Zeros of Random Analytic Functions

by

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University of California, Berkeley

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Abstract

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The dominant theme of this thesis is that random matrix valued analytic functions, generalizing both random matrices and random analytic functions, for many purposes can (and perhaps should) be effectively studied in that level of generality.

We study zeros of random analytic functions in one complex variable. It is known that there is a one parameter family of Gaussian analytic functions with zero sets that are stationary in each of the three symmetric spaces, namely the plane, the sphere and the unit disk, under the corresponding group of isometries.

We show a way to generate non Gaussian random analytic functions whose zero sets are also stationary in the same domains. There are particular cases where the exact distribution of the zero set turns out to belong to an important class of point processes known as determinantal point processes.

Apart from questions regarding the exact distribution of zero sets, we also study certain asymptotic properties. We show asymptotic normality for smooth statistics applied to zeros of these random analytic functions. Lastly, we present some results on certain large deviation problems for the zeros of the planar and hyperbolic Gaussian analytic functions.
To all my teachers.
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Even more influential on my development as a person and on my attitude towards learning, were my parents, my teachers at school and college, and my relatives and friends (the public opinion is that I could stand much more development, but I claim for myself all credit for that). It would be silly to even try to adequately acknowledge in words, their roles in my making. My father’s huge répertoire of stories fired my imagination in my early childhood and made me think beyond everyday concerns. Apart from many other things, my mother saw to it that I paid due attention to my studies, till the time came when I realized that it was a pleasure. My brother and sister, my aunts and uncles and my friends were quite as important in shaping me. Almost none of them is a mathematician, but they have that high respect (even without full comprehension) for knowledge that is so widespread in India. I greatly value my friends for their great company and for never quite giving up on me, even though I have always been most irregular in returning their e-mails or phone calls. I am sure that all of them will feel happy on seeing my thesis.
Chapter 1

Introduction

1.1 Introductory remarks

Random analytic functions on the one hand and random matrices on the other are two well studied topics in probability theory and mathematical physics. One of the chief interests to a probabilist in these objects is the kind of point processes one gets, by taking the set of zeros or the set of eigenvalues, as the case may be. Both these kinds of point processes typically have the property of “repulsion”, meaning that the points distribute themselves more evenly than they would if they were thrown down independently. That is an appealing feature because, while there are ways to construct point processes that are more clumped than independent points, there are not many natural ways in which to generate point processes with less clumping.

This fact and several others (more empirical than mathematical) have led to a folk wisdom that random analytic functions and random matrices share many similarities. Differing responses to this statement have been heard, including one that points out the obvious tautology here (after all the characteristic polynomial of a random matrix is a random polynomial), and another that says that there is not much similarity but instead evokes an “anthropic” reasoning (the same set of people work on both these fields). Without denying the validity of these explanations, in this thesis we take a more positive approach attempting to provide a unifying framework that includes both random matrices and random analytic functions (see caveats below).
This is the simple but seemingly useful idea of considering random matrix-valued analytic functions, and the set of points where it becomes singular (i.e., the zeros of the determinant). The linear polynomials reduce to random matrices and the $1 \times 1$ matrices correspond to random analytic functions.

A little explanation is in order. When we talk of random analytic functions, we tacitly mean that we are somehow specifying the distribution of coefficients or some closely related quantities (otherwise any random set of points would be the zeros of a random analytic function). Furthermore it is usually difficult to analyse a random analytic function (especially to get exact properties) except in the case of Gaussian coefficients. So the essence of the above paragraph is that the determinant of a Gaussian matrix-valued analytic function is a non-Gaussian analytic function in itself, but nevertheless amenable to analysis because it is built out of Gaussian analytic functions.

Secondly, the earlier claim about random matrices falling within our framework should be toned down. The chief, although not the whole, emphasis in random matrix theory is on the study of Hermitian random matrices and their (real) eigenvalues, for physical as well as mathematical reasons. When we go to higher polynomials there is perhaps no natural way to get the zeros to lie on the real line. This may explain why these objects have not been studied before. What we study here are zeros in the complex plane, for which of course there is no such problem. Nevertheless we believe that it is also interesting mathematically to study polynomials with random Hermitian or random unitary coefficients (we do not do this here) even though the zeros are spread out in the complex plane.

We now outline the contents of the thesis briefly.

- In the remaining sections of this chapter, we give a quick introduction to the basic notions of a point process, correlation functions and Gaussian analytic functions. Most importantly, we recall the three canonical families of Gaussian analytic functions on the plane, the sphere and the unit disk (hyperbolic plane).

- In Chapter 2 we give a recipe for generating a slew of (non-Gaussian) random analytic functions whose zeros are stationary in the plane, the sphere and the unit disk. We make some basic computations on the distribution of zeros
that will be used later.

- In Chapter 3 we recall the notion of a determinantal point process, and characterize the stationary determinantal point processes in the three fundamental domains. Of these the planar ones are known to be (limits of) the distribution of eigenvalues of certain random matrices (the Ginibre ensemble) while the processes on the sphere and disk are new (these processes themselves have been considered before in [5], but an independent probabilistic meaning was not known).

- In Chapter 4 we present an evocative analogy which suggests that the determinantal point processes on the sphere and the disk, introduced in Chapter 3, are in fact the singular points of certain random matrix-valued analytic functions that were introduced in Chapter 2.

- In Chapter 5 we prove that the stationary determinantal processes on the sphere introduced in Chapter 3 are the singular points of the random matrix analytic function $zA - B$ (in this case, more simply, the eigenvalues of $A^{-1}B$).

- In Chapter 6 we give partial proof that the determinantal processes on the disk introduced in Chapter 3 are the singular points of the random matrix analytic function $A_0 +-zA_1 + z^2A_2 + \ldots$.

- In Chapter 7 we show asymptotic normality for smooth statistics applied to the zeros of random analytic functions introduced in Chapter 2 following a method of Sodin and Tsirelson who showed the same for the canonical models of Gaussian analytic functions.

- In Chapter 8 and Chapter 9 we move away from the line of presentation so far, and return to canonical Gaussian analytic functions. We deal with two large deviation type problems for zeros of the planar Gaussian analytic function, one posed by Yuval Peres, which we solve fully and another due to Mikhail Sodin, which we solve partially.
1.2 Basic notions and definitions

1.2.1 Point processes, Correlation functions

A point process in a locally compact Polish space $\Omega$ is a random integer-valued positive Radon measure $\mathcal{X}$ on $\Omega$. (Recall that a Radon measure is a Borel measure which is finite on compact sets.) If $\mathcal{X}$ almost surely assigns at most measure 1 to singletons, it is a simple point process; in this case $\mathcal{X}$ can be identified with a random discrete subset of $\Omega$, and $\mathcal{X}(D)$ represents the number of points of this set that fall in $D$.

The distribution of a point process can, in most cases, be described by its correlation functions (also known as joint intensities) w.r.t a fixed Radon measure $\mu$ on $\Omega$.

**Definition 1.2.1.** The correlation functions of a point process $\mathcal{X}$ w.r.t. $\mu$ are functions (if any exist) $\rho_n : \Omega^n \rightarrow [0, \infty)$ for $n \geq 1$, such that for any family of mutually disjoint Borel subsets $D_1, \ldots, D_k$ of $\Omega$, and for any non-negative integers $n_1, \ldots, n_k$,

$$
E \left[ \prod_{i=1}^{k} \left( \frac{\mathcal{X}(D_i)}{n_i} \right)^{n_i} \right] = \int_{\prod D_i^{n_i}} \rho_n(x_1, \ldots, x_n) d\mu(x_1) \ldots d\mu(x_n), \quad (1.2.1)
$$

where $n = \sum_{i=1}^{k} n_i$.

**Remark 1.2.2.** It is a natural question to ask for conditions that guarantee the existence of correlation functions and conditions under which they determine the distribution of the point process. Such conditions do exist, see Lenard’s (23), (24), (25) or the survey by Soshnikov (37). But the conditions are too complicated and not relevant for our purposes. In any case, when the joint distribution of $\mathcal{X}(D_1), \ldots, \mathcal{X}(D_k)$ is determined by its moments, the correlation functions determine the distribution of $\mathcal{X}$.

**Remark 1.2.3.** For overlapping sets, the situation is more complicated. Restricting attention to simple point processes, $\rho_n$ is not the intensity measure of $\mathcal{X}^n$, but that of $\mathcal{X}^\wedge n$, the set of ordered $n$-tuples of distinct points of $\mathcal{X}$. Indeed, (1.2.1)
implies (see [23; 24; 28]) that for any Borel set $B \subset \Omega^n$ we have

$$E \#(B \cap \mathcal{X}^n) = \int_B \rho_n(x_1, \ldots, x_n) \, d\mu(x_1) \ldots d\mu(x_n).$$

(1.2.2)

Assuming that $\mathcal{X}$ is simple, the correlation functions may be interpreted as follows:

- If $\Omega$ is finite and $\mu = \text{counting measure}$ then $\rho_k(x_1, \ldots, x_k)$ is the probability that $x_1, \ldots, x_k \in \mathcal{X}$.
- If $\Omega$ is open in $\mathbb{R}^d$ and $\mu = \text{Lebesgue measure}$, if $\rho_n$ exist and are continuous, then

$$\rho_k(x_1, \ldots, x_k) = \lim_{\epsilon \to 0} \frac{P[\mathcal{X} \text{ has a point in each of } B_{\epsilon}(x_j)\]}{(\text{Vol}(B_{\epsilon}))^k}. \quad (1.2.3)$$

Conversely, if for every $k \geq 1$, the right hand side of (1.2.3) exists and is continuous in $x_i$, $1 \leq i \leq k$, then it is the $k$-point correlation functions of $\mathcal{X}$.

For us $\Omega$ will always be an open subset of the plane (or the sphere $S^2$) and $\mathcal{X}$ will be a simple point process. $\mu$ may always be taken to be the Lebesgue measure on $\Omega$, but we often find it convenient to use some other measure that is mutually absolutely continuous with the Lebesgue measure.

### 1.2.2 Complex Gaussian distribution

A **standard complex Gaussian** is a complex-valued random variable with probability density $\frac{1}{\pi} e^{-|z|^2}$ w.r.t the Lebesgue measure on the complex plane. Equivalently, one may define it as $X + iY$, where $X$ and $Y$ are i.i.d. $N(0, \frac{1}{2})$ random variables.

Let $a_k, 1 \leq k \leq n$ be i.i.d. standard complex Gaussians. Let $a$ denote the column vector $(a_1, \ldots, a_n)^t$. Then if $B$ is an $m \times n$ matrix, $Ba + \mu$ is said to be an $m$-dimensional complex Gaussian vector with mean $\mu$ (an $m \times 1$ vector) and covariance $\Sigma = BB^*$ (an $m \times m$ matrix). We denote its distribution by $CN_m(\mu, \Sigma)$.

Here are some basic properties of complex Gaussian random variables.

- If $a$ is a complex Gaussian, its distribution is determined by $\mu = E[a]$ and $\Sigma = E[(a - \mu)(a - \mu)^*]$. All moments of the form

$$E[(a_k - \mu_k)(a_j - \mu_j)], \quad 1 \leq k, j \leq n,$$
vanish. This is the case even for \( j = k \).

- If \( a \) is a standard complex Gaussian, then \(|a|^2\) and \( \frac{a}{|a|} \) are independent, and have exponential distribution with mean 1 and uniform distribution on the circle \( \{ z : |z| = 1 \} \), respectively.

- If \( a_n, n \geq 1 \) are i.i.d. \( \mathbb{C}N(0, 1) \), then by an easy application of Borel-Cantelli,

\[
\limsup_{n \to \infty} |a_n|^{\frac{1}{n}} = 1, \quad \text{almost surely.} \tag{1.2.4}
\]

In fact, equation (1.2.4) is valid for any i.i.d. sequence of complex-valued random variables \( a_n \), such that \( \mathbb{E}[\max\{\log |a_1|, 0\}] < \infty \). Equation (1.2.4) is useful to compute the radii of convergence of random power series with independent coefficients.

- **Wick Expansions:** The Wick or the Feynman diagram expansion is an expansion of \( L^2 \) functions of a Gaussian measure in an orthonormal basis consisting of polynomials of the underlying Gaussians. Following the presentation in the book by Janson [17], we state the essential facts in the limited context that we shall need later. More details and complete proofs of the assertions can be found in [17].

Let \( a_1, \ldots, a_p \) be i.i.d. standard complex Gaussians. Consider the collection of all monomials \( \prod_k a_k^{m_k} \overline{a}_k^{n_k} \) in these \( p \) variables, and orthonormalise them by projecting the polynomials of degree \( N \) on the orthogonal complement of the polynomials of degree \( N - 1 \).

This procedure is the same as applying Gram-Schmidt to the monomials after arranging them in increasing order of the degree (how we order monomials of the same degree is immaterial because distinct monomials of the same degree are clearly orthogonal). Thus we get an orthonormal basis of all square integrable functions of the \( a_k \)s, and the basis elements, termed Wick powers, are denoted by

\[
\prod_k \frac{a_k^{m_k} \overline{a}_k^{n_k}}{\sqrt{m_k!n_k!}} := \prod_k \frac{a_k^{m_k} \overline{a}_k^{n_k}}{\sqrt{m_k!n_k!}}; \\
\]

the equality a consequence of the independence of \( a_k \)s.
These Wick polynomials are known explicitly (see (17))-

\[ :a^m \overline{a}^n := \sum_{r=0}^{m \land n} (-1)^r r! \binom{m}{r} \binom{n}{r} a^{m-r} \overline{a}^{n-r}, \]

although this is not particularly important to us. It is quite well known that products of random variables that are jointly Gaussian can be described by summing over the weights of certain combinatorial entities. There is a similar formula (known as Wick formula or Feynman diagram formula) for expectation of product of Wick powers. We shall only need the following special case.

**Wick/Feynman diagram formula:** Let \((b_1, \ldots, b_s)\) have a complex Gaussian distribution with mean zero. Then

\[
E \left[ \prod_{j=1}^{s} :b_j^m \overline{b}_j^n : \right] = \sum_{\gamma} \nu(\gamma), \quad (1.2.5)
\]

where the sum is over all complete Feynman diagrams \(\gamma\) without self interaction (henceforth we shall just say *Feynman diagram*). To define this, consider a collection of \(\sum_j (m_j + n_j)\) vertices with \(m_j\) of the vertices labeled \(j\) and \(n_j\) of the vertices labeled \(\overline{j}\), for \(1 \leq j \leq s\). All the vertices labeled \(j\) are also supposed to be distinguishable although we shall not introduce any more notation to distinguish them. Now, each \(\gamma\) is a matching of these vertices (a subgraph in which each vertex has degree 1), such that each edge in \(\gamma\) connects a vertex labeled \(i\) to a vertex labeled \(\overline{j}\) for some \(i \neq j\).

The value \(\nu(\gamma)\) of the diagram is the product of the weights of all the edges in \(\gamma\), and the weight of an edge joining a vertex labeled \(i\) to a vertex labeled \(\overline{j}\) (\(i \neq j\)) is \(E[b_i \overline{b}_j]\).

**Example 1.2.4.** Let \(s = 2\). Then we must consider Feynman diagrams on the labels \(\{1, \overline{1}, 2, \overline{2}\}\) with \(m_1\) vertices labeled 1, \(m_2\) vertices labeled 2, \(n_1\) vertices labeled \(\overline{1}\) and \(n_2\) vertices labeled \(\overline{2}\). Since a Feynman diagram (in our terminology as explained above) must connect 1s to \(\overline{2}\)s and vice-versa, and must give every vertex degree one, there are no Feynman diagrams unless
$m_1 = n_2$ and $m_2 = n_1$, in which case there are $m_1!n_1!$ such diagrams. Thus

$$\mathbb{E}\left[\prod_{j=1}^{2} b_j^{m_j} \bar{b}_j^{n_j}\right] = \begin{cases} m_1!n_1! \mathbb{E}[b_1 \bar{b}_2]^m \mathbb{E}[b_2 \bar{b}_1]^n & \text{if } m_1 = n_2, m_2 = n_1, \\ 0 & \text{otherwise.} \end{cases}$$

### 1.2.3 Gaussian analytic functions

Endow the space of analytic functions on a region $\Omega$ with the topology of uniform convergence on compact sets. This makes it a complete separable metric space which is the standard setting for doing probability theory. To see completeness, if $\{f_n\}$ is a Cauchy sequence, then $f_n$ converges uniformly on compact sets to some continuous function $f$. Then it is easy to see that $f$ must be analytic because its integral on any closed contour is zero since $\int_{\gamma} f = \lim_{n \to \infty} \int_{\gamma} f_n$ and the latter vanishes for every $n$, by analyticity of $f_n$.

**Definition 1.2.5.** Let $f$ be a random variable taking values in the space of analytic functions on a region $\Omega \subset \mathbb{C}$. We say $f$ is a Gaussian analytic function (GAF) on $\Omega$ if $(f(z_1), \ldots, f(z_n))$ has a mean zero complex Gaussian distribution for every $z_1, \ldots, z_n \in \Omega$.

It is easy to see the following properties of GAFs.

- $\{f^{(k)}\}$ are jointly Gaussian, i.e., the joint distribution of $f$ and finitely many derivatives of $f$ at finitely many points,

$$\left\{f^{(k)}(z_j) : 0 \leq k \leq n, 1 \leq j \leq m\right\},$$

has a (mean zero) complex Gaussian distribution.

- The distribution of a Gaussian analytic function is determined by its covariance kernel $\left(\mathbb{E}\left[f(z)\bar{f}(w)\right]\right)_{z,w \in \Omega}$ denoted by $K_f(z, w)$ or just $K(z, w)$ if there is no ambiguity as to which $f$ is under consideration.

### 1.2.4 Stationary zero sets of Gaussian analytic functions

Our interest is in the zero set of a random analytic function. Unless one’s intention is to model a particular physical phenomenon by a point process, there is one criterion that makes some point processes more interesting than
others, namely, *stationarity* under a large group of transformations (stationarity of a random process means invariance of its distribution under a group action. It is also called *invariance*, especially when the stationarity is in “space” rather than “time”, but we use both terms interchangeably). There are three particular two dimensional domains on which the group of conformal automorphisms act transitively (There are two others that we do not consider here, the cylinder or the punctured plane, and the two dimensional torus). We introduce these domains now.

- **The Complex Plane** $\mathbb{C}$: The group of transformations

  $$\varphi_{\lambda,\beta}(z) = \lambda z + \beta, \quad z \in \mathbb{C}$$  

  (1.2.6)

  where $|\lambda| = 1$ and $\beta \in \mathbb{C}$, is nothing but the Euclidean motion group. These transformations preserve the Euclidean metric $ds^2 = dx^2 + dy^2$ and the Lebesgue measure $dm(z) = dx\,dy$ on the plane.

- **The Sphere** $S^2 = \mathbb{C} \cup \{\infty\}$: The group of transformations

  $$\varphi_{\alpha,\beta}(z) = \frac{\alpha z + \beta}{-\beta z + \alpha}, \quad z \in \mathbb{C} \cup \{\infty\}$$  

  (1.2.7)

  where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$, is the group of linear fractional transformations mapping $\mathbb{C} \cup \{\infty\}$ to itself bijectively. These transformations preserve the spherical metric $ds^2 = \frac{dx^2 + dy^2}{(1+|z|^2)^2}$ and the spherical area measure $\frac{dm(z)}{(1+|z|^2)^2}$.

  We call it the spherical metric because it is the push forward of the usual metric on the sphere inherited from $\mathbb{R}^3$, onto $\mathbb{C} \cup \{\infty\}$ under the stereographic projection, and the measure is the push forward of the spherical area measure. The transformations (1.2.7) are just the rotations of the sphere under this identification with $\mathbb{C} \cup \{\infty\}$.

- **The Hyperbolic Plane** $\mathbb{D} = \{z : |z| < 1\}$: The group of transformations

  $$\varphi_{\alpha,\beta}(z) = \frac{\alpha z + \beta}{\beta z + \alpha}, \quad z \in \mathbb{D}$$  

  (1.2.8)

  where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 - |\beta|^2 = 1$, is the group of linear fractional transformations mapping the unit disk $\mathbb{D} = \{z : |z| < 1\}$ to itself bijectively. These
transformations preserve the hyperbolic metric \( ds^2 = \frac{dx^2 + dy^2}{(1-|z|^2)^2} \) and the hyperbolic area measure \( \frac{dm(z)}{(1-|z|^2)^2} \) (this normalization differs from the usual one, with curvature \(-1\), by a factor of \(4\), but it makes the analogy with the other two cases more formally similar). This is one of the many models discovered by Poincaré for the hyperbolic geometry of Bolyai, Gauss and Lobachevsky (see [6] for an introduction).

