On the $\nu$-zeros of the Bessel functions of purely imaginary order

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Abstract

The $\nu$-zeros of the Bessel functions of purely imaginary order are examined for fixed argument $x > 0$. In the case of the modified Bessel function of the second kind $K_{i\nu}(x)$, it is known that it possesses a countably infinite sequence of real $\nu$-zeros described by $\nu_n \sim \pi n / \log n$ as $n \to \infty$. Here we apply a unified approach to determine asymptotic estimates of the $\nu$-zeros of the modified Bessel functions $L_{i\nu}(x) \equiv I_{i\nu}(x) + L_{-i\nu}(x)$ and $K_{i\nu}(x)$ and the ordinary Bessel functions $J_{i\nu}(x) \pm J_{-i\nu}(x)$. Numerical results are presented to illustrate the accuracy of the expansions so obtained.

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1. Introduction

Bessel functions of purely imaginary order play an important role in quantum mechanics in the solution of the one-dimensional Schrödinger equation with exponential potentials. The Bessel function $K_{i\nu}(x)$ stands alone among the standard Bessel functions since it is real when the argument $x$ is positive and decays as $x \to +\infty$; the Bessel functions $J_{i\nu}(x)$, $Y_{i\nu}(x)$ and the modified Bessel function $I_{i\nu}(x)$ are all complex when $\nu$ and $x$ are real and nonzero.

A comprehensive discussion of the properties of the Bessel functions of imaginary order has been given by Dunster in [3]. In particular, he introduced the real function [3, (2.2)]

$$L_{i\nu}(x) = \frac{\pi}{2 \sinh \pi \nu} \{I_{i\nu}(x) + I_{-i\nu}(x)\} \quad (\nu \neq 0),$$

which is an appropriate numerically satisfactory companion to $K_{i\nu}(x)$ when $\nu$ is real and nonzero and $x > 0$. This may be contrasted with the standard definition of $K_{i\nu}(x)$ given by

$$K_{i\nu}(x) = \frac{\pi i}{2 \sinh \pi \nu} \{I_{i\nu}(x) - I_{-i\nu}(x)\}. \quad (1.2)$$

For the $J$-Bessel function of imaginary order, Dunster [3, (3.4)] introduced the real functions (when $x > 0$)

$$F_{i\nu}(x) = \frac{1}{2 \cosh 2 \pi \nu} \{J_{i\nu}(x) + J_{-i\nu}(x)\},$$

(1.3)
\[ G_{\nu}(x) = \frac{1}{2 \sinh \frac{\pi \nu}{2}} \{ J_{\nu}(x) - J_{-\nu}(x) \}, \quad (1.4) \]

which clearly satisfy \( F_{-\nu}(x) = F_{\nu}(x) \) and \( G_{-\nu}(x) = G_{\nu}(x) \). He obtained several properties satisfied by these functions, such as connection formulas, Wronskians, integral representations and the behaviour at singularities. In addition, he considered the zeros of \( L_{\nu}(x), F_{\nu}(x) \) and \( G_{\nu}(x) \) as a function of the real argument \( x \).

In a recent paper, Bagirova and Khanmamedov [1] considered the \( \nu \)-zeros of \( K_{\nu}(x) \) when \( x > 0 \) is fixed (that is, the zeros considered as a function of \( \nu \) rather than the argument \( x \)). They have shown that \( K_{\nu}(x) \) has a countably infinite number of (simple) real zeros in \( \nu \) when \( x > 0 \) is fixed. We label the zeros \( \nu_n \) \( (n = 1, 2, \ldots) \) and observe that it is sufficient to consider only the case \( \nu > 0 \) since \( K_{-\nu}(x) = K_{\nu}(x) \). By transforming the differential equation satisfied by \( K_{\nu}(x) \) into a one-dimensional Schrödinger equation with an exponential potential, these authors employed the well-known quantisation rule to deduce the leading asymptotic behaviour of the \( n \)th zero given by

\[ \nu_n \sim \frac{\pi n}{\log n} \quad (n \to +\infty). \quad (1.5) \]

This approximation was further discussed and refined in [5]. The \( \nu \)-zeros of the Hankel functions have been considered in [2, 3]. The numerical computation of the Bessel functions of purely imaginary order has been discussed in [6].

