EXISTENCE AND STABILITY OF PATTERNS IN A REACTION-DIFFUSION-ODE SYSTEM WITH HYSTERESIS IN NON-UNIFORM MEDIA

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Dedicated to the memory of the late Professor Yuzo Hosono
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Abstract. This paper is concerned with the existence and stability of steady states of a reaction-diffusion-ODE system arising from the theory of biological pattern formation. We are interested in spontaneous emergence of patterns from spatially heterogeneous environments, hence assume that all coefficients in the equations can depend on the spatial variable. We give some sufficient conditions on the coefficients which guarantee the existence of far-from-the-equilibrium patterns with jump discontinuity and then verify their stability in a weak sense. Our conditions cover the case where the number of equilibria of the kinetic system (i.e., without diffusion) changes from one to three in the spatial interval, which is not obtained by a small perturbation of constant coefficients. Moreover, we consider the asymptotic behavior of steady states as the diffusion coefficient tends to infinity. Some examples and numerical simulations are given to illustrate the theoretical results.

1. Introduction. In the pioneering paper [12], Turing proposed that the diffusion-driven-instability (DDI, for short) might account for the spontaneous formation of patterns in developmental biology. Here, DDI is the destabilization of spatially homogeneous state caused by the interaction of two chemical substances with different diffusion rates. Since then DDI has been one of the frameworks of understanding pattern formation in many branches of science. See, for instance, Murray’s monograph [9] for various models in biology, and [13] for mathematical analyses of reaction–diffusion systems exhibiting concentration phenomena.

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However, DDI explains only the onset of pattern as already pointed out by Turing. In fact, patterns appear as a result of nonlinear interactions, while DDI is a local property of constant steady states. Moreover, patterns are formed not only in classical reaction-diffusion systems where all species diffuse, but also in degenerate systems where some species do not diffuse. The latter systems are modeled by reaction-diffusion-ODE systems (see e.g., [1, 14] in ecology, [10] in developmental biology). Sherrat et al. [10] proposed a model consisting of free receptors, bound receptors and ligands, which describes coupling of cell-localized processes with cell-to-cell communication via diffusion in a cell assembly. Free and bound receptors are located on the cell surface and hence do not diffuse. Ligands diffuse and act by binding themselves to receptors and thereby trigger intracellular reactions, leading to cell differentiation. Their model has built-in spatial heterogeneity, which triggers patterning. Marciniak-Czochra [5, 6] proposed two types of receptor-based models without spatial heterogeneity; they involve an auto-catalytic reaction in the free receptor production [5], or hysteresis (existence of two “stable” branches of a nullcline) [6]. These spatially uniform receptor-based models are capable of producing patterns as a result of nonlinear interactions.

In [3, 4], we considered the reaction-diffusion-ODE system of the form

\[
\begin{align*}
  u_t &= -\mu_1(x)u - \gamma(x)uv + m_1(x)\frac{u^2}{1+ku^2}, & x \in [0, L], & t > 0, \\
  v_t &= Dv_{xx} - \mu_2(x)v - \gamma(x)uv + m_2(x)\frac{u^2}{1+ku^2}, & x \in (0, L), & t > 0, \\
  v_x(0, t) &= v_x(L, t) = 0, & t > 0, \\
  (u(x, 0), v(x, 0)) &= (u_0(x), v_0(x)), & x \in (0, L),
\end{align*}
\]

in the case where, in addition to \( D \) and \( k \), all of \( \mu_1(x), \mu_2(x), m_1(x), m_2(x) \) and \( \gamma(x) \) are positive constants. This is also a receptor-based model for pattern formation. For the biological aspects of the model, see [3] and the references therein. Here, we only remark that \( u \) and \( v \) stand for the density of free receptors and ligands, respectively. We interpret that the status of cell changes at the place where \( u \) is large. This model was devised to exhibit both DDI and hysteresis. Assuming that all coefficients are positive constants, let us define

\[
\begin{align*}
  f(u, v) &= -\mu_1u - \gamma uv + m_1\frac{u^2}{1+ku^2}, \\
  g(u, v) &= -\mu_2v - \gamma uv + m_2\frac{u^2}{1+ku^2}.
\end{align*}
\]

Under appropriate conditions on the coefficients, the nullclines \( f(u, v) = 0 \) and \( g(u, v) = 0 \) intersect (a) at exactly one point (the monostable case; see Figure 2 (a) in Section 6), (b) at exactly three points with two on the middle branch of \( f = 0 \) (the DDI case; see Figure 2 (b)) or (c) at exactly three points with one on the middle branch, the other on the right branch of \( f = 0 \) (the bistable case; see Figure 2 (c)). In [3], the DDI case was treated, and the existence of steady states with jump discontinuities in \( u \) was established and their stability in Weinberger’s sense [14] was proved. Also in [4], bifurcation of nonconstant steady states from the constant solution and their instability were proved.

Once we identify substances consisting of the system, the interactions among them are determined. However, the coefficients in the equations represent the speed of reaction, and hence they may depend on the environment. Therefore, when
we study the situation where the size of system, e.g., the size of cell assembly, is relatively large, it is reasonable to consider the coefficients to be dependent on the spatial variable.

In this paper, we are concerned with the case where coefficients depend on the spatial variable $x$ and ask whether problem (1.1) still has steady states with jump discontinuities in $u$. In particular, we are interested in the situation where all of the monostable, DDI and bistable cases coexist in the interval $[0, L]$.

The main results of the paper are described as follows:

First, we fix a positive constant $\beta$ in some interval and try to find a stationary solution $(u(x), v(x))$ of (1.1) for which $u(x)$ undergoes jump discontinuity at every point $x_\beta$ such that $v(x_\beta) = \beta$. The number of points of discontinuity for $u(x)$ is called the mode of $(u(x), v(x))$. In particular, a mode one solution means $v(x)$ is either monotone increasing or monotone decreasing on $[0, L]$. We impose two assumptions (H0) and (H) (see Sections 2 and 3): (H0) ensures that the hysteresis occurs in the first quadrant $u \geq 0$ and $v \geq 0$ for every $x \in [0, L]$. The second condition (H) concerns the monotonicity of $v'(x_\beta)$ in $x_\beta$.

We prove the existence of mode one solution under some assumptions in Theorem 3.1. Also, we prove that, under certain conditions that guarantee the existence of both types of mode one solutions, we can construct stationary solutions of arbitrary mode $n$ (Theorem 3.2). Moreover, in Theorem 3.3, we consider the situation where no monotone solutions exist and prove that there are infinitely many stationary solutions of sufficiently large mode. Combining these results, we may conclude that, under assumptions (H0) and (H), system (1.1) always has a stationary solution $(u(x), v(x))$ with jump discontinuity in $u(x)$.

In Section 3, we prove these results in several steps. In the constant coefficient case, we first construct a monotone increasing solution $v(x)$ on a subinterval $[0, l] \subset [0, L]$ subject to homogeneous Neumann boundary conditions and $v(l - x)$ gives rise to a monotone decreasing solution on $[0, L]$. Then by reflection and periodic extension, we obtain a solution of arbitrary mode due to the translation invariance and the homogeneous Neumann boundary condition. Moreover, construction of monotone solution is relatively easy if we make use of the solution formula for the equation $v'' + f(v) = 0$ (see [3, 7]), which is not applicable to the variable coefficient case. In this paper, in order to construct monotone solutions, we develop the method devised by Mimura, Tabata and Hosono [8], which was originally used to treat the constant coefficient case. We extracted condition (H) from [8], where they checked (H) always holds in the constant coefficient case. It is a contribution of the present paper to single out a class of equations with variable coefficients (satisfying (H)) that admit stationary solutions with jump discontinuity, and this class contains equations with coefficients far from constant (see Example 2 in Section 6). We point out that a new approach is required when we construct higher mode solutions and this makes our arguments much more complicated than in the constant coefficient case.

Furthermore, we consider the case where the diffusion coefficient $D$ is sufficiently large. We prove in Theorem 4.1 that under assumption (H0) only, there are monotone stationary solutions whenever $D$ is sufficiently large. Moreover, we study the asymptotic behavior of solutions as $D \to \infty$ (Theorem 4.4).

Since our results include the case of constant coefficients, we have therefore proved the existence of stationary solutions with jump discontinuity in $u(x)$ for the monostable case and the bistable case, which were not considered in [3].
In Section 5, we prove that all stationary solutions with jump discontinuities in $u(x)$ constructed in this paper are stable in a weak sense introduced by Weinberger (Theorem 5.3).

In Section 6, some examples which satisfy our assumptions on the coefficients are given together with numerical simulations. Example 2 shows that we have a monotone solution even when the monostable, DDI and bistable cases coexist in the interval $[0, L]$. It is to be emphasized that the coexistence of these three cases cannot be achieved by small perturbation of constant coefficients.

2. Preliminaries.

2.1. Equilibria of the kinetic system. In order to study steady states of (1.1), we first study equilibria of the kinetic system

\[
\begin{align*}
  u_t &= f(u, v, x), \\
  v_t &= g(u, v, x),
\end{align*}
\]

where

\[
\begin{align*}
  f(u, v, x) &= -\mu_1(x)u - \gamma(x)uv + m_1(x)\frac{u^2}{1 + ku^2}, \\
  g(u, v, x) &= -\mu_2(x)v - \gamma(x)uv + m_2(x)\frac{u^2}{1 + ku^2}.
\end{align*}
\]

We always assume that $k > 0$, $\mu_1(x) > 0$, $\mu_2(x) > 0$, $m_1(x) > 0$, $m_2(x) > 0$ and $\gamma(x) > 0$. The coefficient $K_d = 1/k$ is called the dissociation constant, and $\sqrt{K_d}$ represents the ligand concentration producing half occupation of receptors; $K_d$ may depend on physical parameters such as temperature. In this paper we assume that $K_d$ is constant throughout the interval $0 \leq x \leq L$, and take the effect of environmental heterogeneity into account through the coefficients $m_1(x)$, $m_2(x)$, as well as $\mu_1(x)$, $\mu_2(x)$ and $\gamma(x)$.

Clearly, if $u = 0$, then the equation $g(u, v, x) = 0$ reduces to $-\mu_2(x)v = 0$, i.e., $v = 0$. Therefore, $(u, v) = (0, 0)$ is an equilibrium of (2.1) for each $x \in [0, L]$. If $u \neq 0$, then $f(u, v, x) = 0$ implies that

\[
m_1(x)\frac{u}{1 + ku^2} = \mu_1(x) + \gamma(x)v,
\]

so that

\[
v = \frac{1}{\gamma(x)} \left( m_1(x)\frac{u}{1 + ku^2} - \mu_1(x) \right). \tag{2.2}
\]

It follows from $g(u, v, x) = 0$ that

\[
v = \frac{m_2(x)}{\mu_2(x) + \gamma(x)u} \frac{u^2}{1 + ku^2}. \tag{2.3}
\]

Putting (2.2) and (2.3) together yields

\[
\frac{1}{\gamma(x)} \left( m_1(x)\frac{u}{1 + ku^2} - \mu_1(x) \right) = \frac{m_2(x)}{\mu_2(x) + \gamma(x)u} \frac{u^2}{1 + ku^2},
\]

namely,

\[
k\mu_1(x)\gamma(x)u^3 + (k\mu_1(x)\mu_2(x) + (m_2(x) - m_1(x))\gamma(x))u^2
\]

\[
+ (\mu_1(x)\gamma(x) - m_1(x)\mu_2(x))u + \mu_1(x)\mu_2(x) = 0. \tag{2.4}
\]
As to the existence of positive roots of (2.4), we can classify the possibilities as follows. First we introduce some notations. Let
\[ b_0(x) = k\mu_1(x)\gamma(x) > 0, \quad b_1(x) = k\mu_1(x)\mu_2(x) + (m_2(x) - m_1(x))\gamma(x), \]
\[ b_2(x) = \mu_1(x)\gamma(x) - m_1(x)\mu_2(x), \quad b_3(x) = \mu_1(x)\mu_2(x) > 0, \]
\[ \Delta(x) = b_2^2(x) - 3b_0(x)b_2(x), \quad u_2(x) = -\frac{b_1(x) + \sqrt{\Delta(x)}}{3b_0(x)}, \]
\[ P(x) = \frac{-2}{9b_0(x)}\Delta(x)u_2(x) + b_3(x) - \frac{b_1(x)b_2(x)}{9b_0(x)}. \]

Then we have the following lemma. For the proof, see Appendix.

**Lemma 2.1.** Let \( \mu_1(x), \mu_2(x), m_1(x), m_2(x) \) and \( \gamma(x) \) be positive functions defined on \([0, L]\).

(i) If one of the following conditions (a)–(e) hold, then \((u_{e,0}(x), v_{e,0}(x)) = (0, 0)\) is the only nonnegative steady state of (2.1):

(a) \( b_2(x) < 0, P(x) > 0 \).
(b) \( b_2(x) = 0, b_1(x) < 0 \) and \( P(x) > 0 \).
(c) \( b_2(x) \geq 0 \) and \( b_1(x) \geq 0 \).
(d) \( b_2(x) > 0, b_1(x) < 0 \) and \( \Delta(x) < 0 \).
(e) \( b_2(x) > 0, b_1(x) < 0, \Delta(x) \geq 0 \) and \( P(x) > 0 \).

(ii) If one of the following conditions (f)–(h) hold, then (2.1) has two positive solutions \((u_{e,1}(x), v_{e,1}(x))\) and \((u_{e,2}(x), v_{e,2}(x))\), which satisfy

\[ 0 < u_{e,2}(x) = u_{e,1}(x), \quad 0 < v_{e,2}(x) \leq v_{e,1}(x) < \nu_r(x) \]

with

\[ \nu_r(x) = \frac{(m_1(x) - 2\sqrt{k\mu_1(x)})/(2\sqrt{k}\gamma(x))}{(m_1(x) - \sqrt{2k\mu_1(x) + \gamma(x)v})} : \quad (2.5) \]

(f) \( b_2(x) < 0 \) and \( P(x) \leq 0 \).
(g) \( b_2(x) = 0, b_1(x) < 0 \) and \( P(x) \leq 0 \).
(h) \( b_2(x) > 0, b_1(x) < 0, \Delta(x) \geq 0 \) and \( P(x) \leq 0 \).

The equality \((u_{e,1}(x), v_{e,1}(x)) = (u_{e,2}(x), v_{e,2}(x))\) occurs if and only if \( P(x) = 0 \).

### 2.2. Properties of nonlinear functions \( f \) and \( g \)

In this subsection, we present some properties of the functions \( f \) and \( g \), which will be used in later sections.

**Proposition 2.2.** Under the assumption that \( \nu_r(x) > 0 \), the equation \( f(u, v, x) = 0 \) defines three distinct branches of roots: \( u = h_0(v, x), \ u = h_1(v, x) \) and \( u = h_2(v, x) \), where

\[ h_0(v, x) = 0 \quad \text{for} \quad v \in J_0, \]
\[ h_1(v, x) = \frac{m_1(x)}{2k\mu_1(x) + \gamma(x)v} + \frac{\sqrt{m_1^2(x) - 4k\mu_1(x) + \gamma(x)v}}{2k\mu_1(x) + \gamma(x)v} \quad \text{for} \quad v \in J_1, \]
\[ h_2(v, x) = \frac{m_1(x)}{2k\mu_1(x) + \gamma(x)v} - \frac{\sqrt{m_1^2(x) - 4k\mu_1(x) + \gamma(x)v}}{2k\mu_1(x) + \gamma(x)v} \quad \text{for} \quad v \in J_2. \]

Here, \( J_0 = (-\infty, +\infty) \), \( J_1 = (-\mu_1(x)/\gamma(x), \nu_r(x)) \) and \( J_2 = (-\mu_1(x)/\gamma(x), \nu_r(x)) \).

**Proof.** For notational convenience, we let

\[ p(u, v, x) = -\mu_1(x) - \gamma(x)v + m_1(x) \frac{u}{1 + ku^2}. \]

Obviously, \( f(u, v, x) = 0 \) if and only if either \( u = 0 \) or \( p(u, v, x) = 0 \). It is easy to see that \( p(u, v, x) = 0 \) has two distinct positive roots: \( u = h_1(v, x) \) for \( v \in (-\mu_1(x)/\gamma(x), \nu_r(x)) \) and \( u = h_2(v, x) \) for \( v \in (-\nu_r(x), \nu_r(x)) \).
Proposition 2.3. Let \( m_2(x) \geq m_1(x) \) for all \( x \in [0, L] \). Then there exist two subintervals \( J_i^* \subset J_i \) (\( i = 0, 1 \)) such that

(i) \( g(h_0(v,x),v,x) < 0 \) and \( \frac{\partial}{\partial v} g(h_0(v,x),v,x) < 0 \) for \( v \in J_0^* \);

(ii) \( g(h_1(v,x),v,x) > 0 \) and \( \frac{\partial}{\partial v} g(h_1(v,x),v,x) < 0 \) for \( v \in J_1^* \);

where \( J_0^* = (0, +\infty) \), \( J_1^* = (\frac{-\mu_1(x)}{\gamma(x)}, A(x)) \), \( A(x) \) will be defined in the proof (immediately after (2.6)).