Note that in each case, the group of transformations acts transitively on the corresponding space, i.e., for every \( z, w \) in the domain, there is a transformation \( \varphi \) such that \( \varphi(z) = w \). This means that in these spaces every point is just like every other point. Now we introduce three families of GAFs whose relation to these symmetric spaces will be made clear in Proposition[1.2.7]

In each case, the domain of the random analytic function can be found from (1.2.4). Indeed, (1.2.4) implies that when \( a_n \) are i.i.d. standard complex Gaussians, \( \sum_n a_n c_n z^n \) has the same radius of convergence as \( \sum_n c_n z^n \).

- **The Complex Plane** \( \mathbb{C} \): Define for \( L > 0 \),
  \[
  f(z) = \sum_{n=0}^{\infty} a_n \frac{\sqrt{L^n}}{\sqrt{n!}} z^n. 
  \]
  (1.2.9)
  For every \( L > 0 \), this is a random analytic function in the entire plane.

- **The Sphere** \( S^2 \): Define for \( L \in \mathbb{N} = \{1, 2, 3, \ldots\} \),
  \[
  f(z) = \sum_{n=0}^{L} a_n \frac{\sqrt{L(L-1)\ldots(L-n+1)}}{\sqrt{n!}} z^n. 
  \]
  (1.2.10)
  For every \( L \in \mathbb{N} \), this is a random analytic function on \( S^2 = \mathbb{C} \cup \infty \) with a pole at \( \infty \) (i.e., it is a polynomial).

- **The Hyperbolic Plane** \( \mathbb{D} \): Define for \( L > 0 \),
  \[
  f(z) = \sum_{n=0}^{\infty} a_n \frac{\sqrt{L(L+1)\ldots(L+n-1)}}{\sqrt{n!}} z^n. 
  \]
  (1.2.11)
  For every \( L > 0 \), this is a random analytic function in the unit disk \( \mathbb{D} = \{ z : |z| < 1 \} \).
Remark 1.2.6. Although we wrote (1.2.9) for every $L > 0$, they are identical up to a scaling of the complex plane. However, the functions in (1.2.10) and (1.2.11) are truly different for different $L$, i.e., there is no transformation of the $S^2$ and $\mathbb{D}$, that makes $f_L$ and $f_{L'}$ the same, for $L \neq L'$. This is particularly obvious for the sphere, because $L$ then denotes the number of zeros of $f$.

We just quote the following proposition from (35). (The proof is contained in the proof of Proposition 2.1.1). These random analytic functions were discovered in several stages and (partially) by several authors. The main contributions are due to Bogomolny, Bohigas and Leboeuf (1) and (2), Kostlan (22), Shub and Smale (32). Some of them are natural generalizations (to complex coefficients) of random polynomials studied by Mark Kac in his founding papers starting with (19). The special case $L = 2$, in the unit disk was derived also by Diaconis and Evans (8) as the limit of the logarithmic derivative of characteristic polynomials of random unitary matrices. The uniqueness in Proposition 1.2.7 also was perhaps known, but a much stronger form of uniqueness (that the first intensity of zeros of any Gaussian analytic function determines the distribution of the Gaussian analytic function itself, up to multiplication by arbitrary deterministic non-vanishing analytic functions) was found by Sodin (33).

**Proposition 1.2.7.** The zero sets of the GAF $f$ in equations (1.2.9), (1.2.10) and (1.2.11) are invariant (in distribution) under the transformations defined in equations (1.2.6), (1.2.7) and (1.2.8) respectively. This holds for every allowed value of the parameter $L$, namely $L > 0$ for the plane and the disk and $L \in \mathbb{N}$ for the sphere.

Moreover, these are the only Gaussian analytic functions (up to multiplication by deterministic non-vanishing analytic functions) with stationary zero sets in these domains.
Chapter 2

Stationary zero sets of random analytic functions

As we saw in Proposition 1.2.7 on each of the three domains $\mathbb{C}/S^2/\mathbb{D}$, there is a one parameter family of Gaussian analytic functions whose zero sets are stationary under the corresponding group of isometries. Moreover, these are the only Gaussian analytic functions on these domains with these properties. Indeed Hannay [13] likens the uniqueness of the Gaussian analytic function in (1.2.9) to that of the Poisson process or the thermal blackbody radiation.

Here we stick to the three domains $\mathbb{C}/S^2/\mathbb{D}$ and ask for random analytic functions whose zero sets are stationary. By Proposition 1.2.7 we must necessarily seek among non-Gaussian analytic functions. A natural idea might be to replace i.i.d. Gaussians in the coefficients by i.i.d. complex-valued random variables from some other distribution. However, these seem difficult to analyse. Gaussian analytic functions have the nice property that the evaluations of the function and its derivatives are all Gaussian with distributions that we can explicitly work with and this fails in other cases. In fact we do not know of another example of a power series with i.i.d. coefficients whose zero set is stationary (on any of these three domains). We resolve this deadlock by constructing non-Gaussian analytic functions using Gaussian analytic functions as building blocks.
2.1 A recipe for stationary zero sets of random analytic functions

Let \( Q \) be a (non-random) homogeneous polynomial in \( k \) variables with complex coefficients and let \( f \) be any Gaussian analytic function (not necessarily one of the canonical models defined in Section 1.2.4). Then if \( f_i, i \leq k \) are i.i.d. copies of \( f \), then \( Q(f_1, \ldots, f_k) \) is a random analytic function on the same domain as \( f \).

**Proposition 2.1.1.** Let \( Q \) be a homogeneous polynomial of degree \( d \) in \( k \) variables with complex coefficients, and let \( f \) be one of the canonical models of Gaussian functions in \((1.2.9), (1.2.10)\) or \((1.2.11)\). If \( f_i, 1 \leq i \leq k \) are i.i.d. copies of \( f \), then the zero set of the random analytic function

\[
F(z) := Q(f_1(z), \ldots, f_k(z))
\]

is stationary under the same group of isometries as the zero set of \( f \).

**Proof.** First we recall the proof of invariance of the zero set of the Gaussian analytic functions in \((1.2.9), (1.2.10)\) and \((1.2.11)\). Fix an isometry \( \varphi \) of \( \Omega \) (given in \((1.2.6), (1.2.7)\) and \((1.2.8)\)). In each of the three cases, there is a deterministic non-vanishing function \( \Delta_{\varphi,L} \) such that

\[
f(z) \overset{d}{=} \Delta_{\varphi,L}(z)f(\varphi(z)),
\]

where in fact

\[
\Delta_{\varphi,L}(z) = \begin{cases} 
  e^{Lz\lambda^2 + \frac{1}{2}L|\beta|^2} & \text{domain } = \mathbb{C}. \\
  \varphi'(z)^\frac{L}{2} & \text{domain } = \mathbb{S}^2. \\
  \varphi'(z)^{-\frac{L}{2}} & \text{domain } = \mathbb{D}.
\end{cases}
\]

Note that the equality in \((2.1.1)\) is for the entire process, not just for a fixed \( z \). Therefore, the zero set of \( f \) is invariant in distribution under the action of \( \varphi \). (To prove equation \((2.1.1)\), just compute the covariance kernels of the Gaussian processes on the left and right hand sides).

Coming back to \( F \), we see that

\[
\Delta(z)^d F(\varphi(z)) = Q(\Delta(z)f_1(\varphi(z)), \ldots, \Delta(z)f_k(\varphi(z)))
\]

\[
\overset{d}{=} Q(f_1(z), \ldots, f_k(z)) \quad \text{from } (2.1.1)
\]

\[
= F(z).
\]
This implies that the zero set of $F$ is invariant in distribution under the action of $\varphi$.

This is a very simple observation, but note that while $F$ is built in a simple manner out of copies of $f$, the zero set of $F$ is by no means a simple transformation of the zero sets of $f_1, \ldots f_k$ (except in trivial cases such as when $Q(\zeta_1, \zeta_2) = \zeta_1 \zeta_2$). Thus the sets of zeros that we get are genuinely new point processes, but have the advantage of being based on Gaussian analytic functions, and therefore amenable to analysis. We illustrate this next, by computing the first and second correlations (joint intensities) for the zeros of $F$. The tool that we use to study functions such as $F$ is the Wick expansion, suggested to us by Mikhail Sodin (see the paper by Sodin and Tsirelson [35] for a use of Wick expansions in the context of Gaussian analytic functions). We call random analytic functions of the kind described in Proposition 2.1.1 as polygafs.

2.2 How to study the zeros of a polygaf?

If $F$ is any analytic function (not random) on $\Omega$, let $dn_F$ denote the counting measure, with appropriate multiplicities, on the zeros of $F$. Then,

$$\frac{1}{2\pi} \Delta \log |F(z)| = dn_F(z)$$

in the sense of distributions. This just means that for any $\varphi \in C^\infty_0(\Omega),$

$$\int_\Omega \varphi(z)dn_F(z) = \int_\Omega \Delta \varphi(z) \frac{1}{2\pi} \log |F(z)| dm(z),$$

where $m$ is the Lebesgue measure. Therefore when $F$ is any random analytic function, understanding the distribution of the zero set depends on being able to do computations with $\log |F|$ (When $F$ is Gaussian, there are other approaches to studying the zero set of $F$, but it appears that the approach outlined here is the only one that is equally convenient for our more general setting. The other methods make use of the probability density of $F$ evaluated at several points in the domain etc, which are not available to us here).
Now from (2.2.2), if \( \varphi_i, 1 \leq i \leq k \) are smooth functions with disjoint supports in \( \Omega \), we get that

\[
E \left[ \prod_{i=1}^{k} \varphi_i(z) dm_F(z) \right] = (2\pi)^{-k} \int_{\Omega^k} \prod_{i=1}^{k} \Delta_{z_i} \varphi_i(z_i) \left[ \prod_{i=1}^{k} \log |F(z_i)| \right] \prod_{i=1}^{k} dm(z_i)
\]

In the last line we integrated by parts.

In 1.2.1 we defined the correlation functions in terms of the moments of the joint counts of the number of points falling in several regions. Fixing \( k \) distinct points \( w_1, \ldots, w_k \) in \( \Omega \) and letting \( \varphi_i \) be a bump function in a small neighbourhood of \( z_i \), by elementary measure theoretical arguments one can deduce that the \( k \)-point correlation function \( \rho_k \) of the zero set of \( F \), w.r.t Lebesgue measure is given by

\[
\rho_k(w_1, \ldots, w_k) = \frac{1}{(2\pi)^k} \left( \prod_{i=1}^{k} \Delta_{w_i} \right) \left[ \prod_{i=1}^{k} \log |F(w_i)| \right], \quad \text{(2.2.3)}
\]

for distinct \( w_1, \ldots, w_k \).

**Remark 2.2.1.** Hammersley [12] gave a formula for the correlation functions of zeros of random polynomials in terms of the distribution of the coefficients. (2.2.3) is an alternative way of expressing the same. In this form, for \( k = 1 \) it is sometimes called Edelman-Kostlan formula.

The way to analyse \( \log |F| \) is via Wick expansions that were outlined in Chapter 1.

**Example 2.2.2.** The particular example of Wick expansions that is of interest to us is the following: Let \( Q \) be a homogeneous polynomial in \( k \) variables with complex coefficients. If \( a_i, i \leq k \) are i.i.d. \( \mathbb{C}N(0,1) \) random variables, then

\[
E \left[ \left| \log |Q(a_1, \ldots, a_k)| \right|^p \right] < \infty
\]

for every finite \( p \). Hence we can expand \( \log |Q(a_1, \ldots, a_k)| \) in Wick powers as

\[
\log |Q(a_1, \ldots, a_k)| = \sum_{m.n \in \mathbb{Z}_+^k} \frac{C_{m.n}}{\sqrt{m!n!}} \prod_{j=1}^{k} a_j^{m_j} a_j^{-n_j}, \quad \text{(2.2.4)}
\]
where \( m = (m_1, \ldots, m_k), n = (n_1, \ldots, n_k), m! = \prod_{j=1}^{k} m_j! \) and

\[
C_{m,n} = \frac{1}{\sqrt{m!n!}} \mathbb{E} \left[ \log |Q(a_1, \ldots, a_k)| \prod_{j=1}^{k} a_j^{m_j} a_j^{n_j} \right],
\]

and the equality in (2.2.4) is in the \( L^2 \) sense (it could be better, of course).

We record two observations for later use.

- \( C_{m,n} = C_{n,m} \) for all \( m, n \in \mathbb{Z}_+^k \), because \( \alpha_k \) are also i.i.d. \( \mathbb{C}N(0,1) \).

- \( C_{m,n} = 0 \) unless \( m = n \), where \( m := \sum_j m_j \). To see this, note for any \( \lambda \) with \( |\lambda| = 1 \), \( \lambda a_j \) are also i.i.d. \( \mathbb{C}N(0,1) \) and hence, by the homogeneity of \( Q \), it is also true that \( \log |Q(\lambda a_1, \ldots, \lambda a_k)| = \log |Q(a_1, \ldots, a_k)| \). Therefore, from the equation above for \( C_{m,n} \), we see that \( C_{m,n} = \lambda^{m-n} C_{m,n} \), which cannot be true unless \( m = n \) or \( C_{m,n} = 0 \).

### 2.3 Distribution of the zero set of a polygaf

Now let \( f_i \) be i.i.d. copies of \( f \), a Gaussian analytic function on a domain \( \Omega \) (not necessarily one of the canonical GAFs on the plane, sphere or disk). As before \( Q \) is a homogeneous polynomial.

Define \( F(z) = Q(f_1(z), \ldots, f_k(z)) \). If \( K(z,w) \) is the covariance kernel of \( f \), then set

\[
\hat{F}(z) = \frac{F(z)}{K(z,z)^{d/2}} = Q \left( \hat{f}_1(z), \ldots, \hat{f}_k(z) \right),
\]

where \( \hat{f}_j(z) = \frac{f_j(z)}{\sqrt{K(z,z)}} \) and \( d \) is the degree of \( Q \). Then from (2.2.4) we can write,

\[
\log |\hat{F}(z)| = \sum_{m,n \in \mathbb{Z}_+^k} \frac{C_{m,n}}{\sqrt{m!n!}} \prod_{j=1}^{k} : f_j(z)^{m_j} \hat{f}_j(z)^{n_j} :
\]

We work with \( \hat{F} \) rather than \( F \), because \( \hat{f}_j(z) \) are i.i.d standard Gaussians for any \( z \), and so we can directly use the Wick formulas that we stated in the previous chapter.

**First Intensity:** The first intensity (or 1-point correlation function) as given by (2.2.3) is

\[
\rho_1(z) = \frac{1}{2\pi} \Delta_z \mathbb{E} [\log |F(z)|].
\]
Therefore,

\[ \rho_1(z) = \frac{1}{2\pi} \Delta_z \mathbb{E} [\log |F(z)|] \]
\[ = \frac{1}{2\pi} \Delta_z \mathbb{E} [\log |\hat{F}(z)|] + \frac{1}{2\pi} \Delta_z \log K(z, z)^{d/2} \]
\[ = \frac{1}{2\pi} \Delta_z C_{0,0} + \frac{1}{2\pi} \Delta_z \log K(z, z)^{d/2} \]

from (2.2.4). Therefore we obtain

\[ \rho_1(z) = \frac{d}{4\pi} \Delta_z \log K(z, z). \quad (2.3.2) \]

As a special case, set \( k = 1 \), \( Q(\zeta) = \zeta \) in formula (2.3.2) to deduce that the intensity of zeros of \( f \) is \( \frac{1}{4\pi} \Delta_z \log K(z, z) \) (This is known as the Edelman-Kostlan formula). Thus we see that the intensity of zeros of \( F \) is \( d \) times the intensity of zeroes of \( f \). This simple relationship between the intensities is surprisingly not quite obvious from the definition.

**Two point Correlations:** Again from (2.2.3) we get the 2-point correlation. It is easy to see that

\[ \rho_2(z, w) - \rho_1(z)\rho_1(w) = \frac{1}{(2\pi)^2} \Delta_z \Delta_w \mathbb{E} [\log |\hat{F}(z)| \log |\hat{F}(w)|] \]

From (2.2.4) the right hand side can be written as

\[ \frac{1}{(2\pi)^2} \Delta_z \Delta_w \sum_{m, n, m', n'} C_{m,n} C_{m', n'} \frac{1}{\sqrt{m!n!m'!n'!}} \prod_{j=1}^{k} \mathbb{E} [\hat{f}_j(z)^{m_j} \hat{f}_j(z)^{n_j} \hat{f}_j(z)^{m'_j} \hat{f}_j(z)^{n'_j}] . \]

This is precisely the situation elucidated in Example 1.2.4. Thus, only terms with \( m = n' \), \( n = m' \) survive.

Now make use of the observations made earlier- (1) \( C_{m,n} = \overline{C}_{n,m} \) and (2) \( C_{m,n} = 0 \) unless \( m_* = n_* \). Grouping together terms by \( p = m_* \), we get,

\[ \rho_2(z, w) - \rho_1(z)\rho_1(w) = \frac{1}{(2\pi)^2} \sum_{p=0}^{\infty} |\tilde{C}_p|^2 \Delta_z \Delta_w \frac{|K(z, w)|^{2p}}{K(z, z)^{p} K(w, w)^p} \]

where \( |\tilde{C}_p|^2 = \sum_{m_* = n_* = p} |C_{m,n}|^2 \).

**Remark 2.3.1.** Equation (2.3.3) has the appealing feature that the effects of the two ingredients of \( F \), namely the polynomial \( Q \) and the Gaussian analytic function \( f \), are clearly separated. \( |\tilde{C}_p|^2 \) depends only on \( Q \) while \( \frac{|K(z, w)|^{2p}}{K(z, z)^{p} K(w, w)^p} \) depends only
on \( f \). This observation is crucially used in the next chapter, when we compute the correlations for specific polynomials \( Q \). One can write analogous but more complicated expressions for higher correlations, with \( \Delta_{z_k}, 1 \leq k \leq m \) applied to a sum over Feynman diagrams.
Chapter 3

Stationary determinantal point processes

In this chapter we move away from random analytic functions and talk about a different class of point processes. In the next chapter, we return to zeros of random analytic functions and show that there are point processes in the intersection of the two classes.

One of the main qualitative properties of zero sets of random analytic functions is that they have the property of “repulsion”, also called “negative correlation”, at short ranges. This terminology is a little misleading because correlations are never negative! The precise meaning of negative correlations is that \( \rho_2(x,y) < \rho_1(x)\rho_1(y) \) for \( x,y \) that are sufficiently close. There is another class of point processes that has this repulsion property at all distances in a very strong sense. These point processes were introduced by Macchi [26] and are known as **Determinantal (Fermionic) point processes**. See Figure 3 for a visual comparison of zeros and eigenvalues with a Poisson process.

**Definition 3.0.2.** A point process \( \mathcal{X} \) on \( \Omega \) is said to be a **determinantal process** with kernel \( \mathbb{K} : \Omega^2 \rightarrow \mathbb{C} \), if it is simple (i.e., there are no coincident points almost surely) and its correlation functions w.r.t a measure \( \mu \) satisfy:

\[
\rho_k(x_1, \ldots, x_k) = \det (\mathbb{K}(x_i, x_j))_{1 \leq i,j \leq k},
\]

for every \( k \geq 1 \) and \( x_1, \ldots, x_k \in \Omega \). We shall always assume that \( \mathbb{K} \) is the projection
Figure 3.1: Samples of a translation invariant determinantal process (left) and zeros of a Gaussian analytic function (right). Determinantal processes exhibit repulsion at all distances, and the zeros repel at short distances only. The picture in the center shows a Poisson process (picture due to Bálint Virág).

kernel of a closed subspace of $L^2$, i.e., that $\mathbb{K}(x, y) = \sum_j \varphi_j(z)\overline{\varphi_j(y)}$ for a sequence of functions (there may be infinitely many of them) $\{\varphi_j\}$ orthonormal in $L^2(\mu)$. The distribution is determined by giving the kernel $\mathbb{K}$ or the Hilbert space on which it is a projection.