A more detailed investigation of the asymptotic behaviour of the \( n \)th zero of \( K_{\nu}(x) \) for fixed \( x > 0 \) has been given by the author in [5]. This relied on use of the well-known asymptotic expansion of \( K_{\nu}(x) \) for \( \nu \to +\infty \) and yielded very accurate results from a three-term expansion for \( \nu \). Here we consider the \( \nu \)-zeros of the functions \( L_{\nu}(x), K_{\nu}(x), F_{\nu}(x) \) and \( G_{\nu}(x) \) employing a simpler, unified approach. Although this approach does not produce results as accurate as those obtained in [5], it nevertheless yields very satisfactory agreement (from a three-term expansion) with the numerically computed \( \nu \)-zeros of these functions.

\section{The modified Bessel functions}

The modified Bessel function \( I_{\nu}(x) \) is defined by

\[ I_{\nu}(x) = \frac{(\frac{x}{2})^\nu}{\Gamma(1+i\nu)} \sum_{k=0}^{\infty} \frac{(\frac{x}{2})^{2k}}{k!(1+i\nu)_k}, \quad (2.1) \]

where \( (a)_n = \Gamma(a+n)/\Gamma(a) \) is the Pochhammer symbol. We can obtain the expansion of \( I_{\nu}(x) \) for \( \nu \to +\infty \) by making use of the well-known expansion

\[ \frac{1}{\Gamma(1+i\nu)} \sim \frac{1}{\sqrt{2\pi i\nu}} e^{i\nu} e^{-\nu} \sum_{k=0}^{\infty} \frac{\gamma_k}{(i\nu)^k} \quad (\nu \to +\infty), \]

where the first few Stirling coefficients \( \gamma_k \) are

\[ \gamma_0 = 1, \quad \gamma_1 = -\frac{1}{12}, \quad \gamma_2 = \frac{1}{288}, \quad \gamma_3 = \frac{139}{51840}, \quad \gamma_4 = -\frac{571}{2488320}, \quad \gamma_5 = -\frac{163879}{209018880}. \]

We observe that the sum appearing in (2.1) for fixed \( x \) can be written in the form

\[ \sum_{k=0}^{\infty} \frac{(\frac{x}{2})^{2k}}{k!(1+i\nu)_k} \sim \sum_{k=0}^{\infty} \frac{C_k(\chi)}{(i\nu)^k}, \quad \chi := \frac{x^2}{4} \quad (\nu \to +\infty), \]

where the coefficients \( C_k(\chi) \) for \( 0 \leq k \leq 5 \) are

\[ C_0(\chi) = 1, \quad C_1(\chi) = \chi, \quad C_2(\chi) = \frac{\chi}{2}(-2 + \chi), \]
\[ C_3(\chi) = \frac{\chi}{6}(6 - 9\chi + \chi^2), \quad C_4(\chi) = \frac{\chi}{24}(-24 + 84\chi - 24\chi^2 + \chi^3), \quad C_5(\chi) = \frac{\chi}{120}(120 - 900\chi + 500\chi^2 - 50\chi^3 + \chi^4). \] (2.2)

Then we obtain the large-\(\nu\) expansion of \(I_{\nu}(x)\) in the form
\[
I_{\nu}(x) \sim \frac{e^{\frac{i}{2}x}}{\sqrt{2\pi\nu}} e^{-i\Phi} \sum_{k=0}^{\infty} \frac{\gamma_k}{(i\nu)^k} = \frac{e^{\frac{i}{2}x}}{\sqrt{2\pi\nu}} e^{-i\Phi} \sum_{k=0}^{\infty} \frac{a_k}{(i\nu)^k},
\] (2.3)

where
\[
\Phi := \nu \log \nu - \nu + \nu \log \frac{1}{2} + \frac{\pi}{4} = \nu \log \lambda + \frac{\pi}{4}, \quad \lambda := \frac{2}{e^x}.
\] (2.4)

