Lectures on Topological Quantum Field Theory

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Abstract

In these lectures we present a general introduction to topological quantum field theories. These theories are discussed in the framework of the Mathai-Quillen formalism and in the context of twisted $N = 2$ supersymmetric theories. We discuss in detail the recent developments in Donaldson-Witten theory obtained from the application of results based on duality for $N = 2$ supersymmetric Yang-Mills theories. This involves a description of the computation of Donaldson invariants in terms of Seiberg-Witten invariants. Generalizations of Donaldson-Witten theory are reviewed, and the structure of the vacuum expectation values of their observables is analyzed in the context of duality for the simplest case.

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1 Introduction

Topological quantum field theory (TQFT) emerged in the eighties as a new relation between mathematics and physics. This relation connected some of the most advanced ideas in the two fields. The nineties have been characterized by its development, originating unexpected results in topology and testing some of the most fundamental ideas in quantum field theory and string theory.

The first TQFT was formulated by Witten [87] in 1988. He constructed the theory now known as Donaldson-Witten theory, which constitutes a quantum field theory representation of the theory of Donaldson invariants [32, 33]. His work was strongly influenced by M. Atiyah [13]. In 1988 Witten formulated also another two TQFTs which have been widely studied during the last ten years: topological sigma models in two dimensions [88] and Chern-Simons gauge theory [89] in three dimensions. These theories are related, respectively, to Gromov invariants [45] and to knot and link invariants as the Jones polynomial [50] and its generalizations.
TQFT has provided an entirely new approach to study topological invariants. Being a quantum field theory, TQFT can be analyzed from different points of view. The richness inherent to quantum field theory can be exploited to obtain different perspectives on the topological invariants involved in TQFT. This line of thought has shown to be very successful in the last years and new topological invariants as well as new relations between them have been obtained.

TQFTs have been studied from both, perturbative and non-perturbative points of view. In the case of Chern-Simons gauge theory, non-perturbative methods have been applied to obtain properties \cite{89, 41} of knot and link invariants, as well as general procedures for their computation \cite{83, 53, 68}. Perturbative methods have also been studied for this theory \cite{46, 7, 17, 2, 5} providing integral representations for Vassiliev invariants \cite{84}. In Donaldson-Witten theory perturbative methods have proved its relation to Donaldson invariants. Non-perturbative methods have been applied \cite{92} after the work by Seiberg and Witten \cite{79} on $N=2$ supersymmetric Yang-Mills theory. The outcome of this application is a totally unexpected relation between Donaldson invariants and a new set of topological invariants called Seiberg-Witten invariants. One of the main purposes of these lectures is to describe the general aspects of this result.

Donaldson-Witten theory is a TQFT of cohomological type. TQFTs of this type can be formulated in a variety of frameworks. The most geometric one corresponds to the Mathai-Quillen formalism \cite{74}. In this formalism a TQFT is constructed out of a moduli problem \cite{15}. Topological invariants are then defined as integrals of a certain Euler class (or wedge products of the Euler class with other forms) over the resulting moduli space. A different framework is the one based on the twisting of $N=2$ supersymmetry. In this case, information on the physical theory can be used in the TQFT. Indeed, it has been in this framework where Seiberg-Witten invariants have shown up. After Seiberg and Witten worked out the low energy effective action of $N=2$ supersymmetric Yang-Mills theory it became clear that a twisted version of this effective action could lead to topological invariants related to Donaldson invariants. These twisted actions \cite{51, 1, 10} revealed a new moduli space, the moduli space of abelian monopoles \cite{92}. Its geometric structure has been derived in the context of the Mathai-Quillen formalism \cite{61}. Invariants associated to this moduli space should be related to Donaldson invariants. This turned out to be the case. The relevant invariants for the case of $SU(2)$ as gauge group are the Seiberg-Witten invariants.

Donaldson-Witten theory has been generalized after studying its coupling to topological matter fields \cite{71, 1, 10}. The resulting theory can be regarded as a twisted form of $N=2$ supersymmetric Yang-Mills theory coupled to hypermultiplets, or, in the context of the Mathai-Quillen formalism, as the TQFT associated to the moduli space of non-abelian monopoles \cite{74}. Perturbative and non-perturbative methods have been applied to this theory for the case of $SU(2)$ as gauge group and one hypermultiplet of matter in the fundamental representation \cite{53}. In this case, again, it turns out
that the generalized Donaldson invariants can be written in terms of Seiberg-Witten invariants. It is not known at the moment which one is the situation for other groups and representations but one would expect that in general the invariants associated to non-abelian monopoles could be expressed in terms of some other simpler invariants, being Seiberg-Witten invariants just the first subset of the full set of invariants.

In Table 1 we have depicted the present situation in three and four dimensions relative to Chern-Simons gauge theory and Donaldson-Witten theory, respectively. These theories share some common features. Their topological invariants are labeled with group-theoretical data: Wilson lines for different representations and gauge groups (Jones polynomial and its generalizations), and non-abelian monopoles for different representations and gauge groups (generalized Donaldson polynomials); these invariants can be written in terms of topological invariants which are independent of the group and representation chosen: Vassiliev invariants and Seiberg-Witten invariants. This structure leads to the idea of universality classes [63, 72] of topological invariants. In this respect Vassiliev invariants constitute a class in the sense that all Chern-Simons or quantum group knot invariants for semi-simple groups can be expressed in terms of them. Similarly, Seiberg-Witten invariants constitute another class since generalized Donaldson invariants associated to several moduli spaces can be written in terms of them. This certainly holds for the two cases described above but presumably it holds for other groups. It is very likely that Seiberg-Witten invariants are the first set of a series of invariants, each defining a universality class.

|                | \(d = 3\) | \(d = 4\) |
|----------------|-----------|-----------|
| perturbative   | Vassiliev | Donaldson |
| non-perturbative| Jones     | Seiberg-Witten |

Table 1: Topological invariants in the perturbative and the non-perturbative regimes for \(d = 3\) and \(d = 4\).

These lectures are organized as follows. In sect. 2 we present a general introduction to TQFT from a functional integral point of view. In sect. 3 we review the Mathai-Quillen formalism and we discuss it in the context of supersymmetric quantum mechanics and topological sigma models. In sect. 4 we introduce Donaldson-Witten theory in the framework of the Mathai-Quillen formalism, and from the point of view of twisting \(N = 2\) supersymmetric Yang-Mills theory. We then discuss the computation of its observables from a perturbative point of view, showing its relation to Donaldson invariants, and then from a non-perturbative one obtaining its expression in terms of Seiberg-Witten invariants. This last analysis is done in two different approaches: in the abstract approach, based on the structure of \(N = 1\) supersymmetric Yang-Mills theory and valid only for Kähler four-manifolds with \(H^{(2,0)} \neq 0\), and in the concrete approach, based on the structure of \(N = 2\) supersymmetric Yang-Mills theory and valid for any four-manifold. Explicit expressions for Donaldson invariants are collected for the case of \(SU(2)\) as gauge group and simply-connected four-manifolds.
with $b_2^+ > 1$. In sect. 5 we describe the generalizations of Donaldson-Witten theory and review, for simply-connected four-manifolds with $b_2^+ > 1$, the structure of the vacuum expectation values of its observables for the case of $SU(2)$ as gauge group and one hypermultiplet in the fundamental representation. Finally, in sect. 6 we include some final remarks.

Before entering into the core of these lectures let us recall that excellent reviews on TQFTs are already available [21, 30, 24, 37]. In these lectures we have mainly concentrated on subjects not covered in those reviews though, trying to be self-contained, some overlapping is unavoidable. There are also good reviews on Seiberg-Witten invariants [34, 69, 75, 96] from a mathematical perspective.
2 Topological Quantum Field Theory

In this section we present the general structure of TQFT from a functional integral point of view. As in ordinary quantum field theory, the functional integration involved is not in general well defined. Similarly to that case this has led to the construction of an axiomatic approach [14]. In these lectures, however, we are not going to describe this approach. We will concentrate on the functional integral point of view. Although not well defined in general, this is the approach which has shown to be more successful.

Our basic topological space will be an \( n \)-dimensional Riemannian manifold \( X \) endowed with a metric \( g_{\mu\nu} \). On this space we will consider a set of fields \( \{\phi_i\} \), and a real functional of these fields, \( S(\phi_i) \), which will be regarded as the action of the theory. We will consider operators, \( O_\alpha(\phi_i) \), which will be in general arbitrary functionals of the fields. In TQFT these functionals are labeled by some set of indices \( \alpha \) carrying topological or group-theoretical data. The vacuum expectation value (vev) of a product of these operators is defined as the following functional integral:

\[
\langle O_{\alpha_1}O_{\alpha_2}\cdots O_{\alpha_p} \rangle = \int[D\phi_i]O_{\alpha_1}(\phi_i)O_{\alpha_2}(\phi_i)\cdots O_{\alpha_p}(\phi_i) \exp(-S(\phi_i)).
\] (2.1)

A quantum field theory is considered topological if it possesses the following property:

\[
\frac{\delta}{\delta g_{\mu\nu}}(O_{\alpha_1}O_{\alpha_2}\cdots O_{\alpha_p}) = 0,
\] (2.2)

i.e., if the vacuum expectation values (vevs) of some set of selected operators remain invariant under variations of the metric \( g_{\mu\nu} \) on \( X \). If such is the case these operators are called observables.

There are two ways to guarantee, at least formally, that condition (2.2) is satisfied. The first one corresponds to the situation in which both, the action, \( S \), as well as the operators \( O_\alpha \), are metric independent. These TQFTs are called of Schwarz type [21]. In the case of Schwarz-type theories one must first construct an action which is independent of the metric \( g_{\mu\nu} \). The method is best illustrated by considering an example. Let us take into consideration the most interesting case of this type of theories: Chern-Simons gauge theory [89]. The data in Chern-Simons gauge theory are the following: a differentiable compact three-manifold \( M \), a gauge group \( G \), which will be taken simple and compact, and an integer parameter \( k \). The action is the integral of the Chern-Simons form associated to a gauge connection \( A \) corresponding to the group \( G \):

\[
S_{CS}(A) = \int_M \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A).
\] (2.3)

Observables are constructed out of operators which do not contain the metric \( g_{\mu\nu} \). In gauge invariant theories, as it is the case, one must also demand for these operators invariance under gauge transformations. An important set of observables in
Chern-Simons gauge theory is constituted by the trace of the holonomy of the gauge connection $A$ in some representation $R$ along a 1-cycle $\gamma$, i.e., the Wilson loop:

$$\text{Tr}_R(\text{Hol}_\gamma(A)) = \text{Tr}_R P \exp \int_\gamma A.$$  \hfill (2.4)

The vevs are labeled by representations, $R_i$, and embeddings, $\gamma_i$, of $S^1$ into $M$:

$$\langle \text{Tr}_{R_1} P e^{\int_{\gamma_1} A} \ldots \text{Tr}_{R_n} P e^{\int_{\gamma_n} A} \rangle = \int \left[ D\phi \right] \text{Tr}_{R_1} P e^{\int_{\gamma_1} A} \ldots \text{Tr}_{R_n} P e^{\int_{\gamma_n} A} e^{ik_4 \frac{ik}{4\pi} S_{\text{CS}}(A)}. \hfill (2.5)$$

A non-perturbative analysis of Chern-Simons gauge theory \cite{82} shows that the invariants associated to the observables (2.5) are knot and link invariants of polynomial type as the Jones polynomial \cite{50} and its generalizations (HOMFLY \cite{39}, Kauffman \cite{52}, Akutsu-Wadati \cite{9}, etc.). The perturbative analysis \cite{46, 17, 2, 5} has also led to this result and has shown to provide a very useful framework to study Vassiliev invariants \cite{54, 20, 18, 3, 4} (see \cite{56} for a brief review).

An important set of theories of Schwarz type are the BF theories \cite{48}. These theories can be formulated in any dimension and are considered, as Chern-Simons gauge theory, exactly solvable quantum field theories. We will not describe them in these lectures. They have acquired importance recently since it has been pointed out that four-dimensional Yang-Mills theories could be regarded as a deformation of these theories \cite{29}. It is important also to remark that Chern-Simons gauge theory plays an important role in the context of the Ashtekar approach \cite{11} to the quantization of four-dimensional gravity \cite{16, 42}.

The second way to guarantee (2.2) corresponds to the case in which there exists a symmetry, whose infinitesimal form will be denoted by $\delta$, satisfying the following properties:

$$\delta \mathcal{O}_\alpha(\phi_i) = 0, \quad T_{\mu\nu}(\phi_i) = \delta G_{\mu\nu}(\phi_i), \hfill (2.6)$$

where $T_{\mu\nu}(\phi_i)$ is the energy-momentum tensor of the theory, i.e.,

$$T_{\mu\nu}(\phi_i) = \frac{\delta}{\delta g^{\mu\nu}} S(\phi_i), \hfill (2.7)$$

and $G_{\mu\nu}(\phi_i)$ is some tensor.

The fact that $\delta$ in (2.6) is a symmetry of the theory means that the transformations, $\delta \phi_i$, of the fields are such that both, $\delta S(\phi_i) = 0$, and, $\delta \mathcal{O}_\alpha(\phi_i) = 0$. Conditions (2.6) lead, at least formally, to the following relation for vevs:

$$\frac{\delta}{\delta g^{\mu\nu}} \langle \mathcal{O}_{\alpha_1} \mathcal{O}_{\alpha_2} \cdots \mathcal{O}_{\alpha_p} \rangle = - \int [D\phi_i] \mathcal{O}_{\alpha_1}(\phi_i) \mathcal{O}_{\alpha_2}(\phi_i) \cdots \mathcal{O}_{\alpha_p}(\phi_i) T_{\mu\nu} \exp (- S(\phi_i))$$

$$= - \int [D\phi_i] \delta \left( \mathcal{O}_{\alpha_1}(\phi_i) \mathcal{O}_{\alpha_2}(\phi_i) \cdots \mathcal{O}_{\alpha_p}(\phi_i) G_{\mu\nu} \exp (- S(\phi_i)) \right) = 0, \hfill (2.8)$$

which implies that the quantum field theory can be regarded as topological. In (2.8) it has been assumed that the action and the measure $[D\phi_i]$ are invariant under
the symmetry $\delta$. We have assumed also in (2.8) that the observables are metric-independent. This is a common situation in this type of theories, but it does not have to be necessarily so. In fact, in view of (2.8), it would be possible to consider a wider class of operators satisfying:

$$\frac{\delta}{\delta g_{\mu\nu}} O_\alpha(\phi_i) = \delta O_\alpha^{\mu\nu}(\phi_i),$$  

(2.9)

with $O_\alpha^{\mu\nu}(\phi_i)$ a certain functional of the fields of the theory.

This second type of TQFTs are called cohomological or of Witten type [21, 90]. One of its main representatives is Donaldson-Witten theory [87], which can be regarded as a certain twisted version of $N = 2$ supersymmetric Yang-Mills theory. It is important to remark that the symmetry $\delta$ must be a scalar symmetry. The reason is that, being a global symmetry, the corresponding parameter must be covariantly constant and for arbitrary manifolds this property, if it is satisfied at all, implies strong restrictions unless the parameter is a scalar.

Most of the TQFTs of cohomological type satisfy the relation:

$$S(\phi_i) = \delta \Lambda(\phi_i),$$  

(2.10)

for some functional $\Lambda(\phi_i)$. This has far-reaching consequences, for it means that the topological observables of the theory (in particular the partition function itself) are independent of the value of the coupling constant. Indeed, let us consider for example the vev:

$$\langle O_{\alpha_1} O_{\alpha_2} \cdots O_{\alpha_p} \rangle = \int [D\phi_i] O_{\alpha_1}(\phi_i) O_{\alpha_2}(\phi_i) \cdots O_{\alpha_p}(\phi_i) \exp \left( -\frac{1}{g^2} S(\phi_i) \right).$$  

(2.11)

Under a change in the coupling constant, $1/g^2 \rightarrow 1/g^2 - \Delta$, one has (assuming that the observables do not depend on the coupling), up to first order in $\Delta$:

$$\langle O_{\alpha_1} O_{\alpha_2} \cdots O_{\alpha_p} \rangle \rightarrow \langle O_{\alpha_1} O_{\alpha_2} \cdots O_{\alpha_p} \rangle + \Delta \int [D\phi_i] \delta \left[ O_{\alpha_1}(\phi_i) O_{\alpha_2}(\phi_i) \cdots O_{\alpha_p}(\phi_i) \Lambda(\phi_i) \exp \left( -\frac{1}{g^2} S(\phi_i) \right) \right]$$  

$$= \langle O_{\alpha_1} O_{\alpha_2} \cdots O_{\alpha_p} \rangle.$$

(2.12)

Hence, observables can be computed either in the weak coupling limit, $g \rightarrow 0$, or in the strong coupling limit, $g \rightarrow \infty$.

So far we have presented a rather general definition of TQFT and made a series of elementary remarks. Now we will analyze some aspects of its structure. We begin pointing out that given a theory in which (2.6) holds one can build correlators which correspond to topological invariants (in the sense that they are invariant under deformations of the metric $g_{\mu\nu}$) just by considering the operators of the theory which are invariant under the symmetry. We will call these operators observables. Actually,
to be more precise, we will call observables to certain classes of those operators. In virtue of eq. (2.8), if one of these operators can be written as a symmetry transformation of another operator, its presence in a correlation function will make it vanish. Thus we may identify operators satisfying (2.4) which differ by an operator which corresponds to a symmetry transformation of another operator. Let us denote the set of the resulting classes by \( \{ \Phi \} \). Actually, in general, one could identify bigger sets of operators since two operators of which one of them does not satisfy (2.6) could lead to the same invariant if they differ by an operator which is a symmetry transformation of another operator. For example, consider \( O \) such that \( \delta O = 0 \) and \( O + \delta \Gamma \). Certainly, both operators lead to the same observables. But it may well happen that \( \delta^2 \Gamma \neq 0 \) and therefore we have operators which do not satisfy (2.6) that must be identified. The natural way out is to work *equivariantly*, which in this context means that one must consider only operators which are invariant under both, \( \delta \) and \( \delta^2 \). In turns out that in most of the cases (and in particular, in all the cases that we will be considering) \( \delta^2 \) is a gauge transformation, so in the end all that has to be done is to restrict the analysis to gauge-invariant operators, a very natural requirement. Hence, by restricting the analysis to the appropriate set of operators, one has that in fact,

\[
\delta^2 = 0.
\]

(2.13)

Property (2.13) has striking consequences on the features of TQFT. First, the symmetry must be odd which implies the presence in the theory of commuting and anticommuting fields. For example, the tensor \( G_{\mu
u} \) in (2.6) must be anticommuting. This is the first appearance of an odd non-spinorial field in TQFT. Those kinds of objects are standard features of cohomological TQFTs. Second, if we denote by \( Q \) the operator which implements this symmetry, the observables of the theory can be described as the cohomology classes of \( Q \):

\[
\{ \Phi \} = \frac{\ker Q}{\text{Im} Q}, \quad Q^2 = 0.
\]

(2.14)

Equation (2.6) means that in addition to the Poincare group the theory possesses a symmetry generated by an odd version of the Poincare group. The corresponding odd generators are constructed out of the tensor \( G_{\mu\nu} \) in much the same way as the ordinary Poincare generators are built out of \( T_{\mu\nu} \). For example, if \( P_{\mu} \) represents the ordinary momentum operator, there exists a corresponding odd one \( G_{\mu} \) such that,

\[
P_{\mu} = \{ Q, G_{\mu} \}.
\]

(2.15)

Let us discuss the structure of the Hilbert space of the theory in virtue of the symmetries that we have just described. The states of this space must correspond to representations of the algebra generated by the operators in the Poincare groups and by \( Q \). Furthermore, as follows from our analysis of operators leading to (2.14), if one
is interested only in states $|\Psi\rangle$ leading to topological invariants one must consider states which satisfy,

$$Q|\Psi\rangle = 0,$$

and two states which differ by a $Q$-exact state must be identified. The odd Poincare group can be used to generate descendant states out of a state satisfying (2.16). The operators $G_\mu$ act non-trivially on the states and in fact, out of a state satisfying (2.16) we can build additional states using this generator. The simplest case consists of,

$$\int_{\gamma_1} G_\mu |\Psi\rangle,$$

(2.17)

where $\gamma_1$ is a 1-cycle. One can easily verify using (2.6) that this new state satisfies (2.16): 

$$Q \int_{\gamma_1} G_\mu |\Psi\rangle = \int_{\gamma_1} \{Q, G_\mu\} |\Psi\rangle = \int_{\gamma_1} P_\mu |\Psi\rangle = 0.$$

(2.18)

Similarly, one may construct other invariants tensoring $n$ operators $G_\mu$ and integrating over $n$-cycles $\gamma_n$: 

$$\int_{\gamma_n} G_{\mu_1} G_{\mu_2} \ldots G_{\mu_n} |\Psi\rangle.$$

(2.19)

Notice that since the operator $G_\mu$ is odd and its algebra is Poincare-like the integrand in this expression is an $n$-form. It is straightforward to prove that these states also satisfy condition (2.16). Therefore, starting from a state $|\Psi\rangle \in \ker Q$ we have built a set of partners or descendants giving rise to a topological multiplet. The members of a multiplet have well defined ghost number. If one assigns ghost number $-1$ to the operator $G_\mu$ the state in (2.15) has ghost number $-n$ plus the ghost number of $|\Psi\rangle$. Of course, $n$ is bounded by the dimension of the manifold $X$. Among the states constructed in this way there may be many which are related via another state which is $Q$-exact, i.e., which can be written as $Q$ acting on some other state. Let us try to single out representatives at each level of ghost number in a given topological multiplet.

