3x + 1 DYNAMICS ON RATIONALS
WITH FIXED DENOMINATOR

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Abstract. We propose the existence of an infinite class of exact analogues of the 3x+1 conjecture for rational numbers with fixed denominators. For some other denominators, there are several attracting cycles, which exhibit scaling and covariance phenomena. We analyze these phenomena in terms of results of Bohm, Sontacchi and Lagarias.

1. Introduction

We consider dynamical systems on sets of rational numbers with fixed denominator under the 3x + 1 function. We present numerical evidence that there is a (probably infinite) class of these dynamical systems satisfying an exact analogue of the 3x + 1 conjecture. We also discuss regularities displayed by attractors among dynamical systems of this kind that do not happen to satisfy a 3x + 1 conjecture.

The 3x + 1 conjecture may be stated as follows. Let $T: \mathbb{Z} \to \mathbb{Z}$ satisfy

$$T(x) =\begin{cases} x/2, & \text{for } x \text{ even} \\ (3x + 1)/2, & \text{for } x \text{ odd.} \end{cases}$$

Let $T^n(k)$ denote the $n$th iterate of $T$ at the integer $k$. The pair $(1, 2)$ is an attracting cycle for $T$. The conjecture states that the basin of attraction of this cycle is the entire set of positive integers. In other words, if $k \in \mathbb{Z}^+$, then $T^n(k) = 1$ for some $n$. The literature on this problem was surveyed in [Lagarias 1985] and [Lagarias 1998]. Generalizations are discussed in those papers and also in [Matthews 2001].

It is natural to consider the effect of $T$ on non-positive integers. There have been only a few other cycles found in this way, namely, those containing 0, −1, −5, and −17. It is also natural to consider extensions of $T$ to rational numbers. The set of attracting cycles under such extensions is the subject of [Lagarias 1990]. It is also discussed in [Matthews 2001]. Some observations in [Matthews 2001] are very much in the spirit of the present paper. Results in [Lagarias 1990] are used below to explain some of our own observations. We use the notation of [Lagarias 1990] when it is convenient.

In the sequel, fractions are understood to be in lowest terms unless otherwise noted. Following Lagarias, we define $\mathbb{Q}[(2)]$ as the ring consisting of fractions $j/k$.
with $k$ odd, and consider $j/k \in \mathbb{Q}[(2)]$ to be even or odd according to the parity of $j$. Then $T$ has an extension taking $\mathbb{Q}[(2)]$ to $\mathbb{Q}[(2)]$, which we also denote as $T$.

Let $D_k \subset \mathbb{Q}[(2)]$ be the subset consisting of the fractions $j/k$, $j \in \mathbb{Z}^+$. We say that $j/k$ has “denominator $k$”. The function $T$ preserves $D_k$ when $k \equiv 1$ or 5 mod 6; we restrict our attention to these cases. The $3x + 1$ conjecture tells us that $T$ has exactly one attracting cycle on $D_1$, the pair $(1, 2)$, and that the basin of attraction of $(1, 2)$ contains $D_1$. (This way of putting it depends upon the convention, which we follow throughout, that we count cycles that are irreducible in the sense of [Lagarias 1990]—cycles for which the cyclic permutations are distinct.) The behavior of $T$ on the sets $D_k$ for certain $k > 1$ provides exact analogues of the original conjecture. For example:

**Conjecture 1.1.**

i. The cycle of length 4 containing $5/7$ is the only attractor for $D_7$, and its basin of attraction contains $D_7$.

ii. The cycle of length 11 containing $5/19$ is the only attractor for $D_{19}$, and its basin of attraction contains $D_{19}$.

iii. The cycle of length 23 containing $13/31$ is the only attractor for $D_{31}$, and its basin of attraction contains $D_{31}$.

The data also support, in our opinion,

**Conjecture 1.2.**

The set of positive integers $k$ such that $D_k$ lies in the basin of exactly one attractor is infinite.

We present numerical evidence for the two conjectures in the final section. We will not call the following a conjecture:

**Question 1.3.**

Are there divergent orbits for any of the $D_k$?

No divergent orbits have turned up in our experiments. On the other hand, no argument for a negative answer is known to the present writer.

The behavior of $T$ on $D_k$ for certain other values of $k$ is more complex, and exhibits a structure. For these $D_k$ there are several families of attractors, the members of which have equal length, and the lengths associated to certain of these families are multiples of the lengths associated to others. The number of odd elements exhibits the same scaling phenomenon as the length. Our analysis will explain this scaling, although some questions remain. For example, within the range of our observations, usually, but not always, the number of odd elements of an attracting cycle and the length of the cycle vary together. This covariance remains enigmatic.

