Quantum Wire Network with Magnetic Flux

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The single channel transport properties of a quantum wire network in an ambient electromagnetic field are investigated. The network is made of three semi-infinite external leads connected to a ring which is crossed by a magnetic flux. Two different kinds of bulk dynamics, associated with the Schrödinger and Dirac equations, both minimally coupled to the external electromagnetic field, are analyzed. Connecting the external leads to heat reservoirs with different temperatures and/or chemical potentials, the system is driven away from equilibrium. We derive the conductance and the noise of this configuration, describing in details the impact of the magnetic flux. In the case of weak coupling between the ring and the reservoirs, a resonant tunneling effect is observed. We also discover a new power law behaviour of the pure thermal noise which differs from the usual linear Johnson-Nyquist law when the magnetic flux is nonzero.

I. INTRODUCTION

Quantum wire networks are expected to play a fundamental role in future nano-electronics. For this reason the transport properties of such networks attract much attention \cite{1-35}. Since a quantum wire network is composed from junctions and extremely thin (of the order of few nanometers) leads, it is natural to represent it by a graph, as shown in Fig. 1 where the vertices \(V_j\) and edges \(E_j\) model the junctions and the wires respectively. In this way one is led to the study of electron transport along graphs. This problem has been extensively studied in the last three decades (see for instance \cite{36,37} and references therein) with a recently growing interest \cite{3-35} in the universal features of the above systems. Despite great progress in this direction, the role and the physical implications of the different possible boundary conditions at the junctions are not yet fully understood.

The dynamics away from equilibrium needs further investigation as well. In this context we focus below on three particular aspects. These are the interplay between bulk dynamics and vertex interactions, the construction of an equivalent total scattering matrix relative to all external edges, as shown in Fig. 1. More precisely, we investigate the conductance and the noise of this junction as a function of the transmission \(t\) in the ring, the flux \(\phi\), the temperatures \(\beta_i\) and chemical potentials \(\mu_i\) of the external edges, as shown in Fig. 2 below. Aharonov-Bohm type oscillations with \(\phi\) occur in both the conductance and the noise. The period of these oscillations equals the elementary flux quantum \(\phi_0 = 2\pi\hbar c/e\) associated with a single charge \(e\). The pure thermal \((\mu_i = 0)\) noise has a \(\phi\)-dependent power law behaviour at small temperatures, with discontinuities in the points \(\phi = n\phi_0, n \in \mathbb{Z}\). As functions of \(\mu_i\), the current and the shot noise have an interesting plateaux structure in the regime \(t \sim 1\). The fundamental and essentially unique input for deriving these results is the requirement of self-adjointness of the Schrödinger and Dirac Hamiltonians on the ring graph of Fig. 2 in external magnetic field. This requirement determines a family of admissible point-like interactions at the vertices on the ring.
We focus on the scale invariant ones because they incorporate the universal features of the system while being simple enough to be analyzed explicitly. In particular, no phenomenological potentials are introduced.

![Diagram](image)

**FIG. 3.** (Color online) Y-junction connected at infinity to thermal reservoirs with inverse temperature $\beta_i$ and chemical potential $\mu_i$.

The paper is organized as follows. In the next section we discuss the relation between bulk dynamics and point-like interactions at the vertices. The main tool here is the use of the self-adjoint extensions of the Hamiltonian from the bulk to the whole graph. In section III we reconstruct the global scattering matrix $S^o$ from the local $S_i$ matrices in the presence of magnetic flux $\phi$ crossing the ring junction in Fig. 2. We describe the basic features of $S^o$ and establish the breaking of time reversal invariance induced by $\phi$. In section IV we investigate the non-equilibrium dynamics of the Y-junction with external leads connected to thermal reservoirs with a different temperature and chemical potential, as shown in Fig. 3. The conductance and the noise are explicitly derived. The helicity transport in the Dirac case is also described. Section V is devoted to the conclusions and a discussion of some further developments. The appendices collect some technical details.

Let us mention finally that the wires, displayed in the figures of this paper, are planar and in most of the cases have straight line edges. However, the discussion below is completely general and applies to segments of arbitrary smooth curves in $\mathbb{R}^3$ as well. It is also extremely important to realize that in the mapping, presented in Fig. 2, the Y-junction with scattering matrix $S^o$ is a convenient way to model the ring junction with matrices $S_i$ and is not the result of taking a zero size limit of the ring while maintaining a fixed magnetic flux. In spite of the fact that we adopt scale invariant local scattering matrices $S_i$, the total scattering matrix $S^o$ encodes explicitly the finite size of the ring and is not scale invariant. The ring junction in the left hand side of Fig. 2 and the point like Y-junction in the right hand side are completely equivalent and we use these names interchangeably in the rest of the paper with this understanding. In this respect our approach is very different from the one using boundary conformal field theory.

## II. Interplay Between Bulk Dynamics and Vertex Interactions

The fundamental inputs for applying transport theory on graphs are essentially two:

(i) dynamics in the bulk (edges);

(ii) interaction at the vertices.

In most of the papers on the subject it is implicitly assumed that (i) and (ii) are two independent ingredients. Some recent developments in the spectral theory of operators on graphs ("quantum graphs") have shown however that this is not the case, if one requires unitary time-evolution of the system. It turns out that this basic physical condition relates the bulk and vertex interactions. The point is that the bulk dynamics is described by a Hermitian Hamiltonian, which becomes self-adjoint only by imposing special boundary conditions at the vertices. These conditions generate particular point-like interactions, which are described by specific (and not arbitrary) scattering matrices $S_i$, associated with each vertex $V_i$ of the graph. Since the form of $S_i$ has relevant physical implications, it is instructive to describe the phenomenon in more details. We consider for this purpose the simplest example of fermion excitations, which propagate freely in the bulk.

