Reduction of a family of metric gravities

With highlights on conservation laws in metric formulations, consistency before dynamics, and a fresh view on the unity of Newtonian and Einsteinian gravity

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Abstract. A recent proposal by Shuler regarding a postulate-based derivation of a family of metrics describing the gravitational field outside a static spherically symmetric mass distribution is reviewed. All of Shuler’s gravities agree with the Schwarzschild solution in the weak-field limit, but they differ in the strong-field domain, i.e., close enough to a sufficiently compact source of the field. It is found that the evoked postulates of i) momentum conservation and ii) consistency of field strength measurement are satisfied in all metric theories of gravity compatible with the Einstein equivalence principle, no matter what the form of the metric. Therefore, they cannot be used, within any correct deduction, to derive a particular metric. Shuler’s derivations are based on an inconsistent set of correspondences between local and distant quantities. Furthermore, it is shown here that out of the family of possible metrics given by Shuler only one member, the Schwarzschild metric, satisfies a standard relativistic generalization of Newton’s law of gravitation, suggesting the others to be unphysical.

Key words. Metric gravity, conservation laws, kinematic consistency, postulational approach, Schwarzschild metric, meaning of Einstein’s equation

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1 Introduction

There are a number of reasons why one might wish to derive the metric around a spherically symmetric mass distribution without using Einstein’s field equations. One is teaching: several appealing physical results, including those referring to the four classical tests of general relativity, are derivable from that metric. This may then be motivating enough for students to tackle the mathematics of the field equations, whereas plunging into difficult differential geometry first might rather demotivate them. Another is historical interest: it is amusing to speculate how Einstein, had he been in the possession of an exact solution before the field equations, might have avoided some of the tedious detours (such as the “Entwurf theory” of 1913) that he actually took [1], before finally settling on the correct field equations in a race with Hilbert. Third, a successful shortcut towards the Schwarzschild solution may be helpful in approaching a deeper understanding of the theory of gravity, both for the teacher and students.

These were some of the motivations for a series of papers [2,3,4] I wrote in the last few years, exploring roads to the Schwarzschild metric with as few postulates as possible beyond the standard ingredients, two of which come from the “easy” sector of general relativity (GR) describing motion of matter in the gravitational field, viz. the equivalence principle and special relativity (SR), the third being the Newtonian limit (NL), required to apply far enough from matter distributions. In Ref. [2], the single postulate was used that a static gravitational field should be source-free in vacuum, which made a certain divergence vanish and provided a relationship between the two functions that may be chosen freely in a time independent spherically symmetric metric after coordinates have been fixed. This condition

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[1] The most general form of a stationary spherically symmetric metric contains four independent functions of the radial coordinate [3], of which two can be eliminated by requiring, first, the radial coordinate of an event to be the length of the circumference, divided by 2π, of a circle about the symmetry center through it and, second, the time coordinate to be orthogonal to the spatial foliation.
was not sufficient to fully determine the metric $g^{rr}$ so the known experimental result on the perihelion precession of Mercury was used to fix the $g_{rr}$ coefficient to lowest nontrivial order in $1/r$, which then yielded a weak-field metric that was sufficiently accurate to predict the famous factor of two between Einstein’s 1911 result for light-bending by the Sun and his final prediction. In the second paper $[3]$, another postulate was introduced, this time a dynamical one, requiring gravity to not deform the shape of a spherical (scalar) wave about the symmetry center, in a freely falling frame. This postulate is based on stipulating mathematical simplicity of a law of physics. To the large majority of physicists, it may seem less convincing than the first postulate with its clear physical contents. While these two postulates were sufficient to derive the full Schwarzschild metric, the mathematical nature of the second and the mere fact that two postulates were needed rather than a single one, may have impaired the persuasive power of the approach. This was remedied in the final article $[4]$, in which a single requirement on the behavior of dust balls in a gravitational field turned out to enable a derivation of the exact Schwarzschild metric. Moreover, this requirement is, as I will argue in this article, nothing but a straightforward generalization, according to well-accepted rules, of Newton’s universal law of gravitation (in vacuum) to the relativistic case. Conceptually, things cannot get simpler than this, and the approach is convincing independent of the question whether to interpret GR formulas as referring to space-time itself or to tensor fields on some unexplored background.

Some care was taken in these articles to avoid the fallacies of preceding attempts at a simple derivation of the Schwarzschild metric. In particular, the postulates employed apply to the limit of a parallel gravitational field as well and allow one to thus derive the Rindler metric, a test that any postulatory approach to the Schwarzschild metric should have to pass $[3,4]$. Moreover, they are not derivable from the Einstein equivalence principle (EEP), special relativity and/or the limit of Newtonian behavior of the gravitational field at infinity. After all, it has been demonstrated in some mathematical detail that these three ingredients are insufficient to rigorously justify the Schwarzschild metric $[5]$. This should be obvious: the field-theoretic sector of GR describing the generation of gravity by mass-energy contains additional physics beyond the matter sector characterizing motion of particles in an existing gravitational field. The physics of the latter arises from SR augmented by the EEP, but the solution to the question of how gravitational fields are generated required additional ideas.

There have been many attempts at straightforward derivations of the Schwarzschild metric or some of its consequences (such as the correct description of light bending) in the history of modern physics. These were probably motivated by the simple form of the metric in Schwarzschild coordinates. An early idea by Lenz $[6]$ and Schiff $[7]$ became quite influential, as modern authors repeated their mistake $[8,9]$ in spite of Rindler’s demonstration $[10]$, via a counterexample, that Schiff’s argument does not work and notwithstanding the mathematical analysis given by Gruber et al. $[5]$. In particular, Rindler shows that while the $g_{tt}$ term of the metric may be inferred from time dilation (as observable via photon redshift) on the basis of appropriate clock dropping experiments and that a procedure involving the dropping of rulers may be made to work for the definition of lateral coordinates leading to $g_{rr}$ and $g_{r\varphi}$, a similar method for radially oriented rulers will not produce a result on $g_{rr}$.

The first approach obtaining the Schwarzschild metric via postulates taking the place of the field equations seems to be due to Tangherlini $[11]$. While it was favorably received by Sacks and Ball $[12]$ in yet another rebuttal of Schiff-like arguments, Rindler later demonstrated $[13]$ that one of postulates of Ref. $[11]$ (termed “strong version of the principle of equivalence” by its author) was in fact a coordinate postulate that failed when applied in attempting to derive the Rindler metric from it. Hence the postulate was not convincing, as its validity seemed to be restricted to the spherically symmetric case.

A remarkably concise postulate-based derivation of the Schwarzschild metric has been given recently by Dadhich $[14]$. Unfortunately, one of his postulates is without plausible foundation in the absence of the field equations. To wit, he requires the “acceleration of a photon” $\tilde{r}$ to be zero on radial geodesics. Here, $r$ is the circumferential radial coordinate and the double derivative is with respect to an affine parameter on null geodesics. If, instead, the metric had been written in isotropic form (see, e.g., $[15]$) with a different radial coordinate (often denoted as $\rho$), the radial acceleration of a photon would have been nonzero. So the postulate remains unconvincing as long as no argument is given, why it is the circumferential coordinate that should lead to vanishing coordinate acceleration. Dadhich uses this postulate to derive $g_{rr}g_{rr} = -1$ in the Schwarzschild metric. In fact, it has been shown before that, in a static metric, a sufficient and necessary condition for this property to hold is that the radial coordinate is an affine parameter on a radial null geodesic $[15]$. Hence, it holds true in the Schwarzschild metric but its analog does not hold in the standard form of the Rindler metric. Dadhich’s approach thus fails the test of Rindler’s counterexample as well.

Finally, Shuler published, in this journal $[17]$, an alleged derivation of a “family of metric gravities”, including the Schwarzschild solution, from conservation principles instead of the field equations. To obtain the Schwarzschild metric proper, he requires in fact three postulates, which looks like a step backward in comparison with Ref. $[4]$, where just a single one is needed.

The purpose of the present paper is to show, on the one hand, that out of this family only a single member satisfies a straightforward generalization of Newton’s law of gravitation to the relativistic case, using well-known rules of how to transform kinematic laws from their prerelativistic form into a correct relativistic version. No surprise, the surviving
member of the family is just the one predicted by GR. On the other hand, I would like to point out that the two postulates on which Shuler bases the central part of his derivation, i.e., conservation of momentum and consistency of field strength measurement are satisfied in any metric theory due to the equivalence principle and thus do not constrain the metric. They are not independent postulates but incorporated in the Lagrangian equations of motion following from the metric.

That Shuler nonetheless obtains a condition for the coefficients of the metric, is due to his use of inconsistent expressions connecting distant with local quantities. This leads him to the erroneous belief that the inverse proportionality of $g_{tt}$ and $g_{rr}$ is a physical principle rather than a result of coordinate choice, in spite of the mathematical analysis of Ref. [16] proving the contrary.

A postulate with physical content used by Shuler (beyond the EEP, SR and NL) is the requirement of a form of Gauss’s law (effectively stating that the field has no sources in vacuum). This indeed constrains the metric by establishing a relationship between the two radial functions appearing in it. However, Shuler discards this – most sensible – postulate in favor of far less convincing alternatives to obtain a family of metrics.

The rest of this article is organized as follows. In sect. 2, the meaning of conservation of energy and momentum in the framework of a metric description is discussed. It is shown that these conservation laws lead to the Lagrangian equations of motion, aka geodesic equations, but do not constrain the metric in any way. Section 3 deals with the question how certain quantities pertaining to local phenomena at one place in the metric can be defined operationally for a distant observer (at some other place in the metric). In the absence of uniqueness, there is some choice in doing so, but a sanity requirement is that any set of definitions must be internally consistent. It is argued that Shuler’s set is inconsistent. Moreover, consistency of field strength measurement is shown not to lead to constraints on the metric. Shuler’s idea that $g_{tt}g_{rr} = -1$ reflects a property of space-time rather than one of coordinates is refuted. Finally, it is demonstrated in sect. 4 that a particular form of Newton’s law of gravitation may be generalized to a relativistically valid form in a standard straightforward manner. This generalization yields a criterion to pick out a single member of Shuler’s family of gravities. Section 5 summarises and concludes the paper.

