Estimates for $p$-Laplace type equation in a limit case

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Abstract

We study the Dirichlet problem for a $p$–Laplacian type operator in the setting of the Orlicz–Zygmund space $L^q \log^{-\alpha} L(\Omega, \mathbb{R}^n)$, $q > 1$ and $\alpha > 0$. More precisely, our aim is to establish which assumptions on the parameter $\alpha > 0$ lead to existence, uniqueness of the solution and continuity of the associated nonlinear operator.

Keywords: Dirichlet problems, $p$–Laplace operators, existence, uniqueness, continuity, Orlicz–Sobolev spaces.

Mathematics Subject Classification (2000): 35J60

1 Introduction

Let $\Omega$ be a bounded Lipschitz domain of $\mathbb{R}^N$, $N \geq 2$. We consider the Dirichlet problem

\begin{equation}
\begin{cases}
\text{div} \, A(x, \nabla u) = \text{div} \, f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\end{equation}
where $A : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory vector field satisfying the following assumptions for a.e. $x \in \Omega$ and all $\xi, \eta \in \mathbb{R}^N$

$$
\langle A(x, \xi), \xi \rangle \geq a|\xi|^p \quad (1.2)
$$

$$
|A(x, \xi) - A(x, \eta)| \leq b|\xi - \eta|(|\xi| + |\eta|)^{p-2} \quad (1.3)
$$

$$
\langle A(x, \xi) - A(x, \eta), \xi - \eta \rangle \geq a|\xi - \eta|^2(|\xi| + |\eta|)^{p-2} \quad (1.4)
$$

where $p \geq 2$, $0 < a \leq b$.

Let $f = (f^1, f^2, \ldots, f^N)$ be a vector field of class $L^s(\Omega, \mathbb{R}^N)$, $1 \leq s \leq q$ where $q$ is the conjugate exponent of $p$, i.e. $pq = p + q$.

**Definition 1.1.** A function $u \in W^{1,r}_0(\Omega)$, $p - 1 \leq r \leq p$, is a solution of (1.1) if

$$
\int_\Omega \langle A(x, \nabla u), \nabla \varphi \rangle \, dx = \int_\Omega \langle f, \nabla \varphi \rangle \, dx, \quad (1.5)
$$

for every $\varphi \in C_0^\infty(\Omega)$.

By a routine argument, it can be seen that the identity (1.5) still holds for functions $\varphi \in W^{1,\frac{p-1}{p-1-r}}(\Omega)$ with compact support. We shall refer to such a solution as a distributional solution or (as some people say) as a very weak solution [17, 20].

We point out that, if $r < p$, such a solution may have infinite energy, i.e. $|\nabla u| \notin L^p(\Omega)$. The existence of a solution $u \in W^{1,p-1}_0(\Omega)$ to problem (1.1) is obtained in [1] when $\text{div} \, f$ belongs to $L^1(\Omega, \mathbb{R}^N)$. It is well known that the uniqueness of solutions to (1.1) in the sense of Definition 1.1 generally fails [26, 1]. Then, other possible definitions have been introduced, as the so-called duality solutions [27], the approximation solutions (SOLA) [5], the entropy solutions [25, 19, 6]. Recent results for the regularity of such solutions are given in [21, 22]. However, these ideas do not apply if one wants to investigate the uniqueness of a distributional solution. At the present time the problem remains unclear, unless for $p = 2$ [11] and $p = N$ [1]. In the case $p = 2$ the range of exponents $r$ allowing for a comprehensive theory is known, see [2, 18]. In the general case, uniqueness is proved in the setting of the grand Sobolev space (see [12]).

Our goal in the present paper is to study problem (1.1) assuming that the datum $f$ lies in the Orlicz–Zygmund space $L^q\log^{-\alpha} L(\Omega, \mathbb{R}^n)$, $\alpha > 0$. More precisely, our aim is to establish under which assumptions on the parameter $\alpha > 0$ we can define a continuous operator

$$
\mathcal{H} : L^q\log^{-\alpha} L(\Omega, \mathbb{R}^n) \to L^p\log^{-\alpha} L(\Omega, \mathbb{R}^n) \quad (1.6)
$$
which carries a given vector field $f$ into the gradient field $\nabla u$.

In the case $\alpha \leq 0$, in the literature there are several results on the continuity of the operator defined in (1.6) [23, 8, 14]. Moreover, as a consequence of the results in [10] and [4] and the interpolation theorem of [3], when $p = 2$ the operator $\mathcal{H}$ is Lipschitz continuous for any $-\infty < \alpha < \infty$. Actually, for $p = 2$ and suitable $\alpha > 0$, the existence for problem (1.1) is also ensured for not uniformly elliptic equations [24].

Here we consider the case $p > 2$. Our main results are the following.

**Theorem 1.1.** For each $f \in L^q \log^{-\alpha} L(\Omega, \mathbb{R}^n)$, $1 < q < 2$ and $0 < \alpha < \frac{p}{p-2}$, the problem (1.1) admits a unique solution $u : \Omega \rightarrow \mathbb{R}$ such that

$$\|\nabla u\|_{L^p \log^{-\alpha} L(\Omega, \mathbb{R}^n)} \leq C \|f\|_{L^q \log^{-\alpha} L}$$

Moreover, the operator $\mathcal{H}$ is continuous.

**Theorem 1.2.** There exists a constant $C > 0$ depending on $n, p, \alpha, a$ and $b$ such that, if $f$ and $g$ belong to $L^q \log^{-\alpha} L(\Omega, \mathbb{R}^n)$, $1 < q < 2$ and $0 < \alpha < \frac{p}{p-2}$, then

$$\|\mathcal{H}f - \mathcal{H}g\|_{L^p \log^{-\alpha} L} \leq C \left( \|f\|_{L^q \log^{-\alpha} L}^{q(1-\gamma)} \|f\|_{L^q \log^{-\alpha} L} \right) \left( \|f\|_{L^q \log^{-\alpha} L}^{\gamma} \right),$$

where $\gamma = \frac{\alpha p}{p-2}$.

We point out that Theorem 1.1 improves the result of [12] in two different directions. First of all, when $0 < \alpha < \frac{p}{p-2}$, it gives higher integrability of the solutions found in [12]. On the other hand, the case $\alpha = \frac{p}{p-2}$ is not covered by [12].

