One-bit compressive sensing with norm estimation

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Abstract

Compressive sensing typically involves the recovery of a sparse signal $x$ from linear measurements $\langle a_i, x \rangle$, but recent research has demonstrated that recovery is also possible when the observed measurements are quantized to sign($\langle a_i, x \rangle$) ∈ {±1}. Since such measurements give no information on the norm of $x$, recovery methods geared towards such one-bit compressive sensing measurements typically assume that $\|x\|_2 = 1$. We show that if one allows more generally for quantized affine measurements of the form sign($\langle a_i, x \rangle + b_i$), and if such measurements are random, an appropriate choice of the affine shifts $b_i$ allows norm recovery to be easily incorporated into existing methods for one-bit compressive sensing. Alternatively, if one is interested in norm estimation alone, we show that the fraction of measurements quantized to +1 (versus −1) can be used to estimate $\|x\|_2$ through a single evaluation of the inverse Gaussian error function, providing a computationally efficient method for norm estimation from 1-bit compressive measurements. Finally, all of our recovery guarantees are universal over sparse vectors, in the sense that with high probability, one set of measurements will successfully estimate all sparse vectors simultaneously.

1 Introduction

Compressive sensing, as introduced in [3] [6], [4], concerns the approximation of a sparse (or approximately sparse) vector $x \in \mathbb{R}^n$ from linear measurements of the form

$$y_i = \langle a_i, x \rangle, \quad i = 1, 2, \ldots, m.$$  

To allow processing using digital computers, the measurements $y_i$ must be quantized to a finite number of bits in practical compressive sensing architectures. In the extreme case, it is of interest to consider the one-bit compressive sensing problem, as introduced in [2], which concerns the approximation of a sparse (or almost sparse) vector $x \in \mathbb{R}^n$ from one-bit quantized measurements of the form

$$y_i = \text{sign}(\langle a_i, x \rangle), \quad i = 1, 2, \ldots, m,$$  

where sign($t$) = 1 when $t \geq 0$ and sign($t$) = −1 when $t < 0$. In practice a comparator (one-bit quantizer) is easy to build, fast, and consumes relatively little power, so one-bit measurements

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may even be *preferable* in situations where finer quantization is expensive relative to additional measurements. One-bit measurements may carry added benefits such as robustness to certain nonlinearities in the signal acquisition process (saturation, for example). Additionally, recent research indicates that in some settings, recovery from one-bit measurements may even out-perform multi-bit compressed sensing [10].

In [12], Plan and Vershynin provided the first computationally tractable method for reconstructing effectively sparse vectors from one-bit measurements of the form (1) via convex optimization [12] (cf. [11]). In particular, they prove the following:

**Theorem 1** (Theorem 1.1 of [12]). Let \(a_i \in \mathbb{R}^n, i = 1, \ldots, m\) be random vectors with independent and identically distributed standard Gaussian entries and suppose 
\[
m > C\delta^{-5} s \log^2(2n/s).
\]
With probability exceeding \(1 - C \exp(-c\delta m)\), the following holds for every \(x \in \mathbb{R}^n\) with \(\|x\|_2 = 1\) and \(\|x\|_1 \leq \sqrt{s}\): The solution \(x^\#\) to the optimization problem

\[
\min_{x^' \in \mathbb{R}^n} \|x^'\|_1 \text{ subject to } \sum_{i=1}^{m} |\langle a_i, x^' \rangle| = m \text{ and } \text{sign}(\langle a_i, x^' \rangle) = \text{sign}(\langle a_i, x \rangle), \quad i \in [m] \tag{2}
\]
satisfies

\[
\left\| x - \frac{x^\#}{\|x^\#\|_2} \right\|_2 \leq \delta.
\]

Above, \(C\) and \(c\) are universal constants, independent of all other parameters.

A limitation of this result, and in prior results treating the one-bit compressive sensing problem ([2], [1], [9], [11], [12], [13]) is that \(\|x\|_2 = 1\) must be assumed *a priori* to guarantee any accuracy in the reconstructed solution. If one considers only quantized linear measurements \(y_i = \langle a_i, x \rangle\), then such an assumption must be made: quantized linear measurements give no information about the magnitude of the underlying vector \(x\). As we will show, this problem can be resolved if one allows more generally for quantized affine linear measurements \(y_i = \text{sign}(\langle a_i, x \rangle + b_i)\).

**Contributions of this paper.** We are concerned with the scenario where the norm of \(x\) is not known a priori, and must be estimated along with the direction, from one-bit compressive measurements. Because measurements of the form \(\langle a_i, x \rangle\) give no information about the norm, we consider the reconstruction of \(x \in \mathbb{R}^n\) from more general one-bit measurements of the form

\[
y_i = \text{sign}(\langle a_i, x \rangle + b_i), \quad i = 1, 2, \ldots, m, \tag{3}
\]
where \(b = (b_i)_{i=1}^m\) is known (such affine shifts should not incur any additional difficulties in hardware design). We note that the application of affine shifts before quantization was recently shown to produce more accurate reconstructions in the area of one-bit matrix completion [5], but towards a different purpose. To our knowledge, nonzero thresholds have not been considered previously within the context of 1-bit compressive sensing. For reconstructing \(x\) from the measurements (3), we propose two algorithms:

- **Augmented convex programming approach.** When the shifts \(b_i\) are standard Gaussian variables and the measurement vectors \(a_i\) have i.i.d. standard Gaussian entries, we can rewrite the affine measurements (3) as augmented linear measurements

\[
y_i = \text{sign}(\langle a_i, x \rangle + b_i) = \text{sign}(\langle \tilde{a}_i, \tilde{x} \rangle),
\]
where \( \tilde{a}_i = (a_i, b_i) \) and \( \tilde{x} \in \mathbb{R}^{n+1} \) is given by \( \tilde{x} = (x, 1) \). We use a standard 1-bit compressed sensing recovery method, such as that of Plan and Vershynin (2) to give us an estimate \( \tilde{x}^\delta \) of \( \tilde{x} \), albeit without magnitude information. Defining \([n] := \{1, ..., n\}\) and denoting by \( x_T \) the restriction of \( x \) to \( T \subset [n] \), we note that if each ratio \( x^\delta_j/\tilde{x}_j \) is roughly the same then the ratio of the norms \( \|x^\delta_{[n]}\|_2/\|x\|_2 \) should be close to the known ratio \( x^\delta_{n+1}/\tilde{x}_{n+1} = x^\delta_{n+1}/1 \). Rearranging, this gives

\[
\|x\|_2 \approx \frac{\|x^\delta_{[n]}\|_2}{x^\delta_{n+1}}.
\]

In section 3.1, we formalize this intuition and derive theoretical guarantees for this method.

