ANALOGUE OF WEIL REPRESENTATION FOR ABELIAN SCHEMES

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This paper is devoted to the construction of projective actions of certain arithmetic groups on the derived categories of coherent sheaves on abelian schemes over a normal base \( S \). These actions are constructed by mimicking the construction of Weil in [27] of a projective representation of the symplectic group \( \text{Sp}(V^* \oplus V) \) on the space of smooth functions on the lagrangian subspace \( V \). Namely, we replace the vector space \( V \) by an abelian scheme \( A/S \), the dual vector space \( V^* \) by the dual abelian scheme \( \hat{A} \), and the space of functions on \( V \) by the (bounded) derived category of coherent sheaves on \( A \) which we denote by \( D^b(A) \). The role of the standard symplectic form on \( V \oplus V^* \) is played by the line bundle \( \mathcal{B} = p_{14}^* \mathcal{P} \otimes p_{23}^* \mathcal{P}^{-1} \) on \((\hat{A} \times S A)^2\) where \( \mathcal{P} \) is the normalized Poincaré bundle on \( \hat{A} \times A \). Thus, the symplectic group \( \text{Sp}(V^* \oplus V) \) is replaced by the group of automorphisms of \( \hat{A} \times S A \) preserving \( \mathcal{B} \). We denote the latter group by \( \text{SL}_2(A) \) (in [20] the same group is denoted by \( \text{Sp}(\hat{A} \times S A) \)). We construct an action of a central extension of certain "congruent-subgroup" \( \Gamma(A,2) \subset \text{SL}_2(A) \) on \( D^b(A) \). More precisely, if we write an element of \( \text{SL}_2(A) \) in the block form

\[
\begin{pmatrix}
    a_{11} & a_{12} \\
    a_{21} & a_{22}
\end{pmatrix}
\]

then the subgroup \( \Gamma(A,2) \) is distinguished by the condition that elements \( a_{12} \in \text{Hom}(A, \hat{A}) \) and \( a_{21} \in \text{Hom}(\hat{A}, A) \) are divisible by 2. We construct autoequivalences \( F_g \) of \( D^b(A) \) corresponding to elements \( g \in \Gamma(A,2) \), such the composition \( F_g \circ F_{g'} \) differs from \( F_{gg'} \) by tensoring with a line bundle on \( S \) and a shift in the derived category. Thus, we get an embedding of the central extension of \( \Gamma(A,2) \) by \( \mathbb{Z} \times \text{Pic}(S) \) into the group of autoequivalences of \( D^b(A) \). The 2-cocycle of this central extension is described by structures similar to the Maslov index of a triple of lagrangian subspaces in a symplectic vector space. However, the situation here is more complicated since the construction of the functor \( F_g \) requires a choice of a Schrödinger representation of certain Heisenberg group \( G_g \) associated with \( g \). The latter is a central extension of a finite group scheme \( K_g \) over \( S \) by \( \mathbb{G}_m \) such that the commutation pairing \( K_g \times K_g \to \mathbb{G}_m \) is non-degenerate. If the order of \( K_g \) is \( d^2 \) then a Schrödinger representation of \( G_g \) is a representation of \( G_g \) in a vector bundle of rank \( d \) over \( S \) such that \( \mathbb{G}_m \subset G_g \) acts naturally. Any two such representations differ by tensoring with a line bundle on \( S \). The classical example due to D. Mumford arises in the situation when there is...
a relatively ample line bundle $L$ on an abelian scheme $\pi : A \to S$. Then the vector bundle $\pi_* L$ on $S$ is a Schrödinger representation of the Mumford group $G(L)$ attached to $L$, this group is a central extension of the finite group scheme $K(L)$ by $\mathbb{G}_m$ where $K(L)$ is the kernel of the symmetric homomorphism $\phi : A \to \hat{A}$ associated with $L$. Our Heisenberg groups $G_g$ are subgroups in some Mumford groups and there is no canonical choice of a Schrödinger representation for them in general (this ambiguity is responsible for the $\text{Pic}(S)$-part of our central extension of $\Gamma(A,2)$). Moreover, unless $S$ is a spectrum of an algebraically closed field, it is not even obvious that a Schrödinger representation for $G_g$ exists. Our main technical result that deals with this problem is the existence of a ”Schrödinger representation” for finite symmetric Heisenberg group schemes of odd order established in section 2. Further we observe that the obstacle to existence of such a representation is an element $\delta(G_g)$ in the Brauer group of $S$, and that the map $g \mapsto \delta(G_g)$ is a homomorphism. This allows us to use the theory of arithmetic groups to prove the vanishing of $\delta(G_g)$.

When an abelian scheme $A$ is equipped with some additional structure (such as a symmetric line bundle) one can sometimes extend the above action of a central extension of $\Gamma(A,2)$ on $D^b(A)$ to an action of a bigger group. The following two cases seem to be particularly interesting. Firstly, assume that a pair of line bundles on $A$ and $\hat{A}$ is given such that the composition of the corresponding isogenies between $A$ and $\hat{A}$ is the morphism of multiplication by some integer $N > 0$. Then we can construct an action of a central extension of the principal congruenz-subgroup $\Gamma_0(N) \subset \text{SL}_2(\mathbb{Z})$ on $D^b(A)$ (note that in this situation there is a natural embedding of $\Gamma_0(N)$ into $\text{SL}_2(A)$ but the image is not necessarily contained in $\Gamma(A,2)$). Secondly, assume that we have a symmetric line bundle $L$ on $A$ giving rise to a principal polarization. Then there is a natural embedding of $\text{Sp}_{2n}(\mathbb{Z})$ into $\text{SL}_2(A^n)$, where $A^n$ denotes the $n$-fold fibered product over $S$, which is an isomorphism when $\text{End}(A) = \mathbb{Z}$. In this case we construct an action of a central extension of $\text{Sp}_{2n}(\mathbb{Z})$ on $D^b(A^n)$. The main point in both cases is to show the existence of relevant Schrödinger representations. Both these situations admit natural generalizations to abelian schemes with real multiplication treated in the same way. For example, we consider the case of an abelian scheme $A$ with real multiplication by a ring of integers $R$ in a totally real number field, equipped with a symmetric line bundle $L$ such that $\phi_L : A \to \hat{A}$ is an $R$-linear principal polarization. Then there is an action of a central extension of $\text{Sp}_{2n}(R)$ by $\mathbb{Z} \times \text{Pic}(S)$ on $D^b(A^n)$. When $R = \mathbb{Z}$ we determine this central extension explicitly using a presentation of $\text{Sp}_{2n}(\mathbb{Z})$ by generators and relations. It turns out that the $\text{Pic}(S)$-part of this extension is induced by a non-trivial central extension of $\text{Sp}_{2n}(\mathbb{Z})$ by $\mathbb{Z}/2\mathbb{Z}$ via the embedding $\mathbb{Z}/2\mathbb{Z} \hookrightarrow \text{Pic}(S)$ given by an element $(\pi_* L)^\otimes 4 \otimes \overline{\omega}_A^2$ where $\overline{\omega}_A$ is the restriction of the relative canonical bundle of $A/S$ to the zero section. Also we show that the restriction of this central extension to certain congruenz-subgroup $\Gamma_{1,2} \subset \text{Sp}_{2n}(\mathbb{Z})$ splits.

In the case when $S$ is the spectrum of the algebraically closed field the constructions
of this paper were developed in [22] and [20]. In the latter paper the \( \mathbb{Z} \)-part of the above central extension is described in the analytic situation. Also in [22] we have shown that the above action of an arithmetic group on \( D^b(A) \) can be used to construct an action of the corresponding algebraic group over \( \mathbb{Q} \) on the (ungraded) Chow motive of \( A \). In the present paper we extend this to the case of abelian schemes and their relative Chow motives.

Under the conjectural equivalence of \( D^b(A^n) \) with the Fukaya category of the mirror dual symplectic torus (see [12]) the above projective action of \( \text{Sp}_{2n}(\mathbb{Z}) \) should correspond to a natural geometric action on the Fukaya category. The central extension by \( \mathbb{Z} \) appears in the latter situation due to the fact that objects are lagrangian subvarieties together with some additional data which form a \( \mathbb{Z} \)-torsor (see [12]).

The paper is organized as follows. In the first two sections we study finite Heisenberg group schemes (\textit{non-degenerate theta groups} in the terminology of [14]) and their representations. In particular, we establish a key result (Theorem 2.3) on the existence of a Schrödinger representation for a symmetric Heisenberg group scheme of odd order. In section 3 we consider another analogue of Heisenberg group: the central extension \( H(\hat{A} \times A) \) of \( \hat{A} \times S \) by the Picard groupoid of line bundles on \( S \). We develop an analogue of the classical representation theory of real Heisenberg groups for \( H(\hat{A} \times A) \). Schrödinger representations for finite Heisenberg groups enter into this theory as a key ingredient for the construction of intertwining operators. In section 4 we construct a projective action of \( \Gamma(A, 2) \) on \( D^b(A) \), in section 5 — the corresponding action of an algebraic group over \( \mathbb{Q} \) on the relative Chow motive of \( A \). In section 6 we study the group \( \tilde{\text{SL}}_2(A) \), the extension of \( \text{SL}_2(A) \) which acts on the Heisenberg groupoid. In section 7 we extend the action of \( \Gamma(A, 2) \) to that of a bigger group in the situation of abelian schemes with real multiplication. In section 8 we study the corresponding central extension of \( \text{Sp}_{2n}(\mathbb{Z}) \).

All the schemes in this paper are assumed to be noetherian. The base scheme \( S \) is always assumed to be connected. We denote by \( D^b(X) \) the bounded derived category of coherent sheaves on a scheme \( X \). For a morphism of schemes \( f : X \to Y \) of finite cohomological dimension we denote by \( f_* : D^b(X) \to D^b(Y) \) (resp. \( f^* : D^b(Y) \to D^b(X) \)) the derived functor of direct (resp. inverse) image. For any abelian scheme \( A \) over \( S \) we denote by \( e : S \to A \) the zero section. For an abelian scheme \( A \) (resp. morphism of abelian schemes \( f \)) we denote by \( \hat{A} \) (resp. \( \hat{f} \)) the dual abelian scheme (resp. dual morphism). For every line bundle \( L \) on \( A \) we denote by \( \phi_L : A \to \hat{A} \) the corresponding morphism of abelian schemes (see [17]). When this is reasonable a line bundle on an abelian scheme is tacitly assumed to be rigidified along the zero section (one exception is provided by line bundles pulled back from the base). For every integer \( n \) and a commutative group scheme \( G \) we denote by \( [n] = [n]_G : G \to G \) the multiplication by \( n \) on \( G \), and by \( G_n \subset G \) its kernel. We use freely the notational analogy between sheaves and functions writing in particular \( \mathcal{F}_x = \int_Y \mathcal{G}_{y,x} \, dy, \) where \( x \in X, y \in Y, \mathcal{F} \in D^b(X), \mathcal{G} \in D^b(Y \times X) \), instead of \( \mathcal{F} = p_{2*}(\mathcal{G}) \).
1. Heisenberg group schemes

Let $K$ be a finite flat group scheme over a base scheme $S$. A finite Heisenberg group scheme is a central extension of group schemes

$$0 \to \mathbb{G}_m \to G \xrightarrow{p} K \to 0$$

such that the corresponding commutator form $e : K \times K \to \mathbb{G}_m$ is a perfect pairing. Let $A$ be an abelian scheme over $S$, $L$ be a line bundle on $A$ trivialized along the zero section. Then the group scheme $K(L) = \{ x \in A \mid t^*_x L \simeq L \}$ has a canonical central extension $G(L)$ by $\mathbb{G}_m$ (see [17]). When $K(L)$ is finite, $G(L)$ is a finite Heisenberg group scheme.

A symmetric Heisenberg group scheme is an extension $0 \to \mathbb{G}_m \to G \to K \to 0$ as above together with an isomorphism of central extensions $G \cong [-1]^*G$ (identical on $\mathbb{G}_m$), where $[\cdot]^*G$ is the pull-back of $G$ with respect to the inversion morphism $[-1] : K \to K$. For example, if $L$ is a symmetric line bundle on an abelian scheme $A$ (i.e. $[\cdot]^*L \simeq L$) with a symmetric trivialization along the zero section then $G(L)$ is a symmetric Heisenberg group scheme.

For any integer $n$ we denote by $G^n$ the push-forward of $G$ with respect to the morphism $[n] : \mathbb{G}_m \to \mathbb{G}_m$. For any pair of central extensions $(G_1, G_2)$ of the same group $K$ we denote by $G_1 \otimes G_2$ their sum (given by the sum of the corresponding $\mathbb{G}_m$-torsors). Thus, $G^n \simeq G^\otimes n$. Note that we have a canonical isomorphism of central extensions

$$G^{-1} \simeq [-1]^*G^{op}$$

where $[-1]^*G^{op}$ is the pull-back of the opposite group to $G$ by the inversion morphism $[-1] : K \to K$. In particular, a symmetric extension $G$ is commutative if and only if $G^2$ is trivial.

**Lemma 1.1.** For any integer $n$ there is a canonical isomorphism of central extensions

$$[n]^*G \simeq G^{\frac{n(n+1)}{2}} \otimes [-1]^*G^{\frac{n(n-1)}{2}}$$

where $[n]^*G$ is the pull-back of $G$ with respect to the multiplication by $n$ morphism $[n] : K \to K$. In particular, if $G$ is symmetric then $[n]^*G \simeq G^{m^2}$.

**Proof.** The structure of the central extension $G$ of $K$ by $\mathbb{G}_m$ is equivalent to the following data (see e.g. [3]): a cube structure on $\mathbb{G}_m$-torsor $G$ over $K$ and a trivialization of the corresponding biextension $\Lambda(G) = (p_1 + p_2)^*G \otimes p_1^*G^{-1} \otimes p_2^*G^{-1}$ of $K^2$. Now for any cube structure there is a canonical isomorphism (see [3])

$$[n]^*G \simeq G^{\frac{n(n+1)}{2}} \otimes [-1]^*G^{\frac{n(n-1)}{2}}$$

which is compatible with the natural isomorphism of biextensions

$$([n] \times [n])^*\Lambda(G) \simeq \Lambda(G)^{n^2} \simeq \Lambda(G)^{\frac{n(n+1)}{2}} \otimes ([1] \times [1])^*\Lambda(G)^{\frac{n(n-1)}{2}}.$$
The latter isomorphism is compatible with the trivializations of both sides when $G$ arises from a central extension. 

**Remark.** Locally one can choose a splitting $K \to G$ so that the central extension is given by a 2-cocycle $f : K \times K \to \mathbb{G}_m$. The previous lemma says that for any 2-cocycle $f$ the functions $f(nk, nk')$ and $f(k, k')^{-\frac{n(n+1)}{2}} f(-k, -k')^{-\frac{n(n-1)}{2}}$ differ by a canonical coboundary. In fact this coboundary can be written explicitly in terms of the functions $f(mk, k)$ for various $m \in \mathbb{Z}$.

**Proposition 1.2.** Assume that $K$ is annihilated by an integer $N$. If $N$ is odd then for any Heisenberg group $G \to K$ the central extension $G^N$ is canonically trivial, otherwise $G^{2N}$ is trivial. If $G$ is symmetric and $N$ is odd then $G^N$ (resp. $G^{2N}$ if $N$ is even) is trivial as a symmetric extension.

**Proof.** Combining the previous lemma with (1.2) we get the following isomorphism:

$$[n]^*G \simeq G^{\frac{n(n+1)}{2}} \otimes (G^{op})^{\frac{n(n-1)}{2}} \simeq G^n \otimes (G \otimes G^{op-1})^{\frac{n(n-1)}{2}}.$$  

Now we remark that $G \otimes G^{op-1}$ is given by a trivial $\mathbb{G}_m$-torsor over $K$ with the group law induced by the commutator form $e : K \times K \to \mathbb{G}_m$ considered as 2-cocycle. It remains to note that $e^{\frac{n(n-1)}{2}} = 1$ for $n = 2N$ (resp. for $n = N$ if $N$ is odd). Hence, the triviality of $G^n$ in these cases.

**Corollary 1.3.** Let $G \to K$ be a symmetric Heisenberg group such that the order of $K$ over $S$ is odd. Then the $\mathbb{G}_m$-torsor over $K$ underlying $G$ is trivial.

**Proof.** The isomorphism (1.2) implies that the $\mathbb{G}_m$-torsor over $K$ underlying $G^2$ is trivial. Together with the previous proposition this gives the result.

If $G \to K$ is a (symmetric) Heisenberg group scheme, such that $K$ is annihilated by an integer $N$, $n$ is an integer prime to $N$ then $G^n$ is also a (symmetric) Heisenberg group. When $N$ is odd this group depends only on the residue of $n$ modulo $N$ (due to the triviality of $G^N$).

