Research Article

Existence of Solutions of a Nonlocal Elliptic System via Galerkin Method

Alberto Cabada¹ and Francisco Julio S. A. Corrêa²

¹ Departamento de Análise Matemática, Facultade de Matemáticas, Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain
² Unidade Académica de Matemática e Estatística, Centro de Ciências e Tecnologia, Universidade Federal de Campina Grande, 58.109-979 Campina Grande, Brazil

Correspondence should be addressed to Alberto Cabada, alberto.cabada@usc.es

Received 2 February 2012; Accepted 28 February 2012

Academic Editor: Pavel Drábek

Copyright © 2012 A. Cabada and F. J. S. A. Corrêa. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

By means of the Galerkin method and by using a suitable version of the Brouwer fixed-point theorem, we establish the existence of at least one positive solution of a nonlocal elliptic N-dimensional system coupled with Dirichlet boundary conditions.

1. Introduction

This paper is devoted to the study of the nonlocal elliptic system

$$
\begin{align*}
-\Delta u(x) + a(x) \int_{\Omega} b(y)v^p(y) \, dy &= g(u(x), v(x)), & x \in \Omega, \\
-\Delta v(x) + c(x) \int_{\Omega} d(y)u^q(y) \, dy &= h(u(x), v(x)), & x \in \Omega, \\
\quad u(x), v(x) &> 0, & x \in \Omega, \\
\quad u(x) = v(x) &= 0, & x \in \partial \Omega.
\end{align*}
$$

(1.1)

Here, $\Omega \subset \mathbb{R}^N, N \geq 2$, is a bounded smooth domain, $p, q$ are positive numbers, $a, b, c, d \in C(\overline{\Omega})$, and the nonlinearities $g$ and $h$ will be defined later.

The one-dimensional counterpart of this problem has been considered by Cabada et al. in [1]. There the authors, by using dual variational methods and Leray-Schauder degree, with $p = q = 1, b \equiv d \equiv 1$, and $a = c$ a real parameter, and under suitable assumptions on
the nonlinear functions $g(u,v) \equiv g(v)$ and $h(u,v) \equiv h(u)$, show the existence of solution (not necessarily positive) depending upon the parameter $a$.

In this case, we present a different point of view from the one used in [1]. We note that, among other things, we assume $N \geq 2$.

Motivated by its many applications and the richness of the methods employed to solve it, this kind of problems has been studied by different authors when only one equation is considered, see, among others, [2–9]. Indeed, there is a lot of phenomena that may be modeled by equations of the form

$$u_t - \Delta u = f(x, u, B(u)), \quad (1.2)$$

where $B$ is a nonlocal operator which, in some applications, is written in the form

$$B(u) = \int_{\Omega} b(x)[u(x)]^\beta dx. \quad (1.3)$$

See [10–13] for some surveys on these equations.

In particular, steady-state solutions deliver us to elliptic equations such as

$$-\Delta u(x) = f(x, u(x), B(u)), \quad x \in \Omega,$$

$$u(x) = 0, \quad x \in \partial\Omega, \quad (1.4)$$

which, in several cases, have a behaviour quite different from the local one

$$-\Delta u(x) = f(x, u(x)), \quad x \in \Omega,$$

$$u(x) = 0, \quad x \in \partial\Omega. \quad (1.5)$$

One of the most significant differences between these two types of problems is the nonexistence, in some particular cases, of maximum principles. For instance, Allegretto and Barabanova [4] consider the one-dimensional problem

$$-u''(x) + \eta \int_0^1 u(y)dy = \sin(\pi x), \quad 0 < x < 1,$$

$$u(0) = u(1) = 0, \quad \eta > 0. \quad (1.6)$$

It is not difficult to verify that the explicit solution of this problem is given by the expression

$$u(x) = \frac{12\eta(x^2 - x)}{(\pi^3(12 + \eta))} + \frac{\sin(\pi x)}{\pi^2}. \quad (1.7)$$

So, when the values of the positive parameter $\eta$ are small, the solution is positive. However, if $\eta$ is large enough, function $u$ becomes negative near the end points $x = 0$ and
with $f$.

where $\delta, \gamma > 0$ are constants and $f(x, u)$ is a given function. Taking $B = \delta(-\Delta + \gamma)^{-1}$, under

Dirichlet boundary condition, problem (1.9) is equivalent to the integrodifferential problem

$$
-\Delta u(x) + Bu = f(x, u), \quad x \in \Omega,
$$

$$
u(x) = 0, \quad x \in \partial\Omega.
$$

Consider now the problem

$$
-\Delta u(x) + Bu - \lambda u = f(x), \quad x \in \Omega,
$$

$$
u(x) = 0, \quad x \in \partial\Omega,
$$

with $f \in L^2(\Omega)$ and $f \geq 0$ in $\Omega$.

Let $\lambda_1$ be the first eigenvalue of operator $-\Delta$ in the space $H^1_0(\Omega)$, and assume that

$\lambda_1 > \sqrt{\delta} - \gamma$. Then, for all $\lambda \in (2\sqrt{\delta} - \gamma, \lambda_1 + (\delta/(\gamma + \lambda_1)))$, problem (1.11) satisfies a maximum principle, that is, under the above assumptions, the solution $u$ of (1.11) satisfies $u \geq 0$ a.e. in $\Omega$. See [15] for the proof of this result.

