Kinetics of electron cooling in metal films at low temperatures and revision of the two-temperature model

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Abstract

The two-temperature model (2TM) introduced by Kaganov, Lifshitz, and Tanatarov is widely used to describe the energy relaxation in the electron–phonon system of a metallic film. At the same time, the accuracy of the description of the electron–phonon system in terms of the 2TM, i.e. on the basis of the electron and phonon temperatures, has not been considered in detail until now. In this paper we present a microscopic theory of cooling of instantly heated electrons in metallic films. In framework of this theory the main features of electron cooling in thick and thin films were found, and an analysis of the accuracy of the 2TM in the low-temperature region was carried out. We consider a more accurate three-temperature model, which (in contrast to the 2TM) explicitly takes into account phonons with angles of incidence exceeding the angle of total internal reflection. The contribution of these phonons to the kinetics of electron relaxation can be significant if the sound velocities in the film and the substrate are quite different.

Keywords: hot electrons, heat transfer, metal film, energy relaxation

1. Introduction

The two-temperature model (2TM) proposed by Kaganov, Lifshitz, and Tanatarov in 1956 [1] is still the main model for analyzing experiments on the energy relaxation of excited electrons in metals. It is sufficient to notice that on the basis of this model, the effect of high-power laser radiation on the metal surface was considered [2, 3], and the kinetics of electron cooling in the experiments on the heating of metal films by femtosecond laser pulses was analyzed [4–6]. The essence of 2TM is as follows. It is believed that excited electrons lose their energy as a result of the emission of phonons. In the process, both the electron and phonon subsystems are assumed to be thermalized, i.e. having temperatures \( T_e \) and \( T_p \), respectively. The energy transfer from electrons to phonons, \( P_{ep} \), is expressed in terms of the phonon–electron collision integral, in which the electron distribution function is a Fermi function with temperature \( T_e \), and the phonon distribution function is a Bose function with temperature \( T_p \). In the deformation potential approximation, a direct calculation yields the relation \( P_{ep} = f(T_e) - f(T_p) \) with the function \( f(T) = GT \) at temperatures much higher the Debye temperature \( \Theta_D \), and \( f(T) = \Sigma T^5 \) for \( T \ll \Theta_D \). The constants \( G \) and \( \Sigma \), which depend on the material, characterize the strength of the electron–phonon interaction at the high and low temperature ranges.

In the work of one of the authors [7], the two-temperature approach was generalized to the case of an arbitrary electron dispersion law and an arbitrary frequency dependence of the Eliashberg function \( \alpha^2 F(\omega) \), describing the interaction of electrons and phonons. With the help of a similar generalization, Allen [8] deduced the linear relation between the cooling rate of a thin film and the electron–phonon coupling constant \( \lambda \) which is well known in the theory of superconductivity. The relation established by Allen made it possible to obtain the values of \( \lambda \) for a number of metals and compounds from the experiment (see, for example, [9]).

Despite the progress in understanding the peculiarities of the relaxation of heated electrons in metals, the validity and...
the range of the applicability of 2TM have not yet been fully elucidated [10–14] and require a special analysis especially in the low-temperature region, where 2TM is used to describe the work of hot electron bolometers [15, 16]. A microscopic theory based on kinetic equations for the electron and phonon distribution functions can provide reliability of such an analysis. A kinetic approach to the analysis of steady-state heat transfer between a metal layer and insulator slabs was developed in [7, 17]. Subsequently, this approach was extended to the case of unsteady heating of a film deposited on a dielectric substrate [18] (see also [19]). The equations obtained in [18] will be used below to compare the predictions of 2TM with the results of a microscopic model. This will make it possible to determine whether the model based on the assumption of phonon and electron temperatures adequately describes the energy relaxation of electrons. Thus, the purpose of this paper is twofold: firstly, to obtain a fairly complete physical picture of the energy relaxation of heated electrons from the microscopic theory, and secondly, to find out to what degree the predictions of 2TM correspond to this picture. The analysis is limited to a region of low temperatures \( T < T_\text{cr} \), where electron–electron collisions dominate electron–phonon collisions, which allows us to consider electrons as being thermalized. Because the temperature \( T_\text{e} \sim \Theta_\text{D}/\tau_\text{F} \), for typical metals we have \( T_\text{e} \sim 1 \text{ K} \). It should also be noted that in sufficiently ‘dirty’ metals the temperature \( T_\text{e} \lesssim 10 \text{ K} \) due to the enhancement of the electron–electron and the weakening of the electron–phonon interaction [20]. Helium temperatures are the operating temperatures for hot electron bolometers and single-photon optical detectors. We note that a decrease in temperature increases the quantum efficiency of these devices and reduces noise equivalent power [21, 22].

Unlike electrons, phonons are not thermalized at low temperatures, since at \( T \lesssim \Theta_D \) the frequency of phonon–phonon collisions is several orders of magnitude lower than the collision frequency of phonons with electrons. This condition is sufficient to avoid introducing the phonon temperature and to describe the phonon subsystem in terms of the distribution function. The exact solution of the kinetic equation for the phonon distribution function obtained in [18] allows us to represent the heat balance equation for electrons in the form of a nonlinear integro-differential dynamic equation for the electron temperature. This equation is the basis for analyzing the cooling of instantly heated electrons in metallic films at low temperatures. In this paper we consider the case of relatively weak heating, when the equation can be solved using the operational method (the Laplace transformation). Comparison of Laplace transforms of the electron temperature, found from microscopic theory and 2TM, allows us to establish the range of applicability of 2TM and to analyze its accuracy.

The energy relaxation is described in 2TM by two linear differential equations—for the electron and the phonon temperatures. However, 2TM does not take into account the fact that the phonons are divided into two groups by their ability to pass through the interface between the film and the substrate. Indeed, if \( \theta_\text{cr} \) is the critical angle of total internal reflection, then phonons with angles of incidence \( \theta < \theta_\text{cr} \) can pass to the substrate with some probability \( \alpha(\theta) \), and phonons with \( \theta_\text{cr} < \theta < \pi/2 \) are completely reflected from the boundary with the substrate (see section 2). The three-temperature model (3TM) assigns the temperature \( T_\text{e} \) to the electrons, and the temperatures \( T_\text{ph1} \) and \( T_\text{ph2} \) to the above-mentioned phonon groups, respectively. Below we show that the 3TM significantly improves the description of the relaxation of the electron temperature compared with 2TM and in a number of cases it yields results practically not differing from the results of the microscopic model. Notice that 3TM is formulated in accordance with the so-called ‘multi-temperature approach’, when each subsystem of particles participating in the energy relaxation is assigned its own temperature (see, for example, [23–25]).

The structure of this article is as follows. In section 2 for the convenience of the reader, we reproduce the main points of the derivation of the dynamic equation for the electron temperature [18]. In section 3 on the basis of this equation, we analyze the dynamics of cooling of electrons in a linear regime, i.e. when the initial heating of the electrons is small in comparison with the temperature of the thermostat. Here, the case of a heat-insulated film is considered. The heat transfer to the substrate is taken into account in section 4, which presents the analytical results describing the cooling of electrons in pure films. In the same section, a three-temperature model is considered and a comparison of the predictions of the microscopic model, 2TM, and 3TM is made. Section 5 contains a discussion of the results of the work and the main conclusions.

