Differentially Private Hypothesis Testing, Revisited

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ABSTRACT

Hypothesis testing is different from traditional applications of differential privacy in that one needs an accurate estimate of how the noise affects the result (i.e., a \( p \)-value). Previous approaches to differentially private hypothesis testing either used output perturbation techniques that generally had large sensitivities (hence risked swamping the data with noise), or input perturbation techniques that resulted in highly unreliable \( p \)-values (and hence invalid statistical conclusions). In this paper, we develop a variety of practical hypothesis tests that address these problems. Using a different asymptotic regime that is more suited to hypothesis testing with privacy, we show a modified equivalence between chi-squared tests and likelihood ratio tests. We then develop differentially private likelihood ratio and chi-squared tests for a variety of applications on tabular data (i.e., independence, homogeneity, and goodness-of-fit tests). An open problem is whether new test statistics specialized to differential privacy could lead to further improvements. To aid in this search, we further propose a permutation-based testbed that can allow experimenters to empirically estimate the behavior of new test statistics for private hypothesis testing before fully working out their mathematical details (such as approximate null distributions). Experimental evaluations on small and large datasets using a wide variety of privacy settings demonstrate the practicality and reliability of our methods.

Keywords
Differential Privacy; Hypothesis Testing

1. INTRODUCTION

Hypothesis testing is an important aspect of experiment analysis and data mining. It is used to separate findings into those that are likely the result of spurious noise and those that are worth pursuing. While other aspects of data mining have seen the development of privacy-preserving mining algorithms, hypothesis testing is largely unexplored to the extent that existing methods can only be used reliably in rare circumstances. Our paper addresses these limitations, but first let us examine what the difficulties are.

Suppose we collected the voter data in Table 1a. One may be interested in determining if it provides statistical evidence that voting behavior and gender are not independent.

In the classical (non-private) setting, this question is typically answered by performing chi-squared (\( \chi^2 \)) or likelihood ratio tests of independence. This would involve computing the chi-squared or likelihood ratio test statistics (reviewed in Section 2) from the data, and applying statistical theory which states that if the table were sampled from a distribution in which gender and voting were independent (i.e., a null distribution), these test statistics will behave approximately like one sample of a chi-squared random variable with 1 degree of freedom [5]. The \( p \)-value is the probability that such a chi-squared random variable would be larger than the actual test statistic, so small \( p \)-values indicate that there is statistical evidence against independence. For example, the likelihood ratio statistic for Table 1a is 2.918. This corresponds to a \( p \)-value of 0.0876, which is generally not considered strong enough to rule out independence.

Now let us consider problems raised by earlier applications of differential privacy to hypothesis testing [11, 15, 21].

**Example 1.1 (Input Perturbation)**. The first proposed algorithm [11] adds independent Laplace(\( b \)) noise with density \( f(x; b) = \frac{1}{2b} e^{-|x|/b} \) (and \( b = 2/\epsilon \)) to each cell of the table to achieve \( \epsilon \)-differential privacy [9]. When we added this noise to Table 1a, we obtained Table 1b. The next step [11] is to simply run the noisy table through off-the-shelf statistical software (which is unaware of this added noise). Intuitively, this seems a bit dangerous as the point of hypothesis testing is to determine how an analysis is affected by noise in the observed data. On the other hand, theoretical arguments [11] showed that asymptotically (i.e., when the original data are large enough), off-the-shelf software would be reliable. What happens in practice? The \( p \)-values produced by this method are extremely biased and will often lead to false conclusions.

For example, the likelihood ratio statistic for noisy Table 1b

|          | vote | not vote |          | vote | not vote |
|----------|------|----------|----------|------|----------|
| male     | 238  | 262      | female   | 560  | 625      |
| female   | 265  | 235      |          | 575  | 100      |

(a) original table                                (b) 0.2-differential privacy
is equal to 6.939 and off-the-shelf software would return an estimated p-value of 0.0084, which is often considered statistically significant and clearly contradicts the likelihood ratio test on the original data.\footnote{For reference, our proposed methods yield a p-value of 0.0511 for Table 1b, which is closer to the analysis over the original data in Table 1a.} This mismatch between theoretical arguments and empirical results also points out the need for a more reliable privacy-preserving statistical theory.

In Section 4, we show how to appropriately adjust private statistical theory so that asymptotic results become a good approximation of what happens in practice. Our analysis also implies that the algorithms of \cite{11} are only reliable when the standard deviation of the added noise (i.e., $2\sqrt{2}/\epsilon$) is a small fraction (say, 0.01) of the sampling error (approximately $\sqrt{n}$). When the privacy parameter $\epsilon$ is 0.2, this means that the table would need roughly $n = 2,000,000$ people to produce a reliable $p$-value (this is many orders of magnitudes larger than Table 1a).

So does the problem go away with big data? The answer is no. First, not all datasets are large and we would still like reliable statistical analysis for them. Second, when there are many records, there is a pressure to use much smaller $\epsilon$ values (so that privacy guarantees become strong). Third, big data is only possible when data about individuals are easy to collect. This means that an individual’s data will probably belong to many datasets and this could lead to a privacy breach through composition \cite{8} unless the $\epsilon$ values are small.

**Example 1.2** (Output Perturbation). The unreliability of the algorithm in \cite{11} was noticed by \cite{15, 21}, which proposed an output perturbation method: compute the chi-squared statistic on the original data (here it is 2.916), determine the sensitivity $S$ (the worst-case change in chi-squared values due to the alteration of one individual’s data), and add Laplace($b$) noise with $b = S/\epsilon$, and then use a different asymptotic distribution for computing the $p$-value. For $2 \times 2$ tables with $n = 1,000$ (as in our example), the sensitivity $S$ is at least\footnote{achieved by the worst-case tables $(\begin{smallmatrix} 1 & 0 \\ 0 & 0.999 \end{smallmatrix})$ and $(\begin{smallmatrix} 1 & 0 \\ 0 & 0.998 \end{smallmatrix})$ with $\chi^2$ values 1000 and 499.5, respectively.} 500. This noise has standard deviation at least $500\sqrt{2}/\epsilon$ and completely overwhelms the original value (i.e., 2.916). For this reason, \cite{15, 21} apply their work only to tables whose column sums are known (or have been released in a non-private manner), which is a special case in which the amount of added noise can be reduced.

Our contributions are as follows:

- We introduce likelihood ratio and $\chi^2$ tests for independence, homogeneity, and goodness of fit that properly account for the privacy noise added to the input table. These tests produce reliable results for much smaller data sizes/privacy parameters than prior work and are compatible with a variety of added noise distributions. To do this, we explore a different privacy-aware asymptotic statistical regime under which asymptotic distributions for the null hypothesis better match empirical observations (i.e. it fixes problems observed in Example 1.1).

- It is likely that new test statistics could lead to even better differentially private hypothesis tests. However, working out the mathematical details, such as approximate null distributions and numerical computation routines is a difficult task. Therefore, we propose a permutation-based testbed that would help researchers focus their efforts and identify promising test statistics for private hypothesis testing. This testbed would allow them to estimate the quality of the test statistic on real data before they work out all of the mathematical details. The testbed itself is differentially private in restricted scenarios, but we explain how results in the testbed would be expected to carry over to actual applications. By exploring this testbed, we find that test statistics that do not perform well in the classical case become much more appealing in the context of privacy.

- Extensive empirical evaluation on small and large data sets with a variety of privacy parameter settings validates our proposed approaches to private hypothesis testing.

We introduce notation and terminology in Section 2, discuss related work in Section 3, and present our various statistical tests on privacy-enhanced tables in Section 4. We present our permutation-based testbed in Section 5. Experiments appear in Section 6 and conclusions in Section 7.

## 2. PRELIMINARIES AND NOTATION

Let $D$ be a set of $n$ records $\{x_1, \ldots, x_n\}$ where all of the continuous attributes have been discretized. We can turn $D$ into a table of counts (contingency table) in several ways:

- Create a multidimensional histogram and (for convenience of notation) store it as a vector $T[..]$ with $T[i]$ being the count in bucket $i$.

- If it is important to distinguish between two attributes, say $R$ and $C$, form a two-dimensional table $T[., .]$ where $T[i, j]$ is the number of records with $R = i$ and $C = j$. We will use the shorthand $T*[., j] = \sum_i T[i, j]$ and $T[i, *] = \sum_j T[i, j]$, as well as $T[*, *] = \sum_i \sum_j T[i, j]$.

The size of a table ($T[..]$ or $T[*, .]$) is the sum of its counts.

### 2.1 Review of Hypothesis Testing

There are several common types of statistical hypotheses that practitioners like to test:

- **Goodness of fit**: given a table $T[..]$ of size $n$ and a probability vector $\theta$, test whether $T[..]$ can be modeled as a Multinomial($n, \theta$) distribution.

- **Homogeneity**: given tables $T[..]$ of size $n_1$ and $S[..]$ of size $n_2$, test whether they are samples from the same distribution.

- **Independence**: given a table $T[., .]$ of size $n$, test whether the rows and columns (i.e. attributes $R$ and $C$) are independent.

These hypotheses are commonly tested in the following ways:

### Definition 1 (Goodness of fit test). Compute either the likelihood ratio statistic $LR$ or chi-squared statistic $\chi^2$ as follows:

$$LR = 2 \sum_{i=1}^N T[i] \log \left( \frac{T[i]}{E[i]} \right)$$ (1)

$$\chi^2 = \sum_{i=1}^N \frac{(T[i] - E[i])^2}{E[i]}$$ (2)
where \( E[i] = n \theta[i]\) are estimated expected null hypothesis cell counts. Under the null hypothesis (i.e. \( T[i] \) is generated from a Multinomial \((n, \theta)\) distribution) the asymptotic distribution of both \( LR \) and \( \chi^2 \) is a chi-squared random variable with \( r - 1 \) degrees of freedom (where \( r \) is the number of cells in \( T \)). Thus the p-value can be approximated as the probability that the chi-squared random variable exceeds the chosen test statistic computed from \( T \).\(^3\)

**Definition 2 (Homogeneity Test).** Given \( T[\cdot,\cdot] \) of size \( n_1 \) and \( S[\cdot,\cdot] \) of size \( n_2 \), compute \( LR \) or \( \chi^2 \) as follows:

\[
LR = 2 \sum_{i=1}^{r} \frac{T[i]}{E[i]} \log \left( \frac{T[i]}{E[i]} \right) + 2 \sum_{i=1}^{r} \frac{S[i]}{E[i]} \log \left( \frac{S[i]}{E[i]} \right)
\]

\[
\chi^2 = \sum_{i=1}^{r} \sum_{j=1}^{c} \frac{(T[i,j] - E[i,j])^2}{E[i,j]}, \quad (3)
\]

where \( E[i,j] = n_1 (S[i] + T[i])/n_1 + n_2 \) and \( E_2[i,j] = n_2 (S[i] + T[i])/n_1 + n_2 \) are estimated expected null hypothesis cell counts. Under the null hypothesis (i.e. \( T[\cdot,\cdot] \) and \( S[\cdot,\cdot] \) are samples from the same multinomial distribution) the asymptotic distribution of both \( LR \) and \( \chi^2 \) is a chi-squared random variable with \( r - 1 \) degrees of freedom (where \( r \) is the number of cells in \( T \)). The p-value can be approximated as the probability that the chi-squared random variable exceeds the chosen test statistic computed from \( T \) and \( S \).\(^3\)

**Definition 3 (Test of Independence).** Given table \( T[\cdot,\cdot] \) with \( r \) rows and \( c \) columns, compute \( LR \) or \( \chi^2 \) as:

\[
LR = 2 \sum_{i=1}^{r} \sum_{j=1}^{c} T[i,j] \log \left( \frac{T[i,j]}{E[i,j]} \right)
\]

\[
\chi^2 = \sum_{i=1}^{r} \sum_{j=1}^{c} \frac{(T[i,j] - E[i,j])^2}{E[i,j]}, \quad (6)
\]

where \( E[i,j] = T[i,\cdot] T[\cdot,j]/T[\cdot,\cdot] \) are estimated expected null hypothesis cell counts. Under the null hypothesis (i.e. the rows and columns of \( T[\cdot,\cdot] \) are independent) the asymptotic distribution of both \( LR \) and \( \chi^2 \) is a chi-squared random variable with \( (r - 1)(c - 1) \) degrees of freedom. The p-value can be approximated as the probability that the chi-squared random variable exceeds the chosen test statistic computed from \( T \).

A low p-value (say, 0.01, depending on the application) indicates strong evidence against the null hypothesis while a larger p-value indicates absence of evidence. As can be observed from Definitions 1, 2, 3, the likelihood ratio and chi-squared tests are asymptotically equivalent \([5]\). However, as we explain in Section 4, differences will emerge due to privacy constraints.

### 2.2 Review of Differential Privacy

Differential privacy \([3]\) is a set of restrictions that guarantee that any data associated with an individual will have little impact on the result of a computation.\(^4\)

\(^4\)Alternatively, we can sample many tables from the Multinomial \((n, \theta)\) distribution and compute the test statistic of each one. The p-value could then be approximated as the fraction of sampled tables whose test statistic is \( \geq \) the test statistic of the actual table.