In the particular case that the vector field $A$ takes the form

$$A(x, \xi) = \langle A(x)\xi, \xi \rangle^{\frac{p-2}{2}} A(x)\xi$$

(1.9)

where $A : \Omega \rightarrow \mathbb{R}^{N \times N}$ is a measurable, symmetric, uniformly elliptic matrix field, we also prove a stability theorem for solutions to problem (1.1) in terms of the characteristic of $A$ (see Section 3). The characteristic of the symmetric matrix field $A : \Omega \rightarrow \mathbb{R}^{N \times N}$ (see [13]) is defined as the quantity

$$K_A = \text{ess sup}_{x \in \Omega} \left( 1 + |A(x) - I|^{\frac{p}{2}} \right).$$

(1.10)

Observe that $K_A \geq 1$ and $K_A = 1$ if and only if $A$ is the identity matrix.
Theorem 1.3. Assume that $A: \Omega \to \mathbb{R}^{N \times N}$ is a measurable symmetric matrix field satisfying the ellipticity bounds
\[
a^\frac{2}{p} |\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq b^\frac{2}{p} |\xi|^2, \tag{1.11}\]
for a.e. $x \in \Omega$, for every $\xi \in \mathbb{R}^N$. There exists a constant $C > 0$ depending on $n, p, \alpha, a$ and $b$ such that, if $u, v \in W^{1, L^p \log^{-\alpha}}(\Omega)$, with $0 < \alpha < \frac{1}{p-2}$, verify
\[
\begin{cases}
\text{div} \left( \langle A(x)\nabla u, \nabla u \rangle^{\frac{p-2}{2}} A(x)\nabla u \right) = \text{div} \left( |\nabla v|^{p-2} \nabla v \right) & \text{in } \Omega, \\
u = v & \text{on } \partial \Omega,
\end{cases}
\tag{1.12}
\]
then
\[
\|\nabla u - \nabla v\|_{L^p \log^{-\alpha} L^p} \leq C (K_A - 1)^{\gamma(1-\gamma)} K_A^{\gamma(\gamma+1)} \| |\nabla u| + |\nabla v|\|_{L^p \log^{-\alpha} L^p} \tag{1.13}
\]
where $\gamma = \frac{p-2}{p}$.

The main tool to prove our results is the Hodge decomposition and fine properties of the norm in the Zygmund spaces developed in Section 2.

2 Preliminary results

2.1 Basic notation

We indicate that quantities $a, b \geq 0$ are equivalent by writing $a \sim b$; namely, $a \sim b$ will mean that there exist constants $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$. Similarly, $a \lesssim b$ ($a \gtrsim b$ respectively) will mean that there exists $c > 0$ such that $a \leq cb$ ($a \geq cb$ respectively).

From now on, $\Omega$ will denote a bounded Lipschitz domain in $\mathbb{R}^N$. For a function $v \in L^p(\Omega)$ with $1 \leq p < \infty$ we set
\[
\|v\|_p = \left( \int_\Omega |v|^p \, dx \right)^{\frac{1}{p}}.
\]
Barred integrals denote averages, namely $\bar{f}_\Omega = \frac{1}{|\Omega|} \int_\Omega f$. 

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2.2 Marcinkiewicz Spaces

For $0 < p < \infty$, the Marcinkiewicz space weak-$L^p(\Omega)$, also denoted by $L^{p, \infty}(\Omega)$, consists of all measurable functions $g : \Omega \to \mathbb{R}$ such that

$$
\|g\|_{L^{p, \infty}(\Omega)}^p = \|g\|_{p, \infty}^p = \sup_{t > 0} t^p |\{ x \in \Omega : |g(x)| > t \}| < \infty.
$$

A useful property of the Marcinkiewicz norm is given by the following identities

$$
\|g^\alpha\|_{p, \infty}^p = \|g\|_{\alpha p, \infty}^{\alpha p} \quad \text{for } \alpha > 0. \tag{2.1}
$$

For $1 < q < p$ one has

$$
L^{p, \infty}(\Omega) \subset L^q(\Omega).
$$

We shall appeal to the following Hölder type inequality

$$
\|v\|_{L^q(E)} \leq \left( \frac{p}{p - q} \right)^{\frac{1}{q}} |E|^{-\frac{1}{q}} |\Omega|^{\frac{1}{q} - \frac{1}{p}} \|v\|_{L^{p, \infty}(\Omega)} \tag{2.2}
$$

which holds true for $v \in L^{p, \infty}(\Omega)$, $E \subset \Omega$ and $q < p$.

2.3 Grand Lebesgue and grand Sobolev Spaces

For $1 < p < \infty$ we denote by $L^p(\Omega)$ the grand–Lebesgue space $L^p(\Omega)$ consisting of all functions $v \in \bigcap_{0 < \varepsilon \leq p-1} L^{p-\varepsilon}(\Omega)$ such that

$$
\|v\|_p = \sup_{0 < \varepsilon \leq p-1} \varepsilon^{\frac{1}{p}} \left( \int_{\Omega} |v|^{p-\varepsilon} \, dx \right)^{\frac{1}{p-\varepsilon}} < \infty. \tag{2.3}
$$

Moreover

$$
\|v\|_p \sim \sup_{0 < \varepsilon \leq p-1} \varepsilon^{\frac{1}{p}} \left( \int_{\Omega} |v|^{p-\varepsilon} \, dx \right)^{\frac{1}{p-\varepsilon}}. \tag{2.4}
$$

The Marcinkiewicz class weak – $L^p(\Omega)$ is contained in $L^p(\Omega)$ (see [15, Lemma 1.1]).

More generally, if $\alpha > 0$ we denote by $L^{\alpha,p}(\Omega)$ the grand–Lebesgue space consisting of all functions $v \in \bigcap_{0 < \varepsilon \leq p-1} L^{p-\varepsilon}(\Omega)$ such that

$$
\|v\|_{\alpha,p} = \sup_{0 < \varepsilon \leq p-1} \varepsilon^{\frac{\alpha}{p}} \left( \int_{\Omega} |v|^{p-\varepsilon} \, dx \right)^{\frac{1}{p-\varepsilon}} < \infty. \tag{2.5}
$$
2.4 Zygmund spaces

We shall need to consider the Zygmund space \( L^q \log^{-\alpha} L(\Omega) \), for \( 1 < q < \infty \), \( \alpha > 0 \). This is the Orlicz space generated by the function

\[
\Phi(t) = t^q \log^{-\alpha}(a + t), \quad t \geq 0,
\]

where \( a \geq e \) is a suitably large constant, so that \( \Phi \) is increasing and convex on \([0,\infty[\). The choice of \( a \) will be immaterial. More explicitly, for a measurable function \( f \) on \( \Omega \), \( f \in L^q \log^{-\alpha} L(\Omega) \) simply means that

\[
\int_{\Omega} |f|^q \log^{-\alpha}(a + |f|) \, dx < \infty.
\]

It is customary to consider the Luxemburg norm

\[
[f]_{L^q \log^{-\alpha} L} = \inf \left\{ \lambda > 0 : \int_{\Omega} \Phi(|f|/\lambda) \, dx \leq 1 \right\},
\]

and \( L^q \log^{-\alpha} L(\Omega) \) is a Banach space. However, we shall introduce an equivalent norm, which involves the norms in \( L^{q-\varepsilon}(\Omega) \), for \( 0 < \varepsilon \leq q - 1 \), and is more suitable for our purposes. For \( f \) measurable on \( \Omega \), we set

\[
\|f\|_{L^q \log^{-\alpha} L} = \left\{ \int_0^{\varepsilon_0} \varepsilon^{q-1} \|f\|_{L^{q-\varepsilon}}^q \, d\varepsilon \right\}^{1/q} \tag{2.6}
\]

Here \( \varepsilon_0 \in (0, q - 1] \) is fixed. The following is a refinement of a result of [11].

