Asymmetric Probability Mass Function for Count Data Based on the Binomial Technique: Synthesis and Analysis with Inference

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Abstract: In this article, a new probability mass function for count data is proposed based on the binomial technique. After introducing the methodology of the newly model, some of its distributional characteristics are discussed in-detail. It is found that the newly model has explicit mathematical expressions for its statistical and reliability properties, which is not the case with many well-known discrete models. Moreover, it can be used as an effectively probability tool for modeling asymmetric over-dispersed data with leptokurtic shapes. The parameters estimation through the classical point of view have been done via utilizing the technique of maximum likelihood and Bayesian approaches. A MCMC simulation study is carried out to examine the performance of the estimators. Finally, two distinct real data sets are analyzed to prove the flexibility and notability of the newly model.

Keywords: binomial technique; count data; bayesian analysis; MCMC simulation

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1. Introduction

In today’s competitive era, the data generated from various fields such as engineering, economics, medical sciences, etc., is becoming more complex day by day. As a result, for modeling such data, we need distributions that are best suited for analytical studies of these multidimensional and complex data. For these reasons, over the past three decades, the development of new probability distribution has become the center of the statistical research. However, a large part of this research has been devoted to the development of continuous probability distributions. However, there may be situations where discrete distributions may be more appropriate for data modeling or the generated data may be naturally discrete; for example, in the field of reliability analysis, the lifetime of an on/off switching device is a discrete random variable, see [1], among others. These circumstances demand suitable discrete distributions that can adequately model such data. Therefore, in recent years, the derivation of discrete distributions has also received the attention of researchers.

The first name in this episode comes from [2], who gave a discretized version of the continuous Weibull distribution. Then afterward, ref. [3] obtained discrete inverse Weibull distribution. The paper by [4] provided an excellent review of the development of discrete distributions until 2014. Thereafter, many important discrete distributions have evolved in the literature. Ref. [5] derived discrete generalized Rayleigh distribution, ref. [6] suggested discrete Weibull geometric distribution, and discrete additive Perks–Weibull distribution was proposed by [7]. Recently, to fit discrete increasing failure and count data, ref. [8] developed a discrete Perks distribution, ref. [9] introduced a new two-parameter exponentiated discrete Lindley distribution with bathtub-shaped hazard rate characteristics.
ref. [10] developed a discrete Gompertz-G family for over and under-dispersed data, ref. [11] presented a discrete Burr–Hatke distribution with the associated count regression model, ref. [12] discussed discrete Bilal distribution with properties and applications on integer-valued autoregressive processes, and [13] proposed a new three-parameter discrete Lindley distribution with an associated INAR (1) process, among others.

An important notable thing about the above-cited articles is that most of them are discretized versions of the continuous probability distribution, and they have evolved from the techniques described in [4]. Apart from these methods, another important approach to obtaining discrete models is due to Hu et al. [14]. This approach allows us to generate new discrete distributions by compounding two probability functions. According to this methodology, if $X$ and $M$ are two discrete random variables with probability mass functions (PMFs) $f(x)$ and $w(m)$, respectively, then the PMFs of these two random variables are connected by the binomial decay transformation, i.e.,

$$
\Pr[X = x] = \sum_{m=x}^{\infty} \binom{m}{x} p^x (1 - p)^{m-x} w(m); \quad x = 0, 1, 2, \ldots
$$

(1)

Here, it is notable that the expression in Equation (1) is a proper PMF with an attenuating coefficient $p \in [0, 1]$. Ref. [14] considered $M$ as a Poisson variate with parameter $\lambda > 0$, and then using Equation (1), they obtained $\Pr[X = x]$ as a Poisson distribution with parameter $\lambda p > 0$. Ref. [15] derived uniform Poisson distribution by applying a similar idea to that of [14]. Ref. [16] proposed uniform-geometric distribution by replacing binomial distribution with uniform distribution and setting $w(m)$ as geometric distribution in Equation (1). Recently, Ref. [17] introduced binomial-discrete Lindley distribution by substituting $w(m)$ as discrete Lindley distribution in Equation (1). In this paper, the authors propose a new three-parameter binomial new Poisson-weighted exponential (BNPWE) distribution for modeling count data by substituting $w(m)$ as a new Poisson-weighted exponential (NPWE) distribution. This procedure has been followed by a very large number of researchers in order to look for new distributions that allow capturing “certain” properties of “certain” data sets. Such an abundance of this type of procedure has made it banal, with the consequent obtaining of uninteresting results. However, in the case of this paper, this procedure is applied to discrete distributions, which is not at all frequent. Thus, there is still a way to go here, which favors the interest of the paper under analysis. The proposed model is created and developed by deducing formulas and studying properties for its parameters. From the properties of these parameters, they try to intuit and justify their suitability for application to certain types of data, in particular for medical applications.

