STOCHASTIC VOLterra equations of nonscalar type in Hilbert space

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Abstract

In the paper stochastic Volterra equations of non scalar type in Hilbert space are studied. The aim of the paper is to provide some results on stochastic convolution and mild solutions to those Volterra equations. The motivation of the paper comes from a model of aging viscoelastic materials. The pseudo-resolvent approach is used.

Keywords: Stochastic Volterra equation of non scalar type, pseudo-resolvent, stochastic convolution, mild solution.

1 Introduction

Assume that $H$ is a separable Hilbert space with a scalar product $\langle \cdot , \cdot \rangle_{H}$ and a norm $| \cdot |_{H}$. We are concerned with a stochastic linear Volterra equation of non scalar type in $H$ of the form

$$X(t) = X_0 + \int_{0}^{t} A(t - \tau)X(\tau)d\tau + \int_{0}^{t} \Psi(\tau)dW(\tau), \quad t \in [0, T]. \quad (1)$$

Let $G$ denote another separable Hilbert space such that $G \hookrightarrow d H$ ($G$ is densely imbedded in $H$). In (1), $A \in L_{\text{loc}}^{1}(\mathbb{R}_{+}; B(G, H))$, where $B(G, H)$ denotes the space of all bounded linear operators acting from $G$ into $H$. We assume that $W$ is a cylindrical Wiener process with values in some another Hilbert space $U$, and $\Psi$ is a stochastic process specified below.

Observe that Volterra equation of so called scalar type (see (Prüss 1993)) corresponds to the operator $A(t)$ of the form $A(t) = a(t)A$, where $a \in L_{\text{loc}}^{1}(\mathbb{R}_{+})$, and $A$ is a closed linear unbounded operator densely defined in $H$ and $G = (D(A), | \cdot |_{A})$, where $D(A)$ denotes the domain of $A$ equipped with the graph norm $| \cdot |_{A}$ of $A$.

The setting for (1) when $A(t)$ are bounded seems to cover many abstract treatments of Volterra equations although this is not the most general approach. However, it is very difficult to develop a reasonable theory in very general case even for deterministic equation.

As we have already written, the paper has been inspired by a model of a hereditarily-elastic anisotropic aging body with a straight crack, see (Costabel et al. 2004).

The equation (1) is a stochastic version of the deterministic Volterra equation

$$u(t) = \int_{0}^{t} A(t - \tau)u(\tau)d\tau + f(t), \quad t \geq 0, \quad (2)$$

studied extensively in (Prüss 1993).
Definition 1 A family $(\tilde{S}(t))_{t \geq 0} \subset B(H)$ is called pseudo-resolvent for $\mathcal{F}$ if the following conditions are satisfied:

(S1) $\tilde{S}(t)$ is strongly continuous in $H$ on $\mathbb{R}_+$, and $\tilde{S}(0) = I$;

(S2) $U(t) = \int_0^t \tilde{S}(\tau)d\tau$ is leaving $G$ invariant, and $(U(t))_{t \geq 0} \subset B(G)$ is locally Lipschitz on $\mathbb{R}_+$;

(S3) the following resolvent equations hold

$$\tilde{S}(t)y = y + \int_0^t A(t-\tau)dU(\tau)y, \quad t \geq 0, \quad y \in G,$$

$$\tilde{S}(t)y = y + \int_0^t \tilde{S}(t-\tau)A(\tau)y d\tau, \quad t \geq 0, \quad y \in G.$$  

- Equations (3) and (4) are called the first, resp. second resolvent equation for $\mathcal{F}$.

- A pseudo-resolvent $\tilde{S}(t)$ is called resolvent for $\mathcal{F}$ if in addition

(S4) for $y \in G$, $\tilde{S}(-y)g = y$ a.e. and $\tilde{S}(\cdot)y$ is Bochner-measurable in $G$ on $\mathbb{R}_+$.

