INFINITE TIME BLOW-UP OF MANY SOLUTIONS TO A GENERAL QUASILINEAR PARABOLIC-ELLiptIC KELLER–SEGEL SYSTEM

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ABSTRACT. We consider a parabolic-elliptic chemotaxis system generalizing

\[
\begin{align*}
    u_t &= \nabla \cdot ((u + 1)^{m-1}\nabla u) - \nabla \cdot (u(u + 1)^{\sigma-1}\nabla v) \\
    0 &= \Delta v - v + u
\end{align*}
\]

in bounded smooth domains \( \Omega \subset \mathbb{R}^N, N \geq 3 \), and with homogeneous Neumann boundary conditions. We show that

- solutions are global and bounded if \( \sigma < m - \frac{N-2}{N} \)
- solutions are global if \( \sigma \leq 0 \)
- close to given radially symmetric functions there are many initial data producing unbounded solutions if \( \sigma > m - \frac{N-2}{N} \).

In particular, if \( \sigma \leq 0 \) and \( \sigma > m - \frac{N-2}{N} \), there are many initial data evolving into solutions that blow up after infinite time.

1. Introduction. Whereas diffusion has an equilibrating effect, cross-diffusive terms appearing in chemotaxis models like

\[
\begin{align*}
    u_t &= \nabla \cdot ((u + 1)^{m-1}\nabla u) - \nabla \cdot (u(u + 1)^{\sigma-1}\nabla v) \\
    0 &= \Delta v - v + u
\end{align*}
\]

tend to lead to the exact opposite, to aggregation. It is therefore of interest to characterize which of these mechanisms is more decisive for the solution behaviour, in dependence on their relative strengths as given by the size of the exponents \( m \) and \( \sigma \) in (1). Are all solutions global and bounded? Do some solutions blow up? If so, in finite or in infinite time? Indeed, there are studies showing for the related parabolic-parabolic chemotaxis system (where \( 0 \) in the second equation of (1) is replaced by \( v_t \)) that for different choices of \( m \) and \( \sigma \), any of these qualitatively different behaviours can be observed. One of the main tools for proving the existence of unbounded solutions is the use of an energy functional together with the construction of suitable initial data \( u_0, v_0 \) – which makes this one of the few respects in which the fully parabolic system is easier to deal with than the parabolic-elliptic “simplification” (1). After all, there it is possible to choose \( u_0 \) and \( v_0 \) independently of each other, whereas in (1) only \( u_0 \) can be selected, providing us with much less freedom for the construction. It is the parabolic–elliptic setting we are going to consider here; mainly being interested in \( \sigma \leq 0 \).

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Before we do so, let us briefly recall some of the known results in related models:

We will begin with the **parabolic-parabolic model**

\[
\begin{align*}
    u_t &= \nabla \cdot (D(u) \nabla u) - \nabla \cdot (S(u) \nabla v) \\
    v_t &= \Delta v - v + u
\end{align*}
\]  

(2)

in bounded domains \( \Omega \subset \mathbb{R}^N \) with no-flux boundary conditions and for easier comparison state the results for \( D(u) = (u + 1)^{m-1} \) and \( S(u) = u(u + 1)^{\sigma-1} \) or \( S(u) = u^\sigma \) and let

\[\alpha = m - \sigma - 1,\]

so that \( \frac{\alpha}{\sigma} \) approximately has the form of \( u^\alpha \). If \( m = 1 = \sigma \), then (2) becomes the classical Keller–Segel model, where it is known that all solutions exist globally and are bounded if \( N = 1 \) ([27]), for \( N = 2 \), smallness of the initial mass \( \int_{\Omega} u(\cdot, 0) \) is sufficient to guarantee boundedness ([11, 25]), whereas for large initial mass in \( N = 2 \) ([13, 28], cf. also [12]) and any mass in \( N \geq 3 \) ([35]) there are initial data leading to unbounded solutions. It has also been shown that this blow-up occurs within finite time for a “large” set of (radially symmetric) initial data ([38, 22]).

Retaining linear diffusion (\( m = 1 \)) but varying \( \sigma \), it turns out that \( -\alpha < \frac{2}{N} \) leads to global existence, but on the other hand if \( -\alpha \) is slightly larger than \( \frac{2}{N} \), then there are unbounded radially symmetric solutions, [14]. If both diffusion and sensitivity are allowed to be nonlinear, it is the same condition distinguishing global existence from possible blow-up: If \( -\alpha > \frac{2}{N} \), then there are unbounded solutions ([36]), whereas complementarily [31] asserts global boundedness under the condition that \( -\alpha < \frac{2}{N} \). (For analogous boundedness and blow-up results in a two-species model see [32].) Also, it was for models of this kind that the convexity assumption on domains often used in earlier works on chemotaxis models was removed in [16], where again \( -\alpha < \frac{2}{N} \) is the condition ensuring global boundedness of the solutions.

If \( \Omega \) is 1-dimensional and \( \sigma = 1 \), the solutions also remain bounded in the case \( \alpha = -2 = -\frac{2}{N} \) as has been shown very recently in [4].

In the presence of logistic source terms, one condition ensuring global existence again is \( -\alpha < \frac{2}{N} \) – another would be sufficient strength of the consumptive part of the logistic source (for precise conditions refer to [21, 33]); blow-up results have not been obtained.

The case of degenerate diffusion (\( D(u) = u^{m-1} \) instead of \( D(u) = (u + 1)^{m-1} \)) requires additional technical care (and restrictions such as \( m \geq 1 \)), but finally the same conditions on \( \alpha \) are recovered, for boundedness ([17, 29, 30]) as well as for blow-up ([15, 18]).

With the exception of [38], the works mentioned up to this point do not help in distinguishing blowup in finite time from that occurring after infinite time. Building on the method of [38], in [6] Cieslak and Stinner showed that finite-time blowup occurs if \( N \geq 3: \sigma \geq 1, m \in \mathbb{R}, -\alpha > \frac{2}{N} \). Results pertaining to 2-dimensional domains can be found in [7]. The more recent extension [8] of [6] showed finite-time blow-up if \( -\alpha > \frac{2}{N} \), \( m \geq 1 \) \( N \geq 3 \) or \( \sigma > \frac{2}{N} \), \( -\alpha > \frac{2}{N} \) (and \( m \geq 1 \) if \( \sigma < 1 \)) and infinite-time blow-up under the condition that \( \frac{2}{N} < -\alpha < \frac{2}{N} - \sigma \). Similarly, solutions blow-up after infinite time if \( \sigma \leq 0 \) and \( -\alpha > \frac{2}{N} \) ([39]). These papers also show that blow-up occurs for “many” initial data. For a different class of diffusivity and sensitivity functions, consult [40], which gives conditions ensuring blow-up in infinite time for \( D \) and \( S \) being of exponential type.
Another relative of (1) is the further simplified system

\[ u_t = \nabla \cdot (D(u) \nabla u) - \nabla \cdot (S(u) \nabla v) \]
\[ 0 = \Delta v - \int_\Omega u_0 + u. \]

It has the convenient property that the analysis can be performed on a single scalar parabolic equation for the cumulated mass \( w(r,t) = \int_0^r \rho u(r,t) \, dr \), which – in contrast to (1) – is accessible for comparison arguments. For the classical case of 2-dimensional domains, \( m = \sigma = 1 \), Jäger and Luckhaus ([19]) thereby showed existence of radially symmetric initial data such that \( u \) explodes in the center of the domain after finite time. On the other hand, solutions rising from initial data with small mass exist globally ([2]).

If \( \sigma = 1 \) and \( m \) is such that \( -\alpha < \frac{2}{N} \), then solutions exist globally and are bounded, whereas \( -\alpha > \frac{2}{N} \) may incur blow-up within finite time ([9]). If \( \sigma \) may vary and \( m \leq 1 \), [41] again asserts boundedness under the condition that \( -\alpha < \frac{2}{N} \) and the possibility of blow-up if \( -\alpha > \frac{2}{N} \), provided \( \sigma > 0 \).

