ASYMPTOTIC BEHAVIOR OF STRONG FELLER SEMIGROUPS

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Abstract. We prove that a weakly ergodic, strong Feller semigroup on the space of measures converges strongly to a projection onto its fixed space. In contrast to a recent result of Gerlach [5, Theorem 3.6] we do not assume the semigroup to be stochastically continuous.

1. Introduction

Important in the study of the asymptotic behavior of transition semigroups of Markov processes is the strong Feller property. A celebrated theorem of Doob asserts that a stochastically continuous Markovian semigroup which is irreducible, has the strong Feller property and admits an invariant measure \( \mu^* \) is stable, i.e. the limit \( \lim_{t \to \infty} T(t) \mu \) exists for all probability measures \( \mu \) on the state space of the Markov process. This limit is to be understood in the total variation norm. Various versions and proofs of Doob’s theorem can be found in [3, 7, 8, 14, 15].

We note that if a semigroup satisfies the hypothesis of Doob’s theorem, then the irreducibility assumption implies that the fixed space of the semigroup is one dimensional, whence we have \( \lim_{t \to \infty} T(t) \mu = \mu^* \) for all probability measures \( \mu \) in this situation.

Recently, Gerlach [5] proved that a stochastically continuous Markovian semigroup with the strong Feller property is stable if and only if it is weakly ergodic, i.e. the Cesàro averages of every orbit converge in the weak topology induced by the bounded, continuous functions. In this case, we have \( \lim_{t \to \infty} T(t) \mu = P \mu \), where \( P \) is the associated ergodic projection. For more information on weak ergodicity of Markovian semigroups we refer to [6].

The objective of the present article is to generalize Gerlach’s result in two ways. First, we weaken the assumption of stochastic continuity to a certain integrability condition. Second, we also drop the assumption that the semigroup is Markovian, merely asking that it is sub-Markovian. This allows us to apply our results also to semigroups whose orbits converge to zero.

Our motivation to consider also Markovian semigroups which fail to be stochastically continuous stems from a recent example in [1]. There, given a bounded, open and Dirichlet regular subset \( \partial \) of \( \mathbb{R}^d \), the authors studied a diffusion process subject to certain nonlocal boundary conditions on the state space \( E := \overline{\partial} \). As it turns out the corresponding Markov semigroup is not stochastically continuous. We should point out here, that in the case where the state space is compact, a strong Feller semigroup is immediately compact and immediately norm continuous (see [1] for details). As a consequence, the asymptotic behavior of the semigroup can be studied by different means than presented here and even stronger results are obtained. However, if the state space is not compact, then a strong Feller semigroup need neither be immediately norm continuous nor immediately compact. Thus, when extending the results of [1] to unbounded domains different tools are needed. In that situation the results obtained here still apply.

2010 Mathematics Subject Classification. 47D07, 60J35.
Key words and phrases. Markovian semigroup, strong Feller property, stability, mean ergodic.
This article is organized as follows. In Section 2 we recall basic definitions and results concerning Markov semigroups which are needed later on. In Section 3 we prove our generalization of Gerlach’s result (Theorem 3.3). In the concluding Section 4, we discuss consequences of this result for the adjoint semigroup and explain how our results can be used in the study of the asymptotic behavior of partial differential equations.

2. Markovian semigroups

Throughout, $E$ denotes a Polish space and $B(E)$ denotes its Borel $\sigma$-algebra. We denote by $\mathcal{M}(E)$, $B_b(E)$ and $C_b(E)$ the spaces of signed measures on $B(E)$, the space of bounded, Borel-measurable functions on $E$ and the space of bounded, continuous function on $E$ respectively. The canonical duality between $\mathcal{M}(E)$ and $B_b(E)$ is denoted by $\langle \cdot, \cdot \rangle$.

A kernel is a map $k: E \times \mathcal{M}(E) \to \mathbb{C}$ such that

(i) for every $x \in E$ the map $A \mapsto k(x, A)$ is a measure on $\mathcal{B}(E)$,

(ii) for every $A \in \mathcal{B}(E)$ the function $x \mapsto k(x, A)$ is measurable and

(iii) $\sup_{x \in E} |k(x, E)| < \infty$, where $|k|(x, \cdot)$ is the total variation of the measure $k(x, \cdot)$.