Comment Because $A(t) \in B(G,H)$ the convolution $A \star dU$ in (3) is well-defined though the function $A(\cdot)y$ is not assumed to be continuous. Indeed, for $y \in G$ the function $g(t) := U(t)y$ is locally Lipschitz in $G$ by the condition (S2) and then $g \in BV_{\text{loc}}(\mathbb{R}_+; G)$. For details, see (Prüss 1993, Section 6).

Let us emphasize that the pseudo-resolvents are always unique. We shall assume in the paper that the equation (1) is well-posed in this sense that (1) admits a pseudo-resolvent $\tilde{S}(t)$, $t \geq 0$. Precise definition of well-posedness is given in (Prüss 1993). That definition is a direct extension of well-posedness of Cauchy problems and Volterra equations of scalar type. The lack of well-posedness leads to distribution resolvents, see e.g. (DaPrato and Iannelli 1984).

In sections 2 and 3 we shall use the following VOLTERA ASSUMPTIONS (abbr. (VA)):

1. $A \in L^1_{\text{loc}}(\mathbb{R}_+; B(G,H));$

2. $\tilde{S}(t)$, $t \geq 0$, are pseudo-resolvent operators corresponding to $\mathcal{F}$.

2 Probabilistic background

We are given a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, $t \geq 0$, with normal filtration and a Wiener process $W$ with the positive linear covariance operator $Q$ in $U$. Assume that $W$ is a cylindrical process, that is $\text{Tr}Q = +\infty$ (if $\text{Tr}Q < +\infty$ then $W$ is a genuine Wiener process). We will need the subspace $U_0 := Q^{1/2}(U)$ of the space $U$, which endowed with the inner product $\langle u, v \rangle_{U_0} := \langle Q^{-1/2}u, Q^{-1/2}v \rangle_U$ forms a Hilbert space.

A linear, bounded operator $B$ acting from $U_0$ into $H$ is called a Hilbert-Schmidt if $\sum_{k=1}^{+\infty} |Bu_k|_H^2 < +\infty$, where $\{u_k\} \subset U_0$ is a base in $U_0$. The set $L_2(U_0, H)$ of all Hilbert-Schmidt operators from $U_0$ into $H$, equipped with the norm $|B|_{L_2(U_0, H)} := (\sum_{k=1}^{+\infty} |Bu_k|_H^2)^{1/2}$, is a separable Hilbert space. For abbreviation we shall denote $L^0_2 := L_2(U_0, H)$.

Assume that $\Psi$ belongs to the class of $L^0_2$-predictable processes satisfying condition

$$P \left( \int_0^T |\Psi(\tau)|^2_{L_2^0} d\tau < +\infty \right) = 1.$$  

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Such processes are called \textit{stochastically integrable} on $[0, T]$ and create a linear space which will be denoted by $\mathcal{N}(0, T; L^0_2)$.

Let $\Phi(t), t \in [0, T]$, be a measurable $L^0_2$-valued process. Define the norms
\[
||\Phi||_t := \left\{ \mathbb{E}\left( \int_0^t |\Phi(\tau)|^2_{L^2_0} d\tau \right) \right\}^{\frac{1}{2}} = \left\{ \mathbb{E} \int_0^t \left[ \text{Tr}(\Phi(\tau)Q^2)(\Phi(\tau)Q^2)^* \right] d\tau \right\}^{\frac{1}{2}}, t \in [0, T].
\]

By $\mathcal{N}^2(0, T; L^0_2)$ we shall denote a Hilbert space of all $L^0_2$-predictable processes $\Phi$ such that $||\Phi||_T < +\infty$.

In the whole paper we shall use the following \textbf{Probability Assumptions} (abbr. (PA)):

1. $X_0$ is an $H$-valued, $\mathcal{F}_0$-measurable random variable;
2. $\Psi \in \mathcal{N}^2(0, T; L^0_2)$, where the interval $[0, T]$ is fixed.

