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Inclusion Relations for Dini Functions Involving Certain Conic Domains

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Abstract: In recent years, special functions such as Bessel functions have been widely used in many areas of mathematics and physics. We are essentially motivated by the recent development; in our present investigation, we make use of certain conic domains and define a new class of analytic functions associated with the Dini functions. We derive inclusion relationships and certain integral preserving properties. By applying the Bernardi-Libera-Livingston integral operator, we obtain some remarkable applications of our main results. Finally, in the concluding section, we recall the attention of curious readers to studying the \( q \)-generalizations of the results presented in this paper. Furthermore, based on the suggested extension, the \( (p,q) \)-extension will be a relatively minor and unimportant change, as the new parameter \( p \) is redundant.

Keywords: analytic function; univalent function; starlike and convex function; subordination; quasi-convex functions; Bessel functions; Dini functions

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1. Introduction, Definitions and Motivation

The theory of special functions is an important component in most branches of mathematics. Special functions are applied in complicated mathematical calculations by engineers and scientists. The applications are covering a wide range of fields in physics, engineering and computer sciences. In the field of computer science, special functions known as activation functions have a very important place. The widespread usage of these functions has attracted a large number of researchers to work in many areas. The study of the geometric properties of special functions such as Bessel functions, hypergeometric functions, Mittag-Leffler functions, Struve functions, Wright functions and other related functions is a continuing aspect of geometric function theory research. Some geometric properties of these functions can be found in [1–3].

Bessel functions are a set of solutions to a second-order differential equation that can appear in a variety of contexts. Bessel functions were initially employed by Bessel to explain three-body motion, with the Bessel functions appearing in the series expansion on planetary perturbation [4]. Euler, Lagrange and the Bernoullis also examined the Bessel functions. Now, consider the equation
\[
\zeta^2 w''(\zeta) + \zeta w'(\zeta) + \left(\zeta^2 - u^2\right)w(\zeta) = 0, \quad u, \zeta \in \mathbb{C},
\]
(1)

which is a second-order linear homogeneous differential equation. The Equation (1) is well-known Bessel’s differential equation. Its solution is known as the Bessel function and is represented by \( J_u(\zeta) \), where \( J_u(\zeta) \) is given as follows:

\[
J_u(\zeta) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+u+1)} \left(\frac{\zeta}{2}\right)^{u+2n}, \quad \zeta \in \mathbb{C},
\]
(2)

where \( \Gamma \) represents the well-known Euler Gamma function; also, the well-known Pochhammer symbol \((\gamma)_n\) is given by the relation

\[
(\gamma)_n = \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)},
\]

or

\[
(\gamma)_n = \begin{cases} 
1 & (n = 0) \\
\gamma(\gamma + 1)(\gamma + 2)...(\gamma + n - 1) & (n \in \mathbb{N} \text{ and } \gamma \in \mathbb{C}).
\end{cases}
\]

See [5] for a thorough examination of the first-order Bessel function.

The Dini function is a kind of special function and is the combination of the Bessel function of the first kind and is defined by

\[
r_u(\zeta) = (1-u)J_u(\zeta) + \zeta J'_u(\zeta)
\]

Moreover, the normalized Dini functions \( d_u(\zeta) : E \to \mathbb{C} \) can be defined as (see [6])

\[
d_u(\zeta) = 2^u \Gamma(u+1) \zeta^{1-\frac{u}{2}} \left(1 - u\right) \phi_u \left(\sqrt{\zeta}\right) + \sqrt{\zeta} \psi_u \left(\sqrt{\zeta}\right)
\]

\[
= \zeta + \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1) \Gamma(u+1) \zeta^{n+1}}{4^n n! \Gamma(u+n+1)},
\]

where

\[
u \in \mathbb{R}, \quad n \in \mathbb{N} \quad \text{and} \quad k_1 = u \notin \mathbb{Z}_0^{-1} = \{0, -1, -2, \ldots\}.
\]

