Asymptotic behaviour of a class of inhomogeneous scalar field cosmologies

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Abstract

The asymptotic behaviour of a class of inhomogeneous scalar field cosmologies with a Liouville type of potential is studied. We define a set of new variables for which the phase space of the system of Einstein equations is bounded. This allows us to perform a complete analysis of the evolution of these cosmologies. We also discuss the extension of the cosmic no-hair theorem.
I Introduction

Since the proposal by Misner [1] of the “chaotic cosmology program”, the idea that the Universe emerged from a highly irregular state and that the inhomogeneities and the anisotropies were washed away giving place to a highly symmetric universe, has been one of the most attractive ideas in cosmology. In spite of the theorem proved by Collins and Hawking [2] which states that only a subclass of measure zero of the space of homogeneous solutions approach isotropy and the alternative approaches that postulated, according to the second law of thermodynamics [3], that the universe began in a highly regular state, Misner’s idea has come back due to the success of the inflationary scenarios.

The cosmic “no-hair” theorems of Wald [4] for homogeneous models and of Jensen and Stein-Schabes [5] for inhomogeneous spacetimes pointed out how the introduction of a cosmological constant, which can be considered as induced by a scalar field (inflaton), allows the models to isotropize approaching the de Sitter solution. This situation, however, may dramatically change if one takes into account the dynamical behaviour of the scalar field. Heusler [6], for example, extending the theorem of Collins and Hawking to the case of convex and positive potentials of the scalar field possessing a local minimum.
has shown that only the homogeneous Bianchi models which admit a FRW solution as a particular case approach isotropy. Later Kitada and Maeda [7] and Ibáñez et al. [8] have shown that when one assumes a Liouville type of potential for the scalar field (exponential potential), the Wald’s theorem for homogeneous solutions still applies if the exponential potential is quite flat.

Although Jensen and Stein-Schabes [5] extended the Wald theorem to inhomogeneous solutions, little is known about the effect of the dynamics of the scalar field on the asymptotic behaviour of the models. The first attempts to incorporate the effect of the dynamical evolution of the scalar field was made by Goldwirth and Piran [9] using numerical studies of inhomogeneous models and, later on by Calzetta and Sakellariadou [10] by studying the evolution of a family of inhomogeneous Cauchy data. The first inhomogeneous scalar field exact solutions with exponential potential of the Einstein field equations were obtained by Feinstein and Ibáñez [11] and it was shown there that the scalar field does not guarantee by itself that the model inflates or isotropize.

Although there are some exact inhomogeneous solutions obtained up to now, we are lacking a result similar to Heusler’s. In particular, for the exponential potential, it would be interesting to study the conditions on the scalar field leading to inflation and isotropization.
The asymptotic behaviour of homogeneous but anisotropic solutions for a perfect fluid has been widely studied. By using the kinematical quantities of the fluid as variables, the field equations can be written as an autonomous system of ordinary differential equations \[12\]. One of the most distinctive features is that the equilibrium points of these systems are self-similar solutions. This analysis has recently been extended to the case when the matter source is a scalar field with exponential potential \[13\]. Since the scalar field is homogeneous, one can globally associate with it a perfect fluid, and then by using the kinematical variables of the fluid, the Einstein field equations decouple and the phase space of the system becomes bounded.

The main difficulty in analyzing the asymptotic behaviour of inhomogeneous solutions is related with the fact that the field equations involve partial derivatives. The study of such systems was initiated in \[14\], where a particular family of self-similar solutions with perfect fluid, for which the Einstein equations reduced to an autonomous system, was studied. Since the source of the metric was a perfect fluid, the kinematical quantities could be used again to analyse the asymptotic behaviour of the system, as in the homogeneous case.

In dealing with an inhomogeneous scalar field one can not apply the
analysis developed in [14] due to the fact that one cannot globally interpret
the scalar field as a perfect fluid. Therefore in order to investigate whether
the inhomogeneous scalar field cosmologies undergo an inflationary epoch
leading to the homogenization and the isotropization of the spacetime, one
does have to look for a different way to tackle this problem.

