STATISTICAL AND DYNAMICAL MEASURES OF SIMPLE IRREVERSIBLE PROCESSES

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A simple model of an irreversible process is introduced. The equation of iterations in the model includes a noise generation term. We study the properties of the system when the noise generation term is a stochastic process (e.g. a random number generator) or a deterministic process (e.g. a chaotic map). We compare the time series obtained from the above implementations of the model by use of statistical methods (such as Detrended Fluctuation Analysis). The conclusion is that using statistical methods the two versions of the model are indistinguishable. The advantage of this observation is that we may calculate the Lyapunov exponent for the model. As a result we obtain an equation relating the DFA exponents (a statistical measure) with the Lyapunov exponent for such models. On the other hand, typical statistical properties can also be calculated, as for example the diffusion coefficient for a particle, which movement is defined by the above model.

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1. Introduction

Usually deterministic and stochastic analysis are not considered compatible. On one hand we are often limited to statistical analysis when dealing with experimental time series. On the other hand a purely stochastic approach to data analysis may lead to an erroneous interpretation about the genesis of the analyzed system.

In this work we investigate, whether it is possible to substitute the Lyapunov exponent with a statistical measure, at least in simple 1-D systems. This may be valuable in the case when we do not know what is the equation behind the process we observe. It may also be useful when it is technically difficult to calculate the Lyapunov exponent [1, 2].

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In this paper, we use the so-called general random walk as our main model \cite{3,4,5}. We analyze the properties of this model when the noise components of the walk are generated by a stochastic process or a deterministic process (e.g. the tent map). Using Detrended Fluctuation Analysis (DFA) \cite{6,7,8} we compare the time series obtained from the above implementations of the model. The conclusion is that both implementations of the model lead to the same results. As a consequence of this observation, we calculate the Lyapunov exponent for the deterministic version of model. The result is an equation relating the DFA exponents (a statistical measure) with the Lyapunov exponent for our model, which is practically independent of the method used to generate noise. Using the Lyapunov exponent we may calculate the diffusion coefficient for a particle, which movement is defined by the model. By doing this we also doublecheck that the deterministic implementation of the model has the same statistical properties as the purely stochastic one.

1.1. The method

Detrended Fluctuation Analysis (DFA) has been originally applied to the DNA walk \cite{6}. Therefore in general one needs to define a walk that is related to the input series analogously \cite{7}. To do this, the input series $\{X(k)\}$ of length $N$ is integrated after subtracting their average value. The series $\{y(k)\}$ is then:

$$y(k) = \sum_{i=1}^{k} (X(i) - \langle X \rangle),$$

where $X(i)$ is the $i$-th point of a discrete time series and $\langle X \rangle$ is the average value of the data. Next, the integrated series is divided into subintervals of equal length $n$, and for each subinterval a linear least squares fit to the $y(k)$, denoted $y_n(k)$, is made (fig. 1). The RMS fluctuation around the regression line is then given by the equation:

$$F(n) = \sqrt{\frac{1}{N} \sum_{k=1}^{N} [y(k) - y_n(k)]^2}.$$  

The dependence of $F$ on the length $n$ is examined via a plot of log $F(n)$ versus log $n$ (fig. 2). When scaling occurs, the overall slope of the line in the double-logarithmic scale is equal to the DFA exponent and denoted by $\alpha = \log(e^{-b}F(n))/(\log n)^{-1}$, where $b$ is the intercept of the approximated trend.
Generally, when a scaling $F(n) \propto n^\alpha$ is observed, the scaling exponent in the range $0.5 < \alpha < 1$ indicates positive long-range power-law correlations (in other words, persistence) and $0 < \alpha < 0.5$ infers anticorrelations (antipersistence) [6, 7, 8]. $\alpha = 1.5$ is obtained for the Brownian walk. The exponent $\alpha = 0.5$ (fig. 2a) corresponds to uncorrelated data.

If there are short-range correlations, the slope for low $n$ may differ from $0.5$ but it will approach this value for large $n$. A crossover occurs when for different ranges of $n$ we observe different slopes [7]. Crossovers in the linear dependence of the exponent on window size $n$ have been observed in detrended fluctuation analysis of complex data (e.g. biological data) and may be an important indicator characterizing the underlying process [6, 7].