2 Conservation laws

Shuler [17] mentions Noether’s theorem implying energy conservation as a consequence of the homogeneity of time and momentum conservation as a consequence of the homogeneity of space. He then talks about choosing homogeneous observer coordinates, albeit without considering whether these are possible at all. Clearly, there are situations where no homogeneous (global) coordinates can be introduced. The surface of a sphere is a homogeneous (and isotropic) two-dimensional space. Nevertheless, no finite coordinate patch on a sphere is homogeneous. The line element on a sphere with radius $a$, written in standard spherical coordinates,

$$\mathrm{d}s^2 = a^2 \left( \mathrm{d}\vartheta^2 + \sin^2 \vartheta \, \mathrm{d}\varphi^2 \right),$$

is inhomogenous in the coordinate $\vartheta$ and still describes a homogeneous space. So whether space is homogeneous or not is not a question of the homogeneity of coordinates. Homogeneity means invariance under spatial translations and is a coordinate independent property.

It is then a somewhat bizarre claim by Shuler that if the circumferential radial coordinate $r = C/2\pi$ is used as one of the coordinates in a spherically symmetric metric, the latter will be homogeneous in $r$. This is not even true for flat space! Let us write out the Minkowski metric in Cartesian and in spherical coordinates:

$$\mathrm{d}s^2 = -c^2 \mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = -c^2 \mathrm{d}t^2 + \mathrm{d}r^2 + r^2 \left( \mathrm{d}\vartheta^2 + \sin^2 \vartheta \, \mathrm{d}\varphi^2 \right).$$

A Lagrangian describing the motion of a free particle in this metric can be obtained setting $L = (\mathrm{d}s/\mathrm{d}\tau)^2$, where $\tau$ is the proper time of the particle. Multiplication of a Lagrangian by a constant produces a new valid Lagrangian and $L' = \frac{1}{2}m (\mathrm{d}s/\mathrm{d}\tau)^2$ is a form that reduces, in the nonrelativistic limit, to the classical Lagrangian up to an additive constant (see Ref. [2]). We read off [4] that the Lagrangian does not depend on either $x$, $y$ or $z$, hence the canonical momenta $p_x$, $p_y$, and $p_z$ are conserved and since these are the Cartesian components of the momentum $\mathbf{p}$, a free particle will retain its initial momentum in this metric. Momentum is conserved.

However, we also read off that the Lagrangian does depend on $r$ explicitly, meaning that the radial momentum component $p_r = \mathbf{p} \cdot \mathbf{e}_r$ is not a conserved quantity, in general. ($e_r = \sin \vartheta \cos \varphi \, e_x + \sin \vartheta \sin \varphi \, e_y + \cos \vartheta \, e_z$ is a unit vector in the radial direction.) Indeed, this can be easily verified by way of an example. Consider a particle moving parallel to the $z$ axis (but not along it). Its momentum may then be written $\mathbf{p} = pe_z$. Since momentum is conserved, $p$ is constant in time. We have $p_r = p e_z \cdot \mathbf{e}_r = p \cos \vartheta$ and the angle $\vartheta$ varies, as the particle moves from the region $z < 0$ to $z > 0$, from a value larger than $\pi/2$ to a value smaller than $\pi/2$. Hence $p_r$ is not conserved, and this is so, precisely, because space is not homogeneous in $r$ (i.e., the metric is not independent of $r$).
If we treat gravity in a Newtonian framework as a one-body problem, i.e., we consider how a test body moves in the gravitational field of a much more massive one, space is clearly not homogeneous, as we have a gravitational field, and fields are properties of space, by definition. Of course, the momentum of such a test body will change under the gravity of the massive body. While we may still state that the total momentum of the test body and the gravitating one will be conserved (if the ensemble is embedded in homogeneous space) and a similar statement would be true in the general relativistic framework, this is not of relevance for the question of momentum conservation of the test body alone. If Shuler nonetheless speaks of momentum conservation, he must have something different in mind. As it happens, people often refer to the momentum balance equation \( \mathbf{F} = \frac{dp}{dt} \) (i.e., Newton’s second axiom) as momentum conservation, and this might also be Shuler’s meaning. A relativistic statement of this would seem to require us to specify the relativistic meaning of force.

Actually, here we are in a fortunate situation due to the fact that relativists prefer an interpretation, in which gravity is \textit{not} a force. If there is no force, we must have some sort of momentum conservation. And indeed, the equivalence principle reveals this (plus energy conservation) to be true and, consequently, provides us with the full set of equations of motion in a given metric. The EEP says that a sufficiently small freely falling system is a local inertial system, in which the physics of SR applies. That is, the motion of test particles not subjected to additional forces is governed by the metric (2) and hence, momentum conservation applies. However, the momentum will change as soon as the particle has moved far enough for the local inertial system to lose its validity or after enough time has passed for that to happen (as locality is to be required not only in space but also in time). This momentum change is due to the fact that in the new local inertial system that may be constructed around the particle at a later time, momentum will have a different value, for momentum typically changes under a change of the frame of reference. Momentum is a conserved quantity but not an invariant.

Now we can see immediately, how the equations of motion in a metric follow from the equivalence principle and local energy and momentum conservation. Local energy and momentum conservation imply that the Lagrangian from the Minkowski metric (2) (or one equivalent to it) governs the motion. The existence of a local frame of reference in which this applies is secured by the EEP. Since Lagrangian equations of motion are derivable from Hamilton’s principle, which is coordinate free, they take the same general form in whatever coordinates we may choose. So if we transform back to the global coordinates describing the metric in an extended patch of space-time, the equations of motion must still be the Lagrangian equations of motion (following from the form of the metric in the global coordinates). This transformation is trivial, as we constructed our Lagrangian as an invariant (both ds and dr are invariants) under coordinate transformations, so it is still given by \((ds/dr)^2\), now with the line element written in terms of the global metric. The equations of motion so obtained are the geodesic equations \[18\].

Hence, the geodesic equations are a direct consequence of energy and momentum conservation in local freely falling systems, due to the equivalence principle, for \textit{any form of the metric}. Therefore, local energy or momentum conservation cannot constrain the form of the metric.

It might be added that the distinction between momentum and proper momentum made by Shuler is not particularly useful. Proper momentum would have to be defined either as the spatial part of four-momentum, in which case it would just be momentum, or as the momentum of an object in its own, co-moving, frame of reference, in which case it would always be zero.

### 3 Consistency of field strength measurement

Shuler’s second postulate is consistency of field strength measurement, abbreviated CFS. In words, it requires, reasonably, that a force, applied to a tether to counter any weight at the bottom of the tether must be equal to the force that we obtain when using the tether as a measuring device, exploiting known relationships between the energy at the bottom end of a tether and its perception at the top. The tether argument was introduced in Ref. [2] to avoid (or solve) the problem of comparing radial lengths at different radii in a spherically symmetric metric. So the requirement is entirely rational, but the question arises immediately whether it should not be automatically satisfied in any sensible theory. It should therefore not be necessary to ask for it in a separate postulate. Stated differently, a theory that is inconsistent at the kinematic level and needs a dynamic law (in the form of an additional postulate) to repair this, is not well-formulated to begin with.

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3. The new frame usually has a different velocity from the old one.
4. Whereas I invented this argument for myself, it is so simple that there can be little doubt that similar arguments were developed many times before.


3.1 How to measure distant properties?

To discuss Shuler’s approach, we need some notation. Let us write the line element for a static spherically symmetric space-time as

$$ds^2 = g_{tt}(r) c^2 dt^2 + g_{rr}(r) dr^2 + g_{\vartheta\vartheta} d\vartheta^2 + g_{\varphi\varphi} d\varphi^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (\mu, \nu = t, r, \vartheta, \varphi), \tag{3}$$

where $g_{tt} = r^2$, $g_{\vartheta\vartheta} = r^2 \sin^2 \vartheta$, off-diagonal elements are zero, and we use the Einstein summation convention in the second formula. Note that $dx^\mu = c dt$, so our $g_{tt}$ differs by a factor of $c^{-2}$ from Shuler’s, who introduces additional notation, setting $\varsigma = 1/\sqrt{-g_{tt}}$ and $\rho_{rr} = \sqrt{g_{rr}}$.

Then Shuler writes [his equations (3)] a number of “coordinate transformations” for locally measured quantities at some point $r (\vartheta, \varphi)$ in the metric, allegedly relating them to the “observer reference” far from the point $r$.

Clearly, this is a misnomer. A (passive) coordinate transformation is just a relabeling of the coordinates associated with events in space-time. It does not affect the object to which coordinate labels are attached. Also, it is local, i.e., the transformation describes how the coordinates of a particular point in space-time are changed going from one labeling to the other. So it cannot be used to connect distant points in space-time.

Active coordinate transformations describe some motion of objects, e.g., a rotation. In Euclidean space, we are entitled to believe that the object does not change its properties under this motion, due to the symmetries of the space. Objects are freely movable. This is still true in spaces of constant curvature. As soon as the curvature is different at different points, the motion of an object may affect its geometric properties. Stresses may build up, to which the object has to react and, therefore, we cannot be sure it will remain the same after the transformation. Doing experiments (e.g., rotating objects consisting of different materials in the same way), we may find out about such effects and quantify them. Shuler’s relationships, to which no operational meaning is given, are not of this type either. So they are not coordinate transformations.

