ON THE DEGREE 2 MAP FOR A SPHERE

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Abstract. The purpose of this article is to compare the two self-maps of $\Omega^k S^{2n+1}$ given by $\Omega^k[2]$ the $k$-fold looping of a degree 2 map and $\Psi^k(2)$ the $H$-space squaring map. The main results give that in case $2n + 1 \neq 2^t - 1$, these maps are frequently not homotopic and also that their homotopy theoretic fibres are not homotopy equivalent.

The methods are a computation of an unstable secondary operation constructed by Brown and Peterson in the first case and the Nishida relations in the second case.

One question left unanswered here is whether the maps $\Omega^k S^{2n+1}[2]$ and $\Psi^k S^{2n+1}(2)$ are homotopic on the level of $\Omega^k_0 S^{2n+1}$. A natural conjecture is that these two maps are homotopic.

1. Introduction and statement of results

Consider the two natural self-maps of $\Omega^k S^{2n+1}$ given by the $k$-fold looping of a degree 2 map

$$\Omega^k[2] : \Omega^k S^{2n+1} \to \Omega^k S^{2n+1}$$

and if $k \geq 1$, the $H$-space squaring map

$$\Psi^k(2) : \Omega^k S^{2n+1} \to \Omega^k S^{2n+1}.$$

Furthermore, let

$$\Omega^k S^{2n+1}\{[2]\}$$

denote the homotopy theoretic fibre of $\Omega^k[2]$ and

$$\Omega^k S^{2n+1}\{\Psi\}$$

denote the homotopy theoretic fibre of $\Psi^k(2)$. The purpose of this article is to compare

- the maps $\Omega^k[2]$ and $\Psi^k(2)$, as well as
- the spaces $\Omega^k S^{2n+1}\{\Psi\}$ and $\Omega^k S^{2n+1}\{[2]\}$.

The main point of this article is that the previous comparison is a further illustration of the dichotomy between spheres of dimension $2^t - 1$ and spheres of other dimensions. Namely, Theorems 1.1 and 1.3 imply that if $\Omega^k[2]$, and $\Psi^k(2)$ are homotopic, then the values of $k$ are monotonically increasing with $n$ for certain choices of $2n + 1$ which are not equal to $2^t - 1$. On the other hand, the strong form of the Kervaire invariant conjecture implies that the maps $\Omega^k[2]$ and $\Psi^k(2)$ are homotopic for $k = 2$ in case $2n + 1 = 2^t - 1$. Further discussion concerning this last point is given in Proposition 1.6 below in which the dimensions of the spheres are $2n + 1 = 2^t - 1$.

Theorem 1.1. Assume that the two self-maps of $\Omega^k S^{2n+1}$ given by

$$\Omega^k[2], \Psi^k(2) : \Omega^k S^{2n+1} \to \Omega^k S^{2n+1}$$

are homotopic.

(1) If $2n + 1 = 2^{t+1} + 2^t - 1$ for $t \geq 1$, then $k \geq 2t + 1$.

(2) If $2n + 1 = 2^{t+1} + 1$ for $t \geq 1$, then $k \geq 2t + 1$.

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Corollary 1.2. There does not exist a finite integer \( k \) such that for all \( n > 0 \), the two self-maps of \( \Omega^k S^{2n+1} \) given by

\[ \Omega^k[2], \Psi^k(2) : \Omega^k S^{2n+1} \to \Omega^k S^{2n+1} \]

are homotopic.

Similar results are satisfied if the map \( \Omega^k[2] \) is replaced by \( \Omega^k[-1] \) where \([-1]\) is a map of degree \(-1\) and the map \( \Psi^k(2) \) is replaced by \( \Psi^k(-1) \), any choice of the loop inverse for \( \Omega^k S^{2n+1} \).

In addition, if the two self-maps of \( \Omega^k S^{2n+1} \) given by \( \Omega^k[2] \), and \( \Psi^k(2) \) are homotopic, then their homotopy theoretic fibres are homotopy equivalent. However, the converse does not appear to be evident. The next theorem, a direct consequence of the Nishida relations, implies that these fibres are frequently not homotopy equivalent in case \( n \) is divisible by 8. Thus the maps \( \Omega^k[2] \) and \( \Psi^k(2) \) are not homotopic in these cases. It seems likely that if the maps \( \Omega^k[2] \) and \( \Psi^k(2) \) are homotopic on \( \Omega^k S^{2n+1} \) and \( n \) is restricted to values such that \( 2n + 1 \neq 2^j - 1 \), then \( \lim_{n \to \infty} k = \infty \).

Theorem 1.3. If \( n > 1 \) and \( q \geq 1 \), then the mod-2 cohomology of \( \Omega^{2n} S^{q2^{n+2}+1}\{\Psi\} \) and \( \Omega^{2n} S^{q2^{n+2}+1}\{[2]\} \) are not isomorphic as modules over the mod-2 Steenrod algebra and thus these spaces are not homotopy equivalent.

Corollary 1.4. If \( n > 1 \) and \( q \geq 1 \) then the maps

\[ \Omega^{2n}[2], \Psi^{2n}(2) : \Omega^{2n} S^{q2^{n+2}+1} \to \Omega^{2n} S^{q2^{n+2}+1} \]

are not homotopic.

Remark: Theorem 1.4 sometimes gives stronger information than Theorem 1.3 concerning the minimum values of \( k \) such that the maps

\[ \Omega^k[2], \Psi^k(2) : \Omega^k S^{2n+2+1} \to \Omega^k S^{2n+2+1} \]

are possibly homotopic. Both results can be improved in special cases.

One example is given in Table 1 below for the case of \( \Omega^k S^{17} \). The stated values of \( k \) in Table 1 are obtained as consequences of the techniques used to prove Theorem 1.1 rather than the explicit statement of either 1.1 or 1.3. In the case of \( \Omega^k S^{17} \), it follows that \( k \geq 15 \) by an application of a secondary operation obtained from the relation

\[ Sq^{18} = Sq^8(Sq^4 Sqq^4 + Sq^5 Sqq) + Sq^8 Sqq^2 + Sq^{16} Sqq^2 + Sq^{15} Sq^3 + Sq^{14} Sq^4. \]

The requisite verifications are sketched in section 2 here after the proof of Theorem 1.1.

Table 1. Results for \( S^{2n+1}, n = 2, 4, 8, 5, 6 \)

| If \( \Omega^k[2] \simeq \Psi^k(2) : \Omega^k S^5 \to \Omega^k S^5 \), then \( k \geq 3 \). | If \( \Omega^k[2] \simeq \Psi^k(2) : \Omega^k S^9 \to \Omega^k S^9 \), then \( k \geq 7 \). |
| --- | --- |
| If \( \Omega^k[2] \simeq \Psi^k(2) : \Omega^k S^{17} \to \Omega^k S^{17} \), then \( k \geq 15 \). | If \( \Omega^k[2] \simeq \Psi^k(2) : \Omega^k S^{11} \to \Omega^k S^{11} \), then \( k \geq 5 \). |
| If \( \Omega^k[2] \simeq \Psi^k(2) : \Omega^k S^{13} \to \Omega^k S^{13} \), then \( k \geq 7 \). |
Proposition 1.5. If the two self-maps of $\Omega^k S^{2n+1}$ given by

$$\Omega^k [2], \Psi^k (2) : \Omega^k S^{2n+1} \to \Omega^k S^{2n+1}$$

are homotopic, then

1. $\Omega^k [\tau_{2n+1}, \tau_{2n+1}] \circ \Omega^k (h_2) : \Omega^k S^{2n+1} \to \Omega^k S^{2n+1}$ is null-homotopic and
2. the composite denoted $\lambda(n, k)$

$$\Sigma^{2n+1} (\mathbb{RP}^{2n}/\mathbb{RP}^{2n-k}) = \Sigma^{2n+1} (\mathbb{RP}^{2n-k+1}) \xrightarrow{\text{pinch}} S^{4n+1} \xrightarrow{\mu_{2n+1}} S^{2n+1}$$

is null-homotopic.

