Log-Harnack Inequality and Exponential Ergodicity for Distribution Dependent CKLS and Vasicek Model*

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Abstract

In this paper, Wang’s log-Harnack inequality and exponential ergodicity are derived for two types of distribution dependent SDEs: one is the CKLS model, where the diffusion coefficient is a power function of order $\theta$ with $\theta \in \left[\frac{1}{2}, 1\right)$; the other one is Vasicek model, where the diffusion coefficient only depends on distribution. Both models in the distribution independent case can be used to characterize the interest rate in mathematical finance.

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1 Introduction

The SDE

(1.1) \[ dX_t = (\alpha - \delta X_t)dt + |X_t|^\theta dW_t, \quad X_0 \geq 0, \]

with $\alpha \geq 0, \delta \geq 0, \theta \in \left[\frac{1}{2}, 1\right)$ is called CKLS model, which was introduced in [9]. It can be used to characterize the evolution of the interest rate in finance. By the Yamada-Watanabe

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approximation [12], (1.1) is strongly well-posed. In particular, when \( \theta = \frac{1}{2} \), it is called Cox-Ingersoll-Ross (CIR) model [4, Section 4.6]. For CIR model, one can refer to [5, 8, 17, 20, 21] for more introductions, applications, the convergence rate of various numerical methods and functional inequalities. Recently, [11] has proved Wang’s Harnack inequality and super Poincaré inequality for (1.1) with \( \theta \in (\frac{1}{2}, 1) \).

On the other hand, there are many results on the distribution dependent SDEs, also named McKean-Vlasov SDEs or mean field SDEs, in which the coefficients depend on the law of the solution, see for instance, [3, 6, 7, 10, 13, 15] and references therein. [2] investigated the strong well-posedness and propagation chaos of McKean-Vlasov SDEs with Hölder continuous diffusion coefficients, and the diffusion is assumed to be distribution free.

In this paper, we will first consider the distribution dependent version of (1.1), i.e. mean field CKLS model:

\[
(1.2) \quad dX_t = (\alpha - \delta X_t)dt + \gamma \mathbb{E}(X_t)dt + |X_t|^\theta dW_t,
\]

where \( \frac{1}{2} \leq \theta < 1 \), \( \alpha, \delta \geq 0 \), \( \gamma \geq 0 \) and \( W_t \) is a one-dimensional Brownian motion on some complete filtration probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \). Noting that the diffusion in (1.2) is degenerate at 0, we cannot directly use coupling by change of measure as in [19] to derive the log-Harnack inequality. Instead, we will adopt Girsanov’s transform together with the method of coupling by change of measure to obtain the desired log-Harnack inequality. To this end, we will study the log-Harnack inequality for the decoupled SDEs. The crucial trick is to estimate \( \mathbb{E} \int_0^t |X_s|^{-2\theta} ds \), an upperbound of which will be provided in Lemma 2.3 below by constructing appropriate test functions. Moreover, the exponential ergodicity in \( L^1 \)-Wasserstein distance is also proved by the Yamada-Watanabe approximation in the case \( \delta > \gamma \).

In addition, the Vasicek model

\[
(1.3) \quad dX_t = (\gamma - \beta X_t)dt + \sigma dW_t
\]

with \( \gamma, \beta, \sigma \in \mathbb{R} \) can also be used to characterize the interest rate and it was proposed in [16]. Compared with (1.1), the solution to (1.3) can take negative values. Let \( \mathcal{P} \) be the collection of all probability measures on \( \mathbb{R} \) equipped with the weak topology. Consider the distribution dependent case of (1.3):

\[
(1.4) \quad dX_t = (\gamma - \beta X_t)dt + b(L_{X_t})dt + \sigma (L_{X_t})dW_t,
\]

where \( b, \sigma : \mathcal{P} \to \mathbb{R} \) are measurable. Noting that the diffusion in (1.4) depends on distribution, which produces essential difficulty to study the log-Harmanck inequality since the coupling by change of measure is unavailable. Fortunately, by observing the fact that the solution to (1.4) follows Gaussian distribution, we can estimate the relative entropy between two solutions from different initial distributions, which is equivalent to the log-Harnack inequality.

The paper is organized as follows: In Section 2, we give results on the distribution dependent CKLS model (1.2): the log-Harnack inequality and the exponential ergodicity in \( L^1 \)-Wasserstein distance; The log-Harnack inequality as well as exponential ergodicity in \( L^2 \)-Wasserstein distance and in relative entropy for distribution dependent Vasicek model (1.4) will be given in Section 3.
2 Distribution Dependent CKLS Model

2.1 Log-Harnack Inequality

The monograph [18] gives many applications of dimension-free Harnack inequality and a lot of models for it being true. For \( p \in [1, \infty) \), let

\[ P_p := \left\{ \mu \in \mathcal{P} : \mu(| \cdot |^p) := \int_{\mathbb{R}} |x|^p \mu(dx) < \infty \right\}. \]

\( P^+_p \) is the subset of \( P_p \) with support on \([0, \infty)\). It is well known that \( P_p \) is a Polish space under the \( L^p \)-Wasserstein distance

\[ W_p(\mu_1, \mu_2) := \inf_{\pi \in \mathcal{C}(\mu_1, \mu_2)} \left( \int_{\mathbb{R} \times \mathbb{R}} |x - y|^p \pi(dx, dy) \right)^{1/p}, \quad \mu_1, \mu_2 \in P_p, \]

where \( \mathcal{C}(\mu_1, \mu_2) \) is the set of all couplings for \( \mu_1 \) and \( \mu_2 \). In this section, we investigate the log-Harnack inequality for (1.2). By Lemma 2.2 below, (1.2) with \( \alpha, \delta, \gamma \geq 0 \) and \( X_0 \geq 0 \) is equivalent to

\[ dX_t = (\alpha - \delta X_t)dt + \gamma \mathbb{E}(X_t)dt + X_0^\theta dW_t. \]

Noting that \( |x^\theta - y^\theta| \leq |x - y|^\theta, x, y \geq 0, \) [2, Theorem 1.2] yields that (2.1) is well-posed. For any \( \mu_0 \in P_1^+ \), let \( P_t^* \mu_0 \) be the distribution of the solution to (2.1) with initial distribution \( \mu_0 \). Define

\[ P_t f(\mu_0) = \int_{\mathbb{R}} f(x)(P_t^* \mu_0)(dx), \quad \mu_0 \in P_1^+, t \geq 0, f \in \mathcal{B}_b([0, \infty)). \]

For any \( \mu, \nu \in \mathcal{P} \), the relative entropy between \( \mu, \nu \) is defined as

\[ \text{Ent}(\nu | \mu) = \begin{cases} \nu(\log(\frac{d\nu}{d\mu})), & \nu \ll \mu; \\ \infty, & \text{otherwise}. \end{cases} \]

