tilde{\Omega}} |u(x)| (\phi(x))^2 \, dx + C_0 |f, u\varphi^2|, 
\]

where \( C_0 \) is an absolute constant.

**Proposition 2.8** (\[4\] Proposition 2.3). Let \( 0 < r < R \). If \( u \in \mathbb{H}^s (B(0, r)) \), then there exists a positive constant \( C (N, s, \frac{R}{r}) \) such that

\[
\left( \int_{B(0, r)} |u(x)|^{2s} \, dx \right)^{\frac{2}{2s}} \leq C \left( N, s, \frac{R}{r} \right) \int_{B(0, R) \times B(0, R)} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy,
\]

where the constant \( C (N, s, \frac{R}{r}) \) goes to \( \infty \) as \( R \) goes to \( r \).

In order to describe the behaviour of Palais-Smale sequences for the variational problem in (2.15), we define Lévy’s concentration function for any \( u \in L^2_s (\Omega) \) as follows,

\[
Q_u (r) := \sup_{z \in \mathbb{R}^N} \int_{B(z, r)} |u(x)|^{2s} \, dx.
\]

We collect here some facts about Lévy’s concentration function.

**Proposition 2.9.** Let \( \Omega \) be a bounded domain of \( \mathbb{R}^N \). If \( u \in L^2_s (\Omega) \) with \( u(x) = 0 \) a.e. for \( x \in \mathbb{R}^N \setminus \Omega \), then the following statements hold.

(i) For any \( \delta \) with \( 0 < \delta < \int_{\Omega} |u(x)|^{2s} \, dx \), there exists \( r > 0 \), \( z_{\text{max}} \in \mathbb{R}^N \) such that

\[
\int_{B(z_{\text{max}}, r)} |u(x)|^{2s} \, dx = \delta,
\]

where \( \text{dist}(z_{\text{max}}, \Omega) \leq r \).

(ii) For any \( r > 0 \) and \( \xi \in \mathbb{R}^N \), we have

\[
Q_{u, \xi} (1) = Q_u (r),
\]

where \( u_{r, \xi}(x) = r^{\frac{N-2s}{2}} u(rx + \xi) \).

(iii) For any \( 0 < r < R \), we have

\[
Q_u (R) \leq \left( (N + 1) \frac{R}{r} \right) Q_u (r).
\]

**Proof.** (i) The proof is standard, please refer to [6] Lemma 3.1 for the details.

(ii) For any \( z \in \mathbb{R}^N \), we have by the change of variables that

\[
\int_{B(z, 1)} |u_{r, \xi}(x)|^{2s} \, dx = \int_{B(rz + \xi, r)} |u(x)|^{2s} \, dx.
\]

(2.21)
Taking the supremum over $\mathbb{R}^N$ on both sides of (2.22), we obtain that
\[
Q_{u,\xi}(1) = \sup_{z \in \mathbb{R}^N} \int_{B(z,1)} |u_{r,\xi}(x)|^{2^*} \, dx
\]
\[
= \sup_{z \in \mathbb{R}^N} \int_{B(rz+\xi,r)} |u(x)|^{2^*} \, dx
\]
\[
= \sup_{z \in \mathbb{R}^N} \int_{B(z,r)} |u(x)|^{2^*} \, dx
\]
\[
= Q_u(r).
\]

(iii) Let $x \in \mathbb{R}^N$ and $R > 0$, there exists $[(N+1) \frac{R}{r}]$ balls $B(z_1, r)$, $B(z_2, r)$, $\ldots$, $B\left(\frac{z_{[(N+1) \frac{R}{r}]}}{r}\right)$ in $\mathbb{R}^N$ such that their union cover $B(z,R)$ (see for instance [20, Corollary 1.3]), therefore
\[
\int_{B(z,R)} |u(x)|^{2^*} \, dx \leq \sum_{j=1}^{[(N+1) \frac{R}{r}]} \int_{B(z_j,r)} |u(x)|^{2^*} \, dx,
\]
which implies that
\[
\int_{B(z,R)} |u(x)|^{2^*} \, dx \leq [(N+1) \frac{R}{r}] Q_u(r).
\]
Taking the supremum to the left hand side of (2.23), we can obtain (2.21).

This ends the proof of Proposition 2.9. $\square$

3. Palais-Smale sequences

As a consequence of a general minimax method, using Ekeland’s $\varepsilon$-variational principle (see, for instance [2, 33]), we have the following well-known result.

Lemma 3.1. Suppose that the assumptions of Lemma 2.1 hold. Let $m(\Omega)^\sigma_G$ be defined by (2.20). Then there exists a sequence $\{u_n\}_{n=1}^\infty \subseteq \dot{H}^s(\Omega)^\sigma_G$ such that
\[
E(u_n; \Omega) \to m(\Omega)^\sigma_G, \quad \text{and} \quad \nabla E(u_n; \Omega) \to 0 \text{ in } \left(\dot{H}^s(\Omega)^\sigma_G\right)', \quad \text{as} \quad n \to \infty.
\]

Proof. By Ekeland’s $\varepsilon$-variational principle, there exists a sequence $\{u_n\}_{n=1}^\infty \subseteq \dot{H}^s(\Omega)^\sigma_G$ satisfying
\[
E(u_n; \Omega) \to m(\Omega)^\sigma_{MP,G}, \quad \text{and} \quad \nabla E(u_n; \Omega) \to 0 \text{ in } \left(\dot{H}^s(\Omega)^\sigma_G\right)', \quad \text{as} \quad n \to \infty.
\]
By (2.20), we obtain that
\[
E(u_n; \Omega) \to m(\Omega)^\sigma_G, \quad \text{and} \quad \nabla E(u_n; \Omega) \to 0 \text{ in } \left(\dot{H}^s(\Omega)^\sigma_G\right)', \quad \text{as} \quad n \to \infty.
\]
This ends the proof of Lemma 3.1. $\square$

By the change of variables, we have the following result.