Consider an $(n-1)$-cycle which is the boundary of an $n$-dimensional surface, $\gamma_{n-1} = \partial S_n$. If one builds a state taking such a cycle one finds $(P_\mu = -i\partial_\mu)$,

$$\int_{\gamma_{n-1}} G_{\mu_1} G_{\mu_2} \ldots G_{\mu_{n-1}} |\Psi\rangle = i \int_{S_n} P_{[\mu_1} G_{\mu_2} G_{\mu_3} \ldots G_{\mu_{n-1}]} |\Psi\rangle = iQ \int_{S_n} G_{\mu_1} G_{\mu_2} \ldots G_{\mu_n} |\Psi\rangle,$$

(2.20)

i.e., it is $Q$-exact. The symbols $[~]$ in (2.20) denote that all indices between them must by antisymmetrized. In (2.20) use has been made of (2.13). This result tells us that the representatives we are looking for are built out of the homology cycles of the manifold $X$. Given a manifold $X$, the homology cycles are equivalence classes among cycles, the equivalence relation being that two $n$-cycles are equivalent if they differ by a cycle which is the boundary of an $n+1$ surface. Thus, knowledge on the homology of the manifold on which the TQFT is defined allows us to classify the
representatives among the operators (2.13). Let us assume that \( X \) has dimension \( d \) and that its homology cycles are \( \gamma_{i_n}, i_n = 1, \ldots, d_n, \) \( n = 0, \ldots, d, \) being \( d_n \) the dimension of the \( n \)-homology group, and \( d \) the dimension of \( X \). Then, the non-trivial partners or descendants of a given \( \vert \Psi \rangle \) highest-ghost-number state are labeled in the following way:
\[
\int_{\gamma_{i_n}} G_{\mu_1} G_{\mu_2} \ldots G_{\mu_n} \vert \Psi \rangle, \quad i_n = 1, \ldots, d_n, \quad n = 0, \ldots, d. \tag{2.21}
\]

A similar construction to the one just described can be made for fields. Starting with a field \( \phi(x) \) which satisfies,
\[
[Q, \phi(x)] = 0, \tag{2.22}
\]
one can construct other fields using the operators \( G_{\mu} \). These fields, which we will call partners or descendants are antisymmetric tensors defined as,
\[
\phi_{\mu_1 \mu_2 \ldots \mu_n}^{(n)}(x) = \frac{1}{n!} \{G_{\mu_1}, [G_{\mu_2}, \ldots, [G_{\mu_n}, \phi(x)] \ldots] \}, \quad n = 1, \ldots, d. \tag{2.23}
\]
Using (2.13) and (2.22) one finds that these fields satisfy the so-called topological descent equations:
\[
d\phi^{(n)} = i[Q, \phi^{(n+1)}], \tag{2.24}
\]
where the subindices of the forms have been suppressed for simplicity, and the highest-ghost-number field \( \phi(x) \) has been denoted as \( \phi^{(0)}(x) \). These equations enclose all the relevant properties of the observables which are constructed out of them. They constitute a very useful tool to build the observables of the theory. Let us consider an \( n \)-cycle and the following quantity:
\[
W_{\phi}^{(\gamma_n)} = \int_{\gamma_n} \phi^{(n)}. \tag{2.25}
\]
The subindex of this quantity denotes the highest-ghost-number field out of which the form \( \phi^{(n)} \) is generated. The superindex denotes the order of such a form as well as the cycle which is utilized in the integration. Using the topological descent equations (2.24) it is immediate to prove that \( W_{\phi}^{(\gamma_n)} \) is indeed an observable:
\[
[Q, W_{\phi}^{(\gamma_n)}] = \int_{\gamma_n} [Q, \phi^{(n)}] = -i \int_{\gamma_n} d\phi^{(n-1)} = 0. \tag{2.26}
\]
Furthermore, if \( \gamma_n \) is a trivial homology cycle, \( \gamma_n = \partial S_{n+1} \), one obtains that \( W_{\phi}^{(\gamma_n)} \) is \( Q \)-exact,
\[
W_{\phi}^{(\gamma_n)} = \int_{\gamma_n} \phi^{(n)} = \int_{S_{n+1}} d\phi^{(n)} = i \int_{S_{n+1}} [Q, \phi^{(n+1)}] = i[Q, \int_{S_{n+1}} \phi^{(n+1)}], \tag{2.27}
\]
and therefore its presence in a vev makes it vanish. Thus, similarly to the previous analysis leading to (2.21), the observables of the theory are operators of the form (2.23):

\[ W^{(\gamma_n)}_{\phi}, \quad i_n = 1, ..., d_n, \quad n = 0, ..., d, \quad (2.28) \]

where, as before, \( d_n \) denotes the dimension of the \( n \)-homology group. Of course, these observables are a basis of observables but one can make arbitrary products of them leading to new ones.

One may wonder at this point how it is possible that there may be observables which depend on the space-time position \( x \) and nevertheless lead to topological invariants. For example, an observable containing the zero form \( \phi^{(0)}(x) \) seems to lead to vacuum expectation values which depend on \( x \) since the space-time position \( x \) is not integrated over. A closer analysis, however, shows that this is not the case. As follows from the topological descent equation (2.24), the derivative of \( \phi^{(0)}(x) \) with respect to \( x \) is \( Q \)-exact and therefore such a vacuum expectation value is actually independent of the space-time position.

The structure of observables described here is common to all cohomological TQFTs. In these lectures we will review the cases of topological sigma models, and Donaldson-Witten theory and its generalizations. In the first case the highest-ghost-number observables are built out of the cohomology of the target manifold of the sigma model. In the second case they are obtained from the independent Casimirs of the gauge group under consideration. Once the highest-ghost-number observables are identified, their families are constructed solving the topological descent equations (2.24).

We will finish this section pointing out that there is a special set \([23, 31, 25]\), of cohomological TQFTs which play an important role when trying to analyze Euler characters of some moduli spaces. The feature which characterizes these theories is that they possess two topological symmetries. One of the most interesting examples of this kind of theories is a twisted version of \( N = 4 \) supersymmetric Yang-Mills, which has been used recently to carry out a test of \( S \)-duality in four dimensions.
3 The Mathai-Quillen formalism

In the rest of these lectures we will restrict ourselves to cohomological TQFTs. These theories can be constructed from supersymmetric theories. In fact, the first examples of TQFTs of this type, four-dimensional Donaldson-Witten theory \cite{87} and two-dimensional topological sigma models \cite{88}, were constructed starting from four-dimensional $N = 2$ supersymmetric gauge theory and two-dimensional $N = 2$ supersymmetric sigma models, respectively. We will discuss their origin from supersymmetric theories in a forthcoming section. In this section we will introduce them within a more mathematical framework, the Mathai-Quillen formalism, which we first discuss in its simplest form. We will follow a presentation similar to the one by Blau and Thompson \cite{24}.

TQFTs of cohomological type are characterized by three basic data: fields, symmetries, and equations \cite{90, 21, 30}. The starting point is a configuration space $X$, whose elements are fields $\phi_i$ defined on some Riemannian manifold $X$. These fields are generally acted on by some group $G$ of local transformations (gauge symmetries, or a diffeomorphism group, among others), so one is naturally led to consider the quotient space $X/G$. Within this quotient space, a certain subset or moduli space, $\mathcal{M}$, is singled out by a set of equations $s(\phi_i) = 0$:

$$\mathcal{M} = \{ \phi_i \in X | s(\phi_i) = 0 \}/G.$$  \hspace{1cm} (3.1)

Within this framework, the topological symmetry $\delta$ furnishes a representation of the $G$-equivariant cohomology on the field space. When $G$ is the trivial group, $\delta$ is nothing but the de Rham operator on the field space.

The next step consists of building the topological theory associated to this moduli problem. We will do this within the framework of the Mathai-Quillen formalism \cite{74}. This formalism is the most geometric one among all the approaches leading to the construction of TQFTs. It can be applied to any Witten-type theory. It was first implemented in the context of TQFT by Atiyah and Jeffrey \cite{15}, and later further developed in a series of works \cite{22, 30}. The basic idea behind this formalism is the extension to the infinite-dimensional case of ordinary finite-dimensional geometric constructions. Soon after the formulation of the first TQFTs it became clear that the partition function of these theories was related to the Euler class of a certain bundle associated to the space of solutions of the basic equations of the theory. In the finite-dimensional case there are many different, though equivalent, forms of thinking on the Euler class, which we will recall below. The Mathai-Quillen formalism basically consists of generalizing one of these forms to the infinite-dimensional case. In what follows we will give a brief account on the fundamentals of the construction. For further details, we refer the reader to \cite{30, 24}, where excellent reviews on this approach are presented.
3.1 Finite-dimensional case

Let $X$ be an orientable, boundaryless, compact $n$-dimensional manifold. Let us consider an orientable vector bundle $\mathcal{E} \to X$ of rank $\text{rk}(\mathcal{E}) = 2m \leq n$ over $X$. For completeness we recall that a vector bundle $\mathcal{E}$, with a $2m$-dimensional vector space $F$ as fibre, over a base manifold $X$, is a topological space with a continuous projection, $\pi : \mathcal{E} \to X$, such that, $\forall x \in X$, $\exists U_x \subset X$, open set, $x \in U_x$, $\mathcal{E}$ is a product space, $U_x \times F$, when restricted to $U_x$. This means that there exists a homeomorphism $\varphi : U_x \times F \to \pi^{-1}(U)$ which preserves the fibres, $i.e.$, $\pi(\varphi(x,f)) = x$, with $f \in F$.

There exist two complementary ways of defining the Euler class of $\mathcal{E}$, $e(\mathcal{E}) \in H^{2m}(X)$:

1. In terms of sections. A section $s$ of $\mathcal{E}$ is a map $s : X \to \mathcal{E}$ such that $\pi(s(x)) = x$. A generic section is one which is transverse to the zero section, and which therefore vanishes on a set of dimension $n - 2m$. In this context $e(\mathcal{E})$ shows up as the Poincare dual (in $X$) of the homology class defined by the zero locus of a generic section of $\mathcal{E}$.

2. In terms of characteristic classes. The approach makes use of the Chern-Weil theory, and gives a representative $e_{\nabla}(\mathcal{E})$ of $e(\mathcal{E})$ associated to a connection $\nabla$ in $\mathcal{E}$,

$$e_{\nabla}(\mathcal{E}) = (2\pi)^{-m}\text{Pf}(\Omega_{\nabla}), \quad (3.2)$$

where $\text{Pf}(\Omega_{\nabla})$ stands for the Pfaffian of the curvature $\Omega_{\nabla}$, which is an anti-symmetric matrix of two-forms. The representative $e_{\nabla}(\mathcal{E})$ can be written in “field-theoretical” form:

$$e_{\nabla}(\mathcal{E}) = (2\pi)^{-m}\int d\chi \chi^a \Omega_a^{\chi b} \chi_b, \quad (3.3)$$

by means of a set of real Grassmann-odd variables $\chi_a$, $a = 1, \ldots, 2m$, satisfying the Berezin rules of integration:

$$\int d\chi_a \chi_b = \delta_{ab}. \quad (3.4)$$

If $\text{rk}(\mathcal{E}) = 2m = n = \dim(X)$, one can evaluate $e(\mathcal{E})$ on $X$ to obtain the Euler number of $\mathcal{E}$ in two different ways:

$$\chi(\mathcal{E}) = \sum_{x_k : s(x_k) = 0} (\pm 1),$$

$$\chi(\mathcal{E}) = \int_X e_{\nabla}(\mathcal{E}). \quad (3.5)$$
In the first case, one counts signs at the zeroes of a generic section. In the second case, an integration of the differential form (3.2) is performed. Of course, both results coincide, and do not depend either on the section \( s \) (as long as it is generic) or on the connection \( \nabla \). When \( 2m < n \) one can evaluate \( e(E) \) on \( 2m \)-cycles or equivalently take the product with elements of \( H^{n-2m}(X) \) and evaluate it on \( X \).

In the particular case that \( E \equiv TX \) the expression \( \chi(E) = \sum_{x : s(x) = 0} (\pm 1) \), which gives the Euler number of the base manifold \( X \), can be generalized to a non-generic vector field \( V \) (which is a section of the tangent bundle),

\[
\chi(X) = \chi(X_V),
\]

where \( X_V \) is the zero locus of \( V \), which is not necessarily zero-dimensional.

In this framework the Mathai-Quillen formalism gives a representative of the Euler class, \( e_{s,\nabla}(E) \), which interpolates between the two approaches sketched above. It depends explicitly on both, a section, \( s \), and a connection, \( \nabla \), on \( E \):

\[
e_{s,\nabla}(E) \in [e(E)],
\]

\[
\chi(E) = \int_X e_{s,\nabla}(E), \quad (\text{if } 2m = n).
\]

The construction of \( e_{s,\nabla} \) is given by the formalism. First, it provides an explicit representative of the Thom class \([27], \Phi(E)\), of \( E \). Let \( E \to X \) be a vector bundle of rank \( 2m \) with fibre \( F \), and let us consider the cohomology of forms with Gaussian decay along the fibre. By integrating the form along the fibre one has an explicit isomorphism (the Thom isomorphism) between \( k \) forms over \( E \) and \( k - 2m \) forms over \( X \). This isomorphism can be made explicit with the aid of the Thom class, whose representative, \( \Phi(E) \), is a closed \( 2m \)-form over \( E \) with Gaussian decay along the fibre such that its integral over the fibre is unity. In terms of this form, and given any arbitrary \( p \)-form \( \omega \) over \( X \), its image under the Thom isomorphism is the \( p + 2m \) form \( \pi^*(\omega) \wedge \Phi(E) \), which by construction has Gaussian decay along the fibre. \( \pi^*(\omega) \) is the pull-back of \( \omega \) by the projection \( \pi : E \to X \). If \( s \) is any section of \( E \), the pull-back of the Thom form under \( s \), \( s^*\Phi(E) \), is a closed form in the same cohomology class as the Euler class \( e(E) \). If \( s \) is a generic section, then \( s^*\Phi(E) \) is the Poincare dual of the zero locus of \( s \). Mathai and Quillen constructed an explicit representative, \( \Phi_{\nabla}(E) \), of the Thom form in terms of a connection \( \nabla \) in \( E \). Its pullback by a section \( s \), \( e_{s,\nabla}(E) = s^*\Phi_{\nabla}(E) \), is represented as a Grassmann integral:

\[
e_{s,\nabla}(E) = (2\pi)^{-m} \int d\chi e^{-\frac{1}{2}|s|^2 + \frac{1}{2}\chi_{a}\Omega^{ab}\chi_b + i\nabla^a\chi_{a}}.
\]

As a consistency check note that, as follows from (3.3), \( e_{s=0,\nabla}(E) = e_{\nabla}(E) \), \textit{i.e.}, the pull-back of the Mathai-Quillen representative by the zero section gives back the Euler class of \( E \). \( e_{s,\nabla}(E) \) is a closed \( 2m \)-form. This can be verified after integrating over the
Grassmann-odd variables $\chi_a$. It is closed because the exponent is invariant under the transformations:
\[ \delta s = \nabla s, \quad \delta \chi_a = is_a. \] (3.9)

It is possible to find a nice physics-like form for $e_{s,\nabla}(\mathcal{E})$. To this end we introduce grassmann odd real variables $\psi^\mu$ with the correspondence:
\[ dx^\mu \leftrightarrow \psi^\mu, \] (3.10)
\[ \omega = \frac{1}{p!} \omega_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \leftrightarrow \omega(\psi) = \frac{1}{p!} \omega_{\mu_1 \cdots \mu_p} \psi^{\mu_1} \cdots \psi^{\mu_p}. \] (3.11)

The integral over $X$ of a top-form, $\omega^{(n)}$, is therefore given by a simultaneous conventional integration over $X$ and a Berezin integration over the $\psi$'s:
\[ \int_X \omega^{(n)} = \int_X dx \int d\psi \omega^{(n)}(\psi). \] (3.12)

In this language, the Mathai-Quillen representative (3.8) can be rewritten as:
\[ e_{s,\nabla}(\mathcal{E})(\psi) = (2\pi)^{-m} \int d\chi e^{-\frac{1}{2}|s|^2 + \frac{1}{2} \chi_a \Omega^{ab}(\psi) \chi_b + i \nabla s^a(\psi) \chi_a}, \] (3.13)

and, for example, in the case $n = 2m$, one has the following expression for the Euler number of $\mathcal{E}$:
\[ \chi(\mathcal{E}) = (2\pi)^{-m} \int_X dxd\psi d\chi e^{-\frac{1}{2}|s|^2 + \frac{1}{2} \chi_a \Omega^{ab}(\psi) \chi_b + i \nabla s^a(\psi) \chi_a}. \] (3.14)

It is worth to remark that (3.14) looks like the partition function of a field theory whose “action” is:
\[ A(x, \psi, \chi) = \frac{1}{2} |s|^2 - \frac{1}{2} \chi_a \Omega^{ab}(\psi) \chi_b - i \nabla s^a(\psi) \chi_a. \] (3.15)

This action is invariant under the transformations:
\[ \delta x^\mu = \psi^\mu, \quad \delta \psi^\mu = 0, \quad \delta \chi_a = is_a. \] (3.16)

We mentioned above that the Mathai-Quillen representative interpolates between the two different approaches to the Euler class of a vector bundle. This statement can be made more precise as follows. The construction of $e_{s,\nabla}(\mathcal{E})$ is such that it is cohomologous to $e_{\nabla}(\mathcal{E})$ for any choice of a generic section $s$. Take for example the case $n = 2m$, and rescale $s \rightarrow \gamma s$. Nothing should change, so in particular:
\[ \chi(\mathcal{E}) = \int_X e_{\gamma s,\nabla}(\mathcal{E}). \] (3.17)

We can now study (3.17) in two different limits:
1. Limit $\gamma \to 0$: after using (3.3), $\chi(\mathcal{E}) = (2\pi)^{-m} \int \text{Pf}(\Omega)$. 