We shall introduce the intrinsic metric:

\[ \rho(x, y) = \int_{x \wedge y}^{x \vee y} \frac{dr}{r^\theta} = \frac{(x \vee y)^{1-\theta} - (x \wedge y)^{1-\theta}}{1-\theta} = \sqrt{\frac{(x^{1-\theta} - y^{1-\theta})^2}{(1-\theta)^2}}, \quad x, y \in [0, \infty), \]

and the \( L^2 \)-Wasserstein distance induced by \( \rho \):

\[ W_{2, \rho}(\mu, \nu) = \inf_{\pi \in \mathcal{C}(\mu, \nu)} \left( \int_{[0, \infty) \times [0, \infty)} \rho(x, y)^2 \pi(dx, dy) \right)^{1/2}, \quad \mu, \nu \in P_1^+. \]

**Theorem 2.1.** Assume \( \delta > 0 \) and \( \gamma \geq 0 \). Then the following assertions hold.
(1) Assume $\frac{1}{2} < \theta < 1$ and $\alpha \geq \frac{\theta}{2}$. For any $T > 0$, $f \in \mathcal{B}_0^+([0, \infty))$ with $f > 0$, $\mu_0, \nu_0 \in \mathcal{P}_1^+$ with $\mu_0(|\cdot|^{1-2\theta}) < \infty$, the log-Harnack inequality holds, i.e.

\[
P_T \log f(\nu_0) \leq \log P_T f(\mu_0) + \frac{2(1-\theta)(\delta - \frac{\theta}{2})\mathbb{W}_{2,\rho}(\mu_0, \nu_0)^2}{(e^{2(1-\theta)(\delta - \frac{\theta}{2})T} - 1)} + \gamma^2(e^{-2(\delta - \gamma)T} + 1)\mathbb{W}_1(\mu_0, \nu_0)^2 \Gamma(T, \delta, \alpha, \theta, \mu_0, \nu_0),
\]

where

\[
\Gamma(T, \delta, \alpha, \theta, \mu_0, \nu_0) = \inf_{\varepsilon \in (0, \frac{\theta}{2})} \left\{ \frac{1}{2\theta - 1} \mu_0(\cdot)^{1-2\theta} + \left((\delta^+)^{2\eta} \varepsilon^{1-2\eta} + \varepsilon^{-\frac{1}{2\theta - 1}}\right)T + \frac{\varepsilon^{-1} e^{2(1-\theta)(\delta - \frac{\theta}{2})\mathbb{W}_{2,\rho}(\mu_0, \nu_0)^2}}{e^{2(1-\theta)(\delta - \frac{\theta}{2})T} - 1} \right\}.
\]

(2) Assume $\theta = \frac{1}{2}$ and $\alpha > \frac{1}{2}$. Then for any $T > 0$, $f \in \mathcal{B}_0^+([0, \infty))$ with $f > 0$, $\mu_0, \nu_0 \in \mathcal{P}_1^+$ satisfying $\mu_0(|\log(\cdot)|) < \infty$, the log-Harnack inequality holds, i.e.

\[
P_T \log f(\nu_0) \leq \log P_T f(\mu_0) + \frac{(\delta - \frac{1}{2})\mathbb{W}_{2,\rho}(\mu_0, \nu_0)^2}{(e^{(\delta - \frac{1}{2})T} - 1)} + \gamma^2(e^{-2(\delta - \gamma)T} + 1)\mathbb{W}_1(\mu_0, \nu_0)^2 \Gamma(T, \delta, \alpha, \theta, \mu_0, \nu_0),
\]

where

\[
\Gamma(T, \delta, \alpha, \mu_0, \nu_0) = \inf_{\varepsilon \in (0, \alpha - \frac{1}{2})} \frac{\mu_0(|\log(\cdot)|^{1/2}) + (\alpha + \delta^+)T + \varepsilon^{-1} e^{(\delta - \frac{1}{2})\mathbb{W}_{2,\rho}(\mu_0, \nu_0)^2}}{e^{(\delta - \frac{1}{2})T} - 1}.
\]

### 2.2 Proof of Theorem 2.1

Before giving the proof of Theorem 2.1, we make some preparations. The first lemma tells us that the solution to (1.2) with non-negative initial value is non-negative.

**Lemma 2.2.** Assume $\alpha, \gamma \geq 0$. Let $X_t$ be the solution to (1.2) with $\mathcal{F}_0$-measurable non-negative initial value $X_0$. Then $\mathbb{P}$-a.s.

\[
X_t \geq 0, \quad t \geq 0.
\]

Moreover, it holds

\[
\mathbb{E}(X_t) = e^{-(\delta - \gamma)t}\mathbb{E}(X_0) + \frac{\alpha}{\delta - \gamma}(1 - e^{-(\delta - \gamma)t}), \quad t \geq 0,
\]

where $\frac{\alpha}{\delta - \gamma}(1 - e^{-(\delta - \gamma)t}) = \alpha t$ if $\delta = \gamma$. 


Proof. For $\varepsilon \in (0, 1)$, noting that $\int_{\varepsilon/e}^{\varepsilon} \frac{1}{x} \, dx = 1$, there exists a continuous function $\psi_\varepsilon : [0, \infty) \to [0, \infty)$ with the support $[\varepsilon/e, \varepsilon]$ such that

$$0 \leq \psi_\varepsilon(x) \leq \frac{2}{x}, \quad x \in [\varepsilon/e, \varepsilon], \quad \int_{\varepsilon/e}^{\varepsilon} \psi_\varepsilon(r) \, dr = 1.$$  

Define

$$\mathbb{R} \ni x \mapsto V^0_\varepsilon(x) := \int_0^{x^-} \int_0^y \psi_\varepsilon(z) \, dz \, dy.$$  

It is not difficult to see that

$$V^0_\varepsilon(x) = 0, x \geq -\varepsilon/e, \quad x^- - \varepsilon \leq V^0_\varepsilon(x) \leq x^-, \quad x \in \mathbb{R},$$

and

$$0 \leq (V^0_\varepsilon)'(x) \leq \frac{2}{x^-} 1_{[\varepsilon/e, \varepsilon]}(x^-), \quad x \in \mathbb{R}.$$  

By Itô’s formula, we get

$$dV^0_\varepsilon(X_t) = (V^0_\varepsilon)'(X_t)(\alpha - \delta X_t + \gamma \mathbb{E}(X_t)) \, dt + (V^0_\varepsilon)'(X_t) |X_t|^\theta \, dW_t + \frac{1}{2} (V^0_\varepsilon)''(X_t) |X_t|^{2\theta} \, dt.$$  

For any $n \geq 1$, let $\tau_n = \inf\{t \geq 0 : |X_t| \geq n\}$. We arrive at

$$\mathbb{E}V^0_\varepsilon(X_{s \wedge \tau_n}) = \mathbb{E}V^0_\varepsilon(X_0) + \mathbb{E} \int_0^{s \wedge \tau_n} (V^0_\varepsilon)'(X_t)(\alpha - \delta X_t + \gamma \mathbb{E}(X_t)) \, dt + \frac{1}{2} \mathbb{E} \int_0^{s \wedge \tau_n} (V^0_\varepsilon)''(X_t) |X_t|^{2\theta} \, dt.$$  

