SPACES OF POLYNOMIAL KNOTS IN LOW DEGREE

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Abstract

We show that all knots up to 6 crossings can be represented by polynomial knots of degree at most 7; among which except for 5_2, 5_2^*, 6_1, 6_1^*, 6_2, 6_2^* and 6_3 all are in their minimal degree representation. We provide concrete polynomial representation of all these knots. Durfee and O’shea had asked a question: Is there any 5 crossing knot in degree 6? In this paper we try to partially answer this question. We define the set \( P_d \) to be the set of all polynomial knots given by \( t \mapsto (f(t), g(t), h(t)) \) where \( f, g \) and \( h \) are real polynomials with \( \deg(f) < \deg(g) < \deg(h) \leq d \). This set can be identified with a subset of \( \mathbb{R}^{3d} \) and thus it is equipped with the natural topology which comes from the usual topology \( \mathbb{R}^{3d} \). In this paper we determine a lower bound on the number of path components of \( P_d \) for \( d \leq 7 \). We define path equivalence between polynomial knots in the space \( P_d \) and show that path equivalence is stronger than the topological equivalence.

Keywords: double points, crossing data, path equivalence

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1 Introduction

The idea of representing a long knot by polynomial embeddings was discussed by Arnold [23]. Later as an attempt to settle a long lasting conjecture of Abhyankar [22] in algebraic geometry Shastri [1] proved that every long knot is isotopic to an embedding given by \( t \mapsto (f(t), g(t), h(t)) \) where \( f, g \) and \( h \) are real polynomials. These kind of embeddings are referred as polynomial knots.

In his paper Shastri produced a choice of very simple polynomials \( f, g \) and \( h \) to represent the trefoil knot and the figure eight knot. He was hoping that once

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there are more examples available to represent various knot types, the conjecture of Abhyankar may be solved. This motivated the study of polynomial knots in a more rigorous and constructive manner. Explicit examples were constructed to represent a few classes of knots such as torus knots (see [18] and [20]) and two bridge knots (see [11] and [14]). To make the polynomials as simple as possible the notion of degree sequence and the minimal degree sequence were introduced. The minimality was with respect to the lexicographic order in \( \mathbb{N}^3 \). In this respect minimizing such a degree became a concern.

Around the same time Vassiliev [24] studied and discussed the topology of the spaces \( V_d \) for \( d \in \mathbb{N} \); where each of the space \( V_d \) is space (with natural topology coming from \( \mathbb{R}^{3d-3} \)) of all polynomial knots \( t \mapsto (f(t), g(t), h(t)) \) such that \( f, g \) and \( h \) are monic polynomials without constant term and having same degree \( d \). Later, Durfee and O’shea [2] studied the spaces \( K_d \), where the space \( K_d \) is the space of all polynomial knots of degree \( d \). Note that a polynomial knot is an embedding from \( \mathbb{R} \) to \( \mathbb{R}^3 \). If we compose it by linear transformation of the form \( (x, y, z) \mapsto (x - \alpha z, y, z) \) (\( \alpha \) and \( \beta \) being some suitable real number) we get a polynomial knot \( t \mapsto (f_1(t), g_1(t), h(t)) \) with \( \deg(f_1) \) and \( \deg(g_1) \) being at most \( d - 1 \), which by further composing with a linear transformation of the type \( (x, y, z) \mapsto (x - \gamma y, y, z) \) gives a polynomial knot \( t \mapsto (f_2(t), g_1(t), h(t)) \) with \( \deg(f_2) \) at most \( d - 2 \). These transformations are orientation preserving and hence the new polynomial knots obtained upon composition are topologically equivalent to the old one. Thus if a knot can be represented as a polynomial knot of degree \( d \) then it is equivalent to a polynomial knot with degree sequence \((d_1, d_2, d_3)\), where \( d_1 < d_2 < d_3 \leq d \). For a particular knot type, determining a polynomial representation with degree sequence \((d_1, d_2, d_3)\) such that \( d_1 < d_2 < d_3 \) and \( d_3 \) is least such number, is still an unsolved problem. On the other hand another important question that can be asked is: given any positive integer \( d \) how many knots can be realized as a polynomial knot in degree \( d \)? Here it can be seen that for \( d \leq 4 \) there is only one knot namely the unknot that can be realized. There are three nonequivalent knots that can be realized in \( d = 5 \) namely, the unknot, the right hand trefoil and the left hand trefoil and \( d = 5 \) is the least degree for the trefoils. Note that if a knot is realized in degree \( d \) it can be realized in degrees higher than \( d \). For degree 6 we found an additional knot the figure eight knot which has 4 crossings.

In this connection Durfee and O’shea asked: are there any 5 crossing knot in degree 6? We note that there are only two knots with 5 crossings denoted as \( 5_1 \) and \( 5_2 \) in the Rolfsen’s table. Using a knot invariant known as superbridge index, we can prove that \( 5_1 \) cannot be represented in degree 6. For \( 5_2 \) knot the superbridge index is not known. We show that there exists a projection of \( 5_2 \) knot
given by \( t \mapsto (f(t), g(t)) \) with \( \deg(f) = 4 \) and \( \deg(g) = 5 \), we in fact produce one. We also show that for any generic choice of a projection of \( 5_2 \) with degrees \((4, 5)\) there does not exist a polynomial \( h \) in degree 6 such that \( t \mapsto (f(t), g(t), h(t)) \) is a polynomial representation of \( 5_2 \). We conjecture that there are no 5 crossing knots in degree 6. This will be ascertained once a conjecture of Jin and Jeon [?] regarding superbridge index is proved. We show that both 5 crossing knots and all 6 crossing knots including the composite knots are realized in degree 7. We also look at the space \( \mathcal{P}_d \) of all polynomial knots \( t \mapsto (f(t), g(t), h(t)) \) such that \( fghio \) and define two polynomial knots in this space to be path equivalent if they belong to the same path component in this space.

This paper is organized as follows: In section 2 we discuss polynomial knots and introduce the space \( \mathcal{P}_d \), for \( d \geq 2 \), of all polynomial knots given by \( t \mapsto (f(t), g(t), h(t)) \) with \( \deg(f) < \deg(g) < \deg(h) \leq d \). For various values of \( 1 \leq d \leq 7 \), we discuss the different knots those belong to the space \( \mathcal{P}_d \). We separately take the case \( d = 6 \) and show that knots with crossing number 5 or more are not likely to be realized in \( \mathcal{P}_6 \). We show that all knots up to 6 crossings belong to the space \( \mathcal{P}_7 \).

In section 3, we concentrate on the path components in the space \( \mathcal{P}_d \), for \( d \geq 2 \). We prove that if \( d \) is minimal for a knot \( \phi = (f, g, h) \) to belong to \( \mathcal{P}_d \), then the knot \( \phi \) and \( \psi = (f, g, -h) \) belong to the different path components of \( \mathcal{P}_d \). Similarly if any other polynomial \( f \) or \( g \) has minimal degree for a particular knot then this might lead to more path components. We prove that, \( \mathcal{P}_2 \) has four path components; and \( \mathcal{P}_3 \) and \( \mathcal{P}_4 \) are path connected. We estimate the lower bounds on the number of path components of \( \mathcal{P}_5, \mathcal{P}_6 \) and \( \mathcal{P}_7 \). We conclude in section 4 by mentioning few remarks for the spaces \( \mathcal{P}_d \), for \( d > 7 \).

2 Polynomial knots

**Definition 2.1.** A long (non compact) knot is a proper smooth embedding \( \phi : \mathbb{R} \to \mathbb{R}^3 \) such that a map \( t \mapsto ||\phi(t)|| \) of \( \mathbb{R} \) into itself is strictly monotone outside a closed interval and \( ||\phi(t)|| \to \infty \) as \( |t| \to \infty \).

It is clear that, using the stereographic projection \( \pi : S^3 \setminus \{(0,0,0,1)\} \to \mathbb{R}^3 \), we can identify the one point compactification of \( \mathbb{R}^3 \) with \( S^3 \). Thus by this identification, any long knot \( \phi : \mathbb{R} \to \mathbb{R}^3 \) has a unique extension as a continuous embedding \( \tilde{\phi} : S^1 \to S^3 \) such that \( \tilde{\phi} \) takes the north pole \((0,1)\) of \( S^1 \) to the north pole \((0,0,0,1)\) of \( S^3 \). The map \( \tilde{\phi} \) is tame knot and it is smooth everywhere except at the north pole where it has algebraic singularity (see [2], proposition 1).
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Definition 2.2. Two long knots $\phi, \psi : \mathbb{R} \to \mathbb{R}^3$ are said to be topologically equivalent if there exist orientation preserving diffeomorphisms $F : \mathbb{R} \to \mathbb{R}$ and $H : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\psi = H \circ \phi \circ F$.

Proposition 2.1. For two long knots $\phi$ and $\psi$, the following three statements are equivalent:

a) $\phi$ and $\psi$ are topologically equivalent.

b) $\phi$ and $\psi$ are ambient isotopic.

c) The extensions $\tilde{\phi} : S^1 \to S^3$ and $\tilde{\psi} : S^1 \to S^3$ are ambient isotopic.

Definition 2.3. A smooth embedding $\phi = (f, g, h) : \mathbb{R} \to \mathbb{R}^3$; where $f, g$ and $h$ are univariate real polynomials; is called a polynomial knot.

Remark 1. A polynomial knot is long knot and it has been proved ([1] and [19]) that each long knot is topologically equivalent to some polynomial knot. Thus each (classical) knot $K$ is ambient isotopic to the extension $\tilde{\phi} : S^1 \to S^3$. In other words, each polynomial knot $\phi$ of degree $d$ is topologically equivalent to a polynomial knot $\psi = (f_1, g_1, h_1)$, where $\deg(f_1) < \deg(g_1) < \deg(h_1) \leq d$ and none of the degree lie in the semi group generated by the other two (see [2], Section 5).

Definition 2.4. For a polynomial knot $\phi = (f, g, h) : \mathbb{R} \to \mathbb{R}^3$, a number $\deg(\phi) = \max\{\deg(f), \deg(g), \deg(h)\}$ is called the degree of the polynomial knot $\phi$.