2. Limit $\gamma \to \infty$: the curvature term in (3.14) can be neglected, leading to $\chi(\mathcal{E}) = \sum_{x_k: s(x_k) = 0}(\pm 1)$. These signs are generated by the ratio of the determinants of $\nabla s$ and its modulus, which result from the Gaussian integrations after expanding around each zero $x_k$.

Hence, we recover from this unified point of view the two complementary ways to define the Euler class described at the beginning of the section.

Let us work out an explicit example. To be definite we will consider $\mathcal{E} = TX$. The section $s$ is taken as a vector field $V$ on $X$, which we assume to be generic. The action (3.15), after rescaling $V \to \gamma V$, takes the form:

$$A(x, \psi, \chi) = \frac{1}{2} \gamma^2 g_{\mu\nu} V^\mu V^\nu - \frac{1}{4} \chi_a R^{ab}_{\mu\nu} \psi^\mu \psi^\nu \chi_b - i \gamma \nabla_\mu V^\nu \psi^\mu e^a_\nu \chi_a.$$  (3.18)

To compute (3.14) in the limit $\gamma \to \infty$ we expand around the zeroes, $x_k$, of $V$ ($V^\mu(x_k) = 0$):

$$\chi = \sum_{x_k} \int \frac{dxd\psi d\chi}{x} (2\pi)^{-m} e^{-\frac{1}{2} \gamma^2 g_{\mu\nu} \partial_\nu \psi^\mu \partial_\nu V^\mu} e^{-\frac{1}{2} \gamma^2 g_{\mu\nu} H_{\mu\nu}^{(k)} \psi^\mu \chi_\mu}.$$  (3.19)

Next, we rescale the variables in the following way:

$$x \to \gamma^{-1} x, \quad dx \to \gamma^{-1} dx, \quad \psi \to \gamma^{-\frac{1}{2}} \psi, \quad d\psi \to \gamma^\frac{1}{2} d\psi, \quad \chi \to \gamma^{-\frac{1}{2}} \chi, \quad d\chi \to \gamma^\frac{1}{2} d\chi.$$  (3.20)

Notice that the measure is invariant under this rescaling. Using the shorthand notation for the Hessian, $H^{(k)}_{\sigma} = \partial_\sigma V^\mu|_{x_k}$, one finds, after taking the limit $\gamma \to \infty$:

$$\chi = \sum_{x_k} \frac{1}{(2\pi)^{2m}} \sqrt{g} \det H^{(k)} (2m)! \det H^{(k)} = \sum_{x_k} \frac{1}{(2m)!} \det H^{(k)}.$$  (3.21)

which indeed corresponds to the Euler number of $X$ in virtue of the Poincare-Hopf theorem.

It is possible to introduce auxiliary fields in the formulation. In the example under consideration, after using:

$$e^{-\frac{1}{2} \gamma^2 g_{\mu\nu} V^\mu V^\nu} = \frac{1}{(2\pi)^m} \int dB e^{-\frac{1}{2} \gamma^2 (g_{\mu\nu} B^\mu B^\nu + 2B_\mu V^\mu)},$$  (3.22)
being $B^\mu$ an auxiliary field, the Euler number resulting from (3.18) can be rewritten as:

$$\chi = \int d^4x \psi d\bar{\psi} dB \frac{\gamma^2 \sqrt{g}}{(2\pi)^2 m} e^{-\gamma^2 (g_{\mu\nu} B^\mu B^\nu + 2i B^\mu V^\nu) + \frac{1}{4} \chi_a \mathbb{R}^{ab}_{\mu\nu} \psi^\mu \psi^\nu \chi_b + i \gamma \nabla^\mu V^\nu \psi^\mu \psi^\nu}. \quad (3.23)$$

Making the redefinitions:

$$\begin{align*}
\psi^\mu &\rightarrow \gamma \frac{1}{2} \psi^\mu, \\
d\psi &\rightarrow (\gamma - \frac{1}{2}) 2m \sqrt{g} d\psi, \\
\chi_a &\rightarrow \gamma \frac{1}{2} e_{a\mu} \bar{\psi}^\mu, \\
d\chi &\rightarrow (\gamma - \frac{1}{2}) 2m \sqrt{g} d\bar{\psi},
\end{align*}$$

(3.24)

one obtains:

$$\chi = \int d^4x \psi d\bar{\psi} dB \frac{1}{(2\pi)^2 m} e^{-\gamma^2 \left(\frac{1}{2} g_{\mu\nu} (B^\mu B^\nu + 2i B^\mu V^\nu) - \frac{1}{4} \mathbb{R}^{\rho\sigma}_{\mu\nu} \bar{\psi}^\rho \psi^\sigma \psi^\mu \psi^\nu + i \nabla^\mu V^\nu \psi^\mu \psi^\nu\right)}.$$

(3.25)

This looks like the partition function of a topological quantum field theory, in which $g = 1/\gamma$ plays the role of the coupling constant. Furthermore, the exponent of (3.25) is invariant under the symmetry:

$$\begin{align*}
\delta x^\mu &= \psi^\mu, \\
\delta \dot{\psi}^\mu &= 0, \\
\delta \dot{\bar{\psi}}^\mu &= B^\mu, \\
\delta B^\mu &= 0.
\end{align*}$$

(3.26)

Notice that $\delta^2 = 0$. In fact, one easily finds that the exponent is indeed $\delta$-exact:

$$\chi = \int d^4x \psi d\bar{\psi} dB \frac{1}{(2\pi)^2 m} e^{-\gamma^2 \left(\frac{1}{2} g_{\mu\nu} (B^\mu B^\nu + 2i B^\mu V^\nu) + \Gamma^\sigma_{\mu\nu} \bar{\psi}^\sigma \psi^\mu \psi^\nu + i \nabla^\mu V^\nu \psi^\mu \psi^\nu\right)}.$$

(3.27)

This result makes possible to use field-theoretical arguments to conclude that $\chi$ is independent of the coupling $\gamma$ and of the metric $g_{\mu\nu}$.

The Mathai-Quillen formalism can be recasted in a conventional BRST language in which $\Psi = \frac{1}{2} \bar{\psi}^\mu (B^\mu + 2i V^\mu + \Gamma^\sigma_{\mu\nu} \bar{\psi}^\sigma \psi^\nu g^{\nu\tau})$ plays the role of a gauge fermion, and the exponent of (3.27) can be regarded as an action $A = \delta \Psi$. In many situations one has the general pattern:

$$\Psi = \chi_a \left( \frac{s^a}{\text{section}} + \frac{\theta^{ab}}{\text{connection}} \chi_b + \frac{B^a}{\text{auxiliary}} \right).$$

(3.28)
In order to fully understand the construction, let us be more specific with our example. Consider the two-sphere \( S^2 \) with the standard parametrization:

\[
\alpha : (0, \pi) \times (0, 2\pi) \rightarrow \mathbb{R}^3,
\]

\[
(\theta, \varphi) \rightarrow (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta).
\]  

(3.29)

In terms of these coordinates, we have the relations:

\[
ds^2 = d\theta^2 + \sin^2 \theta d\varphi^2, \quad g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \theta \end{pmatrix},
\]

(3.30)

and the following values for the Christoffel symbol (\( \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} (\partial_\mu g_{\nu\sigma} - \partial_\nu g_{\mu\sigma}) \)):

\[
\Gamma_\theta\theta = \Gamma_\varphi\varphi = \Gamma_\theta\varphi = 0, \\
\Gamma^\theta_{\varphi\varphi} = -\sin \theta \cos \theta, \quad \Gamma^\varphi_{\theta\varphi} = \frac{\cos \theta}{\sin \theta}.
\]  

(3.31)

Let us pick an orthonormal frame:

\[
e^\mu_a = \begin{pmatrix} 1 & 0 \\ 0 & \sin \theta \end{pmatrix}, \quad e^a_\mu = \begin{pmatrix} 1 & 0 \\ 0 & \sin \theta \end{pmatrix},
\]

(3.32)

where the vielbeins satisfy the standard relations: \( e^\mu_a e^\nu_b g_{\mu\nu} = \delta_{ab}, \ e^a_\mu e^b_\nu \delta_{ab} = g_{\mu\nu} \). The Riemann curvature tensor (\( R^\lambda_{\mu\nu\kappa} = \partial_\nu \Gamma^\lambda_{\mu\kappa} + \Gamma^\lambda_{\tau\mu} \Gamma^\tau_{\nu\kappa} \)) in \((\theta, \varphi)\) coordinates is given by:

\[
R^\theta_{\varphi\varphi\theta} = \sin^2 \theta,
\]

(3.33)

while the curvature two-form \( \Omega^{ab} \) takes the form:

\[
\Omega^{12} = R^{12}_{\varphi\theta} d\varphi \wedge d\theta = e^1_\varphi e^2_\theta g^{\varphi\varphi} R^\theta_{\varphi\theta} d\varphi \wedge d\theta = \sin \theta d\varphi \wedge d\theta.
\]  

(3.34)

Next let us consider the vector field\footnote{This vector field is actually equivalent to the one considered in \cite{24} but with a better choice of coordinates for our purposes.}:

\[
V^a = (\sin \varphi, \cos \varphi \cos \theta) \rightarrow V^\mu = (\sin \varphi, \cos \varphi \cot \theta).
\]  

(3.35)

This vector field has zeroes at \( \varphi = 0, \theta = \frac{\pi}{2} \) and \( \varphi = \pi, \theta = \frac{\pi}{2} \). The components of the form \( \nabla V^a \) are:

\[
\nabla_\theta V^\varphi = -\cos \varphi, \quad \nabla_\varphi V^\theta = \sin^2 \theta \cos \varphi, \quad \nabla_\theta V^\theta = \nabla_\varphi V^\varphi = 0,
\]

(3.36)

or, alternatively,

\[
\nabla_\theta V^a = e^a_\mu \nabla_\theta V^\mu = (0, -\cos \varphi \sin \theta), \quad \nabla_\varphi V^a = e^a_\mu \nabla_\varphi V^\mu = (\cos \varphi \sin \theta, 0).
\]  

(3.37)
and therefore,
\[ \nabla V^a = (\sin^2 \theta \cos \varphi d\varphi, -\cos \varphi \sin \theta d\theta). \]  
(3.38)

The Euler class representative,
\[ e_{V, \nabla}(TS^2) = \frac{1}{2\pi} \int d\chi_1 d\chi_2 e^{-\frac{i}{2} V^a V^a + \frac{i}{2} \chi_a \Omega^a_{\psi b} \chi_b + i \nabla V^a \chi_a}, \]  
(3.39)
after performing the rescaling \( V^a \to \gamma V^a \),
\[ -\frac{1}{2} V^a V^a + \frac{1}{2} \chi_a \Omega^a_{\psi b} \chi_b + i \nabla V^a \chi_a \to \]  
\[ -\frac{\gamma^2}{2} (\sin^2 \theta + \cos^2 \varphi \cos^2 \theta) - \chi_1 \chi_2 \sin \theta d\theta \wedge d\varphi \]  
\[ + i \gamma (\sin^2 \theta \cos \varphi d\varphi \chi_1 - \cos \varphi \sin \theta d\theta \chi_2), \]  
(3.40)
becomes:
\[ e_{V, \nabla}(TS^2) = \frac{1}{2\pi} e^{-\frac{\gamma^2}{2} (\sin^2 \theta + \cos^2 \varphi \cos^2 \theta) \sin \theta (1 + \gamma^2 \cos^2 \varphi \sin^2 \theta) d\theta \wedge d\varphi}. \]  
(3.41)

The Euler number of \( S^2 \) is given by the integral:
\[ \chi(S^2) = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin \theta e^{-\frac{\gamma^2}{2} (\sin^2 \theta + \cos^2 \varphi \cos^2 \theta) (1 + \gamma^2 \cos^2 \varphi \sin^2 \theta)}. \]  
(3.42)

Although \( \gamma \) appears explicitly in this expression, the result of the integration should be independent of \( \gamma \). The reader is urged to prove it (we do not know of any analytical proof, we have only numerical evidence). One can perform, however, two independent checks. On the one hand, in the limit \( \gamma \to 0 \), (3.42) gives trivially the correct result, \( \chi(S^2) = 2 \). On the other hand, one can explore the opposite limit, \( \gamma \to \infty \), where the integral,
\[ \chi(S^2) = \int_{S^2} dx \int d\psi_1 d\psi_2 \int d\chi_1 d\chi_2 \frac{1}{2\pi} e^{-A(x, \psi, \chi)}, \]  
(3.43)
with \( A(x, \psi, \chi) = \frac{1}{2} V^a V^a - \frac{1}{2} \chi_a \Omega^a_{\psi b} \chi_b - i \nabla V^a \chi_a \), is dominated by the zeroes of \( V^a \). We expand around them:

(a) \( \theta = \frac{\pi}{2} + x, \varphi = 0 + y \). We get:
\[ V^a V^a = \sin^2 \varphi + \cos^2 \varphi \cos^2 \theta = x^2 + y^2 + \cdots \]  
\[ \frac{1}{4} \chi_a R_{\mu \nu}^{ab} \psi^\mu \psi^\nu \chi_b = \chi_1 \chi_2 \psi^\theta \psi^\varphi R^{12}_{\theta \varphi} = \sin \theta \chi_1 \chi_2 \psi^\theta \psi^\varphi = (1 - \frac{x^2}{2} + \cdots) \chi_1 \chi_2 \psi^\theta \psi^\varphi \]  
\[ \nabla_\mu V^\nu \psi^\mu e^a_{\psi} \chi_a = -\cos \varphi \sin \theta \psi^\theta \chi_2 + \cos \varphi \sin^2 \theta \psi^\varphi \chi_1 = -\psi^\theta \chi_2 + \psi^\varphi \chi_1 + \cdots \]  
(3.44)
Next, performing the rescaling:

\[
x, y \rightarrow \gamma^{-1} x, \gamma^{-1} y,
\]

\[
\psi^\mu \rightarrow \gamma^{-\frac{1}{2}} \psi^\mu,
\]

\[
\chi_a \rightarrow \gamma^{-\frac{1}{2}} \chi_a,
\]

we obtain:

\[
\frac{1}{2\pi} \int dx dy e^{-\frac{1}{2}(x^2 + y^2)} \int d\chi_1 d\chi_2 d\psi^\theta d\psi^\varphi e^{i(-\psi^\theta \chi_2 + \psi^\varphi \chi_1)} = 1.
\]

(3.46)

(b) Similarly, expanding around the second zero, \( \theta = \frac{\pi}{2} + x, \varphi = \pi + y \), one finds the contribution:

\[
\frac{1}{2\pi} \int dx dy e^{-\frac{1}{2}(x^2 + y^2)} \int d\chi_1 d\chi_2 d\psi^\theta d\psi^\varphi e^{i(\psi^\theta \chi_2 - \psi^\varphi \chi_1)} = 1.
\]

(3.47)

Therefore, in the limit \( \gamma \rightarrow \infty \) we have reproduced the behavior described in (3.21),

\[
\chi(S^2) = \left( \frac{\det \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}{\det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} + \frac{\det \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}{\det \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} \right) = 2.
\]

(3.48)

### 3.2 Infinite-dimensional case

We now turn into the study of the infinite-dimensional case. The main complication that one finds in this case is that \( e(\mathcal{E}) \) is not defined. By taking advantage of what we have learned so far, we could try to use the Mathai-Quillen formalism to define something analogous to an Euler class for \( \mathcal{E} \). It turns out that this is actually possible. The outcome of the construction is what is called a regularized Euler number for the bundle \( \mathcal{E} \). Unfortunately, it depends explicitly on the section chosen for the construction, so it is important to make good selections.

The outline of the construction is as follows. First recall that, as stated in (3.6), in the finite-dimensional case \( \chi(X) = \chi(X_V) \) when \( V \) is non-generic, i.e., when its zero locus, \( X_V \), has dimension \( \dim(X_V) < 2m \). For \( X \) infinite dimensional the idea is to introduce a vector field \( V \) with finite-dimensional zero locus. The regularized Euler number of \( \mathcal{E} \) would be then defined as:

\[
\chi_V(X) = \chi(X_V),
\]

(3.49)

which explicitly depends on \( V \). By analogy with the finite-dimensional case one expects that:

\[
\chi_V(X) = \int_X e_V(TX),
\]

(3.50)
as a functional integral, where \( e_{s,\nabla}(TX) \) is meant to be the Mathai-Quillen representative for the corresponding Euler class.

In general, the regularized Euler number \( \chi_s(\mathcal{E}) \) of an infinite-dimensional vector bundle \( \mathcal{E} \) is given by:

\[
\chi_s(\mathcal{E}) = \int_X e_{s,\nabla}(\mathcal{E}),
\]

(3.51)

where \( e_{s,\nabla}(\mathcal{E}) \) is given by the Mathai-Quillen formalism. The construction of \( e_{s,\nabla}(\mathcal{E}) \) will be illustrated by the description of several examples. This construction follows the pattern of the finite dimensional case. Before entering into the discussions of these examples it is important to remark that equation (3.51) makes sense when the zero locus of \( s \), \( X_s \), is finite dimensional. \( \chi_s(\mathcal{E}) \) is the Euler number of some finite-dimensional vector bundle over \( X_s \), and it corresponds to the regularized Euler number of the infinite-dimensional bundle \( \mathcal{E} \). Of course, \( \chi_s(\mathcal{E}) \) depends on \( s \), but if \( s \) is naturally associated to \( \mathcal{E} \) one expects to obtain interesting topological information.

