A COUNTEREXAMPLE TO WEITZENBÖCK’S THEOREM
IN CHARACTERISTIC $p$

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Abstract. In this paper we give a counterexample to Weitzenböck’s Theorem in positive characteristic. Namely we show that if $k$ is an algebraically closed field of characteristic $p$, there is an action of $G_a$ on $\mathbb{A}_k^n$, induced by a linear representation of $G_a$, such that the ring of invariants $k[x_1,\ldots,x_n]^{G_a}$ is not finitely generated.

1. Introduction

By results of [4] and [6], Hilbert’s fourteenth problem was essentially settled for reductive groups. Weitzenböck established in an Acta Mathematica article [10] in 1932 that for a linear representation of $G_a$ of dimension $n$, $\mathbb{C}[x_1,\ldots,x_n]^{G_a}$ is finitely generated. Thus the question of whether Hilbert’s 14th problem could be answered in the affirmative reduced to whether finite generation of invariants was true for unipotent groups. Nagata showed that this was false. Mukai brought the dimension of the vector group used in Nagata’s counterexample down to three in [5]. Later activity focused on whether the ring of invariants $k[x_1,\ldots,x_n]^{G_a}$ is finitely generated for a non-linear action of $G_a$ on $\mathbb{A}_k^n$. The answer again is no. Roberts, Freudenberg, and both Daigle and Freudenberg established that the ring of invariants was not always finitely generated in [8], [3], and [1]. In 1962 the author of [9] gave a useful survey of results related to Weitzenböck’s theorem, and extended the result. The computation of invariants in characteristic $p$ has always been problematic. The author of [2] computed the ring of invariants for what I shall call the thrice twisted representation of $G_a$ to be

$$k[x_1, x_2^p - x_5 x_1^{p^3-1}, x_3^p - x_5^p x_1^{p^3-1} - x_4^p - x_1^{p^3-2} x_5^p].$$

However, the invariant $x_3^{p^2} - x_5 x_1^{p^2-1}$ is not in this ring, although its $p$-th power is. The goal of this paper is to settle the question of whether it is true that for every linear action of $G_a \cong \text{Spec}(k[t])$ on

$$\mathbb{A}_k^n \cong \text{Spec}(k[x_1,\ldots,x_n]),$$

over an algebraically closed field $k$ of characteristic $p$ the ring of invariants $k[x_1,\ldots,x_n]^{G_a}$ is finitely generated, by providing a counterexample.
We describe the main results of this paper in Section 2. In Section 3 we give our conventions for this paper, and review some preliminary information about $G_a$ representations, and the invariant rings of unipotent groups. One of the important preliminary ideas is that linear $G_a$ representations in characteristic zero are determined by a locally nilpotent derivation, while in positive characteristic they are determined by a finite collection of commuting operators. The invariant ring of $G_a$ in positive characteristic is more like that of a vector group, and unlike what we see in characteristic zero. In Section 5 we give a review of Nagata’s work on the $a$-transform. We display results about an auxiliary ring in Section 6 that will be useful in the proof of the main result. In Section 7 we prove the remaining necessary results and end with the counterexample.

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2. Main Results

The thrice twisted representation of $G_a \cong \text{Spec}(k[t])$ is a representation of dimension five such that for the induced action $\mu : G_a \times \mathbb{A}_k^5 \to \mathbb{A}_k^5$ the coaction is:

\begin{align*}
    x_1 &\mapsto x_1 \\
    x_2 &\mapsto x_2 + tx_1 \\
    x_3 &\mapsto x_3 + t^p x_1 \\
    x_4 &\mapsto x_4 + t^p^2 x_1 \\
    x_5 &\mapsto x_5 + t^p^3 x_1.
\end{align*}

Note that the characteristic of $k$ must be $p > 0$. The following theorem is the main result.

**Theorem 2.1.** The invariant ring $k[x_1, \ldots, x_5]^{G_a}$ of the thrice twisted representation of $G_a$ is not finitely generated.

To aid in the proof of the main result, we consider the following ring

$$A := k[w_1, \ldots, w_5]/\langle (w_2^p - w_3)(w_2^{p^2} - w_4) - w_1^{p^3 - p}w_5 \rangle,$$

that we may embed as a subring of the ring of invariants of the thrice twisted representation of $G_a$. We will also review the work of Nagata on the $a$-transform.

