Global conformal gauge in string theory

M. O. Katanaev *

Steklov Mathematical Institute,
ul. Gubkina, 8, Moscow, 119991, Russia

August 3, 2021

Abstract

It was supposed for fifty years that the conformal gauge in string theory exists globally on the whole string world sheet. In fact, almost all results were obtained under this assumption. However, this statement was proved only locally in some neighbourhood of an arbitrary point on the worlds sheet, and its extension is far from being obvious. In the present paper we prove that the conformal gauge does exists globally on the string worlds sheet represented by an infinite strip with straight parallel boundaries.

Introduction. The (super)string theory is one of the main fields of research in mathematical physics for the last fifty years (see, e.g., [1, 2, 3]). The basic assumption in this model is that the conformal gauge exists globally on the whole string worlds sheet represented by infinite strip with straight boundaries. For example, this assumption provides a basis for covariant and light cone quantization. In fact, we can say that the existence of the global conformal gauge is crucial for the theory of strings.

After fixing the conformal gauge the (super)string becomes a consistent fruitful and very interesting model which deserves analysis by itself. However the question remains: is there a solution of the original covariant equations of motion of the Nambu–Goto string which cannot be brought to the conformally flat form? In the present paper, we prove the existence of the global conformal gauge on the whole string worlds sheet represented by infinite strip with straight boundaries. This justifies the assumption made long ago.

The local existence of the conformal gauge is well known for a long time (see, e.g., [4, 5]). It is proved by writing down equations for transformation functions and considering their integrability conditions which guarantee the existence of solution in some neighbourhood of an arbitrary point. Its global extension is far from being obvious. The transition from local to global considerations in the present paper is based on the global existence theorem for the solution of the Cauchy problem for a two-dimensional hyperbolic differential equations with varying coefficients (see, e.g., [7]). This theorem is highly nontrivial, but allows one to make global statements.

The bosonic string. Consider two manifolds: a plane $\mathbb{R}^2$ with arbitrary global coordinates $x = (x^\alpha) := (x^0, x^1) := (\tau, \sigma)$, $\alpha = 0, 1$, and $D$-dimensional Minkowskian
space $\mathbb{R}^{1,D-1}$ with Cartesian coordinates $X = (X^\lambda)$, $\lambda = 0, 1, \ldots, D-1$, $D \geq 2$, and the Lorentz metric $\eta_{ab} := \text{diag}(+-\ldots)$. Let there be a smooth embedding

$$X : \mathbb{R}^2 \supset \overline{U} \ni (\tau, \sigma) \mapsto (X^\lambda(\tau, \sigma)) \in \mathbb{R}^{1,D-1},$$

(1)

of some closed subset $\overline{U}$ of a plane where $U$ is a connected and simply connected open subset in $\mathbb{R}^2$. This embedding defines symmetric quadratic form with components

$$h_{\alpha\beta} := \partial_\alpha X^a \partial_\beta X^b \eta_{ab} = \partial_\alpha X^\lambda \partial_\beta X_\lambda$$

(2)

on the string worldsheet $M := X(\overline{U})$. We assume that the embedding is such that

$$(\partial_0 X)^2 := \dot{X}^2 := \dot{X}^\lambda \dot{X}^\mu \eta_{\lambda\mu} > 0,$$

$$(\partial_1 X)^2 := X'^2 := X'^\lambda X'^\mu \eta_{\lambda\mu} < 0,$$

(3)

where the dot and prime denote differentiations with respect to $\tau$ and $\sigma$, respectively, on $U$. Here and in what follows indices $\lambda, \mu, \ldots$ are often omitted. So global coordinates $\tau, \sigma$ on $U$ are timelike and spacelike, respectively. Then the determinant of the induced quadratic form is negative

$$h := \det h_{\alpha\beta} = \dot{X}^2 X'^2 - (\dot{X}, X')^2 < 0,$$

(4)

where parenthesis denote the usual scalar product in $\mathbb{R}^{1,D-1}$. Now the embedding (1) defines the Lorentzian metric on the string worldsheet interior $U$ with signature $(+\ldots)$. Open string is the embedding (1) of the closed straight strip

$$-\infty < \tau < \infty, \quad 0 \leq \sigma \leq \pi$$

(5)

with properties (3). This strip is vertical if $\tau$ and $\sigma$ coordinate axes are depicted by vertical and horizontal straight lines on a plain $\mathbb{R}^2$, respectively.

