CONTAINING ALL PERMUTATIONS

Michael Engen and Vincent Vatter
Department of Mathematics
University of Florida
Gainesville, Florida USA

Numerous versions of the question “what is the shortest object containing all permutations of a given length?” have been asked over the past fifty years: by Karp (via Knuth) in 1972; by Chung, Diaconis, and Graham in 1992; by Ashlock and Tillotson in 1993; and by Arratia in 1999.

The preponderance of questions of this form, which have hitherto been considered in isolation, stands in stark contrast to the dearth of answers to them. We survey and synthesize these questions and their partial answers, introduce infinitely more related questions, and then establish new upper bounds for one family of these questions.

1. Introduction

What is the shortest object containing all permutations of length \( n \)? As we shall describe, there are a variety of such problems, going by an assortment of names including superpatterns and superpermutations. This diversity of universal permutation problems stems from the multiple possible definitions of the terms involved. Throughout, we call all such questions universal permutation problems.

A word is simply a finite sequence of letters or entries, drawn from some alphabet. The length of the word \( w \), denoted \( |w| \) throughout, is the number of its letters, and if \( w \) is a word of length at least \( i \) then we denote by \( w(i) \) its \( i \)th letter. For our purposes (which do not involve group theory), we take a permutation of length \( n \) to be a word consisting of the letters \( [n] = \{1, 2, \ldots, n\} \), each occurring precisely once. Thus permutations are a special type of words over \( \mathbb{P} \), where \( \mathbb{P} \) denotes the positive integers. Two words \( u, v \in \mathbb{P}^n \) (that is, both of length \( n \), with positive integer letters) are order-isomorphic if

\[
u(i) > u(j) \iff v(i) > v(j)
\]

for all \( i, j \in [n] \).

In all universal permutation problems considered here, the object that is to contain all permutations, called the universal object, is a word, but there are two different types of containment. Sometimes
we insist that the word \( w \) contain each permutation as a contiguous subsequence, or \textit{factor}, by which we mean that \( w \) can be expressed as a concatenation \( w = upv \) where the word \( p \) is order-isomorphic to \( \pi \). At other times we merely insist that the word \( w \) contain each permutation \( \pi \) as a \textit{subsequence}, by which we mean that there are indices \( 1 \leq i_1 < i_2 < \cdots < i_n \leq |w| \) so that the word \( p = w(i_1)w(i_2)\cdots w(i_n) \) is order-isomorphic to \( \pi \).

These notions of containment give rise to two different universal permutation problems. To obtain infinitely many, we vary the size of the alphabet that the letters of the universal word \( w \) can be drawn from. In the strictest form, we insist that \( w \) is a word over the alphabet \([n]\), meaning that \( w \) is only allowed the symbols of the permutations it must contain. In this case, the notion of order-isomorphism reduces to equality: a word \( p \in [n]^n \) is order-isomorphic to a permutation \( \pi \) of length \( n \) if and only if \( p = \pi \). At the other end of the spectrum, we allow the letters of \( w \) to be arbitrary positive integers. Between these extremes, another interesting case stands out: when the alphabet is \([n+1]\), thus allowing one more symbol than the permutations. Table 1 displays the best upper bounds established to-date (or, in the rightmost two cells of the upper row, the known formulas) for the six versions of this question that have garnered the most interest.

Note that the bounds shown in Table 1 weakly decrease as we move from left to right (a word over \([n]\) is also a word over \([n+1]\), which is also a word over \(\mathbb{P} \)) and also as we go from top to bottom (factors are also subsequences). Another notable feature of this table is that the lengths of the shortest universal words over the alphabet \([n]\) seem to be significantly greater than the lengths of the shortest universal words over the alphabet \([n+1]\), whose lengths seem to be either equal or close to those of the shortest universal words over the largest possible alphabet, \(\mathbb{P} \).

We remark that some researchers in this area have sought a universal \textit{permutation} instead of a universal word, but this is in fact equivalent to finding a universal word over \(\mathbb{P} \), as we briefly explain. The word \( u \in \mathbb{P}^n \) is \textit{order-homomorphic} to the word \( v \in \mathbb{P}^n \) if

\[
u(i) > u(j) \implies v(i) > v(j)
\]

for all indices \( i, j \in [n] \). It is clear that every word over \(\mathbb{P} \) is order-homomorphic to at least one permutation (one simply needs to “break ties” among the letters of the word), and it is not hard to see that if \( u \) contains the permutation \( \pi \) (as a factor or subsequence) and \( u \) is order-homomorphic to \( v \), then \( v \) also contains \( \pi \) (in the same sense as \( u \) does). Thus finding a universal permutation, in either the factor or subsequence setting, is equivalent to finding a universal word over \(\mathbb{P} \).

Each of the subsequent five sections of this paper is devoted to the examination of one of the cells of Table 1 (except for Section 3, which considers both the upper-center and upper-right cells). In

|                | words over \([n]\) | words over \([n+1]\) | words over \(\mathbb{P} \) |
|----------------|-------------------|----------------------|--------------------------|
| factor         | \(n! + (n-1)! + (n-2)!\) + (n-3)! + n - 3 | \(n! + n - 1\) | \(n! + n - 1\) |
| subsequence    | \(\left\lceil \frac{n^2 - 7}{3}n + \frac{19}{3} \right\rceil\) | \(\frac{n^2 + n}{2}\) | \(\left\lceil \frac{n^2 + 1}{2} \right\rceil\) |

\(\text{Table 1: Known upper bounds for the lengths of the shortest universal words in six flavors of the problem of containing all permutations. Note that the two bounds in the leftmost column are the best that are known only for } n \text{ large (} n \geq 7 \text{ in the factor case and } n \geq 13 \text{ in the subsequence case).} \)
the conclusion we briefly describe some other directions which lead even more universal permutation problems.

2. As Factors, Over $[n]$  

The case in the upper-left of Table 1 dates to a 1993 paper of Ashlock and Tillotson [5], and can be restated as follows.

What is the length of the shortest word over the alphabet $[n]$ which contains each permutation of length $n$ as a factor?

This version of the universal permutation has recently attracted a surprising amount of attention, including a pair of articles in Quanta Magazine [21, 28], and investigations are very much ongoing.

We call a word over the alphabet $[n]$ which contains all permutations of length $n$ as factors an $n$-superpermutation. A (not particularly good) lower bound on the length of $n$-superpermutations is easy to establish by observing that every word $w$ has at most $|w| \cdot n$ many factors of length $n$.

**Observation 2.1.** Every $n$-superpermutation has length at least $n! + n - 1$.

The first two lengths are easy to compute: the word 1 meets the demands for $n = 1$ and the word 121 is as short as possible for $n = 2$. The shortest 3-superpermutation has length 9—one more than the above lower bound, but may be shown to be optimal with a slightly more delicate argument, which we now present. First, there is a word of length 9, \[ 123121321, \]

that contains all permutations of length 3 as factors. Now suppose the word $w$ over the alphabet $[3]$ contains all permutations of length 3 as factors. We say that the letter $w(i)$ is wasted if the factor $w(i - 2)w(i - 1)w(i)$ is not equal to a new permutation of length 3 (either because not all of the letters are defined, or because it contains a repeated letter, or because that permutation occurs earlier in $w$). Thus we have

\[ |w| = n! + (\# \text{ of wasted letters in } w). \]

Clearly the first 2 letters of $w$ are wasted. If $w$ contains an additional wasted letter, then its length must be at least 9. Suppose then that $w$ does not contain any additional wasted letters. Thus each of the factors

\[ w(1)w(2)w(3), \ w(2)w(3)w(4), \ w(3)w(4)w(5), \ and \ w(4)w(5)w(6) \]

must be equal to different permutations. However, the only way for these factors to be equal to permutations at all is to have $w(4) = w(1)$, $w(5) = w(2)$, and $w(6) = w(3)$, and this implies that $w(4)w(5)w(6) = w(1)w(2)w(3)$, a contradiction.