3. Notation and Conventions

In this work, when we say the word variety we mean an integral, separated, scheme of finite type over an algebraically closed field $k$. We use the notation $:= $ to denote the phrase “is defined to equal.” If $f : Z \to Y$ is a map of schemes, then we will follow Grothendieck’s convention of denoting the corresponding map $\mathcal{O}_Y \to f_*(\mathcal{O}_Z)$, by $f^*$. We denote the ring $k[x_1, \ldots, x_n]$
by \( k[X] \), and \( f(x_1, \ldots, x_n) \) by \( f(X) \). We reserve the letter \( \mu \) for the action of \( G_a \) on \( A^n_k \). For \( t_0 \in k \), we denote the coaction of \( G_a \) evaluated at \( t_0 \) on \( f(X) \in k[X] \), by \( t_0 \cdot f(X) \), or by \( \mu^*(f(X))(t_0) \). If the coordinate ring of \( G_a \) is \( k[t] \), then we denote the image of \( f(X) \) under the coaction of \( G_a \) by \( \mu^*(f(X))(t) \), or by \( t \cdot f(X) \). If for any \( t_0 \in k \), \( t_0 \cdot x_i - x_i \) is a homogeneous, degree one polynomial in \( k[x_1, \ldots, x_i-1] \); then we call an action of \( G_a \) on \( A^n_k \) linear. We denote the action of \( G_a \) on itself by \( \mu_G \), and the co-multiplication by \( \mu^\# \). We denote \( a_{j_1, \ldots, j_n}(\prod_{i=1}^n x_i^{j_i}) \) by \( a_{J X} \).

4. Actions of Unipotent Groups on \( A^n_k \)

Let \( V \) be a vector space and \( A^n_k \cong \text{Spec}(\text{Sym}(V^*)) \). Let \( \mu : A^n_k \times G_a \to A^n_k \) be a linear action of \( G_a \) on \( A^n_k \). The co-action \( \mu^\# \) must satisfy the following equation:

\[
\mu^\#_{k[x_1, \ldots, x_n]}(x_m) = \sum_{i=0}^{d_m} a_i(x_m) \otimes t^i
\]

for all \( x_m \), where \( m \in \{1, \ldots, n\} \), \( d_m \in \mathbb{N}_0 \) and \( a_i \in \text{End}(V^*) \). If \( A^n_k = \text{Spec}(k[x_1, \ldots, x_n]) \), then the following diagram commutes, which commutes, implies certain relations among the \( a_i \):

\[
\begin{array}{ccc}
  k[X, t] & \xrightarrow{\mu^\#} & k[X] \\
  \downarrow \text{id} \otimes \mu^\#_{a} & & \downarrow \text{id} \\
  k[X, t, r] & \xleftarrow{\mu^\# \otimes \text{id}_a} & k[X, r]
\end{array}
\]

In (1), \( \epsilon \) is the co-identity of \( G_a \). The following Proposition describes the collection of endomorphisms \( a_i \).

**Proposition 4.1.** Let \( \text{char}(k) = p > 0 \). If \( \mu \) is a linear action of \( G_a \) on \( A^n_k \), then \( \mu \) is determined by an almost everywhere zero collection \( \{a_{p^i} \in \text{End}(V^*)\}_{i \in \mathbb{N}_0} \) such that each \( a_{p^i} \) is \( p \)-nilpotent for each \( i \in \mathbb{N}_0 \), and \( a_{p^i} a_{p^j} = a_{p^{i+j}} \).

**Proof.** It is clear that \( a_0 \) is the identity, since if \( m \in \{1, \ldots, n\} \),

\[
(\text{id}_{k[X]} \otimes \epsilon) \circ a_0(x_m) = x_m.
\]

We wish to utilize the commutativity of (1) to produce the identities necessary to prove the proposition. The top and left parts of the square in (1)
show that

\[
\left( \text{id}_{k[X]} \otimes \mu_{G_m}^p \right) \circ \mu_{k[X]}^p(x_m) = \left( \text{id}_{k[X]} \otimes \mu_{G_m}^p \right) \left( \sum_{i=0}^{\infty} a_i(x_r) \otimes t^i \right)
\]

\[
= \sum_{i=0}^{\infty} a_i(x_m)(t + r)^i,
\]

while by the right and bottom parts of the square in (1)

\[
\left( \mu_{k[X]}^p \otimes \text{id}_{k[t]} \right) \circ \mu_{k[X]}^p(x_m) = \left( \mu_{k[X]}^p \otimes \text{id}_{k[t]} \right) \left( \sum_{i=0}^{\infty} a_i(x_m) \otimes t^i \right)
\]

\[
= \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} a_{i+j}(a_i(x_m)) t^i r^j.
\]

The sum \( \sum_{i=0}^{\infty} a_i(x_m)(t + r)^i \) equals

\[
\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i+j}(x_m) \binom{i+j}{j} t^i r^j.
\]

After equating the coefficients of \( t^i r^j \), we see that

\[
a_j a_i = a_{i+j} \binom{i+j}{j}.
\]

We obtain the identity \( a_i^2 = a_{2i} \binom{2i}{i} \) via (2) with \( i = j \). By induction the following identities are true:

\[
a_p^p = a_p a_{p-1} \prod_{j=0}^{p-1} \binom{jp^j}{p^j}
\]

\[
= a_{p^i+1} \prod_{j=0}^{p^i} \binom{jp^j}{p^j}.
\]

The next lemma and its corollary imply that \( \binom{p^{i+1}}{p^i} \equiv 0 \). After proving this corollary, it is then clear that \( a_p^p \equiv 0 \).

