DBI skyrmion, high energy (large s) scattering and fireball production

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Abstract

We analyze the high energy scattering of hadrons in QCD in an effective theory model inspired from a gravity dual description. The nucleons are skyrmion-like solutions of a DBI action, and boosted nucleons give pions field shockwaves necessary for the saturation of the Froissart bound. Nuclei are analogs of BIon crystals, with the DBI skyrmions forming a fluid with a fixed inter-nucleon distance. In shockwave collisions one creates scalar (pion field) “fireballs” with horizons of nonzero temperature, whose scaling with mass we calculated. They are analogous to the hydrodynamic “dumb holes,” and their thermal horizons are places where the pion field becomes apparently singular. The information paradox becomes then a purely field theoretic phenomenon, not directly related to quantum gravity (except via AdS-CFT).
1 Introduction

In a series of papers \cite{1, 2, 3, 4, 5, 6} it was shown that the high energy, small angle (large \(s\), fixed \(t\)) scattering in QCD can be described through a simple cut-off AdS gravity dual a la Polchinski-Strassler \cite{7}, where the AdS space ends at an IR brane. Above the gauge theory Planck scale \(\hat{M}_P = N_c^{1/4} \Lambda_{QCD}\), the scattering is dominated by black hole creation in the gravity dual. The maximal Froissart regime in QCD

\[ \sigma = \frac{\pi}{m^2 \ln^2 s} \quad (1.1) \]

corresponds in the gravity dual to black hole creation that occurs effectively on the IR brane. Then in the gravity dual we have an effectively 4d scattering in a gravity theory with a mass (the KK mass obtained by reducing gravity onto the IR brane), reducing to gravitational shockwave scattering in the large \(s\) limit, and creating black holes. It was shown in \cite{3} that this model exactly matches the 1952 Heisenberg model for the saturation of the Froissart bound \cite{8}, a model in which one analyzes pion field shockwave scattering. Moreover, one needs a nonlinear DBI-type action for the pion field to obtain saturation of the bound. Heisenberg takes the action

\[ S = T(3) \int d^4x \sqrt{1 + \frac{(\partial_\mu \phi)^2}{\Lambda^2} + m^2 \phi^2} \quad (1.2) \]

Because of the matching of the two descriptions of the Froissart saturation, in \cite{6} it was proposed that there should be a pion field “soliton” created in the QCD collisions, radiating thermally at a given temperature, and this object was identified with the fireball observed at RHIC (see also \cite{9, 10} for a different view of how the RHIC fireballs are related to black holes, in \(\mathcal{N} = 4\) SYM).

A different use of the pion field was put forward in the program of Skyrme-like models \cite{11, 12, 13, 14}. There one tries to find the nucleon as a topological solution of an effective pion field action with higher derivatives. The nonlinear (or linear) sigma model action does not admit solitons, but if one adds a higher order correction solitons can be obtained \cite{11, 13, 12}. The original Skyrme model had a particular correction, but one obtains the same results with a large class of corrections. In particular, Pavlovskii \cite{15} has analyzed a DBI-like action, that reduces to the sigma model at low energies and contains an infinite series of higher order corrections, summarized in the DBI square root. The action

\[ S = \frac{1}{2} \int d^4xf^2_{\pi} \Lambda_{QCD}^2 \left[ \sqrt{1 + (e^{-i \tilde{\phi} \cdot \vec{r}/f_{\pi} \partial_{\mu} e^{i \tilde{\phi} \cdot \vec{r}/f_{\pi}})^2}/(\Lambda_{QCD}^2)^2} - 1 \right] \quad (1.3) \]

admits Skyrme-like topological solitons that could be identified with the nucleons.

In this paper we will try to put these two approaches together, and find an effective field theory description that will encompass both the hadrons at rest -nucleons in a Skyrme-like model- and the high energy scattering of hadrons. For the gravity dual description of the latter we often found direct effective field theory interpretation, so now we would like to use the intuition gained from the gravity dual to set up a purely effective field theory description.
of the scattering. We know that QCD itself should give a good description of the scattering, but a first principle QCD description is very hard, and the gravity dual description is suited to an effective field theory description anyway, so that’s what we will find. However, this will not be a usual effective field theory, since if the Skyrme-like soliton only needs a few correction terms to the action, the large $s$ scattering needs an action valid at energies well above the natural cut-off, $\Lambda_{QCD}$.

We will find an action that admits both a Skyrme-like topological soliton of the type of Pavlovskii and solutions with horizons that radiate thermally at a given temperature, corresponding to the “fireballs” observed at RHIC. The fireball solutions will be very similar to the “dumb holes” of Unruh [16], obtained in sonic booms. The hydrodynamic equations of the sonic boom are exactly analogous to the scalar equations of motion in our theory.

The paper is organized as follows. In section 2 we will review the high energy (large $s$) scattering in gravity duals of QCD, in section 3 we will look at various DBI actions and their BIon solutions. In section 4 we will boost the BIon solutions and compare to the boosting in the gravity dual model for QCD scattering. In section 5 we will analyze fireball-like BIon solutions with horizons and compare them with the “dumb holes”, thus calculating their temperature. In section 6 we will look at SU(2) actions and topological solutions, in particular the Pavlovskii solution. Section 7 contains the bottom line of the paper. We present our proposed DBI action, argue for its form from the gravity dual perspective together with QCD arguments and find its “SkyrBIon” solution. In section 8 we look at high energy scattering in our model and in section 9 we conclude.

2 High energy scattering in QCD gravity duals

In this section we review the description of large $s$, fixed $t$ scattering via gravity duals of QCD.

Following Polchinski and Strassler [7], the amplitude for scattering in QCD can be found from a gravity dual by multiplying with wave functions in the extra dimensions and integrating over these extra dimensions

$$A(p) = \int drd\bar{r}\Omega\sqrt{g_{\text{string}}}(\bar{p})\prod_i \psi_i \quad (2.1)$$

We do not know of course the gravity dual of QCD, but we know that it should look like $AdS_5 \times X_5$ space modified in the IR (and maybe in the UV).

$$ds^2 = \frac{\bar{r}^2}{R^2}dx^2 + \frac{R^2}{\bar{r}^2}d\bar{r}^2 + R^2 ds_X^2 = e^{-2y/R}dx^2 + dy^2 + R^2 ds_X^2 \quad (2.2)$$

Polchinski and Strassler proposed that one could obtain a lot of information just by putting a sharp cut-off in the IR (at $r_{\text{min}} \sim R^2 \Lambda$). In the large $s$, fixed $t$ regime, the scattering in the gravity dual is concentrated close to, but not on the IR cut-off (IR brane) [2].

But high $s$, fixed $t$ scattering in a gravitational theory was shown by ’t Hooft [17] to be described by scattering Aichelburg-Sexl gravitational shockwaves [18]

$$ds^2 = 2dx^+ dx^- + (dx^+)^2 \Phi(x^i)\delta(x^+) + dx^2 \quad (2.3)$$
where the function $\Phi$ satisfies the Poisson equation

$$\Delta_{D-2} \Phi(x^i) = -16 \pi G p_0^{D-2}(x^i)$$

(2.4)

He showed that for $s \leq M_{Pl}$ one can treat one particle as a shockwave and the other as a null probe (geodesic), and proposed that above $M_{Pl}$ both particles should be shockwaves, and one will create black holes. This case of $s \gg M_{Pl}$ was analyzed (and the cross section for black hole production was computed) in flat 4 dimensions in [19] and extended to higher dimensions and curved space in [1]. As suggested in [20] on general arguments, it was found that the cross section in flat space grows like a power law

$$\sigma \sim r_H^2 \sim s^{1/3}$$

(2.5)

One can find solutions for gravitational shockwaves in curved spaces of gravity dual type [4], and we find that the solutions still look like [2, 3] in the given background, where now $\Phi$ satisfies the Poisson equation in the background.

In [2, 5] it was found that the behaviour for the cross section for black hole formation in the gravity dual translates into the same kind of behaviour for the QCD cross section, and we have the following regimes.

Above the Planck scale, which in gauge theories corresponds to $\tilde{M}_P = N_c^{1/4} \Lambda_{QCD}$ with $\Lambda_{QCD}$ the scale of the lightest glueball excitation, and in real QCD would be about 1-2 GeV, in the gravity dual we form small black holes, that “feel” only flat space. Correspondingly, one finds that the shockwave profile $\Phi$ is

$$\Phi = \frac{16 \pi G_D}{\Omega_{D-3}(D-4)r^{D-4}} \sim \frac{1}{r^{D-4}}$$

(2.6)

By applying the formalism for shockwave scattering with black hole formation and translating to QCD with [2, 1] one finds that

$$\sigma \sim s^{1/3}$$

(2.7)

both in the gravity dual and in QCD.

As one increases $s$ beyond $E_R = N_c^2/R$ in the gravity dual and above $\tilde{E}_R = N_c^2 \Lambda_{QCD}$ in the gauge theory, about 10 GeV in real QCD, the black holes produced in the gravity dual start to “feel” the curvature of space. One finds that for consistency, in the $AdS_{d+1} \times X_{\tilde{d}}$ gravity dual $X_{\tilde{d}}$ needs to be large (with scale much larger in the IR than that of AdS), and the shape of the shockwave profile becomes

$$\Phi = \frac{K_1 R_n R_n^n}{r^{n}} \sim \frac{1}{r^{2(d-1)+d}} = \frac{1}{r^{11}}$$

(2.8)

and by applying the formalism one finds

$$\sigma \sim s^{1/n} = s^{1/11}$$

(2.9)

both in the gravity dual and in QCD.
Finally, the last regime corresponds to the maximal Froissart behaviour. Above an unknown energy scale $\hat{E}_F$, that depends on the details of the gravity dual in the IR, but in real QCD the experimental data suggests it should be between 100 GeV and 1 TeV, the black holes created in the gravity dual are so large that they reach the IR cut-off and get stuck there (since the scattering in the dual happens mostly near the IR). Thus the gravity dual scattering effectively happens on the 4d IR brane, and creates 4d black holes. The shockwave profile is

$$\Phi(r, y = 0) \simeq R_s \sqrt{\frac{2\pi R}{r}} C_1 e^{-M_1 r} \sim e^{-M_1 r}$$

(2.10)

where $M_1$ is the mass of the lightest graviton excitation: if we are reducing gravity on the IR brane, gravity has a nonzero KK mass $M_1$. Then one finds the gauge theory cross section

$$\sigma_{\text{gauge}} \simeq \bar{K} \frac{\sqrt{2}}{M_1} \ln[0.5 \sqrt{s} M_1 \hat{G}_4]$$

(2.11)

where $\bar{K}$ is a numerical constant depending on the details of the gravity dual, $M_1$ translates to the mass of the lightest QCD excitation and the constant multiplying $\sqrt{s}$ in the log cannot be taken too seriously, as the subleading behaviour of $\sigma$ is modified anyway.

The description of this last Froissart behaviour was shown in [3] to match exactly to the description of the saturation in Heisenberg’s model [8]. Heisenberg says that at high enough energies the Lorentz-contracted hadrons colliding will look like shockwaves, characterized by a transverse size $\sim 1/M_H$. Moreover, at high enough energies, the hadrons effectively “dissolve” in the pion field, which also becomes Lorentz contracted to a shockwave, with transverse size characterized by $1/m_\pi$.

Heisenberg assumes that the “degree of inelasticity” $\alpha (=\mathcal{E}/\sqrt{s}=\text{energy loss/collision energy})$ behaves like the overlap of pion wavefunctions, and the pion wavefunction behaves like $\psi(x^i) \sim e^{-m_\pi x}$. Then one finds that the behaviour of the QCD cross section is given by

$$\sigma = \pi b_{\text{max}}(s)^2 \simeq \frac{\pi}{m_\pi^2} \frac{\sqrt{s}}{<E_0>} \ln^2 <E_0>$$

(2.12)

where $<E_0>$ is the average emitted pion energy. But the average emitted pion energy is found to increase linearly with energy for a free pion action or for an action of the $\lambda \phi^4$ type. Instead, Heisenberg finds that one needs a nonlinear action in order to get an approximately constant $<E_0>$. He takes the DBI action in (1.2) and then he gets $<E_0> \sim m_\pi$.

The same kind of picture appears in the gravity dual, where we collide particles, characterized by some size, but in the high energy limit we find that the only thing of relevance is the gravity field, and we effectively collide gravitational shockwaves, also characterized by a shockwave profile $\Phi$ that has a characteristic size $M_1$, the scale of the lightest excitation.

Until now, we talked in the gravity dual about pure gravity, corresponding to pure gauge theory. But in reality QCD has pions (Goldstone bosons of chiral symmetry) which are much lighter than the scale of the lightest glueball. The discussion of the gravity dual before the onset of the Froissart behaviour is the same, as it is governed still by gravity producing black holes. But if one has a Goldstone boson in the gauge theory, it can be modelled by a radion in the gravity dual (the position of the IR brane can be made dynamical). If the
radion mass is smaller than $M_1$ (the KK graviton mass), the radion will dominate at large enough $s$ (dominated by the IR). The scattering will then produce local brane bending, and the brane will bend significantly, entering the effective scattering region of the gravity dual before the black holes created in this region will become large enough, and thus the cross section at infinite $s$ will be dominated by brane bending. If the radion is more massive than gravity ($M_1$), the brane bending will be small, and the created black holes will reach the brane before the brane reaches the scattering region, and the cross section at infinite $s$ will still be dominated by black holes.

In the physical case of lighter radion (thus lighter pion), at least heuristically the same Froissart behaviour will apply, as shown in [20]. We do not have at our disposal the rigorous arguments of [2] anymore, since those were based on powerful general relativity theorems about horizons, and the collision of shockwaves in scalar field theory is still too hard to solve explicitly, but we will still rely on our gravity intuition for guidance.

One observes that the action for the radion, considered as a brane moving in one (almost flat) direction, is in fact the DBI action

$$S = T(3) \int d^4x \sqrt{1 + (\partial_\mu X)^2/\Lambda^2}$$

where $X$ is the radion (brane position) and $\Lambda$ is related to the string scale. This is exactly the action taken by Heisenberg (1.2), at zero mass, but one could consider as a radion stabilization mechanism giving it a mass according to (1.2). Moreover, if the IR brane is considered to be a D-brane probe, it will also have a U(1) gauge field on it. On static solutions (time independent) and at zero magnetic field, the action can be taken as

$$S = T(3) \int d^4x \sqrt{1 + (\nabla X)^2/\Lambda_1^2 - (\nabla \phi)^2/\Lambda_2^2}$$

where $\phi$ is an electric potential and $\Lambda_1$ and $\Lambda_2$ can be a priori different. We have to remember though that this is the D-brane action in a flat extra dimension, thus the curvature of the gravity dual space will induce higher order corrections to the DBI action. We will discuss them later.

We are thus driven to study the DBI actions with both signs inside the square root, corresponding to either real scalar or electric potential (coming from the original Born-Infeld action).

### 3 DBI actions and BIon solutions

In this section we look at DBI actions and their BIon solutions and try to connect to the picture from the last section.

We saw that both in the Heisenberg model and in the gravity dual description we are driven to consider DBI actions. In the gravity dual shockwave collisions produced black holes, and equivalently we proposed that in QCD pion field shockwave collisions should produce “fireballs”, solutions with horizons radiating thermally. The collision process is
quite complicated, as we saw in the gravity dual case, where we could prove a black hole forms, but we couldn’t calculate its metric, so we took as an approximation that we create a static spherically symmetric black hole. In reality however we have a dynamic complicated process. Similarly now we will take as an approximation that we create static spherically symmetric “fireballs,” thus in this section we will study such solutions.

Moreover, one can ask also whether one can understand the nucleons as being modelled by solutions to the same DBI actions. We will thus look at all static spherically symmetric solutions of the DBI actions. In this section we will set the DBI scales $\Lambda_1$ and $\Lambda_2$ to 1 for simplicity.