![Diagram](image)

**FIG. 4.** (Color online) A vertex $V$ with $n$ edges $E_i$.

Let us denote by $V$ a generic vertex of the network with $n$ edges $E_i$, oriented as in Fig. 4. We consider first non-relativistic fermions (Schrödinger network). In this case the time evolution in the bulk is described by the Schrödinger equation

$$i\partial_t + \frac{1}{2m}\partial_x^2 \psi(t, x, i) = 0 ,$$

where $x > 0$ is the distance from the vertex $V$. The only non-trivial interaction is localized at $V$ and is encoded in boundary conditions at $x = 0$. To select these conditions, we require that the bulk Hamiltonian $-\partial_x^2$ admits a self-adjoint extension at $x = 0$. In such a way we cover all point-like interactions localized at $V$ and leading to a unitary time evolution of the system. The most general boundary condition, implementing this natural physical condition for the Schrödinger network, is

$$\sum_{j=1}^n [\lambda (I - U)_{ij} \psi(t, 0, j) - i (I + U)_{ij} (\partial_x \psi)(t, 0, j)] = 0 ,$$

where $x > 0$ is the distance from the vertex $V$.
where $U$ is an arbitrary $n \times n$ unitary matrix and $\lambda$ is a real parameter with the dimension of mass. $U = I$ and $U = -I$ generalize to the star graph in Fig. 1 the familiar Neumann and Dirichlet boundary conditions on the half line. It has been established\textsuperscript{34,35} that the point-like interaction, induced by (2), generates
\begin{equation}
S(k) = \frac{\lambda (I - U) - k(I + U)}{\lambda (I - U) + k(I + U)}, \quad k \in \mathbb{R}, \tag{3}
\end{equation}
which is a very special momentum-dependent scattering matrix. An equivalent form of (3), which is more convenient for explicit computations, is given in the appendix A. The representation (A3) shows in particular that $S(k)$ is a meromorphic function with simple poles, all of which located on the imaginary axis and different from 0.

The equation of motion (1) is invariant under the time reversal operation
\begin{equation}
T \psi(t, x, i) T^{-1} = -\eta_T \psi(-t, x, i), \quad |\eta_T| = 1, \tag{4}
\end{equation}
$T$ being an anti-unitary operator. We stress however that the vertex interaction (and therefore $S(k)$) preserves the time reversal symmetry if and only if $U$ is symmetric\textsuperscript{36}, or equivalently
\begin{equation}
S(k)^t = S(k), \tag{5}
\end{equation}
where the apex $t$ indicates transposition. We will apply the condition (5) when discussing below the breaking of time reversal symmetry in the Y-junction with magnetic flux.

Let us consider now relativistic fermions (Dirac network). In this case the dynamics is governed by the Dirac equation (we consider for simplicity the massless case)
\begin{equation}
(\gamma_0 \partial_t - \gamma_x \partial_x) \psi(t, x, i) = 0, \quad x > 0, \tag{6}
\end{equation}
where
\begin{equation}
\psi(t, x, i) = \begin{pmatrix} \psi_1(t, x, i) \\ \psi_2(t, x, i) \end{pmatrix}, \tag{7}
\end{equation}
and
\begin{equation}
\gamma_t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{8}
\end{equation}
The boundary conditions, which define all self-adjoint extensions of the bulk Hamiltonian $\gamma_t \gamma_x \partial_x$ at the vertex $V$, are now\textsuperscript{33,59}
\begin{equation}
\psi_1(t, 0, i) = \sum_{j=1}^n U_{ij} \psi_2(t, 0, j), \tag{9}
\end{equation}
where $U$ is any unitary $n \times n$ matrix. Observing that both the equation of motion (6) and the boundary condition (9) preserve scale invariance, it is not surprising that the scattering matrix, associated with (9), is simply\textsuperscript{35}
\begin{equation}
S(k) = \theta(k) U + \theta(-k) U^{-1}, \tag{10}
\end{equation}
Since the explicit form of $S^\phi$ for a generic ring junction with general $S_i(k)$ is quite complicated, we simplify the considerations by focusing on the case of identical local scattering matrices $S_i(k) \equiv S(k)$ ($i = 1, 2, 3$) and equidistant vertices, separated by a distance $d$ along the ring. In this case the system is invariant under cyclic permutations, implying that $S^\phi$ is a circulant matrix, i.e.

$$S^\phi(k) = \left( \begin{array}{ccc} \sigma_1(k,\phi) & \sigma_2(k,\phi) & \sigma_3(k,\phi) \\ \sigma_3(k,\phi) & \sigma_1(k,\phi) & \sigma_2(k,\phi) \\ \sigma_2(k,\phi) & \sigma_3(k,\phi) & \sigma_1(k,\phi) \end{array} \right). \quad (15)$$

The explicit form of $\sigma_j(k,\phi)$ in terms of the matrix elements $S_{ij}(k)$ of the local $S$-matrix is

$$\sigma_j(k) = \frac{1}{3} \sum_{\ell=1}^{3} e^{i \pi (\ell-1)(j-1)} \lambda \left( k, \phi + \frac{2(\ell-1)}{3} \pi \right), \quad (16)$$

where

$$\lambda(k,\phi) = -\det[S(k)] \frac{e^{ikd} - S_{23}(-k)e^{i\phi} - S_{32}(-k)e^{-i\phi} - \det[S(-k)]S_{11}(k)e^{-ikd}}{e^{-ikd} - S_{23}(k)e^{i\phi} - S_{32}(k)e^{-i\phi} - \det[S(k)]S_{11}(-k)e^{ikd}}. \quad (17)$$

except at the points

$$\phi = 3n\pi, \quad n \in \mathbb{Z}. \quad (24)$$

Taking into account the condition (3), we thus conclude that only the fluxes (24) preserve time reversal invariance. Otherwise, the magnetic field breaks down the time reversal symmetry. In fact, the following relations hold:

$$(S^\phi)^\dagger(k) = S^\phi(-k), \quad (S^\phi)^t(k) = S^{-\phi}(k), \quad (25)$$

so that

$$\sigma_j^*(k,\phi) = \sigma_j(-k,-\phi), \quad (26)$$

and

$$\sigma_1(k,\phi) = \sigma_1(-k,\phi), \quad \sigma_3(k,\phi) = \sigma_2(-k,\phi). \quad (27)$$

We can now define the transmission amplitudes

$$\tau_+(k,\phi) \equiv |\sigma_2(k,\phi)|, \quad \tau_-(k,\phi) = |\sigma_3(k,\phi)|, \quad (28)$$

and the reflection amplitude

$$\varrho(k,\phi) \equiv |\sigma_1(k,\phi)|, \quad (29)$$

associated with (15). They satisfy

$$\tau_\pm(k,\phi) = \tau_\pm(-k,-\phi), \quad \tau_-(k,\phi) = \tau_+(k,-\phi), \quad (30)$$

and

$$\varrho(k,\phi) = \varrho(-k,\phi) = \varrho(k,-\phi), \quad (31)$$

which reflects the symmetry structure of our ring junction.