Lemma 3.2. Under the assumptions of Lemma 2.1, for any positive number $\lambda$ and any point $\xi$ in $\mathbb{R}^N$ satisfying $G \cdot \xi = \{\xi\}$, let us define the domain
\[
\Omega_{\xi,\lambda} \doteq \left\{ x \big| x \in \frac{1}{\lambda} (\Omega - \xi) \right\},
\]
and the function
\[
u_{\xi,\lambda}(x) \doteq \lambda^{\frac{N-2s}{2}} u(\lambda x + \xi).
\]
Then the following statements hold:

(I) \( u \in \dot{H}^s (\Omega)_G^{\sigma} \) if and only if \( u_{\xi,\lambda} \in \dot{H}^s (\Omega_{\xi,\lambda})_G^{\sigma} \), moreover, we have

\[
\|u\| = \|u_{\xi,\lambda}\|, \quad \text{and} \quad \int_{\Omega} |u(x)|^{2^*_s} \, dx = \int_{\Omega_{\xi,\lambda}} |u_{\xi,\lambda}(x)|^{2^*_s} \, dx.
\]

(II) if \( u \in \dot{H}^s (\Omega)_G^{\sigma} \), then we have

\[
\sup_{\phi \in H^s(\Omega)_G^{\sigma}} \langle \nabla \mathcal{E} (u ; \Omega) , \phi \rangle = \sup_{\psi \in H^s(\Omega)_G^{\sigma}} \langle \nabla \mathcal{E} (u_{\xi,\lambda} ; \Omega_{\xi,\lambda}) , \psi \rangle.
\]

The following geometrical lemma will be used to show Lemma 3.4.

**Lemma 3.3** ([10] [11]). Let \( G \) be a closed subgroup of the group \( O (N) \), \( \{x_n\}_{n=1}^{\infty} \) be a sequence of \( \mathbb{R}^N \) and \( \{\lambda_n\}_{n=1}^{\infty} \) be a sequence of \( (0, \infty) \). If \( G \) satisfies \((G1)\) then,

either there exist a subsequence (still denoted by \( \{x_n\}_{n=1}^{\infty} \)) and a sequence \( \{\xi_n\}_{n=1}^{\infty} \) of \( (\mathbb{R}^N)^G \), such that

\[
\frac{1}{\lambda_n} \text{dist} (Gx_n , \xi_n) \leq N_0,
\]

where \( N_0 \) is some positive integer independent of \( n \);

or for any integer \( q \), there exist \( \delta > 0 \) and elements \( g_1, g_2, \ldots, g_q \) of \( G \), such that for \( 1 \leq i \neq k \leq q \),

\[
\frac{1}{\lambda_n} |g_jx_n - g_kx_n| \to \infty \quad \text{as} \quad n \to \infty.
\]

Now, we are able to describe the compactness properties of Palais-Smale sequences for \((2,20)\).

**Lemma 3.4.** Let \( G \) be closed subgroup of the group \( O (N) \), \( \Omega \) be a \( G \)-invariant bounded smooth domain in \( \mathbb{R}^N \) and \( \sigma : G \to \mathbb{Z}_2 \) be a continuous group homomorphism. Suppose the group \( G \) satisfies \((G1),(G2)\). If \( \{u_n\}_{n=1}^{\infty} \) is a sequence of \( \dot{H}^s (\Omega)_G^{\sigma} \) satisfying

\[
\mathcal{E} (u_n ; \Omega) \to \mathcal{m} (\Omega)_G^{\sigma}, \quad \text{and} \quad \nabla \mathcal{E} (u_n ; \Omega) \to 0 \quad \text{in} \quad \left( \dot{H}^s (\Omega)_G^{\sigma} \right)' \quad \text{as} \quad n \to \infty, \quad (3.1)
\]

then, up to a subsequence (still denoted by \( \{u_n\}_{n=1}^{\infty} \)),

(PS1) either there exists a nontrivial \( U \in \mathcal{N} (\Omega)_G^{\sigma} \) such that

\[
\lim_{n \to \infty} \|u_n - U\| = 0.
\]

(PS2) or there exist a sequence \( \{\xi_n\}_{n=1}^{\infty} \subseteq (\mathbb{R}^N)^G \), a sequence \( \{\lambda_n\}_{n=1}^{\infty} \subseteq (0, \infty) \) and a nontrivial function \( V \in \mathcal{N} (\mathcal{H})_G^{\sigma} \) satisfying

\[
(- \triangle)^s V - |V|^{2^*_s - 2} V = 0, \quad \text{in} \quad \mathcal{H},
\]

such that

\[
\lim_{n \to \infty} \|u_n (\cdot) - \frac{1}{\lambda_n^{\frac{N}{2}} \nu} V \left( \frac{\cdot - \xi_n}{\lambda_n} \right)\| = 0,
\]

where \( \mathcal{H} \) is either a half space of \( \mathbb{R}^N \) or \( \mathcal{H} = \mathbb{R}^N \).