### 3.2.1 Supersymmetric quantum mechanics

Let \( X \) be a smooth, orientable, Riemannian manifold with metric \( g_{\mu\nu} \). The loop space, \( LX \), is defined by the set of smooth maps:

\[
x : S^1 \to X,
\]

\[
t \in [0,1] \to x^\mu(t),
\]

\[
x^\mu(0) = x^\mu(1).
\]

(3.52)

Let us denote by \( T(LX) \) the tangent vector bundle to \( LX \), with fibre \( \mathcal{F} = T_x(LX) = \Gamma(x^*(TX)) \). A vector field over \( LX \) has the form:

\[
V(x) = \int dt V^\mu(x(t)) \frac{\partial}{\partial x^\mu(t)}.
\]

(3.53)

The metric on \( X \) provides a natural metric for \( T_x(LX) \): let \( V_1, V_2 \in T_x(LX) \), then,

\[
g_x(V_1, V_2) = \int dt g_{\mu\nu}(x(t)) V_1^\mu(x(t)) V_2^\nu(x(t)).
\]

(3.54)

The Levi-Civita connection on \( LX \) is the pullback connection from \( X \):

\[
\nabla V = \int dt_1 dt_2 \left[ \frac{\delta V^\mu(x(t_1))}{\delta x^\nu(t_2)} + \Gamma^\mu_{\nu\lambda}(x(t_2)) V^\lambda(x(t_1)) \delta(t_1 - t_2) \right] \frac{\partial}{\partial x^\mu(t_1)} \otimes \dot{x}^\nu(t_2),
\]

(3.55)

where \( \{ \partial/\partial x^\mu(t) \} \) is a basis of \( T_x(LX) \), and \( \{ \dot{x}^\nu(t) \} \) is a basis of \( T^*_x(LX) \). Let us consider the vector field:

\[
V^\mu(x) = \frac{d}{dt} x^\mu \equiv \dot{x}^\mu.
\]

(3.56)
The zero locus of $V$ is the space of constant loops, $(LX)_{V} = X$. Therefore, the regularized Euler number of $LX$ is the Euler number of $X$ itself:

$$\chi_{V}(LX) = \chi((LX)_{V}) = \chi(X). \quad (3.57)$$

Let us construct the Mathai-Quillen representative for this Euler number following the same procedure as in the finite-dimensional case:

Finite Dimensional Case
\[
\begin{align*}
\chi &= \int \frac{dx d\psi d\bar{\psi} dB}{(2\pi)^{2m}} e^{-\gamma^2 \delta \Psi(x,\psi,\bar{\psi},B)}, \\
\Psi(x,\psi,\bar{\psi},B) &= \frac{1}{2} \bar{\psi}_\mu (B^\mu + 2iV^\mu + \Gamma^\sigma_{\tau\nu} \bar{\psi}_\sigma \psi_\tau g^{\mu\tau}), \\
\delta x^\mu &= \psi^\mu, \quad \delta \psi^\mu = 0, \\
\delta \bar{\psi}_\mu &= B_\mu, \quad \delta B_\mu = 0.
\end{align*}
\]

Supersymmetric Quantum Mechanics
\[
\begin{align*}
\chi_V &= \int \frac{dx d\psi d\bar{\psi} dB}{(2\pi)^{2m}} e^{-\gamma^2 \delta \Psi(x,\psi,\bar{\psi},B)}, \\
\Psi(x,\psi,\bar{\psi},B) &= \frac{1}{2} \int dt \bar{\psi}_\mu (B^\mu + 2i\dot{x}^\mu + \Gamma^\sigma_{\tau\nu} \bar{\psi}_\sigma \psi_\tau g^{\mu\tau}), \\
\delta x^\mu(t) &= \psi^\mu(t), \quad \delta \psi^\mu(t) = 0, \\
\delta \bar{\psi}_\mu(t) &= B_\mu(t), \quad \delta B_\mu(t) = 0.
\end{align*}
\]

In order to evaluate $\chi_V(LX)$ we first integrate out the auxiliary fields in the action in (3.58). One finds:

$$\chi_V = \int \frac{dx d\psi d\bar{\psi}}{(2\pi)^m} \frac{1}{\sqrt{g}} e^{-\gamma^2 \int dt \left[ \frac{1}{2g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} - i\bar{\psi} \nabla t \psi - \frac{1}{4} R_{\mu\nu} \bar{\psi}_\mu \bar{\psi}_e \psi^e \psi^\nu \right]}, \quad (3.59)$$

where,

$$\nabla_t \psi^\mu = \dot{\psi}^\mu + \Gamma^\sigma_{\mu\nu} \bar{\psi}^\sigma \dot{x}^\nu. \quad (3.60)$$

The action in the exponential of (3.59) is precisely the action corresponding to supersymmetric quantum mechanics [86]. The δ-transformations become:

$$\delta x^\mu = \psi^\mu, \quad \delta \psi^\mu = 0, \quad \delta \bar{\psi}_\mu = -ig_{\mu\nu} \dot{x}^\nu - \Gamma^\sigma_{\mu\nu} \bar{\psi}_\sigma \psi^\nu, \quad (3.61)$$

which close only on-shell, i.e., $\delta^2 = 0$, modulo field equations. As discussed below, they can be regarded as supersymmetry transformations.

At this point it is convenient to discuss an additional symmetry which is present in the systems under consideration: the ghost number symmetry. The δ-symmetry is compatible with the following ghost number assignment:

| δ | x | ψ | ψ | B |
|---|---|---|---|---|
| 1 | 0 | 1 | -1 | 0 |
The action is ghost number invariant, as it is the measure itself. In fact, \( \#\psi\)-zero modes = \( \#\bar{\psi}\)-zero modes, i.e., \( \dim(\ker \nabla_t) = \dim(\text{coker} \nabla_t) \). \( \ker \nabla_t \) corresponds to the tangent space at a constant bosonic mode, i.e., the tangent space at the zero-locus of a section or moduli space: if \( \dot{x}^\mu = 0 \) and \( x^\mu \to x^\mu + \delta x^\mu \) then,

\[
\frac{d}{dt}\delta x^\mu = 0 \iff \nabla_t \delta x^\mu |_{\dot{x}^\mu = 0} = 0.
\]

Thus \( \ker \nabla_t \) provides the directions in which a given bosonic zero-mode can be deformed into a nearby bosonic zero-mode. The ghost number symmetry is potentially anomalous. In this case:

\[
\text{Ghost number anomaly} = \dim(\ker \nabla_t) - \dim(\text{coker} \nabla_t) = 0,
\]

but in general it does not vanish.

Let us compute \( \chi_V(LX) \) in the limit \( \gamma \to \infty \). In this limit the exact result is obtained very simply by considering the expansion of the exponential around bosonic and fermionic zero modes:

- **Bosonic part**: \( \dot{x}^\mu = 0 \to x^\mu \) constant
- **Fermionic part**: \( \{ \psi^\mu(t) = \psi^\mu + \text{non-zero modes} \}
  \{ \bar{\psi}^\mu(t) = \bar{\psi}^\mu + \text{non-zero modes} \}

The integration over the non-zero modes is trivial since the \( \delta \) symmetry implies that the ratio of determinants is equal to 1. The integration over the zero modes gives:

\[
\begin{align*}
\chi_V &= \int_X dx \left( \frac{2\pi}{\gamma} \right)^{-m} \int \left[ \prod_{\mu=1}^{2m} d\psi^\mu \right] \left[ \prod_{\nu=1}^{2m} d\bar{\psi}^\nu \right] e^{\gamma^2 \frac{1}{4} \Omega^{\mu\nu} \bar{\psi}_\mu \psi_\nu} \\
&= \int_X \left( \frac{2\pi}{\gamma} \right)^{-m} \int \left( \prod_{a=1}^{2m} d\chi_a \right) e^{\frac{\gamma^2}{2} \bar{\chi}_a \Omega^{ab} \chi_b} = \int_X \left( \frac{2\pi}{\gamma} \right)^{-m} \text{Pf}(\Omega^{ab}) \\
&= \chi(X)
\end{align*}
\]

where \( \chi_a = e^\mu_a \bar{\psi}_\mu \).

In the general case, the measure is not ghost-number invariant. To get a non-vanishing functional integral one needs to introduce operators with non-zero ghost number. Knowledge of the ghost number anomaly gives information on the possible topological invariants, i.e., on the possible non-vanishing vacuum expectation values of the theory, so in the end it provides a selection rule. In the most interesting situations the ghost number anomaly is given by the index of some operator in the theory. In general, the bosonic zero-modes provide the zero-locus of the section, whereas the fermionic zero-modes are related to the possible deformations of the bosonic zero-modes.
To finish this quick tour through supersymmetric quantum mechanics, it is interesting to recall that $\chi_V$ can be computed using Hamiltonian methods [5, 6, 40]. The expression (3.59) possesses a second $\delta$-like symmetry, $\bar{\delta}$:
\[
\begin{align*}
\delta x^\mu &= \psi^\mu, \\
\delta \psi^\mu &= 0, \\
\bar{\delta} \bar{\psi}_\mu &= -ig_{\mu\nu} \dot{x}^\nu - \Gamma^\sigma_{\mu\nu} \bar{\psi}_\sigma \psi^\nu, \\
\bar{\delta} \psi^\mu &= 0, \\
\bar{\delta} \bar{\psi}_\mu &= -ig_{\mu\nu} \dot{x}^\nu - \Gamma^\sigma_{\mu\nu} \psi_\sigma \bar{\psi}^\nu,
\end{align*}
\]

One finds, after using the field equations, that:
\[
\begin{align*}
\delta^2 &= 0, \\
\bar{\delta}^2 &= 0, \\
\delta \bar{\delta} + \bar{\delta} \delta &= \frac{d}{dt},
\end{align*}
\]
which, in terms of operators,
\[
\begin{align*}
\delta &\leftrightarrow Q, \\
\bar{\delta} &\leftrightarrow \bar{Q}, \\
\frac{d}{dt} &\leftrightarrow H,
\end{align*}
\]
($H$ stands for the Hamiltonian operator) implies that:
\[
Q^2 = \bar{Q}^2 = 0, \quad \{Q, \bar{Q}\} = Q\bar{Q} + \bar{Q}Q = H,
\]
which is the standard supersymmetry algebra for 0 + 1-supersymmetric field theories. We can carry out explicitly the canonical quantization of the theory by imposing the canonical commutation relations:
\[
\begin{align*}
\{\bar{\psi}^\mu, \psi^\nu\} &= g^{\mu\nu}, \\
\{\psi^\mu, \psi^\nu\} &= \{\bar{\psi}^\mu, \bar{\psi}^\nu\} = 0.
\end{align*}
\]
From these equations it is natural to interpret $\bar{\psi}$ as fermion creation operators. In view of this, we have the following structure on the Hilbert space:

\[
\begin{cases}
-\text{States with one fermion: } \omega_\mu(x)\bar{\psi}^\mu|\Omega\rangle, \\
-\text{States with two fermions: } \omega_{\mu\nu}(x)\bar{\psi}^\mu\bar{\psi}^\nu|\Omega\rangle, \\
\vdots \\
-\text{States with } n \text{ fermions: } \omega_{\mu_1,\ldots,\mu_n}(x)\bar{\psi}^{\mu_1}\ldots\bar{\psi}^{\mu_n}|\Omega\rangle,
\end{cases}
\]
being $|\Omega\rangle$ the Clifford vacuum. The Hilbert space of our system is thus $\Omega^*(X)$, the set of differential forms on $X$. $Q$ and $\bar{Q}$ are represented on this Hilbert space by the exterior derivative and its adjoint,
\[
Q \leftrightarrow d, \quad \bar{Q} \leftrightarrow d^+,
\]
therefore, the Hamiltonian is the Hodge-de Rham Laplacian on $X$:
\[
H = dd^+ + d^+d = \Delta.
\]
The zero-energy states are in one-to-one correspondence with the harmonic forms on $X$. After rescaling the parameter $t$ and the fermionic fields by,

$$t \rightarrow \gamma^2 t, \quad \bar{\psi} \rightarrow \gamma^{-1 \bar{\psi}}, \quad \psi \rightarrow \gamma^{-1} \psi,$$

the partition function (3.59) takes the form:

$$\chi_{V} = \int \frac{dx d\psi d\bar{\psi}}{(2\pi)^m} \sqrt{g} e^{-\frac{1}{\gamma^2} \int dt \left[ \frac{i}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - i \bar{\psi} \nabla_t \psi - \frac{1}{4} R^{\mu\nu\rho\sigma} \bar{\psi} \psi \bar{\psi} \psi \right]},$$

(3.74)

Using heat-kernel techniques [6, 40] one finds:

$$\chi_{V} = \text{Tr} \left[ (-1)^F e^{-\frac{1}{\gamma} H} \right],$$

(3.75)

where $F$ is the fermion number operator. In the limit $\gamma \rightarrow 0$ only the zero-modes of $H$ survive and therefore one must count harmonic forms with signs, which come from $(-1)^F$, leading to the result:

$$\chi_{V} = \sum_{k=0}^{2m} (-1)^k b_k = \chi(X),$$

(3.76)

($b_k$ are the Betti numbers of $X$) in perfect agreement with our previous calculation.

Actually, due to supersymmetry, for each non-zero energy bosonic mode there is a fermionic one with the same energy which cancels its contribution to (3.75). Therefore, the computation performed in the Hamiltonian formalism holds for any $\gamma$.

### 3.2.2 Topological sigma models

Our next example of TQFT was introduced by Witten [88] as a twisted version of the $N = 2$ supersymmetric sigma model in two dimensions. Here we will briefly analyze it within the framework of the Mathai-Quillen formalism. The model, in its more general form, is defined in terms of a smooth, almost-hermitian manifold $X$, with metric $G_{mn}$ and almost-complex structure $J_{ij}$ satisfying:

$$J_{ij} J_{jk} = -\delta^i_k, \quad G_{ij} J^i_k J^j_m = G_{km}.$$  

(3.77)

Let us consider the set of smooth maps from a Riemann surface $\Sigma$ to $X$, $\phi : \Sigma \rightarrow X$, and the vector bundle $E \rightarrow \Sigma$ with fibre $\mathcal{F} = \Gamma(T^*\Sigma) \otimes \phi^*(TX))^+$, where by $^+$ we denote the self-dual part, i.e., if $\tilde{\phi}^i_\alpha \in \mathcal{F}$, then $\tilde{\phi}^i_\alpha$ is self-dual, i.e., $\tilde{\phi}^i_\alpha J^j_\beta \epsilon^\beta_\alpha = \tilde{\phi}^j_\beta$. The choice of section in $E$ is the following:

$$s(\phi)^i_\alpha = \partial_\alpha x^i + J^j_\beta \epsilon^\beta_\alpha \partial_\beta x^j.$$

(3.78)

Notice that it satisfies the self-duality condition $s(\phi)^j_\beta = s(\phi)^i_\alpha J^j_\beta \epsilon^\alpha_\beta$. 

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We will restrict the discussion to the simplest case in which the manifold \( X \) is Kähler. Following the general pattern (3.28), the gauge fermion is given by:

\[
\Psi(\phi, \chi, \varrho, B) = \frac{1}{2} \int_\Sigma d^2\sigma \sqrt{h} \left[ \varrho^\alpha (B_\alpha^\mu + 2i \varrho_\alpha^\mu + \Gamma^\mu_{\nu\sigma} \chi^\nu \varrho^\sigma_\alpha) \right].
\] (3.79)

The model is invariant under the symmetry transformations:

\[
\delta x^\mu = \chi^\mu, \quad \delta \varrho_\alpha^\mu = B_\alpha^\mu, \quad \delta \chi^\mu = 0, \quad \delta B_\alpha^\mu = 0.
\] (3.80)

After integrating out the auxiliary fields the action \( A = \delta \Psi \) reads:

\[
A(\phi, \chi, \varrho) = \int_\Sigma d^2\sigma \sqrt{h} \left( \frac{1}{2} G_{\mu\nu} h^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x^\nu + \frac{1}{2} \epsilon^{\alpha\beta} J_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu - ih^{\alpha\beta} G_{\mu\nu} \varrho^\mu_\alpha D^\beta_\chi \right)
\] (3.81)

where \( D_\alpha \chi^\mu = \partial_\alpha \chi^\mu + \Gamma^\mu_{\nu\sigma} \partial_\alpha x^\nu \chi^\sigma \).

Rewriting (3.78) in terms of holomorphic indices, \( \alpha \rightarrow (z, \bar{z}) \), and \( i \rightarrow (I, \bar{I}) \), the equation for the zero locus of the section becomes:

\[
\partial_\alpha x^i + J^i_\beta \epsilon_\alpha^\beta \partial_\beta x^\bar{z} = 0 \rightarrow \partial_\alpha x^I = 0, \quad (3.82)
\]

i.e., it corresponds to holomorphic instantons. In order to study the dimension of this moduli space one must study the possible deformations of the solutions of (3.82):

\[
x^i \rightarrow x^i + \delta x^i, \quad (D_\alpha \delta x^i)^+ = 0. \quad (3.83)
\]

This is precisely the field equation for the field \( \chi \),

\[
(D_\alpha \chi)^+ = 0, \quad (3.84)
\]

which clarifies the role played by the \( \chi \)-zero modes. The dimension of the moduli space of holomorphic instantons can be obtained with the help of an index theorem, and in many situations coincides with the ghost-number anomaly of the theory. Contrary to the case of supersymmetric quantum mechanics, this anomaly is in general not zero. This implies that, in general, one is forced to insert operators to obtain non-trivial results.

The observables are obtained from the analysis of the \( \delta \)-cohomology associated to the symmetry (3.80). The highest-ghost-number ones turn out to be [88]:

\[
O^{(0)}_A = A_{i_1 \ldots i_p} \chi^{i_1} \chi^{i_2} \cdots \chi^{i_p}, \quad A \in \Omega^*(X),
\] (3.85)

and satisfy the relation:

\[
\{Q, O^{(0)}_A \} = O^{(0)}_{dA},
\] (3.86)
where $Q$ denotes the generator of the symmetry $\delta$. This relation allows to identify the $Q$-cohomology classes of the highest-ghost-number observables with the (de Rham) cohomology classes of $X$.

The topological descent equations (2.24) now take the form:

$$d\mathcal{O}_A^{(0)} = \{Q, \mathcal{O}_A^{(1)}\}, \quad d\mathcal{O}_A^{(1)} = \{Q, \mathcal{O}_A^{(2)}\}. \quad (3.87)$$

They are easily solved:

$$\mathcal{O}_A^{(1)} = A_{i_1, \ldots, i_p} \partial_{\alpha} x^{i_1} \chi^{i_2} \cdots \chi^{i_p} d\sigma^\alpha, \quad \mathcal{O}_A^{(2)} = \frac{1}{2} A_{i_1, \ldots, i_p} \partial_\alpha x^{i_1} \partial_\beta x^{i_2} \chi^{i_3} \cdots \chi^{i_p} d\sigma^\alpha \wedge d\sigma^\beta. \quad (3.88)$$

With the help of these operators one completes the family of observables which, as expected, are labeled by homology classes of the two-dimensional manifold $\Sigma$:

$$\int_\gamma \mathcal{O}_A^{(1)}, \quad \int_\Sigma \mathcal{O}_A^{(2)}. \quad (3.89)$$

The topological sigma model which has been described in this section is called of type A. It turns out that there are two possible ways to twist $N = 2$ supersymmetric sigma models. One of the possibilities leads to type-A models while the other generates what are called type-B models [58, 91]. The existence of these two models is linked [91] to mirror symmetry in the context of string theory. The type-B model constitutes a special kind of TQFT which does not fall into any of the two kinds described in the previous section. Type-A models depend on the Kähler class of the target manifold and are independent of the complex structure. On the contrary, type-B models depend on the complex structure and are independent of the Kähler class. Type-B models have been generalized to accommodate Kodaira-Spencer deformation theory [60]. They have been also analyzed from other points of view [71].

Topological sigma models have been generalized including potential terms [82, 57]. The resulting theories for the case of type A have been understood recently in the context of the Mathai-Quillen formalism after the construction of equivariant extensions [72].
Donaldson-Witten theory was historically the first TQFT to be introduced. It was constructed by Witten [87] in 1988 using some insight from Floer theory [36] and twisting $N = 2$ supersymmetric Yang-Mills theory. The vacuum expectation values of its observables are Donaldson invariants for four-manifolds [32, 33, 35]. The theory was later analyzed by Atiyah and Jeffrey from the viewpoint of the Mathai-Quillen formalism [15]. See [67] for other early approaches to this theory.

Donaldson invariants were introduced [32] by S. Donaldson in 1983. They are topological invariants for four-manifolds which depend on the differentiable structure of the manifold. They are very important in topology because they are helpful in the classification of differentiable four-manifolds. Contrary to the case of dimensions two and three, for higher dimensions there are topological obstructions for the existence of smooth structures. Though the origin of this problem is well understood in dimensions five and higher, the situation in four dimensions is quite different. Donaldson invariants constitute a very promising tool to improve our knowledge in the case of four dimensions.