From the unitarity of $S^\phi$ we deduce the expected relation

$$\varrho^2(k,\phi) + \tau_+^2(k,\phi) + \tau_-^2(k,\phi) = 1, \quad (32)$$

among probabilities.
One can verify that both $\tau_{\pm}$ and $\rho$ are periodic in $\phi$ with period $\phi_0 = 2\pi$. Note, that this is not the case of $\sigma_2$ and $\sigma_3$, which are only $6\pi$-periodic. The period $\phi_0$ has a deep physical meaning. In fact, recalling our convention $c = \hbar = e = 1$, $\phi_0$ equals precisely the elementary flux quantum $2\pi\hbar c/e$ associated with a single charge $e$. This flux quantum appears naturally in the context of Aharonov-Bohm type oscillations in non simply connected mesoscopic systems. It is not surprising therefore that the $\phi_0$-periodicity shows up below in the currents, the conductance and noise of the ring junction.

We emphasize finally that both $\rho$ and $\tau_{\pm}$ are $k$-dependent, in spite of the fact that the local scattering matrices \[ A \] are constant. This dependence is a direct consequence of the finite size of the ring. In fact, the momentum $k$ enters $\rho$ and $\tau_{\pm}$ only through the dimensionless combination $k d$. It follows from \[ \tau \] that $\rho$ and $\tau_{\pm}$ are $2\pi/d$-periodic in $k$, the shape of the oscillations being strongly influenced by the transmission $t$ and the flux $\phi$. The behaviour of the probability $\tau_{\pm}^2$ for $d = 1$, shown in Fig. \[ \text{FIG. 6} \] confirms this statement. The dashed (black) and the continuous (red) lines describe the oscillations for $t = 0.5$ and $t = 0.99$ respectively. As already observed, for $t \sim 1$ the external edges are almost isolated. Accordingly, in this regime, one expects very small transmission amplitudes. This is indeed the case with the exception of the momenta $k = \frac{\pm \phi + 2\pi n}{3d}$, characterized by the appearance of sharp peaks with maximum close to $4/9$ and corrections of order $1 - t$. For $\phi \neq n\pi$, there are six of them in each $k$ interval of length $2\pi/d$ and only three if $\phi = n\pi$. Fig. \[ \text{FIG. 4} \] illustrates the phenomenon for $\phi = 0$ (left) and $\phi = \pi/4$ (right). Because of \[ \text{FIG. 5} \], the behaviour of the reflection amplitude $\rho$ is complementary.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5}
\caption{(Color online) $\tau_{\pm}^2(k, \phi, d = 1)$ for $t = 0.5$ (dashed black line) and $t = 0.99$ (continuous red line) for $\phi = 0$ (left) and $\phi = \pi/4$ (right).}
\end{figure}

We stress that at $t = 1$ the amplitudes $\tau_{\pm}$ become actually discontinuous in $k$. One has indeed

\[
\lim_{t \to 1} \tau_{\pm}(k, \phi) = \begin{cases} 
\frac{2}{3}, & k = \frac{\pm \phi + 2\pi n}{3d}, \quad n \in \mathbb{Z}, \\
0, & k \neq \frac{\pm \phi + 2\pi n}{3d}.
\end{cases}
\]

(33)

The impact of the peculiar $k$-dependence of $\tau_{\pm}$ for $t \sim 1$ on the current and the shot noise is discussed in the next section.

It is worth mentioning that similar phenomena show up in the other limit $t \sim 0$ of almost disconnected external edges. Also in that case the transmission amplitudes are negligible, apart around the points $k = n\pi$, where the nontrivial behaviour, displayed in Fig. \[ \text{FIG. 6} \] is observed.

Another type of discontinuities of $\tau_{\pm}$, which involves the magnetic flux and also deserves attention, is described by

\[
\lim_{k \to \frac{2\pi}{\phi_0} = \frac{2\pi}{\phi_0}^{\nu}} \tau_{\pm}(k, \phi) = \begin{cases} 
\frac{2}{3}, & \phi = n\phi_0, \quad l, n \in \mathbb{Z}, \\
0, & \phi \neq n\phi_0.
\end{cases}
\]

(34)

We will show below that \[ \text{Eqs. (34)} \] has important consequences for the behaviour of the pure thermal noise at small temperatures.

Let us conclude this section with a remark about the Dirac junction. According to the observation at the end of Appendix \[ \text{A} \] the matrices $S^\phi_{\pm}$ apply both to the Schrödinger and Dirac junctions, the possible differences in the behaviour of the conductance and the noise in these cases being related essentially to the different dispersion relations.

\section{IV. CURRENTS, CONDUCTANCE AND NOISE AWAY FROM EQUILIBRIUM}

In this section we apply the results of Ref. \[ \text{[33]} \] about non-equilibrium transport on star graphs to the $Y$-junction in Fig. \[ \text{FIG. 3} \]. The interaction in the vertex is described by the flux dependent scattering matrix $S^\phi$ and the leads are connected at infinity to thermal reservoirs (heat baths) with (inverse) temperature $\beta_i$ and chemical potential

\[
\mu_i = k_F - V_i,
\]

(35)

where $k_F$ defines the Fermi energy and $V_i$ is the external voltage applied to the edge $E_i$. In what follows we keep $k_F$ fixed, varying eventually the gate voltages $V_i$. The system is away from equilibrium if $S^\phi$ admits at least one non-trivial transmission coefficient among edges with different $\beta_i$ and/or $\mu_i$. It turns out \[ \text{[33]} \] that the corresponding non-equilibrium dynamics can be implemented by a steady state $\Omega_{\beta,\mu}$, characterized by non-vanishing time-independent charge and heat currents circulating along the leads. Nevertheless, the contact with
the heat baths ensures energy conservation. In this sense $\Omega_{\beta,\mu}$ describes an ideal system without dissipation. The construction of $\Omega_{\beta,\mu}$ involves the scattering matrix $S^\phi$ and fully takes into account both the minimal coupling with the external magnetic field and the vertex interactions. We denote in what follows the expectation values in the state $\Omega_{\beta,\mu}$ by $\langle \cdots \rangle_{\beta,\mu}$ and consider the Schrödinger and the Dirac cases separately. We stress that the current correlation functions $\langle j_x(t, x, i) \rangle_{\beta,\mu}$ and $\langle j_x(t_1, x_1, i_1)j_x(t_2, x_2, i_2) \rangle_{\beta,\mu}$ used below are exact. No approximations, like linear response theory, are adopted.