Furthermore, the difference $\Omega^k [2] - \Psi^k (2) = \Delta_k$ restricted to the second May-Milgram filtration of $\Omega^k S^{2n+1}$ is null-homotopic if and only if the composite $\lambda(n, k)$

$$\Sigma^{2n+1} (\mathbb{RP}^{2n-k+1}) \xrightarrow{\text{pinch}} S^{4n+1} \xrightarrow{\mu_{2n+1}} S^{2n+1}$$

is null-homotopic. Thus if $\lambda(n, k)$ is null-homotopic, there is a homotopy commutative diagram

$$\begin{array}{ccc}
S^{4n+1} & \xrightarrow{\mu_{2n+1}} & \Sigma^{2n+2} (\mathbb{RP}^{2n-1}) \\
\downarrow & & \downarrow \\
S^{2n+1} & \xrightarrow{1} & S^{2n+1}
\end{array}$$

for some map $\bar{w}_{2n+1}$.

It is reasonable to ask whether the two natural self-maps of $\Omega^2 S^{2n+1}$ given by

- the 2n + 1-fold looping of a degree 2 map, $\Omega^2 S^{2n+1} [2]$, and
- the $\mathcal{H}$-space squaring map $\Psi^{2n+1} (2)$

are homotopic, or whether the mod-2 cohomology algebras of $\Omega^2 S^{2n+1} [2]$, and $\Omega^0 S^{2n+1} [2]$ are isomorphic. The few known cases occur for $2n+1 = 1, 3, 7, 15, 31, 63$ as listed in [3] with the case $2n+1 = 63$ based on the computations in [3]. If these maps are homotopic, then the degree 2 map on $S^{2n+1}$ induces multiplication by 2 on the level of homotopy groups.

Proposition 1.6. Assume that $n \geq 0$.

1. The two self-maps of $\Omega S^{2n+1}$ given by $\Omega [2]$, and $\Psi^1 (2)$ are homotopic if and only if $w_{2n+1} = 0$. Thus these two self-maps are homotopic if and only if $2n+1$ equals 1, 3, or 7.
2. The two self-maps of $\Omega^2 S^{2n+1}$ given by $\Omega^2 [2]$, and $\Psi^2 (2)$ are homotopic if and only if the Whitehead square $w_{2n+1} = [\tau_{2n+1}, \tau_{2n+1}]$ is divisible by 2. Thus these maps
   (a) are homotopic for $n = 1, 3, 7, 15, 31, 63$ and
   (b) are not homotopic when $2k+1$ is not $2^j - 1$ for some $j$. 

The main results concerning these maps are closely tied to features of the Whitehead square

$$[\tau_{2n+1}, \tau_{2n+1}] : S^{2n+2l-1} \to S^{n+l}$$

denoted $w_{2n+1}$, as well as the classical James-Hopf invariant map

$$h_2 : \Omega S^{n+l} \to \Omega S^{2n+2l-1}.$$ 

Most of the next result is proven in [3] Proposition 11.3 (in which there is a misprint where $\Omega^2 (\phi)$ should be $\Omega^2 (\phi)$) and a mildly different proof is included in section 4 here for the convenience of the reader.
Further information concerning the divisibility of the Whitehead square is listed next. In case $2n + 1 = 2^j - 1$, that the Whitehead square is divisible by 2 is known as the strong form of the Kervaire invariant one conjecture and is known to admit a positive solution in case $2n + 1$ is $1, 3, 7, 15, 31$, or $63 \ [8\ [13]$. The Whitehead square is not divisible by 2 in case $2n + 1 \neq 2^j - 1$. Little is known about the answer in general in case $2n + 1 = 2^j - 1 > 63$. A reformulation of this topology question in terms of the Lie group $G_2$ and the zero divisors in a classical construction of L. E. Dickson concerning a (usually non-associative) multiplication on $\mathbb{R}^{2^n}$ is given in [14] with additional information given in [11].

In view of these examples, it appears that the cohomology of $\Omega_{2n+1}^k S^{2n+1} \{ \Psi \}$ and $\Omega_{2n+1}^k S^{2n+1} \{ [2] \}$ might be interesting as algebras over the Steenrod algebra. The homology of $\Omega_{2n+2}^k S^{2n+1}$ has been worked out by T. Hunter [6].

The proof of Theorem 1.1 depends on factorizations of $\Sigma_{2n+2}^k$ for $2n + 2 \neq 2^j$ which give lower bounds on $k$ via the method of evaluation of secondary operations of Brown, and Peterson [1]. For example, $\Sigma_{10}^k$ can be factored in the following two ways

$$\Sigma_{10}^k = \Sigma_4 \Sigma_2 \Sigma_4^2 + \Sigma_8 \Sigma_2^2 + \Sigma_4 \Sigma_6 \Sigma_1,$$
and $$\Sigma_{10}^k = \Sigma_2 \Sigma_8 + \Sigma_9 \Sigma_1^2$$

An application of the Brown and Peterson secondary operation with the first factorization gives that $\Omega^6 \{ [2] \} \neq \Psi^6 (2) : \Omega^6 S^9 \to \Omega^6 S^9$, while an application of the operation with the second factorization gives that $\Omega^2 \{ [2] \} \neq \Psi^2 (2) : \Omega^2 S^9 \to \Omega^2 S^9$. Hence the first factorization gives a stronger result.

The best lower bounds for $k$ using the methods above occur for the smallest value of $k$ such that $\Sigma_{2n+2}^k$ is in the left ideal of the Steenrod algebra given by

$$L(k) = A \{ \Sigma_1, \Sigma_2, \ldots, \Sigma_k \}.$$ 

In the example above, $\Sigma_{10}^k$ is in $L(2)$, and in fact $k = 2$ is the smallest such $k$. The smallest value of $k$ such that $\Sigma_{2n+2}^k$ is in $L(k)$ is given in [7], and is described next.

**Theorem 1.7.** If the two self-maps of $\Omega^k S^{2n+1}$ given by $\Omega^k \{ [2] \}$ and $\Psi^k (2)$ are homotopic, then

$$k \geq F(2n + 2),$$

where the function $F$ is defined below.