The coefficients \(a_k\) are
\[
a_0 = 1, \quad a_1 = C_1(\chi) + \gamma_1 C_0(\chi), \quad a_2 = C_2(\chi) + \gamma_1 C_1(\chi) + \gamma_2 C_0(\chi),
\]
and in general,
\[
a_k = C_k(\chi) + \sum_{r=1}^{k} \gamma_r C_{k-r}(\chi) \quad (k \geq 1).
\]

The same procedure applied to \(I_{-\nu}(x)\) leads to the expansion
\[
I_{-\nu}(x) \sim \frac{e^{-\frac{i}{2}x}}{\sqrt{2\pi\nu}} e^{i\Phi} \sum_{k=0}^{\infty} \frac{a_k}{(-i\nu)^k} \quad (\nu \to +\infty).
\] (2.5)

The expansions (2.3) and (2.5) can now be employed in the definitions of the modified Bessel functions \(L_{\nu}(x)\) and \(K_{\nu}(x)\) defined in (1.1) and (1.2). We then obtain
\[
L_{\nu}(x) \sim \frac{\pi}{\sinh \pi\nu} \frac{e^{\frac{i}{2}x}}{\sqrt{2\pi\nu}} \left\{ \cos \Phi \left( 1 - \frac{a_2}{\nu^2} + \frac{a_4}{\nu^4} - \cdots \right) - \sin \Phi \left( \frac{a_1}{\nu} - \frac{a_3}{\nu^3} + \frac{a_5}{\nu^5} - \cdots \right) \right\}
\] (2.6)

and
\[
K_{\nu}(x) \sim \frac{\pi}{\sinh \pi\nu} \frac{e^{\frac{i}{2}x}}{\sqrt{2\pi\nu}} \left\{ \sin \Phi \left( 1 - \frac{a_2}{\nu^2} + \frac{a_4}{\nu^4} - \cdots \right) + \cos \Phi \left( \frac{a_1}{\nu} - \frac{a_3}{\nu^3} + \frac{a_5}{\nu^5} - \cdots \right) \right\}
\] (2.7)

for \(\nu \to +\infty\).

### 2.1 Determination of the zeros

We first consider the \(\nu\)-zeros of \(L_{\nu}(x)\) and, in (2.3), accordingly set
\[
\Phi = (n + \frac{1}{2})\pi - \epsilon,
\] (2.8)

where \(n\) is a large positive integer and \(\epsilon\) is a small quantity. From (2.6), the \(\nu\)-zeros are determined by
\[
\sin \epsilon \left( 1 - \frac{a_2}{\nu^2} + \frac{a_4}{\nu^4} - \cdots \right) - \cos \epsilon \left( \frac{a_1}{\nu} - \frac{a_3}{\nu^3} + \frac{a_5}{\nu^5} - \cdots \right) = 0,
\] (2.9)

