Wick Power Series in Indefinite Metric Field Theories

A. G. Smirnov and M. A. Soloviev

I. E. Tamm Department of Theoretical Physics, P. N. Lebedev Physics Institute,
Leninsky prosp. 53, Moscow 117924, Russian Federation

Abstract

The analytic aspects of the operator realization of Wick power series of infrared singular free fields are considered. Taking advantage of the holomorphy properties of the two-point correlation function and its Hilbert majorant in x-space, we solve in a general and model independent way the problem of finding the adequate test function space on which a given Wick series is convergent. Substantial attention is paid to the proper formulation of the spectral condition in case the suitable test functions are entire analytic in momentum space.

1 Introduction

This work is devoted to the functional–analytic aspects of the operator realization of the Wick power series

$$\sum_k d_k \phi^k(x)$$

(1)

of a free field $\phi$ whose two-point vacuum expectation value $w(x_1-x_2) = \langle \Psi_0, \phi(x_1)\phi(x_2)\Psi_0 \rangle$ does not satisfy the positivity condition. The construction of such series exhibits some peculiarities which have no analogues in the positive metric case and should be taken into account in the covariant quantization of gauge theories. Of particular interest are the problems of adequate choice of test functions and of appropriate generalization of the spectral condition, which have been raised by Moschella and Strocchi [1]. As shown in [2, 3], a simple way of overcoming these problems is the use of a relevant Paley–Wiener–Schwartz–type theorem for analytic functionals. Here we outline the development [4, 5] of this approach based on exploiting in a systematic manner the analytic properties of the Hilbert majorant of the two-point function $w(x_1-x_2)$. We present a general and model independent criterion which enables one to find easily the adequate test functions on which a given Wick series is convergent. Particular emphasis is given to formulating the spectral condition for those quantum fields that are well defined only under smearing with test functions analytic in $p$-space and showing that it is satisfied for the sums of Wick series with arbitrarily singular infrared behavior.

2 Analytic Properties of the Hilbert Majorant

We begin by recalling some of the standard facts concerning the indefinite metric formalism [8, 9], restricting our consideration to the case of a single neutral scalar field $\phi(x)$ in
\(d\)-dimensional space-time \((d \geq 2)\). In this formalism, the state space \(\mathcal{H}\) is endowed by a pseudo-Hilbert structure. This means that \(\mathcal{H}\) is provided with two sesquilinear forms \(\langle \cdot, \cdot \rangle\) and \((\cdot, \cdot)\). Because of a failure in positivity, the vacuum expectation values \(\langle \Psi_0, \phi(x_1) \ldots \phi(x_n) \Psi_0 \rangle\) are expressible in terms of the indefinite metric \(\langle \cdot, \cdot \rangle\), whereas the positive scalar product \((\cdot, \cdot)\) defines the notion of convergence in \(\mathcal{H}\). The forms \(\langle \cdot, \cdot \rangle\) and \((\cdot, \cdot)\) are connected by the relation

\[
(\Phi, \eta \Psi) = \langle \Phi, \Psi \rangle, \quad \Phi, \Psi \in \mathcal{H},
\]

where \(\eta\) is a self-adjoint involutory operator, which is called the Krein operator. As a consequence,

\[
|\langle \Phi, \Psi \rangle| \leq \sqrt{(\Phi, \Phi)} \sqrt{(\Psi, \Psi)}, \quad \Phi, \Psi \in \mathcal{H}.
\]

