The multi-harmonic signal frequencies estimation in finite time

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Abstract. The paper presents a method to estimate the frequencies of a multi-harmonic signal in finite time. We use parameterization based on applying delay operators to a measurable signal. The result is a linear regression model with an unknown vector which depends on the signal parameters. We use Dynamic Regressor Extension and Mixing method to replace the nth order regression model with scalar regressions. After that, we estimate the parameters separately using the standard gradient descent method. In the last step, we find algebraically the finite-time parameter estimates. The set of numerical simulations demonstrates the efficiency of the proposed approach.

1. Introduction

Online estimation of frequencies for a multi-harmonic signal is one of the fundamental theoretical problems, and it is widespread in many practical applications: disturbance rejection systems [1], dynamic positioning systems for vessels under external disturbances such as waves, winds, and currents [2]. Online parameter estimation is used in power systems for load balancing, fault detection, and power quality [3, 4]. The sensorless speed estimation approach is based on online parameter estimation [5].

At present, there are various methods of multi-harmonic parameter estimation. One of the approaches is the construction of an adaptive observer of the minimum dimension 3n, which provides global exponential convergence [6, 7], as well as a hybrid observer [8] for the synthesis of multi-harmonic signal frequency estimates with saturation. Alternative approaches are the algorithms using the Dynamic Regressor Extension and Mixing (DREM) method [9], which allows improving noise immunity and transient characteristics simultaneously, and the approach bases on the application of the Volterra integral operator [10].

This paper extends the previously proposed method of sinusoidal signal estimation [11] to unbiased multi-sinusoidal signals. We use parameterization with delay operators and obtain the linear regression model. Using DREM, we split this model into scalar regressions, which improves transition behavior and constructs estimates at a predefined finite time, using a scheme described in [11].
The paper is organized as follows. Section 2 formulates the problem and the basic assumption. In Section 3, we describe the structure of the linear regression model for the general case, and show its validity for the dyadic sinusoid signal. Section 3 presents the finite-time estimation scheme for unknown parameters of the regression models. Section 4 introduces the finite-time frequencies estimation. In Section 5, simulation results are presented, illustrating the approach efficiency for a signal of three harmonics.

2. Problem statement
Consider a measurable multi-harmonic signal:

\[ y(t) = \sum_{i=1}^{n} A_i \sin(\omega_i t + \phi_i), \]

where \( \omega_i \in \mathbb{R}_+ \) are frequencies \( \omega_i \neq \omega_j, i, j = 1, \ldots, n \); \( \phi_i \) are phases; \( A_i \in \mathbb{R}_+ \) are amplitudes; \( n \) is the number of harmonics in the signal \( y(t) \). The parameters \( \omega_i \), \( \phi_i \) and \( A_i \) are unknown.

The objective is to find the estimates \( \tilde{\omega}_i(t) \) of the frequencies \( \omega_i \), which provide convergence of the errors \( \tilde{\omega}_i(t) = \omega_i - \tilde{\omega}_i(t) \) to zero at the predefined finite time \( t_{\text{fft}} > 0 \), i.e.

\[ |\tilde{\omega}_i(t)| = 0, \quad \text{for } \forall t \geq t_{\text{fft}}. \]

Our basic assumption is the following.

Assumption A1. The upper bound on the signal frequencies \( \omega_i \) is known and is equal to \( \omega \).

3. Parameterization
In this section, we aim to obtain a linear regression model with measurable variables and a constant vector depending on the signal frequencies.

Consider the discrete analog of the signal (1):

\[ y_d[k] = \sum_{i=1}^{n} A_i \sin(\omega_i kT + \phi_i), \]

where \( T > 0 \) is the sampling period, \( t = kT, k \in \mathbb{Z}_+ \).

As shown in [12], we can express the value of a discrete signal (3) at the time \( kT \) as a linear combination of \( 2n \) previous values of \( y_d[(k-1)T], \ldots, y_d[(k-2n)T] \). We will get a similar result for a continuous signal. For this purpose, we introduce an \( h \)-second delay operator:

\[ \left( Z (\cdot) \right)(t) = \begin{cases} 0, & t < h, \\ (\cdot)(t-h), & t \geq h, \end{cases} \]

where \( h \in \mathbb{R}_+ \) is a chosen delay value, \( h < \frac{\pi}{2\omega} \).