\textbf{Definition 2} Assume that conditions (VA) and (PA) hold. An $H$-valued predictable process $X(t), t \in [0, T]$, is said to be a \textbf{strong solution} to (1), if $X$ has a version such that
\[
P\left( X(t) \in G \right) = 1
\]
for almost all $t \in [0, T]$, $\int_0^T |A(t - \tau)X(\tau)|_H d\tau < +\infty$, $P$-a.s. and for any $t \in [0, T]$ the equation (1) holds $P$-a.s.

\textbf{Definition 3} Let conditions (VA) and (PA) hold. An $H$-valued predictable process $X(t), t \in [0, T]$, is said to be a \textbf{mild solution} to the stochastic Volterra equation (1), if
\[
\mathbb{E}\left( \int_0^t |\tilde{S}(t - \tau)\Psi(\tau)|^2_{L^2_0} d\tau \right) \leq +\infty \text{ for } t \leq T
\]
and, for arbitrary $t \in [0, T]$,
\[
X(t) = \tilde{S}(t)X_0 + \int_0^t \tilde{S}(t - \tau)\Psi(\tau) dW(\tau),
\]
where $\tilde{S}(t)$ is the pseudo-resolvent for the equation (2).

\textbf{Comment} The well-posedness of (1) implies existence and uniqueness of the resolvent $\tilde{S}(t), t \geq 0$, and in the consequence, existence and uniqueness of the mild solution to the equation (1).

\textbf{Proposition 1} Assume that (VA) hold, $B$ is a linear bounded operator acting from $U$ into the space $H$ and
\[
\int_0^T |\tilde{S}(\tau)B|_{L^2_0}^2 d\tau = \int_0^T \text{Tr}[\tilde{S}(\tau)BQB^*\tilde{S}^*(\tau)] d\tau < +\infty,
\]
where $B^*, \tilde{S}^*(\tau)$ are appropriate adjoint operators.

Then
(i) the process $\tilde{W}^B := \int_0^t \tilde{S}(t - \tau)BdW(\tau)$ is Gaussian, mean-square continuous on $[0, T]$ and then has a predictable version;

(ii) \[
\text{Cov} \tilde{W}^B(t) = \int_0^t |\tilde{S}(\tau)BQB^*\tilde{S}^*(\tau)| d\tau, \quad t \in [0, T]; \tag{8}
\]
Proposition 1 is analogous to the known result obtained in the semigroup case. Gaussianity of the process \( \tilde{W}^B \) follows from the definition and properties of stochastic integral. Part (ii) comes from theory of stochastic integral while (iii) follows from the definition of \( \tilde{W}^B \) and the assumption [7].

**Proposition 2** Assume that conditions (VA) hold with operator \( A \in W^{1,1}(0,T; \mathcal{B}(G,H)) \). Let \( X \) be a strong solution to the equation [7] in the case \( \Psi(t) = B \), where \( B \in \mathcal{B}(U,H) \) and trajectories of \( X \) are integrable w.p. 1 on \( [0,T] \). Then, for any function \( \xi \in C^1([0,T]; H^*) \) and \( t \in [0,T] \), the following formula holds

\[
\langle X(t), \xi(t) \rangle_H = \langle X_0, \xi(0) \rangle_H + \int_0^t \langle (\hat{A} \ast X)(\tau) + A(0)X(\tau), \xi(\tau) \rangle_H d\tau
+ \int_0^t \langle \xi(\tau), BdW(\tau) \rangle_H + \int_0^t \langle X(\tau), \xi(\tau) \rangle_H d\tau,
\]

where dots above \( A \) and \( \xi \) mean time derivatives and \( \ast \) means the convolution.

**Proof** First, we consider functions \( \xi(\tau) := \xi_0 \varphi(\tau), \tau \in [0,T] \), where \( \xi_0 \in H^* \) and \( \varphi \in C^1[0,T] \). Denote \( F_{\xi_0}(t) := \langle X(t), \xi_0 \rangle_H, \quad t \in [0,T] \).

