MATHEMATICAL MODELING ABOUT NONLINEAR DELAYED HYDRAULIC CYLINDER SYSTEM AND ITS ANALYSIS ON DYNAMICAL BEHAVIORS

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Abstract. In this paper, we study dynamics in delayed nonlinear hydraulic cylinder equation, with particular attention focused on several types of bifurcations. Firstly, basing on a series of original equations, we model a nonlinear delayed differential equations associated with hydraulic cylinder in glue dosing processes for particleboard. Secondly, we identify the critical values for fixed point, Hopf, Hopf-zero, double Hopf and tri-Hopf bifurcations using the method of bifurcation analysis. Thirdly, by applying the multiple time scales method, the normal form near the Hopf-zero bifurcation critical points is derived. Finally, two examples are presented to demonstrate the application of the theoretical results.

1. Introduction. A hydraulic cylinder (also called a linear hydraulic motor) is a mechanical actuator that is used to give a unidirectional force through a unidirectional stroke (see Fig. 1). In hydraulic systems, cylinders are crucial component converting the fluid power into linear motion and force. It has many applications, notably in construction equipment (engineering vehicles), manufacturing machinery, and civil engineering. Hydraulic cylinders get their power from pressurized hydraulic fluid, which is typically oil. The hydraulic cylinder consists of a cylinder barrel, in which a piston connected to a piston rod moves back and forth. The barrel is closed on one end by the cylinder bottom (also called the cap) and the other end by the cylinder head (also called the gland) where the piston rod comes out of the cylinder. The piston has sliding rings and seals. The piston divides the inside of the cylinder into two chambers, the bottom chamber (cap end) and the piston rod side chamber (rod end / head end) [7, 6].

Prepressing and hot-pressing are two important processes during the particleboard production, which are driven by hydraulic cylinder. Hydraulic drive unit is a small system of servovalve-controlled symmetrical cylinder, and the principle of hydraulic drive unit is shown in Fig. 2. It is a common power machine that

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hydraulic cylinder is controlled by four-way valve, which is consist of three fundamental equations, that is, flow equation of hydraulic control valve, continuity equation of hydraulic cylinder and force-balance equation of hydraulic cylinder and load.

Consider the valve-motor combination shown schematically in Fig. 2 and let us first determine the servovalve equations. Assuming constant supply pressure, the linearized servovalve flow equations are

\[ q_1 = K_q x_v - 2K_c p_1, \]
\[ q_2 = K_q x_v + 2K_c p_2, \]

where

- \( q_1, q_2 \) = forward and return flows, respectively, \( m^3/sec \);
- \( p_1, p_2 \) = forward and return pressures, respectively, \( N/m^2 \);
- \( x_v \) = valve displacement from neutral, \( m \);
- \( K_q \) = valve flow gain, \( m^2/sec \);
- \( K_c \) = valve flow-pressure coefficient, \( m^5/(N\cdot sec) \).

Adding the servovalve flow equations, we obtain the usual form of the linearized flow equation of the servovalve as follows:

\[ q_L = K_q x_v - K_c p_L, \]

where

- \( q_L = (q_1 + q_2)/2 \) = load flow, \( m^3/sec \);
- \( p_L = p_1 - p_2 \) = load pressure difference, \( N/m^2 \).

Let us now turn to the motor chambers and assume that the pressure in each chamber is everywhere the same and does not saturate or cavitate, fluid velocities in the chambers are small so that minor losses are negligible, line phenomena are absent, and temperature and density are constant. Applying the continuity equation...
to each of the piston chambers yields

\[ q_1 - C_{ip}(p_1 - p_2) - C_{ep}p_1 = \frac{dv_1}{dt} + \frac{v_1}{\beta_e} \frac{dp_1}{dt}, \]

\[ C_{ip}(p_1 - p_2) - C_{ep}p_2 - q_2 = \frac{dv_2}{dt} + \frac{v_2}{\beta_e} \frac{dp_2}{dt}, \]

where

- \( t = \) time, sec;
- \( v_1 = \) volume of forward chamber (includes valve, connecting line, and piston volume), \( m^3 \);
- \( v_2 = \) volume of return chamber (includes valve, connecting line, and piston volume), \( m^3 \);
- \( C_{ip} = \) internal or cross-port leakage coefficient of piston, \( m^5/(N \cdot \text{sec}) \);
- \( C_{ep} = \) external leakage coefficient of piston, \( m^5/(N \cdot \text{sec}) \);
- \( \beta_e = \) effective bulk modulus of system (includes oil, entrapped air, and mechanical compliance of chambers), \( N/m^2 \).

The volumes of the piston chambers may be written

\[ v_1 = v_{01} + A_p x_p, \]
\[ v_2 = v_{02} - A_p x_p, \]

where

- \( A_p = \) area of piston, \( m^2 \);
- \( x_p = \) displacement of piston, \( m \);
- \( v_{01} = \) initial volume of forward chamber, \( m^3 \);
- \( v_{02} = \) initial volume of return chamber, \( m^3 \).

In contrast with the rotary motor, the initial chamber volumes are not necessarily equal for the piston. However, it will be assumed that the piston is centered such that these volumes are equal, that is, \( v_{01} = v_{02} = v_0 \). Therefore,

\[ v_t = v_1 + v_2 = v_{01} + v_{02} = 2v_0, \]

where \(v_t = \) total volume of fluid under compression in both chambers, \( m^3 \).

Then, the volume and continuity expressions can be combined to yield

\[ q_L = A_p \frac{dx_p}{dt} + C_{ip}p_L + \frac{v_t}{4\beta_e} \frac{dp_L}{dt}, \]  \( (2) \)

where \( C_{tp} = C_{ip} + C_{ep}/2 = \) total leakage coefficient of piston, \( m^5/(N \cdot \text{sec}) \), which is the usual form of the continuity equation. To be specific, \( A_p \frac{dx_p}{dt} \) means the flow driving cylinder piston, and \( C_{tp}p_L \) means total flow, and \( \frac{v_t}{4\beta_e} \frac{dp_L}{dt} \) means total compression flow.

Note that there exists a delay, denoted by \( \tau \), describing the time that the flow is leaked from cylinder, thus, equation \( (2) \) becomes

\[ q_L = A_p \frac{dx_p}{dt} + C_{ip}p_L(t - \tau) + \frac{v_t}{4\beta_e} \frac{dp_L}{dt}. \]  \( (3) \)

Considering the load characteristics of hydraulic drive unit, the output force and load force of servo cylinder balance equation is determined as follows:

\[ A_p p_L = m_t \frac{d^2x_p}{dt^2} + B_p \frac{dx_p}{dt} + Kx_p + F_c + F, \]  \( (4) \)

where

- \( m_t = \) total mass of piston and load referred to piston, kg;
- \( B_p = \) viscous damping coefficient of piston and load, \((m \cdot N \cdot \text{sec})/\text{rad}\);
$K$ = load spring gradient, $(N \cdot sec)/rad$;
$F_c$ = coulomb friction, $N$;
$F$ = load force, $N$.

Generally speaking, load force is nonlinear, and approximately follows $F = K_0 x_p^2$, where $K_0$ is load force coefficient.

