FREE SPECIAL GELFAND—DORFMAN ALGEBRA

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ABSTRACT. A Gelfand—Dorfman algebra is called special if it can be embedded into a differential Poisson algebra. We find a new basis of the free Novikov algebra. With its help, we construct the monomial basis of the free special Gelfand—Dorfman algebra.

1. Introduction

A vector space $A$ with a bilinear product $\circ$ satisfying the identities

$$(x_1 \circ x_2) \circ x_3 - x_1 \circ (x_2 \circ x_3) = (x_2 \circ x_1) \circ x_3 - x_2 \circ (x_1 \circ x_3),$$

and

$$(x_1 \circ x_2) \circ x_3 = (x_1 \circ x_3) \circ x_2,$$  

is called a Novikov algebra. Novikov algebras were introduced in the study of Hamiltonian operators concerning integrability of certain partial differential equations [8]. Later, Novikov algebras appeared in the study of Poisson brackets of hydrodynamic type [2].

It is well-known that given a commutative algebra $A$ with a derivation $d$, the space $A$ under the product $x_1 \circ x_2 = x_1d(x_2)$ is a Novikov algebra. Moreover, all identities fulfilled in $(A, \circ)$ are consequences of (1) and (2). Applying the rooted trees, the monomial basis of the free Novikov algebra in terms of $\circ$ was constructed in [7]. In terms of Young diagram, the basis was constructed in [6].

An algebra $A$ satisfying only the identity (1) is called a left-symmetric algebra. Left-symmetric algebras have been studying since 1960s, they have applications in affine geometry, ring theory, vertex algebras etc, see the survey [3]. Left-symmetric algebras embeddable under the operation $x_1 \circ x_2 = x_1d(x_2)$ into permutative algebras were studied in [12].

Note that every associative algebra is left-symmetric one, and every left-symmetric algebra under the commutator $[a, b] = a \circ b - b \circ a$ is a Lie algebra. For that reason, every Novikov algebra under the commutator is a Lie algebra satisfying an additional

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identity of degree 5 of the following form:
\[
\sum_{\sigma \in S_4} (-1)^\sigma [x_{\sigma(1)}, [x_{\sigma(2)}, [x_{\sigma(3)}, [x_{\sigma(4)}, x_5]]]] = 0.
\]

To find all special identities for Novikov algebras considered under the commutator is still an open problem. It is equivalent to the same question formulated for a commutative algebra \(C\) with a derivation \(d\) considered under the product
\[
[x_1, x_2] = x_1 d(x_2) - x_2 d(x_1),
\]
which is called Wronskian bracket.

Given a Poisson algebra \((P, \cdot, \{,\})\) with a derivation \(d\) due to both products, define on \(P\) new operations as follows,
\[
x_1 \circ x_2 = x_1 d(x_2), \quad [x_1, x_2] = \{x_1, x_2\}.
\]
Recall that the variety of Poisson algebras is defined by the identities,
\[
\begin{align*}
    x_1 \cdot x_2 &= x_2 \cdot x_1, \quad (x_1 \cdot x_2) \cdot x_3 = x_1 \cdot (x_2 \cdot x_3), \\
    \{x_1, x_2\} &= -\{x_2, x_1\}, \quad \{\{x_1, x_2\}, x_3\} + \{\{x_2, x_3\}, x_1\} + \{\{x_3, x_1\}, x_2\} = 0,
\end{align*}
\]

The algebra \(P^{(d)} := (P, \circ, \{,\})\) has a Novikov product \(\circ\), a Lie product \(\{,\}\), and moreover, the following identities hold,
\[
\begin{align*}
    x_2 \circ [x_1, x_3] &= [x_1, x_2 \circ x_3] - [x_3, x_2 \circ x_1] + [x_2, x_1] \circ x_3 - [x_2, x_3] \circ x_1, \\
    [x_1, (x_2 \circ x_3) \circ x_4] &= [x_1, x_2 \circ x_3] \circ x_4 + [x_1, x_2 \circ x_4] \circ x_3 - ([x_1, x_2] \circ x_3) \circ x_4, \\
    [x_3 \circ x_1, x_4 \circ x_2] &= [x_4 \circ x_1, x_3 \circ x_2] + [x_3, x_4 \circ x_1] \circ x_2 - [x_4, x_3 \circ x_2] \circ x_1 \\
    &\quad - [x_4, x_3 \circ x_1] \circ x_2 + [x_3, x_4 \circ x_2] \circ x_1 + 2([x_4, x_3] \circ x_1) \circ x_2.
\end{align*}
\]

There may exist identities of degree greater than 5 fulfilled in \(P\) which are independent from \((6)\)–\((8)\).

