Secrecy Outage Capacity of Fading Channels

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Abstract

This paper considers point to point secure communication over flat fading channels under an outage constraint. More specifically, we extend the definition of outage capacity to account for the secrecy constraint and obtain sharp characterizations of the corresponding fundamental limits under two different assumptions on the transmitter CSI (Channel state information). First, we find the outage secrecy capacity assuming that the transmitter has perfect knowledge of the legitimate and eavesdropper channel gains. In this scenario, the capacity achieving scheme relies on opportunistically exchanging private keys between the legitimate nodes. These keys are stored in a key buffer and later used to secure delay sensitive data using the Vernam’s one time pad technique. We then extend our results to the more practical scenario where the transmitter is assumed to know only the legitimate channel gain. Here, our achievability arguments rely on privacy amplification techniques to generate secret key bits. In the two cases, we also characterize the optimal power control policies which, interestingly, turn out to be a judicious combination of channel inversion and the optimal ergodic strategy. Finally, we analyze the effect of key buffer overflow on the overall outage probability.

I. INTRODUCTION

Secure communication is a topic that is becoming increasingly important thanks to the proliferation of wireless devices. Over the years, several secrecy protocols have been developed and incorporated in several wireless standards; e.g., the IEEE 802.11 specifications for Wi-Fi. However, as new schemes are being developed, methods to counter the specific techniques also appear. Breaking this cycle is critically dependent on the design of protocols that offer provable secrecy guarantees. The information theoretic secrecy paradigm adopted here, allows for a systematic approach for the design of low complexity and provable secrecy protocols that fully exploit the intrinsic properties of the wireless medium.

Most of the recent work on information theoretic secrecy is, arguably, inspired by Wyner’s wiretap channel [2]. In this setup, a passive eavesdropper listens to the communication between two legitimate nodes over a separate communication channel. While attempting to decipher the message, no limit is imposed on the computational resources available to the eavesdropper. This assumption led to defining perfect secrecy capacity as the maximum achievable rate subject to zero mutual information rate between the transmitted message and the signal received by the eavesdropper. In the additive Gaussian noise scenario [3], the perfect secrecy capacity turned out to be the difference between the capacities of the legitimate and eavesdropper channels. Therefore, if the eavesdropper channel has a higher channel gain, information theoretic secure communication is not possible over the main

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channel. Recent works have shown how to exploit multipath fading to avoid this limitation \cite{4}, \cite{5}, \cite{7}. The basic idea is to opportunistically exploit the instants when the main channel enjoys a higher gain than the eavesdropper channel to exchange secure messages. This opportunistic secrecy approach was shown to achieve non-zero ergodic secrecy capacity even when on average the eavesdropper channel has favorable conditions over the legitimate channel. Remarkably, this result still holds even when the channel state information of the eavesdropper channel is not available at the legitimate nodes.

The ergodic result in \cite{4} applies only to delay tolerant traffic, e.g., file downloads. Early attempts at characterizing the delay limited secrecy capacity drew the negative conclusion that non-zero delay limited secrecy rates are not achievable, over almost all channel distributions, due to secrecy outage events corresponding to the instants when the eavesdropper channel gain is larger than the main one \cite{6}, \cite{8}. Later, it was shown in \cite{14} that, interestingly, a non-zero delay limited secrecy rate could be achieved by introducing private key queues at both the transmitter and the receiver. These queues are used to store private key bits that are shared opportunistically between the legitimate nodes when the main channel is more favorable than the one seen by the eavesdropper. These key bits are used later to secure the delay sensitive data using the Vernam one time pad approach \cite{1}. Hence, secrecy outages are avoided by simply storing the secrecy generated previously, in the form of key bits, and using them whenever the channel conditions are more advantageous for the eavesdropper. However, this work stopped short of proving sharp capacity results or deriving the corresponding optimal power control policies. These results can be recovered as special cases of the secrecy outage capacity and power control characterization obtained in the sequel. In particular, this work investigates the outage secrecy capacity of point-to-point block fading channels. We first consider the scenario where perfect knowledge about the main and eavesdropper channels are available a-priori at the transmitter. The outage secrecy capacity and corresponding optimal power control policy is obtained and then the results are generalized to the more practical scenario where only the main channel state information (CSI) is available at the transmitter. Finally, the impact of the private key queue overflow on secrecy outage probability is studied. Overall, our results reveals interesting structural insights on the optimal encoding and power control schemes as well as sharp characterizations of the fundamental limits on secure communication of delay sensitive traffic over fading channels.

The rest of this paper is organized as follows. We formally introduce our system model in Section II. In Section III, we obtain the capacity results for the full and main CSI scenarios. The optimal power control policies, for both cases, are derived in Section IV. The effect of key buffer overflow on the outage probability is investigated in Section V. We provide simulations to support our main results in Section VI. Finally, Section VII offers some concluding remarks. To enhance the flow of the paper, the proofs are collected in the Appendices.

II. System Model

We study a point-to-point wireless communication link, in which a transmitter is trying to send information to a legitimate receiver, under the presence of a passive eavesdropper. We divide time into discrete slots, where blocks are formed by $N$ channel uses, and $B$ blocks combine to form a super-block. Let the communication period consist
of $S$ super-blocks. We use the notation $(s, b)$ to denote the $b^{th}$ block in the $s^{th}$ super-block. We adopt a block fading channel model, in which the channel is assumed to be constant over a block, and changes randomly from one block to the next. Within each block $(s, b)$, the observed signals at the receiver and at the eavesdropper are:

\[ Y(s, b) = G_m(s, b)X(s, b) + W_m(s, b) \]

and

\[ Z(s, b) = G_e(s, b)X(s, b) + W_e(s, b), \]

respectively, where $X(s, b) \in \mathbb{C}^N$ is the transmitted signal, $Y(s, b) \in \mathbb{C}^N$ is the received signal by the legitimate receiver, and $Z(s, b) \in \mathbb{C}^N$ is the received signal by the eavesdropper. $W_m(s, b)$ and $W_e(s, b)$ are independent noise vectors, whose elements are drawn from standard complex normal distribution. We assume that the channel gains of the main channel $G_m(s, b)$ and the eavesdropper channel $G_e(s, b)$ are i.i.d. complex random variables. The power gains of the fading channels are denoted by $H_m(s, b) = |G_m(s, b)|^2$ and $H_e(s, b) = |G_e(s, b)|^2$. We sometimes use the vector notation $H(\cdot) = [H_m(\cdot) \ H_e(\cdot)]$ for simplicity, and also use the notation $H^{s,b} = \{H\}_{s'=1, b'=1}^{s,b}$ to denote the set of channel gains $H(s', b')$ observed until block $(s, b)$. We use similar notation for other signals as well, and denote the sample realization sequences with lowercase letters. We assume that the probability density function of instantaneous channel gains, denoted as $f(h)$, is well defined, and is known by all parties. We define channel state information (CSI) as one’s knowledge of the instantaneous channel gains. We define full transmitter CSI as the case in which the transmitter has full causal knowledge of the main and eavesdropper channel gains. We define main transmitter CSI as the case in which that the transmitter only knows the CSI of the legitimate receiver. In both cases, the eavesdropper has complete knowledge of both the main and the eavesdropper channels. Let $P(s, b)$ denote the power allocated at block $(s, b)$. We consider a long term power constraint (or average power constraint) such that,

\[
\limsup_{S,B \to \infty} \frac{1}{SB} \sum_{s=1}^{S} \sum_{b=1}^{B} P(s, b) \leq P_{\text{avg}}
\]

for some $P_{\text{avg}} > 0$.

Let $\{W(s, b)\}_{s=1, b=1}^{S,B}$ denote the set of messages to be transmitted with a delay constraint. $W(s, b)$ becomes available at the transmitter at the beginning of block $(s, b)$, and needs to be securely communicated to the legitimate receiver at the end of that particular block. We consider the problem of constructing $(2^{N_R} N)$ codes to communicate message packets $W(s, b) \in \{1, \cdots, 2^{N_R}\}$ of equal size, which consists of:

1) A stochastic encoder that maps $(w(s, b), x^{s-1,b-1})$ to $x(s, b)$ based on the available CSI, where $x^{s,b-1}$ summarizes the previously transmitted signals, and

2) A decoding function that maps $y^{s,b}$ to $\hat{w}(s, b)$ at the legitimate receiver.

Note that we consider the current block $x(s, b)$ to be a function of the past blocks $x^{s,b-1}$ as well. This kind of generality allows us to store shared randomness to be exploited in the future to increase the achievable secrecy rate.

\footnote{An exception is for $b = 1$, in which case the previous signals are summarized by $x^{s-1,b}$.}
Define the error event with parameter $\delta$ at block $(s, b)$ as

$$E(s, b, \delta) = \{ \hat{W}(s, b) \neq W(s, b) \} \cup \{ \frac{1}{N} \|X(s, b)\|^2 > P(s, b) + \delta \}$$

which occurs either when the decoder makes an error, or when the power expended is greater than $P(s, b) + \delta$. The equivocation rate at the eavesdropper is defined as the entropy rate of the message at block $(s, b)$, conditioned on the received signal by the eavesdropper during the transmission period, and available eavesdropper CSI, which is equal to $\frac{1}{N} H(W(s, b) | Z^{SB}, h^{SB})$. The secrecy outage event at rate $R$ with parameter $\delta$ at block $(s, b)$ is defined as

$$O_{\text{sec}}(s, b, R, \delta) = O_{\text{eq}}(s, b, R, \delta) \cup O_{\text{ch}}(s, b, R)$$

where the equivocation outage

$$O_{\text{eq}}(s, b, R, \delta) = \left\{ \frac{1}{N} H(W(s, b) | Z^{SB}, h^{SB}) < R - \delta \right\}$$

occurs if the equivocation rate at block $(s, b)$ is less than $R - \delta$, and channel outage

$$O_{\text{ch}}(s, b, R) = \left\{ \frac{1}{N} I(X(s, b); Y(s, b)) < R \right\}$$

occurs if channel at block $(s, b)$ is unsuitable for reliable transmission at rate $R$. Defining $\bar{O}_{\text{sec}}(\cdot)$ as the complement of the event $O_{\text{sec}}(\cdot)$, we now characterize the notion of $\epsilon$-achievable secrecy capacity.

**Definition 1:** Rate $R$ is achievable securely with at most $\epsilon$ probability of secrecy outage if, for any fixed $\delta > 0$, there exist $S, B$ and $N$ large enough such that the conditions

$$\mathbb{P}(E(s, b, \delta) | \bar{O}_{\text{sec}}(s, b, R, \delta)) < \delta$$

$$\mathbb{P}(O_{\text{sec}}(s, b, R, \delta)) < \epsilon + \delta$$

are satisfied for all $(s, b), s \neq 1$.

We call such $R$ an $\epsilon$-achievable secrecy rate. Note that the security constraints are not imposed on the first super-block.

**Definition 2:** The $\epsilon$-achievable secrecy capacity is the supremum of $\epsilon$-achievable secrecy rates $R$.

**Remark 1:** The notion of secrecy outage was previously defined and used in [6], [8]. However, those works did not consider the technique of storing shared randomness for future use, and in that case, secrecy outage depends only on the instantaneous channel states. In our case, secrecy outage depends on previous channel states as well. Note that we do not impose a secrecy outage constraint on the first superblock $(s = 1)$. We refer to the first superblock as an initialization phase used to generate initial common randomness between the legitimate nodes. Note that this phase only needs to appear once in the communication lifetime of that link. In other words, when a session (which consists of $S$ superblocks) between the associated nodes is over, they would have sufficient number of common key bits for the subsequent session, and would not need to initiate the initialization step again.
III. Capacity Results

In this section, we investigate this capacity under two different cases; full CSI and main CSI at the transmitter. Before giving the capacity results, we define the following quantities. For a given power allocation function $P(s, b)$, let $R_m(s, b)$ and $R_s(s, b)$ be as follows,

$$R_m(s, b) = \log(1 + P(s, b)H_m(s, b))$$

$$R_s(s, b) = \text{max}(\log(1 + P(s, b)H_m(s, b)) - \log(1 + P(s, b)H_e(s, b)))$$

where $[\cdot]^+ = \max(\cdot, 0)$. Note that, $R_m(\cdot)$ is the supremum of achievable main channel rates, without the secrecy constraint. Also, $R_s(\cdot)$ is the non-negative difference between main channel and eavesdropper channel’s supremum achievable rates. We show in capacity proofs that the outage capacity achieving power allocation functions lie in the space of stationary power allocation functions that are functions of instantaneous transmitter CSI. Hence for full CSI, we constrain ourselves to the set $\mathcal{P}$ of stationary power allocation policies that are functions of $\mathbf{h}(s, b) = [h_m(s, b) \ h_e(s, b)]$. For simplicity, we drop the block index $(s, b)$, and use the notation $P(\mathbf{h})$ for the stationary power allocation policy. Similarly, with main CSI we consider the power allocation policies that are functions of $h_m(s, b)$, and use the notation $P(h_m)$ for the stationary power allocation policy. In both cases, since the secrecy rate $R_s(s, b)$, and the main channel rate $R_m(s, b)$ are completely determined by the stationary power allocation functions $P(\cdot)$ and channel gains $\mathbf{h}$, we will interchangeably use the notations $R_s(s, b) \equiv R_s(\mathbf{h}, P)$ and $R_m(s, b) \equiv R_m(\mathbf{h}, P)$.