Albeit only in higher dimensions (\( N \geq 5 \)) and for small \( k > 1 \), in (3) with additional source \(+ u - \mu u^k\), finite-time blow-up was shown despite the logistic growth restriction in [43] by extension of [37] if \( m \in (0, 2 - \frac{2}{N}) \) and \( \sigma \in (1, 2 + 2m) \) ([43, Theorem 1.2]). Again, largeness of \( \mu \) and \( k \) or \( -\alpha < \frac{2}{N} \) ensure global existence.

As to the parabolic-elliptic system (1), the only available blow-up results deal with the classical model with \( m = \sigma = 1 \), where finite-time blow-up has been shown to occur in two-dimensional domains for radial initial data with sufficiently large mass that are concentrated in the sense that their second moment is small [24, 23], or in higher dimensional domains, where a higher moment seemed decisive [24] (for a corresponding result in \( \Omega = \mathbb{R}^N \) and condition on the second moment see [3]).

On the other hand, for other choices of \( m \) and \( \sigma \), \( -\alpha < \frac{2}{N} \) again ensures global boundedness (see [5, Thm 5.3] for \( N = 3 \), [21, 42] for a general system also including logistic source terms, or [34] for a closely related parabolic-elliptic-elliptic attraction-repulsion system).

**Results.** For functions

\[ D(u) = (u + 1)^{m-1}, \quad S(u) = u(u + 1)^{\sigma-1} \]

in

\[ u_t = \nabla \cdot (D(u) \nabla u) - \nabla \cdot (S(u) \nabla v) \quad \text{in } \Omega \times (0, T_{\text{max}}) \]
\[ 0 = \Delta v - v + u \quad \text{in } \Omega \times (0, T_{\text{max}}) \]
\[ u(\cdot, 0) = u_0 \quad \text{in } \Omega \]
\[ \partial_v u|_{\partial \Omega} = \partial_v v|_{\partial \Omega} = 0 \quad \text{in } (0, T_{\text{max}}) \]

we will attempt to characterize, which exponents spawn which kind of solution behaviour. Slightly generalizing \( D \) and \( S \) if compared to (4), we will assume that

\[ D, S \in C^1(\mathbb{R}) \] are such that \( D(\xi) > 0 \) for all \( \xi \geq 0 \), \( S(\xi) > 0 \) for all \( \xi > 0 \),

and will usually assume that, in addition, with \( c_D > 0 \), \( C_S > 0 \),

\[ D(u) \geq c_D (u + 1)^{m-1} \quad \text{for } u > 0, \]

and

\[ S(u) \leq C_S u(u + 1)^{\sigma-1} \quad \text{for } u > 0. \]
Defining
\[ G(u) := \int_1^u \int_1^s \frac{D(\xi)}{S(\xi)} \, d\xi \, ds, \quad u \in (0, \infty), \] (9)
we will furthermore assume
\[ G(\zeta) \leq C_G (1 + \zeta^{2+\alpha}) \] (10)
for some \( \alpha \in \mathbb{R} \) and all \( \zeta > 0 \) (which is consistent with the assumption that \( \frac{D}{S} \approx u^\alpha \)
from the first part of the introduction and, in the case of (4) is satisfied with \( \alpha = m - \sigma - 1 \).

Our first result will then be to recover the conditions for global existence and boundedness of solutions:

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^N, N \geq 1 \), be a bounded domain with smooth boundary. Let \( \sigma \in \mathbb{R} \) and \( m \in \mathbb{R} \) satisfy
\[ \sigma < m - \frac{N - 2}{N} \] (11)
and assume that \( D \) and \( S \) fulfill (6) as well as (7) and (8) with some \( c_D > 0 \) and \( C_S > 0 \). Then for every \( \beta \in (0,1) \) and every nonnegative function \( u_0 \in C^\beta(\overline{\Omega}) \) the solution \((u,v)\) to (5) exists globally and is bounded.

This will be the consequence of a differential inequality for \( \int_\Omega u^\mu \), a small change in which can also be used to show global existence of solutions for nonpositive \( \sigma \):

**Theorem 1.2.** Let \( \Omega \subset \mathbb{R}^N, N \geq 1 \), be a bounded domain with smooth boundary. Let \( m \in \mathbb{R}, \sigma \leq 0, \beta \in (0,1) \). Assume that \( D, S \) satisfy (6) and (7), (8) with some \( c_D > 0 \) and \( C_S > 0 \). Then for every \( 0 \leq u_0 \in C^\beta(\overline{\Omega}) \), the solution \((u,v)\) to (5) exists globally.

The most exciting part, however, will be the detection of unbounded solutions. Here we will rely on
\[ F(u,v) := \frac{1}{2} \int_\Omega |\nabla v|^2 + \frac{1}{2} \int_\Omega v^2 - \int_\Omega uv + \int_\Omega G(u), \quad (u,v) \in C^0(\overline{\Omega}) \times C^1(\overline{\Omega}), \] (12)
which has been known to be an energy functional for (2) and (1) for a long time (see [26, 11, 1, 38]) and lies at the core of unboundedness results in the parabolic-parabolic setting ([14, 38, 6, 7, 8], see above), where it is known that initial data \((u_0,v_0)\) with sufficiently negative energy \( F(u_0,v_0) \) yield unbounded solutions, if \( S \) and \( D \) satisfy
\[ \int_{s_0}^s \frac{\tau D(\tau)}{S(\tau)} \, d\tau \leq \frac{N - 2 - \delta}{N} \int_{s_0}^s \int_{s_0}^\sigma \frac{D(\tau)}{S(\tau)} \, d\tau \, d\sigma + Ks \quad \text{for all } s \geq s_0, \] (13)
with some \( \delta > 0, s_0 \geq 1 \) and \( K \geq 0 \).

**Remark 1.3.** Condition (13) is, in particular, satisfied if \( u^\beta \frac{D(u)}{S(u)} \to c_0 > 0 \) as \( u \to \infty \) for some \( \beta > \frac{2}{N} \) ([36, Cor. 5.2]). If \( D(u) = (u + 1)^{m-1} \) and \( S(u) = u(u + 1)^{\sigma-1} \), then \( \beta = -\alpha \).

In stark contrast to the parabolic-parabolic case, in our search for suitable initial data, we will have to ensure that \( u_0 \) and \( v_0 \) “fit”. (Since no initial data for \( v \) are part of (5), we have to define \( v_0 \) by \( 0 = \Delta v_0 - v_0 + u_0 \), but are at least justified in using these functions by Lemma 3.5.) The corresponding construction will be what Section 4 will be devoted to.
Not satisfied with having found one function \( u_0 \) that leads to blow-up, we will then proceed to show that there are actually “many” choices of initial data with this property:

**Theorem 1.4.** Let \( \Omega = B_R \subset \mathbb{R}^N \), \( N \geq 3 \). Let \( S, \, D \) be such that (6), \( S(0) = 0 \) and (13) with some \( s_0 > 0 \), \( K > 0 \), \( \delta > 0 \) as well as (8) and (7) with some \( C_D > 0 \), \( C_S > 0 \) and \( m \in \mathbb{R} \) and \( \sigma \leq 0 \). Assume that \( G \) as defined in (9) satisfies (10) with some \( \alpha \in \mathbb{R} \) and \( C_G > 0 \). If \( -\alpha > \frac{2}{N} \), the following holds:

Let \( p \in [1, \frac{2N}{N-2}] \) if \( \alpha \leq -\frac{4}{N-2} \) and \( p \in [1, -\frac{4N}{2}] \) if \( \alpha > -\frac{4}{N-2} \). Given radially symmetric \( u_0 \in C^3(\overline{\Omega}) \) for some \( \beta > 0 \), there are radially symmetric functions \( u_\eta, v_\eta \) such that \( 0 = \Delta v_\eta - v_\eta + u_\eta, \partial_\nu v_\eta|_{\partial \Omega} = 0 \) for any \( \eta \in (0, 1) \), and

\[
\|u_\eta - u_0\|_{L^p(\Omega)} \to 0 \quad \text{as } \eta \searrow 0.
\]

and that the solutions to (5) for these initial data \( u(\cdot, 0) := u_\eta \) blow up.