1.2. The model

We define the model by the iterative equation:

$$x_{n+1} = x_n + \xi_{n+1} - \xi_{n+1-M}. \quad (3a)$$

Here, $n$ is the iteration index (a natural number), $x_n$ is the variable value at the $n$-th iteration and $\xi_n$ is the noise term. The model equation (3a) is equivalent to:

$$x_n = \sum_{i=0}^{M-1} \xi_{n-i}. \quad (3b)$$

As explicitly written in equation (3b), our model is a system with memory, the range of which is defined by parameter $M$. This is an analogy to the general random walk process [3, 4].

We use two processes to generate the noise $\xi_n$ - a stochastic and a deterministic one. In the first approach we use a stochastic process that has a uniform distribution with $\langle \xi \rangle = 0.5$ and $\langle \xi_i \xi_k \rangle = \delta_{ik} \langle \xi_i^2 \rangle$.

As a second approach we generate noise using chaotic maps. The following equation defines the tent map (also known as the symmetric triangular map) [1, 9]:

$$\xi_{n+1} = f(\xi_n) = 1 - |1 - 2 \xi_n|. \quad (4)$$

It is well-known [1, 9, 10] that iterating eq. (4) generates data which is equivalent to statistically uncorrelated noise with a uniform i.i.d. distribution. In other words, the natural invariant density [1, 9] for the tent map (4) is equal to $\rho(\xi) = 1$.

In this paper we also consider the generation of noise using the Ulam map (i.e. the logistic map at fully developed chaos) [11]:

$$\xi_{n+1} = f(\xi_n) = 4\xi_n(1 - \xi_n). \quad (5)$$
The main difference between the maps (5) and (4) is that the natural invariant density for the Ulam map does not correspond to the uniform distribution [9], instead it is equal to:

$$\rho(\xi) = \frac{1}{\pi \sqrt{\xi(1-\xi)}}.$$  \hspace{1cm} (6)

As we will show further, the differences between the maps (4) and (5) will not affect the results presented in this paper. To obtain a simple correspondence with the parameter $x_n$ we define $\xi_n$ as:

$$\xi_{n+1} = f(x_n mod 1).$$  \hspace{1cm} (7)

What is important for our study, this approach does not alter the slope of the chaotic map therefore it preserves the value of the Lyapunov exponent (though it introduces singularities).

2. Analysis and comparison of the models

2.1. The stochastic model

First, let us focus on the stochastic version of the model given by eq. (3). As it can be seen in fig. 2 we observe a crossover for series obtain with parameter $M \neq 1$. The point of crossover is found to be dependent on the parameter $M$. The dependence is a power-law and for larger $M$ it occurs at larger $n$ (fig. 3). A similar relation between the correlation range and the crossover point was assumed in [7] (without proof or reference to a model) and applied to the analysis of the heart rate variability series.

We can find this dependence between the crossover point $n_c$ and parameter $M$ from numerical data (fig. 3):

$$\log n_c = 0.4 \log M + 0.4.$$  \hspace{1cm} (8)

Therefore, it is possible to obtain the parameter $M$ of the model (3) only by analyzing the time series (experimental data), e.g. using the statistical method DFA.

For a pure random walk ($M \rightarrow \infty$) it is easy to find the diffusion coefficient $D$, which in the general case is defined as [3, 11]:

$$D = \lim_{n \rightarrow \infty} \frac{\langle (x_n - x_0)^2 \rangle}{2n}.$$  \hspace{1cm} (9)

As $M \rightarrow \infty$, eq. (3) becomes

$$x_n = \sum_{j=0}^{n} \xi_j.$$  \hspace{1cm} (10a)
and, equivalently
\[ x_{i+1} = x_i + \xi_i + 1. \]  
(10b)

Therefore the diffusion coefficient of the pure random walk (10) is equal to:
\[ D_s = \frac{\langle \xi^2 \rangle + \langle \xi \rangle^2}{2} = \frac{7}{24}. \]  
(11)