Assuming that \(\phi\) is a tempered operator–valued distribution defined on Schwartz’s test function space \(S\), we can formulate as usual all Wightman axioms with the exception of the spectral condition. In the indefinite metric case, the implementers of the translation group \(U(\xi)\) are not unitary but pseudounitary and are, as a rule, unbounded operators. For this reason, the standard formulation of the spectral condition in terms of the generators of translations becomes meaningless. However, one can express the spectral condition in terms of the matrix elements of \(U(\xi)\):

\[
\text{supp} \int \langle \Phi, U(\xi) \Psi \rangle e^{-ip\xi} d\xi \subset \bar{V}_+,
\]

where \(\bar{V}_+\) is the closed upper light cone, and \(\Phi, \Psi\) are arbitrary vectors in the domain of \(\phi\). Equivalently, the spectral condition can be expressed in terms of the support properties of the vector-valued functional

\[
\Psi(f) = \int \phi(x_1) \ldots \phi(x_n) f(x_1, \ldots, x_n) dx_1 \ldots dx_n \Psi_0,
\]

Namely, the inclusion

\[
\text{supp} \hat{\Psi} \subset K_n^- = \{(p_1, \ldots, p_n) : p_m + \ldots + p_n \in \bar{V}_-, \quad \forall m = 1, \ldots, n\}
\]

is valid, where \(\hat{\Psi}\) is the Fourier transform of \(\Psi(f)\). This form of the spectral condition can be used as long as there are test functions of compact support in \(p\)-space. The case when test functions in \(p\)-space are analytic is considered in Section 4.

The two-point function \(w(x_1 - x_2)\) serves as the kernel of the sesquilinear form \(\langle f, g \rangle_S = \langle \phi(f) \Psi_0, \phi(g) \Psi_0 \rangle\) on \(S(\mathbb{R}^d) \times S(\mathbb{R}^d)\):

\[
\langle f, g \rangle_S = \int w(x_1 - x_2) \bar{f}(x_1) g(x_2) dx_1 dx_2,
\]
while the form \((f, g)_S = (\phi(f)\Psi_0, \phi(g)\Psi_0)\) defines the distribution \(w_{maj}\), which is called the majorant of \(w\):

\[
(f, g)_S = \int w_{maj}(x_1, x_2) f(x_1) g(x_2) \, dx_1 dx_2. 
\]

(8)

For a free scalar field \(\phi\), the factorization property of the \(n\)-point Wightman functions implies the relation

\[
\langle : \phi(f_1) \cdots \phi(f_m) : \Psi_0, : \phi(g_1) \cdots \phi(g_n) : \Psi_0 \rangle = \delta_{mn} \prod_j \langle f_j, g_{\pi(j)} \rangle_S, 
\]

where \(f_i, g_j \in S\), \(\pi\) runs over all permutations of indices, and \(::\) denotes the Wick-ordered product defined in terms of Wightman functions by the recursive relation

\[
: \phi(f_1) \cdots \phi(f_n) := \phi(f_1) : \phi(f_2) \cdots \phi(f_n) : - \sum_{j=2}^n \langle \Psi_0, \phi(f_1)\phi(f_j)\Psi_0 \rangle_S : \phi(f_2) \cdots \phi(f_{j-1})\phi(f_{j+1}) \cdots \phi(f_n) : .
\]

A formula analogous to (9) holds for the positive scalar product:

\[
\langle : \phi(f_1) \cdots \phi(f_m) : \Psi_0, : \phi(g_1) \cdots \phi(g_n) : \Psi_0 \rangle = \delta_{mn} \prod_j \langle f_j, g_{\pi(j)} \rangle_S. 
\]

(10)

It means that the state space \(\mathcal{H}\) is obtainable by the usual Fock procedure from the one-particle subspace \(\mathcal{H}^1 = \{\Psi: \Psi = \phi(f)\Psi_0, f \in S\}\), where bar stands for the closure in the topology defined by \(\langle \cdot, \cdot \rangle\). In particular, the relation (10) implies that \(\mathcal{H}^1\) is mapped onto itself by the Krein operator \(\eta\). Indeed, by (3) and (10), the vector \(\eta\phi(f)\Psi_0\) is \(\langle \cdot, \cdot \rangle\)-orthogonal to all vectors of the form \(\phi(g_1) \cdots \phi(g_m)\):\(\Psi_0\) with \(m \neq 1\). Because of (10) and the cyclicity of the vacuum, the linear span of such vectors is dense in the orthogonal complement \((\mathcal{H}^1)^\perp\) and therefore \(\eta\phi(f)\Psi_0\) lies in \((\mathcal{H}^1)^\perp = \mathcal{H}^1\). For a continuous mapping, the image of the closure of a set is contained in the closure of its image, whence \(\eta\mathcal{H}^1 \subset \mathcal{H}^1\). Since \(\eta\) is an involutory operator, this inclusion implies \(\mathcal{H}^1 = \eta \mathcal{H}^1 \subset \eta \mathcal{H}^1\), and so \(\eta \mathcal{H}^1 = \mathcal{H}^1\).