**Lemma 4.2.** Let \( k \) be a field of characteristic \( p \). If \( N = \sum_{i=0}^{j} r_i p^i \) is the \( p \)-adic expansion of the integer \( N \), then the following identity holds:

\[
(x + 1)^N \equiv \prod_{i=0}^{j} \sum_{\ell=0}^{r_i} \binom{r_i}{\ell} x^{\ell}.
\]
Proof. Because of the $p$-adic expansion of the integer $N$, the following identities hold:

$$
\sum_{m=0}^{N} \binom{N}{m} x^m = (x + 1)^N,
$$

$$
= (x + 1)^{\sum_{i=0}^{j} r_i p^i},
$$

(4)

Since

$$(x + 1)^{p^i} \equiv x^{p^i} + 1,$$

the identity (4) implies that

(5)

$$
(x + 1)^N \equiv \prod_{i=0}^{j} \left(x^{p^i} + 1\right)^{r_i}.
$$

By the binomial theorem and (5)

$$
\prod_{i=0}^{j} \left(x^{p^i} + 1\right)^{r_i} = \prod_{i=0}^{j} \sum_{\ell=0}^{r_i} \left(\frac{r_i}{\ell}\right) x^{p^i \ell}.
$$

\[\Box\]

Corollary 4.3 (Lucas). If an integer $m$ has the $p$-adic expansion $m = \sum_{i=0}^{j} c_i p^i$, then

(6)

$$
\binom{N}{m} \equiv \prod_{i=0}^{j} \left(\frac{r_i}{c_i}\right) \mod p
$$

Proof. The coefficient of $x^m$ is $\binom{N}{m}$, but it is also congruent to $\prod_{i=0}^{j} \left(\frac{r_i}{c_i}\right)$ by (3).

By (2) and Corollary 4.3

$$
a_{r_i}^{p^i} = a_{r_i p^i} \prod_{\ell=1}^{r_i} \left(\frac{\ell p^i}{p^i}\right),
$$

(7)

$$
\equiv r_i! a_{r_i p^i}.
$$

Definition 4.4. The set $R_\ell$ is the set of $j$ such that $\sum_{j \in R_\ell} r_{\ell,j} p^j = \ell$ and every coefficient $r_{\ell,j}$ is nonzero.

It is clear that:

$$
a_N = a_{\sum_{i \in R_N} r_{N,i} p^i} \equiv \prod_{i \in R_N} \frac{a_{r_{N,i}}^{p^i}}{r_{N,i}!},
$$

where we derive the last part of the earlier inequalities by repetitious use of (2) and (7). Also $a_{p^i}^p \equiv 0$. Since the co-action’s effect on each $x_i$
determines its effect on $k[x_1, \ldots, x_n]$, a finite collection of commuting $p$-nilpotent elements of $\text{End}(V^*)$, $(a_1, a_p, a_{p^2}, a_{p^3}, \ldots)$ determines the action completely. □

**Theorem 4.5.** If $G$ is a connected group, without characters, over an algebraically closed field, acting on $\mathbb{A}^n$, and $k[X]^G$ is finitely generated, then $k[X]^G$ is a UFD.

**Proof.** Let $f(X)$ be irreducible in $k[X]^G$. The ideal $\langle f(X) \rangle k[X]$ factors into $\prod_{i=1}^s u_i(X)^{e_i}$. If each element $u_i(X)$ is invariant, then there are no nontrivial factors, or else $f(X)$ would not be irreducible in $k[X]^G$. Since $f(X)$ is invariant, $G \times V(u_i(X)) \to V(u_j(X))$. Because $G$ is connected, $e \cdot V(u_i(X)) = V(u_i(X))$ implies that $j = i$, or else the image would not be irreducible. Therefore, $g \cdot u_i(X) = c_i(g)u_i(X)$. Since $G$ has no characters, $c_i(g) = 1$. Because each $u_i(X) \in k[X]^G$, $f(X)$ is irreducible in $k[X]$. Since $f(X)$ is irreducible in $k[X]$, it is prime in $k[X]$, which means it is prime in $k[X]^G$. Therefore, all irreducible elements of $k[X]^G$ are prime. So, $k[X]^G$ is a UFD if it is finitely generated. □

5. Some Results of Nagata

If the reader wishes to find more about this work they may consult [7]; our presentation will be self contained and will suffice for our purposes. The following are some results of [7].

**Definition 5.1.** Let $R$ be an integral domain with quotient field $L$. Let $a$ be an ideal of $R$. The set

$$S(a, R) = \{f \mid f \in L, fa^n \subseteq R \text{ for some } n\}.$$ 

**Definition 5.2.** With the same assumptions as before

For any integer $n \geq 0$, set $a^{-n} = \{f \mid f \in L, fa^n \subseteq R\}$.

Then $S(a, R) = \bigcup_{n \geq 0} a^{-n}$.

