A Bound on Vorticity

Brett McInnes
National University of Singapore
email: matmcinn@nus.edu.sg

ABSTRACT

When nuclei collide peripherally at relativistic velocities, a Quark-Gluon Plasma may be formed with much larger values of vorticity than in any other known physical system: the STAR collaboration has reported observations of global polarization of $\Lambda$ and $\bar{\Lambda}$ hyperons, generated in such collisions, corresponding to a value of the local angular velocity about $9 \pm 1 \times 10^{21} \cdot s^{-1}$ (averaged over impact energies). We show that, in a gauge-gravity model of QGP vorticity, one should nevertheless expect the vorticity to be bounded above in a collision with a given collision energy and centrality; furthermore, this bound is by a quantity which decreases with increasing impact energy. That is, the prediction is that QGP vorticity should be suppressed in collisions at high energy: we predict that it should be barely detectable at $\sqrt{s_{NN}} = 200$ GeV, but much more evident in the lower-energy collisions studied in the various Beam Energy Scans. This apparently paradoxical prediction is in fact supported by observational data. The bound is approximately attained in the cases where hyperon polarization is observable; if this is always the case in such experiments, then the model agrees with recent suggestions that these observations are related to a strong dependence of the relevant plasma moment of inertia on impact energy.
1. How Vortical Can The QGP Be?

Very general considerations [1–9] indicate that sufficiently energetic peripheral heavy-ion collisions generate Quark-Gluon Plasmas (QGP) with substantial vorticities. This was predicted to be observable in the form of global polarization of Λ and Λ̅ hyperons, and indeed the STAR collaboration [10] at the RHIC facility has announced [11] the first direct experimental data of this kind: for discussions see for example [12–21].

This major discovery has an unusual history. For these observations were not the first in which evidence of this effect were sought: in [22] the STAR collaboration itself reported such an attempt, involving collisions at energies considerably higher (√sNN = 62.4, 200 GeV) than those in which QGP vorticity was first detected [11]. No statistically significant polarization effect was observed at that time. Similarly, such an effect has not been seen at the still higher collision energies studied in the ALICE experiment at the LHC [23]. On the other hand, more recently, the STAR collaboration [24–26] has in fact succeeded in detecting hyperon polarization in plasmas produced in 200 GeV collisions; but this feat was only achieved by using a data set about 150 times larger than the one used in [22], and the computed total polarization is correspondingly tiny.

In other words, it seems that the polarization is most readily observed at (relatively) low impact energies, most clearly between 19.6 and 39 GeV; it becomes steadily more difficult to detect as the impact energy increases from those values. This very remarkable observation means that vorticity could be a major topic of investigation for experiments involving even lower impact energies, such as the NA61/SHINE experiment [28] and of course the Beam Energy Scan programmes [29, 30] being conducted at various facilities [31–33, 10], which will be reporting further data in the future.

The angular momentum imparted to the plasma in high impact energy collisions is undoubtedly much larger than in their low-energy counterpart. The apparently paradoxical observation that hyperon polarization is nevertheless most clearly observable at low impact energies is currently a matter of intense theoretical interest: see for example [34–35, 7, 12–17, 36–40]. Several explanations have been proposed, including the “depolarizing” effects of higher temperatures in higher energy collisions [34], a possible strong effect of higher energies on fluid moments of inertia [35], longer lifetime of the fluid phase in the high energy case (leading to a dilution of vorticity), subtle effects of high energy on the rapidity distribution of the plasma’s angular momentum [14], asymmetries in the fireball which are concealed by the higher rate of its expansion in the high-energy case [13, 38–39], and so on; and it may well be that several or all of these effects (or indeed some entirely new phenomenon) play a role to various degrees.

Here we wish to propose that the observations and their explanations can be organised according to the following simple scheme. It is intuitively reasonable to suppose that there must be some upper bound on the vorticity in a sample of any fluid, since its centrifugal effect will tend to disrupt the vortex or modify its structure (quantified through a relevant

---

1A similar pattern has been predicted [27] for collisions of nuclei other than gold and lead, for example for copper-copper collisions.

2The angular momentum can be computed, using specific models of the nucleus, from the impact parameter of the collision; this in turn is deduced by means of the concept of centrality. “Centrality p% to q%”, p < q, means that p% of collisions produce more tracks observed in the detector of a given experiment than the collisions under consideration, while q% produce fewer.
moment of inertia). It is therefore natural to seek some understanding of our problem in terms of such a “vorticity bound” on the QGP produced in peripheral collisions.

Some perspective on this situation is provided by the remark of Keane [32] that, in view of the extremely small size of these systems and the fact that they are produced in the aftermath of collisions at speeds extremely close to the speed of light, it is not surprising that angular velocities on the order of $10^{22} \cdot s^{-1}$ are produced. Elaborating on this, we note that, for a disc of radius (say) 1 femtometre, such an angular velocity (which is in any case an over-estimate for most collisions) corresponds to a maximal tangential velocity less than one-tenth of the speed of light. Even allowing for the effects of viscosity, which in any case is generally described as “small” (see [41] for an attempt to be precise about this), from this point of view one could argue that the observed vorticity is actually rather small for a system of this size. (Another way of looking at this is to note that an angular velocity of around $10^{22} \cdot s^{-1}$ implies that the system executes, during the lifetime of the plasma (on the order of $10^{-23} - 10^{-22}$ seconds), far less than one complete revolution [42].)

In summary, it appears not only that QGP vorticity is unobservable in very high energy collisions, but that it is not particularly large even in relatively low-energy collisions. This is consistent with the fact that it has been so hard to detect; equally, it is consistent with the idea that there is some kind of restrictive upper bound on this vorticity. In this work, we will use a simple gauge-gravity model to propose an explicit bound of this kind. If this upper bound is responsible for “holding down” the QGP vorticity, then one would expect the bound to be approximately attained; we will see that this is in fact the case.

2. A Vorticity Bound From Holography

The experimental evidence for QGP vorticity was discovered at the RHIC facility by lowering the impact energies in convenient steps (to be described). As with the other Beam Energy Scan experiments mentioned earlier, this takes us into the realm of very strongly coupled plasmas. It also takes us into an unfamiliar region of the quark matter phase diagram, a region of large baryonic chemical potentials, and also of extremely intense magnetic fields [43–49] induced in the plasma by the spectator nucleons (this could be relevant here because of possible non-trivial interactions between magnetic fields and vorticity: see for example [50, 51]). To analyse the behaviour of the vortical QGP, we therefore need technical tools capable of handling strong coupling, large baryonic chemical potentials and magnetic fields, and so on.

In this domain, the techniques of gauge-gravity duality [52–54] (“holography”) are suitable, and they have played the useful role of directing attention to features of the QGP that might otherwise have escaped attention. The method is particularly valuable in suggesting the existence of certain bounds on important quantities: most famously, it imposes, under well-understood circumstances, a (lower) bound on the plasma’s dynamic viscosity, a bound which depends on its entropy density but also on other physical parameters [55–58]. Other examples involve parameter-dependent bounds on cosmic magnetic fields [59] and on the QGP Reynolds number [41].

The simplest holographic models of the QGP involve thermal black holes in an asymptotically AdS spacetime: the physics at infinity should correspond, at least qualitatively
and perhaps to some extent quantitatively, to the physics of the QGP. In the case where
the system at infinity has a large angular momentum, holographic duality requires that
the bulk black hole should itself have a non-zero angular momentum: it must be an AdS-
Kerr black hole (first used in this general context, though in a four-dimensional bulk, in [60].) The angular momentum of the black hole controls (in a highly non-trivial way)
the rotational parameters of matter on the boundary, which models the QGP vortices we
wish to study.

Even in the simplest models (which are nevertheless fully adequate for analysing
present data, as in [11]), the polarization percentage is proportional not only to the
vorticity, but also, inversely, to the temperature: see equation (25), below. This does
not, however, solve the problem: as impact energies increase, so do temperatures, but
only very slowly — far too slowly to account for the effect we are discussing here. In
these models, then, one has to look for an explanation in terms of the behaviour of the
QGP vorticity itself: it must vary in some unusual (non-linear) way as the impact energy
— and, with it, the angular momentum imparted to the plasma — increases. (A non-
linear relation between angular momentum and angular velocity is possible, but only if
the relevant moment of inertia of the plasma sample is affected by the impact energy, as
suggested in general terms in [35]; related ideas have been put forward in [13, 38, 39].)

The crucial point here is that, in the physics of rotating black holes, there is in fact
a highly unusual relationship between the angular momentum and the angular velocity of
particles moving around the black hole, connected with the well-known phenomenon of
frame dragging. This relation is certainly not one of direct proportionality (zero angular
momentum does not entail zero angular velocity, for example). Thus we can hope that
the gauge-gravity duality might supply the relation we need here.

We will show that the simple holographic model based on the geometry of a five-
dimensional AdS-Kerr black hole spacetime actually realises this programme. It leads
directly to an explicit non-linear relation between vorticity and angular momentum, which
does in fact impose an upper bound on the possible vorticity in a boundary theory, a
particular configuration in the dual $\mathcal{N} = 4$ super-Yang-Mills theory that represents the
QGP produced by a heavy-ion collision with a given impact energy and centrality. The
bound takes the explicit and simple form

$$\omega \leq \kappa \frac{\varepsilon}{\alpha} \approx 0.2782 \frac{\varepsilon}{\alpha},$$

where $\omega$ is the vorticity (equal to the local angular velocity in the convention, used here, in
which vorticity is half the curl of the velocity field$^3$), $\varepsilon$ is the energy density of the plasma,
$\alpha$ is its angular momentum density, and $\kappa$ is a certain dimensionless constant with the
indicated approximate value$^4$. The two densities can be computed from the impact energy
and the centrality of the collision, as explained earlier, and so one can think of the bound
as being fixed by those data$^5$.

$^3$Note that many references define it as the curl itself, but here we follow [8] and [11].
$^4$Its value can readily be computed numerically to any required precision; it is derived from one of the
roots of a certain octic polynomial.
$^5$We shall use, throughout, natural (not Planck) units; however, since most of the parameters have
interpretations in terms of lengths and volumes, it is convenient to take as our base unit the femtometre
or fm (1 fm$^{-1} \approx 197.327$ MeV). Then $\omega$ has units of fm$^{-1}$, $\varepsilon$ has units of fm$^{-4}$, and $\alpha$ has units of fm$^{-3}$. More familiar units for $\omega$ can be restored by multiplying by $c = 3 \times 10^{23}$ fm·s$^{-1}$.
This upper bound is remarkable in three ways. First, it depends only on two readily
computed quantities, \( \varepsilon \) and \( \alpha \), and only on their ratio: each of these quantities, as a
density, varies strongly as the fireball expands, but their ratio is much less affected.

