DISCRIMINANTS OF MORPHISMS OF SHEAVES

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Abstract. The aim of this paper is to give a unified definition of a large class of discriminants arising in algebraic geometry using the discriminant of a morphism of locally free sheaves. The discriminant of a morphism of locally free sheaves has a geometric definition in terms of grassmannian bundles, tautological sequences and projections and is a simultaneous generalization of the discriminant of a morphism of schemes, the discriminant of a linear system on a smooth projective scheme and the classical discriminant of degree $d$ polynomials. We study the discriminant of a morphism in various situations: The discriminant of a finite morphism of schemes, the discriminant of a linear system on the projective line and the discriminant of a linear system on a flag variety. The main result of the paper is that the discriminant of any linear system on any flag variety is irreducible.

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1. Introduction

The aim of this paper is to give a unified definition of a large class of discriminants arising in algebraic geometry. We define the discriminant of an arbitrary morphism of locally free finite rank sheaves. This discriminant is defined for any morphism

$$\phi : u^* \mathcal{E} \to \mathcal{F}$$

of locally free finite rank $\mathcal{O}_X$-modules on an arbitrary scheme $X$ defined over an arbitrary base scheme $S$ where $u : X \to S$ is any quasi compact morphism of schemes. The discriminant of a morphism of locally free sheaves has a geometric definition in terms of grassmannian bundles, tautological sequences and projections. It is a simultaneous generalization of the discriminant of a morphism of schemes, the discriminant of a linear system on a smooth projective scheme and the classical discriminant of degree $d$ polynomials. We study this discriminant in the case of...
linear systems on projective spaces and flag varieties and use a previous result on
the $P$-module structure of the jet bundle (see [7]) to give a candidate for a resolution
of the ideal sheaf $I$ of the $k$'th discriminant $D^k(\mathcal{O}(d))$ of the linear system defined
by $\mathcal{O}(d)$ on $\mathbb{P}(V^*)$. We also study the discriminant of a linear system on any flag
variety $SL(E)/P$ and prove it is irreducible in general.

In section two we give the general definition of the discriminant of a morphism
of locally free sheaves. We prove the following general result: Assume we are given
a morphism $\phi : u^*\mathcal{E} \to \mathcal{F}$ of locally free sheaves on $X$ where $X$ is irreducible and
quasi compact over $S$ and $\text{Coker}(\phi^*)$ is locally free. It follows the discriminant
$D^1(\phi)$ is irreducible. This is Corollary 2.6. We also prove it is a simultaneous
generalization of the discriminant of a morphism of schemes and the discriminant
of a linear system on a smooth projective scheme (see Example 2.12 and 3.8). We
prove in Example 2.18 that the discriminant $D^1(\mathcal{O}(d))$ on the projective line is the
classical discriminant parametrizing degree $d$ polynomials with multiple roots.

In section three of the paper we consider the discriminant $\text{Discr}(P(t))$ of a poly-
nomial $P(t)$ in $A[t]$ where $A$ is any commutative ring. We relate the discriminant
to properties of the ring extension $A \subseteq A[t]/P(t)$ and give precise criteria for the
extension to be etale in the case when $P(t)$ is a monic polynomial.

In section four we study the discriminant $D^l(\mathcal{O}(d))$ of a line bundle $\mathcal{O}(d)$ on
$\mathbb{P}^1_K$ where $K$ is an arbitrary field. Using the Taylor morphism, jet bundles and
projections we prove in Theorem 4.5 that $D^l(\mathcal{O}(d))$ is an irreducible local complete
intersection for all $1 \leq l \leq d$.

In section five of the paper we prove some general results on jet bundles on
projective space and higher cohomology groups of exterior powers of jet bundles
on projective space. We give a complete description (see Theorem 5.2) of the
$SL(V)$-module structure of all higher cohomology groups of all exterior powers of
jet bundles and dual jet bundles on projective space. We also calculate the higher
direct images of a class of twisted jet bundles with an $SL(V)$-linearization (see
Theorem 5.10). We also prove that any $G$-module $W$ may be realized as the global
sections of a $G$-linearized locally free $\mathcal{O}_{G/P}$-module $\mathcal{E}(\rho)$ (see Proposition 5.4). We
study the discriminant of a linear system on projective space and prove existence
of a complex of locally free sheaves which is a candidate for a resolution of the ideal
sheaf of the discriminant (see Example 5.11). We finally consider discriminants of
linear systems on flag varieties (see Example 5.12) and prove in Theorem 5.13 all
such discriminants are irreducible.

Much research has been devoted to the study of discriminants and syzygies of
discriminants (see [3] and [18]). The novelty of the approach in this paper is the
introduction of a functorial discriminant valid for a map of locally free sheaves
relative to a quasi compact family of schemes. This gives a unified definition of
a large class of discriminants appearing in algebraic geometry. All definitions are
intrinsic and all discriminants have a canonical scheme structure.

2. Discriminants of morphisms of sheaves

In this section we introduce the discriminant of an arbitrary morphism of locally
free sheaves on an arbitrary scheme. We prove the discriminant of a morphism of
locally free sheaves is a simultaneous generalization of the discriminant of a mor-
phism of schemes and the discriminant of a linear system on a smooth projective
scheme.
Let in the following $u : X \to S$ be an arbitrary quasi compact morphism of schemes and let $E$ be a locally free finite rank $O_S$-module and $F$ a locally free finite rank $O_X$-module. Assume
\[
\phi : u^*E \to F
\]
is an arbitrary morphism of $O_X$-modules. Let $G_n(u^*E^*)$ be the grassmannian bundle of the locally free sheaf $u^*E$. The grassmannian has the following properties:

- There is a projection morphism $\tilde{\pi} : G_n(u^*E^*) \to X$ with the following properties: Let $Y = G_n(u^*E^*)$. There is an isomorphism $Y \cong G_n(E^*) \times_S X$ giving a commutative diagram

\[
\begin{array}{ccc}
Y & \longrightarrow & X \\
\downarrow q & & \downarrow u \\
G_n(E^*) & \longrightarrow & S.
\end{array}
\]

Here $p = \tilde{\pi}$ is the projection morphism.

- There is on $G_n(E^*)$ a tautological sequence

\[
(2.0.1) \quad 0 \to S \to \pi^*E \to Q \to 0
\]
of locally free sheaves with $rk(S) = n$. The locally free sheaf $S$ is the tautological subbundle on $G_n(E^*)$. The sequence $2.0.1$ reflects the fact that the grassmannian $G_n(E^*)$ is the scheme representing the grassmannian functor $\text{Grass}_n(E^*)$: Via the Yoneda Lemma it follows the grassmannian functor $\text{Grass}_n(E^*)$ is represented by a scheme $G_n(E^*)$ and a universal object. The universal object is given by the subbundle

\[
0 \to S \to \pi^*E.
\]

**Example 2.1.** The tautological line bundle.

In the case when $n = 1$ it follows $S = \mathcal{O}(-1)$ and we get the sequence of the tautological sub-bundle

\[
0 \to \mathcal{O}(-1) \to \pi^*E
\]
on $\mathbb{P}(E^*)$.

**Proposition 2.2.** Let $u : X \to S$ be a quasi compact morphism of schemes and let $E$ be a locally free $O_S$-module of rank $m$. It follows

\[
\pi : G_n(E^*) \to S
\]

and

\[
q : G_n(u^*E^*) \to G_n(E^*)
\]
are quasi compact morphisms.

**Proof.** Since $u$ is quasi compact it follows for any open affine subscheme $V = \text{Spec}(A) \subseteq S$ the inverse image $U = u^{-1}(V) \subseteq X$ is a finite union of open affine schemes:

\[
U = \text{Spec}(B_1) \cup \cdots \cup \text{Spec}(B_k).
\]

This is a general fact: A scheme over an affine scheme is quasi compact if and only if it is a finite union of open affine sub schemes. Let $\pi_V : V \to \text{Spec}(\mathbb{Z})$ and $\pi_U : \text{Spec}(A) \to S$.
Let \( U \to \Spec(Z) \) be the structure morphisms. Assume \( E|_V = A\{e_1, \ldots, e_m\} = \pi_v^* W \) where

\[
W = Z\{e_1, \ldots, e_m\}.
\]

Let

\[
W^* = Z\{x_1, \ldots, x_m\}.
\]

It follows

\[
\pi^{-1}(V) = G_n(\mathcal{E}^*|_V) \cong G_n(\pi_v^* W^*) \cong G_n(W^*) \times_Z \Spec(A).
\]

Since \( G_n(W^*) = \bigcup_{i=1}^t \Spec(A_i) \) is a finite union of affine open schemes \( \Spec(A_i) \) it follows

\[
\pi^{-1}(V) = G_n(W^*) \times_Z \Spec(A) = \bigcup_{i=1}^t \Spec(A_i) \times_Z \Spec(A) \cong \bigcup_{i=1}^t \Spec(A_i \otimes_Z A)
\]

is a finite union of affine open schemes. It follows \( \pi \) is a quasi compact morphism of schemes.

Pick the open set

\[
\Spec(A_i \otimes_Z A) \subseteq \pi^{-1}(V) = G_n(W^*) \times_Z \Spec(A).
\]

It follows

\[
q^{-1}(\Spec(A_i \otimes_Z A)) = \Spec(A_i) \times_V u^{-1}(V) = \bigcup_{j=1}^k \Spec(A_i) \times_V \Spec(B_j) = \bigcup_{j=1}^k \Spec(A_i \otimes_Z B_j).
\]

Hence

\[
q^{-1}(\Spec(A_i \otimes_Z A))
\]

is a finite union of open affine schemes. The open sets \( \Spec(A_i \otimes_Z A) \) cover \( G_n(W^*) \times_Z \Spec(A) \) hence it follows \( q \) is a quasi compact morphism, and the Proposition is proved.

Note: For any quasi compact morphism \( u : X \to S \) of schemes and any closed subscheme \( Z \subseteq X \) we get an induced morphism \( v : Z \to S \) which is quasi compact. The map of structure sheaves

\[
v^\#: \mathcal{O}_X \to v_* \mathcal{O}_Z
\]

gives rise to an ideal sheaf

\[
\mathcal{I} = \ker(v^\#) \subseteq \mathcal{O}_S.
\]

The ideal sheaf \( \mathcal{I} \subseteq \mathcal{O}_S \) corresponds to a subscheme \( u(Z) \subseteq S \): the schematic image of \( Z \) via \( u \). Hence if \( v : Z \to S \) is a closed morphism it follows we get a canonical structure of closed subscheme on the schematic image \( u(Z) \subseteq S \). This structure is not necessarily the reduced induced structure on the topological space \( u(Z) \) viewed as a closed subspace of \( S \).

The sequence \([2.0.1]\) has the following property: Assume \( s \in \pi^{-1}(z) \) is a \( \kappa(z) \)-rational point. If we take the fiber of \([2.0.1]\) at \( s \) we get an exact sequence of \( \kappa(z) \)-vector spaces

\[
0 \to \mathcal{S}(s) \to \pi^* \mathcal{E}(s) \to Q(s) \to 0
\]

and \( \pi^* \mathcal{E}(s) = \mathcal{E}(\pi(s)) \otimes_{\kappa(\pi(s))} \kappa(s) \cong \mathcal{E}(z) \). We get a canonical \( n \)-dimensional \( \kappa(z) \) sub vector space

\[
S(s) \subseteq \mathcal{E}(z)
\]

for each \( s \in \pi^{-1}(z)(\kappa(z)) \). By functoriality the following holds:

\[
\pi^{-1}(z) \cong G_n(\mathcal{E}(z)^\ast)
\]
hence the fiber $\pi^{-1}(z)$ is canonically isomorphic to the Grassmannian parametrizing $n$-planes in the $\kappa(z)$-vector space $E(z)$. The tautological sequence \text{2.0.1} gives for each $\kappa(z)$-rational point in $G_n(E(z)^\ast)$ its corresponding $n$-plane in $E(z)$. We get a one to one correspondence

$$\{\kappa(z) - \text{rational points } s \in G_n(E(z)^\ast)\} \cong \{n - \text{planes } W \subseteq E(z)\}$$

given by

$$s \in G_n(E(z)^\ast) \mapsto S(s) \subseteq E(z).$$

We will use the tautological sequence to define the discriminant of a morphism of locally free sheaves. Let $Y = G_n(E^\ast) \times_S X$. Let $S_Y = q^* S$, $E_Y = q^* \pi^* E = p^* u^* E$ and $F_Y = p^* F$. We get a morphism

$$S_Y \rightarrow E_Y \rightarrow p^* \phi^* F_Y.$$

Let $\tilde{\phi}$ be the composed morphism

$$\tilde{\phi} : S_Y \rightarrow F_Y.$$

Let $Z(\tilde{\phi})$ be the zero scheme of the morphism $\tilde{\phi}$. If $U = \text{Spec}(A)$ is a trivialization of $S_Y$ and $F_Y$ and $\tilde{\phi} = (a_{ij})$ with $a_{ij} \in A$ it follows the ideal of $Z(\tilde{\phi})$ is generated by $a_{ij}$ on the open set $U$.

Since $\pi$ and $q$ are quasi compact morphisms we may define the following:

**Definition 2.3.** Let $I^n(\phi) = Z(\tilde{\phi})$ be the $n$-incidence scheme of $\phi$. The scheme $D^n(\phi) = q(I(\phi))$ is the $n$-discriminant of $\phi$. The scheme $\text{Discr}^n(\phi) = \pi(D^n(\phi))$ is the direct image $n$-discriminant of the morphism $\phi$.

We get a diagram

$$\begin{array}{ccc}
I^n(\phi) & \xrightarrow{i} & G_n(E^\ast) \times_S X & \xrightarrow{p} & X \\
\downarrow q & & \downarrow p & & \downarrow u \\
D^n(\phi) & \xrightarrow{j} & G_n(E^\ast) & \xrightarrow{\pi} & S \\
\end{array}$$

where $i$, $j$ and $k$ are inclusions of schemes.

Assume $\psi : E \rightarrow F$ is a map of locally free $O_X$-modules.

**Lemma 2.4.** The following holds: $x \in Z(\psi)$ if and only if $\psi(x) = 0$.

**Proof.** Let $U = \text{Spec}(A) \subseteq X$ be an open subset where $E$ and $F$ trivialize. It follows $E$ is the sheafification of $A^m$ and $F$ the sheafification of $A^n$ for some integers $m, n \geq 1$. Let $x \in \text{Spec}(A)$ correspond to a prime ideal $p_x \subseteq A$. It follows $x \in Z(\psi)$ if and only of $a_{ij} \in p_x$ for all $i, j$ where $(a_{ij}) = \psi|_U$ and $a_{ij} \in A$ are the coefficients of $\psi$ over $U$. It follows $x \in Z(\psi)$ if and only if the fiber map $\psi(x)$ is zero and the Lemma is proved. \hfill $\square$

Let $\phi^* : F^* \rightarrow u^* E^*$ be the dual of $\phi$ and consider the exact sequence

\begin{equation}
\begin{array}{ccc}
F^* & \rightarrow & \phi^* u^* E^* \rightarrow \text{Coker}(\phi^*) \rightarrow 0 \\
\end{array}
\end{equation}
of coherent $\mathcal{O}_X$-modules. Assume $\text{Coker}(\phi^*)$ is locally free and let $n = 1$. We get a closed immersion

$$\mathbb{P}(\text{Coker}(\phi^*)) \subseteq \mathbb{P}(u^*\mathcal{E}) \cong \mathbb{P}(\mathcal{E}^*) \times_S X$$

of schemes.