An algebra \((G, \circ, [,])\) such that \((G, \circ)\) is Novikov, \((G, [,])\) is Lie, and the identity \((6)\) holds is called a Gelfand–Dorfman algebra (GD-algebra) \([14, 15]\). GD-algebras appeared in \([8]\) as a source of Hamiltonian operators: with the help of the structure constants of a Gelfand–Dorfman algebra one may construct a differential operator. In \([15]\), it was shown that GD-algebras are closely related with Lie conformal algebras.

It is worth to note that the identities \((7)\) and \((8)\) are not fulfilled in the free GD-algebra, thus, these identities are called special identities. Moreover, the identities \((7)\) and \((8)\) are mutually independent \([10, 11]\).

A Gelfand–Dorfman algebra \(G\) is called special if there exists a Poisson algebra \(P\) with a derivation \(d\) such that \(G\) injectively embeds into \(P^{(d)}\). It is known that the class of special GD-algebras forms a variety \([11]\), thus, we may consider the free special Gelfand–Dorfman algebra \(\text{SGD}(X)\) generated by a set \(X\). By \(\text{ComDer}(X)\)
and PoisDer$\langle X \rangle$ we denote the free commutative and the free Poisson algebra with a derivation in the signature generated by $X$, respectively.

In [11] it was proved that every 2-dimensional GD-algebra is special. Another interesting result is that a GD-algebra such that $[a, b] = a \circ b - b \circ a$ is special [9]. Therefore, a natural problem arises: To construct a monomial basis of the free SGD-algebra in terms of $\circ$ and $[,]$. We solve this problem. For the solution, we construct a new monomial basis of the free Novikov algebra.

For all mentioned varieties we have the following diagram:

$$
\begin{array}{ccc}
\text{Nov}(X) & \hookrightarrow & \text{ComDer}(X) \\
\downarrow & & \downarrow \\
\text{SGD}(X) & \hookrightarrow & \text{PoisDer}(X)
\end{array}
$$

In §2, we construct new basis of the free Novikov algebra (Theorem 2). In §3, we define what is a canonical form of monomials PoisDer$\langle X \rangle$ of weight $-1$. Finally, in §4, the linear basis of the free special GD-algebra is constructed (Theorem 3). For simplicity, we identify the element $d(x)$ with $x'$. In this paper, all algebras are defined over a field of characteristic 0.

2. Basis of free Novikov algebra

Let $X = \{x_i \mid i \in I\}$, where $I$ is well ordered set. The free commutative algebra ComDer$\langle X, d \rangle$ with a derivation $d$ in the signature has a standard linear basis consisting of monomials

$$x_{i_1}^{(r_1)} \ldots x_{i_k}^{(r_k)}, \quad x_j \in X, \quad i_1 \leq \ldots \leq i_k, \quad r_j \geq 0.$$ 

Here $x_j^{(0)} = x_j$, $x_j^{(n+1)} = (x_j^{(n)})'$. Thus, the elements $x^{(r)}$, where $x \in X$ and $r \in \mathbb{N}$, generate ComDer$\langle X, d \rangle$ as a commutative algebra. For simplicity, we will denote ComDer$\langle X, d \rangle$ as ComDer$\langle X \rangle$ omitting the symbol $d$.

**Definition 1.** Let $u$ be a monomial from the standard basis of ComDer$\langle X, d \rangle$. Define the weight function $\text{wt}(u) \in \mathbb{Z}$ by induction as follows,

$$\text{wt}(x) = -1, \quad x \in X;$$

$$\text{wt}(d(u)) = \text{wt}(u) + 1; \quad \text{wt}(uv) = \text{wt}(u) + \text{wt}(v).$$

We may consider the space ComDer$\langle X \rangle$ under the product $u \circ v = ud(v)$, denote the obtained Novikov algebra as ComDer$\langle X \rangle^{(d)}$. Let ComDer$\langle X \rangle_{-1}$ be a span of monomials from the standard basis of ComDer$\langle X \rangle$ of weight $-1$. Note that ComDer$\langle X \rangle_{-1}$ is closed under the Novikov product $\circ$, so, it is a Novikov subalgebra of ComDer$\langle X \rangle^{(d)}$.

Let us recall the well-known results related to the free Novikov algebra.
Theorem 1. a) \[\comder(X)_{-1} \circ \cong \nov(X)\]  
\[\]  
b) \[\]  
Every Novikov algebra can be embedded into a free differential commutative algebra.

Let us define an order on the elements \(x^{(r)}\) of \(\comder(X)\) as follows: \(x^{(r_m)} > x^{(r_n)}\) if \(r_m > r_n\) or \(r_m = r_n\) and \(i > j\).