**DBI action for 4d YM.** The original action of Born and Infeld [21] is

$$L = \sqrt{-\det(\eta_{\mu\nu} + \frac{F_{\mu\nu}}{\sqrt{2}})} = \sqrt{1 + \frac{1}{2} (\frac{F_{\mu\nu} F^{\mu\nu}}{2} - (\frac{F_{\mu\nu} \ast F^{\mu\nu}}{2}))^2} = \sqrt{1 - \vec{E}^2 + \vec{B}^2 - (\vec{E} \cdot \vec{B})^2}$$

Here $E_i = F_{0i}$, $F_{ij} = \epsilon_{ijk} B_k$. One defines in the usual way

$$\vec{D} = \frac{\partial L}{\partial \vec{E}}$$

Then at $B=0$, the field equation is

$$\nabla \cdot \vec{D} = \rho$$

as in electromagnetism (Maxwell), just that now we have a different definition for $\vec{D}$

$$\vec{D} = -\frac{\vec{E}}{\sqrt{1 - \vec{E}^2}} \Rightarrow \vec{E} = -\frac{\vec{D}}{\sqrt{1 + \vec{D}^2}}$$

Defining the electric potential $\phi$ by $\vec{E} = -\nabla \phi$ we get the DBI-electric lagrangean

$$L = \sqrt{1 - (\nabla \phi)^2}$$

**4d scalar DBI.** We follow the above analysis closely. In the static case ($\partial_t = 0$), we define

$$\vec{F} = \vec{\nabla} X$$

The Lagrangean is then

$$L = \sqrt{-\det(\eta_{\mu\nu} + \partial_\mu X \partial_\nu X)} = \sqrt{1 + (\partial_\mu X)^2} = \sqrt{1 + \vec{F}^2}$$

Defining

$$\vec{C} = \frac{\partial L}{\partial \vec{F}}$$

the equation of motion is

$$\nabla \cdot \vec{C} = \rho$$

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as for the free theory! (analogous to the fact that for BI we had the same equation as for
the Maxwell case) except now the definition of $\vec{C}$ is different

$$\vec{C} = \frac{\vec{F}}{\sqrt{1 + \vec{F}^2}} \Rightarrow \vec{F} = \frac{\vec{C}}{\sqrt{1 - \vec{C}^2}} \quad (3.10)$$

**BIons and catenoids**

For a treatment of solutions to the scalar and electric DBI action see [22, 23, 24]. The
electric BIon (solution to electric DBI), originally found by Born and Infeld, is given by

$$\phi(r) = C \int_{r}^{\infty} \frac{dx}{\sqrt{C^2 + x^4}} \quad (3.11)$$

with the asymptotics (eqs 72-75 in [23])

$$\phi(r) \simeq C \cdot \frac{r}{r \to \infty}; \quad \phi(r) \simeq \text{const.} - r, \quad r \to 0 \quad (3.12)$$

From the asymptotics at infinity, we see that $C = q =$ electric charge. The BIon is found by
noting that the solution to (3.3) with delta function source is the same as for the Maxwell
theory in terms of $\vec{D}$, and then finding $\phi$. As Born and Infeld noted, the advantage is now
that the solution is nonsingular in terms of $\vec{E}$, which reaches its maximum of $\vec{E} = 1$ at $r=0$.
Consequently, also the energy of this solution (with delta function source) is finite, unlike
the case of Maxwell theory.

Analogously, one finds the “catenoid” solution, i.e. solution to the scalar DBI action, as

$$X(r) = \bar{C} \int_{r}^{\infty} \frac{dx}{\sqrt{x^4 - \bar{C}^2}} \quad (3.13)$$

But this has a horizon at $r = r_0 = \sqrt{C}$, where $\vec{F} = \vec{\nabla}X$ diverges, even though $X$ remains
finite.

**Solutions to the D-brane action**

As shown in [22], one can embed the previous solutions into the U(1) D-brane action,
that on static solutions (and at zero magnetic field) is

$$S = T_{(3)} \int d^4x \sqrt{(1 - (\nabla \phi)^2)(1 + (\nabla X)^2) + (\vec{\nabla} \phi \cdot \vec{\nabla} X)^2} \quad (3.14)$$

For a review of solutions to this action, see also [24, 23]. On top of those, the D-brane action
also has BPS solutions, which were shown to correspond to ($C = \bar{C} \equiv q$)

$$\phi = X = \frac{q}{r} \quad (3.15)$$

and are understood in string theory as fundamental strings attached to the D brane and
stretching all the way to infinity (BPS BIons). It has a singularity at $r=0$, where the
fundamental string is attached, and has infinite energy, because the fundamental string has
finite tension and infinite length.
The BIon solution however has no singularity nor a horizon at \( r = \sqrt{C} \). One solves the equation for \( \vec{D} \) in the usual way (\( \vec{D} = \vec{\nabla} (C/r) \)), and at \( r = \sqrt{C} \), \( |\vec{D}| = 1 \) and \( |\vec{E}| = 1/\sqrt{2} \), and there is no singularity: the electric potential \( \phi \) is continuous and its derivative is also. The BIon has a finite total energy, which is the reason Born and Infeld put it forward in the first place, as they wanted to have an electron (solution with delta function source) with finite classical self-energy. In string theory it is not clear what its interpretation is, but one might not worry about that too much, since it carries no topological charge, so one could maybe have doubts about its stability. The simplest possibility for an interpretation is of a string going through the D-brane, such as not to excite the scalar field.

An interesting property of BIONS is that like charges repel each other, and opposite charges attract (just as in Maxwell theory), and one is able to write down explicitly a Bionic crystal solution (first done by Hoppe [25], see also [23]), in which each charge is surrounded by opposite charges, like in a NaCl crystal of Maxwell theory.

For the “catenoid” solution, at \( r = \sqrt{C} \), \( X \) is finite but not its derivative, and we will call this surface a horizon. We will note later on that this name is justified, and it is very like the horizon of a black hole, but for the moment it just indicates the apparent singularity of the solution. Putting back momentarily an energy scale \( \beta^2 \), i.e. from
\[
S = \beta^{-2} \int \left[ \sqrt{1 + \beta^2 \left( \partial X \right)^2} - 1 \right] + \int X (\bar{C} \delta(r)) \tag{3.16}
\]
we have
\[
C_i = \partial_i \frac{\bar{C}}{r}; \quad F_i = \frac{C_i}{\sqrt{1 - \beta^2 \bar{C}^2}} \tag{3.17}
\]
We can ask the question: can we continue the solution beyond this apparent singularity?

Unlike for a black hole, it was argued in [22] that in this case the only analytic continuation that makes sense is to glue another solution with a different asymptotic space, creating an Einstein-Rosen bridge. In string theory, this corresponds to a brane-antibrane pair, connected by a fundamental string of length \( L \), as in fig.1.

But if we think of the catenoid as being a metastable solution, created in a collision of some type, as we will want later on, one should not have a full extra anti-D-brane, but one should at least be able to continue into something that looks locally as an extra brane (a brane “bubble”) as in fig.2 if not in a different way.

For a black hole, when we reach the horizon, we can analytically continue in two ways: we can either continue to a singularity, OR glue another asymptotic region to create an Einstein-Rosen bridge. So if the catenoid is a scalar analog of a black hole, we would think we should be able to continue the solution to \( r = 0 \) without making an Einstein-Rosen bridge.

However if we naively continue \( \vec{F} \) through \( r = \sqrt{C} \) using the same relation relating it to \( \bar{C} \) (where \( \bar{C} \) is the solution to \( \nabla \cdot \bar{C} = \bar{C} \delta(r) \)), it becomes complex, thus \( X \) becomes complex. So the only possibility of keeping \( \vec{F} \) and \( X \) real is to change the sign of \( (\partial X)^2 \) in the action, effectively changing \( X \) into a \( \phi \) (electric potential)-type variable. Then the continuation is analytic in the sense that \( \bar{C} \) has the same (analytic) expression inside and outside \( r = \sqrt{C} \) (for it, nothing interesting happens at \( r = \sqrt{C} \)). But it is not clear what would be the physical significance of changing the sign of \( (\partial X)^2 \) in the action at the horizon.
Figure 1: D-brane anti-D-brane system connected by a string: two catenoid solutions with different asymptotic regions connected on the horizon

Figure 2: A single D-brane, the continuation past the horizon can be a brane “bubble”
Note that
\[ \phi(\sqrt{C}) = C \int_{\sqrt{C}}^\infty \frac{dx}{\sqrt{x^4 + C^2}} = \sqrt{C} \frac{2\Gamma(5/4)^2}{\sqrt{\pi}} = C \int_0^{\sqrt{C}} \frac{dx}{\sqrt{x^4 + C^2}} \] (3.18)
thus \( \phi(0) = 2\phi(\sqrt{C}) \), whereas
\[ X(\sqrt{C}) = \bar{C} \int_{\sqrt{C}}^\infty \frac{dx}{\sqrt{x^4 - C^2}} = \sqrt{C} \sqrt{\pi} \frac{\Gamma(5/4)}{\Gamma(3/4)} = \sqrt{2} \phi(\sqrt{C})|_{C=\bar{C}} \] (3.19)
so gluing the \( X \) solution to a \( \phi \)-type solution inside the horizon would give a finite displacement \( X(0) \) at \( r=0 \) (\( X \) is the position of the D-brane in the transverse direction).

For the BIon action \( \sqrt{1 - (\vec{\nabla} \phi)^2} \), we have also the solution
\[ \phi(r) = C \int_r^\infty \frac{dx}{\sqrt{C^2 - x^4}} \] (3.20)
which is just the continuation of the “catenoid” solution to \( r < r_0 \), and redefining \( X \rightarrow iX \), thus effectively changing the action. This solution also has a horizon at \( r = r_0 \), where \( \phi \) is finite, but \( \phi' \) is infinite and negative. However, this solution has imaginary action (or energy) (the square root is negative), but is a real solution to the equations of motion.

Finally, note that the BIon and catenoid solutions are defined up to a sign (\( \pm \)), and an additive constant. One could construct more solution that are not real. For instance, \( i \times \) the BIon solution is a solution to the scalar (“catenoid”) action, but that is a trivial observation, reflecting the fact that redefining the scalar by an \( i \) takes us from one action to the other.

For a general static solution of the D-brane action studied in \cite{22}, the equations of motion can be written as
\[ \vec{\nabla} \cdot g_{\mu\nu} \vec{I} = \vec{\nabla} \cdot [\frac{\vec{E}(1 + (\vec{\nabla} X)^2) - \vec{\nabla} X (\vec{E} \cdot \vec{\nabla} X)}{\sqrt{(1 - \vec{E}^2)(1 + (\vec{\nabla} X)^2) + (\vec{E} \cdot \vec{\nabla} X)^2}}] = 0 \]
\[ \vec{\nabla} \cdot [\frac{\vec{\nabla} X + g_{\mu} \vec{I}(\vec{I} \cdot \vec{\nabla} X)}{\sqrt{1 + (\vec{\nabla} X)^2 + g_{\mu}^2 \vec{I}^2 + g_{\mu}^2 (\vec{I} \cdot \vec{\nabla} X)^2}}] = 0 \text{ or} \]
\[ \vec{\nabla} \cdot [\frac{\vec{\nabla} X(1 - \vec{E}^2) + \vec{E} (\vec{E} \cdot \vec{\nabla} X)}{\sqrt{(1 - \vec{E}^2)(1 + (\vec{\nabla} X)^2) + (\vec{E} \cdot \vec{\nabla} X)^2}}] = 0 \] (3.21)

On a spherically symmetric solution (with \( \vec{E} = -\vec{\nabla} \phi \)), the equations become
\[ \frac{r^2 \phi'}{\sqrt{1 - \phi'^2 + X'^2}}' = 0; \quad \frac{r^2 X'}{\sqrt{1 - \phi'^2 + X'^2}}' = 0 \] (3.22)

The solutions of these equations are obtained by integrating them with constants \( C \) and \( \bar{C} \), respectively:
\[ \phi'^2 = \frac{1 + X'^2}{1 + r^4/C^2}; \quad X'^2 = \frac{1 - \phi'^2}{r^4/C^2 - 1} \] (3.23)
from which we obtain the general solution

\[ X' = \frac{\bar{C}}{\sqrt{r^4 + C^2 - \bar{C}^2}} \]

\[ \phi' = \frac{C}{\sqrt{r^4 + C^2 - \bar{C}^2}} \] (3.24)

Thus we see that for \( C > \bar{C} \), both \( \phi \) and \( X \) look like the BIon, being everywhere defined and having finite derivatives at zero. For \( C < \bar{C} \), both \( \phi \) and \( X \) look like the catenoid, having a horizon at a finite \( r \), equal to \((C^2 - \bar{C}^2)^{1/4}\). At \( C = \bar{C} \equiv q \) we have the BPS BIon, which blows up at \( r=0 \).

At \( r=0 \), we can specify \( \bar{\phi}(0) \equiv \phi'(r = 0) \) and \( \bar{X}(0) \equiv X'(r = 0) \), but we have to satisfy

\[ (1 - \bar{\phi}(0)^2 + \bar{X}(0)^2)\phi(0) = (1 - \bar{\phi}(0)^2 + \bar{X}(0)^2)\bar{X}(0) = 0 \] (3.25)

i.e. either \( \bar{\phi}(0) = \bar{X}(0) = 0 \), or \( \bar{\phi}(0)^2 - \bar{X}(0)^2 = 1 \) (for the BPS BIon \( \bar{\phi}(0) = \bar{X}(0) = \infty \), so that solves it). We can check explicitly from the general solution that this equation is true.

The BIon has \( \bar{\phi}(0) = 1 \), thus we can think of any solution with \( X \) as a perturbation around the BIon. Note that this means that the square root in the action is equal to zero at \( r = 0 \). Only for the BPS BIon the square root at \( r = 0 \) is equal to 1.

At \( r = \infty \), by putting

\[ \bar{\phi} \sim \frac{a}{r^n} \quad \bar{X} \sim \frac{b}{r^m} \] (3.26)

we find from the equations of motion that \( n = m = 2 \), as for the separate \( \phi \) and \( X \) theories, thus the interaction doesn’t change that, as it should.

As we saw, if we put \( \bar{\phi} = 0 \), any solution will have a horizon (the unique solution is of catenoid type, which has a horizon). We check that indeed, we can’t have a real solution defined at \( r=0 \), since then \(-\bar{X}(0)^2 = 1 \). However, if \( \bar{\phi} \neq 0 \), we see that the solution has a finite \( \bar{X}(0) = \sqrt{\bar{\phi}(0)^2 - 1} \), and the solution can be extended to infinity without encountering a horizon. This is due to the fact that we can think of these solutions as adding more electric field at \( r=0 \) \((\bar{\phi}(0) - 1)\) to the BIon, in order to compensate for the added scalar \( (\bar{X}(0)) \).

The Hamiltonian of a static configuration \((\bar{X} = 0)\) is

\[ H = \frac{1}{g_p} \int d^3x \left[ \frac{1}{\sqrt{(1 - \vec{E}^2)(1 + (\vec{\nabla} \bar{X})^2) + (\vec{E} \cdot \vec{\nabla} \bar{X})^2}} - 1 \right] \] (3.27)

For the general spherically symmetric 4d static configurations above it is

\[ E = \frac{1}{g} \int d^3x \left[ \frac{1 + X'^2}{\sqrt{1 - \phi'^2 + X'^2}} - 1 \right] = \frac{1}{g} \int d^3x \left[ \frac{r^4 + C^2}{\sqrt{r^4(r^4 + C^2 - \bar{C}^2)}} - 1 \right] \] (3.28)

Specifically, for the catenoid (purely scalar),

\[ gE_X = \int_{r_0}^{\infty} r^2 dr (\sqrt{1 + X'^2} - 1) = \bar{C}^{3/2} I \] (3.29)
where

\[ I = \int_1^\infty x^2 \text{d}x \left[ \frac{x^2}{\sqrt{x^4 - 1}} - 1 \right] \approx 0.770343 \quad (3.30) \]

Note that then \( r_0 = \sqrt{C} \propto (E_X)^{1/3} \), same as for an object of constant energy density. However, the energy density diverges near the horizon, showing that a significant fraction of the energy is concentrated near it. For the BIon we have

\[ gE_\phi = \int_0^\infty r^2 \text{d}r \left[ \frac{1}{\sqrt{1 - \phi'^2}} - 1 \right] = C^{3/2} \int_0^\infty \frac{\text{d}x}{x^2 + \sqrt{x^4 + 1}} = \frac{(\Gamma[1/4])^2}{6\sqrt{\pi}} C^{3/2} \quad (3.31) \]

and now the energy density diverges at \( r=0 \), thus a significant portion of the energy is situated near the origin. Then for the general solution with horizon \( \bar{C} > C \) we have

\[ gE = r_0^3 I + \frac{C^2}{r_0} \sqrt{\frac{\Gamma[5/4]}{\Gamma[3/4]}}; \quad r_0^4 \equiv \bar{C}^2 - C^2 \quad (3.32) \]

and for the general solution with no horizon \( C > \bar{C} \)

\[ gE = \frac{(\Gamma[1/4])^2}{2\sqrt{\pi}r_0} \left[ \frac{r_0^4}{3} + \frac{\bar{C}^2}{2} \right]; \quad \bar{r}_0^4 \equiv C^2 - \bar{C}^2 \quad (3.33) \]

For the BIon \( C \) is an asymptotic U(1) charge, thus will be quantized in the quantum theory, and for the catenoid \( \bar{C} \) is an asymptotic scalar charge, thus again we expect it to be quantized. We see that both the catenoid and the BIon energies go like charge to the power \( 3/2 \), thus higher charge objects will be unstable towards decay onto the lower charge ones \( E(Q_1 + Q_2) > E(Q_1) + E(Q_2) \). This is also valid for the general solution with both \( C \) and \( \bar{C} \) nonzero (if \( C \propto \bar{C} \propto n \), then \( E \propto n^{3/2} \) always).