In this paper we initiate the study of the asymptotic behaviour of the
scalar field cosmologies by considering, as a first step, a 2 parameter family
of $G_2$ self-similar solutions. The fact that the scalar field is not equivalent
to a perfect fluid prevents the use of the kinematical quantities to describe
the evolution of the solutions. Despite of the lack of a preferred timelike
congruence in the spacetime we manage to find a set of new variables, in terms
of which the phase space of the system becomes bounded. This allows us to
perform a complete analysis of the asymptotic behaviour of these spacetimes.
In addition, the way to introduce the set of new variables is a promising
method for dealing with more general solutions.

The plan of the paper is as follows: in Section 2 we present the metric
and introduce the new variables. Section 3 is devoted to the analysis of the
phase space and in Section 4 we discuss the results.
II The metric and the compactified phase space

We will consider solutions with one-dimensional inhomogeneity. These metrics are described by the generalized Einstein-Rosen spacetimes which admit an Abelian group of isometries $G_2$. If the two Killing vectors are hypersurface orthogonal the line element can be written as

$$ds^2 = e^F \left(-dt^2 + dz^2\right) + G \left(e^h dx^2 + e^{-h} dy^2\right),$$

where the metric functions depend on $t$ and $z$ and the Killing vectors are $\partial_x$ and $\partial_y$.

The matter source for the metric is that of a minimally coupled scalar field with potential, for which the stress-energy tensor is given by

$$T_{ab} = \ddot{\phi}_a \ddot{\phi}_b - g_{ab} \left(\frac{1}{2} \dddot{\phi}_c \dddot{\phi}^c + V(\dddot{\phi})\right),$$

(latin indices run from 0 to 3) with the Liouville type of the potential

$$V(\dddot{\phi}) = \Lambda e^{k\dddot{\phi}}, \quad \Lambda \geq 0.$$  

It is well known [13] that as long as the gradient of the scalar field remains timelike, (2) can be rewritten as a perfect fluid stress-energy tensor.
To simplify the equations we will concentrate in this paper on the class of solutions for which the element of transitivity surface $G$ is homogeneous

$$G = G(t), \quad (4)$$

which is suitable to a description of cosmological models.

It was shown in [16] that if one assumes separability of the metric and separability, in the additive sense, of the scalar field one obtains, from the field equations, that the dependence of these functions on the variable $z$ is linear

$$h(t, z) = p(t) + az, \quad F(t, z) = f(t) + cz, \quad \phi(t, z) = \phi(t) + bz, \quad (5)$$

where $a$, $b$ and $c$ are arbitrary constants which drive the inhomogeneity.

This linear behaviour for the inhomogeneity has been recently considered, in a different context, by Vilenkin [17]. The general solution for this class of metrics and the study of few particular examples were given in [11].

Before going to the field equations it is worth mentioning that the metric (1)-(5) admits a homothetic vector field given by

$$H = \frac{2}{c} \frac{\partial}{\partial z} + \left(1 - \frac{a}{c}\right) x \frac{\partial}{\partial x} + \left(1 + \frac{a}{c}\right) y \frac{\partial}{\partial y}. \quad (6)$$

It was conjectured in [14] that the self-similar solutions could be the attractors of the $G_2$ cosmologies and in this sense the metric (1)-(5) would play
an important role in the study of the asymptotic behaviour of the inhomogeneous solutions. The metric (1)-(5) is similar to the metric studied by Hewitt et al [14] (it is obtained interchanging \( t \) by \( z \) in (5)). Both metrics, however, differ in the character of the orbits of the similarity group and in the type of the source. In our case the orbits of (6) are spacelike given by \( t = \text{constant} \) and the scalar field is inhomogeneous and, therefore, it is not equivalent, in general, to a perfect fluid.