We define a normal form of monomials \(\comder(X)\) of weight \(-1\) as follows,

\[
x_{i_1} x_{i_2} \ldots x_{i_{n-1}} x_{i_n}^{(r_n)} \ldots x_{j_2}^{(r_2)} x_{j_1}^{(r_1)} x_{k_m}^{r_m} \ldots x_{k_2}^{(r_2)} x_{k_1}^{(r_1)},
\]

where

\[
n \geq 1, \quad r_1, \ldots, r_n \geq 2, \quad l = r_1 + r_2 + \ldots + r_n - n,
\]

\[
x_{i_1} \leq \ldots \leq x_{i_l}, \quad x_{j_1}^{(r_1)} \geq \ldots \geq x_{j_1}^{(r_1)}, \quad x_{k_m} \geq \ldots \geq x_{k_1}.
\]

Denote by \(N(X)\) the set of all normal forms \((9)\) of monomials from the standard basis of \(\comder(X)\) of weight \(-1\).

For \(a \in N(X)\), put

\[
L(a) = (r_1, \ldots, r_n), \quad M(a) = (i_1, \ldots, i_l), \quad R(a) = (k_m, \ldots, k_1),
\]

and define \(S(a) = (L(a), R(a), M(a))\). Given \(a, b \in N(X)\), we say that \(a < b\) if and only if \(S(a) < S(b)\), we compare all tuples involved lexicographically.

Denote by \(\magma(X)\) the free magma algebra with binary operation \(\circ\) generated by \(X\). We define a linear map \(\varphi: \comder(X)_{-1} \rightarrow \magma(X)\). By linearity it is enough to define \(\varphi\) on the set \(N(X)\), we do it inductively as follows,

\[
\varphi\left(x_{i_1} x_{i_2} \ldots x_{i_{n-1}} x_{i_n}^{(n-1)}\right) = x_{i_{n-1}} \circ (x_{i_{n-2}} \circ \ldots \circ (x_{i_1} \circ x_{i_n}) \ldots),
\]

\[
\varphi\left(x_{i_1} x_{i_2} \ldots x_{i_j} x_{j_n}^{(r_n)} x_{j_2} x_{j_1}^{(r_1)} x_{k_m}^{r_m} \ldots x_{k_2} x_{k_1}^{(r_1)}\right) = \varphi\left(x_{i_1} x_{i_2} \ldots x_{i_{j-1}} x_{j_n}^{(r_n)} B_1 x_{j_{n-1}}^{r_{n-1}} \ldots x_{j_2} x_{j_1}^{(r_1)} x_{k_m}^{r_m} \ldots x_{k_2} x_{k_1}^{(r_1)}\right),
\]

where \(B_1 = \varphi(x_{i_{j+1}} x_{i_{j+2}} \ldots x_{i_{j+n}} x_{j_n}^{(r_n)})\) is a new letter, so on this step we extend the generating set \(X\) to \(X_1 = X \cup \{B_1\}\). Thus, we define \(\text{wt}(B_1) = -1\) and \(x < B_1\) for all \(x \in X\). On each step \(i\), we add a new letter \(B_i\) to the generating set, i.e. \(X_i = X_{i-1} \cup \{B_i\}\), we define \(\text{wt}(B_i) = -1\) and \(y < B_i\) for all \(y \in X_{i-1}\). Calculating by the given rule, finally, we get

\[
\varphi\left(x_{i_1} x_{i_2} \ldots x_{i_j} x_{j_n}^{(r_n)} x_{j_2} x_{j_1}^{(r_1)} x_{k_m}^{r_m} \ldots x_{k_2} x_{k_1}^{(r_1)}\right) = (\ldots ((B_{n-1} \circ (x_{i_{r_{n-1}}} \circ \ldots (x_{i_1} \circ x_{j_1}) \ldots)) \circ x_{k_m}) \ldots) \circ x_{k_1},
\]

where all previous letters \(B_1, \ldots, B_{n-2}\) are inside \(B_{n-1}\).

Example 1. Let \(x, y, z, t, q \in X, y > x, t > q\), then

\[
\varphi(xyzt^{(2)}t'q') = \varphi(B_1t'q') = \varphi(B_2q') = B_2 \circ q = (B_1 \circ t) \circ q = ((y \circ (x \circ z)) \circ t) \circ q.
\]
Define a homomorphism

\[ \tau: \text{Magma}(X) \to \text{ComDer}(X)_{-1} \]

by the formula \( \tau(x) = x, \) \( x \in X; \) the last algebra is considered as a Novikov one. For example, if \( x, y, z \in X, \) then \( \tau(x \circ (y \circ z)) = x(yz)' = xy'z' + xyz(2). \)