**Theorem 2.5.** There is an equality $I^1(\phi) = \mathbb{P}(\text{Coker}(\phi^*))$ as subschemes of $\mathbb{P}(u^*\mathcal{E}^*)$.

**Proof.** Let $Y = \mathbb{P}(u^*\mathcal{E}^*) = \mathbb{P}(\mathcal{E}^*) \times_S X$ and let $\mathcal{I}, \mathcal{J} \subseteq \mathcal{O}_Y$ be the ideal sheaves of $I^1(\phi)$ and $\mathbb{P}(u^*\mathcal{E})$. We want to prove there is an equality $\mathcal{I} = \mathcal{J}$ of ideal sheaves. Consider the diagram

$$\begin{array}{ccc}
Y & \xrightarrow{p} & X \\
\downarrow{q} & & \downarrow{u} \\
\mathbb{P}(\mathcal{E}^*) & \xrightarrow{\pi} & S
\end{array}$$

Let

$$\alpha : \mathcal{O}(-1) \to \pi^*\mathcal{E}$$

be the tautological sub-bundle on $\mathbb{P}(\mathcal{E}^*)$. Pull this and the morphism $\phi : u^*\mathcal{E} \to \mathcal{F}$ back to $Y$ to get the morphism

$$\tilde{\phi} : \mathcal{O}(-1)_Y \to \mathcal{E}_Y \to \mathcal{F}_Y.$$  

By definition $Z(\tilde{\phi}) = I^1(\phi)$. We want to show $Z(\tilde{\phi}) = \mathbb{P}(\text{Coker}(\phi^*))$ is an equality of schemes. We prove there is an equality of ideal sheaves. Assume $V = \text{Spec}(A) \subseteq S$ is an affine open subscheme where $\mathcal{E}$ trivialize. Let $\pi_V : \text{Spec}(A) \to \text{Spec}(\mathcal{Z})$ be the structure morphism and let

$$W = \mathcal{Z}\{e_0, ..., e_l\}.$$  

It follows

$$E|_V = \pi_V^*W = A \otimes_{\mathcal{Z}} \mathcal{Z}\{e_0, ..., e_l\}.$$  

Let $U = \text{Spec}(B) \subseteq u^{-1}(V)$ be an open set where $\mathcal{F}$ trivialize and let $\pi_U : \text{Spec}(B) \to \text{Spec}(\mathcal{Z})$ be the structure morphism. Let $Z = \mathcal{Z}\{f_0, ..., f_m\}$. It follows

$$\mathcal{F}|_U = \pi_U^*Z = B \otimes_{\mathcal{Z}} \mathcal{Z}\{f_0, ..., f_m\}.$$  

Let $x_i = e_i^*$ and $y_j = f_j^*$. Pull $\mathcal{E}$ back to $U$ to get

$$u^*\mathcal{E}|_U = B \otimes_{\mathcal{Z}} W = B \otimes_{\mathcal{Z}} \mathcal{Z}\{e_0, ..., e_l\} = B\{e_0, ..., e_l\}.$$  

Restrict the morphism $\phi$ to $U$ to get

$$\phi|_U : B\{e_0, ..., e_l\} \to B\{f_0, ..., f_m\}$$

with $\phi|_U = (b_{ij})$ with $b_{ij} \in B$. Consider the morphism

$$p : \mathbb{P}(u^*\mathcal{E}^*) \to X.$$  

It follows

$$p^{-1}(U) = \mathbb{P}(u^*\mathcal{E}|_U) = \mathbb{P}(W^*) \times_{\mathcal{Z}} \text{Spec}(B).$$

Hence $p^{-1}(U)$ may be covered by open affine schemes on the form

$$U_i = D(x_i) \times_{\mathcal{Z}} \text{Spec}(B) = \text{Spec}(B[x_0/x_i, ... , x_l/x_i])$$
Lemma 2.7. If induces an inclusion canonical inclusion It follows the composed map is the zero map. Hence a claim of the Corollary follows. □

Proof. Assume is a point with residue field The matrix of We want to calculate generators for the image sequence for with The matrix of is irreducible. It follows is irreducible and the ideal sheaf is generated by the elements on the open set . Consider the exact sequence

\[ \mathcal{F}^* \to \phi^* u^* \mathcal{E}^* \to \text{Coker}(\phi^*) \to 0. \]

We want to calculate generators for the image \( \text{Im}(\phi^*) \subseteq u^* \mathcal{E}^* \) on the open set \( U_i \). The matrix of \( \phi^*|_{U_i} \) is the transpose of the matrix \( \phi|_{U_i} \) and one checks that on \( U_i \) the following holds:

\[ \phi^*|_{U_i}(yk) = b_{k,0}t_0 + b_{k,1}t_1 + \cdots b_{k,i} + \cdots + b_{k,t_i} = c_{k,i}. \]

It follows \( \text{Im}(\phi^*|_{U_i}) \) is generated by the elements \( c_{k,i} \) hence the ideal sheaf \( \mathcal{I} \) of \( \mathbb{P}(\text{Coker}(\phi^*)) \) is on \( U_i \) generated by \( c_{k,i} \). It follows \( \mathcal{I} = \mathcal{J} \) and the claim of the Theorem follows.

Corollary 2.6. If \( X \) is irreducible and \( \text{Coker}(\phi^*) \) is locally free it follows \( D^1(\phi) \) is irreducible.

Proof. Since \( X \) is irreducible and \( I^1(\phi) = \mathbb{P}(\text{Coker}(\phi^*)) \) is a projective bundle on \( X \) it follows \( I^1(\phi) \) is irreducible. It follows \( D^1(\phi) = q(I^1(\phi)) \) is irreducible and the claim of the Corollary follows.

We give an interpretation of \( I^n(\phi), D^n(\phi) \) and \( \text{Discre}^n(\phi) \) in terms of points. Assume \( z \in S \) is a point with residue field \( \kappa(z) \). Assume \( y = (s, x) \in Y \) with \( u(x) = \pi(s) = z \). It follows \( \kappa(y) = \kappa(s) = \kappa(x) = \kappa(z) \). By Lemma 2.4 it follows \( y \in I^n(\phi) \) if and only if \( \phi_Y(y) = 0 \). We get

\[ \mathcal{S}_Y(y) = \mathcal{S}(s) \otimes_{\kappa(s)} \kappa(y) \cong \mathcal{S}(s) \]

and

\[ \mathcal{E}_Y(y) = \mathcal{E}(z) \otimes_{\kappa(z)} \kappa(y) \cong \mathcal{E}(z) \]

It follows the composed map

\[ \tilde{\phi}(y) : \mathcal{S}(s) \to \mathcal{E}(z) \to \mathcal{F}(x) \]

is the zero map. Hence a \( \kappa(z) \)-rational point \( y = (s, x) \) is in \( I^n(\phi) \) if and only if the canonical inclusion

\[ \mathcal{S}(s) \subseteq \mathcal{E}(z) \]

induces an inclusion

\[ \mathcal{S}(s) \subseteq \text{Ker}(\phi(x)). \]

Lemma 2.7. If \( y = (s, x) \in I^n(\phi)(\kappa(z)) \) it follows \( \dim_{\kappa(z)} \text{Ker}(\phi(x)) \geq n \).
Proof. By the discussion above there is an inclusion
\[ S(s) \subseteq Ker(\phi(x)) \]
and since \( \dim_{K(x)} S(s) = n \) the claim of the lemma is proved. \( \square \)

Hence \( s \in D^n(\phi) \) with \( \pi(s) = z \) if and only if there is a point \( x \in X \) with \( u(x) = \pi(s) \) such that the canonical inclusion
\[ S(s) \subseteq \mathcal{E}(z) \]
induce an inclusion
\[ S(s) \subseteq Ker(\phi(x)). \]

Example 2.8. On the case \( n = 1 \):

Consider the tautological subbundle on \( \mathbb{P}(\mathcal{E}^*) = \mathbb{G}_1(\mathcal{E}^*) \)
\[ 0 \to \mathcal{O}(-1) \to \pi^* \mathcal{E}. \]
Pull this sequence back to \( Y = \mathbb{P}(\mathcal{E}^*) \times X \) to get a sequence
\[ \mathcal{O}(-1)_Y \to \mathcal{E}_Y. \]
Similarly pull back the sequence \( \phi : u^* \mathcal{E} \to \mathcal{F} \) to \( Y \) to get the sequence
\[ \tilde{\phi} : \mathcal{O}(-1)_Y \to \mathcal{E}_Y \to \mathcal{F}_Y. \]

Since
\[ \tilde{\phi} \in \text{Hom}(\mathcal{O}(-1)_Y, \mathcal{F}_Y) \cong \text{Hom}(\mathcal{O}, \mathcal{O}(1)_Y \otimes \mathcal{F}_Y) = H^0(Y, \mathcal{O}(1)_Y \otimes \mathcal{F}_Y) \]
It follows we have described the 1-incidence scheme \( I^1(\phi_Y) \) as the zero locus of \( \tilde{\phi} \)
viewed as a global section of the locally free sheaf \( \mathcal{O}(1)_Y \otimes \mathcal{F}_Y. \)

Lemma 2.9. There is an exact sequence on \( Y \)
\[ \mathcal{O}(-1)_Y \otimes \mathcal{F}_Y^* \to \mathcal{O}_Y \to \mathcal{O}_{Z(\tilde{\phi})} \to 0. \]

Proof. The proof is left to the reader as an exercise. \( \square \)

Consider for \( j \geq 1 \) the \( j \)'th exterior product
\[ \wedge^j(\mathcal{O}(-1)_Y \otimes \mathcal{F}_Y^*) \cong \mathcal{O}(-j)_Y \otimes \wedge^j \mathcal{F}_Y^*. \]
The ideal sheaf \( \mathcal{I} \) of \( Z(\tilde{\phi}) \) is locally generated by a regular sequence hence from [1] we get a Koszul-resolution
\[ 0 \to \mathcal{O}(-r)_Y \otimes \wedge^r \mathcal{F}_Y^* \to \mathcal{O}(-r+1)_Y \otimes \wedge^{r-1} \mathcal{F}_Y^* \to \cdots \]
\[ \cdots \to \mathcal{O}(-j)_Y \otimes \wedge^j \mathcal{F}_Y^* \to \cdots \]
\[ \cdots \mathcal{O}(-2)_Y \otimes \wedge^2 \mathcal{F}_Y^* \to \mathcal{O}(-1)_Y \otimes \mathcal{F}_Y^* \to \mathcal{O}_Y \to \mathcal{O}_{Z(\tilde{\phi})} \to 0 \]
of the structure sheaf \( \mathcal{O}_{Z(\tilde{\phi})} \) of the incidence scheme \( Z(\tilde{\phi}) \). Here \( r = rk(\mathcal{F}^*). \)

Proposition 2.10. There is an isomorphism of \( \mathcal{O}_{\mathbb{P}(\mathcal{E}^*)} \)-modules
\[ R^iq_*(\mathcal{O}(-j)_Y \otimes \wedge^j \mathcal{F}_Y^*) \cong \mathcal{O}(-j) \otimes \pi^* R^i u_*(\wedge^j \mathcal{F}^*). \]

Proof. By the projection formula and base change we get
\[ R^iq_*(\mathcal{O}(-j)_Y \otimes \wedge^j \mathcal{F}_Y^*) \cong R^iq_*(q^* \mathcal{O}(-j) \otimes \wedge^j q^* \mathcal{F}^*) \cong \mathcal{O}(-j) \otimes R^iq_*(p^* \wedge^j \mathcal{F}^*) \]
and the claim of the Proposition is proved. \( \square \)
We get a double complex of $\mathcal{O}_{\mathcal{P}(\mathcal{E})}$-modules on $\mathbb{P}(\mathcal{E}^*)$ defined by

$$\mathcal{C}(\phi)^{i,j} = \mathcal{O}(-j) \otimes \pi^* \mathcal{R}^i u_*(\wedge^j \mathcal{F}^*).$$

A natural question to ask is if the total complex

$$\text{Tot}(\mathcal{C}(\phi)^{i,j})_n = \oplus_{i+j=n} \mathcal{O}(-j) \otimes \pi^* \mathcal{R}^i u_*(\wedge^j \mathcal{F}^*)$$

may be used to construct a resolution of the ideal sheaf of $D(\phi) \subseteq \mathbb{P}(\mathcal{E}^*)$.

**Definition 2.11.** Let the total complex $\text{Tot}(\mathcal{C}(\phi)^{i,j})_n$ be the discriminant complex of $\phi$.

**Example 2.12.** Discriminants of morphisms of schemes.

Assume $f : U \to V$ is a quasi compact map of smooth schemes of finite type over a field $F$. We get a map of sheaves of differentials

$$(2.12.1) \quad df : f^* \Omega^1_V \to \Omega^1_U.$$

Since $U, V$ are smooth over $F$ it follows $\Omega^1_V, \Omega^1_U$ are locally free sheaves of finite rank. We use the cotangent sequence [2.12.1] to give a set theoretic definition of the discriminant of the morphism $f$.

**Definition 2.13.** Let $\operatorname{Discr}^3(f) \subseteq V$ be the set of points $s \in V$ such that there is an $x \in U$ with $f(x) = s$ and $\operatorname{Ker}(df(x)) \neq 0$. We say $\operatorname{Discr}^3(f)$ is the set theoretic discriminant of the morphism $f$.

Let $I^n(df)$ be the $n$-incidence scheme of $df$. We get a diagram of maps of schemes where by Proposition 2.12 $q$ and $\pi$ are quasi compact morphisms:

$$\begin{array}{ccc}
I^n(df) & \xrightarrow{i} & \mathbb{G}_n((\Omega^1_U)^*) \times U \\
\downarrow q & & \downarrow f \\
\mathbb{G}_n((\Omega^1_U)^*) & \xrightarrow{\pi} & V \\
\downarrow \bar{q} & & \\
D^n(df) & \rightarrow & \operatorname{Discr}^n(df)
\end{array}$$

**Definition 2.14.** The scheme $D^n(df)$ is the $n$-discriminant of the morphism $f$. The scheme $\operatorname{Discr}^n(df)$ is the direct image $n$-discriminant of the morphism $f$.