The following notation is used to define the function $F$. Given any positive integer $n$, let $[n]$ denote the dyadic expansion of $n$ viewed as an ordered sequence of zeroes and ones with right lexicographical ordering. That is if

$$n = 2^{j_n} + 2^{j_{n-1}} + \cdots + 2^{j_1} + 2^{j_0}$$

with

$$j_n > j_{n-1} > \cdots > j_1 > j_0 \geq 0,$$

then the dyadic expansion of $n$ is ambiguously denoted

$$\alpha = (\alpha_q, \alpha_{q-1}, \ldots, \alpha_1, \alpha_0)$$

for which

$$\alpha_s = \begin{cases} 1 & \text{if } s = j_m \text{ for } t \geq m \geq 0, \\ 0 & \text{if } s \neq j_m \text{ for } t \geq m \geq 0. \end{cases}$$

Notice that $(\alpha_q, \alpha_{q-1}, \ldots, \alpha_1, \alpha_0)$ and $(0, \alpha_q, \alpha_{q-1}, \ldots, \alpha_1, \alpha_0)$ are dyadic expansions of the same integer, and are both equal to $[n]$.

Given a binary string $\alpha$,

1. let $|\alpha|$ denote the integer with binary expansion $\alpha$,
2. let $\text{len}(\alpha)$ denote the length of the binary string $\alpha$, and
(3) let $z(\alpha)$ denote the number of non-trailing zeroes in a string $\alpha$, thus if

$$\alpha = (\alpha_q, \alpha_{q-1}, \ldots, \alpha_1, \alpha_0)$$

as above, then $z(\alpha) = \text{len}(\alpha) - t - j_0$. (For example, 

$$|010010000| = 2^7 + 2^4 = |10010000|,$$

$$\text{len}(010010000) = 9,$$

and $z(010010000)$ is 3.)

Given binary strings $\alpha$ and $\beta$, let $\alpha\beta$ denote their concatenation. With this notation the function $F$ is defined on a positive integer $n$ as follows. Write $[n] = \alpha\beta$ such that $|\alpha| < z(\beta)$ and $\text{len}(\beta)$ is minimal. Then

$$F(n) = n - 2^{\text{len}(\beta) - 2} + 1.$$  

There are two main computations in this article. One is the evaluation of an unstable secondary operation due to Brown and Peterson which gives a proof of Theorem 1.1. The second is a computation with the Nishida relations which gives a proof of Theorem 1.3.

A table of contents of this paper is as follows:

1: Introduction and statement of results
2: Unstable secondary operations related to the Whitehead product and the proof of Theorem 1.1
3: The Nishida relations and the proof of Theorem 1.3
4: On the Proof of Proposition 1.5

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2. **Unstable secondary operations related to the Whitehead product, and the proof of Theorem 1.1**

To prove Theorem 1.1 it suffices to check the statement that if the two self-maps of $\Omega^k S^{2n+1}$ given by $\Omega^k[2]$, $\Psi^k(2) : \Omega^k S^{2n+1} \to \Omega^k S^{2n+1}$ are homotopic, then

- for the case $2n + 1 = 2^t + 1$ with $t \geq 1$, it follows that $k \geq 2^t + 1$ and
- for the case $2n + 1 = 2^t + 1 + 1$ for $t \geq 1$, it follows that $k \geq 2^t + 1$.

The steps in the strategy of the proof are as follows.

1. Consider the difference of the two maps $\Omega^k[2]$ and $\Psi^k(2)$ restricted to the second May-Milgram filtration of $\Omega^k S^{2n+1}$.
2. Observe that this difference factors through the suspension of a truncated projective space.
3. Compute a non-trivial unstable secondary operation in the cohomology of the suspension of a truncated projective space (for the special cases listed directly above).
4. Conclude that the difference is essential when restricted to the suspension of a truncated projective space as given in step 3.
5. Conclude by Proposition 1.5 that the two maps fail to be homotopic when restricted to the second May-Milgram filtration of $\Omega^k S^{2n+1}$ in these special cases.
The proof of Theorem 1.1 thus reduces to an evaluation of certain unstable secondary cohomology operations constructed by Brown, and Peterson[1].

The Adem relations are $Sq^iSq^j = \sum_{0 \leq s \leq [i/2]} \binom{j-1}{i-2s} Sq^{i+j-s}Sq^s$ in case $i < 2j$. Thus if $2n+2$ is not a power of 2, $Sq^{2n+2}$ appears in some Adem relation which can be rewritten as

$$Sq^{2n+2} = \sum_{t_i \neq 2n+2} a_i Sq^{t_i}$$

for which $a_i$ is in the Steenrod algebra. The special cases given below suffice for the purposes here with more complete answers given in[2].

The secondary cohomology operations, devised by Brown, and Peterson[1] to detect the Whitehead square on spheres not of dimension $2^k - 1$ are described next in order to set up the context for the applications here. The results of[1] also give tertiary operations which detect the Whitehead square for spheres of dimension $2^k - 1$ with $k > 3$, but these operations are not used here. Additional information concerning secondary operations is listed in[3].

Consider the Eilenberg-Mac Lane space $K(\mathbb{Z}/2\mathbb{Z}, n)$ together with factorizations of $Sq^{2n+2} = \sum_{t_i \neq 2n+2} a_i Sq^{t_i}$ to obtain

$$K(\mathbb{Z}/2\mathbb{Z}, 2n+2) \xrightarrow{\Pi_{t_i \neq 2n+2}Sq^{t_i}} \Pi_{t_i \neq 2n+2} K(\mathbb{Z}/2\mathbb{Z}, 2n+2 + t_i)$$

with homotopy theoretic fibre denoted ambiguously by $E_{2n+2}$ (depending on the choice of factorization of $Sq^{2n+2}$). Thus there is a fibration

$$\Pi_{t_i \neq 2n+2} K(\mathbb{Z}/2\mathbb{Z}, 2n+1 + t_i) \to E_{2n+2} \to K(\mathbb{Z}/2\mathbb{Z}, 2n+2)$$

for which $t_{2n+2}$ denotes the fundamental cycle for the base, and $x_{2n+1+t_i}$ denotes the fundamental cycle for each Eilenberg-Mac Lane space in the fibre. The transgression of the cohomology class $\Sigma_i a_i x_{2n+1+t_i}$ is

$$\Sigma_i a_i Sq^{t_i}(x_{2n+2}) = Sq^{2n+2}(x_{2n+2}) = t_{2n+2}^2.$$ 

Next, consider the looped fibration above to obtain an analogous fibration

$$\Pi_{t_i \neq 2n+2} K(\mathbb{Z}/2\mathbb{Z}, 2n + t_i) \to \Omega E_{2n+2} \to K(\mathbb{Z}/2\mathbb{Z}, 2n)$$

The analogous cohomology class

$$\Sigma_i a_i x_{2n+2 + t_i}$$

obtained for this last fibration is an infinite cycle in the Serre spectral sequence for this fibration. Thus there is a choice of cohomology class $\Phi(t_{2n+1})$ in the cohomology of $\Omega E_{2n+2}$ which restricts to $\Sigma_i a_i x_{2n+2 + t_i}$ in the cohomology of $\Pi_{t_i \neq 2n+2} K(\mathbb{Z}/2\mathbb{Z}, 2n + t_i)$. Brown, and Peterson[1] show that the reduced coproduct for $\Phi(t_{2n+1})$ is non-trivial, and is thus given by

$$t_{2n+1} \otimes t_{2n+1}$$

by degree considerations.

The first non-vanishing homotopy group of $E_{2n+2}$ is given by $\pi_{2n+2}(E_{2n+2}) = \mathbb{Z}/2\mathbb{Z}$ with a choice of representative for a generator denoted

$$\lambda : S^{2n+2} \to E_{2n+2}.$$

The loop map

$$f = \Omega(\lambda) : \Omega S^{2n+2} \to \Omega E_{2n+2}$$

induces an isomorphism $f_* : H_{2n+1}(\Omega S^{2n+2}) \to H_{2n+1}(\Omega E_{2n+2})$.