**Lemma 1.** Let \( a \in N(X). \) Then \( \tau(\varphi(a)) = a + \sum_j b_j, \) where \( b_j < a \) for all \( j. \)

**Proof.** By the definition of \( \tau, \) it is enough to prove the statement for

\[ a = x_{i_1}x_{i_2} \ldots x_{i_n}x_{j_1}^{(r_1)} \ldots x_{j_2}^{(r_2)}x_{j_1}. \]

By the Leibniz rule fulfilled for \( d, \) we have

\[
\tau(\varphi(x_{i_1}x_{i_2} \ldots x_{i_n}x_{j_1}^{(r_1)} \ldots x_{j_2}^{(r_2)}x_{j_1}^{(r_1)})) = \tau(B_{n-1} \circ (x_{i_1-1} \circ \ldots \circ (x_{i_1} \circ x_{j_1}) \ldots))
\]

\[
= \tau(B_{n-1} \circ (x_{i_1-1} \circ \ldots \circ (x_{i_1} \circ x_{j_1}) \ldots))'
\]

\[
= \tau(B_{n-1})x_{i_1} \ldots x_{i_{r_1}}x_{j_1}^{(r_1)} + \sum_{p < r_1} \tau(B_{n-1}) \ldots x_{j_1}^{(p)},
\]

and all summands are less than \( x_{i_1} \ldots x_{i_{r_1}}x_{j_1}^{(r_1)} \) due to the above defined order on \( N(X). \) Analogously, we deal with \( \tau(B_{n-1}) \) and so on. \( \square \)

Define \( N_\varphi = \{ \varphi(a) \mid a \in N(X) \}. \)

**Theorem 2.** The set \( N_\varphi \) forms a basis of the free Novikov algebra \( \text{Nov}(X). \)

**Proof.** By Theorem I, we identify \( \text{Nov}(X) \) with the Novikov algebra \( \text{ComDer}(X)_{-1}. \) Let \( L \) be a linear span of \( N_\varphi \) in \( \text{Magma}(X). \) We want to show that \( \tau \) is an isomorphism of \( L \) and \( \text{Nov}(X) \) considered as vector spaces.

By Lemma 1 we have that \( \tau(\varphi(a)) = a + \sum_j b_j \) with \( b_j < a \) for every \( a \in N(X). \) Thus, we derive that \( \tau(N_\varphi) \) is linearly independent.

Suppose that \( \tau(N_\varphi) \) is not complete, so, the set \( M = \{ a \in N(X) \mid a \) is not expressed through \( \tau(N_\varphi) \} \) is not empty. Choose a minimal \( a \in M \) due to the order \( <, \) such element exists, since the set of tuples \( S(a) \) is well-ordered. We have \( a - \tau([a]) = \sum_j b_j. \) By the assumption, all \( b_j \) are expressed via \( \tau(N_\varphi), \) so, \( a \) is expressed too, a contradiction.

So, \( \tau: L \to \text{Nov}(X) \) is an isomorphism of vector spaces. Thus, we may define the product on \( L \) by the formula \( n \circ m = \tau^{-1}(\tau(n)\tau(m)'), \) where \( n, m \in N_\varphi. \) Since \( \tau \) is also a homomorphism between algebras \( L \) and \( \text{Nov}(X), \) we have proved the statement. \( \square \)

Let \( n \) be a positive integer. We consider Young diagrams corresponding to the partitions

\[ \lambda_1 + \ldots + \lambda_k = n, \quad \lambda_1 > \lambda_2 \geq \ldots \geq \lambda_k \geq 1. \]
We fill the Young diagrams by elements of $X$:

\[
\begin{array}{cccc}
  x_{i_1} & x_{i_2} & \cdots & x_{i_{\lambda_1-1}} & x_{t_1} \\
  \vdots & & & & \\
  x_{i_r} & \cdots & x_{i_{r\lambda_r-1}} & x_{t_r} \\
  x_{t_{r+1}} & & & \\
  \vdots & & & \\
  x_{t_{r+p}} & & & 
\end{array}
\]

Here

\[
i_{1\lambda_1-1} \geq \ldots \geq i_1 \geq \ldots \geq i_{r\lambda_r-1} \geq \ldots \geq i_r, \quad t_{r+1} \geq \ldots \geq t_{r+p},
\]

\[
t_1 \geq t_2 \text{ if } \lambda_1 = \lambda_2 + 1, \text{ and } t_s \geq t_{s+1} \text{ if } \lambda_s = \lambda_{s+1} \text{ for } s = 2, \ldots, r - 1.
\]

For the diagram with exactly one row $(x_{i_1}, x_{i_2}, \ldots, x_{i_{\lambda_1-1}}, x_{t_1})$ we attach a monomial of $\text{Nov}(X)$ as follows,

\[
u_1 := x_{i_{\lambda_1-1}} \circ (\ldots \circ (x_{i_2} \circ (x_{i_1} \circ x_{t_1})) \ldots).
\]

For the diagram with $m$ rows we attach a monomial of $\text{Nov}(X)$ inductively,

\[
u_m := x_{i_m \lambda_m-1} \circ (\ldots \circ (x_{i_{m-1}} \circ (x_{i_{m-2}} \circ (x_{i_{m-1}} \circ x_{t_m})) \ldots)).
\]

The set of the constructed Young diagrams with the corresponding monomials coincides with the set $N_\phi$.