By composing a polynomial knot $\phi = (f, g, h)$ of degree $d$ with an orientation preserving tame polynomial automorphism of $\mathbb{R}^3$ acquires the form $\psi = (f_1, g_1, h_1)$ such that $\deg(f_1) < \deg(g_1) < \deg(h_1) \leq d$ and none of the degree lie in the semigroup generated by the other two.

For an arbitrary but fixed positive integer $d \geq 2$, consider a set $A_d$ of all maps $\phi = (f, g, h) : \mathbb{R} \to \mathbb{R}^3$, where $f, g$ and $h$ are real polynomials and $\deg(\phi) \leq d - 2$, $\deg(g) \leq d - 1$ and $\deg(h) \leq d$. An element of this set will be typically a map $t \mapsto (a_{d-2}t^{d-2} + \cdots + a_1t + a_0, b_{d-1}t^{d-1} + \cdots + b_1t + b_0, c_dt^d + \cdots + c_1t + c_0)$ where $a_i$’s, $b_i$’s and $c_i$’s are real numbers. This type of a map can be identified with a tuple $(a_0, \ldots, a_{d-2}, b_0, \ldots, b_{d-1}, c_0, \ldots, c_d) \in \mathbb{R}^{3d}$. Thus the set $A_d$ can be identified with $\mathbb{R}^{3d}$, and hence it has natural topology coming from the usual topology of $\mathbb{R}^{3d}$. Let $P_d$ be the set of all polynomial knots $\phi = (f, g, h)$ such that $\deg(f) < \deg(g) < \deg(h) \leq d$. It is proper subset of $A_d$ and thus it has subspace.

1A tame polynomial automorphism is composition of a) orientation preserving affine transformations and b) maps which add a real multiple of a positive power of one row to another row.
In other words, $P_d$ can be thought of as a topological subspace of $\mathbb{R}^{3d}$ by the identification of $A_d$ with $\mathbb{R}^{3d}$. Hence we may think elements of $P_d$ as ordered $3d$-tuples of real numbers.

Let us note that whenever we talk about the spaces $A_d$ and $P_d$, we assume that $d$ is a fixed positive integer greater than or equal to $2$.

**Remark 2.** A polynomial map whose at least one of the component is linear, is clearly an embedding (i.e. polynomial knot).

**Definition 2.5.** A (classical) knot $K$ is said to have a polynomial representation in degree $d$ (or is said to be represented in $P_d$), if there is a polynomial knot $\phi$ in $P_d$ whose extension as an embedding of $S^1$ in $S^3$ is topologically equivalent to $K$. In this case $\phi$ is called a polynomial representation of $K$.

**Definition 2.6.** A positive integer $d$ is said to be the minimal polynomial degree (or just polynomial degree) of a (classical) knot $K$, if $d$ is the least positive integer such that there is a polynomial representation $\phi$ of the knot $K$ in degree $d$. In this case $\phi$ is called a minimal polynomial representation of $K$.

**Remark 3.** Clearly the minimal polynomial degree of a knot is a knot invariant.

**Remark 4.** If a (classical) knot $K$ is represented by a polynomial knot $\phi = (f, g, h)$ then the knots $(-f, g, h), (f, -g, h), (f, g, -h)$ and $(-f, -g, -h)$ represent mirror images of $K$. Thus a knot and its mirror image have same polynomial degree and hence the minimal degree cannot detect the chirality of knots.

Certain numerical knot invariants can be inferred from the polynomial degree of a knot and vice-versa. In this connection some known useful results are summarized in the following proposition.

**Proposition 2.2.** Let $K$ be a (classical) knot having minimal polynomial degree $d$, then we have the following:

a) $c[K] \leq \frac{(d-2)(d-3)}{2}$,

b) $b[K] \leq \frac{(d-1)}{2}$ and

c) $s[K] \leq \frac{(d+1)}{2}$;

where $c[K]$, $b[K]$, $s[K]$ denote the minimal crossing number, bridge index and superbridge index respectively of the ambient isotopy class of the knot $K$.

The part a) can be proved using the Bezout’s theorem. Also the proofs of the parts b) and c) are trivial. To get an idea about the proofs of all the parts, one can refer to [2], propositions 12, 13 and 14.
From the results mentioned in the proposition 2.2, it is clear that in order to represent a knot with certain number of crossings the degree of its polynomial representation has a lower bound. However, knots with same number of crossings may have different crossing pattern, i.e., over and under crossing information. The result below tells us how the degree relates to the nature of the crossings.

**Theorem 2.3.** Suppose a polynomial knot \( \phi \) has a regular projection \( t \mapsto (f(t), g(t)) \) with \( n \) transversal double points and the crossing data of the knot is such that there are \( r \) changes from under crossing to over crossing (together with, from over crossing to under crossing) as we move along the knot. Let \( d_1 = \text{deg}(f) \leq n \) and \( d_2 = \text{deg}(g) \leq n+1 \) and assume that no any \( k = 1, 2, \ldots, n+2 \) (except \( d_1 \) and \( d_2 \)) does belong to the semigroup generated by \( d_1 \) and \( d_2 \). Let \( d \) be the minimal number for which there exists a polynomial \( h \) of degree \( d \) such that the embedding \( \psi = (f, g, h) \) represents (is topologically equivalent to) the knot \( \phi \), then \( d \leq \min\{n+2, r\} \).

**Proof.** The double points of the projection \( t \mapsto (f(t), g(t)) \) can be obtained by finding the real roots of the resultant \( Q(s) \) of the polynomials \( F(s, t) = \frac{f(s) - f(t)}{s - t}, s \neq t \) and \( G(s, t) = \frac{g(s) - g(t)}{s - t}, s \neq t \). Let \([a, b]\) be an interval that contains all these roots. Let us call these roots as the crossing points. We divide the interval \([a, b]\) into sub intervals \( a = a_0 < a_1 < \cdots < a_r = b \) in such a way that the \( a_i \)'s are not from the crossing points and within a sub interval \([a_{i-1}, a_i]\) all the crossing points are either under crossing points or over crossing points. Let \( h_1(t) = \prod_{i=1}^r (t - a_i) \). Clearly \( h_1(t) \) is a polynomial of degree \( r \) that has opposite signs at under crossing and over crossing points and thus \( \phi_1 = (f, g, h_1) \) represents the knot \( \phi \).

Let for \( i = 1, 2, \ldots, n; (s_i, t_i) \), with \( s_i < t_i \) and \( s_1 < s_2 < \cdots < s_n \), be the pairs of parametric values at which the projection has double points. Let \( C_{n+2}t^{n+2} + C_{n+1}t^{n+1} + \cdots + C_1t \) be a polynomial of degree \( n + 2 \), where \( C_i \)'s are unknowns which we have to find by solving the following system

\[
\begin{align*}
C_{n+2}(t_1^{n+2} - s_1^{n+2}) + C_{n+1}(t_1^{n+1} - s_1^{n+1}) + \cdots + C_1(t_1 - s_1) &= e_1 \\
C_{n+2}(t_2^{n+2} - s_2^{n+2}) + C_{n+1}(t_2^{n+1} - s_2^{n+1}) + \cdots + C_1(t_2 - s_2) &= e_2 \\
& \vdots \\
C_{n+2}(t_n^{n+2} - s_n^{n+2}) + C_{n+1}(t_n^{n+1} - s_n^{n+1}) + \cdots + C_1(t_n - s_n) &= e_n
\end{align*}
\]

of \( n \) linear equations in \( n + 2 \) unknowns \( C_1, C_2, \ldots, C_{n+2} \). Where \( e_i \)'s are arbitrary but fixed non-zero real numbers and they are positive or negative according to which the crossing is over crossing or under crossing. Since no any \( k = 1, 2, \ldots, n+2 \) (except \( d_1 \) and \( d_2 \)) does belong to the semigroup generated by \( d_1 \) and \( d_2 \); so the
conditions $f(s_i) = f(t_i)$ and $g(s_i) = g(t_i)$ for $i = 1, 2, \ldots, n$; imply that only two columns of the coefficient matrix

$$M = \begin{bmatrix}
    t_1^{n+2} - s_1^{n+2} & t_1^{n+1} - s_1^{n+1} & \cdots & t_1 - s_1 \\
    t_2^{n+2} - s_2^{n+2} & t_2^{n+1} - s_2^{n+1} & \cdots & t_2 - s_2 \\
    \vdots & \vdots & \ddots & \vdots \\
    t_n^{n+2} - s_n^{n+2} & t_n^{n+1} - s_n^{n+1} & \cdots & t_n - s_n
\end{bmatrix}$$

of the above system are dependent and remaining $n$ columns are linearly independent. Thus it has infinitely many solutions and let $C_1 = c_1, C_2 = c_2, \ldots, C_{n+2} = c_{n+2}$ be any one of the solution. Hence the polynomial $h_2(t) = c_{n+2}t^{n+2} + c_{n-1}t^{n-1} + \cdots + c_1t$ will be the polynomial such that the embedding $\phi_2 = (f, g, h_2)$ represents the knot $\phi$.

Thus we have the polynomials $h_1$ and $h_2$ with degrees $r$ and $n+2$ respectively such that the corresponding knots $\phi_1$ and $\phi_2$ represent the knot $\phi$. Hence the minimal number $d$ has to be at most $\min\{n+2, r\}$. \hfill $\square$

In connection with polynomial representation of knots, the following questions are of interest namely:

**Question 1.** Given a knot $K$ what is the least degree $d$ such that $K$ has a polynomial representation in $P_d$?

**Question 2.** Given a positive integer $d$ what are the knots that can have a polynomial representation in $P_d$?

**Question 3.** Given a positive integer $d$ estimate the number of path components in the space $P_d$.

We have answered all the three questions for $2 \leq d \leq 5$; and we have partial answers for $d = 6, 7$. In general, all these problems are difficult and answering one helps in answering the other two.

**Proposition 2.4.** The unknot is the only knot that can be represented as a polynomial knot in the space $P_d$, for $d \leq 4$.