Letting $n \to \infty$, the dominated convergence theorem yields

$$\mathbb{E}V^0_\varepsilon(X_s) = \mathbb{E}V^0_\varepsilon(X_0) + \mathbb{E} \int_0^{s} (V^0_\varepsilon)'(X_t)(\alpha - \delta X_t + \gamma \mathbb{E}(X_t)) \, dt + \frac{1}{2} \mathbb{E} \int_0^{s} (V^0_\varepsilon)''(X_t) |X_t|^{2\theta} \, dt.$$  

This together with (2.4)-(2.6), $\gamma \geq 0, X_0 \geq 0, \varepsilon \in (0, 1)$ implies

$$\mathbb{E}V^0_\varepsilon(X_s) \leq \mathbb{E}V^0_\varepsilon(X_0) + \mathbb{E} \int_0^{s} (V^0_\varepsilon)'(X_t)(\alpha - \delta X^+_t + \gamma \mathbb{E}(X^+_t)) \, dt + \mathbb{E} \int_0^{s} |\delta| \mathbb{E}(X^-_t) \, dt + \int_0^{s} 1_{[\varepsilon/e, \varepsilon]}(X^-_t) \, dt.$$  

(2.7)
\[
\leq \int_0^s (|\delta| + \gamma)\mathbb{E}(X_t^-) dt + \int_0^s 1_{[\varepsilon/e, \varepsilon]}(X_t^-) dt.
\]

Letting \( \varepsilon \to 0 \), the dominated convergence theorem, (2.4), (2.8) and Gronwall’s inequality yield
\[
\mathbb{E}(X_t^-) = 0, \ t \geq 0.
\]

This combined with the continuity of \( X_t \) in \( t \) implies that \( \mathbb{P} \)-a.s.
\[
X_t \geq 0, \ t \geq 0.
\]

Finally, by the same argument to obtain (2.7), we have
\[
\mathbb{E}(X_t) = \mathbb{E}(X_0) + \alpha t - \int_0^t (\delta - \gamma)\mathbb{E}(X_s) ds.
\]

This implies (2.2) immediately. So, we complete the proof. \( \Box \)

With the above preparations in hand, we are in the position to complete the proof of Theorem 2.1.

**Proof of Theorem 2.1.** Let \( \mu_t = P_t^*\mu_0, \nu_t = P_t^*\nu_0 \). We divide the proof into three steps.

**(I)** For any \( x > 0 \), consider
\[
(2.9) \quad dX_t^{x,\mu} = (\alpha - \delta X_t^{x,\mu}) dt + \gamma \mu_t(\cdot) dt + (x^{x,\mu})^\theta dW_t, \quad X_0^{x,\mu} = x.
\]

For simplicity, we denote \( X_t = X_t^{x,\mu} \). For any \( m \geq 1 \), define
\[
(2.10) \quad \beta_m = \inf \left\{ t \geq 0 : X_t \leq \frac{1}{m} \right\}.
\]

Then by Lemma 2.3 below with \( \alpha_t = \alpha + \gamma \mu_t \) and \( \zeta(t) = 0 \), \( \mathbb{P} \)-a.s. \( \lim_{m \to \infty} \beta_m = \infty \) and so \( \mathbb{P} \)-a.s. \( X_t > 0, t \geq 0 \). Letting
\[
(2.11) \quad \alpha_t' = \alpha + \gamma \nu_t(\cdot), \ t \geq 0,
\]

we have \( \alpha_t' \geq \alpha \) due to \( \gamma \nu_t(\cdot) \geq 0 \). We rewrite (2.9) as
\[
(2.12) \quad dX_t = (\alpha_t' - \delta X_t) dt + X_t^\theta d\tilde{W}_t, \quad X_0 = x,
\]

here
\[
\tilde{W}_t = W_t + \int_0^t X_s^\theta (\gamma \mu_s(\cdot) - \gamma \nu_s(\cdot)) ds.
\]

Let
\[
R_s = \exp \left\{ -\int_0^s X_t^{-\theta}(\gamma \mu_t(\cdot) - \gamma \nu_t(\cdot)) dW_t - \frac{1}{2} \int_0^s |X_t^{-\theta}(\gamma \mu_t(\cdot) - \gamma \nu_t(\cdot))|^2 dt \right\}, \quad s \in [0, T],
\]

\[
\tilde{W}_t = W_t + \int_0^t X_s^\theta (\gamma \mu_s(\cdot) - \gamma \nu_s(\cdot)) ds.
\]
(2.2) implies that for any \( m \geq 1 \), \((R_{s \wedge \beta_m})_{s \in [0,T]}\) is a martingale and Girsanov’s theorem yields that \((\tilde{W}_{s \wedge \beta_m})_{s \in [0,T]}\) is a one-dimensional Brownian motion under \( \mathbb{Q}_T^m = R_{T \wedge \beta_m} \mathbb{P} \). Moreover, it follows from (2.2) that

\[
\mathbb{E}(R_{s \wedge \beta_m} \log R_{s \wedge \beta_m}) \\
\leq \frac{1}{2} \mathbb{E}_{\mathbb{Q}_T^m} \int_0^{s \wedge \beta_m} X_t^{-2\theta} |\gamma \mu_t(\cdot) - \gamma \nu_t(\cdot)|^2 dt \\
\leq \frac{1}{2} \gamma^2 \mathbb{E}(X_0 - Y_0)^2 (e^{-2(\delta - \gamma)s} + 1) E_{\mathbb{Q}_T^m} \int_0^{s \wedge \beta_m} X_t^{-2\theta} dt, \quad s \in [0,T].
\]

By Lemma 2.3 below for \( W_t = \tilde{W}_{t \wedge \beta_m}, \zeta = 0 \) and \( \mathbb{P} = \mathbb{Q}_{T}^m \), we have

\[
\sup_{m \geq 1} \mathbb{E}_{\mathbb{Q}_T^m} \int_0^T X_t^{-2\theta} dt < \infty,
\]

which yields

\[
\sup_{m \geq 1} \mathbb{E}(R_{s \wedge \beta_m} \log R_{s \wedge \beta_m}) < \infty, \quad s \in [0,T].
\]

Then it follows from the martingale convergence theorem and the fact \( \mathbb{P} \)-a.s. \( \lim_{m \to \infty} \beta_m = \infty \) that \( \mathbb{E}R_s = 1, s \in [0,T] \), which means that \( \{R_s\}_{s \in [0,T]} \) is a martingale.