A. Schrödinger junction

1. Current

The charge transport in this case is associated with the conserved current

$$j_x(t, x, i) = \frac{i}{2m} \left[ \psi^\dagger (\partial_x \psi) - (\partial_x \psi^\dagger) \psi \right] (t, x, i).$$

Using the explicit construction of the steady state $\Omega_{\beta,\mu}$, one finds the Landauer-Büttiker formula

$$\langle j_x(t, x, i) \rangle_{\beta,\mu} = \int_0^\infty \frac{dk}{2\pi m} \sum_{j=1}^3 \left[ \delta_{ij} - |S^\phi_{ij}(k)|^2 \right] d_j(k),$$

where

$$d_j(k) = \frac{e^{-\beta_\xi} [\omega(k) - \mu_\xi]}{1 + e^{-\beta_\xi} [\omega(k) - \mu_\xi]}, \quad \omega(k) = \frac{k^2}{2m},$$

is the familiar Fermi distribution. Unitarity of $S^\phi$ implies the Kirchhoff’s rule

$$\sum_{i=1}^3 \langle j_x(t, x, i) \rangle_{\beta,\mu} = 0.$$ (39)

Introducing the variable $\xi = k^2/2m$ and using that the Fermi distribution (39) approaches the Heaviside step function $\theta(\mu_\xi - \xi)$ in the zero temperature limit $\beta_\xi \to \infty$, one obtains from (37)

$$J_i \equiv \langle j_x(t, x, i) \rangle_{\mu} = \theta(\mu_\xi) \frac{\mu_\xi}{2\pi} - \sum_{j=1}^3 \theta(\mu_j) \int_0^{\mu_j} \frac{dk}{2\pi} \left| S^\phi_{ij} \left( \sqrt{2m\xi} \right) \right|^2.$$ (40)

We will investigate the behaviour of (40), comparing $J_i$ to the shot noise in subsection 3 below.

2. Conductance

The standard definition

$$\langle j_x(t, x, i) \rangle_{\beta,\mu} = \sum_{j=1}^3 G_{ij} V_j$$

and (37) determine the conductance tensor

$$G_{ij} = \frac{1}{mV_j} \int_0^\infty \frac{dk}{2\pi} k \left[ \delta_{ij} - |S^\phi_{ij}(k)|^2 \right] d_j(k).$$ (42)

The periodicity of $|\sigma_j(k)|$ in $\phi$, established in the previous section, indicates that $G_{ij}$ oscillate with period $\phi_0$. The unitarity of $S^\phi$ implies the following $t$-independent bound

$$|G_{ij}| \leq \frac{1}{m|V_j|} \int_0^\infty \frac{dk}{2\pi} k d_j(k) = \frac{1}{2\pi \beta_j |V_j|} \log \left[ 1 + e^{(k_F-V_j)\beta_j} \right],$$ (43)

on the amplitude of the oscillations.

FIG. 7: (Color online) $G_{11}(\phi, t)$ for $\beta_1 = 1$ (left) and $\beta_1 = 0.2$ (right).

FIG. 8: (Color online) Sections of the plots in Fig. 7 for fixed $\phi$ (first line) and fixed $t$ (second line) at $\beta_1 = 1$ (left) and $\beta_1 = 0.2$ (right).

In order to get a more precise idea on the dependence of $G_{ij}$ on $\phi$, the transmission $t$ and the temperature $\beta$, we concentrate on (42). The $k$-integration can not be performed in a closed analytic form, but being well defined, the integral can be computed numerically. The following plots illustrate the result of this computation, in which the values $d = 1$, $m = 1/2$, $k_F = 2$ have been
fixed. Typical contour plots of $\mathcal{G}_{11}$ in the $\phi-t$ plane are displayed in Fig. [7] at two different temperatures ($\beta_1 = 1$ and $\beta_1 = 0.2$) for the voltage $V_1 = 1$. As usual, higher regions are shown in lighter shades. Fig. [8] collects some sections of the contour plots. The first line displays the sections at $\phi = 0$ (dashed line) and $\phi = 2\pi/3$ (continuous line). As expected, the conductance vanishes at $t = 0,1$, when the external edges of the Y-junction are isolated from each other. We see also that the position of the maximum of $G_{11}$ depends on the flux $\phi$. In the second line of Fig. [3] we report the sections at $t = 0.4$ (dashed) and $t = 0.7$ (continuous), which show the expected oscillation in $\phi$ with period $\phi_0$. The continuous (red) lines illustrate the impact of the higher harmonics $n\phi_0$, which become relevant for $t > 0.5$. Note that the oscillations of $\mathcal{G}_{11}$ are symmetric with respect to $\phi = \pi$, which is actually a general feature of the reflection amplitude $\rho$. The off-diagonal elements do not share this property, which is evident from Fig. [9] showing the behaviour of $\mathcal{G}_{12}$.