**Definition 4 (Differential Privacy \([3]\)).** A randomized algorithm \( A \) satisfies \( \epsilon \)-differential privacy if for all contingency tables \( T \) and \( T' \) that are derived from datasets that differ on the value of one record, and for all \( V \subseteq range(A) \),

\[
P(A(T) \in V) \leq e^\epsilon P(A(T') \in V)
\]

The concept of sensitivity is used by a simple algorithm called the Laplace mechanism to create \( \epsilon \)-differentially private outputs \([3]\).

**Definition 5 (Sensitivity \([3]\)).** Let \( h \) be a function over contingency tables (the output of \( h \) can be either a scalar or a vector). The sensitivity of \( h \), denoted by \( S(h) \), is defined as \( S(h) = \max_{T \rightarrow T'} ||h(T) - h(T')||_1 \), where the maximum is over all pairs of tables that are derived from datasets that differ on the value of one record.

**Definition 6 (Laplace Mechanism \([3]\)).** Given a (vector or scalar valued) function \( h \), privacy parameter \( \epsilon \), contingency table \( T \), and upper bound \( S \geq S(h) \), the Laplace mechanism adds independent Laplace \((S/\epsilon) \) random variables (with density \( f(x) = \frac{1}{2S} \exp(-|x|/S) \)) to each component of \( h(T) \).

### 3. RELATED WORK

Genome-wide association studies (GWAS) use statistical tests for finding associations between diseases and single-nucleotide polymorphisms (SNPs). The need for privacy became evident after Homer et al. \([10]\) raised the possibility of identifying individual participants in GWAS based on published SNP data. With GWAS in mind, Johnson and Shmatikov \([11]\) proposed differentially private algorithms for independence testing (e.g., \( \chi^2 \)-tests) using input perturbation as discussed in Examples 1.1. This method often produces unreliable conclusions except for extreme data sizes, as noted by Uhler et al. \([15]\) (and our own experiments).

Similar types of negative results produced by running off-the-shelf statistical analyses after input perturbation were reported by Vu and Slavkovic \([16]\) and Fienberg et al. \([6]\). Thus, Uhler et al. \([15]\) instead proposed computing the true \( \chi^2 \) statistic, adding noise to it, and then adjusting the asymptotic distribution used to compute p-values. Their method was limited to \( 3 \times 2 \) contingency tables where each of the 2 columns added up to \( n/2 \). Yu et al. \([21]\) later removed these restrictions but still required the column-sums to be released exactly (i.e., in a non-private way). In both cases, under the null hypothesis of independence, collecting more data will not result in convergence to the non-private analysis over the original data.

Independent efforts by Gaboardi et al. \([7]\) consider only private chi-squared goodness-of-fit and independence tests. The two works are complementary in the following sense. The analytical results that they derive focus on the special case of Gaussian noise added to data (thus satisfying the less stringent privacy definition \( (\epsilon, \delta) \)-differential privacy) because of the special interactions between this type of noise and the Central Limit Theorem. Our work handles asymptotics for more general noise distributions (we only require finite variance) and so can handle the pure version of differential privacy. Furthermore, our modified privacy-aware equivalence between likelihood ratio and chi-squared allows all of their results to be translated to likelihood ratio tests as well. We also propose a permutation-based experimental testbed that allows us to evaluate various test statistics before computing their asymptotic distributions.
In a more general setting, Smith [14] studied statistical estimators that are known to have asymptotically normal distributions and provided differentially-private versions of those estimators that are also asymptotically normal. As with the tests proposed by [11] (discussed in Example 1.1), very large data sizes are needed to observe approximate normality.

Differential privacy has also been applied to other statistical tasks such as computing commonly used robust statistical estimators [2] and computing private M-estimators [13]. Chaudhuri and Hsu [1] established a formal connection between differential privacy and robust statistics by deriving convergence rates in terms of a concept called gross error sensitivity. Dwork et al. [4] presented a framework for controlling the false discovery rate of a large sequence of hypothesis tests. The method relies on injecting noise directly into \( p \)-values, which removes their guarantee that they must be (approximately) uniformly distributed under the null hypothesis. Specific instantiations of the framework for various statistical tests are not given and its empirical performance is unknown. Wasserman and Zhou [18] studied rates of convergence between true distributions and differentially private estimates of distributions. They found that the provable convergence rates under differential privacy were often slower than in the non-private case.

4. PRIVATE HYPOTHESIS TESTING

We consider the following setting: a data owner has a table of counts (e.g., \( T[i] \) or \( T[\cdot, j] \)) and obtains noisy tables (e.g., \( \tilde{T[i]} \), \( \tilde{T[\cdot, j]} \)) by adding independent 0-mean noise with finite variance to each table cell. The table size \( n \), the density function of the noise, and the noisy tables themselves are publicly released. If the noise follows a Laplace(\( 2\epsilon \)) distribution, then this output satisfies \( \epsilon \)-differential privacy.

Our goal is to conduct goodness-of-fit, homogeneity, and independence tests using such noisy data. We feel this is a natural setting as such releases of noisy counts do not force the end-users into any particular data mining task.

We first justify our chosen asymptotic regime (Section 4.1), present the modified equivalence between chi-squared and likelihood ratio (Section 4.2) then derive asymptotic distributions for our tests (Section 4.3). We use these distributions for \( p \)-value computation in Section 4.4.

4.1 The Asymptotic Regime

The key to approximating the null distribution of a test statistic in the classical (non-private) case is the Central Limit Theorem (CLT), which, for example, states that as \( n \to \infty \),

\[
\frac{\tilde{T}[i] - np_i}{\sqrt{np_i(1 - p_i)}} \to N(0, 1) \quad \text{in distribution (where} \quad np_i
\]

is the expected value of \( T[i] \) and \( N(0, 1) \) is the standard Gaussian. In practice, this Gaussian approximation works well even if \( n \) is not large.

What happens in the CLT if we replace the true count \( T[i] \) with a noisy count \( \tilde{T}[i] = T[i] + V[i] \) where, for example, \( V[i] \) is a Laplace(\( 2\epsilon \)) random variable with standard deviation \( \sqrt{8}/\epsilon \)?

\[
\frac{\tilde{T}[i] - np_i}{\sqrt{np_i(1 - p_i)}} = \frac{T[i] - np_i}{\sqrt{np_i(1 - p_i)}} + \frac{V[i]}{\sqrt{np_i(1 - p_i)}} \quad (7)
\]

The first term on the right hand side is well-approximated by the standard Gaussian distribution even if \( n \) is not large. What about the second term? If \( \sqrt{n} \) is much larger than the standard deviation of \( V[i] \), then this term is close to 0 (since its variance is close to 0). As discussed in Section 1, this is an unlikely situation. Thus, when making \( n \to \infty \) so that the CLT can apply to the first term, we need to make sure the second term does not get wiped out. We do this by also adding the condition that \( \text{std}(V[i]) / \sqrt{n} = \kappa \) (a constant) as \( n \to \infty \). In the case of Laplace noise, this is equivalent to setting \( \epsilon \sqrt{n} = \sqrt{8}/\kappa \).

4.2 Relations between Likelihood Ratio and Chi-Squared Statistics

In classical statistics, likelihood ratio tests and chi-squared tests are asymptotically equivalent (i.e. the test statistics have the same asymptotic distributions [5]). In the privacy-preserving case, this equivalence is modified somewhat.

**Theorem 1.** Suppose the probabilities under the true null hypothesis are nonzero. Consider the noisy table \( \tilde{T} = T + V\kappa\sqrt{n} \) where \( V \) is a 0-mean random variable with fixed variance. Let \( \tilde{\chi}^2 \) denote the chi-squared statistic where we replace \( T \) with the noisy \( \tilde{T} \) and compute the expected counts \( E \) from \( \tilde{T} \) instead of \( T \) (or use any \( E \) such that \( E/n \) converges in probability to the true null distribution). Let \( LR \) denote the likelihood ratio statistic with the same substitutions and from each term we subtract \( 2(\tilde{T}[i] - E[i])^2 \). Then \( \tilde{\chi}^2 \) and \( LR \) have the same asymptotic distribution as \( n \to \infty \).

For proof, see Appendix A. The usefulness of this theorem is that sometimes it is easier to work with the likelihood ratio statistic and sometimes it is easier to work with \( \chi^2 \) when deriving asymptotic distributions.

4.3 The Asymptotic Distributions

In this section we derive asymptotic distributions for our various tests. We phrase the results in terms of added Laplace noise for differential privacy, but the results hold when the noise \( V \) follows any 0-mean distribution with finite variance. We use these distributions for \( p \)-value computation in Section 4.4.

**Theorem 2.** (Independence testing). Let \( T[\cdot, j] \) be a contingency table sampled from a Multinomial\((n, \theta_0)\) distribution. Consider the noisy table \( \tilde{T} = T + V\kappa\sqrt{n} \) where \( V \) is a table of independent Laplace(\( 2\epsilon \)) random variables. If the rows and columns under \( \theta_0 \) are independent and if no cells have probability 0, then as \( n \to \infty \), the chi-squared statistic and the likelihood ratio statistic (Definition 3) computed from \( \tilde{T} \) (instead of \( T \)) asymptotically have the distribution of the random variable:

\[
\sum_{ij} \frac{(A[i, j] + \kappa V^*[i, j])^2}{\theta_0[i, j]} - \sum_{j} \frac{(A[\cdot, j] + \kappa V^*[\cdot, j])^2}{\theta_0[\cdot, j]} + \frac{(A[\cdot, \cdot] + \kappa V^*[\cdot, \cdot])^2}{1}
\]

Clearly \( \kappa \) is known since \( n \) and the noise distribution are publicly released by the data owner.

If there are two tables \( S, T \) then there are two batches of random variables which we can denote by \( V_1 \) and \( V_2 \).

Thus the noise is \( V\kappa\sqrt{n} \) and so \( \text{std}(V\kappa\sqrt{n}) / \sqrt{n} = \text{constant} \), as required by our asymptotic regime.

If the log is not defined (i.e. \( \tilde{T}[i] < 0 \)), do not subtract the offset, simply replace that term with the corresponding chi-squared term (e.g., \( (\tilde{T}[i] - E[i])^2 / E[i] \)).
where $V^*$ has the same distribution as $V$, and the vectorized version $\text{vec}(A) \sim \mathcal{N}(0, \text{diag}(\text{vec}(\theta_0)) - \text{vec}(\theta_0)\text{vec}(\theta_0)^T)$. It is asymptotically equivalent to the quantity we get by replacing $\theta_0[i,j]$ with $\frac{T[i,j]}{T[i+]^2}$.

For proof, see Appendix B.

**Theorem 3. (Homogeneity testing).** Let $T[\cdot]$ and $S[\cdot]$ be samples from Multinomial($n_1, \theta_0$) and Multinomial($n_2, \theta_0$) distributions, respectively. Consider the noisy versions $\tilde{T} = T + V^1_{\kappa_1/\sqrt{n_1}}$ and $\tilde{S} = S + V^2_{\kappa_2/\sqrt{n_2}}$ where $V^1, V^2$ are vectors of independent Laplace($2/\epsilon$) random variables. If no cells have probability $0$, then as $n_1, n_2 \to \infty$ the chi-squared statistic and the likelihood ratio statistic (Definition 2) computed from $\tilde{T}$ and $\tilde{S}$ (instead of $T$ and $S$) asymptotically have the distribution of the random variable:

$$\sum_j \left[ \sqrt{\frac{n_1}{n_1+n_2}} (A_1[j] + \kappa_11^* V[j]) - \sqrt{\frac{n_2}{n_1+n_2}} (A_2[j] + \kappa_22^* V[j]) \right]^2_0$$

where $V^1, V^2$ are independent with the same distribution as $V^1, V^2$ and $A_1, A_2 \sim \mathcal{N}(0, \text{diag}(\theta_0) - \theta_0\theta_0^T)$, and $A_1, A_2$ are independent. It is asymptotically equivalent to the quantity we get by replacing $\theta_0[i,j]$ with $(\tilde{T}[i,j] + \tilde{S}[i,j])/(n_1 + n_2)$.

For proof, see Appendix C.

**Theorem 4. (Goodness-of-fit).** Let $T[\cdot]$ be a sample from a Multinomial($n, \theta_0$) distribution. Consider the noisy version $\tilde{T} = T + V_{\kappa/\sqrt{n}}$ where $V$ is a vector of independent Laplace($2/\epsilon$) random variables. If no cells have probability $0$, then as $n \to \infty$, then the statistics:

$$\chi^2 = \sum_j (\tilde{T}[j] - n\theta_0[j])^2/(n\theta_0[j])$$

$$LR = 2 \sum_j (\tilde{T}[j] \log(\tilde{T}[j]/n\theta_0[j]) - \tilde{T}[j] + n\theta_0[j])$$

asymptotically have the same distribution as:

$$\sum_j (A[j] + \kappa V[j])^2/\theta_0[j]$$

where $V$ has the same distribution as $V$ and $A \sim \mathcal{N}(0, \text{diag}(\theta_0) - \theta_0\theta_0^T)$.

For proof, see Appendix D.

