SIMPLICITY OF RIGHT-ANGLED HECKE $C^*$-ALGEBRAS

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Abstract. By exploiting properties of boundaries associated with Coxeter groups we obtain a complete characterization of simple right-angled multi-parameter Hecke $C^*$-algebras. This extends previous results by Caspers, Larsen and the author. Based on work by Raum and Skalski, we further describe the center and the character space of right-angled Hecke $C^*$-algebras.

Introduction

(Iwahori) Hecke algebras are deformations of the group algebra of Coxeter groups depending on a deformation (multi-)parameter $q$. They can be viewed as an abstraction of certain endomorphism rings which naturally appear in the representation theory of finite groups of Lie type [5] and are particularly well studied in the case of spherical and affine Coxeter groups (see [23], [26], [2], [27]). For other Coxeter groups they appear in the context of buildings and Kac-Moody groups acting on them [35].

Hecke algebras of a given Coxeter system $(W,S)$ can be naturally represented on the Hilbert space $\ell^2(W)$ of square-summable functions on $W$. They complete to $C^*$-algebras $C^*_{r,q}(W)$ and von Neumann algebras $N_{r,q}(W)$. The study of these operator algebras gave insight in the cohomology of associated buildings and its $\ell^2$-Betti numbers (see [17], [14]) and they are related to Dykema’s interpolated free group factors, which play an important role in the treatment of the infamous free factor problem (see [15], [32], [19]). Much earlier Hecke operator algebras of spherical and affine type Coxeter systems have been studied in [31].

Despite their ubiquitousness, until now Hecke operator algebras are well understood only in the case of spherical and affine type Coxeter groups. In particular, the natural question for a characterization of the simplicity (i.e. triviality of the ideal structure) and the trace-uniqueness of $C^*_{r,q}(W)$ and the factoriality (i.e. triviality of the center) of $N_{r,q}(W)$ is wide open. Recently, the investigation of right-angled Hecke $C^*$-algebras and right-angled Hecke-von Neumann algebras made some progress. In [19] Garnarek characterized the factoriality of single-parameter Hecke-von Neumann algebras. Complementing his ideas with a new combinatorial approach, the result was later extended to the multi-parameter case by Raum and Skalski [33]. In [10] Caspers, Larsen and the author studied the $C^*$-algebraic setting and proved, using classical averaging techniques, simplicity and trace-uniqueness results for right-angled Hecke $C^*$-algebras and certain ranges of deformation parameters $q$. As remarked in [10] Subsection 5.4], Dykema’s results on free products of finite dimensional abelian $C^*$-algebras in [16] imply a complete description of the ideal structure and the trace-uniqueness of Hecke $C^*$-algebras of free products of right-angled abelian Coxeter groups. It is further known that the reduced group $C^*$-algebra of an irreducible Coxeter system is simple if and only if the corresponding Coxeter system is of non-affine type (see [18], [21], [12]). Other relevant references treating non-affine Hecke operator algebras are [9], [11], [34].

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A notion that was famously used by Kalantar and Kennedy \cite{Kalantar_Kennedy} to solve the longstanding question regarding the C*-simplicity of a given (discrete) group is that of Furstenberg’s boundary and (topological) boundary actions. The authors established a link between dynamical properties of the Furstenberg boundary of a given group and the structure of the corresponding group C*-algebra, which led to results on simplicity, uniqueness of trace and tightness of nuclear embeddings of group C*-algebras (see e.g. \cite{Kalantar_Kennedy}, \cite{Brown_Jr}, \cite{Kalantar_r}) and inspired various generalizations (see e.g. \cite{Brown}, \cite{Brown2}, \cite{Kalantar}). Inspired by the approach in \cite{Kalantar_Kennedy}, our present work goes into a similar direction. In \cite{Kalantar_Kennedy} the author introduced and studied topological boundaries and compactifications associated with connected rooted graphs. These are topological spaces that reflect combinatorial properties of the underlying graph and which are particularly tractable in the case of (Cayley graphs of) Coxeter groups. In the latter context the spaces have been considered earlier by Caprace-Lécureux \cite{Caprace-Lecureux} and Lam-Thomas \cite{Lam_Thomas}. The striking advantage of the construction is its close connection to the Hecke operator algebras of the corresponding Coxeter system (see Section 1 for more details), which has been utilized in \cite{Kalantar_Kennedy, Section 4}. We will further exploit the implications of this connection by using it to characterize the simplicity of right-angled Hecke C*-algebras, thus extending the results in \cite{Kalantar_Kennedy} and (partially) answering \cite[Question 5.13]{Kalantar}. This leads to a full classification of the simplicity in the right-angled case which is the main result of this paper.

**Theorem.** Let \((W,S)\) be an irreducible, right-angled Coxeter system with \(\#S < \infty\) and \(q = (q_s)_{s \in S} \in \mathbb{R}_{>0}^{(W,S)}\) a multi-parameter. Let further \(\mathcal{R}\) be the region of convergence of the growth series \(\sum_{w \in W} z_w\), set \(\mathcal{R}' := \{(q_s')_{s \in S} \mid q \in \mathcal{R} \cap \mathbb{R}_{>0}^{(W,S)}, \epsilon \in \{-1,1\}^{(W,S)}\}\) and define \(\overline{\mathcal{R}'}\) to be the closure of \(\mathcal{R}'\) in \(\mathbb{R}_{>0}^{(W,S)}\). Then the Hecke C*-algebra \(C_{r,q}^*(W)\) is simple if and only if \(q \in \mathbb{R}_{>0}^{(W,S)}\setminus \overline{\mathcal{R}}\).

**Corollary.** Let \((W,S)\) be an irreducible, right-angled Coxeter system with \(\#S = \infty\) and \(q = (q_s)_{s \in S} \in \mathbb{R}_{>0}^{(W,S)}\) a multi-parameter. Then the Hecke C*-algebra \(C_{r,q}^*(W)\) is simple if and only if there exists a finite subset \(T \subseteq S\) such that the Hecke C*-algebra \(C_{r,q_T}^*(W_T)\) of the special subgroup \(W_T \subseteq W\) with \(q_T := (q_t)_{t \in T}\) is simple.

Using a Haagerup-type inequality from \cite{Kalantar}, we will also prove that the central projections of right-angled Hecke-von Neumann algebras considered by Raum and Skalski in \cite{Raum_Skalski} are already contained in the corresponding Hecke C*-algebras. This leads to a decomposition of \(C_{r,q}^*(W)\) which is analogous to the one of \(N_q(W)\) and can be used to characterize the space of characters (i.e. unital, multiplicative, linear functionals) of \(C_{r,q}^*(W)\).

**Structure.** In Section 1 we recall general facts about Coxeter groups, the boundaries introduced in \cite{Kalantar_Kennedy} and multi-parameter Hecke algebras (resp. their operator algebraic counterparts). In Section 2 a number of technical statements related to certain C*-algebras associated with Coxeter systems are proved, which allow to translate group algebraic arguments into the \(q\)-deformed setting. This will be used in Section 3 and 4.3 where the center, the character space and the simplicity of right-angled Hecke C*-algebras are characterized.

1. Preliminaries and notation

1.1. **General notation.** We will write \(\mathbb{N} := \{0, 1, 2, \ldots\}\) and \(\mathbb{N}_{\geq 1} := \{1, 2, \ldots\}\) for the natural numbers. Scalar products of Hilbert spaces are linear in the first variable and we denote the bounded operators on a Hilbert space \(\mathcal{H}\) by \(\mathcal{B}(\mathcal{H})\). For a C*-algebra \(A\) we will write \(S(A)\) for the state space of \(A\) and endow it with the weak-* topology. Further, if \((A,G,\alpha)\) is a dynamical
system we write \( g.\alpha := \alpha_g(a) \) where \( a \in A, g \in G \). Similar notation is being used for (continuous) group actions on topological spaces. The symbol \( \otimes \) denotes the algebraic tensor product of \( * \)-algebras, \( \otimes \) is the minimal tensor product of \( C^* \)-algebras, \( \boxtimes \) denotes the tensor product of von Neumann algebras and we write \( \rtimes \) for reduced \( (C^* \text{-algebraic)} \) crossed products. Further, the neutral element of a group is always denoted by \( e \) and for a set \( S \) we write \( \#S \) for the number of elements in \( S \) and \( \chi_S \) for the characteristic function on \( S \).

1.2. Coxeter groups. A Coxeter group \( W \) is a group generated by a (possibly infinite) set \( S \) of the form

\[
W = \langle S \mid \forall s, t \in S: (st)^{m_{st}} = e \rangle ,
\]

where \( m_{st} \in \{1, 2, \ldots, \infty\} \) with \( m_{ss} = 1 \) and \( m_{st} \geq 2 \) for all \( s \neq t \). The condition \( m_{st} = \infty \) means that no relation of the form \( (st)^m = 1, m \in \mathbb{N} \) is imposed. The pair \((W, S)\) is called a Coxeter system. It is right-angled if \( m_{st} \geq 2 \) and \( m_{st} = \infty \) for all \( s \neq t \). If the generating set \( S \) is finite the system \((W, S)\) has finite rank. The data of \((W, S)\) is usually encoded in its Coxeter diagram whose vertex set is \( S \) and whose edge set is given by \( \{(s, t) \mid m_{st} \geq 3\} \) where every edge between two vertices \( s, t \in S \) is labeled by the corresponding exponent \( m_{st} \).

For a subset \( T \subseteq S \) the special subgroup \( W_T \) of \( W \) generated by \( T \) is also a Coxeter group with the same exponents as \( W \) (see [13, Theorem 4.1.6]). The system \((W, S)\) is irreducible if its Coxeter diagram is connected. This is the case if and only if \( W \) does not decompose as a non-trivial direct product of special subgroups.