We call a flat subgroup scheme $I \subset K$ $G$-isotropic if the central extension (1.1) splits over $I$ (in particular, $e|_{I \times I} = 1$). If $\sigma : I \to G$ is the corresponding lifting, then we have the reduced Heisenberg group scheme

$$0 \to \mathbb{G}_m \to p^{-1}(I^\perp)/\sigma(I) \to I^\perp/I \to 0$$

where $I^\perp \subset K$ is the orthogonal complement to $I$ with respect to $e$. If $G$ is a symmetric Heisneberg group, then $I \subset K$ is called symmetrically $G$-isotropic if the restriction of the central extension (1.1) to $I$ can be trivialized as a symmetric extension. If $\sigma : I \to G$ is the corresponding symmetric lifting them the reduced Heisenberg group $p^{-1}(I^\perp)/\sigma(I)$ is also symmetric.
Let us define the Witt group \( WH_{\text{sym}}(S) \) as the group of isomorphism classes of finite symmetric Heisenberg groups over \( S \) modulo the equivalence relation generated by 
\[ [G] \sim [p^{-1}(I^\perp)/\sigma(I)] \]
for a symmetrically \( G \)-isotropic subgroup scheme \( I \subset K \). The (commutative) addition in \( WH_{\text{sym}}(S) \) is defined as follows: if \( G_i \rightarrow K_i \) \((i = 1, 2)\) are Heisenberg groups with commutator forms \( e_i \) then their sum is the central extension
\[
0 \rightarrow \mathbb{G}_m \rightarrow G_1 \times \mathbb{G}_m G_2 \rightarrow K_1 \times K_2 \rightarrow 0
\]
so that the corresponding commutator form on \( K_1 \times K_2 \) is \( e_1 \oplus e_2 \). The neutral element is the class of \( \mathbb{G}_m \) considered as an extension of the trivial group. The inverse element to \([G]\) is \([G^{-1}]\). Indeed, there is a canonical splitting of \( G \times \mathbb{G}_m G^{-1} \rightarrow K \times K \) over the diagonal \( K \subset K \times K \), hence the triviality of \([G] + [G^{-1}]\). We define the order of a finite Heisenberg group scheme \( G \rightarrow K \) over \( S \) to be the order of \( K \) over \( S \) (specializing to a geometric point of \( S \) one can see easily that this number has form \( d^2 \)). Let us denote by \( WH'_{\text{sym}}(S) \) the analogous Witt group of finite Heisenberg group schemes \( G \) over \( S \) of odd order. Let also \( WH(S) \) and \( WH'(S) \) be the analogous groups defined for all (not necessarily symmetric) finite Heisenberg groups over \( S \) (with equivalence relation given by \( G \)-isotropic subgroups).

Remark. Let us denote by \( W(S) \) the Witt group of finite flat group schemes over \( S \) with non-degenerate symplectic \( \mathbb{G}_m \)-valued forms (modulo the equivalence relation given by global isotropic flat subgroup schemes). Let also \( W'(S) \) be the analogous group for group schemes of odd order. Then we have a natural homomorphism \( WH(S) \rightarrow W(S) \) and one can show that the induced map \( WH'_{\text{sym}} \rightarrow W'(S) \) is an isomorphism. This follows essentially from the fact that a finite symmetric Heisenberg group of odd order is determined up to an isomorphism by the corresponding commutator form, also if \( G \rightarrow K \) is a symmetric finite Heisenberg group with the commutator form \( e, I \subset K \) is an isotropic flat subgroup scheme of odd order, then there is a unique symmetric lifting \( I \rightarrow G \).

**Theorem 1.4.** The group \( WH_{\text{sym}}(S) \) (resp. \( WH'_{\text{sym}}(S) \)) is annihilated by 8 (resp. 4).

**Proof.** Let \( G \rightarrow K \) be a symmetric finite Heisenberg group. Assume first that the order \( N \) of \( G \) is odd. Then we can find integers \( m \) and \( n \) such that \( m^2 + n^2 \equiv -1 \mod(N) \). Let \( \alpha \) be an automorphism of \( K \times K \) given by a matrix \( \begin{pmatrix} m & -n \\ n & m \end{pmatrix} \).

Let \( G_1 = G \times \mathbb{G}_m G \) be a Heisenberg extension of \( K \times K \) representing the class \( 2[G] \in WH'_{\text{sym}}(S) \). Then from Lemma 1.1 and Proposition 1.2 we get \( \alpha^*G_1 \simeq G_1^{-1} \), hence \( 2[G] = -2[G] \), i.e. \( 4[G] = 0 \) in \( WH'(S) \).

If \( N \) is even we can apply the similar argument to the 4-th cartesian power of \( G \) and the automorphism of \( K^4 \) given by an integer \( 4 \times 4 \)-matrix \( Z \) such that \( Z^4 = Z \).
(2N − 1) id. Such a matrix can be found by considering the left multiplication by a quaternion a + bi + cj + dk where a^2 + b^2 + c^2 + d^2 = 2N − 1.

2. Schrödinger representations

Let G be a finite Heisenberg group scheme of order d^2 over S. A representation of G of weight 1 is a locally free \( O_S \)-module together with the action of G such that \( \mathbb{G}_m \subset G \) acts by the identity character. We refer to chapter V of [14] for basic facts about such representations. In this section we study the problem of existence of a Schrödinger representation for G, i.e. a weight-1 representation of G of rank d (the minimal possible rank). It is well known that such a representation exists if S is the spectrum of an algebraically closed field (see e.g. [14], V, 2.5.5).

Another example is the following. As we already mentioned one can associate a Schrödinger representation to Theorem V, 2.4.2 of [14] for any weight-1 representation W. Namely, to each finite Heisenberg group scheme G of order m acts by the identity character. We refer to chapter V of [14] for the main result of this section is that for symmetric Heisenberg group schemes of odd order a Schrödinger representation exists after some smooth base change. The basic facts about such representations. In this section we study the problem of existence of a Schrödinger representation for G (called the Schrödinger representations for G) is a Cech 2-cocycle with values in \( L \) on an abelian scheme \( S \). Then \( L \) is a Cech 2-cocycle with values in \( L \). Choose an open covering \( U_i \) such that there exist Schrödinger representations \( V_i \) for G over \( U_i \). For a sufficiently fine covering we have G-isomorphisms \( \phi_{ij} : V_i \to V_j \) on the intersections \( U_i \cap U_j \), and \( \phi_{jk} \phi_{ij} = \alpha_{ijk} \phi_{ik} \) on the triple intersections \( U_i \cap U_j \cap U_k \) for some functions \( \alpha_{ijk} \in \mathcal{O}^*(U_i \cap U_j \cap U_k) \). Then \( \alpha_{ijk} \) is a Cech 2-cocycle with values in \( \mathbb{G}_m \) whose cohomology class \( e(G) \in H^2(S, \mathbb{G}_m) \) doesn’t depend on the choices made. Furthermore, by definition \( e(G) \) is trivial if and only if there exists a global weight-1 representation we are looking for.

Using the language of gerbs (see e.g. [8]) we can rephrase the construction above without fixing an open covering. Namely, to each finite Heisenberg group scheme G we can associate the \( \mathbb{G}_m \)-gerb \( \text{Schr}_G \) on S such that \( \text{Schr}_G(U) \) for an open set \( U \subset S \) is the category of Schrödinger representations for G over U. Then \( \text{Schr}_G \) represents the cohomology class \( e(G) \in H^2(S, \mathbb{G}_m) \).

Notice that the class \( e(G) \) is actually represented by an Azumaya algebra \( \mathcal{A}(G) \) which is defined as follows. Locally, we can choose a Schrödinger representation V for G and put \( \mathcal{A}(G) = \text{End}(V) \). Now for two such representations V and V’ there is a canonical isomorphism of algebras \( \text{End}(V) \simeq \text{End}(V’) \) induced by any G-
isomorphism \( f : V \to V' \) (since any other \( G \)-isomorphism differs from \( f \) by a scalar), hence these local algebras glue together into a global Azumaya algebra \( \mathcal{A}(G) \) of rank \( d^2 \). In particular, \( d \cdot e(G) = 0 \) (see e.g. [9], prop. 1.4).

Now let \( W \) be a global weight-1 representation of \( G \) which is locally free of rank \( l \cdot d \) over \( S \). Then we claim that \( \text{End}_G(W) \) is an Azumaya algebra with the class \(-e(G)\). Indeed, locally we can choose a representation \( V \) of rank \( d \) as above and a \( G \)-isomorphism \( W \simeq V^l \) which induces a local isomorphism \( \text{End}_G(W) \simeq \text{Mat}_l(O) \).

Now we claim that there is a global algebra isomorphism
\[
\mathcal{A}(G) \otimes \text{End}_G(W) \simeq \text{End}(W).
\]
Indeed, we have canonical isomorphism of \( G \)-modules of weight 1 (resp. \(-1\)) \( V \otimes \text{Hom}_G(V,W) \simeq W \) (resp. \( V^* \otimes \text{Hom}_G(V^*,W^*) \simeq W^* \)). Hence, we have a sequence of natural morphisms
\[
\text{End}(W) \simeq W^* \otimes W \simeq V^* \otimes V \otimes \text{Hom}_G(V^*,W^*) \otimes \text{Hom}_G(V,W) \to \\
\text{End}(V) \otimes \text{Hom}_{G \times G}(V^* \otimes V,W^* \otimes W) \to \text{End}(V) \otimes \text{End}_G(W)
\]
— the latter map is obtained by taking the image of the identity section \( id \in V^* \otimes V \) under a \( G \times G \)-morphism \( V^* \otimes V \to W^* \otimes W \). It is easy to see that the composition morphism gives the required isomorphism. This leads to the following statement.

**Proposition 2.1.** For any finite Heisenberg group scheme \( G \) over \( S \) a canonical element \( e(G) \in \text{Br}(S) \) is defined such that \( e(G) \) is trivial if and only if a Schrödinger representation for \( G \) exists. Furthermore, \( d \cdot e(G) = 0 \) where the order of \( G \) is \( d^2 \), and if there exists a weight-1 \( G \) representation which is locally free of rank \( l \cdot d \) over \( S \) then \( l \cdot e(G) = 0 \).

**Proposition 2.2.** The map \( [G] \mapsto e(G) \) defines a homomorphism \( \text{WH}(S) \to \text{Br}(S) \).

**Proof.** First we have to check that if \( I \subset K \) is a \( G \)-isotropic subgroup, \( \bar{I} \subset G \) its lifting, and \( \bar{G} = p^{-1}(\bar{I} \perp) / \bar{I} \) then \( e(\bar{G}) = e(G) \). Indeed, there is a canonical equivalence of \( G \)-gerbs \( \text{Schr}_G \to \text{Schr}_{\bar{G}} \) given by the functor \( V \mapsto V^{\bar{I}} \) where \( V \) is a (local) Schrödinger representation of \( G \). Next if \( G = G_1 \times G_2 \) then for every pair \( (V_1, V_2) \) of weight-1 representations of \( G_1 \) and \( G_2 \) there is a natural structure of weight-1 \( G \)-representation on \( V_1 \otimes V_2 \), hence we get an equivalence of \( G \)-gerbs \( \text{Schr}_{G_1} + \text{Schr}_{G_2} \to \text{Schr}_G \) which implies the equality \( e(G) = e(G_1) + e(G_2) \). At last, the map \( V \to V^* \) induces an equivalence \( \text{Schr}_G^{op} \to \text{Schr}_{G^{-1}} \) so that \( e(G^{-1}) = -e(G) \). \( \square \)

**Theorem 2.3.** Let \( G \) be a symmetric finite Heisenberg group scheme of odd order. Then \( e(G) = 0 \), that is there exists a global Schrödinger representation for \( G \).
Proof. Let \([G] \in WH_{\text{sym}}(S)\) be a class of \(G\) in the Witt group. Then \(4[G] = 0\) by Theorem 1.4, hence \(4e(G) = 0\) by Proposition 2.2. On the other hand, \(d \cdot e(G) = 0\) by Proposition 2.1 where \(d\) is odd, therefore, \(e(G) = 0\). □

Let us give an example of a symmetric finite Heisenberg group scheme of even order without a Schrödinger representation. First let us recall the construction from [23] which associates to a group scheme \(G\) over \(S\) which is a central extension of a finite commutative group scheme \(K\) by \(\mathbb{G}_m\), and a \(K\)-torsor \(E\) over \(S\) a class \(e(G, E) \in H^2(S, \mathbb{G}_m)\). Morally, the map

\[ H^1(S, K) \to H^2(E, \mathbb{G}_m) : E \mapsto e(G, E) \]

is the boundary homomorphism corresponding to the exact sequence

\[ 0 \to \mathbb{G}_m \to G \to K \to 0. \]

To define it consider the category \(C\) of liftings of \(E\) to to a \(G\)-torsor. Locally such a lifting always exists and any two such liftings differ by a \(\mathbb{G}_m\)-torsor. Thus, \(C\) is a \(\mathbb{G}_2\)-gerb over \(S\), and by definition \(e(G, E)\) is the class of \(C\) in \(H^2(S, \mathbb{G}_m)\). Note that \(e(G, E) = 0\) if and only if there exists a \(G\)-equivariant line bundle \(L\) over \(E\), such that \(\mathbb{G}_m \subset G\) acts on \(L\) via the identity character.

A \(K\)-torsor \(E\) defines a commutative group extension \(G_E\) of \(K\) by \(\mathbb{G}_m\) as follows. Choose local trivializations of \(E\) over some covering \((U_i)\) and let \(\alpha_{ij} \in K(U_i \cap U_j)\) be the corresponding 1-cocycle with values in \(K\). Now we glue \(G_E\) from the trivial extensions \(\mathbb{G}_m \times K\) over \(U_i\) by the following transition isomorphisms over \(U_i \cap U_j\):

\[ f_{ij} : \mathbb{G}_m \times K \to \mathbb{G}_m \times K : (\lambda, x) \mapsto (\lambda e(x, \alpha_{ij}), x) \]

where \(e : K \times K \to \mathbb{G}_m\) is the commutator form corresponding to \(G\). It is easy to see that \(G_E\) doesn’t depend on a choice of trivializations. Now we claim that if \(G\) is a Heisenberg group then

\[ e(G, E) = e(G \otimes G_E) - e(G). \]

This is checked by a direct computation with Čech cocycles. Notice that if \(E^2\) is a trivial \(K\)-torsor then \(G_E^2\) is a trivial central extension of \(K\), hence \(G_E\) is a symmetric extension. Thus, if \(G\) is a symmetric Heisenberg group, then \(G \otimes G_E\) is also symmetric. As was shown in [23] the left hand side of (2.1) can be non-trivial. Namely, consider the case when \(S = A\) is a principally polarized abelian variety over an algebraically closed field \(k\) of characteristic \(\neq 2\). Let \(K = A_2 \times A\) considered as a (constant) finite group scheme over \(A\). Then we can consider \(E = A\) as a \(K\)-torsor over \(A\) via the morphism \([2] : A \to A\). Now if \(G \to A_2\) is a Heisenberg extension of \(A_2\) (defined over \(k\)) then we can consider \(G\) as a constant group scheme over \(A\) and the class \(e(G, E)\) is trivial if and only if \(G\) embeds into the Mumford group \(G(L)\) of some line bundle \(L\) over \(A\) (this embedding should be the identity on \(\mathbb{G}_m\)). When \(\text{NS}(A) = \mathbb{Z}\) this means, in particular, that the commutator form \(A_2 \times A_2 \to \mathbb{G}_m\) induced by \(G\) is proportional
to the symplectic form given by the principal polarization. When \( \dim A \geq 2 \) there is a plenty of other symplectic forms on \( A \), hence, \( e(G, E) \) can be non-trivial.

Now we are going to show that one can replace \( A \) by its general point in this example. In other words, we consider the base \( S = \text{Spec}(k(A)) \) where \( k(A) \) is the field of rational functions on \( A \). Then \( E \) gets replaced by \( \text{Spec}(k(A)) \) considered as an \( A_2 \)-torsor over itself corresponding to the Galois extension

\[
[2]^* : k(A) \to k(A) : f \mapsto f(2)
\]

with the Galois group \( A_2 \). Note that the class \( e(G, E) \) for any Heisenberg extension \( G \) of \( A_2 \) by \( k^* \) is annihilated by the pull-back to \( E \), hence, \( e(G, E) \) is represented by the class of Galois cohomology \( H^2(A_2, k(A)^*) \subset \text{Br}(k(A)) \) where \( A_2 \) acts on \( k(A) \) by translation of argument. It is easy to see that this class is the image of the class \( e_G \in H^2(A_2, k^*) \) of the central extension \( G \) under the natural homomorphism \( H^2(A_2, k^*) \to H^2(A_2, k(A)^*) \).

From the exact sequence of groups

\[
0 \to k^* \to k(A)^* \to k(A)^*/k^* \to 0
\]

we get the exact sequence of cohomologies

\[
0 \to H^1(A_2, k(A)^*/k^*) \to H^2(A_2, k^*) \to H^2(A_2, k(A)^*)
\]

(note that \( H^1(A_2, k(A)^*) = 0 \) by Hilbert theorem 90). It follows that central extensions \( G \) of \( A_2 \) by \( k^* \) with trivial \( e(G, E) \) are classified by elements of \( H^1(A_2, k(A)^*/k^*) \).

**Lemma 2.4.** Let \( A \) be a principally polarized abelian variety over an algebraically closed field \( k \) of characteristic \( \neq 2 \). Assume that \( \text{NS}(A) = \mathbb{Z} \). Then \( H^1(A_2, k(A)^*/k^*) = \mathbb{Z}/2\mathbb{Z} \).