Using Itô’s formula to the process \( F_{\xi_0}(t) \varphi(t) \), we obtain

\[
d[F_{\xi_0}(t) \varphi(t)] = \varphi(t) dF_{\xi_0}(t) + \dot{\varphi}(t) F_{\xi_0}(t) dt, \quad t \in [0,T].
\]

Then

\[
dF_{\xi_0}(t) = \langle \int_0^t \dot{A}(t - \tau)X(\tau) d\tau + A(0)X(t), \xi_0 \rangle_H dt + \langle BdW(t), \xi_0 \rangle_H
= \langle (\hat{A} \ast X)(t) + A(0)X(t), \xi_0 \rangle_H dt + \langle BdW(t), \xi_0 \rangle_H.
\]

From (10) and (11),

\[
F_{\xi_0}(t) \varphi(t) = F_{\xi_0}(0) \varphi(0) + \int_0^t \varphi(s) \langle (\hat{A} \ast X)(s) + A(0)X(s), \xi_0 \rangle_H ds
+ \int_0^t \langle \varphi(s)BdW(s), \xi_0 \rangle_H + \int_0^t \dot{\varphi}(s) \langle X(s), \xi_0 \rangle_H ds
= \langle X_0, \xi(0) \rangle + \int_0^t \langle (\hat{A} \ast X)(s) + A(0)X(s), \xi(s) \rangle_H ds
+ \int_0^t \langle BdW(s), \xi(s) \rangle_H + \int_0^t \langle X(s), \dot{\xi}(s) \rangle_H ds.
\]

Hence, we proved the formula (9) for functions \( \xi \) of the form \( \xi(s) = \xi_0 \varphi(s), s \in [0,T] \). Because such functions form a dense subspace in \( C^1([0,T]; H^*) \), the proposition is true. ■
3 Properties of stochastic convolution

In this section we study mild solution to the equation $\Psi$. We use the following notation

$$\tilde{W}^\Psi(t) := \int_0^t \tilde{S}(t-\tau)\Psi(\tau)\,dW(\tau), \quad t \in [0,T], \quad (12)$$

where the condition (5) holds.

**Proposition 3** Assume that $\tilde{S}(t)$, $t \geq 0$, are the pseudo-resolvent operators to the Volterra equation (2). Then, for arbitrary $\Psi \in \mathcal{N}(0,T;L^0_2)$, the process $\tilde{W}^\Psi(t)$, $t \geq 0$, given by (12) has a predictable version.

**Proof** Proof is analogous to construction of stochastic integral, see e.g. (Liptser and Shiryayev 1973).

The process $\tilde{S}(t-\tau)\Psi(\tau)$, $\tau \in [0,T]$, belongs to $\mathcal{N}(0,T;L^0_2)$, because $\Psi \in \mathcal{N}(0,T;L^0_2)$. We may use the following estimate: for arbitrary $a > 0$, $b > 0$ and $t \in [0,T],

$$P(|\tilde{W}^\Psi(t)|_H > a) \leq \frac{b}{a^2} + P \left( \int_0^t |\tilde{S}(t-\tau)\Psi(\tau)|^2_{L^0_2} \,d\tau > b \right). \quad (13)$$

Since pseudo-resolvent operators $\tilde{S}(t)$, $t \geq 0$, are uniformly bounded on compact intervals, there exists a constant $C > 0$, such that $|\tilde{S}(t-\tau)\Psi(\tau)|^2_{L^0_2} \leq C^2 |\Psi(\tau)|^2_{L^0_2}$, $\tau \in [0,T]$.

Then (13) reads

$$P(|\tilde{W}^\Psi(t)|_H > a) \leq \frac{b}{a^2} + P \left( \int_0^t |\Psi(\tau)|^2_{L^0_2} \,d\tau > \frac{b}{C^2} \right). \quad (14)$$

We prove predictability of the process $\tilde{W}^\Psi$ in two steps. In the first step $\Psi$ is an elementary process, so the process $\tilde{W}^\Psi$ has a predictable version by Proposition 1 part (i).

