ON BRAID MONODROMY FACTORIZATIONS

V. Kharlamov and Vik. S. Kulikov

Abstract. We introduce and develop a language of semigroups over the braid groups for a study of braid monodromy factorizations (bmfs) of plane algebraic curves and other related objects. As an application we give a new proof of Orevkov’s theorem on realization of a bmf over a disc by algebraic curves and show that the complexity of such a realization can not be bounded in terms of the types of the factors of the bmf. Besides, we prove that the type of a bmf is distinguishing Hurwitz curves with singularities of inseparable types up to H-isotopy and J-holomorphic cuspidal curves in \( \mathbb{CP}^2 \) up to symplectic isotopy.

Introduction.

In this paper, we deal with algebraic curves and other related objects in the projective plane, such as J-holomorphic and Hurwitz curves (the definition of Hurwitz curves is given in Section 3), which imitate the behavior of plane algebraic curves with respect to pencils of lines. A common feature of all these geometric objects is that for each of them there can be defined so called braid monodromy factorizations (bmfs, in short), which are known to be a powerful tool for study the topology of embedding of curves in \( \mathbb{CP}^2 \). Since founding works by O. Chisini [6],[7], these braid monodromy factorizations are considered as genuine factorizations in the braid groups and studied up to some moves, called in our days Hurwitz moves. We propose to study them by means of suitable semigroups over the braid groups, so that Hurwitz equivalent factorizations become represented by elements of these semigroups. In some cases (such as that of topological Hurwitz curves, see Section 3) it is useful to go up to a second level, i.e., to consider factorization semigroups over semigroups of the first level.

The first author is a member of Research Training Networks EDGE and RAAG, supported by the European Human Potential Program. The second author is partially supported by INTAS-00-00259, 00-00269 and RFBR 02-01-00786.
As it seems to us, the language of semigroups simplifies constructing and study of the objects defined by bmfs. As an example we give an almost pure algebraic proof of a recent result due to S. Orevkov [19], which states that any bmf over a disc can be realized by an algebraic curve (and which generalizes Rudolf’s theorem [20] on algebraic realization of quasi-positive braids). In our construction the curve seats in a ruled surface and has only the simplest ramifications outside the disc. For arbitrary, non necessary round, discs the construction is explicit so that the degree of the curve and the ruling can be bounded. On the other hand, we show, by means of Moishezon examples [17], that for round discs there does not exist any bound in terms of the types of the factors of the bmf.

A braid monodromy factorization of a projective curve in a ruled surface is a factorization of $\Delta^{2N}$, where $\Delta$ is the so called Garside element and $N$ is the degree of the ruling. Its factors correspond to the critical values of the restriction of the projection to the curve. As is known, contrary to over disc factorizations not any projective bmf can be realized by an algebraic curve (see [17]). On the other hand, any projective bmf can be realized in the class of Hurwitz curves. We prove that the type of a braid monodromy factorization (by the type we mean the orbit under the natural conjugacy action of the braid group, see Section 1) is distinguishing Hurwitz curves with singularities of types $w^k = z^n$ up to $H$-isotopy (i.e., isotopy in the class of Hurwitz curves) at least in the case when the critical values in distinct critical points are distinct. And we show that any such Hurwitz curve is $H$-isotopic to an almost-algebraic one, i.e., a one which can be given by an algebraic equation over a disc containing all the critical values of the projection.

In Section 4 we give few remarks on isotopies of $J$-holomorphic curves in $\mathbb{C}P^2$. In particular, using known results on $J$-holomorphic curves and the results of Section 3, we show how various symplectic isotopy problems are reduced to a pure algebraic, in a sense, study of braid factorization types. In this section we show that two cuspidal $J$-holomorphic curves are symplectically isotopic if and only if they have the same braid monodromy factorization type. We also give a bmf-characterization
of algebraicity for nodal symplectic surfaces in $\mathbb{CP}^2$: a nodal symplectic surface is symplectically isotopic to an algebraic curve if and only if its bmf is a partial re-degeneration of a factorization whose factors are conjugates of the squares of standard generators of the braid group.

The paper is organized as follows. Section 1 is devoted to factorization semigroups: we give basic constructions, investigate their functorial properties, and apply them to the braid groups. There we introduce the notion of stable equivalence in semigroups over a braid group and prove that two elements with conjugated factors are stably equivalent if and only if they factorize the same element of the braid group. This result is then used in Section 2 for the proof of the generalized Rudolf theorem mentioned above. Section 3 is devoted to $H$-istopies. There, besides Hurwitz and almost-algebraic curves, which have algebraic singularities, we consider what we call topological Hurwitz curves allowing them to have arbitrary cone singularities. Then, we introduce a class of cone singularities of inseparable type and prove that two topological Hurwitz curves of the same degree having singularities of inseparable types any two of which lie in different fibers of the projection are $H$-isotopic if and only if these curves have the same braid monodromy factorization type. Section 4 deals with the symplectic case.

Acknowledgements. We are grateful to E. Artal-Bartolo, S. Nemirovski and V. Schevchishin for useful discussions and proposals. This research was started during the stay of the second author in Strasbourg university and finished within the frame of RiP programs in Mathematisches Forschungsinstitut Oberwolfach.

§1. Factorization calculus.

1.1. Factorization semigroups. A collection $(S, B, \alpha, \lambda)$, where $S$ is a semigroup, $B$ is a group, and $\alpha : S \to B$, $\lambda : B \to \text{Aut}(S)$ are homomorphisms, is called a semigroup $S$ over a group $B$ if for all $s_1, s_2 \in S$

$$s_1 \cdot s_2 = \lambda(\alpha(s_1))(s_2) \cdot s_1 = s_2 \cdot \rho(\alpha(s_2))(s_1),$$

where $\rho(g) = \lambda(g^{-1})$. If we are given two semigroups $(S_1, B_1, \alpha_1, \lambda_1)$ and $(S_2, B_2, \alpha_2, \lambda_2)$ over, respectively, groups $B_1$ and $B_2$, we call a pair $(h_1, h_2)$ of homomor-
phisms $h_1 : S_1 \to S_2$ and $h_2 : B_1 \to B_2$ a homomorphism of semigroups over groups if

(i) $h_2 \circ \alpha_{S_1} = \alpha_{S_2} \circ h_1,$

(ii) $\lambda_{B_2}(h_2(g))(h_1(s)) = h_1(\lambda_{B_1}(g))(s)$ for all $s \in S_1$ and all $g \in B_1.$

The factorization semigroups defined below constitute the principal, for our purpose, examples of semigroups over groups.

Let $\{g_i\}_{i \in I}$ be a set of elements of a group $B.$ For each $i \in I$ denote by $O_{g_i} \subset B$ the set of all the elements in $B$ conjugated to $g_i$ (the orbit of $g_i$ under the action of $B$ by inner automorphisms). Call their union $X = \cup_{i \in I}O_{g_i} \subset B$ the full set of conjugates of $\{g_i\}_{i \in I}$ and the pair $(B, X)$ an equipped group.

For any full set of conjugates $X$ there are two natural maps $r = r_X : X \times X \to X$ and $l = l_X : X \times X \to X$ defined by $r(a, b) = b^{-1}ab$ and $l(a, b) = aba^{-1}$ respectively. For each pair of letters $a, b \in X$ denote by $R_{a, b; r}$ and $R_{a, b; l}$ the relations defined in the following way:

$R_{a, b; r}$ stands for $a \cdot b = b \cdot r(a, b)$ if $b \neq 1$ and $a \cdot 1 = a$ otherwise;

$R_{a, b; l}$ stands for $a \cdot b = l(a, b) \cdot a$ if $a \neq 1$ and $1 \cdot b = b$ otherwise.

Now, put

$\mathcal{R} = \{R_{a, b; r}, R_{a, b; l} \mid (a, b) \in X \times X, a \neq b \text{ if } a \neq 1 \text{ or } b \neq 1\}$

and introduce the semigroup

$S(B, X) = \langle x \in X : R \in \mathcal{R} \rangle$

by means of this relation set $\mathcal{R}.$ Introduce also a homomorphism $\alpha_X : S(B, X) \to B$ given by $\alpha_X(x) = x$ for each $x \in X.$

Next, we define two actions $\lambda$ and $\rho$ of the group $B$ on the set $X$:

$x \in X \mapsto \rho(g)(x) = g^{-1}xg \in X$

and

$x \in X \mapsto \lambda(g)(x) = gxg^{-1} \in X.$
As is easy to see, the above relation set $R$ is preserved by the both actions and, therefore, $\rho$ and $\lambda$ define an anti-homomorphism $\rho : B \to \text{Aut}(S(B, X))$ (right action) and a homomorphism $\lambda : B \to \text{Aut}(S(B, X))$ (left or conjugation action). The action $\lambda(g)$ on $S(B, X)$ is called simultaneous conjugation by $g$. Put $\lambda_S = \lambda \circ \alpha_X$ and $\rho_S = \rho \circ \alpha_X$.

Claim 1.1. For any $g \in B$ and any $x_i, x_j \in X$ we have

(i) $\lambda(g) = \rho(g^{-1})$;

(ii) $\alpha_X(r(x_i, x_j)) = x_j^{-1}x_ix_j$;

(iii) $\alpha_X(l(x_i, x_j)) = x_ix_jx_i^{-1}$;

(iv) $\rho_S(x_i)(x_j) = r(x_j, x_i)$;

(v) $\lambda_S(x_i)(x_j) = l(x_i, x_j)$;

(vi) $\rho(\alpha_X(x_i)^{-1})(x_j) = l(x_i, x_j)$;

(vii) $\lambda(\alpha_X(x_i)^{-1})(x_j) = r(x_j, x_i)$.

Proof. Straightforward. □

It follows from Claim 1.1 that $(S(B, X), B, \alpha_X, \lambda_S)$ is a semigroup over $B$. We call such semigroups the factorization semigroups over $B$. When $B$ is fixed, we abbreviate $S(B, X)$ to $S_X$. By $x_1 \cdot \ldots \cdot x_n$ we denote the element in $S_X$ defined by a word $x_1 \ldots x_n$.

Notice that $S : (B, X) \mapsto (S(B, X), B, \alpha_X, \lambda)$ is a functor from the category of equipped groups to the category of the semigroups over groups. In particular, if $X \subset Y$ are two full sets of conjugates in $B$, then the identity map $id : B \to B$ defines an embedding $id_{X,Y} : S(B, X) \to S(B, Y)$. So that, for each group $B$, the semigroup $S_B = S(B, B)$ is an universal factorization semigroup over $B$, which means that each semi-group $S_X$ over $B$ is canonically embedded in $S_B$ by $id_{X,B}$.

Since $\alpha_X = \alpha_B \circ id_{X,B}$, we make no difference between $\alpha_X$ and $\alpha_B$ and denote the both simply by $\alpha$.

Denote by $B_X$ the subgroup of $B$ generated by the image of $\alpha : S(B, X) \to B$, and for each $s \in S_X$ denote by $B_s$ the subgroup of $B$ generated by the images $\alpha(x_1), \ldots, \alpha(x_n)$ of the elements $x_1, \ldots, x_n$ of a factorization $s = x_1 \cdot \ldots \cdot x_n$. 
Claim 1.2. *The subgroup $B_s$ of $B$ does not depend on the presentation of $s$ as a word in letters $x_i$ of $X$.*

*Proof.* It follows from (ii) and (iii) of Claim 1.1. □

**Proposition 1.1.** For any $X$ and any $s \in S_X$ as above,

(i) $\ker \lambda$ coincides with the centralizer $C_X$ of $B_X$ in $B$;

(ii) if $\alpha(s)$ belongs to the center $C(B_s)$ of $B_s$, then $\lambda(g)$ leaves fixed $s \in S_X$ whatever is $g \in B_s$.

*Proof.* (i) is evident.

(ii) The group $B_s$ is generated in $B$ by $\alpha(x_1), \ldots, \alpha(x_n)$, where $s = x_1 \ldots x_n$ with $x_i \in X$. Therefore, to prove (ii) it is sufficient to show that $\lambda_S(x_i)(s) = s$ for each $i = 1, \ldots, n$ as soon as $\alpha(s) \in C(B_s)$. Using the relations $x_j \cdot x_i = x_i \cdot r(x_j, x_i)$, we can move $x_i$ to the left and obtain a presentation of $s$ in the form

$$s = x_i \cdot \tilde{x}_1 \ldots \cdot \tilde{x}_{n-1} = x_i \cdot \tilde{s}.$$ 

If $\alpha(s) \in C(B_s)$, then

$$\lambda_S(s)(x_i) = x_i.$$ 

Finally,

$$s = l(\tilde{x}_1, x_i) \ldots l(\tilde{x}_{n-1}, x_i) \cdot x_i = \lambda_S(x_i)(\tilde{x}_1 \ldots \tilde{x}_{n-1}) \cdot x_i =$$

$$= \lambda_S(x_i)(\tilde{s}) \cdot x_i = \lambda_S(\lambda_S(x_i)(\tilde{s}))(x_i) \cdot \lambda_S(x_i)(\tilde{s}) = \lambda_S(x_i \cdot \tilde{s})(x_i) \cdot \lambda_S(x_i)(\tilde{s}) =$$

$$= x_i \cdot \lambda_S(x_i)(\tilde{s}) = \lambda_S(x_i)(x_i) \cdot \lambda_S(x_i)(\tilde{s}) = \lambda_S(x_i)(s).$$

□

Consider two full sets of conjugates $X_1, X_2$ in $B$ and the semigroups $S_{X_1}$ and $S_{X_2}$ associated with them. A map $\psi : X_2 \to S_{X_1}$ can be extended to a homomorphism $\psi : S_{X_2} \to S_{X_1}$ if and only if for all $x_i, x_j \in X_2$ the equalities

$$\psi(x_i) \cdot \psi(x_j) = \psi(x_j) \cdot \psi(r(x_i, x_j))$$
and
\[ \psi(x_i) \cdot \psi(x_j) = \psi(l(x_i, x_j)) \cdot \psi(x_i) \]
hold in \( S_{X_1} \).

We say that the homomorphism \( \psi \) is defined over \( B \) if \( \alpha_{X_2}(x) = \alpha_{X_1}(\psi(x)) \)
for all \( x \in X_2 \).

**Example 1.1.** Let \( X_1 \) be the set of the conjugates of an element \( x_1 \in B \) and \( X_2 \) the set of the conjugates of \( x_1^2 \). Assume that the map \( \phi : X_1 \to X_2 \) given
by \( \phi(x) = x^2 \) for \( x \in X_1 \) is a bijection. Then the map \( \psi : X_2 \to S_{X_1} \) given by
\[ \psi(x) = \phi^{-1}(x) \cdot \phi^{-1}(x) \]
defines a homomorphism \( \psi : S_{X_2} \to S_{X_1} \) over \( B \).

**Example 1.2.** Example 1.1 can be generalized as follows. Pick a \( n \)-set \( \{x_1, \ldots, x_n\} \)
of elements in a full conjugate set \( X_1 \subset B \) and a \( k \)-set of products \( s_j(x_1, \ldots, x_n) = x_{i_1(j)} \cdot \cdots \cdot x_{i_m(j)(j)} \in S_{X_1}, j = 1, \ldots, k \). Consider \( X_2 = O_{\bar{s}_1} \cup \cdots \cup O_{\bar{s}_k} \) with \( O_{\bar{s}_j} \)
being the full set of conjugates of \( \bar{s}_j = \alpha(s_j) \in B \). Assume that \( \bar{s}_i \) and \( \bar{s}_j \) are not
conjugated in \( B \) for \( i \neq j \). Then, the map \( X_2 \to S_{X_1} \) given by \( g\bar{s}_j g^{-1} \mapsto \lambda(g)(s_j) \in S_{X_1} \) can be extended uniquely to a homomorphism \( r : S_{X_2} \to S_{X_1} \) defined over \( B \). Such a homomorphism \( r \) is called re-degeneration of the set \( \{s_j\} \).

In Section 4 we use a kind of generalization of this notion (which no more takes
a form of a homomorphism). It looks as follows. In notation of Example 1.2, put
\( Z = X_1 \cup X_2 \) and consider an element \( z = z_1 \cdot z_2 \in S_Z \) where \( z_1 \in S_{X_2}, z_2 \in S_Z \).
The element
\[ \tau = r(z_1) \cdot z_2 \in S_Z \]
is called a partial re-degeneration of \( z \).

The construction of the semigroups \( S(B, X) \) can be iterated. Namely, one can
consider the conjugation action of \( B \) on \( S(B, X) \), pick any set \( Y \) which is a union
of orbits of this action and introduce the semigroup \( S(S(B, X), Y) \) as a semigroup
generated by the letters \( s \in Y \) and being subject to the relations
\[ s_i s_j = s_j \rho_S(s_j)(s_i) \]
and
\[ s_i s_j = \lambda_S(s_i)(s_j)s_i \]
for all $s_i, s_j \in Y$.

One can introduce, in addition, the homomorphisms $\beta_S : S(S(B,X), Y) \to S(B, X)$ sending $s = s_1 \ldots s_n \in S(S(B,X), Y)$ to $s_1 \cdot \ldots \cdot s_n \in S(B,X)$ and the conjugation actions $\lambda : B \to \text{Aut}(S(S(B,X), Y))$ ($\lambda(g)$ is acting as simultaneous conjugation by $g$), as well as associated with them the homomorphisms $\beta = \alpha \circ \beta_S$ and the actions $\lambda_S = \lambda \circ \alpha_S : S(B,X) \to \text{Aut}(S(S(B,X), Y))$, $\lambda_{S,S} = \lambda \circ \beta : S(S(B,X), Y) \to \text{Aut}(S(S(B,X), Y))$ of, respectively, $B$, $S(B,X)$, and $S(S(B,X), Y)$ on $S(S(B,X), Y)$. The right actions $\rho, \rho_S$, and $\rho_{S,S}$ of, respectively, $B$, $S(B,X)$, and $S(S(B,X), Y)$ on $S(S(B,X), Y)$ are defined in a similar way.

On the other hand, if $X$ is a subset of $Y$, then there is the natural embedding of $S_X = S(B,X)$ into $S(S_X, Y)$. Moreover, there is the natural embedding of $S(S_X, Y)$ into the universal semigroup $S_S(B) = S(S_B, S_B)$ over $S_B$. Thus, any semigroup $S_X$ over $B$ can be considered as a subsemigroup of $S_S(B)$ and we may, without any confusion, denote the operation in $S_S(B)$ by $\cdot$. Note that $S(S_X, X)$ is naturally isomorphic to $S_X$.