2. The BNPWE Distribution

The NPWE distribution has recently been introduced by [18]. He obtained this discrete model by compounding Poisson and a new weighted exponential distribution of [19] and showed its applicability in a first-order integer-valued autoregressive process. The survival function (SF) of NPWE distribution can be expressed as

$$
S(x; \alpha, \theta) = (1 + \alpha + \alpha \theta)^{-(x+1)}; \quad x \in \mathbb{N}_0,
$$

(2)

where $\mathbb{N}_0 = \{0, 1, 2, \ldots, \nu\}$ for $0 < \nu < \infty$, $\alpha > 0$, and $\theta > 0$. The PMF corresponding to Equation (2) is

$$
p(x; \alpha, \theta) = \alpha (1 + \theta)(1 + \alpha + \alpha \theta)^{-(x+1)}; \quad x \in \mathbb{N}_0.
$$

(3)

The PMF in Equation (1) can be represented as

$$
\Pr[X = x] = \sum_{m=x}^{\infty} \Pr(X = x|M = m) w(m),
$$

(4)
where $X|M = m$ has the binomial $(m, p)$ distribution. Now, let us consider that $X|M = m$ follows binomial $(m, \beta)$ distribution and $w(m)$ has the NPWE distribution given in Equation (3). Then, using Equation (4), the PMF of $X$ can be expressed as

$$
\Pr[X = x; \alpha, \beta, \theta] = \frac{\alpha(1 + \theta)\beta^x}{(\alpha + \beta + \alpha\theta)^{x+1}},
$$

where $x \in \mathbb{N}_0$, $\alpha > 0$, $0 < \beta < 1$, and $\theta > 0$. If random variable $X$ has the PMF in Equation (5), then it is known as BNPWE distribution and it is symbolized by $X \sim \text{BNPWE}(\alpha, \beta, \theta)$. The corresponding CDF to Equation (5) is

$$
F(x; \alpha, \theta, \beta) = 1 - \left(\frac{\beta}{\alpha + \beta + \alpha\theta}\right)^{x+1}; x \in \mathbb{N}_0.
$$

Based on Equation (6), the quantile function can be formulated as

$$
x_q = \frac{\ln(1 - q)}{\ln\left(\frac{\beta}{\alpha + \beta + \alpha\theta}\right)} - 1; 0 < q < 1.
$$

Figure 1 shows the PMF plots for various values of the BNPWE parameters.

2.1. Moments, Skewness, Kurtosis and Index of Dispersion

Let $X$ be a BNPWE random variable. Then, the probability-generating function (PrGF) can be proposed in closed form as

$$
\Pi_X(s) = \frac{\alpha(1 + \theta)}{a\theta - \beta s + \alpha + \beta}.
$$

On replacing $s$ by $e^s$ in Equation (8), we get the moment-generating function (MGF). The first four derivatives of the MGF, with respect to $s$ at $s = 0$, yield the first four moments (FFM) about origin. Thus, the FFM values of the BNPWE distribution, respectively, are

$$
E(X) = \frac{\beta}{\alpha(1 + \theta)},
$$

$$
E(X^2) = \frac{\beta(\alpha\theta + 2\beta + \alpha)}{\alpha^2(1 + \theta)^2},
$$

$$
E(X^3) = \frac{6\beta\left(\frac{1}{6}\alpha^2(1 + \theta)^2 + \beta\alpha(1 + \theta) + \beta^2\right)}{\alpha^3(1 + \theta)^3}
$$

and

$$
E(X^4) = \frac{24\beta\left(\frac{1}{12}\alpha^2(1 + \theta)^2 + \beta\alpha(1 + \theta) + \beta^2\left(\frac{1}{2}\alpha(1 + \theta) + \beta\right)\right)}{\alpha^4(1 + \theta)^4}.
$$
Based on the FFM of the BNPWE distribution about origin, the skewness, kurtosis, and index of dispersion (IxOD) can be derived in explicit forms. The IxOD is defined as variance to mean ratio; it indicates whether a certain distribution is suitable for over or (under)-dispersed data. If IxOD $> (\prec) 1$, the model is under (over-dispersed). Table 1 lists some descriptive statistics for the BNPWE distribution for various values of the model parameters.

| Parameter | Measure $\downarrow$ | $\alpha$ |
|-----------|-----------------|---------|
|           | 0.01 0.1 0.7 1.0 1.5 2.0 2.5 3.5 |
| Mean      | 29.4117 2.9411 0.4201 0.2941 0.1960 0.1470 0.1176 0.0840 |
| Variance  | 894.4636 11.5916 0.5967 0.3806 0.2345 0.1686 0.1314 0.0910 |
| IxOD      | 30.4117 3.9411 1.4201 1.2941 1.1960 1.1470 1.1176 1.0840 |
| Skewness  | 2.0002 2.0214 2.3824 2.5743 2.8747 3.1509 3.4066 3.8700 |
| Kurtosis  | 9.0011 9.0862 10.6758 11.6272 13.2639 14.9282 16.6052 19.9775 |

It is clear that the BNPWE distribution is appropriate for modeling over-dispersed data. Furthermore, it is capable of modeling leptokurtic and positively skewed data.

2.2. Mean Residual Life (MRL) and Mean Past Life (MPL)

The MRL is a helpful tool to analyze the burn-in and maintenance policies. In the discrete setting, the MRL is defined as

$$\Lambda(i) = E(X - i|X \geq i) = \frac{1}{1 - F(i - 1; x; \alpha, \theta, \beta)} \sum_{j=i+1}^{\infty} [1 - F(j - 1; x; \alpha, \theta, \beta)]; \; i \in \mathbb{N}_0,$$

Let $X$ be a BNPWE random variable, then the MRL can be reported in a closed form as

$$\Lambda(i) = \frac{\beta + \alpha (1 + \theta)}{\alpha (1 + \theta)} \left( \frac{\beta}{\alpha + \beta + a \theta} \right). \quad (13)$$

