ON THE INTERSECTION OF RATIONAL TRANSVERSAL SUBTORI

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Abstract. We show that under a suitable transversality condition, the intersection of two rational subtori in an algebraic torus \((\mathbb{C}^*)^n\) is a finite group which can be determined using the torsion part of some associated lattice. Applications are given to the study of characteristic varieties of smooth complex algebraic varieties. As an example we discuss A. Suciu’s line arrangement, the so-called deleted \(B_3\)-arrangement.

1. Introduction

Let \(L\) be a free \(\mathbb{Z}\)-module of finite rank \(n\), and let \(A \subset L\) and \(B \subset L\) be two primitive sublattices, i.e. \(A\) and \(B\) are subgroups such that \(\text{Tors}(L/A) = \text{Tors}(B/A) = 0\). Consider the associated \(\mathbb{C}\)-vector spaces

\[ V = L \otimes \mathbb{Z} \mathbb{C}, \quad V_A = A \otimes \mathbb{Z} \mathbb{C} \quad \text{and} \quad V_B = B \otimes \mathbb{Z} \mathbb{C}. \]

Let \(\exp_L : V \rightarrow T = L \otimes \mathbb{Z} \mathbb{C}^*\) be the associated exponential map given by

\[ \exp_L = 1_L \otimes \exp \]

where \(1_L : L \rightarrow L\) is the identity and \(\exp : \mathbb{C} \rightarrow \mathbb{C}^*\) is defined by \(t \mapsto \exp(2\pi it)\). Then \(\exp_L\) is a surjective group homomorphism with kernel \(L = L \otimes \mathbb{Z} \mathbb{C}^* \subset V\). If a \(\mathbb{Z}\)-basis of \(L\) is chosen, then one has obvious identifications \(L = \mathbb{Z}^n\), \(V = \mathbb{C}^n\), \(T = (\mathbb{C}^*)^n\) and \(\exp_L : \mathbb{C}^n \rightarrow (\mathbb{C}^*)^n\) is given by

\[ (t_1, \ldots, t_n) \mapsto (\exp(2\pi it_1), \ldots, \exp(2\pi it_n)). \]

The main result of this note is the following.

Theorem 1.1. With the above notation, if in addition \(V_A \cap V_B = 0\), then there is a group isomorphism

\[ \text{Tors}(L/(A + B)) \rightarrow \exp_L(V_A) \cap \exp_L(V_B). \]

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In fact any algebraic subtorus $S \subset T$, (i.e. $S$ is a closed algebraic subset and a subgroup in $T$), comes from a primitive lattice $A(S) \subset L$, see Lemma 2.1 in Section II in Arapura’s paper [1]. Hence Theorem 1.1 applies to any pair of such algebraic subtori.

This Theorem is proved in the second section. In the third section we show how to use Theorem 1.1 to describe the intersections of the irreducible components of the characteristic varieties of smooth complex algebraic varieties. A specific example coming from the hyperplane arrangement theory concludes the paper.

2. The proof

Let $n = \text{rank} L$, $a = \text{rank} A$ and $b = \text{rank} B$. Consider the quotient group $L' = L/A$, which is again a lattice, of rank $n - a$. The composition $B \to L \to L'$ of the inclusion $B \to L$ and the projection $L \to L'$ gives rise to an injective morphism $\iota : B \to L'$ identifying $B$ to the sublattice $B' = \iota(B) \subset L'$.

Then there is a basis $e'_1, ..., e'_{n-a}$ of the lattice $L'$ such that $B'$ is the subgroup spanned by $d_1 e'_1, ..., d_b e'_b$ for some positive integers $d_j$. Moreover there is an integer $m$ with $1 \leq m \leq b + 1$ such that

\begin{equation}
1 = d_1 = ... = d_{m-1} < d_m \leq ... \leq d_b \quad \text{and} \quad d_m | d_{m+1} | ... | d_b.
\end{equation}

It follows that

\begin{equation}
\text{Tors}(L/(A + B)) = \text{Tors}(\frac{L/A}{(A + B)/A}) = \text{Tors}(L'/B')
\end{equation}

and hence

\begin{equation}
\text{Tors}(L/(A + B)) = \mathbb{Z}/d_m \mathbb{Z} \oplus \mathbb{Z}/d_{m+1} \mathbb{Z} \oplus ... \oplus \mathbb{Z}/d_b \mathbb{Z}.
\end{equation}

Let $e_1, ..., e_{n-a}$ be any lifts of the vectors $e'_j$’s to $L$ and let $f_1, ..., f_a$ be a $\mathbb{Z}$-basis of $A$. Then $B = \{e_1, ..., e_{n-a}, f_1, ..., f_a\}$ is a $\mathbb{Z}$-basis of $L$.