Proof. (i) From the definitions of \( g(u,v) \) and \( h_0(v,x) \), we immediately obtain

\[
g(h_0(v,x), v,x) = -\mu_2(x) v < 0 \quad \text{for} \quad v \in (0, +\infty),
\]

\[
\frac{\partial}{\partial v} g(h_0(v,x), v,x) = -\mu_2(x) < 0 \quad \text{for} \quad v \in (0, +\infty).
\]

(ii) In view of \( m_1(x) u/(1 + ku^2) = \mu_1(x) + \gamma(x)v \), it is straightforward to obtain

\[
g(h_1(v,x), v,x) = -\mu_2(x) v - \gamma(x) h_1(v,x)v + \frac{m_2(x)}{m_1(x)} (\mu_1(x) + \gamma(x)v) h_1(v,x),
\]

and

\[
\frac{\partial}{\partial v} g(h_1(v,x), v,x) = -\mu_2(x) - \gamma(x) \frac{\partial h_1}{\partial v}(v,x)v + \frac{m_2(x)}{m_1(x)} \mu_1(x) \frac{\partial h_1}{\partial v}(v,x)
\]

\[
- \gamma(x) h_1(v,x) + \frac{m_2(x)}{m_1(x)} \gamma(x) \left( h_1(v,x) + \frac{\partial h_1}{\partial v}(v,x)v \right). \tag{2.6}
\]

For the simplicity of notations, we denote \( A(x) = v_r(x) \), if the kinetic system (2.1) has no positive equilibria or the two positive equilibria are on the branch \( u = h_2(v,x) \); while \( A(x) = v_{r,1}(x) \), if one of the positive equilibria is on the branch \( u = h_1(v,x) \), and the other is on \( u = h_2(v,x) \). Moreover, let

\[
r(v,x) = \frac{m_1(x)}{2k(\mu_1(x) + \gamma(x)v)}, \quad s(v,x) = \frac{\sqrt{\Delta_1(x)}}{2k(\mu_1(x) + \gamma(x)v)},
\]

where \( \Delta_1(x) = m_1^2(x) - 4k(\mu_1(x) + \gamma(x)v)^2 \). By direct calculation, we obtain

\[
\frac{\partial r}{\partial v}(v,x) = -\frac{m_1(x) \gamma(x)}{2k(\mu_1(x) + \gamma(x)v)^2}, \quad \frac{\partial s}{\partial v}(v,x) = -\frac{m_1^2(x) \gamma(x)}{2k(\mu_1(x) + \gamma(x)v)^2 \sqrt{\Delta_1(x)}}.
\]

Note that

\[
h_1(v,x) = r(v,x) + s(v,x), \quad \frac{\partial h_1}{\partial v}(v,x) = \frac{\partial r}{\partial v}(v,x) + \frac{\partial s}{\partial v}(v,x).
\]

Substituting these into (2.6), we obtain

\[
\frac{\partial}{\partial v} g(h_1(v,x), v,x) = -\frac{\mu_1(x) m_1(x) \gamma(x)}{2k(\mu_1(x) + \gamma(x)v)^2} - \frac{m_1^2(x) \mu_1(x) \gamma(x)}{2k(\mu_1(x) + \gamma(x)v)^2 \sqrt{\Delta_1(x)}}
\]

\[
-\mu_2(x) - \frac{2(m_2(x) - m_1(x))(\mu_1(x) + \gamma(x)v) \gamma(x)}{m_1(x) \sqrt{\Delta_1(x)}}.
\]

Recalling that \( m_2(x) \geq m_1(x) \), we conclude that \( \frac{\partial}{\partial v} g(h_1(v,x), v,x) < 0 \) for all \( v \in (-\frac{\mu_1(x)}{\gamma(x)}, A(x)) \). To verify \( g(h_1(v,x), v,x) > 0 \), we consider the following three cases, separately:

**Case I.** \( f(u,v) = g(u,v,x) = 0 \) have no positive intersection point, namely, \((0,0)\) is the only equilibrium of (2.1). In this case, we observe that the curve
We thus have completed the proof of Proposition 2.3.

Case II. If \( f(u, v, x) = g(u, v, x) = 0 \) have two positive intersection points and both of them are on the branch \( u = h_2(v, x) \). Since \( \frac{\partial g}{\partial v} h_1(v, x, v, x) < 0 \), we have \( g(h_1(v, x), v, x) \) is strictly decreasing in \( \left( -\frac{\mu_1(x)}{\gamma(x)}, A(x) \right) \). This means that

\[
g(h_1(v, x), v, x) > g(h_1(v, x), v, x) = g(h_2(v, x), v, x).
\]

On the other hand,

\[
g(h_2(0, x), 0, x) = m_2(x) \frac{(h_2(0, x))^2}{1 + k(h_2(0, x))^2} > 0,
\]

and if there exist two equilibria on the branch \( u = h_2(v, x) \), then \( g(h_2(v, x), v, x) > 0 \). Hence, \( g(h_1(v, x), v, x) > 0 \) for \( v \in \left( -\frac{\mu_1(x)}{\gamma(x)}, A(x) \right) \).

Case III. If \( f(u, v, x) = g(u, v, x) = 0 \) have two positive intersection points (\( u_{e,1}(x), v_{e,1}(x) \)) and (\( u_{e,2}(x), v_{e,2}(x) \)) where \( (u_{e,1}(x), v_{e,1}(x)) \) is on the branch \( u = h_1(v, x) \), and \( (u_{e,2}(x), v_{e,2}(x)) \) is on \( u = h_2(v, x) \). In view of the monotonicity of \( g(h_1(v, x), v, x) \), we are led to

\[
g(h_1(v, x), v, x) > g(h_1(v, x), v, x) = g(h_2(v, x), v, x).
\]

Since \( g(h_1(v, x), v, x) = 0 \), we have \( g(h_1(v, x), v, x) > 0 \) for \( v \in \left( -\frac{\mu_1(x)}{\gamma(x)}, A(x) \right) \).

We thus have completed the proof of Proposition 2.3.

In what follows, we always assume that \( m_1(x), m_2(x) \) and \( \mu_1(x) \) satisfy

\[ \text{(H0) } m_2(x) \geq m_1(x) > 2\sqrt{k}\mu_1(x) \quad \text{for all } x \in [0, L]. \]

Therefore, \( v_e(x) > 0 \) for all \( x \in [0, L] \), and hence we have \( A(x) > 0 \) for all \( x \in [0, L] \).

3. Existence of steady states with jump discontinuity. In this section, we construct various types of stationary solutions under the assumption (H0). The stationary problem associated with (1.1) reads as

\[
\begin{dcases}
 f(u, v) = 0, & x \in [0, L], \\
 Dv'' + g(u, v) = 0, & x \in (0, L), \\
 v'(0) = v'(L) = 0.
\end{dcases}
\]

From the first equation of (3.1), three different relations, \( u = h_0(v, x) \), \( u = h_1(v, x) \) and \( u = h_2(v, x) \) are obtained. Using the functions \( h_0(v, x) \) and \( h_1(v, x) \) in the second equation of (3.1), we reduce (3.1) to a single equation involving a discontinuous nonlinearity, i.e.,

\[
\begin{dcases}
 Dv'' + g^\beta(v, x) = 0, & x \in (0, L), \\
v'(0) = v'(L) = 0,
\end{dcases}
\]

where

\[
g^\beta(v, x) = \begin{cases} 
 g(h_0(v, x), v, x) & \text{for } v < \beta, \\
 g(h_1(v, x), v, x) & \text{for } v > \beta,
\end{cases}
\]

and \( \beta \) is an arbitrarily fixed number in \( (0, d) \) with \( d = \min_{x \in [0, L]} A(x) \). For simplicity of notation, we denote

\[
g_0(v, x) = g(h_0(v, x), v, x), \quad g_1(v, x) = g(h_1(v, x), v, x).
\]
Throughout the paper, by a solution $v$ of (3.2), we mean that $v \in C^1([0, L])$, $v''$ is piecewise continuous on $[0, L]$ and $v$ satisfies the first equation of (3.2) at all points of continuity of $v''$.

Before stating our main results, we first extend the domain of definition of the functions $\mu_1(x)$, $\mu_2(x)$, $m_1(x)$, $m_2(x)$ and $\gamma(x)$ to $\mathbb{R}$ in the following way: $\mu_i \in C^1(\mathbb{R})$, $\mu_i(x) \in [\frac{1}{2} \mu_i(0), \frac{3}{2} \mu_i(0)]$ for $x < 0$, $\mu_i(x) \in [\frac{1}{2} \mu_i(L), \frac{3}{2} \mu_i(L)]$ for $x > L$, $i = 1, 2$. In addition, we require that (H0) is satisfied for all $x \in \mathbb{R}$. As to $m_1(x)$, $m_2(x)$ and $\gamma(x)$, we extend them in the same way as $\mu_i(x)$. Then, taking a $\lambda$ in the interval $[0, L]$, we consider the following two initial value problems:

$$
\begin{align*}
Dv'' + g_0(v, x) &= 0, && x \in (0, L), \\
v'(\lambda) &= 0, && v(\lambda) = a_0,
\end{align*}
$$

(3.3)

and

$$
\begin{align*}
Dv'' + g_1(v, x) &= 0, && x \in (0, L), \\
v'(\lambda) &= 0, && v(\lambda) = a_1,
\end{align*}
$$

(3.4)

where $0 < a_0 < \beta < a_1 < d$.

Let $V_0(x; a_0, \lambda)$ and $V_1(x; a_1, \lambda)$ be unique solutions of (3.3) and (3.4), respectively. Note that $V_0''(x; a_0, \lambda) > 0$ and $V_1''(x; a_1, \lambda) < 0$. Hence, $V_{0,+}(x) = V_0(x; a_0, 0)$ and $V_{1,+}(x) = V_1(x; a_1, L)$ are monotone increasing and $V_{0,-}(x) = V_0(x; a_0, L)$ and $V_{1,-}(x) = V_1(x; a_1, 0)$ are monotone decreasing, as long as they exist.

We can prove that there exists at least one pair of points $\{x_{1,0}, y_{1,0}\}$ with $x_{1,0} > 0$ and $y_{1,0} < L$ such that $V_{0,+}(x_{1,0}) = V_{1,+}(y_{1,0}) = \beta$ and $V_{0,+}(x_{1,0}) = V_{1,+}(y_{1,0})$. We define $\{X_{1,0}, Y_{1,0}\}$ to be the maximal pair in the following sense: $x_{1,0} \leq X_{1,0}$ and $Y_{1,0} \leq y_{1,0}$ for any pair of $\{x_{1,0}, y_{1,0}\}$. Similarly, we define $\{x_{2,0}, y_{2,0}\}$ by $V_{0,-}(x_{2,0}) = V_{1,-}(y_{2,0}) = \beta$ and $V_{0,-}(x_{2,0}) = V_{1,-}(y_{2,0})$ with $x_{2,0} < L$ and $y_{2,0} > 0$. Moreover, we let $\{X_{2,0}, Y_{2,0}\}$ be the maximal in the sense that $X_{2,0} \leq x_{2,0} \leq y_{2,0}$ and $y_{2,0} \geq y_{2,0}$ for any $\{x_{2,0}, y_{2,0}\}$. For the existence of the pair $\{x_{i,0}, y_{i,0}\}$ ($i = 1, 2$), see the paragraph immediately after Lemma 3.7 below. Next we set

$$
\begin{align*}
\xi_{0, +}(x_{1,0}) &= \frac{\partial V_{0,+}}{\partial x}(x_{1,0}), && \eta_{0, +}(x_{1,0}) = \frac{\partial V_{0,+}}{\partial a_1}(x_{1,0}), \\
\xi_{1, +}(y_{1,0}) &= \frac{\partial V_{1,+}}{\partial x}(y_{1,0}), && \eta_{1, +}(x_{1,0}) = \frac{\partial V_{1,+}}{\partial a_1}(y_{1,0}), \\
\xi_{0, -}(x_{2,0}) &= \frac{\partial V_{0,-}}{\partial x}(x_{2,0}), && \eta_{0, -}(x_{2,0}) = \frac{\partial V_{0,-}}{\partial a_1}(x_{2,0}), \\
\xi_{1, -}(y_{2,0}) &= \frac{\partial V_{1,-}}{\partial x}(y_{2,0}), && \eta_{1, -}(y_{2,0}) = \frac{\partial V_{1,-}}{\partial a_1}(y_{2,0}).
\end{align*}
$$

and impose the following conditions:

(H) (i) \( \left( \frac{\partial \xi_{0, +}}{\partial x} \eta_{0, +} - \frac{\partial \eta_{0, +}}{\partial x} \xi_{0, +} \right) \bigg|_{x = x_{1,0}} > 0 \), \( \left( \frac{\partial \xi_{0, -}}{\partial x} \eta_{0, -} - \frac{\partial \eta_{0, -}}{\partial x} \xi_{0, -} \right) \bigg|_{x = x_{2,0}} > 0 \),

(ii) \( \left( \frac{\partial \xi_{1, +}}{\partial x} \eta_{1, +} - \frac{\partial \eta_{1, +}}{\partial x} \xi_{1, +} \right) \bigg|_{x = y_{1,0}} < 0 \), \( \left( \frac{\partial \xi_{1, -}}{\partial x} \eta_{1, -} - \frac{\partial \eta_{1, -}}{\partial x} \xi_{1, -} \right) \bigg|_{x = y_{2,0}} < 0 \).

We say that a solution $v(x)$ of (3.2) is of mode $n$ if $v'(x)$ has precisely $n - 1$ zeros in the interval $(0, L)$.

The following three theorems are main results of this section and combined together, they imply that under assumptions (H0) and (H) system (1.1) has at least
one steady-state solution with jump discontinuity in $u$. First, we give a sufficient condition for the existence of a monotone increasing (or decreasing) solution.

**Theorem 3.1.** (Monotone solutions) Suppose that (H0) and (H) are satisfied.
(i) If $X_{1, \beta} \geq Y_{1, \beta}$, then problem (3.2) has a unique increasing solution.
(ii) If $X_{2, \beta} \leq Y_{2, \beta}$, then problem (3.2) has a unique decreasing solution.

Next, we show that if both of monotone solutions exist, we have in fact multimode solutions.

**Theorem 3.2.** (Mode $n$ solutions for any $n$) Suppose that (H0) and (H) are satisfied. If $X_{1, \beta} \geq Y_{1, \beta}$ and $X_{2, \beta} \leq Y_{2, \beta}$, then problem (3.2) has two types of mode $n$ solutions for any positive integer $n$.

Finally, we prove that even if neither of monotone solutions exist, we have multimode solutions.

**Theorem 3.3.** (Mode $n$ solutions for sufficiently large $n$) Suppose that (H0) and (H) are satisfied. If $X_{1, \beta} < Y_{1, \beta}$ and $X_{2, \beta} > Y_{2, \beta}$, then problem (3.2) has two types of mode $n$ solutions for any sufficiently large positive integer $n$. 

---

**Figure 1.** The relationship between $X_{i, \beta}$ and $Y_{i, \beta}$. Cases (a) and (b) are exclusive each other; and cases (c) and (d) are exclusive each other. Theorem 3.2 treats cases (a) and (c); Theorem 3.1 (i) deals with case (a); Theorem 3.1 (ii) deals with case (c) and Theorem 3.3 treats cases (b) and (d).
See Section 6 for examples which satisfy (H0) and (H). In the following, we prove Theorems 3.1-3.3 in several steps.

**Lemma 3.4.** Let (H0) be satisfied. (i) The initial value problem (3.3) has a unique solution \( V_0(x; a_0, \lambda) \) in \( \mathbb{R} \). Moreover, let \( \xi_0(x; a_0, \lambda) = \frac{\partial V_0}{\partial x}(x; a_0, \lambda) \) and \( \eta_0(x; a_0, \lambda) = \frac{\partial^2 V_0}{\partial x^2}(x; a_0, \lambda) \). Then for any \( a_0 \in (0, d) \), we have

(a) \( \xi_0(x; a_0, \lambda) > 0 \) and \( \eta_0(x; a_0, \lambda) > 1 \) for all \( x \in (\lambda, +\infty) \);
(b) \( \xi_0(x; a_0, \lambda) < 0 \) and \( \eta_0(x; a_0, \lambda) > 1 \) for all \( x \in (-\infty, \lambda) \).

(ii) The initial value problem (3.4) has a unique solution \( V_1(x; a_1, \lambda) \), in the maximal existence interval \( I_{M, \lambda} = [X_1(a_1, \lambda), X_2(a_1, \lambda)] \), where \( V_1(x; a_1, \lambda) > 0 \) if \( X_1(a_1, \lambda) < x < X_2(a_1, \lambda) \) and \( V_1(X_1(a_1, \lambda); a_1, \lambda) = 0 \) for \( i = 1, 2 \). Moreover, let \( \xi_1(x; a_1, \lambda) = \frac{\partial V_1}{\partial x}(x; a_1, \lambda) \), \( \eta_1(x; a_1, \lambda) = \frac{\partial^2 V_1}{\partial x^2}(x; a_1, \lambda) \). Then for any \( a_1 \in (0, d) \), there exist \( 0 \leq \tilde{X}_1(a_1, \lambda) < L \) and \( 0 < \tilde{X}_2(a_1, \lambda) \leq L \) such that

(a) \( \xi_1(x; a_1, \lambda) > 0 \) and \( \eta_1(x; a_1, \lambda) > 1 \) for all \( x \in [\tilde{X}_1(a_1, \lambda); \lambda) \);
(b) \( \xi_1(x; a_1, \lambda) < 0 \) and \( \eta_1(x; a_1, \lambda) > 1 \) for all \( x \in (\lambda, \tilde{X}_2(a_1, \lambda)] \).

