Yang-Lee zeros of the one-dimensional $Q$-state Potts model

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Abstract

The distributions of the Yang-Lee zeros of the ferromagnetic and antiferromagnetic $Q$-state Potts models in one dimension are studied for arbitrary $Q$ and temperature. The Yang-Lee zeros of the Potts antiferromagnet have been fully investigated for the first time. The distributions of the Yang-Lee zeros show a variety of different shapes. Some of the Yang-Lee zeros lie on the positive real axis even for $T > 0$. For the ferromagnetic model this happens only for $Q < 1$, while there exist some zeros of the antiferromagnetic model on the positive real axis both for $Q < 1$ and for $Q > 1$.

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I. INTRODUCTION

The $Q$-state Potts model \cite{1,2,3,4,5} is a generalization of the Ising ($Q = 2$) model. The $Q$-state Potts model exhibits a rich variety of critical behavior and is very fertile ground for the analytical and numerical investigations of first- and second-order phase transitions. The Potts model is also related to other outstanding problems in physics and mathematics. Fortuin and Kasteleyn \cite{6,7} have shown that the $Q$-state Potts model in the limit $Q \to 1$ defines the problem of bond percolation. They \cite{7} also showed that the problem of resistor network is related to a $Q = 0$ limit of the partition function of the Potts model. In addition, the zero-state Potts model describes the statistics of treelike percolation \cite{8}, and is equivalent to the undirected Abelian sandpile model \cite{9}. The $Q = \frac{1}{2}$ state Potts model has a connection to a dilute spin glass \cite{10}. The $Q$-state Potts model with $0 \leq Q < 1$ describes transitions in the gelation and vulcanization processes of branched polymers \cite{11}. The partition function of the Potts model is also known as the Tutte dichromatic polynomial \cite{12} or the Whitney rank function \cite{13} in graph theory and combinatorics of mathematics.

By introducing the concept of the zeros of the partition function in the complex magnetic-field plane, Yang and Lee \cite{14} proposed a mechanism for the occurrence of phase transitions in the thermodynamic limit and yielded a new insight into the unsolved problem of the ferromagnetic Ising model in an arbitrary nonzero external magnetic field. It has been known \cite{14,15,16,17,18,19,20,21,22,23,24} that the distribution of the zeros of a model determines its critical behavior. Lee and Yang \cite{15} also formulated the celebrated circle theorem which states that the partition function zeros of the Ising ferromagnet lie on the unit circle in the complex magnetic-field ($x = e^{\beta H}$) plane. However, for the ferromagnetic $Q$-state Potts model with $Q > 2$ the Yang-Lee zeros in the complex $x$ plane lie close to, but not on, the unit circle with the two exceptions of the critical point $x = 1$ ($H = 0$) itself and the zeros in the limit $T = 0$ \cite{19,25,26,27,28}. It has been shown \cite{27,28} that the distributions of the ferromagnetic Yang-Lee zeros for $Q > 1$ have similar properties independent of dimension.

Recently, the exact results on the Yang-Lee zeros of the ferromagnetic Potts model have been found using the one-dimensional model. Mittag and Stephen \cite{29} studied the Yang-Lee zeros of the three-state Potts ferromagnet in one dimension. Glumac and Uzelac \cite{30} found the eigenvalues of the transfer matrix of the one-dimensional Potts model for general $Q$.\[2\]
In particular, they have shown that the ferromagnetic Yang-Lee zeros can lie on the real axis for \( Q < 1 \). However, the properties of the Yang-Lee zeros of the antiferromagnetic Potts model have never been known except for the one-dimensional Ising antiferromagnet [31]. In this paper, we study the ferromagnetic and antiferromagnetic Yang-Lee zeros of the one-dimensional \( Q \)-state Potts model.