2. Microscopic derivation of the dynamic equation for the electron temperature

In order to examine the features of the relaxation dynamics of the electron temperature \( T_\text{e} \), we use a simple microscopic model that allows obtaining the main results in an analytical form. So, let us take the spectrum of electrons as quadratic and isotropic, \( \varepsilon_\text{p} = \hbar^2 q^2/(2m) \) (\( \hbar \) is the quasimomentum of the electron, \( m \) is its effective mass) and suppose that phonons have a longitudinal acoustic branch only. Since we confine ourselves to a temperature region well below the Debye temperature \( \Theta_D \), the dispersion law for phonons can be considered linear: \( \omega_q = s q \), where \( s \) is the velocity of longitudinal sound, \( q \) is the modulus of the phonon wave vector. For the electron–phonon interaction, we use the deformation potential approximation, modified in such a way as to take into account the renormalization of this interaction by impurities. Finally, the heat exchange between the metal film and the dielectric substrate will be described in terms of the well-known acoustic mismatch model [26–28].

In the low-temperature region, electron–electron collisions dominate over electron–phonon collisions, and this means that the electronic subsystem of the film is thermalized. In the absence of a transport current, the electron distribution function is the isotropic Fermi function. Due to the high thermal conductivity of the electron gas, the electron temperature \( T_\text{e} \) does not depend on the coordinate \( z \) perpendicular to the film, but depends only on the time \( t \) [17]. The phonon distribution function \( N_\text{ph}(z,t) \) must be found from the kinetic equation
\[
\frac{\partial N_q}{\partial t} + s_i \frac{\partial N_q}{\partial z} = \nu_q \{ n_q[T_e(t)] - N_q(z,t) \},
\]
where \(s_i\) is the projection of the phonon velocity on the axis \(z\), \(\nu_q\) is the phonon–electron collision frequency, \(n_q(T) = [\exp(\hbar \omega_q/T) - 1]^{-1}\) is the Bose distribution function (\(k_B = 1\)). A fairly simple form of the integral of phonon–electron collisions (the right-hand side of the kinetic equation) is a consequence of the Fermi distribution of electrons. For pure films, in the deformation potential approximation, the frequency of phonon–electron collisions is given by
\[
\nu_q = \frac{m^2 \mu^2}{2 \pi \hbar^3 \rho_q s} \omega_q.
\]
Here \(\mu\) is the deformation potential constant, which is of the order of the Fermi energy \(\epsilon_F\); \(\rho_q\) is the film density. The linear dependence of \(\nu_q\) on \(\omega_q\) corresponds to the quadratic dependence of the Eliashberg function on the phonon frequency, \(\alpha^2 F(\omega) \propto \omega^2\), and to the cubic dependence of the electron–phonon collision frequency on temperature: \(\tau_e^{-1} \propto T_e^3\). [29–32].

In dirty films, where the electron mean free path \(l\) is much smaller than the wavelength of the thermal phonons \(\lambda_T\) (\(\lambda_T = 2 \pi \hbar s / T_e\)), the dependencies as \(\tau_e^{-1} \propto T_e^4\) [33–36] and \(\tau_e^{-1} \propto T_e^3\) [36–40] were observed experimentally. In order to extend our analysis to the case of dirty films, we generalize equation (2) and write down the frequency of phonon–electron collisions in the form
\[
\nu_q = \frac{m^2 \mu^2}{2 \pi \hbar^3 \rho_q s} (\hbar \omega_q)^{1+r}.
\]
where \(\mu_r\) is a phenomenological parameter, and \(r\) is a number. Equation (3) leads to a power-law frequency dependence of the Eliashberg function: \(\alpha^2 F(\omega) \propto \omega^{2+r}\) and to the power-law temperature dependence of the electron–phonon frequency collisions of \(\tau_e^{-1} \propto T_e^{3+r}\). So, equation (3) describes both groups of experiments on dirty films, if we take \(r = 1\) and \(r = -1\), respectively. As follows from the theory [41], the dependence \(\tau_e^{-1} \propto T_e^4\) occurs due to the scattering of electrons by light point defects that oscillate together with the crystal lattice, whereas \(\tau_e^{-1} \propto T_e^3\) corresponds to the dominant scattering of electrons by heavy impurity atoms or grain boundaries, which are weakly involved in lattice vibrations.

Boundary conditions should be added to the differential equation (1). As one of them we take the condition of specular reflection of phonons on the free boundary of the film (at \(z = d\)):
\[
N_q(d,t) = N_q(d,t).
\]
Here \(q = (q_x, q_y, q_z > 0)\) is the wave vector of the phonon incident on the boundary \(z = d\), and \(q' = (q_x, q_y, -q_z)\) is the wave vector of the specularly reflected phonon. At the interface between the film and the substrate (\(z = 0\), the following condition is fulfilled:
\[
N_q(0, t) = \alpha n_q(T_B) + \beta N_q(0, t),
\]
where \(\alpha\) is the probability of the phonon passing the film/substrate interface, and \(\beta = 1 - \alpha\) is the probability of the phonon reflection from the interface. The condition (5) means that both phonons reflected from the interface with the substrate and phonons that have passed from the substrate to the film fall into the state with the wave vector \(q\); \(T_B\) is the temperature of the substrate. Note that the condition (5) assumes that phonons that leave the film do not return back. Such a picture corresponds to experiments on fairly narrow films deposited on single-crystal substrates with high thermal conductivity.

In the acoustic mismatch model [26–28] the probability \(\alpha\) depends on the angle of incidence of the phonon and the acoustic impedances of the film and substrate:
\[
\alpha(\theta_1) = \frac{4Z' \cos \theta_1 \cos \theta_2}{(Z \cos \theta_2 + Z' \cos \theta_1)^2}.
\]
Here \(Z = \rho_s s (Z' = \rho' s')\) is the acoustic impedance of the film (substrate); the angles of incidence and refraction are related by the relation \(s' \sin \theta_1 = s \sin \theta_2 (\rho' \rho)\) denote the density and velocity of the longitudinal sound of the substrate). The dependence of \(\alpha(\theta_1)\) for a copper film on a quartz substrate is shown in Figure 1. The angle of total internal reflection is \(\theta_{\text{cr}} = \arcsin s/s'\). Since \(s = 4.7\) km s\(^{-1}\), \(s' = 5.9\) km s\(^{-1}\), the angle \(\theta_{\text{cr}}\) is approximately 0.92.

The solution of the kinetic equation (1) with the boundary conditions (4) and (5) can be obtained by performing the Fourier transformation with respect to time, and then solving the ordinary differential equation for \(N_q(z,\omega)\) (see the details of calculations in [18]). The inverse Fourier transformation gives
\[
N_q(0, t) = \alpha [1 - \beta \chi(0)]^{-1} \exp[-\nu_q s \nu_q(T_B)]
+ \int_{-\infty}^{t} dt' n_q[T_e(t')] \nu_q \exp[-\nu_q(t - t')] \beta^{{\nu}'(1 - z/2d)},
\]

Figure 1. The probability of the passage of phonons through the Cu/SiO\(_2\) interface. The following parameter values were used to calculate \(\alpha(\theta_1)\): \(s_{\text{Cu}} = 4.7\) km s\(^{-1}\), \(s_{\text{SiO}_2} = 5.9\) km s\(^{-1}\), \(\rho_{\text{Cu}} = 8.9\) g cm\(^{-3}\) and \(\rho_{\text{SiO}_2} = 2.2\) g cm\(^{-3}\).
\[
N_q(z, t) = \alpha[1 - \beta \chi(0)]^{-1} \exp[-(2d - z)\nu_q/|z|]n_q(T_B)
+ \int_{-\infty}^{t} d't n_q(T_e(t'))\nu_q \exp[-\nu_q(t - t')][\beta^{|\tau'|}/2d],
\]
with \(\chi(0) = \exp(-2d\nu_q/|z|)\) and \(\tau = |z|/(t' - t)/2d\). The square brackets in the exponent \(\beta\) denote the integer part of the number.