**Proof.** (i) We remark that the initial value problem (3.3) has a unique solution for all \( x > 0 \) since \( g_0(V_0, x) = -\mu_2(x)V_0 \). Therefore, \( V_0''(x) > 0 \), which implies that \( V_0''(x) \) is strictly increasing in \( x \). Due to the initial condition \( V_0'(\lambda) = 0 \), we find that \( \xi_0(x; a_0, \lambda) > 0 \) for \( x \in (\lambda, +\infty) \), while \( \xi_0(x; a_0, \lambda) < 0 \) for \( x \in (-\infty, \lambda) \). Furthermore, \( \eta_0(x; a_0, \lambda) \) is a solution of

\[
\begin{cases}
D\eta_0'' + \frac{\partial g_0}{\partial V}(V_0, x)\eta_0 = 0, & x \in (0, L), \\
\eta_0'(\lambda; a_0, \lambda) = 0, & \eta_0(\lambda; a_0, \lambda) = 1.
\end{cases}
\]

Note that \( \frac{\partial g_0}{\partial V}(V_0, x) < 0 \) by Proposition 2.3 (i). Therefore,

\[
\eta_0''(\lambda; a_0, \lambda) = -\frac{1}{D} \frac{\partial g_0}{\partial V}(V_0, \lambda)\eta_0(\lambda; a_0, \lambda) > 0.
\]

We expand \( \eta_0(x; a_0, \lambda) \) in \( x \) near \( \lambda \):

\[
\eta_0(x; a_0, \lambda) = \eta_0(\lambda; a_0, \lambda) + \eta_0'(\lambda; a_0, \lambda)(x - \lambda) + \frac{1}{2} \eta_0''(\lambda + \theta(x - \lambda); a_0, \lambda)(x - \lambda)^2
\]

\[
= \eta_0(\lambda; a_0, \lambda) + \frac{1}{2} \eta_0''(\lambda + \theta(x - \lambda); a_0, \lambda)(x - \lambda)^2
\]

\[
> 0,
\]

where \( 0 < \theta < 1 \). Hence, as long as \( \eta_0(x; a_0, \lambda) > 0 \), we have \( D\eta_0'' = -\frac{\partial g_0}{\partial V}(V_0, x)\eta_0 > 0 \), which means that \( \eta_0''(x; a_0, \lambda) \) is strictly increasing in \( x \). Combining this with the initial condition \( \eta_0''(\lambda; a_0, \lambda) = 0 \), we have \( \eta_0''(x; a_0, \lambda) > 0 \) for \( x \in (\lambda, +\infty) \) while \( \eta_0''(x; a_0, \lambda) < 0 \) for \( x \in (-\infty, \lambda) \). It follows that \( \eta_0(x; a_0, \lambda) \) is strictly increasing in \( x \) for \( x \in (\lambda, +\infty) \), whereas it is decreasing in \( x \) for \( x \in (-\infty, \lambda) \). From \( \eta_0(\lambda; a_0, \lambda) = 1 \), we immediately see that \( \eta_0(x; a_0, \lambda) > 1 \) for all \( x \in (-\infty, \lambda) \cup (\lambda, +\infty) \).

(ii) From \( g_1(V_1, x) > 0 \), we obtain \( V_1''(x) < 0 \). Since \( V_1'(\lambda) = 0 \), we see that \( V_1' > 0 \) for \( x \in [X_1(a_1, \lambda), \lambda) \), while \( V_1' < 0 \) for \( x \in (\lambda, X_2(a_1, \lambda)] \). Therefore, there are four cases to be considered.

**Case I.** \( X_1(a_1, \lambda) < 0 \).

**Case II.** \( 0 \leq X_1(a_1, \lambda) < \lambda \).

**Case III.** \( X_2(a_1, \lambda) > L \).

**Case IV.** \( \lambda < X_2(a_1, \lambda) \leq L \).
We define $\tilde{X}_1(a_1, \lambda) = 0$ in Case I, $\tilde{X}_1(a_1, \lambda) = X_1(a_1, \lambda)$ in Case II, and $\tilde{X}_2(a_1, \lambda) = L$ in Case III, $\tilde{X}_2(a_1, \lambda) = X_2(a_1, \lambda)$ in Case IV. This means that $\xi_1(x; a_1, \lambda) > 0$ for $x \in [\tilde{X}_1(a_1, \lambda), \lambda]$, whereas $\xi_1(x; a_1, \lambda) < 0$ for $x \in (\lambda, \tilde{X}_2(a_1, \lambda)]$. Moreover, it is easy to check that $\eta_1(x; a_1, \lambda)$ satisfies

$$
\begin{cases}
D\eta_1'' + \frac{\partial g_1}{\partial v}(V_1, x)\eta_1 = 0, & x \in (0, L), \\
\eta_1'(\lambda; a_1, \lambda) = 0, & \eta_1(\lambda; a_1, \lambda) = 1.
\end{cases}
$$

Recall that $\frac{\partial g_1}{\partial v}(V_1, x) < 0$ from Proposition 2.3 (ii). In the same way as above, we obtain $\eta_1(x; a_1, \lambda) > 1$ for $x \in (\tilde{X}_1(a_1, \lambda), \lambda) \cup (\lambda, \tilde{X}_2(a_1, \lambda))$. \hfill \Box

Lemma 3.5. Assume that (H0) is satisfied. For $i = 0, 1$, let $x = l_{i,+}(a_i; \lambda, \beta)$ be the unique solution of $V_i(x; a_i, \lambda) = \beta$ in the interval $(\lambda, +\infty)$, and $x = l_{i,-}(a_i; \lambda, \beta)$ be the unique solution of $V_i(x; a_i, \lambda) = \beta$ in the interval $(-\infty, \lambda)$. Then

(i) $l_{0,+}$ is decreasing with respect to $a_0$, while $l_{0,-}$ is increasing with respect to $a_0$;

(ii) $l_{1,+}$ is increasing with respect to $a_1$, while $l_{1,-}$ is decreasing with respect to $a_1$.

Proof. Here we only give the proof of properties of $l_{0,+}$ and $l_{1,-}$, since we can treat $l_{0,-}$ and $l_{1,+}$ in the same way. By differentiating $V_0(l_{0,+}(a_0); a_0, \lambda) = \beta$ and $V_1(l_{1,-}(a_1); a_1, \lambda) = \beta$ with respect to $a_0$ and $a_1$, respectively, we derive

$$
\frac{\partial V_0}{\partial x} \frac{\partial l_{0,+}}{\partial a_0} + \frac{\partial V_0}{\partial a_0} = 0 \quad \text{and} \quad \frac{\partial V_1}{\partial x} \frac{\partial l_{1,-}}{\partial a_1} + \frac{\partial V_1}{\partial a_1} = 0,
$$

which yield

$$
\frac{\partial l_{0,+}}{\partial a_0} = \frac{\partial V_0}{\partial a_0} / \frac{\partial V_0}{\partial x} = -\frac{\eta_0(x; a_0, \lambda)}{\xi_0(x; a_0, \lambda)} < 0, \\
\frac{\partial l_{1,-}}{\partial a_1} = -\frac{\partial V_1}{\partial a_1} / \frac{\partial V_1}{\partial x} = -\frac{\eta_1(x; a_1, \lambda)}{\xi_1(x; a_1, \lambda)} < 0.
$$

This completes the proof of the lemma. \hfill \Box

From Lemma 3.5, we know that for $i = 0, 1$, the functions $l_{i,+}(a_i; \lambda, \beta)$ have the inverses $a_i = a_i(l_{i,+}; \lambda, \beta)$. Set $\psi_{i,+}(l_{i,+}; \lambda, \beta) = \frac{\partial a_i}{\partial x}(l_{i,+}; a_i(l_{i,+}; \lambda, \beta), \lambda)$ and $\psi_{i,+}^*(a_i; \lambda, \beta) = \frac{\partial a_i}{\partial x}(l_{i,+}(a_i; \lambda, \beta); a_i, \lambda)$. The following condition (H1) is frequently used in the proof of our theorems and is reduces to (H) when $\lambda = 0$ or $\lambda = L$:

(H1) \quad (i) \left( \frac{\partial a_i}{\partial x} \eta_0 - \frac{\partial a_i}{\partial x} \eta_0 \right)_{(l_{i,0}; \lambda, \beta)} > 0; \quad (ii) \left( \frac{\partial a_i}{\partial x} \eta_1 - \frac{\partial a_i}{\partial x} \xi_1 \right)_{(l_{i,1}; \lambda, \beta)} < 0.

Lemma 3.6. Suppose that (H0) and (H1) are satisfied. Then

(i) the function $\psi_{0,+}(l_{0,+}; \lambda, \beta)$ is positive and increasing with respect to $l_{0,+}$, while $\psi_{0,+}^*(a_0; \lambda, \beta) > 0$ and it is decreasing with respect to $a_0$;

(ii) the function $\psi_{0,-}(l_{0,-}; \lambda, \beta)$ is negative and increasing with respect to $l_{0,-}$, the function $\psi_{0,-}^*(a_0; \lambda, \beta)$ is negative and also increasing with respect to $a_0$;

(iii) the function $\psi_{1,-}(l_{1,-}; \lambda, \beta)$ is positive and decreasing with respect to $l_{1,-}$, while $\psi_{1,-}^*(a_1; \lambda, \beta) > 0$ and it is increasing with respect to $a_1$;

(iv) the function $\psi_{1,+}(l_{1,+}; \lambda, \beta)$ is negative and decreasing with respect to $l_{1,+}$, and $\psi_{1,+}^*(a_1; \lambda, \beta) < 0$ and it is decreasing with respect to $a_1$.

Here, by decreasing and increasing we mean strictly decreasing and strictly increasing, respectively.
Proof. We only prove the assertions for \( \psi_{0,+}(l_{0,+}; \lambda, \beta) \) and \( \psi_{1,-}(l_{1,-}; \lambda, \beta) \). Integrating \( V''_0 = -\frac{1}{D}g_0(V_0, x) \) from \( \lambda \) to \( l_{0,+} \), we have
\[
\int_{\lambda}^{l_{0,+}} V''_0 \, dx = -\frac{1}{D} \int_{\lambda}^{l_{0,+}} g_0(V_0, x) \, dx > 0,
\]
which means that \( \psi_{0,+}(l_{0,+}; \lambda, \beta) > 0 \). Noting that \( \xi_0 = \frac{\partial V_0}{\partial x}, \eta_0 = \frac{\partial V_0}{\partial a_0} \), we differentiate \( \psi_{0,+}(l_{0,+}; \lambda, \beta) \) by \( l_{0,+} \) and obtain
\[
\frac{d}{dl_{0,+}} \psi_{0,+}(l_{0,+}; \lambda, \beta) = \frac{\partial \xi_0}{\partial x} + \frac{\partial \xi_1}{\partial a_1} \frac{\partial a_0}{\partial l_{0,+}} = \frac{\partial \xi_0}{\partial x} - \frac{\partial \xi_1}{\partial a_1} \eta_0(l_{0,+}; \lambda, \beta) = \frac{1}{\eta_0(l_{0,+}; \lambda, \beta)} \left( \frac{\partial \xi_0}{\partial x} \eta_0 - \frac{\partial \eta_0}{\partial x} \xi_0 \right)_{(l_{0,+}; \lambda, \beta)}.
\]
By (H1) (i), we obtain \( \frac{d}{dl_{0,+}} \psi_{0,+}(l_{0,+}; \lambda, \beta) > 0 \). Hence, \( \psi_{0,+}(l_{0,+}; \lambda, \beta) \) is increasing with respect to \( l_{0,+} \). Similarly, we find that \( \psi_{0,+}(a_0; \lambda, \beta) > 0 \) and it is decreasing with respect to \( a_0 \).

Now we integrate \( V''_1 = -\frac{1}{D}g_1(V_1, x) \) from \( l_{1,-} \) to \( \lambda \) and obtain
\[
\int_{l_{1,-}}^{\lambda} V''_1 \, dx = -\frac{1}{D} \int_{l_{1,-}}^{\lambda} g_1(V_1, x) \, dx < 0.
\]
This implies that \( \psi_{1,-}(l_{1,-}; \lambda, \beta) > 0 \). In a similar fashion,
\[
\frac{d}{dl_{1,-}} \psi_{1,-}(l_{1,-}; \lambda, \beta) = \frac{\partial \xi_1}{\partial x} - \frac{\partial \xi_1}{\partial a_1} \eta_1(l_{1,-}; \lambda, \beta) = \frac{1}{\eta_1(l_{1,-}; \lambda, \beta)} \left( \frac{\partial \xi_1}{\partial x} \eta_1 - \frac{\partial \eta_1}{\partial x} \xi_1 \right)_{(l_{1,-}; \lambda, \beta)}.
\]
The assumption (H1) (ii) yields \( \frac{d}{dl_{1,-}} \psi_{1,-}(l_{1,-}; \lambda, \beta) < 0 \). Consequently, \( \psi_{1,-}(l_{1,-}; \lambda, \beta) \) is decreasing with respect to \( l_{1,-} \). In the same way, we see that \( \psi_{1,-}(a_1; \lambda, \beta) > 0 \) and it is increasing with respect to \( a_1 \).

From Lemma 3.6, we know that the inverse functions with respect to the first variable of \( \psi_{i,+}(l_{i,+}; \lambda, \beta) \) (\( i = 0, 1 \)) do exist. Therefore, we denote the inverses of \( \alpha_{i,\pm} = \psi_{i,\pm}(l_{i,\pm}; \lambda, \beta) \) by \( l_{i,\pm} = s_{i,\pm}(\alpha_{i,\pm}; \lambda, \beta) \) for \( i = 0, 1 \).

**Lemma 3.7.** The functions \( s_{0,+}(\alpha_{0,+}; \lambda, \beta) \) and \( s_{0,-}(\alpha_{0,-}; \lambda, \beta) \) are increasing with respect to \( \alpha_{0,+} \) and \( \alpha_{0,-} \), respectively, while \( s_{1,+}(\alpha_{1,+}; \lambda, \beta) \) and \( s_{1,-}(\alpha_{1,-}; \lambda, \beta) \) are decreasing with respect to \( \alpha_{1,+} \) and \( \alpha_{1,-} \), respectively.

**Proof.** We only prove the assertions for \( s_{0,+} \) and \( s_{1,-} \). Obviously,
\[
\frac{ds_{0,+}}{d\alpha_{0,+}}(\alpha_{0,+}; \lambda, \beta) = \frac{d\psi_{0,+}^{-1}}{d\alpha_{0,+}}(\alpha_{0,+}; \lambda, \beta) = \frac{1}{\frac{d\psi_{0,+}}{d\alpha_{0,+}}(\alpha_{0,+}; \lambda, \beta)} > 0,
\]
\[
\frac{ds_{1,-}}{d\alpha_{1,-}}(\alpha_{1,-}; \lambda, \beta) = \frac{d\psi_{1,-}^{-1}}{d\alpha_{1,-}}(\alpha_{1,-}; \lambda, \beta) = \frac{1}{\frac{d\psi_{1,-}}{d\alpha_{1,-}}(\alpha_{1,-}; \lambda, \beta)} < 0.
\]
In order to construct solutions of (3.2), we introduce some notations. Set
\[
\tilde{l}_{0,\pm}(0; \lambda, \beta) = \limsup_{a_0 \to 0} l_{0,\pm}(a_0; \lambda, \beta), \quad \tilde{l}_{0,\pm}(0; \lambda, \beta) = \liminf_{a_0 \to 0} l_{0,\pm}(a_0; \lambda, \beta),
\]
\[
\tilde{l}_{1,\pm}(d; \lambda, \beta) = \liminf_{a_1 \to d} l_{1,\pm}(a_1; \lambda, \beta), \quad \tilde{l}_{1,\pm}(d; \lambda, \beta) = \limsup_{a_1 \to d} l_{1,\pm}(a_1; \lambda, \beta),
\]
\[
\bar{\psi}_{i,\pm} (\lambda, \beta) = \lim_{l_{i,\pm} \to l_{i,\pm}} \psi_{i,\pm} (l_{i,\pm}; a_i, \lambda, \beta) \quad \text{for } i = 0, 1.
\]
\[
\bar{a}_0 (\lambda, \beta) = \min \{ \bar{\psi}_{0,\pm} (\lambda, \beta), \bar{\psi}_{1,\pm} (\lambda, \beta) \},
\]
\[
\bar{a}_1 (\lambda, \beta) = \max \{ \bar{\psi}_{0,\pm} (\lambda, \beta), \bar{\psi}_{1,\pm} (\lambda, \beta) \},
\]
\[
\bar{s}_{0,\pm} (\lambda, \beta) = \lim_{a_{0,\pm} \to \bar{a}_0} s_{0,\pm} (a_{0,\pm}; \lambda, \beta), \quad \bar{s}_{1,\pm} (\lambda, \beta) = \lim_{a_{1,\pm} \to \bar{a}_1} s_{1,\pm} (a_{1,\pm}; \lambda, \beta),
\]
\[
\bar{s}_{0,\pm} (\lambda, \beta) = \lim_{a_{0,\pm} \to \bar{a}_0} s_{0,\pm} (a_{0,\pm}; \lambda, \beta), \quad \bar{s}_{1,\pm} (\lambda, \beta) = \lim_{a_{1,\pm} \to \bar{a}_1} s_{1,\pm} (a_{1,\pm}; \lambda, \beta).
\]
As a matter of fact, \( \tilde{l}_{0,\pm} = +\infty, \tilde{l}_{0,\pm} = -\infty \). However, \( \tilde{l}_{1,\pm}, \bar{\psi}_{i,\pm} \) and \( \bar{a}_i \) are finite. Therefore, for all \( l_1 \in (\lambda, \min \{ L, \bar{s}_{0,\pm} (\lambda, \beta) \}) \) and \( l_2 \in (\max \{ 0, \bar{s}_{1,\pm} (\lambda, \beta) \}, L) \), we can construct monotone increasing solutions \( V_0(x; a_0, \lambda) \) and \( V_1(x; a_1, \lambda) \) such that \( V_0(1; a_0, \lambda) = V_1(l_2; a_1, \lambda) = \beta \), while for all \( l_3 \in (\max \{ 0, \bar{s}_{0,\pm} (\lambda, \beta) \}, L) \) and \( l_4 \in (\lambda, \min \{ L, \bar{s}_{1,\pm} (\lambda, \beta) \}) \), we can construct monotone decreasing solutions \( V_0(x; a_0, \lambda) \) and \( V_1(x; a_1, \lambda) \) such that \( V_0(l_3; a_0, \lambda) = V_1(1; a_1, \lambda) = \beta \). This proves, in particular, the existence of \( \{ x_{i,\beta}, y_{i,\beta} \} \) for \( i = 1, 2 \), introduced at the beginning of the present section. We now come to the proofs of Theorems 3.1-3.3. Since \( \beta \) is fixed, we suppress \( \beta \) below for the simplicity of notations.