Computations by hand become more difficult at $n = 4$, but we invite the reader to check that the word

\[ 12341231423124132413214321 \]

of length 33 is a 4-superpermutation, and that no shorter word suffices.
As Ashlock and Tillotson [5] noticed, the lengths of these superpermutations are, respectively, $1! = 1$, $2! + 1! = 3$, $3! + 2! + 1! = 9$, and $4! + 3! + 2! + 1! = 33$. They also established the following two results.

**Proposition 2.2** (Ashlock and Tillotson [5, Theorem 3 and Lemma 5]). If there is an $(n-1)$-superpermutation of length $m$ then there is an $n$-superpermutation of length $m + n!$.

**Proposition 2.3** (Ashlock and Tillotson [5, proof of Theorem 18]). Every $n$-superpermutation has length at least

\[ n! + (n - 1)! + n - 2. \]

Proposition 2.2 implies that there is an $n$-superpermutation of length $n! + \cdots + 2! + 1!$. Given this construction and the lengths they had been able to compute, Ashlock and Tillotson made the natural conjecture that the shortest $n$-superpermutation has length $n! + \cdots + 2! + 1!$ for all $n$. They further conjectured that all of the shortest $n$-superpermutations were unique up to the relabeling of their letters.

For about twenty years, very little progress seemed to have been made on the two conjectures of Ashlock and Tillotson, although the conjectures were rediscovered many times on Internet forums such as MathExchange and StackOverflow (references to some of these rediscoveries are cited in Johnston’s article [26]). Then in 2013, Johnston [26] constructed many distinct $n$-superpermutations of length $n! + \cdots + 2! + 1!$ for all $n \geq 5$, proving that at least one of Ashlock and Tillotson’s two conjectures must be false, although giving no hint of which one. A year later, Benjamin Chaffin verified the $n = 5$ case of the length conjecture by computer (see Johnston’s blog post [27] for details), showing that no word of length less than $153 = 5! + 4! + 3! + 2! + 1!$ is a 5-superpermutation. This showed, via Johnston’s constructions, that Ashlock and Tillotson’s uniqueness conjecture was certainly false.

The next case of the length conjecture to be verified would be $n = 6$, where the conjectured bound was $6! + 5! + 4! + 3! + 2! + 1! = 873$. However, only weeks after Chaffin’s verification of the length conjecture for $n = 5$, Houston [22]—by viewing the problem as an instance of the traveling salesman problem—found a 6-superpermutation of length only 872. It is still not known if this is the shortest 6-superpermutation, but regardless, Houston’s construction and Proposition 2.2 lower the upper bound on the length of the shortest $n$-superpermutation to $n! + \cdots + 3! + 2!$ for all $n \geq 6$.

After breaking the length conjecture of Ashlock and Tillotson in 2014, Houston created a discussion group (on Google Groups) called Superpermutators, where those who were interested in the problem could work on it in a loose and unofficial Polymath-esque manner, and most of the subsequent research mentioned here has been communicated there.

The next breakthrough was made shortly after the mathematical physicist John Baez tweeted about Houston’s construction in September 2018. This tweet caused Greg Egan, who is known for his science fiction novels (coincidentally including one called *Permutation City* [12]), to become interested in the problem. Egan found inspiration in an unpublished manuscript of Williams [41]. In that paper, Williams had shown how to construct Hamiltonian paths and cycles in the Cayley graph on the symmetric group $S_n$ generated by the two permutations denoted $(12 \cdots n)$ and $(12)$ in cycle notation (see Sawada and Williams [39] for a published, streamlined construction). Williams’ construction had solved a forty year-old conjecture of Nijenhuis and Wilf [36] (which was later included by Knuth as an exercise with a difficulty rating of 48/50 in Volume 4A of the *Art of Computer Programming* [30, Problem 71 of Section 7.2.1.2]), and in October 2018, Egan showed how it could be adapted to prove the following.
Theorem 2.4 (Egan [13]). For all \( n \geq 4 \), there is an \( n \)-superpermutation of length at most
\[
n! + (n - 1)! + (n - 2)! + (n - 3)! + n - 3.
\]

For \( n = 6 \), the construction of Theorem 2.4 is worse than Houston’s (Theorem 2.4 gives a 6-superpermutation of length 873), but for \( n \geq 7 \) this bound is strictly less than the bound of \( n! + \cdots + 3! + 2! \) implied by Houston’s construction and Proposition 2.2.

For several years before this, it had been known that there was an argument (on a website devoted to anime) which claimed to improve on the lower bound provided by Proposition 2.3. In particular, this argument was mentioned in a 2013 blog post of Johnston [25]. However, the argument was far from what most mathematicians would consider a proof, and there had been no efforts to make it rigorous, in part because the claimed lower bound was so far from what was thought to be the correct answer at the time. However, Egan’s breakthrough quickly inspired several participants of the Superpermutators group to write a rigorous version of the lower bound. In the process, it was realized that the argument did not originate on the anime website where it was found, but was merely copied there from a series of anonymous posts in 2011 on the somewhat-notorious Internet forum 4chan. Thus we have the following result.

Theorem 2.5 (Anonymous 4chan poster). Every \( n \)-superpermutation has length at least
\[
n! + (n - 1)! + (n - 2)! + n - 3.
\]

For general \( n \), Theorems 2.4 and 2.5 are the best bounds established so far. There had been some hope in the Superpermutators group that perhaps Egan’s construction could be made one letter shorter for \( n \geq 7 \) while the lower bound could be increased by \( (n - 3)! - 1 \), so that the two met at
\[
n! + (n - 1)! + (n - 2)! + (n - 3)! + n - 4,
\]
but this has also been shown to be false in the \( n = 7 \) case. In this case, the original length conjecture of Ashlock and Tillotson suggests that the length of the shortest 7-superpermutation should be \( 7! + 6! + 5! + 4! + 3! + 2! + 1! = 5913 \), while Egan’s Theorem 2.4 gives a 7-superpermutation of length \( 7! + 6! + 5! + 4! + 4 = 5908 \). In February 2019, Bogdan Coanda made several theoretical improvements to the computer search for superpermutations and used these to find a 7-superpermutation of length \( 7! + 6! + 5! + 4! + 3 = 5907 \), thus matching the wishful thinking above. (Continuing the tradition of “publishing” progress on this problem in unorthodox places, Coanda announced his construction in the comments to a YouTube video). However, shortly thereafter, Egan and Houston modified Coanda’s approach to construct a 7-superpermutation of length \( 7! + 6! + 5! + 4! + 2 = 5906 \).

