Design of a fretboard using the stiff string equation

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Abstract

Guitar fretboards are designed based on the equation of the ideal string. That is, it neglects several factors as nonlinearities and bending stiffness of the strings. Due to this fact, intonation of guitars along the whole neck is not perfect, and guitars have right tuning just in an average sense. There are commercially available fretboards that differ from the traditional design.¹ As a final application of this work we would like to redesign the fretboard layout considering the effects of bending stiffness.

The main goal of this project is to analyze the differences between the solutions of vibrations of the ideal string and a stiff string. These differences should lead to changes in the fret distribution for a guitar, and, hopefully improve the overall intonation of the instrument. We will start analyzing the ideal string equation and after a good understanding of this analytical solution we will proceed with the, more complex, stiff equation. Topics like separation of variables, Fourier transforms, and Perturbation analysis might prove useful during the course of this project.

1 Modeling

The ideal string equation is a second order partial differential equation, while the stiff equation includes an extra term that turns it into a fourth order partial differential equation.¹

¹One example is the [11] by the Company True Temperament AB, where each fretboard is made using CNC processes.
differential equation. This additional term takes into account the bending stiffness that is normally neglected in the ideal string.\(^2\)

Let’s take a small segment with small displacements in the vertical direction \(w\). The body diagram is shown in Figure 1, where we assumed that the tension is constant along the length element.

![Forces diagram over an element of the string with length \(dx\).](image)

**Figure 1:** Forces diagram over an element of the string with length \(dx\).

Summing forces in the normal to the centerline of the element yields

\[
T \sin \left( \theta + \frac{\partial \theta}{\partial x} \, dx \right) - T \sin \theta + V - \left( V + \frac{\partial V}{\partial x} \, dx \right) + p \, dx = (\rho A \, dx) \frac{\partial^2 w}{\partial t^2}.
\]

If we assume that the angles are small, then \(\sin \theta \approx \theta\), thus

\[
T \left( \theta + \frac{\partial \theta}{\partial x} \, dx \right) - T \sin \theta + V - \left( V + \frac{\partial V}{\partial x} \, dx \right) + p \, dx = (\rho A \, dx) \frac{\partial^2 w}{\partial t^2},
\]

expanding and dividing by \(dx\)

\[
T \frac{\partial \theta}{\partial x} - \frac{\partial V}{\partial x} + p = \rho A \frac{\partial^2 w}{\partial t^2}.
\]

We know that \(\tan \theta = \frac{\partial w}{\partial x}\), then \(\sec^2 \theta \frac{\partial \theta}{\partial x} = \frac{\partial^2 w}{\partial x^2}\), and since we considered small angles \(\sec^2 \theta \approx 1\), replacing this in the equation we obtain

\[
T \frac{\partial^2 w}{\partial t^2} - \frac{\partial V}{\partial x} + p = \rho A \frac{\partial^2 w}{\partial t^2}.
\]

\(^2\)This term is probably more important for electric guitars, since they use steel strings. And also for bass guitars, since they have thicker strings.
Summing moments about the center of the element and omitting higher order differentials, yields

\[ M + \left( M + \frac{\partial M}{\partial x} \, dx \right) - V \, dx = 0, \]

that after expansion reads

\[ V = \frac{\partial M}{\partial x}. \]

We can use the same assumptions that are done in the Euler-Bernoulli beams modeling, i.e., that the bending moment is proportional to the linearized curvature

\[ M = EI \frac{\partial^2 w}{\partial x^2}, \]

that leads to

\[ \frac{\partial V}{\partial x} = \frac{\partial^2 M}{\partial x^2} = \frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 w}{\partial x^2} \right), \]

and replacing in (1), yields

\[ T \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 w}{\partial x^2} \right) + p = \mu \frac{\partial^2 w}{\partial t^2}, \]

(2)

with \( \mu = \rho A \) the linear mass density. If we assume that the bending stiffness \( EI \) is constant, and consider that there are no body forces, we obtain

\[ T \frac{\partial^2 w}{\partial x^2} - EI \frac{\partial^4 w}{\partial x^4} = \mu \frac{\partial^2 w}{\partial t^2}. \]