**Proof.** We divide the proof into several steps.
Step 1. We claim that the sequence \( \{u_n\}_{n=1}^{\infty} \) satisfying (3.1) is uniformly bounded in \( \dot{H}^s (\Omega)_G^\sigma \). By (3.1), we have

\[
o(1) \|u_n\| = \langle \nabla E (u_n ; \Omega) , u_n \rangle = \|u_n\|^2 - \int_{\Omega} |u_n (x)|^{2^*_s} \, dx,
\]

we obtain that

\[
\frac{S}{N} \|u_n\|^2 = E (u_n ; \Omega) - \frac{1}{2^*_s} \langle \nabla E (u_n ; \Omega) , u_n \rangle = m (\Omega)_G^\sigma + o (1) \|u_n\|,
\]

and

\[
\frac{S}{N} \int_{\Omega} |u_n (x)|^{2^*_s} \, dx = E (u_n ; \Omega) - \frac{1}{2} \langle \nabla E (u_n ; \Omega) , u_n \rangle = m (\Omega)_G^\sigma + o (1) \|u_n\|, \tag{3.2}
\]

which implies that the sequence \( \{u_n\}_{n=1}^{\infty} \) is uniformly bounded in \( \dot{H}^s (\Omega)_G^\sigma \). Therefore, by the Rellich-Kondrachov theorem, after passing to a subsequence if necessary, we obtain that

\[
u_n \to U \quad \text{ weakly in } \dot{H}^s (\Omega)_G^\sigma; \tag{3.3}
\]

\[
u_n \to U \quad \text{ in } L^p (\Omega) \text{ for all } 1 \leq p < 2^*_s; \tag{3.4}
\]

\[\nu_n \to U \quad \text{ almost everywhere in } \Omega. \]

Hence, for each \( \varphi \in \dot{H}^s (\Omega)_G^\sigma \), one has

\[
\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u_n (x) - u_n (y)) (\varphi (x) - \varphi (y))}{|x - y|^{N+2s}} \, dx \, dy \to \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(U (x) - U (y)) (\varphi (x) - \varphi (y))}{|x - y|^{N+2s}} \, dx \, dy. \tag{3.5}
\]

Moreover, by [33] Theorem A.2] and (3.4) with \( p = 2^*_s - 1 \), we obtain that

\[|u_n|^{2^*_s - 2} u_n \to |U|^{2^*_s - 2} U \quad \text{ in } L^1 (\Omega). \]

Therefore, we have for any \( \phi \in C_c^\infty (\Omega) \) that

\[
\int_{\Omega} |u_n (x)|^{2^*_s - 2} u_n (x) \phi (x) \to \int_{\Omega} |U (x)|^{2^*_s - 2} U (x) \phi (x). \tag{3.6}
\]

By density of \( C_c^\infty (\Omega) \) in \( \dot{H}^s (\Omega)_G^\sigma \), we obtain for any \( \varphi \in \dot{H}^s (\Omega)_G^\sigma \) that

\[
\int_{\Omega} |u_n (x)|^{2^*_s - 2} u_n (x) \varphi (x) \to \int_{\Omega} |U (x)|^{2^*_s - 2} U (x) \varphi (x), \tag{3.6}
\]

which together with (3.1) and (3.5) implies for any \( \varphi \in \dot{H}^s (\Omega)_G^\sigma \) that

\[
\langle \nabla E (U ; \Omega) , \varphi \rangle = 0. \tag{3.7}
\]

Step 2. For the case \( U \neq 0 \), we claim that [PS1] holds. In fact, on the one hand, by choosing \( \varphi = U \) in (3.7), we obtain that

\[N (U ; \Omega) = 0,\]

which means \( U \in \mathcal{N} (\Omega)_G^\sigma \). By (2.12) and (2.20), we obtain that

\[
\frac{S}{N} \|U\|^2 = E (U ; \Omega) - \frac{1}{2^*_s} N (U ; \Omega) \geq m (\Omega)_G^\sigma. \tag{3.8}
\]
On the other hand, by the weak lower semicontinuity of the norm \( \| \cdot \| \), we have
\[
\| U \|^2 \leq \lim_{n \to \infty} \| u_n \|^2 = \frac{N}{s} m(\Omega)^s_G.
\] (3.9)

Therefore, we get
\[
\lim_{n \to \infty} \| u_n \| = \| U \|,
\]
which together with (3.3) implies that
\[
\lim_{n \to \infty} \| u_n - U \| = 0.
\]

Hence \((\text{PS}1)\) holds.

**Step 3.** For the case \( U = 0 \). By (3.2), up to a subsequence (we still denote by \( \{u_n\}_{n=1}^\infty \)), we get
\[
\int_\Omega |u_n(x)|^{2^*} \, dx \geq \frac{N}{2s} m(\Omega)^s_G.
\] (3.10)

Let us choose \( \delta \) satisfying \( 0 < \delta < \frac{N}{2s} m(\Omega)^s_G \) to be determined later. By Proposition 2.9 there exist a sequence \( \{z_n\}_{n=1}^\infty \) of \( \mathbb{R}^N \) and a sequence \( \{\lambda_n\}_{n=1}^\infty \) of \( (0, +\infty) \), such that
\[
\delta = \int_{B(z_n, \lambda_n)} |u_n(x)|^{2^*} \, dx.
\]

**Step 3a.** We claim that after passing to another subsequence if necessary, there exists a sequence \( \{\xi_n\}_{n=1}^\infty \) of \( (\mathbb{R}^N)^G \) such that
\[
\frac{1}{\lambda_n} \text{dist} (G \cdot z_n, \xi_n) \leq N_0,
\] (3.11)
where the constant \( N_0 \) is the positive integer in Lemma 3.3. We suppose by contradiction that (3.11) does not hold. On the one hand, by Lemma 3.3 for each \( q \in \mathbb{N} \), one can find \( q \) elements \( g_1, g_2, \cdots, g_q \) of \( G \), such that, for \( n \) sufficiently large,
\[
|g_jz_n - g_kz_n| \geq 2\lambda_n, \text{ if } j \neq k,
\]
which implies that
\[
B(g_jz_n, \lambda_n) \cap B(g_kz_n, \lambda_n) = \emptyset.
\] (3.12)

