On duality and fractionality of multicommodity flows in directed networks

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ABSTRACT

In this paper we address a topological approach to multiflow (multicommodity flow) problems in directed networks. Given a terminal weight \( \mu \), we define a metrized polyhedral complex, called the directed tight span \( T_\mu \), and prove that the dual of the \( \mu \)-weighted maximum multiflow problem reduces to a facility location problem on \( T_\mu \). Also, in case where the network is Eulerian, it further reduces to a facility location problem on the tropical polytope spanned by \( \mu \).

By utilizing this duality, we establish the classifications of terminal weights admitting a combinatorial min–max relation (i) for every network and (ii) for every Eulerian network. Our result includes the Lomonosov–Frank theorem for directed free multiflows and Ibaraki–Karzanov–Nagamochi’s directed multiflow locking theorem as special cases.

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1. Introduction

A network \((G, S, c)\) is a triple of a directed graph \( G = (VG, EG) \), a specified set \( S \subseteq VG \) of nodes called terminals, and a nonnegative integer-valued edge-capacity \( c : EG \rightarrow \mathbb{Z}_+ \). An \( S \)-path is a (directed) path joining distinct terminals. A multiflow (multicommodity flow) is a pair \((\mathcal{P}, \lambda)\) of a set \( \mathcal{P} \) of \( S \)-paths and a nonnegative flow-value function \( \lambda : \mathcal{P} \rightarrow \mathbb{R}_+ \) satisfying the capacity constraint: \( \sum \{ \lambda(P) | P \in \mathcal{P}, P \text{ contains } e \} \leq c(e) \) for \( e \in EG \). Given a nonnegative terminal weight \( \mu : S \times S \rightarrow \mathbb{R}_+ \), the flow-value \( \text{val}(\mu, f) \) of multiflow \( f = (\mathcal{P}, \lambda) \) is defined by \( \sum \{ \lambda(P)\mu(s_P, t_P) | P \in \mathcal{P} \} \), where \( s_P \) and \( t_P \) denote the start node and the end node of \( P \), respectively. Then the \( \mu \)-weighted maximum multiflow problem is formulated as:

\[ \text{\( \mu \)-MFP: Maximize } \text{val}(\mu, f) \text{ over all multiflows } f \text{ in } (G, S, c) \text{.} \]

For a special terminal weight \( \mu \), the \( \mu \)-MFP has a nice integrality property. For example, consider \( S = \{s, t\} \) and \((\mu(s, t), \mu(t, s)) = (1, 0)\). Then the maximum multiflow problem says that the maximum flow value is equal to the minimum \( (s, t) \)-cut value and there always exists an integral maximum flow (maximum flow \((\mathcal{P}, \lambda)\) for which \( \lambda \) is integer-valued). Consider the case where \( \mu(s, t) = 1 \) for all distinct \( s, t \in S \), and network is Eulerian. Lomonosov [1] and Frank [2] independently proved that the maximum flow value is equal to the sum of the minimum \( (s, S \setminus s) \)-cut value over \( s \in S \) and there exists an integral maximum multiflow.
The goal of this paper is to classify weight functions $\mu : S \times S \to \mathbb{R}_+$ for which $\mu$-MFP possesses such a combinatorial min–max relation. This classification problem, called the fractionality problem, was raised by Karzanov for the undirected $\mu$-MFP ($G$ is undirected and $\mu$ is symmetric); see [3]. It is well known that the LP-dual to $\mu$-MFP is a linear optimization over metrics on node set $V_G$. In the 90’s, Karzanov [4,5] found a remarkable fact that all possible candidates of optimal metrics are embedded into a metric space on a polyhedral complex associated with $\mu$. This polyhedral complex is known as the tight span, which was earlier introduced by Isbell [6] and Dress [7] independently. Then the LP-dual reduces to a facility location problem on the tight span. Furthermore, if the tight span has a sufficiently nice geometry (dimension at most two), then one can obtain a combinatorial min–max relation from its shape. Otherwise (dimension at least three), one can conclude that $\mu$-MFP has no such a combinatorial duality relation. Recently, this beautiful theory was further extended by the first author, and the fractionality problem for the undirected $\mu$-MFP was roughly settled [8–10].

Our previous paper [11] started to develop an analogous duality theory for directed multiflows. In the directed case, the LP-dual is a linear optimization over possibly asymmetric metrics, which we call directed metrics. We introduced a directed version $T_\mu$ of the tight span (directed tight span). In the case of metric $\mu$-MFP ($\mu$ is a directed metric), we showed that the LP-dual reduces to a facility location problem on $T_\mu$; see [11, Section 4]. Moreover, in the case where a network is Eulerian, this LP-dual further reduces to a facility location problem on the tropical polytope $\mathcal{Q}_\mu$, spanned by $\mu$, which was earlier introduced by Develin–Sturmfels [12] in the context of the tropical geometry.

The main contribution of this paper extends this duality theory for possibly nonmetric weights and solves the fractionality problems (i) for $\mu$-MFP and (ii) for Eulerian $\mu$-MFP (which is $\mu$-MFP on an Eulerian network). In Section 2, we establish a general duality relation for $\mu$-MFP with a possibly nonmetric weight $\mu$. As well as the metric case, the LP-dual reduces to a facility location problem on the directed tight span $T_\mu$ (Theorem 2.2). However, in the Eulerian case, we need a more careful treatment for the nonmetricity of $\mu$. We introduce the slimmed tropical polytope $\mathcal{Q}_\mu^{\text{slim}}$, which is a certain subset of the tropical polytope and coincides with it if $\mu$ is a metric. Then we prove that the LP-dual to an Eulerian $\mu$-MFP reduces to a facility location on $\mathcal{Q}_\mu^{\text{slim}}$ (Theorem 2.4). In Section 3, we show the integrality theorem (Theorem 3.1) that (i) if $\dim T_\mu \leq 1$, then every $\mu$-MFP has an integral optimal multiflow, and (ii) if $\dim \mathcal{Q}_\mu^{\text{slim}} \leq 1$ then every Eulerian $\mu$-MFP has an integral optimal multiflow. We remark that the former result can be proved by a reduction to the minimum cost circulation. The second result includes the Lomonosov–Frank theorem for directed free multiflows [1,2] and Ibaraki–Karzanov–Nagamochi’s directed version of the multiflow locking theorem [13] as special cases. We give a combinatorial characterization of weights $\mu$ with $\dim \mathcal{Q}_\mu^{\text{slim}} \leq 1$ in terms of oriented trees (Theorem 3.4), and explain a relationship among these results. In Section 4, we show that the one-dimensionality of the directed tight span and the slimmed tropical polytope are best possible for the integrality. Theorem 4.1 says that if $\dim T_\mu \geq 2$, then there is no positive integer $k$ such that every $\mu$-MFP has a $1/k$-integral optimal multiflow, and if $\dim \mathcal{Q}_\mu^{\text{slim}} \geq 2$, then there is no positive integer $k$ such that every Eulerian $\mu$-MFP has a $1/k$-integral optimal multiflow.

**Notation.** The sets of real numbers and nonnegative real numbers are denoted by $\mathbb{R}$ and $\mathbb{R}_+$, respectively. The set of functions from a set $X$ to $\mathbb{R}$ (resp. $\mathbb{R}_+$) is denoted by $\mathbb{R}^X$ (resp. $\mathbb{R}_+^X$). For a subset $Y \subseteq X$, the characteristic function $1_Y \in \mathbb{R}^X$ is defined by $1_Y(x) = 1$ for $x \in Y$ and $1_Y(x) = 0$ for $x \notin Y$. We particularly denote by $1$ the all–one function in $\mathbb{R}^X$. For $p, q \in \mathbb{R}^X, p \leq q$ means $p(x) \leq q(x)$ for each $x \in X$, and $p < q$ means $p(x) < q(x)$ for each $x \in X$. For $p \in \mathbb{R}^X, (p)_+ \in \mathbb{R}^X$ is defined by $(p)_+(x) = \max\{p(x), 0\}$ for each $x \in X$. For a set $P$ in $\mathbb{R}^X$, a point $p$ in $P$ is said to be *minimal* if there is no other point $q \in P \setminus \{p\}$ with $p \leq q$.

For a set $S$, a nonnegative real-valued function $d$ on $S \times S$ having zero diagonals $d(s, s) = 0(s \in S)$ is called a directed distance. We regard a terminal weight $S \times S \to \mathbb{R}_+$ as a directed distance. A directed distance $d$ on a set $S$ is called a directed metric if it satisfies the triangle inequality $d(s, t) + d(t, u) \geq d(s, u)$ for every triple $s, t, u \in S$. A directed metric space is a pair $(S, \mu)$ of a set $S$ and a directed metric $d$ on $S$. For a directed metric $d$ on $V$, and two subsets $A, B \subseteq V$, let $d(A, B)$ denote the minimum distance from $A$ to $B$:

$$d(A, B) = \inf\{d(x, y) \mid (x, y) \in A \times B\}.$$

In our theory, the following directed metric $D^\infty_\mu$ on $\mathbb{R}^X$ is particularly important:

$$D^\infty_\mu(p, q) = \| (q - p)_+ \|_\infty \quad (= \max_{x \in X} (q(x) - p(x))_+) \quad (p, q \in \mathbb{R}^X).$$

We remark that $D^\infty_\mu(p, q) = 0$ whenever $p \geq q$.

For a directed or undirected graph $G$, its node set and edge set are denoted by $V_G$ and $E_G$, respectively. If directed, an edge with tail $x$ and head $y$ is denoted by $xy$. If undirected, we do not distinguish $xy$ and $yx$. In a network $(G, S, c)$, a non-terminal node is called an inner node. For a node $x \in V_G$, we say “$x$ fulfills the Eulerian condition" if the sum of the capacities $c(xy)$ over edges $xy$ leaving $x$ is equal to that over edges entering $x$. A network $(G, S, c)$ is said to be Eulerian if every inner node fulfills the Eulerian condition, and is said to be totally Eulerian if every node fulfills the Eulerian condition.

A directed distance and directed metric is often simply called a distance and a metric, respectively.
2. Duality

Let \((G, S, c)\) be a network and let \(\mu\) be a directed distance on \(S\). We denote by \(\text{MFP}^*(\mu; G, S, c)\) the optimal value of \(\mu\)-MFP for \((G, S, c)\). The linear programing dual to \(\mu\)-MFP is given by

\[
\text{LPD} : \quad \text{Minimize } \sum_{xy \in EG} c(xy)d(x, y) \\
\text{subject to} \quad d \text{ is a directed metric on } VG, \\
d(s, t) \geq \mu(s, t) \quad (s, t \in S).
\]

We are going to represent LPD as a facility location problem on a metrized polyhedral complex associated with \(\mu\). Let \(S^c\) and \(S^r\) be copies of \(S\). For an element \(s \in S\), the corresponding elements in \(S^c\) and \(S^r\) are denoted by \(s^c\) and \(s^r\), respectively. We denote \(S^c \cup S^r\) by \(S^\mu\). For a point \(p \in \mathbb{R}^{S^\mu}\), the restrictions of \(p\) to \(S^c\) and \(S^r\) are denoted by \(p^c\) and \(p^r\), respectively, i.e., \(p = (p^c, p^r)\). Consider the following unbounded polyhedron in \(\mathbb{R}^{S^\mu}\):

\[
P_\mu = \{ p \in \mathbb{R}^{S^\mu} \mid p(s^c) + p(t^r) \geq \mu(s, t) \quad (s, t \in S) \}.
\]

Let \(D^\infty\) be a directed metric on \(\mathbb{R}^{S^\mu}\) defined as

\[
D^\infty(p, q) = \max\{D^c_\infty(p^c, q^c), D^r_\infty(q^r, p^r) \} \quad (p, q \in \mathbb{R}^{S^\mu}).
\]

We endow \(P_\mu^+\) and its subsets with this directed metric. For a subset \(R\) in \(P_\mu^+\), we denote by \((R)^+ = R^+\) the set of nonnegative points in \(R\). Also for \(s \in S\) we denote by \((R)_s = R_s\) the set of points \(p \in R\) with \(p(s^c) + p(s^r) = \mu(s, s) = 0\); if \(R \subseteq \mathbb{R}^{S^\mu}\) then \(R_s\) is the set of points \(p \in R\) with \(p(s^c) = p(s^r) = 0\).

For a subset \(R \subseteq P_\mu^+\), consider the following facility location problem on \(R\):

\[
\text{FLP} : \quad \text{Minimize } \sum_{xy \in EG} c(xy)D^\infty(\rho(x), \rho(y)) \\
\text{subject to} \quad \rho : VG \rightarrow R, \\
\rho(s) \in R_s \quad (s \in S).
\]

Let \(\text{FLP}^*(R; G, S, c)\) denote the minimum value of this problem. Then the following weak/strong duality holds:

**Lemma 2.1.** For a network \((G, S, c)\) and a directed distance \(\mu\) on \(S\), we have

1. \(\text{MFP}^*(\mu; G, S, c) \leq \text{FLP}^*(R; G, S, c)\) for any subset \(R \subseteq P_\mu^+\), and
2. \(\text{MFP}^*(\mu; G, S, c) = \text{FLP}^*(P_\mu^+; G, S, c)\).

**Proof.** We first note the following property:

For \(s, t \in S\) and \((p, q) \in R_s \times R_t\), we have \(D^\infty(p, q) \geq \mu(s, t)\).

Indeed, \(D^\infty(p, q) \geq (q(s^c) - p(s^c))^+ \geq (q(s^c) + q(t^r) \geq \mu(s, t)\), where we use \(p(s^c) = q(t^r) = 0\).