A kernel is called Markovian if $k(x, \cdot)$ is a probability measure for every $x \in E$. If $k(x, \cdot)$ is a positive measure with $k(x, E) \leq 1$ for every $x \in E$, then $k$ is called sub-Markovian.

Given a kernel $k$, we can define bounded linear operators $T \in \mathcal{L}(\mathcal{M}(E))$ and $T' \in \mathcal{L}(B_b(E))$ by

$$\langle T\mu, f \rangle := \int_E k(x, A) \, d\mu(x) \quad \text{and} \quad \langle T'f, x \rangle := \int_E f(y)k(x, dy).$$

With respect to the canonical duality $\langle \cdot, \cdot \rangle$ these operators are adjoint to each other, i.e. we have $\langle T\mu, f \rangle = \langle \mu, T'f \rangle$ for all $\mu \in \mathcal{M}(E)$ and $f \in B_b(E)$. Operators which are given by equation (2.1) for some kernel $k$ are called kernel operators (on $\mathcal{M}(E)$ resp. on $B_b(E)$). The kernel $k$ is called the associated kernel.

A (sub-)Markovian operator on $\mathcal{M}(E)$ is a kernel operator that is associated with a (sub-)Markovian kernel. A Markovian operator is called strong Feller if we have $T'f \in C_b(E)$ for all $f \in B_b(E)$. It is called ultra Feller if $T'$ maps bounded subsets of $B_b(E)$ to equicontinuous subsets of $C_b(E)$, i.e. for every $c > 0$ the set

$$\{T'f : f \in B_b(E), \|f\|_\infty \leq c\}$$

is equicontinuous. It is well known, see [13, §1.5], that the product of two strong Feller operators is ultra Feller.

We now turn to semigroups. A (sub-)Markovian semigroup on $\mathcal{M}(E)$ is a family $T = (T(t))_{t \geq 0}$ of (sub-)Markovian kernel operators such that

(1) $T(t+s) = T(t)T(s)$ for all $t, s > 0$ and

(2) $t \mapsto \langle T(t)\mu, f \rangle$ is measurable for all $\mu \in \mathcal{M}(E)$ and $f \in B_b(E)$.

A (sub-)Markovian semigroup is called strong Feller if it consists of strong Feller operators. Note that by the semigroup law and the above mentioned result a strong Feller semigroup actually consists of ultra Feller operators.

It follows from Lemma 6.1 and Theorem 6.2 of [11] (on a separable metric space the weak continuity assumption made in [11] Theorem 6.2 can be replaced by the weak measurability assumption (2) in the above definition) that a (sub-)Markovian semigroup $T$ has a Laplace transform consisting of kernel operators in the sense that there exists a family $(R(\lambda))_{\lambda > 0}$ of kernel operators such that

$$\langle R(\lambda)\mu, f \rangle = \int_0^\infty e^{-\lambda t} \langle T(t)\mu, f \rangle \, dt$$
for all \( \lambda \in \mathbb{C}_+ := \{ z \in \mathbb{C} : \text{Re} z > 0 \} \). Moreover, the family \( (R(\lambda))_{\text{Re} \lambda > 0} \) is a pseudoresolvent, i.e. for \( \lambda_1, \lambda_2 \in \mathbb{C}_+ \) the resolvent identity
\[
R(\lambda_1) - R(\lambda_2) = (\lambda_2 - \lambda_1)R(\lambda_1)R(\lambda_2)
\]
holds. Unfortunately \( R(\lambda) \) is not injective in general. However, there exists a unique multivalued operator \( \mathcal{G} \) such that \( (\lambda - \mathcal{G})^{-1} = R(\lambda) \) for all \( \lambda \in \mathbb{C}_+ \), see, e.g., \cite{9} Proposition A.2.4. As usual, a multivalued operator on \( \mathcal{M}(E) \) is a linear subset \( \mathcal{G} \subset \mathcal{M}(E) \times \mathcal{M}(E) \). We write \( \mu \in D(\mathcal{G}) \) and \( \nu \in \mathcal{G}x \) to indicate that \( (\mu, \nu) \in \mathcal{G} \). The operator \( \mathcal{G} \) can be viewed as the generator of the semigroup \( T \) and several important properties which hold true for generators of strongly continuous semigroups have natural generalizations to our setting, see for example \cite{11} Proposition 5.7.