**Remark 5.3.** When $\text{ht}(a) = 1$, the rational functions $f \in a^{-n}$ are those with poles of order less than or equal to $n$ along $V(a)$. If $D = V(a)$, then the inclusion $R \subset a^{-1}$ induces the isomorphism

$$S(a, R) = \lim_{n \in \mathbb{N}_0} H^0(\text{Spec}(R), \mathcal{L}(nD)).$$

**Definition 5.4.** Let $V$ be a variety. An affine variety $V'$ is called an associated affine variety of $V$ if

- The variety $V' \supseteq V$.
- The set of divisors of $V'$ coincides with that of $V$ i.e. the set of local rings of rank 1 of $V$ and $V'$ are the same.

**Remark 5.5.** The set $S(a, R)$ is a ring.
Proposition 5.6. Let $\mathfrak{a}$ be an ideal of a Noetherian domain $R$. Let $S$ be the $\mathfrak{a}$-transform of $R$ and $R'$ a subring of $S$ containing $R$. Then the $\mathfrak{a}R'$-transform of $R'$ is $S$.

Definition 5.7. Let $R$ be an integral domain and let $\mathfrak{a}$ be an ideal of $R$. We say that the $\mathfrak{a}$-transform of $R$ is finite if $S(\mathfrak{a}, R) = R[\mathfrak{a}^{-n}]$ for some $n \geq 0$.

Remark 5.8. The $\mathfrak{a}$-transform of $R$ is finite if and only if it is finitely generated.

Proposition 5.9. Let $F$ be a proper closed subset of an affine variety $V$, and let $\mathfrak{a}$ be an ideal which defines $F$ in the affine ring $R$ of $V$. Then $V \setminus F$ has an associated affine variety if and only if the $\mathfrak{a}$-transform $S$ of $R$ is finite; in this case, $S$ defines an associated affine variety and $S$ contains and is integral over the affine ring of any associated affine variety of $V \setminus F$.

Proposition 5.10. Let $V$ be an affine variety defined by an affine ring $R$ and let $F$ be a closed set defined by an ideal $\mathfrak{a}$.

The variety $V \setminus F$ is affine if and only if $1 \in \mathfrak{a}S$, where $S$ is the $\mathfrak{a}$-transform of $R$. In this case $F$ is pure of codimension 1 and $S$ is the affine ring of $V \setminus F$.

6. An Auxiliary Ring

In this section, we shall establish certain results about the ring

$$A = k[w_1, \ldots, w_5]/\langle (w_2^p - w_3)(w_2^{p^2} - w_4) - w_1^{p^3-p}w_5 \rangle.$$  

This ring appears as a subring of the ring of invariants of the thrice twisted representation of $G_2$.

Lemma 6.1. The element

$$f(W) := (w_2^p - w_3)(w_2^{p^2} - w_4) - w_1^{p^3-p}w_5$$

is an irreducible element of $k[w_1, \ldots, w_5]$.

Proof. The polynomial $f(W)$ is an element of $k[w_1, \ldots, w_4][w_5]$. Thus, $f(W) = h_1(W) + w_5h_2(W)$, where

$$h_1(W) = (w_2^p - w_3)(w_2^{p^2} - w_4)$$

$$h_2(W) = -w_1^{p^3-p}.$$
Denote \( h(W) \in k[w_1, \ldots, w_4] \) by \( h(W_5) \). If \( f(W) \) is reducible, then \( f(W) = u(W)v(W) \) where

\[
\begin{align*}
u(W) &= \sum_{i=0}^{s} u_i(W_5^i) w_5^i \\
v(W) &= \sum_{j=0}^{r} v_j(W_5^j) w_5^j.
\end{align*}
\]

The top degree of \( f(W) \) in \( w_5 \) is one, so without loss of generality, let \( s = 0 \) and \( r = 1 \). Since

\[
u_0(W_5)v_1(W_5) = h_2(W) = -w_1^{p^3-p},
\]
either \( w_1 \mid u_0(W_5) \), or \( u(W) \) is a unit. If \( w_1 \mid u_0(W_5) \), then \( w_1 \mid f(W) \). However \( w_1 \nmid f(W) \). Therefore \( u(W) \) is a constant, and \( f(W) \) is irreducible. \( \square \)

**Proposition 6.2.** The ring

\[
A := k[w_1, \ldots, w_5]/((w_2^p - w_3)(w_2^{p^2} - w_4) - w_1^{p^3-p}w_5),
\]
is an integral domain.