It suffices to prove (2). Take a map \(\rho : VG \rightarrow P_\mu^+\) feasible to \(\text{FLP}\). Let \(d\) be a metric on \(VG\) defined by \(d(x, y) = D^\infty(\rho(x), \rho(y))\). By (2.1), we have \(d(s, t) = D^\infty(\rho(s), \rho(t)) \geq \mu(s, t)\) for \(s, t \in S\). Thus \(d\) is feasible to LPD with the same objective value. Conversely, take a metric \(d\) feasible to LPD. Define \(\rho : VG \rightarrow P^S_\mu\) by \((\rho(s))(s^c) = d(s, x)\) for \(s \in S\). Then \((\rho(s))(s^c) + (\rho(s))(t^r) = d(s, x) + d(x, t) \geq d(s, t) \geq \mu(s, t)\) holds. Hence \((\rho(s))(s^c) \geq (\rho(s))(s^c) = d(s, s) = 0\). Thus \(\rho\) is feasible to \(\text{FLP}\) for \(R = P_\mu^+\). By the triangle inequality we have \(D^\infty(\rho(s), \rho(s')) = \max\{\max_{t \in S}(d(s, t) - d(s, x)), \max_{t \in S}(d(x, t) - d(y, t))\} \leq d(x, y)\). Since \(c\) is nonnegative, we have \(\sum_{xy \in EG} c(xy)D^\infty(\rho(x), \rho(y)) \leq \sum_{xy \in EG} c(xy)d(x, y)\).

\[\square\]

In the following, we are going to determine “reasonably small” subsets \(R \subseteq P_\mu^+\) for which the strong duality holds (i) for general networks and (ii) for Eulerian networks. In the next subsection (Section 2.1), we introduce the directed tight span \(T_\mu\) and a fiber \(Q_\mu\) of the tropical polytope as subsets in \(P_\mu\), and list their fundamental properties, shown by our previous paper [11]. In Section 2.2, we show that the strong duality holds for \(R = T_\mu\) in every network (Theorem 2.2). We introduce the notion of a slimmed section and show that the strong duality holds for a slimmed section \(R \subseteq Q_\mu\) in every Eulerian network (Theorem 2.4).

### 2.1. Preliminaries: Tight spans and tropical polytopes

Consider the following (non-convex) polyhedral subsets in \(P_\mu^+\):

\[
T_\mu = \text{the set of minimal elements of } P_\mu^+, \\
Q_\mu = \text{the set of minimal elements of } P_\mu.
\]

We call \(T_\mu\) the directed tight span. The polyhedron \(P_\mu\) has the linearity space \((1, -1)\mathbb{R}\). The projection \(\tilde{Q}_\mu := Q_\mu/(1, -1)\mathbb{R}\) is known as the tropical polytope generated by matrix \((-\mu(s, t) \mid s, t \in S)\); see Devlin and Sturmfels [12]. We note the
relation: $Q_μ ⊆ Q_μ^+ ⊆ T_μ ⊆ P_μ$. In the inclusions, $T_μ$ is a subcomplex (i.e., a union of faces) of $P_μ$, and $Q_μ^+$ is a subcomplex of $T_μ$. A subset $R ⊆ Q_μ$ is called a section if the projection $p ∈ R \mapsto \bar{p} ∈ \bar{Q}_μ$ is bijective. A subset $R ⊆ R^{5\mu}$ is said to be balanced if there is no pair $p, q$ of points in $R$ such that $p' < q'$ or $p' < q''$. In fact, the projection $Q_μ^+ \mapsto \bar{Q}_μ$ is surjective and there always exists a balanced section in $Q_μ^+$ [11, Lemma 2.4]. Fig. 1 illustrates $Q_μ$, $Q_μ^+$, and $\bar{Q}_μ$ for the all–one distance on a 3-set $\{s, t, u\}$. In this case, $T_μ = Q_μ^+$ holds, $Q_μ$ consists of three infinite strips with a common side, $Q_μ^+$ is a folder consisting of three triangles, and $\bar{Q}_μ$ is a star of three leaves.

The rest of this subsection is devoted to listing basic properties of these polyhedral sets. They were proved in [11, Section 2]. The most important property for us is the existence of nonexpansive retractions among them.

A. Nonexpansive retractions. For two directed metric spaces $(V, d)$ and $(V', d')$, a map $ϕ : V \mapsto V'$ is said to be nonexpansive if $d'(ϕ(x), ϕ(y)) ≤ d(x, y)$ for all pairs $x, y \in V$. By a cycle $C$ of $V$ we mean a cyclic permutation $(x_1, x_2, \ldots, x_n)$ of a finite (multi-)set in $V$. Its length $d(C)$ is defined by $d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n) + d(x_n, x_1)$. Also, $ϕ : V \mapsto V'$ is said to be cyclically nonexpansive if $d'(ϕ(C)) ≤ d(C)$ for all cycles $C$ in $V$. A map from a set $V$ to its subset $S \subseteq V$ is said to be a retraction if it is the identity on $S$.

(1) There exists a nonexpansive retraction $ϕ : P^+_μ \mapsto T_μ$ with $ϕ(p) ≤ p$ for $p ∈ P^+_μ$. (2.2)

(2) There exists a cyclically nonexpansive retraction $ϕ : T_μ \mapsto Q_μ^+$. (2.3)

(3) For any balanced section $R ⊆ Q_μ$, the retraction $ϕ_R : Q_μ \mapsto R$ determined by the relation

$$\varphi_R(p) - p ∈ (1, -1)R \ (p ∈ Q_μ)$$

is cyclically nonexpansive.

See Fig. 2 for the retraction in (3).

B. Geodesics and embedding. A path $P ⊆ R^{5\mu}$ is the image of a continuous map $ϕ : [0, 1] \mapsto R^{5\mu}$. The length of $P$ from $ϕ(0)$ to $ϕ(1)$ is defined by the supremum of $\sum_{i=0}^{n-1} D(ϕ(t_i), ϕ(t_{i+1}))$ over all $n > 0$ and $0 = t_0 < t_1 < \cdots < t_n = 1$. For simplicity, we restrict $ϕ$ to be sufficiently nice: $ϕ$ is injective and its length is finite. A subset $R ⊆ R^{5\mu}$ is said to be geodesic if each pair $p, q ∈ R$ of points is joined by a path in $R$ having length $D_∞(p, q)$ from $p$ to $q$.

$$T_μ, Q_μ, Q_μ^+ and any balanced section in Q_μ are all geodesic. (2.3)$$

For $s ∈ S$, let $μ_s$ be the point in $R^{5\mu}$ defined by

$$μ_s(t) = (μ(t, s), μ(s, t)) \ (t ∈ S). (2.4)$$
Namely, $\mu_s$ is composed by the $s$-th column and $s$-th row vectors of $\mu$ (as a matrix).

1. For any balanced section $R$ in $Q^+_\mu$ we have $R_1 = (Q^+_\mu)_s = (T_{\mu})_s$.
2. For $s, t \in S$ we have $D_\infty(T_{\mu}), (T_{\mu})_t = \mu(s, t)$.
3. If $\mu$ is a metric, then $(T_{\mu})_s = \{\mu_s\}$ for each $s \in S$.

In particular, if $\mu$ is a metric, then metric space $(S, \mu)$ is isometrically embedded into any balanced section $R$ in $Q^+_\mu$ by $s \mapsto \mu_s$.

C. Tight extensions. For a metric $\mu$ on $S$, an extension of $\mu$ is a metric $d$ on $V$ with $S \subseteq V$ and $d(s, t) = \mu(s, t)$ for $s, t \in S$, and it is said to be tight if there is no other extension $d'$ on $V$ with $d' \neq d$ and $d' \leq d$. Also an extension $d$ on $V$ of $\mu$ is said to be cyclically tight if there is no other extension $d'$ on $V$ such that $d'(C) \leq d(C)$ for all cycles $C$ in $V$ and $d'(C) < d(C)$ for some cycle $C$. Every cyclically tight extension is a tight extension. The converse is not true. For example, $S = \{s, t\}$, $\mu(s, t) = \mu(t, s) = 1$, $V = \{s, t, u, v\}$, and consider two extensions $d, d'$ on $V$ defined by

\[
\begin{array}{cccc}
s & t & u & v \\
0 & 1 & 1 & 0 \\
t & 1 & 0 & 1 \\
u & 0 & 0 & 0 \\
v & 1 & 1 & 0 \\
\end{array} \quad \begin{array}{cccc}
s & t & u & v \\
0 & 1 & 1 & 1 \\
t & 1 & 0 & 1 \\
u & 0 & 0 & 0 \\
v & 0 & 0 & 0 \\
\end{array}
\]

Then $d$ is tight, but not cyclically tight. Indeed, for every pair $(x, y) \in V \times V \setminus S \times S$ if $d(x, y) > 0$, then $d(s, t) = d(s, x) + d(x, y) + d(y, t)$ or $d(s, t) = d(t, s) = d(t, x) + d(x, y) + d(y, s)$; this means that we cannot decrease $d(x, y)$ keeping the triangle inequality. Compare $d'$ with $d$. Then $d'(C) \leq d(C)$ for all cycles $C$, and $0 = d'(u, v) + d'(v, u) < d(u, v) + d(v, u) = 1$. Thus $d'$ is not cyclically tight.

Every tight extension and cyclically tight extension are embedded into $(T_{\mu}, D_\infty)$ and $(Q^+_\mu, D_\infty)$, respectively. We use this fact in Section 4.

Let $\mu$ be a metric on $S$ and $d$ its extension on $V$.

1. $d$ is tight if and only if there is an isometric embedding $\rho : V \rightarrow T_{\mu}$ such that $\rho(s) = \mu_s$ for each $s \in S$.
2. $d$ is cyclically tight if and only if there is an isometric embedding $\rho : V \rightarrow Q^+_\mu$ such that $\rho(s) = \mu_s$ for each $s \in S$ and $\rho(V)$ is balanced.

Here an isometric embedding from $(V, d)$ to $(V', d')$ is a map $\rho : V \rightarrow V'$ satisfying $d'((\rho(x), \rho(y)) = d(x, y)$ for all $x, y \in V$.

D. Further technical stuff. For a point $p \in P_{\mu}$, let $K_{\mu}(p) = K(p)$ denote the bipartite graph on $S^\ast$ with edge set $\{s't' \mid p(s') = p(t') = \mu(s, t)\}$.

1. A point $p \in P^+_\mu$ belongs to $T_{\mu}$ if and only if $K(p)$ has no isolated node $u$ with $p(u) > 0$.
2. A point $p \in P^+_\mu$ belongs to $Q^+_\mu$ if and only if $K(p)$ has no isolated node.
3. For $p \in T_{\mu}$, the dimension of the minimal face of $T_{\mu}$ containing $p$ is equal to the number of components in $K(p)$ having no node $u$ with $p(u) = 0$.
4. For $p \in Q^+_\mu$, the dimension of the minimal face of $Q^+_\mu$ containing $p$ is equal to the number of components in $K(p)$ [12, Proposition 17].
5. Any $k$-dimensional face $F$ in $(T_{\mu}, D_\infty)$ is isometric to a $k$-dimensional polytope in $(R^k, D_\infty)$.
6. $D_\infty(p, q) = D_\infty(p^\ast, q^\ast) = D_\infty(q^\ast, p^\ast)$ holds for $p, q \in Q^+_\mu$ and for $p, q \in T_{\mu}$.
7. $T_{\mu}$ has dimension at most 1 if and only if $T_{\mu}$ is a path isometric to a segment in $(R, D_\infty)$.

The property (3) follows from [11, (2.1)].

Our technical arguments use a method of perturbing a point $p \in Q^+_\mu$ to another point $p' \in Q^+_\mu$. For a node subset $U$ in a graph $K(p)$, the set of nodes in $S^\ast \setminus U$ incident to $U$ is denoted by $N_p(U) = N(U)$. The following consideration is a basis for our perturbation method, which has a similar flavor of manipulating dual variables in bipartite matching problems:

For $p \in Q^+_\mu$, $X \subseteq S$, let $p' := p - \epsilon 1_{X'}$ for small $\epsilon > 0$, and $Y^\prime := N_p(X^\prime)$.

- $p(s^\prime) > 0$ for small $\epsilon > 0$ and $Y^\prime := N_p(X^\prime)$.
- Put $p' \leftarrow p' + \epsilon 1_{Y^\prime}$ so that $p' \in P_{\mu}$.
- Then all edges joining $X^\prime \setminus X^\prime$ and $Y^\prime$ vanish in $K(p')$.
- Therefore, $p'$ belongs to $Q^+_\mu$ if and only if each node in $S^\ast \setminus X^\prime$ is joined to $S^\ast \setminus Y^\prime$ in $K(p)$.
- We can increase $\epsilon$ until some coordinate of $p'$ in $X^\prime$ reaches zero or there appears an edge joining $X^\prime$ and $S^\ast \setminus Y^\prime$. 
Here the fourth implication uses \((2.7)(1')\).

2.2. Duality relations

First we establish strong duality relations for general networks and for inner Eulerian networks, which are easy consequences of the existence of nonexpansive retractions \((2.2)\).

**Theorem 2.2.** Let \(\mu\) be a directed distance on \(S\).

1. MFP\(^\ast\)(\(\mu\); \(G\), \(S\), \(c\)) = FLP\(^\ast\)(\(T_\mu\); \(G\), \(S\), \(c\)) holds for every network \((G, S, c)\).
2. MFP\(^\ast\)(\(\mu\); \(G\), \(S\), \(c\)) = FLP\(^\ast\)(\(R\); \(G\), \(S\), \(c\)) holds for every balanced section \(R\) in \(Q_\mu^+\) and every inner Eulerian network \((G, S, c)\).