A more precise idea about the meaning of these relations is gleaned by inspecting them more closely. Most often, their right-hand side contains a local quantity such as a time or length interval, a velocity or an acceleration at some position in the field, to which for definiteness I will assign the radial coordinate $r_1$. For these quantities, there exist well-defined measuring procedures using devices that can be set up in close vicinity of the object to be measured. The measurement result will, within the accuracy of the device, not depend on the kind of apparatus used. Moreover, we have clear recipes of expressing the results in terms of the coordinates appearing in (4). For example, a radial velocity is given by the increment of radial proper length element divided by the proper time of the coordinate stationary observer at $r_1$:

$$v_{\text{rad}} = \frac{df_{\text{rad}}}{dt} = \frac{\sqrt{|g_{rr}(r_1)|} dr}{\sqrt{-g_{tt}(r_1)} dt} \bigg|_{r_1} = \frac{2r_1}{-g_{rr}} \frac{dr}{dt} \bigg|_{r_1} \ . \tag{4}$$

On the left-hand side of Shuler’s equations, we have what these measurements should correspond to for the distant observer (whom I will place at $r_2$). This observer does not have any local device near the object to be measured nor would it be of any use to him without a means of reading off the measurement result from the distance. A crucial point is now that there is no unique way of relating or connecting all “measurements at a distance” made by the observer at $r_2$ with the local measurements made by the observer at $r_1$.

According to Bunn and Hogg, “the inability to compare vectors at different points is the definition of a curved space-time” and this applies to velocity (four-)vectors as well.

To clarify this issue of non-uniqueness a little, let me give a well-known example from cosmology. Consider the recession velocity of sufficiently distant galaxies. Whether this velocity is superluminal or not, is a matter of interpretation. If we define that velocity as the change of the proper distance of the galaxy from the Milky Way per cosmological time, then superluminal velocities will result and we will find that all galaxies with a redshift exceeding 1.46 move away from us faster than light, which however does not impede us from seeing them. One advantage of this definition is that it allows us to calculate (or define!) the size of the current universe. On the other hand, if we interpret the recession velocity in terms of parallel transport of the velocity four-vector of the galaxy along the null geodesic taken by its light to reach us, i.e., if we parallel transport this vector to the position of the Milky Way and

5 The velocity of a billiard ball rolling across a table could be measured using a high-speed camera to take pictures at well-defined time intervals or by two light barriers connected to a stop watch or else by reflecting sound from the ball and measuring the frequency Doppler shift. All well-designed methods should give the same velocity.

6 Scalar quantities are unproblematic, but most measurements refer to more complicated entities. Energy, for example is not a scalar but a component of a four-vector.

7 Certain observations such as looking through a telescope or taking the spectrum of light from the distant object may obviously be made from a distance. But their translation to quantities such as velocities or accelerations is not immediate.

8 As obtained by adding up the proper distance increments of local coordinate stationary intermediate observers.
then extract the three-velocity from it, we will always find a subluminal velocity. This has the advantage that redshift and recession velocity are connected via the standard special relativistic Doppler formula \[19\], so this velocity is easily measurable by evaluation of the redshift.

Equipped with this understanding, we must interpret Shuler’s set of relationships (3) mostly as definitions of their left-hand sides giving properties “as seen by the distant observer”. Shuler does not give any operational justification of these definitions, hence some checking is necessary to see whether they are meaningful.

His first equation, not yet a comparison between distant and local quantities, is

\[
dt = c d\tau ,
\]

i.e., \(d\tau = \sqrt{-g_{tt}} dt\). This merely relates a proper time interval of a coordinate stationary observer to an interval of the global coordinate time, both pertaining to the same pair of events. Hence, it is an equality between coordinate differentials evaluated at the same location. Nevertheless, it is possible to use this to establish a connection between local and distant times on the basis of “slices of simultaneity” defined by the metric (3). This works as follows. To enable a comparison of time intervals at positions \(r_1\) and \(r_2\), intersect the world lines of the two observers at these positions by (3D) spatial sections corresponding to global times \(t_A\) and \(t_B\). The intervals of global time corresponding to the so-defined pieces of the world lines are \(\Delta t_1 = t_B - t_A = \Delta t_2\). Making that infinitesimal, we obtain

\[
\frac{1}{\sqrt{-g_{tt}(r_1)}} dr_1 = dt_1 = dt_2 = \frac{1}{\sqrt{-g_{tt}(r_2)}} dr_2 \quad \Rightarrow \quad dr_2 = \frac{\sqrt{-g_{tt}(r_2)}}{\sqrt{-g_{tt}(r_1)}} dr_1 \quad \Rightarrow \quad \frac{1}{r_2 \rightarrow \infty} \frac{1}{\sqrt{-g_{tt}(r_1)}} dr_1 = c d\tau_1.
\]

Dropping the labels 1 and 2, the relationship becomes identical in form to (5). But it means something different. Confusion about the meaning of coordinate differentials is a major source of errors in attempts by non-professionals to prove or disprove a thing about relativity. Equation (3) compares two times at the same position \(r\), whereas in eq. (6) times at different positions are related to each other. It would therefore actually be better not to drop the position labels 1 and 2.

In principle, the relationship (6) is not unique — different global time coordinates produce different simultaneity relationships, and not all of them will lead to the same value of \(g_{tt}(r_1)\) in (6). However, a few requirements such as stationarity of the metric plus a choice of spatial coordinates that make coordinate stationary observers the same as those in the metric (3) will suffice to fix the ratio. It is the same, for example, in Schwarzschild [21] and Painlevé-Gullstrand [22][23] forms of the metric. Moreover, there is a physical experiment establishing this ratio of local and distant times via the redshift of photons \(^9\). So this relationship is definitely acceptable.

While simultaneity slices are based on a coordinization, the resulting comparison between times at a distance arguably is not purely abstract. In fact, Rindler gives, in his book on GR [25], a procedure for the operational (though not practical) realization of the coordinate grid corresponding to a stationary metric, using rigid rods (outside of matter) and clocks that are running fast (or slow) with respect to standard clocks, by a factor that may depend on position but is constant in time.

Next, Shuler has

\[
d\ell = \rho_r dr = \sqrt{g_{rr}} dr.
\]

Again, this is, in the first place, a relationship between quantities near the same space-time position — a proper distance interval \(d\ell\) and a coordinate interval \(dr\), both at radius \(r\). To achieve a similar relationship between distant observers as for times, we might try to implement

\[
\frac{1}{\sqrt{g_{rr}(r_1)}} d\ell_1 = dr_1 = dr_2 = \frac{1}{\sqrt{g_{rr}(r_2)}} d\ell_2 \quad \Rightarrow \quad d\ell_2 = \frac{\sqrt{g_{rr}(r_2)}}{\sqrt{g_{rr}(r_1)}} d\ell_1 \quad \Rightarrow \quad \frac{1}{r_2 \rightarrow \infty} \frac{1}{\sqrt{g_{rr}(r_1)}} d\ell_1 = d\ell_1 / \rho_{rr},
\]

\(^9\) This is always possible outside the event horizon. What is important is that the two world lines are intersected by the same surfaces of constant time, so only two of these are necessary to delimit intervals on both lines.

\(^{10}\) Corresponding time intervals at the emitter and absorber of an electromagnetic wave train are, of course, not determined by coordinate simultaneity. Rather their beginnings and ends are determined by the null geodesics connecting the pairs of initial and final emission and absorption events of the sequence of photons [23]. The time dilation factor resulting from this is, however, the same in a stationary metric as the one obtained from the simultaneity relation.
i.e., just as we equated the global $t$ coordinates of world lines of distant observers, we would now have to equate $r$ values of corresponding ends of a spatial (i.e., spacelike) line segment through each observer. However, this does not work, as the (hyper)surfaces of constant $r$ in the neighborhood of observer 1 do not extend to observer 2 and vice versa. Rather, one with a smaller radius is completely contained in another with a larger radius. Hence, while $r$ is a global coordinate, the $r$ values of the distant observer have no definite relationship to those of the local one. To delimit $dr_1$ and $dr_2$, we need four surfaces, we cannot make do with two. This is a relevant difference between the time and the radial coordinates.

On the other hand, Shuler does not use (8) anyway. His connection between a distant and a local radial velocity,

$$v_{\text{radial}} = v_{r-\text{radial}} \rho_{rr}/\varsigma$$

would read, in our notation,

$$v_{\text{radial}}(r_2) = \frac{d\ell_2}{dr_2} \bigg|_{r_2 \to \infty} = \frac{dr_2}{dt_2} = v_{\text{radial}}(r_1) \sqrt{g_{rr}(r_1)} \sqrt{-g_{tt}(r_1)} = \frac{d\ell_1}{dt_1} \sqrt{g_{rr}(r_1)} \sqrt{-g_{tt}(r_1)} = g_{rr}(r_1) \frac{dr_1}{dt_1},$$

(10)

The radial prefactor gets squared when we express the measured velocity by the coordinate velocity (whereas the temporal one is canceled out). This certainly does not look right. There seems to be one prefactor $\sqrt{g_{rr}(r_1)}$ too many and this may be due to a confusion between coordinate velocities and measured velocities. Clearly, the observer at $r_1$ cannot measure $dr_1/dt_1$ (nor $dr_1/d\tau_1$) by only local means, because to establish the difference $dr_1$, she has to measure two circumferences, which is a nonlocal operation, whereas $d\ell_1$ is what is directly measured by local rulers.

Hence, Shuler’s relationships between distant and local measurements do not seem convincing. At least this author cannot find a rational way of justifying them. Nevertheless, since they may be viewed as definitions of the distant measurement results, pointing out the absence of an obvious operational realization is not sufficient. There might still be a counterintuitive one, easily to be overlooked. What may be done, however, is to to rule them out on the grounds that they form an inconsistent set. This will be shown below.

But then it will be necessary to develop our own consistent set of rules establishing measurements at a distance. On the way, it will turn out useful to calculate the proper acceleration of a coordinate stationary observer in the metric (8). The tether argument will be outlined and exhibited to give the same proper acceleration and a useful definition of force at a distance. This will then be used to demonstrate that CFS is automatically satisfied in any metric (12).