The equation for the electron temperature follows from the Boltzmann kinetic equation for the electron distribution function. Multiplying this equation by the energy of the electron and integrating over the momenta, we obtain the heat balance equation

\[
c_e(T_e) \frac{dT_e}{dt} = W(t) - P_{ep}(t),
\]
with the electronic heat capacity \(c_e = (\pi^2/3)N(0)T_e\), where \(N(0) = mp_e/\pi^2h^2\) is the density of states on the Fermi surface. The symbol \(W(t)\) represents the specific power of the heat sources in the film. The average specific power transferred from the electrons to the phonons is written as follows:

\[
P_{ep} = \frac{1}{d} \int_{0}^{d} dz \int \frac{d^3q}{(2\pi)^3} \hbar\omega_q \nu_q \{n_q[T_e(t)] - N_q(z, t)]\}.
\]

Since \(N_q(z, t)\) is known, further calculations are not difficult. Substituting \(P_{ep}\) in equation (9), we obtain a nonlinear integro-differential equation for the electron temperature

\[
c_e(T_e) \frac{dT_e}{dt} = W(t) - 2 \int_{q, > 0} \frac{d^3q}{(2\pi)^3} \hbar\omega_q \nu_q \left\{n_q(T_e(t))
- n_q(T_B) - \int_{-\infty}^{t} d't n_q(T_e(t')) - n_q(T_B)\right\}
\times \nu_q \exp[-\nu_q(t - t')][\beta^{|\tau'|}(1 - \alpha(\tau))],
\]
where \(|\tau|\) and \(\{\tau\}\) denote the integer and fractional part of \(\tau\), respectively. The appearance of these quantities is a consequence of multiple reflections of phonons from the boundaries of the film. We notice that the phonon contribution to the heat balance equation consists of two parts. The first of them, local in time, describes the radiation of nonequilibrium phonons at time \(t\), and the second part, integral with respect to time, represents the absorption of nonequilibrium phonons emitted at earlier moments \(t' < t\).

In what follows, we shall consider the kinetics of cooling of electrons heated during the period much less than the characteristic times of thermal relaxation. If such heating ends at the time \(t = 0\), then in equation (11) one can take \(W(t) = w(t')\), where the absorbed energy \(w\) determines the electron temperature jump from temperature of the thermostat \(T_B\) to \(T_e(0)\) according to the equality \(2w = c_e(T_e(0))T_e(0) - c_e(T_B)T_B\). After the heating of the electronic subsystem, its energy relaxation begins. The dynamics of relaxation is described by equation (11), where we should put \(W(t) = 0\) and take the lower limit of time integration equal to zero.

The integro-differential equation (11) can be linearized in the case of weak heating, when \(\Delta T_e(t) = T_e(t) - T_B \ll T_B\). The linearized equation, having the form

\[
c_e(T_B) \frac{d\Delta T_e(t)}{dt} = -2 \int_{q, > 0} \frac{d^3q}{(2\pi)^3} \hbar\omega_q \nu_q \frac{dn_q(T_B)}{dT_B} \left\{\Delta T_e(t)
- \int_{0}^{t} d't \Delta T_e(t')\nu_q \exp[-\nu_q(t - t')][\beta^{|\tau'|}(1 - \alpha(\tau))]\right\},
\]
is the basis for subsequent analysis.

We notice that the case of weak heating is quite meaningful, since it allows us to find all the cooling regimes of electrons and their characteristic times, and also to compare the predictions of the two-temperature and microscopic models. Moreover, it is possible to modify 2TM and formulate a 3TM, which gives a picture of the energy relaxation of electrons very close to the predictions of the microscopic theory.

3. Relaxation of the electron temperature in a thermally insulated film

First of all, let us consider the case when the electrons cool down in the absence of heat escape from the film to the substrate. Since in reality the thermal insulation of the film from the substrate cannot be ideal, in section 4 we establish the conditions under which the film can be considered sufficiently insulated. Now we just put \(\alpha = 0\) and \(\beta = 1\) in equation (12) and obtain

\[
c_e(T_B) \frac{d\Delta T_e(t)}{dt} = - \int \frac{d^3q}{(2\pi)^3} \hbar\omega_q \nu_q \frac{dn_q(T_B)}{dT_B} \left\{\Delta T_e(t)
- \int_{0}^{t} d't \Delta T_e(t')\nu_q \exp[-\nu_q(t - t')][\beta^{|\tau'|}(1 - \alpha(\tau))]\right\}.
\]

The natural method for solving this equation is the operational method (the Laplace transformation). For the Laplace transform of the electron temperature

\[
\Delta \tilde{T}_e(p) = \int_{0}^{\infty} \Delta T_e(t) e^{-pt} dt,
\]
the operational method gives

\[
\Delta \tilde{T}_e(p) = \frac{c_e \Delta T_e(0)}{c_e p + a - \tilde{K}(p)}.
\]
Here \(p\)-independent quantity \(a\) is determined by the integral over the phonon wave vectors as

\[
a = \int \frac{d^3q}{(2\pi)^3} \hbar\omega_q \nu_q \frac{dn_q(T_B)}{dT_B} \frac{\nu_q^2}{p + \nu_q},
\]
and \(\tilde{K}(p)\) is the Laplace transform of the kernel of equation (13):

\[
\tilde{K}(p) = \int \frac{d^3q}{(2\pi)^3} \hbar\omega_q \frac{dn_q(T_B)}{dT_B} \frac{\nu_q^2}{p + \nu_q}.
\]
Converting the Laplace transformation, we obtain the required time dependence of the electron temperature in the form of an integral in the complex $p$-plane

$$
\Delta T_e(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{e^{pt} c_e \Delta T_e(0) \, dp}{p^{\epsilon + cp} + p^{\epsilon + cp}}. \quad (18)
$$

We choose the value $\sigma$ so that all singularities of the integrand are to the left of the path of integration.

Substituting equation (3) in equation (18) and integrating over the angles of the vector $q$, we obtain

$$
\Delta T_e(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} dp \, e^{pt} \Delta T_e(0) \left( -\frac{1}{p^{\epsilon + cp} + p^{\epsilon + cp}} \right) + \frac{1}{\zeta_1} \int_0^\infty \frac{x^{3+r}}{\sinh^2(x/2) |x^{1+r} + p/\nu(T_B)|} \, dx \right)^{-1}, \quad (19)
$$

where the dimensionless variable $x = \hbar \omega \nu / T_B$, and the characteristic collision frequency of thermal phonons with electrons $\nu(T_B) = \frac{\pi^2 \nu}{2\pi} \nu^{1+r}$. The constant $\zeta_1 = 16 \pi^4 c_e/(15 c_p)$ is determined by the ratio of the electron and phonon heat capacities at the temperature of the thermostat. For the phonon heat capacity in the model with one (acoustic) mode of oscillations, we have the expression $c_p = (2\pi^2 T_B^3)/(15 \hbar^4 x^4)$. It is not difficult to see that the case $r = -1$ (for which $\tau_e^{-1} \propto T_B^2$) is special. Indeed, integrations over $x$ and $p$ are separated for $r = -1$ only, and this, as will be seen later, leads to an exponential dependence of $T_e$ on time. The case $r = -1$ will be considered separately, but for now we shall assume $r \neq -1$.