Let $\frac{dx_p(t)}{dt} = y_p$, then according to [1], [3] and [4], a model for hydraulic cylinder in supplying glue system can be described mathematically as follows:

\[
\begin{align*}
\frac{dx_p(t)}{dt} &= y_p, \\
\frac{dy_p(t)}{dt} &= \frac{1}{m_p} [A_p p_L - B_p y_p - Kx_p - F - K_0 x_p^2], \\
\frac{dp_L(t)}{dt} &= \frac{4\beta_t}{v_t} [K_q x_v - K_c p_L - A_p y_p - C_{tp} p_L(t - \tau)].
\end{align*}
\]

(5)

It is well known that delay differential equations (DDE) may exhibit higher codimension singularities more frequently than that in ordinary differential equations (ODE) [2, 10, 11, 12, 13]. In this paper, by applying the local stability theory, we investigate the existence of several types of bifurcations for this hydraulic cylinder system (5). Then we derive the normal form associated with Hopf-zero critical point.

The rest of the paper is organized as follows. In Section 2, we consider the stability of the equilibria and the existence of fixed point, Hopf, Hopf-zero, double Hopf bifurcation or higher codimension bifurcations into the system, thus makes system more complicated. By applying the multiple time scales (MTS) method [8, 3, 9], we derive the normal form near Hopf-zero critical point.

In this section, system (5) is considered. First of all, we determine the equilibria of this system.

When $K^2 - 4K_0 (F_c - \frac{A_p K_q x_v}{K_c + C_{tp}}) > 0$, system (5) has two equilibria:

\[
(x_{p1}^*, 0, p_L^*) = \left( -\frac{-K - \sqrt{K^2 - 4K_0 (F_c - \frac{A_p K_q x_v}{K_c + C_{tp}})}}{2K_0}, 0, \frac{K_q x_v}{K_c + C_{tp}} \right),
\]

(6)

\[
(x_{p2}^*, 0, p_L^*) = \left( -\frac{-K + \sqrt{K^2 - 4K_0 (F_c - \frac{A_p K_q x_v}{K_c + C_{tp}})}}{2K_0}, 0, \frac{K_q x_v}{K_c + C_{tp}} \right).
\]

(6)

When $K^2 - 4K_0 (F_c - \frac{A_p K_q x_v}{K_c + C_{tp}}) = 0$, system (5) has one equilibrium:

\[
(x_{p3}^*, 0, p_L^*) = \left( -\frac{K}{2K_0}, 0, \frac{K_q x_v}{K_c + C_{tp}} \right).
\]

(7)

For convenience, we assume the equilibrium of system (5) is $(x_p^*, 0, p_L^*)$. Transferring the equilibrium to the origin, that is, let $\hat{x}_p = x_p - x_p^*, \hat{y}_p = y_p, \hat{p}_L = p_L - p_L^*$, then drop the hat, we obtain the following equations:
The trivial equilibrium of (8) is \((x_p, y_p, p_L) = (0, 0, 0)\).

The characteristic equation of (8), evaluated at origin, is given by
\[
\lambda^3 + M_2\lambda^2 + M_1\lambda + M_0 + e^{-\lambda\tau}(N_2\lambda^2 + N_1\lambda + N_0) = 0,
\]
where \(M_2 = \frac{B_2}{m_i} + \frac{4\beta_0K_c}{v_i}\), \(M_1 = \frac{4K_0x^*_p + 4\beta_0K_cB_0 + A^2_0}{m_iv_i}\), \(M_0 = \frac{4\beta_0K_c}{m_iv_i}(K + 2K_0x^*_p)\), \(N_2 = \frac{-2\beta_0C_p}{v_i}\), \(N_1 = \frac{B_2}{m_i}N_2\), \(N_0 = \frac{(K + 2K_0x^*_p)}{m_i}N_2\).

Case 1. Fixed point bifurcation.

Note that system (5) has one equilibrium \((-\frac{K}{2K_0}, 0, \frac{K_0x^*_p}{K_c+C_{tp}})\) when \(K^2 = 4K_0(F_c - \frac{A_pK_cx^*_p}{K_c+C_{tp}}) = 0\), and \(K + 2K_0x^*_p = 0\), \(M_0 = N_0 = 0\), that is, the characteristic equation of (8), is given by
\[
\lambda[\lambda^2 + M_2\lambda + M_1 + e^{-\lambda\tau}(N_2\lambda + N_1)] = 0,
\]
then the characteristic equation (10) has one zero root \(\lambda = 0\), and system (5) undergoes a fixed point bifurcation when \(K^2 = 4K_0(F_c - \frac{A_pK_cx^*_p}{K_c+C_{tp}}) = 0\). When \(\tau = 0\), Eq. (10) becomes
\[
\lambda[\lambda^2 + (M_2 + N_2)\lambda + M_1 + N_1] = 0,
\]
all the roots of Eq. (11) have negative real part expect one zero root when \(M_1 + N_1 > 0\) and \(M_2 + N_2 > 0\).

Case 2. Hopf bifurcation.

Next, we consider the local bifurcations appears only with purely imaginary roots, such as Hopf and double Hopf bifurcations for \(K^2 = 4K_0(F_c - \frac{A_pK_cx^*_p}{K_c+C_{tp}}) > 0\). When \(\tau = 0\), Eq. (9) becomes
\[
\lambda^3 + (M_2 + N_2)\lambda^2 + (M_1 + N_1)\lambda + M_0 + N_0 = 0.
\]
If \(M_2 + N_2 > 0\), \(M_0 + N_0 > 0\) and \((M_2 + N_2)(M_1 + N_1) - M_0 - N_0 > 0\), all the roots of (12) have negative real parts, and equilibrium \((x^*_p, 0, p^*_L)\) is stable, where \((x^*_p, 0, p^*_L)\) is given by (9).

To find possible periodic solutions, which may bifurcate from a Hopf critical point, let \(\lambda = i\omega \ (i^2 = -1, \ \omega > 0)\) be a root of Eq. (9). Substituting \(\lambda = i\omega\) into (9) and separating the real and imaginary parts yields
\[
\begin{align*}
M_0 - M_2\omega^2 &= \cos(\omega\tau)N_2\omega^2 - \sin(\omega\tau)N_1\omega - N_0\cos(\omega\tau), \\
\omega^3 - M_1\omega &= \sin(\omega\tau)N_2\omega^2 + \cos(\omega\tau)N_1\omega - \sin(\omega\tau)N_0.
\end{align*}
\]
Adding the square of two equations of (13). Let \(z = \omega^2\), then it follows from (13) that
\[
F_1(z) := z^3 + (M_2^2 - 2M_1 - N_2^2)z^2 + (M_1^2 - 2M_0M_2 - N_1^2 + 2N_0N_2)z + M_0^2 - N_0^2 = 0,
\]
and the derivative of $F_1(z)$ is
\[ F'_1(z) := 3z^2 + 2(M_2^2 - 2M_1 - N_2^2)z + (M_1^2 - 2M_0M_2 - N_1^2 + 2N_0N_2) = 0. \] (15)

When $\Delta_1 := (M_2^2 - 2M_1 - N_2^2)^2 - 3(M_1^2 - 2M_0M_2 - N_1^2 + 2N_0N_2) > 0$, equation $F'_1(z) = 0$ has two real roots:
\[ z^*_1 = \frac{-(M_2^2 - 2M_1 - N_2^2) - \sqrt{\Delta_1}}{3}, \quad z^*_2 = \frac{-(M_2^2 - 2M_1 - N_2^2) + \sqrt{\Delta_1}}{3}. \] (16)

Therefore, we give the following assumptions:

- (H1) $\Delta_1 > 0$, $M_2^2 - N_2^2 < 0$, $z^*_2 < 0$, $F_1(z^*_2) < 0$.
- (H2) $\Delta_1 > 0$, $M_2^2 - N_2^2 > 0$, $z^*_2 > 0$, $F_1(z^*_2) < 0$.
- (H3) $\Delta_1 > 0$, $M_2^2 - N_2^2 < 0$, $z^*_1 > 0$, $F_1(z^*_1) > 0$, $F_1(z^*_2) < 0$.
- (H4) $\Delta_1 > 0$, $M_2^2 - N_2^2 > 0$, $z^*_2 < 0$.