**Proof of Theorem 3.1.** (i) We notice that \( X_{1,\beta} \geq Y_{1,\beta} \) is equivalent to \( \bar{s}_{1,\pm} (L) \leq \bar{s}_{0,\pm} (0) \). Let \( b = \max \{ 0, \bar{s}_{1,\pm} (L) \} \), \( e = \min \{ L, \bar{s}_{0,\pm} (0) \} \). Then we divide \( [0, L] \) into two subintervals \( [0, A_0] \) and \( [A_0, L] \) such that \( b < A_0 < e \). By virtue of Lemma 3.5, for every \( A_0 \in (b, e) \), there exists unique solutions \( V_0(x; a_0, 0) \) on \([0, A_0] \) and \( V_1(x; a_1, L) \) on \([A_0, L] \) such that
\[
V_0(A_0; a_0(A_0; 0), 0) = V_1(A_0; a_1(A_0; L), L) = \beta.
\]
Our goal is to connect \( V_0(x; a_0, 0) \) and \( V_1(x; a_1, L) \) at \( x = A_0 \) in \( C^1 \)-sense. To do this, we define
\[
\Phi_0(A_0) = V_0''(A_0; a_0(A_0; 0), 0) - V_1''(A_0; a_1(A_0; L), L).
\]
From Lemma 3.6 (i) and (iii), it holds that \( V_0''(A_0; a_0(A_0; 0), 0) \) is increasing in \( A_0 \), while \( V_1''(A_0; a_1(A_0; L), L) \) is decreasing in \( A_0 \). Hence, \( \Phi_0(A_0) \) is strictly increasing in \( A_0 \). In addition, by the definitions of \( \bar{s}_{0,\pm} (0) \) and \( \bar{s}_{1,\pm} (L) \), we know that
\[
V_0''(\bar{s}_{0,\pm} (0); a_0(\bar{s}_{0,\pm} (0); 0), 0) = V_1''(\bar{s}_{1,\pm} (L); a_1(\bar{s}_{1,\pm} (L); L), L).
\]
Since \( \bar{s}_{1,\pm} (L) \leq \bar{s}_{0,\pm} (0) \), we have
\[
V_0''(\bar{s}_{1,\pm} (L); a_0(\bar{s}_{1,\pm} (L); 0), 0) \leq V_0''(\bar{s}_{0,\pm} (0); a_0(\bar{s}_{0,\pm} (0); 0), 0),
\]
\[
V_1''(\bar{s}_{0,\pm} (0); a_1(\bar{s}_{0,\pm} (0); L), L) \leq V_1''(\bar{s}_{1,\pm} (L); a_1(\bar{s}_{1,\pm} (L); L), L),
\]
which imply that \( \Phi_0(\bar{s}_{0,\pm} (0)) \geq 0 \geq \Phi_0(\bar{s}_{1,\pm} (L)) \). Moreover, we have
\[
\lim_{A_0 \to L} \Phi_0(A_0) = \lim_{A_0 \to L} (V_0''(A_0; a_0(A_0; 0), 0) - V_1''(A_0; a_1(A_0; L), L))
\]
\[
= V_0''(L; a_0(L), 0) - 0 > 0,
\]
\[
\lim_{A_0 \to 0} \Phi_0(A_0) = \lim_{A_0 \to 0} (V_0''(A_0; a_0(A_0; 0), 0) - V_1''(A_0; a_1(A_0; L), 0))
\]
\[
= 0 - V_1''(0; a_1(0; L), 0) < 0.
\]
Proof of Theorem 3.2. Observe that 

\[
\Phi_0(e) \geq 0 \geq \Phi_0(b).
\]

The two equalities hold if and only if \( \bar{s}_{0,+}(0) = \hat{s}_{1,-}(L) \). By the intermediate value theorem, there exists a unique \( \Lambda^*_0 \) satisfying \( \Phi_0(\Lambda^*_0(L)) = 0 \). Since \( \Lambda^*_0 \) is uniquely determined by \( L \), we denote it by \( \Lambda^*_0(L) \).

Hence,

\[
v_1(x) = \begin{cases} 
V_0(x; a_0(\Lambda^*_0(L); 0), 0) & \text{for } x \in (0, \Lambda^*_0(L)), \\
V_1(x; a_1(\Lambda^*_0(L); L), L) & \text{for } x \in (\Lambda^*_0(L), L),
\end{cases}
\]

is a unique increasing solution of problem (3.2), which satisfies

\[
v'_1(0) = v'_1(L) = 0,
\]

\[
V'_0(\Lambda^*_0(L); a_0(\Lambda^*_0(L); 0), 0) = V'_1(\Lambda^*_0(L); a_1(\Lambda^*_0(L); L), L),
\]

\[
V_0(\Lambda^*_0(L); a_0(\Lambda^*_0(L); 0), 0) = V_1(\Lambda^*_0(L); a_1(\Lambda^*_0(L); L), L),
\]

\[
\beta.
\]

(ii) Note that \( X_{2,\beta} \leq Y_{2,\beta} \) is equivalent to \( \bar{s}_{1,-}(0) \geq \bar{s}_{0,-}(L) \). In a similar way to (i), we can prove that there exists a unique decreasing solution, say \( v_2(x) \), where

\[
v_2(x) = \begin{cases} 
V_1(x; a_1(\Lambda_{0,*}(L); 0), 0) & \text{for } x \in (0, \Lambda_{0,*}(L)), \\
V_0(x; a_0(\Lambda_{0,*}(L); L), L) & \text{for } x \in (\Lambda_{0,*}(L), L),
\end{cases}
\]

and \( v_2(x) \) satisfies

\[
v'_2(0) = v'_2(L) = 0,
\]

\[
V'_1(\Lambda_{0,*}(L); a_1(\Lambda_{0,*}(L); 0), 0) = V'_0(\Lambda_{0,*}(L); a_0(\Lambda_{0,*}(L); L), L),
\]

\[
V_1(\Lambda_{0,*}(L); a_1(\Lambda_{0,*}(L); 0), 0) = V_0(\Lambda_{0,*}(L); a_0(\Lambda_{0,*}(L); L), L),
\]

\[
\beta.
\]

\( \square \)

Proof of Theorem 3.2. Observe that \( X_{1,\beta} \geq Y_{1,\beta} \) and \( X_{2,\beta} \leq Y_{2,\beta} \) are equivalent to \( \bar{s}_{1,-}(L) \leq \bar{s}_{0,+}(0) \) and \( \hat{s}_{1,+}(0) \geq \hat{s}_{0,-}(L) \). We remark that under these conditions, for any subinterval \([L', L''] \subset [0, L] \), it is not difficult to see that \( \bar{s}_{1,-}(L'') \leq \bar{s}_{0,+}(L') \) and \( \hat{s}_{1,+}(L') \geq \hat{s}_{0,-}(L'') \) are also satisfied. Therefore, we can construct both monotone increasing and decreasing solutions on \([L', L'']\). Next, we will show that for each integer \( n \geq 2 \), we can find a partition of \([0, L]\) into \( n \)-subintervals, \([0, L_1], [L_1, L_2], \ldots, [L_{n-1}, L]\) such that problem (3.2) has a solution \( v(x) \), which is monotone in each of the subintervals \([L_i, L_{i+1}]\). Moreover, in Step 3, we will prove that as \( n \) increases, \( a_{0,n} = v(0) \) increases; if \( n = 2i \), then \( a_{0,n}^* = v(L) \) increases, while if \( n = 2i + 1 \), then \( a_{1,n} = v(0) \) decreases. In addition, \( a_{0,n} \to \beta, a_{0,n}^* \to \beta \) and \( a_{1,n} \to \beta \), as \( n \to \infty \).

We only prove the existence for \( n = 2 \) and \( n = 3 \), since we can treat other cases in exactly the same way.

Step 1. If \( n = 2 \), let \( L_1 \) be any number in \((0, L)\). We apply Theorem 3.1 to the intervals \([0, L_1]\) and \([L_1, L]\) and obtain a monotone increasing solution \( \tilde{v}_1(x) \) in the interval \([0, L_1]\), and a monotone decreasing solution \( \tilde{v}_2(x) \) in the interval \([L_1, L]\), respectively, where

\[
\tilde{v}_1(x) = \begin{cases} 
V_0(x; a_0(\Lambda^*_0(L_1); 0), 0) & \text{for } x \in (0, \Lambda^*_0(L_1)), \\
V_1(x; a_1(\Lambda^*_0(L_1); L), L) & \text{for } x \in (\Lambda^*_0(L_1), L_1),
\end{cases}
\]

\[
\tilde{v}_2(x) = \begin{cases} 
V_1(x; a_1(\Lambda_{0,*}(L_1); L_1), L_1) & \text{for } x \in (L_1, \Lambda_{0,*}(L_1)), \\
V_0(x; a_0(\Lambda_{0,*}(L_1); L), L) & \text{for } x \in (\Lambda_{0,*}(L_1), L),
\end{cases}
\]

and \( \tilde{v}_1(x), \tilde{v}_2(x) \) satisfy

\[
\tilde{v}'_1(0) = \tilde{v}'_1(L_1) = \tilde{v}'_2(L_1) = \tilde{v}'_2(L) = 0,
\]

\[
\tilde{v}_1(L_1) = a_1(\Lambda^*_0(L_1)), \quad \tilde{v}_2(L_1) = a_1(\Lambda_{0,*}(L_1)).
\]
Now, we define \( \Phi_1(L_1) \) by
\[
\Phi_1(L_1) = a_1(\Lambda_0^*(L_1)) - a_1(\Lambda_{0,*}(L_1)).
\]

We first show that if \( L_1 < L'_1 \), then \( \Lambda_0^*(L_1) < \Lambda_0^*(L'_1) \) and \( L_1 - \Lambda_0^*(L_1) < L'_1 - \Lambda_0^*(L'_1) \). We prove this by contradiction. If we increase \( L_1 \), at least one of \( \Lambda_0^*(L_1) \) and \( L_1 - \Lambda_0^*(L_1) \) must increase. Without loss of generality, we may suppose that \( \Lambda_0^*(L_1) > \Lambda_0^*(L'_1) \), while \( L_1 - \Lambda_0^*(L_1) \leq L'_1 - \Lambda_0^*(L'_1) \). By Lemma 3.7, we see that \( \frac{d\Lambda_0}{da_{0,+}} > 0 \) and \( \frac{d(L_1 - \Lambda_0^*)}{da_{1,+}} > 0 \), which imply that
\[
V_0'(\Lambda_0^*(L'_1); a_0(\Lambda_0^*(L'_1); 0), 0) < V_0'(\Lambda_0^*(L_1); a_0(\Lambda_0^*(L_1); 0), 0),
\]
\[
V_1'(\Lambda_0^*(L'_1); a_1(\Lambda_0^*(L'_1); L'_1), L'_1) \geq V_1'(\Lambda_0^*(L_1); a_1(\Lambda_0^*(L_1); L_1), L_1).
\]
This means that
\[
V_1'(\Lambda_0^*(L'_1); a_1(\Lambda_0^*(L'_1); L'_1), L'_1) > V_0'(\Lambda_0^*(L'_1); a_0(\Lambda_0^*(L'_1); 0), 0),
\]
a contradiction. Therefore, \( \Lambda_0^*(L_1) \) and \( L_1 - \Lambda_0^*(L_1) \) must increase at the same time. Since \( L_1 - \Lambda_0^*(L_1) \) is increasing in \( a_1 \) in virtue of Lemma 3.5 (ii), we conclude that \( a_1(\Lambda_0^*(L_1)) \) is strictly increasing in \( L_1 \). Similarly, we can prove that if we increase \( L_1 \), then \( L = L_1 - \Lambda_0^*(L_1) \) is increasing in \( L_1 \) and \( L - \Lambda_0^*(L_1) \) have to decrease at the same time. By applying Lemma 3.5 (ii), we know that \( \Lambda_{0,*}(L_1) = L_1 \) is increasing with respect to \( a_1(\Lambda_{0,*}(L_1)) \), which means that \( a_1(\Lambda_{0,*}(L_1)) \) is strictly decreasing in \( L_1 \). Consequently, \( \Phi_1(L_1) \) is strictly increasing in \( L_1 \). In addition, since \( 0 < L_1 < L \), we see that
\[
\lim_{L_1 \to 0} \Phi_1(L_1) = -a_1(\Lambda_{0,*}(0)) < 0, \quad \lim_{L_1 \to L} \Phi_1(L_1) = a_1(\Lambda_0^*(L)) > 0.
\]
Therefore, there exists a unique \( L_1^* \) such that \( \Phi_2(L_1^*) = 0 \). Thus,
\[
w_1(x) = \begin{cases} \hat{\tilde{v}}_1(x) & \text{for } x \in (0, L_1^*), \\ \hat{\tilde{v}}_2(x) & \text{for } x \in (L_1^*, L) \end{cases}
\]
is a solution of problem (3.2), which satisfies
\[
w_1'(0) = w_1'(L) = 0, \quad \hat{\tilde{v}}_1'(L_1^*) = \hat{\tilde{v}}_2'(L_1^*) = 0, \quad \hat{\tilde{v}}_1(L_1^*) = \hat{\tilde{v}}_2(L_1^*).
\]
Observe that \( L_1^* < L \). According to the above analysis, we have \( \Lambda_0^*(L_1^*) < \Lambda_0^*(L) \). Therefore, we conclude that
\[
v_1(0) = a_0(\Lambda_0^*(L)) < a_0(\Lambda_0^*(L_1^*)) = w_1(0).
\]

**Step 2.** If \( n = 3 \), it follows from the case \( n = 2 \) that for each \( 0 < L_2 < L \), we can construct a mode 2 solution \( \tilde{w}_1(x) \) in the interval \([0, L_2]\), where
\[
\tilde{w}_1(x) = \begin{cases} \hat{\tilde{v}}_1'(x) & \text{for } x \in (0, \tilde{L}_1^*), \\ \hat{\tilde{v}}_2'(x) & \text{for } x \in (\tilde{L}_1^*, L_2). \end{cases}
\]
Moreover, \( \tilde{w}_1(x) \) satisfies
\[
\tilde{w}_1'(0) = \tilde{w}_1'(L_2) = 0, \quad \tilde{w}_1(L_2) = a_0(\Lambda_{0,*}(L_2)).
\]
By virtue of Theorem 3.1 (i), we know that there exists a unique monotone increasing solution \( w_2(x) \) in the interval \([L_2, L]\), which satisfies
\[
w_2'(L_2) = w_2'(L) = 0, \quad w_2(L_2) = a_0(\Lambda_0^*(L_2)),
\]
where \( \Lambda_0^*(L_2) \) is such that
\[
V_0(\Lambda_0^*(L_2); a_0(\Lambda_0^*(L_2); L_2), L_2) = V_1(\Lambda_0^*(L_2); a_1(\Lambda_0^*(L_2); L), L) = \beta.
\]
It remains to show that there exists a unique \( L_2^* \) for which the matching condition \( a_0(\Lambda_{0,*}(L_2^*)) = a_0(\Lambda_0^*(L_2^*)) \) is satisfied. To prove this, we put
\[
\Phi_2(L_2) = a_0(\Lambda_{0,*}(L_2)) - a_0(\Lambda_0^*(L_2)).
\]
Indeed, as \( L_2 \) increases, both \( \Lambda_{0,*}(L_2) \) and \( L_2 - \Lambda_{0,*}(L_2) \) increase. It follows from Lemma 3.5 (i) that \( a_0(\Lambda_{0,*}(L_2)) \) is strictly decreasing in \( L_2 \). On the other hand, \( L - L_2, \Lambda_0^*(L_2) - L_2 \) and \( L - \Lambda_0^*(L_2) \) are decreasing in \( L_2 \). Lemma 3.5 (i) yields that \( a_0(\Lambda_0^*(L_2)) \) is strictly increasing in \( L_2 \). This means that \( \Phi_2(L_2) \) is strictly decreasing in \( L_2 \). Notice that \( 0 < L_2 < L \), and
\[
\lim_{L_2 \to 0} \Phi_2(L_2) = \beta - a_0(\Lambda_0^*(L_2)) > 0, \quad \lim_{L_2 \to L} \Phi_2(L_2) = a_0(\Lambda_{0,*}(L_2)) - \beta < 0.
\]
Hence, there exists a unique \( L_2^* \) such that \( \Phi_2(L_2^*) = 0 \), and we conclude that
\[
z_1(x) = \begin{cases}
\tilde{w}_1(x) & \text{for } x \in (0, L_2^*), \\
\tilde{w}_2(x) & \text{for } x \in (L_2^*, L)
\end{cases}
\]
is a mode 3 solution of problem (3.2), which is composed of three monotone solutions. Moreover, \( z_1(x) \) satisfies
\[
z_1^*(0) = z_1^*(L) = 0, \quad \tilde{w}_1^*(L_2^*) = \tilde{w}_2^*(L_2^*) = 0, \quad \tilde{w}_1(L_2^*) = \tilde{w}_2(L_2^*).
\]

**Step 3.** The method in Steps 1 and 2 can be summarized as follows: First we take a \( \Lambda \in (0, L) \) and construct a mode \( k \) solution \( v_k(x; \Lambda) \) on \( [0, \Lambda] \) and a monotone solution \( W(x; \Lambda) \) on \( [\Lambda, L] \). Here, \( W(x; \Lambda) \) is increasing if \( V_k(\Lambda; \Lambda) < \beta \), and is decreasing if \( V_k(\Lambda; \Lambda) > \beta \). We prove (i) the monotonicity of the function \( \Lambda \mapsto V_k(\Lambda; \Lambda) - W(\Lambda; \Lambda) \), (ii) \( V_k(\Lambda; \Lambda) - W(\Lambda; \Lambda) \) change signs for \( \Lambda \) sufficiently small and for \( \Lambda \) sufficiently close to \( L \). Then we obtain a unique mode \( k + 1 \) solution on \( [0, L] \). Therefore, by induction on \( n \), we can prove the existence of a mode \( n \) solution \( v(x) \) for all \( n \geq 1 \). The fine point of the proof is included in Step 1 and hence we omit the detail. Next, we consider the behaviors of \( a_{0,n}, a_{0,n}^\ast \) and \( a_{1,n} \) as \( n \) increases (for the definition of \( a_{0,n}, a_{0,n}^\ast \) and \( a_{1,n} \), see the first paragraph of Proof of Theorem 3.2). The following two observations will play an important role in the proof of Theorem 3.3.