### 2.2. The deterministic model

The dependence of the properties of chaotic maps on the control parameter is usually described by the Lyapunov exponent, which is a measure of the memory of the initial conditions. The Lyapunov exponent for one-dimensional iterated maps is calculated as [9, 10, 11]:

\[
\lambda = \lim_{N \to \infty} \lambda(N, x_0) = \lim_{N \to \infty} \frac{1}{N} \ln \left| \frac{dF(x_0)}{dx_0} \right| = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \ln |F'(x_i)|,
\]  
(12)

where \( x_i \) corresponds to the \( i \)-th iteration of the map \( F(\cdot) \) (in our case defined by eq. (3)) and \( x_0 \) is the initial condition. A negative value of \( \lambda \) indicates periodic states, a positive \( \lambda \) - chaotic states. In general the Lyapunov exponent is dependent of the initial condition \( x_0 \), but in the case of the map defined by eq. (3) \( \lambda = \lambda(x_0) \).

Instead of calculating the Lyapunov exponent using the long-time limit \( \lim_{N \to \infty} \lambda(N) \) as in eq. (12), this exponent may be calculated as the average of all the possible values of the one-step finite-time Lyapunov exponent \( \lambda_i \) [2, 12]:

\[
\lambda = \langle \lambda_i \rangle ; \quad \lambda_i = \ln |F'(x_i)|.
\]  
(13)

First, we will consider the case of the pure random walk, \textit{i.e.} when \( M \to \infty \). The Lyapunov exponent (eq. (12)) for the above general walk can be easily found if we note, that the inversion of the tent map (4) gives two symmetric preimages \( \xi_n \) for each \( \xi_{n+1} \) [1, 10].

To calculate the exponent \( \lambda \) in our model we need to find the product of \( 2^M \) preimages. However, eq. (13) is the average of one-step Lyapunov exponents \( \lambda_i \). Due to the symmetry of the preimages of (4), there are two equally probable (in the statistical sense, as \( n \) tends to infinity) values of \( \lambda_i \), namely \( \ln 1 \) and \( \ln 3 \). Therefore the Lyapunov exponent for the pure random walk is equal to:

\[ \lambda = \frac{1}{2} \ln 3. \]  
(14)

We obtain the same calculation in the case of the Ulam map [5].
Now we will derive the relation between the Lyapunov exponent and the memory parameter M for the general random walk. At first, this seems a difficult task, because this time we have to find the values of the derivatives of all the preimages of $x_{n+1}$ up to $2^M$ possible values of $x_{n-M+1}$. This step is a little similar to the construction of the Julia set, in the way that one needs to find all the preimages values (compare [1], p.104). Yet it is well-known that the Julia set is quite complex. On the contrary, in this case, after simple calculations, a finite sum of a geometrical progression is obtained, and eq. (13) yields:

$$\lambda = \frac{1}{4} \ln \left[ (3 + \frac{1}{2^M-1})(3 - \frac{1}{2^M-1})(1 + \frac{1}{2^M-1})(1 - \frac{1}{2^M-1}) \right]$$

In the above derivation we use the fact, that in the long-time limit (i.e. when averaging over infinitely many iterations) all values of the one-step Lyapunov exponents are equally probable. As we stated earlier, this is due to the symmetry of the two preimages in both the logistic map and the tent map.

Concluding, we see that the Lyapunov equation depends only on the parameter $M$ and can be calculated from the equation:

$$\lambda = \frac{1}{4} \ln \left[ (9 - 2^{2-2M})(1 - 2^{2-2M}) \right].$$

(15)

Note, that in the limit $M \to \infty$ we obtain eq. (14) which was calculated earlier for more clarity.

The important observation now is that using the DFA method we have obtained the relation between the the crossover point $n_c$ and $M$ (eq. (8), see fig. 3). This relation is valid for both the statistical model of eq. (3) and the deterministic models of eqs. (4) and (5).