The key observation, which is central to our approach and serves as a starting point in [4], is that the majorant inherits the momentum–space support properties of the initial two-point function and, consequently, its analyticity properties in \(x\)-space. The following lemma shows that the roots of this fact lie in the general structure of the indefinite metric formalism.

**Lemma.** Let \(\phi(x)\) be a free field acting in the corresponding Fock–Hilbert–Krein space. Then

\[
\text{supp } \hat{w}_{maj}(p_1, p_2) \subset (\text{supp } \hat{w}) \times (-\text{supp } \hat{w}).
\]

**Proof.** The null space \(N_{<>}\) of the form \(\langle \cdot, \cdot \rangle_S\) coincides with the null space \(N_0\) of the form \(\langle \cdot, \cdot \rangle_S\). Indeed, let \(f \in N_{<>}\). Then \(\langle \phi(f)\Psi_0, \phi(g)\Psi_0 \rangle = 0\) for any \(g \in S\) and, by continuity,
\[ \langle \phi(f) \Psi_0, \Phi \rangle = (\phi(f) \Psi_0, \eta \Phi) = 0 \text{ for any } \Phi \in \mathcal{H}^1. \] Since \( \eta \mathcal{H}^1 = \mathcal{H}^1 \), we see that \( f \in N_0 \), and so \( N_{<} \subset N_0 \). Analogously, \( N_0 \subset N_{<} \).

Now suppose the lemma is false and fix a point \((q_1, q_2) \in \text{supp} \, \hat{w}_{\text{maj}}\) which does not belong to \((\text{supp} \, \hat{w}) \times (\text{supp} \, \hat{w})\). For definiteness, let \( q_1 \notin \text{supp} \, \hat{w} \). To prove the statement, it is sufficient to find a test function \( f \in N_{<} \) which does not belong to \( N_0 \).

The relation \( \langle f, g \rangle_S = \int \hat{w}(p)\hat{f}(p)\hat{g}(p)\,dp/(2\pi)^d \), which is obtained by rewriting (7) in momentum space, implies that all test functions such that \( \text{supp} \, \hat{f} \cap \text{supp} \, \hat{w} = \emptyset \) belong to \( N_{<} \). Let \( O \) be an open neighbourhood of \( q_1 \) such that \( O \cap \text{supp} \, \hat{w} = \emptyset \). By assumption, there exists a test function \( H \in \mathcal{D}(O \times \mathbb{R}^d) \) such that \( \hat{w}_{\text{maj}}(H) \neq 0 \). Since \( \mathcal{D}(O) \otimes \mathcal{D}(\mathbb{R}^d) \) is dense in \( \mathcal{D}(O \times \mathbb{R}^d) \), one can assume that \( H(p_1, p_2) = F(p_1)G(p_2) \), where \( F \in \mathcal{D}(O) \) and \( G \in \mathcal{D}(\mathbb{R}^d) \). Taking into account the relation (8) in the form \( \langle f, g \rangle_S = \int \hat{w}_{\text{maj}}(p_1, p_2)\hat{f}(p_1)\hat{g}(p_2)dp_1dp_2/(2\pi)^{2d} \) and setting \( \tilde{f}(p) = F(p) \), \( \hat{g}(p) = G(p) \), we obtain that \( \langle f, g \rangle_S \neq 0 \) and hence \( f \notin N_0 \). However, \( \text{supp} \, \hat{f} \cap \text{supp} \, \hat{w} = \emptyset \) and \( f \in N_{<} \). The lemma is proved.