Every element \( w \in W \) decomposes as a product \( w = s_1 \ldots s_n \) of generators \( s_1, \ldots, s_n \in S \). The expression \( s_1 \ldots s_n \) is called reduced if it has minimal length. The word length of \( w \), denoted by \( |w| \), is then defined to be the number of generators appearing in a reduced expression for \( w \), where \( |e| := 0 \). One says that \( w \) starts (resp. ends) with \( v \in W \) if \( |v^{-1}w| = |w| - |v| \) (resp. \( |wv^{-1}| = |w| - |v| \)). In that case we write \( v \leq_R w \) (resp. \( v \leq_L w \)). This defines a partial order on \( W \) which is called the weak right (resp. weak left) Bruhat order of \((W, S)\). It turns \( W \) into a complete meet-semilattice (see [3, Theorem 3.2.1]). To simplify the notation we will usually write \( \leq \) instead of \( \leq_R \).

In the right-angled case, cancellation of the form \( s_1 \ldots s_n = s_1 \ldots \hat{s}_i \ldots \hat{s}_j \ldots s_n \) for \( s_1, \ldots, s_n \in S \) implies that \( s_i = s_j \) and that \( s_i \) commutes with \( s_{i+1} \ldots \hat{s}_{j-1} \) (this follows from [13, Lemma 3.3.3]). Here \( s_1 \ldots s_n = s_1 \ldots \hat{s}_i \ldots \hat{s}_j \ldots s_n \) means that \( s_i \) and \( s_j \) are removed from the expression \( s_1 \ldots s_n \). We will use this frequently without further mention. Another useful property is the following.

**Proposition 1.1** (see [3, Proposition 3.1.2 (vi)]). Let \((W, S)\) be a Coxeter system, \( v, w \in W \) and \( s \in S \) with \( s \leq v, s \leq w \). Then, \( v \leq w \) if and only if \( sv \leq sw \).

1.3. Topological boundaries of Coxeter groups. In [29] topological boundaries and compactifications associated with connected rooted graphs were introduced and studied. These topological spaces are particularly useful in the case of (Cayley graphs of) Coxeter groups. For these, the spaces have been introduced earlier by Caprace and Lécureux in [8] and by Lam and Thomas in [30] in a different setting, using different formalisms. The construction that we will follow in this paper coincides with the one in [29], but we will restrict to the case of Cayley graphs of Coxeter groups. For more details and the general construction see [29].

Let \((W, S)\) be a finite rank Coxeter system and denote by \( K := \operatorname{Cay}(W, S) \) the Cayley graph of \( W \) with respect to the generating set \( S \), i.e. the graph with vertex set \( W \) and edge set \( \{(v, w) \in W \times W \mid v^{-1}w \in S\} \). The metric \( d: W \times W \to \mathbb{R}_{\geq 0} \) defined by \( d(v, w) := |v^{-1}w| \) turns (the vertex set of) \( K \) into a metric space. A geodesic path \( \alpha \) in \( K \) is a (possibly infinite)
sequence $\alpha_0\alpha_1...$ of vertices with $d(\alpha_i, \alpha_j) = |i - j|$ for all $i, j$. Without further comments we will often extend a finite geodesic path $\alpha_0...\alpha_n$ to an infinite path via $\alpha_0...\alpha_n\alpha_n\alpha_1...$ and still call it (finite) geodesic. For a geodesic path $\alpha$ and $w \in W$ we write $w \leq \alpha$ if $w \leq \alpha_i$ for all large enough $i$ and we write $w \not\leq \alpha$ if $w \not\leq \alpha_i$ for all large enough $i$. Now, define an equivalence relation $\sim$ on the set of all infinite geodesic paths in $K$ via $\sim$ if and only if for every $w \in W$ the implications $w \leq \alpha \iff w \leq \beta$ hold. Write $\partial(W, S)$ for the set of corresponding equivalence classes. This set is called the boundary of $(W, S)$ and $(W, S) := W \cup \partial(W, S)$ is called the compactification of $(W, S)$. The weak right Bruhat order naturally extends to a partial order $\leq$ on $(W, S)$ (see [29, Lemma 2.2]).

We then equip $(W, S)$ with the topology generated by the subbase of sets of the form

$$\mathcal{U}_w := \left\{ z \in (W, S) \mid w \leq z \right\} \quad \text{and} \quad \mathcal{U}_w := \left\{ z \in (W, S) \mid w \not\leq z \right\},$$

where $w \in W$. This turns $\partial(W, S)$ and $(W, S)$ into metrizable compact spaces and $W$ naturally embeds as a dense discrete subset into $(W, S)$. Further, the left action of $W$ on itself induces a (continuous) action $W \curvearrowright (W, S)$ with $W.(\partial(W, S)) = \partial(W, S)$. This action has some desirable properties, one of which will play a role in the characterization of the simplicity of right-angled Hecke $C^*$-algebras.

**Theorem 1.2** ([29, Theorem 3.20 and Proposition 3.26]). Let $(W, S)$ be a right-angled irreducible Coxeter system with $3 \leq \#S < \infty$. Then the action $W \curvearrowright \partial(W, S)$ is a boundary action, meaning that the following statements hold:

- **Minimality:** For every $z \in \partial(W, S)$ the $W$-orbit $W.z := \{w.z \mid w \in W\}$ is dense in $\partial(W, S)$;
- **Strong proximality:** For every probability measure $\nu \in \text{Prob}(\partial(W, S))$ the weak-$*$ closure of the $W$-orbit $W.\nu$ contains a point mass $\delta_z \in \text{Prob}(\partial(W, S))$ for some $z \in \partial(W, S)$.

Further, the action is topologically free, i.e. for every $w \in W \setminus \{e\}$ the set $(\partial(W, S))_w := \{z \in \partial(W, S) \mid w.z = z\}$ has no inner points.

### 1.4. Multi-parameter Hecke algebras

For a Coxeter system $(W, S)$ define $\mathbb{R}^{(W, S)}_{>0}$ to be the set of all multi-parameters $q = (q_s)_{s \in S} \in \mathbb{R}^S_{>0}$ for which $q_s = t_q$ for all $s, t \in S$ which are conjugate to each other. The sets $C_p^{(W, S)}$ and $\{-1, 1\}^{(W, S)}$ are defined in a similar way. For a tuple $q = (q_s)_{s \in S} \in \mathbb{R}^{(W, S)}_{>0}$, $s \in S$ and a reduced expression $w = s_1...s_n$ of $w \in W$ write

$$q_w := q_{s_1}...q_{s_n} \quad \text{and} \quad p_s(q) := q_s^{-\frac{1}{2}}(q_s - 1).$$

Then $q_w$ does not depend on the choice of the reduced expression for $w$ (see [13, Chapter 17.1]). Following the notation in [33] we also write $q_{s, \epsilon} := \epsilon q_{s}^{\epsilon}$ and $q_{w, \epsilon} := q_{s_1, \epsilon}...q_{s_n, \epsilon}$ for $q \in \mathbb{R}^{(W, S)}_{>0}$, $\epsilon \in \{-1, 1\}^{(W, S)}$.

By [13, Proposition 19.1.1] for $q \in \mathbb{R}^{(W, S)}_{>0}$ there exists a unique (unital) $*$-algebra $C_q[W]$ spanned by a linear basis $\{T_w(q) \mid w \in W\}$ such that for $s \in S$, $w \in W$ one has

$$T_s(q)T_w(q) = \begin{cases} T_w(q) & \text{if } s \not\leq w \\ T_s(q) + p_s(q)T_w(q) & \text{if } s \leq w \\ \end{cases}$$

(1.1)

and

$$T_w(q)^* = T_{w^{-1}}(q).$$
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This ∗-algebra is called the (Iwahori) Hecke algebra of \((W, S)\) with parameter \(q\). Here we use a different normalization of the generators than in \([13]\), which coincides with the notation in \([19, 9, 11, 10, 33]\) and \([29]\). The equality \((1.1)\) in particular implies that \(T^{(q)}_{w} = T^{(q)}_{s_{1}} \ldots T^{(q)}_{s_{n}}\) for a reduced expression \(w = s_{1} \ldots s_{n}\) of \(w \in W\). The ∗-algebra \(C_{q}[W]\) can be represented on \(ℓ^{2}(W)\) by bounded operators via

\[
T^{(q)}_{s} \delta_{w} = \begin{cases} 
\delta_{sw}, & \text{if } s \leq w, \\
\delta_{sw} + p_{s}(q)\delta_{w}, & \text{if } s < w,
\end{cases}
\]

where \((\delta_{w})_{w \in W}\) denotes the canonical orthonormal basis of \(ℓ^{2}(W)\). This defines a faithful ∗-representation \(C_{q}[W] \hookrightarrow B(ℓ^{2}(W))\), therefore we will view \(C_{q}[W]\) as a ∗-subalgebra of \(B(ℓ^{2}(W))\).

The norm closure \(C^{∗}_{r,q}(W) := \overline{C_{q}[W]}^{||||}\) is called the (reduced) Hecke C*-algebra and the weak closure \(N_{q}(W) := \overline{C_{q}[W]}^{w.o.}\) is called the Hecke-von Neumann algebra. Note that for \(q_{s} = 1, s \in S\),

\[
C_{q}[W] = C[W], \quad C^{∗}_{r,q}(W) = C^{∗}_{r}(W), \quad N_{q}(W) = L(W),
\]

are the group algebra, reduced group C* algebra and group-von Neumann algebra of \(W\). Further, for every \(q \in \mathbb{R}^{(W,S)}_{>0}\) the vector state \(x \mapsto \langle x\delta_{t}, \delta_{r} \rangle\) restricts to a faithful tracial state \(\tau_{q}\) on \(C^{∗}_{r,q}(W)\) and \(N_{q}(W)\) with \(\tau_{q}(T^{(q)}_{w}) = 0\) for all non-trivial \(w \in W\).