A combination of Theorem 1.2 and Theorem 1.4 in particular entails

**Corollary 1.5.** Let \( \Omega = B_R \subset \mathbb{R}^N \), \( N \geq 3 \), let \( S, \, D \) satisfy (6), \( S(0) = 0 \) and (13) with some \( s_0 > 0 \), \( K > 0 \), \( \delta > 0 \) as well as (8) and (7) with some \( C_D > 0 \), \( C_S > 0 \) and \( m \in \mathbb{R} \) and \( \sigma \leq 0 \). Assume that \( G \) as defined in (9) satisfies (10) with some \( \alpha \in \mathbb{R} \) and \( C_G > 0 \). If \( -\alpha > \frac{2}{N} \), let \( p \in [1, \frac{2N}{N-2}] \) if \( \alpha \leq -\frac{4}{N-2} \) and \( p \in [1, -\frac{4N}{2}] \) if \( \alpha > -\frac{4}{N-2} \). Given radially symmetric \( u_0 \in C^3(\overline{\Omega}) \) for some \( \beta > 0 \), there are radially symmetric functions \( u_\eta, v_\eta \) such that \( 0 = \Delta v_\eta - v_\eta + u_\eta, \partial_\nu v_\eta|_{\partial \Omega} = 0 \) for any \( \eta \in (0, 1) \), and

\[
\|u_\eta - u_0\|_{L^p(\Omega)} \to 0 \quad \text{as } \eta \searrow 0.
\]

and that the solutions to (5) for these initial data \( u(\cdot, 0) := u_\eta \) exist globally, but blow up at time \( \infty \).

In particular, with this we have detected a wide range of parameters \( m, \sigma \) for which infinite-time blow-up is, in some sense, the typical behaviour of radially symmetric solutions to (5).

2. Global existence and boundedness. This section is devoted to the results on global existence and boundedness. We begin the preparations by recalling a statement on local existence including an extensibility criterion. A similar result can be found, for example, in [21, Lemma 2.1]. Note, however, that the present lemma gives a stronger assertion concerning the regularity of \( v \) at time 0, which will be crucial for our purpose.

**Lemma 2.1.** Let \( S, \, D \in C^1(\mathbb{R}) \) be such that \( D(s) > 0 \) for all \( s \geq 0 \), let \( \beta \in (0, 1) \). Then for any nonnegative \( u_0 \in C^3(\overline{\Omega}) \) there is \( T_{\max} > 0 \) and a unique pair of functions \( (u, v) \in (C^0(\overline{\Omega} \times [0, T_{\max}]) \cap C^2(\overline{\Omega} \times (0, T_{\max}))) \times (C^0([0, T_{\max}]), C^1(\overline{\Omega})) \cap C^2(\overline{\Omega} \times (0, T_{\max}))) \) (hereafter: “classical solution”) that satisfies (5) and is such that

\[
\text{either } T_{\max} = \infty \text{ or } \limsup_{t \to T_{\max}} \|u(\cdot, t)\|_\infty = \infty.
\]

Moreover, \( u \) and \( v \) are nonnegative in \( \overline{\Omega} \times (0, T_{\max}) \).

**Proof.** We begin the proof with the assertion on uniqueness and assume that, for some fixed \( T > 0 \), \((u_1, v_1), (u_2, v_2) \in (C^0(\overline{\Omega} \times [0, T]) \cap C^2(\overline{\Omega} \times (0, T))) \times (C^0([0, T]), C^1(\overline{\Omega})) \cap C^2(\overline{\Omega} \times (0, T))) \) both solve (5) with the same nonnegative initial data \( u_1(\cdot, 0) = u_0 = u_2(\cdot, 0) \). We note that this also implies \( v_1(\cdot, 0) = v_2(\cdot, 0) \),
because these functions solve $0 = \Delta v_i(\cdot, 0) - v_i(\cdot, 0) + u_0$ in a weak sense due to $v_i \in C^0([0, T), C^1(\Omega))$ and (5b), and the weak solution of this equation is unique.

We pick an arbitrary $T' \in (0, T)$ and let $c_1 > 0, c_2 > 0, c_3 > 0, c_4 > 0$ and $c_5 > 0$ be such that

$$0 \leq u_i(x, t) \leq c_1$$ for all $(x, t) \in \Omega \times (0, T')$, $i \in \{1, 2\},$

$$\sup_{\xi \in (0, c_1)} D(\xi) \leq c_2, \quad \|\nabla v_1(\cdot, t)\|_{L^\infty(\Omega)} \leq c_3$$ for all $t \in (0, T')$,

$$\sup_{\xi \in (0, c_1)} |S'(\xi)| \leq c_4, \quad S(u_2(x, t)) \leq c_5$$ for all $(x, t) \in \Omega \times (0, T')$.

We have that

$$0 = \Delta(v_1 - v_2) - (v_1 - v_2) + (u_1 - u_2) \quad \text{in } \Omega \times (0, T')$$

and hence obtain

$$\frac{1}{2} \frac{d}{dt} \left( \int_\Omega |\nabla(v_1 - v_2)|^2 + \int_\Omega (v_1 - v_2)^2 \right) = -\int_\Omega \nabla(D(u_1) - D(u_2)) \nabla(v_1 - v_2) + \int_\Omega (S(u_1) \nabla v_1 - S(u_2) \nabla v_2) \nabla(v_1 - v_2)$$

in $(0, T')$. By the mean value theorem and the condition that $D(s) \geq c_6 := \inf_{\xi \in (0, c_1)} D(\xi) > 0$, we have that $(D(u_1) - D(u_2))(u_1 - u_2) \geq c_3(u_1 - u_2)^2$ and that $(D(u_1) - D(u_2))^2 \leq c_2^2(u_1 - u_2)^2$. Integration by parts, (5b) and Young's inequality show that in $(0, T')$

$$-\int_\Omega \nabla(D(u_1) - D(u_2)) \nabla(v_1 - v_2)$$

$$= \int_\Omega (D(u_1) - D(u_2)) \Delta(v_1 - v_2)$$

$$= \int_\Omega (D(u_1) - D(u_2))(v_1 - v_2) - \int_\Omega (D(u_1) - D(u_2))(u_1 - u_2)$$

$$\leq \frac{c_2^2}{2c_6} \int_\Omega (v_1 - v_2)^2 + \frac{c_6}{2c_2} \int_\Omega (D(u_1) - D(u_2))^2 - \int_\Omega (D(u_1) - D(u_2))(u_1 - u_2)$$

$$\leq \frac{c_2^2}{2c_6} \int_\Omega (v_1 - v_2)^2 + \frac{c_2^2c_6}{2c_2} \int_\Omega (u_1 - u_2)^2 - c_6 \int_\Omega (u_1 - u_2)^2$$

$$= \frac{c_2^2}{2c_6} \int_\Omega (v_1 - v_2)^2 - \frac{c_6}{2} \int_\Omega (u_1 - u_2)^2.$$
whereas the last term in (15) can be estimated according to
\[
\int_{\Omega} (S(u_1)\nabla v_1 - S(u_2)\nabla v_2) \nabla (v_1 - v_2)
\]
\[
= \int_{\Omega} (S(u_1)\nabla v_1 - S(u_2)\nabla v_1 + S(u_2)\nabla (v_1 - v_2)) \nabla (v_1 - v_2)
\]
\[
\leq c_3 \int_{\Omega} |S(u_1) - S(u_2)||\nabla (v_1 - v_2)| + c_5 \int_{\Omega} |\nabla (v_1 - v_2)|^2
\]
\[
\leq c_3 c_4 \int_{\Omega} |u_1 - u_2||\nabla (v_1 - v_2)| + c_5 \int_{\Omega} |\nabla (v_1 - v_2)|^2
\]
\[
\leq \frac{c_6}{2} \int_{\Omega} |u_1 - u_2|^2 + \left( \frac{c_3 c_4}{2c_6} + c_5 \right) \int_{\Omega} |\nabla (v_1 - v_2)|^2 \quad \text{in } (0, T').
\]