Closed string is the embedding (1) of the closed straight vertical strip (fundamental domain)

$$-\infty < \tau < \infty, \quad -\pi \leq \sigma \leq \pi$$

(6)

with identified boundaries. There are many ways to identify smoothly the boundaries (6). In string theory, we, first, impose the conformal gauge on the metric on the same strip (6) and, second, impose the smooth periodicity conditions

$$\partial_k X^\lambda|_{\sigma=-\pi} = \partial_k X^\lambda|_{\sigma=\pi}, \quad \forall \lambda, \forall \tau, \quad k = 0, 1, 2, \ldots,$$

(7)

up to the needed order. It is the prime aim of the present paper to prove that the conformal gauge on the same strips does exist.

If infinite strips in the $\tau, \sigma$ plane have curved boundaries, then all of them are diffeomorphic to strips (5) or (6). Thus we did not loose generality by specifying the coordinate range on the $\tau, \sigma$ plane.

The dynamics of the Nambu–Goto string is governed by the action proportional to the string worldsheet area

$$S_{\text{NG}} := -\int_{\sigma} dx \sqrt{|h|} = -\int_{\sigma} d\tau d\sigma \sqrt{(\dot{X}, X')^2 - \dot{X}^2 X'^2}.$$  

(8)
It implies the Euler–Lagrange equations
\[
\frac{1}{\sqrt{|h|}} \frac{\delta S_{NG}}{\delta X^\alpha} = \Box_{(h)} X^\alpha = h^\alpha{}_{\beta} \nabla_\alpha \nabla_\beta X^\Lambda = \frac{1}{\sqrt{|h|}} \partial_\alpha \left( \sqrt{|h|} h^\alpha{}_{\beta} \partial_\beta X^\Lambda \right) = 0, \tag{9}
\]
where the two-dimensional wave operator \(\Box_{(h)}\) is built by the induced metric \(h_{\alpha\beta}\) and \(\nabla_\alpha\) is the covariant derivative with respective Christoffel’s symbols.

We assume that ends of an open string are free, and then the action (8) implies also the boundary conditions
\[
s^\beta \partial_\beta X^\alpha |_{\sigma=0,\pi} = 0, \tag{10}
\]
where \(s^\alpha\) are components of the spacelike vector which is perpendicular to the boundaries with respect to the induced metric. The action (8) does not yield any boundary condition for a closed string. Instead, we have periodicity conditions (7) imposed by hands.

In string theory, the crucial role is played by the possibility to impose global conformal gauge
\[
h_{\alpha\beta} = e^{2\phi} \eta_{\alpha\beta}, \quad \eta_{\alpha\beta} := \text{diag}(+-), \tag{11}
\]
where \(\phi(x)\) is some sufficiently smooth function, on the whole string worldsheet. The aim of the present paper is to prove that this conformal gauge can be imposed on the same strips (5) and (6) both for open and closed strings with the same straight boundaries.

**The idea of the proof.** We construct two orthogonal vector fields: the timelike \(t = t^\alpha \partial_\alpha\) and spacelike \(s = s^\alpha \partial_\alpha\) with properties
\[
(t,s) = 0, \quad t^2 + s^2 = 0, \quad t^2 > 0, \quad \forall x \in U, \tag{12}
\]
where
\[
(t,s) := t^\alpha s^\beta h_{\alpha\beta}, \quad t^2 := (t,t), \quad s^2 := (s,s).
\]
Then we find conditions for commutativity of these vector fields: \([t,s] = 0\). The next step is to find two families of integral curves \(x^\alpha(\bar{\tau},\bar{\sigma})\) defined by the system of equations
\[
\frac{\partial x^\alpha}{\partial \bar{\tau}} = t^\alpha, \quad \frac{\partial x^\alpha}{\partial \bar{\sigma}} = s^\alpha, \tag{13}
\]
where \(\bar{\tau}\) and \(\bar{\sigma}\) are parameters along integral curves of vector fields \(t\) and \(s\), respectively. The integrability conditions for this system are fulfilled on the whole \(U\):
\[
\frac{\partial^2 x^\alpha}{\partial \bar{\tau} \partial \bar{\sigma}} - \frac{\partial^2 x^\alpha}{\partial \bar{\sigma} \partial \bar{\tau}} = \frac{\partial s^\alpha}{\partial \bar{\tau}} - \frac{\partial t^\alpha}{\partial \bar{\sigma}} = t^\beta \partial_\beta s^\alpha - s^\beta \partial_\beta t^\alpha = [t,s]^\alpha = 0,
\]
due to the commutativity of vector fields. Consequently, there is a nondegenerate coordinate transformation \((\tau,\sigma) \mapsto (\bar{\tau},\bar{\sigma})\) on the whole worldsheet \(U\).