**Remark 3.0.3.** This definition may look artificial at first sight, but the motivation comes from quantum mechanics, where the probability densities are given by the absolute square of a complex-valued function called the amplitude. If $\varphi_1, \ldots, \varphi_n$ (assumed to be orthogonal) are single particle wave functions of electrons, the most natural $n$-particle wave function is not $\prod_k \varphi_k(x_k)$ (as would have been the case for “independence”) because firstly it has no symmetry in $x_1, \ldots, x_n$, and secondly it shows no properties such as repulsion. Hence the idea is to anti-symmetrize this, to get $\det (\varphi_i(x_j))$ (This is for Fermions. For Bosons, one symmetrizes and gets the permanent of $(\varphi_i(x_j))$, but that is another story). The probability density is given by the absolute square of this, which can be written as $\det (\mathbb{K}(x_i, x_j))_{1 \leq i, j \leq n}$, with $\mathbb{K}(x, y) = \sum_{j=1}^n \varphi_j(x)\overline{\varphi_j(y)}$. To generalize this notion to infinite particle systems, it is necessary to formulate the definition in terms of correlation functions as done in Definition 3.0.2. For correlation functions to be non-negative in (3.0.1) a natural assumption is to take $\mathbb{K}$ to be a Hermitian non-negative definite kernel. Then it turns out that $0 \leq \mathbb{K} \leq 1$ is a necessary condition for there to exist a determinantal process with kernel $\mathbb{K}$ (see (37) or (14)). And then, any such
process can be expressed as a mixture of determinantal processes with projection kernels (see \cite{30,31,14}). The rank of the projection is the number of points in the process. Observe that \( \rho_2(x, y) - \rho_1(x)\rho_1(y) = -|K(x, y)|^2 \) is negative. More general inequalities like this are a clear consequence of the determinantal form of the correlation functions.

**Remark 3.0.4.** Another justification for this definition is the huge number of instances of determinantal processes that have arisen so far in random matrix theory and combinatorics, many of them predating the definition. To name a few,

- Non-intersecting random walks on the line (Karlin and McGregor \cite{20,18}).
- Eigenvalues of a random unitary matrix chosen according to the Haar measure (Dyson \cite{9}). (There are many more random matrix examples).
- Subset of edges of a finite graph present in a uniformly chosen spanning tree (Burton and Pemantle \cite{4}).
- (An encoding of) Young diagrams sampled from the poissonised Plancherel measure of the symmetric group (Borodin, Okounkov and Olshanski \cite{3}).

The extensive survey of Soshnikov \cite{37} gives many more examples and details.

While determinantal processes on \( \mathbb{R} \) or \( \mathbb{Z} \) have been studied extensively because they arise in random matrix theory and combinatorics, in two dimensions they seem to have been largely untouched.

To get a determinantal point process is trivial. Just take a reproducing kernel Hilbert space \( H \) of functions on \( \Omega \), and let \( K \) be the reproducing kernel of \( H \). Then there exists a determinantal point process with kernel \( K \) (trivial when \( \dim(H) < \infty \), otherwise it can be constructed by taking limits of finite dimensional ones. See \cite{37} or \cite{14} for details). However to deserve our attention, the process must have some attractive features in addition to its determinantal nature. We adhere to two guiding principles.

- That the process be invariant in distribution (i.e., stationary) under a rich group of transformations.
- That the process arise in a natural way probabilistically.
Although imprecise, the first principle suggests that we consider the same three domains \( \mathbb{C}, \mathbb{S}^2, \mathbb{D} \). And since we would ultimately want to relate these to zeros of random analytic functions, we consider Hilbert spaces of analytic functions on these spaces. We show that the Hilbert spaces that give rise to stationary determinantal processes are precisely the well known Bargmann-Fock spaces of analytic functions. But before going into this, we should point out that these determinantal processes were already studied by Caillol under the name “One component plasma on the sphere” (the two-component version was studied by Forrester, Jancovici, and Madore, and Jancovici and Téllez from the point of view of constructing Coulomb gases on these spaces. We arrived at these processes prompted by a question of Bálint Virág as to what natural determinantal processes can be defined on the two dimensional sphere. As we remarked earlier, one can define a determinantal process (which can be regarded as Coulomb gas at a particular temperature \( \beta = 2 \)) by choosing one’s favourite Hilbert space with a kernel. What is new here is that we show that these are unique in a certain sense, and most importantly, we show in the next chapter how to get these determinantal processes as zeros of random analytic functions.

**Theorem 3.0.5.** Let \( \Omega \) be one of \( \mathbb{C}, \mathbb{S}^2, \mathbb{D} \) and let \( \mu \) be an arbitrary radially symmetric Radon measure on \( \Omega \). Let \( \varphi_k \) be complex analytic functions on \( \Omega \) and belong to \( L^2(\mu) \). Then the determinantal process with kernel

\[
K(z, w) = \sum_{k=1}^{N} \varphi_k(z) \bar{\varphi}_k(w)
\]

is invariant in distribution under the corresponding group of isometries if and only if it is one of the following.

- \( \Omega = \mathbb{C}, \, d\mu(z) = e^{-\alpha|z|^2} \, dm(z), \, N = \infty, \, \varphi_k(z) = \sqrt{\frac{\pi}{\alpha}} z^k, \) where \( \alpha \in (0, \infty) \), and the kernel is

\[
K^C_{\alpha}(z, w) = \frac{\alpha}{\pi} e^{\alpha z \bar{w}}.
\]

The Hilbert space is the space of analytic functions in \( L^2(\mathbb{C}, \frac{1}{\pi} e^{-\alpha|z|^2}) \). We call this process Det-$\mathbb{C}$-$\alpha$.

- \( \Omega = \mathbb{S}^2, \, d\mu(z) = \frac{dm(z)}{(1+|z|^2)^{\alpha+1}}, \, N = \alpha, \, \varphi_k(z) = \sqrt{\frac{\pi}{\alpha}} z^k, \) where \( \alpha \in \{1, 2, 3, \ldots\} \), and
the kernel is
\[ K^{S^2}_\alpha(z, w) = \frac{\alpha}{\pi} (1 + z\overline{w})^{\alpha-1}. \quad (3.0.4) \]

The Hilbert space is the space of analytic functions in \( L^2(S^2, \frac{1}{\pi(1+|z|^2)^{\alpha+1}}) \). We call this process Det-\( S^2-\alpha \).

- \( \Omega = \mathbb{D}, \ d\mu(z) = (1 - |z|^2)\frac{1}{\pi} dm(z), \ N = \infty, \ \varphi_k(z) = \sqrt{\frac{\pi}{\alpha}} z^k, \) where \( \alpha \in (0, \infty) \), and the kernel is
\[ K^{\mathbb{D}}_\alpha(z, w) = \frac{\alpha}{\pi} \frac{1}{(1 - z\overline{w})^{\alpha+1}}. \quad (3.0.5) \]

The Hilbert space is the space of analytic functions in \( L^2(\mathbb{D}, \frac{1}{\pi}(1 - |z|^2)^{\alpha-1}) \). We call this process Det-\( \mathbb{D}-\alpha \).

**Remark 3.0.6.** Note the similarity to the classification of Gaussian analytic function with stationary zeros in Proposition 1.2.7. Just as there, here too, Det-\( \mathbb{C}-\alpha \) processes are all identical up to scale, whereas the Det-\( S^2-\alpha \) and Det-\( \mathbb{D}-\alpha \) are genuine one parameter families of point processes.

There are (at least) two ways of using a positive definite kernel in probability theory. One is to use it as the covariance kernel of a Gaussian process and another is to use it as the kernel of a determinantal process (it has to be a projection kernel for the latter). We are not aware of any probabilistic connection between the two.

When we use the reproducing kernels of the Bargmann-Fock spaces as covariance kernels of Gaussian processes, we get the Gaussian analytic functions introduced in (1.2.9), (1.2.10) and (1.2.11). When we use them as kernels for determinantal processes we get stationary point processes in these domains.

**Proof of Theorem 3.0.5** Let
\[ \rho^2(z) = \begin{cases} \frac{1}{\pi} & \text{if } \Omega = \mathbb{C}, \\ \frac{1}{\pi(1+|z|^2)^2} & \text{if } \Omega = S^2, \\ \frac{1}{\pi(1-|z|^2)^2} & \text{if } \Omega = \mathbb{D}. \end{cases} \]

Then \( \rho^2(z) dm(z) \) is the unique invariant measure (up to multiplication by a constant) on \( \Omega \).

If \( \mathcal{X} \) is a determinantal process with kernel \( K \) on \( \Omega \), with distribution invariant under the corresponding isometry group, then the first intensity of the process, \( K(z, z) d\mu(z) \) must be equal to \( \alpha \rho^2(z) dm(z) \) for some \( \alpha > 0 \).
Express the correlation functions of $\mathcal{X}$ w.r.t the measure $\rho^2(z)dm(z)$ instead of the Lebesgue measure. Then the kernel becomes

$$\frac{\alpha K(z, w)}{\sqrt{K(z, z)} \sqrt{K(w, w)}}.$$ 

Invariance of the second correlation function implies that

$$\frac{\alpha \|K(\varphi(z), \varphi(w))\|^2}{K(\varphi(z), \varphi(z))K(\varphi(w), \varphi(w))} = \text{Const}(z, w), \quad (3.0.6)$$

for every isometry $\varphi$ of $\Omega$, where by $\text{Const}(z, w)$ we mean that it does not depend on $\varphi$.

The idea is this. We differentiate equation $\text{(3.0.6)}$ w.r.t $\varphi$ and equate to zero. The derivatives w.r.t $\varphi$ can be written as derivatives w.r.t $z, w$. That gives us differential equations for $K$ that are easy to solve.

Firstly, fix any $(z_0, w_0)$ such that $K(z_0, w_0) \neq 0$. Without loss of generality take $z_0 = 0 = w_0$. Then there is a neighbourhood $N$ of 0 in the Complex plane such that if $z, w \in N$, and $\varphi$ is close to identity, then $|K(z, w) - K(0, 0)| < \frac{1}{2}|K(0, 0)|$. Let $S$ be the disk of radius $\frac{1}{2}|K(0, 0)|$ centered at $K(0, 0)$. Then $S$ cannot intersect both the positive and negative parts of real axis because $S$ is convex and does not contain 0. Moreover $\overline{S}$ and $S$ intersect the real line at the same points. Therefore by removing the positive or the negative half line, we can define a continuous branch of logarithm on $S \cup \overline{S}$. Henceforth “log” will denote this function.

Taking logarithms in Equation $(3.0.6)$ we get

$$\log K(\varphi(z), \varphi(w)) + \log K(\varphi(w), \varphi(z)) - \log K(\varphi(z), \varphi(z)) - \log K(\varphi(w), \varphi(w)) \quad (3.0.7)$$

is equal to const$(z, w)$, not depending on $\varphi$. To differentiate w.r.t $\varphi$ we parameterize it with complex numbers as follows.

- **Complex plane** Write $\varphi(z) = \lambda z + \alpha$, where $\alpha \in \mathbb{C}$, $|\lambda| = 1$.

- **Sphere** Write $\varphi(z) = \frac{az + \beta}{-\beta z + \alpha^\prime}$, where $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$.

- **Disk** Write $\varphi(z) = \frac{az + \beta}{\beta z + \alpha}$, where $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 - |\beta|^2 = 1$.

Let us deal with the planar case first.
Complex plane Write \( \varphi_\alpha(z) = z + \alpha \), where \( \alpha \in C \). Then for \( \epsilon \) is small enough, if \( |\alpha| < \epsilon \) and \( z, w \in N \), then \( K(\varphi_\alpha(z), \varphi_\alpha(w)) \in S \). Apply \( \frac{\partial^2}{\partial z \partial w} \) to equation (3.0.7) and evaluate at \( \alpha = 0 \). We get

\[
0 = \frac{\partial^2}{\partial z \partial w} \log K(z, w) + \frac{\partial^2}{\partial w \partial \bar{z}} \log K(w, z) - \frac{\partial^2}{\partial z \partial \bar{z}} \log K(z, z) - \frac{\partial^2}{\partial w \partial \bar{w}} \log K(w, w)
\]

Where \( Q(z, w) = \frac{\partial^2}{\partial z \partial w} \log K(z, w) \). This is well defined for \( z, w \in N \) and is analytic in \( z, \bar{w} \). Applying \( \frac{\partial^2}{\partial z \partial w} \) to \( Q(z, w) + Q(w, z) - Q(z, z) - Q(w, w) \) we deduce that \( \frac{\partial^2}{\partial z \partial w} Q(z, w) = 0 \). By expanding \( Q \) locally as a power series in \( z, \bar{w} \), and observing the symmetry \( Q(z, w) = \overline{Q(w, z)} \), we conclude that \( Q(z, w) = h(z) + \overline{h}(w) \) for some \( h \) holomorphic on \( N \). Then

\[
\log K(z, w) = \overline{w} h_1(z) + z \overline{h_1}(w) + C, \quad \forall z, w \in N,
\]

where \( h'_1 = h \) and \( C \) is a constant.

Now again consider (3.0.7) and apply \( \frac{\partial}{\partial w} \) to it. We get

\[
0 = \frac{\partial}{\partial z} \log K(z, w) + \frac{\partial}{\partial w} \log K(w, z) - \frac{\partial}{\partial z} \log K(z, z) - \frac{\partial}{\partial w} \log K(w, w)
\]

\[
= \overline{w} h(z) + \overline{h_1}(w) + \overline{\bar{z}} h(w) + h_1(z) - \overline{\bar{z}} h(z) - \overline{\bar{w}} h(w) - \overline{\bar{h}_1}(w)
\]

\[
= (\overline{z} - \overline{w})(h(w) - h(z)).
\]

Since this has to hold for every \( z, w \in N \), we must have \( h \equiv 0 \). That means that \( h_1(z) = az + b \) for some \( a, b \). Therefore,

\[
\log K(z, w) = \overline{w} (az + b) + z(aw + b) + C
\]

\[
= (a + \overline{w})(z + \frac{b}{a + \overline{a}})(w + \frac{b}{a + \overline{a}}) + C'
\]

Making a change of variables we get \( K(z, w) = e^{az \overline{w}} \). Then \( \alpha \) has to be positive. This concludes the planar case.

Sphere In this case, we again differentiate Equation (3.0.6) w.r.t \( \alpha, \beta \) and their conjugates. However \( \varphi_{\alpha, \beta} \) depends on the parameters and their conjugates as well, and that makes the equation longer. A simplification is obtained by noting the following:

Let \( g(z, w) \) be analytic in \( z, \overline{w} \). Then with \( \varphi = \varphi_{\alpha, \beta} \),

\[
(\frac{\partial}{\partial \beta} + \overline{w} \frac{\partial}{\partial \alpha})(\frac{\partial}{\partial \beta} + z \frac{\partial}{\partial \alpha}) g(\varphi(z), \varphi(w)) = \frac{(1 + z \overline{w})^2}{(-\beta z + \overline{w})(-\beta \overline{w} + \alpha)} (\partial_1 \overline{\partial_2} g)(\varphi(z), \varphi(w)).
\]

(3.0.8)
Here \( \partial_1, \partial_2 \) denote the derivatives w.r.t the first and second arguments.

Apply \( \left( \frac{\partial}{\partial \beta} + \bar{w} \frac{\partial}{\partial \alpha} \right) \left( \frac{\partial}{\partial \beta} + z \frac{\partial}{\partial \alpha} \right) \) to Equation (3.0.7) and evaluate at \( \alpha = 1, \beta = 0 \). Then (3.0.8) yields,

\[
0 = Q(z, w) + Q(w, z) - Q(z, z) - Q(w, w),
\]

where, this time \( Q(z, w) = (1 + z\bar{w})^2 \frac{\partial^2}{\partial z \partial w} \log K(z, w) \). But all the considerations that applied to \( Q \) in the Planar case also apply here and we get \( K(z, w) = (1 + z\bar{w})^\alpha \) (perhaps after a change of variables). Now \( \alpha \) will have to be a positive integer (because integrating \( K(z, z) \) over the whole space should give \( N! \), where \( N \) is the total number of points in the process).

**Disk** Analogous to the spherical case, here we observe that for any \( g \) analytic in \( z \) and anti-analytic in \( w \),

\[
\left( \frac{\partial}{\partial \beta} - \bar{w} \frac{\partial}{\partial \alpha} \right) \left( \frac{\partial}{\partial \beta} - z \frac{\partial}{\partial \alpha} \right) g(\phi_{\alpha, \beta}(z), \phi_{\alpha, \beta}(w)) = \frac{(1 - z\bar{w})^2}{(\beta z + \alpha \bar{w})^2} \left( \partial_1 \partial_2 g \right)(\phi_{\alpha, \beta}(z), \phi_{\alpha, \beta}(w)).
\]  
(3.0.9)

Applying \( \left( \frac{\partial}{\partial \beta} - \bar{w} \frac{\partial}{\partial \alpha} \right) \left( \frac{\partial}{\partial \beta} - z \frac{\partial}{\partial \alpha} \right) \) to (3.0.7) and using equation (3.0.9) gives us

\[
0 = Q(z, w) + Q(w, z) - Q(z, z) - Q(w, w),
\]

where, this time \( Q(z, w) = (1 - z\bar{w})^2 \frac{\partial^2}{\partial z \partial w} \log K(z, w) \). As before this leads us to \( K(z, w) = (1 - z\bar{w})^\alpha \) and \( \alpha \) will have to be positive.

This completes the proof of the theorem. \( \square \)
Chapter 4

Random matrix-valued analytic functions

In Chapter 2, we saw that by choosing a Gaussian analytic function $f$ with a stationary zero set and a (non-random) homogeneous polynomial $Q$, we could construct a random analytic function with stationary zeros. There are two complementary questions that arise naturally.

1. Can one study these random analytic functions in this generality without having to appeal to special $Q$ and $f$?

2. Are there particular examples of $Q$ and $f$ that are somehow special?

The answer to both these questions is yes. Regarding the first question, we already saw in Chapter 2 that the correlation functions of the zero set can be computed in a general fashion. We shall use these computations in Chapter 7 to prove asymptotic normality for the zero sets in general. In the current chapter and the next two, we answer the second question and show that zeros of random analytic functions sometimes (but far from frequently, let alone always) turn out to be determinantal point processes.
4.1 Determinantal processes that are zeros of RAFs: Known results

We remarked earlier that the focus in random matrix theory has been on Hermitian random matrices. In the preface to his book “Random matrices”, Mehta (27) says “The theory of non-Hermitian random matrices, though not applicable to any physical problems, is a fascinating subject and must be studied for its own sake. In this direction an impressive step [has been taken by] Ginibre ...” Ginibre found the exact distribution of eigenvalues of three (two, strictly speaking) ensembles of non-Hermitian random matrices. We quote the one that is relevant to us.

**Theorem 4.1.1 (Ginibre(1965) (11) ).** Let $A$ be an $n \times n$ matrix with i.i.d. standard complex Gaussian entries. Then the eigenvalues of $A$ have density

$$
\rho_n(z_1, \ldots, z_n) = \frac{1}{\pi^n} \prod_{k=1}^{n-1} k! \ e^{-\sum_{k=1}^n |z_k|^2} \prod_{i<j} |z_i - z_j|^2.
$$

(4.1.1)

Equivalently, one may say that the eigenvalues of $A$ form a determinantal point process with kernel

$$
K_n(z, w) = \sum_{k=0}^{n-1} \frac{(zw)^k}{k!},
$$

(4.1.2)

w.r.t the reference measure $d\mu(z) = \frac{1}{\pi} e^{-|z|^2}$. The corresponding Hilbert space $H = \text{span}\{1, z, \ldots, z^{n-1}\} \subset L^2(\mathbb{C}, e^{-|z|^2} dm(z)).$

Despite the enthusiastic response, as shown by Mehta’s quote above, there do not seem to be any significant exact results beyond Ginibre’s. The following beautiful result of Peres and Virág (28) seems to be the next such.

**Theorem 4.1.2 (Peres and Virág(2003) (28) ).** Let $f$ be the random power series whose coefficients are i.i.d. standard complex Gaussians (this is the case $\Omega = \mathbb{D}, L = 1$ in (1.2.11). Then the zeros of $f$ form a determinantal point process on the unit disk $\mathbb{D}$ with the kernel (the Bergman kernel of the unit disk)

$$
K(z, w) = \frac{1}{\pi(1 - zw)^2},
$$


w.r.t the reference measure $d\mu(z) = \frac{1}{\pi} dm(z)$ on $\mathbb{D}$. The corresponding Hilbert space $H = \text{span}\{1, z, z^2, \ldots\} \subset L^2(\mathbb{D}, \frac{dm(z)}{\pi})$ is the space of all analytic functions in $L^2(\mathbb{D})$.

**Remark 4.1.3.** Observing that Theorem 4.1.2 identifies the distribution of zeros of the Gaussian analytic function (with $L = 1$) defined in (1.2.11) as being $\text{Det-}\mathbb{D} - 1$ (recall the definition of $\text{Det-}\mathbb{D} - 1$ from (3.0.5)), one is tempted to guess that the Gaussian analytic functions defined in (1.2.9), (1.2.10) and (1.2.11) might have zeros distributed like the $\text{Det-}\mathbb{C} - L$, $\text{Det-}\mathbb{S}^2 - L$ and $\text{Det-}\mathbb{D} - L$ (defined in (3.0.3), (3.0.4) and (3.0.5), respectively). However these canonical Gaussian analytic functions do not have determinantal zero sets. Indeed, it was observed by Peres and Virág in their paper that these zero sets do not have negative correlations at large distances and hence, cannot be determinantal). Therefore this beautiful might-have-been story is completely false! Nevertheless, to quote Einstein, “The Lord is subtle, but not malicious”. In the next section we shall see how a completely different but equally compelling picture might well be true.