Under (H1), Eq. (14) has one positive root $z_1$, and $F'_1(z_1) > 0$. Under (H2), Eq. (14) has two positive roots, $z_1$ and $z_2$. Suppose $z_1 < z_2$, then $F'_1(z_1) < 0$, $F'_1(z_2) > 0$. Under (H3), Eq. (14) has three positive roots, $z_1$, $z_2$, and $z_3$. Suppose $z_1 < z_2 < z_3$, then $F'_1(z_1) > 0$, $F'_1(z_2) > 0$ and $F'_1(z_3) > 0$. Under (H4), Eq. (14) does not have positive root.

Without loss of generality, we assume that Eq. (14) may have positive roots: $z_k$ ($k = 1, 2, 3$), thus, $\omega_k = \sqrt{z_k}$. Actually, if Eq. (14) has only one (or two) positive root $z_1$ (or $z_1, z_2$), we only need to fix $k = 1$ (or $k = 1, 2$). Due to Eq. (13), we obtain
\[ Q_k := \sin(\omega_k \tau) = \frac{(N_2\omega_k^2 - N_0)\omega_k^2(w_k^2 - M_1) - N_1\omega_k(M_0 - M_2\omega_k^2)}{(N_2\omega_k^2 - N_0)^2 + N_1^2\omega_k^2}, \]
\[ P_k := \cos(\omega_k \tau) = \frac{(M_0 - M_2\omega_k^2)(N_2\omega_k^2 - N_0) + N_1\omega_k^2(\omega_k^2 - M_1)}{(N_2\omega_k^2 - N_0)^2 + N_1^2\omega_k^2}. \] (17)

The time delay $\tau$ can be determined from (17) as
\[ \tau_k^{(j)} = \frac{1}{\omega_k} \left[ \arccos(P_k) + 2j\pi \right], \quad Q_k \geq 0, \]
\[ \frac{1}{\omega_k} \left[ 2\pi - \arccos(P_k) + 2j\pi \right], \quad Q_k < 0. \] (18)

Let $\lambda(\tau) = a(\tau) + i\omega(\tau)$ be the root of (9) satisfying $a(\tau^{(j)}) = 0$, $\omega(\tau^{(j)}) = \omega_k$, $k = 1, 2, 3$; $j = 0, 1, 2, \ldots$. Then, we have the transversality conditions:
\[ \text{Sign}[\text{Re}(\frac{d\lambda}{d\tau_k^{(j)}})]^{-1} = \text{Sign}[\frac{\omega_k^2 F'_1(z_k)}{N_1^2\omega_k^4 + (N_0 - N_2\omega_k^2)^2 \omega_k^2}] = \text{Sign}[F'_1(z_k)], \] (19)
where $k = 1, 2, 3$; $j = 0, 1, 2, \ldots$.

**Case 3. Hopf-zero bifurcation.**

When $K^2 - 4K_0(F_c - \frac{A_2K_0F_c}{K_0 - x_0}) = 0$, characteristic equation (9) has one zero root. To find possible periodic solutions, which may bifurcate from a Hopf critical point, let $\lambda = i\omega$ ($i^2 = -1$, $\omega > 0$) be a root of Eq. (10). Substituting $\lambda = i\omega$ into (10) and separating the real and imaginary parts yields
\[ \omega^2 - M_1 = \sin(\omega \tau)N_2\omega + \cos(\omega \tau)N_1, \]
\[ M_2\omega = \sin(\omega \tau)N_1 - \cos(\omega \tau)N_2\omega. \] (20)

Let $z = \omega^2$. Then it follows from (20) that
\[ F_2(z) := z^2 + (M_2^2 - 2M_1 - N_2^2)z + M_1^2 - N_1^2 = 0. \] (21)
Denote $\Delta_2 = (M_2^2 - 2M_1 - N_2^2)^2 - 4(M_1^2 - N_1^2)$, and we give the following assumptions:

(H5) $M_1^2 - N_1^2 < 0$.
(H6) $M_1^2 - N_1^2 > 0$, $\Delta_2 > 0$, $M_2^2 - 2M_1 - N_2^2 < 0$.
(H7) $M_1^2 - N_1^2 > 0$, $M_2^2 - 2M_1 - N_2^2 > 0$.

Under (H5), Eq. (21) has one positive root $z_1 = \frac{1}{2}(2M_1 + N_2^2 - M_2^2 + \sqrt{\Delta_2})$, then $\omega_1 = \sqrt{z_1}$ and $F'_2(z_1) > 0$. Under (H6), Eq. (21) has two positive roots $z_{1,2} = \frac{1}{2}(2M_1 + N_2^2 - M_2^2 \pm \sqrt{\Delta_2})$, then $\omega_{1,2} = \sqrt{z_{1,2}}$ and $F'_2(z_1) < 0$, $F'_2(z_2) > 0$ with $z_1 < z_2$. Under (H7), Eq. (21) does not have positive root.

When Eq. (21) has one or two positive roots, without loss of generality, we assume that (21) may have positive roots: $z_k$ ($k = 1, 2$), thus, $\omega_k = \sqrt{z_k}$. Actually, if Eq. (21) has only one positive root $z_1$, we only need to fix $k = 1$. Due to (20), we obtain

$$Q_k := \sin(\omega_k \tau) = \frac{N_2 \omega_k(\omega_k^2 - M_1) + N_1 M_2 \omega_k}{N_2 \omega_k^2 + N_1^2},$$
$$P_k := \cos(\omega_k \tau) = \frac{N_1(\omega_k^2 - M_1) - N_2 M_2 \omega_k^2}{N_2 \omega_k^2 + N_1^2}.$$  \hspace{1cm} (22)

The time delay $\tau$ can be determined from (22) as

$$\tau_k^{(j)} = \begin{cases} \frac{1}{\omega_k}[\arccos(P_k) + 2j\pi], & Q_k \geq 0, \\ \frac{1}{\omega_k}[2\pi - \arccos(P_k) + 2j\pi], & Q_k < 0. \end{cases}$$  \hspace{1cm} (23)

Let $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ be the root of (10) satisfying $\alpha(\tau_k^{(j)}) = 0$, $\omega(\tau_k^{(j)}) = \omega_k$, $k = 1, 2$; $j = 0, 1, 2, \cdots$. Then, we have the transversality conditions:

$$\text{Sign}[\text{Re}\left(\frac{d\lambda}{d\tau_k^{(j)}}\right)^{-1}] = \text{Sign}\left[\frac{\omega_k^2 F'_2(z_k)}{N_1^2 \omega_k^4 + N_2^2 \omega_k^6}\right] = \text{Sign}[F'_2(z_k)],$$ \hspace{1cm} (24)

where $k = 1, 2$; $j = 0, 1, 2, \cdots$. Combining the above results, we have the following theorem.

