Global unique solvability of the initial-boundary value problem for one-dimensional barotropic equations of viscous compressible bifluids

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Abstract. We consider the equations of a multi-velocity model of a binary mixture of viscous compressible fluids (two-fluid medium) in the case of one-dimensional barotropic motions. We prove the global (in time) existence and uniqueness of a strong solution to the initial-boundary value problem describing the motion in a bounded spatial domain.

1. Introduction

The paper is devoted to the problem of unique solvability for the equations of motion of viscous compressible fluid mixtures (multifluid media, multicomponent media, multifluids). Regarding the origin of the model and its physical meaning, we refer to [1–4]. A survey of possible formulations of the model and known results can be found in [5, 6]. Related multi-velocity models are discussed in [7–11]. As the first results on the solvability of multifluid models in the multidimensional case (but in approximate formulations), we can indicate [12–14].

Weak solutions in the multidimensional case for the considered version of the model were constructed in [15–17]. Similar results for related models were obtained in [18–23]. For a number of reasons, including the goal of constructing more regular solutions, one-dimensional formulations are of interest. A detailed discussion of these formulations and an overview of the results can be found in [24]. In short, in multi-fluid models there are two dimensions, namely, the number of spatial variables and the number of components of the mixture. The equality of one of these parameters to one does not “weaken” the multidimensionality of the other parameter. Therefore, the study of one-dimensional flows of a multicomponent medium is of interest. Solvability for related one-dimensional models is shown in [25–27].

The peculiarity of the paper is that a variant of the model with non-diagonal (namely, triangular) viscosity matrix is considered. Mathematically, we are talking about a system of equations of a mixed type, obtained by combining compressible Navier—Stokes-type systems, but additionally interconnected in the higher order terms due to the viscosity matrix and therefore not allowing the automatic transfer of the known results of the classical (single-fluid) theory developed in such works as [28–31].

Solvability results for a model with a diagonal viscosity matrix are obtained in [32, 33]. However, it should be noted that we do not use here the technique of these works, but we directly
extend the technique developed for a single-component viscous gas to the multi-component case with a non-diagonal viscosity matrix.

2. Statement of the Problem and the Main Result

We consider the system of equations of motion of binary viscous compressible fluid mixtures in the case of one spatial variable \((i = 1, 2)\):

\[
\begin{align*}
\partial_t \rho_i + \partial_x (\rho_i u_i) &= 0, \\
\rho_i (\partial_t u_i + u_i \partial_x u_i) + K_i \partial_x (\rho_i) &= \sum_{j=1}^{2} \mu_{ij} \partial_{xx} u_j + (-1)^{i+1} a(u_2 - u_1).
\end{align*}
\]

Here \(\rho_i\) (the density of the \(i\)-th constituent of the mixture) and \(u_i\) (the velocity of the \(i\)-th component) are unknown values, and we are given physical constants \(a > 0, \ K_i > 0, \ \gamma_i > 1, \ i = 1, 2\), and constant viscosity coefficients \(\{\mu_{ij}\}_{i,j=1}^2\), which form the matrix \(\mathbf{M} > 0\), i. e. \((\mathbf{M} \xi, \xi) \geq M_0 |\xi|^2\) for all \(\xi \in \mathbb{R}^2\) with some constant \(M_0(\mathbf{M}) > 0\).

Let us consider the system (1), (2) in the rectangular \(Q_T\) (here and below \(Q_t = (0, 1) \times (0, t)\)) with an arbitrary finite positive height \(T\) complemented by the following initial and boundary conditions \((i = 1, 2)\):

\[
\begin{align*}
\rho_i|_{t=0} &= \rho_{0i}(x), \quad u_i|_{t=0} = u_{0i}(x), \quad x \in [0, 1], \\
&
\rho_i|_{x=0} = \rho_i|_{x=1} = 0, \quad t \in [0, T].
\end{align*}
\]