whence
\[
\tan \epsilon = \frac{\frac{a_1}{\nu} - \frac{a_3}{\nu^3} + \frac{a_5}{\nu^5} - \cdots}{1 - \frac{a_2}{\nu^2} + \frac{a_4}{\nu^4} - \cdots} = \frac{a_1}{\nu} + \frac{a_1 a_2 - a_3}{\nu^3} + \frac{a_1 a_2 - a_2 a_3 - a_1 a_4 + a_5}{\nu^5} + \cdots.
\]
Inversion of the tangent then yields
\[ \epsilon = \frac{A_0}{\nu} + \frac{A_1}{\nu^3} + \frac{A_2}{\nu^5} + \cdots \] (2.10)
for \( \nu \to +\infty \), where
\[ A_0 = a_1, \quad A_1 = a_1a_2 - a_3 - \frac{1}{3}a_1^3, \]
\[ A_2 = a_1^2 + a_2^3 - a_3^2 - a_2a_3 - a_1a_4 + a_5 + \frac{1}{5}a_1^5. \] (2.11)
From (2.4) and (2.8), we finally obtain the equation describing the large-\( n \) zeros of \( L_{i\nu}(x) \) given by
\[ \nu \log \lambda \nu = (n + \frac{1}{4})\pi - \epsilon = m_+ - \frac{A_0}{\nu} - \frac{A_1}{\nu^3} - \frac{A_2}{\nu^5} - \cdots, \] (2.12)
where for convenience we have put \( m_+ = (n + \frac{1}{4})\pi \).
To solve this equation we now expand \( \nu \equiv \nu_n \) as
\[ \nu_n = \xi + \frac{b_0}{\xi} + \frac{b_1}{\xi^3} + \frac{b_2}{\xi^5} + \cdots, \]
where the \( b_k \) are constants to be determined and we suppose that \( \xi \) is large as \( n \to \infty \).
Substitution in (2.12) then produces
\[ \xi \log \lambda \xi + \frac{b_0(1 + \log \lambda \xi)}{\xi} + \frac{b_1(1 + \log \lambda \xi)}{\xi^3} + \frac{b_2(1 + \log \lambda \xi)}{\xi^5} + \cdots = m_+ - \left( \frac{A_0}{\xi} + \frac{A_1 - A_0b_0}{\xi^3} + \frac{A_2 - 3A_1b_0 + A_0(b_0^2 - b_1)}{\xi^5} + \cdots \right). \]
Equating coefficients of like powers of \( \xi \), we obtain
\[ \xi \log \lambda \xi = m_+, \] (2.13)
and
\[ b_0 = \frac{-A_0}{1 + \log \lambda \xi}, \quad b_1 = \frac{A_0 - A_1b_0 - \frac{1}{2}b_0^2}{1 + \log \lambda \xi}, \]
\[ b_2 = \frac{3A_1b_0 - A_0(b_0^2 - b_1) - A_2 - b_0b_1 + \frac{1}{6}b_0^3}{1 + \log \lambda \xi}. \]
The solution of (2.13) for the lowest-order term \( \xi \) can be expressed in terms of the Lambert \( W \) function, which is the (positive) solution\(^3\) of \( W(z)e^{W(z)} = z \) for \( z > 0 \). Rearrangement of (2.13) shows that
\[ \frac{m_+}{\xi} e^{m_+ / \xi} = \lambda m_+, \]
whence
\[ \frac{m_+}{\lambda} = \frac{m_+}{W(\lambda m_+)} \] (2.14)
If we define \( \chi := \xi / m_+ \), so that by (3.1) \( 1 + \log \lambda \xi = (1 + \chi) / \chi \) and introduce the coefficients
\[ B_0 = \frac{-A_0}{1 + \chi}, \quad B_1 = \frac{A_0b_0 - A_1 - \frac{1}{2}b_0^2}{\chi^2(1 + \chi)}, \quad B_2 = \frac{3A_1b_0 - A_0(b_0^2 - b_1) - A_2 - b_0b_1 + \frac{1}{6}b_0^3}{\chi^4(1 + \chi)}, \] (2.15)
then we finally have the result:
\[^3\text{In [7 p. 111] this is denoted by } W_{\nu}(z).\]
Theorem 1. The expansion for the $n$th $\nu$-zero of $L_{\nu}(x)$ for fixed $x > 0$ is
\[ \nu_n \sim \frac{m_+}{W(\lambda m_+)} + \frac{B_0}{m_+^3} + \frac{B_1}{m_+^5} + \cdots \quad (n \to \infty), \] (2.16)
where $m_+ = (n + \frac{1}{4})\pi$, $\lambda = 2/(ex)$ and the coefficients $B_k$ are given in (2.15).

The same procedure can be applied to determine the $\nu$-zeros of $K_{\nu}(x)$. In (2.7), we now set
\[ \Phi = n\pi - \epsilon, \]
where again $n$ is a large positive integer and $\epsilon$ is a small quantity, to produce the same equation given in (2.9). Consequently, the value of $\epsilon$ is given by (2.10) with the same coefficients $A_k$. Then
\[ \nu \log \lambda \nu = (n - \frac{1}{4})\pi - \epsilon = m_- - \frac{A_0}{\nu} - \frac{A_1}{\nu^3} - \frac{A_2}{\nu^5} - \cdots, \]
where $m_- = (n - \frac{1}{4})\pi$ and so we obtain the final result:

Theorem 2. The expansion for the $n$th $\nu$-zero of $K_{\nu}(x)$ for fixed $x > 0$ is
\[ \nu_n \sim \frac{m_-}{W(\lambda m_-)} + \frac{B_0}{m_-^3} + \frac{B_1}{m_-^5} + \cdots \quad (n \to \infty), \] (2.17)
where $m = (n - \frac{1}{4})\pi$ and $\lambda = 2/(ex)$ and the coefficients $B_k$ are given in (2.15).