### 4.2 Determinantal processes that are zeros of RAf:s: An analogy

First let us list all the Gaussian analytic functions whose zero sets we know to be determinantal. This includes Theorem 4.1.2 and two trivial cases (a one-point point process is always determinantal!).

- $z - a = 0$: One zero, with standard complex Gaussian distribution on $\mathbb{C}$. $H = \text{span}\{1\}$ in $L^2(\mathbb{C}, \frac{e^{-|z|^2}}{\pi} dm(z))$.

- $za - b = 0$: One zero, distributed uniformly on $\mathbb{S}^2$ upon stereographic projection from the plane. $H = \text{span}\{1\}$ in $L^2(\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}, \frac{1}{\pi(1+|z|^2)} dm(z))$.

- $a_0 + za_1 + z^2 a_2 + \cdots = 0$: Peres and Virág (28): Infinitely many zeroes in the disk. A determinantal point process with kernel $\frac{1}{\pi} (1 - |z|^2)^{-2}$ w.r.t Lebesgue measure on the unit disk. Equivalently, $H = \text{span}\{1, z, z^2, \ldots\}$ in $L^2(\mathbb{D}, \frac{1}{\pi} dm(z))$.

Note that Ginibre’s result (Theorem 4.1.1) can be seen as regarding the zeros of the random analytic function $zI - A$, which can be thought of as a matrix version of the first of the above examples. This suggests that we consider the matrix versions of the other two, i.e., we look at
- $\det(zA - B) = 0$, where $A, B$ are $n \times n$ independent matrices with i.i.d. standard complex Gaussian entries.

- $\det (A_0 + zA_1 + z^2A_2 + \ldots) = 0$, where $A_k$ are independent $n \times n$ matrices with each one having i.i.d. complex Gaussian entries.

The analogy strongly suggests that the solutions to these equations should give us the determinantal point processes corresponding to the Bargmann-Fock spaces on the sphere and the unit disk (but only for integer values of the parameter, since the size of the matrix, namely $n$, is discrete). Before going into the statements and proofs, we make some big-picture remarks and connect these objects to the random analytic functions studied in Chapter 2.

**Remark 4.2.1.** Note that here we are looking at the set of $z$ for which a random matrix-valued analytic function ($zA - B$ or $A_0 + zA_1 + z^2A_2 + \ldots$) becomes singular. This concept is an obvious generalization of both random matrices (which correspond to the case when the analytic function is linear) and Gaussian analytic functions (which correspond to the case when the matrices have size $1 \times 1$). In spite of this natural appeal, the concept of a random matrix-valued analytic function does not seem to have been considered in the literature.

One possible reason could be that the focus in random matrix theory has been almost entirely on eigenvalues in one dimension (real line or the circle) for physical reasons as well as the strong mathematical connections with orthogonal polynomials, representation theory etc. Moreover the eigenvalues have a physical meaning in quantum mechanics. Note the difficulty of forcing the zeros to lie on the real line, except by considering eigenvalues of a Hermitian matrix. Nevertheless, the idea of matrixifying seems to be useful, not only as suggested above with Gaussian matrix coefficients, but also polynomials with coefficients that are Haar-distributed unitary matrices.

**4.3 Matrix-valued GAFs and polygafs**

Now we want to point out the connection with homogeneous polynomials applied to i.i.d. copies of Gaussian analytic functions (polygafs, that is).
Consider $\det(zA - B)$. This is the same as applying the homogeneous polynomial $Q = $"det" in $n^2$ variables, to $n^2$ i.i.d. copies of the Gaussian analytic function $f(z) = az - b$ (which is the case $\Omega = S^2, L = 1$ in (1.2.10)).

Similarly $\det \left(A_0 + zA_1 + z^2A_2 + \ldots\right)$ is the homogeneous polynomial $Q =$"det" in $n^2$ variables, applied to $n^2$ i.i.d. copies of the Gaussian analytic function $f(z) = a_0 + a_1z + a_2z^2 + \ldots$ (which is the case $\Omega = D, L = 1$ in (1.2.11)).

In other words, we have already shown in Proposition 2.1.1 that the zero sets of these RAFs are stationary in $\Omega$. In the next two chapters we investigate the distributions in greater depth. We shall show that in the first case ($\Omega = S^2$) we do get determinantal processes, whereas in the second, we show partial results in this direction. The precise statements of the conjectures are as follows:

**Conjecture 4.3.1.** Let $A, B$ be i.i.d. $n \times n$ matrices with i.i.d. standard complex Gaussian entries. The zeros of $\det(zA - B)$ form a determinantal point process with kernel

$$K(z, w) = \frac{n}{\pi} \frac{(1 + zw)^{n-1}}{(1 + |z|^2)^{n+1}} \frac{(1 + |w|^2)^{n+1}}{2},$$

w.r.t the Lebesgue measure on $\mathbb{C}$. Equivalently, $K$ is the projection kernel on the subspace of analytic functions in $L^2(S^2, \frac{n}{\pi(1+|z|^2)^{n+1}}dm(z))$.

**Conjecture 4.3.2.** Let $A_k$ be i.i.d. $n \times n$ matrices with i.i.d. standard complex Gaussian entries. The zeros of $\det \left(A_0 + zA_1 + z^2A_2 + \ldots\right)$ form a determinantal point process on $D$ with kernel

$$K(z, w) = \frac{n}{\pi} \frac{(1 - |z|^2)^{n+1}}{(1 - zw)^{n+1}} \frac{(1 - |w|^2)^{n+1}}{2},$$

w.r.t the Lebesgue measure on $D$. Equivalently, $K$ is the projection kernel on the subspace of analytic functions in $L^2(D, \frac{n}{\pi(1-|z|^2)^{n+1}}dm(z))$.

**Remark 4.3.3.** The Det-$\mathbb{C}$ - 1 process are obtained from Ginibre’s theorem 4.1.1 by letting $n \to \infty$. So from the point of view of determinantal processes, our problems can be stated as finding a probabilistic meaning to Det-$S^2 - \alpha$ and Det-$D \rightarrow \alpha$. 
Chapter 5

Matrix analytic functions on the sphere

In this section we prove Conjecture 4.3.1 stated at the end of Chapter 4, i.e., we show that the processes Det-$S^2$-$\alpha$ arise as the singular points of the matrix GAF $zA - B$ or equivalently, zeros of the polygaf $\det (zA - B)$. Recall that for $\Omega = S^2$, $\alpha$ is a positive integer (the number of points in the process). We shall denote it by $n$ in this section.

**Theorem 5.0.4.** Let $A, B$ be independent $n \times n$ random matrices with i.i.d. standard complex Gaussian entries. Then the set $X$ of zeros of $\det (zA - B) = 0$, i.e., the eigenvalues of $A^{-1}B$, has the distribution Det-$S^2$-$n$.

We need the following lemma.

**Lemma 5.0.5.** Let $X$ be a point process on $\mathbb{C}$ with $n$ points almost surely. Assume that the $n$-point correlation function (equivalently the density) of $X$ has the form

$$p(z_1, \ldots, z_n) = |\Delta(z_1, \ldots, z_n)|^2 V(|z_1|^2, \ldots, |z_n|^2).$$

Here $\Delta(z_1, \ldots, z_n)$ denotes the Vandermonde factor $\prod_{i<j} (z_j - z_i)$.

Suppose also that $X$ has a distribution invariant under automorphisms of the sphere $S^2$, i.e., under the transformations $\varphi_{\alpha, \beta}(z) = \frac{\alpha z + \beta}{-\beta z + \alpha}$ for any $\alpha, \beta$ satisfying $|\alpha|^2 + |\beta|^2 = 1$. Then

$$V(|z_1|^2, \ldots, |z_n|^2) = \text{Const} \prod_{k=1}^n \frac{1}{(1 + |z_k|^2)^{n+1}}.$$  \hspace{1cm} (5.0.1)
Proof of Lemma 5.0.5. The claim is that the probability density of the \( n \) points of \( X \) (in exchangeable random order) is

\[
q(z_1, \ldots, z_n) := \text{Const.} \left| \Delta(z_1, \ldots, z_n) \right|^2 \prod_{k=1}^{n} \frac{1}{(1 + |z_k|^2)^{n+1}}
\]

First let us check that the density \( q \) is invariant under the isometries of \( S^2 \). For this let \( \alpha, \beta(z) = \alpha z + \beta \), with \( \alpha, \beta \) satisfying \( |\alpha|^2 + |\beta|^2 = 1 \). Then,

\[
\varphi'(z) = \frac{1}{(-\beta z + \alpha)^2}.
\]

\[
1 + |\varphi(z)|^2 = \frac{1 + |z|^2}{|\beta z + \alpha|^2}.
\]

\[
\varphi(z) - \varphi(w) = \frac{z - w}{(-\beta z + \alpha)(-\beta w + \alpha)}.
\]

From (5.0.2), (5.0.3) and (5.0.4), it follows that

\[
q(\varphi(z_1), \ldots, \varphi(z_n)) \prod_{k=1}^{n} |\varphi'(z_k)|^2 = q(z_1, \ldots, z_n),
\]

which shows the invariance of \( q \).

Invariance of \( X \) means that \( \forall \alpha, \beta \) with \( |\alpha|^2 + |\beta|^2 = 1 \), and for every \( z_1, \ldots, z_n \), we have

\[
p(\varphi(z_1), \ldots, \varphi(z_n)) \prod_{k=1}^{n} |\varphi'(z_k)|^2 = p(z_1, \ldots, z_n).
\]

Set \( W(z_1, \ldots, z_n) = \frac{p(z_1, \ldots, z_n)}{q(z_1, \ldots, z_n)} \). Then, from (5.0.6) and (5.0.5), we get

\- \( W(z_1, \ldots, z_n) \) is a function of \( |z_k|^2 \), \( 1 \leq k \leq n \), only.

\- \( W(\varphi(z_1), \ldots, \varphi(z_n)) = W(z_1, \ldots, z_n) \) for every \( z_1, \ldots, z_n \).

We claim that these two statements imply that \( W \) is a constant. To see this fix \( z_k = r_k e^{i\theta_k} \), \( 1 \leq k \leq n \), such that \( r_1 < r_k \) for \( k \geq 2 \). Let \( \alpha = \frac{1}{\sqrt{1 + r_1^2}}, \beta = -\frac{z_1}{\sqrt{1 + r_1^2}} \). Then \( |\alpha|^2 + |\beta|^2 = 1 \) and so \( \varphi_{\alpha, \beta} \) is an isometry of \( S^2 \). From the above stated properties of \( W \), we deduce,

\[
W(z_1, \ldots, z_n) = W(\varphi(z_1), \ldots, \varphi(z_n))
\]

\[
= W\left(0, \frac{r_2 e^{i\theta_2} - z_1}{1 + r_2 e^{i\theta_2} z_1}, \ldots, \frac{r_n e^{i\theta_n} - z_1}{1 + r_n e^{i\theta_n} z_1}\right).
\]
Take $z_1 = 1$ and $1 < r_k < 1 + \epsilon$. Then as $\theta_k$, $2 \leq k \leq n$ vary independently over $[0, 2\pi]$, the quantities $|\frac{r_k e^{i\theta_k} - z_1}{1 + r_k e^{i\theta_k} z_1}|$ vary over the intervals $[\frac{r_k - 1}{r_k + 1}, \frac{r_k + 1}{r_k - 1}]$. By our choice of $r_k$s, this means that

$$W(0, t_2, \ldots, t_n) = \text{Constant} \quad \forall t_k \in \left[\epsilon, \frac{1}{\epsilon}\right].$$

$\epsilon$ is arbitrary, hence $W(0, t_2, \ldots, t_n)$ is constant. Therefore $W(z_1, \ldots, z_n)$ is constant.

This shows that $p(z_1, \ldots, z_n) = \text{Const} \cdot q(z_1, \ldots, z_n)$.

Proof of Theorem 5.0.4. Firstly we observe that the distribution of $X$ is invariant under conformal automorphisms of $S^2$. This is a direct consequence of Proposition 2.1.1, with $Q = \"\det\"$ and $f(z) = az - b$. Still we give another simple direct proof. Let $\alpha, \beta$ be such that $|\alpha|^2 + |\beta|^2 = 1$. Set

$$C = \alpha A + \beta B \quad \text{and} \quad D = -\beta A + \alpha B.$$

Then $C$ and $D$ are i.i.d. matrices with i.i.d. standard complex Gaussian entries. Therefore the eigenvalue set of $C^{-1}D$ is also distributed as $X$.

Now $X$ is the set of solutions to the equation

$$\det (zA - B) = 0,$$

say $X = \{z_1, \ldots, z_n\}$. By our observation $X$ also has the same distribution as the set of solutions to

$$0 = \det (zC - D) = \det ((z\alpha + \beta)A - (\beta z + \alpha)B),$$

which is precisely $\{\varphi_{\alpha, \beta}(z_k)\}$. This proves the invariance.

Now by Lemma 5.0.5 it suffices to show that the density of points in $X$ is of the form given in (5.0.1).

We use the following well known matrix decomposition. **Schur decomposition**: Any diagonalizable matrix $M \in GL(n, \mathbb{C})$ can be written as

$$M = U(Z + T)U^*, \quad (5.0.7)$$

where $U$ is unitary, $T$ is strictly upper triangular and $Z$ is diagonal. Moreover the decomposition is almost unique, in the following sense:
$M = V(W + S)V^*$ in addition to (5.0.7), with $V, S, W$ being respectively unitary, strictly upper triangular, and diagonal, if and only if the element of $W$ are a permutation of the elements of $Z$, and if this permutation is identity, then $V = U\Theta$ and $\Theta S\Theta^* = T$ for some $\Theta$ that is both diagonal and unitary, i.e., $\Theta = \text{Diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})$.

Corresponding to this matrix decomposition (5.0.7), Ginibre (11) proved the measure decomposition

**Ginibre’s measure decomposition:** If $M$ is decomposed as in (5.0.7), with the elements of $Z$ in a uniformly randomly chosen order, then

$$\prod_{i,j} dm(M_{ij}) = \left( \prod_{i<j} |z_i - z_j|^2 \prod_k dm(z_k) \right) \left( \prod_{i<j} dm(T_{ij}) \right) dv(U) \quad (5.0.8)$$

where $\nu$ is a finite measure on the unitary group $U(n)$ such that $d\nu(U\Theta) = d\nu(U)$ for every diagonal unitary $\Theta$.

Conditional on $A$, the matrix $M := A^{-1}B$ has the density

$$e^{-\text{tr}(M^*A^*AM)} |A|^{2n}$$

w.r.t. the Lebesgue measure on $GL(n, \mathbb{C}) \subset \mathbb{C}^{n^2}$. From the measure decomposition (5.0.8) we get the density of $Z, T, U, A$ to be

$$\left( \prod_{i<j} |z_i - z_j|^2 \prod_k dm(z_k) \right) e^{-\text{tr}(A^*A(I+MM^*))} |A|^{2n} dv(U) \prod_{i<j} dm(T_{ij}) \prod_{i,j} dm(A_{ij}).$$

(We have omitted constants entirely) Thus the density of $Z$ is obtained by integrating over $T, U, A$. Now write $Z = \Theta R$ where $\Theta$ and $R$ are diagonal matrices with the polar and radial parts of $z_k$, respectively. Then

$$MM^* = U\Theta(R + \Theta^*T)(R + \Theta^*T)^*\Theta^*U^*.$$

As stated earlier, $d\nu(U\Theta) = d\nu(U)$. The elements of $\Theta^*T$ are the same as elements of $T$, but multiplied by complex numbers of absolute value 1. Hence, $\Theta^*T$ has the same “distribution” as $T$. Thus replacing $U$ by $\Theta^*U$ and $T$ by $\Theta^*T$ we see that the density of $Z$ is of the form $\prod_{i<j} |z_i - z_j|^2 V(R)$. This is the form of the density required to apply Lemma 5.0.5. Thus we conclude that the eigenvalue density is

$$\text{Const.} \prod_{i<j} |z_i - z_j|^2 \prod_k \frac{1}{(1 + |z_k|^2)^{n+1}}. \quad (5.0.9)$$
To compute the constant, note that

\[
\left\{ \sqrt{\frac{n}{\pi}} \binom{n-1}{k} \frac{z^k}{(1 + |z|^2)^{\frac{n+1}{2}}} \right\}_{0 \leq k \leq n-1}
\]

is an orthonormal set. Projection on the Hilbert space generated by these functions gives a determinantal process whose kernel is as given in the definition of \( \text{Det} - S^2 - n \). Writing out the density shows that this is the same as the eigenvalue density that we have determined. Hence the constants must match. \( \square \)
Chapter 6

Matrix analytic functions on the disk

Conjecture 4.3.2 at the end of Chapter 4 asserted that the singular points of the matrix GAF $A_0 + zA_1 + z^2 A_2 + \ldots$ or equivalently, the zeros of the polygaf $\det(A_0 + zA_1 + z^2 A_2 + \ldots)$ are distributed as $\text{Det} - \mathbb{D} - n$. In other words they form a determinantal point process with kernel

$$K_n(z, w) = \frac{n}{\pi} \frac{1}{(1 - zw)^{n+1}}$$

w.r.t the reference measure $d\mu(z) = (1 - |z|^2)^{\frac{1}{2}(n-1)} dm(z)$ on $\mathbb{D}$. Here $A_k$ are i.i.d. $n \times n$ matrices with i.i.d. standard complex Gaussian entries.

As already emphasized, Proposition 2.1.1 applies to show that the zeros of the polygaf $A_0 + zA_1 + z^2 A_2 + \ldots$ are stationary on the unit disk. In this chapter we shall prove that the first and second correlation functions agree with those of the determinantal process with kernel $K_n$.

**First intensity:** In the notation of Chapter 2 we have $Q =$determinant, a homogeneous polynomial in $n^2$ variables $\zeta_{ij}, i, j \leq n$ and $f_{ij}(z)$ i.i.d. copies of the power series $\sum_{n=0}^{\infty} a_n z^n$.

From (2.3.2), the intensity of zeros is $\frac{n}{4\pi} \Delta \log(1 - |z|^2)^{-1}$. By an elementary computation this comes out to be $\frac{n}{\pi(1 - |z|^2)^{n+1}}$. Since this is the same as $K_n(z, z)$, it follows that the polygaf under consideration has the same intensity of zeros as the determinantal process with kernel $K$ (we omit $n$ in the subscript often).
2-point correlations: We shall prove that
\[ \rho_2(z, w) - \rho_1(z)\rho_1(w) = -|K(z, w)|^2, \quad (6.0.1) \]
which shows that the 2-point correlations for the zeros of the polygaf agree with those of the determinantal process with kernel \( K \). (Henceforth correlations are expressed w.r.t. the Lebesgue measure).

We prove \( 6.0.1 \) by using Theorem 5.0.4 and a change of variables!

Let \( g_{ij}(z) = a_{ij}z + b_{ij} \), where \( a_{ij}, b_{ij} \) are all i.i.d. \( \mathbb{C}N(0, 1) \). Then \( K(z, w) = 1 + z\overline{w} \). Consider the polygaf \( G(z) = \text{det}(g_{ij}(z)) \). This is precisely the polygaf considered in Chapter 5. Thus we know from Theorem 5.0.4 (which we proved by certain matrix decompositions, not at all by using the formulas for correlation functions of polygafs) that the zeros of \( G \) are determinantal with kernel \[
\frac{n}{\pi} \frac{(1 + z\overline{w})^{n-1}}{(1 + |z|^2)^{n+1/2}(1 + |w|^2)^{n+1/2}}.
\]

But the the general formula \( 2.3.3 \) for two-point correlations of zeros of polygafs applies to \( G \) also and hence it must be the case that
\[
\frac{16}{(2\pi)^2} \frac{\partial}{\partial z} \frac{\partial}{\partial w} \sum_{p=0}^\infty |\tilde{C}_p|^2 \frac{(1 + z\overline{w})^p(1 + w\overline{z})^p}{(1 + z\overline{z})^p(1 + w\overline{w})^p} = -\frac{n^2}{\pi^2} \frac{(1 + z\overline{w})^{n-1}(1 + w\overline{z})^{n-1}}{(1 + |z|^2)^{n+1/2}(1 + |w|^2)^{n+1/2}}, \quad (6.0.2)
\]
where \( \tilde{C}_p \) depend only on \( Q \), not on the GAFs that we feed in.