**Proof.** Let $K$ be a knot which is represented as a polynomial knot in the space $P_d$, for $d \leq 4$. Then by the proposition \[2.2\](a), the minimal number of crossings $c[K]$ for $K$ will satisfy $c[K] \leq \frac{(d-2)(d-3)}{2} \leq \frac{(4-2)(4-3)}{2} = 1$ (since $d \leq 4$). Thus $K$ must be the unknot. \hfill $\square$

**Proposition 2.5.** The unknot, the trefoil knot and the mirror image of the trefoil knot are the only knots those can be represented as polynomial knots in $P_5$.\hfill $7$
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Proof. Let $K$ be a knot which is represented as a polynomial knot in $\mathbb{P}_5$. Then by the proposition 2.2, the minimal number of crossings $c[K]$ of $K$ satisfies $c[K] \leq \frac{(5-2)(5-3)}{2} = 3$. Hence $K$ must be either the unknot or the trefoil knot or the mirror image of the trefoil knot.

Proposition 2.6. There exist polynomials $f$ and $g$ of degrees 4 and 5 respectively such that the map $t \mapsto (f(t), g(t))$ represents a regular projection of $5_2$ knot.

Proof. Consider a plane curve $C$ given by the parametric equation $(X(t), Y(t)) = (t^4, t^5)$. This curve has an isolated singularity at the origin. For such plane curves, there are two important numbers that remain invariant under any formal isomorphisms of plane curves. The first one is the Milnor number $\mu$ and the other is the $\delta$ invariant (see [10]). For a single component plane curve they satisfy the relation $2\delta = \mu$.

In the present case it turns out that the $\delta$ invariant is equal to $\frac{(5-1)(4-1)}{2} = 6$. The $\delta$ invariant of a plane curve which is singular at the origin measures the number of double points that can be created in a neighborhood of the origin. Note that, we have $\delta \geq 5$. Using a result of Daniel Pecker [9] we can deform the curve $C : (X(t), Y(t)) = (t^4, t^5)$ into a new curve $\tilde{C} : (f(t), g(t)) = (a_4t^4 + a_3t^3 + a_2t^2 + a_1t + a_0, b_5t^5 + b_4t^4 + b_3t^3 + b_2t^2 + b_1t + b_0)$ such that $\tilde{C}$ has 5 real nodes and 1 imaginary node. By continuity argument, we can choose the coefficients $a_i$’s and $b_i$’s such that the nodes occur in the order they are in the regular projection of the given knot.

In fact we have obtained a choice of $f$ and $g$ using mathematica: $(x(t), y(t)) = (2(t - 2)(t + 4)(t^2 - 11), t(t^2 - 6)(t^2 - 16))$ which gives us a regular projection of $5_2$ knot shown in the following figure:

Figure 1: Projection of $5_2$ knot
We consider the following sets:

\[ U_1 = \{(a_0, a_1, a_2, a_3, a_4, b_0, b_1, b_2, b_3, b_4, b_5) \in \mathbb{R}^{11} : \text{a map } t \mapsto (a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0, b_5 t^5 + b_4 t^4 + b_3 t^3 + b_2 t^2 + b_1 t + b_0) \text{ of } \mathbb{R} \text{ into } \mathbb{R}^2 \text{ is a regular projection of } 5_2 \text{ knot with } 5 \text{ crossings}\} \]

\[ U_2 = \{(a_0, a_1, a_2, a_3, a_4, b_0, b_1, b_2, b_3, b_4, b_5) \in \mathbb{R}^{11} : \text{a map } t \mapsto (a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0, b_5 t^5 + b_4 t^4 + b_3 t^3 + b_2 t^2 + b_1 t + b_0) \text{ of } \mathbb{R} \text{ into } \mathbb{R}^2 \text{ is a regular projection of } 5_2 \text{ knot with } 6 \text{ crossings}\} \]

For a typical element of \( U = U_1 \cup U_2 \) we must have both \( a_4 \) and \( a_5 \) non zero, otherwise by an application of the Bezout’s theorem there would be less than five crossings for the curve \( t \mapsto (a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0, b_5 t^5 + b_4 t^4 + b_3 t^3 + b_2 t^2 + b_1 t + b_0) \) (see [2], lemma 4); and thus it will not be a projection of \( 5_2 \) knot. We have shown that \( U \) is nonempty set. For a projection \( t \mapsto (f(t), g(t)) \), where \( f \) and \( g \) are polynomials of degrees 4 and 5 respectively; we would like to find a polynomial \( h \) of least possible degree such that \( t \mapsto (f(t), g(t), h(t)) \) represents \( 5_2 \) knot. We have the following theorem:

**Theorem 2.7.** For a generic choice of \( a_i \)'s and \( b_i \)'s such that the curve \( t \mapsto (f(t), g(t)) = (a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0, b_5 t^5 + b_4 t^4 + b_3 t^3 + b_2 t^2 + b_1 t + b_0) \) is a regular projection of \( 5_2 \) knot, there does not exist a polynomial \( h \) of degree 6 such that \( t \mapsto (f(t), g(t), h(t)) \) represents \( 5_2 \) knot.

**Proof.** Let \( t \mapsto (f(t), g(t)) = (a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0, b_5 t^5 + b_4 t^4 + b_3 t^3 + b_2 t^2 + b_1 t + b_0) \) be regular projection of \( 5_2 \) knot and \( h(t) = c_6 t^6 + c_5 t^5 + c_4 t^4 + c_3 t^3 + c_2 t^2 + c_1 t + c_0 \) be a degree 6 polynomial such that \( t \mapsto (f(t), g(t), h(t)) \) represents \( 5_2 \). Composing this embedding with a suitable affine transformation, we can assume that the coefficients \( c_5, c_4 \) and the constant term \( c_0 \) are zero. Thus we can take \( h(t) = c_6 t^6 + c_3 t^3 + c_2 t^2 + c_1 t \). Note that, the projection has either 5 or 6 double points (crossings). We consider the following two cases.

Case i) If the projection has 5 crossings:
Let for \( i = 1, 2, \ldots, 5 \); \((s_i, t_i)\) with \( s_i < t_i \) and \( s_1 < s_2 < \cdots < s_5 \) be the pairs of parametric values at which the crossings occur in the curve \( t \mapsto (f(t), g(t)) \). Since \( c_5, c_4, c_2 \) and \( c_1 \) such that for some choice of \( r_i > 0, i = 1, 2, \ldots, 5 \); \( h(t_i) - h(s_i) = r_i \) for \( i \) odd and \( h(t_i) - h(s_i) = -r_i \) for \( i \) even. This gives us a system of 5 linear equations.
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in 4 unknowns as follows:

\[
\begin{align*}
&c_0(t_1^6 - s_1^6) + c_3(t_1^3 - s_1^3) + c_2(t_1^2 - s_1^2) + c_1(t_1 - s_1) = r_1 \\
&c_0(t_2^6 - s_2^6) + c_3(t_2^3 - s_2^3) + c_2(t_2^2 - s_2^2) + c_1(t_2 - s_2) = -r_2 \\
&c_0(t_3^6 - s_3^6) + c_3(t_3^3 - s_3^3) + c_2(t_3^2 - s_3^2) + c_1(t_3 - s_3) = r_3 \\
&c_0(t_4^6 - s_4^6) + c_3(t_4^3 - s_4^3) + c_2(t_4^2 - s_4^2) + c_1(t_4 - s_4) = -r_4 \\
&c_0(t_5^6 - s_5^6) + c_3(t_5^3 - s_5^3) + c_2(t_5^2 - s_5^2) + c_1(t_5 - s_5) = r_5
\end{align*}
\]

The rank of the coefficient matrix

\[
A = \begin{bmatrix}
  t_1^6 - s_1^6 & t_1^3 - s_1^3 & t_1^2 - s_1^2 & t_1 - s_1 \\
  t_2^6 - s_2^6 & t_2^3 - s_2^3 & t_2^2 - s_2^2 & t_2 - s_2 \\
  t_3^6 - s_3^6 & t_3^3 - s_3^3 & t_3^2 - s_3^2 & t_3 - s_3 \\
  t_4^6 - s_4^6 & t_4^3 - s_4^3 & t_4^2 - s_4^2 & t_4 - s_4 \\
  t_5^6 - s_5^6 & t_5^3 - s_5^3 & t_5^2 - s_5^2 & t_5 - s_5
\end{bmatrix}
\]

of the above system of linear equations is at most 4. The system has a solution if and only if the rank of \( A \) is equal to the rank of the augmented matrix \( \tilde{A} \). In other words, the system has no solution if \( \text{det}(\tilde{A}) \neq 0 \).

Let for \( j = 1, 2, \ldots, 5; A_j \) be a submatrix (of order 4) of \( A \) obtained by deleting the \( j^{th} \) row of \( A \). Then we have, \( \text{det}(\tilde{A}) = r_1 \text{det}(A_1) + r_2 \text{det}(A_2) + r_3 \text{det}(A_3) + r_4 \text{det}(A_4) + r_5 \text{det}(A_5) \). Note that for each \( j \), \( \text{det}(A_j) \) is an algebraic function of \( t_i \)'s and \( s_i \)'s which are actually analytic functions of the coefficients \( a_k \)'s of \( f \) and the coefficients \( b_k \)'s of \( g \). Thus \( \text{det}(\tilde{A}) \) is a non-constant analytic function of \( a_k \)'s, \( b_k \)'s and \( r_k \)'s. Hence the set

\[
V_1 = \{ (a_0, \ldots, a_4, b_0, \ldots, b_5, r_1, \ldots, r_5) \in U_1 \times (\mathbb{R}^+)^5 : \text{det}(\tilde{A}) \neq 0 \}
\]

is an open and dense subset of \( U_1 \times (\mathbb{R}^+)^5 \). Also for any choice of an element in \( V_1 \), the above system has no solution. Hence a polynomial \( h \) does not exist for generic choice of regular projection \( t \mapsto (f(t), g(t)) \) with 5 crossings, where \( \text{deg}(f) = 4 \) and \( \text{deg}(g) = 5 \), such that \( t \mapsto (f(t), g(t), h(t)) \) represents 5_2 knot.