**Step (II).** By Step (I), we know that \((\tilde{W}_t)_{t \in [0,T]}\) is a one-dimensional Brownian motion under the probability measure \( \mathbb{Q}_T = R_T \mathbb{P} \). Let \( X_t^{y,\nu} \) solve (2.9) with \((y, \nu)\) replacing \((x, \mu)\) for \( y \geq 0 \). Let \( Y_t \) solve

\[
dY_t = (\alpha_t' - \delta Y_t) dt + Y_t^{\theta} d\tilde{W}_t + Y_t^{\theta} 1_{[0,\tau)}(t) \xi_t dt, \quad Y_0 = y,
\]

where

\[
\xi_t = \frac{2(\delta - \frac{\theta}{2}) (x^{1-\theta} - y^{1-\theta}) e^{(1-\theta)(\delta - \frac{\theta}{2}) t}}{e^{2(1-\theta)(\delta - \frac{\theta}{2}) T} - 1}, \quad t \geq 0
\]

and \( \tau = \inf\{t \geq 0 : X_t = Y_t\} \). Set \( Y_t = X_t, t \geq \tau \). According to the proof of [11, Theorem 2.1(1)], \( \mathbb{Q}_T(\tau \leq T) = 1 \) and \( \{\tilde{W}_t\}_{t \in [0,T]} \) with \( \tilde{W}_t = \tilde{W}_t + \int_0^t \xi_s 1_{[0,\tau)}(s) ds \) is a one-dimensional Brownian motion under \( \mathbb{Q}_T = R_T \mathbb{Q}_T \), where

\[
\tilde{R}_t = \exp \left\{ - \int_0^{t \wedge \tau} \xi_s d\tilde{W}_s - \frac{1}{2} \int_0^{t \wedge \tau} |\xi_s|^2 ds \right\}, \quad t \in [0,T].
\]

Moreover, we have \( \mathcal{L}_{Y_t}\mathbb{Q}_T = \mathcal{L}_{X_t^{y,\nu}}, t \in [0,T], \mathbb{Q}_T\)-a.s. \( X_T = Y_T \) and

\[
\mathbb{E}_{\mathbb{Q}_T} \log(\tilde{R}_t) = \frac{1}{2} \int_0^T |\xi_s|^2 ds \leq \frac{(1 - \theta)(\delta - \frac{\theta}{2}) \rho(x,y)^2}{(e^{2(1-\theta)(\delta - \frac{\theta}{2}) T} - 1)}.
\]

**Step (III).** Noting that

\[
dW_t = d\tilde{W}_t - X_t^{-\theta}(\gamma \mu_t(\cdot) - \gamma \nu_t(\cdot)) dt = d\tilde{W}_t - \xi_t 1_{[0,\tau)}(t) dt - X_t^{-\theta}(\gamma \mu_t(\cdot) - \gamma \nu_t(\cdot)) dt,
\]
we conclude that

\[ E^{Q_T} \log R_T = E^{Q_T} \int_0^T X_t^{-\theta} (\gamma \mu_t(\cdot) - \gamma \nu_t(\cdot)) \xi_t 1_{[0,\tau]}(t) \, dt \]

\[ + \frac{1}{2} E^{Q_T} \int_0^T X_t^{-2\theta} |\gamma \mu_t(\cdot) - \gamma \nu_t(\cdot)|^2 \, dt \]

\[ \leq E^{Q_T} \int_0^T X_t^{-2\theta} |\gamma \mu_t(\cdot) - \gamma \nu_t(\cdot)|^2 \, dt + \frac{1}{2} E^{Q_T} \int_0^T |\xi_t|^2 \, dt. \]

So, (2.14) and (2.15) yield

\[ E^{Q_T} \log(\bar{R}_\tau R_T) \leq \frac{2(1 - \theta)(\delta - \frac{\theta}{2}) \rho(x, y)^2}{(e^{2(1-\theta)(\delta - \frac{\theta}{2}) T} - 1)} \]

\[ + \frac{1}{2} E^{Q_T} \int_0^T X_t^{-2\theta} |\gamma \mu_t(\cdot) - \gamma \nu_t(\cdot)|^2 \, dt. \]

Applying Young’s inequality, for any \( f \in \mathcal{B}_b^+(\mathbb{R}_0^+) \) with \( f > 0 \), we have

\[ E \log f(X_T^{y,\nu}) = E^{Q_T} \log f(Y_T) = E^{Q_T} \log f(X_T) \leq \log E f(X_T^{y,\mu}) + E^{Q_T} \log(\bar{R}_\tau R_T). \]

Rewrite (2.12) as

\[ dX_t = (\alpha_t^\nu - \delta X_t) dt - X_t^\nu \xi_t 1_{[0,\tau]}(t) dt + X_t^\nu dW_t, \quad X_0 = x. \]

Applying Lemma 2.3 below for \( W_t = W_t, \xi(t) = \xi_t 1_{[0,\tau]}(t) \) and \( \mathbb{P} = \tilde{Q}_T \), combining (2.2) and (2.16)-(2.17), when \( \frac{1}{2} < \theta < 1 \) and \( \alpha \geq \frac{\theta}{2} \),

\[ E \log f(X_T^{y,\nu}) \leq \log E f(X_T^{y,\mu}) + \frac{2(1 - \theta)(\delta - \frac{\theta}{2}) \rho(x, y)^2}{(e^{2(1-\theta)(\delta - \frac{\theta}{2}) T} - 1)} \]

\[ + \gamma^2 (e^{-2(\delta - \gamma) T} + 1) \mathbb{W}_1(\mu_0, \nu_0)^2 \Gamma(T, \delta, \alpha, \theta, \delta_x, \delta_y), \]

and when \( \theta = \frac{1}{2} \) and \( \alpha > \frac{1}{2} \),

\[ E \log f(X_T^{y,\nu}) \leq \log E f(X_T^{y,\mu}) + \frac{(\delta - \frac{1}{2}) \rho(x, y)^2}{(e^{2(\delta - \frac{1}{2}) T} - 1)} \]

\[ + \gamma^2 (e^{-2(\delta - \gamma) T} + 1) \mathbb{W}_1(\mu_0, \nu_0)^2 \Gamma(T, \delta, \alpha, \delta_x, \delta_y). \]

Noting that both \( \mu_0[|\cdot|^{1-2\theta}] < \infty \) and \( \mu_0(|\log(\cdot)|) < \infty \) yield

\[ P^*_T \mu_0 = \int_{(0,\infty)} \mathcal{L}_{X_T^{y,\nu}} \mu_0(dx), \quad P^*_T \nu_0 = \int_{(0,\infty)} \mathcal{L}_{X_T^{y,\nu}} \nu_0(dx), \]

for any \( \pi \in \mathcal{C}(\mu_0, \nu_0) \), taking expectation with respect to \( \pi \), using Jensen’s inequality, and then taking infimum in \( \pi \) on the two sides of (2.18) and (2.19), we complete the proof.
Let $\alpha_t$ be a measurable function from $[0, \infty)$ to $[\alpha, \infty)$, $\zeta(t)$ be a progressively measurable process with $\mathbb{E} \int_0^T |\zeta(s)|^2 ds$ locally bounded in $t$ and $X_t^\zeta$ be a non-negative solution to the SDE

$$
\text{(2.20)} \quad dX_t = (\alpha_t - \delta X_t) dt - X_t^\theta \zeta(t) dt + X_t^\theta dW_t, \quad X_0 = x > 0.
$$

For any $m \geq 1$, let $\beta_m^\zeta$ be defined in (2.10) with $X_t^\zeta$ replacing $X_t$.