3. As Factors, Over \([n+1]\) and \(\mathbb{P}\)

In moving from the previous universal permutation problem to this one, we see for the first time the dramatic effect of adding a letter to the alphabet. Not only does the addition of a single letter seem to significantly shorten the universal words, but it changes the problem from one which remains wide open to one solved a decade ago.

Recall that a de Bruijn word of order \( n \) over the alphabet \([k]\) is a word \( w \) of length \( k^n \) such that every word in \([k]^n\) occurs exactly once as a cyclic factor in \( w \), or equivalently, every such word occurs exactly once as a factor in the longer word
\[
w(1)w(2)\cdots w(k^n)\, w(1)w(2)\cdots w(n - 1).
\]
These words were (mis)named for de Bruijn (see [10]) because in addition to establishing that such words exist, he showed that there are precisely $(k!)^{n-1}/kn$ of them. An example of a de Bruijn word, written cyclically, is shown on the left of Figure 2.

In their highly influential 1992 paper, Chung, Diaconis, and Graham [8] explored generalizations of de Bruijn words to other types of objects, including permutations. (In fact, Diaconis and Graham [11, Chapter 4] state that their motivation was a magic trick.) As they defined it, a universal cycle (frequently shortened to ucycle) for the permutations of length $n$ would be a word $w$ of length $n!$ (over some alphabet) such that every permutation of length $n$ is order-isomorphic to a cyclic factor of $w$, or equivalently, to a factor of $w(1)w(2)\cdots w(n!) w(1)w(2)\cdots w(n-1)$. An example of a universal cycle over $[5]$, written cyclically, for the permutations of length 4 is shown on the right of Figure 2.

If such a universal cycle $w$ were to exist (which was the question they were interested in, leaving enumerative concerns for later), then the word $w(1)w(2)\cdots w(n!) w(1)w(2)\cdots w(n-1)$ would be, in our terms, a shortest possible answer to the universal permutation problem for factors over the alphabet $\mathbb{P}$. In this way, their universal cycle of length $4! = 24$ for the permutations of length 4 shown on the right of Figure 2 is converted (starting at noon and proceeding clockwise) into the universal word

$$1234125341532413254123$$

of length $4! + 4 - 1 = 27$. Thus, together with the trivial lower bound of $n! + n - 1$ noted in Observation 2.1, the answer to the question posed in the upper-right-hand cell of Table 1 is implied by the following result.

**Theorem 3.1** (Chung, Diaconis, and Graham [8]). *For all positive integers $n$, there is a universal cycle over the alphabet $[6n]$ for the permutations of length $n$.*

Chung, Diaconis, and Graham left open the question of whether the alphabet $[6n]$ could be shrunk. Proposition 2.3 shows that for $n \geq 3$, there cannot be a universal cycle over the alphabet $[n]$ for the permutations of length $n$. Therefore the result below, established by Johnson in 2009, is best possible.

**Theorem 3.2** (Johnson [24]). *For all positive integers $n$, there is a universal cycle over the alphabet $[n+1]$ for the permutations of length $n$.*

In terms of universal permutation problems, Theorem 3.2 establishes that there is a word of length $n! + n - 1$ over the alphabet $[n+1]$ which contains every permutation of length $n$ as a factor.
4. AS SUBSEQUENCES, OVER \([n]\)

The universal permutation problem for subsequences over the alphabet \([n]\) pre-dates the others by 20 years. In a 1972 Stanford technical report entitled “Selected Combinatorial Research Problems” and edited together with Chvátal and Klarner, Knuth [9, Problem 36] stated the following problem, which he attributed to Richard Karp:

What is the shortest string of \(\{1, 2, \ldots, n\}\) containing all permutations on \(n\) elements as subsequences? (For \(n = 3, 1213121\); for \(n = 4, 123412314321\); for \(n = 5, M.\) Newey claims the shortest has length 19.)

Let \(a(n)\) denote the length of the shortest word over the alphabet \([n]\) which contains a subsequence order-isomorphic to every permutation of length \(n\). Still to this day, the only exact values of \(a(n)\) that are known are those for \(n \leq 7\), which were computed by Newey to be 1, 3, 7, 12, 19, 28, and 39 in his 1973 Stanford technical report [35]. From Newey’s computation one can see that 
\[ a(n) = n^2 - 2n + 4 \]

for \(3 \leq n \leq 7\). Indeed, Newey gives a construction of universal words of this length for all \(n \geq 3\), and thus it is an upper bound on the answer to this universal permutation problem. While Newey remarked that it is an “obvious conjecture” that the length of the shortest universal word in this case is \(n^2 - 2n + 4\), he also suggested a competing conjecture which would imply that \(a(n)\) grows like \(n^2 - n \log_2(n)\).

Simpler constructions of universal words of length \(n^2 - 2n + 4\) were presented in a 1974 paper of Adleman [1], a 1975 paper of Koutas and Hu [31], and a 1976 paper of Galbiati and Preparata [15]. The latter two constructions were given a common generalization in the 1980 paper of Mohanty [34]. Interestingly, of these four papers, only Koutas and Hu were bold (or foolish) enough to conjecture that \(n^2 - 2n + 4\) is the true answer (it isn’t).

After this initial flurry of activity, the problem lay dormant until the surprising 2011 work of Zălinescu [42], who lowered the upper bound by 1 for \(n \geq 10\), constructing a word of length \(n^2 - 2n + 3\) which contains all permutations of length \(n\) as subsequences. However, his upper bound stood for just over one year before being improved upon by the following.

**Theorem 4.1** (Radomirović [37]). For all \(n \geq 7\), there is a word over the alphabet \([n]\) of length 
\[ n^2 - 7n/3 + 19/3 \]
containing subsequences order-isomorphic to every permutation of length \(n\).

Kleitman and Kwiatkowski remain the only researchers to establish nontrivial lower bounds, in their 1976 paper [29]. Their main result gives a lower bound of \(n^2 - c_n n^{7/4 + \epsilon}\), where the constant \(c_\epsilon\) depends on \(\epsilon\). While their bound lacks in concreteness, it does establish that the growth of the length of the shortest universal permutation is asymptotic to \(n^2\).

5. AS SUBSEQUENCES, OVER \([n + 1]\)

As in the factor case, by adding a single symbol to our alphabet, we again see a dramatic decrease in the bound on the length of the shortest universal word. While this version of the problem has not been explicitly studied in the literature before, it was studied implicitly in the 2009 work of Miller [33], where she established the following bound.

**Theorem 5.1** (Miller [33]). For all \(n\), there is a word over the alphabet \([n + 1]\) of length \((n^2 + n)/2\) containing subsequences order-isomorphic to every permutation of length \(n\).
To establish this result, define the infinite zigzag word to be the word formed by alternating ascending runs of the odd positive integers (135 ⋅⋅⋅ ) and descending runs of the even positive integers (⋅⋅⋅ 642). (While this object does not conform to most definitions of the word word in combinatorics, we hope the reader forgives us the slight expansion of the definition adopted here.) We are interested in the leftmost embeddings of words over \( \mathbb{P} \) in the infinite zigzag word.