- **Empirical distribution function approach.** When the shifts \( b_i \) are all set to a common, non-random value \( \tau \), we may use a method based on the empirical cumulative distribution function to estimate the norm of \( x \). This method is motivated by the observation that \( \langle a_i, x \rangle \) is a Gaussian random variable with mean zero and standard deviation of \( \|x\| \), which is the quantity we wish to estimate. Thus, the fraction of measurements \( \text{sign}(\langle a_i, x \rangle - \tau) \) that are negative should approximate the cumulative distribution function of a \( \mathcal{N}(0, \|x\|^2) \) random variable evaluated at \( \tau \). The accuracy of the empirical cumulative distribution function (empirical cdf or EDF) is guaranteed by the Dvoretzky-Keifer-Wolfowitz (DKW) Inequality from [7], and we use the value of the empirical cdf at \( \tau \) to obtain an estimate for \( \|x\|_2 \). Section 3.1 presents our theoretical results on this method.

Both methods assume a known upper bound on the norm of \( x \) and the EDF method further assumes a known lower bound on \( \|x\|_2 \). For each method we present sufficient conditions on \( m \) for universal sparse signal recovery to hold with high probability (to within a desired accuracy \( \delta > 0 \)). We show that the performance of the augmented convex programming approach scales like \( \|x - x^\delta\|_2 \lesssim 1/m^{1/5} \), similar to the theoretical rate given in [12] in the case where \( \|x\|_2 = 1 \) is assumed. We show that the EDF method is guaranteed to do at least this well, and in certain regimes even achieves the scaling \( \|x - x^\delta\|_2 \lesssim 1/m^{1/2} \). Finally, we include numerical experiments comparing the accuracy of each norm recovery method, and find that empirically, the performance of both methods scales like \( \|x - x^\delta\|_2 \lesssim 1/m \), matching a known lower bound for the performance for one-bit compressive sensing [9]. The numerical experiments suggest that the EDF method is more sensitive to the choice of parameters such as the lower and upper bounds on \( \|x\|_2 \). At the same time, for norm estimation alone, the EDF method is much more computationally efficient than solving a convex program, requiring only a single evaluation of the inverse Gaussian error function, and not requiring one to store the measurement matrix \( A \) but rather just the proportion of measurements which are quantized to \( -1 \) (versus \( +1 \)).

# 2 Preliminaries

Throughout, we use \( C, c, C_1, \) etc. to denote absolute constants whose values may change from line to line. For integer \( n \) we denote \([n] = \{1, 2, ..., n\}\). Vectors are written in bold italics, e.g. \( x \), and their coordinates written in plain text so that the \( i \)th component of \( x \) is \( x_i \). The \( \ell_1 \) and \( \ell_2 \) norms of a vector \( x \in \mathbb{R}^n \) are defined as \( \|x\|_1 = \sum_{i=1}^n |x_i| \) and \( \|x\|_2 = (\sum_{i=1}^n x_i^2)^{1/2} \). The number of nonzero coordinates of \( x \) is denoted by \( \|x\|_0 = |\text{supp}(x)| \).
To prove our main results, we will need a few lemmas. The first lemma is a simple geometric inequality concerning the norm of the difference between two vectors.

**Lemma 2.** Consider vectors \( \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n \) and positive scalars \( t_1, t_2, \alpha, \eta \in \mathbb{R} \) satisfying \( t_1 \geq \alpha > \eta, \quad \|\mathbf{x}_1\|^2 + t_1^2 = 1, \quad \|\mathbf{x}_2\|^2 + t_2^2 \leq 1, \quad \text{and} \quad \|\mathbf{x}_1 - \mathbf{x}_2\|^2 + (t_1 - t_2)^2 \leq \eta^2. \) Then

\[
\left\| \frac{\mathbf{x}_1}{t_1} - \frac{\mathbf{x}_2}{t_2} \right\|^2 \leq \frac{4\eta^2}{\alpha^2(\alpha - \eta)^2}
\]

**Proof.** First, define \( \varepsilon = 1 - \|\mathbf{x}_2\|^2 - t_2^2. \) Since

\[
\sqrt{\|\mathbf{x}_1\|^2 + t_1^2} = \sqrt{\|\mathbf{x}_2\|^2 + t_2^2} \leq \sqrt{(\mathbf{x}_1 - \mathbf{x}_2)^2 + (t_1 - t_2)^2} \leq \eta,
\]

it follows that \( 1 - \sqrt{1 - \varepsilon} \leq \eta, \) so \( (1 - \eta)^2 \leq 1 - \varepsilon \) and finally \( \varepsilon \leq 2\eta - \eta^2 \leq 2\eta. \) Also by the reverse triangle inequality, \( t_2 \geq t_1 - \eta \geq \alpha - \eta. \)

Next we note that \( \|\mathbf{x}_1 - \mathbf{x}_2\|^2 + (t_1 - t_2)^2 \leq \eta^2 \) implies

\[
-2\langle \mathbf{x}_1, \mathbf{x}_2 \rangle \leq \eta^2 - (t_1 - t_2)^2 - \|\mathbf{x}_1\|^2 - \|\mathbf{x}_2\|^2
= \eta^2 - (t_1^2 + \|\mathbf{x}_1\|^2) - (t_2^2 + \|\mathbf{x}_2\|^2) + 2t_1t_2
= \eta^2 - 2 + \varepsilon + 2t_1t_2
\]