On the other hand, using the fact that \( |u(gx)| = |u(x)| \) for all \( u \in \dot{H}^{s}(\Omega)^s_G \), by the change of variables, we have,
\[
\int_{B(g_jz_n, \lambda_n)} |u_n(x)|^{2^*} \, dx = \int_{B(z_n, \lambda_n)} |u_n(x)|^{2^*} \, dx, \text{ for all } j = 1, 2, \cdots, q.
\] (3.13)

Now, combining (3.12) with (3.13), we obtain for each \( q \in \mathbb{N} \) that,
\[
q\delta = \sum_{j=1}^q \int_{B(g_jz_n, \lambda_n)} |u_n(x)|^{2^*} \, dx \leq \int_{\Omega} |u_n(x)|^{2^*} \, dx,
\]
which contradicts with the uniform boundedness of \( \int_{\Omega} |u_n(x)|^{2^*} \, dx \), see (3.2). Therefore, (3.11) holds. For each \( z_n \), there exist \( g_n \in G \) and \( \xi_n \in (\mathbb{R}^N)^G \) such that \( |g_nz_n - \xi_n| < N_0\lambda_n \), hence we have
\[
\delta \leq \int_{B(\xi_n, (N_0 + 1)\lambda_n)} |u_n(x)|^{2^*} \, dx,
\] (3.14)
which implies that
\[
|\Omega \cap B(\xi_n, (N_0 + 1)\lambda_n)| > 0.
\] (3.15)
However, by Proposition 2.9 we obtain that
\[ \int_{B(\xi_n,(N_0+1)\lambda_n)} |u_n(x)|^{2^*} \, dx \leq Q_{u_n}((N_0 + 1)\lambda_n) \]
\[ \leq (N + 1)(N_0 + 1)Q_{u_n}(\lambda_n) \]
\[ \leq (N + 1)(N_0 + 1)\delta. \]  
(3.16)

From now on, we consider the new sequence \((v_n)_{n=1}^\infty\) which is defined by
\[ v_n(x) := \frac{x-x_n}{\lambda_n} u_n(\lambda_n x + \xi_n). \]

By letting \(\Omega_n = \left\{ x \mid x \in \frac{1}{\lambda_n}(\Omega - \xi_n) \right\}\), we have \(v_n \in \dot{H}^s(\Omega_n)\), and
\[ \|v_n\| = \|u_n\|, \]
\[ \int_{\Omega_n} |v_n(x)|^{2^*} \, dx = \int_{\Omega} |u_n(x)|^{2^*} \, dx. \]
(3.17)

Moreover, by (3.14) and (3.16), we have
\[ \delta \leq \int_{B(0,N_0+1)} |v_n(x)|^{2^*} \, dx \leq (N + 1)(N_0 + 1)\delta. \]
(3.18)

By the Rellich-Kondrachov theorem, after passing to a subsequence if necessary, we have
\[ v_n \rightharpoonup V \quad \text{weakly in } \dot{H}^s(\mathbb{R}^N); \]
\[ v_n \to V \quad \text{in } L^p_{\text{loc}}(\mathbb{R}^N) \text{ for all } 1 \leq p < 2^*_s; \]
\[ v_n \to V \quad \text{almost everywhere in } \mathbb{R}^N. \]

**Step 3b.** We claim that \(V \neq 0\). Arguing by contradiction, we assume that \(V = 0\). Let \(h \in C_\infty(\mathbb{R}^N)\) be a radially symmetric function satisfying
\[ h(x) = \begin{cases} 1, & \text{if } x \in B(0, N_0 + 1), \\ 0, & \text{if } x \notin B(0, 2(N_0 + 1)). \end{cases} \]

By the fact that \(v_n \in \dot{H}^s(\Omega_n)\), and (3.18), we have
\[ \Omega_n \cap B(0, N_0 + 1) \neq \emptyset. \]

Therefore, by replacing \(\Omega, \Omega, u \text{ and } f \text{ with } \Omega_n, B(0, 4(N_0 + 1)), v_n \text{ and } |v_n|^{2^*-2}v_n + \nabla \mathcal{E}(v_n; \Omega_n)\) respectively in Lemma 2.7, we obtain that
\[ \int_{B(0,4(N_0+1))} \int_{B(0,4(N_0+1))} \frac{|v_n(x) h(x) - v_n(y) h(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \]
\[ \leq C_0 \int_{B(0,4(N_0+1))} \int_{B(0,4(N_0+1))} \frac{|h(x) - h(y)|^2}{|x-y|^{N+2s}} \left( |v_n(x)|^2 + |v_n(y)|^2 \right) \, dx \, dy \]
\[ + C_0 \left( \sup_{y \in B(0,2(N_0+1))} \int_{\mathbb{R}^N \setminus B(0,4(N_0+1))} \frac{|v_n(x)|}{|x-y|^{N+2s}} \, dx \right) \int_{\mathbb{R}^N} |v_n(x)| h^2(x) \, dx \]
\[ + C_0 \int_{\mathbb{R}^N} h^2(x) |v_n(x)|^{2^*} \, dx \]
\[ + C_0 \langle \nabla \mathcal{E}(v_n; \Omega_n), h^2 v_n \rangle. \]
(3.19)
Since for any \( x \in B(0, 4(N_0 + 1)) \),
\[
\int_{B(0,4(N_0+1))} \frac{1}{|x-y|^{N+2s-2}} \, dy = \int_{B(-x,4(N_0+1))} \frac{1}{|y|^{N+2s-2}} \, dy \leq \int_{B(0,8(N_0+1))} \frac{1}{|y|^{N+2s-2}} \, dy \leq C_1(N, N_0),
\]
where \( C_1(N, N_0) \) is a constant depending only on \( N \) and \( N_0 \). Using (3.24), we get
\[
\begin{align*}
(3.20) & \leq 2C_0 \|\nabla h\|_L^2 \int_{B(0,4(N_0+1))} |v_n(x)|^2 \left( \int_{B(0,4(N_0+1))} \frac{1}{|x-y|^{N+2s-2}} \, dy \right) \, dx \\
& \leq 2C_0 \|\nabla h\|_L^2 C_1(N, N_0) \|v_n\|_{L^2(B(0,4(N_0+1))} \\
& = o(1),
\end{align*}
\]
where we used that \( \lim_{n \to \infty} \int_{B(0,4(N_0+1))} |v_n(x) - V(x)|^2 \, dx = 0 \) and \( V = 0 \).