After that, it is proved in [16], by using semigroup theory, that this maximum principle does not hold for all $\lambda < 2\sqrt{\delta} - \gamma$. Indeed, the approach used in [16] may be used to prove that a general maximum principle for the problem (1.6) is not valid. In view of this, the method of sub- and supersolution should be used carefully by considering a relation between the growth of the nonlinearity and the parameters of the problem.
Remark 1.2. It is worthy to remark that problem (1.1) has no variational structure even in the scalar case. So the most usual variational techniques cannot be used to study it.

To attack problem (1.1), we will use the Galerkin method through the following version of the Brouwer fixed-Point Theorem whose proof may be found in Lions [17, Lemma 4.3].

**Proposition 1.3.** Let $F : \mathbb{R}^m \to \mathbb{R}^m$ be a continuous function such that

$$
\langle F(\xi), \xi \rangle > 0 \quad \text{if} \quad |\xi| = r,
$$

(1.12)

for some $r > 0$, where $\langle \cdot, \cdot \rangle$ is the Euclidian Scalar product and $| \cdot | = \langle \cdot, \cdot \rangle^{1/2}$ is the corresponding Euclidian norm in $\mathbb{R}^m$. Then, there exists $\xi_0 \in \mathbb{R}^m$, $|\xi_0| \leq r$ such that $F(\xi_0) = 0$.

### 2. A Sublinear Problem

In this section, we consider the problem

$$
\begin{align*}
-\Delta u(x) + \lambda \int_{\Omega} v^p(y) dy &= u^\alpha(x) + v^\beta(x), \quad x \in \Omega, \\
-\Delta v(x) + \lambda \int_{\Omega} u^q(y) dy &= u^\gamma(x) + v^\delta(x), \quad x \in \Omega, \\
u(x), v(x) &> 0, \quad x \in \Omega, \\
u(x) = v(x) &= 0, \quad x \in \partial \Omega.
\end{align*}
$$

(2.1)

Here, $\lambda$ is a real parameter and $\alpha, \beta, \gamma, \delta$ are positive constants whose properties will be precised later.

In order to use Proposition 1.3, we have to introduce a suitable setup. First of all, we consider an orthonormal Hilbertian basis $B = \{\varphi_1, \varphi_2, \ldots\}$ in $H^1_0(\Omega)$ whose norm is the usual one

$$
\|u\|^2 = \int_{\Omega} |\nabla u(y)|^2 \, dy, \quad \forall u \in H^1_0(\Omega).
$$

(2.2)

Next, let $\mathbb{V}_m$ be the finite dimensional vector space

$$
\mathbb{V}_m = [\varphi_1, \ldots, \varphi_m] \subset B,
$$

(2.3)

equipped with the norm induced by the one in $H^1_0(\Omega)$.

Thus, if $u \in \mathbb{V}_m$, there is a unique $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m$ such that

$$
u = \sum_{j=1}^{m} \xi_j \varphi_j
$$

(2.4)
Abstract and Applied Analysis
and, as a consequence,
\[ \|u\|^2 = |\xi|^2. \] (2.5)

So, the spaces \( V_m \) and \( \mathbb{R}^m \) are isomorphic and isometric by
\[ V_m \leftrightarrow \mathbb{R}^m, \]
\[ u = \sum_{j=1}^{m} \xi_j \varphi_j \leftrightarrow \xi = (\xi_1, \ldots, \xi_m). \] (2.6)

From now on, we identify, with no additional comments, \( u \leftrightarrow \xi \) via this isometry.
In order to obtain a nontrivial solution of problem (2.1), let \( \varepsilon > 0 \) be a constant and
consider the auxiliary problem
\[
\begin{align*}
-\Delta u(x) + \lambda \int_{\Omega} (v^+)^p (y) \, dy &= (u^+)^a (x) + (v^+)^\beta (x) + \varepsilon, & x \in \Omega, \\
-\Delta v(x) + \lambda \int_{\Omega} (u^+)^q (y) \, dy &= (u^+)^\gamma (x) + (v^+)^\delta (x), & x \in \Omega,
\end{align*}
\]
\[ u(x), v(x) > 0, \quad x \in \Omega, \]
\[ u(x) = v(x) = 0, \quad x \in \partial \Omega. \] (2.7)

**Theorem 2.1.** Assume that \( \Omega \) is a bounded \( C^2 \) domain of \( \mathbb{R}^N \) and the constants \( p, q, \alpha, \beta, \gamma, \delta \in (0, 1) \). Then, for all \( \lambda < 0 \), problem (2.1) has at least one solution in \( (C^2(\Omega) \cap C(\overline{\Omega})) \times (C^2(\Omega) \cap C(\overline{\Omega})) \).

**Proof.** First of all, we consider a map \((F, G) : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^m, (F, G) = (F_1, \ldots, F_m, G_1, \ldots, G_m)\), defined, for all \( i = 1, \ldots, m \), as
\[
\begin{align*}
F_i(\xi, \eta) &= \int_{\Omega} \nabla u \nabla \varphi_i + \lambda \int_{\Omega} (v^+)^p \nabla \varphi_i - \int_{\Omega} (u^+)^a \varphi_i - \int_{\Omega} (v^+)^\beta \varphi_i - \varepsilon \int_{\Omega} \varphi_i, \\
G_i(\xi, \eta) &= \int_{\Omega} \nabla v \nabla \varphi_i + \lambda \int_{\Omega} (u^+)^q \nabla \varphi_i - \int_{\Omega} (u^+)^\gamma \varphi_i - \int_{\Omega} (v^+)^\delta \varphi_i,
\end{align*}
\] (2.8)
where we are identifying \((u, v) \in V_m \times V_m, u = \sum_{j=1}^{m} \xi_j \varphi_j, v = \sum_{j=1}^{m} \eta_j \varphi_j, \) with \((\xi, \eta) \in \mathbb{R}^m \times \mathbb{R}^m, \xi = (\xi_1, \ldots, \xi_m), \eta = (\eta_1, \ldots, \eta_m)\).