3. Normal form of monomials of weight $-1$ in $\text{PoisDer}(X)$

Let $Y$ be a well-ordered set with respect to an order $<$, and let $Y^*$ be the set of all associative words in the alphabet $Y$ (including the empty word denoting by 1). Extend the order to $Y^*$ by induction on the word length as follows. Put $u < 1$ for every nonempty word $u$. Further, $u < v$ for $u = y_i u', v = y_j v'$, $y_i, y_j \in Y$ if either $y_i < y_j$ or $y_i = y_j, u' < v'$. In particular, the beginning of every word is greater than the whole word.

Definition 2. A word $w \in Y^*$ is called an associative Lyndon–Shirshov word if for arbitrary nonempty $u$ and $v$ such that $w = uv$, we have $w > vu$.

For example, a word $aabac$ is an associative Lyndon–Shirshov word when $a > b > c$.

Consider the set $Y^+$ of all nonassociative words in $Y$, here we exclude the empty word from consideration.
**Definition 3.** A nonassociative word \([u] \in Y^+\) is called a nonassociative Lyndon–Shirshov word (an LS-word, for short) provided that

1. (LS1) the associative word \(u\) obtained from \([u]\) by eliminating all parentheses is an associative Lyndon–Shirshov word;
2. (LS2) if \([u] = [[u_1],[u_2]]\), then \([u_1]\) and \([u_2]\) are LS-words, and \(u_1 > u_2\);
3. (LS3) if \([u_1] = [[u_{11}],[u_{12}]]\), then \(u_2 \geq u_{12}\).

These words appeared independently for the algebras and groups [4, 13]. In [13], it was proved that the set of all LS-words in the alphabet \(Y\) is a linear basis for a free Lie algebra generated by \(Y\). Moreover, each associative Lyndon–Shirshov word \(w\) possesses the unique arrangement of parentheses which gives an LS-word \([w]\).

We consider the free Poisson algebra \(\text{Pois}\langle X \rangle\) generated by \(X\). Here we denote the operations by \(x\cdot y\) and \(\{x, y\}\). Since \(\text{Pois}\langle X \rangle = \text{Com}(\text{Lie}\langle X \rangle)\), the set of commutative words

\[
A_1 A_2 \ldots A_n, \quad A_1 \leq \ldots \leq A_n,
\]

where \(A_i\) are Lyndon–Shirshov words in \(\text{Lie}\langle X \rangle\) forms a standard basis of \(\text{Pois}\langle X \rangle\).

By [10], for the free Poisson algebra generated by a set \(X\) with a derivation \(d\), we have the equality \(\text{PoisDer}\langle X \rangle = \text{Pois}\langle X_\infty \rangle\), where \(X_\infty = \{x_i^{(n)} \mid i \in I, n \in N\}\).

Define an order on \(X_\infty\) as follows: \(x_i^{(m)} > x_j^{(n)}\) if \(m > n\) or \(m = n, i > j\).

We define an order on \(X\) as follows. At first, we compare two Lie words \(A_1\) and \(A_2\) by degree, i.e., \(A_1 > A_2\) if \(\text{deg}\ A_1 > \text{deg}\ A_2\). If \(\text{deg}\ A_1 = \text{deg}\ A_2\), then we compare corresponding associative Lyndon–Shirshov words as it was defined above.

Also, we define \(A > x_k^{(m)} > B\), where \(A\) and \(B\) are LS-words on \(X_\infty\) of degree at least two and \(A\) but not \(B\) involves \(d\) in its notation.

Recall the definition of the weight function [10, Definition 2] on basic monomials [12] with Lie words taken from \(H(X, d)\) of \(\text{PoisDer}\langle X \rangle\),

\[
\text{wt}(x) = -1, \quad x \in X;
\]

\[
\text{wt}(d(u)) = \text{wt}(u) + 1; \quad \text{wt}\{u, v\} = \text{wt}(u) + \text{wt}(v) + 1; \quad \text{wt}(uv) = \text{wt}(u) + \text{wt}(v).
\]

Due to [10], we have \((\text{PoisDer}\langle X \rangle, \circ, [\cdot, \cdot]) \cong \text{SGD}(X)\), and the linear map \(\xi: \text{PoisDer}\langle X \rangle \rightarrow \text{SGD}(X)\) defined by the formulas \(\xi(a \circ b) = ab', \xi([a, b]) = \{a, b\}\) provides the isomorphism.