3. Noise

The zero frequency noise power is defined as usual\textsuperscript{[44,45]} by

$$P_{ij} = \lim_{\nu \to 0^+} \int_{-\infty}^{\infty} dt e^{i\nu t} \langle j_x(t, x_1, i) j_x(0, x_2, j) \rangle^{\text{conn}}_{\beta, \mu},$$

where $\langle j_x(t_1, x_1, i) j_x(t_2, x_2, j) \rangle^{\text{conn}}_{\beta, \mu}$ is the connected current-current correlation function in the state $\Omega_{\beta, \mu}$. It turns out\textsuperscript{[45]} that $P_{ij}$ is $x_{1,2}$-independent and is given by

$$P_{ij} = \frac{1}{m} \int_0^\infty \frac{dk}{2\pi} k \left\{ k_{ij} d_i(k) c_i(k) - |S^0_{ij}(k)|^2 d_j(k) c_j(k) - |S^0_{ji}(k)|^2 d_i(k) c_j(k) - \frac{1}{2} \sum_{l,m=1}^3 S_{il}^0(k) \bar{S}_{jl}^0(k) S_{jm}^0(k) \bar{S}_{im}^0(k) [c_i(k) d_m(k) + c_m(k) d_i(k)] \right\},$$

where $c_i(k) \equiv 1 - d_i(k)$. $P$ is a symmetric matrix\textsuperscript{[43]}. If we assume now $\mu_i = \mu$ and $\beta_i = \beta$, so that the setup of Fig. [3] respects the cyclic symmetry among the vertices on the ring, then $P$ is also a circulant matrix. Combining this fact with the Kirchhoff rule

$$\sum_{j=1}^3 P_{ij} = 0, \quad i = 1, 2, 3,$$

we obtain that

$$P = P_{11} \begin{pmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{pmatrix},$$

with

$$P_{11} = \frac{2}{m} \int_0^\infty \frac{dk}{2\pi} k d(k) c(k) \left[ \tau^2_+(k, \phi) + \tau^2_-(k, \phi) \right].$$

(48)

Like the conductance, $P_{11}$ oscillates in $\phi$ with period $\phi_0$. The bound on the amplitude, following from unitarity is now

$$0 \leq P_{11} \leq \frac{2}{m} \int_0^\infty \frac{dk}{2\pi} k d(k) c(k) = \frac{1}{\pi \beta} \frac{1}{(1 + e^{-\beta \mu})}.$$

(49)

Let us investigate the pure thermal noise, obtained from (48) by setting $\mu = 0$ (i.e. a voltage $V = k_F$). Using the variable $T \sim 1/\beta$, the numerical study confirms that at large temperatures

$$P_{11} \sim T, \quad \text{for } T \to \infty,$$

(50)

independently of the flux $\phi$, as suggested by [49]. In this regime one recovers therefore the well-known Johnson-Nyquist formula. The situation changes drastically at small temperatures, where we find a $\phi$-dependent power law behaviour

$$P_{11} \sim T^{g(\phi)}, \quad \text{for } T \to 0,$$

(51)

where $g$ is plotted in Fig. [10]. In absence of magnetic flux one observes a linear behaviour at small temperatures as well. However, as soon as the magnetic flux is switched on, the exponent $g(\phi)$ rapidly increases from
the value 1 to 2, leading to a quadratic power law. We see therefore that the presence of a magnetic flux in the Y-junction implies a relevant small temperature modification of the standard Johnson-Nyquist thermal noise behaviour. This new feature provides an interesting signature of a physical effect that hopefully can be observed experimentally.

The above numerical results are supported by an exact analytic argument, reported in a more formal accompanying paper[59], which gives

$$g(\phi) = \begin{cases} 1, & \phi = n\phi_0, \quad n \in \mathbb{Z}, \\ 0, & \phi \neq n\phi_0. \end{cases}$$

Strictly speaking, the power $g$ is discontinuous in the points $\phi = n\phi_0$. One can show[59] that the origin of this discontinuity is the peculiar behaviour of the transmission probabilities $\tau_{\pm}$ in the integrand (48), namely the lack of commutativity of the two limits

$$\lim_{\phi \to n\phi_0} \lim_{k \to 0} \tau_{\pm}^2(k) \neq \lim_{k \to 0} \lim_{\phi \to n\phi_0} \tau_{\pm}^2(k).$$

In fact, it follows from (49) that

$$\lim_{\phi \to n\phi_0} \lim_{k \to 0} \tau_{\pm}^2(k) = 0,$$

whereas

$$\lim_{k \to 0} \lim_{\phi \to n\phi_0} \tau_{\pm}^2(k) = \frac{4}{9}.$$

The apparently smooth behaviour of $g(\phi)$ around the points $\phi = 0, \phi_0$ in Fig. 10 is clearly an artifact of the numerical computation, which is performed unavoidably with a finite precision.

The situation is more complicated in the presence of a non-zero chemical potential $\mu > 0$. The parameter controlling the behaviour of the noise $P_{11}$ in this case is $\mu/T$. For the complete study we refer to Ref. [59], where various ranges of this parameter are considered.

Let us investigate now the shot noise defined by eq. (45). For this purpose we set $\beta_i = \beta$ and take the $\beta \to \infty$ limit, keeping $\mu_i > 0$ arbitrary. Adopting the variable $\xi = k^2/2m$ one gets

$$P_{ij} = \frac{1}{2} \sum_{l \neq m = 1}^{3} \varepsilon(\mu_l - \mu_m) \int_{\mu_m}^{\mu_l} \frac{d\xi}{2\pi} F_{ijlm} \left(\sqrt{2m\xi}\right),$$

where

$$F_{ijlm}(k) = S^\phi_{il}(k)S^\phi_{jl}(k)S^\phi_{jm}(k)S^\phi_{im}(k),$$

and $\varepsilon$ is the sign function. Note that since $P_{ij}$ is symmetric and satisfies (46), we only need to compute the diagonal elements $\{P_{ii} : i = 1, 2, 3\}$ in order to reconstruct the complete matrix. Indeed, for $i, j, k$ mutually distinct,

$$P_{ij} = \frac{1}{2}(P_{kk} - P_{ii} - P_{jj}).$$

The diagonal elements read

$$P_{ii} = \sum_{l < m = 1}^{3} \varepsilon(\mu_l - \mu_m) \int_{\mu_m}^{\mu_l} \frac{d\xi}{2\pi} F_{iilm} \left(\sqrt{2m\xi}\right).$$