1.2. Hurwitz equivalence. Let, as above, $Y$ be a union of orbits of the conjugation action $\lambda$ of $B$ on $S(B,X)$. An ordered set

$$\{y_1, \ldots, y_n \mid y_i \in Y\}, n \in \mathbb{Z}$$

is called a factorization of $g = \beta(y_1) \ldots \beta(y_n) \in B$ in $Y$. Denote by $F_{X,Y} \subset \bigcup_n Y^n$ the set of all possible factorizations of the elements of $B$ in $Y$ over all $n \in \mathbb{N}$. There is a natural map $\varphi : F_{X,Y} \to S(S(B,X), Y)$, given by

$$\varphi(\{y_1, \ldots, y_n\}) = y_1 \cdot \ldots \cdot y_n.$$ 

The transformations which replace in $\{y_1, \ldots, y_n\}$ some two neighboring factors $(y_i, y_{i+1})$ by $(y_{i+1}, \rho_S(y_j)(y_{i+1}))$ or $(\lambda_S(y_i)(y_{i+1}), y_i)$ and preserve the other factors are called Hurwitz moves. Two factorizations are Hurwitz equivalent if one can be obtained from the other by a finite sequence of Hurwitz moves.

Claim 1.3. Two factorizations $y = \{y_1, \ldots, y_n\}$ and $z = \{z_1, \ldots, z_n\}$ are Hurwitz equivalent if and only if $\varphi(y) = \varphi(z)$. 
Proof. Evident. □

Remark 1.1. Below, according with Claim 1.3, we identify classes of Hurwitz equivalent factorizations in $Y$ with their images in $S(S_X, Y)$. And when $Y = X$, we identify $S(S_X, Y)$ with $S_X$.

1.3. Semigroups over the braid group and stable equivalence. In this subsection, $B = B_m$ is the braid group with $m$ strings. We fix a set $\{a_1, \ldots, a_{m-1}\}$ of so called standard generators, i.e., generators being subject to the relations

\begin{align*}
(1.1) & \quad a_i a_{i+1}a_i = a_{i+1}a_ia_{i+1} & 1 \leq i \leq n - 1, \\
(1.2) & \quad a_i a_k = a_k a_i & |i - k| \geq 2.
\end{align*}

We denote by $B_m^+$ the semi-group defined by the same generating letters and relations.

Garside’s Theorem. [8] The natural homomorphism $i : B_m^+ \rightarrow B_m$ is an embedding.

Following this theorem, we identify $B_m^+$ with its image $i(B_m^+)$ in $B_m$ and call the images $i(g)$ of the elements $g \in B_m^+$ positive elements of the group $B_m$.

Denote by $A_k = A_k(m)$, $k \geq 0$, the full set of conjugates of $a_i^{k+1}$ in $B_m$ (recall that all the generators $a_1, \ldots, a_{m-1}$ are conjugated to each other). Consider the semigroup $S_{A_0}$ as a subsemigroup of the universal semi-group $S_{B_m}$ over $B_m$.

A positive word $g = a_{i_1} \ldots a_{i_n}$ in the alphabet $\{a_1, \ldots, a_n\}$ defines an element \( \overline{g}(a_1, \ldots, a_{m-1}) = a_{i_1} \cdot \ldots \cdot a_{i_n} \in S_{A_0} \). On the other hand, $g$ defines an element $\tilde{g} = a_{i_1} \ldots a_{i_n}$ of $B_m^+$.

Lemma 1.1. [3] A map $\nu : B_m^+ \rightarrow S_{A_0}$ given by $\nu(\tilde{g}) = \overline{g}$ is a well-defined injective homomorphism of semi-groups.

Proof. To show that $\nu$ is a well-defined homomorphism, it is sufficient to check that the relations (1.1) and (1.2) hold in $S_{A_0}$. We have

\begin{align*}
\quad & a_i \cdot a_{i+1} \cdot a_i = a_{i+1} \cdot (a_i^{-1}a_ia_{i+1}) \cdot a_i = a_{i+1} \cdot a_i \cdot (a_i^{-1}a_i^{-1}a_ia_{i+1}a_i) = \\
& = a_{i+1} \cdot a_i \cdot (a_i^{-1}a_{i+1}a_ia_{i+1}) = a_{i+1} \cdot a_i \cdot a_{i+1}
\end{align*}
for $1 \leq i \leq n - 1$ and
\[ a_i \cdot a_k = a_k \cdot (a_k^{-1} a_i a_k) = a_k \cdot a_i \]
for $|i - k| \geq 2$. The homomorphism $\nu$ is injective, since $\alpha_B \circ \nu$ is the identity map by Garside’s Theorem. □

Let $\Delta = \Delta_m$ be the so-called Garside element:
\[ \Delta = (a_1 \ldots a_{m-1}) \ldots (a_1 a_2 a_3)(a_1 a_2) a_1. \]
As is well-known,
\[ \Delta^2 = (a_1 \ldots a_{m-1})^m \]
is the generator of the center of $B_m$. Denote by $\delta^2 = \delta^2_m$ the element in $S_{A_0} \subset S_{B_m}$ equal to
\[ \delta^2 = (a_1 \cdot \ldots \cdot a_{m-1})^m. \]

**Lemma 1.2.** The element $\delta^2$ is fixed under the conjugation action of $B_m$ on $S_{B_m}$, i.e., $\rho(g)(\delta^2) = \delta^2$ for any $g \in B_m$.

**Proof.** It follows from $\alpha(\delta^2) = \Delta^2$ and Proposition 1.1 (ii) applied to $s = \delta^2$ (for which $(B_m)_s = B_m$). □

In our study of topological Hurwutz surfaces (see section 3.2) we use an extension $\tilde{S}_{B_m}$ of $S_{B_m}$ which is defined as follows. To each element $I$ belonging to the set $\mathcal{I}$ of all the subsets of $\{1, \ldots, m\}$, let us associate a letter $1_I$. Consider a semigroup $\tilde{S}_{B_m}$ generated by the pairs $(g, 1_I)$, $g \in B_m$ and $I \in \mathcal{I}$, and being subject to the relations
\[
(g_1, 1_{I_1}) \cdot (g_2, 1_{I_2}) = (g_1 g_2 g_1^{-1}, 1_{\sigma(g_1)(I_2)}) \cdot (g_1, 1_{I_1}),
\]
\[
(g_1, 1_{I_1}) \cdot (g_2, 1_{I_2}) = (g_2, 1_{I_2}) \cdot (g_2^{-1} g_1 g_2, 1_{\sigma(g_2^{-1})(I_1)})
\]
for all $g_1, g_2 \in B_m$ and all $I_1, I_2 \in \mathcal{I}$; here $\sigma$ is an action on $\mathcal{I}$ induced by the natural homomorphism from $B_m$ to the symmetric group $\Sigma_m$ acting on $\{1, \ldots, m\}$. To extend the actions $\lambda$ and $\rho$ of $B_m$ on $S_{B_m}$ to the action on $\tilde{S}_{B_m}$ we put
\[ \lambda(b)((g, 1_I)) = (\lambda(b)(g), 1_{\sigma(b)(I)}) \]
and $\rho(g) = \lambda(g^{-1})$ for $b, g \in B_m$ and $I \in \mathcal{I}$. Also we extend the homomorphism $\alpha$ by

$$
\alpha((g, 1_I)) = g \in B_m
$$

for all $I$. Note that the map $g \mapsto (g, 1_{\emptyset})$ is extended to an embedding of $S_{B_m}$ in $\tilde{S}_{B_m}$ over $B_m$.

Denote by $B_{k,i}$, $k + i \leq m$, a subgroup of the braid group $B_m$ generated by a part $a_{i+1}, \ldots, a_{i+k-1}$ of a fixed set of standard generators $a_1, \ldots, a_{m-1}$ of $B_m$. We say that an element $b \in B_m$ has the interlacing number $l(b) = k$ if $k$ is the smallest number such that $b$ is conjugated in $B_m$ to an element in $B_{k,0}$. For a pair $(B_{n,i}, b)$ with $b \in B_{n,i}$ and $l(b) = k$, an element $\tilde{g} = (\bar{b}, 1_{\{i+i+1, \ldots, i+n\}}) \in \tilde{S}_{B_m}$ is called a standard tbmf-form of $b \in B_{n,i}$ if $\bar{b} \in B_{k,i} \subset B_{n,i}$ is conjugated to $b$ in $B_m$ (if $i + k \geq n$ then $\{i + k + 1, \ldots, i + n\} = \emptyset$).

Now to each finite sequence of integers $k_1, \ldots, k_t$ such that $k_1 + \cdots + k_t \leq m$, $k_1 \geq 2, \ldots, k_t \geq 2$, let associate a sequence of subgroups $B_{k_1, k_1} \cdots B_{k_t, k_t}$, $1 \leq i \leq t$, of $B_m$. Introduce also the subsets $T_{k_1, \ldots, k_t}$ of $\tilde{S}_{B_m}$ consisting of the products $s = \tilde{g}_1 \cdots \tilde{g}_t$, where $\tilde{g}_i \in \tilde{S}_{B_m}$ are the standard tbmf-forms of elements in $B_{k_1, k_1} \cdots B_{k_t, k_t}$ for each $1 \leq i \leq t$. Then, define $T = T_m$ to be the union $\bigcup \lambda(g)T_{k_1, \ldots, k_t}$ over all $g \in B_m$ and all sequences of integers $k_1, \ldots, k_t$ such that $k_1 + \cdots + k_t \leq m$. Note that for any permutation $\sigma \in \Sigma_t$ and any $\tilde{g}_1 \cdots \tilde{g}_t \in T_{k_1, k_1} \cdots B_{k_t, k_t}$, one has

$$
\tilde{g}_1 \cdots \tilde{g}_t = \tilde{g}_{\sigma(1)} \cdots \tilde{g}_{\sigma(t)}.
$$

The elements of $T$ are called tbm factorizations and the semigroup $\mathcal{T} = \mathcal{T}_m = S(\tilde{S}_{B_m}, T)$ is called the tbm factorization semigroup. Two tbm factorizations are said of the same factorization type if they belong to the same orbit under the conjugation action of $B_m$.

The group $B_m$ as the set can be represented as the disjoint union over $k$, $1 \leq k \leq m$, of the orbits of $T_{k, \emptyset} = \{(b, 1_{\emptyset}) \mid l(b) = k\} \subset T_k$ under the conjugation action of $B_m$. This presentation defines an imbedding $i : S_{B_m} \to \mathcal{T}$ of semigroups over $B_m$. Thus, when it can not lead to a confusion, we identify $S_{B_m}$ with its image $i(S_{B_m}) \subset \mathcal{T}$. 
We say that an element \( s_1 \in \mathcal{T} \) (in particular, \( s_1 \in S_{B_m} \)) is \textit{stably equal} to an element \( s_2 \in \mathcal{T} \) \((s_2 \in S_{B_m}, \text{respectively})\) if there is an integer \( n \geq 1 \) such that
\[
s_1 \cdot (\delta^2)^n = s_2 \cdot (\delta^2)^n
\]
in \( \mathcal{T} \).

**Theorem 1.1.** Let \( O_{x_1}, \ldots, O_{x_n} \) be the orbits of elements \( x_1, \ldots, x_n \in \mathcal{T} \) under the conjugation action of \( B_m \) on \( \mathcal{T} \). Then for any \( y_i \in O_{x_i}, 1 \leq i \leq n \), and for any permutation \( \sigma \in \Sigma_n \), the elements \( s_1 = x_1 \cdot \ldots \cdot x_n \) and \( s_2 = y_{\sigma(1)} \cdot \ldots \cdot y_{\sigma(n)} \) are stably equal in \( \mathcal{T} \) if and only if \( \beta(s_1) = \beta(s_2) \).

**Remark 1.2.** The above theorem remains true if \( \mathcal{T} \) is replaced by any sets \( X \subset S_{B_m}, \) containing \( A_0 \) and \( \mathcal{T} \) by \( S(S_{B_m}, X) \).

**Proof of Theorem 1.1.** It is evident that if \( s_1 \) and \( s_2 \) are stably equal in \( \mathcal{T} \) then \( \beta(s_1) = \beta(s_2) \).

Let \( \beta(s_1) = \beta(s_2) \). Since any permutation is a product of transpositions, we can assume that \( \sigma = \text{id} \). Indeed, for each \( g_1, g_2 \in \mathcal{T} \) we have the relation \( g_1 \cdot g_2 = g_2 \cdot \rho(g_2)(g_1) \) as a relation in \( \mathcal{T} \) and in which \( g_1 \) and \( \rho(g_2)(g_1) \) belong to the same orbit. So, applying these relations we get \( s_2 = \tilde{y}_1 \cdot \ldots \cdot \tilde{y}_n \) with the factors \( \tilde{y}_i \in O_{x_i} \).

**Lemma 1.3.** ([8]) For any \( g \in B_m \) there are positive elements \( r_1, r_2 \in B_m \) and integers \( k, p \in \mathbb{Z}, p \geq 1 \), such that
\[
(i) \quad g = \Delta^{2k} r_1;
(ii) \quad gr_2 = \Delta^{2p}.
\]

**Proof.** It follows from Theorem 5 in [8]. \( \square \)

By Lemma 1.3 (i), since \( x_i \) and \( \tilde{y}_i \) belong to the same orbit \( O_{x_i} \) and \( \Delta^2 \) belongs to the center of \( B_m \), we may assume that there are positive elements \( g_i \) such that \( \tilde{y}_i = \rho(g_i^{-1})(x_i) \). Applying Lemma 1.3 (ii) to each \( g_i \), we can find positive elements \( r_i \) and positive integers \( p_i \) such that \( g_i r_i = \Delta^{2p_i} \). By Garside’s Theorem, Claim 1.1 and Lemma 1.1, \( \tilde{y}_i \cdot \tilde{r}_i = (\delta^2)^{p_i} \) in \( S_{A_0} \subset S_{B_m} \subset \mathcal{T} \). Put \( p = p_1 + \cdots + p_n \). By
Proposition 1.1 (ii), it follows that for each \( x \in S_{B_m} \), we have \( x \cdot \delta^2 = \delta^2 \cdot x \) in \( S_{B_m} \).

In addition, \( \rho(g_i^{-1})(x_i) \cdot \bar{g}_i = \bar{g}_i \cdot x_i \). Therefore,

\[
s_2 \cdot (\delta^2)^p = \rho(g_1^{-1})(x_1) \cdot \ldots \cdot \rho(g_n^{-1})(x_n) \cdot (\delta^2)^p = \\
= \rho(g_1^{-1})(x_1) \cdot (\delta^2)^{p_1} \cdot \ldots \cdot \rho(g_n^{-1})(x_n) \cdot (\delta^2)^{p_n} = \\
= \rho(g_1^{-1})(x_1) \cdot \bar{g}_1 \cdot \bar{r}_1 \cdot \ldots \cdot \rho(g_n^{-1})(x_n) \cdot \bar{g}_n \cdot \bar{r}_n = \\
= \bar{g}_1 \cdot x_1 \cdot \bar{r}_1 \cdot \ldots \cdot \bar{g}_n \cdot x_n \cdot \bar{r}_n.
\]

Consider in the beginning the case when all \( x_i \) are standard generators. Then, applying the Garside theorem to \( \alpha(s_1)(\Delta^2)^p = \alpha(s_2)(\Delta^2)^p \) in \( B_m \) we get

\[
s_1 \cdot (\delta^2)^p = \bar{g}_1 \cdot x_1 \cdot \bar{r}_1 \cdot \ldots \cdot \bar{g}_n \cdot x_n \cdot \bar{r}_n
\]

in \( \nu(B_{m_1}^+) \), which gives \( s_1 \cdot (\delta^2)^p = s_2 \cdot (\delta^2)^p \) in \( \nu(B_{m_1}^+) \).

In general case, all \( \bar{g}_i \) and \( \bar{r}_i \) belong to \( \nu(B_{m_1}^+) \). Applying the relations \( a_i \cdot x_j = x_j \cdot \rho_{S,S}(x_j)(a_i) \), we can move to the left all \( x_i \) and obtain that \( s_2 \cdot (\delta^2)^p = s_1 \cdot s_3 \),

where \( s_3 = \prod(t_i^{-1}z_i t_i) \) and each \( z_i \) is a letter of the alphabet \( \{a_1, \ldots, a_{m-1}\} \). Thus there is a positive integer \( q \) such that \( s_3 \cdot (\delta^2)^q = (\prod z_i) \cdot (\delta^2)^q \), and therefore

\[
s_2 \cdot (\delta^2)^{p+q} = s_2 \cdot (\delta^2)^p \cdot (\delta^2)^q = s_1 \cdot s_3 \cdot (\delta^2)^q = s_1 \cdot (\prod z_i) \cdot (\delta^2)^q = s_1 \cdot (\delta^2)^{p+q},
\]

since \((\delta^2)^{p+q}\) and \((\prod z_i) \cdot (\delta^2)^q\) belong to \( \nu(B_{m_1}^+) \) and \( \alpha((\prod z_i) \cdot (\delta^2)^q) = (\Delta^2)^{p+q} \). \([\square]\)

Let us address two problems which seem to be open.

**Garside problem.** Is \( \alpha : S_{A_0(m)} \rightarrow B_m \) an embedding for any \( m \)? In particular, does the equation \( \alpha(s) = \Delta^2_m \) have only one solution, \( s = \delta^2_m \)?

**Word problem.** Does the word problem for \( T_m \) (respectively, for \( S_{B_m}, S_{A_{\leq 2}} \) with \( A_{\leq 2} = \bigcup_{k \leq 2} A_k(m) \)) have the positive solution?