To calculate the integral over $p$ in equation (19), we need to continue analytically the integrand to the region located to the left of the integration contour, i.e. into the domain $\text{Im}(p) < \sigma$. The function of the complex variable $p$, given by an integral over $x$, has singularities for negative values of $p$. An analytic continuation of such a function is performed by cutting a complex $p$-plane along the negative part of the real axis. In this case, the integral over $p$ reduces to the contribution of the pole $p = 0$ and to integrals along the edges of the cut. Making the substitutions $x^{1+r} = y, p = -p_1 \nu(T_B)$ and using the well-known formula

$$
\lim_{\delta \rightarrow +0} \int_{y = y_0 - i\delta}^{y = y_0 + i\delta} f(y) dy = V_p. \int f(y) dy + i \pi f(y_0), \quad (20)
$$

we obtain

$$
\Delta T_e(t) = \frac{c_e \Delta T_e(0)}{c_e + c_p} + \int_0^\infty \frac{\zeta_1 (1 + r) \, p_1^{(1+r)/(1+r)} e^{-p_1 \nu(T_B) \Delta T_e(0)} \, dp_1}{\sinh^2\left(\frac{\nu(T_B) \Delta T_e(0)}{2}\right) \left[ \zeta_1 (1 + r) + V_p \int_0^\infty \frac{x^{3/2} \, dx}{\sinh^2\left(\frac{\nu(T_B) \Delta T_e(0)}{2}\right)} \right]} \left[ \frac{\pi \nu(T_B) \Delta T_e(0)}{\sinh^2\left(\frac{\nu(T_B) \Delta T_e(0)}{2}\right)} \right], \quad (21)
$$

From equation (21), it is not difficult to find the behavior of $\Delta T_e(t)$ for large times, when $\nu(T_B) \gg 1$. In this limit, the main contribution to the integral over $p_1$ is given by $p_1 \ll 1$, that allows us to neglect $p_1$ in the integral over $y$, and neglect the last term in curly brackets. Replacing the hyperbolic sine by its argument, we obtain the expression

$$
\Delta T_e(t) \approx \frac{c_e \Delta T_e(0)}{c_e + c_p} + \frac{5}{4\pi^2} \Gamma\left(\frac{1}{1+r}\right) c_e c_p \Delta T_e(0) \int_0^\infty \frac{\nu(T_B) \Delta T_e(0)}{\sinh^2\left(\frac{\nu(T_B) \Delta T_e(0)}{2}\right)} \, dp_1 \left[ \left(\frac{\nu(T_B) \Delta T_e(0)}{2}\right) \right]^{3/2} \left(\frac{\nu(T_B) \Delta T_e(0)}{2}\right) \, dp_1, \quad (22)
$$

where $\Gamma$ is the gamma function. It follows from equation (22) that for large times the relaxation of the electron temperature has a power-law character. The term $c_e \Delta T_e(0)/(c_e + c_p)$ determines the electron temperature, which is established in a thermally insulated film at $t \rightarrow \infty$, when electrons and phonons come to thermodynamic equilibrium.

For small times, the asymptotic behavior of $\Delta T_e(t)$ is convenient to find from equation (18) by expanding the integrand in powers of $p^{-1}$. If we retain the first two terms of the expansion, we obtain

$$
\Delta T_e(t) \approx \Delta T_e(0) (1 - t/\tau_e), \quad (23)
$$

where the average frequency of electron–phonon collisions

$$
\tau_e^{-1} = \frac{1}{c_e} \int d^3 q \frac{\hbar \omega}{2\pi^2} \nu(T_B) \nu_T \frac{d\nu_T(T_B)}{dT_B}. \quad (24)
$$

In our opinion, the slow power-law relaxation of the electron temperature in thermally insulated films at large times is a very unexpected result of section 3, since this behavior of $\Delta T_e(t)$ is qualitatively different from the exponential relaxation predicted by 2TM (see the appendix). The reason for the power-law relaxation of the electron temperature in a thermally insulated metal film is the dependence of the phonon–electron collision frequency on the phonon energy (see equation (3)). When summing over all phonon modes, such a distribution of the phonon–electron collision times leads to a power-law dependence $\Delta T_e(t)$. In other words, the reason for the power-law relaxation of the electron temperature is the absence of a single relaxation time for all phonon modes. (On the contrary, the so-called ‘r-approximation’ implies such a single relaxation time.)

### 4. Heat removal to substrate, and three-temperature model (3TM)

The removal of heat from the film to the substrate substantially complicates the picture of cooling of the heated electrons. With a relatively weak heating and $\alpha \neq 0$, the dynamics of the electron temperature is described by a linear equation (12), which can be solved by the operational method, just like equation (13). For the Laplace transform of the deviation of the electron temperature from $T_B$, we obtain
\[ \Delta T_e(p) = c_e \Delta T_e(0) \left\{ c_p p + 2 \int_{|q|>0} \frac{d^3q}{(2\pi)^3} \hbar \omega_q \nu_q \frac{d\alpha_q(T_B)}{dT_B} \right\} \]

\[ \times \left[ p + \frac{\alpha_s \nu_q}{2d(p + \nu_q)} \right]^{-1}, \]  

(25)

with \( \chi(p) = \exp[-2d(p + \nu_q)/s] \). We notice immediately that in the case \( \alpha \neq 0 \) equation (25) has two new features compared to \( \Delta T_e(p) \), obtained in section 3. First, the pole \( p = 0 \) disappears, so that \( \Delta T_e(t) \to 0 \) as \( t \to \infty \). Second, equation (25) contains a new parameter, namely, the film thickness \( d \). Further we shall see that the nature of the thermal relaxation of the electronic subsystem depends essentially on the value of \( d \). Since equation (25) describes the relaxation of the electron temperature for systems with arbitrary \( r \) and arbitrary parameter values, it turns out to be rather complicated. Such an analysis is greatly simplified for \( r = -1 \), when the frequency of phonon–electron collisions does not depend on the wave vector of the phonon. It is the case that will be considered in the next subsection. Recall that \( r = -1 \) corresponds to the dependence \( \tau_e^r \propto T_e^2 \), which is often observed in the experiments [36–40].

4.1. The frequency of phonon–electron collisions does not depend on the wave vector of the phonon

It follows from equation (3) that the frequency of phonon–electron collisions \( \nu_q \) loses its dependence on the phonon wave vector \( q \) for \( r = -1 \). This circumstance substantially facilitates the integration over modulus \( q \). To simplify the integration over the angles of the vector \( q \), we assume that the probability \( \alpha(\theta) \) is equal to the constant \( \alpha_0 \) for \( \theta < \theta_{ct} \) and is equal to zero for \( \theta_{ct} < \theta < \pi/2 \). Recall that the angle \( \theta_{ct} \) is the angle of total internal reflection, which is determined by the sound velocities in the film and in the substrate according to the equation \( \sin(\theta_{ct}) = (s/s') \). The obtained stepwise approximation has all the characteristic features of the real dependence \( \alpha(\theta) \) (see figure 1) if we take \( \alpha_0 \approx \alpha(0) \). Now we take into account that for sufficiently different velocities \( s \) and \( s' \) the angle \( \theta_{ct} \) is not too close to \( \pi/2 \), and hence the projection of the phonon velocity \( s \) of the order \( s \). Replacing \( s \) by \( s - 1 \) and using the stepwise approximation \( \alpha(\theta) \), we obtain for \( r = -1 \)

\[ \Delta T_e(p) = c_e \Delta T_e(0) \left\{ c_p p + \frac{\nu_0}{(p + \nu_0)} + c_p p \right\} \]

\[ \times \left\{ \frac{\nu_0^2}{(p + \nu_0)^2} \right\} \left[ 1 - \exp\left(-2d(p + \nu_0)/s \right) \right]^{1-1}. \]  

(26)

Here \( \nu_0 \) is the frequency of phonon–electron collisions that does not depend on the wave vector of the phonon:

\( \nu_0 = m^2 \mu^2_{\perp} \) / \( (2\pi \hbar^3/\rho_f) \).