Since \( (R(\lambda))_{\text{Re} \lambda > 0} \) consists of kernel operators, we can also consider the adjoints \( (R(\lambda)')_{\text{Re} \lambda > 0} \) on the space \( B_b(E) \). Also this is a pseudoresolvent and we write \( \mathcal{G}' \) for the multivalued operator for which \( (\lambda - \mathcal{G}')^{-1} = R(\lambda)' \).

In our main result concerning ergodic behavior, we assume the semigroup \( T \) to be weakly ergodic. We recall the relevant definitions and results in the situation of strong Feller semigroups from \cite{6}.

For a strong Feller semigroup \( T \) we denote by \( A_t \) the Cesàro means, i.e. \( A_t \) is the unique operator for which
\[
\langle A_t \mu, f \rangle = \frac{1}{t} \int_0^t \langle T(s)\mu, f \rangle \, ds.
\]
It follows from the results of \cite{11} that \( A_t \) is well-defined and itself a strong Feller operator. The strong Feller semigroup \( T \) is called weakly ergodic if the limit \( \lim_{t \to \infty} A_t \mu \) exists in the \( \sigma(\mathcal{M}(E), C_b(E)) \)-topology for every \( \mu \in \mathcal{M}(E) \) and the limit \( \lim_{t \to \infty} A_t f \) exists in the \( \sigma(C_b(E), \mathcal{M}(E)) \)-topology for every \( f \in C_b(E) \). In this case, there exists a kernel operator \( P \) such that \( \langle A_t \mu, f \rangle \to \langle P \mu, f \rangle \) for all \( \mu \in \mathcal{M}(E) \) and all \( f \in C_b(E) \). The operator \( P \) is called ergodic projection.

We now have the following characterizations.

**Theorem 2.1.** Let \( T \) be a (sub-) Markovian semigroup which is strong Feller. The following are equivalent.

1. \( T \) is weakly ergodic.
2. For every \( x \in E \) the net \( (A_t \delta_x) \) has a \( \sigma(\mathcal{M}(E), C_b(E)) \)-cluster point as \( t \to \infty \). Here, \( \delta_x \) denotes Dirac measure in \( x \).
3. The fixed space \( \text{fix}(T) := \{ \mu \in \mathcal{M}(E) : T(t)\mu = \mu, \forall t > 0 \} \) separates the fixed space \( \text{fix}(T') := \{ f \in C_b(E) : T(t)'f = f, \forall t > 0 \} \), i.e. for every \( f \in \text{fix}(T') \) there exists some \( \mu \in \text{fix}(T) \) with \( \langle \mu, f \rangle \neq 0 \).

In that case, the ergodic projection \( P \) is a projection onto \( \text{fix}(T) \) and \( P' \) is a projection onto \( \text{fix}(T') \).

**Proof.** This follows from \cite{6} Theorem 5.7], noting that a strong Feller semigroup satisfies Hypothesis 5.1 assumed there, which was shown in \cite{6} Proposition 5.4.

### 3. Asymptotic behavior on the space of measures

Consider a Markovian semigroup \( T \) on \( \mathcal{M}(E) \). A probability measure \( \nu \) is called invariant if \( T(t)\nu = \nu \) for all \( t > 0 \). Note that for any measure \( \mu \) we can identify the function \( f \in L^1(\mu) \) with the measure \( \mu f := f \, d\mu \). This identification yields an isometry between \( L^1(\mu) \) and a closed subspace of \( \mathcal{M}(E) \). We may thus view \( L^1(\mu) \) a subspace of \( \mathcal{M}(E) \). For an invariant measure \( \nu \) we have \( T(t)L^1(\nu) \subset L^1(\nu) \) for all \( t > 0 \), see \cite{10} Corollary 4.3]. Moreover, writing \( T(t)(f \, d\nu) = (T_1(t)f) \, d\nu \) we have \( \|T_1(t)f\|_1 \leq \|f\|_1 \) for all \( f \in L^1(\mu) \).

In the proof of Gerlach’s theorem, the assumption that \( T \) is stochastically continuous is only used to prove that if the restriction \( T_1 \) of \( T \) to \( L^1(\nu) \) for an invariant
measure $\nu$ (such that $T_1$ is irreducible) is strongly continuous. This follows from [10] Theorem 4.6. In the following result we prove that this is also true without the assumption of stochastic continuity. This is our main contribution in generalizing Gerlach’s result. We also believe that this result is interesting in its own right.