**Proof.** To show that \( A \) is an integral domain, define a map

\[
\tau : k[w_1, \ldots, w_5] \to k(w_1, \ldots, w_4)
\]
by \( \tau(w_5) = (w_2^p - w_3)(w_2^{p^2} - w_4)/w_1^{p^3-p} \), and \( \tau(w_i) = w_i \) for \( i = 1, \ldots, 4 \). The inverse image of 0 is prime. By Lemma 6.1 \((w_2^p - w_3)(w_2^{p^2} - w_4) - w_1^{p^3-p}w_5\) is irreducible, height one, and is also a prime contained in the kernel. So it is the kernel. Therefore, \( A \) is an integral domain. \( \square \)

**Lemma 6.3.** If \( R \) is a ring and \( f \in R \), then \( D(f) \) denotes the primes of \( \text{Spec}(R) \) that do not contain \( f \). The coordinate rings of \( D(w_1), D(w_2^p - w_3), \) and \( D(w_2^{p^2} - w_4) \) of

\[
A = k[w_1, \ldots, w_5]/((w_2^p - w_3)(w_2^{p^2} - w_4) - w_1^{p^3-p}w_5),
\]
are all UFD’s.

**Proof.** Observe that \( A_{w_1} \cong k[w_1, \ldots, w_4]_{w_1} \), and thus is a UFD. In \( A_{w_2^p - w_3} \),

\[
w_4 = w_2^p - \frac{w_1^{p^3-p}w_5}{(w_2^p - w_3)},
\]
and in \( A_{w_2^{p^2} - w_4} \)

\[
w_3 = w_2^p - \frac{w_1^{p^3-p}w_5}{(w_2^{p^2} - w_4)}.
\]
So
\[ A_{w_2^p - w_3} \cong k[w_1, w_2, w_3, w_5]_{w_2^p - w_3} \]
\[ A_{w_2^p - w_4} \cong k[w_1, w_2, w_4, w_5]_{w_2^p - w_4}, \]
and both rings are UFDs. \hfill \square

**Proposition 6.4.** The ideals \( p := \langle w_1, w_2^p - w_3 \rangle \) and \( p_1 := \langle w_1, w_2^p - w_4 \rangle \) of
\[ A := k[w_1, \ldots, w_5]/((w_2^p - w_3)(w_2^p - w_4) - w_5^{p^3 - p} w_5) \]
are height one prime ideals of \( A \) such that \( \text{Spec}(A) \setminus V(p) \) and \( \text{Spec}(A) \setminus V(p_1) \) are not affine.

**Proof.** Both \( w_2^p - w_3 \) and \( w_2^p - w_4 \) are irreducible polynomials in the quotient \( k[W]/\langle w_1 \rangle \). Therefore, \( \langle w_1, w_2^p - w_3 \rangle \) and \( \langle w_1, w_2^p - w_4 \rangle \) are prime ideals of height less than or equal to two in \( k[W] \). Both of these ideals strictly contain \((w_2^p - w_3)(w_2^p - w_4) - w_5^{p^3 - p} w_5\), and so they have height one in \( A \), and height two in \( k[W] \).

Suppose that \( Y_1 := \text{Spec}(A) \setminus V(p) \) is affine. If this is true, then \( H^1(Y_1, \mathcal{O}_{\text{Spec}(A)}) = 0 \). The open set \( Y_1 \) has a covering by two open affine sets, namely \( D(w_1) \) and \( D(w_2^p - w_3) \). The Cech complex of \( Y_1 \) with coefficients in \( \mathcal{O}_{\text{Spec}(A)} \) ends in the terms
\[ A_{w_1} \times A_{w_2^p - w_3} \longrightarrow A_{w_1(w_2^p - w_3)} \longrightarrow 0. \]

\[ (s_1(W), s_2(W)) \longrightarrow s_1(W) - s_2(W) \longrightarrow 0. \]

If \( A_{w_1} \times A_{w_2^p - w_3} \) maps surjectively onto \( A_{w_1(w_2^p - w_3)} \), then let \( g(w_1, \ldots, w_5) \notin p \), and write \( s_1(W) = u_1(W)/w_1^{\ell_1} \) and \( s_2(W) = u_2(W)/(w_2^p - w_3)^{\ell_2} \). Here we may assume that \( u_1 \nmid u_1(W) \) and \( w_2^p - w_3 \nmid u_2(W) \) by Lemma 6.3. If
\[ \frac{g(w_1, \ldots, w_5)}{(w_1(w_2^p - w_3))} = s_1(W) - s_2(W), \]
then
\[ g(w_1, \ldots, w_5) = (u_1(W)(w_2^p - w_3)^{\ell_2} - u_2(W)w_1^{\ell_1})/(w_1^{\ell_1 - 1}(w_2^p - w_3)^{\ell_2 - 1}). \]
Both \( \ell_i \geq 1 \). If \( \ell_1 = \ell_2 = 1 \), then \( g(W) \in p \). Without loss of generality, if \( \ell_1 > 1 \), then \( u_1(W)(w_2^p - w_3)^{\ell_2} \equiv 0 \) mod \( \langle w_1 \rangle \). Therefore \( u_1 | u_1(X) \), which is a contradiction. Therefore, \( H^1(Y_1, \mathcal{O}_{\text{Spec}(A)}) \neq 0 \).