For $j = 1, ..., b$, let $g_j \in B$ be vectors such that their classes $g'_j$ in $L'$ satisfy $g'_j = d_j e'_j$. It follows that $g_j = d_j e_j + a_j$ for some vectors $a_j \in A$. Write now

\begin{equation}
a_j = \sum_{i=1,a} \alpha_{ji} f_i
\end{equation}

for some $\alpha_{ji} \in \mathbb{Z}$. By replacing $e_j$ by $e_j + r_j$ for suitable vectors $r_j \in A$, we may and will assume in the sequel that

\begin{equation}
0 \leq \alpha_{ji} < d_j
\end{equation}

for all $i = 1, ..., a$ and $j = 1, ..., b$. In particular $a_j = 0$ for $j = 1, ..., m - 1$.

Lemma 2.1. The vectors $g_1, ..., g_b$ form a $\mathbb{Z}$-basis of the lattice $B$.
Proof. Note that the vectors $g_1, ..., g_b$ are all contained in $B$ and, on the other hand, their images under $i$ span the lattice $B'$.

Assume now that $\exp_L(v_A) = \exp_L(v_B)$ for some vectors

$$v_A = p_1 f_1 + ... + p_a f_a \in V_A$$

and

$$v_B = q_1 g_1 + ... + q_b g_b \in V_B$$

where $p_i, q_j \in \mathbb{C}$. It follows that $v_A - v_B \in \ker \exp_L = L$. More precisely, we get

$$q_j d_j \in \mathbb{Z} \text{ for } j = 1, ..., b$$

and

$$z_i := p_i - \sum_{j=1}^{b} q_j \alpha_{ji} \in \mathbb{Z} \text{ for } i = 1, ..., a.$$ 

It follows that $q_j = k_j / d_j$ and we may and will assume that $0 \leq k_j < d_j$, since the value of $\exp_L(v_B)$ is not changed when the coefficients $q_j$ are modified by integers. Note that with this choice one has $k_j = 0$ for $j = 1, ..., m - 1$. In this way we get a surjective group homomorphism

$$\theta : \mathbb{Z} / d_m \mathbb{Z} \oplus \mathbb{Z} / d_{m+1} \mathbb{Z} \oplus ... \oplus \mathbb{Z} / d_b \mathbb{Z} \to \exp_L(V_A) \cap \exp_L(V_B)$$

given by

$$\hat{k} = (\hat{k}_m, ..., \hat{k}_b) \mapsto \exp_L(\frac{k_m}{d_m} g_m + ... + \frac{k_b}{d_b} g_b).$$

This morphism $\theta$ is indeed correctly defined since for any choice of the $q_j$’s as above we may use the defining equation of $z_i$ above, set $z_i = 0$ and determine the values for $p_i$’s, i.e. find a vector $v_A$ such that $\exp_L(v_A) = \exp_L(v_B)$.

To show that $\theta$ is injective, we have to show that $\ker \theta = 0$.

Since $B$ is primitive so on the same lines we can take the set

$$\{g_1, \cdots, g_b, h_1, \cdots, h_{n-b}\}$$

as a $\mathbb{Z}$-basis of $L$, where $h_1', \cdots, h_{n-b}'$ is a $\mathbb{Z}$-basis for the lattice $L/B$. Let $\hat{k} \in \ker \theta$. Then $\theta(\hat{k}) = \exp_L(\frac{k_m}{d_m} g_m + \cdots + \frac{k_b}{d_b} g_b) = 1$, which implies that $\frac{k_i}{d_i} \in \mathbb{Z}$, for all $m \leq i \leq b$. Therefore, $\hat{k} = (\hat{k}_m, ..., \hat{k}_b) = (\hat{0}, ..., \hat{0})$ i.e., $\ker \theta = 0$.