**Theorem 3.1.** Let $T$ be a Markovian semigroup and $\nu$ be an invariant measure such that $T_1 := T|_{L^1(\nu)}$ is irreducible, i.e. if $I$ is a closed ideal in $L^1(\mu)$ which is invariant under $T_1$, then either $I = \{0\}$ or $I = L^1(\nu)$. Then $T_1$ is strongly continuous.

**Proof.** We denote the Laplace transform of $T$ by $(R(\lambda))_{\lambda > 0}$ and by $\mathcal{G}$ the generator of $T$, i.e. the multivalued operator with $(\lambda - \mathcal{G})^{-1} = R(\lambda)$ for all $\lambda \in \mathbb{C}_+$. Note that $rgR(\lambda) = D(\mathcal{G})$ for every $\lambda \in \mathbb{C}_+$. We denote by $\mathcal{G}'$ the generator of $T'$.

**Step 1:** $D(\mathcal{G}')$ is $\sigma(L^\infty(\nu), L^1(\nu))$ dense in $L^\infty(\nu)$.

For clarity’s sake, in this step we write $[f]$ for the equivalence class of a function $f$ modulo equality $\nu$-almost everywhere. Let $[f] \in L^1(\nu)$ be such that $\langle [g], [f] \rangle_{L^\infty, L^1} = 0$ for all $g \in D(\mathcal{G'})$. We write $\nu_f := f d\nu$. Then $\langle [g], [f] \rangle_{L^\infty, L^1} = \langle g, \nu_f \rangle$ for all $g \in D(\mathcal{G'})$.

Pick $\lambda > 0$. Since $R(\lambda)'h \in D(\mathcal{G'})$ for all $h \in B_b(E)$ and since $T'$ maps $D(\mathcal{G'})$ into itself (see [11] Lemma 5.4), we have $\langle R(\lambda)'h, \nu_f \rangle = \langle T(t)'R(\lambda)'h, \nu_f \rangle = 0$ for all $h \in B_b(E)$ and $t > 0$. Taking the difference of these two expressions and bringing $T(t)'R(\lambda)'$ to the other side, we find

$$\langle h, T(t)(R(\lambda)\nu_f - R(\lambda)\nu_f) \rangle = 0$$

for all $h \in B_b(E)$. By [11] Proposition 5.7 this implies that $0 \in \mathcal{G}(R(\lambda)\nu_f)$.

On the other hand, $\mathcal{G}(R(\lambda)\nu_f) \ni \lambda R(\lambda)\nu_f - \nu_f$, as is easy to see. By linearity, it follows that $\lambda R(\lambda)\nu_f - \nu_f - 0 = R(\lambda)\nu_f = 0 = \ker R(\lambda)$. In particular, $\lambda R(\lambda)\nu_f - \nu_f \in D(\mathcal{G'})$.

It follows from linearity that $\nu_f \in D(\mathcal{G'})$. Moreover,

$$(\lambda - \mathcal{G})\nu_f = (\lambda - \mathcal{G})(\nu_f - \lambda R(\lambda)\nu_f) + \lambda(\lambda - \mathcal{G})R(\lambda)\nu_f$$

$$= (\lambda - \mathcal{G})^2R(\lambda)(\nu_f - \lambda R(\lambda)\nu_f) + \lambda(\lambda - \mathcal{G})R(\lambda)\nu_f$$

$$\geq 0 + \lambda\nu_f,$$

as $(\nu_f - \lambda R(\lambda)\nu_f) \in \ker R(\lambda)$. Consequently, $0 \in \mathcal{G}\nu_f$ and thus $\nu_f \in \text{fix}(T')$. As $\nu_f \ll \nu$, actually $\nu_f \in \text{fix}(T_1)$. But $T_1$ is irreducible, which implies that $\dim \text{fix}(T_1) = 1$ so that $\nu_f = c\nu$. Testing against $1 \in D(\mathcal{G'})$, we find that $0 = \langle 1, \nu_f \rangle = c\langle 1, \nu \rangle = c$.