Now,

\[
\left\| \frac{\mathbf{x}_1}{t_1} - \frac{\mathbf{x}_2}{t_2} \right\|^2 = \frac{\|\mathbf{x}_1\|^2}{t_1^2} + \frac{\|\mathbf{x}_2\|^2}{t_2^2} - 2\frac{\langle \mathbf{x}_1, \mathbf{x}_2 \rangle}{t_1t_2}
\leq \frac{1 - t_1^2}{t_1^2} + \frac{1 - \varepsilon - t_2^2}{t_2^2} + \frac{\eta^2 - 2 + \varepsilon + 2t_1t_2}{t_1t_2}
= \frac{(t_1 - t_2)^2 - \varepsilon t_1^2 + \eta^2 t_1t_2 + \varepsilon t_1t_2}{t_1^2t_2^2}
= \frac{(t_1 - t_2)^2}{t_1^2t_2^2} + \frac{\eta^2 t_1t_2}{t_1^2t_2^2} + \frac{\varepsilon t_1(t_2 - t_1)}{t_1^2t_2^2}
\leq \frac{4\eta^2}{t_1^2t_2^2}
\]

where in the final inequality we use that \( 0 < t_1 \leq 1, \) \( 0 < t_2 \leq 1, \) \( t_2 - t_1 < \eta, \) and \( \varepsilon < 2\eta. \) Over the range \( \alpha \leq t_1 \leq 1, \) and \( t_1 - \eta \leq t_2 < 1, \) this expression attains its maximum at \( t_1 = \alpha, \) \( t_2 = \alpha - \eta. \) Substituting these values for \( t_1 \) and \( t_2 \) results in the bound stated in the lemma.

\( \square \)

The next lemma gives a bound on the variation of a function with an inverse dependence on the Gaussian error function.

**Lemma 3.** Let \( \text{erf} : \mathbb{R} \to [-1, 1] \) be the Gaussian error function, and define \( h : [0, 1] \to \mathbb{R} \) by

\[
h(u) = \begin{cases} \frac{1}{\text{erf}^{-1}(2u - 1)}, & u \in (0, 1) \\ 0, & u \in \{0, 1\} \end{cases}
\]

For \( \eta > 0 \) and \( a, b \in \left[ \frac{1}{2} + \eta, \frac{1}{2}(\text{erf}(1) + 1) \right], \) we have \( |h(a) - h(b)| \leq |h'(\frac{1}{2} + \eta)(b - a)|. \)
Proof. As the derivative of the inverse error function is \( \frac{d}{du} \text{erf}^{-1}(u) = \frac{1}{2} \sqrt{\pi} \exp \left( \left( \text{erf}^{-1}(u) \right)^2 \right) \), the derivative of \( h \) is given by
\[
h'(u) = -\sqrt{\pi} \exp \left( \frac{(\text{erf}^{-1}(2u - 1))^2}{(\text{erf}^{-1}(2u - 1))^2} \right),
\]
which is negative and decreasing in absolute value on the interval \( \left( \frac{1}{2}, \frac{1}{2} \text{erf}(1) + 1 \right) \). Thus, for any \( \eta > 0 \) and \( a, b \) in \( \left[ \frac{1}{2} + \eta, \frac{1}{2} (\text{erf}(1) + 1) \right] \), we have \( |h(a) - h(b)| \leq |h'(\frac{1}{2} + \eta)| |b - a| \).

3 Main results

Here we describe and give guarantees for two methods by which the norm of an unknown vector \( x \in \mathbb{R}^n \) can estimated, possibly along with the direction, from one-bit compressive measurements \( y_i = \text{sign}(\langle a_i, x \rangle + b_i) \). The first method augments the convex program (1) to retrieve norm as well as directional information about the unknown vector, and inherits the error guarantees for that approach. The second method estimates the norm directly from the measured proportion \#\{\( i : y_i = -1 \)\}/\( m \), using the shift \( b_i = \tau \) and the Gaussianity of \( a_i \) to obtain a consistent estimator for \( \|x\|_2 \) which is analyzed using the Dvoretzky-Keifer-Wolfowitz inequality. This method is very efficient to implement compared to the convex programming approach, requiring only a single evaluation of the inverse Gaussian error function. At the same time, it is less robust to parameter uncertainty, as can be seen from numerical experiments.

3.1 Augmented convex programming

Our first main result is a bound on the accuracy of approximating a \( x \in \mathbb{R}^n \) with assumed structural constraint \( \|x\|_1/\|x\|_2 \leq \sqrt{s} \) from one-bit measurements
\[
y_i = \text{sign}(\langle a_i, x \rangle + b_i), \quad i = 1, 2, \ldots, m. \tag{4}
\]
Note that the constraint \( \|x\|_1/\|x\|_2 \leq \sqrt{s} \) describes a class of “approximately” \( s \)-sparse vectors. For reconstruction, we consider an augmented version of the convex program (2) specified below.

**Theorem 4.** For \( i = 1, \ldots, m \), let the random vectors \( a_i \in \mathbb{R}^n \) be independent and identically distributed with \( \mathcal{N}(0,1) \) entries and let \( b_i \) be independent \( \mathcal{N}(0, R) \) scalars. If for some specified \( \delta < \min\{1, R/2\} \),
\[
m \geq C\delta^{-5} R^5 s \log^2 \left( \frac{2n}{s} \right),
\]
then with probability exceeding \( 1 - C \exp \left( -c \frac{m \delta}{R} \right) \) the following holds uniformly for all vectors \( x \in \mathbb{R}^n \) with \( \|x\|_2 \leq R \): the solution \( (x^\sharp, t^\sharp) \) to the optimization problem
\[
\min_{z \in \mathbb{R}^n, u \in \mathbb{R}} \|z, u\|_1 \text{ subject to } \sum_{i=1}^{m} |\langle a_i, z \rangle + ub_i| = m \text{ and } \text{sign}(\langle a_i, z \rangle + ub_i) = \text{sign}(\langle a_i, x \rangle + b_i), \quad i \in [m]
\]
satisfies
\[
\|Rx^\sharp/t^\sharp - x\|_2 \leq 4\sqrt{2}\delta.
\]
Proof. Let $\eta = \delta/(R\sqrt{2})$. Having $m \geq C\eta^{-5}s \log^2 \left(\frac{2m}{s}\right) \geq C\eta^{-5}(s+1) \log^2 \left(\frac{2(n+1)}{s+1}\right)$ implies by Theorem 1 that