A. Full CSI

**Theorem 1:** Let the transmitter have full CSI. Then, for any $\epsilon$, $0 \leq \epsilon < 1$, the $\epsilon$-achievable secrecy capacity is identical to

$$C^\epsilon_P = \max_{P(h) \in \mathcal{P}} \frac{\mathbb{E}[R_s(\mathbf{h}, P)]}{1 - \epsilon}$$

where the set $\mathcal{P} \subseteq \mathcal{P}$ consists of power control policies $P(\mathbf{h})$ that satisfies the following conditions.

$$\mathbb{P}\left(R_m(\mathbf{h}, P) < \frac{\mathbb{E}[R_s(\mathbf{h}, P)]}{1 - \epsilon}\right) \leq \epsilon$$

$$\mathbb{E}[P(\mathbf{H})] \leq P_{\text{avg}}$$

A detailed proof of achievability and converse part is provided in Appendix A. Here, we briefly justify the result. For a given $P(\mathbf{h})$, $R_s(\mathbf{h}, P)$ the supremum of the secret key generation rates within a block that experiences channel gains $\mathbf{h}$. This implies that the expected achievable secrecy rate $\mathbb{E}[R_s(\mathbf{H}, P)]$ without the outage constraint. With the outage constraint, the fluctuations of $R_s(\mathbf{H}, P)$ due to fading are unacceptable, since $R_s(\mathbf{H}, P)$ can go below the desired rate when the channel conditions are unfavorable (e.g., when $H_m < H_e$, $R_s(\mathbf{H}, P) = 0$). Hence, we utilize secret key buffers to smoothen out these fluctuations to provide secrecy rate of $\mathbb{E}[R_s(\mathbf{H}, P)]$ at each block. The generated secrecy is stored in secret key buffers of both the transmitter and receiver, and is utilized to secure data of same size using Vernam’s one-time pad technique. With the allowable amount of secrecy outages, this
rate goes up to $\mathbb{E}[R_s(H, P)]/(1 - \epsilon)$. The channel outage constraint (8) on the other hand is a necessary condition to satisfy the secrecy outage constraint in (4) due to (2).

**Example 1:** Consider a four state system, where $H_m$ and $H_e$ takes values from the set $\{1, 10\}$ and the joint probabilities are as given in Table I. Let the average power constraint be $P_{\text{avg}} = 0.5$, and there is no power control, i.e., $P(h) = P_{\text{avg}} \forall h$. The achievable instantaneous secrecy rate at each state is given in Table II. According to the pessimistic result in [6,8], any non-zero rate cannot be achieved with a secrecy outage probability $\epsilon < 0.6$ in this case. However, according to Theorem 1, rate $R = 0.8$ can be achieved with $\epsilon$ secrecy outage probability, since $\mathbb{E}[R_s(H, P_{\text{avg}})] = 0.8$. A sample path is provided for both schemes in Figure 1 and it is shown how our scheme avoids secrecy outage in the second block.

| $P(h)$ | 1 | 10 |
|--------|---|----|
| 1      | 0.1 | 0.1 |
| 10     | 0.4 | 0.4 |

| $R_s(H, P_{\text{avg}})$ | 1 | 10 |
|--------------------------|---|----|
| 1 | 0 | 0 |
| 10 | 2 | 0 |

**TABLE I**

**TABLE II**

![Fig. 1. A sample path. With strategy 2, secrecy outage can be avoided for block $t = 2$ via the use of key bits.]

**B. Main CSI**

**Theorem 2:** Let the transmitter have main CSI. Then, for any $\epsilon$, $0 \leq \epsilon < 1$, the $\epsilon$-achievable secrecy capacity is identical to

$$C_{M}^{\epsilon} = \max_{P(h_m) \in \mathcal{P}'} \frac{\mathbb{E}[R_s(H, P)]}{1 - \epsilon}$$

Although Theorem 1 is stated for the case where random vector $H$ is continuous, the result similarly applies to discrete $H$ as well.
where the set $\mathcal{P}'' \subseteq \mathcal{P}$ consists of power control policies $P(h_m)$ that satisfies the following conditions.

$$P \left( R_m(\mathbf{H}, P) < \frac{\mathbb{E}[R_s(\mathbf{H}, P)]}{1 - \epsilon} \right) \leq \epsilon$$

(11)

$$\mathbb{E}[P(H_m)] \leq P_{av}$$

(12)

Although the problems (7)-(9) and (10)-(12) are of the same form, due to the absence of eavesdropper CSI, the maximization in this case is over power allocation functions $\mathcal{P}''$ that depend on the main channel state only. Hence, $C_e' \leq C_e''$. A detailed proof of achievability and converse is provided in Appendix E. As in the full CSI case, our achievable scheme uses similar key buffers and Vernam’s one time pad technique to secure the message. The main difference is the generation of secret key bits. Due to the lack of knowledge of $H_e(s, b)$ at the transmitter, secret key bits cannot be generated within a block. Instead, using the statistical knowledge of $H_e(s, b)$, we generate keys over a super-block. Roughly, over a superblock the receiver can reliably obtain $\frac{E[NB]}{e^{H_e(s, b)}}$ bits of secret key can be extracted by using a universal hash function.

Now, we show that power allocation policy has minimal impact on the performance in the high power regime.

**Theorem 3:** For any $\epsilon > 0$, the $\epsilon$-achievable secrecy capacities with full CSI and main CSI converge to the same value

$$\lim_{P_{av} \to \infty} C_e' = \lim_{P_{av} \to \infty} C_e'' = \frac{\mathbb{E}[H_m > H_e] \log (H_m / H_e)}{(1 - \epsilon)}$$

(13)

**Proof:** For $h \equiv [h_m, h_e]$ such that $h_m > h_e$, we can see from (6) that $\lim_{P(h) \to \infty} R_s(h, P) = \log \left( \frac{h_m}{h_e} \right)$, and for $h_m \leq h_e$, $R_s(h, P) = 0$. Furthermore, for $h_m > 0$, we can see from (5) that $\lim_{P(h) \to \infty} R_m(h, P) = \infty$. Let $P(h) = P_{av}$ (no power control), which does not require any CSI. Then, we get

$$\lim_{P_{av} \to \infty} \mathbb{E}[R_s(\mathbf{H}, P)] = \mathbb{E}[H_m > H_e] \log (H_m / H_e) < \infty$$

Combining the last 2 equations, we get

$$\lim_{P_{av} \to \infty} P \left( R_m(\mathbf{H}, P) < \frac{\mathbb{E}[R_s(\mathbf{H}, P)]}{1 - \epsilon} \right) = P(H_m = 0)$$

and $P(H_m = 0) = 0$, since probability density function of $\mathbf{H}$ is well defined. Hence, channel outage constraints (8) and (11) are not active in the high power regime. Therefore, $P(h) \in \mathcal{P}'$, and $P(h) \in \mathcal{P}''$. From (7)-(9) and (10)-(12), we conclude that $C_e' = C_e'' = \frac{\mathbb{E}[H_m > H_e] \log (H_m / H_e)}{(1 - \epsilon)}$.

Our simulation results also illustrate that the power allocation policy has minimal impact on the importance in the high power regime. On the other hand, when the average power is limited, the optimality of the power allocation function is of critical importance, which is the focus of the following section.

**IV. Optimal Power Allocation Strategy**

**A. Full CSI**

The optimal power control strategy, $P^*(h)$ is the stationary strategy that solves the optimization problem (7)-(9). In this section, we will show that $P^*(h)$ is a time-sharing between the channel inversion power policy, and the
secure waterfilling policy. We first introduce the channel inversion power policy, \( P_{\text{inv}}(h, R) \), which is the \textit{minimum} required power to maintain main channel rate of \( R \). For \( h = [h_m h_e] \),

\[
P_{\text{inv}}(h, R) = \frac{2R - 1}{h_m}
\]  

(14)

Next we introduce \( P_{\text{wf}}(h, \lambda) \),

\[
P_{\text{wf}}(h, \lambda) = \frac{1}{2} \left[ \sqrt{\left( \frac{1}{h_e} - \frac{1}{h_m} \right)^2 + 4 \frac{1}{\lambda} \left( \frac{1}{h_e} - \frac{1}{h_m} \right) - \left( \frac{1}{h_e} + \frac{1}{h_m} \right)^2} \right] + ,
\]  

(15)

We call it the ‘secure waterfilling’ power policy because it maximizes the ergodic secrecy rate without any outage constraint, and resembles the ‘waterfilling’ power control policy. Here, the parameter \( \lambda \) determines the power expended on average. Now, let us define a time-sharing region

\[
G(\lambda, k) = \left\{ h : [R_s(h, P_{\text{inv}}) - R_s(h, P_{\text{wf}})]^+ - \lambda [P_{\text{inv}}(h, b) - P_{\text{wf}}(h, \lambda)]^+ \geq k \right\}
\]  

(16)

which is a function of parameters \( \lambda \) and \( k \).

\textit{Theorem 4:} \( P^*(h) \) is the unique solution to

\[
P^*(h) = P_{\text{wf}}(h, \lambda^*) + 1 \{ h \in G(\lambda^*, k^*) \} (P_{\text{inv}}(h, c_F^*) - P_{\text{wf}}(h, \lambda^*))^+
\]  

(17)

subject to: \( k^* \leq 0, \lambda^* > 0 \)

\[
C_F^* = \mathbb{E}[R_s(H, P^*)]/(1 - \epsilon)
\]  

(18)

\[
\mathbb{P}(H \in G(\lambda^*, k^*)) = 1 - \epsilon
\]  

(19)

\[
\mathbb{E}[P^*(H)] = P_{\text{avg}}
\]  

(20)

where \( \mathbb{E}[R_s(H, P^*)] \) is the expected secrecy rate under the power allocation policy \( P^*(h) \).

\textit{Proof:} Define a sub-problem

\[
\mathbb{E}[R_s(H, P^R)] = \max_{P(h)} \mathbb{E}[R_s(H, P)]
\]  

(21)

subject to: \( P(h) \geq 0, \forall h \)

\[
\mathbb{E}[P(H)] \leq P_{\text{avg}},
\]  

(22)

\[
\mathbb{P}(R_m(H, P) < R) \leq \epsilon
\]  

(23)

Let \( P^R(h) \) be the power allocation function that solves this sub-problem. Note that for \( R = \mathbb{E}[R_s(H, P^R)]/(1 - \epsilon) \), this problem is identical to (7)-(9), hence giving us \( P^*(h) \). We will prove the existence and uniqueness of such \( R \).

\textit{Lemma 1:} There exists a unique \( R_{\text{max}} > 0 \) such that the sub-problem (21)-(23) has a solution for all \( R \leq R_{\text{max}}, \) which is found by solving

\[
P_{\text{avg}} = \int_{h_m \geq c} P_{\text{inv}}(h, R_{\text{max}}) f(h) dh
\]  

(24)

for \( h = [h_m h_e] \), where the constant \( c \) is chosen such that \( \mathbb{P}(H_m \leq c) = \epsilon \).
Proof is provided in Appendix C-A.

**Lemma 2:** For any \( R \leq R_{\text{max}} \),

\[
P^R(h) = P_{\text{w}}(h, \lambda) + 1 \ (h \in G(\lambda, k)) (P_{\text{inv}}(h, R) - P_{\text{w}}(h, \lambda))
\]

where \( k \in (-\infty, 0] \) and \( \lambda \in (0, +\infty) \) are parameters that satisfy (22) and (23) with equality.

Proof is provided in Appendix D. It is left to show there exists a unique \( R \) that satisfies \( R = \mathbb{E}[R_s(H, P^R)]/(1 - \epsilon) \).

**Lemma 3:** \( \mathbb{E}[R_s(H, P^R)] \) is a continuous non-increasing function of \( R \).

Proof is provided in Appendix C-B.

**Lemma 4:** There exists a unique \( R \), \( 0 \leq R \leq R_{\text{max}} \), which satisfies \( R = \mathbb{E}[R_s(H, P^R)]/(1 - \epsilon) \).

Proof is provided in Appendix C-C. This concludes the proof of the theorem.

Due to (17), the optimal power allocation function is a time-sharing between the channel allocation power allocation function and secure waterfilling; a balance between avoiding channel outages, hence secrecy outages, and maximizing the expected secrecy rate. The time sharing region \( G(\lambda, k) \) determines the instants \( h \), for which avoiding channel outages are guaranteed through the choice of \( P(h) = \max(P_{\text{inv}}(h, R), P_{\text{w}}(h, \lambda)) \). (19) ensures that channel outage probability is at most \( \epsilon \), and (20) ensures that average power constraint is met with equality. (18), on the other hand, is an immediate consequence of (7).

Note that, an extreme case is \( P^*(h) = P_{\text{w}}(h, \lambda^*) \ \forall h \), which occurs when \( P_{\text{inv}}(h, R) \leq P_{\text{w}}(h, \lambda^*) \) for any \( h \in G(\lambda^*, k^*) \), which translates into the fact that the secure waterfilling solution itself satisfies the channel outage probability in (3). However, that the other extreme \( (P^*(h) = P_{\text{inv}}(h, R^*) \ \forall h) \) cannot occur for any non-zero \( \epsilon \) due to (17). The parameter \( C_F^\epsilon \) can be found graphically as shown in Figure 2 by plotting \( \mathbb{E}[R_s(H, P^R)] \) and \( (1 - \epsilon)R \) as a function of \( R \). The abcissa of the unique intersection point is \( R = C_F^\epsilon \).