**Proof.** Take an optimal map \(\rho\) for FLP with \(R = P_\mu^+\). Take a nonexpansive retraction \(\rho : P_\mu^+ \to T_\mu\) in \((2.2)(1)\). Consider the composition \(\phi \circ \rho : VG \to T_\mu\). Then \(\phi \circ \rho\) is also feasible, and does not increase the objective value. Thus we have \((1)\).

Next we show \((2)\). Suppose that \((G, S, c)\) is inner Eulerian. Then the capacity function \(c : EG \to \mathbb{Z}_+\) is decomposed into the sum of the incidence vectors of cycles \(C_1, C_2, \ldots, C_m\) and \(\mathcal{S}\)-paths \(P_1, P_2, \ldots, P_n\) (possibly repeating). Take an optimal map \(\rho\) for FLP with \(R = T_\mu\). Then we have

\[
\sum_{\substack{x \in EG}} c(xy)D_\infty(\rho(x), \rho(y)) = \sum_{i=1}^m D_\infty(C_i) + \sum_{j=1}^n D_\infty(\rho(P_j)).
\]

We can take a cyclically nonexpansive retraction \(\rho : T_\mu \to R\) by \((2),(3)\) in \((2.2)\). Since \(\phi \circ \rho(s) = \rho(s) \in \{T_\mu\}_s = R_s\) for each \(s \in S\) by \((2.5)(1)\), \(\phi \circ \rho\) is an identity on \(\{T_\mu\}_s\) and thus \(\phi \circ \rho\) is also feasible. Moreover, by cyclic nonexpansiveness, 

\[
D_\infty(\phi \circ \rho(C_i)) \leq D_\infty(\rho(C_i))
\]

holds also. Hence \(\phi \circ \rho\) is also optimal. \(\square\)

We give some examples. Consider the all-one distance \(\mu\) on a 3-set \(\{s, t, u\}\); recall Fig. 1. Then the FLP is a location problem on a directed metric space on a folder consisting of three triangles, each of which is isometric to triangle \(\{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq x \leq 1\}\) in \((\mathbb{R}^2, D_\infty^\mu)\). Suppose that the network is inner Eulerian. We can take a balanced section \(R\) in \(Q_\mu^+\), which is a tree. By a cyclically nonexpansive retraction from \(Q_\mu^+\) to \(R\), the FLP reduces to a location problem on the tree \(R\); see Fig. 2. Consider the 2-commodity flow case; let \(S = \{s, t, t', u\}\), and let \(\mu(s, t) = \mu(s', t') = 1\) and let the other distances be zero. Then \(T_\mu\) is given by \(\{1_{(t,t')}, \alpha(1_{(t',t)} - 1_{(t,t')}) + \beta(1_{(t,t') - 1_{(t',t')}}) \mid 0 \leq \alpha, \beta \leq 1\}\), which is isometric to a square \(\{(x,y) \in \mathbb{R}^2 \mid 0 \leq x, y \leq 1\}\) in \((\mathbb{R}^2, D_\infty^\mu)\). Terminal regions \((T_\mu)_s, (T_\mu)_t, (T_\mu)_{t'}, (T_\mu)_u\) correspond to four sides as in Fig. 3. In this case, \(T_\mu = Q_\mu^+\) holds, and moreover \(Q_\mu^+\) itself is a balanced section. In contrast to the previous example, the region contraction in FLP does not occur if the inner Eulerian condition is imposed.

**Slipped sections and Eulerian condition on terminals.** Next we consider the case where some of the terminals fulfill the Eulerian condition. In this case, the strong duality holds for further smaller subsets in \(Q_\mu\), called slipped sections. To define a slipped section, we need several (somewhat technical) notions. Recall notions of \(K(p)\) and \(N_p(s)\) associated with \(p \in Q_\mu\); see Section 2.1.1. Let \(\delta_0\) be the set of subsets \(X\) of \(S\) such that \(\mu(s, t) = 0 = 0\) for all \((s, t) \in X \times X\); obviously \(\{s\} \in \delta_0\). For \(X \in \delta_0\), let \(Q_{\mu,X}\) denote the set of points \(p \in Q_\mu\) with \(s's' \notin EK(p)\) for \(s \in X\) and \(s's' \notin EK(p)\) for \(s \notin X\); in particular \(Q_{\mu,X} = \bigcap_{s \in X} (Q_{s})_s \bigcup \bigcup_{s \notin X} (Q_{s})_s\). A point \(p \in Q_{\mu,X}\) is called a fat relative to \(X\) if \(p \in Q_{\mu,X}\) and \(N_p(S' \setminus X') \subseteq S' \setminus X'\) for \(S' \setminus X' \subseteq S' \setminus X'\). The degenerate set \(Q_{\mu,X}^\text{deg}\) relative to \(X\) is the set of points \(p \in Q_{\mu,X}\) with \(N_p(S' \setminus X') = S' \setminus X'\) or \(N_p(S' \setminus X') = S' \setminus X'\). Any point in a degenerate set is a fat. A proper fat is a fat not belonging to any degenerate set. Let \(Q_\mu^\text{lim}\) be the subset of \(Q_\mu\) obtained by deleting all proper fats. We consider the following equivalence relation \(\sim\) on \(Q_\mu^\text{lim}\): \(p \sim q\) if \(p - q \in (1, -1)\mathbb{R}\), or for some \(X \in \delta_0\), both \(p\) and \(q\) belong to \(Q_{\mu,X}^\text{deg}\) and \(p - q \in (1, -1)\mathbb{R} + (1_{X'}, -1_{X'})\). The quotient
Proposition 2.3. If \( \mu \) is a metric, then \( Q_\mu \) has no fat, and hence \( \tilde{Q}_{\text{slim}}^\mu = \tilde{Q}_\mu \).

Proof. By \( (Q_\mu)_s \supset Q_{\mu,X} \) for \( s \in X \) and (2.5) (3), \( Q_{\mu,X}^+ \) is a single point \( \mu_s \) for \( s \in X \). In \( K(\mu_s) \), node \( s' \) is incident to all nodes in \( S' \), and \( s' \) is incident to all nodes in \( S' \); thus \( \mu_s \) is never a fat. \square

Again we consider special sections in \( Q_{\mu, \text{slim}}^+ \), called slimmed sections. Here a section is a subset of \( Q_{\mu, \text{slim}}^+ \) bijectively projected into \( \tilde{Q}_{\mu, \text{slim}}^+ \). We first define a slimmed section \( R \) in \( (Q_{\mu, \text{slim}}^+)^+ \), which is a section such that it is balanced, and for each \( X \in \delta_0 \), there is no pair \( p, q \in (Q_{\mu, \text{deg}}^+) \) such that \( p(s') < q(s') \) for all \( s' \in S' \setminus X' \) or \( p(s') < q(s') \) for all \( s' \in S' \setminus X' \). Next, a slimmed section \( R \) in \( Q_{\mu, \text{slim}}^+ \) is a section such that it is balanced and \( R/(1, -1)R = R'/(1, -1)R \) for some slimmed section \( R' \) in \( (Q_{\mu, \text{slim}}^+)^+ \) (recall that the projection from \( Q_{\mu, \text{slim}}^+ \) to \( \tilde{Q}_\mu \) is surjective).

**Fig. 4.** Two examples of \( Q_{\mu, \text{slim}}^+ \).

**Fig. 5.** Retraction from \( Q_{\mu, \text{slim}}^+ \) to a slimmed section.

\( Q_{\mu, \text{slim}}/\sim \) is called the slimmed tropical polytope associated with \( \mu \), and is denoted by \( \tilde{Q}_{\mu, \text{slim}} \). The tropical polytope and the slimmed tropical polytope are the same if \( \mu \) is a metric.

Theorem 2.4. Let \( \mu \) be a directed distance on \( S \).
Lemma 2.5. For any slimmed section \( R \) in \( Q^+_\mu \), there exists a cyclically nonexpansive retraction \( \varphi \) from \( Q^+_\mu \) to \( \varphi((Q^+_\mu)^+) \subseteq (Q^+_\mu)^+ \) for each \( s \in S \).

The proof uses the following new retraction lemma:

**Lemma 2.5.** For any slimmed section \( R \) in \( Q^+_\mu \), there exists a cyclically nonexpansive retraction \( \varphi \) from \( Q^+_\mu \) to \( \varphi((Q^+_\mu)^+) \subseteq (Q^+_\mu)^+ \) for each \( s \in S \).

The proof of this lemma is given in the end of this subsection. Assuming Lemma 2.5, we prove Theorem 2.4. Let \( S^* \) be the set of proper terminals. Take an optimal map \( \rho : VG \to Q^+_\mu \) for FLP. As in the proof of Theorem 2.2, there are cycles \( C_i \) and \( S^* \)-paths \( P_i \) such that (2.9) holds. Take a cyclically nonexpansive retraction \( \varphi \) in Lemma 2.5. Then \( \varphi \circ \rho : VG \to R \) is feasible to FLP with \( R \). Since \( \varphi \) is cyclically nonexpansive, \( D_\infty(\varphi \circ \rho(C_i)) \leq D_\infty(\rho(C_i)) \) and \( D_\infty(\varphi \circ \rho(P_i)) \leq D_\infty(\rho(P_i)) \), where the second inequality follows from the fact that \( \varphi \) is the identity on \( R \) for each proper terminal \( s \in S^* \). Thus \( \varphi \circ \rho \) and \( \rho \) have the same objective value. The statement (2) follows from (1) and (2.2)(3).

Fig. 5 illustrates cyclically nonexpansive retractions in the examples of Fig. 4. Again the 2-commodity tight span in Fig. 3 has no fat; the region contraction in FLP does not occur even if the totally Eulerian condition is imposed.

As a corollary, we obtain topological properties of slimmed sections.

**Corollary 2.6.** Let \( R \subseteq Q^*_\mu \) be a slimmed section.

1. \( R \) is contractible and geodesic, and so is \( R_s \) for \( s \in S \).
2. If \( R \subseteq Q^*_\mu \), then \( \mu(s, t) = D_\infty(R_s, R_t) \) for \( s, t \in S \).

**Proof.** (1). A cyclically nonexpansive map is continuous in the Euclidean topology [11, Remark 2.4]. So \( R \) is homotopy equivalent to convex set \( P^*_\mu \), which is contractible. Since \( P^*_\mu \) is geodesic, so is \( R \); see [11, Section 2.3]. Since \( R_s \) is a retract of a face of \( P^*_\mu \), it is contractible and geodesic by the same argument.

(2). Consider the Eulerian network \( (G, S, c) \) such that \( c(st) = c(ts) = 1 \) and the other capacities are zero. Obviously \( MFP^*(G, S, c) = \mu(s, t) + \mu(t, s) \). By \( MFP^*(\mu, G, S, c) = FLP^*(R; G, S, c) \), there is \( (p, q) \in R \times R \) with \( D_\infty(p, q) + D_\infty(q, p) = \mu(s, t) + \mu(t, s) \). Necessarily \( D_\infty(p, q) = \mu(s, t) \) and \( D_\infty(q, p) = \mu(t, s) \) by (2.1).

**Proof of Lemma 2.5.** In the proof, we denote by \( Q^+_\mu \) and \( Q^+_\mu \) the projections of \( Q^*_\mu \) to \( \mathbf{R}^S \) and \( \mathbf{R}^T \), respectively. These projections are bijective and isometric by (2.7)(4). For \( q \in Q^*_\mu \), we can lift \( p \in Q^*_\mu \) with \( p^c = q \) by \( p(t^c) = \max_{c^c \in S^c} (\mu(s, t) - q(s^c)) \).

We remark that \( p(s^c) = p(s^c) = 0 \) for \( p \in Q^+_\mu \) and \( s \in X \) in \( \delta_0 \). Let \( B \) be any balanced section in \( Q^+_\mu \). By \( Q^*_\mu \subseteq \bigcup_{s \in X}(Q^*_\mu)_s \) and (2.5)(1), \( B \) includes \( Q^+_\mu \) for all \( X \) in \( \delta_0 \). One can verify from (2.8) that a point \( p \in Q^+_\mu \) is a fat if and only if for small \( \epsilon > 0 \) we have \( p^c - \epsilon \mathbf{1}_{(s^c) \in X^c} \in (Q^+_\mu)^c \) or \( p^c - \epsilon \mathbf{1}_{(s^c) \in X^c} \in (Q^+_\mu)^c \) (use the property that any node \( u \) with \( p(u) = 0 \) is incident to \( X^c \) by nonnegativity of \( \mu \)).

Based on this, for \( \epsilon > 0 \), consider the map \( \psi^c_{X^c} : B \) by the following process. For each \( p \in Q^+_\mu \), add \( -\epsilon \mathbf{1}_{(s^c) \in X^c} \) to \( p^c \), where \( \epsilon^* \) is the maximum nonnegative real in \( [0, \epsilon] \) such that \( p^c - \epsilon \mathbf{1}_{(s^c) \in X^c} \) belongs to the closure of \( (Q^+_\mu)^c \). Then lift the resulting point \( q \) to \( q^c \in Q^*_\mu \) with \( (p^c) = q \) and define \( \psi^c_{X^c}(q) := p^c \). Extend \( \psi^c_{X^c} \) to a map \( B \to B \) by defining it to be the identity on the points not in \( Q^+_\mu \).