Since for any \( x \in R^N \setminus B(0, 4(N_0 + 1)) \) and any \( y \in B(0, 2(N_0 + 1)) \), we have \( \frac{1}{2} \leq |x-y| \leq \frac{3}{2}|x| \), there exists a positive constant \( C_2(N, N_0) \) depending only on \( N \) and \( N_0 \) such that
\[
\int_{R^N \setminus B(0, 4(N_0+1))} \frac{1}{|x-y|^{(N+2s)\frac{2s-1}{2}}(2s-1)} \, dx \leq C_2(N, N_0),
\]
which together with the Hölder inequality yields that
\[
(3.21) = C_0 \left( \sup_{y \in B(0,2(N_0+1))} \int_{R^N \setminus B(0,4(N_0+1))} \frac{|v_n(x)|}{|x-y|^{N+2s}} \, dx \right) \int_{R^N} |v_n(x)|h^2(x) \, dx \\
\leq C_0 C_2(N, N_0) \left( \int_{\Omega_n} |v_n(x)|^{2s} \, dx \right)^{\frac{2s-1}{2s}} \int_{B(0,2(N_0+1))} |v_n(x)| \, dx \\
= o(1),
\]
where we used the uniform boundedness of \( \int_{\Omega_n} |v_n(x)|^{2s} \, dx \), \( \lim_{n \to \infty} \int_{B(0,2(N_0+1))} |v_n(x) - V(x)| \, dx = 0 \) and \( V = 0 \).

Next, combining [6, Lemma A.1] with (2.4) and Lemma 3.2 we obtain that
\[
(3.23) = o(1) \|h^2v_n\| = o(1) \|v_n\|.
\]

Now, we deal with (3.22). By the Hölder inequality and Proposition 2.8, we have
\[
\int_{R^N} h^2(x) |v_n(x)|^{2s} \, dx \leq \left( \int_{B(0,2(N_0+1))} |v_n(x)|^{2s} \, dx \right)^{\frac{2s-2}{2s}} \left( \int_{R^N} |h(x)v_n(x)|^{2s} \, dx \right)^{\frac{2s}{2s}} \\
\leq C(N, s, 2) \left( \int_{B(0,2(N_0+1))} |v_n(x)|^{2s} \, dx \right)^{\frac{2s-2}{2s}} \|h v_n\|_{H^s(B(0,4(N_0+1))}^2 \\
\leq C(N, s, 2) (2(N_0 + 1)(N + 1)\delta)^{\frac{2s-2}{2s}} \|h v_n\|_{H^s(B(0,4(N_0+1))}^2.
\]
Next, by choosing

$$\delta = \min \left\{ \frac{N}{2s} \min (\Omega)^{\sigma}, \frac{1}{2(N_0 + 1)} \frac{1}{(N + 1)} \left( \frac{1}{2C (N, s, 2)} \right)^{\frac{2s}{N-2}} \right\},$$

we have

$$\|h v_n\|^2_{H^s (B(0, \frac{1}{2} N_0 + 1))} = o(1).$$

Therefore,

$$\int_{B(0, (N_0 + 1))} |v_n (x)|^{2s} \, dx \leq \int_{B(0, 2(N_0 + 1))} |h (x) v_n (x)|^{2s} \, dx \leq \|h v_n\|^2_{H^s (B(0, \frac{1}{2} N_0 + 1))}$$

which contradicts with (3.18). Therefore $V \neq 0$.

**Step 3c.** Without loss of generality, one may assume, after passing to another subsequence if necessary, as $n \to \infty$

$$\xi_n \to \xi^0, \quad \lambda_n \to \lambda^0,$$

with $\xi^0 \in (\mathbb{R}^N)^{G}$ and $\lambda^0 \geq 0$. Now, we distinguish two cases according to the fact whether $\frac{1}{\lambda_n} \lim_n \text{dist} (\xi_n, \partial \Omega)$ is finite or not.