**Lemma 2.1.** We have \( f \in L^q \log^{-\alpha} L(\Omega) \) if and only if

\[
\|f\|_{L^q \log^{-\alpha} L} < \infty. \tag{2.7}
\]

Moreover, \( \|f\|_{L^q \log^{-\alpha} L} \) is a norm equivalent to the Luxemburg one, that is, there exist constants \( C_i = C_i(q, \alpha, a, \varepsilon_0) \), \( i = 1, 2 \), such that for all \( f \in L^q \log^{-\alpha} L(\Omega) \)

\[
C_1 \, [f]_{L^q \log^{-\alpha} L} \leq \|f\|_{L^q \log^{-\alpha} L} \leq C_2 \, [f]_{L^q \log^{-\alpha} L}.
\]

**Proof.** It is easy to check that \( \|f\|_{L^q \log^{-\alpha} L} \) defined by (2.6) is a norm.

Let \( f \) be a measurable function defined in \( \Omega \). We clearly have

\[
|f|^q(a + |f|)^{-\varepsilon} \leq |f|^{q-\varepsilon} \leq 2^{q-1}[a^q + |f|^q(a + |f|)^{-\varepsilon}],
\]

for a.e. in \( \Omega \), hence integrating

\[
\int_{\Omega} |f|^q(a + |f|)^{-\varepsilon} \, dx \leq \|f\|_{L^{q-\varepsilon}}^q \leq 2^{q-1}a^q + 2^{q-1} \int_{\Omega} |f|^q(a + |f|)^{-\varepsilon} \, dx.
\]
This in turn implies
\[
\int_0^{\varepsilon_0} \varepsilon^{a-1} \left[ \int_{\Omega} |f|^q (a + |f|)^{-\varepsilon} \, dx \right] d\varepsilon \leq \int_0^{\varepsilon_0} \varepsilon^{a-1} \|f\|^{q-\varepsilon}_{q-\varepsilon} \, d\varepsilon
\]
(2.8)
Moreover,
\[
\int_0^{\varepsilon_0} \varepsilon^{a-1} (a + |f|)^{-\varepsilon} \, d\varepsilon = \log^{-\alpha}(a + |f|) \int_0^{\varepsilon_0 \log(a + |f|)} \tau^{a-1} e^{-\tau} \, d\tau
\]
and
\[
\int_0^{\varepsilon_0 \log a} \tau^{a-1} e^{-\tau} \, d\tau \leq \int_0^{\varepsilon_0 \log(a + |f|)} \tau^{a-1} e^{-\tau} \, d\tau \leq \int_0^{\infty} \tau^{a-1} e^{-\tau} \, d\tau
\]
Therefore from (2.8) we get
\[
C_3 \int_{\Omega} |f|^q \log^{-\alpha}(a + |f|) \, dx \leq \int_0^{\varepsilon_0} \varepsilon^{a-1} \|f\|^{q-\varepsilon}_{q-\varepsilon} \, d\varepsilon
\]
\[
\leq C_4 \left[ 1 + \int_{\Omega} |f|^q \log^{-\alpha}(a + |f|) \, dx \right]
\]
(2.9)
for some positive constants.
Assume now that \( f \) satisfies (2.7). As
\[
\|f\|^{q-\varepsilon}_{q-\varepsilon} \leq \|f\|^{q}_{q-\varepsilon} + 1
\]
we see that the first term of (2.9) is finite, so \( f \in L^q \log^{-\alpha} \mathcal{L}(\Omega) \). Furthermore, if \( \|f\|_{L^q \log^{-\alpha} \mathcal{L}} = 1 \), then (2.9) implies
\[
\int_{\Omega} |f|^q \log^{-\alpha}(a + |f|) \, dx \leq C_5
\]
for a constant independent of \( f \). By homogeneity,
\[
[f]_{L^q \log^{-\alpha} \mathcal{L}} \leq C_5 \|f\|_{L^q \log^{-\alpha} \mathcal{L}}
\]
(2.10)
for all \( f \).
In case \( f \in L^q \log^{-\alpha} \mathcal{L}(\Omega) \), since the Zygmund space is continuously embedded in the gran Lebesgue space \( \mathcal{L}^{\alpha,q} \) (see [15]), there exists a constant \( C_6 > 0 \) such that
\[
\|f\|_{q-\varepsilon} \leq C_6 \varepsilon^{-\alpha/q} [f]_{L^q \log^{-\alpha} \mathcal{L}},
\]

thus
\[ \|f\|_{q-\varepsilon}^q = \|f\|^q_{q-\varepsilon} \leq \|f\|_{q-\varepsilon}^q C_7 \|f\|_L^{q \log -\alpha} \]
and by (2.9) we get (2.7). In fact, if \( \|f\|_{L^q \log -\alpha} = 1 \), then we have
\[ \|f\|_{L^q \log -\alpha} \leq C_8 \]
and by homogeneity we conclude with the reverse inequality to (2.10).

