Doubly heavy tetraquarks in an extended chromomagnetic model

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Using an extended chromomagnetic model, we perform a systematic study of the masses of the doubly heavy tetraquarks. We find that the ground states of the doubly heavy tetraquarks are dominated by color-triplet \([qq\overline{q}q]/(Q\bar{Q})}\) configuration, which is opposite to that of the fully heavy tetraquarks. The combined results suggest that the color-triplet configuration becomes more important when the mass difference between the quarks and antiquarks increases. We find three stable states which lie below the thresholds of two pseudoscalar mesons. They are the \(IJ^P = 01^+ nn\bar{b}\) tetraquark, the \(IJ^P = 00^+ n\bar{c}\bar{b}\) tetraquark and the \(J^P = 1^+ n\bar{s}\bar{b}\) tetraquark.

I. INTRODUCTION

Besides the conventional meson and baryon composed of quark-antiquark pair and three quarks, there also exist hadrons composed of more than three quarks, or gluons. These states are called exotic states, such as the tetraquark [1, 2], pentaquark [3, 4], molecule [5–7], glueball [8, 9], hybrid [10, 11], etc. In 2003, the first charmoniumlike state \(X(3872)\) was observed by the Belle Collaboration in the exclusive \(B^\pm \rightarrow \pi^\pm \pi^\pm \pi^- J/\psi\) decays [12]. Its quantum number is \(I^G,J^{PC} = 0^+1^{++}\) [13]. The discovery of \(X(3872)\) opens a new era of the hadron spectroscopy. Lots of charmoniumlike and bottomoniumlike states were found since then, such as the \(Y(4260)\) [14], \(Z_c(3900)\) [15, 16], \(Z_b(10610)\) and \(Z_b(10650)\) [17] states. In the fully heavy sector, the LHCb collaboration observed a narrow structure and a wide structure in the \(J/\psi\)-pair invariant mass spectrum in the range of 6.2 ~ 7.2 GeV, which could be all-charm hadrons [18]. More details can be found in Refs. [19–26] and references therein.

The heavy quarks in these states are all in hidden flavor(s). In 2017, the LHCb Collaboration observed the \(\Xi_{cc}^{++}\) in the \(\Lambda_{cc}^+ K^- \pi^+\pi^-\) decay channel [27]. Its mass was determined to be 3621.40 ± 0.72(stat.) ± 0.27(syst.) ± 0.14(ΔΣ+c) MeV. This is the first doubly heavy baryon observed in experiment. The \(\Xi_{cc}^{++}\) baryon gives implications for the doubly heavy tetraquarks [28, 29], which are exotic states with open heavy flavors. Recently, the LHCb Collaboration observed a very narrow state in the \(D^0\bar{D}^0\pi^+\pi^-\) mass spectrum [30–33]. Under the \(J^P = 1^+\) assumption, its mass with respect to the \(D^+\bar{D}^0\) and width are

\[
\delta m_{BW} = -273 \pm 61 \pm 5^{+11}_{-14} \text{ keV},
\]

and

\[
\Gamma_{BW} = 410 \pm 165 \pm 43^{+38}_{-38} \text{ keV},
\]

respectively. The statistical significance for the signal is over 10σ, while that for \(\delta m_{BW} < 0\) is 4.3σ. This structure is consistent with the \(DD^*\) molecule interpretation predicted by Li et al. within the one-boson-exchange (OBE) model [34]. The discovery of the \(T_{cc}^+\) inspired many related studies [35–44]. Actually, the doubly heavy tetraquarks have been studied extensively in the literature, with models like the quark model [1, 45–72], QCD sum rules [73–83], Lattice QCD [84–102], the OBE potentials [34, 103–107], chiral perturbation theory [108–111], etc. Their production has also been studied (for instance, see Refs. [112, 113]). The studies suggest that the masses of some of the doubly heavy tetraquarks are lighter than the thresholds of two mesons, which makes them stable against the strong and electromagnetic decays. For example, Du et al. [76] studied the \(QQq\overline{q}'\) (\(Q = c,b\) and \(q,q' = u,d,s\)) in the QCD sum rules. They found that the \(bbq\overline{q}'s\) are stable. The stableness of doubly heavy tetraquarks is also supported by the lattice QCD calculations. Leskovec et al. [100] used lattice QCD to investigate the spectrum of the \(bbud\) four-quark system with quantum numbers \(I(J^P) = 0(1^+)\), and obtained a binding energy of \((-128 \pm 24 \pm 10)\) MeV, corresponding to the mass 10476 ± 24 ± 10 MeV. Mohanta and Basak [102] studied the \(bbud\) states on lattice using non relativistic QCD (NRQCD) action for bottom and highly improved staggered quark (HISQ) action for the light up/down quarks. They got the binding energy for the \(1^+ bbud\) tetraquark system to be \(-189(18)\) MeV compared to the \(BB^*\). Using \(\Xi_{cc}^{++}\) mass as input, Karliner and Rosner [29] predicted that the mass of the ground state of \(IJ^P = 01^+\) doubly charm tetraquark (\(T_{cc}\)) to be 3882.2 ± 12 MeV in the chromomagnetic model.

In the quark model [114–118], the mass of hadron can be decomposed into the quark masses, the kinetic energy and the potentials which include the color-independent Coulomb and confinement interactions, and the hyperfine interactions like the spin-spin, spin-orbit, and tensor terms. If we restrict to the S-wave states, the spin-orbit and tensor interactions do not contribute. We can use the extended chromomagnetic model [1, 58, 118–127]. In this model, the masses of S-wave hadrons consist of effective

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quark masses, the color interaction and the chromomagnetic interaction. This simplified model gives good account of all S-wave mesons and baryons [125]. In this work, we use the extended chromomagnetic model to study the S-wave doubly heavy tetraquarks. With the wave function obtained, we further use a simple method to estimate the partial decay ratios of the tetraquark states. In Sec. II we introduce the methods of present work. The numerical results are presented and discussed in Sec. III. We conclude in Sec. IV.

II. FORMALISM

A. Hamiltonian

In the chromomagnetic model, the Hamiltonian of the S-wave hadron reads [123, 125–131]

\[
H = \sum_i m_i + H_{\text{CE}} + H_{\text{CM}}
\]

where \(m_i\) is the effective mass of ith quark, \(H_{\text{CE}}\) is the chromoelectric (CE) interaction [123, 125–127]

\[
H_{\text{CE}} = -\sum_{i<j} a_{ij} F_i \cdot F_j,
\]

and \(H_{\text{CM}}\) is the chromomagnetic (CM) interaction [1, 26, 120–122]

\[
H_{\text{CM}} = -\sum_{i<j} v_{ij} S_i \cdot S_j F_i \cdot F_j.
\]

Here, \(a_{ij}\) and \(v_{ij} \propto \langle \alpha_s(r_{ij}) \delta(r_{ij}) \rangle / m_i m_j\) are effective coupling constants which depend on the constituent quark masses and the spatial wave function. \(S_i = \sigma_i/2\) and \(F_i = \lambda_i/2\) are the quark spin and color operators. For the antiquark,

\[
S_{\bar{q}} = -S^*_{\bar{q}}, \quad F_{\bar{q}} = -F^*_{\bar{q}}.
\]

Since

\[
\sum_{i<j} (m_i + m_j) F_i \cdot F_j = \left( \sum_i m_i F_i \right) \cdot \left( \sum_i F_i \right) - \frac{4}{3} \sum_i m_i,
\]

and the total color operator \(\sum_i F_i\) nullifies any color-singlet physical state, we can rewrite the Hamiltonian as [125–127]

\[
H = -\frac{3}{4} \sum_{i<j} m_{ij} V^C_{ij} - \sum_{i<j} v_{ij} V^\text{CM}_{ij},
\]

where

\[
m_{ij} = (m_i + m_j) + \frac{4}{3} \delta_{ij},
\]

is the quark pair mass parameter. \(V^C_{ij} = F_i \cdot F_j\) and \(V^\text{CM}_{ij} = S_i \cdot S_j F_i \cdot F_j\) are the color and CM interactions between quarks.

B. Wave function

To investigate the masses of the tetraquarks, we need to construct the wave functions. The total wave function is a direct product of the orbital, color, spin and flavor wave functions. Here, the orbital wave function is symmetric since we only consider the S-wave states. Since the Hamiltonian does not contain a flavor operator explicitly, we first construct the color-spin wave function, and then incorporate the flavor wave function to account for the Pauli principle.

The spins of the tetraquarks can be 0, 1 and 2. In the \(qq\bar{q}\bar{q}\) configuration, the possible color-spin wave functions \(\{\alpha^J_i\}\) are listed as follows,

1. \(J^P = 0^+\):

\[
\begin{align*}
\alpha_1^0 &= |(q_1 q_2)_{1/2}^{6} (\bar{q}_3 \bar{q}_4)_{0/2}^{6} \rangle, \\
\alpha_2^0 &= |(q_1 q_2)_{1/2}^{6} (\bar{q}_3 \bar{q}_4)_{0/2}^{-6} \rangle, \\
\alpha_3^0 &= |(q_1 q_2)_{1/2}^{6} (\bar{q}_3 \bar{q}_4)_{0/2}^{6} \rangle, \\
\alpha_4^0 &= |(q_1 q_2)_{1/2}^{6} (\bar{q}_3 \bar{q}_4)_{0/2}^{-6} \rangle.
\end{align*}
\]

2. \(J^P = 1^+\):

\[
\begin{align*}
\alpha_1^1 &= |(q_1 q_2)_{1/2}^{6} (\bar{q}_3 \bar{q}_4)_{1/2}^{6} \rangle, \\
\alpha_2^1 &= |(q_1 q_2)_{1/2}^{6} (\bar{q}_3 \bar{q}_4)_{1/2}^{-6} \rangle, \\
\alpha_3^1 &= |(q_1 q_2)_{1/2}^{6} (\bar{q}_3 \bar{q}_4)_{1/2}^{6} \rangle, \\
\alpha_4^1 &= |(q_1 q_2)_{1/2}^{6} (\bar{q}_3 \bar{q}_4)_{1/2}^{-6} \rangle.
\end{align*}
\]

3. \(J^P = 2^+\):

\[
\begin{align*}
\alpha_1^2 &= |(q_1 q_2)_{1/2}^{6} (\bar{q}_3 \bar{q}_4)_{1/2}^{6} \rangle, \\
\alpha_2^2 &= |(q_1 q_2)_{1/2}^{6} (\bar{q}_3 \bar{q}_4)_{1/2}^{-6} \rangle.
\end{align*}
\]

where the superscript 3, 3, 6 or 6 denotes the color, and the subscript 0, 1 or 2 denotes the spin.