In conclusion, in $(0, T')$ we obtain
\[
\frac{1}{2} \frac{d}{dt} \left( \int_{\Omega} |\nabla (v_1 - v_2)|^2 + \int_{\Omega} (v_1 - v_2)^2 \right)
\]
\[
\leq \left( \frac{c_3 c_4}{2c_6} + \frac{c_3 c_4}{2c_6} + c_5 \right) \left( \int_{\Omega} |\nabla (v_1 - v_2)|^2 + \int_{\Omega} (v_1 - v_2)^2 \right),
\]
which by Grönwall’s inequality and $v_1(\cdot, 0) = v_2(\cdot, 0)$ shows that $v_1 = v_2$ in $\Omega \times (0, T')$ and hence in $\Omega \times (0, T)$ by arbitrariness of $T'$. By (5b), this entails that $u_1 = u_2$.

For sufficiently small $T > 0$ (where the precise meaning of “sufficiently small” depends on $\|u_0\|_{L^\infty(\Omega)}$ and $\|u_0\|_{C^3(\bar{\Omega})}$), the map $S$ defined on the set
\[
\Omega := \{ u \in C^0(\bar{\Omega} \times [0, T]) | \|u\|_{L^\infty(\Omega \times (0, T))} \leq \|u_0\|_{L^\infty(\Omega)} + 1, u(\cdot, 0) = u_0 \}
\]
by $Su = \tilde{u}$, with $\tilde{u}$ being the solution of
\[
\tilde{u}_t = \nabla \cdot (D(\tilde{u})\nabla \tilde{u} - S(\tilde{u})\nabla \tilde{v}), \quad \partial_{\nu}\tilde{u}|_{\partial\Omega} = 0, \quad \tilde{u}(\cdot, 0) = u_0,
\]
where $\tilde{v}$ solves
\[
0 = \Delta \tilde{v} - \tilde{v} + \tilde{u}, \quad \partial_{\nu}\tilde{v}|_{\partial\Omega} = 0,
\]
(16) can be seen to be a continuous and compact map of $X$ into $X$ and to hence have a fixed point $u$ according to Schauder’s theorem. The corresponding calculations rely on the well-known elliptic regularity estimate for any $p \in (1, \infty)$ asserting the existence of a constant $C_p > 0$ such that all solutions of (16) satisfy
\[
\|\tilde{v}\|_{W^{2,p}(\Omega)} \leq C_p \|\tilde{u}\|_{L^p(\Omega)}
\]
(17) (which can, e.g. be obtained from \[10, \text{Thm. 19.1}\] in combination with the estimate $\|\tilde{u}\|_{L^p(\Omega)} \leq \|\tilde{u}\|_{L^p(\Omega)}$ that results from (16) by testing with (an approximation of) $v^{p-1}$) and on parabolic regularity statements that can be found in \[20, \text{Lemma 2.1, parts iii) and iv}], which also guarantee $u \in C^0(\bar{\Omega} \times [0, T]) \cap C^{2,1}(\bar{\Omega} \times (0, T))$. We let $v$ be the solution of (16) for $\tilde{u} = u$. As particular consequence of (17) applied to some $p > N$ and linearity of (16) let us note that
\[
\|v\|_{C^0([0,T];C^1(\Omega))} \leq c_6 \|v\|_{C^0([0,T];W^{2,p}(\Omega))} \leq C_p c_6 \|u\|_{C^0([0,T];L^p(\Omega))}
\]
and hence $v \in C^0(0, T), C^1(\Omega))$. The extensibility criterion (14) can be concluded from the dependence of $T$ on $\|u_0\|_{L^\infty(\Omega)}$ and $\|u_0\|_{C^3(\bar{\Omega})}$ in combination with \[20, \text{Lemma 2.1 iv}] prohibiting blow-up of $\|v\|_{C^p(\bar{\Omega})}$ while $\|u\|_{L^\infty(\Omega)}$ remains bounded. Nonnegativity is obtained from classical comparison theorems._suite
In order to show boundedness of $u$, it suffices to estimate the norm of $u$ in a suitable $L^p(\Omega)$-space, with some large, but finite $p$.

**Lemma 2.2.** Let $q_1 > N + 2$ and $q_2 > \frac{N+2}{2}$, $m \in \mathbb{R}$, $\sigma \in \mathbb{R}$, $\beta \in (0, 1)$ and assume that $S$ and $D$ satisfy (6), (7) and (8) and let $u_0 \in C^{\beta}(\overline{\Omega})$. Let

$$p > \max \left\{ \frac{N}{2}(1-m), q_1\sigma, 1 - m \frac{(N + 1)q_1 - (N + 2)}{q_1 - (N + 2)}, 1 - \frac{m}{Nq_2} \right\}.$$

Then for every $K > 0$ there is $C > 0$ such that whenever $(u, v) \in (C^{2,1}(\overline{\Omega} \times (0, T)) \cap C^0(\overline{\Omega} \times [0, T]))^2$ solves (5) in $\Omega \times (0, T)$ for some $T > 0$ and satisfies

$$\|u(\cdot, t)\|_{L^p(\Omega)} \leq K \text{ for all } t \in (0, T),$$

then

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C \text{ for all } t \in (0, T).$$

**Proof.** Since $p > N$, $\|v(\cdot, t)\|_{W^{1,\infty}(\Omega)}$ can be controlled by $\|u(\cdot, t)\|_{L^p(\Omega)}$ for $t \in (0, T)$ by elliptic regularity estimates (cf. (17)). With $f := S(u)\nabla v$ and $g = 0$ we hence have that $\|f\|_{L^p(\Omega)} \leq C_S \|\nabla v\|_{L^\infty(\Omega)} \|(u + 1)^\sigma\|_{L^p(\Omega)}$ is bounded in $(0, T)$ and [31, Lemma A.1] is applicable.

According to the previous lemma and (14), global existence and boundedness can be shown by ensuring that $\int_\Omega u^p$ is bounded locally or globally in time, respectively, for some large $p$. These assertions will rest on the following differential inequality.

**Lemma 2.3.** Let $m \in \mathbb{R}$, $\sigma \in \mathbb{R}$, $\beta \in (0, 1)$. Let $D$ and $S$ satisfy (6) as well as (7) and (8) with some $c_D > 0$ and $C_S > 0$ and let $u_0 \in C^{\beta}(\overline{\Omega})$ be nonnegative. Let $p \in (1, \infty)$ satisfy $p > 1 - \sigma$. Then the solution $(u, v)$ of (5) satisfies

$$\frac{1}{p} \frac{d}{dt} \int_\Omega (u + 1)^p \leq - \frac{4c_D(p - 1)}{(m + p - 1)^2} \int_\Omega |\nabla (u + 1)|^{\frac{m+p-1}{p}} + \frac{C_S(p - 1)}{p + \sigma - 1} \int_\Omega (u + 1)^{p+\sigma}$$

in $(0, T_{max})$.