### 3.2 Accelerations and forces in the spherically symmetric metric

The four-acceleration $a$ is given by

$$a^\lambda = \frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{d\tau}.\tag{11}$$

For a freely falling particle, all its components are zero and these equations reduce to the geodesic equations for the coordinate accelerations. With a coordinate stationary particle (or observer) instead, $x^\lambda$ is constant for $\lambda = r, \vartheta, \varphi$, so the above equations simplify to

$$a^t = \frac{d^2 t}{d\tau^2} + \Gamma^r_t c^2 \left(\frac{dt}{d\tau}\right)^2,$$

$$a^\lambda = \Gamma^\lambda_{tt} c^2 \left(\frac{dt}{d\tau}\right)^2, \quad \lambda = r, \vartheta, \varphi.\tag{12}$$

There is no need to evaluate $a^t$ from its formula, because we know that the four-acceleration must be perpendicular to the four-velocity $u^\lambda$, which for a coordinate stationary observer has only a non-vanishing component along the time direction, i.e., $u^\lambda = 0$ for $\lambda = r, \vartheta, \varphi$. Hence, $a^t$ must be zero. The fastest way to obtain the needed Christoffel symbols $\Gamma^r_t$ is to read them off the Lagrangian equations of motion

$$\frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial x^\mu} = 0,$$

(13)

11 Effectively, Shuler employs $dr_2 = \rho_{rr} d\ell_1$, which looks incompatible with (7).

12 The demonstration will be restricted to the spherically symmetric metric, but the result should be generalizable.

13 $u^t$ must be nonzero then, since the magnitude of the four-velocity is a non-vanishing constant ($c$).
where an overdot means differentiation with respect to proper time and the Lagrangian is $L = \dot{s}^2$, as before, i.e.,

$$L = g_{tt}c^2\dot{t}^2 + g_{rr}\dot{r}^2 + \dot{r}^2 \left( \dot{\vartheta}^2 + \sin^2 \varphi \dot{\varphi}^2 \right) = -c^2 .$$

(14)

For $\mu = r$, we have $\partial L/\partial \dot{r} = 2g_{rr}\dot{r}$ and obtain, after division by $2g_{rr}$ (a prime denotes a derivative w.r.t. $r$)

$$\ddot{r} - \frac{g'_{tt}(r)}{2g_{rr}(r)}c^2\dot{t}^2 + \frac{g'_{rr}(r)}{2g_{rr}(r)}\dot{r}^2 - \frac{r}{g_{rr}(r)} \left( \dot{\vartheta}^2 + \sin^2 \varphi \dot{\varphi}^2 \right) = 0 ,$$

(15)

from which we can read off $\Gamma^r_{tt}$ as well as $\Gamma^r_{r\vartheta}$, $\Gamma^r_{\vartheta\vartheta}$, and $\Gamma^r_{\varphi\varphi}$ (and see that $\Gamma^r_{\mu\nu} = 0$ for $\mu \neq \nu$). Here, we need only

$$\Gamma^r_{tt} = -\frac{g'_{tt}(r)}{2g_{tt}(r)} \right) .$$

(16)

Since the $\dot{t}$ term of the Lagrangian has no $\vartheta$ or $\varphi$ dependent prefactor, the equations for $\vartheta$ and $\varphi$ just give $\Gamma^\vartheta_{tt} = \Gamma^\varphi_{tt} = 0$. So the only non-vanishing component of the four-acceleration is in the $r$ direction. Moreover, from (14) we have $g_{tt}c^2\dot{t}^2 = -c^2$, hence $\dot{t}^2 = -1/g_{tt}$, and finally,

$$a^r = \frac{g'_{tt}(r)c^2}{2g_{tt}(r)g_{rr}(r)} .$$

(17)

This is the (outward directed) radial component of the four-acceleration in the coordinate basis of the metric. The proper acceleration $a$ is just the length of this vector, hence

$$a = \sqrt{a^\lambda a_\lambda} = \sqrt{g_{rr}(a^r)^2} = \frac{g'_{tt}(r)c^2}{2g_{tt}(r)\sqrt{g_{rr}(r)}} .$$

(18)

What will be the acceleration “seen” by a distant observer, corresponding to this?

Let us find out what parallel transport gives. Since the answer will depend on the path along which we do the transport, we should choose a plausible path to get a meaningful result. A choice that suggests itself is transport along a radial geodesic. We know by symmetry that radial straight lines can be particle trajectories, i.e., geodesics of space-time. Transporting the vector $a^\lambda$ from $r = r_1$ to $r = r_2$, we must then require the covariant derivative with respect to $r$ to vanish along the path:

$$a^\lambda_r = a^\lambda_r + \Gamma^\lambda_{rt}a^t = 0 .$$

(19)

For $\lambda = t, \vartheta, \varphi$, the $\Gamma^\lambda_{rt}$ are zero, so we find $a^\lambda_r = 0$, if $a^\mu = 0$ for $\mu \neq r$. Since these components were zero to begin with, they remain zero along the path. The only component that may change is $a^r$. Here, we obtain

$$a^r_r = a^r_r + \Gamma^r_{rr}a^r = \frac{\partial a^r}{\partial r} + \frac{g'_{rr}}{2g_{rr}}a^r = 0 .$$

(20)

This first-order differential equation can be integrated using the initial condition at $r_1$, which yields

$$a^r(r) = \frac{g'_{tt}(r_1)c^2}{2\sqrt{g_{rr}(r_1)}g_{rr}(r_1)}g_{tt}(r_1) ,$$

(21)

and the length of this vector is

$$a(r) = \frac{g'_{tt}(r_1)c^2}{2g_{tt}(r_1)\sqrt{g_{rr}(r_1)}} ,$$

(22)

which is independent of $r$. Note that this is precisely the same result as [18], because the radial coordinate $r$ in that equation is the initial value $r_1$. So the proper acceleration of the observer at $r_1$ is the same for observers at any intermediate $r$ between $r_1$ and $r_2$ and at $r_2$ itself. In the end, this is not surprising: The proper acceleration is a relativistic invariant, so it must be the same for all observers. However, this is not necessarily what we would like the local acceleration to translate into for the distant observer. The fact that an observer at larger $r$ sees activities happening at smaller $r$ slowed down in time should show up in a smaller value of an acceleration for the distant observer than for the local one. Hence, we might consider options different from parallel transport to give meaning to the notion of “acceleration at a distance”.

8 Klaus Kassner: Reduction of a family of metric gravities
One way to proceed is to start from an uncontroversial relationship such as the frequency change of a photon emitted at \( r_1 \) with frequency \( \nu_1 \) and received at \( r_2 \) with frequency \( \nu_2 \),

\[
\nu_2 = \frac{\sqrt{-g_{tt}(r_1)}}{\sqrt{-g_{tt}(r_2)}} \nu_1 ,
\]

and to use well-devised thought experiments to develop a description for the distant observer at \( r_2 \) that is consistent with the local description at \( r_1 \). The standard interpretation of eq. (23) is in terms of time dilation, i.e., we would consider it equivalent to eq. (14). In fact, some authors \[25\] consider this the only legitimate interpretation, saying that it is not the frequency of the photon that changes but the perception of the frequency via measuring devices that are subject to different rates of proper time. The alternative view that the photon energy and, hence, its frequency decreases as the photon rises in a potential well is considered unacceptable by these authors. However, it is an uncontroversible fact that an electromagnetic wave emitted with frequency \( \nu_1 \) at \( r_1 \), will be found to have frequency \( \nu_2 \) at \( r_2 \) by any experiment capable of determining these frequencies. So to discuss the frequency shift as something that did not “really” happen, seems moot.

What should be realized here is that the frequency is not a property of the photon alone. Frequency is number of oscillations per time. But the photon does not have a time. The time in the definition of frequency is an observer time. So the photon frequency is a property of the relationship between the photon and the observer. It is then certainly o.k. to say that during the free fall of the photon from \( r_1 \) to \( r_2 \), its internal properties do not change. However, its frequency is determined not by its internal properties alone but also by the frame of reference of the observer. In fact, the same monochromatic electromagnetic wave will be observed to have different frequencies by two observers at the same place, if these have different velocities (Doppler effect), even though the photons making up the wave are identical due to its monochromatic nature. Just as frequency, energy is also a frame dependent quantity. Therefore, the point of view seems legitimate that the energy change of the photon as it moves in a gravitational field is due to the continually changing frame of reference of local observers along its path. Since there exist frames, in which the energy of the photon remains constant (e.g., the frame with the global time, because that time is homogeneous), it is possible to define a potential that does the book keeping vis-a-vis such a frame. Via this potential, energy conservation can also be implemented in the sequence of local frames passed by the photon.\[14\]

Once we have established a relationship for the transformation of local frequencies to distant ones via eq. (23), it is easy to set up a transformation rule for energies. Since photon energies differ from their frequencies only by a constant factor \((\hbar)\), we have, for these energies:

\[
E_2 = \frac{\sqrt{-g_{tt}(r_1)}}{\sqrt{-g_{tt}(r_2)}} E_1 .
\]

But the same relationship must hold for any form of energy, not just for photons, if we require conservation of energy on converting one form of energy into another. Otherwise, a perpetuum mobile may be constructed \[2\]. Indeed, suppose that there are two forms of energy, labeled by subscripts \( a \) and \( b \), for which the conversion factors between local and distant energies would be different. Let the total energy be \( E_1 = E_{a1} + E_{b1} \) in the local system at \( r_1 \) and assume that \( E_2 = E_{a1}/\varsigma_a + E_{b1}/\varsigma_b \) with \( \varsigma_b > \varsigma_a \). Convert the energy of type \( b \) into type \( a \) in the local system, so we obtain \( \hat{E}_{a1} = E_{a1} + E_{b1} \) due to local energy conservation. According to the distant observer, the energy is now \( \hat{E}_2 = \hat{E}_{a1}/\varsigma_a = \hat{E}_2 + E_{b1}(1/\varsigma_a - 1/\varsigma_b) > E_2 \), so for her, energy conservation does not hold. We may take the view that the energy-containing system does not suffer any internal changes when transported in free flight to \( r_2 \), so it will arrive there with energy \( \hat{E}_2 \). We convert this completely to energy of type \( b \). Then we have, due to local energy conservation at \( r_2 \), \( \hat{E}_b = \hat{E}_2 \) and this corresponds to an energy \( \hat{E}_1 = \hat{E}_2\varsigma_b = (E_{a1} + E_{b1})\varsigma_b/\varsigma_a \). Moving the energy on a free-fall trajectory back to \( r_1 \) completes the cycle with \( \hat{E}_1 = \hat{E}_1\varsigma_b/\varsigma_a > E_1 \) at the disposal of the observer at \( r_1 \). To exclude such a possibility, we must assume \[23\] to hold for any kind of energy.