In the above process, \( p, \epsilon, \epsilon^* \), the following key property holds:

\[
D_\infty(\psi^c_{X^c}(p), q) \leq D_\infty(p, q) + \epsilon^*,
\]
\[
D_\infty(\psi^c_{X^c}(p), q) = D_\infty(p, q) - \epsilon^*.
\]

**Proof.** By (2.7)(4), we may consider \( p^c, q^c, D^c_\infty \) instead of \( p, q, D_\infty \). The first relation is obvious from the definition of \( D^c_\infty \); see [11, p.8]. We show the second. We claim that \( K(q) \) has an edge \( s^c t^c \) joining \( S^c \) and \( X^c \). Suppose not. Then \( X^c \supseteq N_q(X^c) \). Since \( X \in \delta_0 \), \( \mu(s, t) = 0 \) holds for all \( (s^c, t^c) \in N_q(X^c) \times X^c \). Hence we have \( q(u) = 0 \) for \( u \in N_q(X^c) \cup X^c \) and \( q(s^c) > 0 \) for \( s^c \in S^c \setminus X^c \). Since \( q = \psi^c_{X^c}(q) \), necessarily \( X^c \supseteq N_q(X^c) \) (proper inclusion), and \( q \) is a proper fat relative to \( Y \) with \( Y^c = N_q(X^c) \subset X^c \); a contradiction.

For an edge \( s^c t^c \in E(K(q) \text{ joining } S^c \setminus X^c \text{ and } X^c \), we have \( \epsilon^* \leq p(s^c) - q(s^c) \leq p(s^c) - q(s^c) - q(t^c) \leq \mu(s, t) - p(s^c) - q(s^c) - q(t^c) = p(s^c) - q(s^c) - q(t^c) \), where we use \( q(s^c) + q(t^c) = \mu(s, t) \) by \( s^c t^c \in E(K(q) \text{ and } q(t^c) = 0 \). Therefore we have \( (p(s^c) - q(s^c) - q(t^c)) = (p(t^c) - q(t^c)) \leq \epsilon^* \).

\[
D^c_\infty(q^c, p^c - \epsilon^* \mathbf{1}_{(s^c) \in X^c}) = \| (p^c - \epsilon^* \mathbf{1}_{(s^c) \in X^c})_c - q^c_c \| \leq \max_{c^c \in S^c} \{ (p(t^c) - q(t^c))_c - q^c_c \} = D^c_\infty(q^c, p^c) - \epsilon^*,
\]
where the second equality uses \( p(t^c) = 0 \) for all \( t^c \in X^c \) and the third uses \( (\ast) \).
We can define $\varphi^{c}_{X,e} : B \rightarrow B$ by changing roles of $c$ and $r$, and an analogous property holds. Let $\varphi^{c}_{X} := \lim_{e \rightarrow \infty} \varphi^{c}_{X,e}$ and $\varphi^{r}_{X} := \lim_{e \rightarrow \infty} \varphi^{r}_{X,e}$ (well-defined). Next we study the image of $\varphi^{c}_{X}$. Let $p^{*} := \varphi^{c}_{X}(p)$ for $p \in Q^{+}_{\mu,X}$. Then $K(p^{*})$ necessarily has an edge joining $S' \setminus X^c$ and $X'$. Therefore if $p^{*}$ is a fat, then it is a fat relative to $Y \supset X$ (proper inclusion), or $p^{*}$ is $r$-maximal in $(Q^{\text{deg}}_{\mu,X})^{+}$ in the sense that $p^{*} - \epsilon(1_{S(X')}, -1_{S(X')}) \notin (Q^{\text{deg}}_{\mu,X})^{+}$ for every $\epsilon > 0$. Also if $p$ belongs to $(Q^{\text{deg}}_{\mu,X})^{+}$, then $p^{*}$ is an $r$-maximal point with $p - p^{*} \in (1_{S(X')}, -1_{S(X')}) \mathbb{R}$; see the right of Fig. 5. Again an analogous property holds for $\varphi^{r}_{X}$ by changing roles of $r$ and $c$. Let $\varphi^{c}_{X} := \varphi^{c}_{X} \circ \varphi^{r}_{X}$. Then the image $\varphi^{c}_{X}(B)$ does not contain a proper fat relative to $X$.

Let $B^{\mathcal{S}}_{c}$ be the subset of $B$ obtained by deleting all proper fats and replacing each $(Q^{\text{deg}}_{\mu,X})^{+}$ by the set of its $c$-maximal points. Order all subsets $X_{1}, X_{2}, \ldots, X_{m}$ in $\mathcal{S}$ so that $X_{i} \subseteq X_{j}$ implies $i \leq j$. Then the composition $\varphi := \varphi_{X_{m}} \circ \varphi_{X_{m-1}} \circ \cdots \circ \varphi_{X_{1}}$ is a retraction from $B$ to $B^{\mathcal{S}}_{c}$.

We show that $\varphi : B \rightarrow B^{\mathcal{S}}_{c}$ is cyclically nonexpansive. Let $X := X_{i}$. Take a cycle $C$ in $\varphi_{X_{i-1}} \circ \varphi_{X_{i-2}} \circ \cdots \circ \varphi_{X_{1}}(B)$. We prove that $g(\epsilon) := D_{\infty}(\varphi^{c}_{X_{i}}(C)) - D_{\infty}(C)$ is a monotone nonincreasing function. It suffices to show $g(\epsilon) \leq 0$ for small $\epsilon > 0$. By construction, $C$ does not contain proper fats relative to any $Y \subset X$. By (2.10), for a consecutive pair $(p, q)$ in $C$ we have

$$D_{\infty}(\varphi^{c}_{X_{i}}(p, q)) - D_{\infty}(p, q) \begin{cases} \leq \epsilon & \text{if } p \neq \varphi^{c}_{X_{i}}(q), \\ = -\epsilon & \text{if } p = \varphi^{c}_{X_{i}}(q), \\ = 0 & \text{if } p = \varphi^{c}_{X_{i}}(q), \\ \end{cases}$$

(2.11)

Since the number of consecutive pairs $(p, q)$ with $p \neq \varphi^{c}_{X_{i}}(q)$ is equal to that with $p = \varphi^{c}_{X_{i}}(q)$, $q \neq \varphi^{c}_{X_{i}}(q)$, summing up (2.11) over all consecutive pairs yields $g(\epsilon) \leq 0$. The argument for $\varphi^{r}_{X}$ is similar. Thus $\varphi$ is cyclically nonexpansive.

We next verify that $B^{\mathcal{S}}_{c}$ is slimmed. Indeed, take an arbitrary pair, $p^{0}$ of $c$-maximal points in $(Q^{\text{deg}}_{\mu,X})^{+}$. Then there are edges $s't' \in EK(p)$, $s''t'' \in EK(p')$ joining $X^c$ and $X'$. Hence $p'(s'') + p'(t'') = p'(t') \geq \mu(s, t') = p(t')$, and $p(t') \geq \mu(s, t', t'') = p'(t'')$.

Finally we construct a cyclically nonexpansive retraction from $Q^{\mu}$ to any slimmed section. Any slimmed section $R$ in $(Q^{\mu})^{+}$ is obtained from a balanced section $B$ by deleting all proper fats and replacing each $(Q^{\text{deg}}_{\mu,X})^{+}$ by a subset $R_{X}$ with the properties that (i) there is no pair $p, q \in R_{X}$ with $p(s') < q(s')$ for all $s' \in S \setminus X^c$ or $p(s') < q(s')$ for all $s' \in S \setminus X'$, and (ii) for each $p' \in (Q^{\text{deg}}_{\mu,X})^{+}$ there uniquely exists $p \in R_{X}$ with $p - p' \in (1_{S(X')}, -1_{S(X')}) \mathbb{R}$. It suffices to give a cyclically nonexpansive map from $B^{\mathcal{S}}_{c}$ to $R$. For each $X \in \mathcal{S}$, we can define a map $\varphi^{X}_{c}$ on $B^{\mathcal{S}}_{c}$ as: For each $p \in (Q^{\text{deg}}_{\mu,X})^{+}$, define $\varphi^{X}_{c}(p)$ to be the point $p'$ in $R_{X}$ determined by the relation $p' - p \in (1_{S(X')}, -1_{S(X')}) \mathbb{R}$, and to be the identity on the other points. So it suffices to prove that $\varphi^{X}_{c}$ is cyclically nonexpansive; consider the composition of $\varphi^{X}_{c}$ for all $X \in \mathcal{S}$. One can verify this fact in the essentially same way as above. The projection of $(Q^{\text{deg}}_{\mu,X})^{+}$ to $R^{(S(X'))^{c}}$ is isometry, and the image of $R_{X}$ is a balanced set in $R^{(S(X'))^{c}}$. So we can apply the method in the proof of [11, Lemma 2.7]; the details are left to the readers.

3. Integrality

The geometry of $T_{\mu}$ and $\mathcal{G}^{\mathcal{S}}_{\mu}$ crucially affects the integrality of $\mu$-MFP. The dimension of $T_{\mu}$ is defined by the largest dimension of faces of $T_{\mu}$. The dimension of $\mathcal{G}^{\mathcal{S}}_{\mu}$ is defined by the largest dimension of faces $F$ of $\mathcal{G}^{\mathcal{S}}_{\mu}$ in modulo $\sim$; intuitively, it is the dimension of its slimmed section. The main goal of this section is to prove the following integral theorem:

**Theorem 3.1.** Let $\mu$ be a directed distance on $S$.

(1) If $\dim T_{\mu} \leq 1$, then $\mu$-MFP has an integral optimal multiflow for every network $(G, S, c)$.

(2) If $\dim \mathcal{G}^{\mathcal{S}}_{\mu} \leq 1$, then $\mu$-MFP has an integral optimal multiflow for every properly inner Eulerian network $(G, S, c)$ (relative to $\mu$).

The first statement (1) is reducible to the minimum cost circulation. So we mainly concentrate on the second statement (2) and its consequences. The rest of this section is organized as follows. In Section 3.1, we give basic definitions for cuts, cut distances, and oriented-tree realizations. Then, in Section 3.2, we prove Theorem 3.1(2). In Section 3.3, we give a useful “combinatorial version” of Theorem 3.1(2), and derive (slight) extensions of the Lomonosov–Frank theorem for directed free multiflows and Ibaraki–Karzanov–Nagamochi’s directed version of the multiflow locking theorem. In Section 3.4, we prove (1) by a reduction to the minimum cost circulation.

3.1. Preliminary: partial cuts, cut distances, and oriented trees

A partial cut on a set $S$ is an ordered pair $(A, B)$ of disjoint subsets $A, B \subseteq S$. We particularly call $(A, B)$ a cut if $A \cup B = S$. For a partial cut $(A, B)$ on $S$, the cut distance $\delta_{A,B} : S \times S \rightarrow \mathbb{R}_{+}$ is defined by

$$\delta_{A,B}(s, t) = \begin{cases} 1 & \text{if } (s, t) \in A \times B, \\ 0 & \text{otherwise,} \\ \end{cases} \quad (s, t \in S).$$
In a network $(G, S, c)$, for a node subset $X \subseteq VG$, let $\partial X$ denote the set of edges leaving $X$. For a partial cut $(A, B)$ on $S$, the following relation is nothing but the max-flow min-cut theorem:

$$\text{MFP}^*(\delta_{A,B}; G, S, c) = \min \{ c(\partial X) \mid A \subseteq X \subseteq VG \setminus B \}. \quad (3.1)$$

An oriented tree $\Gamma'$ is a directed graph whose underlying undirected graph is a tree. For a nonnegative edge length $\alpha : E\Gamma' \rightarrow \mathbf{R}_+$, we define directed metric $D_{\Gamma',\alpha}$ on $\Gamma'$ as follows. For two nodes $u, v$, the distance $D_{\Gamma',\alpha}(u, v)$ is defined by the sum of edge-length $\alpha(e)$ over edges $e = pq$ such that the unique walk from $u$ to $v$ passes through $pq$ in order $u \rightarrow p \rightarrow q \rightarrow v$. Namely $D_{\Gamma',\alpha}$ does not count the edge-length of edges with the opposite direction. A subgraph of $\Gamma'$ is a subgraph whose underlying undirected graph is a tree. For a directed distance $\mu$ on $S$, an oriented-tree realization $(\Gamma, \alpha; \{F_i\}_{i \in S})$ of $\Gamma'$ is a tree, a nonnegative edge-length $\alpha$, and a family $\{F_i\}_{i \in S}$ of subtrees indexed by $S$ such that

$$\mu(s, t) = D_{\Gamma',\alpha}(F_s, F_t) \quad (s, t \in S).$$

Deletion of an edge $e = uv$ in $\Gamma'$ decomposes $\Gamma'$ into two connected components $\Gamma'_e$, $\Gamma''_e$ so that $\Gamma'_e$ contains $u$. This yields a partial cut $(A_e, B_e)$ of $S$ by $A_e := \{ s \in S \mid F_s \text{ belongs to } \Gamma'_e \}$ and $B_e := \{ s \in S \mid F_s \text{ belongs to } \Gamma''_e \}$. From the definition of $D_{\Gamma',\alpha}$, one can easily see

$$\mu = \sum_{e \in \Gamma'} \alpha(e)\delta_{A_e, B_e}. \quad (3.2)$$

\subsection{Proof of Theorem 3.1(2)}

Suppose $\dim \tilde{Q}_\text{lim} < 1$. Then we can take a slimmed section $R$ represented as a union of one-dimensional facets of $(Q^\text{lim}_0)^+$; see the proof of Lemma 2.5. By (3) and (4) in (2.7), each segment in $R$ is isometric to a segment in $R, D^\pm_\text{lim}$. Since $R$ is contractible (Corollary 2.6), the 1-skeleton graph $\Gamma$ of $R$ is a tree. Orient this 1-skeleton graph $\Gamma'$ so that for each edge $pq$ (segment $[p, q]$), $p$ is oriented to $q \iff D_{\alpha}(p, q) > 0$ (and $D_{\alpha}(q, p) = 0$). Also let $\alpha(pq) := D_{\alpha}(p, q)$ (oriented) edge $pq \in E\Gamma'$. Then we obtain an oriented tree $\Gamma'$ with edge-length $\alpha$. Let Vert$_R$ be the set of vertices (endpoints of segments) of $R$. Since $R$ is geodesic (Corollary 2.6(1)), Vert$_R, D_{\alpha}(\Gamma')$ is isometric to $\Gamma, D_{\alpha}$. For $s \in S$, let $F_s$ be the subgraph induced by $R_s$ (well-defined since $R_s$ is a subcomplex of $R$). Since $R_s$ is also contractible (Corollary 2.6(1)), $F_s$ is a subtree. Summarizing these facts together with Corollary 2.6(2), we can conclude that $(\Gamma', \alpha; \{F_i\}_{i \in S})$ is an oriented-tree realization of $\mu$.