Secondly, as we will see, the derivation follows in a quite elementary way from the
structure of the geometry imposed on conformal infinity by the bulk black hole geometry,
without any further assumptions: it does not, for example, depend on subtle stability
analyses like those employed to derive the bounds in [59] and [41]. Its status as a prediction
of the gauge-gravity duality is correspondingly robust.

Finally, and most importantly: if we fix the centrality but increase the impact energy,
both \( \varepsilon \) and \( \alpha \) increase; but \( \alpha \) increases far more rapidly than \( \varepsilon \), so, since the right side of
(1) depends on \( \alpha \) inversely, the bound becomes ever more stringent as the impact energy
increases. In particular, as we will show, the bound reduces the vorticity below any
observable level for high-energy collisions (at present, any impact energy beyond \( \sqrt{s_{\text{NN}} = 200 \text{ GeV}} \)). On the other hand, the bound is much less stringent at lower impact energies,
and (allowing for the large theoretical and observational uncertainties) permits clearly
observable hyperon polarization, particularly at impact energies in the \( \sqrt{s_{\text{NN}} = 20 – 40 \text{ GeV}} \) range.

We find, comparing with the data reported in [11], that our bound is satisfied, in fact
approximately attained, in all cases except one, where the impact energy is so low that
it is questionable whether a plasma is in fact actually produced (and where the reported
global polarization for \( \Lambda \) hyperons is anomalously large). We will show that, if we assume
that the bound is attained, then the model gives an explicit account of the changes in the plasma moment of inertia that explain how the angular velocity can be small at high
impact energies.

A useful way of thinking about the vorticity bound is that it tells us precisely what
it means, for collisions at a given impact energy and centrality, to say that the resulting
vorticity is “large”: it means that the vorticity is of the order of the value on the right side
of the inequality (1). We will show that the vorticities reported in [11] are certainly large
in this sense. Since these vortices are sustained by the strong(est) interaction, it may be
that these reported values are the largest possible for vortices in any substance.

We begin by setting up the holographic model.

3. The Holographic Model of Vorticity

The bulk metric here is that of a five-dimensional asymptotically AdS\(_5\) black hole with a
topologically spherical event horizon; this allows us to exploit the gauge-gravity duality in
its most familiar form, in which the physics in an asymptotically AdS\(_5\) spacetime is dual
to that of an \( \mathcal{N} = 4 \) super-Yang-Mills theory defined on the four-dimensional conformal
boundary. We need the black hole to rotate, and the corresponding metrics were given
in [61] (but see also [62] and [63] for important discussions of the formulae for the physical
mass and angular momentum). In five dimensions, the black hole can rotate around two

---

6In principle we can also endow the black hole with electric charge, as in [64], in order to model
a non-zero baryonic chemical potential. However, just as the metric at conformal infinity for the
four-dimensional AdS-Kerr-Newman metric is formally independent of the electric charge, so also the inclusion
of electric charge in this case will not modify the form of the boundary metric given in equation (12).
distinct axes simultaneously, so in general one has a pair of rotation parameters, \((a, b)\); but here we only need one axis, corresponding at conformal infinity to the axis of a vortex; so we set one of the rotation parameters equal to zero. The AdS$_5$-Kerr metric in this simplified case takes the form

\[
g(\text{AdSK}_{(a,0)}^{(b)}) = -\frac{\Delta_r}{\rho^2} \left[ dt - \frac{a}{\Xi} \sin^2 \theta \, d\phi \right]^2 + \frac{\rho^2}{\Delta_r} \, dr^2 + \frac{\rho^2}{\Delta_\theta} \, d\theta^2
\]

\[
+ \frac{\sin^2 \theta \, \Delta_\theta}{\rho^2} \left[ a \, dt - \frac{r^2 + a^2}{\Xi} \, d\phi \right]^2 + r^2 \cos^2 \theta \, d\psi^2,
\]

where

\[
\rho^2 = r^2 + a^2 \cos^2 \theta,
\]

\[
\Delta_r = (r^2 + a^2) \left( 1 + \frac{r^2}{L^2} \right) - 2M,
\]

\[
\Delta_\theta = 1 - \frac{a^2}{L^2} \cos^2 \theta,
\]

\[
\Xi = 1 - \frac{a^2}{L^2}.
\]

Here \(L\) is the asymptotic AdS curvature length scale, and \(a\) and \(M\) are (positive) parameters\(^3\) describing the geometry of the spacetime; they are related (but not equal) respectively to the black hole specific angular momentum (angular momentum per unit physical mass) \(A\) and the physical mass\(^4\) \(m\).

In fact, setting \(b = 0\) in the formulae given in \([63]\), we have (if \(j\) denotes the black hole’s physical angular momentum)

\[
m = \frac{\pi M (2 + \Xi)}{4 \ell_B^3 \Xi^2}, \quad j = \frac{\pi M a}{2 \ell_B^3 \Xi^2},
\]

where \(\ell_B\) is the gravitational length scale in the bulk. Thus the specific angular momentum is given by

\[
A = \frac{2a}{2 + \Xi} = \frac{2a}{3 - (a^2/L^2)}.
\]

Note carefully that this differs from the four-dimensional case, where \(a\) itself is the specific angular momentum.

Other important characteristics of the black hole can be computed from its geometry: for example, the Hawking temperature is given \([63]\) by

\[
T = \frac{r_H \left( 1 + \frac{r_H^2}{L^2} \right)}{2\pi (r_H^2 + a^2)} + \frac{r_H}{2\pi L^2},
\]

below (which is all we need to deduce the vorticity bound). Note in this connection that the rate of fall-off of the electric contribution to the metric is larger in five than in four dimensions.

\(^7\)In the system of natural units we use here, \(a\) has units of length, while \(M\) has units of squared length.

\(^8\)In natural units, \(A\) has units of length, \(m\) of inverse length.
where \( r_H \) denotes the horizon radius (which can be regarded as a function of \( M \) and \( a \) through its definition as the largest root of \( \Delta_r \)), and the entropy by

\[
S = \frac{\pi^2 (r_H^2 + a^2) r_H}{2\ell_B^2 \Xi}.
\]  

(7)

Notice that \( a \) must always be strictly smaller than \( L \), since otherwise the spacetime signature will not be Lorentzian when \( \theta \) is sufficiently near to 0. Since this implies that \( \Xi \) must be positive, it is clear from equation (5) that \( A \) is always smaller than \( a \) and \( L \) (though it approaches \( a \), and therefore also \( L \), as \( a \) approaches \( L \)). In short, we always have

\[
A < a < L.
\]  

(8)

We see that \( L \) must have some physical meaning in the bulk, in addition to its geometric interpretation there: it is a quantity which, in some way we need to explain, is related to or perhaps determined by the specific angular momentum of the black hole. In the application to the QGP, holographic duality implies that \( L \) must have a physical interpretation in terms of the plasma physics, connected in some way with the specific angular momentum of the plasma. In any case it is clear that, since \( A \) is fixed separately for each class of collisions (specifically, by the impact energy and centrality), so also \( L \) must be influenced by those parameters, and perhaps determined by them; otherwise it will not be possible to ensure that (8) holds under all circumstances.

In short, the inequalities (8) are of basic importance, because they essentially imply that, in order for us to make use of the AdS-Kerr black holes in holography, we must elucidate the physical meaning of \( L \). We will see that \( L \) is omnipresent in our calculations, and so it is essential to have a good understanding of the consequences of varying it. We will return to this point several times.

Finally, we draw the reader’s attention to the fact that, contrary to appearances, the angular coordinates \( \theta, \phi, \psi \) used in [61], which we follow here, are not the familiar polar coordinates on the three-sphere: for example, one sees that, when \( A = 0 \), the angular part of the metric is \( d\theta^2 + \sin^2 \theta d\phi^2 + \cos^2 \theta d\psi^2 \), which is indeed the usual round metric on \( S^3 \), but not in polar coordinates. Instead, these are the coordinates usually used to describe \( S^3 \) when it is regarded as a principal \( U(1) \)-bundle over \( S^2 \), that is, as the Hopf bundle [65]. The two coordinates \( \phi \) and \( \psi \) both run from 0 to \( 2\pi \), while \( \theta \) runs from 0 to \( \pi/2 \); thus in fact each fixed value of \( \theta \) corresponds to a two-torus, except in the “degenerate” cases \( \theta = 0, \pi/2 \), which correspond to one-dimensional circles. In particular, the single condition \( \theta = \pi/2 \) reduces the dimensionality by two: it means that we are on the equator of \( S^3 \), described by a single angular coordinate, \( \phi \).

---

\( ^9 \)In terms of the usual coordinates \( (x_0, x_1, x_2, x_3) \) on Euclidean \( \mathbb{R}^4 \), we have, for a three-sphere of radius \( r \),

\[
x_0 = r \cos \psi \cos \theta \\
x_1 = r \sin \psi \cos \theta \\
x_2 = r \cos \phi \sin \theta \\
x_3 = r \sin \phi \sin \theta.
\]

Notice however that, if we set \( \psi = 0 \), then the remaining coordinates can be interpreted as polar coordinates on a hemisphere of \( S^2 \).
The physics of this black hole determines, according to the gauge-gravity duality \cite{52}, that of a field theory on the boundary \((r \to \infty)\); fields in the latter approximate, to some extent, to the QGP. Thus, the Hawking temperature \(T\) of the black hole corresponds to the temperature of the plasma, the ratio of the black hole’s entropy to its (physical) mass is equal to the ratio of the plasma entropy density \(s\) to its energy density \(\varepsilon\), and similarly the black hole specific angular momentum parameter \(A\) is interpreted as the ratio of the QGP angular momentum density \(\alpha\) to its energy density.