**Theorem 2.1.** For system (3), we have the following conclusions:

(1) When $K^2 - 4K_0(F_c - \frac{A_p K_{x_{\tau}}} {K_{x_{\tau}} + C_{\tau p}}) = 0$, system (3) may undergo a fold bifurcation from equilibrium $(-\frac{K_{x_{\tau}}}{2K_0}, 0, \frac{K_{x_{\tau}}}{K_{x_{\tau}} + C_{\tau p}})$.

If $M_1 + N_1 > 0$ and $M_2 + N_2 > 0$, where $M_1, M_2, N_1, N_2$ are given by (9), we have

(a) Under (H5) or (H6), system (3) undergoes a fold bifurcation from equilibrium $(-\frac{K_{x_{\tau}}}{2K_0}, 0, \frac{K_{x_{\tau}}}{K_{x_{\tau}} + C_{\tau p}})$ for $\tau \in \{\tau \mid \tau \geq 0, \tau \neq \tau_k^{(j)}\}$, where $\tau_k^{(j)}$ is given by (23). In particular, when $\tau \in [0, \tau_0]$ with $\tau_0 = \min\{\tau_k^{(0)}\}$, Eq. (10) has a zero root, and all the other roots have negative real part.

(b) Under (H7), Eq. (10) has a zero root, and all the other roots have negative real part for $\tau \geq 0$.

(2) When $K^2 - 4K_0(F_c - \frac{A_p K_{x_{\tau}}} {K_{x_{\tau}} + C_{\tau p}}) > 0$, system (3) undergoes a Hopf bifurcation from equilibrium $((x_p^0)^k, 0, p_{\tau}^k)$ at $\tau = \tau_k^{(j)}$ ($k = 1, 2, 3; j = 0, 1, 2, \cdots$), where $(x_p^0, 0, p_{\tau}^k)$ is given by (10), and $\tau_k^{(j)}$ is given by (18).

If $M_2 + N_2 > 0$, $M_0 + N_0 > 0$ and $(M_2 + N_2)(M_1 + N_1) - M_0 - N_0 > 0$, we have
(a) Under (H1), Eq. (14) has one positive root, then equilibrium \((x_p^*, 0, p_p^*)\) of system (5) is local asymptotically stable for \(\tau \in (0, \tau_1^{(0)})\), and unstable for \(\tau \in (\tau_1^{(0)}, +\infty)\).

(b) Under (H2), Eq. (14) has two positive roots, then there exists \(m \in N\) such that \(0 < \tau_2^{(0)} < \tau_1^{(0)} < \tau_1^{(1)} < \tau_1^{(2)} < \cdots < \tau_2^{(m-1)} < \tau_1^{(m-1)} < \tau_2^{(m)} < \tau_1^{(m+1)}\), and equilibrium \((x_p^*, 0, p_p^*)\) of system (5) is local asymptotically stable for \(\tau \in [0, \tau_2^{(0)}) \cup \bigcup_{l=0}^{m-1} (\tau_1^{(l)}, \tau_2^{(l+1)})\) and unstable for \(\tau \in \bigcup_{l=0}^{m} (\tau_2^{(l)}, \tau_1^{(l+1)})\).

(c) Under (H3), Eq. (14) has three positive roots, then there exists a family of \(\tau_k\) such that \(0 < \tau_1 < \tau_2 < \tau_3 < \tau_4 < \tau_5 < \cdots\), and there also exists \(m \in N\) and a family of \(k_1, k_2, \cdots, k_j \in N\), then the equilibrium \((x_p^*, 0, p_p^*)\) of system (5) is local asymptotically stable for \(\tau \in [0, \tau_1) \cup \bigcup_{j=1}^{m} (\tau_{k_j-1}, \tau_{k_j})\) and unstable for \(\tau \in \bigcup_{j=1}^{m} (\tau_{k_j}, \tau_{k_j+1})\).

(d) Under (H4), Eq. (14) does not have positive root, then the equilibrium \((x_p^*, 0, p_p^*)\) of system (5) is local asymptotically stable for \(\tau \geq 0\).

(3) When \(K^2 - 4K_0(F_c - \frac{A_pKc}{Kc+C_c0}) = 0\), system (5) undergoes a Hopf-zero bifurcation from equilibrium \((-K, 0, -\frac{Kc \pm \sqrt{Kc}}{Kc+C_c0})\) at \(\tau = \tau_k^{(j)}\) \((j = 0, 1, 2, \cdots)\), where \(\tau_k^{(j)}\) is given by (27).

When \(M_1 + N_1 > 0\) and \(M_2 + N_2 > 0\), we have

(a) Under (H5), Eq. (21) has only one positive root, then equilibrium \((-\frac{K}{2K_0}, 0, -\frac{Kc \pm \sqrt{Kc}}{Kc+C_c0})\) of system (5) is unstable for \(\tau \in (\tau_1^{(0)}, +\infty)\). When \(\tau \in (0, \tau_1^{(0)})\), characteristic equation (10) has one zero root and all the other roots have negative real part.

(b) Under (H6), Eq. (27) has two positive roots, then there exists \(m \in N\) such that \(0 < \tau_2^{(0)} < \tau_1^{(0)} < \tau_1^{(1)} < \tau_1^{(2)} < \cdots < \tau_2^{(m-1)} < \tau_1^{(m-1)} < \tau_2^{(m)} < \tau_1^{(m+1)}\), and equilibrium \((-\frac{K}{2K_0}, 0, -\frac{Kc \pm \sqrt{Kc}}{Kc+C_c0})\) of system (5) is unstable for \(\tau \in \bigcup_{l=0}^{m-1} (\tau_2^{(l)}, \tau_1^{(l+1)})\)\(\cup (\tau_2^{(m)}, +\infty)\). When \(\tau \in [0, \tau_2^{(0)}) \cup \bigcup_{l=0}^{m-1} (\tau_1^{(l)}, \tau_2^{(l+1)})\), characteristic equation (10) has one zero root and all the other roots have negative real part.

From Theorem 2.1, a possible multiple-Hopf bifurcation may occur when two such families of curves intersect. Thus, we obtain the following theorem.

**Theorem 2.2.**

1. When \(K^2 - 4K_0(F_c - \frac{A_pKc}{Kc+C_c0}) > 0\), if (H2) holds, and \(\tau_1^{(0)} = \tau_2^{(1)}\), then system (5) occurs double Hopf bifurcation, where \(\tau_k^{(j)}\) \((k = 1, 2; j, l = 0, 1, 2, \cdots)\) is given by (18).

2. When \(k^2 - 4k_0(F_c - \frac{A_pKc}{Kc+C_c0}) > 0\), if (H3) holds, and \(\tau_1^{(0)} = \tau_2^{(0)} = \tau_3^{(k)}\), then system (5) occurs tri-Hopf bifurcation, where \(\tau_k^{(j)}\) \((k = 1, 2; j, l = 0, 1, 2, \cdots)\) is given by (18).

3. When \(k^2 - 4k_0(F_c - \frac{A_pKc}{Kc+C_c0}) = 0\), if (H6) holds, and \(\tau_1^{(0)} = \tau_2^{(0)}\), then system (5) occurs 2-Hopf-1-zero bifurcation, where \(\tau_k^{(j)}\) \((k = 1, 2; j, l = 0, 1, 2, \cdots)\) is given by (23).