The following statement is well-known. A proof can be found in \([10\) Proposition 4.7].

**Proposition 1.3.** Let \((W, S)\) be a Coxeter system, \(q = (q_{s})_{s \in S} \in \mathbb{R}^{(W,S)}_{>0}\) and \(\epsilon = (\epsilon_{s})_{s \in S} \in \{-1, 1\}^{(W,S)}\). Set \(q' := (q_{s}^{∗})_{s \in S}\). Then \(C^{∗}_{r,q}(W) \cong C^{∗}_{r,q'}(W)\) via \(T^{(q)}_{s} \mapsto \epsilon_{s}T^{(q')}_{s}\).

The following decomposition follows from the universal property of the Hecke algebra and \([10\) Lemma 1.1].

**Lemma 1.4.** Let \((W, S)\) be a Coxeter system which admits a non-trivial decomposition of the form \((W, S) = (W_{T} \times W_{T'}, T \sqcup T')\). Set \(q_{T} := (q_{t})_{t \in T}\) and \(q_{T'} := (q_{t})_{t \in T'}\). Then for every \(q \in \mathbb{R}^{(W,S)}_{>0}\) the corresponding Hecke algebra decomposes as an algebraic tensor product \(C_{q}[W] \cong C_{q_{T}}[W_{T}] \otimes C_{q_{T'}}[W_{T'}]\) via

\[
T^{(q)}_{t} \mapsto \begin{cases} 
T^{(q_{T})}_{t} \otimes 1, & \text{if } t \in T, \\
1 \otimes T^{(q_{T'})}_{t}, & \text{if } t \in T'.
\end{cases}
\]

This induces C*-algebraic and von Neumann-algebraic isomorphisms \(C^{∗}_{r,q}(W) \cong C^{∗}_{r,q_{T}}(W_{T}) \otimes C^{∗}_{r,q_{T'}}(W_{T'})\) and \(N_{q}(W) \cong N_{q_{T}}(W_{T}) \otimes N_{q_{T'}}(W_{T'})\).

Proposition \([13]\) and Lemma \([14]\) allow to restrict in the treatment of the question for the simplicity of Hecke C*-algebras to irreducible Coxeter systems and multi-parameters \(q = (q_{s})_{s \in S} \in \mathbb{R}^{(W,S)}_{>0}\) with \(0 < q_{s} \leq 1\). Since simplicity of C*-algebras is preserved by inductive limits, it further suffices to consider finite rank Coxeter groups, as the following lemma illustrates. It easily follows from \([13\) Lemma 19.2.2].

**Lemma 1.5.** Let \((W, S)\) be a Coxeter system, \(q = (q_{s})_{s \in S} \in \mathbb{R}^{(W,S)}_{>0}\), \(T_{0} \subseteq S\) finite and \(S := \{T \subseteq S \mid T \text{ finite with } T_{0} \subseteq T\}\). For \(T \in S\) let \(q_{T} := (q_{t})_{t \in T}\). Then

\[
\{(C^{∗}_{r,q_{T}}(W_{T}), \phi_{T,T'}) \mid T, T' \in S \text{ with } T \subseteq T'\}
\]
with $\phi_{T,T'}(T^{(qT)}) := T^{(qT)}_t$ for $t \in T$ defines an inductive system with $C^*_{r,q}(W) \cong \lim C^*_{r,qT}(W_T)$.

2. The C*-algebras $\mathcal{D}(W,S)$ and $\mathfrak{A}(W)$

The aim of this section is to recall the construction of the C*-algebras $\mathcal{D}(W,S)$ and $\mathfrak{A}(W)$ associated with a given Coxeter system $(W,S)$ which appears in [29, Section 4]. We will further prove a number of technical statements which will play a role in the later sections.

Let $(W,S)$ be a finite rank Coxeter system and define for $w \in W$, $P_w \in \ell^\infty(W) \subseteq B(\ell^2(W))$ to be the orthogonal projection onto $\text{Span} \{\delta_v \mid v \in W \text{ with } w \leq v\} \subseteq \ell^2(W)$. Note that $P_e = 1$.

Remark 2.1. Recall that $W$ equipped with the weak right Bruhat order defines a complete meet-semilattice. If existent, denote the corresponding join of two elements $v,w \in W$ by $v \vee w$. We then have $P_v P_w = P_{v \vee w}$ for all $v, w \in W$ where we assume that $P_{v \vee w} = 0$ if the join $v \vee w$ does not exist. In particular, the equalities $P_s P_t = 0$ (i.e. $P_s$ and $P_t$ are orthogonal to each other) if $m_{st} = \infty$ and $P_s P_t = P_{st}$ if $m_{st} = 2$ hold (this follows for instance from [13, Lemma 4.3.3]).

Denote the quotient map of $B(\ell^2(W))$ onto $B(\ell^2(W))/K$ by $\pi$ where $K := K(\ell^2(W))$ is the ideal of compact operators in $B(\ell^2(W))$ and write $P_w := \pi(P_w)$ for $w \in W$. It was shown in [29, Proposition 2.6] that the commutative C*-algebra $\mathcal{D}(W,S)$ generated by all projections $P_w$, $w \in W$ identifies with $C((W,S))$ via $P_w \mapsto \chi_{\partial w}$. Similarly, by [29, Lemma 2.3 and Proposition 2.6], $\pi(D(W,S)) \cong C(\partial(W,S))$ via $\tilde{P}_w \mapsto \chi_{\partial w \cap \partial(W,S)}$. Further, let $\mathfrak{A}(W)$ be the $C^*$-subalgebra of $B(\ell^2(W))$ generated by the reduced group C*-algebra $C^*_r(W)$ and $\mathcal{D}(W,S)$. Since for $q = (q_s)_{s \in S} \in \mathbb{R}_{\geq 0}^{(W,S)}$ and $s \in S$ the operator $T^{(q)}_s$ decomposes as $T^{(1)}_s + p_s(q)P_s$, we have an inclusion of the corresponding Hecke C*-algebra $C^*_r(W) \subseteq \mathfrak{A}(W)$. In fact, $\mathfrak{A}(W)$ is the smallest $C^*$-subalgebra of $B(\ell^2(W))$ containing all Hecke C*-algebras of $(W,S)$. It naturally identifies with the reduced crossed product C*-algebra of the action $W \rtimes (W,S)$ via

$$\iota : \mathfrak{A}(W) \cong C((W,S)) \rtimes W, \quad P_w \mapsto \chi_{\partial w} \text{ and } T^{(1)}_w \mapsto \lambda_w,$$

where $\lambda$ denotes the left-regular representation of $W$. In a similar way $\pi(\mathfrak{A}(W))$ identifies with $C(\partial(W,S)) \rtimes W$ via

$$\kappa : \pi(\mathfrak{A}(W)) \cong C(\partial(W,S)) \rtimes W, \quad \tilde{P}_w \mapsto \chi_{\partial w \cap \partial(W,S)} \text{ and } \pi(T^{(1)}_w) \mapsto \lambda_w.$$

These maps are $W$-equivariant with respect to the action of $W$ on $\mathfrak{A}(W)$ defined by $w.x := T^{(1)}_w x T^{(1)}_{w^{-1}}$ for $w \in W$, $x \in \mathfrak{A}(W)$ and the action of $W$ on $\pi(\mathfrak{A}(W))$ defined by $w.x := \pi(T^{(1)}_w) x \pi(T^{(1)}_{w^{-1}})$ for $w \in W$, $x \in \pi(\mathfrak{A}(W))$. Denote the region of convergence of the multivariate growth series $W(z) := \sum_{w \in W} z_w$ by

$$\mathcal{R} := \{z \in \mathbb{C}^{(W,S)} \mid W(z) \text{ converges} \},$$

set

$$\mathcal{R}' := \{(q^*_s)_{s \in S} \mid q \in \mathcal{R} \cap \mathbb{R}_{\geq 0}^{(W,S)}, \epsilon \in \{-1,1\}^{(W,S)}\}$$

and let $\overline{\mathcal{R}'}$ be the closure of $\mathcal{R}'$ in $\mathbb{R}_{\geq 0}^{(W,S)}$. By [29, Corollary 4.4], for $q \in \mathbb{R}_{\geq 0}^{(W,S)} \setminus \mathcal{R}'$ the restriction of $\kappa \circ \pi$ to $C^*_r,q(W)$ factors to an embedding of $C^*_r,q(W)$ into $C(\partial(W,S)) \rtimes W$. We will therefore often view $C^*_r,q(W)$ with $q \in \mathbb{R}_{\geq 0}^{(W,S)} \setminus \mathcal{R}'$ as a $C^*$-subalgebra of $C(\partial(W,S)) \rtimes W$ and of $\pi(\mathfrak{A}(W))$. 

2.1. Elementary properties of the action \( W \curvearrowright \mathcal{D}(W, S) \). Let us proceed with some technical statements which will play a role in the following sections.

**Proposition 2.2.** Let \((W, S)\) be a right-angled Coxeter system and \( w \in W, s \in S \). Then the following equalities hold:

1. \( sP_w = P_w \) if \( w \notin C_W(s) \);
2. \( sP_w = P_{sw} - P_w \) if \( w \in C_W(s) \) and \( s \leq w \);
3. \( sP_w = P_w \) if \( w \in C_W(s) \) and \( s \notin w \).

Here \( C_W(s) := \{ v \in W \mid sv = vs \} \) denotes the centralizer of \( s \) in \( W \).

**Proof.** First observe that by Proposition 1.1 for all \( s \in S \) and \( v, w \in W \) with \( s \leq w, s \notin v \) or \( s \notin w, s \leq v \),

\[
(sP_w)\delta_v = T_s^{(1)}P_w\delta_{sv} = \begin{cases} 
\delta_v, & \text{if } w \leq sv \\
0, & \text{if } w \notin sv
\end{cases} = \begin{cases} 
\delta_v, & \text{if } sw \leq v \\
0, & \text{if } sv \notin v
\end{cases} = P_{sw}\delta_v.
\]

(2.1)

We will cover the remaining cases in the following.