**Proof.** We introduce

$$\tilde{S}(u) = \int_0^u (\zeta + 1)^{p-2} S(\zeta) d\zeta \quad \text{for } u \geq 0$$

and note that according to (8)

$$\tilde{S}(u) \leq C_S \int_0^u (\zeta + 1)^{p+\sigma - 2} d\zeta \leq \frac{C_S}{p + \sigma - 1} (u + 1)^{p+\sigma - 1} \text{ for any } u \geq 0. \quad (19)$$

We then use the first equation of (5) together with integration by parts and the estimates (7) and $-\Delta v = u - v \leq u$ to obtain

$$\frac{1}{p} \frac{d}{dt} \int_\Omega (u + 1)^p = \int_\Omega (u + 1)^{p-1} \nabla \cdot (D(u)\nabla u - S(u)\nabla v)$$

$$= -(p - 1) \int_\Omega D(u)(u + 1)^{p-2} |\nabla u|^2 + (p - 1) \int_\Omega (u + 1)^{p-2} S(u)\nabla u \cdot \nabla v$$
which shows that \( \text{Lemma 2.2 and (14) then results in global existence.} \)

so that \( \text{Lemma 2.3 guarantees} \ y \sigma < m \)

even neglecting the dissipative term in (18). If \( \text{Young's inequality to find} \)

\[ \begin{align*}
2.1. \text{Letting} & \quad \text{Proof of Theorem 1.2.} \\
\text{Local existence of a solution} & \quad \text{\Omega} \\
(18). & \quad \text{\Omega} \\
\text{We begin the preparation of the corresponding proof with the following different} & \quad \text{\Omega} \\
\text{estimate of} & \quad \text{\Omega} \\
\text{be achieved, finally leading to boundedness of solutions, regardless of the sign of} & \quad \text{\Omega} \\
\text{\sigma} & \quad \text{\Omega} \\
\text{Let} & \quad \text{\Omega} \\
\text{Lemma 2.4.} & \quad \text{\Omega} \\
\text{Apparently, the estimate underlying this proof of Theorem 1.2 is rather rough,} & \quad \text{\Omega} \\
\text{even neglecting the dissipative term in (18). If} \quad \text{\Omega} \\
\text{\sigma < m - \frac{N-2}{N}, better estimates can be achieved, finally leading to boundedness of solutions, regardless of the sign of} & \quad \text{\Omega} \\
\text{\sigma.} & \quad \text{\Omega} \\
\text{We begin the preparation of the corresponding proof with the following different} & \quad \text{\Omega} \\
\text{estimate of} & \quad \text{\Omega} \\
\text{\sigma < m - \frac{N-2}{N}.} & \quad \text{\Omega} \\
\text{Let} & \quad \text{\Omega} \\
p > \max\{1,2-m,1-\sigma,(\frac{N}{2}-1)\sigma + \frac{N}{2}(1-m)\} & \quad \text{\Omega} \\
\text{and} & \quad \text{\Omega} \\
c_D, C_S > 0. & \quad \text{\Omega} \\
\text{Then for} & \quad \text{\Omega} \\
\text{any} & \quad \text{\Omega} \\
K > 0 & \quad \text{\Omega} \\
\text{there is} & \quad \text{\Omega} \\
C > 0 & \quad \text{\Omega} \\
\text{such that every nonnegative function} & \quad \text{\Omega} \\
w \in C^2(\Omega) & \quad \text{\Omega} \\
\text{which satisfies} & \quad \text{\Omega} \\
\text{\Omega} & \quad \text{\Omega} \\
\text{fulfills} & \quad \text{\Omega} \\
\frac{C_S(p-1)}{p+\sigma-1} \int_{\Omega} (w+1)^{p+\sigma} & \quad \text{\Omega} \\
\text{\Omega} & \quad \text{\Omega} \\
\leq & \quad \text{\Omega} \\
\frac{2c_D(p-1)}{(m+p-1)^2} \int_{\Omega} |\nabla (w+1)^{\frac{m+p-1}{2}}|^2 + C. & \quad \text{\Omega} \\
\text{Proof.} & \quad \text{\Omega} \\
\text{We let} & \quad \text{\Omega} \\
a := & \quad \text{\Omega} \\
\frac{N}{2} & \quad \text{\Omega} \\
\frac{(p+m-1)(1 - \frac{1}{p+\sigma})}{1 + \frac{N}{2}(p+m-2)}, & \quad \text{\Omega} \\
\text{so that} & \quad \text{\Omega} \\
-\frac{N(p+m-1)}{2(p+\sigma)} & \quad \text{\Omega} \\
= & \quad \text{\Omega} \\
(1 - \frac{N}{2})a - \frac{N(p+m-1)}{2}(1-a). & \quad \text{\Omega} \\
\text{By the conditions on} & \quad \text{\Omega} \\
p & \quad \text{\Omega} \\
\text{positivity} & \quad \text{\Omega} \\
of \ a & \quad \text{\Omega} \\
\text{is obvious. Moreover, we have} & \quad \text{\Omega} \\
m - 1 & \quad \text{\Omega} \\
> & \quad \text{\Omega} \\
\frac{2}{N} + (1 - \frac{2}{N})\sigma, & \quad \text{\Omega} \\
\text{which means} & \quad \text{\Omega} \\
p + m - 1 & \quad \text{\Omega} \\
> & \quad \text{\Omega} \\
p+\sigma - & \quad \text{\Omega} \\
\frac{2p}{N} - \frac{2\sigma}{N} = (1 - \frac{\sigma}{N})(p+\sigma) & \quad \text{\Omega} \\
\text{and hence} & \quad \text{\Omega} \\
0 < \frac{p+\sigma-1}{p+\sigma} & \quad \text{\Omega} \\
\text{Therefore,} & \quad \text{\Omega} \\
p + m - 1 - & \quad \text{\Omega} \\
\frac{p+m-1}{p+\sigma} & \quad \text{\Omega} \\
< & \quad \text{\Omega} \\
\frac{2}{N} + p + m - 2 & \quad \text{\Omega} \\
i.e. & \quad \text{\Omega} \\
\frac{N}{2}(p+m-1)(1 - \frac{1}{p+\sigma}) & \quad \text{\Omega} \\
< 1 + \frac{N}{2}(p+m-2), & \quad \text{\Omega} \\
\text{which shows that} & \quad \text{\Omega} \\
a < 1.
Let Lemma 2.5.

\begin{align*}
\int_{\Omega} (w + 1)^{p+\sigma} &= \int_{\Omega} (w + 1)^\frac{p+\sigma-1}{p+m-1} \frac{2(p+\sigma)}{p+m-1} \\
&= \| (w + 1)^\frac{p+\sigma-1}{2} \|_{L^2(\Omega)} \| (w + 1)^\frac{p+\sigma-1}{p+m-1} \|_{L^{2(p+\sigma)}(\Omega)} \\
&\leq c_1 \| \nabla (w + 1)^\frac{p+\sigma-1}{2} \|_{L^2(\Omega)} (w + 1)^\frac{p+\sigma-1}{p+m-1} + c_1 \| (w + 1)^\frac{p+\sigma-1}{2} \|_{L^{2(p+\sigma)}(\Omega)}(1-a) \\
&+ c_1 \| (w + 1)^\frac{p+\sigma-1}{p+m-1} \|_{L^{2(p+\sigma)}(\Omega)}(1-a) \\
&\leq c_1 \| \nabla (w + 1)^\frac{p+\sigma-1}{2} \|_{L^2(\Omega)} (w + 1)^\frac{p+\sigma-1}{p+m-1} + c_1 \| (w + 1)^\frac{p+\sigma-1}{2} \|_{L^{2(p+\sigma)}(\Omega)}(1-a) \\
&+ c_1 \| (w + 1)^\frac{p+\sigma-1}{p+m-1} \|_{L^{2(p+\sigma)}(\Omega)}(1-a) \\
&\leq c_1 \| \nabla (w + 1)^\frac{p+\sigma-1}{2} \|_{L^2(\Omega)} (w + 1)^\frac{p+\sigma-1}{p+m-1} + c_1 \| (w + 1)^\frac{p+\sigma-1}{2} \|_{L^{2(p+\sigma)}(\Omega)}(1-a) \\
&+ c_1 \| (w + 1)^\frac{p+\sigma-1}{p+m-1} \|_{L^{2(p+\sigma)}(\Omega)}(1-a) \\
&\leq c_1 \| \nabla (w + 1)^\frac{p+\sigma-1}{2} \|_{L^2(\Omega)} (w + 1)^\frac{p+\sigma-1}{p+m-1} + c_1 \| (w + 1)^\frac{p+\sigma-1}{2} \|_{L^{2(p+\sigma)}(\Omega)}(1-a) \\
&+ c_1 \| (w + 1)^\frac{p+\sigma-1}{p+m-1} \|_{L^{2(p+\sigma)}(\Omega)}(1-a)
\end{align*}

