Boundary stress tensor and counterterms for weakened AdS$_3$ asymptotic in New Massive Gravity

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Resorting to the notion of a stress-tensor induced on the boundary of a spacetime, we compute the conserved charges associated to exact solutions of New Massive Gravity that obey weakened versions of AdS$_3$ asymptotic boundary conditions. The computation requires the introduction of additional counterterms, which play the rôle of regularizing the semiclassical stress-tensor in the boundary theory. We show that, if treated appropriately, different ways of prescribing asymptotically AdS$_3$ boundary conditions yield finite conserved charges for the solutions. The consistency of the construction manifests itself in that the charges of hairy asymptotically AdS$_3$ black holes computed by this holography-inspired method exactly match the values required for the Cardy formula to reproduce the black hole entropy. We also consider new solutions to the equations of motion of New Massive Gravity, which happen to fulfill Brown-Henneaux boundary conditions despite not being Einstein manifolds. These solutions are shown to yield vanishing boundary stress-tensor. The results obtained in this paper can be regarded as consistency checks for the prescription proposed in [1].

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I. INTRODUCTION

Twenty five years have passed since Brown and Henneaux discovered that the asymptotic dynamics of Einstein gravity in three-dimensional Anti-de Sitter space (AdS$_3$) is generated by the two-dimensional conformal algebra $\mathfrak{c}_2$, that is, by two copies of the Virasoro algebra with non-vanishing central extension. By the middle of 90’s, it became clear that this observation meant much more than an intriguing matching of symmetries. In 1995, Coussaert, Henneaux, and Van Driel proved that the asymptotic dynamics of Einstein gravity in AdS$_3$ is actually governed by a two-dimensional conformal field theory (CFT$_2$), at that time identified as the Liouville field theory [3]. Later, with the advent of AdS/CFT correspondence, in 1997, all these observations acquired a natural framework and were understood from a more general perspective [4].

Over time we learned that the holographic description of the asymptotic AdS$_3$ dynamics of Einstein gravity in terms of Liouville theory suffers from some flaws and is not fully satisfactory [5]; in particular, in what regards to the statistical description of the thermodynamical properties of Bañados-Teitelboim-Zanelli black holes [6]. However, this did not prevent the experts from addressing the problem of black hole thermodynamics [7] and other fundamental problems of quantum gravity [8]. In fact, since the beginning, three-dimensional gravity has proven to be a fruitful testing ground for AdS/CFT correspondence.

More recently, in the last three years, the interest on three-dimensional gravity in AdS space has been renewed, mainly due to the work of Witten [9] in which a candidate to be the CFT dual of three-dimensional Einstein’s general relativity was presented. The proposal in [9] was that Einstein gravity with negative cosmological constant about AdS$_3$ space could be dual to an extremal self-dual two-dimensional conformal field theory, which exhibits the property of being holomorphically factorizable. This proposal, in its original form, was subsequently criticized in different works [10–12], in particular in what concerns to the validity of the construction for large values of the central charge. Then, it became immediately clear that certain questions on three-dimensional gravity are far from being clear. One such question is, for instance, the question about the non-perturbative configurations of Einstein’s gravity, or the question on whether the degenerated and complex saddle points play any important rôle at quantum level.

After Witten’s paper in 2007, a new proposal for a consistent theory of quantum gravity in three-dimensional appeared. In [13], Li, Song and Strominger pointed out that, at a very special point of the space of parameters, the
Topologically Massive Gravity [14] with negative cosmological constant seems to lose its local degree of freedom and, at the same time, the central charge of the left-moving sector of the asymptotic symmetry algebra vanishes. This observation led the authors of [15] to conjecture that, for a specific choice of the coupling constant, Topologically Massive Gravity is dual to a holomorphic (chiral) conformal field theory. This proposal is usually referred to as the "Chiral Gravity Conjecture", and it was extensively discussed in the recent literature [15]. Subsequently we learned that the realization of the ideas of [15] sensibly depends on the way the asymptotic boundary conditions are prescribed. Actually, this is not surprising: after all, it is well established that the asymptotic AdS boundary conditions may differ from one theory to another [16], and, besides, a given theory may admit more than one set of consistent boundary conditions. Therefore, the discussion on the validity of the proposal in [15] resulted in a discussion on how to define what "asymptotically AdS₃ space" actually means in this context. This issue was eventually clarified in [17], where it was pointed out that two different theories, both defined by the same Lagrangian but imposing two alternative sets of boundary conditions for each one, seem to exist. While one of these theories turns out to be dual to a chiral CFT, the other one, defined by imposing weakened boundary conditions, is believed to be dual to a Logarithmic CFT [17, 19]. This is, indeed, far from being a minor difference, as a Logarithmic CFT is necessarily non-unitary, giving raise to the question on whether the dual CFT picture really makes sense if weakened asymptotic conditions are imposed. Therefore, the lesson we learn from the recent discussions on three-dimensional chiral gravity was actually instructive: It provides us with a concise example that shows how dependent on the precise choice of asymptotic boundary conditions the details of AdS/CFT correspondence can be.

More recently, a new theory of massive gravity in three dimensions was proposed by Bergshoeff, Hohm, and Townsend [20]. This theory, usually referred to as "New Massive Gravity", also seems to offer a possibility for formulating a consistent theory of quantum gravity in three dimensions. At the linearized level, the new theory coincides, after field redefinition, with the Fierz-Pauli massive model, which turns out to be unitary. Besides, its action presents other interesting properties [21-23] and exhibits a rich and interesting catalog of solutions [24-28]. As in the case of Topologically Massive Gravity (TMG), New Massive Gravity (NMG) also has a point in the space of parameters at which the central charge of the dual theory vanishes. And the properties of the dual CFT also seem to depend on the precise prescription of AdS₃ boundary conditions. Actually, in NMG there exist more than one way of relaxing Brown-Henneaux boundary conditions (BH) of three-dimensional Einstein theory. This is associated to the existence of a massive parity-invariant graviton modes in the theory. In this paper, we are interested precisely in studying how relaxing BH asymptotic conditions in different ways may affect the definition of a regularized stress tensor in the boundary CFT that would be dual to NMG in AdS₃ space. More precisely, we aim to go a step further in the discussion on whether relaxing BH boundary conditions leads to a well defined CFT in the boundary or not. Our contribution is to show that, indeed, all the ways of deforming BH boundary conditions known so far lead to regularizable stress tensor that can be used to compute conserved charges of asymptotically AdS exact solutions. Holographic renormalization in NMG was recently studied in [29].