Case ii) If the projection has 6 crossings:
Let for \( i = 1, 2, \ldots, 6; (s_i, t_i) \) with \( s_i < t_i \) and \( s_1 < s_2 < \cdots < s_6 \) be the pairs of parametric values at which the crossings occur in the curve \( t \mapsto (f(t), g(t)) \). Let \( e = (e_1, e_2, \ldots, e_6) \) be a pattern such that this together with the projection \( t \mapsto (f(t), g(t)) \) describe the 5_2 knot, where \( e_i \) is either 1 or -1 respectively according to which the \( i^{th} \) crossing is under crossing or over crossing. Let \( U_e \) be a set of elements \((a_0, a_1, a_2, a_3, a_4, b_0, b_1, b_2, b_3, b_4, b_5) \) of \( U_2 \) such that the projection
t \to (a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0, b_5 t^5 + b_4 t^4 + b_3 t^3 + b_2 t^2 + b_1 t + b_0)$ together with the pattern $e$ describe the $5_2$ knot.

We have to find the values of the coefficients $c_6, c_3, c_2$ and $c_1$ such that for some choice of $r_i > 0$, $i = 1, 2, \ldots, 5$; $h(t_i) - h(s_i) = e_i r_i$. This gives us a system of 6 linear equations in 4 unknowns as follows:

\[
\begin{align*}
&c_6(t_1^6 - s_1^6) + c_3(t_1^3 - s_1^3) + c_2(t_1^2 - s_1^2) + c_1(t_1 - s_1) = e_1 r_1 \\
&c_6(t_2^6 - s_2^6) + c_3(t_2^3 - s_2^3) + c_2(t_2^2 - s_2^2) + c_1(t_2 - s_2) = e_2 r_2 \\
&c_6(t_3^6 - s_3^6) + c_3(t_3^3 - s_3^3) + c_2(t_3^2 - s_3^2) + c_1(t_3 - s_3) = e_3 r_3 \\
&c_6(t_4^6 - s_4^6) + c_3(t_4^3 - s_4^3) + c_2(t_4^2 - s_4^2) + c_1(t_4 - s_4) = e_4 r_4 \\
&c_6(t_5^6 - s_5^6) + c_3(t_5^3 - s_5^3) + c_2(t_5^2 - s_5^2) + c_1(t_5 - s_5) = e_5 r_5 \\
&c_6(t_6^6 - s_6^6) + c_3(t_6^3 - s_6^3) + c_2(t_6^2 - s_6^2) + c_1(t_6 - s_6) = e_6 r_6
\end{align*}
\]

The rank of the coefficient matrix

\[
B = \begin{bmatrix}
  t_1^6 - s_1^6 & t_1^3 - s_1^3 & t_1^2 - s_1^2 & t_1 - s_1 \\
  t_2^6 - s_2^6 & t_2^3 - s_2^3 & t_2^2 - s_2^2 & t_2 - s_2 \\
  t_3^6 - s_3^6 & t_3^3 - s_3^3 & t_3^2 - s_3^2 & t_3 - s_3 \\
  t_4^6 - s_4^6 & t_4^3 - s_4^3 & t_4^2 - s_4^2 & t_4 - s_4 \\
  t_5^6 - s_5^6 & t_5^3 - s_5^3 & t_5^2 - s_5^2 & t_5 - s_5 \\
  t_6^6 - s_6^6 & t_6^3 - s_6^3 & t_6^2 - s_6^2 & t_6 - s_6
\end{bmatrix}
\]

of the above system is at most 4. The system has a solution if and only if the rank of $B$ is equal to the rank of the augmented matrix $\bar{B}_e$ (where the subscript $e$ denotes the dependence of $\bar{B}_e$ on the pattern $e$). Thus the system has no solution if $\bar{B}_e$ has full rank (i.e. $\text{rank}(\bar{B}_e) = 5$).

Note that for each $i$, $t_i$ and $s_i$ are analytic functions of the coefficients $a_k$’s of $f$ and the coefficients $b_k$’s of $g$. Thus $\bar{B}_e$ (as function from $\mathbb{R}^{17}$ to $\mathbb{R}^{30}$) is a non-constant analytic function of $a_k$’s, $b_k$’s and $r_k$’s. Hence the set

\[
V_e = \{(a_0, \ldots, a_4, b_0, \ldots, b_5, r_1, \ldots, r_6) \in U_e \times (\mathbb{R}^+)^6 : \bar{B}_e \text{ has full rank}\}
\]

is an open and dense subset of $U_e \times (\mathbb{R}^+)^6$. It is clear that, for any choice of an element in $V_e$, the above system has no solution.

Now consider the disjoint union $V_2 = \bigsqcup_e V_e$ which is clearly an open and dense subset of disjoint union $U_3 = \bigsqcup_e U_e \times (\mathbb{R}^+)^6$, where both the unions are taken over all the patterns $e$. Note that, $U_2 \times (\mathbb{R}^+)^6 \subseteq U_3$. It is easy to see that, for any choice of an element in $V_2$, the corresponding system of linear equations has no solution. Hence a polynomial $h$ does not exist for generic choice of regular
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The projection $t \mapsto (f(t), g(t))$ with 6 crossings, where $\deg(f) = 4$ and $\deg(g) = 5$, such that $t \mapsto (f(t), g(t), h(t))$ represents $5_2$ knot.

The cases $i)$ and $ii)$ together prove the theorem. \qed

Conjecture 2.1. $5_2$ knot can not be realized in $P_6$.

It is conjectured that the only 3-superbridge knots are $3_1$ and $4_1$, and if it is proved; then by the proposition 2.2(c), the above conjecture will follow immediately. Similarly we conjecture that the knots $6_1, 6_2$ and $6_3$ cannot be realized in $P_6$.

3 Path components in the spaces $P_d$

For any positive integer $d$, Vassiliev [24] has discussed the topology (it is inherited from $\mathbb{R}^{3(d+1)}$) of the space $K_d$ of polynomial knots $\phi = (f, g, h)$ where $\deg(\phi) = d$, i.e. degree of each of $f, g, h$ is at most $d$ and at least one of them has degree $d$.

Let $P$ denote the set of all polynomial knots. Clearly

\[ P = \bigcup_{d \geq 1} K_d. \]

Thus $P$ can be given the inductive limit topology and we can talk of $P$ as the space of all polynomial knots. The spaces $P_d$ and $K_n$, for $d \geq 2$ and $n \geq 1$, become subspaces of $P$.

**Definition 3.1.** Two polynomial knots $\phi$ and $\psi$ in $P$ are said to be polynomially isotopic if there exists a one parameter family $\Phi = \{ \Phi_s \in P : s \in [0, 1] \}$ of polynomial knots such that $\Phi_0 = \phi$ and $\Phi_1 = \psi$. In this case, $\Phi$ is called as a polynomial isotopy from $\phi$ to $\psi$.

In the definition above one has to note that the map $\Psi : [0, 1] \to P$ defined by $t \mapsto \Phi_t$ is continuous, i.e. it defines a path in $P$. Thus being polynomially isotopic is an equivalence relation in $P$ and the equivalence classes are nothing but the path components of the space $P$. It is easy to note that, if two polynomial knots are polynomially isotopic then they are topologically equivalent as long knots and also the converse was proved in [19]. Hence two polynomial knots are topologically equivalent as long knots if and only if they lie in the same path component of $P$.

For two topologically equivalent (i.e. polynomially isotopic) polynomial knots $\phi$ and $\psi$ in $P_d$, the polynomial isotopy may pass through polynomial knots of various degrees and it might be possible that the isotopy in $P$ passes through polynomial knots which are not members of $P_d$. Hence two topologically equivalent knots in
$P_d$ may not belong to the same path component of $P_d$. So with this in mind, we define a path equivalence of two polynomial knots in $P_d$ as follows:

**Definition 3.2.** Two polynomial knots in $P_d$ are said to be path equivalent in $P_d$ if they belong to the same path component of $P_d$.

Also path equivalence in $K_d$ is defined in a similar way, i.e. two polynomial knots in $K_d$ are said to be path equivalent in $K_d$ if they belong to the same path component of $K_d$. Using advanced techniques of differential topology Durfee and O’shea gave a proof (see [2], proposition 9) for the following fact:

**Proposition 3.1.** If two polynomial knots in $K_d$ are path equivalent in $K_d$, then they are topologically equivalent.

Later they asked: Is the converse true? Note that if we do not insist on fixed degree then we do have a converse. However, two topologically equivalent polynomial knots in the same degree $d$ may belong to the different path components of $K_d$. We have the following corollary:

**Corollary 3.1.1.** Any path equivalent polynomial knots in $P_d$ are topologically equivalent.

*Proof.* Let $\phi$ and $\psi$ be two polynomial knots belonging to the same path component of $P_d$. Since $P_d \subset K_d$, so $\phi$ and $\psi$ are members of $K_d$ belonging to its same path component. Thus by the proposition 3.1, they are topologically equivalent. $\square$

**Theorem 3.2.** Let $\phi = (f, g, h) \in P_d$ be a minimal polynomial representation of a classical knot $K$ having polynomial degree $d$. Then $\phi$ and its mirror image $\psi = (f, g, -h)$ belong to the different path components of $P_d$.