**Definition 1.** By a strong solution to the problem (1)–(4) we mean a collection of functions \((\rho_1, \rho_2, u_1, u_2)\) such that the equations (1), (2) are valid almost everywhere in \(Q_T\), the initial data (3) are accepted for almost all \(x \in (0, 1)\), the boundary conditions (4) are valid for a. a. \(t \in (0, T)\), and the following inequalities and inclusions hold \((i = 1, 2)\):

\[
\begin{align*}
\rho_i &\geq \text{const} > 0 \quad \text{a. e. in } Q_T, \\
\rho_i &\in L_{\infty}(0, T; W^1_2(0, 1)), \\
\partial_t \rho_i &\in L_{\infty}(0, T; L^1_2(0, 1)), \\
u_i &\in L_{\infty}(0, T; W^1_2(0, 1)) \cap L^2_2(0, T; W^2_2(0, 1)), \\
\partial_t u_i &\in L^2_2(Q_T);
\end{align*}
\]

thus, (1) are understood as equalities in \(L_{\infty}(0, T; L^2_2(0, 1))\), and (2) in \(L^2_2(Q_T)\).

**Theorem 2.** Let the initial data in (3) satisfy the conditions

\[
\rho_{0i} \in W^1_2(0, 1), \quad \rho_{0i} > 0, \quad u_{0i} \in W^0_2(0, 1), \quad i = 1, 2,
\]

and \(\mu_{12} = 0\). Then there exists the unique strong solution to the problem (1)–(4) in the sense of Definition 1.

**Proof of Theorem 2.** The existence of a unique strong solution to the problem (1)–(4) in a small time interval \([0, t_0]\) is proved using the Galerkin method, see [27]. To extend the solution from the segment \([0, t_0]\) to the entire segment \([0, T]\) under consideration, it is necessary to obtain a priori estimates for the local solution in the classes (5), the limiting constants in which depend only on the input data and on the value of \(T\), but not on the parameter \(t_0\). Therefore, we will focus on obtaining global estimates.

Consider a hypothetical solution \((\rho_1, \rho_2, u_1, u_2)\) of the problem (1)–(4), which has all the necessary differential properties, and such that the densities \(\rho_i, i = 1, 2\) are strictly positive and bounded (see Definition 1).

2
3. Lagrangian Mass Coordinates

We transform the problem (1)–(4) introducing two variants of the Lagrangian coordinates associated with the velocity of each component of the mixture, with the number \( m \in \{1, 2\} \).

In other words, we take \( y_m(x, t) = \int_0^t \rho_m(s, t) \, ds \) and \( t \) as new independent variables. Then the system (1), (2) takes the form

\[
\frac{\partial_t \rho_i}{\rho_m} + \rho_m(u_i - u_m)\partial_{y_m} \rho_i + \rho_i \rho_m \partial_{y_m} u_i = 0, \quad i, m = 1, 2, \tag{7}
\]

\[
\frac{\rho_i}{\rho_m} \partial_t u_i + \rho_i (u_i - u_m) \partial_{y_m} u_i + K_i \partial_{y_m} \rho_i = \sum_{j=1}^{2} \mu_{ij} \partial_{y_m} (\rho_m \partial_{y_m} u_j) + \frac{(-1)^{i+1} a}{\rho_m} (u_2 - u_1), \quad i, m = 1, 2. \tag{8}
\]

The domain \( Q_T \), during this transformation, maps into the rectangular \((0, d_m) \times (0, T)\), where \( d_m = \int_0^1 \rho_{0m} \, dx > 0 \), initial and boundary conditions are converted to the form (here \( i, m = 1, 2 \))

\[
\rho_i |_{t=0} = \tilde{\rho}_0(y_1), \quad u_i |_{t=0} = \tilde{u}_0(y_1), \quad y_1 \in [0, d_1] \tag{9}
\]

(in the first Lagrangian coordinate system), or

\[
\rho_i |_{t=0} = \tilde{\rho}_2(y_2), \quad u_i |_{t=0} = \tilde{u}_0(y_2), \quad y_2 \in [0, d_2] \tag{10}
\]

(in the second one), and in both cases

\[
u_i |_{y_m=0} = \nu_i |_{y_m=d_m} = 0, \quad t \in [0, T]. \tag{11}
\]