## 2. Invariants of Attracting Cycles

With a rational number $x \in \mathbb{Q}[(2)]$, we associate the sequence

$$b(x) = \{b_j(x)\}_{j=0,1,...} \in \{0, 1\}^{\mathbb{Z}_{\geq 0}}$$

by setting $b_j(x) = 0$ or 1, according to whether $T^j(x)$ is even or odd. This is the **parity sequence associated to** $x$. 

Similarly, with any vector $c = (x_0, x_1, ..., x_{n-1})$ representing a cycle on $\mathbb{Q}[2]$, we associate a vector $v = v(c) = (v_0, v_1, ..., v_{n-1}) \in \{0, 1\}^n$ (which we term, following Lagarias, a “0–1 vector”) by setting $v_i = 0$ if $x_i$ is even, and $v_i = 1$ if $x_i$ is odd. We term this the parity vector associated to $c$. This association is one-to-one. The following uniqueness theorem (according to Lagarias, “essentially due to Böhm and Sontacchi” [Böhm and Sontacchi 1978]) appears in [Lagarias 1990] (p. 36). We will refer to it as the

**BSL Theorem.** Given any 0–1 vector $v = (v_0, v_1, ..., v_{n-1})$ there is a unique $x$ in $\mathbb{Q}[2]$ which is periodic of period $n$ under iteration by the 3x + 1 function $T$, and whose parity sequence starts with $v$. It is given by

$$x = x(v) = (2^n - 3^{v_0 + ... + v_{n-1}} - 1 \sum_{j=0}^{n-1} v_j 3^{v_j + 1 + ... + v_{n-1}}2^j.$$  

(2.1)

Evidently, $n$ is the length of $v$. We introduce the notation $\lambda(v) := n$. We introduce two more invariants: $\omega(v) := \sum_{j=0}^{n-1} v_j$, and $\rho(v) := \sum_{j=0}^{n-1} v_j 3^{v_j + 1 + ... + v_{n-1}}2^j$. Thus (suppressing the dependence on $v$) we have

$$x = \frac{\rho}{2^\lambda - 3^{\omega}}.$$  

(2.2)

3. **The Number of Attractors for a Given Denominator**

Let $\alpha(k)$ be the number of attractors for $T$ on $D_k$, and let $\alpha_n(k)$ be the number of such attractors of length $n$. The BSL Theorem allows an interpretation of the numbers $\alpha(k)$ and $\alpha_n(k)$. Let $V(k)$ be the set of 0–1 vectors $v$ such that $k$ = the denominator of $x(v)$, i.e. $k = (2^\lambda - 3^\omega)/(\rho, 2^\lambda - 3^\omega)$. Let $\nu(k)$ be the size of $V(k)$. Let $V_n(k)$ be the subset of $V(k)$ with $\lambda = n$, and let $\nu_n(k)$ be the size of $V_n(k)$. Then

**Proposition 3.1.**

1. $\alpha(k)$ is finite if and only if $\nu(k)$ is finite.
2. $\alpha(k) \neq 0$ if and only if $\nu(k) \neq 0$.
3. $\alpha_n(k) = \nu_n(k)/n$.
4. $\alpha(k) = \sum_{n=1}^{\infty} \nu_n(k)/n$.

**Proof** Only the third claim needs an argument. If $v \in V_n(k)$, the cyclic permutations of $v$ are distinct. Otherwise, distinct cyclic permutations of the attractor containing $x(v)$ correspond to non-distinct cyclic permutations of $v$, and this is ruled out by the BSL Theorem. Furthermore, the cyclic permutations of $v$ all lie in $V_n(k)$, because they correspond to cyclic permutations of the attractor starting with $x(v)$, and, if $k \equiv 1 \text{ or } 5 \mod 6$, all numbers in the cycle corresponding to $v$ have denominator $k$. □

We can use Proposition 3.1 and a result of Lagarias to derive an identity for $\nu_n$. Let $I(n)$ count the number of irreducible cycles of length $n$. It is easy to see that the denominators $k$ of cycles in $\mathbb{Q}[2]$ must all satisfy $k \equiv 1 \text{ or } 5 \mod 6$, so

$$I(n) = \sum_{k=1}^{\infty} \alpha_n(k) = \sum_{k=1}^{\infty} \nu_n(k)/n.$$  

Now, Eq. 2.5 of [Lagarias 1990] (p. 38) states:

$$I(n) = \frac{1}{n} \sum_{d|n} \mu(d)2^{n/d}.$$  

By combining these facts, we arrive at
Proposition 3.2.

\[ \sum_{k=1}^{\infty} \nu_n(k) = \sum_{d\mid n} \mu(d)2^{n/d}. \]

The quantity \( \nu_n(k), \ n \) fixed, vanishes for large \( k \) (obvious). It would be interesting to have a good bound on \( k \) to replace the upper bound in the left-hand sum.