**Case I.** $\frac{1}{\lambda_n} \lim_n \text{dist} (\xi_n, \partial \Omega) = \infty$. In this case, by (3.15), we have $\xi_n \in \Omega$. Therefore, for each compact subset $F$ of $\mathbb{R}^N$, there exists $n_0$ such that

$$F \subseteq \Omega_n \quad \text{for all } n \geq n_0.$$ 

On the one hand, for each $\varphi \in C_c^\infty (\mathbb{R}^N)^{G}$, for $n$ sufficiently large, we obtain that

$$\langle \nabla \mathcal{E} (v_n; \mathbb{R}^N), \varphi \rangle = \langle \nabla \mathcal{E} (v_n; \Omega_n), \varphi \rangle = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(v_n (x) - v_n (y)) (\varphi (x) - \varphi (y))}{|x - y|^{N+2s}} \, dx \, dy - \int_{\Omega_n} |v_n (x)|^{2s} \varphi (x) \, dx \leq \int_{\Omega} |u_n (x)|^{2s} - 2u_n (x) \varphi (x) \, dx$$

$$= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left[ \frac{u_n (x) - u_n (y)}{|x - y|^{N+2s}} \right] \varphi \left( \frac{x - \xi_n}{\lambda_n} \right) \, dx \, dy - \int_{\Omega} \lambda_n^{\frac{N-2s}{2}} \varphi \left( \frac{x - \xi_n}{\lambda_n} \right) \, dx$$

$$= \langle \nabla \mathcal{E} (u_n; \Omega), \lambda_n^{\frac{N-2s}{2}} \varphi \left( \frac{\cdot - \xi_n}{\lambda_n} \right) \rangle = o (1) \|\lambda_n^{\frac{N-2s}{2}} \varphi \left( \frac{\cdot - \xi_n}{\lambda_n} \right) \|$$

On the other hand, for each $\varphi \in C_c^\infty (\mathbb{R}^N)^{G}$, one has

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(v_n (x) - v_n (y)) (\varphi (x) - \varphi (y))}{|x - y|^{N+2s}} \, dx \, dy \to \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(V (x) - V (y)) (\varphi (x) - \varphi (y))}{|x - y|^{N+2s}} \, dx \, dy,$$

(3.25)
moreover, by Theorem A.2 and (3.4) with $p = 2s - 1$, we can obtain that
\[ |v_n|^{2s-2}v_n \to |V|^{2s-2}V \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^N), \]
which implies that
\[ \int_{\mathbb{R}^N} |v_n(x)|^{2s-2}v_n(x) \varphi(x) \, dx \to \int_{\mathbb{R}^N} |V(x)|^{2s-2}V(x) \varphi(x) \, dx, \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^N)\sigma. \]

Therefore
\[ \langle \nabla \mathcal{E}(V; \mathbb{R}^N), \varphi \rangle = 0, \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^N)\sigma, \]
hence
\[ \langle \nabla \mathcal{E}(V; \mathbb{R}^N), \psi \rangle = 0, \quad \text{for all } \psi \in \dot{H}^s(\mathbb{R}^N)\sigma. \]
and
\[ N(V; \mathbb{R}^N) = 0. \]

So $V \in N(\mathbb{R}^N)_G$, and
\[ \|v_n - V\| \to 0. \]

**Case II.** $\frac{1}{\lambda_n^{1/2}} \lim_{n \to \infty} \text{dist}(\xi_n, \partial \Omega) = \rho$ with $\rho \in (0, \infty)$.

When $\{\xi_n\}_{n=1}^\infty \subseteq \overline{\Omega}$, let
\[ \mathcal{H} := \{ x \in \mathbb{R}^N \mid x \cdot \nu > -\rho \}, \]
where $\nu$ is the inward pointing unit normal to $\partial \Omega$ at $\xi^0$. Since $\xi^0 \in (\mathbb{R}^N)_G$, so is $\nu$. Thus, we have $\mathcal{H}$ is $G$-invariant.

When $\{\xi_n\}_{n=1}^\infty \not\subseteq \overline{\Omega}$, let
\[ \mathcal{H} := \{ x \in \mathbb{R}^N \mid x \cdot \nu > \rho \}. \]

Similarly, we also have $\mathcal{H}$ is $G$-invariant.