In the new coordinate system, the induced metric \(\tilde{h}_{\alpha\beta}\) is conformally flat due to Eqs.(12):
\[
\tilde{h}_{00} = h_{\alpha\beta} \frac{\partial x^\alpha}{\partial \bar{\tau}} \frac{\partial x^\beta}{\partial \bar{\tau}} = t^2, \\
\tilde{h}_{01} = h_{\alpha\beta} \frac{\partial x^\alpha}{\partial \bar{\tau}} \frac{\partial x^\beta}{\partial \bar{\sigma}} = (t,s) = 0, \\
\tilde{h}_{11} = h_{\alpha\beta} \frac{\partial x^\alpha}{\partial \bar{\sigma}} \frac{\partial x^\beta}{\partial \bar{\sigma}} = s^2 = -t^2. \tag{14}
\]
The final step is the analysis of domains of the definition of parameters \( \tilde{\tau} \) and \( \tilde{\sigma} \) of integral curves (13) which are new coordinates.

**Infinite string.** First, we consider embedding (1) where \( U = \mathbb{R}^2 \). Arbitrary timelike and spacelike tangent vectors \( T \) and \( S \) to the string worldsheet in the embedding space \( \mathbb{R}^{1,D-1} \) can be decomposed on \( X \) and \( X' \):

\[
T = A(\cosh \varphi \dot{X} + \sinh \varphi X'), \\
S = B(\sinh \psi \dot{X} + \cosh \psi X'),
\]

where \( A(x) > 0, B(x) > 0 \) and \( \varphi(x), \psi(x) \in \mathbb{R} \) are some functions.

**Lemma 0.1.** Vector fields \( T \) and \( S \) on \( U \) satisfy equalities

\[
(T, S) = 0, \quad T^2 + S^2 = 0,
\]

if and only if vector field \( S \) is given by Eq.(15) with arbitrary functions \( B > 0 \) and \( \psi \in \mathbb{R} \), and the second vector field has the form

\[
T = -\frac{B}{\sqrt{|h|}} \left[ \cosh \psi \dot{X}^2 + \sinh \psi (\dot{X}, X') \right] \dot{X} + \frac{B}{\sqrt{|h|}} \left[ \sinh \psi \dot{X}^2 + \cosh \psi (\dot{X}, X') \right] X'.
\]

The proof is given in [6].

The differential map of the embedding \( U \hookrightarrow \mathbb{R}^{1,D-1} \) acts on vectors as follows

\[
T(U) \ni t^\alpha \partial_\alpha, \quad s^\alpha \partial_\alpha \mapsto T = t^\alpha \partial_\alpha X^\alpha \partial_\alpha, \quad S = s^\alpha \partial_\alpha X^\alpha \partial_\alpha \in T(\mathbb{R}^{1,D-1}),
\]

where \( t \) and \( s \) are vector fields on \( U \). Comparing the above formulae with Eqs.(15) and (17) allows us to define vector fields on \( U \):

\[
t = -\frac{B}{\sqrt{|h|}} \left[ \cosh \psi \dot{X}^2 + \sinh \psi (\dot{X}, X') \right] \partial_0 + \frac{B}{\sqrt{|h|}} \left[ \sinh \psi \dot{X}^2 + \cosh \psi (\dot{X}, X') \right] \partial_1, \\
s = B \sinh \psi \partial_0 + B \cosh \psi \partial_1.
\]

These vectors can be easily rewritten in the form

\[
t = \varepsilon^{\alpha\beta} s_\beta \partial_\alpha, \quad s = s^\alpha \partial_\alpha,
\]

where \( \varepsilon^{\alpha\beta} \) is the totally antisymmetric second rank tensor, \( \varepsilon^{01} = -1/\sqrt{|h|} \), and components \( s^\alpha \) are arbitrary.