### 4.4 p-Value Algorithms

To apply Theorems 2, 3, 4, we set $\kappa = 1/\sqrt{n}$, where $n$ is the actual table size (in the case of the homogeneity test, we set $\kappa_1 = 1/\sqrt{n_1}$ and $\kappa_2 = 1/\sqrt{n_2}$). The recipe for computing $p$-values is relatively simple:

1. Compute the appropriate test statistic from the noisy tables (e.g., $\tilde{T}[-,-]$, $\tilde{T}[\cdot,-]$ and/or $\tilde{S}[\cdot,-]$). Let $t^*$ be its value.
   - For goodness of fit, the test statistics are given by Equations 8 or 9.
   - For homogeneity, they are obtained from Definition 2 by replacing the true tables $T[\cdot, \cdot]$ and $S[\cdot, \cdot]$ with their noisy versions $\tilde{T}[\cdot, \cdot]$ and $\tilde{S}[\cdot, \cdot]$. The $E_1$ and $E_2$ values from the definition must also be computed from the noisy tables.
   - For independence, they are obtained from Definition 3 by replacing the true table $T[\cdot, \cdot]$ with the noisy version $\tilde{T}[\cdot, \cdot]$ and the $E[i,j]$ values from the definition are computed as $\tilde{T}[i,j]/\tilde{T}[\cdot, \cdot]$.

2. Sample $m$ reference points $t_1, \ldots, t_m$ (discussed next).

3. Set the $p$-value to be $|\{t_i : t_i \geq t^*\}|/m$.

The $m$ reference points $t_1, \ldots, t_m$ are obtained from Algorithm 1 for independence testing and Algorithm 2 for homogeneity testing. Goodness-of-fit testing is actually a tiny concept.

**Algorithm 1: Sampling for Independence Test**

```plaintext```
input : Noisy table $\tilde{T}[\cdot, \cdot]$, $\epsilon > 0$. 
1 for each $i, j$ do 
2 $\theta_0[i,j] \leftarrow \frac{\tilde{T}[i]\tilde{T}[j]}{\tilde{T}[\cdot, \cdot]^2}$ 
3 end 
4 for $\ell = 1, \ldots, m$ do 
5 $\text{vec}(A) \sim \mathcal{N}(0, \text{diag}(\theta_0)) - \text{vec}(\theta_0)\text{vec}(\theta_0)^T$
6 Reshape $A$ to dimensions of $\tilde{T}$
7 $V^\epsilon[.., \cdot] \sim \text{Laplace}(2/\epsilon)$ // fresh noise
8 $X \leftarrow A + V^\epsilon/\sqrt{n}$
9 generate $t_{\ell} = \sum_{ij} X[i,j]^2/n_{ij} - \sum_{ij} X[i,j]^2/n_{ij} + X[i,j]^2$
10 end
```

**Algorithm 2: Sampling for Homogeneity Test**

```plaintext```
input : Noisy tables $\tilde{T}[\cdot, \cdot]$ and $\tilde{S}[\cdot, \cdot]$, $\epsilon > 0$. 
1 for each $i$ do 
2 $\theta_0[i] \leftarrow (\tilde{T}[\cdot, i] + \tilde{S}[\cdot, i])/(n_1 + n_2)$ 
3 end 
4 for $\ell = 1, \ldots, m$ do 
5 $A_1, A_2 \sim \mathcal{N}(0, \text{diag}(\theta_0) - \theta_0\theta_0^T)$
6 $V^\epsilon[.., \cdot] \sim \text{Laplace}(2/\epsilon)$ // fresh noise
7 $X_1 \leftarrow A_1 + V^\epsilon/\sqrt{n_1}$; $X_2 \leftarrow A_2 + V^\epsilon/\sqrt{n_2}$
8 generate $t_{\ell} = \sum_{ij} \left( \sqrt{\frac{n_1}{n_1+n_2}} X_1[i,j] - \sqrt{\frac{n_2}{n_1+n_2}} X_2[i,j] \right)^2$ 
9 end
```

3. Set the $p$-value to be $|\{t_i : t_i \geq t^*\}|/m$.

The $m$ reference points $t_1, \ldots, t_m$ are obtained from Algorithm 1 for independence testing and Algorithm 2 for homogeneity testing. Goodness-of-fit testing is actually a tiny modification of a well-known simulation trick of applied statisticians (also noted by [7]): since goodness-of-fit tests whether the table came from a Multinomial($n, \theta_0$) distribution, for a prespecified $\theta_0$, one simply generates $m$ tables $Q_1, \ldots, Q_m$ from this distribution, adds fresh independent Laplace($2/\epsilon$) noise to each table to get $\tilde{Q}_1, \ldots, \tilde{Q}_m$, sets $t_i$ to be the value of the test statistic computed from $\tilde{Q}_i$, and compares all the $t_i$ to the test statistic $t^*$ on $\tilde{T}$.

**Theorem 5.** The $p$-value algorithms satisfy $\epsilon$-differential privacy.

**Proof.** The algorithms only use the differentially private noisy tables, the table sizes (which are assumed to be known according to the definition of differential privacy in Definition 4), and the density of the noise distribution (which is also public knowledge). At no point do they access the true data or the true noise that was added to it. As such, our algorithms are strictly post-processing algorithms and hence satisfy differential privacy due to the post-processing property of differential privacy [3].

In Section 6 we show that this approach to compute $p$-values is much more reliable than prior work.

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*asymptotics are not needed at all, they are just a proof of concept.
5. PERMUTATION TESTBED

The likelihood ratio and $\chi^2$ statistics are general-purpose statistical tools that can be adapted to a variety of tests. However, it is likely that some unknown privacy-specific test statistics could outperform them. Finding such test statistics is an open problem and, as we have seen, approximating their null distribution will generally not be easy. It would be very helpful to be able to estimate how well a new test statistic could perform on real data before figuring out all of these mathematical details. This is the purpose of our permutation-based testbed.

First, we need to choose an application that is rich enough to exhibit variations between various test statistics. Naively, one could select the goodness-of-fit test since it is possible to exactly sample from its null distribution (see Section 4). However, goodness-of-fit is too simple: rejecting the null hypothesis is the same as rejecting 1 possible distribution for the data. On the other hand, independence testing is a much richer scenario. The null hypothesis (independence of rows and columns) consists of infinitely many possible distributions (those where rows and columns are independent) and rejecting the null hypothesis means rejecting all of these distributions.

We argue that the ideal testbed is differentially-private independence testing when row and column sums are known. It has several appealing properties (which we explain in this section):

- The null distribution can be sampled exactly (which is important for privacy).
- Real-data experimental results on input-perturbation methods would carry over straightforwardly to the unrestricted case (i.e., unknown row and column sums).
- Real-data experimental results on output-perturbation methods would be a lower bound on the unrestricted case, hence allowing experimenters to rule out statistics that do not perform well in the testbed.

First, we explain how to do differentially-private hypothesis testing when row and column sums are known in Section 5.1. Then we discuss how experimental results would carry over to the unrestricted case in Section 5.2. We illustrate its use in Section 5.3. We present experiments in Section 6 to compare likelihood ratio and $\chi^2$ with other statistics to conﬁrm our intuition that there exist other test statistics that are better suited for privacy. Earlier work by Uhler et al. [15] and Yu et al. [21] studied differential privacy when only exact column sums are known – this scenario would not be suited for a testbed as the exact null distribution cannot be sampled from.

5.1 Private Independence Testing with Known Marginals

Consider a set of records $D = \{x_1, \ldots, x_n\}$ and suppose the records have two distinguished categorical attributes, which we call $R$ and $C$. We can construct a table $T[r, \cdot]$ where $T[i, j]$ is the number of records with $R = i$ and $C = j$. Although $T$ is not public knowledge, suppose its row and column sums are known (i.e., for any $i, j$, $T[\cdot, j]$ and $T[i, \cdot]$ are public). We are interested in publishing the rest of the information in $T$ in a private manner that does not leak any more information beyond what the row and column sums already revealed. The privacy definition that allows us to do this was proposed by Kifer and Machanavajjhala [12] – it ends up being a variant of differential privacy based on a concept called "marginal neighbors."
5.2 Translating Experimental Results

Let us compare input perturbation noise for $\epsilon$-differential privacy and for $\epsilon$-min-differential privacy. We need $\text{Laplace}(2/\epsilon)$ noise for the former and $\text{Laplace}(4/\epsilon)$ noise for the latter (i.e., the tables are twice as noisy). Thus $\epsilon$-differential privacy and 2-\text{min}-differential privacy use the same amount of noise and therefore would generate the same values for the test statistic $t^*$ (in the case of input perturbation), so the p-value one gets under this testbed using 2-\text{min}-differential privacy should correspond to the p-value an experimenter would have gotten under $\epsilon$-differential privacy (had statistical details, such as approximating the null distribution with unknown row/columns sums, been worked out in advance).

For output perturbation, we add $\text{Laplace}(s_h/\epsilon)$ noise for $\epsilon$-min differential privacy (where $s_h$ is defined in Lemma 1) and $\text{Laplace}(S(h)/\epsilon)$ noise for $\epsilon$-differential privacy (where $S(h)$ is defined in Definition 5). Thus the noise added under this testbed using $\frac{\text{Laplace}(s_h)}{\epsilon}$-min differential privacy is equal to the noise added under $\epsilon$-differential privacy (and hence sets up the correspondence between the resulting p-values). What if one hasn’t yet fully worked out the sensitivity $S(h)$ under $\epsilon$-differential privacy? In this case, the statements are slightly less precise, but by comparing the following:

$$s_h = \max_{T_1, T_2} ||h(T_1) - h(T_2)||_1$$
$$s^* = \max_{T_1, T_2: \text{underlying datasets differ on 2 records}} ||h(T_1) - h(T_2)||_1$$
$$S(h) = \max_{T_1, T_2: \text{underlying datasets differ on 1 record}} ||h(T_1) - h(T_2)||_1$$

it follows directly from the definitions (and the triangle inequality applied to the $L1$ norm) that $s_h \leq s^* \leq 2S(h)$. This means $\text{Laplace}(s_h/\epsilon)$ has less variance than $\text{Laplace}(S(h)/\epsilon)$ and so the quality of p-values for output perturbation under 2-min-differential privacy are expected to be a lower bound on the quality for $\epsilon$-differential privacy.11

5.3 Usage

We will use our permutation testbed to compare the likelihood ratio and $\chi^2$ statistics (with both input and output perturbation) to two other statistics that are rarely, if ever, used for independence testing in the non-private case, log-likelihood (LL) and absolute difference between actual count and expected count (Diff):

$$LL = \left[ \sum_{i=1}^{r} \log(T[i, \ast]) + \sum_{j=1}^{c} \log(T[\ast, j]) \right] - \log(n!) - \sum_{i,j} \log(T[i,j]!$$

$$\text{Diff} = \sum_{i,j} \left| T[i,j] - \frac{T[i, \ast]T[\ast, j]}{n} \right|$$

An important point of distinction is that for input perturbation, we will be using the noisy values $\tilde{T}[i,j], \tilde{T}[i, \ast]$, equal to $\sum_{i,j} \tilde{T}[i,j]$ and $\tilde{T}[,j]$, to compute the statistics instead of the true values $T[i,j], T[i, \ast], T[,j]$. This is because in the unrestricted case, the true values will not be available once the input has been perturbed.

In contrast, for output perturbation, we will use the true values of $T[i, \ast], T[,j]$ to compute the statistics (since they are public in $\epsilon$-min-differential privacy). This is because the output perturbation results would be a lower bound to the quality we should expect in the unrestricted case, and the sensitivity $s_h$ using this method is easier to compute (another advantage of this testbed).

We now provide $s_h$ calculations for the output perturbation versions of the test statistics. In some cases, $s_h$ depends on the dimensions of $T$. Proofs can be found in Appendix E. One important fact to note is that the Diff statistic has lowest $s_h$ and so, intuitively, is expected to perform well in the privacy setting.

**Theorem 7.** The $s_h$ value of the $\chi^2$-statistic for a $2 \times 2$ contingency tables is:

$$C = \frac{n^2}{T[1,1]T[2,1]T[1,2]T[2,2]}$$

The Proof of Theorem 7 is provided in Appendix E. Note that table margins are $O(n)$, so the constant $C$ in Theorem 7 is $O(1/n^2)$ and so the $s_h$ value is $O(1)$ for $2 \times 2$ tables. However, the chi-squared statistic does not grow with $n$ under the null hypothesis, so the noise can be a significant part of the output.

**Theorem 8.** The $s_h$ value of the $\chi^2$-statistic for an $r \times c$ table $T$ with $r \geq 3, c \geq 3$ is:

$$C = \frac{n^2}{T[1,1]T[2,1]T[1,2]T[2,2]}$$

The proof of Theorem 8 is provided in Appendix E. A simple analysis shows that the $s_h$ in Theorem 8 is $O(1)$.

**Theorem 9.** The $s_h$ value of the likelihood ratio statistic for $2 \times 2$ contingency tables is

$$2 \times \max\left\{ \begin{array}{l} \log \left( \frac{T[1,1]T[2,2]}{(T[1,1]+T[2,1]+T[1,2]+T[2,2])^2} \right) + \log \left( \frac{T[1,2]T[2,1]}{(T[1,1]+T[2,1]+T[1,2]+T[2,2])^2} \right) \\
\log \left( \frac{T[1,1]T[2,2]}{(T[1,1]+T[2,1]+T[1,2]+T[2,2])^2} \right) + \log \left( \frac{T[1,2]T[2,1]}{(T[1,1]+T[2,1]+T[1,2]+T[2,2])^2} \right) \\
\log \left( \frac{T[1,1]T[2,2]}{(T[1,1]+T[2,1]+T[1,2]+T[2,2])^2} \right) + \log \left( \frac{T[1,2]T[2,1]}{(T[1,1]+T[2,1]+T[1,2]+T[2,2])^2} \right) \\
\log \left( \frac{T[1,1]T[2,2]}{(T[1,1]+T[2,1]+T[1,2]+T[2,2])^2} \right) + \log \left( \frac{T[1,2]T[2,1]}{(T[1,1]+T[2,1]+T[1,2]+T[2,2])^2} \right) \end{array} \right. $$

11We note that typically, $s_h$ can be much smaller than $s^*$.2/
The proof of Theorem 9 is provided in Appendix E.