The tether then is a means to establish a relationship between local and distant forces using the energy relationship, with the basic arrangement shown in fig. 1. A mass \( m \) is suspended from a massless inextensible tether and we ask what force has to be exerted at its top to balance the gravitational force pulling on the mass at its bottom. Neither masslessness nor inextensibility are achievable in a strict sense, but they can be attained with increasing accuracy at the price of increasing effort (taking lighter and stronger materials). We can do the experiment quasistatically, so special relativistic length contraction will not play any role. Once the tether relationships are confirmed over some finite distance experimentally, they can be extended to arbitrary distances via the thought experiment. Due to the

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\[14\] This is particularly evident for kinetic energy, because the velocities that enter its definition change with a change of the frame of reference.

\[15\] This would then make the aforementioned energy-loss interpretation an acceptable alternative to the interpretation only in terms of time dilation.
of the frame of reference, the transformation law for forces must be the same as that for energies: \( E = mc^2 \) because we know that the energy of a mass at rest is calculated from the energy via \( F = -dE/d\ell \). Since \( d\ell \) is a proper length element and hence invariant under changes of the frame of reference, the transformation law for forces must be the same as that for energies:

\[
F_2 = \frac{\sqrt{-g_{tt}(r_1)} \sqrt{-g_{rt}(r_2)}}{\sqrt{-g_{tt}(r_2)}} F_1. \tag{25}
\]

In fact, the tether relationship allows us to fully calculate the radial local force \( F(r) \) in terms of the metric coefficients, because we know that the energy of a mass at rest is \( E = mc^2 \) in its local frame. Setting \( r_1 = r \), we have, from (24)

\[
E_2(r_2, r) = \frac{\sqrt{-g_{tt}(r)}}{\sqrt{-g_{tt}(r_2)}} mc^2, \tag{26}
\]

hence

\[
F_2(r_2, r) = -\frac{dE_2}{d\ell} = -\frac{dE_2}{dr} \frac{dr}{d\ell} = -\frac{1}{\sqrt{-g_{tt}(r_2)}} \frac{-g_{tt}(r)}{2\sqrt{-g_{tt}(r)}} mc^2 \frac{1}{\sqrt{g_{rr}(r)}}. \tag{27}
\]

From this, we may obtain the local force by use of (25)

\[
F_1(r_1) = \frac{\sqrt{-g_{tt}(r_2)}}{\sqrt{-g_{tt}(r_1)}} F_2(r_2, r_1) = -m \frac{g_{tt}(r_1)c^2}{2g_{tt}(r_1)\sqrt{g_{rr}(r_1)}} \tag{28}
\]

and this is just the mass times minus the proper acceleration from (22), that is, in the local frame, Newton’s law holds with the local acceleration given by the negative proper acceleration. This corresponds to expectations; the force equation (25) gives the correct answer for the local result. We have consistency so far. Shuler actually uses eq. (27), with \( r_2 = \infty \) and the abbreviation \( g = |g_{tt}| c^2 \) as the force countering the accelerating force from the point of view of a distant observer. This would suggest consistency with the development presented here. However, the derivation of eq. (27) is not based on Shuler’s approach to lengths at a distance. The tether argument suggests

\[
d\ell_1 = d\ell_2 \quad \Leftrightarrow \quad \sqrt{g_{rr}(r_1)} dr_1 = \sqrt{g_{rr}(r_2)} dr_2, \tag{29}
\]

which for \( r_2 \to \infty \) reduces to \( dr_2 = d\ell_1 \). This can be readily seen from fig. 1. The crosses at \( A_2 \) and \( A_1 \) denote points marked on the tether before spooling out a length increment \( d\ell \). After spooling, \( A_2 \) has moved to \( B_2 \) and \( A_1 \) to \( B_1 \), and the proper distance traveled by the point \( A_2 \) that a nearby coordinate stationary observer at \( r_2 \) assigns to this procedure is \( d\ell_2 = d\ell \), that assigned by a similar observer near \( A_1 \) to the motion of that point \( d\ell_1 = d\ell \). This gives eq. (29), so the tether in fact is a tool for comparing lengths at a distance, too. Note, moreover, that it can easily be extended to a comparator for arbitrarily oriented length elements. This is demonstrated by the contraption of fig. 2 where a few frictionless pulleys have been added to show that length elements at a distance are always directly related by equations without any factors \( \rho_{rr} \), when comparison is effected via a tether.

\[\text{Fig. 1. Tether used for an operational definition of forces and lengths at a distance.}\]

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16 Force and proper length element carry a sign here. The positive direction is given by \( e_r \), so a negative \( F \) is pointing “downward” and \( d\ell \) has the same sign as \( dr \).
3.3 An inconsistency

This then suggests that Shuler’s scaling relations between distant lengths and his force calculation are mutually inconsistent. It is easy to show that this is indeed the case. Shuler states the following transformation rules for radial velocities, accelerations and forces [his equations (3) and (5), with slightly abbreviated notation]

\[
\begin{align*}
v_{2\text{rad}} &= v_{1\text{rad}} \rho_{rr}/\varsigma \\
a_{2\text{rad}} &= a_{1\text{rad}} \rho_{rr}/\varsigma^2 \\
F_{2\text{rad}} &= F_{1\text{rad}}/\varsigma ,
\end{align*}
\]

and he obviously also agrees with \( E_{2\text{rad}} = E_{1\text{rad}}/\varsigma \), which follows, via the photon frequency argument, from his \( dt = \varsigma d\tau \). Since this must also hold for kinetic energies and we are entitled to consider slowly moving particles, for which the kinetic energy is given by \( m_{\text{in}}v^2/2 \) we can derive a scaling law for the inertial mass \( m_{\text{in}} \) in this relationship:

\[
E_{2\text{kin}} = \frac{1}{2}m_{2\text{in}}v_{2\text{rad}}^2 = \frac{1}{2}m_{2\text{in}}v_{1\text{rad}}^2 \rho_{rr}/\varsigma^2 = E_{1\text{kin}}/\varsigma = \frac{1}{2}m_{1\text{in}}v_{1\text{rad}}^2/\varsigma \\
\Rightarrow \quad m_{2\text{in}} = m_{1\text{in}} \frac{\varsigma}{\rho_{rr}} .
\]

This says that the observed radial inertial mass at infinity is reduced by a factor \( \varsigma/\rho_{rr}^2 \) with respect to its local value. (The factor \( \rho_{rr} \) is missing for the transverse inertial masses, so the inertial mass will acquire a tensorial character, to which no \textit{a priori} objections are made.)

However, we may also derive the relationship between local and distant inertial masses from the force and acceleration equations

\[
F_{2\text{rad}} = m_{2\text{in}}a_{2\text{rad}} = m_{2\text{in}}a_{1\text{rad}}\rho_{rr}/\varsigma^2 = F_{1\text{rad}}/\varsigma = m_{1\text{in}}a_{1\text{rad}}/\varsigma \\
\Rightarrow \quad m_{2\text{in}} = m_{1\text{in}} \frac{\varsigma}{\rho_{rr}} ,
\]

which gives a \textit{different} factor now between the two masses. The results (31) and (32) are incompatible with each other, unless \( \rho_{rr} = 1 \). But Shuler uses \( \rho_{rr} \) values different from one in the rest of his paper. His reasoning is based on inconsistent equations.

3.4 Field strength measurement

We are now in a position to discuss consistency of field strength measurement. It will be useful to first verify what this means in Newtonian physics. The force “measured” by a tether at its upper end, \( r_2 \), when a mass was fastened to its lower end, \( r_1 \), is the sum of the weight of the mass at \( r_1 \) and the weights of the length elements of the tether at their respective positions in the gravitational field. But we required the tether to be massless, so the total force will be due to the weight of the mass at the bottom only. The tension in the tether will be a constant and so will be
the force measured by it, as long as the position \( r_1 \) is not changed. Now let an observer at an intermediate position \( r (r_1 < r < r_2) \) pick up the tether with a force just sufficient to counterbalance the weight at the bottom. The tether above \( r \) will go slack and can even be cut off without destroying mechanical equilibrium. CFS then shows up in the force to be applied by the observer at \( r \) being constant, i.e., independent of \( r \), and equal to the weight of the mass at \( r_1 \).

