A FREE BOUNDARY PROBLEM FOR AN ELASTIC MATERIAL

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Abstract. In this paper we investigate the dynamics of an elastic material, for example, a spring with some weight. Such dynamics is usually represented by the ordinary differential equation for the length of the spring or the partial differential equation with the linear strain on a fixed domain. The main purpose of this paper is to propose a new free boundary problem with a nonlinear strain as a mathematical model for an elastic material. Also, we establish the well-posedness for initial boundary value problem with the nonlinear strain on the cylindrical domain.

1. Introduction. The following ordinary differential equation is well-known as a mathematical model for a spring with some weight:

$$m \frac{d^2 \ell}{dt^2} = -\frac{\kappa}{L}(\ell - L),$$

where $\ell = \ell(t)$ is the length of the spring for $t > 0$, $m$ is the mass of the weight, $\kappa$ is a positive constant from Hooke’s law and $L$ is the original length of the spring. When we adopt this model, we suppose that the dynamics of the spring is independent of changes of some variables in space so that the unknown function $\ell$ does not depend on the space variable $x$. However, in order to deal with a one-dimensional material made of a shape memory alloy we usually consider a mathematical model including parameters depending on the space variable. Moreover, we must consider a free boundary problem, because the length of the material may change. These facts are the motivations of the present paper.

From now on the mathematical model will be given. Let $u = u(t, x)$ be the displacement at $(t, x)$, that is, $u(t, x) = x - X(t, x)$ where $X(t, x)$ is the position of the point at $t = 0$ (see Figure 1). Also, we denote by $\varepsilon(t, x)$ the strain at $(t, x)$. Then, easily, we obtain

$$\varepsilon = \lim_{\Delta X \to 0} \frac{\Delta x - \Delta X}{\Delta X} = \lim_{\Delta X \to 0} \frac{u(t, x + \Delta x) - u(t, x)}{\Delta x} - 1 \quad \text{so that} \quad \frac{\partial u}{\partial x} \frac{\partial x}{\partial X} = \frac{\partial x}{\partial X} - 1 \quad \text{and} \quad \frac{\partial x}{\partial X} = \frac{1}{1 - u_x}.$$
If \( u_x = 1 \), then \( \frac{\partial X}{\partial x} = 0 \). This shows that the different points have moved to the same place and it is impossible, physically. Usually, we suppose that \( u_x \) is sufficiently small so that by using Taylor’s expansion we regard the definition of the strain as \( \varepsilon = u_x \).

Next, we calculate the velocity \( v \) as follows:

\[
v(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \lim_{\Delta t \to 0} \frac{(x + \Delta x) - x}{\Delta t} = \lim_{\Delta t \to 0} \frac{u(t + \Delta t, x + \Delta x) - u(t, x)}{\Delta t} = u_t + vu_x.
\]

Hence, we have \( v = \frac{u_t}{1 - u_x} \). Furthermore, in our problem the density \( \rho \) is not a constant function. Then, by mass conservation law we have

\[
\rho(t, x) = \frac{\partial X}{\partial x} \rho(0, X(t, x)) = (1 - u_x) \rho(0, X(t, x)).
\]

Here, we note the momentum balance law:

\[
(\rho v)_t = \rho f + \sigma_x,
\]

where \( f \) is the internal force. Thus we obtain

\[
\rho_0(x) u_{tt} = \rho f + \sigma_x,
\]

where \( \rho_0(x) = \rho(0, x) \). Therefore, Hooke’s law \( \sigma = \kappa \varepsilon \) implies that

\[
\rho_0 u_{tt} = \rho f + \kappa \left( \frac{u_x}{1 - u_x} \right). \tag{2}
\]