From \( 6.0.2 \) we get
\[
\frac{16}{(2\pi)^2} \frac{\partial}{\partial x_1} \frac{\partial}{\partial y_1} \frac{\partial}{\partial x_2} \frac{\partial}{\partial y_2} \sum_{p=0}^\infty |\tilde{C}_p|^2 \frac{(1 + x_1 y_2)^p(1 + x_2 y_1)^p}{(1 + x_1 y_1)^p(1 + x_2 y_2)^p} = -\frac{n^2}{\pi^2} \frac{(1 + x_1 y_2)^{n-1}(1 + x_2 y_1)^{n-1}}{(1 + x_1 x_2)^{n+1/2}(1 + y_1 y_2)^{n+1/2}}, \quad (6.0.3)
\]
This is because, both sides of \( 6.0.3 \) are analytic in \( x_1, x_2 \) and anti-analytic in \( y_1, y_2 \) and moreover, \( 6.0.2 \) says that the two are equal on the diagonal \( \{x_1 = y_1, x_2 = y_2\} \). Thus, by a standard (and elementary) fact that can be found in any introductory book on several variable complex analysis, see for example Rudin [29], the two sides must be equal for all \( x_1, x_2, y_1, y_2 \).

Now make the substitution \( x_1 = iz, x_2 = iw, y_1 = i\overline{w}, y_2 = i\overline{z} \) to get
\[
\frac{16}{(2\pi)^2} \frac{\partial}{\partial z} \frac{\partial}{\partial w} \sum_{p=0}^\infty |\tilde{C}_p|^2 \frac{(1 - z\overline{z})^p(1 - w\overline{w})^p}{(1 - z\overline{w})^p(1 - w\overline{z})^p} = -\frac{n^2}{\pi^2} \frac{(1 - z\overline{z})^{n-1}(1 - w\overline{w})^{n-1}}{(1 - z\overline{w})^{n+1/2}(1 - w\overline{z})^{n+1/2}}, \quad (6.0.4)
\]
But again using (2.3.3) the left hand side is precisely what we get for $\rho_2(z, w) - \rho_1(z)\rho_1(w)$, when $Q$ is the determinant of $n^2$ variables and $f_{ij}$ are i.i.d. copies of $f(z) = \sum_{n=0}^{\infty} a_n z^n$, with $a_n$ being i.i.d. standard complex Gaussians. And the right side of (6.0.4) is $K(z, w)K(w, z)$.

We already know that $\rho_1(z) = K(z, z)$. Therefore, the two point correlation $\rho_2(z, w)$ is

$$\det \begin{bmatrix} K(z, z) & K(z, w) \\ K(w, z) & K(w, w) \end{bmatrix}.$$
Chapter 7

Asymptotic Normality

7.1 Background: Results for Gaussian analytic functions

Sodin and Tsirelson \(^{(35)}\) proved asymptotic normality for smooth \((C^2_c)\) statistics applied to the zeros of the three canonical models of Gaussian analytic functions in \([1.2.9]\), \([1.2.10]\) and \([1.2.11]\), as the density parameter \(L \to \infty\). More precisely, they showed that for any real valued \(\varphi \in C^2_c(\Omega)\), if

\[
Z_L(\varphi) = \sum_{z \in \Omega^{-1}(0)} \varphi(z),
\]

then,

\[
\frac{Z_L(\varphi) - \mathbb{E}[Z_L(\varphi)]}{\sqrt{\text{Var}(Z_L(\varphi))}} \to N(0,1),
\]

and also that

\[
\text{Var}(Z_L(\varphi)) = \frac{\kappa}{L} \|\Delta^* \varphi\|_{L^2(m^*)}^2 + o(L^{-1}),
\]

for a constant \(\kappa\) that is described explicitly and the same for all the three geometries. (Note that as \(L \to \infty\), the variance goes to zero!) Here \(m^*\) is the invariant measure on \(\Omega\) and \(\Delta^*\) is the invariant Laplacian. In other words

\[
dm^*(z) = \begin{cases} 
  \frac{dm(z)}{1+|z|^2} & \text{if } \Omega = \mathbb{C} \\
  \frac{dm(z)}{(1-|z|^2)^2} & \text{if } \Omega = S^2 \\
  \frac{dm(z)}{(1-|z|^2)^2} & \text{if } \Omega = \mathbb{D}
\end{cases}
\]

(7.1.1)
and

\[ \Delta^* = \begin{cases} \Delta & \text{if } \Omega = \mathbb{C}, \\ (1 + |z|^2)\Delta & \text{if } \Omega = S^2, \\ (1 - |z|^2)\Delta & \text{if } \Omega = D. \end{cases} \]  

(7.1.2)

### 7.2 Our results: for polygafs

In this article we modify the method of Sodin and Tsirelson to obtain central limit theorems for smooth statistics of zeros of polygafs. One point of this exercise is to demonstrate that the random analytic functions can be studied at the level of generality introduced in Chapter 2. (Recall that polygafs include matrix GAFs as very special cases.)

**Theorem 7.2.1.** Let \( F_L(z) = Q(f_1^{(L)}(z), \ldots, f_p^{(L)}(z)) \) where,

- \( Q \) is a fixed non-random homogeneous polynomial in \( p \) complex variables,
- \( f_i^{(L)} \) are i.i.d. GAFs in (1.2.9) or (1.2.10) or (1.2.11),
- \( \varphi : \Omega \to \mathbb{R} \) is a \( C^2_{\text{c}} \) function on \( \Omega \).

Set

\[ Z_L(\varphi) = \sum_{z \in F_L^{-1}\{0\}} \varphi(z). \]

Then

\[ \frac{Z_L(\varphi) - \mathbf{E}[Z_L(\varphi)]}{\sqrt{\text{Var}(Z_L(\varphi))}} \to N(0, 1). \]

Moreover

\[ \text{Var}(Z_L(\varphi)) = \frac{\kappa(Q)}{L} \|\Delta^* \varphi\|_{L^2(m^*)}^2 + o(L^{-1}), \]

(7.2.1)

for a constant \( \kappa(Q) \) that is described explicitly and is the same for all the three geometries.

Let \( \hat{Z}_L(\varphi) = Z_L(\varphi) - \mathbf{E}[Z_L(\varphi)] \). Let \( \chi \) denote a standard (real) normal random variable. The idea of the proof is to show that

\[ \mathbf{E}\left[ \hat{Z}_L(\varphi)^s \right] = \mathbf{E}\left[ \hat{Z}_L(\varphi)^2 \right]^\frac{s}{2} \mathbf{E}[\chi^s] + \mathbf{E}\left[ \hat{Z}_L(\varphi)^2 \right]^\frac{s}{2} o(1) \quad \text{as } L \to \infty, \]

(7.2.2)
for \( s = 1, 2, 3, \ldots \). Then the moments of \( \frac{Z_L(\varphi) - \mathbb{E}[Z_L(\varphi)]}{\sqrt{\text{Var}[Z_L(\varphi)]}} \) converge to those of \( \chi \) and convergence in distribution follows. To show (7.2.2), we need to compute the moments of \( \hat{Z}_L(\varphi) \).

Recall the formula (2.2.2)

\[
\int_\Omega \varphi(z)dn_F(z) = \int_\Omega \Delta \varphi(z) \frac{1}{2\pi} \log |F(z)|dm(z).
\]

(7.2.3)

From this we can also write

\[
\hat{Z}_L(\varphi) = \int_\Omega \Delta \varphi(z) \frac{1}{2\pi} \log |\hat{F}(z)|dm(z),
\]

where \( \hat{F}(z) = \frac{1}{K(z,z)^{d/2}}F(z) \) as defined in Section 2.3. Then one can write the moments of \( \hat{Z}_L(\varphi) \) as

\[
\mathbb{E} \left[ \hat{Z}_L(\varphi)^s \right] = (2\pi)^{-s} \int_{\Omega^s} \left( \prod_{j=1}^s \Delta z_j \varphi(z_j) \right) \mathbb{E} \left[ \prod_{j=1}^s \log |\hat{F}_L(z_j)| \right] \prod_{j=1}^s dm(z_j)
\]

(7.2.4)

for \( s = 1, 2, \ldots \).

### 7.2.1 Central moments of \( Z_L(\varphi) \)

From the homogeneity of \( Q \), we can write \( \hat{F}_L(z) = Q(g_1^{(L)}(z), \ldots, g_p^{(L)}(z)) \), where \( g_i(z) = \frac{f_i(z)}{\sqrt{K(z,z)}} \). These \( g_i^{(L)} \) are no longer analytic functions, but they are independent complex Gaussian processes on \( \Omega \) with constant variance 1. Thus for any fixed \( z \), we have that \( g_i^{(L)}(z) \), \( 1 \leq i \leq p \) are i.i.d. standard complex Gaussians, and from (2.2.4) it follows that

\[
\log |\hat{F}_L(z)| = \sum_{m,n \in \mathbb{Z}_+} \frac{C_{m,n}}{\sqrt{m!n!}} \prod_{k=1}^p :g_k^{m_k}(z)g_k^{n_k}(z) :.
\]

with coefficients \( C_{m,n} \) that are the same as in (2.2.4) and depend only on \( Q \) but not on \( z \) or even the GAF \( f \). We get

\[
\mathbb{E} \left[ \prod_{j=1}^s \log |\hat{F}_L(z_j)| \right] = \sum_{\{m_j,n_j\}_{1 \leq j \leq s}} \left( \prod_{j=1}^s \frac{C_{m_j,n_j}}{\sqrt{m_j!n_j!}} \right) \prod_{k=1}^p \mathbb{E} \left[ \prod_{j=1}^s :g_k^{m_{kj}}(z_j)g_k^{n_{kj}}(z) : \right].
\]

Here \( m_j = (m_{1j}, \ldots, m_{pj}) \) and likewise for \( n_j \).
Each of the expectations on the right hand side of the above equation can be “evaluated” by the Feynman diagram formula \((1.2.5)\). We denote by \(\hat{K}_L(z, w)\), the quantity \(E[g(z)g(w)] = \frac{K(z, w)}{\sqrt{K(z, z)K(w, w)}}\). Also we write \(v(\gamma; z_1, \ldots, z_s)\) for the value of a Feynman diagram \(\gamma\), with edge weights given by the covariance matrix of the the Gaussian vector \((g(z_1), \ldots, g(z_s))\). Then, we get

\[
E \left[ \prod_{j=1}^{s} \log |\hat{F}_L(z_j)| \right] = \sum_{\{m_j, n_j\}_{1 \leq j \leq s}} \left( \prod_{j=1}^{s} \frac{C_{m_j, n_j}}{\sqrt{m_j!n_j!}} \right) \sum_{\gamma_1, \ldots, \gamma_p} \prod_{k=1}^{p} v(\gamma_k; z_1, \ldots, z_s) \\
= \sum_{\gamma_1, \ldots, \gamma_p} \left( \prod_{j=1}^{s} \frac{C_{m_j, n_j}}{\sqrt{m_j!n_j!}} \right) \prod_{k=1}^{p} v(\gamma_k; z_1, \ldots, z_s) \\
= \sum_{\gamma_1, \ldots, \gamma_p} \left( \prod_{j=1}^{s} \frac{C_{m_j, n_j}}{\sqrt{m_j!n_j!}} \right) v(\cup_{k=1}^{p} \gamma_k; z_1, \ldots, z_s).
\]

Here in the last sum, \(\gamma_k, 1 \leq k \leq p\) vary over all possible Feynman diagrams on labels \(\{1, \bar{1}, \ldots, s, \bar{s}\}\) and \(m_j, n_j \in \mathbb{Z}_+^p\) are such that the number of vertices labeled \(j\) in \(\gamma_k\) is \(m_{kj}\) and the number of vertices labeled \(\bar{j}\) in \(\gamma_k\) is \(n_{kj}\).

Put this together with \((7.2.4)\) to deduce that \(E[\hat{Z}_L(\varphi)^s]\) is equal to

\[
(2\pi)^{-s} \sum_{\gamma_1, \ldots, \gamma_p} \left( \prod_{j=1}^{s} \frac{C_{m_j, n_j}}{\sqrt{m_j!n_j!}} \right) \int_{\Omega^s} \left( \prod_{j=1}^{s} \Delta_{z_j} \varphi(z_j) \right) v(\cup_{k=1}^{p} \gamma_k; z_1, \ldots, z_s) \prod_{j=1}^{s} dm(z_j).
\]

(7.2.5)

Here again, the sum is over all legal diagrams \(\gamma_k\).

Our goal is to prove \((7.2.2)\). To that end, we now consider the second moment, which is obtained by setting \(s = 2\). We get

\[
E \left[ \prod_{j=1}^{2} \log |\hat{F}_L(z_j)| \right] = \sum_{\gamma_1, \ldots, \gamma_p} \left( \prod_{j=1}^{2} \frac{C_{m_j, n_j}}{\sqrt{m_j!n_j!}} \right) v(\cup_{k=1}^{p} \gamma_k; z_1, z_2),
\]

(7.2.6)

where, each \(\gamma_k\) is now a Feynman diagram on vertices labeled \(\{1, \bar{1}, 2, \bar{2}\}\).

Now suppose \(s\) is even. Write \((7.2.6)\) with \(z_1, z_2\) replaced by \(z_{2k-1}, z_{2k}\) for \(k \leq \frac{s}{2}\) and multiply them together. On the right hand side, we get (the product of the values of Feynman diagrams is the value of the union of the Feynman diagrams)

\[
\sum_{\gamma_1, \ldots, \gamma_p} \left( \prod_{j=1}^{s} \frac{C_{m_j, n_j}}{\sqrt{m_j!n_j!}} \right) v(\cup_{k=1}^{p} \gamma_k; z_1, \ldots, z_s),
\]
where the sum is over all Feynman diagrams $\gamma_k$ on vertices labeled by $\{1, \bar{1}, \ldots, s, \bar{s}\}$, such that, when all the vertices labeled by $j, \bar{j}$ are identified for each $j$, then $\bigcup_{k=1}^p \gamma_k$ has connected components $\{1, 2\}, \{3, 4\}, \ldots, \{s-1, s\}$.

To get $E \left[ Z_L(\varphi)^2 \right]^{1/2}$, integrate against $\frac{1}{(2\pi)^2} \prod_{j=1}^s \Delta_{z_j} \varphi(z_j)$ w.r.t Lebesgue measure over $\Omega^s$. This yields that $E \left[ Z_L(\varphi)^2 \right]^{1/2}$ is equal to

\[
(2\pi)^{-s} \sum_{\gamma_1, \ldots, \gamma_p} \left( \prod_{j=1}^s \frac{C_{m_j, n_j}}{\sqrt{m_j!n_j!}} \right) \int_{\Omega^s} \left( \prod_{j=1}^s \Delta_{z_j} \varphi(z_j) \right) v(\bigcup_{k=1}^p \gamma_k; z_1, \ldots, z_s) \prod_{j=1}^s dm(z_j). \tag{7.2.7}
\]

Here again the sum is over all diagrams that have connected components $\{1, 2\}, \{3, 4\}, \ldots, \{s-1, s\}$ (upon merging vertices labeled $j, \bar{j}$).

Instead of pairing $\{1, 2, \ldots, s\}$ as $\{1, 2\}, \ldots, \{s-1, s\}$, we could use any other matching. Write the expression analogous to (7.2.7) for each matching, and add them all up. Recall that the number of matchings of $\{1, 2, \ldots, s\}$ is $E[\chi^s]$. Thus we deduce that $E \left[ Z_L(\varphi)^2 \right]^{1/2} E[\chi^s]$ is equal to

\[
(2\pi)^{-s} \sum_{\gamma_1, \ldots, \gamma_p} \left( \prod_{j=1}^s \frac{C_{m_j, n_j}}{\sqrt{m_j!n_j!}} \right) \int_{\Omega^s} \left( \prod_{j=1}^s \Delta_{z_j} \varphi(z_j) \right) v(\bigcup_{k=1}^p \gamma_k; z_1, \ldots, z_s) \prod_{j=1}^s dm(z_j), \tag{7.2.8}
\]

where the sum is over all diagrams $\gamma_k$, $1 \leq k \leq p$ such that $\bigcup \gamma_k$ “splits”, i.e., when vertices $j, \bar{j}$ are merged, we get $s/2$ component each of size 2.

Compare (7.2.8) with (7.2.5) (the expressions are incomplete without the commentaries that follows after the equations!). The terms on the right hand side of (7.2.5) that are absent in (7.2.8) are precisely those, for which $\bigcup_{k=1}^p \gamma_k$ does not split into $s/2$ components. The proof of (7.2.2) will be complete once we show that these terms together contribute a negligible amount compared to $E \left[ Z_L(\varphi)^2 \right]^{1/2}$.

### 7.2.2 Estimating the second moment

This is the case $s=2$, which we already dealt with in detail, in Chapter 2. Particularly, from (2.3.3) we can write

\[
E \left[ \hat{Z}_L(\varphi)^2 \right] = \frac{1}{(2\pi)^2} \sum_{p=0}^\infty |\tilde{C}_p|^2 \int_{\Omega^2} \Delta \varphi(z) \Delta \varphi(w) |\hat{K}_L(z, w)|^{2p} dm(z) dm(w), \tag{7.2.9}
\]
where $|\tilde{C}_p|^2 = \sum_{m_*=n_*=p} |C_{m,n}|^2$. It is convenient to express everything in terms of the invariant quantities of $\Omega$. From (7.1.1) and (7.1.2) we rewrite (7.2.9) as

$$E\left[\hat{Z}_L(\varphi)^2\right] = \sum_{p=0}^{\infty} \frac{|\tilde{C}_p|^2}{(2\pi)^2} \int_{\Omega^2} \Delta^* \varphi(z) \Delta^* \varphi(w) |\hat{K}_L(z, w)|^{2p} dm^*(z) dm^*(w).$$

Now split $\Delta^* \varphi(z) \Delta^* \varphi(w)$ as $(\Delta^* \varphi(z))^2 + (\Delta^* \varphi(w))^2 - (\Delta^* \varphi(z) - \Delta^* \varphi(w))^2$. We get a sum of three integrals. The first two integrals are equal by symmetry. We shall argue that the last integral is negligible. Firstly note that

$$|\hat{K}_L(z, w)|^2 = \begin{cases} 
  e^{-L|z-w|^2} & \text{if } \Omega = \mathbb{C} \\
  \frac{|1+|z|^2|^L (1+|w|^2)^L}{(1-|z|^2)^L (1-|w|^2)^L} & \text{if } \Omega = S^2 \\
  \frac{|1-|z|^2|^L (1-|w|^2)^L}{(1+|z|^2)^L (1+|w|^2)^L} & \text{if } \Omega = \mathbb{D}. 
\end{cases} \quad (7.2.10)$$

Thus $|\hat{K}_L(z, w)|^2$ is 1 when $z = w$ and decays rapidly as $(z, w)$ moves away from the diagonal. Since $(\Delta^* \varphi(z) - \Delta^* \varphi(w))^2$ vanishes on the diagonal, it is easy to calculate that

$$\int_{\Omega^2} (\Delta^* \varphi(z) - \Delta^* \varphi(w))^2 |\hat{K}_L(z, w)|^{2p} dm^*(z) dm^*(w) = o \left( \frac{1}{L^p} \right). \quad (7.2.11)$$

The first two integrals give us

$$\int_{\Omega^2} (\Delta^* \varphi(z))^2 |\hat{K}_L(z, w)|^{2p} dm^*(z) dm^*(w). \quad (7.2.12)$$

Since $\hat{K}_L$ is also invariant under isometries of $\Omega$, fixing $z$ and integrating w.r.t $w$ we get

$$\left( \int_{\mathcal{M}} |\hat{K}_L(0, w)|^{2p} dm^*(w) \right) \left( \int_{\Omega} (\Delta^* \varphi(z))^2 dm^*(z) \right).$$

Now from (7.2.10) it can be checked by direct computation that

$$\int_{\mathcal{M}} |\hat{K}_L(0, w)|^{2p} dm^*(w) = \begin{cases} 
  \frac{0}{L^p} & \text{if } \Omega = \mathbb{C} \\
  \frac{\pi}{L^p + 1} & \text{if } \Omega = S^2 \\
  \frac{\pi}{L^p - 1} & \text{if } \Omega = \mathbb{D}. 
\end{cases} \quad (7.2.13)$$

From (7.2.11), (7.2.12) and (7.2.13), we get

$$E\left[\hat{Z}_L(\varphi)^2\right] = \left( \sum_{p=0}^{\infty} \frac{|\tilde{C}_p|^2}{4\pi L^p} \right) \|\Delta^* \varphi\|_{L^2(m^*)}^2 + o \left( \frac{1}{L} \right). \quad (7.2.14)$$
This shows (7.2.1) with
\[ \kappa(Q) = \sum_{p=0}^{\infty} \frac{|\tilde{C}_p|^2}{4\pi p}. \]

### 7.2.3 Estimating the values of non-split diagrams

We want to show that the contribution of non-split diagrams to \( E[\hat{Z}_L(\phi)^s] \) is negligible. Note that this includes all the diagrams in the case when \( s \) is odd.