In the second step $\Psi \in \mathcal{N}(0,T;L^0_2)$. There exists a sequence $(\Psi_n)$ of elementary processes that for arbitrary $c > 0$,

$$P \left( \int_0^T |\Psi(\tau) - \Psi_n(\tau)|^2_{L^0_2} \,d\tau > c \right) \rightarrow_{n \rightarrow +\infty} 0. \quad (15)$$

Because, by the previous part of the proof, the sequence $(W_n^{\Psi})$ converges in probability, it has subsequence converging almost surely. This fact implies the predictability of $\tilde{W}^\Psi(t)$, $t \in [0,T]$. $\blacksquare$

**Proposition 4** Assume that $\Psi \in \mathcal{N}^2(0,T;L^0_2)$. Then the process $\tilde{W}^\Psi(t)$, $t \geq 0$, defined by (12) has square integrable trajectories.

**Proof** From Fubini’s theorem and boundness of operators $\tilde{S}(t)$ we obtain

$$\mathbb{E} \int_0^T \left[ \int_0^t |\tilde{S}(t-\tau)\Psi(\tau)dW(\tau)|^2_H \right] \,dt = \int_0^T \mathbb{E} \left[ \int_0^t |\tilde{S}(t-\tau)\Psi(\tau)dW(\tau)|^2_H \right] \,dt = \int_0^T \int_0^t |\tilde{S}(t-\tau)\Psi(\tau)|^2_{L^0_2} \,d\tau \,dt \leq M \int_0^T \int_0^t |\Psi(\tau)|^2_{L^0_2} \,d\tau \,dt < +\infty, \quad t \in [0,T].$$

$\blacksquare$
\textbf{Theorem 1} Assume that (VA) and (PA) hold with \( A \in W^{1,1}(0, T), B(G, H) \), and \( X \) is an \( H \)-valued predictable process. Let \( \bar{S}(t) \) be a weak analytic resolvent for \( \bar{A} \). Then strong solution to (1) is a mild solution, that is
\[
X(t) = X_0 + \tilde{W}(t) \quad t \in [0, T].
\]

\textbf{Proof} By Proposition 2, for any \( \xi \in C^1([0, T], H^*) \) and \( t \in [0, T] \), the following equation holds
\[
\langle X(t), \xi(t) \rangle_H = \langle X_0, \xi(0) \rangle_H + \int_0^t \langle (\ddot{A} \ast X)(\tau) + A(0)X(\tau), \xi(\tau) \rangle_H d\tau
\]
\[
+ \int_0^t \langle \Psi(\tau)dW(\tau), \xi(\tau) \rangle_H + \int_0^t \langle X(\tau), (\tilde{S}^*(t - \tau)\xi)' \rangle_H d\tau, \quad P - \text{a.s.}
\]

Let us take \( \xi(\tau) := \tilde{S}^*(t - \tau)\zeta \), for \( \zeta \in H^*, \tau \in [0, t] \).

Now, the equation (17) may be written like
\[
\langle X(t), \tilde{S}^*(0)\zeta \rangle_H = \langle X_0, \tilde{S}^*(0)\zeta \rangle_H + \int_0^t \langle (\ddot{A} \ast X)(\tau) + A(0)X(\tau), \tilde{S}^*(t - \tau)\zeta \rangle_H d\tau
\]
\[
+ \int_0^t \langle \Psi(\tau)dW(\tau), \tilde{S}^*(t - \tau)\zeta \rangle_H + \int_0^t \langle X(\tau), (\tilde{S}^*(t - \tau)\zeta)' \rangle_H d\tau,
\]
where derivative (\( \cdot \)'\) in the last term is taken over \( \tau \).

Since \( \tilde{S}^*(0) = I \), we obtain
\[
\langle X(t), \zeta \rangle_H = \langle S(t)X_0, \zeta \rangle_H + \int_0^t \langle \tilde{S}(t - \tau) \int_0^\tau \dot{A}(\tau - \sigma)X(\sigma)d\sigma + A(0)X(\tau) \rangle_H d\tau
\]
\[
+ \int_0^t \langle \tilde{S}(t - \tau)\Psi(\tau)dW(\tau), \zeta \rangle_H + \int_0^t \langle \tilde{S}(t - \tau)X(\tau), \zeta \rangle_H d\tau
\]
for any \( \zeta \in H^* \).