**Remark 2.2.** We examine the dependence of \( \|f\|_{L^q \log -\alpha} \) defined by (2.6), on the parameter \( \varepsilon_0 \). For fixed \( 0 < \varepsilon_0 \leq \varepsilon_1 \leq q - 1 \), by Hölder's inequality we have
\[ \|f\|_{q-\varepsilon} \leq \|f\|_{q-\varepsilon_0/\varepsilon_1} \]
and hence
\[ \int_{\varepsilon_0}^{\varepsilon_1} \varepsilon^{\alpha-1} \|f\|_{q-\varepsilon}^q d\varepsilon \leq \int_{\varepsilon_0}^{\varepsilon_1} \varepsilon^{\alpha-1} \|f\|_{q-\varepsilon}^q d\varepsilon \leq \left( \frac{\varepsilon_1}{\varepsilon_0} \right)^\alpha \int_{\varepsilon_0}^{\varepsilon_1} \varepsilon^{\alpha-1} \|f\|_{q-\varepsilon}^q d\varepsilon. \]  
(2.11)

**Remark 2.3.** It is clear that (2.7) implies \( f \in L^{\alpha,q}(\Omega) \). We remark that the norm (2.6) compares in a very simple way with \( \|f\|_{L^{\alpha,q}} \). Indeed, as \( \varepsilon \mapsto \|f\|_{q-\varepsilon} \) is decreasing, for all \( \sigma \in [0, q - 1] \) we have
\[ \left\{ \int_0^\sigma \varepsilon^{\alpha-1} \|f\|_{q-\varepsilon}^q d\varepsilon \right\}^{1/q} \geq \|f\|_{q-\sigma} \left( \frac{\sigma^{\alpha}}{\alpha} \right)^{1/q}, \]  
(2.12)
hence by (2.11)
\[ \|f\|_{L^{\alpha,q}} \leq \left( \frac{q - 1}{\varepsilon_0} \right)^{\alpha/q} \|f\|_{L^q \log -\alpha}. \]  
(2.13)

Moreover, using (2.6), the inclusion \( L^{\alpha,q}(\Omega) \subset L^q \log ^{-\beta}(\Omega) \) for \( \beta > \alpha \) (see (11)) is trivial:
\[ \int_{\varepsilon_0}^{\varepsilon_1} \varepsilon^{\alpha-1} \|f\|_{q-\varepsilon}^q d\varepsilon = \int_{\varepsilon_0}^{\varepsilon_1} \varepsilon^{\alpha} \|f\|_{q-\varepsilon}^{q(\varepsilon^{\beta-\alpha} - 1)} d\varepsilon \]
and then
\[ \|f\|_{L^q \log ^{-\beta}} \leq \left( \frac{\varepsilon_0^{\beta-\alpha}}{\beta - \alpha} \right)^{1/q} \|f\|_{L^{\alpha,q}}. \]
We point out that a simple application of the Lebesgue dominated convergence theorem proves that

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{\alpha/q} \|f\|_{q-\varepsilon} = 0,$$

(2.14)

for all \( f \in L^q \log^{-\alpha} L(\Omega) \), see [11]. Actually, (2.14) follows directly from (2.7), since it implies that the left hand side of (2.12) tends to 0 as \( \sigma \downarrow 0 \).

We stress that (2.14) does not hold uniformly, as \( f \) varies in a bounded set of \( L^q \log^{-\alpha} L(\Omega) \). Indeed, for each \( \varepsilon > 0 \) sufficiently small so that \( \Phi(e^{1/\varepsilon}) > 1 \), we choose a measurable subset \( E \subset \Omega \) verifying (\( \Omega \) has no atoms)

$$|E| = |\Omega| e^{-q/\varepsilon} \log^{-\alpha}(a + e^{1/\varepsilon}) = |\Omega|/\Phi(e^{1/\varepsilon})$$

and set

$$f = f_\varepsilon = e^{1/\varepsilon} \chi_E.$$

Then we find that \([f]_{L^q \log^{-\alpha} L} \equiv 1\), while

$$\|f\|_{q-\varepsilon} = e^{1/\varepsilon} e^{-1/q \varepsilon/(q-\varepsilon)} \log^{\alpha/(q-\varepsilon)}(a + e^{1/\varepsilon})$$

$$= e^{-1/(q-\varepsilon)} \log^{\alpha/(q-\varepsilon)}(a + e^{1/\varepsilon})$$

and

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{\alpha/q} \|f\|_{q-\varepsilon} = e^{-1/q}.$$

**Lemma 2.4.** For each relatively compact subset \( M \subset L^q \log^{-\alpha} L(\Omega) \), condition (2.14) holds uniformly for \( f \in M \), that is

$$\lim_{\varepsilon \downarrow 0} \left( \sup_{f \in M} \varepsilon^{\alpha/q} \|f\|_{q-\varepsilon} \right) = 0.$$

**Proof.** For simplicity, we assume \( \varepsilon_0 = q - 1 \). As \( M \) is totally bounded, fixed arbitrarily \( \sigma > 0 \) we find a finite number of elements \( f_1, \ldots, f_k \in M \) with the property that, \( \forall f \in M, \exists j \in \{1, \ldots, k\} \):

$$\varepsilon^{\alpha/q} \|f - f_j\|_{q-\varepsilon} \leq \|f - f_j\|_{L^\alpha} \leq \alpha^{1/q} \|f - f_j\|_{L^q \log^{-\alpha} L} < \sigma,$$

for all \( \varepsilon \in [0, \varepsilon_0] \). Above, we used (2.13). Moreover, \( \exists \varepsilon_\sigma \in [0, \varepsilon_0] \) such that

$$\varepsilon^{\alpha/q} \|f_j\|_{q-\varepsilon} < \sigma, \quad \forall \varepsilon \in [0, \varepsilon_\sigma], \forall j \in \{1, \ldots, k\}.$$

Therefore, we conclude easily for any \( f \in M \) and \( \varepsilon \in [0, \varepsilon_\sigma] \)

$$\varepsilon^{\alpha/q} \|f\|_{q-\varepsilon} \leq \varepsilon^{\alpha/q} (\|f_j\|_{q-\varepsilon} + \|f - f_j\|_{q-\varepsilon}) < 2\sigma.$$

In particular, if \((f_n)_{n \in \mathbb{N}}\) is a converging sequence in \( L^q \log^{-\alpha} L(\Omega) \), then

$$\lim_{\varepsilon \downarrow 0} \left( \sup_n \varepsilon^{\alpha/q} \|f_n\|_{q-\varepsilon} \right) = 0.$$

9
2.5 Sobolev space $W^{1,L^p\log^{-\alpha}L_0}(\Omega)$

For a bounded domain $\Omega \subset \mathbb{R}^N$, let $W^{1,L^p\log^{-\alpha}L_0}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\| = \|\nabla u\|_{L^q\log^{-\alpha}L^1}. $$

3 Proof of Theorem 1.1 and Theorem 1.2

Our goal in this section is to prove Theorem 1.1.