*Proof.* Let us assume contrary that $\phi$ and $\psi$ belong to the same path component of $P_d$. Since $\phi \in P_d$ and $\deg(\phi) = d$, so we have $\deg(f) \leq d - 2$, $\deg(g) \leq d - 1$ and $\deg(h) = d$. Let

\[
\begin{align*}
  f(t) &= a_{d-2}t^{d-2} + \cdots + a_1 t + a_0, \\
  g(t) &= b_{d-1}t^{d-1} + \cdots + b_1 t + b_0 \text{ and} \\
  h(t) &= c_d t^d + \cdots + c_1 t + c_0.
\end{align*}
\]

Clearly $c_d \neq 0$. Suppose $\Phi : [0,1] \to P_d$ be a path from $\phi$ to $\psi$. For any $s \in [0,1]$; let $\Phi_s = \Phi(s)$ and let $\Phi_s$ is given by

\[
\Phi_s(t) = (\alpha_{d-2}(s)t^{d-2} + \cdots + \alpha_1(s) t + \alpha_0(s), \beta_{d-1}(s)t^{d-1} + \cdots + \beta_1(s) t + \beta_0(s), \\
\gamma_d(s)t^d + \cdots + \gamma_1(s) t + \gamma_0(s));
\]

13
for all $t \in \mathbb{R}$. Here for each $i$, $\alpha_i$, $\beta_i$, and $\gamma_i$ are continuous functions from $[0, 1]$ into $\mathbb{R}$. Since $\Phi_0 = \phi = (f, g, h)$ and $\Phi_1 = \psi = (f, g, -h)$, thus for each $i$ we have $\alpha_i(0) = \alpha_i(1) = a_i$, $\beta_i(0) = \beta_i(1) = b_i$, $\gamma_i(0) = c_i$ and $\gamma_i(1) = -c_i$. In particular, $\gamma_d(0) = c_d$ and $\gamma_d(1) = -c_d$. Since $\gamma_d$ is continuous function, so by the intermediate value theorem $\gamma_d(s_0) = 0$ for some $s_0 \in (0, 1)$; and so $\Phi_{s_0}$ is polynomial knot in $P_d$ of degree strictly less than $d$. By suitable parameterization of the interval $[0, 1]$, we will have a path in $P_d$ from $\Phi_0 = \phi$ to $\Phi_{s_0}$. In other words, $\Phi_{s_0}$ is a polynomial representation of $K$ having degree strictly less than $d$. This contradicts the fact that $d$ is the minimal polynomial degree for $K$. Hence the polynomial knot $\phi = (f, g, h)$ and its mirror image $\psi = (f, g, -h)$ lie in the different path components of $P_d$.

**Corollary 3.2.1.** For a classical knot $K$ having minimal polynomial degree $d$, we have the following:

i) If $K$ is achiral, then $P_d$ has at least two path components corresponding to $K \simeq K^*$.

ii) If $K$ is chiral, then $P_d$ has at least four path components corresponding to $K$ and its mirror image $K^*$.

**Proof.** i) If $K$ is achiral, then the proof follows trivially from the theorem 3.2.

ii) If $K$ is chiral and it is represented by $\phi_1 = (f, g, h) \in P_d$, then $\phi_2 = (-f, g, -h)$ also represents $K$, and $\phi_3 = (f, g, -h)$ and $\phi_4 = (-f, g, h)$ will represent $K^*$. Thus by the corollary 3.1.1, $\{\phi_1, \phi_2\}$ and $\{\phi_3, \phi_4\}$ can not belong to the same path component of $P_d$. Now by the argument similar to the argument used in the proof of the theorem 3.2, one can see that $\phi_1$ and $\phi_2$ belong to two distinct path components of $P_d$. Similarly, $\phi_3$ and $\phi_4$ belong to two distinct path components of $P_d$. Hence $\phi_1, \phi_2, \phi_3$ and $\phi_4$ belong to four distinct path components of $P_d$.

**Corollary 3.2.2.** Let $\phi = (f, g, h) \in P_d$ be minimal polynomial representation of a classical knot $K$ having minimal polynomial degree $d$. Also assume that either $\deg(f)$ or $\deg(g)$ is minimal in the sense that by reducing the degree will result into impossibility of representing $K$. Then we have the following:

i) If $K$ is achiral, then $P_d$ has at least four path components corresponding to $K \simeq K^*$.

ii) If $K$ is chiral, then $P_d$ has at least eight path components corresponding to $K$ and its mirror image $K^*$.
Proof. Without loss of generality assume that deg(f) is minimal.

i) If $K$ is achiral; then by the arguments similar to the argument used in the proof of the theorem 3.2, $\phi_1 = \phi = (f, g, h), \phi_2 = (f, g, -h), \phi_3 = (-f, g, h)$ and $\phi_4 = (-f, g, -h)$ belong to four distinct path components of $P_d$.

ii) If $K$ is chiral. Then (a) $\psi_1 = \phi = (f, g, h), \psi_2 = (f, -g, -h), \psi_3 = (-f, g, -h)$ and $\psi_4 = (-f, -g, h)$ represent $K$; and (b) $\psi_5 = (f, g, -h), \psi_6 = (f, -g, h), \psi_7 = (-f, g, h)$ and $\psi_8 = (-f, -g, -h)$ will represent $K^\ast$. So by the corollary 3.1.1, the number of path components corresponding to $K$ and its mirror image $K^\ast$.

Corollary 3.2.3. Let $\varphi = (f, g, h) \in P_d$ be minimal polynomial representation of a classical knot $K$ having minimal polynomial degree $d$. Also assume that deg(f) and deg(g) are minimal in the sense that by reducing any one of the degree will result into impossibility of representing $K$. Then $P_d$ has at least eight path components corresponding to $K$ and its mirror image $K^\ast$.

Proof. By the arguments similar to the argument used in the proof of the theorem 3.2, it is easy to see that $\varphi_1 = \varphi = (f, g, h), \varphi_2 = (f, g, -h), \varphi_3 = (f, -g, h), \varphi_4 = (f, -g, -h), \varphi_5 = (-f, g, h), \varphi_6 = (-f, g, -h), \varphi_7 = (-f, -g, h)$ and $\varphi_8 = (-f, -g, -h)$ belong to eight distinct path components of $P_d$.

Remark 5. In view of the corollary 3.1.1, the number of topologically distinct knots in $P_d$ provides us a lower bound on the number of path components of $P_d$. Also the corollaries 3.2.1, 3.2.2 and 3.2.3 helps to estimate a lower bound on the number of path components of $P_d$.

3.1 Spaces $P_2, P_3$ and $P_4$

Proposition 3.3. The space $P_2$ is open in $A_2$ and it has exactly four path components.

Proof. By definition of $P_2$ and remark 3.2, it is easy to see that:

$$P_2 = \{ (a_0, b_1t + b_0, c_2t^2 + c_1t + c_0) \in A_2 : b_1 \neq 0 \text{ and } c_2 \neq 0 \}$$

$$\cong \{ (a_0, b_0, b_1, c_0, c_1, c_2) \in \mathbb{R}^{3d} : b_1 \neq 0 \text{ and } c_2 \neq 0 \}.$$
This is a Zariski open set. Hence $P_2$ is open in $A_2$ and it has four path components as given below:

\[ U_1 = \{ (a_0, b_1t + b_0, c_2t^2 + c_1t + c_0) \in A_2 : b_1 > 0 \text{ and } c_2 > 0 \}, \]
\[ U_2 = \{ (a_0, b_1t + b_0, c_2t^2 + c_1t + c_0) \in A_2 : b_1 > 0 \text{ and } c_2 < 0 \}, \]
\[ U_3 = \{ (a_0, b_1t + b_0, c_2t^2 + c_1t + c_0) \in A_2 : b_1 < 0 \text{ and } c_2 > 0 \} \text{ and} \]
\[ U_4 = \{ (a_0, b_1t + b_0, c_2t^2 + c_1t + c_0) \in A_2 : b_1 < 0 \text{ and } c_2 < 0 \}. \]

**Definition 3.3.** A polynomial map $\phi = (f, g, h) \in A_d$ is said to have degree sequence $(d_1, d_2, d_3)$ if $d_1 = \deg(f), d_2 = \deg(g)$ and $d_3 = \deg(h)$.

**Lemma 3.4.** Let $X$ be a topological space and $F = \{ U_\alpha \}_{\alpha \in \Gamma}$ is an arbitrary collection (with at least two members) of non-empty subsets of $X$ which forms a covering for it. Suppose $U_{\alpha_1}$ and $U_{\alpha_2}$ be any two distinct members of $F$ which satisfy the following two conditions:

\begin{itemize}
  \item [i)] For any $x \in U_{\alpha_1}$ and any $y \in U_{\alpha_2}$, there is a path from $x$ to $y$.
  \item [ii)] For any $U_\alpha$ and any $z \in U_\alpha$, there exists an element $w \in U_{\alpha_1} \cup U_{\alpha_2}$ such that there is a path from $z$ to $w$.
\end{itemize}

Then $X$ is path connected.

**Theorem 3.5.** The space $P_3$ is path connected.

**Proof.** We consider the following sets:

\[ U_1 = \{ \varphi \in A_3 : \varphi \text{ has degree sequence } (0, 1, 2) \}, \]
\[ U_2 = \{ \varphi \in A_3 : \varphi \text{ has degree sequence } (0, 1, 3) \}, \]
\[ U_3 = \{ \varphi \in A_3 : \varphi \text{ has degree sequence } (0, 2, 3) \} \cap P_3 \text{ and} \]
\[ U_4 = \{ \varphi \in A_3 : \varphi \text{ has degree sequence } (1, 2, 3) \}. \]

It is easy to note that, these are pairwise disjoint nonempty subsets of $A_3$. By remark 2, one can see that, each of them is subset of $P_3$. Also

\[ P_3 = \bigcup_{1 \leq i \leq 4} U_i. \]

To prove this proposition, we use the lemma 3.4.

\begin{itemize}
  \item [a)] Let $\phi \in U_1$ and $\psi \in U_4$ be arbitrary elements. Let $\Phi : [0, 1] \to A_3$ be given by $\Phi(s) = \Phi_s$ for all $s \in [0, 1]$; where
  \[ \Phi_s(t) = (1 - s) \phi(t) + s \psi(t) \]
\end{itemize}
for all $t \in \mathbb{R}$. It is clear that, $\Phi_0 = \phi$ and $\Phi_1 = \psi$. Also for any $s \in (0,1]$, $\Phi_s$ has
degree sequence $(1,2,3)$. Thus $\Phi_s \in U_4$ for all $s \in (0,1]$. Hence $\Phi$ is a path in $\mathcal{P}_3$
joining $\phi \in U_1$ and $\psi \in U_4$.

b) Let $\varphi = (f,g,h)$ be an arbitrary element of $U_2$. Let $\Psi : [0,1] \to A_3$ be
given by $\Psi(s) = \Psi_s$ for all $s \in [0,1]$; where

$$\Psi_s(t) = \left( (1-s)f(t) + st, (1-s)g(t) + st^2, h(t) \right)$$

for all $t \in \mathbb{R}$. It is easy to see that, $\Psi_0 = \varphi$ and $\Psi_1 = \tau$; where $\tau$ is given by
$\tau(t) = (t, t^2, h(t))$ for all $t \in \mathbb{R}$. One can see that, $\Psi_s \in U_4$ for all $s \in (0,1]$. Thus
$\Psi$ is a path in $\mathcal{P}_3$ joining $\varphi \in U_2$ and $\tau \in U_4$.

c) Let $\sigma = (x,y,z)$ be any element of $U_3$. Let $\Upsilon : [0,1] \to A_3$ be given by
$\Upsilon(s) = \Upsilon_s$ for all $s \in [0,1]$; where

$$\Upsilon_s(t) = \left( (1-s)x(t) + st, y(t), z(t) \right)$$

for all $t \in \mathbb{R}$. Clearly $\Upsilon_0 = \sigma$ and $\Upsilon_1 = \upsilon$; where $\upsilon$ is given by $\upsilon(t) = (t, y(t), z(t))$
for all $t \in \mathbb{R}$. It is to see that, $\Upsilon_s \in U_4$ for all $s \in (0,1]$. Hence $\Upsilon$ is a path in $\mathcal{P}_3$
joining $\sigma \in U_3$ and $\upsilon \in U_4$.

