ON A GENERALIZATION OF GROTHENDIECK’S THEOREM

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ABSTRACT. A wide generalization of the classical theorem of A. Grothendieck asserting that for any faithfully flat extension of commutative rings, the corresponding relative Picard group and the Amitsur 1-cohomology group with values in the units-functor are isomorphic, is obtained. This implies some known results that are concerned with extending to non-commutative rings of Grothendieck’s theorem.

1. Introduction

One of the fundamental results in descent theory is Grothendieck’s theorem (see Corollary 4.6. in [3]) establishing an isomorphism between the relative Picard group $\text{Pic}(S/B)$ of a faithfully flat extension $i: B \rightarrow S$ of commutative rings and the Amitsur 1-cohomology group $H^1(S/B, U)$ of the extension with values in the units-functor $U$.

Grothendieck’s result was generalized in [7] to non-commutative rings as follows: Let $i: B \rightarrow S$ be an extension of non-commutative rings, let $\text{Inv}_B(S)$ denote the group of invertible $B$-subbimodules of $S$, and $\text{Aut}_{A-\text{cor}}(S\otimes B S)$ the group of $B$-coring automorphisms of the Sweedler’s canonical $B$-coring $S\otimes B S$. Masouka defined a group homomorphism $\Gamma: \text{Inv}_B(S) \rightarrow \text{Aut}_{A-\text{cor}}(S\otimes B S)$ and showed that if either (a) $S$ is faithfully flat as a right or left $B$-module, or (b) $B$ is a direct summand of $S$ as a $B$-bimodule, then $\Gamma$ is an isomorphism of groups.

This has been further generalized by L. El Kaoutit and J. Gómez-Torrecillas [4], considering extensions of non-commutative rings of the form $B \rightarrow \text{End}_A(M)$, where $M$ is a $B$-$A$-bimodule with $M_A$ finitely generated and projective.

In the present paper, we obtain a more general result that includes the above results as particular cases.

We refer to [1] for terminology and general results on (co)monads, and to [2] for a comprehensive introduction to the theory of corings and comodules.

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2. Preliminaries

We begin by recalling that a comonad \( \mathbf{G} \) on a given category \( \mathcal{B} \) is an endofunctor \( G : \mathcal{B} \to \mathcal{B} \) equipped with natural transformations \( \epsilon : 1 \to G \) and \( \delta : G \to G^2 \) such that the diagrams

\[
\begin{array}{ccc}
G^2 & \xrightarrow{\delta G} & G^3 \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
G & \xrightarrow{\delta} & G^2 \\
\end{array}
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
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\((U(\theta_b), \eta_{U(b)})\) has an equalizer in \(\mathcal{A}\) - one then finds \(R_{\mathcal{G}}(b, \theta_b)\) as the equalizer

\[
\begin{array}{ccc}
R_{\mathcal{G}}(b, \theta_b) & \xrightarrow{e(b, \theta_b)} & U(b) \\
& & \xrightarrow{U(\theta_b)} \eta_{U(b)} \\
& & U F U(b)
\end{array}
\]

2. Assuming the existence of \(R_{\mathcal{G}}, K_{\mathcal{G}}\) is an equivalence of categories (in other words, \(F\) is comonadic) iff the functor \(F\) is conservative (=isomorphism-reflecting) and preserves (or equivalently, preserves and reflects) the equalizer \(2.1\) for each \((b, \theta_b) \in \mathcal{B}_{\mathcal{G}}\).

Let \(i : B \to S\) be an arbitrary extension of (non-commutative) rings, \(\mathcal{A}\) be the category \(\mathcal{B}\text{-Mod}\) of left \(B\)-modules, \(\mathcal{B}\) be the category \(\mathcal{S}\text{-Mod}\) of left \(S\)-modules,

\(F_S = S \otimes_B : \mathcal{B}\text{-Mod} \to \mathcal{S}\text{-Mod}\)

and

\(U_S : \mathcal{S}\text{-Mod} \to \mathcal{B}\text{-Mod}\)

be the restriction-of-scalars functor. It is well known that \(F_S\) is left adjoint to \(U_S\) and that the unit \(\eta\) of this adjunction is given by

\(\eta_X : X \to S \otimes_B X, \eta_X(x) = 1 \otimes_B x\).