### 4. Scaling, Covariance and Repetition

The following table illustrates typical behavior of the parameters we studied. The denominator is specified by the parameter \( k \). We specify attractors \( c \) by the numerators of their smallest members. For example, the first row conveys the following: there is a cycle for \( D_5 \) with smallest member \( 1/5 \), length three and one odd element. Since the parameters \( \lambda(v) \) and \( \omega(v) \) are invariant under cyclic permutation of \( v \), they are, in effect, invariants of \( c(v) \), namely, the length of \( c \) and the number of odd elements of \( c \), respectively. We treat them as such in this table and later.

| \( k \) | \( c \) | \( \lambda(c) \) | \( \omega(c) \) |
|---|---|---|---|
| 5  | 1  | 3  | 1  |
| 5  | 19 | 5  | 3  |
| 5  | 23 | 5  | 3  |
| 5  | 187| 27 | 17 |
| 5  | 347| 27 | 17 |
| 7  | 5  | 4  | 2  |
| 11 | 1  | 6  | 2  |
| 11 | 13 | 14 | 8  |
| 13 | 1  | 4  | 1  |
| 13 | 131| 24 | 15 |
| 13 | 211| 8  | 5  |
| 13 | 227| 8  | 5  |
| 13 | 259| 8  | 5  |
| 13 | 251| 8  | 5  |
| 13 | 283| 8  | 5  |
| 13 | 287| 8  | 5  |
| 13 | 319| 8  | 5  |

We will refer to a search among the fractions \( j/k, 1 \leq j \leq N \) as a search of depth \( N \). For the given \( k \), the table lists all attractors \( c \) for \( D_k \) such that some of the fractions surveyed in a search of depth 500 fall into \( c \).

The \( k = 13 \) rows illustrate the phenomenon we term scaling: (1) for a given \( D_k \) and a more or less numerous set of attractors of equal length, the ratios \( \lambda/\omega \) are identical and are equal to the corresponding ratio for a single additional attractor of greater length. (2) For a given denominator, when two attractors \( c_1 \) and \( c_2 \) satisfy \( \lambda(c_1)/\omega(c_1) = \lambda(c_2)/\omega(c_2) \), it is also the case that \( \lambda(c_1)/\lambda(c_2) \) and \( \omega(c_1)/\omega(c_2) \) are integers. The \( k = 5, 11 \) rows exhibit dynamical systems without scaling. Of course
it is possible that sufficiently deep searches would turn up scaling for every $k$. This would refute both Conjectures 1 and 2. Covariance is our term for the apparent interdependence for a fixed $D_k$ of $\omega(c)$ and $\lambda(c)$: they seem to vary together.

There are exceptions to this behavior, but its frequent occurrence seems to require an explanation. The smallest examples in which covariance fails are given in the following table.

| $k$ | $c$ | $\lambda(c)$ | $\omega(c)$ |
|-----|-----|--------------|-------------|
| 511 | 11  | 54           | 24          |
| 511 | 293 | 45           | 24          |
| 757 | 43  | 84           | 37          |
| 757 | 85  | 84           | 46          |

Finally, we see in our data repetition of identical $\lambda$ and $\omega$ values for distinct attractors of a single denominator. Our first table supplies several examples. In a search of depth 50 among all denominators $k \equiv 1$ or $5 \mod 6, 1 \leq k \leq 1501$, we found 121 examples of scaling and 205 examples of repetition. There were 83 denominators exhibiting both of these behaviors.

Some of this we can explain, but not all of it. Even if our conjectures are correct, we can ask whether there are $D_k$ with more than one attractor that do not exhibit scaling. We do not know the answer. Below, we explain the existence of scaling, but not the pattern in which a single long attractor shares the same $\lambda/\omega$ ratio as a larger set of short attractors. We cannot explain covariance. We offer speculation instead. From the BSL Theorem, it is clear that the parity vectors for different $c$ are never related by cyclic permutations, yet some families of attracting cycles do appear to be related by non-cyclic permutations of their parity vectors; superficially this is what covariance comes down to. We can ask: do these permutations form a group or some other structure? A positive answer might create a situation in which we could predict the existence of cycles missed by our searches (because of their insufficient depth) to fill out empty slots in such a structure.