The geometric framework for Donaldson-Witten theory is the following. Let us consider a compact oriented four-dimensional manifold $X$ endowed with a metric $g_{\mu\nu}$. Over this manifold $X$ we construct a principal bundle, $P \rightarrow X$, with group $G$ which will be assumed to be simple and compact. The automorphism group of the bundle $P$, $\mathcal{G}$, is the infinite dimensional gauge group, whose Lie algebra will be denoted by $\text{Lie}(\mathcal{G}) = \Gamma(\text{ad}P) = \Omega^0(X, \text{ad} P)$. A connection in $P$ will be denoted by $A$ and the corresponding covariant derivative and self-dual part of its curvature by $D_\mu$ and $F^+ = p^+(dA + A \wedge A)$, respectively.

The aim of Donaldson-Witten theory is to reformulate, in a field-theoretic language, the theory proposed by Donaldson [32, 33] which characterizes diffeomorphism classes of four-manifolds in terms of cohomology classes built on the moduli space of the anti-self-dual (ASD) $G$-instantons of Atiyah, Hitchin and Singer [12, 38]. The main ingredient of the theory is therefore the instanton equation:

$$F^+(A) = \frac{1}{2} \left( F(A) + *F(A) \right) = 0. \quad (4.1)$$

Starting from the instanton equation (4.1) we would like to build the topological field theory which is associated to these equations in the framework of the Mathai-Quillen formalism. According to our previous discussion, we have to specify (i) a field space, and (ii) a vector space with the quantum numbers of equation (4.1). The configuration or field space of the theory is just the space of $G$-connections on $P$, $\mathcal{A}$. The vector space $\mathcal{F}$ is the space of self-dual two-forms on $X$ with values in the adjoint bundle $\text{ad}P$, $\Omega^{2,+}(X, \text{ad} P)$. There is a natural action of the group of gauge transformations $\mathcal{G}$ on both $\mathcal{A}$ and $\mathcal{F}$, which allows us to introduce the principal bundle $\mathcal{A} \rightarrow \mathcal{A}/\mathcal{G}$, and the associated vector bundle $\mathcal{E}_+ = \mathcal{A} \times_\mathcal{G} \Omega^{2,+}(X, \text{ad} P)$. In this context, equation
is regarded as defining a section of $E_s$, $s: \mathcal{A}/\mathcal{G} \to E_s$,

$$s(A) = F^+(A).$$ \hfill (4.2)

The zero locus of this section gives precisely the moduli space of ASD instantons.

In order to complete the construction we must specify the field content of the theory. Let us introduce the following set of fields:

$$\chi_{\mu \nu}, \ G_{\mu \nu} \in \Omega^{2^+}(X, \text{ad}P), \ \psi_\mu \in \Omega^1(X, \text{ad}P), \ \eta, \ \lambda, \ \phi \in \Omega^0(X, \text{ad}P).$$ \hfill (4.3)

The ghost number carried by each of the fields is the following:

| $A_\mu$ | $\chi_{\mu \nu}$ | $G_{\mu \nu}$ | $\psi_\mu$ | $\eta$ | $\lambda$ | $\phi$ |
|--------|-----------------|----------------|----------|--------|---------|--------|
| 0      | -1             | 0              | 1        | -1     | -2      | 2      |

This field content is bigger than the standard one described in the previous section in our discussion of supersymmetric quantum mechanics and topological sigma models. The reason for this is that for situations in which a gauge symmetry is present the Mathai-Quillen formalism must be modified so that pure gauge degrees of freedom are projected out. We will not discuss these aspects here. We refer the reader to [30] for details. The outcome of the analysis is that now the gauge fermion decomposes into two parts: $\Psi = \Psi_{\text{loc}} + \Psi_{\text{proj}}$. The first one enforces the localization into the moduli space while the second one takes care of the projection.

In (4.3) the Grassmann-odd self-dual two-form $\chi_{\mu \nu}$ is the fibre antighost, while $G_{\mu \nu}$ is its bosonic partner (it is an auxiliary field). The Grassmann-odd one-form $\psi_\mu$ lives in the (co)tangent space to the field space and is to be understood as providing a basis for differential forms on $\mathcal{A}$, whereas the scalar bosonic field $\phi$ –or rather its expectation value $\langle \phi \rangle$– plays the role of the curvature two-form of the bundle $\mathcal{A} \to \mathcal{A}/\mathcal{G}$. The Grassmann-odd scalar field, $\eta$, together with its bosonic partner, $\lambda$, enforce the horizontal projection [30].

The scalar symmetry which characterizes the theory has the form:

\[
\begin{align*}
\delta A_\mu &= \psi_\mu, & \delta \chi_{\mu \nu} &= G_{\mu \nu}, \\
\delta \psi &= d_A \phi, & \delta G_{\mu \nu} &= i[\chi_{\mu \nu}, \phi], \\
\delta \phi &= 0, & \delta \lambda &= \eta, & \delta \eta &= i[\lambda, \phi],
\end{align*}
\] \hfill (4.4)

where $\delta^2$ = gauge transformation with gauge parameter $\phi$, so one is led to study the $\mathcal{G}$-equivariant cohomology of $\delta$. The action of the theory is $\delta$-exact. The appropriate gauge fermions are:

\[
\begin{align*}
\Psi_{\text{loc}} &= \int d^4x \sqrt{|g|} \text{Tr} \left[ 2\chi_{\mu \nu}(F^+_{\mu \nu} - \frac{1}{2}G^{\mu \nu}) \right], \\
\Psi_{\text{proj}} &= \int d^4x \sqrt{|g|} \text{Tr} [i\lambda D_\mu \psi^\mu]. \hfill (4.5)
\end{align*}
\]
After integrating out the auxiliary fields the action reads:

$$\delta(\Psi_{\text{loc}} + \Psi_{\text{proj}}) \rightarrow \int_M d^4x \sqrt{g} \text{Tr} \left( F^{\mu \nu} - i \chi^{\mu \nu} D_\mu \psi_\nu + i \eta D_\mu \psi^\mu + \frac{1}{4} \phi \{ \chi_{\mu \nu}, \chi^{\mu \nu} \} + \frac{i}{4} \lambda \{ \psi_\mu, \psi^\mu \} - \lambda D_\mu D^\mu \phi \right).$$ (4.6)

The moduli space associated to the theory is the space of solutions of (4.1) modulo gauge transformations. This is the space of ASD instantons, which is finite dimensional and will be denoted by $\mathcal{M}_{\text{ASD}}$. To obtain its dimension one has to study the number of independent perturbations to the equations (4.1), modulo gauge transformations. They are given by the equations:

$$A \rightarrow A + \delta A \Rightarrow (D_\mu \delta A_\nu)^+ = 0,$$
$$d_A^* \delta A = 0.$$ (4.7)

The second equation just says that $\delta A$ is orthogonal to the vertical directions (gauge orbits) tangent to the field space, which are of the form $d_A \omega$, $\omega \in \Omega^0(X, \text{ad}P)$. The equations above are precisely the $\psi_\mu$-field equations as derived from the action (4.6): $(D_\mu \psi_\nu)^+ = 0$, $D_\mu \psi^\mu = 0$, ($\psi_\mu$ zero modes). The dimension of the moduli space is calculated from (4.7) with the aid of an index theorem [12]. For $G = SU(2)$, the result is:

$$\dim \mathcal{M}_{\text{ASD}} = 8k - \frac{3}{2} (\chi + \sigma),$$ (4.8)

where $k$ is the instanton number, and $\chi$ and $\sigma$ are, respectively, the Euler characteristic and the signature of the manifold $X$. The ghost-number anomaly equals precisely the dimension of the moduli space, which is generically not zero. This implies that observables must be introduced to obtain non-vanishing vacuum expectation values. The observables of the theory are obtained from the analysis of the $G$-equivariant cohomology of $\delta$ (recall $\delta^2 = \text{gauge transformation}$). We will come back to this issue later. Now we will construct the theory from a different point of view.

### 4.1 Twist of $N = 2$ supersymmetry

We have described Donaldson-Witten theory from the point of view of the Mathai-Quillen formalism. The construction results rather compact and geometric. However, this approach was not available in the early days of TQFT, and, in fact, the theory was originally constructed in the less geometric way that we will review now. This alternative formulation, though being less transparent from the geometric point of view, provides an explicit link to four-dimensional $N = 2$ supersymmetric Yang-Mills theory which has proved to be very fruitful to perform explicit calculations.

Let us begin with a review of some generalities concerning $N = 2$ supersymmetry in four-dimensions. The global symmetry group of $N = 2$ supersymmetry in $\mathbb{R}^4$
is $H = SU(2)_L \otimes SU(2)_R \otimes SU(2)_I \otimes U(1)_R$ where $\mathcal{K} = SU(2)_L \otimes SU(2)_R$ is the rotation group and $SU(2)_I \otimes U(1)_R$ is the internal (chiral) symmetry group. The supercharges, $Q^i_\alpha$ and $\overline{Q}_{i\dot{\alpha}}$, which generate $N = 2$ supersymmetry have the following transformations under $H$:

$$Q^i_\alpha \left( \frac{1}{2}, 0, 0, \frac{1}{2} \right)^1, \quad \overline{Q}_{i\dot{\alpha}} \left( 0, \frac{1}{2}, 1, \frac{1}{2} \right)^{-1}, \quad (4.9)$$

where the superindex denotes the $U(1)_R$ charge and the numbers within parentheses the representations under each of the factors in $SU(2)_L \otimes SU(2)_R \otimes SU(2)_I$. The supercharges (4.9) satisfy:

$$\{Q^i_\alpha, Q^j_\beta\} = \delta^i_j P_{\alpha\beta}. \quad (4.10)$$

The twist consists of considering as the rotation group the group, $\mathcal{K}' = SU(2)'_L \otimes SU(2)_R$, where $SU(2)'_L$ is the diagonal subgroup of $SU(2)_L \otimes SU(2)_I$. This implies that the isospin index $i$ becomes a spinorial index $\alpha$: $Q^i_\alpha \rightarrow Q^\beta_\alpha$ and $\overline{Q}_{i\dot{\beta}} \rightarrow G_{\alpha\dot{\beta}}$. Precisely the trace of $Q^\beta_\alpha$ is chosen as the generator of the scalar symmetry: $Q = Q^\alpha_\alpha$. Under the new global group $H' = \mathcal{K}' \otimes U(1)_R$, the symmetry generators transform as:

$$G_{\alpha\dot{\beta}} \left( \frac{1}{2}, \frac{1}{2} \right)^{-1}, \quad Q(\alpha\beta) \left( 1, 0 \right)^1, \quad Q \left( 0, 0 \right)^1. \quad (4.11)$$

Notice that we have obtained a scalar generator $Q$. It is important to stress that as long as we stay on a flat space (or one with trivial holonomy), the twist is just a fancy way of considering the theory, for in the end we are not changing anything. However, the appearance of a scalar symmetry makes the procedure meaningful when we move to an arbitrary four-manifold. Once the scalar symmetry is found we must study if, as stated in (2.6), the energy-momentum tensor is exact, i.e., if it can be written as the transformation of some quantity under $Q$. The $N = 2$ supersymmetry algebra gives a necessary condition for this to hold. Indeed, after the twisting, this algebra becomes:

$$\{Q^i_\alpha, \overline{Q}_{j\dot{\beta}}\} = \delta^i_j P_{\alpha\beta} \rightarrow \{Q, G_{\alpha\beta}\} = P_{\alpha\beta}, \quad \{Q, Q\} = 0, \quad (4.12)$$

where $P_{\alpha\dot{\beta}}$ is the momentum operator of the theory. Certainly (4.12) is only a necessary condition for the theory to be topological. However, up to date, for all the $N = 2$ supersymmetric models whose twisting has been studied the relation on the right hand side of (4.12) has become valid for the whole energy-momentum tensor. Notice that (4.12) are the basic equations (2.13) and (2.15) of our general discussion on TQFT. It is important to remark that twisted theories are considered as Euclidean theories. This implies that the twisting procedure is often accompanied by some changes on the complex nature of the fields. This delicate issue has been treated recently by Blau and Thompson [23].

In $\mathbb{R}^4$ the original and the twisted theories are equivalent. However, for arbitrary manifolds they are certainly different due to the fact that their energy-momentum
tensors are not the same. The twisting changes the spin of the fields in the theory, and therefore their couplings to the metric on \( X \) become modified. This suggests an alternative way of looking at the twist. All that has to be done is: gauge the internal group \( SU(2)_I \), and identify the corresponding \( SU(2) \) connection with the spin connection on \( X \). This process changes the spin connection and therefore the energy-momentum tensor of the theory, which in turn modifies the couplings to gravity of the different fields of the theory. This alternative point of view to the twisting procedure has been recently reviewed in this context in [65].

Under the twist, the field content is modified as follows:

\[
A_{\alpha\dot{\alpha}} \left( \frac{1}{2}, \frac{1}{2}, 0 \right) \longrightarrow A_{\alpha\dot{\alpha}} \left( \frac{1}{2}, \frac{1}{2} \right)^0,
\]

\[
\lambda_{\alpha i} \left( \frac{1}{2}, 0, \frac{1}{2} \right) \leftarrow \chi_{\alpha\beta} \left( 1, 0 \right)^{-1}, \eta(0, 0)^{-1},
\]

\[
\lambda_{\dot{\alpha}} \left( 0, \frac{1}{2}, \frac{1}{2} \right) \leftarrow \psi_{\alpha\dot{\alpha}} \left( \frac{1}{2}, \frac{1}{2} \right)^1,
\]

\[
B \left( 0, 0, 0 \right)^{-2} \longrightarrow \lambda \left( 0, 0 \right)^{-2},
\]

\[
B^* \left( 0, 0, 0 \right)^2 \longrightarrow \phi \left( 0, 0 \right)^2,
\]

\[
D_{ij} \left( 0, 0, 1 \right)^0 \longrightarrow G_{\alpha\beta} \left( 1, 0 \right)^0.
\]

In the process of twisting, the \( U(1)_R \) symmetry becomes the \( U(1) \)-like symmetry associated to the ghost number of the topological theory. The ghost number anomaly is thus naturally related to the chiral anomaly of \( U(1)_R \). The twisted action differs from the action (4.6) obtained in the Mathai-Quillen formalism by a term of the form,

\[
\int_M d^4 x \sqrt{g} \text{Tr} \left( \frac{i}{2} \phi \{ \eta, \eta \} + \frac{1}{8} [\lambda, \phi]^2 \right).
\]

This term turns out to be \( Q \)-exact (\( \sim \{ Q, \int \eta[\phi, \lambda] \} \)) and therefore it can be ignored.

Associated to each of the independent Casimirs of the gauge group \( G \) it is possible to construct highest-ghost-number operators. For example, for the quadratic Casimir this operator is:

\[
W_0 = \frac{1}{8\pi^2} \text{Tr}(\phi^2),
\]

and it generates the following family of operators:

\[
W_1 = \frac{1}{4\pi^2} \text{Tr}(\phi \psi), \quad W_2 = \frac{1}{4\pi^2} \text{Tr}(\frac{1}{2} \psi \land \psi + \phi \land F), \quad W_3 = \frac{1}{4\pi^2} \text{Tr}(\psi \land F).
\]

These operators are easily obtained by solving the descent equations, \( \delta W_i = dW_{i-1} \). From them one defines the following observables:

\[
\mathcal{O}^{(k)} = \int_{\gamma_k} W_k,
\]
where \( \gamma_k \in H_k(M) \). The descent equations imply that they are \( \delta \)-invariant and that they only depend on the homology class \( \gamma_k \).

The functional integral corresponding to the topological invariants of the theory has the form:

\[
\langle \mathcal{O}^{(k_1)} \mathcal{O}^{(k_2)} \cdots \mathcal{O}^{(k_p)} \rangle = \int \mathcal{O}^{(k_1)} \mathcal{O}^{(k_2)} \cdots \mathcal{O}^{(k_p)} \exp(-S/g^2),
\]

(4.18)

where the integration has to be understood on the space of field configurations modulo gauge transformations and \( g \) is a coupling constant. The standard arguments described in sect. 2 show that due to the \( \delta \)-exactness of the action \( S \), the quantities obtained in (4.18) are independent of \( g \). This implies that the observables of the theory can be obtained either in the limit \( g \to 0 \), where perturbative methods apply, or in the limit \( g \to \infty \), where one is forced to consider a non-perturbative approach. The crucial point is to observe that the \( \delta \)-exactness of the action implies, at least formally, that in either case the values of the vevs must be the same.

4.2 Perturbative approach

The previous argument for \( g \to 0 \) implies that the semiclassical approximation of the theory is exact. In this limit the contributions to the functional integral are dominated by the bosonic field configurations which minimize \( S \). These turn out to be given by the equations:

\[
F^+ = 0, \quad D_\mu D^\mu \phi = 0.
\]

(4.19)

Let us assume that in the situation under consideration there are only irreducible connections (this is true in the case \( b_2^+ = \dim \Omega^{2+}(X) > 1 \)). In this case the contributions from the even part of the action are given entirely by the solutions of the equation \( F^+ = 0 \), i.e., by instanton configurations. Being the connection irreducible there are no non-trivial solutions to the second equation in (4.19).

The zero modes of the field \( \psi \) come from the solutions to the equations,

\[
(D_\mu \psi_\nu)^+ = 0, \quad D_\mu \psi^\mu = 0,
\]

(4.20)

which are precisely the ones that define the tangent space to the space of instanton configurations. The number of independent solutions of these equations determine the dimension of the instanton moduli space \( \mathcal{M}_{\text{ASD}} \). As stated in (4.8), for \( SU(2) \),

\[
d_{\mathcal{M}_{\text{ASD}}} = 8k - 3(\chi + \sigma)/2.
\]

The fundamental contribution to the functional integral (4.18) is given by the elements of \( \mathcal{M}_{\text{ASD}} \) and by the zero-modes of the solutions to (4.20). Once these have been obtained they must be introduced in the action and an expansion up to quadratic terms in non-zero modes must be performed. The fields \( \phi \) and \( \lambda \) are integrated out.
originating a contribution which is equivalent to the replacement of the field $\phi$ in the operators $O^{(k)}$ by,

$$\langle \phi^a \rangle = \int d^4 y \sqrt{g} G^{ab}(x, y) [\psi_\mu(y), \psi^\mu(y)]^b; \quad (4.21)$$

where $G^{ab}(x, y)$ is the inverse of the Laplace operator,

$$D_\mu D^\mu G^{ab}(x, y) = \delta^{ab} \delta^{(4)}(x - y). \quad (4.22)$$

These are the only relevant terms in the limit $g \to 0$. The resulting gaussian integrations then must be performed. Due to the presence of the $\delta$ symmetry these come in quotients whose value is $\pm 1$. The functional integral (4.18) takes the form:

$$\langle O^{(k_1)} O^{(k_2)} \cdots O^{(k_p)} \rangle = \int_{\mathcal{M}_{\text{ASD}}} d\psi_1 \cdots d\psi_{d_{\mathcal{M}_{\text{ASD}}}} O^{(k_1)} O^{(k_2)} \cdots O^{(k_p)} (-1)^{\nu(a_1, \ldots, a_{d_{\mathcal{M}_{\text{ASD}}}})}, \quad (4.23)$$

where $\nu(a_1, \ldots, a_{d_{\mathcal{M}_{\text{ASD}}}}) = 0, 1$. The integration over the odd modes leads to a selection rule for the product of observables. This selection rule is better expressed making use of the ghost numbers of the fields. For the operators in (4.17) one finds: $U(O^{(k)}) = 4 - k$, and the selection rule can be written as $d_{\mathcal{M}_{\text{ASD}}} = \sum_{i=1}^p U(O^{(k_i)})$.