**Theorem 10.** The $s_h$ of the likelihood ratio statistic LR on $r \times c$ ($r \geq 3$, $c \geq 3$) contingency tables is

$$2 \times \max_{i_1 \neq i_2, j_1 \neq j_2} \max_{d, b} \left( \left\{ \begin{array}{ll}
\frac{a^b}{c-1} + \log \frac{a^b}{c-1} & \text{if } T[1,1] \leq T[1,c], T[c,1] \geq T[2,2], \\
\frac{a^b}{c-1} + \log \frac{a^b}{c-1} & \text{if } T[1,1] \geq T[1,c], T[c,1] \leq T[2,2], \\
\frac{a^b}{c-1} + \log \frac{a^b}{c-1} & \text{otherwise}
\end{array} \right. \right)$$

where $a = \min(T[i_1,j_1], T[i_1,j_2], T[i_2,j_1])$, $d = \min(T[i_2,j_1], T[i_2,j_2])$, $b = \min(T[i_1,j_1], T[i_1,j_2]) - 1$, $c = \min(T[i_2,j_1], T[i_2,j_2]) - 1$. $s_h$ is the value of the statistic should grow with one of them that has a constant sensitivity. Clearly proofs can be found in [17]. The Diff statistic is the only one that does not grow with $n$ under the null hypothesis.

**Theorem 11.** The $s_h$ value of the log-likelihood statistic based on $2 \times 2$ contingency tables is

$$\frac{1}{2} \left\{ \begin{array}{ll}
\log(T[1,1] - T[1,2] + 1) - \log(T[1,1] - T[1,2]) & \text{if } T[1,1] \leq T[1,2], T[2,1] \geq T[2,2], \\
\log(T[1,1] - T[1,2] + 1) - \log(T[1,1] - T[1,2]) & \text{if } T[1,1] \geq T[1,2], T[2,1] \leq T[2,2], \\
\log(T[1,1] - T[1,2] + 1) - \log(T[1,1] - T[1,2]) & \text{otherwise}
\end{array} \right. \}
$$

The proof of Theorem 11 is provided in Appendix E.

**Theorem 12.** The $s_h$ value of the LL statistic (from Equation 10) for $r \times c$ tables ($r \geq 3$, $c \geq 3$) is

$$\max_{i_1 \neq i_2, j_1 \neq j_2} \max_{d, b} \left\{ \begin{array}{ll}
\frac{a^b}{c-1} + \log \frac{a^b}{c-1} & \text{if } T[1,1] \leq T[1,c], T[c,1] \geq T[2,2], \\
\frac{a^b}{c-1} + \log \frac{a^b}{c-1} & \text{if } T[1,1] \geq T[1,c], T[c,1] \leq T[2,2], \\
\frac{a^b}{c-1} + \log \frac{a^b}{c-1} & \text{otherwise}
\end{array} \right. \}
$$

The proof of Theorem 12 is provided in Appendix E. Here $s_h$ also grows logarithmically.

**Theorem 13.** The $s_h$ value of the Diff statistic (from Equation 11) is equal to 4.

The goal of the experiments is to determine at what point the methods break down. This will serve to identify the frontier for future research. Generally, we found that the tests work extremely well on large data even with large amounts of privacy noise. The tests work well on small datasets (e.g., $n \approx 1,000$) where the non-private $p$-value strongly rejects the null hypothesis (e.g., $p \leq 0.01$). Beyond that, the power of the tests starts to degrade (a significant improvement over prior work [11, 15, 21] in terms of reliability and power).

We use five real data sets from which we obtain seven contingency tables. The first dataset was used in [6]. Its purpose is to study the risk of getting coronary thrombosis for Czech car factory workers. Its sample size is $n = 1841$. The second dataset was also used in [6] and contains data from a survey investigating the relationship between a wife’s shopping and husband’s unemployment in Rochdale. Its sample size is 665. The third data set was used in [20]. The dataset contains information about home zone, work zone and income category of individuals and was extracted from a 2000 census database. Its sample size is 2291. The fourth data set was used in [19] and contains data from a research survey about white Christians’ attitude toward abortion. Its sample size is 1055. The previous datasets are relatively small (and hence very challenging for privacy-preserving statistical testing). For a large dataset, which allows us to explore very small $c$ settings, we used the 2014 NYC Taxi data\footnote{http://www.nyc.gov/html/tlc/html/about/trip_record_data.shtml} which contains trip records for all NYC yellow taxis. This dataset has a sample size of $n = 165,114,361$. Our experiments use 10,000 reference points to compute $p$-values and results are averaged over 100 runs (of privacy-preserving table perturbations).

We evaluate the reliability of our methods in Section 6.1. We evaluate the quality of the $p$-values on the real datasets in Section 6.2. Then, in Section 6.3, we explore our permutation test with the additional test statistics discussed in Section 5. As a proof of concept, these results indicate that new test statistics could further improve statistical performance, but the necessary test statistics (and their $p$-value algorithms) are an open problem.

### 6.1 Reliability of the Asymptotic Distributions

A $p$-value is a strong statistical statement: under the null hypothesis, $P(p$-value $\leq q) = q$. In other words, under the null hypothesis, $p$-values must be uniformly distributed. This criterion allows us to evaluate the reliability of tests: pick a null distribution, sample data from it, compute $p$-values from each sampled dataset, and plot the quantiles of the resulting $p$-values against the uniform distribution. The result is a Q-Q plot and a perfect match will be a diagonal line. We compare the reliability of our $p$-value computations for the $\chi^2$ and likelihood ratio statistics (denoted by $\chi^2$ and $LR$) against the reliability of the earlier method proposed by [11], which simply ran noisy tables through off-the-shelf software (we denote their results by $\chi^2$-JS and LR-JS). Here we present reliability Q-Q plots for the test of independence. When no noise is added to the tables, both methods match the ideal diagonal line and so are reliable, as expected and shown in Figure 3 for independence testing and Figure 4 for homogeneity testing.
Figure 1: Test: independence, Attributes: Passenger Count, Payment Type with $n = 165, 114, 361$, $r = 4$, $c = 3$ (left); Test: homogeneity, Attribute: Payment Type (first half year and second half year) with $n_1 = 85, 480, 239$, $n_2 = 79, 634, 122$, $r = 2$, $c = 3$ (middle); Test: Goodness-of-fit, Attribute: Payment Type (second half of year) with $n = 79, 634, 122$, $r = 1$, $c = 3$, $\theta_0 = [0.00823912, 0.5762475, 0.41551338]$ estimated from first half of year (right).

Figure 2: Independence testing on $2 \times 2$ tables. Tables used: Czech AD with $n = 1841$ (left), Czech BC with $n = 1841$ (middle), Rochdale AB with $n = 665$ (right).

Figure 3: Q-Q plots against the uniform distribution with $\epsilon = \infty$ (no noise) for test of independence. $n = 1000$, $r = c = 2$, $P_{row} = P_{col} = [1/2, 1/2]$. Very similar results are obtained for several other parameter settings. Both methods are expected to perform well for this sanity check.

Figure 4: Q-Q plots against the uniform distribution with $\epsilon = \infty$ (no noise) for test of homogeneity. $n_1 = 1200$, $n_2 = 2800$, table dimension $= 2$, $\theta_0 = [1/2, 1/2]$. Very similar results are obtained for several other parameter settings. Both methods are expected to perform well for this sanity check.

Figure 5 shows the Q-Q plots under a variety of settings, such as sample size $n$, number of rows $r$, number of columns $c$ and the type of null distribution: the null distribution probability of table entry $T_{i,j}$ is set to $P_{row}[i]P_{col}[j]$. As can be seen from Figure 5, our methods remain reliable throughout these settings while the competitors returned unreliable $p$-values. The upward bend of the reliability curve for LR-JS (and $\chi^2$-JS) indicates strong bias towards producing small $p$-values and hence would lead to many false discoveries if applied in practice.

Reliability plots for the test of homogeneity can be found in Figure 6 and goodness-of-fit in Figure 7. Recall that homogeneity tests whether two tables come from the same multinomial distribution. In the figure, we report the settings of the two table sizes $n_1$ and $n_2$, the number of cells in each table, and the null distribution Multinomial probability vector $\theta_0$ used to generate tables for the reliability plot. The various settings for the goodness-of-fit reliability plots are presented in Figure 7.

6.2 Statistical Power on Real Data

Now we evaluate the $p$-values generated by our methods using real datasets and compare them to non-private $p$-values. Please note that the experimental settings are challenging as the standard deviation of the added noise ($\sqrt{\epsilon}/\epsilon$) is substantial compared to the standard deviation of the data itself ($O(\sqrt{n})$). For example, the taxi data has sample size $n = 165, 114, 361$ and we used $\epsilon = 0.0001$. Here $\sqrt{n} = 12, 849.7$ and noise std$= 28, 284.3$ so a loss in power is expected. Nevertheless, we generally find that when the null hypothesis is strongly rejected (non-private $\chi^2$ or LR $p$-value $\leq 0.01$), our private tests $\tilde{\chi}^2$ and $\tilde{LR}$ also reject the null hypothesis at level 0.01. The output perturbation methods of
Figure 5: Q-Q plots against the uniform distribution with $\epsilon = 0.2$ for test of independence. $n = 1000, r = c = 2$, $P_{row} = P_{col} = [1/2, 1/2]$ (left); $n = 4000, r = c = 2$, $P_{row} = P_{col} = [1/2, 1/2]$ (middle left); $n = 4000, r = c = 3$, $P_{row} = P_{col} = [1/3, 1/3, 1/3]$ (middle right); $n = 4000, r = c = 3$, $P_{row} = P_{col} = [0.1, 0.1, 0.8]$ (right).

[15, 21] required too much noise and are omitted to avoid skewing the graphs (we evaluate them in the permutation testbed, where their noise requirement is smaller). Figure 1 shows our results for various tests on large NYC taxi data. Despite the large amount of noise, the $p$-values showed very good performance.

Figure 8 shows the power results for independence tests with Census data. Again, as we move from left to right we observe smaller data sizes and less (nonprivate) evidence against the null hypothesis serve to reduce power of the private tests. Figure 9 shows the power results for homogeneity tests with Czech car worker and Rochdale data. Again, as we move from left to right we observe smaller data sizes and less (nonprivate) evidence against the null hypothesis serve to reduce power of the private tests. The extreme cases is again the Rochdale data where the small non-private test statistic gets dominated by noise. The resulting $p$-value is closer to being uniformly distributed as noise gets larger.

Next we move to extremely challenging cases with small sample sizes but high relative noise. We arrange the figures so that from left to right there is a decrease in sample size and a reduction in statistical significance of non-private analysis. Figures 2 and Figure 10 show results for independence and homogeneity testing, respectively. The results show good power when the null hypothesis is strongly rejected (non-private $p \leq 0.01$) with sample sizes of 1,800 or more. Power decreases as sample size decreases and non-private $p$-value increases. Of particular interest is the Rochdale data (Figure 2, right). It is a small dataset with a very high $p$-value and small test statistic that is dominated by noise. Since the noise obscures any statistical signal, the $p$-value behaves more like a uniform random variable (hence we add 80% error bars on the plot). This is expected, and note that the null hypothesis would be erroneously rejected only in rare circumstances (which is the expected behavior of $p$-values).

For goodness of fit tests, we used extremely small sample sizes (on the order of a few hundred) with large standard deviation of Laplace noise relative to the standard deviation of the data. The results are shown in Figures 11 and 12. The private $p$-values are still good quality but their quality degrades as the sample size is further diminished. Again, the Rochdale data with only 79 data points has a very small non-private $\chi^2$ and LR value and so is completely dominated by noise, leading to large variance but no false conclusions except in rare cases (as is allowed by the definition of $p$-value).

6.3 Permutation Testbed

Now we experiment with our permutation testbed to compare the $\chi^2$ and likelihood ratio LR statistics to the nontraditional LL and Diff statistics considered in Section 5.3. We compare the non-private versions (evaluated on actual data) to the input perturbation (with suffix “-in”) and output perturbation (with suffix “-out”) versions. The method of [15, 21] is denoted $\chi^2$-out and has lower noise requirements in the testbed than in general.

