I briefly review advances in the understanding and modeling of relativistic stellar dynamics around massive black holes (MBHs) in galactic nuclei, following the inclusion of coherent relaxation and of secular processes in a new formal analytic description of the dynamics.

1 Relaxation in galactic nuclei

The dense, centrally concentrated stellar cluster that exists around most MBHs offers opportunities for strong, possibly destructive interactions between it and stars. These include direct plunges, leading to tidal disruption flares or gravitational waves (GW) flares, inspiral processes leading to quasi-periodic GW emission from extreme mass ratio inspiral events (EMRIs), tidally powered stars ("squeezars"), strong tidal scattering or capture by massive accretion disks. These processes affect MBH growth and may create exotic stellar populations around MBHs.

This naturally leads to the question "How do stars closely interact with, and fall into a MBH, and at what rates?" This is known as the stellar dynamical "loss-cone problem". It is a non-trivial problem, in spite of the presence of so many stars so close to the MBH, because the phase space volume of unstable orbits is minute. The few stars initially on such orbits quickly fall into the MBH on the short dynamical timescale, and then the rates would drop to zero, if it were not for dynamical processes that deflect additional stars from stable orbits to those with velocity vectors that point toward the MBH, within the loss-cone (Fig. 1 left). Thus, the loss-cone question is essentially the question: "how do galactic nuclei randomize and relax?"

1.1 Non-coherent 2-body relaxation (NR)

The discreteness of stellar systems leads to non-coherent 2-body relaxation (NR) (Fig. 1 left). This guarantees a minimal relaxation rate, on a timescale $T_{NR} \sim Q^2 P(r)/N_s(r) \log Q$, where $Q = M_*/M_\bullet$ is the MBH/star mass ratio, $P(r)$ is the radial orbital period, and $N_s(r)$ the number of stars inside $r$. Because the impact parameter $b$ of these point-point interactions can be small, NR is boosted by the Coulomb factor $\log(b_{\text{max}}/b_{\text{min}}) = \log Q$. $T_{NR}$ is the timescale for changes of order unity in energy, $T_E$. It is however easier to drive a star into the MBH by reducing its angular momentum $L$ and making its orbit more radial, than by reducing the orbital energy $E < 0$, and shrinking the orbit. The timescale for changing $j = L/L_c(a) = \sqrt{1 - e^2}$ from $j$ to 0 is $T_L = j^2 T_E$ ($L_c = \sqrt{GM_\bullet a}$ is the circular $L$, $a$ the sma and $e$ the eccentricity).

In the absence of dissipation, stars with $j \ll 1$ are deflected by $L$-scattering at nearly constant $a$ to the innermost stable orbit (ISO), at $j_{\text{iso}} = 4\sqrt{r_g/a}$ ($r_g = GM_*/c^2$) and then plunge directly into the MBH (Fig. 1 center). When a dissipative mechanism is present (e.g. GW), phase space is divided in two (Fig. 1 left). Below some critical sma $a_c$, all stars eventually cross the
“inspiral line” where $E$-dissipation is faster than $j$-scattering, and then inspiral gradually into the MBH as EMRIs. Stars above $a_c$ plunge directly. The respective rates of plunges and inspirals can then by estimated by the ratio of number of stars on the relevant scales (the MBH radius of influence $r_h$ for plunges (e.g tidal disruptions), and $a_c$ for inspirals) over $T_{NR}$ on that scale:

$$R_p \sim N_*(r_h)/T_{NR} \log(L_c/L_{\text{iso}})$$

and $R_i \sim N_*(a_c)/T_{NR} \log(L_c/L_{\text{iso}})$. Because $N_*(a_c) \ll N_*(r_h)$, the inspiral rate is much lower than the plunge rate, typically $R_i \sim 0.01 R_p$.

### 1.2 Coherent resonant relaxation (RR) in nearly-spherical systems

Resonant relaxation is a process of rapid $L$-relaxation that occurs when the gravitational potential is symmetric enough to restrict the evolution of orbits on timescales much longer than the orbital time (e.g. nearly-fixed Keplerian ellipses in the nearly-Keplerian potential close to a MBH, where the stellar mass is negligible, but far enough so that GR effects are weak). In that case, a test orbit will feel a residual torque from the static, orbit-averaged background of stellar “mass wires”, which persists for a long coherence time $T_c$, until small deviations from symmetry accumulate and randomize the background. These orbit-orbit interactions randomize the angular momentum on a timescale $T_{RR} \sim Q^2 P(r)^2/N_*(r) T_c$. Unlike NR, these extended objects do not undergo close interactions. Rather, RR is boosted by the long coherence time.

RR is relevant for the loss-cone problem because it is possible to have $T_c \gg P$ in the symmetric potential near a MBH, so that $T_{RR}/T_{NR} \sim \log(Q) P/T_c \ll 1$. That is, angular momentum evolution, and in particular that leading to $j \to 0$ and strong interactions with the MBH, can be greatly accelerated. Unchecked, RR will completely suppress EMRIs by driving all stars into plunge orbits (Fig. 2 center). However, very eccentric orbits undergo GR in-plane (Schwarzschild) precession, which quenches RR by rapidly alternating the direction of the residual torque on the orbit. This motivated the “fortunate coincidence conjecture”: The $O(\beta^2 j^{-2})$ GR precession becomes significant before $O(\beta^5 j^{-7} Q^{-1})$ GW dissipation, and this may allow EMRIs to proceed unperturbed, decoupled from the background stars.