I. First we prove the following min–max relation:

$$\text{MFP}^*(\mu; G, S, c) = \text{FLP}^*(\text{Vert}_R; G, S, c) = \min \left\{ \sum_{xy \in EG} c(xy)D_{\Gamma',\alpha}(\rho(x), \rho(y)) \mid \rho : VG \rightarrow \Gamma', \rho(s) \in VF_s \ (s \in S) \right\}. \quad (3.3)$$

This means that FLP becomes a discrete location problem on $\Gamma'$. By (3.1) and (3.2) in (2.7), each segment in $R$ is isometric to a segment in $R, D^\pm_\text{lim}$. Since $R$ is contractible (Corollary 2.6), the 1-skeleton graph $\Gamma$ of $R$ is a tree. Orient this 1-skeleton graph $\Gamma'$ so that for each edge $pq$ (segment $[p, q]$), $p$ is oriented to $q \iff D_{\alpha}(p, q) > 0$ (and $D_{\alpha}(q, p) = 0$). Also let $\alpha(pq) := D_{\alpha}(p, q)$ (oriented) edge $pq \in E\Gamma'$. Then we obtain an oriented tree $\Gamma'$ with edge-length $\alpha$. Let Vert$_R$ be the set of vertices (endpoints of segments) of $R$. Since $R$ is geodesic (Corollary 2.6(1)), Vert$_R, D_{\alpha}(\Gamma')$ is isometric to $\Gamma, D_{\alpha}$. For $s \in S$, let $F_s$ be the subgraph induced by $R_s$ (well-defined since $R_s$ is a subcomplex of $R$). Since $R_s$ is also contractible (Corollary 2.6(1)), $F_s$ is a subtree. Summarizing these facts together with Corollary 2.6(2), we can conclude that $(\Gamma', \alpha; \{F_i\}_{i \in S})$ is an oriented-tree realization of $\mu$.

II. Second we derive the following min-cut expression:

$$\text{MFP}^*(\mu; G, S, c) = \sum_{e \in \Gamma'} \alpha(e)\min \{ c(\partial X) \mid A_e \subseteq X \subseteq VG \setminus B_e \}. \quad (3.4)$$

(\leq) follows from LHS $\leq \sum_{e \in \Gamma'} \alpha(e)\text{MFP}^*(\delta_{A_e, B_e}; G, S, c) = \text{RHS}$, where the inequality follows from (3.2), and the equality follows from the max-flow min-cut theorem (3.1). Let $\rho^* : V \rightarrow \Gamma'$ be an optimal map in (3.3). Let $d^*$ be the metric on $VG$ defined by $d^*(x, y) = D_{\Gamma',\alpha}(\rho^*(x), \rho^*(y))$ for $x, y \in VG$. Then $d^*$ has an oriented-tree realization $(\Gamma', \alpha; \{\rho(x)\}_{x \in VG})$. Again the deletion of edge $e$ yields a cut $(X_e, Y_e)$ of $VG$ with $A_e \subseteq X_e \subseteq V \setminus B_e$, and $d'' = \sum_{e \in \Gamma'} \alpha(e)\delta_{X_e, Y_e}$. Thus MFP$^*(\mu; G, S, c) = \sum_{xy \in EG} c(xy)d''(x, y) = \sum_{e \in \Gamma'} \alpha(e)\sum_{xy \in EG} c(xy)\delta_{X_e, Y_e}(x, y) = \sum_{e \in \Gamma'} \alpha(e)c(\partial X_e)$. This proves claim (3.3).
III. Finally, we show the existence of an integral optimal multiflow. We use the splitting-off technique. By multiplying edges, we may assume that each edge has unit capacity. For a pair \((xy, yz)\) of consecutive edges, the splitting-off operation is to delete \(xy\) and \(yz\) and add a new edge from \(x\) to \(z\) (of unit capacity). If the splitting-off operation does not decrease the optimal multiflow value, then from any optimal multiflow in the new network after the splitting-off we obtain an optimal multiflow in the initial network, and we can apply the inductive argument (on the number of edges). Consider any optimal (fractional) multiflow \(f = (\mathcal{P}, \lambda)\). Suppose that there is a pair \((xy, yz)\) of consecutive edges such that some path in \(\mathcal{P}\) with nonzero flow-value passes through \(xy, yz\) in order. If such a pair does not exist, then \(f\) is already an integral multiflow. We show that the splitting-off at \((xy, yz)\) is successful. Suppose that the splitting-off decreases the optimal flow-value. By (3.4), there are \(e \in E\) and \(X^*\) attaining the minimum of \(\min\{c(\partial X) \mid A \subseteq X \subseteq VG \setminus B_e\}\) such that \((*)\) \(x, z \in X^*\) \iff \(y \in X^*\) \or \(y \notin X^*\), \(z \notin X^*\). Since \(f\) is an optimal multiflow for weight \(\delta_{A, B_e}\), i.e., a maximum (single commodity) \((A_e, B_e)\)-flow, each path in \(\mathcal{P}\) (with nonzero flow-value) must meet \(\partial X^*\) at most once. This contradicts (*).

3.3. Combinatorial min–max relations

We have already shown that if \(\dim \widehat{\mathcal{Q}}_{\mu}^{\text{lim}} \leq 1\), then we obtain an oriented-tree realization of \(\mu\) by \(\widehat{\mathcal{Q}}_{\mu}^{\text{lim}}\), and the min–max relation (3.3) from this realization. The next theorem states that if \(\mu\) is realized by an oriented tree, then one can get such a min–max relation directly (without calculating \(\mathcal{Q}_{\mu}^{\text{lim}}\)). Let \(\text{IMFP}^*(\mu; G, S, c)\) denote the maximum flow-value with respect to \(\mu\) over all integral multiflows in \((G, S, c)\).

**Theorem 3.2.** Let \(\mu\) be a directed distance on \(S\) having an oriented-tree realization \((\Gamma, \alpha; \{F_s\}_{s \in S})\), and let \((G, S, c)\) be an inner Eulerian network such that the Eulerian condition is fulfilled by each terminal \(s\) with \(F_s\) being neither a single node nor a directed path. Then the following relation holds:

\[
\text{MFP}^*(\mu; G, S, c) = \text{IMFP}^*(\mu; G, S, c) = \min \left\{ \sum_{xy \in E} c(xy)D_{\Gamma, \alpha}(\rho(x), \rho(y)) \mid \rho : VG \to V\Gamma, \rho(s) \in VF_s (s \in S) \right\}
\]

\[
= \sum_{e \in E\Gamma} \alpha(e) \min\{c(\partial X) \mid A_e \subseteq X \subseteq VG \setminus B_e\},
\]

(3.5)

where \((A_e, B_e)\) is a partial cut on \(S\) determined by the deletion of edge \(e \in E\Gamma\).

The proof uses the next proposition, which says that \((\mathcal{Q}_{\mu}^{\text{lim}})^+\) is (essentially) a geometric realization of an oriented-tree realization \((\Gamma, \alpha; \{F_s\}_{s \in S})\).

**Proposition 3.3.** Suppose that \(\mu\) has an oriented-tree realization \((\Gamma, \alpha; \{F_s\}_{s \in S})\) so that \(\{F_s\}_{s \in S}\) contains all single-node subtrees. Let \(S_0 \subseteq S\) consist of elements \(s\) such that \(F_s\) is a single node \(v_s\). Then the following holds:

1. \((\mathcal{Q}_{\mu}^{\text{lim}})^+= \bigcup\{[\mu_t, \mu_u] \mid s, t \in S_0, v_tv_s \in E\Gamma\}\).
2. \((\mathcal{Q}_{\mu}^{\text{lim}})_{s} = \bigcup\{[\mu_t, \mu_u] \mid t, u \in S_0, \forall v \in EF_s \} \text{ for } s \in S\).
3. \((\mathcal{Q}_{\mu}^{\text{lim}})^+\) itself is a slimmed section.
4. \(Q_{\mu_s}\) has no fat if \(F_s\) is a single node or a directed path.

See Section 2.1 B for definition of \(\mu_s\). The proof is a routine verification, but rather technical. So the proof is given in the end of this subsection. In (4) the converse (only-if part) also holds. However we omit the proof, which is also a lengthy verification.

Assuming Proposition 3.3, we complete the proof of Theorem 3.2. Suppose that \(\mu\) is realized by \((\Gamma, \alpha; \{F_s\}_{s \in S})\). We can add isolated terminals to \((G, S, c)\) so that \(\{F_s\}_{s \in S}\) includes all single-node subtrees. Thus we may assume that \((\Gamma, \alpha; \{F_s\}_{s \in S})\) fulfills the hypothesis in Proposition 3.3. Consider a slimmed section \(R = (\mathcal{Q}_{\mu}^{\text{lim}})^+\). Then the 1-skeleton graph of \(R\) coincides with \(\Gamma\). Hence we can apply the arguments (e.g., (3.3), (3.4)) in the previous subsection.

We give characterizations of a class of distances \(\mu\) with \(\dim \mathcal{Q}_{\mu}^{\text{lim}} \leq 1\). Two partial cuts \((A, B)\) and \((A', B')\) are said to be laminar if \(A \subseteq A'\), \(B \supseteq B'\) or \(A \supseteq A'\), \(B \subseteq B'\) or \(A \supseteq A'\), \(B \subseteq B'\), \(B \subseteq A'\). A family \(\mathcal{A}\) of partial cuts is said to be laminar if every pair in \(\mathcal{A}\) is laminar.

**Theorem 3.4.** For a directed distance \(\mu\) on \(S\), the following conditions are equivalent:

1. \(\dim \mathcal{Q}_{\mu}^{\text{lim}} \leq 1\).
2. \(\mu\) has an oriented-tree realization.
3. There are a laminar family \(\mathcal{A}\) of partial cuts on \(S\) and a positive weight \(\alpha : \mathcal{A} \to \mathbb{R}_+\) such that

\[
\mu = \sum_{(A, B) \in \mathcal{A}} \alpha(A, B)\delta_{A, B}.
\]
Proof. We have already seen (1) \(\Rightarrow\) (2) in Section 3.2. Theorem 4.1(2) in Section 4 says that if \(\dim \bar{Q}_{n}^{\text{slim}} \geq 2\) then there is no integer \(k\) such that \(\mu\)-MFP has a \(1/k\)-integral multiflow for every Eulerian network. Therefore, by Theorem 3.2, if \(\mu\) has an oriented-tree realization, then \(\dim \bar{Q}_{n}^{\text{slim}} \leq 1\) necessarily holds. Thus we have (2) \(\Rightarrow\) (1). The equivalence (2) \(\Leftrightarrow\) (3) is not difficult, and is essentially obtained by [14] (in an undirected version).

Directed multiflow locking theorem. Let \(A\) be a set of partial cuts on terminal set \(S\) in a network. We say that a multiflow \(f\) locks \(A\) if \(f\) is simultaneously a maximum \((A, B)\)-flow for all partial cuts \((A, B)\) in \(A\). In the case where \(A\) is laminar, there are an oriented-tree \(T\) and a family \(\{F_{i}\}_{i \in S}\) of subtrees such that \(A\) coincides with the set \(\{(A_{i}, B_{i})\}_{i \in E}\) of partial cuts on \(S\). Consider distance \(\mu := \sum_{(A, B) \in A} \delta_{A, B}\). Then \(\mu\) is realized by \((T, 1; \{F_{i}\}_{i \in S})\). Here \(F_{i}\) is a directed path if (and only if) there is no pair \((A, B), (A', B') \in A\) with \(s \not\in A \cup B \cup A' \cup B'\) and \(A \subseteq B', A' \subseteq B\) or \(B \subseteq A', B' \subseteq A\). Apply Theorem 3.2 to \(\mu\). From the last equality in (3.5), an optimal multiflow is necessarily optimal to \(\delta_{A, B}\)-MFP for each \((A, B) \in A\); see the argument after (3.4). This implies the following:

**Theorem 3.5.** Let \(A\) be a laminar family of partial cuts on \(S\), and let \((G, S, c)\) be an inner Eulerian network. If the Eulerian condition is fulfilled by each terminal \(s\) having a pair \((A, B), (A', B') \in A\) with \(s \in A \cup B \cup A' \cup B'\) and \(A \subseteq B', A' \subseteq B\) or \(B \subseteq A', B' \subseteq A\), then there is an integral multiflow locking \(A\).