The equality \(\tau_c = \tau_1^{(0)} = \tau_2^{(0)}\) \((\tau_c = \tau_1^{(j)} = \tau_2^{(l)} = \tau_3^{(k)})\) implies that the linearized system on the equilibrium has two (three) pairs of purely imaginary eigenvalues.
and one pair of pure imaginary roots \( \pm i\omega_2 \) when \( \tau = \tau_c \). For some \( \tau \), if Eq. \( 6 \) has two (three) pairs of purely imaginary roots \( \pm i\omega_1 \), \( \pm i\omega_2 \), \( \pm i\omega_3 \) and all the other roots have non-zero real part, then system \( 5 \) undergoes a double Hopf bifurcation (tri-Hopf bifurcation).

3. Normal form of Hopf-zero bifurcation. In this section, we only derive the normal form of Hopf-zero bifurcation by using the multiple time scales method, and the normal forms of other bifurcations can be derived similarly. When \( K^2 - 4K_0(F_c - \frac{A_pK_x}{K_x + C_{tp}}) = 0 \) and \( \tau = \tau^{(i)}_k \), the characteristic equation \( 9 \) has one zero root \( \lambda = 0 \) and one pair of pure imaginary roots \( \lambda = \pm i\omega \), where \( \tau^{(i)}_k \) \( (k = 1, 2; j, l = 0, 1, 2, \ldots) \) is given by \( 23 \). We treat the total leakage flow \( C_{tp} \) and the leakage delay \( \tau \) as two bifurcation parameters. Suppose system \( 8 \) undergoes a Hopf-zero bifurcation from the trivial equilibrium at the critical point: \( C_{tp} = C_{tpc}, \tau = \tau_c \). Further, we write Eq. \( 8 \) as follows:

\[
\dot{X}(t) = AX(t) + BX(t - \tau) + F(X(t), X(t - \tau)), \quad (25)
\]

where \( X(t) = (x_p(t), y_p(t), p_L(t))^T \), \( X(t - \tau) = (x_p(t - \tau), y_p(t - \tau), p_L(t - \tau))^T \),

\[
A = \begin{pmatrix}
0 & 1 & 0 \\
0 & -\frac{B_p}{m_i} & A_p \\
0 & -\frac{2B_p}{m_i} & \frac{A_p}{m_i}
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -\frac{4\beta c_1}{v_t}
\end{pmatrix}.
\]

\[
F(X(t), X(t - \tau)) = \begin{pmatrix}
0 \\
0 \\
-\frac{K_0\tau^2}{m_i}
\end{pmatrix}.
\]

Define the linearized equation of \( 26 \) as

\[
\dot{X}(t) = AX(t) + BX(t - \tau) := L_c(X(t), X(t - \tau)),
\]

whose characteristic equation has a pair of purely imaginary roots \( \pm i\omega \) and a zero root, and no other roots with zero real part. Let \( h_1 = (h_{11}, h_{12}, h_{13})^T \) and \( h_2 = (h_{21}, h_{22}, h_{23})^T \) be the two eigenvectors of the linear operator \( L_c \) corresponding to the eigenvalues \( i\omega \) and 0, respectively. Further, let \( h_1^* = (h^*_{11}, h^*_{12}, h^*_{13})^T \) and \( h_2^* = (h^*_{21}, h^*_{22}, h^*_{23})^T \) be the two normalized eigenvectors of the adjoint operator \( L_c^* \) of the linear operator \( L_c \) corresponding to the eigenvalues \( -i\omega \) and 0, respectively, satisfying the inner product

\[
\langle h_1^*, h_i \rangle = \langle h_2^*, h_i \rangle = 1, \quad i = 1, 2.
\]

By a simple calculation, we have

\[
h_1 = (h_{11}, h_{12}, h_{13})^T = (1, i\omega, \frac{iwm_i}{A_p}(i\omega + \frac{B_p}{A_p}))^T,
\]

\[
h_2 = (h_{21}, h_{22}, h_{23})^T = (1, 0, 0)^T,
\]

\[
h_1^* = (h^*_{11}, h^*_{12}, h^*_{13})^T = \frac{4\beta c_1}{v_t}A_p, i\omega - \frac{B_p}{m_i}v_t^T,
\]

\[
h_2^* = (h^*_{21}, h^*_{22}, h^*_{23})^T = \frac{2\beta c_1}{v_t}A_p + \frac{4\beta c_1B_p(K_c + C_{tpc})}{m_i v_t} - \frac{4\beta c_1(K_c + C_{tpc})}{v_t} A_p^2 m_i v_t^T.
\]

where \( d_1 = \left(\frac{B_p - i\omega}{A_p}\right)^2 m_i v_t - \frac{4\beta c_1 i\omega}{m_i v_t} - 1 \), \( d_2 = \frac{m_i v_t}{4\beta c_1 A_p^2 + 4\beta c_1 B_p(K_c + C_{tpc})} \).
Next, we use the multiple time scales (MTS) method to derive the normal form of system (25) associated with Hopf-zero bifurcation. The solution of (25) is assumed to take the form:

\[
X(t) = X(T_0, T_1, T_2, \cdots) = \sum_{k=1}^{\infty} \epsilon^k X_k(T_0, T_1, T_2, \cdots),
\]

(27)

where

\[
X(T_0, T_1, T_2, \cdots) = \left( x_{p}(T_0, T_1, T_2, \cdots), y_{p}(T_0, T_1, T_2, \cdots), pL(T_0, T_1, T_2, \cdots) \right)^T,
\]

\[
X_k(T_0, T_1, T_2, \cdots) = \left( x_{pk}(T_0, T_1, T_2, \cdots), y_{pk}(T_0, T_1, T_2, \cdots), p_{Lk}(T_0, T_1, T_2, \cdots) \right)^T.
\]

The derivative with respect to \( t \) is now transformed into

\[
\frac{d}{dt} = \frac{\partial}{\partial T_0} + \epsilon \frac{\partial}{\partial T_1} + \epsilon^2 \frac{\partial}{\partial T_2} + \cdots = D_0 + \epsilon D_1 + \epsilon^2 D_2 + \cdots,
\]

where the differential operator \( D_i = \frac{\partial}{\partial T_i}, \ i = 0, 1, 2, \cdots \).