Moreover, let \( \ell(t), \ell(t) > 0 \), be the length of the one-dimensional elastic material and fix the material at \( x = 0 \). This leads to the homogeneous Dirichlet boundary \( u(t, 0) = 0 \). On the other hand, on the free boundary \( x = \ell(t) \) it holds that \( u(t, \ell(t)) = \ell(t) - X(t, \ell(t)) = \ell(t) - \ell_0 \) and \( mL''(t) = mg - \sigma(t, \ell(t)) \), where \( \ell_0 = \ell(0) \).
From the above argument we get the following system:

\[
\begin{aligned}
\rho_0 u_{tt} &= \rho f + \kappa \left( \frac{u_x}{1-u_x} \right)_{xx} \quad \text{in } Q_{\ell}(T), \\
u(t, 0) &= 0 \quad \text{for } 0 \leq t \leq T, \\
u(t, \ell(t)) &= \ell(t) - \ell_0 \quad \text{for } 0 \leq t \leq T, \\
m\ell''(t) &= mg - \kappa \left( \frac{u_x}{1-u_x} \right) \quad \text{for } 0 \leq t \leq T, \\
u(0, x) &= u_0(x), \quad u_t(0, x) = v_0(x), \quad \ell(0) = \ell_0, \quad \ell'(0) = \ell_0, 
\end{aligned}
\]

where \(Q_{\ell}(T) = \{(t, x)|0 < t < T, 0 < x < \ell(t)\}\), and \(u_0\) and \(v_0\) are initial functions, \(\ell_0\) is an initial value of the velocity of the free boundary.

Here, we clarify the connection between the ordinary differential equation (1) and the above system (3), briefly. The connection means that the neglect of the mass of the spring, \(\rho = 0\), leads to (1). In fact, first we have \(\kappa \left( \frac{u_x}{1-u_x} \right)_{xx} = 0\) on \(Q_{\ell}(T)\) so that \(\frac{u_x}{1-u_x} = c(t)\) on \([0, \ell]\) for each \(t \in [0, T]\), where \(c(t)\) is independent of \(x\). Then, we get \(u(t) = \frac{c(t)}{1+c(t)}\) for each \(t \in [0, T]\). According to the boundary condition for \(u\) we obtain \(u(t, x) = \frac{c(t)\ell - \ell_0}{c(t)}\). By substituting this function \(u\) into the free boundary condition \(m\ell''(t) = mg - \kappa \left( \frac{u_x}{1-u_x} \right)\) we can get (1).

It is very difficult to solve the above system (3), directly. Then as the first step of this research we consider the following initial boundary value problem (P1) on the cylindrical domain \(Q(T) = (0, T) \times (0, 1), T > 0:\)

\[
\begin{aligned}
u_{tt} + \gamma u_{xxxx} - \mu u_{xx} - \kappa \left( \frac{u_x}{1-u_x} \right)_{xx} &= 0 \quad \text{in } Q(T), \\
u(t, 0) &= u(t, 1) = 0, \quad u_{xx}(t, 0) = u_{xx}(t, 1) = 0 \quad \text{for } 0 < t < T, \\
u(0, x) &= u_0(x), \quad u_t(0, x) = v_0(x) \quad \text{for } 0 < x < 1,
\end{aligned}
\]

where \(\gamma\) and \(\mu\) are positive constants. Since our final goal of this research is to investigate free boundary problem for shape memory alloys and the equation \(u_{tt} + \gamma u_{xxxx} - \mu u_{xx} = f + \sigma x\) is well known as one of mathematical models for shape memory alloys (cf. [4]), we adopt (4) as the approximation of (2).

In Section 2 we give a precise definition of a solution of (P1) and show a theorem concerned with the existence and the uniqueness of a solution.

The other purpose of the present paper is to propose the following free boundary problem (P2) with the linear strain: The problem is to find \(\ell = \ell(t)\) and \(u = u(t, x)\) on \(Q_{\ell}(T), T > 0, Q_{\ell}(T) = \{(t, x)|0 < x < \ell(t), 0 < t < T\}\), such that

\[
\begin{aligned}
u_{tt} + \gamma u_{xxxx} - \mu u_{xx} - \kappa u_{xx} &= f \quad \text{in } Q_{\ell}(T), \\
u(t, 0) &= u_{xx}(t, 0) = 0 \quad \text{for } 0 < t < T, \\
u(t, \ell(t)) &= \ell(t) - \ell_0, \quad u_{xx}(t, \ell(t)) = 0 \quad \text{for } 0 < t < T, \\
u(0) &= u_0, \quad u_t(0) = v_0 \quad \text{on } [0, \ell_0], \\
\ell''(t) &= g(t) - \kappa u_x(t, \ell(t)), \\
\ell(0) &= \ell_0, \quad \ell'(0) = \ell_0.
\end{aligned}
\]

In author’s forthcoming paper [2] we shall discuss the free boundary problem (P2).