This includes Ibaraki–Karzanov–Nagamochi’s result for laminar cuts.

**Theorem 3.6 ([13, Theorem 5]).** Let \(A\) be a laminar family of cuts on \(S\). For every inner Eulerian network \((G, S, c)\), there is an integral multiflow locking \(A\).

0–1 distances and commodity graphs. Suppose the case where \(\mu\) is \(\{0, 1\}\)-valued. In this case, \(\mu\) can be identified with a commodity graph \(H\) by \(st \in EH \iff \mu(s, t) = 1\). For a commodity graph \(H\) on \(S\), let \(\mu_{H}\) denote the corresponding 0–1 distance on \(S\) defined by \(\mu_{H}(s, t) = 1 \iff st \in EH\). In the case where \(H\) is a complete digraph, Lomonosov and Frank independently established the following min–max relation:

**Theorem 3.7 ([1,2]).** Let \(H\) be a complete digraph on \(S\). For every inner Eulerian network \((G, S, c)\), we have

\[
\text{MFP}^{*}(\mu_{H}; G, S, c) = \text{IMFP}^{*}(\mu_{H}; G, S, c) = \sum_{s \in S} \min\{c(\partial X) \mid s \in X \subseteq VG \setminus (S \setminus s)\}.
\]

This theorem can be regarded as a special case of Theorem 3.2. Indeed, the all–one distance is realized by a star with the sink (or source) as its center. So we can extend this theorem to a class of commodity graphs having oriented-tree realizations.

A quasi-complete digraph \(H\) is a simple digraph having a node subset \(T\) such that

1. all edges are incident to \(T\),
2. the subgraph induced by \(T\) is a complete digraph, and
3. all edges between \(T\) and \(V \setminus T\) leave \(T\) or enter \(T\).

The node set \(T\) is said to be the complete part, and \(H\) is said to be source-type if the edges between \(T\) and \(V \setminus T\) enter \(T\) and is said to be sink-type otherwise. For a quasi-complete digraph \(H\) with complete part \(T = \{x_{1}, x_{2}, \ldots, x_{m}\}\), the corresponding \(\{0, 1\}\)-valued distance \(\mu_{H}\) has an oriented-tree realization by a star \(\Gamma\) of \(m\) leaves \(v_{1}, v_{2}, \ldots, v_{m}\) such that the center \(v_{0}\) is a source if \(H\) is source-type, and is a sink if \(H\) is sink-type. Indeed, for \(i = 1, 2, \ldots, m\), let \(R_{i}\) be the subtree consisting of one node \(v_{i}\). For a node \(s \in V \setminus T\), if \(s\) is joined to \(x_{j_{1}}, x_{j_{2}}, \ldots, x_{j_{r}}\), then let \(R_{i}\) be the subtre consisting of nodes \(\{v_{0}, v_{1}, v_{2}, \ldots, v_{m}\} \setminus \{v_{j_{1}}, v_{j_{2}}, \ldots, v_{j_{r}}\}\). Then one can verify that \((\Gamma, 1; \{R_{i}\}_{i \in V})\) is a required realization. In particular, each node \(s\) having at least \(m - 1\) edges is associated with a single node or a directed path in \(\Gamma\).

**Theorem 3.8.** Let \(H\) be a quasi-complete digraph on \(S\) with complete part \(T\), and let \((G, S, c)\) be an inner Eulerian network such that the Eulerian condition is fulfilled by each terminal \(s\) incident to at most \(|T| - 2\) edge in \(H\). Then the following holds:

\[
\text{MFP}^{*}(\mu_{H}; G, S, c) = \text{IMFP}^{*}(\mu_{H}; G, S, c) = \begin{cases} \sum_{s \in T} \min\{c(\partial X) \mid s \in X \subseteq VG \setminus N_{H}(s)\} & \text{if } H \text{ is a sink-type,} \\ \sum_{s \in T} \min\{c(\partial X) \mid N_{H}(s) \subseteq X \subseteq VG \setminus s\} & \text{if } H \text{ is a source-type,} \\ \end{cases}
\]

where \(N_{H}(s)\) is the set of nodes incident to \(s\) in \(H\).

A multipartite extension of a graph \(H\) is a graph obtained by replacing each node \(v\) by a node subset \(U_{v}\) and joining each pair \((x, y) \in U_{x} \times U_{y}\) exactly when \(vu \in EH\). Trivially we can further extend this relation (3.6) to the case where \(H\) is a multipartite extension of a quasi-complete digraph (by super sink/source argument).
Also we easily see from Theorem 3.4 the following:

**Proposition 3.9.** For a simple digraph \( H \) on \( S \), the following conditions are equivalent:

(a) \( \dim \bar{Q}_{\mu, \lim} = 1 \).

(b) \( H \) is a multipartite extension of a quasi-complete digraph.

**Proof of Proposition 3.3.** Take an arbitrary \( s \in S_0 \). We first claim \( \mu_s \in (Q_{\mu, \lim}^+) \). Since \( F_i \) is a single node \( v_i \), we have \( \mu_i(t^s) + \mu_s(u^i) = \mu(t, s) + \mu(s, u) = D_{\mu, \alpha}(F_i, v_i) + D_{\mu, \alpha}(v_i, F_0) \geq D_{\mu, \alpha}(F_i, F_0) = \mu(t, u) \) for \( t, u \in S \). Thus \( \mu_s \in P^\mu \).

Next we give a description of \( K(\mu_i) \). Delete \( v_i \) from \( I \). Let \( I_1, I_2, \ldots, I_k \) be the resulting connected components. For \( i = 1, 2, \ldots, k \), let \( U_i \) be the set of elements \( t \in S \) such that \( F_i \) contains \( v_i \). Then \( (W, U_1, U_2, \ldots, U_k) \) is a partition of \( S \). A pair \( (t^u, t^v) \in \Gamma \) has an edge in \( K(\mu_i) \) if and only if a shortest path from \( F_i \) to \( F_0 \) can pass through the node \( v_i \). We remark that tracing an edge in reverse direction takes zero length. Then we see the following:

(a) Pair \( (u^t, v^t) \in U_i^t \times U_j^t \) has an edge if and only if \( i \neq j \).

(b) Each pair \( (u^t, t^v) \in W^t \times W^t \) has an edge.

(c) \( u^t \) is incident to each element in \( S^t \) and \( v^t \) is incident to each element in \( S^v \).

So there is no isolated node, and thus we have \( \mu_s \in Q_{\mu, \lim}^+ \). By (c), \( \mu_s \) is not a fat. Thus \( \mu_s \in (Q_{\mu, \lim}^+) \) and in particular \( \mu_s \in Q_{\mu, \lim}^+ \). \( \square \)

If true, then we obtain the first statement (1) (since \( (Q_{\mu, \lim}^+) \) is connected). Perturb \( \mu_s \) into \( p \) so that \( p \in (Q_{\mu, \lim}^+) \) and \( E(K(\mu_s)) \leq E(K(p)) \) (i.e., \( p \) belongs to a face containing \( \mu_s \)). Let \( X^- \) be the set of nodes \( u \in S^t \) with \( p(u) < \mu_s(u) \), and let \( X^+ \) be the set of nodes \( u \in S^t \) with \( p(u) > \mu_s(u) \). Recall (2.8). Necessarily \( X^+ = N_{\mu_s}(X^-) \); otherwise there is an isolated node in \( K(p) \).

Claim (3.7) is true. We claim \( X^- = \bigcup_{i \in I} U_i^t \) or \( X^+ = \bigcup_{i \in I} U_i^t \) for some \( I \subseteq \{ 1, 2, \ldots, k \} \).

Thus we can verify \( p \) is a proper fat relative to \( W_0 \); a contradiction. Also \( U_i^t \cup U_j^t \subseteq X^+ \) is impossible by (d). \( \square \)

In the argument above, we can see that the perturbed \( p \) never belongs to any degenerate set; so \( \bar{Q}_{\mu, \lim} \) has no degenerate set. This implies (3). The claim (2) can be verified in a straightforward manner.

(4) Let \( t \) be a terminal such that \( F_i \) is a single node or a directed path. Take any \( s \in S_0 \) with \( v_i \) belonging to \( F_i \). Then \( \mu_s \) belongs to \( \bar{Q}_{\mu, \lim}^+ \). Again perturb \( \mu_s \) into \( p \in (Q_{\mu, \lim}^+) \). It suffices to show \( p \in \bar{Q}_{\mu, \lim}^+ \). In the partition \( |W, U_1, U_2, \ldots, U_k| \) for \( K(\mu_s) \), \( t \) belongs to \( W \). As above, consider \( X^-, X^+ \). Then \( X^- = \bigcup_{i \in I} U_i^t \) or \( X^+ = \bigcup_{i \in I} U_i^t \) for some \( I \subseteq \{ 1, 2, \ldots, k \} \). From the assumption that \( F_i \) is a single node or a directed path, one can see that \( t^+ \) is incident to all nodes in \( S^t \) except \( U_i^t \) for which \( I \) includes the tail of \( F_i \), and that \( t^- \) is incident to all nodes in \( S^t \) except \( U_i^t \) for which \( I \) includes the head of \( F_i \). From this fact, either \( I = \{ i \} \) or \( \{ j \} \); otherwise edge \( t^+ t^- \) vanishes in \( K(p) \) and this implies \( p \notin \bar{Q}_{\mu, \lim}^+ \). Thus we can verify \( p \in \bar{Q}_{\mu, \lim}^+ \). \( \square \)

4.3. Case \( \dim T_\mu \leq 1 \) — Reduction to minimum cost circulation

Suppose \( \dim T_\mu \leq 1 \). In this case, \( T_\mu \) is also a tree. Thus the argument in Section 3.2 is applicable. However, by (2.7)/(5) \( T_\mu \) is a path isometric to a segment in \( (R, D^\mu) \). Therefore by (2.5)/(2) there is a family \( \{ a_t, b_t \} | s \in S \} \) of segments in \( R \) such that

\[
\mu(s, t) = (a_t - b_t)_+ \quad (s, t \in S).
\]

By using this expression, we show that \( \mu \)-MFP is reducible to the minimum cost circulation. Let \( (G, S, c) \) be a network. For each terminal pair \( (s, t) \) with \( \mu(s, t) = (a_t - b_t)_+ > 0 \), add a new edge \( (\text{terminal edge}) tWs \) with edge-cost \( -\mu(s, t) \). Then
consider the minimum cost circulation problem on the new network; this is a relaxation of \( \mu \)-MFP. As is well known, there is an integral minimum cost circulation. This circulation can be decomposed into the sum of the incidence vectors for some (possibly repeating) cycles. If each cycle contains at most one terminal edge, then we obtain an integral optimal multiofflow by deleting the terminal edge from each cycle. So suppose that there is a cycle \( C \) containing at least two terminal edges. Then \( C \) is the union of terminal edges \( t_0 t_1, t_1 t_2, \ldots, t_{k-1} t_k \) and \( S \)-paths \( P_{1,2}, P_{3,4}, \ldots, P_{k-2,k-1}, P_{k,0} \), where \( k \) is an odd integer, and \( P_{i,i+1} \) is an \((t_i, t_{i+1})\)-path. We claim

\[
\sum_{i=0,2,4, \ldots, k-1} \mu(t_{i+1}, t_i) \leq \sum_{i=0,2,4, \ldots, k-1} \mu(t_{i-1}, t_i), \tag{3.8}
\]

where we let \( t_{-1} = t_k \). The LHS \( \equiv (\mu(C)) \) is the negative of the cost of the cycle \( C \) and the RHS is the total flow-value of \( S \)-paths \( \{P_{i,i+1}\}_{i=1,3,5, \ldots} \) (with unit flow-values). Suppose that the claim (3.8) is true. By decomposing each cycle into \( S \)-paths as above, we obtain an integral multiofflow \( f \) whose total flow-value \( \text{val}(\mu, f) \) is at least the negative of the total cost of the mincost relaxation problem. So \( f \) is optimal.

The claim (3.8) can be seen as follows. Move point \( x \) in \( R^s \) as \( a_0 \rightarrow b_1 \rightarrow a_2 \rightarrow b_3 \rightarrow \cdots \rightarrow b_k \rightarrow a_0 \). In each odd step, the point \( x \) moves in the negative direction since \( a_i > b_{i+1} \). In particular the total move over odd steps coincides with \( \mu(C) \). Since the point \( x \) returns to the initial point, \( \mu(C) \) is at most the total moves in the positive direction over even steps, which equals the RHS in (3.8).

### 4. Unbounded fractionality

The integrality theorem (Theorem 3.1) in the previous section is best possible. The goal of this section is to establish the unbounded fractionality property:

**Theorem 4.1.** Let \( \mu \) be a directed distance on \( S \).

1. If \( \dim T_\mu \geq 2 \), then there is a positive integer \( k \) such that \( \mu \)-MFP has a \( 1/k \)-integral optimal multiofflow for every network \((G, S, c)\).
2. If \( \dim G_\mu \geq 2 \), then there is a positive integer \( k \) such that \( \mu \)-MFP has a \( 1/k \)-integral optimal multiofflow for every Eulerian network \((G, S, c)\).

In the following, the edge set of a complete digraph (without loops) on a set \( V \) is denoted by \( E_\mu \). We regard a function \( g : V \times V \rightarrow R^s \) with zero diagonals \( g(x, x) = 0 \) for \( x \in V \) as \( E_\mu \rightarrow R_4^s \); we simply denote \( g(x, y) \) by \( g(xy) \).