Let us analyze the right hand side of (18). From the properties of convolution and (S3),
\[
\int_0^t \dot{\tilde{S}}(t - \tau)X(\tau)d\tau = -\int_0^t \tilde{S}(t - \tau)X(t - \tau)d\tau = -\int_0^t \left[ \int_0^\tau \dot{A}(\tau - s)U(s) \right]'X(t - \tau)d\tau
\]
\[
= -\int_0^t \left[ \int_0^\tau \dot{A}(\tau - s)dU(s) + A(0)\tilde{S}(\tau) \right]X(t - \tau)d\tau.
\]
We have
\[
\int_0^t A(0)\tilde{S}(t - \tau)X(\tau)d\tau = \int_0^t A(0)\tilde{S}(\tau)X(t - \tau), \quad d\tau
\]
and, from (S2),
\[
\int_0^\tau \dot{A}(\tau - s)dU(s) = \int_0^\tau \dot{A}(\tau - s)U'(s)ds = \int_0^\tau \dot{A}(\tau - s)\tilde{S}(s)ds.
\]
Hence
\[
\int_0^t \tilde{S}(t - \tau) \left[ \int_0^\tau \dot{A}(\tau - \sigma)X(\sigma)d\sigma \right]d\tau = \int_0^t \tilde{S}(t - \tau)(\dot{A} \ast X)(\tau)d\tau
\]
\[
= (\tilde{S} \ast (\dot{A} \ast X))(t) = ((\tilde{S} \ast \dot{A})(\tau) \ast X)(t) =
\]
\[
= \int_0^t (\tilde{S} \ast \dot{A})(\tau)X(t - \tau)d\tau = \int_0^t \left[ \int_0^\tau \tilde{S}(\tau)\dot{A}(\tau - s)ds \right]X(t - \tau)d\tau.
\]
So, the right hand side of (18) reduces to \( \int_0^t \langle \tilde{S}(t-\tau)\Psi(\tau)dW(\tau), \zeta \rangle_H \), what means that (16) holds.

**Theorem 2** Assume that \( A \in L^1_{\text{loc}}([0,T]; \mathcal{B}(G,H)) \) and \( \tilde{S}(t), \; t \geq 0, \) is a resolvent to the equation (2). If \( \Psi \in N^2(0,T; L^0_2) \), then the stochastic convolution \( \tilde{W}\Psi \) fulfills the following equation

\[
\tilde{W}\Psi(t) = \int_0^t A(t-\tau) \tilde{W}\Psi(\tau)\,d\tau + \int_0^t \Psi(\tau)\,dW(\tau). \tag{19}
\]

**Proof** Let us notice that the process \( \tilde{W}\Psi \) has integrable trajectories. Then, from Dirichlet’s formula and stochastic Fubini’s theorem

\[
\int_0^t A(t-\tau) \tilde{W}\Psi(\tau)\,d\tau = \int_0^t A(t-\tau) \tilde{S}(\tau-\sigma)\Psi(\sigma)\,dW(\sigma)\,d\tau =
\]

\[
= \int_0^t \left[ \int_0^t A(t-\tau) \tilde{S}(\tau-\sigma)\,d\tau \right] \Psi(\sigma)dW(\sigma) =
\]

\[
(z := \tau-\sigma) = \int_0^t \left[ \int_0^{t-\sigma} A(t-\sigma-z) \tilde{S}(z)\,dz \right] \Psi(\sigma)dW(\sigma) =
\]

\[
= \int_0^t (A \ast \tilde{S})(t-\sigma)\Psi(\sigma)\,dW(\sigma) = \int_0^t \tilde{S}(t-\sigma-I)\Psi(\sigma)\,dW(\sigma) =
\]

\[
= \int_0^t \tilde{S}(t-\sigma)\Psi(\sigma)\,dW(\sigma) - \int_0^t \Psi(\sigma)dW(\sigma).
\]

What gives the theorem.