Signals with multiple delays can be represented using this delay operator, as \( y(t - kh) = Z^k y(t), \)

\( k = 1, 2n. \)

Let us write an expression where the continuous signal \( y(t) \) with \( n \) harmonics is expressed via \( 2n \) delayed signals \( y(t - h), \ldots, y(t - 2nh) \):

\[ \left[ Z^2 + 1 - 2Zc_1 \right] \cdots \left[ Z^2 + 1 - 2Zc_n \right] y(t) = 0, \quad \text{for } \forall t \geq 2nh, \]

where \( c_i := \cos\omega_i t \) are constants, \( i = 1, n \); \( Z \) is the delay operator (4).

Using (5), we get the regression model

\[ \psi(t) = \varphi^T(t) \theta, \quad t \geq 2nh, \]

where \( \psi(t) \in \mathbb{R}^1 \) is a regressand, \( \varphi^T(t) = [\varphi_1(t) \ldots \varphi_n(t)] \in \mathbb{R}^n \) is a regressor, \( \theta^T = [\theta_1 \ldots \theta_n] \in \mathbb{R}^n \) is an unknown parameter vector:
\[\psi(t) = (Z^2 + 1)^n y(t) = \sum_{i=0}^{n} C_n^i y(t - 2h(n - i)),\]
\[\varphi_1(t) = 2Z(Z^2 + 1)^{n-1} y(t) = 2\sum_{i=0}^{n-1} C_n^{i-1} y(t - h(2(n - i) - 1)),\]
\[\varphi_2(t) = 2Z^2(Z^2 + 1)^{n-2} y(t) = 2^2 \sum_{i=0}^{n-2} C_n^{i-2} y(t - 2h(n - i - 1)),\]
\[\ldots\]
\[\varphi_n(t) = 2^n Z^n y(t) = 2^n y(t - nh),\]
\[
\begin{align*}
\theta_1 &= -c_1 - c_2 - \cdots - c_n, \\
\theta_2 &= c_1 c_2 + c_1 c_3 + \cdots + c_{n-1} c_n, \\
\theta_n &= (-1)^n c_1 c_2 \cdots c_n,
\end{align*}
\]
where \(C_n^i = \frac{n!}{i!(n-i)!}\).

It is not difficult to show that the expressions (5)–(6) are valid for a single sinusoid signal \(y(t) = A_1 \sin(\omega_1 t + \phi_1)\). Let us show correctness of (5) for a signal with two sinusoids:
\[y(t) = A_1 \sin(\omega_1 t + \phi_1) + A_2 \sin(\omega_2 t + \phi_2).\] (8)

Rewrite expression (8) in matrix form:
\[
y(t) = \mu^T \sigma(t),
\]
\[\mu^T = [1 \ 0 \ 1 \ 0],
\]
\[\sigma(t) = [\alpha_1(t) \ \beta_1(t) \ \alpha_2(t) \ \beta_2(t)]^T,
\]
where \(\alpha_i(t) = A_i \sin(\omega_i t + \phi_i), \beta_i(t) = A_i \cos(\omega_i t + \phi_i), i = 1, 2\).

We use the notation (9) to write down signals with multiple \(h\) delay as follows:
\[
y(t - kh) = \alpha_1(t) \cos(k\omega_1 h) - \beta_1(t) \sin(k\omega_1 h) + \alpha_2(t) \cos(k\omega_2 h) - \beta_2(t) \sin(k\omega_2 h) = \\
= \left[\cos(k\omega_1 h) \ -\sin(k\omega_1 h) \ \cos(k\omega_2 h) \ -\sin(k\omega_2 h)\right] \sigma(t), \quad k = \frac{1}{h}.
\] (10)