The paper is organized as follows: In Section II, after reviewing the New theory of Massive Gravity of [20], we discuss the definition of the boundary stress tensor associated to asymptotically AdS solutions in NMG recently given in [1]. In Section III, we undertake the computation of conserved charges of solutions with weakened falling-off in AdS₃. In the first subsection of Section III, as a warming up, we discuss the simplest example of weakened asymptotically AdS boundary conditions: We consider the logarithmic deformation of the extremal BTZ solution found in [26, 27]. This is a solution that emerges at the so-called chiral point of theory of massive gravity and, despite of that, carries non-vanishing conserved charges. The mass and angular momentum obtained by using the boundary stress tensor agree with the results obtained with other methods in the literature. Then, in the second subsection of Section III, we consider a different set of relaxed boundary conditions: We analyze the rotating hairy black hole geometry found in [28]. For this geometry, we show that it is actually possible to regularize the boundary stress tensor by adding local counterterms in the boundary. The need of new counterterms is due to the soften falling-off that the gravitational field exhibits, which is the imprint of the gravitational hair. The calculation we perform yields finite results for both the mass and the angular momentum of the black hole solution. We explicitly show that the conserved charges obtained in this way are precisely the ones required for the computation of the hairy black hole entropy. This reinforces the definition of conserved charges given in [30]. In the third subsection of Section III, we consider a new solution of NMG that happens to fulfill Brown-Henneaux boundary conditions but not being an Einstein manifold. This solution is shown to have vanishing conserved charges when computing with the same method. Section IV contains our conclusions, which can be summarized as follows: The boundary CFT dual description of NMG in asymptotically AdS₃ seems to make sense even when relaxed asymptotic boundary conditions are considered. Alternatively, the consistency of the results obtained in this paper can be regarded as a non-trivial consistency check of the holographic prescription proposed in [1].
II. REVIEW OF NEW MASSIVE GRAVITY

A. Bulk action and equations of motion

The action of three-dimensional massive gravity is \[ S = \frac{1}{16\pi G} \int \Sigma d^3x \sqrt{-g} (R - 2\Lambda) + \frac{1}{32\pi G \mu} \int \Sigma d^3x \varepsilon^\alpha\beta\gamma \Gamma_{\alpha\beta\gamma} (\partial_\alpha R^\gamma_{\beta\gamma} + \frac{2}{3} \varepsilon^\beta_\gamma \Gamma^\gamma_{\beta\rho} + \frac{1}{16\pi G m^2} \int \Sigma d^3x \sqrt{-g} (R_{\mu\nu} R_{\mu\nu} - \frac{3}{8} R^2). \] (1)

Here we are omitting boundary terms; see (15) below.

The field equations coming from varying (1) with respect to the metric are

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{1}{2m^2} K_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = 0, \] (2)

where the Cotton tensor \( C_{\mu\nu} \) is given by

\[ C_{\mu\nu} = \frac{1}{2} \varepsilon_\rho^\alpha \nabla_\alpha R_{\rho\nu} + \frac{1}{2} \varepsilon_\rho^\alpha \nabla_\alpha R_{\mu\rho}. \] (3)

while the tensor \( K_{\mu\nu} \) is given by

\[ K_{\mu\nu} = 2\Box R_{\mu\nu} - \frac{1}{2} \nabla_\mu \nabla_\nu R - \frac{1}{2} \Box g_{\mu\nu} + 4 R_{\mu\nu\rho\beta} R^{\rho\beta} - \frac{3}{2} R R_{\mu\nu} - R_{\alpha\beta} R^{\alpha\beta} g_{\mu\nu} + \frac{3}{8} R^2 g_{\mu\nu}. \] (4)

The tensor \( K_{\mu\nu} \) obeys the remarkable property \( g^{\mu\nu} K_{\mu\nu} = R_{\mu\nu} R^{\mu\nu} - \frac{3}{4} R^2 \), i.e. its trace equals the Lagrangian from which it comes. Cotton tensor, on the other hand, is traceless, and, in some sense, it can be thought of as the three-dimensional analogue of the Weyl tensor.

In this paper we are mainly interested in pure New Massive Gravity, namely the theory defined by setting \( \mu = \infty \) in (1). Nevertheless, some of the remarks hold for the general theory. In the case \( \mu = \infty \), an alternative way of writing action (1) exists [1]. This amounts to introduce an second-rank auxiliary field \( f_{\mu\nu} \) and consider the alternative action

\[ S_A = \frac{1}{16\pi G} \int \Sigma d^3x \sqrt{-g} (R - 2\Lambda + f^{\mu\nu} (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) - \frac{1}{4} m^2 (f_{\mu\nu} f^{\mu\nu} - f^2)), \] (5)

where \( f = g^{\mu\nu} f_{\mu\nu} \). This permits to turn the problem into one of second order. It is easy to verify that the equations of motion derived from (5) coincides with those coming from (1) with \( \mu = \infty \). In fact, varying (5) with respect to the non-dynamical field \( f_{\mu\nu} \) one finds that the auxiliary field on-shell is proportional to the Schouten tensor; namely

\[ f_{\mu\nu} = \frac{2}{m^2} (R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R), \] (6)

and, then, plugging this into the field equations that come from varying (5) with respect to \( g_{\mu\nu} \), one recovers equations (2)-(4).

The equations of motion (2)-(4) admit three-dimensional Anti-de Sitter space-time (AdS\(_3\)) as exact solutions. Written in Poincaré coordinates, the metric of AdS\(_3\) reads

\[ ds^2 = - \left( \frac{r^2}{l^2} + 1 \right) dt^2 + \left( \frac{r^2}{l^2} + 1 \right)^{-1} dr^2 + r^2 d\phi^2, \] (7)

where \( l \) is the "radius" of the space, which is given in terms of \( \Lambda, \mu, \) and \( m \). Throughout this paper we will mainly use Poincaré coordinates [17], which only cover one patch of AdS\(_3\) space (this global obstruction is not important for our purpose). In this system of coordinates, the boundary of the space is located at \( r = \infty \) (plus a point at \( r = 0 \)). We will consider the convention such that \( x^0 = t, x^1 = \phi \) and \( x^2 = r \), with the Latin indices labeling coordinates \( i, j = 0, 1 \), and the Greek indices labeling all the coordinates \( \mu, \nu = 0, 1, 2 \).