Let $b_{2n+1}$ denote the fundamental cycle in $H_{2n+1}(\Omega S^{2n+2})$. Recall that $(b_{2n+1})^2$ is non-zero in the Pontrjagin ring. Thus $f^*(\Phi(t_{2n+1}))$ evaluates non-trivially on $(b_{2n+1})^2$.
with the natural pairing $< f^*(\Phi(t_{2n+1})), (b_{2n+1})^2 >= 1$. Consequently, $f$ induces a non-trivial map

$$f_* : H_{4n+2}(\Omega S^{2n+2}) \to H_{4n+2}(\Omega E_{2n+2}).$$

Furthermore, since the indeterminacy of the operation is given by any choice of map to the fibre

$$\Omega S^{2n+2} \to \bigoplus_{i \neq 2n+2} K(\mathbb{Z}/2\mathbb{Z}, 2n + t_i),$$

the indeterminacy is always trivial. Since the attaching map for the $4n + 2$ cell in a minimal cell decomposition of $\Omega S^{2n+2}$ is the Whitehead product $w_{2n+1}$, the operation of $\mathfrak{H}$ detects this element as long as $2n + 2 \neq 2^r$. These operations will be applied to the following context.

Recall $F_s$ the $s$-th May-Milgram filtration $\Omega^k S^{2n+1}$ which was exploited earlier by Toda [15] in the special case of $s = 2$. The inclusion $F_{s-1}$ in $F_s$ is a cofibration. The filtration quotient

$$F_2/F_1$$

is homotopy equivalent to

$$\Sigma^{2n+1-2k}(\mathbb{RP}^{2n}/\mathbb{RP}^{2n-2k}) = \Sigma^{2n+1-2k}(\mathbb{RP}^{2n+1-2k}).$$

Next consider the cofibre sequence

$$\Sigma^{2n+1-2k}(\mathbb{RP}^{2n-2k+1}) \xrightarrow{\text{inclusion}} \Sigma^{2n+1-2k}(\mathbb{RP}^{2n-2k+1}) \xrightarrow{K} S^{4n+1-2k}$$

for which

$$K : \Sigma^{2n+1-2k}(\mathbb{RP}^{2n-2k+1}) \to S^{4n+1-2k}$$

denotes the natural collapse map with an induced isomorphism (in mod 2 homology)

$$K_* : H_{4n+1} \to H_{4n+1}(S^{4n+1-2k}).$$

In addition, there is a “boundary” map obtained from the Barratt-Puppe sequence

$$\delta : S^{4n+1} \to \Sigma^{2n+2-2k}(\mathbb{RP}^{2n-2k+1}).$$

Consider the self-map of $\Omega^k S^{2n+1}$ given by the difference

$$\Delta_k = \Omega^k [2] - \Psi^k (2)$$

restricted to the second filtration $F_2 = F_2(\Omega^k S^{2n+1})$. By Proposition 1.5, the difference $\Delta_k = \Omega^k [2] - \Psi^k (2)$ restricted to the second May-Milgram filtration of $\Omega^k S^{2n+1}$ is null-homotopic if and only if the natural composite

$$\Sigma^{2n+1}(\mathbb{RP}^{2n-2k+1}) \xrightarrow{K} S^{4n+1} \xrightarrow{w_{2n+1}} S^{2n+1}$$

denoted $\lambda(n,k)$ is null-homotopic in which case, there is a homotopy commutative diagram

$$\begin{array}{ccc}
S^{4n+1} & \xrightarrow{w_{2n+1}} & \Sigma^{2n+2}(\mathbb{RP}^{2n-2k+1}) \\
\downarrow & & \downarrow \\
S^{2n+1} & \xrightarrow{1} & S^{2n+1}.
\end{array}$$

for some map $\bar{w}_{2n+1}$.

The next step in the proof of Theorem 1.1 is an examination of certain values of $n$ and $k$ such that the composite $\lambda(n,k)$ is essential. By definition, there is a homotopy commutative diagram

$$\begin{array}{ccc}
\Sigma^{2n+1}(\mathbb{RP}^{2n-2k+1}) & \xrightarrow{\lambda(n,k)} & S^{2n+1} \\
\downarrow K & & \downarrow 1 \\
S^{4n+1} & \xrightarrow{w_{2n+1}} & S^{2n+1}
\end{array}$$

together with an induced morphism of cofibre sequences.
Thus the action of the Steenrod algebra for the cohomology of \( \Sigma X \) is not necessarily unique up to homotopy. (Note: The choice of any choice of map \( J \) from the action on the cohomology of a truncated projective space.

Furthermore, any choice of map

\[
g : X(n, k) \to J_2(S^{2n+1})
\]

obtained from a morphism of cofibre sequences induces an isomorphism

\[
g_* : H_*(X(n, k)) \to H_*(J_2(S^{2n+1}))
\]

for \( i = 2n+1, 4n+2 \) by inspection of the long exact sequence in homology obtained from a cofibre sequence. Furthermore, there is a homotopy equivalence after one suspension,

\[
\Sigma (S^{2n+1} \vee S^{2n+2}(\mathbb{RP}^{2n-2k+1})) \to \Sigma X(n, k).
\]

Thus the action of the Steenrod algebra for the cohomology of \( \Sigma X(n, k) \) is obtained from the action on the cohomology of a truncated projective space.

Consider the composite

\[
X(n, k) \xrightarrow{g} J_2(S^{2n+1}) \xrightarrow{i} \Omega S^{2n+2}
\]

denoted

\[
G : X(n, k) \to \Omega S^{2n+2}
\]

for which \( i \) is an equivalence through the \( 6n+2 \) skeleton. Recall the map

\[
f : \Omega S^{2n+2} \to \Omega E_{2n+2}
\]

defined earlier in this proof. Since

\[
< f^*(\Phi(i_{2n+1})), (b_{2n+1}^+)^2 > = 1,
\]

it follows that

\[
< (G \circ f)^*(\Phi(i_{2n+1})), e_{4n+2} > = 1
\]

where \( e_{4n+2} \) is the unique non-trivial class in \( H_{4n+2}(X(n, k)) \). Thus, if the indeterminacy of the choice of map \( G \circ f : X(n, k) \to \Omega E_{2n+2} \) is zero, then \( \lambda(n, k) \) is essential as the resulting cohomology operation is itself non-trivial. Vanishing of the indeterminacy will be checked next for special cases.

Recall the Adem relations \( Sq^i Sq^j = \sum_{0 \leq s \leq [i/2]} \binom{j-s-1}{i-2s} Sq^{i+j-s}Sq^s \). There are two such relations to consider in the proof of Theorem 1.14. The first case is handled by the choice of operation arising from the relation

\[
Sq^{2t} Sq^{2t+1} = \sum_{0 \leq s \leq 2t-1} \binom{2t+1-s-1}{2t-2s} Sq^{2t+2t+1-s}Sq^s
\]

for \( t \geq 1 \). It follows that

\[
Sq^{2t+1} = Sq^{2t} Sq^{2t+1} + \sum_{1 \leq s \leq 2t-1} \binom{2t+1-s-1}{2t-2s} Sq^{2t+1+2t+1-s}Sq^s
\]

with

1. \( 2n+2 = 2t+1 + 2t \) for \( t \geq 1 \),
2. \( 2t - 2s \geq 0 \), and
3. \( 2t + 2t+1 - s \geq 2t + 2t+1 - 2t-1 = 2t-1 + 2t+1 \).
To check that this operation has zero indeterminacy and thus that the map \( \lambda(n, k) \) is essential, it suffices to check that the operation \( a_s = Sq^{2^t+2^{t+1} - s} \) vanishes on the cohomology of \( X(n, k) \) for \( 2n + 2 = 2^{t+1} + 2^t \), \( t \geq 1 \) and \( 2^t - 2s \geq 0 \) with \( s > 0 \).