(3)

1.1 Nondimensional form

If we start from equation 3 we can rewrite the equation in non-dimensional form as

\[ \frac{\partial^2 u}{\partial \xi^2} - \epsilon \frac{\partial^4 u}{\partial \xi^4} - \alpha^2 \frac{\partial^2 u}{\partial \tau^2} = 0, \]

(4)

with \( u = w/L, \xi = x/L, \tau = \omega t, \alpha^2 = L^2 \mu \omega^2 / T = L^2 \omega^2 / c^2, \epsilon = EI / (L^2 T), L \)

the length of the string, \( \omega \) a characteristic frequency of the system, \( c \) the phase speed for the ideal string. When, \( EI \) is small the equation can be rewritten as

\[ \frac{\partial^2 u}{\partial \xi^2} - \alpha^2 \frac{\partial^2 u}{\partial \tau^2} = 0, \]

(5)

meaning that we are neglecting the bending stiffness of the string.
1.2 Solution of the PDE

If we take the Fourier transform of equation 4 we obtain

\[ \frac{\partial^2 U}{\partial \xi^2} - \epsilon \frac{\partial^4 U}{\partial \xi^4} - \kappa^2 U = 0, \tag{6} \]

with \( u = w/L, \xi = x/L, \tau = \omega t, \kappa^2 = L^2 \mu \omega^2 / T = L^2 \omega^2 / c^2, \) \( \epsilon = EI/(L^2 T), \) \( L \) the length of the string, \( \omega \) a characteristic frequency of the system, \( c \) the phase speed for the ideal string, and \( U \) is the Fourier transform of \( u. \) We know that the solution for the time part of the PDE is of the form \( A \sin(\omega t) + B \cos(\omega t), \) but we are more interested in the spatial part. The reason for this interest is to find the eigenvalues of the differential equation, that, ultimately, leads to the eigenfrequencies.

The solution of the resulting differential equation is

\[ u(s) = C_1 \cos \left( \frac{\sqrt{2}s}{2\sqrt{\epsilon}} \sqrt{\alpha - 1} \right) + C_2 \sin \left( \frac{\sqrt{2}s}{2\sqrt{\epsilon}} \sqrt{\alpha - 1} \right) + C_3 \sinh \left( \frac{\sqrt{2}s}{2\sqrt{\epsilon}} \sqrt{\alpha + 1} \right) + C_4 \cosh \left( \frac{\sqrt{2}s}{2\sqrt{\epsilon}} \sqrt{\alpha + 1} \right), \]

with \( \alpha = \sqrt{1 + 4 \epsilon \kappa^2}. \) With boundary conditions

\[ u(0) = 0, \quad \frac{du}{ds} = 0 \]
\[ u(1) = 0, \quad \frac{du}{ds} = 0, \]

If we solve for \( C_3 \) and \( C_4 \) the first two equations we find

\[ C_4 = -C_1 \quad C_3 = -\frac{C_2 \sqrt{\alpha - 1}}{\sqrt{\alpha + 1}} \]

giving

\[ u(s) = C_1 \cos \left( \frac{\sqrt{2}s}{2\sqrt{\epsilon}} \sqrt{\alpha - 1} \right) - C_1 \cosh \left( \frac{\sqrt{2}s}{2\sqrt{\epsilon}} \sqrt{\alpha + 1} \right) - \frac{C_2 \sqrt{\alpha - 1}}{\sqrt{\alpha + 1}} \sinh \left( \frac{\sqrt{2}s}{2\sqrt{\epsilon}} \sqrt{\alpha + 1} \right) + C_2 \sin \left( \frac{\sqrt{2}s}{2\sqrt{\epsilon}} \sqrt{\alpha - 1} \right). \]