To state the result we derive Theorem 5.1, we need two additional definitions. First, given a word \( p \in \mathbb{P}^* \), we define the word \( p^{+1} \in \mathbb{P}^* \) to be the word formed by adding 1 to each letter of \( p \), so \( p^{+1}(i) = p(i) + 1 \) for all indices \( i \) of \( p \). Next we say that the word \( p \in \mathbb{P}^* \) has an immediate repetition if there is an index \( i \) with \( p(i) = p(i + 1) \), i.e., if \( p \) contains a factor equal to \( \ell \ell \) for some letter \( \ell \in \mathbb{P} \).

**Proposition 5.2.** If the word \( p \in \mathbb{P}^n \) has no immediate repetitions then either \( p \) or \( p^{+1} \) occurs as a subsequence of the first \( n \) runs of the infinite zigzag word.

Before proving Proposition 5.2, note that permutations do not have immediate repetitions. Thus if \( \pi \) is a permutation of length \( n \), Proposition 5.2 implies that either \( \pi \) or \( \pi^{+1} \) occurs as a subsequence in the first \( n \) runs of the infinite zigzag word. Since \( \pi^{+1} \) is order-isomorphic to \( \pi \) and both \( \pi \) and \( \pi^{+1} \) are words over \([n+1]\), this implies that the restriction of the first \( n \) runs of the infinite zigzag word to the alphabet \([n+1]\) contains every permutation of length \( n \). For example, in the case of \( n = 5 \) we obtain the universal word

\[
135 642 135 642 135
\]

of length 15 over the alphabet \([6]\).

The restriction of the infinite zigzag word described above consists of \( n \) runs of average length \((n + 1)/2\): if \( n \) is odd, then all runs are of this length, while if \( n \) is even, then half are of length \( n/2 \) and half are of length \( n/2 + 1 \). Thus Proposition 5.2 implies Theorem 5.1.

**Proof of Proposition 5.2.** We define the score of the word \( p \in \mathbb{P}^* \), denoted \( s(p) \), to be the minimum number of runs in an initial segment of the zigzag containing \( p \) minus the length of \( p \). Thus our goal is to show that for every word \( p \in \mathbb{P}^* \) without immediate repetitions, either \( s(p) \leq 0 \) or \( s(p^{+1}) \leq 0 \). In fact we show that for such words we have \( s(p) + s(p^{+1}) = 1 \), which implies this.

We prove this claim by induction on the length of \( p \). For the base case, we see that words consisting of a single odd letter are contained in the first run of the infinite zigzag word (thus corresponding to scores of 0) while words consisting of a single even letter are contained in the second run (corresponding to scores of 1). Thus for every \( \ell \in \mathbb{P} \) we have \( s(\ell) + s(\ell^{+1}) = 1 \), as desired. Now suppose that the claim is true for all words \( p \in \mathbb{P}^n \) without immediate repetitions and let \( \ell \in \mathbb{P} \) denote a letter. We see that, for any \( p \in \mathbb{P}^n \),

\[
s(p\ell) - s(p) = \begin{cases} -1 & \text{if } \ell < p(n) \text{ and both entries are even or} \\
0 & \text{if } \ell \text{ and } p(n) \text{ are of different parity; or} \\
+1 & \text{if } \ell = p(n), \text{ or}
\end{cases}
\]

\[
\text{if } \ell > p(n) \text{ and both entries are odd;}
\]

Because our words do not have immediate repetitions, we can ignore the possibility that \( \ell = p(n) \). In the other cases, it can be seen by inspection that

\[
(s(p\ell) - s(p)) + (s((p\ell)^{+1}) - s(p^{+1})) = 0.
\]
Thus if \( p\ell \) has no immediate repetitions then
\[
s(p\ell) + s((p\ell)^+1) = s(p) + s(p^+1) = 1,
\]
completing the proof of the inductive claim, and thus also of the proposition.

We conclude our consideration of this case by providing a lower bound. Suppose that the word \( w \) over the alphabet \([n+1]\) contains a subsequence order-isomorphic to every permutation of length \( n \). For each letter \( \ell \in [n+1] \), let \( r_\ell \) denote the number occurrences of the letter \( \ell \) in \( w \). To create a subsequence of \( w \) which is order-isomorphic to a permutation, we must choose a letter of the alphabet \([n+1]\) to omit and then choose precisely one occurrence of each of the other letters. Thus the number of permutations that can be contained in \( w \) is at most
\[
\sum_{\ell \in [n+1]} r_1 \cdots r_{\ell-1} r_{\ell+1} \cdots r_{n+1} = r_1 \cdots r_{n+1} \sum_{\ell \in [n+1]} \frac{1}{r_\ell}.
\]
(†)

Setting \( m = |w| = \sum r_\ell \), we see that the maximum of (†) over all \((r_1, \ldots, r_{n+1}) \in \mathbb{R}^{n+1}\) occurs when each \( r_\ell \) is equal to \( m/(n+1) \), and in that case the number of permutations contained in \( w \) is at most
\[
(n+1) \left( \frac{m}{n+1} \right)^n.
\]

For the quantity above to be greater than or equal to \( n! \) (which it must be, if \( w \) is to contain all permutations of length \( n \)), an application of Stirling’s Formula shows that, asymptotically, we must have \( m \geq n^2/e \).

6. As Subsequences, Over \( \mathbb{P} \)

For the final cell of Table 1, we seek a word over the positive integers \( \mathbb{P} \) which contains all permutations of length \( n \) as subsequences. As remarked in the introduction, this is equivalent to seeking a permutation which contains all permutations of length \( n \), and such a permutation is sometimes called an \( n \)-superpattern (for example, in Bóna’s textbook *Combinatorics of Permutations* [7, Chapter 5, Exercises 19–22 and Problems Plus 9–12]). At the end of Section 5 of their 1985 paper [40], Simion and Schmidt computed the number of 3-universal permutations (in this sense of the problem) of length \( m \geq 5 \) to be
\[
m! - 6C_m + 5 \cdot 2^m + 4 \left( \frac{m}{2} \right) - 2F_m - 14m + 20,
\]
where \( C_m \) denotes the \( m \)th Catalan number and \( F_m \) denotes the \( m \)th combinatorial Fibonacci number (so \( F_0 = F_1 = 1 \) and \( F_m = F_{m-1} + F_{m-2} \) for \( m \geq 2 \)). However, the first to study this version of the universal permutation problem in its own right was Arratia [4], in 1999.

As our alphabet has only expanded from the version of the problem discussed in the previous section, the upper bound of \((n^2 + n)/2 \) established in Theorem 5.1 also holds for the version of the problem discussed in this section. It should be noted that before Miller [33] established Theorem 5.1 in 2009, Eriksson, Eriksson, Linusson, and Wästlund [14] had established an upper bound for this problem asymptotically equal to \( 2n^2/3 \).