### 1.3 The Schwarzschild Barrier

The first full PN2.5 $N$-body simulations revealed a surprising result: not only does GR precession quench RR before the GW-dominated regime, as conjectured, but there appears to be some kind of barrier in phase space, dubbed the Schwarzschild Barrier (SB), which prevents the orbits from evolving to $j \to 0$. Instead, they appear to linger for roughly $T_c$ near the SB, where their orbital parameters oscillate at the GR precession frequency, and then they evolve back to $j \to 1$. An early analysis suggested that this behavior is related to precession under the influence of a residual dipole-like residual force. However, a full self-consistent explanation of the SB was lacking, and its very existence and nature remained controversial.
I now describe briefly a new formal framework for expressing coherent relaxation and secular processes in galactic nuclei, and discuss implications for the steady state phase space structure and loss rates.

2 Hamiltonian dynamics with correlated background noise

Two key insights inform the new advances in understanding coherent relativistic dynamics, which lie in the difficult-to-treat interface between deterministic Hamiltonian dynamics and stochastic kinetic theory. (1) The effect of the background stars on a test stars should be described by a correlated noise model $\eta(t)$, whose degree of smoothness (differentiability) determines dynamics on short timescales. (2) The long-term steady-state remains (unavoidably) the maximal entropy configuration, irrespective of the details of the nature of the relaxation processes.

This formal treatment of PN1 dynamics in the presence of correlated (RR) noise $\eta(t)$ (a 3-vector in $L$-space) allows to write a phase-averaged leading-order ($\ell = 1$) Hamiltonian $H_1$ and derive stochastic EOMs for the orbital elements of a test star, $x \equiv (j, \phi, \cos \theta)$ and the argument of periapse $\psi$, which precesses at frequency $\nu_p(j)$, where $\nu_r$ is the RR torque frequency. This $\eta$ formalism allows to evolve a test star in time for a given realization of the noise. Moreover, even though $\eta$ is time-correlated, it is possible to derive (and validate with the stochastic EOMs) approximate diffusion coefficients (DCs) $D_{1,2}$, which allow to evolve in time the probability density $\rho(j)$ with the Fokker-Planck (FP) equation,

$$\frac{\partial \rho}{\partial t} = \frac{1}{2} \left\{ jD_2 \frac{\partial}{\partial j} \left[ \rho \right] \right\}, \text{ where } D_2 = |\nu_{r,j}|^2 \mathcal{F}_{C(t)}[\nu_p(j)] \text{ and } D_1 = \frac{1}{2j} \frac{\partial jD_2}{\partial j}. \quad (2)$$

$\mathcal{F}_{C(t)}$ is the Fourier transform of $\eta$’s auto-correlation function (ACF). The explicit dependence of $D_2$ on the spectral power of the noise at the precession frequency is an expression of adiabatic invariance (AI). If, and only if the noise has an upper frequency cutoff, as it must if it is smooth (this is physically expected, since the background noise is the superposition of continuous orbital motions), then there is a critical $j_0$ such that for $j < j_0$ the precession is fast enough so that $D_2(j) \rightarrow 0$, and the star decouples from the background resonant torques (Fig. 3, left, center). This describes the dynamics of the SB: it is not a reflecting boundary, but a locus in phase space where diffusion rapidly drops due to AI. Since diffusion to yet lower $j$ slows further down, while diffusion to higher $j$ speeds further up, orbits statistically seem to bounce away from the SB.
Figure 3 – The smoothness of the noise model and diffusion dynamics. **Left:** Three \( \eta \) models and their ACF: discontinuous steps \((C^0)\), continuous but not continuously differentiable \((C^1\) with exponential ACF), smooth \((C^\infty\) with Gaussian ACF). **Center:** The corresponding \( D_2 \); note the steep cutoff at \( j < 0.1 \) for the smooth noise model. **Right:** The MC simulations of \( j \)-only evolution reproduce the AI/SB limit at \( j < j_0 \) in the absence of NR, but NR erases this feature completely on timescale \( t \to T_{NR} \).

3 The steady state loss-cone

NR is impervious to AI. When \( t \to T_{NR} \), the system approaches the maximal entropy solution \((dN/dj = 2j, \text{Fig. } 3 \text{ right})\). Monte Carlo (MC) simulations of the probability density, branching ratios and loss rates with the \( \eta \) formalism (Fig. 2 right) show that the RR-dominated region in phase space is well separated from the plunge and inspiral loss-lines, so the effect of RR on the loss rates is small \((< \times 2 - 3)\). Specifically, we conclude that GR quenching of RR is effective, so the EMRI rates remain largely unaffected by RR. RR can be significant for processes whose loss-line crosses the RR-dominated region, e.g. destruction by interaction with an accretion disk.

4 summary

NR, RR, GW dissipation and secular precession can be treated analytically as effective diffusion with correlated noise. The steady state depends mostly on NR, which erases AI. RR can be important in special cases. The \( \eta \) formalism provides stochastic EOMs for evolving test particles and an FP/MC diffusion procedure for evolving the probability density. This makes it possible to model the relativistic loss-cone in galactic nuclei with realistic \( N_\star \gg 1 \), (unlike direct \( N \)-body simulations), and obtain the branching ratios, loss rates and steady state stellar distributions.

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