Consider any \( p \)-tuple of diagrams \((\gamma_1, \ldots, \gamma_p)\) on labels \( \{1, \overline{1}, \ldots, s, \overline{s}\} \) such that \( \bigcup_k \gamma_k \) does not split into pairs when \( j, \overline{j} \) are merged. We bound
\[
\int_{\Omega^s} \left( \prod_{j=1}^{s} \Delta_{z_j}(z_j) \right) \nu(\bigcup_{k=1}^{p} \gamma_k; z_1, \ldots, z_s) \prod_{j=1}^{s} dm(z_j),
\]
in absolute value by the obvious
\[
\|\Delta \varphi\|_\infty \int_{A^s} |\nu(\bigcup_{k=1}^{p} \gamma_k; z_1, \ldots, z_s)| \prod_{j=1}^{s} dm(z_j),
\]
where \( A \) is the support of \( \varphi \). Then split the integral of \( |\nu(\bigcup_{k=1}^{p} \gamma_k; z_1, \ldots, z_s)| \) as a product of integrals over the connected components of \( \bigcup_k \gamma_k \).

Without loss of generality, let \( \{1, 2, \ldots, r\} \) be a connected component of \( \bigcup_k \gamma_k \) (when \( j, \overline{j} \) are merged). Since the weights of edges are bounded by 1, deleting some edges will only increase the integral that we want to bound. We delete enough edges to get a spanning tree \( T \) on \( \{1, 2, \ldots, r\} \). So we are left with an integral of the form
\[
\int_{A^r} \prod_{(i,j) \in T} |\tilde{K}_L(z_i, z_j)| dm(z_1) \ldots dm(z_r).
\]
Integrate inwards starting with the leaves. From (7.2.13) we get that
\[
\int_{A^r} \prod_{(i,j) \in T} |\tilde{K}_L(z_i, z_j)| dm(z_1) \ldots dm(z_r) < CL^{-r}.
\]
Multiplying the contribution from each component, we get
\[
\int_{A^s} \nu(\bigcup_{k=1}^{p} \gamma_k; z_1, \ldots, z_s) \prod_{j=1}^{s} dm(z_j) < CL^{-s+n. \text{of components of } \bigcup \gamma_k}.
\]
Non-split diagrams are precisely those that have less than \( \frac{s}{2} \) components, whence the right hand side is \( O(L^{-\frac{s-1}{2}}) \).
Now we want to bound the total contribution of all unsplit diagrams. This can be done in the following manner.

Approximate \( \log |F(z)| \) by polynomials (by truncating the Wick expansion). For integration of \( \Delta \phi \) against a polynomial, our cruder bound on a single Feynman diagram suffices, since there are only finitely many terms and each of them goes to zero. So we get asymptotic normality for \( \Delta \phi \) integrated against polynomials. From this, one can deduce asymptotic normality for \( \mathcal{E}_L(\varphi) \). We skip the details (see [35]).

To summarize, we have argued that

\[
\mathbb{E} \left[ \hat{\mathcal{E}}_L(\varphi)^s \right] - \mathbb{E} \left[ \hat{\mathcal{E}}_L(\varphi)^2 \right]^{\frac{s}{2}} \mathbb{E} [\chi^s] = o(L^{-s/2})
\]

and proved that \( \mathbb{E} \left[ \hat{\mathcal{E}}_L(\varphi)^2 \right]^{\frac{s}{2}} \) is of order \( L^{-s/2} \). Thus (7.2.2) follows.
Chapter 8

Overcrowding Problems

8.1 Statements of the problems

In this chapter we go back to the canonical models of Gaussian analytic functions defined in Chapter 1. Namely, consider the following Gaussian analytic functions (GAFs):

- **Planar GAF**: The function defined in (1.2.9),
  \[ g(z) = \sum_{n=0}^{\infty} \frac{a_n z^n}{\sqrt{n!}} \]
  where \( a_n \) are i.i.d. standard complex Gaussian random variables.

- **Hyperbolic GAFs**: For each \( L > 0 \) the function defined in (1.2.11)
  \[ f_L(z) = \sum_{n=0}^{\infty} \left( \frac{-L}{n} \right)^{1/2} a_n z^n \]
  where as before \( a_n \) are i.i.d. standard complex Gaussians. Almost surely, \( f_L \) is an analytic function in the unit disk (and no more).

We denote the zero set by \( \mathcal{Z} \). Let \( n(r) \) denote the number of points of \( \mathcal{Z} \) in the disk of radius \( r \) around 0 (The GAF will be clear from the context). By the invariance of the zero sets, the results carry over to disks centered elsewhere.

Yuval Peres asked the following question and conjectured that the probability decays as \( e^{-cm^2 \log(m)} \) in the planar case (personal communication).
Figure 8.1: Samples of the zero set of \( g \). Left: The zero process sampled under certain sufficient conditions (see the conditions in the lower bound of the proof of Theorem 8.1.1). Take \( \alpha = 0.5, r = 2, m = 16 \) on the coefficients forcing 16 zeros in the disk of radius 2. Right: The unconditioned zero process.

**Question:** Fix \( r > 0, (r < 1 \text{ in the Hyperbolic case}) \). Estimate \( P [n(r) > m] \) as \( m \to \infty \).

One motivation for such a question is in Figure 8.1. There one can see the distribution of the zero set under certain conditions on the coefficients that force large number of zeros in the disk of radius 2 (this is not the zero set conditioned to have overcrowding - that seems harder to simulate). The picture suggests that the distribution of the conditioned process may be worth studying on its own. A large deviation estimate of the kind we derive will presumably be a necessary step in such investigations.

The answer is different in the two settings. We prove-

**Theorem 8.1.1.** Consider the planar GAF \( g \). For any \( \epsilon > 0 \), \( \exists \) a constant \( C_2 \) (depending on \( \epsilon, r \)) such that for every \( m \geq 1 \),

\[
e^{-\frac{1}{2}m^2 \log(m) + O(m^2)} \leq P [n(r) \geq m] \leq C_2 e^{-(\frac{1}{2}-\epsilon)m^2 \log(m)}.
\]

In particular, \( P [n(r) \geq m] = e^{-\frac{1}{2}m^2 \log(m)(1+o(1))} \).

**Theorem 8.1.2.** Fix \( L > 0 \) and consider the GAF \( f_L \). For any fixed \( r < 1 \), there are
constants $\beta, C_1, C_2$ (depending on $L$ and $r$) such that for every $m \geq 1$,

$$C_1(r)e^{-\frac{m^2}{\log(r)}} \leq P[n(r) \geq m] \leq C_2(r)e^{-\beta(r)m^2}.$$ 

We prove Theorem 8.1.1 in Section 8.2, Theorem 8.1.2 in Section 8.3.

### 8.2 Overcrowding - The planar case

In this section we prove Theorem 8.1.1. Before that we explain why one expects the constant $\frac{1}{2}$ in the exponent in Theorem 8.1.1 by analogy with the Ginibre ensemble.

#### 8.2.1 Ginibre ensemble

The Ginibre ensemble is the determinantal point process (earlier we denoted this by Det-$C-1$) in the plane with kernel

$$K(z, w) = \frac{1}{\pi}e^{-\frac{1}{2}|z|^2-\frac{1}{2}|w|^2+zw}.$$ (8.2.1)

This process is of interest because it is the limit in distribution, as $n \to \infty$, of the point process of eigenvalues of an $n \times n$ matrix with i.i.d. standard complex Gaussian entries (Theorem 4.1.1).

The Ginibre ensemble has many similarities to the zero set of $g$. In particular, the Ginibre ensemble is invariant in distribution under Euclidean motions, has constant intensity $\frac{1}{\pi}$ in the plane and has the same negative correlations as $Z_g$ at short distances. Therefore there are other similarities too, for instance, see (7). There are also differences between the two point processes. For instance, the Ginibre ensemble has all correlations negative, whereas for the zero set of $g$, long-range two-point correlations are positive. However, in our problem, since we are considering a fixed disk and looking at the event of having an excess of zeros in it, it seems reasonable to expect the same behaviour for both these point processes, since it is the short range interaction that is relevant. In case of the Ginibre ensemble, the overcrowding problem is easy to solve.

**Theorem 8.2.1.** Let $n_G(r)$ be the number of points of the Ginibre ensemble in the disk of radius $r$ around 0 (by translation invariance, the same is true for any disk
of radius $r$). Then for a fixed $r > 0$,

$$P[n_G(r) \geq m] = e^{-\frac{1}{2}m^2 \log(m)(1+o(1))}.$$  

Proof. By Kostlan (21), the set of absolute values of the points of the Giniibre ensemble has the same distribution as the set \{\(R_1, R_2, \ldots\), where \(R_n\) are independent, and \(R_n^2\) has Gamma\((n, 1)\) distribution for every \(n\). Hence \(R_n^2 \overset{d}{=} \xi_1 + \ldots + \xi_n\), where \(\xi_k\) are i.i.d. Exponential random variables with mean 1, and it follows that

$$P[R_n^2 < r^2] \geq \prod_{k=1}^{n} P[\xi_k < \frac{r^2}{n}] \geq \left(\frac{r^2}{2n}\right)^n,$$

as long as \(n \geq r^2\), because \(P[\xi_1 < x] \geq \frac{x}{2}\) for \(x < 1\). Therefore we get

$$P[n_G(r) \geq m] \geq \prod_{n=1}^{m} P[R_n^2 < r^2] \geq \prod_{n=1}^{m} \left(\frac{r^2}{2n}\right)^n \geq \left(\frac{r^2}{2}\right)^{\frac{m(m+1)}{2}} e^{-\sum_{n=1}^{m} n \log(n)}.$$  

Here and elsewhere we shall encounter the term \(\sum_{n=1}^{m} n \log(n)\). We compute its asymptotics now.

$$n \log(n) \leq x \log(x) \leq (n+1) \log(n+1) \quad \text{for } n \leq x \leq n+1$$

Integrate from 1 to \(m+1\) and note that

$$\int_1^{a} x \log(x) dx = \frac{1}{2} a^2 \log(a) - \frac{a^2}{4} + \frac{1}{4},$$

to get

$$\sum_{n=1}^{m} n \log(n) \leq \frac{1}{2}(m+1)^2 \log(m+1) - \frac{(m+1)^2}{4} + \frac{1}{4} \leq \sum_{n=1}^{m+1} n \log(n).$$  

(8.2.5)

Thus (8.2.4) gives

$$P[n_G(r) \geq m] \geq e^{-\frac{1}{2} \frac{(m+1)^2}{2} \log(m+1) + \frac{(m+1)^2}{4} - \frac{1}{4} + \frac{m(m+1)}{2} \log(r^2/2)} = e^{-\frac{1}{2} m^2 \log(m) + O(m^2)}.$$
To prove the inequality in the other direction, note that

\[
P[n_G(r) \geq m] \leq \sum_{n=1}^{m^2} 1(R_n^2 < r^2) \geq m + \sum_{n=m^2+1}^{\infty} P[R_n^2 < r^2]
\]

\[
\leq \left(\frac{m^2}{m}\right) \prod_{n=1}^{m} P[R_n^2 < r^2] + \sum_{n>m^2} e^{-n \log(n)(1+o(1))}.
\]

In the second line, for the first summand we used the fact that \(R_n^2\) are stochastically increasing and for the second term we used the well known fact \(P[R_n^2 < r^2] = P[\text{Pois}(r^2) \geq n]\) and then the usual bound on the tail of a Poisson random variable, namely \(P[\text{Poisson}(\theta) \geq a] \leq e^{-a \log(a/\theta) + a - \theta}\).

Using the same idea to bound \(P[R_n^2 < r^2]\) in the first summand, we obtain

\[
P[n_G(r) \geq m] \leq \left(\frac{m^2}{m}\right) \prod_{n=1}^{m} e^{-n \log(n/r^2) - r^2 + n} + e^{-m^2 \log(m^2)(1+o(1))}
\]

\[
\leq \left(\frac{m^2}{m}\right) e^{\frac{m(m+1)}{2}(1+\log(r^2)) - mr^2 - \sum_{n=m}^{m^2} n \log(n)} + e^{-m^2 \log(m^2)(1+o(1))}
\]

\[
= e^{-\frac{1}{2}m^2 \log(m)(1+o(1))} \quad \text{(using (8.2.5) again)}.
\]

In the last line we used \(\left(\frac{m^2}{m}\right) < m^{2m}\). This completes the proof. \(\square\)

### 8.2.2 Proof of Theorem 8.1.1

Our method of proof is largely based on that of Sodin and Tsirelson \[36\]. (They estimate the “hole probability”, \(P[n(r) = 0]\) as \(r \to \infty\).)

**Proof of Theorem 8.1.1** Lower Bound  Suppose the \(m\)th term dominates the sum of all the other terms on \(\partial D(0; r)\), i.e., suppose

\[
|a_m z^m| > \left| \sum_{n \neq m} a_n z^n \right| \quad \text{whenever } |z| = r. \tag{8.2.6}
\]

Then, by Rouche’s theorem \(g(z) = \frac{a_M z^m}{\sqrt{m!}}\) and \(\frac{a_m z^m}{\sqrt{m!}}\) have the same number of zeros in \(D(0; r)\). Hence \(n(r) = m\). Now we want to find a lower bound for the probability of the event in (8.2.6). Note that the left side of (8.2.6) is identically equal to \(\frac{|a_m|^m}{\sqrt{m!}}\).

Now suppose the following happen-

1. \(|a_n| \leq n \forall n \geq m + 1\).
2. $|a_m| \geq (\alpha + 1)m$ where $\alpha$ will be chosen shortly.

3. $|a_n| \frac{x^n}{\sqrt{n!}} < \frac{x^m}{\sqrt{m!}}$ for every $0 \leq n \leq m - 1$.

Then the right hand side of (8.2.6) is bounded by

\[
\text{RHS of (8.2.6)} \leq \sum_{n=0}^{m-1} |a_n| \frac{r^n}{\sqrt{n!}} + \sum_{n=m+1}^{\infty} |a_n| \frac{r^n}{\sqrt{n!}} \\
\leq \sum_{n=0}^{m-1} \frac{r^m}{\sqrt{m!}} + \sum_{n=m+1}^{\infty} n r^n \frac{m!}{\sqrt{n!}} \\
\leq m \frac{r^m}{\sqrt{m!}} + C \frac{m!}{\sqrt{n!}} \\
= (C + 1)m \frac{r^m}{\sqrt{m!}} \\
\leq |a_m| \frac{r^m}{\sqrt{m!}}
\]

if $\alpha = C$. Thus if the above three events occur with $\alpha = C$, then the $m^{th}$ term dominates the sum of all the other terms on $\partial D(0; r)$. Also these events have probabilities as follows.

1. $P[|a_n| \leq n \forall n \geq m + 1] \geq 1 - \sum_{n=m+1}^{\infty} e^{-n^2} \geq 1 - C'e^{-m^2}$.

2. $P[|a_m| \geq (C + 1)m] = e^{-(C+1)^2m^2}$.

3. The third event has probability as follows. Recall again that $P[\xi < x \geq \frac{x}{e}]$ if $x < 1$ and $\xi$ has exponential with mean 1. We apply this below with $x = \left(\frac{r^m - n \sqrt{m}}{\sqrt{m!}}\right)^2$. This is clearly less than 1 if $n \geq r^2$. Therefore if $m$ is sufficiently large it is easy to see that for all $0 \leq n \leq m - 1$, the same is valid. Thus

\[
P \left[ a_n \leq \frac{r^{m-n} \sqrt{n!}}{\sqrt{m!}} \forall n \leq m - 1 \right] = \prod_{n=0}^{m-1} P \left[ a_n \leq \frac{r^{m-n} \sqrt{n!}}{\sqrt{m!}} \right] \\
\geq \prod_{n=0}^{m-1} \frac{2^{-2m-2n} n!}{2m!} \\
= \frac{r^{m(m+1)} e^{ \frac{m^2}{2} \log(m) } 2^{-m} e^{-m^2 \log(m)} + O(m^2)}{r^{m(m+1)} e^{ \frac{m^2}{2} \log(m) } 2^{-m} e^{-m^2 \log(m)} + O(m^2)} \\
= e^{-\frac{1}{2} m^2 \log(m) + O(m^2)}.
\]

Since these three events are independent, we get the lower bound in the theorem.
Upper Bound  By Jensen’s formula, for any $R > r$ we have
\[
n(r) \log \left( \frac{R}{r} \right) \leq \int_r^R \frac{n(u)}{u} \, du = \int_0^{2\pi} \log |g(Re^{i\theta})| \frac{d\theta}{2\pi} - \int_0^{2\pi} \log |g(re^{i\theta})| \frac{d\theta}{2\pi}. \tag{8.2.7}
\]

Let $R = R_m = \sqrt{m}$. Sodin and Tsirelson (36) show that
\[
\mathbb{P} \left[ \log M(t) \geq \left( \frac{1}{2} + \epsilon \right) t^2 \right] \leq e^{-e^{t^2}} \tag{8.2.8}
\]
where $M(t) = \max \{|g(z)| : |z| \leq t\}$.

Now suppose $n(r) \geq m$ and $\log M(R_m) \leq \left( \frac{1}{2} + \epsilon \right) m$ for some $\epsilon > 0$. Then by (8.2.7) we have
\[
- \int_0^{2\pi} \log |g(re^{i\theta})| \frac{d\theta}{2\pi} \geq m \log \left( \frac{\sqrt{m}}{r} \right) - \left( \frac{1}{2} + \epsilon \right) m
= \frac{1}{2} m \log(m) - m \log(r) - \left( \frac{1}{2} + \epsilon \right) m
= \frac{1}{2} m \log(m) - O(m).
\]

Thus $\mathbb{P}[n(r) \geq m]$ is bounded by
\[
\mathbb{P} \left[ \log M(R_m) \geq \left( \frac{1}{2} + \epsilon \right) m \right] + \mathbb{P} \left[ - \int_0^{2\pi} \log |g(re^{i\theta})| \frac{d\theta}{2\pi} \geq \frac{1}{2} m \log(m) - O(m) \right]
\leq e^{-e^{m}} + \mathbb{P} \left[ - \int_0^{2\pi} \log |g(re^{i\theta})| \frac{d\theta}{2\pi} \geq \frac{1}{2} m \log(m)(1 + o(1)) \right] \quad \text{by 8.2.7}
\]

From Lemma 8.2.2, we deduce that for any $\delta > 0$, there is a constant $C_2$ such that
\[
\mathbb{P} \left[ - \int_0^{2\pi} \log |g(re^{i\theta})| \frac{d\theta}{2\pi} \geq \frac{1}{2} m \log(m)(1 + o(1)) \right] \leq C_2 e^{-\left(2 - \delta\right)(\frac{m}{2} \log(m))^2 / \log(\frac{m}{2} \log(m))}
\leq C_2 e^{-\left(\frac{1}{2} - \frac{\delta}{4}\right)m^2 \log(m)(1 + o(1))}.
\]

From this, the upper bound follows. □

Lemma 8.2.2.  For any given $\delta > 0$, there is a constant $C_2$ such that for every $m$,
\[
\mathbb{P} \left[ - \int_0^{2\pi} \log |g(re^{i\theta})| \frac{d\theta}{2\pi} \geq m \right] \leq C_2 e^{-\left(\frac{(2 - \delta)m^2}{2 \log(m)}\right)}.
\]
Proof. Let $P$ be the Poisson kernel on $D(0; r)$. Fix $\epsilon > 0$ and let $A_\epsilon = \sup\{P(re^{i\theta}, w) : |w| = \epsilon, \theta \in [0, 2\pi]\}$ and $B_\epsilon = \inf\{P(re^{i\theta}, w) : |w| = \epsilon, \theta \in [0, 2\pi]\}$. Since $\log|g|$ is a subharmonic function, for any $w$ with $|w| = \epsilon$, we get

$$\log |g(w)| \leq \int_0^{2\pi} \log |g(re^{i\theta})|P(re^{i\theta}, w)\frac{d\theta}{2\pi}$$

$$\leq A_\epsilon \int_0^{2\pi} \log_+ |g(re^{i\theta})|\frac{d\theta}{2\pi} + B_\epsilon \left(\int_0^{2\pi} \log |g(re^{i\theta})|\frac{d\theta}{2\pi} - \int_0^{2\pi} \log_+ |g(re^{i\theta})|\frac{d\theta}{2\pi}\right)$$

$$\leq B_\epsilon \int_0^{2\pi} \log |g(re^{i\theta})|\frac{d\theta}{2\pi} + A_\epsilon \log_+ M(r).$$

This implies $\log M(\epsilon) \leq B_\epsilon \int_0^{2\pi} \log |g(re^{i\theta})|\frac{d\theta}{2\pi} + A_\epsilon \log_+ M(r)$.