The part a) satisfies the first assumption; and the parts b) and c) together
satisfy the second assumption of the lemma 3.4. Hence by that lemma, $\mathcal{P}_3$ is path
connected. 

\textbf{Proposition 3.6.} For real numbers $a, b$ and $c$; a polynomial map $\phi \in A_4$ given by
$t \mapsto (t^2 + at, t^3 + bt, t^4 + ct)$ is embedding if and only if both the conditions

\begin{itemize}
  \item[i)] $3a^2 + 4b = 0$ and $a^3 - 2c = 0$ and
  \item[ii)] $3a^2 + 4b < 0$ and $a^3 + 2ab + c = 0$
\end{itemize}
do not hold.

\textbf{Proof.} It is easy to check: The polynomial map $\phi$ is not an embedding \iff for some
real numbers $s_0$ and $t_0$, $\phi(s_0) = \phi(t_0)$ (if $s_0 \neq t_0$) or $\phi'(t_0) = 0$ (if $s_0 = t_0$) \iff
$(s,t) = (s_0,t_0)$ is the common solution of the equations

\begin{equation}
(t + s) + a = 0, \tag{3.1}
\end{equation}

\begin{equation}
(t^2 + st + s^2) + b = 0 \text{ and} \tag{3.2}
\end{equation}

\begin{equation}
(t^3 + t^2 s + ts^2 + s^3) + c = 0. \tag{3.3}
\end{equation}

Now using 3.1 in 3.2 we get

\begin{equation}
t^2 + at + (a^2 + b) = 0 \tag{3.4}
\end{equation}
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This quadratic equation has solutions \( t = t_1 \) and \( t = t_2 \); where

\[
t_1 = \frac{-a - \sqrt{-3a^2 - 4b}}{2} \quad \text{and} \quad t_2 = \frac{-a + \sqrt{-3a^2 - 4b}}{2}.
\]

Since we just want the real solutions, so we must have \( 3a^2 + 4b \leq 0 \). So we consider the cases when \( 3a^2 + 4b = 0 \) and \( 3a^2 + 4b < 0 \).

Case \( a \): If \( 3a^2 + 4b = 0 \), then \( t = -a/2 \) is the only solution of 3.4; and hence \((s, t) = (-a/2, -a/2)\) is the only common solution of 3.1 and 3.2. So using this value of \((s, t)\) in 3.3, we get \( a^3 - 2c = 0 \).

Case \( b \): If \( 3a^2 + 4b < 0 \), then \( t = t_1 \) and \( t = t_2 \) are two distinct solutions of 3.4. Thus it is easy to see that, \((s, t) = (t_1, t_2)\) and \((s, t) = (t_2, t_1)\) are the only common solutions of 3.1 and 3.2. Using any one of the value of \((s, t)\) in 3.3, we get \( a^3 + 2ab + c = 0 \).

In other words, cases \( a \) and \( b \) together say that; 3.1, 3.2 and 3.3 has a common solution \( \iff \) any one of the condition \( i) \) or \( ii) \) does hold. Hence \( \phi \) is not an embedding \( \iff \) any one of the condition \( i) \) or \( ii) \) does hold. \( \Box \)

Corollary 3.6.1. For \( e_1, e_2, e_3 \in \{-1, 1\} \) and real numbers \( a, b, c \); a polynomial map \( \psi \in \mathcal{A}_4 \) given by \( t \mapsto (e_1 t^2 + at, e_2 t^3 + bt, e_3 t^4 + ct) \) is embedding if and only if both the conditions

i) \( 3a^2 + 4e_2 b = 0 \) and \( e_1 a^3 - 2e_3 c = 0 \)

ii) \( 3a^2 + 4e_2 b < 0 \) and \( e_1 a^3 + 2e_1 e_2 ab + e_3 c = 0 \)

do not hold.

Proof. It is easy to note that, \( \psi \) is an embedding \( \iff \tau \in \mathcal{A}_4 \) given by

\[
t \mapsto (e_1(e_1 t^2 + at), e_2(e_2 t^3 + bt), e_3(e_3 t^4 + ct))
\]

is an embedding. But we have, \( \tau(t) = (t^2 + e_1 at, t^3 + e_2 bt, t^4 + e_3 ct) \) for all \( t \in \mathbb{R} \). Now using the proposition 3.6 for \( \tau \); we get, \( \tau \) is an embedding \( \iff \) both conditions \( i) \) and \( ii) \) in the statement of the corollary do not hold. Hence same is true for the polynomial map \( \psi \).

\( \Box \)

Proposition 3.7. For \( e_1, e_2, e_3 \in \{-1, 1\} \) and real numbers \( a, b, c \); a polynomial knot \( \varphi \in \mathcal{P}_4 \) given by \( t \mapsto (e_1 t^2 + at, e_2 t^3 + bt, e_3 t^4 + ct) \) is path equivalent to

a) a polynomial knot \( \tau \in \mathcal{P}_4 \) given by \( t \mapsto (0, e_2 t, e_3 t^2) \) or

b) a polynomial knot \( \psi \in \mathcal{P}_4 \) given by \( t \mapsto (0, -e_2 t, -2e_3 t^2) \)
in the space $\mathcal{P}_4$.

**Proof.** We consider two cases as follows:

Case $a$): If $3a^2 + 4e_2b \geq 0$. Let $\Phi : [0, 1] \to \mathcal{A}_4$ be given by $\Phi(s) = \Phi_s$ for all $s \in [0, 1]$; where

$$\Phi_s(t) = \left( (1-s)(e_1t^2 + at), (1-s)(e_2t^3 + bt) + e_2st, (1-s)(e_3t^4 + ct) + e_3st^2 \right)$$

for all $t \in \mathbb{R}$. Clearly $\Phi_0 = \varphi$ and $\Phi_1 = \tau$. Now for each $s \in (0, 1); \Phi_s$ is an embedding $\iff$ a polynomial map $\frac{1}{1-s}\Phi_s \in \mathcal{A}_4$ which is given by

$$t \mapsto \left( e_1t^2 + at, e_2t^3 + \left( b + \frac{e_2s}{1-s} \right) t, e_3t^4 + \left( \frac{e_3s}{1-s} t^2 + ct \right) \right)$$

is embedding $\iff$ a map $\Psi_s \in \mathcal{A}_4$ given by

$$t \mapsto \left( e_1t^2 + at, e_2t^3 + \left( b + \frac{e_2s}{1-s} \right) t, e_3t^4 + \left( c - \frac{e_1e_3as}{1-s} \right) t \right)$$

is embedding. For each $s \in (0, 1)$, since $3a^2 + 4e_2b + \frac{4s}{1-s} > 0$. Thus by the corollary 3.6.1 for each $s \in (0, 1), \Psi_s$ is embedding and hence so is $\Phi_s$. This shows that there is path in $\mathcal{P}_4$ joining $\varphi$ and $\tau$.

Case $b$): If $3a^2 + 4e_2b < 0$. Let $\Upsilon : [0, 1] \to \mathcal{A}_4$ be given by $\Upsilon(s) = \Upsilon_s$ for all $s \in [0, 1]$; where

$$\Upsilon_s(t) = \left( (1-s)(e_1t^2 + at), (1-s)(e_2t^3 + bt) - e_2st, (1-s)(e_3t^4 + ct) - 2e_3st^2 \right)$$

for all $t \in \mathbb{R}$. It is easy to see that, $\Upsilon_0 = \varphi$ and $\Upsilon_1 = \psi$. Now for each $s \in (0, 1); \Upsilon_s$ is an embedding $\iff$ a map $\frac{1}{1-s}\Upsilon_s \in \mathcal{A}_4$ which is given by

$$t \mapsto \left( e_1t^2 + at, e_2t^3 + \left( b - \frac{e_2s}{1-s} \right) t, e_3t^4 - \frac{2e_3s}{1-s} t^2 + ct \right)$$

is an embedding $\iff$ a polynomial map $\Gamma_s \in \mathcal{A}_4$ given by

$$t \mapsto \left( e_1t^2 + at, e_2t^3 + \left( b - \frac{e_2s}{1-s} \right) t, e_3t^4 + \left( c + \frac{2e_1e_3as}{1-s} \right) t \right)$$

is an embedding. Since $\varphi$ is an embedding; so for each $s \in (0, 1),

$$e_1a^3 + 2e_1e_2a \left( b - \frac{e_2s}{1-s} \right) + e_3 \left( c + \frac{2e_1e_3as}{1-s} \right) = e_1a^3 + 2e_1e_2ab + e_3c \neq 0.$$ 

Thus by the corollary 3.6.1 for each $s \in (0, 1), \Gamma_s$ is embedding and hence so is $\Upsilon_s$. This shows that there is path in $\mathcal{P}_4$ joining $\varphi$ and $\psi$.  \qed

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Proposition 3.8. Let \( \phi \in \mathcal{P}_4 \) be a polynomial knot with degree sequence \((2, 3, 4)\). Then for some \( e_1, e_2, e_3 \in \{-1, 1\} \) and some real numbers \( a, b, c \); a polynomial knot \( \psi \) given by \( t \mapsto (e_1 t^2 + a t, e_2 t^3 + b t, e_3 t^4 + c t) \) is path equivalent to \( \phi \) in the space \( \mathcal{P}_4 \).