Due to the nonlinearity of the action, it is hard to construct explicit (separated) multicenter solutions. The classical interaction potential between two solutions would be then given by the difference between the energy of the two-center solutions and the individual energies of the single center solutions,

\[ E(R) = E(1, r_0 = 0; 2, r_0 = R) - E_1 - E_2 \quad (3.34) \]

Given that we can’t construct the multicenter solutions, we can’t calculate the form of the potential, but we can say something about the asymptotic features. At large distances, the interaction becomes (free) Maxwell electromagnetism plus free scalar, thus the potential between two BIONS will be

\[ E_\phi(R) \simeq \frac{Q_1Q_2}{R} \quad (3.35) \]

where \( Q_1 \) and \( Q_2 \) are the electric charges \( (C_1 \) and \( C_2) \), thus repulsive if \( Q_1Q_2 > 0 \) and attractive if \( Q_1Q_2 < 0 \). And the potential between two catenoids will be

\[ E_X(R) \simeq -\frac{\bar{Q}_1\bar{Q}_2}{R} \quad (3.36) \]
thus attractive for like scalar charges $\bar{Q}_1$ and $\bar{Q}_2$ ($\bar{C}_1$, $\bar{C}_2$). For a general solution, it should be the sum of the two potentials.

However, at small enough distances, the two center solution will look approximately like a single center solution with charges equal to the sum of the individual charges, thus the potential will always be repulsive! That is true both for the Bions, and for the catenoids, thus for the general solution, since in all cases the energy goes like charge to the power $3/2$. This is of course for the case when the interacting objects are of the same type (same ratio $\bar{C}/\bar{C}$).

The only exception is the BPS BIon, for which the energy of two center BPS objects exactly equals the sum of the individual BPS objects, i.e. the potential is zero. In that case, the potential is zero at infinity ($E_\phi$ cancels against $E_X$), but what about at zero?

The energy of the BPS BIon is

$$gE = \int d^3x (\vec{\nabla}X)^2 = \int_0^\infty r^2dr \left[ \frac{r^4 + C^2}{r^4} - 1 \right] = C^2 \int_0^\infty \frac{dr}{r^2}$$

(3.37)

thus divergent, and the divergence is the same as in the free scalar theory (or free electromagnetism). Because the energy is divergent, we can’t draw any conclusions from the fact that it behaves like the square of the charge. In fact, the energy of the multicenter solution is exactly equal to the sum of the individual solutions, due to the BPS property. The point is here that one has to regularize the infinite energy, and if one introduces a lower cut-off for $r$, $r \geq \delta$, while keeping also the value of the scalar field, $X(\delta)$, fixed, one gets

$$gE \sim \frac{C^2}{\delta} \propto CX(\delta)$$

(3.38)

thus energy that is linear in the charge C. The divergence signals the fact that we have to take into account quantum theory, and at the quantum level we know the BPS BIon is a BPS string, thus stable, and in string theory $X(\delta)$ has physical significance. Thus the assumption is that $X(\delta)$ is kept fixed, i.e. $X_1(\delta) = X_2(\delta) = X_{1+2}(\delta)$, and one just adds the coefficient C. For a catenoid we have also $X(\bar{r}_0) = \bar{C}^{1/2}\sqrt{\pi}\Gamma[5/4]/\Gamma[3/4] \simeq 1.31103\bar{C}^{1/2}$, thus $E \propto CX(\bar{r}_0)$, but now the energy is finite, and we can treat this object classically, and there is no reason to keep $X(\bar{r}_0)$ fixed, but we can let it vary with $\bar{r}_0$, and there is no interpretation in terms of a string ($\bar{r}_0$ is just a finite “thickness” of the solution).

The same situation of diverging energy would happen if we tried to apply the above logic to a free theory like Maxwell electromagnetism or free scalar theory. For example, for electromagnetism, the classical potential would be

$$E(R) = \int d^4x [E^2(1, r = 0; 2, r = R) - \vec{E}_1^2 - \vec{E}_2^2]$$

$$= \int r^2dr \left[ \frac{Q_1 \hat{r}}{r^2} \cdot \frac{Q_2 (\vec{r} - \vec{R})/|\vec{r} - \vec{R}|}{(\vec{r} - \vec{R})^2} \right]^2 - \left( \frac{Q_1 \hat{r}}{r^2} \right)^2 - \left( \frac{Q_2 (\vec{r} - \vec{R})/|\vec{r} - \vec{R}|}{(\vec{r} - \vec{R})^2} \right)^2$$

$$= 2Q_1Q_2 \int r^2dr \frac{\hat{r} \cdot (\vec{r} - \vec{R})/|\vec{r} - \vec{R}|}{r^2(\vec{r} - \vec{R})^2}$$

(3.39)
and this diverges, and we can only draw the conclusion that $E(R) \sim Q_1 Q_2 / R$ by scaling, which is correct, but nothing about the coefficient or its sign. In fact exactly the same calculation applies for the free scalar, but for scalars the true sign is opposite!

## 4 Boosting BIons; comparison with gravity

In order to have a consistent effective field theory picture for high energy scattering, we should boost the solutions corresponding to the nucleons, creating shockwaves in the pion field. The collision of such shockwaves should create “fireball”-like objects. In this section we will thus analyze the boosting of the DBI solutions from last section. We will then compare with the gravity dual case.

**Boosted “catenoid”**

If one boosts a scalar static solution $f(x, y, z)$, by definition the boosted solution is $g(x', y', z', t') = f(x, y, z)$, thus for a spherically symmetric solution $f(\sqrt{x^2 + y^2 + z^2})$ we have

$$g(x, y, z, t) = f(\sqrt{x^2 + y^2 + \frac{(z - vt)^2}{1 - v^2}}) \quad (4.1)$$

and it thus satisfies the equations

$$\partial_{x'} g = \gamma \partial_z f \big|_{z = \gamma (z' - vt)}; \quad \partial_{y'} g = \partial_y f; \quad \partial_{z'} g = \partial_z f; \quad \partial_{t'} g = -\gamma v \partial_z f \quad (4.2)$$

Let’s then boost the “catenoid”,

$$X(r) = \int_r^\infty \frac{\tilde{C} \ dx}{\sqrt{x^4 - \beta^2 \tilde{C}^2}} \quad (4.3)$$

This is a true scalar, so we just replace $r$ with

$$r = \sqrt{\tilde{r}^2 + \gamma^2 (z - vt)^2}; \quad \tilde{r}^2 = x^2 + y^2 \quad (4.4)$$

Then after the boost

$$\frac{\partial X}{\partial \tilde{r}} = \frac{\tilde{C}}{\sqrt{\tilde{r}^4 - \beta^2 \tilde{C}^2 \tilde{r}}}; \quad \frac{\partial X}{\partial z} = \frac{\tilde{C}}{\sqrt{\tilde{r}^4 - \beta^2 \tilde{C}^2}} \gamma^2 (z - vt) \frac{\gamma^2 (z - vt)}{r};$$

$$\frac{\partial X}{\partial t} = -\frac{\tilde{C}}{\sqrt{\tilde{r}^4 - \beta^2 \tilde{C}^2}} \gamma^2 v (z - vt) \frac{\gamma^2 v (z - vt)}{r} \Rightarrow (\partial X)^2 = \frac{\tilde{C}^2}{r^4 - \beta^2 \tilde{C}^2} \quad \text{as before boost (4.5)}$$

Thus in the limit $v \to 1$ we have the shockwave-like solution

$$X = X(\tilde{r}), \quad z = t \quad 0; \quad \text{otherwise} \quad (4.6)$$

(one could write $X = X(\tilde{r}) \delta(z - t)/\delta(0)$). But at finite $v$, we have

$$\frac{\partial X/\partial z}{\partial X/\partial \tilde{r}} = \gamma \frac{\gamma (z - vt)}{\tilde{r}} \rightarrow \gamma \rightarrow \infty \quad (\text{if as } v \to 1, \gamma (z - vt) \sim \tilde{r}) \quad (4.7)$$
Let’s check the Lorentz invariance of the field distribution by looking at its energy, and proving $E = \gamma E_0$. From

$$S = - \int d^3x \ dt(\sqrt{1 + (\partial_\mu X)^2} - 1) \quad (4.8)$$

we find

$$H = \int d^3x \left[ \frac{1 + (\nabla X)^2}{\sqrt{1 + (\nabla X)^2 - \dot{X}^2}} - 1 \right] \quad (4.9)$$

Then before the boost,

$$E_0 = \int 4\pi r^2 dr \left[ \sqrt{1 + \frac{C^2}{r^4 - \beta^2 C^2}} - 1 \right] = \int 2\pi \tilde{r} d\tilde{r} d\tilde{z} \left[ \sqrt{1 + \frac{C^2}{r^4 - \beta^2 C^2}} - 1 \right] \quad (4.10)$$

and after the boost (changing the variable of integration from $z$ to $z' = \gamma(z - vt)$), we get

$$E = \frac{1}{\gamma} \int 2\pi \tilde{r} d\tilde{r} d\tilde{z}' \left[ \sqrt{1 + \frac{C^2}{r^4 - \beta^2 C^2}} \frac{1}{1 + \frac{\tilde{C}^2 \tilde{r}^2}{r^4 - \beta^2 C^2}} + \gamma \frac{\tilde{z}^2}{r^4 - \beta^2 C^2} - 1 \right] \quad (4.11)$$

where $r^2 = \tilde{r}^2 + \tilde{z}'^2$. We can see at most that at large $\gamma$, $E$ goes like $\gamma^2/\gamma = \gamma$, as it should.

**Boosted BIon**

Now the scalar is actually the electric field, the zeroth component of a vector, thus the action is

$$S = - \int d^3x \ dt(\sqrt{1 - (\nabla \phi - \dot{\vec{A}})^2 + (\nabla \times \vec{A})^2 - ((\nabla \phi - \dot{\vec{A}}) \cdot (\nabla \times \vec{A}))^2} - 1) \quad (4.12)$$

and becomes

$$S = - \int d^3x \ dt(\sqrt{1 - (\partial \phi)^2} - 1) \quad (4.13)$$

only when $\vec{A} = 0$, $\dot{\phi} = 0$, thus we have to boost differently, taking into account that $\phi$ is not a scalar anymore, but now $(\phi, \vec{A})$ is a vector.

Then on a static, purely electric configuration like the BIon ($F_{0i} = E_i, F_{ij} = \epsilon_{ijk} B_k$) we get

$$E'_z(x', y', z', t') = E_z(x, y, z); \ E'_x(x'_\mu) = \gamma E_x(x_\mu); \ E'_y(x'_\mu) = \gamma E_y(x_\mu) \quad (4.14)$$

$$B'_y(x'_\mu) = \gamma v E_x(x_\mu); \ B'_x(x'_\mu) = -\gamma v E_y(x_\mu); \ B'_z = 0$$

Taking $E_i = -\partial_t f(\sqrt{x^2 + y^2 + z^2})$ and boosting it to obtain $E'_i = \partial_{t'} A'_i - \partial_{x'} g(x', y, z', t')$, and if we also boost $(f, \vec{A})$ as a vector (i.e. giving an extra condition, that we don’t change gauge when boosting, which is a nontrivial condition, as this is not the case for gravity, see below), we obtain

$$g(x, y, z, t) = A'_0 = \gamma (A_0 + v A_z) = \gamma A_0 = \gamma f(\sqrt{x^2 + y^2 + \frac{(z - vt)^2}{1 - v^2}}) \quad (4.15)$$
Specifically, for the BIon we have

\[
E'_z = \frac{C}{\sqrt{r^4 + \beta^2 C^2}} \frac{\gamma(z - vt)}{r}; \quad E'_\tilde{r} = \frac{C}{\sqrt{r^4 + \beta^2 C^2}} \frac{\gamma \tilde{r}}{r}; \quad B'_y = \frac{C}{\sqrt{r^4 + \beta^2 C^2}} \frac{\gamma vx}{r} \tag{4.16}
\]

thus in the large \( \gamma \) limit,

\[
\frac{E'_\tilde{r}}{E'_z} = \frac{\tilde{r}}{\gamma(z - vt)} \cdot \gamma \to \infty \text{ if } \gamma(z - vt) \sim \tilde{r}; \quad B'_y \to E'_r \tag{4.17}
\]

If we boost \( \phi \) as the 0 component of a vector, at \( v \to 1 \) we get that \( \phi' = 0 \) if \( z \neq t \), but now \( \phi(z = t) \sim \gamma \to \infty \). Moreover, the width in \( z - vt \) is \( \sim 1/\gamma \), thus

\[
\phi'(\tilde{r}, z, t) = a\phi(\tilde{r})\delta(z - t) \tag{4.18}
\]

with \( a \) a number.

The extremal BIon of Callan and Maldacena \cite{22} is a solution to the action in (3.14). The static BPS solution is

\[
X = \frac{q}{|\tilde{r} - \tilde{r}_0|^{p-2}}; \quad \tilde{E} = \tilde{\nabla} X \tag{4.19}
\]

thus both \( X \) and \( \tilde{E} \) are singular at \( r=0 \). Boosting this will generate the same \( X \) as for the catenoid (with horizon at \( r=0 \)), but now we will have both electric and magnetic fields as well (thus the action in (3.14) is not valid anymore, it needs to be “boosted” as well, i.e. the \( \tilde{B} \) dependence specified).

Note that when we boost the black hole solution to obtain the A-S solution, we write

\[
\begin{align*}
g_{tt}(x', y', z', t') &= 1 - \frac{2MG}{\sqrt{x'^2 + y'^2 + (z' - vt')^2}}; \quad g_{rr} = 1/g_{tt} \tag{4.20} \\
\end{align*}
\]

and then we transform \( g_{tt}, g_{rr} \) to \( g_{\mu'\nu'} \) (and take the limit \( v \to 1 \)), which would be the equivalent of calculating only \( \tilde{E}' \), not \( g \). Moreover, one then has to transform the coordinates also to reach the system where the A-S looks simple. It could be that the same is required here (for BIons and catenoids), but we will not pursue this further.

## 5 BIon solutions and “dumb holes” as fireballs

In \cite{16} it was found that there are analogs of black holes in hydrodynamics, dubbed “dumb holes”. When a configuration of fluid moves at ultrasonic speed it creates horizons that radiate thermally, analogous to the thermal horizons of black holes. The hydrodynamics equations are written for a potential flow, in terms of a scalar potential \( \Phi \). On the other hand, we want to obtain also a thermal horizons in the collision of scalar (\( \Phi \)) shockwaves in DBI theory. We will see that in fact the equations in the two cases are completely similar. We will thus first review the “dumb holes” and then completely parallel the calculation for the DBI scalar, calculating the temperature of scalar “fireballs”. For the “fireballs” we
will take the static spherically symmetric solutions studied in section 3, and we will discuss towards the end whether this is a good approximation to the real dynamical situation.

For ultrasonic fluid flow, the surface where \( v = c \) (the velocity of particles reaches the sound velocity) is a horizon that radiates particles thermally, and can be mapped to a black hole. The fluid has an equation of state \( p = p(\rho) \) and the fluid motion is irrotational \( \vec{\nabla} \times \vec{v} = 0 \), such that one can write \( \vec{v} = \vec{\nabla} \Phi \) (potential flow). The speed of sound is defined by \( c^2 = dp/d\rho \).