Define

\[
X_j = (x_{pj}, y_{pj}, p_{Lj})^T = X_j(T_0, T_1, T_2, \cdots),
\]

\[
X_{j, \tau_c} = (x_{pj, \tau_c}, y_{pj, \tau_c}, p_{Lj, \tau_c})^T = X_j(T_0 - \tau_c, T_1, T_2, \cdots),
\]

with \( j = 1, 2, \cdots \). From (27), we obtain

\[
X(t) = \epsilon D_0 X_1 + \epsilon^2 D_1 X_1 + \epsilon^3 D_2 X_1 + \epsilon^2 D_0 X_2 + \epsilon^3 D_1 X_2 + \epsilon^3 D_0 X_3 + \cdots.
\]

(28)

We take perturbations as \( C_{lp} = C_{lp} + \epsilon \mu_1 \) and \( \tau = \tau_c + \epsilon \mu_2 \) in system (25), where \( \mu_1 \) and \( \mu_2 \) are called detuning parameters [9]. To deal with the delayed terms, we expand \( X_j(t - \tau) \) at \( X_{j, \tau_c} \) for \( j = 1, 2, 3, \cdots \), particularly,

\[
p_{L}(t - \tau) = \epsilon p_{L1, \tau_c} + \epsilon^2 p_{L2, \tau_c} + \epsilon^3 p_{L3, \tau_c} - \epsilon^2 \mu_1 D_0 p_{L1, \tau_c} - \epsilon^3 \mu_1 D_0 p_{L2, \tau_c} - \epsilon^2 \tau_c D_1 p_{L1, \tau_c} - \epsilon^3 \tau_c D_1 p_{L2, \tau_c} - \epsilon^3 \tau_c D_1 p_{L3, \tau_c} + \cdots.
\]

(29)

where \( p_{Lj, \tau_c} = p_{Lj}(T_0 - \tau_c, T_1, T_2, \cdots) \) with \( j = 1, 2, \cdots \).

Then, substituting the solutions with the multiple scales (27)–(29) into (25) and balancing the coefficients of \( \epsilon^1 \) \( (j = 1, 2, 3, \cdots) \) yields a set of ordered linear differential equations. First, for the \( \epsilon^1 \)-order terms, we have

\[
D_0 x_{p1} - y_{p1} = 0,
\]

\[
D_0 y_{p1} + \frac{1}{m_t} (A_{p} p_{L1} - B_{p} y_{p1}) = 0,
\]

\[
D_0 p_{L1} + \frac{4 \beta_c}{v_t} (K_c p_{L1} + A_{p} y_{p1} + C_{lp} p_{L1, \tau_c}) = 0.
\]

(30)

Since \( \pm i \omega \) and 0 are the eigenvalues of the linear part of (25), the solution of (30) can be expressed in the form of

\[
X_1(T_1, T_2, \cdots) = G_1(T_1, T_2, \cdots) e^{i\omega T_0} h_1 + \bar{G}_1(T_1, T_2, \cdots) e^{-i\omega T_0} \bar{h}_1
\]

\[
+ G_2(T_1, T_2, \cdots) h_2,
\]

(31)

where \( h_1 \) and \( h_2 \) are given by (26).
Next, for the \( e^2 \)-order terms, we obtain

\[
D_0 x_{p2} - y_{p2} = -D_1 x_{p1},
\]

\[
D_0 y_{p2} - \frac{1}{m_t}(A_p p_{L2} - B_p y_{p2}) = -D_1 y_{p1} - \frac{k_0}{m_t} x_{p1},
\]

\[
D_0 p_{L2} + \frac{4\beta_e}{v_t}(K_e p_{L2} + A_p y_{p2} + C_{tpe} p_{L2,\tau_e})
\]

\[
= -D_1 p_{L1} + \frac{4\beta_e}{v_t}(C_{tpe} \mu_1 D_0 p_{L1,\tau_e} + C_{tpe} \tau_e D_1 p_{L1,\tau_e} - \mu_2 p_{L1,\tau_e}).
\]

Nonhomogeneous equation (32) has a solution if and only if the so-called solvability condition is satisfied [8]. That is, the right-hand side of nonhomogeneous equation (32) is orthogonal to every solution of the adjoint homogeneous problem. Substituting solution (31) into the right expression of (32), we obtain the coefficient of term \( e^{i\omega T_1} \), denoted as \( m_1 \), and the constant term, denoted as \( m_2 \). As a matter of fact, finding the solvability conditions is equivalent to finding the conditions resulted from eliminating the secular terms. Let \( (h^1_1, m_1) = 0 \) and \( (h^2_2, m_2) = 0 \), where \( h^j_j \) \((j = 1, 2)\) is given by [26]. Then \( \frac{\partial G_1}{\partial T_1} \) and \( \frac{\partial G_2}{\partial T_1} \) are solved to yield

\[
\frac{\partial G_1}{\partial T_1} = \frac{1}{L_1} \left[ \frac{-8k_0\beta_e A_p}{m_t v_t} G_1 G_2 + (i\omega + \frac{B_p}{m_t})^2 \frac{4\beta_e m_t}{v_t A_p} e^{-i\omega \tau_e} (C_{tpe} \mu_1 \omega + \mu_2 i) G_1 \right],
\]

\[
\frac{\partial G_2}{\partial T_1} = -\frac{(K_e + C_{tpe}) K_0 (G_2^2 + 2G_1 G_1)}{A_p^2 + B_p (K_e + C_{tpe})},
\]

where \( L_1 = i\omega \frac{4\beta_e A_p}{v_t} + (i\omega + \frac{B_p}{m_t})^2 \frac{m_t}{A_p} (4\beta_e C_{tpe} \tau_e i e^{-i\omega \tau_e} - i\omega) \).

Then, the particular solution of \( X_2(t) \) is obtained from the resulting equation of (32) as

\[
x_{p2} = a_1 e^{i\omega T_0} + b_1 e^{2i\omega T_0} + c.c. + c_1, \\
y_{p2} = a_2 e^{i\omega T_0} + b_2 e^{2i\omega T_0} + c.c. + c_2, \\
p_{L2} = a_3 e^{i\omega T_0} + b_3 e^{2i\omega T_0} + c.c. + c_3,
\]

where \( c.c. \) stands for the complex conjugate of the preceding terms, and

\[
a_1 = \frac{1}{i\omega} \left( a_2 - \frac{\partial G_1}{\partial T_1} \right),
\]

\[
a_2 = \frac{A_p L v_t - (i\omega m_t \frac{\partial G_1}{\partial T_1}) + 2K_0 G_1 G_2 (i\omega v_t + 4\beta_e K_e + 4\beta_e C_{tpe} e^{-i\omega \tau_e}) + 4\beta_e A_p^2}{(i\omega m_t + B_p) (i\omega v_t + 4\beta_e K_e + 4\beta_e C_{tpe} e^{-i\omega \tau_e}) + 4\beta_e A_p^2},
\]

\[
a_3 = \frac{(i\omega m_t + B_p) L v_t + 4\beta_e A_p (i\omega m_t \frac{\partial G_1}{\partial T_1}) + 2K_0 G_1 G_2}{(i\omega m_t + B_p) (i\omega v_t + 4\beta_e K_e + 4\beta_e C_{tpe} e^{-i\omega \tau_e}) + 4\beta_e A_p^2},
\]

\[
L = -i\omega (i\omega + \frac{B_p}{m_t}) \frac{m_t}{A_p} \frac{\partial G_1}{\partial T_1} + \frac{4\beta_e e^{-i\omega \tau_e} m_t (i\omega + \frac{B_p}{m_t})}{A_p v_t} (C_{tpe} \tau_e i \omega \frac{\partial G_1}{\partial T_1} - C_{tpe} \mu_1 \omega^2 G_1 - \mu_2 i \omega G_1),
\]

\[
\]
\[ b_1 = \frac{b_2}{2\omega}, \]
\[ b_2 = -\frac{K_0 G_i^2(\omega v_t + 2\beta_c K_e + 2\beta_c C_{tpc} e^{-2i\omega \tau})}{(2\omega m_t + B_p)(\omega v_t + 2\beta_c K_e + 2\beta_c C_{tpc} e^{-2i\omega \tau}) + 2\beta_c A_p^2}, \]
\[ b_3 = \frac{2\beta_c A_p K_0 G_i^2}{(2\omega m_t + B_p)(\omega v_t + 2\beta_c K_e + 2\beta_c C_{tpc} e^{-2i\omega \tau}) + 2\beta_c A_p^2}, \]
\[ c_1 = 0, \quad c_2 = \frac{\partial G_2}{\partial T_1}, \quad c_3 = -\frac{A_p c_2}{K_e + C_{tpc}}. \]