In both cases, it is easy to see that $\mathcal{H}$ is a half plane in $\mathbb{R}^N$, therefore if $X$ is compact and $X \subseteq \mathcal{H}$, there exists $n_0$ such that $X \subseteq \Omega_n$ for all $n \geq n_0$, moreover if $X$ is compact and $X \subseteq \mathbb{R}^N \setminus \mathcal{H}$, then $X \subseteq \mathbb{R}^N \setminus \Omega_n$ for $n$ large enough. As $V_n \to V$ a.e. in $\mathbb{R}^N$, this implies $V \neq 0$ in $\Omega_n$, in particular, $V = 0$ a.e. in $\mathbb{R}^N \setminus \mathcal{H}$. So $V \in \dot{H}^s(\mathcal{H})_G$. Moreover, for each $\varphi \in C_c^\infty(\mathcal{H})_G$, for $n$ sufficiently large, we have
\[
\langle \nabla \mathcal{E}(v_n; \mathcal{H}), \varphi \rangle \\
= \langle \nabla \mathcal{E}(v_n; \Omega_n), \varphi \rangle \\
= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(v_n(x) - v_n(y)) (\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \, dx \, dy - \int_{\Omega_n} |v_n(x)|^{2s-2}v_n(x) \varphi(x) \, dx \\
= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left[ u_n(x) - u_n(y) \right] \left[ \lambda_n^{\frac{N-2s}{2}} \varphi \left( \frac{x-\xi_n}{\lambda_n} \right) - \lambda_n^{\frac{N-2s}{2}} \varphi \left( \frac{y-\xi_n}{\lambda_n} \right) \right] \, dx \, dy \\
- \int_{\Omega} |u_n(x)|^{2s-2}u_n(x) \left( \lambda_n^{\frac{N-2s}{2}} \varphi \left( \frac{x-\xi_n}{\lambda_n} \right) \right) \, dx \\
= \beta \|\lambda_n^{\frac{N-2s}{2}} \varphi \left( \cdot - \xi_n \right) \| \\
= o(1) \|\lambda_n^{\frac{N-2s}{2}} \varphi \left( \cdot - \xi_n \right) \|. \]
On the other hand, for each \( \varphi \in C^\infty_c(\mathcal{H})^\sigma_G \), one has
\[
\int \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(v_n(x) - v_n(y)) (\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \, dx \, dy \to \int \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(V(x) - V(y)) (\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \, dx \, dy,
\]
and by [33, Theorem A.2], we can see that
\[
|v_n|^{2s-2} v_n \to |V|^{2s-2} V \quad \text{in } L^1_{\text{loc}}(\mathcal{H}),
\]
which implies that
\[
\int_{\mathcal{H}} |v_n(x)|^{2s-2} v_n(x) \varphi(x) \, dx \to \int_{\mathcal{H}} |V(x)|^{2s-2} V(x) \varphi(x) \, dx, \quad \text{for all } \varphi \in C^\infty_c(\mathcal{H})^\sigma_G.
\]
Therefore
\[
\langle \nabla \mathcal{E}(V; \mathcal{H}), \varphi \rangle = 0, \quad \text{for all } \varphi \in C^\infty_c(\mathcal{H})^\sigma_G,
\]
hence
\[
\langle \nabla \mathcal{E}(V; \mathcal{H}), \psi \rangle = 0, \quad \text{for all } \psi \in \dot{H}^s(\mathcal{H})^\sigma_G.
\]
and
\[
\mathcal{N}(V; \mathcal{H}) = 0.
\]
So \( V \in \mathcal{N}(\mathcal{H})^\sigma_G \), we obtain that
\[
\|v_n - V\| \to 0.
\]
After a change of variable, we have
\[
\|u_n(x) - \lambda_n \frac{(N-2s)}{2} \left( V\left( \frac{x}{\lambda_n} \right) \right) \|^2
\]
\[
= \int \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_n(x) - u_n(y) - \lambda_n \frac{(N-2s)}{2} V\left( \frac{x}{\lambda_n} \right) + \frac{(N-2s)}{2} V\left( \frac{y}{\lambda_n} \right)|^2}{|x-y|^{N+2s}} \, dx \, dy
\]
\[
= \int \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\lambda_n \frac{(N-2s)}{2} u_n(\lambda_n x + \xi_n) - \frac{(N-2s)}{2} u_n(\lambda_n y + \xi_n) - V(x) + V(y)|^2}{|x-y|^{N+2s}} \, dx \, dy
\]
\[
= \int \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|v_n(x) - v_n(y) - V(x) + V(y)|^2}{|x-y|^{N+2s}} \, dx \, dy
\]
\[
= \|v_n - V\|^2,
\]
therefore we obtain that
\[
\lim_{n \to \infty} \|u_n(\cdot) - \frac{1}{\lambda_n^{(N-2s)/2}} V\left( \frac{\cdot - \xi_n}{\lambda_n} \right)\| = 0,
\]
which completes the proof. \(\square\)

4. Entire nodal solutions

In this section we prove the main result.

Lemma 4.1. Let \( G \) be a closed subgroup of \( O(N) \) and \( \sigma : G \to Z_2 \) be a continuous homomorphism and is surjective, and the group \( G \) satisfies \((G1)\) \((G2)\). Then \( \mathcal{E}(u; \mathbb{R}^N) \) attains its minimum on \( \mathcal{N}(\mathbb{R}^N)^\sigma_G \). Consequently, the problem \((1.1)\) has a nontrival \( \sigma \)-equivariant solution in \( \dot{H}^s(\mathbb{R}^N) \).
Proof. The unit ball $\Omega := \{x \in \mathbb{R}^N : |x| < 1\}$ is $G$-invariant for every $G$, as $0 \in \Omega$, we have that $\Omega^G \neq \emptyset$. Then, by Lemma 2.6 we know $m(\Omega)_G^\sigma = m(\mathbb{R}^N)_G^\sigma$.

By Lemma 3.1 there exists a subsequence $\{u_n\}_{n=1}^\infty$ satisfying

$$\mathcal{E}(u_n ; \Omega) \to m(\Omega)_G^\sigma, \text{ and } \nabla \mathcal{E}(u_n ; \Omega) \to 0 \text{ in } \left(\overline{H^s(\Omega)_G^\sigma}\right)' \text{ as } n \to \infty.$$

Then Lemma 3.4 asserts that there are two possibilities:

- **Case i.** there exists $u \in \mathcal{N}(\Omega)_G^\sigma$, such that $\mathcal{E}(u ; \Omega) = m(\Omega)_G^\sigma$.
- **Case ii.** there exists $V \in \mathcal{N}(\mathcal{H})_G^\sigma$, such that $\mathcal{E}(V ; \mathcal{H}) = m(\mathcal{H})_G^\sigma$.

As $\mathcal{N}(\Theta)_G^\sigma \subset \mathcal{N}(\mathbb{R}^N)_G^\sigma$, for every $G$-invariant domain $\Theta$ in $\mathbb{R}^N$, from the above two possibilities we can get $\mathcal{E}(\cdot ; \mathbb{R}^N)$ attains its minimum on $\mathcal{N}(\mathbb{R}^N)_G^\sigma$. \qed

In order to prove Theorem 1.1 it suffices to show that there are $n$ groups with the properties stated in the following lemma.