$$\left\| \frac{(x, R)}{\|x\|_2^2 + R^2} - \frac{(x^\sharp, t^\sharp)}{\|x\|_2^2 + t^\sharp^2} \right\|_2 \leq \eta$$

with high probability on the draw of the matrix, uniformly for all $x$ satisfying the assumptions of the theorem. Since $\|x\|_2 \leq R$ by assumption, we choose $\alpha = \frac{1}{\sqrt{2}} \leq t_1 := \frac{R}{\sqrt{\|x\|_2^2 + R^2}}$ and apply Lemma 2 with $x_1 = \frac{x}{\sqrt{\|x\|_2^2 + R^2}}$, $x_2 = \frac{x^\sharp}{\sqrt{\|x\|_2^2 + t^\sharp^2}}$, and $t_2 = \frac{t^\sharp}{\sqrt{\|x\|_2^2 + t^\sharp^2}}$. With our choice of $\eta$ this yields

$$\|x - Rx^\sharp/t^\sharp\|_2^2 = R^2\|x/R - x^\sharp/t^\sharp\|_2^2 \leq R^2 \frac{4\delta^2}{R^2(1/\sqrt{2} - \delta/(R\sqrt{2}))^2} = \frac{8\delta^2}{(1 - \delta/R)^2} \leq 32\delta^2.$$ 

To obtain the last inequality above, we used the assumption $\delta < R/2$. It follows that

$$\|x - Rx^\sharp/t^\sharp\|_2 \leq 4\sqrt{2}\delta. \quad (5)$$

A few remarks are in order.

**Remark 5** (Alternative scaling of shift). Recovery in an analogous fashion is also possible if for $\tau \geq \delta$ we sample $b_i \sim \mathcal{N}(0, \tau)$ instead of $b_i \sim \mathcal{N}(0, R)$, as would be necessary if $R$ is unknown a priori. The proof runs similarly, but we now estimate $x$ with $\tau x^\sharp/t^\sharp$ and apply Lemma 1 with $\eta = \frac{\delta}{\sqrt{R^2 + \tau^2}} \leq \alpha = \frac{\tau}{\sqrt{R^2 + \tau^2}} \leq t_1 := \frac{R}{\sqrt{\|x\|_2^2 + \tau^2}}$, $x_1 = \frac{x}{\sqrt{\|x\|_2^2 + \tau^2}}$, $x_2 = x^\sharp$, and $t_2 = t^\sharp$. Then the error bound (5) is replaced with the conclusion that (with high probability) $\|x - \tau x^\sharp/t^\sharp\|_2 \leq \frac{2\delta}{(\tau - \delta)} \sqrt{R^2 + \tau^2}$.

**Remark 6** (Tightness). For fixed $n$, $s$, and $R$, the parameter $\lambda := \delta^{-5}$ plays the role of an over-sampling parameter and appears in the rate of decay of the reconstruction error as $\|x - Rx^\sharp/t^\sharp\|_2 \lesssim \lambda^{-1/5}$. Compared to the known lower bound of $\|x - x^\sharp/t^\sharp\|_2 \gtrsim \lambda^{-1}$ for the 1-bit compressive sensing problem in the case $\|x\|_2 = 1$ and $\|x\|_0 \leq s$, this rate is suboptimal [9]. On the other hand, this rate matches the error rate achievable using the convex optimization method (2).

**Remark 7** (Alternative reconstruction methods). The above theorem can be easily adapted to alternate reconstruction methods and inherits their associated error decay rates. For example, using the non-uniform recovery method of [11], one obtains an error of $\delta$ at number of measurements $m \gtrsim \delta^{-1}R^4s \log n/s$. This improves the dependence of the number of measurements on $\delta$, $R$, and $\log n$ at the expense of losing the uniform recovery guarantee.
3.2 Estimating $\|x\|_2$ using the empirical distribution function

In this section, we consider an alternate approach to 1-bit compressive sensing with built-in norm estimation, where now we estimate $\|x\|_2$ given measurements $y = \text{sign}(Ax - b)$ with constant (non-random) $b = \tau = (\tau, ..., \tau) \in \mathbb{R}^m$ and $\tau \neq 0$. Unlike the previous approach, the method in this section only approximates the norm of $x$, and gives no information about its direction.

We consider $m$ measurement vectors $a_i \in \mathbb{R}^n$ whose entries $a_{i,j}$ are i.i.d. $\mathcal{N}(0,1)$. Note that $\langle a_i, x \rangle \sim \mathcal{N}(0, \|x\|_2^2)$, and so $\|x\|_2$ is the standard deviation of $\langle a_i, x \rangle$. Since we only have access to the signs of the samples $\langle a_i, x \rangle - \tau$, and not the samples themselves, we cannot simply estimate $\|x\|_2$ via the sample standard deviation of $\{\langle a_i, x \rangle\}_{i=1}^m$. Instead, we will make use of the empirical cumulative distribution function defined by

$$F_m(\tau) := \frac{\# \{i : y_i = -1\}}{m}, \quad (6)$$

which gives the proportion of the $m$ measurements $\{\langle a_i, x \rangle\}_{i=1}^m$ satisfying $\langle a_i, x \rangle \leq \tau$. As $m$ increases, the random variable $F_m(\tau)$ will approach $F(\tau) = \frac{1}{2}(1 + \text{erf}(\frac{\tau}{\sqrt{2} \|x\|_2}))$, where $F$ is the cumulative distribution function of a Gaussian random variable with mean 0 and variance $\|x\|_2^2$. Indeed, the empirical distribution function $F_m(\tau)$ is a consistent estimator of $F(\tau)$. We note that for $F(\tau) \neq \frac{1}{2}$, we may invert the expression for $F(\tau)$ to get $\|x\|_2 = \sqrt{2} \text{erf}^{-1}(2F(\tau) - 1)$, which motivates, as an approximation of $\|x\|_2$, the estimator

$$\Lambda = \Lambda_m(\tau) := \frac{\tau}{\sqrt{2} \text{erf}^{-1}(2F_m(\tau) - 1)}. \quad (7)$$

To help estimate the accuracy of $\Lambda$ as an approximation to $\|x\|_2$, we turn to the Dvoretzky-Keifer-Wolfowitz Inequality [7], which gives the following bound on the difference between the cumulative distribution function and empirical cumulative distribution function.