Pick $y = (d, s) \in I^n(df)$. Assume $\pi(d) = f(s) = z$ and $\kappa(y) = \kappa(d) = \kappa(s) = \kappa(z)$. Since $d \in \pi^{-1}(z) = \mathbb{G}_n(\Omega^1_V(z)^*)$ the tautological sequence on the grassmanian gives a canonical $n$-dimensional vector subspace

$$\mathcal{S}(d) \subseteq \Omega^1_V(z).$$

We get a composed map

$$\mathcal{S}(d) \to \Omega^1_V(z) \to df(s) \Omega^1_V(s)$$

and since the composed map is the zero map it follows

$$\mathcal{S}(d) \subseteq \operatorname{Ker}(df(s)).$$

It follows $\dim_{\kappa(z)} \operatorname{Ker}(df(s)) \geq n$ hence $df(s)$ is not injective at $s$.

It follows the underlying set of points of $\operatorname{Discr}^n(df)$ is the set of $s = f(x) \in V$ with $x \in U$ and $\dim(\operatorname{Ker}(df(x))) \geq n$. We see the discriminant of a morphism
of locally free sheaves generalize the set theoretic discriminant of a morphism of schemes in the sense that the underlying set of points of \( D^1(df) \) is a lifting of \( \text{Discr}^1(f) \subseteq V \) to the projectivization of the cotangent bundle \( \Omega^1_U \). We get a sequence of subschemes

\[
\cdots \subseteq \text{Discr}^n(df) \subseteq \cdots \subseteq \text{Discr}^1(df) \subseteq V.
\]

In this case the total complex \( 2.10.1 \) becomes

\[
\text{Tot}(\mathcal{C}(f)^{i,j})_n = \bigoplus_{i+j=n} \mathcal{O}(-j) \otimes \pi^* f_* (\wedge^j (\Omega^1_U)^*)^*).
\]

**Example 2.15.** Discriminants of linear systems on projective schemes.

Let \( X \subseteq \mathbb{P}^d_F \) be a smooth projective scheme over a field \( F \) and let \( \mathcal{L} \in \text{Pic}(X) \). Let \( u : X \to \text{Spec}(F) \) be the structure morphism and let \( W = H^0(X, \mathcal{L}) \). There is a morphism of locally free sheaves

\[
u^* W \to T^k \mathcal{J}^k_X(\mathcal{L}).\]

Here \( T^k \) is the \( k \)th Taylor map and \( \mathcal{J}^k_X(\mathcal{L}) \) is the \( k \)th jet bundle of \( \mathcal{L} \). We get a diagram of maps of schemes

\[
\begin{array}{ccc}
T^1(T^k) & \xrightarrow{i} & \mathbb{P}(W^*) \times X \\
\downarrow & & \downarrow p \\
D^1(T^k) & \xrightarrow{j} & \mathbb{P}(W^*)
\end{array}
\]

\[
\begin{array}{ccc}
\mathbb{P}(W^*) \times X & \xrightarrow{u} & X \\
\downarrow q & & \downarrow u \\
\mathbb{P}(W^*) & \xrightarrow{\pi} & \text{Spec}(F)
\end{array}
\]

**Definition 2.16.** The scheme \( D^k(\mathcal{L}) = D^1(T^k) \) is the \( k \)th discriminant of the linear system defined by \( \mathcal{L} \).

We see the discriminant of a map of locally free sheaves generalize the discriminant of a linear system on a smooth projective scheme. The classical discriminant \( D^k(\mathcal{L}) \) equals the 1-discriminant \( D^1(T^k) \) of the \( k \)th Taylor morphism.

In this case the total complex \( 2.10.1 \) becomes

\[
\text{Tot}(\mathcal{C}(T^k)^{i,j})_n = \bigoplus_{i+j=n} \mathcal{O}(-j) \otimes \pi^* H^i(X, \wedge^j \mathcal{J}^k(\mathcal{L})^*)^*).
\]

To study the total complex for discriminants of linear systems we need information on the higher cohomology of exterior powers of duals of jet bundles. In the next section we will study higher cohomology groups of exterior powers of \( SL(V) \)-linearized jet bundles and the total complex in the situation where \( X = \mathbb{P}(V^*) \) and \( \mathcal{L} = \mathcal{O}(d) \) for \( d \geq 1 \).

Note: The definition given in 2.16 was communicated to the author by D. Laksov.

**Example 2.17.** Invertible sheaves on projective space.

We interpret the rational points of \( D^k(\mathcal{O}(d)) \) on \( \mathbb{P}(V^*) \) where \( V \) is an \( N+1 \)-dimensional vector space over any field \( F \). Let \( W = H^0(\mathbb{P}(V^*), \mathcal{O}(d)) \). Consider the diagram

\[
\begin{array}{ccc}
\mathbb{P}(W^*) \times \mathbb{P}(V^*) & \xrightarrow{p} & \mathbb{P}(V^*) \\
\downarrow q & & \downarrow \pi \\
\mathbb{P}(W^*) & \xrightarrow{\pi} & \text{Spec}(F)
\end{array}
\]
Let \( Y = \mathbb{P}(W^*) \times \mathbb{P}(V^*) \) and consider the following sequence of locally free \( \mathcal{O}_Y \)-modules
\[
\phi_Y : \mathcal{O}(-1)_Y \rightarrow W \otimes \mathcal{O}_Y \rightarrow \mathcal{J}^k(\mathcal{O}(d))_Y.
\]
It follows the \( F \)-rational points of the incidence scheme \( Z(\phi_Y) \) have the following interpretation: Pick an \( F \)-rational point \( x = (s, y) \in Y \) \( \kappa(x) = \kappa(s, y) = F \). It follows there is an equality of residue fields \( \kappa(s) = \kappa(y) = \kappa(s, y) = F \). The point \( x \) is by Lemma 2.4 in \( Z(\phi_Y) \) if and only if \( \phi_Y(x) = 0 \). We interpret this equation in terms of fibers: We get
\[
\mathcal{O}(-1)_Y(s, y) \rightarrow W \otimes \mathcal{O}_Y(s, y) \rightarrow \mathcal{J}^k(\mathcal{O}(d))_Y(s, y)
\]
which becomes
\[
\mathcal{O}(-1)(s) \otimes_{\kappa(s)} \kappa(s, y) \rightarrow W \rightarrow \mathcal{J}^k(\mathcal{O}(d))(y) \otimes_{\kappa(y)} \kappa(s, y)
\]
which becomes
\[
\mathcal{O}(-1)(s) \rightarrow W \rightarrow \mathcal{T}^k(y) \mathcal{J}^k(\mathcal{O}(d))(y).
\]
Write \( \tilde{s} = \mathcal{O}(-1)(s) \subseteq W \). It follows
\[
\mathcal{T}^k(y)(\tilde{s}) = 0 \text{ in } \mathcal{J}^k(\mathcal{O}(d))(y).
\]
Hence an \( F \)-rational \( (s, y) \in \mathbb{P}(W^*) \times \mathbb{P}(V^*) \) is in \( Z(\phi_Y) \) if and only if its corresponding section \( \tilde{s} \subseteq W \) satisfies
\[
\mathcal{T}^k(y)(\tilde{s}) = 0.
\]
Hence the points \( s \in D^k(\mathcal{O}(d))(F) \) are described in terms of the Taylor map \( T^k \) at some point \( y \in \mathbb{P}(V^*)(F) \). In local coordinates the Taylor map \( T^k(y) \) formally takes the form: \( \tilde{s} \in W \).

**Example 2.18. Linear systems on the projective line.**

Assume \( \mathbb{P}(V^*) = \mathbb{P}^1 \) is the projective line where \( V = F\{e_0, e_1\} \) and \( V^* = F\{x_0, x_1\} \). Let \( W = \mathbb{H}^0(\mathbb{P}^1, \mathcal{O}(d)) \). The Taylor map
\[
T^1 : W \otimes \mathcal{O}_{\mathbb{P}^1} \rightarrow \mathcal{J}^1(\mathcal{O}(d))
\]
is defined as follows: Let \( s_i = x_0^{d-i}x_1^i \) for \( i = 0, \ldots, d \) be the global sections of \( \mathcal{O}(d) \). Let \( y_i = s_i^*, t = \frac{1}{y_i} \) and \( \frac{1}{t} = \frac{d}{x_1} \). Let \( U_{ij} = D(y_i) \times D(x_j) \) be an open cover of \( \mathbb{P}(W^*) \times \mathbb{P}^1 \). Let \( u_j = y_i \). On \( D(x_0) \) we get the following
\[
T^1 : F[t]\{s_i\} \rightarrow F[t]\{1 \otimes x_0^d, dt \otimes x_0^d\}
\]
with
\[
T^1(s_i) = T^1(x_0^{d-i}x_1^i) = t^i \otimes x_0^d + it^{i-1}dt \otimes x_0^d.
\]
On \( D(x_1) \) we get
\[
T^1 : F[1/t]\{s_i\} \rightarrow F[1/t]\{1 \otimes x_1^d, d(1/t) \otimes x_1^d\}
\]
with
\[
T^1(s_i) = T^1(x_0^{d-i}x_1^i) = (1/t)^i \otimes x_1^d + i(1/t)^{i-1}dt \otimes x_1^d.
\]
Consider the following map on \( Y = \mathbb{P}(W^*) \times \mathbb{P}^1 \)
\[
(2.18.1) \quad \mathcal{O}(-1)_Y \rightarrow \alpha W_Y \rightarrow \mathcal{T}^1(\mathcal{O}(d))_Y
\]
and restrict to \( U_{i0} = D(y_i) \times D(x_0) \). We get
\[
\alpha|_{U_{i0}} : F[u_j^*, t] \frac{1}{y_i} \rightarrow F[u_j^*, t] \otimes_F F\{s_0, \ldots, s_d\} \rightarrow \mathcal{T}^1 F[u_j^*, t]\{1 \otimes x_0^d, dt \otimes x_0^d\}
\]
given by
\[ T^1(\alpha(1/y_i)) = T^1(u_0^i \otimes s_0 + \cdots + u_d^i \otimes s_d) = u_0^i \otimes x_0^d + u_1^i(t+dt) \otimes x_1^d + \cdots + u_d^i(t+dt)^d \otimes x_d^d \]
where
\[ f_i(t) = u_0^i + u_1^i t + \cdots + u_d^i t^d. \]
Let
\[ a(t) = y_0 + y_1 t + \cdots + y_d t^d \]
it follows \( y_i f_i(t) = a(t) \). Restrict the map \(2.18.1\) to \( U_{i1} = D(y_i) \times D(x_1) \). We get
\[ \alpha|_{U_{i1}} : F[u_j^i, 1/t] \frac{1}{y_i} \rightarrow F[u_j^i, 1/t] \otimes F \{s_i\} \rightarrow T^1 F[u_j^i, 1/t] \{1 \otimes x_1^d, d(1/t) \otimes x_1^d\} \]
given by
\[ T^1(\alpha(1/y_i)) = T^1(u_0^i \otimes s_0 + \cdots + u_d^i \otimes s_d) =
\]
\[ u_0^i(\frac{1}{t})^d \otimes x_1^d + (d-1)(\frac{1}{t})^{d-1} d(\frac{1}{t}) \otimes x_1^d + \cdots +
\]
\[ ((\frac{1}{t})^{d-i} \otimes x_1^d + (d-i)(\frac{1}{t})^{d-i-1} d(\frac{1}{t}) \otimes x_1^d) + \cdots + u_d^i \otimes x_d^d =
\]
given by
\[ g_i(\frac{1}{t}) = u_0^i(\frac{1}{t})^d + \cdots + (\frac{1}{t})^{d-i} + \cdots + u_d^i. \]
Let
\[ b(t) = y_0(\frac{1}{t})^d + \cdots + y_i(\frac{1}{t})^{d-i} + \cdots + y_d. \]
It follows \( b(\frac{1}{t}) = y_i g_i(\frac{1}{t}) \). Let \( U_{ij} = D(y_i) \times D(x_j) \) and let \( Y_{ij} = U_{ij} \cap I^j(\mathcal{O}(d)) \). It follows \( q(Y_{ij}) = Z_{ij} \subseteq D(y_i) \). Let \( j = 0 \) and let \( I_{i0} = (f_i(t), f_j^i(t)) \). We get a map
\[ q : Y_{i0} \rightarrow D(y_i) \]
which gives a map
\[ q^* : \mathcal{O}_{D(y_i)} \rightarrow q_* \mathcal{O}_{Y_{i0}} \]
given by
\[ q^* : F[u_j^i] \rightarrow F[u_j^i, t]/I_{i0}. \]
It follows
\[ \mathcal{I}_{Z_{i0}} = \ker(q^*) = Res(f_i(t), f_j^i(t)) = Res(a(t), a^j(t))|_{D(y_i)}. \]
Let \( j = 1 \) and let \( I_{i1} = (g_i(\frac{1}{t}), g_j^i(\frac{1}{t})) \). We get a map
\[ q : Y_{i1} \rightarrow D(y_i) \]
which gives a map
\[ q^* : \mathcal{O}_{D(y_i)} \rightarrow q_* \mathcal{O}_{Y_{i1}} \]
given by
\[ q^* : F[u_j^i] \rightarrow F[u_j^i, \frac{1}{t}]/I_{i1}. \]
It follows
\[ \mathcal{I}_{Z_{i1}} = \ker(q^*) = Res(b_i(\frac{1}{t}), b_j^i(\frac{1}{t}))|_{D(y_i)}. \]
Let $\mathcal{I} = (\text{Res}(a(t), a'(t)))$ and $\mathcal{J} = \text{Res}(b(t), b'(t))$. It follows
\[
\mathcal{I}_{D^1(\mathcal{O}(d))}|_{Z_{\mathcal{I}}} = \mathcal{I}_{Z_{\mathcal{I}}} = \text{Res}(a(t), a'(t))|_{D(y_i)} = \mathcal{I}|_{D(y_i)}
\]
and
\[
\mathcal{I}_{D^1(\mathcal{O}(d))}|_{Z_{\mathcal{J}}} = \mathcal{J}_{Z_{\mathcal{J}}} = \text{Res}(b(t), b'(t))|_{D(y_i)} = \mathcal{J}|_{D(y_i)}.
\]
We get an equality of ideal sheaves in $\mathcal{O}_{\mathcal{P}(W^*)}$
\[
\mathcal{I}_{D^1(\mathcal{O}(d))} = \mathcal{I} = \mathcal{J}.
\]
It follows the ideal sheaf $\mathcal{I}_{D^1(\mathcal{O}(d))}$ is generated by the irreducible polynomial
\[
\text{Res}(a(t), a'(t)) = \text{Res}(b(t), b'(t))
\]
hence $D^1(\mathcal{O}(d))$ is the discriminant scheme of degree $d$ polynomials parametrizing degree $d$ polynomials
\[
a(t) = y_0 + y_1t + \cdots + y_dt^d
\]
in the variable $t$ with multiple roots. It follows $D^1(\mathcal{O}(d))$ is a determinantal scheme. We get a filtration of closed subschemes
\[
D^d(\mathcal{O}(d)) \subseteq \cdots \subseteq D^i(\mathcal{O}(d)) \cdots \subseteq D^1(\mathcal{O}(d)) \subseteq \mathcal{P}(W^*).\]
One may ask if $D^i(\mathcal{O}(d))$ is a determinantal scheme for $1 < i < d$. There is work in progress on this problem: One wants to check if $D^k(\mathcal{L}(\lambda))$ is a determinantal scheme where $\mathcal{L}(\lambda) \in \text{Pic}^G(G/P)$. Here $G$ is a semi simple linear algebraic group and $P \subseteq G$ a parabolic sub group.