Since there is a homotopy equivalence
\[
\Sigma(S^{2n+1} \vee \Sigma^{2n+2}(\mathbb{R}P^{2n-2-k+1})) \to \Sigma X(n, k),
\]
it suffices to check that the operation \( a_s = Sq^{2^t+2^{t+1} - s} \) vanishes on the cohomology of \( \mathbb{R}P^{2n-2-k+1} \). The assumption that \( k < 1 < 2^t \) gives that this operation has zero indeterminacy by a check of degrees and thus the map \( \lambda(n, k) \) is essential.

The assumption that the two self-maps of \( \Omega^kS^{2n+1} \) given by \( \Omega^k[2] \), and \( \Psi^k(2) \) are homotopic implies that \( \Delta_k \), and hence \( \lambda(n, k) \) is null in case \( n = 2^t + 2^{t-1} - 1 \), with \( k < 1 < 2^t \), contradicting Proposition \cite{13}. Hence, \( k < 1 \geq 2^t \) and Theorem \cite{13} part 1, follows.

The second case is handled by the choice of operation arising from the relation
\[
Sq^{2^t} Sq^{2^t + 2^t} = \Sigma_{0 \leq s \leq 2^{t-1}} \left( \frac{2^{t+1} - s}{2^t - 2s} \right) Sq^{2^t+2^{t+1} - s} Sq^s
\]
for \( t \geq 1 \). It follows that
\[
Sq^{2^t + 2^{t+1}} = Sq^{2^t} Sq^{2^t + 2^t} + \Sigma_{1 \leq s \leq 2^{t-1}} \left( \frac{2^{t+1} - s}{2^t - 2s} \right) Sq^{2^t+2^{t+1} - s} Sq^s
\]
with
1. \( 2n + 2 = 2 + 2^{t+1} \) for \( t \geq 1 \),
2. \( 2^t \geq 2s \), and
3. \( 2 + 2^{t+1} - s \geq 2 + 2^{t+1} - 2^{t-1} = 2 + 2^t + 2^{t-1} \).

Thus if \( k < 1 + 2^t \) with \( n = 2^t \) and \( 2^{t-1} \geq s \) as above, the operation \( a_s = Sq^{2^t+2^{t+1} - s} \) vanishes on the cohomology of \( \mathbb{R}P^{2n-2-k+1} \) and thus \( a_s \) vanishes on the cohomology of \( X(n, k) \). Hence if \( k < 1 + 2^t \), the associated operation has zero indeterminacy in the cohomology of \( X(n, k) \) and so the map \( \lambda(n, k) \) is essential. The rest of the proof in this case is analogous to that for the first case and is omitted. It follows that \( k \geq 1 + 2^t \) and Theorem \cite{13} part 2, follows.

The values of \( k \) given in Table 1 follow from an analogous secondary operation obtained from an iteration of the Adem relations as listed next rather than the explicit estimates in Theorem \cite{13}. For example, the relation
\[
Sq^{18} = Sq^8 [Sq^4(Sq^4 Sq^4 + Sq^5 Sq^1) + Sq^8 Sq^2] + Sq^{16} Sq^2 + Sq^{15} Sq^3 + Sq^{14} Sq^4
\]
gives a stronger result than that stated in Theorem \cite{13}. The indeterminacy of the associated secondary operation is zero by an inspection of the action of the Steenrod operations on the cohomology of \( X(8, 14) \) a space which satisfies the property that \( \Sigma X(8, 14) \) is homotopy equivalent to \( \Sigma(S^{17} \vee \Sigma^{18}(\mathbb{R}P^{16})) \). The relations listed next are used to give the results in Table 1 by a direct check that the indeterminacy vanishes in these cases. The details are omitted.

1. \( Sq^6 = Sq^2 Sq^4 + Sq^5 Sq^1 \).
2. \( Sq^{10} = Sq^4 Sq^6 + Sq^5 Sq^2 = Sq^4(Sq^2 Sq^4 + Sq^5 Sq^1) + Sq^8 Sq^2 \).
3. \( Sq^{18} = Sq^8 Sq^{10} + Sq^{16} Sq^2 + Sq^{15} Sq^3 + Sq^{14} Sq^4 \) and thus \( Sq^{18} = Sq^8(Sq^2 Sq^4 + Sq^5 Sq^1) + Sq^8 Sq^2) + Sq^{16} Sq^2 + Sq^{15} Sq^3 + Sq^{14} Sq^4 \).
4. \( Sq^{12} = Sq^4 Sq^8 + Sq^{11} Sq^1 + Sq^{10} Sq^2 \).
5. \( Sq^{14} = Sq^6 Sq^8 + Sq^{13} Sq^1 + Sq^{11} Sq^3 \).

3. The Nishida relations and the proof of Theorem \cite{13}

Information concerning the mod-2 homology of the spaces \( \Omega^kS^n\{[2] \} \) and \( \Omega^kS^n\{\Psi \} \) is given below. This information is used to show that the action of the Steenrod operations on the mod-2 cohomology of the spaces in Theorem \cite{13} differ, thus proving
the Theorem. References are [2, 3]. In what follows below, assume that $1 < k < n - 3$ with homology always taken with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

The mod-2 homology $H_\ast \Omega^k S^n \{[2]\}$ is isomorphic to $H_\ast \Omega^k S^n \otimes H_\ast \Omega^{k+1} S^n$ as a Hopf algebra in case $1 < k < n - 3$ with the natural map

$$H_\ast \Omega^k S^n \{[2]\} \to H_\ast \Omega^k S^n$$

induced by a $k$-fold loop map. This map is an epimorphism of Hopf algebras over the mod-2 Steenrod algebra. Thus there is a unique class $x_{n-k}$ in $H_{n-k} \Omega^k S^n \{[2]\}$, which projects to a class by the same name in $H_\ast \Omega^k S^n$. There is a unique non-zero class $x_{n-k-1}$ in $H_{n-k-1} \Omega^k S^n \{[2]\}$ which is in the image of the natural map

$$H_\ast \Omega^{k+1} S^n \to H_\ast \Omega^k S^n \{[2]\}$$

such that $Sq^i(x_{n-k}) = x_{n-k-1}$. The Nishida relations are given by

$$Sq^i Q_r(x) = \Sigma_{0 \leq t \leq i, r + q - 2t + 2i} Q_{r-2t+2i} Sq^t(x)$$

for any class $x$ of degree $q$ with binomial coefficients given by

$$(a, b) = (a + b)!/a! \cdot b!$$

for $a, b \geq 0$. The Steenrod operations in $H_\ast \Omega^k S^n \{[2]\}$ then follow by specialization. Examples of Steenrod operations acting on the classes