In view of the inclusion \( \text{supp} \, \hat{w} \subset \mathbb{V}_+ \), which is implied by the spectral condition, it immediately follows from the proved lemma that \( w_{\text{maj}} \) is the boundary value of an analytic function \( w_{\text{maj}}(z, z') \) which is holomorphic in the tubular domain \( \{ z, z' : y = \text{Im} \, z \in \mathbb{V}_-, y' = \text{Im} \, z' \in \mathbb{V}_+ \} \). Moreover, for all \( y \in \mathbb{V}_+ \) the following bound holds

\[ |w(x - x' - 2iy)|^2 \leq |w_{\text{maj}}(x - iy, x + iy)| \cdot |w_{\text{maj}}(x' - iy, x' + iy)|, \tag{11} \]

where \( w(z) \) is the Wightman analytic function whose boundary value is \( w \). Indeed, taking \( f(\xi) = (\nu/\sqrt{\pi})^d e^{-\nu^2(\xi - x - iy)^2} \) and \( g(\xi) = (\nu/\sqrt{\pi})^d e^{-\nu^2(\xi - x' - iy)^2} \), setting \( \Phi = \phi(f)\Psi_0, \Psi = \phi(g)\Psi_0, \) and writing the left- and right-hand sides in (3) as integrals over a plane in the analyticity domain and passing to the limit as \( \nu \to \infty \), we immediately obtain (11).

### 3 The Convergence Criterion for Wick Series

When finding those test functions under smearing with which the Wick power series is convergent, we make use of the generalized Gelfand-Shilov spaces of type \( S \). Let us recall that the space \( S_a^b \) in this class is determined by two non-decreasing sequences of positive numbers \( a_k \) and \( b_l \) and consists of smooth functions on \( \mathbb{R}^n \) that satisfy the bounds

\[ \sup_{x} \sup_{|\kappa| \leq k} \sup_{|\lambda| \leq l} |x^\kappa \partial^\lambda f(x)| \leq CA^k B^l a_k b_l, \]

with constants \( A, B, C \) depending on \( f \). From the functional analysis standpoint it is reasonable to impose the conditions

\[ a_k^2 \leq a_{k-1} a_{k+1}, \quad a_{k+l} \leq C_1 h_1^{k+l} a_k a_l, \quad b_k^2 \leq b_{k-1} b_{k+1}, \quad b_{k+l} \leq C_2 h_2^{k+l} b_k b_l. \]

\[ ^1 \text{As usual, for an open set } O \text{ we denote by } \mathcal{D}(O) \text{ the space composed of smooth functions whose support is contained in } O. \]
where $C_{1.2}$ and $h_{1.2}$ are constants. The indicator functions $a(r) = \sup_k r^k/a_k$, and $b(s) = \sup_k s^k/b_k$ characterize the behavior of the test functions and of their Fourier transforms at infinity, while in the context of QFT they indicate the infrared and ultraviolet behavior of the fields defined on $S^b_a$. The spaces defined by $a_k = k^{\alpha k}$, $b_k = \Gamma^{\beta l} \ (\alpha, \beta \geq 0)$ are most often used in applications, but even a wider framework is preferable for the problem under consideration because it allows more precisely characterizing that behavior.

It is reasonable to require $d_1 \neq 0$ and to construct the operator realization not only of the series (11) but also of all series $\sum_k d_k^j \phi^k(x)$ which are subordinate to the series (11) in the sense that $|d_k^j| \leq C|d_k|$, $1 \leq j \leq n$. Conversely, as is shown in [4], if all series of the form (12) are convergent, then (11) and all its subordinate series averaged with test functions in $S^b_a(\mathbb{R}^d)$ converge in the sense of a strong graph limit (see [8] for the definition), and their sums are well defined on a common dense invariant domain.