Conversely, given observational data on \(T\), \(s/\varepsilon\), and \(\mathcal{A} = \alpha/\varepsilon\) for a particular sample of the QGP, one can solve for the three parameters \(M, a, L\) and so reconstruct the bulk geometry. Again we stress that \(L\) is in principle computable from physical data on the boundary\(^{11}\). In practice, this cannot be done in the absence of data regarding the entropy density of the vortical plasma (which may well differ greatly from its value for plasmas produced in central collisions). We will see however that there is another way of deducing the value of \(L\).

\section*{4. Vorticity and Angular Momentum in the Holographic Model}

As with the asymptotically flat Kerr geometry, the bulk geometry here is characterized by the remarkable phenomenon of \textit{frame dragging}. There is a fundamental difference between the asymptotically AdS and asymptotically flat cases, however, and this proves to be crucial for our purposes; so let us briefly review how frame dragging works in our case.

Consider a free particle (always with non-zero rest mass, henceforth) in the five-dimensional AdS-Kerr spacetime. Because this spacetime, unlike the Schwarzschild-Tangherlini spacetime, does not have a symmetry in the \(\theta\) direction, even free particles can have complicated orbits; so we will (throughout) focus on particles on or near the equator. As explained above, in the Hopf coordinates on \(S^3\) this is a circle obtained by setting \(\theta\) equal to a fixed constant, \(\pi/2\). Fixing also \(r\) and \(\psi\), we can then focus on angular momentum associated with the \(\phi\) direction, which is all we need to set up a model of vorticity in the QGP. (Under these assumptions, the particle angular momenta associated with rotation around the \(\theta\) and \(\psi\) directions vanish, which is reasonable in view of the fact that, by assumption, the black hole is not rotating around those directions. Note that this statement would not hold for the \(\phi\) direction: a particle with zero angular velocity in that direction does not have vanishing angular momentum in that direction.)

Recall now that, in a spacetime like this one which is not flat, angular momentum is not defined by the familiar formula which makes it proportional to the angular velocity;

\footnote{We have equation (6) for the temperature; \(s/\varepsilon\) is expressed in terms of a ratio computed by combining equations (4) and (7); and equation (5) gives us an expression for \(\alpha/\varepsilon\). None of the resulting equations involves \(\ell_B\), so indeed we now have three equations for the three unknowns \(M, a, L\). One can, in principle, compute \(\ell_B\) from \(L\), using the holographic dictionary \cite{52}, if one knows the number of colours in the boundary theory.}

\footnote{There is an extensive literature (for example, see \cite{66–68}) on the interpretation of the AdS cosmological constant as defining a pressure (in five dimensions, \(1/(2\pi\ell_B^2L^2)\)), with a consequent (very subtle) interpretation in terms of the pressure of the matter in the boundary theory. A computation of \(L\) and \(\ell_B\) in terms of these other QGP parameters therefore indicates that this pressure might also be computable in principle.}
for that quantity would not, in general, be conserved for a free particle even in the presence of a rotational symmetry.

Instead one proceeds by noting that an angular Killing vector field — in this case the one associated with the \( \phi \) direction, normalised so that \( \phi \) has period \( 2\pi \) — allows us to define, along the geodesics corresponding to free particles, a conserved quantity. This is the angular momentum per unit mass: the inner product of this normalised Killing field with the unit tangent to the particle worldline, \( \dot{t}\partial_t + \dot{\phi}\partial_\phi \) (where dots denote differentiation with respect to the proper time of the particle).

Thus, for a massive particle on an equatorial orbit, outside the event horizon, with zero angular momentum, we have (from equation (2))

\[
\frac{a}{r^2\Xi} \left( \Delta_r - [r^2 + a^2] \right) \dot{t} + \frac{1}{r^2\Xi^2} \left( -a^2\Delta_r + [r^2 + a^2]^2 \right) \dot{\phi} = 0,
\]

and so the angular velocity of such a particle relative to this system, \( \Omega_0 \) (the subscript referring to the angular momentum per unit mass), takes the value

\[
\Omega_0 = \frac{d\phi}{dt} = \left( \frac{1 - \frac{r^2 + a^2}{\Delta_r}}{\left( a^2 - \frac{[r^2 + a^2]^2}{\Delta_r} \right)} \right) a\Xi.
\]

This is non-zero in general, and so we have the phenomenon of frame dragging: even a particle with zero angular momentum has a non-zero angular velocity in general.

The great difference between the asymptotically flat and asymptotically AdS cases, however, is that frame dragging in the latter case is omnipresent: even at arbitrarily large distances from the black hole, one still has a residual angular velocity (denoted by \( \omega_0 \)), obtained by letting \( r \to \infty \) in (10) (note that \( \Delta_r \) is asymptotic to \( r^4/L^2 \)):

\[
\omega_0 = -\frac{a}{L^2}.
\]

This does not happen in the asymptotically flat case, and it represents an important departure. For it means that the rotation of the black hole leaves, in the AdS case, a direct physical imprint on the region “near infinity”. When we construct the conformal boundary, as we need to do to use holography\(^{12}\), we must take this into account.

The geometry of this boundary is fixed by means of a conformal re-scaling of the metric \( g(\text{AdSK}^{(a,0)}_5) \) as the limit is taken to infinity. Mathematically we are entitled to use any conformal factor with the appropriate asymptotic behaviour, and clearly there is a wide choice of representatives of the conformal structure at infinity so defined. From a physical point of view, however, it makes sense to choose a representative which reflects the persistence of frame dragging to points in the spacetime arbitrarily distant from the black hole.

With this in mind, we take the boundary metric to be

\[
g(\text{AdSK}^{(a,0)}_5)_\infty = -dt^2 + \frac{2a\sin^2\theta \, dt \, d\phi}{\Xi} + \frac{L^2 \, d\theta^2}{1 - (a/L)^2 \cos^2\theta} + \frac{L^2 \sin^2\theta \, d\phi^2}{\Xi} + L^2 \cos^2\theta \, d\psi^2;
\]

\(^{12}\)For the AdS/CFT duality in this situation see [69].
obtained from $g(\text{AdS}K_5^{(a,0)})$ by extracting a conformal factor $r^2/L^2$, taking the limit $r \to \infty$, and then doing some algebraic simplifications.

Notice that, with this metric, the time coordinate $t$ represents proper time for a stationary observer at infinity located at one of the poles ($\theta = \psi = 0$); we can take this observer to be an outside observer, fixed in the laboratory; so, in each case we consider below (particles with zero, respectively non-zero angular momentum), $d\phi/dt$ represents an angular velocity as measured by this observer. Notice too that a massive particle on the equator moving with angular velocity $\omega_0$ has, as the notation suggests, zero angular momentum: from equation (12), the angular momentum per unit mass is

$$\frac{ia}{\Xi} + \frac{\dot{\phi}L^2}{\Xi} = \frac{i}{\Xi} \left(a + \omega_0L^2\right) = 0,$$

so, with this choice of metric, the angular velocity at infinity matches the residual frame dragging angular velocity in the bulk. One might say that “the space at infinity is rotating” at a constant angular velocity, but this is rather misleading: we should bear in mind that it means no more than that $\omega_0$ is the angular velocity of massive particles having angular momentum per unit mass equal to zero (and not to $A$, for example).

We propose to use the spacetime with metric $g(\text{AdS}dyKN)_\infty$ to connect the rotation of the bulk black hole with the angular momentum of the plasma in a QGP with non-zero vorticity. As usual in collision physics, the matter produced by the collision is described by focusing on a two-dimensional reaction plane or “$x-z$ plane” containing the axis of the collision (conventionally denoted $z$), a perpendicular axis measuring distance away from the axis (conventionally denoted by $x$) and a $y$ axis pointing in the approximate direction of the angular momentum vector. We regard the particles constituting a QGP vortex as executing circular motion in the reaction plane, and we model this holographically by studying particles moving on the equator of the conformal boundary of the black hole we introduced above. The connection is made by requiring these particles to have the same angular momentum to mass ratio as the particles in the actual vortex.

We stress that we are exclusively interested in modelling the angular motion of a single particle moving (approximately) on a single typical circular orbit, not in constructing a model of an entire vortex. This is for two reasons. First, the data refer only to measured vorticities: the actual linear velocity of particles in the vortices are poorly constrained, and in any case they obviously vary with distance from the centre in a way that is still more poorly constrained. Second, an attempt to deal with the variations in the radial direction would lead into the subtleties associated with the relativistic rotating disc, the subject of the well-known “Ehrenfest paradox” (see [70], page 90, for a clear modern treatment). Rather than become involved in questions such as whether the rotating QGP is “Born rigid”, and so on, we prefer to focus only on the angular velocity, which is much simpler.

To resume: we have seen that, due to the frame dragging effect inherited at infinity from the bulk black hole, a particle with zero angular momentum there rotates relative to the coordinate system we used above. Here, however, we are not concerned with matter having zero angular momentum: by hypothesis, the angular momentum of the vortical QGP is some non-zero, in fact large, value. The obvious way to represent this in our model is to consider particles on the equator not with zero angular momentum, but rather with an angular momentum to mass ratio given precisely by $A$, the ratio of the QGP angular
momentum density to its energy density. These will of course rotate not at $\omega_0$ but at a different angular velocity, $\omega_A$, relative to this coordinate system.

The physical angular velocity $\omega$ is computed as the difference between the angular velocity of particles with non-zero angular momentum and that of fictitious particles with zero angular momentum: we have $\omega = \omega_A - \omega_0$. This will correspond to the vorticity of the plasma observed in the laboratory. ($\omega_0$ and $\omega_A$, by contrast, do not correspond to anything observable and should be regarded as mere formal constructs.)