It is also well known that the Eilenberg-Moore category \((\mathcal{S}\text{-Mod})_{\mathcal{G}}\) of \(\mathcal{G}\)-coalgebras, \(\mathcal{G}\) being the comonad on \(\mathcal{S}\text{-Mod}\) associated to the adjunction \(F_S \dashv U_S\), is equivalent to the category \(S \otimes_B (\mathcal{S}\text{-Mod})\) of left comodules over the Sweedler canonical \(B\)-coring \(S \otimes_B S\) corresponding to the ring extension \(i\), by an equivalence which identifies the comparison functor \(K_{\mathcal{G}} : \mathcal{B}\text{-Mod} \to (\mathcal{S}\text{-Mod})_{\mathcal{G}}\) with the functor

\(K_S : \mathcal{B}\text{-Mod} \to S \otimes_B (\mathcal{S}\text{-Mod}), \ K_S(X) = (S \otimes_B X, \theta_{S \otimes_B X}),\)

where \(\theta_{S \otimes_B X} = S \otimes_B \eta_X\) for all \(X \in \mathcal{B}\text{-Mod}\). (Note that a left \(S \otimes_B S\)-comodule is a pair \((Y, \theta_Y)\) with \(Y \in \mathcal{S}\text{-Mod}\) and \(\theta_Y : Y \to S \otimes_B Y\) a left \(A\)-module morphism for which the diagrams

\[
\begin{array}{ccc}
Y & \xrightarrow{\theta_Y} & S \otimes_B Y \\
\alpha_Y & & \phi_Y \\
Y & \xrightarrow{\theta_Y} & S \otimes_B Y
\end{array}
\]

and

\[
\begin{array}{ccc}
Y & \xrightarrow{\theta_Y} & S \otimes_B Y \\
\phi_Y & & \phi_Y \\
S \otimes_B Y & \xrightarrow{S \otimes_B \eta_Y} & S \otimes_B Y
\end{array}
\]

where \(\alpha_Y\) denotes the left \(S\)-module structure on \(Y\), are commutative.)

So, to say that the functor \(F_S = S \otimes_B \) is comonadic is to say that the functor \(K_S\) is an equivalence of categories. Applying Beck’s theorem and using that \(\mathcal{B}\text{-Mod}\) has all equalizers, we get:
Theorem 2.2. The functor $F_S = S \otimes_B - : \text{BMod} \to \text{sMod}$ is comonadic if and only if

(i) the functor $F_S$ is conservative, or equivalently, the ring extension $i : B \to S$ is a pure morphism of right $B$-modules;

(ii) for any $(Y, \theta_Y) \in S \otimes_B^S(\text{sMod})$, $F_S$ preserves the equalizer

\[ R_S(Y, \theta_Y) \xrightarrow{e_{(Y, \theta_Y)}} Y \xrightarrow{\eta_Y} S \otimes_B Y, \]

where $R_S : S \otimes_B^S(\text{sMod}) \to \text{BMod}$ is the right adjoint of the comparison functor $K_S : \text{BMod} \to S \otimes_B^S(\text{sMod})$.

Let $A$ be a ring and $\Sigma$ be an $A$-coring. Let us write $\text{End}_{A-\text{cor}}(\Sigma)$ (resp. $\text{Aut}_{A-\text{cor}}(\Sigma)$) for the monoid (resp. group) of $A$-coring endomorphisms (resp. automorphisms) of $\Sigma$. Recall that any $g \in \text{End}_{A-\text{cor}}(\Sigma)$ induces functors:

$g(-) : \Sigma(A\text{Mod}) \to \Sigma(A\text{Mod}),$

defined by $g(Y, \theta_Y) = (Y, (g \otimes_A 1) \circ \theta_Y)$, and

$(-)_g : \text{Mod}^\Sigma \to \text{Mod}^\Sigma$

defined by $(Y', \theta_{Y'})_g = (Y', (1 \otimes_A g) \circ \theta_{Y'})$.