Consider the same situation in the spherically symmetric metric \([3]\). We cannot now assume that the local tension in the tether is constant, because that will be prevented by time dilation. The force that an observer at \( r \) must balance to pick up the weight is obviously given by eq. (25) with \( r_2 \) replaced by \( r \):

\[
F(r) = \frac{\sqrt{-g_{tt}(r_1)}}{\sqrt{-g_{tt}(r)}} F_1. \tag{33}
\]

This is a local force at \( r \) and if \(|g_{tt}(r)|\) decreases with decreasing \( r \), the magnitude of this force will increase towards the bottom of the tether, an effect that is due to time dilation. What will the value of this force be in the frame of the observer at \( r_2 \), according to the rules of force transmission via tethers? We just have to apply the same formula, but now between \( r \) and \( r_2 \):

\[
\tilde{F}(r_2, r) = \frac{\sqrt{-g_{tt}(r_2)}}{\sqrt{-g_{tt}(r)}} F(r) = \frac{\sqrt{-g_{tt}(r_1)}}{\sqrt{-g_{tt}(r_2)}} F_1 = F(\tilde{r}_2, \tilde{r}_1). \tag{34}
\]

For the observer at \( r_2 \), the force does not depend on \( r \), it appears constant along the tether and it agrees with the weight he assigned to the mass before \([via \text{eq. (26)}]\).

Hence, CFS is satisfied the same way it is in Newtonian physics, regardless of the values taken by the metric coefficients \( g_{tt}(r) \) and \( g_{rr}(r) \).

A brief discussion of what is wrong with Shuler’s argument may be in order. He sets \( F = mg/(\rho_{tt} \rho_{rr}) \) (where \( g \) is \(|g_{tt}(r)|c^2/2|\) and interprets this \( g \) as the acceleration seen by the distant observer. Shuler then simply requires \( \rho_{tt} \rho_{rr} = \text{const} \) to keep the formula \( F \propto mg \). If he had correctly taken into account the scaling of inertial mass obtained from his force relationship according to \([22]\), i.e., \( m_{\text{in}} = m/(\rho_{tt} \rho_{rr}) \) and \( F = m_{\text{in}} g \), he would have obtained no condition on the metric coefficients. Another flaw of his argument is that the variations of \( r \) considered in his discussion are actually variations of the position of the mass, i.e., of \( r_1 \), rather than variations of an observer position, as they should. It certainly cannot be assumed that the force in the tether does not change when the mass moves farther into, or out of, the inhomogeneous gravitational field.

### 3.5 Conservation of space-time and \( g_{tt}g_{rr} = -1 \)

Shuler then goes on to state that the relationship \( g_{tt}g_{rr} = -1 \) “for a century […] has been thought to be an artifact of coordinate choice” \([17]\). He neglects to mention that indeed there are mathematical proofs that this relationship is a consequence of coordinate choice, such as the one given in Ref. \([16]\). To portray it as a mere belief is a misrepresentation.

What is more, his argument being faulty, he himself fails to prove anything about the structure of space-time.

In fact, it is difficult to understand why Shuler did not check his argument by applying it to different metrics such as the Rindler one

\[
ds^2 = \frac{g^2 x^2}{c^4} c^2 dt^2 + dx^2 + dy^2 + dz^2,
\]

(describing a parallel gravitational field\([15]\)) The tether argument is known to work there as well \([3]\). If it provides a restriction on the product of a temporal and a spatial coefficient of the metric because of an underlying structural feature of space-time, we should also obtain a restriction between the coefficients \( g_{tt} \) and \( g_{xx} \) here. In the example, this cannot be discussed away by declaring the coordinates artificial. The spatial coordinates are Cartesian and directly related to length measurements in the Rindler frame. The time coordinate is Einstein synchronized just as in the Schwarzschild metric. Nevertheless, we have no constancy of the product \( g_{tt}g_{xx}g_{yy}g_{zz} \) which Shuler’s arguments suggest to be a property for any set of “physical” coordinates. Hence, Shuler’s approach fails to the same counterexample as those of Schiff and Tangherlini.

Moreover, the way Shuler discusses his “conservation principle” of space-time as derived from CPM and CFS, suggests that he does not seem to be clearly aware of the difference between a conservation law and an invariance

\[\text{\footnotesize\[17\]}\text{Moreover, he might assign a constant tension to the tether, a somewhat dubious procedure. Whether the tether breaks or not depends only on the local string tension, not on the one assigned from a distance, which can be much lower.}

\[\text{\footnotesize\[18\]}\text{\( \bar{g} \) is the proper acceleration of the observer at \( x = c^2/\bar{g} \).} \]
principle. Invariance of space-time under Lorentz transformations makes sure that its four-dimensional volume element does not change due to length contraction or time dilation. But space-time being an objective entity in geometrical theories, its volume element is even invariant under arbitrary coordinate transformations. This does not, however, restrict the metric coefficients. It just requires them to appear in the correct way in the representation of a 4D volume element $dV$, i.e., we have

$$dV = \sqrt{|\det(g_{\mu\nu})|} \, d^4x$$

in arbitrary coordinates $x^\mu$.

To complete the discussion of the reasons for the inverse relationship between $g_{tt}$ and $g_{rr}$ let us consider the equations for a radial null geodesic in the metric $^{(3)}$. We may obtain these from a Lagrangian constructed from the metric with the help of an affine parameter instead of the proper time (which is not defined for light). Call the affine parameter $\omega$ and assume the geodesic to be radial, so $d\varphi/d\omega = 0$ and the Lagrangian reduces to

$$L = \left(\frac{ds}{d\omega}\right)^2 = g_{tt}c^2\dot{t}^2 + g_{rr}r^2 = 0,$$  \hspace{1cm} (37)

where now the dot signifies a derivative with respect to $\omega$. Since $t$ is a cyclic coordinate, we have energy conservation

$$g_{tt}\dot{t} = A = \text{const}$$  \hspace{1cm} (38)

and instead of the equation of motion for $r$, we use the line element itself to describe the dynamics of $r$:

$$g_{tt}c^2\dot{t}^2 + g_{rr}r^2 = 0 \Rightarrow \frac{A^2c^2}{g_{tt}} = -g_{rr}r^2$$

$$\dot{r}^2 = -\frac{A^2c^2}{g_{tt}g_{rr}}.$$  \hspace{1cm} (39)

We see that constancy of the denominator on the right-hand side is equivalent to $r$ being a linear function of $\omega$: $r = B\omega + C \Rightarrow \dot{r} = B = \text{const}$. This proves the claim that for a metric of the form $^{(3)} g_{tt}g_{rr}$ is a constant (which must be -1, if the metric becomes Minkowskian for $r \to \infty$), iff $r$ is an affine parameter on radial null geodesics itself (which means it must be a linear function of any other affine parameter). Clearly, this is a coordinate condition.

$x$ in the Rindler metric is not an affine parameter and therefore, we do not have $g_{tt}g_{xx} = \text{const}$. However, introducing the alternative coordinate $\xi = (g/2c^2) x^2$, we can rewrite the Rindler metric as

$$ds^2 = -\frac{2g\xi}{c^2} c^2\, dt^2 + \frac{c^2}{2g\xi} \, d\xi^2 + dy^2 + dz^2,$$  \hspace{1cm} (40)

and herein, $\xi$ is of course an affine parameter on null geodesics in the $x$ direction. However, $\xi$ is not a coordinate having a direct interpretation in terms of length measurements by Rindler observers.

### 4 Generalization of Newton’s universal law of gravitation and its consequences

Up to this point, my criticism of Shuler’s article was rather destructive. There is little merit in this kind of endeavor although it is necessary, if science is not to lose its reputation as a self-correcting enterprise. A mere comment might have been in order to just point out the errors in Shuler’s paper. But then the length restrictions of comments make it difficult to achieve the clarity needed in refuting imprecise notions or inaccurate chains of reasoning. Still, it might not have been worth bothering with, unless there was the possibility to present a constructive aspect, too. This is what I would like to do now.

Shuler did not give any good reasons for why we should have $g_{tt}g_{rr} = -1$, but we know of course that it is true for the final result. Let us now pretend that sufficient arguments for this point have been given and the metric has been reduced to depending on a single radial function. Shuler then uses various different assumptions to construct a family of metrics, in which $g_{tt}$ takes the particular forms

$$g_{tt} = -\left(1 + \frac{GM}{rc^2}\right)^{-2/n},$$  \hspace{1cm} (41)

with $n = -2$ (the Schwarzschild solution), $n = -1$ or $n = 1$. The $n = 1$ solution does not have an event horizon. Shuler suggests to consider certain strong-field experiments to distinguish between them experimentally. Instead, I will give a theoretical argument strongly favoring one of these solutions.
The basic idea is to find an appropriate relativistic generalization of Newton’s universal law of gravitation and then to use this to restrict the metric. Unfortunately, the best-known form of Newton’s law

\[ F_g = -G \frac{mM}{r^2} e_r \]  

(42)
gives the gravitational force via an action at a distance (even though it can be reinterpreted in local terms and Newton was convinced that such an interpretation is the only sensible one). One might try to gain intuition from Maxwell’s equations generalizing Coulomb’s law to the relativistic case, but to start from (42) does not look too promising. Moreover, the relativistic transformation laws for forces are not trivial, which is a second obstacle.

The standard local form of the law,

\[ \Delta \Phi_g(r) = 4\pi G \rho(r) , \]  

(43)
a Poisson equation with the mass density \( \rho(r) \) as source term, introduces a potential for the force, but a simple and straightforward generalization does not suggest itself, the equation being static and potentials only slightly less difficult than forces. Moreover, with hindsight we know that general relativity has more than one potential (each element of the metric may be considered one) and it is not clear how to go from one to several potentials in a compelling manner.

A variant of (43) that is physically much more transparent reads

\[ \Phi_g(r_c) = \overline{\Phi}_g(\{r \mid |r - r_c| = \epsilon\}) - \frac{G}{2\epsilon} m(r_c, \epsilon) . \]  

(44)

It gives the potential at the center \( r_c \) of a small ball with radius \( \epsilon \) as the average over the potential on the surface of the ball and a correction term containing the gravitating mass in the ball. Of course, eq. (44) is just an integrated version of (43), obtained with the help of the free-space Green’s function, but it is much easier to interpret than (43).