The variance residual life (VRL) function can be defined in a closed form as

$$\Omega_{VRL}(i) = E(X^2|X \geq i) - [E(X|X \geq i)]^2$$

$$= \frac{2[\beta + \alpha(1 + \theta)][\beta + \alpha(1 + \theta)]}{\alpha^2(1 + \theta)^2} \left( \frac{\beta}{\alpha + \beta + a \theta} \right) - (2i - 1)\Lambda(i) - \Lambda^2(i). \quad (14)$$

Thus, $X$ is increasing (decreasing) VRL if

$$\Omega_{VRL}(i + 1) \geq (\prec) \Lambda(i) [1 + \Lambda(i + 1)]. \quad (15)$$

The residual coefficient of variation (RCV) can be listed in a closed form where

$$RCV(i) = \sqrt{\Omega_{VRL}(i)} / \Lambda(i).$$

Another measure of interest in reliability theory is MPL. It measures the time elapsed since the failure of $X$ given that the system/component has failed sometime before $i$. In the discrete setting, the MPL is defined as

$$\delta(i) = E(i - X|X < i) = \frac{1}{F(i - 1; x, \theta, \beta)} \sum_{m=1}^{i} F(m - 1; x, \theta, \beta); \; i \in \mathbb{N}_0 - \{0\},$$
where $\delta(0) = 0$. Let $X$ be a BNPWE random variable; then, the MPL can be represented in a closed form as
\[
\delta(i) = \left[1 - \left(\frac{\beta}{\alpha + \beta + \alpha \theta}\right)^i\right]^{-1} \left[i + \beta^{i+1}(\alpha + \beta + \alpha \theta)^{-i} - \beta\right].
\] (16)

For $i \in \mathbb{N}_0$, we get $\delta(i) \leq i$. The mean of the model can be listed as
\[
\text{Mean} = i - \delta(i)F(i - 1) + \Lambda(i)[1 - F(i - 1)]; i \in \mathbb{N}_0 - \{0\}. \tag{17}
\]

The reversed hazard rate function (RHRF) and the MPL are related as
\[
r(i; \theta) = 1 - \frac{\delta(i + 1)}{\delta(i)}; i \in \mathbb{N}_0 - \{0\}. \tag{18}
\]

If $X$ is a BNPWE random variable, then the CDF can be recovered by the MPL as
\[
F(k; \alpha, \theta, \beta) = F(0)_{i=1}^{k} \left[\frac{\delta(i)}{\delta(i + 1) - 1}\right]; k \in \mathbb{N}_0 - \{0\}, \tag{19}
\]

where $F(0) = \left(\sum_{i=1}^{\infty} \frac{\delta(i)}{\delta(i + 1) - 1}\right)^{-1}$. Tables 2–4 list some numerical computations of reliability concepts for different values of the model parameter $\alpha$, $\beta$, and $\theta$, respectively, at time $i = 10$ h.

### Table 2. Some numerical computations of reliability concepts for various values of $\alpha$.

| Measure | $\alpha | \beta = 0.5, \theta = 0.7$ |
|---------|------------------|
|        | 0.01  | 0.1   | 0.7   | 1.0   | 1.5   | 2.0   | 2.5   | 3.5   |
| MRL    | 29.417 | 2.9411 | 0.4201 | 0.2941 | 0.1960 | 0.1470 | 0.1176 | 0.0840 |
| VRL    | 894.4636 | 11.5916 | 0.5967 | 0.3806 | 0.2345 | 0.1686 | 0.1314 | 0.0910 |
| RCV    | 1.0168 | 1.1575 | 1.8384 | 2.0976 | 2.4698 | 2.7928 | 3.0822 | 3.5916 |
| MPL    | 5.7753 | 7.6248 | 9.5798 | 9.7058 | 9.8039 | 9.8529 | 9.8823 | 9.9159 |

### Table 3. Some numerical computations of reliability concepts for various values of $\beta$.

| Measure | $\beta | \alpha = 0.5, \theta = 0.7$ |
|---------|------------------|
|        | 0.01  | 0.1   | 0.2   | 0.3   | 0.5   | 0.7   | 0.9   | 0.99  |
| MRL    | 0.0117 | 0.1176 | 0.2352 | 0.3529 | 0.5882 | 0.8235 | 1.0588 | 1.1647 |
| VRL    | 0.0119 | 0.1314 | 0.2906 | 0.4775 | 0.9342 | 1.5017 | 2.1799 | 2.5212 |
| RCV    | 9.2736 | 3.0822 | 2.2912 | 1.9578 | 1.6431 | 1.4880 | 1.3944 | 1.3632 |
| MPL    | 9.9882 | 9.8823 | 9.7647 | 9.6470 | 9.4122 | 9.1800 | 8.9541 | 8.8556 |

### Table 4. Some numerical computations of reliability concepts for various values of $\theta$.

| Measure | $\theta | \alpha = 0.5, \beta = 0.5$ |
|---------|------------------|
|        | 0.01  | 0.1   | 0.7   | 1.0   | 1.5   | 2.0   | 2.5   | 3.5   |
| MRL    | 0.9900 | 0.9090 | 0.5882 | 0.4999 | 0.4000 | 0.3333 | 0.2857 | 0.2222 |
| VRL    | 1.9703 | 1.7355 | 0.9342 | 0.7500 | 0.5600 | 0.4444 | 0.3673 | 0.2716 |
| RCV    | 1.4177 | 1.4491 | 1.6431 | 1.7320 | 1.8708 | 1.9999 | 2.1213 | 2.3452 |
| MPL    | 9.0192 | 9.0969 | 9.4122 | 9.5001 | 9.6000 | 9.6666 | 9.7142 | 9.7777 |
From Tables 2–4, it is clear that:

1. The MRL and VRL decrease for (fixed \( \beta \) and \( \theta \) with \( \alpha \) grows) and (fixed \( \alpha \) and \( \beta \) with \( \theta \) grows), whereas for fixed \( \alpha \) and \( \theta \) with \( \beta \to 1 \), the MRL and VRL increase.
2. The RCV and MPL increase for (fixed \( \beta \) and \( \theta \) with \( \alpha \) grows) and (fixed \( \alpha \) and \( \beta \) with \( \theta \) grows), whereas for fixed \( \alpha \) and \( \theta \) with \( \beta \to 1 \), the RCV and MPL decrease.

3. Parameter Estimation

3.1. Classical Estimation Using Method of Maximum Likelihood

In this section, we determine the MLEs of the model parameter based on a complete sample. Assume that \( X_1, X_2, \ldots, X_n \) is a random sample of size \( n \) from the BNPWE distribution. The likelihood function \( (L) \) can be expressed as follows

\[
L = \frac{a^n(1 + \theta)^n \beta^\sum_{i=1}^{n} x_i}{(a + \beta + a\theta)^\sum_{i=1}^{n} (x_i + 1)}. \tag{20}
\]

The log-likelihood function \( (LLF) \) corresponding to Equation (20) is

\[
\ln L = n \ln[a(1 + \theta)] + \ln \beta \sum_{i=1}^{n} x_i - \ln(a + \beta + a\theta) \sum_{i=1}^{n} [x_i + 1]. \tag{21}
\]

By differentiating Equation with respect to the parameters \( \alpha, \beta, \) and \( \theta \), we get the non-linear likelihood equations. These equations cannot be solved analytically; therefore, an iterative procedure such as Newton–Raphson is required to solve these non-linear equations numerically.

3.2. Bayesian Estimation

The significance of Bayesian analysis has grown enormously over the last several decades not only because Bayesian estimators have become much simpler to compute, but also because it is one of the most acceptable methods of computing estimates for complicated models. Given this, the current section is dedicated to the Bayesian estimation of unknown parameters of the BNPWE distribution under informative priors (IPs) and non-informative priors (NIPs).

Suppose that the prior distributions of the parameters \( \alpha, \beta, \) and \( \theta \) are Gamma \((a_1, a_2)\), Beta \((b_1, b_2)\), and Gamma \((c_1, c_2)\), respectively. Then, the joint prior distribution of \( \alpha, \beta, \) and \( \theta \) is given by

\[
g(\alpha, \beta, \theta) \propto \alpha^{a_1-1} \beta^{b_1-1} \theta^{c_1} \exp(-\alpha a_1 - \beta b_2 - \theta c_2), \tag{22}
\]

where \((\alpha, \theta) > 0, 0 < \beta < 1, (a_1, b_1, a_2, b_2, c_1, c_2) > 0\) are the hyperparameters of the prior densities and can be fixed based on the amount of available prior information. If we fixed \( a_1 = b_1 = a_2 = b_2 = c_1 = c_2 = 0 \), the joint prior density becomes non-informative prior density.

Using the application of Bayes theorem, the unnormalized joint posterior distribution of the parameters \( \alpha, \beta, \) and \( \theta \) given data can be obtained via the likelihood function and the joint prior density as

\[
P(\alpha, \beta, \theta|X) \propto \frac{a^n \beta^\sum_{i=1}^{n} x_i + b_1 - 1 \theta^{c_1} (1 + \theta)^n \exp(-\alpha a_1 - \beta b_2 - \theta c_2)}{(a + \beta + a\theta)^\sum_{i=1}^{n} (x_i + 1)}. \tag{23}
\]

Loss functions (LFs) play a vital role when researchers are not only interested in choosing the right decision but also consider the economic consequences that arise through a Bayes estimate of the unknown parameter. In our case, we use a well-known symmetric loss function called squared error function (SELF). In this loss function, positive and negative errors are equally penalized. Under SELF, the Bayes estimator of a parameter is simply the expectation of that parameter with respect to its posterior distribution.
Now, the Bayes estimator of a function of parameters $\alpha$, $\beta$, and $\theta$, say $\bar{w}(\alpha, \beta, \theta)$ is

$$\bar{w}(\alpha, \beta, \theta) = \int_\alpha \int_\beta \int_\theta w(\alpha, \beta, \theta)P(\alpha, \beta, \theta|\mathbf{x})d\alpha d\beta d\theta. \quad (24)$$

The integral is difficult to obtain in the closure form due to the complex form of the posterior distribution. Therefore, we use a mixture of two famous Monte Carlo simulation techniques under a hybrid algorithm. These techniques are Gibbs sampler, see [20] and the Metropolis–Hastings algorithm, see [21,22]. The first technique allows us to simulate posterior samples from full conditional marginal posterior density instead of the joint posterior distribution, whereas the latter one enables us to draw required samples when conventional methods fail to generate samples. For this purpose, the full conditional marginal posterior densities of the parameters $\alpha$, $\beta$, and $\theta$ are, respectively as

$$P_{11}(\alpha|\beta, \theta, \mathbf{x}) \propto \alpha^{n_1+a_1-1}\exp\left(-a_1\sum_{j=1}^{n_1}(x_j)\right), \quad (25)$$

$$P(\beta|\alpha, \theta, \mathbf{x}) \propto \beta^{\sum_{j=1}^{n_2}(x_j)+b_2-1}\exp\left(-\beta b_2\right), \quad (26)$$

$$P(\theta|\alpha, \beta, \mathbf{x}) \propto (\alpha + \beta + \theta)^{n}\exp\left(-\theta c_2\right). \quad (27)$$