2. Initial boundary value problem (P1). Throughout this paper we put \(H = L^2(0, 1), X = H^1_0(0, 1)\), and write \(X^*\) as the dual space of \(X\), \((\cdot, \cdot)\) as the inner
product of $H$ and $\langle \cdot, \cdot \rangle$ as the pairing between $X^*$ and $X$. Moreover, we set $\beta : (-\infty, 1) \to \mathbb{R}$, $R := (-\infty, \infty)$, $\beta(r) = \frac{1}{1-r}$ for $r < 1$. Clearly, $\beta$ is the maximal monotone graph on $R \times R$ and $\partial \beta(r) = \beta(r)$, where $\beta(r) = -r - \log(1-r)$ if $r < 1$, $= \infty$ otherwise. We note that $\hat{\beta}$ is proper, convex and lower semi-continuous on $R$. We quote the book by Brezis [3] for definitions and basic properties of convex functions and subdifferentials.

First, we give a definition of a solution of (P1) as follows:

**Definition 1.** Let $u$ be a function on $Q(T)$. We call that $u$ is a weak solution of (P1) on $[0, T)$, $T > 0$, if the conditions (S1) $\sim$ (S4) hold.

(S1) $u \in S_w(T)$, where $S_w(T) := \{ u \in W^{2,2}(0, T; X^*) \cap W^{1,\infty}(0, T; X); u \in L^{\infty}(0, T; H^3(0, 1)) \cap W^{1,2}(0, T; H^2(0, 1)), u_{xx} \in L^{\infty}(0, T; X) \}$. 

(S2) There exists a positive constant $\delta$ such that $1 - u_x \geq \delta$ on $Q(T)$.

(S3) It holds that

$$\langle u_t, \eta \rangle - \gamma(u_{xxx}, \eta_x) - \mu(u_{xx}, \eta) + \langle \beta(u_x), \eta_x \rangle = 0 \quad \text{for} \quad \eta \in X \quad \text{and a.e.} \quad t \in [0, T].$$

(S4) $u(t, \cdot) = u_0$ and $u_t(t, \cdot) = v_0$.

Moreover, if $u \in S_s(T) := S_w(T) \cap W^{2,2}(0, T; V) \cap W^{1,\infty}(0, T; H^2(0, 1)) \cap L^{\infty}(0, T; H^2(0, 1)) \cap W^{1,2}(0, T; H^3(0, 1))$ and (4) holds a.e. on $Q(T)$, then we say that $u$ is a strong solution of (P1) on $[0, T]$.

The next theorem guarantees the well-posedness of (P1).

**Theorem 1.** Let $T > 0$, $u_0 \in H^3(0, 1) \cap X$ with $u_{0xx} \in X$ and $v_0 \in H^1_0(0, 1)$. Then there exists a positive constant $c$ independent of $T$ such that if

$$|v_0|_{L^2(0, 1)}^2 + \frac{\gamma}{2} |u_{0xx}|_{L^2(0, 1)}^2 + \int_0^1 \hat{\beta}(u_{0x}) dx \leq c,$$

then (P1) has a unique weak solution on $[0, T]$ for any $T > 0$. Moreover, if $u_0 \in H^4(0, 1)$ and $v_0 \in H^2(0, 1)$, then the weak solution is a strong solution.

We can prove the uniqueness under the more general assumptions.