Note: For determinantal schemes much is known about their syzygies (see [6]).

3. Discriminants and standard etale morphisms

In this section we study the discriminant of a polynomial and its relationship with finite and standard etale morphisms. We relate the discriminant of a monic polynomial $P(t)$ in $A[t]$ to properties of the integral ring extension $A \subseteq A[t]/P(t)$. We give an explicit proof of the fact that
\[
\text{Spec}(A[t]/P(t)) \to \text{Spec}(A)
\]
is generically etale using the discriminant $\text{Discr}(P(t))$. We prove in Theorem 3.18 that any etale morphism of schemes is locally on the form
\[
\text{Spec}(A[t]/P(t)) \to \text{Spec}(A)
\]
where $P(t) \in A[t]$ is a polynomial with $\text{Discr}(P(t))$ a unit in $A$. We also prove a general result (see Theorem 3.20 on properties of the ring extension $A \subseteq A[t]/P(t)$) when $P(t)$ is an arbitrary (not necessarily monic) polynomial.

Let $X = \text{Spec}(B)$ and $Y = \text{Spec}(A)$ where $A, B$ are commutative unital rings. Let $f : \text{Spec}(B) \to \text{Spec}(A)$ be a finite map of affine schemes. Hence $A \to B$ is an integral extension of rings. Let $p \in X$ be a point with $q = f(p)$.

**Definition 3.1.** We say $f$ is unramified at $p$ if the following two conditions holds:

The canonical map

(3.1.1) $f^\# : \mathcal{O}_{Y,q} \to f_*\mathcal{O}_{X,p}$ satisfies $m_q\mathcal{O}_{X,p} = m_p$.

(3.1.2) The field extension $\kappa(q) \subseteq \kappa(p)$ is a finite separable extension.
We say the morphism $f$ is etale at $p$ if it unramified at $p$ and the ring homomorphism

\[(3.1.3) \quad f^* : \mathcal{O}_{Y,q} \rightarrow f_* \mathcal{O}_{X,p}\]

is flat. The morphism $f : X \rightarrow Y$ is an etale morphism if it is etale at $p$ for all $p \in X$. The morphism $f$ is ramified if for all $q \in Y$ there is a $p \in f^{-1}(q)$ where $f$ is not etale at $p$.

The following result is well known.

**Lemma 3.2.** The morphism $f$ is etale if and only if it is flat and for every $q \in Y$ the fiber $f^{-1}(q)$ is the disjoint union of reduced points $p \in f^{-1}(q)$ with $\kappa(q) \subseteq \kappa(p)$ a finite separable extension of fields.

**Proof.** See [4] or [14].

For a finite etale morphism $f : \text{Spec}(B) \rightarrow \text{Spec}(A)$ it is well known the number of points $p$ in $f^{-1}(q)$ is constant. We let $d = \# f^{-1}(q)$ be the degree of $f$.

Recall the following general result: Let $A_1, \ldots, A_l$ be commutative rings with unit and let $S_i \subseteq A_i$ be multiplicatively closed subsets for $i = 1, \ldots, l$. Let $A = \oplus_{i=1}^l A_i$ be the direct sum of the commutative rings $A_i$ and let $S = \oplus_{i=1}^l S_i$. It follows $S \subseteq A$ is a multiplicatively closed subset.

**Lemma 3.3.** There is an isomorphism

\[
(S_1 \oplus \cdots \oplus S_l)^{-1}(A_1 \oplus \cdots \oplus A_l) \cong S_1^{-1}A_1 \oplus \cdots \oplus S_l^{-1}A_l
\]

of commutative rings.

**Proof.** We prove this by induction on $l$. Assume $l = 2$. We want to prove the isomorphism

\[
(S \oplus T)^{-1}(A \oplus B) \cong S^{-1}A \oplus T^{-1}B.
\]

Define the following morphism

\[
g : A \oplus B \rightarrow S^{-1}A \oplus T^{-1}B
\]

by

\[
g(a,b) = (a/1, b/1).
\]

It follows for all $(s, t) \in S \oplus T$ the element $g(s, t)$ is invertible. Moreover if $g(a, b) = (a/1, b/1) = 0$ it follows $a/1 = 0 = b/1$ hence there is an element $(s, t) \in S \oplus T$ with $sa = tb = 0$. It follows $(s,t)(a,b) = (sa, tb) = 0$. Finally any element $(a/s, b/t) \in S^{-1}A \oplus T^{-1}B$ may be written as

\[
g(a,b)g(s,t)^{-1}.
\]

It follows there is a canonical isomorphism

\[
S^{-1}A \oplus T^{-1}B \cong (S \oplus T)^{-1}(A \oplus B)
\]

of rings and the claim is proved. The Lemma now follows by induction. □

Let $K$ be an arbitrary field and let $P(t) = t^d + a_1 t^{d-1} + \cdots + a_{d-1} t + a_d \in K[t]$ be a polynomial with coefficients in $K$. Recall the following notion: The polynomial $P(t)$ is separable if its roots in the algebraic closure $\overline{K}$ of $K$ are all distinct. The polynomial $P(t)$ is inseparable if it has multiple roots. Recall the following well known result: Let $X = \text{Spec}(K[t]/P(t)$ and $Y = \text{Spec}(K)$. Let $f : X \rightarrow Y$ be the structure morphism.
Proposition 3.4. The morphism $f$ is etale if and only if $P(t) \in K[t]$ is a separable polynomial. The morphism $f$ is ramified if and only if $P(t)$ is inseparable.

Proof. Assume $P(t)$ is separable. It follows $P(t)$ has $d$ distinct roots $\alpha_1, \ldots, \alpha_d \in \overline{K}$. Let $P(t) = Q_1^{d_1} \cdots Q_m^{d_m}$ be a decomposition of $P(t)$ in $K[t]$ where $Q_i$ are irreducible polynomials in $K[t]$. Since $P(t)$ is a separable polynomial it follows $q_i = 1$ for $i = 1, \ldots, m$. Let $X_i = \text{Spec}(K[t]/Q_i(t))$. It follows by the Chinese Remainder Theorem that

$$K[t]/P(t) \cong K[t]/Q_1(t) \oplus \cdots \oplus K[t]/Q_m(t).$$

We get an isomorphism

$$X \cong X_1 \cup \cdots \cup X_m$$

of schemes. We get a map

$$f : X_1 \cup \cdots \cup X_m \to Y$$

induced by the natural map $K \to K[t]/P(t)$. Let $L = K[t]/P(t)$ and $L_i = K[t]/Q_i(t)$. It follows $L_i$ is a separable field extension of $K$. By the Kunneth formula the following holds: There is an isomorphism

$$H^0(X, \mathcal{O}_X) \cong H^0(X_1, \mathcal{O}_{X_1}) \oplus \cdots \oplus H^0(X_m, \mathcal{O}_{X_m})$$

of rings. One also sees $L_i = H^0(X_i, \mathcal{O}_{X_i})$. Let $m_i$ be the following ideal:

$$m_i = L_1 \oplus \cdots \oplus L_{i-1} \oplus \{0\} \oplus L_{i+1} \oplus \cdots \oplus L_m.$$ 

It follows

$$H^0(X, \mathcal{O}_X)/m_i \cong H^0(X_i, \mathcal{O}_{X_i}) = L_i$$

hence $m_i \subseteq H^0(X, \mathcal{O}_X)$ is a maximal ideal. The ideals $m_1, \ldots, m_m$ are all maximal ideals in $H^0(X, \mathcal{O}_X)$. Consider $m_i \in X$ and look at the map

$$f^\#: \mathcal{O}_{Y, f(m_i)} \to \mathcal{O}_{X, m_i}.$$ 

It is given by the natural map $K \to L_{m_i}$. Let $S_i \subseteq H^0(X, \mathcal{O}_X)$ be the multiplicatively closed subset defined by $S_i = H^0(X, \mathcal{O}_X) - m_i$. It follows

$$S_i = L_1 \oplus \cdots \oplus L_i^* \oplus \cdots \oplus L_m$$

hence we get by lemma 3.3 an isomorphism

$$L_{m_i} \cong S_i^{-1} H^0(X, \mathcal{O}_X) \cong L_1^{-1} L_1 \oplus \cdots \oplus (L_i^*)^{-1} L_i \oplus \cdots \oplus L_m^{-1} L_m \cong L_i.$$ 

Hence the map

$$f^\#: \mathcal{O}_{Y, f(m_i)} \to \mathcal{O}_{X, m_i}$$

is the map

$$K \to L_i.$$ 

The field extension

$$\kappa(f(m_i)) = K \subseteq L_i = \kappa(m_i)$$

is separable. It follows the map $f$ is flat and unramified at $m_i$ hence $f$ is etale. Conversely assume $f : X \to Y$ is etale and $P(t)$ inseparable. It follows $P(t) = Q_1^{d_1} \cdots Q_m^{d_m}$ where all polynomials $Q_i$ are irreducible and one of the following holds:

(3.4.1) There is an $i$ with $q_i > 1.$

(3.4.2) All $q_i = 1$ and there is an $i$ where $K \subseteq K[t]/Q_i$ is inseparable.
Let $\mathfrak{m}_i$ be the maximal ideal corresponding to $i$. It follows $f$ is not etale at $i$ which is a contradiction, and the first part of the Proposition is proved. The second part of the claim is obvious and the Proposition is proved.

Let

$$F(t) = a_m t^m + a_{m-1} t^{m-1} + \cdots + a_1 t + a_0$$

and

$$G(t) = b_n t^n + b_{n-1} t^{n-1} + \cdots + b_1 t + b_0$$

be polynomials in $A[t]$ where $A$ is an arbitrary commutative ring with unit and $a_m, b_n \neq 0$. Make the following definition

**Definition 3.5.** Let $\text{Res}_{m,n}(F(t), G(t))$ be the following determinant

$$\begin{vmatrix} a_m & a_{m-1} & \cdots & a_1 & a_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_m & a_{m-1} & \cdots & a_1 & a_0 & 0 & \cdots & 0 & 0 \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & a_1 & a_0 \\ b_n & b_{n-1} & \cdots & b_1 & b_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & b_n & b_{n-1} & \cdots & b_1 & b_0 & 0 & \cdots & 0 & 0 \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & b_1 & b_0 \end{vmatrix}$$

We say the element $\text{Res}_{m,n}(F(t), G(t))$ is the resultant of the polynomials $F(t)$ and $G(t)$.

The matrix contains $n$ rows with $a_i$’s and $m$ rows with $b_j$’s. It is a square matrix of rank $m + n$. We often write $\text{Res}(F, G)$ instead of $\text{Res}_{m,n}(F(t), G(t))$. From the definition it is immediate $\text{Res}(F, G)$ is an element of the ring $A$.

The following result is well known:

**Proposition 3.6.** Let $A = K$ be an arbitrary field, and let $F(t), G(t)$ be polynomials in $K[t]$ of degree $m$ and $n$ with $a_m, b_n \neq 0$. Let $\alpha_1, \ldots, \alpha_m$ and $\beta_1, \ldots, \beta_n$ be the roots of $F$ and $G$ in an algebraic closure $\overline{K}$ of $K$. Assume $F'(t)$ is a polynomial of degree $m' \leq m$. The following holds:

(3.6.1) \[ \text{Res}(F, G) = a_m b_n \prod_{i,j} (\alpha_i - \beta_j) \]

(3.6.2) \[ \text{Res}(F, G) = 0 \iff F \text{ and } G \text{ have a common root in } \overline{K} \]

(3.6.3) \[ \text{Res}(F, G) = (-1)^{mn} \text{Res}(G, F) \]

(3.6.4) \[ \text{Res}_{m,n}(FF', G) = \text{Res}_{m,n}(F, G) \text{Res}_{m',n}(F', G) \]

**Proof.** For a proof of the facts (3.6.1),(3.6.4) see [3], Section 12. \[ \square \]

Let $A$ be an arbitrary commutative ring with unit and let $P(t) = t^d + a_{d-1} t^{d-1} + \cdots + a_1 t + a_0 \in A[t]$ be any degree $d$ monic polynomial. The formal derivative $P'(t)$ is again a polynomial in $A[t]$.

**Definition 3.7.** We let $\text{Discr}(P(t)) = \text{Res}_{d,d-1}(P(t), P'(t)) \in A$ be the discriminant of the polynomial $P(t)$. We say $P(t)$ is separable if $\text{Discr}(P(t))$ is a unit in $A$. We say $P(t)$ is inseparable if $\text{Discr}(P(t))$ is nilpotent.

**Proposition 3.8.** Assume $A = K$ is a field. It follows $\text{Discr}(P(t)) = 0$ if and only if $P(t)$ has a root $\alpha \in \overline{K}$ of multiplicity greater than 2.
Lemma 3.10. Let \( P(t) \) be defined by \( P(t) = (t - \alpha)^2 Q(t) \) for some polynomial \( Q(t) \in \mathbb{K}[t] \), hence \( \alpha \) has multiplicity greater than 2. Conversely, if \( P(t) = (t - \alpha)^2 Q(t) \) with \( Q(t) \mathbb{K}[t] \) it follows \( P(t) \) and \( P'(t) \) have a common root, hence \( \text{Discr}(P(t)) = 0 \) and the claim of the Proposition follows.

\[ \square \]

Corollary 3.9. The following holds: \( \text{Discr}(P(t)) \neq 0 \) if and only if all roots of \( P(t) \) have multiplicity one.

Proof. Since \( \text{Discr}(P(t)) \neq 0 \) it follows from Proposition \( 3.8 \) that all roots of \( P(t) \) in \( \mathbb{K} \) are of multiplicity one. Conversely if all roots of \( P(t) \) have multiplicity one it follows \( \text{Discr}(P(t)) \neq 0 \). The Corollary is proved.

Proof. Assume \( \text{Discr}(P(t)) = 0 \) hence \( P(t) \) and \( P'(t) \) have a common root \( \alpha \in \mathbb{K} \). It follows \( P(t) = (t - \alpha)^2 Q(t) \) for some polynomial \( Q(t) \in \mathbb{K}[t] \), hence \( \alpha \) has multiplicity greater than 2. Conversely, if \( P(t) = (t - \alpha)^2 Q(t) \) with \( Q(t) \mathbb{K}[t] \) it follows \( P(t) \) and \( P'(t) \) have a common root, hence \( \text{Discr}(P(t)) = 0 \) and the claim of the Proposition follows.