(i) As \( n \) increases, \( a_{0,n} \) and \( a_{0,n}^\ast \) increase, while \( a_{1,n} \) decreases. We use the same notations as in Steps 1 and 2 for the interval \([0, L_3]\) and compare mode 1 solution \( v_1(x) \), mode 2 solution \( w_1(x) \) and mode 3 solution \( z_1(x) \) on \([0, L_3]\). Since \( L_2^* < L \), we have \( \tilde{L}_1^* < \tilde{L}_1 < L \). By monotonicity, we obtain
\[
w_1(0) = a_0(\Lambda_0^*(L_1^*)) < a_0(\Lambda_0^*(\tilde{L}_1)) = z_1(0),
\]
which means that \( v_1(0) < w_1(0) < z_1(0) \). We can repeat this argument to show that \( v(0) = a_{0,n} \) increases as \( n \) increases. On the other hand, \( L - L_2^* < L \) yields
\[
v_1(L) = a_1(\Lambda_0^*(L)) > a_1(\Lambda_0^*(L_2^*)) = z_1(L).
\]
This procedure can be repeatedly applied to the case where \( n \) is odd. For \( n \) even, we can apply repeatedly the reasoning for \( x = 0 \) to that for \( x = L \).

(ii) As \( n \to \infty, a_{0,n} \to \beta, a_{0,n}^\ast \to \beta, a_{1,n} \to \beta \). We prove the assertion only for \( n = 2i + 1 \). Let \( v(x) \) be a mode \( n \) solution which is increasing in the subinterval \([0, L_1]\) and \( x_\beta \) be the number in \((0, L_1)\) such that \( v(x_\beta) = \beta \). We expand \( v(x) \) near \( x = 0 \):
\[
v(x) = v(0) + v'(0)x + \frac{1}{2}v''(0)x^2 = a_{0,n} - \frac{1}{2D}g_0(v(\theta x), \theta x)x^2,
\]
where $0 < \bar{\theta} < 1$. Substituting $x = x_\beta$ into the above equality, we obtain
\[
\beta = v(x_\beta) = a_{0,n} - \frac{1}{2D}g_0(v(\bar{\theta}x_\beta), \bar{\theta}x_\beta)x_\beta^2.
\]
Since there are positive constants $m_0$, $M_0$ such that $m_0 \leq g_0(v,x) \leq M_0$ for all $v \geq a_{\min}$ and $x \in [0,L]$, where $a_{\min} = v_*(0)$ and $v_*(x)$ is the increasing solution on $[0,L]$, we find that
\[
\sqrt{\frac{2D}{M_0}}(\beta - a_{0,n}) \leq x_\beta \leq \sqrt{\frac{2D}{m_0}}(\beta - a_{0,n}).
\]
Similarly,
\[
\sqrt{\frac{2D}{M_1}}(c_{1,n} - \beta) \leq L_1 - x_\beta \leq \sqrt{\frac{2D}{m_1}}(c_{1,n} - \beta),
\]
where $m_1$, $M_1$ are positive constants such that $m_1 \leq g_1(v,x) \leq M_1$ for $v \leq a_{\max}$ and $x \in [0,L]$, where $a_{\max} = v_*(L)$. However, from the expression
\[
v'(x_\beta) - v'(x_\beta) + \frac{1}{D} \int_{x_\beta}^s g_1(v(t), t) \, dt = 0,
\]
we obtain $\int_{x_\beta}^{L_1} g_1(v(t), t) \, dt = Dv'(x_\beta)$. Hence,
\[
(L_1 - x_\beta)m_1 \leq Dv'(x_\beta) \leq (L_1 - x_\beta)M_1.
\]
Assume that $\beta - a_{0,n} \geq \delta_0 > 0$ for all $n \geq 0$. Then clearly, $v'(x_\beta) \geq \gamma_0 > 0$ for some constant $\gamma_0$ independent of $n$, since $x_\beta \geq \sqrt{2DM_0^{-1}(\beta - a_{0,n})}$. Hence, $v'(x_\beta) = -D^{-1} \int_{x_\beta}^{s_\beta} g_0(v(t), t) \, dt$. We thus find that there exists a positive constant $\omega_0$ independent of $n$ such that
\[
x_\beta \geq \omega_0, \quad L_1 - x_\beta \geq \omega_0.
\]
Hence, $L_1 \geq 2\omega_0$. Repeating these arguments, we obtain $L_{i+1} - L_i \geq 2\omega_0$ for all $i \leq n - 1$, so that $L \geq 2n\omega_0$ for any $n \geq 2$, a contradiction. Therefore, $a_{0,n} \to \beta$. In the same way, we can prove that $a_{0,n} \to \beta$ and $a_{1,n} \to \beta$, as $n \to \infty$.

Now we have proved that for any integer $n \geq 2$, if we start from $V_0$, then there exists a unique solution $v_{n,+}(x)$ of mode $n$ such that $v_{n,+}(x) > 0$ in the interval $(0,L_1)$. On the other hand, in exactly the same way, we can verify that if we start from $V_1$, then there exists also a unique solution $v_{n,-}(x)$ of mode $n$ such that $v_{n,-}(x) < 0$ in the interval $(0,L_1)$. The proof of Theorem 3.2 is now complete.

**Proof of Theorem 3.3.** Notice that $X_{1,\beta} < Y_{1,\beta}$ and $X_{2,\beta} > Y_{2,\beta}$ are equivalent to $\bar{s}_{1,-}(L) > \bar{s}_{0,+}(0)$ and $\bar{s}_{1,+}(0) < \bar{s}_{0,-}(L)$, and there is a sufficiently small positive number $\varepsilon_0$ such that if $L'' - L' < \varepsilon_0$, then in $[L', L''] \subset [0,L]$, the conditions $\bar{s}_{1,-}(L'') \leq \bar{s}_{0,+}(L')$ and $\bar{s}_{1,+}(L') \geq \bar{s}_{0,-}(L'')$ are satisfied. The main idea of the proof of this theorem is described as follows: for sufficiently large positive integer $n$, we first divide $[0,L]$ into $n$-subintervals $[L_0, L_1], [L_1, L_2], \ldots, [L_{n-1}, L_n]$, where $L_0 = 0$, $L_n = L$ and $L_{i+1} - L_i \leq \varepsilon_0$ for $i = 0, 1, \ldots, n-1$. Then in each of the subintervals $[L_i, L_{i+1}]$, we have
\[
\bar{s}_{1,-}(L_{i+1}) \leq \bar{s}_{0,+}(L_i), \quad \bar{s}_{1,+}(L_i) \geq \bar{s}_{0,-}(L_{i+1}).
\]

Next, we will prove that in the first two intervals, we can always construct a multimode solution. Since $\bar{s}_{1,-}(L_1) \leq \bar{s}_{0,+}(0)$ and $\bar{s}_{1,-}(L_2) \leq \bar{s}_{0,+}(L_1)$, by Theorem 3.1 (i), we can construct monotone increasing solutions $w_{1,+}(x)$, $w_{1,-}(x)$ in the intervals $[0, L_1]$ and $[L_1, L_2]$, respectively. Due to $\bar{s}_{1,+}(0) \geq \bar{s}_{0,-}(L_1)$ and $\bar{s}_{1,+}(L_1) \geq \bar{s}_{0,-}(L_2)$, we obtain
\(\tilde{s}_{0,-}(L_2)\), by Theorem 3.1 (ii), we can construct monotone decreasing solutions \(v_{1,-}(x), w_{1,-}(x)\) in the intervals \([0, \tilde{L}_1]\) and \([\tilde{L}_1, L_2]\), respectively.

If we happen to have \(v_{1,+}(\tilde{L}_1) = w_{1,-}(\tilde{L}_1)\), then

\[
\tilde{z}_1(x) = \begin{cases} v_{1,+}(x) & \text{for } x \in (0, \tilde{L}_1), \\ w_{1,-}(x) & \text{for } x \in (\tilde{L}_1, L_2) \end{cases}
\]

is a mode 2 solution in the interval \([0, \tilde{L}_2]\).

If \(v_{1,+}(\tilde{L}_1) \neq w_{1,-}(\tilde{L}_1)\), without loss of generality, we may assume that \(v_{1,+}(\tilde{L}_1) < w_{1,-}(\tilde{L}_1)\). Then by extending \(v_{1,+}(x)\) to the right of \(\tilde{L}_1\), we can construct a monotone decreasing solution \(\tilde{w}_{1,-}(x)\) on \([\tilde{L}_1, L^*]\) such that

\[
\tilde{w}'_{1,-}(\tilde{L}_1) = \tilde{w}'_{1,-}(L^*) = 0, \quad \tilde{w}_{1,-}(\tilde{L}_1) = v_{1,+}(\tilde{L}_1),
\]

where \(L^*\) is uniquely determined by \(v_{1,+}(\tilde{L}_1)\). We claim that \(\tilde{L}_1 < L^* < \tilde{L}_2\). Indeed, let \(x_\gamma\) be a positive number such that \(V_1(x_\gamma; \tilde{w}_{1,-}(\tilde{L}_1), \tilde{L}_1) = \beta\). By Lemma 3.5 (ii), we know that

\[
V'_1(x_\gamma; \tilde{w}_{1,-}(\tilde{L}_1), \tilde{L}_1) > V'_1(\Lambda_0, \tilde{w}_{1,-}(\tilde{L}_1), \tilde{L}_1).
\]

If we assume \(L^* \geq \tilde{L}_2\), then \(L^* - x_\gamma > \tilde{L}_2 - \Lambda_0, \tilde{L}_1\), where \(\Lambda_0, \tilde{L}_1\) satisfies

\[
V_1(\Lambda_0, \tilde{w}_{1,-}(\tilde{L}_1); w_{1,-}(\tilde{L}_1), \tilde{L}_1) = \beta.
\]

Therefore

\[
V_0'(x_\gamma; \tilde{w}_{1,-}(L^*), L^*) < V_0'(\Lambda_0, \tilde{w}_{1,-}(\tilde{L}_1); w_{1,-}(\tilde{L}_2), \tilde{L}_2).
\]

Notice that

\[
V'_1(\Lambda_0, w_{1,-}(\tilde{L}_1); w_{1,-}(\tilde{L}_1), \tilde{L}_1) = V'_0(\Lambda_0, w_{1,-}(\tilde{L}_1); w_{1,-}(\tilde{L}_2), \tilde{L}_2),
\]

which implies that

\[
V_0'(x_\gamma; \tilde{w}_{1,-}(L^*), L^*) < V'_1(x_\gamma; \tilde{w}_{1,-}(\tilde{L}_1), \tilde{L}_1).
\]

This is a contradiction, and hence we must have \(L^* < \tilde{L}_2\).

Then in the interval \([L^*, \tilde{L}_2]\), we construct a monotone increasing solution \(w_{1,+}^*(x)\). If we happen to have \(\tilde{w}_{1,-}(L^*) = w_{1,+}^*(L^*)\), then

\[
\tilde{z}_2(x) = \begin{cases} v_{1,+}(x) & \text{for } x \in (0, \tilde{L}_1), \\ \tilde{w}_{1,-}(x) & \text{for } x \in (\tilde{L}_1, L^*), \\ w_{1,+}^*(x) & \text{for } x \in (L^*, \tilde{L}_2) \end{cases}
\]

is a mode 3 solution in the interval \([0, \tilde{L}_2]\).

Next we consider the situation \(\tilde{w}_{1,-}(L^*) \neq w_{1,+}^*(L^*)\). First we explain how to treat the case \(\tilde{w}_{1,-}(L^*) < w_{1,+}^*(L^*)\).

**Case I.** \(v_{1,-}(\tilde{L}_1) > w_{1,+}(\tilde{L}_1)\). Since \([0, \tilde{L}_1]\) satisfies all the conditions of Theorem 3.2, we can construct a mode 2 solution \(v_{2,+}(x)\) on \([0, \tilde{L}_1]\) and according to (i) in Theorem 3.2, we see that \(v_{2,+}(\tilde{L}_1) > v_{1,-}(\tilde{L}_1) > w_{1,+}(\tilde{L}_1)\). Next, let \(\Lambda_1\) be any number in \([\tilde{L}_1, L^*]\). Then in the interval \([\Lambda_1, \tilde{L}_2]\), we can construct a monotone increasing solution \(w_{1,+}(x)\). Next, we prove that we can construct a mode 2 solution \(w_{2,+}(x)\) in the interval \([0, \Lambda_1]\). To do this, we use the following lemma:

**Lemma 3.8.** There exists a sufficiently small positive constant \(\kappa\) such that the following assertion holds true. Let \(0 < K < L\). Assume that in \([0, K]\), we have a mode \(m\) solution \(\xi_{m,+}(x; K)\) and a partition of the interval \([0, K]\)

\[
[0, K] = [K_0, K_1] \cup [K_1, K_2] \cup \cdots \cup [K_{m-1}, K_m],
\]


such that \( \zeta_{m, +}(x; K) \) is monotone in each of the subintervals \([K_{i-1}, K_i]\), where \( K_0 = 0, K_m = K \); \( \delta_i = K_i - K_{i-1} < \varepsilon_0 \) for \( 1 \leq i \leq m \). Here, \( \varepsilon_0 \) is the constant we have chosen at the beginning of the proof of Theorem 3.3. Moreover, we assume that \( |\zeta_{m, +}(x; K) - \beta| < \kappa \) for all \( x \in [0, K] \). Then for every \( K' < K \), there exists a mode \( m \) solution \( \zeta_{m, +}(x; K') \) in the interval \([0, K']\).

Proof of Lemma 3.8. We prove this lemma by mathematical induction on \( m \).

If \( m = 1 \), then we have a monotone solution on \([0, K']\) because of \( K' < K < \varepsilon_0 \).

Assume the assertion holds true for \( m - 1 \). Then we prove the assertion for \( m \).

We distinguish the following two cases.

Case A. \( K' \in [K_{m-1}, K_m] \).

If this case occurs, then in the interval \([K_{m-1}, K_m] \), we can construct a monotone increasing solution \( y_{1, +}(x; K') \). Moreover, \( \zeta_{m, +}(K_{m-1}; K) < y_{1, +}(K_{m-1}; K') \) since \( \zeta_{m, +}(x; K) \) gives rise to a monotone increasing solution on \([K_{m-1}, K_m] \) because of the monotonicity here.

Proof. For \( m = 1 \), we have \( \hat{\zeta}_{m, +}(K_{m-1}; K) > \beta \). Now, since \( \beta = \kappa \), we have \( \zeta_{m, +}(x; K') \) is strictly decreasing with respect to \( K' \). Therefore, by the intermediate value theorem, there exists a mode \( m \) solution on \([0, K']\) as in Step 1 of the proof of Theorem 3.2, although we do not use the monotonicity here.

Case B. \( K' \in [0, K_{m-1}] \).