II. PARTITION FUNCTION

The \( Q \)-state Potts model in an external magnetic field \( H_q \) on a lattice \( G \) with \( N_s \) sites and \( N_b \) bonds is defined by the Hamiltonian

\[
\mathcal{H}_Q = -J \sum_{\langle i,j \rangle} \delta(\sigma_i, \sigma_j) - H_q \sum_k \delta(\sigma_k, q),
\]

(1)

where \( J \) is the coupling constant (ferromagnetic model for \( J > 0 \) and antiferromagnetic model for \( J < 0 \)), \( \langle i, j \rangle \) indicates a sum over nearest-neighbor pairs, \( \sigma_i = 1, 2, ..., Q \), and \( q \) is a fixed integer between 1 and \( Q \). The partition function of the model is

\[
Z_Q = \sum_{\{\sigma_n\}} e^{-\beta \mathcal{H}_Q},
\]

(2)

where \( \{\sigma_n\} \) denotes a sum over \( Q^{N_s} \) possible spin configurations and \( \beta = (k_B T)^{-1} \). The partition function can be written as

\[
Z(a, x, Q) = \sum_{E=0}^{N_b} \sum_{M=0}^{N_s} \Omega_Q(E, M)a^E x^M,
\]

(3)

where \( a = y^{-1} = e^{\beta J} \), \( x = e^{\beta H_q} \), \( E \) and \( M \) are positive integers \( 0 \leq E \leq N_b \) and \( 0 \leq M \leq N_s \), respectively, and \( \Omega_Q(E, M) \) is the number of states with fixed \( E \) and fixed \( M \). The states with \( E = 0 \) (\( E = N_b \)) correspond to the antiferromagnetic (ferromagnetic) ground states. For ferromagnetic interaction \( J > 0 \) the physical interval is \( 1 \leq a \leq \infty \) (\( \infty \geq T \geq 0 \)), whereas for antiferromagnetic interaction \( J < 0 \) the physical interval \( 0 \leq a \leq 1 \) (\( 0 \leq T \leq \infty \)). The parameter \( Q \) enters the Potts model as an integer. However, the study of the \( Q \)-state Potts model has been extended to continuous \( Q \) due to the Fortuin-Kasteleyn representation of the partition function [6, 7] and its extension [32].

For the one-dimensional Potts model in an external field the eigenvalues of the transfer matrix were found by Glumac and Uzelac [30]. The eigenvalues are \( \lambda_{\pm} = (A \pm iB)/2 \), where
\( A = a(1 + x) + Q - 2 \) and \( B = -i\sqrt{[a(1 - x) + Q - 2]^2 + 4(Q - 1)x}, \) and \( \lambda_0 = a - 1 \) which is \((Q-2)\)-fold degenerate. The partition function of the one-dimensional model \((N = N_s = N_b)\) is given by
\[
Z_N(a, x, Q) = \lambda_+^N + \lambda_-^N + (Q - 2)\lambda_0^N. \tag{4}
\]

**III. PARTITION FUNCTION ZEROS**

When \( \lambda_+ \) and \( \lambda_- \) are two dominant eigenvalues, the partition function becomes
\[
Z_N \simeq \lambda_+^N + \lambda_-^N \tag{5}
\]
for large \( N \). If we define \( A = 2C \cos \psi \) and \( B = 2C \sin \psi \), where \( C = \sqrt{(a - 1)(a + Q - 1)x} \), then \( \lambda_\pm = C \exp(\pm i\psi) \), and the partition function is
\[
Z_N = 2C^N \cos N\psi. \tag{6}
\]
The zeros of the partition function are then given by
\[
\psi = \psi_k = \frac{2k + 1}{2N} \pi, \quad k = 0, 1, 2, ..., N - 1. \tag{7}
\]
In the thermodynamic limit the locus of the partition function zeros is determined by the solution of
\[
A = 2C \cos \psi, \tag{8}
\]
where \( 0 \leq \psi \leq \pi \). In the special case \( Q = 2 \) the contribution by the eigenvalue \( \lambda_0 \) disappears from the partition function, Eq. (4), and the equation (8) determines all the locus even for finite systems. From Eq. (8) the locus of the Yang-Lee zeros for any \( Q \) is obtained to be
\[
x_1(\psi) = \frac{1}{a^2} \left[ \sqrt{f_1 \cos \psi + i\sqrt{f_2}} \right]^2, \tag{9}
\]
where \( f_1 = (a - 1)(a + Q - 1) \) and \( f_2 = f_1 \sin^2 \psi + Q - 1 \). The edge zeros of \( x_1(\psi) \) are given by
\[
x_\pm = \frac{1}{a^2} \left[ \sqrt{(a - 1)(a + Q - 1)} \pm \sqrt{1 - Q} \right]^2 \tag{10}
\]
from \( x_1(0) \) and \( x_1(\pi) \). If \( f_1 > 0 \) and \( f_2 > 0 \) or \( f_1 < 0 \) and \( f_2 < 0 \), it is easily verified that
\[
|x_1(\psi)| = \sqrt{x_+ x_-} = \left| \frac{a + Q - 2}{a} \right|. \tag{11}
\]
From Eq. (11) we see that the zeros of $x_1(\psi)$ lie on a circle in the complex $x$ plane. The one point of the circle $x_1(\psi)$,