Next we consider the flavor wave function. There are six types of total wave functions when we consider the Pauli principle:

1. Type A: \(\varphi_A = \{(nn\bar{Q}Q)^{t=1}, ss\bar{Q}Q\}\)

\[
\begin{align*}
\Psi_A^{1^+} &= \varphi_A \otimes \alpha_2^0, \\
\Psi_A^{2^+} &= \varphi_A \otimes \alpha_4^0.
\end{align*}
\]

(b) \(J^P = 1^+\):

\[
\Psi_A^{1^+} = \varphi_A \otimes \alpha_1^1.
\]
(c) \( J^P = 2^+ \):

\[ \Psi_{A}^{2+} = \varphi_A \otimes \alpha_2^2, \quad (15) \]

2. Type B: \( \varphi_B = \{(nn\bar{Q}Q)^I=0\} \)

(a) \( J^P = 1^+ \):

\[ \Psi_{B1}^{1+} = \varphi_B \otimes \alpha_1^1, \]
\[ \Psi_{B2}^{1+} = \varphi_B \otimes \alpha_0^1, \quad (16) \]

3. Type C: \( \varphi_C = \{(nn\bar{c}\bar{b})^{I=1, ss\bar{c}\bar{b}}\} \)

(a) \( J^P = 0^+ \):

\[ \Psi_{C1}^{0+} = \varphi_C \otimes \alpha_2^0, \]
\[ \Psi_{C2}^{0+} = \varphi_C \otimes \alpha_3^0, \quad (17) \]

(b) \( J^P = 1^+ \):

\[ \Psi_{C1}^{1+} = \varphi_C \otimes \alpha_1^1, \]
\[ \Psi_{C2}^{1+} = \varphi_C \otimes \alpha_1^1, \]
\[ \Psi_{C3}^{1+} = \varphi_C \otimes \alpha_5^1, \quad (18) \]

(c) \( J^P = 2^+ \):

\[ \Psi_{C}^{2+} = \varphi_C \otimes \alpha_2^2, \quad (19) \]

4. Type D: \( \varphi_D = \{(nn\bar{c}\bar{b})^{I=0}\} \)

(a) \( J^P = 0^+ \):

\[ \Psi_{D1}^{0+} = \varphi_D \otimes \alpha_1^0, \]
\[ \Psi_{D2}^{0+} = \varphi_D \otimes \alpha_2^0, \quad (20) \]

(b) \( J^P = 1^+ \):

\[ \Psi_{D1}^{1+} = \varphi_D \otimes \alpha_1^1, \]
\[ \Psi_{D2}^{1+} = \varphi_D \otimes \alpha_2^1, \]
\[ \Psi_{D3}^{1+} = \varphi_D \otimes \alpha_6^1, \quad (21) \]

(c) \( J^P = 2^+ \):

\[ \Psi_{D}^{2+} = \varphi_D \otimes \alpha_2^2, \quad (22) \]

5. Type E: \( \varphi_E = \{ns\bar{Q}Q\} \)

(a) \( J^P = 0^+ \):

\[ \Psi_{E1}^{0+} = \varphi_E \otimes \alpha_2^0, \]
\[ \Psi_{E2}^{0+} = \varphi_E \otimes \alpha_3^0, \quad (23) \]

(b) \( J^P = 1^+ \):

\[ \Psi_{E1}^{1+} = \varphi_E \otimes \alpha_1^1, \]
\[ \Psi_{E2}^{1+} = \varphi_E \otimes \alpha_4^1, \]
\[ \Psi_{E3}^{1+} = \varphi_E \otimes \alpha_6^1, \quad (24) \]

(c) \( J^P = 2^+ \):

\[ \Psi_{E}^{2+} = \varphi_E \otimes \alpha_2^2, \quad (25) \]

6. Type F: \( \varphi_F = \{ns\bar{c}\bar{b}\} \)

\[ \Psi_{F}^{1+} = \varphi_F \otimes \alpha_1^1, \quad (26) \]

Diagonalizing the Hamiltonian [Eq. (8)] in these bases, we can obtain the masses and eigenvectors of the doubly heavy tetraquarks.

**C. Partial decay rates**

Next we consider the strong decay properties of the tetraquarks. There are various methods for studying the tetraquark decays, such as the dimeson decay through the quark interchange model [132–135] and the dibaryon decay through the \(^3P_0\) model [136–139]. These models require the dynamical structure of the hadrons, which is beyond the power of the chromomagnetic model. Here we adopt a simple method to estimate the partial decay ratios of the tetraquark states.

In Sec. II B we have constructed the wave function in the \(qq\otimes\bar{q}\bar{q}\) configuration, the tetraquark states are superposition of the bases. The tetraquark states can also be written as the linear superposition of the bases in the \(qq\otimes\bar{q}\bar{q}\) configuration (see Appendix A). Normally, the \(qq\) component in the tetraquark can be either of colorsinglet or of color-octet. The former one can easily dissociate into two S-wave mesons in relative S wave, which is called “Okubo-Zweig-Iizuka- (OZI-)superallowed” decays. The recoupling coefficient tell us the overlap between the tetraquark and a particular meson × meson state. Then we can determine the decay amplitude of the tetraquark into that particular meson × meson channel. The latter one can only fall apart through the gluon exchange [120, 140]. In this work, we will focus on the “OZI-superallowed” decays.

For each decay mode, the branching fraction is proportional to the square of the coefficient \(c_i\) of the corresponding component in the eigenvectors, and also depends on the phase space. For two body decay through \(L\)-wave, the partial decay width reads [126, 141]

\[ \Gamma_i = \gamma_i \alpha \frac{k^{2L+1}}{m^{2L}} |c_i|^2, \quad (27) \]

where \(m\) is the mass of the initial state, \(k\) is the momentum of the final states in the rest frame of the initial state,
\( \alpha \) is an effective coupling constant, and \( \gamma_i \) is a quantity determined by the decay dynamics. Generally, \( \gamma_i \) is determined by the spatial wave functions of both initial and final states, which are different for each decay process. In the quark model, the spatial wave functions of the pseudoscalar and vector mesons are the same. Thus for each tetraquark, we have

\[
\gamma M_1 M_2 = \gamma M'_1 M'_2 = \gamma M'_1 M'_2
\]

where \( M_i \) and \( M'_i \) are pseudoscalar and vector mesons respectively. Then we can estimate the partial decay width ratios of the tetraquark states.

### III. NUMERICAL RESULTS

#### A. Parameters

To calculate the tetraquark masses, one needs to estimate the parameters \( \{m^i_{ij}, v^i_{ij}\} \). In Ref. [125] we used the meson and baryon masses to extract the parameters \( \{m^m_{1q}, v^m_{1q}\} \) and \( \{m^b_{1q}, v^b_{1q}\} \). The baryon parameters \( \{m^b_{Q_1Q_2}, v^b_{Q_1Q_2}\} \) between two heavy quarks cannot be fitted from baryons because of the lack of experimental data. For this reason, we adopted the assumptions

\[
\delta m^b_{q_1q_2} = a^b_{q_1q_2} - a^m_{q_1q_2} \approx 0
\]

and

\[
R^b_{q_1q_2} = \frac{v^b_{q_1q_2}}{v^m_{q_1q_2}} = 2.3 \pm 0.30
\]

to estimate them from the meson parameters \( \{m^m_{Q_1Q_2}, v^m_{Q_1Q_2}\} \). The resulting parameters are listed in Table I. Since the CM interaction strength \( v^i_{ij} \)'s are inversely proportional to the quark masses, the meson parameters \( \{v^i_{cc}, v^i_{cb}, v^i_{bb}\} \) between heavy flavors are quite small. Thus the large uncertainty of the ratio \( R^i_{q_1q_2} \) does not have much effects on the baryon parameters \( \{v^i_{cc}, v^i_{cb}, v^i_{bb}\} \) and the mass spectrum of the doubly heavy tetraquarks. As shown in Ref. [125], the introduction of the first assumption makes the difference \( \delta m^b_{q_1q_2} \approx m^b_{q_1q_2} - m^m_{q_1q_2} \) separable over the two quarks

\[
\delta m^b_{q_1q_2} \approx \delta m^b_{q_1} + \delta m^b_{q_2}
\]

where \( \delta m^b_{q_1} \approx m^b_{q_1} - m^m_{q_1} \) is the difference of the effective quark mass extracted from the baryon and meson. In this way, the ten \( \delta m^b_{q_1q_2} \)'s reduce to four \( \delta m^b_{q_1} \)'s. Actually, such property can be achieved by a weaker assumption. Namely, we assume that the difference \( a^b_{q_1q_2} - a^m_{q_1q_2} \) is separable over the two quarks

\[
a^b_{q_1q_2} - a^m_{q_1q_2} \approx \delta a^b_{q_1q_2} + \delta a^m_{q_2}
\]

Then we have

\[
\delta m^b_{q_1q_2} \approx \delta m^b_{q_1} + \delta m^b_{q_2}
\]

which includes the quark mass difference and the differences between color interactions. We again reduce the ten \( \delta m^b_{q_1q_2} \)'s into four degrees of freedom. All results are unchanged except that we reinterpret the \( \delta m^b_{q_1q_2} \) as \( \delta m^b_{q_1} \) (see Table II or Table VI of Ref. [125]).

Now we consider the tetraquarks. In Ref. [127], we used the following scheme to estimate the masses of the fully heavy tetraquarks

\[
m^t_{q_1q_2} \approx m^b_{q_1q_2}
\]

\[
m^t_{q_1q_2} \approx m^m_{q_1q_2}
\]

\[
v^t_{q_1q_2} \approx v^b_{q_1q_2}
\]

\[
v^t_{q_1q_2} \approx v^m_{q_1q_2}
\]

Within this scheme, we found that the ground states of the fully heavy tetraquarks are dominated by color-sextet configurations, which is consistent with the dynamical calculations [142, 143]. Nonetheless, this scheme ignores the difference of the spatial configurations between the tetraquarks and the normal hadrons, which will evidently cause large uncertainties [1, 143, 144]. To appreciate the uncertainty, we introduce three additional schemes for comparison (see Table III). The scheme III (IV) differs from the scheme I (II) by

\[
v^t_{q_1q_2} \approx v^m_{q_1q_2} \quad \Rightarrow \quad v^t_{q_1q_2} \approx v^b_{q_1q_2}
\]

Due to the smallness of \( v^b_{qQ} \) and \( v^m_{qQ} \), the results in scheme I (II) are very similar to those in scheme III (IV). Thus we will focus on the scheme I and scheme II.

#### B. The \( nnQQ \) systems

1. The \( nncc \) and \( nmbb \) tetraquarks

Inserting the parameters into the Hamiltonian, we can determine the tetraquark masses. The masses and eigenvectors of the \( nnQQ \) tetraquarks are listed in Table IV. Here, we assume that the SU(2) flavor symmetry is exact and denote \( u, d \) quarks collectively as \( n \). In the following, we will use \( T_i(nnQQ, m, I, J^P) \) to represent the \( nnQQ \) tetraquarks, where the subscript \( i \) denotes the particular scheme of the parameters. In Figs 1–2, we plot the relative position of the \( nnQQ \) tetraquarks and their meson-meson thresholds.