![Finding C_F^\epsilon on Graph](image.png)

**Example 2:** Consider the same system model in Example 1. We have found that for \( R = 0.8 \) bits/channel use is achievable with \( \epsilon \) probability of secrecy outage with no power control, i.e., \( P(h) = 0.5 \ \forall h \). Let \( \epsilon = 0.2 \), we will see if we can do better than \( R = 1 \) with power control. Solving the problem (17)-(20), we can see that the

\[3\] Although Theorem 4 assumes \( H \) is a continuous random vector, the results similarly hold for the discrete case as well.
time-sharing, and power expended in each state are as given in Tables III and IV. For \( h \equiv [h_m \ h_e] = [10 \ 1] \), i.e., the legitimate channel has a better gain, secure waterfilling is used and when \( h = [10 \ 10] \), secret key bits cannot be generated, but channel inversion is used to guarantee a main channel rate of \( R \), which is secured by the excess keys generated during the state \( h = [10 \ 1] \). As a result, we can see that a rate of \( C_{\text{sec}}^{\text{EC}} = 1.26 \text{ bits/ per channel use} \) is achievable, which corresponds to 26% increase with respect to no power control. As mentioned in Theorem 3, this gain diminishes at the high power regime, i.e., when \( P_{\text{avg}} \to \infty \).

TABLE III

| Time Sharing Regions |
|----------------------|
| \( h_m \ \hbar_e \to \) | 1 | 10 |
| 1 | wf | wf |
| 10 | wf | inv |

TABLE IV

| \( h_m \ \hbar_e \to \) | 1 | 10 |
|----------------------|
| 1 | 0 | 0 |
| 10 | 1.11 | 0.14 |

B. Main CSI

Here, we find the optimal power control strategy \( P^*(h_m) \), which solves the optimization problem (10)-(12). Let us define \( P_w(h_m, \lambda) \) as the maximum of 0, and the solution of the following equation

\[
\frac{\partial \mathbb{E}[R_s(H, P)]}{\partial P(h_m)} = \frac{h_m \mathbb{P}(h_e \leq h_m)}{1 + h_m P(h_m)} - \int_0^{h_m} \left( \frac{h_e}{1 + h_e P(h_m)} \right) f(h_e) dh_e - \lambda = 0
\]

(25)

\( P_w(h_m, \lambda) \) will replace \( P_{\text{wf}}(h, \lambda) \) in the full CSI case.

**Theorem 5:** \( P^*(h_m) \) is the unique solution to

\[
P^*(h_m) = P_w(h_m, \lambda^*) + 1(h_m \geq c) \left( P_{\text{inv}}(h_m, C_{\text{M}}^*) - P_w(h_m, \lambda^*) \right)
\]

(26)

subject to: \( \lambda^* > 0 \)

\[
C_{\text{M}}^* = \mathbb{E}[R_s(H, P^*)]/(1 - \epsilon)
\]

(27)

\[
\mathbb{P}(H_m \geq c) = 1 - \epsilon
\]

(28)

\[
\mathbb{E}[P^*(H_m)] = P_{\text{avg}}
\]

(29)

where \( \mathbb{E}[R_s(H, P^*)] \) is the expected secrecy rate under the power allocation policy \( P^*(h_m) \).

**Proof:** The proof follows the approach in Full CSI case, hence we omit the details for brevity. Define the sub-problem

\[
\mathbb{E}[R_s(H, P^R)] = \max_{P(h_m)} \mathbb{E}[R_s(H, P)]
\]

(30)

subject to: \( P(h_m) \geq 0 \), \( \forall h_m \)

\[
\mathbb{E}[P(H_m)] \leq P_{\text{avg}},
\]

(31)

\[
\mathbb{P}(R_m(H, P) < R) \leq \epsilon
\]

(32)
Let $P^R(h_m)$ be the power allocation function that solves this sub-problem. Lemmas 1 and 4 also hold in this case.

**Lemma 5:** For any $R \leq R_{\text{max}}$,

$$P^R(h_m) = P_w(h_m, \lambda) + 1(h_m > c) (P_{\text{inv}}(h_m, R) - P_w(h_m, \lambda))^+$$

where $c$ is a constant that satisfies $\mathbb{P}(H_m \geq c) = 1 - \epsilon$, and $\lambda \in (0, +\infty)$ is a constant that satisfies (31) with equality.

The proof is similar to the proof of Lemma 2, and is provided in Appendix E.

The graphical solution in Figure 2 to find $C^{\epsilon}_F$ also generalizes to the main CSI case.

### V. Sizing the Key Buffer

The proofs of the capacity results of Section III assume availability of infinite size secret key buffers at the transmitter and receiver, which mitigate the effect of fluctuations in the achievable secret key bit rate due to fading. Finite-sized buffers, on the other hand will lead to a higher secrecy outage probability due to wasted key bits by the key buffer overflows. We revisit the full CSI problem, and we consider this problem at ‘packet’ level, where we assume a packet is of fixed size of $N$ bits. We will prove the following result.

**Theorem 6:** Let $\epsilon' > \epsilon$. Let $MC^{\epsilon'}_F(\epsilon')$ be the buffer size (in terms of packets) sufficient to achieve rate $C^{\epsilon'}_F$ with at most $\epsilon'$ probability of secrecy outage. Then,

$$\lim_{\epsilon' \to \epsilon} \frac{MC^{\epsilon'}_F(\epsilon') - C^{\epsilon'}_F}{\var{R_s(H, P^{C^{\epsilon'}_F})}+(C^{\epsilon'}_F)^2(1-\epsilon)} \leq 1$$

Before providing the proof, we first interpret this result. If buffer size is infinite, we can achieve rate $C^{\epsilon'}_F$ with $\epsilon$ probability of secrecy outage. With finite buffer, we can achieve the same rate with $\epsilon'$ probability of secrecy outage. Considering this difference to be the price that we have to pay due to the finiteness of the buffer, we can see that the buffer size required scales with $O\left(\frac{1}{\epsilon' - \epsilon} \log \frac{1}{\epsilon' - \epsilon}\right)$, as $\epsilon' - \epsilon \to 0$.

**Proof:** Achievability follows from simple modifications to the capacity achieving scheme described in Appendix A. We will first study the key queue dynamics, then using the heavy traffic limits, we provide an upper bound to the key loss ratio due to buffer overflows. Then, we relate key loss ratio to the secrecy outage probability, and conclude the proof.

For the key queue dynamics, we use a single index $t$ to denote the time index instead of the double index $(s, b)$, where $t = sB + b$. We consider transmission at outage secrecy rate of $R$, and use power allocation function $P^R(h)$, which solves the problem (21)-(23). Let us define $\{Q_M(t)\}_{t=1}^{\infty}$ as the key queue process with buffer size $M$, and let $Q_M(1) = 0$. Then, during each block $t$,

1. The transmitter and receiver agree on secret key bits of size $R_s(t)$ using privacy amplification, and store the key on their secret key buffers.
2. The transmitter pulls key bits of size $R$ from its secret key buffer to secure the message stream of size $R$ using one time pad, and transmits over the channel.
as explained in Appendix A. The last phase is skipped if outage (\(\mathcal{O}_{\text{enc}}(t)\)) is declared, which is triggered by one of the following events

- **Channel Outage (\(\mathcal{O}_{\text{ch}}(t)\)):** The channel cannot support reliable transmission at rate \(R\), i.e. \(R_m(t) < R\).
- **Key Outage (\(\mathcal{O}_{\text{key}}(t)\)):** There are not enough key bits in the key queue to secure the message at rate \(R\). This event occurs when \(Q_{M}(t) + R_s(t) - R < 0\).
- **Artificial outage (\(\mathcal{O}_a(t)\)):** Outage is artificially declared, even though reliable transmission at rate \(R\) is possible.

Due to the definition of \(P_R(h)\), \(P(\mathcal{O}_{\text{ch}}(t)) \leq \varepsilon \forall t\), and the set \(\{\mathcal{O}_{\text{ch}}(t)\}\) of events indexed by \(t\) are i.i.d. We choose \(\{\mathcal{O}_a(t)\}\) such that \(\mathcal{O}_x(t) = \mathcal{O}_{\text{ch}}(t) \cup \mathcal{O}_a(t)\) is i.i.d. as well, and
\[
P(\mathcal{O}_x(t)) = \varepsilon, \; \forall t
\]

The dynamics of the key queue can therefore be modeled by
\[
Q_{M}(t+1) = \min(M, Q_{M}(t) + R_s(t) - 1(\mathcal{O}_{\text{enc}}(t))R)
\]  

(34)

Note that \(Q_{M}(t) \geq 0 \forall t\), due to the definition of \(\mathcal{O}_{\text{key}}(t)\).

Let \(L^T(M)\) be the time average loss ratio over the first \(T\) blocks, for buffer size \(M\), which is defined as the ratio of the amount of loss of key bits due to overflows, and the total amount of input key bits
\[
L^T(M) = \frac{\sum_{t=1}^{T} (Q_{M}(t) + R_s(t) - 1(\mathcal{O}_{\text{enc}}(t))R - M)^+}{\sum_{t=1}^{T} R_s(t)}
\]  

(35)

Then, we can see that \(\forall T > 0\),
\[
(1 - L^T(M)) \sum_{t=1}^{T} R_s(t) = Q_{M}(T) + \sum_{t=1}^{T} R(\mathcal{O}_{\text{enc}}(t))
\]  

(36)

follows from (34), (35), and the fact that \(Q_{M}(1) = 0\).

**Lemma 6:** \(Q_{M}(t)\) converges in distribution to an almost surely finite random variable.

The proof is provided in Appendix F-A. This implies that \(\lim_{T \to \infty} P(\mathcal{O}_{\text{enc}}(t))\) exists. Now, we provide our asymptotic result for the key loss ratio. We define the drift and variance of this process as
\[
\mu_R = \mathbb{E}[R_s(H, P^R) - R1(\mathcal{O}_x(t))] = \mathbb{E}[R_s(H, P^R)] - R(1 - \varepsilon)
\]  

(37)

and
\[
\sigma^2_R = \text{Var}[R_s(H, P^R) - R1(\mathcal{O}_x(t))]
\]

respectively, where (37) follows from the definition of \(\mathcal{O}_x(t)\).

**Lemma 7:** For any \(M > 0\), the key loss ratio satisfies the following asymptotic relationship
\[
\lim_{R \to C_p} \lim_{T \to \infty} L^T(M \frac{\sigma^2_R}{|\mu_R|}) \left( \frac{2|\mu_R| \mathbb{E}[R_s(H, P^R)] e^{-2|\mu_R| \sigma^2_R}}{\sigma^2_R} \right) \leq e^{-2M}
\]  

(38)

The proof is provided in Appendix F-B.
Lemma 8: If \( \lim_{t \to \infty} P(O_{\text{enc}}(t)) = \epsilon' \), then \( \epsilon' \) secrecy outage probability \((41)\) is satisfied.

Proof: Find \( B \) such that \( P(O_{\text{enc}}(t)) = \epsilon' + \delta \) for any \( t > B \). In \( 2 \)-index notation \((s, b)\) with \( t = sB + b \), it corresponds to \( P(O_{\text{enc}}(s, b, R)) = \epsilon' + \delta, \forall(s, b) \neq 1 \). Then,

\[
\begin{align*}
\mathbb{P}(O_{\text{sec}}(s, b, R, \delta)) &\leq \mathbb{P}(O_{\text{sec}}(s, b, R, \delta)|\bar{O}_{\text{enc}}(s, b, R)) + \mathbb{P}(O_{\text{enc}}(s, b, R)) \\
&\leq \mathbb{P}(O_{\text{enc}}(s, b, R)) \\
&\leq \epsilon' + \delta
\end{align*}
\]

(39) 

(40) 

(41) 

Here, \((39)\) follows from the union bound, and second term follows from the equivocation analysis \((54)\) and \((55)\) in Appendix A, which shows that there exists some packet size \( N \) large enough such that \( \mathbb{P}(O_{\text{sec}}(s, b, R, \delta)|\bar{O}_{\text{enc}}(s, b, R)) = 0 \). Equation \((41)\) implies that \( \epsilon' \) secrecy outage probability \((41)\) is satisfied.