1. Assume that \( w \notin C_W(s) \). If \( s \leq w \) and \( s \leq v \), then \( w \notin sv \) and \( sw \notin v \). Indeed, if we assume that \( w \leq sv \), then \( s \leq sv \) in contradiction to \( s \notin sv \). Further, if we assume that \( sw \leq v \), then there exists \( u \in W \) with \( v = (sw)u \) and \( |v| = |sw| + |u| \). Since \((W, S)\) is right-angled \( s \leq v \) implies that \( s \leq u \) and \( sw \in C_W(s) \). But then \( w \in C_W(s) \) in contradiction to the assumption \( w \notin C_W(s) \). We get that \((sP_w)\delta_v = 0 = P_{sw}\delta_v \).

If \( s \notin w \) and \( s \notin v \), then one obtains in the same way \( w \notin sv \) and \( sw \notin v \) which implies \((sP_w)\delta_v = 0 = P_{sw}\delta_v \). With (2.1) this covers all possible cases. Hence, \( sP_w = P_{sw} \).

2. Assume that \( w \in C_W(s) \) and \( s \leq w \). If \( s \notin v \), then (2.1) implies \((sP_w)\delta_v = P_{sw}\delta_v \) and hence \((sP_w)(1 - P_s) = P_{sw}(1 - P_s) \). If \( s \leq v \), then \( w \notin sv \) implies \((sP_w)\delta_v = 0 \) and hence \((sP_w)P_s = 0 \). Combined this leads to

\[
sP_w = (sP_w)(1 - P_s) + (sP_w)P_s = P_{sw}(1 - P_s) = P_{sw} - P_{sw\vee s} = P_{sw} - P_w.
\]

3. Assume that \( w \in C_W(s) \) and \( s \notin w \). If \( s \leq v \), then \((sP_w)\delta_v = P_{sw}\delta_v \) and hence \( P_{sw}P_s\delta_v = P_{sw}\delta_v \) by (2.1). So consider the case where \( s \notin v \). If \( w \leq v \), then \( v = wu \) for some \( u \in W \) with \(|v| = |w| + |u| \) and \( s \notin u \). Hence, \( sv = wu \) for some \( u \in W \) with \(|sv| = |w| + |u| \) and \( s \leq u \). We get that \( v = s(sv) = w(su) \geq w \).

Together this gives \((sP_w)\delta_v = P_{sw}\delta_v \), i.e. \( sP_w = P_w \) as claimed.

\[\Box\]

**Remark 2.3.** Let \((W, S)\) be a right-angled Coxeter system, \( q \in \mathbb{R}_{>0}^{(W, S)} \) and \( s \in S, w \in W \). Recall that \( T_s(q) = T_s^{(1)} + p_s(q)P_s \) for \( q \in \mathbb{R}_{>0}^{(W, S)} \). In combination with Remark 2.1 the Proposition 2.2 leads to a description of the conjugation of the generating projections in \( \mathcal{D}(W, S) \) with the Hecke operators \( T_s(q) \), \( s \in S \). In particular, for \( s \in S, w \in W \) with \( w \notin C_W(s) \) and \( s \notin w \) the identities

\[
T_s^{(q)}(1 - P_s)T_s^{(q)} = T_s^{(1)}(1 - P_s)T_s^{(1)} = P_s
\]

and

\[
T_s^{(q)}P_wT_s^{(q)} = T_s^{(1)}P_wT_s^{(1)} = P_{sw}
\]

hold.
2.2. **Paths in the Coxeter diagram of right-angled Coxeter groups.** To simplify the statements and proofs of later sections, we introduce the following notion which already implicitly appears in the proof of [29, Theorem 3.20].

**Definition 2.4.** Let \((W, S)\) be a right-angled finite rank Coxeter system. A *path* \(s_1...s_n \in W\) *in the Coxeter diagram of* \((W, S)\) *is a product of generators* \(s_1, ..., s_n \in S\) *with* \(m_{s_is_{i+1}} = \infty\) *for* \(i = 1, ..., n - 1\). *We say that the path is closed if* \(m_{s_is_n} = \infty\ *and that the path covers the whole graph if* \(\{s_1, ..., s_n\} = S\).

**Remark 2.5.** Let \((W, S)\) be a right-angled finite rank Coxeter system. For a closed path \(g := s_1...s_n \in W\) in the Coxeter diagram of \((W, S)\) that covers the whole graph we have that \(|sg| > |g|\) for every \(s \in S \setminus \{s_1\}\) and \(C_W(g) = \{g^i \mid i \in \mathbb{Z}\}\). In particular, \(|g^n| = |n||g|\) for every \(n \in \mathbb{Z}\). These facts have been crucial in the proof of [29, Theorem 3.20].

In the single-parameter case the following lemma appears in [9, Lemma 2.7]. The proof presented there translates verbatim to the multi-parameter case. Therefore we omit it here.

**Lemma 2.6 ([9, Lemma 2.7]).** Let \((W, S)\) be a right-angled Coxeter system. Denote the set of *subsets of* \(W\) *whose elements pairwise commute (including the empty set) by* \(Clig\) *and write* \(P_\Gamma := \prod_{s \in \Gamma} P_s\) *for* \(\Gamma \in Clig\). *Further, for* \(w \in W\) *let* \(A_w\) *be the set of triples* \((w', \Gamma, w'')\) *with* \(w', w'' \in W\) *and* \(\Gamma \in Clig\) *such that* \(w = w'(\prod_{s \in \Gamma} s)w''\), \(|w| = |w'| + |\prod_{s \in \Gamma} s| + |w''|\) *and* \(|w'| > |w|\) *for all* \(t \in S\) *with* \(m_{st} = 2\) *for all* \(s \in S\). *Then the operator* \(T^{(q)}_{w}\) *decomposes as*

\[
T^{(q)}_{w} = \sum_{(w', \Gamma, w'') \in A_w} \left(\prod_{s \in \Gamma} p_s(q)\right) T^{(1)}_{w'} P_1 T^{(1)}_{w''}.
\]

**Corollary 2.7.** Let \((W, S)\) be a right-angled Coxeter system, \(q = (q_s)_{s \in S} \in \mathbb{R}_{\geq 0}^{(W,S)}\), \(l \in \mathbb{N}\) and let \(g := s_1...s_n \in W\) be a closed path in the Coxeter diagram of \((W, S)\). Then there exists an operator \(x \in A(W)\) such that \(T^{(q)}_{g^l}\) decomposes as \(T^{(q)}_{g^l} = T^{(1)}_{g^l} + P_{s_1} x\).

**Proof.** Write \(t_1...t_m\) for the reduced expression \(g^l = (s_1...s_n)(s_1...s_n)...(s_1...s_n)\) of \(g^l\) where \(m = nl\). By Lemma 2.6 the operator \(T^{(q)}_{g^l}\) decomposes as

\[
T^{(q)}_{g^l} = T^{(1)}_{g^l} + \sum_{i=1}^{m} p_{t_i}(q) T^{(1)}_{t_1...t_{i-1}} P_{t_i} T^{(1)}_{t_{i+1}...t_m},
\]

where the first summand corresponds to the triple \((e, [], g^l)\) \(A^l_{g^l}\) and the other summands correspond to triples \((t_1...t_{i-1}, t_i, t_{i+1}...t_m)\) \(A^l_{g^l}\) with \(i = 1, ..., m\) (since \(g\) is a closed path in the Coxeter diagram of \((W, S)\) all triples in \(A^l_{g^l}\) are of these forms). Using the description in Proposition 2.2 we get that

\[
T^{(q)}_{g^l} = T^{(1)}_{g^l} + \sum_{i=1}^{m} p_{t_i}(q) T^{(1)}_{t_1...t_{i-2}} P_{t_{i-1}} T^{(1)}_{t_{i-1}t_{i+1}...t_m} = ... = T^{(1)}_{g^l} + \sum_{i=1}^{m} p_{t_i}(q) P_{t_1...t_{i-1}} T^{(1)}_{t_{i+1}...t_m},
\]

so the claim follows by setting \(x := \sum_{i=1}^{m} p_{t_i}(q) P_{t_1...t_{i-1}} T^{(1)}_{t_{i+1}...t_m}\). 

**Lemma 2.8.** Let \((W, S)\) be an irreducible right-angled, finite rank Coxeter system, let \(g := s_1...s_n \in W\) be a path in the Coxeter diagram of \((W, S)\) that covers the whole graph and let \(q = (q_s)_{s \in S} \in \mathbb{R}_{>0}^{(W,S)}\). Then the series \(\sum_{w \in W} q_w\) converges if and only if the series \(\sum_{w \in W: g \leq w^{-1}} q_w\) converges.
where the fifth equality follows from the fact that partial sums of $g$ since $2.3$. States on $D$.