The leading behaviour $\nu_n \sim \xi$ can be seen from the asymptotic expansion of $W(z)$ for $z \to +\infty$ [7, (4.13.10)]
\[ W(z) \sim \log z - \log \log z + \frac{\log \log z}{\log z} + \cdots, \]
from which it follows that
\[ \frac{1}{W(z)} \sim 1 \log z \left(1 + \frac{\log \log z}{\log z} + \cdots\right). \]

Then, for the Bessel functions $L_{\nu}(x)$ and $K_{\nu}(x)$, we have the leading behaviour of the $\nu$-zeros given by
\[ \nu_n \sim \frac{m_{\pm}}{\log \lambda m_{\pm}} \quad (n \to \infty), \] (2.18)
respectively.

3. The Bessel functions $F_{\nu}(x)$ and $G_{\nu}(x)$

The $J$-Bessel function of purely imaginary order is defined by
\[ J_{\nu}(x) = \frac{(\frac{1}{2}x)^i\nu}{\Gamma(1 + i\nu)} \sum_{k=0}^{\infty} (-i)^k (\frac{1}{2}x)^{2k}/k!(1 + i\nu)_k, \]
from which we obtain in an analogous manner to that described in Section 2 the large-$\nu$ expansion
\[ J_{\pm i\nu}(x) \sim \frac{e^{\mp i\nu\Phi}}{\sqrt{2\pi\nu}} e^{\mp i\Phi} \sum_{k=0}^{\infty} \frac{\hat{a}_k}{(\pm i\nu)^k} \quad (\nu \to +\infty), \] (3.1)
where $\Phi$ is defined in (2.4). The coefficients $\hat{a}_k$ are given by

$$\hat{a}_0 = 1, \quad \hat{a}_k = C_k(-\chi) + \sum_{r=1}^k \gamma_k C_{r-k}(-\chi) \quad (k \geq 1),$$

with the first few $C_k(\chi)$ defined in (2.2).

If the expansion (3.1) is substituted into the definitions of $F_{i\nu}(x)$ and $G_{i\nu}(x)$ in (1.3) and (1.4), we obtain

$$\cosh \frac{1}{2} \pi \nu F_{i\nu}(x) \sim \frac{\epsilon^{\frac{1}{2} \pi \nu}}{\sqrt{2\pi \nu}} \left\{ \cos \Phi \left( 1 - \frac{\hat{a}_2}{\nu^2} + \frac{\hat{a}_4}{\nu^4} + \cdots \right) - \sin \Phi \left( \frac{\hat{a}_1}{\nu} - \frac{\hat{a}_3}{\nu^3} + \frac{\hat{a}_5}{\nu^5} - \cdots \right) \right\}$$

and

$$\sinh \frac{1}{2} \pi \nu G_{i\nu}(x) \sim -\frac{\epsilon^{\frac{1}{2} \pi \nu}}{\sqrt{2\pi \nu}} \left\{ \sin \Phi \left( 1 - \frac{\hat{a}_2}{\nu^2} + \frac{\hat{a}_4}{\nu^4} + \cdots \right) + \cos \Phi \left( \frac{\hat{a}_1}{\nu} - \frac{\hat{a}_3}{\nu^3} + \frac{\hat{a}_5}{\nu^5} - \cdots \right) \right\}$$

as $\nu \to +\infty$. Proceeding as in Section 2, we set $\Phi = (n + \frac{1}{2})\pi - \epsilon$ for $F_{i\nu}(x)$ and $\Phi = n\pi - \epsilon$ for $G_{i\nu}(x)$. This produces

$$\epsilon = \frac{A_0}{\nu} + \frac{A_1}{\nu^3} + \frac{A_2}{\nu^5} + \cdots,$$

where the coefficients $A_k$ are specified in (2.11) with the $a_k$ replaced by $\hat{a}_k$.