In the case of NMG (i.e. \( m \neq \infty \) and \( \mu = \infty \)), we have that the radius of AdS\(_3\) may take the values

\[ l^2_\pm = -\frac{1}{2\Lambda} \left( 1 \pm \sqrt{1 + \frac{\Lambda}{m^2}} \right). \] (8)
This shows that NMG has two different maximally symmetric “vacua”, each of them having a different effective cosmological constant $\frac{\Lambda}{l^2}$.

As it happens in the case of Einstein gravity in $\text{AdS}_3$, the charges associated to the asymptotic isometries for NMG in $\text{AdS}_3$ expand two copies of the Virasoro algebra. The central charge in this case is given by \[ c = \frac{3l}{2G} \left( 1 + \frac{1}{2m^2l^2} \right), \] which reduces to Brown-Henneaux central charge of general relativity \[ c \] in the limit $m \to \infty$. If $\mu < \infty$, the theory defined by \[ \mu \] also admits $\text{AdS}_3$ as a solution. This is evident once one knows that the Cotton tensor vanishes if and only if the metric is conformally flat, as $\text{AdS}_3$ is. Since parity gets broken when $\mu < \infty$, it happens that the two copies of Virasoro algebra that generate the asymptotic diffeomorphisms acquire different central charges. These are \[ c_L = \frac{3l}{2G} \left( 1 - \frac{1}{\mu l} + \frac{1}{2m^2l^2} \right) \quad \text{and} \quad c_R = \frac{3l}{2G} \left( 1 + \frac{1}{\mu l} + \frac{1}{2m^2l^2} \right), \] which agree with \[ \mu = \infty \]. Returning to the case with $\mu = \infty$, it is worth mentioning that there exist two special points in the space of parameters at which the theory exhibits special properties. One such point is \[ m^2l^2 = \frac{1}{2}. \] At this point, we have $l_\mu^2 = l_\gamma^2$, implying that $\Lambda = -1/(2l^2)$. This is the point of the moduli space where hairy BTZ-like black hole solutions are admitted as exact solutions \[ 28 \]. Due to the presence of the gravitational hair, the falling-off of the gravitational field is weaker than the BH asymptotic exhibited by the BTZ solution. In turn, the question arises as to whether this weakened asymptotic behavior is also consistent with the existence of a dual CFT description in the boundary. This is precisely the question we want to address in this paper. We will analyze this in the following section.

The other point of the space of parameters at which something special happens is \[ m^2l^2 = -\frac{1}{2}. \] At this point, which corresponds to the value $\Lambda = -3/(2l^2)$, the central charge \[ \mu \] vanishes, resembling what happens in TMG at the chiral point (i.e. $m = \infty$ with $\mu = 1/l$). In fact, one can talk about a “generalized chiral point” for the theory \[ 10 \], which happens at \[ \frac{1}{\mu l} - 1 = \frac{1}{2m^2l^2}, \] i.e. where $c_L$ in \[ 10 \] vanishes. We will also analyze this point in the following Section.

**B. Boundary action and Brown-York stress tensor**

Now, let us move to discuss the definition of a boundary stress tensor for NMG in asymptotically $\text{AdS}_3$ space. This was actually done recently in Ref. \[ 33 \], where the Brown-York type of construction was considered. The idea, as always one tries to define a holographic stress tensor in this context, is to start from the Brown-York stress tensor \[ 33 \] and then "push" the whole quantity towards to the boundary. In the process, in addition to the standard boundary terms, it becomes necessary to add terms in the action to cancel the divergences that appear at large $r$. These terms are constructed with intrinsic quantities of the boundary, preserving the spirit of the holographic correspondence.

Actually, from the perspective of $\text{AdS}/\text{CFT}$ correspondence, one gives new meaning to the whole idea of constructing a boundary stress tensor in such a way \[ 33 \]. In fact, holography provides us with a physical interpretation for such observable: It can be regarded as the expectation value $< T^i_{ij} >$ of the stress tensor of the dual two-dimensional CFT that is formulated on the boundary. This gives to $T^i_{ij}$ a more concise physical meaning. From this boundary field theory point of view, the additional terms required to cancel divergences at large $r$ are thought of as counterterms, being part of the regularization method. This holographic stress tensor can then be used, in particular, to compute conserved charges associated to "localized" solutions in the bulk of $\text{AdS}_3$, or even used to read the value of the central charge of the boundary CFT$_2$. The latter can be accomplished by considering fluctuations of the boundary metric and computing the expectation value of the trace through the Weyl anomaly $< T^{i j}_{\gamma \gamma} > = \frac{c}{12} \gamma^{ij} R_{ij}$. In fact, bulk transformations consistently translate into the anomalous term in the transformation of $< T^i_{ij} >$. 
The boundary tensor for NMG proposed in [1] was proven to work not only for asymptotically AdS₃ metrics, but also for asymptotically Lifshitz metrics. This is interesting in the context of the recent proposal for a non-relativistic holography [35]; see also [36]. Nevertheless, in the case of Lifshitz spacetimes it happens that additional counterterms are needed in order to get a finite result, in contrast to what happens in the case of AdS₃ where only a boundary cosmological constant needs to be added. This suggests that, when trying to define a boundary stress tensor for configurations of weakened asymptotic in AdS₃, one has to be open to the possibility of including additional boundary terms.