Now, we have that, for all \( i = 1, \ldots, m \), the following equations hold:
\[
\begin{align*}
F_i(\xi, \eta) \cdot \xi_i &= \int_{\Omega} \nabla u \cdot \nabla (\xi_i \varphi_i) + \lambda \int_{\Omega} (v^+)^p \xi_i \varphi_i - \int_{\Omega} (u^+)^a (\xi_i \varphi_i) \\
&\quad - \varepsilon \int_{\Omega} (v^+)^\beta (\xi_i \varphi_i) - \varepsilon \int_{\Omega} \xi_i \varphi_i, \\
G_i(\xi, \eta) \cdot \eta_i &= \int_{\Omega} \nabla v \cdot \nabla (\eta_i \varphi_i) + \lambda \int_{\Omega} (u^+)^q \eta_i \varphi_i - \int_{\Omega} (u^+)^\gamma (\eta_i \varphi_i) - \int_{\Omega} (v^+)^\delta (\eta_i \varphi_i).
\end{align*}
\] (2.9)
Therefore,

\[
\langle F, G \rangle(\xi, \eta), (\xi, \eta) \rangle = \int_\Omega |\nabla u|^2 + \int_\Omega |\nabla v|^2 + \lambda \int_\Omega (v^+)^p \int_\Omega u - \int_\Omega (u^+)q u
\]
\[
- \int_\Omega (v^+)^p u - e \int_\Omega u + \lambda \int_\Omega (u^+)^q \int_\Omega v - \int_\Omega (u^+)\gamma v - \int_\Omega (v^+)\delta v.
\]

(2.10)

Denoting as \(\| (x, y) \|^2 = |x|^2 + |y|^2\) for all \(x, y \in \mathbb{R}^m\), using the isometry between \(V_m\) and \(\mathbb{R}^m\), and the inequalities of Hölder, Poincaré, and Sobolev, we arrive at the following estimations:

\[
\left( \int_\Omega (v^+)^p \right) \int_\Omega u \leq C \left( \int_\Omega (|v|^2)^{p/2} \right)^{p/2} \cdot \| u \| \leq C \| (u, v) \|^{p+1},
\]

(I1)

\[
\int_\Omega (v^+)^q u \leq \left( \int_\Omega |v|^{2q} \right)^{1/2} \| u \| \leq C \| (u, v) \|^{q+1},
\]

(I2)

and, in a similar way,

\[
\int_\Omega (u^+)\alpha u \leq C \| (u, v) \|^\alpha,\]

(I3)

\[
\int_\Omega (u^+)^q \int_\Omega v \leq C \| (u, v) \|^q,
\]

(I4)

\[
\int_\Omega (u^+)\gamma v \leq \| (u, v) \|^\gamma,
\]

(I5)

\[
\int_\Omega (v^+)\delta v \leq C \| (u, v) \|^\delta.
\]

(I6)

We note that the positive constant \(C\) depends on \(\Omega\), but it does not depend on the rest of the parameters involved in problem (2.7).
Using the previous estimations \((I_1)-(I_6)\) together with the fact that \(\lambda < 0\), we deduce that
\[
\langle (F, G)(\xi, \eta), (\xi, \eta) \rangle \geq \|(u, v)\|^2 + \lambda C \|(u, v)\|^{p+1} - C \|(u, v)\|^{\beta+1} \\
- \epsilon C \|(u, v)\| + \lambda C \|(u, v)\|^{q+1} - C \|(u, v)\|^{\gamma+1} \\
- C \|(u, v)\|^{\delta+1} - C \|(u, v)\|^{\alpha+1}.
\] (2.11)

Now, since \(0 < p, q, \alpha, \beta, \gamma, \delta < 1\), there is \(r > 0\) such that
\[
\langle (F, G)(\xi, \eta), (\xi, \eta) \rangle > 0 \quad \text{if} \quad \|(\xi, \eta)\| > r.
\] (2.12)

In view of Proposition 1.3, there exists \(r > 0\) that does not depend on \(m\), and a pair \((u_m, v_m) \in \mathbb{V}_m \times \mathbb{V}_m\), such that \(\|u_m\|, \|v_m\| \leq r\), and satisfies the following equalities for all \(i = 1, \ldots, m\):
\[
\int_\Omega \nabla u_m \nabla \varphi_i + \lambda \int_\Omega (v_m^r)^p \int_\Omega \varphi_i - \int_\Omega (u_m^r)^{\alpha} \varphi_i - \int_\Omega (v_m^r)^{\beta} \varphi_i - \epsilon \int_\Omega \varphi_i = 0, \\
\int_\Omega \nabla v_m \nabla \varphi_i + \lambda \int_\Omega (u_m^s)^q \int_\Omega \varphi_i - \int_\Omega (u_m^s)^{\gamma} \varphi_i - \int_\Omega (v_m^s)^{\delta} \varphi_i = 0.
\] (2.13)