Therefore if $\int_0^{2\pi} \log |g(re^{i\theta})|\frac{d\theta}{2\pi} \leq -m$, then one of the following must happen:

Either $\{\log M(\epsilon) \leq -B_\epsilon m + \sqrt{m}\}$ or $\{A_\epsilon \log_+ M(r) > \sqrt{m}\}$.

Using Lemma 8.2.8, since $M(r) < M \left(Cm^{\frac{1}{4}}\right)$ for any $C$, we see that

$$P \left[\log_+ M(r) > \sqrt{m} A_\epsilon \right] \leq e^{-e^{Cm}}$$

for some constant $C$ depending on $\epsilon$. Hence

$$P \left[\int_0^{2\pi} \log |g(re^{i\theta})|\frac{d\theta}{2\pi} \leq -m\right] \leq e^{-e^{Cm}} + P \left[\log M(\epsilon) \leq -B_\epsilon m + \sqrt{m}\right]$$

$$\leq e^{-e^{Cm}} + e^{-2B_\epsilon^2 \frac{m^2}{\log(m)}(1+o(1))}$$

where in the last line we have used Lemma 8.2.3.

As $\epsilon \to 0$, $B_\epsilon \to 1$ and hence the proof is complete.

Now we prove the upper bound on the maximum modulus in a disk of radius $r$ that was used in the last part of the proof of Lemma 8.2.2. For possible future use we prove a lower bound too.
Lemma 8.2.3. Fix \( r > 0 \). There are constants \( \alpha, C_1, C_2 \) such that
\[
C_1 e^{-\frac{m^2}{\log(m)}} \leq \mathbb{P}[\log M(r) \leq -m] \leq C_2 e^{-\frac{2m^2}{\log(m)}(1+o(1))}.
\]

**Proof. Lower bound** By Cauchy-Schwarz, \( M(r) \leq \left( \sum_{n=0}^{k-1} |a_n|^2 \right)^{1/2} e^{r^2/2} + \sum_{n=k}^{\infty} |a_n|^n \sqrt{n!} \).
We shall choose \( k \) later. We will bound from below the probability that each of these summands is less than \( e^{-m} \).

Let \( \varphi_k \) denote the density of \( \Gamma(k,1) \).
\[
\mathbb{P} \left[ \left( \sum_{n=0}^{k-1} |a_n|^2 \right)^{1/2} e^{r^2/2} \leq \frac{e^{-m}}{2} \right] = \mathbb{P} \left[ \sum_{n=0}^{k-1} |a_n|^2 \leq \frac{e^{-2m} e^{r^2}}{4} \right] \\
\geq \varphi_k \left( \frac{e^{-2m} e^{r^2}}{8} \right) \frac{e^{-2m} e^{r^2}}{8} \\
= e^{-2mk-k \log(k)+O(k)}
\]

Also if \( |a_n| \leq n^2 \ \forall n \geq k \), then the second summand
\[
\sum_{n=k}^{\infty} |a_n|^n \sqrt{n!} \leq C r^{k+1} \leq C e^{-k \log(k)/3}
\]

Also the event \( \{ |a_n| \leq n^2 \ \forall n \geq k \} \) has probability at least \( 1 - \sum_{n=k+1}^{\infty} e^{-n^4} \geq 1 - C e^{-k^4} \).

Thus if we set \( k = \frac{\gamma m}{\log(m)} \) for a sufficiently large \( \gamma \), then both the terms are less than \( e^{-m^2} \) with probability at least \( e^{-2\gamma m^2/\log(m)} \).

**Upper bound** By Cauchy’s theorem,
\[
a_n = \frac{\sqrt{n!}}{2\pi i} \int_{C_r} \frac{\mathbb{g}(\zeta)}{\zeta^n+1} d\zeta,
\]
where \( C_r \) is the curve \( C_r(t) = re^{it}, 0 \leq t \leq 2\pi \). Therefore,
\[
|a_n| \leq \frac{M(r)\sqrt{n!}}{r^n}.
\]

Thus we get
\[
\mathbb{P}[M(r) \leq e^{-m}] \leq \prod_{n=0}^{\infty} \mathbb{P} \left[ |a_n| \leq \frac{e^{-m} \sqrt{n!}}{r^n} \right].
\]

\( |a_n|^2 \) are i.i.d. exponential random variables with mean 1. Therefore,
\[
\mathbb{P} \left[ |a_n| \leq \frac{e^{-m} \sqrt{n!}}{r^n} \right] \leq \frac{e^{-2m n!}}{r^{2n}}.
\]
Using this bound for $n \leq k := \frac{3m}{\log(m)}$, we get

$$P[M(r) \leq e^{-m}] \leq \prod_{n=0}^{k} \frac{e^{-2mn!}}{r^{2n}} \leq Ce^{-2nk + \frac{\beta^2}{2}\log(k) + O(k^2)} \leq Ce^{\left(-\beta + \frac{\beta^2}{2}\right) \frac{m^2}{\log(m)} + O\left(\frac{m^2}{\log(m)}\right)}. $$

$-2\beta + \frac{\beta^2}{2}$ is minimized when $\beta = 2$ and we get,

$$P[M(r) \leq e^{-m}] \leq e^{-2\frac{m^2}{\log(m)}(1+o(1)).} \quad (8.2.9)$$

8.3 Overcrowding - The hyperbolic case

8.3.1 The determinantal case

We give a quick proof of Theorem [8.1.2] in the special case $L = 1$, as it is much easier and moreover we get matching upper and lower bounds. The proof is similar to the case of the Ginibre ensemble dealt with in Theorem [8.2.1] and is based on the fact that the set of absolute values of the zeros of $f_1$ is distributed the same as a certain set of independent random variables. The reason for this similarity between the two cases owes to the fact that both of them are determinantal. The zero set of $f_1$ is a determinantal process with the Bergman kernel for the unit disk, namely

$$K_B(z, w) = \frac{1}{\pi (1 - zw)^2},$$

as discovered by Peres and Virág [28].

Proof of Theorem [8.1.2] for $L = 1$. By the result of Peres and Virág quoted in Theorem [4.1.2], the set of absolute values of the zeros of $f_1$ has the same distribution as the set $\{U_n^{1/2n}\}$ where $U_n$ are i.i.d. uniform$[0, 1]$ random variables. Therefore,

$$P[n(r) \geq m] \geq \prod_{n=1}^{m} P[U_n^{1/2n} < r] = \prod_{n=1}^{m} r^{2n} = r^{m(m+1)}.$$
To prove the inequality in the other direction, note that

\[
P[n(r) \geq m] \leq \sum_{n=1}^{m^2} P[U_{1/2n}^1 < r] \geq m \]  
\leq \sum_{n=m^2+1}^{\infty} P[U_{1/2n}^1 < r] + \sum_{n>m^2} r^{2n} 
= \left( \frac{m^2}{m} \right) \prod_{n=1}^{m^2} P[U_{1/2n}^1 < r] + \sum_{n>m^2} r^{2n} 
= \left( \frac{m^2}{m} \right) r^{m(m+1)} + \frac{r^{2m+2}}{1-r^2} 
= r^{m(m+1)} \left( 1 + O \left( e^{m \log(m)} \right) \right).
\]

This completes the proof of the theorem for \( L = 1 \).

8.3.2 All values of \( L \)

**Remark 8.3.1.** Overall, the idea of proof is the same as in that of Theorem 8.1.1. However we do not get matching upper and lower bounds in the present case, the reason being that in the hyperbolic analogue of Lemma 8.2.3, the leading term in the exponent of the upper bound does depend on \( r \), unlike in the planar case. (An examination of the proof of Theorem 8.1.1 reveals that we get a matching upper bound only because replacing \( r \) by \( \epsilon \) does not affect the leading term in the exponent in the upper bound in Lemma 8.2.3). However we still expect that the lower bound in Theorem 8.1.2 is tight. (See remark after the proof).

**Proof of Theorem 8.1.2** Lower Bound As before we find a lower bound for the probability that the \( m \)th term dominates the rest. Note that if \( |z| = r \),

\[
|f_L(z) - \left( \frac{-L}{m} \right)^{1/2} a_m z^m| \leq \sum_{n=0}^{m-1} |a_n| \left( \frac{-L}{n} \right)^{1/2} r^n + \sum_{n=m+1}^{\infty} |a_n| \left( \frac{-L}{n} \right)^{1/2} r^n \tag{8.3.1}
\]

Now suppose the following happen-

1. \(|a_n| \leq \sqrt{n} \ \forall n \geq m + 1.\)
2. \(|a_m| \geq (\alpha + 1) \sqrt{m} \) where \( \alpha \) will be chosen shortly.
3. \(|a_n| \left( \frac{-L}{n} \right)^{1/2} r^n < \frac{1}{\sqrt{m}} \left( \frac{-L}{m} \right)^{1/2} r^m \) for every \( 0 \leq n \leq m - 1.\)
Then the right hand side of (8.3.1) is bounded by

\[
\text{RHS of (8.3.1)} \leq \sum_{n=0}^{m-1} |a_n| \left( -\frac{L}{n} \right)^{1/2} r^n + \sum_{n=m+1}^{\infty} |a_n| \left( -\frac{L}{n} \right)^{1/2} r^n
\]

\[
\leq \sum_{n=0}^{m-1} \frac{1}{\sqrt{m}} \left( -\frac{L}{m} \right)^{1/2} r^n + \sum_{n=m+1}^{\infty} \sqrt{m} \left( -\frac{L}{n} \right)^{1/2} r^n
\]

\[
\leq \sqrt{m} \left( -\frac{L}{m} \right)^{1/2} r^m + C \sqrt{m} \left( -\frac{L}{m} \right)^{1/2} r^m = (C + 1) \sqrt{m} \left( -\frac{L}{m} \right)^{1/2} r^m
\]

\[
\leq |a_m| \left( -\frac{L}{m} \right)^{1/2} r^m
\]

if \( \alpha = C \). Thus if the above three events occur with \( \alpha = C \), then the \( m \)th term dominates the sum of all the other terms on \( \partial D(0; r) \). Also these events have probabilities as follows.

1. \( \mathbb{P}[|a_n| \leq \sqrt{n} \forall n \geq m + 1] \geq 1 - \sum_{n=m+1}^{\infty} e^{-n} \geq 1 - C' e^{-m} \).

2. \( \mathbb{P}[|a_m| \geq (\alpha + 1)\sqrt{m}] = e^{-(\alpha + 1)^2 m} \).

3. The third event has probability as follows. Recall again that \( \mathbb{P} [\xi < x] \geq \frac{x}{2} \) if \( x < 1 \) and \( \xi \) has exponential distribution with mean 1. We apply this below with \( x = \frac{(m+1) \cdots (m+L-1) \sqrt{m} \left( -\frac{L}{m} \right)^{1/2}}{(n+1) \cdots (n+L-1)} \). This is clearly less than 1. Thus

\[
\mathbb{P} \left[ a_n \leq \frac{(\frac{-L}{m})^{1/2} r^{m-n}}{\sqrt{m} \left( -\frac{L}{m} \right)^{1/2}} \forall n \leq m - 1 \right] = \prod_{n=0}^{m-1} \mathbb{P} \left[ a_n \leq \frac{(\frac{-L}{m})^{1/2} r^{m-n}}{\sqrt{m} \left( -\frac{L}{n} \right)^{1/2}} \right]
\]

\[
\geq \prod_{n=0}^{m-1} \frac{(\frac{-L}{m})^{2m-2n}}{2m \left( -\frac{L}{n} \right)}
\]

\[
= r^{m(m+1)} m^{-m} \prod_{n=0}^{m-1} \frac{(m+1) \cdots (m+L-1)}{(n+1) \cdots (n+L-1)}
\]

\[
\geq r^{m(m+1)} m^{-m} \prod_{n=0}^{m-1} \frac{m^L}{(n+L)^L}
\]

\[
\geq r^{m(m+1)} + O(m \log(m)).
\]

Since these three events are independent, we get the lower bound in the theorem.
Upper Bound  The proof will proceed along the same lines as in Theorem 8.1.1. We need the following analogue of Lemma 8.2.3.

Lemma 8.3.2. Fix \( r < 1 \). Let \( M(r) = \sup_{z \in D(0; r)} |f_L(z)| \). Then

\[
P[M(r) \leq e^{-m}] \leq e^{-\frac{m^2}{\log(1/r)}} (1 + o(1)).
\]

Proof. By Cauchy’s theorem, for every \( n \geq 0 \),

\[
a_n \left( -\frac{L}{n} \right)^{1/2} = 1 \frac{1}{2\pi i} \int_{rT} f(\zeta) \frac{d\zeta}{\zeta^{n+1}}.
\]

From this we get

\[
|a_n|^2 \leq \frac{M(r)^2}{(-L/n)^2r^{2n}}.
\]

Since \( (-L/n) \geq \frac{n^{L-1}}{(L+1)^1} \), we obtain

\[
P[M(r) \leq m] \leq \prod_{n} P[|a_n|^2 \leq \frac{\Gamma(L+1)e^{-2m}}{n^{L-1}r^{2n}}] \\
\leq \prod_{n=0}^{m} \frac{\Gamma(L+1)e^{-2m}}{r^{2n}n^{L-1}} \\
\leq e^{-\frac{2m^2}{\log(1/r)}} \left( \frac{m}{\log(1/r)} \right)^2 \log(r) + O(m \log(m)) \\
= e^{-\frac{m^2}{\log(1/r)}} + O(m \log(m)).
\]

\[\square\]

Coming back to the proof of the upper bound in the theorem, fix \( R \) such that \( r < R < 1 \). Then by Jensen’s formula,

\[
n(r) \log \left( \frac{R}{r} \right) \leq \int_{r}^{R} \frac{n(u)}{u} du = \int_{rT}^{R} \log |f(Re^{i\theta})| \frac{d\theta}{2\pi} - \int_{rT} \log |f(re^{i\theta})| \frac{d\theta}{2\pi} \quad (8.3.2)
\]

Now consider the first summand in the right hand side of (8.2.7).

\[
P \left[ \int_{rT} \log |f(Re^{i\theta})| \frac{d\theta}{2\pi} > \sqrt{m} \right] \leq P \left[ \log M(R) \geq \sqrt{m} \right].
\]
Now suppose that $|a_n| < \lambda^n$ $\forall n \geq m + 1$ where $1 < \lambda < 1/R$. This has probability at least $C_1 e^{-\lambda^{2m}/2}$. Then,

$$M(R) \leq \sum_{n=0}^{\infty} |a_n| \left( \frac{-L}{n} \right)^{1/2} R^n \leq \left( \sum_{n=0}^{m} |a_n|^2 \right)^{1/2} C_R + C_{R'}$$

for some constants $C_R$ and $C_{R'}$.

Thus if $M(R) > e^{\sqrt{m}}$ then either $\sum_{n=0}^{m} |a_n|^2 > Ce^{2\sqrt{m}}$ or else $|a_n| > \lambda^n$ for some $n \geq m + 1$. Thus

$$P[M(R) > \sqrt{m}] \leq e^{-e^{\sqrt{m}}}.$$

This proves that

$$P \left[ \int_{R^2} \log |f(re^{i\theta})| \frac{d\theta}{2\pi} > \sqrt{m} \right] \leq e^{-e^{\sqrt{m}}}.$$

Fix $\delta > 0$ and $R$ close enough to 1 such that $\log(R) > -\delta$. Then with probability $\geq 1 - e^{-e^{\sqrt{m}}}$, we obtain from (8.3.2),

$$- \int_{R^2} \log |f(re^{i\theta})| \frac{d\theta}{2\pi} \geq m \left( \log \left( \frac{1}{r} \right) - \delta \right) - \sqrt{m}.$$

Now the calculations in the proof of Lemma 8.2.2 show that

$$\log M(\epsilon) \leq B_{\epsilon} \int_{0}^{2\pi} \log |f(re^{i\theta})| P(re^{i\theta}, w) \frac{d\theta}{2\pi} + A_{\epsilon} \log_+ M(r).$$

Here $0 < \epsilon < r$ is arbitrary and $A_{\epsilon}, B_{\epsilon}$ are as defined in Lemma 8.2.2. By the same computations as in that Lemma, we obtain, we obtain the inequality

$$P \left[ \int_{0}^{2\pi} \log |f(re^{i\theta})| \frac{d\theta}{2\pi} \leq -m(\log r | - \delta) + \sqrt{m} \right] \leq e^{-B_{\epsilon}^2 m^2 \log^2(r)(1 - \delta)} + e^{-e^{\epsilon m}}.$$

Therefore, by (8.3.2)

$$P \left[ n(r) \geq m \right] \leq e^{-\kappa m^2 \log^2(r)(1 + o(1))},$$

where $\kappa = \sup \left\{ \frac{B_{\epsilon}^2}{|\log(\epsilon)|} : 0 < \epsilon < r \right\}$. However it is clear that this cannot be made to match the lower bound by any choice of $\epsilon$. \qed
Remark 8.3.3. If we could prove

\[ P \left[ \int_0^{2\pi} \log |f(re^{i\theta})| \frac{d\theta}{2\pi} \leq -x \right] \leq \frac{e^{-x^2}}{\log(r)}, \]

that would have given us a matching upper bound. Now, one way for the event \( \int_0^{2\pi} \log |f(re^{i\theta})| \frac{d\theta}{2\pi} \leq -x \) to occur is to have \( \log M(r) < -x \) which, by Lemma 8.3.2, has probability at most \( e^{-x^2/\log(t)} \). One way to proceed could be to show that if the integral is smaller than \( -x \), so is \( \log M(s) \) for \( s \) arbitrarily close to \( r \) (with high probability). Alternately, if we could bound the coefficients directly by the bound on the integral (as in Lemma 8.3.2), that would also give us the desired bound. For these reasons, and keeping in mind the case \( L = 1 \), where we do have a matching upper bound, we believe that the lower bound in Theorem 8.1.2 is tight.
Chapter 9

Moderate and very large deviations for zeros of the planar GAF

Inspired by the results obtained (using not entirely rigorous physical arguments) by Jancovici, Lebowitz and Manificat (15) for Coulomb gases in the plane (e.g., Ginibre ensemble), M. Sodin (34) conjectured the following.

**Conjecture 9.0.4 (Sodin).** Let \( n(r) \) be the number of zeroes of the planar GAF \( g \) in the disk \( D(0, r) \). Then, as \( r \to \infty \)

\[
\frac{\log \log \left( \frac{1}{P[|n(r) - r^2| > r^\alpha]} \right)}{\log r} \to \begin{cases} 
2\alpha - 1, & \frac{1}{2} \leq \alpha \leq 1; \\
3\alpha - 2, & 1 \leq \alpha \leq 2; \\
2\alpha, & 2 \leq \alpha.
\end{cases} \tag{9.0.1}
\]

The idea here is that the deviation probabilities undergo a qualitative change in behaviour when the deviation under consideration becomes comparable to the perimeter (\( \alpha = 1 \)) or to the area (\( \alpha = 2 \)) of the domain.

Sodin and Tsirelson (36) had already settled the case \( \alpha = 2 \) by showing that for any \( \delta > 0 \), \( \exists c_1(\delta), c_2(\delta) \) such that

\[
e^{-c_1(\delta)r^4} \leq P[|n(r) - r^2| > \delta r^2] \leq e^{-c_2(\delta)r^4}.
\]

Here we consider \( P[n(r) - r^2 > r^0] \) and prove that a “phase transition” in
the exponent occurs at $\alpha = 2$. More precisely we prove that the conjecture holds for $\alpha > 2$ and show the lower bound for $1 < \alpha < 2$.

**Theorem 9.0.5.** Fix $\alpha > 2$. Then
\[
P \left[ n(r) \geq r^2 + \gamma r^\alpha \right] = e^{-\left(\frac{\alpha}{2} - 1\right)\gamma^2 r^{2\alpha} \log r \left(1+o(1)\right)}.
\]

**Theorem 9.0.6.** Fix $1 < \alpha < 2$. Then for any $\gamma > 0$,
\[
P \left[ n(r) \geq r^2 + \gamma r^\alpha \right] \geq e^{-\gamma^3 r^{3\alpha - 2} \left(1+o(1)\right)}.
\]

We prove Theorem 9.0.5 in Section 9.1 and Theorem 9.0.6 in Section 9.2. Taken together these show that the asymptotics of $P \left[ n(r) \geq r^2 + \gamma r^\alpha \right]$ does undergo a qualitative change at $\alpha = 2$.

