On gravitational and electromagnetic memory

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ABSTRACT. We present a unified investigation of memory effect in Einstein-Maxwell theory. Our result recovers the two known gravitational memory effects and the two known electromagnetic memory effects.

1 Introduction

In the last few years there has been a renewed interest on gravitational and electromagnetic memory effects. Although the latter were investigated long time ago [1–8] (see also [9–13] for the realization in experimental detections), the new enthusiasm comes from a purely theoretical side. In 2014 Strominger and Zhiboedov discovered a fundamental connection between the gravitational memory effect and Weinberg’s soft graviton theorem [14]. They are mathematically equivalent. This equivalence was shortly extended to gauge theories [15–17]. Inspired by this fascinating equivalence, new gravitational [18] and electromagnetic [19] memory effects were reported.

The investigation in literatures on memory effects are performed independently for different theories. A unified treatment of different types of memory effects in a coupled theory is still missing[1]. The aim of the present note is to provide a unified treatment for gravitational and electromagnetic memory effects in Einstein-Maxwell theory.

In this work, we study Einstein-Maxwell theory in Newman-Penrose (NP) formalism [21]. We work in NP formalism since it makes the geometrical property of the spacetime more transparent and it has natural connection with the spinor formalism which is the most satisfactory way of investigating fermion coupled theories. We obtain the general asymptotic solutions which generalizes the result of [22,23] to the case of an arbitrary conformal factor. Then we examine the memory effect via studying the motion of a charged test particle. By solving the equations of motion of a free falling charged particle, we find that the charged particle, which is initially static, is forced to orbit due to

[1] Memory effect was investigated in [20] in Einstein-Maxwell theory. But only gravitational effect was involved.
the presence of the gravitational [24–34] and electromagnetic [8] radiations. The charged particle receives a time delay due to massive objects with or without electric charge in the space-time [35–37], gravitational radiation [33, 34] and electromagnetic radiation. The change of the velocity of the charged particle is the leading memory effect. The memory induced by gravitational and electromagnetic radiations recovers the gravitational and electromagnetic memory formulas respectively. The position displacement of the charged particle is the sub-leading memory effect. The gravitational and electromagnetic contributions recover the spin memory formula in [18] and the new electromagnetic memory formula in [19] respectively. The gravitational and electromagnetic memory effects happen at the same order while the contribution of electromagnetic radiation to the time delay of the charged particle shows up at one order higher than gravitational radiation.

2 Einstein-Maxwell theory in the NP formalism

In NP formalism, two real null vectors \( e_1 = l, \ e_2 = n \), one complex null vector \( e_3 = m \) and its complex conjugate vector \( e_4 = \bar{m} \) are chosen as the basis vectors. As directional derivatives, the basis vectors are designated with special symbols

\[
D = l^\mu \partial_\mu, \quad \Delta = n^\mu \partial_\mu, \quad \delta = m^\mu \partial_\mu. \tag{2.1}
\]

The connection coefficients are called spin coefficients in the NP formalism with special Greek symbols (we will follow the convention of [38]). The Lagrangian of four-dimensional Einstein-Maxwell theory is

\[
\mathcal{L} = \sqrt{-g} \left[ R - \frac{1}{2} F^2 \right], \quad F = dA. \tag{2.2}
\]

The metric is constructed from the basis vectors as

\[
g_{\mu\nu} = n_\mu l_\nu + n_\mu n_\nu - m_\mu \bar{m}_\nu - \bar{m}_\mu m_\nu. \tag{2.3}
\]

In a hyperbolic Riemannian manifold, it is always possible to introduce a coordinate system \((u, r, x^A)\) where \((A = 3, 4)\) [21] such that the basis vectors and the co-tetrad must have the form

\[
n = \partial_u + U \partial_r + X^A \partial_{x^A}, \quad l = \partial_r, \quad m = \omega \partial_r + L_A \partial_{x^A},
\]

\[
n = \left[ - U - X^A (\omega L_A + \omega \bar{L}_A) \right] du + dr + \left( \omega \bar{L}_A + \omega L_A \right) dx^A, \quad l = du, \quad m = -X^A L_A du + L_A dx^A. \tag{2.4}
\]

\[\text{The displacement effect is from a single test particle while the displacement discovered in [1] is a relative displacement of nearby observers. So they are different types of memory effect.}\]
where $L_AL^A = 0$, $L_AL^A = -1$. In addition, the freedom of the rotation of the basis vectors will set the spin coefficients

$$\pi = \kappa = \epsilon = 0, \ \rho = \bar{\rho}, \ \tau = \bar{\alpha} + \beta. \quad (2.5)$$