**Lemma 0.2.** Vector fields \( t \) and \( s \) on \( U \) related by equalities (19) commute if and only if

\[
t_\alpha = \frac{\partial_\alpha \chi}{\partial \chi^2}, \quad \partial \chi^2 := h^{\alpha\beta} \partial_\alpha \chi \partial_\beta \chi > 0,
\]

where \( \chi \) is a nontrivial solution of the wave equation

\[
\Box_{(h)} \chi := h^{\alpha\beta} \nabla_\alpha \nabla_\beta \chi = 0,
\]

satisfying \( \partial \chi^2 > 0 \). For any nontrivial solution of Eq.(21) satisfying the condition \( \partial \chi^2 > 0 \), vector fields (19) and (20) commute.
The proof is given in [6].

Thus commuting vector fields $t$ and $s$ with properties (12) have generally the following form

$$
t = h^{\alpha\beta} \frac{\partial}{\partial \chi^\beta} \partial_\alpha, \quad s = \varepsilon^{\alpha\beta} \frac{\partial}{\partial \chi^\beta} \partial_\alpha, \tag{22}$$

where $\chi$ is an arbitrary solution of the wave equation (21) such that $\partial \chi^2 > 0$.

Suppose that the determinant of the induced metric $h_{\alpha\beta}$ in nonzero on the whole plane $(\tau, \sigma) \in \mathbb{R}^2$ and separated from 0 and $\pm \infty$ at infinity:

$$0 < \epsilon \leq \lim_{\tau^2 + \sigma^2 \to \infty} |\det h_{\alpha\beta}| \leq M < \infty, \tag{23}$$

where $\epsilon$ and $M$ are some constants. It is well known that the Cauchy problem for the hyperbolic equation (21) has unique solution $\chi$ on the whole plane, if the Cauchy data are given on a spacelike curve, say, $\tau = 0$ (see, e.g., [7], book IV, ch. I). It is easily verified that there exist such Cauchy data that the inequality $\partial \chi^2 > 0$ holds everywhere. This implies that nontrivial solution of the wave equation (21) exists on the whole plane $\mathbb{R}^2$. There are many such solutions, and they are parameterized by the Cauchy data.

Thus the vector fields $s$ and $t$ are given on the whole plane $\mathbb{R}^2$. The inequality $t^2 > 0$ (12) implies that component $t^0$ is separated from zero and bounded on the plane including infinity. Thus Eqs. (13) imply

$$\frac{\partial \tau}{\partial \tilde{\tau}} = t^0 \quad \Rightarrow \quad \tilde{\tau} \sim \int_{t^0}^\infty \frac{d\tau}{t^0}.$$  

The last integral is divergent and thus the coordinate $\tilde{\tau}$ runs over the whole real line $\mathbb{R}$. Similar statement is valid for the space coordinate $\tilde{\sigma}$. Consequently, new coordinates cover the whole plane $(\tilde{\tau}, \tilde{\sigma}) \in \mathbb{R}^2$.

Moreover, it is known that any surface with a Lorentzian metric can be globally isometrically embedded in flat Minkowskian space $\mathbb{R}^{1,D-1}$ of sufficiently large dimension. Since the dimension $D$ in Lemma 0.1 is not fixed, we obtain

**Theorem 0.1.** Let an arbitrary metric $h_{\alpha\beta}$ of Lorentzian signature be given on the whole plane $\mathbb{R}^2$. Let it be nondegenerate at infinity (23). Then there exists a surjective diffeomorphism on the plane

$$\mathbb{R}^2 \ni (x^\alpha) \mapsto (\tilde{x}^\alpha(x)) \in \mathbb{R}^2 \tag{24}$$

such that metric $h_{\alpha\beta}$ in new coordinate system has conformally flat form

$$\tilde{h}_{\alpha\beta} := h_{\gamma\delta} \frac{\partial x^\gamma}{\partial \tilde{x}^\alpha} \frac{\partial x^\delta}{\partial \tilde{x}^\beta} = e^{2\phi} \eta_{\alpha\beta}, \tag{25}$$

where $\phi(\tilde{x})$ is some function on $\mathbb{R}^2$ separated from $\pm \infty$ at infinity $\tilde{\tau}^2 + \tilde{\sigma}^2 \to \infty$.