Let the symbol \( A \) mean the class of all analytic functions \( t \) in the open unit disk:

\[
E = \{ \zeta : \zeta \in \mathbb{C} \text{ and } |\zeta| < 1 \}
\]

and having a Taylor-Maclaurin form as

\[
t(\zeta) = \zeta + \sum_{n=2}^{\infty} a_n \zeta^n.
\]
(3)

Let \( S \subset A \), consisting of univalent functions in \( E \) and with the condition

\[
t(0) = 0 = t'(0) - 1.
\]

Moreover, all normalized univalent functions in \( E \) are contained in the set \( A \subset S \). For two given functions \( g_1, g_2 \in A \), we say that \( g_1 \) is subordinate to \( g_2 \), written symbolically as \( g_1 \preceq g_2 \), if there exists a Schwarz function \( w \), which is holomorphic in the open unit disk \( E \) with

\[
w(0) = 0 \quad \text{and} \quad |w(\zeta)| < 1,
\]
so that

\[ g_1(\zeta) = g_2(w(\zeta)) \quad (\zeta \in E). \]

Moreover, if the function \( g_2 \) is univalent in \( E \), then the following equivalence hold true:

\[ g_1 \prec g_2 \iff g_1(0) = g_2(0) \text{ and } g_1(E) \subset g_2(E). \quad (4) \]

For two given functions \( t \) and \( g \) from the class \( \mathcal{A} \), the convolution (or Hadamard product) of these functions is given by

\[ (t \ast g)(\zeta) = \zeta + \sum_{n=2}^{\infty} a_n b_n g^n \]

Let \( \mathcal{P} \) be the class of Carathéodory functions, consisting of all analytic functions \( p \) satisfying the conditions

\[ \Re(p(\zeta)) > 0 \quad (\zeta \in E) \]

and

\[ p(\zeta) = 1 + \sum_{n=1}^{\infty} c_n g^n. \]

Robertson [7] introduced and studied the classes of starlike \( \mathcal{S}^*(\gamma) \) and convex \( \mathcal{C}(\gamma) \) functions of order \( \gamma \) as follows

\[ t \in \mathcal{A} \text{ and } \Re\left( \frac{\zeta t'(\zeta)}{t(\zeta)} \right) > \gamma \quad (0 \leq \gamma < 1) \]

and

\[ t \in \mathcal{A} \text{ and } \Re\left( 1 + \frac{\zeta t''(\zeta)}{t'(\zeta)} \right) > \gamma \quad (0 \leq \gamma < 1), \]

respectively. Note that \( t \in \mathcal{C}(\gamma) \) if and only if \( \zeta t' \in \mathcal{S}^*(\gamma) \) (see also Srivastava and Owa [8]).

In 1964, Libera [9] introduced the class \( \mathcal{K}(\gamma, \alpha) \) of close to convex functions of order \( \gamma \) and type \( \alpha \) \((0 \leq \alpha < 1)\), which is defined by

\[ \mathcal{K}(\gamma, \alpha) = \left\{ t \in \mathcal{A} \text{ and } g \in \mathcal{S}^*(\alpha) : \Re\left( \frac{\zeta t'(\zeta)}{g(\zeta)} \right) > \gamma \right\} \quad (0 \leq \gamma < 1). \]

Many researchers have recently examined different classes of analytic and univalent functions in various areas; see for more information [10–18]. By taking inspiration from the above-cited work, Shams et al. [19] introduced the domain \( \Omega_{k,\gamma} \) for \( 0 \leq k < \infty \) as follows:

\[ \Omega_{k,\gamma} = \left\{ u + iv : (u - \gamma)^2 > k^2((u-1)^2 + v^2) \right\}. \]

Note that, for \( 0 < k < 1 \),

\[ \Omega_{k,\gamma} = \left\{ u + iv : \left(\frac{u + \frac{2\gamma - \gamma^2}{1 - \gamma}}{k \left( \frac{1}{1 - \gamma} \right)} \right)^2 - \left( \frac{v}{\frac{1 - \gamma}{\sqrt{1-\gamma}}} \right) > 1 \right\}, \]

for \( k > 1 \),

\[ \Omega_{k,\gamma} = \left\{ u + iv : \left(\frac{u + \frac{2\gamma - \gamma^2}{1 - \gamma}}{k \left( \frac{1}{1 - \gamma} \right)} \right)^2 - \left( \frac{v}{\frac{1 - \gamma}{\sqrt{1-\gamma}}} \right) < 1 \right\}. \]