**Lemma 2.3.** The following assertions hold.

1. Assume $\theta \in (\frac{1}{2}, 1), \alpha > 0$, we have $\lim_{m \to \infty} \beta_m^\zeta = \infty$ and

$$
\mathbb{E} \int_0^T (X_t^\zeta)^{-2\theta} dt \leq \inf_{\varepsilon_1 \in (0, \frac{\alpha}{1})} \frac{1}{2\theta - 1} \left( \frac{\delta^+}{\varepsilon_1} \right)^{\frac{1}{2\theta - 1}} + \left( \frac{\delta^+}{\varepsilon_1} \right)^{1-\theta} + \left( \frac{\delta^+}{\varepsilon_1} \right)^{T} + \varepsilon_1 \mathbb{E} \int_0^T \zeta(t)^2 dt.
$$

2. Assume $\theta = \frac{1}{2}$ and $\alpha > \frac{1}{2}$, we obtain $\lim_{m \to \infty} \beta_m^\zeta = \infty$ and

$$
\mathbb{E} \int_0^T (X_t^\zeta)^{-1} dt \leq \inf_{\varepsilon_1 \in (0, \alpha - \frac{1}{2})} \frac{\log(\frac{\alpha + 1}{2})}{\alpha - \frac{1}{2} - \varepsilon_1} + \left( \frac{\delta^+}{\varepsilon_1} \right)^{T} + \varepsilon_1 \mathbb{E} \int_0^T \zeta(t)^2 dt.
$$

**Proof.** For simplicity, we omit the superscript, i.e. we denote $X_t = X_t^\zeta$ and $\beta_m = \beta_m^\zeta$.

1. Define

$$
V(x) = \frac{1}{2\theta - 1} x^{1-2\theta}, \quad x > 0.
$$

Then it is clear that

$$
\text{(2.21)} \quad V(x) > 0, \quad V'(x) = -x^{-2\theta}, \quad V''(x) = 2\theta x^{-2\theta - 1}, \quad x > 0, \quad \lim_{x \to 0} V(x) = \infty.
$$

It follows from Itô’s formula, (2.21) and $\alpha_t \geq \alpha$ that

$$
dV(X_t) \leq (\alpha - \delta X_t - X_t^\theta \zeta(t))(-X_t^{-2\theta}) dt + X_t^\theta(-X_t^{-2\theta}) dW_t + \theta X_t^{-2\theta - 1} X_t^\theta dt, \quad t \leq \beta_m.
$$

So, we have

$$
V(X_{s\wedge \beta_m}) - V(x) \leq \int_0^{s\wedge \beta_m} (-\alpha X_t^{-2\theta} + \delta^+ X_t^{-2\theta + 1} + \theta X_t^{-1} + X_t^{-\theta} |\zeta(t)|) dt - \int_0^{s\wedge \beta_m} X_t^{-\theta} dW_t.
$$

Noting that $-2 \leq -2\theta < -1, \alpha > 0$, Young’s inequality implies that for any $\varepsilon_1 \in (0, \frac{\alpha}{3})$,

$$
\delta^+ X_t^{-2\theta + 1} = (\delta^+)^{\frac{\alpha + 1}{\alpha - 2\theta}} \left( \varepsilon_1 X_t^{-2\theta} \right)^{\frac{2\theta - 1}{\alpha - 2\theta}} \leq (\delta^+)^{\frac{\alpha + 1}{\alpha - 2\theta}} \left( \varepsilon_1 X_t^{-2\theta} \right)^{2\theta - 1},
$$

$$
\theta X_t^{-1} = (\theta^{\frac{2\theta - 1}{\alpha - 2\theta}} \varepsilon_1^\frac{\alpha - 2\theta}{\alpha - 2\theta}) \left( \varepsilon_1 X_t^{-2\theta} \right)^{\frac{2\theta - 1}{\alpha - 2\theta}} \leq \varepsilon_1^{-\frac{\alpha - 2\theta}{\alpha - 2\theta}} + \varepsilon_1 X_t^{-2\theta},
$$




\[ |\zeta(t)| X_t^{-\theta} = (|\zeta(t)|^2 \varepsilon_1)^{\frac{1}{2}} (\varepsilon_1 X_t^{-2\theta})^{\frac{1}{2}} \leq |\zeta(t)|^2 \varepsilon_1^{-1} + \varepsilon_1 X_t^{-2\theta}. \]

Combining (2.22)-(2.23), we conclude that for any \( \varepsilon_1 \in (0, \frac{\alpha}{3}) \), it holds
\[
\mathbb{E}V(X_{s \land \beta_m}) \leq V(x) + (\delta^+) 2^\theta \varepsilon_1^{1-2\theta} s + \varepsilon_1 \frac{1}{2\theta-1} s + \mathbb{E} \int_0^s |\zeta(t)|^2 \varepsilon_1^{-1} dt, \quad s \geq 0.
\]

This implies that
\[
\mathbb{P}(\beta_m \leq s) \leq (2\theta - 1) m^{1-2\theta} \mathbb{E}[V(X_{s \land \beta_m}) 1_{\{\beta_m \leq s\}}]
\leq (2\theta - 1) m^{1-2\theta} \left( V(x) + (\delta^+) 2^\theta \varepsilon_1^{1-2\theta} s + \varepsilon_1 \frac{1}{2\theta-1} s + \mathbb{E} \int_0^s |\zeta(t)|^2 \varepsilon_1^{-1} dt \right), \quad s \geq 0.
\]

So, \( \mathbb{P} \) a.s. \( \lim_{m \to \infty} \beta_m = \infty \) and thus \( \mathbb{P} \) a.s. \( X_t > 0, t \geq 0 \). Moreover, substituting (2.23) into (2.22) and taking expectation, we get
\[
\mathbb{E} \int_0^{T \land \beta_m} X_t^{-\theta} dt \leq \inf_{\varepsilon_1 \in (0, \frac{\alpha}{3})} \frac{1}{2\theta-1} X_t^{1-2\theta} \left( (\delta^+) 2^\theta \varepsilon_1^{1-2\theta} T + \varepsilon_1 \frac{1}{2\theta-1} T + \varepsilon_1^{-1} \mathbb{E} \int_0^T |\zeta(t)|^2 dt \right).
\]

Letting \( m \to \infty \), Fatou’s lemma derives (1).