Let \( \lim_{t \to \infty} P(O_{\text{enc}}(t)) = \epsilon' \). Since \( P(O_{\text{enc}}(t)) = \epsilon \) and \( O_{\text{enc}}(t) = O_X(t) \cup O_{\text{key}}(t) \), we have \( \lim_{t \to \infty} P(O_{\text{key}}(t)) > 0 \). This implies that \( \lim_{T \to \infty} \sum_{t=1}^{T} Q_{M}(T) = 0 \) (since otherwise, key outage probability would be zero), which, due to \((36)\) implies

\[
(1 - \lim_{T \to \infty} L^T(M))E[R_s(H, P^R)] = (1 - \lim_{t \to \infty} P(O_{\text{enc}}(t)))R = (1 - \epsilon')R
\]

(42) 

Here, due to the choice of power allocation function \( P^R(h) \), we have \( E[R_s(H, P^R)] = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} R_s(t) \). Plugging the result of Lemma \(7\) into \((42)\), we obtain the required key buffer size to achieve \( \epsilon' \) probability of secrecy outage

\[
\lim_{R \to C_F} \frac{\frac{\sigma^2_b}{2|R_R|} \log \left( \frac{M_R(\epsilon') - R}{\frac{\sigma^2_h}{2|R_R|} (1 - (1 - \epsilon')R)} \right)}{2|R_R|} \leq 1
\]

(43) 

We know from \((7)\) that \( \epsilon \) and \( \epsilon' \)-achievable secrecy capacities satisfy the conditions \( C_F' \big(1 - \epsilon'\big) = E[R_s(H, P^R)]\big|_{R=C_F'} \) and \( C_F' \big(1 - \epsilon\big) = E[R_s(H, P^R)]\big|_{R=C_F'} = E[R_s(H, P^+)] \), respectively. By Lemma \(3\) we know that \( E[R_s(H, P^R)] \) is a continuous function of \( R \), hence for any given \( \epsilon' > \epsilon \), there exists an \( R \) such that \( C_F' < R < C_F' \), and \( E[R_s(h, P^R)] = (1 - \frac{\epsilon'}{2})R \). Furthermore, as \( \epsilon' \to \epsilon \), \( C_F' \to C_F' \). Let us define a monotonically decreasing sequence \((\epsilon'(1), \epsilon'(2), \ldots)\), such that \( \lim_{i \to \infty} \epsilon'(i) = \epsilon \). For any \( i \in \mathbb{N} \), find \( R(i) \) such that \( C_F' < R(i) < C_F' \), and \( E[R_s(h, P^{R(i)})] = (1 - \frac{\epsilon + \epsilon'(i)}{2})R(i) \), therefore \( \mu_{R(i)} = (\epsilon - \epsilon')/(2R(i)) \). From \((43)\), we get

\[
\lim_{i \to \infty} \frac{\sigma^2_{R(i)}(\epsilon'(i)) - R(i)}{(\epsilon - \epsilon')R(i) \log \left( \frac{\sigma^2_{R(i)}(\epsilon'(i))}{(\epsilon - \epsilon')R(i)} \right)} \leq 1
\]

Since as \( i \to \infty \), \( R(i) \to C_F' \), \( \epsilon'(i) \to \epsilon \) and \( \sigma^2_{R(i)} \to \sigma^2_{C_F'} \), where

\[
\sigma^2_{C_F'} = \text{Var}[R_s(H, P^{C_F'}) - C_F' \mathbf{1}(\bar{O}_X(t))] \\
\leq \text{Var}[R_s(H, P^{C_F'})] - C_F'(1 - \epsilon)\epsilon
\]

The last inequality induces the upper bound \((33)\), which concludes the proof.
VI. Numerical Results

In this section, we conduct simulations to illustrate our main results with two examples. In the first example, we analyze the relationship between $\epsilon$-achievable secrecy capacity and average power. We assume that both the main channel and eavesdropper channel are characterized by Rayleigh fading, where the main channel and eavesdropper channel power gains follow exponential distribution with means 2 and 1, respectively. Since Rayleigh channel is non-invertible, maintaining a non-zero secrecy rate with zero secrecy outage probability is impossible. In Figure 3, we plot the $\epsilon$-achievable secrecy capacity as a function of the average power, for $\epsilon = 0.02$ outage probability, for both full CSI and main CSI cases. It can be clearly observed from the figure that the gap between capacities under full CSI and main CSI vanishes as average power increases, which support the result of Theorem 3.

![Outage Capacity vs Power](image)

Fig. 3. The $\epsilon$-achievable secrecy capacities as a function of average power, $P_{\text{avg}}$

In the second example, we study the relationship between the buffer size, key loss ratio and the outage probability. We assume that both the main and eavesdropper channel gains follow a chi-square distribution of degree 2, but with means 2 and 1, respectively. We focus on the full CSI case, and consider the scheme described in Section V.

We consider transmission at secrecy rate of $R$ with the use of the power allocation policy $P^R(h)$ that solves the problem \( (21)-(23) \). For $\epsilon = 0.02$, and the average power $P_{\text{avg}} = 1$, we plot the key loss ratio \( (35) \), as a function of buffer size $M$ in Figure 4, for $R = C_F^\epsilon, R = 1.01C_F^\epsilon$ and $R = 1.02C_F^\epsilon$, where $C_F^\epsilon$ is the $\epsilon$-achievable secrecy capacity. It is shown in Lemma 7 of Section V that expect the key loss ratio $L^T(M)$ decreases as $R$ increases, which is observed in Figure 4. Finally, we study the relationship between the secrecy outage probability and the buffer size for a given rate. In Figure 5, we plot the secrecy outage probabilities, denoted as $\epsilon'$, as a function of buffer size $M$ for the same encoder parameters. On the same graph, we also plot our asymptotic result given in Theorem 6 which provides an upper bound on the required buffer size to achieve $\epsilon'$ outage probability for rate $C_F^\epsilon$, with the assumption that \( (33) \) is an equality for any $\epsilon'$. We can see that, this theoretical result serves as an upper bound on the required buffer size when $\epsilon' - \epsilon$, which is the additional secrecy outages due to key buffer overflows,
is very small. Another important observation from Figures 4 and 5 is that, for a fixed buffer size, although the key loss ratio decreases as $R$ increases, secrecy outage probability increases. This is due to the fact that key bits are pulled from the key queue at a faster rate, hence the decrease in the key loss ratio does not compensate for the increase of the rate that key bits are pulled from the key queue, therefore the required buffer size to achieve same $\epsilon'$ is higher for larger values of $R$.

![Fig. 4. Relationship between buffer size $M$, and key loss ratio $L^L(M)$](image1)

![Fig. 5. Relationship between buffer size $M$, and outage probability $\epsilon'$](image2)

VII. CONCLUSIONS

This paper obtained sharp characterizations of the secrecy outage capacity of block flat fading channels under the assumption full and main CSI at the transmitter. In the two cases, our achievability scheme relies on opportunistically exchanging private keys between the legitimate nodes and using them later to secure the delay sensitive information. We further derive the optimal power control policy in each scenario revealing an interesting structure based by judicious time sharing between time sharing and the optimal strategy for the ergodic. Finally, we investigate the effect of key buffer overflow on the secrecy outage probability when the key buffer size is finite.

APPENDIX A

PROOF OF THEOREM 1

First, we prove the achievability. Consider a fixed power allocation function $P(h) \in \mathcal{P}'$. Let us fix $R < \mathbb{E}[R_s(H, P)]/(1 - \epsilon)$. We show that for any $\delta > 0$, there exist some $B$ and $N$ large enough such that the constraints in (3) and (4) are satisfied, which implies that any $R < \mathbb{E}[R_s(H, P)]/(1 - \epsilon)$ is an $\epsilon$-achievable secrecy rate. The outage capacity is then found by maximizing $\mathbb{E}[R_s(H, P)]/(1 - \epsilon)$ over the set $\mathcal{P}'$ of power allocation functions.

Our scheme utilizes secret key buffers at both the transmitter and legitimate receiver. Then,

i) At the end of every block $(s, b)$, using privacy amplification, legitimate nodes (transmitter and receiver) generate...
N(R(s, b) − δ) bits of secret key from the transmitted signal in that particular block, and store it in their secret key buffers. We denote the generated secret key at the transmitter as V(s, b), and at the receiver as \(\hat{V}(s, b)\).

ii) At every block (s, b), s ≠ 1, the transmitter pulls NR bits from its secret key buffer to secure the outage constrained message of size \(H(W(s, b)) = NR\), using Vernam’s one time pad. The receiver uses the same key to correctly decode the message. We denote the pulled key at the transmitter as K(s, b), and at the receiver as \(\hat{K}(s, b)\). For simplicity in analysis, we assume that keys generated at s − 1’th superblock are used only in the s’th superblock. This stage is skipped in the first super-block, and when ‘encoder’ outage \(O_{enc}(s, b, R)\) occurs, which is the union of the following events:

- Channel outage \(O_{ch}(s, b, R)\): Channel is not suitable for reliable transmission at rate R, i.e., \(R_{en}(s, b) < R\).
- Key outage \(O_{key}(s, b, R)\): There are not enough key bits in the key queue to secure \(W(s, b)\), i.e.,
  \[
  \left(\sum_{b' = 1}^{B} H(V(s - 1, b')) - \sum_{b' = 1}^{b} H(K(s, b'))\right) < 0
  \]
- Artificial outage \(O_{a}(s, b, R)\): The transmitter declares ‘outage’, even though reliable secure transmission of \(W(s, b)\) is possible. This is introduced to bound the probability of key outages, and is explained in the outage analysis.

**Encoding:**

Our random coding arguments rely on an ensemble of codebooks generated according to a zero mean Gaussian distribution with variance \(P(s, b)\).

1) When \(O_{enc}(s, b, R)\) does not occur, the message is secured with the secret key bits pulled from the key queue, using one time pad.

\[
W_{sec}(s, b) = W(s, b) \oplus K(s, b)
\]  

(44)

Clearly, \(W_{sec}(s, b) \in \mathcal{W}_{sec} = \{1, \ldots , 2^{NR}\}\). Furthermore, let \(\mathcal{W}_{x1}(s, b) = \{1, \ldots , 2^{N(R_{en}(s, b)-R-\delta)}\}\). To transmit the one time padded message \(w_{sec}(s, b)\), the encoder randomly and uniformly chooses \(w_{x1}(s, b)\) among \(\mathcal{W}_{x1}(s, b)\), and transmits to codeword \(x(s, b)\) indexed by \((w_{sec}(s, b), w_{x1}(s, b))\) over the channel.

2) When \(O_{enc}(s, b, R)\) occurs, \(W(s, b)\) is not transmitted. Let \(\mathcal{W}_{x2} = \{1, \ldots , 2^{N(R_{en}(s, b)-\delta)}\}\). The encoder randomly and uniformly chooses \(w_{x2}(s, b)\) among \(\mathcal{W}_{x2}\), and transmits to codeword \(x(s, b)\) indexed by \(w_{x2}(s, b)\) over the channel.

The reason for transmitting \(w_{x1}(s, b)\) and \(w_{x2}(s, b)\) is to confuse the eavesdropper to the fullest extent in the privacy amplification process.

**Decoding:**

1) When \(O_{enc}(s, b, R)\) does not occur, the receiver finds the jointly typical \((\hat{w}_{sec}(s, b), \hat{w}_{x1}(s, b), y(s, b))\) pair, where \(y(s, b)\) denotes the received signal at block (s, b). Then, using one-time pad, the receiver obtains \(\hat{w}(s, b) = \hat{w}_{sec}(s, b) \oplus \hat{w}_{x1}(s, b) \oplus y(s, b)\).

\[\text{Note that, it is also possible to use a finite number of codebooks by partitioning the set } \{h\} \text{ of channel gains, and using a different Gaussian codebook for every partition.}\]

\[\text{We assume that both the message and the key are converted to binary form in this process.}\]
\(w_{\text{sec}}(s, b) \oplus \hat{k}(s, b)\).

2) When \(O_{\text{enc}}(s, b, R)\) occurs, the receiver finds the jointly typical \((\hat{w}_{x,2}(s, b), y(s, b))\).

Define the error events
\[
E_1(s, b) = \{(\hat{w}_{\text{sec}}(s, b), \hat{w}_{x,1}(s, b)) \neq (w_{\text{sec}}(s, b), w_{x,1}(s, b))\}
\]
\[
E_2(s, b) = \{\hat{w}_{x,2}(s, b) \neq w_{x,2}(s, b)\}
\]
\[
E_3(s, b, \delta) = \left\{ \frac{1}{N} \|X(s, b)\|^2 > P(s, b) + \delta \right\}
\]

Independent of whether the event \(O_{\text{enc}}(s, b, R)\) occurs or not, the encoding rate is equal to \(R_m(s, b) - \delta\), which is below the supremum of achievable main channel rates. Furthermore, each element of \(X(s, b)\) is independently drawn from Gaussian distribution of mean 0 and variance \(P(s, b)\). Therefore, random coding arguments guarantee us that \(\forall B > 0, \exists N_1 > 0\) such that \(\forall N > N_1, \mathbb{P}(E_1(s, b)) \leq \frac{\delta}{3B}, \mathbb{P}(E_2(s, b)) \leq \frac{\delta}{3B}\) and \(\mathbb{P}(E_3(s, b)) \leq \frac{\delta}{3}\).

**Privacy Amplification:** At the end of every block \((s, b)\), the transmitter and receiver generate secret key bits, by applying a universal hash function on the exchanged signals in that particular block. First, we provide the definition of a universal hash function.

**Definition 3:** (9) A class \(G\) of functions \(A \rightarrow B\) is universal, if for any \(x_1 \neq x_2\) in \(A\), the probability that \(g(x_1) = g(x_2)\) is at most \(\frac{1}{2}\) when \(g\) is chosen at random from \(G\) according to a uniform distribution.