**Proof.** Interpreting \( k(A)^*/k^* \) as the group of divisors linearly equivalent to zero we obtain the exact sequence

\[
0 \to k(A)^*/k^* \to \text{Div}(A) \to \text{Pic}(A) \to 0,
\]

where \( \text{Div}(A) \) is the group of all divisors on \( A \). Note that as \( A_2 \)-module \( \text{Div}(A) \) is decomposed into a direct sum of modules of the form \( \mathbb{Z}^{A_2/H} \) where \( H \subset A_2 \) is a subgroup. Now by Shapiro lemma we have \( H^1(A_2, \mathbb{Z}^{A_2/H}) \simeq H^1(H, \mathbb{Z}) \), and the latter group is zero since \( H \) is a torsion group. Hence, \( H^1(A_2, \text{Div}(A)) = 0 \). Thus, from the above exact sequence we get the identification

\[
H^1(A_2, k(A)^*/k^*) \simeq \text{coker}(\text{Div}(A)^{A_2} \to \text{Pic}(A)^{A_2}).
\]

Now we use the exact sequence

\[
0 \to \text{Pic}^0(A) \to \text{Pic}(A) \to \text{NS}(A) \to 0,
\]

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where $\text{Pic}^0(A) = \hat{A}(k)$. Since the actions of $A_2$ on $\text{Pic}^0(A)$ and $\text{NS}(A)$ are trivial we have the induced exact sequence

$$0 \to \text{Pic}^0(A) \to \text{Pic}(A^{A_2}) \to \text{NS}(A).$$

The image of the right arrow is the subgroup $2\text{NS}(A) \subset \text{NS}(A)$. Note that $\text{Pic}^0(A) = [2]^*\text{Pic}^0(A)$, hence this subgroup belongs to the image of $[2]^*\text{Div}(A) \subset \text{Div}(A)^{A_2}$. Thus, we deduce that

$$H^1(A_2, k(A)^*/k^*) \simeq \text{coker}([2]^*\text{Div}(A)^{A_2} \to 2\text{NS}(A)).$$

Let $[L] \subset \text{NS}(A)$ be the generator corresponding to a line bundle $L$ of degree 1 on $A$. Then $L^4 = [2]^*L$, hence $4 \cdot [L] = [L^4]$ belongs to the image of $\text{Div}(A)^{A_2}$. On the other hand, it is easy to see that there is no $A_2$-invariant divisor representing $[L^2]$, hence

$$H^1(A_2, k(A)^*/k^*) \simeq \mathbb{Z}/2\mathbb{Z}.$$

3. Representations of the Heisenberg groupoid

Recall that the Heisenberg group $H(W)$ associated with a symplectic vector space $W$ is a central extension

$$0 \to T \to H(W) \to W \to 0$$

of $W$ by the 1-dimensional torus $T$ with the commutator form $\exp(B(\cdot, \cdot))$ where $B$ is the symplectic form. In this section we consider an analogue of this extension in the context of abelian schemes (see [22], sect. 7, [23]). Namely, we replace a vector space $W$ by an abelian scheme $X/S$. Bilinear forms on $W$ get replaced by biextensions of $X^2$. Recall that a biextension of $X^2$ is a line bundle $\mathcal{L}$ on $X^2$ together with isomorphisms

$$\mathcal{L}_{x+x',y} \simeq \mathcal{L}_{x,y} \otimes \mathcal{L}_{x',y},$$

$$\mathcal{L}_{x,y+y'} \simeq \mathcal{L}_{x,y} \otimes \mathcal{L}_{x,y'}$$

— this is a symbolic notation for isomorphisms $(p_1 + p_2, p_3)^*\mathcal{L} \simeq p_{13}^*\mathcal{L} \otimes p_{23}^*\mathcal{L}$ and $(p_1, p_2 + p_3)^*\mathcal{L} \simeq p_{12}^*\mathcal{L} \otimes p_{13}^*\mathcal{L}$ on $X^3$, satisfying some natural cocycle conditions (see e.g. [3]). The parallel notion to the skew-symmetric form on $W$ is that of a skew-symmetric biextension of $X^2$ which is a biextension $\mathcal{L}$ of $X^2$ together with an isomorphism of biextensions $\phi: \sigma^*\mathcal{L} \Rightarrow \mathcal{L}^{-1}$, where $\sigma : X^2 \to X^2$ is the permutation of...
factors, and a trivialization $\Delta^*\mathcal{L} \simeq \mathcal{O}_X$ of $\mathcal{L}$ over the diagonal $\Delta : X \to X^2$ compatible with $\phi$. A skew-symmetric biextension $\mathcal{L}$ is called \textit{symplectic} if the corresponding homomorphism $\psi_{\mathcal{L}} : X \to \hat{X}$ (where $\hat{X}$ is the dual abelian scheme) is an isomorphism. An \textit{isotropic} subscheme (with respect to $\mathcal{L}$) is an abelian subscheme $Y \subset X$ such that there is an isomorphism of skew-symmetric biextensions $\mathcal{L}|_{Y \times Y} \simeq \mathcal{O}_{Y \times Y}$.

This is equivalent to the condition that the composition $Y \xrightarrow{i} X \xrightarrow{\psi_{\mathcal{L}}} \hat{X} \xrightarrow{\hat{i}} \hat{Y}$ is zero. An isotropic subscheme $Y \subset X$ is called \textit{lagrangian} if the morphism $Y \to \ker(\hat{i})$ induced by $\psi_{\mathcal{L}}$ is an isomorphism. In particular, for such a subscheme the quotient $X/Y$ exists and is isomorphic to $\hat{Y}$.

Note that to define the Heisenberg group extension it is not sufficient to have a symplectic form $B$ on $W$: one needs a bilinear form $B_1$ such that $B(x, y) = B_1(x, y) - B_1(y, x)$. In the case of the real symplectic space one can just take $B_1 = B/2$, however in our situation we have to simply add necessary data. An \textit{enhanced} symplectic biextension $(X, B)$ is a biextension $B$ of $X^2$ such that $L := B \otimes \sigma^*B^{-1}$ is a symplectic biextension. The standard enhanced symplectic biextension for $X = \hat{A} \times A$, where $A$ is any abelian scheme, is obtained by setting

$$B = p_1^*\mathcal{P} \in \text{Pic}(\hat{A} \times A \times \hat{A} \times A),$$

where $\mathcal{P}$ is the normalized Poincaré line bundle on $A \times \hat{A}$.

Given an enhanced symplectic biextension $(X, B)$ one defines the \textit{Heisenberg groupoid} $H(X) = H(X, B)$ as the stack of monoidal groupoids such that $H(X)(S')$ for an $S$-scheme $S'$ is the monoidal groupoid generated by the central subgroupoid $\text{Pic}(S')$ of $\mathbb{G}_m$-torsors on $S'$ and the symbols $T_x, x \in X(S')$ with the composition law

$$T_x \circ T_{x'} = B_{x,x'} T_{x+x'}.$$

The Heisenberg groupoid is a central extension of $X$ by the stack of line bundles on $S$ in the sense of Deligne [4].

In [22] we considered the action of $H(\hat{A} \times A)$ on $\mathcal{D}^b(A)$ which is similar to the standard representation of the Heisenberg group $H(W)$ on functions on a lagrangian subspace of $W$. Below we construct similar representations of the Heisenberg groupoid $H(X)$ associated with lagrangian subschemes in $X$. Further, we construct intertwining functors for two such representations corresponding to a pair of lagrangian subschemes, and consider the analogue of Maslov index for a triple of lagrangian subschemes that arises when composing these intertwining functors.

To define an action of $H(X)$ associated with a lagrangian subscheme one needs some auxiliary data described as follows. An \textit{enhanced} lagrangian subscheme (with respect to $B$) is a pair $(Y, \alpha)$ where $Y \subset X$ is a lagrangian subscheme with respect to $X$, $\alpha$ is a line bundle on $Y$ with a rigidification along the zero section such that an isomorphism of symmetric biextensions $\Lambda(\alpha) \simeq B|_{Y \times Y}$ is given, where $\Lambda(\alpha) =$
\[(p_1 + p_2)^* \alpha \otimes p'^* \alpha^{-1} \otimes p''^* \alpha^{-1}.\] Note that an enhanced lagrangian subscheme is a particular case of an isotropic pair as defined in [22] II, 7.3.

With every enhanced lagrangian subscheme \((Y, \alpha)\) one can associate a representation of \(H(X)(S)\) as follows (see [22],[23]). Let \(D(Y, \alpha)\) be the category of pairs \((\mathcal{F}, a)\) where \(\mathcal{F} \in D^b(X)\), \(a\) is an isomorphism in \(D^b(Y \times X)\):

\[(3.1) \quad a : (i_Y p_1 + p_2)^* \mathcal{F} \cong \mathcal{B}^{-1}|_{Y \times X} \otimes p'^* \alpha^{-1} \otimes p''^* \mathcal{F}\]

where \(i_Y : Y \hookrightarrow X\) is the embedding, such that \((e \times id)^* a = id\). These data should satisfy the following cocycle condition:

\[(p_1 + p_2, p_3)^* a = (p_2, p_3)^* a \circ (p_1, i_Y p_2 + p_3)^* a\]

in \(D^b(Y \times Y \times X)\). Then there is a natural action of the Heisenberg groupoid \(H(X)(S)\) on the category \(D(Y, \alpha)\) such that a line bundle \(M\) on \(S\) acts by tensoring with \(p^* M\) and a generator \(T_x\) acts by the functor

\[(3.2) \quad \mathcal{F} \mapsto B|_{X \times x} \otimes t_x^*(\mathcal{F}).\]

If \(S'\) is an \(S\)-scheme then this action is compatible with the action of \(H(X)(S')\) on \(D(Y_{S'}, \alpha_{S'})\) via pull-back functors.

Let \(\delta_{Y, \alpha} \in D(Y, \alpha)\) be the following object (delta-function at \((Y, \alpha)):\)

\[(3.3) \quad \delta_{Y, \alpha} = i_Y^* (\alpha^{-1})\]

where \(i_Y : Y \to X\) is the embedding. It is easy to see that \(\delta_{Y, \alpha}\) has a canonical structure of an object of \(D(Y, \alpha)\) and for \(y \in Y\) one has \(T_y(\delta_{Y, \alpha}) \simeq \alpha_y^{-1} \delta_{Y, \alpha}\).

Let \((Y, \alpha), (Z, \beta)\) be a pair of enhanced lagrangian subschemes in \(X\), such that \(Y \cap Z\) is finite over \(S\). Then the natural morphism \(Y \to X/Z \simeq \hat{Z}\) is an isogeny, hence, \(Y \cap Z\) is flat over \(S\). Note that we have isomorphisms of biextensions \(\Lambda(\alpha|_{Y \cap Z}) \simeq \Lambda(\beta|_{Y \cap Z}) \cong B|_{(Y \cap Z)^2}\), hence the trivialization of \(\Lambda(\beta|_{Y \cap Z} \otimes \alpha^{-1}|_{Y \cap Z})\). Thus, the \(\mathbb{G}_m\)-torsor \(G_{Y, Z} = \beta|_{Y \cap Z} \otimes \alpha^{-1}|_{Y \cap Z}\) has a natural structure of a central extension of \(Y \cap Z\) by \(\mathbb{G}_m\). Furthermore, the corresponding commutator form \((Y \cap Z)^2 \to \mathbb{G}_m\) is non-degenerate since it corresponds to the canonical duality between \(Y \cap Z = \ker(Y \to \hat{Z})\) and \(Y \cap Z = \ker(Z \to \hat{Y})\) (see [23], remark after Prop. 3.1). Thus, \(G_{Y, Z}\) is a finite Heisenberg group scheme over \(S\). If the line bundles \(\alpha\) and \(\beta\) are symmetric then so is \(G_{Y, Z}\).

Let \(V\) be a Schrödinger representation of \(G_{Y, Z}\). Generalizing the construction of [23] we define the \(H(X)(S)\)-intertwining operator

\[R(V) : D(Y, \alpha) \to D(Z, \beta) : \mathcal{F} \mapsto \text{Hom}_{G_{Y, Z}}(V, p_{2*}(\mathcal{B}|_{Z \times X} \otimes p'^* \beta \otimes (i_Z p_1 + p_2)^* \mathcal{F})).\]

Here \(p_1\) and \(p_2\) are the projections of the product \(Z \times_S X\) onto its factors. The \(G_{Y, Z}\)-module structure on \(p_{2*}(\mathcal{B}|_{Z \times X} \otimes p'^* \beta \otimes (i_Z p_1 + p_2)^* \mathcal{F})\) comes from the natural \(G_{Y, Z}\)-action on \(I(\mathcal{F}) = \mathcal{B}|_{Z \times X} \otimes p'^* \beta \otimes (i_Z p_1 + p_2)^* \mathcal{F}\) which is compatible with the
action of $Y \cap Z$ on $Z \times X$ by the translation of the first argument and arises from the canonical isomorphism

$$I(F)_{(z+u,x)} \simeq \beta_u \alpha_u^{-1} I(F)_{(z,x)}$$

where $z \in Z$, $x \in X$, $u \in Y \cap Z$ (one should consider this as an isomorphism in $\mathcal{D}^b((Y \cap Z) \times Z \times X)$). When $V$ is the representation associated with a Lagrangian subgroup scheme $H \subset G_{Y,Z}$ this functor coincides with the one defined in [23].

Let us call an enhanced Lagrangian subscheme $(Y, \alpha)$ admissible if the projection $X \to X/Y$ splits. For such a subscheme we have an equivalence $\mathcal{D}(Y, \alpha) \simeq \mathcal{D}^b(X/Y)$. Namely, let $s_{X/Y} : X \to Y$ be a splitting of the canonical projection $q_{X/Y} : X \to X/Y$. Let $q_Y = \text{id} - s_{X/Y} q_{X/Y} : X \to Y$ be the corresponding projection to $Y$. Then the functors $\mathcal{F} \mapsto s_{X/Y}^* \mathcal{F}$ and $\mathcal{G} \mapsto (q_Y, s_{X/Y} q_{X/Y})^* \mathcal{B}^{-1} \otimes q_Y^* \alpha^{-1} \otimes s_{X/Y}^* \mathcal{G}$ where $\mathcal{F} \in \mathcal{D}(Y, \alpha)$, $\mathcal{G} \in \mathcal{D}^b(X/Y)$ give the required equivalence. When $(Y, \alpha)$ and $(Z, \beta)$ are both admissible we can represent the above functor $R(V) : \mathcal{D}^b(X/Y) \to \mathcal{D}^b(X/Z)$ in the standard "integral" form.

**Lemma 3.1.** Assume that $(Y, \alpha)$ and $(Z, \beta)$ are admissible, $Y \cap Z$ is finite. Then $R(V)(\mathcal{G}) \simeq p_2_* (p_1^* \mathcal{G} \otimes \mathcal{K}(V))$ where $p_i$ are the projections of $X/Y \times X/Z$ on its factors, $\mathcal{K}(V)$ is the following vector bundle on $X/Y \times X/Z$:

$$\mathcal{K}(V) = (p_1 - q_{X/Y} s_{X/Z} p_2)^* E(V) \otimes \alpha^{-1} \otimes s_{X/Y} q_{X/Y} p_1 - s_{X/Z} p_2, s_{X/Y} p_2)^* \mathcal{B} \otimes (s_{X/Y} (p_1 - q_{X/Y} s_{X/Z} p_2), q_Y s_{X/Z} p_2)^* \mathcal{B} \otimes (q_Y s_{X/Z} p_2)^* \alpha^{-1}$$

where $s_{X/Y} : X/Y \to X$ (resp. $s_{X/Z} : X/Z \to X$) is the splitting of the projection $q_{X/Y} : X \to X/Y$ (resp. $q_{X/Z} : X \to X/Z$), $q_Y = \text{id} - s_{X/Y} q_{X/Y}$, $E(V)$ is the following bundle on $X/Y$:

$$E(V) = \text{Hom}_{G_{Y,Z}}(V, (q_{X/Y} i_Z)_*, (\beta \otimes (q_Y i_Z)^* \alpha^{-1} \otimes (i_Z, s_{X/Y} q_{X/Y} i_Z)^* \mathcal{B}^{-1}))$$

where $i_Y : Y \to X$, $i_Z : Z \to X$ are the embeddings.

**Proof.** By definition we have

$$R(V)(\mathcal{G})_{\bar{x}} \simeq \text{Hom}(V, \int_{\bar{x}} \beta_{\bar{x}} \mathcal{B}_{\bar{x}} \otimes \alpha_{\bar{x}}^{-1} \mathcal{G}_{q_X/Y(z+s_{X/Z}(\bar{x}))})$$

where $\bar{x} \in X/Z$, $z \in Z$. Using the isomorphism $\alpha_{q_Y(z+s_{X/Z}(\bar{x}))} \simeq \alpha_{q_Y(z)} \alpha_{q_Y s_{X/Z}(\bar{x})} \mathcal{B}_{q_Y(z)} q_Y s_{X/Z}(\bar{x})$
and collecting together terms depending only on $\bar{z} = q_{X/Y}(z)$ we get
\[
R(V)(G)_{\bar{z}} \simeq \text{Hom}(V, \int_{\bar{z}} \beta_{z}^{-1} \alpha_{q_{V}(z)}^{-1} \mathcal{B}_{s_{X/Y}(\bar{z}+q_{X/Y} s_{X/Z}(\bar{z}))} \mathcal{B}_{s_{X/Y}(\bar{z}+q_{X/Y} s_{X/Z}(\bar{z}))}^{\alpha_{q_{V}(z)}^{-1}} d\bar{z}) \simeq \\
\int_{X/Y} E(V)_{\bar{z}} \mathcal{B}_{s_{X/Y}(\bar{z}+q_{X/Y} s_{X/Z}(\bar{z}))}^{\alpha_{q_{V}(z)}^{-1}} d\bar{z}
\]

where $\bar{z}$ is now considered as a variable on $X/Y$. Making the change of variables $\bar{z} \mapsto \bar{z} - q_{X/Y} s_{X/Z}(\bar{x})$ we arrive to the formula (3.5). □

**Theorem 3.2.** Assume that $(Y, \alpha)$ and $(Z, \beta)$ are admissible and $Y \cap Z$ is finite, then $R(V)$ is an equivalence of categories. Let $(T, \gamma)$ be an admissible enhanced lagrangian subscheme such that $Y \cap T$ and $Z \cap T$ are finite, $W$ (resp. $U$) be a Schrödinger representation for $G_{Y,T}$ (resp. $G_{Z,T}$). Then

\[
R(U) \circ R(V) \simeq R(W) \otimes M[n]
\]

for some line bundle $M$ on $S$ and some integer $n$.

**Proof.** The direct computation shows that the kernel $K(V) \in \mathcal{D}^{b}(X/Y \times X/Z)$ constructed above satisfies the "uniform" intertwining property (with respect to $H(X)$-action) defined in [22]. Hence, the analogue of Schur lemma for the action of $H(X)$ on $\mathcal{D}^{b}(X/Y)$ where $Y$ is an admissible lagrangian subscheme (see [22] Thm 7.9) implies that

\[
p_{13\ast}(p_{12}^{\ast}K(V) \otimes p_{23}^{\ast}K(V^{\ast})) \simeq \Delta_{\ast}(F)
\]

where $K(V^{\ast})$ is the similar kernel on $X/Z \times X/Y$ giving rise to the functor $R(V^{\ast}) : \mathcal{D}^{b}(X/Z) \to \mathcal{D}^{b}(X/Y)$, $p_{ij}$ are the projections of $X/Y \times X/Z \times X/Y$ on the pairwise products, $\Delta : X/Y \to (X/Y)^{2}$ is the diagonal embedding. In the case when $S$ is the spectrum of a field we know that $F \simeq N[n]$ for some line bundle $N$ on $S$ and some integer $n$ (see [22]). By Prop.1.7 of [16] this implies that the same is true when $S$ is connected. Therefore, in this case the composition $R(V^{\ast}) \circ R(V)$ is isomorphic to the tensoring with $N[n]$. Repeating this for the composition $R(V) \circ R(V^{\ast})$ we conclude that $R(V)$ is an equivalence. Similar argument works for the proof of the second assertion. □

**Remarks.** 1. Most probably, one can extend this theorem to the case of arbitrary enhanced lagrangian subschemes. However, it seems that the definition of $\mathcal{D}(Y, \alpha)$ should be modified in this case (one should start with appropriate category of complexes and then localize it).
2. An integer \( n \) and a line bundle \( M \) on \( S \) appearing in the above theorem should be considered as analogues of the Maslov index (see [13]) for a triple \((Y, Z, T)\). Note that different choices of Schrödinger representations \( V, W, \) and \( U \) above affect \( M \) but not \( n \), hence the function \( n(Y, Z, T) \) behaves very much like the classical Maslov index (cf. [20]).