**Remark 1.** If $u_1$ and $u_2$ satisfy the following (S1'), (S2'), (S3') and (S4), then $u_1 = u_2$ on $Q(T)$.

(S1') $u, u_x \in L^{\infty}(Q(T))$;

(S2') $1 - u_x \geq 0$ a.e. on $Q(T)$ and $\frac{1}{1-u_x} \in L^4(Q(T))$;

(S3') It holds that

$$\int_{Q(T)} u(\eta_t + \gamma \eta_{xxxx} + \mu \eta_{xx}) dx dt + \int_{Q(T)} \beta(u_x) \eta_x dx dt = \int_0^1 v_0(0) dx - \int_0^1 u_0 \eta_t(0) dx$$

for $\eta \in S_s(T)$ with $\eta(T) = \eta_t(T) = 0$.

In Section 3 we prove the existence part of Theorem 1 by using Banach’s fixed point theorem and standard approximation method. In section 4 the uniqueness is proved by the dual equation method. The dual equation method was already discussed in Chapter 3 of [5] and was applied to one-dimensional Shape memory alloy problem called Falk model in [1].

3. The existence of solutions. The aim of this section is to prove the existence of a solution. First, we recall the following elementary lemma:
Lemma 1. Assume that \( u_0 \in H^3(0,1) \cap X \), \( u_{0xx} \in X \), \( v_0 \in X \) and \( f \in L^2(0,T;H) \). Then there exists one and only one \( u \in S_w(T) \) such that

\[
(u_t, \eta) - \gamma (u_{xxx}, \eta_x) - \mu (u_{xx}, \eta) = (f, \eta) \quad \text{for} \quad \eta \in X \quad \text{and} \quad \text{a.e. on} \ [0,T],
\]

\( u(0) = u_0 \) and \( u_t(0) = v_0 \), and 
\[
t \to \frac{1}{2} |u_t(t)|^2_H + t \to \frac{1}{2} |u_{xx}(t)|^2_H + \frac{\gamma}{2} |u_{xxx}(t)|^2_H \quad \text{are absolutely continuous on} \ [0,T], \text{ and}
\]

\[
\frac{1}{2} \frac{d}{dt} |u_t(t)|^2_H = (u_t(t), u_t(t)),
\]

and 
\[
\frac{d}{dt} \left( \frac{1}{2} |u_{xx}(t)|^2_H + \frac{\gamma}{2} |u_{xxx}(t)|^2_H \right) + \mu |u_{xx}(t)|^2_H \leq \frac{1}{2\mu} |f(t)|^2_H \quad \text{for a.e.} \ t \in [0,T].
\]

Moreover, if \( u_0 \in H^4(0,1) \), \( v_0 \in H^2(0,1) \) and \( f \in L^2(0,T;X) \), then \( u \in S_w(T) \) and

\[
u_{tt} + \gamma u_{xxx} - \mu u_{xx} = f \quad \text{a.e. on} \ Q(T).
\]

In order to establish the existence we approximate \( \beta \) by the Yosida approximation \( \beta_\lambda \ (\lambda > 0) \). More precisely,

\[
\beta_\lambda(r) = \frac{r - \lambda - 1 + \sqrt{(r - \lambda - 1)^2 + 4r\lambda}}{2\lambda} \quad \text{for} \ r \in R. \tag{5}
\]

As to the approximate problem we have:

Lemma 2. Let \( \lambda \in (0,1] \). If \( u_0 \in H^3(0,1) \cap X \), \( u_{0xx} \in X \), \( v_0 \in X \), then the approximate problem has a unique solution, that is, there exists one and only one \( u_\lambda \in S_w(T) \) such that

\[
(u_{\lambda t}, \eta) - \gamma (u_{\lambda xxx}, \eta_x) - \mu (u_{\lambda xx}, \eta) = (\beta_\lambda(u_{\lambda x})_x, \eta) \quad \text{for} \quad \eta \in X, \text{a.e. on} \ [0,T], \tag{6}
\]

\( u_\lambda(0) = u_0 \) and \( u_{\lambda t}(0) = v_0 \). Moreover, if \( u_0 \in H^4(0,1) \) and \( v_0 \in H^2(0,1) \), then \( u_\lambda \in S_w(T) \) and

\[
u_{\lambda t} + \gamma u_{\lambda xxx} - \mu u_{\lambda xx} = \beta_\lambda(u_{\lambda x})_x \quad \text{a.e. on} \ Q(T).
\]