Proof. Since all summands of the series are positive it is clear that the convergence of $\sum_{w \in W} q_w$ implies the convergence of $\sum_{w \in W: g \leq w^{-1}} q_w$. So assume that $\sum_{w \in W: g \leq w^{-1}} q_w$ converges. For every $i, j \in \mathbb{N}$ with $i < j$ we have that

\[
\left| \sum_{w \in W: |w| \leq i} q_w - \sum_{w \in W: |w| \leq j} q_w \right| = \sum_{w \in W: i < |w| \leq j} q_w
\]

\[
= \sum_{w \in W: i < |w| \leq j, s_n \leq w^{-1}} q_w + \sum_{w \in W: i < |w| \leq j, s_n \notin w^{-1}} q_w
\]

\[
\leq (1 + q_{s_n}) \sum_{w \in W: i-1 < |w| \leq j, s_n \notin w^{-1}} q_w
\]

\[
= \frac{1 + q_{s_n}}{q_{g^{-1}}} \sum_{w \in W: i-1 < |w| \leq j, s_n \notin w^{-1}} q_{g^{-1}w}
\]

\[
= \frac{1 + q_{s_n}}{q_g} \sum_{w \in W: i-1 < |g^{-1}w^{-1}| \leq j, g \leq w^{-1}} q_w
\]

\[
= \frac{1 + q_{s_n}}{q_g} \sum_{w \in W: i-1+n < |w| \leq j+n, g \leq w^{-1}} q_w,
\]

where the fifth equality follows from the fact that

\[
\{ w g^{-1} \mid w \in W \text{ with } s_n \notin w^{-1} \} = \{ w \in W \mid g \leq w^{-1} \}
\]

since $g = s_1 \ldots s_n$ is a path in the Coxeter diagram of $(W, S)$. That implies that the sequence of partial sums of $\sum_{w \in W} q_w$ is a Cauchy sequence and hence that the series converges. \qed

2.3. States on $\mathfrak{A}(W)$ and $\pi(\mathfrak{A}(W))$. Our solution of the simplicity question for right-angled Hecke C*-algebras is inspired by Haagerup’s approach to the unique trace property of group C*-algebras in [20]. The translation of the techniques into the deformed setting requires the study of states on the C*-algebras $\mathfrak{A}(W)$ and $\pi(\mathfrak{A}(W))$.

Lemma 2.9. Let $(W, S)$ be a right-angled, finite rank Coxeter system. For every $u \in W$ and $0 < q < 1$ the operator $Q^u_q$ on $\ell^2(W)$ defined by

\[
Q^u_q := \sum_{l=|u|}^{\infty} \sum_{w \in W: |w| = l, u \leq w^{-1}} q^l P_w
\]

exists (where the limit is taken with respect to the operator norm) and is contained in the C*-algebra $D(W, S) \subseteq \mathcal{B}(\ell^2(W))$.

Proof. It suffices to show that the sequence $(Q^u_{q, i})_{i \geq |u|}$ with

\[
Q^u_{q, i} := \sum_{l=|u|}^{i} \sum_{w \in W: |w| = l, u \leq w^{-1}} q^l P_w \in D(W, S)
\]

exists (where the limit is taken with respect to the operator norm) and is contained in the C*-algebra $D(W, S) \subseteq \mathcal{B}(\ell^2(W))$. \qed
is a Cauchy sequence. For \( v \in W \) and \( l \in \mathbb{N} \) set \( \kappa_l(v) := \# \{ w \in W \mid w \leq v \text{ and } |w| = l \} \). It has been shown in [9, Lemma 4.4] that \( \kappa_l(v) \leq C l^{#S-2} \) for some constant \( C > 0 \). Using this in the third line of the following inequalities, we get that for \( i < j \)

\[
\| (Q_{q,j} - Q_{q,i}) \xi \|_2 = \left\| \sum_{v \in W} \left( \sum_{j=i+1}^l \sum_{w \in W : |w| = l, w \leq v, u \leq w^{-1}} q^j \right)^{\frac{1}{2}} \right\|_2 \xi(v) \delta_v \leq \left\| \sum_{v \in W} \left( \sum_{j=i+1}^l q^j \kappa_v(l) \right)^{\frac{1}{2}} \right\|_2 \xi(v)^2 \leq C \left( \sum_{j=i+1}^l q^j l^{#S-2} \right)^{\frac{1}{2}} \| \xi \|_2.
\]

For \( 0 < q < 1 \) the series \( \sum_{l=1}^{\infty} q^l l^{#S-2} \) converges. This implies the claim. \( \square \)

The following proposition will play a crucial role in Subsections 3 and 4.3. Recall that \( \bar{R'} \) is the closure of \( R' := \{ (q_s^s)_{s \in S} \mid q \in R \cap \mathbb{R}_{>0}^{(W,S)}, \epsilon \in \{-1,1\}^{(W,S)} \} \) in \( \mathbb{R}_{>0}^{(W,S)} \), where \( R \) denotes the region of convergence of the growth series of \( (W,S) \).

**Proposition 2.10.** Let \( (W,S) \) be a right-angled, irreducible, finite rank Coxeter system, \( q = (q_s)_{s \in S} \in \mathbb{R}_{>0}^{(W,S)} \setminus \bar{R'} \) and let \( g := s_1 \cdots s_n \in W \) be a path in the Coxeter diagram of \( (W,S) \) that covers the whole graph. Then, for every state \( \phi \) on \( \mathcal{A}(W) \) there exists a sequence \( (w_i)_{i \in \mathbb{N}} \subseteq W \) of group elements with increasing word length such that \( g \leq w_i^{-1} \) for all \( i \in \mathbb{N} \) and \( q_{w_i}^{-1} \phi(P_{w_i}) \to 0 \).

The same statement holds, if one replaces \( \mathcal{A}(W) \) by \( \pi(\mathcal{A}(W)) \) and \( P_{w_i} \) by \( \hat{P}_{w_i} \).

**Proof.** The set \( \mathbb{R}_{>0}^{(W,S)} \setminus \bar{R'} \) is open in \( \mathbb{R}_{>0}^{(W,S)} \), so there exist positive real numbers \( q', \lambda \in (0,1) \) such that \( q' q := (q' q_s)_{s \in S} \in \mathbb{R}_{>0}^{(W,S)} \setminus \bar{R'} \) and \( \lambda q' q := (\lambda q' q_s)_{s \in S} \in \mathbb{R}_{>0}^{(W,S)} \setminus \bar{R'} \). In particular, Lemma 2.8 implies that the series \( \sum_{|w| \leq 1} (q' q)_w \) diverges. By the root test criterium for convergence,

\[
\limsup_l \left( \sum_{w \in W : |w| = l, g \leq w^{-1}} (q' q)_w \right)^{1/l} \geq 1
\]

and hence

\[
\limsup_l \left( \sum_{w \in W : |w| = l, g \leq w^{-1}} (q' q)_w \right)^{1/l} > 1.
\]

One can thus find a strictly increasing sequence \( (l_i)_{i \in \mathbb{N}} \subseteq \mathbb{N} \) and a constant \( C > 1 \) such that for all \( i \in \mathbb{N} \),

\[
\left( \sum_{w \in W : |w| = l_i, g \leq w^{-1}} (q' q)_w \right)^{1/l_i} \geq C. \tag{2.2}
\]

For \( w \in W \) define the set

\[
C_w := \{ v \in W \mid g \leq v^{-1} \text{ and } z_v = z_w \text{ for all } z = (z_s)_{s \in S} \in \mathbb{C}^{(W,S)} \}
\]
and note that the elements in \( C_w \) all have the same length. Choose for every \( i \in \mathbb{N} \) an element \( w_i \in W \) with \( \# C_{w_i}(q'q)_{w_i} = \max_{|w|=l_i, g \leq w^{-1}} \# C_w(q'q)_w \) that has length \( l_i \) and satisfies \( g \leq w_i^{-1} \). Since by the definition of \( C_w \), the equality \( \# C_v(q'q)_v = \# C_w(q'q)_w \) holds for all \( v \in C_w \), this element can be chosen in such a way that \( \phi(P_{w_i}) \leq \phi(P_v) \) for all \( v \in C_{w_i} \). Now, by picking a suitable subset \( M \subseteq W \) of elements \( w \in W \) with length \( l_i \) and \( g \leq w^{-1} \), the sum \( \sum_{w \in W: |w|=l_i, g \leq w^{-1}} (q'q)_w \) can be written as \( \sum_{w \in M} \# C_w(q'q)_w \). By the choice of \( w_i \) we hence have

\[
\sum_{w \in W: |w|=l_i, g \leq w^{-1}} (q'q)_w \leq (l_i + 1)^{\# S} \# C_{w_i}(q'q)_{w_i},
\]

which implies in combination with (2.2) that for all \( i \in \mathbb{N} \),

\[
\# C_{w_i}(q'q)^{l_i} q_{w_i} \geq \frac{C^{l_i}}{(l_i + 1)^{\# S}}.
\]

It follows from Lemma 2.9 that the series \( \sum_{w \in W: g \leq w^{-1}} (q')^{|w|} \phi(P_w) \) converges. By the same argument as above we hence have that

\[
\limsup_l \left( \sum_{w \in W: |w|=l_i, g \leq w^{-1}} (q')^{l_i} \phi(P_w) \right)^{1/l_i} \leq L
\]

for all \( i \in \mathbb{N} \) where \( 0 < L < 1 \). But then, by the choice of \( w_i \),

\[
\# C_{w_i}(q'q)^{l_i} \phi(P_{w_i}) \leq (q')^{l_i} \sum_{w \in C_{w_i}} \phi(P_w) \leq \sum_{w \in W: |w|=l_i, g \leq w^{-1}} (q')^{l_i} \phi(P_w) \leq L^{l_i}
\]

and thus with (2.3)

\[
0 \leq q_{w_i}^{-1} \phi(P_{w_i}) < \frac{L^{l_i}}{\# C_{w_i}(q'q)^{l_i} q_{w_i}} \leq (l_i + 1)^{\# S} \left( \frac{L}{C} \right)^{l_i} \to 0.
\]

This implies the first part of the statement. The second part is an immediate consequence, since \( \pi(\mathfrak{A}(W)) \) is a quotient of \( \mathfrak{A}(W) \). That finishes the proof. \( \square \)

**Remark 2.11.** The proof of Proposition 2.10 significantly simplifies in the case of single-parameters \( q \). Indeed, if we follow the notation of Proposition 2.10 and assume that \( q_s = q_t \) for all \( s, t \in S \), Lemma 2.9 implies that for \( i \in \mathbb{N} \) and \( 0 < q' < 1 \),

\[
\sum_{w \in W: |w|=i, g \leq w^{-1}} (q')^i \phi(P_w) \leq \phi(Q_{q'}^g).
\]

One can thus find an element \( w_i \) of length \( i \) with \( g \leq w_i^{-1} \) such that \( \phi(P_{w_i}) \leq (\# L_i^{\# S} (q')^i)^{-1} \phi(Q_{q'}^g) \) where \( L_i^{\# S} := \{ w \in W \mid |w| = i, g \leq w_i^{-1} \} \). We get that

\[
q_{w_i}^{-1} \phi(P_{w_i}) \leq \frac{\phi(Q_{q'}^g)}{\# L_i^{\# S} (q'q)_{w_i}}.
\]
The Cauchy-Hadamard formula (for radii of convergence of power series) implies that for increasing $i$, if $q'$ is close enough to 1, the expression on the right approaches 0.