In this case, we first choose \( K' \) sufficiently close to \( K_{m-1} \). Then in the intervals \([0, K']\) and \([K', K_{m-1}]\), we can construct a mode \( m - 1 \) solution \( \zeta_{m-1, +}(x; K') \) and a monotone increasing solution \( y_{1, +}(x; K') \), respectively. By the continuous dependence of \( \zeta_{m-1, +}(x; K') \) on \( K' \), we have \( \zeta_{m-1, +}(K'; K') \rightarrow \zeta_{m, +}(K_{m-1}; K') \) as \( K' \uparrow K_{m-1} \). On the other hand, \( \zeta_{m-1, +}(K'; K') \rightarrow \beta \) as \( K' \uparrow K_{m-1} \). Therefore, \( \zeta_{m-1, +}(K'; K') < \beta \) if \( K' < K_{m-1} \) is sufficiently close to \( K_{m-1} \).

Next, let \( K_1^* \) satisfy \( s_0, + (K_1^*) = \bar{s}_1, - (K_{m-1}) \). Using the same method as in Case A, we see that \( \zeta_{m-1, +}(K_1^*; K_1^*) > \bar{y}_{1, +}(K_1^*; K_1^*) \), where \( \zeta_{m-1, +}(x; K') \) and \( \bar{y}_{1, +}(x; K') \) are monotone solutions and monotone increasing solutions in the intervals \([0, K_1^*] \) and \([K_1^*, K_{m-1}] \), respectively. Therefore, by the intermediate value theorem, there exists a mode \( m \) solution on \([0, K']\) for \( K' \in [K_1^*, K_{m-1}] \). Then repeating this procedure, we obtain a finitely many \( K_1^* > K_2^* > \cdots > K_r^* > K_{r+1}^* \) such that \( \bar{s}_1, -(K_{r+1}^*) \leq \bar{s}_0, +(0) \) and \( \bar{s}_1, +(0) \geq \bar{s}_1, -(K_{r+1}^*) \) (notice that there is an \( \bar{k} > 0 \) such that \( K_i^* - K_{i+1}^* \geq \bar{k} \) for any \( i \geq 1 \)). Therefore, for any \( K' \in [K_{r+1}^*, K_{m-1}] \), there exists a mode \( m \) solution on \([0, K']\). By Theorem 3.2, we can construct a mode \( m \) solution on \([0, K_{r+1}^*] \). Consequently, there exists a mode \( m \) solution on \([0, K']\) for all \( K' \in [0, K_{m-1}] \).

Hence, by Lemma 3.8, we can construct a mode 2 solution \( v_{2, +, \Lambda_1}(x) \) in the interval \([0, \Lambda_1]\) for any \( \Lambda_1 \in [\tilde{L}_1, L^*] \). Now, we would like to choose an appropriate \( \Lambda_1 \) so that the matching condition \( v_{2, +, \Lambda_1}(\Lambda_1) = w_{1, +, \Lambda_1}(\Lambda_1) \) is satisfied. To this end, we define

\[
\Phi_1^*(\Lambda_1) = v_{2, +, \Lambda_1}(\Lambda_1) - w_{1, +, \Lambda_1}(\Lambda_1).
\]

We see in the same way as in \( \Phi_2(L_2) \) that \( \Phi^*(\Lambda_1) \) is strictly decreasing with respect to \( \Lambda_1 \), where \( \Phi_2(L_2) \) is defined in Step 2 of the proof of Theorem 3.2. In addition,

\[
\Phi_1^*(L^*) = \tilde{w}_{1, +}(L^*) - w_{1, +}(L^*) < 0, \quad \Phi_1^*(\tilde{L}_1) = v_{2, +}(\tilde{L}_1) - w_{1, +}(\tilde{L}_1) > 0.
\]
By the intermediate value theorem, there exists a unique \( \Lambda_1^* \) such that \( \Phi_1^*(\Lambda_1^*) = 0 \). Hence, starting from \( V_0 \), we can construct a mode 3 solution in the interval \([0, \bar{L}_2]\).

**Case II.** \( v_{1-}(L_1) < w_{1+}(L_1) \).

We start from \( w_{1+}(L_1) \) and construct a monotone decreasing solution \( \hat{v}_{1-}(x) \) in the interval \([L_1^*, \bar{L}_1]\), where \( L_1^* \) is such that \( 0 < L_1^* < L_1 \) and \( \hat{v}_{1-}'(L_1^*) = 0 \). Then in the interval \([0, L_1^*]\), we construct a monotone increasing solution \( v_{1+}(x) \).

(i) If \( v_{1+}^*(L_1^*) = \hat{v}_{1-}(L_1^*) \) happens to hold, then

\[
\hat{z}_3(x) = \begin{cases} 
  v_{1+}^*(x) & \text{for } x \in (0, L_1^*), \\
  \hat{v}_{1-}(x) & \text{for } x \in (L_1^*, \bar{L}_1), \\
  w_{1+}(x) & \text{for } x \in (\bar{L}_1, L_2)
\end{cases}
\]

is a mode 3 solution in the interval \([0, \bar{L}_2]\).

(ii) If \( v_{1+}^*(L_1^*) < \hat{v}_{1-}(L_1^*) \), then as in Case I, we let \( \Lambda_2 \) be any number in \([L_1^*, \bar{L}_1]\). Then in the interval \([0, \Lambda_2]\), we can construct a monotone increasing solution \( v_{1+}(x) \) and in the interval \([\Lambda_2, \bar{L}_2]\), by Lemma 3.8, we can construct a mode 2 solution \( w_{2-}(x) \). Set

\[
\Phi_2^*(\Lambda_2) = v_{1+}(\Lambda_2) - w_{2-}(\Lambda_2).
\]

We would like to choose an appropriate \( \Lambda_2 \) so that the condition \( \Phi_2^*(\Lambda_2) = 0 \) is satisfied. This is possible. Indeed, recall that \([\bar{L}_1, \bar{L}_2]\) satisfies all the conditions of Theorem 3.2. Therefore, there is a mode 2 solution \( w_{2-}(x) \) on \([\bar{L}_1, \bar{L}_2]\). Moreover, we have \( w_{2-}(\bar{L}_1) < v_{1+}(\bar{L}_1) \). Notice that \( \Phi_2^*(\Lambda_2) \) is strictly increasing in \( \Lambda_2 \), and

\[
\Phi_2^*(L_1^*) = v_{1+}^*(L_1^*) - \hat{v}_{1-}(L_1^*) < 0, \quad \Phi_2^*(\bar{L}_1) = v_{1+}(\bar{L}_1) - w_{2-}(\bar{L}_1) > 0.
\]

Therefore, starting from \( V_0 \), we can construct a mode 3 solution in the interval \([0, \bar{L}_2]\).

(iii) If \( v_{1+}^*(L_2^*) > \hat{v}_{1-}(L_2^*) \), then we extend \( \hat{v}_{1-}(x) \) to the left of \( L_2^* \) and construct a monotone increasing solution \( \hat{v}_{1+}(x) \) in the interval \([L_2^*, L_1^*]\), where \( L_2^* \) satisfies \( 0 < L_2^* < L_1^* \) and \( \hat{v}_{1+}'(L_2^*) = 0 \). Next, we construct a monotone decreasing solution \( v_{1-}^*(x) \) on \([0, L_2^*]\).

(a) If we happen to have \( v_{1-}^*(L_2^*) = \hat{v}_{1+}(L_2^*) \), then

\[
\hat{z}_4(x) = \begin{cases} 
  v_{1-}^*(x) & \text{for } x \in (0, L_2^*), \\
  \hat{v}_{1+}(x) & \text{for } x \in (L_2^*, L_1^*), \\
  \hat{v}_{1-}(x) & \text{for } x \in (L_1^*, \bar{L}_1), \\
  w_{1+}(x) & \text{for } x \in (\bar{L}_1, L_2)
\end{cases}
\]

is a mode 4 solution in the interval \([0, \bar{L}_2]\).

(b) If \( v_{1-}^*(L_2^*) > \hat{v}_{1+}(L_2^*) \), then using the same approach as in (ii), we can find a mode 4 solution in the interval \([0, \bar{L}_2]\) starting from a monotone decreasing solution \( V_1 \). Indeed, let \( v_{1-}(x) \) be the monotone decreasing solution on \([0, \Lambda_3]\), \( L_2^* \leq \Lambda_3 \leq \bar{L}_1 \), and \( w_{3+}(x) \) be the mode 3 solution on \([\Lambda_3, \bar{L}_2]\). Define

\[
\Phi_3^*(\Lambda_3) = v_{1-}(\Lambda_3) - w_{3+}(\Lambda_3) \quad \text{for } \Lambda_3 \in [L_2^*, \bar{L}_1].
\]

Then \( \Phi_3^*(\Lambda_3) \) is strictly decreasing in \( \Lambda_3 \). Furthermore,

\[
\Phi_3^*(L_2^*) = v_{1-}^*(L_2^*) - \hat{v}_{1+}(L_2^*) > 0, \quad \Phi_3^*(\bar{L}_1) = v_{1-}(\bar{L}_1) - w_{3+}(\bar{L}_1) < 0,
\]

where \( w_{3+}(x) \) is a mode 3 solution on \([\bar{L}_1, \bar{L}_2]\) and \( w_{3+}(\bar{L}_1) > w_{1+}(\bar{L}_1) > v_{1-}(\bar{L}_1) \). Therefore, there is a unique \( \Lambda_4^* \in (L_2^*, \bar{L}_1) \) such that \( \Phi_3^*(\Lambda_4^*) = 0 \).
(c) Finally, we consider the case \( v_{1,-}^*(L_2^*) < \hat{v}_{1,+}(L_2^*) \). Since we will extend solutions many times, we change notations to describe the procedure clearly. Let \( \varphi_{m,\pm}(x;[a,b]) \) denote mode \( m \) solutions in the interval \([a,b]\) such that \( \varphi'_{m,+}(x;[a,b]) > 0 \) and \( \varphi'_{m,-}(x;[a,b]) < 0 \) for \( x-a > 0 \) sufficiently small.

Note that we have constructed \( \varphi_{3,+}(x;[L_2^*, \bar{L}_2]) \) by using \( \hat{v}_{1,+}(x) \), \( \hat{v}_{1,-}(x) \) and \( w_{1,+}(x) \). Then we extend \( \varphi_{3,+}(x;[L_2^*, \bar{L}_2]) \) to \( \varphi_{4,-}(x;[L_2^*, \bar{L}_2]) \) with \( L_2^* < \bar{L}_2 \) and construct \( \varphi_{1,+}(x;[0, L_2^*]) \). We compare \( \varphi_{1,+}(x;[0, L_2^*]) \) with \( \varphi_{4,-}(x;[L_2^*, \bar{L}_2]) \) at \( x = L_2^* \). Suppose that \( \varphi_{1,+}(L_2^*;[0, L_2^*]) \leq \varphi_{4,-}(L_2^*;[L_3^*, \bar{L}_2]) \). Recall that

\[
\varphi_{1,+}(L_1;[0, \bar{L}_1]) > \varphi_{2,-}(L_1;[L_1, \bar{L}_2]) > \varphi_{4,-}(L_1;[L_1, \bar{L}_2]).
\]

Here, the first inequality is nothing but \( w_{2,-}(\bar{L}_1) < v_{1,+}(\bar{L}_1) \), while the second inequality is due to the increase of mode. Now from \( \varphi_{1,+}(L_1^*;[0, L_2^*]) \leq \varphi_{4,-}(L_2^*;[L_3^*, \bar{L}_2]) \) and \( \varphi_{1,+}(L_1;[0, \bar{L}_1]) \geq \varphi_{4,-}(L_1;[L_1, \bar{L}_2]) \), we conclude that there exists a unique mode 5 solution \( \varphi_{5,+}(x;[0, L_2^*]) \).

Otherwise, we extend \( \varphi_{4,-}(x;[L_2^*, \bar{L}_2]) \) to \( \varphi_{5,+}(x;[L_2^*, \bar{L}_2]) \) with \( L_2^* < \bar{L}_2 \), and argue as above. We claim that this extension process of \( \varphi_{k,+}(x;[L_{k-1}^*, \bar{L}_2]) \) ends in finite times. If not, \( L_{k-1}^* \downarrow L_\infty \geq 0 \) as \( k \to \infty \) and we have a sequence of solutions of increasing modes on \([L_{k-1}^*, \bar{L}_2]\). On the other hand, from (ii) in Step 3 of the proof of Theorem 3.2, we know that \( \varphi_{k,\pm}(x;[L_{k-1}^*, \bar{L}_2]) \) converges to \( \beta \) uniformly on \([L_{k-1}^*, \bar{L}_2]\). This is impossible because \( \varphi_{k,\pm}(L_2^*;[L_{k-1}^*, \bar{L}_2]) = \varphi_{1,+}(L_2^*;[L_1, \bar{L}_2]) \), which is a constant independent of \( k \). Therefore, for some \( k \in \mathbb{N} \), we have \( |\varphi_{1,\pm}(L_{k-1}^*;[0, L_{k-1}^*]) - \beta| < |\varphi_{k,\pm}(L_{k-1}^*;[L_{k-1}^*, \bar{L}_2]) - \beta| \) and this gives us a mode \( k+1 \) solution \( \varphi_{k+1,\pm}(x;[0, L_2^*]) \).

For the case \( v_{1,+}(\bar{L}_1) > w_{1,-}(\bar{L}_1) \) or the case \( v_{1,+}(\bar{L}_1) < w_{1,-}(\bar{L}_1) \) and \( \hat{w}_{1,-}(L^*) < w_{1,+}(L^*) \), we can apply the same method to obtain a solution in the interval \([0, \bar{L}_2]\).

Up to now, we have already constructed a solution on \([0, L_2]\). Then following the same argument as in the first two intervals, we can construct a solution on \([0, L_3]\). We repeat this procedure and obtain a mode \( n \) solution of problem (3.2) for some \( n \) sufficiently large.

If we have a mode \( n \) solution of (3.2), then Lemma 3.8 guarantees that we have a mode \( n \) solution on every subinterval \([0, L''] \subset [0, L] \). Therefore, we can apply the arguments in the proof of Lemma 3.8 to obtain a mode \( n+1 \) solution on \([0, L] \). \qed

4. **Asymptotic behavior of positive solution as \( D \to \infty \).** In this section, we study the limiting behavior of solutions of (3.2) as \( D \) tends to infinity. We first prove that if \( D \) is sufficiently large, then (3.2) always has a solution. Then we shall show that any solution of (3.2) converges to \( \beta \) uniformly as \( D \to \infty \).

**Theorem 4.1.** Assume that \((\text{H0})\) is satisfied. Then there exists \( \hat{D} > 0 \) such that problem (3.2) has a monotone solution \( v_D(x) \) if \( D > \hat{D} \). Moreover, if, in addition, \((\text{H1})\) is satisfied, then problem (3.2) has two types of mode \( n \) solutions for any \( n \geq 1 \).

**Proof.** We divide the proof into two steps.

**Step 1.** We first consider the following three initial value problems for \( i = 0, 1 \) and for \( \lambda = 0 \) or \( \lambda = L \):

\[
\begin{cases}
D_i v'' + g_i(v, x) = 0, & x \in (0, L), \\
v(\lambda) = b_i, & v'(\lambda) = 0, 
\end{cases}
\]  

(4.1)

\[
\begin{cases}
D_i v'' + g_i(v, x) = 0, & x \in (0, L), \\
v(\lambda) = b_i, & v'(\lambda) = 0, 
\end{cases}
\]  

(4.2)
and
\[
\begin{cases}
D_1 v'' + g_i(v, x) = 0, & x \in (0, L), \\
v(\lambda) = c_i, & v'(\lambda) = 0.
\end{cases}
\]

The following comparison lemma is crucial.

**Lemma 4.2.** For \( i = 0, 1 \), let \( v_{i,D_1}(x) \), \( v_{i,D_2}(x) \), \( \tilde{v}_{i,D_1}(x) \) be the unique solutions of problems (4.1), (4.2), (4.3), respectively. Then the following statements hold true.

(i) If \( D_1 < D_2 \), then for \( \lambda = 0, i = 0 \), we have \( v_{0,D_1}(x) > v_{0,D_2}(x) \) for all \( x \in [0, L] \), while for \( \lambda = L, i = 1 \), we have \( v_{1,D_1}(x) < v_{1,D_2}(x) \) for all \( x \in [0, L] \).

(ii) If \( b_i < c_i, \ i = 0, 1 \), then for \( \lambda = 0, i = 0 \), we have \( v_{0,D_1}(x) < \tilde{v}_{0,D_1}(x) \) for all \( x \in [0, L] \), and for \( \lambda = L, i = 1 \), we have \( v_{1,D_1}(x) < \tilde{v}_{1,D_1}(x) \) for all \( x \in [0, L] \).

**Proof.** We only prove the assertion for \( \lambda = 0, i = 0 \), since the rest are treated in the same way.

(i) Set \( W_1(x) = v_{0,D_1}(x) - v_{0,D_2}(x) \). Then \( W_1(x) \) satisfies
\[
\begin{cases}
W_1'' + \left( \frac{1}{D_2} - \frac{1}{D_1} \right) g_0(v_{0,D_1}(x)) = 0, & x > 0, \\
W_1(0) = 0, & W_1'(0) = 0,
\end{cases}
\]
where \( \tilde{\theta}_1 \in (0, 1) \). From (4.4), we know that
\[
W_1''(0) + \left( \frac{1}{D_2} - \frac{1}{D_1} \right) g_0(v_{0,D_1}(0), 0) = 0.
\]
Since \( g_0(v_{4,D_1}(0), 0) < 0 \), \( D_1 < D_2 \) and \( W_1(0) = 0 \), we have \( W_1''(0) > 0 \). On the other hand, we expand \( W_1(x) \) around 0 as follows:
\[
W_1(x) = W_1(0) + W_1'(0)x + \frac{1}{2} W_1''(0)x^2 + o(x^2)
\]
\[
= \frac{1}{2} W_1''(0)x^2 + o(x^2),
\]
which implies that \( W_1(x) > 0 \) for all \( x \) near 0.