(1) $Q_i(x_{n-k})$ for $0 \leq i \leq k - 1$

(2) $Q_j(x_{n-k-1})$ for $0 \leq j \leq k$

will be considered next. Specialize to the case $H_\ast \Omega^{2n} Sq^{2n+2} \{[2]\}, q \geq 1$. Thus there are classes

$$v = x_{q_{2n+2}-2n+1}$$

and

$$u = x_{q_{2n+2}-2n}$$

with $Sq^i x_{q_{2n+2}-2n+1} = Sq^i(v) = u = x_{q_{2n+2}-2n}$ as given by the above remarks (for which reference to degrees is deliberately omitted in the cases of $u$, and $v$). Notice that

(1) $Sq^2 Q_{2n-1}(v) = \Sigma_{0 \leq i \leq 2n-2i, 2n-1 + |v| - 2n+1 + 2i} Q_{2n+2i} Sq^i(v)$

(2) $Sq^2 Q_{2n-1}(v) = (2n - 2, 2n - 1 + |v| - 2n+1 + 2) Q_1 Sq^i(v)$

and

(3) $Sq^2 Q_{2n-1}(v) = (2n - 2, 2n+2 - 2n+1 + 2) Q_1 u$

By [2], the module of primitive elements in $H_\ast \Omega^k S^n \{[2]\}$ is spanned by $Q_{i_1} Q_{i_2} \cdots Q_{i_k}(v)$ and $Q_{j_1} Q_{j_2} \cdots Q_{j_m}(u)$ for $1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n - 1$ and $1 \leq j_1 \leq j_2 \leq \cdots \leq j_m \leq n$ as well as their 2$\ast$-th. The degree of $Q_1(x)$, $|Q_1(x)|$, is given by $i + 2|x|$. If $2 < 2k < n - 2$, then the next result follows by induction and a standard degree count [2, 3].

**Lemma 3.1.** Assume that

$$2 < 2k < n - 2.$$

There are unique non-trivial primitive elements

$$v \in H_{n-k} \Omega^k S^n \{[2]\}$$

and

$$u \in H_{n-k-1} \Omega^k S^n \{[2]\}.$$

A basis for the module of primitives $PH_\ast \Omega^k S^n \{[2]\}$ in degrees less than $3(n - k - 1)$ is

$$\{u, v, Q_i(u), Q_j(v) | 0 \leq i \leq n - k, 0 \leq j \leq n - k - 1\}.$$
Furthermore, the element $Q_{k-1}(v)$ is the unique non-trivial primitive in $H_{2n-k-1}\Omega^k S^n\{[2]\}$. Thus there is exactly one non-trivial primitive element in $H_{q2^{n+3} - 2^n - 1}\Omega^{2^n} S^{q2^{n+2}+1}\{[2]\}$ given by $Q_{2^n-1}(v)$.

The action of certain Steenrod operations is given next.

**Lemma 3.2.** If $n > 1$ and $q \geq 1$, then $(2^n - 2, q2^{n+2} - 2^n + 1) = 0$ modulo 2. Thus in $H_* \Omega^{2^n} S^{q2^{n+2}+1}\{[2]\}$, there are unique non-trivial primitive elements $v = x_{q2^{n+2} - 2^n + 1}$ and $u = x_{q2^{n+2} - 2^n}$ with $Sq^1_1 v = u$ and $Sq^1_2 Q_{2^n-1}(v) = 0$ for the unique non-zero primitive element $Q_{2^n-1}(v)$ in degree $q2^{n+3} - 2^n - 1$.

**Proof.** To prove Lemma 3.2, recall the well-known method for evaluating binomial coefficients via $p$-adic expansions [14]. Let $p$ be a prime. Assume that $a$ and $b$ are strictly positive integers for which there are choices of $p$-adic expansions $a = \sum_{i=0}^m a_i p^i$ and $b = \sum_{i=0}^m b_i p^i$, $0 \leq a_i, b_i < p$. Then

$$\binom{a}{b} = \prod_{i=0}^m \binom{a_i}{b_i} \pmod{p}. $$

Thus, consider the mod 2 reduction of the binomial coefficient given by

$$(2^n - 2, q2^{n+2} - 2^n + 1) = \binom{\alpha}{\beta}$$

for $\alpha = q2^{n+2} - 2^n$ and $\beta = 2^n - 2$. Notice that

$$q2^{n+2} - 2^n = (q - 1)2^{n+2} + 2^{n+2} - 2^n = (q - 1)2^{n+2} + 2^n + 1 + 2^n$$

where $q - 1$ is a non-negative integer. In this case, the 2-adic expansion for $q2^{n+2} - 2^n = \sum_{i=0}^m a_i 2^i$ has $a_i = 0$ for $i \leq n - 1$ and $a_n = a_{n+1} = 1$. The 2-adic expansion for $2^n - 2 = \sum_{i=0}^m b_i 2^i$ has $b_i = 1$ for $1 \leq i \leq n - 1$ and all other $b_i$ are 0. Hence, it follows that $1 \leq i \leq n - 1$

$$\binom{a_i}{b_i} = \binom{q}{1} \equiv 0 \pmod{2},$$

and thus

$$\binom{2^n - 2}{q - 2^n} = \prod_{i=0}^m \binom{a_i}{b_i} = 0 \pmod{2}. $$

Features of the homology of $\Omega^k S^n\{\Psi\}$ were worked out in [2] for $1 < k < n - 3$ using the stabilization map $E : S^n \to QS^n$ to obtain a map

$$\gamma : \Omega^k S^n\{\Psi\} \to (\Omega^k QS^n)\{\Psi\}.$$ Computation with the Steenrod operations will be given using $\gamma$. The following properties are satisfied in these cases by [2].

(1) The mod-2 homology of $\Omega^k S^n\{\Psi\}$ is isomorphic to

$$H_* \Omega^k S^n \otimes H_* \Omega^{k+1} S^n$$

as a Hopf algebra. The natural map

$$H_* \Omega^k S^n\{\Psi\} \to H_* \Omega^k S^n$$

induced by a $k - 1$-fold loop map and is an epimorphism of Hopf algebras over the mod-2 Steenrod algebra. Thus there are unique (non-trivial) primitive elements $x_j$ in $H_* \Omega^k S^n\{\Psi\}$ for $j$ equal to either $n - k$, or $n - k - 1$.

It is convenient for the computations below to abbreviate the names of two elements in the following way: $u = x_{n-k-1}$ and $v = x_{n-k}$.
(2) The mod-2 homology of \((\Omega^kQ^n)\{\Psi}\) is isomorphic to

\[ H_*\Omega^kQ^n \cong H_*\Omega^{k+1}Q^n \]

as a Hopf algebra. The natural map

\[ H_*(\Omega^kQ^n)\{\Psi\} \rightarrow H_*\Omega^kQ^n \]

induced by an infinite loop map is an epimorphism of Hopf algebras over the mod-2 Steenrod algebra. Thus there are unique (non-trivial) primitive elements \(y_j\) in \(H_j((\Omega^kQ^n)\{\Psi}\)) for \(j\) equal to either \(n - k\), or \(n - k - 1\).

It is again convenient for the computations below to abbreviate the names of two elements in the following way: \(w = y_{n-k-1}\) and \(z = y_{n-k}\).