If we combine (10) in matrix form, we get
\[
Y(t, h) = T(h) \sigma(t),
\]
\[
Y(t, h) = \begin{bmatrix} y(t - h) \\
y(t - 2h) \\
y(t - 3h) \\
y(t - 4h) \end{bmatrix}, \quad \sigma(t) = \begin{bmatrix} \alpha_1(t) \\
\beta_1(t) \\
\alpha_2(t) \\
\beta_2(t) \end{bmatrix},
\]
\[T(h) = \begin{bmatrix} \cos(\omega_1 h) & -\sin(\omega_1 h) & \cos(\omega_2 h) & -\sin(\omega_2 h) \\
\cos(2\omega_1 h) & -\sin(2\omega_1 h) & \cos(2\omega_2 h) & -\sin(2\omega_2 h) \\
\cos(3\omega_1 h) & -\sin(3\omega_1 h) & \cos(3\omega_2 h) & -\sin(3\omega_2 h) \\
\cos(4\omega_1 h) & -\sin(4\omega_1 h) & \cos(4\omega_2 h) & -\sin(4\omega_2 h) \end{bmatrix},
\]
where \(T \in \mathbb{R}^{4 \times 4}\) is constant matrix and \(T(h) = [T_1(h) \ T_2(h) \ T_3(h) \ T_4(h)]\), \(T_i(h)\) is the \(i\)-th column of the matrix \(T(h)\), \(i = 1, 4\).

Multiplying both sides of (11) on the left by the inverse matrix \(T^{-1}(h)\) and using the equality
\[T^{-1}(h) = \frac{1}{\det T(h)} \text{adj} T(h),\]
we obtain
\[
\text{adj} T(h) Y(t, h) = \det T(h) \sigma(t),
\] (12)
where \(\text{adj} \{\cdot\}\) is the adjugate matrix, \(\det \{\cdot\}\) is the determinant.

Multiplying both sides of (12) on the left by the constant vector \(\mu^T\) and using the equality (9), we get
\[
\mu^T \text{adj} T(h) Y(t, h) = \det T(h) \mu^T \sigma(t),
\]
\[\mu^T x(t, h) = \det T(h) y(t),
\] (13)
where $x(t, h) = \text{adj}(T(h)) Y(t, h) = [x_1(t, h) \quad x_2(t, h) \quad x_3(t, h) \quad x_4(t, h)]^T$.

Using the Cramer’s rule, we get expressions for the first and third $x(t, h)$ vector components:

$$
\begin{align*}
    x_1(t, h) &= \det[Y(t, h) \quad T_2(h) \quad T_3(h) \quad T_4(h)], \\
    x_3(t, h) &= \det[T_1(h) \quad T_2(h) \quad Y(t, h) \quad T_4(h)].
\end{align*}
$$

(14)

Substituting (9) and (14) in (11), we obtain

$$
\det[Y(t, h) \quad T_2(h) \quad T_3(h) \quad T_4(h)] + \det[T_1(h) \quad T_2(h) \quad Y(t, h) \quad T_4(h)] = \det[T_1(h) \quad T_2(h) \quad T_3(h) \quad T_4(h)] y(t).
$$

(15)

Let us calculate the determinant $\det[T_1(h) \quad T_2(h) \quad T_3(h) \quad T_4(h)]$ on the right side of the expression (15):

$$
\det T(h) = \begin{vmatrix}
    \cos(\omega_1 t) & -\sin(\omega_1 t) & \cos(\omega_2 t) & -\sin(\omega_2 t) \\
    \cos(2\omega_1 t) & -\sin(2\omega_1 t) & \cos(2\omega_2 t) & -\sin(2\omega_2 t) \\
    \cos(3\omega_1 t) & -\sin(3\omega_1 t) & \cos(3\omega_2 t) & -\sin(3\omega_2 t) \\
    \cos(4\omega_1 t) & -\sin(4\omega_1 t) & \cos(4\omega_2 t) & -\sin(4\omega_2 t)
\end{vmatrix} = \begin{vmatrix}
    c_1 & -s_1 & c_2 & -s_2 \\
    2c_1^2 - 1 & -2s_1c_1 & 2c_2^2 - 1 & -2s_2c_2 \\
    4c_1^3 - 3c_1 & -4s_1^2 + 3 & 4c_2^3 - 3c_2 & -4s_2^2 + 3 \\
    8c_1^4 - 8c_1^2 + 1 & -8s_1^2c_1 + 4c_1 & 8c_2^4 - 8c_2^2 + 1 & -8s_2^2c_2 + 4c_2
\end{vmatrix},
$$

where the notation $c_i = \cos(\omega_i t)$, $s_i = \sin(\omega_i t)$, $i \in \{1, 2\}$ are used to shorten the record.