Let us review the construction in [1] in more detail: As mentioned, [1] follows the standard steps in the construction of a holographic stress tensor in the context of AdS₃/CFT₂. First, boundary terms are added to the action for the variational principle to be defined in such a way that it is sufficient to set the variation of the fields to zero in the boundary. These boundary terms resemble the Gibbons-Hawking term in Einstein gravity, although the fact the variational principle to be defined in such a way that it is sufficient to set the variation of the fields to zero in the boundary. These boundary terms resemble the Gibbons-Hawking term in Einstein gravity, although the fact the theory under consideration is of fourth order introduces some differences. Here, we will adopt the prescription of [1]. Then, once the action is properly defined, one considers a two-dimensional foliation of the space, defined at constant boundary. These boundary terms resemble the Gibbons-Hawking term in Einstein gravity, although the fact the variational principle to be defined in such a way that it is sufficient to set the variation of the fields to zero in the boundary. These boundary terms resemble the Gibbons-Hawking term in Einstein gravity, although the fact the theory under consideration is of fourth order introduces some differences. Here, we will adopt the prescription of [1]. Then, once the action is properly defined, one considers a two-dimensional foliation of the space, defined at constant values of a preferable radial coordinate \( r \). For this purpose it is convenient to consider the ADM decomposition of the metric, in which the three-dimensional metric on the three-manifold \( \Sigma \) reads

\[
ds^2 = N^2 dr^2 + \gamma_{ij}(dx^i + N^i dr)(dx^j + N^j dr),
\]

where \( N^2 \) is the "radial" lapse function, and \( \gamma_{ij} \) is the two-dimensional space-time metric on the constant-\( r \) surfaces with coordinates \( x^i \) with \( i = 0, 1 \).

The next step to construct \( T_{ij} \) is to find out the appropriate boundary terms that come to play the rôle of the Gibbons-Hawking term of general relativity. For the case of NMG, this was done in [1] with the help of the formulation in terms of the auxiliary field \( f^i_\mu \). The boundary action then reads

\[
S_B = \frac{1}{16\pi G} \int_{\partial \Sigma} d^2 x \sqrt{-\gamma} \left( -2K - \hat{f}^{ij} K_{ij} + \hat{f} K \right),
\]

where we are using the conventions of [1]; namely, Latin indices refer to the constant-\( r \) surfaces \( i, j = 0, 1 \), being \( K \) the trace of the extrinsic curvature, \( K = \gamma^{ij} K_{ij} \). In ADM variables \(14\) the extrinsic curvature reads \( K_{ij} = -\frac{1}{2N} \left( \partial_r \gamma_{ij} - \nabla_i N_j - \nabla_j N_i \right) \). The field \( \hat{f}_{ij} \) in \(15\) is defined as follows

\[
\hat{f}_{ij} = f_{ij} + 2h^{ij} N_j sN^i + N^2 N^j
\]

and in the boundary action \(15\) we have \( \hat{f} = \gamma^{ij} \hat{f}_{ij} \).

One can actually verify that the boundary action above leads to a well defined variational principle; see [1] for details. The boundary stress tensor is thus defined by varying \( S_A + S_B \) with respect to \( \gamma_{ij} \). That is,

\[
T_{ij} = \frac{2}{\sqrt{-\gamma}} \frac{\delta}{\delta \gamma_{ij}} (S_A + S_B).
\]

And it takes the form

\[
16\pi G \ T_{ij} = K_{ij} - \gamma_{ij} K - \frac{1}{2} \hat{f} K_{ij} + \nabla_i (\hat{h} j) - \nabla_j (\hat{h} i) + \frac{1}{2} D_r \hat{f}_{ij} - K_{kl} \hat{f}^{kl} + \frac{1}{2} \hat{h} K_{ij} + \gamma_{ij} (\hat{h} h - \frac{1}{2} \hat{f} K + \frac{1}{2} D_r \hat{f})\]

where \( h^i \) and \( s \) are given by the components of \( f^{\mu \nu} \) that have at least one radial coordinate, namely \( h^i = f^{ir} \) (with \( i = 0, 1 \)) and \( s = f^{rr} \), in terms of which we have also defined \( \hat{h}^{ij} = N (h^i + sN^i) \) and \( \hat{h} = N^2 s \). The covariant radial derivative \( D_r \) in the expression above is defined in such a way it acts on the fields \( \hat{f} \) and \( \hat{f}^j \) as follows

\[
D_r \hat{f}_{ij} = \frac{1}{N} \left( \partial_r \hat{f}_{ij} - N^k \partial_k \hat{f}_{ij} + \hat{f}_i^k \partial_k N_j + \hat{f}_j^k \partial_k N_i \right), \quad D_r \hat{f} = \frac{1}{N} \left( \partial_r \hat{f} - N^k \partial_k \hat{f} \right).
\]

Here, in order to make the reading easier, we are using the conventions of the original reference [1], to which we refer for the details.

stress tensor \(13\) yields the notion of conserved charges associated to a given Killing vector \( \xi \). Writing the two-dimensional metric as

\[
\gamma_{ij} \ dx^i \ dx^j = - \hat{N}^2 dt^2 + R^2 (d\phi + \hat{N}_\phi dt)^2,
\]

1 Recall the conventions here: \( x^0 = t, \ x^1 = \phi \), with \( i, j = 0, 1 \).
the charge is defined by \[ Q[\xi] = \int d^2x \; R \; T_{ij} u^i \xi^j, \tag{20} \]

where \( R \) plays the rôle of the one-dimensional metric of the constant-\( t \) lines, and \( u^i \) is a time-like vector normal to these lines. This enables to define the mass \( M \) and the angular momentum \( J \) of an asymptotically AdS_3 solution as the conserved charges associated to the Killing vectors \( \xi^0 = \partial_t \) and \( \xi^3 = \partial_\phi \), respectively. This yields

\[ Q[\partial_t] = M = \int d^2x \; R \; T_{i0} u^i, \quad Q[\partial_\phi] = J = \int d^2x \; R \; T_{i\phi} u^i. \tag{21} \]

Now, we are in a position to compute the conserved charges associated to specific NMG solutions. We are specially interested in solutions that, while being asymptotically AdS_3 in a sense, do not necessarily obey BH boundary conditions. As we anticipated, to accomplish this we will probably need to add boundary terms to the action (15) and thus improve the definition (17)-\( 18 \) in order to regularize (21). More precisely, we have to supplement the boundary action \( S_B \) by adding an additional piece \( S_C \) which would contain quantities constructed by intrinsic quantities of the boundary. The variation of \( S_C \) with respect to the boundary metric is what provides the terms that ultimately cancel the near-boundary divergences. In NMG, because of the feasibility of formulating the theory in terms of the auxiliary field \( f_\mu^\nu \), the set of intrinsic quantities of the boundary among which we can choose those counterterms gets considerably enhanced with respect to the case of general relativity. For instance, we have at hand the following selection of counterterms

\[ S_C = \int d^2x \sqrt{\gamma} \left( \alpha_0 + \alpha_1 \hat{f} + \alpha_2 f^2 \right), \tag{22} \]

where the coefficients \( \alpha_i \) are to be fixed to obtain a finite result. In turn, the renormalized stress tensor \( T_{ij}^{(\text{ren})} \) turns out to be defined by

\[ T_{ij}^{(\text{ren})} = T_{ij} + \delta \frac{\delta}{\delta \gamma_{ij}} S_C. \tag{23} \]