3. Central projections in Hecke $C^*$-algebras

In [33] Raum and Skalski, generalizing the single-parameter results by Garnercarek [19], proved that for a right-angled, irreducible, finite rank Coxeter system $(W, S)$ with at least three generators and $q = (q_s)_{s \in S} \in \mathbb{R}_{>0}^{(W, S)}$ the corresponding Hecke-von Neumann algebra $N_q(W)$ decomposes as $N_q(W) \cong M \oplus \bigoplus_{\epsilon \in \{-1, 1\}^{(W, S)}} |w|, r \in \mathcal{R}_e$ $C$ where $q_\epsilon := (\epsilon_s q_s^\epsilon)_{s \in S}, |q_\epsilon| := (q_s^\epsilon)_{s \in S}$ and where $M$ is a factor. In particular, $N_q(W)$ is a factor if and only if $q \in \mathbb{R}_{>0}^{(W, S)} \setminus \mathcal{R}'$. It is a natural question whether for $q \in \mathcal{R}'$ the central projections in $N_q(W)$ are already contained in the corresponding Hecke $C^*$-algebra $C_{\epsilon q}^*(W)$. We will prove this by using a Haagerup-type inequality from [10]. We will further characterize the characters (i.e. unital, linear, multiplicative functionals) of right-angled Hecke $C^*$-algebras.

3.1. The center of right-angled Hecke $C^*$-algebras.

**Theorem 3.1** ([10] Theorem 3.4). Let $(W, S)$ be a right-angled, finite rank Coxeter system and let $q = (q_s)_{s \in S} \in \mathbb{R}_{>0}^{(W, S)}$. Then there exists a constant $C > 0$ such that for every $l \in \mathbb{N}_{\geq 1}$ and $x \in C_{\epsilon q}^*(W)$ of the form $x := \sum_{w \in W : |w| = l} c_w T_w^{(q)}$ with coefficients $c_w \in \mathbb{C}$ we have $\|x\| \leq C l \|x\|_2$.

We will further need the following easy lemma.

**Lemma 3.2.** Let $(W, S)$ be a finite rank Coxeter system. Then the intersection $\mathcal{R} \cap \mathbb{R}_{>0}^{(W, S)}$ of the region of convergence $\mathcal{R}$ of the growth series $W(z) = \sum_{w \in W} z^w$ with $\mathbb{R}_{>0}^{(W, S)}$ is open in $\mathbb{R}_{>0}^{(W, S)}$.

**Proof.** Assume that the set $\mathcal{R} \cap \mathbb{R}_{>0}^{(W, S)}$ is not open in $\mathbb{R}_{>0}^{(W, S)}$ and let $q \in \mathcal{R} \cap \mathbb{R}_{>0}^{(W, S)}$ be a point on its boundary. Since $\sum_{w \in W} q_w$ converges, the power series $f(z) := \sum_{w \in W} q_w z^{|w|}$ absolutely converges for all $z \in \mathbb{C}$ with $|z| \leq 1$. But the radius of convergence of $f$ coincides with the distance of the origin to the closest pole of $f$, hence there exists $\lambda > 1$ such that $\sum_{w \in W} q_w z^{|w|}$ absolutely converges for all $z \in \mathbb{C}$ with $|z| < \lambda$. This implies that $(2^{-1}(1 + \lambda) q_s)_{s \in S} \in \mathcal{R} \cap \mathbb{R}_{>0}^{(W, S)}$ which contradicts the choice of $q$. \hfill $\square$

**Proposition 3.3.** Let $(W, S)$ be a right-angled, finite rank Coxeter system, let $q = (q_s)_{s \in S} \in \mathbb{R}_{>0}^{(W, S)}$ with $0 < q_s \leq 1$ for all $s \in S$ be a multi-parameter and let $W(z) = \sum_{w \in W} z^w$ be the growth series of $(W, S)$. Further, let $\epsilon \in \{-1, 1\}^{(W, S)}$, $q_\epsilon := (\epsilon_s q_s^\epsilon)_{s \in S}$ and assume that $|q_\epsilon| := (q_s^\epsilon)_{s \in S} \in \mathcal{R}$. Then the operator

$$E_{q, \epsilon} = \frac{1}{W(|q_\epsilon|)} \sum_{i=0}^{\infty} \sum_{w \in W : |w| = i} (\sqrt{q})_{w, \epsilon} T_w^{(q)}$$

exists (where the limit is taken with respect to the operator norm), it is a projection in $C_{q, \epsilon}^*(W)$ and it satisfies $T_w^{(q)} E_{q, \epsilon} E_{q, \epsilon}^* T_w^{(q)} = c_{\epsilon s} q_s^\epsilon E_{q, \epsilon}$ for all $s \in S$. For distinct $\epsilon, \epsilon' \in \{-1, 1\}^{(W, S)}$ with $|q_\epsilon|, |q_{\epsilon'}| \in \mathcal{R}$ the projections $E_{q, \epsilon}$ and $E_{q, \epsilon'}$ are orthogonal to each other.
Proof. By assumption $|q_s| \in \mathbb{R}$, so Lemma 3.2 implies that there exists $\lambda > 1$ such that still $|\lambda q_s| := (\lambda q_s^s)_{s \in S} \in \mathbb{R}$. Using the root test criterion for convergence,

$$\limsup_{l} \left( \sum_{w \in W : |w| = l} \lambda^l |q_{w,e}| \right)^{1/l} \leq 1$$

and hence

$$\limsup_{l} \left( \sum_{w \in W : |w| = l} |q_{w,e}| \right)^{1/l} < 1.$$

One can therefore find $l_0 \in \mathbb{N}$ and $0 < L < 1$ such that for all $l \geq l_0$,

$$\sum_{w \in W : |w| = l} |q_{w,e}| < L^l. \quad (3.1)$$

Now set $E_{q,e}^{(i)} := (W(\{q_s\})^{-1} \sum_{l=0}^{i} \sum_{w : |w| = l} \left( \sqrt{l} \right)_{w,e} T_{w}^q$. For $i,j \in \mathbb{N}$ with $i < j$ and $i \geq l_0$ we have by Theorem 3.1 and the inequality (3.1) that

$$\left\| E_{q,e}^{(j)} - E_{q,e}^{(i)} \right\| \leq \frac{1}{W(\{q_s\})} \sum_{l=i+1}^{j} \left\| \sum_{w : |w| = l} \left( \sqrt{l} \right)_{w,e} T_{w}^q \right\|$$

$$\leq \frac{1}{W(\{q_s\})} \sum_{l=i+1}^{j} C_1 \sqrt{\sum_{w : |w| = l} |q_{w,e}|}$$

$$< \frac{1}{W(\{q_s\})} \sum_{l=i+1}^{j} C_1 L^l \frac{l}{2}.$$

The series $\sum_{l=0}^{\infty} L^l \frac{l}{2}$ converges, so $(E_{q,e}^{(i)})_{i \in \mathbb{N}} \subseteq C^*_r(W)$ converges to $E_{q,e} \in C^*_r(W)$. The remaining statements follow from short calculations (compare also with [13, Lemma 19.2.5], [19, Theorem 5.3] and [33, Proposition 2.2]) that we omit here. \qed

Remark 3.4. In [33] Raum and Skalski introduced the notion of Hecke eigenvectors (for the parameter $q$). These are non-zero elements $\eta \in L^2(W)$ satisfying $T^q_s \eta = \eta$ for all $s \in S$. The central projections considered in Proposition 3.3 are exactly the orthogonal projections onto the Hecke eigenspaces. Note that they are always of finite rank.

The following corollary follows from [33, Theorem A] and Proposition 1.3.

**Corollary 3.5.** Let $(W,S)$ be a right-angled, finite rank Coxeter system with $\#S \geq 3$ and let $q \in \mathbb{R}^{(W,S)}$. Then the center of the Hecke $C^*$-algebra $C^*_r(W)$ coincides with the center of the Hecke-von Neumann algebra $N_q(W)$.

One other immediate consequence is that right-angled Hecke $C^*$-algebras admit a decomposition which is analogous to the one of their von Neumann-algebraic counterparts.

**Corollary 3.6.** Let $(W,S)$ be a right-angled, finite rank Coxeter system with $\#S \geq 3$ and let $q = (q_s)_{s \in S} \in \mathbb{R}^{(W,S)}$. Then the corresponding Hecke $C^*$-algebra $C^*_r,q(W)$ decomposes as

$$C^*_r,q(W) \cong \pi(C^*_r(W)) \oplus \bigoplus_{e \in \{-1,1\}^{(W,S)}} \mathbb{C},$$

where $\pi$ is the canonical representation.
where \(\pi\) denotes the quotient map of \(B(\ell^2(W))\) onto \(B(\ell^2(W))/K(\ell^2(W))\).