The other boundary conditions lead to the system of equations

\[ \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = 0, \]
with
\[ A_{11} = \cos \left( \frac{\sqrt{2}\sqrt{\alpha - 1}}{2\sqrt{\epsilon}} \right) - \cosh \left( \frac{\sqrt{2}\sqrt{\alpha + 1}}{2\sqrt{\epsilon}} \right) \]
\[ A_{12} = -\frac{\sqrt{\alpha - 1}}{\sqrt{\alpha + 1}} \sinh \left( \frac{\sqrt{2}\sqrt{\alpha + 1}}{2\sqrt{\epsilon}} \right) + \sin \left( \frac{\sqrt{2}\sqrt{\alpha - 1}}{2\sqrt{\epsilon}} \right) \]
\[ A_{21} = -\frac{\sqrt{2}\alpha - 1}{2\sqrt{\epsilon}} \sin \left( \frac{\sqrt{2}\sqrt{\alpha - 1}}{2\sqrt{\epsilon}} \right) - \frac{\sqrt{2}\alpha + 1}{2\sqrt{\epsilon}} \sinh \left( \frac{\sqrt{2}\sqrt{\alpha + 1}}{2\sqrt{\epsilon}} \right) \]
\[ A_{22} = \frac{\sqrt{2}\alpha - 1}{2\sqrt{\epsilon}} \cos \left( \frac{\sqrt{2}\sqrt{\alpha - 1}}{2\sqrt{\epsilon}} \right) - \frac{\sqrt{2}\alpha - 1}{2\sqrt{\epsilon}} \cosh \left( \frac{\sqrt{2}\sqrt{\alpha + 1}}{2\sqrt{\epsilon}} \right) \]

To obtain solutions that are nontrivial, we need to make \( \det(A) = 0 \), i.e.,
\[ \sqrt{\alpha - 1} \cos \left( \frac{\sqrt{2}\sqrt{\alpha - 1}}{2\sqrt{\epsilon}} \right) \cosh \left( \frac{\sqrt{2}\sqrt{\alpha + 1}}{2\sqrt{\epsilon}} \right) - \sqrt{\alpha - 1} \]
\[ - \frac{1}{\sqrt{\alpha + 1}} \sin \left( \frac{\sqrt{2}\sqrt{\alpha - 1}}{2\sqrt{\epsilon}} \right) \sinh \left( \frac{\sqrt{2}\sqrt{\alpha + 1}}{2\sqrt{\epsilon}} \right) = 0 \]
or
\[ \sqrt{\alpha - 1} \cos \left( \frac{\sqrt{2}\sqrt{\alpha - 1}}{2\sqrt{\epsilon}} \right) - \frac{\sqrt{\alpha - 1}}{\cosh \left( \frac{\sqrt{2}\sqrt{\alpha + 1}}{2\sqrt{\epsilon}} \right)} \]
\[ - \frac{1}{\sqrt{\alpha + 1}} \sin \left( \frac{\sqrt{2}\sqrt{\alpha - 1}}{2\sqrt{\epsilon}} \right) \tanh \left( \frac{\sqrt{2}\sqrt{\alpha + 1}}{2\sqrt{\epsilon}} \right) = 0 \] (7)

This is the characteristic equation of our problem. We need to solve this equation numerically to find the roots.

1.3 Perturbation solution

We can solve equation 7 for the eigenvalues using numerical methods. From a design point of view, it would be easier to have some analytic expressions that could be used to compute the desired parameters. Thus, we can use a perturbation method to find an approximated solution to this problem [5], and try to obtain some analytic expressions.

Let us assume solutions of the form
\[ u(s) = u_0(s) + \epsilon u_1(s) + \epsilon^2 u_2(s) + \cdots \]
\[ \kappa^2 = \kappa_0^2 + \epsilon \kappa_1^2 + \epsilon^2 \kappa_2^2 + \cdots \]
If we substitute these in the differential equations and group by powers of \( \epsilon \), keeping the first two powers, we obtain

\[
\begin{align*}
\left[ \frac{\partial^2}{\partial s^2} + \kappa_0^2 \right] u_0 &= 0 \\
\left[ \frac{\partial^2}{\partial s^2} + \kappa_0^2 \right] u_1 &= \frac{\partial^4 u_0}{\partial s^4} - \kappa_1^2 u_0 .
\end{align*}
\]