$$x_1\left(\frac{\pi}{2}\right) = -\frac{a + Q - 2}{a},$$

always lies on the real axis. However, if $f_1 > 0$ and $f_2 < 0$ or $f_1 < 0$ and $f_2 > 0$, the zeros of $x_1(\psi)$ lie on the real axis. Recently, Ghulghazaryan et al. tried to understand the properties of Eq. (9) in the antiferromagnetic ($a \leq 1$) region.

On the other hand, when $\lambda_+$ and $\lambda_0$ are two dominant eigenvalues, the partition function can be written as

$$Z_N \simeq \lambda_+^N + (Q - 2)\lambda_0^N$$

for large $N$. The partition function zeros are then determined by

$$\frac{\lambda_+}{\lambda_0} = (2 - Q)^{1/N} \exp(i\phi),$$

where

$$\phi = \phi_k = \frac{2\pi k}{N}, \quad k = 0, 1, 2, ..., N - 1.$$  \hspace{1cm} (15)

In the thermodynamic limit the locus of the partition function zeros is determined by the solution of

$$a^2 x^2 + (Q - 1)x - axA + (a - 1)Ae^{i\phi} - (a - 1)^2 e^{2i\phi} = 0,$$  \hspace{1cm} (16)

where $0 \leq \phi \leq 2\pi$. The equation (16) also determines the locus of the zeros when $\lambda_-$ and $\lambda_0$ are two dominant eigenvalues. Eq. (16) gives the second locus of the Yang-Lee zeros

$$x_2(\phi) = \frac{e^{i\phi}[a + Q - 2 - (a - 1)e^{i\phi}]}{a + Q - 1 - ae^{i\phi}},$$  \hspace{1cm} (17)

which is the unit circle in the limit $a \to \infty$ and a type of limaçon of Pascal for $a = 0$. The two points of the locus $x_2(\phi)$,

$$x_2(0) = 1$$  \hspace{1cm} (18)

and

$$x_2(\pi) = -\frac{2a + Q - 3}{2a + Q - 1},$$  \hspace{1cm} (19)

can be lie on the real axis. From $|\lambda_+| = |\lambda_0|$ we also obtain

$$x_* = \frac{a - 1}{a + Q - 1}.$$  \hspace{1cm} (20)

The second locus, Eq. (17), of the Yang-Lee zeros, although it is very important in understanding the $Q$-state Potts model, has never been considered in the literature until now.
IV. FERROMAGNETIC ($a \geq 1$) YANG-LEE ZEROS

In this section we study the Yang-Lee zeros of the ferromagnetic $Q$-state Potts model for $a \geq 1$.

A. $Q > 1$

In this case $f_1$ and $f_2$ are always positive and the Yang-Lee zeros $x_1(\psi)$ lie on a circle where the eigenvalues satisfy

$$\frac{|\lambda_+|}{|\lambda_0|} = \sqrt{\frac{(a + Q - 1)(a + Q - 2)}{a(a - 1)}} > 1,$$

which implies that the locus $x_2(\phi)$ does not appear. For $1 < Q < 2$ the zeros lie inside the unit circle while for $Q > 2$ the zeros lie outside the unit circle, as shown in [27, 30]. In the special case $Q = 2$ we of course find that $|x_1(\psi)| = 1$, as proved by Lee and Yang [15].