3.1 A fundamental lemma

Assume that $A = A(x, \xi)$ satisfies (1.2)–(1.4). For $\psi \in W^{1,L^p}(\Omega)$, we consider the equations

$$\text{div } A(x, \nabla u) = \text{div } f \text{ in } \Omega, \quad (3.1)$$
$$\text{div } A(x, \nabla v) = \text{div } g \text{ in } \Omega, \quad (3.2)$$

with $f, g \in L^{q-\varepsilon}(\Omega, \mathbb{R}^n)$, $0 < \varepsilon < 1$. Let $u, v \in \mathcal{W}^{1,p-\varepsilon}(\Omega)$ be solutions to (3.1) and (3.2) respectively such that

$$u - v \in \mathcal{W}^{1,p-\varepsilon}_0(\Omega)$$

Then

**Lemma 3.1.** There exists $0 < \varepsilon_p(n) < 1/p$ and a constant $C > 0$ depending on $n, p, \alpha, a$ and $b$ such that the following uniform estimate holds

$$\|\nabla u - \nabla v\|_{p-\varepsilon}^p \leq C \left( \varepsilon_p(n)^{\frac{p-1}{p}} \|\nabla u\| + \|\nabla v\|_{p-\varepsilon}^p + \|f - g\|_{q-\varepsilon}^q \right), \quad (3.3)$$

for every $0 < \varepsilon < \varepsilon_p(n)$.

**Proof of Lemma 3.1.** The proof is achieved with a similar argument as in [12]. We sketch it for the sake of completeness.

Since $\Omega$ is Lipschitz, we may use the Hodge decomposition of the vector field $|\nabla u - \nabla v|^{-\varepsilon}(\nabla u - \nabla v) \in \mathcal{L}^{\frac{p-\varepsilon}{p}}(\Omega)$ (see [15, 16]), namely

$$|\nabla u - \nabla v|^{-\varepsilon}(\nabla u - \nabla v) = \nabla \varphi + h, \quad (3.4)$$
for some \( \varphi \in \mathcal{W}_0^{1,1-\varepsilon p} (\Omega) \) and some divergence free vector field \( h \in \mathcal{L}^{p-\varepsilon p} (\Omega) \). Moreover, fixed \( 0 < \varepsilon_p(n) < 1/p \), for every \( 0 < \varepsilon < \varepsilon_p(n) \) the following estimates hold (see [10])

\[
\| \nabla \varphi \|_{p-\varepsilon p} \leq C(n,p) \| \nabla u - \nabla v \|_{p-\varepsilon p} \quad (3.5)
\]

\[
\| h \|_{p-\varepsilon p} \leq C(n,p) \varepsilon \| \nabla u - \nabla v \|_{p-\varepsilon p} \quad (3.6)
\]

From condition (1.4) we obtain

\[
\| \nabla u - \nabla v \|_{p-\varepsilon p} \leq \int_{\Omega} \left( |\nabla u| + |\nabla v| \right)^{p-2} |\nabla u - \nabla v \|^{\varepsilon p} \ dx
\]

\[
\leq \frac{1}{a} \int_{\Omega} \left\langle A(x, \nabla u) - A(x, \nabla v), |\nabla u - \nabla v \|^{\varepsilon p} (\nabla u - \nabla v) \right\rangle \ dx
\]

(3.7)

By Definition 1.1, we are legitimate to use \( \varphi \) as a test function for equations in both (3.1) and (3.2) respectively. Then

\[
\| \nabla u - \nabla v \|_{p-\varepsilon p} \leq \frac{1}{a} \left[ \int_{\Omega} \left\langle f - g, \nabla \varphi \right\rangle \ dx + \int_{\Omega} \left\langle A(x, \nabla u) - A(x, \nabla v), h \right\rangle \ dx \right]
\]

(3.8)

With the aid of condition (1.3) and the Hölder’s inequality, we get

\[
\| \nabla u - \nabla v \|_{p-\varepsilon p} \leq \frac{1}{a} \left[ \| f - g \|_{q-\varepsilon q} \| \nabla \varphi \|_{p-\varepsilon p}^{q-\varepsilon q} \right.
\]

\[
\left. + b \| \nabla u \| + |\nabla v|^{p-2} \| \nabla u - \nabla v \|_{p-\varepsilon p} \| h \|_{p-\varepsilon p} \right]
\]

(3.9)

which, in view of (3.5) and (3.6), yields

\[
\| \nabla u - \nabla v \|_{p-\varepsilon p} \leq C \left[ \| f - g \|_{q-\varepsilon q} \| \nabla u - \nabla v \|_{p-\varepsilon p}^{1-\varepsilon p} \right.
\]

\[
\left. + \varepsilon \| \nabla u \| + |\nabla v|^{p-2} \| \nabla u - \nabla v \|_{p-\varepsilon p}^{2-\varepsilon p} \right]
\]

(3.10)

where \( C = C(n,p,a,b) \). With the aid of Young’s inequality we obtain

\[
\| \nabla u - \nabla v \|_{p-\varepsilon p}^{p-1} \leq C \| f - g \|_{q-\varepsilon q} + C \| \nabla u \| + |\nabla v|^{p-2} \| \nabla u - \nabla v \|_{p-\varepsilon p}
\]

\[
\leq C \left( \| f - g \|_{q-\varepsilon q} + \varepsilon \| \nabla u \| + |\nabla v|^{p-1} \right)
\]

\[
+ \frac{1}{(p-1)2^{p-1}} \| \nabla u - \nabla v \|_{p-\varepsilon p}^{p-1}
\]

(3.11)
Once the latter term is absorbed by the left hand side, we have
\[ \| \nabla u - \nabla v \|_{p-\varepsilon p} \leq C \left( \| f - g \|_{q-\varepsilon q} + \varepsilon^{\frac{p-1}{p}} \| \nabla u \| + \| \nabla v \|^{p-1}_{p-\varepsilon p} \right) \] (3.12)
which corresponds to the estimate we wanted to prove. \qed

**Corollary 3.2.** Under the assumptions of Lemma 3.1, if \( u = v \) on \( \partial \Omega \), there exists \( 0 < \varepsilon_0 < 1/p \) and a constant \( C > 0 \) depending on \( n, p, \alpha, a \) and \( b \) such that, for any \( 0 < \varepsilon < \varepsilon_0 \) the following uniform estimate holds
\[ \| \nabla u - \nabla v \|_{p-\varepsilon p} \leq C \left( \varepsilon^{\frac{p-1}{p-2}} \| f \|_{q-\varepsilon q} + \| f - g \|_{q-\varepsilon q} \right), \] (3.13)

**Proof.** For \( g = 0 \) and \( v = 0 \), estimate (3.3) reduces to
\[ \| \nabla u \|_{p-\varepsilon p} \leq C \left( \| f \|_{q-\varepsilon q} + \varepsilon^{\frac{p-1}{p}} \| \nabla u \|^{p-1}_{p-\varepsilon p} \right) \] (3.14)
which gives, for \( C \varepsilon^{\frac{p-1}{p-2}} < 1 \)
\[ \| \nabla u \|_{p-\varepsilon p}^{p-1} \leq C \| f \|_{q-\varepsilon q}^{q-1} \] (3.15)
Similarly, one has
\[ \| \nabla v \|_{p-\varepsilon p}^{p-1} \leq C \| g \|_{q-\varepsilon q}^{q-1} \] (3.16)
Inserting (3.15) and (3.16) into (3.12), we finally get (3.13). \qed

### 3.2 Uniqueness

Under the assumptions of Theorem 1.1, if \( f = g \), estimate (3.13) reduces to
\[ \| \nabla u - \nabla v \|_{p-\varepsilon p} \leq C \varepsilon^{\frac{p-1}{p}} \| f \|_{q-\varepsilon q} \] (3.17)
Then, if \( f \in L^q \log^{-\alpha} L(\Omega, \mathbb{R}^n) \), \( 0 < \alpha \leq p/(p-2) \), uniqueness follows from (2.14) letting \( \varepsilon \to 0^+ \) in (3.17). Actually, we can prove a stronger uniqueness result.