The equations of motion are the local pressure equation and the continuity equation, i.e.

\[
\rho \left( \frac{\dot{v} + (\vec{v} \cdot \vec{\nabla})\vec{v}}{2} \right) = -\vec{\nabla}p(\rho); \quad \dot{\rho} + \vec{\nabla} \cdot (\rho\vec{v}) = 0
\]

(5.1)

The first equation is integrated to the Bernoulli equation

\[
\Phi + \frac{\vec{v} \cdot \vec{v}}{2} + h(\rho) = 0; \quad h(\rho) = \int \frac{dp}{\rho}
\]

(5.2)

If we now calculate the fluctuation equations, in variables \( \phi = \delta \Phi \) and \( \psi = \delta \rho/\rho \), we get

\[
\dot{\phi} + \vec{v} \cdot \vec{\nabla} \phi + c^2 \psi = 0
\]

\[
(\frac{d}{dt} + \vec{v} \cdot \vec{\nabla})\psi + (\vec{\nabla} \phi) \cdot \vec{\nabla} \ln \rho + \vec{\nabla}^2 \phi = 0
\]

(5.3)

Eliminating \( \psi \) we get

\[
\frac{\rho}{c^2}(\frac{d}{dt} + \vec{v} \cdot \vec{\nabla})\phi = 0
\]

(5.4)

which is exactly the equation of motion for a scalar field in a curved spacetime, \( \partial_\mu \sqrt{g} g^{\mu\nu} \partial_\nu \phi = 0 \) if the metric is given by

\[
\sqrt{g} g^{\mu\nu} = \rho \left( \frac{1}{c^2} + \frac{v^i v^j}{c^2} - \delta^{ij} \right)
\]

(5.5)

which implies in 4d (after finding \( g_{\mu\nu} \) and defining a new time coordinate by \( d\tau = dt + v^i dx^i/(c^2 - v^2) \))

\[
ds^2 = \rho c [(c^2 - v^2) d\tau^2 - (\delta^{ij} + \frac{v^i v^j}{c^2 - v^2}) dx^i dx^j]
\]

(5.6)

or, in the case of a radial flow

\[
ds^2 = \rho c [(1 - v^2/c^2) c^2 d\tau^2 - \frac{dr^2}{1 - v^2/c^2} - r^2 d\Omega^2]
\]

(5.7)

In the new coordinates, the scalar wave equation is

\[
\partial_\tau \frac{\rho/c^2}{1 - v^2/c^2} \partial_\tau \phi + \partial_i \rho (\frac{v^i v^j}{c^2} - \delta^{ij}) \partial_j \phi = 0
\]

(5.8)

At constant (or negligible variation of) \( \rho \), the fluctuation equation in the original coordinates can be written as

\[
[\partial_t + (\vec{\nabla} \Phi) \cdot \vec{\nabla} + \vec{\nabla}^2 \Phi] [\partial_t + (\vec{\nabla} \Phi) \cdot \vec{\nabla}] \delta \Phi - c^2 \vec{\nabla}^2 \delta \Phi = 0
\]

(5.9)
The horizon is the surface where $v = c \ ((\nabla \Phi)^2 = c^2)$, thus the metric becomes singular and then in the above equation we have 

\[
[\partial_t^2 + ((\tilde{\nabla} \Phi)^2 - c^2)\tilde{\nabla}^2 + ...] \delta \Phi = 0
\]  

(5.10)

thus the coefficient of $\vec{k}^2$ in a fluctuation mode $\vec{k}$ changes sign.

For static fluctuations ($\partial_t = 0$), the equation is

\[-c^2\nabla^2 \delta \Phi + (\nabla^2 \Phi)(\tilde{\nabla} \Phi) \cdot \tilde{\nabla} \delta \Phi + (\tilde{\nabla} \Phi) \cdot \tilde{\nabla}((\tilde{\nabla} \Phi) \cdot \tilde{\nabla} \delta \Phi) = 0
\]  

(5.11)

DBI Lagrangean

The DBI scalar has a Lagrangean $\sqrt{1 + (\tilde{\nabla} \Phi)^2 - (\partial_t \Phi)^2}$. Here we denote the scalar by $\Phi$ just to underline the analogy with the hydrodynamics case. It gives the fluctuation equation on static solutions ($\partial_t \Phi = 0$, $\partial_t \delta \Phi \neq 0$):

\[
\delta[\tilde{\nabla} \frac{\tilde{\nabla} \Phi}{\sqrt{1 + (\tilde{\nabla} \Phi)^2}}] - \frac{1}{\sqrt{1 + (\tilde{\nabla} \Phi)^2}} \partial_t^2 \delta \Phi \\
\approx -\frac{1}{\sqrt{1 + (\tilde{\nabla} \Phi)^2}} \partial_t^2 \delta \Phi + \tilde{\nabla} \cdot (\frac{\tilde{\nabla} \delta \Phi}{\sqrt{1 + (\tilde{\nabla} \Phi)^2}}) - \\
-\tilde{\nabla} \cdot \left(\frac{\tilde{\nabla} \Phi}{(1 + (\tilde{\nabla} \Phi)^2)^{3/2}}\right)((\tilde{\nabla} \Phi) \cdot \tilde{\nabla} \delta \Phi) = 0
\]  

(5.12)

We see that now the equation is very similar to the one above. The horizon is again where the coefficient of $\nabla^2 \delta \Phi$ changes sign, which now means

\[
\frac{1}{\sqrt{1 + (\tilde{\nabla} \Phi)^2}} = \frac{(\tilde{\nabla} \Phi)^2}{(1 + (\tilde{\nabla} \Phi)^2)^{3/2}}
\]  

(5.13)

whose solution is $|\nabla \Phi| = \infty$, which is what we called a horizon for the “catenoid” anyway! Before we thought of the $|\nabla \Phi| = \infty$ solution as a horizon just because of the singularity, but now we see that it is the exact analog of the dumb hole horizon, thus of the black hole horizon.

In fact, we can do more. The fluctuation equation (5.12) can be rewritten in a suggestive form as (using that $\partial_t \Phi = 0$)

\[-\partial_\nu \frac{1}{\sqrt{1 + (\tilde{\nabla} \Phi)^2}} \partial_\nu \delta \Phi + \partial_\nu \frac{1}{\sqrt{1 + (\tilde{\nabla} \Phi)^2}} (\delta^{ij} - \frac{\partial^i \Phi}{\sqrt{1 + (\tilde{\nabla} \Phi)^2}} \frac{\partial^j \Phi}{\sqrt{1 + (\tilde{\nabla} \Phi)^2}}) \partial_\nu \delta \Phi = 0
\]  

(5.14)

which is the same as the black hole equation in the new coordinates (5.8) with the identification

\[c^2 = 1 + (\tilde{\nabla} \Phi)^2, \quad \rho = \frac{1}{\sqrt{1 + (\tilde{\nabla} \Phi)^2}}; \quad \frac{c}{v} = \frac{\partial^i \Phi}{\sqrt{1 + (\tilde{\nabla} \Phi)^2}} \quad \text{(i.e., } v^i = \partial^i \Phi) \]  

(5.15)
Of course, just because $v = c$ when $|\nabla \Phi| = \infty$ it is not a good reason for it to be a thermal horizon. There are many hydrodynamic surfaces where $v = c$ that do not radiate thermally as “dumb holes”, just as there are horizons of finite area, for extremal black holes, that nevertheless are at $T = 0$. The key point is that the variation at the horizon, $dv/dr|_{v=c}$ for dumb holes and $\partial_r g_{tt}|_{r=r_H} \propto k$ for black holes that gives the temperature of the horizon. But the point is that neither for the black holes or the dumb holes, the presence of a singularity is not required, the thermality is a property of the horizon itself. What lies behind the horizon is of no consequence.

**Information trapping and horizons**

One should observe though that the horizon of the scalar fireball doesn’t have the black hole horizon property that nothing can escape from the horizon. It has that property only for scalar field excitations and only in a very limited sense. In general, excitations cannot come out of the horizon because of the same reason that the horizon radiates thermally (the coefficient of $\nabla^2 \phi$, proportional to $v^2 - c^2$, becomes zero at the horizon in a specific way). This is seen as follows.

For a Schwarzschild black hole, the infinite time delay necessary for a massless particle to come out, from the perspective of an outside observer is due to equating $ds^2 = 0$ and finding that then

$$\int dt = \int^R \frac{dr}{c(1 - 2M/r)} \to \infty \text{ as } R \to r_H$$

(5.16)

For the metric in (5.7) this becomes $\int dr/[c(1 - v^2/c^2)]$ being divergent at the horizon, which is easily seen to be due to the fact that $d[c(1 - v^2/c^2)]/dr$ is a nonzero constant at the horizon. This is the same condition that we will find below for the nonzero horizon temperature, if $\rho$ is a constant or satisfies $\rho c(1 - v^2/c^2) \neq$ constant (in our case $\rho$ is not constant but satisfies the latter property). Thus the infinite time delay and finite nonzero temperature are almost always related, as expected, and if it’s true, scalar excitations cannot get out. However, in our case, $d[c(1 - v^2/c^2)]/dr$ is infinite at the horizon, implying a finite time delay, and we will find below an infinite temperature as well.

But any other (nonscalar) excitation can always come out. We might ask how come $\Phi$ fluctuations can’t get out if $d[c(1 - v^2/c^2)]/dr$ is nonzero and finite at the horizon, after all, we are in flat space so it takes a finite time for radiation to come from the horizon. But the point is that the scalar fluctuation equation is the same as for a black hole, meaning the *characteristic* equation for a scalar field perturbation will propagate as a function of the asymptotic coordinates $r, t$ the same way as from a black hole. In other words, the phase and group velocities ($c_{ph}$ and $c_{gr}$) of scalar excitations would tend to zero in those cases and information could not be exchanged with the inside of the horizon.

In our case, substituting $\delta \Phi = A \exp(i(\omega t - kr))$ (spherical waves) in the perturbation equation (5.12) one obtains

$$\omega^2 = \frac{1}{1 + \Phi^2}(k^2 - 3k \frac{\Phi' \Phi''}{1 + \Phi^2}) = k^2 \frac{r^4 - r_0^4}{r^4} + 6k \frac{r_0^4}{r^5}$$

(5.17)

thus we see that the coefficient of $k^2$ becomes zero at the horizon and the coefficient of $k$ is finite. One can easily check that if for our solution we had $\Phi' \sim 1/(r - r_0)$ near the horizon
(which would imply both finite nonzero temperature and infinite “geodesic” time delay) we would get also the coefficient of $k$ to go to zero at the horizon, but slower than the coefficient of $k^2$. As it is, we get the phase and group velocities

$$c_{ph}^2 = \frac{\omega^2}{k^2} = \frac{r^4 - r_0^4}{r^4} + \frac{6 r_0^4}{k r^5} \equiv a + \frac{b}{k} \to \frac{6}{k r_0} \quad \text{as } r \to r_0$$

$$c_{gr} = \frac{d\omega}{dk} = \frac{a + b/(2k)}{a + b/k} \to \frac{1}{2} \sqrt{\frac{b}{k}} \to \frac{1}{2} \sqrt{\frac{6}{k r_0}} \quad \text{as } r \to r_0$$

and we see that infinite k modes have zero phase and group velocities. This is not a relativistic formula ($\omega = ck$, $c = \text{constant}$), but not a nonrelativistic one either (which would generically be of the type $c = c_0 + ak + bk^2 + ...$), but comes from the gravitational (black hole) background. The fact that one could apparently have $c_{ph} > 1$ and $c_{gr} > 1$ is an artefact of our approximation of spherical waves, which clearly only works at the horizon if $k \gg 1/r_0$ ($\lambda \ll r_0$).

Thus high energy perturbations have phase and group velocities going to zero as $1/\sqrt{k}$, hence for them the horizon is indeed impenetrable, and information can be exchanged with increasing difficulty with the inside of the horizon. If we would have infinite “geodesic” time delay ($\Phi' \sim 1/(r - r_0)$ near the horizon), then the phase and group velocities would be zero at the horizon, thus all scalar information would be stuck at the horizon, exactly as for a black hole. Then the phase and group velocities would be again proportional to $1/\sqrt{k}$, but would be multiplied by a factor that vanishes at the horizon.

Finally, notice that if we take the original electromagnetic DBI action for $\phi$ (for the BIon, with the minus sign in the square root), the sign of the two terms in (5.12) is the same, thus one can never cancel them against each other. Thus the original DBI action never admits horizons!

**Temperature calculation**

For a black hole in flat space, the horizon temperature is

$$T = \frac{k}{2\pi}$$

where $k$ is the “surface gravity” of the horizon. For a static, spherically symmetric solution with only $g_{rr}(r)$ and $g_{tt}(r)$ nontrivial (and possibly an $r$-dependent conformal factor for the sphere metric), one can easily calculate that

$$(2k)^2 = \lim_{\text{horizon}} g_{rr}^r (\partial_r g_{tt})^2$$

For a Schwarzschild black hole, $g_{rr} = g_{tt}$ and we calculate that $k = 1/(4MG)$, as known. For the “dumb hole”, using the above map to a curved spacetime (5.5), we get that

$$(2k)^2 = \left\{ \frac{1}{\rho} \partial_r [\rho c(1 - v^2/c^2)] \right\}^2_{v=c} \Rightarrow T = \frac{1}{4\pi \rho} \partial_r [\rho c(1 - v^2/c^2)] |_{v=c}$$

Note that in Unruh’s case, where $\rho$ and $c$ are nonzero and finite at the horizon, one gets $T = (dv/dr|_{v=c})/(2\pi)$, but in our case that is not true.
For the catenoid, we have
\[ \Phi' = \frac{D}{\sqrt{1 - D^2}} = \frac{\bar{C}/r^2}{\sqrt{1 - (\bar{C}/r^2)^2}} \]  
(5.22)
and as we saw we map to the “dumb hole” calculation by defining the velocities (5.15)

\[ \frac{v^2}{c^2} = \frac{(\vec{\nabla}\Phi)^2}{(1 + (\vec{\nabla}\Phi)^2)}; \quad c^2 = 1 + \Phi'^2; \quad \rho = \frac{1}{\sqrt{1 + \Phi'^2}} \]  
(5.23)
and then
\[ 2k = \left| \sqrt{1 + \Phi'^2} \frac{d}{dr} \left[ \frac{1}{1 + \Phi'^2} \right] \right|_{r=r_0} = \left| \sqrt{1 + \Phi'^2} \frac{d}{dr} \left[ \frac{\bar{C}^2}{r^4} \right] \right|_{r=r_0} = \sqrt{\frac{r_0}{4(r - r_0)r_0}} \frac{4}{r_0} \]
\[ \Rightarrow T = \sqrt{\frac{r_0}{4(r - r_0)}} \frac{1}{\pi r_0} \]  
(5.24)
thus the temperature is infinite!

Obviously, the temperature of a system cannot be infinite. There are several possible interpretations. First, we notice that the infinite square root factor in front of the temperature comes from \( \sqrt{1 + \Phi'^2} \), which is equal to \( c \) using the map to black holes, and if \( c \) was a constant, we could have rescaled the time coordinate to get rid of it in the wave equation. As it is, we cannot rescale it away. But \( \sqrt{1 + \Phi'^2} \) is also the energy density of the solution, which cannot become infinite, so there will either be higher order corrections to the action preventing that, or else for the real dynamical (time dependent) process of high energy collisions one cannot approximate by the production of a static spherically symmetric solution. Thus at the classical level, the temperature could be made finite either by higher order corrections in the action, or by deviations from time independence and sphericity. As an example, the fluctuation equation around a time dependent solution is

\[ \frac{\partial_i}{\sqrt{1 + (\vec{\nabla}\Phi)^2 - \Phi^2}} (\delta^{ij} - \frac{\partial^i \Phi}{\sqrt{1 + (\vec{\nabla}\Phi)^2}} \frac{\partial^j \Phi}{\sqrt{1 + (\vec{\nabla}\Phi)^2}} \frac{\partial^i \Phi}{\sqrt{1 + (\vec{\nabla}\Phi)^2}}) \partial_j \delta \Phi \\
+ \frac{\nabla^i \Phi}{\sqrt{1 + (\vec{\nabla}\Phi)^2 - \Phi^2}} \frac{1}{1 + (\vec{\nabla}\Phi)^2 - \Phi^2} \partial_i \delta \Phi \\
+ \partial_i \frac{\nabla^i \Phi}{\sqrt{1 + (\vec{\nabla}\Phi)^2 - \Phi^2}} \frac{1}{1 + (\vec{\nabla}\Phi)^2 - \Phi^2} \nabla^i \delta \Phi \\
- \partial_i \frac{\nabla^i \Phi}{\sqrt{1 + (\vec{\nabla}\Phi)^2 - \Phi^2}} \frac{1 + \Phi^2}{1 + (\vec{\nabla}\Phi)^2 - \Phi^2} \partial_i \delta \Phi = 0 \]  
(5.25)
which will have a modified map to the black hole fluctuation equation. Finally, besides all this possible classical resolutions, an infinite temperature implies an infinite particle production,
thus quantum mechanics could also regulate this behaviour. In any case, we will assume that the infinite factor $\sqrt{1 + \Phi'^2}$ which is proportional, as noted, to the energy density of the solution, is regulated to some large but finite value, and calculate the mass scaling of the temperature coming from $r_0$.