Let \((Y, \alpha), (Z, \beta)\) and \((T, \gamma)\) be a triple of enhanced lagrangian subschemes in \( X \). Let us denote by \( K = K(Y, Z, T) \) the kernel of the homomorphism \( Y \times Z \times T \to X : (y, z, t) \mapsto y + z + t \). Let \( p_Y : K \to Y, p_Z : K \to Z \) and \( p_T : K \to T \) be the restrictions to \( K \) of the natural projections from \( Y \times Z \times T \) to its factors. Consider the following line bundle on \( K \):

\[
(3.7) \quad M(Y, Z, T) = (-p_Y)^* \alpha^{-1} \otimes p^*_Z \beta \otimes p^*_T \gamma \otimes (p_Z, p_T)^* \mathcal{B}|_{Z \times T}.
\]

Then \( M(Y, Z, T) \) has a canonical cube structure induced by that of \( \alpha, \beta, \gamma \) and \( \mathcal{B} \).

**Lemma 3.3.** There are canonical isomorphisms of line bundles with cube structures on \( K \)

\[
M(Y, Z, T) \simeq M(Z, T, Y) \simeq M(T, Y, Z).
\]

There is a canonical isomorphism of biextensions of \( K \times K \):

\[
\Lambda(M(Y, Z, T)) \simeq (p_Z p_2, p_T p_1)^* \mathcal{L}
\]

where \( p_i \) are the projections of \( K \times K \) on its factors.

**Proof.** We have

\[
(M(Z, T, Y) \otimes M(Y, Z, T)^{-1})_{y,z,t} = \alpha_y \alpha_{-y} \beta_z^{-1} \beta_{-z}^{-1} \mathcal{B}_{t,y} \mathcal{B}_{z,t}^{-1} \simeq \mathcal{B}_{g,y} \mathcal{B}_{g,z}^{-1} \mathcal{B}_{t,y} \mathcal{B}_{t,z}^{-1}
\]

where \( y + z + t = 0 \) (here we used the isomorphism \( \alpha_y \alpha_{-y} \simeq \mathcal{B}_{g,y} \mathcal{B}_{g,-y} \simeq \mathcal{B}_{g,y}^{-1} \) and the similar isomorphism for \( \beta \)). It is easy to see that when we substitute \( t = -y - z \) the right hand side becomes trivial.

The second isomorphism is obtained as follows:

\[
\Lambda(M(Y, Z, T))_{(y,z,t),(y',z',t')} \simeq \mathcal{B}_{-y,-y} \mathcal{B}_{z,z} \mathcal{B}_{t,t} \mathcal{B}_{z',z'} \mathcal{B}_{t,t}^{-1}
\]

If we substitute \( -y = z + t, -y' = z' + t' \) the right hand side becomes \( \mathcal{B}_{t,t}^{-1} \mathcal{B}_{z',z'} \simeq \mathcal{L}_{z',t} \). \( \Box \)

Consider the embedding \( Z \cap T \hookrightarrow K : u \mapsto (0, -u, u) \). Then the previous lemma implies that \( \Lambda(M(Y, Z, T)) \) is trivial over \((Z \cap T) \times K \). Hence, \( M(Y, Z, T)|_{Z \cap T} \) has a structure of central extension and the action of \( Z \cap T \) on \( K \) by translations lifts to an action of this central extension on \( M(Y, Z, T) \). Moreover, we have a canonical isomorphism of central extensions

\[
M(Y, Z, T)_{(0, -u, u)} \simeq \gamma_u \beta_{-u} \mathcal{B}_{-u, u} \simeq \gamma_u \beta_u^{-1} = (G_{Z,T})_u.
\]

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Hence, there is an action of \( G_{Z,T} \) on \( M(Y,Z,T) \) compatible with the action of \( Z \cap T \) on \( K \) by translations. Using cyclic permutation we get embeddings of \( Y \cap T \) and \( Y \cap Z \) into \( K \) and it is easy to see that the images of the three embeddings are independent so that we get an embedding \((Y \cap Z) \times_S (Z \cap T) \times_S (Y \cap T) \hookrightarrow K \) and the compatible action of \( G_{Y,Z} \times_{G_m} G_{Z,T} \times_{G_m} G_{T,Y} \) on \( M(Y,Z,T) \).

**Theorem 3.4.** With the notation and assumptions of Theorem 3.2 we have

\[
M[n] \simeq \text{Hom}_{G_{Y,Z,T}}(V_{Y,Z,T}, p_* M(Y,Z,T))
\]

where \( G_{Y,Z,T} = G_{Y,Z} \times_{G_m} G_{Z,T} \times_{G_m} G_{T,Y} \), \( V_{Y,Z,T} = V \otimes U \otimes W^* \), \( p : K \rightarrow S \) is the projection.

**Proof.** Let us compare the restrictions of \( R(U) \circ R(V)(\delta) \) and \( R(W)(\delta) \) to the zero section, where \( \delta = \delta_{Y,\alpha} \in \mathcal{D}(Y,\alpha) \) is the delta-function at \( Y \) defined by (3.3). On the one hand, we have

\[
R(U) \circ R(V)(\delta)_0 \simeq \text{Hom}_{G_{Y,Z,T}}(V \otimes U, \int_{Z \times T} \gamma_{U,\beta} \mathcal{B}_{z,t} \delta_{z,t} dz dt) \simeq \text{Hom}_{G_{Y,Z,T}}(V \otimes U, \int_K M(Y,Z,T)).
\]

On the other hand,

\[
R(W)(\delta)_0 \simeq \text{Hom}_{G_{Y,T}}(W, \int_{Y \cap T} \gamma_a \alpha^{-1} du) \simeq W^*
\]

since \( f_{Y \cap T} G_{Y,T} \simeq W^* \otimes W \) by [14] V 2.4.2. Therefore,

\[
\int_K M(Y,Z,T) \simeq V \otimes U \otimes W^* \otimes M[n]
\]

as a representation of \( G_{Y,Z,T} \). \( \square \)

Consider the following example. Let \( X = \hat{A} \times A, B = p_{14}^* \mathcal{P}, (Y,\alpha) = (A, \mathcal{O}_A), (T,\gamma) = (\hat{A}, \mathcal{O}_{\hat{A}}), \) and \((Z,\beta) = (Z_{\phi,m},\beta)\) where \( \phi = \phi_L : A \rightarrow \hat{A} \) is the symmetric isogeny associated with a rigidified line bundle \( L \) on \( A \), \( Z_{\phi,m} = (\phi,m \text{id}_A)(A) \simeq A/\ker(\phi_m) \) where \( \phi_m = \phi|_{A_m} \), \( \beta \) is obtained from \( L^m \) by descent (such \( \beta \) always exists if \( m \) is odd, since \( \ker(\phi_m) \) is isotropic with respect to \( e^{L_m} \)). Then \( K(Y,Z,T) \simeq Z, M(Y,Z,T) \simeq \beta, Y \cap T = 0, Y \cap Z \simeq \ker(\phi)/\ker(\phi_n) \), and \( Z \cap T \simeq A_n/\ker(\phi_n) \). Hence, if we take \( W = \mathcal{O}_S \) we get

\[
M[n] \simeq \text{Hom}_{G_{Y,Z,T}}(V \otimes U, p_* \beta).
\]

In particular, when \( m = 1 \) we have \( Z \simeq A, \beta = L, Z \cap T = 0 \), and \( G_{Y,Z} = G(L) \). Thus, if we take \( U = W = \mathcal{O}_S \) we obtain \( M[n] \simeq \text{Hom}_{G(L)}(V, p_* L) \).

Note that if one of the pairwise intersections of \( Y, Z \) and \( T \) is trivial then \( K(Y,Z,T) \) is an abelian scheme over \( S \). More precisely, if say \( Y \cap Z = 0 \) then \( K(Y,Z,T) \simeq T \).
and it is easy to see from the above considerations that in this case we have an isomorphism of Heisenberg groups

\[ G(M(Y, Z, T)) \cong G_{Y,Z} \times_{G_m} G_{Z,T} \times_{G_m} G_{T,Y}. \]  

(3.8)

4. WEIL REPRESENTATION ON THE DERIVED CATEGORY OF AN ABELIAN SCHEME

In this section the base scheme \( S \) is always assumed to noetherian, normal and connected. Let \( K \) denotes the field of rational functions on \( S \).

**Lemma 4.1.** Let \( A \) and \( A' \) be abelian schemes over \( S \), \( A_K \) and \( A'_K \) be their general fibers which are abelian varieties over \( K \). Then the restriction map

\[ \text{Hom}_S(A, A') \to \text{Hom}_K(A_K, A'_K) : f \mapsto f|_K \]

is an isomorphism. The morphism \( f \) is an isogeny if and only if \( f|_K \) is an isogeny.

**Proof.** The proof of the first assertion is similar to the proof of the fact that an abelian scheme over a Dedekind scheme is a Néron model of its generic fiber (see [2] 1.2.8). We have to check that any homomorphism \( f_K : A_K \to A'_K \) extends to a homomorphism \( f : A \to A' \). Let \( \phi : A \to A' \) be the rational map defined by \( f_K \).

Since \( A \) is normal, by a valuative criterion of properness \( \phi \) is defined in codimension \( \leq 1 \). Let \( V \subset A \) be a non-empty subscheme over which \( \phi \) is defined. Then since the projection \( p : A \to S \) is flat and of finite presentation it is open. Thus, \( U = p(V) \) is open, and \( \phi_U : A_U \to A'_U \) is a \( U \)-rational map in the terminology of [2]. By Weil’s theorem (see [2] 4.4.1) \( \phi_U \) is defined everywhere, hence we get a homomorphism \( f_U : A_U \to A'_U \) extending \( f_K \). It remains to invoke Prop. I 2.9 of [6] to finish the proof.

The part concerning isogenies can be proven by exactly the same argument as in [2] 7.3 prop. 6: starting with an isogeny \( f|_K : A_K \to A'_K \) we can find another isogeny \( g|_K : A'_K \to A_K \) such that the composition \( g|_K f|_K \) is the multiplication by an integer \( l \) on \( A_K \). By the first part we can extend \( g_K \) to a homomorphism \( g : A' \to A \). This implies that \( gf = l_A \) — the multiplication by \( l \) morphism on \( A \). It follows that the restriction of \( f \) to each fiber is an isogeny, hence \( f \) is an isogeny itself.

For an abelian scheme \( A \) the group \( \text{SL}_2(A) \) is defined as the subgroup of automorphisms of \( \hat{A} \times A \) preserving the line bundle \( p_{14}^* \mathcal{P} \otimes p_{23}^* \mathcal{P}^{-1} \) on \( (\hat{A} \times A)^2 \). More explicitly, if we write an automorphism of \( \hat{A} \times A \) as a matrix \( g = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \) where \( a_{11} \in \text{Hom}(\hat{A}, \hat{A}) \), \( a_{12} \in \text{Hom}(A, \hat{A}) \) etc., then \( g \in \text{SL}_2(A) \) if and only if the inverse automorphism \( g^{-1} \) is given by the matrix \( \begin{pmatrix} \tilde{a}_{22} & -\tilde{a}_{12} \\ -\tilde{a}_{21} & \tilde{a}_{11} \end{pmatrix} \). It follows from Lemma 4.1 that when the base \( S \) is normal we have \( \text{SL}_2(A) \cong \text{SL}_2(A_K) \).
Now similarly to the classical picture one has to consider the group of automorphisms of the Heisenberg extension $H(X)$ corresponding to $X = \hat{A} \times A$ with the structure of enhanced symplectic biextension given by $B = p_1^* \mathcal{P}$. Namely, we define $\tilde{\text{SL}}_2(A)$ as the group of triples $g = (\tilde{g}, L^g, M^g)$ where $\mathcal{F} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \text{SL}_2(A)$, $L^g$ (resp. $M^g$) is a line bundle on $\hat{A}$ (resp. $A$) rigidified along the zero section, such that

$$\phi_{L^g} = \hat{a}_{11}a_{21}, \quad \phi_{M^g} = \hat{a}_{22}a_{12},$$

where for a line bundle $L$ on an abelian scheme $B$ we denote by $\phi_L : B \to \hat{B}$ the symmetric homomorphism corresponding to the symmetric biextension $\Lambda(L)$. The group law on $\tilde{\text{SL}}_2(A)$ is defined uniquely from the condition that the there is an action of $\tilde{\text{SL}}_2(A)$ on the stack of Picard groupoids $\mathcal{H}(A)$ such that an element $g = (\tilde{g}, L^g, M^g)$ acts by the functor which is identical on $\text{Pic}$ and sends the generator $T_{(x,y)}$ (where $(x, y) \in \hat{A} \times A$) to $L_x \otimes M_y \otimes \mathcal{P}_{(a_{12}y, a_{21}a)} \otimes T_{\tilde{g}(x,y)}$. We refer to [22] for explicit formulas for the group law in $\text{SL}_2(A)$. It is easy to see that the natural projection $\tilde{\text{SL}}_2(A) \to \text{SL}_2(A)$ is a homomorphism with the kernel isomorphic to $A(S) \times \hat{A}(S)$.

Consider the subgroup $\tilde{\text{SL}}_2(A) \subset \text{SL}_2(A)$ consisting of triples with symmetric $L^g$ and $M^g$. Then we have an isomorphism $\tilde{\text{SL}}_2(A) \simeq \text{SL}_2(A_K)$ since any symmetric line bundle on $A_K$ extends to a symmetric line bundle on $A$ (see [14], II.3.3).

Let $\Gamma(A) = \Gamma(A_K)$ be the image of the projection $\tilde{\text{SL}}_2(A_K) \to \text{SL}_2(A_K)$. Then $\Gamma(A)$ has finite index in $\text{SL}_2(A_K)$ since it contains the subgroup $\Gamma(A, 2) = \Gamma(A_K, 2) \subset \text{SL}_2(A_K)$ consisting of matrices with $a_{12}$ and $a_{21}$ divisible by 2.

In the case when $S$ is the spectrum of an algebraically closed field it was shown in [22] that there exist intertwining functors between representations of Heisenberg groupoid corresponding to the natural action of $\tilde{\text{SL}}_2(A)$ on the Heisenberg groupoid $H(\hat{A} \times A)$ which are analogous to the operators of Weil-Shale representation. We are going to extend this construction to the case of a normal base scheme.

Recall (see [22], sect. 10) that there is a natural action of $\tilde{\text{SL}}_2(A)$ on the set of enhanced lagrangian subvarieties in $X = \hat{A} \times A$ such that a triple $g = (\tilde{g}, L^g, M^g) \in \tilde{\text{SL}}_2(A)$ maps $(\hat{A}, \mathcal{O}_\hat{A})$ to $\tilde{g}(\hat{A}) = (a_{11}, a_{21})(\hat{A})$ with the line bundle corresponding to $L^g \in \text{Pic}(\hat{A})$. Furthermore, there is a natural equivalence of categories

$$\mathcal{F}_* : \mathcal{D}(\hat{A}, \mathcal{O}_\hat{A}) \to \mathcal{D}(\mathcal{F}(\hat{A}), L^g) : \mathcal{F} \mapsto \mathcal{F}_* \mathcal{F}$$

such that the standard $H(X)$-action on $\mathcal{D}(\hat{A}, \mathcal{O}_\hat{A})$ corresponds to the $g$-twisted $H(X)$-action on $\mathcal{D}(\mathcal{F}(\hat{A}), L^g)$. On the other hand, if $\hat{A} \cap \mathcal{F}(\hat{A}) = \ker(a_{21})$ is finite (hence, flat) over $S$ and there exists a Schrödinger representation $V$ for the corresponding Heisenberg extension $G_g := G_{\hat{A}, \mathcal{F}(\hat{A})}$ of $\ker(a_{21})$ then the construction of the previous
section gives another equivalence

$$D(\mathcal{F}(\hat{A}), L^g)) \to D(\hat{A}, \mathcal{O})$$

compatible with the standard $H(X)$-actions. Composing it with the previous equivalence we get an equivalence $\rho \Rightarrow \rho^g$ where $\rho$ is the representation of $H(X)$ on $D(\hat{A}, \mathcal{O}) \simeq D(A)$ given by (3.2), $\rho^g = \rho \circ g$ is the same representation twisted by $g$. Using (3.5) it is easy to compute that the kernel on $A \times_S A$ corresponding to this equivalence has form

$$(4.2) \quad \mathcal{K}(g, V) = (p_2 - a_{22}p_1)^* E \otimes (a_{12} \times \text{id})^* \mathcal{P}^{-1} \otimes (-p_1)^* M^g$$

where

$$E = \text{Hom}_{G_g^{-1}}(V^*, a_{21*}(\mathcal{L}^g)^{-1}))$$

Note that here $G_g^{-1}$ is the restriction of the Mumford’s extension $G((\mathcal{L}^g)^{-1}) \to \ker(a_{11}a_{21})$ to $\ker(a_{21})$.