Let us note that the equations (7) and (8) lead to the relations

\[
\partial_t \left( \frac{\rho_i}{\rho_m} \right) + \partial_{y_m} (\rho_i (u_i - u_m)) = 0, \quad i, m = 1, 2,
\]

\[
\partial_t \left( \frac{\rho_i}{\rho_m} u_i \right) + \partial_{y_m} (\rho_i (u_i - u_m) u_i) + K_i \partial_{y_m} \rho_i = \sum_{j=1}^{2} \mu_{ij} \partial_{y_m} (\rho_m \partial_{y_m} u_j) + \frac{(-1)^{i+1} a}{\rho_m} (u_2 - u_1), \quad i, m = 1, 2.
\]

**Remark 3.** After introducing the Lagrangian coordinates associated with the velocity of each component, the original (one) system of equations (1), (2) is transformed into two systems (7), (8), and each of the systems is defined in its own spatial region, which significantly complicates their analysis and, therefore, a simplifying assumption was made on the triangularity of the viscosity matrix. However this statement of the problem is an essential advance in the theory, since (as it was mentioned above) so far, the results were obtained only for the diagonal matrix of viscosities [32, 33].

Further, in Sections 4–6, we reproduce a standard technique for estimating the solutions of one-dimensional viscous gas equations with small variations according to bifluid specificity. However, this work needs to be done. Starting from Section 7, the differences from the one-component case become more significant.
4. First A Priori Estimate

It is not difficult to obtain the first a priori estimate

\[
\sum_{i=1}^{2}\int_{0}^{t}\left(\frac{1}{2}\rho_{i}u_{i}^{2} + \frac{K_{i}}{\gamma_{i} - 1}\rho_{i}^{\gamma_{i}}\right)dx + M_{0}\sum_{i=1}^{2}\int_{0}^{t}\int_{0}^{t}\left|\partial_{x}u_{i}\right|^{2}dx\,d\tau + \nonumber
\]

\[
+a\int_{0}^{t}\int_{0}^{t}|u_{1} - u_{2}|^{2}\,dx\,d\tau \leq \sum_{i=1}^{2}\int_{0}^{t}\left(\frac{1}{2}\rho_{0_i}u_{0_i}^{2} + \frac{K_{i}}{\gamma_{i} - 1}\rho_{0_i}^{\gamma_{i}}\right)dx =: B_{1}. \tag{12}
\]

The estimate (12) obviously provide the inequalities

\[
\|u_{i}\|_{L_{2}(0,T;L_{\infty}(0,1))} \leq \sqrt{\frac{B_{1}}{M_{0}}}, \quad i = 1, 2. \tag{13}
\]

In the Lagrangian coordinates \((y_{m}, t)\) the formula (12) takes the form

\[
\sum_{i=1}^{2}\int_{0}^{t}\int_{0}^{t}\left(\frac{1}{2}\rho_{m}u_{m}^{2} + \frac{K_{m}}{\gamma_{m} - 1}\rho_{m}^{\gamma_{m}}\right)dy_{m} + M_{0}\sum_{i=1}^{2}\int_{0}^{t}\int_{0}^{t}\rho_{m}|\partial_{y_{m}}u_{i}|^{2}dy_{m}\,d\tau + \nonumber
\]

\[
+a\int_{0}^{t}\int_{0}^{t}\frac{|u_{1} - u_{2}|^{2}}{\rho_{m}}dy_{m}\,d\tau \leq B_{1}, \quad t \in [0, T], \quad m = 1, 2, \tag{14}
\]

from which, in particular, we have

\[
\int_{0}^{t}\int_{0}^{t}\left(\frac{u_{m}^{2}}{2} + \frac{K_{m}}{\gamma_{m} - 1}\rho_{m}^{\gamma_{m}-1}\right)dy_{m} + M_{0}\int_{0}^{t}\int_{0}^{t}\rho_{m}|\partial_{y_{m}}u_{m}|^{2}dy_{m}\,d\tau + \nonumber
\]

\[
+a\int_{0}^{t}\int_{0}^{t}\frac{|u_{1} - u_{2}|^{2}}{\rho_{m}}dy_{m}\,d\tau \leq B_{1}, \quad t \in [0, T], \quad m = 1, 2. \tag{15}
\]