Omitting the details, we finally obtain the result:

**Theorem 3.** The expansion for the $n$th $\nu$-zero of $F_{i\nu}(x)$ for fixed $x > 0$ is

$$\nu_n \sim \frac{m_+}{W(\lambda m_+)} + \frac{B_0}{m_+} + \frac{B_1}{m_+^3} + \frac{B_2}{m_+^5} + \cdots \quad (n \to \infty), \quad (3.2)$$

and that for $G_{i\nu}(x)$ is

$$\nu_n \sim \frac{m_-}{W(\lambda m_-)} + \frac{B_0}{m_-} + \frac{B_1}{m_-^3} + \frac{B_2}{m_-^5} + \cdots \quad (n \to \infty), \quad (3.3)$$

where $m_\pm = (n \pm \frac{1}{2})\pi$ and $\lambda = 2/(ex)$. The coefficients $B_k$ are given in (2.14) and the $A_k$ in (2.11) with the $a_k$ replaced by $\hat{a}_k$.

4. Numerical results

In this section we present results to illustrate the accuracy of the expansions in Theorems 1–3. In numerical calculations it is found more accurate to use the expression for $\xi$ in terms of the Lambert function in (2.14), rather than the asymptotic estimate, since the scale in this latter series is log $\lambda m_\pm$ and so requires an extremely large value of $n$ to attain reasonable accuracy.

We present numerical results in Table 1 showing the zeros of $L_{i\nu}(x)$ and $K_{i\nu}(x)$ computed using the FindRoot command in Mathematica compared with the asymptotic values determined from the expansions (2.10) and (2.17) with coefficients $B_k$, $k \leq 2$, where $\xi$ is evaluated from (2.14). It is seen that there is satisfactory agreement with the computed zeros, even for $n = 1$. However, it should be remarked that the values for the $\nu$-zeros of $K_{i\nu}(x)$ obtained in [8] using the standard asymptotic expansion of $K_{i\nu}(x)$ for large $\nu$ yielded far more accurate values from a three-term expansion. For example, when $n = 10$ (and $x = 1$) there was agreement to 8dp, whereas in Table 1 there is agreement to only 3dp. A similar set of results is shown in Table 2 for the Bessel functions $F_{i\nu}(x)$ and $G_{i\nu}(x)$. 
Table 1: Values of the zeros of $L_{i\nu}(x)$ and $K_{i\nu}(x)$ and their asymptotic estimates when $x = 1$.

| $n$ | Zero $\nu_n$ | Asymptotic | Zero $\nu_n$ | Asymptotic |
|-----|--------------|------------|--------------|------------|
| 1   | 3.790205     | 3.786398   | 2.962549     | 2.962961   |
| 2   | 5.225963     | 5.223461   | 4.534491     | 4.531277   |
| 3   | 6.505143     | 6.503534   | 5.879867     | 5.877888   |
| 4   | 7.691206     | 7.690065   | 7.107584     | 7.106243   |
| 5   | 8.812990     | 8.812124   | 8.258936     | 8.257949   |
| 10  | 13.861303    | 13.860936  | 13.385883    | 13.385492  |
| 20  | 22.620755    | 22.620598  | 22.207659    | 22.207497  |
| 50  | 45.082187    | 45.082135  | 44.732940    | 44.732888  |

Table 2: Values of the zeros of $F_{i\nu}(x)$ and $G_{i\nu}(x)$ and their asymptotic estimates when $x = 1$.

| $n$ | Zero $\nu_n$ | Asymptotic | Zero $\nu_n$ | Asymptotic |
|-----|--------------|------------|--------------|------------|
| 1   | 3.850274     | 3.844515   | 3.045668     | 3.031436   |
| 2   | 5.265045     | 5.263499   | 4.581762     | 4.578794   |
| 3   | 6.534299     | 6.534022   | 5.913240     | 5.912492   |
| 4   | 7.714536     | 7.714724   | 7.133494     | 7.133503   |
| 5   | 8.832476     | 8.832846   | 8.280167     | 8.280467   |
| 10  | 13.872097    | 13.872514  | 13.397175    | 13.397602  |
| 20  | 22.626541    | 22.626791  | 22.213581    | 22.213837  |
| 50  | 45.084649    | 45.084747  | 44.735426    | 44.735525  |

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