Reducing the products of Wick monomials to the totally normally ordered form by means of the Wick theorem and applying the Fock structure condition (9), we can express the scalar products in (13) in terms of the two-point function $w$ and its majorant $w_{\text{maj}}$. It is also convenient to pass in (13) to the summation over the multi-indices $K = \{k_{j,m}, 1 \leq j < m \leq 2n\}$ with nonnegative integer components which have the sense of the number of pairings between $\phi^k$: and $\phi^{k_m}$. As a result, the series (13) takes the form

$$\sum K D_K |W^K (\hat{f} \otimes f)|, \tag{14}$$

where $f = f_1 \otimes \ldots \otimes f_n$ and

$$W^K = \prod_{1 \leq j < m \leq n} w(x_m - x_j)^{k_{j,m}} \prod_{n+1 \leq j < m \leq 2n} w(x_j - x_m)^{k_{j,m}} \prod_{n+1 \leq j \leq n} w_{\text{maj}} (x_j, x_m)^{k_{j,m}},$$

$$D_K = \frac{k!}{K!} \prod_{1 \leq j \leq 2n} |d_{k_j}|, \quad k_j = k_{1.j} + \ldots + k_{j-1,j} + k_{j,j+1} + \ldots + k_{j,2n}.$$
From the results of Section 2, it follows that all distributions $W^K$ are the boundary values of analytic functions $W^K(z)$ which are holomorphic in the same tubular domain. For this reason, we can study the convergence of the series (14) by means of the following theorem proven in [1].

**Theorem 1.** Let $V$ be an open convex cone and let $(v_K)$ be a countable family of tempered distributions which are the boundary values of functions $v_K(z)$ holomorphic in the tubular domain $T^V = \{ z : \text{Im} \, z \in V \}$. If there exists a vector $\eta \in V$ such that

$$\sum_K \inf_{0 < t < \delta} e^{st} \int \frac{|v_K(x + it\eta)|}{a(|x|/A)} \, dx \leq C_{\delta, \epsilon, A} b(\epsilon s)$$

for every positive $\delta$, $\epsilon$ and $A$, then the family $(v_K)$ is unconditionally summable in the space $S_a^b$ which is dual of $S_a^b$.

We characterize the infrared and ultraviolet behavior of the majorant by a pair of monotone nonnegative functions $w_{\text{IR}}$ and $w_{\text{UV}}$ increasing as their arguments tend to infinity and to zero, respectively, and satisfying the bound

$$|w_{\text{maj}}(z, z')| \leq C_1 w_{\text{IR}}(|z| + |z'|) + C_2 w_{\text{UV}}(|y| + |y'|), \tag{15}$$

where $y = \text{Im} \, z$ and $y' = \text{Im} \, z'$ belong to the negative and positive $y_0$-semi-axes, respectively. On the coefficients $d_k$ of the series (1) we impose the restrictions

$$\lim_{k \to \infty} (k!)^{-1/k} |d_{2k}| = 0, \quad |d_{k+l}| \leq Ch^{k+l}|d_k||d_l|. \tag{16}$$

The first condition ensures that the Wightman functions of the sum of (1) are analytic in the usual domain of local QFT, whereas the second one permits passing from multiple series over the multi-index $K$ to one-tuple series over $|K| = \sum_{1 \leq j < m \leq 2n} k_{jm}$.

The following convergence criterion is formulated in terms of the characteristics $w_{\text{IR}}$ and $w_{\text{UV}}$ and therefore solves the convergence problem for Wick series in a model independent way.