We first consider the \( nncc \) tetraquarks. The quantum number of its lightest state is \( IJ^P = 0^+ \), namely the
FIG. 1. Mass spectra of the $I = 0$ (solid) and $I = 1$ (dashed) $nn\bar{c}\bar{c}$ and $nn\bar{b}\bar{b}$ tetraquark states in scheme I (black) and scheme II (blue). The dotted lines indicate various meson-meson thresholds. The masses are all in units of MeV.
TABLE I. Parameters of the $q\bar{q}$ pairs for mesons and of the $qq$ pairs for baryons [125] (in units of MeV).

| Parameter | $m_{m_s}$ | $m_{n_s}$ | $m_{n_s}$ | $m_{n_s}$ | $m_{n_s}$ | $m_{n_s}$ | $m_{n_s}$ | $m_{n_s}$ | $m_{n_s}$ |
|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| Value     | 615.95    | 794.22    | 936.40    | 1973.22   | 2076.14   | 3068.53   | 5313.35   | 5403.25   | 6322.27   | 9444.97   |

| Parameter | $v_{n_s}$ | $v_{n_s}$ | $v_{n_s}$ | $v_{n_s}$ | $v_{n_s}$ | $v_{n_s}$ | $v_{n_s}$ | $v_{n_s}$ | $v_{n_s}$ |
|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| Value     | 477.92    | 298.57    | 249.18    | 106.01    | 107.87    | 85.12     | 33.89     | 36.43     | 47.18     | 45.98     |

| Parameter | $v_{n_s}$ | $v_{n_s}$ | $v_{n_s}$ | $v_{n_s}$ | $v_{n_s}$ | $v_{n_s}$ | $v_{n_s}$ | $v_{n_s}$ | $v_{n_s}$ | $v_{n_s}$ |
|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| Value     | 724.85    | 906.65    | 1049.36   | 2079.96   | 2183.68   | 3171.51   | 5412.25   | 5494.80   | 6416.07   | 9529.57   |

| Parameter | $v_{n_s}$ | $v_{n_s}$ | $v_{n_s}$ | $v_{n_s}$ | $v_{n_s}$ | $v_{n_s}$ | $v_{n_s}$ | $v_{n_s}$ | $v_{n_s}$ | $v_{n_s}$ |
|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| Value     | 305.34    | 212.75    | 195.30    | 62.81     | 70.63     | 56.75     | 19.92     | 8.47      | 31.45     | 30.65     |

FIG. 2. Mass spectra of the $I = 0$ (solid) and $I = 1$ (dashed) $mn\bar{c}$ tetraquark states in scheme I (black) and scheme II (blue). The dotted lines indicate various meson-meson thresholds. The masses are all in units of MeV.
The $nn\bar{c}\bar{c}$, $3749.8, 0, 1^+$ state or $T_{11}(nn\bar{c}\bar{c}, 3868.7, 0, 1^+)$ state. The other isoscalar state is $T_I(nn\bar{c}\bar{c}, 3976.1, 1, 1^+)$ or $T_{11}(nn\bar{c}\bar{c}, 4230.8, 0, 1^+)$. We find that the scheme II always gives larger masses than the scheme I. The reason is that the two schemes choose different value of $m_{\bar{q},q}^i$, which results in different values of the color interaction. More precisely, the difference of the color interaction between the two schemes is

$$
\Delta H_C = H_C^{II} - H_C^{I} = -\frac{3}{4} \sum_{i,j} \langle m_{ij}^I - m_{ij}^II \rangle V_{C}^{C} = -\frac{3}{4} \sum_{i,j} \sum_{i,j} \delta m_{ij}^{bb} V_{C}^{C} \approx -\frac{3}{8} \sum_{i,j} \delta m_{ij}^{bb} \sum_{i,j} \delta m_{ij}^{bb} F_i \cdot F_j
$$

where in the last line we have ignored the terms proportional to $\sum_i F_i$. Note that both $\langle (q_1 q_2)^6, (q_3 q_4)^6 \rangle$ and $\langle (q_1 q_2)^3, (q_3 q_4)^3 \rangle$ are eigenstates of $(F_1 + F_2)^2$, with eigenvalues $10/3$ and $4/3$ respectively. In other words,

$$
\langle \Delta H_C \rangle = \begin{cases} 
\frac{5}{6} \sum_i \delta m_{ii}^{bb}, & \text{for } \langle (q_1 q_2)^6, (q_3 q_4)^6 \rangle, \\
\frac{1}{2} \sum_i \delta m_{ii}^{bb}, & \text{for } \langle (q_1 q_2)^3, (q_3 q_4)^3 \rangle.
\end{cases}
$$

For the $nn\bar{c}\bar{c}$ system, $\sum_i \delta m_{ii}^{bb} = 212.9$ MeV. The ground state $T_I(nn\bar{c}\bar{c}, 3749.8, 0, 1^+)$ is dominated by the color-triplet configuration, and its mass is increased by about 118.9 MeV. While the mass of the color-sextet configuration dominated state $T_I(nn\bar{c}\bar{c}, 3976.1, 0, 1^+)$ is increased by 254.7 MeV. The deviation from Eq. (41) is caused by the color mixing. In the isovector sector, we have four tetraquark states. They are all above the corresponding S-wave decay channels. It is interesting to note that the $T_{11}(nn\bar{c}\bar{c}, 3686.7, 0, 1^+)$ in scheme II is quite close to the newly observed $T_{cc}^+$ state.

The $nn\bar{b}b$ tetraquarks is very similar to the $nn\bar{c}\bar{c}$ tetraquarks. Its lightest state also have quantum number $I^J = 0^+$, namely the $T_I(nn\bar{b}b, 10291.6, 0, 1^+)$ or $T_{11}(nn\bar{b}b, 10390.9, 0, 1^+)$. In both schemes, this state lies below the $BB$ threshold and is stable against strong decays. In scheme I, the $T_I(nn\bar{b}b, 10468.9, 1, 0^+)$, $T_I(nn\bar{b}b, 10485.3, 1, 1^+)$ and $T_I(nn\bar{b}b, 10507.9, 1, 2^+)$ also lie below the the $BB$ threshold. But they are not stable in scheme II. Thus we cannot draw a definite conclusion.

Besides the masses, the eigenvectors also help understand the nature of the tetraquarks. Within the four possible quantum numbers, the $I^J = 10^+$ one and the $I^J = 0^+$ one are of particular interest because they both have two possible color configurations, namely the color-sextet $| (qq)^6 \otimes (QQ)^6 \rangle$ and color-triplet $| (qq)^3 \otimes (QQ)^3 \rangle$. For simplicity, we denote them as $6_e \otimes 6_c$ and $3_c \otimes 3_c$. As pointed out by Wang et al. [142], there are two competing effects in determining whether the $6_e \otimes 6_c$ or $3_c \otimes 3_c$ dominates the tetraquark’s ground state. In the one-gluon-exchange (OGE) model, the color interactions in color-triplet diquark are attractive, while those in color-sextet diquark are repulsive. On the other hand, the attractions between $6_e$ diquark and $6_c$ antidiquark and between the $3_c \otimes 3_c$ counterpart are both attractive, and the former one is much stronger. The authors of Refs. [127, 142, 143] found that the color-sextet configuration has more net attractions for most fully heavy tetraquarks. Thus the ground states contain more color-sextet components than the color-triplet one. The only exception is the $ccb\bar{b}$ tetraquark in model II of Ref [142], whose ground state has 53% of the $3_c \otimes 3_c$ component. It is also interesting to note that, when the mass ratio between quarks and antiquarks deviates from one, the color-triplet configuration becomes more important in the ground states. For example, Ref. [127] found that the $T(bbb\bar{b}, 18836.1, 0^+)$ and $T(ccee, 6044.9, 0^+)$ have 18.5% and 30.5% of the $3_c \otimes 3_c$ components, while the $T(cbb\bar{b}, 12596.3, 0^+)$ has 48.4%. This tendency also exists in the doubly heavy tetraquarks. As shown in Table IV, the $3_c \otimes 3_c$ components become dominant in ground states of the $nnQQ$ tetraquarks. This phenomenon can also be explained by the color interaction Hamiltonian.

$$
\langle H_C (nnQQ) \rangle = -\frac{3}{4} \left( m_{nn}^{t}V_{C}^{C} + m_{nQ}^{t}V_{34}^{C} + m_{nQ}^{t}V_{13}^{C} + V_{23}^{C} \right)
$$

$$
= -\frac{3}{4} \left( m_{nQ}^{t} \sum_{i<j} V_{ij}^{C} + 2\delta m (V_{12}^{C} + V_{34}^{C}) \right)
$$

$$
= 2m_{nQ}^{t} - \frac{3}{2} \delta m (V_{12}^{C} + V_{34}^{C})
$$
TABLE IV. Masses and eigenvectors of the $nn\bar{c}$, $nn\bar{b}$ and $nn\bar{b}$ tetraquarks. The masses are all in units of MeV.

| System      | $J^P$ | Scheme I          | Scheme II          |
|-------------|-------|-------------------|--------------------|
|             |       | Mass  | Eigenvector       | Mass  | Eigenvector       |
| ($nn\bar{c}$)$^{f=1}$ | 0^+   | 3833.2 | {0.515, 0.857} | 3969.2 | {0.350, 0.937} |
|             |       | 4127.4 | {0.857, −0.515} | 4364.9 | {0.937, −0.350} |
|             | 1^+   | 3946.4 | {1}           | 4053.2 | {1}           |
|             | 2^+   | 4017.1 | {1}           | 4123.8 | {1}           |
| ($nn\bar{b}$)$^{f=1}$ | 0^+   | 10468.8 | {0.123, 0.992} | 10569.3 | {0.086, 0.996} |
|             |       | 10808.9 | {0.992, −0.123} | 11054.6 | {0.996, −0.086} |
|             | 1^+   | 10485.3 | {1}           | 10584.2 | {1}           |
|             | 2^+   | 10507.9 | {1}           | 10606.8 | {1}           |
| ($nn\bar{b}$)$^{f=0}$ | 1^+   | 3749.8 | {0.354, −0.935} | 3868.7 | {0.212, −0.977} |
|             |       | 3976.1 | {0.935, 0.354} | 4230.8 | {0.977, 0.212} |
| ($nn\bar{b}$)$^{f=1}$ | 0^+   | 7189.5 | {0.366, 0.931} | 7305.6 | {0.232, 0.973} |
|             |       | 7440.9 | {0.931, −0.366} | 7684.7 | {0.973, −0.232} |
|             | 1^+   | 7211.0 | {−0.311, −0.648, 0.696} | 7322.5 | {−0.180, −0.687, 0.704} |
|             |       | 7264.2 | {−0.048, 0.742, 0.669} | 7367.3 | {−0.029, 0.719, 0.694} |
|             |       | 7417.0 | {0.949, −0.175, 0.262} | 7665.1 | {0.983, −0.104, 0.150} |
|             | 2^+   | 7293.2 | {1}           | 7396.0 | {1}           |
| ($nn\bar{b}$)$^{f=0}$ | 0^+   | 7003.4 | {0.440, 0.898} | 7124.6 | {0.266, 0.964} |
|             |       | 7220.3 | {0.898, −0.440} | 7459.0 | {0.964, −0.266} |
|             | 1^+   | 7046.2 | {0.228, −0.219, 0.949} | 7158.0 | {0.122, −0.133, 0.984} |
|             |       | 7232.9 | {0.899, −0.327, −0.292} | 7482.4 | {0.910, −0.381, −0.165} |
|             |       | 7329.3 | {−0.374, −0.919, −0.122} | 7584.9 | {−0.397, −0.915, −0.074} |
|             | 2^+   | 7353.2 | {1}           | 7610.3 | {1}           |

TABLE V. The eigenvectors of the $nn\bar{c}$ tetraquark states in the $n\bar{c}\otimes n\bar{c}$ configuration. The masses are all in units of MeV.