Here, \( Q_{k,\gamma}(\zeta) \) are the extremal functions for the conic domain, \( \Omega_{k,\gamma} \), and are given below:
\[ Q_{k,\gamma}(\zeta) = \begin{cases} 
Q_1(\zeta) & (k = 0) \\
Q_2(\zeta) & (k = 1) \\
Q_3(\zeta) & (0 < k < 1) \\
Q_4(\zeta) & (k > 1) 
\end{cases} \] (5)

where

\[ Q_1(\zeta) = \frac{1 + (1 - 2\gamma)\zeta}{1 - \zeta}, \]

\[ Q_2(\zeta) = 1 + \frac{2(1 - \gamma)}{\pi^2} \left( \log \left( \frac{1 + \sqrt{\zeta}}{1 - \sqrt{\zeta}} \right) \right)^2, \]

\[ Q_3(\zeta) = 1 + \frac{2(1 - \gamma)}{1 - k^2} \times \sinh^2 \left\{ \left( \frac{\pi}{2} \arccos k \right) \right\} \]

\[ Q_4(\zeta) = \frac{(1 - \gamma) \sin \left( \frac{\pi}{2K(i)} \int_0^{\frac{\pi}{2K(i)}} \frac{1}{\sqrt{1 - x^2}} \sqrt{1 - (ix)^2} \, dx \right)}{k^2 - 1} + \frac{k^2 - \gamma}{k^2 - 1}, \]

and \( i \in (0, 1), k = \cosh \left( \frac{\pi K'(i)}{K(i)} \right), K(i) \) is the first kind of Legendre’s complete elliptic integral.

We now give the following well-known subclasses of the analytic function.

**Definition 1** (see [19]). A function \( t \in A \) is said to be in the class \( t \in k-\mathcal{ST}(\gamma) \) if and only if

\[ \frac{\zeta t'(\zeta)}{t(\zeta)} < Q_{k,\gamma}(\zeta). \] (6)

**Definition 2** (see [19]). A function \( t \in A \) is said to be in the class \( t \in k-\mathcal{UCV}(\gamma) \) if and only if

\[ 1 + \frac{\zeta t''(\zeta)}{t'(\zeta)} < Q_{k,\gamma}(\zeta). \]

We note that

\[ \Re(p(\zeta)) > \Re(Q_{k,\gamma}(\zeta)) > \frac{k + \gamma}{k + 1}. \]

**Definition 3** (see [20]). A function \( t \in A \) is said to be in the class \( k-\mathcal{UCC}(\gamma) \) if and only if

\[ \frac{\zeta t'(\zeta)}{g(\zeta)} < Q_{k,\gamma}(\zeta), \text{ for some } g(\zeta) \in k-\mathcal{ST}(\gamma) \quad (0 \leq \gamma < 1). \]

**Definition 4** (see [20]). A function \( t \in A \) is said to be in the class \( k-\mathcal{UQC}(\gamma) \) if and only if

\[ \frac{(\zeta t'(\zeta))'}{g'(\zeta)} < Q_{k,\gamma}(\zeta), \text{ for some } g(\zeta) \in k-\mathcal{UCV}(\gamma) \quad (0 \leq \gamma < 1). \]

The functions defined in (5) play the role of extremal functions for each of the above-defined function classes.

**Remark 1.** First of all, it is easy to see that

\[ 0-\mathcal{UCV}(\gamma) = \mathcal{C}(\gamma) \quad \text{and} \quad 0-\mathcal{ST}(\gamma) = \mathcal{S}^*(\gamma), \]

where \( \mathcal{C}(\gamma) \) and \( \mathcal{S}^*(\gamma) \) are the functions classes introduced and studied by Robertson (see [7]).