The main conditions of approaching flatness at infinity are $\Psi_0 = \frac{\Psi_0}{r^5} + O(r^{-6})$ and $\phi_0 = \frac{\phi_0}{r^5} + O(r^{-4})$. The solutions of the NP equations (see in Appendix A) in asymptotic expansions in the stereographic coordinates $(z, \bar{z})$ with arbitrary conformal factor is given by:

$$\Psi_0 = \frac{\Psi_0^0(u, z, \bar{z})}{r^5} + \frac{\Psi_1^0(u, z, \bar{z})}{r^6} + O(r^{-7}), \ \phi_0 = \frac{\phi_0^0(u, z, \bar{z})}{r^3} + \frac{\phi_1^0(u, z, \bar{z})}{r^4} + O(r^{-5}),$$

$$\Psi_1 = \frac{\Psi_1^0(u, z, \bar{z})}{r^4} + \frac{3\phi_0^0u - \bar{\phi}_0^0}{r^5} + O(r^{-6}), \ \phi_1 = \frac{\phi_1^0(u, z, \bar{z})}{r^2} - \frac{\bar{\phi}_0^0}{r^3} + O(r^{-4}),$$

$$\Psi_2 = \frac{\Psi_2^0(u, z, \bar{z})}{r^3} + \frac{\phi_0^2 + \bar{\phi}_1^0}{r^4} + \frac{1}{2r^5} \left[ \lambda^0 \Psi_0^0 + \bar{\lambda}^0 \bar{\Psi}_0^0 + 3\sigma^0 \bar{\sigma}_0^0 \Psi_2^0 + 4\Psi_1^0 \bar{\sigma}_0^0 + \sigma^0 \bar{\sigma}_1^0 \right] + O(r^{-6}),$$

$$\phi_2 = \frac{\phi_2^0(u, z, \bar{z})}{r} - \frac{\bar{\phi}_0^0}{r^2} + \lambda^0 \phi_0^0 + \sigma^0 \bar{\sigma}_0^0 \phi_2^0 + 2\phi_1^0 \bar{\sigma}_0^0 + \sigma^0 \bar{\sigma}_1^0 + \bar{\sigma}_0^0 \phi_1^0 + O(r^{-4}),$$

$$\Psi_3 = \frac{\Psi_3^0}{r^2} + \frac{\phi_0^3 + \bar{\phi}_2^0}{r^3} + O(r^{-4}), \ \Psi_4 = \frac{\Psi_4^0}{r} - \frac{\bar{\Psi}_3^0}{r^2} + O(r^{-3}),$$

$$\rho = -\frac{1}{r} - \frac{\sigma^0 \bar{\sigma}_0^0}{r^3} + \frac{\sigma^0 \Psi_0^0 + \bar{\sigma}_0^0 \bar{\Psi}_0^0}{r^4} - \frac{\Psi_1^0}{3r^5} + O(r^{-6}),$$

$$\sigma = \frac{\sigma^0 (u, z, \bar{z})}{r^2} + \frac{\sigma^0 \Psi_0^0 - \frac{1}{2} \Psi_0^0}{r^4} - \frac{\Psi_1^0}{3r^5} + O(r^{-6}),$$

$$\alpha = \frac{\alpha^0}{r} - \frac{\sigma^0 \bar{\sigma}_0^0 \alpha^0}{r^2} + \frac{\sigma^0 \bar{\sigma}_0^0 \alpha^0}{r^3} + \frac{6\sigma^0 \bar{\sigma}_0^0 (\bar{\sigma}_0^0 \bar{\sigma}_0^0)^2 - \sigma^0 \bar{\sigma}_0^0 \Psi_0^0 + \sigma^0 \bar{\sigma}_0^0 \bar{\Psi}_0^0 - 2\phi_1^0 \bar{\phi}_0^0}{6r^4} + O(r^{-5}),$$

$$\beta = -\frac{\alpha^0}{r} + \frac{\sigma^0 \bar{\sigma}_0^0 \alpha^0}{r^2} - \frac{\sigma^0 \bar{\sigma}_0^0 \alpha^0}{r^3} + \frac{\sigma^0 \bar{\sigma}_0^0 \alpha^0}{6r^4} + O(r^{-5}),$$