Restoring the energy scale $\hat{M}_P$, we get

$$T \propto \frac{1}{r_0} = \frac{1}{\sqrt{C}} \propto \frac{\hat{M}_P^{4/3}}{M_1^{1/3}}$$

(\sqrt{C} is related to the mass of the catenoid by $M = \hat{M}_P^{3/2}/g$, as we saw in section 3.)

This calculation is valid as long as $r_0 < m_\phi$, thus for $M < \hat{M}_P^4/m_\phi^3$. After that we have to take $m_\phi$ into account. Similarly we then get

$$T = \frac{1}{4\pi} |\sqrt{1 + \Phi'^2} + m^2\Phi^2| \frac{d}{dr} \frac{1 + m^2\Phi^2}{1 + \Phi'^2 + m^2\Phi^2} |_{r=r_1}$$

Near the horizon, the equation of motion for the massive DBI scalar becomes

$$r\Phi''(1 + m^2\Phi^2) + 2\Phi'^2 = 0$$

giving the approximate solution

$$\Phi(r \simeq r_1) \simeq \Phi_1 + \sqrt{r_1(1 + m^2\Phi_1^2)(r - r_1)}$$

This gives then

$$T \simeq \sqrt{1 + m^2\Phi_1^2} \sqrt{\frac{r_1}{4(r - r_1)}} \frac{1}{\pi r_1}$$

At large distances, $\Phi \sim Ce^{-mr}/r$, with $C$ being some power law of the mass. Thus perturbatively, the horizon will be where $\Phi$ becomes of order 1, and if we would apply this perturbative formula we would get $r_1 \sim 1/m\ln(Cm)$. Considering again that the infinite factor $\sqrt{1 + \Phi'^2 + m^2} = \sqrt{1 + m^2\Phi_1^2} \sqrt{r_1/(4(r - r_1))}$, proportional to the energy density at the horizon, will be regulated to a finite value and calculating only the mass scaling with $r_1$ we obtain

$$T \propto \frac{m}{\ln(Cm)} \propto m$$

as we argued in [6] we should get.

Finally, if the temperature does become finite as we argued, by the arguments we already gave, light will take an infinite time to reach the horizon, a fact observed from geodesic arguments in the black hole background and from the vanishing of $c_{ph}$ and $c_{gr}$ in the scalar background.

### 6 SU(2) DBI actions and Pavlovskii’s topological soliton

Based purely on phenomenological grounds, Pavlovskii [15] looked at a DBI action for the SU(2) pions of QCD. It contains an infinite number of higher derivative corrections to the
nonlinear sigma model action, that have a chance of having a Skyrme-like topological solit. Such a soliton was indeed found. Here we will extend that discussion and look for modifications of the SU(2) actions that can fit our purposes. We will carefully look at the numerical solutions in order to generalize in the next section to the case of interest.

The action \[ \mathcal{L} = f^2 Tr \beta^2 \left[ \sqrt{1 + \frac{1}{2\beta^2} L_\mu L^\mu} - 1 \right] \] (6.1)

where \( L_\mu = U^{-1} \partial_\mu U, U = \exp(i \vec{\phi}_I / f) \), and chooses a spherically symmetric configuration ("hedgehog")

\[ U = e^{iF(r)\vec{n} \cdot \vec{\tau}}; \quad \vec{n} = \frac{\vec{r}}{r} \] (6.2)

One should note that the sign inside of the square root is different in [15] due to a local sign mistake. In the spherically symmetric solution used afterwards he has the same signs in terms of \( F \), which is correct but because of the \( i \)'s in \( U \), the sign changes between the above action and the action in terms of \( F \) (i.e. \( L_i L_i = -F'^2 + ... \)).

Then

\[ U = e^{i\vec{n} \cdot \vec{\sigma} F(r)} = \cos F(r) + i\vec{n} \cdot \vec{\sigma} \sin F(r) \] (6.3)

and then for such a static configuration

\[ L_i = U^+ \partial_i U = in_i \vec{n} \cdot \vec{\sigma} (F' - \frac{\sin F \cos F}{r}) + i\sigma_i \frac{\sin F \cos F}{r} - i \frac{\sin^2 F}{r} n_j \epsilon_{ijk} \sigma_k \Rightarrow \]

\[ L_i^2 = -F'^2 - 2 \frac{\sin^2 F}{r^2} \] (6.4)

and the energy functional is \( (E = -S) \) for static solutions

\[ E = 8\pi f^2 \beta^2 \int_0^\infty (1 - R)r^2 dr; \quad R = \sqrt{1 - \frac{1}{\beta^2} \left( \frac{F'^2}{2} + \frac{\sin^2 F}{r^2} \right)} \] (6.5)

And the equation of motion is

\( (r^2 \frac{F'}{R})' = \frac{\sin 2F}{R} \) (6.6)

Explicitly, it gives

\[ (r^2 - \frac{1}{\beta^2} \sin^2 F)F'' + (2rF' - \sin 2F) \]

\[-\frac{1}{\beta^2} (rF'^3 - F'^2 \sin 2F + \frac{3}{r} F' \sin^2 F - \frac{1}{r^2} \sin 2F \sin^2 F) = 0 \] (6.7)

For a good solution, \( F \) should go to zero at infinite \( r \), thus the large \( r \) expansion of such a solution is found to be

\[ F(r) = \frac{a}{r^n} + \frac{b}{r^{n+m}} + \frac{c}{r^{n+p}} \] (6.8)
and then from the equations of motion

\[ F(r) = \frac{a}{r^2} - \frac{a^3}{21r^6} - \frac{a^3}{3\beta^2r^8} \quad (6.9) \]

The first 2 terms in \( F(r) \) appear just from the equation \( r^2F'' + 2rF' - \sin 2F = 0 \), i.e. would appear also from the “linear” action (from the SU(2) sigma model)

\[ S = \int \left[ \frac{(\partial F)^2}{2} + \frac{\sin^2 F}{r^2} \right] = -\int Tr \frac{L_i^2}{2} \quad (6.10) \]

The first term appears when we approximate the sin to first order, and the second when we approximate to second order.

By comparison, for the single scalar DBI action, the first two terms in the expansion would come from the equation \( r^2F'' + 2rF' - rF'^3/\beta^2 \), with the solution

\[ F = \frac{a}{r} - \frac{a^3}{20\beta^2r^5} \quad (6.11) \]

Notice that in first order,

\[ L_\mu L_\mu \simeq - (\vec{\sigma} \cdot \partial_\mu \vec{\pi})^2 = \partial_\mu \pi_i \partial_\mu \pi_i \quad (6.12) \]

which gives on a spherically symmetric solution (the above ansatz)

\[ -F'^2 - 2F^2/r^2 \quad (6.13) \]

Thus for three perturbative scalars, with action

\[ S = \frac{1}{2} \int d^4x \partial_\mu \pi^i \partial_\mu \pi^i \quad (6.14) \]

if one chooses a spherically symmetric (“hedgehog”) solution, with \( \pi_i = n_i F(r), \vec{n} = \vec{r}/r \), one gets

\[ S = \frac{1}{2} \int d^4x (F'^2 + 2F^2/r^2) \quad (6.15) \]

thus the difference in behaviour at infinity between the SU(2) case and the single scalar case comes not from the nonlinearity of the action (it couldn’t), but rather from the spherically symmetric ansatz.

The above action has solutions that have \( F(0) = N\pi \). Such a solution with \( N \geq 1 \), if it also goes to zero at infinity, will have a nonzero topological charge

\[ B = \frac{1}{24\pi^2} \int d^3x \epsilon_{ijk} Tr(L_i L_j L_k) \quad (6.16) \]

If \( F(0) = N\pi \), then we can find from the equations of motion that \( F'(0) = a \) is not constrained, thus parametrizing a set of solutions with \( F = N\pi + ar + o(r^3) \). However, for a single value of \( a \) (negative) is the solution going to 0 at infinity, thus being topological.
In [15], the topological solution was found numerically, by imposing the good behaviour at infinity. For most solutions, one encounters a point where the solution becomes numerically unstable, at a nonzero \( r = r_0 \) satisfying \( F(r_0) = \pm \arcsin(\beta r_0) \), where we can check that \( F'(r_0) = 0 \). Physically, it is clear that these solutions will be continued to \( F(0) = 0, F'(0) > 0 \), thus carrying no topological charge.

We have also analyzed numerically what happens when the asymptotics at zero are correct, i.e. \( F(0) = \pi \) (the other N’s are similar). Numerically, by trial and error, (everything is in units of \( \beta \)) we find that the topological solution has \( a \simeq -0.809466 \). Specifically, for \( F(0.01) = \pi - 0.00809466, F'(0.01) = -0.809466 \). Then \( F(300) \simeq 0 \). Specifically, \( F(10) = 0.0706502 = b/r^2 \Rightarrow b = 7.065 \) and then \( F'(10) = -0.014102 \simeq -2b/r^3 \), but the relation is not true anymore at \( r=100 \), showing that this is an approximate solution. This topological solution is given in fig.[9] with a detail in fig.[1].

Note that this value for \( a \) is very close to the value for which the square root argument becomes negative, thus the energy becomes imaginary:

\[
F \simeq \pi - ar + \ldots \Rightarrow \sqrt{1 - \left(\frac{F'^2}{2} + \frac{\sin^2 F}{r^2}\right)} \simeq \sqrt{1 - \frac{3a^2}{2}} \Rightarrow |a| \leq \sqrt{\frac{2}{3}} = 0.8164965809... \quad (6.17)
\]

For \( a \) between the topological solution value \( a \simeq -0.809466 \) and the minimum value \( a = -0.8164965809... \) \( F \) is negative at infinity, but the numerics are unstable, so it is not clear when it is actually reached. As an example, for \( F(0.01) = \pi - 0.008096, F'(0.01) = -0.8096, F(250) \simeq -0.5 \) and falling (see fig.5), and for \( a = -0.8164965 \) we get \( F(300) \) flattening out in the neighbourhood of \(-25\) (see fig.6). However, from the equation of motion, we see that the allowed values at infinity are only \( F(\infty) = k\pi/2 \), with \( k \) an integer. Of course, for odd \( k \), we don’t have a solution with topological charge.

For \( a \) larger than the topological value, \( F(\infty) \) is positive, and the same remarks apply. For example, for \( a = -0.8093, F(300) \simeq 0.8 \) and increasing (see fig.6).

Note that from the point of view of the equations of motion there is nothing wrong with \( |a| > 0.8164965809... \), just that the energy (or action) of the solution is imaginary. Then numerically, we can check that we obtain horizons: \( F \) becomes more and more negative, until at a finite point \( F'(r_0) = -\infty \). For example, for \( a = -0.905 \) this happens at about \( r = 0.449 \), for \( a = -0.816502 \) it happens for at about \( r = 18.84 \), etc. (see fig.9).

In conclusion, we see that Pavlovskii’s action admits topological solutions. Most solutions are not topological, thus unstable, and there are no solutions with horizons that have both real energy and good behaviour at infinity.

One can ask what happens when we change the sign of \( \beta^2 \) in the action (6.1). We find that for the solution with the same asymptotics at infinity as Pavlovskii’s topological solution (the first two terms in the asymptotics at infinity do not depend on \( \beta^2 \)), i.e. with \( b \simeq 7.05 \) at \( r = 10 \), specifically \( F(10) = 0.07, F'(10) = -0.0141 \) (see fig.6), we get a horizon at \( r = 1.8310126459463 \) (computer goes out of memory). Reversely, for the same asymptotics at zero as Pavlovskii’s solution, i.e. with \( a = -0.809466 \), for our action, \( F \) goes down to about 1.36 and then goes back up to a maximum slightly above 1.6, asymptoting to \( F(\infty) = \pi/2 = 1.5707963268 \) (see fig.10). The same qualitative behaviour (down to a minimum below \( \pi/2 \), up to a maximum above \( \pi/2 \) and asymptoting to \( \pi/2 \)) is valid for a whole range of
Figure 3: Topological solution

Figure 4: Topological solution, detail

Figure 5: Slightly below topological solution
Figure 6: Limit solution

Figure 7: Slightly above topological solution

Figure 8: Solution with imaginary energy
values of $F'(0) = a$. We can easily check that $F' = 0$ implies $F''(r_0) = sin(2F(r_0))/r_0^2$, thus we have minima below $F = \pi/2$ and maxima above it.

We have tried a large range of parameters $a$ (for asymptotics at $r=0$) and $b$ (for asymptotics at $r = \infty$), but always the same result: If the asymptotics at zero are good, then the solution tends to $\pi/2$ instead of 0 at infinity, thus being non-topological, i.e. unstable. If the asymptotics at infinity are good, then the solution develops a horizon at finite $r$ ($F'(r)$ is infinite), and within computer accuracy, $F$ seems finite, thus having a square root singularity, as expected. A detail of the square root horizon, for $F(r \to \infty) \approx 0.002/r$ is given in fig.11.

We have tried values of $b$ (at $r=10$) varying between 0.02 and 7.05. A few examples: $b=0.02$ has horizon at about $r=0.292$, $b=1$ at about $r=1.044$, $b=2$ at about 1.290, $b=4$ at about 1.574.

Let us now check under which conditions can we have a horizon, with $F(r)$ finite but $F'(r) = \infty$ at a finite $r = r_1$. From the equations of motion one finds

$$F(r \approx r_1) \approx F(r_1) + \sqrt{2(sin^2 F(r_1) - \beta^2 r_1^2)/r_1} (r - r_1) + ...$$ (6.18)

We see that if $\beta^2 > 0$ (Pavlovskii’s case), the solutions with good asymptotics at infinity indeed cannot have horizons. For them $(sin^2 F(r_1) - \beta^2 r_1^2) < 0$ at infinity and it becomes equal to zero at the point where the solution becomes indeterminate. Horizons can be obtained if $r < r_1$, i.e. $F'(r_1) = \infty$ from below, thus for solutions that have good asymptotics at zero. Moreover, for this we needed solutions with imaginary energy (or action). Exactly like for the single DBI scalar whose solutions we described analytically in section 3, these are actually solutions to the action with $\beta^2 < 0$, analytically continued through their horizons.

Indeed, we see now explicitly that in the $\beta^2 < 0$ case, any solution with good asymptotics at infinity will have a horizon at some $r_1$, exactly like in the case of the catenoids studied in section 3. As mentioned, if we analytically continue these solutions past their horizons we obtain solutions for the $\beta^2 > 0$ case with imaginary energy.