Since \( \frac{\partial g_0}{\partial v} < 0 \) by Proposition 2.3 (i), we find that if \( W_1(x) > 0 \), then
\[
W_1'' = -\left( \frac{1}{D_2} - \frac{1}{D_1} \right) g_0(v_{0,D_1}, x) > 0,
\]
which means that \( W_1(x) \) is strictly increasing in \( x \) as long as \( W_1(x) > 0 \). Due to the initial condition \( W_1(0) = 0 \), we obtain \( W_1''(x) > 0 \) if \( x > 0 \). It follows that \( W_1(x) \) is also strictly increasing in \( x \). Since \( W_1(0) = 0 \), we conclude that \( W_1(x) > 0 \) for all \( x > 0 \) whenever \( W_1(x) \) exists.

(ii) Similarly, we let \( W_2(x) = v_{0,D_1}(x) - \tilde{v}_{0,D_1}(x) \). It is easy to check that \( W_2(x) \) is a solution of
\[
\begin{cases}
D_1 W_2'' + (g_0)v_{0,D_1} + \tilde{\theta}_2 W_2, x = 0, & x \in (0, L), \\
W_2(0) < 0, & W_2'(0) = 0,
\end{cases}
\]
where \( \tilde{\theta}_2 \in (0, 1) \). Clearly, for \( x \) close to 0, we have
\[
W_2(x) = W_2(0) + W_2'(0)x + o(x) = W_2(0) + o(x) < 0.
\]
Hence, if \( W_2(x) < 0 \), we are led to
\[
D_1 W_2'' = -g_0(\tilde{v}_{0,D_1} + \tilde{\theta}_2 W_2, x) < 0.
\]
which implies that \( W_2'(x) \) is strictly decreasing in \( x \). If follows from \( W_2'(0) = 0 \) and \( W_2(0) < 0 \) that \( W_2(x) < 0 \) for all \( x > 0 \) as long as \( W_2(x) \) exists. This completes the proof of the lemma.

**Step 2.** For \( b_0 \in (0, \beta) \), \( b_1 \in (\beta, d) \), we study the two over-determined problems:

\[
\begin{align*}
\begin{cases}
Dv'' + g_1(v, x) = 0, & x \in (0, L), \\
v(0) = b_0, & v'(0) = 0, \\
v(L) = \beta,
\end{cases} \\

\begin{cases}
Dv'' + g_0(v, x) = 0, & x \in (0, L), \\
v(0) = b_0, & v'(0) = 0, \\
v(L) = \beta,
\end{cases}
\] (4.5)

From Lemma 4.2 (ii), we know that for any \( b_0 \in (0, \beta) \), \( b_1 \in (\beta, d) \), there exist unique \( D = D_1^* \) and \( D = D_2^* \) for which the problems (4.5) and (4.6) have a solution, respectively.

**Lemma 4.3.** Let \( v_{0,D_1^*}(x) \), \( v_{0,D_2^*}(x) \) be solutions of the following two initial-value problems, respectively:

\[
\begin{align*}
\begin{cases}
D_1^*v'' + g_0(v, x) = 0, & x \in (0, L), \\
v'(0) = 0, & v(0) = b_0 + \tilde{\delta}_0, \\
v(L) = \beta,
\end{cases} \\

\begin{cases}
D_2^*v'' + g_1(v, x) = 0, & x \in (0, L), \\
v'(0) = 0, & v(0) = b_0, \\
v(L) = b_1 - \tilde{\delta}_0.
\end{cases}
\] (4.6)

Then for any sufficiently small positive number \( \tilde{\delta}_0 \), there exist \( \tilde{\delta}_1 > 0 \), \( \tilde{\delta}_2 > 0 \) such that \( v_{0,D_1^*}(L - \tilde{\delta}_1) = v_{1,D_2^*}(\tilde{\delta}_2) = \beta \), and \( \tilde{\delta}_1 \to 0 \), \( \tilde{\delta}_2 \to 0 \) as \( \tilde{\delta}_0 \to 0 \).

**Proof.** This lemma is a direct consequence of Lemma 4.2 (ii), therefore we omit the details.

Let \( \bar{L} = L - \tilde{\delta}_1 \). We now study the problem

\[
D_1^*v'' + g_1(v, x) = 0, \quad x \in (\bar{L}, L),
\] (4.9)

with the boundary conditions

\[
v(\bar{L}) = \beta, \quad v'(L) = 0.
\] (4.10)

We know that this problem has a unique solution \( v_{1,D_1^*}(x) \) and we let \( \tilde{b}_1 = v_{1,D_1^*}(L) \). Furthermore, by applying Lemma 3.5 (ii), we see that \( \bar{L} \) is decreasing in \( \tilde{b}_1 \), which means that \( \bar{\delta}_1 \) is increasing in \( \tilde{b}_1 \). We note that \( \tilde{b}_1 \to \beta \) as \( \tilde{\delta}_0 \to 0 \). In addition, if we integrate \( v_{1,D_1^*}' = -\frac{1}{D}g_1(v_{1,D_1^*}, x) \) from \( \bar{L} \) to \( L \), then

\[
v_{1,D_1^*}'(\bar{L}; a_1(\bar{L}; L), L) = \frac{1}{D} \int_\bar{L}^L g_1(v_{1,D_1^*}, x) \, dx \to 0 \quad \text{as} \quad \bar{L} \uparrow L.
\]

Therefore, by the definition of \( \Phi_0 \) (see (3.5)), we obtain

\[
\Phi_0(\bar{L}) = v_{0,D_1^*}'(\bar{L}; a_0(\bar{L}; 0), 0) - v_{1,D_1^*}'(\bar{L}; a_1(\bar{L}; L), L) > 0.
\]

Similarly, we have

\[
\Phi_0(\tilde{\delta}_2) = v_{0,D_2^*}'(\tilde{\delta}_2; a_0(\tilde{\delta}_2; 0), 0) - v_{1,D_2^*}'(\tilde{\delta}_2; a_1(\tilde{\delta}_2; L), L) < 0.
\]
where \( v_{0,D_2}(x) \) is a solution of \( D_2^2 v'' + g_0(v, x) = 0 \), \( x \in (0, \delta_2) \), under the conditions \( v'(0) = 0 \), \( v(\delta_2) = \beta \). Let \( D = \max \{D_1^*, D_2^*\} \). By virtue of Theorem 3.1 (i), for all \( D > D_0 \), we can construct a monotone increasing solution of problem (3.2).

On the other hand, starting from \( V_1 \) and using the same argument, we can also construct a monotone decreasing solution for large \( D \). Moreover, if we assume that \((H)\) is satisfied, then all the conditions of Theorem 3.2 are satisfied, and we conclude that if \( D \) is sufficiently large, then we can obtain two types of modes for any positive integer \( n \).

We state and prove a result on the asymptotic behavior of solutions of (3.2) as \( D \to \infty \).

**Theorem 4.4.** Suppose that \((H)\) is satisfied. Let \( v_D(x) \) be a solution of (3.2) for \( D \) sufficiently large. Then the solution \( v_D(x) \to \beta \) as \( D \to \infty \) uniformly on \([0, L]\). More precisely, \( \bar{v}_D = \frac{1}{L} \int_0^L v_D(x) dx \to \beta \) and \( \|v_D(x) - \beta\|_{\infty} = O(\frac{1}{D}) \).

**Proof.** To verify this, we decompose \( v_D(x) \) into two parts: \( v_D(x) = \bar{v}_D + \varphi_D(x) \), where \( \bar{v}_D = \frac{1}{L} \int_0^L v_D(x) dx \) and \( \varphi_D(x) \in C^1[0, L] \). Then we have \( \int_0^L \varphi_D(x) dx = 0 \).

By the mean value theorem for definite integrals, we see that there exists a \( y \in [0, L] \) such that \( \varphi_D(y) = 0 \). On the other hand, \( \varphi_D(x) \) satisfies

\[
\begin{aligned}
&\quad \left\{ \begin{array}{ll}
D \varphi''_D + g_\beta(\bar{v}_D + \varphi_D(x), x) = 0, & x \in (0, L), \\
\varphi'_D(0) = \varphi_D(L) = 0.
\end{array} \right.
\end{aligned}
\]

Therefore, \( \varphi_D(x) = -\frac{1}{D} \int_0^x g_\beta(\bar{v}_D + \varphi_D(\tau), \tau) d\tau \). Integrating this equation from \( y \) to \( x \), we obtain

\[
\varphi_D(x) = \varphi_D(y) - \frac{1}{D} \int_y^x \int_0^t g_\beta(\bar{v}_D + \varphi_D(s), s) ds dt
\]

\[
= -\frac{1}{D} \left( (x - y) \int_y^x g_\beta(\bar{v}_D + \varphi_D(s), s) ds + \int_y^x (x - t) g_\beta(\bar{v}_D + \varphi_D(t), t) dt \right).
\]

Note that \( |g_\beta(\bar{v}_D + \varphi_D(x), x)| \leq \|g_\beta\|_\infty \). This implies that

\[
|\varphi_D(x)| \leq \frac{1}{D} \left( |x - y| \|g_\beta\|_\infty y + \|g_\beta\|_\infty \int_y^x |x - t| dt \right)
\]

\[
= \frac{\|g_\beta\|_\infty}{D} \left( |x - y| + \frac{(x - y)^2}{2} \right)
\]

\[
= \frac{\|g_\beta\|_\infty}{D} |x - y| \left( 1 + \frac{|x - y|}{2} \right) \leq \frac{3L^2}{2D} \|g_\beta\|_\infty,
\]

which means that \( \varphi_D(x) = O(\frac{1}{D}) \). Therefore, \( v_D(x) = \bar{v}_D + O(\frac{1}{D}) \).

Next, we show that \( \bar{v}_D \to \beta \) as \( D \to \infty \). Assume that there exist a sequence \( D_i \to +\infty \) and a sequence of solutions \( \bar{v}_{D_i} \) corresponding to \( D_i \) such that \( |\bar{v}_{D_i} - \beta| \geq \delta_0 \) for some \( \delta_0 > 0 \). Since \( 0 < v_{D_i}(x) \leq d \), we have \( 0 < \bar{v}_{D_i} \leq d \). Therefore, there exists a subsequence of \( \bar{v}_{D_i} \), we still denote by \( \bar{v}_{D_i} \), such that \( \bar{v}_{D_i} \to \gamma \) and \( \gamma \) satisfies \( \gamma \geq \beta + \delta_0 \) or \( \gamma \leq \beta - \delta_0 \). We may assume that \( \gamma \geq \beta + \delta_0 \). Then

\[
v_{D_i}(x) = \bar{v}_{D_i} + \varphi_{D_i}(x), \quad |\varphi_{D_i}(x)| \leq \frac{C}{D_i}
\]

For \( i \) sufficiently large, we have

\[
v_{D_i}(x) \geq \bar{v}_{D_i} - |\varphi_{D_i}(x)| \geq \beta + \delta_0 - \frac{\delta_0}{2} = \beta + \frac{\delta_0}{2} \quad \text{for all} \ x \in [0, L].
\]
It follows that
\[
\int_0^L g^2(v_{D_1}, x) \, dx = \int_0^L g_1(v_{D_1}, x) \, dx > 0.
\]
On the other hand, integrating the first equation of (3.2) from 0 to L, we obtain
\[
\int_0^L g^2(v_{D_1}, x) \, dx = 0.
\]
This is a contradiction. Similarly, if \( \gamma \leq \beta - \delta_0 \), we can also get a contradiction. Therefore, \( \bar{v}_D \to \beta \) as \( D \to \infty \), which implies that \( v_D(x) \to \beta \) as \( D \to \infty \) uniformly on \([0, L]\).

**Remark 4.1.** For \( D > 0 \) sufficiently large, the distribution of ligands \( v(x) \) is almost uniform. However, the distribution of free receptors \( u(x) \) exhibits a sharp heterogeneity. This model has a mechanism which enhances small charges in \( v(x) \) to clearly recognizable patterns in \( u(x) \).

5. **Stability.** In this section, under a special topology, we consider the stability of steady states with jump discontinuity we obtained above. To begin with, let us recall the definition of \((\varepsilon_0, A)\)-stability introduced by Weinberger [14] and the stability conditions (see Theorem 2.3 in [3]).

**Definition 5.1.** A stationary solution \((\bar{u}, \bar{v})\) of system (1.1) is said to be \((\varepsilon_0, A)\)-stable for positive constants \( \varepsilon_0 \) and \( A \), if the following statement holds: Let initial functions \((u_0, v_0)\) satisfy
\[
\|u_0 - \bar{u}\|_{L^\infty(I_0)}^2 + \|v_0 - \bar{v}\|_{H^1(I)}^2 < \varepsilon^2
\]
for some \( I_0 \subset I = (0, L) \) with \( \text{meas}(I \setminus I_0) \leq \varepsilon^4 \) and for some \( \varepsilon \in (0, \varepsilon_0) \). Then
\[
\|u(t, \cdot) - \bar{u}\|_{L^\infty(I_0)}^2 + \|v(t, \cdot) - \bar{v}\|_{H^1(I)}^2 < A\varepsilon^2
\]
for all \( t > 0 \).

**Lemma 5.2.** Consider the initial-boundary value problem for a system of equations
\[
\begin{aligned}
&u_t = f(u, v, x), \quad x \in [0, L], \quad t > 0, \\
v_t = Dv_{xx} + g(u, v, x), \quad x \in (0, L), \quad t > 0, \\
v_x(0, t) = v_x(L, t) = 0.
\end{aligned}
\]
Let \((\bar{u}, \bar{v})\) be a steady state with finitely many jump discontinuities of \( \bar{u} \). Denote the Jacobi matrix of the kinetic system at the steady state by
\[
Q(x) = \begin{pmatrix}
f_u(\bar{u}, \bar{v}) & f_v(\bar{u}, \bar{v}) \\
g_u(\bar{u}, \bar{v}) & g_v(\bar{u}, \bar{v})
\end{pmatrix}
\]
and assume that the following inequalities hold for all \( x \in (0, L) \):
\[
\begin{aligned}
f_u(\bar{u}, \bar{v}) &\leq -C_0 < 0, \\
g_v(\bar{u}, \bar{v}) &\leq -C_0 < 0, \\
\det Q(x) &> 0,
\end{aligned}
\]
where \( C_0 \) is a positive constant independent of \( x \). Then \((\bar{u}, \bar{v})\) is \((\varepsilon_0, A)\)-stable for a pair \((\varepsilon_0, A)\) with some positive constants \( \varepsilon_0 \) and \( A \).

**Theorem 5.3.** The steady state \((\bar{u}, \bar{v})\) with jump discontinuity obtained in Theorems 3.1–3.3 and Theorem 4.1 are \((\varepsilon_0, A)\)-stable.
From Proposition 2.3, we see that
\[ \frac{\partial}{\partial x} \]
If \( \tilde{u}(x) = h_0(\tilde{v}(x), x) = 0 \), since \( \tilde{v} \) is positive on \([0, L]\), we obtain
\[ q_{11}(x) = -\mu_1(x) - \gamma(x)\tilde{v} < 0, \quad q_{12}(x) = 0, \]
\[ q_{21}(x) = -\gamma(x)\tilde{v} < 0, \quad q_{22}(x) = -\mu_2(x) < 0. \]
Since \( \tilde{v}(x) \) is bounded \( 0 < \tilde{v}(x) < d = \min_{x \in [0, L]} A(x) \), we find a positive constant \( C_1 \) such that
\[ q_{11}(x) \leq -C_1 < 0, \quad q_{22}(x) \leq -C_1 < 0, \quad \det Q(x) > 0 \quad \text{for all} \ x \in [0, L]. \]
If \( \tilde{u}(x) = h_1(\tilde{v}(x), x) \), since \( \frac{m_1(x)\tilde{u}}{1 + k\tilde{u}^2} = \mu_1(x) + \gamma(x)\tilde{v} \), we obtain
\[ q_{11}(x) = -\mu_1(x) + \gamma(x)\tilde{v} + \frac{2}{1 + k\tilde{u}^2}(\mu_1(x) + \gamma(x)\tilde{v}) \]
\[ = 1 - \frac{k\tilde{u}^2}{1 + k\tilde{u}^2}(\mu_1(x) + \gamma(x)\tilde{v}). \]
Since \( \tilde{u} = h_1(\tilde{v}(x), x) > \frac{1}{\sqrt{k}} \) for all \( x \in [0, L] \), we obtain \( \max_{x \in [0, L]} q_{11}(x) < 0. \) Obviously,
\[ q_{22}(x) = -\mu_2(x) - \gamma(x)h_1(\tilde{v}, x) \leq -\min_{x \in [0, L]} \mu_2(x) < 0. \]
It remains to consider the sign of \( \det Q(x) \). Note that \( f(h_1(\tilde{v}, x), \tilde{v}, x) = 0 \). By the implicit function theorem, we have
\[ \frac{\partial h_1}{\partial v}(\tilde{v}, x) = -\frac{q_{12}(x)}{q_{11}(x)}. \]
On the other hand,
\[ \frac{\partial}{\partial v} g(h_1(v, x), v, x) \bigg|_{v=\tilde{v}} = q_{21}(x) \frac{\partial h_1}{\partial v}(\tilde{v}, x) + q_{22}(x) = \frac{\det Q(x)}{q_{11}(x)}. \]
From Proposition 2.3, we see that \( \frac{\partial}{\partial v} g(h_1(v, x), v, x) \bigg|_{v=\tilde{v}} < 0. \) Since \( q_{11}(x) < 0 \), we have
\[ \det Q(x) = q_{11}(x) \frac{\partial}{\partial v} g(h_1(v, x), v, x) \bigg|_{v=\tilde{v}} > 0. \]
Therefore, all the conditions in Lemma 5.2 are satisfied and we conclude that the steady state \((\tilde{u}, \tilde{v})\) is \((\varepsilon_0, A)\)-stable. \( \square \)

6. **Examples and numerical simulations.** In this section, we give a few examples for which the assumption (H1) is satisfied and illustrate the patterns obtained by numerical simulations. Recall that
\[ g_0(v, x) = -\mu_2(x)v, \]
\[ g_1(v, x) = -\mu_2(x)v - \gamma(x)h_1(v, x)v + \frac{m_2(x)}{m_1(x)}(\mu_1(x) + \gamma(x)v)h_1(v, x). \]
Therefore,
\[ \frac{\partial g_0}{\partial x}(v, x) = -\mu_2'(x)v, \]
where \( \tilde{\kappa} \) is of the boundary value problem that (6.1) in Case I, (6.2), (6.3) in Case II are satisfied.