The Einstein equations and the Klein-Gordon equation for the scalar field with the ansatz (5) are given by

\[
\frac{\ddot{G}}{G} - 2 V(\phi) e^F = 0, \quad (7)
\]
\[
\ddot{p} + \frac{\dot{G}}{G} \dot{p} = 0, \quad (8)
\]
\[
ap\dot{p} - c\frac{\dot{G}}{G} + 2b\dot{\phi} = 0, \quad (9)
\]
\[
\frac{\ddot{G}}{G} - \frac{1}{2} \left(\frac{\dot{G}}{G}\right)^2 - \ddot{\phi} - \frac{1}{2} \dot{G}^2 + \phi^2 + \frac{1}{2} a^2 + b^2 = 0, \quad (10)
\]
\[
\ddot{f} - \frac{1}{2} \left(\frac{\dot{G}}{G}\right)^2 + \frac{1}{2} \ddot{\phi}^2 + \phi^2 - \frac{1}{2} a^2 - b^2 = 0, \quad (11)
\]
\[
\ddot{\phi} + \frac{\dot{G}}{G} \dot{\phi} + kV(\phi)e^F = 0, \quad (12)
\]

For a massless scalar field, \( V = 0 \), the former system of equations turns out to be easily integrable giving the following solution

\[
p(t) = A \ln t, \quad \phi(t) = B \ln t, \quad G(t) = t,
\]
\( f(t) = \left( \frac{1}{2} A^2 + B^2 - \frac{1}{2} \right) \ln t + \frac{1}{2} \left( \frac{1}{2} a^2 + b^2 \right) t^2, \) \hfill (13)

where \( A \) and \( B \) are constants subject to the condition

\[ aA - c + 2bB = 0. \] \hfill (14)

When \( V \neq 0 \), from (7) or (12), and due to the exponential form of the potential

\[ c = -kb. \] \hfill (15)

In this case if the constant \( b = 0 \) then, from (9), either \( \dot{p} = 0 \) and the metric is Bianchi type VI\(_b\), or \( a = 0 \) and the metric is Bianchi type I. Let’s note that whatever value \( V \) takes, when \( a = b = 0 \) the metric is Bianchi type I.

In order to look for a suitable set of new variables to compactify the phase space we write the generalized Friedman equation which is obtained from (7) and (10)

\[-\frac{1}{2} \left( \frac{\dot{G}}{G} \right)^2 - f \frac{\dot{G}}{G} + \frac{1}{2} \dot{p}^2 + \dot{\phi}^2 + \frac{1}{2} a^2 + b^2 + 2V(\phi)e^F = 0. \] \hfill (16)

This equation suggests the introduction of the following set of variables

\[ \beta = \frac{\dot{p}}{\dot{G}/G + f}, \quad \Psi = \sqrt{2} \frac{\dot{\phi}}{\dot{G}/G + f}, \]

\[ \Phi = \frac{\dot{f}}{\dot{G}/G + f}, \quad \Gamma = \frac{2\sqrt{V}e^F}{\dot{G}/G + f}. \] \hfill (17)
In terms of these new variables (16) is written as

\[ 1 - \beta^2 - \Psi^2 - \Phi^2 - \Gamma^2 = \frac{(a^2 + 2b^2)}{(\dot{G}/G + \dot{f})^2} \geq 0. \]  

(18)

Therefore the space of solutions described by the variables (17) is bounded to the inside of the 4-sphere (18). Let’s note that, from (16), \( \dot{G}/G + \dot{f} \) has to be different from zero, unless \( \Lambda \) be negative.