So, for all \(\varphi \in \mathbb{V}_m\), it is satisfied that
\[
\int_\Omega \nabla u_m \nabla \varphi + \lambda \int_\Omega (v_m^r)^p \int_\Omega \varphi - \int_\Omega (u_m^r)^{\alpha} \varphi - \int_\Omega (v_m^r)^{\beta} \varphi - \epsilon \int_\Omega \varphi = 0, \\
\int_\Omega \nabla v_m \nabla \varphi + \lambda \int_\Omega (u_m^s)^q \int_\Omega \varphi - \int_\Omega (u_m^s)^{\gamma} \varphi - \int_\Omega (v_m^s)^{\delta} \varphi = 0.
\] (2.14)

In view of the boundedness of the approximate solutions \(u_m, v_m\) in \(H^1_0(\Omega)\), we have, perhaps for subsequences, that \(u_m \rightharpoonup u_c\) and \(v_m \rightharpoonup v_c\) in \(H^1_0(\Omega)\).

Fixing \(k < m\) and making \(m \to +\infty\) in the last two equalities, we obtain, after using Sobolev immersions, that the following equalities hold for all \(\varphi \in \mathbb{V}_k\):
\[
\int_\Omega \nabla u_c \nabla \varphi + \lambda \int_\Omega (v_c^r)^p \int_\Omega \varphi - \int_\Omega (u_c^r)^{\alpha} \varphi - \int_\Omega (v_c^r)^{\beta} \varphi - \epsilon \int_\Omega \varphi = 0, \\
\int_\Omega \nabla v_c \nabla \varphi + \lambda \int_\Omega (u_c^s)^q \int_\Omega \varphi - \int_\Omega (u_c^s)^{\gamma} \varphi - \int_\Omega (v_c^s)^{\delta} \varphi = 0.
\] (2.15)

Since \(k\) is arbitrary, the last two identities are valid for all \(\varphi \in H^1_0(\Omega)\). Consequently, \((u_c, v_c) \in H^1_0(\Omega) \times H^1_0(\Omega)\) is a weak solution of the auxiliary problem (2.7).
Now, since $\lambda < 0$ and $\epsilon > 0$, we have, from (2.7), that $-\Delta u_\epsilon \geq \epsilon$ and $-\Delta v_\epsilon \geq 0$ on $\Omega$. This fact, together with the Dirichlet boundary conditions, says that $u_\epsilon, v_\epsilon > 0$ on $\Omega$. In consequence, the following equalities hold for all $\varphi \in H^1_0(\Omega)$:

$$
\int_{\Omega} \nabla u_\epsilon \nabla \varphi + \lambda \int_{\Omega} v_\epsilon \varphi = \int_{\Omega} u^\alpha \varphi - \int_{\Omega} v^\beta \varphi - \epsilon \int_{\Omega} \varphi = 0,
$$

(2.16)

$$
\int_{\Omega} \nabla v_\epsilon \nabla \varphi + \lambda \int_{\Omega} u_\epsilon \varphi = \int_{\Omega} u^\gamma \varphi - \int_{\Omega} v^\delta \varphi = 0.
$$

That is to say, $(u_\epsilon, v_\epsilon)$ is a weak solution of problem

$$
-\Delta u(x) + \lambda \int_{\Omega} v^\rho(y) dy = u^\alpha(x) + v^\delta(x) + \epsilon, \quad x \in \Omega,
$$

$$
-\Delta v(x) + \lambda \int_{\Omega} u^\tau(y) dy = u^\gamma(x) + v^\delta(x), \quad x \in \Omega,
$$

(2.17)

$$
u(x), \varphi(x) > 0, \quad x \in \Omega,
$$

$$
u(x) = \varphi(x) = 0, \quad x \in \partial \Omega.
$$

In particular, it satisfies the following inequalities:

$$
-\Delta u_\epsilon(x) \geq u_\epsilon^\alpha(x), \quad x \in \Omega,
$$

$$
-\Delta v_\epsilon(x) \geq v_\epsilon^\delta(x), \quad x \in \Omega,
$$

(2.18)

$$
u_\epsilon(x), \nu_\epsilon(x) > 0, \quad x \in \Omega,
$$

$$
u_\epsilon(x) = \nu_\epsilon(x) = 0, \quad x \in \partial \Omega.
$$

Let $u_\alpha, v_\delta > 0$ be the unique solution of the problem

$$
-\Delta u(x) = u^\alpha(x), \quad x \in \Omega,
$$

$$
-\Delta v(x) = v^\delta(x), \quad x \in \Omega,
$$

(2.19)

$$
u(x), \varphi(x) > 0, \quad x \in \Omega,
$$

$$
u(x) = \varphi(x) = 0, \quad x \in \partial \Omega.
$$

We point out that the existence and uniqueness of the solutions of each of the above equations follow from [18, 19] because $0 < \alpha, \delta < 1$.

In the sequel, we use the following comparison result, due to Ambrosetti, Brezis, and Cerami.
Lemma 2.2. [20, Lemma 3.3]. Assume that \( f(t) \) is a continuous function such that \( t^{-1}f(t) \) is decreasing for \( t > 0 \). Let \( v \) and \( w \) satisfy

\[
-\Delta v(x) \leq f(v(x)), \quad x \in \Omega, \\
v(x) > 0, \quad x \in \Omega, \\
v(x) = 0, \quad x \in \partial\Omega, \\
-\Delta w(x) \geq f(w(x)), \quad x \in \Omega, \\
w(x) > 0, \quad x \in \Omega, \\
w(x) = 0, \quad x \in \partial\Omega.
\]

Then, \( w(x) \geq v(x) \) for all \( x \in \Omega \).