**Theorem 2.** Let $\phi$ be a free field acting in the pseudo-Hilbert space $\mathcal{H}$, and let the positive majorant of its correlation function satisfy the inequality (15) in which $w_{\text{IR}}$ and $w_{\text{UV}}$ are monotone. Under the conditions (16) on the coefficients, the sums of the series (1) and all its subordinate series are well defined as operator-valued generalized functions on every space $S_a^b$ whose indicator functions satisfy the inequalities

$$\sum_k L^k k! |d_{2k}| w_{\text{IR}}(r)^k \leq C_{L, \epsilon} a(\epsilon r), \quad \inf_{t > 0} e^{st} \sum_k L^k k! |d_{2k}| w_{\text{UV}}(t)^k \leq C_{L, \epsilon} b(\epsilon s)$$

for arbitrarily large $L > 0$ and arbitrarily small $\epsilon > 0$.

The proof can be obtained by applying Theorem 1 to the series of distributions $\sum_K D_K W^K(x)$ because the unconditional convergence of this series in $S_a^b$ is equivalent to the convergence of (14) for any $f \in S_a^b$. The details of the proof can be found in [1]. Here we only note that because of (11), the bound (13) on the majorant proves to be sufficient.
for estimating $W^K(z)$, although the explicit expression for $W^K$ contains not only $w_{maj}$ but also the two-point function $w$.

**Remark.** Since $w_{maj}$ is a tempered distribution, its singularities are no worse than polynomial or logarithmic ones. Therefore, it can be assumed that the inequalities

$$w_{IR}(\lambda r) \leq C_\lambda w_{IR}(r), \quad w_{UV}(t/\lambda) \leq C'_\lambda w_{UV}(t),$$

hold for any $\lambda > 0$, at least in the limiting sense, i.e., for $r > R(\lambda)$ and $t < \delta(\lambda)$. In this case, the conditions specified in Theorem 2 take a simpler form

$$\sum_k L^k k! d_{2k} w_{IR}(r)^k \leq C L a(r), \quad \inf_{t>0} e^{st} \sum_k L^k k! d_{2k} w_{UV}(t)^k \leq C L b(s),$$

where $L > 0$ is arbitrarily large. In the most important case of the normal exponential $\exp \{i g \phi : (x)\}$ we arrive at

$$\exp \{L w_{IR}(r)\} \leq C L a(r), \quad \inf_{t>0} \exp \{st + L w_{UV}(t)\} \leq C L b(s).$$

### 4 Generalized Spectral Condition

Simple models treated in [1] show that the local and covariant formulation of gauge theories may require, for infrared reasons, the use of test function spaces consisting of functions analytic in $p$-space. The operator realization of the Wick exponential of the dipole ghost field is a typical example. In this case, the definitions (4) and (6) as well as the very notion of support of a generalized function break down. In [3, 5] a generalization of the spectral condition was proposed which is applicable to the case of arbitrarily high infrared singularity. In order to formulate this generalization, we need an alternative description of test function spaces in terms of complex variables. Namely, let us introduce the following definition.

**Definition 1.** Let $\alpha(s)$ and $\beta(s)$ be nonnegative continuous functions indefinitely increasing on the half-axis $s \geq 0$, let $\alpha(s)$ be convex and differentiable for $s > 0$, and let $\beta(s)$ be convex with respect to $\ln s$ and satisfy the condition $2 \beta(s) \leq \beta(hs)$ with a constant $h > 1$. We define $E^{a,b}_{\alpha,A}$ to be the inductive limit $\lim_{A,B \to 0} E_{\alpha,A}^{a,b}$, where the Banach spaces $E^{a,b}_{\alpha,A}$ consist of entire analytic functions on $\mathbb{C}$ with the finite norm