In the case in which $d_{\mathcal{M}_{\text{ASD}}} = 0$, the only observable is the partition function, which takes the form:

$$\langle 1 \rangle = \sum_i (-1)^{\nu_i}, \quad (4.24)$$

where the sum is over isolated instantons, and $\nu_i = \pm 1$. In general, the integration of the zero-modes in (4.23) leads to an antisymmetrization in such a way that one ends with the integration of a $d_{\mathcal{M}_{\text{ASD}}}$-form on $\mathcal{M}_{\text{ASD}}$. The resulting real number is a topological invariant. Notice that in the process a map

$$H_k(M) \to H^k(\mathcal{M}_{\text{ASD}}) \quad (4.25)$$

has been constructed. The vevs of the theory provide polynomials in $H_{k_1}(M) \times H_{k_2}(M) \times \cdots \times H_{k_p}(M)$ which are precisely the Donaldson polynomials invariants.

### 4.3 Non-perturbative approach

The study of Donaldson-Witten theory from a perturbative point of view proved that the vevs of the observables of this theory are related to Donaldson invariants. However, it did not provide a new method to compute these invariants since the functional integral leads to an integration over the moduli space of instantons, which is precisely the step where the hardest problems to compute these invariants appear.
In the context of quantum field theory there exists the possibility of studying the form of these observables from a non-perturbative point of view, i.e., in the strong coupling limit $g \to \infty$. This line of research seemed difficult to implement until very recently. However, in 1994, after the work by Seiberg and Witten [79], important progress was made in the knowledge of the non-perturbative structure of $N = 2$ supersymmetric Yang-Mills theories. Their results were immediately applied to the twisted theory leading to explicit expressions for the topological invariants in a variety of situations [92]. But perhaps the most important outcome of this approach is the emergence of the existence of a relation between the moduli space of instantons and other moduli spaces such as the moduli space of abelian monopoles which will be introduced below.

$N = 2$ supersymmetric Yang-Mills theory is asymptotically free. This means that the effective coupling constant becomes small at large energies. The perturbative methods which have been used are therefore valid at these energies. At low energies, however, those methods are not valid and one must use non-perturbative techniques. Before 1994 the infrared behavior of the $N = 2$ supersymmetric theory was not known and the non-perturbative approach seemed to be out of reach. However, the infrared behavior of $N = 1$ supersymmetric Yang-Mills was known, and, in 1993, Witten [93] was able to make explicit calculations for the Donaldson invariants on Kähler manifolds with $H^{(2,0)} \neq 0$ using information concerning these theories. This approach is known as the abstract approach while the one based on the infrared behavior of $N = 2$ supersymmetric Yang-Mills is referred to as the concrete approach. We will discuss them now in turn.

### 4.3.1 Donaldson invariants: abstract approach

The key ingredient of this approach is the following observation due to Witten [93]: on a Kähler manifold with $H^{(2,0)} \neq 0$ Donaldson-Witten theory can be perturbed by a mass term preserving the topological character of the theory. The theory can then be regarded as a twisted $N = 1$ super Yang-Mills theory with matter fields. The infrared behavior of this theory is known: it has a mass gap and undergoes confinement and chiral symmetry breaking. Moreover, the $\mathbb{Z}_{2h}$ subgroup ($h$ is the dual Coxeter number of the gauge group $G$) of $U(1)_R$ which is preserved by instantons is believed to be spontaneously broken to $\mathbb{Z}_2$, which allows fermion masses, giving rise to an $h$-fold degeneracy of the vacuum. Vacuum expectation values are written as a sum over contributions from each of the vacua, these contributions being related by the broken symmetry $\mathbb{Z}_h = \mathbb{Z}_{2h}/\mathbb{Z}_2$. Witten studied the case of $SU(2)$ and he proved that the vevs have the structure first found by Kronheimer and Mrowka [53]. We shall now briefly review the fundamentals of his construction.

On a four-dimensional Kähler manifold the holonomy is reduced according to the pattern:

$$SU(2)_L \otimes SU(2)_R \longrightarrow U(1)_L \otimes SU(2)_R.$$  \hspace{1cm} (4.26)
The 2 of $SU(2)_L$ decomposes as a sum of one-dimensional representations of $U(1)_L$. In particular, for the $N = 2$ supersymmetric charges $Q^i_\alpha$ we have:

$$Q^i_\alpha \rightarrow Q^i_1 \oplus Q^i_2.$$  \hspace{1cm} (4.27)

After twisting these give rise to two independent scalar charges, each transforming under definite $U(1)_L$ transformations:

$$Q_1 = Q^1_1, \quad Q_2 = Q^2_2.$$  \hspace{1cm} (4.28)

These charges satisfy the relations:

$$Q = Q_1 + Q_2, \quad (Q_1)^2 = 0 = (Q_2)^2, \quad \{Q_1, Q_2\} = 0.$$  \hspace{1cm} (4.29)

It is important to remark that from the point of view of the untwisted theory $Q_1$ can be regarded in the context of $N = 1$ superspace as a derivative with respect to $\theta_1$.

The field content of $N = 2$ super Yang-Mills theory consists of a gauge (or vector) multiplet, which is represented by a constrained chiral spinor super field $W_\alpha(A, \lambda^1)$, and a scalar multiplet, which is represented by a chiral $N = 1$ superfield $\Psi(B, \lambda^2)$. The action in $N = 1$ superspace takes the form:

$$S = \int d^4x d^2\theta d^2\bar{\theta}(\Psi^i e^V \Psi) + \int d^4x d^2\theta \text{Tr}(W^\alpha W_\alpha) + \int d^4x d^2\bar{\theta} \text{Tr}(\bar{W}_\dot{\alpha} \bar{W}^{\dot{\alpha}}).$$  \hspace{1cm} (4.30)

In this expression $V$ is the vector superpotential, related to $W$ by $W \sim \bar{D}^2 e^{-V} D e^V$, and $D$ and $\bar{D}$ are superspace covariant derivatives. It is well known that $\text{Tr}W^2|_{\theta^2}$ and $\text{Tr}\bar{W}^2|_{\bar{\theta}^2}$ coincide up to a $\theta$-term. This implies that the action (4.31) is $Q_1$-exact modulo a shift in the $\theta$-angle (since $Q_1$ can be regarded as a derivative with respect to $\theta_1$). This shift can be absorbed in a chiral rotation which implies a rescaling of the observables.

When $X$ is simply connected, and for gauge group $SU(2)$, the only relevant observables in Donaldson-Witten theory are the ones associated to even forms in (4.15) and (4.16):

$$\mathcal{O} = \frac{1}{8\pi^2} \text{Tr}(\phi^2),$$

$$I(\Sigma) = \frac{1}{4\pi^2} \int_\Sigma \text{Tr}(\phi F + \frac{1}{2} \psi \wedge \psi),$$  \hspace{1cm} (4.31)

where $\Sigma$ is a two-cycle on the manifold $X$. Both are $Q_1$-invariant. Following Witten [93] we perturb the theory introducing a mass term for the $\Psi$ superfield:

$$\Delta S = - \int \omega \wedge d^2z d^2\theta \text{Tr}(\Psi^2) + \text{h.c.}, \quad \omega \in H^{(2,0)}(X),$$  \hspace{1cm} (4.32)
which breaks the symmetry from $N = 2$ down to $N = 1$. This mass term is not $Q_1$-exact. However, it turns out that the perturbed action has the following form:

$$S + \Delta S = S + I(\bar{\omega}) + \{Q_1, \ldots\} \tag{4.33}$$

being $\bar{\omega}$ the Poincare dual to $\omega$. Since $I(\bar{\omega})$ is after all an observable, the perturbation only introduces a relabeling of the observables themselves. To see this, consider the generating function for the Donaldson polynomials:

$$\langle e^{\sum a \alpha_a I(\Sigma_a) + \lambda \mathcal{O}} \rangle, \tag{4.34}$$

where $\{\Sigma_a\}_{a=1,...,b_2(X)}$ is a basis of $H_2(X)$, and $\alpha_a$ and $\lambda$ are constant parameters. The perturbation (4.33) just amounts to a shift in the $\alpha_a$ parameters.

Summarizing, we have shown that for Kähler manifolds with $H^{(2,0)} \neq 0$ there exists a TQFT, which can be regarded as a twisted version of $N = 1$ supersymmetric Yang-Mills theory, whose vev (4.34) differ from the corresponding ones in Donaldson-Witten theory by a shift in the parameters $\alpha_a$. We will now use the knowledge on the infrared behavior of $N = 1$ supersymmetric Yang-Mills theory to compute (4.34) in the new topological theory.

The first step consists of a rescaling of the metric, $g_{\mu\nu} \rightarrow tg_{\mu\nu}$. In the limit $t \rightarrow \infty$ one expects that a description in terms of the degrees of freedom of the vacuum states of the physical theory in $\mathbb{R}^4$ is valid. Hence, the idea is to compute the observables of the twisted theory on each vacuum of the $N = 1$ supersymmetric gauge theory.

If $\omega$ does not vanish, the untwisted theory possesses $h$ vacuum states. Take $G = SU(2)$, $h = 2$. Standard arguments based on general properties of TQFTs and $N = 1$ supersymmetry lead to the following result for the generating functional (4.34):

$$\langle e^{\sum a \alpha_a I(\Sigma_a) + \lambda \mathcal{O}} \rangle = C_1 e^{(\eta_1^2 \sum_{a,b} \alpha_a \alpha_b \#(\Sigma_a \cap \Sigma_b) + \lambda \xi_1)}$$

$$+ C_2 e^{(\eta_2^2 \sum_{a,b} \alpha_a \alpha_b \#(\Sigma_a \cap \Sigma_b) + \lambda \xi_2)}, \tag{4.35}$$

(notice that each term in (4.35) comes from each of the two vacuua) where,

$$C_i = e^{(a_i \chi + b_i \sigma)} \tag{4.36}$$

is the partition function of the theory in the $i$ vacuum, and $a_i$, $b_i$, $\eta_i$ and $\xi_i = \langle \mathcal{O} \rangle_i$ ($i = 1, 2$) are universal constants independent of the manifold $X$. The symmetry $\mathbb{Z}_2$ gives relations among the constants:

$$C_2 = i^\Delta C_1, \quad \eta_2 = -\eta_1, \quad \xi_2 = -\xi_1, \tag{4.37}$$

with $\Delta = \frac{1}{2}d_M = 4k - \frac{3}{4}(\chi + \sigma)$. The relation between $\eta_1$ and $\eta_2$, and $\xi_1$ and $\xi_2$, results very simply from the $\mathbb{Z}_2$ transformations of the observables. The relation between $C_1$
and $C_2$ is the result of taking into consideration the gravitational anomaly associated to that symmetry. Notice that $i^\Delta$ is independent of the instanton number $k$.

If $\omega$ vanishes along some regions, each vacuum is further split up into two along each region [93]. We will not discuss this more general case in these lectures. The result agrees with the general structure found by Kronheimer and Mrowka [55]. The unknown parameters in (4.36), (4.37) are universal, i.e., independent of $X$, and they can be fixed by comparison to the known values of (4.34) for some manifolds. The success of this approach has an outstanding importance, for the agreement found between the results of the calculation and previously known mathematical results gives support to the conjectured picture in the physical $N = 1$ supersymmetric gauge theory.

### 4.3.2 Donaldson invariants: concrete approach

As explained at the beginning of this subsection, in 1994, Seiberg and Witten, using arguments based on duality, obtained exact results [79] for many $N = 2$ supersymmetric Yang-Mills theories, determining their moduli space of vacua. These physical results were immediately applied [92] to the corresponding twisted theories obtaining a new expression for Donaldson invariants in terms of a new set of invariants: the Seiberg-Witten invariants.

The general argument which explains why the exact results on the infrared behavior of $N = 2$ supersymmetric Yang-Mills theories can be used in the context of TQFT is the following. In the twisted theory the presence of the coupling $g$ can be regarded as a rescaling of the metric. In the limit $g \to \infty$ the rescaling of the metric is arbitrarily large and one expects that calculations can be done in terms of the vacua corresponding to $\mathbb{R}^4$. Recall that $N = 2$ supersymmetric Yang-Mills theories are asymptotically free and their large distance behavior is equivalent to their low energy one. This argument is summarized in Fig. [1].

From the work by Seiberg and Witten [79] follows that at low energies $N = 2$ supersymmetric Yang-Mills theories behave as abelian gauge theories. For the case of gauge group $SU(2)$, which will be the case considered in this discussion, the effective low energy theory is parametrized by a complex variable $u$ which labels the vacuum structure of the theory. At each value of $u$ the effective theory is an $N = 2$ supersymmetric abelian gauge theory coupled to $N = 2$ supersymmetric matter fields. One of the most salient features of the effective theory is that there are points in the $u$-complex plane where some matter fields become massless. These points are singular points of the vacuum moduli space and they are located at $u = \pm \Lambda^2$, where $\Lambda$ is the dynamically generated scale of the theory. At $u = \Lambda^2$ the effective theory consists of an $N = 2$ supersymmetric abelian gauge theory coupled to a massless monopole, while at $u = -\Lambda^2$ it is coupled to a dyon. The effective theories at each singular point are related by an existing $\mathbb{Z}_2$ symmetry in the $u$-plane. This symmetry relates the
behavior of the theory around one singularity to its behavior around the other.

At this point it is convenient to recall some facts about the exact solution of $N = 2$ supersymmetric Yang-Mills found by Seiberg and Witten [79] (see [14] for some reviews on this topic and Alvarez-Gaume’s lectures [8] in this volume). One of the most important features of $N = 2$ supersymmetric Yang-Mills theory is that its lagrangian can be written in terms of a single holomorphic function, the prepotential $\mathcal{F}$. This prepotential is holomorphic in the sense that it depends only on the $N = 2$ chiral superfield $\Psi$ which defines the theory, and not on its complex conjugate. The microscopic theory is defined by a classical quadratic prepotential:

$$\mathcal{F}_{cl}(\Psi) = \frac{1}{2} \tau_{cl} \Psi^2,$$

$$\tau_{cl} = \frac{\theta_{bare}}{2\pi} + \frac{4\pi i}{g_{bare}^2}. \quad (4.38)$$

In terms of this prepotential the lagrangian is given by the following expression in $N = 1$ superspace:

$$\mathcal{L} = \frac{1}{4\pi} \text{ImTr} \left[ \int d^4\theta \frac{\partial \mathcal{F}(A)}{\partial A} \tilde{A} + \int d^2\theta \frac{1}{2} \frac{\partial^2 \mathcal{F}(A)}{\partial A^2} W^\alpha W_\alpha \right], \quad (4.39)$$

where $A$ is a chiral $N = 1$ superfield containing the fields $(\phi, \psi)$, and $W$ is a constrained chiral spinor superfield containing the non-abelian gauge field and its $N = 1$ superpartner $(A_\mu, \lambda)$. The lagrangian $(4.39)$ is equivalent to the one entering $(4.30)$ after replacing the $N = 1$ superfield $\Psi$ there by the $N = 1$ superfield $A$. All the

---

\[2\text{This } N = 2 \text{ chiral superfield should not be confused with the } N = 1 \text{ chiral superfield used in subsection 4.3.1.}\]
fields take values in the adjoint representation of the gauge group, which we take to be $SU(2)$. The potential term for the complex scalar $\phi$ is:

$$V(\phi) = \text{Tr} \left( [\phi, \phi^\dagger]^2 \right).$$

(4.40)

The minimum of this potential is attained at field configurations of the form $\phi = \frac{1}{2} a \sigma^3$, which define the classical moduli space of vacua. A convenient gauge invariant parametrization of the vacua is given by $u = \text{Tr} \phi^2$, which equals $\frac{1}{2} a^2$ semiclassically. For $u \neq 0$, $SU(2)$ is spontaneously broken to $U(1)$. The spectrum of the theory splits up into two massive $N = 2$ vector multiplets, which accommodate the massive $W^\pm$ bosons together with their superpartners, and an $N = 2$ abelian multiplet which accommodates the $N = 2$ photon together with its superpartners. For $u = 0$, the full $SU(2)$ symmetry is (classically) restored.

To study the quantum vacua Seiberg and Witten analyzed the structure of the low energy theory, whose effective lagrangian up to two derivatives is given, after integrating out the massive modes, by an expression like (4.39) but with a new effective prepotential depending only on an abelian multiplet. The result of their analysis can be summarized as follows:

- At the quantum level the $SU(2)$ symmetry is never restored. The theory stays in the Coulomb phase throughout the $u$-plane.
- The moduli space of vacua ($u$-plane) is a complex one-dimensional Kähler manifold.
- At the points $u = \pm \Lambda^2$, the prepotential $F$ has singularities.
- The singularities correspond to the presence of a massless monopole (at $u = \Lambda^2$) and a massless dyon (at $u = -\Lambda^2$).
- Near each of the singularities the effective action should include together with the $N = 2$ abelian vector multiplet, a massless monopole or a dyon hypermultiplet.

A summary of the main features of both, the classical and the quantum moduli spaces, is depicted in Fig. 2.

We will describe now the computation of observables. As we did in the perturbative approach, we will consider the theory on manifolds $X$ with $b_2^+ > 1$. In the limit $g \to 0$ one has to take into account the classical moduli space. Since for $b_2^+ > 1$ there are not abelian instantons the only contribution come from $u = 0$ and one has to go through the analysis carried out in our discussion of the perturbative approach. As described there, one is led to the standard approach to Donaldson invariants via integration over the moduli space of non-abelian instantons. In the limit $g \to \infty$, since the supersymmetric theory is asymptotically free, we are in the infrared regime,
Figure 2: Classical and quantum moduli spaces.
and the contributions come from the quantum moduli space. In the case under consideration ($b_2^+ > 1$) there are no abelian instantons. Since the abelian gauge field is the only massless field away from the singularities, the only contributions come from the singular points, $u = \pm \Lambda^2$, where there are additional massless fields. Near each of these points, $N = 2$ supersymmetry dictates the form of the weakly coupled effective theory. Since the observables of the twisted theory are independent of the coupling constant, one expects that Donaldson invariants can be expressed in terms of vevs of some operators in the twisted effective theories around each singular point. This analysis has been summarized in Fig. 3.

The theory around the monopole singularity is an $N = 2$ supersymmetric abelian gauge theory coupled to a massless hypermultiplet. This theory has a twisted version which has been constructed in \cite{7,1}, from the point of view of twisting $N = 2$ supersymmetry, and in \cite{5} using the Mathai-Quillen formalism. It has been addressed in other works \cite{10,43}. The structure of this theory is similar to the one of Donaldson-Witten theory. The resulting action is $\delta$-exact and therefore one can study the theory in the weak coupling limit, which, being the theory abelian, corresponds to the low energy limit.