Assuming for definiteness that $\mu_1 < \mu_2 < \mu_3$, one obtains from (59)

$$P_{11} = \int_{\mu_1}^{\mu_2} \frac{d\xi}{2\pi} \phi^2(1 - \phi^2) + \int_{\mu_2}^{\mu_3} \frac{d\xi}{2\pi} \tau_{+}^2(1 - \tau_{+}^2),$$

$$P_{22} = \int_{\mu_1}^{\mu_2} \frac{d\xi}{2\pi} \tau_{-}^2(1 - \tau_{-}^2) + \int_{\mu_2}^{\mu_3} \frac{d\xi}{2\pi} \tau_{+}^2(1 - \tau_{+}^2),$$

$$P_{33} = \int_{\mu_1}^{\mu_2} \frac{d\xi}{2\pi} \tau_{+}^2(1 - \tau_{+}^2) + \int_{\mu_2}^{\mu_3} \frac{d\xi}{2\pi} \phi^2(1 - \phi^2),$$

where $\phi$ and $\tau_{\pm}$ are computed at $k = \sqrt{2m\xi}$. The behaviour of $\phi$ and $\tau_{\pm}$ implies that $P_{ii}$ oscillate in $\phi$ with period $\phi_0$. The amplitude is subject to the obvious unitarity bound

$$0 \leq P_{ii} \leq \mu_3 - \mu_1.$$

The plots of the shot noise $P_{11}$ in Fig. 11 obtained for $\mu_1 = 1, \mu_2 = 2, \mu_3 = 3$ and $m = 1/2$ confirm the oscillations and the impact of the higher harmonics at $t = 0.8$ (continuous red line).

We study finally the behaviour of the shot noise as a function of the chemical potentials $\mu_i$, or equivalently, the voltages $V_i$ in (35). It is instructive to do this, comparing $P_{ii}$ with the zero-temperature steady current $J_i$ given by (40), and the transmission amplitude $\tau_{\pm}$. For this purpose we fix $m = 1/2, \mu_1 = \mu_2 = d = 1$ and vary $\mu_3$. In this regime

$$J_1(\phi) = -\int_{1}^{\mu_3} \frac{d\xi}{2\pi} \tau_{+}^2 = J_2(-\phi), \quad J_3 = -J_1 - J_2.$$
FIG. 11: (Color online) \( P_{11}(\phi,t) \) and its sections at \( t = 0.3 \) (dashed line) and \( t = 0.8 \) (continuous line).

FIG. 12: (Color online) Plots of \( P_{11} \) (black dashed), \( J_1 \) (blue dotted) and \( \tau^2/2\pi \) (red continuous) at \( t = 0.99 \) for \( \phi = 0 \) (left) and \( \phi = \pi/4 \) (right).

\[
P_{11}(\phi) = \int_1^{\mu_3} \frac{d\xi}{2\pi} \tau^2(1 - \tau^2) = P_{22}(-\phi),
\]
\[
P_{33} = \int_1^{\mu_3} \frac{d\xi}{2\pi} \varrho^2(1 - \varrho^2).
\]

An interesting resonant tunneling effect is observed for \( t \sim 1 \). This corresponds to the situation where the external edges are weakly coupled to the ring. The peaks in the transmission amplitudes \( \tau_+ \), discovered in section III, can be interpreted as resonances corresponding to eigenstates of the ring. A similar situation was discussed in the case of the ring with two external edges. As the voltage is increased, these resonances generate plateaux in the shot noise \( P_1 \) and the current \( J_1 \). This fact is illustrated in Fig. 12, where we plotted \( \tau_-(\sqrt{\mu_3})/2\pi \) (continuous red curve), \( P_{11}(\mu_3) \) (dashed black curve) and \( J_1(\mu_3) \) (dotted blue curve). Switching on the magnetic field changes the location of the peaks and hence the location of the jumps from one plateau to the next.

For a fixed potential \( \mu_3 \), the amount of current flowing in the external edges can vary with the magnetic flux. This is shown in Fig. 13. We fixed \( \mu_3 = 20 \) which is the approximate value where \( J_1 \) jumps from one plateau to the next in Fig. 12 for \( \phi = \pi/4 \), in correspondence with the fourth peak in \( \tau_- \). However, if \( \phi < \pi/4 \) roughly, the peak is moved to the left, so that \( J_1 \) takes on the higher value. If \( \phi > \pi/4 \) roughly, the peak moves to the right and \( J_1 \) takes on the lower value.

B. Dirac junction

The Dirac junction \([6,9]\) can be treated in the same way as the Schrödinger one. For this reason we omit the details and report only the basic equations, discussing the differences. First of all, in the Dirac case both particles (with chemical potential \( \mu_1 \)) and antiparticles (with chemical potential \( \bar{\mu}_1 \)) are present. The relative steady state \( \Omega_{\beta,\mu,\bar{\mu}} \) has been constructed in Ref. 33. The charge transport is associated with the current

\[
j_x(t, x, i) = \langle [ : \psi_1^i \psi_1^\dagger : - : \psi_2^i \psi_2^\dagger : ](t, x, i), \quad (67)
\]

where \( : \cdots : \) indicates the normal product. Introducing the Fermi distributions for particles and antiparticles

\[
f_i(k) = \frac{e^{-\beta_i(|k| - \mu_1)}}{1 + e^{-\beta_i(|k| - \mu_1)}}, \quad \bar{f}_i(k) = \frac{e^{-\beta_i(|k| + \bar{\mu}_1)}}{1 + e^{-\beta_i(|k| + \bar{\mu}_1)}}, \quad (68)
\]

one finds this in the case the following steady current

\[
\langle j_x(t, x, i) \rangle_{\beta,\mu,\bar{\mu}} = \int_0^\infty \frac{dk}{2\pi} \sum_{j=1}^3 \left[ \delta_{ij} - |S_{ij}(k)|^2 \right] [f_j(k) - \bar{f}_j(k)], \quad (69)
\]

conductance

\[
\mathbb{G}_{ij} = \frac{1}{V_j} \int_0^\infty \frac{dk}{2\pi} \left[ \delta_{ij} - |S_{ij}(k)|^2 \right] [f_j(k) - \bar{f}_j(k)], \quad (70)
\]

and zero frequency noise power
\[ P_{ij} = \int_0^\infty \frac{dk}{2\pi} \left\{ \delta_{ij} F_{ii}(k) - |S_{ij}^\phi(k)|^2 F_{jj}(k) - |S_{ji}^\phi(k)|^2 F_{ii}(k) + \frac{1}{2} \sum_{l,m=1}^3 S_{il}^\phi(k) S_{jl}^\phi(k) S_{jm}^\phi(k) S_{im}^\phi(k) [F_{lm}(k) + F_{ml}(k)] \right\}, \]  

where

\[ F_{ij}(k) = f_i(k)[1 - f_j(k)] + \bar{f}_i(k)[1 - \bar{f}_j(k)]. \]  