The argument $\theta$ of $x_1(\psi) (= |x_1(\psi)|e^{i\theta(\psi)})$ is given by

$$\cos \frac{\theta}{2} = \sqrt{\frac{(a - 1)(a + Q - 1)}{a(a + Q - 2)}} \cos \psi. \quad (22)$$

As $a \to \infty$ ($T \to 0$), the zeros approach the unit circle. For $a = 1$ ($T = \infty$), $\cos(\theta/2) = 0$, so $\theta(\psi) = \pi$ and all the zeros lie at $1 - Q$. As $Q \to \infty$, the radius $|x_1(\psi)|$ increases without bound. The Yang-Lee edge zero for $T > 0$ is given by

$$\theta_0 = 2 \cos^{-1} \sqrt{\frac{(a - 1)(a + Q - 1)}{a(a + Q - 2)}} > 0, \quad (23)$$

while $\theta_0 = 0$ at $T = 0$. Therefore, we conclude that no zero lie on the positive real axis for any $T > 0$.

B. $Q < 1$

At $a = 1$ ($T = \infty$), all the Yang-Lee zeros lie at the point $1 - Q$ on the positive real axis. For $1 < a \leq 1 - \frac{Q}{2} + \frac{\sqrt{Q}}{2}$, all the zeros lie on the positive real axis between $x_-$ and $x_+$ ($0 < x_- < x_+ < 1$), as pointed out by Glumac and Uzelac [30, 33, 34]. For $a > 1 - \frac{Q}{2} + \frac{\sqrt{Q}}{2}$, the locus consists of the loop $x_2(\phi)$ ($\phi_* \leq \phi \leq 2\pi - \phi_*$) and the line $x_1(\psi)$ ($0 \leq \psi \leq \psi_*$) on
the positive real axis between \( x_* = x_1(\psi_*) \) and \( x_+ = x_1(0) \) (\( 0 < x_* < x_+ < 1 \)). The loop \( x_2 \) meets with the line \( x_1 \) at the point

\[
x_* = x_2(\phi_*) = x_2(2\pi - \phi_*),
\]

(24)

where \( |\lambda_+| = |\lambda_-| = |\lambda_0| \). The loop \( x_2(\phi) \) cuts the real axis at two points \( x_2(\pi) \) and \( x_* (> x_2(\pi)) \). The sign of \( x_2(\pi) \) is positive for \( 1 - \frac{Q}{2} + \sqrt{Q^2} < a < \frac{3-Q}{2} \) and negative for \( a > \frac{3-Q}{2} \).

Figure 1 shows the locus of the Yang-Lee zeros for \( Q = \frac{1}{2} \) and \( a = \frac{6}{5} \) from which we obtain \( x_2(\pi) = \frac{1}{19}, x_* = \frac{2}{7}, \) and \( x_+ = 0.8119 \).

At \( Q = 0 \), the line \( x_1(\psi) \) shrink to the point

\[
x_* = x_+ = x_2(0) = 1,
\]

(25)

and all the zeros lie on the loop \( x_2(\phi) \) (\( 0 \leq \phi \leq 2\pi \)) for \( a > 1 \). As \( Q \to 1 \), two points \( x_* \) and \( x_+ \) approach \( (a - 1)/a \) together, and the line \( x_1(\psi) \) disappears. At \( a = \infty \) (\( T = 0 \)), the line \( x_1(\psi) \) again shrink to the point \( x_* = x_+ = 1 \), and the loop \( x_2(\phi) \) becomes the unit circle centered at the origin.

V. ANTIFERROMAGNETIC (\( 0 \leq a \leq 1 \)) YANG-LEE ZEROS

In this section we investigate the Yang-Lee zeros of the antiferromagnetic \( Q \)-state Potts model for \( 0 \leq a \leq 1 \).