We denote the fraction of (‘trapped’) phonons with angles \( \theta_{ct} < \theta < \pi/2 \) via \( \eta_0 \). A simple calculation of this fraction yields \( \eta_0 = \cos(\theta_{ct}) \). Accordingly, the fraction of phonons that can leave the film is \( 1 - \eta_0 \). It is this value that appears in equation (26).

In order to distinguish limiting cases, we introduce the dimensionless quantities \( p = p/\nu_0 \), \( d = 2d\nu_0/s \), and \( \zeta = c_e/\nu_0 \). The Laplace transform of the electron temperature takes the form

\[ \Delta T_e(p) = \zeta \Delta T_e(0)/[\nu_0 F(p)], \]

(27)

with

\[ F(p) = \frac{\bar{p} + (1 - \eta_0) \alpha_0}{(p + 1)d} \left[ 1 - \exp\left(-d[p + 1] \right) \right]. \]  

(28)

The graph of \( F(p) \) for fairly general (not large or small) values of the parameters \( \zeta = 1, \eta_0 = 0.2, \bar{d} = 0.5, \alpha_0 = 0.5 \) is shown in figure 2. We see that the function \( F(p) \) has three zeros. These zeros correspond to the poles of the complex function \( \Delta T(p) \), which means that the dependence \( \Delta T(t) \) will be a linear combination of three exponentials with exponents \( (p_i \nu_0) \), where \( i = 1, 2, 3 \). To find out the meaning of the poles, consider how \( F(p) \) changes with decreasing \( \zeta \) and with increasing \( \bar{d} \). Let us take \( \zeta < 0.25 \) and \( \bar{d} = 3 \), and leave the rest of the parameters unchanged. The dependence \( F(p) \) for new parameter values is presented in figure 3. It is clearly seen (and this will be shown analytically) that the decrease of \( \zeta \) strongly affects the pole \( p_1 \), shifting it to the left in the region of large values \( |p| \), while an increase of \( \bar{d} \) shifts the pole \( p_3 \) to the right to the point \( p = 0 \).

For small \( \zeta \), the pole \( p_1 \) must be searched in the domain \(|p| \gg 1 \). In this region, we have \( F(p) \approx \zeta \bar{p} + (1 - 1/p) \). The root of this expression is \( p_1 \approx -1/\zeta - 1 \), whence it is clear that small \( \zeta \) corresponds to large \( |p_1| \). The pole \( p_1 \) describes the initial stage of relaxation of the electron temperature, when hot electrons emit nonequilibrium phonons. This stage is characterized by the exponential \( e^{\zeta \nu/p_1 s} \). Here we used the equality \( \nu_0 c_p / c_e = 1/\tau_e \), from which it follows that for \( \zeta \ll 1 \) the relaxation is determined by the time \( \tau_e \), since \( \tau_e \ll 1/\nu_0 \).

The pole \( p_3 \) is easy to find if we take into account that for large film thicknesses \( \bar{d} \gg 1 \) the value \( |p_3| \ll 1 \). In the domain of small \( |p| \), the function \( F(p) \approx (\zeta + 1)p + (1 - \eta_0) \alpha_0 / \bar{d} \), from which we have \( p_3 \approx -(1 - \eta_0) \alpha_0 / [(\zeta + 1) \bar{d}] \). The pole \( p_3 \) describes the late stages of relaxation of the electron temperature. The corresponding time dependence has the form \( \Delta T_e(t) \propto e^{-t/\tau_e s} \), where \( \tau_{es} = (1 - \eta_0) \alpha_0 s / [(\zeta + 1)2d] \).
The microscopic analysis of the relaxation of the electron temperature, carried out in section 4.1, showed that for $r = -1$ the relaxation process is described by a linear combination of three exponentials. A different result, namely, the dependence $\Delta T_e(t)$ is determined by two exponentials, was obtained in the appendix, where the electron temperature relaxation is analyzed on the basis of 2TM. The latter is not surprising since in 2TM the energy relaxation is described by only two linear differential equations for the electron and phonon temperatures. The reason for the qualitative difference between the two relaxation scenarios is quite simple: 2TM does not take into account the fact that the phonons are divided into two groups by their ability to pass through the interface between the film and the substrate. In fact, if phonons with angles of incidence $\theta < \theta_c$ can pass into the substrate with a probability $\alpha_0$, then the phonons with $\theta_c < \theta < \pi/2$ are completely reflected from the interface. In this subsection we consider a 3TM, which assigns the temperature $T_p$ to the electrons and assigns the temperatures $T_{p1}$ and $T_{p2}$ to the phonon groups noticed above. The 3TM gives the three-exponential behavior of $\Delta T_e(t)$, which improves the description of the relaxation of the electron temperature and, in a number of cases, yields results that are practically not different from the results of the microscopic model (MM) presented in section 4.1.

Thus, within the framework of 3TM, the process of relaxation of the electron–phonon system in a metallic film is described by the following set of equations:

$$
\frac{d\Delta T_e}{dr} = -\Sigma_5(1-\eta_0)(T_{p3}^5 - T_{p1}^5) - \Sigma_5\eta_0(T_{p3}^5 - T_{p2}^5),
$$

$$
\frac{d\Delta T_{p1}}{dr} = \Sigma_5(T_{p3}^5 - T_{p1}^5) - K(T_{p1}^4 - T_{B}^4),
$$

$$
\frac{d\Delta T_{p2}}{dr} = \Sigma_5(T_{p3}^5 - T_{p2}^5),
$$

with $\eta_0 = \cos(\theta_c)$. The constants $\Sigma_5$ and $K$ describe the heat transfer between electrons and phonons in the film and between the phonon subsystems of the film and the substrate, respectively. (For simplicity, we consider the case of pure films). If the initial heating of the electrons is small, the set of equations can be linearized in $\Delta T_e$, $\Delta T_{p1}$ and $\Delta T_{p2}$, which are small deviations of the temperatures $T_e$, $T_{p1}$, and $T_{p2}$ from the bath temperature $T_B$. Introducing the energy relaxation time of electrons $\tau_e = c_e/(5T_B^4\Sigma_5)$ and the time of phonon escape from the film $\tau_{es} = c_p/(4K\tau_B)$, we obtain a set of linear equations

$$
\frac{d\Delta T_e}{dr} = -\frac{1}{\tau_e}(1-\eta_0)(\Delta T_e - \Delta T_{p1}) - \frac{1}{\tau_{es}}\eta_0(\Delta T_e - \Delta T_{p2}),
$$