In dealing with perfect fluid models new variables were defined as the kinematical quantities of the fluid divided by an appropriate power of the rate of expansion \( \theta \). This assures a good behaviour of these variables near the initial singularity. Let’s note that our new variables (17) are divided by the quantity \( \dot{G}/G + \dot{f} \) which is not related with the expansion of any timelike congruence. Nevertheless, from the general behaviour near the initial singularity found by Belinskii et al [18] and from the work of Isenberg and Moncrief [19] we can assume that the metric (1)-(5), near the initial singularity behaves, for each value of the coordinate \( z \), like a Kasner model. Therefore, when \( t \rightarrow 0 \)

\[ G \sim t, \quad f \sim p \sim \phi \sim \ln t, \]  

(19)

and the variables (17) remain bounded when \( t \) tends to zero.

From (18) we see that the points on the surface of the 4-sphere represent
either homogeneous Bianchi type I solutions (when constants $a$ and $b$ are zero) or the initial singularity of the models.

By using (17), (9) is written as

$$a\beta + b\sqrt{2}\Psi - c(1 - \Phi) = 0.$$  \hspace{1cm} (20)

Except for the trivial case when $a = b = 0$, this equation gives a constraint for the constants $a$ and $b$. Alternatively, if $a$ and $b$ are fixed one can look at this equation as giving a plane, intersection of which with the sphere (18) describes the phase space. In the study of the equilibrium points in the next Section $a$ and $b$ will be arbitrary chosen constrained by (20).

Using the variables (17), (7) becomes

$$\ddot{G} = \frac{1}{2} \left( \frac{\Gamma}{1 - \Phi} \right)^2 \left( \frac{\dot{G}}{G} \right)^2.$$  \hspace{1cm} (21)

This equation decouples from the rest of field equations if we introduce a new time coordinate

$$\frac{d\tau}{dt} = \frac{\dot{G}}{G} + \ddot{f} = \frac{1}{1 - \Phi} \frac{\dot{G}}{G},$$  \hspace{1cm} (22)

Near the initial singularity

$$\frac{d\tau}{dt} \sim \frac{1}{t} \Rightarrow \tau \to -\infty,$$  \hspace{1cm} (23)

thus $\tau$ varies from $-\infty$ to $+\infty$. In terms of this new time the field equations
are written as

\[ \beta' = -\beta \left(1 - \beta^2 - \Psi^2 - \Phi^2\right), \]  
(24)

\[ \Psi' = -\Psi \left(1 - \beta^2 - \Psi^2 - \Phi^2\right) - \frac{k}{2\sqrt{2}} \Gamma^2, \]  
(25)

\[ \Phi' = (1 - \Phi) \left(1 - \beta^2 - \Psi^2 - \Phi^2\right) - \frac{1}{2} \Gamma^2, \]  
(26)

\[ \Gamma' = -\Gamma \left(-\beta^2 - \Psi^2 - \Phi^2 + \frac{1}{2} \Phi - \frac{k}{2\sqrt{2}} \Psi\right), \]  
(27)

where \( \prime \) means derivative with respect the new time \( \tau \). Differentiating (20) with respect to \( \tau \) one can easily see that the constraint equation (20) holds for all values of \( \tau \), as long as the initial conditions verify the equation (20). Therefore (24)-(27) along with the constraint equation (20) describe the evolution of the metric (1)-(5).

III The Equilibrium Points and The Invariant Sets

In this Section we shall study the qualitative behaviour of the trajectories of the system (24)-(27). Let’s first note that the system admits the discrete symmetry \( \Gamma \rightarrow -\Gamma \), and, therefore, without loss of generality the study of the equilibrium points will be restricted to \( \Gamma \geq 0 \).
Equilibrium points

The equilibrium points of an autonomous system play an important role in describing the qualitative behaviour of its solutions. The local stability of the equilibrium points are given by the eigenvalues of the linearized differential equations. The equilibrium points of the system (24)-(27) can be found explicitly and we now give them and their character.

$$\begin{align*}
\beta = 0 & \quad \Psi = -\frac{k\sqrt{2}}{2 + k^2} \\
\Phi = \frac{k^2}{2 + k^2} & \quad \Gamma = \frac{2\sqrt{2}}{2 + k^2}
\end{align*}$$