Taking out common factor $-s_1$ of the second column and $-s_2$ of the fourth column, respectively, we have

$$
\det T(h) = s_1s_2 \begin{vmatrix}
    c_1 & 1 & c_2 & 1 \\
    2c_1^2 - 1 & 2c_1 & 2c_2^2 - 1 & 2c_2 \\
    4c_1^3 - 3c_1 & -4s_1^2 + 3 & 4c_2^3 - 3c_2 & -4s_2^2 + 3 \\
    8c_1^4 - 8c_1^2 + 1 & -8s_1^2c_1 + 4c_1 & 8c_2^4 - 8c_2^2 + 1 & -8s_2^2c_2 + 4c_2
\end{vmatrix}.
$$

(16)

Interchanging $s_1^2$ and $1 - c_1^2$, $i \in \{1, 2\}$, we can rewrite (16) as

$$
\det T(h) = s_1s_2 \begin{vmatrix}
    c_1 & 1 & c_2 & 1 \\
    2c_1^2 - 1 & 2c_1 & 2c_2^2 - 1 & 2c_2 \\
    4c_1^3 - 3c_1 & 4c_1^2 - 1 & 4c_2^3 - 3c_2 & 4c_2^2 - 1 \\
    8c_1^4 - 8c_1^2 + 1 & 8c_1^3 - 4c_1 & 8c_2^4 - 8c_2^2 + 1 & 8c_2^3 - 4c_2
\end{vmatrix}.
$$

Subtracting the second column multiplied by $c_1$ from the first column and the fourth column multiplied by $c_2$ from the third column, we obtain

$$
\det T(h) = s_1s_2 \begin{vmatrix}
    0 & 1 & 0 & 1 \\
    1 & 2c_1 & -1 & 2c_2 \\
    -4c_1^2 + 1 & 8c_1^3 - 4c_1 & -4c_2^2 + 1 & 8c_2^3 - 4c_2
\end{vmatrix}.
$$

Now let us add a second column multiplied by $2c_1$ to the second column and, similarly, add a third column multiplied by $2c_2$ to the fourth column:

$$
\det T(h) = s_1s_2 \begin{vmatrix}
    0 & 1 & 0 & 1 \\
    -1 & 0 & -1 & 0 \\
    -4c_1^2 + 1 & -2c_1 & -4c_2^2 + 1 & -2c_2
\end{vmatrix}.
$$

Subtracting the second column multiplied by $2c_1$ from the first column of the determinant and subtracting the fourth column multiplied by $2c_2$ from the third column, we get
Adding the second determinant row to the fourth row and, similarly, adding the third row to the first row, we conclude that

\[ \det T(h) = s_1 s_2 \begin{vmatrix} -2c_1 & 1 & -2c_2 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & -2c_1 & 1 & -2c_2 \end{vmatrix}. \]

Thus, it is enough to find an expression for the first determinant

\[ \Delta_1(t, h, \omega_1, \omega_2) = \det \begin{vmatrix} y(t - h) & -\sin(\omega_1 h) & \cos(\omega_2 h) & -\sin(\omega_2 h) \\ y(t - 2h) & -\sin(2\omega_1 h) & \cos(2\omega_2 h) & -\sin(2\omega_2 h) \\ y(t - 3h) & -\sin(3\omega_1 h) & \cos(3\omega_2 h) & -\sin(3\omega_2 h) \\ y(t - 4h) & -\sin(4\omega_1 h) & \cos(4\omega_2 h) & -\sin(4\omega_2 h) \end{vmatrix} = 4s_1 s_2 (c_1 - c_2)^2. \]