Our first batch of results is shown in Figure 13 and illustrates three separate interesting phenomena. The left table has $p$-values that, in the non-private case, are generally considered highly significant. The output perturbation methods (including the perturbed $\chi^2$ statistic) generally have high variance and perform much worse than the input perturbation methods due to the amount of noise they require. The exception is the Diff statistic, whose output perturbation version requires the least amount of noise. The middle table has $p$-value that is often considered borderline for rejecting the null hypothesis. Again, we see the same pattern, with slightly more variance even for the input perturbation results. The reason for this is that the higher the non-private $p$-value, the smaller the value of the non-private test statistic. When that is small and when $n$ is small, the resulting value of the test statistic is easily dominated by the noise (creating large variance), but does not lead to unsupported rejections of the null hypothesis. This behavior is actually expected and desired. Even in the non-private case, when the null hypothesis is true, the $p$-values should be uniformly distributed while if the null hypothesis is false, the $p$-values...
Figure 6: Q-Q plots against the uniform distribution with $\epsilon = 0.2$ for test of homogeneity. $n_1 = 400, n_2 = 600$, table dimension = 2, $\theta_0 = [1/2, 1/2]$ (left); $n_1 = 1200, n_2 = 2800$, table dimension = 2, $\theta_0 = [1/2, 1/2]$ (middle left); $n_1 = 1200, n_2 = 2800$, table dimension = 3, $\theta_0 = [1/3, 1/3, 1/3]$ (middle right); $n_1 = 1200, n_2 = 2800$, table dimension = 3, $\theta_0 = [0.1, 0.1, 0.8]$ (right).

Figure 7: Q-Q plots against the uniform distribution for goodness of fit test. L\_Chi (top), CHI2 (bottom). $n = 1000$, $r = 1$, $c = 4$, $\theta_0 = [1/4, 1/4, 1/4, 1/4], \epsilon = \infty$ (left); $n = 500$, $r = 1$, $c = 4$, $\theta_0 = [1/4, 1/4, 1/4, 1/4], \epsilon = 0.2$ (middle left); $n = 1000$, $r = 1$, $c = 4$, $\theta_0 = [1/4, 1/4, 1/4, 1/4], \epsilon = 0.2$ (middle right); $n = 1000$, $r = 1$, $c = 4$, $\theta_0 = [0.1, 0.2, 0.3, 0.4], \epsilon = 0.2$ (right).

Figure 8: Independence tests. Tables used: Home Zone and Income Category with $n = 2291$, $r = 4$, $c = 16$ (left), Religion and Attitudes with $n = 1055$, $r = c = 3$ (middle), Religion and Education with $n = 1055$, $r = c = 3$ (right).
Figure 9: Homogeneity tests (Table dimension = 2). Tables used: Czech A with $n_1 = 1054$, $n_2 = 787$ (left), Czech B with $n_1 = 1581$, $n_2 = 260$ (middle), Rochdale A with $n_1 = 586$, $n_2 = 79$ (right).

Figure 10: Homogeneity test. Attributes used: Home Zone with $n_1 = 1286$, $n_2 = 1005$, Table dim = 4 (left), Attitudes with $n_1 = 453$, $n_2 = 602$, Table dim = 3 (middle), Education with $n_1 = 453$, $n_2 = 602$, Table dim = 3 (right).

Figure 11: Goodness of fit on $1 \times 2$ tables. Tables used: Czech A with $n = 787$, $\theta_0=[0.4886148, 0.5113852]$ (left), Czech B with $n = 260$, $\theta_0=[0.58760278, 0.41239722]$ (middle), Rochdale A with $n = 79$, $\theta_0=[0.77986348, 0.22013652]$ (right).

Figure 12: Goodness of fit tests. Attributes used: Home Zone with $n = 1005$, $r = 4$, $c = 4$, $\theta_0=[0.50155521, 0.05987558, 0.21772939, 0.22983981]$ (left), Attitudes with $n = 602$, $r = 1$, $c = 3$, $\theta_0=[0.37748344, 0.21633554, 0.40618102]$ (middle), Education with $n = 602$, $r = 1$, $c = 3$, $\theta_0=[0.14569536, 0.53421634, 0.3200883]$ (right).
should gravitate towards very small values.

Figure 14 shows typical results. Generally, the output perturbation methods have high variance because of their noise requirements. Meanwhile, input perturbation methods perform reasonably well even in these tough noise scenarios. The exception is the output perturbation of the Diff statistic, which is clearly the best and requires the least noise of all. The Diff statistic grows with \( n \), and so would not have an asymptotic distribution in the unrestricted case, however, it is a promising starting point upon which other privacy-aware statistics could be built.

7. CONCLUSIONS

In this paper, we revisited the topic of \( \epsilon \)-differentially private hypothesis testing. We provided \( p \)-value algorithms that, for the first time, allow reliable private hypothesis tests for data sizes often used in social sciences, as well as reliable tests with very strong privacy protections (i.e. small \( \epsilon \) values) for large data sizes. We believe that new test statistics tailored for the privacy domain may yield even further improvement, but those test statistics are open problems. We also proposed a permutation-based testbed to help estimate the quality of new test statistics on real data before fully working out their mathematical details.

8. REFERENCES

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APPENDIX

A. PROOF OF THEOREM 1

THEOREM 1. Suppose the probabilities under the true null hypothesis are nonzero. Consider the noisy table \( \widetilde{T} = T + V\kappa \sqrt{n} \) where \( V \) is a 0-mean random variable\(^{13}\) with fixed variance.\(^{14}\) Let \( \tilde{\chi}^2 \) denote the chi-squared statistic where we replace \( T \) with the noisy \( \widetilde{T} \) and compute the expected counts \( E \) from \( \widetilde{T} \) instead of \( T \) (or use any \( E \) such that \( E/n \) converges in probability to the true null distribution). Let \( \bar{L} \bar{R} \) denote the likelihood ratio statistic with the same substitutions and from each term i subtract \( 2(\widetilde{T}[i] - E[i]) \).\(^{15}\) Then \( \tilde{\chi}^2 \) and \( \bar{L} \bar{R} \) have the same asymptotic distribution as \( n \to \infty \).

Without loss of generality, assume all of the noisy tables have been stuffed into one vector \( \widetilde{T} \) and similarly for the expected counts \( E \).

Note that asymptotically, the correction of terms where \( \widetilde{T}[i] < 0 \) will not be needed since \( \widetilde{T}[i]/n \) converges in probability to the true parameter \( \theta[i] \).

PROOF. We will use the Taylor series expansion of \( \log(1 + x) \) around \( x = 0 \):

\[
\log(1 + x) = x - \int_0^1 \int_0^1 u \frac{1}{(1 + ux^2)^2} du \, dv
\]

\[
2 \sum_i \left( \overline{T}[i] \log \frac{\overline{T}[i]}{E[i]} - \overline{T}[i] + E[i] \right) = 2 \sum_i \overline{T}[i] \left( \frac{\overline{T}[i]}{E[i]} - 1 \right) - 2 \sum_i \overline{T}[i] \int_0^1 \int_0^1 u \left( \frac{\overline{T}[i]}{E[i]} - 1 \right)^2 \frac{1}{(1 + uv \left( \frac{\overline{T}[i]}{E[i]} - 1 \right))^2} du \, dv
\]

\[
= 2 \sum_i \overline{T}[i] - E[i] \left( \frac{\overline{T}[i]}{E[i]} - 1 \right) - 2 \sum_i \overline{T}[i] \int_0^1 \int_0^1 u \left( \frac{\overline{T}[i]}{E[i]} - 1 \right)^2 \frac{1}{(1 + uv \left( \frac{\overline{T}[i]}{E[i]} - 1 \right))^2} du \, dv
\]

\[
= 2 \sum_i \frac{(\overline{T}[i] - E[i])^2}{E[i]} - 2 \sum_i \overline{T}[i] \int_0^1 \int_0^1 u \left( \frac{\overline{T}[i]}{E[i]} - 1 \right)^2 \frac{1}{(1 + uv \left( \frac{\overline{T}[i]}{E[i]} - 1 \right))^2} du \, dv
\]

Now, both \( \overline{T}/n \) and \( E/n \) converge in probability to the true (nonzero) null distribution and so \( \overline{T}/E \) converges to 1 in probability. Thus, an application of Slutsky’s theorem \(^{5}\) allows us to conclude that the term containing the integral converges in distribution to the same limit as \( 2 \sum_i \int_0^1 \int_0^1 u \frac{\overline{T}[i] - E[i]}{E[i]} \, du \, dv = \sum_i \frac{(\overline{T}[i] - E[i])^2}{E[i]} \).

Thus \( 2 \sum_i \left( \overline{T}[i] \log \frac{\overline{T}[i]}{E[i]} - \overline{T}[i] + E[i] \right) \) and \( \frac{(\overline{T}[i] - E[i])^2}{E[i]} \) converge in distribution to the same limit. \( \square \)

B. PROOF OF THEOREM 2

THEOREM 2. (Independence testing). Let \( T[\cdot, \cdot] \) be a contingency table sampled from a Multinomial(\( n, \theta_0 \)) distribution. Consider the noisy table \( \widetilde{T} = T + V\kappa \sqrt{n} \) where \( V \) is a table of independent Laplace(2/\( \epsilon \)) random variables. If the rows and columns under \( \theta_0 \) are independent and if no cells have probability 0, then as \( n \to \infty \), the chi-squared statistic and the likelihood ratio statistic (Definition 3) computed from \( \overline{T} \) (instead of \( T \)) asymptotically have the distribution of the random variable:

\[
\sum_{i,j} \frac{(A[i,j] + \kappa V^*[i,j])^2}{\theta_0[i,j]} - \sum_{i,j} \frac{(A[i,j] + \kappa V^*[i,j])^2}{\theta_0[i,j]} - \sum_j \frac{(A[\cdot, j] + \kappa V^*[\cdot, j])^2}{\theta_0[\cdot, j]} + \frac{(A[\cdot, \cdot] + \kappa V^*[\cdot, \cdot])^2}{1}
\]

where \( V^* \) has the same distribution as \( V \), and the vectorized version \( vec(A) \sim N(0, \text{diag}(vec(\theta_0)) - vec(\theta_0)vec(\theta_0)^\top) \). It is asymptotically equivalent to the quantity we get by replacing \( \theta_0[i,j] \) with \( \frac{\overline{T}[i,j]}{\overline{T}[\cdot, \cdot]} \).

We first need the following Lemma 2.

---

\(^{13}\)If there are two tables \( S, T \) then there are two batches of random variables which we can denote by \( V_1 \) and \( V_2 \).

\(^{14}\)Thus the noise is \( V\kappa \sqrt{n} \) and so \( \text{std}(V\kappa \sqrt{n})/\sqrt{n} = \text{constant} \), as required by our asymptotic regime.

\(^{15}\)If the log is not defined (i.e. \( \overline{T}[i] < 0 \)), do not subtract the offset, simply replace that term with the corresponding chi-squared term (e.g., \( (\overline{T}[i] - E[i])^2 / E[i] \)).
THEOREM 2. Let $T$ be a contingency table sampled from a Multinomial$(n, \theta)$ distribution with no entries having 0 probability. Let $V_\epsilon$ be a table (with same dimensions as $T$) of independent Laplace$(2/\epsilon)$ random variables. Let $\bar{T} = T + V\kappa\sqrt{n}$. Then as $n \to \infty$, $\frac{\bar{T} - n\theta}{\sqrt{n}}$ converges in law to the distribution of the random variable $A + \kappa V^*$, where $V^*$ has the same distribution as $V_\epsilon$ and $\text{vec}(A) \sim N(0, \text{diag}(\text{vec}(\theta)) - \text{vec}(\theta)\text{vec}(\theta)^t)\)\)

PROOF. Since $T$ and $V_\epsilon$ are independent, the result follows from the Central limit theorem and a variation of Slutsky’s theorem [5]. □

The proof of Theorem 2 is provided below.

PROOF. When convenient, we will treat $\bar{T}$, $T$, and $\theta$ as either vectors (with one index) or 2-d arrays with two indices (e.g., $\theta[i, j]$). The conversion is simple: $\theta[(i - 1) \cdot c + j] = \theta[i, j]$.

We will consider the noisy likelihood ratio statistic as it is easier to work with (we apply Theorem 1 and note that in the theorem, $E[i, j] = \bar{T}[i, \cdot]\bar{T}[\cdot, j]/\bar{T}[\cdot, \cdot]$ and so $\sum_{ij} E[i, j] - \sum_{ij} \bar{T}[i, j] = 0$):

$$LR = 2 \left( \sum_{i} \sum_{j} \bar{T}[i, j] \log \left( \frac{\bar{T}[i, j]}{E[i, j]} \right) \right)$$

$$= 2 \left( \sum_{i} \sum_{j} \bar{T}[i, j] \log \left( \frac{\bar{T}[i, j]}{\sum_{i=1}^{r} \sum_{j=1}^{c} \bar{T}[i^*, j^*]} \right) - 2 \sum_{i} \sum_{j} \bar{T}[i, j] \log \left( \frac{\sum_{i=1}^{r} \sum_{j=1}^{c} \bar{T}[i^*, j^*]}{\sum_{i=1}^{r} \sum_{j=1}^{c} \bar{T}[i^*, j^*]} \right) - 2 \sum_{i=1}^{r} \sum_{j=1}^{c} \bar{T}[i, j] \log \left( \frac{\sum_{i=1}^{r} \sum_{j=1}^{c} \bar{T}[i^*, j^*]}{\sum_{i=1}^{r} \sum_{j=1}^{c} \bar{T}[i^*, j^*]} \right) \right)$$

$$= 2 \sum_{i=1}^{r} \sum_{j=1}^{c} \bar{T}[i, j] \log \left( \bar{T}[i, j] \right) + 2 \sum_{i=1}^{r} \sum_{j=1}^{c} \bar{T}[i, j] \log \left( \sum_{i=1}^{r} \sum_{j=1}^{c} \bar{T}[i^*, j^*] \right) - 2 \sum_{i=1}^{r} \sum_{j=1}^{c} \bar{T}[i, j] \log \left( \bar{T}[i, j] \right)$$

We will

- use the second order taylor expansion to expand these quantities around $n\theta_0[i, j]$, $n\theta_0[i, \cdot]$, $n\theta_0[\cdot, j]$ and $n\theta_0[\cdot, \cdot]$. That is, $f(x) = f(x_0) + (x - x_0)^t \nabla f(x_0) + (x - x_0)^t \int_0^1 \nabla^2 \upsilon f(x_0 + \upsilon(x - x_0)) d\upsilon (x - x_0)$.