This point is outside the sphere when $k^2 < 2$ and it is on the surface of the sphere when $k^2 = 2$. The constraint equation (20) is trivially satisfied for all values of the constants $a$ and $b$. The corresponding metric and the scalar field when $k^2 > 2$ are

$$\begin{align*}
(ds)^2 &= Ce^{k^2 At/2 - kbz} \left(-dt^2 + dz^2\right) + e^{At} \left(e^{az} dx^2 + e^{-az} dy^2\right), \\
\phi &= -\frac{k}{2} At + bz,
\end{align*}$$

(28)

where

$$C = \frac{A^2}{\Lambda}, \quad A = \sqrt{\frac{2(a^2 + 2b^2)}{k^2 - 2}}.$$  

(29)

This metric was obtained in [11] and is of Bianchi type VI. The eigenvalues
of the linearized system are

\[
\frac{-2}{k^2 + 2}, \quad \frac{-2}{k^2 + 2}, \quad \frac{-1 + \sqrt{5 - 2k^2}}{k^2 + 2}, \quad \frac{-1 - \sqrt{5 - 2k^2}}{k^2 + 2}.
\]

(30)

When \(k^2 < \frac{5}{2}\) the equilibrium point is a stable node and when \(k^2 > \frac{5}{2}\) a stable focus.

II \[ \begin{cases} \beta = 0 & \Psi = -\frac{k}{2\sqrt{2}} & \Phi = \frac{1}{2} & \Gamma = \frac{\sqrt{6 - k^2}}{2\sqrt{2}} \end{cases} \]

This equilibrium point is on the surface of the sphere. It disappears when \(k^2 > 6\) and coincides with the former equilibrium point when \(k^2 = 2\).

Substituting the former values of the variables of the equilibrium point into (17) one easily gets that the solution corresponds to a homogeneous Bianchi type I solution \((a = b = 0)\). The line element and the scalar field are

\[
\begin{align*}
ds^2 &= Ct^m \left(-dt^2 + dz^2\right) + t^m \left(dy^2 + dz^2\right), \\
\phi &= -\frac{2k}{k^2 - 2} \ln t,
\end{align*}
\]

(31)

where

\[
C = \frac{2(6 - k^2)}{\Lambda (k^2 - 2)^2}.
\]

(32)

and the metric represents the FRW universe with massless minimally coupled
scalar field as a source. The eigenvalues of the linearized system are

\[-\frac{1}{8}(6 - k^2), \quad -\frac{1}{4}(2 - k^2), \quad -\frac{1}{8}(6 - k^2), \quad -\frac{1}{8}(6 - k^2).\]  

(33)

When \(k^2 < 2\) this equilibrium point is a stable node but when \(2 < k^2 < 6\) is a saddle point.

It is important to note that the constraint equation (20) is satisfied not only because \(a\) and \(b\) vanish but because \(\beta = 0\) and \(\sqrt{2}\Phi + k(1 - \Phi) = 0\). If we look into the evolution in time of a particular solution with \(a\) and \(b\) different from zero, and with \(k^2 < 2\), \(\beta, \Psi\) and \(\Phi\) will take values such that the constraint equation will be always satisfied and as \(t \to \infty\) the solution will approach this equilibrium point becoming, therefore, homogeneous and isotropic.

III \(\{\beta^2 + \Psi^2 + \Phi^2 = 1, \quad \Gamma = 0\}\)

This ring of equilibrium points belongs to the surface of the sphere and represents, therefore, Bianchi type I solutions with a minimally coupled massless scalar field. By choosing a particular point on the surface, i.e. \(\beta = a_1, \Psi = a_2\) and \(\Phi = a_3\) with \(a_1^2 + a_2^2 + a_3^2 = 1\), the eigenvalues of the
linearized system are

\[ \frac{1}{2} \left( 2 - a_3 + \frac{k}{\sqrt{2}a_2} \right), \quad 2(1 - a_3), \quad 0, \quad 0. \quad (34) \]

Hence, any of these points are unstable, except when Φ = 1 \((a_3 = 1)\), which corresponds to the Minkowski spacetime.