| System      | $J^P$ | Scheme I          | Scheme II          |
|-------------|-------|-------------------|--------------------|
|             |       | Mass  | $\bar{D}^* \bar{D}$ | $\bar{D}^* \bar{D}$ | $\bar{D} \bar{D}$ | $\bar{D} \bar{D}$ |
| ($nn\bar{c}$)$^{f=1}$ | 0^+   | 3833.2 | 0.116         | 0.639         | 3969.2 | −0.023       | 0.611       |
|             |       | 4127.4 | 0.755         | 0.093         | 4364.9 | 0.763       | 0.207       |
|             | 1^+   | 3946.4 | 0.408         | 0.408         | 4053.2 | 0.408       | 0.408       |
|             | 2^+   | 4017.1 | 0.577         |               | 4123.8 | 0.577       |             |
| ($nn\bar{c}$)$^{f=0}$ | 1^+   | 3749.8 | −0.177       | 0.415        | −0.415 | 3868.7       | −0.277     | 0.369       | −0.369     |
|             |       | 3976.1 | 0.685       | 0.280        | −0.280 | 4230.8       | 0.651      | 0.338       | −0.338     |
TABLE VI. The values of $k \cdot |c_i|^2$ for the $n\bar{n}c\bar{c}$ tetraquarks (in unit of MeV).

| System | $J^P$ | Scheme I | Scheme II |
|--------|-------|----------|-----------|
| $\bar{D}^*\bar{D}^*\bar{D}D$ | $0^+$ | 3833.2 | 251.3 |
| | | 4127.4 | 7.6 | 4364.9 | 497.6 | 48.5 |
| $\bar{D}^*\bar{D}^*\bar{D}D$ | $1^+$ | 3946.4 | 61.9 | 4053.2 | 98.8 |
| | $2^+$ | 4017.1 | | 4123.8 | 155.3 |
| $\bar{D}^*\bar{D}^*\bar{D}D$ | $I^=1$ | 3749.8 | 34.7 | 4230.8 | 281.1 | 96.8 |

TABLE VII. The partial width ratios for the $n\bar{n}c\bar{c}$ tetraquarks. For each state, we choose one mode as the reference channel, and the partial width ratios of the other channels are calculated relative to this channel. The masses are all in unit of MeV.

| System | $J^P$ | Scheme I | Scheme II |
|--------|-------|----------|-----------|
| $\bar{D}^*\bar{D}^*\bar{D}D$ | $0^+$ | 3833.2 | 1 |
| | | 4127.4 | 1 | 4364.9 | 10.3 | 1 |
| $\bar{D}^*\bar{D}^*\bar{D}D$ | $1^+$ | 3946.4 | 1 | 4053.2 | 1 |
| | $2^+$ | 4017.1 | | 4123.8 | 1 |
| $\bar{D}^*\bar{D}^*\bar{D}D$ | $I^=0$ | 3749.8 | | 3868.7 | | 1 |
| | | 3976.1 | | 4230.8 | 1.5 | 1 |

TABLE VIII. The eigenvectors of the $n\bar{n}b\bar{b}$ tetraquark states in the $n\bar{n}b\bar{b}$ configuration. The masses are all in units of MeV.

| System | $J^P$ | Scheme I | Scheme II |
|--------|-------|----------|-----------|
| $\bar{B}^*\bar{B}^*\bar{B}\bar{B}$ | $0^+$ | 10468.8 | 0.546 | 10569.3 | -0.227 | 0.533 |
| | | 10808.9 | 0.737 | 11054.6 | 0.729 | 0.364 |
| $\bar{B}^*\bar{B}^*\bar{B}\bar{B}$ | $1^+$ | 10485.3 | 0.408 | 10584.2 | 0 | 0.408 | 0.408 |
| | $2^+$ | 10507.9 | 0.577 | 10606.8 | 0.577 |
| $\bar{B}^*\bar{B}^*\bar{B}\bar{B}$ | $I^=0$ | 10291.6 | -0.374 | 10390.9 | -0.383 | 0.306 | -0.306 |
| | | 10703.4 | 0.600 | 10950.3 | 0.594 | 0.395 | -0.395 |
and the partial width ratios of the other channels are calculated relative to this channel. The masses are all in unit of MeV.

| System   | $J^P$ | Scheme I | Scheme II |
|----------|-------|----------|-----------|
|          | Mass  | $B^+B^*$ | $BB^*$ $BB$ | Mass  | $B^+B^*$ | $BB^*$ $BB$ |
| $(nn\bar{b})^{I=1}$ | $0^+$ | 10468.8  | $\times$ $\times$ | 10569.3 | $\times$ | 66.5 |
|          | 10808.9 | 503.0   | 136.5 | 11054.6 | 788.7 | 216.6 |
|          | $1^+$ | 10485.3  | $\times$ | 10584.2 | $\times$ |
|          | 10507.9 | $\times$ | 10606.8 | $\times$ |
| $(nn\bar{b})^{I=0}$ | $1^+$ | 10291.6  | $\times$ $\times$ | 10390.9 | $\times$ $\times$ |
|          | 10703.4 | 193.6 | 111.0 | 10950.3 | 450.3 | 213.6 |

TABLE X. The partial width ratios for the $nn\bar{b}$ tetraquarks. For each state, we choose one mode as the reference channel, and the partial width ratios of the other channels are calculated relative to this channel. The masses are all in unit of MeV.

| System   | $J^P$ | Scheme I | Scheme II |
|----------|-------|----------|-----------|
|          | Mass  | $B^+B^*$ | $BB^*$ $BB$ | Mass  | $B^+B^*$ | $BB^*$ $BB$ |
| $(nn\bar{b})^{I=1}$ | $0^+$ | 10468.8  | $\times$ $\times$ | 10569.3 | $\times$ | 66.5 |
|          | 10808.9 | 3.7   | 1 | 11054.6 | 3.6 | 1 |
|          | $1^+$ | 10485.3  | $\times$ | 10584.2 | $\times$ |
|          | 10507.9 | $\times$ | 10606.8 | $\times$ |
| $(nn\bar{b})^{I=0}$ | $1^+$ | 10291.6  | $\times$ $\times$ | 10390.9 | $\times$ $\times$ |
|          | 10703.4 | 0.9 | 1 | 10950.3 | 1.1 | 1 |

TABLE XI. The eigenvectors of the $nn\bar{c}$ tetraquark states in the $n\bar{c}\otimes n\bar{b}$ configuration. The masses are all in units of MeV.
TABLE XII. The values of $k \cdot |c_i|^2$ for the $nn\bar{c}$ tetraquarks (in unit of MeV).

| System | $J^P$ | Scheme I | Scheme II |
|--------|-------|----------|-----------|
|        |       | $\bar{D}B^*$ | $D^*B$ | $\bar{D}B$ | $\bar{D}B^*$ | $D^*B$ | $\bar{D}B$ |
| $(nn\bar{c})^{I=1}$ | 0$^+$ | 7189.5 $\times$ | 130.4 | 7305.6 $\times$ | 226.3 |
|        |       | 7440.9 329.3 | 35.6 | 7684.7 590.5 | 99.9 |
|        | 1$^+$ | 7211.0 $\times$ $\times$ | 80.7 | 7322.5 $\times$ 0.004 | 188.4 |
|        |       | 7264.2 $\times$ $\times$ | 3.7 | 7367.3 22.4 | 123.6 | 4.7 |
|        |       | 7417.0 213.2 90.8 | 46.4 | 7665.1 397.6 | 172.5 | 117.9 |
|        | 2$^+$ | 7293.2 $\times$ | | 7396.0 143.3 | |
| $(nn\bar{c})^{I=0}$ | 0$^+$ | 7003.4 $\times$ | $\times$ | 7124.6 $\times$ $\times$ | $\times$ |
|        |       | 7220.3 $\times$ | | 7459.0 169.3 | | 347.6 |
|        | 1$^+$ | 7046.2 $\times$ $\times$ $\times$ | 7158.0 $\times$ $\times$ | 7482.4 55.0 | 132.7 | 367.1 |
|        |       | 7329.3 $\times$ 107.5 9.6 | | 7584.9 271.9 | 320.1 | 16.2 |
|        | 2$^+$ | 7353.2 161.3 | | 7610.3 610.5 | |

TABLE XIII. The partial width ratios for the $nn\bar{c}$ tetraquarks. For each state, we choose one mode as the reference channel, and the partial width ratios of the other channels are calculated relative to this channel. The masses are all in unit of MeV.

| System | $J^P$ | Scheme I | Scheme II |
|--------|-------|----------|-----------|
|        |       | $\bar{D}B^*$ | $D^*B$ | $\bar{D}B$ | $\bar{D}B^*$ | $D^*B$ | $\bar{D}B$ |
| $(nn\bar{c})^{I=1}$ | 0$^+$ | 7189.5 $\times$ | 1 | 7305.6 $\times$ | 1 |
|        |       | 7440.9 9.2 | 1 | 7684.7 5.9 | 1 |
|        | 1$^+$ | 7211.0 $\times$ $\times$ 1 | 7322.5 $\times$ 0.00002 | 1 |
|        |       | 7264.2 $\times$ $\times$ 1 | 7367.3 4.8 | 26.5 | 1 |
|        |       | 7417.0 4.6 2.0 1 | 7665.1 3.4 | 1.5 | 1 |
|        | 2$^+$ | 7293.2 $\times$ | | 7396.0 1 |
| $(nn\bar{c})^{I=0}$ | 0$^+$ | 7003.4 $\times$ | $\times$ | 7124.6 $\times$ $\times$ | $\times$ |
|        |       | 7220.3 $\times$ | 1 | 7459.0 0.5 | 1 |
|        | 1$^+$ | 7046.2 $\times$ $\times$ $\times$ | 7158.0 $\times$ $\times$ $\times$ | 7482.4 0.1 | 0.4 | 1 |
|        |       | 7329.3 $\times$ 11.2 1 | 7584.9 16.8 | 19.7 | 1 |
|        | 2$^+$ | 7353.2 1 | | 7610.3 1 | |
\[
\delta m = \frac{1}{2} \left( m_{nn} + m_{QQ} - m_{nQ} \right) \tag{43}
\]
Taking scheme I as an example, we have
\[
\delta m (nn\bar{c}) = -12.52 \text{ MeV},
\]
\[
\delta m (nn\bar{b}) = -93.07 \text{ MeV},
\]
while for the fully heavy tetraquarks
\[
\delta m (bb\bar{b}) = +42.30 \text{ MeV},
\]
\[
\delta m (cc\bar{c}) = +51.49 \text{ MeV},
\]
\[
\delta m (cc\bar{b}) = +15.15 \text{ MeV}. \tag{48}
\]

As the ratios \(m_{nQ}/m_{n}\)'s increase, the \(3_c \otimes 3_c\) components become more important in the ground states.

Another interesting conclusion from the Hamiltonian is that the color interaction does not mix the \(6_c \otimes 6_c\) and \(3_c \otimes 3_c\) configurations. Actually, this conclusion applies for all S-wave tetraquarks with \(q_1 = q_2\) or \(q_3 = q_4\). Let's consider the matrix element of color interaction \(\langle \alpha | H_{CE} | \beta \rangle\). Note that the color interaction is independent of the spin operator, and thus is a rank-0 tensor in the \(q_1q_2\) spin space. Its matrix elements over different \(q_1q_2\) spin states always vanish. If \(q_1 = q_2\), the Pauli principle further renders the matrix elements vanish unless the bases \(\alpha\) and \(\beta\) possess the same color symmetry over \(q_1q_2\). The same argument works for \(q_3q_4\) as well. In summary, the color interaction does no mix the \(6_c \otimes 6_c\) and \(3_c \otimes 3_c\) color configurations if \(q_1 = q_2\) or \(q_3 = q_4\).