**Lemma 4.2.** Let $N = 4n + m$ with $n \geq 1$ and $m \in \{0, 1, 2, 3\}$, then for each $j = 1, \ldots, n$, there exists a closed subgroup $G_j$ of $O(N)$ and a continuous homomorphism $\sigma_j : G_j \to \mathbb{Z}_2$ with the following properties:

- (a) $G_j$ and $\sigma_j$ satisfy $(G1)$ and $(G2)$
- (b) If $u, v : \mathbb{R}^N \to \mathbb{R}$ are nontrivial functions, $u$ is $\sigma_i$-equivariant and $v$ is $\sigma_j$-equivariant with $i \neq j$, then $u \neq v$.

**Proof.** Let $\Gamma$ be the group generated by $\{e^{i\theta} : \theta \in [0, 2\pi]\}$, acting on $\mathbb{C}^2$ by

$$e^{i\theta}(\zeta_1, \zeta_2) := (e^{i\theta}\zeta_1, e^{i\theta}\zeta_2), \quad g(\zeta_1, \zeta_2) := (-\bar{\zeta}_2, \bar{\zeta}_1) \quad \text{for } (\zeta_1, \zeta_2) \in \mathbb{C}^2,$$

and let $\sigma : \Gamma \to \mathbb{Z}_2$ be the homomorphism given by $\sigma(e^{i\theta}) := 1$ and $\sigma(g) := -1$. Note that the $\Gamma$-orbit of a point $z \in \mathbb{C}^2$ is the union of two circles that lie in orthogonal planes if $z \neq 0$, and it is 0 if $z = 0$.

Define $\Lambda_j := O(N - 4j)$ if $j = 1, \ldots, n - 1$ and $\Lambda_n := 1$, then $\Lambda_j$-orbit of a point $y \in \mathbb{R}^{N-4j}$ is an $(N-4j-1)$-dimensional sphere if $j = 1, \ldots, n$, and it is a single point if $j = n$.

Define $G_j = \Gamma_j \times \Lambda_j$, acting coordinatewise on $\mathbb{R}^N \equiv (\mathbb{C}^2)^j \times \mathbb{R}^{N-4j}$, i.e.,

$$(\gamma_1, \ldots, \gamma_j, \eta)(z_1, \ldots, z_j, y)^T = (\gamma_1z_1, \ldots, \gamma_jz_j, \eta y)^T, \quad j = 1, \ldots, n, \quad \text{and } \sigma_j : G_j \to \mathbb{Z}_2$$

where $\gamma_i \in \Gamma, \eta \in \Lambda_j, z_i \in \mathbb{C}^2$ and $y \in \mathbb{R}^{N-4j}$, and $\sigma_j : G_j \to \mathbb{Z}_2$ be the homomorphism

$$\sigma_j(\gamma_1, \ldots, \gamma_j, \eta) = \sigma(\gamma_1)\sigma(\gamma_2)\cdots\sigma(\gamma_j).$$  

(4.2)

Obviously $\sigma_j$ is surjective.

We firstly show that $G_j$ and $\sigma_j$ satisfy $(G1)$ and $(G2)$. On one hand, let $x = (x_1, \ldots, x_j, y) \in \mathbb{R}^N$, and if $G_jx \neq \{x\}$, there exists $(\gamma_1, \ldots, \gamma_j, \eta)$ and $(\beta_1, \ldots, \beta_j, \alpha) \in G_j$ such that

$$(\gamma_1, \gamma_2, \ldots, \gamma_j, \eta)x \neq (\beta_1, \beta_2, \ldots, \beta_j, \alpha)x \in G_jx, \quad j = 1, \ldots, n, \quad \text{therefore } \dim(G_jx) > 0 \text{ and } \dim(G_jx) = \begin{cases} N - 2j - 1, & \text{if } j = 1, \ldots, n - 1, \\ 2j + 1, & \text{if } j = n. \end{cases}$$

(4.3)

On the other hand, let $x = (x_1, \ldots, x_j, 0)$, suppose $g(\xi) = \xi$, it is easy to see

$$g = \left(e^{ik_1\theta}q_{4n_1}, e^{ik_2\theta}q_{4n_2}, \ldots, e^{ik_j\theta}q_{4n_j}, 1\right),$$

where $k_j \in \mathbb{N}, n_j \in \mathbb{N}$ and $\theta = 0$. 

Therefore, we obtain that
\[ \sigma_j(g) = \sigma(e^{ik_1}e^{n_1}) \sigma(e^{ik_2}e^{n_2}) \cdots \sigma(e^{ik_j}e^{n_j}) = 1. \]

Therefore, (G2) hold.

Secondly, we prove (b). Suppose \( u \neq 0 \) is \( \sigma_i \)-equivariant and \( v \neq 0 \) is \( \sigma_j \)-equivariant with \( i < j \), and \( u(x) = v(x) \) for some \( x = (z_1, \ldots, z_j, y) \in (\mathbb{C}^2)^j \times \mathbb{R}^{N-4j} \). Then
\[
\begin{align*}
    u(z_1, \ldots, g_j z_j, y) &= u(z_1, \ldots, z_j, y) \\
    &= v(z_1, \ldots, z_j, y) \\
    &= -v(z_1, \ldots, g_j z_j, y),
\end{align*}
\]

which implies that \( u(z_1, \ldots, g_j z_j, y) \neq v(z_1, \ldots, g_j z_j, y) \). Therefore \( u \neq v \).

We are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let \( N = 4n + m \) with \( n \geq 1 \) and \( m \in \{0, \ldots, 3\} \), for each \( j = 1, \ldots, n \), let \( G_j \) be the closed subgroup of \( O(N) \) and \( \sigma_j \) be the continuous homomorphism given by Lemma 1.2. The Lemma 1.1 tell us that there be \( \sigma_j \)-equivariant solution \( u_j \) to the problem (I.1). Lemma 4.2 shows that the solutions \( u_1, \ldots, u_n \) are pairwise distinct.

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