**Proposition 8** (Dvoretzky-Keifer-Wolfowitz [7]). Let $X_1, X_2, ..., X_m$ be i.i.d. random variables with cumulative distribution function $F$, and let $F_m$ be the associated empirical cumulative density function $F_m(\tau) := \frac{1}{m} \sum 1_{X_i \leq \tau}$. Then for any $\gamma > 0$,

$$\text{Prob} \left( \sup_{\tau} |F_m(\tau) - F(\tau)| > \gamma \right) \leq 2 \exp(-2m\gamma^2).$$

The DKW inequality will allow us to bound the accuracy of $\Lambda_m(\tau)$ as an estimate of $\|x\|_2$.

**Lemma 9.** Fix $0 < \delta < 1/5$, and let $x \in \mathbb{R}^n$ be such that $r \leq \|x\|_2 \leq R$ for known positive constants $r$ and $R$. Let $A \in \mathbb{R}^{m \times n}$ be a matrix with independent identically distributed $\mathcal{N}(0,1)$ entries. Set $\tau = r$ and compute $F_m(\tau)$ from $y = \text{sign}(Ax - \tau)$ via (6). If

$$m \geq \pi r^2 R^2 \delta^{-2} \log \frac{2}{\varepsilon},$$

then with probability at least $1 - \varepsilon$ it holds that

$$|F(\tau) - F_m(\tau)| < \frac{\delta}{\sqrt{2\pi} R}.$$
and 
\[ F(\tau) \text{ and } F_m(\tau) \in \left[ \frac{1}{2} \left( 1 + \text{erf} \left( \frac{(1-\delta)r}{\sqrt{2}R} \right) \right), \frac{1}{2} \left( 1 + \text{erf} \left( \frac{(1+\delta)r}{\sqrt{2}R} \right) \right) \right]. \]

Proof. By Proposition 8, we have for any choice of $\gamma > 0$, that $|F(\tau) - F_m(\tau)| \leq \gamma$ with probability at least $1 - 2 \exp(-2m\gamma^2)$. Set $\tau = r$ and note that 
\[ F(\tau) = \frac{1}{2} \left( 1 + \text{erf} \left( \frac{\tau}{\|x\|_2 \sqrt{2}} \right) \right) \in \left[ \frac{1}{2} \left( 1 + \text{erf} \left( \frac{r}{\sqrt{2}R} \right) \right), \frac{1}{2} \left( 1 + \text{erf} \left( \frac{1}{\sqrt{2}} \right) \right) \right]. \]

For $\delta < 1$, set 
\[ \eta = \frac{1}{2} \text{erf} \left( \frac{(1-\delta)r}{\sqrt{2}R} \right) \]
and 
\[ \gamma = \frac{1}{2} \left( \text{erf} \left( \frac{r}{\sqrt{2}R} \right) - \text{erf} \left( \frac{(1-\delta)r}{\sqrt{2}R} \right) \right). \]
Noting that 
\[ \frac{d\text{erf}(x)}{dx} = \frac{2}{\sqrt{\pi}} \exp(-x^2), \]
we have for $0 \leq a \leq b$ 
\[ (b - a) \frac{2}{\sqrt{\pi}} \exp(-b^2)x \leq \text{erf}(b) - \text{erf}(a) \leq (b - a) \frac{2}{\sqrt{\pi}} \exp(-a^2). \]

Consequently 
\[ \frac{\delta}{\sqrt{2\pi eR}} \leq \gamma \leq \frac{\delta}{\sqrt{2\pi R}}. \]

By the DKW inequality, with probability exceeding 
\[ 1 - 2 \exp(-2m\gamma^2) \geq 1 - 2 \exp \left( -\frac{\delta^2r^2}{\pi eR^2}m \right), \]
we have 
\[ F_m(\tau) \in \left[ \frac{1}{2} + \eta, \frac{1}{2} + \frac{1}{2} \text{erf} \left( \frac{1}{\sqrt{2}} \right) + \frac{1}{2} \left( \text{erf} \left( \frac{r}{\sqrt{2}R} \right) - \text{erf} \left( \frac{(1-\delta)r}{\sqrt{2}R} \right) \right) \right]. \]

This yields the conclusion of the lemma provided 
\[ 2\gamma = \text{erf} \left( \frac{r}{\sqrt{2}R} \right) - \text{erf} \left( \frac{(1-\delta)r}{\sqrt{2}R} \right) \leq \text{erf} \left( \frac{1}{\sqrt{2}} \right), \]
which holds when $\delta \leq 1/5$, and hence $\gamma \leq \frac{\delta r}{\sqrt{2\pi eR}} \leq \frac{\delta r}{5 \sqrt{2\pi}} \leq \frac{1}{2} \text{erf} \left( \frac{1}{\sqrt{2}} \right)$.