Proof. a) Let \( \phi \) is given by

\[
t \mapsto (a_2 t^2 + a_1 t + a_0, b_3 t^3 + b_2 t^2 + b_1 t + b_0, c_4 t^4 + c_3 t^3 + c_2 t^2 + c_1 t + c_0).
\]

Note that \( a_2, b_3 \) and \( c_4 \) all are nonzero. Let \( \Phi : [0, 1] \to \mathcal{A}_4 \) be given by \( \Phi(s) = \Phi_s \) for all \( s \in [0, 1] \); where

\[
\Phi_s(t) = (a_2 t^2 + a_1 t + (1-s) a_0, b_3 t^3 + b_2 t^2 + b_1 t + (1-s) b_0, c_4 t^4 + c_3 t^3 + c_2 t^2 + c_1 t + (1-s) c_0)
\]

for all \( t \in \mathbb{R} \). It is easy to check that, \( \Phi_s \in \mathcal{P}_4 \) for each \( s \in [0, 1] \). Let \( f(t) = a_2 t^2 + a_1 t, g(t) = b_3 t^3 + b_2 t^2 + b_1 t \) and \( h(t) = c_4 t^4 + c_3 t^3 + c_2 t^2 + c_1 t \) for all \( t \in \mathbb{R} \); and let \( \tau = (f, g, h) \). Clearly \( \Phi_0 = \phi \) and \( \Phi_1 = \tau \). Thus we have a path in \( \mathcal{P}_4 \) joining \( \phi \) and \( \tau \).

b) Let \( \Psi : [0, 1] \to \mathcal{A}_4 \) be given by \( \Psi(s) = \Psi_s \) for all \( s \in [0, 1] \); where

\[
\Psi_s(t) = \left( \begin{array}{c} f(t), g(t) - \frac{b_2}{a_2} s f(t), h(t) + \frac{b_2 c_3 - b_3 c_2}{a_2 b_3} s f(t) - \frac{c_3}{b_3} s g(t) \end{array} \right)
\]

for all \( t \in \mathbb{R} \). Note that, \( \Psi_s \in \mathcal{P}_4 \) for each \( s \in [0, 1] \). Let

\[
g_1(t) = b_3 t^3 + b_1 t = b_3 t^3 + \left( b_1 - \frac{a_1 b_2}{a_2} \right) t \quad \text{and} \quad h_1(t) = c_4 t^4 + c_1 t = c_4 t^4 + \left( c_1 + \frac{a_1 b_2 c_3 - a_1 b_3 c_2}{a_2 b_3} - \frac{b_1 c_3}{b_3} \right) t
\]

for all \( t \in \mathbb{R} \); and let \( \sigma = (f, g_1, h_1) \). It is easy to check that, \( \Psi_0 = \tau \) and \( \Psi_1 = \sigma \). So we have a path in \( \mathcal{P}_4 \) joining \( \tau \) and \( \sigma \).

c) Let \( \Upsilon : [0, 1] \to \mathcal{A}_4 \) be given by \( \Upsilon(s) = \Upsilon_s \) for all \( s \in [0, 1] \); where

\[
\Upsilon_s(t) = \left( \begin{array}{c} \left( 1 - s + \frac{s}{|a_2|} \right) f(t), \left( 1 - s + \frac{s}{|b_3|} \right) g_1(t), \left( 1 - s + \frac{s}{|c_4|} \right) h_1(t) \end{array} \right)
\]

for all \( t \in \mathbb{R} \). For all \( s \in [0, 1] \); we have \( 1 - s + \frac{s}{|a_2|} > 0 \), \( 1 - s + \frac{s}{|b_3|} > 0 \) and
1 − s + \frac{s}{|c_4|} > 0. Thus for each \( s \in [0, 1] \), \( \Upsilon_s \) is a polynomial knot in \( P_4 \). Let

\[
\begin{align*}
    f_2(t) &= e_1 t^2 + at = \frac{a_2 t^2 + a_1 t}{|a_2|}, \\
    g_2(t) &= e_2 t^2 + bt = \frac{b_3 t^3 + b_1 t}{|b_3|} \quad \text{and} \\
    h_2(t) &= e_3 t^2 + ct = \frac{c_4 t^4 + c_1 t}{|c_4|}
\end{align*}
\]

for all \( t \in \mathbb{R} \); and let \( \psi = (f_2, g_2, h_2) \). It is easy to see that, \( \Upsilon_0 = \sigma \) and \( \Upsilon_1 = \psi \). So we get a path in \( P_4 \) joining \( \sigma \) and \( \psi \).

The parts a), b) and c) together give a path in \( P_4 \) joining \( \phi \) and \( \psi \).

**Corollary 3.8.1.** Any polynomial knot in \( P_4 \) with degree sequence \((2, 3, 4)\) is path equivalent in \( P_4 \) to a polynomial knot with degree sequence \((0, 1, 2)\).

The proof of this corollary follows trivially by the proposition 3.7 and proposition 3.8.

**Theorem 3.9.** The space \( P_4 \) is path connected.

**Proof.** Consider the following sets:

- \( V_1 = \{ \phi \in A_4 : \phi \text{ has degree sequence } (0, 1, 2) \} \),
- \( V_2 = \{ \phi \in A_4 : \phi \text{ has degree sequence } (0, 1, 3) \} \),
- \( V_3 = \{ \phi \in A_4 : \phi \text{ has degree sequence } (0, 1, 4) \} \),
- \( V_4 = \{ \phi \in A_4 : \phi \text{ has degree sequence } (0, 2, 3) \} \cap P_4 \),
- \( V_5 = \{ \phi \in A_4 : \phi \text{ has degree sequence } (0, 2, 4) \} \cap P_4 \),
- \( V_6 = \{ \phi \in A_4 : \phi \text{ has degree sequence } (0, 3, 4) \} \cap P_4 \),
- \( V_7 = \{ \phi \in A_4 : \phi \text{ has degree sequence } (1, 2, 3) \} \),
- \( V_8 = \{ \phi \in A_4 : \phi \text{ has degree sequence } (1, 2, 4) \} \),
- \( V_9 = \{ \phi \in A_4 : \phi \text{ has degree sequence } (1, 3, 4) \} \) and
- \( V_{10} = \{ \phi \in A_4 : \phi \text{ has degree sequence } (2, 3, 4) \} \cap P_4 \).

Note that, these are pairwise disjoint nonempty subsets of \( A_4 \). By remark 2 each of them is subset of \( P_4 \). Also

\[
P_4 = \bigcup_{1 \leq i \leq 10} V_i.
\]

By the argument similar to the argument used in the part a) of the proof of the theorem 3.5, one can show that there is path in \( P_4 \) joining an arbitrary element of
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$V_i$ to an arbitrary element of $V_9$. Also by the arguments similar to the arguments used in the parts $b)$ and $c)$ of the proof of that theorem; it is easy to produce a path from an arbitrary element of $V_i$, for $i = 2, 3, \cdots, 8$; to some element of $V_9$. Also by the corollary [3.8.1] there is path from an arbitrary element of $V_10$ to some element of $V_1$. Thus both the assumptions of the lemma [3.4] are satisfied and hence $P_4$ is path connected.

3.2 Space $P_5$

By the proposition [2.5] we can realize the unknot and the trefoil knot in degree 5. In fact, Shastri [1] had shown a degree 5 realization of trefoil. It can be easily seen that if a knot $K$ is represented by $t \mapsto (f(t), g(t), h(t))$ then its mirror image $K^*$ can be represented by $t \mapsto (f(t), g(t), -h(t))$. Thus we can realize at most 3 knots in degree 5 namely: the unknot, the right hand trefoil and the left hand trefoil.

A mathematica plot of the Shastri’s trefoil $t \mapsto (t^3 - 3t, t^4 - 4t^2, t^5 - 10t)$ and its mirror image $t \mapsto (-t^3 - 3t, t^4 - 4t^2, t^5 - 10t)$ is shown in the following figure below:

![Figure 2: Representation of trefoil knot and its mirror image](image)

From the corollary [3.2.3] in the space $P_5$, the right hand trefoil and left hand trefoil both have at least 4 path components each. For example $t \mapsto (t^3 - 3t, t^4 - 4t^2, t^5 - 10t), \quad t \mapsto (t^3 - 3t, -(t^4 - 4t^2), -(t^5 - 10t)),$ $t \mapsto (-t^3 - 3t, t^4 - 4t^2, -(t^5 - 10t))$ and $t \mapsto (-t^3 - 3t), -(t^4 - 4t^2), t^5 - 10t)$ are each topologically equivalent to the same handed trefoil and lie in the different path components of the space $P_5$. An estimation of the lower bound on number of path components for each possible knot type in $P_5$ is given in the table below:
Thus $P_5$ has at least 9 path components.

### 3.3 Space $P_6$

All knots that have polynomial representation in degree 5 naturally have their representation in degree 6 as well. By the proposition 2.2(a), a knot $K$ that has a polynomial representation in degree 6, the minimal crossing number $c[K]$ must be less than or equal to 6. But by the proposition 2.2(c), the knots $5_1, 5_1^*, 3_1#3_1, 3_1^*#3_1^*$ and $3_1#3_1^*$ cannot be represented in degree 6 (since they are 4-superbridge). Also, by theorem 2.7, almost there is no possibility to represent the knots $5_2$ and $5_2^*$ in $P_6$; and same is true for the knots $6_1, 6_1^*, 6_2, 6_2^*$ and $6_3$. While we have a degree 6 polynomial representation $t \mapsto (f(t), g(t), h(t))$ of $4_1$ knot with degree sequence $(4, 5, 6)$; where

$$f(t) := (-4.8 + t) (-0.3 + t) (3.6 + t) (10 + t),$$
$$g(t) := (-4.8 + t) (-3.3 + t) (-0.3 + t) (2.3 + t) (4.6 + t) \text{ and}$$
$$h(t) := 0.5 t (-0.19 + t) (21.22 - 9.19 t + t^2) (17.78 + 8.42 t + t^2).$$

A *mathematica* plot of this representation of $4_1$ is shown in the figure below:

![Figure 3: Representation of $4_1$](image-url)

By the proposition 2.5, it is easy to argue that one can not represent $4_1$ using polynomial knot of degree less than 6. Hence minimal polynomial degree of $4_1$
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is 6. Also one can see that deg\((f)\) and deg\((g)\) are minimal, i.e. no any one can be reduced. Thus by the corollary 3.2.3 there are at least eight distinct path components in the space \(\mathcal{P}_6\) corresponding to the knot \(4_1\). Also the knots \(0_1, 3_1\) and \(3_1^*\) are realized in \(\mathcal{P}_6\) and each of them represents at least 1 path component. A comprehensive table of an estimation of the lower bound on number of path components for each possible knot type in \(\mathcal{P}_6\) is given below:

| Sr. No. | Knot Type | Polynomial degree of knot type | Lower bound on the number of path components of \(\mathcal{P}_6\) |
|---------|-----------|-------------------------------|----------------------------------|
| 1.      | 0\(_1\)   | 1                             | 1                                |
| 2.      | \(3_1\)   | 5                             | 1                                |
| 3.      | \(3_1^*\) | 5                             | 1                                |
| 4.      | \(4_1\)   | 6                             | 8                                |
|         |           |                               | 11                               |

So the space \(\mathcal{P}_6\) has at least 11 path components.