$$\lambda = \frac{\chi^0}{r^2} + \frac{\sigma^0 \bar{\sigma}_0^0 \chi^0}{r^3} - \frac{\bar{\sigma}_0^0 \Psi_1^0 - \chi^0 \bar{\phi}_0^0}{3r^4} + O(r^{-5}),$$

$$\gamma = \gamma^0 - \frac{\chi^0}{r^2} + \frac{2\bar{\sigma}_0^0 \chi^0}{r^3} - \frac{6\bar{\phi}_2^0 \bar{\phi}_0^0}{3r^4} + O(r^{-5}),$$
\[-3(\gamma^0 + 3\gamma^0)\phi_0^0\phi_0^0 + 12\phi_0^0\phi_0^0 + 24\phi_0^0\phi_0^0 + 9\phi_0^0\phi_0^0 - 3\phi_0^0\phi_0^0 \right] + O(r^{-5}),
\]
\[\nu = \nu^0 - \frac{\nu_3^0}{r} + \frac{\bar{\partial}\Psi_0^0 - 2\phi_0^0\phi_0^0}{6r^2} + O(r^{-3}),\]

\[X^z = \frac{\bar{P}\Psi_0^0}{6r^3} + \frac{\bar{P}}{12r^4} \left( -\bar{\partial}\Psi_0^0 - 2\sigma^0\Psi_0^0 + 4\phi_0^0\phi_0^0 \right) + O(r^{-5}),\]

\[\omega = \frac{\bar{\partial}\sigma^0}{r} - \frac{\sigma^0\bar{\partial}\Psi_0^0 + \frac{1}{2}\bar{\partial}\Psi_0^0 + \bar{\partial}\Psi_0^0}{6r^3} + O(r^{-4}),\]

\[U = -r(\gamma^0 + \gamma^0) + \mu^0 - \frac{\Psi_2^0 + \bar{\Psi}_2^0}{2r} + \frac{\bar{\partial}\Psi_0^0 - \bar{\partial}\Psi_1^0 + 6\phi_0^0\phi_1^0}{6r^2} - \frac{1}{2r^3} \left[ \lambda^0\Psi_0^0 + \lambda^0\bar{\Psi}_0^0 \right]
\]

\[(2.7)\]

\[L^z = -\frac{\bar{P}(u, z, \bar{z})}{r^2} \left( \frac{\sigma^0}{2}\sigma^0 - \frac{1}{6}\Psi_0^0 \right) + \frac{\bar{P}\Psi_0^0}{12r^3} + O(r^{-6}),\]

\[L^\bar{z} = \frac{P(u, z, \bar{z})}{r^2} + \frac{\sigma^0\bar{\partial}\Psi_0^0}{6r^3} + \frac{P}{12r^5} \left( 12(\sigma^0\sigma^0)^2 + \phi_0^0\phi_0^0 - 2\sigma^0\bar{\Psi}_0^0 - \sigma^0\bar{\Psi}_0^0 \right) + O(r^{-6}),\]

\[L_z = -\frac{r}{P} + \frac{\Psi_0^0 + \phi_0^0\bar{\Psi}_0^0}{12Pr^3} + O(r^{-4}),\]

\[L_{\bar{z}} = -\frac{\sigma^0}{P} + \frac{\Psi_0^0 + \phi_0^0\bar{\Psi}_0^0}{6Pre^2} + \frac{\bar{P}^4}{12P^3} + O(r^{-4}),\]

where

\[\alpha^0 = \frac{1}{2} \bar{P}\hat{c}_z \ln P, \quad \mu^0 = -\frac{1}{2} P\bar{P}\hat{c}_z \ln \bar{P},\]

\[\lambda^0 = \hat{c}_u\bar{\sigma}^0 + \sigma^0(3\gamma^0 - \gamma^0),\]

\[\gamma^0 = -\frac{1}{2} \hat{c}_u \ln \bar{P}, \quad \nu^0 = \bar{\partial}(\gamma^0 + \gamma^0),\]

\[\psi^0_2 - \bar{\psi}^0_2 = \bar{\partial}\sigma^0 - \bar{\partial}\bar{\sigma}^0 - \bar{\partial}\sigma^0 - \sigma^0\chi^0,\]