Proof. Let \( \psi : A \to B \) be a map of commutative rings. We get an induced map

\[ \psi_t : A[t] \to B[t] \]

defined by

\[ \psi_t(P(t)) = \psi(b_n)t^n + \psi(b_{n-1})t^{n-1} + \cdots + \psi(b_1)t + \psi(b_0). \]

Let \( P_\psi(t) = \psi_t(P(t)) \).

We get the following result:

Lemma 3.10. There is an equality

\[ \psi(\text{Discr}(P(t))) = \text{Discr}(P_\psi(t)) \]

in \( B \).

Proof. By definition

\[ P(t) = t^n + a_{d-1}t^{d-1} + a_{d-2}t^{d-2} + \cdots + a_1t + a_0 \]

has coefficients \( a_i \) in \( A \) and

\[ P'(t) = dt^{d-1} + (d-1)a_{d-1}t^{d-2} + \cdots + 2a_2t + a_1. \]

The discriminant \( \text{Discr}(P(t), P'(t)) \) is by Definition \( 3.7 \) given by the following determinant:

\[ |M| = \begin{vmatrix} 1 & a_{d-1} & \cdots & a_1 & a_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & a_{d-1} & \cdots & a_1 & a_0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots \\ d & (d-1)a_{d-1} & \cdots & a_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & d & (d-1)a_{d-1} & \cdots & a_1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 2a_2 & a_1 \end{vmatrix}. \]

Assume \( \psi : A \to B \) is a ring homomorphism. It follows

\[ \psi(\text{Discr}(P(t))) = \psi(|M|) = |\psi(M)| = \text{Discr}(P_\psi(t)) \]

and the Lemma is proved.

Proof. Let \( b = \text{Discr}(P(t)) \in A \). Consider the natural morphism

\[ \pi : \text{Spec}(A[t]/P(t)) \to \text{Spec}(A) \]
Moreover, the natural map is the map of rings is ramified and claim 3.11.2 follows. We prove claim 3.11.3: There is an isomorphism $p \in \text{Spec}(A[t]/P(t))$ is by Proposition 3.4 an etale map. It follows from Lemma 3.2 the map of affine schemes.

Proof. We first prove 3.11.1. We want to show

$$\pi : \pi^{-1}(D(b)) \to D(b)$$

is an etale morphism. Pick $p \in \text{Spec}(A)$ with $p \in D(b)$. It follows $b \not\in p$. Let $\psi : A \to \kappa(p)$ be the natural map where $\kappa(p)$ is the residue field of $p$. The induced map on the fiber $\pi^{-1}(p) \to \text{Spec}(\kappa(p))$ is the natural map

$$\text{Spec}(\kappa(p)[t]/P_{\psi}(t)) \to \text{Spec}(\kappa(p)).$$

Since $b = \text{Discr}(P(t)) \not\in p$ it follows

$$\psi(\text{Discr}(P(t))) = \text{Discr}(P_{\psi}(t)) \neq 0$$

in the residue field $\kappa(p)$. It follows from Corollary 3.11.2 $P_{\psi}(t)$ is separable in $\kappa(p)[t]$ hence the map

$$\pi : \text{Spec}(\kappa(p)[t]/P_{\psi}(t)) \to \text{Spec}(\kappa(p))$$

is by Proposition 3.4 an etale map. It follows from Lemma 3.2 the map

$$\pi : \pi^{-1}(D(b)) \to D(b)$$

is etale and claim 3.11.1 is proved. We prove claim 3.11.2. Pick a point $p \in V(b)$ and consider the morphism

$$\psi : \pi^{-1}(p) = \text{Spec}(\kappa(p)[t]/P_{\psi}(t)) \to \text{Spec}(\kappa(p)).$$

Since $\psi(b) \in p$ it follows $\text{Discr}(P_{\psi}(t)) = 0$ in the residue field $\kappa(p)$. By Proposition 3.8 it follows $P_{\psi}(t)$ is inseparable over $\kappa(p)$. It follows from Proposition 3.4 for each $p \in V(b)$ the morphism

$$\pi^{-1}(p) \to \text{Spec}(\kappa(p))$$

is ramified and claim 3.11.2 follows. We prove claim 3.11.3. There is an isomorphism of rings

$$S^{-1}(A[t]/P(t)) \cong S^{-1}(A[t])/P_{\phi}(t) \cong A_b[t]/P_{\phi}(t).$$

Hence

$$\pi^{-1}(D(b)) \cong \text{Spec}(S^{-1}(A[t]/P(t))) \cong \text{Spec}(A_b[t]/P_{\phi}(t)).$$

Moreover, the natural map

$$\text{Spec}(A_b[t]/P_{\phi}(t)) \to \text{Spec}(A_b)$$

is the map

$$\pi^{-1}(D(b)) \to D(b).$$
and the Proposition is proved.

Example 3.12. An inseparable polynomial.

Let \( a_1, \ldots, a_d \in A \) be elements with \( a_i - a_j \) nilpotent for some \( i \neq j \). It follows the polynomial

\[
P(t) = (t - a_1)(t - a_2) \cdots (t - a_d) \in A[t]
\]

is an inseparable polynomial: Let \( p \subseteq A \) be a prime ideal. It follows \( a_i - a_j \in p \) for all \( i \neq j \). Let \( \phi : A \rightarrow \kappa(p) \) be the canonical map. It follows \( P_\phi(t) \) has multiple roots hence \( \text{Discr}(P_\phi(t)) = \phi(\text{Discr}(P(t))) = 0 \) in \( \kappa(p) \). It follows \( \text{Discr}(P(t)) \) is nilpotent since \( \text{Discr}(P(t)) \in p \) for all primes \( p \). It follows \( P(t) \) is an inseparable polynomial.

Example 3.13. A separable polynomial.

Let \( a_1, \ldots, a_d \in A \) be elements with \( a_i - a_j \) not nilpotent for all \( i \neq j \). It follows the polynomial

\[
P(t) = (t - a_1)(t - a_2) \cdots (t - a_d) \in A[t]
\]

is a separable polynomial: Let \( p \subseteq A \) be a prime ideal. It follows \( a_i - a_j \in \kappa(p) \) for all \( i \neq j \). Let \( \phi : A \rightarrow \kappa(p) \) be the canonical map. It follows \( P_\phi(t) \) is a separable polynomial for all \( p \). It follows \( \text{Discr}(P_\phi(t)) = \phi(\text{Discr}(P(t))) \neq 0 \) in \( \kappa(p) \) for all primes \( p \). Hence \( \text{Discr}(P(t)) \) is a unit in \( A \) and \( P(t) \) is a separable polynomial.

Corollary 3.14. The open set \( U = D(b) \subseteq \text{Spec}(A) \) is the maximal open subset \( U \subseteq \text{Spec}(A) \) where \( \pi : \pi^{-1}(U) \rightarrow U \) is etale.

Proof. This follows from Proposition 3.11 Claim 3.11.1 and 3.11.2.

Corollary 3.15. The morphism \( \pi : \text{Spec}(A[t]/P(t)) \rightarrow \text{Spec}(A) \) is etale if and only if \( \text{Discr}(P(t)) \) is a unit in \( A \).

Proof. By Proposition 3.11 3.11.2 it follows the morphism \( \pi \) is etale if and only if \( V(\text{Discr}(P(t))) = \emptyset \). This is if and only if \( \text{Discr}(P(t)) \) is a unit, and the Corollary follows.

Corollary 3.16. The morphism \( \pi : \text{Spec}(A[t]/P(t)) \rightarrow \text{Spec}(A) \) is ramified if and only if \( \text{Discr}(P(t)) \) is nilpotent in \( A \).

Proof. Let \( p \in \text{Spec}(A) \) and let \( \phi : A \rightarrow \kappa(p) \) be the canonical map. It follows \( \pi \) is ramified if and only if \( P_\phi(t) \in \kappa(p)[t] \) is inseparable for all primes \( p \). This is if and only if \( \text{Discr}(P_\phi(t)) = \phi(\text{Discr}(P(t))) \) is zero in the residue field \( \kappa(p) \) for all primes \( p \). This is if and only if \( \text{Discr}(P(t)) \in p \) for all primes \( p \subseteq A \). This is if and only if \( \text{Discr}(P(t)) \) is nilpotent in \( A \), and the Corollary follows.

It follows the morphism \( \pi \) is etale if and only if \( P(t) \) is separable. The morphism \( \pi \) is ramified if and only if \( P(t) \) is inseparable. Hence the discriminant \( \text{Discr}(P(t)) \) measures when the morphism

\[
\pi : \text{Spec}(A[t]/P(t)) \rightarrow \text{Spec}(A)
\]

is etale.

Definition 3.17. The scheme \( D(\pi) = V(\text{Discr}(P(t))) \subseteq \text{Spec}(A) \) is the discriminant of the morphism \( \pi \) where \( \pi : \text{Spec}(A[t]/P(t)) \rightarrow \text{Spec}(A) \) is the canonical morphism.
The underlying set of points of $D(\pi)$ equals the set theoretical discriminant $\text{Discr}^1(\pi)$ from Definition 2.13. By Proposition 3.11 $D(\pi)$ is the largest closed subscheme of $\text{Spec}(A)$ with the property that the map $\pi : \pi^{-1}(D(\pi)) \to D(\pi)$ is ramified.

**Theorem 3.18.** Let $f : X \to Y$ be an etale morphism of degree $d$ and let $p \in X$ with $q = f(p)$. There is an open affine neighborhood $U = \text{Spec}(A) \subseteq Y$ with $q \in U$ and $f^{-1}(U) = \text{Spec}(B)$ where $B = A[t]/P(t)$ where $P(t)$ is a monic degree $d$ polynomial with $\text{Discr}(P(t))$ a unit in $A$.

**Proof.** By [4] it follows there are affine open sets $p \in U = \text{Spec}(B) \subseteq X$ and $V = \text{Spec}(A) \subseteq Y$ with $f(U) \subseteq V$ and a commutative diagram

\[
\begin{array}{ccc}
U & \xrightarrow{\text{Spec}(A[t]/P(t))} & V \\
\downarrow f & & \downarrow f \\
V & \cong & \text{Spec}(A)
\end{array}
\]

where $j$ is an open immersion,

$\phi : A[t] \to A[t]/P(t)$

is the canonical map and $P(t) \in A[t]$ is a monic degree $d$ polynomial. Since $U \subseteq \text{Spec}(A[t]/P(t)) = D(b)$ it follows by Proposition 3.11 $f(U) \subseteq D(b)$ where $b = \text{Discr}(P(t)) \in A$. Since the extension $A \subseteq A[t]/P(t)$ is faithfully flat it follows

$\text{Spec}(A[t]/P(t)) \to \text{Spec}(A)$

is open. It follows the morphism $f : U \to V$ is open hence $f(U) \subseteq V$ is an open set. We may therefore choose a basic open set $\text{Spec}(A) \subseteq f(U)$ with $q = f(p) \in \text{Spec}(A)$. It follows the map

$f : f^{-1}(\text{Spec}(A)) \to \text{Spec}(A)$

equals the map

$f : \text{Spec}(A[t]/P(t)) \to \text{Spec}(A)$

where

$\phi : A[t] \to A[t]$ is the canonical map. Since $\text{Spec}(A) \subseteq D(b)$ it follows by Proposition 3.11 and Corollary 3.15 $\text{Discr}(P(t))$ is a unit, and the claim of the Theorem follows. \hfill \Box

**Definition 3.19.** A map

$\text{Spec}(A[t]/P(t)) \to \text{Spec}(A)$

where $P(t) \in A[t]$ is a monic polynomial with $\text{Discr}(P(t))$ a unit is called a standard etale morphism.

By Theorem 3.18 it follows any etale morphism is locally a given by a standard etale morphism.

Note: The definition of standard etale morphism given in Definition 3.19 differs from the one given in the litterature (see [4] and [14]). In the litterature a standard etale morphism is a morphism on the form

$\pi : \text{Spec}(A[t]/P(\pi)) \to \text{Spec}(A)$

where $P(\pi) \in A[t]$ is a monic polynomial with $\text{Discr}(P(\pi))$ a unit in $A$.
where
\[ \phi : A[t] \to A[t]_{P(t)} \]
is the canonical map.
Let \( P(t) \in A[t] \) be any monic degree \( d \) polynomial where \( A \) is an arbitrary commutative ring. Let \( b = \text{Discr}(P(t)) \in A \) be the discriminant. By Proposition 3.11 we get a diagram of maps of schemes
\[ \xymatrix{ \pi^{-1}(V(b)) \ar[r]^i \ar[d]_{\pi} & \Spec(A[t]/P(t)) \ar[d]_{\pi} & \Spec(A_b[t]/P_\phi(t)) \ar[l]_j \ar[d]_{\pi} \\
V(b) \ar[r]^k & \Spec(A) & \Spec(A_b) \ar[l]_t } \]
where \( \phi : A \to A_b \) is the canonical morphism. It follows by Corollary 3.15 the morphism
\[ \Spec(A_b[t]/P_\phi(t)) \to \Spec(A_b) \]
is standard etale. Hence any morphism
\[ \Spec(A[t]/P(t)) \to \Spec(A) \]
decompose into a standard etale morphism and a ramified morphism. The open set \( D(b) = \Spec(A_b) \subseteq \Spec(A) \) is dense, hence the morphism
\[ \Spec(A[t]/P(t)) \to \Spec(A) \]
is generically etale.
Let \( P(t) = a_d t^d + a_{d-1} t^{d-1} + \cdots + a_1 t + a_0 \in A[t] \) be any polynomial with coefficients in \( A \). Assume \( a_d \neq 0 \). Assume we are given integers \( d > d_1 > d_2 > \cdots > d_k \geq 0 \). Let \( A_i = A/(a_d, a_{d_1}, \ldots, a_{d_i}) \) where \( a_{d_i} \) is the \( d_i \)th coefficient of \( P(t) \). and \( A^i = (A_{i-1})_{a_{d_i}} \). Let \( U_i = \Spec(A^i) \) and \( V_i = \Spec(A_i) \).

**Theorem 3.20.** There exists unique integers \( d > d_1 > d_2 > \cdots > d_k \geq 0 \) satisfying the following: We may write \( \Spec(A) = V_0 \cup U_0 \) and for all \( i = 1, \ldots, k \) \( V_i = V_{i+1} \cup U_{i+1} \). Moreover for all \( i \) there is an element \( b_i \in A^i \) giving a disjoint union
\[ \Spec(A^i) = V(b_i) \cup D(b_i) \]
with the following properties: The natural morphism
\[ \pi : \Spec(A^i[t]/P(t)) \to \Spec(A^i) \]
satisfy
\begin{align*}
(3.20.1) \quad & \pi : \pi^{-1}(D(b_i)) \to D(b_i) \text{ is standard etale of degree } d_i. \\
(3.20.2) \quad & \pi : \pi^{-1}(V(b_i)) \to V(b_i) \text{ is ramified.}
\end{align*}
The open set \( D(b_i) \subseteq \Spec(A^i) \) is the largest open subscheme with property \( 3.20.1 \).