We will see what happens to the temperature of the horizon with respect to the calculation in the previous section. We take a more general case of a Lagrangean of the type $\sqrt{f(\Phi) + (\vec{\nabla}\Phi)^2 - \Phi^2}$ such that near the horizon $r_1$ we have $\Phi' \approx \sqrt{f(\Phi)r_1/(2(r - r_1))}$, as is the case here. Then we can easily calculate the fluctuation equation and find that near the horizon it looks the same way as in section 5, with $\sqrt{1 + (\vec{\nabla}\Phi)^2}$ replaced by $\sqrt{f(\Phi) + (\vec{\nabla}\Phi)^2}$, but now we also get a “mass term” for the fluctuation, with

$$m^2 = \frac{f''(\Phi)}{2} - \frac{(f'(\Phi))^2}{4(f(\Phi) + (\vec{\nabla}\Phi)^2)} + \sqrt{f(\Phi) + (\vec{\nabla}\Phi)^2} \vec{\nabla}[\frac{\vec{\nabla}\Phi}{2(f(\Phi) + (\vec{\nabla}\Phi)^2)^{3/2}} f'(\Phi)]$$ (6.19)

Then near the horizon

$$v^2/c^2 = \frac{\Phi'^2}{f(\Phi) + \Phi'^2} \approx \frac{r_1}{4(r - r_1) + r_1}$$ (6.20)

and we get the same result for the temperature as in the single scalar case, namely $T \propto (r_1)^{-1}$ (up to an infinite factor of the energy density), just that $r_1$ is now different.
Figure 9: Solution with horizon ($\beta^2 < 0$); asymptotics at infinity of topological solution

Figure 10: The $\beta^2 < 0$ case; asymptotics at zero of topological solution

In conclusion, for $\beta^2 > 0$ (Pavlovskii’s case) we have a topological solution, but no solutions with horizons, whereas for $\beta^2 < 0$ all solutions with good asymptotics at infinity have horizons, and there are no topological solutions. But we know from physical considerations that we would like an effective field theory for QCD that gives both.

7 Proposed effective DBI action for QCD and “Skyrme-BIon” solution for nucleons

QCD effective action

In fact, the action for a probe U(1) D-brane in the gravity dual geometry, representing the dynamics of the IR cut-off of the geometry, has both a scalar and an electric potential, thus having both signs in the square root. As we saw in section 3, this action admits both BIon solutions, that are localized lumps of electric field analogous to the Skyrme-like soliton in the previous section, and catenoid solutions, that have thermal horizons, whose temperature we have calculated in section 5. The BIons have arbitrary charges (given by their delta
Figure 11: Detail of horizon

function source), but in the quantum theory we expect those to be quantized. Their analog for Pavlovskii’s SU(2) theory, the topological soliton, has already quantized charges, due to topology.

The fact that the IR behaviour of the gravity dual can be encoded in a D-brane probe with a U(1) gauge field on it is not a priori obvious. However, this procedure has been used before to calculate meson masses.

The usual procedure for having topological solitons of Skyrme type representing the nucleons is to write a nonlinear extension of the sigma model action for the SU(2) pions. But then as we saw there seems to be impossible to have solutions with horizons as well, as we know we should, from the gravity dual picture. An action similar to the one of the U(1) D-brane probe would give the correct physics, but why? We have neglected an important fact, namely that the action we are interested in has to be valid at energies beyond $\Lambda_{QCD}$.

We know that in the analysis of QCD confinement one doesn’t hope that the pions are the only effective degrees of freedom. They should be the only relevant degrees of freedom only at large enough distances (comparable with, or larger than $1/m_\pi$). ‘t Hooft \textsuperscript{26} proposed an idea that is now the standard way to understand confinement, the “dual superconductor”. Out of the SU(3) QCD action one, one uses a partial gauge condition that keeps only the abelian projection, to the maximal abelian subgroup, $U(1) \times U(1)$. The normal vacuum of QCD is then a “dual superconductor” background: after a duality transformation on the abelian subgroup $U(1) \times U(1)$ one is in a superconductor background, where the dual gauge field (monopole field in QCD) is in the broken (massive) phase, due to interaction with a dual Higgs field, corresponding to a monopole-antimonopole pair condensate of QCD. This dual Higgs field is of course also massive in the confining vacuum.

Then a quark-antiquark pair of QCD (forming a meson) will serve as endpoints for tubes of confined electric flux, as in a dual type II superconductor, giving a linear potential. In the dual abelian theory, monopole-antimonopole pairs (solitons of that $U(1) \times U(1)$ theory) will have between them magnetic field flux lines confined to tubes. In this dual theory, the gauge field will be nonzero inside the flux tube and zero outside (where it is massive), and the Higgs field will be zero at the core of the flux tube and go to the symmetry-breaking value
on the outside. Nucleons will have three quarks, thus are harder to explain, and one has to invoke some spherically-symmetric version of the meson mechanism. Also, in this picture the quarks are just sources, but if they are fundamental, as in real QCD, they introduce extra light degrees of freedom, the SU(2) pions (in fact, the lightest). Also, the confinement picture changes, as quark-antiquark pairs can break off an electric flux line.

Seiberg and Witten [27] provided the first example of such a mechanism, for the SU(2) \( \mathcal{N} = 2 \) SYM theory broken to the confining \( \mathcal{N} = 1 \) SYM by a soft-breaking susy mass term \( mTr\Phi^2 \). The low energy effective action for the SU(2) \( \mathcal{N} = 2 \) SYM theory is an abelian (U(1)) \( \mathcal{N} = 2 \) susy theory. At the point on the moduli space where monopoles become massless, one has to make a duality transformation and go to a description in terms of monopoles. When one adds the susy-breaking mass term, one is driven to the (former moduli space) point where monopoles are massless, and one finds that the dual Higgs field (monopole-antimonopole field) condenses, making the dual photon massive. This is the picture expected from the 't Hooft idea, just that it is valid only at low energies (on the moduli space). Seiberg and Witten calculate exactly only the low energy theory, given by \( \int d^4x d^4\theta \mathcal{F}(\Psi) \), but higher order corrections (coming from terms like \( \int d^4x d^4\theta d^4\bar{\theta}\mathcal{H}(\Psi, \bar{\Psi}) \), for instance) are not calculated.

So for the effective theory of QCD in the confining vacuum we expect to have a dual abelian (probably \( U(1) \times U(1) \)) theory for a gauge field interacting with a Higgs that gives the photons mass, and also the SU(2) pions (almost Goldstone bosons for the chiral symmetry introduced with the light fundamental quarks). The low energy \( U(1) \times U(1) \) gauge theory is probably complicated (as seen in the Seiberg-Witten example), and the low energy SU(2) pion action is the usual nonlinear sigma model action

\[
S = \int d^4x \frac{f_\pi^2}{4} [TrL_\mu L^\mu + m_\pi^2 Tr(U + U^+ - 2)]
\]  

where as before \( L_\mu = U^{-1}\partial_\mu U \). At high energies we don’t have much information, except from the gravity dual theory. The action for a single scalar pion was of DBI type, as it corresponded to the position of a brane (IR cut-off) in an extra dimension. We know that the QCD pions are an SU(2) triplet, probably corresponding in the dual gravity theory to fluctuations of the IR cut-off (physical brane) in an SU(2)-valued set of directions, like on an \( S^3 \). Thus we want to have an SU(2) generalization of the pion DBI action that reduces to (7.1) at low energies.

For the dual gauge degrees of freedom, we will take a single U(1) (probably the correct theory involves both U(1)’s, but we take the minimal assumption). We will also assume that for the problems at hand, the dual Higgs will only give mass to the dual photon. If the dual superconductor is correct in all details, the Higgs should change from zero inside the nucleons to the symmetry breaking value outside. However, from the gravity dual point of view an action for the Higgs would imply making assumptions about the actual model for gravity dual of QCD. Since we don’t have such a model, we don’t have any idea how such an action would look like. The minimal assumption then is to hope that the dynamics of the Higgs is irrelevant for the nucleons, and either it is approximately constant, or at the surface of the nucleon, where the Higgs would vary, the pion field is more important than
the Higgs. In that case, we could just replace the Higgs with a mass term for the U(1) gauge field. But what would be the high energy behaviour of the U(1) gauge action? From the dual picture, if we can think of the U(1) as the gauge field on a probe D-brane, the action should also be of DBI type. It is also hard to see what else could it be, as the DBI action is the unique nonlinear correction of the Maxwell theory that is both causal and has only one characteristic surface (generically, there are two) [28].

However, that can’t be the gauge action for arbitrary energies either, we expect that above some energy the simplest description of the gauge fields should be in terms of the original gluons. Thus this action should be valid only for U(1) energies smaller than some scale \( \bar{M} \), above which we cannot say anything. The simplest assumption that we will use afterwards, is that effectively \( |\nabla \phi| \leq \bar{M} \), i.e. after it reaches \( \bar{M} \) it stays fixed at that value, but in reality it should be a more complicated condition. The gravity dual picture suggests something similar, as we will discuss in more detail later. Indeed, we have seen that at \( \hat{E}_R \) in the gravity dual the created black holes start feeling the curvature of space. Thus at this energy gravity (dual to the gauge fields) starts feeling new terms in the action, in agreement with having a relation between \( \bar{M} \) and \( \hat{E}_R \).

Thus we propose the action (at zero magnetic field)

\[
S = \frac{f_π^2 M_1^2}{2} \int d^4x \left[ \frac{E^2}{M_2^2} - \left( 1 - \frac{L_i^2}{M_1^2} \right) + \frac{L_0^2}{M_1^2} + \frac{m_π^2}{M_1^2} (U + U^+ - 2) + M_A^2 \phi^2 - 1 \right]
\]

where \( E = \nabla \phi \) and \( M_A \) is the gauge field mass, and \( E_{\text{max}} = \bar{M} \).

The most conservative ansatz for the mass scales is \( M_1 = M_2 = M_A = \Lambda_{\text{QCD}} \), but they need to be only of the same order of magnitude from physical considerations. Most of the time we will assume this ansatz to be true however.

A priori there should be some Chern-Simons term in the action as well, that reduces to the one in the original Skyrme model [12], and since also the D-brane action in the gravity dual will have in general such a term. But we know that it will not contribute for spherically symmetric solutions (which we are interested in), so we will ignore it in the following.

**D-brane action toy model**

To understand the issues better, let us look at the simpler model of the single scalar pion, described by the U(1) D-brane action. This is the case analyzed in the gravity dual, with the D-brane scalar being the position of the IR cut-off (brane) in AdS space. Moreover, we analyzed its solutions in section 3, so now we can interpret them. In section 3 we looked at the massless action, which had as solutions the BIon, the catenoid, the BPS BIon and general solutions interpolating between these three.

In the gravity dual, the cleanest case is the case when the scalar is also absent (pure gravity, the IR cut-off is fixed, i.e. the radion has a very large mass). That would correspond to putting the DBI scalar to zero. Then the BIon should stand for a glueball-type solution (excitation of the theory). The constant \( C \) in the BIon solution can be taken as electric charge (we identify it as such by the behaviour at infinity), thus is quantized, as it should. But if
the gauge field is massive, as we have argued it has to be (being in a dual superconductor background), we will not measure an electric charge at infinity. However we also know that by colliding two glueballs we create black holes in the gravity dual, and eventually these black holes live on the IR cut-off, being effectively 4 dimensional. Thus eventually (in the Froissart regime) in the 4 dimensional collision of two field theory glueballs at very high energy we also expect to create a fireball type solution with a horizon. But we saw that the pure electric action (no scalar) has no solutions with horizons. It must follow that the description of the pure Yang Mills case at large energies cannot be in terms of the pure U(1) DBI action, but must include a true scalar (for the description of a scalar glueball field), which scalar can generate solutions with horizons. This scalar should be relevant above some energy scale $\hat{E}_F$ (the unknown Froissart energy scale for glueballs). It could be for instance that the Higgs field giving mass to the U(1) (which we ignored in this analysis) has also a DBI-like kinetic term with energy scale $\hat{E}_F$. But there is one more modification that one expects to the U(1) DBI action: at energies above the soft Pomeron scale $\hat{E}_R$, in the gravity dual the created black holes start feeling the curvature of the space, thus the effective action for U(1) should get new terms. They will modify the value of $\phi$ only very slightly, but enough to ensure that for instance the energy density at the center of the BIon doesn’t diverge anymore, but stays finite (the energy density is proportional to $1/\sqrt{1-\phi'^2}$).

In the case that we have a nonzero single scalar pion (in the gravity dual the radion has a small mass), the BIon would again correspond to a glueball solution. Now a BIon-like general solution with $C > \bar{C}$ will be also a type of dressed glueball, and by colliding two such solutions (at least for two solutions with opposite charge C so that the result of the collision is neutral) one should create a scalar pion fireball, with a horizon. Such a solution is the catenoid, and catenoid-like solutions with $\bar{C} > C$ if the total charge of the colliding objects is nonzero. But now there should be not only glueballs in the theory, but also hadrons, which have a scalar pion profile and quantized scalar charge (toy version of the baryon charge). We need then $C \geq \bar{C}$ to avoid horizons, and if both have unit values, $C = \bar{C} = 1$, which would imply the BPS BIon is such a hadron. However, we don’t want for hadrons solutions with diverging scalar at zero, so we can either consider $C > \bar{C}$ and find a solution with a given finite value of the scalar at the origin $X(0) = X_0$ (which specifies $\bar{C}$ if C=1), or consider that the action has higher order corrections that limit the value of the derivative $X'(0)$, such that the BPS BIon becomes also finite. In this toy model, the baryon charge is not well defined, but we will see that in the real case we will have a combination of the two proposed solutions, namely we will have a given value at zero $X(0)$ and the scalar derivative $X'(0)$ will be limited in value.

**Solutions and SkyrBIon as nucleon**

We will first neglect the mass terms ($m_\pi^2$ and $m_A^2$ terms) in (7.2). Note first that for $\vec{E} = 0$ we get the Pavlovskii action with $\beta^2 < 0$, case we analyzed in section 6, and for $U=1$ we get the original action of Born and Infeld. Since the fields appear quadratically in the action, the above truncations are consistent, and we can embed any solutions we found in those cases in the action (7.2) with $m_\pi = M_A = 0$.

We can easily check that $(n_i L_i)^2 = -F'^2$, thus for a spherically symmetric solution, for
which $E_i = -\partial_i \phi(r) \sim n_i$, $(E_i L_i)^2 = -\phi^2 F' r^2$ and the action on the solution becomes

$$S = -\int dt \ r^2 dr \sqrt{(1 - \phi'^2)(1 + F'^2 + 2 \frac{\sin^2 F}{r^2}) + \phi'^2 F'^2 - 1}$$

$$= -\int dt \ r^2 dr \sqrt{(1 - \phi'^2)(1 + 2 \frac{\sin^2 F}{r^2}) + F'^2 - 1}$$  \hspace{1cm} (7.3)$$

with equations of motion

$$\left[\frac{r^2 \phi'}{R} + 2 \phi' \sin^2 F \right]' = 0; \quad \left[\frac{r^2 F'}{R} \right]' = \frac{(1 - \phi^2) \sin 2F}{R}$$  \hspace{1cm} (7.4)$$

where $R$ is the square root in the action.

The first equation can be easily integrated with a constant, giving

$$\phi' = \frac{C}{\sqrt{C^2 + r^4(1 + 2 \frac{\sin^2 F}{r^2})}} \sqrt{1 + \frac{F'^2}{1 + 2 \frac{\sin^2 F}{r^2}}}$$

$$R = \sqrt{\frac{r^4(1 + 2 \frac{\sin^2 F}{r^2})}{C^2 + r^4(1 + 2 \frac{\sin^2 F}{r^2})}} \sqrt{1 + F'^2 + 2 \frac{\sin^2 F}{r^2}}$$  \hspace{1cm} (7.5)$$

Then the second equation is (6.7) (for $\beta^2 = -1/2$) with a modified right hand side, i.e.

$$(r^2 + 2 \sin^2 F) F'' + (2r F' - \sin 2F)$$

$$+ 2(r F'^3 - F'^2 \sin 2F + \frac{3}{r} F' \sin^2 F - \frac{1}{r^2} \sin 2F \sin^2 F) =$$

$$= \frac{1 + F'^2 + 2 \sin^2 F/r^2}{1 + 2 \sin^2 F/r^2} \frac{C^2}{C^2 + r^4(1 + 2 \sin^2 F/r^2)} \times$$

$$\times \left[2r F' - \sin 2F + 2 \sin^2 F \left(\frac{F'}{r} - \frac{\sin 2F}{r^2}\right)\right]$$  \hspace{1cm} (7.6)$$

We can now find the behaviour of solutions at $r = 0$ and $r = \infty$. Let’s first assume that $F(0) = N \pi$ is a good vacuum at $r = 0$ and expand around it. Defining $\bar{\phi}_0 = \bar{\phi}(0) = \phi'(0)$ and $\bar{F}_0 = \bar{F}(0) = F'(0)$, we get from (7.5) that

$$\bar{\phi}_0^2 = \frac{1 + 3 \bar{F}_0^2}{(1 + 2 \bar{F}_0^2)}$$  \hspace{1cm} (7.7)$$

Note that this is the condition that $R$, the square root in the action, is zero at $r=0$, the same condition we obtained for the U(1) D-brane action in section 3, in particular for the BIon solution. Thus as for the DBI action studied in section 3, $\bar{\phi}_0 = 1, \bar{F}_0 = 0$ is a solution, and is actually the same solution, the BIon, since for $F=0$ we have the same action, and this is a consistent truncation of the original action.
However, when we try to satisfy (7.6) at nonzero $F$, we see that it is impossible. A Taylor expansion doesn’t work, as the $O(r)$ terms on the left hand side of (7.6) cancel, whereas on the right hand side we get

$$\frac{1 + 3F_0^2}{1 + 2F_0^2} [-2rF_0^3]$$

(7.8)

so that would imply $F_0 = 0$. But then we would get

$$(r^2F'')(0) = -2(rF'^3)(0)$$

(7.9)

which doesn’t have any solution for any nonzero coefficient in the Taylor expansion. A singular behaviour of the type $F - N\pi = ar^\alpha$, with $0 < \alpha < 1$ doesn’t work either, as it will give the equation

$$a^3r^{3\alpha-2}\alpha(\alpha - 1 + i\sqrt{3})(\alpha - 1 - i\sqrt{3}) = 0$$

(7.10)

with no real solution between 0 and 1. The other possible singular behaviour, $F' = a\ln r, F - N\pi \sim ar(\ln r - 1)$ doesn’t give a solution either, the left hand side being of the order $r\ln^2 r$ and the right hand side of the order $r\ln^3 r$.