So we conclude that \( u_\epsilon \geq u_\delta > 0 \) and \( v_\epsilon \geq v_\delta > 0 \) in \( \Omega \).

Taking limits on both members of (2.16) and (2.6) as \( \epsilon \to 0^+ \), we deduce that \( u_\epsilon \rightharpoonup u \) and \( v_\epsilon \rightharpoonup v \), for some \( u, v \in H^1_0(\Omega) \) such that \( u, v > 0 \) in \( \Omega \).

Proceeding as before, by using Sobolev embeddings and elliptic regularity, we conclude that \((u, v)\) is a classical solution of the system (2.1).

Remark 2.3. We note that, from the fact that problem (1.6) is a one-dimensional particular case of problem (2.1), when \( \lambda > 0 \), we cannot ensure that, in general, the problem (2.1) has a positive solution in \( \Omega \).

Remark 2.4. We should point out that we may consider a more general system than (2.1). To be more precise, we may consider a system like

\[
-\Delta m u(x) + \lambda \int_\Omega v^m(y) \, dy = u^m(x) + v^m(x), \quad x \in \Omega, \\
-\Delta n v(x) + \lambda \int_\Omega u^n(y) \, dy = u^n(x) + v^n(x), \quad x \in \Omega, \\
u(x), v(x) > 0, \quad x \in \Omega, \\
u(x) = v(x) = 0, \quad x \in \partial\Omega,
\]

where \(-\Delta_m u = - \text{div} (|\nabla u|^{m-2} \nabla u)\) and \(-\Delta_n u = - \text{div} (|\nabla u|^{n-2} \nabla u)\) are, respectively, the \( m \)-Laplacian and \( n \)-Laplacian. Although the proof of the existence of solution follows similar ideas as those used in Theorem 2.1, the calculations are more complicated because we have to work with Schauder’s basis in \( W^{1,m}_0(\Omega) \) and \( W^{1,n}_0(\Omega) \) and these spaces do not enjoy a Hilbert space structure.
If we would like to dare a little more, we may consider a system like

\[-\Delta_{m(x)} u(x) + \lambda \int_\Omega v^p(x) \, dy = u^\alpha(x) + v^\beta(x), \quad x \in \Omega,\]

\[-\Delta_{n(x)} v(x) + \lambda \int_\Omega u^q(x) \, dy = u^\gamma(x) + v^\delta(x), \quad x \in \Omega,\]  \hspace{1cm} (2.22)

\[u(x), \, v(x) > 0, \quad x \in \Omega,\]

\[u(x) = v(x) = 0, \quad x \in \partial \Omega,\]

where

\[-\Delta_{m(x)} u = -\text{div} \left( |\nabla u|^{m(x)-2} \nabla u \right),\]

\[-\Delta_{n(x)} u = -\text{div} \left( |\nabla u|^{n(x)-2} \nabla u \right)\]  \hspace{1cm} (2.23)

are, respectively, the \( m(x) \)-Laplacian and \( n(x) \)-Laplacian and \( p(x), \alpha(x), \beta(x), n(x), q(x), \gamma(x), \delta(x) \) are continuous functions on \( \Omega \) satisfying suitable conditions. In this case, we have to work in the generalized Lebesgue-Sobolev spaces \( W^{1,m(x)}_0(\Omega) \) and \( W^{1,n(x)}_0(\Omega) \). See, for example, [21, 22] and the references therein, for more detailed information on this subject.

3. A Singular Problem

The Galerkin method may also be used to attack a singular version of the problem (2.1). More precisely, let us consider a simple version of a singular problem as

\[-\Delta u(x) + \lambda \int_\Omega v^p(x) \, dy = \frac{1}{u^\alpha(x)}, \quad x \in \Omega,\]

\[-\Delta v(x) + \lambda \int_\Omega u^q(x) \, dy = v^\delta(x), \quad x \in \Omega,\]  \hspace{1cm} (3.1)

\[u(x), v(x) > 0, \quad x \in \Omega,\]

\[u(x) = v(x) = 0, \quad x \in \partial \Omega\]

with \( \alpha, \delta > 0 \).
Proof. Reasoning as in the proof of Theorem 2.1 we obtain, for each $\epsilon > 0$, a solution $(u_\epsilon, v_\epsilon)$ of problem (3.1). So, since $\lambda < 0$, we obtain

$$-\Delta u_\epsilon(x) \geq \frac{1}{((u_\epsilon^* (x)) + \epsilon)^\alpha} > 0, \quad x \in \Omega,$$

$$-\Delta v_\epsilon(x) \geq (v_\epsilon^* (x))^\delta \geq 0, \quad x \in \Omega,$$

$$u_\epsilon(x) = v_\epsilon(x) = 0, \quad x \in \partial \Omega.$$

However, for the sake of simplicity and to illustrate the method, we restrict our discussion to the problem (3.1).