\[\psi^0_3 = \bar{\partial}\mu^0 - \bar{\partial}\chi^0, \quad \psi^0_4 = \bar{\partial}\nu^0 - \hat{c}_u\chi^0 - 4\gamma^0\chi^0,\]

\[\hat{c}_u\phi_0^0 + (\gamma^0 + 3\gamma^0)\phi_0^0 = \hat{c}_0^0 + \sigma^0\phi_0^0,\]

\[\hat{c}_u\phi_0^1 + 2(\gamma^0 + \gamma^0)\phi_0^1 = \hat{c}_0^2,\]

\[\hat{c}_u\psi_0^0 + (\gamma^0 + 3\gamma^0)\psi_0^0 = \hat{c}_0^1 + 3\sigma^0\psi_2^0 + 3\phi_0^0\phi_2^0,\]

\[\hat{c}_u\psi_0^1 + 2(\gamma^0 + 2\gamma^0)\psi_0^0 = \hat{c}_0^1 + 2\sigma^0\psi_3^0 + 2\phi_0^0\phi_2^0,\]

\[\hat{c}_u\psi_0^2 + 3(\gamma^0 + 3\gamma^0)\psi_0^0 = \hat{c}_0^1 + 3\sigma^0\psi_4^0 + \phi_0^0\phi_2^0,\]

\[\hat{c}_u\psi_0^3 + 2(2\gamma^0 + \gamma^0)\psi_0^0 = \hat{c}_0^1 + 2\sigma^0\psi_4^0 + \phi_0^0\phi_2^0.\]
The “$\partial$” operator is defined as

\[
\partial \eta^s = PP^{-s} \partial_z (P^s \eta^s) = P \partial_z \eta^s + 2s \alpha^0 \eta^s, \\
\bar{\partial} \eta^s = \bar{P} P^s \partial_z (P^{-s} \eta^s) = \bar{P} \partial_z \eta^s - 2s \alpha^0 \eta^s,
\]

where $s$ is the spin weight of the field $\eta$. The spin weights of relevant fields are listed in Table 1.

| $s$ | $\bar{\partial}$ | $\partial_u$ | $\gamma^0$ | $\nu^0$ | $\mu^0$ | $\sigma^0$ | $\lambda^0$ | $\Psi^0_4$ | $\Psi^0_3$ | $\Psi^0_2$ | $\Psi^0_1$ | $\Psi^0_0$ | $\phi^0_2$ | $\phi^0_1$ | $\phi^0_0$ |
|-----|-----------------|--------------|-----------|--------|---------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|--------|---------|
| 1   | 0               | 0            | -1        | 0      | 2       | -2        | -2        | 0         | 1         | 2         | -1        | 0         | 1         | 0       | 1       |

We will work in retarded radial gauge $A_r = 0$. The Maxwell-tensor is determined as

\[
F_{\mu\nu} = (\phi_1 + \bar{\phi}_1)(n_\mu l_\nu - l_\mu n_\nu) + (\phi_1 - \bar{\phi}_1)(m_\mu \bar{m}_\nu - \bar{m}_\mu m_\nu) \\
+ \phi_2(l_\mu m_\nu - m_\mu l_\nu) + \bar{\phi}_2(l_\mu \bar{m}_\nu - \bar{m}_\mu l_\nu) \\
+ \phi_0(\bar{m}_\mu n_\nu - n_\mu \bar{m}_\nu) + \bar{\phi}_0(m_\mu n_\nu - n_\mu m_\nu).
\]

In terms of the gauge fields $A_\mu$,

\[
A^0_u = -(\phi_1^0 + \bar{\phi}_1^0), \quad \partial_u A^0_z = -\frac{\phi_0^0}{P}, \quad A^1_z = -\frac{\bar{\phi}_0^0}{P}, \quad (\partial_z A^0_z - \partial_z A^1_z) = \frac{\phi_0^0 - \bar{\phi}_1^0}{PP},
\]

\[
\partial_u \left( \frac{A^0_u}{PP} \right) = \partial_u (\partial_z A^0_z + \partial_z A^1_z),
\]

where

\[
A_u = \frac{A^0_u(u, z, \bar{z})}{r} + O(r^{-2}), \quad A_z = A^0_z(u, z, \bar{z}) + \frac{A^1_z(u, z, \bar{z})}{r} + O(r^{-2}).
\]

