Reciprocity in dyadic interactions is common and a topic of interest across disciplines. In some cases, reciprocity may be expected to be more or less prevalent among certain kinds of dyads. In response to interest among researchers in estimating dyadic reciprocity as a function of covariates, this paper proposes an extension to the multilevel Social Relations Model. The outcome variable is assumed to be a binomial proportion, as is commonly encountered in observational and archival research. The approach draws on principles of multilevel modeling to implement random intercepts and slopes that vary among dyads. The corresponding variance function permits the computation of a dyadic reciprocity correlation. The modeling approach can potentially be integrated with other statistical models in the field of social network analysis.

Keywords Social Network Analysis · Dyadic Reciprocity · Social Relations Model

1 Introduction

In recent decades, methods and concepts from social network analysis have been incorporated into diverse academic disciplines, ranging from sociology to ecology to political science and physics [1, 2]. Although social network analysis is often associated with survey research, observational and archival data present opportunities to study dyadic behavior, arguably without the biases that characterize survey research with informants [3, 4, 5, 6, 7, 8]. In many settings, particularly studies of animal behavior, observational research may represent the only viable option for obtaining data on social relationships. Often, the datasets from these studies are characterized by observations of events over a period of time. For instance, ecologists may document whether individual i helped individual j during an observation.

For ecologists, there is often substantive interest in the extent of dyadic reciprocity between individuals in a population. This interest relates in part to theoretical models of cooperation, where contingent reciprocity represents one possible path toward helping behavior among individuals [9]. By extension, researchers have considered the extent to which dyadic reciprocity may vary as a function of another variable. For instance, individuals who share a kinship tie may exhibit greater reciprocity than unrelated individuals [10].

The statistical analysis of relational data poses well-known challenges, however, and multiple statistical approaches have been advanced to address the structural dependencies that typify these data [11]. One approach is the multilevel Social Relations Model (SRM), which employs random effects to partition data into separate giving, receiving, and relational components [12, 13]. Originally developed for continuous responses, the SRM can be adapted for discrete responses, such as binary and count outcomes [14, 15, 16].

Conventional applications of the SRM, however, assume that dyadic reciprocity is constant across all dyads in the sample. In other words, the conventional SRM cannot be used to assess the extent to which dyadic reciprocity varies as a function of a covariate. Here I advance an extension of the multilevel SRM that incorporates varying slopes to permit such analyses. The model assumes that dyads have been observed multiple times and that the outcome variable is a binomial proportion.
2 The Statistical Model

Imagine an outcome variable, \( y_{ij} \), in which the response reflects a binomial proportion where the numerator is the number of times that node \( i \) directed a behavior toward node \( j \) and the denominator represents the total number of opportunities for the behavior to have been directed. Further imagine a predictor variable, \( x_{ij} \), that represents a dyadic characteristic, such as the degree of relatedness between individuals \( i \) and \( j \) [17].

A binomial regression model can then be specified:

\[
y_{ij} \sim \text{binomial}(n_{ij}, p_{ij}) \\
\operatorname{logit}(p_{ij}) = \alpha + a_i + b_j + \beta(x_{ij}) + u_{ij|} + v_{ij|}(x_{ij}) + d_{ij}
\]

where \( \alpha \) represents the intercept, \( a_i \) is a node-level random intercept for the sending node \( i \), \( b_j \) is an analogous random intercept for incoming events to \( j \), \( \beta(x_{ij}) \) is a conventional fixed effect parameter for the predictor, \( u_{ij} \) and \( v_{ij}(x_{ij}) \) are the respective random intercept and slope for dyad \( ij \), and \( d_{ij} \) is a parameter that captures possible overdispersion beyond the conditional expectation for the binomial distribution.

The random effects for sending and recipient nodes are assumed bivariate normally distributed with zero means and homogeneous covariance matrix

\[
\begin{pmatrix} a_i \\ b_i \end{pmatrix} \sim \text{Normal}\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_a^2 & \sigma_{ab} \\ \sigma_{ab} & \sigma_b^2 \end{pmatrix} \right\}
\]

As in other applications of the SRM [13], the correlation between the respective effects is known as the "generalized reciprocity correlation."

The random effects for the dyad-level intercepts and slopes are likewise assumed bivariate normally distributed with zero means and homogeneous covariance matrix

\[
\begin{pmatrix} u_{ij} \\ v_{ij} \end{pmatrix} \sim \text{Normal}\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_u^2 & \sigma_{uv} \\ \sigma_{uv} & \sigma_v^2 \end{pmatrix} \right\}
\]

Importantly, the notation of \( |ij| \) is an indicator for a symmetric dyadic relationship. That is, within each dyad, the relationship from \( i \) to \( j \) and the relationship from \( j \) to \( i \) share the same index [19].