3. On the intersection of irreducible components of characteristic varieties

3.1. Local systems, characteristic and resonance varieties. Let $M$ be a quasi-projective smooth complex algebraic variety. The rank one local systems on $M$ are parameterized by the algebraic group

$$\mathbb{T}(M) = \text{Hom}(H_1(M), \mathbb{C}^*)$$
The connected component $T^0(M)$ of the unit element $1 \in T(M)$ is an algebraic torus, i.e. it is isomorphic to $(\mathbb{C}^*)^n$, where $n \in \mathbb{N}$ is the first betti number of $M$, i.e., $n = b_1(M)$. It is clear that $T^0(M) = T(M)$ if and only if the integral homology group $H_1(M)$ is torsion free. For $\rho \in T(M)$, we denote by $\mathcal{L}_\rho$ the corresponding local system on $M$.

The computation of the twisted cohomology groups $H^j(M, \mathcal{L}_\rho)$ is one of the major problems in many areas of topology. To study these cohomology groups, one idea is to study the \textit{characteristic varieties} defined by

\begin{equation}
V^j_m(M) = \{ \rho \in T(M) \mid \dim H^j(M, \mathcal{L}_\rho) \geq m \}.
\end{equation}

To simplify the notation, we set $V_m(M) = V^1_m(M)$. It is known that the following holds, see Beauville \cite{Beauville} and Simpson \cite{Simpson} in the proper case and Arapura \cite{Arapura} in the quasi-projective case.

\textbf{Theorem 3.2.} The positive dimensional irreducible components of $V_m(M)$ are translated subtori in $T(M)$ by elements of finite order. More precisely, for each positive dimensional irreducible component $W$ of $V_m(M)$ we can write $W = \rho \cdot f^*(\mathbb{T}(S)),$ where $f : M \to S$ is a surjective regular mapping to a curve $S$, having a connected general fiber and $\rho \in T(M)$ is a finite order character.

If $1 \in W$, then we can take $\rho = 1$ in the above equality. Let $T_1W$ denote the tangent space to $W$ at the identity $1$ in such a case. Theorem 2, (b) in \cite{Kollar} gives the following.

\textbf{Theorem 3.3.} Let $M$ be a quasi-projective smooth complex algebraic variety. Let $W_1$ and $W_2$ be two distinct irreducible components of the characteristic variety $V_1(M)$ such that $1 \in W_1 \cap W_2$. Then $T_1W_1 \cap T_1W_2 = 0$.

Note that any such tangent space $T_1W \subset H^1(M, \mathbb{C})$ is rationally defined, i.e. there is a primitive lattice $L \subset H^1(M, \mathbb{Z})$ such that $T_1W = L \otimes_{\mathbb{Z}} \mathbb{C}$ under the identification $H^1(M, \mathbb{C}) = H^1(M, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$. Indeed, one can take $L$ to be the primitive sublattice $f^*(H^1(S, \mathbb{Z}))$, in view of the functoriality of the exponential mapping $\exp : T_1T(M) = H^1(M, \mathbb{C}) \to T(M)$ and of the following.

\textbf{Lemma 3.4.} Let $f : M \to S$ be a surjective regular mapping to a curve $S$, having a connected general fiber. Then $f^*(H^1(S, \mathbb{Z}))$ is a primitive sublattice in $H^1(M, \mathbb{Z})$.

\textbf{Proof.} In these conditions, it is well known that the morphism $f_* : H_1(M, \mathbb{Z}) \to H_1(S, \mathbb{Z})$ is surjective. Let $L_0$ be a primitive sublattice in $H_1(M, \mathbb{Z})$ such that $H_1(M, \mathbb{Z}) = \ker f_* \oplus L_0$. Then $f^*(H^1(S, \mathbb{Z}))$ can be identified to the dual $L_0^\vee = \text{Hom}(L_0, \mathbb{Z}) = \{ u \in \text{Hom}(H_1(M, \mathbb{Z}), \mathbb{Z}) : u|_{\ker f_*} = 0 \}$. Moreover $H^1(M, \mathbb{Z}) = H_1(M, \mathbb{Z})^\vee = (\ker f_*)^\vee \oplus L_0^\vee$, which completes the proof of the claim.