3 The memory effects

The memory effects are all encoded in the solution space derived in the previous section. To specify the observational effects, we will examine the motion of a charged time-like particle. The charged particle will be constrained to a fixed radial distance $r_0$ that is very far from the gravitational and electromagnetic source. The $r = r_0$ hypersurface is time-like, its induced metric can be derived easily from the solution space in the previous
section. The induced metric in series expansions is given by

\[ ds^2 = \left[ 1 + \frac{\Psi_2^0 + \Psi_2^0}{r_0} - \frac{\overline{\sigma} \Psi_1^0 + \overline{\Psi}_1^0}{3 r_0^2} - 6 \phi_1^0 \overline{\phi}_1^0 + O(r^{-3}) \right] du^2 \]

\[ - 2 \left[ \frac{\overline{\sigma} \sigma_0^0}{P_s} - \frac{2 \overline{\Psi}_1^0}{3 P_s r_0} + O(r^{-2}) \right] dudz - 2 \left[ \frac{\overline{\sigma} \sigma_0^0}{P_s} - \frac{2 \Psi_0^0}{3 P_s r_0} + O(r^{-2}) \right] du \bar{z} \]

\[ - \left[ \frac{\overline{\sigma} r^0}{P_s^2} - \frac{\Psi_0^0}{3 P_s^2 r_0} + O(r^{-2}) \right] dz^2 - \left[ \frac{\sigma^0 r^0}{P_s^2} - \frac{\Psi_0^0}{3 P_s^2 r_0} + O(r^{-2}) \right] d\bar{z}^2 \]

\[ - 2 \left[ \frac{\sigma_0^0}{P_s^2} + \frac{\sigma^0 \sigma_0^0}{P_s^2} + O(r^{-2}) \right] d\bar{z} \bar{z}. \] (3.1)

We now work in the unit 2-sphere case where \( P_s = \frac{1 + \bar{z} \bar{z}}{\sqrt{2}} \). The induced Maxwell field on the \( r = r_0 \) hypersurface is

\[ F_{uz} = -\frac{\phi_0^0}{P_s} + \frac{\overline{\phi}_0^0 - \overline{\sigma} \phi_0^0}{P_s r_0} + O(r_0^{-2}), \quad F_{u \bar{z}} = -\frac{\overline{\phi}_0^0}{P_s} + \frac{\overline{\sigma} \phi_0^0 - \sigma^0 \phi_0^0}{P_s r_0} + O(r_0^{-2}), \]

\[ F_{\bar{z} \bar{z}} = \frac{\phi_0^0 - \overline{\phi}_0^0}{P_s^2} + \frac{\overline{\sigma} \phi_0^0 - \overline{\sigma} \phi_0^0}{P_s^2 r_0} + O(r_0^{-2}). \] (3.2)

Free falling charged particle on this hypersurface will of course not travel along geodesic. The tangent vector \( V \) of the particle worldline satisfies

\[ V^\nu (\overline{\nabla}_\nu V^\mu + q F_{\nu \mu}) = 0, \] (3.3)

where \( \overline{\nabla} \) is the covariant derivative on this 3 dimensional hypersurface and \( q \) is the charge of the particle.

Following \[33\], we impose that \( V \) has the following asymptotic expansion

\[ V^u = 1 + \sum_{a=1}^{\infty} \frac{V^u_a}{r^a}, \quad V^z = \sum_{a=2}^{\infty} \frac{V^z_a}{r^a}. \] (3.4)

Then we can solve (3.3) order by order. The solution up to relevant order is

\[ V_1^u = -\frac{\Psi_2^0 + \overline{\Psi}_2^0}{2}, \] (3.5)

\[ V_2^u = \frac{1}{6} (\overline{\sigma} \Psi_1^0 + \overline{\Psi}_1^0) - \overline{\sigma} \sigma_0^0 \phi_0^0 + \frac{3}{8} (\Psi_2^0 + \overline{\Psi}_2^0)^2 - \phi_1^0 \overline{\phi}_1^0 + q^2 P_s^2 A_z^0 A_z^0, \] (3.6)

\[ V_2^z = -P_s \overline{\sigma} \sigma^0 + q P_s^2 A_z^0, \] (3.7)