We now proceed precisely as in the computation of frame dragging. Again we evaluate the conserved quantity, the inner product of the unit tangent with the normalised Killing vector field, but now we have

$$\dot{t}a + \dot{\phi}L^2 + \frac{\dot{\phi}}{\Xi} = A.$$  \hspace{1cm} (14)

The situation now is a little more complicated than before, since this equation cannot be solved for $\omega_A = d\phi/dt$ directly; we need also to use the fact that $\dot{t}\partial_t + \dot{\phi}\partial_\phi$ is indeed a unit vector, that is,

$$-\dot{t}^2 + 2t\dot{\phi}a + \frac{\dot{\phi}^2L^2}{\Xi} = -1.$$  \hspace{1cm} (15)

Using this equation, we can eliminate $\dot{t}$, so obtaining a quadratic equation for $\omega_A$:

$$L^2\omega_A^2 + 2a\omega_A + \frac{a^2}{L^2} \left( 1 - \frac{4a^2}{L^2} \Xi^2 \right) = 0. \hspace{1cm} (16)$$

Solving this and subtracting $\omega_0$, we obtain two equal and oppositely signed values for $\omega$, corresponding of course to the two possible directions of rotation; taking the positive value we have finally,

$$\omega = \frac{A}{L^2}\sqrt{\frac{\Xi}{1 + \frac{4A^2}{L^2}}}, \hspace{1cm} (17)$$

where, through equation (15), we can regard $\Xi$ as the function of $A$ given by

$$\Xi = \sqrt{1 + \frac{3A^2}{L^2} - 1 - \frac{A^2}{L^2}}. \hspace{1cm} (18)$$

We see that the relation between angular velocity and angular momentum (per unit mass) is not simple in this model; the two are not proportional unless $A/L$ is known to be very small and $L$ is fixed. In fact we see that $\omega$ is small both when $A$ is small and when it is large (that is, close to its maximum possible value $L$: for in that case, $a$ too is close to $L$, and then the factor of $\Xi$ in the numerator in (17) is small). In short, equations (17) and (18) supply the promised non-linear relation which we hope to use to codify the peculiar fact that, observationally, large angular momenta do not seem to lead to large hyperon polarizations.

It is important to understand that the structure of equation (17) ensures that causality is never violated here. To see this, let us compute the linear velocity of objects on the

---

13We remind the reader that $L$ is not a “constant” under all circumstances: as an upper bound on $A$, it depends on the impact energy and centrality of the collision producing the plasma.
equator having angular velocity $\omega$. From equation (12) we see that the circumference is $2\pi L/\sqrt{\Xi}$ (which is much larger than $2\pi L$, when $A$, and therefore $a$, are close to $L$). For an object with angular velocity $\omega$, the linear velocity is therefore

$$v_\omega = \frac{\omega L}{\sqrt{\Xi}}, \quad (19)$$

and this will exceed unity, violating causality, as $a$ and $A$ approach $L$ unless $\omega$ tends to zero sufficiently rapidly in the same limit. In short, the angular velocity must decrease towards zero as the angular momentum increases in order to avoid violating causality. In fact, substituting equation (17) into (19) we find

$$v_\omega = \frac{A}{L} \sqrt{\frac{1}{1 + \frac{a^2}{L^2} \Xi}}, \quad (20)$$

and (in view of inequality (8)) this does always satisfy causality, because the factor of $\sqrt{\Xi}$ in the denominator of (19) has been cancelled.

Let us consider the origin of the crucial quantity $\Xi$. It appears in the penultimate term in equation (12) simply because it is inherited from the black hole metric in equation (2). In both cases it is necessarily present in order to ensure that the geometry should be regular: without it, there would be conical singularities at the poles (that is, the ratio of the circumference of a circle centred on a pole to its radius would not tend to $2\pi$ as the radius tends to zero).

To summarize, then: due to the (geometrically necessary) presence of the $\Xi$ factor, the length of the equator in this geometry depends on the angular momentum, and this enforces a non-linear relation between angular velocity and angular momentum. This in turn allows $\omega$ to approach zero as $A$ becomes large (approaching $L$). This seems odd on one side of the holographic duality, but by no means on the other; for it is obvious that the Kerr-like bulk geometry does change in response to changes in the angular momentum of the black hole. Equally, the holographic duality reconciles us to the possibility that angular momentum need not be linearly related to angular velocity, for, around a black hole, this is precisely the normal situation.

Equation (17) gives the relation between the angular velocity and the specific angular momentum of a free particle on a circular orbit in the conformal infinity of the five-dimensional AdS-Kerr black hole (rotating exclusively around one of the possible two axes). Holographically, it should relate the vorticity of a “QGP-like” field theory to the ratio of its angular momentum and energy densities. The relationship is mediated by the quantity $L$, which, as explained earlier, we know how to compute in principle, but not in practice. Leaving that difficulty to one side for the present, we see that when $A$ is small relative to $L$, as is the case for nearly central collisions, then (from equation (18)) $\Xi$ is close to unity, so from equation (17) we have $L \approx \sqrt{A/\omega}$; that is, for approximately central collisions $L$ plays the role played classically by the radius of gyration, the square

---

14This is particularly clear in the boundary geometry: there, the circumference of a circle of the form $\theta = \theta_0 = \text{constant}$, located on the two-dimensional hemisphere $\psi = 0$, is $2\pi L \sin \theta_0/\sqrt{\Xi}$, while its radius, measured from the pole (of both the 3-sphere and the two-dimensional hemisphere) along the hemisphere, is $L \int_0^{\theta_0} \frac{d\theta}{\sqrt{1 - (a^2/L^2) \cos^2 \theta}}$, the ratio only tends to $2\pi$ as $\theta_0 \to 0$ because the $\sqrt{\Xi}$ factor is present.
root of the moment of inertia of a rigid system (around some axis) divided by the mass of the system. The radius of gyration describes the distribution of rotating matter relative to the axis of rotation: it is the link between angular momentum and angular velocity, and as such is clearly of central interest in any effort to understand the peculiar observed relation between the angular momentum of the QGP and its vorticity.

When \( A \) is not small relative to \( L \), then the latter can no longer be identified exactly with the radius of gyration, but clearly it plays the same role in linking angular momentum to vorticity: it determines an “effective radius of gyration”, \( k_{\text{eff}} \equiv \sqrt{\frac{A}{\omega}} \) (since \( L \) fixes \( \omega \) when \( A \) is given). We can therefore think of different values of \( L \), for given \( A \), as having a physical meaning in terms of variations of \( k_{\text{eff}} \), that is, in terms of various possible distributions of the plasma around the axis of a vortex. We will return to this later.

The unfortunate fact remains, however, that we do not know how to compute \( L \) from the actual data on the vortical QGP: thinking of it as a quantity computable from an effective radius of gyration does not help, since there are no relevant data. Fortunately, there is a simple way to circumvent this problem.

5. The Bound

As we know, the black hole parameter \( A \) has a holographic interpretation as the ratio of the plasma angular momentum density to its energy density; it can therefore be computed explicitly from physical data, and we will explain how to do this shortly. Knowing \( A \), we may use equations (17) and (18) to regard \( \omega \) as a function of \( L \): as we have seen, this has an explicit physical interpretation in terms of the variation of angular velocity with the effective radius of gyration. We know (from the inequality (8)) the domain on which \( L \) can vary: it is the open interval \(( A, \infty )\). We now ask more specifically: for a given value of \( A \), how does \( \omega \) behave, as a function of \( L \), on this domain?

Equations (17) and (18) answer this question. Clearly \( \omega \) vanishes as \( L \to \infty \), as expected, since this is the regime of small angular momentum per unit mass (or of large effective radius of gyration, as above). More surprisingly, we have seen that it also vanishes as \( L \) tends down to \( A \). Now since \( \omega \) is continuous and positive as a function of \( L \), it follows that \( \omega \) is bounded above.

In fact, \( \omega (L) \) can be completely described analytically; the analysis is rather intricate but of course essentially elementary. One finds that this function has a unique maximum, \( \omega_{\text{max}} \), at \( L = L_{\omega_{\text{max}}} \), where

\[
L_{\omega_{\text{max}}} = \sigma A, \quad \text{(21)}
\]

where \( \sigma \) is a pure number which can be computed to any desired precision: approximately, it is

\[
\sigma \approx 1.2499. \quad \text{(22)}
\]

We can now compute \( \omega_{\text{max}} \) by substituting \( L_{\omega_{\text{max}}} \), that is, \( \sigma A \), into equation (18), to

---

\( ^{15} \)One computes the derivative and sets it equal to zero; a somewhat elaborate algebraic manipulation reduces this condition to solving the octic \( 24\sigma^8 + 63\sigma^6 - 64\sigma^4 - 176\sigma^2 + 48 = 0 \), where \( \sigma \) is defined in equation (21). This is a quartic in \( \sigma^2 \) and so it can be solved explicitly and exactly if desired. One finds that there are two real positive solutions for \( \sigma \), of which however only one, given approximately in (22), allows the crucial inequality (8) to be satisfied.
find \( \Xi \) in this case; then equation (17) gives

\[
\omega_{\text{max}} = \frac{\kappa}{A} \approx \frac{0.2782}{A},
\]

(23)

where \( \kappa \) is a pure number, which can be computed to any desired precision, having the indicated approximate value.

That is, no matter how \( L \) is determined by the underlying physical effects, \( \omega \) can never exceed this value, once \( A \) has been fixed. With the holographic interpretation of \( A \) as \( \alpha/\varepsilon \), this is the bound given in Section 2 above.

An example is shown in Figure 1: this shows \( \omega(L) \) in the case of collisions at 27 GeV per pair, with 20% centrality (see below for the details). Here \( \alpha/\varepsilon \approx 17.37 \text{ fm}; \) there is a clear maximum at \( L \approx 21.72 \text{ fm}, \) and the value of \( \omega \) at that maximum is \( \approx 0.0160 \text{ fm}^{-1}; \) both values are in agreement with (21), (22), and (23). (In more conventional units, this upper bound is around \( 4.8 \times 10^{21} \text{ s}^{-1}; \) this is to be compared with the observed value (averaged over impact energies, so including considerably larger angular velocities at impact energies below 27 GeV) reported in [11], that is, \( 9 \pm 1 \times 10^{21} \text{ s}^{-1}. \))

Figure 1: Prediction of the holographic model for angular velocity (units fm\(^{-1}\)) as a function of \( L \) (units fm); impact energy = 27 GeV, centrality = 20%.

To summarize: we have proposed that the observational data can be described by a non-linear relation between vorticity and angular momentum. The gauge-gravity duality
provides an explicit relation of this kind: equation (17) (with $\Xi$ given by (18)). If this equation does describe the situation more or less accurately, then it follows that QGP vorticities are bounded above, as in equation (23).

Notice that, even when the particles in the model with angular velocity $\omega$ have the maximal possible angular velocity for a given value of $A$, they move at a velocity $v_\omega(\omega_{\text{max}})$, given by substituting $L/A = \sigma$ into equation (20), from which we obtain

$$v_\omega(\omega_{\text{max}}) \approx 0.750,$$

that is, well below light speed. Thus causality does not explain the existence of the upper bound — an important point.