- use the fact that $\theta_0[i, j] = \theta_0[i, \cdot] \theta_0[\cdot, j]$.

- use convergence in probability of $\bar{T}/n$ to $\theta_0$ to deduce that $n/[n\theta_0[i, j] + \upsilon((\bar{T}[i, j] - n\theta_0[i, j]))] \to 1/\theta_0[i, j]$ in probability.
\( \Xi(\tilde{T}_n) = 2 \sum_{ij} n\theta_{0[i,j]} \log(n\theta_{0[i,j]}) - 2 \sum_{i} n\theta_{0[i,\cdot]} \log(n\theta_{0[i,\cdot]}) - 2 \sum_{j} n\theta_{0[\cdot,j]} \log(n\theta_{0[\cdot,j]}) + 2n\theta_{0[\cdot,\cdot]} \log(n\theta_{0[\cdot,\cdot]}) \\
+ 2 \sum_{ij} (\tilde{T}[i,j] - n\theta_{0[i,j]})(1 + \log(n\theta_{0[i,j]})) \\
- 2 \sum_{j} (\tilde{T}[\cdot,j] - n\theta_{0[\cdot,j]})(1 + \log(n\theta_{0[\cdot,j]})) \\
+ 2(\tilde{T}[\cdot,\cdot] - n\theta_{0[\cdot,\cdot]})(1 + \log(n\theta_{0[\cdot,\cdot]})) \\
= 2 \sum_{ij} \int_{0}^{1} \int_{0}^{1} v n\theta_{0[i,j]} + uv(\tilde{T}[i,j] - n\theta_{0[i,j]}) dv \, du - 2 \sum_{i} \int_{0}^{1} \int_{0}^{1} v n\theta_{0[i,\cdot]} + uv(\tilde{T}[i,\cdot] - n\theta_{0[i,\cdot]}) dv \, du \\
+ 2 \sum_{j} \int_{0}^{1} \int_{0}^{1} v n\theta_{0[\cdot,j]} + uv(\tilde{T}[\cdot,j] - n\theta_{0[\cdot,j]}) dv \, du + 2 \sum_{i} \int_{0}^{1} \int_{0}^{1} v n\theta_{0[\cdot,\cdot]} + uv(\tilde{T}[\cdot,\cdot] - n\theta_{0[\cdot,\cdot]}) dv \, du \\
\sim \sum_{ij} \left( \frac{\tilde{T}[i,j] - n\theta_{0[i,j]}}{\sqrt{n}} \right)^2 \frac{1}{\theta_{0[i,j]}} - \sum_{i} \frac{1}{\sqrt{n}} \left( \frac{\tilde{T}[i,\cdot] - n\theta_{0[i,\cdot]}}{\sqrt{n}} \right)^2 \frac{1}{\theta_{0[i,\cdot]}} \\
- \sum_{j} \frac{1}{\sqrt{n}} \left( \frac{\tilde{T}[\cdot,j] - n\theta_{0[\cdot,j]}}{\sqrt{n}} \right)^2 \frac{1}{\theta_{0[\cdot,j]}} - \sum_{j} \frac{(A[i,j] + n\epsilon V^*[j])^2}{\theta_{0[i,j]}} - \sum_{i} \frac{(A[i,\cdot] + n\epsilon V^*[\cdot,j])^2}{\theta_{0[i,\cdot]}} + \frac{(A[i,\cdot] + n\epsilon V^*[\cdot,\cdot])^2}{\theta_{0[\cdot,\cdot]}} \\
\sim \sum_{ij} \frac{(A[i,j] + n\epsilon V^*[j])^2}{\theta_{0[i,j]}} + \sum_{j} \frac{(A[i,\cdot] + n\epsilon V^*[\cdot,j])^2}{\theta_{0[i,\cdot]}} + \frac{(A[i,\cdot] + n\epsilon V^*[\cdot,\cdot])^2}{\theta_{0[\cdot,\cdot]}}. \\
\end{align*}

Where the last two lines follow from: (a) noting that under the null hypothesis \( \theta_{0[i,j]} = \theta_{0[i,\cdot]} \theta_{0[\cdot,j]} \), (b) convergence in probability (e.g., \( \sum_{ij} \tilde{T}[i,j] \to \theta_0[1,1] \)), (c) Lemma 2 and (d) Slutsky’s theorem [5].

The theorem follows for the likelihood ratio statistic. The asymptotic equivalence follows from convergence in probability. Together with Theorem 1, the theorem also follows for the chi-squared statistic. \( \square \)

### C. PROOF OF THEOREM 3

**Theorem 3.** (Homogeneity testing). Let \( T[\cdot] \) and \( S[\cdot] \) be samples from Multinomial(\( n_1, \theta_0 \)) and Multinomial(\( n_2, \theta_0 \)) distributions, respectively. Consider the noisy versions \( \tilde{T} = T + V_1^* \kappa_1 \sqrt{n} \) and \( \tilde{S} = S + V_2^* \kappa_2 \sqrt{n} \) where \( V_1, V_2 \) are vectors of independent Laplace(2/\( \epsilon \)) random variables. If no cells have probability 0, then as \( n_1, n_2 \to \infty \) the chi-squared statistic and the likelihood ratio statistic (Definition 2) computed from \( \tilde{T} \) and \( \tilde{S} \) (instead of \( T \) and \( S \)) asymptotically have the distribution of the random variable:

\[
\sum_{j} \left[ \frac{n_2}{n_1 + n_2} (A[j] + \kappa_1 V_1[j]) - \frac{n_1}{n_1 + n_2} (A[j] + \kappa_2 V_2[j]) \right]^2 \frac{1}{\theta_{0[j]}}
\]

where \( V_1, V_2 \) are independent with the same distribution as \( V_1, V_2 \) and \( A_1, A_2 \sim N(0, \text{diag}(\theta_0) - \theta_0 \theta_0') \), and \( A_1, A_2 \) are independent. It is asymptotically equivalent to the quantity we get by replacing \( \theta_{0[j]} \) with \( (\tilde{T}[j] + \tilde{S}[j])/(n_1 + n_2) \).

**Proof.** Note that \( E_1[i] = \frac{n_1}{n_1 + n_2} (\tilde{T}[i] + S[i]) \) and \( E_2[i] = \frac{n_2}{n_1 + n_2} (\tilde{T}[i] + S[i]) \) and \( \sum_i E_1[i] + \sum_j E_2[j] = \sum_i \tilde{T}[i] + \sum_j \tilde{S}[j] \) (which we use when applying Theorem 1).

According to Definition 2, the chi-squared statistic based on \( \tilde{T} \) and \( \tilde{S} \) is:

\[
\tilde{\chi}^2 = \sum_j \frac{(\tilde{T}[j] - \frac{n_1}{n_1 + n_2} (\tilde{T}[j] + \tilde{S}[j])}^{2} + \sum_j \sqrt{n_2(T[j] + S[j])} \]
\[ \tilde{\chi}^2 = \sum_j \frac{n_1 + n_2}{n_1(T[j] + S[j])} \left( \tilde{T}[j] - \frac{n_1}{n_1 + n_2}(T[j] + \bar{S}[j]) \right)^2 + \frac{n_1 + n_2}{n_2(T[j] + S[j])} \left( \bar{S}[j] - \frac{n_2}{n_1 + n_2}(T[j] + \bar{S}[j]) \right)^2 \]

\[ \tilde{\chi}^2 = \sum_j \left( \frac{1}{n_1(n_1 + n_2)(T[j] + S[j])} + \frac{1}{n_2(n_1 + n_2)(T[j] + S[j])} \right) \left( n_2T[j] - n_1\bar{S}[j] \right)^2 \]

\[ \tilde{\chi}^2 = \sum_j \left( \frac{n_1n_2}{(T[j] + S[j])} \left( \frac{T[j] - n_n_1\theta_0[j]}{\sqrt{n_1}} \right) - \frac{1}{\sqrt{n_2}} \left( \frac{S[j] - n_2\theta_0[j]}{\sqrt{n_2}} \right) \right)^2 \]

where the last line follows from a) convergence in probability \( \frac{\tilde{T}[j] + \bar{S}[j]}{n_1 + n_2} \rightarrow \theta_0[j] \), (b) Lemma 2 in Appendix B, and (c) Slutsky’s theorem [5].

The theorem follows for the chi-squared statistic. Together with Theorem 1, the theorem also follows for the likelihood ratio statistic. The asymptotic equivalence also follows from convergence in probability. \( \square \)

D. PROOF OF THEOREM 4

**Theorem 4.** (Goodness-of-fit). Let \( T[\cdot] \) be a sample from a Multinomial(\( n, \theta_0 \)) distribution. Consider the noisy version \( \tilde{T} = T + V_\epsilon \sqrt{n} \) where \( V \) is a vector of independent Laplace(2/\( \epsilon \)) random variables. If no cells have probability 0, then as \( n \rightarrow \infty \), then the statistics:

\[ \tilde{\chi}^2 = \sum_j \frac{\tilde{T}[j] - n\theta_0[j]^2}{n\theta_0[j]} \] (8)

\[ \tilde{L}R = 2\sum_j \left( \tilde{T}[j] \log(\tilde{T}[j]/n\theta_0[j]) - \tilde{T}[j] + n\theta_0[j] \right) \] (9)

asymptotically have the same distribution as:

\[ \sum_j (A[j] + \nu^* \nu[j])^2 / n\theta_0[j] \] where \( \nu^* \) has the same distribution as \( V \) and \( A \sim N(0, \text{diag}(\theta_0) - \theta_0\theta_0^*) \).

**Proof.** According to Definition 1, the chi-squared statistic based on \( \tilde{T} \) is:

\[ \tilde{\chi}^2 = \sum_j \frac{(\tilde{T}[j] - n\theta_0[j])^2}{n\theta_0[j]} = \sum_j \left( \frac{\tilde{T}[j] - n\theta_0[j]}{\sqrt{n}} \right)^2 \] \( \sim \sum_j \frac{(A[j] + \nu^* \nu[j])^2}{n\theta_0[j]} \)

because of Lemma 2 Appendix B)

The theorem follows for the chi-squared statistic. Together with Theorem 1, the theorem also follows for the likelihood ratio statistic. \( \square \)

E. PROOF OF SENSITIVITIES

**Theorem 7.** The \( n \)th value of the \( \chi^2 \)-statistic for a 2 x 2 contingency tables is:

\[ C = \max \left\{ \begin{array}{ll} C[n - 2T, 1, 2T[1, \cdot]] & \text{if } T[1, \cdot] \leq T[\cdot, 1], \ T[2, \cdot] \geq T[\cdot, 2] \\ C[n - 2T, 1, 1T[2, \cdot]] & \text{if } T[1, \cdot] > T[\cdot, 1], \ T[2, \cdot] < T[\cdot, 2] \\ C[n - 2T, 1, 1T[1, \cdot]] & \text{if } T[1, \cdot] > T[\cdot, 1], \ T[2, \cdot] > T[\cdot, 2] \\ C[n - 2T, 2, 2T[\cdot, 2]] & \text{if } T[1, \cdot] < T[\cdot, 1], \ T[2, \cdot] < T[\cdot, 2] \end{array} \right. \]

where \( C = \frac{n^2}{T[1, \cdot]T[\cdot, 1] - T[1, \cdot]T[2, \cdot]} \).
Proof. From Definition 3, the $\chi^2$ statistic based on a $2 \times 2$ contingency table $T$ with fixed marginals $T[1, \cdot], T[2, \cdot], T[\cdot, 1], T[\cdot, 2]$ is

$$\chi^2(T) = C/n \times (T[1,1]T[2,2] - T[1,2]T[2,1])^2$$

From Definition 7, the neighboring contingency table $T'$ of $T$ has cell counts $T[1,1] - 1, T[1,2] + 1, T[2,1] + 1, T[2,2] - 1$. This implies the conditions $T[1,1] \geq 1$ and $T[2,2] \geq 1$ \footnote{The case of incrementing $T[1,1], T[2,2]$ and decrementing $T[1,2], T[2,1]$ is symmetric because we can exchange $T$ and $T'$. This is also true for all other neighboring contingency tables with fixed marginals.}. From Definition 5, the sensitivity equals

$$\max_{T[1,1],T[1,2],T[2,1],T[2,2]} \left| \chi^2(T) - \chi^2(T') \right| = \max_{T[1,1],T[1,2],T[2,1],T[2,2]} C \left| 2T[1,1]T[2,2] - 2T[1,2]T[2,1] - n \right|$$

There are two ways to solve the above problem, that is, either maximize the formula inside the absolute value, or minimize it. Note we have the constraints $1 \leq T[1,1] \leq \min(T[1,\cdot],T[\cdot,\cdot]), 0 \leq T[1,2] \leq \min(T[1,\cdot],T[\cdot,\cdot]) - 1, 0 \leq T[2,1] \leq \min(T[2,\cdot],T[\cdot,\cdot]) - 1, 1 \leq T[2,2] \leq \min(T[2,\cdot],T[\cdot,\cdot]),$ and $T[1,\cdot] + T[2,\cdot] = T[\cdot,1] + T[\cdot,2] = n$. Since the marginals are fixed, we only have four variables.