$$
\frac{d\Delta T_{p1}}{dr} = \frac{c_e}{c_p\tau_e}(\Delta T_e - \Delta T_{p1}) - \frac{1}{\tau_{es}}\Delta T_{p1},
$$

$$
\frac{d\Delta T_{p2}}{dr} = \frac{c_e}{c_p\tau_e}(\Delta T_e - \Delta T_{p2})
$$

with the initial conditions $\Delta T_e(t = 0) = \Delta T_e(0)$ and $\Delta T_{p1}(t = 0) = \Delta T_{p1}(0)$ and $\Delta T_{p2}(t = 0) = 0$. In order to be able to
compares the predictions of 3TM with the results of the MM and 2TM, we solve the set of equations (34)–(36) by the operational method. Introducing \( \nu_e = 1/\tau_e, \nu_p = c_p \nu_e/\epsilon_p \) and \( \nu_{es} = 1/\tau_{es} \) for Laplace transforms \( \Delta \tilde{T}_e(p), \Delta \tilde{T}_{p1}(p) \) and \( \Delta \tilde{T}_{p2}(p) \), we obtain

\[
\Delta \tilde{T}_e(p) = \frac{(p + \nu_p)(p + \nu_p + \nu_{es}) \Delta \tilde{T}_e(0)}{D}, \quad (37)
\]

\[
\Delta \tilde{T}_{p1}(p) = \frac{(p + \nu_p)\nu_p \Delta \tilde{T}_e(0)}{D}, \quad (38)
\]

\[
\Delta \tilde{T}_{p2}(p) = \nu_p(p + \nu_p + \nu_{es}) \Delta \tilde{T}_e(0)/D, \quad (39)
\]

with the following determinant of the equations:

\[
D = p^3 + (\nu_e + 2\nu_p + \nu_{es})p^2 + (\nu_p + \nu_{es})(\nu_p + \nu_e)p + (1 - \eta_0)\nu_e \nu_p \nu_{es}.
\]

In the low-temperature regime, where the phonons are not thermalized the microscopic theory describes the relaxation of the electron–phonon system in terms of the electron temperature and the phonon distribution function. We notice that the phenomenological models (3TM and 2TM) also give their own expressions for the electron temperature, in which phenomenological parameters appear. Comparing these expressions with the results of the microscopic theory (see equations (27) and (28)), we can relate microscopic and phenomenological parameters, and then compare the predictions of MM with 3TM and 2TM.

In order to express the phenomenological frequencies \( \nu_e \) and \( \nu_p \) via microscopic parameters, it is convenient to take the limiting case of a thermally insulated film. For \( \nu_{es} \), it is convenient to use the case of a weak thermal contact between the film and the substrate. Comparing equations (27) and (28) with equations (37) and (40), it immediately follows that \( \nu_p = \nu_0 \), and \( \nu_e = \epsilon_p \nu_0/\epsilon_p \). For a bad heat sink (when \( \nu_{es} < \nu_e, \nu_p \)), the determinant D has the root \( p_3 = -(1 - \eta_0)\nu_e \nu_p/(\nu_e + \nu_p) \). Comparing it with the root \( p_3 \), found in section 4.1, we have \( \nu_{es} = \alpha_0 \eta_0/(2d_0) \).

Now we can compare the dynamics of the electron temperature, obtained on the basis of the microscopic theory, with the results of 3TM and 2TM. To do this, we rewrite the Laplace transforms of the electron temperature, given by equations (37) and (40), in the dimensionless quantities introduced in section 4.1. We have for 3TM

\[
\Delta \tilde{T}_e(p) = \frac{(p + 1)p + 1 + \alpha_0/d)\zeta \Delta \tilde{T}_e(0)}{\nu_0(p^3 + (1 + 2\zeta + \alpha_0/d)p^2 + (1 + \zeta)(1 + \alpha_0/d)p + (1 - \eta_0)\alpha_0/d)}.
\]

A similar transformation of equation (A.5) yields for 2TM

\[
\Delta \tilde{T}_e(p) = \frac{(p + 1 + \alpha_0/d)\zeta \Delta \tilde{T}_e(0)}{\nu_0(p^3 + (1 + \zeta + \alpha_0/d)p + \alpha_0/d)}.
\]

Comparison of the predictions of the three models is much simpler and more obvious if we consider not the time dependence of the electron temperature, but the dependence of \( \Delta T(0) \zeta/|\Delta \tilde{T}_e(p)\nu_0| \) on the dimensionless variable \( p \). Figure 4 shows the dependence of \( \Delta T(0) \zeta/|\Delta \tilde{T}_e(p)\nu_0| \) on \( p \) for films of different thicknesses. The results of MM are given by a solid line, and the results of 3TM and 2TM are given by dashed and dotted lines, respectively. It is seen from figure 4(a) that for effectively thick films \( (d_{ef} \gg 1) \), the results of MM and 3TM are practically coincident, whereas 2TM does not reproduce the root \( p_2 \). As for films with a thickness of \( d_{ef} = 1 \) (see figure 4(b)), such films are fairly well described by 3TM, but very badly by 2TM. Only for effectively thin films with \( d_{ef} \ll 1 \), 3TM does not satisfactorily describe the early stages of the relaxation of the electron temperature. However, 3TM describes these stages no worse than 2TM. Thus, we can conclude that, in comparison with the well-known 2TM, the phenomenological 3TM, which takes into account the locking (in the film) of phonons with angles of incidence \( \theta > \theta_c \), improves the description of the electron temperature relaxation.

### 4.3. Kinetics of electron cooling in pure films

In section 4.2, we examined the case \( r = -1 \) when the frequency of phonon–electron collisions \( \nu_q \) does not depend on the wave vector of the phonon. Now we consider the more general situation when \( \nu_q \) depends on \( q \). Let us take a case of pure, i.e. \( r = 0 \). In this case, the dependence of \( \nu_q \) on \( q \) is linear, and this simplifies calculations and gives a more clear picture of the energy relaxation of electrons without cluttering it with nonessential details. As in section 4.1, we will consider the dependence \( \alpha(\theta) \) as stepwise, so that \( \alpha = \alpha_0 \) for \( \theta < \theta_c \) and \( \alpha = 0 \) for \( \theta > \theta_c \). In addition, we set \( s = s \). Under these assumptions, the Laplace transform of the deviation of the electron temperature has the form

\[
\Delta \tilde{T}_e(p_1) = \frac{\Delta \tilde{T}_e(0)}{p} \left\{ p_1 + \frac{1}{\xi_1} \int_0^\infty \frac{x^4dx}{\sinh^2(x/2)} \left( (p_1 + x) + \frac{(1 - \eta_0)\alpha_0}{\xi_1 d_1} \right) \sqrt{1 - \exp(-d_1(p_1 + x))} \right\}^{-1},
\]

with dimensionless quantities \( p_1 = p/p_0/T_B \), \( x = \hbar \omega_q/T_B \), \( d_1 = 2d_0/T_B \), and \( \xi_1 = 16\alpha_0 \epsilon_c/15cp \).