**Theorem 3.3.** Assume (1.2) - (1.4) hold. There exist \( s \in (p-1/p, p) \) depending only on \( n, p, a \) and \( b \), such that if \( u, v \in W^{1,1}(\Omega) \) satisfy \( u - v \in W^{0,1,1}(\Omega) \), \( \nabla u \in L^s \log^{-\alpha} L(\Omega, \mathbb{R}^n) \), \( 0 < \alpha \leq p/(p-2) \), \( \nabla v \in L^s(\Omega, \mathbb{R}^n) \) and
\[ \text{div} \, A(x, \nabla u) = \text{div} \, A(x, \nabla v) \] (3.18)
then \( u = v \) in \( \Omega \).
Proof of Theorem 3.3. Arguing as in Lemma 3.1, we decompose the vector field $|\nabla u - \nabla v|^{\varepsilon_p} (\nabla u - \nabla v) \in \mathcal{L}^{\frac{p}{p-\varepsilon_p}}(\Omega)$ and for $f = g$ we get the following estimate

$$\|\nabla u - \nabla v\|_{p-\varepsilon_p}^p \leq C\varepsilon^{\frac{p}{p-\varepsilon_p}} \|\nabla u\| + |\nabla v| \|p-\varepsilon_p\|^p \tag{3.19}$$

which yields

$$\|\nabla u - \nabla v\|_{p-\varepsilon_p}^p \leq C\varepsilon^{\frac{p}{p-\varepsilon_p}} \left( \|\nabla u - \nabla v\|_{p-\varepsilon_p}^p + \|\nabla u\|_{p-\varepsilon_p}^p \right) \tag{3.20}$$

for $0 < \varepsilon < \varepsilon_p(n)$ and $C = C(n, p, a, b)$. Now, if $0 < \varepsilon < \min \{ \varepsilon_p(n), 1/C^{\frac{p-2}{p}} \}$, the first term in the right hand side can be absorbed by the left hand side of (3.20) and so

$$\|\nabla u - \nabla v\|_{p-\varepsilon_p}^p \leq \left( \frac{C\varepsilon}{1-C\varepsilon} \right)^{\frac{p}{p-2}} \|\nabla u\|_{p-\varepsilon_p}^p \tag{3.21}$$

The conclusion of Theorem 3.3 follows by (2.14), as $\varepsilon \to 0^+$ in (3.21). \qed

The previous theorem improves the uniqueness result of [12], which does not cover the case $\alpha = p/(p-2)$. We point out that our result also improves the result in [7], since the Marcinkiewicz space $weak - \mathcal{L}^p$ is contained in $\mathcal{L}^{p,\infty}(\Omega)$ when $1 < \alpha \leq p/(p-2)$. Actually, estimate (3.3) allows us to give a simple proof of [7, Theorem 4.2]. Arguing as in Theorem 3.3, we arrive at (3.21) for $|\nabla u| \in \mathcal{L}^{p,\infty}(\Omega)$ and $|\nabla v| \in \mathcal{L}^p(\Omega)$. Then, by Hölder’s inequality (2.2) we get

$$\|\nabla u - \nabla v\|_{p-\varepsilon_p}^p \leq C\varepsilon^{\frac{p}{p-2}-1} \|\nabla u\|_{p,\infty}^p \tag{3.22}$$

and letting $\varepsilon \to 0^+$ we have $u = v$ in $\Omega$.

3.3 Existence

Let $f \in \mathcal{L}^q \log^{-\alpha} \mathcal{L}(\Omega, \mathbb{R}^N), 1 < q < 2$ and $0 < \alpha \leq p/(p-2)$. The aim of this subsection is to prove the existence in Theorem 1.1. As a preliminary step, we show that, if $(f_n)_n$ is a converging sequence in $\mathcal{L}^q \log^{-\alpha} \mathcal{L}(\Omega, \mathbb{R}^N)$, such that for each $n$

$$\begin{cases} 
\text{div} \mathcal{A}(x, \nabla u_n) = \text{div} f_n \\
u_n = 0 \quad \text{on} \ \partial \Omega 
\end{cases} \tag{3.23}
$$

then $(\nabla u_n)_n$ is a Cauchy sequence in $\mathcal{L}^p \log^{-\alpha} \mathcal{L}(\Omega, \mathbb{R}^N)$. To prove this, we first note that, by Lemma 2.4, if we fix a $\sigma > 0$, we find $\vartheta \in ]0, 1]$ such that, if $0 < \varepsilon < \vartheta\varepsilon_p(n)$, then

$$\varepsilon^\alpha \|f_m\| + |f_n|^{q-\varepsilon_p} < \sigma,$$
for all $m, n \in \mathbb{N}$. Hence (3.13) with $f_m, f_n$ in place of $f, g$, and $u_m, u_n$ in place of $u, v$, respectively, yields

\[
\|\nabla u_m - \nabla u_n\|_{p-\varepsilon p}^p \lesssim \sigma + \|f_m - f_n\|_{q-\varepsilon q}^q. \tag{3.24}
\]

We multiply both sides by $\varepsilon^{\alpha-1}$ and integrate with respect to $\varepsilon$ on $(0, \vartheta \varepsilon_p(n))$. For $\delta = \varepsilon p/\vartheta \geq \varepsilon p$, we have

\[
\|\nabla u_m - \nabla u_n\|_{p-\varepsilon p} \geq \|\nabla u_m - \nabla u_n\|_{p-\varepsilon},
\]

hence

\[
\int_0^{\vartheta \varepsilon_p(n)} \varepsilon^{\alpha-1} \|\nabla u_m - \nabla u_n\|_{p-\varepsilon p}^p d\varepsilon \geq \left(\frac{\vartheta}{p}\right)^\alpha \int_0^{\varepsilon_0} \delta^{\alpha-1} \|\nabla u_m - \nabla u_n\|_{p-\varepsilon}^p d\delta, \tag{3.25}
\]

where $\varepsilon_0 = p\varepsilon_p(n)$. On the other hand,

\[
\int_0^{\vartheta \varepsilon_p(n)} \varepsilon^{\alpha-1} d\varepsilon = \frac{(\vartheta \varepsilon_p(n))^\alpha}{\alpha}
\]

and (setting here $\delta = \varepsilon q$)

\[
\int_0^{\vartheta \varepsilon_p(n)} \varepsilon^{\alpha-1} \|f_m - f_n\|_{q-\varepsilon q}^q d\varepsilon \leq q^{-\alpha} \int_0^{\varepsilon_0} \delta^{\alpha-1} \|f_m - f_n\|_{q-\varepsilon}^q d\delta. \tag{3.26}
\]