In order to improve on the Miller’s upper bound, we further restrict the infinite zigzag word and then consider a specific order-homomorphic image of that restriction. To this end, we define the
word $z_n$ to be the restriction of the first $n$ runs of the infinite zigzag word to the alphabet $[n]$. When $n$ is even, each run of $z_n$ has length $n/2$. When $n$ is odd, $z_n$ consists of $(n + 1)/2$ ascending odd runs, each of length $(n + 1)/2$, and $(n - 1)/2$ descending even runs, each of length $(n - 1)/2$. Thus we have

$$|z_n| = \begin{cases} \frac{n^2}{2} & \text{if } n \text{ is even}, \\ \frac{n^2 + 1}{2} & \text{if } n \text{ is odd}. \end{cases}$$

Next we choose a specific permutation, $\zeta_n$, such that $z_n$ is order-homomorphic to $\zeta_n$. Recall that this means that for all indices $i$ and $j$,

$$z_n(i) > z_n(j) \implies \zeta_n(i) > \zeta_n(j).$$

In constructing $\zeta_n$, we have the freedom to break ties between equal letters of $z_n$. That is to say, if $z_n(i) = z_n(j)$ for $i \neq j$, then in constructing $\zeta_n$ we may choose whether $\zeta_n(i) < \zeta_n(j)$ or $\zeta_n(i) > \zeta_n(j)$ arbitrarily without affecting any other pair of comparisons and thus without losing any occurrences of permutations. We choose to break these ties by replacing all instances of a given letter $k \in [n]$ in $z_n$ by a decreasing subsequence in $\zeta_n$. Thus for indices $i < j$, we have

$$z_n(i) = z_n(j) \implies \zeta_n(i) > \zeta_n(j).$$

This choice uniquely determines $\zeta_n$ (up to order-isomorphism), as all comparisons between its letters are determined either in $z_n$, if the corresponding letters of $z_n$ are distinct, or by the rule above, if the corresponding letters of $z_n$ are equal. Figure 3 shows an example.

In the following sequence of results we show that $\zeta_n$ is almost universal. In fact, we show that $\zeta_n$ fails to be universal only for even $n$, and in that case, the only missing permutation is the decreasing permutation $n \cdots 21$. The first of these results, below, covers almost all permutations (in fact, Proposition 6.2 implies that this result handles all but $2^{n-1}$ permutations of length $n$).

We say that two entries $\pi(j)$ and $\pi(k)$ form a inverse-descent if $j < k$ and $\pi(j) = \pi(k) + 1$. (As the name is meant to indicate, if a pair of entries forms an inverse-descent in $\pi$, then the corresponding entries of $\pi^{-1}$ form a descent.) If $\pi(j)$ and $\pi(k)$ form an inverse-descent and they are not adjacent in $\pi$ (so $k \geq j + 2$) then we say that they form a distant inverse-descent.
Proposition 6.1. If the permutation $\pi$ of length $n$ has a distant inverse-descent, then $\zeta_n$ contains a subsequence order-isomorphic to $\pi$.

Proof. Suppose that the entries $\pi(a)$ and $\pi(b)$ form a distant inverse-descent in $\pi$, meaning that $\pi(a) = \pi(b) + 1$ and $b \geq a + 2$. We define the word $p \in [n-1]^n$ by

$$p(i) = \begin{cases} 
\pi(i) & \text{if } \pi(i) \leq \pi(b), \\
\pi(i) - 1 & \text{if } \pi(i) \geq \pi(a) = \pi(b) + 1.
\end{cases}$$

The word $p$ has two occurrences of the letter $\pi(b)$, but since $\pi(a)$ and $\pi(b)$ form a distant inverse-descent, these two occurrences of $\pi(b)$ in $p$ do not constitute an immediate repetition. Thus Proposition 5.2 shows that either $p$ or $p^+$ occurs as a subsequence in the first $n$ runs of the infinite zigzag word. As $p$ and $p^+$ are both words over $[n]$, whichever of these words occurs in the first $n$ runs of the infinite zigzag word also occurs as a subsequence of $z_n$. Suppose that this subsequence occurs in the indices $1 \leq i_1 < i_2 < \cdots < i_n \leq |z_n|$, so $z_n(i_1)z_n(i_2)\cdots z_n(i_n)$ is equal to either $p$ or $p^+$, and thus for $j, k \in [n]$ we have

$$z_n(i_j) > z_n(i_k) \iff p(j) > p(k).$$

Because $z_n$ is order-homomorphic to $\zeta_n$, this implies that for all pairs of indices $j, k \in [n]$ except the pair $\{a, b\}$, we have

$$\zeta_n(i_j) > \zeta_n(i_k) \iff p(j) > p(k) \iff \pi(j) > \pi(k).$$

Furthermore, since $p(a) = p(b)$, we have $z_n(a) = z_n(b)$, and so by our construction of $\zeta_n$ it follows that $\zeta_n(a) > \zeta_n(b)$, while we know that $\pi(a) > \pi(b)$ because those entries form an inverse-descent. This verifies that $\zeta_n(i_1)\zeta_n(i_2)\cdots \zeta_n(i_n)$ is order-isomorphic to $\pi$, completing the proof. 

To describe the permutations that Proposition 6.2 does not apply to, we need the notions of sums of permutations and layered permutations. Given permutations $\pi$ and $\sigma$ of respective lengths $m$ and $n$, their (direct) sum is the permutation $\pi \oplus \sigma$ of length $m + n$ defined by

$$(\pi \oplus \sigma)(i) = \begin{cases} 
\pi(i) & \text{if } 1 \leq i \leq m, \\
\pi(j - m) + m & \text{if } m + 1 \leq i \leq m + n.
\end{cases}$$

Pictorially, it is convenient to identify a permutation $\pi$ with its plot, i.e., the set $\{(i, \pi(i))\}$ in the plane. The plot of $\pi \oplus \sigma$ then consists of the plot of $\pi$ placed above and to the right of the plot of $\sigma$, as shown on the left of Figure 4. A permutation is said to be layered if it can be expressed as the sum of any number of decreasing permutations, and in this case, these decreasing permutations are themselves called the layers. An example of a layered permutation is shown on the right of Figure 4.

Proposition 6.2. The permutation $\pi$ is layered if and only if it does not have a distant inverse-descent.
Proof. One direction is trivial: if \( \pi \) is layered then all of its inverse-descents are between consecutive entries, so it does not have a distant inverse-descent. The other direction follows by induction. Let \( n \) denote the length of \( \pi \). If \( n \leq 2 \) then \( \pi \) is necessarily layered, so there is nothing to prove. Suppose now that \( n \geq 3 \), that \( \pi \) is a permutation of length \( n \) without distant inverse-descents, and that all shorter permutations without inverse-descents are layered.

We claim that \( \pi = \delta \oplus \sigma \) where \( \delta \) is a nonempty decreasing permutation. To see this, consider \( \pi(1) \). If \( \pi(1) = 1 \), then \( \pi = 1 \oplus \sigma \) for some \( \sigma \). Otherwise \( \pi(1) > 1 \), and because \( \pi \) does not have a distant inverse-descent, the entry \( \pi(1) - 1 \) must occur immediately to the right of \( \pi(1) \). Continuing in this fashion, the claim is established. As \( \sigma \) does not have any distant inverse-descents, it is layered by induction, completing the proof.

Having characterized the permutations to which Proposition 6.1 does not apply, we now show that almost all of them are nonetheless contained in \( \zeta_n \).

**Proposition 6.3.** If the permutation \( \pi \) of length \( n \) is layered and not a decreasing permutation of even length, then \( \zeta_n \) contains a subsequence order-isomorphic to \( \pi \).