This means that we can find the Lyapunov exponent for our model using only statistical analysis by comparing eqs. (15) and (8):

$$\lambda = \frac{1}{4} \ln \left[ \left(1 - \frac{1}{2^{2n_c^{5/2}}-2}\right)^2 \left(1 + \frac{1}{2^{n_c^{5/2}}-1}\right)^2 \right].$$

(16)

We obtained a quite complicated relation, therefore, to doublecheck its correctness we show that the Lyapunov exponent approaches the limit:

$$\lim_{M \to \infty} \lambda = \frac{1}{2} \ln 3$$

(17)
already for $M > 10$ (fig. 4). This is in agreement with the previous finding in eq. (14) obtained for pure random walk case.

The diffusion coefficient may be calculated using the second term of the cumulant expansion of the one-step Lyapunov exponent \[2\]:

$$D = \lim_{n \to \infty} \langle [\lambda_n - \langle \lambda_n \rangle]^2 \rangle$$  \hspace{1cm} (18)

Equation (18) for our model yields:

$$D = \frac{1}{4} \left[ \ln^2 \frac{3 + 2^{1-M}}{\Delta_M} + \ln^2 \frac{3 - 2^{1-M}}{\Delta_M} + \ln^2 \frac{1 + 2^{1-M}}{\Delta_M} + \ln^2 \frac{1 - 2^{1-M}}{\Delta_M} \right], \hspace{1cm} (19)$$

where $\Delta_M = \sqrt[4]{(9 - 2^{2-2M})(1 - 2^{2-2M})}$. The above equation in the case of the pure random walk ($M \to \infty$) yields:

$$D_{\lambda} = \lim_{M \to \infty} D = \frac{1}{4} \ln^2 3$$  \hspace{1cm} (20)

Comparing the above result to eq. (11) obtained for the stochastic model:

$$D_{\lambda} - D_s \approx 0.01. \hspace{1cm} (21)$$

Thus, we obtained a value which is close to the stochastic diffusion coefficient \[11\]. By comparing the basic statistical properties of stochastically and deterministically generated series we concluded earlier that the series are statistically equivalent. Eq. (21) may indicate that although the diffusion coefficient is only a statistical measure (as in eq. (9)), it may be slightly sensitive to the dynamical genesis of the system. Still, it would be very difficult to use such a method even numerically, as usually the statistical errors are greater than the difference found in eq. (21).

3. Summary

We studied the properties of the model defined by eq. \[3\] when the noise generation term is either a stochastic process or a deterministic process. We compared the time series obtained from the above implementations of the model by use of DFA. The conclusion is that the two versions of the model are indistinguishable using such statistical methods. The advantage of this observation is that we may calculate the Lyapunov exponent for the model defined by eq. \[3\]. We obtain an equation relating the DFA exponents (a statistical measure) with the dynamical Lyapunov exponent for such models. On the other hand, typical statistical properties can be calculated in two ways: using statistical analysis or using equations of deterministic
non-equilibrium mechanics. As an example, we calculated the diffusion coefficient (20) and obtained a value which is very close to the stochastic diffusion coefficient (11).

The DFA software used in the calculations, is generously available from the authors of the method at the Physionet Database World Wide Web site (http://www.physionet.org [13]).

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Figure captions

FIG. 1. In detrended fluctuation analysis the integrated series \( y(k) \) is linearly approximated in each window of size \( n \). Here, two cases of approximation are shown on a fragment of a test series: for \( n = 500 \) (continuous lines) and \( n = 2000 \) (dashed lines).

FIG. 2. DFA plots for time series from the model defined by eq. (3) at different parameter \( M \) values. The length of the time series were \( N = 10^5 \). 

a): \( M = 1 \), this is equivalent to uncorrelated noise series 

b): \( M = 2 \), 

c): \( M = 5 \), a crossover occurs - we witness two slopes. 

d): \( M = 100 \), the crossover is shifted in respect to \( M = 5 \). For window sizes \( n \) smaller than the crossover point \( n_c \), the slope value is near to 1.5 (corresponding to Brownian noise), for window sizes larger than \( n_c \) it approaches 0.5 (white noise).

FIG. 3. The equation (8) obtained from numerical simulations.

FIG. 4. The dependence of the Lyapunov exponent \( \lambda \) and its variance on the parameter \( M \).