Let us describe the structure of the twisted $N = 2$ supersymmetric abelian gauge theory coupled to a twisted hypermultiplet. We will assume that the four-dimensional manifold $X$ is a spin manifold. The analysis naturally extends to the case of manifolds
which are not spin as shown in [32]. A hypermultiplet is built out of two chiral \( N = 1 \) superfields, \( Q \) and \( \tilde{Q} \),
\[
Q(q^1, \psi_{q\alpha}), \quad Q^\dagger(q_1^\dagger, \bar{\psi}_{q\dot{\alpha}}), \quad \tilde{Q}(q_2^\dagger, \psi_{\tilde{q}\dot{\alpha}}), \quad \tilde{Q}^\dagger(q^2, \bar{\psi}_{\tilde{q}\dot{\alpha}}). \tag{4.41}
\]

After the twisting these fields become:
\[
q^i(0, 0, \frac{1}{2})^0 \rightarrow M^\alpha (\frac{1}{2}, 0)^0, \\
\psi_{q\alpha}(\frac{1}{2}, 0, 1)^1 \rightarrow \mu_\alpha (\frac{1}{2}, 0)^1, \\
\tilde{\psi}_{\tilde{q}\dot{\alpha}}(0, \frac{1}{2}, 0)^{-1} \rightarrow \nu_\dot{\alpha} (0, \frac{1}{2})^{-1}, \\
q_i^\dagger(0, 0, \frac{1}{2})^0 \rightarrow \overline{M}_\alpha (\frac{1}{2}, 0)^0, \tag{4.42}
\]
\[
\tilde{\psi}_{\tilde{q}\dot{\alpha}}(0, \frac{1}{2}, 0)^{-1} \rightarrow \bar{\nu}_\dot{\alpha} (0, \frac{1}{2})^{-1}, \\
\psi_{\tilde{q}\dot{\alpha}}(\frac{1}{2}, 0, 1)^1 \rightarrow \bar{\mu}_\dot{\alpha} (\frac{1}{2}, 0)^1.
\]

The twisted fields \( M_\alpha, \mu_\alpha, \) and \( \nu_\dot{\alpha} \) belong, respectively, to \( \Gamma(S^+ \otimes L) \) and \( \Gamma(S^- \otimes L) \), where \( S^\pm \) are the positive/negative chirality spin bundles and \( L \) is a complex line bundle. The action of the twisted abelian effective theory around the monopole singularity is given by [61]:
\[
S_{AM} = \int_X \sqrt{g} [g^{ij} D_i M^\alpha D_j M_\alpha + \frac{1}{4} R M^\alpha M_\alpha + \frac{1}{2} F^{+\alpha\beta} F_{\alpha\beta}^+ - \frac{1}{8} \overline{M}^{(\alpha} M^{\beta)} M_{(\alpha M_\beta)}] \\
+ i \int_X (\lambda \wedge * d^* d\phi - \frac{1}{\sqrt{2}} \chi \wedge * d^\psi + \eta \wedge * d^* \psi) \\
+ \int_X \epsilon(i\phi \lambda \overline{M}^\alpha M_\alpha + \frac{1}{2\sqrt{2}} \chi^{\alpha\beta}(\overline{M}_{(\alpha} M_{\beta)} + \bar{\mu}_{(\alpha} M_{\beta)}) - \frac{i}{2}(v^\dot{\alpha} D_{\alpha} \mu^\alpha - \bar{\mu}^\alpha D_{\alpha} v^\dot{\alpha}) \\
- \frac{1}{2}(\overline{M}^\alpha \psi_{\dot{\alpha} \dot{\alpha}} v^\dot{\alpha} - \bar{v}^\dot{\alpha} \psi_{\dot{\alpha} \dot{\alpha}} M^\alpha) + \frac{1}{2} \eta (\bar{\mu}^\alpha M_\alpha - M^\alpha \mu_\alpha) + \frac{i}{4} \phi \bar{v}^\dot{\alpha} v_{\dot{\alpha}} - \lambda \bar{\mu}^\alpha \mu_\alpha]. \tag{4.43}
\]

This action is invariant under the following scalar symmetry:
\[
[Q, M_\alpha] = \mu_\alpha, \quad \{Q, M_\alpha\} = -i\phi M_\alpha, \\
\{Q, \nu_{\dot{\alpha}}\} = h_{\dot{\alpha}}, \quad [Q, h_{\dot{\alpha}}] = -i\phi \nu_{\dot{\alpha}}. \tag{4.44}
\]

We only list the transformations for the matter fields, the transformations for the rest of the fields are the abelianized version of the ones in (4.4). The action \( S_{AM} \) is \( Q \)-exact and therefore the semiclassical approximation is exact. The main contribution
to the functional integral coming from the bosonic part of the action is given by the solutions to the equations:

$$F^{+}_{\alpha\beta} + \frac{i}{2} \mathcal{M}_{(\alpha M_\beta)} = 0, \quad D_{\alpha\dot{\alpha}} M^\alpha = 0. \quad (4.45)$$

These equations are known as monopole equations [92]. The tangent space to the moduli space, $\mathcal{M}_{AM}$, defined by these equations is given by the linearization of (4.45), which happen to be the field equations:

$$(d\psi)^+_{\alpha\beta} + \frac{i}{2}(\mathcal{M}_{(\alpha\mu\beta)} + \bar{\mu}_{(\alpha M_\beta)}) = 0,$$

$$D_{\alpha\dot{\alpha}} \psi^\alpha + i\psi_{\alpha\dot{\alpha}} M^\alpha = 0. \quad (4.46)$$

The dimension of the moduli space can be calculated from (4.46) by means of an index theorem, and turns out to be [92],

$$d_{\mathcal{M}_{AM}} = \left(c_1(L)\right)^2 - \frac{2\chi + 3\sigma}{4}. \quad (4.47)$$

The only contributions to the partition function come from $d_{\mathcal{M}_{AM}} = 0$ (isolated monopoles). Introducing the shorthand notation, $x = -2c_1(L)$, we have:

$$d_{\mathcal{M}_{AM}} = 0 \Leftrightarrow x^2 = 2\chi + 3\sigma. \quad (4.48)$$

As in the case of Donaldson-Witten theory, the integration over the quantum fluctuations around the background (4.45) gives an alternating sum over the different monopole solutions for a given class $x$:

$$n_x = \sum_i \epsilon_{i,x}, \quad \epsilon_{i,x} = \pm 1. \quad (4.49)$$

The $n_x$ are the partition functions of the twisted abelian theory for a fixed class $x$ (compare to (4.24)). Those classes such that (4.48) holds and $n_x \neq 0$ are called basic classes. The quantities $n_x$ turn out to constitute a new set of topological invariants for four-manifolds known as Seiberg-Witten invariants.

To fix ideas let us analyze in certain detail the outline of the calculation of the partition function of Donaldson-Witten theory on a manifold $X$ with $b^+_2 > 1$ and for gauge group $SU(2)$. Recall that we are dealing with a TQFT which corresponds to a twisted version of $N = 2$ supersymmetric Yang-Mills theory. In the weak coupling limit, $g \to 0$, the partition function is dominated by $SU(2)$ instanton configurations as in (4.24):

$$Z = \sum_{k=0}^{\infty} \delta(8k - \frac{3}{2}(\chi + \sigma)) Z_k, \quad (4.50)$$

44
where,

\[ Z_k = \sum_{\text{solutions}} (-1)^{\nu_i}, \quad \nu_i = 0, 1. \]  \hspace{1cm} (4.51)

In the strong-coupling limit, \( g \to \infty \), or, \( \tilde{g} = 1/g \to 0 \), by analogy with the physical theory, we expect that the correct description is given by a sum over the partition functions of effective TQFTs which are twisted versions of the corresponding effective description of the physical theory at the points \( u = \pm \Lambda^2 \):

\[ Z = c(Z_{u=\Lambda^2} + Z_{u=-\Lambda^2}), \]  \hspace{1cm} (4.52)

where \( c \) is a factor to be fixed. The partition functions of the twisted theories are dominated by configurations satisfying the monopole equations (4.45) for classes \( x \) satisfying (4.48). For \( Z_{u=\Lambda^2} \) we have:

\[ Z_{u=\Lambda^2} = \sum_x n_x \delta(x^2 - 2\chi - 3\sigma), \quad x = -2c_1(L), \]  \hspace{1cm} (4.53)

with the Seiberg-Witten invariants given by

\[ n_x = \sum_{\text{solutions}} (-1)^{\mu_i}, \quad \mu_i = 0, 1. \]  \hspace{1cm} (4.54)

The partition function at the dyon singularity \( Z_{u=-\Lambda^2} \) is related to the previous one by a \( Z_2 \) transformation. This transformation is anomalous on a gravitational background, and therefore there is a contribution from the measure when comparing \( Z_{u=\Lambda^2} \) and \( Z_{u=-\Lambda^2} \). We will discuss the details of this issue below. Now we take for granted that the relation between both partition functions is given by:

\[ Z_{u=-\Lambda^2} = i^{\chi+\sigma} \frac{1}{4} Z_{u=\Lambda^2}. \]  \hspace{1cm} (4.55)

Being the theory topological, the result obtained in both limits should be the same. We then obtain the following relation:

\[ Z = \sum_{k=0}^{\infty} \delta(8k - \frac{3}{2}(\chi + \sigma))Z_k = c \sum_{\text{basic classes}} \delta(x^2 - 2\chi - 3\sigma) \left[n_x + i^{\chi+\sigma} n_x\right]. \]  \hspace{1cm} (4.56)

The quantity \( c \) is fixed comparing both sides of this equation for different four-manifolds \( X \) with \( b_2^+ > 1 \). It turns out that,

\[ c = 2^{1+\frac{7}{4}(\chi+11\sigma)}. \]  \hspace{1cm} (4.57)

This quantity should be computable from field-theoretical arguments but, to our knowledge, it is not known at the moment how to do it. Some steps to determine it have been given in reference [94].
Equation (4.56) constitutes a field-theory prediction which has been verified on all manifolds in which it has been tested. Its content is very important because it relates topological information coming from two apparently unrelated moduli spaces. On the left hand side the contributions are given by non-abelian instanton configurations. However, on the right hand side the contributions come from abelian monopole configurations. It is also important to remark that the fact that eq. (4.56) holds constitutes a very important test for Seiberg-Witten theory. Indeed, for example, if the quantum moduli space for $SU(2)$ had had a number of singularities different from two, the right hand side of (4.56) would have been different, spoiling the agreement.

The partition function is not the only observable that can be computed by making use of Seiberg-Witten theory. One can indeed compute the full generating function (4.34). The steps needed to carry out this computation are the following:

1. Work out the form of the observables in the variables of the effective theory around the monopole singularity, $u = \Lambda^2$.

2. Work out the contribution from the dyon singularity, $u = -\Lambda^2$, using the $Z_2$ symmetry present in the $u$-plane.

3. Sum over all basic classes $x$.

We will go now through these steps in turn.

1. As in the case of the abstract approach, let us assume that $X$ is simply connected. Recall that in this case the relevant observables are (4.31). We reproduce them here:

$$
\mathcal{O} = \frac{1}{8\pi^2} \text{Tr}(\phi^2),
$$

$$
I(\Sigma_a) = \frac{1}{4\pi^2} \int_{\Sigma_a} \text{Tr}(\phi F + \frac{1}{2} \psi \wedge \psi),
$$

being $\{\Sigma_a\}_{a=1,...,b_2(X)}$ a basis of $H_2(X)$. These observables are the ingredients of the generating function (4.34),

$$
\left\langle \exp \left( \sum_a \alpha_a I(\Sigma_a) + \lambda \mathcal{O} \right) \right\rangle,
$$

which is the goal of our computation. Recall that $\lambda$ and $\alpha_a$ are arbitrary parameters.

In computing (4.59) we must address the question of what is the form of the observables of Donaldson-Witten theory in terms of operators of the effective abelian theory. To answer this question we will use the expansion of the observables in the untwisted, physical theory, together with the descent equations in the topological abelian theory. We follow the argument presented in [66] which is originally due to...
Witten. The descent equations for the abelian monopole theory can be found in [61]. Near the monopole singularity, the $u$ variable has the expansion [79]:

$$u(a_D) = \Lambda^2 + \left(\frac{du}{da_D}\right)_0 a_D + \text{higher order},$$

(4.60)

where $(du/da_D)_0 = -2i\Lambda$, while “higher order” stands for operators of higher dimensions in the expansion. The field $a_D$ corresponds to the field $\phi_D$ of the topological abelian theory [61], while the gauge-invariant parameter $u$ corresponds to the observable (4.15). In terms of observables of the corresponding twisted theories, the expansion (4.60) reads,

$$\mathcal{O} = \langle \mathcal{O} \rangle - \frac{1}{\pi} \langle V \rangle \phi_D + \text{higher order},$$

(4.61)

where $\langle \mathcal{O} \rangle$, $\langle V \rangle$ are real $c$-numbers which should be related to the values of $u(0)$, $(du/da_D)_0$ in the untwisted theory. From the observable $\mathcal{O}$ one can obtain the observable $\mathcal{O}^{(2)}$ by the descent procedure. By applying it in the abelian TQFT to the right hand side of (4.61), we obtain:

$$\mathcal{O}^{(2)} = -\frac{1}{\pi} \langle V \rangle F_D + \text{higher order},$$

(4.62)

where $F_D$ is the dual electromagnetic field strength (associated to the magnetic monopole). In particular, taking into account that $x = -c_1(L^2) = -[F_D]/\pi$, where $[F_D]$ denotes the cohomology class of the two-form $F_D$, we finally obtain:

$$I(\Sigma) = \langle V \rangle (\Sigma \cdot x) + \text{higher order},$$

(4.63)

where the dot denotes the pairing between 2-cohomology and 2-homology. From the point of view of TQFT, higher dimensional terms should not contribute to the expansion because of the invariance of the theory under rescalings of the metric. Notice that the lower order terms in (4.61) and (4.63) are $c$-numbers, i.e., they have zero-ghost numbers. This means that the only contributions to (4.59) will come from configurations such that the dimension of the moduli space of abelian monopoles, $d_{\text{MAM}}$, vanishes. This dimension is given in (4.47). Thus the only classes contributing to (4.59) are again the basic classes, i.e., classes, $x$, satisfying $x^2 = 2\chi + 3\sigma$ and $n_x \neq 0$. Manifolds for which this holds are called of simple type.

We are now in the position to evaluate the correlation function (4.59) at the monopole singularity. This is rather simple since (4.61) and (4.63) are $c$-numbers. Only additional contact terms for the operators $I(\Sigma)$ appear (see reference [53] for a discussion on this point). These terms have the following form. Let us introduce $v = \sum_a \alpha_a I(\Sigma_a)$, then the contribution turns out to be [53]:

$$\gamma v^2 = \gamma \sum_{a,b} \alpha_a \alpha_b (\Sigma_a, \Sigma_b),$$

where $\gamma$ is a real number. Taking into account this term and eq. (4.61) and (4.63) one finds that the contribution at the monopole singularity is:

$$C_1 \exp(\gamma v^2 + \lambda \langle \mathcal{O} \rangle) \sum_x n_x e^{\langle V \rangle v \cdot x},$$

(4.64)
where $C_1$ is the factor $c$ which appeared in (4.52) and turned out to be (4.57).

2. Next, we work out the contribution from the dyon singularity, $u = -\Lambda^2$. This contribution is related to the one from $u = \Lambda^2$ by a $Z_2$ transformation, which is the anomaly-free symmetry on the $u$-plane which remains after the breaking of the chiral symmetry $U(1)_R$. Let us begin by recalling the transformations of the fields entering the observables under the $U(1)_R$-transformations:

\[
\begin{align*}
\psi^1_\alpha & \rightarrow e^{-i\varphi} \psi^1_\alpha, \\
\psi^2_\alpha & \rightarrow e^{-i\varphi} \psi^2_\alpha, \\
B & \rightarrow e^{-2i\varphi} B.
\end{align*}
\]

Instanton effects break this symmetry down to $Z_8$ ($4N_c - 2N_f$ in the general case of $SU(N_c)$ gauge group with $N_f$ hypermultiplets in the fundamental representation). Under this anomaly-free $Z_8$,

\[
B \rightarrow e^{-2i\left(\frac{2\pi}{8}\right)} B = e^{-i\pi/2} B,
\]

and therefore,

\[
u = \text{Tr}(B^2) \rightarrow e^{-i\pi} \nu = -\nu,
\]

which gives a $Z_2$ symmetry on the $u$-plane. This $Z_2$ symmetry relates the contributions to the vevs from $u = \Lambda^2$ to the ones from $u = -\Lambda^2$. Under the $Z_8$ symmetry, the observables (4.31) transform as follows:

\[
I(\Sigma_a) = \frac{1}{4\pi^2} \int_{\Sigma_a} \text{Tr} \left( \phi F + \frac{1}{2} \psi \wedge \psi \right) \rightarrow e^{-\frac{i\pi}{2}} I(\Sigma_a) = -i I(\Sigma_a),
\]

\[
O = \frac{1}{8\pi^2} \text{Tr}(\phi)^2 \rightarrow e^{-i\pi} O = -O,
\]

hence, using (4.64) one finds:

\[
\begin{align*}
u = \Lambda^2, & \quad C_1 \exp \left( \gamma v^2 + \lambda \langle O \rangle + \langle V \rangle v \cdot x \right), \\
u = -\Lambda^2, & \quad C_2 \exp \left( -\gamma v^2 - \lambda \langle O \rangle - i\langle V \rangle v \cdot x \right).
\end{align*}
\]

The quantities $C_2$ and $C_1$ are related because on a curved background the $Z_8$ transformation, while being preserved by gauge instantons, picks anomalous contributions from the measure due to gravitational anomalies. The contribution is of the form $\exp i\pi \Delta$, where $\Delta$ is the index of the Dirac operator. For a basic class, $\dim M_{\text{AM}} = 0$, and therefore, from (4.47), $(c_1(L))^2 = \frac{2\chi + 3\sigma}{4}$. The index of the Dirac operator is,

\[
\Delta = -\frac{\sigma}{8} + \frac{1}{2}(c_1(L))^2 = \frac{\chi + \sigma}{4} \in \mathbb{Z},
\]
and therefore the anomaly can be written as $i^\Delta$, with $\Delta = \frac{\chi + \sigma}{4}$. Then,

$$C_2 = i^\Delta C_1.$$  \hfill (4.70)

3. Finally, we take both contributions and sum over basic classes. The final form of the generating function of vevs of observables turns out to be:

$$\langle e^{\sum a I(\Sigma_a) + \lambda O} \rangle = C_1 \left[ e^{(\gamma v^2 + \langle O \rangle) \lambda} \sum_x n_x e^{i v . x} + i^\Delta e^{(-\gamma v^2 - \langle O \rangle) \lambda} \sum_x n_x e^{-iv . x} \right].$$  \hfill (4.71)

By comparing to known results by Kronheimer and Mrowka \cite{55} the unknown constants in \hfill (4.71) are fixed to be:

$$\gamma = \frac{1}{2}, \quad \langle O \rangle = 2, \quad \langle V \rangle = 1, \quad C_1 = 2^{1+\frac{1}{4}(7\chi + 11\sigma)}.$$  \hfill (4.72)

The ratio between $\langle V \rangle$ and $\langle O \rangle$ is predicted by Seiberg-Witten theory since both originated from eq. \hfill (4.60). It agrees with \hfill (4.72). Notice that these constants, $\gamma$, $\langle V \rangle$, and $\langle O \rangle$, coming from the structure of the physical theory in $\mathbb{R}^4$, should be universal, i.e., entirely independent of the manifold $X$. This turns out to be the case according to the values \hfill (4.72), a very important test of our arguments. Different aspects of Seiberg-Witten solution are reflected in \hfill (4.71). The fact that this formula fits all known mathematical results for simply-connected manifolds with $b_2^+ > 1$ is rather satisfactory from the physical point of view.