Furthermore, (20) entails \( \frac{N}{2}(\sigma-m+1) < 1 \), so that \( \frac{N}{2}(p+\sigma-1) < 1 + \frac{N}{2}(p-m) \) and hence

\[
\frac{p+\sigma}{p+m-1} = \frac{\frac{N}{2}(p+\sigma-1)}{1 + \frac{N}{2}(p-m)} < 1.
\]

We can therefore apply Young’s inequality to (22) and accounting for (21) we obtain

\[
C = C(K) > 0 \text{ such that}
\]

\[
C_S(p-1) \left( \frac{p+\sigma}{p+m-1} \right) \int_{\Omega} (w + 1)^{p+\sigma} \leq \frac{2(p-1)c_D}{(m+p-1)^2} \| \nabla (w + 1)^\frac{p+\sigma-1}{2} \|_{L^2(\Omega)}^2 + C.
\]

We have seen that under the condition (20) it is possible to estimate \( \int_{\Omega} \| \nabla (u+1)^{\frac{m+p-1}{2}} \|^2 \) and a constant. This would transform (18) into a statement of the form \( y' \leq c_1 - c_2 \int_{\Omega} \| \nabla (u+1)^{\frac{m+p-1}{2}} \|^2 \). In order to derive boundedness of \( \int_{\Omega} u^p \) from this, we shall also require control of \( \int_{\Omega} w \) by means of \( \int_{\Omega} \| \nabla (u+1)^{\frac{m+p-1}{2}} \|^2 \). If \( \sigma > 0 \), clearly the statement of Lemma 2.4 is even stronger than that. Since our interest in this paper mainly lies in the case of \( \sigma < 0 \), we prepare the following

**Lemma 2.5.** Let \( \Omega \subset \mathbb{R}^N \), \( N \geq 1 \), be a bounded domain with smooth boundary. Let \( m \in \mathbb{R} \), \( c_D > 0 \) and \( p > \max\{1, 2 - m, \frac{N}{2}(1 - m)\} \). Then for every \( K > 0 \) there is \( C > 0 \) such that every nonnegative function \( w \in C^2(\Omega) \) satisfying \( \int_{\Omega} w^2 \leq K \) fulfills

\[
\left( \int_{\Omega} (w + 1)^p \right)^{\frac{m+p-1}{p}} \leq \frac{2c_D(p-1)}{(m+p-1)^2} \int_{\Omega} \| \nabla (w + 1)^{\frac{m+p-1}{2}} \|^2 + C.
\]

**Proof.** We let

\[
b = \frac{N(m+p-1)}{2p}(1 - \frac{1}{p})
\]

so that \( -\frac{N(m+p-1)}{2p} = (1 - \frac{N}{2})b - \frac{N(m+p-1)}{2}(1 - b) \) and that, by the conditions imposed on \( p, b \) is clearly positive and

\[
\frac{1}{p}(m+p-1) = \frac{1-m}{p} - 1 < \frac{2}{N} - 1 = \frac{2}{N}(1 + \frac{N}{2}(m+p-2 - (m+p-1))),
\]
Showing that \( \frac{N}{2} (m + p - 1)(1 - \frac{1}{p}) < 1 + \frac{N}{2} (m + p - 2) \) and hence also \( b < 1 \). From the Gagliardo–Nirenberg inequality we then obtain \( c_1 > 0 \) such that

\[
\left( \int_{\Omega} (w + 1)^p \right)^{\frac{m+p-1}{p}} = \left( \int_{\Omega} (w + 1)^{\frac{m+p-1}{2}} \right)^{\frac{m+p-1}{p}} \leq c_1 \left( \int_{\Omega} \left| \nabla (w + 1) \right|^2 \right)^{\frac{m+p-1}{2p}} \leq (w + 1)^{\frac{m+p-1}{2p}} \left( \int_{\Omega} \left| \nabla (w + 1) \right|^2 \right)^{\frac{m+p-1}{2p}}
\]

holds for every \( w \in C^2(\Omega) \), and, thanks to \( b < 1 \) and \( \int_\Omega (w + 1) \leq K \), (23) follows via an application of Young’s inequality.

With the help of this estimate, we have reduced the proof of boundedness by means of Lemma 2.3 to the following elementary situation.

**Lemma 2.6.** Let \( f : \mathbb{R} \to \mathbb{R} \) be such that there exists \( x_0 \in \mathbb{R} \) with \( f(x) < 0 \) for any \( x > x_0 \). Let \( y \in C^0([0, T)) \cap C^1((0, T)) \) for some \( T > 0 \) be such that

\[
y'(t) \leq f(y(t)) \quad \text{for any } t \in (0, T).
\]

Then \( y(t) \leq \max\{y(0), x_0\} \) for any \( t \in (0, T) \).

**Proof.** Assuming \( t \in (0, T) \) to be given, we let \( t_0 := \sup\{s \in [0, t] \mid y(s) \leq x_0\} \) (or \( t_0 = 0 \) in case this set is empty). By definition, we have that \( y(s) > x_0 \) for all \( s \in (t_0, t) \) and \( y(t_0) \leq \max\{x_0, y(0)\} \). Hence

\[
y(t) = y(t_0) + \int_{t_0}^t y'(s)ds \leq y(t_0) + \int_0^t f(y(s))ds \leq y(t_0) + \int_0^t 0 \, ds = y(t_0) \leq \max\{x_0, y(0)\} \]

**Proof of Theorem 1.1.** Local existence of solutions is guaranteed by Lemma 2.1. If we then combine the differential inequality from Lemma 2.3 with the estimates of Lemma 2.4 and Lemma 2.5, for any sufficiently large \( p \) we obtain \( c_1 > 0 \) and \( c_2 > 0 \) such that

\[
\frac{d}{dt} \int_{\Omega} (u + 1)^p \leq c_1 - c_2 \left( \int_{\Omega} (u + 1)^p \right)^{\frac{m+p-1}{p}},
\]

which, according to Lemma 2.6, shows boundedness of \( \int_{\Omega} (u + 1)^p \) and hence, by Lemma 2.2 boundedness of \( u \).

3. **The energy functional – and unboundedness of solutions.** As announced in the introduction, the proof of unboundedness of solutions relies on use of the functional (12), namely on the fact that it decreases along solution trajectories, in the case of global bounded solutions cannot decrease below its lowest value for radially symmetric steady states, but, depending on the initial data, might start from an even lower number.

We begin by recalling that \( \mathcal{F} \) actually is an energy functional.