A. \( Q > 1 \)

Because \( f_1 \) is always negative, the zeros of \( x_1(\psi) \) lie on a circle if \( f_2 < 0 \) (\( \psi_0 < \psi < \pi - \psi_0 \)) and on the real axis if \( f_2 > 0 \) (\( 0 \leq \psi < \psi_0 \) or \( \pi - \psi_0 < \psi \leq \pi \)). For \( a < 1 - \frac{Q}{2} \), the locus consists of the line \( x_1(\psi) \) (\( 0 \leq \psi \leq \psi_0 \) and \( \pi - \psi_0 \leq \psi \leq \psi_* \)) on the negative real axis between \( x_+ = x_1(0) \) and \( x_* = x_1(\psi_*) \) (\( x_+ < x_* < 0 \)), the circle \( x_1(\psi) \) (\( \psi_0 \leq \psi \leq \pi - \psi_0 \)) with the radius \( |x_1(\psi)| \), and the loop \( x_2(\phi) \) (\( 0 \leq \phi \leq \phi_* \) and \( 2\pi - \phi_* \leq \phi \leq 2\pi \)), inside the circle \( x_1 \). The loop \( x_2 \) again meets with the line \( x_1 \) at the point

\[
x_* = x_2(\phi_*) = x_2(2\pi - \phi_*),
\]

(26)
where $|\lambda_+| = |\lambda_-| = |\lambda_0|$. The circle cuts the real axis at two points $x_1^0$ and $x_1(\pi/2) (= -x_1^0 > 1)$, where the point $x_1^0$ is defined by

$$x_1^0 = x_1(\psi_0) = x_1(\pi - \psi_0) = \frac{a + Q - 2}{a}. \quad (27)$$

Similarly, the loop also cuts the real axis at two points $x_*$ ($< 0$) and $x_2(0) (= 1$). At $a = 0$ ($T = 0$), the circle $x_1(\psi)$ disappears for $1 < Q < 2$, and the locus consists of the line $x_1(\psi)$ on the real axis between $-\infty$ and $x_*$ ($< 0$) and the loop $x_2(\phi)$. For example, figure 2 shows the locus for $Q = \frac{11}{10}$ and $a = \frac{1}{10}$. In this case we obtain $x_+ = -54.83$, $x_1^0 = -8$, $x_* = -\frac{9}{2}$, the radius $|x_1| = 8$ for the circle, $\psi_0 = 48.19^\circ$, $\psi_* = 134.07^\circ$, and $\phi_* = 46.02^\circ$.

At $a = 1 - \frac{Q}{2}$, two points $x_*$ and $x_1^0$ on the real axis meet at

$$x_* = x_1^0 = -1, \quad (28)$$

other two points $x_1(\pi/2)$ and $x_2(0)$ on the real axis also meet at

$$x_1(\pi/2) = x_2(0) = 1, \quad (29)$$

and the circle $x_1(\psi)$ and the loop $x_2(\phi)$ become the identical locus as the unit circle. On this unit circle, three eigenvalues have the same magnitude

$$|\lambda_+| = |\lambda_-| = |\lambda_0| = \frac{Q}{2}. \quad (30)$$

Therefore, the locus consists of the line $x_1(\psi)$ on the real axis between $x_+$ and $x_*$ and the circle.