**Proof.** Let \( \pi \) denote an arbitrary layered permutation of length \( n \). To prove the result, we compute the score of \( \pi \) as in the proof of Proposition 5.2, show that this score can only take on the values 0 or \( \pm 1 \), and then describe an alternative embedding of \( \pi \) in \( \zeta_n \) in the case where the score of \( \pi \) is 1, except when \( \pi \) is a decreasing permutation of even length.

Recall that the score of any word \( \pi \), \( s(\pi) \), is defined as the number of initial runs of the infinite zigzag word necessary to contain \( \pi \) minus the length of \( \pi \). As observed in the proof of Proposition 5.2, the score of a word does not change upon reading a letter of opposite parity. This implies that, while reading a layered permutation, the score changes only when transitioning from one layer to the next and thus we compute the score of \( \pi \) layer-by-layer.

The change in score when moving from one layer of \( \pi \) to the next is determined by the parity of the last entry of the layer we are leaving and the first entry of the layer we are entering. Specifically, the score changes by \(-1\) if both of these entries are odd and \(+1\) if both are even. This shows that in order to compute the score of the layered permutation \( \pi \), we simply need to know the parities of the first and last entries of each of its layers. This information is represented by the labels of the nodes of the directed graph shown in Figure 5.

Moreover, not all transitions between these nodes are possible, because the last entry of a layer is precisely 1 greater than the first entry of the preceding layer. This is why there are only eight edges shown in Figure 5. In this figure, each of those edges is labeled by the change in the score function. Note that the first layer must end with 1 (an odd entry), and its first entry must be either odd (for a
score of 0) or even (for a score of 1); this is equivalent to starting our walk on the graph in Figure 5 at the node labeled even \cdots even before any layers are read.

From this graphical interpretation of the scoring process, it is apparent that the score of a layered permutation can take on only three values: -1 if it ends at the node odd \cdots even; 0 if it ends at either node even \cdots even or odd \cdots odd; or 1 if it ends at the node even \cdots odd. Except in this final case, we are done.

Now suppose that we are in the final case, so the ultimate layer of $\pi$ is of even \cdots odd type. The first entry of this layer is the greatest entry of $\pi$, so we know that $\pi$ has even length. If $\pi$ were a decreasing permutation then there would be nothing to prove (as we have not claimed anything in this case), so let us further suppose that $\pi$ is not a decreasing permutation, and thus that $\pi$ has at least two layers. We further divide this case into two cases. In both cases, as in the proof of Proposition 6.1, we construct a word $p \in [n-1]^n$ such that if $z_n$ contains $p$ then $\zeta_n$ contains $\pi$.

First, suppose that the penultimate layer of $\pi$ is of even \cdots odd type and that this layer begins with the entry $\pi(b)$. This implies that the penultimate layer of $\pi$ has at least two entries (because its first and last entries have different parities). In this case, we define $p$ by

$$p(i) = \begin{cases} 
\pi(i) & \text{if } \pi(i) < \pi(b), \\
\pi(i) - 1 & \text{if } \pi(i) \geq \pi(b).
\end{cases}$$

In other words, to form $p$ from $\pi$ we decrement the first entry of the penultimate layer and all entries of the ultimate layer. Because the penultimate layer of $\pi$ has at least two entries, performing this operation creates an immediate repetition (of the entry $\pi(b) - 1$) at the beginning of this layer. For example, if $\pi = 21\,6543\,87$ then $\pi(b) = 6$ and we decrement the 6, 8, and 7 to obtain the word $p = 21\,5543\,76$.

As with our previous constructions, note that if $z_n$ contains an occurrence of $p$ then $\zeta_n$ will contain a copy of $\pi$. We now establish that $z_n$ contains $p$ by showing that $s(p) = 0$, which requires a further bifurcation into subcases. In both subcases, the scoring of $p$ is computed by considering its score in the antepenultimate (second-from-last) layer, the score change when reading the (newly lowered) first entry of the penultimate layer, the score penalty of +1 because $p$ contains an immediate repetition (namely, $\pi(b) - 1$ occurs twice in a row), and finally the score change between the penultimate and ultimate layers. We label these cases by the final three nodes of the directed graph from Figure 5 visited while computing the score of $\pi$.

- The final three layers are of type (even \cdots even)(even \cdots odd)(even \cdots odd). Note that this case includes the possibility that $\pi$ has only two layers. If it exists, the score while reading the antepenultimate layer is 0 and the ascent between it and the penultimate layer is of different parity (even to odd), contributing 0 to the score. In either case, the score of $p$ is 0 upon reading the first entry of the penultimate layer. The immediate repetition in the penultimate layer contributes +1 to the score, while the ascent between the penultimate and ultimate layers is odd and thus contributes -1. Thus we have $s(p) = 0$.

- The final three layers are of type (even \cdots odd)(even \cdots odd)(even \cdots odd). The score while reading the antepenultimate layer is +1. Here, the ascent between the antepenultimate and penultimate layers is odd, so it contributes -1 to the score, the immediate repetition contributes +1, and the ascent between the penultimate and ultimate layers is odd and thus contributes -1, giving $s(p) = 0$. 
It remains to treat the case where the penultimate layer of $\pi$ is of even $\cdots$ even type. Note that this case includes the possibility that the penultimate layer consists of a single entry. Suppose that the penultimate layer ends with the entry $\pi(p)$. We define $p$ by

$$p(i) = \begin{cases} 
\pi(i) & \text{if } \pi(i) < \pi(a) \text{ or } \pi(i) = n, \\
\pi(i) + 1 & \text{if } \pi(i) \geq \pi(a) \text{ and } \pi(i) \neq n.
\end{cases}$$

Thus in forming $p$ from $\pi$ we increment all entries of the penultimate layer and all but the first entry of the ultimate layer. For example, if $\pi = 21 3 6 5 4 8 7$ then we increment the 6, 5, 4, and 7 to obtain the word $p = 21 3 7 6 5 8 8$.

As before, if $z_n$ contains an occurrence of $p$ then $\zeta_n$ will contain a copy of $\pi$. Thus we need only show that $s(p) = 0$, which we do, as in the previous case, by considering the scoring of the final three layers. As in that case, we identify two subcases.

- The final three layers are of type $(\text{odd} \cdots \text{even})(\text{even} \cdots \text{even})(\text{even} \cdots \text{odd})$. The score while reading the antepenultimate layer is $-1$. The ascent between the antepenultimate and penultimate layers is of different parity (even to odd) and thus contributes 0 to the score, the ascent between the penultimate and ultimate layers is of different parity (odd to even) and thus contributes 0 to the score, and finally, the immediate repetition at the beginning of the ultimate layer contributes $+1$ to the score, so $s(p) = 0$.

- The final three layers are of type $(\text{odd} \cdots \text{odd})(\text{even} \cdots \text{even})(\text{even} \cdots \text{odd})$. The score while reading the antepenultimate layer is 0. The ascent between the antepenultimate and penultimate layers contributes $-1$ to the score (as both entries are now odd), the ascent between the penultimate and ultimate layers is of different parity (odd to even) and thus contributes 0 to the score, and finally, the immediate repetition at the beginning of the ultimate layer contributes $+1$ to the score, so $s(p) = 0$.

Now that we have handled all of the cases, the proof is complete.