Secondly, we have
\[ k \cdot \text{UCV}(0) = k \cdot \text{UCV} \quad \text{and} \quad 0 \cdot \text{ST}(\gamma) = k \cdot \text{ST}, \]

where \( k \cdot \text{UCV} \) and \( k \cdot \text{ST} \) are the functions classes introduced and studied by Kanas and Wiśniewska (see [21]). Thirdly, we have

\[ 1 \cdot \text{UCV}(0) = \text{UCV} \quad \text{and} \quad 1 \cdot \text{ST}(0) = \text{ST}, \]

where \( \text{UCV} \) and \( \text{ST} \) are the function classes introduced by Goodman [22] and also studied in [23,24].

Here, we remark that quantum calculus or simply \( q \)-calculus has many applications in different branches of mathematics and physics. Geometric function theory is one among them. In geometric function theory, the role of \( q \)-calculus is very important. Many new subclasses of analytic functions have been generalized and studied by giving it \( q \)-extension. Furthermore, by using the \( q \)-approach, deferent subclasses, which involve the conic domains, have been generalized. Some significant works on \( q \)-calculus in geometric function theory can be found in [25–33]. In particular, we call the attention of curious readers to the prospect of studying the \( q \)-generalizations of the results presented in this paper, influenced by a newly published survey-cum-expository review article by Srivastava [17]; see also [16].

The theory of the operator plays a vital role in the development of geometric function theory. Many new convolution operators have been defined and studied by well-known mathematicians. We are motivated by the above-mentioned works and the recent research; here, we now use the idea of convolution and introduce the operator \( d_{k_1} \) as follows:

\[ d_{k_1} t(\zeta) = d_{k_1} = \varphi_u(\zeta) * t(\zeta) = \zeta + \sum_{n=1}^{\infty} \frac{(-1)^n(2n+1)}{4^n n!(k_1)_n} a_{n+1} \zeta^{n+1}. \quad (7) \]

The following recursive relation can be easily verified from the definition (7):

\[ \zeta (d_{k_1+1} t(\zeta))' = k_1 d_{k_1} t(\zeta) - (k_1 - 1) d_{k_1+1} t(\zeta), \quad (8) \]

where \( k_1 = u \notin \mathbb{Z}_0^- \).

Here, in [34], the generalized Bernardi-Libera-Livingston integral operator is defined as follows:

\[ \mathcal{L}_\lambda(t) = \frac{\lambda + 1}{\zeta^\lambda} \int_0^\zeta \frac{t(t)}{t^{1-\lambda}} dt, \quad (\lambda > -1). \]

In our present investigation, we make use of certain conic domains and define a new class of analytic functions associated with the Dini functions. We derive inclusion relationships and certain integral preserving properties. By applying the Bernardi-Libera-Livingston integral operator, we obtain some remarkable applications of our main results.

2. Preliminaries Results

In order to prove our main results, we need the following Lemmas.

**Lemma 1** (In [35]). In For a convex univalent function \( h \) in \( E \) with the following conditions,

\[ h(0) = 1 \quad \text{and} \quad \Re(\nu h(\zeta) + \mu) > 0 \quad (\nu, \mu \in \mathbb{C}). \]

If the function \( p \) with \( p(0) = 1 \) is analytic in \( E \), then we have

\[ p(\zeta) + \frac{\zeta p'(\zeta)}{vp(\zeta) + \mu} < h(\zeta) \Rightarrow p(\zeta) < h(\zeta) \quad (\zeta \in E). \]
Lemma 2 (In [36]). Let \( h \) be convex in the open unit disk \( E \), let \( U \geq 0 \) and \( B(\zeta) \) is analytic in \( E \) with 
\[
\Re(B(\zeta)) > 0.
\]
If \( g(\zeta) \) is analytic in \( E \) with \( h(0) = g(0) \), then 
\[
U\xi^2g''(\zeta) + B(\zeta)g(\zeta) \prec h(\zeta) \implies g(\zeta) \prec h(\zeta).
\]