To approach problem (3.1), we consider a nonsingular perturbation as

$$-\Delta u(x) + \lambda \int_\Omega (\nu^*)^\delta (y) dy = \frac{1}{((u^*(x)) + \epsilon)^\alpha}, \quad x \in \Omega,$$

$$-\Delta v(x) + \lambda \int_\Omega (u^*)^\delta (y) dy = (\nu^*(x))^\delta, \quad x \in \Omega,$$

$$u(x) = v(x) = 0, \quad x \in \partial \Omega,$$

with $\epsilon > 0$, to obtain approximate solutions $(u_\epsilon, v_\epsilon)$. In this way, we avoid the singular term.

We have the following result.

**Theorem 3.1.** Let $0 < \alpha, \delta < 1$ be real numbers and $\lambda < 0$ a real parameter. Then, problem (3.1) possesses a positive solution $(u, v) \in (C^2(\Omega) \cap C(\overline{\Omega})) \times (C^2(\Omega) \cap C(\overline{\Omega}))$.

**Proof.** Reasoning as in the proof of Theorem 2.1 we obtain, for each $\epsilon > 0$, a solution $(u_\epsilon, v_\epsilon)$ of problem (3.4). So, since $\lambda < 0$, we obtain

$$-\Delta u_\epsilon(x) \geq \frac{1}{((u_\epsilon^* (x)) + \epsilon)^\alpha} > 0, \quad x \in \Omega,$$

$$-\Delta v_\epsilon(x) \geq (v_\epsilon^* (x))^\delta \geq 0, \quad x \in \Omega,$$

$$u_\epsilon(x) = v_\epsilon(x) = 0, \quad x \in \partial \Omega.$$
In view of the maximum principle, we obtain that $u_\epsilon > 0$ in $\Omega$ and $v_\epsilon \geq 0$ in $\Omega$. Consequently,

$$-\Delta u_\epsilon(x) + \lambda \int_\Omega v_\epsilon^p(y)\,dy = \frac{1}{(u_\epsilon(x) + \epsilon)^\alpha}, \quad x \in \Omega,$$

$$-\Delta v_\epsilon(x) + \lambda \int_\Omega u_\epsilon^q(y)\,dy = v_\epsilon^\delta(x), \quad x \in \Omega,$$

$$u_\epsilon(x) = v_\epsilon(x) = 0, \quad x \in \partial \Omega.$$ \hfill (3.6)

If $v_\epsilon \equiv 0$, we obtain, in view of (3.6), that $\int_\Omega u_\epsilon^q(y)\,dy = 0$, which contradicts the fact that $u_\epsilon > 0$ in $\Omega$. Thus, $v_\epsilon \not\equiv 0$ and, because $v_\epsilon \geq 0$, the maximum principle gives $v_\epsilon > 0$ in $\Omega$.

The solution of problem (3.1) will be obtained by studying the limit when $\epsilon \to 0$. Thus, we may suppose that $0 < \epsilon < 1$. In view of this, we obtain from (3.6) that

$$-\Delta u_\epsilon(x) > \frac{1}{(u_\epsilon(x) + 1)^\alpha}, \quad x \in \Omega,$$

$$u_\epsilon(x) = 0, \quad x \in \partial \Omega.$$ \hfill (3.7)

Let $\omega_\epsilon$ be the unique positive solution of

$$-\Delta \omega_\epsilon(x) = \frac{1}{(u_\epsilon(x) + 1)^\alpha}, \quad x \in \Omega,$$

$$\omega_\epsilon(x) = 0, \quad x \in \partial \Omega.$$ \hfill (3.8)

From the maximum principle, we deduce that $u_\epsilon > \omega_\epsilon > 0$ in $\Omega$. Since $1/(u_\epsilon + 1)^\alpha$ is bounded, by using interior elliptic regularity, we obtain that there is $\omega \in C^2(\Omega)$ such that $\omega_\epsilon \to \omega$ in $C^2(\Omega)$, for all $\Omega' \subset\subset \Omega$.

Using the Galerkin method and reasoning as in the proof of Theorem 2.1, we deduce that the approximate solutions $u_\epsilon, v_\epsilon$ are uniformly bounded in $H_0^1(\Omega)$ with respect to $0 < \epsilon < 1$. In consequence, we conclude that $u_\epsilon \to u$ and $v_\epsilon \to v$ in $H_0^1(\Omega)$ in the weak sense. Hence, in view of (3.8), we obtain

$$-\Delta \omega(x) = \frac{1}{(u(x) + 1)^\alpha}, \quad x \in \Omega,$$

$$\omega(x) = 0, \quad x \in \partial \Omega.$$ \hfill (3.9)

Invoking again the maximum principle, we get $\omega > 0$ in $\Omega$. Since $u_\epsilon > \omega_\epsilon$ in $\Omega$, we conclude that $u \geq \omega > 0$ in $\Omega$. Reasoning as before, $v > 0$ in $\Omega$, and $(u, v)$ is a classical solution of problem (3.1), which finishes the proof of this result. \hfill $\Box$
4. On a Superlinear Problem

At last we will make some remarks on a superlinear problem. In order to simplify the exposition, let us consider the one equation case

\[-\Delta u(x) + \lambda \int_{\Omega} u^p(y)dy = u^\alpha(x) + f(x), \quad x \in \Omega,\]

\[u(x) > 0, \quad x \in \Omega,\]

\[u(x) = 0, \quad x \in \partial \Omega,\]

where \(1 < p, \alpha < (N + 2)/(N - 2), \quad N \geq 3\) and \(f \in L^2(\Omega), f \geq 0, f \neq 0\). Such a function \(f\) is introduced in order to ensure that the solution is nonnegative and nontrivial. Note that \(u \equiv 0\) is a solution of

\[-\Delta u(x) + \lambda \int_{\Omega} u^p(y)dy = u^\alpha(x), \quad x \in \Omega,\]

\[u(x) = 0, \quad x \in \partial \Omega,\]

and we do not have at our disposal the results in [19, 20].