Proof. First, we put \( N(T) = L^2(0,T;H^2(0,1)) \). Let \( w \in N(T) \). Then, Lemma 1 implies that there exists a unique solution \( u \in S_w(T) \) of the following problem since \( \beta_\lambda(w_x) \in L^2(Q(T)) \):

\[
(u_{tt}, \eta) - \gamma (u_{xx}, \eta_x) - \mu (u_{xx}, \eta) = (\beta_\lambda(w_x)_x, \eta) \quad \text{for} \quad \eta \in X \quad \text{and} \quad \text{a.e. on} \ [0,T],
\]

\( u(0) = u_0 \) and \( u_t(0) = v_0 \). We denote this mapping by \( \Lambda \), that is, \( \Lambda : N(T) \to N(T) \), \( \Lambda(w) = u \).

Let \( w_1, w_2 \in N(T) \), \( u_i = \Lambda(w_i) \ (i = 1,2) \), \( w = w_1 - w_2 \) and \( u = u_1 - u_2 \). Also, we have

\[
(u_{tt}, u_t) - \gamma (u_{xx}, u_{xx}) - \mu (u_{xx}, u_t) = (\beta_\lambda(w_{1x}) - \beta_\lambda(w_{2x}), u_{xx}) \quad \text{and} \quad \text{a.e. on} \ [0,T].
\]

Then, since \( \beta_\lambda \) is Lipschitz continuous, we see that

\[
\frac{1}{2} \frac{d}{dt} |u_t(t)|^2_H + \frac{\gamma}{2} \frac{d}{dt} |u_{xx}(t)|^2_H + \mu |u_{xx}(t)|^2_H \\
\leq \frac{1}{\lambda} |w_x(t)|_H |u_{xx}(t)|_H \\
\leq \frac{\mu}{2} |u_{xx}(t)|^2_H + \frac{1}{2\mu \lambda^2} |w_x(t)|^2_H \quad \text{for a.e.} \ t \in [0,T].
\]
Obviously, we obtain
\[
\frac{1}{2}|u_\lambda(t)|_H^2 + \frac{\gamma}{2}|u_{\lambda xx}(t)|_H^2 + \frac{\mu}{2} \int_0^t |u_{\tau x}(\tau)|_H^2 d\tau \leq \frac{1}{2\mu \lambda^2} \int_0^t |w_{\tau x}(\tau)|_H^2 d\tau \quad \text{for } 0 \leq t \leq T.
\]

Therefore,
\[
\int_0^T |u(t)|^2_{H^2(0,1)} dt \leq KT \int_0^T |w(t)|^2_{H^2(0,1)} dt,
\]
where \( K \) is a positive constant. This implies that \( \Lambda : N(T_0) \to N(T_0) \) is the contraction mapping for small \( T_0 > 0 \). Hence, by using Banach’s fixed point theorem \( \Lambda \) has a fixed point. Moreover, since the choice of \( T_0 \) is independent of initial values, the approximate problem has a unique solution on the whole interval \([0, T]\). \( \square \)

**Proof of the existence.** Let \( u_\lambda \) be a solution of the approximate problem for \( \lambda \in (0, 1] \). Because of \( u_{\lambda, t}(t) \in X \) for a.e. \( t \in [0, T] \) we can substitute \( u_{\lambda, t}(t) \) into (6) as the test function. Easily, we obtain
\[
\frac{d}{dt} \left( \frac{1}{2} |u_{\lambda, t}(t)|_H^2 + \frac{\gamma}{2} |u_{\lambda xx}(t)|_H^2 + \int_0^1 \hat{\beta}_\lambda(u_{\lambda, x})(t) dx + \mu |u_{\lambda, tx}(t)|_H^2 \right) \leq 0 \quad \text{for a.e. } t \in [0, T],
\]
where \( \hat{\beta}_\lambda \) is the primitive of \( \beta_\lambda \). By integrating this inequality we see that
\[
\frac{1}{2} |u_{\lambda, t}(t)|_H^2 + \frac{\gamma}{2} |u_{\lambda xx}(t)|_H^2 + \int_0^1 \hat{\beta}_\lambda(u_{\lambda, x})(t) dx + \mu \int_0^t |u_{\lambda, tx}(\tau)|_H^2 d\tau \leq \frac{1}{2} |u_0|_H^2 + \frac{\gamma}{2} |u_{\lambda xx}|_H^2 + \int_0^1 \hat{\beta}(u_{\lambda, x}) dx \quad \text{for } t \in [0, T].
\]