Next, for the \( \epsilon^3 \)-order terms, we obtain
\[ D_0 x_{p3} - y_{p3} = -D_2 x_{p1} - D_1 x_{p2}, \]
\[ D_0 y_{p3} - \frac{1}{m_t} (A_p pL_3 - B_p y_{p3}) = -D_2 y_{p1} - D_1 y_{p2} - \frac{2K_0}{m_t} x_{p1} x_{p2}, \]
\[ D_0 pL_3 + \frac{4\beta_c}{v_t} (K_e pL_3 + A_p y_{p3} + C_{tpc} pL_3, \tau_c) = -D_2 pL_1 - D_1 pL_2 \]
\[ + \frac{4\beta_c}{v_t} (C_{tpc} \mu_1 D_0 pL_2, \tau_c + C_{tpc} \mu_1 D_1 pL_1, \tau_c + C_{tpc} \tau_c D_2 pL_1, \tau_c + C_{tpc} \tau_c D_1 pL_2, \tau_c \]
\[ - \mu_2 pL_2, \tau_c + \mu_1 \mu_2 D_0 pL_1, \tau_c + \mu_2 \tau_c D_1 pL_1, \tau_c). \]

Substituting solution (29), (31), (33) and (34) into the right expression of (36), by using solvability condition, then \( \frac{\partial G_1}{\partial T_2} \) and \( \frac{\partial G_2}{\partial T_2} \) are solved to yield
\[ \frac{\partial G_1}{\partial T_2} = \left\{ -\frac{4\beta_c m_t}{v_t} \frac{\partial G_1}{\partial T_1} \right\}, \]
\[ \frac{\partial G_2}{\partial T_2} = \left\{ \frac{4\beta_c A_p m_t}{v_t} \frac{\partial G_2}{\partial T_1} - \frac{2K_0}{m_t} C_{tpc} \tau_c \frac{\partial G_2}{\partial T_1} \right\}. \]

The norm form associated with Hopf-zero bifurcation is given as follows:
\[ \dot{G}_1 = \epsilon \frac{\partial G_1}{\partial T_1} + \epsilon^2 \frac{\partial^2 G_1}{\partial T_2} + \cdots, \]
\[ \dot{G}_2 = \epsilon \frac{\partial G_2}{\partial T_1} + \epsilon^2 \frac{\partial^2 G_2}{\partial T_2} + \cdots. \]

Note that \( \frac{\partial G_1}{\partial T_1} \) and \( \frac{\partial G_2}{\partial T_1} \) are \((j+1)\)-order linear homogeneous polynomial involving \( G_1 \) and \( G_2 \). With the use of backwards scaling \( \epsilon \to 1/\epsilon \), the above equation
which is the normal form derived using the MTS method, where \( \frac{\partial G_1}{\partial T_1} \), \( \frac{\partial G_1}{\partial T_2} \) (\( k = 1, 2 \)) are given by (33) and (37), respectively.

4. Examples. In this section, we choose two groups of parameters used in actual system, then show the results associated with stability and bifurcation analysis.

Example 1.

Let \( A_p = 0.1256m^2 \), \( x_v = 0.01m \), \( K_c = 1.25 \times 10^{-4}m^5/(N \cdot s) \), \( K_p = 7.4 \times 10^{-4}m^5/s \), \( C_{tp} = 5 \times 10^{-16}m^5/(N \cdot s) \), \( m_t = 1500kg \), \( K = 1.25 \times 10^9(N \cdot s)/rad \), \( K_0 = 10^8N/m^2 \), \( F_c = 2 \times 10^6N \), \( B_p = 2.25 \times 10^6(m \cdot N \cdot s)/rad \), \( \beta_c = 7 \times 10^8N/m^2 \), \( v_t = 3.768 \times 10^{-3}m^3 \). Obviously, we have \( K^2 - 4K_0(F_c - \frac{A_pK_y}{K_p}) = 1.5545 \times 10^{18} > 0 \). According to (6), system (5) has two equilibria \((x_{p1}^*, 0, p_L^*) = (-1.2484, 0, 0.0592)\) and \((x_{p2}^*, 0, p_L^*) = (-0.0016, 0, 0.0592)\). Note that \( M_2 + N_2 = 9.2889 \times 10^7 > 0 \), \( M_0 + N_0 = 7.7208 \times 10^{13} \), \( M_2 + N_2(M_1 + N_1) - M_0 - N_0 = 1.2943 \times 10^9 > 0 \) for the equilibrium \((x_{p2}^*, 0, p_L^*)\) is local asymptotically stable for \( \tau = 0 \) in system (5) (see Fig. 4). \( \Delta_1 = 7.4444 \times 10^{31} > 0 \), \( M_0^2 - N_0^2 > 0 \), \( z_2^* = -6 \times 10^6 < 0 \), that is (H4) holds, then Eq. (14) does not have positive real root. Actually, by computing, Eq. (14) becomes \( F_1^*(z) = z^3 + 8.6281 \times 10^5z^2 + 3.3759 \times 10^{22}z + 5.9610 \times 10^{27} = 0 \), the three roots of Eq. (14) are: \( z_{1,2} = -2.9393 \times 10^5 \pm 7.7749 \times 10^5i \), \( z_3 = -8.628082713 \times 10^{15} \), and all of the three roots of Eq. (14) have negative real parts.

Based on Theorem 2.1 (2)(d), the equilibrium \((x_{p2}^*, 0, p_L^*)\) of system (5) is local asymptotically stable for \( \tau \geq 0 \). Then \( M_2 + N_2 = 9.2889 \times 10^7 > 0 \), \( M_0 + N_0 = -7.7208 \times 10^{13} \), \( (M_2 + N_2)(M_1 + N_1) - M_0 - N_0 = 1.2943 \times 10^9 > 0 \) for the equilibrium \((x_{p1}^*, 0, p_L^*) = (-1.2484, 0, 0.0592)\), then the equilibrium \((x_{p1}^*, 0, p_L^*)\) is unstable for \( \tau = 0 \) in system (5). \( \Delta_1 = 7.4444 \times 10^{31} > 0 \), \( M_0^2 - N_0^2 > 0 \), \( z_2^* = -6 \times 10^6 < 0 \), that is (H4) holds, then Eq. (14) does not have positive real root. Thus, the equilibrium \((x_{p1}^*, 0, p_L^*)\) of system (5) is unstable for \( \tau \geq 0 \).

Remark 1. Note that system (5) is a stiff differential equation, and they differ quite a bit in the order of magnitude among the parameters. When \( \tau = 0 \), system (5) is an ordinary differential equation, and we can show the numerical simulations in Matlab by using the command “ode15s”, which is used to solve stiff ordinary differential equation (see Fig. 4). However, when \( \tau \neq 0 \), there does not exist the command in Matlab to solve stiff delayed differential equation, thus we can not show the simulation results for \( \tau \neq 0 \).

Example 2.