**Corollary.** Let

$$\tilde{U}_0 := \{(\tilde{\tau}, \tilde{\sigma}) \in \mathbb{R}^2 : \quad \tilde{\sigma} \in [\tilde{\sigma}_1, \tilde{\sigma}_2], \quad \tilde{\tau} \in \mathbb{R}\} \tag{26}$$

be closed vertical strip with straight boundaries on the plane of new coordinates $\tilde{\tau}, \tilde{\sigma}$ and assumptions of theorem 0.1 hold. Then there exists diffeomorphism (24) of a closed domain $(\tau, \sigma) \in \tilde{U} \subset \mathbb{R}^2$ bounded by integral curves $x(\tau, \tilde{\sigma}_1, 2)$:

$$\frac{\partial x(\tilde{\tau}, \tilde{\sigma}_{1,2})}{\partial \tilde{\tau}} = t_{1,2},$$

where $t_{1,2}$ are inverse images of vector fields $\partial/\partial \tilde{\tau}$ to the boundaries of $\tilde{U}_0.$
To clarify the arbitrariness in coordinates $\tilde{\tau}, \tilde{\sigma}$ defined by the function $\chi$ we consider

**Example.** Let the induced metric be conformally flat:

$$h_{\alpha\beta}dx^\alpha dx^\beta = e^{2\phi}(d\tau^2 - d\sigma^2) = e^{2\phi}d\xi d\eta,$$

where light cone coordinates $\xi := \tau + \sigma, \eta := \tau - \sigma$ are introduced. Then the wave equation (21) is reduced to the flat d’Alembert equation $(\partial_0^2 - \partial_1^2)\chi = 0$. Its general solution is given by two arbitrary sufficiently smooth functions

$$\chi = F(\xi) + G(\eta).$$

We choose only the functions which satisfy inequality

$$\partial^2 \chi = 4 e^{-2\phi} F' G' > 0 \quad \Rightarrow \quad F' G' > 0,$$

where prime denotes differentiation by the corresponding argument. Then the metric is

$$\tilde{h}_{\alpha\beta}d\tilde{x}^\alpha d\tilde{x}^\beta = \frac{e^{2\phi}}{4F' G'}d\tilde{\xi} d\tilde{\eta}.$$

It corresponds to the conformal transformation

$$\tilde{\xi} := 2F(\xi), \quad \tilde{\eta} := 2G(\eta).$$

We see that the arbitrariness in definition of the vector fields described in Lemmas 0.1 and 0.2 corresponds to conformal transformations on the string worldsheet.

Thus to find the diffeomorphism (24) in explicit form for a given metric $h_{\alpha\beta}$, we have

(i) find a nontrivial solution of the wave equation (21),
(ii) construct the vector fields $t$ and $s$ using Eqs. (20), (19), and
(iii) find a general solution of the system of equations (13).

We have proved that this problem does have many solutions (the whole arbitrariness is contained in the choice of nontrivial solution of the wave equation).

**Open string.** Now we consider an open string which worldsheet $\overline{U}$ is an infinite strip on the plane $(\tau, \sigma) \in \mathbb{R}^2$ with two, probably, curved left $\gamma_l$ and right $\gamma_r$ boundaries. The induced metric on the boundaries is degenerate, and results of the previous section must be revised. First, we assume that metric is not degenerate and return to this problem later.

If the metric is nondegenerate on $\overline{U}$ including the boundaries, then we continue it on the whole plain in some sufficiently smooth manner. As the consequence of theorem 0.1 there is a diffeomorphism (24) after which the metric becomes conformally flat. The problem is that the boundaries $\gamma_{l,r}$ on the plain $\tilde{\tau}, \tilde{\sigma}$ may be not straight vertical lines. However there are residual diffeomorphisms in the form of conformal maps of $\tilde{\tau}, \tilde{\sigma}$ coordinates. We now show that it is enough to straighten the strip.