In particular, \( |u_{\lambda xx}(t)|^2_H \leq \frac{1}{\gamma} C_1 \) for \( t \in [0, T] \) so that \( |u_{\lambda, x}(t)|_{L^\infty(0,1)} \leq \frac{1}{\gamma} |u_{\lambda xx}(t)|_H \leq \frac{\sqrt{2}}{\gamma \sqrt{C_1}} \) for any \( t \in [0, T] \). Hence, for \( 0 < \delta < 1 \) if \( \frac{\sqrt{2}}{\gamma \sqrt{C_1}} \leq 1 - \delta \), then \( |u_{\lambda, x}(t)|_{L^\infty(0,1)} \leq 1 - \delta \) for any \( t \in [0, T] \), that is, \( 1 - u_{\lambda, x} \geq \delta > 0 \) on \( Q(T) \).

On account of (5) this estimate implies \( |\beta_\lambda(u_{\lambda, x})| \leq C_2 \) on \( Q(T) \) for some positive constant \( C_2 \), which is independent of \( \lambda \). Moreover, \( |\beta^*_\lambda(u_{\lambda, x})| \leq \frac{1}{\delta} \) on \( Q(T) \).

Then, immediately, we infer that \( |\beta_\lambda(u_{\lambda, x})| \leq \frac{1}{\gamma} |u_{\lambda xx}| \) on \( Q(T) \) so that the set \( \{\beta_\lambda(u_{\lambda, x})\} \) is bounded in \( L^\infty(0, T; H) \). This together with Lemma 1 shows that for some positive constant \( C_3 \)
\[
|u_{\lambda, tx}(t)|_H^2 \leq C_3, \quad |u_{\lambda xx}(t)|_H^2 \leq C_3 \quad \text{for } 0 \leq t \leq T, 0 < \lambda \leq 1,
\]
\[
\int_0^T |u_{\lambda, tx}(t)|_H^2 dt \leq C_3 \quad \text{for } 0 < \lambda \leq 1.
\]

Moreover, from (6) it follows that \( \{u_{\lambda, t}\} \) is bounded in \( L^2(0, T; X^*) \).

Accordingly, by the above uniform estimates we can take a subsequence \( \{\lambda_j\} \) of \( \{\lambda\} \) and \( u \in S_u(T) \) such that
\[
\begin{align*}
    u_j := u_{\lambda_j} & \rightharpoonup u \quad \text{weakly* in } L^\infty(0, T; H^3(0, 1)), \\
    u_{jtt} & \rightharpoonup u_{tt} \quad \text{weakly in } L^2(0, T; X^*), \\
    u_j & \rightharpoonup u \quad \text{weakly* in } L^\infty(0, T; X) \quad \text{and } L^2(0, T; H^2(0, 1)), \\
    u_j & \to u, u_{j xx} \rightharpoonup u_x \quad \text{in } C(Q(T)), \quad \text{as } j \to \infty.
\end{align*}
\]