The steps of the above-mentioned hybrid algorithm are as follows:

1. Start with the initial values of $\alpha$, $\beta$, and $\theta$ as $(\alpha(0), \beta(0), \theta(0))$ and set $j = 0$.
2. Generate $\alpha(j)$ from $P_{11}(\alpha|\beta(j-1), \theta(j-1), \mathbf{x})$ using the following steps:
   (a) Generate the proposal point $\alpha^*$ from the proposal distribution $N(\alpha(j-1), \sigma_\alpha^2)$ and $u_1$ from Uniform (0,1) distribution. Here, the variance $\sigma_\alpha^2$ is suitably chosen.
   (b) Calculate the acceptance probability (AP) $\rho_\alpha = \min\left[1, \frac{P_{11}(\alpha^*|\beta(j-1), \theta(j-1), \mathbf{x})}{P_{11}(\alpha(j)|\beta(j-1), \theta(j-1), \mathbf{x})}\right]$, and if $u_1 \leq \rho_\alpha$, then record $\alpha(j) = \alpha^*$; otherwise, sustain $\alpha(j) = \alpha(j-1)$.
3. Generate $\beta(j)$ from $P_{12}(\beta|\alpha(j), \theta(j-1), \mathbf{x})$ using the following steps:
   (a) Generate the proposal value $\beta^*$ from $N(\beta(j-1), \sigma_\beta^2)$ and $u_2$ from Uniform (0,1) distribution. The variance $\sigma_\beta^2$ is appropriately selected.
   (b) Compute the AP $\rho_\beta = \min\left[1, \frac{P_{12}(\beta^*|\alpha(j), \theta(j-1), \mathbf{x})}{P_{12}(\beta(j)|\alpha(j), \theta(j-1), \mathbf{x})}\right]$, and if $u_2 \leq \rho_\beta$, then record $\beta(j) = \beta^*$; otherwise, store $\beta(j) = \beta(j-1)$.
4. Generate $\theta(j)$ from $P_{13}(\theta|\alpha(j), \beta(j), \mathbf{x})$ using the following steps:
   (a) Generate the proposal point $\theta^*$ from $N(\theta(j-1), \sigma_\theta^2)$ and $u_3$ from Uniform (0,1) distribution. The variance $\sigma_\theta^2$ is well-chosen here.
   (b) Calculate the AP $\rho_\theta = \min\left[1, \frac{P_{13}(\theta^*|\alpha(j), \beta(j), \mathbf{x})}{P_{13}(\theta(j)|\alpha(j), \beta(j), \mathbf{x})}\right]$, and if $u_3 \leq \rho_\theta$, then record $\theta(j) = \theta^*$; otherwise, sustain $\theta(j) = \theta(j-1)$.
5. Set $j = j + 1$.
6. Rerun the steps 2–5, a large number of times, say $N$ times, and obtain $\alpha(j)$, $\beta(j)$, and $\theta(j)$, $j = 1, 2, \ldots, N$.

To avoid the effect of initial values, the first $m$ draws are eliminated. After the convergence diagnostic of the generated chains through various graphical and statistical tests, the values $\alpha(j), \beta(j)$, and $\theta(j)$, $j = m + 1, m + 2, \ldots, N$, represent the required posterior sam-
ples, which can be utilized to compute the Bayes estimators of the unknown parameters. Hence, the Bayes estimators of \( \alpha, \beta, \) and \( \theta \) under SELF are, respectively, obtained as

\[
\hat{\alpha}_B = \frac{1}{N - m} \sum_{j=m+1}^{N} \alpha(j), \quad \hat{\beta}_B = \frac{1}{N - m} \sum_{j=m+1}^{N} \beta(j), \quad \text{and} \quad \hat{\theta}_B = \frac{1}{N - m} \sum_{j=m+1}^{N} \theta(j). \tag{28}
\]

Notably, by fixing \( a_1 = b_1 = a_2 = b_2 = c_1 = c_2 = 0 \) and applying the above-mentioned procedure, we can obtain the Bayes estimators under NIPs.

4. A Monte Carlo Simulation Study

To discuss the behavior of the considered estimation methods based on various values of \( n \) and true parametric combination, we perform an MCMC simulation study. This assessment consists of the following steps: Draw 2000 samples of sizes \( n = 25, 50, 100, \) and \( 200 \) from the newly model with \( (\alpha, \beta, \theta) = (0.5, 0.4, 0.5) \) and \( (0.5, 0.4, 1.01) \). Calculate the MLE and BE (under IP and NIP with SELF) for the 2000 samples, say \( \hat{\tau}_j^\varphi, \hat{\tau}_j^\varphi, \hat{\tau}_j^\varphi; \tau = \alpha, \beta, \theta; j = 1, 2, \ldots, 2000; \varphi = \text{ML and Bayes.} \) Compute the mean squared error (MSE) and average absolute error (ABE) for all point estimates, where

\[
\text{MSE} = \frac{1}{2000} \sum_{j=1}^{2000} (\hat{\tau}_j^\varphi - \tau)^2 \quad \text{and} \quad \text{ABE} = \frac{1}{2000} \sum_{j=1}^{2000} |\hat{\tau}_j^\varphi - \tau|. \tag{29}
\]

The empirical results are shown in Tables 5–7. To assess the convergence of generated chains, we observe the MCMC, histogram, and auto-correlation plots. These plots show that all chains reach their stationery distributions. Due to space constraints, we have included convergence diagnostic plots only for the parametric combination \( (0.5, 0.4, 0.5) \) under IPs and NIPs, and they can be viewed in Figures 2 and 3, respectively.