We have seen that the only $\sigma(L^\infty(\nu), L^1(\nu))$-continuous functional which vanishes on $D(\mathcal{G'})$ is 0. Thus, by the Hahn-Banach theorem, $D(\mathcal{G'})$ is $\sigma(L^\infty(\nu), L^1(\nu))$-dense in $L^\infty(\nu)$.

**Step 2:** For $f \in L^1(\nu) \cap L^\infty(\nu)$, the orbit $t \mapsto T_1(t)f$ is $\sigma(L^1(\nu), L^\infty(\nu))$-continuous.

To see this, we recall that a bounded subset of $L^1(\nu)$ is relatively weakly compact if and only if it is uniformly integrable, see [2] Theorem 4.7.18. If $f \in L^1(\nu)^+ \cap L^\infty(\nu)$, then for any measurable set $A \subset E$, we have

$$\int_A T_1(t)f \, d\nu \leq \int_A T_1(t)\|f\|_{\infty}1 \, d\nu = \|f\|_{\infty}\nu(A),$$

since $\nu$ is invariant. This implies that the set $\{T_1(t)f : t \geq 0\}$ is uniformly integrable, hence relatively weakly compact. In particular, if $t_n \to t$, then the sequence $T_1(t_n)f$ has a weak accumulation point. Passing to a subsequence if necessary, we assume that $T_1(t_n)f$ converges weakly to $g$. Writing $\nu_g = gd\nu$, $\nu_f = fd\nu$ and noting that for $\varphi \in D(\mathcal{G'})$ the orbit $t \mapsto T(t)\nu'$ is $\|\cdot\|_{\infty}$-continuous (this follows from [11]...
Proposition 5.7], we find
\[ \langle \varphi, \nu_g \rangle = \langle \varphi, g \rangle_{L^\infty(\nu), L^1(\nu)} = \lim_{n \to \infty} \langle \varphi, T_1(t_n)f \rangle_{L^\infty(\nu), L^1(\nu)} \]
\[ = \lim_{n \to \infty} \langle T(t_n)\varphi, \nu_f \rangle = \langle T(t)\varphi, \nu_f \rangle = \langle \varphi, T(t)\nu_f \rangle \]
for all \( \varphi \in D(\mathcal{G}) \). As this set is \( \sigma(L^\infty(\nu), L^1(\nu)) \)-dense in \( L^\infty(\nu) \) by the first step, it separates the points in \( L^1(\nu) \) and it follows that \( g = T_1(t)f \). We have thus proved that \( T_1(t_n)f \to T_1(t)f \) weakly.

**Step 3:** We finish the proof.

By what was done so far, \( t \mapsto T_1(t)f \) is weakly continuous for all \( f \) in a dense subset of \( L^1(\nu) \). As in the proof of [4] Theorem II.5.8 one can show that this already implies that \( T_1 \) is strongly continuous. In an effort of being self-contained, we provide the details.

For \( f \in L^1(\nu) \cap L^\infty(\nu) \), the orbit \( s \mapsto T_1(s)f \) is weakly continuous, hence weakly measurable. Since \( E \) is Polish, the space \( L^1(\nu) \) is separable. It follows from Pettis’ measurability theorem, that the orbit \( s \mapsto T_1(s)f \) is strongly measurable. Since \( \|T_1(s)\| \leq 1 \) for all \( s \geq 1 \) the integrals
\[ f_r := \frac{1}{r} \int_0^r T_1(s)f \, ds \]
exist as Bochner integrals. Note that \( f_r \to f \) weakly as \( r \to 0 \). Consider the vector space
\[ D := \text{span}\{f_r : r > 0, f \in L^1(\nu) \cap L^\infty(\nu)\} \]
Then the weak closure of \( D \) contains \( L^1(\nu) \cap L^\infty(\nu) \) and, by the Hahn–Banach theorem, equals the norm closure of \( D \). It follows that \( D \) is dense in \( L^1(\nu) \). In view of the uniform boundedness of the operators \( T_1(t) \) it thus suffices to prove that \( T(t)f \to f_r \) as \( t \to 0 \). This follows immediately from the estimate
\[ \|T(t)f_r - f_r\| = \frac{1}{r} \left| \int_0^r T(t + s)f \, ds - \int_0^r T(s)f \, ds \right| \]
\[ \leq \frac{1}{r} \left( \int_0^r \|T(s)f\| \, ds + \int_0^r \|T(s)f\| \, ds \right) \leq \frac{2t}{r} \|f\| \to 0 \]
as \( t \to 0 \). This finishes the proof. \( \square \)

It is an immediate question whether one needs the assumption that \( T_1 \) be irreducible. For strong Feller semigroups we obtain the following.