The helicity transport, which has attracted some attention recently\[^{20}\] is described in our case by

\[ h_x(t, x, i) = [ : \psi_i^\dagger \psi_1 : + : \psi_i^\dagger \psi_2 : ](t, x, i). \]  

Note the change of sign of the second term in the right hand side of (75) with respect to the charge current (67). This sign difference propagates in the helicity steady current, which reads

\[ \langle h_x(t, x, i) \rangle_{\beta, \mu, \bar{\mu}} = \int_0^\infty \frac{dk}{2\pi} \sum_{j=1}^3 \left[ \delta_{ij} + |S_{ij}^\phi(k)|^2 \right] f_j(k) - \bar{f}_j(k). \]  

Unitarity implies

\[ \sum_{i=1}^3 \langle h_x(t, x, i) \rangle_{\beta, \mu, \bar{\mu}} = \frac{1}{\pi} \sum_{i=1}^3 \frac{1}{\beta_i} \ln \left( \frac{1 + e^{\beta_i \mu_i}}{1 + e^{-\beta_i \mu_i}} \right), \]  

which shows that the relative Kirchhoff rule is not automatically satisfied\[^{21}\]. Differently from the electric charge, which is always conserved for a unitary \( \mathcal{S}^\phi \), the helicity is a conserved quantum number in the state \( \Omega_{\beta, \mu, \bar{\mu}} \) if and only if the temperatures and the chemical potentials of the heat baths satisfy

\[ \sum_{i=1}^3 \frac{1}{\beta_i} \ln \left( \frac{1 + e^{\beta_i \mu_i}}{1 + e^{-\beta_i \mu_i}} \right) = 0. \]  

This is for instance the case if \( \mu_i = -\bar{\mu}_i \) with generic \( \beta_i \).

\[ \text{V. OUTLOOK AND CONCLUSIONS} \]

Combining a scattering approach of quantum field theory with results arising from the spectral theory of operators on graphs, we discussed transport properties of fermions on a ring connected to thermal reservoirs. In the bulk the fermions interact only with an external ambient electromagnetic field. At the vertices on the ring we consider the most general scale invariant local interactions which generate unitary time evolution. The formalism applies equally well to the Schrödinger or Dirac case and conceptually uses only the self-adjointness of the relative Hamiltonians on the ring graph in Fig. 2 with magnetic flux \( \phi \).

As is well-known, the crucial ingredient in the scattering approach to transport theory is the scattering matrix of the "sample" under investigation. In our case, this
matrix takes into account all the above mentioned interactions. The systematic method, originally developed in [51], can be generalized [52] to the presence of an external magnetic field and provides us with a method to compute the relative scattering matrix $S^\phi$ explicitly. Hence, all the required analytic structure of reflection and transmission amplitudes is known and its dependence on $\phi$ and the size of the ring is easily incorporated.

With this last ingredient now available, this paper represents a further step in exploring the interesting interplay between spectral theory on graphs and quantum field theory on graphs with applications to condensed matter problems, as initiated in the series of papers [39,43]. The non-equilibrium dynamics, generated by the contact of heat baths with different temperatures and chemical potentials, is captured by steady states $\Omega_{\beta,\mu}$, which incorporate $S^\phi$ the scattering matrix $S^\phi$. The expectation values of the current in $\Omega_{\beta,\mu}$ reproduce the Landauer–Büttiker steady current and noise formulae [52,53,55] in terms of $S^\phi$. Using these formulas, we found a resonant tunneling effect in the case where the ring is weakly coupled to the external leads. The case with two external leads was originally discussed in [57] and we confirmed that the effect takes place also for three leads. In fact, using our general method, it is not hard to see that this effect will persist with more external leads [59], the number of resonances available growing with the number of leads. It is worth mentioning that the resonant tunneling is a finite size effect. This is a reason why it has not been detected in the boundary conformal field theory approach [10,15] to Y-junctions.

The most striking effect, discovered in this paper, is the sharp influence of the magnetic flux on the temperature power law of the thermal noise and the typical signature of the Schrödinger and Dirac cases. The usual Johnson–Nyquist linear behaviour is radically modified at small temperatures to a quadratic (Schrödinger case) and cubic (Dirac case) behaviour as soon as the magnetic flux is switched on. That could not have been seen with previous methods available, since the effect is based on a detailed examination of the limiting behaviour of the fundamental eigenvalue of the scattering matrix $S^\phi$.

As already explained, this paper represents a benchmark where the tools of quantum field theory on graphs have been successfully tested and shown to produce reliable and new results with relatively little effort. This opens the way to a range of future investigations. One is the study of quantum transport in more complicated three-dimensional multiply connected networks. In principle, this can be done as easily as the present case, the essential features being captured by the scattering matrix of these structures. Another important extension is the use of bosonization techniques to describe quantum transport of self-interacting fermions in the bulk of the graph. This has already been implemented successfully for the Tomonaga-Luttinger model on a star-graph [20,21]. The next natural step is to investigate such a system on a ring. In this respect, it is important to realize that the present framework does not rely on conformal symmetry, thus allowing to investigate finite size effects, which is usually problematic with conformal techniques. A quantitative comparison with the exotic phase diagram of the Tomonaga-Luttinger liquid on graphs, studied in Refs. [10,22,24,27,28,30], is an interesting issue for future investigation.