Now let us consider two determinants \( \Delta_1(t, h, \omega_1, \omega_2) := \det \begin{vmatrix} Y(t, h) & T_2(h) & T_3(h) & T_4(h) \end{vmatrix} \) and \( \Delta_2(t, h, \omega_1, \omega_2) := \det \begin{vmatrix} T_1(h) & T_2(h) & Y(t, h) & T_4(h) \end{vmatrix} \) in the left part of the expression (15). It is not difficult to notice that the determinants have the same structure with the accuracy to replace the frequency values \( \omega_1 \) and \( \omega_2 \):

\[ \Delta_2(t, h, \omega_1, \omega_2) = \Delta_1(t, h, \omega_2, \omega_1). \]

Thus, it is enough to find an expression for the first determinant \( \Delta_1(t, h, \omega_1, \omega_2) \), then we can found the value of the second determinant \( \Delta_2(t, h, \omega_1, \omega_2) \) from it. Let us compute \( \Delta_1(t, h, \omega_1, \omega_2) \):

\[ \Delta_1(t, h, \omega_1, \omega_2) = s_1 s_2 \begin{vmatrix} y(t - h) & 1 & -2c_2 & 1 \\ y(t - 2h) & 2c_1 & -1 & 0 \\ y(t - 3h) & 4c_1^2 - 1 & 0 & -1 \\ y(t - 4h) & 8c_1^3 - 4c_1 & 1 & -2c_2 \end{vmatrix} = 2y(t - h)s_1 s_2 (c_1 - c_2)(4c_1^2 - 1) - 4y(t - 2h)s_1 s_2 (c_1 - c_2)(c_1 - c_2 + 4c_1^2 c_2) + 2y(t - 3h)s_1 s_2 (c_1 - c_2)(4c_1^2 + 4c_1 c_2 - 1) - 4y(t - 4h)s_1 s_2 (c_1 - c_2)c_1. \]

Computing \( \Delta_1(t, h, \omega_1, \omega_2) + \Delta_2(t, h, \omega_1, \omega_2) \) and applying \( \Delta_2(t, h, \omega_1, \omega_2) = \Delta_1(t, h, \omega_2, \omega_1) \), we get

\[ \Delta_1(t, h, \omega_1, \omega_2) + \Delta_2(t, h, \omega_1, \omega_2) = 8y(t - h)s_1 s_2 (c_1 - c_2)^2(c_1 + c_2) - 8y(t - 2h)s_1 s_2 (c_1 - c_2)^2(1 + 2c_1 c_2) + 8y(t - 3h)s_1 s_2 (c_1 - c_2)^2(c_1 + c_2) + 8y(t - 4h)s_1 s_2 (c_1 - c_2)^2(c_1 - c_2)^2. \]

Equating the left and right parts of the expression (15), we obtain

\[ 2y(t - h)(c_1 + c_2) - 2y(t - 2h)(1 + 2c_1 c_2) + 2y(t - 3h)(c_1 + c_2) - y(t - 4h) = y(t). \]
\[ [Z^2 + 1 - 2Zc_1] \cdot [Z^2 + 1 - 2Zc_2] y(t) = 0, \text{ for } \forall t \geq 4h. \]

Finally, let us write down the relation (17) as a second-order linear regression model the same as in (6):

\[ \psi(t) = \varphi(t)^T \theta, \quad (18) \]

where

\[
\psi(t) = -y(t) - 2y(t - 2h) - y(t - 4h),
\]
\[
\varphi(t) = \begin{bmatrix} 2y(t - h) + 2y(t - 3h) \\ 4y(t - 2h) \end{bmatrix},
\]
\[
\theta = \begin{bmatrix} -c_1 - c_2 \\ c_1 c_2 \end{bmatrix}. \quad (19)
\]

4. Finite-time frequencies estimation

In the previous section, we have constructed the \( n \)-th order regression model, and its parameters vector \( \theta \) depends on unknown signal frequencies \( \omega_i \). In this section we aim to obtain finite-time frequencies estimation, using the DREM method [9] in combination with finite-time estimation approach [13].