Hence, if $\ker(a_{21})$ is finite over $S$ we get a functor from the gerb of Schrödinger representations for $G_g$ to the stack of $H(X)$-equivalences $\text{Isom}_{H(X)}(\rho, \rho^g)$. More precisely, the category of intertwining operators between $\rho$ and $\rho^g$ is defined in terms of kernels in $D^b(A \times_S A)$ (see [22]) and the glueing property is satisfied because the kernels corresponding to equivalences are actually vector bundles (perhaps, shifted). Indeed, the latter property is local with respect to the $\text{fppf}$ topology on the base $S$ and locally a Schrödinger representation for $G_g$ exists and give rise to the kernel (4.2) in $\text{Isom}_{H(X)}(\rho, \rho^g)$ which is a vector bundle up to shift. Now any other object of $\text{Isom}_{H(X)}(\rho, \rho^g)$ is obtained from a given one by tensoring with a line bundle on $S$ and a shift.

Thus, when $\ker(a_{21})$ is finite the obstacle for the existence of a global equivalence between $\rho$ and $\rho^g$ is given by the class $e(G_g) \in \text{Br}(S)$. Let $U \subset SL_2(A)$ be the subset of matrices such that $a_{21}$ is an isogeny. It turns out that similarly to the case of real groups one can deal with $U$ instead of the entire group when defining representation of $SL_2(A)$. This observation can be formalized as follows. Let us call a subset $B$ of a group $G$ big if for any triple of elements $g_1, g_2, g_3 \in G$ the intersection $B^{-1} \cap Bg_1 \cap Bg_2 \cap Bg_3$ is non-empty. This condition first appeared in [27] IV. 42, while the term is due to D. Kazhdan. The reason for introducing this notion is the following lemma.

**Lemma 4.2.** Let $B \subset G$ be a big subset. Then $G$ is isomorphic to the abstract group generated by elements $[b]$ for $b \in B$ modulo the relations $[b_1][b_2] = [b]$ when $b, b_1, b_2 \in B$ and $b = b_1b_2$. If $c : G \times G \to C$ is a 2-cocycle (where $C$ is an abelian group with the trivial $G$-action) such that $c(b_1, b_2) = 0$ whenever $b_1, b_2, b_1b_2 \in B$ then $c$ is a coboundary.
Proof. For the proof of the first statement we refer to [27], IV, 42, Lem. 6. Let \( c : G \times G \rightarrow C \) be a 2-cocycle, \( H \) be the corresponding central extension of \( G \) by \( C \). Consider the group \( \widetilde{H} \) generated by the central subgroup \( C \) and generators \([b]\) for \( b \in B \) subject to relations \([b_1][b_2] = c(b_1, b_2)[b_1 b_2]\), where \( b_1, b_2, b_1 b_2 \in B \). Then \( \widetilde{H}/C \cong G \), hence the natural homomorphism \( \widetilde{H} \rightarrow H \) is an isomorphism. If \( c(b_1, b_2) = 0 \) whenever \( b_1, b_2, b_1 b_2 \in B \), then the extension \( \widetilde{H} \cong H \rightarrow G \) splits, hence \( c \) is a coboundary. \( \square \)

At this point we need to recall some results from [22], sect. 9 concerning the group \( \text{SL}_2(A) \). Since \( \text{SL}_2(A) = \text{SL}_2(A_K) \) we can work with abelian varieties over a field. First note that this group can be considered as a group of \( \mathbb{Z} \)-points of an algebraic group \( \text{SL}_2 \). It turns out that the corresponding algebraic group \( \text{SL}_2, A, \mathbb{Q} \) over \( \mathbb{Q} \) is very close to be semi-simple. Namely, if we fix a polarization on \( A \) then the latter group is completely determined by the algebra \( \text{End}(A) \otimes \mathbb{Q} \) and the Rosati involution on it. Decomposing (up to isogeny) \( A \) into a product \( A_1^{n_1} \times \ldots A_l^{n_l} \) where \( A_i \) are different simple abelian varieties and choosing a polarization compatible with this decomposition it is easy to see that

\[
\text{SL}_2, A, \mathbb{Q} \cong \prod_i \text{R}_{K_i, 0/Q} U_{2n_i, F_i}^\ast
\]

where \( F_i = \text{End}(A_i) \otimes \mathbb{Q} \), \( K_i \) is the center of \( F_i \), \( K_{i, 0} \subset K_i \) is the subfield of elements stable under the Rosati involution, \( U_{2n_i, F_i}^\ast \) is the group of \( F_i \)-automorphisms of \( F_i^{2n_i} \) preserving the standard skew-hermitian form, \( \text{R}_{K_i, 0/Q} \) denotes the restriction of scalars from \( K_{i, 0} \) to \( \mathbb{Q} \). Thus, the only case when the group \( U_{2n_i, F_i}^\ast \) is not semi-simple is when the Rosati involution on \( F_i \) is of the second kind, i.e. \( K_{i, 0} \neq K_i \). In the latter case, \( U_{2n_i, F_i}^\ast \) is a product of the semi-simple subgroup \( \text{SU}_{2n_i, F_i}^\ast \) (defined using the determinant with values in \( K_i \)) and the central subgroup \( K_i^0 = \{ x \in K_i | N_{K_i/K_{i, 0}} = 1 \} \) consisting of diagonal matrices. Furthermore, the intersections of these two subgroup is finite. It follows that the group \( \text{SL}_2, A, \mathbb{Q} \) always has an almost direct decomposition into a product of the semi-simple subgroup \( H = \prod_i \text{R}_{K_i, 0/Q} U_{2n_i, F_i}^\ast \) and a central subgroup \( Z \) consisting of diagonal matrices. Now we can prove the following result.

**Lemma 4.3.** Let \( \Gamma \subset \text{SL}_2(A) \) be a subgroup of finite index. Then the subset \( \Gamma \cap U \) is big.

**Proof.** By definition \( U \) is an intersection of a Zariski open subset \( U \) in the irreducible algebraic group \( \text{SL}_2, A_K, \mathbb{Q} \) with \( \Gamma \). If \( \Gamma \) were Zariski dense in \( \text{SL}_2, A_K, \mathbb{Q} \) the proof would be finished. This is not always true, however, we claim that \( \Gamma \cap H \) is dense in \( H \) where \( H \) is the subgroup of \( \text{SL}_2, A_K, \mathbb{Q} \) introduced above. Indeed, this follows from the fact that \( H \) is semi-simple and the corresponding real groups have no compact factors (see [22] 9.4). On the other hand, the set \( U \) is \( Z \)-invariant (since \( Z \) consists of diagonal matrices). It follows that for any \( g \in \Gamma \) the intersection \( U g \cap H \) is a
non-empty Zariski open subset of $H$ (as a preimage of a non-empty Zariski set under the isogeny $H \rightarrow \text{SL}_2(A_{K,Q}/Z)$). Therefore, for any triple of elements $g_1, g_2, g_3 \in \Gamma$ the intersection $U \cap U g_1 \cap U g_2 \cap U g_3 \cap H$ is a non-empty open subset in $H$, hence it contains an element of $\Gamma \cap H$. As $U = U^{-1}$ this shows that $U$ is big. \[\square\]

**Proposition 4.4.** There exists a homomorphism $\delta_A : \Gamma(A) \rightarrow \text{Br}(S)$ such that for any $g \in \hat{\text{SL}}_2(A)$ lying over $\overline{g} \in \Gamma(A)$ there exists a global object in $\text{Isom}^0_{H(X)}(\rho, \rho^g)$ if and only if $\delta_A(\overline{g}) = 0$. If the $a_{21}$-entry of $\overline{g}$ is an isogeny then $\delta_A(\overline{g}) = e(G_g)$.

**Proof.** For any $g \in \hat{\text{SL}}_2(A)$ let us denote by $\text{Isom}^0_{H(X)}(\rho, \rho^g) \subset \text{Isom}_{H(X)}(\rho, \rho^g)$ the full subcategory of kernels that belong to the core of the standard $t$-structure on $\mathcal{D}^b(A \times_S A)$. We claim that $\text{Isom}^0_{H(X)}(\rho, \rho^g)$ is a gerb. Indeed, we know this when $\overline{g} \in \Gamma(A) \cap U$. Now by Lemma 4.3 any element in $\hat{\text{SL}}_2(A)$ can be written as a product $gg'$ where $g$ and $g'$ lie over $\Gamma(A) \cap U$. Locally over $S$ there exist Schrödinger representations for $G_g$ and $G_{g'}$, hence (4.2) defines the corresponding kernels $K(g) \in \text{Isom}_{H(X)}(\rho, \rho^g)$ and $K(g') \in \text{Isom}_{H(X)}(\rho, \rho^{g'})$. Now their composition $K(gg') = p_{13} \circ (p_{12}K(g') \otimes p_{23}K(g))$ is an element of $\text{Isom}_{H(X)}(\rho, \rho^{gg'})$ and Prop. 1.7 of [16] implies that $K(gg')$ has only one non-zero sheaf cohomology which is flat over $S$ (since this is so in the case of an algebraically closed field considered in [22]). It follows that any object of $\text{Isom}_{H(X)}(\rho, \rho^{gg'})$ over any open subset of $U$ is a pure $S$-flat sheaf (perhaps shifted), hence, the gluing axiom is satisfied.

Now we can define $\delta_A(\overline{g}) \in H^2(S, \mathbb{G}_m)$ for an element $\overline{g} \in \Gamma(A)$ as the class of the gerb $\text{Isom}^0_{H(X)}(\rho, \rho^g)$ where $g \in \hat{\text{SL}}_2(A)$ is any element lying over $\overline{g}$. When $\overline{g} \in U$ this class is equal to $e(G_g) \in \text{Br}(S)$. Clearly, $\delta_A$ is a homomorphism $\Gamma(A) \rightarrow H^2(S, \mathbb{G}_m)$. Since $\Gamma(A)$ is generated by $\Gamma(A) \cap U$ we have $\delta_A(\overline{g}) \in \text{Br}(S)$ for any $\overline{g} \in \Gamma(A)$. \[\square\]

**Lemma 4.5.** Let $g = (\overline{g}, L^g, M^g)$ be an element of $\hat{\text{SL}}_2(A)$ such that $\ker(a_{21})$ is flat over $S$, its neutral component $\ker(a_{21})^0$ is an abelian subscheme of $\hat{A}$, and $L^g|_{\ker(a_{21})^0}$ is trivial. Then $\delta_A(\overline{g}) = 0$ if and only if there exists a Schrödinger representation for the Heisenberg extension $G_g$ of $\pi_0(\ker(a_{21})) = \ker(a_{21})/(\ker(a_{21}))^0$ induced by $G(L^g)|_{\ker(a_{21})}$.

**Proof.** Let $V$ be a Schrödinger representation for $G_g$. Then we can define $K(g, V)$ by the formula (4.2) where $E$ is defined as follows. First we descend $L^g$ to a line bundle $\mathcal{L}$ on $\hat{A}/\ker(a_{21})^0$, then we set

\[E = \text{Hom}_{G_g^1}(V^*, \mathcal{L}^{-1}))\]

where $\mathcal{L} = \hat{A}/\ker(a_{21})^0 \rightarrow A$ is the finite map induced by $a_{21}$. Note that when $S$ is the spectrum of an algebraically closed field this kernel $K(g, V)$ coincides with the
one defined in [22], 12.3. By definition \( K(g, V) \) is the direct image of a bundle on an abelian subscheme

\[
\text{supp } K(g, V) = \{(x_1, x_2) \in A^2 \mid x_2 - a_{22} x_1 \in a_{21}(\hat{A})\}
\]

(note that \( a_{21}(\hat{A}) \subset A \) is an abelian subscheme since \( \ker(a_{21}) \) is flat). Applying this to \( g^{-1} \) we get

\[
\text{supp } K(g^{-1}, V^*) = \{(x_1, x_2) \in A^2 \mid x_2 - \hat{a}_{11} x_1 \in \hat{a}_{21}(\hat{A})\}.
\]

Hence, the sheaf \( p_1^* K(g^{-1}, V^*) \otimes p_2^* K(g, V) \) on \( A^3 \) is supported on the abelian subscheme

\[
X(g) = \{(x_1, x_2, x_3) \in A^3 \mid x_3 - a_{22} x_2 \in a_{21}(\hat{A}), x_2 - \hat{a}_{11} x_1 \in \hat{a}_{21}(\hat{A})\}.
\]

Note that for \((x_1, x_2, x_3) \in X(g)\) we have

\[
x_3 - a_{22} \hat{a}_{11} x_1 \in a_{21}(\hat{A}) + a_{22} \hat{a}_{21}(\hat{A}) = a_{21}(\hat{A})
\]

since \( a_{22} \hat{a}_{21} = a_{21} \hat{a}_{22} \). Now \( a_{22} \hat{a}_{11} x_1 \equiv x_1 \mod(a_{21}(\hat{A})) \), hence \( x_3 \equiv x_1 \mod(a_{21}(\hat{A})) \). Thus, we have an isomorphism

\[
X(g) \rightarrow \{(x, x') \mid x - x' \in a_{21}(\hat{A})\} \times \hat{a}_{21}(\hat{A}) : \\
(x_1, x_2, x_3) \mapsto ((x_1, x_3), x_2 - \hat{a}_{11} x_1).
\]

It follows that the restriction of the projection \( p_{13} \) to \( X(g) \) is flat and surjective. Therefore, applying Prop. 1.7 of [16] we conclude as in the proof of Theorem 3.2 that \( \overline{T}(V) \) is an equivalence. Thus, the map \( V \mapsto K(g, V) \) gives a functor from \( \text{Schr}_{G_g} \) to \( \text{Isom}_{H(X)}(\rho, \rho^g) \).

\[ \square \]

**Proposition 4.6.** Let \( A \) and \( B \) be abelian schemes over the same base \( S \), then there is a natural embedding \( i_A : \Gamma(A) \to \Gamma(A \times B) \) such that

\[
\delta_A = \delta_{A \times B} \circ i_A.
\]

**Proof.** It is sufficient to check this identity on elements of \( \Gamma(A) \cap U \), in which case this follows immediately from Lemma 4.5. \( \square \)

Consider the subgroup of finite index \( \Gamma_0(A) \subset \Gamma(A) \) consisting of matrices

\[
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
\]

in \( \Gamma(A) \) for which \( a_{21} \) is divisible by 2. Note that \( \Gamma_0(A) \) contains \( \Gamma(A, 2) \).

**Lemma 4.7.** Let \( A \) be an abelian variety over a field such that there exists a symmetric line bundle \( L \) on \( A \) which induces an isogeny \( f : A \to \hat{A} \) of odd degree. Then the subgroup of \( \Gamma(A) \) generated by its elements

\[
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}, \text{ for which } a_{21} \text{ is an isogeny of odd degree, contains } \Gamma_0(A).
\]

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Proof. Recall that $U \subset \text{SL}_2(A)$ is the subset defined by the condition that $a_{21}$ is an isogeny. Consider the matrix $\gamma_f = \begin{pmatrix} \text{id} & 0 \\ f & \text{id} \end{pmatrix} \in \Gamma(A)$. Let $U_1 = U \cap U \gamma_f^{-1}$. Then the argument similar to that of Lemma 4.3 shows that $\Gamma_0(A) \cap U_1$ is a big subset in $\Gamma_0(A)$, in particular, $\Gamma_0(A)$ is generated by $\Gamma_0(A) \cap U_1$. Now let $\gamma \in \Gamma_0(A) \cap U_1$, then $\gamma \gamma_f \in U$ and its $a_{21}$-entry is an isogeny of odd degree which implies the statement. 

**Proposition 4.8.** The restriction of the homomorphism $\delta_A : \Gamma(A) \to \text{Br}(S)$ to $\Gamma_0(A)$ is trivial.

**Proof.** Recall that if $a_{21}$ is an isogeny then $\delta_A(g)$ is defined as an obstacle for the existence of a Schrödinger representation of the (symmetric) Heisenberg extension $G_g$ of $\ker(a_{21})$ attached to $g$, where $g$ projects to a matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \text{SL}_2(A)$. Hence, by Theorem 2.3 $\delta_A(g) = 0$ if $a_{21}$ is an isogeny of odd degree. In particular, if $A$ is principally polarized then Lemma 4.7 implies that the restriction of $\delta_A$ to $\Gamma_0(A)$ is trivial.

By Zarhin's trick (see [28]) for any abelian scheme $A$ over $S$ there exists an abelian scheme $B$ over $S$ such that $A \times B$ admits a principal polarization. Now by Proposition 4.6 we have $\delta_A = \delta_{A \times B} \circ i_A$. Therefore, the restriction of $\delta_A$ to $\Gamma_0(A)$ is trivial.

**Remark.** It is easy to see that the kernel of $\delta_A$ is in general bigger than $\Gamma_0(A)$. Namely, it contains also matrices for which $a_{21}$ (or $a_{12}$) is an isogeny of odd degree, those for which $a_{12}$ is divisible by 2, and those for which $a_{11}, a_{22} \in \mathbb{Z}$. Sometimes, these elements together with $\Gamma_0(A)$ generate the entire group $\Gamma(A)$, however, it is not clear whether $\delta_A$ is always trivial. In the section 7 we will prove it in some special cases.

Let $\hat{\Gamma}_0(A)$ be the preimage of $\Gamma_0(A)$ in $\hat{\text{SL}}_2(A)$. In other words, this is the subgroup of elements $g \in \hat{\text{SL}}_2(A)$ such that for the corresponding matrix in $\text{SL}_2(A)$ the $a_{21}$-entry is divisible by 2. We say that there is a faithful action of a group $G$ on a category $C$ if there is an embedding of $G$ into a group of autoequivalences of $C$ (considered up to isomorphism).

**Theorem 4.9.** For any abelian scheme $A$ over a normal connected noetherian base $S$ there is a faithful action of a central extension of the group $\hat{\Gamma}_0(A)$ by $\mathbb{Z} \times \text{Pic}(S)$ on $\mathcal{D}^b(A)$.