**Corollary 3.2.** Let \( T \) be a Markovian semigroup which is strong Feller. If \( \nu \) is an invariant measure, then the restriction of \( T \) to \( L^1(\mu) \) is strongly continuous.

**Proof.** It follows from [4] Theorem 3.1 taking into account Theorem 2.6 of that article, that there exist at most countably many disjoint \( T \)-invariant measures \( \nu_n \) such that \( \nu = \sum \nu_n \) and such that the restriction of \( T \) to \( L^1(\nu_n) \) is irreducible for every \( n \). By Theorem 3.1 the restriction \( T_n \) of \( T \) to the space \( L^1(\nu_n) \) is strongly continuous for every \( n \). Now let \( 0 \leq f \in L^1(\nu) \) be given. Then \( f = \sum f_n \) where the function \( f_n \) belongs to \( L^1(\nu_n) \). Let us denote the restriction of \( T \) to \( L^1(\nu) \) by \( T_1 \). To prove strong continuity of \( T_1(t)f \) at 0, we can first pick \( n_0 \) so large that \( \sum_{n \geq n_0} \|f_n\|_1 \leq \varepsilon \). Since \( \|T_n(t)f_n\|_1 \leq \|f_n\|_1 \), the invariance of \( L^1(\nu_n) \) yields
\[ \|T(0)f - f_n\|_1 = \sum_{n \geq n_0} \|T_n(t)f_n - f_n\|_1 \leq \sum_{n \leq n_0} \|T_n(t)f_n - f_n\| + 2\varepsilon. \]
Since $T_n$ is strongly continuous for every $n$, we can pick $t_0$ so small that $\|T_n(t)f_n - f_n\|_1 \leq \varepsilon/n_0$ for $0 < t \leq t_0$. Thus $\|T(t)f - f\|_1 \leq 2\varepsilon$ for such $t$, proving the strong continuity.

We now come to the announced generalization of Gerlach’s result.

**Theorem 3.3.** Let $T$ be a (sub-)Markovian semigroup which is strongly Feller and weakly ergodic with ergodic projection $P$. Then we have $\lim_{t \to \infty} T(t)\mu = P\mu$ in the total variation norm.

*Proof.* We first note that we can assume without loss of generality that $T$ is Markovian. Indeed, otherwise we attach an extra isolated point $\dagger \notin E$ to $E$, setting $\tilde{E} := E \cup \{\dagger\}$. Then also $\tilde{E}$ is a Polish space. Denoting the kernel associated with $T(t)$ by $k_t$, we define

$$\tilde{k}_t(x, A) := k_t(x, A \cap E) + (1 - k_t(x, E))\delta_\dagger (A)$$

for $x \in E$ and $\tilde{k}_t(\dagger, \cdot) = \delta_\dagger$. We denote the operator associated with $\tilde{k}_t$ by $\tilde{T}(t)$. Then $(\tilde{T}(t))$ is a Markovian, strong Feller semigroup. Moreover,

$$\text{fix}(\tilde{T}) = \text{span} \left( \text{fix}(T) \cup \{\delta_\dagger\} \right) \quad \text{and} \quad \text{fix}(\tilde{T}') = \text{span} \left( \text{fix}(T') \cup \{1_{\{\dagger\}}\} \right).$$

Thus, the fixed spaces have gained exactly one dimension. By assumption, $T$ is weakly ergodic, whence $\text{fix}T$ separates $\text{fix}T'$. Since $\langle \delta_\dagger, 1_{\{\dagger\}} \rangle = 1 \neq 0$, it follows that $\text{fix}\tilde{T}$ separates $\text{fix}\tilde{T}'$ whence $\tilde{T}$ is weakly ergodic as a consequence of Theorem 2.1.