In the region $1 - \frac{Q}{2} < a < 1 - \frac{Q}{2} + \frac{\sqrt{Q}}{2}$, the circle $x_1(\psi)$ disappears, and the line $x_1(\psi)$ ($0 \leq \psi \leq \psi_*$) on the negative real axis between $x_+$ and $x_*$ ($x_+ < x_* < 0$) again meets with the loop $x_2(\phi)$ ($\phi_* \leq \phi \leq 2\pi - \phi_*$) at the point $x_*$. The loop cuts the real axis at two points $x_*$ and $x_2(\pi)$. The sign of $x_2(\pi)$ is positive ($0 < x_2(\pi) < 1$) for $1 - \frac{Q}{2} < a < \frac{3-Q}{2}$ and negative for $\frac{3-Q}{2} < a < 1 - \frac{Q}{2} + \frac{\sqrt{Q}}{2}$. Figure 3 shows the locus of the Yang-Lee zeros for $Q = 3$ and $a = \frac{1}{10}$ which give $x_2(\pi) = -\frac{1}{11}$, $x_* = -\frac{3}{7}$, and $x_+ = -777.84$. At $a = 0$ ($T = 0$), the locus still consists of the line $x_1(\psi)$ on the real axis between $-\infty$ and $x_*$ ($< 0$) and the loop $x_2(\phi)$ for $2 < Q < 4$. At $T = 0$, the sign of $x_2(\pi)$ is positive ($0 \leq x_2(\pi) < 1$) for $2 < Q \leq 3$ and negative for $3 < Q < 4$. As $Q \to 1$, two points $x_+$ and $x_*$ approach $(a - 1)/a$ together, and the line $x_1(\psi)$ disappears.

At $a = 1 - \frac{Q}{2} + \frac{\sqrt{Q}}{2}$, $x_* = x_-$, and the loop $x_2(\phi)$ shrinks to the point $x_*$. For $a > 1 - \frac{Q}{2} + \frac{\sqrt{Q}}{2}$, the loop disappears, and the only locus is the line $x_1(\psi)$ on the negative real axis between
$x_+$ and $x_−$ ($x_+ < x_- < 0$). As $Q \rightarrow \infty$, $x_\pm \rightarrow -\infty$ for $a < 1$. At $a = 1$ ($T = \infty$), two edge zeros $x_+$ and $x_-$ meet, and the line $x_1(\psi)$ shrinks to the point

$$x_+ = x_- = 1 - Q$$

(31)

for $Q > 1$. In the special case $Q = 2$, $f_1 < 0$ and $f_2 > 0$, and the line $x_1(\psi)$ is the only locus of the Yang-Lee zeros. Therefore, all the Yang-Lee zeros of the antiferromagnetic Ising model lie on the negative real axis between $x_+$ and $x_-$ ($> x_+$) for $a < 1$, as shown by Yang [31]. At $a = 0$ ($T = 0$), all the zeros lie on the negative real axis between two edge zeros,

$$x_a = -\infty$$

(32)

and

$$x_b = \frac{(Q - 2)^2}{4(1 - Q)} \quad (x_b \leq 0),$$

(33)

for $Q = 2$ and $Q \geq 4$.

B. $Q < 1$

For $0 \leq a < 1 - \frac{Q}{2} - \frac{\sqrt{Q}}{2}$, the locus consists of the loop $x_2(\phi)$ ($0 \leq \phi \leq \phi_*$ and $2\pi - \phi_* \leq \phi \leq 2\pi$) and the line $x_1(\psi)$ ($0 \leq \psi \leq \psi_*$) on the positive real axis between $x_* = x_1(\psi_*)$ and $x_+ = x_1(0)$ ($1 < x_* < x_+$). The loop $x_2$ cuts the real axis at two points $x_2(0) (= 1)$ and $x_*$ ($> 1$). The loop again meets with the line $x_1$ at the point

$$x_* = x_2(\phi_*) = x_2(2\pi - \phi_*),$$

(34)

where $|\lambda_+| = |\lambda_-| = |\lambda_0|$. At $Q = 0$, two points $x_2(0)$ and $x_*$ meet, the loop $x_2(\phi)$ shrinks to the point $x_2(0) = x_*$, and all the Yang-Lee zeros lie on the positive real axis between $x_−$ ($= x_* = 1$) and $x_+$ ($\geq x_-)$ for $a \leq 1$.