Gathering all the preceding results we obtain the final expression for the generating function of Donaldson invariants:

$$\langle e^{\sum a I(\Sigma_a) + \lambda O} \rangle = 2^{1+\frac{1}{4}(7\chi + 11\sigma)} \left[ e^{\left(\frac{v^2}{2} + 2\lambda\right)} \sum_x n_x e^{i v . x} + i^\Delta e^{\left(-\frac{v^2}{2} - 2\lambda\right)} \sum_x n_x e^{-iv . x} \right].$$  \hfill (4.73)

The expression above verifies the so-called simple type condition:

$$\left( \frac{\partial^2}{\partial \lambda^2} - 4 \right) \langle e^{\sum a I(\Sigma_a) + \lambda O} \rangle = 0.$$  \hfill (4.74)

All simply-connected four-manifolds with $b_2^+ > 1$ for which \hfill (4.73) is known verify this property.
5 Generalized Donaldson-Witten Theory

So far we have discussed two different moduli problems in four-dimensional topology, one defined by the ASD instanton equations and another one defined by the Seiberg-Witten monopole equations. There is a natural generalization of these moduli problems which involves a non-abelian gauge group and also includes spinor fields. It is the moduli problem defined by the non-abelian monopole equations, introduced in reference [62] in the context of the Mathai-Quillen formalism and as a generalization of Donaldson theory. It has been also considered in reference [49, 10], as well as in the mathematical literature [76, 81, 78, 28].

In order to introduce these equations in the case of $G = SU(N)$ and the monopole fields in the fundamental representation $N$ of $G$, let us consider a Riemannian four-manifold $X$ together with a principal $SU(N)$-bundle $P$ and a vector bundle $E$ associated to $P$ through the fundamental representation. Suppose for simplicity that the manifold is spin, and consider a section $M^i_\alpha$ of $S^+ \otimes E$. The non-abelian monopole equations read in this case:

$$F^{+ij}_{\alpha\beta} + i \left( \partial^j M^i_\alpha - \frac{\delta^i}{N} \frac{\delta j}{M^k_\alpha M^k_\beta} \right) = 0,$$

$$(D^E_\alpha M_\alpha)^i = 0. \quad (5.1)$$

Starting from these equations it is possible to build the associated topological field theory within the Mathai-Quillen formalism. Not surprisingly, the resulting theory is the non-abelian version of the topological theory of abelian monopoles, that is, a twisted version of $N = 2$ super Yang-Mills coupled to one massless hypermultiplet. The field content is just the non-abelian version of that of the abelian monopole theory. In addition to the fields in Donaldson-Witten theory, we have the following matter fields:

$$M_\alpha, \mu_\alpha \in \Gamma(S^+ \otimes E), \quad \nu_\alpha \in \Gamma(S^- \otimes E), \quad (5.2)$$

together with their corresponding complex conjugates. The action for the model takes the form:

$$S_{\text{NAM}} = \int_X e^{\left[ g^{\mu\nu} D_\mu \overline{M}^\alpha D_\nu M_\alpha \right.} \frac{1}{4} R \overline{M}^\alpha M_\alpha$$

$$- \frac{1}{4} \text{Tr} \left( F^{+\alpha\beta} F^{+\alpha\beta} \right) + \frac{1}{4} \left( \overline{M}^\alpha T^a M_\beta \right) \left( \overline{M}^\alpha T^a M_\beta \right) \right]$$

$$+ \int_X \left[ \frac{i}{2} \eta \wedge \ast d^A \psi + \frac{i}{2 \sqrt{2}} \chi^{\alpha\beta} (p^+ (d_A \psi)) \right] + \frac{i}{8} \chi^{\alpha\beta} \left[ \chi_{\alpha\beta}, \phi \right]$$

$$+ \frac{i}{2} \chi \wedge \ast d^A \phi - \frac{1}{2} \chi \wedge \ast [\ast \psi, \psi]$$

$$+ \int_X e^{\left[ \frac{i}{2} \overline{M}^\alpha \{ \phi, \lambda \} M_\alpha - \frac{1}{\sqrt{2}} (\overline{M} \chi^{\alpha\beta} \mu_\beta - \chi^{\alpha\beta} \overline{M} \chi_{\alpha\beta} M_\beta) \right] \right.}$$

$$50$$
The gauge fermions of the theory are:

\[ -\frac{i}{2}(\overline{\gamma}_\alpha D^\alpha \mu + \overline{\mu}^\alpha D_{\alpha \dot{\alpha}} v^{\dot{\alpha}}) + \frac{1}{2}(\overline{\mathcal{M}}^i \psi_{a \dot{a}} v^{\dot{a}} + \overline{v}_{\dot{a}} \psi^{\dot{a}} M_{a}) \]
\[ -\frac{1}{2}(\overline{\mu}^\alpha \eta M_{a} + \overline{M}_{\alpha}^a \eta \mu_{a}) + \frac{i}{4} \overline{v}_{\dot{a}} \phi v_{\dot{a}} - \overline{\mu}^\alpha \lambda \mu_{\alpha}. \]  

This action can be derived either by applying the Mathai-Quillen formalism, as we discuss below, or by twisting the corresponding action for the physical \( N = 2 \) supersymmetric theory. However, as it happens in Donaldson-Witten theory, the action which comes directly from the twisting and that in (5.3) differ by the term (4.14), which, being of the form \( \sim \{ Q, \int \eta [\phi, \lambda] \} \), can be safely ignored.

From the monopole eq. (5.1) follows that the appropriate geometric setting is the following. The field space is \( \mathcal{A} \times \Gamma(X, S^+ \otimes E) \), which is the space of gauge connections on \( P \) and positive chirality spinors in the representation \( N \) of \( G \). The vector bundle has as fibre, \( \mathcal{F} = \Omega^{2+}(X, \text{ad} P) \oplus \Gamma(X, S^+ \otimes E) \), as dictated by the quantum numbers of the monopole equations. These equations are arranged into a section of the vector bundle \( (\mathcal{A} \times \Gamma(X, S^+ \otimes E)) \times \mathcal{F} \):

\[ s(A, M) = \left( \frac{1}{\sqrt{2}}(F_{\alpha \beta}^{ij} + i(M_{\alpha}^i M_{\beta}^j - \frac{\delta_{ij}}{N} M_{\alpha}^k M_{\beta}^k)), (D^\alpha M_{\alpha})^i, \right) \]

in such a way that the zero locus of this section gives precisely the desired moduli space. The action (5.3) is exact with respect to the following transformations:

\[ [Q, A] = \psi, \quad [Q, M_{\alpha}^i] = \mu_{\alpha}^i, \]
\[ \{Q, \psi\} = d_A \phi, \quad \{Q, \mu_{\alpha}^i\} = -i \phi^{ij} M_{\alpha}^j, \]
\[ [Q, \phi] = 0, \quad \{Q, \psi_{\dot{a}}\} = h_{\dot{a}}, \]
\[ \{Q, \lambda\} = \eta, \quad \{Q, \lambda\} = i[\lambda, \phi], \]
\[ [Q, \chi_{\mu \nu}] = H_{\mu \nu}, \quad \{Q, h_{\dot{a}}\} = -i \phi^{ij} \psi_{\dot{a}}, \]
\[ [Q, \eta] = i[\lambda, \phi], \quad \{Q, \eta\} = i[\lambda, \phi], \]  

The gauge fermions of the theory are:

\[ \Psi_{\text{loc}} = \int_X e^{\frac{i}{2} \xi \alpha \beta j i} \left( \frac{1}{\sqrt{2}}(F_{\alpha \beta}^{ij} + i(M_{\alpha}^i M_{\beta}^j - \frac{\delta_{ij}}{N} M_{\alpha}^k M_{\beta}^k)) - \frac{i}{4} H_{\alpha \beta}^{ji} \right) \]
\[ -\frac{i}{2}(\overline{v}_{\dot{a}} D^\alpha M_{\alpha} - \overline{M}_{\alpha}^a D_{\alpha \dot{a}} v^{\dot{a}}) - \frac{1}{8}(\overline{v}_{\dot{a}} h_{\dot{a}}^\dot{\alpha} + \overline{h}_{\dot{a}} v^{\dot{a}}), \]
\[ \Psi_{\text{proj}} = -\frac{1}{2} \int_X [i \text{Tr}(\lambda \wedge \ast d^*_A \psi) + e(\mu^\alpha \lambda M_{\alpha} - \overline{M}^\alpha \lambda \mu_{\alpha})]. \]  

The dimension of the moduli space of non-abelian monopoles is given by the dimension of the corresponding tangent space, which in turn is defined by the linearized version of the monopole equations:

\[ \frac{p^+(d_A \psi)^{ij}_{\alpha \beta}}{2} + \frac{i}{2}(\overline{M}_{\alpha}^i H_{\beta}^j + \overline{\mu}_{\alpha}^i M_{\beta}^j) - \frac{\delta_{ij}}{N}(\overline{M}^k_{\alpha} H_{\beta}^k + \overline{\mu}_{\alpha}^k M_{\beta}^k) = 0, \]
\[ (D^\alpha M_{\alpha})^i + i \psi_{\alpha}^{ij} M_{\alpha}^j = 0. \]  

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The dimension of the moduli space counts essentially the number of independent solutions to these equations modulo gauge transformations, and it is given by a suitable index theorem, with the result:

\[
\dim \mathcal{M}_{NA} = \dim \mathcal{M}_{ASD} + 2 \text{ index } D_E \\
= (4N - 2)c_2(E) - \frac{N^2 - 1}{2}(\chi + \sigma) - \frac{N}{4}\sigma,
\]

(5.8)

Notice that $\mathcal{M}_{ASD} \subset \mathcal{M}_{NA}$. In addition to this, the usual conditions to have a well-defined moduli problem (like the reducibility) are essentially the same as in Donaldson theory.

The observables of the theory are the same as in Donaldson-Witten theory since no non-trivial observables involving matter fields have been found. The topological invariants are then given by correlation functions of the form (4.18):

\[
\langle \mathcal{O}^{(k_1)} \mathcal{O}^{(k_2)} \cdots \mathcal{O}^{(k_p)} \rangle = \int \mathcal{O}^{(k_1)} \mathcal{O}^{(k_2)} \cdots \mathcal{O}^{(k_p)} \exp(-S/g^2),
\]

(5.9)

In the perturbative regime, $g \to 0$, one finds the same pattern as in ordinary Donaldson-Witten theory. There is a map like in (4.25), $H_k(X) \to H^k(\mathcal{M}_{NA})$, which implies that the vevs of the theory provide a new set of polynomials in $H_{k_1}(X) \times H_{k_2}(X) \times \cdots \times H_{k_p}(X)$. As in the case of ordinary Donaldson-Witten theory, the perturbative approach does not provide any further insight into the precise form of these topological invariants. Fortunately, it is again possible to apply the results of Seiberg and Witten on $N = 2$ supersymmetric theories to analyze the model at hand in the non-perturbative regime, in much the same way as it has been done in the case of Donaldson-Witten theory.

We will now discuss the non-perturbative approach. We will follow the same strategy as in the case of Donaldson-Witten theory. This has been depicted in Fig. [4]. The physical theory underlying the theory of non-abelian monopoles is an $N = 2$ supersymmetric Yang-Mills theory coupled to one massless hypermultiplet in the fundamental representation of the gauge group, which we take to be $SU(2)$. This theory is asymptotically free. Hence, it is weakly coupled ($g \to 0$) in the ultraviolet, and strongly coupled ($g \to \infty$) in the infrared. The infrared behavior of this theory has been also determined by Seiberg and Witten [73]. Their results can be summarized as follows:

- The quantum moduli space of vacua is a one-dimensional complex Kähler manifold (the $u$-plane).
- For any $u$ there is an unbroken $U(1)$ gauge symmetry (Coulomb phase).
- At a generic point on the $u$-plane the only light degree of freedom is the $U(1)$ gauge field (together with its $N = 2$ superpartners).
Figure 4: Schematic diagram of the concrete approach for non-abelian monopoles.

- There are three singularities at finite values of \( u \).
- Near each of these singularities a magnetic monopole or dyon becomes massless and weakly coupled to a dual \( U(1) \) gauge field.

A scheme of the quantum moduli space is presented in Fig. 5.

For \( X \) such that \( b_2^+ > 1 \) (there are no abelian instantons) the only contributions come from the three singularities. Following the same arguments as in the abelian case one finds the following general result:

\[
\langle e^{\left( \sum_{\alpha} \alpha \alpha I(\Sigma_{\alpha}) - \lambda \phi \right)} \rangle = \sum_{i=1}^{3} C_i e^{\left( \frac{2\eta_i}{4} v^2 + \xi_i \lambda \right)} \sum_x n_x e^{\zeta_i v \cdot x}.
\]  
(5.10)

Relations among the quantities \( C_i, \eta_i, \zeta_i \) and \( \xi_i \) are obtained by using the broken \( U(1)_R \) symmetry. Recall that:

\[
\begin{align*}
\psi^1_\alpha &\rightarrow e^{-i\varphi} \psi^1_\alpha, \\
\psi^2_\alpha &\rightarrow e^{-i\varphi} \psi^2_\alpha, \\
B &\rightarrow e^{-2i\varphi} B, \\
\psi_{qi} &\rightarrow e^{i\varphi} \psi_{qi}, \\
\psi_{\tilde{q}i} &\rightarrow e^{i\varphi} \psi_{\tilde{q}i}.
\end{align*}
\]
(5.11)

Instanton effects break this symmetry down to \( \mathbb{Z}_6 \). Under this \( \mathbb{Z}_6 \),

\[
B \rightarrow e^{-\frac{2i\varphi}{3}} B,
\]
(5.12)
and therefore
\[ u = \text{Tr}(B^2) \longrightarrow e^{-\frac{4\pi}{3}} u = e^{\frac{2\pi}{3}} u, \] (5.13)
which generates a $\mathbb{Z}_3$ symmetry on the $u$-plane which interchanges the three singularities. Under this symmetry the observables transform as follows:

\[ I(\Sigma_a) \longrightarrow e^{-\frac{2\pi}{3}} I(\Sigma_a), \]
\[ \mathcal{O} \longrightarrow e^{\frac{2\pi}{3}} \mathcal{O}. \] (5.14)

This implies for the unknown constants in (5.10) the following set of relations:

\[ \eta_2 = e^{-\frac{2\pi}{3}} \eta_1, \quad \eta_3 = e^{\frac{2\pi}{3}} \eta_1, \]
\[ \xi_2 = e^{\frac{2\pi}{3}} \xi_1, \quad \xi_3 = e^{-\frac{2\pi}{3}} \xi_1, \]
\[ \zeta_2 = e^{-\frac{2\pi}{3}} \zeta_1, \quad \zeta_3 = e^{-\frac{4\pi}{3}} \zeta_1. \] (5.15)

The relations among the $C_i$ are obtained by working out the contribution from the measure due to gravitational anomalies. The anomaly comes from the fields $\psi, \chi, \eta, \mu, \nu$, and implies the relations:

\[ C_2 = e^{-\frac{16\pi}{9}} C_1, \quad C_3 = e^{-\frac{16\pi}{9}} C_1. \] (5.16)

Denoting $C = C_1$, $\eta = \eta_1$, $\zeta = \zeta_1$ and $\xi = \xi_1$, one has the final result for manifolds with $b_2^+ > 1$ [33]:

\[ \langle \exp(\sum_a \alpha_a I(\Sigma_a) + \lambda \mathcal{O}) \rangle \]
\[= C \left( \exp \left( \frac{\eta}{2} v^2 + \lambda \xi \right) \sum_x n_x \exp(\zeta v \cdot x) \right) + e^{-\frac{\eta}{2} v^2 + \lambda \xi} \exp \left( -e^{-\frac{\eta}{2} v^2 + \lambda \xi} \sum_x n_x \exp \left( e^{-\frac{4\pi}{3} \zeta v \cdot x} \right) \right) \]

where unknown constants appear as in the pure Donaldson-Witten case. The generating function (5.17) verifies a generalized form of the simple type condition (4.74):

\[
\left( \frac{\partial^3}{\partial \lambda^3} - \xi^3 \right) \exp(\sum_\alpha \alpha_a I(\Sigma_a) + \lambda \mathcal{O}) = 0. \tag{5.18}
\]

Unfortunately, the left-hand side of (5.17) is not known. Thus we can not fix the constants \(\eta\), \(\xi\), \(\zeta\) and \(C\) as we did in the case of Donaldson-Witten theory. That equation has to be regarded as a prediction for those quantities. The result (5.17) suggests that, as stated in the introduction, moduli problems in four-dimensional topology can be classified in universality classes associated to the effective low-energy description of the underlying physical theory. One important question that should be addressed is how large is the set of moduli spaces which admit a description in terms of Seiberg-Witten invariants. It is very likely that in the search for this set new types of invariants will be found leading to new universality classes. The case considered in this section is the simplest of its kind. Other situations should certainly be addressed.
6 Final remarks

We will end these lectures making several remarks. First of all we should mention that there is a rich structure associated to twisted $N = 4$ supersymmetric gauge theories. These theories can be twisted in three non-equivalent ways [95, 70, 25, 59]. Ideas based on duality have been applied [83] to one of the resulting twisted theories proving invariance under the full duality group. The other two twisted theories, as well as many other twisted theories coming from scale invariant $N = 2$ supersymmetric gauge theories, should be addressed from a similar perspective. It is expected that duality is realized for all these theories.

In the context of generalized Donaldson-Witten theory it is worth to notice that one of our assumptions can be released. We have assumed in our discussion that the manifold $X$ is spin. Of course, this is only a simplifying assumption. The theory can be also defined on general four-manifolds using Spin$^c$-structures [49, 76, 28, 78]. It has been recently shown in the context of non-abelian monopoles that, from the point of view of twisted $N = 2$ supersymmetry, the inclusion of Spin$^c$-structures corresponds to an extended twisting procedure associated to the gauging of the baryon number [65].

It is also important to point out that the moduli space of non-abelian monopoles has a natural $U(1)$ action which acts as a rotation on the monopole fields. The fixed points of this action are essentially the moduli space of ASD instantons and the moduli space of abelian Seiberg-Witten monopoles. This has opened the way to a mathematical proof of the equivalence of both theories using localization techniques [78, 76], and some promising and concrete results in this direction have been recently obtained [77]. From the point of view of the Mathai-Quillen formalism the $U(1)$ action makes the bundle $\mathcal{E}$ an $U(1)$-equivariant bundle and one can obtain a general expression for the equivariant extension of the Thom form in this formalism [64]. For the non-abelian monopole theory, the TQFT associated to this extension is precisely twisted $N = 2$ supersymmetric Yang-Mills coupled to one massive flavor. This result is promising for two reasons. Firstly, it demonstrates that it is possible to add mass-like parameters to this type of theories while still retaining the topological character of the theory. Secondly, it is tempting to think that the physics of the massive theory could shed some light on the localization problem.

Another important observation is the one made by Taubes [80] pointing out a relation between Seiberg-Witten invariants and Gromov invariants. As mentioned in sect. 3, Gromov invariants are the topological quantities which appear in two-dimensional topological sigma models. Therefore, what is involved here is a relation between TQFTs in four dimensions and TQFTs theories in two dimensions. It is very likely that an explanation of this relation based on physics will come from string theory. In fact, it is tempting to speculate that string theory will provide a new point of view to understand the relations among the invariants associated to different
moduli spaces. There are several results which point into this direction. It is known, for example, that Donaldson theory shows up in certain compactifications of the heterotic string \cite{47}. Similarly, other topological quantum field theories, as the ones associated to twisted $N = 4$ supersymmetric gauge theory can also be understood from a string theory perspective \cite{19}.

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