We utilize Edmonds–Giles’ lemma for rational polyhedra; see [15, Section 22.1]:

For an integer \( k > 0 \), a rational polyhedron \( P \subseteq R^s \) is \( 1/k \)-integral if and only if \( \min \{ (c, x) \mid x \in P \} \) is a \( 1/k \)-integer vector for some integral vector \( c \in Z^s \) for which the minimum is finite. \( \tag{4.1} \)

Here a polyhedron \( P \) is said be \( 1/k \)-integral if each face of \( P \) contains a \( 1/k \)-integral vector, and \( \langle \cdot, \cdot \rangle \) denotes the standard inner product in \( R^s \). For a finite set \( V \supseteq S \), consider the following two unbounded polyhedra:

\[
\mathcal{D}_{\mu, V} := \{ d : \text{metric on } V \mid d(st) \geq \mu(st) (st \in E_\mu) \} + R_3^V,
\]

\[
\mathcal{D}_{\mu, V} := \mathcal{D}_{\mu, V} + L, \quad \text{where } L := \{ l \in R^V \mid l(C) = 0 \text{ (all cycles } C \text{ in } V) \}. \]

Note that \( \mathcal{D}_{\mu, V} \) is pointed, and \( \mathcal{D}_{\mu, V}^t \) is not pointed. Then \( \min \{ (c, d) \mid d \in \mathcal{D}_{\mu, V} \} \) is finite if and only if \( c \) is nonnegative, and if finite, then it equals \( \text{MFP}^t(\mu; (V, E_\mu), S, c) \). Also \( \min \{ (c, d) \mid d \in \mathcal{D}_{\mu, V}^t \} \) is finite if and only if \( (V, E_\mu), S, c \) is totally Eulerian, and if finite, it equals \( \text{MFP}^t(\mu; (V, E_\mu), S, c) \). Hence it suffices to show:

**Proposition 4.2.** Let \( \mu \) be a directed distance on \( S \), and let \( k \) be any positive integer.

1. If \( \dim T_\mu \geq 2 \), then \( \mathcal{D}_{\mu, V} \) is not \( 1/k \)-integral for some \( V \supseteq S \).
2. If \( \dim G_\mu \geq 2 \), then \( \mathcal{D}_{\mu, V}^t \) is not \( 1/k \)-integral for some \( V \supseteq S \).

Indeed, if a \( 1/k \)-integral optimal multiofflow always exists, then the optimal value is always \( 1/k \)-integral, and \( \mathcal{D}_{\mu, V} \) is \( 1/k \)-integral for all \( V \) by (4.1). The rest of this section is devoted to the proof of this proposition. We note the following relation for two metrics \( d, d' \) on \( V \), which follows from the cycle decomposition of a circulation:

\[
d \equiv d' \mod L \quad \text{if and only if } d(xy) = d'(xy) - p(x) + p(y) (xy \in E_\mu) \quad \text{for some } p : V \rightarrow R. \tag{4.2}
\]

### 4.1. Preliminaries: Minimal and extreme metrics

We begin with preliminary arguments. A metric \( d \in \mathcal{D}_{\mu, V} \) is said to be minimal if there is no other metric \( d' \in \mathcal{D}_{\mu, V} \) with \( d' \neq d \) and \( d' \leq d \), and is said to be \( C \)-minimal if there is no other metric \( d' \in \mathcal{D}_{\mu, V} \) with \( d' \neq d \mod L \) and \( d'(C) \leq d(C) \) for all cycles \( C \).
We first give characterizations of minimal and \( C \)-minimal metrics. Let \( d \) be a metric on \( S \). Then \( d \) defines the equivalence relation on \( S \) by \( x \sim_{d} y \iff d(x,y) = d(y,x) = 0 \). Let \([x]\) denote the equivalence class including \( x \), and let \([xy]\) denote the set of edges from \([x]\) to \([y]\). An edge \( xy \in E_{S} \) is said to be extremal if there is no edge \( st \in E_{S} \setminus \{xy\} \) with \( d(st) = d(x)+d(y)+d(\gamma) \). An edge \( xy \) is extremal if and only if \( x'y' \in [xy] \) is extremal. So a class \([xy]\) is said to be extremal if \( xy \) is extremal. Let \( H_{\mu,d} \) be the directed graph on \( S \) with edge set \( EH_{\mu,d} = \{ st \in E_{S} \mid d(st) = \mu(st) \} \).

**Lemma 4.3.** Let \( d \) be a directed metric in \( D_{\mu,s} \).

1. \( d \) is minimal if and only if every extremal class meets an edge in \( H_{\mu,d} \).
2. \( d \) is \( C \)-minimal if and only if every extremal class meets a cycle in \( H_{\mu,d} \).

We note that \( d(xy) = d(x'y') \) for \( x'y' \in [xy] \) and \( xy \in H_{\mu,d} \) if \([x] = [y]\). In particular, \( xy \) is never extremal if \([x] = [y]\) (and \( d \neq 0 \)). In most cases, we may consider the case where each class consists of one edge.

**Proof.** (1). If part: By condition we cannot decrease \( d \) on extremal classes. Consequently, we cannot decrease \( d \) on non-extremal classes by the triangle inequality. Only-if part: suppose that some extremal class \([xy]\) fulfills \( d(uv) > \mu(uv) \) for \( uv \in [xy] \). We can decrease \( d \) on \([xy]\) with keeping triangle inequality.

(2). For \( p : S \to R \), let \( d \ast p \) be defined by \( (d \ast p)(xy) = d(xy) - p(x) + p(y) \) for \( xy \in E_{S} \). By definition and (4.2), we see:

\[ (*) \quad d \ast p \] is \( C \)-minimal if and only if \( d \ast p \) is minimal for every \( p : S \to R \) with \( d \ast p \in D_{\mu,s} \).

(\#2) The set of extremal edges of \( d \) is the same as that of \( d \ast p \).

First we show the if part (of (2)). Take any \( p : S \to R \) with \( d \ast p \in D_{\mu,s} \), and take any extremal class \([st]\) of \( d \ast p \). Since \([st]\) is also an extremal class for \( d \) by (\#2), \([st]\) meets a cycle \( C \) in \( H_{\mu,d} \). Therefore \( \sum_{xy \in C} (d \ast p)(xy) = \sum_{xy \in C} d(xy) = \sum_{xy \in C} \mu(xy) \). By \( d(xy) = p(x) + p(y) = (d \ast p)(xy) \), \( p \) is constant on \( C \); in particular \( (d \ast p)(xy) = d(xy) = \mu(xy) \) on \( C \). This means that \( d \ast p \) is minimal by (1). Thus \( d \ast p \) is \( C \)-minimal by (\#1).

We show the only-if part. Suppose that some extremal class \([st]\) does not meet any cycle. In this case, there is a node subset \( U \subseteq S \) such that \([s] \subseteq U \), \([t] \subseteq S \setminus U \), and there is no edge entering \( U \) (consider the strong component decomposition of \( H_{\mu,d} \)). For a sufficiently small \( \epsilon > 0 \), let \( p : S \to R \) be defined by \( p(u) = 0 \) for \( u \in U \) and \( p(u) = \epsilon \) for \( u \notin U \). Then \( d \ast p \in D_{\mu,s} \) and \( (d \ast p)(uv) = d(uv) + \epsilon > \mu(uv) \) for \( uv \in [st] \). Thus \( d \ast p \) is not minimal, and \( d \) is not \( C \)-minimal.

Second we recall the notion of extreme metrics. A metric \( d \) on a finite set \( V \) is called extreme if \( d \) lies on an extreme ray the polyhedral cone \( M_{V} \) formed by all metrics on \( V \). Also \( d \) is \( C \)-extreme if the projection \( d/\mathcal{L} \) is extreme in \( M_{V}/\mathcal{L} \). We give a family of extreme metrics. Consider the refinement sequence of triangulations of a plane triangle \((x,y) \in R^{2} | 0 \leq x \leq 1 \) in \((R^{2}, D_{\infty}^{+}) \) by congruent triangles, as in Fig. 6. Let \( \gamma_{n} \) be the metric obtained by restricting \((R^{2}, D_{\infty}^{+}) \) to the vertex set of the \( n \)-th triangulation.

**Lemma 4.4.** \( \gamma_{n} \) is extreme and \( C \)-extreme.

**Proof.** One can easily verify that \( \gamma_{1} \) is extreme. Suppose \( \gamma_{n} = d' + d'' \) for some metrics \( d', d'' \). We show \( \gamma' = \alpha \gamma_{n} \) for some positive \( \alpha \). We observe that the restriction of \( \gamma_{n} \) to each triangle is isometric to \((1/\alpha) \gamma_{1} \), which is extreme. Since the triangulation is connected, we can take a common positive \( \alpha \) such that \( d'(pq) = \alpha \gamma_{n}(pq) \) for \( p, q \) in any triangle. For an arbitrary pair \( p, q \) of vertices, there is a path \( p = p_{1}, p_{2}, \ldots, p_{m} = q \) lying on the triangulation graph such that \( \gamma_{n}(pq) = \sum_{i=1}^{m-1} \gamma_{n}(p_{i}p_{i+1}) \). By \( \gamma_{n} = d' + d'' \), this equality must hold for \( d' \). Hence, we have \( d'(pq) = \sum_{i} d'(p_{i}p_{i+1}) = \alpha \sum_{i} \gamma_{n}(p_{i}p_{i+1}) = \alpha \gamma_{n}(pq) \). Thus we have \( d' = \alpha \gamma_{n} \).

Next we consider the \( C \)-extremality. As above, the \( C \)-extremality of \( \gamma_{n}(n \geq 2) \) reduces to that of \( \gamma_{1} \) by the following observations: For an arbitrary cycle \( C \) there is a cycle \( C' \) in the graph with \( d(C) = d(C') \), and for an arbitrary cycle \( C' \) in the graph there are cycles \( C_{1}, C_{2}, \ldots, C_{m} \) each of which belongs to a triangle such that \( d(C') = \sum_{i=1}^{m} \pm d(C_{i}) \). The \( C \)-extremality of \( \gamma_{1} \) also follows from a routine calculation, and is left to the readers. Sketch: suppose \( \gamma_{1} \equiv d' + d'' \) mod \( \mathcal{L} \). By (4.2), \( \gamma_{1}(xy) + \gamma_{1}(yz) = \gamma_{1}(xz) \) implies the same equality for \( d' + d'' \), which in turn implies the same equality for \( d' \). By using it, we can show \( d'(C) = \alpha \gamma_{1}(C) \) for \( \alpha := d'(xy) + d'(yx) \). \( \square \)
4.2. Proof (metric case)

Suppose that $\mu$ is a metric. In this case, $D_{\mu,V}$ is represented as

$$D_{\mu,V} = \{d : \text{metric on } V | d(st) = \mu(st) \ (st \in E_S)\} + R^+_{xy}.$$

Recall the notions in Section 2.1C. Then, metric $d$ is minimal in $D_{\mu,V}$ if and only if $d$ is a tight extension of $\mu$. Also $d$ is $C$-minimal in $D_{\mu,V}$ if and only if $d$ is a cyclically tight extension of $\mu$.

We first prove Proposition 4.2(2). $Q^-_{\mu} = \bar{Q}_{\mu}$ by Proposition 2.3. We can take a balanced section $R$ in $Q^+_{\mu}$ containing a 2-dimensional face $F$, which is isometric to a polygon in $(R^2, D^+_\infty)$ by (2.7)(3). Therefore we can take a subset $U$ in $F$ whose metric induced by $D^+_\infty$ is isometric to $\beta|u$, for some $\beta > 0$. Fix an integer $k > 0$. By (2.6)(2), for an arbitrary integer $n > 0$, we can take a cyclically tight extension $d$ on $V$ having $\beta|u$ as a submetric. Take a sufficiently large $n$. Since $d/\mathcal{L}$ belongs to a bounded face of $D_{\mu,V}/\mathcal{L}$, we can decompose $d$ into a convex combination of cyclically tight extensions $d_1, d_2, \ldots, d_m$ in modulo $\mathcal{L}$ such that $d_i/\mathcal{L}$ is an extreme point in $D_{\mu,V}/\mathcal{L}$ for $i = 1, 2, \ldots, m$. Since $d$ has $\beta|u$ as a submetric and $\gamma|u$ is $C$-extreme, some $d_i$ has a submetric $\gamma$ with $\gamma \equiv \alpha\beta|u \mod \mathcal{L}$ for some $\alpha > 0$. Therefore $d_i(pq) + d_i(qp) = \alpha\beta/n$ for some $pq \in E_V$. Since $d_i$ is cyclically tight, it is embedded into $(Q^+_{\mu}, D^+_\infty)$. Therefore $\alpha\beta$ is bounded by the diameter of $Q^+_{\mu}$ (bounded set). Since $n$ is sufficiently large, we have $\alpha\beta/n < 1/k$. Hence face $d_i + \mathcal{L}$ has no $1/k$-integer vector.

Proposition 4.2(1) can be shown in a similar manner. Since $T_{\mu}$ has a 2-dimensional face, we can take a tight extension $d$ having $\beta|u$ as a submetric. Since $D_{\mu,V}$ is pointed and $d$ belongs to a bounded face in $D_{\mu,V}$ by minimality, we can decompose $d$ into a convex combination of extremal points in $D_{\mu,V}$. For a sufficiently large $n$, one of the summands is not $1/k$-integer as above.