Easily, we have \( 1 - u_x \geq \delta > 0 \) on \( Q(T) \), \( \beta_{\lambda_j}(u_{j xx}) \to \beta(u_x) \) in \( C(Q(T)) \) and \( \beta_{\lambda_j}(u_{j xx}) \to \beta(u_x) \) weakly in \( L^2(Q(T)) \) as \( j \to \infty \). Here, it holds that
\[
\int_0^T \left( (u_{jtt}, \eta) - \gamma(u_{jxxx}, \eta_x) - \mu(u_{jxxx}, \eta) \right) dt = \int_0^T (\beta_{\lambda_j}(u_{j xx}), \eta) dt
\]
for $\eta \in L^2(0,T;X)$ and each $j$. Letting $j \to \infty$, we observe that
\[
\int_0^T (\langle u_{tt}, \eta \rangle - \gamma(u_{xxx}, \eta_x) - \mu(u_{txxx}, \eta))dt = \int_0^T (\beta(u_x), \eta)dt \quad \text{for } \eta \in L^2(0,T;X).
\]
Thus we have proved the existence of a weak solution. By using Lemma 1 we can show that a weak solution is the strong solution under some assumptions.

4. The uniqueness. In this section we shall show that the problem (P1) has at most one solution. It is clear that the weak solution of (P1) satisfies (S1'), (S2'), (S3') and (S4). Hence, we shall prove Remark 1.

Proof of Remark 1. Let $u_1$ and $u_2$ be functions satisfying (S1'), (S2'), (S3') and (S4). Immediately, we have
\[
\int_{Q(T)} u(\eta t + \gamma \eta_{xxx} + \mu \eta_{txxx})dxdt + \int_{Q(T)} (\beta(u_1_x) - \beta(u_1_x))\eta_x dxdt = 0
\]
for $\eta \in S_*(T)$ with $\eta(T) = \eta_0(T) = 0$, where $u = u_1 - u_2$. Here, we put
\[
F = \frac{1}{1 - u_{1x}} \frac{1}{1 - u_{2x}} \quad \text{on } Q(T).
\]
Then, we have
\[
\int_{Q(T)} u(\eta t + \gamma \eta_{xxx} + \mu \eta_{txxx})dxdt + \int_{Q(T)} Fu_x \eta_x dxdt = 0 \quad \text{(7)}
\]
and $F \in L^2(Q(T))$ because of (S3'). Here, we can take a sequence $\{F_j\} \subset C^\infty_0(Q(T))$ such that $F_j \to F$ in $L^2(Q(T))$ as $j \to \infty$.

Let $z \in C^\infty_0(Q(T))$ and consider the following problem for each $j$.
\[
\begin{cases}
\eta_{tt} + \gamma \eta_{xxx} + \mu \eta_{txxx} - (F_j \eta_{tx})_x = z & \text{in } Q(T), \\
\eta_j(t,0) = \eta_j(t,1) = \eta_{jxx}(t,0) = \eta_{jxx}(t,1) = 0 & \text{for } 0 < t < T, \\
\eta_j(T) = \eta_j(T) = 0.
\end{cases}
\]
By applying Lemma 1 there exists a unique solution $\eta_j \in S_*(T)$ of the above problem.

From now on, we shall show some uniform estimates for $\eta_j$ with respect to $j$. Here, we put $\eta_j(t,x) = \eta_j(T-t,x)$ for $(t,x) \in Q(T)$. Obviously, we have
\[
\begin{aligned}
\eta_j(t) + \gamma \eta_j_{xxx} - \mu \eta_j_{txxx} - (\hat{F}_j \eta_j_{tx})_x = \hat{z} & \quad \text{in } Q(T), \\
\eta_j(t,0) = \eta_j(t,1) = \eta_{jxx}(t,0) = \eta_{jxx}(t,1) = 0 & \quad \text{for } 0 < t < T, \\
\eta_j(0) = \hat{\eta}_j(0) = 0,
\end{aligned}
\]
where $\hat{F}(t,x) = F(T-t,x)$ and $\hat{z}(t,x) = z(T-t,x)$ for $(t,x) \in Q(T)$. We multiply (8) by $\eta_j$ and integrate it over $(0,1)$. Then, it yields that
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} |\eta_j(t)|^2_H + \gamma \frac{d}{dt} |\eta_{jxx}(t)|^2_H + \mu |\eta_{jtx}(t)|^2_H & = \langle \hat{z}(t), \eta_j(t) \rangle - \langle \hat{F}_j(t)\eta_j(t), \eta_{jtx}(t) \rangle \\
& \leq \frac{1}{2} |\hat{z}(t)|^2_H + \frac{1}{2} |\eta_j(t)|^2_H + \mu |\eta_{jtx}(t)|^2_H + \frac{1}{2\mu} |\hat{F}_j(t)\eta_j(t)|^2_H \\
& \leq \frac{1}{2} |\hat{z}(t)|^2_H + \frac{1}{2} |\eta_j(t)|^2_H + \mu |\eta_{jtx}(t)|^2_H + \frac{1}{2\mu} |\hat{F}_j(t)\eta_j(t)|^2_H \quad \text{for a.e. } t \in [0,T].
\end{align*}
\]
It is clear that
\[
\frac{1}{2} \frac{d}{dt} |\hat{\eta}_{jx}(t)|_{H}^{2} + \frac{\gamma}{2} \frac{d}{dt} |\hat{\eta}_{jxxx}(t)|_{H}^{2} + \frac{\mu}{2} |\hat{\eta}_{jtxx}(t)|_{H}^{2} \\
\leq \frac{1}{2} |\hat{\varepsilon}(t)|_{H}^{2} + \frac{1}{2} |\hat{\eta}_{jx}(t)|_{H}^{2} + \frac{1}{2\mu} |\hat{F}_{j}(t)|_{H}^{2} |\hat{\eta}_{jxx}(t)|_{H}^{2} \text{ for a.e. } t \in [0,T].
\] (9)