So if we would only consider (7.5) we would obtain that, as in section 3, $\bar{\phi}_0 = 1 \Rightarrow \bar{F}_0 = 0$, but now moreover $3/2 \geq \bar{\phi}_0^2 \geq 1$, and $\bar{\phi}_0$ increases monotonically with $\bar{F}_0$. Based on the toy model of section 3, we would expect to find solutions corresponding to all these values, however as we saw, (7.6) implies that in fact that all solutions must blow up before reaching $r=0$, as in the single scalar case! (Even in the pure pion case, at $\phi = 0$, we had solutions that went to $F = N\pi$, though they had the wrong asymptotics at infinity, but now even those are excluded).

For the behaviour at $r = \infty$, we find as before, that the noninteracting asymptotics is not modified: Putting (and assuming $F(\infty) = 0$)

$$\bar{\phi} \sim \frac{a}{r^n}; \quad \bar{F} \sim \frac{b}{r^m}$$

(7.11)

we find $n = 2, m = 3$ as for the case where the scalars don’t interact with each other.

For $\phi = 0$ we have as we mentioned Pavlovskii’s action for $\beta^2 < 0$, for which we saw that any solution that has good asymptotics at infinity will have a horizon.

For a general solution (at nonzero $\phi$) with a good behaviour at infinity for $F$, let’s understand the formation of the horizon. If the horizon would be only in $F$, i.e. $F' = \infty$, $F$ finite at $r$ finite, and $\phi'$ finite, we would get

$$F \simeq F_0 + \sqrt{\frac{(r_1^2 + 2\sin^2 F(r_1))(1 - \phi'^2)}{r_1}(r - r_1)}$$

(7.12)

which is equal at $\phi' = 0$ to the condition we had for the Pavlovskii case. However, in fact we can see from (7.3) that if we have $F$ finite, $F'$ infinite at $r$ finite, we also have $\phi'$ infinite. Then we get

$$F \simeq F_0 + \sqrt{\frac{(r_1^2 + 2\sin^2 F(r_1))}{r_1(1 - a)}(r - r_1)}$$

(7.13)
where
\[ a = \frac{C^2}{C^2 + r_1^4(1 + 2\sin^2 F(r_1)/r_1^2)} \left( \frac{1}{1 + 2\sin^2 F(r_1)/r_1^2} \right) < 1 \] (7.14)
thus we always have a solution, as in the Pavlovskii case.

We also deduce that there can be no horizons of this type at \( r=0 \), where \( \phi' \leq \sqrt{3}/2 \) (with \( F \) finite and \( F' \) infinite, since that would need \( \bar{\phi}_0 \leq 1 \), which is excluded), as we have already argued.

Finally, we have done an extensive numerical solution search for the equation (7.6) confirming that there are no solutions that go to \( F = N\pi \) at zero, and the solutions with good asymptotics at infinity have horizons at finite \( r \). As the parameter \( b \) in the asymptotics at infinity of \( F \) in (7.4) is decreased, the position of the horizon \( r_1 \) decreases, and \( F \) increases until a maximum around \( F \approx 1.478 \) for \( b \approx 0.29 \) and \( r_1 \approx 0.172 \) (all this choosing \( C=1 \); see the horizon detail of this solution in fig.12). An example of horizon solution with \( b = 0.6, r_1 \approx 0.48 \) and \( F_m \approx 1.31 \) is given in fig.13 and an example with \( b = 0.02, r_1 \approx 0.001 \) and \( F_m \approx 0.28 \) us given in fig.14 both of them having \( C=1 \). As one increases \( C \), the maximum of \( F \) increases as well: for instance at \( C=3 \), for \( b \approx 1.8 \) we have \( F \approx 2.18 \) at \( r_1 \approx 0.36 \).

The conclusion is that this action (7.2) does not have topological solutions, contrary to the intuition that we gained in the single pion case, the toy model D-brane action. However, we have to remember that we said the action (7.2) is only valid up to \( |\vec{E}_{max}| = \bar{M} \). At \( F=0 \) there is no problem, as then \( |\vec{E}| \) is bounded by \( M_2 \), but in the presence of diverging \( F' \), \( |\vec{E}| \) would diverge as well.

As we mentioned, the simplest assumption is to say \( |\vec{E}| \) becomes constant and equal to \( \bar{M} \) (in reality it could increase at a slower rate and saturate at a higher energy value). But by the relation between \( F' \) and \( \phi' \), \( F' \) will be forced to saturate as well. Indeed, then the first equation in (7.4) can't be easily integrated to give (7.5) anymore, but the modified second equation can be seen to still imply that \( F' \) is infinite only if \( \phi' \) is infinite. We observe from (7.3), valid before saturation, that the closer we are to \( r=0 \) when \( \phi' \) saturates, the larger the saturation value of \( F' \).
Figure 13: Solution with larger $b$ than the maximum

Figure 14: Solution with smaller $b$ than the maximum
Finally, this means that we will nevertheless have a topological solution, reaching \( F = \pi \) because instead of the horizon in \( F \) at finite \( r \) of the action (7.2), we can continue with \( F' = F'_0 > \bar{M} \) down to \( r=0 \). Numerically, this will work if \( \bar{M} \sim 10 \) (in units of \( M_1 = M_2 \)), since as we saw, for the maximum, \( \pi - F_{\text{max}} \simeq 1.7 \) and \( r_1 \simeq 0.17 \) (and moreover, for larger \( r_1 \), \( F_{\text{max}} \) decreases only slightly). Without a complete high energy action, we can’t derive the exact solution unfortunately.

We will call this solution the skyrBIon and it will represent a nucleon.

Before we analyze it further, let us look at a point that may be of concern. Since we avoided the horizon in \( F \) at small \( r \), why do we still have solutions with horizons at all (as we claimed, the solutions with horizons correspond to fireballs created in collisions)? The answer is that we have treated until now the massless case. But of course, we have nonzero masses for the pion and the gauge field. The mass of the gauge field should be around the first glueball mass, thus around \( M_1, M_2 \). The mass term will not affect the solutions at \( r < M_A \sim M_1 \sim 1 \) (in our units), but at larger distances, \( \phi \) will decreases exponentially to a negligible value. Thus if \( F' \) diverges at \( r > 1 \) it will not yet be coupled to the (very small) \( \phi \) field and thus \( F' \) will not be bounded. Only at \( r < 1 \) as for the skyrBIon solution this can happen.

**Interpretation**

Since the proposed skyrBIon solution will represent a nucleon, it has to have the right properties. The topological charge it carries, \( B \), will be interpreted as baryon number, as usual. It can also be positive or negative, corresponding to particles and antiparticles, as usual. It is not clear from what we derived whether there are solutions with higher baryon number, that will depend crucially on the high energy modification of the action. What seems certain is that there aren’t any solutions with arbitrarily high baryon number, there will be a maximum \( B \). That is so if the slope \( F' \) cannot increase indefinitely, since the point \( r_1 \) at which it starts saturating cannot be larger than \( 1/\Lambda_{QCD} \). That is in accordance with the real world, where stable particles (spherically symmetric configurations) of arbitrarily high \( B \) don’t exist, and solutions of \( B > 1 \) if present could represent possible metastable particles. This situation is to be contrasted with, for instance, the case of SU(2) DBI pions of “wrong” sign studied by Pavlovskii, where an arbitrarily high \( B \) was possible. We should mention that by contrast, nuclei (which have higher \( B \)), will not be spherically symmetric configurations, as is true in the real world. We will discuss them shortly.

The skyrBIons have also a (quantized) U(1) electric charge. Due to the mass of the gauge field, it is not a charge detectable at infinity, but we will have skyrBIons of negative U(1) charge with the same \( (\phi, F) \) profiles (the equations of motion are invariant under \( \phi \) reflection), but different U(1) interactions. But that just accounts for the fact that the proton is different than the neutron, even once you take the electromagnetic interactions out! In other words, p-p vs. p-n scattering differs more than by the purely electromagnetic exchange diagrams. So the massive U(1) charge +1 and -1 particles with \( B=+1 \) are the \( p \) and the \( n \).

We also have the BIons in the theory, purely U(1) solutions with an arbitrary quantized charge, but these are easily seen to be understood as scalar glueball excitations, since the massive U(1) should be dual to the SU(3) gauge fields.
Nuclei as BIonic crystals

Finally, how do we understand the nuclei in this picture? It was understood for a long time that in Skyrme-like models, the nuclei should be bound states of nucleons forming sort of a crystalline structure and not by a higher B spherically symmetric soliton. This is indeed what we obtain, as we shall see, with the added bonus that it seems impossible to even have arbitrarily high B solitons anyway.

First, notice that BIons solutions of the massless BI action (U(1) gauge field) can form crystals \[23, 25\] of the NaCl type, i.e. with positive and negative U(1) charges alternating in the crystal. This is maybe not so surprising, but one can write down an exact solution for this. If the BIons in our theory represent glueballs as argued, it would be interesting to see what this fact could imply experimentally for glueballs. Of course, as we said, the gauge field is actually massive, so the BIon crystal would be modified in our theory anyway.

More importantly, let’s see what will happen for the skyrBIon solution. They come with massive U(1) charges +1 and -1, that we argued should correspond to the proton and neutron. At large distances \((r > M_A)\) only the pion interaction is relevant, and it is an attractive interaction, being mediated by perturbative scalars. That is, a two skyrBIon solution at large distance will have a smaller energy than two individual skyrBIons.

As we argued at the end of section 3, at small distances, as long as the energy is finite, we can figure out if the potential is attractive or repulsive from the behaviour as a function of charge. The energy of the static spherically symmetric system in the case of zero mass and no higher order corrections is

\[
H = \int r^2 dr \left[ \frac{1 + F''^2 + 2 \sin^2 \frac{F}{r}}{R} - 1 \right] \tag{7.15}
\]

We see that only the \(\sin^2 \frac{F}{r^2}\) terms stop the action from scaling as charge to the 3/2 power as in section 3, and if the radius where \(F'\) and \(\phi'\) reach their maximum is not too small, this term will not affect too much the scaling. We also see that before even reaching radii smaller than \(1/M_A \sim 1/\Lambda_{QCD}\) and the U(1) field becoming relevant, the pion scalar field becomes nonlinear, and given the analysis in section 3 could already become repulsive. We can thus say that the interaction will certainly turn repulsive at some distance inside \(1/\Lambda_{QCD}\), maybe even before.

Finally, we can say that once we reach the zone of saturation of the derivative of \(\phi\) (at \(r \approx r_1\)) the potential will be clearly repulsive. Indeed, it is probably safe to assume that the energy density will be approximately constant \((\rho_0)\) inside it (for \(r \leq r_1\)). If \(F'\) is approximately constant inside \(r_1\), then \(r_1\) is proportional to the topological charge \(B=N\) (at \(r=0, F = N\pi\)), thus the energy in the core will increase like \(\rho_0 r_1^3 \sim N^3\), thus two unit charge baryons will be strongly repulsed.

Given this picture it is clear that there will be a minimum \(r_C\) for the potential between two nucleons that is at least for radii inside \(1/\Lambda_{QCD} < 0.2 fm\) if not outside it (larger). Moreover, we talked about the interaction of two identical nucleons, but there will clearly be a difference between the U(1) like charge and opposite charge interactions at least at distances of the order of \(r_1\).

All of the above features are in accordance with the real world, where a meson (pion)
interaction potential works down to about 0.4fm after which the interaction potential rises dramatically.

The structure of nuclei is known to be well described by the Bethe-Weizsacker formula for the binding energy of the nucleus

\[ B(A, Z) = a_V A - a_s A^{2/3} - a_C \frac{Z^2}{A^{1/3}} + a_P \delta A^{-3/4} - a_{\text{sym}} \frac{(N - Z)^2}{A} \]  

(7.16)

where \( \delta = +1 \) for \((N,Z)\) even-even, \(-1\) for odd-odd and \(0\) for even-odd or odd-even. The first three terms (volume term, surface tension term and Coulomb interaction) are known to be deduced from a liquid drop model, and the \(a_P\) term is quantum in nature (the nucleons are fermions of spin 1/2). For the liquid drop model a nucleon-nucleon potential with a minimum is probably enough, and that was qualitatively derived already. The last term in the binding energy can also be understood qualitatively in our model. As stated before, in the \(U(1)\) D-brane toy model, the BIons form NaCl-like crystals, with alternating +1 and -1 charges. Thus the minimum energy in this case is given by equal number of +1 charges and -1 charges. The same should be true if we add some (constant, positive) scalar charge to all BIons, such that we keep \(C > \bar{C}\).

It is then very likely that for skyrBIons also the minimum energy configuration is given by equal numbers of \(U(1)\) charges +1 and -1, that we equated with \(n\) and \(p\), thus for \(N=Z\). Thus we expect indeed a term that drops with \((N - Z)^2\) in the binding energy (the fact that it is \((N - Z)^2\) and not some other power is just because a perturbative \(U(1)\) interaction is proportional to the charge squared).

8 High energy scattering models

In this section we will see what we can derive about the high energy scattering of hadrons that are modelled by boosted field theory solutions. We will look both at the toy model solitons and at the skyrBIons.

We will start with a few general observations. If we take two boosted hadrons, represented by boosted solitons, and collide them, we can follow Heisenberg’s argument and find again the cross section:

If we have a coefficient of inelasticity (energy loss is \(E = \alpha \sqrt{s}\))

\[ \alpha \sim \frac{1}{b^n} \]  

(8.1)

then, provided that the action is of DBI type, Heisenberg showed that \(b_{\text{max}}\) is obtained when \(E = \langle E_0 \rangle \sim m_\pi\) (for a linear action he gets \(\langle E_0 \rangle \sim \gamma \sim \sqrt{s}\), thus gets a \(b_{\text{max}}\) independent of \(s\), so it’s not relevant for the high energy behaviour of real hadrons).

Then Heisenberg suggests that \(\alpha\) should be proportional to the pion wavefunction overlap, and the wavefunction is proportional to \(e^{-mr}\), thus \(\alpha \sim e^{-mb}\):

\[ \int d^3 r \psi_1(r) \psi_2(r) \sim \int d^3 r e^{-mr} e^{-m(b-r)} \sim V e^{-mb} \]  

(8.2)
after which one obtains $\sigma \sim \ln^2 s$. In our case though, if $\alpha \sim 1/b^p$, then we find $\sigma \sim s^{1/p}$. But now if $\alpha$ is due to a wavefunction, pion or otherwise, and it satisfies $\psi(r) \sim 1/r^n$, then

$$\alpha \sim \int d^3r \psi_1(r)\psi_2(r) \sim \int d^3r \frac{1}{r^n|\vec{r} - \vec{b}|^n} \sim \frac{1}{b^{2n-3}}$$

(8.3)

thus $p = 2n - 3$, contrasted with the case when $\alpha \sim \psi(b)$ which gives $\alpha \sim 1/r^n$, thus $p = n$. Note that in Heisenberg’s case there is no difference in between the two hypotheses about $\alpha$, nor if we also take derivatives on the wavefunctions, in all cases we get $\alpha \sim e^{-m_b}$. Now however, we get different results. Moreover, there is no good reason why the wavefunction overlap behaviour with $s$ would dominate over the dynamics of the theory (in the Froissart regime, the overlap is exponential, thus dominates).