It tells us that when there is no mass density around the point considered \( (r_c) \), then the gravitational potential, if it is not constant in the ball, does not have a minimum at \( r_c \), as there must be a smaller potential somewhere on the surface of the ball. So no test particle will have a stable position anywhere in vacuum. Second, if there is a mass density, it will reduce the potential, so the gravitational force is attractive. Actually, the variant (44) is given by Feynman in a YouTube video Many Mathematical Representations and Resulting Paradoxes. He argues that physicists ought to keep in mind different mathematical representations of the same law of nature, because they do not know ahead of time, which one is the best to generalize when something breaks due to more precise experiments and a new theory is needed.

Neither of the three representations of Newton’s law is too well-suited for generalization, so I will suggest a fourth, the goal being a statement that not only describes gravitation at a single point in space but provides information about tidal forces, which, after all, are an essential feature of gravity. Let us introduce the concept of a dust ball. This is a cloud of test particles, i.e., particles so small that they do not disturb the gravitational field and do not interact gravitationally with each other. Their motion will of course be affected by an external gravitational field. We will assume our dust balls to be small and initially spherical or ellipsoidal, which means that after a sufficiently small deformation they will still be ellipsoids. Then I claim that Newton’s law can be formulated as follows: Given a sufficiently small freely falling dust ball, the particles of which are initially at rest with respect to each other, the rate at which it will start to shrink is proportional to its volume times the mass density at its center. Quantitatively:

\[ \frac{\dot{V}}{V} \bigg|_{t=0} = -4\pi G \rho(r) . \]  

(45)

The formulation is a minor modification of a similar law given by Baez and Bunn in an article about the meaning of Einstein’s equation [27]. A few comments may be useful. For convenience, the initial time has been set equal to zero. Since the particles are assumed to be initially not moving with respect to each other, the volume rate of change is zero to linear order in \( t \), so we need to consider the quadratic order, leading to a second derivative in time (a dot is a local derivative). We could, instead of invoking volume and density, have simply taken \( \ddot{\epsilon} \), but it gives the potential at the center

\[ \Phi_g(r) = \frac{4\pi G}{2\epsilon} \int_{|r| < \epsilon} \rho(r') d^3r' . \]  

(43)

This is true as long as deformations are small enough for changes of higher than quadratic order in their amplitudes to be negligible. The ellipsoid may then change the position of its center and its orientation as well as the lengths of its semi-major axes. But it will remain an ellipsoid.
mass to be structureless, so its gravitational field will have spherical symmetry, i.e., it will be a central force field. Also, the mass is assumed to be constant, so the field will not be time dependent. It is known that time and velocity independent central forces are conservative, so the field has a potential $\Phi(r)$. The initial rest state means that each test particle is falling on a radial trajectory towards the center. Call the semi-axes of the ball along the directions of a spherical coordinate system $\delta \ell_r$, $\delta \ell_\vartheta$, and $\delta \ell_\varphi$. For symmetry reasons, the orientation of these axes will not change during a short interval of fall. The volume of the ellipsoid is $V = \frac{4}{3} \pi \delta \ell_r \delta \ell_\vartheta \delta \ell_\varphi$, its rate of change may then be expressed as

$$\frac{\dot{V}}{V} \bigg|_{t=0} = \frac{\delta \dot{\ell}_r}{\delta \ell_r} + \frac{\delta \dot{\ell}_\vartheta}{\delta \ell_\vartheta} + \frac{\delta \dot{\ell}_\varphi}{\delta \ell_\varphi} \bigg|_{t=0},$$

(46)

because first-order time derivatives of the $\delta \ell_\mu$ vanish. Writing $\delta \dot{\ell}_r = \delta \ddot{r}$, $\delta \ddot{\ell}_\vartheta = r \delta \ddot{\vartheta}$ und $\delta \ddot{\ell}_\varphi = r \sin \vartheta \delta \ddot{\varphi}$, we obtain, taking the time derivatives and dropping first order derivatives at the end

$$\frac{\delta \ddot{\ell}_r}{\delta \ell_r} = \frac{\delta \ddot{r}}{\delta r},$$

(47)

$$\frac{\delta \ddot{\ell}_\vartheta}{\delta \ell_\vartheta} = \frac{\ddot{r}}{r} + \delta \ddot{\vartheta},$$

(48)

$$\frac{\delta \ddot{\ell}_\varphi}{\delta \ell_\varphi} = \frac{\ddot{r}}{r} + \cot \vartheta \delta \ddot{\varphi} + \frac{\delta \ddot{\varphi}}{\delta \varphi}.$$  

(49)

Because particles fall along straight radial lines towards the center, $\delta \vartheta$ and $\delta \varphi$ are time independent — the $\vartheta$ and $\varphi$ coordinates of a particle do not change during fall. Therefore, the last two formulas will reduce to just $\ddot{r}/r$ on their right-hand sides and we obtain for the rate of volume change the simple result

$$\frac{\dot{V}}{V} \bigg|_{t=0} = \frac{\delta \ddot{r}}{\delta r} + 2 \frac{\ddot{r}}{r}.$$  

(50)

According to our dust ball form of Newton’s law (DBNL), eq. (45), we must have

$$\frac{\delta \ddot{r}}{\delta r} + 2 \frac{\ddot{r}}{r} = 0,$$

(51)

because outside of the attracting point mass, the mass density is zero. (The density of the test particles is negligible.) It is then easy to express $\ddot{r}$ and $\delta \ddot{r}$ with the help of the potential (a prime denotes a derivative w.r.t. $r$):

$$\ddot{r} = -\Phi'(r), \quad \delta \ddot{r} = -\Phi'(r) \delta r + \Phi'(r) = -\Phi''(r) \delta r.$$  

(52)
Substituting this into (51), we find

$$-\Phi''(r) - \frac{2\Phi'(r)}{r} = 0.$$ (53)

This is a linear differential equation for the potential, the general solution of which is straightforward to find:

$$\Phi(r) = \frac{c_1}{r} + c_2.$$ (54)

The constant $c_2$ is additive and may be chosen equal to zero. To determine $c_1$, we write out the DBNL with a $\delta$ function mass density:

$$-\Phi''(r) - \frac{2\Phi'(r)}{r} = -\Delta \Phi(r) = -4\pi G M \delta(r).$$ (55)

Here, we have exploited that the differential operator in $r$ applied to $\Phi$ on the left-hand side is just the radial part of the Laplacian. We integrate over a small sphere with radius $\varepsilon > 0$ about the origin and use Gauss’s divergence theorem to find

$$\oint_{r=\varepsilon} \nabla \Phi \cdot dS = \int_{r\leq\varepsilon} 4\pi G M \delta(r) \, d^3r = 4\pi G M \Rightarrow c_1 = -GM.$$ (56)

Hence, the potential is $\Phi(r) = -GM/r$, leading to a force on a test mass $m$ given by $F = -GmM/r^2 e_r$. Since the calculation works both ways, i.e., we may derive the standard form of Newton’s law from DBNL and we may start from the potential of Newton’s law and trace the calculation backwards from (56) to (45), the equivalence of DBNL and Newton’s universal law of gravitation has been shown. DBNL is just another way to formulate Newton’s gravitational law.

Generalization of (45) to a relativistic equation is still not easy because of its right-hand side. In Newton’s theory, the field source is a scalar density. With hindsight, we know that its generalization will lead to the divergence of a four-tensor. However, our interest is to obtain results on a special solution or class of solutions of the full theory without using knowledge from the field equations (which would immediately allow us to conclude what the generalization must look like). So we eschew attempts to generalize DBNL in its full glory. But in vacuum the right-hand side is zero, eq. (45) looks like a purely kinematic relation and we know how to deal with those.

What do we have to do? We replace the Newtonian absolute time by the proper time of the test particles or more precisely by the proper time of an appropriately chosen representative (the center particle of the ball). Moreover, we require the law to hold in a local inertial frame rather than in the global Newtonian one, of which absolute space is a representative. The relativistic dust ball law for motion in a gravitational field in vacuum (DBV) then takes the form

$$\ddot{\delta V}/\delta V|_{\tau=0} = 0,$$ (57)

all quantities being evaluated in the comoving local frame of the dust ball center particle. Overdots now mean derivatives with respect to proper time again.

Let us now assume that the central prerequisite for the derivation of Shuler’s family of gravities, the relationship $g_{tt}g_{rr} = \text{const}$ has found some sound justification or we have simply been told that it is true. Can we then use (57) to further constrain the metric?

In terms of the rates of changes of the semi-axes, the volume rate of change is still given by (46), with the time being replaced by the proper time of the center particle. In the metric (3), $\delta \ell_r = \sqrt{g_{rr}} \delta r$, so instead of eq. (47), we obtain

$$\frac{\delta \dot{\ell}_r}{\delta \ell_r} = \frac{\delta \dot{r}}{\delta r} + \frac{g'_{rr}}{2g_{rr}} \dot{r}.$$ (58)

The equations for the rates of change of $\delta \ell_\theta$ and $\delta \ell_\phi$ keep the forms (48) and (49), for obvious reasons. These are quantities calculated in the coordinate stationary frame given by the metric. What we need, however, are the

20 Speaking of a local inertial frame in which tidal forces are being felt, is tricky. Gravitation is treated like any other field quantity this way. The point will be reconsidered below.
corresponding quantities in the frame of the falling center particle of the dust ball. At first sight, it might seem that
these must be the same, because that falling frame is at rest with respect to the stationary frame at time $t = \tau = 0$
and its velocity grows linearly in $t$ or $\tau$, is therefore negligible as $\tau \to 0$. However, we have to evaluate a second
derivative of a position-like quantity and if that quantity grows as $r^2$, i.e., remains small for small $\tau$, the second
derivative with respect to $\tau$ will still give a finite contribution at $\tau = 0$. To evaluate this contribution, a local Lorentz
transformation from the coordinate stationary to the momentarily comoving freely falling frame may be performed [4]. As it
turns out, this leaves the formulas for the polar and azimuthal semi-axes [18] and [19] unchanged (the velocity of the center
particle does not change in time along these two directions) but modifies (58) into

$$\frac{\delta \tilde{\ell}_{cr}}{\delta \ell_{sr}} = \frac{\delta \tilde{r}}{\delta r} + \frac{g'_{rr}}{2g_{rr}} \tilde{r} - \frac{g_{rr} \tilde{r}^2}{c^2},$$

(59)

where the additional subscript $c$ is a reminder that this is a quantity referring to the local freely falling frame of the
center particle.