**Proof.** By Proposition 2.2 (as well as Remark 2.3), one has

\[
A = C^*_r,q(W) \cong \bigoplus_{\epsilon \in \{-1,1\}^{(W,S)}} (1 - E_{q,\epsilon}) \subseteq B(\ell^2(W)).
\]

By [33] Theorem A the von Neumann algebra \(A' \subseteq B(\ell^2(W))\) is a factor, necessarily of type II\(_1\), so \(A\) contains no compact operators. This implies that \(A \cong \pi(C^*_r,q(W))\) from which the claim follows. \(\square\)

### 3.2. Characters on Hecke C*-algebras

The operators appearing in Proposition 3.3 are projections onto one-dimensional subspaces of \(\ell^2(W)\) and thus induce characters on the right-angled Hecke C*-algebras. Let us prove that all characters on right-angled Hecke C*-algebras arise in such a manner.

**Proposition 3.7.** Let \((W, S)\) be a right-angled, irreducible, finite rank Coxeter system and \(q = (q_s)_{s \in S} \in \mathbb{R}_{> 0}^{(W,S)}\). Then the set of characters of the corresponding Hecke C*-algebra \(C^*_r,q(W)\) is given by

\[
\{ \chi_{q_s} \mid \epsilon \in \{-1,1\}^{(W,S)} \text{ with } |q_s| \in \overline{R'} \},
\]

where \(|q_s| := (q_{s,\epsilon})_{s \in S} \in \mathbb{R}_{> 0}^{(W,S)}\) and \(\chi_{q_s}\) satisfies \(\chi_{q_s}(T^{(q)}_{s,\epsilon}) := e_s q_{s,\epsilon}^{-\frac{1}{2}}\) for all \(s \in S\).

**Proof.** By Proposition 1.3 we can assume that \(0 < q_s \leq 1\) for all \(s \in S\). Arguing exactly as in the proof of [10] Lemma 5.3 it follows from Proposition 3.3 (or also from [33] Proposition 2.2) that for every \(\epsilon \in \{-1,1\}^{(W,S)}\) with \(|q_s| \in \overline{R'}\) the character \(\chi_{q_s}\) exists. Conversely, let \(\chi\) be a state on \(\mathcal{A}(W)\) which restricts to a character on \(C^*_r,q(W)\). For \(s \in S\) the Hecke relation

\[
(T^{(q)}_s)^2 = 1 + p_s(q)T^{(q)}_s \implies (\chi(T^{(q)}_s))^2 - p_s(q)\chi(T^{(q)}_s) = 1 \text{ and hence } \chi(T^{(q)}_s) \in \{ q_{s,\epsilon}^{-\frac{1}{2}}, -q_{s,\epsilon}^{-\frac{1}{2}} \}.
\]

One can thus find \(\epsilon \in \{-1,1\}^{(W,S)}\) with \(\chi = \chi_{q_s}\). Now assume that \(|q_s| \notin \overline{R'}\), fix \(s \in S\) and choose a path \(g := s_1 \ldots s_n \in W\) in the Coxeter diagram of \((W, S)\) that covers the whole graph and for which \(m_{s_1s_1} = \infty\). By Proposition 2.10 there exists a sequence \((w_i)_{i \in \mathbb{N}} \subseteq W\) of increasing word length with \(g \leq w_i^{-1}\) for all \(i \in \mathbb{N}\) and \(|qw_i^{-1}\epsilon\chi(P_{w_i})| \to 0\). We further have that \(P_{w_i} w_i \leq P_{w_i}\), so \(\chi(P_{w_i} w_i) \leq \chi(P_{w_i})\). Using that \(T^{(q)}_{w_i}\) and \(T^{(q)}_{w_i^{-1}}\) lie in the multiplicative domain of \(\chi\) (see for instance [7] Proposition 1.5.7) one has

\[
\left| \chi(T^{(q)}_s) \right| = \left| q_{w_i,\epsilon}^{-1} \chi(T^{(q)}_{w_i^{-1}}(1 - P_s)T^{(q)}_{w_i^{-1}}) + \chi(T^{(q)}_{w_i} P_s T^{(q)}_{w_i^{-1}}) \right| \leq \left| q_{w_i,\epsilon}^{-1} \chi(T^{(q)}_{w_i^{-1}}(1 - P_s)T^{(q)}_{w_i^{-1}}) \right| + \left| q_{w_i,\epsilon}^{-1} \chi(T^{(q)}_{w_i} P_s T^{(q)}_{w_i^{-1}}) \right|.
\]

By Proposition 2.2 (as well as Remark 2.3),

\[
(1 - P_s)T^{(q)}_{w_i^{-1}} = T^{(1)}_{w_i^{-1}} P_{w_i} \quad \text{and} \quad T^{(q)}_{w_i} P_s = P_{w_i} T^{(1)}_{w_i},
\]

so

\[
\left| \chi(T^{(q)}_s) \right| \leq \left| q_{w_i,\epsilon}^{-1} \chi(T^{(q)}_{w_i^{-1}} P_{w_i}) \right| + \left| q_{w_i,\epsilon}^{-1} \chi(P_{w_i} T^{(1)}_{w_i} T^{(q)}_{w_i^{-1}}) \right| = \left| q_{w_i,\epsilon}^{-1/2} \chi(T^{(1)}_{w_i^{-1}} P_{w_i}) \right| + \left| q_{w_i,\epsilon}^{-1/2} q_{w_i,\epsilon}^{-1} \chi(P_{w_i} T^{(1)}_{w_i}) \right|.
\]
The Cauchy-Schwarz inequality then implies
\[
|\chi(T_s^{(q)})| \leq (1 + q_s^{-1/2})\sqrt{|q_{w,s}^{-1}\chi(P_{w,s})|} \to 0.
\]
This contradicts $\chi(T_s^{(q)}) \in \{q_s^0, -q_s^{-1}\}$. \hfill $\square$

4. Simplicity of right-angled Hecke $C^*$-algebras

In this last section we study the simplicity of right-angled Hecke $C^*$-algebras (recall that a $C^*$-algebra is simple if it contains no non-trivial, two-sided, closed ideal). Our approach is inspired by \cite{20}. It requires the following lemma which immediately follows from \cite[Lemma 4.2]{29}.

**Lemma 4.1.** Let $(W,S)$ be a Coxeter system, $q = (q_s)_{s \in S} \in \mathbb{R}_{\geq 0}^{(W,S)}$ and $w \in W$. Let further $w = s_1...s_n$ with $s_1, ..., s_n \in S$ be a reduced expression for $w$. Then,
\[
\prod_{i=1}^{n} \min \{q_s^{\pm 1}\} \leq (T_w^{(q)})^*T_w^{(q)} \leq \prod_{i=1}^{n} \max \{q_s^{\pm 1}\}.
\]

Recall that for a finite rank Coxeter system $(W,S)$ and $q \in \mathbb{R}_{\geq 0}^{(W,S)} \setminus R'$ the Hecke $C^*$-algebra $C_{r,q}^*(W)$ can be viewed as a $C^*$-subalgebra of $\pi(\mathfrak{A}(W))$ (see Section 2). We will use this observation frequently.

**Proposition 4.2.** Let $(W,S)$ be a right-angled, irreducible, finite rank Coxeter system, $q = (q_s)_{s \in S} \in \mathbb{R}_{\geq 0}^{(W,S)} \setminus R'$ with $0 < q_s \leq 1$ for all $s \in S$ and let $I \neq C_{r,q}^*(W)$ be an ideal of $C_{r,q}^*(W)$ where we view $C_{r,q}^*(W)$ as a $C^*$-subalgebra of $\pi(\mathfrak{A}(W))$. Then, for every two elements $s,t \in S$ with $m_{st} = \infty$ there exists a state $\phi$ on $\pi(\mathfrak{A}(W))$ that vanishes on $I$ and which satisfies $\phi(P_s) = 1$, $\phi(P_t) = 0$.

**Proof.** Choose a state on $C_{r,q}^*(W)$ that vanishes on $I$. We can extend it to a state $\psi$ on $\pi(\mathfrak{A}(W))$. Further let $g := s_1...s_n \in W$ with $s_1 := s$, $s_2 := t$ be a path in the Coxeter diagram of $(W,S)$ that covers the whole graph and let $(w_i)_{i \in \mathbb{N}} \subseteq W$ be a sequence as in Proposition 2.11 i.e. the $w_i$ have increasing word length, $g \leq w_i^{-1}$ for all $i \in \mathbb{N}$ and $q_{w_i}^{-1}\psi(\tilde{P}_{w_i}) \to 0$. Note that $\psi(T_w^{(q)}T_{w_i}^{-1}) = 0$ is not possible since then Lemma 4.1 and the Cauchy-Schwarz inequality would imply
\[
0 = \psi(T_w^{(q)}T_{w_i}^{-1}) \geq q_{w,s}^{-1}\psi((T_w^{(q)})^2) \geq q_{w,s}^{-1}|\psi(T_s^{(q)})|^2
\]
and thus $\psi((T_s^{(q)})^2) = \psi(T_s^{(q)}) = 0$. This contradicts the identity $(T_s^{(q)})^2 = 1 + p_s(q)T_s^{(q)}$. With Proposition 2.12 (as well as Remark 2.3 and Lemma 4.1) we get that for $i \in \mathbb{N}$,
\[
\left|\frac{\psi(T_w^{(q)}\tilde{P}_i T_w^{(q)\frac{1}{w_i}}) - 1}{\psi(T_w^{(q)}T_{w_i}^{-1})} \right| = \frac{|\psi(T_w^{(q)}(\tilde{P}_i - 1)T_{w_i}^{(q)})|}{\psi(T_w^{(q)}T_{w_i}^{-1})} \leq q_{w_i}^{-1}\psi(\tilde{P}_{w_i}) \to 0.
\]
The weak-* compactness of the state space $\mathcal{S}(\pi(\mathfrak{A}(W)))$ implies that we can find a subsequence of
\[
\left(\psi(T_w^{(q)}T_{w_i}^{(q)\frac{1}{w_i}})\right)_{i \in \mathbb{N}} \subseteq \mathcal{S}(\pi(\mathfrak{A}(W)))
\]
that weak-* converges to a state $\phi$. By construction, this state vanishes on the ideal $I$, we have $\phi(\tilde{P}_t) = 1$ and hence also $\phi(\tilde{P}_t) = 0$ since $0 \leq \tilde{P}_t \leq 1 - \tilde{P}_s$. \hfill $\square$
Recall that the inner action of the group $W$ on $\pi(\mathfrak{A}(W))$ defined by $w.x := T_w^{(1)}xT_{w^{-1}}^{(1)}$ for $w \in W$, $x \in \pi(\mathfrak{A}(W))$ induces an action of $W$ on the state space of $\pi(\mathfrak{A}(W))$ via $(w.\phi)(x) := \phi(T_w^{(1)}xT_{w^{-1}}^{(1)})$ for $\phi \in \mathcal{S}(\pi(\mathfrak{A}(W)))$, $w \in W$ and $x \in \pi(\mathfrak{A}(W))$.