For example, for asymptotically AdS_3 solutions that obey the BH boundary conditions, it is sufficient to consider a cosmological boundary term \( \alpha_0 = -\frac{1}{2\pi Gl}(1 + \frac{1}{2\pi Gl}) \) with no additional contributions (i.e. with \( \alpha_1 = \alpha_2 = 0 \)) to cancel the divergences [1]. In that case, in turn we have \( T_{ij}^{(\text{ren})} = T_{ij} - \frac{1}{2\pi Gl}(1 + \frac{1}{2\pi Gl}) \gamma_{ij} \). In contrast, for asymptotically Lifshitz spaces (with critical exponent \( z \neq 1 \)) additional terms in (23) have to be turned on [1]. As we will see, even in AdS_3, if one relax BH asymptotic, in general one has to consider non-vanishing \( \alpha_1 \) and \( \alpha_2 \) to get a finite result. We will discuss this in the following Section.

III. CONSERVED CHARGES AND WEAKENED ADS_3 BOUNDARY CONDITIONS

A. A first example: Logarithmic deformation of extremal BTZ

As a warming up, let us begin by considering a simple example of relaxed boundary conditions. This example was already analyzed in [1]. Consider the theory [4] at the point of the space of parameters such that \( c_L = 0 \). That is, consider the relation (13). At this point, there exist exact solutions that are asymptotically AdS_3 in the sense of the weakened boundary conditions defined by Grumiller and Johansson in Ref. [18] which, however, do not obey the BH boundary conditions\(^2\). In fact, it is not hard to verify that equations of motion (2)-(4) admit the following metric as a solution if (13) holds

\[ ds^2 = -N^2(r) dt^2 + \frac{dr^2}{N^2(r)} + r^2(N^\phi(r) dt - d\phi)^2 + N^2_\phi(r)(dt - l d\phi)^2 \tag{24} \]

\(^2\) More precisely, the solution we consider in this subsection have an "intermediate" asymptotic; weaker than Brown-Henneaux but still stronger than Grumiller-Johansson, and consequently consistent with the latter.
where
\[ N^2(r) = \frac{r^2}{l^2} - 4GM + \frac{4C^2M^2l^2}{r^2}, \quad N_\phi(r) = \frac{2GMl}{r^2}, \]
(25)
and
\[ N_k^2(r) = k \log |r^2 - 2GMl^2|. \]
(26)

This metric is a perturbation of the extremal BTZ black hole, which is recovered setting \( k = 0 \). It can be shown that for \( k \neq 0 \) the space is locally equivalent to a pp-wave in AdS space [24]. More properties of the solution were studied in references [26, 27].

The asymptotic AdS\(_3\) boundary conditions that the metric \([24]-[26]\) fulfills are given by the following next-to-leading behavior,
\[ g_{tt} \simeq -\frac{r^2}{l^2} + \mathcal{O}(\log(r)), \quad g_{rr} \simeq \frac{l^2}{r^2} + \mathcal{O}(r^{-4}), \quad g_{\phi t} \simeq \mathcal{O}(\log(r)), \]
(27)
\[ g_{\phi r} \simeq \mathcal{O}(1), \quad g_{\phi\phi} \simeq r^2 + \mathcal{O}(\log(r)), \quad g_{rt} \simeq \mathcal{O}(1), \]
(28)
which is evidently weaker than the standard BH boundary conditions. In particular, the presence of contributions of order \( \mathcal{O}(\log(r)) \) makes the falling-off of the components \( g_{\phi t} \) weaker. Nevertheless, these boundary conditions are still consistent with the definition of asymptotic charges that realize the boundary two-dimensional conformal algebra, and this was studied in [18] for the case of TMG \((\eta = \infty, \mu < \infty)\). What we want to emphasize here is that, despite the weaker falling-off \([27]-[28]\) that the metric presents, it is still consistent with the definition of a boundary stress tensor as done in [1]. Moreover, this boundary stress tensor can be used to calculate the conserved charges associated to the metric \([24]-[26]\) in a very simple way. To see this, let us consider the case \( m^2l^2 = -1/2 \) and \( \mu = \infty \), for convenience, i.e. consider \( \mu = \infty \) in \([13]\). Then, since the value of the boundary cosmological constant required to regularize \( T_{ij} \)
in NMG in AdS\(_3\) happens to be proportional to the quantity \( 1 + \frac{1}{4GM} \), it turns out that at the point \([11]\) there is no need to add a regularizator term in this case, and the computation of the mass and the angular momentum \([21]\) then yields
\[ M = \frac{2k}{G}, \]
(29)
\[ J = \frac{2k}{G}. \]
(30)
This agrees with the result obtained by other methods, like the Super Angular Momentum method considered in [27, 37, 38]. The holographic computation of \([29]-[30]\) was already done in [1]; here, we have reviewed it as a first working example to emphasize that, at least in some cases, relaxing the asymptotic may lead to a well defined solution. However, one could argue that having obtained a finite result for the charges in this case is not quite surprising since, after all, \([24]-[25]\) is a solution that occurs at the chiral point where special things happen. Therefore, a less simple example would be to consider solutions that, while exhibiting weakened asymptotic, appear for \( m^2l^2 \neq -1/2 \). Such a solution exists, and we will study it in the next Subsection.