**Invariant sets**

Besides the equilibrium points, the existence of invariant sets help to describe the qualitative behaviour of the solutions of an autonomous system. In our case, there are three invariant sets, two of them describing massless and exponential potential scalar field solutions respectively while the third gives the Bianchi type I solutions.

The points with \(\Gamma = 0\) (massless scalar field spacetimes) compose an invariant set of the dynamical system (24)-(27) whose solutions are given by (13) and (14). The dynamics of the points of this subspace can be easily studied: when \(\Gamma = 0\) the dynamical system reduces to

\[
\begin{align*}
\beta' &= -\beta \left( 1 - \beta^2 - \Psi^2 - \Phi^2 \right), \\
\Psi' &= -\Psi \left( 1 - \beta^2 - \Psi^2 - \Phi^2 \right), \\
\Phi' &= (1 - \Phi) \left( 1 - \beta^2 - \Psi^2 - \Phi^2 \right),
\end{align*}
\]

(35)
with straight lines as solutions

\[
\Phi = \frac{\Phi_0 - 1}{\beta_0} \beta + 1, \quad \Psi = \frac{\Psi_0}{\beta_0} \beta.
\]  

(36)

These lines start on the surface of the sphere \(\beta^2 + \Psi^2 + \Phi^2 = 1\) and intersect at the point \(\beta = \Psi = 0, \Phi = 1\). Thus the solutions evolve from the Kasner initial singularity to the Minkowski spacetime.

The second invariant set is given by the points with \(\Gamma > 0\) (and by symmetry \(\Gamma < 0\)), with equilibrium points being the points I and II described above.

Finally the third invariant set is given by the points on the surface of the sphere \((\beta^2 + \Psi^2 + \Phi^2 + \Gamma^2 = 1)\). In this case the dynamical system is reduced to

\[
\begin{align*}
\beta' &= -\beta \left(1 - \beta^2 - \Psi^2 - \Phi^2\right), \\
\Psi' &= - \left(\Psi + \frac{k}{2\sqrt{2}}\right) \left(1 - \beta^2 - \Psi^2 - \Phi^2\right), \\
\Phi' &= \left(\frac{1}{2} - \Phi\right) \left(1 - \beta^2 - \Psi^2 - \Phi^2\right).
\end{align*}
\]

(37)

The solutions of this system are again straight lines.

\[
\Phi = \left(\Phi_0 - \frac{1}{2}\right) \frac{\beta}{\beta_0} + 1, \quad \Psi = \left(\Psi_0 + \frac{k}{2\sqrt{2}}\right) \frac{\beta}{\beta_0} - \frac{k}{2\sqrt{2}},
\]

(38)

and the equilibrium points of the system (37) are the points II and III.
The behaviour of the system (24)-(27) can be visualized in Fig.1 where the phase space for $\beta = 0$ and $\Gamma \geq 0$ is depicted for different values of the constant $k$. The positions of the equilibrium point I as function of $k$ are represented by the dashed line. All the solutions start on the circle $\Phi^2 + \Psi^2 = 1, \Gamma = 0$. When $\Gamma = 0$ (massless scalar field), for all the cases, the solutions tend to the point $\Phi = 1, \Psi = 0$. When $k^2 < 2$ the point I is outside the sphere and the solutions tend to the equilibrium point II. When $2 < k^2 < 6$ point I is inside the sphere and the solutions evolve either to the point I (those that go inside the sphere) or to the point II (those that lie on the boundary of the sphere). When $k^2 > 6$ the only attracting point is point I (inside the sphere). The trajectories on the surface start and finish on the circle $\Phi^2 + \Psi^2 = 1, \Gamma = 0$ in such a way that their projections on the plane $\Gamma = 0$ are straight line directed to the point $\Psi = -\frac{k}{2\sqrt{2}} \Phi = \frac{1}{2}$ which is outside the surface.