It remains only to conclude. The length of $\zeta_n$ is $(n^2 + 1)/2$ when $n$ is odd and $n^2/2$ when $n$ is even. When $n$ is odd, we have established that $\zeta_n$ is universal. However, Proposition 6.3 shows that $\zeta_n$ need not be universal when $n$ is even (indeed, it can be checked that $\zeta_n$ is not universal when $n$ is even). However, in this case we know that $\zeta_n$ contains the decreasing permutation $(n - 1) \cdots 21$ (for instance because it contains the permutation $(n - 1) \cdots 21 \oplus 1$). Thus we obtain a universal permutation by prepending a new maximum entry to $\zeta_n$, giving us the following bound.

**Theorem 6.4.** There is a word over $\mathbb{P}$ of length $\left\lceil (n^2 + 1)/2 \right\rceil$ containing subsequences order-isomorphic to every permutation of length $n$.

A computer search reveals that the bound in Theorem 6.4 is best possible for $1 \leq n \leq 5$. However, for $n = 6$ the bound in the Theorem 6.4 is 19, but Arnar Arnarson [private communication] has found that the permutation

$$6 \ 14 \ 10 \ 2 \ 13 \ 17 \ 5 \ 8 \ 3 \ 12 \ 9 \ 16 \ 1 \ 7 \ 11 \ 4 \ 15$$

of length 17 is universal for the permutations of length 6. Computations have shown that no shorter permutation is universal for the permutations of length 6.
The best lower bound in this case is still the one given by Arratia [4] in his initial work on the problem. Note that if the word $w$ of length $m$ over the alphabet $P$ is to contain a subsequence order-isomorphic to each permutation of length $n$ then we must have

$$\binom{m}{n} \geq n!.$$  

As in the lower bound of the previous section, an application of Stirling’s Formula shows that, asymptotically, we must have $m \geq n^2/e^2$. In fact, Arratia [4, Conjecture 2] conjectured that the length of the shortest universal permutation in this case is asymptotic to $n^2/e^2$.

7. Further Variations

In case the infinitely many problems discussed up to now are not enough, we conclude by briefly describing further variants that have been studied by other researchers.

1. As observed in Section 2, there is no universal cycle over the alphabet $[n]$ for the permutations of length $n$. However, Jackson [23] proved that there is a universal cycle over the alphabet $[n]$ for all shorthand encodings of permutations of length $n$, where the shorthand encoding of the permutation $\pi$ of length $n$ is the word $\pi(1) \cdots \pi(n-1)$. This result and some extensions are discussed in Volume 4A of The Art of Computer Programming [30, Section 7.2.1.2, Exercises 111–113], where Knuth asked for an explicit construction of such a universal cycle (Jackson’s proof was non-constructive). Knuth’s request was answered by Ruskey and Williams [38]. Further constructions have been given by Holroyd, Ruskey, and Williams [19, 20].

2. Gupta [17] considered a subsequence version of a universal cycle for permutations. A rosary is a word $w$ over the alphabet $[n]$ such that every permutation of length $n$ is contained as a subsequence of the word $w(k)w(k+1) \cdots w(|w|)w(1)w(2) \cdots w(k-1)$ for some value of $k$ (if we think of the letters as being arranged in a circle as on the left of Figure 6, we can start anywhere we like, but must traverse the rosary clockwise, and cannot return to where we started). Gupta conjectured that one could always construct a rosary of length at most $n^2/2$. This conjecture was included in Guy’s book Unsolved Problems in Number Theory [18, Problem E22] and proved in the case where $n$ is even by Lecouturier and Zmitkou [32]. Gupta also considered
the variant where one is allowed to traverse the rosary both clockwise and counterclockwise (see the right of Figure 6); he conjectured that one can always construct a rosary of length at most $3n^2/8 + 1/2$ in this version of the problem.

3. Albert and West [3] studied the existence of universal cycles in the sense of Section 3 for permutation classes, making no restrictions on the size of the alphabet. To describe their results, we define a partial order on the set of all finite permutations where $\sigma \leq \pi$ if $\pi$ contains a subsequence which is order-isomorphic to $\sigma$. If $\sigma \nleq \pi$ then we say that $\pi$ avoids $\sigma$. A permutation class is a downset in this order. Every permutation class can be specified by giving the set of minimal elements not in the class (this set is called the basis of the class), and when presented in this form, we use the notation

$$\text{Av}(B) = \{\pi : \pi \text{ avoids all } \beta \in B\}.$$ 

Most of Albert and West’s results are negative in nature, but some classes they consider, such as $\text{Av}(132, 312)$, do have universal cycles (over the alphabet $\mathbb{P}$). They say that a permutation class with such a universal cycle is value cyclic.

4. At the end of his paper, Arratia [4] defines $t(n)$ to be the least integer $m$ such that at least half of all permutations of length $m$ are $n$-universal, and he states that Noga Alon has conjectured that $t(n)$ is asymptotic to $n^2/4$. Figure 7 shows the plots of the proportion of $n$-universal permutations of lengths $0 \leq m \leq 40$ for $n = 3, 4, 5,$ and $6$. For $n = 3$, we compute these proportions exactly using the formula of Simion and Schmidt [40] mentioned at the beginning of Section 6, while for $n \geq 4$, these plots are obtained by random sampling to a high level of confidence. This data and further computations suggest the following values for $t(n)$ for $1 \leq n \leq 8$:

| $n$ | $t(n)$ |
|-----|--------|
| 1   | 1      |
| 2   | 3      |
| 3   | 7      |
| 4   | 13     |
| 5   | 20     |
| 6   | 28     |
| 7   | 36     |
| 8   | 48     |

While the first six values of $t(n)$ above might lead the reader to suspect that $t(n)$ is the nearest integer to $\pi n^2/4$, this seems not to hold for $n = 7, 8$. We leave it to the reader to decide
whether these values support or undermine Alon’s conjecture that \( t(n) \sim n^2/4 \).

5. Universal words over \( P \) containing, as subsequences, all permutations of length \( n \) from a proper permutation class have also been studied. Bannister, Cheng, Devanny, and Eppstein [6] construct a universal word of length \( n^2/4 + \Theta(n) \) for the permutations of length \( n \) in the class \( \text{Av}(132) \), and they show that every proper subclass \( C \subseteq \text{Av}(132) \) has a universal word of length at most \( O(n \log^2 n) \). In [6], among other results, Bannister, Devanny, and Eppstein find a universal word of length at most \( 22n^{3.2} + \Theta(n) \) for the class \( \text{Av}(321) \). Finally, Albert, Engen, Pantone, and Vatter [2] considered the class of layered permutations, \( \text{Av}(231, 312) \) in this notation. In addition to verifying a conjecture of Gray [16], they showed that the length of the shortest universal word over \( P \) containing all layered permutations of length \( n \) as subsequences is given precisely by

\[
a(n) = (n + 1)[\log_2(n + 1)] + 1
\]

for \( n \geq 1 \).

Acknowledgements. We are grateful to Jay Pantone for his assistance in verifying that no permutation of length 16 or less is universal for the permutations of length 6.

References

[1] Adleman, L. Short permutation strings. *Discrete Math.* 10, 2 (1974), 197–200.