\textit{□}

Applying Theorem \cite{Kollar} to this setting, we get the following.
Corollary 3.5. Let $W_1$ and $W_2$ be two distinct irreducible components of the characteristic variety $\mathcal{V}_1(M)$ such that $1 \in W_1 \cap W_2$. Let $L_1$ and $L_2$ be the primitive sublattices in $H^1(M, \mathbb{Z})$ associated to $W_1$ and $W_2$ respectively by the above construction. Then there is a group isomorphism

$$Tors(H^1(M, \mathbb{Z})/(L_1 + L_2)) = W_1 \cap W_2.$$ 

Remark 3.6. Let $W_1$ and $W_2$ be two distinct irreducible components of the characteristic variety $\mathcal{V}_1(M)$, at least one of them, say $W_1$ translated, i.e. $1 \notin W_1$, and meeting at a point $\rho$. Then we may write $W_1 = \rho \cdot W_1'$ and $W_2 = \rho \cdot W_2'$, where $W_j'$ are subtori in $\mathbb{T}(M)$.

Assume that $\dim W_j > 1$ for $j = 1, 2$ (the claim is obvious when one of the two components is one dimensional) and that $M$ is a hyperplane arrangement complement. Then $T_1W_1' \cap T_1W_2' = 0$, since $W_1'$ and $W_2'$ are again two distinct irreducible components of the characteristic variety $\mathcal{V}_1(M)$, see [7]. Moreover, one can see exactly as above that the tangent spaces $T_1W_j'$ are rationally defined and the above Corollary yields a set bijection

$$Tors(H^1(M, \mathbb{Z})/(L_1' + L_2')) = W_1 \cap W_2$$

where $L_j'$ is the primitive sublattice associated to $W_j'$ by the above construction.

Note that any character in such an intersection $W_1 \cap W_2$ has finite order. Indeed, let $W_1 = \rho_1W_1'$ and $W_2 = \rho_2W_2'$, where $\rho_1$ and $\rho_2$ are finite order characters, i.e. $\rho_1^{m_1} = 1$ and $\rho_2^{m_2} = 1$. Then $\rho \in W_1 \cap W_2$ implies that $\rho = \rho_1w_1 = \rho_2w_2$, where $w_j \in W_j'$. Let $m = \text{lcm}(m_1, m_2)$. Then $\rho^m = \rho_1^m w_1^m$ and $\rho^m = \rho_2^m w_2^m$ which implies that $\rho^m = w_1^m = w_2^m \Rightarrow \rho^m \in W_1' \cap W_2'$ so by Corollary 3.3 $\rho^m$ is of finite order. Thus, $\rho$ is a finite order character.

A completely different proof of the finiteness of the intersection $W_1 \cap W_2$ of two distinct irreducible components of the first characteristic variety was given in [6].

Let $H^*(M, \mathbb{C})$ be the cohomology algebra of the variety $M$ with $\mathbb{C}$-coefficients. Right multiplication (cup-product) by an element $z \in H^1(M, \mathbb{C})$ yields a cochain complex $(H^*(M, \mathbb{C}), \mu_z)$. The resonance varieties of $M$ are the jumping loci for the cohomology of this complex, namely

$$\mathcal{R}_m^j(M) = \{z \in H^1(M, \mathbb{C}) \mid \dim H^j(H^*(M, \mathbb{C}), \mu_z) \geq m\}.$$ 

To simplify the notation, we set $\mathcal{R}_m(M) = \mathcal{R}_m^1(M)$

One of the main results relating the characteristic and resonance varieties is the following.

Theorem 3.7. Let $M$ be a hypersurface arrangement complement. The exponential map $\exp : H^1(M, \mathbb{C}) \to \mathbb{T}^0(M)$ induces for any $m, j \geq 1$ an isomorphism of analytic germs

$$(\mathcal{R}_m^j(M), 0) \simeq (\mathcal{V}_m^j(M), 1).$$
This equality of germs implies that the resonance variety $\mathcal{R}_m(M)$ is exactly the tangent cone at 1 of the characteristic variety $\mathcal{V}_m(M)$, a fact established by Cohen and Suciu [5] and which can also be derived from [5]. See also Theorem 3.7 in [4]. It was claimed by A. Libgober that this property holds for any smooth quasi-projective variety, see [9], but now there are counter-examples to this claim, see [5].