**1.4. ”Pure nodal” semigroup.** In this subsection we work with the semigroup \( S_{A_1(m)} \). Let fix a set of standard generators \( \{a_1, \ldots, a_{m-1}\} \) of \( B_m \) and consider \( B_{m-1} \) as a subgroup of \( B_m \) generated by \( \{a_1, \ldots, a_{m-2}\} \). Put

\[
\tilde{\delta}^2_m = \prod_{l=m}^{2} \prod_{k=1}^{l-1} z_{k,l}^2 \in S_{A_1},
\]
where \( z_{k,l} = (a_{l-1} \ldots a_{k+1})a_k(a_{l-1} \ldots a_{k+1})^{-1} \) for \( k < l \) (the notation \( \prod_{a}^b \) states for the left to the right product from \( a \) to \( b \)). As is known, see for example [18], \( \alpha(\tilde{\delta}_m^2) = \Delta_m^2 \) and

\[
\tilde{\delta}_m^2 = \prod_{k=1}^{m-1} z_{k,m}^2 \cdot \tilde{\delta}_{m-1}^2.
\]

**Lemma 1.4.**

\((i)\) \( a_k z_{j,k}^2 a_k^{-1} = z_{j,k+1}^2; \)
\((ii)\) \( a_k z_{k,m}^2 a_k^{-1} = z_{k+1,m}^2; \)
\((iii)\) \( a_k z_{j,r}^2 a_k^{-1} = z_{j,r}^2 \) if either \( r < k \), or \( j < k \), or \( j > k + 1 \);
\((iv)\) \( a_k z_{k+1,m}^2 a_k^{-1} = z_{k+1,m}^2 z_{k,m}^2 z_{k+1,m}^2. \)

**Proof.** It follows from relations (1.1) and (1.2) and the definition of the elements \( z_{k,j}^2 \). \( \square \)

**Proposition 1.2.** The element \( \tilde{\delta}_m^2 \in S_{A_1} \) is a fixed element under the conjugation action of \( B_m \).

**Proof.** It is sufficient to show that \( \lambda(a_i)(\tilde{\delta}_m^2) = \tilde{\delta}_m^2 \) for \( i = 1, \ldots, m - 1 \). These equalities will be proved by induction on \( m \).

By induction hypothesis and Lemma 1.4, we have that for \( i < m - 1 \)

\[
\lambda(a_i)(\tilde{\delta}_m^2) = \lambda(a_i)(\prod_{k=1}^{m-1} z_{k,m}^2 \cdot \tilde{\delta}_{m-1}^2) = \\
= \prod_{k=1}^{i-1} \lambda(a_i)(z_{k,m}^2) \cdot \lambda(a_i)(z_{i,m}^2) \cdot \lambda(a_i)(z_{i+1,m}^2) \cdot \prod_{k=i+2}^{m-1} \lambda(a_i)(z_{k,m}^2) \cdot \lambda(a_i)(\tilde{\delta}_{m-1}^2) = \\
= \prod_{k=1}^{i-1} z_{k,m}^2 \cdot z_{i+1,m}^2 \cdot \prod_{k=i+2}^{m-1} \lambda(a_i)(z_{k,m}^2) \cdot \lambda(a_i)(\tilde{\delta}_{m-1}^2) = \tilde{\delta}_m^2.
\]
Applying again Lemma 1.4, we have that for $i = m - 1$

$$
\lambda(a_{m-1})(\delta^2_m) = \lambda(a_{m-1})(\prod_{k=1}^{m-1} z^2_{k,m} \cdot \prod_{k=1}^{m-2} z^2_{k,m-1} \cdot \delta^2_{m-2}) =
$$

$$
= \prod_{k=1}^{m-2} \lambda(a_{m-1})(z^2_{k,m}) \cdot \lambda(a_{m-1})(z^2_{m-1,m}) \cdot \prod_{k=1}^{m-2} \lambda(a_{m-1})(z^2_{k,m-1}) \cdot \lambda(a_{m-1})(\delta^2_{m-2}) =
$$

$$
= \prod_{k=1}^{m-2} (a_{m-1} z^2_{k,m} a_{m-1}) \cdot z^2_{m-1,m} \cdot \prod_{k=1}^{m-2} (a_{m-1} z^2_{k,m-1} a_{m-1}) \cdot \delta^2_{m-2} =
$$

$$
= z^2_{m-1,m} \cdot \prod_{k=1}^{m-2} (a^{-1}_{m-1} z^2_{k,m} a_{m-1}) \cdot \prod_{k=1}^{m-2} z^2_{k,m} \cdot \delta^2_{m-2} =
$$

$$
= z^2_{m-1,m} \cdot \prod_{k=1}^{m-2} z^2_{k,m-1} \cdot \prod_{k=1}^{m-2} z^2_{k,m} \cdot \delta^2_{m-2}.
$$

To complete the proof of the Proposition, it is sufficient to show that

$$(1.3) \quad \prod_{k=1}^{m-2} z^2_{k,m} \cdot z^2_{m-1,m} \cdot \prod_{k=1}^{m-2} z^2_{k,m-1} = z^2_{m-1,m} \cdot \prod_{k=1}^{m-2} z^2_{k,m-1} \cdot \prod_{k=1}^{m-2} z^2_{k,m}.$$ 

We have

$$
t_m = \alpha(z^2_{m-1,m} \cdot \prod_{k=1}^{m-2} z^2_{k,m-1}) = a^2_{m-1} a_{m-2} \cdots a^2_2 a_1 a_2 \cdots a_{m-2}
$$

and to prove equality (1.3) it is sufficient to show that

$$
t_m z^2_{k,m} t_m^{-1} = z^2_{k,m}
$$

for $k = 1, \ldots m - 2$. By induction on $m$, applying relations (1.1) and (1.2), we have
for $k \leq m - 3$

$$t_m z_{k,m}^{-1} = (a_{m-1}^2 a_{m-2} t_m a_m - 2)^2 z_{k,m}^2 (a_{m-1}^2 a_{m-2} t_m a_m - 2)^{-1} =$$

$$= (a_{m-1}^2 a_{m-2} t_m a_m - 2)^2 z_{k,m}^2 (a_{m-1}^2 a_{m-2} t_m a_m - 2)^{-1} =$$

$$= (a_{m-1}^2 a_{m-2} t_m a_m - 2)^2 z_{k,m}^2 (a_{m-1}^2 a_{m-2} t_m a_m - 2)^{-1} =$$

$$= (a_{m-1}^2 a_{m-2} t_m a_m - 2)^2 z_{k,m}^2 (a_{m-1}^2 a_{m-2} t_m a_m - 2)^{-1} =$$

$$= (a_{m-1}^2 a_{m-2} t_m a_m - 2)^2 z_{k,m}^2 (a_{m-1}^2 a_{m-2} t_m a_m - 2)^{-1} =$$

$$= (a_{m-1}^2 a_{m-2} t_m a_m - 2)^2 z_{k,m}^2 (a_{m-1}^2 a_{m-2} t_m a_m - 2)^{-1} =$$

$$= (a_{m-1}^2 a_{m-2} t_m a_m - 2)^2 z_{k,m}^2 (a_{m-1}^2 a_{m-2} t_m a_m - 2)^{-1} =$$

$$= (a_{m-1}^2 a_{m-2} t_m a_m - 2)^2 z_{k,m}^2 (a_{m-1}^2 a_{m-2} t_m a_m - 2)^{-1} =$$

$$= (a_{m-1}^2 a_{m-2} t_m a_m - 2)^2 z_{k,m}^2 (a_{m-1}^2 a_{m-2} t_m a_m - 2)^{-1} =$$

$$= (a_{m-1}^2 a_{m-2} t_m a_m - 2)^2 z_{k,m}^2 (a_{m-1}^2 a_{m-2} t_m a_m - 2)^{-1} =$$

$$= (a_{m-1}^2 a_{m-2} t_m a_m - 2)^2 z_{k,m}^2 (a_{m-1}^2 a_{m-2} t_m a_m - 2)^{-1} =$$

$$= (a_{m-1}^2 a_{m-2} t_m a_m - 2)^2 z_{k,m}^2 (a_{m-1}^2 a_{m-2} t_m a_m - 2)^{-1} =$$

$$= (a_{m-1}^2 a_{m-2} t_m a_m - 2)^2 z_{k,m}^2 (a_{m-1}^2 a_{m-2} t_m a_m - 2)^{-1} =$$

$$= (a_{m-1}^2 a_{m-2} t_m a_m - 2)^2 z_{k,m}^2 (a_{m-1}^2 a_{m-2} t_m a_m - 2)^{-1} =$$

$$= (a_{m-1}^2 a_{m-2} t_m a_m - 2)^2 z_{k,m}^2 (a_{m-1}^2 a_{m-2} t_m a_m - 2)^{-1} =$$

$$= (a_{m-1}^2 a_{m-2} t_m a_m - 2)^2 z_{k,m}^2 (a_{m-1}^2 a_{m-2} t_m a_m - 2)^{-1} =$$

Using the same calculations as above, one can show that $t_m z_{m-2}^2 t_m^{-1} = z_{m-2}^2$. □

The following theorem is a consequence of Proposition 1.2 and Corollary 4.1, Theorem 3.1, Remark 4.1 which will be proven in sections 3 and 4.

**Theorem 1.2.** Let $A_1 \subset B_m$ be the full set of conjugates of $a_1^2$. Then, $\tilde{\delta}_m^2$ is the only element $s \in S_{A_1}$ such that $\alpha(s) = \Delta_m^2$. □

§2. Existence of polynomials with given braid monodromy factorizations over a disc.

2.1. Local braid monodromy over a point (unique germ). Here and further we denote by $(z, w)$ the standard coordinates in $\mathbb{C}^2$. The proofs of the (well known) statements used here and the further references can be found, for example, in [4].

If a germ $(C, 0) \subset (\mathbb{C}^2, 0)$ of a reduced complex analytic curve does not contain the germ $z = 0$ then it is given by an equation

\[(2.1) \quad P(z, w) = 0,\]
\[ P(z, w) = w^k + \sum_{i=1}^{k} q_i(z)w^{k-i}, \]

where \( q_i(z) \) are convergent power series (i.e., \( q_i(z) \in \mathbb{C}\{z\} \)), \( q_i(0) = 0 \) and the polynomial \( w^k + \sum q_i(z)w^{k-i} \in \mathbb{C}\{z\}\{w\} \) has no multiple factors. Therefore, one can choose a small polydisc \( D = D_1 \times D_2 \subset \mathbb{C}^2 \), \( D_1 = D_1(\varepsilon_1) = \{ z \in \mathbb{C} \mid |z| < \varepsilon_1 \} \) and \( D_2 = D_2(\varepsilon_2) = \{ w \in \mathbb{C} \mid |w| < \varepsilon_2 \} \), such that: \( C \) is an analytic set at each point of the closure \( \text{Cl} D \) of \( D \), the projection on the \( z \)-factor \( \text{pr} = \text{pr}_1 : C \cap D \rightarrow D_1 \) is a proper finite map of degree \( k \), and \( (z, w) = 0 \) is the unique critical point of \( \text{pr}|_{C \cap \text{Cl} D} \). Reciprocally, if \( D \) is a polydisc and \( C \) is \( W \)-prepared in \( D \), i.e., if \( C \) is a reduced complex analytic curve with the latter properties with respect to \( \text{pr} \), then it is defined in \( \text{Cl} D \) by an equation of the same type. A \( W \)-prepared germ \((C, o)\) is algebraic (in coordinates \((z, w)\)) if and only if \( q_i \in \mathbb{C}[z] \) for each \( i = 1, \ldots, k \).

To define the braid monodromy, let pick a point \( u \in \partial D_1 \) and put \( D_{2,u} = \text{pr}^{-1}(u), K = K(u) = \{ w_1, \ldots, w_k \} = D_{2,u} \cap C \). The loop \( \partial D_1 \) oriented counterclockwise and starting at \( u \) lifts to \( \partial D \cap C \) as a motion \( \text{pr}_2(\{w_1(t), \ldots, w_k(t)\}) \) of \( k \) distinct points in \( D_2 \) starting and ending at \( K \). This motion defines a braid \( b_{(C, o)} \in B_k = B_k[D_2, K] \) which is called the braid monodromy of \((C, o)\) with respect to \( \text{pr} \). Note that \( l(b_{(C, o)}) = k \).

The link of \((C, 0)\) is an iterated positive torus link. This link is determined by the Puiseaux pairs of irreducible components of \((C, 0)\) and the mutual intersection numbers of the components. Therefore, as soon as we choose in \( B_k \) as standard generators the half-twists \( a_1, \ldots, a_{k-1} \), the braid \( b_{(C, o)} \) becomes an element of \( B_k^+ \subset B_k \) (a positive braid) and is called the standard form of the braid monodromy of \((C, o)\) with respect to \( \text{pr} \). The standard generators and the standard form \( b_{(C, 0)} \) are defined uniquely up to conjugation.

The topological type of the triple \((D, C, \text{pr})\) is determined by the standard form of its braid monodromy, and vice versa. Besides, for each triple \((D, C, \text{pr})\) there is a constant \( M = M_{(C, o)} \in \mathbb{N} \) such that the topological type of \((D, C, \text{pr})\) coincides with the topological type of a singularity given by \( \overline{P}(z, w) = w^k + \sum \overline{q}_i(z)w^{k-i} = \)
0, where $\bar{q}_i(z) = q_i(z) + z^M r_i(z)$ and $r_i(z)$ are arbitrary analytic functions. In particular, one can make $C$ algebraic without changing the topological type of $(D, C, \text{pr})$.

The topological type of the triple $(D, C, \text{pr})$, and thus the standard form of its braid monodromy, is determined by a \textit{resolution of singularities relative to} $\text{pr}$. By the latter we mean a sequence of blow ups $\sigma_1: U_1 \to D$, $\ldots$, $\sigma_n: U_n \to U_{n-1}$ with centers at points such that $\sigma^{-1}(C \cup F)$, where $\sigma = \sigma_n \circ \cdots \circ \sigma_1$ and $F = \{z = 0\}$, is a divisor with normal crossings. Let put

$$\sigma^*(C) = C' + \sum_{i=1}^n c_i E_i$$

and

$$\sigma^*(F) = E_0 + \sum_{i=1}^n a_i E_i,$$

where $C'$, $E_0$, and $E_i, 1 \leq i \leq n$ are the strict transforms in $U_n$ of $C$, $F$, and the exceptional divisors of $\sigma_i, 1 \leq i \leq n$, respectively. In this notation, for the constant $M$ mentioned above one can take any $m$ such that

$$m(\sum a_i E_i) - (\sum c_i E_i)$$

is a strictly positive divisor, i.e., if $ma_i - c_i > 0$ for all $i$. Indeed, let $s$ be a singular point of $\sigma^{-1}(C \cup F)_{\text{red}}$, $s \in C' \cap E_i$ for some $i$. Choose local coordinates $(z_i, w_i)$ in a neighbourhood of $s$ such that $z_i = 0$ is an equation of $E_i$ and $w_i = 0$ is an equation of $C'$. We have

$$\sigma^*(z) = z_i^{a_i} \quad \sigma^*(P(z, w)) = z_i^{c_i} w_i$$

up to a function non-vanishing at $s$ and

$$\sigma^*(P(z, w)) = (w_i + z_i^{Ma_i-c_i} \sum \sigma^*(w^k r_j(z))) z_i^{c_i}.$$ 

Therefore the germ given by $P(z, w) = 0$ has the same resolution of singularities as $(C, 0)$ has.
2.2. Local braid monodromy over a point (several germs). Now let $C \in \mathbb{C}^2$ be an affine reduced algebraic curve given in coordinates $(z, w)$ by equation

\[(2.2) \quad w^m + \sum_{i=1}^{m} q_i(z)w^{m-i} = 0, \quad q_i \in \mathbb{C}[z].\]

Notice that any affine algebraic curve is given by such an equation after a suitable linear change of coordinates.

Consider $C_{\varepsilon_1} = C \cap (D_1(\varepsilon_1) \times D_2(\varepsilon_2))$, $\varepsilon_2 = \varepsilon_1^{-1}$. Assuming that $0 < \varepsilon_1 << 1$, the projection $\text{pr}_{|C_{\varepsilon_1}} : C_{\varepsilon_1} \rightarrow D_1$, $D_1 = D_1(\varepsilon_1)$, is a proper map of degree $m$ with the unique critical value $z = 0$ (contrary to the situation in 2.1, here the number of critical points may be more than one). By a traditional abuse of language we speak on $C_{\varepsilon_1}$ as the germ of $C$ over 0 with respect to pr. As in the local case (see 2.1), we fix $u \in \partial D_1$ and put $D_{2,u} = \text{pr}^{-1}(u)$, $K = K(u) = \{w_1, \ldots, w_m\} = D_{2,u} \cap C$. Giving to $\partial D_1$ its counter clock-wise orientation, we get over $\partial D_1$ by means of $\text{pr}_2(C \cap \partial D_1)$ an oriented loop of $m$-tuples in $D_2$ and, thus, a braid $\tilde{b}_{(C_{\varepsilon_1}, o)} \in B_m = B_m[D_2, K]$.

Let $\text{pr}_{|C}^{-1}(0) = \{(0, w_1^0), \ldots, (0, w_s^0)\}$. Then the germ $C_{\varepsilon_1}$ over 0 of $C$ splits into the disjoint union $C_{\varepsilon_1} = \bigsqcup_i C_{\varepsilon_1,i}$ of $W$-prepared germs of singularities of multiplicities $k_i$, $1 \leq i \leq s$, $k_1 + \cdots + k_s = m$ with centers at $(0, w_i)$, $1 \leq i \leq s$. Let $k_1, \ldots, k_i \geq 2$ and $k_{i+1} = \cdots k_s = 1$. We need to select a suitable polydisc for each of these $W$-prepared germs. Therefore, choose $\varepsilon_3 > 0$ and adjust $\varepsilon_1 << \varepsilon_3$ so that each $C_{\varepsilon_1,i} \subset D_1 \times E_i$, where $E_i = \{|w - w_i^0| < \varepsilon_3\}$ and $E_i \cap E_j = \emptyset$ for $i \neq j$. Denote by $K_i = K_i(u) = \{w_{i,1}, \ldots, w_{i,k_i}\} = D_{2,u} \cap C_{\varepsilon_1,i}$ and $E_{i,u} = (D_1 \times E_i) \cap D_{2,u}$. The embeddings $(E_i, K_i) \subset (D_{2,u}, K)$ defines embeddings $\eta_i : B_{k_i}[E_{i,u}, K_i] \subset B_m[D_{2,u}, K]$ so that

$$\tilde{b}_{(C_{\varepsilon_1}, o)} = \prod_{i=1}^{t} \tilde{b}_{(C_{\varepsilon_1,i}, o)} \subset B_m.$$ 

As in 2.1, let choose in each of $B_{k_i}[E_{i,u}, K_i]$ as standard generators the half twists. Then, each $\tilde{b}_{(C_{\varepsilon_1,i}, o)} \in B_{k_i}$ becomes the standard form of braid monodromy of $C_{\varepsilon_1,i}$ with respect to pr. The union of the images of these generators under $\eta_i$ can be extended to a set of standard half-twist generators in $B_m = B_m[D_{2,u}, K]$. Thus,
we get a topological braid monodromy
\[ b(C_{\varepsilon_1}, o) = b(C_{\varepsilon_1}, o) \cdot \ldots \cdot b(C_{\varepsilon_1}, t, o) \subset T \]
(see the definition of \( T \) in section 1.3) and call it the \textit{standard form} of braid monodromy of \( C \) over 0.