Table 5. The MSE, ABE, and bootstrap CI for MLEs.

| Schema | \( n \) | \( \alpha \) | MSE | ABE [bootstrap CI] | \( \beta \) | MSE | ABE [bootstrap CI] | \( \theta \) | MSE | ABE [bootstrap CI] |
|--------|--------|--------|-----|-------------------|--------|-----|-------------------|--------|-----|-------------------|
| I      | 25     | 0.00481 | 0.05600 [0.363,0.566] | 0.00604 | 0.05995 [0.310,0.487] | 0.00189 | 0.02690 [0.455,0.591] |
|        | 50     | 0.00275 | 0.04192 [0.399,0.545] | 0.00285 | 0.04099 [0.319,0.470] | 0.00085 | 0.02495 [0.457,0.547] |
|        | 100    | 0.00155 | 0.03100 [0.413,0.531] | 0.00132 | 0.02855 [0.354,0.451] | 0.00078 | 0.02257 [0.461,0.536] |
|        | 200    | 0.00088 | 0.02215 [0.437,0.512] | 0.00078 | 0.02069 [0.366,0.432] | 0.00069 | 0.02069 [0.467,0.529] |
| II     | 25     | 0.00651 | 0.06331 [0.325,0.576] | 0.00785 | 0.06791 [0.316,0.491] | 0.00039 | 0.04407 [0.876,1.43] |
|        | 50     | 0.00377 | 0.04699 [0.377,0.558] | 0.00405 | 0.04731 [0.324,0.486] | 0.00215 | 0.03758 [0.894,1.39] |
|        | 100    | 0.00214 | 0.03464 [0.395,0.534] | 0.00205 | 0.03328 [0.357,0.461] | 0.00182 | 0.03740 [0.911,1.30] |
|        | 200    | 0.00149 | 0.02702 [0.428,0.510] | 0.00111 | 0.02404 [0.364,0.435] | 0.00171 | 0.03719 [0.951,1.25] |
Table 6. The MSE, ABE, and bootstrap CI for Bayes estimates under IPs.

| Schema | n  | MSE   | ABE [bootstrap CI] | MSE   | ABE [bootstrap CI] | MSE   | ABE [bootstrap CI] |
|--------|----|-------|---------------------|-------|---------------------|-------|---------------------|
| I      | 25 | 0.00121 | 0.02702 [0.311,0.568] | 0.00222 | 0.03811 [0.326,0.507] | 0.00029 | 0.01427 [0.387,0.583] |
|        | 50 | 0.00092 | 0.02570 [0.334,0.557] | 0.00191 | 0.03623 [0.337,0.501] | 0.00191 | 0.01121 [0.394,0.561] |
|        | 100 | 0.00077 | 0.02243 [0.356,0.543] | 0.00140 | 0.03038 [0.342,0.487] | 0.00018 | 0.01106 [0.398,0.553] |
|        | 200 | 0.00048 | 0.01760 [0.377,0.531] | 0.00079 | 0.02311 [0.359,0.423] | 0.00012 | 0.00948 [0.417,0.528] |
| II     | 25 | 0.00178 | 0.03197 [0.339,0.547] | 0.00220 | 0.03847 [0.334,0.489] | 0.00023 | 0.01297 [0.872,1.624] |
|        | 50 | 0.00127 | 0.02933 [0.350,0.541] | 0.00217 | 0.03779 [0.351,0.478] | 0.00018 | 0.01111 [0.911,1.426] |
|        | 100 | 0.00081 | 0.02334 [0.384,0.530] | 0.00135 | 0.03057 [0.369,0.451] | 0.00015 | 0.01009 [0.937,1.340] |
|        | 200 | 0.00052 | 0.01878 [0.390,0.522] | 0.00099 | 0.02197 [0.388,0.430] | 0.00012 | 0.00959 [0.970,1.229] |

Table 7. The MSE, ABE, and bootstrap CI for Bayes estimates under NIPs.