Therefore, recalling definition (2.6), from (3.24) we get

\[
\|\nabla u_m - \nabla u_n\|_{L^p L^{p-\varepsilon p}} \lesssim \sigma + \|f_m - f_n\|_{L^q L^{q-\varepsilon q}}. \tag{3.27}
\]

with no restrictions on $m, n \in \mathbb{N}$. Now, as the sequence $(f_n)_n$ converges in $L^q L^{p-\varepsilon p} L(\Omega)$, we have

\[
\|f_m - f_n\|_{L^q L^{p-\varepsilon p} L} < \sigma,
\]

provided $m$ and $n$ are sufficiently large, hence

\[
\|\nabla u_m - \nabla u_n\|_{L^p L^{p-\varepsilon p} L} \lesssim \sigma
\]

proving that $(\nabla u_n)_n$ is a Cauchy sequence as desired.

Now we are in a position to prove existence of solution for problem (1.1). Indeed, we approximate the vector field $f$ in the right hand side of the equation by $f_n \in L^q(\Omega, \mathbb{R}^N)$, $n = 1, 2, \ldots$, such that $f_n \to f$ in
\( L^q \log^{-\alpha} L(\Omega, \mathbb{R}^N) \), and for each \( n \) we consider the (unique) solution \( u_n \) to the problem

\[
\begin{align*}
\text{div} \mathcal{A}(x, \nabla u_n) &= \text{div} f_n \\
 u_n &\in W^{1,p}_0(\Omega)
\end{align*}
\]

(3.28)

Using what we have seen above, \((u_n)_n\) converges in \( W^{1,L^p \log^{-\alpha} L^p}_0(\Omega) \), that is, there exists \( u \in W^{1,L^p \log^{-\alpha} L^p}_0(\Omega) \) such that \( u_n \to u \). To conclude that \( u \) solves (1.1), we only need to note, that by (1.3) we can pass to the limit as \( n \to \infty \) into the equation of (3.28), getting

\[
\text{div} \mathcal{A}(x, \nabla u) = \text{div} f,
\]

since \( \nabla u_n \to \nabla u \) in \( L^{p-1}(\Omega, \mathbb{R}^N) \) in particular.

The estimate (1.7) follows from (3.15), by the same argument used above, by integrating with respect to \( \varepsilon \).

Also continuity of the operator \( \mathcal{H} \) follows. Indeed, clearly \( f_n \to f \) in \( L^q \log^{-\alpha} L^q(\Omega, \mathbb{R}^n) \).

Proof of Theorem 1.2: Let now \( 0 < \alpha < p/(p-2) \) and let \( f, g \in L^q \log^{-\alpha} L^q(\Omega, \mathbb{R}^n) \).

Denote by \( u \) and \( v \) the solutions of (3.1) and (3.2), of class \( W^{1,L^p \log^{-\alpha} L^p}_0(\Omega) \), respectively. To prove (1.8), we multiply both sides of (3.13) by \( \varepsilon^{\alpha-1} \) and integrate with respect to \( \varepsilon \) on \( (0, \vartheta \varepsilon_p(n)) \), for fixed \( \vartheta \in [0,1] \). Similarly as for (3.25) and (3.26), we have

\[
\int_0^{\varepsilon_p(n)} \varepsilon^{\alpha-1} \|
abla u - \nabla v\|_{p-\varepsilon}^p d\varepsilon \geq \left( \frac{\vartheta}{p} \right) \int_0^{\varepsilon_0} \delta^{\alpha-1} \|
abla u - \nabla v\|_{p-\delta}^p d\delta,
\]

(3.29)

\[
\int_0^{\varepsilon_p(n)} \varepsilon^{\alpha-1} ||f - g||_{q-\varepsilon}^q d\varepsilon \leq q^{-\alpha} \int_0^{\varepsilon_0} \delta^{\alpha-1} ||f - g||_{q-\delta}^q d\delta.
\]

(3.30)

respectively. On the other hand,

\[
\int_0^{\varepsilon_p(n)} \varepsilon^{\frac{p}{p-2}+\alpha-1} ||f||_{L^p \log^{-\alpha} L^p} + ||g||_{L^p \log^{-\alpha} L^p} \geq \left( \frac{\varepsilon_p(n)}{q} \right)^{\frac{p}{p-2}} \int_0^{\varepsilon_0} \delta^{\alpha-1} ||f||_{L^p \log^{-\alpha} L^p} + ||g||_{L^p \log^{-\alpha} L^p} d\delta.
\]

(3.31)

and therefore we get

\[
\|
abla u - \nabla v\|_{L^p \log^{-\alpha} L^p} \leq \varepsilon^{\frac{p}{p-2}-\alpha} ||f||_{L^p \log^{-\alpha} L^p} + ||g||_{L^p \log^{-\alpha} L^p} \leq \varepsilon^{\frac{p}{p-2}-\alpha} ||f||_{L^q \log^{-\alpha} L^q} + ||g||_{L^q \log^{-\alpha} L^q}.
\]

(3.32)

For

\[
\varepsilon^{\frac{p}{p-2}} = \frac{||f||_{L^q \log^{-\alpha} L^q} + ||g||_{L^q \log^{-\alpha} L^q}}{||f||_{L^p \log^{-\alpha} L^p} + ||g||_{L^p \log^{-\alpha} L^p}}
\]

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we obtain estimate (1.8). In particular, for \( g = 0 \) and \( v = 0 \),
\[
\|\nabla u\|_{L^p \log^{-\alpha} \mathcal{L}}^p \lesssim \|f\|_{L^q \log^{-\alpha} \mathcal{L}}^q .
\] (3.33)

**Remark 3.4.** Assume \( f \) and \( g \) in \( L^q \log^{-\alpha} \mathcal{L}(\Omega, \mathbb{R}^N) \), and let \( u \) and \( v \) in \( \mathcal{W}^{1, p} \log^{-p/2} \mathcal{L}_0(\Omega) \) solve (3.1) and (3.2) respectively. For \( 0 \leq \alpha < p/(p - 2) \), we can prove that
\[
\|\nabla u - \nabla v\|_{L^p \log^{-\alpha} \mathcal{L}}^p \lesssim \|f - g\|_{L^q}^q .
\] (3.34)