**Theorem 10.** Fix $0 < \delta < \frac{2\sqrt{e}}{9} R$, and let $x \in \mathbb{R}^n$ be such that $r \leq \|x\|_2 \leq R$ for known strictly positive constants $r$ and $R$. Let $A \in \mathbb{R}^{m \times n}$ be a matrix with independent identically distributed $\mathcal{N}(0,1)$ entries. Set $\tau = r$ and compute $F_m(\tau)$ and $\Lambda = \Lambda_m(\tau)$ from $y = \text{sign}(Ax - \tau)$ via (6) and (7) respectively. If 
\[ m \geq 4\pi r^2 e^2 \delta^{-2} \log \frac{2}{\varepsilon}, \]
then with probability at least $1 - \varepsilon$ it holds that

$$\left|\|x\|_2 - \Lambda_m(\tau)\right| \leq \delta.$$  

**Proof.** Define the function $h : [0, 1] \rightarrow \mathbb{R}$ as in Lemma 3 by

$$h(u) = \begin{cases} \frac{1}{\text{erf}(2u-1)}, & u \in (0, 1) \\ 0, & u \in \{0, 1\} \end{cases}$$

Then $\|x\|_2 - \Lambda_m(\tau) = \frac{\tau}{\sqrt{2}} |h(F(\tau)) - h(F_m(\tau))|$. If $F(\tau)$ and $F_m(\tau)$ are in $[.5 + \eta, \frac{1}{2}(\text{erf}(1) + 1)]$ for some $\eta > 0$, then by Lemma 3 we have:

$$\|x\|_2 - \Lambda_m(\tau) = \frac{\tau}{\sqrt{2}} |h(F(\tau)) - h(F_m(\tau))| \leq \frac{\tau}{\sqrt{2}} |h'(0.5 + \eta)||F(\tau) - F_m(\tau)|.$$  

Indeed, provided $\delta_0 := \frac{1}{2R\sqrt{e}} \delta < 1/5$, by Proposition 9 we have that $|F(\tau) - F_m(\tau)| \leq \gamma := \frac{\delta_0}{\sqrt{2\pi} R}$, and that $F_m(\tau)$ and $F(\tau)$ do satisfy the assumptions of Lemma 3 with probability at least $1 - \varepsilon$. So, using Lemma 3 and the definitions of $\gamma, \tau$ we conclude that

$$\|x\|_2 - \Lambda_m(\tau) = \frac{\tau}{\sqrt{2}} |h(F(\tau)) - h(F_m(\tau))| \leq \frac{\tau}{\sqrt{2}} |h'(0.5 + \eta)||F(\tau) - F_m(\tau)| \leq 2R\sqrt{e}\delta_0 = \delta.$$  

The previous theorem gave a bound for norm estimation for a particular fixed $x$, and we assumed no particular structural constraints on $x$. We now provide a universal norm estimation bound akin to Theorem 4 for the class of $s$-sparse vectors: $x \in \mathbb{R}^n$ satisfying $\|x\|_0 \leq s, r \leq \|x\|_2 \leq R$.

**Theorem 11.** For $i = 1, \ldots, m$, let the random vectors $a_i \in \mathbb{R}^n$ have i.i.d. $\mathcal{N}(0, 1)$ entries. Fix $0 < \delta < 1/5$ and positive constants $r \leq R$. If

$$m \geq C_1 \frac{R^4}{r^2} \delta^{-2} s \log \left( \frac{Rn}{\delta rs} \right)$$

then with probability exceeding $1 - C_2 \exp(-c_3 m)$,

$$\left|\|x\|_2 - \Lambda_m\left(\frac{3r}{5}\right)\right| \leq 2\delta$$

holds uniformly for all vectors $x \in \mathbb{R}^n$ in the set $\{r \leq \|x\|_2 \leq R\} \cap \{\|x\|_0 \leq s\}$. Here, $\Lambda_m\left(\frac{3r}{5}\right)$ is the estimator computed from $y = \text{sign}(Ax - \tau)$ via (7) with $\tau = \frac{3r}{5}$. 


Proof. The idea of the proof is to first show that Theorem 10 holds uniformly over a sufficiently fine net of points contained in the set of \( s \)-sparse vectors of bounded norm. We then leverage concentration properties of the matrix \( A \) and monotonicity of the function \( h \) as defined in Lemma 3 to extend to a bound which holds uniformly for \( s \)-sparse vectors with norm bounded by \( R \).

It will be helpful below to define \( F_m \) and \( \Lambda \) now as functions of more than one argument: 

\[
F_m(\tau, z) := \#\{i : (a_i, z) \leq \tau\}, \quad \Lambda_m(\tau, z) := \sqrt{2\text{erf}^{-1}(2F_m(\tau, z) - 1)}.
\]

First, consider a finite set of points \( Q \) such that 

\[
Q \subseteq S := \{ x \in \mathbb{R}^n : \|x\|_2 \leq R, \|x\|_0 \leq s \},
\]

and 

\[
\min_{q \in Q} \|x - q\|_2 \leq \frac{\xi}{4} \quad \text{for each } x \in S. \tag{10}
\]

By a well-known result in the literature on covering numbers (see, e.g., [8] Appendix C.2) such a set exists. Let \( B_2^n \) denote the unit Euclidean ball in \( \mathbb{R}^n \). Given a fixed \( s \)-dimensional linear subspace \( T \) of \( \mathbb{R}^n \), there exists a finite set of points \( Q_T \) in \( B_2^n \cap T \) such that \( \max_{x \in B_2^n \cap T} \min_{q \in Q_T} \|x - q\|_2 \leq \xi/4 \), and such that \( \#(Q_T) \leq (12/\xi)^s \). Picking such a set of points for each of the \( \binom{n}{s} \) \( s \)-dimensional linear subspaces \( T \) whose union is \( \{ x \in \mathbb{R}^n : \|x\|_0 \leq s \} \), and rescaling, we arrive at a set of points \( Q \) in \( S \) of size \( \#(Q) \leq (\frac{n}{s})^s(12R/\xi)^s \) satisfying (10).

Now note that there exists a constant \( C \) so that with probability exceeding \( 1 - 2\exp(-\frac{\delta^2}{2C}m) \), where \( m > 2C\delta^{-2}s\log(n/s) \), the normalized matrix \( \frac{1}{\sqrt{m}}A \in \mathbb{R}^{m \times n} \) has the restricted isometry property of order \( 2s \) at level \( \delta \), that is,

\[
(1 - \delta)\|x\|_2 \leq \frac{1}{\sqrt{m}}\|Ax\|_2 \leq (1 + \delta)\|x\|_2 \quad \forall x : \|x\|_0 \leq 2s
\]

For more details on the restricted isometry property, we refer the reader to [8], Theorem 9.2.