It is easy to see that the left $S$-module $S$ is a left $(S \otimes_B S)$-comodule with left coaction

$s\theta : S \to S \otimes_B S, \quad s \mapsto s \otimes_B 1,$

and that $g(S, s\theta) = (S, g \circ s\theta)$. Symmetrically, the right $S$-module $S$ is a right $(S \otimes_B S)$-comodule with the right action

$\theta_S : S \to S \otimes_B S, \quad s \mapsto 1 \otimes_B s,$

and that $(S, \theta_S)_g = (S, g \circ \theta_S)$.

For a given injective homomorphism $i : B \to S$ of rings, let

- $I_B(S)$ denote the monoid of all $B$-subbimodules of $S$, the multiplication being given by

$IJ = \{ \sum_{k \in K} i_k \cdot j_k, \quad I, J \in I_B(S), \quad i_k \in I, \quad j_k \in J, \text{ and } K \text{ is a finite set} \};$
• $I^L_B(S)$ (resp. $I^R_B(S)$) denote the submonoid of $I_B(S)$ consisting of those $I \in I_B(S)$ for which the map

\[ m^L_I : S \otimes_B I \to S, \ m \otimes_B i \to mi, \]

(resp. $m^R_I : I \otimes_B S \to S, \ i \otimes_B m \to im$)
is an isomorphism;

• $J(g) = \{ s \in S \mid g(s \otimes_B 1) = 1 \otimes_B s \}$ for $g \in \text{End}_{B-\text{cor}}(S \otimes_B S)$ and let $i_g : J(g) \to S$ be the canonical embedding;

• $J'(g) = \{ s \in S \mid s \otimes_B 1 = g(1 \otimes_B s) \}$ for $g \in \text{End}_{B-\text{cor}}(S \otimes_B S)$ and let $i'_g : J'(g) \to S$ be the canonical embedding.

It is clear that $J(g), J'(g) \in I_B(S)$ for all $g \in \text{End}_{B-\text{cor}}(S \otimes_B S)$.

The following result is verified directly:

**Proposition 2.3.** For any $g \in \text{End}_{B-\text{cor}}(S \otimes_B S)$, $R_S(g(S, S \theta)) \simeq J(g)$.

### 3. Main Results

In this section we present our main results.

We begin with

**Proposition 3.1.** For any $g \in \text{End}_{B-\text{cor}}(S \otimes_B S)$, the following conditions are equivalent:

(i) $J(g) \in I^L_B(S)$;

(ii) the $S(S, S \theta)$-component of the counit $\varepsilon : K_S R_S \to 1$ of the adjunction $K_S \dashv R_S$ is an isomorphism;

(iii) the functor $S \otimes_B - : B\text{Mod} \to S\text{Mod}$ preserves the equalizer

\[
J(g) \xrightarrow{i_g} S \xrightarrow{\eta_S} S \otimes_B S;
\]

(iv) the morphism $S \otimes_B i_g : S \otimes_B J(g) \to S \otimes_B S$ is a monomorphism.

**Proof.** It is well known (see, for example, [1]) that, for any $(Y, \theta_Y) \in S \otimes_B S'(S\text{Mod})$, the diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\theta_Y} & S \otimes_B Y \\
& & S \otimes_B Y \\
& \xrightarrow{S \otimes_B \theta_Y} & S \otimes_B S \otimes_B Y
\end{array}
\]
is an equalizer and that the \((Y, \theta_Y)\)-component \(\varepsilon_{(Y,\theta_Y)}\) of \(\varepsilon\) appears as the unique factorization of the morphism \(S \otimes_B e_{(Y,\theta_Y)}\) through the morphism \(\theta_Y\):

\[
\begin{array}{ccc}
\varepsilon_{(Y,\theta_Y)} & \downarrow & S \otimes_B R_S(Y, \theta_Y) \\
\varepsilon_{(Y,\theta_Y)} & \Downarrow & S \otimes_B e_{(Y,\theta_Y)} \\
Y & \theta_Y \downarrow & S \otimes_B Y \\
& \theta_Y \downarrow & S \otimes_B Y \\
& & S \otimes_B S \otimes_B Y.
\end{array}
\]