**Proof.** Consider
\[ P(t) = a_d t^d + \cdots + a_1 t + a_0 \in A[t]. \]
We may write \( \Spec(A) = V(a_d) \cup D(a_d) \) as a disjoint union. It follows \( V(a_d) = \Spec(A_0) \) and \( D(a_d) = \Spec(A^0) \). On \( D(a_d) \) it follows the leading coefficient \( a_d \) of \( P(t) \) is invertible. Let
\[ \phi : A \to A_{a_d} = A^0 \]
be the canonical map of rings. It follows $P_\alpha(t) \in A_{n_\alpha}[t]$ has a unit as leading coefficient. Let $b_0 = Discr(P_\alpha(t)) \in A^0$ it follows $Spec(A^0) = V(b_0) \cup D(b_0)$. We get a canonical morphism

$$\pi : \pi^{-1}(D(b_0)) = Spec(A^0[t]/P_\alpha(t)) \to Spec(A^0) = D(b_0)$$

and by Proposition 3.11 it follows the morphism

$$\pi : \pi^{-1}(D(b_0)) \to D(b_0)$$

is standard etale of degree $d$. Again by Proposition 3.11 it follows the morphism

$$\pi : \pi^{-1}(V(b_0)) \to V(b_0)$$

is ramified. By Corollary 3.14 it follows $D(b_0) \subseteq Spec(A^0)$ is the maximal open subscheme with this property. The Theorem now follows by Proposition 3.11 and an induction.

4. Discriminants of Linear Systems on the Projective Line

In this section we study the discriminant $D^l(O(d))$ of the line bundle $O(d)$ on $P^1_K$ where $K$ is any field.

Let $E = K\{e_0, e_1\}$ be a $K$-vector space of dimension two where $K$ is any field and let $E^* = K\{x_0, x_1\}$ be its dual. Let $X = Spec(K)$ and $\mathbb{P} = \mathbb{P}(E^*) = \mathbb{P}_K^1 \times \mathbb{Z} Spec(K)$ be the projective line over $K$. Cover $\mathbb{P}$ by the two standard open subsets $D(x_1)$ and let $t = \frac{x_0}{x_1}$ and $s = \frac{x_1}{x_0}$. Let $O(d)$ be the invertible sheaf corresponding to the graded $K[x_0, x_1]$-module $K[x_0, x_1](d)$, and let $W = H^0(\mathbb{P}, O(d))$ be its global sections. It follows $W = Sym^d(E^*)$ hence $W$ is a free $K$-module on the global sections $s_i = x_0^{i-1}x_1$. Let $y_i = s_i^*$. It follows $\mathbb{P}(W^*)$ is the proj of the graded ring $K[y_0, \ldots, y_d]$ and there is a canonical structure morphism

$$\pi : \mathbb{P}(W^*) \to Spec(K).$$

The tautological sequence on $\mathbb{P}(W^*)$ is the morphism

$$0 \to O(-1) \to \pi^*W.$$  

(4.0.3)

It is the sheafification of the following morphism of graded $K[y_0, \ldots, y_d]$-modules:

$$\alpha : K[y_0, \ldots, y_d](-1) \to K[y_0, \ldots, y_d] \otimes_K K\{s_0, \ldots, s_d\}$$

$$\alpha(1) = \sum_{j=0}^d y_j \otimes s_j.$$

Fix an integer $0 \leq i \leq d$ and let $U_i = D(y_i)$. Restrict (4.0.3) to $U_i$ to get the following map:

$$\alpha|_{U_i} : O(-1)|_{U_i} \to \pi^*W|_{U_i}$$

given by

$$\alpha(1/y_i) = \sum_{j=0}^d \frac{y_j \otimes s_j}{y_i} = \sum_{j=0}^d \frac{y_j}{y_i} \otimes s_j.$$
Let \( u_j = \frac{y_j}{y_i} \) for \( j = 0, \ldots, d \). It follows \( u_i = \frac{y_i}{y_i} = 1 \). Consider the diagram

\[
\begin{array}{ccc}
P(W^*) \times A & \overset{p}{\longrightarrow} & P \\
\downarrow q & & \downarrow u \\
\mathbb{P}(W^*) & \overset{\pi}{\longrightarrow} & \text{Spec}(K)
\end{array}
\]

The \( l \)'th Taylor map for \( \mathcal{O}(d) \) is a map of locally free \( \mathcal{O}_P \)-modules

\[(4.0.4) \quad T^l : u^*W \to \mathcal{J}(\mathcal{O}(d))\]

on \( P = \mathbb{P}(E^*) \). Pull the Taylor map \[4.0.4\] and the tautological sequence back to \( Y = \mathbb{P}(W^*) \times_K P \) to get the sequence

\[
\mathcal{O}(-1)_Y \to \alpha_Y W_Y \to T^l \mathcal{J}(\mathcal{O}(d))_Y
\]

and define \( \phi^l(\mathcal{O}(d)) = T_V^l \circ \alpha_Y \).

**Definition 4.1.** The scheme \( T^l(\mathcal{O}(d)) = Z(\phi^l(\mathcal{O}(d))) \) is the \( l \)'th incidence scheme of \( \mathcal{O}(d) \). The scheme \( D^l(\mathcal{O}(d)) = q(T^l(\mathcal{O}(d))) \) is the \( l \)'th discriminant of \( \mathcal{O}(d) \).

For \( l = 0 \) the Taylor morphism is the evaluation map

\[
T^0 : \mathcal{O}_P \otimes W \to \mathcal{O}(d)
\]

defined locally by

\[
T^0(U) : \mathcal{O}_P(U) \otimes W \to \mathcal{O}(d)(U)
\]

\[
T^0(U)(a \otimes s) = as|_U.
\]

Let \( U_{i,j} = D(y_i) \times D(x_j) \subseteq \mathbb{P}(W^*) \times_K P \) for fixed \( i, j \). Let \( j = 0 \) and \( t = x_1/x_0 \).

Restrict the map \( \phi^0(\mathcal{O}(d)) \) to \( U_{i,0} \) to get the following map

\[
K[u_0, \ldots, u_d][t] \frac{1}{y_i} \to K[u_0, \ldots, u_d][t] \otimes_K K\{s_0, \ldots, s_d\} \to K[u_0, \ldots, u_d][t]x_0^d
\]

defined by

\[
\phi^0(\mathcal{O}(d))(\frac{1}{y_i}) = u_0 + u_1t + u_2t^2 + \cdots + t_i + \cdots + u_dt^d = f(t).
\]

It follows

\[
I^0(\mathcal{O}(d))|_{U_{i,0}} = Z(f(t)) \subseteq \text{Spec}(K[u_0, \ldots, u_d][t]) = U_{i,0}.
\]

We get a map

\[
q|_{U_{i,0}} : I^0(\mathcal{O}(d))|_{U_{i,0}} \to D(y_i).
\]

Let \( Z_{i,j} = I^0(\mathcal{O}(d))|_{U_{i,j}} \).

We get an induced map

\[
q^\# : \mathcal{O}(D(y_i)) \to q_\ast\mathcal{O}_{Z_{i,j}}
\]

defined by the natural map

\[
q^\# : K[u_0, \ldots, u_d] \to K[u_0, \ldots, u_d][t]/(f(t)).
\]

It follows \( Ker(q^\#) = 0 \) hence \( q(Z_{i,0}) = D(y_i) \) for \( i = 0, \ldots, d \). It follows \( D^0(\mathcal{O}(d))|_{D(y_i)} \) is schematically dense in \( D(y_i) \).

Consider the map \( \phi^0(\mathcal{O}(d)) \) on the open set \( U_{i,1} = D(y_i) \times D(x_1) \). Let \( v = x_0/x_1 \).

We get the following map

\[
K[u_0, \ldots, u_d][v] \frac{1}{y_i} \to K[u_0, \ldots, u_d][v] \otimes_K K\{s_0, \ldots, s_d\} \to K[u_0, \ldots, u_d][v]x_1^d
\]
defined by
\[ \phi^0(\mathcal{O}(d))(\frac{1}{y_i}) = u_0v^d + u_1v^{d-1} + u_2v^{d-2} + \cdots + u_d = g(v). \]
It follows
\[ I^0(\mathcal{O}(d))|_{U_{i1}} = \mathbb{Z}(g(v)) \subseteq \text{Spec}(K[u_0, \ldots, u_d][v]) = U_{i1}. \]
We get a map
\[ q|_{U_{i1}} : I^0(\mathcal{O}(d))|_{U_{i1}} \to D(y_i). \]
Let \( Z_{i1} = I^0(\mathcal{O}(d))|_{U_{i1}} \). We get an induced map
\[ q^\#: \mathcal{O}_{D(y_i)} \to q_*\mathcal{O}_{Z_{i1}} \]
defined by the natural map
\[ q^\#: K[u_0, \ldots, u_d] \to K[u_0, \ldots, u_d][v]/(g(v)). \]
It follows \( \text{Ker}(q^\#) = (0) \) hence \( q(Z_{i1}) = D(y_i) \) for \( i = 0, \ldots, d \). It follows \( D^0(\mathcal{O}(d))|_{D(y_i)} \) is schematically dense in \( D(y_i) \).

**Proposition 4.2.** For all \( d \geq 1 \) it follows \( D^0(\mathcal{O}(d)) = \mathbb{P}(W^*) \).

**Proof.** By the above argument it follows \( q(I^0(\mathcal{O}(d))) = \mathbb{P}(W^*) \) and the Proposition is proved. \( \square \)

For all \( 1 \leq l \leq d \) we get on \( \mathbb{P}^1_K \) an exact sequence of locally free sheaves
\[(4.2.1) \quad 0 \to Q_{k,d} \to W \otimes \mathcal{O}_{\mathbb{P}^1_K} \to \mathcal{J}^l(\mathcal{O}(d)) \to 0 \]
where \( rk(\mathcal{J}^l(\mathcal{O}(d))) = l + 1, \) \( rk(Q_{k,d}) = d - k \) and \( rk(W \otimes \mathcal{O}_{\mathbb{P}^1_K}) = d + 1 \). Dualize the sequence \( 4.2.1 \) to get the sequence
\[ 0 \to \mathcal{J}^l(\mathcal{O}(d))^* \to W^* \otimes \mathcal{O}_{\mathbb{P}^1_K} \to Q_{k,d}^* \to 0. \]
Take relative projective space bundle to get a closed immersion
\[ \mathbb{P}(Q_{k,d}) \subseteq \mathbb{P}(W^*) \times_K \mathbb{P}^1_K. \]
It follows from \( \mathbb{P}(Q_{k,d}^*) = I^k(\mathcal{O}(d)) \) hence \( \dim(I^k(\mathcal{O}(d))) = \dim(\mathbb{P}(Q_{k,d})) = d - k - 1 + 1 = d - k. \)

**Lemma 4.3.** Assume \( p : X \to Y \) is any morphism of irreducible schemes with \( U \subseteq X \) and \( V \subseteq Y \) open subschemes with \( p(U) \subseteq V \). Assume the induced morphism \( \bar{p} : U \to V \) is integral. It follows \( \dim(X) = \dim(Y). \)

**Proof.** Since \( p : U \to V \) is integral it follows \( \dim(U) = \dim(V). \) We get since \( U, V \) are dense in \( X \) and \( Y \)
\[ \dim(X) = \dim(U) = \dim(V) = \dim(Y) \]
and the Lemma follows. \( \square \)

**Lemma 4.4.** Assume \( p : X \to Y \) is a proper morphism of irreducible schemes satisfying the conditions of Lemma \( \text{[4.3]} \). Assume \( Z \subseteq X \) is a closed irreducible subscheme. It follows \( \dim(Z) = \dim(p(Z)) \).
Proof. Let $U = \text{Spec}(B) \subseteq X$ and $V = \text{Spec}(A) \subseteq Y$ be open dense subschemes such that $p$ induce an integral morphism

$$\tilde{p} : U \rightarrow V.$$ 

It follows $U \cap Z \subseteq Z$ and $V \cap p(Z) \subseteq p(Z)$ are open dense subsets. We get an induced morphism

$$q : U \cap Z \rightarrow V \cap p(Z)$$

which is integral, hence $\dim(U \cap Z) = \dim(V \cap p(Z))$. It follows $\dim(Z) = \dim(p(Z))$ and the Lemma follows. \[ \square \]

Consider again the diagram

$$Y = \mathbb{P}(W^*) \times_A \mathbb{P}^n \xrightarrow{\pi} \mathbb{P}$$

$$\xrightarrow{\phi} \mathbb{P}(W^*) \rightarrow \text{Spec}(A)$$

when $A = K$ is a field and look at the sequence of locally free sheaves

$$\phi^\dagger(\mathcal{O}(d)) : \mathcal{O}(-1)Y \rightarrow W_Y \rightarrow \mathcal{J}^\dagger(\mathcal{O}(d))Y.$$ 

Consider the Taylor map on $\mathbb{P}^1_K$. Let $U_i = D(x_i)$ and $t = x_1/x_0$, $s = x_0/x_1$. On $U_0$ we get the following calculation:

$$T^i|_{U_0} : \mathcal{O}_{U_0} \otimes W \rightarrow \mathcal{J}^i(\mathcal{O}(d))|_{U_0}$$

looks as follows:

$$T^i|_{U_0} : K[t] \{ x_0^{d-i} x_1 \} \rightarrow K[t] \{ dt^j \otimes x_0^j \}$$

with

$$T^i(x_0^{d-i} x_1) = T^i(t^i x_0^d) = (t + dt)^i \otimes x_0^d.$$ 

Let $U_{ij} = D(y_j) \times D(x_j) \subseteq \mathbb{P}(W^*) \times \mathbb{P}^1$. Let $u_j = y_j/y_i$. The composed morphism

$$\phi^\dagger(\mathcal{O}(d))|_{U_{0j}} : K[u_0, ..., u_d][t] \frac{1}{y_i} \rightarrow K[u_0, ..., u_d][t] \otimes K \{ x_0^{d-i} x_1 \} \rightarrow K[u_0, ..., u_d][t] \{ dt^j \otimes x_0^j \}$$

is the following map:

$$\phi^\dagger(\mathcal{O}(d))|_{U_{0j}}(1/y_i) = (f(t), f'(t)/1!, ..., f^{(i)}(t)/i!)$$

where

$$f(t) = u_0 + u_1 t + \cdots + t^i + \cdots + u_d t^d.$$ 

It follows we get an equality of ideal sheaves on $U_{0j}$

$$\mathcal{I}_{T^i(\mathcal{O}(d))}|_{U_{0j}} = \{ f(t), f'(t), ..., f^{(i)}(t) \}.$$ 