3.4 Space \(\mathcal{P}_7\)

By the proposition 2.2 (a), a knot \(K\) that has a polynomial representation in degree 7, the minimal crossing number \(c[K]\) has to be at most 10. In fact we have produced polynomial representations of the knots \(5_1, 5_1^*, 5_2, 5_2^*, 6_1, 6_1^*, 6_2, 6_2^*, 6_3, 3_1#3_1, 3_1^*#3_1^*, 3_1#3_1^*, 8_{19}\) and \(8_{19}^*\) in \(\mathcal{P}_7\).

1) A polynomial representation \(t \mapsto (u(t), v(t), w(t))\) of \(5_1\) knot with degree sequence \((4, 5, 7)\) is given by

\[
\begin{align*}
u(t) &:= 4 \left(-24.01 + t^2\right) \left(-4 + t^2\right), \\
v(t) &:= t \left(-30.25 + t^2\right) \left(-12.25 + t^2\right) \text{ and} \\
w(t) &:= -0.1 \left(-26.8328 + t^2\right) \left(-13.6702 + t^2\right) \left(0.1135 + t^2\right).
\end{align*}
\]
2) A polynomial representation $t \mapsto (x(t), y(t), z(t))$ of $5_2$ knot with degree sequence $(4, 5, 7)$ is given by

$$x(t) := 20 \left( -17 + t \right) \left( -10 + t \right) \left( 15 + t \right) \left( 21 + t \right),$$

$$y(t) := t \left( -400 + t^2 \right) \left( -121 + t^2 \right) \text{ and}$$

$$z(t) := -0.005 \, t \left( -20.1133216 + t \right) \left( -14.260128 + t \right) \left( 12.2430449 + t \right) \left( 20.5785825 + t \right) \left( 0.0107598 - 0.0343124 \, t + t^2 \right).$$

3) A polynomial representation $t \mapsto (f(t), g(t), h(t))$ of $6_1$ knot with degree sequence $(5, 6, 7)$ is given by

$$f(t) := 60 \left( -43.4 + t \right) \left( -28 + t \right) \left( 5 + t \right) \left( 31.4 + t \right) \left( 47.6 + t \right),$$

$$g(t) := (-49 + t) \left( -38 + t \right) \left( -8 + t \right) \left( -6 + t \right) \left( 28 + t \right) \left( 43.6 + t \right) \text{ and}$$

$$h(t) := -0.07 \left( -45.995024874 + t \right) \left( 5.231021635 + t \right) \left( 19.036560084 + t \right) \left( 758.763745443 - 54.4650519227 \, t + t^2 \right) \left( 2059.948386689 + 90.4819595699 \, t + t^2 \right).$$
4) A polynomial representation \( t \mapsto (u(t), v(t), w(t)) \) of \( 6_2 \) knot with degree sequence \((5, 6, 7)\) is given by

\[
\begin{align*}
  u(t) &:= 4 \left( -39 + t \right) \left( -5 + t \right) \left( 35 + t \right) \left( -625 + t^2 \right), \\
v(t) &:= 0.1 \left( -39 + t \right) \left( -30 + t \right) \left( -10 + t \right) \left( 20 + t \right) \left( 25 + t \right) \left( 41 + t \right) \text{ and} \\
w(t) &:= 0.005 \left( -39.8753791 + t \right) \left( -27.4156408 + t \right) \left( 28.436878 + t \right) \left( 37.25572585 + t \right) \left( 0.002423881 - 0.005429486 t + t^2 \right).
\end{align*}
\]

5) A polynomial representation \( t \mapsto (x(t), y(t), z(t)) \) of \( 6_3 \) knot with degree sequence \((5, 6, 7)\) is given by

\[
\begin{align*}
  x(t) &:= 15 \left( -29 + t \right) \left( -20 + t \right) \left( 10 + t \right) \left( 30 + t \right)^2, \\
y(t) &:= \left( -32 + t \right) \left( -6 + t \right) \left( 4 + t \right) \left( 30 + t \right) \left( -400 + t^2 \right) \text{ and} \\
z(t) &:= -0.06 \left( -33.329044815 + t \right) \left( 376.737563885 \right) \left( 37.8892469397 + t \right) \left( 955.98573648 + 61.56649851 t + t^2 \right) \left( 144.275534095 + 21.404400212 t + t^2 \right) \left( 955.98573648 + 61.56649851 t + t^2 \right).
\end{align*}
\]
6) A polynomial representation $t \mapsto (f(t), g(t), h(t))$ of $3_1 \# 3_1$ knot with degree sequence $(5, 6, 7)$ is given by

\[
\begin{align*}
    f(t) &:= 5 \left( 77.3 - 17.5 t + t^2 \right) \left( 77.3 + 17.5 t + t^2 \right), \\
    g(t) &:= (-102.01 + t^2) (-53.29 + t^2) (-4.84 + t^2) \quad \text{and} \\
    h(t) &:= -0.15 t (-99.695462027 + t^2) (-68.11720396 + t^2) (0.025367747 + t^2).
\end{align*}
\]

7) A polynomial representation $t \mapsto (u(t), v(t), w(t))$ of $3_1 \# 3_1^*$ knot with degree sequence $(5, 6, 7)$ is given by

\[
\begin{align*}
    u(t) &:= 30 (-32.5 + t) (-21.3 + t) (-3.3 + t) (16.2 + t) (28 + t), \\
    v(t) &:= (-34 + t) (-23 + t) (-6.8 + t) (12 + t) (21.7 + t) (33.1 + t) \quad \text{and} \\
    w(t) &:= -0.03 t (-32.807367 + t) (-24.209735 + t) (15.257278 + t) (28.289226 + t) \\
    & \quad (0.0043718 - 0.0082068 t + t^2).
\end{align*}
\]
8) A polynomial representation \( t \mapsto (x(t), y(t), z(t)) \) of \( 8_{19} \) knot with degree sequence \((5, 6, 7)\) is given by

\[
x(t) := t^5 - 5.5 t^3 + 4.5 t , \\
y(t) := t^6 - 7.35 t^4 + 14 t^2 \quad \text{and} \\
z(t) := t^7 - 8.13297 t^5 + 18.5762 t^3 - 10.4337 t. 
\]

The knots \(5_1, 5_1^*, 3_1 \# 3_1, 3_1^* \# 3_1^*, 3_1 \# 3_1^*, 8_{19} \) and \(8_{19}^*\) are 4-superbridge. So by the proposition 2.2\(c\), any of them can not be represented by a polynomial knot with degree less than 7. In other words, their polynomial degree is 7 and hence each of them represents at least two path components of \(\mathcal{P}_7\). Also remaining knots with at most 6 crossings represents at least one path component each. We have summarize this in the following table:
| Sr. No. | Knot Type | Polynomial degree of knot type | Lower bound on the number of path components of $P_7$ |
|---------|-----------|-------------------------------|--------------------------------------------------|
| 1.      | $0_1$     | 1                             | 1                                                |
| 2.      | $3_1$     | 5                             | 1                                                |
| 3.      | $3_1^*$   | 5                             | 1                                                |
| 4.      | $4_1$     | 6                             | 1                                                |
| 5.      | $5_1$     | 7                             | 2                                                |
| 6.      | $5_1^*$   | 7                             | 2                                                |
| 7.      | $5_2$     | 6 or 7                        | 1                                                |
| 8.      | $5_2^*$   | 6 or 7                        | 1                                                |
| 9.      | $6_1$     | 6 or 7                        | 1                                                |
| 10.     | $6_1^*$   | 6 or 7                        | 1                                                |
| 11.     | $6_2$     | 6 or 7                        | 1                                                |
| 12.     | $6_2^*$   | 6 or 7                        | 1                                                |
| 13.     | $6_3$     | 6 or 7                        | 1                                                |
| 14.     | $3_1 \# 3_1$ | 7                          | 2                                                |
| 15.     | $3_1^* \# 3_1^*$ | 7                          | 2                                                |
| 16.     | $3_1 \# 3_1^*$ | 7                          | 2                                                |
| 17.     | $8_{19}$  | 7                             | 2                                                |
| 18.     | $8_{19}^*$| 7                             | 2                                                |

Hence the space $P_7$ has at least 25 path components.

4 Conclusion

We have seen that polynomial knots inside $P_d$ for which $d$ is the minimal degree give rise to many different path components. However, determining the minimal degree of a knot is a challenging problem. In certain cases a few numerical invariants such as superbridge index help in proving the minimality of the degree of a polynomial knot. In $P_7$, the knots $5_2, 5_2^*, 6_1, 6_1^*, 6_2, 6_2^*$ and $6_3$ may give rise to many components if we can prove that 7 is the least degree for these knots. This has been conjectured that all these knots are 4-superbridge. Once this conjecture is proved, it will immediately follow that 7 is the least degree for representing these knots as polynomial knots. This will bring at least 7 more path components in $P_7$. Note that for $d > 7$, many non alternating knots can belong to the space $P_d$ and lead to many more path components.
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