**Case** \( A(t) = a(t)A \)

In this section we consider the case when \( A(t) = a(t)A \) and, in contrary to the previous assumptions, \( A \) is a closed unbounded operator in \( H \) with a dense domain \( D(A) \). Then the equation (11) has the form

\[
X(t) = X_0 + \int_0^t a(t-\tau)A X(\tau)\,d\tau + \int_0^t \Psi(\tau)\,dW(\tau), \; t \in [0,T]. \tag{20}
\]

For details, precise setting and several results concerning (20) we refer to (Karczewska, 2005). Because the results analogous to those from sections 2 and 3 are formulated there, we do not repeat them here.

We denote

\[
W_\Psi(t) := \int_0^t S(t-\tau)\Psi(\tau)dW(\tau), \; t \in [0,T];
\]

now, the operators \( S(t), \; t \geq 0, \) become resolvents (see (Prüss 1993) for definition and properties).

In this case, the equality (19) reads

\[
W_\Psi(t) = \int_0^t a(t-\tau)A W_\Psi(\tau)\,d\tau + \int_0^t \Psi(\tau)\,dW(\tau). \tag{21}
\]

Now, although the equation (20) is formally simpler than (11), the case is interesting because \( A \) is unbounded. In consequence, we are not able to obtain the equality (21) directly like in Theorem 2.
Proposition 5 If $\Psi \in \mathcal{N}^2(0, T; L^2_0)$ and $\Psi(\cdot, \cdot)(U_0) \subset D(A)$, $P$-a.s., then the stochastic convolution $W^\Psi$ fulfills the equation
\[
\langle W^\Psi(t), \xi \rangle_H = \int_0^t \langle a(t-\tau)W^\Psi(\tau), A^*\xi \rangle_H + \int_0^t \langle \xi, \Psi(\tau)dW(\tau) \rangle_H, \quad P - a.s.,
\]
for any $t \in [0, T]$ and $\xi \in D(A^*)$.

Comment Proposition 5 states that in case $A(t) = a(t)A$, a mild solution to (20) is a weak solution to (22).

This is well-known (see, e.g. (Prüss 1993)) that the scalar equation corresponding to (20) is
\[
s(t) + \mu(a * s)(t) = 1. \tag{22}
\]

Definition 4 We say that function $a \in L^1(0, T)$ is completely positive on $[0, T]$ if for any $\mu \geq 0$, the solution to (22) satisfies $s(t) \geq 0$ on $[0, T]$.

Proposition 6 (Clemént and Nohel 1979) Assume that $a \in L^1(0, T)$ and $a$ is nonnegative and nonincreasing on $[0, T]$. Then $a$ is completely positive on $[0, T]$.

Comment If $a \in L^1(0, T)$ and is completely monotonic on $[0, T]$, i.e. $(-1)^ka^{(k)}(t) \geq 0$, $t \in (0, T)$, $k = 0, 1, \ldots$, then $a$ is completely positive on $[0, T]$.

Proposition 7 Assume that $A$ is $m$-accretive operator and function $a$ is completely positive on $[0, T]$. If $\Psi$ and $A\Psi$ belong to $\mathcal{N}^2(0, T; L^2_0)$ and in addition $\Psi(\cdot, \cdot)(U_0) \subset D(A)$, $P$-a.s., then the equality (21) holds.

Proof The operator $A$ is $m$-accretive if an only if it generates the linear strongly continuous semigroup of contractions, see e.g. (Zheng 2004). So, we may use results due to (Clemént and Nohel 1979).