**Proof.** According to Proposition 4.8 for every $g \in \hat{\Gamma}_0(A)$ there exists a global object in $\text{Isom}_{H(X)}(\rho, \rho^g)$. It is defined uniquely up to a shift and tensoring with a line bundle on $S$. Hence, the required action of a central extension. The fact that this action is faithful is clear in the case when the base is a field: for example, one can
use explicit formulas for these functors from Lemma 4.5. Since the action of \( \text{Pic}(S) \) on \( \mathcal{D}^b(A) \) is obviously faithful the general case follows.

\[ \textbf{Corollary 4.10.} \text{With the assumptions of the above theorem, there is a faithful action of a central extension of } \Gamma(A, 2) \text{ by } \mathbb{Z} \times \text{Pic}(S) \text{ on } \mathcal{D}^b(A). \]

\[ \text{Proof.} \text{This action is obtained from the canonical homomorphism } \Gamma(A, 2) \to \widehat{\text{SL}}_2(A) \text{ splitting the projection } \widehat{\text{SL}}_2(A) \to \text{SL}_2(A). \text{Namely, under this splitting the matrix } \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \Gamma(A, 2) \text{ maps to the element } \left( \begin{pmatrix} a_{11} a_{22} / 2, \pi \right), \left( \text{id, } \pi \right) P, \left( \text{id, } \pi \right) P \right) \text{ of } \widehat{\text{SL}}_2(A), \text{where } \mathcal{P} \text{ is the Poincaré line bundle on } A \times \hat{A}. \]

5. \text{THE INDUCED ACTION ON A CHOW MOTIVE}

In this section we will construct a projection action of the algebraic group \( \text{SL}_2(A, \mathbb{Q}) \) on the relative Chow motive of an abelian scheme \( \pi : A \to S \) with rational coefficients. Let us denote by \( \text{Cor}(A) \) the Chow group \( \text{CH}^*(A \times_S A) \otimes \mathbb{Q} \) considered as a \( \mathbb{Q} \)-algebra with multiplication given by the composition of correspondences:

\[ \beta \circ \alpha = p_{13*}(p_{12*}(\alpha) \cdot p_{23*}(\beta)) \]

where \( \alpha, \beta \in \text{CH}^*(A \times_S A) \otimes \mathbb{Q} \), \( p_{ij} \) are the projections from \( A^3 \) to \( A^2 \). The unit of this algebra is \( [\Delta] \in \text{CH}^0(A \times_S A) \) where \( \Delta \subset A \times_S A \) is the relative diagonal, \( g = \dim A \).

Using the Riemann-Roch theorem it is easy to see that the multiplication (5.1) is compatible with the composition law on \( K^0(A \times_S A) \) arising from the interpretation of \( \mathcal{D}^b(A \times_S A) \) as the category of functors from \( \mathcal{D}^b(A) \) to itself considered above, via the map

\[ K^0(A \times_S A) \otimes \mathbb{Q} \to \text{CH}^*(A \times_S A) \otimes \mathbb{Q} : x \mapsto \text{ch}(x) \cdot \pi^* \text{Td}(e^* T_{A/S}) \]

where \( \text{ch} \) is the Chern character. Let us consider the embedding of algebras

\[ \text{CH}(S)_{\mathbb{Q}} \to \text{Cor}(A) : x \mapsto \pi^*(x) \cdot [\Delta] \]

where \( \text{CH}(S)_{\mathbb{Q}} = \text{CH}(S) \otimes \mathbb{Q} \) is equipped with the usual multiplication. In particular, we have an embedding of groups of invertible elements \( \text{CH}(S)_{\mathbb{Q}}^* \subset \text{Cor}(A)^* \). Applying the map (5.2) to the kernels giving projective action of \( \Gamma(A, 2) \) on \( \mathcal{D}^b(A) \) we obtain a homomorphism

\[ \tilde{\rho} : \Gamma(A, 2) \to (\text{Cor}(A))^*/\pm \text{Pic}(S), \]

where \( \text{Pic}(S) \) is embedded into \( \text{CH}(S)_{\mathbb{Q}}^* \) by Chern character (multiplication by \( \pm \) arises from shifts in derived category). Our aim is to approximate this homomorphism.
by a morphism of algebraic groups over $\mathbb{Q}$. More precisely, we have to replace $\tilde{\phi}$ by the induced homomorphism
\[ \phi : \Gamma(A, 2) \to (\text{Cor}(A))^*/\text{CH}(S)^*_\mathbb{Q}. \]

Now we claim that one can replace here source and target by some algebraic groups over $\mathbb{Q}$ such that $\phi$ will be induced by an algebraic homomorphism. Naturally, the source should be replaced by $\text{SL}_{2, A, \mathbb{Q}}$ (see the previous section). To approximate the target we have to replace algebras $\text{Cor}(A)$ and $\text{CH}(S)_\mathbb{Q}$ by their finite-dimensional subalgebras.

**Theorem 5.1.** There exists a finite-dimensional $\mathbb{Q}$-subalgebra $D \subset \text{Cor}(A)$ and a morphism of algebraic $\mathbb{Q}$-groups $\rho : \text{SL}_{2, A, \mathbb{Q}} \to D^*/(D \cap \text{CH}(S))_\mathbb{Q}$ inducing $\phi$ on $\Gamma(A, 2)$.

*Proof.* For a pair of abelian schemes $A$ and $B$ over $S$ let us consider the map
\[ \gamma_{A, B} : \text{Hom}(A, B) \to \text{CH}(A \times_S B) \]
that sends an $S$-morphism $f : A \to B$ to the class $[\Gamma_f]$ of the (relative) graph of $f$. One can extend naturally $\gamma$ to a map
\[ \gamma : \text{Hom}(A) \otimes \mathbb{Q} \to \text{CH}(A \times_S B) \otimes \mathbb{Q} \]
by sending $f/n$ to $\gamma([n]_A)^{-1} \circ \gamma(f)$, where $f \in \text{End}(A)$, $n \neq 0$ — here we take the inverse to $\gamma([n]_A)$ in the algebra $\text{Cor}(A)$ and use its natural action on $\text{CH}(A \times_S B) \otimes \mathbb{Q}$. It is easy to check that $\gamma$ is a polynomial map (see [22], Lemma 13.3, for the case $A = B$).

Now let $\mathfrak{g} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be an element of $\Gamma(A, 2) \cap U$ (recall that $U$ is defined by the condition that $a_{21}$ is an isogeny). From the formula (4.2) we get the following expression for $\phi(\mathfrak{g})$:

\begin{equation}
\phi(\mathfrak{g}) = (p_2 - a_{22}p_1)^*(a_{21} \cdot \text{ch}(L^g)^{-1}) \cdot (a_{12} \times \text{id})^*(\text{ch}(\mathcal{P})) \cdot p_1^*(\text{ch}(M^g)) \mod \text{CH}(S)^*_\mathbb{Q}.
\end{equation}

Note that $\text{ch}(L^g)$ (resp. $\text{ch}(M^g)$) is a polynomial function of $a_{11}a_{21}$ (resp. $a_{22}a_{12}$). Also the functors $f^*$ and $f_*$ can be expressed as compositions with the correspondence given by the graph of $f$. It follows from the above remarks that the right hand side of (5.3) is obtained by evaluating at $\mathfrak{g}$ of a polynomial map $\psi : \text{End}(A \times A) \otimes \mathbb{Q} \to \text{Cor}(A)$. In particular, the image of $\psi$ belongs to a finite-dimensional $\mathbb{Q}$-subspace of $\text{Cor}(A)$.

Let $\mathcal{U} \subset \text{SL}_{2, A, \mathbb{Q}}$ be the Zariski open subset defined by $\deg(a_{21}) \neq 0$. Note that $\mathcal{U}$ is stable under the inversion morphism $g \mapsto g^{-1}$. Let us also denote $\mathcal{U}^{(2)} =$
\[ \mu^{-1}(U) \cap (U \times U) \subset U \times U \] where \( \mu \) is the group law. Consider two polynomial maps
\[ a_1 : U \to \text{Cor}(A) : u \mapsto a_1(u) = \psi(u^{-1}) \circ \psi(u), \]
\[ a_2 : U^{(2)} \to \text{Cor}(A) : (u_1, u_2) \mapsto a_2(u_1, u_2) = \psi(u_1 u_2) \circ \psi(u_2^{-1}) \circ \psi(u_1^{-1}). \]

It is easy to see that the images of both maps belong to the subalgebra \( \text{CH}(S)_\mathbb{Q} \subset \text{Cor}(A) \). This can be done either by direct computation using (5.3) or using the density of \( \Gamma \) in \( H \subset \text{SL}_{2,A,\mathbb{Q}} \) (see the previous section). Also an easy direct computation shows that \( a_1(u) \) is invertible in the algebra \( \text{CH}(S)_\mathbb{Q} \) for all \( u \in U \). This immediately implies that \( a_2(u_1, u_2) \) is invertible for any \( (u_1, u_2) \in U^{(2)} \). Indeed, \( a_2(u_1, u_2) \) is a divisor of \( a_1(u_1) a_1(u_2) a_1(u_2^{-1} u_1^{-1}) \). Note that the components of images of \( a_1 \) and \( a_2 \) span finite-dimensional subspaces in \( \text{CH}(S)^{i}_{\mathbb{Q}} \) for any \( i \). It follows that there exists a finite dimensional subalgebra \( D_S \subset \text{CH}(S)_\mathbb{Q} \) such that the images of \( a_1 \) and \( a_2 \) belong to \( D^*_S \). Now we have
\[ \psi(u_1) \circ \psi(u_2) = a_2(u_1, u_2)^{-1} a_1(u_1) a_1(u_2) \psi(u_1 u_2) \]
for \( (u_1, u_2) \in U^{(2)} \). Let \( D \subset \text{Cor}(A) \) be the \( D_S \)-submodule generated by \( \psi(u) \) with \( u \in U \). Then \( D \) is finite-dimensional as a \( \mathbb{Q} \)-vector space and (5.4) shows that \( \psi(u_1) \circ \psi(u_2) \in D \) for any \( (u_1, u_2) \in U^{(2)} \). Since \( U^{(2)} \) is dense in \( U \times U \) it follows that \( D \) is a subalgebra. Now (5.4) implies that \( \psi \) uniquely extends to a homomorphism \( \text{SL}_{2,A,\mathbb{Q}} \to D^*/D^*_S \). \( \square \)

6. Splittings of the extension \( \widetilde{SL}_2(A) \to SL_2(A) \)

Let \( A/S \) be an abelian scheme with a principal polarization \( \phi : A \to \hat{A} \). Then we have the Rosati involution
\[ \varepsilon_\phi : \text{End}(A) \to \text{End}(A) : f \mapsto \phi^{-1} \circ \hat{f} \circ \phi. \]
The group \( \text{SL}_2(A) \) is completely determined by the algebra \( \text{End}(A) \) with involution \( \varepsilon_\phi \). The definition of the group \( \tilde{\text{SL}}_2(A) \) requires in addition the knowledge of the extension
\[ 0 \to \hat{A}(S) \to \text{Pic}(A) \to \text{Hom}^{\text{sym}}(A, \hat{A}) \]
(6.1) together with the action of the multiplicative monoid of \( \text{End}(A) \) on it. Thus, splittings of the homomorphism \( \tilde{\text{SL}}_2(A) \to \text{SL}_2(A) \) should be related to splittings of (6.1). More precisely, it’s natural to consider splittings compatible with the \( \text{End}(A) \)-action. We’ll show that such splittings of (6.1) correspond to simultaneous splittings of homomorphisms \( \tilde{\text{SL}}_2(A^n) \to \text{SL}_2(A^n) \) for all \( n \), where \( A^n/S \) is the \( n \)-th relative cartesian power of \( A/S \).

More generally, we start with arbitrary subring \( R \subset \text{End}(A) \) stable under the Rosati involution. Let us denote by \( \varepsilon : R \to R \) the restriction of the Rosati involution
to \( R \), let \( R^+ \subset R \) be the subring of elements stable under \( \varepsilon \). Then for any \( n \geq 1 \) we can consider the subgroup \( SL_2(A^n, R) \subset SL_2(A^n) \) consisting of \( 2n \times 2n \) matrices with all entries belonging to \( R \) (we identify \( \hat{A} \) with \( A \) via \( \phi \)). Let \( \widetilde{SL}_2(A^n, R) \subset \widetilde{SL}_2(A^n) \) be the preimage of \( SL_2(A^n, R) \). We are interested in splittings of the natural homomorphisms

\[
(6.2) \quad \widetilde{SL}_2(A^n, R) \to SL_2(A^n, R)
\]

It turns out that the following structure on \( A \) is relevant for this.

**Definition 6.1.** A \( \Sigma_{R,\varepsilon} \)-structure for \( \phi \) is a homomorphism \( R^+ \to \text{Pic}(A) : r_0 \mapsto L(r_0) \) such that

\[
(6.3) \quad \phi_{L(r_0)} = \phi \circ [r_0]_A
\]

for any \( r_0 \in R^+ \) and

\[
(6.4) \quad [r]^* L(r_0) \simeq L(\varepsilon(r) r_0 r)
\]

for any \( r \in R, r_0 \in R^+ \).

Note that (6.4) for \( r = -1 \) implies that all line bundles \( L(r_0) \) are symmetric. In [24] we studied the question of existence of \( \Sigma_{R,\varepsilon} \)-structure for an abelian variety. For example, in the case of a complex elliptic curve \( E \) with complex multiplication such a structure for \( R = \text{End}(E) \) exists if and only if \( R \) is unramified at 2. Another example is the case when \( R \) is a ring of integers in a totally real number field unramified at 2 and \( \varepsilon = \text{id} \) (see next section).

**Theorem 6.2.** A \( \Sigma_{R,\varepsilon} \)-structure on \( A \) induces canonical splittings of the homomorphisms (6.2) for all \( n \).

**Proof.** It is easy to see that a \( \Sigma_{R,\varepsilon} \)-structure on \( A \) induces a similar structure on \( A^n \) with \( R \) replaced by the matrix algebra \( \text{Mat}_n(R) \) and \( \varepsilon \) replaced by the corresponding involution \( (a_{ij}) \mapsto (\varepsilon(a_{ji})) \) of \( \text{Mat}_n(R) \). Hence, it is sufficient to consider the case \( n = 1 \). In this case we define the splitting

\[
\text{SL}_2(A, R) \to \text{SL}_2(A, R) : g = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto (g, (\phi^{-1})^* L(\varepsilon(a_{11}) a_{21}), L(\varepsilon(a_{22}) a_{12}))
\]

\( \square \)

Now we are going to prove that conversely the existence of splitting of (6.2) for \( n = 2 \) implies the existence of \( \Sigma_{R,\varepsilon} \)-structure. We use the following observation. For any abelian scheme \( A \) there is a natural embedding of the semi-direct product \( \text{Aut}(A) \ltimes \text{Hom}^\text{sym}(A, \hat{A}) \) into \( \text{SL}_2(A) \) as the subgroup of matrices with \( a_{21} = 0 \). Thus, a splitting of the homomorphism \( \text{SL}_2(A) \to \text{SL}_2(A) \) restricts to a splitting of the homomorphism of \( \text{Aut}(A) \)-modules \( \text{Pic}(A) \to \text{Hom}^\text{sym}(A, \hat{A}) \). In the situation
considered above we have similar subgroups in $\text{SL}_n(A^n, R)$. Note that the subgroup of $\text{Hom}^\text{sym}(A^n, \hat{A}^n)$ consisting of matrices with entries in $R$ can be identified with the group of hermitian matrices $\text{Mat}^\text{herm}_n(R)$ and the natural right action of $\text{GL}_n(R)$ on it induced by its action on $A^n$ is given by the formula

$$B \mapsto C B C$$

where $C = (c_{ij}) \in \text{GL}_n(R)$, $B \in \text{Mat}^\text{herm}_n(R)$, $C = (\varepsilon(c_{ji}))$. Thus, a splitting of the homomorphism (6.2) induces a splitting of the homomorphism of $\text{GL}_n(R)$-modules

$$\text{Pic}(A^n, R) \to \text{Mat}^\text{herm}_n(R)$$

where $\text{Pic}(A^n, R) \subset \text{Pic}(A^n)$ is the subgroup of line bundles $L$ such that $\phi_L \in \text{Hom}(A^n, \hat{A}^n)$ has entries in $R$.

Now we claim that such a splitting for $n = 2$ leads to a $\Sigma_{R, \varepsilon}$-structure on $A$.

**Theorem 6.3.** Any splitting of the homomorphism of $\text{GL}_2(R)$-modules $\text{Pic}(A^2, R) \to \text{Mat}^\text{herm}_2(R)$ is induced by a unique $\Sigma_{R, \varepsilon}$-structure.