**Lemma 3.1.** If \( (u, v) \) is a classical solution to (5) with some \( D, S \) satisfying (6), then the function \( \mathcal{F} \) defined by (12) satisfies

\[
\frac{d}{dt} \mathcal{F}(u, v) = -D(u, v) \quad \text{on } (0, T_{\text{max}}),
\]

where

\[
D(u, v) := \int_{\Omega} S(u) \left| \frac{D(u)}{S(u)} \nabla u - \nabla v \right|^2.
\]
Proof. We note that with \( G \) as defined in (9)
\[
G'(u) = \int_1^u \frac{D(\xi)}{S(\xi)} d\xi \quad \text{for any } u \in (0, \infty)
\]
and hence
\[
\nabla G'(u) = \frac{D(u)}{S(u)} \nabla u \quad \text{for any positive differentiable function } u.
\]
With (5) and integration by parts, the calculations are straightforward and we give them without further comment:
\[
\frac{d}{dt} \mathcal{F}(u, v) = \int_\Omega \nabla v \nabla v_t + \int_\Omega vv_t - \int_\Omega uv - \int_\Omega \nabla \cdot (D(u) \nabla u - S(u) \nabla v) = -\int_\Omega \Delta v \nabla v_t + \int_\Omega vv_t - \int_\Omega uv - \int_\Omega \nabla \cdot \left( \frac{D(u)}{S(u)} \nabla u - S(u) \nabla v \right)
\]
\[
= -\int_\Omega \left( \Delta v - v + u \right) v_t - \int_\Omega \left( \nabla G'(u) - \nabla v \right) \left( D(u) \nabla u - S(u) \nabla v \right)
\]
\[
= -\int_\Omega \frac{D(u)}{S(u)} |\nabla u|^2 - \int_\Omega \frac{D(u)}{S(u)} S(u) \nabla u \cdot \nabla v \quad \text{on } (0, T_{max}).
\]
We can (and will) simplify the expression for \( \mathcal{F} \) in the particular situation that \( u \) and \( v \) fulfil (5b):

Lemma 3.2. If \((u, v) \in C^0(\Omega) \times C^1(\Omega) \) is such that
\[
0 = \Delta v - v + u,
\]
is satisfied in the weak sense, then
\[
\mathcal{F}(u, v) = \int_\Omega G(u) - \frac{1}{2} \int_\Omega uv.
\]
Proof. If (26) holds, then, upon using \( \frac{1}{2} v \) as test function, we have
\[
\frac{1}{2} \int_\Omega |\nabla v|^2 = -\frac{1}{2} \int_\Omega v^2 + \frac{1}{2} \int_\Omega uv.
\]
If we insert this into the definition of \( \mathcal{F} \), we obtain (27). \( \square \)

If a solution \((u, v)\) is global and bounded, \( \mathcal{F}(u(\cdot, t), v(\cdot, t)) \) converges, at least along a sequence \( t_k \nearrow \infty \).
Lemma 3.3. Assume that \( u_0 \in C^\beta(\overline{\Omega}) \), \( \beta \in (0,1) \), is nonnegative, \( S \) and \( D \) satisfy (6) and \( S(0) = 0 \) and that \((u, v)\) is a global classical solution of (3) which is bounded in the sense that

\[
\sup_{t \in [0, \infty)} \|u(\cdot, t)\|_{L^\infty(\Omega)} < \infty.
\]

Then there are a sequence \((t_k)_{k \in \mathbb{N}} \nearrow \infty\) and \((u_\infty, v_\infty) \in (C^2(\overline{\Omega}))^2\) such that \((u(\cdot, t_k), v(\cdot, t_k)) \to (u_\infty, v_\infty)\) in \((C^2(\overline{\Omega}))^2\) as \( k \to \infty \) and \((u_\infty, v_\infty)\) satisfies

\[
\int_\Omega u_\infty = \int_\Omega u_0, \quad D(u_\infty)\nabla u_\infty = S(u_\infty)\nabla v_\infty, \\
\partial_t v_\infty|_{\partial \Omega} = 0, \quad -\Delta v_\infty + v_\infty = u_\infty.
\]

(28)

If \( u_0 \) is radially symmetric, then also \((u_\infty, v_\infty)\) is radially symmetric.

Proof. The proof closely follows that of [36, Lemma 2.2]: Boundedness of \( u \) makes application of regularity theory possible, yielding \( c_1 > 0 \) such that

\[
\|u\|_{C^{2+\beta,1+\frac{\beta}{2}}([0, t] \times \overline{\Omega})} \leq c_1 \quad \text{and} \quad \|v(\cdot, t)\|_{C^{2+\beta,1+\frac{\beta}{2}}([0, t] \times \overline{\Omega})} \leq c_1
\]

(29)

for every \( t > 0 \). Due to Arzelà–Ascoli’s theorem and \( \int_0^\infty D(u(\cdot, t), v(\cdot, t))dt < \infty \), which is a result of an integration of (24) and (29), we can extract a sequence \((t_k)_{k \in \mathbb{N}} \nearrow \infty\) such that \( D(u(\cdot, t_k), v(\cdot, t_k)) \to 0 \) and \( u(\cdot, t_k) \to u_\infty, v(\cdot, t_k) \to v_\infty \) in \( C^2(\overline{\Omega}) \) as \( k \to \infty \). Apart from

\[
D(u_\infty)\nabla u_\infty = S(u_\infty)\nabla v_\infty,
\]

(30)

the properties asserted in (28) immediately follow from the convergence in \( C^2(\overline{\Omega}) \). In order to show that (30) holds, we fix \( x \in \Omega \). If \( u_\infty(x) = 0 \), then also \( \nabla u_\infty(x) = 0 \) due to the nonnegativity of \( u_\infty \), so that \( S(0) = 0 \) ensures that (30) holds in \( x \).

We have chosen the subsequence such that \( D(u(\cdot, t_k), v(\cdot, t_k)) \to 0 \). Hence for almost every \( x \in \Omega \) with \( u_\infty(x) \neq 0 \) by (6) we have \( \liminf_{k \to \infty} S(u(x, t_k)) > 0 \) and \( S(u(x, t_k))|_{(u(x, t_k))} D(u(x, t_k))\nabla u(x, t_k) - \nabla v(x, t_k)|^2 \to 0 \) as \( k \to \infty \), which shows that \( \frac{D(u_\infty(x))}{S(u_\infty(x))} \nabla u_\infty(x) - \nabla v_\infty(x) = 0 \) and thus asserts that (30) holds almost everywhere in \( \Omega \) and – by virtue of \( u_\infty, v_\infty \in C^2(\overline{\Omega}) \) – in all of \( \Omega \).

On the other hand, it is impossible to achieve arbitrarily low values of \( F \) during convergence as observed in Lemma 3.3.

Lemma 3.4. Let \( N \geq 3 \). Then for any \( M > 0, K > 0, s_0 \geq 1, \delta > 0 \) and \( R > 0 \) there is \( C > 0 \) such that whenever \( S, D \) satisfy (6) as well as (13), then every radially symmetric solution to

\[
\int_\Omega u_\infty = M, \quad -\Delta u_\infty + v_\infty = u_\infty, \quad \partial_t v_\infty|_{\partial \Omega} = 0, \quad D(u_\infty)\nabla u_\infty = S(u_\infty)\nabla v_\infty
\]

in \( B_R \subset \mathbb{R}^N \) satisfies

\[
F(u_\infty, v_\infty) > -C.
\]

Proof. This is [36, Lemma 3.4]. Due to its length we refrain from repeating the proof.

In combination, the previous lemmata mean that
Lemma 3.5. Let $\Omega = B_R$ for some $R > 0$. Let $D$ and $S$ satisfy (6) and $S(0) = 0$. Assume that furthermore (13) is satisfied with some $s_0 \geq 1$, $K > 0$, $\delta > 0$. Then there is $C > 0$ with the following property: If $u_0 \in C^0(\Omega)$ is radially symmetric and such that
\[
F(u_0, v_0) < -C
\]
holds for the function $v_0 \in C^2(\Omega)$ defined by
\[
0 = \Delta v_0 - v_0 + u_0, \quad \partial_\nu v_0 |_{\partial \Omega} = 0,
\]
then the corresponding solution is not globally bounded, i.e. blows up, either after finite or in infinite time.