Indeed, in the case \( \alpha = 0 \), passing to the limit as \( \varepsilon \to 0 \) in (3.3), we find that \( \nabla u - \nabla v \in L^p(\Omega, \mathbb{R}^N) \) and
\[
\|\nabla u - \nabla v\|_{L^p \log^{-\alpha} \mathcal{L}}^p \lesssim \|f - g\|_{L^q}^q .
\] (3.34)

In the case \( 0 < \alpha < p/(p - 2) \), similarly as for (3.31) we find (\( \theta = 1 \))
\[
\int_0^{\varepsilon_p(n)} \frac{\varepsilon^{p/2 - \alpha - 1}}{\theta^\alpha} \|f + |g|\|_{L^q}^q d\varepsilon \leq \left( \frac{\varepsilon_p(n)^\alpha}{\theta^\alpha} \right) \int_0^{\varepsilon_p(n)} \frac{\varepsilon^{p/2 - \alpha - 1}}{\theta^\alpha} \|f + |g|\|_{L^q}^q d\theta .
\] (3.35)

By (3.29), (3.30) and (3.35) we get
\[
\|\nabla u - \nabla v\|_{L^p \log^{-\alpha} \mathcal{L}}^p \lesssim \||f\|_{L^q \log^{-\alpha} \mathcal{L}}^p \lesssim \|f\|_{L^q \log^{-\alpha} \mathcal{L}}^q + \|f - g\|_{L^q \log^{-\alpha} \mathcal{L}}^q .
\] (3.36)

**Proof of Theorem 1.3.** Under assumption (1.11) it is easy to verify that \( A(x, \xi) \) defined in (1.9) satisfies assumptions (1.2)–(1.4) with \( \lambda = a \). By arguing as in the proof of Lemma 3.1, as in [9] we get
\[
\|\nabla u - \nabla v\|_{L^{p \varepsilon_p}}^p \leq C(n, p, a, b) \left\{ \left( K_A - 1 \right) \frac{\varepsilon^{p+1}}{\varepsilon_p^p} ||\nabla v||_{L^{p \varepsilon_p}} + \varepsilon^{p+1} ||\nabla u| + |\nabla v||_{L^{p \varepsilon_p}} \right\}
\] (3.37)

which holds true as long as \( \varepsilon \in (0, \varepsilon_p(n)) \) for some \( \varepsilon_p(n) > 0 \). Let us fix some \( \vartheta \in (0, 1) \) which will be properly chosen later. Let us consider the integrals
\[
I_1 = \int_0^{\varepsilon_p(n)} \varepsilon^{\alpha - 1} \|\nabla u - \nabla v\|_{L^{p \varepsilon_p}} d\varepsilon
\]
\[
I_2 = \int_0^{\varepsilon_p(n)} \varepsilon^{\alpha - 1} \|\nabla v\|_{L^{p \varepsilon_p}} d\varepsilon
\]
\[
I_3 = \int_0^{\varepsilon_p(n)} \varepsilon^{p+1} \|\nabla u| + |\nabla v||_{L^{p \varepsilon_p}} d\varepsilon
\] (3.38)
so that estimate (3.37) infers

\[ I_1 \leq C(n, p, a, b) \left\{ (K_A - 1)^{\frac{p}{p-1}} I_2 + I_3 \right\} \]  \hspace{1cm} (3.39)

We set

\[ \delta = \frac{\varepsilon p}{\vartheta} \]

Since \( \delta \geq \varepsilon p \), a use of Holder’s inequality allow us to obtain

\[ I_1 \geq \int_0^{\vartheta \varepsilon_p(n)} \varepsilon^{\alpha-1} \| \nabla u - \nabla v \|_{p-\delta}^p d\varepsilon = \left( \frac{\vartheta}{p} \right)^\alpha \int_0^{\vartheta \varepsilon_p(n)} \delta^{\alpha-1} \| \nabla u - \nabla v \|_{p-\delta}^p d\delta \]  \hspace{1cm} (3.40)

On the other hand, since \( 0 \leq \vartheta \leq 1 \) we have

\[ I_2 \leq \int_0^{\vartheta \varepsilon_p(n)} \varepsilon^{\alpha-1} \| \nabla v \|_{p-\vartheta}^p d\varepsilon = \frac{1}{p^\alpha} \int_0^{\vartheta \varepsilon_p(n)} \delta^{\alpha-1} \| \nabla v \|_{p-\delta}^p d\delta \]  \hspace{1cm} (3.41)

Similarly,

\[ I_3 \leq (\vartheta \varepsilon_p(n))^{\frac{p}{p-2}} \int_0^{\vartheta \varepsilon_p(n)} \varepsilon^{\alpha-1} \| \nabla u + |\nabla v| \|_{p-\vartheta}^p d\varepsilon = C(n, p, \alpha) \int_0^{\vartheta \varepsilon_p(n)} \delta^{\alpha-1} \| \nabla u + |\nabla v| \|_{p-\delta}^p d\delta \]  \hspace{1cm} (3.42)

Combining (3.40), (3.41) and (3.42) with (3.37) we have

\[ \vartheta^{\alpha} \| \nabla u - \nabla v \|_{\mathcal{L}^p \log^{-\alpha}}^p \leq C \left\{ (K_A - 1)^{\frac{p}{p-1}} \| \nabla v \|_{\mathcal{L}^p \log^{-\alpha}}^p + \vartheta^{\frac{p}{p-2}} \| \nabla u + |\nabla v| \|_{\mathcal{L}^p \log^{-\alpha}}^p \right\} \]  \hspace{1cm} (3.43)

Now, we pick \( \vartheta \) in such a way that

\[ \vartheta^{\frac{p}{p-2}} = \left( \frac{K_A - 1}{K_A} \right)^{\frac{p}{p-1}} \]

Hence, (3.43) may be rewritten as

\[ \| \nabla u - \nabla v \|_{\mathcal{L}^p \log^{-\alpha}}^p \leq C (K_A - 1)^{\frac{p}{p-1}} - \alpha \frac{p-2}{p-1} K_A^{\frac{p}{p-1}} \left\{ \left( \frac{p}{p-1} - \alpha \right) \| \nabla v \|_{\mathcal{L}^p \log^{-\alpha}}^p \right. \]

\[ \left. + \| \nabla u + |\nabla v| \|_{\mathcal{L}^p \log^{-\alpha}}^p \right\} \]  \hspace{1cm} (3.44)

Finally, (1.13) is proved.  \( \square \)

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