The above construction depends only on the numbering of the points \( p_i^{-1}(0) = \{(0, w^0_1), \ldots, (0, w^0_s)\} \) and the extension of the union of the images of the generators under \( \eta_i \) to the sets of generators in \( B_m = B_m[D_{2,u}, K] \). Therefore, the standard form of braid monodromy is \textbf{defined uniquely up to conjugation action} of \( B_m \).

As it follows from the above construction,
\[ \alpha(b(C_{\varepsilon_1}, o)) = \tilde{b}(C_{\varepsilon_1}, o). \]

To conclude this subsection, let us recall a well known elementary transformation replacing a germ of a pencil by a local singularity. Namely, with a germ of \( C \) over 0 given by \((2.2)\) one can associate another germ \( \bar{C}_{\varepsilon_1} \) given by
\[ w^m + \sum_{i=1}^{m} z^i q_i(z)w^{m-i} = 0. \]
It has only one point over 0 and we call it the \textit{associated singularity} of \( C \) over 0.

\textbf{Remark 2.1.} The geometric meaning of the associated singularity is the following. We can consider \( \mathbb{C}^2 \) with coordinates \((z, w)\) as an affine chart in some ruled surface \( \text{pr} : \Sigma_N \to \mathbb{P}^1, N > 0 \), non-intersecting the exceptional section \( E_N \) of \( \Sigma_N \), and \( C \) as an algebraic curve contained in this chart of \( \Sigma_N \). Perform the elementary transformation \( \tau : \Sigma_N \to \Sigma_{N+1} \) with center at the intersection point of \( E_N \) with the fiber \( F_0 \) over the point \( z = 0 \). Then, equation \((2.3)\) is the equation of the image \( \tau(C) \subset \Sigma_{N+1} \) in the corresponding chart of \( \Sigma_{N+1} \). The associated singularity determines uniquely the germ of \( C \): to get it back it is sufficient to perform the inverse transformation \( \tau^{-1} \).

The next claim sets up a relation between a standard braid monodromy form of a germ of a curve and the braid monodromy of the singularity associated to the germ. Since this claim is not used below, we put its proof to the end of this section.
**Claim 2.1.** Let \((C_{\varepsilon_1}, o)\) be a germ of an algebraic curve of degree \(m\) over a point \(o\) and \((\bar{C}_{\varepsilon_1}, o)\) the germ of its associated singularity. Then

\[
b_{(\bar{C}_{\varepsilon_1}, o)} = \alpha(b_{(C_{\varepsilon_1}, o)}) \Delta_m^2.
\]

**Remark 2.2.** Since a germ of a curve is determined by its associated singularity, it follows from this claim that two germs \((C_1, \varepsilon_1, o)\) and \((C_2, \varepsilon_1, o)\) are topologically equivalent if and only if \(\alpha(b_{(C_1, \varepsilon_1, o)}) = \alpha(b_{(C_2, \varepsilon_1, o)})\). Therefore, introducing a definition of braid monodromy factorizations of algebraic curves (see below), we could restrict ourselves to factorization semigroups of the first level. However, in Section 3 we extend the class of curves under investigation from algebraic to topological Hurwitz curves. A braid monodromy factorization of a topological Hurwitz curve requires to consider factorization semigroups of the second level. Therefore, for unity of exposition, we define the braid monodromy factorization of an algebraic curve as an element of some factorization semigroup of the second level over a braid group as well.

### 2.3. Braid monodromy factorization over a disc.

Here, as in subsection 2.2, we consider a polynomial \(P(z, w) = w^m + \sum_{i=1}^{m} q_i(z)w^{m-i} \in \mathbb{C}[z, w]\) having no multiple factors and the curve \(C\) in \(\mathbb{C}^2\) given by \(P(z, w) = 0\).

Pick any \(r > 0\) such that no critical value of \(pr|_C\) belongs to the boundary of the disc \(D_1(r)\). Denote by \(z_1, \ldots, z_n \in D_1(r)\) the critical values of \(pr|_C\) situated in \(D_1(r)\). Choose a positive \(\varepsilon << 1\) such that the discs \(D_{1,i}(\varepsilon) = \{z \in \mathbb{C} \mid |z - z_i| < \varepsilon\}, i = 1, \ldots, n\), would be disjoint. Pick any points \(u_i \in \partial D_{1,i}(\varepsilon), 1 \leq i \leq n\), and a point \(u_0 \in \partial D_1(r)\). Put \(D_{2,u_i} = pr^{-1}(u_i), i = 0, \ldots, n\), and \(K(u_i) = \{w_{i,1}, \ldots, w_{i,m}\} = D_{2,u_i} \cap C\). Select disjoint simple paths \(l_i \subset Cl D_1(r) \cup U_i D_{1,i}(\varepsilon), i = 1, \ldots, n\), starting at \(u_0\) and ending at \(u_i\) and renumber the points, if needed, in a way that the product \(\gamma_1 \ldots \gamma_n\) of the loops \(\gamma_i = l_i \circ \partial D_{1,i}(\varepsilon) \circ l_i^{-1}\) would be equal to \(\partial D_1(r)\) in \(\pi_1(Cl D_1(r) \setminus \{z_1, \ldots, z_n\}, u_0)\) (as usual, \(\partial D\) states for a one counter-clock wise turn loop).

Now, as in subsection 2.2, one can associate with each \(\gamma_i, 1 \leq i \leq n\), an element \(b_i \in T_0 \subset T \subset S_{B_m}, B_m = B_m[D_2, K(u_0)]\), where \(D_2\) is a disc of a big
radius in $w$-plane. For each $1 \leq i \leq n$ the conjugacy class of $b_i$ is the standard form of braid monodromy of the germ of $C$ at $z_i$. The factorization $b_1 \cdot \ldots \cdot b_n \in T$ is called a braid monodromy factorization of the polynomial $P(z, w)$, or that of the curve $C$, over $D_1(r)$. Since, for given $P(z, w)$, the braid monodromy factorizations over $D_1(r)$ coincide up to Hurwitz moves and simultaneous conjugations (see [18]), they all have the same factorization type in the sense of section 1. We denote the factorization type of braid monodromy factorizations of $P(z, w)$ over $D_1(r)$ (respectively, of the curve $C$ given by $P(z, w) = 0$) by $\text{bmt}(P(z, w), D_1(r))$ (respectively, $\text{bmt}(C, D_1(r))$). If $P(z, w)$ has no critical values outside $D_1(r)$, we speak simply on braid modnoromy factorization of $P(z, w)$ (or $C$) and denote it factorization type by $\text{bmt}(P(z, w))$ (respectively, $\text{bmt}(C)$).

Consider all braid monodromy standard forms of all germs over 0 of all affine algebraic curves of degree $m$. Denote by $\mathcal{P}$ the union of their orbits under the conjugation action of $B_m$ on $S_{B_m}$ and put $\mathcal{P} = \mathcal{P}_m = S(S_{B_m}, P)$. Note that $A_k \subset \mathcal{P}$, where $A_k$ is defined in section 1.2 (in particular, the elements belonging to $A_0$ correspond to a simple tangency between $C$ and $z = 0$, belonging to $A_1$ correspond to a node, and belonging to $A_2$ to an ordinary cusp). Since $\mathcal{P} = S(S_{B_m}, P)$ is embedded into $\mathcal{T}$, each braid monodromy factorization of a polynomial $P(z, w)$ over a disc $D_1(r)$ defines an element in the semigroup $\mathcal{P}$ and two braid monodromy factorizations of two polynomials over discs are of the same braid factorization type if and only if the corresponding elements in $\mathcal{P}$ belong to the same orbit of the conjugation action of $B_m$ on $\mathcal{P}$. Therefore, we call the orbits of the conjugation action of $B_m$ on $\mathcal{P}$ geometric braid factorization types. Note that $\beta_S : \mathcal{P} \rightarrow S_{B_m}$ is an embedding. Therefore, often we do not make difference between $\mathcal{P}$ and its image $\beta_S(\mathcal{P})$.

All the notions introduced in this Subsection extend literally to any closed domain in $\mathbb{C}$ diffeomorphic to a closed disc.

2.4. Polynomial realizations. Let $\Sigma_N$, $N \geq 1$, be a relatively minimal ruled rational surface, $\text{pr} : \Sigma_N \rightarrow \mathbb{P}^1$ its ruling, $F$ a fiber of $\text{pr}$ and $E_N$ the exceptional
section, $E_N^2 = -N$. We choose a point $\infty \in \mathbb{P}^1$ and put $F_\infty = p^{-1}(\infty)$.

Consider the linear system

$$\mathbb{P} = \mathbb{P}_{N,m} = \mathbb{P}H^0(\Sigma_N, \mathcal{O}_{\Sigma_N}(mE_N + mNF)).$$

In the chart $\Sigma_N \setminus (F_\infty \cup E_N) \simeq \mathbb{C}^2$ we can choose coordinates $(z, w)$ such that the restriction of $p$ to $\Sigma_N \setminus (F_\infty \cup E_N)$ coincides with the projection $pr : (z, w) \mapsto z$. With respect to these coordinates any element $\bar{C} \in \mathbb{P}$ can be given by equation

$$P(z, w) = 0,$$

where $P(z, w) = \sum_{i=0}^{m} q_i(z)w^{m-i}$ and $q_i(z) = \sum_{j=0}^{iN} a_{i,j}z^j$ are polynomials of degrees $\leq iN$.

If $\bar{C}$ is an irreducible curve, then, since the intersection number $\bar{C}E_N$ is zero, the projection $pr|_{\bar{C}} : \bar{C} \to \mathbb{P}^1$ is a proper map of degree $m$. It has $m(m - 1)N$ critical values (counted with multiplicities) which are found from the equation

$$R(z) = 0,$$

where $R(z) = R_{P,P'}(z) = \sum r_k z^k$ is the resultant of $P(z, w)$ and $P'_w(z, w) = \frac{\partial}{\partial w} P(z, w)$, $\deg R(z) = m(m - 1)N$.

The coefficients $r_k$ of the resultant admit polynomial expressions in coefficients of $P$. These polynomials $r_k \in \mathbb{C}[a_{0,0}, \ldots, a_{m,mN}]$ define a rational map

$$\mathcal{R} : \mathbb{P} \to \text{Sym}_{m(m-1)N} \mathbb{P}^1 \simeq \mathbb{P}^{m(m-1)N},$$

where $\text{Sym}_{m(m-1)N} \mathbb{P}^1$ is the symmetric product of $m(m - 1)N$ copies of $\mathbb{P}^1$. The only non regular points of this map correspond to reducible curves $\bar{C} \in \mathbb{P}$ with a multiple component different from a fiber (here, the regularity at a point is equivalent to the existence of a continuous extension, and the checking of the latter is straightforward).

Below we will need the following lemmas.
Lemma 2.1. Any braid monodromy factorization $b = bmf(\bar{C}) \in \mathcal{P}$ of a generic curve $\bar{C} \in \mathbb{P}$ is equal to $(\delta^2)^N$.

Proof. By Lemma 1.2, $(\delta^2)^N$ is the unique element of its orbit under the conjugation action of $B_m$. Thus, it is sufficient to show that $(\delta^2)^N$ is equal to some braid monodromy factorization. Such an equality is well-known if $N = 1$ (see, for example, [16]). We show how to deduce the general statement from this particular case.

Pick $z_1, \ldots, z_N$ distinct from 0. Consider a curve $\bar{C}_0 \in \mathbb{P}$ given by equation

$$\prod_{j=1}^{m}(wz_2 \ldots z_N + (-1)^{N}a_j(z - z_1)\ldots(z - z_N)) = 0, \quad a_i \neq a_j \text{ if } i \neq j,$$

and a generic curve $\bar{C} \in \mathbb{P}$ sufficiently close to $\bar{C}_0$ (i.e., a curve in the intersection of a small topological neighborhood of $\bar{C}_0$ with the set of curves $C \in \mathbb{P}$ for which the projection $\text{pr}|_C : C \to \mathbb{P}^1$ has only simple critical points and distinct critical values). The set of critical values of $\text{pr}|_C$ splits into $N$ subsets $\{z_{k,1}, \ldots, z_{k,m(m-1)}\}$, $k = 1, \ldots, N$, lying in small disjoint discs $D_{1,k}(\varepsilon) = \{|z - z_k| < \varepsilon\}$, $0 < \varepsilon << 1$. Correspondingly, $b = b_1 \cdot \ldots \cdot b_N$, where $b_k$ is the braid monodromy factorization of $\bar{C}$ over $D_{1,k}(\varepsilon)$.

Let us show that $b_1 = \delta^2$ (the proof of $b_2 = \cdots = b_N = \delta^2$ is the same). Define a path $\bar{C}_0(t), 0 \leq t \leq 1$, in $\mathbb{P}$ by equations

$$\prod_{j=1}^{m}(wz_2(t) \ldots z_N(t) + (-1)^{N}a_j(z - z_1)\ldots(z - z_N(t))) = 0,$$

$$z_j(t) \to \infty \text{ when } t \to 1, \quad z_j(0) = z_j.$$

It connects the curve $\bar{C}_0(0) = \bar{C}_0$ with a curve $\bar{C}_0(1)$ given by equation

$$\prod_{j=1}^{m}(w - a_j(z - z_1)) = 0.$$

Since $\mathcal{R}$ is regular at each point of the path $\bar{C}_0(t)$, we can find a path $\bar{C}(t)$ sufficiently close to it such that: for each $t \neq 1$ the projection $\text{pr} : \bar{C}(t) \to \mathbb{P}^1$ has only simple critical points and distinct critical values, $\bar{C}(0) = \bar{C}$; $\bar{C}(1)$ is given by polynomial
\[ \sum_{i+j=m} a_{i,j}(z-z_1)^i w^j = 0, \] and for each \( 0 \leq t < 1 \) all the \( m(m-1) \) critical points of \( \bar{C}(t) \) lie in \( \{ |z-z_1| < \varepsilon \} \). It implies that the braid monodromy factorization of \( \bar{C} \) over \( D_1(\varepsilon) \) coincides with the braid monodromy factorization of \( \bar{C}(1) \) over \( D_1(\varepsilon) \), which is equal to \( \delta^2 \) (case \( N=1 \)). \hfill \Box

**Lemma 2.2.** Let \( \bar{C}_0 \in \mathbb{P} \) have an ordinary singular point of multiplicity \( m \) at \( z = 0, w = 0 \), i.e., the equation of \( \bar{C}_0 \) is of the form \( P_{\geq m}(z, w) = \sum_{i+j\geq m} a_{i,j} z^i w^j = 0 \) where the polynomial \( \sum_{i+j=m} a_{i,j} w^j = 0 \) has \( m \) distinct roots. Let \( Q(z, w) \in \mathbb{P} \) be a polynomial such that for some \( \varepsilon_0 > 0 \) and for all \( 0 < \varepsilon << 1 \), the curve \( \bar{C}_\varepsilon \) given by equation

\[ P_{\geq m}(z, w) + \varepsilon Q(z, w) = 0 \]

has exactly \( m(m-1) \) different critical values in \( D_1(\varepsilon_0) = \{ |z| < \varepsilon_0 \} \). Then, the braid monodromy factorization of \( \bar{C}_\varepsilon \) over \( D_1(\varepsilon_0) \) is equal to \( \delta^2 \).

**Proof.** Connect \( \bar{C}_0 \) with a curve \( \bar{C}_1 \in \mathbb{P} \) given by equation \( \sum_{i+j=m} a_{i,j} z^i w^j = 0 \) by a path \( \{ \bar{C}_t \}_{0 \leq t \leq 1} \) in \( \mathbb{P} \), where \( \bar{C}_t \) is given by equation

\[ P_t(z, w) = \sum_{i+j=m} a_{i,j} z^i w^j + \sum_{i+j>m} a_{i,j}(t) z^i w^j = 0. \]

Since the leading part of the equations is nondegenerate, for any \( t \) the resultant \( \mathcal{R}(\bar{C}_t) \) has the following roots: \( z_{1,0}(t) = \cdots = z_{m(m-1),0}(t) = 0 \), \( z_{k,0}(t) \neq 0 \) for \( k > m(m-1) \). Hence, we can choose \( \varepsilon_0 > 0 \) such that \( 2\varepsilon_0 < |z_{k,0}(t)| \) for \( k > m(m-1) \) and for all \( t \in [0,1] \). For \( \varepsilon \) small enough the image \( \{ \mathcal{R}(\bar{C}_{t,\varepsilon}) \} \) of the path \( \{ P_t(z, w) + \varepsilon Q(z, w) = 0 \} \) is contained in a neighborhood \( V_{\varepsilon_0} \subset \text{Sym}_{m(m-1)} \mathbb{P}^1 \) of \( \{ \mathcal{R}(\bar{C}_t) \} \),

\[ V_{\varepsilon_0} = \{ (z_1, \ldots, z_{Nm(m-1)}) \mid |z_i| < \varepsilon_0 \text{ for } i \leq m(m-1) \text{ and } |z_i| > \varepsilon_0 \text{ for } i > m(m-1) \}. \]

Changing slightly the coefficients of the polynomial \( Q(z, w) \), we can assume that for \( 0 < \varepsilon << 1 \) and any \( 0 \leq t \leq 1 \), \( \mathcal{R}(\bar{C}_{t,\varepsilon}) \) belongs to \( V_{\varepsilon_0} \cap \{ z_i \neq z_j \text{ for } i, j \leq m(m-1), i \neq j \} \). Under this assumption, all the curves \( \bar{C}_{t,\varepsilon} \) have the same braid monodromy factorization over \( D_{\varepsilon_0} \).
The Viro patch-working, see [23], is based on quasi-homogeneous changes of coordinates exclusively, and thus respects the braid monodromy. Hence, to finish the proof it remains to replace $\bar{C}_{t,\varepsilon}$ by a generic polynomial obtained by patch-working generic polynomials

$$\sum_{i+j<m} a'_{i,j} z^i w^j + \sum_{i+j=m} a_{i,j} z^i w^j + \sum_{i+j>m} a'_{i,j} z^i w^j$$

close to $\bar{C}_1$ and to apply Lemma 2.1 to the first one (case $N = 1$). □

**Lemma 2.3.** Let $\bar{b}$ be the braid monodromy factorization of an affine part $C$ of a projective curve $\bar{C} \in \mathbb{P}_{N_1,m}$ over a disc $D_1(r)$, $r >> 1$. Then, there is $M = M_C \in \mathbb{N}$ such that for any $N_2 \geq M$ there is a projective curve $\tilde{C}_1 \in \mathbb{P}_{N_1+N_2,m}$ such that $\bar{b} = \tilde{b} \cdot (\delta^2)^{N_2}$ is the braid monodromy factorization of its affine part $C_1$ over the disc $D_1(r_1)$, $r_1 >> r$ and $\tilde{b}$ is the braid monodromy factorization of $C_1$ over the disc $D_1(r)$.