If we replace \( w_2^p - w_3 \) with \( w_2^p - w_4 \), and \( p \) with \( p_1 \), in the above argument, then it is clear that \( \text{Spec}(A) \setminus V(p_1) \) is not affine. \hfill \square
7. The Main Result

Throughout this section we consider the thrice twisted representation of $G_a$:

\[
\begin{align*}
x_1 & \mapsto x_1 \\
x_2 & \mapsto x_2 + tx_1 \\
x_3 & \mapsto x_3 + t^p x_1 \\
x_4 & \mapsto x_4 + t^p^2 x_1 \\
x_5 & \mapsto x_5 + t^p^3 x_1.
\end{align*}
\]

We first shall prove a lemma about four important invariants that will prove useful.

**Lemma 7.1.** Let

\[
\begin{align*}
z_1(X) & := x_2^p - x_3 x_1^{p-1} \\
z_2(X) & := x_2^p - x_4 x_1^{p^2-1} \\
z_3(X) & := x_2^p - x_5 x_1^{p^3-1}.
\end{align*}
\]

The invariants $x_1, z_1(X), z_2(X)$ and $z_3(X)$ are algebraically independent.

**Proof.** Suppose there is a polynomial $f(u_1, \ldots, u_4)$ such that $f(x_1, z_1(X), z_2(X), z_3(X)) = 0$. On $V(x_2)$ this relation becomes

\[
f(x_1, -x_3 x_1^p, -x_4 x_1^{p^2}, -x_5 x_1^{p^3}) = 0.
\]

Since $x_1, x_3, x_4$ and $x_5$ are algebraically independent, $x_1, z_1(X), z_2(X)$ and $z_3(X)$ are algebraically independent as well. \(\square\)

**Proposition 7.2.** There is an injective map $\phi^5 : A \to k[X]^{G_a}$ where

\[
A := k[w_1, \ldots, w_5]/((w_2^p - w_3)(w_2^{p^2} - w_4) - w_1^{p^3-p} w_5),
\]

and $k[X]^{G_a}$ is the ring of invariants of the thrice twisted representation of $G_a$.

**Proof.** Define a map $k[W] \to k[X]^{G_a}$ as follows:

\[
\begin{align*}
w_1 & \mapsto x_1 \\
w_2 & \mapsto z_1(X) = x_2^p - x_3 x_1^{p-1} \\
w_3 & \mapsto z_2(X) = x_2^p - x_4 x_1^{p^2-1} \\
w_4 & \mapsto z_3(X) = x_2^p - x_5 x_1^{p^3-1} \\
w_5 & \mapsto (x_2^p - x_4 x_1^{p^2-1})(x_2^p - x_5 x_1^{p^3-1}).
\end{align*}
\]

Because $x_1, z_1(X), z_2(X)$ and $z_3(X)$ are algebraically independent by Lemma 7.1, the kernel of this map of $k[W] \to k[X]^{G_a}$ is $((w_2^p - w_3)(w_2^{p^2} - w_4) - w_1^{p^3-p} w_5)$; so this is an injective map of $A$ into the ring of invariants. \(\square\)
Let \( u_i \) be the rational function \( x_i / x_1 \). The ring
\[
B := k[X]_{x_1} = \oplus_{\nu \in \mathbb{Z}} k[u_2, \ldots, u_5][x_1^\nu].
\]
Let \( B_0 \) be the degree zero part of \( B \). If we assign weights to the \( u_i \) by
\[
\text{wt}(u_2) = 1 \\
\text{wt}(u_3) = p \\
\text{wt}(u_4) = p^2 \\
\text{wt}(u_5) = p^3,
\]
then the ring \( B_0 \) has a grading \( \oplus_{d \in \mathbb{N}_0} B_{0,d} \), where \( B_{0,d} \) is the vector space of polynomials \( f(u_2, u_3, u_4, u_5) \) of weight \( d \). So the ring \( B \) has a double grading
\[
B = \oplus_{\nu \in \mathbb{Z}, d \in \mathbb{N}_0} B_{0,d} x_1^\nu.
\]
This is a direct sum, since if \( \nu_1 = \nu_2 \) and \( d_1 \neq d_2 \), these are disjoint. If \( \nu_1 < \nu_2 \) and
\[
x_1^{\nu_1} g_{d_1}(U) = x_1^{\nu_2} g_{d_2}(U),
\]
then
\[
B_0 \ni g_{d_1}(U) \\
= x_1^{\nu_2 - \nu_1} g_{d_2}(U) \\
\notin B_0,
\]
a contradiction. With respect to this grading, the rational functions
\[
y_1(X) := z_1(X)/x_1^p = \left( \frac{x_2}{x_1} \right)^p - \frac{x_3}{x_1} = u_2^p - u_3\\ny_2(X) := z_2(X)/x_1^{p^2} = \left( \frac{x_2}{x_1} \right)^{p^2} - \frac{x_4}{x_1} = u_2^{p^2} - u_4\\ny_3(X) := z_3(X)/x_1^{p^3} = \left( \frac{x_2}{x_1} \right)^{p^3} - \frac{x_5}{x_1} = u_2^{p^3} - u_5
\]
are of weights \( p, p^2, \) and \( p^3 \) respectively. We make the following definition with these functions in mind.