(2) Define
\[
\tilde{V}(x) = \log(x + 1) - \log x, \quad x > 0.
\]

Then we have
\[
\tilde{V}(x) > 0, \quad \tilde{V}'(x) = (x + 1)^{-1} - x^{-1}, \quad \tilde{V}''(x) = x^{-2} - (x + 1)^{-2}, \quad x > 0, \quad \lim_{x \to 0} \tilde{V}(x) = \infty.
\]

By Itô’s formula, we arrive at
\[
\tilde{V}(X_{s \land \beta_m}) - \tilde{V}(x)
\leq \int_0^{s \land \beta_m} (\alpha - \delta X_t - X_t^{-\frac{1}{2}} \zeta(t)) [(X_t + 1)^{-1} - X_t^{-1}] dt
+ \frac{1}{2} \int_0^{s \land \beta_m} X_t [X_t^{-2} - (X_t + 1)^{-2}] dt + \int_0^{s \land \beta_m} X_t^{\frac{1}{2}} [(X_t + 1)^{-1} - X_t^{-1}] dW_t
\leq \int_0^{s \land \beta_m} \left( -\alpha + \frac{1}{2} \right) X_t^{-1} dt + \frac{(\alpha + \delta^+) s}{2}
+ \int_0^{s \land \beta_m} X_t^{-\frac{1}{2}} |\zeta(t)| dt + \int_0^{s \land \beta_m} X_t^{\frac{1}{2}} [(X_t + 1)^{-1} - X_t^{-1}] dW_t.
\]

Since \( \alpha > \frac{1}{2} \) and
\[
|\zeta(t)| X_t^{-\frac{1}{2}} = (|\zeta(t)|^2 \varepsilon_1^{-1})^{\frac{1}{2}} (\varepsilon_1 X_t^{-1})^{\frac{1}{2}} \leq |\zeta(t)|^2 \varepsilon_1^{-1} + \varepsilon_1 X_t^{-1},
\]
for any \( \varepsilon_1 \in (0, \alpha - \frac{1}{2}) \), (2.24) implies
\[
\mathbb{E}\tilde{V}(X_{s \land \beta_m}) \leq \tilde{V}(x) + (\alpha + \delta^+) s + \mathbb{E} \int_0^s |\zeta(t)|^2 \varepsilon_1^{-1} dt.
\]
As a result, it holds
\[ \mathbb{P}(\beta_m \leq s) \leq \left[ \log\left(\frac{1}{m} + 1\right) - \log\frac{1}{m} \right]^{-1} \mathbb{E}[\bar{V}(X_{s\wedge \beta_m})1_{\{\beta_m \leq s\}}] \]
\[ \leq \left[ \log\left(\frac{1}{m} + 1\right) - \log\frac{1}{m} \right]^{-1} \left( \bar{V}(x) + (\alpha + \delta^+)s + \mathbb{E} \int_0^s |\zeta(t)|^2 \varepsilon^{-1} dt \right), \quad s \geq 0, \]
which implies that \( \mathbb{P}\)-a.s. \( \lim_{m \to \infty} \beta_m = \infty \) and thus \( \mathbb{P}\)-a.s. \( X_t > 0, t \geq 0 \). Finally, it follows from (2.24) that
\[ \mathbb{E} \int_0^{T \wedge \beta_m} X_t^{-1} dt \leq \inf_{\varepsilon \in (0, a - \frac{1}{2})} \frac{\log \left( \frac{x+1}{x} \right) + (\alpha + \delta^+)T + \varepsilon^{-1} \mathbb{E} \int_0^T \zeta(t)^2 dt}{\alpha - \frac{1}{2} - \varepsilon_1}. \]
Letting \( m \to \infty \), Fatou’s lemma completes the proof.

2.3 Exponential Ergodicity in Wasserstein Distance

Recall that \( P_t^* \mu \) is the distribution of the solution to (2.1) with initial distribution \( \mu \in \mathcal{P}^+_1 \).

**Theorem 2.4.** Assume that \( \alpha \geq 0, \delta > \gamma \geq 0 \). Then \( P_t^* \) has a unique invariant probability measure \( \mu \in \mathcal{P}_1 \) satisfying
\[ \mathbb{W}_1(P_t^* \nu, \mu) \leq e^{- (\delta - \gamma) t} \mathbb{W}_1(\nu, \mu), \quad \nu \in \mathcal{P}^+_1. \]

**Proof.** Let \( \psi_\varepsilon \) be defined in (2.3). Define
\[ \mathbb{R} \ni x :\mapsto V_\varepsilon(x) := \int_0^{\|x\|} \int_0^y \psi_\varepsilon(z) dz dy \]
It is not difficult to see that
\[ |x| - \varepsilon \leq V_\varepsilon(x) \leq |x|, \quad \text{sgn}(x) V_\varepsilon'(x) \in [0, 1], \quad x \in \mathbb{R}, \]
and
\[ 0 \leq V_\varepsilon''(x) \leq \frac{2}{|x|} 1_{[\varepsilon/e, e]}(|x|), \quad x \in \mathbb{R}. \]
Let \( X_t \) and \( Y_t \) be solutions to (2.1) with non-negative initial values \( X_0 \) and \( Y_0 \) respectively. For any \( \varepsilon > 0 \), it follows from Itô’s formula that
\[ dV_\varepsilon(X_t - Y_t) = V_\varepsilon'(X_t - Y_t)(-\delta(X_t - Y_t) + \gamma(\mathbb{E}(X_t) - \mathbb{E}(Y_t)))dt 
+ V_\varepsilon'(X_t - Y_t)[X^\theta_t - Y^\theta_t]dW_t 
+ \frac{1}{2} V_\varepsilon''(X_t - Y_t)[X^\theta_t - Y^\theta_t]^2 dt. \]
By (2.26) and the inequality \(|x^\theta - y^\theta| \leq |x - y|^\theta, x, y \geq 0\), we have
\[
\frac{1}{2} V''(X_t - Y_t)|X_t^\theta - Y_t^\theta|^2 \leq \varepsilon^{2\theta-1} 1_{[\varepsilon/e,\varepsilon]}(|X_t - Y_t|).
\]
So, by the same argument to obtain (2.7), (2.25) yields that
\[
\mathbb{E} V'\varepsilon(X_s - Y_s) \leq \mathbb{E} V\varepsilon(X_0 - Y_0) + \int_0^s - (\delta - \gamma) \mathbb{E} |X_t - Y_t| dt + \int_0^s \varepsilon^{2\theta-1} 1_{[\varepsilon/e,\varepsilon]}(|X_t - Y_t|) dt.
\]
Letting \(\varepsilon \to 0\) and using (2.25), we arrive at
\[
\mathbb{E} |X_s - Y_s| \leq \mathbb{E} |X_0 - Y_0| + \int_0^s - (\delta - \gamma) \mathbb{E} |X_t - Y_t| dt.
\]
Gronwall’s inequality implies that
\[
\mathbb{E} |X_s - Y_s| \leq e^{-(\delta - \gamma)s} \mathbb{E} |X_0 - Y_0|.
\]
Since \(\delta > \gamma\), it is standard to prove that \(P^*_t\) has a unique invariant probability \(\mu\) with support on \([0, \infty)\) and satisfying
\[
\mathbb{W}_1(P^*_t \nu, \mu) \leq e^{-(\delta - \gamma)t} \mathbb{W}_1(\nu, \mu), \quad \nu \in \mathcal{P}_1^+,
\]
see [19, Proof of Theorem 3.1(2)].