**Proof.** Let

$$s : \text{Mat}^\text{herm}_2(R) \to \text{Pic}(A^2, R)$$

be such a splitting. Then for $r_0 \in R^+$ one has

$$s\begin{pmatrix} r_0 & 0 \\ 0 & 0 \end{pmatrix} = p_1^* L(r_0) \otimes p_2^* \eta(r_0)$$

for some line bundle $L(r_0)$ and $\eta(r_0)$ on $A$ such that $\phi_{L(r_0)} = \phi \circ [r_0]$, $\eta(r_0) \in \text{Pic}^0(A)$. The compatibility of $s$ with the action of $\text{GL}_2(R)$ means that

$$[C]^* s(B) = s(C B C)$$

where $B \in \text{Mat}^\text{herm}_2(R)$, $C = (c_{ij}) \in \text{GL}_2(R)$, $C = (\varepsilon(c_{ji}))$. Applying this to $C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ we deduce from (6.5) the equality

$$s\begin{pmatrix} 0 & 0 \\ 0 & r_0 \end{pmatrix} = p_2^* L(r_0) \otimes p_1^* \eta(r_0).$$

Also using the identity

$$\begin{pmatrix} 1 & 0 \\ \varepsilon(r) & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & r \\ 0 & \varepsilon(r) \end{pmatrix} = \begin{pmatrix} 1 & r \\ \varepsilon(r) & \varepsilon(r) r \end{pmatrix}$$

for any $r \in R$ we deduce that

$$s\begin{pmatrix} 1 & r \\ \varepsilon(r) & \varepsilon(r) r \end{pmatrix} = \left(\text{id} \begin{pmatrix} r \end{pmatrix} \right)^* (p_1^* L(1) \otimes p_2^* \eta(1)) = (p_1 + [r] p_2)^* L(1) \otimes p_2^* \eta(1).$$

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Combining this with (6.5) and (6.6) one can easily compute that
\[
(6.7) \quad s \left( \begin{array}{cc} 0 & r \\ \varepsilon(r) & 0 \end{array} \right) = (\phi \times [r])^* \mathcal{P} \otimes p_1^* \eta(-\varepsilon(r) r) \otimes p_2^*[r]^* L(1) \otimes L(-\varepsilon(r) r)).
\]

Note that the formulas (6.5), (6.6), and (6.7) completely determine \( s \). Now the identity
\[
\begin{pmatrix} 1 & \varepsilon(r) \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & r_0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} = \begin{pmatrix} \varepsilon(r) r_0 r & \varepsilon(r) r_0 \\ r_0 r & r_0 \end{pmatrix}
\]
which holds for any \( r \in \mathbb{R} \), \( r_0 \in \mathbb{R}^+ \) implies the equality
\[
(6.8) \quad \left( \begin{array}{cc} \text{id} & 0 \\ [r] & \text{id} \end{array} \right)^* (p_2^* \mathcal{L}(r_0) \otimes p_1^* \eta(r_0)) = s \left( \begin{array}{cc} \varepsilon(r) r_0 r & \varepsilon(r) r_0 \\ r_0 r & r_0 \end{array} \right).
\]

Computing the right hand side using (6.5)–(6.7) and restricting to \( A \times 0 \) we obtain the identity
\[
(6.9) \quad [r]^* \mathcal{L}(r_0) = \mathcal{L}(\varepsilon(r) r_0 r) \otimes \eta(-r_0 r \varepsilon(r) r_0).
\]

On the other hand, setting \( r = 1 \) and restricting (6.8) to \( 0 \times A \) we get
\[
(6.10) \quad \mathcal{L}(r_0^2) = [r_0]^* \mathcal{L}(1) \otimes \eta(r_0).
\]

Setting \( r = 1 \) in (6.9) we obtain the triviality of \( \eta(r_0^2) \). Then taking \( r_0 = 1 \) and \( r \in \mathbb{R}^+ \) in (6.9) we obtain that
\[
[r]^* \mathcal{L}(1) = \mathcal{L}(r^2)
\]
for \( r \in \mathbb{R}^+ \). Comparing this with (6.10) we deduce the triviality of \( \eta(r_0) \) for all \( r_0 \in \mathbb{R}^+ \). Now (6.9) implies that \( \mathcal{L}(\cdot) \) gives a \( \Sigma_{R,\varepsilon} \)-structure. \( \square \)

**Corollary 6.4.** A \( \Sigma_{R,\varepsilon} \)-structure for \( \phi \) exists if and only if a splitting of the homomorphism (6.2) for \( n = 2 \) exists.

**Example.** Let \( E = \mathbb{C}/\mathbb{Z}[i] \) be an elliptic curve with complex multiplication by the ring of Gaussian numbers \( R = \mathbb{Z}[i] \), so that the corresponding Rosati involution \( \varepsilon \) is just the complex conjugation. In this situation there is no \( \Sigma_{R,\varepsilon} \)-structure corresponding to the standard polarization of \( E \). Indeed, the corresponding line bundle \( \mathcal{L}(1) \) should be of the form \( \mathcal{O}(x) \) where \( x \) is a point of order 2 on \( E \). Now the identity \( [1 + i]^* \mathcal{L}(1) = \mathcal{L}(2) \) leads to a contradiction (see [24] for details).
7. Abelian schemes with real multiplication

Let $F$ be a totally real number field, $R$ be its ring of integers. Let $A \to S$ be an abelian scheme with real multiplication by $R$, i.e. a ring homomorphism $R \to \text{End}_S(A) : r \mapsto [r]_A$ is given. Then the dual abelian scheme $\hat{A}$ also has a natural real multiplication by $R$. Let $J, \hat{J} \subset F$ be fractional ideals for $R (= \text{non-zero finitely generated } R\text{-submodules of } F)$ such that $\hat{J}J \subset R$.

**Definition 7.1.** An $(J, \hat{J})$-polarization on $A$ is a pair of $R$-module homomorphisms

\[ \lambda_J : J \to \text{Hom}_{\text{sym}}^R(A, \hat{A}), \]
\[ \lambda_{\hat{J}} : \hat{J} \to \text{Hom}_{\text{sym}}^R(\hat{A}, A) \]

where $\text{Hom}_{\text{sym}}^R(A, \hat{A})$ is an $R$-module of symmetric $R$-linear homomorphisms $f : A \to \hat{A}$ (i.e. $\hat{f} = f$ and $f \circ [r]_A = [r]_{\hat{A}} \circ f$ for any $r \in R$), such that $\lambda_J(m) \circ \lambda_{\hat{J}}(l) = [lm]_A$ and $\lambda_J(l)\lambda_{\hat{J}}(m) = [lm]_{\hat{A}}$ for any $l \in J$, $m \in \hat{J}$.

**Remark.** Usually one also imposes some positivity condition on a polarization. In the case of $(J, \hat{J})$-polarizations one can fix an ordering on $J$: this means that for each embedding $\sigma : F \to \mathbb{R}$ an orientation of the line $J \otimes_{R, \sigma} \mathbb{R}$ is chosen. Then one should require that if an element $l \in J$ is totally positive then the homomorphism $\lambda_J(l) : A \to \hat{A}$ is positive (i.e. $\lambda_J(l)$ is a polarization in the classical sense).

Note that the notion of $(J, \hat{J})$-polarization is equivalent to that of $(Jx, \hat{J}x^{-1})$-polarization for any $x \in F^*$. Also an $(J, \hat{J})$-polarization of $A$ is the same as an $(\hat{J}, J)$-polarization of $\hat{A}$. When $\hat{J} = J^{-1}$ we recover the notion of $J$-polarization in the sense of P. Deligne and G. Pappas [5] (except for the positivity condition). Recall that they define an $J$-polarization of an abelian scheme $A$ with real multiplication by $R$ as an $R$-linear homomorphism $\lambda : J \to \text{Hom}^\text{sym}_R(A, \hat{A})$ such that the image of a totally positive element of $J$ under $\lambda$ is positive, and the induced morphism $A \otimes_R J \to \hat{A}$ is an isomorphism. In this case we have also an isomorphism $\hat{A} \otimes_R J^{-1} \cong A$ which induces an $R$-linear homomorphism $\lambda' : J^{-1} \to \text{Hom}^\text{sym}_R(A, A)$, hence we get an $(J, J^{-1})$-polarization in our sense. Conversely, given an $(J, J^{-1})$-polarization as above then the morphism $\mu : A \otimes_R J \to \hat{A}$ induced by $\lambda_J$ and the morphism $\mu' : \hat{A} \to A \otimes_R J$ induced by $\lambda_{J^{-1}}$ are inverse to each other, so that $\lambda_J$ gives an $J$-polarization of $A$ (except for the positivity condition).

For a pair $(J, \hat{J})$ as above we define the subgroup $\Gamma(J, \hat{J}) \subset \text{SL}_2(F)$ as follows:

\[ \Gamma(J, \hat{J}) = \{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \text{SL}_2(F) : a_{11}, a_{12} \in R, a_{12} \in J, a_{21} \in \hat{J} \}. \]
Note that for \( x \in F^* \) the homomorphisms \( \rho_{J,\bar{J}} \) and \( \rho_{Jx,\bar{J}x^{-1}} \) are compatible with the natural isomorphism \( \Gamma(J, \bar{J}) \cong \Gamma(Jx, \bar{J}x^{-1}) \) (induced by the conjugation by \( \begin{pmatrix} x & 0 \\ 0 & x^{-\frac{1}{2}} \end{pmatrix} \)). In particular, in the case \( R = \mathbb{Z} \) the group \( \Gamma(J, \bar{J}) \) is always isomorphic to the principal congruence-subgroup \( \Gamma_0(N) = \Gamma(\mathbb{Z}, N\mathbb{Z}) \subset \text{SL}_2(\mathbb{Z}) \) (for some \( N > 0 \)).

If \( A \) is \((J, \bar{J})\)-polarized then using \( \lambda_J \) and \( \lambda_J \) we can define a homomorphism

\[
\rho_{J,\bar{J}} : \Gamma(J, \bar{J}) \to \text{SL}_2(A) : \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \begin{pmatrix} [a_{11}]_A & \lambda_J(a_{12}) \\ \lambda_J(a_{21}) & [a_{22}]_A \end{pmatrix}
\]

**Lemma 7.2.** For any non-zero element \( r \in R \) (resp. \( l \in J, m \in \bar{J} \)) the corresponding morphism \( [r]_A \) (resp. \( \lambda_J(l), \lambda_J(m) \)) is an isogeny.

**Proof.** There exists a non-zero integer \( N \) such that \( r' = N/r \in R \), so that \( [r']_A \circ [r]_A = [N]_A \). Hence, \( [r]_A \) is an isogeny on each fiber, therefore, it is an isogeny. Similar argument works for \( \lambda_J(l) \) and \( \lambda_J(m) \). \( \square \)

Recall that we denote by \( \Gamma(A) \) the image of the homomorphism \( \tilde{\text{SL}}_2(A) \to \text{SL}_2(A) \). Let assume for simplicity that the image \( \rho_{J,\bar{J}} \) is contained in \( \Gamma(A) \). Otherwise, we can consider a finite flat base change \( S' \to S \) such that the image of \( \Gamma(J, \bar{J}) \) is contained in \( \Gamma(A_{S'}) \) where \( A_{S'} \) is the induced abelian scheme over \( S' \). Indeed, recall that one has an exact sequence

\[
0 \to \hat{A}_2 \to \text{Pic}^\text{sym}(A) \to \text{Hom}^\text{sym}(A, \hat{A}) \to 0
\]

of sheaves in fpqc topology, hence the boundary homomorphism \( J \xrightarrow{\delta} \text{Hom}^\text{sym}(A, \hat{A}) \to H^1(S, \hat{A}_2) \) which can be considered as a \( J^* \otimes \hat{A}_2 \)-torsor over \( S \) where \( J^* = \text{Hom}_{\mathbb{Z}}(J, \mathbb{Z}) \) (similarly, for \( \lambda_J \) we get an \( J^* \otimes A_2 \)-torsor). Now we can take \( S' \) to be the corresponding \( J^* \otimes \hat{A}_2 \times \bar{J}^* \otimes A_2 \)-torsor over \( S \).

By the above lemma we can define an obstruction homomorphism \( \delta_{J,\bar{J}} : \Gamma(J, \bar{J}) \to \text{Br}(S) \) as follows: if \( a_{21} \)-entry of the matrix \( h \in \Gamma(J, \bar{J}) \) is non-zero then the same entry of \( \overline{f} = \rho_{J,\bar{J}}(h) \) is an isogeny and we can put \( \delta_{J,\bar{J}}(h) = e(G_g) \) where \( g \in \tilde{\text{SL}}_2(A) \) lies above \( \overline{f} \). Otherwise, \( \delta_{J,\bar{J}}(h) = 0 \). As in proposition 4.4 one can check that \( \delta_{J,\bar{J}} \) is a homomorphism.

Let \( I \subset R \) be a non-zero ideal. Let us denote by \( \Gamma_I(J, \bar{J}) \) the group of matrices

\[
\begin{pmatrix}
\bar{a}_{11} & \bar{a}_{12} \\
\bar{a}_{21} & \bar{a}_{22}
\end{pmatrix}
\]

where \( \bar{a}_{11}, \bar{a}_{22} \in R/I, \bar{a}_{12} \in J/IJ, \bar{a}_{21} \in \bar{J}/I\bar{J}, \) such that \( \bar{a}_{11}\bar{a}_{22} - \bar{a}_{12}\bar{a}_{21} = 1 \). Here we use the natural homomorphism of \( R/I \)-modules \( J/IJ \otimes \bar{J}/I\bar{J} \to R/IR \) induced by \( J \otimes \bar{J} \to R \).

**Lemma 7.3.** Assume that \( I \subset J \bar{J} \). Then the natural reduction homomorphism \( \pi_I : \Gamma(J, \bar{J}) \to \Gamma_I(J, \bar{J}) \) is surjective.
Proof. Let \( \begin{pmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{pmatrix} \in \Gamma(J, \hat{J}) \) be any element. Choose any non-zero liftings \( a_{12} \in J, a'_{21} \in \hat{J} \) and \( a'_{22} \in R \) of \( \bar{a}_{12}, \bar{a}_{21} \) and \( \bar{a}_{22} \). For an element \( r \in R \) and a finite \( R \)-module \( Q \) we say that \( r \) is relatively prime to \( Q \) if \( Q = rQ \), or equivalently, \( r \not\in \mathfrak{p} \) for any prime ideal \( \mathfrak{p} \) associated with \( Q \). By the Chinese remainder theorem we can lift \( \bar{a}_{11} \) to an element \( a_{11} \in R \) which is relatively prime to \( J/Ra_{12} \). On the other hand, \( a_{11} \) is relatively prime to \( R/J \hat{J} \) since \( a_{11} a'_{22} \equiv 1 \mod(I) \) and \( I \subset J/\hat{J} \). Therefore, \( a_{11} \) is relatively prime to \( J/J a_{12} \) (since \( \text{supp}(J/J a_{12}) \subset \text{supp}(R/J \hat{J}) \cup \text{supp}(J/R a_{12})) \). It follows that \( J = a_{11} J + J a_{12} = (R a_{11} + J a_{12}) J \) which implies the equality \( R = R a_{11} + J a_{12} \). Thus, we can write \( 1 = r a_{11} + m a_{12} \) where \( r \in R, m \in \hat{J} \). Let \( x = a_{11} a'_{22} - a_{12} a'_{21} - 1 \in I \). Then replacing \( a'_{22} \) and \( a'_{21} \) by \( a_{22} = a'_{22} - x r, a_{21} = a'_{21} - x m \) we achieve \( a_{11} a_{22} - a_{12} a_{21} = 1 \). \( \Box \)

**Proposition 7.4.** The homomorphism \( \delta_{J, \hat{J}} : \Gamma(J, \hat{J}) \to \text{Br}(S) \) is trivial.

*Proof.* Consider the reduction homomorphism \( \pi_I : \Gamma(J, \hat{J}) \to \Gamma_I(J, \hat{J}) \) where \( I \subset 2 J \hat{J} \). Then by Lemma 7.3 \( \pi_I \) is surjective. On the other hand, the kernel of \( \pi_I \) is contained in the subgroup \( \Gamma(J, 2 \hat{J}) \subset \Gamma(J, \hat{J}) \). By Proposition 4.8 the restriction of \( \delta_{J, \hat{J}} \) to the subgroup \( \Gamma(J, 2 \hat{J}) \) is trivial. Hence, \( \delta_{J, \hat{J}}(\ker(\pi_I)) = 0 \), so that \( \delta_{J, \hat{J}} = \overline{\sigma} \circ \pi_I \) for some homomorphism \( \overline{\sigma} : \Gamma_I(J, \hat{J}) \to \text{Br}(S) \). Moreover, since by Lemma 7.3 the homomorphism \( \Gamma(J, 2 \hat{J}) \to \Gamma_I(J, 2 \hat{J}) \) is surjective, it follows that \( \overline{\sigma} \) is trivial on matrices with \( \bar{a}_{21} \in 2 \hat{J}/I J \). In particular, \( \overline{\sigma} \) vanishes on any diagonal matrix. Let \( h = \begin{pmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{pmatrix} \) be any element of \( \Gamma_I(J, \hat{J}) \). Then \( \bar{a}_{11} \bar{a}_{22} \equiv 1 \mod(J \hat{J}) \), hence \( \bar{a}_{11} \mod(J \hat{J}) \) is a unit in \( R/J \hat{J} \). Let \( u \in (R/I)^* \) be any unit such that \( u \equiv a_{11} \mod(J \hat{J}) \) (such a unit always exists since \( R/I \) is an artinian ring). Then replacing \( h \) by \( h \cdot \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \) we reduce the problem of showing that \( \overline{\sigma}(h) = 0 \) to the case when \( \bar{a}_{11} \equiv 1 \mod(J \hat{J}) \).

Now we use the result of L. Vaserstein [26] which asserts that if \( F \neq \mathbb{Q} \) then the subgroup of \( \Gamma(J, \hat{J}) \) consisting of matrices with \( a_{11} \equiv 1 \mod(J \hat{J}) \) is generated by elementary matrices, i.e. matrices of the form \( \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \) and \( \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix} \), where \( l \in L, m \in M \). In the case \( F = \mathbb{Q} \) we may assume that \( J = \mathbb{Z}, \hat{J} = N \mathbb{Z} \) for some \( N \in \mathbb{Z} \) and the corresponding assertion for \( \Gamma(\mathbb{Z}, N \mathbb{Z}) = \Gamma_0(N) \) is trivial. Note that \( \delta_{J, \hat{J}} \) vanishes on elementary matrices (the corresponding Heisenberg groups are either Mumford groups of line bundles or trivial, so they admit Schrödinger representations), hence it vanishes on any matrix with \( a_{11} \equiv 1 \mod(J \hat{J}) \) and we are done. \( \Box \)

Let \( \hat{\Gamma}(J, \hat{J}) \) be the preimage of \( \Gamma(J, \hat{J}) \) under the homomorphism \( \hat{\text{SL}}_2(A) \to \text{SL}_2(A) \).
Theorem 7.5. For every $(J, \tilde{J})$-polarized abelian scheme $A$ over $S$ with $R$-multiplication such that the image of $p_{1, j}$ is contained in $\Gamma(A)$ there exists a faithful action of a central extension of the group $\tilde{\Gamma}(J, \tilde{J})$ by $\mathbb{Z} \times \text{Pic}(S)$ on $\mathcal{D}^b(A)$. Without this assumption we always have compatible faithful projective actions of $\Gamma(2J, 2\tilde{J})$ on $\mathcal{D}^b(A)$ and of $\tilde{\Gamma}(J, \tilde{J})$ on $A_S$, for some finite flat base change $S' \to S$.