**Lemma 9:** For any \(B > 0\), there exists \(N_2(B) > 0\) such that, \(\forall N > N_2(B)\), and for any block \((s, b)\),

- When \(O_{\text{enc}}(s, b, R)\) does not occur, the transmitter and receiver can generate secret key bits \(V(s, b) = G([W_{\text{sec}}(s, b) W_{x,1}(s, b)])\) and \(\hat{V}(s, b) = G([\hat{W}_{\text{sec}}(s, b) \hat{W}_{x,1}(s, b)])\) respectively, such that \(V(s, b) = \hat{V}(s, b)\) if the error event \(E_1(s, b)\) does not occur, and

\[
H(V(s, b)) = N(R_s(s, b) - \delta)
\]
\[
\frac{1}{N} I(V(s, b); Z^SB, h^SB, G) \leq \delta / B
\]

- Similarly, when \(O_{\text{enc}}(s, b, R)\) occurs, the transmitter and receiver can generate secret key bits \(V(s, b) = G([W_{x,2}(s, b)])\) and \(\hat{V}(s, b) = G([\hat{W}_{x,2}(s, b)])\) respectively, such that \(V(s, b) = \hat{V}(s, b)\) if the error event \(E_2(s, b)\) does not occur, and (45), (46) are satisfied.

**Proof:** The proof follows the approach of [11], which applies privacy amplification to Gaussian channels. First, we introduce the information theoretic quantities required for the proof. For random variables \(A, B\), define

- Renyi entropy of \(A\) as \(\log \mathbb{E}[P_A(a)]\)
- Min-entropy as \(A_h(A) = \min_a \log \left( \frac{1}{P_A(a)} \right)\).
- Conditional min-entropy of \(A\) given \(B\) as \(H_\infty(A|B) = \inf_b H_\infty(A|B = b)\).
- \(\delta\)-smooth min-entropy of \(A\) as \(H^\delta_\infty(A) = \max_{A', \|P_A - P_{A'}\|_\infty \leq \delta} H_\infty(A')\).

Without loss of generality, we drop the block index \((s, b)\) and \(R\), and focus on the first block \((1, 1)\), and assume the event \(O_{\text{enc}}\) does not occur. Let \(W_X = [W_{\text{sec}} W_{x,1}]\), with sample realization sequences denoted by \(w_{x,1}\). Let
$V = G(W_X)$, where $G$ denotes a random universal hash function that maps $W_X$ to an $r$-bit binary message $V \in \{0, 1\}^r$. Then, it is clear that if error event $E_1$ does not occur, $\hat{V} = V$ since $W_X = \hat{W}_X$, for any choice of $G$. To show that the security constraints (45)-46 are satisfied, we cite the privacy amplification theorem, which is originally defined for discrete channels. For this purpose, we define a quantization function $\phi$, with sensitivity parameter $\Delta = \sup_z |z - \phi(z)|$. Let $Z^\Delta = \phi(Z)$ denote the quantized version of $Z$, where $z^\Delta$ denotes realization sequences. Then, by Theorem 3 of [9] there exists a universal function $G$ such that

$$H(G(W_X)|Z^\Delta = z^\Delta, G) \geq r - \frac{2^{r-H(W_X|Z^\Delta = z^\Delta)}}{\ln 2}$$

Now, we relate this expression to the Shannon entropy of the message, conditioned on eavesdropper’s actual received signal. Using the facts $H_\infty(W_X) \leq R(W_X)$ and $H_\infty(W_X|Z^\Delta, G) \leq H_\infty(W_X|Z^\Delta = z^\Delta, G)$, it is easy to show that

$$H(G(W_X)|Z^\Delta, G) \geq r - \frac{2^{r-H_\infty(W_X|Z^\Delta)}}{\ln 2}$$

Then, due to the asymptotic relationship between continuous random variables and their quantized versions [13], there exists a quantization function $\phi$ such that $\Delta$ is small enough, and

$$H(G(W_X)|G, Z) \geq H(G(W_X)|G, Z^\Delta) - \frac{\delta}{2B}$$

$$\geq r - \frac{2^{r-H_\infty(W_X|Z^\Delta)}}{\ln 2} - \frac{\delta}{2B}$$

(47)

are satisfied. To relate min-entropy to Shannon entropy, we use the result of Theorem 1 of [11]; $\forall \delta' > 0$, $\exists$ a block length $N'$ such that $\forall N > N'$,

$$\frac{1}{N} H(X^\Delta|Z^\Delta) \leq \frac{1}{N} H_\infty(X^\Delta|Z^\Delta) + \delta'/B$$

(48)

Now, we proceed as follows,

$$H_\infty(W_X|Z^\Delta) = \lim_{\delta' \to 0} H_\infty^\delta(W|Z^\Delta)$$

$$\geq H(W_X) - I(W_X;Z^\Delta) - N\delta/B$$

(49)

$$\geq H(W_X) - I(X;Z) - N\delta/B$$

(50)

$$= NR_a - N\delta/B$$

(51)

where (49) follows from (48), and the appropriate choice of $N'$. (50) follows from the fact that $W_X \to X \to Z \to Z^\Delta$ forms a Markov chain. (51) follows from the fact that $H(W_X) = N(R_m - \delta)$, and similarly $I(X;Z) \leq N(R_m - R_s - \delta)$, which is the eavesdropper’s maximum achievable rate. For the choice of $H(V) = r = N(R_s - \delta)$,

---

6We omit $h^{SB}$ in the following parts of the proof of Lemma [9] for notational simplicity.
from (47), (51), and the fact that $V = G(W_X)$, we get

$$I(V; G, Z) = H(G(W_X)) - H(G(W_X) | Z, G)$$

\[
\leq \frac{2^{-N(B-1)/B}}{\ln 2} + \frac{\delta}{2B}
\]

\[
\leq \frac{\delta}{B}
\]

(52)

since there exists some $N''$ such that for all $N > N''$, (52) Hence, for $N > N_2 = \max(N', N'')$, the constraints (45), (46) are satisfied. The proof for the case where $\mathcal{O}_{enc}$ occurs is very similar, and is omitted.

Equivocation Analysis: Secrecy outage probability can be bounded above as

$$P(\mathcal{O}_{sec}(s, b, R, \delta)) = P(\mathcal{O}_{eq}(s, b, R, \delta) \cup \mathcal{O}_{ch}(s, b, R))$$

$$\leq P(\mathcal{O}_{eq}(s, b, R, \delta) \cup \mathcal{O}_{enc}(s, b, R))$$

$$\leq P(\mathcal{O}_{eq}(s, b, R, \delta)) + P(\mathcal{O}_{enc}(s, b, R))$$

(53)

where the first equality follows from the definition of secrecy outage (3), and (53) follows from the union bound.

Now, we upper bound the first term. For the choice $N, B$ such that $N = \max(N_1(B), N_2(B))$, the equivocation at every block $(s, b)$ in case of no encoder outage can be bounded as

$$H(W(s, b) | Z^{SB}, h^{SB}, G, \bar{\mathcal{O}}_{enc}(s, b, R))$$

$$= H(W(s, b) | Z(s - 1, :), h^{SB}, G, \bar{\mathcal{O}}_{enc})$$

$$- I(W(s, b); Z(s, b) | Z(s - 1, :), h^{SB}, G, \bar{\mathcal{O}}_{enc})$$

(54)

$$\geq NR - H(W(s, b)) - I(W(s, b); Z(s, b), W_{sec}(s, b) | Z(s - 1, :), h^{SB}, G, \bar{\mathcal{O}}_{enc})$$

$$\geq NR - H(K(s, b) | Z(s - 1, :), h^{SB}, G, \bar{\mathcal{O}}_{enc}))$$

$$\geq NR - \sum_{b=1}^{B} H(V(s - 1, b) | Z(s - 1, :), h^{SB}, G, \bar{\mathcal{O}}_{enc}))$$

$$\geq N(R - \delta)$$

where we use the notation $Z(s, :) = \{Z(s, i)\}_{i=1}^{B}$, and omit the index $(s, b, R)$ from $\bar{\mathcal{O}}_{enc}(s, b, R)$. Notice that $W(s, b) = W_{sec}(s, b) \oplus K(s, b)$, and due to our encoder structure, $W_{sec}(s, b)$ is transmitted only in $(s, b)$’th block, and similarly, $K(s, b)$ is generated in $s - 1$’th superblock. Then, due to the memoryless property of the channel, $W(s, b) \rightarrow (Z(s, b), Z(s - 1, :)) \rightarrow Z^{SB}$, hence (a) follows. The first term of (b) follows since $W_{sec}(s, b)$ is independent of $Z(s - 1, :)$, and the second term of (b) follows since $W(s, b) \rightarrow W_{sec}(s, b) \rightarrow Z(s, b)$ forms a Markov chain. (c) follows due to $H(W(s, b)) = NR$. (d) follows since there is no encoder outage, hence key outage, and (e) follows since $K(s, b)$ is pulled from the key buffers, which contain the pool of key bits $\{V(s - 1, b)\}_{b=1}^{B}$.
generated during superblock $s - 1$, and (54) follows from (46). Then,

$$
\mathbb{P}(\mathcal{O}_{eq}(s, b, R, \delta)|\hat{\mathcal{O}}_{enc}(s, b, R)) = \mathbb{P}\left(\frac{1}{N}H(W(s, b)|Z^{SB}, h^{SB}, G) < R - \delta\right)
$$

$$
= 0
$$

(55)

Now, we bound the encoder outage probability. By the union bound,

$$
\mathbb{P}(\mathcal{O}_{enc}(s, b, R)) \leq \mathbb{P}(\mathcal{O}_{ch}(s, b, R) \cup \mathcal{O}_{a}(s, b, R)) + \mathbb{P}(\mathcal{O}_{key}(s, b, R))
$$

Since $P(h) \in \mathcal{P}'$, due to the definition in (54), (53), (55), (54), for the choice $s = 1$

$$
\mathbb{P}(\mathcal{O}_{ch}(s, b, R)) = \mathbb{P}(R_m(s, b) < R)
$$

$$
= \mathbb{P}\left(R_m(\mathbf{H}, P) < \frac{\mathbb{E}[R_m(\mathbf{H}, P)]}{1 - \epsilon}\right) \leq \epsilon
$$

where in (f), we interchangeably use $R_m(s, b) \equiv R_m(h, P)$ due to stationarity of $P(h)$. Note that, the events $\mathcal{O}_{ch}(s, b, R)$ indexed by $(s, b)$ are i.i.d. Here, we introduce i.i.d. artificial outages $\mathcal{O}_{a}(s, b, R)$ such that

$$
\mathbb{P}(\mathcal{O}_{ch}(s, b, R) \cup \mathcal{O}_{a}(s, b, R)) = \epsilon, \ \forall(s, b)
$$

This would help us bound the probability of key outage. For $(s, b), s \neq 1$

$$
\mathbb{P}(\mathcal{O}_{key}(s, b, R)) = \mathbb{P}\left(\sum_{i=1}^{B} H(V(s - 1, i)) - \sum_{i=1}^{b} H(K(s, i)) < 0\right)
$$

$$
= \mathbb{P}\left(\sum_{i=1}^{B} N(R_a(s - 1, i) - \delta) - \sum_{i=1}^{b} NR_1(\hat{\mathcal{O}}_{ch}(s, i, R) \cap \hat{\mathcal{O}}_{a}(s, i, R)) < 0\right)
$$

$$
\leq \mathbb{P}\left(\sum_{i=1}^{B} [R_a(s - 1, i) - \delta - R_1(\hat{\mathcal{O}}_{ch}(s, i, R) \cap \hat{\mathcal{O}}_{a}(s, i, R))] < 0\right)
$$

(56)

Note that, the expression in (56) represents a random walk with expected drift $\mu = \mathbb{E}[R_m(\mathbf{H}, P)] - \delta - R(1 - \epsilon)$. For $R \leq \frac{\mathbb{E}[R_m(\mathbf{H}, P)] - \delta}{1 - \epsilon}, \mu > 0$, hence by the law of large numbers, $\exists B_1 > 0$ such that $\forall B > B_1, \mathbb{P}(\mathcal{O}_{key}(s, b, R)) < \delta, \ s \neq 1$. Therefore, due to union bound and (53), (54), (55), for the choice $B = B_1, N = \max(N_1(B_1), N_2(B_1))$, $\mathbb{P}(\mathcal{O}_{sec}(s, b, R, \delta)) \leq \epsilon + \delta$, which satisfies (4).