Let us define a canonical form of monomials \(\text{PoisDer}\langle X \rangle\) of weight \(-1\) as follows:

\[
x_{i_1} \ldots x_{i_k} B_1 \ldots B_m A_n \ldots A_1 x^{(r_1)}_{j_1} \ldots x^{(r_1)}_{j_1},
\]

where \(A_i, B_j\) are Lie-words of degree at least 2 and \(A_i\) but not \(B_j\) involves \(d\) in its notation, moreover,

\[
A_1 \leq A_2 \leq \ldots \leq A_n, \quad B_m \geq B_{m-1} \geq \ldots \geq B_1,
\]

\[
x_{i_k} \geq x_{i_{k-1}} \geq \ldots \geq x_{i_1}, \quad x^{(r_1)}_{j_1} \geq x^{(r_{1-1})}_{j_1} \geq \ldots \geq x^{(r_1)}_{j_1}.
\]

Denote by \(N(X)\) the set of all normal forms [13] of monomials from the standard basis of \(\text{PoisDer}\langle X \rangle\) of weight \(-1\).
Denote by $\text{Magma}_2(X)$ the free algebra with two binary (magma) operations $\circ$ and $[,]$ generated by $X$. We define a linear map $\psi: \text{PoisDer}(X)_{-1} \to \text{Magma}_2(X)$ by induction.

At first, we consider a Lie LS-word corresponding to an associative LS-word $w = x_1^{(r_1)} \ldots x_t^{(r_t)}$ with $k = r_1 + \ldots + r_t \geq 1$. Let $\pi \in S_t$ be a permutation acting on the letters of $w \in X^*_8$ such that in the associative word $w^\pi = x_1^{(p_1)} \ldots x_t^{(p_t)}$ we have $x_j^{(p_m)} \geq x_{j+1}^{(p_{m+1})}$ for $m = 1, \ldots, t - 1$.

Given $c_k \geq \ldots \geq c_1$ such that $\text{wt}(c_i) = -1$ (here by $c_i$ we mean either $x \in X$ or a Lie LS-word in $X$), we put

$$\psi(c_1 \ldots c_k[w]) = [u],$$

where the corresponding associative LS-word $u = v^\pi$ and the word $v = v_1 \ldots v_t$ is defined as follows,

$$v_m = G(c_1, \ldots, c_k, w)_m := \varphi(c_k^{-1} \cdot \ldots \cdot c_{k-p+1} \cdot \ldots \cdot c_k^{-1} \cdot \ldots \cdot c_{k-p+1} \cdot x_j^{(p_m)}),$$

the map $\varphi$ is defined by (10). Here, we add new generators $G(c_1, \ldots, c_k, w)_m$ to $X$ for all $p_m \geq 1$ to get the superset $\tilde{X}$. We compare new generators by the length in terms on the $\circ$ operation and then after elimination of all signs of the $\circ$ operation we compare them as associative words. By this rule, all new letters are greater than elements from $X$. Also, we put $\text{wt}(G(c_1, \ldots, c_k, w)_m) = -1$.

Let $u$ be a word of the form $[13]$. Define $\psi(u)$ as follows,

$$\psi(u) = \psi(c_1 \ldots c_m A_n \ldots A_1 x_j^{(r_1)} \ldots x_j^{(r_1)}) = \psi(c_1 \ldots c_{p_n-1} \tilde{A}_n A_{n-1} \ldots A_1 x_j^{(r_1)} \ldots x_j^{(r_1)}),$$

where $\text{wt}(c_{p_n-1+1} \ldots c_m A_n) = -1$, $\tilde{A}_n = \psi(c_{p_n-1+1} \ldots c_m \{A_n\})$ is a letter of the extended alphabet $\tilde{X}$.