Corollary 7.6. Let $A$ be an abelian scheme over a normal noetherian connected base $S$. Assume that the projection $\tilde{S}\text{L}_2(A) \to \text{SL}_2(A)$ is surjective and $\text{End}_S(A) \simeq R$ is a totally real field. Then there is a faithful action of a central extension of $\tilde{S}\text{L}_2(A)$ by $\mathbb{Z} \times \text{Pic}(S)$ on $\mathcal{D}^b(A)$.

Proof. By Prop. X 1.5 of [7] the general fiber $A_K$ admits an $R$-linear polarization $\lambda : A_K \to \tilde{A}_K$. Hence, $J = \text{Hom}_K(A_K, \tilde{A}_K)$ and $\tilde{J} = \text{Hom}_K(\tilde{A}_K, A_K)$ can be considered as fractional ideals for $R$, such that $J \tilde{J} \subset R$. By definition $\text{SL}_2(A_K) = \Gamma(J, \tilde{J})$ and by Lemma 4.1 we have an $(J, \tilde{J})$-polarization on $A$. □

Now let us consider the case of abelian scheme $A$ with $R$-linear principal polarization $\phi : A \to \tilde{A}$. In this case we have a natural inclusion

$$i_\phi : \text{Sp}_{2n}(R) \to \text{SL}_2(A^n) : \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \mapsto \begin{pmatrix} [M_{11}]_{\tilde{A}} & \phi(n)[M_{12}]_{\tilde{A}} \\ [M_{21}]_{\tilde{A}} \phi^{-1}_n [M_{22}]_{\tilde{A}} \end{pmatrix}$$

where $A^n$ is the relative $n$-th cartesian power of $A$ with the induced polarization $\phi(n)$, $M_{ij} \in \text{Mat}_n(R)$, for every abelian scheme $A$ with multiplication by $R$ we denote the natural map $\text{Mat}_n(R) \to \text{End}(A^n)$ by $M \mapsto [M]_A$.

Now we claim that if $R$ is unramified at $2$ then one can split the extension $\tilde{\text{S}\text{L}_2}(A^n) \to \text{SL}_2(A)$ over $\text{Sp}_{2n}(R)$ provided that a symmetric line bundle $L(1)$ on $A$ is given such that $\phi_{L(1)} = \phi$. According to Theorem 6.2 it is sufficient to construct a $\Sigma_{R, \text{id}}$-structure for $\phi$. Note that since $R$ is unramified at $2$ every element $r \in R$ can be represented in the form $r = a^2 + 2b$ with $a, b \in R$. Now we define $L(r) = [a]^*L(1) \otimes (\phi, [b]_A)^*\mathcal{P}$ where $\mathcal{P}$ is the Poincaré line bundle. It is easy to see that $L(r)$ doesn’t depend on a choice of $a$ and $b$, and satisfies (6.3) and (6.4) with $\varepsilon = \text{id}$. The induced structure for $A^n$ and $\text{Mat}_n(R)$ is given by the homomorphism

$$\text{Mat}_n^{\text{sym}}(R) \to \text{Pic}^{\text{sym}}(A^n) : B = (b_{ij}) \mapsto L(B) = \bigotimes_{i < j} (\phi p_{ij}, [b_{ij}]_A p_{ij})^*\mathcal{P} \otimes \bigotimes_i p^*_i L(b_{ii})$$

where $\text{Mat}_n^{\text{sym}}(R)$ denotes symmetric matrices with entries in $R$, $p_i : A^n \to A$ is the projection on the $i$-th factor. It is easy to see that $\phi_{L(B)} = [B]_A$ and that $[C]_A L(B) \simeq L(C^t B C)$ for any $C \in \text{Mat}_n(R)$. Now we can write the required splitting

$$\text{Sp}_{2n}(R) \to \tilde{\text{S}\text{L}_2}(A^n) : M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \mapsto (i_\phi(M), (\phi^{-1}_n)^*L(\text{tr}^t M_{11} M_{21}), L(\text{tr}^t M_{22} M_{12})).$$
Using the above splitting we can construct a projective action of $\text{Sp}_{2n}(R)$ on $D^b(A^n)$. The vanishing of the obstacle follows in this case from the fact that $\text{Sp}_{2n}(R)$ is generated by elementary matrices established in [1].

**Theorem 7.7.** Let $A$ be an abelian scheme with real multiplication by $R$ over $S$, $L(1)$ be a symmetric line bundle on $A$ rigidified along the zero section such that $\phi_{L(1)}: A \to \hat{A}$ is an $R$-linear isomorphism. Assume that $R$ is unramified at 2. Then there is a canonical faithful action of a central extension of $\text{Sp}_{2n}(R)$ by $\mathbb{Z} \times \text{Pic}(S)$ on $D^b(A^n)$ where $A^n$ is the relative cartesian power of $A$, $n \geq 1$. These actions are compatible via the natural embeddings $D^b(A^n) \to D^b(A^{n+1})$ and $\text{Sp}_{2n}(R) \to \text{Sp}_{2n+2}(R)$.

**Proof.** The same argument as in Proposition 4.4 allows to define an obstacle homomorphism $\delta: \text{Sp}_{2n}(R) \to \text{Br}(S)$ such that $\delta(h) = 0$ if and only if there exists a global object in $\text{Isom}_{H(H)}(\rho, \rho^h)$ where $X = \hat{A} \times_S A^n$, $\rho$ is the representation of the Heisenberg groupoid on $D^b(A^n)$. It is easy to check that $\delta$ vanishes on elementary matrices, hence it is zero. $\square$

8. **The central extension**

In this section we describe explicitly the central extension of $\text{Sp}_{2n}(\mathbb{Z})$ by $\mathbb{Z} \times \text{Pic}(S)$ corresponding to the projective action defined in the previous section.

We are going to use a presentation of $\text{Sp}_{2n}(\mathbb{Z})$ by generators and relations borrowed from [11]. We always use the standard symplectic basis $e_1, \ldots, e_n, f_1, \ldots f_n$ in $\mathbb{Z}^{2n}$ such that $(e_i, f_j) = \delta_{ij}$. First of all let us introduce the relevant elementary matrices following the notation of [11] 5.3.1. Let $S_{2n}$ be the set of pairs $(i, j)$ where $1 \leq i, j \leq 2n$ which are not of the form $(2k-1, 2k)$ or $(2k, 2k-1)$. Then for every $(i, j) \in S_{2n}$ we define an elementary matrix $E_{ij}$ as follows:

$$E_{2k, 2l} = \begin{pmatrix} 1 & 0 \\ \gamma_{k,l} & 1 \end{pmatrix},$$

$$E_{2k-1, 2l-1} = \begin{pmatrix} 1 & -\gamma_{k,l} \\ 0 & 1 \end{pmatrix},$$

$$E_{2k-1, 2l} = \begin{pmatrix} e_{kl} & 0 \\ 0 & e_{lk}^{-1} \end{pmatrix},$$

$$E_{2l, 2k-1} = E_{2k-1, 2l}$$

where $\gamma_{kl}$ has zero $(\alpha, \beta)$-entry unless $(\alpha, \beta) = (k, l)$ or $(\alpha, \beta) = (l, k)$, in the latter case $(\alpha, \beta)$-entry is 1; $e_{kl}$ for $k \neq l$ is the usual elementary matrix with units on the diagonal and at $(k, l)$-entry and zeros elsewhere. Now theorem 9.2.13 of [11] asserts that for $n \geq 3$ the group $\text{Sp}_{2n}(\mathbb{Z})$ has a presentation consisting of the generators $E_{ij} = E_{ji}$ (where $(i, j) \in S_{2n}$) subject to the relations

(1) $[E_{ij}, E_{kl}] = 1$, if $(i, k), (i, l), (j, k), (j, l)$ are in $S_{2n}$. 35
(2) $[E_{ij}, E_{kl}] = E_{il}$, if $(j, k) \not\in S_{2n}$, $j$ is even, and $i$, $j$, $k$, and $l$ are distinct
(3) $[E_{ij}, E_{ki}] = E_{ii}^{2}$, if $(j, k) \not\in S_{2n}$, $j$ is even, and $i$, $j$, and $k$ are distinct
(4) $[E_{ii}, E_{kl}] = E_{ii}E_{kl}^{-1}$ if $(i, k) \not\in S_{2n}$, $i$ is even, and $i$, $k$, and $l$ are distinct
(5) $[E_{ii}, E_{kl}] = E_{ii}^{-1}E_{kl}^{-1}$ if $(i, k) \not\in S_{2n}$, $i$ is odd, and $i$, $k$, and $l$ are distinct
(6) $(E_{11}E_{22}E_{11})^{4} = 1$.

It is convenient to introduce also the symplectic matrix

$$\varphi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$ 

Then one has the following relations

(8.1) $\varphi^{-1}E_{2k-1,2l-1}\varphi = E_{2k,2l}$

for all $1 \leq k, l \leq n$,

(8.2) $\varphi^{-1}E_{2k-1,2l}\varphi = E_{2l-1,2k}$

for all $k \neq l$. In particular, the group $\text{Sp}_{2n}(Z)$ is generated by $\varphi$ and $E_{ij}$ with $i$ odd. The latter set of generators is more convenient from the point of view of our projective representation on $\mathcal{D}(A)$ since the functors corresponding to $E_{ij}$ with $i$ odd are very easy to describe (see the proof of the theorem below).

Let us denote by $\tilde{\text{Sp}}_{2n}(Z)$ the group with generators $E_{ij} = E_{ji}$ ($(i, j) \in S_{2n}$) and one more generator $\epsilon$ subject to relations (1)-(5) above, the commutativity relation $[\epsilon, E_{ij}] = 1$ for all $(i, j) \in S_{2n}$, and the modified relation (6)

$$(E_{11}E_{22}E_{11})^{4} = \epsilon.$$ 

Let $\pi : A \rightarrow S$ be an abelian scheme with a symmetric line bundle $L$ (rigidified along the zero section) which induces a principal polarization $\phi : A \Rightarrow \hat{A}$. Let us also denote $\Delta(L) = 2\pi_{*}L + e^{*}\omega_{A/S} \in \text{Pic}(S)$ where $\omega_{A/S}$ is the relative canonical bundle. It is known that $4 \cdot \Delta(L) = 0$ (see e. g. [6], I, 5.1).

**Theorem 8.1.** Let $n \geq 3$. The group $\tilde{\text{Sp}}_{2n}(Z)$ is a central extension of $\text{Sp}_{2n}(Z)$ by $Z$. The central extension of $\text{Sp}_{2n}(Z)$ by $Z \times \text{Pic}(S)$ corresponding to the projective action on $\mathcal{D}(A)$ is obtained from the $\tilde{\text{Sp}}_{2n}(Z)$ by the push-forward with respect to the homomorphism $Z \rightarrow Z \times \text{Pic}(S) : 1 \mapsto (2g, 2\Delta(L))$

**Proof.** Let us choose the intertwining functors corresponding to the generators $\varphi$ and $E_{ij}$ (with $i$ odd) in the following way. The functor corresponding to $\varphi$ is the composition $\phi^{(i)} \circ F_{A}$ where $F_{A} : \mathcal{D}(A) \rightarrow \mathcal{D}(A)$ is the Fourier-Mukai transform. The functor corresponding to $E_{2k-1,2l-1}$ is simply tensor multiplication with the line bundle $L(\gamma_{k,l})$. Note that $L(\gamma_{k,l}) = (\phi p_{k}, p_{l})^{*}P$ if $k \neq l$ while $L(\gamma_{kk}) = p_{k}^{*}L(1)$. At last the functor corresponding to $E_{2k-1,2l}$ is $[e_{kl}]^{*}A$. We claim that these functors satisfy all the relations (1)-(5). Let $P \subset \text{Sp}_{2n}(Z)$ be the subgroup of matrices of
the form \( \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \). Then there is an obvious action of \( P \) on \( \mathcal{D}^b(A^n) \) such that the element \( \begin{pmatrix} tC^{-1} & 0 \\ 0 & C \end{pmatrix} \) where \( C \in \text{GL}_n(\mathbb{Z}) \), \( B \in \text{Mat}^{\text{sym}}(n, \mathbb{Z}) \), acts by the functor \( [C^{-1}]_A \circ (\cdot \otimes L(-B)) \). It is easy to see that our definition of the functors corresponding to the generators \( E_{ij} \) for \( i \) odd is compatible with this action. This means that all the relations out of (1)–(5) which contain only these generators are satisfied by the corresponding functors. Furthermore, using the relations (8.1), (8.2) one can see that all the relations out of (1)–(5) containing a generator \( E_{ij} \) with \( i \) and \( j \) of opposite parity in the left hand side, are satisfied by our functors. Similarly, the relation (5) follows from (4) using the relations (8.1) and (8.2). It remains to check the relation (1) for \( i \) and \( j \) even, and \( k \) and \( l \) odd, the relations (2), (3) for \( i \) even and \( l \) odd, and the relation (4) for \( l \) odd. This can be done directly applying the both sides of a relation to the object \( e_4 \mathcal{O}_S \in \mathcal{D}^b(A^n) \). Thus, the relations (1)–(5) hold for our functors. Now using that \( F^2_A \simeq \{ -1 \}^g \otimes \omega^1_{\mathcal{A}/[\mathcal{S}]} \), where \( g = \dim A/S \), and that \( F_A(L) \simeq L^{-1} \otimes \pi^*\pi_*L \) one can easily compute that the functor corresponding to \( (E_{11}E_{22}E_{11})^4 \) is \( (\cdot) \otimes \Delta(L)^{\otimes 2}[2g] \).

The \( \mathbb{Z} \)-part of the central extension of \( \tilde{\text{Sp}}_{2n}(\mathbb{Z}) \) acting on \( \mathcal{D}^b(A^n) \) was computed in [20]. Namely, for \( g = 1 \) the corresponding class in \( H^2(\text{Sp}_{2n}(\mathbb{Z}), \mathbb{Z}) \) is a half of the class of the cocycle given by the Malsov index. On the other hand, it is easy to see that the class of the central extension \( \tilde{\text{Sp}}_{2n}(\mathbb{Z}) \) is a generator of \( H^2(\text{Sp}_{2n}(\mathbb{Z}), \mathbb{Z}) \) for sufficiently large \( n \). Indeed, it is known that \( H^2(\text{Sp}_{2n}(\mathbb{Z}), \mathbb{Z}) = \mathbb{Z} \) for large \( n \) while \( \text{Sp}_{2n}(\mathbb{Z}) = [\text{Sp}_{2n}(\mathbb{Z}), \text{Sp}_{2n}(\mathbb{Z})] \) for \( n \geq 3 \). Moreover, the relations (2) and (4) easily imply that the element \( \epsilon \) belongs to \( [\tilde{\text{Sp}}_{2n}(\mathbb{Z}), \tilde{\text{Sp}}_{2n}(\mathbb{Z})] \), hence \( \tilde{\text{Sp}}_{2n}(\mathbb{Z}) = [\text{Sp}_{2n}(\mathbb{Z}), \tilde{\text{Sp}}_{2n}(\mathbb{Z})] \). It follows, that the central extension of \( \text{Sp}_{2n}(\mathbb{Z}) \) by \( \mathbb{Z}/p\mathbb{Z} \) obtained from \( \tilde{\text{Sp}}_{2n}(\mathbb{Z}) \) is non-trivial for every prime \( p \), so our claim follows.

Let \( \Gamma_{1,2} \subset \text{Sp}_{2n}(\mathbb{Z}) \) be the subgroup of matrices \( \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \) such that \( tM_{11}M_{12} \) and \( tM_{22}M_{21} \) have even diagonal entries.

Let \( A \) be a principally polarized abelian scheme over \( S \). Then we have a canonical splitting of the projection \( \tilde{\text{SL}}_2(A^n) \to \text{SL}_2(A^n) \) over \( \Gamma_{1,2} \) which is constructed as in the previous section using line bundles

\[
L(B) = \bigotimes_{i<j}(\phi_{ij}, [b_{ij}]_A p_j)^*\mathcal{P} \otimes \bigotimes_i(\phi_{ii}, [b_{ii}/2] p_i)^*\mathcal{P}
\]

associated with symmetric integer even-diagonal matrices \( B = (b_{ij}) \) (note that this time we don’t need any additional data on \( A \)). It is known (see [19] A.4) that \( \Gamma_{1,2} \) is generated by elements

\[
\varphi, \begin{pmatrix} tC^{-1} & 0 \\ 0 & C \end{pmatrix}, \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}
\]

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where $C \in \text{GL}_n(\mathbb{Z})$, $B$ is symmetric integer with even diagonal. Obviously, this implies vanishing of the obstruction for the projective action of $\Gamma_{1,2}$ on $\mathcal{D}^b(A^n)$ by intertwining operators, hence this leads to a central extension of $\Gamma_{1,2}$ by $\mathbb{Z} \times \text{Pic}(S)$.

**Proposition 8.2.** Let $A/S$ be a principally polarized abelian scheme of dimension $g \geq 3$. Then the central extension of $\Gamma_{1,2}$ by $\text{Pic}(S)$ acting on $\mathcal{D}^b(A^n)$ up to shifts is trivial.

**Proof.** Considering a finite flat covering of $S$ corresponding to a choice of a symmetric line bundle inducing a principal polarization and using Theorem 8.1 one can see that the central extension in question is induced by a central extension of $\Gamma_{1,2}$ by the torsion subgroup $\text{Pic}(S)^{\text{tors}} \subset \text{Pic}(S)$. Note that it is sufficient to prove our assertion in the case when $A$ is the universal abelian scheme over the moduli stack $\mathcal{A}_g$ of principally polarized abelian schemes. It remains to notice that $\text{Pic}(\mathcal{A}_g)^{\text{tors}} = 0$ since $\text{Sp}_{2g}(\mathbb{Z})$ has no abelian quotients for $g \geq 3$ (this is deduced using the Kummer exact sequence — see [18]).

**Corollary 8.3.** The central extension of $\text{Sp}_{2n}(\mathbb{Z})$ by $\mathbb{Z}/2\mathbb{Z}$ obtained by push-forward from $\tilde{\text{Sp}}_{2n}(\mathbb{Z})$ has a splitting over $\Gamma_{1,2}$.

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