**Error Analysis:** For $N, B$ such that $B = B_1, N = \max(N_1(B_1), N_2(B_1)), \forall(s, b), s \neq 1,$

$$
\mathbb{P}(E(s, b, \delta)|\hat{\mathcal{O}}_{enc}(s, b, R)) \leq \mathbb{P}(W(s, b) \neq \hat{W}(s, b)) + \mathbb{P}\left(\frac{1}{N}\|X(s, b)\|^2 > P(s, b) + \delta\right)
$$

$$
= \mathbb{P}(W_{sec}(s, b) \oplus K(s, b) \neq \hat{W}_{sec}(s, b) \oplus \hat{K}(s, b)) + \mathbb{P}\left(\frac{1}{N}\|X(s, b)\|^2 > P(s, b) + \delta\right)
$$

$$
\leq \mathbb{P}(W_{sec}(s, b) \neq \hat{W}_{sec}(s, b)) + \mathbb{P}(K(s, b) \neq \hat{K}(s, b)) + \mathbb{P}\left(\frac{1}{N}\|X(s, b)\|^2 > P(s, b) + \delta\right)
$$

where the first term can be bounded as $\mathbb{P}(W_{sec}(s, b) \neq \hat{W}_{sec}(s, b)) \leq \frac{\delta}{19}$ due to definition of $E_1(s, b)$, and the choice of $N$. Similarly, the third term can be bounded as $\mathbb{P}(W_{sec}(s, b) \neq \hat{W}_{sec}(s, b)) \leq \delta/3$ due to definition of
$E_2(s, b)$, and the choice of $N$. The second term can be bounded as
\[
\mathbb{P}(K(s, b) \neq \hat{K}(s, b)) \leq 1 - \prod_{i=1}^{B} \mathbb{P}(V(s - 1, i) = \hat{V}(s - 1, i))
\]
\[
\leq \sum_{i=1}^{B} \mathbb{P}(V(s - 1, i) \neq \hat{V}(s - 1, i))
\]
\[
\leq \sum_{i=1}^{B} \mathbb{P}(E_1(s - 1, i)) \mathbb{P}(O_{\text{enc}}(s - 1, i, R))
\]
\[
+ \mathbb{P}(E_2(s - 1, i)) \mathbb{P}(O_{\text{enc}}(s - 1, i, R))
\]
\[
\leq B \frac{\delta}{3B}
\]
where $(a)$ follows from the fact that keys used in $s$’th superblock are generated in $s - 1$’th superblock, and $(b)$ follows due to the definitions of $E_1(s, b)$ and $E_2(s, b)$. Therefore, $\mathbb{P}(E(s, b, \delta) | O_{\text{enc}}(s, b, R)) \leq \delta$. Finally,
\[
\mathbb{P}(E(s, b, \delta) | O_{\text{sec}}(s, b, R, \delta)) = \mathbb{P}(O_{\text{enc}}(s, b, R) | O_{\text{sec}}(s, b, R, \delta)) \mathbb{P}(E(s, b, \delta) | O_{\text{enc}}(s, b, R))
\]
\[
+ \mathbb{P}(O_{\text{enc}}(s, b, R) | O_{\text{sec}}(s, b, R, \delta)) \mathbb{P}(E(s, b, \delta) | O_{\text{enc}}(s, b, R))
\]
\[
\leq \mathbb{P}(O_{\text{enc}}(s, b, R) | O_{\text{sec}}(s, b, R, \delta)) + \mathbb{P}(E(s, b, \delta) | O_{\text{enc}}(s, b, R))
\]
\[
\leq \delta
\]
where the last inequality follows from the fact that $\mathbb{P}(O_{\text{enc}}(s, b, R) | O_{\text{sec}}(s, b, R, \delta)) = 0$. This concludes the achievability. Now, we prove the converse. Consider a power allocation function $P(s, b)$, which satisfies the average power constraint
\[
\limsup_{S, B \to \infty} \frac{1}{SB} \sum_{s=1}^{S} \sum_{b=1}^{B} P(s, b) \leq P_{\text{avg}}
\] (57)
Let $\delta > 0$. It follows from the converse proof of ergodic secrecy capacity [4], and law of large numbers that $\exists B_1, N_1$ such that for every $S, B > B_1$, and $N > N_1$, the time-average equivocation rate is bounded as
\[
\frac{1}{SBN} \sum_{s=1}^{S} \sum_{b=1}^{B} H(W(s, b) | Z^{SB}, h^{SB})
\]
\[
\leq \limsup_{S, B \to \infty} \frac{1}{SB} \sum_{s=1}^{S} \sum_{b=1}^{B} R_s(s, b) + \delta
\] (58)
If $R$ is an $\epsilon$ achievable rate, then $\exists B_2, N_2$ such that $\forall B > B_2, N > N_2$
\[
\frac{1}{SBN} \sum_{s=1}^{S} \sum_{b=1}^{B} H(W(s, b) | Z^{SB}, h^{SB})
\]
\[
\geq \sum_{s=1}^{S} \sum_{b=1}^{B} \frac{1}{SB} (R - \delta) \mathbb{1}(O_{\text{sec}}(s, b, R, \delta))
\]
\[
\geq (R - \delta)(1 - \epsilon - \delta)
\] (59)

---

For any reliable code that yields vanishing probability of error as $S, B, N \to \infty$. 

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where (a) follows directly from the definition of the event \( \mathcal{O}_{\text{sec}}(s, b, R, \delta) = \{ H(W(s, b) | Z^{SB}, h^{SB}) \geq R - \delta \} \cap \{ \frac{1}{N} I(X(s, b); Y(s, b)) \geq R \} \), and (b) follows from applying the secrecy outage constraint (4), and the law of large numbers.

From (58), (59), it follows that any \( \epsilon \)-achievable rate \( R \) satisfies

\[
R \leq R^* = \limsup_{S, B \to \infty} \sum_{s=1}^{S} \sum_{b=1}^{B} \frac{1}{SB} R_{m}(s, b)/(1 - \epsilon)
\]  

(60)

Since secrecy outage probability has to be satisfied (4), for \( (s, b), s \neq 1 \) channel outage probability also has to be satisfied, i.e., \( \mathbb{P}(\mathcal{O}_{\text{ch}}(s, b, R)) \leq \epsilon \), which implies

\[
\mathbb{P}(R_{m}(s, b) < R^*) \leq \epsilon
\]  

(61)

Since \( R_{m}(s, b) \) and \( R_{a}(s, b) \) are both deterministic functions of the power \( P(s, b) \) and instantaneous channel gains \( h(s, b) \), it follows that the power allocation function that maximizes \( R^* \) under the constraints (57), (61) is a stationary function of instantaneous channel gains \( h(s, b) \). Interchanging the notations \( P(s, b) \equiv P(h) \), \( R_{a}(s, b) \equiv R_{a}(h, P) \) and \( R_{m}(s, b) \equiv R_{m}(h, P) \), we can see that for any \( \epsilon \) achievable secrecy rate, the constraints (7)-(9) are satisfied, which completes the proof.

**Appendix B**

**Proof of Theorem 2**

The proof is very similar to the proof for full CSI, hence we only point out the differences. For full CSI, key generation occurs at the end of every block, using privacy amplification. Due to lack of eavesdropper channel state at the legitimate nodes, this is no longer possible. However, as shown in [4], it is still possible to generate secret key bits over a superblock. The following lemma replaces Lemma 9 in the full CSI case.

**Lemma 10:** Let us define \( W_X(s) = \{W_X(s, b)\}_{b=1}^{B} \), where

\[
W_X(s, b) = \begin{cases} 
  [W_{sec}(s, b) \ W_{x1}(s, b)], & \mathcal{O}_{\text{enc}}(s, b, R) \text{ does not occur} \\
  [W_{x2}(s, b)], & \mathcal{O}_{\text{enc}}(s, b, R) \text{ occurs}
\end{cases}
\]

and similarly define \( \hat{W}_{X}(s) \). There exists \( N_2 > 0, B_1 > 0 \) such that, \( \forall N > N_2, B > B_1 \), and for any superblock \( s \), the transmitter and receiver can generate secret key bits \( V(s) = G(W_X(s)) \) and \( \hat{V}(s) = G(\{W_X(s)\}) \) respectively, such that \( V(s) = \hat{V}(s) \) if none of the error events \( \{E_i(s, b)\}_{b=1}^{B}, i \in \{1, 2\} \) occur, and

\[
H(V(s)) = NB(\mathbb{E}[R_{a}(H, P)] - \delta)
\]  

(62)

\[
\frac{1}{NB} I(V(s); Z^{SB}, h^{SB}, G) \leq \delta
\]  

(63)

The proof is very similar to the proof of Lemma 9, and is omitted here. Following the same equivocation and error analysis in the full CSI case, we can see that any rate \( R < \mathbb{E}[R_{a}(H, P)]/(1 - \epsilon) \) is achievable. The converse proof is also the same as in full CSI case, and is omitted here.
APPENDIX C 
PROOFS IN SECTION IV-A 

A. Proof of Lemma 7

The parameter $R_{\text{max}}$ is the maximum value for which the problem (21)–(23) has a solution; hence the average power constraint (22) is active. Moreover, the outage constraint (23) is also active, and due to the fact that $R_m(h, P)$ is a concave increasing function of $P(h)$, we have $\mathbb{P}(R_m(H, R_{\text{max}}) = R_{\text{max}}) = (1 - \epsilon)$, since otherwise one can further increase $R_{\text{max}}$ to find a power allocation function that satisfies the equality. Since for a given $h$, the power allocation function that yields $R_{\text{max}}$ is $P_{\text{inv}}(h, R_{\text{max}})$, we have

$$P_{\text{avg}} = \int_{h \in \mathcal{K}} P_{\text{inv}}(h, R_{\text{max}}) f(h) dh$$

where $\mathcal{K}$ the set of channel gains for which the system operates at rate $R_{\text{max}}$, and $\mathbb{P}(H \in \mathcal{K}) = (1 - \epsilon)$. The set $\mathcal{K}$ contains channel gains $h$ for which $P_{\text{inv}}(h, R_{\text{max}})$ takes minimum values, so that the average power constraint is satisfied for the maximum possible $R$. Since $P_{\text{inv}}(h, P) = \frac{h_{m-1}}{k_m}$ is a decreasing function of $h_m$, one can see that the choice of $\mathcal{K}$ that yields $R_{\text{max}}$ is $\mathcal{K} = \{h : h_m \geq \gamma\}$. Since the probability density function of $H$ is well defined, $\mathbb{P}(H_m = 0) = 0$, hence $c > 0$, which, along with $P_{\text{avg}} > 0$, implies that $R_{\text{max}} > 0$.

B. Proof of Lemma 8

Let $R_{\text{max}} > R > R' > 0$. Then, any $P(h)$ that satisfies $\mathbb{P}(R_m(H, P) < R) \leq \epsilon$, would also satisfy $\mathbb{P}(R_m(H, P) < R') \leq \epsilon$. So, the set of power allocation functions that satisfy (23) shrinks as $R$ increases, hence $\mathbb{E}[R_s(H, P^{R})]$ is a non-increasing function of $R$. Now, we prove that $\mathbb{E}[R_s(H, P^{R})]$ is continuous. From Lemma 2 we know that

$$P^{R}(h) = P_{\text{wf}}(h, \lambda_R) + 1(h \in \mathcal{G}(\lambda_R, k_R))(P_{\text{inv}}(h, R') - P_{\text{wf}}(h, \lambda_R))^{+}$$

$$P^{R'}(h) = P_{\text{wf}}(h, \lambda_{R'}) + 1(h \in \mathcal{G}(\lambda_{R'}, k_{R'}))(P_{\text{inv}}(h, R') - P_{\text{wf}}(h, \lambda_{R'}))^{+}$$

where $(\lambda_R, k_R)$ and $(\lambda_{R'}, k_{R'})$ are constants that satisfy (19) and (20) with equality with respect to parameters $R$ and $R'$, respectively. Let us define another power allocation function $\tilde{P}^{R'}(h)$ such that

$$\tilde{P}^{R'}(h) = P_{\text{wf}}(h, \lambda_{R'}) + 1(h \in \mathcal{G}(\lambda_{R'}, k_{R'}))(P_{\text{inv}}(h, R') - P_{\text{wf}}(h, \lambda_{R'}))^{+}$$

It is easy to see that $\mathbb{E}[R_s(H, \tilde{P}^{R'})] \leq \mathbb{E}[R_s(H, P^{R'})]$. Combining the facts i) for any $h$, $P_{\text{inv}}(h, R)$ is a continuous function of $R$ ii) $R_s(h, P)$ is a continuous function of $P$ iii) integration preserves continuity, we can see that $\int R_s(h, P^{R'}) - R_s(h, P_{\text{inv}}^{R'})1(h \in \mathcal{G}(\lambda_R, k_R)) f(h) dh$ is a continuous function of $R'$. Hence, for any $\gamma > 0$, one can find a $\delta > 0$ such that for any $R' < R$, $|R' - R| < \delta$,

$$\mathbb{E}[R_s(H, P^{R})] - \mathbb{E}[R_s(H, P^{R'})] \leq \mathbb{E}[R_s(H, P^{R})] - \mathbb{E}[R_s(H, \tilde{P}^{R'})]$$

$$\leq \int R_s(h, P_{\text{inv}}^{R}) - R_s(h, P_{\text{inv}}^{R'})1(h \in \mathcal{G}(\lambda_R, k_R)) f(h) dh$$

$$\leq \gamma$$
which proves that $\mathbb{E}[R_s(H, P^R)]$ is a left continuous function of $R$. Following a similar approach, it can also be shown that $\mathbb{E}[R_s(H, P^R)]$ is continuous from the right.