We now condition on the event that \( A \) has the restricted isometry property. Let \( Q' = \{ P(q) : q \in Q \} \) where \( P \) projects radially by \( P(q) = q \cdot \max\{ \frac{\delta r}{\sqrt{s}} : \|q\|_2 \} \). For any \( x \in S \cap \{ \|x\|_2 \geq \frac{\delta r}{\sqrt{s}} \} \), consider the point \( q \in Q' \) realizing (10). Note that \( x - q \) is \( 2s \)-sparse for each \( x, q \in S \). Let \( m^* = \#\{ i : |\langle a_i, x - q \rangle|^2 \geq r^2/4 \} \). We have then

\[
\begin{align*}
m^*r^2/4 \leq \|A(x - q)\|_2^2 & \leq m(1 + \delta)^2\|x - q\|_2^2 \\
& \leq m(1 + \delta)^2\xi^2/16.
\end{align*}
\]

It follows that \( m^* \leq \frac{\xi^2}{16}m \). Moreover,

\[
m \cdot F_m(r, x) = \#\{ i : \|a_i, x\| \leq r \}
\geq \#\{ i : \langle a_i, q \rangle \leq r/2 \text{ and } \langle a_i, x - q \rangle \leq r/2 \}
= m - \#\{ i : \langle a_i, q \rangle \geq r/2 \text{ or } \langle a_i, x - q \rangle \geq r/2 \}
\geq m - (m(1 - F_m(r/2, q)) + m^*)
\geq m \left( F_m(r/2, q) - \frac{\xi^2}{r^2} \right)
\]

\[
10
\]
So, repeating this calculation for the upper bound we have

\[ F_m(r/2, q) - \frac{\xi^2}{r^2} \leq F_m(r, x) \leq F_m(3r/2, q) + \frac{\xi^2}{r^2}. \]

We will now choose \( \xi \) small enough to obtain

\[ F_m(r/3, q) \leq F_m(r, x) \leq F_m(5r/3, q). \]  \hspace{1cm} (11)

In particular, for the right-hand side inequality to hold we desire

\[ \xi^2/r^2 \leq F_m(5r/3, q) - F_m(3r/2, q) \]

\[ = (F_m(5r/3, q) - F(5r/3, q)) + (F(5r/3, q) - F(3r/2, q)) + (F(3r/2, q) - F_m(3r/2, q)). \]

Using Lemma 9 to control the first and third terms and (9) for the middle term, this is achieved if

\[ \xi^2/r^2 \leq \frac{r}{6R\sqrt{2\pi e}} - \frac{\delta}{\sqrt{2\pi e}} \frac{19r}{6R}. \]

So if we choose \( \delta < c_1 < \frac{1}{19\sqrt{e}} \) small enough for the right hand side to be positive, it suffices to choose \( \xi \leq c_2 r \sqrt{r/R} \) for (11) to hold. Here \( c_2 = \sqrt{\frac{1-19c_1\sqrt{e}}{6\sqrt{2\pi e}}} \) is a sufficiently small constant and is also sufficient to obtain the left hand side of (11).

Since \( r/3 \) and \( 5r/3 \) are both lower bounds for \( \|q\|_2 \), and since we have invoked Lemma 9, we have by Lemma 3 that \( t \to h(t) \) is decreasing over the range \( t \in [F_m(r/3, q), F_m(5r/3, q)] \). It follows that

\[ \|q\|_2 - \Lambda_m(r, x) \leq \max \{ \|q\|_2 - \Lambda_m(r/3, q), \|q\|_2 - \Lambda_m(5r/3, q) \} \]

and hence

\begin{align*}
\|x\|_2 - \Lambda_m(r, x) &\leq \|x\|_2 - \|q\|_2 + \|q\|_2 - \Lambda_m(r, x) \\
&\leq \|x\|_2 - \|q\|_2 + \max \{ \|q\|_2 - \Lambda_m(r/3, q), \|q\|_2 - \Lambda_m(5r/3, q) \} \\
&\leq \xi/4 + 2R\sqrt{e}\delta \\
&\leq 3R\sqrt{e} \cdot \delta. \hspace{1cm} (12)
\end{align*}

The last inequality is obtained by setting \( \xi \leq 4R\sqrt{e}\delta \).

To obtain the bound on \( m \) and the probability, note that invoking Lemma 9 for each individual \( q \), requires

\[ m \geq \pi e \frac{R^2}{r^2} \delta^{-2} \log \left( \frac{2}{\varepsilon} \right) \]

and yields a probability of failure less than \( \varepsilon \). By our choice of \( \xi \), \( \varepsilon(Q) \leq (\frac{2\pi e}{c_1})^s C_0^s \delta^{-s}(R/r)^{3s/2} \) for some positive constant \( C_0 \), so we pick \( m \geq C_1 \frac{R^2}{r^2} \delta^{-2} \log(\frac{R}{r} \frac{c_3}{\delta}) \). Consequently by a union bound, and by accounting for the probability that \( A \) satisfies the RIP, our probability of failure is less than \( C_2 \exp(-c_3m) \). The full statement of the theorem follows by rescaling \( \delta \) and \( r \).

As noted above, this method of norm estimation does not give us an estimate of the direction \( x \) itself; it only yields an estimate of the norm. In order to recover \( x \), we could easily combine the estimated norm with an estimate of \( x/\|x\|_2 \) recovered as in Proposition 1.
Corollary 12. Let \( x \in \mathbb{R}^n \) be such that \( 0 < r \leq \|x\|_2 \leq R \). Let \( \delta > 0 \) and choose \( \tau \) as in Theorem 10. Suppose we have \( m = m_1 + m_2 \) random Gaussian vectors and we collect \( m_1 \) measurements of the form \( y_i = \sign(\langle a_i, x \rangle - \tau) \), and \( m_2 \) measurements \( y_i = \sign(\langle a_i, x \rangle) \). Suppose \( \Lambda \) is calculated from the \( m_1 \) measurements as in Theorem 10 and let \( x^\sharp \) be the solution to the optimization problem in Proposition 1. If \( m_1 \geq C_0 \delta^{-5} R^5 \left( s \log^2 \left( \frac{2n}{\delta} \right) + \log \left( C/\varepsilon \right) \right) \) and \( m_2 \geq 4 \pi^2 R^4 \delta^{-2} \log(4/\varepsilon) \), then with probability at least \( 1 - \varepsilon \) it holds that \( \| \Lambda x^\sharp - x \|_2 \leq \delta \).