In the gravity dual case, we had a A-S shockwave perturbation in flat space or $AdS_{d+1} \times X_d$, with

$$|\nabla \Psi(r)| \sim \frac{\sqrt{s}}{r^p}$$

(8.4)

and the horizon was situated where $|\nabla \Psi(r)| = 1$, giving $b_{\text{max}} \sim s^{1/(2p)}$, and since $\sigma \sim \pi b_{\text{max}}^2 \sim s^{1/p}$, it is the same kind of analysis as we have now; if we identify $|\nabla \Psi(r)|$ with $E(r) = \sqrt{s}\alpha(r)$ (it would be also with $\sqrt{s}\psi(r)$ if $\alpha \sim \psi$).

We should note that the quantity of interest is $\sqrt{s}\alpha(r)$, which increases with $\sqrt{s}$. The pion field is a scalar, thus as we noted $\psi(r)$ (and thus $\alpha(r)$ when it is only due to the pion wavefunctions) stays constant as one boosts.

From the collision of high energy boosted solitons (BIons or skyrBIons) it is hard to derive the behaviour of $\alpha(r)$, but we can try to understand at least the energy regimes. All energy regimes of interest obey $\sqrt{s} \gg M \simeq \Lambda_{\text{QCD}}$ and then the soliton becomes a shockwave (corresponding in the dual to $\sqrt{s} > M_P$, when particles become gravitational shockwaves).

Now we will adress specific cases. First, when there are no pions at all (in the gravity dual we only have gravity, the IR cut-off is non-dynamical), as we argued in the last section, we should have the massive U(1) field but in order to generate horizons we should have also a scalar field becoming relevant at the Froissart scale $\hat{E}_F$. We will assume for generality that the mass of the gauge field, $M_A$, (as well as the mass of the new scalar field, $M'_A$, that should be of the same order) is smaller than the DBI scale of the gauge field, $M_1$. Then we have a region $1/M_A > r > 1/M_1$ where we can still neglect the masses.

We have seen in section 5 that if we can ignore the masses of the fields, the DBI action has a solution with a horizon and a temperature that goes like $T \sim M^{-1/3}$. In the collision of two glueballs we will create such a scalar field (Higgs?) fireball, thus there should be some evidence of thermalization in the final state (after the decay of the fireball). However, this object will not be dual to a black hole (but rather to black hole creation integrated over the 5th dimension), so its analysis will be more complicated. It is hard to estimate the behaviour of $\alpha(r)$ and thus of $\sigma(s)$ at this point. It should depend on the massive gauge field profile and on the scalar field (Higgs?) profile. As one continues increasing the energy, eventually the core of the BIon will become relevant. As we argued, when the scattering energy reaches the “soft Pomeron scale” $\hat{E}_R$, in the gravity dual the produced black holes start feeling the curvature of the space, and correspondingly the U(1) DBI action should have high energy
corrections that keep the energy density from diverging at the core of the BIon. As one
boosts the Bions above $\hat{E}_R$, part of the interaction is due to the overlap of the BIon core
with the tail of the second BIon, $\phi \sim \hat{C}_2/r$. Thus the high energy corrections to the BIon
become important and the scattering cross section behaviour should be modified too. When
$r_0 \sim 1/M'_A$, thus when $\sqrt{s} \sim M_P^8/M'_A^3$, one will still create a fireball, but the temperature
will be different. The cross section behaviour however still depends on the dynamics of the
theory, at least until an unknown energy scale $\Lambda_{QCD}$.

We have seen that in all cases (above $M_P$, even before the Froissart bound) we will produce
metastable scalar field solutions with thermal horizons. We have also argued in section 5
that the horizon of the fireball acts like the horizon of a black hole with respect to scalar
excitations, i.e. it will infinitely delay the exchange of information with the outside. This is
exactly true if the temperature of the fireball is nonzero and finite, and if the temperature
is infinite, is is true only for high energy modes (the phase and group velocities go like
$1/\sqrt{s}$).

Another important issue to discuss is the issue of transparency of the scalar “fireball”. We
have seen that in all cases (above $M_P$, even before the Froissart bound) we will produce
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exactly true if the temperature of the fireball is nonzero and finite, and if the temperature
is infinite, is is true only for high energy modes (the phase and group velocities go like
$1/\sqrt{s}$). In [6] we have argued that the “jet quenching” observed at RHIC, which we argued
should be in the Froissart regime, is nothing but the information paradox of black holes,
i.e. information (quantum particles) coming in, and thermal radiation coming out. But
we know that “jet quenching” is not absolute, there is information coming out, even in the
RHIC (Froissart) regime, and certainly before that, so how do we reconcile this with the
observation made that fireballs have information absorbing horizons?
In the gravity dual, it is easy to understand this. One integrates over the fifth coordinate, i.e. the black hole produced in the bulk has quantum fluctuations along the fifth dimension. As one increases the energy, the black hole becomes more classical and gets stuck on the IR brane. Thus particles thrown at the interacting region in 4d field theory can miss the black hole in the fifth dimension due to quantum fluctuations, but that becomes increasingly rare as the energy of the produced black hole increases. But how do we understand this phenomenon in field theory?

First off, we saw that for the catenoid (classical, static, spherically symmetric solution) the temperature is infinite and then the velocities at the horizon go like $\frac{1}{\sqrt{k}}$, thus the infinite information delay at the horizon is only approximate, and valid only for high energy modes. Since the RHIC experiments deals with high energy probes anyway (jets, hence the name “jet quenching”), this is exactly what is observed. But we argued that the temperature becomes finite, due either to time dependence, higher order corrections to the action or quantum effects, in which case the horizon should absorb all information. But while it is true that the fireball horizon absorbs scalar field information, it is also essentially (i.e. even classically) unstable: one has to continue the scalar field solution inside the horizon, and the most likely possibility, the “brane bubble” depicted in fig.2 is unstable, and there seems to be no way to continue to a singularity behind the horizon. Another symptom of the classical instability is the infinite temperature of the simple catenoid, for which we said a possible cure would be through time dependence. The fireball also has a horizon, so it is clearly quantum mechanically unstable also due to the thermal radiation. But unlike the black hole, the absence of a singularity that would crush everything makes it unclear why the information would be destroyed in the first place. A scalar particle entering the metastable scalar fireball would linger at the horizon for a long time, but as the fireball horizon would disappear it could presumably continue forward. If the time scale of the fireball existence is long enough, the particle could thermalize, giving rise to “jet quenching”.

Note therefore that the scalar fireball is a much cleaner example of the information paradox, meaning that quantum particles collide and thermal radiation comes out, in a purely quantum mechanical scattering. This underscores the fact believed in string theory, that the black hole really doesn’t destroy information [29], and the information can be retrieved from the almost thermal radiation coming out. The scalar fireball production is also a first step towards finding a field theory formalism where the temperature is not introduced by hand, but appears due to the time evolution (a zero temperature system creates a finite temperature one through time evolution).

9 Discussion and conclusions

In this paper we have analyzed the nucleons (baryons) and their high energy scattering in the fixed $t$, high $s$ regime from the effective field theory point of view, guided by the gravity dual description developed in [1, 2, 3, 5, 6]. At rest, the nucleons are supposed to be described by the Skyrme picture, as topological solitons of the pion field, and at high enough energy, by colliding shockwaves of the pion field, according to Heisenberg’s model for the saturation of the Froissart bound. We wanted a description that can interpolate between these two
cases, with the nucleons being solitons of some effective action for QCD involving the pion field that when boosted to high energies become the colliding shockwaves.

Both in the Heisenberg model and the gravity dual descriptions, the DBI D-brane action played a major role, so we studied its static solutions, the BIon, the catenoid, the BPS BIon and the solutions interpolating between them. The D-brane action was a toy model for the effective action for QCD, with the solutions interpolating between the BIon and the BPS BIon standing for the nucleons, and the solutions interpolating between the BPS BIon and the catenoid standing for fireballs created in the high energy collision of nucleons. Since at the energy scale $\hat{E}_R$, in the gravity dual the action is changed by the fact that the created black holes start feeling the curvature of space, it follows that at high enough energies the same should be true for the U(1) effective gauge field. Its action should be modified at high energies, thus ensuring for instance that the energy density at the center of the BIon doesn’t diverge. We analyzed the boosted solutions and then the fireballs that will be created in their collision. The “dumb holes” of Unruh [16] are an example of field theory (specifically, hydrodynamics equations) solutions that have thermally emitting “horizons” where $v = c$ and $dv/dr$ is finite, such that (in Unruh’s case, when $\rho$ and $c$ are nonzero and finite at the horizon)

$$T = \frac{1}{2\pi} \frac{dv}{dr}$$

They can be in fact mapped to the black hole solutions, hence their name. We have shown that catenoids are also of the same type, we can also map the fluctuation equation to a black hole equation, and the surface where the scalar field diverges, i.e. where $X' \to \infty$, while $X$ is finite, also has $v = c$ and $dv/dr$ is consistent with nonzero temperature. We calculated that the temperature of the static spherically symmetric solution is actually infinite, but argued that the infinite prefactor should be regulated, either classically or quantum mechanically and then the temperature scales as

$$T \propto \frac{1}{r_0} \propto \frac{M_0^{1/3}}{M^{1/3}}$$

where $r_0$ is the horizon position, $M_0$ is the DBI scale, and $M$ is the mass of the solution. We similarly gave a perturbative argument for the case where the scalar field has a mass $m$, that the temperature of the modified catenoid will be asymptotically (for large mass $M$) proportional to $m$. The horizon of the catenoid can be probed with other types of fields, but scalar excitations propagate as in a black hole background, thus the propagation of information, described by their characteristic surfaces, will be obstructed (infinitely time delayed for high energy modes, the phase and group velocities scale as $1/\sqrt{k}$) as for a black hole horizon, even though it takes a photon a finite time to go to the horizon and back (since we are in Minkowski space). We analyzed the propagation of waves in the scalar background, and proved that if the background corresponds to a black hole of finite nonzero temperature, then light takes an infinite geodesic time to reach the black hole horizon, and correspondingly for the scalar, the phase and group velocities go to zero at the horizon for all modes, besides scaling as $1/\sqrt{k}$.

In the real world, the pions transform under a global SU(2), giving the possibility of having topologically stable solutions, that one identifies with the nucleons in the Skyrme
program. In [15] it was shown that a wrong-sign SU(2) generalization of the DBI action has Skyrme-like solitons. We analyzed this action in detail and found that there are no solutions with horizons, only topological solutions. For the action with the correct sign, we have again analyzed in detail and found that any solution with the correct asymptotics at infinity will have a horizon, and there are no topological solutions. The temperature of the horizons is now also \( T \propto (r_1)^{-1} \), but the position of the horizon, \( r_1 \), is modified with respect to the single scalar case.

The creation and decay of the scalar fireball was found to be a clean laboratory for the black hole information paradox, since the same situation seems to appear: in high energy quantum collisions one creates thermally radiating fireballs, apparently violating the unitarity of quantum mechanics. But this shows that the problem of the information paradox is not due to quantum gravity, but rather to the lack of a unifying field theory formalism where finite temperature can appear in the process of purely quantum mechanical scattering at high energies, as we know happens in the RHIC experiments. There is another argument why the information paradox shouldn’t be related to quantum gravity at all. In [30] a bound for shear viscosity over entropy density was proposed, \( \eta/s \geq 1/(4\pi) \), and in [31] it was shown that the bound is saturated by black holes in gravity duals. But in RHIC collisions, \( \eta/s \) is close to the limit value [32], a fact that we used in [6] to support our assertion that RHIC fireballs are gravity dual black holes living on the IR brane. The gravity dual saturation of the bound was derived using apparently quantum gravity arguments for the thermodynamics of black holes, yet the bound itself is independent of the Planck scale \( M_{Pl} \) (this being the reason why it is possible to have the same bound saturated at RHIC). That would suggest that the thermodynamics of black holes might not have anything to do specifically with quantum gravity, but might just come out of usual field theory.

In section 7, we observed that the correct effective description in the real (QCD) vacuum is given by the “dual superconductor” picture of ’t Hooft. The dual of the maximal abelian subgroup \( U(1) \times U(1) \) of \( SU(3) \) is in a type II superconductor phase, the dual \( U(1) \) gauge fields being massive due to a Higgs field, except for flux tubes between dual monopole-antimonopole pairs representing mesons, with baryons being some spherically symmetric version of this picture. Thus the dual effective picture a la Seiberg-Witten for the nucleons in the usual vacuum should involve the \( U(1) \) gauge fields, the Higgs making them massive and the scalar SU(2) pion fields. Assuming that we can neglect the dynamics of the Higgs and postulating the DBI high energy form of the action based on the gravity dual as well as Heisenberg’s model, we wrote down an effective action for QCD in the usual vacuum, as given in [7]. It is a DBI action for the \( U(1) \) massive gauge field coupled to SU(2) light pions. The action has the nonsingular BIon solution, however, as soon as we add a scalar perturbation at infinity, the solution will be singular at a finite radius, having a “horizon”, with finite scalar \( F \), but infinite \( F' \). As a result, there doesn’t seem to be any topological solution to the action, only solutions with horizons. However, as argued, the DBI action must be modified where the energy density of the \( U(1) \) gauge field diverges, and thus we argued that there will be a solution to the modified action, that we dubbed **skyrBlon**.

We argued that the topological charge \( B \) is the baryon charge, as usual, with \( B=+1 \) being baryons, \( B=-1 \) antibaryons and higher \( B \) solutions, if present, representing states unstable
against decay into B=1 states because of energy considerations. The objects of charge +1 and -1 under the massive U(1) should represent the n and the p (the n and the p have different quark content, which will therefore have different gluon interactions). We argued that the potential between two nucleons should have a minimum at an $r_C$ of the order of $1/\Lambda_{QCD} \sim 0.2 fm$. This feature is enough to guarantee that a nucleus can form from individual nucleons, in a liquid drop-like model. Given that BIons form NaCl-like crystals of alternating charges, we argued that the nuclear energy for skyrBIon nuclei should be minimized for $N=Z$, giving a qualitative agreement with the Bethe-Weizsacker formula.

For the high energy (high s, fixed t) scattering of skyrBIons we could find qualitative agreement with the gravity dual picture, by identifying $|\nabla \Psi(r)|$ with $\mathcal{E}(r) = \sqrt{s} \alpha(r)$, but the particular small power laws obtained in the gravity dual cannot be deduced just from the effective DBI action. In the high energy scattering of two baryon-like solutions of the U(1) D-brane toy model one will create a catenoid= scalar fireball of temperature that goes like $T \propto M^{-1/3}$, where $M$ is the total energy of the collision. In this regime, the fireball does not behave yet like a black hole, except for having a temperature. In the case of skyrBIon collisions, the created fireball solution will be similar. We have argued for the existence of a “soft Pomeron scale” and a Froissart onset scale, as we derived from the gravity dual picture, but we can’t calculate them, except to say that both should be larger than $\hat{M}_P/\Lambda_{QCD}^3 = N\Lambda_{QCD}$. The Froissart regime is associated with production of a scalar fireball of size much larger than $1/m_\pi$, and the temperature of the fireball should be proportional to $m_\pi$ in this asymptotic regime, as argued in [6] from a gravity dual point of view. Then the produced scalar fireball is directly mapped to a gravity dual black hole living on the 4 dimensional IR brane.

Thus as we advocated in [6], we have shown that we can describe the gravity dual picture for high energy scattering completely in terms of field theory, but we have seen the limitations of field theory in terms of calculability. The thermal property of black holes in the gravity dual is easily understood as the thermal property of “horizons” in the effective pion field. The colliding pion field shockwaves are seen as being just boosted versions of nucleons. The nucleons are described as solutions of an effective action, and lead to good qualitative features for the description of nuclei.

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