By applying Gronwall’s inequality to (9), we see that
\[
\frac{1}{2} |\hat{\eta}_{jx}(t)|_{H}^{2} + \frac{\gamma}{2} |\hat{\eta}_{jxxx}(t)|_{H}^{2} + \frac{\mu}{2} \int_{0}^{t} |\hat{\eta}_{jtxx}(\tau)|_{H}^{2} d\tau \\
\leq \frac{1}{2} \exp \left( \int_{0}^{t} (1 + \frac{2}{\gamma} |\hat{F}_{j}(\tau)|_{H}^{2}) d\tau \right) \int_{0}^{t} |\hat{\varepsilon}(\tau)|_{H}^{2} d\tau \text{ for } 0 \leq t \leq T.
\]

Hence, the set \( \{\eta_{jx}\} \) is bounded in \( L^{2}(Q(T)) \). Here, by using (7) we observe that
\[
\int_{Q(T)} zu dx dt \\
= \int_{Q(T)} (\eta_{jtt} + \gamma \eta_{jxxxx} + \mu \eta_{jxx} - (F_{j} \eta_{jx})_{x}) u dx dt \\
= - \int_{Q(T)} u_{x} F_{j} \eta_{jx} dx dt + \int_{Q(T)} F_{j} \eta_{jx} u_{x} dx dt \text{ for each } j.
\]

Therefore, it follows that
\[
|\int_{Q(T)} zu dx dt| \leq |F_{j} - F|_{L^{2}(Q(T))} |u_{x}|_{L^{\infty}(Q(T))} |\eta_{jx}|_{L^{2}(Q(T))} \text{ for each } j
\]
so that
\[
\int_{Q(T)} zu dx dt = 0 \text{ for any } z \in C_{0}^{\infty}(\overline{Q(T)}).
\]

This implies the assertion of Remark 1. \( \square \)

REFERENCES

[1] T. Aiki, Weak solutions for Falk’s model of shape memory alloys, Math. Methods Appl. Sci., 23 (2000), 299–319.
[2] T. Aiki, Well-posedness of the free boundary problem for elastic materials in one-dimensional space, in preparation.
[3] H. Brézis, “Opérateurs maximaux monotones et semi-groupes de contractions dans les spaces de Hilbert,” North-Holland, Amsterdam, 1973.
[4] M. Brokate and J. Sprekels, Hysteresis and Phase Transitions, Springer, Appl. Math. Sci., 121 (1996).
[5] O. A. Ladyzenskaja, V. A. Solonnikov and N. N. Ural’ceva, Linear and Quasi-Linear Equations of Parabolic Type, Transl. Math. Monograph 23, Amer. Math. Soc., Providence R. I., 1968.

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