**Remark 9.0.7.** Nazarov, Sodin and Volberg have recently proved all the remaining parts of the conjecture (personal communication).

### 9.1 Very large deviations for the planar GAF

In this section we prove Theorem 9.0.5.

**Remark 9.1.1.** In the case $\alpha \geq 2$, one side of the estimate as asked for in the conjecture (with $\log \log$ of the probability) follows trivially from the results in Sodin and Tsirelson [36]. They prove that for any $\delta > 0$, there exists a constant $c(\delta)$ such that
\[
P \left[ |n(r) - r^2| > \delta r^2 \right] \leq e^{-c(\delta)r^4}.
\]

When $\alpha \geq 2$, clearly $n((1 - \delta)r^{\sqrt{\alpha}}) \geq n(r)$, whence from the above result it follows that
\[
P \left[ n(r) \geq r^2 + r^\alpha \right] \leq P \left[ n((1 - \delta)r^{\sqrt{\alpha}}) \geq r^\alpha \right] \leq e^{-c(\delta)r^{2\alpha}}.
\]

This gives
\[
\limsup_{r \to \infty} \frac{\log \log \left( \frac{1}{P[|n(r) - r^2| > r^\alpha]} \right)}{\log r} \leq 2\alpha.
\] (9.1.1)
The obviously loose inequality \( n((1 - \delta)r^{\sqrt{\alpha}}) \geq n(r) \) that we used, suggests that \([9.1.1]\) can be improved when \( \alpha > 2 \) to Theorem \([9.0.5]\).

**Proof of Theorem 9.0.5 Lower Bound** Let \( m = r^2 + \gamma r^\alpha \). Suppose the \( m \)th term dominates the sum of all the other terms on \( \partial D(0;\, r) \), i.e., suppose
\[
\left| \frac{a_m z^m}{\sqrt{m!}} \right| \geq \left| \sum_{n \neq m} \frac{a_n z^n}{\sqrt{n!}} \right| \quad \text{whenever } |z| = r. \tag{9.1.2}
\]

Now we want to find a lower bound for the probability of the event in \([9.1.2]\). Note that the left side of \([9.1.2]\) is identically equal to \( \frac{|a_m| r^m}{\sqrt{m!}} \).

Now suppose the following happen-

1. \(|a_n| \leq n \forall n \geq m + 1\).
2. \(|a_m| \geq m\).
3. \(|a_n| \frac{r^n}{\sqrt{n!}} < \frac{\gamma r^\alpha}{m} \frac{r^m}{\sqrt{m!}} \) for every \( 0 \leq n \leq m - 1 \).

Then the right hand side of \([9.1.2]\) is bounded by
\[
\text{RHS of } [9.1.2] \leq \sum_{n=0}^{m-1} |a_n| \frac{r^n}{\sqrt{n!}} + \sum_{n=m+1}^{\infty} |a_n| \frac{r^n}{\sqrt{n!}} \\
\leq \sum_{n=0}^{m-1} \frac{\gamma r^\alpha}{m} \frac{r^m}{\sqrt{m!}} + \sum_{n=m+1}^{\infty} \frac{nr^n}{\sqrt{n!}} \\
\leq \frac{m r^m}{\sqrt{m!}} \left( \frac{\gamma r^\alpha}{m} + o(1) \right) \\
\leq \left| a_m \right| \frac{r^m}{m!}
\]

Thus if the above three events occur, then the \( m \)th term dominates the sum of all the other terms on \( \partial D(0;\, r) \). Also these events have probabilities as follows.

1. \( P[|a_n| \leq n \forall n \geq m + 1] \geq 1 - \sum_{n=m+1}^{\infty} e^{-n^2} \geq 1 - C' e^{-m^2} = 1 - o(1) \).
2. \( P[|a_m| \geq m] = e^{-m^2} = e^{-\gamma^2 r^{2\alpha}(1+o(1))} \).
3. The third event has probability as follows. Recall again that \( P[\xi < x] \geq \frac{x}{\bar{\xi}} \) if \( x < 1 \) and \( \xi \) is exponential with mean 1. We apply this below with \( x = \)
This is clearly less than 1 if $n \geq r^2$. Therefore if $m$ is sufficiently large it is easy to see that for all $0 \leq n \leq m - 1$, the same is valid. Thus

$$P \left[ |a_n| \leq \frac{\gamma r^\alpha r^{m-n} \sqrt{n!}}{m^2 \sqrt{m!}} \forall n \leq m - 1 \right] = \prod_{n=0}^{m-1} P \left[ |a_n| \leq \frac{\gamma r^\alpha r^{m-n} \sqrt{n!}}{m^2 \sqrt{m!}} \right]$$

\[
\geq \prod_{n=0}^{m-1} \frac{\gamma^2 r^{2\alpha} \gamma^{2m-2n} n!}{m^2 \sqrt{2m!}} \\
= r^{2\alpha(m+1)+m(m+1)} 2^{-m} m^{-2m} e^{-\sum_{k=1}^{m} k \log k} \\
= e^{m^2 \log(r) - \frac{1}{2} m^2 \log(m)} + O(m^2) \\
= e^{-(\frac{\alpha}{2}-1)\gamma^2 r^{2\alpha} \log(r) + O(r^{2\alpha})}
\]

Since these three events are independent, we get

$$P \left[ n(r) \geq r^2 + r^\alpha \right] \geq e^{-(\frac{\alpha}{2}-1)\gamma^2 r^{2\alpha} \log r + O(r^{2\alpha})}.$$  \quad (9.1.3)

**Upper Bound** We omit the proof of the upper bound, as it follows the same lines as that of Theorem 8.1.1 and we have already seen such arguments again in the proof of Theorem 8.1.2 (In those two cases as well as the present case, we are looking at very large deviations, and that is the reason why the same tricks work).

Moreover note that the lower bound along with (9.1.1) proves the statement in the conjecture.

\[\square\]

### 9.2 Moderate deviations for the planar GAF

In this section we prove Theorem 9.0.6.

**Proof of Theorem 9.0.6** Write $m = r^2 + \gamma r^\alpha$. As usual, we bound $P \left[ n(r) \geq m \right]$ from below by the probability of the event that the $m^{th}$ term dominates the rest of the series.

Firstly, we need a couple of estimates. Consider $\frac{r^{2n}}{n!}$ as a function of $n$. This increases monotonically up to $n = r^2$ and then decreases monotonically. $m = r^2 + \gamma r^\alpha$ is on the latter part. Write $M = r^2 - \gamma r^\alpha$. 
Firstly, observe that \((r^2 - k)(r^2 + k) < (r^2)^2\), for \(1 \leq k \leq \gamma r^\alpha\), whence
\[r^{2m - 2M} > \prod_{j=M+1}^{m-1} j.\] This implies that
\[\frac{r^M}{\sqrt{M!}} < \frac{r^m}{\sqrt{m!}}\] \hspace{1cm} (9.2.1)

Secondly, note that for any \(n = M - p\),
\[\frac{r^{2n}/n!}{r^{2M}/M!} = \prod_{j=0}^{p-1} \frac{M-j}{r^2} = \prod_{j=0}^{p-1} (1 - \gamma r^\alpha - j r^{-2}) - \sum_{j=0}^{p-1} (\gamma r^\alpha + jr^{-2}) \leq e^{-\gamma pr^\alpha - \frac{p(p+1)}{2} r^{-2}}.\]

Now we set \(p = Cr^{2-\alpha}\) with \(C\) so large that \(e^{-\gamma C} \leq \frac{1}{4}\).
Then also note that if \(n < M - kp\), it follows that
\[\frac{r^{2n}/n!}{r^{2m}/m!} \leq \frac{1}{4^k},\] \hspace{1cm} (9.2.2)
where we used (9.2.1) to replace \(M\) by \(m\).

Thirdly, if \(n = m + p\) with \(p \leq r^2 - \gamma r^\alpha\), then,
\[\frac{r^{2n}/n!}{r^{2m}/m!} = \prod_{j=1}^{p} \frac{r^2}{m + j} = \prod_{j=1}^{p} (1 + \gamma r^\alpha - j r^{-2})^{-1} \leq e^{-\frac{1}{2} \sum_{j=1}^{p} (\gamma r^\alpha + jr^{-2})} = e^{-\frac{1}{2} (\gamma pr^\alpha - \frac{p(p+1)}{2} r^{-2})}.\]

If \(p = 2Cr^{2-\alpha}\), where \(C\) was as chosen before, then for \(n > m + kp\), we get
\[\frac{r^{2n}/n!}{r^{2m}/m!} \leq \frac{1}{4^k}\] \hspace{1cm} (9.2.3)

From now on \(p = 2Cr^{2-\alpha}\) is fixed so that (9.2.2) and (9.2.3) are satisfied.

Next we divide the coefficients other than the \(m\)th one into groups:
• $A_k = \{ n : n \in (M - kp, M - (k - 1)p) \text{ for } 1 \leq k \leq \lceil \frac{M}{p} \rceil \}$.

• $D_k = \{ n : n \in [m + (k - 1)p, m + kp) \text{ for } 1 \leq k \leq \lceil \frac{M}{p} \rceil \}$.

• $B = \{ n : n \in [M + 1, m - 1] \}$.

• $C = \{ n : n \in [2r^2, \infty) \}$.

**Remark 9.2.1.** As defined, there is an overlap between $D_{\lceil \frac{M}{p} \rceil}$ and $C$. This is inconsequential, but for definiteness, let us truncate the former interval at $r^2$ (just as $A_{\lceil \frac{M}{p} \rceil}$ is understood to be truncated at $0$).

Now consider the following events.

1. $|a_n| \leq \frac{2^k}{M}$ for $n \in A_k$ for $k \leq \lceil \frac{M}{p} \rceil$.

2. $|a_n| \leq \frac{2^k}{M}$ for $n \in D_k$ for $k \leq \lceil \frac{M}{p} \rceil$.

3. $\sum_{n \in B} |a_n| \frac{r^n}{\sqrt{n!}} \leq 4 \frac{r^m}{\sqrt{m!}}$.

4. $|a_n| < n - 2r^2$ for $n \in C$.

5. $|a_m| \geq 15$.

Suppose all these events occur. Then

1. The event $|a_n| \leq \frac{2^k}{M}$ for $n \in A_k$, $k \leq \lceil \frac{M}{p} \rceil$ gives

\[
\sup\{| \sum_{n=0}^{M} \frac{a_n z^n}{\sqrt{n!}} : |z| = r \} \leq \sum_{k=1}^{\lceil \frac{M}{p} \rceil} \sum_{n=M-kp+1}^{M-(k-1)p} |a_n| \frac{r^n}{\sqrt{n!}}
\]

\[
\leq \sum_{k=1}^{\lceil \frac{M}{p} \rceil} \frac{1}{2^k \sqrt{m!}} \frac{2^k p}{M} \quad \text{by (9.2.2)}
\]

\[
\leq \frac{r^m}{\sqrt{m!}} \sum_{k=1}^{\lceil \frac{M}{p} \rceil} \frac{p}{M}
\]

\[
\leq \frac{r^m}{\sqrt{m!}} \left( 1 + \frac{p}{M} \right)
\]
2. The event $|a_n| \leq \frac{2^k}{M}$ for $n \in D_k$, $k \leq \lceil \frac{M}{p} \rceil$ gives

$$\sup \{| \sum_{n=m+1}^{n} \frac{a_n z^n}{\sqrt{n!}} : |z| = r \} = \sum_{k=1}^{\lceil \frac{M}{p} \rceil} \sum_{n=m+(k-1)p+1}^{M+kp} |a_n| \frac{r^n}{\sqrt{n!}} \quad (9.2.8)$$

$$\leq \sum_{k=1}^{\lceil \frac{M}{p} \rceil} \frac{1}{2^k} \frac{r^m}{\sqrt{m!}} \frac{2^k p}{M} \quad \text{by (9.2.3)} \quad (9.2.9)$$

$$\leq \frac{r^m}{\sqrt{m!}} \left( 1 + \frac{p}{M} \right). \quad (9.2.10)$$

3. The third event gives

$$\sum_{n \in B} |a_n| \frac{r^n}{\sqrt{n!}} \leq 4 \frac{r^m}{\sqrt{m!}}, \quad (9.2.11)$$

by assumption.

4. The event $|a_n| < n - 2r^2$ for $n \in C$: Since $n > 2r^2$,

$$\frac{r^n}{\sqrt{n!}} = \frac{r^m}{\sqrt{m!}} \prod_{k=m+1}^{n} \frac{r}{\sqrt{k}}$$

$$\leq \frac{r^m}{\sqrt{m!}} \prod_{k=2r^2+1}^{n} \frac{r}{\sqrt{k}}$$

$$\leq \frac{r^m}{\sqrt{m!}} \left( \frac{1}{\sqrt{2}} \right)^{n-2r^2}.$$ 

Therefore we get (using $|a_n| < n - 2r^2 \forall n > 2r^2$)

$$\sum_{n \in C} |a_n| \frac{r^n}{\sqrt{n!}} \leq \frac{r^m}{\sqrt{m!}} \sum_{n>2r^2} (n-2r^2) \left( \frac{1}{\sqrt{2}} \right)^{n-2r^2} \quad (9.2.12)$$

$$= \frac{\sqrt{2}}{\sqrt{m!} (\sqrt{2} - 1)^2}. \quad (9.2.13)$$

Putting together the contributions from these four groups of terms, and using $|a_m| > 15$, we get (for large values of $r$)

$$\sum_{n \neq m} |a_n| \frac{r^n}{\sqrt{n!}} \leq |a_m| \frac{r^m}{\sqrt{m!}}.$$

Now we compute the probabilities of the events enumerated above.
1. The event $|a_n| \leq \frac{2^k}{M}$ for $n \in A_k$ for $k \leq \lceil \frac{M}{p} \rceil$. Now for a fixed $k \leq 3 \log_2(r)$, we deduce

$$
P \left[ |a_n| \leq \frac{2^k}{M} \text{ for } n \in A_k \right] \geq P \left[ |a_n| \leq \frac{1}{M} \text{ for } n \in A_k \right] \geq \left( \frac{1}{2M^2} \right)^p.
$$

Therefore

$$
P \left[ |a_n| \leq \frac{2^k}{M} \text{ for } n \in A_k \text{ for every } k \leq 3 \log_2(r) \right] \geq \left( \frac{1}{2M^2} \right)^{3p \log_2(r)} \geq e^{-cr^2 - \alpha (\log(r))^2}.
$$

for some $c$.

Next we deal with $k > 3 \log_2(r)$.

$$
P \left[ |a_n| \leq \frac{2^k}{M} \text{ for } n \in A_k \text{ for every } k > 3 \log_2(r) \right] \geq 1 - \sum_{k > 3 \log_2(r)} pP \left[ |a| > \frac{2^k}{M} \right] = 1 - \sum_{k > 3 \log_2(r)} p e^{-2kM^{-2}}.
$$

Now the summation in the last line has rapidly decaying terms and starts with $p e^{-6 \log_2(r)M^{-2}}$ which is smaller than $pe^{-r^2}$. Thus

$$
P \left[ |a_n| \leq \frac{2^k}{M} \text{ for } n \in A_k \text{ for every } k > 3 \log_2(r) \right] = 1 - o(1).
$$

Thus the event in question has probability at least $e^{-cr^2 - \alpha (\log(r))^2(1+o(1))}$.

2. The event $|a_n| \leq \frac{2^k}{M}$ for $n \in D_k$ for $k \leq \lceil \frac{M}{p} \rceil$. Following exactly the same steps as above we can prove that

$$
P \left[ |a_n| \leq \frac{2^k}{M} \text{ for } n \in D_k \right] \geq e^{-cr^2 - \alpha (\log(r))^2(1+o(1))}.
$$

3. The event $\sum_{n \in B} |a_n| \frac{r^n}{\sqrt{n!}} \leq 4 \frac{r^m}{\sqrt{m!}}$. By Cauchy-Schwarz,

$$
\left( \sum_{n \in B} |a_n| \frac{r^n}{\sqrt{n!}} \right)^2 \leq \left( \sum_{n \in B} |a_n|^2 \right) \left( \sum_{n \in B} \frac{r^{2n}}{n!} \right).
$$
\[ Y = \sum_{n \in B} |a_n|^2 \] has \( \Gamma(|B|, 1) \) distribution. Also \[ \sum_{n \in B} \frac{r^{2n}}{n!} \leq e^{r^2}, \] since the left hand is part of the Taylor series of \( e^{r^2} \). Therefore the event in question has probability,

\[
P[\text{event in question}] \geq P \left[ Y < 16 \frac{r^{2m}}{m!} e^{-r^2} \right] \geq \varphi \left( \frac{r^{2m}}{m!} e^{-r^2} \right) 8 \frac{r^{2m}}{m!} e^{-r^2},
\]

where \( \varphi \) is the density of the \( \Gamma(|B|, 1) \) distribution. This last follows because \( \varphi \) is increasing on \([0, |B|]\) and thus \( P[Y < x] \geq \varphi \left( \frac{x}{2} \right) \), for \( x < |B| \). Continuing,

\[
P[\text{event in question}] \geq \frac{1}{(2\gamma r^\alpha)^\frac{1}{2}} e^{-8 \frac{r^{2m}}{m!} e^{-r^2}} \left( \frac{8 \frac{r^{2m}}{m!} e^{-r^2}}{2^2 r^{2\alpha}} \right)^{2^2 r^{2\alpha}} (9.2.14)
\]

\[
\geq C e^{2r^{\alpha}(m \log(r^2) - r^2 - m \log(m) + m) + O(r^\alpha \log(r))}(9.2.15)
\]

where we used Stirling’s approximation.

The exponent needs simplification. Take the first and third terms in the exponent. We have \(-2\gamma mr^\alpha \log \left( \frac{m}{r^2} \right)\). Recall that \( m = r^2 + \gamma r^\alpha \) and that \( \alpha < 2 \). Therefore by Taylor’s expansion of \( \log(1 + \gamma r^{\alpha - 2}) \) we get

\[-2\gamma mr^\alpha \log \left( \frac{m}{r^2} \right) = \left\{ \begin{array}{l}
-2\gamma^2 r^{2\alpha} + \gamma^3 r^{3\alpha - 2} - \frac{2}{3} \gamma^4 r^{4\alpha - 4} + \ldots \\
-2\gamma^3 r^{3\alpha - 2} + \gamma^4 r^{4\alpha - 4} - \frac{2}{3} \gamma^5 r^{5\alpha - 6} + \ldots
\end{array} \right. (9.2.16)\]

Now consider [9.2.15]. Expand the fourth term in the exponential as \( 2\gamma r^{2\alpha + 2} + 2\gamma^2 r^{2\alpha} \). We get the following terms

(a) \( r^{2\alpha + 2\alpha} (-2\gamma + 2\gamma) = 0 \), from the second and fourth terms (first piece of the fourth term) in the exponential in [9.2.15].

(b) \( r^{2\alpha} (-2\gamma^2 + 2\gamma^2) = 0 \), from the sum of the first term in the expansion [9.2.16] and the second piece of the fourth term in the exponential in [9.2.15].

(c) \( r^{3\alpha - 2} (\gamma^3 - 2\gamma^3) = -\gamma^3 r^{3\alpha - 2} \), from the expansion [9.2.16].

(d) Other terms such as \( r^\alpha \log(m), r^\alpha \log(r), r^\alpha, r^{4\alpha - 4}, r^{5\alpha - 6} \) etc. All these are of lower order than \( r^{3\alpha - 2} \) when \( 1 < \alpha < 2 \).
Hence,

$$P[\text{event in ques}on] \geq e^{-\gamma^3 r^{3\alpha - 2} (1+o(1))}.$$ 

4. The event $|a_n| < n - 2m$ for $n \in C$. This is just an event for a sequence of i.i.d. complex Gaussians. It has a fixed probability $p_0$ (say).

5. The event $|a_m| \geq 15$ also has a constant probability (not depending on $r$, that is).

This completes the estimation of probabilities. Among these five events, the third one, namely $\sum_{n \in B} |a_n| \leq 4 \frac{r^m}{\sqrt{m!}}$ has the least probability (Recall that $1 < \alpha < 2$).

Also these events are all independent, being dependent on disjoint sets of coefficients. Thus $P \left[ n(r) \geq r^2 + \gamma r^\alpha \right] \geq e^{-\gamma^3 r^{3\alpha - 2} (1+o(1))}$. 

$\square$
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