Following the DREM procedure, we introduce a new delay operator similarly to (4):

\[
[H_d(\cdot)](t) = \begin{cases} 0, & t < d, \\ (\cdot)(t - d), & t \geq d, \end{cases} \quad (20)
\]

where \( d \in \mathbb{R}_+ \) is the delay value.

Applying delay operators (20) to the linear regression model (6), we obtain

\[
H^i\{\psi(t)\} = H^i\{\varphi(t)\}^T \theta, \quad i = 1, n, \quad (21)
\]

where \( H^i\{\cdot\} = H[H[... \{H \{\cdot\} \} ...]] \).

Next, we denote the delay signals as

\[
\psi_i(t) = H^i\{\psi(t)\}, \quad \Phi_i(t) = H^i\{\varphi(t)\} \quad \text{and write the extended system from the expressions (6) in a matrix form:}
\]

\[
e^\Psi_f(t) = e^\Phi_f(t) \theta, \quad (22)
\]

where \( e \in \mathbb{R}_+ \) is normalization gain, \( \Psi_f(t) = [\psi_1(t) \ldots \psi_n(t)]^T \in \mathbb{R}^n \) is an extended regressand, \( \Phi_f(t) = [\Phi_1(t) \ldots \Phi_n(t)]^T \in \mathbb{R}^{n \times n} \) is an extended regressor.

At the mixing step of the DREM procedure, we multiply the regression model (22) by the adjugate matrix \( \text{adj}\{\Phi_f(t)\} \), and we get

\[
\Psi(t) = \Delta(t) \theta, \quad (23)
\]

where \( \Psi(t) = \text{adj}\{e^\Phi_f(t)\}e^\Psi_f(t) = [\Psi_1(t) \ldots \Psi_n(t)]^T, \quad \Delta(t) = \det\{e^\Phi_f(t)\}, \quad \text{adj}\{\cdot\} \) is the adjugate matrix, \( \det\{\cdot\} \) is the determinant.

Let us rewrite the equation (23) component by component:

\[
\Psi_i(t) = \Delta(t) \theta_i, \quad i = 1, n, \quad (24)
\]

where \( \Delta(t) \in \mathbb{R}, \Psi(t) \in \mathbb{R} \).

Now we can estimate the parameters \( \theta_i \) from (24) separately.

Let us evolute the first order regression model parameters, using the standard gradient method [14]:

\[
\hat{\theta}_i(t) = \gamma_i \Delta(t) \left( \Psi_i(t) - \Delta(t) \hat{\theta}_i(t) \right), \quad (25)
\]

where \( \hat{\theta}_i(t) \in \mathbb{R} \) is the estimate of \( \theta_i, \gamma_i \in \mathbb{R}_+ \) is a tuning gain.

The error model for \( \hat{\theta}_i(t) = \theta_i - \hat{\theta}_i(t) \) is expressed as
\[
\dot{\theta}(t) = -\gamma_i \Delta^2(t) \dot{\theta}(t).
\]  
(26)

One can easily find the solution for (26):

\[
\tilde{\theta}(t) = \tilde{\theta}(0) e^{-\gamma_i \int_0^t \Delta^2(r) dr}.
\]  
(27)

It is not difficult to show that the function \(\Delta(t)\) is bounded and persistently exciting, then the estimation method (25) provides an exponential convergence of the estimation error \(\dot{\theta}(t)\) to zero according to [14].

Let us replace \(\tilde{\theta}(t)\) with \(\tilde{\theta}(t)\) in equation (27):

\[
\theta - \tilde{\theta}(t) = \theta W(t) - \tilde{\theta}(0) W(t),
\]  
(28)

where \(W(t) = e^{-\gamma_i \int_0^t \Delta^2(r) dr}\).

Expressing the parameters \(\theta\) from (31), we explicitly find

\[
\theta(t) = \theta(1 - W(t)) = \tilde{\theta}(t) - \tilde{\theta}(0) W(t),
\]  
(29)

and for some \(t_{ff} > n(h + d)\)

\[
\theta^f(t) = \frac{1}{1-W(t)} (\tilde{\theta}(t) - \tilde{\theta}(0) W(t)), \quad t \geq t_{ff}.
\]  
(30)

where \(\theta^f(t)\) is the finite-time estimates of the parameter \(\theta\), \(i = 1, n\) of the regression model (6).