Since \(\alpha_Y \cdot \theta_Y = 1\), \(\varepsilon_{(Y,\theta_Y)} = \alpha_Y \cdot (S \otimes_B e_{(Y,\theta_Y)})\). In particular, when \((Y, \theta_Y) = g(S, s\theta)\) we get that \(\varepsilon_{g(S, s\theta)} = m^l_{i(g)}\). So (i) and (ii) are equivalent.

Since the row of the diagram (3.2) is an equalizer, it follows that the morphism \(S \otimes_B e_{(Y,\theta_Y)}\) is an equalizer of the pair of morphisms \((S \otimes_B \theta_Y, S \otimes_B \eta_Y)\) iff \(\varepsilon_{(Y,\theta_Y)}\) is an isomorphism. In other words, the functor \(S \otimes_B -\) preserves the equalizer (2.2) iff \(\varepsilon_{(Y,\theta_Y)}\) is an isomorphism. As a special case we then have that (ii) is equivalent to (iii).

Finally, since the category \(B\text{-Mod}\) is abelian (and hence coexact in the sense of Barr\(^\triangledown\)), and since \(i_g\) is the equalizer of the \((S \otimes_B -)\)-split pair of morphisms \((s\theta, \eta_S)\), it follows from the proof of Duskin’s theorem (see, for example, \(\triangledown\)) that the functor \(S \otimes_B -\) preserves the equalizer (3.1) iff the morphism \(S \otimes_B i_g\) is a monomorphism. So (iii) and (iv) are also equivalent. This completes the proof. \(\square\)

It is shown in \(\triangledown\) that assigning to each \(I \in I^l_B(S)\) (resp. \(I \in I^r_B(S)\)) the composite \(\Gamma = (1 \otimes_B (m^l_I)^{-1} \otimes_B 1) \circ (1 \otimes_B (m^l_I)^{-1})\) (resp. \(\Gamma' = (m^l_I \otimes_B 1) \circ (1 \otimes_B (m^l_I)^{-1})\)) yields an (anti-)homomorphism of monoids \(\Gamma : I^l_B(S) \to \text{End}_{B\text{-cor}}(S \otimes_B S)\) (resp. \(\Gamma' : I^l_B(S) \to \text{End}_{B\text{-cor}}(S \otimes_B S)\)).

We shall need the following easy consequence of Lemma 2.7 of \(\triangledown\):

**Proposition 3.2.** Assume that \(i : B \to S\) is such that any embedding \(I \hookrightarrow J\) of \(B\)-submodules of \(S\) is an isomorphism whenever its image under the functor \(S \otimes_B -\) is such. Then \(\Gamma : I^l_B(S) \to \text{End}_{B\text{-cor}}(S \otimes_B S)\) is an isomorphism of monoids whose inverse is the map \(g \to J(g)\), provided that \(J(g) \in I^l_B(S)\) for all \(g \in \text{End}_{B\text{-cor}}(S \otimes_B S)\).

Putting Propositions 3.1 and 3.2 together, we get:

**Theorem 3.3.** Let \(i : B \to S\) be as in Proposition 3.2. Then \(\Gamma : I^l_B(S) \to \text{End}_{B\text{-cor}}(S \otimes_B S)\) is an isomorphism of monoids if and only if, for any \(g \in \text{End}_{B\text{-cor}}(S \otimes_B S)\), the equivalent conditions of Proposition 3.1 hold.
Proposition 3.4. If the functor $S \otimes_B - : \text{BMod} \to \text{SMod}$ is comonadic, then $J(g) \in I_B^1(S)$ for all $g \in \text{End}_{B-\text{cor}}(S \otimes_B S)$.