**Theorem 4.1.** Suppose that \(1 < p, \alpha < (N + 2)/(N - 2), \quad N \geq 3\) and \(\lambda < 0\). Then, for each \(f \in L^2(\Omega), \quad \|f\|_2\) sufficiently small, problem (4.1) possesses a nonnegative and nontrivial solution.

**Proof.** First of all, let us consider the auxiliary problem

\[-\Delta u(x) + \lambda \int_{\Omega} (u^+)^p(y)dy = (u^+(x))^\alpha + f(x), \quad x \in \Omega,\]

\[u(x) = 0, \quad x \in \partial \Omega.\]

As before, let us consider \(F : \mathbb{R}^m \rightarrow \mathbb{R}^m, \quad F = (F_1, \ldots, F_m)\), defined, for all \(i = 1, \ldots, m\), as

\[F_i(\xi) = \int_{\Omega} \nabla u \nabla \varphi_i + \lambda \int_{\Omega} (u^+)^p \varphi_i - \int_{\Omega} (u^+)^\alpha \varphi_i - \int_{\Omega} f \varphi_i,\]

where we are using the previous identifications.

For all \(i = 1, \ldots, m\), the following equations hold

\[F_i(\xi) \cdot \xi_i = \int_{\Omega} \nabla u \cdot \nabla (\xi_i \varphi_i) + \lambda \int_{\Omega} (u^+)^p \xi_i \varphi_i - \int_{\Omega} (u^+)^\alpha \xi_i \varphi_i - \int_{\Omega} f \xi_i \varphi_i,\]

and so

\[\langle F(\xi), \xi \rangle = \int_{\Omega} \|\nabla u\|^2 + \lambda \int_{\Omega} (u^+)^p \int u - \int_{\Omega} (u^+)^\alpha u - \int_{\Omega} f u.\]
Using the classical immersions, we get
\[ \langle F(\xi), \xi \rangle \geq \|u\|^2 + \lambda C\|u\|^{p+1} - C\|u\|^\alpha - C\|f\|_2\|u\| \tag{4.7} \]
for all \( u \in \mathbb{V}_m \).

We now consider the function
\[ g(t) = t^2 + \lambda C t^{p-1} - C t^{\alpha-1}, \quad t \geq 0, \tag{4.8} \]
and note that
\[ g(t) = t^2 \left( 1 + \lambda C t^{p-1} - C t^{\alpha-1} \right). \tag{4.9} \]

So, we may find a \( t_0 > 0 \), sufficiently small, such that
\[ g(t_0) = t_0^2 \left( 1 + \lambda C t_0^{p-1} - C t_0^{\alpha-1} \right) > 0. \tag{4.10} \]

As consequence, we can choose \( \|f\|_2 \), small enough, such that
\[ g(t_0) = t_0^2 \left( 1 + \lambda C t_0^{p-1} - C t_0^{\alpha-1} \right) > C\|f\|_2 t_0, \tag{4.11} \]
and, taking \( u \in \mathbb{V}_m \) such that \( \|u\| = t_0 \), we get
\[ \langle F(\xi), \xi \rangle > 0, \quad |\xi| = \|u\| = t_0, \quad u \in \mathbb{V}_m, \tag{4.12} \]
and note that such a \( t_0 \) does not depend on \( m \).

Proceeding as in the proof of Theorem 2.1, we obtain, for each \( m \in \mathbb{N} \), an approximate solution \( u_m \in \mathbb{V}_m \) satisfying \( \|u_m\| \leq t_0 \) and, for all \( i = 1, \ldots, m \),
\[ \int_\Omega \nabla u_m \nabla \varphi_i + \lambda \int_\Omega (u_m^+)^p \varphi_i - \int_\Omega (u_m^+)^a \varphi_i - \int_\Omega f \varphi_i = 0. \tag{4.13} \]

Thus, for all \( \varphi \in \mathbb{V}_m \), we have
\[ \int_\Omega \nabla u_m \nabla \varphi + \lambda \int_\Omega (u_m^+)^p \varphi - \int_\Omega (u_m^+)^a \varphi - \int_\Omega f \varphi = 0. \tag{4.14} \]

Since we are working in the superlinear and subcritical case and using the previous arguments, we obtain that there is a function \( u \in H^1_0(\Omega) \) satisfying
\[ \int_\Omega \nabla u \nabla \varphi + \lambda \int_\Omega (u^+)^p \varphi - \int_\Omega (u^+)^a \varphi - \int_\Omega f \varphi = 0 \tag{4.15} \]
for all \( \varphi \in H^1_0(\Omega) \).
We now use the weak maximum principle to conclude that $u \geq 0$ in $\Omega$. From the fact that $f \neq 0$, the solution is not the trivial one. Then, $u$ is a solution of our original problem (4.1).

**Remark 4.2.** Note that, when we are working with a problem with variational structure the function $f$ may be discarded because we may use the Mountain Pass Theorem. In this case the critical level given by this theorem is positive and so we may conclude that the solution obtained in this way is not null. However, in our case, we can not dispose of the variational techniques. In view of this and to show that we obtain a nontrivial solution, we had to introduce the function $f$.