**Proof.** Denote by $z_1, \ldots, z_k$ the critical values of $pr|_C$ and by $(z_i, w_{i,1}), \ldots, (z_k, w_{i,t_i})$ the critical points over $z = z_i$. Attribute to the germ $(C_{i,j}, (z_i, w_{i,j}))$ of $C$ at $(z_i, w_{i,j})$ a number $M_{i,j}$ defined in section 2.1. Put $M_i = \max_j M_{i,j}$ and $M_C = \sum_i M_i$. Choose points $z_{k+i}$, $i = 1, \ldots, N_2$ such that $| z_{k+i} | >> r$ and perform $N_2 > M_C$ elementary transformations with centers at the intersection points of the fibers $z = z_{k+i}$, $1 \leq i \leq N_2$, with the exceptional sections of $\Sigma_{N_1+i-1}$, $i = 1, \ldots, N_2$. Denote by $\tilde{C} \subset \Sigma_{N_1+N_2}$ the strict transform of $\bar{C}$ under the composition $\tau : \Sigma_{N_1} \to \Sigma_{N_1+N_2}$ of these transformations and by

$$\tilde{P}(z, w) = w^m + \sum_{i=1}^m q_i(z) w^{m-i} = 0$$

the equation of its affine part. The only critical values of $pr|_{\tilde{C}}$ are

$$z_1, \ldots, z_k, z_{k+1}, \ldots, z_{k+N_2}.$$ 

Over $z = z_{k+i}$ there is only one point of $\tilde{C}$ and at this point the curve $\tilde{C}$ is equivalent, with respect to $pr$, to

$$\prod_{i=1}^m (w - c_i(z - z_{k+i})) = 0.$$
The braid monodromy factorization over $D_1(r)$ of $\tilde{C}$ coincides with the braid monodromy factorization of $C$, since the centers of the transformation $\tau$ lie over $z_{k+i}$, $1 \leq i \leq N_2$, and $|z_{k+i}| > r$. By the choice of the constants $M_j$ and by Lemma 2.2, a curve $C_\tau$ given by

$$\tilde{P}(z, w) + \prod_{i=1}^{k} (z - z_i)^{M_i} \left( \sum_{j=0}^{m-1} \varepsilon_j w^j \right) = 0,$$

where all $\varepsilon_j$ are generic and close to zero, has the braid monodromy factorization $\tilde{b} = \tilde{b} \cdot (\delta^2)^{N_2}$ over the disc $D_1(r_1)$, $r_1 > r$. Indeed, by Bertini’s theorem, the curves $C_\tau$ are non-singular over $\mathbb{P}^1 \setminus \{z_1, \ldots, z_k\}$ for almost all $\tau$. Moreover, if for some $\tau$ a fiber $F$ over a point lying near some $z_{k+i}$ is tangent the curve $C_\tau$ with multiplicity greater than 2 at some point, then we can choose $\tau_1$ close to $\tau$ such that $C_{\tau_1}$ and $F$ have the only one simplest tangent point. Therefore there is $\varepsilon_0$ such that for almost all $\tau$ sufficiently close to zero, exactly $m(m-1)$ distinct critical values of $pr : C_\tau \rightarrow \mathbb{P}^1$ lie in each neighborhood $\{|z - z_{k+i}| < \varepsilon_0\}$. □

**Lemma 2.4.** If $b \in \mathcal{P}$ is the braid monodromy factorization over a disc $D$ of an affine curve $P(z, w) = w^m + \sum_{i=1}^{m} p_i(z)w^{m-i} = 0$ with $p_i(z) = \sum_{j=0}^{iN} a_{i,j}z^j$, $N \geq 1$, then there exists an affine curve $Q(z, w) = w^m + \sum_{i=1}^{m} q_i(z)w^{m-i} = 0$ such that

(i) $q_i(z) = \sum_{j=0}^{iN} b_{i,j}z^j$;

(ii) the polynomial $w^m + \sum_{i=1}^{m} b_{i,N}w^{m-i}$ has $m$ different roots;

(iii) all critical points of $Q(z, w)$ lying over the complement $\mathbb{C} \setminus D$ are non-degenerate, in particular, the curve $C$ given by $Q(z, w) = 0$ is non-singular over $\mathbb{C} \setminus D$;

(iv) a braid monodromy factorization of $C$ over the disc $D$ is equal to $b$.

**Proof.** The second property, (ii), can be achieved by a generic choice of the fiber at infinity. After that, it is sufficient to apply to

$$Q(z, w) + \prod_{i=1}^{k} (z - z_i)^{M_i} \left( \sum_{j=0}^{m-1} \varepsilon_j w^j \right) = 0,$$

the same inductive correction procedure as at the end of the proof of Lemma 2.3 □
Theorem 2.1. For any \( b \in \mathcal{P} \) there are \( N \in \mathbb{N} \) and a polynomial \( P(z,w) = w^m + \sum_{i=1}^{m} q_i(z)w^{m-i} \) such that

(i) \( q_i(z) = \sum_{j=0}^{iN} a_{i,j}z^j \);

(ii) the polynomial \( w^m + \sum_{i=1}^{m} a_{i,iN}w^{m-i} \) has \( m \) different roots;

(iii) all critical points of \( P(z,w) \) lying over the complement \( \mathbb{C} \setminus D_1(1) \) of the disc \( D_1(1) = \{ z \in \mathbb{C} \mid |z| < 1 \} \) are non-degenerate, in particular, the curve \( C \) given by \( P(z,w) = 0 \) is non-singular over \( \mathbb{C} \setminus D_1(1) \);

(iv) \( b \) is a braid monodromy factorization of \( P(z,w) \) over the disc \( D_1(1) \).

Proof. Let \( b = \rho(g_1^{-1})(b_1) \cdot \ldots \cdot \rho(g_n^{-1})(b_n) \), where \( b_j \in T \) are standard forms of braid monodromies of algebraic germs \( C_j \) over points \( z = z_j, |z_j| < 1 \). Let

\[
q_j(z,w) = w^m + \sum_{i=1}^{m} (z - z_j)^i q_j(z,w) = 0
\]

be an equation of the germ \( C_j \) at \( z = z_j \) and

\[
q_j(z,w) = w^m + \sum_{i=1}^{m} (z - z_j)^i q_j(z,w) = 0
\]

be the equation of the singularity \( \bar{C}_j \) associated with \( C_j \).

Proposition 2.1. There is a polynomial \( Q(z,w) = w^m + \sum_{i=1}^{m} q_i(z)w^{m-i} \) having at each \( (z_j,0) \) the same type of singularity as \( \bar{C}_j \) and satisfying conditions (i)–(iii) of Theorem 2.1 for some \( N \).

Proof. Choose a constant \( M > M_{(\bar{C}_j,z_j)} \) for all \( j = 1, \ldots, n \), where \( M_{(\bar{C}_j,z_j)} \) is the constant defined in section 2.1.

In the beginning we construct a polynomial \( \bar{Q}(z,w) \) of degree \( \deg_w \bar{Q} = m \) having singularities at \( (z_j,0), j = 1, \ldots, n \), of the same types as \( (\bar{C}_j,z_j) \). The polynomial \( \bar{Q}(z,w) \) will be constructed by \( n \) steps. Put \( \bar{Q}_1(z,w) = q_1(z,w) \). Assume that we have constructed a polynomial \( \bar{Q}_k(z,w) \) of degree \( \deg_w \bar{Q}_k = m \) having singularities at \( (z_j,0), j = 1, \ldots, k \), of the same types as \( (\bar{C}_j,z_j) \). Consider a polynomial \( q_{z_1,\ldots,z_k}(z) = ((z - z_1)\ldots(z - z_k))^M. \) We have \( q_{z_1,\ldots,z_k}(z_{k+1}) \neq 0 \). Therefore we can find polynomials \( p_i(z) \) such that the polynomial

\[
\bar{Q}_{k+1}(z,w) = \bar{Q}_k(z,w) + \sum_{i=1}^{m} q_{z_1,\ldots,z_k}(z)p_i(z)w^{m-i}
\]
have the same types of singularities at \((z_j, 0), j = 1, \ldots, k + 1\), as \((\bar{C}_j, z_j)\). Indeed, 
\(\bar{Q}_{k+1}(z, w)\) has the same singularities as \((\bar{C}_j, z_j)\) for \(j = 1, \ldots, k\), whatever is the choice of the polynomials \(p_i\), since \(M\) is big enough, and choosing appropriate coefficients for the polynomials \(p_i(z)\), the polynomial \(\bar{Q}_{k+1}(z, w)\) will have the same type of singularity at \((z_{k+1}, 0)\) as \((\bar{C}_{k+1}, z_{k+1})\).

To complete the prove of Proposition 2.1, it remains to apply Lemma 2.4.  

By Remark 2.1, it follows from Proposition 2.1 that there is a polynomial \(Q(z, w)\) satisfying conditions \((i)–(iii)\) of Theorem 2.1 for some \(N = N_1\) and such that the set of critical values of \(\text{pr} |_{Q(z, w)=0}\) lying in \(D_1(1)\) coincides with \(\{z_1, \ldots, z_n\}\) and over \(z = z_j, j = 1, \ldots, n\), the germ of the curve \(C\) given by \(Q(z, w) = 0\) has the same topological type as \((C_j, z_j)\). Therefore, the braid monodromy factorization of \(Q(z, w)\) over the disc \(D_1(r), r >> 1\), is equal to

\[
\bar{b} = \rho(\bar{g}_1^{-1})(b_1) \cdot \ldots \cdot \rho(\bar{g}_n^{-1})(b_n) \cdot s,
\]

where \(s \in S_{A_0}\). Besides, from conditions \((i)–(iii)\) of Theorem 2.1 it follows that \(C \subset \overline{C}\) is an affine part of a projective curve \(\overline{C} \in \Sigma_{N_1}\), such that all singular points of \(\overline{C}\) belong to \(C\).

It follows from Theorem 1.1, Proposition 2.1 and Lemma 2.3 that there are \(N_2 >> 1\) and a polynomial \(\overline{P}(z, w)\) of \(\text{deg}_w = m\) whose braid monodromy factorization over the disc \(D_1(r), r >> 1\), is equal to \(b \cdot s \cdot (\delta^2)^{N_2}\), where \(s\) is an element belonging to \(S_{A_0}\). Therefore there is a domain \(U \subset \mathbb{C}\) diffeomorphic to the disk \(D_1(1)\) such that the braid monodromy factorization of \(\overline{P}(z, w)\) over \(U\) is equal to \(b\). There is an analytic isomorphism of \(U\) and \(D_1(1)\). Therefore, there is a polynomial \(\tilde{P}(z, w) = w^m + \sum_{i=1}^m q_i(z)w^{m-i}\) with holomorphic in \(D_1(1)\) coefficients \(q_i(z)\) such that the braid monodromy factorization of \(\tilde{P}(z, w)\) over \(D_1(1)\) is equal to \(b\). Now, it remains to approximate the holomorphic functions \(q_i(z)\) by polynomials with the same jets (of sufficiently big degree) at the critical values of \(p\) belonging to \(D_1(1)\) and then to apply Lemma 2.4.  

□
Remark 2.3. For given $b \in \mathcal{P}$, such that $\beta(b) = \Delta^2$, let

$$P(z, w) = w^m + \sum_{i=1}^{m} q_i(z)w^{m-i}$$

be a polynomial whose existence is stated in Theorem 2.1. As it follows from Proposition 3.1, there does not exist an upper bound for degrees of the polynomials $q_i(z, w)$ depending only on the topological types of critical points of $P(z, w) = 0$ with respect to $pr$ even in the case of the types $A_n$, $n \leq 2$ (simple tangents, nodes and ordinary cusps). On the other hand, as it follows from the proof of Theorem 2.1, there is an effective bound on $\deg z \bar{P}$ for the intermediate polynomial $\bar{P}$, which provides a given braid monodromy factorization over some domain.

Remark 2.4. The case $N = 1$ covers the case of algebraic curves in $\mathbb{C}P^2$.

In the latter case to define a braid monodromy factorization one choose a point outside the curve, and the blowing up reduces this case to the case of curves in $\Sigma_1$.

Proof of Claim 2.1. Let realize a germ $(C_{\varepsilon_1}, o)$ of an algebraic curve over a point $o$ as a germ of some projective curve $\tilde{C} \subset \Sigma_N$. Let $b(\tilde{C})$ be a braid monodromy factorization of $\tilde{C}$. Then $b(\tilde{C}) = b(C_{\varepsilon_1}, o) \cdot \tilde{b}$, where $\tilde{b} \in \mathcal{P}$ is the product of the braid monodromies over all the critical values of the projection $pr$ except $o$. We have

$$\beta(b(\tilde{C})) = \beta(b(C_{\varepsilon_1}, o)) \beta(\tilde{b}) = \Delta_m^{2N}.\$$

Perform the elementary transformation $\tau$ with center at $F_o \cap E_\infty$. The curves $\tau^{-1}(\tilde{C})$ and $\tilde{C}$ have the same critical values, the braid monodromy factorization of $\tau^{-1}(\tilde{C})$ is $b(\tau^{-1}(\tilde{C})) = b(C_{\varepsilon_1}, o) \cdot \tilde{b}$, and also

$$\beta(b(\tau^{-1}(\tilde{C}))) = \beta(b(C_{\varepsilon_1}, o)) \beta(\tilde{b}) = \Delta_m^{2(N+1)}.\$$

Therefore, $\beta(b(C_{\varepsilon_1}, o)) = \beta(b(C_{\varepsilon_1}, o)) \Delta_m^2$. It remains to note that $\beta(b(C_{\varepsilon_1}, o)) = b(C_{\varepsilon_1}, o)$ and $\beta(b(C_{\varepsilon_1}, o)) = \alpha(b(C_{\varepsilon_1}, o))$ in view of the identifications made in section 1.3. □

§3. Braid monodromy factorizations of Hurwitz curves.

3.1. Hurwitz curves. As in section 2, let $\Sigma_N$ be a relatively minimal ruled rational surface, $N \geq 1$, $pr : \Sigma_N \to \mathbb{P}^1$ the ruling, $F$ a fiber of $pr$ and $E_N$ the exceptional section, $E_N^2 = -N$. 
Definition 3.1. The image $\tilde{H} = f(\mathcal{S}) \subset \Sigma_N$ of a smooth map $f : \mathcal{S} \to \Sigma_N \setminus E_N$ of an oriented closed real surface $\mathcal{S}$ is called a Hurwitz curve (in $\Sigma_N$) of degree $m$ if there is a finite subset $Z \subset \tilde{H}$ such that:

(i) $f$ is an embedding of the surface $\mathcal{S} \setminus f^{-1}(Z)$ and for any $s \notin Z$, $\tilde{H}$ and the fiber $F_{pr(s)}$ of $pr$ meet at $s$ transversely and with positive intersection number;

(ii) for each $s \in Z$ there is a neighborhood $U \subset \Sigma_N$ of $s$ such that $\tilde{H} \cap U$ is a complex analytic curve, and the complex orientation of $\tilde{H} \cap U \setminus \{s\}$ coincides with the orientation transported from $\mathcal{S}$ by $f$;

(iii) the restriction of $pr$ to $\tilde{H}$ is a finite map of degree $m$.

For any Hurwitz curve $\tilde{H}$ there is one and only one minimal $Z \subset \tilde{H}$ satisfying the conditions from Definition 3.1. We denote it by $Z(\tilde{H})$.

A Hurwitz curve $\tilde{H}$ is called cuspidal if for each $s \in Z(\tilde{H})$ there is a neighborhood $U$ of $s$ and local analytic coordinates $x, y$ in $U$ such that

(iv) $pr|_U$ is given by $(x, y) \mapsto x$;

(v) $\tilde{H} \cap U$ is given by $y^2 = x^k, k \geq 1$.

It is called ordinary cuspidal if $k \leq 3$ in (v) for all $s \in Z(\tilde{H})$, and nodal if $k \leq 2$.

Since $\tilde{H} \cap E_N = \emptyset$, one can define a braid monodromy factorization $b(\tilde{H}) \in \mathcal{P}$ of $\tilde{H}$ as in the algebraic case. For doing this, we fix a fiber $F_\infty$ meeting transversely $\tilde{H}$ and consider $\tilde{H} \cap \mathbb{C}^2$, where $\mathbb{C}^2 = \Sigma_N \setminus (E_N \cup F_\infty)$. Choose $r >> 1$ such that $pr(Z) \subset D_1(r) \subset \mathbb{C} = \mathbb{CP}^1 \setminus pr F_\infty$, $Z = Z(\tilde{H})$. Denote by $z_1, \ldots, z_n$ the elements of $pr(Z)$. Pick $\varepsilon$, $0 < \varepsilon << 1$, such that the discs $D_{1,i}(\varepsilon) = \{z \in \mathbb{C} \mid |z - z_i| < \varepsilon\}$, $i = 1, \ldots, n$, would be disjoint. Select arbitrary points $u_i \in \partial D_{1,i}(\varepsilon)$ and a point $u_0 \in \partial D_1(r)$. Put $D_{2,u_i} = pr^{-1}(u_i)$, $i = 0, \ldots, n$, and $K(u_i) = \{w_{i,1}, \ldots, w_{i,m}\} = D_{2,u_i} \cap H$. Choose disjoint simple paths $l_i \subset Cl D_1(r) \setminus \bigcup_1^n D_{1,i}(\varepsilon)$, $i = 1, \ldots, n$, starting at $u_0$ and ending at $u_i$ and renumber the points in a way that the product $\gamma_1 \ldots \gamma_n$ of the loops $\gamma_i = l_i \circ \partial D_{1,i}(\varepsilon) \circ l_i^{-1}$ would be equal to $\partial D_1(r)$ in $\pi_1(Cl D_1(r) \setminus \{z_1, \ldots, z_n\}, u_0)$.