In the first way, there are two cases.

1. If $T[1,\cdot] \leq T[1,1], T[2,\cdot] \geq T[2,2]$.
   It is easy to see $T[1,1] = T[1,\cdot], T[2,2] = T[2,\cdot], T[1,2] = 0$ and $T[2,1] = T[2,\cdot] - T[2,2]$ maximize the formula inside the absolute value. They give the result $C \cdot (n - 2T[1,\cdot],2T[1,\cdot])$.

2. If $T[1,\cdot] > T[1,1]$, $T[2,\cdot] < T[2,2]$.
   It is easy to see $T[1,1] = T[1,\cdot], T[2,2] = T[2,\cdot], T[1,2] = 0$ and $T[2,1] = T[2,\cdot] - 2T[2,\cdot]$ maximize the formula inside the absolute value. They give the result $C \cdot (n - 2T[1,\cdot],2T[1,\cdot])$.

In the second way, there are also two cases.

1. If $T[1,\cdot] \leq T[1,2], T[2,\cdot] \geq T[2,1]$.
   It is easy to see $T[1,2] = T[1,\cdot] - 1, T[2,1] = T[2,\cdot] - 1, T[1,1] = 1$ and $T[2,2] = T[2,\cdot] - T[1,\cdot] + 1$ minimize the formula inside the absolute value. They give the result $C \cdot (n - 2T[1,\cdot],2T[1,\cdot])$.

2. If $T[1,\cdot] > T[1,2], T[2,\cdot] < T[2,1]$.
   It is easy to see $T[1,2] = T[1,\cdot] - 1, T[2,1] = T[2,\cdot] - 1, T[1,1] = T[1,\cdot] - T[1,\cdot] + 1$ and $T[2,2] = T[2,\cdot] - 1$ minimize the formula inside the absolute value. They give the result $C \cdot (n - 2T[1,\cdot],2T[1,\cdot])$.

Therefore, the maximum value among all cases that apply to the marginals of table $T$ is its sensitivity with fixed marginals, which leads to the result in Theorem 7. \qed

Theorem 8. The $s_h$ value of the $\chi^2$-statistic for an $r \times c$ table $T$ with $r \geq 3, c \geq 3$ is:

$$\max_{i,j} \left\{ \frac{C}{n^2} \left[ 2(T[i,\cdot] - T[i,j]) + T[1,\cdot] + T[\cdot,1] - T[1,1] \right] \right\}$$

where $a = \min(T[i,\cdot],T[\cdot,j]), d = \min(T[i,\cdot],T[j,\cdot]), b = \min(T[i,\cdot],T[\cdot,j]), c = \min(T[i,\cdot],T[\cdot,j]) - 1, e = \min(T[i,\cdot],T[\cdot,j])$ and $C = \frac{T[i,\cdot]T[j,\cdot]}{T[\cdot,\cdot]T[i,\cdot]T[j,\cdot]}$.

Proof. Let $T$ and $T'$ be neighboring contingency tables no smaller than $3 \times 3$ with fixed marginals. Suppose the four different entries between $T$ and $T'$ locate at the intersection of row $i_1, i_2$ and column $j_1, j_2$. We write $T[i_1,j_1], T[i_1,j_2], T[i_2,j_1], T[i_2,j_2]$ as $a, b, c, d$ respectively for short. The corresponding entries in $T'$ are then $a - 1, b + 1, c - 1, d - 1$. Note we have the conditions $a \geq 1$ and $d \geq 1$. By Definition 3 and Definition 5, the sensitivity of $\chi^2$-statistic can be computed by

$$\frac{C}{n^2} \left[ \chi^2(T) - \chi^2(T') \right] = \max_{T, T'} \left\{ \sum_{i,j} \left( T[i,j] - \frac{T[i,\cdot]T[\cdot,j]}{T[\cdot,\cdot]} \right) - \sum_{i,j} \left( T'[i,j] - \frac{T'[i,\cdot]T'[\cdot,j]}{T'[\cdot,\cdot]} \right) \right\}$$

$$= \max_{T, T'} \left\{ \sum_{i,j} \left( 2T[i,j] - T[i,\cdot] - T[\cdot,j] \right) \right\}$$

Since all marginals are known, the only variables in the function are $a, b, c, d$. There are two ways to solve the problem.
1. Maximize the function inside the absolute value of the objective function by choosing large values for $a, d$ and small values for $b, c$.

   $b = c = 0$ is the smallest possible values for them. $a = \min(T[i_1, \cdot], T[\cdot, j_1])$ and $d = \min(T[i_2, \cdot], T[\cdot, j_2])$ are the largest possible values for them (respectively). Since the table is no smaller than $3 \times 3$, these settings can form a valid table. Plugging them back to the objective function gives $C'[2(T[i_2, \cdot] T[i_1, \cdot] a + T[i_1, \cdot] T[j_1] d) - (T[i_1, \cdot] + T[i_2, \cdot])(T[j_1] + T[\cdot, j_2])]/n$. Next, we just find the indices $i_1, i_2, j_1, j_2$ that maximizes it.

2. Minimize the function inside the absolute value of the objective function by choosing large values for $b, c$ and small values for $a, d$.

   $a = d = 1$ is the smallest possible values for them. $b = \min(T[i_1, \cdot], T[\cdot, j_2]) - 1$ and $c = \min(T[i_2, \cdot], T[\cdot, j_1]) - 1$ are the largest possible values for them (respectively). These settings can also form a valid table. Plugging them back to the objective function gives $C'[(T[i_1, \cdot] - T[i_2, \cdot])(T[j_1] - T[\cdot, j_2]) - 2(T[i_2, \cdot] T[i_1, \cdot] b + T[i_1, \cdot] T[\cdot, j_2] c))/n$. Next, we find the indices $i_1, i_2, j_1, j_2$ that maximizes it.

The sensitivity should be the larger one computed from the above two cases, which gives the result in Theorem 8.

\[ \square \]

**Theorem 9.** The $s_k$ value of the likelihood ratio statistic for $2 \times 2$ contingency tables is

\[
2 \times \max \left\{ \log \frac{T[i_1, \cdot]}{T[i_1, \cdot] + T[\cdot, j_1]} + \log \frac{T[j_1]}{T[j_1] + T[i_2, \cdot]} + \log \frac{T[i_2, \cdot]}{T[i_2, \cdot] + T[j_2, \cdot] + 1}, \log \frac{T[i_1, \cdot] + 1}{T[i_1, \cdot] + 1} \right\}
\]

**Proof.** Suppose $2 \times 2$ contingency table $T$ has fixed marginals $T[1, \cdot], T[2, \cdot], T[\cdot, 1], T[\cdot, 2]$. From Definition 7, the neighboring contingency table $T'$ of $T$ has cell counts $T[1, \cdot] - 1, T[1, \cdot] + 1, T[2, \cdot] + 1, T[2, \cdot] - 1$. This implies the conditions $T[1, \cdot] \geq 1$ and $T[2, \cdot] \geq 1$. We also have the conditions $T[1, \cdot] + T[2, \cdot] = T[\cdot, 1] + T[\cdot, 2] = n$. From Definition 3 and 5, the sensitivity equals

\[
\max_{T'} \sum_{i,j} T[i, j] \log \frac{nT[i, j]}{T[i, j] + T'_{i, j}} - \sum_{i,j} T'_i[i, j] \log \frac{nT'_i[i, j]}{T'_i[i, j] + T'_{i, j}}
\]

\[
= \max_{T[1, \cdot], T[2, \cdot], T[\cdot, 1], T[\cdot, 2]} \left( \log \frac{T[1, \cdot]}{T[1, \cdot] + 1} + \log \frac{T[2, \cdot]}{T[2, \cdot] + 1} + \log \frac{T[\cdot, 1]}{T[\cdot, 1] + 1} + \log \frac{T[\cdot, 2]}{T[\cdot, 2] + 1} \right)
\]

There are two ways to solve the above problem, that is, either maximize the formula inside the absolute value of the above objective function, or minimize it. Note we have the constraints $1 \leq T[1, \cdot] \leq \min(T[1, \cdot], T[\cdot, 1]), 0 \leq T[2, \cdot] \leq \min(T[2, \cdot], T[\cdot, 2]) - 1, 0 \leq T[2, \cdot] \leq \min(T[2, \cdot], T[\cdot, 1]) - 1, 1 \leq T[2, \cdot] \leq \min(T[2, \cdot], T[\cdot, 2])$.

The derivative of $\log \frac{T[i_2, \cdot]}{T[i_2, \cdot] + 1}$ with respect to $T[1, \cdot]$ is $\frac{\log(\frac{T[i_1, \cdot]}{T[i_1, \cdot] + 1})}{T[i_1, \cdot] + 1}$. When $T[1, \cdot] > 1$, the term and its derivative are both positive; when $T[1, \cdot] = 1$, the term equals 0. The last term $\log \frac{T[2, \cdot]}{T[2, \cdot] + 1}$ has exactly the same analysis, and so does $T[2, \cdot]$. For the term $\log \frac{T[2, \cdot]}{T[2, \cdot] + 1}$, its derivative with respect to $T[1, \cdot]$ equals $\log(\frac{T[i_1, \cdot]}{T[i_1, \cdot] + 1})$. Both the term and its derivative are negative when $T[1, \cdot] > 0$. When $T[1, \cdot] = 0$, the term equals 0. Similarly, we can apply the same analysis to $T[2, \cdot]$ as what we do to $T[1, \cdot]$. We use this derivative analysis for the two ways of solving the problem.

In the first way, there are two cases. 1. If $T[1, \cdot] \leq T[\cdot, 1], T[2, \cdot] \geq T[\cdot, 2]$. It is easy to see $T[1, \cdot] = T[1, \cdot], T[2, \cdot] = T[\cdot, 2], T[1, \cdot] = 0$ and $T[2, \cdot] = T[\cdot, 2] - T[\cdot, 2]$ maximize the formula inside the absolute value, which leads to $\log(\frac{T[i_1, \cdot]}{T[i_1, \cdot] + 1}) + \log(\frac{T[i_2, \cdot]}{T[i_2, \cdot] + 1}) + \log(\frac{T[j_1]}{T[j_1] + T[\cdot, j_2]})$.
2. If \( T[1,\cdot] > T[\cdot,1], T[2,\cdot] < T[\cdot,2] \)

   It is easy to see \( T[1,1] = T[\cdot,1], T[2,2] = T[2,\cdot], T[2,1] = 0 \) and \( T[1,2] = T[\cdot,2] - T[\cdot,\cdot] \) maximize the formula inside the absolute value, which leads to

   \[
   \log \frac{T_{1,1}^{T[1,1]}}{T_{2,2}^{T[2,2]}} + \log \frac{T_{1,1}^{T[1,1]}}{T_{2,2}^{T[2,2]}} + \log \frac{T_{1,1}^{T[1,1]}}{T_{2,2}^{T[2,2]}}.
   \]

   In the second way, there are also two cases.

   1. If \( T[1,\cdot] \leq T[\cdot,2], T[2,\cdot] \geq T[\cdot,1] \)

      It is easy to see \( T[1,1] = T[\cdot,1] - 1, T[2,1] = T[\cdot,1] - 1 \) and \( T[2,2] = T[\cdot,2] - T[\cdot,\cdot] + 1 \) minimize the formula inside the absolute value, which leads to

      \[
      \log \frac{T_{1,1}^{T[1,1]-1}}{T_{2,2}^{T[2,2]-1}} + \log \frac{T_{1,1}^{T[1,1]-1}}{T_{2,2}^{T[2,2]-1}} + \log \frac{T_{1,1}^{T[1,1]-1}}{T_{2,2}^{T[2,2]-1}}.
      \]

   2. If \( T[1,\cdot] > T[\cdot,2], T[2,\cdot] < T[\cdot,1] \)

      It is easy to see \( T[1,1] = T[\cdot,1] - 1, T[1,2] = T[\cdot,2] - 1 \) and \( T[2,2] = T[\cdot,2] - T[\cdot,\cdot] + 1 \) minimize the formula inside the absolute value, which leads to

      \[
      \log \frac{T_{1,1}^{T[1,1]-1}}{T_{2,2}^{T[2,2]-1}} + \log \frac{T_{1,1}^{T[1,1]-1}}{T_{2,2}^{T[2,2]-1}} + \log \frac{T_{1,1}^{T[1,1]-1}}{T_{2,2}^{T[2,2]-1}}.
      \]

   Therefore, the maximum value among all cases that apply to the marginals of table \( T \) is its sensitivity with fixed marginals, which leads to the result in Theorem 9.

   \[\square\]

   **Theorem 10.** The \( s_h \) of the likelihood ratio statistic \( LR \) on \( r \times c \) \((r \geq 3, c \geq 3)\) contingency tables is

   \[
   2 \times \max \left\{ \begin{array}{l}
   \max_{i_1,i_2,j_1,j_2} \frac{\log \frac{n^a}{(a-1)^{a-1}} + \log \frac{d^d}{(d-1)^{d-1}}}{(a-1)^{a-1}} + \log \frac{b^b}{(b+1)^{b+1}} + \log \frac{c^c}{(c+1)^{c+1}} \\
   \max_{i_1,i_2,j_1,j_2} \frac{\log \frac{n^a}{(a-1)^{a-1}} + \log \frac{d^d}{(d-1)^{d-1}}}{(a-1)^{a-1}} + \log \frac{b^b}{(b+1)^{b+1}} + \log \frac{c^c}{(c+1)^{c+1}} \end{array} \right. 
   \]

   where \( a = \min(T[i,1] \cdot T[j,1]) \), \( d = \min(T[i,2] \cdot T[j,2]) \), \( b = \min(T[i,1] \cdot T[j,2]) - 1 \), \( c = \min(T[i,2] \cdot T[j,1]) - 1 \).