We are now ready to characterize the simplicity of right-angled Hecke $C^*$-algebras.

**Theorem 4.3.** Let $(W,S)$ be an irreducible, right-angled, finite rank Coxeter system and let $q = (q_s)_{s \in S} \in \mathbb{R}^{(W,S)}_{>0}$ be a multi-parameter. Then the Hecke $C^*$-algebra $C_{r,q}(W)$ is simple if and only if $q \in \mathbb{R}^{(W,S)}$.

**Proof.** By Proposition 3.7 the Hecke $C^*$-algebra $C_{r,q}(W)$ is not simple for $q \in \overline{\mathbb{R}^+}$. For the treatment of the case where $q \in \mathbb{R}^{(W,S)}_{>0} \setminus \overline{\mathbb{R}^+}$ by Proposition 1.3 it suffices to consider multi-parameters with $0 < q_s \leq 1$ for all $s \in S$. View $C_{r,q}(W)$ as a $C^*$-subalgebra of $\pi(\mathfrak{A}(W))$ and assume that $I \neq C_{r,q}(W)$ is an ideal in $C_{r,q}(W)$. Further choose a closed path $g := s_1...s_n$ in the Coxeter diagram of $(W,S)$ that covers the whole graph. Proposition 4.2 implies that we can find a state $\phi$ on $\pi(\mathfrak{A}(W))$ that vanishes on $I$ and for which $\phi(P_{s_1}) = 1$, $\phi(P_{s_n}) = 0$. In particular the projections $P_{s_1}, P_{s_n}$ are contained in the multiplicative domain of $\phi$ (see for instance [7 Proposition 1.5.7]).

By the identification $\pi(D(W,S)) \cong C(\partial(W,S))$ and the equality $\phi(P_{s_1}) = 1$ the restriction of $\phi$ to $\pi(D(W,S))$ corresponds to a probability measure $\mu$ on the boundary $\partial(W,S)$ whose support is contained in the set of all $z \in \partial(W,S)$ with $s_1 \leq z$. The sequence $(g^{-k}\mu)_{k \in \mathbb{N}}$ hence weak-* converges to the point mass $g^\infty \in \text{Prob}(\partial(W,S))$ where $g^\infty := \lim g^i\mu \in \partial(W,S)$ and where $\text{Prob}(\partial(W,S))$ denotes the space of all probability measures on $\partial(W,S)$ (compare also with the proof of [29 Theorem 3.20]). This implies that there exists an increasing sequence $(i_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$ for which $(g^{i_k}\mu)_{k \in \mathbb{N}}$ weak-* converges to a state $\psi$ whose restriction to $\pi(D(W,S))$ is multiplicative.

The product $s_n...s_1$ also defines a path in the Coxeter diagram of $(W,S)$. Using Lemma 2.7 and $\phi(P_{s_n}) = 0$ one deduces that for $a \in I$

$$\psi(a) = \lim_k \phi(T_{g^{-i_k}}^{(1)}aT_{g^i_k}^{(1)}) = \lim_k \phi(T_{g^i_k}^{(q)}aT_{g^{-i_k}}^{(q)}) = 0,$$

so $\psi$ vanishes on the ideal $I$.

Now, let $J$ be the ideal in $\pi(\mathfrak{A}(W))$ generated by $I$. Since $\pi(\mathfrak{A}(W))$ identifies with the crossed product $C^*$-algebra $C(\partial(W,S)) \rtimes \omega$, every element in $\pi(\mathfrak{A}(W))$ can be approximated by a finite sum of the form $\sum_{w \in W} f_wT_w^{(1)}$ where $f_w \in \pi(D(W,S))$. Using $T_{s_i}^{(1)} = T_{s_i}^{(q)} - p_{s_i}(q)P_s$ for $s \in S$, one concludes via induction that every such operator can be written as a finite sum $\sum_{w \in W} g_wT_w^{(q)}$ for suitable $g_w \in \pi(D(W,S))$. But for all $a \in I$, $g,h \in \pi(D(W,S))$ and $v,w \in W$ we have that

$$\psi((gT_w^{(q)}aT_v^{(q)})h) = \psi(g)\psi(T_w^{(q)}aT_v^{(q)})\psi(h) = 0$$

since $T_w^{(q)}aT_v^{(q)} \in I$, so the state $\psi$ vanishes on $J$. In particular, since $\psi \neq 0$, $J$ can not coincide with the whole $C^*$-algebra $\pi(\mathfrak{A}(W))$. But $\pi(\mathfrak{A}(W))$ is simple by [29 Corollary 4.11], so $J = 0$. We get that $C_{r,q}(W)$ must be simple as well. This completes the proof.

**Corollary 4.4.** Let $(W,S)$ be an irreducible, right-angled Coxeter system with $\#S = \infty$ and let $q = (q_s)_{s \in S} \in \mathbb{R}^{(W,S)}_{>0}$. Then the Hecke $C^*$-algebra $C_{r,q}(W)$ is simple if and only if there exists a finite subset $T \subseteq S$ such that the Hecke $C^*$-algebra $C_{r,q_T}(W_T)$ with $q_T := (q_t)_{t \in T}$ is simple.

**Proof.** Again, by Proposition 1.3 it suffices to consider multi-parameters with $0 < q_s \leq 1$ for $s \in S$. First assume that for all finite subsets $T \subseteq S$ the Hecke $C^*$-algebra $C_{r,q_T}(W_T)$ is not
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simple. The map \( \chi : T_{W}^{(q)} \rightarrow q^{\frac{1}{2}}_{W}, \ w \in W \) defines a character on \( \mathbb{C}_{q}[W] \). Further, for every element \( x := \sum_{w \in W} x(w)T_{W}^{(q)} \in \mathbb{C}_{q}[W] \) with \( x(w) \in \mathbb{C} \) for all \( w \in W \) there exists a finite subset \( T \subseteq S \) such that the support \( \{ w \in W \mid x(w) \neq 0 \} \) of \( x \) is contained in the special subgroup \( W_{T} \). Recall that \( C_{r,q_{T}}^{*}(W_{T}) \) canonically embeds into \( C_{r,q}^{*}(W) \). Since by the assumption \( C_{r,q_{T}}^{*}(W_{T}) \) is not simple, Theorem 4.3 implies in combination with Proposition 3.7 that the restriction of \( \chi \) to \( C_{r,q_{T}}^{*}(W_{T}) \) continuously extends to a character \( \chi_{T} \) on \( C_{r,q_{T}}^{*}(W_{T}) \). But then, \( |\chi(x)| = |\chi_{T}(x)| \leq \|x\| \), so (as \( x \) was arbitrary) \( \chi \) continuously extends to a character on \( C_{r,q}^{*}(W) \). Hence \( C_{r,q}^{*}(W) \) is not simple.

Conversely assume that there exists a finite subset \( T \subseteq S \) for which \( C_{r,q_{T}}^{*}(W_{T}) \) is simple. Then from Theorem 4.3 it follows that the \( C^{*}\)-algebra \( C_{r,q_{T}}^{*}(W_{T}) \) is simple for all finite subsets \( T' \subseteq S \) with \( T \subseteq T' \). It is a standard fact that inductive limits of simple \( C^{*}\)-algebras are simple (see e.g. [1] II.8.2.5), so the simplicity of \( C_{r,q}^{*}(W) \) follows from Lemma 1.5.

The following example demonstrates that there exist infinitely generated right-angled, irreducible Coxeter systems and corresponding multi-parameters whose respective Hecke \( C^{*}\)-algebras are non-simple.

Example 4.5. Let \( S = \{s_{1}, s_{2}, ...\} \) be a countable set and consider the Coxeter group \( W \) generated by \( S \) subject to the relations defined by \( m_{ss_{t}} = 2 \) for all \( s \in S \) and \( m_{st} = \infty \) for all \( s, t \in S, s \neq t \). Define \( q := (q_{s})_{s \in S} \in \mathbb{R}_{>0}^{(W,S)} \) by \( q_{s} := 2^{-i} \) for \( i \in \mathbb{N}_{\geq 1} \). Then for every finite subset \( T \subseteq S \) one checks that

\[
\sum_{s \in T} \frac{1}{1 + q_{s}} \geq \frac{\#T}{\sum_{i=1}^{\#T} \frac{1}{1 + 2^{-i}}} \geq \#T - 1
\]

and hence, by the analysis in [10] Subsection 5.4], the \( C^{*}\)-algebra \( C_{r,q_{T}}^{*}(W) \) is not simple. Corollary 4.3 then implies that \( C_{r,q}^{*}(W) \) is not simple.

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