Proof. By Theorem 1, we use a convex optimization problem to obtain \( x^\sharp \) such that
\[
\left\| x^\sharp - x/\|x\|_2 \right\|_2 \leq \frac{\delta}{2} R
\]
with probability at least \( 1 - \varepsilon/2 \), using only the first \( m_1 \geq C_0 \delta^{-5} R^5 \left( s \log^2 \left( \frac{2n}{\delta} \right) + \log \left( C/\varepsilon \right) \right) \) measurements. With the remaining \( m_2 \geq C_{R,s} \delta^{-2} \log(8/\varepsilon) \) measurements, we calculate \( \Lambda \) and have by Theorem 10 that with probability at least \( 1 - \varepsilon/2 \), we have \( \|x\|_2 - \Lambda \leq \frac{\delta}{2} \). Hence, with probability at least \( 1 - \varepsilon \leq (1 - \varepsilon/2)^2 \), we have
\[
\| \Lambda x^\sharp - x \|_2 \leq \left\| \Lambda x^\sharp - \|x\|_2 x^\sharp \right\|_2 + \left\| x^\sharp \right\|_2 \|x - x\|_2
\leq \frac{\delta}{2} + \delta \left\| x \right\|_2 / (2R)
\leq \delta.
\]

4 Numerical Experiments

Here we test the performance of the two proposed methods for 1-bit compressive sensing with norm estimation. In all experiments, we consider \( s \)-sparse vectors \( x \in \mathbb{R}^n \) with \( n = 300 \) and \( s = 10 \) that are constructed by a uniform draw from the set \( S = \{ x : r < \|x\|_2 < R, \|x\|_0 < s \} \) for \( r = 10, R = 20 \). We estimate \( \|x\|_2 \) in two ways: (1) using the approximation \( \|\hat{x}\|_2 \) produced as in Theorem 4, and (2) using the Gaussian empirical cumulative distribution function (EDF) as in Theorem 10 (Figures 1a, 2a). The first estimation method is referred to as \( PV_{\text{aug}} \), because it precedes by applying an augmented version of the optimization problem (2) of Plan and Vershynin [12] as in Theorem 4. In a second set of experiments, we estimate \( x \) itself, rather than just its norm \( \|x\|_2 \), with (1) \( \hat{x} \) as in Theorem 4 (\( PV_{\text{aug}} \)), and (2) by partitioning the measurements into two sets, estimating the norm using one set according to the EDF method described in Theorem 10, and estimating the direction using the remaining measurements, as in Corollary 12. (Figures 1b, 2b).

In Figure 1 we plot recovery error for various values of \( m/n \). Note that the oversampled regime \( m > n \) is neither of interest in classical compressive sensing setting (with no quantization), is still potentially useful in the one-bit setting, particularly when measurements are fast or cheap relative to finer quantization. For each value of \( m/n \) we report the average error over a 40 trials in estimating each of \( x \) and \( \|x\|_2 \). The \( PV_{\text{aug}} \) method outperforms the EDF method in the plotted regime, at the cost of more computation time.

We also explore the effect of the choice of threshold \( \tau \) on the accuracy of recovery for both methods. In the EDF method, each measurement is quantized according to whether it is above or below the same threshold \( \tau \). In the \( PV_{\text{aug}} \) method, we consider for the same parameter \( \tau \) the
thresholds $b_i \sim \mathcal{N}(0, \tau)$. In this case, the expected norm of $b$ equals the norm of the threshold vector $\tau = (\tau, \tau, \ldots, \tau)$ used in the EDF method.

We expect reconstruction to be poor when $\tau$ is too large or too small relative to the true norm of $x$. As $\tau$ goes to zero, the proportion of measurements $y_i$ that are $-1$, $\frac{|\{i : y_i = -1\}|}{m}$ will approach $\frac{1}{2}$ for either fixed $\tau$ or random $\mathcal{N}(0, \tau)$ thresholds. On the other hand as $\tau$ gets large, $\frac{|\{i : y_i = -1\}|}{m}$ will go to 0 for the deterministically thresholded measurements, and to $\frac{1}{2}$ for thresholds $\sim \mathcal{N}(0, \tau)$. The poor performance at these two extremes yields the U-shaped error graphs in Figure 2. We find that the EDF method may slightly outperform the PV$_{\text{aug}}$ method at the optimal choice of $\tau$, but that PV$_{\text{aug}}$ is more robust, its error increasing more gradually as $\tau$ is increased away from $\|x\|_2$.

5 Conclusions

In this paper we have shown that norm recovery, while impossible from one-bit measurements $\text{sign}(\langle a_i, x \rangle)$, is indeed possible from one-bit measurements of the form $\text{sign}(\langle a_i, x \rangle + b_i)$ for known nonzero $b_i$ and $a_i$ with i.i.d. standard Gaussian entries. We presented two methods for norm recovery, the first of which also produces an estimate of $x$ and uses randomly chosen $b_i$, and the second of which uses fixed, deterministic $b_i$ and produces estimates of $\|x\|_2$ only. In both cases, we present uniform guarantees of accurate recovery (with high probability) given sufficient a number of measurements, provided we have some prior upper bound (or upper and lower bound) for the norm of $x$.

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Figure 2: Error of the reconstructed norm (left) and reconstructed signal (right), and for values of the thresholding parameter $\tau$ relative to $\|x\|_2$. Here $m/n=6$. Results are averaged over 40 trials.

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