**Acknowledgments**

This work was done while the second author was working as a Visiting Professor in the Postgraduate Program in Mathematics of the Universidade Federal do Pará, Brazil. He would like to express his profound gratitude to Professor Giovany M. Figueiredo for his warm hospitality. A. Cabada was partially supported by FEDER and Ministerio de Educación y Ciencia, Spain, Project MTM2010-15314. F. J. S. A. Corrêa was partially supported by CNPq, Grant 301603/2007-3.

**References**

[1] A. Cabada, J. A. Cid, and L. Sanchez, “Existence of solutions for elliptic systems with nonlocal terms in one dimension,” *Boundary Value Problems*, vol. 2011, Article ID 518431, 12 pages, 2011.

[2] R. P. Agarwal, M. Bohner, and V. B. Shakhmurov, “Linear and nonlinear nonlocal boundary value problems for differential-operator equations,” *Applicable Analysis*, vol. 85, no. 6-7, pp. 701–716, 2006.

[3] W. Allegretto and A. Barabanova, “Existence of positive solutions of semilinear elliptic equations with nonlocal terms,” *Funkcialaj Ekvacioj*, vol. 40, no. 3, pp. 395–409, 1997.

[4] W. Allegretto and A. Barabanova, “Positivity of solutions of elliptic equations with nonlocal terms,” *Proceedings of the Royal Society of Edinburgh. Section A*, vol. 126, no. 3, pp. 643–663, 1996.

[5] P. Freitas, “A nonlocal Sturm-Liouville eigenvalue problem,” *Proceedings of the Royal Society of Edinburgh. Section A*, vol. 124, no. 1, pp. 169–188, 1994.

[6] D. Liang, J. W.-H. So, F. Zhang, and X. Zou, “Population dynamic models with nonlocal delay on bounded domains and their numerical computations,” *Differential Equations and Dynamical Systems*, vol. 11, no. 1-2, pp. 117–139, 2003.

[7] S. Pečiulyte, O. Stikonienė, and A. Štikonas, “Sturm-Liouville problem for stationary differential operator with nonlocal integral boundary condition,” *Mathematical Modelling and Analysis*, vol. 10, no. 4, pp. 377–392, 2005.

[8] N. Sergejeva, “Fučík spectrum for the second order BVP with nonlocal boundary condition,” *Nonlinear Analysis. Modelling and Control*, vol. 12, no. 3, pp. 419–429, 2007.

[9] H. L. Tidke and M. B. Dhakne, “Existence and uniqueness of mild solutions of second order Volterra integrodifferential equations with nonlocal conditions,” *Applied Mathematics E-Notes*, vol. 9, pp. 101–108, 2009.

[10] F. J. S. A. Corrêa, M. Delgado, and A. Suárez, “Some non-local population models with non-linear diffusion,” *Revista Integración. Temas de Matemáticas, Universidad Industrial de Santander*, vol. 28, no. 1, pp. 37–49, 2010.

[11] N. Dodds, *Non-local differential equations*, Doctoral thesis, Division of Mathematics, University of Dundee, Dundee, Scotland, 2005.

[12] P. Freitas, “Nonlocal reaction-diffusion equations,” in *Differential Equations with Applications to Biology*, vol. 21 of *Fields Institute Communications*, pp. 187–204, American Mathematical Society, Providence, RI, USA, 1999.

[13] J. López-Gómez, “On the structure and stability of the set of solutions of a nonlocal problem modeling Ohmic heating,” *Journal of Dynamics and Differential Equations*, vol. 10, no. 4, pp. 537–566, 1998.
[14] A. Cabada and S. Lois, “Existence results for nonlinear problems with separated boundary conditions,” *Nonlinear Analysis*, vol. 35, no. 4, pp. 449–456, 1999.

[15] D. G. de Figueiredo and E. Mitidieri, “A maximum principle for an elliptic system and applications to semilinear problems,” *SIAM Journal on Mathematical Analysis*, vol. 17, no. 4, pp. 836–849, 1986.

[16] F. J. S. D. A. Corrêa, “On the existence of steady-state solutions of a reaction-diffusion system,” *Nonlinear Times and Digest*, vol. 2, no. 1, pp. 95–106, 1995.

[17] J.-L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Gauthier-Villars, Paris, France, 1969.

[18] H. Brezis and S. Kamin, “Sublinear elliptic equations in $\mathbb{R}^N$,” *Manuscripta Mathematica*, vol. 74, no. 1, pp. 87–106, 1992.

[19] H. Brezis and L. Oswald, “Remarks on sublinear elliptic equations,” *Nonlinear Analysis*, vol. 10, no. 1, pp. 55–64, 1986.

[20] A. Ambrosetti, H. Brezis, and G. Cerami, “Combined effects of concave and convex nonlinearities in some elliptic problems,” *Journal of Functional Analysis*, vol. 122, no. 2, pp. 519–543, 1994.

[21] X.-L. Fan and Q.-H. Zhang, “Existence of solutions for $p(x)$-Laplacian Dirichlet problem,” *Nonlinear Analysis*, vol. 52, no. 8, pp. 1843–1852, 2003.

[22] X. Fan and D. Zhao, “On the spaces $L^{p(x)}$ and $W^{m,p(x)}$,” *Journal of Mathematical Analysis and Applications*, vol. 263, no. 2, pp. 424–446, 2001.