Next we consider their decay properties. For the decays of the tetraquark states in this work, the \((k/m)^2\)'s are all of \(O(10^{-2})\) or even smaller. All higher wave decays are suppressed. Thus we will only consider the S-wave decays in this work. First we transform the wave function of \(nn\bar{Q}\bar{Q}\) tetraquarks into the \(n\bar{Q} \otimes n\bar{Q}\) configuration. Then we can calculate the \(k \cdot |c_i|^2\)'s and partial decay width ratios. The corresponding results are listed in Tables V-X. Note that the two schemes give very similar results, we will mainly focus on the scheme I in the following. In the isovector sector, the \(nn\bar{c}\bar{b}\) tetraquarks are mostly above the S-wave decay channels, thus are wide states. Depending on the schemes, the \(J^P = 2^+\) state may lie on or above the \(D^* D^*\) threshold. Namely the \(T_1(nn\bar{c}\bar{b}, 4017.1, 1.2^+)\) or \(T_{11}(nn\bar{c}\bar{b}, 4123.8, 1.2^+)\). A firm conclusion requires more detailed studies. The \(T_{11}(nn\bar{c}\bar{b}, 4127.4, 1.0^+)\) can decay into both \(DD\) and \(D^* D^*\) channels, with partial decay width ratio
\[
\Gamma_{D^* D^*}/\Gamma_{DD} \sim 37.5. \tag{49}
\]

Thus the \(D^* D^*\) mode is dominant. In the isoscalar sector, the \(T_i(nn\bar{c}\bar{c}, 3749.8, 0.1^+)\) lies below the \(DD^*\) threshold, thus it is a narrow state. However, this state lies above the \(DD\) threshold, so it can decay radiatively into the \(DD\gamma\) final states. The \(T_i(nn\bar{c}\bar{c}, 3976.1, 0.1^+)\) is wide because it lies above the \(DD^*\) and \(D^* D^*\) thresholds. The \(T_i(nn\bar{b}\bar{b}, 10808.9, 1.0^+)\) and \(T_i(nn\bar{b}\bar{b}, 10703.4, 0.1^+)\) lie above all the corresponding S-wave decay channel. Their partial decay width ratios are
\[
\frac{\Gamma[T_i(nn\bar{b}\bar{b}, 10808.9, 1.0^+) \rightarrow B^* B^*]}{\Gamma[T_i(nn\bar{b}\bar{b}, 10808.9, 1.0^+) \rightarrow BB]} \sim 3.7 \tag{50}
\]
and
\[
\frac{\Gamma[T_i(nn\bar{b}\bar{b}, 10703.4, 0.1^+) \rightarrow B^* B^*]}{\Gamma[T_i(nn\bar{b}\bar{b}, 10703.4, 0.1^+) \rightarrow BB]} \sim 0.9 \tag{51}
\]
respectively. Other \(nn\bar{b}\) tetraquarks are all narrow states.

2. The \(nn\bar{c}\bar{b}\) tetraquark

Next we consider the \(nn\bar{c}\bar{b}\) tetraquark. We list the masses and eigenvectors of these states in Table IV. Their relative position and possible decay channels are plotted in Fig. 2. There are two possible stable \(nn\bar{c}\bar{b}\) tetraquark states. The first state is \(T_1(nn\bar{c}\bar{b}, 7003.4, 0.0^+)\), which lies below the \(DB\) threshold by more than 100 MeV. Even in scheme II, this state is about 20 MeV below the threshold. The second state is \(T_1(nn\bar{c}\bar{b}, 7046.2, 0.1^+)\). It is about 100 MeV lighter than the \(DB\) threshold. However, this state lies above the threshold in scheme II. Nonetheless, it lies below its S-wave decay mode \(DB^*\), thus should be a narrow state.

Since the two antiquarks do not have to obey the Pauli principle, we have much bigger number of states than the \(nn\bar{c}\bar{b}/nn\bar{c}\bar{b}\) cases. For each isospin, we have two \(0^+\) states, three \(1^+\) states and one \(2^+\) state. From Table IV, we see that for each possible quantum number, the lower mass states are dominated by color-triplet configurations, while the color-sextet configurations are more important in the higher mass states. For example, the two stable states \(T_1(nn\bar{c}\bar{b}, 7003.4, 0.0^+)\) and \(T_1(nn\bar{c}\bar{b}, 7046.2, 0.1^+)\) have 80.6% and 90.0% of \(3_c \otimes 3_c\) components respectively. This can be explained by the color interaction
\[
\langle H_C (nn\bar{c}\bar{b}) \rangle = m_{n\bar{c}} + m_{n\bar{b}} - \frac{3}{2} \delta m' \langle V_{12}^C + V_{34}^C \rangle \tag{52}
\]
where
\[
\delta m' = \frac{1}{4} (m_{nn} + m_{cb} - m_{n\bar{c}} - m_{n\bar{b}}) = -36.41 \text{ MeV} \tag{53}
\]
Note that both \(6_c \otimes \bar{6}_c\) and \(3_c \otimes 3_c\) configurations are eigenstates of \(V_{12}^C + V_{34}^C\), with eigenvalues \(2/3\) and \(-4/3\) respectively. The negative value of \(\delta m'\) indicates that the color interaction favors the \(3_c \otimes 3_c\) configuration.
In Table XI, we transform the $nn\bar{c}\bar{b}$ tetraquarks into the $n\bar{o}nb$ configuration. Then we calculate the values of $k \cdot |c_i|^2$ and relative partial decay widths, as shown in Tables XII--XIII. Besides the two stable states discussed above, two heavier isoscalar states $T_1(nn\bar{c}\bar{b}, 7220.3, 0^+)$ and $T_1(nn\bar{c}\bar{b}, 7232.9, 0^+)$ are above the $D\bar{B}^*$, while $T_1(nn\bar{c}\bar{b}, 7329.3, 0^+)$ can decay into both $D^*B$ and $DB^*$ modes, with relative width
\[
\Gamma_{D^*B} : \Gamma_{DB^*} \sim 11.2 : 1 \tag{54}
\]
In the isovector sector, the lower $0^+$ state can only decay into $DB$ mode, while the higher one can also decay into $D^*B^*$ mode. All $1^+$ states can decay into $DB^*$ in $S$-wave, while only the highest one can decay into $D^*B$ and $D^*B^*$ modes, with partial decay rates
\[
\Gamma_{D^*B} : \Gamma_{D^*B^*} : \Gamma_{DB^*} \sim 4.6 : 2.0 : 1 \tag{55}
\]
There is no doubt that the current results rely on the mass estimation. In scheme II, the higher masses allow the states to have more decay modes. Yet we find that the partial decay width ratios are quite stable in the two schemes.

C. The $ss\bar{Q}\bar{Q}$ systems

We list the numerical results of the $ss\bar{Q}\bar{Q}$ in Table XIV--XXIII. We also plot the relative position and possible decay channels in Figs. 3–4. The pattern of the mass spectrum is very similar to that of the $nn\bar{Q}\bar{Q}$ tetraquarks with isospin $I = 1$.

First we focus on the $ss\bar{c}\bar{c}$ tetraquarks. The ground state is $T_1(ss\bar{c}\bar{c}, 4043.7, 0^+)$. It can decay into $D_s\bar{D}_s$ in $S$-wave, and thus might be a wide state. The most heavy state $T_1(ss\bar{c}\bar{c}, 4311.1, 0^+)$ lies above the $D_s^*\bar{D}_s^*$ threshold. It decays into $D_s\bar{D}_s$ and $D_s^*\bar{D}_s^*$ modes with the ratios
\[
\frac{\Gamma[T_1(ss\bar{c}\bar{c}, 4311.1, 0^+) \rightarrow D_s\bar{D}_s]}{\Gamma[T_1(ss\bar{c}\bar{c}, 4311.1, 0^+) \rightarrow D_s^*\bar{D}_s^*]} \sim 0.0008 . \tag{56}
\]
Thus the $D_s^*\bar{D}_s^*$ mode is dominant. The other two state $T_1(ss\bar{c}\bar{c}, 4192.6, 1^+)$ and $T_1(ss\bar{c}\bar{c}, 4264.5, 2^+)$ can decay into $D_s\bar{D}_s$ and $D_s^*\bar{D}_s^*$ modes respectively.

Next we turn to the $ss\bar{b}\bar{b}$ tetraquarks. In scheme I, the $T_1(ss\bar{b}\bar{b}, 10697.1, 0^+)$ and $T_1(ss\bar{b}\bar{b}, 10718.2, 1^+)$ lie below the $B_sB_s$ threshold, and the $T_1(ss\bar{b}\bar{b}, 10742.5, 2^+)$ lies just above the $B_sB_s$ threshold, which suggests that they are stable states. However, they become heavier than their $S$-wave decay channels in scheme II. A detailed study with dynamical model, or experimental researches, is required to distinguish which of the two schemes gives better description of the $ss\bar{b}\bar{b}$ tetraquarks. In both schemes, the three states are dominated by $3_s\otimes 3_c$ configuration. Actually, their wave functions are nearly the same, except that they have different total spin, which is the reason for their different masses. The highest state can decay into $B_sB_s$ and $B_s^*B_s^*$ modes, with nearly identical partial width ratios
\[
\frac{\Gamma[T_1(ss\bar{b}\bar{b}, 10928.8, 0^+) \rightarrow B_s^*B_s^*]}{\Gamma[T_1(ss\bar{b}\bar{b}, 10928.8, 0^+) \rightarrow B_sB_s]} \sim 4.4 \tag{57}
\]
and
\[
\frac{\Gamma[T_1(ss\bar{b}\bar{b}, 11154.3, 0^+) \rightarrow B_s^*B_s^*]}{\Gamma[T_1(ss\bar{b}\bar{b}, 11154.3, 0^+) \rightarrow B_sB_s]} \sim 4.1 . \tag{58}
\]

From Fig. 4, we see that the $ss\bar{c}\bar{b}$ tetraquarks are all above the $S$-wave decay channels and are probably broad states. Among them, the $T_1(ss\bar{c}\bar{b}, 7534.3, 2^+)$ is slightly above the $D_s^*B_s^*$, its decay width may be relatively narrow. We also calculate the partial decay width ratios of the $ss\bar{c}\bar{b}$ tetraquarks. It is interesting that some of the ratios are different in the two schemes. For example, in scheme I
\[
\frac{\Gamma[T_1(ss\bar{c}\bar{b}, 7597.3, 0^+) \rightarrow D_s^*B_s^*]}{\Gamma[T_1(ss\bar{c}\bar{b}, 7597.3, 0^+) \rightarrow D_sB_s]} \sim 63.2 \tag{59}
\]
and in scheme II
\[
\frac{\Gamma[T_1(ss\bar{c}\bar{b}, 7818.8, 0^+) \rightarrow D_s^*B_s^*]}{\Gamma[T_1(ss\bar{c}\bar{b}, 7818.8, 0^+) \rightarrow D_sB_s]} \sim 9.1 , \tag{60}
\]
which can be used to distinguish the two schemes.