**Remark 3.8.** According to Theorem 1.1 in Section V in Arapura [1], under the assumption that $H^1(M, \mathbb{Q})$ has a pure Hodge structure, the positive dimensional irreducible components of all characteristic varieties $\mathcal{V}_m(M)$ are (translated) subtori. Our Theorem 1.1 applies to this more general setting as well. The major difference with the case of first characteristic varieties is that distinct components do not necessarily meet only at the origin. Here is an example for which we are grateful to Professor A. Suciu.

Consider the central hyperplane arrangement in $\mathbb{C}^4$ defined by the equation

$$xyzw(x + y + z)(y - z + w) = 0.$$  

Then the corresponding resonance variety $\mathcal{R}_2(M)$ has two 3-dimensional components, say $E_1$ and $E_2$, defined respectively by the ideals

$$I_1 = (x_1 + x_2 + x_3 + x_6, x_4, x_5)$$ 

and

$$I_2 = (x_1, x_2 + x_3 + x_4 + x_5, x_6).$$

These two components intersect in the line $D = (x_1, x_2 + x_3, x_4, x_5, x_6)$. This implies that the irreducible components $W_1 = \exp(E_1)$ and $W_2 = \exp(E_2)$ of the characteristic variety $\mathcal{V}_2(M)$ intersect along the 1-dimensional subtorus $\exp(D)$.

The fact that $M$ has a simply-connected compactification implies the following.

**Corollary 3.9.** The irreducible components of $\mathcal{R}_1(M)$ are precisely the maximal linear subspaces $E \subset H^1(M, \mathbb{C})$, isotropic with respect to the cup product on $M$

$$\cup : H^1(M, \mathbb{C}) \times H^1(M, \mathbb{C}) \rightarrow H^2(M, \mathbb{C})$$

and such that $\dim E \geq 2$.

**Proof.** Let $E$ be a component of $\mathcal{R}_1(M)$. By the above Theorem there is a component $W$ in $\mathcal{V}_1(M)$ such that $1 \in W$ and $T_1W = E$. By Arapura’s results in [1] we can write $W = f_E^*(\mathbb{T}(S))$, where $f_E : M \rightarrow S$ is a regular mapping to a curve $S$. Since in our case $S$ is rational, $T_1W = f_E^*(H^1(S, \mathbb{C}))$ is isotropic with respect to the cup product, since the cup product on $H^1(S, \mathbb{C})$ is trivial. Maximality of $E$ comes from the fact that $E$ is a component of $\mathcal{R}_1(M)$. The restriction $\dim E \geq 2$ comes from [1]. A mapping $f_E$ as above is said to be associated to the subspace $E$. □
4. An example: the deleted $B_3$-arrangement

Let $\mathcal{A}$ be the deleted $B_3$-arrangement which is obtained from the $B_3$ reflection arrangement by deleting the plane $x + y - z = 0$. A defining polynomial for $\mathcal{A}$ is $Q = xyz(x - y)(x - z)(y - z)(x - y - z)(x - y + z)$. The decone $d\mathcal{A}$ is obtained by setting $z = 1$. Let $L_1 : \ell_1 = x = 0$, $L_2 : \ell_2 = y = 0$, $L_3 : \ell_3 = x - y = 0$, $L_4 : \ell_4 = x - 1 = 0$, $L_5 : \ell_5 = y - 1 = 0$, $L_6 : \ell_6 = x - y - 1 = 0$, $L_7 : \ell_7 = x - y + 1 = 0$ be the lines of the associated affine arrangement in $\mathbb{A}^2$. Let $L_8 : \ell_8 = z = 0$ be the line at infinity and let $M$ be the complement of $d\mathcal{A}$ in $\mathbb{A}^2$. The resonant variety $R_1(d\mathcal{A})$ has 12 irreducible components of dimension 2 and 3. These components $E$ and their associated maps $f_E : M \to S$ are given below, see [13], [14]. Denote by $e_1, ..., e_7$ the $\mathbb{Z}$-basis of $H^1(M, \mathbb{Z})$ given by $e_j = \frac{1}{2\pi i} \frac{d\ell_j}{\ell_j}$, see [11]. Then each of the components $E$ is the $C$-vector space spanned by a primitive lattice denoted by $E^0$, i.e. $E = E^0 \otimes \mathbb{Z} \mathbb{C}$, so it is enough in each case to indicate a $\mathbb{Z}$-basis of $E^0$.