3 Distribution Dependent Vasicek Model

In this section, we consider the distribution dependent Vasicek model (1.4). Assume that

\textbf{(H1)} There exist constants \(L_b, L_\sigma \geq 0\) such that
\[
|b(\mu) - b(\nu)| \leq L_b \mathbb{W}_2(\mu, \nu), \quad |\sigma(\mu) - \sigma(\nu)| \leq L_\sigma \mathbb{W}_2(\mu, \nu), \quad \mu, \nu \in \mathcal{P}_2.
\]

\textbf{(H2)} There exists a constant \(K \geq 1\) such that
\[
K^{-1} \leq \sigma^2(\mu) \leq K, \quad \mu \in \mathcal{P}_2.
\]
Under \textbf{(H1)}, (1.4) is strongly well-posed according to [18]. For any \(\mu_0 \in \mathcal{P}_2\), let \(P^*_t \mu_0\) be the distribution of the solution to (1.4) with initial distribution \(\mu_0\), and define
\[
P_t f(\mu_0) = \int_{\mathbb{R}} f(x)(P^*_t \mu_0)(dx), \quad \mu_0 \in \mathcal{P}_2, t \geq 0, f \in \mathcal{B}_b(\mathbb{R}).
\]
It is standard from \textbf{(H1)} that
\[
\mathbb{W}_2(P^*_t \mu_0, P^*_t \nu_0) \leq e^{(-\beta + L_b + L_\sigma^2)t} \mathbb{W}_2(\mu_0, \nu_0), \quad t \geq 0.
\]
Theorem 3.1. The log-Harnack inequality holds, i.e.

$$P_t \log f(\mu_0) \leq \log P_t f(\nu_0) + \sum(t)W_2(\mu_0, \nu_0)^2, \quad f \in \mathcal{B}_b(\mathbb{R}), \quad f > 0, \quad t > 0, \quad \mu_0, \nu_0 \in \mathcal{P}_2$$

with

$$\sum(t) = \frac{2\beta K}{e^{2\beta t} - 1} + \frac{2\beta K}{e^{2\beta t} - 1} \left( L_b + \frac{L_2^2}{2} \right)^2$$

$$+ \frac{K + 1}{2} \left( \frac{1 - e^{-2\beta t}}{2\beta} \right)^2 K^3 L_\sigma^2 e^{-4\beta t} \left( \frac{e^{(\beta + L_b + \frac{L_2^2}{2})t} - 1}{(\beta + L_b + \frac{L_2^2}{2})^2} \right),$$

here $e^{\delta t - \frac{1}{\delta}} = t$ when $\delta = 0$.

Proof. For any $x \in \mathbb{R}$, let

$$\Gamma_{t}^{\mu_0, x} = e^{-\beta t} x + \int_0^t e^{-\beta(t-s)} [\gamma + b(P_s^* \mu_0)] d s,
\Sigma_t^{\mu_0} = \int_0^t |e^{-\beta(t-s)} \sigma(P_s^* \mu_0)|^2 d s, \quad t \geq 0$$

and define

$$X_{t}^{\mu_0, x} = \Gamma_{t}^{\mu_0, x} + \int_0^t e^{-\beta(t-s)} \sigma(P_s^* \mu_0) d W_s, \quad t \geq 0.$$ 

Then it is clear that

$$P_t^* \mu_0 = \int_{\mathbb{R}} \mathcal{L}_{X_{t}^{\mu_0, x}} \mu_0(\, d x), \quad t \geq 0, \quad (3.2)$$

and

$$\frac{d \mathcal{L}_{X_{t}^{\mu_0, x}}}{dz}(z) = \frac{1}{\sqrt{2\pi \Sigma_t^{\mu_0}}} \exp \left\{ - \frac{(z - \Gamma_t^{\mu_0, x})^2}{2 \Sigma_t^{\mu_0}} \right\}, \quad t > 0. \quad (3.3)$$

By (H2), we have

$$\frac{1 - e^{-2\beta t}}{2\beta} K^{-1} \leq \Sigma_t^{\mu_0} \leq \frac{1 - e^{-2\beta t}}{2\beta} K, \quad t \geq 0. \quad (3.4)$$

Moreover, (H1)-(H2) and (3.1) imply

$$|\Sigma_t^{\mu_0} - \Sigma_t^{\nu_0}| \leq 2\sqrt{K} L_\sigma \int_0^t e^{-2\beta(t-s)} W_2(P_s^* \mu_0, P_s^* \nu_0) d s,$$

$$\leq 2\sqrt{K} L_\sigma W_2(\mu_0, \nu_0) \int_0^t e^{-2\beta(t-s)} e^{(\beta + L_b + \frac{L_2^2}{2})s} d s$$

$$\leq 2\sqrt{K} L_\sigma W_2(\mu_0, \nu_0) e^{-2\beta t} e^{(\beta + L_b + \frac{L_2^2}{2})t - 1} \frac{1}{\beta + L_b + \frac{L_2^2}{2}}, \quad t \geq 0,$$

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and
\[
|\Gamma_t^{\mu_0,x} - \Gamma_t^{\nu_0,y}|^2 \leq 2e^{-2\beta t}|x-y|^2 + 2L_b^2 \left| \int_0^t e^{-\beta(t-s)}\mathbb{W}_2(P_s^x\mu_0, P_s^y\nu_0)ds \right|^2
\]
\begin{equation}
(3.6)
\end{equation}
\[
\leq 2e^{-2\beta t}|x-y|^2 + 2L_b^2\mathbb{W}_2(\mu_0, \nu_0)^2 \left| \int_0^t e^{-\beta(t-s)}e^{-\beta t}(e^{(L_b+\frac{L_b^2}{2})t} - 1)^2 \right|
\leq 2e^{-2\beta t}|x-y|^2 + 2L_b^2\mathbb{W}_2(\mu_0, \nu_0)^2 e^{-2\beta t}\frac{(e^{(L_b+\frac{L_b^2}{2})t} - 1)^2}{(L_b + \frac{L_b^2}{2})^2}, \ t \geq 0.
\]

It follows from (3.3) that
\[
\text{Ent}(\mathcal{L}_{X_t^{\mu_0,x}} | \mathcal{L}_{X_t^{\nu_0,y}}) = \int_{\mathbb{R}} \log \left( \frac{d\mathcal{L}_{X_t^{\mu_0,x}}}{d\mathcal{L}_{X_t^{\nu_0,y}}} (z) \right) \mathcal{L}_{X_t^{\mu_0,x}}(dz)
\]
\[
= \log \frac{\sqrt{\Sigma^{\mu_0}}}{\sqrt{\Sigma^{\nu_0}}} + \int_{\mathbb{R}} \frac{(\Sigma^{\mu_0} - \Sigma^{\nu_0})(z - \Gamma_t^{\mu_0,x})^2 + \Sigma^{\mu_0}(\Gamma_t^{\mu_0,x} - \Gamma_t^{\nu_0,y})^2}{2\Sigma^{\mu_0}\Sigma^{\nu_0}} \mathcal{L}_{X_t^{\mu_0,x}}(dz)
\]
\[
= \log \frac{\sqrt{\Sigma^{\mu_0}}}{\sqrt{\Sigma^{\nu_0}}} + \frac{(\Sigma^{\mu_0} - \Sigma^{\nu_0})}{2\Sigma^{\mu_0}} + \frac{(\Gamma_t^{\mu_0,x} - \Gamma_t^{\nu_0,y})^2}{2\Sigma^{\mu_0}}, \ t > 0.
\]