The first equation is satisfied by the pairs

\( \kappa_0 = n\pi, \ u_0(x) = C_1 \sin(n\pi s) \),

and for the second equation to have non-singular solution we get that it needs to be orthogonal to the first equation,

\[
\int_0^1 u_0 \left( \frac{\partial^4}{\partial s^4} - \kappa_1^2 \right) u_1 \, ds = 0 ,
\]

i.e.,

\( \kappa_1 = n^2 \pi^2 . \)

If we solve the second differential equation we find that \( u_1(s) = C_3 \sin(n\pi s) + C_4 \cos(n\pi s) \), and applying boundary conditions we find that \( C_4 = 0 \). Thus

\[
\begin{align*}
\kappa^2 &\approx n^2 \pi^2 + \epsilon n^4 \pi^4 , \\
\kappa^2 &\approx n^2 \pi^2 + \epsilon n^4 \pi^4 ,
\end{align*}
\]

or

\[
\begin{align*}
u &\approx (C_1 + C_3 \epsilon) \sin(n\pi s) \\
\kappa^2 &\approx n^2 \pi^2 + \epsilon n^4 \pi^4 ,
\end{align*}
\]

(8a) (8b)

where we absorbed the constants into a single one.

This approximation is valid for small \( \epsilon \). Figure 2 presents this approximation compared with the solution with Newton method using these values as initial points. This figure also shows a solution using Finite Differences with 1001 points. We can see that the approximation are good for small \( \epsilon \) values.
Let us compute its value for a real example. A steel G string in an electric guitar (196 Hz) has a perturbation parameter of

\[
\epsilon \equiv \frac{EI}{TL^2} = \frac{\pi Er^4}{4TL^2} = \frac{\pi (2 \times 10^{11} \text{ Pa})(0.4046 \times 10^{-3} \text{ m})^4}{4(65.508 \text{ N})(0.6477 \text{ m})^2} \approx 9.725 \times 10^{-6},
\]

that gives a small correction for the eigenvalues. What tells us that is a good approximation to consider the string as an ideal string rather than one with bending stiffness.

## 2 Fretboard layouts

This section describes the fretboard layout based on the eigenvalues obtained from the ideal and stiff equations. For that we need to chose the temperament for the instrument. That is, we need to pick the frequencies relationships...
between consecutive notes. We are interested in a guitar with equal tempera-
ment [9], i.e., that the ratio between two consecutive notes is constant. And 
the octaves are made of 13 notes. This let us with a ratio of \( r = \frac{12}{7} \approx 1.0595. \)

2.1 Ideal string

Based on the ideal string equation [8, 7]. We know that the fundamental 
frequency is given by

\[ f_0 = \frac{v}{2L_0}, \quad (9) \]

being \( v = \sqrt{\frac{T}{\mu}} \) the wave speed, \( T \) is the tension on the string, \( \mu \) is the linear 
mass density, and \( L_0 \) is the string length. Then, the frequency for the \( n \)th fret 
is given by

\[ r^n f_0 = \frac{v}{2L_n}, \]

where \( L_n \) is the length of the part of the string that vibrates.

Solving for \( L_n \), we get

\[ L_n = \frac{v}{2r^n f_0} = \frac{v}{2r^n \left( \frac{v}{2L_0} \right)} = \frac{L_0}{r^n}, \]

that is, the length of the vibrating part of the string is inversely proportional 
to the increase in frequency.

The fret layout refers to the distance from the nut, then we want the 
difference between the overall distance and the vibrating part

\[ x_n = L_0 - L_n = L_0 \left( 1 - \frac{1}{r^n} \right). \quad (10) \]

And we can see that the distribution of frets depends solely in the ratio 
between the length of the string and the vibrating length. Figure 3 presents a 
depiction of this distribution for the frets.