3. Main Results

Theorem 1. Let \( c \geq 1 \) and \( h \) be convex univalent in \( E \) with 
\[
h(0) = 1 \quad \text{and} \quad \Re(h(\zeta)) > 0.
\]
If the following condition is satisfied by a function \( t \in A \)
\[
\frac{1}{1 - \gamma} \left( \frac{\xi(d_{k_1}t(\zeta))'}{d_{k_1}t(\zeta)} - \gamma \right) \prec h(\zeta) \quad (0 \leq \gamma < 1 \quad \text{and} \quad \zeta \in E),
\]
then, we have
\[
\frac{1}{1 - \gamma} \left( \frac{\xi(d_{k_1+1}t(\zeta))'}{d_{k_1+1}t(\zeta)} - \gamma \right) \prec h(\zeta) \quad (0 \leq \gamma < 1 \quad \text{and} \quad \zeta \in E),
\]
where \( d_{k_1} \) is defined in (7).

Proof. Suppose that 
\[
p(\zeta) = \frac{1}{1 - \gamma} \left( \frac{\xi(d_{k_1}t(\zeta))'}{d_{k_1}t(\zeta)} - \gamma \right),
\]
where the function \( p \) belongs to the family of analytic functions in \( E \), with the condition 
\[
p(0) = 1.
\]
Now, making use of (8), we have 
\[
\gamma + (1 - \gamma)p(\zeta) = k_1 \left( \frac{\xi d_{k_1}t(\zeta)}{d_{k_1+1}t(\zeta)} \right) - (k_1 - 1).
\]
Upon taking the logarithmic differentiation of the above equation with regard to \( \zeta \), we find that 
\[
p(\zeta) + \frac{\xi p'(\zeta)}{(1 - \gamma)p(\zeta) + \gamma + k_1 - 1} = \frac{1}{1 - \gamma} \left( \frac{\xi d_{k_1}t(\zeta)}{d_{k_1+1}t(\zeta)} - \gamma \right).
\]
Now, by applying Lemma 1 in conjunction with the above equation, one can easily get the required result. \( \square \)

Theorem 2. Let \( t \in A \). If \( d_{k_1}t(\zeta) \in k\cdot\mathcal{S}\mathcal{T}(\gamma) \), then \( d_{k_1+1}t(\zeta) \in k\cdot\mathcal{S}\mathcal{T}(\gamma) \).

Proof. Suppose that 
\[
q(\zeta) = \frac{\xi(d_{k_1+1}t(\zeta))'}{d_{k_1+1}t(\zeta)},
\]
where the function \( q \) is from the family of analytic functions in \( E \) with the condition \( q(0) = 1 \). From (8), we can write 
\[
k_1 \left( \frac{d_{k_1}t(\zeta)}{d_{k_1+1}t(\zeta)} \right) = q(\zeta) + k_1 - 1.
\]
Taking the logarithmic differentiation of the above equation with regard to \( \zeta \), we find that
\[
\frac{\zeta (d_k t(\zeta))'}{d_k t(\zeta)} = q(\zeta) + \frac{\zeta q'(\zeta)}{q(\zeta) + k_1 - 1} < Q_{k, \gamma}(\zeta).
\]
Since \( Q_{k, \gamma}(\zeta) \) is convex univalent in \( E \) and
\[
\Re(Q_{k, \gamma}(\zeta)) > \frac{k + \gamma}{k + 1}.
\]
By making use of by Theorem 1, in conjunction with (6), we can get the desired result. \( \Box \)

**Theorem 3.** Suppose that \( t \in A \). If \( d_k t(\zeta) \in k \cdot \mathcal{UCV}(\gamma) \), then \( d_{k+1} t(\zeta) \in k \cdot \mathcal{UCV}(\gamma) \).

**Proof.** Making use of Definitions 1 and 2 in conjunction with Theorem 2, we have
\[
d_k t(\zeta) \in k \cdot \mathcal{UCV}(\gamma) \iff \zeta (d_k t(\zeta))' \in k \cdot \mathcal{ST}(\gamma)
\]
\[
\iff d_k \zeta' (\zeta) \in k \cdot \mathcal{ST}(\gamma)
\]
\[
\iff d_{k+1} \zeta' (\zeta) \in k \cdot \mathcal{ST}(\gamma)
\]
\[
\iff d_{k+1} t(\zeta) \in k \cdot \mathcal{UCV}(\gamma).
\]
Hence, Theorem 3 is completed. \( \Box \)

**Theorem 4.** Let \( t \in A \). If \( d_k t(\zeta) \in k \cdot \mathcal{UCC}(\gamma) \), then \( d_{k+1} t(\zeta) \in k \cdot \mathcal{UCC}(\gamma) \).