$$\|g\|_{A,B} = \sup_{p,q} |g(p + iq)| \exp \{-\alpha(A|q|) + \beta(|p|/B)\}.$$

From the listed restrictions on $\alpha$ and $\beta$, it follows that the spaces $E^{a,b}_{\alpha} \beta$ form a subclass of the spaces $S^b_\alpha$. Namely, $E^{a,b}_{\alpha}$ coincides with the space $S^b_\alpha$ defined by $a_k = \sup_{r \geq 0} r^k e^{-\alpha(r)}$ and $b_k = \sup_{s \geq 0} s^k e^{-\beta(s)}$, where $\alpha(r) = \sup_{s \geq 0} (rs - \alpha(s))$. What is more, those restrictions ensure that for the generalized functions defined on $E^{a,b}_{\alpha}$, the notion of a carrier cone can be introduced which replaces the usual notion of support, see [3] for proofs. Loosely speaking, a closed cone $K$ is called a carrier cone of a generalized function $u \in E^{a,b}_{\alpha}$ if $u$
decreases outside \( K \) like \( \exp\{-\alpha(|p|)\} \). In a more formal language, this means that \( u \) allows a continuous extension to a test function space whose elements grow outside \( K \) as \( \exp\{\alpha(|p|)\} \). To be precise, with any open cone \( U \) we associate the space \( \mathcal{E}_\beta^\alpha(U) \) which is defined exactly as \( \mathcal{E}_\beta^\alpha \) with the exception that the norm \( \|g\|_{A,B} \) is replaced by

\[
\|g\|_{U,A,B} = \sup_{p,q} |g(p + iq)| \exp\{-\alpha(A|q|) - \alpha \circ \delta_U(Ap) + \beta(|p|/B)}\),
\]

where \( \delta_U(p) \) is the distance from the point \( p \) to the cone \( U \).

**Definition 2.** A closed cone \( K \) is called a carrier cone of \( u \in \mathcal{E}_\beta^\alpha \) if \( u \) has a continuous extension to the space \( \mathcal{E}_\beta^\alpha(K) = \lim_{U \supset K \setminus \{0\}} \mathcal{E}_\beta^\alpha(U) \).

Once we have the definition of carrier cones, the desired generalization of the spectral condition becomes straightforward.

**Definition 3.** We say that a field \( \phi(x) \) defined on the test function space \( \mathcal{E}_\beta^\alpha \) satisfies the generalized spectral condition if the Fourier transforms of the vector–valued functionals \( \mathcal{F} \) are carried by the cones \( K_{n-} \) defined in \( \mathcal{F} \).

In \([3]\), we have proved the following theorem showing the relevance of this definition.

**Theorem 3.** Suppose the assumptions of Theorem 2 are fullfilled and furthermore the space \( S_\alpha^\beta \) belongs to the subclass of spaces described by Definition 1. Then the sums of the series \( \mathcal{F} \) and all its subordinate series satisfy the generalized spectral condition.

We expect that the proposed description of spectral properties may be useful for both the analysis of concrete models with singular infrared behavior and the self-consistent Euclidean formulation of general gauge QFT.

**Acknowledgements.** This work was supported in part by the Russian Foundation for Basic Research Grant No. 99-01-00376 and INTAS Grant No. 99-1-590 (A. G. S.), and in part by RFBR Grants No. 99-02-17916 (M. A. S.) and No. 00-15-96566.

**References**

[1] U. Moschella and F. Strocchi, *Lett. Math. Phys.* 24 (1992) 103.

[2] M. A. Soloviev, *Lett. Math. Phys.* 41 (1998) 265.

[3] M. A. Soloviev, *Theor. Math. Phys.* 105 (1995) 1520.

[4] A. G. Smirnov and M. A. Soloviev, *Theor. Math. Phys.* 123 (2000) 709, math-ph/00100001.

[5] A. G. Smirnov and M. A. Soloviev, *Theor. Math. Phys.* 125 (2000) 1349, math-ph/0101003.

[6] G. Morchio and F. Strocchi, *Ann. Inst. H. Poincaré A* 33 (1980) 251.

[7] N. N. Bogolyubov, A. A. Logunov, A. I. Oksak, I. T. Todorov, *General Principles of Quantum Field Theory*, Kluwer, Dordrecht, 1990.
[8] M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Vol. 1, Academic Press, New York–London, 1972.