C. Proof of Lemma 2

If $\mathbb{E}[R_s(H, P^R)]|_{R=0} = 0$, then the unique solution of $R = \mathbb{E}[R_s(H, P^R)]/(1 - \epsilon)$ is $R = 0$. So, consider $\mathbb{E}[R_s(H, P^R)]|_{R=0} = 0$. It is easy to see that, $\mathbb{E}[R_s(H, P_{R^\text{max}})] = \int_{h_m \geq c} R_s(h, P) f(h) dh \leq R_m(h, P)(1 - \epsilon)$ follows from definition of parameter $c$, and the inequality $R_s(h, P) \leq R_m(h, P)$. Combining the facts that, the function $\mathbb{E}[R_s(H, P_{R^\text{max}})]$ continues and strictly decreasing on $(0, R_{\text{max}}]$, $\lim_{R \to 0+} \mathbb{E}[R_s(H, P_{R^\text{max}})] = \infty$ and $\mathbb{E}[R_s(H, P_{R^\text{max}})] \leq (1 - \epsilon)$, by the intermediate value theorem, there exists a unique $R > 0$, which satisfies $R = \mathbb{E}[R_s(H, P^R)]/(1 - \epsilon)$.

APPENDIX D

PROOF OF LEMMA 2

We use Lagrangian optimization approach to find $P^R(h)$. We can express $\mathbb{E}[R_s(H, P^R)]$ given in (21)-(23) as

$$\max_{P^R(h), \lambda} J(P^R(h))$$

s.t $R_m(h, P) \geq R, \quad \forall h \in \mathcal{G}$

$$\mathbb{P}(H \in \mathcal{G}) = 1 - \epsilon$$

where the Lagrangian $J(P^R(h))$ is given by the equation

$$J(P^R(h)) = \int R_s(h, P) f(h) dh - \lambda \left[ \int P(h) f(h) dh - P_{\text{avg}} \right]$$

Here, $\mathcal{G}$ is a set which consists of $h$ for which $R_m(h, P) \geq R$ must be satisfied. We will show in this proof that it is of the form (16). This problem is identical to (21), since their constraint sets are identical. Hence solution of this problem would also yield $P^R(h)$. In the following two-step approach, we proceed to find $P^R(h)$. Let us fix $\lambda > 0$.

1) For any $\mathcal{G} \subseteq [0, \infty) \times [0, \infty)$, we find $P_{\mathcal{G}}^R(h)$, which is defined as

$$P_{\mathcal{G}}^R(h) = \arg \max_{P(h)} J(P^R(h))$$

s.t $R_m(h, P) \geq R, \quad \forall h \in \mathcal{G}$

Note that we leave the constraint (23) as is, and not include it in $J(P^R(h))$. 

---

Note: The above text is a continuation of the proof from the previous page 23.
2) Using the result of part 1, we find \(P^R(h)\), by finding the set \(G\) that maximizes \(J'(P(H))\), subject to a constraint \(P(H \in G) = 1 - \epsilon\).

We start with step 1. Since both \(\lambda\) and \(R\) are fixed, therefore we drop them from \(P_{inv}(\cdot)\) and \(P_{wf}(\cdot)\), in the following parts to simplify the notation.

**Lemma 11:** If the problem (67) has a feasible solution, then it could be expressed as

\[
P_G(h) = P_{wf}(h) + [P_{inv}(h) - P_{wf}(h)]^+ 1(h \in G)
\]

where \(P_{wf}(h)\) and \(P_{inv}(h)\) are given in (15) and (14), respectively.

**Proof:** We will interchangeably use \(h = [h_m, h_e]\). Due to (67), \(R_m(h, P) = \log(1 + P(h)h_m) \geq R, \forall h \in G\). Hence, there is a minimum power constraint for set \(G\), as

\[
P(h) \geq P_{inv}(h) = \frac{2R - 1}{h_m}, \forall h \in G
\]

Define \(K\) as the set in which the minimum power constraint (69) is not active, i.e.,

\[
K = \{h \in G : P(h) > P_{inv}(h)\} \cup \bar{G}
\]

where \(\bar{G}\) is complement of \(G\). First, we focus on the solution in the nonboundary set. Since the optimal solution must satisfy the Euler-Lagrange equations,

\[
\frac{dJ(P(h))}{dP(h)} = 0, h \in K
\]

For \(h \in K\), we get the following condition

\[
\frac{h_m}{1 + h_mP(h)} - \frac{h_e}{1 + h_eP(h)} - \lambda = 0
\]

whose solution yields

\[
P(h) = \frac{1}{2} \left[ \sqrt{\left(\frac{1}{h_e} - \frac{1}{h_m}\right)^2 + 4 \left(\frac{1}{h_e} - \frac{1}{h_m}\right) - \left(\frac{1}{h_e} + \frac{1}{h_m}\right)} \right]
\]

If for some \(h \in K\), the value \(P(h)\) is negative, then due to the concavity of \(J(P(h))\) with respect to \(P(h)\), the optimal value of \(P(h)\) is zero [4]. Therefore, the solution yields

\[
P(h) = P_{wf}(h), \ \forall h \in K
\]

Combining the result with the minimum power constraint inside set \(G\), the solution of (67) yields (68), which concludes the proof.
Now, we find $P^R(h)$. We proceed by further simplifying the Lagrangian in (66), for the case where $P(h) = P_G(h)$, for a given $G$ as follows.

$$J(P_G(H)) = \int_{h \in G} [R_s(h, P) - \lambda P(h)] f(h) dh + \int_{h \not\in G} [R_s(h, P) - \lambda P(h)] f(h) dh$$

$$= \int [R_s(h, \text{P}_{\text{inv}}) - \lambda \text{P}_{\text{inv}}(h)] f(h) dh + \int \{[R_s(h, \text{P}_{\text{inv}}) - R_s(h, \text{P}_{\text{wf}})]^+ - \lambda [\text{P}_{\text{inv}}(h) - \text{P}_{\text{wf}}(h)]^+ \} f(h) dh$$

(71)

After this simplification, the first term in (71) does not depend on $G$. We conclude the proof by showing that $P^R(h) = P_G^*(h)$ where the set $G^*$ is defined as follows,

$$G^* = \{ h : [R_s(h, \text{P}_{\text{inv}}) - R_s(h, \text{P}_{\text{wf}})]^+ - \lambda [\text{P}_{\text{inv}}(h) - \text{P}_{\text{wf}}(h)]^+ \geq k \}$$

(72)

where the parameter $k$ is a constant that satisfies $P(H \in G^*) = (1 - \epsilon)$. We prove this by contradiction. First define $\xi(h) = [R_s(h, \text{P}_{\text{inv}}) - R_s(h, \text{P}_{\text{wf}})]^+ - \lambda [\text{P}_{\text{inv}}(h) - \text{P}_{\text{wf}}(h)]^+$. Then, it follows from (71) that $G^*$ is the set that maximize (71), so

$$G^* = \arg \max_G \int \xi(h) f(h) dh$$

Assume that some other $G' \neq G^*$ is optimal, where $P(H \in G') = 1 - \epsilon$. However, we have

$$J(P_{G^*}(H)) - J(P_{G'}(H)) = \int_{G^*} \xi(h) f(h) dh - \int_{G'} \xi(h) f(h) dh$$

$$= \int_{G^* \setminus G'} \xi(h) f(h) dh - \int_{G' \setminus G^*} \xi(h) f(h) dh$$

$$\geq 0$$

(73)

since

$$\int_{G^* \setminus G'} f(h) dh = \int_{G' \setminus G^*} f(h) dh$$

and

$$\xi(h)_{h \in G'} \geq \xi(h)_{h \in G^*}, \ \forall h$$

by definition. This contradicts our assumption that $G'$ is optimal. Note that, $G^*$ is identical to (17). This concludes the proof.
The proof goes along similar lines as in Appendix D, so we skip the details here. We solve the problem for a fixed \( \lambda > 0 \). First, for any given \( G \in [0, \infty) \), we define the following problem, the solution of which yields \( P_G(h_m) \).

\[
P_G(h_m) = \arg \max_{P(h_m)} J(P(H_m))
\]

subject to: \( R_m(\{h_m, h_e\}, P) \geq R, \forall h_m \in G \) (75)

**Lemma 12:** If the problem (74) has a feasible solution, then it can be expressed as

\[
P_G(h_m) = P_w(h_m, \lambda) + 1(h_m \in G) (P_{inv}(h_m, R) - P_w(h_m, \lambda))^+
\]

**Proof:** The proof uses the same approach as in proof of Lemma 11. We define the set \( K \) such that for any \( h_m \in K \), the minimum rate constraint in (75) is not active. Since the optimal solution must satisfy the Euler Lagrange equations, we have

\[
\frac{dJ(P(h_m))}{dP(h_m)} = 0, \ h_m \in K
\]

If we solve the equation for any given \( h_m \), we get

\[
\frac{h_m \mathbb{P}(H_e \leq h_m)}{1 + h_m P(h_m)} - \int_0^{h_m} \left( \frac{h_e}{1 + h_e P(h_m)} \right) f(h_e) dh_e - \lambda = 0
\]

If the power allocation function that solves the equation is negative, then by the convexity of the objective function \( J \), the optimal value of \( P(h_m) \) is 0. Hence, we get \( P_w(h, \lambda) \) as the resulting power allocation function. Whenever the minimum rate constraint (72) is active, we get the channel inversion power allocation function, \( P_{inv}(h, R) \).

Now, using Lemma 12 we solve the following problem,

\[
\max_{P(h_m), G} J(P(H_m))
\]

s.t \( R_m(h, P) \geq R, \ \forall h \in G \)

\( \mathbb{P}(H_m \in G) = 1 - \epsilon \)

the solution of which yields \( P^R(h_m) \). Lemma 12 proves that the solution is a time-sharing between \( P_w(h_m, \lambda) \) and \( P_{inv}(h_m, R) \). Now, we find the optimal \( G \).

**Lemma 13:** The solution of (78) is of the form (76), with the set \( G^* = [c, \infty) \), where \( c \) is a constant which solves \( \mathbb{P}(H_m \geq c) = 1 - \epsilon \).

**Proof:** Let \( P_{G^*}(h_m) \) and \( P_{G'}(h_m) \) be the power allocation functions that are solutions of (76) given the sets \( G^* \) and \( G' \), respectively. We show that, any choice of \( G' \neq G^* \), such that \( \mathbb{P}(H_m \in G') = 1 - \epsilon \) is suboptimal, i.e.,

\[
J(P_{G^*}(H_m)) - J(P_{G'}(H_m)) \geq 0
\]
We continue as follows.

\[
J(P_G^\ast(H_m)) - J(P_{G'}(H_m)) = \\
\int \left\{ \int_{h_m \in G'} \left( [R_s([h_m, h_c], P_{inv}) - R_s([h_m, h_c], P_w)]^+ - \lambda [P_{inv}(h_m, R) - P_{wl}(h_m, \lambda)]^+ \right) f(h_m) dh_m \right\} f(h_c) dh_c \\
- \int \left\{ \int_{h_m \in G'} \left( [R_s([h_m, h_c], P_{inv}) - R_s([h_m, h_c], P_w)]^+ - \lambda [P_{inv}(h_m, R) - P_{wl}(h_m, \lambda)]^+ \right) f(h_m) dh_m \right\} f(h_c) dh_c 
\]

Note that, for any \( h_m' \in G' \setminus G' \) and \( h_m'' \in G' \setminus G' \), we have \( h_m' > h_m'' \). Since \( P_w(h_m', \lambda) > P_w(h_m'', \lambda) \) and \( P_{inv}(h_m', \lambda) < P_{inv}(h_m'', \lambda) \), we have

\[
[P_{inv}(h_m', R) - P_w(h_m', \lambda)]^+ \leq [P_{inv}(h_m'', R) - P_w(h_m'', \lambda)]^+ 
\]

Since \( R_s(\cdot, P) \) is a concave increasing function of \( P(\cdot) \) and \( P_w(\cdot, P) \), we have \( \frac{dP_w(\cdot, P)}{dP} = \lambda \). Therefore, for any \( h_c \), we have

\[
[R_s([h_m', h_c], P_{inv}) - R_s([h_m', h_c], P_w)]^+ - \lambda [P_{inv}(h_m', R) - P_w(h_m', \lambda)]^+ \\
- [R_s([h_m'', h_c], P_{inv}) - R_s([h_m'', h_c], P_w)]^+ + \lambda [P_{inv}(h_m'', R) - P_w(h_m'', \lambda)]^+ \geq 0 
\]

Combining this result with the packing arguments following (72) in Appendix D, we get

\[
J(P_G^\ast(H_m)) - J(P_{G'}(H_m)) \geq 0 
\]

hence concluding the proof. Note that, this result can also be proved using the arguments of Section 4 in \[15\].

**APPENDIX F**

**PROOFS IN SECTION VI**

**A. Proof of Lemma 6**

Due to Theorem 1.2 of Section VI in \[17\], it suffices to show that \( Q_M(t) \) is a positive recurrent regenerative process. Note that \( Q_M(t) \) is a Markov process with an uncountable state space \([0, M]\), since \( Q_M(t) \) can be written as \( Q_M(t + 1) = \min(M, Q_M(t) + R_s(t) - 1(\bar{O}_s(t) \cap \bar{O}_{key}(t))) \) where \( R_s(t) \) and \( \bar{O}_s(t) \) are i.i.d., and \( \bar{O}_{key}(t) = \{Q_M(t) + R_s(t) - R \geq 0\} \) depends only on \( Q_M(t) \) and \( R_s(t) \). Therefore, \( Q_M(t + 1) \) is independent of \( \{Q_M(i)\}_{i=1}^{t-1} \) given \( Q_M(t) \), hence Markovity follows. Now, we prove that \( Q_M(t) \) is a recurrent regenerative process where regeneration occurs at times \( t_1, t_2, \ldots \) such that \( Q_M(t_i) = M \). A sufficient condition for this is to show that \( Q_M(t) \) has an accessible atom \[20\].