Let boundary equations after diffeomorphism $(\tau, \sigma) \mapsto (\tilde{\tau}, \tilde{\sigma})$ be

$$\gamma_l : \tilde{\eta} = \tilde{\eta}_l(\tilde{\xi}), \quad \gamma_r : \tilde{\eta} = \tilde{\eta}_r(\tilde{\xi}), \quad \tilde{\xi} \in \mathbb{R},$$

where functions $\tilde{\eta}_{l,r} \in C^1(\mathbb{R})$ have properties:

$$\tilde{\eta}_l > \tilde{\eta}_r, \quad 0 < \epsilon \leq \frac{d\tilde{\eta}_{l,r}}{d\tilde{\xi}} \leq M < \infty, \quad \epsilon, M \in \mathbb{R}$$

for all $\tilde{\xi} \in \mathbb{R}$ including infinite points.
Theorem 0.2. The conformal transformation

\[ \hat{\xi} = F(\tilde{\xi}), \quad \hat{\eta} = G(\tilde{\eta}), \quad F, G \in C^1(\mathbb{R}), \]

such that the boundaries (28) of an open string worldsheet become straight vertical lines

\[ \gamma_L : \quad \hat{\eta} = \hat{\xi}, \quad \gamma_R : \quad \hat{\eta} = \hat{\xi} - 2\pi, \quad \hat{\xi} \in \mathbb{R} \]

(29)
on the plain \( \hat{\xi}, \hat{\eta} \in \mathbb{R}^2 \) exists.

The proof is given in [6].

Now we discuss an open Nambu–Goto string for which the induced metric on the boundaries is degenerate due to boundary conditions. Let us parameterize metric \( h_{\alpha\beta} \) by its determinant \( \varrho^4 \) and “unimodular metric” \( k_{\alpha\beta} \):

\[ h_{\alpha\beta} := \varrho^2 k_{\alpha\beta}, \quad \det k_{\alpha\beta} := -1. \]  

(30)

It implies that the variable \( \varrho \) is the scalar density of degree \(-1/2\) and unimodular metric is the second rank tensor density of degree 1.

The boundary condition (10) has the form

\[ n^0 \dot{X}^\alpha + n^1 X^\alpha = 0 \quad \Rightarrow \quad X^\alpha = -\frac{n^0}{n^1} \dot{X}^\alpha, \]

because the normal vector \((n^0, n^1)\) must be spacelike and consequently \( n^1 \neq 0 \). As the consequence, the metric degenerates on the boundaries:

\[ \det h_{\alpha\beta} = \rho^4 = \dot{X}^2 X'^2 - (\dot{X}, X')^2 \to 0. \]

Therefore vector fields \( t \) and \( s \) (12) become null

\[ t^2 = \rho^2 k_{\alpha\beta} t^\alpha t^\beta \to 0, \quad s^2 = \rho^2 k_{\alpha\beta} s^\alpha s^\beta \to 0 \]

at the ends of the string because \( \rho \to 0 \).

Now we construct new vector fields for the unimodular metric \( k_{\alpha\beta} \) satisfying relations

\[ (t, s) := k_{\alpha\beta} t^\alpha s^\beta = 0, \quad t^2 + s^2 = 0, \]

(31)

where \( t^2 := k_{\alpha\beta} t^\alpha t^\beta \) and \( s^2 := k_{\alpha\beta} s^\alpha s^\beta \).

Equalities (31) are equivalent to original equations (12) in internal points of \( \tilde{U} \) and extended on boundaries \( \partial \tilde{U} \) by continuity. Now we prove that new vector fields \( t \) and \( s \) exist and coincide with the original ones for \( h_{\alpha\beta} \).

Formulae (14) for metric components after the coordinate transformation have the same form. In addition,

\[ \tilde{h}_{00} = -\tilde{h}_{11} = \varrho^2 k_{\alpha\beta} t^\alpha t^\beta, \quad \tilde{h}_{01} = 0. \]

Function \( \chi \) in new variables must satisfy the equation which does not depend on \( \rho \) [6]:

\[ \square_{(k)} \chi := \partial_\alpha (k^{\alpha\beta} \partial_\beta \chi) = 0. \]  

(32)
This wave equation has many solutions on the whole plain $\mathbb{R}^2$ because depends on non-degenerate unimodular metric $k_{\alpha\beta}$. It implies that vector fields $t$ and $s$ exist and do not depend on $\rho$:

$$t^\alpha = \frac{k^{\alpha\beta} \partial_\beta \chi}{k^{\gamma\delta} \partial_\gamma \partial_\delta \chi}, \quad s^\alpha = \frac{\varepsilon^{\alpha\beta} \partial_\beta \chi}{k^{\gamma\delta} \partial_\gamma \partial_\delta \chi},$$

(33)

where

$$\varepsilon^{\alpha\beta} = \rho^2 \varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is the totally antisymmetric tensor density of degree $-1$.