We are thus in the situation of [5, Theorem 3.6], except that $T$ is not assumed to be stochastically continuous. Inspecting the proof of that theorem, we see that the assumption of stochastic continuity is only used to prove that the restriction of $T$ to some $L^1(\nu)$, where $\nu$ is an invariant measure of $T$ such that the restriction $T_1$ of $T$ to $L^1(\nu)$ is irreducible, is strongly continuous. However, in view of Theorem 3.3, this is automatic, hence we can skip that assumption.

4. **Consequences for the asymptotic behavior on function spaces**

We end this article by discussing some consequences of Theorem 3.3 for the adjoint semigroup $T'$ which acts on the space of bounded measurable functions. Even though Doob’s theorem is classically concerned with the semigroup on the space of measures, there are also results which give information about the adjoint semigroup, see for example [5]. However, it seems that, at least so far, these results were rarely used in the study of partial differential equation, even though there the strong Feller property can often be easily established as the operators occurring are frequently given through continuous Green functions.

One possible explanation for this is that in the study of PDE the domain considered is often assumed to be bounded, so that other methods can be used to study the asymptotic behavior and even obtain better results. This issue was discussed in [8] at the end of Section 5, see also our remarks in the introduction.

However, recently there is an increased interest in partial differential equations in unbounded domains or even the whole space, see [12] and the references therein. As it turns out, in these examples often all relevant facts are known to conclude from our results that the semigroups in question are stable. We present some particular cases to advertise the use of our results in the study of partial differential equations.

We begin with the following basic observation.

**Proposition 4.1.** Let $T$ be a sub-Markovian semigroup which is strongly Feller and weakly ergodic with ergodic projection $P$. Then we have $T'(t)f \to P'f$ as $t \to \infty$ uniformly on the compact subsets of $E$.
Proof. As a consequence of Theorem 3.3, we have $T(t)\mu \to P\mu$ in the total variation norm. Now let $f \in B_0(E)$ and $t_n \uparrow \infty$. Since the operator $T(t_1)$ is ultra Feller and the set $\{T'(t_n - t_1)f : n \in \mathbb{N}\}$ is bounded, the set
\[
\{T'(t_n)f : n \in \mathbb{N}\} = T'(t_0)\{T'(t_n - t_0)f : n \in \mathbb{N}\}
\]
is equicontinuous, hence a subsequence $T'(t_{n_k})$ converges uniformly on compact subsets of $E$ to some function $g \in C_0(E)$. By dominated convergence, we find for any measure $\mu$ that
\[
\langle \mu, g \rangle = \lim_{k \to \infty} \langle \mu, T'(t_{n_k})f \rangle = \lim_{k \to \infty} \langle T(t_{n_k})\mu, f \rangle = \langle P\mu, f \rangle = \langle \mu, Pf \rangle.
\]
Since $\mu$ was arbitrary, we must have $g = Pf$. Since the limit does not depend on the subsequence, we must actually have $T'(t_n)f \to Pf$ uniformly on compact subsets of $E$.

If one considers partial differential equations subject to Dirichlet boundary conditions one can often use the maximum principle to show that the kernel of the generator $G$ of $T$ (i.e. the fixed space of $T$) is trivial. In this case we have

Corollary 4.2. Assume that $T$ is a sub-Markovian semigroup which is strongly Feller such that $\text{fix}(T') = \{0\}$. Then $T'(t)f \to 0$ uniformly on compact subsets of $E$ for every $f \in B_0(E)$.

Proof. Clearly, $\text{fix}(T)$ separates $\{0\}$, thus $T$ is weakly ergodic by Theorem 2.1. Moreover, $P = 0$. Now the claim follows from Proposition 3.1.

In the study of partial differential equations with unbounded coefficients it is often possible to use Lyapunov functions to prove that the Cesàro averages of all Dirac measures have weak accumulation points, see e.g. [12] Theorem 7.1.20, so that in particular $\text{fix}(T) \neq \{0\}$. In that situation we have

Corollary 4.3. Let $T$ be a Markovian semigroup which is strongly Feller such that for every $x \in E$ the net $(A_t\delta_x)$ has a $\sigma(\mathcal{M}(E),C_0(E))$-cluster point for $t \to \infty$. In that case, $T$ is weakly ergodic and we have $T'(t)f \to Pf$ uniformly on compact subsets of $E$. Here, $P$ denotes the ergodic projection.

Proof. This is immediate from Theorem 2.1 and Proposition 4.1.

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