(3) There exist primitive elements in \(H_{i+2(n-k)}\Omega^kQ^n\{\Psi\}\) for \(0 \leq i \leq k-1\) denoted \(Q_i(x_{n-k})\) which project to elements \(Q_i(x_{n-k})\) in \(H_*\Omega^kQ^n\).

The elements \(Q_i(x_{n-k})\), for \(0 \leq i \leq k - 2\) are given by the Araki-Kudo-Dyer-Lashof operations \(Q_i\) on \(x_{n-k}\). However, the element \(Q_{k-1}(x_{n-k})\) is not given by an operation. The symbol \(Q_{k-1}(-)\) is a formal bookkeeping device; this symbol does not mean that it is given by an operation.

(4) By \[3\], the formula

\[ \gamma_*(Q_{k-1}(x_{n-k})) = Q_{k-1}(y_{n-k}) + Q_{k+1}(y_{n-k}+1) \]

is satisfied in case \(n\) is not equal to 3 modulo 4.

Using this information, the Steenrod operations on \(Q_{k-1}(x_{n-k})\) in \(H_*\Omega^kQ^n\{\Psi\}\) will be given from the Nishida relations by using the map \(\gamma\) in the special cases \(\Omega^2S^{2n+2}+1\{\Psi\}\) with \(n \geq 1\). Thus \(q^{2n+2}+1\) is not 3 modulo 4, and \[3\] applies. A direct count of degrees analogous to that in Lemma \[5.1\] gives uniqueness of certain primitives.

Lemma 3.3. Assume that

\[ 2 < 2k < n - 2. \]

A basis for the module of primitives \(PH_*\Omega^kQ^n\{\Psi\}\) in degrees less than \(3(n-k-1)\) is

\[ \{x_{n-k-1}, x_{n-k}, Q_1(x_{n-k-1}), Q_j(x_{n-k})|0 \leq i \leq n-k, 0 \leq j \leq n-k-1\}. \]

Furthermore, the element \(Q_{k-1}(x_{n-k})\) is the unique non-trivial primitive of degree \(2n-k-1\). Thus there is exactly one non-trivial primitive element in

\[ H_{q^{2n+3-2n+1}}\Omega^{2n}S^{2n+2+1}\{\Psi\} \]

given by \(Q_{2n-1}(x_{n-k})\).

Lemma 3.4. If \(n > 1\) and \(q \geq 1\), then \((2^n, q^{2n+2} - 2n+1 + 1) = 1\) modulo 2. Hence, the unique non-zero primitive elements \(u = x_{q^{2n+2}-q-2n+1}\) and \(v = x_{q^{2n+2}-2n+1}\) in \(H_*\Omega^{2n}S^{2n+2+1}\{\Psi\}\) satisfy

- (1) \(Sq^2v = u\),
- (2) \(Sq^n Q_{2n-1}(v) \neq 0\) and
- (3) \(Q_{2n-1}(v)\) is the unique non-zero primitive element in

\[ H_{q^{2n+3-2n-1}}(\Omega^{2n}S^{2n+2+1}\{\Psi\}). \]

Proof of \[3.3\] To prove Lemma \[3.3\] first consider the binomial coefficient

\[ (2^n, q^{2n+2} - 2n+1 + 1) = \left(\frac{q^{2n+2} - 2n+1 + 1}{2^n}\right). \]

Assuming \(q \geq 1\) the 2-adic expansion for \(q^{2n+2} - 2n+1 + 1 = (q-1)2n+2 + 2n+1 + 2^n + 1 = \sum_{i=0}^n a_i 2^i\) has \(a_{n+1} = a_n = a_0 = 1\) with \(a_i = 0\) for \(1 \leq i \leq n - 1\). The 2-adic expansion for \(2^n = \sum_{i=0}^n b_i 2^i\) has \(b_n = 1\) with all other \(b_i = 0\). Thus if \(i \neq n\),
\((\binom{n}{i}) = (\binom{n}{i}) = 1\) with \(\binom{n}{0} = (1)\). Thus \(\binom{2n+2q-2n+1+2n+1}{2n} = \prod_{i=0}^{m} \binom{n}{i} = 1 \mod 2\) and the formula for binomial coefficients follows.

Recall the abbreviation of the names of classes as above with \(u = x_{n-k-1}\) in \(H_{n-k-1}(\Omega^k S^n(\Psi))\) and \(v = x_{n-k}\) in \(H_{n-k}(\Omega^k S^n(\Psi))\). A second abbreviation is given by \(w = y_{n-k-1}\) in \(H_{n-k-1}(\Omega^k Q S^n(\Psi))\), and \(z = y_{n-k}\) in \(H_{n-k}(\Omega^k Q S^n(\Psi))\).

Next consider the element
\[
\gamma_s(\bar{Q}_{2^n-1}(v)) = Q_{2^n-1}(z) + Q_{2^n+1}(w).
\]
The next properties follow at once.

1. \(Sq^{2n}\gamma_s(\bar{Q}_{2^n-1}(v)) = Sq^{2n}_s Q_{2^n-1}(z) + Sq^{2n}_s Q_{2^n+1}(w)\) for \(n \leq k = q^{2n+2} - 2^n\).

2. \(Sq^{2n}_s(Q_{2^n-1}(z)) = 0\) by Lemma 3.2.

3. \(Sq^{2n}_s Q_{2^n+1}(w) = (2^n, 2^n+1 + q^{2n+2} - 2^n - 2^{n+1}) Q_{2^n+1-2^n}(w)\).

4. Since the binomial coefficient \((2^n, 2^n+2 - 2^{n+1} + 1)\) is 1 modulo two,
\[
Sq^{2n}_s Q_{2^n+1}(w) = Q_1(w)
\]
and so \(Sq^{2n}_s Q_{2^n+1}(w) \neq 0\).

5. The element \(Sq^{2n}_s \gamma_s(\bar{Q}_{2^n-1}(v))\) is non-zero.

6. By Lemma 3.3, the element \(\bar{Q}_{2^n-1}(v)\) is the unique non-trivial primitive in \(H_{2^n+3-2^n-1}(\Omega^{2n} Sq^{2n+2+1}(\Psi))\).

The lemma follows. \(\square\)

The proof of Theorem 1.3 is given next.

Proof of 1.3: Lemmas 3.2 and 3.4 immediately imply Theorem 1.3 that \(\Omega^{2n} S^{2^n+q+1}\{2\}\), and \(\Omega^{2n} S^{2^n+q+1}\{2\}\) are not homotopy equivalent for \(n > 1\) and \(q \geq 1\) as these spaces have different actions of the Steenrod algebra as follows.

Assume that \(n > 1\) and \(q \geq 1\). By 3.2 there is an unique non-zero primitive element in \(H_{t}(\Omega^{2n} S^{2^n+q+1}\{2\})\) for \(t = 2^{n+3} - 2^n - 1\) given by \(Q_{2^n-1}(v)\). Furthermore, this element satisfies \(Sq^{2n}_s Q_{2^n-1}(v) = 0\).

Again assume that \(n > 1\) and \(q \geq 1\). By 3.4 there is an unique non-zero primitive element in \(H_{t}(\Omega^{2n} S^{2^n+q+1}\{\Psi\})\) for \(t = 2^{n+3} - 2^n - 1\) given by \(\bar{Q}_{2^n-1}(v)\). Furthermore, this element satisfies \(Sq^{2n}_s Q_{2^n-1}(v) \neq 0\).