### Appendix A: Critical local $S$-matrices

Using the general expression (3), we derive here all scale-invariant (critical) $3 \times 3$ local $S$-matrices for which the Y-junction in Fig. 2 is invariant under cyclic permutations. Let us denote by $U$ the unitary matrix diagonalizing $U$ and parametrize

$$U_d = U^{-1} U$$

as follows

$$U_d = \text{diag} \left( e^{2i\alpha_1}, e^{2i\alpha_2}, \ldots, e^{2i\alpha_n} \right), \quad \alpha_i \in \mathbb{R}.$$  \hspace{1cm} (A2)

By means of (4), one easily verifies that $U$ diagonalizes also $S(k)$ for any $k$ and that

$$S_d(k) = U^\dagger S(k) U = \text{diag} \left( \frac{k + i\eta_1}{k - i\eta_1}, \frac{k + i\eta_2}{k - i\eta_2}, \ldots, \frac{k + i\eta_n}{k - i\eta_n} \right),$$

where

$$\eta_i = \lambda \tan(\alpha_i), \quad -\frac{\pi}{2} \leq \alpha_i \leq \frac{\pi}{2}.$$  \hspace{1cm} (A4)

The scale invariant (critical) $S$-matrices are $k$-independent and can be obtained by taking the limits $\eta_i \to 0$ or $\eta_i \to \infty$ in (A3). Therefore, the most general scale-invariant scattering matrix, compatible with a unitary time evolution of the Schrödinger junction, is given by

$$S^c = U S_d U^\dagger, \quad S_d = \text{diag}(\pm 1, \pm 1, \ldots \pm 1),$$

where $U$ is a generic $n \times n$ unitary matrix. From the group-theoretical point of view, any $S^c$ is a point in the orbit of some $S_d$ under the adjoint action of the unitary group $U(n)$. Obviously, one can enumerate the edges in such a way that the first $p$ eigenvalues of $S^c$ are $-1$ and the remaining $n-p$ are $+1$. The cases $p = 0$ and $p = n$ correspond to $S^c = I_n$ and $S^c = -I_n$ describing the Neumann and Dirichlet conditions respectively.

The representation implies that besides being unitary, $S^c$ is also Hermitian,

$$\left( S^c \right)^\dagger = S^c.$$  \hspace{1cm} (A6)

Assuming time-reversal symmetry in the individual local junction, according to [5] one has in addition that $S^c$ is symmetric,

$$\left( S^c \right)^\dagger = S^c.$$  \hspace{1cm} (A7)
Combining (A6, A7) with unitarity we conclude that the critical Schrödinger S-matrices, preserving time-reversal invariance, are the symmetric matrices belonging to the orthogonal group $O(n)$.

Let us concentrate finally on the case $n = 3$ (relevant for the Y-junction in Fig. [2]) and let us assume that the internal edges of each local junction are equivalent as far as transmission and reflection are concerned. Labeling these edges by the indices 2 and 3, the matrix elements $\{S_{ij} : i, j = 1, 2, 3\}$ must be invariant under the exchange $2 \leftrightarrow 3$. All these requirements fully determine two one-parameter families in $O(3)$, which can be represented by

$$S^c_{\pm}(t) = \pm \begin{pmatrix} 1 - 2t & \sqrt{2t(1-t)} & \sqrt{2t(1-t)} \\ \sqrt{2t(1-t)} & t - 1 & t \\ \sqrt{2t(1-t)} & t & t - 1 \end{pmatrix},$$

where $t \in [0,1]$ is the transmission coefficient controlling the local tunneling between the edges 2 and 3 of the junction. Since $\det(S^c_{\pm}) = \pm 1$, the matrices (A8) belong to the two disconnected components of $O(3)$ and to the orbits $p = 2$ and $p = 1$ respectively.

The matrix $S^c(t = 1/2)$ has been introduced in Ref. 50. In Ref. 57, the matrices $S^c_{\pm}(t)$ have been considered for generic $t \in [0,1]$. To our knowledge, the above interpretation of $S^c_{\pm}(t)$ as critical points in the set of all scattering matrices, ensuring self-adjointness of the Schrödinger Hamiltonian on a star graph, is new.

Finally, let us observe that $S^c_{\pm} = \left(S^c_{\pm}\right)^{-1}$. Therefore, replacing $U$ in (10) by $S^c_{\pm}$ one concludes that the matrices (A8) describe critical points in the Dirac case as well.

**Appendix B: Comments on the construction of $S^c$**

The main idea of the general method of Ref. 51 to compute the total scattering matrix of an arbitrary quantum graph from the knowledge of its local scattering matrices and of its metric structure can be summarized schematically as follows (for details in the case with an ambient magnetic field see [59]). One collects the elements of the local scattering matrices in four matrices $S^{(\text{in-out})}$, $S^{(\text{out-in})}$, $S^{(\text{out-out})}$ and $S^{(\text{out-in})}$ depending on whether the element connects two internal edges, an internal edge and an external edge or vice-versa, or two external edges. One also collects the information about the metric structure of the graph through its consequences in terms of phases that the propagating modes pick as they travel along internal edges. In the presence of a magnetic field, one must also take into account the phase acquired by the propagating modes along the external edges. All this information is gathered in a matrix $E$. The form of the total scattering matrix is then

$$S = S^{(\text{out-out})} + S^{(\text{out-in})}(E - S^{(\text{in-in})})^{-1}S^{(\text{in-out})},$$

which has a very natural interpretation. An incoming mode on an external edge $j$ is partly reflected straight after scattering off the vertex to which the external edges is attached ($S^{(\text{out-out})}_{jj}$) and partly after entering inside the graph, undergoing all possible internal reflections and being emitted again on the same external edge ($[S^{(\text{out-in})} (E - S^{(\text{in-in})})^{-1}S^{(\text{in-out})}]_{jj}$). The same mode is also partly transmitted to the external edge $\ell$ after all possible internal reflections ($[S^{(\text{out-in})} (E - S^{(\text{in-in})})^{-1}S^{(\text{in-out})}]_{jj}$).

As expected, the symmetries of the graph allow for simplifications in the actual calculation of the total scattering matrix. This is shown explicitly in the case when the vertices form a regular polygon in [59]. In this paper, we deal with a particular case of this general situation where the three vertices form an equilateral triangle. Assuming identical local scattering matrices $S_i(k) \equiv S(k)$ for all $i = 1, 2, 3$, one can show [59] that $S^c$ is the circulant matrix given by [15, 16, 17].

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What is essential is to have a well defined tangent vector field along the edges (embedded in the three dimensional physical space) in order to define the projection (and therefore the action) of the ambient electromagnetic potential along the wires.