Proof. Consider the left $(S \otimes_B S)$-comodule $(S, s\theta)$. According to Proposition 2.3 and Theorem 2.1, the pair $(J(g), i_g : J(g) \to S)$ appears as the equalizer

$J(g) \xrightarrow{i_g} S \xrightarrow{\eta_S} S \otimes_B S,$

and since the functor $S \otimes_B -$ is assumed to be comonadic, it preserves the equalizer (2.2) for all $(Y, \theta_Y) \in \text{S} \otimes_B S(\text{SMod})$ and in particular considering $(S, s\theta) \in \text{S} \otimes_B S(\text{SMod})$, we see that

$S \otimes_B J(g) \xrightarrow{S \otimes_B i_g} S \otimes_B S \xrightarrow{S \otimes_B \eta_S} S \otimes_B (S \otimes_B S) \xrightarrow{g \otimes_S \theta} S \otimes_B S$

is an equalizer diagram. It now follows from Proposition 3.1 that $J(g) \in I_B^1(S)$. □

Recalling that any comonadic functor is conservative, and putting Theorem 3.3 and Proposition 3.4 together, we obtain:

Theorem 3.5. If the functor $S \otimes_B - : \text{BMod} \to \text{SMod}$ is comonadic, then $\Gamma : I_B^1(S) \to \text{End}_{B-\text{cor}}(S \otimes_B S)$ is an isomorphism of monoids.

There is of course a dual result.

Theorem 3.6. If the functor $- \otimes_B S : \text{Mod}_B \to \text{Mod}_S$ is comonadic, then $\Gamma' : I_B^r(S) \to \text{End}_{B-\text{cor}}(S \otimes_B S)$ is an anti-isomorphism of monoids.

It is known (see [7]) that the monoid morphism

$\Gamma : I_B^1(S) \to \text{End}_{B-\text{cor}}(S \otimes_B S)$

restricts to a group morphism

$\text{Inv}_B(S) \to \text{Aut}_{B-\text{cor}}(S \otimes_B S),$

which is still denoted by $\Gamma$.

Theorem 3.7. If either

(i) the functor $S \otimes_B - : \text{BMod} \to \text{SMod},$ or
(ii) the functor $- \otimes_B S : \text{Mod}_B \to \text{Mod}_S$

is comonadic, then $\Gamma : \text{Inv}_B(S) \to \text{Aut}_{B-\text{cor}}(S \otimes_B S)$ is an isomorphism of groups.

Proof. The same argument as in [4] shows that if either $\Gamma : I_B^1(S) \to \text{End}_{B-\text{cor}}(S \otimes_B S)$ or $\Gamma' : I_B^r(S) \to \text{End}_{B-\text{cor}}(S \otimes_B S)$ is an isomorphism, then the group homomorphism $\Gamma$ is an isomorphism. Theorems 3.5 and 3.6 now complete the proof. □
As a special case of this theorem, we obtain the following result of Masuoka (see [7]):

**Theorem 3.8.** If either

(i) \( B_S \) is faithfully flat, or

(ii) \( B \) is a direct summand of \( S \) as a \( B \)-bimodule,

then \( \Gamma : I_B^l(S) \to \text{End}_{B-	ext{cor}}(S \otimes B S) \) is an isomorphism of monoids.

**Proof.** In both cases, the functor \( S \otimes_B - \colon \text{B-Mod} \to \text{S-Mod} \) is comonadic. Indeed, to say that \( B_S \) is faithfully flat is to say that the functor \( S \otimes_B - \) is conservative and it preserves all equalizers. Thus, according to Beck’s theorem, this functor is comonadic.