**Definition 7.3.** Let \( T_d \) be the vector space of polynomials \( f(y_1(X), y_2(X), y_3(X)) \) of weight \( d \).

Observe that \( x_1, y_1(X), y_2(X), \) and \( y_3(X) \) are invariant rational functions of weights 1, \( p, p^2, \) and \( p^3 \). In the instance that a rational function in \( x_1^{d-m} T_d \) is in \( k[X] \), it is in \( k[X]^{\mathbb{Z}_a} \).

**Definition 7.4.** The vector space \( R_{d,m} \) is the intersection of \( x_1^{d-m} T_d \) and \( k[X] \).
With these definitions in place, the following inclusions hold:
\[ \oplus_{d,m} x_1^{d-m} T_d \subseteq \oplus_{d,m} B_0 dx_1^{d-m}. \]
\[ \oplus_{d,m} R_{d,m} \subseteq k[X]. \]

**Theorem 7.5.** Consider the thrice twisted representation of \( \mathbb{G}_a \). The ring of invariants
\[ k[x_1, \ldots, x_5]^\mathbb{G}_a = \oplus_{d \geq 0, m \in \mathbb{Z}} R_{d,m}. \]

**Proof.** Since \( x_1 \) is invariant, the coaction on \( k[X] \) extends to a coaction on \( B \). Here the coaction on \( B \) is the following:
\[
\begin{align*}
x_1 & \mapsto x_1 \\
u_2 & \mapsto u_2 + t \\
u_3 & \mapsto u_3 + t^p \\
u_4 & \mapsto u_4 + t^{p^2} \\
u_5 & \mapsto u_5 + t^{p^3}.
\end{align*}
\]
If \( f(x_1, x_2, \ldots, x_5) \in k[x_1, \ldots, x_5]^\mathbb{G}_a \), then because \( f(X) \in B \cap k[X] \)
\[ f(X) = \sum_C a_C(x_1)x_1^{c_1} \cdots x_5^{c_5}, \]
and \( \mu^2(f(X))(t) \) equals
\[
\sum_C a_C(x_1)x_1^{c_1} \cdots x_5^{c_5} (u_2 + t)^{c_1} (u_3 + t^p)^{c_2} (u_4 + t^{p^2})^{c_3} (u_5 + t^{p^3})^{c_4}.
\]
Set \( t = -u_2 \) in (8). Because
\[
\begin{align*}
u_3 - u_2^p & = - \left( \frac{x_2^p}{x_1} - \frac{x_3}{x_1} \right) = -y_1(X) \\
u_4 - u_2^{p^2} & = - \left( \frac{x_2^{p^2}}{x_1^2} - \frac{x_4}{x_1} \right) = -y_2(X) \\
u_5 - u_2^{p^3} & = - \left( \frac{x_2^{p^3}}{x_1^3} - \frac{x_5}{x_1} \right) = -y_3(X),
\end{align*}
\]
the polynomial \( \mu^2(f(X))(-u_2) \) is a polynomial in \( k[x_1, y_1(X), y_2(X), y_3(X)] \).
Since \( f(X) \) is invariant, \( f(X) = \mu^2(f(X))(-u_2) \). Note that
\[ k[y_1(X), y_2(X), y_3(X)] = \oplus_{d \in \mathbb{N}_0} T_d[y_1(X), y_2(X), y_3(X)], \]
and \( a_C(x_1) = \sum_{s=0}^{C} a_s x_1^s \). The polynomial \( f(X) \) can be written with only \( C \) such that \( c_1 = 0 \), and therefore \( f(X) \) is equal to
\[
\sum_{C, s} a_s x_1^{c_2+c_3+c_4} (u_3 - u_2^p)^{c_2} (u_4 - u_2^{p^2})^{c_3} (u_5 - u_2^{p^3})^{c_4}.
\]
Let \( d \in \mathbb{N}_0, \ m \in \mathbb{Z} \). Define \( S(d, m) \) to be the set of \((s, C)\) such that \( c_1 = 0, \ s + c_2 + c_3 + c_4 = d - m \) and \( pc_2 + p^2c_3 + p^3c_4 = d \). With this notation in place, let

\[
  f_d(b_1, b_2, b_3) = \sum_{(s, C) \in S(d, m)} (-1)^{c_2+c_3+c_4} a_s b_1^{c_2} b_2^{c_3} b_3^{c_4}.
\]

And so

\[
  f(X) = \sum_{d, m} x_1^{d-m} f_d(y_1(X), y_2(X), y_3(X)),
\]

where \( f_d(y_1(X), y_2(X), y_3(X)) \in T_d \). If there is an integer \( d \) and a non-negative integer \( m \) such that

\[
  x_1^{d-m} f_d(y_1(X), y_2(X), y_3(X)) \notin k[X],
\]

then \( f(X) \in B \setminus k[X] \). Therefore, \( f(X) \in \oplus_{d, m} R_{d, m} \).