Instead, the existence of the vorticity bound is ultimately due to the specific form taken by the black hole metric in the bulk, which involves factors of $\Delta_\theta$ and $\Xi$ (not occurring in the more familiar asymptotically flat Kerr metric) which must be present if the Einstein equation (with a strictly negative cosmological constant) is to be satisfied and (as we saw earlier) conical singularities are to be avoided. The bound is in this sense truly a holographic bound: it is the expression, through the duality, of specific geometric conditions in the bulk.

We now turn to the problem of comparing this bound with the actual data (from [11, 24]).

6. The Data

Before we begin, we should be open regarding the fact that precision is not to be looked for in these computations. As is well known, the field theories considered in the gauge-gravity duality are not (in all regimes) closely similar to QCD: there are important analogies (discussed very clearly in [52]) but there are basic differences. The observational data, too, suffer from large uncertainties: [11] reports a large systematic uncertainty in the given values of the polarization percentages. (The principal difficulty here is that, to relate vorticities, which are not directly observable, to polarizations, one needs to know the precise temperature corresponding to a given energy density; and this is a notoriously difficult problem).

We will nevertheless see that there is much to be learned here; our main priority at this point, however, is to assure ourselves that the vorticity bound predicts values for hyperon polarizations that are at least of a reasonable order of magnitude. Our secondary objective is to explain clearly how the relevant calculations can be done when more precise data become available in future.

We need estimates for the angular momentum of the QGP in peripheral collisions, and also for the energy density, for various impact energies and centralities. For the former, we rely on [35], where estimates using the AMPT (“A Multi-Phase Transport”) model are given; for the latter we use [73], where very detailed computations of many relevant parameters (using a colour string percolation model) are given. In both cases, the values

\[16\text{Of course, in general, the question of causality can indeed be crucial in discussions of (sufficiently large) rapidly rotating systems: see for example [71] and references therein. The point is that this is not the case here.}

\[17\text{Note that, in [35], the convention is used in which $\omega$ is twice as large as in [11] and here.}
given seem reasonable. Other models might be chosen, but the differences are unlikely to be sufficiently large as to modify our general conclusions.

Proceeding in this way, we can (with simple additional assumptions regarding the geometry of the overlapping nuclei) compute the right side of the inequality (11), and thus place an explicit bound on the vorticity in each case. We can relate this to the average polarizations of primary \( \Lambda \) and \( \Lambda' \) hyperons by means of the equation\(^{18}\) (see [8] and [11])

\[
\overline{P}_{\Lambda'} + \overline{P}_{\Lambda} = \frac{\omega}{T},
\]

where natural units are used and \( T \) is the temperature as usual. This equation allows us to express the vorticity bound as a bound on the total polarization, obtaining inequalities of the form

\[
[\overline{P}_{\Lambda'} + \overline{P}_{\Lambda}] (\sqrt{s_{\text{NN}}}, C) \leq \Phi(\sqrt{s_{\text{NN}}}, C),
\]

where \( C \) denotes centrality and where we now know how to compute \( \Phi(\sqrt{s_{\text{NN}}}, C) \) from \( \omega_{\text{max}} \) in each instance; it is expressed as a percentage.

We consider three different impact energy regimes.

### 6.1 Collisions at 62.4 GeV and Higher

Let us begin by considering the highest-energy collisions studied by the STAR collaboration and reported in [22], at 200 GeV per pair; that work also reports results on collisions at 62.4 GeV, to which we return below; we will also discuss the analogous LHC studies at that point.

In [11], the focus is on collisions with 20% to 50% centrality. This means [74,75] that the impact parameters vary from around 6.75 femtometres (fm) up to around 10.5 fm. On this domain, one finds [35] that the angular momentum imparted to the plasma in 200 GeV collisions steadily decreases, from about 110000 (in natural units; in conventional units, 110000 \( \cdot \hbar \)) at \( b = 6.75 \) fm to around 40000 at \( b = 10.5 \) fm. However, the volume of the overlap region also decreases as the impact parameter increases, and we find that, to a good approximation, the two effects cancel for collisions with 20% to 50% centrality: that is, the angular momentum density \( \alpha \) is roughly independent of \( b \) in this range of \( b \) values. We therefore focus on collisions at 20% centrality, since, in this range, these collisions are the least affected by the variations of the nuclear density near the boundary of the nucleus, and other effects associated with the very small volumes of the plasma produced by high-centrality collisions.

The volume of the plasma sample can then be computed in an elementary way, in a “hard sphere” model, using the formula [76] for the volume of the intersection of two spheres. However, as explained in [35], this underestimates the effective volume, both because the “sharp edge” assumption is inadequate (a Woods-Saxon profile is used in [35]) and because, in reality, some nucleons outside the overlap zone contribute to the fireball, effectively increasing the volume. This effect is estimated in [35] to be of order 2 to 3, depending on the impact parameter: in our case it is around 2. In addition, we must of

\(^{18}\)The primes indicate that these are the polarizations for “primary” hyperons, which means that we are neglecting the “feed-down” effect; see [8] for the theory of this, and [11] for a discussion of its effect on the STAR observations.
course take into account relativistic contraction, estimated in [77] to be roughly 7 for the equilibrated plasma.

Taking all this into account, we find that, for $\sqrt{s_{NN}} = 200$ GeV collisions at $C = 20\%$ centrality, the angular momentum density is approximately given by

$$\alpha (\sqrt{s_{NN}} = 200 \text{ GeV}, C = 20\%) \approx \frac{758}{\text{fm}^3}. \quad (27)$$

According to [73], the energy density in this case is approximately $10.55/\text{fm}^4$, and so we compute the maximal vorticity, according to equation (23), as

$$\omega_{\text{max}} (\sqrt{s_{NN}} = 200 \text{ GeV}, C = 20\%) \approx 0.00387 \text{ fm}^{-1}. \quad (28)$$

Using a temperature of approximately 190 MeV [73], we can express the vorticity bound in this case in the form

$$[\overline{P}_{\Lambda'} + \overline{P}_\Xi] (\sqrt{s_{NN}} = 200 \text{ GeV}, C = 20\%) \leq \approx 0.402\%. \quad (29)$$

When the first observations of polarization of $\Lambda$ and $\Xi$ hyperons were announced, such a value was too small to be detected (see the rightmost points in Figure 4 of [11]). Subsequent analysis of a much larger data set [24] has however found evidence of such polarization, at a level which is in remarkably good agreement with the upper bound given here: that is, the reported values are

$$\overline{P}_{\Lambda'} (\sqrt{s_{NN}} = 200 \text{ GeV}, C = 20\%) \approx \overline{P}_\Xi (\sqrt{s_{NN}} = 200 \text{ GeV}, C = 20\%) \approx 0.2\%.$$

This is the most precise vorticity observation thus far reported, so the agreement with the vorticity bound is particularly noteworthy.

If one repeats this calculation for collisions at an impact energy of 62.4 GeV, one finds that the energy density is of course lower (about $7.59/\text{fm}^4$), as is the temperature (about 179 MeV) and that the angular momentum density also drops, but more sharply, to around $236.4/\text{fm}^3$ (it scales approximately linearly with $\sqrt{s_{NN}}$ [35]): this pattern is seen throughout these calculations. The result is a much less stringent bound,

$$[\overline{P}_{\Lambda'} + \overline{P}_\Xi] (\sqrt{s_{NN}} = 62.4 \text{ GeV}, C = 20\%) \leq \approx 0.980\%. \quad (30)$$

which might well be detectable in an analysis similar to that of [24]; unfortunately, with the current data the error bars are large in this case (see the second-from-rightmost points in Figure 4 of [11]), and clear evidence of polarization is yet to be obtained. There is in any case no conflict with our claim that $\overline{P}_{\Lambda'} + \overline{P}_\Xi$ can be no larger than this.

At the other extreme, one can consider the collisions studied in the ALICE experiment: here, in the collisions at 2.76 TeV, the energy density [78] is about 2.3 times larger than in the 200 GeV collisions, but the angular momentum density is about 13.5 times larger for a given centrality; furthermore, the temperature is somewhat higher, roughly 300 MeV. The ALICE investigation [23] considered centrality in two ranges: 15% to 50%, and also 5% to 15%. As before, in the first case we can take 20% to be representative, and then we obtain an extremely severe bound:

$$[\overline{P}_{\Lambda'} + \overline{P}_\Xi] (\sqrt{s_{NN}} = 2.76 \text{ TeV}, C = 20\%) \leq \approx 0.046\%. \quad (31)$$
The other range is interesting, since the data go down to a very low centrality. Here the angular momentum is enormous, but it does not vary monotonically with impact parameter, so this case merits separate investigation. The much larger overlap volume when the impact parameter is small (around 3.5 fm for 5% centrality) makes itself felt here, and we find in this case a slightly less stringent bound despite the higher angular momentum:

$$\left[ \mathcal{P}_{\Lambda'} + \mathcal{P}_{\bar{\Lambda}} \right] (\sqrt{s_{NN}} = 2.76 \text{ TeV}, C = 5\%) \leq \approx 0.055\%.$$  \hspace{1cm} (32)

This interesting relaxation of the bound at low centralities is characteristic of the holographic model, and we will discuss it in more detail elsewhere. For the present we merely note that this is still an extremely low value.

Even with the substantial (theoretical and observational) uncertainties here, it is clear that, in all cases, the vorticity bound is completely inconsistent with any observation of hyperon polarization in these experiments (and of course this prediction is even more firm for the collisions at 5.02 TeV \cite{19}). This is entirely consistent with the reported data \cite{21}.

In summary, the vorticity bound asserts that global polarization of $\Lambda$ and $\bar{\Lambda}$ hyperons should certainly not be observable at impact energies much above 200 GeV. It is consistent with a tiny total polarization at 200 GeV — now observed, at almost exactly the maximum value permitted by the bound. If the uncertainties can be very considerably reduced, and if the bound continues to be saturated (see below), we expect it to be observable in collisions at 62.4 GeV, at a total percentage about double the observed value at 200 GeV.

Let us turn, then, to much lower impact energies.

### 6.2 Collisions at 39, 27, and 19.6 GeV

The STAR collaboration took data at 39, 27, and 19.6 GeV impact energies. We interrupt our investigation at 19.6 GeV because, while data were also taken at still lower impact energies (to be discussed below), it is not completely clear that the QGP is actually formed in those cases; this is discussed in detail in \cite{73}. We will not take a stand on this issue, but we find it clearest to focus first on the cases which are not in doubt.