4.3. Proof (general case)

Suppose that $\mu$ is not a metric. For a distance $g$ on $S$ and a subset $U \subseteq S$, the restriction of $g$ to $U$ is denoted by $g_U$.

**Lemma 4.5.** Let $d$ be a directed metric in $D_{\mu,V}$.

1. If $d_S$ is minimal in $D_{\mu,S}$ and $d$ is a tight extension of $d_S$, then $d$ is minimal in $D_{\mu,V}$.
2. If $d_S$ is $C$-minimal in $D_{\mu,S}$ and $d$ is a cyclically tight extension of $d_S$, then $d$ is $C$-minimal in $D_{\mu,V}$.

**Proof.** (1) is obvious from the definition. (2) is not so obvious. We utilize Lemma 4.3 by extending $\mu$ to $\bar{\mu} : E_V \rightarrow R_+$ by $\bar{\mu}_x := \mu$ and $\bar{\mu}(xy) := 0$ for $xy \notin E_S$. Take an extremal class $[xy]$ of $d$. We show $[xy] \cap E_S \neq \emptyset$. If true, then $[xy] \cap E_S$ is also an extremal class in $d_\mu$ and meets a cycle of $H_{\mu,d}$ and $d_S$ is $C$-minimal in $D_{\mu,V}$. By Lemma 4.3(2), $[xy]$ meets a cycle $C$ in $H_{\mu,d}$. If this cycle belongs to $E_V \setminus E_S$, then by $d(uv) = d(vu) = 0$ for $uv \in C$, the triangle equality holds, and $d = 0$ on $E_V \setminus E_S$. Then we assume that $C$ includes a path $(u, x, y, v)$ with distinct $u, v \in S$. In particular $d(uv) = d(vu) = d(xy) = d(yu) = 0$. Since $d(uv) = d(uv) + d(xy) + d(yv)$ and $xy$ is extremal, $d(xy) = 0$, and thus $[x], [y] = ([u], [v])$. So $[xy] \cap E_S \neq \emptyset$. □

Our final goal is the following:

**Lemma 4.6.** Let $\mu$ be a directed distance on $S$.

1. If $\dim T_{\mu} \geq k$, then there is a minimal metric $d$ in $D_{\mu,S}$ with $\dim T_d \geq k$.
2. If $\dim \bar{Q}^+_{\mu} \geq 2$, then there is a $C$-minimal metric $d$ in $D_{\mu,S}$ with $\dim \bar{Q}_d \geq 2$.

Assuming the validity of this lemma, we complete the proof of Theorem 4.1. We only show (2) in this theorem; again (1) can be shown in a similar way. By this lemma, we can take a $C$-minimal metric $d$ in $D_{\mu,S}$ with $\dim \bar{Q}_d \geq 2$. Take a cyclically tight extension $d'$ of $d$ such that $d'$ contains $\beta|u$ as a submetric for sufficiently large $n$. Since $d'$ is cyclically tight (Lemma 4.5(2)), $d'$ is decomposed, in modulo $\mathcal{L}$, into a convex combination of $C$-minimal metrics $d_1, d_2, \ldots, d_m$ such that each $d_i$ is an extreme point of $D_{\mu,V}/\mathcal{L}$. Some $d_i + \mathcal{L}$ has no $1/k$-integer vector, as in the metric case.

Let us start the proof. For $p \in P_{\mu}$, let $X_p$ be the set of elements $s \in S$ with $p(s^t) + p(s^t) = 0$. Our argument crucially relies on the following claim:

For $p \in P_{\mu}$, there is a minimal metric $d \in D_{\mu,S}$ such that $p \in P_d$, and

$$d(s^t) + p(s^t) = d(st), \quad \text{if } s^t \in E_{K_{\mu}}(p),$$

$$d(s^t) + p(s^t) > d(st), \quad \text{otherwise}, \quad (s, t \in S \setminus X_p).$$

**Proof.** Replacing $p$ by $p + \alpha(1, -1)$ for some $\alpha$ if necessary, we may assume that $p$ is nonnegative. Let $d$ be the distance on $S$ defined by

$$d(st) = p(s^t) + p(t^t), \quad (s, t \in S).$$
Then $d$ is a metric, more precisely, $d$ is realized by a subdivision of a star. We try to decrease $d(st)$ for $s, t \in S \setminus X_p$ with $s \neq t \notin EK(p)$ keeping the triangle inequality. Since one of $p(s')$, $p(s')$ and one of $p(t')$, $p(t')$ are positive, there is no $u \in S \setminus \{s, t\}$ such that $d(du) + d(st) = d(ut)$ or $d(st) + d(du) = d(st)$. Let $d(st) = d(st) - \epsilon$ for small $\epsilon > 0$ and $s, t \in S \setminus X_p$ with $s \neq t \notin EK(p)$; we remark $d(st) > \mu(st) \geq 0$. Then $d$ does not violate the triangle inequality. So $d$ is not minimal, and we can take a minimal metric $d' \in D_{\mu,5}$ with $d' \leq d$. □

**Proof of Lemma 4.6.** (1). By (2.7)(2), we can take a point $p \in T_\mu$ such that $K_\mu(p)$ has at least $k$ components having no $u \in S \setminus X_p$ with $p(\mu(u)) = 0$. Take a minimal metric $d$ in (4.3). Let $U := S \setminus X_p$. Consider the restrictions $\mu_U$ of $\mu$ to $U$ and $p_U$ of $p$ to $U^\sigma$. Then $p_U$ belongs to $T_{\mu_U}$ and $K_{\mu_U}(p_U)$ has $k$ components. By (2.7)(2) we have $dim T_{\mu_U} \geq k$. One can easily see that $T_{\mu_U}$ is the surjective image of the projection of $T_{\mu}$: necessarily dim $T_{\mu} \geq dim T_{\mu_U} \geq k$. □

**Proof of Lemma 4.6.** (2). Suppose that dim $Q_\mu^{slim} = k$ for $k \geq 2$. We try to find a triangle $(U, d, p)$ of $U \subseteq S$, a metric $d$ on $U$, and $p \in Q_\mu$ such that $d$ is $\epsilon$-minimal in $D_{\mu, U}$ and $K_d(p)$ has at least $3$ components, which implies Lemma 4.6(2) by (2.7)(2') and the following claim.

For $U \subseteq S$, let $d$ be a $\epsilon$-minimal metric in $D_{\mu, U}$. Then there is a $\epsilon$-minimal metric $d^*$ in $D_{\mu, 5}$ with $dim Q_\mu^{slim} \geq dim Q_\mu^{slim}$. □

**(4.4)**

**Proof.** Extend $d$ to $d'$ in $D_{\mu, 5}$ with $d' \equiv d$. Then $d'$ may not be $\epsilon$-minimal in $D_{\mu, 5}$. We can take a $\epsilon$-minimal metric $d''$ in $D_{\mu, 5}$ such that $d''(C) \leq d(C)$ for all cycles $C$ in $S$. Since $d$ is $\epsilon$-minimal in $D_{\mu, U}$, we have $d \equiv d'' \mod L$. Then $Q_\mu$ is a translation of $Q_{d''}$, and hence dim $Q_\mu = dim Q_{d''}$. Since every point $p \in Q_{d''}$ can be extended to a point in $Q_{d'}$, we have dim $Q_\mu^{slim} \geq dim Q_\mu^{slim}$. □

We can take $p \in Q_\mu^{slim}$ such that $p/ \sim$ belongs to the interior of $k$ dimensional face in $Q_\mu^{slim}$ for $k \geq 2$. There are three cases:

(i) $K_\mu(p)$ has at least $3$ components and $X_p = \emptyset$.

(ii) $K_\mu(p)$ has at least $3$ components and $X_p \neq \emptyset$ ($p$ belongs to $Q_\mu$).

(iii) $K_\mu(p)$ has at least $4$ components one of which is a (complete bipartite) component of nodes $X_p \Delta X_p' \subseteq X_p$ ($p$ belongs to the interior of $Q_\mu^{deg}$).

In the following, for a distance $g$ on $S$ and a point $p \in \mathbb{R}^{\sigma_S}$, we denote by $H_g(p)$ the directed graph on $S$ with $EH_g(p) := \{st \mid x \neq t \notin EK(p)\}$ (possibly including loops).

We first consider case (i). Suppose that $H_g(p)$ is strongly connected. Since $X_p = \emptyset$, by (4.3) we can take a minimal metric $d \in D_{\mu, 5}$ such that $d''(C) \leq d(C)$ for all cycles $C$ in $S$. Since $d$ is $\epsilon$-minimal in $D_{\mu, U}$, we have $d \equiv d'' \mod L$. Then $H_{d''}$ is also strongly connected, which immediately implies the $\epsilon$-minimality of $d$ by Lemma 4.3(2). Thus $(S, d, p)$ is a required triple.

Suppose that $H_g(p)$ is not strongly connected. Take a chordless cycle $(x_1, x_2, \ldots, x_m)$ (without repeated nodes). Since $X_p = \emptyset$, equivalently, $H_g(p)$ has no loop, the cycle length $m$ is at least two. Suppose $m \geq 3$. Then let $U \leftarrow\{x_1, x_2, \ldots, x_m\}$, and $p \leftarrow p_U$ (the restriction of $p$ to $U^\sigma$). Then $H_{d_U}(p)$ is a cycle of length $m$, and $K_{d_U}(p)$ is a matching of size $m$ (has $m$ components). According to (4.3), we can take a minimal metric $d \in D_{\mu, U}$ with $p \in Q_d$ and $K_{d_U}(p) = K_d(p)$. Thus $H_{d_U,d}$ is strongly connected, and $(U, d, p)$ is a required triple.

Suppose that there is no simple chordless cycle of length at least $3$ in $H_g(p)$. Since each node has both entering and leaving edges, we can take from $H_g(p)$ two disjoint $2$-cycles $(x_1, x_2, y_1, y_2)$ without edges from $\{y_1, y_2\}$ to $\{x_1, x_2\}$. Let $U \leftarrow \{x_1, x_2, y_1, y_2\}$, and $p \leftarrow p_U$. Again we can take a minimal metric $d \in D_{\mu, U}$ with $K_{d_U}(p) = K_d(p)$. For a small positive $\epsilon$, decrease $p$ by $\epsilon$ on $\{y_1, y_2\}^\complement$ and increase $p$ by $\epsilon$ on $\{y_1, y_2\}$. Then $K_d(p)$ has four components consisting of four disjoint edges $x_1x_2, x_2x_1, y_1y_2, y_2y_1$. Suppose that $d$ is not $\epsilon$-minimal (otherwise $(U, d, p)$ is a required triple). So we may assume that $H_{d_U,d}$ has an extremal edge from $\{x_1, x_2\}$ to $\{y_1, y_2\}$, and has no edge from $\{y_1, y_2\}$ to $\{x_1, x_2\}$. Then we can construct a $\epsilon$-minimal metric from $d$ by adding

| $x_1$ | $x_2$ | $y_1$ | $y_2$ |
|---|---|---|---|
| $x_1$ | 0 | 0 | $\epsilon$ | $\epsilon$ |
| $x_2$ | 0 | 0 | $\epsilon$ | $\epsilon$ |
| $y_1$ | $-\epsilon$ | $-\epsilon$ | 0 | 0 |
| $y_2$ | $-\epsilon$ | $-\epsilon$ | 0 | 0 |

for $\epsilon > 0$ and decreasing some of $d(x_1, y_2)$. To keep $p \in Q_d$, decrease $p$ by $\epsilon$ on $\{y_1, y_2\}^\complement$ and increase $p$ by $\epsilon$ on $\{y_1, y_2\}$. Then the resulting $K_d(p)$ is a matching consisting of the four edges. Thus $(U, d, p)$ is a required triple.

Next we consider case (ii). Let $U \leftarrow S \setminus X_p$ and let $p \leftarrow p_U$. Then $K_{d_U}(p)$ has no isolated node. Obviously, $K_{d_U}(p)$ has at least $3$ connected components, and $X_p = \emptyset$. Thus $p \in Q_{\mu_U}$. Therefore, the situation reduces to case (i).

Finally we consider case (iii). We use the same projection idea. Take $x \in X_p$. Let $A' \subseteq S \setminus X_p$ and $B' \subseteq S \setminus X_p'$ be the sets of nodes $u$ covered only by $x'$ and $x'$, respectively. Suppose $A' \cup B' = \emptyset$. Then let $U \leftarrow S \setminus x$, and let $p \leftarrow p_U$. Then
$X_p$ decreases, $p \in Q_{ slu}^{\text{slim}}$, and $K_{ slu}(p)$ has at least 3 connected components. Suppose that both $A^c$ and $B^c$ are nonempty. For a small positive $\epsilon$, decrease $p$ by $\epsilon$ on $A^c \cup B^c$ and increase $p$ by $\epsilon$ on $x^c, x^r$. Then $X_p$ decreases, $p \in Q_{ slu}^{\text{slim}}$, and $K_{ slu}(p)$ has at least 5 connected components. Suppose that one of $A^c$ and $B^c$, say $B^c$, is empty. Then $x^c$ is necessarily incident to $S^c \setminus x^r$; otherwise $p$ is a proper fat relative to $\{x\}$, a contradiction. For a small positive $\epsilon$, decrease $p$ by $\epsilon$ on $A^c$ and increase $p$ by $\epsilon$ on $x^r$. Again $p \in Q_{ slu}^{\text{slim}} (x^c$ is still covered), and $K_{ slu}(p)$ has at least 4 connected components; Repeat it until $X_p = \emptyset$. After that, the situation reduces to case (i).

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