The behaviour described above remains the same when $\beta$ is different from zero.
IV Conclusions

We have studied in this paper the asymptotic behaviour of a particular class of inhomogeneous solutions with a minimally coupled scalar field with an exponential potential. The metric belongs to the class of $G_2$ cosmologies.

We have succeeded to define a set of new variables for which the entire phase space is bounded by a 4-sphere. From the analysis of the dynamical system obtained for this metric, we deduce the following:

i) As in the homogeneous case [13], the dynamical behaviour of the metric depends on the parameter $k$ which is related to the mass of the scalar field. The parameters $a$ and $b$ which drive the inhomogeneity do not play a significant role in this behaviour.

ii) When $k^2 < 2$ the only equilibrium point is that given by the FRW universe. The trajectories evolve from the surface of the 4-sphere, representing the Kasner regime near the initial singularity, towards the isotropic equilibrium point II. When $k^2 > 2$ there are two equilibrium points, one is the homogeneous Bianchi type VI (point I) and the second is the FRW solution which is a saddle point and, therefore, unstable against small changes of the initial conditions. This means that the cosmic no-hair theorem holds in this case provided $k^2 < 2$. 

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The behaviour described above is almost identical to that of homogeneous models [13], indicating that the introduction of the inhomogeneity through the constants $a$ and $b$ does not affect remarkably the dynamical system. It is likely that the presence of a more “strong” inhomogeneity, assuming another dependence on the spatial coordinate, may change the behaviour of the solutions and the conditions to isotropize. We consider, therefore, that it is important to extend this analysis to a more general class of inhomogeneous scalar field solutions.

As to the inflation of the models, for inhomogeneous solutions there is no natural way to see whether the models inflate. The hypersurfaces $\phi =$constant do not define a globally timelike observer and, therefore, one should use a weaker way to specify the inflationary behaviour. It has been suggested [20], for example, to look at the fulfillment of the strong energy condition, the breaking of which is a necessary condition for a model to inflate. The energy density and the pressure of the scalar field are given by

$$
\rho = -\frac{1}{2}\phi_{,a}\phi^{,a} + V, \quad p = -\frac{1}{2}\phi_{,a}\phi^{,a} - V. \quad (39)
$$

In terms of the new variables, the breaking of the energy condition is written
as

\[ 3p + \rho = e^{-F} \left( \frac{\dot{G}}{G} + \dot{f} \right)^2 \left[ \Psi^2 - \frac{1}{2} \Gamma^2 - \frac{2b^2}{(\dot{G}/G + \dot{f})^2} \right] < 0. \]  

(40)

For the equilibrium points I and II the expression \( \Psi^2 - \frac{1}{2} \Gamma^2 \) turns out to be

\[ \frac{2(k^2 - 2)}{(2 + k^2)^2} \quad \text{and} \quad \frac{3(k^2 - 2)}{16} \]  

respectively. That means that if \( k^2 < 2 \) both points describe spacetimes inflating, but if \( k^2 > 2 \) the energy condition could be broken only by the solution represented by the point I depending on the value of \( b \). This behaviour is similar again with that of the homogeneous models [8].

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Figure Captions

Figure 1.- Phase space when $\beta = 0$ and $\Gamma \geq 0$, and for different values of the constant $k$. The equilibrium point I is represented by a square and the equilibrium point II by a circle. The dashed line describes the position of the point I for different values of $k$. Faded lines represent the trajectories of the solutions lying on the surface of the sphere which is an invariant set. The plane $\Gamma = 0$ is an invariant set describing the massless scalar field solutions.
\[ k^2 < 2 \]

- Figure 1.a -
$6 > k^2 > 2$  

Figure 1.b
$k^2 > 6$

– Figure 1.c –