1. The local components There are 7 local components, corresponding to 6 triple points and 1 quadruple point.

For the triple $L_1 \cap L_2 \cap L_3$, we have

$E_1^0 = \langle e_1 - e_3, e_2 - e_3 \rangle$ and $f_{E_1}(x, y) = \frac{2}{y}$, where $S = \mathbb{C}\setminus\{0, 1\}$;

For the triple $L_3 \cap L_4 \cap L_5$, we have

$E_2^0 = \langle e_4 - e_5, e_4 - e_3 \rangle$ and $f_{E_2}(x, y) = \frac{x - 1}{y}$, where $S = \mathbb{C}\setminus\{0, 1\}$;

For the triple $L_2 \cap L_4 \cap L_5$, we have

$E_3^0 = \langle e_4 - e_2, e_5 - e_2 \rangle$ and $f_{E_3}(x, y) = \frac{x - 1}{y}$, where $S = \mathbb{C}\setminus\{0, 1\}$;

For the triple $L_1 \cap L_5 \cap L_7$, we have

$E_4^0 = \langle e_1 - e_7, e_5 - e_1 \rangle$ and $f_{E_4}(x, y) = \frac{y - 1}{y}$, where $S = \mathbb{C}\setminus\{0, 1\}$;

For the triple $L_1 \cap L_4 \cap L_8$, we have

$E_5^0 = \langle e_1, e_4 \rangle$ and $f_{E_5}(x, y) = x$, where $S = \mathbb{C}\setminus\{0, 1\}$;

For the triple $L_2 \cap L_5 \cap L_8$, we have

$E_6^0 = \langle e_5, e_2 \rangle$ and $f_{E_6}(x, y) = y$, where $S = \mathbb{C}\setminus\{0, 1\}$;

For the quadruple $L_3 \cap L_6 \cap L_7 \cap L_8$, we have

$E_7^0 = \langle e_3, e_6, e_7 \rangle$ and $f_{E_7}(x, y) = x - y$, where $S = \mathbb{C}\setminus\{0, \pm 1\}$.

2. The non-local components There are 5 non-local components, corresponding to braid subarrangements:

For $X = (L_1 L_6 | L_3 L_4 | L_2 L_8)$, we have

$E_8^0 = \langle e_1 - e_3 - e_4 + e_6, e_2 - e_4 - e_2 \rangle$ and $f_{E_8}(x, y) = \frac{x(x - y - 1)}{(x - y)(x - 1)}$, where $S = \mathbb{C}\setminus\{0, 1\}$;

For $Y = (L_4 L_8 | L_2 L_3 | L_5 L_6)$, we have

$E_9^0 = \langle -e_2 - e_4 + e_5 + e_6, e_2 + e_3 - e_4 \rangle$ and $f_{E_9}(x, y) = \frac{x - 1}{y(x - y)}$, $S = \mathbb{C}\setminus\{0, 1\}$

For $Z = (L_1 L_5 | L_2 L_4 | L_3 L_8)$, we have

$E_{10}^0 = \langle e_1 - e_2 - e_4 + e_5, e_2 - e_3 + e_4 \rangle$ and $f_{E_{10}}(x, y) = \frac{x(y - 1)}{y(x - 1)}$, where $S = \mathbb{C}\setminus\{0, 1\}$;

For $W = (L_1 L_3 | L_4 L_7 | L_5 L_8)$, we have
\[ E_{11}^0 = \langle e_1 + e_3 - e_5, e_5 - e_7 - e_4 \rangle \quad \text{and} \quad f_{E_{11}}(x, y) = \frac{x(x-y)}{(x-y+1)(x-1)}; \quad \text{where} \quad S = \mathbb{C} \setminus \{0, 1\}; \]

For \( V = (L_1 L_8 | L_2 L_7 | L_3 L_5) \), we have
\[ E_{12}^0 = \langle e_1 - e_2 - e_7, e_3 + e_5 - e_2 - e_7 \rangle \quad \text{and} \quad f_{E_{12}}(x, y) = \frac{x}{y(x-y+1)}, \quad \text{where} \quad S = \mathbb{C} \setminus \{0, 1\}. \]

One way to obtain these 12 irreducible components \( E_j \) is to compute the cup-product
\[ H^1(M, \mathbb{C}) \times H^1(M, \mathbb{C}) \rightarrow H^2(M, \mathbb{C}) \]
and then to use the computer program SINGULAR to list the irreducible components of the determinantal variety corresponding to \( R \).