| Schema | n  | MSE   | ABE [bootstrap CI] | MSE   | ABE [bootstrap CI] | MSE   | ABE [bootstrap CI] |
|--------|----|-------|---------------------|-------|---------------------|-------|---------------------|
| I      | 25 | 0.00245 | 0.03993 [0.366,0.571] | 0.00422 | 0.04865 [0.334,0.498] | 0.00077 | 0.02617 [0.327,0.560] |
|        | 50 | 0.00191 | 0.03470 [0.387,0.562] | 0.00266 | 0.04242 [0.373,0.553] | 0.00082 | 0.02691 [0.335,0.551] |
|        | 100 | 0.00186 | 0.03332 [0.391,0.553] | 0.00210 | 0.03745 [0.382,0.531] | 0.00072 | 0.02604 [0.361,0.531] |
|        | 200 | 0.00159 | 0.03118 [0.399,0.532] | 0.00120 | 0.02791 [0.389,0.511] | 0.00065 | 0.02491 [0.380,0.510] |
| II     | 25 | 0.00232 | 0.0371 [0.353,0.560] | 0.00524 | 0.05610 [0.321,0.558] | 0.00022 | 0.01313 [0.901,1.141] |
|        | 50 | 0.00197 | 0.03627 [0.361,0.555] | 0.00291 | 0.04235 [0.336,0.546] | 0.00019 | 0.01228 [0.903,1.100] |
|        | 100 | 0.00180 | 0.03443 [0.374,0.536] | 0.00206 | 0.03250 [0.371,0.523] | 0.00017 | 0.01146 [0.924,1.090] |
|        | 200 | 0.00117 | 0.02694 [0.394,0.510] | 0.00114 | 0.02308 [0.387,0.514] | 0.00013 | 0.01032 [0.949,1.087] |

Figure 2. The MCMC diagnostic plots under NIPs.
From Tables 5–7, it is noted that:

- Based on MSE, we observe that the parameter $\beta$ is more sensitive as compared to $\alpha$ and $\theta$.
- The MSE and ABE decrease to zero as $n$ tends to infinity. This shows the consistency of the estimators.
- Both estimation procedures perform satisfactorily. However, in overall comparison, Bayes estimators perform better in comparison to MLE, especially in case of IPs.
- For higher values of the parameters $\alpha$, $\beta$, and $\theta$, the generated random samples produce a large number of 0s and 1s, due to which sometimes we face the convergence issue, and also the estimated error associated with an estimate becomes larger.

5. Data Analysis

In this section, we illustrate the importance of the BNPWE distribution by utilizing data from different areas. We shall compare the fits of the BNPWE distribution with some competitive models such as discrete Weibull inverse Weibull (DWIW), discrete inverse Weibull (DIW), discrete log-logistic (DLogL), discrete Burr type II (DB-II), discrete Rayleigh (DR), discrete Bilal (DBL), discrete Pareto (DPa), and discrete Burr–Hatke (DBH). Some statistical criteria have been used such as the negative log-likelihood ($-L$), corrected (Akaike information criterion) C(AIC), Bayesian IC (BIC), Hannan–Quinn IC (HQIC), and Chi-square ($\chi^2$) test with its corresponding $p$-value.

5.1. Data Set I: COVID-19 Data

The data are reported in (https://www.worldometers.info/coronavirus/country/southkorea/ (accessed on 21 March 2022)) and represent the daily new deaths in South Korea from 15 February to 13 June 2020. The initial MS is reported using the nonparametric KME method in Figure 4, and it is noted that the mass function is asymmetric and multimodal. Moreover, Figure 4 shows that some extreme observations were listed.
The MLEs and Std-er for the parameter(s) as well as GOF measures for data set I are reported in Tables 8 and 9.

Table 8. The MLEs and Std-er for data set I.

| Model | α         | Standard error | β         | Standard error | θ         | Standard error | λ         | Standard error |
|-------|-----------|----------------|-----------|----------------|-----------|----------------|-----------|----------------|
| BNPWE | 0.1387    | 0.0148         | 0.4573    | 0.0347         | 0.3870    | 0.1488         | −         | −              |
| DWI   | 0.9524    | 0.0931         | 0.3953    | 0.4810         | 0.0112    | 0.0166         | 2.8785   | 3.5181         |
| DIW   | 0.2338    | 0.0381         | 1.2658    | 0.1134         | −         | −              | −         | −              |
| DLogL | 2.0210    | 0.1890         | 1.7457    | 0.1523         | −         | −              | −         | −              |
| DB-II | 0.6225    | 0.0487         | 2.3359    | 0.3772         | −         | −              | −         | −              |
| DR    | 0.9306    | 0.0061         | −         | −              | −         | −              | −         | −              |
| DBL   | 0.7487    | 0.0143         | −         | −              | −         | −              | −         | −              |
| DPa   | 0.4151    | 0.0332         | −         | −              | −         | −              | −         | −              |
| DBH   | 0.9315    | 0.0269         | −         | −              | −         | −              | −         | −              |

It is clear that the DWI and DLogL distributions work quite well besides the BNPWE distribution. However, the BNPWE distribution is the best model among all tested models.
Figures 5 and 6 support our empirical results where the BNPWE is more fit to analyze this data, whereas Figure 7 shows that the estimator is unique.

**Figure 5.** The estimated PMFs for data set I.

**Figure 6.** The P-P plot for data set I.
According to the MLEs, the EDS for mean, variance, IxOD, skewness and kurtosis are 2.37710, 8.02773, 3.37710, 2.03090, and 9.12456, respectively. The data exhibit overdispersion. Moreover, it is moderately skewed to the right and leptokurtic.

5.2. Data Set II: Biological Data

This data set is the biological experiment data which represent the number of European corn borer (No. ECB) larvae pyrausta in the field, see [23]. It was an experiment conducted randomly on 8 hills in 15 replications, and the experimenter counts the number of borers per hill of corn. For these data, the mean, variance, skewness, and kurtosis equal 1.326, 3.669, 1.976, and 8.984, respectively. The initial MS is reported using the nonparametric KME method in Figure 8, and it is noted that the mass function is asymmetric and multimodal. Moreover, Figure 8 shows that some extreme observations were reported.

The MLEs and Std-er for the parameter(s) as well as GOF measures for data set II are reported in Tables 10 and 11.