   **Proof.** Let \( T \) and \( T' \) be neighboring contingency tables no smaller than \( 3 \times 3 \) with fixed marginals. Suppose the four different entries between \( T \) and \( T' \) locate at the intersection of row \( i_1, i_2 \) and column \( j_1, j_2 \). We write \( T[i_1,j_1], T[i_1,j_2], T[i_2,j_1], T[i_2,j_2] \) as \( a, b, c, d \) respectively for short. The corresponding entries in \( T' \) are then \( a-1, b+1, c+1, d-1 \). Note we have the conditions \( a \geq 1 \) and \( d \geq 1 \). By Definition 3 and Definition 5, the sensitivity of likelihood ratio statistic can be computed by

   \[
   \max_{T,T'} 2 \sum_{i,j} T[i,j] \log \frac{nT[i,j]}{T'[i,j]} - \sum_{i,j} T'[i,j] \log \frac{nT'[i,j]}{T[i,j]} \]

   \[
   = \max_{a,b,c,d} 2 \left[ \log \frac{a^a}{(a-1)^{a-1}} + \log \frac{b^b}{(b+1)^{b+1}} + \log \frac{c^c}{(c+1)^{c+1}} \right] + \log \frac{d^d}{(d-1)^{d-1}}
   \]

   The objective function only contains variables \( a, b, c, d \). So, following from the proof for Theorem 9, we do the same thing.

   That is, either maximize or minimize the formula inside the absolute value from the objective function.

   In the first case, we choose \( b = c = 0, a = \min(T[i,1] \cdot T[j,1]), d = \min(T[i,2] \cdot T[j,2]) \), which gives the result \( \log \frac{a^a}{(a-1)^{a-1}} + \log \frac{d^d}{(d-1)^{d-1}} \). Next, we find the indices \( i_1, i_2, j_1, j_2 \) which maximizes it.

   In the second case, we choose \( a = d = 1, b = \min(T[i,1] \cdot T[j,2]) - 1, c = \min(T[i,2] \cdot T[j,1]) - 1 \), which gives the result \( \log \frac{b^b}{(b+1)^{b+1}} + \log \frac{c^c}{(c+1)^{c+1}} \). Next, we find the indices \( i_1, i_2, j_1, j_2 \) which maximizes it.

   The sensitivity is the larger of the above two cases, which leads to Theorem 10.

   \[\square\]

   **Theorem 11.** The \( s_h \) value of the log-likelihood statistic based on \( 2 \times 2 \) contingency tables is

   \[
   \max \left\{ \begin{array}{l}
   \log(T[2,\cdot] - T[\cdot,2]) + 1 - \log(T[1,\cdot] - \log(T[\cdot,\cdot]) \\
   \quad \text{if } T[1,\cdot] \leq T[\cdot,1], T[2,\cdot] \leq T[\cdot,\cdot] \\
   \log(T[\cdot,2] - T[2,\cdot]) + 1 - \log(T[\cdot,1] - \log(T[\cdot,\cdot]) \\
   \quad \text{if } T[1,\cdot] \leq T[\cdot,1], T[2,\cdot] \leq T[\cdot,\cdot] \\
   \log(T[1,\cdot] + \log(T[\cdot,1]) - \log(T[2,\cdot] - T[\cdot,1]) + 1 \\
   \quad \text{if } T[1,\cdot] \leq T[\cdot,1], T[2,\cdot] \leq T[\cdot,\cdot] \\
   \log(T[2,\cdot] + \log(T[\cdot,2]) - \log(T[1,\cdot] - T[\cdot,2]) + 1 \\
   \quad \text{if } T[1,\cdot] \leq T[\cdot,1], T[2,\cdot] \leq T[\cdot,\cdot] \\
   \end{array} \right. 
   \]

   **Proof.** Suppose \( 2 \times 2 \) contingency table \( T \) has fixed marginals \( T[1,\cdot], T[2,\cdot], T[\cdot,1], T[\cdot,\cdot] \). From Definition 7, the neighboring contingency table \( T' \) of \( T \) has cell counts \( T[1,1] - 1, T[1,2] + 1, T[2,1] + 1, T[2,2] - 1 \). This implies the conditions \( T[1,1] \geq 1 \) and \( T[2,2] \geq 1 \). We also have the conditions \( T[1,\cdot] + T[2,\cdot] = T[\cdot,1] + T[\cdot,\cdot] = n \). From Equation 10 and 5, the sensitivity equals
\[
\max_{T,T'} \left| \sum_i \log(T[i, \cdot]) + \sum_j \log(T[\cdot, j]) - \sum_{i,j} \log(T[i, j]) \right|
- \left[ \sum_i \log(T'[i, \cdot]) + \sum_j \log(T'[\cdot, j]) - \sum_{i,j} \log(T'[i, j]) \right]
\]

\[
= \max_{T[1,1], T[1,2], T[2,1], T[2,2]} \left[ -\log T[1,1] - \log T[1,2] - \log T[2,1] - \log T[2,2] + \log(T[1,1] - 1) \right.
+ \log(T[1,2] + 1) + \log(T[2,1] + 1) + \log(T[2,2] - 1)!
\]
\[
= \max_{T[1,1], T[1,2], T[2,1], T[2,2]} \left[ -\log T[1,1] + \log(T[1,2] + 1) + \log(T[2,1] + 1) - \log T[2,2] \right]
\]

We solve the objective function by either minimizing the formula inside the absolute value of the objective function or maximizing it.

In the first way, there are two cases.

1. If \(T[1, \cdot] \leq T[\cdot, 1], T[2, \cdot] \geq T[\cdot, 2]\)
   It is easy to see \(T[1,1] = T[\cdot, 1], T[2,2] = T[\cdot, 2], T[1,2] = 0 \text{ and } T[2,1] = T[\cdot, 2] - T[\cdot, 2] \text{ minimize it, which leads to }\)

\[
|\log(T[2,\cdot] - T[\cdot, 2]) - \log T[1,\cdot] - \log T[\cdot, 2]|.
\]

2. If \(T[1, \cdot] > T[\cdot, 1], T[2, \cdot] < T[\cdot, 2]\)
   It is easy to see \(T[1,1] = T[\cdot, 1], T[2,2] = T[\cdot, 2], T[1,2] = 0 \text{ and } T[2,1] = T[\cdot, 2] - T[\cdot, 2] \text{ minimize it, which leads to }\)

\[
|\log(T[\cdot, 2] - T[\cdot, 2]) + \log T[1,\cdot] - \log T[\cdot, 1]|.
\]

In the second way, there are also two cases.

1. If \(T[1, \cdot] \leq T[\cdot, 2], T[2, \cdot] \geq T[\cdot, 1]\)
   It is easy to see \(T[1,2] = T[\cdot, 1], T[2,1] = T[\cdot, 1] - 1, T[1,1] = 1 \text{ and } T[2,2] = T[\cdot, 2] - T[\cdot, 2] + 1 \text{ maximize it, which leads to }\)

\[
|\log(T[1,\cdot] + T[\cdot, 1] - T[\cdot, 1])|.
\]

2. If \(T[1, \cdot] > T[\cdot, 2], T[2, \cdot] < T[\cdot, 1]\)
   It is easy to see \(T[1,2] = T[\cdot, 2], T[2,1] = T[\cdot, 1] - 1, T[1,1] = T[\cdot, 1] - T[\cdot, 2] + 1 \text{ and } T[2,2] = T[\cdot, 2] \text{ maximize it, which leads to }\)

\[
|\log(T[\cdot, 2] - T[\cdot, 2]) - \log T[1,\cdot] + \log T[\cdot, 1]|.
\]

Therefore, the maximum value among all cases that apply to the marginals of table \(T\) is its sensitivity with fixed marginals, which leads to the result in Theorem 11.

Theorem 12. The \(s_h\) value of the LL statistic (from Equation 10) for \(r \times c\) tables \((r \geq 3, c \geq 3)\) is

\[
\max \left\{ \max_{i_1, i_2, j_1, j_2} \log \left[ \frac{(b + 1)(c + 1)}{a} \right], \max_{i_1, i_2, j_1, j_2} \log \left( a b ! \right) \right\}
\]

where \(a = \min(T[i_1, \cdot], T[\cdot, j_1]), d = \min(T[i_2, \cdot], T[\cdot, j_2]), b = \min(T[i_1, \cdot], T[\cdot, j_2]) - 1, c = \min(T[i_2, \cdot], T[\cdot, j_1]) - 1.\)

Proof. Let \(T\) and \(T'\) be neighboring contingency tables no smaller than \(3 \times 3\) with fixed marginals. Suppose the four different entries between \(T\) and \(T'\) locate at the intersection of row \(i_1, i_2\) and column \(j_1, j_2\). We write \(T[i_1, j_1], T[i_1, j_2], T[i_2, j_1], T[i_2, j_2]\) as \(a, b, c, d\) respectively for short. The corresponding entries in \(T'\) are then \(a - 1, b + 1, c + 1, d - 1\). Note we have the conditions \(a \geq 1\) and \(d \geq 1\). By Equation 10 and Definition 5, the sensitivity of log-likelihood statistic can be computed by

\[
\max_{T,T'} \left| \sum_i \log(T[i, \cdot]) + \sum_j \log(T[\cdot, j]) - \sum_{i,j} \log(T[i, j]) \right|
- \left[ \sum_i \log(T'[i, \cdot]) + \sum_j \log(T'[\cdot, j]) - \sum_{i,j} \log(T'[i, j]) \right]
\]

\[
= \max_{a,b,c,d} \left[ -\log a! - \log b! - \log c! - \log d! + \log(a - 1)! \right.
+ \log(b + 1)! + \log(c + 1)! + \log(d - 1)!]
\]

\[
= \max_{a,b,c,d} \left[ -\log a + \log(b + 1) + \log(c + 1) - \log d \right]
\]

It is easy to see the objective function is maximized by either \(a = d = 1, b = \min(T[i_1, \cdot], T[\cdot, j_2]) - 1, c = \min(T[i_2, \cdot], T[\cdot, j_1]) - 1\) or \(b = c = 0, a = \min(T[i_1, \cdot], T[\cdot, j_1]), d = \min(T[i_2, \cdot], T[\cdot, j_2])\), which leads to \(\log [(b + 1)(c + 1)] \) and \(\log (a - 1)\) respectively.

Next, we find the indices \(i_1, i_2, j_1, j_2\) that maximizes them separately.

The sensitivity is the larger from the two cases, which leads to Theorem 12.

Theorem 13. The \(s_h\) value of the Diff statistic (from Equation 11) is equal to 4.
Proof. Suppose $T$ and $T'$ are marginal-neighbors defined in Definition 7. So, there are indices $i_1, i_2, j_1, j_2$ such that $T'[i_1, j_1] = T[i_1, j_1] - 1$, $T'[i_1, j_2] = T[i_1, j_2] + 1$, $T'[i_2, j_1] = T[i_2, j_1] + 1$, $T'[i_2, j_2] = T[i_2, j_2] - 1$. Also, recall that both tables have the same marginals. By Equation 11 and 5, the sensitivity for absolute difference statistic equals

$$\max_{T, T'} \left| \sum_{i,j} T[i,j] - \frac{T[i, \cdot] T[\cdot, j]}{n} \right| = \sum_{i,j} |T'[i,j] - \frac{T'[i, \cdot] T'[\cdot, j]}{n}|$$

$$= \max_{T, T'} \left| T[i_1, j_1] - \frac{T[i_1, \cdot] T[\cdot, j_1]}{n} \right| + \left| T[i_1, j_2] - \frac{T[i_1, \cdot] T[\cdot, j_2]}{n} \right| + \left| T[i_2, j_1] - \frac{T[i_2, \cdot] T[\cdot, j_1]}{n} \right| + \left| T[i_2, j_2] - \frac{T[i_2, \cdot] T[\cdot, j_2]}{n} \right|$$

$$\leq |T[i_1, j_1] - T'[i_1, j_1]| + |T[i_1, j_2] - T'[i_1, j_2]| + |T[i_2, j_1] - T'[i_2, j_1]| + |T[i_2, j_2] - T'[i_2, j_2]|$$

$$= 4$$

That is, the sensitivity of the absolute difference statistic based on contingency tables with fixed marginals is 4. \qed