Next, we give an account of scaling and repetition that covers all the examples of these phenomena in our data. We cannot rule out the existence of sporadic examples that come about some other way. Let $k\mid J$ and $J = dk$. To a representation of $J$ in the form $J = 2^s - 3^t$, we associate a possibly empty set $V_{\lambda,\omega,d}$ of $0 - 1$ vectors $v$ with the properties that $\lambda(v) = \lambda, \omega(v) = \omega$, and $(\rho(v), J) = d$. For these $v$, the BSL Theorem provides an attractor $c(v)$ with $\lambda(c(v)) = \lambda$ and $\omega(c(v)) = \omega$, containing $x(v) = \frac{\rho(v)}{J} \in D_{J/d} = D_k$.

If it happens that there is a pair $v_1, v_2$ in $V_{\lambda,\omega,d}$ that are not related by a cyclic permutation, then $c(v_1) \neq c(v_2)$ but $\lambda(c(v_1)) = \lambda(c(v_2))$ and $\omega(c(v_1)) = \omega(c(v_2))$, which is repetition.

Now suppose that $s = \lambda \delta, t = \omega \delta \in \mathbb{Z}^+$. Then $k\mid 2^s - 3^t = M = fk$ (say). If $V_{\lambda,\omega,d}$ is non-empty then fractions $x(v)$ and attractors $c(v)$ exist satisfying the conditions above. If also some $w \in V_{s,t,f}$, then so do fractions $x(w)$ and attractors $c(w)$ with $x(w) \in c(w)$, $\lambda(c(w)) = s, \omega(c(w)) = t$, and $x(w) = \frac{\rho(w)}{M} \in D_{M/f} = D_k$. 


Scaling arises because $s/\lambda = t/\omega$. The integrality we mentioned above arises because $\delta$ is an integer. One may ask whether the integrality in turn reflects unknown properties of the parity vectors.

5. Some evidence for Conjectures 1.1 and 1.2

The evidence we can offer for Conjecture 1.1 is straightforward. We executed a depth $3 \times 10^6$ search for denominator 7, a depth $1.3 \times 10^6$ search for denominator 19 and a depth $10^6$ search for denominator 31 without turning up a second attracting cycle or a divergent orbit.

To test Conjecture 1.2, we searched for denominators $k, 1 \leq k \leq 2000$, with a single attracting cycle. We conducted searches of varying depth and examined the effect of this variation. In the following table, we display the number $A$ of denominators in the given range showing just one attracting cycle as a function of the search depth $N$.

\begin{center}
\begin{tabular}{|c|c|c|c|}
\hline
$N$ & $A$ & $N$ & $A$ \\
\hline
20 & 213 & 800 & 166 \\
50 & 184 & 1600 & 166 \\
100 & 181 & 2400 & 166 \\
200 & 176 & 3200 & 162 \\
400 & 172 & & \\
\hline
\end{tabular}
\end{center}

The data is not obviously inconsistent with a model of the form $A = c_1 + f(N), 0 \leq f(N) \leq (2000 - c_1) \exp(-c_2 N)$, for constants $c_1, c_2$. Of course, this model, if it is correct, tends to confirm Conjecture 1.2. The Mathematica Statistics package NonlinearFit command picks out the values $c_1 = 171.594, c_2 = 0.189263$ for this data set and the cruder model $A = c_1 + (2000 - c_1) \exp(-c_2 N)$, but it is insensitive to large perturbations of the data, and so we need not take the particular choice seriously. However, Mathematica refrained from issuing a warning message, not a bad sign, at least, and the only measure we have of the goodness of the model’s fit.

References

[Böhm and Sontacchi 1978] C. Böhm and G. Sontacchi, *On the existence of cycles of given length in integer sequences like $x_{n+1} = x_n/2$ if $x_n$ even, and $x_{n+1} = 3x_n + 1$ otherwise*, Atti Accad. Naz. Lincei rend. Cl. Sci. Fis. Mat. Natur 64 (1978), 260–264.

[Lagarias 1985] J. C. Lagarias, *The 3x+1 problem and its generalizations*, Amer. Math. Monthly 92 (1985), 3–23.

[Lagarias 1990] J. C. Lagarias, *The set of rational cycles for the 3x + 1 problem*, Act. Arith. LVI (1990), 33–53.

[Lagarias 1998] J. C. Lagarias, *3x+1 problem annotated bibliography*, http://www.research.att.com/~jcl (1998).

[Matthews 2001] K. Matthews, *The generalized 3x+1 mapping*, http://www.maths.uq.edu.au/~krm/survey.pdf (2001).