Consider $D(x_1) \subseteq \mathbb{P}^1$ and let $s = x_0/x_1$. The Taylor map looks as follows

$$T^i : K[s] \otimes \{ x_0^{d-i} x_1 \} \rightarrow K[s] \{ ds^j \otimes x_0^j \}$$

with

$$T^i(x_0^{d-i} x_1) = T^i(s^i x_0^d) = (s + ds)^{d-i} \otimes x_0^d.$$ 

The composed morphism

$$\phi^\dagger(\mathcal{O}(d))|_{U_{1j}} : K[u_0, ..., u_d][s] \frac{1}{y_i} \rightarrow K[u_0, ..., u_d][s] \otimes K \{ x_0^{d-i} x_1 \} \rightarrow K[u_0, ..., u_d][s] \{ ds^j \otimes x_0^j \}$$
is the following map:
\[ \phi^l(\mathcal{O}(d))|_{U_{ij}}(1/y_i) = (g(s), g'(s)/1!, \ldots, g^{(l)}(s)/l!) \]
where
\[ g(s) = u_0 s^d + u_1 s^{d-1} + \ldots + s^{d-l} + \ldots + u_d. \]
We get an equality of ideal sheaves
\[ \mathcal{I}_{I^l(\mathcal{O}(d))}|_{U_{ij}} = \{g(s), g'(s), \ldots, g^{(l)}(s)\}. \]
Let \( V_{ij} = I^l(\mathcal{O}(d))|_{U_{ij}} \) and consider the morphism
\[ q_{ij} : V_{ij} \rightarrow q(V_{ij}) = D^l(\mathcal{O}(d)) \cap D(y_i). \]
Let \( j = 0 \). We get an induced map at rings
\[ q_{00}^\#: K[u_0, \ldots, u_d] \rightarrow K[u_0, \ldots, u_d][t]/(f(t), \ldots, f^{(l)}(t)). \]
Let \( P_j = \text{Res}(f^{(j)}(t), f^{(j+1)}(t)) \). It follows there is an equality
\[ \text{ker}(q_{00}^\#) = (P_0, \ldots, P_{l-1}). \]
We get an equality of ideal sheaves
\[ \mathcal{I}_{D^l(\mathcal{O}(d))}|_{D(y_i)} = \{P_0, \ldots, P_{l-1}\}. \]
Consider the morphism
\[ q_{l1}^\#: K[u_0, \ldots, u_d] \rightarrow K[u_0, \ldots, u_d][t]/(g(s), \ldots, g^{(l)}(s)/l!). \]
Let \( Q_j = \text{Res}(g^{(j)}(s), g^{(j+1)}(s)) \) it follows \( Q_j = P_j \). It follows
\[ \text{ker}(q_{l1}^\#) = (Q_0, \ldots, Q_{l-1}) = (P_0, \ldots, P_{l-1}) \]
hence \( \mathcal{I}_{D^l(\mathcal{O}(d))} \) is locally generated by \( l \) elements.

**Theorem 4.5.** The \( l \)th discriminant \( D^l(\mathcal{O}(d)) \) is an irreducible local complete intersection of dimension \( d - l \).

**Proof.** Since \( I^l(\mathcal{O}(d)) = \mathbb{P}(Q_{k,d}) \) is irreducible it follows \( D^l(\mathcal{O}(d)) \) is irreducible. Let
\[ q^l : I^l(\mathcal{O}(d)) \rightarrow D^l(\mathcal{O}(d)) \]
be the morphism induced by the projection morphism. Consider \( l = 0 \) and the open set \( U_{d,0} = D(y_d) \times D(x_0) \subseteq \mathbb{P}(W^*) \times_K \mathbb{P}_k^1 \). Let \( V_{d,0} = U_{d,0} \cap I^0(\mathcal{O}(d)) \). We get an induced morphism
\[ q^0 : V_{d,0} \rightarrow q(V_{d,0}) \]
of schemes. Let \( W_{d,0} = q(V_{d,0}) \). We get an induced morphism of sheaves
\[ q^\#: \mathcal{O}_{W_{d,0}} \rightarrow q_* \mathcal{O}_{V_{d,0}} \]
given by
\[ q^\#: B = K[u_0, \ldots, u_d] \rightarrow K[u_0, \ldots, u_d][t]/(f(t)) = B[t]/(f(t)) \]
where
\[ f(t) = u_0 + u_1 t + \ldots + u_{d-1} t^{d-1} + t^d \]
It follows the ring extension \( B \subseteq B[t]/(f(t)) \) is an integral extension hence the morphism
\[ q^0 : V_{d,0} \rightarrow q(V_{d,0}) \]
is an integral morphism. By Lemma 4.4 it follows \( \dim(D^i(O(d))) = \dim(I^i(O(d))) = d - i \). Since the ideal sheaf of \( D^i(O(d)) \) is locally generated by \( i \) elements the Theorem follows.

\[ \square \]

5. Discriminants of linear systems on flag varieties

Let in the following \( F \) be a fixed basefield of characteristic zero and let \( V \) be an \( N + 1 \)-dimensional vector space over \( F \). Let \( \mathbb{P}(V^*) \) be the projective space parametrizing lines in \( V \). The space \( \mathbb{P}(V^*) \) has the following property: There is a well-defined left action of \( SL(V) \) on \( \mathbb{P}(V^*) \) and this give an isomorphism \( SL(V)/P \cong \mathbb{P}(V^*) \) where \( P \subseteq SL(V) \) is the parabolic subgroup fixing a line \( L \) in \( V \). The quotient \( SL(V)/P \) is a geometric quotient in the sense of [15] and there is an equivalence of categories between the category of linear finite dimensional representations of \( P \) and the category of locally free \( \mathcal{O}_{\mathbb{P}(V^*)} \)-modules with an \( SL(V) \)-linearization. We want to study the discriminant \( D^k(O(d)) \) and the complex from 2.10.1 in this situation using \( SL(V) \)-modules.

Let \( \mathbb{P} = \mathbb{P}(V^*) \) and let \( \mathcal{I} \subseteq \mathbb{P} \times \mathbb{P} \) be the ideal of the diagonal. Let \( p,q : \mathbb{P} \times \mathbb{P} \to \mathbb{P} \) be the projection morphisms.

**Definition 5.1.** Let \( k \geq 1 \) and \( d \) be integers. We define the \( k \)th order jet bundle of \( O(d) \) as follows:

\[ J^k(O(d)) = p_*(O_{\mathbb{P} \times \mathbb{P}}/\mathcal{I}^{k+1} \otimes q^*O(d)). \]

The invertible sheaf \( O(d) \) has a unique \( SL(V) \)-linearization and by prolongation of this structure it follows \( J^k(O(d)) \) has a canonical \( SL(V) \)-linearization. Hence \( J^k(O(d)) \) corresponds to a unique finite dimensional \( P \)-module. The exterior product \( \wedge^j J^k(O(d)) \) has for all \( j \geq 1 \) a canonical \( SL(V) \)-linearization. It follows the higher cohomology group \( H^i(\mathbb{P}, \wedge^j J^k(O(d))) \) is a finite dimensional \( SL(V) \)-module for all \( i \geq 0 \).

**Theorem 5.2.** There is for any integers \( 1 \leq k < d \) and \( 1 \leq j \leq rk(J^k) \) an isomorphism

\[ H^0(\mathbb{P}, \wedge^j J^k(O(d))) \cong Sym^{j(d-k)}(V^*) \otimes \wedge^j Sym^k(V^*) \]

of \( SL(V) \)-modules. There is an equality \( H^i(\mathbb{P}, \wedge^j J^k(O(d))) = 0 \) if \( i > 0 \). There is an isomorphism of \( SL(V) \)-modules

\[ H^i(\mathbb{P}, \wedge^j J^k(O(d))^*) = Sym^{j(d-k)-n-1}(V) \otimes \wedge^j Sym^k(V^*) \]

if \( i = n \) and \( j(d-k)-n-1 \geq 0 \). If \( i = 0,..,n-1 \) or \( i = n \) and \( j(d-k)-n-1 < 0 \) it follows \( H^i(\mathbb{P}, \wedge^j J^k(O(d))^*) = 0 \).

**Proof.** Let \( \pi : \mathbb{P} \to S = \text{Spec}(F) \) be the structure morphism. The \( SL(V) \)-module \( Sym^k(V^*) \) is a locally free \( \mathcal{O}_S \)-module with an \( SL(V) \)-linearization. Pull back preserves the \( SL(V) \)-linearization hence \( \pi^* Sym^k(V^*) \) is a locally free \( \mathcal{O}_S \)-module with an \( SL(V) \)-linearization. We may consider the following locally free sheaf:

\[ O(d-k) \otimes \pi^* Sym^k(V^*). \]

Its corresponding \( P \)-module is \( Sym^k(V^*) \otimes Sym^{d-k}(L^*) \) hence by [\[16\]], Theorem 2.4 we get an isomorphism of locally free sheaves with \( SL(V) \)-linearization

\[ O(d-k) \otimes \pi^* Sym^k(V^*) \cong J^k(O(d)). \]
We get the following calculation:
\[ \wedge^j \mathcal{J}^k(O(d)) \cong \wedge^j (O(d - k) \otimes \pi^* \text{Sym}^k(V^*)) \cong \]
\[ O(j(d - k)) \otimes \wedge^j \pi^* \text{Sym}^k(V^*) \cong O(j(d - k)) \otimes \pi^* \wedge^j \text{Sym}^k(V^*) \]
of SL(V)-bundles. It follows there is for every \( j \geq 1 \) an isomorphism
\[ \wedge^j \mathcal{J}^k(O(d)) \cong O(j(d - k)) \otimes \pi^* \wedge^j \text{Sym}^k(V^*) \]
of SL(V)-bundles. We get the following calculation:
\[ H^i(\mathbb{P}, \wedge^j \mathcal{J}^k(O(d))) = R^i \pi_* (O(j(d - k)) \otimes \pi^* \wedge^j \text{Sym}^k(V^*)) \cong \]
\[ \wedge^j \text{Sym}^k(V^*) \otimes R^i \pi_* O(j(d - k)). \]
The first part of the Theorem now follows from the calculation of equivariant cohomology of invertible sheaves on projective space (see [5]).

By Theorem 2.4 in [7] we get an isomorphism
\[ \wedge^j \mathcal{J}^k(O(d))^* \cong O(j(k - d)) \otimes \pi^* \wedge^j \text{Sym}^k(V) \]
as SL(V)-bundles. We get using equivariant higher direct images the following calculation:
\[ H^i(\mathbb{P}, \wedge^j \mathcal{J}^k(O(d))^*) = R^i \pi_* (O(j(k - d)) \otimes \pi^* \wedge^j \text{Sym}^k(V)) = \]
\[ \wedge^j \text{Sym}^k(V) \otimes R^i \pi_* O(j(k - d)) = \wedge^j \text{Sym}^k(V) \otimes H^i(\mathbb{P}, O(j(k - d))). \]
The second part of the Theorem now follows from the calculation of equivariant cohomology of invertible sheaves on projective space (see [5]).

In several papers (see [7], [16] and [17]) the structure of the jet bundle on the projective line and projective space has been studied. In the paper [7] the \( P \)-module structure of the jet bundle on projective space was classified. The novelty of the result in Theorem 5.2 is the calculation of the SL(V)-module structure of the higher cohomology groups
\[ H^i(\mathbb{P}, \wedge^j \mathcal{J}^k(O(d))). \]
This is as indicated in the proof above a consequence of the result given in [7], Theorem 2.4 and equivariant projection formulas.

**Example 5.3.** Digression: The Borel-Weil-Bott Theorem.

For all invertible sheaves \( O(d) \) on \( \mathbb{P} \) with \( d \geq 1 \) the SL(V)-module \( H^0(\mathbb{P}, O(d)) = \text{Sym}^d(V^*) \) is irreducible. This is a particular case of the Borel-Weil-Bott Theorem. We see from Theorem 5.2 that this is no longer true if we consider higher rank SL(V)-linearized locally free sheaves on \( \mathbb{P} \). The SL(V)-module
\[ H^0(\mathbb{P}, \wedge^j \mathcal{J}^k(O(d))) = \text{Sym}^{j(d - k)}(V^*) \otimes \wedge^j \text{Sym}^k(V^*) \]
is never irreducible.

Let \( H \subseteq G \) be a closed subgroup. We refer to a \( G \)-module of the form \( H^i(G/H, E(\rho)) \) where \( i \geq 0 \) and \( E(\rho) \) is a \( G \)-linearized locally free \( O_{G/H} \)-module as a geometric \( G \)-module.

The following general result is true: Let \( P \subseteq G \) be a parabolic subgroup of a linear algebraic group of finite type over \( F \). The quotient \( G/P \) is a smooth projective scheme of finite type over \( F \) and \( G \)-linearized locally free \( O_{G/P} \)-modules are in one to one correspondence with rational \( P \)-modules.

**Proposition 5.4.** Let \( W \) be any \( G \)-module. It follows \( W \) is a geometric \( G \)-module.
Proof. Let 
\[ \pi : G/P \to \text{Spec}(F) \]
be the structure morphism. Let \( \rho : G \to \text{GL}(W) \) and consider the \( \mathcal{O}_{G/P} \)-module \( \mathcal{O}_{G/P} \otimes \pi^*W \). It follows
\[ (\mathcal{O}_{G/P} \otimes \pi^*W)(e) \cong F \otimes_F W \cong W \]
hence \( \mathcal{O}_{G/P} \otimes \pi^*W \) is the \( G \)-linearized locally free sheaf \( \mathcal{E}(\rho) \) corresponding to \( \rho \). Since \( W \) is a \( G \)-module it is also a \( P \)-module and the locally free sheaf \( \mathcal{E}(\rho) \) is a \( G \)-linearized \( \mathcal{O}_{G/P} \)-module of rank \( r = \dim(W) \). It is trivial as abstract locally free \( \mathcal{O}_{G/P} \)-module. It follows from the equivariant projection formul a (see [5])
\[ H^0(G/P, \mathcal{E}(\rho)) \cong R^0\pi_*(\mathcal{O}_{G/P} \otimes \pi^*W) \cong W \otimes R^0\pi_*\mathcal{O}_{G/P} \cong W \otimes_F F \cong W \]
since \( H^0(G/P, \mathcal{O}_{G/P}) \) is the trivial rank one \( G \)-module, and the Proposition follows. \( \square \)

Hence any \( G \)-module may be realized as the higher cohomology group of a \( G \)-linearized locally free \( \mathcal{O}_{G/P} \)-module.