The unique non-zero primitive elements in degree \(2^{n+3} - 2^n - 1\) support different actions of \(Sq^{2n}_s\). Hence the mod-2 cohomology of the spaces \(\Omega^{2n} S^{2^n+q+1}\{2\}\), and \(\Omega^{2n} S^{2^n+q+1}\{\Psi\}\) differ. Theorem 1.3 follows.

\(\square\)

4. On the Proof of Proposition 1.5

A proof of the main part of Proposition 1.5 is given below for convenience of the reader. Proposition 1.5 is a restated version Proposition 11.3 of 3 in which there is a misprint where \(\Omega^q(\phi)\) should be \(\Omega^{q-1}(\phi)\) (as stated in section 1 here).

Theorem 4.1. Assume that the composite
\[
\Omega^k S^{2n+1} \xrightarrow{\Omega^{k-1}h_{2^n}} \Omega^k S^{4n+1} \xrightarrow{\Omega^k(w_{2^n+1})} \Omega^k S^{2n+1}
\]
is null-homotopic.

Then the composite
\[
S^{2n+1-k}(\mathbb{R}P^{2n}_{2n-k+1}) \xrightarrow{\text{collapse}} S^{4n+1} \xrightarrow{w_{2n+1}} S^{2n+1}
\]
is null-homotopic.

Remark: The results in this article do not rule out the possibility that the converse of Theorem 4.1 may be satisfied.
Proof of Theorem 4.1. Let \( F_2 = F_2(\Omega^k S^{2n+1}) \) denote the second filtration of the May-Milgram construction for \( \Omega^k S^{2n+1} \) with \( I : F_2 \to \Omega^k S^{2n+1} \) giving the natural inclusion. One fact is that there is a cofibration sequence

\[
S^{2n+1-k} \xrightarrow{\text{inclusion}} F_2 \xrightarrow{\text{collapse}} \Sigma^{2n+1-k}(\mathbb{RP}^{2n}_{2n-k+1}) \xrightarrow{\delta} S^{2n+2-k} \xrightarrow{} \ldots
\]

with the property that the \( k \)-fold suspension of \( \delta, \Sigma^k(\delta) \), is null-homotopic.

Next, consider the commutative diagram

\[
\begin{array}{ccc}
S^{2n+1-k} & \xrightarrow{\text{inclusion}} & F_2 \\
\downarrow \text{id} & & \downarrow \text{id} \\
\Sigma^{2n+1-k}(\mathbb{RP}^{2n}_{2n-k+1}) & \xrightarrow{\delta} & \Omega^k S^{2n+1} \\
\downarrow \text{id} & & \downarrow \Omega^k S^{4n+1} \\
S^{2n+1-k} & \xrightarrow{\text{coll}} & \Omega^k S^{4n+1} \\
\end{array}
\]

Notice that if \( n \geq 1 \), then any map \( S^{2n+1-k} \to \Omega^k S^{4n+1} \) is null and so there is a homotopy commutative diagram

\[
\begin{array}{ccc}
F_2 & \xrightarrow{\text{collapse}} & \Sigma^{2n+1-k}(\mathbb{RP}^{2n}_{2n-k+1}) \\
\downarrow \Omega^k S^{4n+1} & \xrightarrow{\text{id}} & \downarrow \Omega^k S^{4n+1} \\
\Omega^k S^{4n+1} & \xrightarrow{\text{id}} & \Omega^k S^{4n+1} \\
\end{array}
\]

for some map \( \Theta \) by the standard properties of the cofibration sequence above for \( F_2 \).

Observe that there is a homotopy commutative diagram

\[
\begin{array}{ccc}
\Sigma^{2n+1-k}(\mathbb{RP}^{2n}_{2n-k+1}) & \xrightarrow{1} & \Sigma^{2n+1-k}(\mathbb{RP}^{2n}_{2n-k+1}) \\
\downarrow K & & \downarrow \Theta \\
S^{4n+1-k} & \xrightarrow{E^k} & \Omega^k S^{4n+1} \\
\end{array}
\]

as \( \Omega^k S^{4n+1} \) is \( 4n-k \)-connected.

Next assume that

\[
\Omega^k(w_{2n+1}) \circ \Omega^k h_2
\]

is null. This assumption gives that \( \Omega^k(w_{2n+1}) \circ \Omega^k h_2 \circ I \) is also null-homotopic. Thus there is yet another homotopy commutative diagram

\[
\begin{array}{ccc}
F_2 & \xrightarrow{\text{collapse}} & \Sigma^{2n+1-k}(\mathbb{RP}^{2n}_{2n-k+1}) \\
\downarrow \Omega^k S^{4n+1} & \xrightarrow{\text{id}} & \downarrow \Omega^k S^{4n+1} \\
\Omega^k S^{4n+1} & \xrightarrow{\text{id}} & \Omega^k S^{4n+1} \\
\end{array}
\]

where the vertical left-hand composite is null by assumption. Thus the right-hand vertical composite map in this diagram

\[
\begin{array}{ccc}
\Sigma^{2n+1-k}(\mathbb{RP}^{2n}_{2n-k+1}) & \xrightarrow{\Theta} & \Omega^k S^{4n+1} \\
\downarrow \Theta & & \downarrow \Omega^k S^{2n+1} \\
\Omega^k S^{2n+1} & \xrightarrow{\text{id}} & \Omega^k S^{2n+1} \\
\end{array}
\]
factors, up to homotopy, through the cofibre of the natural map

\[ F_2 \to \Sigma^{2n+1-k}(\mathbb{RP}^{2n}_{2n-k+1}) \]

Hence there is a homotopy commutative diagram

\[
\begin{array}{ccc}
\Sigma^{2n+1-k}(\mathbb{RP}^{2n}_{2n-k+1}) & \xrightarrow{\delta} & S^{2n+2-k} \\
\Omega^k (w_{2n+1}) \circ \Theta & \downarrow & \alpha \\
\Omega^k S^{2n+1} & \xrightarrow{\text{identity}} & \Omega^k S^{2n+1}
\end{array}
\]

for some choice of map \( \alpha \).

Since the cofibration sequence

\[
S^{2n+1-k} \xrightarrow{\text{inclusion}} F_2 \xrightarrow{\text{collapse}} \Sigma^{2n+1-k}(\mathbb{RP}^{2n}_{2n-k+1}) \xrightarrow{\delta} S^{2n+2-k}
\]

satisfies the property that \( \Sigma^k(\delta) \) is null, passage to adjoints gives the following homotopy commutative diagram.

\[
\begin{array}{ccc}
\Sigma^k(\Sigma^{2n+1-k}(\mathbb{RP}^{2n}_{2n-k+1})) & \xrightarrow{\Sigma^k(\delta)} & \Sigma^k(S^{2n+2-k}) \\
\Sigma^k(\Omega^k (w_{2n+1}) \circ \Theta) & \downarrow & \Sigma^k(\alpha) \\
\Sigma^k(\Omega^k S^{2n+1}) & \xrightarrow{1} & \Sigma^k(\Omega^k S^{2n+1}) \\
\downarrow \text{evaluation} & & \downarrow \text{evaluation} \\
S^{2n+1} & \xrightarrow{1} & S^{2n+1}
\end{array}
\]

Hence the vertical left-hand composite factors through the null-homotopic map \( \Sigma^k(\delta) \), and the theorem follows.

\[ \square \]

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