**Proof.** Since we see that
\[
d_k t(\zeta) \in k \cdot \mathcal{UCC}(\gamma)
\]
then, we have
\[
\frac{\zeta (d_k t(\zeta))'}{d_k g(\zeta)} < Q_{k, \gamma}(\zeta) \text{ for some } d_k g(\zeta) \in k \cdot \mathcal{ST}(\gamma).
\]
(9)
Letting
\[
h(\zeta) = \frac{\zeta (d_{k+1} t(\zeta))'}{d_{k+1} g(\zeta)} \text{ and } \mathcal{H}(\zeta) = \frac{\zeta (d_{k+1} g(\zeta))'}{d_{k+1} g(\zeta)}.
\]
It could be seen that both functions \( h(\zeta) \) and \( \mathcal{H}(\zeta) \) are analytic in \( E \) with the following condition:
\[
h(0) = \mathcal{H}(0) = 1.
\]
The following relation now holds true according to Theorem 2:
\[
d_{k+1} g(\zeta) \in k \cdot \mathcal{ST}(\gamma) \text{ and } \Re(\mathcal{H}(\zeta)) > \frac{k + \gamma}{k + 1}.
\]
We observed that
\[
\zeta (d_{k+1} t(\zeta))' = h(\zeta) (d_{k+1} g(\zeta)).
\]
(10)
Taking the differentiation of (10), we obtain
\[
\frac{\zeta (\zeta (d_{k+1} t(\zeta))')'}{d_{k+1} g(\zeta)} = \frac{\zeta (d_{k+1} g(\zeta))'}{d_{k+1} g(\zeta)} h(\zeta) + \zeta h'(\zeta)
\]
\[
= \mathcal{H}(\zeta) h(\zeta) + \zeta h'(\zeta)
\]
(11)
By using (8), we obtain
\[
\frac{\zeta (d_{k_1} t(\zeta))'}{d_{k_1} g(\zeta)} = \frac{d_{k_1} \zeta (\zeta)'}{d_{k_1} g(\zeta)}
\]
\[
= \frac{\zeta (d_{k_1+1} \zeta (\zeta)') + (k_1 - 1)d_{k_1+1} (\zeta (\zeta)')}{\zeta (d_{k_1+1} g(\zeta))} + (k_1 - 1)d_{k_1+1} (\zeta (\zeta)')
\]
\[
= \frac{\zeta (d_{k_1+1} \zeta (\zeta)')}{d_{k_1+1} g(\zeta)} + (k_1 - 1)
\]
\[
= \frac{h(\zeta) + \zeta h'(\zeta)}{H(\zeta) + k_1 - 1}. \tag{12}
\]

From (9), (11) and (12), we have
\[
h(\zeta) + \frac{\zeta h'(\zeta)}{H(\zeta) + k_1 - 1} < Q_{k, \gamma}(\zeta)
\]
By taking \(B(\zeta) = \frac{1}{H(\zeta) + k_1 - 1}\) and \(U = 0\), we obtain
\[
\Re(B(\zeta)) = \frac{\Re(H(\zeta) + k_1 - 1)}{|H(\zeta) + k_1 - 1|^2} > 0
\]
By making use of Lemma 2, we have
\[
h(\zeta) < Q_{k, \gamma}(\zeta).
\]
Then, by Definition 3, we get
\[
d_{k_1+1} t(\zeta) \in k\text{-}UCC(\gamma).
\]
Thus, we have now completed the proof of our Theorem 4. \(\Box\)
If we put \(k = 0\) in the above Theorem, we get the following Corollary.

**Corollary 1.** Let \(t \in A\). If \(d_{k_1} t(\zeta) \in 0\text{-}UCC(\gamma)\), then \(d_{k_1+1} t(\zeta) \in 0\text{-}UCC(\gamma)\).