The inverse Laplace transformation of equation (43) leads to an integral along the edges of the cut in the complex \( p \)-plane passing along the negative real part of the axis. This integral can be written as

\[
\Delta T_e(t) = -\frac{1}{\pi} \int_{-\infty}^{0} dp_1 \int_0^\infty \frac{x^4dx}{\sinh^2(x/2)} R(x,p_1) \left[ \Phi'(p_1) \right]^{-1} \Phi''(p_1),
\]

where \( \Phi'(p_1) \) and \( \Phi''(p_1) \) are the real and imaginary parts of the expression enclosed in curly brackets in equation (43). The calculation of these quantities gives

\[
\Phi'(p_1) = p_1 + \frac{1}{\xi_1 Vp} \int_0^\infty \frac{x^4dx}{\sinh^2(x/2)} \left( (p_1 + x) + \frac{(1 - \eta_0)\alpha_0}{\xi_1 \ln(\beta)} \right),
\]

\[
\Phi''(p_1) = Vp \int_0^\infty \frac{x^4dx}{\sinh^2(x/2)} R(x,p_1) + \frac{(1 - \eta_0)\alpha_0}{\xi_1 \ln(\beta)}.
\]
\[
\Phi'(P_1) = \left(1 + \frac{c_e}{c_p}\right) \left(P_1 + \frac{1}{\tau_{es}}\right),
\]
(48)

where
\[
\tau_{es} = \frac{(1 - \eta_0)\alpha_0}{2d \left(c_p + c_e\right)}.
\]
(49)

Under the assumption that the thermal contact of the film with the substrate is not very good, so that \(\tau_{es}' \gg 1\), the dependence of the electron temperature on time has an exponential character,
\[
\Delta T_e(t) = \frac{c_e \Delta T_e(0)}{c_p + c_e} e^{-t/\tau_{es}'}.
\]
(50)

It should be noticed that equation (50) is valid if the time \(t\) is of the order of \(\tau_{es}'\). If \(t \gg \tau_{es}'\), we obtain the power-law dependence
\[
\Delta T_e(t) = \frac{4\Gamma(5)\eta_0 \Delta T_e(0)}{\zeta_1 (1 + \frac{\pi^2}{12})} \left(\frac{1}{(t/\tau_{es}')}^3\right)^2.
\]
(51)

The reason for changing the asymptotic behavior of \(\Delta T_e(t)\) is fairly simple. If equation (50) represents a slow cooling of a system of electrons and phonons that have come into equilibrium with each other, equation (51) describes the relaxation of phonons, which initially had angles of incidence \(\theta > \theta_c\), i.e. they were originally 'locked' in the film. Such phonons can leave the film in the following way: first they are again absorbed by electrons, and then emitted into the angular range \(0 < \theta < \theta_c\). Additional evidence in favor of the fact that the asymptotic behavior (51) is related to the group of 'locked' phonons, is its proportionality \(\eta_0\). If \(\eta_0 = 0\), that is, if all phonons have the ability to leave the film without reabsorbing by electrons, such asymptotic behavior do not arise.

In the late stages, the exponential relaxation of the electron temperature takes place in thin films, in which \(d_1 \ll \alpha_0/\beta_0\). If \(t\) is greater, but of the order \(\tau_{es}'\), and moreover \(\eta_0 \ll 1\), then
\[
\Delta T_e(t) = \Delta T_e(0) e^{-t/\tau_{es}}.
\]
(52)

where the average frequency of electron–phonon collisions \(\tau_{es}^{-1}\) is determined by equation (24).

The last stage (associated with the escape from the film of phonons with angles of incidence \(\theta > \theta_c\) and which takes place when \(t \gg \tau_{es}\)) is characterized by the following power-law dependence:
\[
\Delta T_e(t) = \frac{4\Gamma(5)\eta_0 \Delta T_e(0)}{\zeta_1 (t/\tau_{es})^3}.
\]
(53)

In the early stages of relaxation, the heat transfer from the film is unimportant, and therefore for both thick and thin films the dependence of the electron temperature on time is given by equation (23).

We notice that equation (50) gives a condition under which the thick film can be considered as thermally insulated one. Namely, we can neglect the heat removal to the substrate if the energy relaxation in the electron–phonon system of the film is considered at times substantially shorter than the time \(\tau_{es}'\).
Thus, the results of the section 3 are valid in this time domain under the condition \( \tau_p^0 \gg \tau_e \) and \( \nu(T_B) \tau_p^0 \gg 1 \).

5. Discussion and conclusions

At present, the 2TM, in which electrons and phonons are characterized by their temperatures, is mainly used to describe the thermal relaxation between heated electrons and a cold lattice. The microscopic theory of cooling of electrons presented in section 2 overcomes the limitation of 2TM, which does not take into account a significant group of nonequilibrium phonons that cannot ‘fly’ out of the metal film into the substrate, because their incidence angles \( \theta \) are larger than the critical angle \( \theta_c \). We notice, however, that the microscopic theory (where electrons are characterized by the temperature \( T_e \) and phonons are described by a distribution function) leads to a rather complex integro-differential equation for \( T_e(t) \), which makes it difficult to use. A simpler phenomenological allowance for phonons with \( \theta > \theta_c \) can be based on the introduction of an intrinsic temperature for them, and therefore the phenomenological model that more adequately describes the energy relaxation in a thin-film system turns out to be a three-temperature one (section 4.2). The 3TM improves the description of the energy relaxation (in comparison with 2TM).

At the same time, 3TM has its own field of application as it can be seen from the results obtained in sections 3 and 4.3. The 3TM most accurately describes the case of \( r = -1 \), when the frequency of phonon–electron collisions does not depend on the phonon wave vector (section 4.1). Recall that the case of \( r = -1 \) corresponds to dirty metal films where electrons are scattered on heavy impurity atoms or grain boundaries, which are weakly entrained by lattice oscillations. Such scattering leads to the dependence \( \tau_p^{-1} \propto T_e^2 \), which is often observed in experiments [36–40].

In addition to analyzing the applicability of 2TM, we obtained a number of new results concerning the energy relaxation of hot electrons. The most important of them are the power dependence of \( \Delta T_e(t) \) in a thermally insulated film (equation (22)) and the power dependence of \( \Delta T_e(t) \) in thick and thin films at the final stage of relaxation (equations (51) and (53)). The importance of the relation (22) is that it allows us to find the parameter \( r \) from the time dependence \( \Delta T_e(t) \), and this parameter determines the nature of electron scattering on impurities in the film, as well as the frequency dependence of the Eliashberg function for the low frequencies.

Interestingly, equations (51) and (53) can be used for finding the mean time of elastic phonon scattering in the substrate \( \tau_p \). It was assumed above that the phonons that ‘fly’ out to the substrate propagate there ballistically and do not return back to the film. This is true for narrow films and single-crystal substrates with a high thermal conductivity. However, the return of phonons should be taken into account for non-crystalline substrates with a short mean free path (for example, glass), as well as wide films, and the transmission coefficient \( \langle \alpha \rangle \approx 1 \). Returning phonons lead to the dependence \( \Delta T_e(t) \propto t^{-1/2} \) [42] if the time \( t \) is greater than the characteristic phonon return time \( \tau_r \). A similar feature was observed in experiments on thin niobium films deposited on substrates with low thermal conductivity [39]. Thus, from the crossover time from the dependence (51) (or (53)) to the dependence \( \Delta T_e(t) \propto t^{-1/2} \), we can find the time \( \tau_r \), and hence the elastic scattering time of the phonon in the substrate.