D. The $ns\bar{Q}\bar{Q}$ systems

We list the masses and wave functions of the $ns\bar{Q}\bar{Q}$ in Table XXIV. The ground states of the $ns\bar{c}\bar{c}$ and $ns\bar{b}\bar{b}$ tetraquarks are both of $1^+$. They are strange counterparts of the $IJ^P = 01^+$ $nn\bar{Q}\bar{Q}$ tetraquarks. Among them, the $T_1(ns\bar{c}\bar{c}, 3919.0, 1^+)$ lies above the $DD_s$ threshold, while the $T_1(ns\bar{b}\bar{b}, 10473.1, 1^+)$ lies deeply below the $BB_s$ threshold. In scheme II, the former one lies above its $S$-wave decay channels $D_s^*D_s$ and $DD_s^*$, while the latter one is still stable. We hope the future experiment can reach for this state.

The last class of the doubly heavy tetraquarks is the $ns\bar{c}\bar{b}$ system. It is composed of four different quarks. Similar to the $nn\bar{c}\bar{b}$ tetraquarks, the ground state of the $ns\bar{c}\bar{b}$ tetraquarks has quantum number $0^+$. Depending on the scheme used, it may be a stable state. A full dynamical quark model study is needed to have a better understanding of these states.

We also study the decay properties of the $ns\bar{Q}\bar{Q}$ tetraquarks, which can be found in Tables XXV–XXXIV.
FIG. 3. Mass spectra of the $ss\bar{c}\bar{c}$ and $ss\bar{b}\bar{b}$ tetraquark states in scheme I (black) and scheme II (blue). The dotted lines indicate various meson-meson thresholds. The masses are all in units of MeV.
TABLE XIV. Masses and eigenvectors of the $ss\bar{c}$, $s\bar{s}b$ and $s\bar{s}\bar{b}$ tetraquarks. The masses are all in units of MeV.

| System | $J^P$ | Scheme I | Scheme II |
|--------|-------|----------|-----------|
|        |       | Mass     | Eigenvector | Mass     | Eigenvector |
| $ss\bar{c}$ | $0^+$ | 4043.7   | $\{0.650, 0.760\}$ | 4199.1   | $\{0.442, 0.897\}$ |
|        |       | 4311.1   | $\{0.760, -0.650\}$ | 4532.1   | $\{0.897, -0.442\}$ |
|        | $1^+$ | 4192.6   | $\{1\}$     | 4300.2   | $\{1\}$     |
|        | $2^+$ | 4264.5   | $\{1\}$     | 4372.1   | $\{1\}$     |
| $s\bar{s}b$ | $0^+$ | 10697.1  | $\{0.196, 0.981\}$ | 10792.1  | $\{0.124, 0.992\}$ |
|        |       | 10928.8  | $\{0.981, -0.196\}$ | 11154.3  | $\{0.992, -0.124\}$ |
|        | $1^+$ | 10718.2  | $\{1\}$     | 10809.8  | $\{1\}$     |
|        | $2^+$ | 10742.5  | $\{1\}$     | 10834.1  | $\{1\}$     |
| $s\bar{s}\bar{b}$ | $0^+$ | 7404.4   | $\{0.547, 0.837\}$ | 7531.3   | $\{0.325, 0.946\}$ |
|        |       | 7597.3   | $\{0.837, -0.547\}$ | 7818.8   | $\{0.946, -0.325\}$ |
|        | $1^+$ | 7431.8   | $\{-0.537, -0.551, 0.639\}$ | 7553.6   | $\{-0.265, -0.662, 0.701\}$ |
|        |       | 7503.3   | $\{-0.096, 0.792, 0.603\}$ | 7603.5   | $\{-0.047, 0.735, 0.677\}$ |
|        |       | 7569.4   | $\{0.838, -0.263, 0.478\}$ | 7795.3   | $\{0.963, -0.146, 0.226\}$ |

TABLE XV. The eigenvectors of the $ss\bar{c}$ tetraquark states in the $\bar{s}c\otimes\bar{s}c$ configuration. The masses are all in units of MeV.

| System | $J^P$ | Scheme I | Scheme II |
|--------|-------|----------|-----------|
|        |       | Mass     | Eigenvector | Mass     | Eigenvector |
| $ss\bar{c}$ | $0^+$ | 4043.7   | 0.240     | 6.45     | 0.054     | 0.629       |
|        |       | 4311.1   | 0.725     | -0.015   | 0.762     | 0.145       |
|        | $1^+$ | 4192.6   | 0        | 0.408    | 0.408     | 4300.2      | 0.408       |
|        | $2^+$ | 4264.5   | 0.577     |         |           | 4372.1      | 0.577       |

TABLE XVI. The values of $k \cdot |c_i|^2$ for the $ss\bar{c}$ tetraquarks (in unit of MeV).

| System | $J^P$ | Scheme I | Scheme II |
|--------|-------|----------|-----------|
|        |       | Mass     | Eigenvector | Mass     | Eigenvector |
| $ss\bar{c}$ | $0^+$ | 4043.7   | $\times$   | 192.5    | $\times$   | 289.1       |
|        |       | 4311.1   | 226.4     | 0.2      | 4532.1     | 476.6       | 23.6       |
|        | $1^+$ | 4192.6   | 80.3      | 4300.2   | 113.0      |             |           |
|        | $2^+$ | 4264.5   | 97.5      | 4372.1   | 187.9      |             |           |
TABLE XVII. The partial width ratios for the $ss\bar{c}\bar{c}$ tetraquarks. For each state, we choose one mode as the reference channel, and the partial width ratios of the other channels are calculated relative to this channel. The masses are all in unit of MeV.

| System $J^P$ | Scheme I | Scheme II |
|--------------|----------|-----------|
| $ss\bar{c}\bar{c}$ | 0$^+$ | 4043.7 | 4199.1 |
| | | $\times$ | $\times$ |
| | | 1 | 1 |
| | 0$^+$ | 4311.1 | 4532.1 |
| | | 1185.7 | 20.2 |
| | | 1 | 1 |
| $ss\bar{c}\bar{c}$ | 1$^+$ | 4192.6 | 4300.2 |
| | | 1 | 1 |
| $ss\bar{c}\bar{c}$ | 2$^+$ | 4264.5 | 4372.1 |
| | | 1 | 1 |

TABLE XVIII. The eigenvectors of the $ss\bar{b}\bar{b}$ tetraquark states in the $s\bar{b}\otimes s\bar{b}$ configuration. The masses are all in units of MeV.

| System $J^P$ | Scheme I | Scheme II |
|--------------|----------|-----------|
| $ss\bar{b}\bar{b}$ | 0$^+$ | 10697.1 | 10792.1 |
| | | $-0.144$ | $-0.199$ |
| | | 0.570 | 10792.1 |
| | | 0.302 | 11154.3 |
| | | 0.737 | 0.343 |
| $ss\bar{b}\bar{b}$ | 1$^+$ | 10718.2 | 10809.8 |
| | | 0 | 0.408 |
| | | 0.408 | 0.408 |
| | | 10809.8 | 0 |
| | | 0.408 | 0.408 |
| $ss\bar{b}\bar{b}$ | 2$^+$ | 10742.5 | 10834.1 |
| | | 0.577 | 0.577 |

TABLE XIX. The values of $k \cdot |c_i|^2$ for the $ss\bar{b}\bar{b}$ tetraquarks (in unit of MeV).

| System $J^P$ | Scheme I | Scheme II |
|--------------|----------|-----------|
| $ss\bar{b}\bar{b}$ | 0$^+$ | 10697.1 | 10792.1 |
| | | $10.8$ | 167.6 |
| | | 0.570 | 0.343 |
| | | 93.8 | 178.5 |
| | 1$^+$ | 10718.2 | 10809.8 |
| | | $\times$ | 64.3 |
| | | 10809.8 | 10834.1 |
| | | 44.4 |
| $ss\bar{b}\bar{b}$ | 2$^+$ | 10742.5 | 10834.1 |
| | | $\times$ | 1 |

TABLE XX. The partial width ratios for the $ss\bar{b}\bar{b}$ tetraquarks. For each state, we choose one mode as the reference channel, and the partial width ratios of the other channels are calculated relative to this channel. The masses are all in unit of MeV.

| System $J^P$ | Scheme I | Scheme II |
|--------------|----------|-----------|
| $ss\bar{b}\bar{b}$ | 0$^+$ | 10697.1 | 10792.1 |
| | | $\times$ | $\times$ |
| | | 1 | 1 |
| | 0$^+$ | 10928.8 | 11154.3 |
| | | 4.4 | 4.1 |
| | | 1 | 1 |
| | 1$^+$ | 10718.2 | 10809.8 |
| | | $\times$ | 1 |
| | | 10809.8 | 10834.1 |
| | | 1 |

IV. CONCLUSIONS

In this work, we systematically study the mass spectrum of the doubly heavy $qq\bar{Q}\bar{Q}$ tetraquarks in the frame-
TABLE XXI. The eigenvectors of the $ss\bar{c}\bar{b}$ tetraquark states in the $s\bar{c}\otimes s\bar{b}$ configuration. The masses are all in units of MeV.

| System | $J^P$ | Scheme I | Scheme II |
|--------|-------|----------|-----------|
|        | Mass  | $\bar{D}_s^*B_s^*$ | $\bar{D}_s^*B_s$ | $\bar{D}_sB_s^*$ | $\bar{D}_sB_s$ |
| $ss\bar{c}\bar{b}$ | 0$^+$ | 7404.4 | 0.145 | 0.642 | 7531.3 | -0.043 | 0.606 |
|        |       | 7597.3 | 0.750 | 0.068 | 7818.8 | 0.763 | 0.224 |
| 1$^+$ | 7431.8 | -0.049 | 0.179 | -0.629 | 7553.6 | 0.133 | 0.040 | -0.581 |
| 1$^+$ | 7503.3 | 0.191 | 0.536 | 0.110 | 7603.5 | 0.249 | 0.515 | 0.085 |
| 2$^+$ | 7569.4 | 0.679 | -0.311 | 0.097 | 7795.3 | 0.648 | -0.388 | 0.268 |
| 2$^+$ | 7534.3 | 0.577 |       |       | 7633.8 | 0.577 |       |       |

TABLE XXII. The values of $k \cdot |c_i|^2$ for the $ss\bar{c}\bar{b}$ tetraquarks (in unit of MeV).

| System | $J^P$ | Scheme I | Scheme II |
|--------|-------|----------|-----------|
|        | Mass  | $\bar{D}_s^*B_s^*$ | $\bar{D}_sB_s^*$ | $\bar{D}_sB_s$ |
| $ss\bar{c}\bar{b}$ | 0$^+$ | 7404.4 | 184.9 | 7531.3 | 0.2 | 279.4 |
|        |       | 7597.3 | 260.2 | 4.1 | 7818.8 | 557.2 | 61.0 |
| 1$^+$ | 7431.8 | × | × | 147.8 | 7553.6 | 5.0 | 0.8 | 239.1 |
| 1$^+$ | 7503.3 | × | 78.3 | 7.2 | 7603.5 | 29.9 | 164.1 | 5.9 |
| 2$^+$ | 7569.4 | 165.0 | 51.1 | 6.9 | 7795.3 | 385.6 | 150.0 | 80.7 |
| 2$^+$ | 7534.3 | 47.8 |       |       | 7633.8 | 190.8 |       |       |

Parameters are fitted from the mesons and baryons. Since the spatial configurations of the $qq$ ($\bar{q}\bar{q}$) and $q\bar{q}$ pairs are different in the conventional hadrons and the tetraquarks, applying these parameters to the tetraquarks may cause errors. To appreciate this uncertainty, we adopt two schemes of parameters to study the $qqQ\bar{Q}$ tetraquarks. As indicated in Eq. (41), the scheme II gives larger masses than the scheme I. However, the wave functions and decay properties of the two schemes are very similar for the $qqQ\bar{Q}$ tetraquarks. We find three states which are stable in both schemes. They are the $n\bar{c}b\bar{b}$ tetraquark with quantum number $J^P = 0^+$, the $n\bar{c}b$ tetraquark with quantum number $J^P = 0^+$ and the $n\bar{c}b\bar{b}$ tetraquark with quantum number $J^P = 1^+$. They all lie below the thresholds of two pseudoscalar mesons, which can only decay through weak processes. Meanwhile, many narrow states which lie below $S$-wave decay channels are also found. It shall be interesting to search for these states.