Let \( A_p = 0.25m^2 \), \( x_v = 0.3m \), \( K_c = 5 \times 10^{-11}m^5/(N \cdot s) \), \( K_p = 2 \times 10^{-3}m^2/s \), \( C_{tp} = 1.5 \times 10^{-10}m^5/(N \cdot s) \), \( m_t = 1500kg \), \( K = 10^7(N \cdot s)/rad \), \( K_0 = 10^8N/m^2 \), \( F_c = 10^5N \), \( B_p = 10^6(m \cdot N \cdot s)/rad \), \( \beta_c = 10^5N/m^2 \), \( v_t = 4 \times 10^{-3}m^3 \). Obviously, we have \( K^2 - 4K_0(F_c - \frac{A_pK_y}{K_p}) = 0 \). According to (7), system (5) has only one equilibrium \((x_{p1}^*, 0, p_L^*) = (-0.05, 0.3 \times 10^6)\). Note that \( M_0^2 - N_0^2 = -4.3750 \times 10^9 < 0 \), that is (H5) holds, then Eq. (21) has one positive real root \( z \), and
\[ \omega = \sqrt{\zeta} = 0.0992. \] Computing from (22)-(24), \( \tau^0_1 = 24.3798, \tau^{(1)}_1 = 87.7083, \] Sign[Re(\( \frac{\partial \lambda}{\partial \tau} \))^{-1}] = Sign[F'(z_1)] > 0. Substituting these expressions into (39), and let \( G_1 = \tau e^{i\theta} \) and \( G_2 = z \), yields the normal form of Hopf-zero bifurcation for \((r, z, \theta)\) with \( r \geq 0 \). Note that the equations associated with \( \dot{r} \) and \( \dot{z} \) are independent of the equation with respective of \( \dot{\theta} \), and the equation for \( \dot{\theta} \) describes a rotation around \( z \)-axis with almost constant angular velocity \( \dot{\theta} \approx \omega \), for \( |z| \) small. Thus, to understand the bifurcations, we need to study only the planar system for \((r, z)\) with \( r \geq 0 \) as follows:

\[
\dot{r} = 4.9568 \times 10^{-4} \mu_1 r + 2.2191 \times 10^8 \mu_2 r - 1.5346 \times 10^{-2} rz \\
- 7.4391 \times 10^{-2} \mu_1 rz - 1.3136 \times 10^{10} \mu_2 rz + 8.1543 \times 10^{10} \mu_2^2 r \\
- 3.1367 \times 10^{-2} \mu_1^2 r - 1.9848 \mu_1 r z^2 - 8.5563 r^2 + 1.2602 \times 10^9 \mu_1 \mu_2 r, \tag{40}
\]

\[ \dot{z} = -0.1524 r^2 - 7.6190 \times 10^{-2} z^2 - 9.0703 \times 10^7 \mu_2 z^2 - 5.2094 \times 10^{10} \mu_2^2 r^2 \\
+ 1.5572 \mu_1 r^2 + 15.5074 r^2 z - 3.6723 \times 10^{-2} z^3. \]

First, considering the normal form (40) truncated up to 2-order:

\[
\dot{r} = 4.9568 \times 10^{-4} \mu_1 r + 2.2191 \times 10^8 \mu_2 r - 1.5346 \times 10^{-2} rz, \tag{41}
\]

\[ \dot{z} = -0.1524 r^2 - 7.6190 \times 10^{-2} z^2. \]

Obviously, Eq. (41) has only one equilibrium \((r, z) = (0, 0)\), which is unstable since \( \dot{z} < 0 \) near it. In particular, when \( 4.9568 \times 10^{-4} \mu_1 + 2.2191 \times 10^8 \mu_2 = 0 \), system (41) undergoes a codimension-4 bifurcation, which is very complicated.

**Remark 2.** Note that normal form truncated up to 2-order, Eq. (41) is not locally topologically equivalent near the origin to the original system (5). Thus, we need to consider the higher order normal form (40). However, we can not omit the high-order term with respect to parameters, such as \( \mu_i^2 r, \mu_i^2 z \) and \( \mu_1 \mu_2 r \) \((i = 1, 2)\), since there exist large difference in the order of magnitudes for these parameters, and we can not also show the numerical simulations for this special delay differential equation. Moreover, it is difficult to obtain the complete bifurcation analysis for normal form (40), and we can only show some simple analysis for certain parameters.
Note that $E_0 = (r, z) = (0, 0)$ corresponds to the original trivial equilibrium, and the other ones are

$$E_1 = (r, z) = (0, -2.074721564 - 2.469923481 \times 10^6 \mu_2), \quad E_2 = (r*, z*),$$

where $E_2 = (r*, z*)$ is nontrivial equilibrium which does not have explicit expression.

The characteristic equation of (40), evaluated at origin, is given by

$$\lambda(\lambda - 4.9568 \times 10^{-4} \mu_1 + 2.2191 \times 10^8 \mu_2 + 8.1543 \times 10^{19} \mu_2^2 - 3.1367 \times 10^{-3} \mu_1^2 + 1.2602 \times 10^6 \mu_1 \mu_2) = 0,$$

obviously, $E_0$ may undergo fold bifurcation.

Define $\varphi_1 = \frac{d(r)}{dr}|_{E_1}$, $\varphi_2 = \frac{d(r)}{dz}|_{E_1}$, $\varphi_3 = \frac{d(z)}{dr}|_{E_1}$ and $\varphi_4 = \frac{d(z)}{dz}|_{E_1}$ in (40), which means substitute $E_1 = (r, z) = (0, -2.074721564 - 2.469923481 \times 10^6 \mu_2)$ into preceding expression. The characteristic equation of (40), evaluated at $E_1$, is given by

$$\lambda^2 - (\varphi_1 + \varphi_3)\lambda + \varphi_1 \varphi_4 - \varphi_2 \varphi_3 = 0,$$

obviously, $E_1$ is local asymptotically stable when $\varphi_1 + \varphi_4 < 0$ and $\varphi_1 \varphi_4 - \varphi_2 \varphi_3 > 0$. $E_1$ may undergo fold bifurcation when $\varphi_1 \varphi_4 - \varphi_2 \varphi_3 = 0$. $E_1$ may undergo Hopf bifurcation when $\varphi_1 \varphi_4 - \varphi_2 \varphi_3 > 0$ and $\varphi_1 + \varphi_4 = 0$, here we omit the details. The stability and existence of bifurcations for $E_2$ can be analysis similarly.

In accordance with above theoretical analysis, the motion of hydraulic cylinder system can be controlled in new stable states by adjusting control parameters total leakage flow $C_{1p}$ and the leakage delay $\tau$, and we give the regions of parameters in which system may have a stable equalized motion. Therefore, according to the above theoretical analysis, we can choose proper controller parameters in delayed nonlinear hydraulic cylinder system in order to achieve various applications.

5. **Conclusion.** In this paper, we have modelled a hydraulic cylinder system with delay, and analysed the stability of the equilibria and the existence of several types of bifurcations. To study dynamical motion, we have derived the normal forms associated with Hopf-zero bifurcation by using the multiple time scales method. Two examples, associated with hydraulic cylinder in glue dosing processes for particle-board, are presented to demonstrate the application of the theoretical results. In accordance with above theoretical analysis, reasonable hydraulic cylinder system with proper parameters can be designed in order to achieve various applications.

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