In the case of collisions at 39 GeV, with 20% centrality, we find that the angular momentum density $\alpha$ has dropped to around 147.8/fm$^3$, the energy density $\varepsilon$ to 7.25/fm$^4$, the temperature to 178 MeV, and so the vorticity bound gives us

$$\left[ \mathcal{P}_{\Lambda'} + \mathcal{P}_{\bar{\Lambda}} \right] (\sqrt{s_{NN}} = 39 \text{ GeV}, C = 20\%) \leq \approx 1.51\%;$$  \hspace{1cm} (33)

the corresponding collisions at 27 GeV have $\alpha \approx 102.3/fm^3$, $T \approx 172$ MeV, and $\varepsilon \approx 5.89/fm^4$, and so we have

$$\left[ \mathcal{P}_{\Lambda'} + \mathcal{P}_{\bar{\Lambda}} \right] (\sqrt{s_{NN}} = 27 \text{ GeV}, C = 20\%) \leq \approx 1.83\%.$$  \hspace{1cm} (34)

Finally, for collisions at 19.6 GeV we have a still lower angular momentum density of around 74.2/fm$^3$, $T \approx 171$ MeV, and the energy density is about 5.6/fm$^4$, leading to

$$\left[ \mathcal{P}_{\Lambda'} + \mathcal{P}_{\bar{\Lambda}} \right] (\sqrt{s_{NN}} = 19.6 \text{ GeV}, C = 20\%) \leq \approx 2.42\%.$$  \hspace{1cm} (35)

The agreement with Figure 4 of \cite{11} (sixth pair from left for 39 GeV, fifth from left for 27 GeV, fourth from left for 19.6 GeV) (of course one has to add the two values
shown there at each impact energy) is far better than one was entitled to expect (that is, agreement to within a factor of at best 2). The rate at which the total polarization declines with increasing impact energy is reproduced particularly well.

We notice that the vorticity bound is not only (approximately) satisfied: it appears, in all three cases, to be (approximately) attained, that is, the inequality in (1) could be an equation. It may well be that the bound is attained also in all of the previous cases we examined; it is just that the bound is so low that the polarization, while present, cannot be detected in some cases (particularly the ALICE observations). We will discuss this observation in more detail, below.

In short: at these impact energies, the vorticity bound relaxes quite dramatically, to the point where global polarization of $\Lambda$ and $\bar{\Lambda}$ hyperons should be clearly observable; and so it has proved: these are the impact energies for which the evidence for hyperon polarization arising from QGP vorticity is most clear-cut [11].

Finally, we consider the collisions with the lowest impact energies.

### 6.3 Collisions at 14.5, 11.5, and 7.7 GeV

The reported data [11] on the $\Lambda$ and $\bar{\Lambda}$ hyperon polarizations present a less clear-cut picture than in the case just considered. In particular, the $\bar{\Lambda}$ polarization results appear to be systematically larger than those for $\Lambda$ hyperons, and this suggests that some additional effect may be at work here, making the interpretation of these results somewhat dubious: see [27], [80], and particularly [81]. In addition, at these impact energies (particularly for the 7.7 GeV case), it is open to doubt whether a QGP actually forms. If this is not the case, of course, then a gauge-gravity approach cannot be used.

In any event, the results of the holographic model are as follows.

At 14.5 GeV, $\alpha \approx 54.96/fm^3$, $T \approx 168$ MeV, $\varepsilon \approx 4.56/fm^4$, and then

$$\left[ P_{\Lambda'} + P_{\bar{\Lambda}} \right] (\sqrt{s_{NN}} = 14.5 \text{ GeV}, C = 20\%) \leq \approx 2.73\%; \quad (36)$$

collisions at 11.5 GeV have $\alpha \approx 43.59/fm^3$, $T \approx 164$ MeV, and $\varepsilon \approx 3.97/fm^4$, leading to

$$\left[ P_{\Lambda'} + P_{\bar{\Lambda}} \right] (\sqrt{s_{NN}} = 11.5 \text{ GeV}, C = 20\%) \leq \approx 3.04\%; \quad (37)$$

and finally the 7.7 GeV collisions have $\alpha \approx 29.18/fm^3$, $T \approx 160$ MeV, and $\varepsilon \approx 3.00/fm^4$, giving

$$\left[ P_{\Lambda'} + P_{\bar{\Lambda}} \right] (\sqrt{s_{NN}} = 7.7 \text{ GeV}, C = 20\%) \leq \approx 3.53\%. \quad (38)$$

Except at 7.7 GeV, the agreement with [11] continues to be fairly good. In the 7.7 GeV case, the reported polarization for $\bar{\Lambda}$ hyperons is so much larger than that for $\Lambda$ hyperons that this case should be viewed with particular caution. In any event, in view of the large error bars in these cases, we can still assert agreement with the vorticity bound.

It is noteworthy that, as one proceeds to higher impact energies, the difference between the reported $\Lambda$ and $\bar{\Lambda}$ hyperon polarizations grows steadily smaller, being quite negligible [24] at 200 GeV; at the same time, the agreement of the data with the vorticity bound becomes steadily better. This may not be a coincidence.

Our results are summarized in Figure 2, which should be compared with Figure 4 of [11] by adding together, in the latter, the values corresponding to the two points at each impact energy.
7. Attaining the Bound

It is striking that, for all impact energies we have considered but the lowest, the observed vorticities are actually similar in value to the right side of our basic inequality, \( \Pi \), and not very much smaller than it, as the inequality would permit. Let us take it, as a working hypothesis, that in all experiments in which the impact energy is sufficiently large as to ensure that a QGP is actually formed, and in which the centrality is not extremely small or extremely large (see below), the vorticity attains the largest value permitted by \( \Pi \). We will see that this allows us to forge a link with certain theoretical ideas regarding the relation between vorticity and angular momentum in these systems.

Obviously, the bound cannot be attained in the case of collisions which are almost exactly central, since the angular momentum density must be close to zero in those cases. According to [35], the AMPT model mentioned earlier predicts that the angular momentum imparted to the plasma rises from zero, in exactly central collisions, up to a maximum at an impact parameter of about 4 fm, before declining towards zero for larger impact parameters. However, the angular momentum is still very large even for impact parameters somewhat below 4 fm and also for values considerably larger. Based on our findings here, we conjecture that the vorticity bound should be attained, if the QGP is formed, for centralities between 5\% and 60\%, though a wider range may well be possible.

Consider, for example, collisions at 27 GeV and 5\% centrality. Despite the low centrality, the angular momentum density is still around 80.6 fm\(^{-3}\), significantly higher than the angular momentum density for collisions at 19.6 GeV and 20\% centrality (about 74.2/fm\(^3\)). Since the bound is attained in the latter case, it is reasonable to suppose that it is also
attained in the former.

Let us see where the hypothesis of the bound being always attained, under the stated conditions, leads us.

We know from equation (21) that, when the vorticity bound is attained, $L_{\omega_{\text{max}}}$ is always a fixed multiple, $\sigma \approx 1.2499$, of $A$; and of course we know that $A$ (for fixed centrality) increases steadily with the impact energy. Thus, in these circumstances, $L_{\omega_{\text{max}}}$ too increases markedly with impact energy. But we know (see the end of Section 4 above) that $L_{\omega_{\text{max}}}$ has an interpretation as the parameter that determines the "effective radius of gyration" of the plasma sample, $k_{\text{eff}} \equiv \sqrt{A/\omega}$. The increase of $L_{\omega_{\text{max}}}$ with impact energy therefore has consequences for this quantity.

If the vorticity bound is attained, we have (using equation (23))

$$k_{\text{eff}}(\omega_{\text{max}}) = \frac{A}{\sqrt{\kappa}} \approx 1.8959 \frac{\alpha}{\varepsilon};$$

that is, $k_{\text{eff}}(\omega_{\text{max}})$, like $L_{\omega_{\text{max}}}$, is a constant multiple of $A$ (that is, of $\alpha/\varepsilon$) and so must increase with the impact energy.\(^{19}\)

The increase of $k_{\text{eff}}(\omega_{\text{max}})$ with impact energy, under the assumption that the vorticity bound is attained, can be related to the claim in [35] that increasing impact energy affects the fluid moment of inertia. Our conclusion is rather more precise: essentially we claim that this statement applies not just to the moment of inertia but also to the (effective) radius of gyration (that is, to the moment of inertia per unit "mass"). In the holographic model, then, the relevant change with increasing impact energy is in the distribution of matter in QGP vortices, rather than in the energy density. This agrees with the suggestions put forward in [38, 39], which may explain why the moment of inertia is (in effect) so sensitive to the impact energy. Clearly this point demands clarification.

To summarize: if the bound is always attained in the experiments, then the holographic model explains the observed decline in the global polarizations with impact energy by implying that the effective radius of gyration increases with impact energy at fixed centrality. The precise relation expresses the effective radius of gyration as a linear function of the ratio of the angular momentum density to the energy density; and holography computes the constant of proportionality as in equation (39).

### 8. Conclusion

The existence of some kind of bound on vorticity should occasion no surprise: the vortex has to resist a centrifugal force in order to sustain itself for a reasonably long interval of time. The question is whether the bound can be implemented in practice (it should depend, in some simple way, on a small number of readily accessible observational parameters); also, to be useful, it should be markedly more restrictive than expected on the basis of obvious constraints such as those imposed by causality.

\(^{19}\)Note that $L_{\omega_{\text{max}}}/k_{\text{eff}}(\omega_{\text{max}}) = \sigma \sqrt{\kappa}$, so, from the numerical values of $\kappa \approx 0.2782$ and $\sigma \approx 1.2499$, we have $L_{\omega_{\text{max}}}/k_{\text{eff}}(\omega_{\text{max}}) \approx 0.6593$ and thus $k_{\text{eff}}(\omega_{\text{max}}) > L_{\omega_{\text{max}}} > A$.

\(^{20}\)We stress that this is indeed an additional assumption, beyond the mere existence of a bound: and one notes that, without it, we have no basis for asserting that $L_{\omega_{\text{max}}}$ must increase as $A$ increases. The existence of a vorticity bound and the increase of $L_{\omega_{\text{max}}}$ with $A$ are logically independent.
We have identified a bound, given by inequality (1), which is precisely of this sort. The bound depends on only two readily extracted parameters, the energy density $\varepsilon$ and the angular momentum density $\alpha$; it is very restrictive, and, crucially, it becomes steadily more so as the impact energy increases.