Using the parameter estimates (33), we can reconstruct \(c_i = \cos(\omega_i t)\) from Vieta's formulas (7):

\[
\begin{align*}
\tilde{c}_1^f(t) &= -\dot{c}_1^f(t) - \dot{c}_2^f(t) - \cdots - \dot{c}_n^f(t), \\
\tilde{c}_2^f(t) &= \dot{c}_1^f(t) \dot{c}_2^f(t) + \dot{c}_1^f(t) \dot{c}_3^f(t) + \cdots + \dot{c}_{n-1}^f(t) \dot{c}_n^f(t), \\
&\vdots \\
\tilde{c}_n^f(t) &= (-1)^{n-1} \dot{c}_1^f(t) \dot{c}_2^f(t) \cdots \dot{c}_n^f(t).
\end{align*}
\]  
(31)

Finally, using the identity \(c_i = \cos(\omega_i t)\), we obtain the desired frequency estimates (2) as

\[
\omega_i^f(t) = \frac{1}{h} \arccos(c_i^f(t)), \quad i = 1, n,
\]  
(32)

at the predefined time \(t_{ff}\).

An alternative method of building frequency estimates \(\omega_i^f(t)\) using \(c_i^f(t)\) without the inverse trigonometric function \(\arccos(\cdot)\) is described in the paper [15].

The estimation algorithm (33) guarantees the convergence of frequency estimates \(\omega_i^f(t)\) to the true values \(\omega_i\), \(i = 1, n\) for the finite time \(t_{ff}\).

5. Simulation results

This section presents the results of numerical modelling to illustrate the efficiency of the proposed frequencies estimation algorithm for a multi-harmonic signal with constant parameters. The simulations have been performed in MATLAB Simulink.

Let us consider the signal \(y(t)\) consisting of three harmonics \(n = 3\):

\[
y(t) = 1,5 \sin(10t + \frac{\pi}{4}) + 0,8 \sin(15t + \frac{\pi}{6}) + 0,7 \sin(25t + \frac{\pi}{8}),
\]

the graph of which is shown in Fig. 1.
Let us compare the simulation results of the described finite-time algorithm and the frequencies estimation algorithm based on the DREM and the gradient descent methods (25), which provides exponential convergence of estimation errors to zero. We use the following parameters for simulations: $h = 0.01, d = 0.013, \gamma_i = 50, i = 1, 3, \hat{\theta}(0) = \hat{\theta}^{ft}(0) = [-3 \ 3 \ -1]^T, \epsilon = 10^5$.

The Figures 2–4 show the frequencies estimation results, where $\hat{\omega}_i(t)$ are the frequency estimates, basing on the standard gradient descent method, and $\tilde{\omega}_i^{ft}(t)$ are constructed by the proposed finite-time estimation method, $i = 1, 3$.

As shown in Figures 2–4, the proposed algorithm estimates the parameters for the finite time. At the same time, the gradient method using the DREM procedure provides the exponential convergence of estimation errors to zero in the absence of interference. The frequency estimations transition time for signals $\hat{\omega}_i(t)$ is 0.25 s. The estimation duration for signals $\tilde{\omega}_i^{ft}(t)$ is 0.11 s. It is also possible to note that there is an overshoot of 20% for the frequency estimation $\tilde{\omega}_2(t)$.

6. Conclusion

In this paper, we consider the finite-time frequencies estimation problem for an unbiased multi-harmonic signal. Using the delay operator, we build a linear regression model that expresses a multi-harmonic signal's value through $2n$ delay signals. Applying the DREM method, we transit the $n$ order
model of regression to \( n \) independent first-order regression models. We obtain parameter estimates using the finite-time estimation method. The frequency estimations are restored by the Vieta's formulas and the inverse trigonometric function. The proposed method allows obtaining frequency estimations for the finite predetermined time.

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