It is clear that the DWIW and DLogL distributions work quite well besides the BNPWE distribution. However, the BNPWE distribution is the best model among all tested models. Figures 9 and 10 support our empirical results where the BNPWE more fit to analyze these data, whereas Figure 11 shows that the estimator is unique.
Figure 9. The estimated PMFs for data set II.

Figure 10. The P-P plot for data set II.
Figure 11. The $L$ profiles for the model parameters for data set II.

Table 10. The MLEs and Std-er for data set II.

| Model    | Parameter | $\beta$  | Std-er | $\alpha$ | Std-er | $\theta$ | Std-er |
|----------|-----------|----------|--------|----------|--------|----------|--------|
| BNPWE    |           | 0.586   | 0.059  | 0.258    | 0.030  | 0.532    | 0.180  |
| DivW     |           | 0.345   | 0.043  | 1.541    | 0.156  |          |        |
| DLLc     |           | 1.401   | 0.121  | 1.943    | 0.188  |          |        |
| DBXII    |           | 0.519   | 0.051  | 2.358    | 0.366  |          |        |
| DBHk     |           | 0.865   | 0.039  |          |        |          |        |
| DBI      |           | 0.657   | 0.019  |          |        |          |        |
| DRh      |           | 0.867   | 0.012  |          |        |          |        |
| DIRh     |           | 0.319   | 0.042  |          |        |          |        |
| DPo      |           | 0.329   | 0.034  |          |        |          |        |

Table 11. The GOF measures test for data set II.

| No. | ECB | OF | BNPWE | DIW | DLLc | DBX-II | DBH | DBL | DR | DIR | Dpo |
|-----|-----|----|-------|-----|------|--------|-----|-----|----|-----|-----|
| 0   | 43  |    | 48.318| 41.400| 41.019| 43.836 | 68.099| 32.669| 15.960| 38.280| 64.470|
| 1   | 35  |    | 28.863| 41.844| 38.940| 21.971 | 39.558| 36.236| 51.904| 20.149| 5.645 |
| 2   | 17  |    | 17.241| 15.418| 15.779| 10.513 | 24.294| 34.588| 15.509| 9.684 |
| 3   | 11  |    | 10.299| 7.166 | 8.434 | 5.980  | 12.534| 20.985| 6.036 | 5.645 |
| 4   | 5   |    | 6.152 | 3.942 | 4.486 | 3.751  | 5.991 | 8.464 | 2.909 | 3.679 |
| 5   | 4   |    | 3.675 | 2.422 | 2.631 | 2.374  | 2.751 | 2.681 | 1.612 | 2.578 |
| 6   | 1   |    | 2.195 | 1.607 | 1.664 | 1.561  | 1.746 | 1.234 | 0.594 | 1.903 |
| 7   | 2   |    | 1.311 | 1.128 | 1.116 | 1.088  | 1.256 | 0.546 | 0.097 | 0.642 | 1.459 |
| 8   | 2   |    | 1.946 | 5.073 | 3.931 | 4.793  | 4.18  | 0.423 | 0.013 | 2.125 | 10.433|

According to the MLEs, the EDS for mean, variance, IxOD, skewness, and kurtosis are 1.48258, 3.68063, 2.48258, 2.06680, and 9.27169, respectively. The data exhibit over dispersion. Furthermore, it is moderately skewed to the right and leptokurtic.
5.3. Bayesian Analysis of Real Data Sets I and II

In this section, we derive Bayes estimates for the unknown parameters of the proposed distribution with their posterior standard errors (PSEs). Here, it is worth noting that in this estimation process, we use NIPs because there is no prior knowledge available about the model parameters for the data sets under study. These estimates can be viewed in Table 12. From this table, we can conclude that Bayesian estimation captures real data sets more accurately than the considered classical estimation procedure in terms of estimation errors.

Table 12. Bayes estimates and PSE for real data sets.

| Data Sets | Parameters | Bayes Estimates | PSE  |
|-----------|------------|-----------------|------|
| I         | $\alpha$   | 0.1384          | 0.0125 |
|           | $\beta$    | 0.4562          | 0.0197 |
|           | $\theta$   | 0.3874          | 0.0196 |
| II        | $\alpha$   | 0.2427          | 0.0291 |
|           | $\beta$    | 0.5778          | 0.0564 |
|           | $\theta$   | 0.5293          | 0.1170 |

6. Conclusions

In this article, a binomial new Poisson weighted exponential distribution is developed for analysis of count data. Its various impressive statistical properties, including moments, skewness, kurtosis, index of dispersion, mean residual life, and mean past life, are derived. One of the main features of the proposed model is that it has closed-form expressions for various important distributional characteristics, which is not the case with many well-known discrete distributions. Furthermore, it may be used to model over-dispersed, leptokurtic, and positively skewed data sets. Under classical estimation approach, the method of maximum likelihood is used, whereas in the Bayesian paradigm, we have used informative and non-informative priors with squared error loss function to estimate the unknown parameters. To evaluate the performance of the estimators with respect to the sample size, a thorough simulation study is performed. Finally, two different real data sets are examined to demonstrate the adaptability of the suggested distribution.

A future plan of action regarding the current study might be an examination of the censored data using the proposed model. We may investigate the load share model where the component failure time follows the BNPWE distribution. The stress-strength parameter may be examined using various censored data. In addition, a bivariate extension of the BNPWE distribution can be developed.

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