Thus, by $n$ such steps we exclude all Lie words of degree at least two which involve $d$ in their notation and afterwards apply the map $\varphi$ from Sec. 2:

$$\psi(u) = \psi(c_1 \ldots c_{p_n-2} \tilde{A}_{n-1} A_{n-2} \ldots A_1 x_j^{(r_1)} \ldots x_j^{(r_1)}) = \ldots = \psi(c_1 \ldots c_{p_l} \tilde{A}_2 A_1 x_j^{(r_1)} \ldots x_j^{(r_1)}) = \varphi(c_1 \ldots c_{p_l} \tilde{A}_2 A_1 x_j^{(r_1)} \ldots x_j^{(r_1)}).$$

**Example 2.** We have

$$\psi(x_3 x_4 x_5 \{x_1', x_2''\}) = [x_3 \circ x_1, x_5 \circ (x_4 \circ x_2)],$$

$$\psi(x_5 x_6 x_7 x_8 \{x_3', x_4''\} \{x_1, x_2'\}) = \psi(x_5 x_6 \circ x_3, x_8 \circ (x_7 \circ x_4) \{x_1, x_2'\} x_9) = \psi(x_5 [x_1, x_6 \circ x_3, x_8 \circ (x_7 \circ x_4)] \circ x_2) x_9].$$

**4. Basis of free SGD-algebra**

Given $a, b \in N(X)$, we compare them as elements from $\text{ComDer}(X)$ (see Sec. 2). Define a homomorphism

$$\tau: \text{Magma}_2(X) \to \text{PoisDer}(X)_{-1}.$$

Lemma 2. Let \( a \in N(X) \). Then \( \tau(\psi(a)) = a + \sum_j b_j \), where \( b_j < a \) for all \( j \).

\textbf{Proof.} Let \( u = c_1 \ldots c_m A_n \ldots A_1 x_{j_1}^{(r_1)} \ldots x_{j_1}^{(r_1)} \in N(X) \). By the definition of \( \psi \) and \( \varphi \), we have (see also (14))
\[
\psi(u) = \varphi(c_1 \ldots c_q A_1 x_{j_1}^{(r_1)} \ldots x_{j_1}^{(r_1)}) = \varphi(c_1 \ldots c_q B_{t_1-1} x_{j_1}^{(r_1)}) = \ldots
\]
\[
= \varphi(c_1 \ldots c_q B_{t-1} x_{j_1}^{(r_1)}) = B_{t-1} \circ (c_q \circ (\ldots (c_1 \circ x_{j_1}) \ldots)).
\]
Applying \( \tau \), we get
\[
\tau(\psi(u)) = \tau(B_{t-1})((\tau(c_q \circ (\ldots (c_1 \circ x_{j_1}))\ldots))' \tag{15}
\]
By Lemma 1,
\[
\tau(c_q \circ (\ldots (c_1 \circ x_{j_1})\ldots)) = c_1 \ldots c_q x_{j_1}^{(r_1-1)} + \sum_j b_j,
\]
where \( b_j < c_1 \ldots c_q x_{j_1}^{(r_1-1)} \). Writing down \( \tau(B_{t-1}) \), we get the analogous expression for it as the right-hand side of (15). By this remark and by the induction reasons, it is enough to figure out with \( \tau(A_1) \). Let \( A_1 = [v_1 \ldots v_k] \), where \( v_i = x_{j_i}^{(r_i)} \). We have by Lemma 1,
\[
\tau(A_1) = [\tau(A_2)c_{t_1} \ldots c_{t_1} x_{j_1}^{(t_1)} + d_1, \ldots, c_{t_k} \ldots c_{t_k} x_{j_k}^{(t_k)} + d_k],
\]
where \( d_i \) are less than the corresponding leading terms. Extracting by the Leibniz rule \( \tau(A_2) \) from the described Lie word as well as all \( c_i \), we get the \( a \) and the sum of words less than \( a \) due to the defined order on \( N(X) \).

\textbf{Example 3.} If \( u = x_3 x_4 \{x_1', x_2'\} \), then
\[
\tau(\psi(x_3 x_4 \{x_1', x_2'\})) = \tau([x_3 \circ x_1, x_4 \circ x_2])
\]
\[
= \{x_3 x_4', x_4 x_2'\} = x_3 x_4 \{x_1', x_2'\} + \{x_3, x_4\} x_2' x_1' + x_4 \{x_3, x_2'\} x_1' + x_3 \{x_1', x_4\} x_2'.
\]
Define \( N_\psi = \{\psi(a) \mid a \in N(X)\} \).

\textbf{Theorem 3.} The set \( N_\psi \) forms a basis of the free SGD-algebra \( \text{SGD}(X) \).

\textbf{Proof.} Analogously to the proof of Theorem 2, where we apply Lemma 2.