The agreement of Figure 2 with Figure 4 of [11], apart from one outlier, is surprising; indeed it is remarkable that the trend is even in the correct (downward) direction. The fact that it suggests, correctly, that hyperon polarization associated with QGP vorticity should be readily observable at impact energies up to around 39 GeV, observable only with difficulty at impact energy 200 GeV, and not at all at higher energies, might almost be called startling.

In view of all of the theoretical and observational uncertainties, however, one should not read too much into this. If one could prove that QCD itself has a holographic dual, then our results would give grounds for asserting that the holographic vorticity bound reflects a fundamental property of the QGP; but that is not yet the case. Our results do make it reasonable, however, to conjecture that a vorticity bound does exist, and that its mathematical form is similar to that of (1). Our simple holographic model produces a very specific value for the constant $\kappa$; perhaps this can be improved by more sophisticated considerations. At least we have a concrete basis for further investigations by other methods.

The observation that, where the vorticity bound is satisfied, it is also attained, merits further attention. One might try to understand it by thinking of the vorticity as being “held down” by the bound: that is, the vorticity would have been far larger were it not for whatever physical effect it is that underlies the bound (such as the direct effect of vorticity itself on the effective radius of gyration, as discussed in this work). In any case, the hypothesis that the bound is always attained has useful consequences, to be explored elsewhere.

We close with a more general observation. It has been said [82] that one reason for the importance of the holographic KSS bound [55, 56] on the dynamic viscosity $\eta$ of the QGP given its entropy density $s$ is that it gives a benchmark: saying that $\eta$ is “small” now means that $\eta$ is not far above $s/4\pi$.

We would like to suggest that the vorticity bound might be used to establish a similar benchmark here: instead of saying that the overall value of the vorticity reported in [11], $9 \pm 1 \times 10^{21} \cdot s^{-1}$, is “large” (meaning that it is large when compared with vorticities attained in other, vastly more extended and very different systems), we might assert that the vorticities attained in these experiments are “large” because, as we have shown, they are of the order of $0.2782 \varepsilon/\alpha$ for collisions at nearly all impact energies — in other words, they are about as large as they can be, if the bound is correct. Since it is probable that no other substance can sustain vortices at such high angular momentum densities, one can say that the vorticity bound implies that the vorticities reported by the STAR collaboration may, in this sense, be the largest possible for any substance. In any case, it might possibly be useful, with future, more precise data, to report vorticities at a given impact energy and centrality by comparing them with the putative maximum possible value provided by the vorticity bound.
Acknowledgements

The author thanks Dr Soon Wanmei for valuable discussions.

References

[1] Zuo-Tang Liang, Xin-Nian Wang, Globally polarized quark-gluon plasma in non-central A+A collisions, Phys.Rev.Lett. 94 (2005) 102301, Erratum-ibid. 96 (2006) 039901, arXiv:nucl-th/0410079

[2] F. Becattini, F. Piccinini, J. Rizzo, Angular momentum conservation in heavy ion collisions at very high energy, Phys.Rev.C77:024906,2008, arXiv:0711.1253 [nucl-th]

[3] Xu-Guang Huang, Pasi Huovinen, Xin-Nian Wang, Quark Polarization in a Viscous Quark-Gluon Plasma, Phys. Rev. C84, 054910(2011), arXiv:1108.5649 [nucl-th]

[4] D.J. Wang, Z. Néda, L.P. Csernai, Viscous potential flow analysis of peripheral heavy ion collisions, Phys. Rev. C 87, 024908 (2013), arXiv:1302.1691 [nucl-th]

[5] D.E. Kharzeev, J. Liao, S.A. Voloshin, G. Wang, Chiral magnetic and vortical effects in high-energy nuclear collisions — A status report, Prog.Part.Nucl.Phys. 88 (2016) 1, arXiv:1511.04050 [hep-ph]

[6] Wei-Tian Deng, Xu-Guang Huang, Vorticity in Heavy-Ion Collisions, Phys.Rev. C93 (2016) 064907, arXiv:1603.06117 [nucl-th]

[7] Long-Gang Pang, Hannah Petersen, Qun Wang, Xin-Nian Wang Vortical fluid and Λ spin correlations in high-energy heavy-ion collisions, Phys.Rev.Lett. 117 (2016) no.19, 192301, arXiv:1605.04024 [hep-ph]

[8] F. Becattini, I. Karpenko, M. Lisa, I. Upsal, S. Voloshin, Global hyperon polarization at local thermodynamic equilibrium with vorticity, magnetic field and feed-down, Phys. Rev. C 95, 054902 (2017), arXiv:1610.02506 [nucl-th]

[9] M.A.Zubkov, Hall effect in the presence of rotation, EPL, 121 (2018) 47001, arXiv:1801.05368 [hep-ph]

[10] V.A. Okorokov (on behalf of the STAR Collaboration), Highlights from STAR heavy ion program, EPJ Web Conf. 158 (2017) 01004, arXiv:1710.05313 [nucl-ex]

[11] STAR Collaboration (L. Adamczyk et al.), Global Λ hyperon polarization in nuclear collisions: evidence for the most vortical fluid, Nature 548, 62 (2017), arXiv:1701.06657 [nucl-ex]

[12] Iu. Karpenko, F. Becattini, Study of Lambda polarization in relativistic nuclear collisions at $\sqrt{s_{NN}} = 7.7 - 200$ GeV, Eur. Phys. J. C (2017) 77: 213, arXiv:1610.04717 [nucl-th]
[13] Hui Li, Long-Gang Pang, Qun Wang, Xiao-Liang Xia, Global $\Lambda$ polarization in heavy-ion collisions from a transport model, Phys. Rev. C 96, 054908 (2017), arXiv:1704.01507 [nucl-th]

[14] Iu. Karpenko, F. Becattini, Vorticity in the QGP liquid and $\Lambda$ polarization at the RHIC Beam Energy Scan, Nucl.Phys. A967 (2017) 764, arXiv:1704.02142 [nucl-th]

[15] Hui Li, Hannah Petersen, Long-Gang Pang, Qun Wang, Xiao-Liang Xia, Xin-Nian Wang, Local and global $\Lambda$ polarization in a vortical fluid, Nucl.Phys. A967 (2017) 772, arXiv:1704.03569 [nucl-th]

[16] Qun Wang, Global and local spin polarization in heavy ion collisions: a brief overview, Nucl.Phys. A967 (2017) 225, arXiv:1704.04022 [nucl-th]

[17] Yuji Hirono, Chiral magnetohydrodynamics for heavy-ion collisions, Nucl.Phys. A967 (2017) 233, arXiv:1705.09409 [hep-ph]

[18] Yifeng Sun, Che Ming Ko, Lambda hyperon polarization in relativistic heavy ion collisions from the chiral kinetic approach, Phys.Rev. C96 (2017) no.2, 024906, arXiv:1706.09467 [nucl-th]

[19] Sergei A. Voloshin, Vorticity and particle polarization in heavy ion collisions (experimental perspective), arXiv:1710.08934 [nucl-ex]

[20] F. Becattini, Polarization and Chirality: the quantum features of the Quark Gluon Plasma, arXiv:1810.02706 [nucl-th]

[21] De-Xian Wei, Wei-Tian Deng, Xu-Guang Huang, Thermal vorticity and spin polarization in heavy-ion collisions, arXiv:1810.00151 [nucl-th]

[22] STAR Collaboration: B.I. Abelev, I. Selyuzhenkov, et al., Global polarization measurement in Au+Au collisions, Phys.Rev.C76:024915,2007, arXiv:0705.1691 [nucl-ex]

[23] Bedangadas Mohanty for the ALICE Collaboration, Measurements of spin alignment of vector mesons and global polarization of hyperons with ALICE at the LHC, arXiv:1711.02018 [nucl-ex]

[24] STAR Collaboration (J. Adam et al.), Global polarization of $\Lambda$ hyperons in Au+Au collisions at $\sqrt{s_{NN}} = 200$ GeV, Phys.Rev. C98 (2018) 014910, arXiv:1805.04400 [nucl-ex]

[25] Takafumi Niida (for the STAR Collaboration), Global and local polarization of $\Lambda$ hyperons in Au+Au collisions at 200 GeV from STAR, arXiv:1808.10482 [nucl-ex]

[26] Zhenyu Ye (for the STAR collaboration), Highlights from the STAR experiment, arXiv:1809.05713 [nucl-ex]

[27] Shuzhe Shi, Kangle Li, Jinfeng Liao, Searching for the Subatomic Swirls in the CuCu and CuAu Collisions, arXiv:1712.00878 [nucl-th]
[28] Ludwik Turko, for the NA61/SHINE Collaboration, Looking for the phase transition - recent NA61/SHINE results, Universe 4 (2018) no.3, 52, arXiv:1801.06919 [hep-ex]

[29] Xiaofeng Luo, Nu Xu, Search for the QCD Critical Point with Fluctuations of Conserved Quantities in Relativistic Heavy-Ion Collisions at RHIC: An Overview, Nucl. Sci. Tech. 28, 112 (2017), arXiv:1701.02105 [nucl-ex]

[30] H. Petersen, Beam energy scan theory: Status and open questions, Nucl.Phys. A967 (2017) 145, arXiv:1704.02904 [nucl-th]

[31] CBM Collaboration: T. Ablyazimov et al., Challenges in QCD matter physics - The Compressed Baryonic Matter experiment at FAIR, Eur. Phys. J. A 53 (2017) 60, arXiv:1607.01487 [nucl-ex]

[32] Declan Keane, The Beam Energy Scan at the Relativistic Heavy Ion Collider, J.Phys.Conf.Ser. 878 (2017) no.1, 012015

[33] V.D. Kekelidze, NICA project at JINR: status and prospects, Journal of Instrumentation 12 C06012 (2017)

[34] B. Betz, M. Gyulassy, G. Torrieri, Polarization probes of vorticity in heavy ion collisions, Phys. Rev. C76, 044901 (2007), arXiv:0708.0035 [nucl-th]

[35] Y. Jiang, Z.-W. Lin, J. Liao, Rotating quark-gluon plasma in relativistic heavy ion collisions. Phys. Rev. C94, 044910 (2016), arXiv:1602.06580 [hep-ph]