Now, if \( B \) is a direct summand of \( S \) as a \( B \)-bimodule, it is not hard to see that the unit of the adjunction \( F_S = S \otimes_B - \dashv U_S \) is a split monomorphism and it follows from Theorem 2.2 of [6] that the functor \( F_S \) is comonadic. Theorem 3.7 now completes the proof. \( \square \)

Dually we have:

**Theorem 3.9.** If either

(i) \( S_B \) is faithfully flat, or

(ii) \( B \) is a direct summand of \( S \) as a \( B \)-bimodule,

then \( \Gamma' : I_B^r(S) \to \text{End}_{B-	ext{cor}}(S \otimes B S) \) is an anti-isomorphism of monoids.

**Theorem 3.10.** If either

(i) \( B_S \) or \( S_B \) is faithfully flat, or

(ii) \( B \) is a direct summand of \( S \) as a \( B \)-bimodule,

then \( \Gamma : \text{Inv}_B(S) \to \text{Aut}_{B-	ext{cor}}(S \otimes B S) \) is an isomorphism of groups.

**Proof.** The argument here is the same as in the proof of Theorem 3.7. \( \square \)

We now consider the following situation: Let \( A \) and \( B \) be rings, \( M \) a \( (B,A) \)-bimodule with \( M_A \) finitely generated and projective, \( S = \text{End}_A(M) \) the ring of right \( A \)-endomorphisms of \( M_A \), and \( \Sigma = M^* \otimes_B M \) the comatrix \( A \)-coring corresponding to \( BMA \) (for the notion of comatrix coring see [5]). When \( BMA \) is faithful, in the sense that the canonical morphism

\[
i : B \to S, \ s \mapsto [m \to sm]
\]

is injective, one has a map

\[
\Gamma_0 : I_B^l(S) \to \text{End}_{A-	ext{cor}}(\Sigma)
\]
of sets defining $\Gamma_0^i(I)$, $I \in I_B^i(S)$, to be the endomorphism

$$m^* \otimes_B m \rightarrow \sum_i m^* x_i \otimes_B y_i m,$$

where $(m_I^l)^{-1}(1) = \sum_i x_i \otimes_B y_i \in I_B^i(S)$.

**Theorem 3.11.** Suppose that $BM_A$ is such that the functor

$$S \otimes_B - : BMod \rightarrow sMod$$

is comonadic. Then the map

$$\Gamma_0 : I_B^i(S) \rightarrow \text{End}_{A \text{-cor}}(\Sigma)$$

is in fact an isomorphism of monoids.

**Proof.** First of all, the morphism $i : B \rightarrow S$ is injective (or equivalently, the bimodule $BM_A$ is faithful), since the functor $S \otimes_B -$ is assumed to be comonadic. Next, it is proved in [4] that the assignment

$$g \rightarrow \hat{g} = (\xi \otimes_B \xi) \circ (M \otimes_A g \otimes_A M^*) \circ (\xi^{-1} \otimes_B \xi^{-1}),$$

where $\xi : M \otimes_A M^* \rightarrow S = \text{End}_A(M)$ is the canonical isomorphism, yields an injective morphism of monoids

$$\widetilde{(-)} : \text{End}_{A \text{-cor}}(\Sigma) \rightarrow \text{End}_{B \text{-cor}}(S \otimes_B S).$$

And the same argument as in the proof of Proposition 2.6 of [4] shows that the following diagram of sets

$$\begin{array}{ccc}
I_B^i(S) & \xrightarrow{\Gamma_0} & \text{End}_{A \text{-cor}}(\Sigma) \\
\downarrow{\Gamma} & & \downarrow{\widetilde{-}} \\
\text{End}_{B \text{-cor}}(S \otimes_B S) & \stackrel{(\cdot)}{\leftarrow} & \\
\end{array}$$

is commutative. Now, since the functor $S \otimes_B -$ is assumed to be comonadic, it follows from Theorem 3.5 that $\Gamma$ is an isomorphism of monoids and hence the monoid morphism $\widetilde{(-)}$, being injective, is also an isomorphism. Commutativity of the diagram then gives that $\Gamma_0$ is an isomorphism of monoids. \qed