The next result (Theorem 5) can be proved similarly as we proved Theorem 4; therefore, we have chosen to omit the details involved in the proof of Theorem 5.

**Theorem 5.** Let \(t \in A\). If \(d_{k_1} t(\zeta) \in k\text{-}UQC(\gamma)\), then \(d_{k_1+1} t(\zeta) \in k\text{-}UQC(\gamma)\).

In the next result, for the generalized Bernardi-Libera-Livingston integral operator \(L_\lambda\), certain closure properties are investigated.

**Theorem 6.** Let \(\lambda > -\frac{k+\gamma}{k+r}\). If \(d_{k_1} \in k\text{-}ST(\gamma)\), then \(L_\lambda(d_{k_1}) \in k\text{-}ST(\gamma)\).

**Proof.** It could be seen that the following relation is due to applying the definition of \(L_\lambda(t)\) and the linearity property of the operator \(d_{k_1}\)
\[
\zeta (d_{k_1} L_\lambda(t(\zeta)))' = (\lambda + 1)d_{k_1} t(\zeta) - \lambda d_{k_1} L_\lambda(t(\zeta)). \tag{13}
\]
Substituting
\[ \frac{\zeta(d_k L_\lambda t(\zeta))'}{d_k L_\lambda t(\zeta)} = p(\zeta) \]
in (13), we have
\[ p(\zeta) = (\lambda + 1) \frac{d_k t(\zeta)}{d_k L_\lambda t(\zeta)} - \lambda. \]  
(14)

\( p \) is analytic in \( E \) with \( p(0) = 1. \)

On differentiating (14), we get
\[ \frac{\zeta(d_k (t(\zeta)))'}{d_k t(\zeta)} = \frac{\zeta(d_k L_\lambda t(\zeta))'}{d_k L_\lambda t(\zeta)} + \frac{\zeta p'(\zeta)}{p(\zeta) + \lambda} = p(\zeta) + \frac{\zeta p'(\zeta)}{p(\zeta) + \lambda}. \]

By using Lemma 1, we have
\[ p(\zeta) < Q_{k,\gamma}(\zeta), \]
since
\[ (Q_{k,\gamma}(\zeta) + \lambda) > 0. \]

Hence, the proof of Theorem 6 is completed. \( \Box \)

If we put \( k = 0 \), in the above Theorem, we get the following Corollary.

**Corollary 2.** Let \( \lambda > -\gamma \). If \( d_{k_1} \in 0-ST(\gamma) \), then \( L_\lambda(d_{k_1}) \in 0-ST(\gamma) \).

The following Theorem (Theorem 7) can be proved by using argument similar to those that are used in the proof of Theorem 6. Therefore, we choose to omit the details involved.

**Theorem 7.** Let \( \lambda > -\frac{k + \gamma}{k+1} \). If \( d_{k_1} \in k-UCV(\gamma) \), then \( L_\lambda(d_{k_1}) \in k-UCV(\gamma) \).

**Theorem 8.** Let \( \lambda > -\frac{k + \gamma}{k+1} \). If \( d_{k_1} \in k-UCCC(\gamma) \), then \( L_\lambda(d_{k_1}) \in k-UCCC(\gamma) \).

**Proof.** By definition, we have
\[ \frac{\zeta(d_k (t(\zeta)))'}{d_k g(\zeta)} < Q_{k,\gamma}(\zeta) \quad \text{for some} \quad d_k g(\zeta) \in k-ST(\gamma). \]  
(15)

Now, from (13), we have
\[ \frac{\zeta(d_k (t(\zeta)))'}{d_k g(\zeta)} = \frac{\zeta(d_k L_\lambda (\zeta t'(\zeta)))'}{d_k L_\lambda (\zeta t(\zeta))} + \lambda \frac{d_k L_\lambda (\zeta t'(\zeta))}{d_k L_\lambda (\zeta t(\zeta))} + \lambda \frac{d_k L_\lambda (\zeta t'(\zeta))}{d_k L_\lambda (\zeta t(\zeta))} + \lambda \]
\[ = \frac{\zeta(d_k L_\lambda (g(\zeta)))'}{d_k L_\lambda (g(\zeta))} + \lambda \]
\[ = \frac{\zeta(d_k L_\lambda (g(\zeta)))'}{d_k L_\lambda (g(\zeta))} + \lambda. \]  
(16)

Since \( d_k g(\zeta) \in k-ST(\gamma) \), by Theorem 6, we have \( L_\lambda(d_{k_1} g(\zeta)) \in k-ST(\gamma) \).