The tetraquarks have two possible color configurations, namely the color-sextet configuration |$(qq)^6(\bar{Q}\bar{Q})^6\rangle$ and the color-triplet one |$(qq)^3(\bar{Q}\bar{Q})^3\rangle$. Unlike the fully heavy tetraquarks, the ground states of the doubly heavy tetraquarks favor the color-triplet configurations. Combining the results of fully and doubly heavy tetraquarks, we can clearly see the trend that the color-triplet configuration is more and more important when the mass ratio between the quarks and antiquarks increases.

Besides the mass spectrum, we also estimate the decay properties of the tetraquarks. We hope these states can be searched for by future experiments.

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Appendix A: Wave function in the $q\bar{q}\otimes q\bar{q}$ configuration

To calculate the partial decay rates, we need to construct the tetraquark wave functions in the $q\bar{q}\otimes q\bar{q}$ configuration. The possible color-spin wave functions $\{\beta_i\}$ are listed as follows,

1. $J^P = 0^+$
\[
\beta^0_i = |(q_1\bar{q}_3)^8(q_2\bar{q}_4)^8\rangle_0
\]
and the partial width ratios of the other channels are calculated relative to this channel. The masses are all in unit of MeV.

TABLE XXIII. The partial width ratios for the $ssar{b}$ tetraquarks. For each state, we choose one mode as the reference channel, and the partial width ratios of the other channels are calculated relative to this channel. The masses are all in unit of MeV.

| System $J^P$ | Scheme I | Scheme II |
|---------------|----------|-----------|
|               | $D_s^* B_s^*$ | $D_s B_s^*$ | $D_s B_s$ |
| $ssar{b}$   | $0^+$     | 7597.3 63.2 1 | 7531.3 0.0007 1 | 7818.8 9.1 1 |
|               |           | 7503.3 19.0 1 | 7795.3 4.8 1.9 1 |
| $ssar{b}$   | $1^+$     | 7534.3 1 1 | 7634.8 1 1 |
|               |           | 7531 1 1 |
|               |           | 7432 1 1 |

FIG. 4. Mass spectra of the $ssar{b}$ tetraquark states in scheme I (black) and scheme II (blue). The dotted lines indicate various meson-meson thresholds. The masses are all in units of MeV.

$\beta_2^0 = |(q_1 q_3)^0 (q_2 q_4)^0_{1/0}|
\beta_3^0 = |(q_1 q_3)^0 (q_2 q_4)^0_{1/0}|
\beta_3^0 = |(q_1 q_3)^0 (q_2 q_4)^0_{1/0}|

2. $J^P = 1^+$
\begin{align*}
\beta_1^1 &= |(q_1 q_3)^1 (q_2 q_4)^1_{1/1}|
\beta_2^2 &= |(q_1 q_3)^1 (q_2 q_4)^1_{1/1}|
\beta_3^3 &= |(q_1 q_3)^1 (q_2 q_4)^1_{1/1}|
\beta_4^4 &= |(q_1 q_3)^1 (q_2 q_4)^1_{1/1}|
\end{align*}

3. $J^P = 2^+$
\begin{align*}
\beta_1^2 &= |(q_1 q_3)^2 (q_2 q_4)^2_{1/2}|
\beta_2^2 &= |(q_1 q_3)^2 (q_2 q_4)^2_{1/2}|
\end{align*}

where the superscript 1 or 8 denotes the color, and the subscript 0, 1 or 2 denotes the spin. Among them, the
TABLE XXIV. Masses and eigenvectors of the \( ns\bar{c}c\), \( ns\bar{b}b\) and \( ns\bar{c}\) tetraquarks. The masses are all in units of MeV.

| System \( J^P \) | Scheme I | Scheme II |
|------------------|----------|-----------|
|                  | Mass     | Eigenvector | Mass     | Eigenvector |
| \( ns\bar{c} \) 0⁺ | 3937.6   | \( \{0.606, 0.795\} \) | 4085.7   | \( \{0.410, 0.912\} \) |
|                  | 4209.3   | \( \{0.795, -0.606\} \) | 4436.2   | \( \{0.912, -0.410\} \) |
| 1⁺               | 3919.0   | \( \{-0.534, -0.006, 0.845\} \) | 4051.5   | \( \{0.284, 0.005, -0.959\} \) |
|                  | 4073.0   | \( \{-0.032, -0.999, -0.027\} \) | 4180.2   | \( \{0.003, 0.99997, 0.007\} \) |
|                  | 4086.5   | \( \{0.845, -0.041, 0.534\} \) | 4328.9   | \( \{0.959, -0.005, 0.284\} \) |
| 2⁺               | 4144.3   | \( \{1\} \) | 4251.5   | \( \{1\} \) |
| \( ns\bar{b} \) 0⁺ | 10586.4  | \( \{0.163, 0.987\} \) | 10684.1  | \( \{0.107, 0.994\} \) |
|                  | 10854.6  | \( \{0.987, -0.163\} \) | 11090.2  | \( \{0.994, -0.107\} \) |
| 1⁺               | 10473.1  | \{0.082, 0.005, -0.997\} | 10569.0  | \{0.056, 0.005, -0.998\} |
|                  | 10605.3  | \{0.007, 0.99996, 0.006\} | 10700.5  | \{0.004, 0.99998, 0.005\} |
|                  | 10778.7  | \{0.997, -0.007, 0.082\} | 11016.1  | \{0.998, -0.004, 0.056\} |
| 2⁺               | 10628.7  | \{1\} | 10723.9  | \{1\} |
| \( ns\bar{c} \) 0⁺ | 7156.5   | \{0.647, 0.029, 0.038, 0.761\} | 7296.2   | \{0.375, 0.036, 0.050, 0.925\} |
|                  | 7299.0   | \{0.030, 0.473, 0.876, -0.087\} | 7421.1   | \{0.044, 0.283, 0.955, -0.080\} |
| 1⁺               | 7333.2   | \{-0.762, 0.070, 0.052, 0.642\} | 7547.8   | \{-0.926, 0.055, 0.058, 0.370\} |
|                  | 7506.6   | \{-0.023, -0.878, 0.477, 0.029\} | 7738.5   | \{-0.026, -0.957, 0.287, 0.032\} |
| 2⁺               | 7212.8   | \{0.423, -0.343, 0.031, 0.024, -0.025, 0.838\} | 7363.1   | \{0.181, -0.185, 0.035, 0.023, -0.030, 0.965\} |
|                  | 7323.9   | \{-0.210, 0.057, -0.413, -0.589, 0.635, 0.180\} | 7440.9   | \{-0.044, 0.030, -0.225, -0.675, 0.698, 0.060\} |
|                  | 7330.5   | \{-0.818, 0.168, 0.161, 0.095, -0.225, 0.466\} | 7487.8   | \{-0.089, -0.033, 0.041, -0.720, -0.687, 0.005\} |
|                  | 7386.1   | \{0.004, 0.224, -0.056, 0.749, 0.615, 0.089\} | 7565.3   | \{0.901, -0.351, -0.047, -0.090, -0.010, -0.233\} |
|                  | 7416.2   | \{0.329, 0.894, 0.062, -0.169, -0.144, 0.198\} | 7665.0   | \{-0.381, -0.916, -0.042, 0.039, 0.048, -0.102\} |
|                  | 7479.9   | \{0.013, -0.040, 0.892, -0.233, 0.383, -0.038\} | 7716.8   | \{-0.014, 0.042, -0.971, 0.129, -0.194, 0.037\} |
| 2⁺               | 7415.1   | \{0.297, 0.955\} | 7517.7   | \{0.064, 0.998\} |
|                  | 7439.0   | \{0.955, -0.297\} | 7690.6   | \{0.998, -0.064\} |

TABLE XXV. The eigenvectors of the \( ns\bar{c}c\) tetraquark states in the \( n\bar{c}\circ s\bar{c}\) configuration. The masses are all in units of MeV.

| System \( J^P \) | Scheme I | Scheme II |
|------------------|----------|-----------|
|                  | Mass     | \( D^* \bar{D}^* \) | \( D^* \bar{D}_s \) | \( \bar{D} \bar{D}^*_s \) | Mass     | \( D^* \bar{D}^*_s \) | \( D^* \bar{D}_s \) | \( \bar{D} \bar{D}^*_s \) |
| \( ns\bar{c} \) 0⁺ | 3937.6   | 0.199     | 0.645     | 4085.7   | 0.026     | 0.623     |
|                  | 4209.3   | 0.737     | 0.022     | 4436.2   | 0.763     | 0.168     |
| 1⁺               | 3919.0   | 0.036     | -0.464    | 0.460    | 4051.5   | -0.227    | 0.395     | -0.391    |
|                  | 4073.0   | -0.029    | -0.413    | -0.403   | 4180.2   | -0.005    | -0.408    | -0.409    |
|                  | 4086.5   | 0.706     | 0.174     | -0.207   | 4328.9   | 0.670     | 0.307     | -0.311    |
| 2⁺               | 4144.3   | 0.577     | 0.625     | 0.577    | 4251.5   | 0.577     | 0.625     | 0.577     |
TABLE XXVI. The values of $k: |c_i|^2$ for the $n\bar{s}\bar{c}$ tetraquarks (in unit of MeV).

| System $J^P$ | Scheme I | Scheme II |
|--------------|-----------|-----------|
| $n\bar{s}\bar{c}$ 0$^+$ | 3937.6 × | 185.3 4085.7 × | 273.5 |
| | 4209.3 233.6 | 0.4 4436.2 478.6 | 31.3 |
| 1$^+$ | 3919.0 × × × | 4051.5 × 60.4 58.0 |
| | 4073.0 × 75.1 70.3 | 4180.2 0.008 107.1 106.8 |
| | 4086.5 × 14.2 20.0 | 4328.9 297.3 80.7 82.5 |
| 2$^+$ | 4144.3 73.7 | 4251.5 174.4 |

TABLE XXVII. The partial width ratios for the $n\bar{s}\bar{c}$ tetraquarks. For each state, we choose one mode as the reference channel, and the partial width ratios of the other channels are calculated relative to this channel. The masses are all in unit of MeV.

| System $J^P$ | Scheme I | Scheme II |
|--------------|-----------|-----------|
| $n\bar{s}\bar{c}$ 0$^+$ | 3937.6 × | 1 4085.7 × | 1 |
| | 4209.3 569.1 | 1 4436.2 15.3 | 1 |
| 1$^+$ | 3919.0 × × × | 4051.5 × 1.04 1 |
| | 4073.0 × 1.1 1 | 4180.2 0.00007 1.003 1 |
| | 4086.5 × 0.7 1 | 4328.9 3.6 0.98 1 |
| 2$^+$ | 4144.3 1 | 4251.5 1 |