**Example 5.5.** Decomposition of geometric \( \text{SL}(V) \)-modules.

One may ask the following general question: Assume \( G \) is a semisimple linear algebraic group and \( P \subseteq G \) a parabolic subgroup. Let \( \mathcal{L} \in \text{Pic}^G(G/P) \). One seeks a decomposition
\[ H^i(G/P, \wedge^j \mathcal{J}^k(\mathcal{L})) \cong \bigoplus \lambda V_\lambda \]
of the \( i \)'th cohomology group of \( \wedge^j \mathcal{J}^k(\mathcal{L}) \) into irreducible \( G \)-modules \( V_\lambda \). Since \( G \) is semi simple and \( H^i(G/P, \wedge^j \mathcal{J}^k(\mathcal{L})) \) is a finite dimensional \( G \)-module, such a decomposition always exist by the general theory of representations of semi simple algebraic groups. From Theorem 5.2 one gets such a decomposition on \( \text{SL}(V)/P = \mathbb{P}(V^*) \) by applying well known combinatorial formulas from the theory of Schur-Weyl modules. The \( \text{SL}(V) \)-module
\[ H^0(\mathbb{P}, \wedge^j \mathcal{J}^k(\mathcal{O}(d))) \cong \text{Sym}^{j(d-k)}(V^*) \otimes \wedge^j \text{Sym}^k(V^*) \]
is formed by applying compositions of Schur-Weyl modules to the standard representation \( V \) and its dual \( V^* \). Given two Schur-Weyl modules \( S_\lambda \) and \( S_\mu \) there is in general a decomposition
\[ (5.5.1) \quad S_\lambda(S_\mu(V^*)) = \bigoplus \lambda V_\lambda \]
of \( S_\lambda(S_\mu(V^*)) \) into irreducible \( \text{SL}(V) \)-modules. To calculate this decomposition - referred to as plethysm - is an unsolved problem in general. There are several formulas which are special cases of [5.5.1]. The Clebsch-Gordan formula which describes the decomposition of the tensor product of two irreducible modules into irreducibles, and the Cauchy formula which describe the decomposition of the symmetric product of the tensor product of two standard modules (see [2]). We leave it to the reader as an exercise to calculate this decomposition in the case of higher cohomology groups of exterior powers of jet bundles on projective space.

Let \( W = H^0(\mathbb{P}, \mathcal{O}(d)) \). There is by [2] on \( \mathbb{P} \) an exact sequence of locally free sheaves
\[ 0 \to Q_{k,d} \to \pi^*W \to \mathcal{J}^k(\mathcal{O}(d)) \to 0 \]
where the rightmost map is $T^k$ - the Taylor map. It follows $Q_{k,d}$ is a locally free sheaf. Dualize this sequence to get the short exact sequence
\[ 0 \to J^k(O(d))^* \to \pi^*W^* \to Q_{k,d}^* \to 0. \]
Take relative projective space bundle to get the following closed immersion of schemes:
\[ \mathbb{P}(Q_{k,d}^*) \subseteq \mathbb{P}(\pi^*W^*) \cong \mathbb{P}(W^*) \times \mathbb{P}. \]

**Lemma 5.6.** There is an equality
\[ \mathbb{P}(Q_{k,d}^*) = I^1(T^k) \]
of schemes.

**Proof.** Since $Q_{k,d}^* = \text{Coker}((T^k)^*)$ the claim follows from Theorem 2.5. □

There is a commutative diagram of maps of schemes
\[
\begin{array}{ccc}
P(Q_{k,d}^*) & \xrightarrow{i} & \mathbb{P}(W^*) \times \mathbb{P} \\
\downarrow{\tilde{q}} & & \downarrow{q} \\
D^1(T^k) & \xrightarrow{j} & \mathbb{P}(W^*) \xrightarrow{\pi} S
\end{array}
\]
where $i,j$ are the inclusion morphisms.

**Definition 5.7.** The scheme $D^k(O(d)) = D^1(T^k)$ is the $k$'th discriminant of the invertible sheaf $O(d)$.

**Example 5.8.** Irreducibility of $D^k(O(d))$.

Note: Since $\mathbb{P}(Q_{k,d}^*)$ is a projective bundle on $\mathbb{P}(V^*)$ it is irreducible. It follows $D^k(O(d))$ is irreducible for all $1 \leq k < d$.

The morphism
\[ \tilde{q} : \mathbb{P}(Q_{k,d}^*) \to D^k(O(d)) \]
is by definition a surjective map of schemes. Let $Y = \mathbb{P}(W^*) \times \mathbb{P}$. On $\mathbb{P}(W^*)$ there is an exact sequence of locally free sheaves
\[ 0 \to O(-1) \to \pi^*W. \]
Pull this map back to $Y$ to get the composed map
\[ \psi : O(-1)_Y \to W_Y \to T^k \mathcal{J}^k(O(d))_Y \]

**Proposition 5.9.** There is an equality $Z(\psi) = \mathbb{P}(Q_{k,d}^*)$ of schemes.

**Proof.** The claim follows from Theorem 2.5 since $Q_{k,d}^* = \text{Coker}((T^k)^*)$. □

We get an exact sequence
\[ O(-1)_Y \otimes \mathcal{J}^k(O(d))_Y^* \to O_Y \to O_{Z(\psi)} \to 0. \]

The ideal sheaf of $Z(\psi)$ is locally generated by a regular sequence hence $Z(\psi)$ is a local complete intersection. There is by general results (see [1]) a Koszul complex of sheaves on $Y$:
\[
\begin{align*}
0 & \to \wedge^N O(-1)_Y \otimes \mathcal{J}^k(O(d))_Y^* \to \cdots \\
\cdots & \to \wedge^2 O(-1)_Y \otimes \mathcal{J}^k(O(d))_Y^* \to O(-1)_Y \otimes \mathcal{J}^k(O(d))_Y^* \\
& \to O_Y \to O_{Z(\psi)} \to 0.
\end{align*}
\]
Hence we get a resolution of the sheaf $O_{Z(\psi)}$ on $Y$. There is an isomorphism

$$\mathcal{J}^k(O(d))^* \cong O(k-d) \otimes \pi^* \text{Sym}^k(V)$$

and an isomorphism

$$\wedge^i O(-1)_Y \otimes \mathcal{J}^k(O(d))_Y \cong q^* O(-j) \otimes \wedge^i p^* (O(k-d) \otimes \pi^* \text{Sym}^k(V)) \cong q^* O(-j) \otimes p^* O(j(k-d)) \otimes \pi^* \wedge^i \text{Sym}^k(V) \cong O(-j, j(k-d)) \otimes \pi^* \wedge^i \text{Sym}^k(V).$$

The complex now becomes

$$0 \to O(-N, N(k-d)) \otimes \pi^* \wedge^N \text{Sym}^k(V) \to \cdots$$

$$\cdots \to O(-j, j(k-d)) \otimes \pi^* \wedge^j \text{Sym}^k(V) \to \cdots$$

$$\cdots \to O(-1, k-d) \otimes \pi^* \text{Sym}^k(V) \to O_Y \to O_{Z(\psi)} \to 0.$$ 

There is a projection morphism

$$q : \mathbb{P}(W^*) \times \mathbb{P} \to \mathbb{P}(W^*)$$

and we want to push down the complex above to get a double complex on $\mathbb{P}(W^*)$. When we push the complex down to $\mathbb{P}(W^*)$ we get a double complex with terms given as follows:

**Theorem 5.10.** There is an isomorphism of $\text{SL}(V)$-linearized locally free sheaves

$$R^* q_* (\wedge^i O(-1)_Y \otimes \mathcal{J}^k(O(d))_Y) \cong O(-j) \otimes \pi^* \text{Sym}^{i(d-k)-n-1}(V) \otimes \wedge^j \text{Sym}^k(V^*)$$

if $j(d-k) - n - 1 \geq 0$. If $i = 0, \ldots, n-1$ or $i = n$ and $j(d-k) - n - 1 < 0$ it follows

$$R^i q_* (\wedge^j O(-1)_Y \otimes \mathcal{J}^k(O(d))_Y) = 0.$$ 

There is an isomorphism of $\text{SL}(V)$-linearized locally free sheaves

$$R^0 q_* (\wedge^j O(-1)_Y \otimes \mathcal{J}^k(O(d))_Y) \cong O(-j) \otimes \pi^* \text{Sym}^{j(d-k)}(V^*) \otimes \text{Sym}^k(V^*).$$

If $i > 0$ there is an equality

$$R^i q_* (\wedge^j O(-1)_Y \otimes \mathcal{J}^k(O(d))_Y) = 0.$$

**Proof.** We get by the projection formula and higher direct images for locally free sheaves with $\text{SL}(V)$-linearization (see [5]) the following calculation:

$$R^i q_* (\wedge^j O(-1)_Y \otimes \mathcal{J}^k(O(d))_Y) \cong$$

$$R^i q_* (\wedge^j q^* O(-1) \otimes p^* \mathcal{J}^k(O(d))^*) \cong$$

$$R^i q_* (q^* O(-j) \otimes \wedge^j p^* \mathcal{J}^k(O(d))^*) \cong$$

$$O(-j) \otimes R^i q_* p^* (\wedge^j \mathcal{J}^k(O(d))^*) \cong$$

$$O(-j) \otimes H^i(\mathbb{P}, \wedge^j \mathcal{J}^k(O(d))^*).$$

The first part of the Theorem now follows from Theorem 5.2. We get the following calculation:

$$R^i q_* (\wedge^j O(-1)_Y \otimes \mathcal{J}^k(O(d))_Y) \cong O(-j) \otimes \pi^* H^i(\mathbb{P}, \wedge^j \mathcal{J}^k(O(d))).$$

The Theorem now follows from Theorem 5.2. \qed

**Example 5.11.** Resolutions of ideal sheaves of discriminants.
Assume \(d - k - n - 1 \geq 0\), \(l(j) = j(d - k) - n - 1\) and \(r = rk(\mathcal{J}^k(\mathcal{O}(d)))\). The complex \(\mathcal{O}_P(V^*)\)-modules

\[
0 \to \mathcal{O}(-r) \otimes \text{Sym}^{(r)}(V) \otimes \wedge^r \text{Sym}^k(V^*) \to \cdots
\]

\[
\cdots \to \mathcal{O}(-j) \otimes \text{Sym}^{(j)}(V) \otimes \wedge^j \text{Sym}^k(V^*) \to \cdots
\]

\[
\mathcal{O}(-1) \otimes \text{Sym}^{(1)}(V) \otimes \text{Sym}^k(V^*) \to \mathcal{O}_P(V^*) \to \mathcal{O}_{D^k(\mathcal{O}(d))} \to 0.
\]

The hope is this complex can be used to construct a resolution of \(D^k(\mathcal{O}(d))\).

**Example 5.12.** Discriminants of linear systems on flag varieties.

In the following we use the notation of [12]. Let \(G = \text{SL}(E)\) where \(E\) is a vector space of finite dimension over a field \(F\) of characteristic zero. Let

\[
E_* : 0 \neq E_1 \subset E_2 \subset \cdots \subset E_k \subset E_{k+1} = E
\]

be a flag in \(E\) of type \(d = \{d_1, \ldots, d_k\}\). Let \(n_i = d_1 + \cdots + d_i\). It follows \(\text{dim}(E_i) = n_i\). Let \(P \subset G = \text{SL}(E)\) be the subgroup fixing the flag \(E_*\). It follows \(P\) is a parabolic subgroup and the quotient \(G/P\) is the flag variety of \(E\) of type \(d\). Let \(V_\lambda\) be an irreducible \(\text{SL}(E)\)-module with highest weight

\[
\lambda = \sum_{i=1}^k l_i(L_1 + \cdots + L_{n_i}) = \sum_{i=1}^k l_i \omega_{n_i}.
\]

Let \(\mathcal{L}(\mathfrak{l}) \in \text{Pic}^G(G/P)\) be the line bundle corresponding to \(\mathfrak{l} = \{l_1, \ldots, l_k\} \subset \mathbb{Z}^k = \text{Pic}^G(G/P)\). It follows there is by Theorem 2.2 in [12] an isomorphism

\[
V_\lambda \cong H^0(G/P, \mathcal{L}(\mathfrak{l}))^*
\]

of \(G\)-modules. Let

\[
T^k : H^0(G/P, \mathcal{L}(\mathfrak{l})) \otimes \mathcal{O}_{G/P} \to \mathcal{J}^k(\mathcal{L}(\mathfrak{l}))
\]

be the Taylor map of order \(k\).

**Theorem 5.13.** The discriminant \(D^k(\mathcal{L}(\mathfrak{l})) = D^1(T^k)\) is irreducible for all \(1 \leq k \leq \text{min}\{l_i + 1\}\).

**Proof.** From [12], Theorem 3.7 it follows the Taylor map

\[
T^k : H^0(G/P, \mathcal{L}(\mathfrak{l})) \to \mathcal{J}^k_{G/P}(\mathcal{L}(\mathfrak{l}))(e)
\]

is surjective for \(1 \leq k \leq \text{min}\{l_i + 1\}\). It follows we get an exact sequence of locally free sheaves

\[
0 \to Q \to H^0(G/P, \mathcal{L}(\mathfrak{l})) \otimes \mathcal{O}_{G/P} \to \mathcal{J}^k_{G/P}(\mathcal{L}(\mathfrak{l})) \to 0
\]

on \(G/P\). Dualize this sequence to get the sequence

\[
0 \to \mathcal{J}^k_{G/P}(\mathcal{L}(\mathfrak{l}))^* \to H^0(G/P, \mathcal{L}(\mathfrak{l}))^* \otimes \mathcal{O}_{G/P} \to Q^* \to 0
\]

We get a closed immersion

\[
\mathbb{P}(Q^*) \subset \mathbb{P}(H^0(G/P, \mathcal{L}(\mathfrak{l}))^*) \times G/P
\]

of schemes. Since \(Q\) is locally free \(Q^* = \text{Coker}(T^k)^*\), hence by Corollary 2.6 it follows \(D^k(\mathcal{L}(\mathfrak{l}))\) is irreducible and the Theorem is proved. \(\square\)
The total complex looks as follows:

\[
\text{Tot}(\mathcal{C}(T^k))_{ij} = \oplus_{i+j=n} \mathcal{O}(-j) \otimes \pi^* H^i(G/P, \wedge^j \mathcal{J}^k(\mathcal{L}^*)).
\]

Since \(H^i(G/P, \wedge^j \mathcal{J}^k(\mathcal{L}^*))\) is a finite dimensional \(G\)-module there is a decomposition

\[
(5.13.1) \quad H^i(G/P, \wedge^j \mathcal{J}^k(\mathcal{L}^*)) \cong \oplus \lambda W^\lambda
\]

into irreducible \(G\)-modules. Hence to check if the total complex \(\text{Tot}(\mathcal{C}(T^k))^\bullet\) can be used to give a resolution of the discriminant \(D^k(\mathcal{L})\) one has to calculate the decomposition (5.13.1).

In a series of papers the structure of the jet bundle \(\mathcal{J}^k(\mathcal{L})\) of a line bundle \(\mathcal{L} \in \text{Pic}^{SL(V)}(SL(V)/P)\) as abstract locally free sheaf, as left and right \(\mathcal{O}\)-module and as left and right \(P\)-module has been studied (see [7], [9], [10], [11] and [12]) using algebraic techniques, geometric techniques, algebraic group techniques and techniques from universal enveloping algebras of semi simple Lie algebras. There is work in progress using techniques similar to the ones introduced in this paper and the papers [11] and [12] on the problem of describing resolutions of ideal sheaves of \(D^k(\mathcal{L})\) where \(\mathcal{L}\) is a line bundle on \(SL(V)/P\) for some parabolic subgroup \(P \subseteq SL(V)\). This problem may be studied using the total complex and determinantal schemes.

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