B. A second example: Hairy Rotating BTZ black hole

An example of a NMG solution with weakened AdS\(_3\) asymptotic outside the chiral point was given in [28]. This corresponds to a different deformation of the BTZ geometry that occurs at \([12]\), i.e. if \( m^2l^2 = +1/2 \). The metric of the solutions reads
\[ ds^2 = -N(r)F(r)dt^2 + \frac{dr^2}{F(r)} + r^2 \left( d\phi + N^\phi(r)dt \right)^2, \]
(31)
where \( N(r), N^\phi(r) \) and \( F(r) \) are functions of the radial coordinate \( r \), given by
\[ N(r) = \left[ 1 + \frac{bl^2}{4H(r)} \left( 1 - \eta \right) \right]^2, \quad N^\phi(r) = -\frac{a}{2r^2} \left( 4GM - bH(r) \right), \]
\[ F(r) = \frac{H^2(r)}{l^2} \left[ \frac{H^2(r)}{l^2} + \frac{b}{2} (1 + \eta) H(r) + \frac{b^2l^2}{16} (1 - \eta)^2 - 4GM \eta \right], \]
(32)
\[ H(r) = \left[ r^2 - 2GMl^2 (1 - \eta) - \frac{b^2l^2}{4} (1 - \eta)^2 \right]^{\frac{1}{2}}. \] (33)

Here \( \eta = \sqrt{1 - a^2/r^2} \), and \( a = J/M \) is the rotation parameter. Actually, for certain range of the parameters \( M, J, \) and \( b \) this solution represents a hairy rotating black hole. The parameter \( b \) represents the "gravitational hair" of the solution, and for the solution to be a black hole the bounds \( M > \frac{b^2 l^2}{16G} \) and \( -Ml \leq J \leq Ml \) need to be satisfied.

For computational purposes it may result convenient to redefine the radial coordinate as \( r \rightarrow \hat{r} = H(r) \). This does not change the asymptotic since near the boundary \( r \simeq \hat{r} + O(1/r) \). It is not hard to verify that the solution is asymptotically AdS3 in a weak sense. More precisely, it obeys the following boundary conditions

\[
\begin{align*}
  g_{tt} &\simeq -\frac{r^2}{4} + O(r), \\
g_{rr} &\simeq \frac{l^2}{r^2} + O(r^{-3}), \\
g_{\theta\theta} &\simeq O(1), \\
g_{\theta t} &\simeq O(1),
\end{align*}
\] (34)

Notice that these asymptotic conditions are not the ones in (27)-(28), nor agree with the BH asymptotic conditions. This is the key point here: We will prove that this (new version of) relaxed asymptotic also yields a well defined stress tensor in the boundary. But, first, let us discuss a little more about the geometry (31)-(33). The properties of the hairy black hole solution were studied in [28] and [30]. In particular, its thermodynamics was studied: The solution has Hawking temperature

\[
T_{\text{H}} = \frac{\eta}{\pi l} \sqrt{2G \left( M + \frac{b^2 l^2}{16G} \right) (1 + \eta)^{-1}},
\] (36)

and Bekenstein-Hawking entropy

\[
S_{\text{BH}} = \pi l \sqrt{\frac{2}{G} \left( M + \frac{b^2 l^2}{16G} \right) (1 + \eta)}.
\] (37)

These quantities fulfill the relation

\[
T_{\text{H}} dS_{\text{BH}} = \eta dM + \frac{b^2 l^2}{8G} \eta \, db - \frac{1}{a} (1 - \eta) \left( M + \frac{b^2 l^2}{16G} \right) \, da,
\] (38)

where

\[
\Omega_+ = \frac{1}{a} (\eta - 1)
\] (39)

is the angular velocity of the horizon. We will see below how the correct definition of the black hole mass and angular momentum yields both a statistical explanation for (37) and a physical meaning for (38).

Then, in principle we are ready to compute the conserved charges associated to this black hole solution resorting to the definition of (23). However, when evaluating the integrals in (21) one rapidly notices that, because of the weaker falling-off (34)-(35), the divergences are severer than in the case of BH boundary conditions. This demands to consider more counterterms than a mere boundary cosmological constant term. Besides, it entails an extra difficulty since one also has to analyze the (non)ambiguity of the definition of the charges by choosing different prescriptions to regularize. For instance, if one includes the three terms of (22) and first tries to calculate the mass of the black hole solution (31)-(33) for the particular case \( a = 0 \), then one finds only two conditions for the three couplings \( \alpha_i \), namely \( \alpha_0 = 16\alpha_2 \) and \( \alpha_1 = 1/l + 8\alpha_2 \), and this introduces an undesired ambiguity in the value of the mass. However, one can show that such ambiguity disappears when the rotating solution (i.e. \( a \neq 0 \)) is considered. In that case, demanding the finiteness of the result completely fixes the three coefficients to be \( \alpha_0 = \alpha_2 = 0 \) and \( \alpha_1 = 1/l \), and the computation of (21) using the improved stress tensor (23) yields

\[
M = M + \frac{b^2 l^2}{16G}.
\] (40)

\[\text{3 The parameter } \eta \text{ was introduced here, and it relates to the notation used in [28] by } \eta = \Xi^{1/2}.\]
\[ \mathcal{J} = J - \frac{ab^2 l^2}{16G}. \]  

(41)

It is worth noticing that these values for the charges, considered together with (38), verify the first principle of black hole thermodynamics, which reads

\[ dM = T_H \, dS_{BH} - \Omega_+ \, d\mathcal{J}. \]  

(42)

Furthermore, we can verify that the result obtained for the conserved charges in (40)-(41) is exactly the one required for the dual CFT to account for the black hole entropy. Let us summarize this story here: The first precise observation about the statistical counting of the three-dimensional black hole degrees of freedom in terms of its boundary dual theory was made by Strominger in Ref. [7] for the case of Einstein gravity. Strominger noticed that, in virtue of the results of [2], the formula for the density of states of the boundary CFT exactly reproduces the Bekenstein-Hawking entropy of the BTZ black hole. The computation uses the fact that, for any CFT that satisfies some physically sensible requirements, the density of states \( \rho(\hbar, \bar{\hbar}) \) of a given conformal dimension \( (\hbar, \bar{\hbar}) \) asymptotically grows following a very simple expression, called the Cardy formula [39]. More precisely,

\[ \rho(\hbar, \bar{\hbar}) \simeq e^{2\pi \sqrt{c/6}} e^{2\pi \sqrt{c/6}}, \]  

(43)

where \( c \) is the central charge of the theory. This, in turn, yields a very simple expression for the entropy in the microcanonical ensemble; namely

\[ S_{CFT} = 2\pi \sqrt{\frac{c}{6}} \left( \sqrt{\frac{\hbar}{6}} + \sqrt{\frac{\bar{\hbar}}{6}} \right). \]  

(44)

The quantities \( \mathcal{E} = h + \bar{h} \) and \( \mathcal{J} = h - \bar{h} \) are typically associated to the energy (in units of \( 1/l \)) and the spin of the fields in the CFT, respectively. In the dual description these quantities turn out to be in correspondence with the mass and angular momentum of the asymptotically AdS\(_3\) black holes states.