Applying Theorem 3, we obtain the multiplication table in the free SGD-algebra. If \( a, b \in N_\psi \) then we compute \( a \circ b \) and \( [a, b] \) as follows. Firstly,
\[
\tau(a \circ b) = \sum_i \alpha_i c_i \in \text{PoisDer}(X),
\]
where \( * = \circ \) or \( * = [\cdot, \cdot] \), \( \alpha_i \in F \) and \( c_i \in N(X) \). Secondly, by Theorem 3, we have \( c_i = \tau(\sum_j \beta_{ij} d_{ij}) \) for some \( \beta_{ij} \in F \) and \( d_{ij} \in N_\psi \), which gives
\[
a \circ b = \sum_i \alpha_i \left( \sum_j \beta_{ij} d_{ij} \right).
\]
Example 4. We have that \([x_1, (x_2 \circ x_3) \circ x_4] \notin N_\psi\) and
\[
\tau([x_1, (x_2 \circ x_3) \circ x_4]) = \{x_1, x_2\}x'_3x'_4 + \{x_1, x'_3\}x_2x'_4 + \{x_1, x'_4\}x_2x'_3
\]
\[
\tau((x_1, x_2) \circ x_3) \circ x_4 + [x_1, x_2 \circ x_3] \circ x_4 - ([x_1, x_2] \circ x_3) \circ x_4
\]
\[
+ [x_1, x_2 \circ x_4] \circ x_3 - ([x_1, x_2] \circ x_3) \circ x_4),
\]
which gives the identity (7).
Also, \([x_2 \circ x_3, x_1 \circ x_4] \notin N_\psi\) and
\[
\tau([x_4 \circ x_1, x_3 \circ x_2]) = x_3x_4\{x'_1, x_2\} + \{x_3, x_4\}x'_2x'_1 + x_3\{x_4, x'_2\}x'_1 + x_4\{x'_1, x_3\}x'_2
\]
\[
\tau([x_3 \circ x_1, x_4 \circ x_2] - ([x_3, x_4] \circ x_2) \circ x_1 - [x_3, x_4 \circ x_2] \circ x_1 + ([x_3, x_4] \circ x_2) \circ x_1
\]
\[
- [x_3 \circ x_1, x_4] \circ x_2 + ([x_3, x_4] \circ x_2) \circ x_1 + ([x_3, x_4] \circ x_2) \circ x_1
\]
\[
+ [x_4, x_3 \circ x_2] \circ x_1 - ([x_4, x_3] \circ x_2) \circ x_1 + [x_4 \circ x_1, x_3] \circ x_2 - ([x_4, x_3] \circ x_2) \circ x_1),
\]
which gives the identity (8).

References

[1] L. A. Bokut, Y. Chen, Z. Zhang, Gröbner–Shirshov bases method for Gelfand–Dorfman–Novikov algebras, J. Algebra Appl. (1) 16 (2017), 1750001, 22 pp.
[2] A. A. Balinskii, S. P. Novikov, Poisson brackets of hydrodynamic type, Frobenius algebras and Lie algebras, Sov. Math. Dokl. 32 (1985), 228–231.
[3] D. Burde, Left-symmetric, or pre-Lie algebras in geometry and physics, Cent. Eur. J. Math., 4 (3), 325–357 (2006).
[4] K. T. Chen, R. H. Fox, and R. C. Lyndon, Free differential calculus. IV: The quotient groups of the lower central series, Ann. Math. (2) 68 (1958), 81–95.
[5] B. A. Duisengaliyeva, U. U. Umirbaev, A wild automorphism of a free Novikov algebra, Sib. Electron. Math. Rep. 15 (2018), 1671–1679.
[6] A. Dzhumadil’daev, N. Ismailov, \(S_n\)- and \(GL_n\)-module structures on free Novikov algebras, J. Algebra 416 (2014), 287–313.
[7] A. S. Dzhumadil’daev, C. Löfwall, Trees, free right-symmetric algebras, free Novikov algebras and identities, Homology, Homotopy Appl. (2) 4 (2002), 165–190.
[8] I. M. Gelfand, I. Ya. Dorfman, Hamilton operators and associated algebraic structures, Funct. Anal. its Appl. (4) 13 (1979), 13–30.
[9] P. S. Kolesnikov, A. Panasenko, Novikov commutator algebras are special, Algebra i Logika (6) 58 (2020), 804–807.
[10] P. S. Kolesnikov, B. Sartayev, A. Orazgaliev, Gelfand–Dorfman algebras, derived identities, and the Manin product of operads, J. Algebra 539 (2019), 260–284.
[11] P. S. Kolesnikov, B. K. Sartayev, On the special identities of Gelfand–Dorfman algebras, Exp. Math., doi:10.1080/10586458.2022.2041134.
[12] P. S. Kolesnikov, B. Sartayev, On the embedding of left-symmetric algebras into differential Perm-algebras, Communications in Algebra, 50 (2022), 3246–3260.
[13] A. I. Shirshov, On free Lie rings, Mat. Sb. (2) 45 (1958), 113–122.
[14] J. Wen, Y. Hong, Extending structures for Gelfand-Dorfman bialgebras. arXiv:2202.10674
[15] X. Xu, Quadratic Conformal Superalgebras, J. Algebra 231 (2000), 1–38.

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