[2] Albert, M. H., Engen, M. T., Pantone, J. T., and Vatter, V. R. Universal layered permutations. *Electron. J. Combin.* 25, 3 (2018), Paper #P3.23, 5 pp.

[3] Albert, M. H., and West, J. Universal cycles for permutation classes. *Discrete Math.* Theor. Comput. Sci. Proc. AK (2009), 39–50.

[4] Arratia, R. A. On the Stanley–Wilf conjecture for the number of permutations avoiding a given pattern. *Electron. J. Combin.* 6 (1999), Note 1, 4 pp.

[5] Ashlock, D. A., and Tillotson, J. Construction of small superpermutations and minimal injective superstrings. *Congr. Numer.* 93 (1993), 91–98.

[6] Bannister, M. J., Cheng, Z., Devanny, W. E., and Eppstein, D. A. Superpatterns and universal point sets. *J. Graph Algorithms Appl.* 18, 2 (2014), 177–209.

[7] Bóna, M. *Combinatorics of Permutations*, second ed. Discrete Mathematics and its Applications. CRC Press, Boca Raton, Florida, 2012.

[8] Chung, F.-R. K., Diaconis, P. W., and Graham, R. L. Universal cycles for combinatorial structures. *Discrete Math.* 110, 1-3 (1992), 43–59.

[9] Chvátal, V., Klarner, D. A., and Knuth, D. E. Selected combinatorial research problems. Tech. Rep. STAN-CS-72-292, Stanford University, 1972.
[10] de Bruijn, N. G. Acknowledgement of priority to C. Flye Sainte-Marie on the counting of circular arrangements of \(2^n\) zeros and ones that show each \(n\)-letter word exactly once. Tech. Rep. T.H.-Report 75-WSK-06, Technische Hogeschool Eindhoven Nederland, 1975.

[11] Diaconis, P. W., and Graham, R. L. *Magical Mathematics: The Mathematical Ideas That Animate Great Magic Tricks*. Princeton University Press, Princeton, New Jersey, 2012.

[12] Egan, G. *Permutation City*. Orion Books Ltd., London, England, 1994.

[13] Egan, G. Superpermutations, Oct. 2018. http://www.gregegan.net/SCIENCE/Superpermutations/Superpermutations.html.

[14] Eriksson, H., Eriksson, K., Linusson, S., and Wästlund, J. Dense packing of patterns in a permutation. *Ann. Comb.* 11, 3-4 (2007), 459–470.

[15] Galbiati, G., and Preparata, F. P. On permutation-embedding sequences. *SIAM J. Appl. Math.* 30, 3 (1976), 421–423.

[16] Gray, D. A. Bounds on superpatterns containing all layered permutations. *Graphs Combin.* 31, 4 (2015), 941–952.

[17] Gupta, H. On permutation-generating strings and rosaries. In *Combinatorics and Graph Theory*, S. B. Rao, Ed., vol. 885 of *Lecture Notes in Math.* Springer-Verlag, Berlin, West Germany, 1981, pp. 272–275.

[18] Guy, R. K. *Unsolved Problems in Number Theory*, third ed. Problem Books in Math. Springer-Verlag, New York, New York, 2004.

[19] Holroyd, A., Ruskey, F., and Williams, A. Faster generation of shorthand universal cycles for permutations. In *Computing and Combinatorics*, M. T. Thai and S. Sahni, Eds., vol. 6196 of *Lecture Notes in Comput. Sci.* Springer-Verlag, Berlin, Germany, 2010, pp. 298–307.

[20] Holroyd, A. E., Ruskey, F., and Williams, A. Shorthand universal cycles for permutations. *Algorithmica* 64, 2 (2012), 215–245.

[21] Honner, P. Unscrambling the hidden secrets of superpermutations. *Quanta Mag.* (Jan. 16, 2019). https://www.quantamagazine.org/unscrambling-the-hidden-secrets-of-superpermutations-20190116/.

[22] Houston, R. Tackling the minimal superpermutation problem. arXiv:1408.5108 [math.CO].

[23] Jackson, B. W. Universal cycles of \(k\)-subsets and \(k\)-permutations. *Discrete Math.* 117, 1–3 (1993), 141–150.

[24] Johnson, J. R. Universal cycles for permutations. *Discrete Math.* 309, 17 (2009), 5264–5270.

[25] Johnston, N. The minimal superpermutation problem, Apr. 2013. http://www.njohnston.ca/2013/04/the-minimal-superpermutation-problem/.

[26] Johnston, N. Non-uniqueness of minimal superpermutations. *Discrete Math.* 313, 14 (2013), 1553–1557.
[27] Johnston, N. All minimal superpermutations on five symbols have been found, Aug. 2014. http://www.njohnston.ca/2014/08/all-minimal-superpermutations-on-five-symbols-have-been-found/.

[28] Klarreich, E. Mystery math whiz and novelist advance permutation problem. Quanta Mag. (Nov. 5, 2018). https://www.quantamagazine.org/sci-fi-writer-greg-egan-and-anonymous-math-whiz-advance-permutation-problem-20181105/.

[29] Kleitman, D. J., and Kwiatkowski, D. J. A lower bound on the length of a sequence containing all permutations as subsequences. J. Combin. Theory Ser. A 21, 2 (1976), 129–136.

[30] Knuth, D. E. The Art of Computer Programming, vol. 4A. Addison-Wesley, Upper Saddle River, New Jersey, 2011.

[31] Koutas, P. J., and Hu, T. C. Shortest string containing all permutations. Discrete Math. 11, 2 (1975), 125–132.

[32] Lecouturier, E., and Zmiaikou, D. On a conjecture of H. Gupta. Discrete Math. 312, 8 (2012), 1444–1452.

[33] Miller, A. B. Asymptotic bounds for permutations containing many different patterns. J. Combin. Theory Ser. A 116, 1 (2009), 92–108.

[34] Mohanty, S. P. Shortest string containing all permutations. Discrete Math. 31, 1 (1980), 91–95.

[35] Newey, M. Notes on a problem involving permutations as subsequences. Tech. Rep. STAN-CS-73-340, Stanford University, 1973.

[36] Nijenhuis, A., and Wilf, H. S. Combinatorial Algorithms. Academic Press, New York, New York, 1975.

[37] Radomirović, S. A construction of short sequences containing all permutations of a set as subsequences. Electron. J. Combin. 19, 4 (2012), Paper 31, 11 pp.

[38] Ruskey, F., and Williams, A. An explicit universal cycle for the \((n−1)\)-permutations of an \(n\)-set. ACM Trans. Algorithms 6, 3 (2010), Art. 45, 12 pp.

[39] Sawada, J., and Williams, A. A Hamilton path for the sigma-tau problem. In Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA). SIAM, Philadelphia, Pennsylvania, 2018, pp. 568–575.

[40] Simion, R. E., and Schmidt, F. W. Restricted permutations. European J. Combin. 6, 4 (1985), 383–406.

[41] Williams, A. Hamiltonicity of the Cayley digraph on the symmetric group generated by \(σ = (12\cdots n)\) and \(τ = (12)\). arXiv:1307.2549 [math.CO].

[42] Zălinescu, E. Shorter strings containing all \(k\)-element permutations. Inform. Process. Lett. 111, 12 (2011), 605–608.