The observation made in [7] was that, if one considers the Brown-Henneaux central charge \( c = \frac{3l}{2G} \) for general relativity, and identify the mass and the angular momentum of an asymptotically \( \text{AdS}_3 \) solution appropriately, then Cardy formula (44) happens to match the entropy of the BTZ black hole. This is a remarkable observation, which merely relies on general aspects of the asymptotic symmetry of Einstein gravity in \( \text{AdS}_3 \). What we want to point out here is that, if we relate the mass and the angular momentum as

\[ lM = h + \bar{h}, \quad \mathcal{J} = h - \bar{h}, \]  

(45)

then (44) exactly reproduces the entropy of the hairy rotating black hole too; see also [30] for analogous computation. This is quite remarkable since, in contrast to BTZ black hole, the metric of the hairy black hole solution (31) does not satisfy the Brown-Henneaux boundary condition, but a relaxed version of them. Actually, considering that for (11) the central charge takes the value

\[ c = \frac{3l}{G}, \]  

and putting together (40), (41), (44), and (45), one finally recovers the entropy (37). Namely,

\[ S_{BH} = S_{CFT}. \]  

(46)

It is worth emphasizing that having obtained this matching is not trivial: If the second term in (41) (and/or in (11)) were not enter in the definition of the black hole mass (and angular momentum), then Cardy formula would have not reproduced the Bekenstein-Hawking entropy, cf. [28]. This is why having obtained such a dependence on \( b \) in the mass formula is pleasant. The argument in [30] to include such a \( b \)-dependent term was that the absence of a chemical potential associated to the hair parameter \( b \) makes possible to absorb its variation by redefining the conserved charges in the first principle of black hole thermodynamics. This naturally leads to consider the solution with \( M + \frac{b^2 l^2}{16G} = 0 \) as the ”ground state”. Here, we have obtained this result in a completely independent way, confirming that the conserved charges (40) and (41) are the correct result, and thus completing the argument of [30].

\[ ^4 \text{Cardy formula strongly relies on modular invariance at one loop, and on certain assumptions on the gap in the spectrum that permit to use a saddle point approximation. See [5] for a discussion on Cardy formula in this context.} \]

\[ ^5 \text{Notice this is twice the Brown-Henneaux central charge for general relativity.} \]
C. A third example: Non-Einstein deformation of BTZ geometry with Brown-Henneaux asymptotic

So far, we have considered two different deformations of the BTZ geometry, each of them representing different ways of relaxing Brown-Henneaux boundary conditions. We have seen how, in both cases, a boundary stress tensor can be actually defined and used to compute finite conserved charges. While solution (24)-(26) satisfies boundary conditions (27)-(28) and did require no counterterms for its charges (29)-(30) to be computed, solution (31)-(33) satisfies the asymptotic (34)-(35) and has conserved charges (40)-(41) which were computed by regularizing the stress tensor. Now, let us consider a third way of perturbing the BTZ geometry; one preserving BH asymptotic but not being a solution of general relativity. Consider again the point of the parameter space (11), at which the central charge (9) vanishes. And consider the following perturbation of the zero-mass BTZ solution (here we set \( l = 1 \))

\[
ds^2 = -r^2 dt^2 + \frac{dr^2}{r^2} + r^2 d\phi^2 + N_\gamma^2 (dt + d\phi)^2
\]

with

\[
N_\gamma^2 = \left( \alpha + \beta t + \frac{\gamma}{r^4} \right).
\]

It can be verified that this ansatz solves the field equations (2)-(4) for \( m^2 l^2 = -1/2 \), \( \mu = \infty \), and if \( \gamma = -\beta^2/72 \).

The metric exhibits the following asymptotic conditions

\[
g_{tt} = -r^2 + \alpha + \beta t + \frac{\gamma}{r^4}, \quad g_{rr} = \frac{1}{r^2}, \quad g_{\phi\phi} = r^2 + \alpha + \beta t + \frac{\gamma}{r^4}, \quad g_{\phi t} = \alpha + \beta t + \frac{\gamma}{r^4},
\]

where \( \alpha, \beta, \gamma \) are real numbers. In turn, this represents a NMG solution that fulfills Brown-Henneaux boundary conditions. What is remarkable is that the metric (47)-(48) is an asymptotically AdS\(_3\) solution of NMG in the sense of BH but it is not an Einstein manifold\(^6\). Interestingly enough, this localized solution yields an expression for the (unrenormalized) stress tensor (18) that vanishes identically. Analogous result is obtained (47)-(48). A similar solution was found for the case of chiral gravity in [40]; we refer to that work for a discussion on the properties of (47)-(48) and its physical relevance. Let us say here that the fact

\[
T_{ij} = 0
\]

when evaluated on (47)-(48) could be relevant for the discussion on which are the sectors to be taken into account for computing a partition function in massive gravity on AdS\(_3\) at the point \( c = 0 \). This is because (50)-(51) are consistent with the BH asymptotic (2) even when metric (47)-(48) is not locally AdS\(_3\), and consequently it raises the question whether this solution should be included or not as a saddle point of the AdS\(_3\) sector. The fact such a solution of massive gravity carry vanishing charges is interesting.