At $a = 1 - \frac{Q}{2} - \frac{\sqrt{Q}}{2}$, $x_* = x_+$, and the line $x_1(\psi)$ shrinks to the point $x_*$. In the region $1 - \frac{Q}{2} - \frac{\sqrt{Q}}{2} < a < 1 - \frac{Q}{2}$, the line disappears, and the locations of Yang-Lee zeros are completely determined by the locus $x_2(\phi)$ ($0 \leq \phi \leq 2\pi$) which cuts the real axis at two points $x_2(0)$ ($= 1$) and $x_2(\pi)$ ($> 1$). The shape of the locus $x_2(\phi)$ is changed from a waterdrop-like shape for $1 - \frac{Q}{2} - \frac{\sqrt{Q}}{2} < a < 1 - Q$ through a circle with center $\frac{1}{1 - Q}$ and radius $\frac{Q}{1 - Q}$ for $a = 1 - Q$ to a crescent-like shape for $1 - Q < a < 1 - \frac{Q}{2}$. Figure 4 shows the locus of the Yang-Lee zeros for $Q = \frac{9}{10}$ and $a = \frac{2}{5}$ which give $x_2(\pi) = \frac{13}{7}$. 9
At $a = 1 - \frac{Q}{2}$, the loci $x_1(\psi)$ and $x_2(\phi)$ become identical to be the unit circle on which three eigenvalues again have the same magnitude

$$|\lambda_+| = |\lambda_-| = |\lambda_0| = \frac{Q}{2}. \quad (35)$$

For $a > 1 - \frac{Q}{2}$, all the Yang-Lee zeros lie on the circle $x_1(\psi)$ ($0 \leq \psi \leq \pi$) which cuts the positive real axis at the point $x_1\left(\frac{\pi}{2}\right)$ ($0 < x_1\left(\frac{\pi}{2}\right) < 1$). At $a = 1$ ($T = \infty$), this circle shrinks to the point $x_1(\psi) = 1 - Q$ for $Q < 1$.

VI. ONE-STATE ($Q=1$) POTTS MODEL

At $Q = 1$, the partition function becomes

$$Z_N(a, x, Q = 1) = (ax)^N. \quad (36)$$

Therefore, all the Yang-Lee zeros lie on the point $x = 0$. On the other hand, in the limit $Q \to 1$, the line $x_1(\psi)$ disappears, and the circle $x_1(\psi)$ is the uniform distribution of the zeros because the circle approaches

$$x_1(\psi) = \left|\frac{a}{a - 1}\right| e^{2i\psi}. \quad (37)$$

The loop $x_2(\phi)$ also approaches

$$x_2(\phi) = \frac{a}{a - 1} e^{i\phi}, \quad (38)$$

which is the same as the circle $x_1(\psi)$.

VII. CONCLUSION

We have studied the interesting properties of the ferromagnetic and antiferromagnetic Yang-Lee zeros of the one-dimensional $Q$-state Potts model for arbitrary $Q$ and temperature. It has been shown that the distributions of the Yang-Lee zeros have a variety of different shapes. In particular, the antiferromagnetic Yang-Lee zeros of the Potts model have been fully investigated for the first time. One of the most interesting results is that some of the Yang-Lee zeros lie on the positive real axis even for $T > 0$. For the ferromagnetic Yang-Lee zeros this happens only for $Q < 1$, while there exist some Yang-Lee zeros of the antiferromagnetic Potts model on the positive real axis both for $Q < 1$ and for $Q > 1$. 
The results obtained from the one-dimensional Potts model may be considered as a road to the full understanding of the Yang-Lee zeros in higher dimensions in that for $Q > 1$ the distributions of the ferromagnetic Yang-Lee zeros have similar properties independent of dimension \[27, 28\].

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FIG. 1: The locus of the Yang-Lee zeros in the complex $x = e^{3Hq}$ plane for $Q = \frac{1}{2}$ and $a = e^{3J} = \frac{6}{5}$. For comparison, the zeros for a finite-size system ($N = 100$) are also shown (open circles).

FIG. 2: The locus of the Yang-Lee zeros for $Q = \frac{11}{10}$ and $a = \frac{1}{10}$. 
FIG. 3: The locus of the Yang-Lee zeros for $Q = 3$ and $a = \frac{1}{10}$. The zeros on the real axis lie between $x_\ast = -\frac{3}{7}$ and $x_+ = -777.84$. Most of them are omitted in the figure.

FIG. 4: The locus of the Yang-Lee zeros for $Q = \frac{9}{10}$ and $a = \frac{2}{5}$.