[36] Iurii Karpenko, Francesco Becattini, Study of Lambda polarization at RHIC BES and LHC energies, EPJ Web Conf. 171 (2018) 17001, arXiv:1710.09726 [nucl-th]

[37] F. Becattini, Polarization in relativistic heavy ion collisions: a theoretical perspective, arXiv:1711.08780 [nucl-th]

[38] Xiao-Liang Xia, Hui Li, Qun Wang, Some aspects of global Lambda polarization in heavy-ion collisions, PoS CPOD2017 (2018) 023, arXiv:1712.02677 [nucl-th]

[39] Xiao-Liang Xia, Hui Li, Ze-bo Tang, Qun Wang, Probing vorticity structure in heavy-ion collisions by local Λ polarization, Phys. Rev. C 98, 024905 (2018), arXiv:1803.00867 [nucl-th]

[40] Yu. B. Ivanov, A. A. Soldatov, Vortex rings in fragmentation regions in heavy-ion collisions at $\sqrt{s_{NN}} = 39$ GeV, Phys. Rev. C 97, 044915 (2018), arXiv:1803.01525 [nucl-th]

[41] Brett McInnes, Holography of the QGP Reynolds Number, Nucl.Phys. B921 (2017) 39, arXiv:1702.02276 [hep-th]

[42] Hans Bantilan, Takaaki Ishii, Paul Romatschke, Holographic Heavy-Ion Collisions: Analytic Solutions with Longitudinal Flow, Elliptic Flow and Vorticity, Phys.Lett. B785 (2018) 201, arXiv:1803.10774 [nucl-th]
[43] V. Skokov, A.Yu. Illarionov, V. Toneev, Estimate of the magnetic field strength in heavy-ion collisions, Int. J. Mod. Phys. A24 (2009) 5925, arXiv:0907.1396 [nucl-th]

[44] Wei-Tian Deng, Xu-Guang Huang, Event-by-event generation of electromagnetic fields in heavy-ion collisions, Phys. Rev. C 85 044907 (2012), arXiv:1201.5108 [nucl-th]

[45] *Strongly interacting matter in magnetic fields*, Eds. D. Kharzeev, K. Landsteiner, A. Schmitt, H.-U. Yee, Lect. Notes Phys. 871, 1 (2013), arXiv:1211.6245 [hep-ph]

[46] Kirill Tuchin, Particle production in strong electromagnetic fields in relativistic heavy-ion collisions, Adv. High Energy Phys. 2013 (2013) 490495, arXiv:1301.0099 [hep-ph]

[47] Gergely Endrodi, Critical point in the QCD phase diagram for extremely strong background magnetic fields, JHEP 1507 (2015) 173, arXiv:1504.08280 [hep-lat]

[48] Xu-Guang Huang, Electromagnetic fields and anomalous transports in heavy-ion collisions — A pedagogical review, Rept. Prog. Phys. 79 (2016) 076302, arXiv:1509.04073 [nucl-th]

[49] Koichi Hattori, Xu-Guang Huang, Novel quantum phenomena induced by strong magnetic fields in heavy-ion collisions, Nucl. Sci. Tech. 28 (2017) 26, arXiv:1609.00747 [nucl-th]

[50] Brett McInnes, A Rotation/Magnetism Analogy for the Quark Plasma, Nucl. Phys. B911 (2016) 173, arXiv:1604.03669 [hep-ph]

[51] Ashutosh Dash, Victor Roy, Bedangadas Mohanty, Magneto-Vortical evolution of QGP in heavy ion collisions, DAE Symp. Nucl. Phys. 62 (2017) 852, arXiv:1705.05657 [nucl-th]

[52] Makoto Natsuume, AdS/CFT Duality User Guide, Lect. Notes Phys. 903 (2015), arXiv:1409.3575 [hep-th]

[53] Veronika E. Hubeny, The AdS/CFT Correspondence, Class. Quant. Grav. 32 (2015) 124010, arXiv:1501.00007 [gr-qc]

[54] Oliver DeWolfe, TASI Lectures on Applications of Gauge/Gravity Duality, PoS TASI2017 (2018) 014, arXiv:1802.08267 [hep-th]

[55] P. Kovtun, D. T. Son, A. O. Starinets, Viscosity in Strongly Interacting Quantum Field Theories from Black Hole Physics, Phys. Rev. Lett. 94 (2005) 111601, arXiv:hep-th/0405231

[56] D. T. Son, A. O. Starinets, Viscosity, Black Holes, and Quantum Field Theory, Ann. Rev. Nucl. Part. Sci. 57:95-118, 2007, arXiv:0704.0240 [hep-th]

[57] Jussi Auvinen, Iurii Karpenko, Jonah E. Bernhard, Steffen A. Bass, Revealing the collision energy dependence of $\eta/s$ in RHIC-BES Au+Au collisions using Bayesian statistics, Nucl. Phys. A967 (2017) 784, arXiv:1704.04643 [nucl-th]
[58] Steffen A. Bass, Jonah E. Bernhard, J. Scott Moreland, Determination of Quark-Gluon-Plasma Parameters from a Global Bayesian Analysis, Nucl.Phys. A967 (2017) 67, arXiv:1704.07671 [nucl-th]

[59] Brett McInnes, A Holographic Bound on Cosmic Magnetic Fields, Nucl. Phys. B 892 (2015) 49, arXiv:1409.3663 [hep-th]

[60] A. Nata Atmaja, K. Schalm, Anisotropic Drag Force from 4D Kerr-AdS Black Holes, JHEP 1104:070,2011, arXiv:1012.3800 [hep-th]

[61] S.W.Hawking, C.J.Hunter, M.M.Taylor-Robinson, Rotation and the AdS/CFT correspondence, Phys.Rev.D59:064005,1999, arXiv:hep-th/9811056

[62] Marco M. Caldarelli, Guido Cognola, Dietmar Klemm, Thermodynamics of Kerr-Newman-AdS Black Holes and Conformal Field Theories, Class.Quant.Grav. 17 (2000) 399, arXiv:hep-th/9908022

[63] G.W. Gibbons, M.J. Perry, C.N. Pope, The First Law of Thermodynamics for Kerr-Anti-de Sitter Black Holes, Class.Quant.Grav.22:1503-1526,2005, arXiv:hep-th/0408217

[64] Z.-W. Chong, M. Cvetic, H. Lu, C.N. Pope, General Non-Extremal Rotating Black Holes in Minimal Five-Dimensional Gauged Supergravity, Phys.Rev.Lett.95:161301,2005, arXiv:hep-th/0506029

[65] Norman Steenrod, The Topology of Fibre Bundles, PMS 14, Princeton University Press (1951)

[66] David Kastor, Sourya Ray, Jennie Traschen, Enthalpy and the Mechanics of AdS Black Holes, Class.Quant.Grav.26:195011,2009, arXiv:0904.2765 [hep-th]

[67] Clifford V. Johnson, Holographic Heat Engines, Class. Quant. Grav. 31 (2014) 205002, arXiv:1404.5982 [hep-th]

[68] Brian P. Dolan, Black holes and Boyle’s law – the thermodynamics of the cosmological constant, Mod.Phys.Lett. A30 (2015) no.03n04, 1540002, arXiv:1408.4023 [gr-qc]

[69] Sayantani Bhattacharyya, Subhaneil Lahiri, R. Loganayagam, Shiraz Minwalla, Large rotating AdS black holes from fluid mechanics, JHEP 0809:054,2008, arXiv:0708.1770 [hep-th]

[70] Oyvind Gron, Sigbjorn Hervik, Einstein’s General Theory of Relativity: With Modern Applications in Cosmology, Springer Science and Business Media, 2007

[71] M. N. Chernodub, Shinya Gongyo, Effects of rotation and boundaries on chiral symmetry breaking of relativistic fermions Phys. Rev. D 95, 096006 (2017), arXiv:1702.08266 [hep-th]

[72] Wit Busza, Krishna Rajagopal, Wilke van der Schee, Heavy Ion Collisions: The Big Picture, and the Big Questions, arXiv:1802.04801 [hep-ph]
[73] Pragati Sahoo, Swatantra Kumar Tiwari, Sudipan De, Raghu nath Sahoo, Rolf P. Scharenberg, Brijesh K. Srivastava, Beam Energy Dependence of Thermodynamic and Transport Properties of Strongly Interacting Matter in a Color String Percolation Approach at the Relativistic Heavy Ion Collider, arXiv:1708.06689 [hep-ph]

[74] W. Broniowski, W. Florkowski, Geometric relation between centrality and the impact parameter in relativistic heavy-ion collisions, Phys.Rev. C65 (2002) 024905, arXiv:nucl-th/0110020

[75] Sruthy Jyothi Das, Giuliano Giacalone, Pierre-Amaury Monard, Jean-Yves Ollitrault, Relating centrality to impact parameter in nucleus-nucleus collisions, Phys. Rev. C97, 014905 (2018), arXiv:1708.00081 [nucl-th]

[76] Weisstein, Eric W. "Sphere-Sphere Intersection." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/Sphere-SphereIntersection.html

[77] B.B. Back et al., The PHOBOS perspective on discoveries at RHIC, Nucl.Phys.A757:28-101,2005, arXiv:nucl-ex/0410022.

[78] ALICE Collaboration, Measurement of transverse energy at midrapidity in Pb-Pb collisions at $\sqrt{s_{NN}}=2.76$ TeV, Phys. Rev. C 94, 034903 (2016), arXiv:1603.04775 [nucl-ex]

[79] Alessandro Grelli (for the ALICE Collaboration), ALICE Overview, arXiv:1711.03369 [hep-ex]

[80] E.E. Kolomeitsev, V.D. Toneev, V. Voronyuk, Vorticity and hyperon polarization at NICA energies, Phys. Rev. C 97, 064902 (2018), arXiv:1801.07610 [nucl-th]

[81] L. P. Csernai, J. I. Kapusta, T. Welle, $\Lambda$ and $\bar{\Lambda}$ spin interaction with meson fields generated by the baryon current in high energy nuclear collisions, arXiv:1807.11521 [nucl-th]

[82] Andreas Karch, Recent Progress in Applying Gauge/Gravity Duality to Quark-Gluon Plasma Physics, AIP Conf.Proc. 1441 (2012) 95, arXiv:1108.4014 [hep-ph]