For each \( f_E \) in the list above we can use the method described in [7] and we get that there is no translated component in \( V_1(dA) \) associated to such an \( f_E \).

It was discovered by A. Suciu (again by using some computer computations) that \( V_1(dA) \) has one 1-dimensional translated component \( W \) associated to the mapping \( f : M \rightarrow \mathbb{C}^* \) defined as, in affine coordinates,
\[ f(x, y) = \frac{x(x-1)(x-y-1)^2}{(x-1)y(x-y+1)^2} \]
and with \( \rho_W = (1, -1, -1, -1, 1, 1, 1) \in (\mathbb{C}^*)^7 \), see [13], [14]. In other words, \( W = \rho_W \otimes \{(t, t^{-1}, 1, t^{-1}, t, t^2, t^{-2}, 1) \mid t \in \mathbb{C}^*\} \).

Let \( V_i \) be the component of \( V_1(dA) \) corresponding to each \( E_i \) for \( i = 1, \ldots, 12 \), i.e. \( V_i = \exp(E_i) \). Then it is known that
\[ W \cap V_8 \cap V_9 \cap V_{10} = \rho_W \]
and
\[ W \cap V_{10} \cap V_{11} \cap V_{12} = \rho_W', \]
where \( \rho_W' = (-1, 1, -1, -1, 1, 1, 1) \in (\mathbb{C}^*)^7 \), see [13], [14]. Since these results were obtained as a result of computer computations, it is useful to provide a direct proof based on Corollary 3.5.

Let \( A = E_8^0 \) and \( B = E_9^0 \) be the primitive lattices in \( H^1(M, \mathbb{Z}) \) introduced above and apply to them the construction explained in Section 2 with \( L = H^1(M, \mathbb{Z}) \).

Here \( n = 7 \), \( a = b = 2 \). The basis \( e'_1, \ldots, e'_5 \) can be chosen as given by the following equivalence classes
\[ e'_1 = [e_1 - e_3 - e_4 + e_6], \quad e'_2 = [e_3], \quad e'_3 = [e_2], \quad e'_4 = [e_5], \quad e'_5 = [e_7]. \]

Then \( m = 1 \) and \( d_2 = 2 \). Let \( f_1 = e_2 + e_3 - e_5 - e_6 \) and \( f_2 = e_4 - e_2 - e_3 \). Then \( B = \{ e_1 - e_3 - e_4 + e_6, e_3, e_2, e_5, e_7, f_1, f_2 \} \) is a \( \mathbb{Z} \)-basis of \( L \) (the coefficient matrix is unimodular) and we can take \( g_1 = e_1 - e_3 - e_4 + e_6 \) and \( g_2 = 2e_3 + f_2 \). Therefore
\[ \text{Tors}(H^1(M, \mathbb{Z})/(E_8^0 + E_9^0)) = \mathbb{Z}_2. \]
Now, by the morphism $\theta : \mathbb{Z}_2 \to V_8 \cap V_9$ used in section [2]

$$\hat{1} \mapsto \exp L \left( \frac{1}{2} (g_2) \right) = \exp L \left( \frac{1}{2} (-e_2 + e_3 + e_4) \right) = (1, -1, -1, 1, 1, 1) = \rho_W.$$  

By Corollary 3.5 it follows that

$$V_8 \cap V_9 = \theta(\mathbb{Z}_2) = \{ 1, \rho_W \}.$$

In exactly the same way one can show that $V_8 \cap V_{10} = V_9 \cap V_{10} = \{ 1, \rho_W \}$. Since clearly $\rho_W \in W$, it follows that the four components $V_8, V_9, V_{10}$ and $W$ meet exactly in one point.

Similarly one shows that $W \cap V_{10} \cap V_{11} \cap V_{12} = \rho'^W_W$ and that all the other intersections of two irreducible components are trivial.

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