As in Section 2, each $\gamma_i$ defines an element $b_i \in T \subset S_{B_m}$. The factorization
$b_1 \ldots b_n \in \mathcal{T}$ is called a braid monodromy factorization of $\bar{H}$. In fact, each $b_i$ is conjugated to a braid monodromy standard form of some algebraic germ over $z_i$. Hence, $b(\bar{H}) = b_1 \ldots b_n$ belongs to $\mathcal{P}$ (see Section 2.3 for the definition of $\mathcal{P}$). The orbit of this element under the conjugation action of $B_m$ in $\mathcal{P}$ is called the geometric braid factorization type and denoted by $bmt$.

**Lemma 3.1.** For a Hurwitz curve $\bar{H} \subset \Sigma_N$ it holds

$$\beta(b(\bar{H})) = \Delta^{2N}.$$ 

*Proof.* Over $\mathbb{P}^1 \setminus D_1(r)$ the curve $\bar{H}$ is the union of $m = \deg \bar{H}$ pairwise disjoint sections of $pr$. By an isotopy of sections they can be transformed into the sections defined, in affine coordinates, by $w = v_i z^N, 1 \leq i \leq m, v_i \in \mathbb{C}$. The latter sections form the braid $\Delta^{2N}$ over $\partial D_1(r)$. □

The converse statement is also proved in a straightforward way.

**Theorem 3.1.** ([17]) For any $b = b_1 \ldots b_n \in \mathcal{P}$ such that $\beta(b) = \Delta^{2N}$ there is a Hurwitz curve $\bar{H} \subset \Sigma_N$ with a braid monodromy factorization $b(\bar{H})$ equal to $b$.

### 3.2. Hurwitz isotopies.

The following definition generalizes the notion of Hurwitz curves. This generalization corresponds to replacement of $\mathcal{P}$ by $\mathcal{T}$.

**Definition 3.2.** The image $\bar{H} = f(\mathcal{G}) \subset \Sigma_N$ of a continuous map $f : \mathcal{G} \to \Sigma_N \setminus E_N$ of a smooth oriented closed real surface $\mathcal{G}$ is called a topological Hurwitz curve (in $\Sigma_N$) of degree $m$ if there is a finite subset $Z \subset \bar{H}$ such that:

1. $f$ is a smooth embedding of the surface $\mathcal{G} \setminus f^{-1}(Z)$ and for any $s \notin Z$, $\bar{H}$ and the fiber $F_{pr(s)}$ of $pr$ meet at $s$ transversely and with positive intersection number;

2. the restriction of $pr$ to $\bar{H}$ is a finite map of degree $m$. (We call a map finite if the preimage of each point is finite.)

As in the case of Hurwitz curves, there is one and only one minimal $Z$, which we denote by $Z(\bar{H})$. We say that $\bar{H}$ is $Z$-generic (with respect to $pr$) if the restriction of $pr$ to $Z(\bar{H})$ is injective.
Definition 3.3. Two Hurwitz (respectively, topological Hurwitz) curves $\bar{H}_1$ and $\bar{H}_2 \subset \Sigma_N$ are called $H$-isotopic if there is a fiberwise continuous isotopy $\phi_t : \Sigma_N \to \Sigma_N$, $t \in [0, 1]$, smooth outside the the fibers $F_{\text{pr}(s)}$, $s \in Z(\bar{H}_1)$, and such that

(i) $\phi_0 = \text{Id}$;

(ii) $\phi_t(\bar{H}_1)$ is a Hurwitz (respectively, topological Hurwitz) curve for all $t \in [0, 1]$;

(iii) $\phi_1(\bar{H}_1) = \bar{H}_2$;

(iv) $\phi_t(E_N) = E_N$ for all $t \in [0, 1]$.

If $\bar{H}_1$ is a topological Hurwitz curve, at any $p \in Z(\bar{H}_1)$ there is a well-defined ($W$-prepared) germ $(D, H_1 = \bar{H}_1 \cap D, \text{pr})$ of this curve in a bi-disc $D = D_1 \times D_2$, $D_1 = D_1(\epsilon_1)$, $D_2 = D_2(\epsilon_2)$, $0 < \epsilon_1 << \epsilon_2$, centered at $p$ and such that the restriction of $\text{pr}$ to $H_1$ is a proper map of finite degree. If $\epsilon_1, \epsilon_2$ are sufficiently small, then: $F_{\text{pr}(p)} \cap H_1 = p$; the above degree does not depend on $\epsilon_1, \epsilon_2$; and, in the same way as in the algebraic case, the link $\partial D \cap H_1$ defines a unique, up to conjugation, braid $b \in B_k$, where $k$ is the above degree. So that, we may speak on a $tH$-singularity $(D, H_1, \text{pr})$ of degree $k$, of type $b$ and of interlacing number $l = l(b)$.

When we are given a link $L \subset \partial D_1 \times D_2$ realizing a braid $b \in B_k$, we associate with it a standard conical model of a topological singularity of type $b$. It is given by $H = C(L)$,

$$C(L) = \{(rz, rw) \mid 0 \leq r \leq 1, (z, w) \in L\}.$$

As is well-known, if $(D, C)$ is a germ of a $W$-prepared analytic singularity then the germ $(D, C, \text{pr})$ is homeomorphic to the cone singularity of type $b = \text{pr}^{-1}(\partial D_1 \cap C)$.

It is convenient to describe a $tH$-singularity $(D, H_1, \text{pr})$ by means of a flow. To make the corresponding formulae more transparent let put $\epsilon_1 = \epsilon_2 = 1$ and move $H_1$ by a suitable (homothety like) fiber preserving $H$-isotopy into the cone body $|w| < \rho |z|$, $\rho = 1 - \epsilon$. Consider a smooth map $v$ from $D_1 \setminus \{0\}$ to the space of smooth vector fields on $D_2$ such that $v(z)(w) = -w$ for any $|w| \geq \rho |z|$. Then the
flow defined by
\[ \frac{dz}{dt} = -z, \quad \frac{dw}{dt} = v(z)(w) \]
transforms any braid \( L \subset \partial D_1(1) \times D_2(\rho) \) into a \( tH \)-singularity. Reciprocally, any \( tH \)-singularity \((D, H_1, pr)\) can be represented in such a way as soon as \( H_1 \) is contained in \(|w| < \rho|z|\). For example, the standard conical model \( C(L) \) is given by any pair \((v, \rho)\) like above with \( v(z)(w) = -w \) for \(|w| < \rho|z|\).

**Claim 3.1.** For any \( tH \)-singularity \((D = D_1 \times D_2, H_1, pr)\) there is an \( H \)-isotopy (preserving each fiber) identical over \( \partial D \) and transforming the singularity in its standard conical model \( C(L) \), \( L = H_1 \cap (\partial D_1 \times D_2) \).

**Proof.** Let represent the \( tH \)-singularity \((D = D_1 \times D_2, H_1, pr)\) and its conical model \( C(L) \) by two flows as above: one flow is associated with a pair \((v_0, \rho)\) and another with \((v_1, \rho)\). Consider the family of \( tH \)-singularities given by the pairs \((v_t = tv_1 + (1 - t)v_0, \rho)\) and denote by \( \Phi_{t, z} \) the imbeddings \( \{z/|z|\} \times D_2 \rightarrow \{z\} \times D_2 \) given by the flow associated with \( v_t \). To accompany the constructed family of \( tH \)-singularities by some ambient \( H \)-isotopy \( \phi_t \) connecting \( H_1 \) with \( C(L) \) over \( D_1 \), it is sufficient to take the flow associated with the vertical vector fields \( \frac{d}{dt}(\Phi_{t, z} \circ \Phi_{1, z}^{-1}) \) extending them for each \( t \) by 0 to the whole \( D_1 \times D_2 \). □

**Remark 3.1.** In fact, the procedure we used in the proof of Claim 3.1 gives, as well, an \( H \)-isotopy between any two \( tH \)-singularities with the same link \( L \). If the both singularities are singularities of Hurwitz curves, they remain singularities of Hurwitz curves (i.e., algebraic singularities) during the isotopy. Certainly, this isotopy is not necessary smooth at the singular point.

The definitions of braid monodromy factorizations and braid factorization types extend literally from Hurwitz to topological Hurwitz curves.

Note only that if \((D, H, pr)\) is a \( tH \)-singularity, then due to Claim 3.1 it is determined uniquely, up to \( H \)-isotopy identical on \( \partial D \), by its boundary closed braid \( b = H \cap pr^{-1}(\partial D_1) \) and respectively by its standard tbmf-form (see section 1.3). Therefore the braid monodromy of a germ of a topological Hurwitz curve
over a point is naturally defined as an element of $T$, and the braid monodromy factorizations of topological Hurwitz curves as elements of $T$. They satisfy the relation $\beta(b(\bar{H})) = \Delta^{2N}$. The inverse statement, that each factorization $b \in T$ with this property is realized by a topological Hurwitz curve, has the same proof as Theorem 3.1.

The braid group $B_k = B[D_2, \{w_1, \ldots, w_k\}]$, with $w_1, \ldots, w_k \in D_2$ acts on $\pi_1 = \pi_1(D_2 \setminus \{w_1, \ldots, w_k\})$ in a natural way. We say that the action of $b \in B_k$ on $\pi_1$ is \textit{inseparable} if only the elements of the subgroup of $\pi_1$ generated by $\partial D_2$ are fixed under the action of $b$. Standard generators $a_1, \ldots, a_{k-1}$ of $B_k$ being fixed, we mean by a \textit{geometric base} of $\pi_1(D_2 \setminus \{w_1, \ldots, w_k\})$ any set of generators $\{x_1, \ldots, x_k\}$ in which the natural action of $B_k$ is given by standard formulas, i.e., $a_i(x_j) = x_j$ for $j \neq i, i+1$ and $a_i(x_{i+1}) = x_i$, $a_i(x_i) = x_ix_{i+1}x_i^{-1}$ for $i \leq k - 1$. For any such base, $\partial D_2 = x_1 \ldots x_k$.

**Lemma 3.2.** Let $b \in B_k$ be an inseparable element and $\{x_1, \ldots, x_m\}$ a geometric base of $\pi_1(D_2 \setminus \{w_1, \ldots, w_m\})$. Regard $b$ as an element of

$$B_{k,0} = B[D_2, \{w_1, \ldots, w_k\}] \subset B[D_2, \{w_1, \ldots, w_m\}] = B_m$$

and consider the induced action of $b$ on $\pi_1 = \pi_1(D_2 \setminus \{w_1, \ldots, w_m\})$. Then, the subgroup $F(b)$ of the fixed elements in $\pi_1$ under the action of $b$ is generated by $l = x_1 \ldots x_k$ and $x_{k+1}, \ldots, x_m$.

**Proof.** Evidently, $l$ and $x_{k+i}$ with any $i \geq 1$ belong to $F(b)$. Let $g \in F(b)$. Write $g$ as a reduced word in letters $\{x_1, \ldots, x_m\}$ and their inverses

$$g = s_1s_2\ldots s_n,$$

where $s_i$ are reduced words which are non-empty if $i \neq 1$ and which are words in $\{x_1, \ldots x_k\}$ and their inverses if $i$ is odd, and in $\{x_{k+1}, \ldots x_m\}$ and their inverses if $i$ is even. Since such a representation is unique, we deduce from $g \in F(b)$ that each $s_i$ belongs to $F(b)$. Now the result follows from the definition of inseparable elements. □
A $tH$-singularity $(D,H,\text{pr})$ is said to have an \textit{inseparable type} if its type $b \in B_k$ is inseparable. Note that a $tH$-singularity of inseparable type $b$ has a tbmf-form $(b, 1_b)$.

**Lemma 3.3.** The singularities of algebraic curve given by $w^k = z^n$ are inseparable for all $n \geq 1$ and $k \geq 1$.

**Proof.** The type of such singularity equals to $b = (a_1 \ldots a_{k-1})^n \in B_k$, where $a_1, \ldots, a_{k-1}$ is a set of standard generators of $B_k$. The equality $\Delta_k^2 = (a_1 \ldots a_{k-1})^k$ implies $b^k = \Delta_k^{2n}$. On the other hand, the action of $\Delta_k^2$ coincides with the conjugation action by the element $\partial D_2 \in \pi_1$. Now it remains to note that the centralizer of any $g \neq 1$ in a free group coincides with the maximal infinite cyclic subgroup containing $g$. □

One can show that Claim 3.1 can not be extended literally to any germ of a topological Hurwits curve over a point. In fact, it does not hold at least if the germ consists of several connected components one of which is not of inseparable type. The following example is a simplest one.

**Example 3.1.** Put $m = 3$ and consider $H_1, H_2$ such that: $\text{bmf}(H_1) = \text{bmf}(H_2) = (1, 1_{\{1,2\}})$; $H_1$ and $H_2$ are rotation symmetric (i.e., invariant under $(z,w) \mapsto (e^{i\phi}z,w)$); in $H_1$ one, figure-V, component is obtained by rotation of a cone over 2 points (lying in $\{\text{Im } z = 0\} \times D_2$; this component corresponds to $1_{\{1,2\}}$) and the other one by rotation of a string not linked with the above cone, while in $H_2$ the latter string is linked with one, and only one, branch of the cone.

**Theorem 3.2.** $Z$-generic topological Hurwitz curves with singularities of inseparable types are $H$-isotopic if and only if they have the same braid monodromy type.

**Proof.** The necessity part is obvious.

Assume that $Z$-generic topological Hurwitz curves $\bar{H}_1, \bar{H}_2$ have singularities of inseparable type and that $\text{bmt}(\bar{H}_1) = \text{bmt}(\bar{H}_2)$. The latter implies that there is an $H$-isotopy which transforms $\bar{H}_1$ into $\bar{H}_2$ everywhere except over a union of disjoint disc neighborhoods of the points of $\text{pr } Z(\bar{H}_1)$, which are transformed by this isotopy.
in a union of disjoint disc neighborhoods of $pr\ Z(\bar{H}_2)$, cf. [11]. It remains to solve the problem of extension of an $H$-isotopy from a boundary of a disc inside the disc. Here, the assumption on the type of singularities is essential.

So, assume that the traces $H_1, H_2$ of $\bar{H}_1, \bar{H}_2$ over a disc $D_1 = D_1(\epsilon)$ are contained in $D = D_1 \times D_2(\epsilon)$, coincide over the boundary of $D_1$, that their braid monodromy $b \in T$ over $\partial D_1$ is of inseparable type and that $D_1$ contains only one point of $Z(H_1)$ and only one point of $Z(H_2)$. Without loss of generality, we may assume that $Z(H_1) = Z(H_2) = \{(0,0)\}$. Now, it is sufficient to construct an $H$-isotopy identical over boundary which transforms $H_1, H_2$ at least over one interval connecting the boundary of $D_1$ with 0, say over the radius $I_− = \{Re z \leq 0, \text{Im } z = 0\} \subset D_1$, and to extend it over a neighborhood of 0. After that the $H$-isotopy can be easily extended to the remaining part of $D_1$, since it is a topological disc over which there are no singular points.

Let $l \geq 2$ be the interlacing number equal to the degree of the singularity, since $\bar{H}_i$ have only the singularities of inseparable type. Due to the definition of $T$, there is a standard tbmf-form $\tilde{(b_1, 1_0)}$, $b_1 \in B_{l,0}$, conjugated to $b$, and due to the definition of the braid monodromy factorization, $H_1$ and $H_2$ split each into $k+1 = m-l+1$ connected components $H_{i,1}, \ldots, H_{i,k+1}, i = 1, 2$, and these splittings coincide over $\partial D_1$. Each component is a topological cone; $H_{i,1}$ is a cone over the braid $b_1 \in B_{l,0}$ and the other are cones over 1-braids $b_2 \in B_{l,1}, \ldots, b_{k+1} \in B_{1,m-1}$.

By Claim 3.1, we may assume that $H_{1,1} = H_{2,1} = C(L)$ where $L$ is the trace of $H_{1,1} = H_{2,1}$ in $\partial D_1 \times D_2(\epsilon)$. In addition, we may assume that the trace of $C(L)$ over some $D_1(\epsilon')$, $\epsilon' \ll \epsilon$, is contained in $D_1(\epsilon') \times \text{Int } D_2(\epsilon')$, $\epsilon' \ll \epsilon$, and that $H_{1,2}, \ldots, H_{1,k+1}$ (respectively, $H_{2,2}, \ldots, H_{2,k+1}$) are distinct constant sections $w = w_j, l+1 \leq j \leq m$ (respectively, $w = w'_j$) of $pr$ over $D_1(\epsilon')$ which are contained in $D_1(\epsilon') \times A_2$, $A_2 = \{r' \leq |w| \leq r\} \subset D_2(\epsilon)$. By an $H$-isotopy the values $w_j$ (respectively, $w'_j$) of these constant sections can be arbitrary continuously changed following arbitrary braid not going through $D_2(\epsilon')$; we call such a change twisting-untwisting. Note that, in particular, when $w_j = w'_j$ for all $j \geq l+1$ we achieve
\(H_1 = H_2\) over a neighborhood of 0. Thus, it is sufficient to show that there exists an \(H\)-isotopy identical on the boundary which transforms \(H_1\) in \(H_2\) over \(I_-\).

To describe the action of \(b_1\) on \(\pi_1(D_2(r) \setminus \{w_1, \ldots, w_l\}, w_0, w_0 \in \partial D_2(r)\), let us replace the standard projection \(pr_2: T \to D_2(r), T = A_1 \times D_2(r), A_1 = \{\epsilon' \leq |z| \leq \epsilon\} \subset D_1\), by a fibration \(pr': T \to D_2(r)\) which coincides with \(pr_2\) on \(\partial D_1(\epsilon') \times D_2(r) \cup A_1 \times \partial D_2(r)\), with the standard projection \(C(L) \to L\) on \(C(L) \cap T\) given by cone structure, and which is constant on each of \(H_{1,j}, j \geq 2\). Such a fibration is given by the flow defined by any vector field on \(T\) which is tangent to \(C(L)\), all \(H_{1,j}, j \geq 2\), and \(A_1 \times \partial D_2(r)\) and obtained as a lift of a rotation invariant radial nowhere zero vector field on \(A_1\). Then \(pr'(C_-), C_- = C(L) \cap pr_1^{-1} I_-\), consists of \(l\) points, which we denote by \(w_1, \ldots, w_l\). Let \(W\) be the element of \(\pi_1(T \setminus C(L), (-\epsilon', w_0))\) given by the constant section \(w = w_0\) over \(\partial D_1(\epsilon')\). If \(x = pr'_*(y)\) where \(y \in \pi_1(T \setminus C(L), (-\epsilon', w_0))\) is realized by a loop lying over \(I_-\), the image \(x^b\) of \(x\) under the action of \(b_1\) is determined by \(x^b = pr'_*(y^b), yW = Wy^b\).