\[ V_3^z = P_s \left[ 2 \overline{\sigma} \sigma^0 \phi_0^0 + \frac{2}{3} \Psi_1^0 + \frac{1}{2} \overline{\sigma} \overline{\sigma} (\Psi_2^0 + \overline{\Psi}_2^0) \right] - P_s \int dv \frac{\overline{\sigma} (\Psi_2^0 + \overline{\Psi}_2^0 + 2 q A_z^0)}{2} - 2 q P_s^2 \sigma^0 A_z^0 + q P_s^2 A_z^1. \] (3.8)
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We have set all integration constants of $u$ to zero as we require the charged particle is static initially. At $r_0^{-2}$ order, $V$ has angular components due to the presence of gravitational waves characterized by $\sigma^0$ and electromagnetic waves characterized by $A_z^0$. In other words, gravitational and electromagnetic radiations force free falling charged particle to rotate. The leading memory effect is the change of the velocity of the charged particle

$$\Delta V^z = -\frac{1}{r_0^2} (P_s \overline{\sigma} \Delta \sigma^0 - q P_s^2 \Delta A_z^0) + O(r_0^{-3}).$$

(3.9)

It includes two parts, namely the gravitational contribution $-P_s \overline{\sigma} \Delta \sigma^0$ [24–34] and electromagnetic contribution $q P_s^2 \Delta A_z^0$ [8].

The sub-leading memory effect is a position displacement determined by

$$\Delta z = \int V^z du = -\frac{1}{r_0^2} \int du (P_s \overline{\sigma} \sigma^0 - q P_s^2 A_z^0) + O(r_0^{-3}).$$

(3.10)

The relevance of the gravitational contribution $-\int (P_s \overline{\sigma} \sigma^0) du$ and electromagnetic contribution $\int (q P_s^2 A_z^0) du$ to sub-leading soft graviton theorem and soft photon theorem were proven in [18] and [19], respectively.

Another sub-leading observational memory effect in gravitational theory is a time delay of the observer [33, 34]. The time delay of a charged particle will also be affected by electromagnetic radiation. Since $V$ is time-like, the infinitesimal change of the proper time of the charged particle can be derived from the co-vector

$$d\chi = \left[1 + \frac{1}{2r_0} (\Psi_2^0 + \Psi_2^1) - \frac{1}{r_0^2} \left( \frac{1}{8} (\Psi_2^0 + \Psi_2^1)^2 + \frac{1}{6} (\overline{\sigma} \overline{\sigma}_1 + \overline{\sigma} \overline{\sigma}_1') - \overline{\sigma} \overline{\sigma}_1 \overline{\sigma} \sigma^0 \\
- \phi_1^0 \phi_1^0 + q^2 P_s^2 A_z^0 A_z^0 \right) \right] du + O(r_0^{-3}).$$

(3.11)

Clearly, the electromagnetic contribution $(\phi_1^0 \phi_1^0 - q^2 P_s^2 A_z^0 A_z^0)$ comes one order higher than the gravitational contribution $\frac{1}{2} (\Psi_2^0 + \Psi_2^1)$ in the $r_0^{-1}$ expansion.

4 Discussion

In this work, the gravitational and electromagnetic memory effects are investigated in a unified fashion by examining the motion of a charged particle. Some interesting applications may cross the reader’s mind. Since our motivation is to provide a unified treatment of memory effect in coupled theories. It would be of interest to test our treatment in generic theories with more matter fields coupled and in various ways of coupling. Another interesting point is about double soft theorems (see, e.g. [32, 40]). Hopefully, our

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3We have used the fact that $du = d\chi + O(r_0^{-1})$, where $\chi$ is the proper time.

4We have used the fact that $dz = \frac{\chi}{r_0} du + O(r_0^{-3})$. 
treatment can shine light on the understanding of the memory effect which is connected to double soft theorem.

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## A NP equations

**Radial equations**

\[
D\rho = \rho^2 + \sigma\overline{\sigma} + \phi_0 \overline{\phi}_0, \quad \text{(A.1)}
\]

\[
D\sigma = 2\rho\sigma + \Psi_0, \quad \text{(A.2)}
\]

\[
D\tau = \tau\rho + \overline{\tau}\sigma + \Psi_1 + \phi_0 \overline{\phi}_1, \quad \text{(A.3)}
\]

\[
D\alpha = \rho\alpha + \beta\overline{\sigma} + \phi_1 \overline{\phi}_0, \quad \text{(A.4)}
\]