For symmetry reasons, $\vartheta$ and $\varphi$ are constant for each falling particle. The equation determining $\tilde{r}$ now follows from
the Lagrangian [14], where the last two terms may again be omitted for radial geodesics. $t$ is cyclic and as an equation
for $r$ we may take the definition of the Lagrangian. This leads to

$$g_{tt} = A = \text{const},$$

$$g_{tt} r^2 + g_{rr} \tilde{r}^2 = -c^2 \Rightarrow A^2 c^2 + g_{tt} (g_{rr} \tilde{r}^2 + c^2) = 0.$$  

(60)

Differentiating the last equation with respect to $\tau$, we get rid of the constant $A$ and find

$$\tilde{r} = -c^2 \frac{g_{tt}}{2g_{tt} g_{rr}} - \frac{(g_{tt} g_{rr})'}{2g_{tt} g_{rr}} \tilde{r}^2.$$  

(61)

Using $g_{tt} g_{rr} = -1$, this simplifies enormously,

$$\tilde{r} = \frac{c^2}{2} g_{tt}' \Rightarrow \delta \tilde{r} = \frac{c^2}{2} g_{tt}' \delta r.$$  

(62)

Another consequence of the value $-1$ of this product is that the second and the third terms on the right-hand side
of (60) cancel each other, so we have $\delta \ell_{sr}/\delta \ell_{cr} = \delta \tilde{r}/\delta r$. Evaluating this together with Eqs. [48] and [49] in terms of
(62) and inserting into the DBV [67], we obtain

$$c^2 \frac{g_{tt}'}{2} + c^2 \frac{g_{tt}'}{r} = 0,$$  

(63)

which is basically the same equation as the one we had for the potential $\Phi(r)$ [eq. [53]], hence $g_{tt}(r) = c_1/r + c_2$. With
the boundary condition $\lim_{r \to \infty} g_{tt}(r) = -1$, we obtain $c_2 = -1$. To determine the constant of integration $c_1$, we may
refer to the first equation of [62] and require that this equation reduces to its Newtonian limit for large $r$, hence

$$\tilde{r} = -\frac{c^2}{2} \frac{c_1}{r^2} \sim -\frac{GM}{r^2} \quad (r \to \infty),$$  

(64)

leading to $c_1 = 2GM/c^2$ and

$$g_{tt}(r) = -\left(1 - \frac{2GM}{c^2 r}\right), \quad g_{rr}(r) = \left(1 - \frac{2GM}{c^2 r}\right)^{-1},$$  

(65)

which is the exact result for the Schwarzschild metric. This is therefore the only one out of Shuler’s family of solutions
that is compatible with the generalization [77] of Newton’s gravitational law. Apparently, the others can be ruled out
without strong-field considerations.

It should be added that the assumption $g_{tt} g_{rr} = -1$ is unnecessary. Rather, this relation can be derived as part of
the solution using the DBV law [57]. All that has to be done is to consider a second (independent) configuration of a
dust ball freely falling in the metric, a circular orbit suggesting itself as the simplest choice. The full calculation has
been presented in Ref. [4].

In Ref. [4], the DBV law was given as a physical statement of Einstein’s field equations in vacuum, because Baez
and Bunn had shown it to be just this [27]. Here, it is justified as a postulate by pointing out that it is a relativistic
generalization of Newton’s law in vacuum, obtained by the usual heuristic rules. In fact, its statement reminds of an
application of the equivalence principle to a Newtonian formulation, valid in a comoving local frame. However, DBV
goes beyond at least the EEP. What the EEP says is that in describing local non-gravitational physics, we may find, given a certain level of accuracy of our measuring devices, a freely falling comoving system in which, if we choose its spatial dimensions sufficiently small and restrict experiments to a sufficiently short duration, all experimental results will be the same no matter what the velocity or the spatiotemporal location of the system. The theory predicting these results is special relativity, so the freely falling system is a local inertial system. But DBV talks about tidal forces which are not predicted by special relativity. In a way, DBV pretends that gravitational fields can be incorporated into the theory much the same way as other fields, and are not different in this respect from, say, electromagnetic fields, for which it is no problem to formulate a law in a local inertial frame. But gravity, if detectable, destroys the inertial nature of the local system... Alternatively (and preferably), we could see DBV as transcending the limits of the notion of local inertial system. It tells us what we should expect to see, if we increase the accuracy of our measuring devices, in what used to be a local inertial frame, sufficiently to detect tidal forces, which makes the system non-inertial. Instead, we could increase the size of the system (or wait long enough) until we see tidal forces with the current accuracy of our devices. It then appears that DBV is somewhat extending the boundaries of the system to match the physics described by the equivalence principle to the physics of the “world at large”.

An interesting question to ponder is whether, given Newton’s law as a non-relativistic description of gravity, the DBV law is a requirement of the strong equivalence principle, stating that the results of sufficiently local experiments, including gravitational ones, do not depend on the velocity and spatiotemporal location of the freely falling frame in which they are performed \[18\]. Tentatively, this author would answer this in the affirmative.

Now, it is certainly conceivable that general relativity becomes incorrect for sufficiently strong gravitational fields. The field equations are derived from the Einstein-Hilbert action, in which the Lagrangian density is just the scalar curvature. If it were a nonlinear function(al) of the scalar curvature instead, starting with the same linear term, then weak-field predictions of the theory would remain unchanged whereas the strong-field sector would look different. It appears, however, that the form of DBV would not change, because curvature effects can be made arbitrarily small by sufficient reduction of the size of the local freely falling system (including its temporal extension). This then suggests that the Schwarzschild geometry would still be a solution of such a modified theory of gravity\[21\]. Even when additional fields enter, such as the scalar field in the Brans-Dicke theory \[28\], this may remain true. Indeed, the Schwarzschild result is an exact black-hole solution of the Brans-Dicke theory, with constant scalar field. It just does not describe the field outside a finite-sized spherical mass distribution correctly anymore. Generally speaking, it would seem that the study of stationary vacuum solutions is not the best way to discriminate between different strong-field variants of metric theories of gravity. Rather, alternate theories should generate predictions for dynamic solutions to be tested experimentally.

While it is a positive feature of Shuler’s article that it proposes experiments by which to make a distinction between the members of his family that all behave the same in the weak-field case, he does not really do justice to the strong-field experiments that are already available. Observations of the decrease of the orbital period of the Hulse-Taylor pulsar \[29\] cannot really be called weak-field probes anymore, and they agree well with the predictions of general relativity. The direct observation of gravitational waves \[30\] definitely tests the strong-field domain of the theory and it is somewhat moot to point out that inaccuracies of the determination of mass parameters of the partners of a black-hole merger still exceed 10% typically. The wave form and phase of the gravitational wave signal are highly specific and this in itself constitutes a quantitative test of the theory at strong fields. Obviously, Shuler’s results cannot be readily tested against this, because he does not have a field theory and cannot predict dynamical situations in any detail. Still, it appears that one of the members of his family (the “proximity-gravity” one, \(n = 1\)) may be immediately ruled out due to the presence of a ring-down signal with an approximately constant frequency at the end of the gravitational wave train \[31\]. This signal is a consequence of the formation of a horizon of the final black hole that is quickly damped to constant shape. Since no horizon would be formed in the “proximity-gravity” model, instead a continuing gravitational wave signal from the final massive object (that is still moving and producing a time dependent quadrupole moment) would have to be expected. Other results than a final black hole (such as worm holes) would lead to echoes in the gravitational wave signal. These are actively being searched for. But most of the more than ten detection events reported so far \[31\] are readily interpreted in terms of a final black hole emerging, in quantitative agreement with GR predictions.

5 Conclusions

In summary, it has been shown that local energy and momentum conservation are an integral feature of the dynamic equations for particles from any metric theory of gravity. Therefore, momentum conservation considerations are not

\[21\] Since the scalar curvature \(R\) of the Schwarzschild solution is exactly zero everywhere outside the central singularity, it is perfectly possible for that solution to be the stationary point of other actions formulated in terms of the scalar curvature, besides the Einstein-Hilbert one. After all, nonlinear terms in \(R\) of these actions will become negligible both for the Schwarzschild solution and small variations about it.
helpful in replacing the field equations when it comes to restricting the metric to a physically possible solution. This is at least true, if the conservation law does not refer to the momentum of the gravitational field itself (i.e., momentum carried by gravitational waves).

Consistency of field strength measurement is another requirement that is automatically satisfied by any metric, provided measurements at a distance are interpreted in a consistent way. Shuler’s manner of establishing force measurements at a distance corresponds to the tether approach discussed, whereas his connection of distant with local velocities corresponds to a different convention of comparing lengths at a distance, incompatible with the force definition established by tethers.

No physical fact can be identified corresponding to Shuler’s “conservation of space-time”. Invariance of space-time volume elements is of course satisfied in GR and other metric theories.

The exact Schwarzschild metric may be derived using the EEP, SR, NL and a generalization of Newton’s universal law of gravitation in vacuum to a general relativistic setting, by standard rules applicable to kinematic laws, i.e., the replacement of time by a local proper time and the replacement of Newton’s absolute space frame by a local freely falling frame. Only one member of Shuler’s family of gravities satisfies the constraints on the metric following from this approach, so the others should be discarded. The surviving gravitational theory is just ordinary GR.

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