Now, consider a loop \(\bar{y}_j, l + 1 \leq j \leq m\), which starts at \((-\epsilon', w_0)\) descends to \((-\epsilon', w_j)\) without entering into \(D_2(r')\), goes along \(H_{1,j-l+1}\) over \(I_-\), returns along \(H_{2,j-l+1}\) and goes up to \((-\epsilon', w_0)\) without entering into \(D_2(r')\). Rotating \(I_-\) we get a map of torus for which \(\bar{y}_j\) is a meridian-map, and \(W\) is given by a parallel-map. Hence, \(y_j\) realized by \(\bar{y}_j\) and \(W\) commute, so that \(x_j = pr'_* y_j\) is invariant under the action of \(b_1\). Since \(b_1\) is inseparable, for any \(j\) there is an isotopy of \(D_2(r) \setminus \{w_1, \ldots, w_l\}\) identical on the boundary which transforms \(pr' \bar{y}_j\) in a loop outside \(D_2(r')\).

Finally, to transform all \(H_{1,j}\) in \(H_{2,j}\) over \(I_-\) by an \(H\)-isotopy which is identical on \(\partial D \cup C(L)\) let apply the following induction procedure. At the first step by an \(H\)-isotopy identical on \(\partial D\) and \(C(L)\) we transform \(pr' \bar{y}_2\) in a loop outside \(D_2(r')\), and then after defined by this loop twisting-untwisting transform it in a constant loop, i.e., identify \(H_{2,2}\) with \(H_{1,2}\) over \(I_-\). When \(H_{1,j} = H_{2,j}\) over \(I_-\) for any \(j \leq i\) \((i < m-l+1)\), we consider the element realized in \(\pi_1(D_2(r) \setminus \{w_1, \ldots, w_l, \ldots, w_{i-1}\})\) by \(pr' \bar{y}_{i+1}\). Since, by the same argument as above, it is also invariant under the natural
action of $b_1$. Therefore by Lemma 3.2, it is a product of elements corresponding
to $\partial D_2(e')$ and loops around $w_j$, $j > l$, not entering into $D_2(e')$. Thus, again
as above, we can apply an $H$-isotopy realizing the homotopy of $\text{pr}' \bar{y}_{i+1}$ in a loop
outside $D_2(r')$ and then after twisting-untwisting an $H$-isotopy transforming it in
a constant loop. □

**Corollary 3.1.** Two Hurwitz curves $\bar{H}_1, \bar{H}_2 \subset \Sigma_N$ with singularities of inseparable
types are isotopic if $\beta_S(bmt(\bar{H}_1)) = \beta_S(bmt(\bar{H}_2))$.

*Proof.* By isotopy, we can slightly move the points belonging to $Z(\bar{H})$ to get a
$Z$-generic Hurwitz curve having $\text{bmt} = \beta_S(bmt(\bar{H}))$. Now the proof follows from
Theorem 3.2. □

**Corollary 3.2.** $Z$-generic Hurwitz curves with singularities of types $w^k = z^n$ are
$H$-isotopic if and only if they have the same braid monodromy type. □

*Proof.* It follows from Lemma 3.3. □

As is shown in [11], in the case of $Z$-generic cuspidal Hurwitz curves the $H$-
isotopy can be made smooth. The theorem below is a generalization of this result
to the non generic case.

**Theorem 3.3.** Let $\bar{H}_1$ and $\bar{H}_2 \subset \Sigma_N$ be cuspidal Hurwitz curves. Then $\bar{H}_1$ and $\bar{H}_2$
are smoothly $H$-isotopic if and only if $\bar{H}_1$ and $\bar{H}_2$ have the same braid monodromy
type.

*Outline of proof.* The braid monodromy standard form of a germ of a cuspidal curve
over a point having singularities of types $w^2 = z^{n_i}$ is defined by the collection of
exponents $\{n_1, \ldots, n_t\}$ and it is of the form

$$(a_1^{n_1}, 1_\emptyset) \cdot (a_3^{n_2}, 1_\emptyset) \cdot \ldots \cdot (a_{2t-1}^{n_t}, 1_\emptyset),$$

where $2t \leq m$.

It is not hard to prove (we omit the proof) that the centralizer $C(b) \subset B_m$ of

$$b = a_1^{n_1}a_3^{n_2}\ldots a_{2t-1}^{n_t}$$
is generated by \( a_1, a_3, \ldots, a_{2t-1}, a_{2t+1}, a_{2t+2}, \ldots, a_{m-1} \) and some elements \( c_i, 1 \leq i \leq t \), and \( d_{i,l}, i \neq l, 1 \leq i, l \leq t \), where

\[
c_i = a_{2t} \cdots a_{2i+1} (a_{2i} a_{2i-1}^2 a_{2i}) a_{2i+1}^{-1} \cdots a_{2t}^{-1}
\]

and

\[
d_{i,l} = (a_{2i} a_{2i-1}) \cdots (a_{2i-2} a_{2i-3}) (a_{2i} a_{2i-1} a_{2i+1} a_{2i}) (a_{2i-2} a_{2i-3}^{-1}) \cdots (a_{2i} a_{2i-1})^{-1}
\]

if \( l < i \) and

\[
d_{i,l} = (a_{2i-2} a_{2i-1}) \cdots (a_{2i+2} a_{2i+1}) (a_{2i} a_{2i+1} a_{2i-1} a_{2i}) (a_{2i-2} a_{2i-1})^{-1} \cdots (a_{2i-2} a_{2i-1})^{-1}
\]

if \( l > i \).

Now as in the proof of Theorem 3.2, by an \( H \)-isotopy, we identify \( \tilde{H}_1 \) and \( \tilde{H}_2 \) over the complement of the union of small discs around the points of \( \text{pr}(Z(\tilde{H}_1)) \). In each such disc \( D_1(\epsilon) \) pick a much more small disc \( D_1(\epsilon') \) over which we can identify \( \tilde{H}_1 \) and \( \tilde{H}_2 \) with the same disjoint union of several sections and several singularities given by \( (w - w_i)^2 = z^{n_i} \). Select a trivialization of \( \text{pr} \) over \( D_1(\epsilon) \setminus D_1(\epsilon') \) with respect to which the trace \( H_1 \) of \( \tilde{H}_1 \) is radially constant. Recall that \( H_2 \) and \( H_1 \) coincide over the boundary of \( D_1(\epsilon) \setminus D_1(\epsilon') \). Since \( H_1 \) and \( H_2 \) have the same \( tbmf \)-form

\[
\tilde{b} = (a_1^{n_1}, 1_\emptyset) \cdot (a_3^{n_2}, 1_\emptyset) \cdots (a_{2t-1}^{n_t}, 1_\emptyset),
\]

the braid \( g = \hat{r} \cap H_2 \in B_m \), where \( \hat{r} = \{ z \in D_1(\epsilon) \mid z \in \mathbb{R}, \epsilon' \leq z \leq \epsilon \} \), commutes with

\[
b = a_1^{n_1} a_3^{n_2} \cdots a_{2t-1}^{n_t}.
\]

According to the description of the centralizer \( C(b) \) of \( b \), the element \( g \in C(b) \) can be written as a word \( \hat{g} \) in the alphabet consisting of the generators of \( C(b) \) and their inverses. The first letter of this word can be removed by one of the twisting-untwisting isotopies: any \( c_i \) and \( a_l \) with \( l \geq 2t + 1 \) (and their inverse) can be removed by twisting-untwisting isotopy used in the proof of Theorem 3.2, \( a_{2i-1} \) with \( 1 \leq i \leq t \) can be removed by twisting-untwisting isotopy used in [11],...
and every $d_{i,t}$ can be removed by a “composition” of these two isotopies. After a sequence of such isotopies the connected components of $H_2$ will not be linked with the connected components of $H_1$, i.e., when we will have $g = 1$. Now the end of the proof coincides with the corresponding part of the proof of Theorem 3.2. □

3.3. Almost-algebraic curves. By Theorem 2.1, for any element $b \in \mathcal{P}$ there are $N \in \mathbb{N}$ and an algebraic curve $\bar{C} \subset \Sigma_N$ not intersecting $E_N$ (and intersecting transversely $F_\infty$) and such that its affine part $C \subset \Sigma_N \setminus (E_N \cup F_\infty)$ has over $D_1(1)$ the braid monodromy factorization equal to $b$. For $b \in \mathcal{P}$, the minimal number $N = N(b)$ such that $b$ is a braid monodromy factorization of an algebraic curve $\bar{C} \subset \Sigma_N$ over $D_1(r) \subset \mathbb{P}^1$ with some $r > 0$ is called the realizing degree of $b$. If $\beta(b) = \Delta^{2k}$, the number

$$d(b) = N(b) - k$$

is called the deficiency of $b$.

**Definition 3.4.** A Hurwitz curve $\bar{H} \subset \Sigma_N$ is called an almost-algebraic curve if $\bar{H}$ coincides with an algebraic curve $C$ over $D_1(r)$ and with the union of $m$ pairwise disjoint sections $H_{\infty,1}, \ldots, H_{\infty,m}$ of $pr$ over $\mathbb{P}^1 \setminus D_1(r)$.

The following theorem is a direct corollary of Theorems 2.1, 3.2, and 3.3.

**Theorem 3.4.**

(i) For each $b \in \mathcal{P}$ with $\beta(b) = \Delta^{2N}$ there is an almost algebraic Hurwitz curve $\bar{H} \subset \Sigma_N$ whose braid monodromy factorization $b(\bar{H})$ is equal to $b$.

(ii) Any $Z$-generic Hurwitz curve $\bar{H} \subset \Sigma_N$ with singularities of types $w^k = z^n$ (in general, with singularities of inseparable type) is $H$-isotopic to an almost-algebraic curve. If the Hurwitz curve is cuspidal, this isotopy can be chosen smooth.

□

A braid monodromy factorization $b(\bar{H}) = b_1 \cdot \ldots \cdot b_n \in \mathcal{P}$ of a generic ordinary
cuspidal Hurwitz curve $\bar{H} \subset \Sigma_N$ has the following form:

$$b(\bar{H}) = \prod_{i=1}^{n}(q_i a_i^{r_i} q_i^{-1}) \in \mathcal{P},$$

where $r_i$ depends on the type of singularity $s_i \in \mathbb{Z}$: $r_i = 3$ if $s_i$ is a cusp (i.e., $\bar{H}$ is given locally at $s_i$ by $y_i^2 = x_i^3$), $r_i = 2$ if $s_i$ is a node (i.e., given by $y_i^2 = x_i^2$), and $r_i = 1$ if $s_i$ is a tangency point (i.e., given by $y_i^2 = x_i$).

**Proposition 3.1.** There is an infinite sequence $\bar{H}_i \subset \Sigma_1$, $i \in \mathbb{N}$, of generic ordinary cuspidal almost-algebraic curves of degree 54 with exactly 378 cusps and 756 nodes and of pairwise distinct braid monodromy types. In particular, they are not $H$-isotopic to each other, almost all of them are not isotopic to an algebraic cuspidal curve and, moreover,

$$\lim_{i \to \infty} d(b(\bar{H}_i)) = \infty.$$

**Proof.** The existence of a sequence of $\bar{H}_i$ like in the first part of the statement follows from Theorems 2.1 and 3.4 applied to the series of pairwise distinct cuspidal braid factorization types of $\Delta^2_{54}$ found by Moishezon, see [17] Theorem 1.

Let fix $N$ and introduce a space $B$ of algebraic curves

$$C \in \mathbb{P} = \mathbb{P}H^0(\Sigma_N, \mathcal{O}_{\Sigma_N}(54E_N + 54NF))$$

which do not contain $E_N$, have no critical points over the boundary of the disc and have: $54(54 - 1)$ critical values lying in $D_1(1) \subset \mathbb{P}^1$ (counting with multiplicities); among them 378 critical values corresponding to the cusps, 756 values corresponding to the nodes and all the other corresponding to simple tangencies. To prove that $\lim d(b(\bar{H}_i)) = \infty$, it is sufficient to show that $B$ consists of finite number of components. Note that $B$ is contained in the space $\mathcal{M}$ of all curves in $\mathbb{P}$ having at least 378 cusps and 756 nodes, which is a quasi-projective variety.

As in the proof of Lemma 2.2, consider the map $\mathcal{R} : \mathbb{P} \to \text{Sym}_{54(54-1)}^\mathbb{P} \simeq \mathbb{P}^{54(54-1)N}$. It is well-defined at each point of $B$ and $\mathcal{R}(B)$ is contained in

$$V = \{(z_1, \ldots, z_{54(54-1)}) | | z_i | < 1 \text{ for } i \leq 54(54 - 1) \text{ and } | z_i | > 1 \text{ for } i > 54(54 - 1)\}.$$
so that $B \subset \mathcal{R}^{-1}(V) \cap \mathcal{M}$. The set $\mathcal{R}^{-1}(V) \cap \mathcal{M}$ is semi-algebraic. Let consider the semi-algebraic stratification of $\mathcal{M}$ according equisingularity. Intersecting it with $\mathcal{R}^{-1}(V)$ we get a finite (semi-algebraic) stratification of $\mathcal{R}^{-1}(V)$. The intersection of $B$ with each stratum of the latter stratification is open and closed in the stratum, hence $B$ has only finite number of connected components. □

**Problem.** Whether any nodal almost-algebraic curve $\bar{H} \in \Sigma_N$ has the deficiency $d(b(\bar{H})) = 0$, i.e., $H$-isotopic to an algebraic nodal curve $\bar{C} \in \Sigma_N$?

3.4. Several remarks. Let

$$b(\bar{H}) = \prod_{i=1}^{n} \lambda(q_i)(b_i) \in \mathcal{P}$$

be a braid monodromy factorization of a Hurwitz curve $\bar{H} \subset \Sigma_N$, $\deg \bar{H} = m$, where each $b_i$ is a standard form of braid monodromy of an algebraic curve over a point. Every $b_i$ is factorized into a product

$$b_i = b_{i,1} \cdot \ldots \cdot b_{i,t_i},$$

where: $b_{i,1} \in B_{k_{i,1},0}^+ \subset B_m$, $B_{k_{i,1},0}^+$ stating for the semigroup generated by the elements $a_1, \ldots, a_{k_{i,1}-1}$; $b_{i,2} \in B_{k_{i,2},k_{i,1}}^+ \subset B_{k_{i,2},k_{i,1}}^+$ being generated by $a_{k_{i,1}+1}, \ldots, a_{k_{i,1}+k_{i,2}-1}; \ldots; b_{i,t_i} \in B_{k_{i,t_i},k_{i,1}+\ldots+k_{i,t_i-1}}^+ \subset B_{k_{i,t_i},k_{i,1}+\ldots+k_{i,t_i-1}}^+$ being generated by $a_{k_{i,1}+\ldots+k_{i,t_i-1}+1}, \ldots, a_{k_{i,1}+\ldots+k_{i,t_i}-1}$. Recall that $k_{i,j} \geq 2$ for $1 \leq j \leq t_i$.

By means of this factorization one can describe the homotopy type of the complement $\Sigma_N \setminus (\bar{H} \cup E_N \cup F_\infty)$ (cf., [13]). Namely, it has the homotopy type of the two dimensional Cayley complex of the following (van Kampen - Zariski) presentation of the fundamental group

$$\pi_1(\Sigma_N \setminus (\bar{H} \cup E_N \cup F_\infty)) = \langle x_1, \ldots, x_m : R_{i,j,k} \rangle;$$

in terms of the canonical action of $B_m$ on the free group with generators $x_1, \ldots, x_m$, the defining relations $R_{i,j,k}$ ($1 \leq i \leq n, 1 \leq j \leq t_i, k_{i,1} + \ldots + k_{i,j-1} + 1 \leq k \leq k_{i,1} + \ldots + k_{i,j} - 1$) take the form

$$q_i^{-1}(b_{i,j}(x_k)) = q_i^{-1}(x_k).$$
Notice also that using the arguments from [1] one can show that the braid monodromy type of topological Hurwitz curves in \( \Sigma_N \) is preserved by some special ambient homeomorphisms. Namely, let associate with each topological Hurwitz curve \( \tilde{H} \) a stratified union \( \tilde{H} = \tilde{H} \cup (\bigcup_{s \in \mathbb{Z}} F_{\text{pr}(s)}) \cup F_\infty \subset \Sigma_N \) (and call it an *equipped curve*); two topological Hurwitz curves \( \tilde{H}_1, \tilde{H}_2 \) has the same braid monodromy type if (and only if) there is a homeomorphism of oriented pairs \( h : (\Sigma_N, \tilde{H}_1) \to (\Sigma_N, \tilde{H}_2) \) which transforms \( \tilde{H}_1 \) in \( \tilde{H}_2 \) and keeps invariant \( F_\infty \).

The condition that \( h \) preserves orientation on the components of topological Hurwitz curves is essential. This follows from the examples ([10]) of irreducible (ordinary) cuspidal algebraic curves \( C_1, C_2 \subset \Sigma_1 \) having different braid monodromy types, but for which there exists a diffeomorphism \( c : (\Sigma_1, \tilde{C}_1) \to (\Sigma_1, \tilde{C}_2) \) not preserving orientation on the components of the equipped curves (in these examples, \( c \) is the complex conjugation).

§4. Braid monodromy factorizations of symplectic surfaces.

In this section we consider singular symplectic surfaces in \( \mathbb{C}P^2 \). As is known, the symplectic structure on \( \mathbb{C}P^2 \) is unique up to symplectomorphisms and multiplication by a constant factor. On the other hand, up to our knowledge, its uniqueness up to isotopy and constant factor is an open question. Moreover, rescaling of the symplectic structure is an important ingredient of our proof of Proposition 4.2 below. These are the reasons why, in what follows, we speak on isotopy classes of symplectic structures.