\[
D\beta = \alpha\sigma + \rho\beta + \Psi_1, \quad \text{(A.5)}
\]

\[
D\gamma = \tau\alpha + \overline{\tau}\beta + \Psi_2 + \phi_1 \overline{\phi}_1, \quad \text{(A.6)}
\]

\[
D\lambda = \rho\lambda + \overline{\sigma}\mu + \phi_2 \overline{\phi}_0, \quad \text{(A.7)}
\]

\[
D\mu = \rho\mu + \sigma\lambda + \Psi_2, \quad \text{(A.8)}
\]

\[
D\nu = \overline{\tau}\mu + \tau\lambda + \Psi_3 + \phi_2 \overline{\phi}_1, \quad \text{(A.9)}
\]

\[
D U = \overline{\tau}\sigma + \overline{\tau}\overline{\sigma} - (\gamma + \overline{\gamma}), \quad \text{(A.10)}
\]

\[
D X^A = \overline{\tau} L^A + \tau \overline{\bar{L}}^A, \quad \text{(A.11)}
\]

\[
D\omega = \rho\omega + \sigma\overline{\omega} - \tau, \quad \text{(A.12)}
\]

\[
D L^A = \rho L^A + \sigma \overline{L}^A, \quad \text{(A.13)}
\]

\[
D\Psi_1 - \overline{\delta}\Psi_0 = 4\rho\Psi_1 - 4\alpha\Psi_0 + \overline{\phi}_1 D\phi_0 - \overline{\phi}_0 D\phi_0 - 2\sigma\phi_0 \overline{\phi}_0 + 2\beta\phi_0 \overline{\phi}_0, \quad \text{(A.14)}
\]

\[
D\Psi_2 - \overline{\delta}\Psi_1 = 3\rho\Psi_2 - 2\alpha\Psi_1 - \lambda\Psi_0
\]

\[
+ \overline{\phi}_1 D\phi_0 - \overline{\phi}_0 D\phi_0 - 2\alpha\phi_0 \overline{\phi}_1 + 2\rho\phi_1 \overline{\phi}_1 + 2\gamma\phi_0 \overline{\phi}_0 - 2\tau\phi_0 \overline{\phi}_0, \quad \text{(A.15)}
\]

\[
D\Psi_3 - \overline{\delta}\Psi_2 = 2\rho\Psi_3 - 2\lambda\Psi_1 + \overline{\phi}_1 D\phi_2 - \overline{\phi}_0 D\phi_2 + 2\mu\phi_1 \overline{\phi}_0 - 2\beta\phi_2 \overline{\phi}_0, \quad \text{(A.16)}
\]

\[
D\Psi_4 - \overline{\delta}\Psi_3 = \rho\Psi_4 + 2\alpha\Psi_3 - 3\lambda\Psi_2
\]

\[
- \overline{\phi}_0 D\phi_2 + \overline{\phi}_1 D\phi_2 + 2\alpha\phi_2 \overline{\phi}_1 + 2\nu\phi_1 \overline{\phi}_0 - 2\gamma\phi_2 \overline{\phi}_0 - 2\lambda\phi_1 \overline{\phi}_1, \quad \text{(A.17)}
\]

\[
D\phi_1 - \overline{\delta}\phi_0 = 2\rho\phi_1 - 2\alpha\phi_0, \quad \text{(A.18)}
\]

\[
D\phi_2 - \overline{\delta}\phi_1 = \rho\phi_2 - \lambda\phi_0. \quad \text{(A.19)}
\]
Non-radial equations

\[ \Delta \lambda = \delta \nu - (\mu + \overline{\mu}) \lambda - (3 \gamma - \overline{\gamma}) \lambda + 2 \alpha \nu - \Psi_4, \]  
(20)

\[ \Delta \rho = \delta \tau - \rho \overline{\mu} - \sigma \lambda - 2 \alpha \tau + (\gamma + \overline{\gamma}) \rho - \Psi_2, \]  
(21)

\[ \Delta \alpha = \delta \gamma + \rho \nu - (\tau + \beta) \lambda + (\overline{\gamma} - \gamma - \overline{\mu}) \alpha - \Psi_3, \]  
(22)

\[ \Delta \mu = \delta \nu - \mu^2 - \lambda \overline{\lambda} - (\gamma + \overline{\gamma}) \mu + 2 \beta \nu - \phi_2 \overline{\phi}_2, \]  
(23)