TABLE XXVIII. The eigenvectors of the $n\bar{s}\bar{b}$ tetraquark states in the $n\bar{b}\otimes s\bar{b}$ configuration. The masses are all in units of MeV.

| System $J^P$ | Scheme I | Scheme II |
|--------------|-----------|-----------|
| $n\bar{s}\bar{b}$ 0$^+$ | 10586.4 −0.170 | 0.560 10684.1 −0.212 | 0.541 |
| | 10854.6 0.745 | 0.321 11090.2 0.734 | 0.353 |
| 1$^+$ | 10473.1 −0.360 0.323 −0.319 | 10569.0 −0.375 0.313 −0.309 |
| | 10605.3 −0.006 −0.409 −0.407 | 10700.5 −0.004 −0.408 −0.408 |
| | 10778.7 0.609 0.380 −0.386 | 11016.1 0.599 0.390 −0.393 |
| 2$^+$ | 10628.7 0.577 | 10723.9 0.577 |
TABLE XXX. The partial width ratios for the $ns\bar{b}b$ tetraquarks. For each state, we choose one mode as the reference channel, and the partial width ratios of the other channels are calculated relative to this channel. The masses are all in unit of MeV.

| System $J^P$ | Scheme I | Scheme II |
|--------------|----------|-----------|
| $ns\bar{b}$ | \begin{tabular}{c|ccc|c}
$0^+$ & 10586.4 & $\times$ & $\times$ & 10684.1 \\ 
& 10854.6 & 436.2 & 109.4 & 11090.2 \\ 
& 10473.1 & $\times$ & $\times$ & 10569.0 \\ 
& 10605.3 & $\times$ & $\times$ & 10700.5 \\ 
& 10778.7 & 168.9 & 99.0 & 10116.1 \\ 
& 10628.7 & $\times$ & $\times$ & 10723.9 \\
\end{tabular} | \begin{tabular}{c|ccc|c}
$0^+$ & 36.6 & 36.6 & 36.6 & 36.6 \\
& 28.9 & 28.9 & 28.9 & 28.9 \\
$1^+$ & 1.3 & 1.3 & 1.3 & 1.3 \\
& 1 & 1 & 1 & 1 \\
$2^+$ & 1 & 1 & 1 & 1 \\
\end{tabular} |

TABLE XXXI. The eigenvectors of the $ns\bar{b}b$ tetraquark states in the $n\bar{c}\bar{s}b$ configuration. The masses are all in units of MeV.

| System $J^P$ | Scheme I | Scheme II |
|--------------|----------|-----------|
| $ns\bar{b}$ | \begin{tabular}{c|ccc|c}
$0^+$ & 7156.5 & 0.126 & 0.708 & 7296.2 \\ 
& 7299.0 & $-0.026$ & $-0.627$ & 7421.1 \\ 
& 7333.2 & $-0.666$ & 0.299 & 7547.8 \\ 
& 7506.6 & $-0.735$ & $-0.128$ & 7738.5 \\
\end{tabular} | \begin{tabular}{c|ccc|c}
$0^+$ & 7212.8 & 0.152 & $-0.148$ & 7336.1 \\ 
& 7323.9 & 0.127 & $-0.039$ & 7440.9 \\ 
& 7330.5 & 0.289 & $-0.630$ & 7487.8 \\ 
& 7386.1 & 0.384 & 0.574 & 7565.3 \\ 
& 7416.2 & 0.574 & 0.362 & 7665.0 \\ 
& 7479.9 & 0.633 & $-0.346$ & 7716.8 \\
\end{tabular} |

TABLE XXXII. The eigenvectors of the $nsc\bar{b}$ tetraquark states in the $n\bar{c}\bar{s}b$ configuration. The masses are all in units of MeV.

| System $J^P$ | Scheme I | Scheme II |
|--------------|----------|-----------|
| $nsc\bar{b}$ | \begin{tabular}{c|ccc|c}
$0^+$ & 7156.5 & 0.126 & 0.708 & 7296.2 \\ 
& 7299.0 & $-0.026$ & $-0.627$ & 7421.1 \\ 
& 7333.2 & $-0.666$ & 0.299 & 7547.8 \\ 
& 7506.6 & $-0.735$ & $-0.128$ & 7738.5 \\
\end{tabular} | \begin{tabular}{c|ccc|c}
$0^+$ & 7212.8 & 0.152 & $-0.148$ & 7336.1 \\ 
& 7323.9 & 0.127 & $-0.039$ & 7440.9 \\ 
& 7330.5 & 0.289 & $-0.630$ & 7487.8 \\ 
& 7386.1 & 0.384 & 0.574 & 7565.3 \\ 
& 7416.2 & 0.574 & 0.362 & 7665.0 \\ 
& 7479.9 & 0.633 & $-0.346$ & 7716.8 \\
\end{tabular} |
TABLE XXXII. The eigenvectors of the \( ns\bar{c} \) tetraquark states in the \( n\bar{b} \otimes s\bar{c} \) configuration. The masses are all in units of MeV.

| System | \( J^P \) | Scheme I | Scheme II |
|--------|---------|---------|-----------|
|        | Mass   | \( B^*\bar{D}_s^* \) | \( B^*\bar{D}_s \) | \( B\bar{D}_s \) | Mass   | \( B^*\bar{D}_s^* \) | \( B^*\bar{D}_s \) | \( B\bar{D}_s \) |
| \( ns\bar{c} \) | \( 0^+ \) | 7156.5 | 0.107 | 0.646 | 7296.2 | -0.298 | -0.493 | 7299.0 | 0.137 | 0.635 | 7421.1 | -0.018 | 0.584 | 7333.2 | -0.598 | 0.407 | 7547.8 | -0.541 | 0.599 | 7506.6 | 0.783 | 0.112 | 7738.5 | 0.786 | 0.238 |
|        | \( 1^+ \) | 7212.8 | -0.136 | 0.596 | -0.128 | 7336.1 | -0.279 | 0.426 | -0.236 | 7323.9 | -0.086 | 0.500 | -0.261 | 7440.9 | 0.114 | 0.549 | -0.048 | 7330.5 | -0.286 | -0.576 | -0.447 | 7487.8 | -0.240 | 0.042 | 0.443 | 7386.1 | 0.054 | -0.169 | -0.438 | 7565.3 | 0.266 | 0.649 | 0.465 | 7416.2 | -0.621 | -0.116 | 0.634 | 7665.0 | 0.566 | 0.140 | -0.612 | 7479.9 | 0.710 | -0.146 | 0.351 | 7716.8 | -0.679 | 0.273 | -0.395 | 7415.1 | -0.308 | 7517.7 | -0.524 |
|        | \( 2^+ \) | 7439.0 | 0.951 | 7690.6 | 0.852 |
(a) $ns\bar{c}\bar{c}$ states

(b) $ns\bar{b}\bar{b}$ states

FIG. 5. Mass spectra of the $ns\bar{c}\bar{c}$ and $ns\bar{b}\bar{b}$ tetraquark states in scheme I (black) and scheme II (blue). The dotted lines indicate various meson-meson thresholds. The masses are all in units of MeV.
TABLE XXXIII. The values of $k \cdot |c_i|^2$ for the $ns\bar{c}\bar{b}$ tetraquarks (in unit of MeV).

| System $J^P$ | Scheme I | Scheme II |
|--------------|-----------|-----------|
|              | $\bar{D}' \bar{B}' \bar{D} \bar{B}$ | $\bar{D}' \bar{B}' \bar{D} \bar{B}$ |
| $ns\bar{c}\bar{b}$ $0^+$ | 7156.5 | 7296.2 |
|              | 7299.0 | 167.7 |
|              | 7333.2 | 47.1 |
|              | 7506.6 | 14.5 |
| 1$^+$ | 7212.8 | 7336.1 |
|              | 7323.9 | 159.4 |
|              | 7330.5 | 20.6 |
|              | 7386.1 | 58.5 |
|              | 7416.2 | 45.3 |
|              | 7479.9 | 66.8 |
| 2$^+$ | 7415.1 | 7517.7 |
|              | 7439.0 | 7690.6 |

| System $J^P$ | Scheme I | Scheme II |
|--------------|-----------|-----------|
|              | $B'^* \bar{D}'* \bar{D}' \bar{D}$ | $B'^* \bar{D}'* \bar{D}' \bar{D}$ |
| $ns\bar{c}\bar{b}$ $0^+$ | 7156.5 | 7296.2 |
|              | 7299.0 | 155.3 |
|              | 7333.2 | 82.7 |
|              | 7506.6 | 11.0 |
| 1$^+$ | 7212.8 | 7336.1 |
|              | 7323.9 | 740.9 |
|              | 7330.5 | 109.2 |
|              | 7386.1 | 14.8 |
|              | 7416.2 | 8.0 |
|              | 7479.9 | 182.6 |
| 2$^+$ | 7415.1 | 7517.7 |
|              | 7439.0 | 7690.6 |
TABLE XXXIV. The partial width ratios for the $ns\bar{c}\bar{b}$ tetraquarks. For each state, we choose one mode as the reference channel, and the partial width ratios of the other channels are calculated relative to this channel. The masses are all in unit of MeV.

| System | $J^P$ | Scheme I | Scheme II |
|--------|-------|----------|-----------|
|        |       | $D^*B_s^*$ | $DB_s^*$ | $DB_s$ |
| $ns\bar{c}\bar{b}$ | $0^+$ | 7156.5 | 7296.2 | 1 |
| | | 7299.0 | 7421.1 | 1 |
| | | 7333.2 | 7547.8 | 0.9 |
| | | 7506.6 | 7738.5 | 6.6 |
| | $1^+$ | 7212.8 | 7336.1 | 1 |
| | | 7323.9 | 7440.9 | 0.04 |
| | | 7330.5 | 7487.8 | 3.2 |
| | | 7386.1 | 7565.3 | 0.3 |
| | | 7416.2 | 7665.0 | 22.3 |
| | | 7479.9 | 7716.8 | 3.3 |
| | $2^+$ | 7415.1 | 7517.7 | 1 |
| | | 7439.0 | 7690.6 | 1 |

| System | $J^P$ | Scheme I | Scheme II |
|--------|-------|----------|-----------|
|        |       | $B^*D_s^*$ | $BD_s^*$ | $BD_s$ |
| $ns\bar{c}\bar{b}$ | $0^+$ | 7156.5 | 7296.2 | 1 |
| | | 7299.0 | 7421.1 | 1 |
| | | 7333.2 | 7547.8 | 0.5 |
| | | 7506.6 | 7738.5 | 8.7 |
| | $1^+$ | 7212.8 | 7336.1 | 1 |
| | | 7323.9 | 7440.9 | 0.007 |
| | | 7330.5 | 7487.8 | 16.7 |
| | | 7386.1 | 7565.3 | 0.1 |
| | | 7416.2 | 7665.0 | 13.0 |
| | | 7479.9 | 7716.8 | 5.1 |
| | $2^+$ | 7415.1 | 7517.7 | 1 |
| | | 7439.0 | 7690.6 | 1 |

$1_c\otimes 1_c$ bases can also be written as combinations of two mesons. For example, $|(n_1\bar{c}_3)^{1}_1 (n_2\bar{c}_4)^{1}_1)_{J}\equiv |\bar{D}^*\bar{D}^*\rangle_{J}$.

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