Denote by $A_\lambda := \frac{1}{\lambda}(I - J_\lambda)$, where $J_\lambda = (I + \lambda A)^{-1}$, $\lambda \geq 0$, the Yosida approximation of the operator $A$. By $S_\lambda(t), t \geq 0$, we denote the resolvent operators corresponding to the Volterra equation (20) with the operator $A_\lambda$ instead of the operator $A$. The paper (Clemént and Nohel 1979) provides the convergence $\lim_{\lambda \to 0^+} S_\lambda(t)x = S(t)x$ for $t \in [0, T]$ and $x \in D(A)$.

Let us recall, that the formula (21) holds for any bounded operator $A$. Then it holds for operators $A_\lambda$, too:
\[
W^\Psi_\lambda(t) = \int_0^t A_\lambda W^\Psi_\lambda(\tau)d\tau + \int_0^t \Psi(\tau)dW(\tau),
\]
where
\[
W^\Psi_\lambda(t) := \int_0^t S_\lambda(t - \tau)\Psi(\tau)dW(\tau).
\]
Because $A_\lambda x = AJ_\lambda x$ for $x \in D(A)$, then
\[
A_\lambda W^\Psi_\lambda(t) = A_\lambda \int_0^t S_\lambda(t - \tau)\Psi(\tau)dW(\tau) = J_\lambda \int_0^t S_\lambda(t - \tau)A\Psi(\tau)dW(\tau).
\]
Now, from the properties of stochastic integral and by the Lebesgue dominated convergence theorem, we obtain
\[
\lim_{\lambda \to 0^+} \sup_{t \in [0,T]} \mathbb{E}|W^\Psi_\lambda(t) - W^\Psi(t)|_H^2 = 0
\]
and
\[
\lim_{\lambda \to 0^+} \sup_{t \in [0,T]} \mathbb{E}|A_\lambda W^\Psi_\lambda(t) - AW^\Psi(t)|_H^2 = 0
\]
what gives the required result.

Definition 5 Suppose \(S(t), t \geq 0\), is a resolvent for (24). \(S(t)\) is called **exponentially bounded** if there are constants \(M \geq 1\) and \(w \in \mathbb{R}\) that
\[
||S(t)|| \leq Me^{wt} \text{ for all } t \geq 0.
\]

In contrary to the case of semigroups, resolvents (if they exist) need not to be exponentially bounded even if the kernel \(a\) belongs to \(L^1(\mathbb{R}_+)\). Existence of such resolvents is given, e.g. by Theorem 1.3 in (Prüss 1993). An important class of kernels providing exponentially bounded resolvent are \(a(t) = t^{\alpha - 1}/\Gamma(\alpha), \alpha \in (0, 2)\) or the class of completely monotonic functions.

Proposition 8 Assume that \(A\) is the infinitesimal generator of a \(C_0\)-semigroup which is exponentially bounded and the kernel \(a\) is a completely positive function. If \(\Psi\) and \(A\Psi\) belong to \(N^2(0,T;L_2^0)\) and in addition \(\Psi(\cdot, \cdot)(U_0) \subset D(A), P\text{-a.s.}\), then the equality (21) holds.

The proof bases on the following auxiliary result.

Lemma 1 (Karczewska and Lizama 2005) Let \(A\) be the infinitesimal generator of a \(C_0\)-semigroup \(T(t), t \geq 0\), satisfying \(||T(t)|| \leq Me^{\omega t}\) with \(M \geq 1, \omega \in \mathbb{R}\) and suppose that \(a \in L^1_{\text{loc}}(\mathbb{R}_+)\) is a completely positive function. Then \((A,a)\) admits an exponentially bounded resolvent \(S(t)\). Moreover, there exists bounded operators \(A_n\), such that \((A_n,a)\) admits resolvent family \(S_n(t)\) satisfying \(||S_n(t)|| \leq Me^{\omega_0 t}, \omega_0 \in \mathbb{R}\), for all \(t \geq 0, n \in \mathbb{N}\), and \(S_n(t)x \to S(t)x\) for all \(x \in X, t \geq 0\). Moreover, the convergence is uniform in \(t\) on every compact subset of \(\mathbb{R}_+\).

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