Since our work uses a simple model, which, in particular, does not take into account the interaction of electrons with transverse phonons, it is important to understand how obtained results depend on the model. As it follows from the results of sections 3 and 4, the cooling kinetics of heated electrons is determined by the following characteristic times: the electron–phonon collision time \( \tau_e \), the phonon–electron collision time \( \tau_p \) (which is \( \nu_0^{-1} \) for \( r = -1 \) and is equal to \( \nu(T_B)^{-1} \) for \( r \neq -1 \)), as well as the average time of phonon escape from the film \( \tau_{es} \). Allowance for transverse phonons will lead to a renormalization of the time \( \tau_e \), since along with longitudinal phonons electrons can also emit transverse phonons. The time \( \tau_p \) should be replaced by the phonon–electron collision time, averaged over the phonon modes. In addition, \( \tau_{es} \) should be understood as the averaged time of departure of longitudinal and transverse phonons from the film [26]. In general, the picture of the energy relaxation of electrons with allowance for transverse phonons will be preserved, and in this respect it does not depend on the model.

It should also be noticed that the formalism presented in section 2, in principle, makes it possible to include transverse phonon modes into the model, having formulated the boundary conditions (4) and (5) in such a way that they take into account the conversion of longitudinal and transverse modes of lattice vibrations at the boundaries of the film [28]. At the same time, however, the expressions describing the dynamics of the electron temperature become significantly more complicated and less transparent.

Since the time-varying electron temperature cannot be measured directly, low-temperature experiments usually analyze the dynamics of the voltage across the film where a small measuring current of a fixed value is passed. Due to the temperature dependence of the phonon contribution to the resistance, the voltage across the film is a function of the electron temperature, and hence it can serve as a thermometer for the electron temperature [18, 19].

The examination of the kinetics of electron cooling in metallic films at low temperatures makes it possible to draw the following conclusions.

(1) In a microscopic theory (which is the main one for this paper), electrons are characterized by the temperature \( T_e \), and phonons are described by a distribution function. The dynamic equation for \( T_e(t) \) obtained in microscopic theory turns out to be integro-differential equation (11), which makes its analysis difficult and in general requires numerical methods. In a relatively simple case of weak heating, the linearized equation for \( \Delta T_e(t) \) equation (12) has an analytic solution which, at \( r \neq -1 \), gives a power-law relaxation in a heat-insulated film, and a power-law
\( \Delta T_e(t) \) in the final stage of relaxation in the presence of a heat sink.

(2) In the low-temperature region, 2TM quantitatively describes the kinetics of the electron–phonon system of the metal film only in case of \( r = -1 \) when the phonon–electron collision frequency does not depend on the phonon wave vector. This condition is fulfilled when electrons are mainly scattered on heavy impurity atoms or grain boundaries, which are weakly involved in the lattice vibration. As can be seen from a comparison with microscopic theory, at \( r \neq -1 \) 2TM does not give a qualitatively correct (power-law) relaxation character \( T_e(t) \) at the late stages.

(3) The 2TM ignores a large group of phonons, in which the angles of incidence on the ‘film–substrate’ interface exceed the angle of total internal reflection \( (\theta > \theta_{cr}) \). This disadvantage is overcome by 3TM, which takes into account both phonons with angles of incidence \( \theta < \theta_{cr} \) and phonons with angles of incidence \( \theta > \theta_{cr} \). The 3TM improves the description of the energy relaxation in a thin-film system (in comparison with 2TM) and at \( r = -1 \) gives results that are close to the results of the MM.

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Appendix. Relaxation of the electron–phonon system in the framework of 2TM

The 2TM introduced in [1] describes the relaxation of interacting electrons and phonons in terms of their temperatures \( T_e \) and \( T_p \). At low temperatures, the dynamic equations for these temperatures have a simple form

\[
\frac{d \Delta T_e}{dt} = -\Sigma \delta (T_e^5 - T_p^5), \quad \frac{d \Delta T_p}{dt} = \Sigma \delta (T_e^5 - T_p^5) - K(T_p^4 - T_B^4),
\]

where the constants \( \Sigma \delta \) and \( K \) describe the heat transfer between electrons and phonons in the film and between the phonon subsystems of the film and substrate. For weak initial heating of the electrons, equations (A.1) and (A.2) can be linearized in \( \Delta T_e \) and \( \Delta T_p \), i.e. in small deviations of temperatures \( T_e \) and \( T_p \) from the thermostat temperature \( T_B \). Using the notation \( \tau_{e} = \frac{c_e}{5\Sigma \delta \Sigma K} \) and \( \tau_{es} = \frac{c_p}{4\Sigma K T_B^3} \), we arrive at a set of linear equations

\[
\frac{d \Delta T_e}{dt} = -\frac{1}{\tau_{e}} (\Delta T_e - \Delta T_p), \quad \frac{d \Delta T_p}{dt} = \frac{c_p}{c_p + c_e} (\Delta T_e - \Delta T_p) - \frac{1}{\tau_{es}} \Delta T_p,
\]

with the initial conditions \( \Delta T_e(0) = \Delta T(0) \) and \( \Delta T_p(0) = 0 \). The set of equations (A.3) and (A.4) is solved by the operational method. For the Laplace transforms \( \Delta \tilde{T}_e(p) \) and \( \Delta \tilde{T}_p(p) \), we obtain

\[
\Delta \tilde{T}_e(p) = \frac{(p + \nu_e + \nu_{es}) \Delta T(0)}{p^2 + (\nu_e + \nu_p + \nu_{es}) p + \nu_p \nu_{es}}, \quad \Delta \tilde{T}_p(p) = \frac{\nu_p \Delta T(0)}{p^2 + (\nu_e + \nu_p + \nu_{es}) p + \nu_p \nu_{es}},
\]

where \( \nu_e = 1/\tau_e, \nu_p = c_e \nu_e/c_p \) and \( \nu_{es} = 1/\tau_{es} \). We denote the roots of the common denominator in equations (A.5) and (A.6) by \( \kappa_1, \kappa_2 \). Their values are as follows:

\[
\kappa_{1,2} = -\frac{1}{2} (\nu_e + \nu_p + \nu_{es}) \pm \frac{1}{2} (\nu_e^2 + \nu_p^2 + \nu_{es}^2 + 2\nu_e \nu_p + 2\nu_p \nu_{es} - 2\nu_e \nu_{es})^{1/2}.
\]

It is convenient to write the solution of the set of equations (A.3) and (A.4) via \( \kappa_1 \) and \( \kappa_2 \). We have

\[
\Delta T_e(t) = \frac{\Delta T(0)}{\tau_e (\kappa_1 - \kappa_2)} \left[ (\kappa_1 \kappa_1 + 1) e^{\kappa_1 t} - (\kappa_1 \kappa_2 + 1) e^{\kappa_2 t} \right],
\]

\[
\Delta T_p(t) = \frac{\Delta T(0)}{\tau_e (\kappa_1 - \kappa_2)} (\kappa_1 \kappa_1 + 1) (\kappa_1 \kappa_2 + 1) \left( e^{\kappa_1 t} - e^{\kappa_2 t} \right).
\]

In the main part of the paper we considered the case of a poor thermal bond between a film and a substrate, when the strong inequality \( \nu_e, \nu_p \gg \nu_{es} \) holds. In this case, 2TM gives for the electron temperature the expression

\[
\Delta T_e(t) = \frac{c_p}{c_p + c_e} e^{-(\nu_e + \nu_p)t} + \frac{c_e}{c_p + c_e} e^{-\nu_{es} t/(1 + \zeta)}. \quad \text{(A.10)}
\]

Here we use the relation \( c_e \nu_e = c_p \nu_p \) and the notation \( \zeta = c_e/c_p \).

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