Dually, one can define a map

$$\Gamma_0' : I_B^r(S) \rightarrow \text{End}_{A \text{-cor}}(\Sigma)$$

that sends $I \in I_B^r(S)$ to the endomorphism

$$m^* \otimes_B m \rightarrow \sum_i m^* y_i \otimes_B x_i m$$

of the $A$-coring $\text{End}_{A \text{-cor}}(\Sigma)$, where $(m_I^r)^{-1}(1) = \sum_i y_i \otimes_B x_i \in I \otimes_B S$. 
**Theorem 3.12.** Suppose that $B M_A$ is such that the functor 
$$- \otimes_B S : \text{Mod}_B \to \text{Mod}_S$$

is comonadic. Then 
$$\Gamma'_0 : I_B^r(S) \to \text{End}_{A-\text{cor}}(\Sigma)$$
is an anti-isomorphism of monoids.

It is not hard to check that the map 
$$\Gamma_0 : I_B^l(S) \to \text{End}_{A-\text{cor}}(\Sigma)$$
of sets restricts to a map 
$$\text{Inv}_B(S) \to \text{Aut}_{A-\text{cor}}(\Sigma)$$
which we still call $\Gamma_0$. As in [4], it follows from Theorems 3.11 and 3.12 that

**Theorem 3.13.** If either

(i) the functor $S \otimes_B -$, or

(ii) the functor $- \otimes_B S$

is comonadic, then the map 
$$\Gamma_0 : \text{Inv}_B(S) \to \text{Aut}_{A-\text{cor}}(\Sigma)$$
is actually an isomorphism of groups.

It is shown in [8] that the functor $S \otimes_B : \text{BMod} \to \text{SMod}$ (resp. $- \otimes_B S : \text{Mod}_B \to \text{Mod}_S$) is comonadic iff the functor $M \otimes_B - : \text{BMod} \to \text{AMod}$ (resp. $- \otimes_B M^* : \text{Mod}_B \to \text{Mod}_A$) is. So we have:

**Theorem 3.14.** If either

(i) the functor $M \otimes_B -$, or

(ii) the functor $- \otimes_B M^*$

is comonadic, then the map 
$$\Gamma_0 : \text{Inv}_B(S) \to \text{Aut}_{A-\text{cor}}(\Sigma)$$
is an isomorphism of groups.

From the last theorem one obtains the following result of L. El Kaoutit and J. Gómez-Torrecillas (see Theorem 2.5 in [4]):

**Theorem 3.15.** If

(i) $B M$ is faithfully flat, or

(ii) $M^*_B$ is faithfully flat, or

(i) $B M_A$ is a separable bimodule,
then
\[ \Gamma_0 : \text{Inv}_B(S) \to \text{Aut}_{A_{\text{cor}}}(\Sigma) \]
is an isomorphism of groups.

Proof. (i) and (ii). To say that \( B^M \) (resp. \( M_B^* \)) is faithfully flat is to say that the functor \( M \otimes_B - : B\text{Mod} \to A\text{Mod} \) (resp. \( - \otimes_B M^* : \text{Mod}_B \to \text{Mod}_A \)) is conservative and preserves all equalizers. Then the functor \( S \otimes_B - : B\text{Mod} \to B\text{Mod} \) (resp. \( S - \otimes_B S : \text{Mod}_B \to \text{Mod}_S \)) is comonadic by a simple application of the Beck theorem. Applying the previous theorem, we see that \( \Gamma \) is an isomorphism of groups.

(iii). If \( B^M_A \) is a separable bimodule, then the ring extension \( i : B \to S \) splits (see, for example, [9]), i.e. \( B \) is a direct summand of \( S \) as a \( B \)-bimodule. But we have already seen (see the proof of Theorem 3.8) that in this case, the functor \( S \otimes_B - : \text{Mod}_B \to \text{Mod}_S \) is comonadic, and Theorem 3.13 shows that \( \Gamma \) is an isomorphism of groups. \( \square \)

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