\[ \Delta \beta = \delta \gamma - \mu \tau + \sigma \nu + \beta (\gamma - \overline{\gamma} - \mu) - \alpha \overline{\lambda} - \phi_1 \overline{\phi}_1, \]  
(24)

\[ \Delta \sigma = \delta \tau - \sigma \mu - \rho \overline{\lambda} - 2 \beta \tau + (3 \gamma - \overline{\gamma}) \sigma - \phi_0 \overline{\phi}_2, \]  
(25)

\[ \Delta \omega = \delta U + \overline{\nu} - \overline{\lambda} \overline{\omega} + (\gamma - \overline{\gamma} - \mu) \omega, \]  
(26)

\[ \Delta L^A = \delta X^A - \overline{\lambda} \overline{L}^A + (\gamma - \overline{\gamma} - \mu) L^A, \]  
(27)

\[ \delta \rho - \delta \sigma = \rho \tau - \sigma (3 \alpha - \beta) - \Psi_1 + \phi_0 \overline{\phi}_1, \]  
(28)

\[ \delta \alpha - \delta \beta = \mu \rho - \lambda \sigma + \alpha \overline{\alpha} + \beta \overline{\beta} - 2 \alpha \beta - \Psi_2 + \phi_1 \overline{\phi}_1, \]  
(29)

\[ \delta \lambda - \delta \mu = \mu \tau + \lambda (\overline{\tau} - 3 \beta) - \Psi_3 + \phi_2 \overline{\phi}_1, \]  
(30)

\[ \delta \overline{\omega} - \delta \omega = \mu - \overline{\mu} - (\alpha - \beta) \omega + (\overline{\alpha} - \beta) \overline{\omega}, \]  
(31)

\[ \delta \overline{L}^A - \delta L^A = (\overline{\alpha} - \beta) \overline{L}^A - (\alpha - \beta) L^A, \]  
(32)

\[ \Delta \Psi_0 - \delta \Psi_1 = (4 \gamma - \mu) \Psi_0 - (4 \tau + 2 \beta) \Psi_1 + 3 \sigma \Psi_2 - \phi_0 \overline{D} \phi_0 + \overline{\phi}_1 \delta \phi_0 - 2 \beta \phi_0 \overline{\phi}_1 + 2 \sigma \phi_1 \overline{\phi}_1, \]  
(33)

\[ \Delta \Psi_1 - \delta \Psi_2 = \nu \Psi_0 + (2 \gamma - 2 \mu) \Psi_1 - 3 \tau \Psi_2 + 2 \sigma \Psi_3 + \overline{\phi}_1 \Delta \phi_0 - \overline{\phi}_2 \delta \phi_0 - 2 \rho \phi_1 \overline{\phi}_1 - 2 \gamma \phi_0 \overline{\phi}_1 + 2 \tau \phi_1 \overline{\phi}_1 + 2 \alpha \phi_0 \overline{\phi}_2, \]  
(34)

\[ \Delta \Psi_2 - \delta \Psi_3 = 2 \nu \Psi_1 - 3 \mu \Psi_2 + (2 \beta - 2 \tau) \Psi_3 + \sigma \Psi_4 - \overline{\phi}_2 \Delta \phi_2 + \overline{\phi}_1 \delta \phi_2 - 2 \mu \phi_1 \overline{\phi}_1 + 2 \beta \phi_2 \overline{\phi}_1, \]  
(35)

\[ \Delta \Psi_3 - \delta \Psi_4 = 3 \nu \Psi_2 - (2 \gamma + 4 \mu) \Psi_3 + (4 \beta - \tau) \Psi_4 + \overline{\phi}_1 \Delta \phi_2 - \overline{\phi}_2 \delta \phi_2 - 2 \alpha \phi_2 \overline{\phi}_2 - 2 \nu \phi_1 \overline{\phi}_1 + 2 \gamma \phi_2 \overline{\phi}_1 + 2 \lambda \phi_1 \overline{\phi}_2, \]  
(36)

\[ \Delta \phi_0 - \delta \phi_1 = (2 \gamma - \mu) \phi_0 - 2 \tau \phi_1 + \sigma \phi_2, \]  
(37)

\[ \Delta \phi_1 - \delta \phi_2 = \nu \phi_0 - 2 \mu \phi_1 - (\overline{\alpha} - \beta) \phi_2. \]  
(38)

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