One can easily verify that in new variables the Euler–Lagrange equations for bosonic string (9) do not depend on $\rho$ too.

They must be solved with the boundary condition

$$n^\alpha \partial_\alpha X^A \big|_{\tau = \pm} = 0,$$

(35)

which does not depend on $\rho$ too.

Consequently, the problem is reduced to solution of Eqs.(34) with boundary conditions (35) for an open Nambu–Goto string. To make the transformation of coordinates $(\tau, \sigma) \mapsto (\hat{\tau}, \hat{\sigma})$ we have to find the unimodular metric $k_{\alpha\beta}$ for a given metric $h_{\alpha\beta}$, choose a solution $\chi$ of the wave equation (32) satisfying the condition $k^{\gamma\delta} \partial_\gamma \partial_\delta \chi > 0$, construct the vector fields $t$ and $s$ using formulae (33), and, finally, integrate Eqs.(13). Therefore the corollary of theorem 0.1 is valid also for an open string. If needed, after solution of this problem, one can compute the conformal factor for the induced metric (25) which goes to zero at the boundaries. Thus we proved the existence of the global conformal gauge for an open string.

**Closed string.** In the initial coordinates $\tau, \sigma \in \mathbb{R}^2$, the worldsheet of a closed string is given by an infinite strip with timelike boundaries which are identified. The identification can be performed in many ways and therefore requires definition. Here we describe the method adopted in string theory.

We showed in the previous section that there is the global diffeomorphism $(\tau, \sigma) \mapsto (\hat{\tau}, \hat{\sigma})$ which maps an arbitrary infinite strip with timelike boundaries on the vertical strip where metric takes the conformally flat form. The same procedure can be performed for the fundamental domain of a closed string. Without loss of generality, we assume that boundaries go through points $\hat{\sigma} = \pm \pi$. Then the boundary identification is written as the periodicity condition:

$$\frac{\partial^k X^A}{\partial \hat{\sigma}^k} \big|_{\hat{\sigma} = -\pi} = \frac{\partial^k X^A}{\partial \hat{\sigma}^k} \big|_{\hat{\sigma} = \pi}, \quad \forall A, \forall \hat{\tau}, \quad k = 0, 1, 2, \ldots.$$  

(36)

In the initial coordinate system this condition is written in the covariant form

$$\nabla_s^k X^A \big|_{\hat{\sigma} = -\pi} = \nabla_s^k X^A \big|_{\hat{\sigma} = \pi}.$$  

(37)

where $\nabla_s := s^\alpha \nabla_\alpha$ is the covariant derivative for the Levi–Civita connection along the normal vector field $s$ which is the pullback of the vector field $\partial/\partial \hat{\sigma}$ under the diffeomorphism $(\tau, \sigma) \mapsto (\hat{\tau}, \hat{\sigma})$. However it is not clear at the beginning for which value of $\tau$ on the
left and right the identification takes place, because we have to find the diffeomorphism 
\((\tau, \sigma) \mapsto (\hat{\tau}, \hat{\sigma})\) explicitly. Simply speaking we firstly transform the metric to conformally flat form and afterwards perform the natural gluing.

**Conclusion.** It was assumed for many years that there exists the global conformal gauge in string theory though this statement was proved only locally. We proved the global existence of the conformal gauge for infinite, open, and closed strings. As a byproduct, we proved global existence of the conformal gauge for a general two-dimensional Lorentzian metric which is not necessarily induced by an embedding.

**References**

[1] M. B. Green, J. H. Schwarz, and E. Witten. *Superstring theory*, volume 1,2. Cambridge U.P., Cambridge, 1987.

[2] L. Brink and M. Henneaux. *Principles of String Theory*. Plenum Press, New York and London, 1988.

[3] B. M. Barbashov and V. V. Nesterenko. *Introduction to the relativistic string theory*. World Scientific, Singapore, 1990.

[4] I. G. Petrovsky. *Lectures on Partial Differential Equations*. Dover Publications Inc., New York – London, 1992.

[5] V. S. Vladimirov. *Equations of Mathematical Physics*. Marcel Dekker, New York, 1971.

[6] M. O. Katanaev. On the existence of the global conformal gauge in string theory. arXiv:1912.08052 [physics.gen-ph].

[7] J. Hadamard. *Le problème de Cauchy et les équations aux dérivées partielles linéaires hyperboliques*. Hermann, Paris, 1932.