**Theorem 7.6.** The map \( \phi^* \), (see Proposition 7.2) induces an isomorphism of

\[
  \text{Frac}(A) \cong \text{Frac}\left( k[x_1, \ldots, x_5]^{G_a} \right) = k(x_1, x_2, x_3, x_4, x_5).
\]

**Proof.** The result is a corollary of Theorem 7.5.

**Proposition 7.7.** Let \( G_a \) act via the thrice twisted representation of \( G_a \). Let \( D_1 := \phi^{-1}(V(p)) \) (see Proposition 7.2 and Lemma 6.4), and

\[
  Y := \text{Spec}(k[x_1, \ldots, x_5]^{G_a}) \setminus D_1.
\]

If \( k[x_1, \ldots, x_5]^{G_a} \) is finitely generated, then there is an isomorphism

\[
  S(pk[X]^{G_a}, k[X]^{G_a}) \cong H^0(Y, \mathcal{O}_{\text{Spec}(k[X]^{G_a})}) \cong S(p, A).
\]

**Proof.** Let \( \phi^* : \text{Frac}(A) \cong \text{Frac}(k[X]^{G_a}) \). Since a rational function \( f(W) \) has a pole of order \( m \in \mathbb{N}_0 \) at \( V(p) \) if and only if \( \phi^*(f(W)) \) has one of the same order at \( D_1 \),

\[
  S(pk[X]^{G_a}, k[X]^{G_a}) \cong H^0(Y, \mathcal{O}_{\text{Spec}(k[X]^{G_a})}).
\]

If in addition \( k[X]^{G_a} \) is finitely generated, then \( D_1 := \phi^{-1}(V(p)) \) is a principal divisor \( V(u(X)) \). Hence \( H^0(Y, \mathcal{O}_{\text{Spec}(k[X]^{G_a})}) \cong k[X]^{G_a}_{\text{Spec}(k[X]^{G_a})} \). Therefore \( Y \) has an associated affine variety, and \( S(pk[X]^{G_a}, k[X]^{G_a}) \) is integral over \( k[X]^{G_a}_{\text{Spec}(k[X]^{G_a})} \). Each of \( k[X]^{G_a}_{\text{Spec}(k[X]^{G_a})} \) and \( S(pk[X]^{G_a}, k[X]^{G_a}) \) have \( \text{Frac}(k[X]^{G_a}) \) as their field of fractions, and \( S(pk[X]^{G_a}, k[X]^{G_a}) \) is integral over \( k[X]^{G_a}_{\text{Spec}(k[X]^{G_a})} \), where the latter is integrally closed. Therefore,

\[
  S(pk[X]^{G_a}, k[X]^{G_a}) \cong H^0(Y, \mathcal{O}_{\text{Spec}(k[X]^{G_a})}) \cong k[X]^{G_a}_{\text{Spec}(k[X]^{G_a})}.
\]

\( \square \)
Theorem 7.8 (Main Result). Consider the thrice twisted representation of $G_a$, whose co-action is:

$$
\begin{align*}
  x_1 & \mapsto x_1 \\
  x_2 & \mapsto x_2 + tx_1 \\
  x_3 & \mapsto x_3 + t^3 x_1 \\
  x_4 & \mapsto x_4 + t^3 x_1 \\
  x_5 & \mapsto x_5 + t^3 x_1.
\end{align*}
$$

The ring $k[x_1, \ldots, x_5]^{G_a}$ is not finitely generated.

**Proof.** If $k[x_1, \ldots, x_5]^{G_a}$ is finitely generated, then $k[x_1, \ldots, x_5]^{G_a}$ is a UFD by Theorem 4.5 and $D_1 = \phi^{-1}(V(p))$ is a principal divisor that we denote by $V(u(X))$. The open sub-scheme $Y := \text{Spec}(k[X]^{G_a}) \setminus D_1$ defined in Lemma 7.7 would be affine with coordinate ring $k[x_1, \ldots, x_5]^{G_a}_{u(X)}$. This would mean that

$$
1 \in p k[X]_{u(X)} S(p k[X]^{G_a}, k[X]^{G_a}).
$$

We can simplify this expression by noting that

$$
k[X]^{G_a}_{u(X)} \cong S(p k[X]^{G_a}, k[X]^{G_a})
\cong S(p, A).
$$

However, since $k[X]^{G_a} \subset k[X]^{G_a}_{u(X)}$, this implies that $1 \in p S(p, A)$, and $\text{Spec}(A) \setminus V(p)$ would be affine. This is not the case by Lemma 6.4, so $k[x_1, \ldots, x_5]^{G_a}$ is not finitely generated. \qed

Remark 7.9. The curious reader might wonder what would happen if instead of using $V(p)$ we used $V(p_1)$. A similar development is possible.

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