**Example 1.** The derivatives \( \mu_1'(x) \), \( \mu_2'(x) \), \( m_1'(x) \), \( m_2'(x) \) and \( \gamma'(x) \) are so small in the maximum norm that (6.1) in Case I, (6.2), (6.3) in Case II are satisfied.

**Case I.** \( \bar{d} = \min_{x \in [0,L]} v_r(x) \).

Hence, either \( \sup_{x \in [0,L]} u_{e,1}(x) < \frac{1}{\sqrt{k}} \) or \( u_{e,1}(x) \) does not exist. In this case, we assume that \( \mu_2(x) \equiv \mu_2 \). Then \( \frac{\partial g_1}{\partial x}(v, x) = 0 \) and we have

\[
\left( \frac{\partial \xi_0}{\partial x} \frac{\partial \eta_0}{\partial x} \right)_{(0, \pm; \lambda, \beta)} = \frac{\partial \xi_0}{\partial x} \frac{\partial \eta_0}{\partial x} \left( \lambda; \lambda, \beta \right) = \frac{1}{D} g_0(a_0, \lambda) > 0
\]

whenever \( \beta \in (0, d) \), \( 0 < a_0 < \beta \) and \( \lambda \in [0, L] \).

For \( i = 1 \), we assume that

\[
\left\| \frac{\partial g_1}{\partial x} \right\|_{L^\infty([\beta, d] \times [0,L])} < \frac{\min_{x \in [0,L]} g_1(d, x)}{L \cosh \left( \sqrt{\frac{M_1}{D}} L \right)},
\]

where \( M_1 = \left\| \frac{\partial g_1}{\partial x} \right\|_{L^\infty([\beta, d] \times [0,L])} \). Then it is easy to see that

\[
\left( \frac{\partial \xi_1}{\partial x} \frac{\partial \eta_1}{\partial x} \right)_{(1, \pm; \lambda, \beta)} < \left( \frac{\partial \xi_1}{\partial x} \frac{\partial \eta_1}{\partial x} \right)_{(\lambda; \lambda, \beta)} + \frac{1}{D} \left\| \frac{\partial g_1}{\partial x} \right\|_{L^\infty([\beta, d] \times [0,L])} \left\| \eta_1 \right\|_{L^\infty([0,L])} \left( \frac{\min_{x \in [0,L]} g_1(d, x)}{L} \right) \left( \sqrt{\frac{M_1}{D}} L \right)
\]

since \( 1 \leq \eta_1 \leq \cosh \left( \sqrt{\frac{M_1}{D}} L \right) \). Therefore, both conditions in (H1) are satisfied.

**Case II.** \( d = \min_{x \in [0,L]} v_r(x) \).

Hence, \( \inf_{x \in [0,L]} u_{e,1}(x) > \frac{1}{\sqrt{k}} \). In this case, we first fix \( \beta \in (0, d) \) and choose \( \tilde{k}_0, \tilde{k}_1 \) \( (0 < \tilde{k}_0 < \beta < d - \tilde{k}_1) \) so that whenever the monotone increasing solution \( v(x) \) of the boundary value problem

\[
Dv'' + g^\beta(v, 0) = 0, \quad 0 < x \leq L; \quad v'(0) = v'(\lambda) = 0
\]
exists and satisfies $\tilde{\kappa}_0 < v(0) < \beta$, we have $\beta < v(\lambda) < d - \tilde{\kappa}_1$. Next we assume that

$$\|\frac{\partial g_0}{\partial x}\|_{L^\infty([\tilde{\kappa}_0, \beta] \times [0, L])} < \frac{-\max_{x \in [0, L]} g_0(\tilde{\kappa}_0, x)}{L \cosh\left(\sqrt{\frac{M}{D}} L\right)}, \quad (6.2)$$

$$\|\frac{\partial g_1}{\partial x}\|_{L^\infty([\beta, d-\tilde{\kappa}_1] \times [0, L])} < \frac{\min_{x \in [0, L]} g_1(d - \tilde{\kappa}_1, x)}{L \cosh\left(\sqrt{\frac{M}{D}} L\right)}, \quad (6.3)$$

where $\tilde{M}_0 = \|\frac{\partial g_0}{\partial x}\|_{L^\infty([\tilde{\kappa}_0, \beta] \times [0, L])}$ and $\tilde{M}_2 = \|\frac{\partial g_1}{\partial x}\|_{L^\infty([\beta, d-\tilde{\kappa}_1] \times [0, L])}$. Similarly to Case I, we easily obtain

$$\left(\frac{\partial \xi_0}{\partial x}\eta_0 - \frac{\partial \eta_0}{\partial x}\xi_0\right) \bigg|_{(l_0, \pm; \lambda, \beta)} > \left(\frac{\partial \xi_0}{\partial x}\eta_0 - \frac{\partial \eta_0}{\partial x}\xi_0\right) \bigg|_{(\lambda; \lambda, \beta)} - \frac{1}{D} \left\|\frac{\partial g_0}{\partial x}\right\|_{L^\infty([\tilde{\kappa}_0, \beta] \times [0, L])} \left\|\eta_1\right\|_{L^\infty[0, L]} L$$

$$\leq -\frac{1}{D} g_0(a_0, \lambda) - \frac{1}{D} \left(\max_{x \in [0, L]} g_0(\tilde{\kappa}_0, x)\right) > 0,$$

$$\left(\frac{\partial \xi_1}{\partial x}\eta_1 - \frac{\partial \eta_1}{\partial x}\xi_1\right) \bigg|_{(l_1, \pm; \lambda, \beta)} < \left(\frac{\partial \xi_1}{\partial x}\eta_1 - \frac{\partial \eta_1}{\partial x}\xi_1\right) \bigg|_{(\lambda; \lambda, \beta)} + \frac{1}{D} \left\|\frac{\partial g_1}{\partial x}\right\|_{L^\infty([\beta, d-\tilde{\kappa}_1] \times [0, L])} \left\|\eta_1\right\|_{L^\infty[0, L]} L$$

$$\leq -\frac{1}{D} g_1(a_1, \lambda) + \frac{1}{D} \min_{x \in [0, L]} g_1(d - \tilde{\kappa}_1, x) < 0,$$

since $1 \leq \eta_0 \leq \cosh\left(\sqrt{\frac{M_0}{D}} L\right)$ and $1 \leq \eta_1 \leq \cosh\left(\sqrt{\frac{M_1}{D}} L\right)$. Therefore, both conditions in (H1) are satisfied.

**Example 2.** The functions $\mu_1(x)$, $m_1(x)$, $\gamma(x)$ are constants, and we assume that either (i) $m_0'(x) \leq 0 \leq \mu_\lambda(x)$ for all $x \in [0, L]$ or (ii) $\mu_\lambda'(x) \leq 0 \leq m_0'(x)$ for all $x \in [0, L]$.

If (i) holds, then we can construct a monotone increasing solution in $[0, L]$, while if (ii) holds, then we can construct a monotone decreasing solution in $x \in [0, L]$.

Here, we only prove the former case. Since $\mu'_2(x) \geq 0$, we have

$$\left(\frac{\partial \xi_0}{\partial x}\eta_0 - \frac{\partial \eta_0}{\partial x}\xi_0\right) = \frac{1}{D} \mu'_2(x) v \eta_0 \geq 0,$$

which implies that

$$\left(\frac{\partial \xi_0}{\partial x}\eta_0 - \frac{\partial \eta_0}{\partial x}\xi_0\right) \bigg|_{(l_0, 0; \mu_\lambda, \beta)} \geq \left(\frac{\partial \xi_0}{\partial x}\eta_0 - \frac{\partial \eta_0}{\partial x}\xi_0\right) \bigg|_{(0, 0; \beta)} = -\frac{1}{D} g_0(a_0, 0) > 0.$$

Note that $h_1(v, x)$ is independent of $x$ in the case. Therefore,

$$\left(\frac{\partial \xi_1}{\partial x}\eta_1 - \frac{\partial \eta_1}{\partial x}\xi_1\right) = \frac{1}{D} \mu'_2(x) v \eta_1 - \frac{1}{D} \frac{m'_2(x)}{m_1} (\mu_\lambda + \gamma v) h_1(v) \geq 0,$$

which means that

$$\left(\frac{\partial \xi_0}{\partial x}\eta_0 - \frac{\partial \eta_0}{\partial x}\xi_0\bigg|_{(l_0, 0; \lambda, \beta)} \geq \left(\frac{\partial \xi_1}{\partial x}\eta_1 - \frac{\partial \eta_1}{\partial x}\xi_1\bigg|_{(L, \lambda, \beta)} = -\frac{1}{D} g_1(a_1, L) < 0.$$

Therefore, both conditions in (H1) are satisfied in $[0, L]$, and we can construct a monotone increasing solution.
Example 3. $D$ is sufficiently large.

From Section 4, we see that if $D$ is sufficiently large, there exists a monotone increasing solution in $[0,L]$, as well as a monotone decreasing solution.

In the numerical simulations below, at each increment of the time variable, we first solved the kinetic system by the Runge-Kutta method, then solved the diffusion equation for $v$ by the backward Euler method. All codes are run in Processing 3.

Figure 2: Nullclines $f(u,v) = 0$ and $g(u,v) = 0$. Here, we fix the parameters $k = 0.02$, $\mu_1 = 1$, $\gamma = 1$, $m_1 = 1.2$, $m_2 = 1.6$ and only change $\mu_2$. In (a), $\mu_2 = 1.7$, we see that $f(u,v) = g(u,v) = 0$ have no intersection points except $(0,0)$, and we call this the monostable case; in (b), $\mu_2 = 3.5$, we see that $f(u,v) = g(u,v) = 0$ have two positive intersection points and they are both on the branch $u = h_2(v)$, which we call the DDI case; in (c), $\mu_2 = 7$, we see that $f(u,v) = g(u,v) = 0$ have two positive intersection points and one of them is on the branch $u = h_1(v)$ and the other is on $u = h_2(v)$, we call this the bistable case.

Figure 3: Pattern formation in Example 1. Here, we fix the parameters $k = 0.009$, $\mu_1 = 1$, $\gamma = 1$, $D = 3$, $\mu_2 = 4.15$ and let $m_1(x) = 1.6 - 0.03 \cos \left( \frac{2\pi x}{L} \right)$, $m_2(x) = 2.25 - 0.03 \cos \left( \frac{2\pi x}{L} \right)$. Under these conditions, $f(u,v) = g(u,v) = 0$ have two positive intersection points and both of them on the branch $u = h_2(v,x)$. Moreover, we observe that $m_1(x)$, $m_2(x)$, $\mu_2(x)$ are nearly constants, and their
Figure 3. Pattern formation in Example 1. The red curve represents receptor; the blue curve represents ligand; light blue is $\mu_2$; green is $m_1(x)$ and brown is $m_2(x)$.

Figure 4. Pattern formation in Example 2. The red curve represents receptor; the blue curve represents ligand; light blue is $\mu_2(x)$ and brown is $m_2(x)$.

derivative are close to zero. Therefore, the assumption (H1) is satisfied and we can construct a multi-mode solution.

Figure 4: Pattern formation in Example 2. Here, we fix the parameters $k = 0.02$, $\mu_1 = 1$, $\gamma = 1$, $m_1 = 1.2$, $D = 3$ and set $m_2(x) = 1.6 - 0.005x$, $\mu_2(x) = 1.4 + 1.2x$. Under these conditions, from Fig. 1, we see that the number of equilibria of the kinetic system varies from zero to three, i.e., it includes all the situations in Fig. 2. Furthermore, $\mu_2'(x) > 0$ and $m_2'(x) < 0$ in $[0, L]$. Therefore, the assumption (H1) is satisfied in $[0, L]$ and we can construct a monotone increasing solution.

Figure 5: Pattern formation in Example 3. Here, we fix the parameters $k = 0.1$, $\mu_1 = 1.6$, $\gamma = 2$, $D = 80$ and set $m_1(x) = 1.25 - 0.15 \cos(\frac{2\pi x}{L})$, $m_2(x) = 2.25 - 0.35 \cos(\frac{2\pi x}{L})$, $\mu_2(x) = 1.75 + 0.15 \cos(\frac{\pi x}{L})$. Under these conditions, $f(u, v) = g(u, v) = 0$ have no positive intersection points. Since $D = 80$, from the analysis in Example 3, we can construct a monotone increasing solution in $[0, L]$.

Concluding remarks. We have explored the possibility of building a theory of pattern formation in non-uniform media for a specific model in developmental biology. There are successful results in this direction, which may be found, e.g., in Chapter 6 of [2] where spatial segregation in competition systems is considered. Those results in ecology seem to be interested in how spatial heterogeneity helps the system have stable nontrivial steady states in a situation where only spatially trivial steady states would be stable if it were not for spatial heterogeneity. On the other hand, our question is whether it is possible or not to produce de novo patterns which overcome the existing spatial heterogeneity. We have found an example which allow us an affirmative answer.

Many questions remain to be answered. For example, can we locate the point of discontinuity in $u(x)$ in terms of the coefficients? In a nonlinearity different from
the one treated in this paper, the maximum point of the solution exhibiting a point-condensation phenomenon is related to a critical point of the locator function defined by the coefficients appearing in the equation (see [11]). Although our problem is quite different, we should look for a means to locate jump points.

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Appendix: Proof of Lemma 2.1. Let
\[ Q(u) = b_0(x)u^3 + b_1(x)u^2 + b_2(x)u + b_3(x). \]
Then we have
\[ Q'(u) = 3b_0(x)u^2 + 2b_1(x)u + b_2(x). \]

If \( \Delta(x) < 0 \), then \( Q'(u) > 0 \) for all \( u \in \mathbb{R} \). Combining this with \( Q(0) = b_3(x) > 0 \), we have \( Q(u) > 0 \) for all \( u \geq 0 \), which implies that (2.1) has no positive equilibrium.

If \( \Delta(x) \geq 0 \), then \( Q'(u) = 0 \) has two real roots \( u = u_1(x) \) and \( u = u_2(x) \), which satisfy \( u_1(x) + u_2(x) = -\frac{2b_1(x)}{3b_0(x)} \), \( u_1(x)u_2(x) = \frac{b_2(x)}{3b_0(x)} \), where
\[ u_1(x) = \frac{-b_1(x) - \sqrt{\Delta(x)}}{3b_0(x)}, \quad u_2(x) = \frac{-b_1(x) + \sqrt{\Delta(x)}}{3b_0(x)}. \]
We consider the following two cases:  

Case I. \( Q'(u) \geq 0 \) for all \( u \geq 0 \).

If this case occurs, then (2.1) has no positive equilibrium. Observe that Case I implies that \( u_1(x) + u_2(x) < 0 \) and \( u_1(x)u_2(x) \geq 0 \), which mean that \( b_1(x) > 0 \) and \( b_2(x) \geq 0 \).

Case II. \( Q'(u) < 0 \) for some \( u > 0 \).

Clearly, Case II yields \( u_2(x) > 0 \), which implies that either \( b_2(x) < 0 \) or \( b_2(x) \geq 0 \) and \( b_1(x) < 0 \). On the other hand, by simple calculation, we see that
\[ Q(u) = \left( \frac{1}{3} u + \frac{b_1(x)}{9b_0(x)} \right) Q'(u) + \frac{2}{9} \left( \frac{3b_2(x) - b_1(x)b_0(x)}{b_0(x)} \right) u + b_3(x) - \frac{b_1(x)b_2(x)}{9b_0(x)}, \]
$Q(u_2(x)) = P(x)$. It is easy to check that (2.1) has positive solutions if and only if $u_2(x) > 0$ and $P(x) \leq 0$. Otherwise, there is no positive solutions of (2.1). We thus complete the proof of Lemma 2.1.

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