![Fret layout for an instrument with 24 frets.](image)

**Figure 3:** Fret layout for an instrument with 24 frets.
2.2 Stiff string

Based on equation 8b we can conclude that the frequency of oscillation of a stiff string is given by

$$f_0 = \frac{1}{DL} \sqrt{\frac{T}{\pi \rho}} \sqrt{1 + \pi^2 \epsilon_0} = \frac{1}{DL} \sqrt{\frac{T}{\pi \rho}} \sqrt{1 + \frac{\pi^3 ED^4}{64 L_0^2 T}}$$

with $\epsilon = \frac{\pi ED^4}{64 L_0 T}$. We want a string with vibrating length $L_n$ and frequency $r^n f_0$. This leads to the equation

$$r^n f_0 = \frac{1}{DL} \sqrt{\frac{T}{\pi \rho}} \sqrt{1 + \frac{\pi^3 ED^4}{64 L_0^2 T}},$$

or

$$r^n \frac{L_0^2}{L_n^2} [1 + \pi^2 \epsilon_0] = 1 + \pi^2 \epsilon_0 \frac{L_0^2}{L_n^2}.$$

(Solving equation 11 for $L_0^2$ we obtain

$$\frac{L_0^2}{L_n^2} = 1 \pm \sqrt{1 + 4 r^n \gamma (1 + \gamma)}$$

with $\gamma = \pi^2 \epsilon_0$. Only the solution with plus sign is of interest since it has as limit case the ideal string result when $\gamma \to 0$. Thus

$$L_n = L_0 \left[1 + \sqrt{1 + 4 r^n \gamma (1 + \gamma)}\right]^{1/2}$$

and

$$x_n = L_0 - L_n = L_0 \left(1 - \left[1 + \sqrt{1 + 4 r^n \gamma (1 + \gamma)}\right]^{1/2}\right).$$

Although the expression for the fret layout has been cast in a similar fashion than equation 10, it should be noted that the parameter $\gamma$ depends on both, geometric and material parameters. Particularly, $\gamma$ is a function of $L_0$ itself.

Since this new fret layout depends on properties of the material and the length scale $L_0$, we need to consider a particular set of strings. We now focus our attention in the strings ESXL110 [1] from D’Addario, the diameters and tensions are presented in table 1. We also consider a Young modulus of steel ($E = 200$ GPa), this is not completely right, since the thicker strings are not made of a single material, but have a core made of one material and are
| String | Note | Diameter (mm) | Tension (N) |
|--------|------|---------------|-------------|
| 1      | E    | 0.2540        | 72.128      |
| 2      | B    | 0.3302        | 68.404      |
| 3      | G    | 0.4318        | 73.696      |
| 4      | D    | 0.6604        | 81.732      |
| 5      | A    | 0.9144        | 84.672      |
| 6      | E    | 1.1684        | 75.166      |

**Table 1:** Diameter and tensions for D’Addario Nickel wound string ESXL110 [1].

wound with another material that mostly add mass but not bending stiffness.

The corrections needed for each string are presented in Figure 4. We considered a length scale of 635 mm (25 in).

![Figure 4: Correction for fret positioning for string of length scale $L_0 = 635$ mm.](image)

Based on these corrections we depicted the the new fret layout that is presented in Figure 12, as expected, the larger corrections appear for thicker strings.
Figure 5: Fret layout for an instrument with 24 frets and strings with length scale $L_0 = 635$ mm.

3 Conclusions

We developed a model for the design of fretted instruments layout that considers the bending stiffness of strings. Higher accuracy can be achieved using numerical methods but the use of analytical ones allows to write explicit formulas for the fret positioning.

Geometric (length scale and diameter), material (Young modulus), and loading (tension) parameters appear explicitly in the expression for fret positioning. Surprisingly, the mass density of the material does not appear in it.

As expected the strings with larger corrections are the thicker ones. This conclusion might be misleading, since thicker strings are commonly made of an inner core of steel (in electric guitars) or nylon (in classical guitars) and have a wounding to add mass (and then lower the pitch). These strings can be viewed as composite strings, and the model developed in the present document does not cover this case. Fletcher presents a method to modify the equations to include these effects in reference [4].

There are different causes for inharmonicities such as large amplitudes in the motion of the strings that lead to non-linear responses, and changes in tension along the string, and they are not considered. Nevertheless, these effects have been considered in different studies in the past [10, 6, 13].

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