5. Estimate of Strict Positivity and Boundedness of the Density \(\rho_{1}\)

Similar to the one-component model, one can obtain the estimates

\[
\|\partial_{y_{1}}\ln\rho_{1}(t)\|_{L_{2}(0,d_{1})} \leq B_{2} \quad \forall t \in [0, T], \tag{16}
\]

\[
0 < B_{3}^{-1} \leq \rho_{1}(y_{1}, t) \leq B_{3} < \infty \quad \text{as} \quad (y_{1}, t) \in [0, d_{1}] \times [0, T], \tag{17}
\]

\[
0 < B_{3}^{-1} \leq \rho_{1}(x, t) \leq B_{3} < \infty \quad \text{as} \quad (x, t) \in [0, 1] \times [0, T]. \tag{18}
\]

6. Estimates for the Derivatives of \(\rho_{1}\) and \(u_{1}\)

After the boundedness and positivity of the density \(\rho_{1}\) is shown, a priori estimates for the derivatives of \(\rho_{1}\) and \(u_{1}\) can be obtained in the Eulerian variables \((x, t)\). So, for example, from (16) and (18) we deduce

\[
\|\partial_{x}\rho_{1}(t)\|_{L_{2}(0,1)} \leq B_{4} \quad \forall t \in [0, T]. \tag{19}
\]
Next, we square the equation (2) with the number $i = 1$, and obtain the estimate
\[ \| \partial_x u_1 \|_{L^\infty(0,T;L^2(0,1))} + \| \partial_{xx} u_1 \|_{L^2(Q_T)} + \| \partial_t u_1 \|_{L^2(Q_T)} \leq B_5. \]  
(20)

Then from the continuity equation (1) with the number $i = 1$ and the estimates (18), (19), (20) it follows that
\[ \| \partial_t \rho_1 \|_{L^\infty(0,T;L^2(0,1))} \leq B_6. \]  
(21)

7. Estimate of Strict Positivity and Boundedness of the Density $\rho_2$

The next crucial point is to obtain the estimate
\[ \| \partial_y \ln \rho_2(t) \|_{L^2(0,d_2)} \leq B_7 \quad \forall \ t \in [0,T]. \]  
(22)

Hence,
\[ | \ln \rho_2(y_2,t) | \leq B_8, \]  
(23)

and consequently
\[ 0 < B_9^{-1} \leq \rho_2(y_2,t) \leq B_9 < \infty \quad \text{as} \quad (y_2,t) \in [0,d_2] \times [0,T]. \]  
(24)

As a result,
\[ 0 < B_9^{-1} \leq \rho_2(x,t) \leq B_9 < \infty \quad \text{as} \quad (x,t) \in [0,1] \times [0,T]. \]  
(25)

8. Estimates for the Derivatives of $\rho_2$ and $u_2$

It follows from (22) and (25) that
\[ \| \partial_x \rho_2(t) \|_{L^2(0,1)} \leq B_{10} \quad \forall \ t \in [0,T]. \]  
(26)

We square the equation (2) with the number $i = 2$ and obtain the estimate
\[ \| \partial_x u_2 \|_{L^\infty(0,T;L^2(0,1))} + \| \partial_{xx} u_2 \|_{L^2(Q_T)} + \| \partial_t u_2 \|_{L^2(Q_T)} \leq B_{11}. \]  
(27)

Now from the continuity equation (1) with the number $i = 2$ and the estimates (25), (26), (27) it follows that
\[ \| \partial_t \rho_2 \|_{L^\infty(0,T;L^2(0,1))} \leq B_{12}. \]  
(28)

Thus, all estimates in the classes (5) for $(\rho_1, \rho_2, u_1, u_2)$ are obtained for arbitrary $T \in (0, +\infty)$.

9. Uniqueness of Solution

Uniqueness of solution is shown in a fairly standard way, and this completes the proof of Theorem 2.

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