We treat only the case of surfaces with isolated singularities and define a *singular symplectic surface* \( C \subset \mathbb{C}P^2 \) as a triple \((C, J, \omega)\) such that: \( C \) is the image \( C = f(\mathcal{S}) \) of an almost everywhere injective \( J \)-holomorphic map \( f : \mathcal{S} \to \mathbb{C}P^2 \) of a closed Riemann surface \( \mathcal{S} \); and \( J \) is an almost complex structure defined on a neighborhood of \( C \) and tamed by a given symplectic structure \( \omega \) of \( \mathbb{C}P^2 \). Two singular symplectic surfaces \((C_0 = f_0(\mathcal{S}), J_0^{\text{loc}}, \omega_0), (C_1 = f_1(\mathcal{S}), J_1^{\text{loc}}, \omega_1)\) are said *weakly symplectically (smoothly) isotopic* if \((f_0, J_0^{\text{loc}}, \omega_0), (f_1, J_1^{\text{loc}}, \omega_1)\) can be included in a continuous (respectively, smooth) family of tamed almost complex and symplec-
tic structures $J_{loc}^t$, $\omega_t$ and that of $J_{loc}^t$-holomorphic maps $f_t : \mathcal{G} \times [0, 1] \to \mathbb{C}P^2$ such that the maps $\phi_t : C_0 \to C_t$ given by $f_0(s) \mapsto f_t(s), s \in \mathcal{G}$, are well-defined homeomorphisms for all $t \in [0, 1]$. The same definitions are applied to symplectic surfaces in any manifold. When the ambient symplectic structure is not changing one speaks on \textit{symplectic isotopy}. In particular, two singular symplectic surfaces are symplectically isotopic, if there is a symplectic diffeotopy transforming one into another.

\textbf{Remark 4.1.} In the case of arbitrary symplectic manifolds it is natural to expect that the classes of symplectically isotopic surfaces and the classes of weakly symplectically isotopic surfaces are not the same in general. By contrary, in the case of symplectic structures on $\mathbb{C}P^2$ these notions coincide. Indeed, in this case any weakly symplectic isotopy can be rescaled $\omega_t \mapsto \omega'_t = \lambda_t \omega_t, \lambda_t \in \mathbb{R}_+$, (without changing $J_t$) to an isotopy with $\omega'_t$ not changing their cohomology class; then it remains to apply the Mozer theorem to get a diffeotopy making $\omega'_t$ constant; the same diffeotopy is then applied to $f_t$ and $J_{loc}^t$. Therefore, when we need to prove that some symplectic surfaces in $\mathbb{C}P^2$ are symplectically isotopic, it is sufficient for us to check that they are weakly symplectically isotopic. It is this strategy that we adopt below.

\textbf{Remark 4.2.} The space of almost complex structures on a finite dimensional real vector space which are tamed by a given symplectic form is contractible, see [9]. It implies that any $\omega$-tamed almost complex structure $J_{loc}^t$ defined in an open subset $U$ of a symplectic manifold $(V, \omega)$ can be extended from a smaller neighborhood $U_0 \subset U$ to a $\omega$-tamed almost complex structure $J$ on the whole $V$. Moreover, by the same reason, if $U = U_0, J_{loc}^t = J_{loc}^t, \omega = \omega_t$ depend smoothly on one or several parameters $t$ the extensions from $U_0 = U_{0,t}$ to $\mathbb{C}P^2$ can be chosen depending smoothly on $t$. Therefore, one gets the same notions if in the above definitions the locally defined $J$-structures are replaced by tamed almost structures defined on the whole $\mathbb{C}P^2$. The choice we made is motivated by simplifications in some of the proofs. Note also that the above extension properties remain true if, in addition,
we will restrict ourself to almost complex structures for which a given symplectic
surface is $J$-holomorphic. In particular, in the definition of weakly symplectically
isotropic surfaces it is sufficient to have $J^*_t$ to be defined only near the singular
points of $C_t$.

As is known (see [21], [15]), the singularities of pseudoholomorphic curves in an
almost complex four-manifold are equivalent to the singularities of genuine complex
curves in complex surfaces up to $C^1$ coordinate change.

If a singular symplectic surface $C$ is given together with a symplectic structure
$\omega$ and a $\omega$-tamed almost complex structure $J^0$ on a neighborhood $U$ of $C$ in $CP^2$,
we extend $J^0$ from a smaller neighborhood $U_0 \subset U$ to a $\omega$-tamed almost complex
structure $J$ on $CP^2$, then consider a generic pencil $L$ of $J$-lines, see [9] and [22],
and define the associated braid factorization type $\text{bmt}(C,\omega,J,L)$ as in section 2.3
in the algebraic case.

Note that any $b \in P$ such that $\alpha(b) = \Delta^2$ (and only the such ones) can be
realized as a braid monodromy factorization of a symplectic surface with respect to
a (non necessarily generic) pencil. Such a realization is obtained from an almost-
algebraic curve given by Theorem 3.4 by rescaling the standard pencil where the
curve is situated.

**Proposition 4.1.** The braid factorization type $\text{bmt}(C,\omega,J,L)$ with respect to a
generic $L$ depends only on the symplectic isotopy class of $C$. (In particular, it does
not depend on the extension $J$ of $J^0$ to the whole $CP^2$.)

**Proof.** Let $C_0, C_1$ be weakly symplectically isotopic symplectic surfaces equipped
with $\omega_i (i = 0, 1)$-tamed $J$-structures, $J^0_i$, $J^1_i$, their extensions $J_0, J_1$, and generic
pencils, $L_0, L_1$, as in the definitions of singular symplectic surface and its braid mon-
odromy factorization, respectively. Pick a weakly symplectic isotopy $(C_t, \omega_t, J^0_t)$
and extend $J^0_t$ to an isotopy $J_t$ on the whole $CP^2$. For each $t$, the space $(CP^2)^*$
of $J_t$-lines is diffeomorphic to $CP^2$ and is equipped with the canonical dual $E$
structure, see [22]. The lines tangent to $C_t$ form the dual $E$-curve $C^*_t$. It has a
finite number of singular points (which correspond to double or excess tangents)
and the dual to $C_t^*$ is $C_t$. Hence, the nongeneric pencils correspond to a choice of the center of the pencil $L_t$ belonging neither to $C_t$ nor to the finite number of $J_t$-lines which are dual to the singular points of $C_t^*$. Therefore, there is a path $L_t$ of generic pencils connecting $L_0$ with $L_1$. Clearly, the braid monodromy factorization $C_t$ defined by $L_t$ is not depending on $t$ and the result follows. □

Recall that the braid monodromy factorization type of $m$ complex lines in $\mathbb{C}P^2$ in general position coincides with $\widetilde{\delta}_m^2$ defined in section 1.4 (see, for example, [18]).

**Corollary 4.1.** The braid monodromy factorization type of a generic family of $m$ pseudo-lines is equal to $\widetilde{\delta}_m^2$.

**Proof.** According to Barraud [2], any generic family of $m$ pseudo-lines is symplectically isotopic to $m$ true lines in general position. □

**Corollary 4.2.** There are infinitely many ordinary cuspidal symplectic plane curves of the same degree with the same number of cusps and the same number of nodes but two-by-two not symplectically isotopic.

**Proof.** It follows from Propositions 3.1 and 4.1. □

**Remark 4.3.** The statement reverse to Proposition 4.1 holds at least if the symplectic isotopy is replaced by a topological one and if, in addition, it is assumed that all singularities are of inseparable types. In deed, as it follows from [12], pencils determined by two almost complex structures tamed by a same symplectic structure or, more generally, by symplectic structures in a continuous family, are isotopic. Together with Theorem 3.2 it implies that, if braid factorization types of two singular symplectic (with respect to a same structure or with respect to structures from the same connected component) surfaces are equal and the singularities of the surfaces are of inseparable types, then the surfaces are topologically isotopic. There is some confusion in replacing a topological isotopy by a smooth one. This is because in the smooth category a smooth isotopy is supposed to be an ambient one, contrary to our choice in the definition of smooth symplectic isotopies. Certainly, the ambient isotopy can be made smooth outside the singular points, and
by contrary, in general, can not be made smooth at the singular points, since they
can have moduli even with respect to smooth changes of coordinates.

**Remark 4.4.** In the above Corollary, the pseudo-lines can be replaced by
Hurwitz curves (recall that according to our definitions, Hurwitz curves are situated
in a pencil of ordinary lines) realizing the generator of $H_2(\mathbb{C}P^2)$, since after rescaling
a transversal pencil, the Hurwitz surfaces becomes $J$-curves, so becomes pseudo-
lines. As Schevchishin communicated to us, Barraud result can be generalized to
nodal symplectic curves (without negative nodes) of genus $\leq 3$. Then, the above
arguments are applied to such curves as well.

The following proposition is a partial inverse of Proposition 4.1.

**Proposition 4.2.** Two symplectic, with respect to the Fubini-Studi symplectic
structure $\omega_0$, ordinary cuspidal surfaces are symplectically $C^1$-smoothly isotopic
in $\mathbb{C}P^2$ if and only if they have the same braid factorization type with respect to a
generic pencil.

*Proof.* The necessity part, i.e., coincidence of braid monodromy factorization types,
follows from Proposition 4.1.

According to Remark 4.1, it is sufficient to find a weakly symplectic isotopy.

Let $C_0$ be an ordinary cuspidal symplectic surface, which is a $J_0$-holomorphic
curve where $J_0$ is an almost complex structure on $\mathbb{C}P^2$ compatible with $\omega_0$. By
a continuous variation of $J_0$ with a support in a neighborhood of some point $p$,
make $J_0$ integrable in a smaller neighborhood $U'_0$. Then, consider a generic pencil
$L_0$ of pseudo-lines with a center $p_0 \in U'_0$. Choose local coordinates $x, y$ near the
critical points of $C_0$ (by critical points we mean the singular points of $C_0$ and the
points of tangency between $C_0$ and $L_0$) so that locally the elements of the $J_0$-pencil
are given by the fibers of $(x, y) \mapsto x$ and $C_0$ is defined by equations $y^2 = x^k$ with
$k = 1, 2, 3$, respectively to the cases of tangency points, nodes, and cusps; in all
the cases the direction $y = 0$ can be chosen symplectic. It allows us to replace $J_0$
by a $\omega_0$-tamed almost complex structure $J'_0$ with respect to which the above local
coordinates become $J$-holomorphic (so that $J'_0$ is integrable near the critical points
and near $p_0$) and $C_0$, as well as the ruling $L_0$, remain $J$-holomorphic. Since $J_0$ and $J'_0$ are $\omega_0$-tamed and since $C_0$ is $J$-holomorphic with respect to both of them, they can be joined by a homotopy almost complex structures keeping $C_0$ to be $J$-holomorphic. By Proposition 4.1, $\text{bmt}(C_0)$ does not change. The next weakly symplectic isotopy consists in a continuous variation of $\omega_0$, it has a support in $U'_0$ and replaces the pair $(\omega_0, J'_0)$ by a pair $(\omega'_0, J'_0)$ which is standard (i.e., Kähler flat) in $U_0 \subset U'_0$. Such a variation is given in [14], Lemma 5.5B.

On the other hand, by Theorem 3.1, there exists a Hurwitz curve $C_1 \subset \mathbb{CP}^2$ whose braid monodromy type (with respect to a generic pencil $L_1$ of ordinary lines) $\text{bmt}(C_1)$ is equal to $\text{bmt}(C_0)$ (recall that the latter is defined by means of a generic pencil of $J_0$-lines). By rescaling a generic pencil $L_1$ of ordinary lines and moving its center to $p_0$, we can assume that $C_1$ is a $J_1$-holomorphic curve, where $J_1$ is a suitable almost complex structure tamed by $\omega_0$ and identical with the standard one in a neighborhood of $p_0$ and in some neighborhoods of the critical points of $C_1$.

In addition, as before, we can replace $(\omega_0, J_1)$ by a pair $(\omega'_1, J'_1)$, $J'_1 = J_1$, standard near $p_0$. To proof Proposition 4.2, it is sufficient to show that $C_0$ and $C_1$ are weakly symplectically $C^1$-smoothly isotopic in $\mathbb{CP}^2$.

Then, proceed as in the proof of Proposition 4.1: pick a path $(\omega'_t, J'_t)$ where the almost complex structures $J'_t$ are tamed by $\omega'_t$ and integrable near $p_0$, and consider a family of $J'_t$-holomorphic pencils of $J'_t$-lines connecting $(J'_i, L_i)$, $i = 0, 1$. Make, by a continuous variation, the almost complex structures $J'_t$ integrable near the critical points. Now, it remains to construct an isotopy between $C'_0$ and $C_1$ and to enhance it so that it becomes a weakly symplectic isotopy. We construct the isotopy in two steps.

At the first step, by a diffeotopy of the pencils holomorphic near $p_0$ and near the critical points we get a diffeotopy $C'_t$ of the curve $C_0 = C'_0$. It provides us with surfaces $C'_t$ which are holomorphic near the critical points and gives a kind of $H$-isotopy: outside the critical points each surface $C'_t$ meets the pseudo-lines transversely and with positive intersection number; the projection of $C'_t$ to the base
of its pencil is a finite ramified covering; near the critical points it is a complex
analytic curve which in local analytic coordinates \( x, y \) such that the projection is
given by \( (x, y) \mapsto x \) is defined by equation \( y^2 = x^k \) with \( k = 1, 2, 3 \). The resulting
curve \( C'_1 \) is a genuine Hurwitz curve, since \( L_1 \) is a pencil of ordinary lines. Obviously,
\( \text{bmt}(C'_1) = \text{bmt}(C_0) \).

In the second step, the arguments from [11] can be applied to the two surfaces
\( C'_1 \) and \( C_1 \) having the same braid monodromy factorization type with respect to \( L_1 \)
to construct an \( H \)-isotopic family of Hurwitz curves \( H_t \), connecting \( H_0 = C'_1 \) and
\( H_1 = C_1 \), with all the properties enumerated above.

To produce a weakly symplectic isotopy by means of the combined isotopy \( (C'_t \)
followed by \( H_t ) \) constructed above, let apply to \( (C'_t, \omega'_t) \) followed by \( (H_t, \omega'_t) \) the
following rescaling of \( \omega'_t \). Like at the beginning of the proof, use the construction
from [14] to make, by a continuous variation of \( \omega'_t \), the pairs \( (\omega'_t, J'_t) \) coinciding with
the standard flat pair \( (\Omega, i) \) in a small ball \( B(\delta), \delta > 0 \), around \( p_0 \). Then, consider
\( \mathbb{C}^1 \)-fibrations \( h_t : \mathbb{C}P^2 \setminus \{p_0\} \to S^2 \) whose fibers \( h_t^{-1}(v), v \in S^2 \), are symplectic and
which coincide with the \( J'_t \)-rulings \( \text{pr} : \mathbb{C}P^2 \setminus \{p_0\} \to S^2 \) outside \( B(\delta) \) and with
the ordinary lines ruling in a smaller ball \( B(\delta') \). As soon as such fibrations are
given, it remains to replace \( \omega'_t \); outside a smaller ball \( B(\delta') \) by \( \omega'_t + Nh_t^* \omega_{S^2} \), where
\( N \geq 0 \) is a sufficiently big constant, \( \omega_{S^2} \) is a volume form on \( S^2 \); and inside \( B(\delta') \)
by some \( \Omega_N \) with \( \Omega_N = \omega'_t + Nh_t^* \omega_{S^2} = \Omega + Nh^* \omega_{S^2} \) near \( \partial B(\delta') \) (\( h \) states for the
standard ruling). Such a family \( \Omega_N \) is found in [14], Proposition 5.1B. A missing
weakly symplectic isotopy between \( (C'_0, \omega'_0) \) and \( (C'_0, \omega'_0 + N \text{pr}^* \omega_{S^2}) \), as well as
that between \( (H_1, \omega'_1 + N \text{pr}^* \omega_{S^2}) \) and \( (H_1, \omega'_1) \), can be given by variation of the
symplectic structure only: \( \theta \mapsto \omega'_t + \theta \text{pr}^* \omega_{S^2}, i = 1, 2 \).

The existence of \( h_t \) with the properties enumerated above can be proven in
the following way. Let number the pseudo-lines going through \( p_0 \) by points in \( S^2 \),
which we identify with the projective line of complex directions at \( p_0 \). Then, to
construct a desired \( h_t \) we fix a parametric representation of a pseudo-line going
through \( p_0 \) as follows. We choose its intersection with a selected pseudo-line not
going through $p_0$ as the value at $\infty$, $p_0$ as the value at 0 and fix the derivative of the parametrization at 0 (a unit tangent vector $\xi$ to $\mathbb{CP}^2$ at $p_0$). By means of such parametrizations, $\phi_{\xi,t} : \mathbb{CP}^1 \to \mathbb{CP}^2$, we introduce the fibers of $h_t$ replacing $\phi_{\xi,t}$ in a disc $0 \in D_1(\epsilon) \subset \mathbb{C} \subset \mathbb{CP}^1$ by $(1 - \delta(r))\Xi + \delta(r)\phi_{\xi,t} : D_1(\epsilon) \to B(\delta)$, where $\Xi$ is the linear part of $\phi_{\xi,t}$ at 0 and $\delta : [0, \epsilon] \to [0, 1]$ is a bump function which is taking value 0 near 0 and 1 near $\epsilon$. The resulting maps $h_t$ have the enumerated above properties if $\epsilon$ is sufficiently small and $r\delta'(r) < 1$ for any $r \in [0, \epsilon]$. (One can check that a fibre in direction $\xi$ is symplectic by means of straightforward calculations with $idh \wedge \overline{dh} \wedge \Omega$ in affine complex coordinates $z, w$ near $p_0$ with $\Re \frac{\partial}{\partial z} = \xi$, $\Re$ states for the real part, using the complex analyticity of the partial derivatives with respect to local coordinates $s$ in $S^2$ of equation $w = sz + \phi_2(s)z^2 + \ldots$ defining the pseudo-lines. the calculations with $h$ one can use the above equation to get an implicit equation for $h$: $h = v - \delta(r)[\phi_2(h)z + \ldots], w = vz$.) □

**Corollary 4.3.** A nodal symplectic, with respect to the Fubini-Studi symplectic structure, surface is symplectically $C^1$-smoothly isotopic to an algebraic curve if and only if its braid factorization type $bmt$ with respect to a generic pencil is a partial re-degeneration of some element from $S_{A_1}$: $bmt = r(z_1) \cdot z_2$, where $r : S_{A_1} \to S_{A_0}$ is the re-degeneration (see Example 1 in section 1.1), $A_0$ is the full set of conjugates of the generator $a_1 \in B_m$, $A_1$ is the full set of conjugates of $a_1^2$, and $z_1, z_2 \in S_{A_1}$.

**Proof.** First, notice that $\alpha(bmt) = \Delta_m^2$, where $m$ is the degree of the symplectic surface. So, according to Theorem 1.2, $z_1 \cdot z_2 = \delta_m^2$. The latter element is the braid monodromy factorization of $m$ lines in general position. It remains to smooth the corresponding nodes of this algebraic curve without deforming the other nodes, which is possible, for example, by Bruzotti theorem [5]. □

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