IV. CONCLUSIONS

In this paper we have considered different solutions of NMG in asymptotically AdS\(_3\) spaces, each of them incarnating a different notions of what ”asymptotically AdS\(_3\)” means. All the solutions discussed here represent different ways of deforming the BTZ geometry. We considered both solutions satisfying Brown-Henneaux boundary conditions and solutions with relaxed asymptotic, having computed the conserved charges associated to all of them. In particular, we studied the logarithmic deformation of the BTZ geometry [26,27] that appears at the chiral point, which obeys the asymptotic conditions proposed by Grumiller and Johansson in [18]. We also considered the hairy rotating black hole solution of [28], for which we reconsidered its thermodynamics in light of the holographic computation of the conserved charges. We also studied a new solution to NMG, which obeys BH asymptotic not being a solution of Einstein gravity. For all the asymptotically AdS\(_3\) solutions studied in this paper it was possible to define a boundary stress tensor; even for those that do not exhibit BH asymptotic. It is likely that NMG formulated in asymptotically AdS\(_3\) space,

\(^6\) This solution was first found by S. de Buyl, G. Compère, and S. Detournay for the case of TMG at the chiral point [40]. We found the NMG analogue inspired in their unpublished work, where the properties of the geometry and its relevance for the chiral gravity conjecture are analyzed.
considering weakened version of BH boundary conditions, is dual to a two-dimensional conformal field theory. In fact, evidence suggesting that AdS/CFT correspondence resists such a relaxation of the asymptotic conditions exists. In the literature we find the following two suggestive observations:

- The group of asymptotic symmetry defined with weaker version of Brown-Henneaux boundary conditions is also generated by two copies of Virasoro algebra with a central extension. The central charge turns out to be exactly the same as if Brown-Henneaux conditions had been imposed \[ \frac{3}{2} \text{[28, 31]}, \] although the other properties of the CFT\(_2\) hardly remain unaltered by the change in the boundary conditions.

- The structure exhibited by two- and three-point functions with relaxed falling-off at \( c = 0 \) seem to be consistent with the functional form of a (presumably Logarithmic) conformal field theory \[ \text{[41]}. \]

In this paper, we added the following piece of evidence:

- It is possible to define a boundary stress tensor for NMG in AdS\(_4\) even if relaxed asymptotic is considered. In contrast to the case of Brown-Henneaux boundary conditions, additional counterterms are needed to regularize the divergences in the stress tensor. These divergences are induced by the soften falling-off of the bulk gravitational field. The conserved charges computed with the regularized stress tensor exactly match the values required for the Cardy formula in the boundary CFT\(_2\) to reproduce the Bekenstein-Hawking entropy of the hairy black hole in AdS\(_3\).

All this seems to suggest that a dual CFT\(_2\) description still exists if weakened asymptotic is prescribed. As it happens in TMG at the chiral point, the cost of relaxing AdS\(_3\) boundary conditions in NMG may be that of losing unitarity in the dual CFT\(_2\). This raises the question on whether it makes sense to consider such a way of formulating AdS/CFT. However, from a less conservative point of view, one can argue that AdS/CFT correspondence could make sense even for non-unitary CFTs. After all, in condensed matter applications non-unitary CFTs play an important rôle, and having a gravity dual for Logarithmic conformal field theory could be very interesting in this context.

Let us emphasize that having found appropriate counterterms to regularize \(< T_{ij} \) in the case of weakened AdS\(_3\) asymptotic is non-trivial, as a priori there is no guarantee to achieve so. A good example is given by trying to proceed in the same way for the case of the asymptotically Warped-AdS\(_3\) spaces in NMG. In fact, there is no obvious manner of defining a regularized boundary stress tensor in that case. In TMG, for which the Brown-York construction was studied in \[ \text{[34]}, \] the situation is actually similar. However, the fact of not being able to fully regularize all the components of \( T_{ij} \) should not prevent us from employing the holographic-inspired method to compute at least some conserved quantities of asymptotically Warped-AdS\(_3\) solutions. To be more precise, let us illustrate this by analyzing the case of asymptotically Warped-AdS\(_3\) black holes in TMG \[ \text{[37]}. \] The metric of these black holes is \[ \text{[42, 44]} \]

\[
\frac{ds^2}{l^2} = dt^2 + \frac{dr^2}{(\nu^2 + 3)(r - r_+)(r - r_-)} + \left(2\nu r - \sqrt{r+r_-(\nu^2 + 3)}\right) dt d\theta + \frac{r}{4} \left(3(\nu^2 - 1)r + (\nu^2 + 3)(r_+ + r_-) - 4\nu \sqrt{r+r_-(\nu^2 + 3)} \right) d\theta^2
\]

where \( r_\pm \) are the location of the horizons of the rotating solution; here we use the conventions of Ref. \[ \text{[43]}, \] defining \( \nu = \mu l/3 \) and considering now \( m = \infty \). It is possible to show that, by adding a boundary cosmological counterterm\(^7\)

\[
S_C = -\frac{\sqrt{\nu^2 + 3}}{16\pi G l} \int d^2 x \sqrt{-\gamma},
\]

one obtains a finite result for the mass of the Warped-AdS\(_3\) black hole. Remarkably, the finite result found for the mass happens to be the correct value, which in the conventions of \[ \text{[42]} \] reads

\[
M = \frac{(\nu^2 + 3)}{48Gl} \left[r_+ + r_- - \frac{1}{\nu} \sqrt{r+r_-(\nu^2 + 3)}\right].
\]

\(^7\) Notice that the value of the boundary cosmological constant matches the value \( 1/l \) for the case \( \nu = 1 \), where the Warped-AdS\(_3\) space coincides with AdS\(_3\) space. The introduction of a boundary cosmological constant to regularize the boundary stress-tensor for asymptotically Warped-AdS\(_3\) spaces was also considered by Daniel Grumiller and Niklas Johansson. The authors thank Alan Garbarz for suggesting this idea to them.
functions. This could be particularly important for the theory at the special point. An expression for the boundary tensor of a dual two-dimensional conformal field theory. For instance, the question remains whether the proposed gravitational field. Nevertheless, further evidence is needed to confirm the interpretation of (18) as being the stress tensor proposed in \[1\].

Going back to the case of AdS\(_3\): The main result of this paper was showing that the definition of a boundary stress tensor proposed in \[1\] can be extended to the case of weakened AdS\(_3\) asymptotic. Our computation turns out to be in accordance with AdS/CFT, whose validity seems to be robust under the relaxation of the falling-off of the gravitational field. Nevertheless, further evidence is needed to confirm the interpretation of (18) as being the stress tensor of a dual two-dimensional conformal field theory. For instance, the question remains whether the proposed expression for the boundary \(\langle T_{ij}\rangle\) can be used to calculate other quantities of the boundary theory, like correlation functions. This can be particularly important for the theory at the special point \(c = 0\), which has been conjectured to be a Logarithmic CFT\(_2\), and the two-point function of the stress tensor would then exhibit a special form. Studying this and other aspects of the dual conformal model in the case of weakened asymptotic is matter of further study.

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