**Definition 4:** An accessible atom \( M \) is a state that is hit with positive probability starting from any state, i.e.,

\[
\sum_{i=1}^{\infty} P(Q_M(t) = M | Q_M(1) = i) > 0 \forall i. 
\]

**Lemma 14:** \( Q_M(t) \) has an accessible atom \( M \).
Proof: Assume $Q_M(1) = i, i \in [0, M]$. Note that, $R_s(t)$ and $O_x(t)$ are both i.i.d. Also note that, $\mathbb{P}(R_s(t) - R1(\bar{O}_x(t)) > 0) > 0 \ \forall t$. Find $\gamma > 0$ such that $\mathbb{P}(R_s(t) - R1(\bar{O}_x(t)) > \gamma) = \gamma \ \forall t$. Let $\eta_i = \lceil \frac{M-i}{\gamma} \rceil$. Then,

$$ \mathbb{P}(Q_M(\eta_i + 1) = M|Q_M(1) = i) \geq \prod_{t=1}^{\eta_i} \mathbb{P}(R_s(t) + 1(\bar{O}_x(t)) > \delta) \geq \gamma^{\eta_i} > 0 $$

Since $Q_M(t)$ is a regenerative process, we know that $t_2 - t_1, t_3 - t_2, \cdots$ are i.i.d. random variables. Define a random variable $\tau$, with distribution identical to $t_{i+1} - t_i$. Now we show that $Q_M(t)$ is positive recurrent, by showing $\mathbb{E}[\tau] < \infty$. Consider another recursion

$$ Q_M'(t + 1) = \min(M, Q_M'(t) + R_s(t) - R1(\bar{O}_x(t)))^+ $$

(79)

with $Q_M'(1) = Q_M(1)$. It is clear that $Q_M'(t)$ is also regenerative, where regeneration occurs at $\{t_i\}$, where $Q_M(t_i) = M$, and let $\tau'$ be equal in distribution to $t'_{i+1} - t'_i$.

**Lemma 15:**

$$ \mathbb{E}[\tau] \leq \mathbb{E}[\tau'] $$

Proof: It suffices to show that when $Q_M(t) \neq M, Q_M'(t) \leq Q_M(t)$. By induction, assuming $Q_M'(t) \leq Q_M(t)$, we need to verify that $Q_M'(t + 1) \leq Q_M(t + 1)$. Consider $Q_M(t + 1) < M$. Then,

$$ Q_M(t + 1) = \left( Q_M(t) + R_s(t) - R1(\bar{O}_x(t) \cap \bar{O}_{\text{key}}(t)) \right)^+ \geq (Q_M(t) + R_s(t) - R1(\bar{O}_x(t)))^+ \geq (Q_M'(t) + R_s(t) - R1(\bar{O}_x(t)))^+ = Q_M'(t + 1) $$

Note that $Q_M'(t)$ is regenerative both at states 0 and $M$. Let $\mathbb{E}[\tau'_1]$ denote the expected time for the process $Q_M'(t)$ to hit 0 from $M$, and $\mathbb{E}[\tau'_2]$ denote the expected time to hit $M$ from 0. Then,

$$ \mathbb{E}[\tau'] \leq \mathbb{E}[\tau'_1] + \mathbb{E}[\tau'_2] $$

(80)

Since the key queue has a negative drift, i.e., $\mu_R = \mathbb{E}[R_s(H, P) - R1(\bar{O}_x(t))] < 0$, it is clear that $\mathbb{E}[\tau'_1] < \infty$. Now, we show that $\mathbb{E}[\tau'_2] < \infty$. Following the approach of Lemma 14 find $\gamma > 0$ such that $\mathbb{P}(R_s(t) - R1(\bar{O}_x(t)) > \gamma) = \gamma$.

9Since considering otherwise would lead to the uninteresting scenario where there are no buffer overflows (since the key queue cannot grow), hence any buffer size $M > C'_k$, is sufficient to achieve $\epsilon'$ secrecy outage probability.
∀t. Let \( \eta = \lceil M/\gamma \rceil \). Then, \( \mathbb{P}(Q_M(\eta + 1) = M|Q_M(1) = 0) \geq \gamma^\eta > 0 \), and

\[
\mathbb{E}[\tau_2'] \leq \sum_{i=0}^{\infty} (\eta + i(\mathbb{E}[\tau_1'] + \eta))\gamma^\eta(1 - \gamma^\eta)^i
\]

\[
\leq \eta\gamma^\eta \sum_{i=0}^{\infty} (1 - \gamma^\eta)^i + \sum_{i=0}^{\infty} (1 - \gamma^\eta)^i(\mathbb{E}[\tau_1'] + \gamma^\eta)
\]

\[
< \infty
\]

The first inequality follows from the fact that with probability \( \gamma^\eta \), \( Q_M(t) \) hits \( M \) at \( \eta \)th block and with probability \( (1 - \gamma^\eta) \), key queue back to state 0 at \( (\mathbb{E}[\tau_1'] + \gamma^\eta)' \)th block (on average). The last inequality follows from \( 0 < \gamma^\eta < 1 \), and ratio test. This result, along with (80) and Lemma 15 concludes that \( Q_M(t) \) is a positive recurrent regenerative process, which concludes the proof.

B. Proof of Lemma 16

We follow an indirect approach to prove the lemma. Let \( \{Q(t)\}_{t \geq 1} \) denote the key queue dynamics of the same system for the infinite buffer case \( (M = \infty) \). First, we use the heavy traffic results in [18] to calculate the overflow probability of the infinite buffer queue. Then, we relate the overflow probability of infinite buffer system to the loss ratio of the finite buffer queue. The dynamics of the infinite buffer queue is characterized by

\[
Q(t + 1) = Q(t) + R_s(t) - 1(\bar{O}_{enc}(t))R
\]

where \( Q(1) = 0 \). The heavy traffic results we will use are for queues that have a stationary distribution. Since it is not clear whether \( Q(t) \) is stationary or not, we will upper bound \( Q(t) \) by another stationary process \( Q'(t) \), and the buffer overflow probability result we will get for \( Q'(t) \) will serve as an upper bound for \( Q(t) \).

Let \( \{Q'(t)\}_{t \geq 1} \) be the process that satisfies the following recursion

\[
Q'(t + 1) = (Q'(t) + R_s(t) - R1(\bar{O}_{\bar{x}}(t))^+) + \left(\frac{\mathbb{E}[\tau_1'] + \gamma^\eta}{\gamma^\eta}\right)
\]

with \( Q'(1) = 0 \). First, we relate \( Q'(t) \) to \( Q(t) \).

Lemma 16:

\[
Q(t) \leq Q'(t) + R, \forall t
\]

Proof: Assuming \( Q(t) \leq Q'(t) + R \), we need to show by induction that \( Q(t + 1) \leq Q'(t + 1) + R \). There are two different scenarios.

1) If \( Q'(t) + R_s(t) - R1(\bar{O}_{\bar{x}}(t)) \geq 0 \), then, using the facts \( \bar{O}_{enc}(t) = \bar{O}_{\bar{x}}(t) \cap \bar{O}_{\bar{x}}(t) \) and \( Q'(t) \leq Q(t) \), we obtain

\[
Q(t) + R_s(t) - R1(\bar{O}_{enc}(t)) \geq Q'(t) + R_s(t) - R1(\bar{O}_{\bar{x}}(t)) \\
\geq 0
\]
which, using the described key queue recursions in (81), implies

\[ Q(t + 1) = Q(t) + R_s(t) - R1 (\bar{\theta}_s(t)) \]  

(84)

Observe that, by (82),

\[ Q'(t + 1) = Q'(t) + R_s(t) - R1 (\bar{\theta}_s(t)) \]

which, in conjunction with (84) and \( Q(t) \leq Q'(t) + R \), yields \( Q(t + 1) \leq Q'(t + 1) + R \).

2) If \( Q'(t) + R_s(t) - R1 (\bar{\theta}_s(t)) < 0 \), then \( Q'(t + 1) = 0 \). We further consider two cases. First, if \( Q(t) + R_s(t) - R \geq 0 \), then,

\[
(Q(t) + R_s(t) - R1 (\bar{\theta}_s(t)))^+ \\
\leq (Q'(t) + R + R_s(t) - R1 (\bar{\theta}_s(t)))^+ \leq R \\
= Q'(t + 1) + R
\]

(85)

Next, if \( Q(t) + R_s(t) - R < 0 \), then

\[ Q(t + 1) = Q(t) + R_s(t) < R = Q'(t + 1) + R \]

which, combined with (85), yields

\[ Q(t + 1) \leq Q'(t + 1) + R \]

Now, we show that \( Q'(t) \) converges in distribution to an almost surely finite random variable \( Q' \). First, we need to show that the expected drift of \( Q'(t) \) is negative. It is clear from (82) that the expected drift of the process \( Q'(t) \) is equal to \( \mu_R = \mathbb{E}[R_s(H, P^R)] - R(1 - \epsilon) \).

**Lemma 17:** For \( R > C_F' \), we have \( \mu_R < 0 \), and \( \mu_R \) is a continuous decreasing function of \( R \).

**Proof:** From Lemma 3 in Section IV-A, we know that \( \mathbb{E}[R_s(H, P^R)] \) is a non-increasing continuous function of \( R \). Therefore, \( \mu_R \) it is a continuous function of \( R \). Furthermore, by definition of \( C_F' \) in (7), \( \mu_{C_F'} = 0 \). Combining these two facts, we conclude that \( \mu_R < 0 \), for \( R > C_F' \).

**Lemma 18:** There exists an almost surely finite random variable \( Q' \) such that, for all \( x \),

\[
\limsup_{t \to \infty} \mathbb{P}(Q(t) > x) \leq \mathbb{P}(Q' + R > x)
\]

(86)

**Proof:** Combining Lemma 17 with the classic results by Loynes [16], we can see that \( Q'(t) \) converges in distribution to an almost surely finite random variable \( Q' \) such that

\[
\lim_{t \to \infty} \mathbb{P}(Q'(t) > x) = \mathbb{P}(Q' > x)
\]

Using (83), we finish the proof of the lemma.

Now, we characterize the tail distribution of the key queue.
Lemma 19: For any given $M \geq 0$,
\[
\lim_{R \to \infty} \limsup_{t \to \infty} P \left( \frac{\|R\|}{\sigma_R} (Q(t) - R) > M \right) \leq e^{-2M} \tag{87}
\]

Proof: First, we prove that
\[
\lim_{R \to \infty} \mathbb{P} \left( \frac{\|R\|}{\sigma_R} Q' > y \right) = e^{-2y}, \tag{88}
\]
which is based on the heavy traffic limit for queues developed in [18], see also Theorem 7.1 in [17]. In order to prove (88), we only need to verify the following three conditions: i) $\lim_{R \to \infty} \|R\| = 0$; ii) $\lim_{R \to \infty} \sigma_R^2 > 0$; and iii) the set $\{(R_s(h, P^R) - R1(\bar{O}_x(t)))^2\}$ of random variables indexed by $R$ is uniformly integrable.

i) From Lemma [17] we obtain $\lim_{R \to \infty} \|R\| = 0$.

ii) Since $R_s(h, P^R) - C_F(\bar{O}_x(t))$ is not a constant random variable, almost surely
\[
\lim_{R \to \infty} \sigma_R^2 = \text{Var}[R_s(h, P^R) - C_F(\bar{O}_x(t))] > 0
\]

iii) Note that, $R$ lies on the interval $[0, R_{\max}]$, where $R_{\max}$, defined in Lemma [1] then we have
\[
(R_s(h, P^R) - R1(\bar{O}_x(t)))^2 = R_s(h, P^R)^2 - 2R_s(h, P^R)R1(\bar{O}_x(t)) + R^21(\bar{O}_x(t)) \\
\leq R_s(h, P^R)^2 + R_{\max}^2
\]
Since $R_s(h, P)$ is a continuous function of $P(h)$, and for any $R$ on the interval $[0, R_{\max}]$, $\lim_{c \to \infty} \mathbb{P}(P^R(h) > c) = 0$, hence we can see that $\lim_{c \to \infty} \mathbb{P}(R_s(h, P^R) > c) = 0$. Therefore, this class of random variables is uniformly integrable. This completes the proof of (88). This result, in conjunction with Lemma [18] completes the proof.

Using Lemma 1 in [19], we relate the loss ratio of our finite buffer queue $Q_M(t)$ to the overflow probability of the infinite buffer queue $Q(t)$ as follows
\[
\mathbb{E}[R_s(h, P^R)] \limsup_{T \to \infty} L_T^T(M) \leq \int_{x=M}^{\infty} \limsup_{t \to \infty} \mathbb{P}(Q(t) > x)dx \tag{89}
\]
Combining Lemma [19] with (89), the proof is complete.

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