Using Lemma 3.3 below for \( a = \sqrt{\Sigma^{\mu_0}} \) and \( b = \sqrt{\Sigma^{\nu_0}} \) and submitting (3.4)-(3.6) into (3.7), we get
\[
\text{Ent}(\mathcal{L}_{X_t^{\mu_0,x}} | \mathcal{L}_{X_t^{\nu_0,y}}) \leq K + \frac{1}{2} \left( 1 - e^{-2\beta t} \right)^{-1} K^3 L_b^2 e^{-4\beta t} \frac{(e^{(\beta + L_b + \frac{L_b^2}{2})t} - 1)^2}{(\beta + L_b + \frac{L_b^2}{2})^2} \mathbb{W}_2(\mu_0, \nu_0)^2
\]
\[
+ \left( 1 - e^{-2\beta t} \right)^{-1} K \left( e^{-2\beta t}|x-y|^2 + L_b^2\mathbb{W}_2(\mu_0, \nu_0)^2 e^{-2\beta t}\frac{(e^{(L_b+\frac{L_b^2}{2})t} - 1)^2}{(L_b + \frac{L_b^2}{2})^2} \right), \ t > 0.
\]

According to [18, Theorem 1.4.2(2)], for any \( f \in \mathcal{B}(\mathbb{R}) \) with \( f > 0 \), it holds
\[
\mathbb{E} \log f(X_t^{\mu_0,x}) \leq \log \mathbb{E} f(X_t^{\mu_0,y})
\]
\[
\leq K + \frac{1}{2} \left( 1 - e^{-2\beta t} \right)^{-1} K^3 L_b^2 e^{-4\beta t} \frac{(e^{(\beta + L_b + \frac{L_b^2}{2})t} - 1)^2}{(\beta + L_b + \frac{L_b^2}{2})^2} \mathbb{W}_2(\mu_0, \nu_0)^2
\]
\[
+ \left( 1 - e^{-2\beta t} \right)^{-1} K \left( e^{-2\beta t}|x-y|^2 + L_b^2\mathbb{W}_2(\mu_0, \nu_0)^2 e^{-2\beta t}\frac{(e^{(L_b+\frac{L_b^2}{2})t} - 1)^2}{(L_b + \frac{L_b^2}{2})^2} \right), \ t > 0.
\]

Taking expectation with respect to any \( \pi \in \mathcal{C}(\mu_0, \nu_0) \) on both sides of the above inequality firstly, utilizing (3.2) and Jensen’s inequality and then taking infimum in \( \pi \in \mathcal{C}(\mu_0, \nu_0) \), we complete the proof. \( \square \)
Remark 3.2. When $L_b = L_b = 0$, Theorem 3.1 reduces to the classical log-Harnack inequality with $\Sigma(t) = \frac{2\beta K t}{e^{2\beta t} - 1}$, see [18] for more distribution independent models. Moreover, the method in the proof of Theorem 3.1 is also available for multidimensional distribution dependent Ornstein-Uhlenbeck process, where the diffusion coefficient only depends on the distribution.

Lemma 3.3. The following inequality holds

$$-\log \left( \frac{b}{a} \right) + \frac{b^2 - a^2}{2a^2} \leq \frac{K + 1}{2} \frac{(b - a)^2}{a^2}, \quad \sqrt{\frac{1 - e^{-2\beta t}}{2\beta}} \sqrt{K - 1} \leq a, b \leq \sqrt{\frac{1 - e^{-2\beta t}}{2\beta}} \sqrt{K}.$$  

Proof. Let $\frac{b-a}{a} = y$, then $b = a(1 + y), K^{-1} - 1 \leq y \leq K - 1$. So, it is sufficient to prove

$$-\log(1 + y) + \frac{y^2 + 2y}{2} \leq \frac{K + 1}{2} y^2, \quad K^{-1} - 1 \leq y \leq K - 1.$$  

Define

$$F(y) = -\log(1 + y) + \frac{y^2 + 2y}{2} - \frac{K + 1}{2} y^2, \quad K^{-1} - 1 \leq y \leq K - 1.$$  

It is easy to see that

$$F'(y) = \frac{1}{1 + y} + 1 + y - (K + 1)y = \frac{Ky(K^{-1} - 1 - y)}{1 + y}, \quad K^{-1} - 1 \leq y \leq K - 1.$$  

Since $y \geq K^{-1} - 1$, we conclude that $F(y)$ takes maximum value at $y = 0$, i.e.

$$F(y) \leq F(0) = 0, \quad K^{-1} - 1 \leq y \leq K - 1.$$  

Therefore, (3.8) holds and we complete the proof.

As an application of Theorem 3.1, we present the exponential ergodicity of $P_t^*$ in relative entropy.

Theorem 3.4. Assume that (H1) – (H2) hold with $\beta > L_b + \frac{L^2}{2}$. Then $P_t^*$ has a unique invariant probability measure $\mu \in \mathcal{P}_2$ with

$$\max(\mathbb{W}_2(P_t^* \nu, \mu)^2, \text{Ent}(P_t^* \nu|\mu)) \leq K(t)e^{-2(\beta - L_b - \frac{L^2}{2})t} \min(\mathbb{W}_2(\nu, \mu)^2, \text{Ent}(\nu|\mu)), \quad \nu \in \mathcal{P}_2, t > 0$$  

for some function $K : (0, \infty) \to [0, \infty)$.

Proof. When $\beta > L_b + \frac{L^2}{2}$, it is standard to derive from (3.1) that $P_t^*$ has a unique invariant probability measure $\mu \in \mathcal{P}_2$ with

$$\mathbb{W}_2(P_t^* \nu, \mu)^2 \leq e^{-2(\beta - L_b - \frac{L^2}{2})t} \mathbb{W}_2(\nu, \mu)^2.$$  

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see [19, Proof of Theorem 3.1(2)]. Consider classical SDE:

\[
\begin{aligned}
\,dX_t &= (\gamma - \beta X_t)\,dt + b(\mu)\,dt + \sigma(\mu)\,dW_t.
\end{aligned}
\]  
(3.9)

Since \( \beta > 0 \), it is clear that \( \mu \) is the unique invariant probability measure of (3.9). Repeating the proof of [14, (4.2)], we can get the log-Sobolev inequality

\[
\mu(f^2 \log f^2) \leq c\mu(|\nabla f|^2), \, f \in C^1_b(\mathbb{R}), \, \mu(f^2) = 1
\]

for some constant \( c > 0 \). According to [1], this implies the Talagrand inequality

\[
\mathbb{W}_2(\nu, \mu)^2 \leq c\text{Ent}(\nu | \mu), \, \nu \in \mathcal{P}_2.
\]

Combining [14, Theorem 2.1] and Theorem 3.1, the proof is completed. \( \square \)

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