The inclusion of the parameter for additive overdispersion follows Browne et al. [20]. As in standard applications of multilevel modeling, these effects are assumed to be normally distributed:

\[
d_{ij} \sim \text{Normal}(0, \sigma_d^2)
\]

There may be binomial data that exhibit minimal overdispersion, in which case this parameter could potentially be omitted.

3 Estimating Dyadic Reciprocity

As noted, this approach estimates symmetric dyad effects. When this approach is used in a conventional SRM for continuous data, Snijders and Kenny [12] show that dyadic reciprocity, \( \rho \), can be estimated as:

\[
\rho = \frac{\sigma_u^2}{\sigma_u^2 + \sigma_v^2}
\]

where \( \sigma_u^2 \) is the variance of the symmetric dyad effects and \( \sigma_v^2 \) is the residual variance (implicitly corresponding to the directed dyadic observations, which are the unit of analysis). This parameterization constrains dyadic reciprocity to be positively correlated, which may be a reasonable assumption in many cases.

The same logic can be used to estimate dyadic reciprocity in a binomial SRM. However, these models lack the constant residual variance of Gaussian regression models. As an alternative, latent parameterizations of binomial models may assume that the corresponding variance is \( \pi^2/3 \), or 3.29 [21]. This quantity can be substituted into the denominator along with the variance for overdispersion effects.

Unlike a conventional SRM with symmetric dyadic effects, the dyadic variance in the above model is not constant. Rather, it varies as a function of the predictor variable, \( x_{ij} \). Snijders [21] notes that in multilevel models with random slopes, the variance follows a quadratic function of the predictor:

\[
\sigma_u^2 + 2\sigma_{uv}x_{ij} + \sigma_v^2x_{ij}^2
\]
This expression can take the place of the dyadic variance, yielding a corresponding calculation for dyadic reciprocity:

$$\rho = \frac{\sigma^2_u + 2\sigma_{uv}x_{ij} + \sigma^2_vx^2_{ij}}{\sigma^2_x + 2\sigma_{ux}x_{ij} + \sigma^2_xx^2_{ij} + \sigma^2_d + 3.29}$$

Once the parameters from the statistical model have been estimated, the dyadic reciprocity correlation can be calculated for different values of \( x \).

4 Discussion

This paper introduces a binomial Social Relations Model that permits the estimation of dyadic reciprocity as a function of a predictor variable. This is accomplished via the parameterization of random effects that correspond to a varying intercept and a varying slope in conventional applications of multilevel modeling [22]. The dyadic variance can then vary as a function of the predictor variable and compared against the residual variance to calculate reciprocity. The present model could be expanded further with the inclusion of other covariates and parameters.

The model here assumes that the outcome is composed of binomial proportions. Such data may be common among researchers who use observational or archival data. In principle, the data could be disaggregated so that the response represent a single binary observation, in which case a comparable random effects structure above could be adapted while permitting analogous estimates of dyadic reciprocity [23].

The use of the SRM often assumes a "round robin" design in which the dyadic relationships among the individuals or nodes are fully observed. In observational studies, however, individuals may not be simultaneously present. In experimental studies, occasionally a subset of dyadic interactions are precluded by the research design [24, 25]. As with broader applications of the Social Relations Model to "block" designs [26], the above parameterization could likewise be employed by omitting such unobserved dyads from the analysis.

Whereas the SRM partitions variance according to givers, receivers, and relational components, a number of researchers have observed that social networks often exhibit additional structural dependencies, such as transitivity and block structures [27, 28, 29]. In this literature, dyadic reciprocity are often a secondary consideration relative to other aspects of network structure (cf. [30]). It may be possible for these statistical approaches to network data to be modified to examine dyadic reciprocity in greater detail alongside other aspects of the data-generating process.

An advantage of the modeling approach above is that it builds on principles that are common in the multilevel modeling literature, namely random intercepts and slopes and the corresponding variance function. One consequence of this approach is that the model is highly parameterized, as two parameters are estimated for each dyad. Another consideration is that the dyadic reciprocity correlation is constrained to be positive. As an alternative, Koster et al. [19] suggested estimating the reciprocity correlation via a tanh link function that allows the dyadic variance and covariance to be modeled as a function of predictors [33]. In some cases, that latter approach could be more advantageously implemented in a Social Relations Model than the present alternative, particularly in contexts where negative reciprocity correlations might be expected.

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1Other modeling approaches emphasize the temporal dynamics in longitudinal dyadic data, with relatively greater attention to reciprocity [31, 32].
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