Let
\[ \frac{\zeta(d_k L_\lambda (g(\zeta)))'}{d_k L_\lambda (g(\zeta))} = \mathcal{H}(\zeta) \]
and
\[ \Re(\mathcal{H}(\zeta)) > \frac{k + \gamma}{k+1}. \]
Now,
\[ h(\zeta) = \frac{\zeta (d_k L_\lambda (t(\zeta)))'}{d_k L_\lambda (g(\zeta))}. \]

We now have that
\[ \zeta (d_k L_\lambda (t(\zeta)))' = h(\zeta) d_k L_\lambda (g(\zeta)). \quad (17) \]

Taking the differentiation of both sides of Equation (17), we have
\[ \frac{\zeta (d_k (\zeta L_\lambda (t(\zeta))))'}{d_k L_\lambda (g(\zeta))} = \zeta h'(\zeta) + h(\zeta) \frac{\zeta (d_k L_\lambda (g(\zeta)))'}{d_k L_\lambda (g(\zeta))} \]
\[ = \zeta h'(\zeta) + h(\zeta) H(\zeta). \quad (18) \]

Thus, from (16) and (18), we arrive at the following:
\[ \frac{\zeta (d_k t(\zeta))'}{d_k g(\zeta)} = \frac{\zeta h'(\zeta) + h(\zeta) H(\zeta) + \lambda h(\zeta)}{H(\zeta) + \lambda}. \]

By using the above equation along with (15), we have
\[ h(\zeta) + \frac{\zeta h'(\zeta)}{H(\zeta) + \lambda} < Q_{k,\gamma}(\zeta). \quad (19) \]

By supposing \( U = 0 \) along with the following
\[ B(\zeta) = \frac{1}{H(\zeta) + \lambda} \quad \text{and} \quad \Re(B(\zeta)) > 0 \quad \text{as} \quad \lambda > -\frac{k + \gamma}{k + 1}. \]

Finally, by making use of Lemma 2, we have our desired result. \( \square \)

If we put \( k = 0 \), in the above Theorem, we have the following Corollary.

**Corollary 3.** Let \( \lambda > -\gamma \). If \( d_k \in 0-\mathrm{UCC}(\gamma) \), then \( L_\lambda (d_k) \in 0-\mathrm{UCC}(\gamma) \).

The following Theorem (Theorem 9) can be proved by using arguments similar to those that are used in the proof of Theorem 6. Therefore, we choose to omit the details involved.

**Theorem 9.** Let \( \lambda > -\frac{k + \gamma}{k + 1} \). If \( d_k \in k-\mathrm{QC}(\gamma) \), then \( L_\lambda (d_k) \in k-\mathrm{QC}(\gamma) \).

4. **Concluding Remarks and Observations**

In our present investigation, we have studied some remarkable subclasses of analytic functions involving the Dini functions in conic domains. We have derived certain inclusion type results and the integral preserving properties for our defined function classes. We have also applied the well-known integral operator Bernardi-Libera-Livingston and have discussed some interesting applications of our main results.

Moreover, we recall the attention of curious readers to the prospect of studying the \( q \)-generalizations of the results reported in this paper, influenced by a newly published survey-cum-expository review article by Srivastava [17]; see also [3,16,37–39]. Furthermore, based on the suggested extension, the \((p,q)\)-extension will be a relatively minor and unimportant change, as the new parameter \( p \) is redundant (see, for details, Srivastava [17], p. 340). Furthermore, in light of Srivastava’s recent result [5], the interested reader’s attention is brought to further investigation into the \((k,s)\)-extension of the Riemann-Liouville fractional integral.
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