Proof. Part of Lemma 2.1 ensures that the map $\varphi: t \mapsto F(u(\cdot, t), v(\cdot, t))$ belongs to $C^0([0, T_{\text{max}}])$. That (5b) is satisfied, together with the regularity of $(u, v)$ asserted in Lemma 2.1, serves to show (31) with $v_0 := (C_1(\Omega) - \lim)_{\tau \to 0} v(\cdot, t)$, firstly in a weak sense, then, by elliptic theory, even classically. According to Lemma 3.1, $\varphi$ is decreasing. Assuming global boundedness of $(u, v)$, the use of Lemma 3.3 leads to $F(u_0, v_0) \geq -C$ by 3.4 (with $C$ as given there).

4. Constructing initial data and estimating $F$. Now that we have established that initial data “with sufficiently negative energy” lead to unbounded solutions, what remains to be shown is that such initial data, in fact, do exist and, even more, that there are many of these in any neighbourhood of given initial data. The difficulty, if compared to previous studies of the parabolic-parabolic model, is that $v_0$ can no longer be chosen arbitrarily, but has to fit with $u_0$; this can already be seen from the statement of Lemma 3.5.

The goal of this section is to construct one family of functions that causes arbitrarily negative values of $F$ if a parameter tends to zero. We will later add these functions to given initial data in order to find many nearby initial data that yield blow-up solutions.

All functions in this section will be radially symmetric; as usual, we will identify radial functions $u: \Omega = B_R \to \mathbb{R}$ and $\tilde{u}: [0, R) \to \mathbb{R}$ if $u(x) = \tilde{u}(|x|)$, $x \in \Omega$, and will use the same symbol $u$ to denote both of these functions.

We fix $\gamma > 0$ and $\delta \in (0, 1)$, both of which will be subject to further conditions later, see (44), (47). For any $\eta \in (0, 1)$ we let $r_\eta := \eta^{\delta}$, define the nonnegative Lipschitz-continuous function
\[
u_{\eta}(r) := \begin{cases} (r^2 + \eta^2)^{-\frac{\gamma}{2}} - (r_\eta^2 + \eta^2)^{-\frac{\gamma}{2}}, & r \leq r_\eta, \\ 0, & r > r_\eta, \end{cases}
\]
and let $v_{\eta}$ be the corresponding solution of
\[-\Delta v_{\eta} + v_{\eta} = u_{\eta} \quad \text{in } \Omega, \quad \partial_\nu v_{\eta} |_{\partial \Omega} = 0,
\]
that is,
\[-r^{1-N}(r^{N-1} v_{\eta r})_r = u_{\eta} - v_{\eta} \quad \text{in } (0, R) \quad \text{with } v_{\eta r}(0) = 0, \quad v_{\eta r}(R) = 0,
\]
where $v_{\eta r}(R) = 0$ results from the Neumann boundary condition in (33) and $v_{\eta r}(0) = 0$ is a consequence of the radial symmetry of $v_{\eta}$, which in turn is implied by radial symmetry of $u_{\eta}$ and uniqueness of solutions to (33).
4.1. **Representation of** $v_\eta$. Let us first derive a representation formula for $v_\eta$, on which all estimates will be based.

Integration of (34) shows that

$$r^{N-1}v_{\eta r} = \int_0^r s^{N-1}v_\eta(s)ds - \int_0^r s^{N-1}u_\eta(s)ds$$

and hence, due to $v_{\eta r}(0) = 0$

$$v_{\eta r}(r) = r^{1-N} \int_0^r s^{N-1}v_\eta(s)ds - r^{1-N} \int_0^r s^{N-1}u_\eta(s)ds,$$

which by another integration is turned into

$$v_\eta(r) = v_\eta(R) + \int_r^R s^{1-N} \int_0^s \sigma^{N-1}u_\eta(\sigma)d\sigma ds - \int_r^R s^{1-N} \int_0^s \sigma^{N-1}v_\eta(\sigma)d\sigma ds.$$

Here we can determine $v_\eta(R)$ from the fact that – by integration of (33) – the $L^1(\Omega)$-norms of $u_\eta$ and $v_\eta$ have to coincide. Using that hence

$$\frac{1}{\omega_N} \|u_\eta\|_{L^1(\Omega)} = \int_0^R t^{N-1}v_\eta(t)dt = \frac{R^N}{N} v_\eta(R) + \int_0^R t^{N-1} \int_0^R s^{1-N} \int_0^s \sigma^{N-1}v_\eta(\sigma)d\sigma ds dt$$

we obtain the following representation for $v_\eta$:

$$v_\eta(r) = NR^{-N} \omega_N^{-1} \|u_\eta\|_{L^1(\Omega)} + NR^{-N} \int_0^R t^{N-1} \int_0^R s^{1-N} \int_0^s \sigma^{N-1}v_\eta(\sigma)d\sigma ds dt$$

$$- NR^{-N} \int_0^R t^{N-1} \int_0^R s^{1-N} \int_0^s \sigma^{N-1}u_\eta(\sigma)d\sigma ds dt$$

$$+ \int_r^R s^{1-N} \int_0^s \sigma^{N-1}u_\eta(\sigma)d\sigma ds$$

$$- \int_r^R s^{1-N} \int_0^s \sigma^{N-1}v_\eta(\sigma)d\sigma ds.$$  \hspace{1cm} (35)

4.2. **Estimates of** $v_\eta$ **from above.** Our aim is $\mathcal{F}(u_\eta, v_\eta) \to -\infty$ as $\eta \to 0$. According to Lemma 3.2, $\mathcal{F}(u_\eta, v_\eta) = \int_\Omega G(u_\eta) - \frac{1}{2} \int_\Omega u_\eta v_\eta$, so that we should prove largeness of $\int_\Omega u_\eta v_\eta$. Estimates of $v_\eta$ from below would be beneficial to this purpose. Due to the last term in (35), which contains $-v_\eta$, we begin this search for such estimates with an attempt to estimate $v_\eta$ from above.

Regardless of whether an estimate from above or below is desired, the first three terms on the right of (35) have a negligible contribution to the size of $v_\eta(r)$ if $\eta$ is small, at least provided $\gamma < N$:

$$\|v_\eta\|_{L^1(\Omega)} = \|u_\eta\|_{L^1(\Omega)} = \omega_N \int_0^R r^{N-1}u_\eta(r)dr \leq \omega_N \int_0^{r_0} r^{N-1}(r^2 + \eta^2)^{\frac{\gamma}{2}} dr \leq \omega_N \int_0^{r_0} r^{N-1-\gamma} dr.$$
\[ \frac{\omega_N}{N-\gamma} r_t^{N-\gamma} = \frac{\omega_N}{N-\gamma} \eta^{\delta(N-\gamma)}, \quad (36) \]

where we have used the obvious estimate
\[ (r^2 + \eta^2)^{-\frac{1}{2}} \leq r^{-\gamma}. \]

With this,
\[
\int_0^R t^{N-1} \int_t^R s^{1-N} \int_0^s \sigma^{N-1} u_\eta(\sigma) d\sigma ds dt \\
\leq \int_0^R t^{N-1} \int_t^R s^{1-N} \int_0^R \sigma^{N-1} u_\eta(\sigma) d\sigma ds dt \\
= \frac{1}{\omega_N} \|u_\eta\|_{L^1(\Omega)} \int_0^R t^{N-1} \int_t^R \sigma^{1-N} \sigma d\sigma ds dt \\
\leq \frac{1}{\omega_N} \|u_\eta\|_{L^1(\Omega)} \frac{1}{N-2} \int_0^R t^{N-1} \int_0^1 t^{2-N} dt \\
= \frac{1}{2(N-2)\omega_N} \|u_\eta\|_{L^1(\Omega)} R^2 \\
\leq \frac{R^2}{2(N-2)(N-\gamma)} \eta^{\delta(N-\gamma)}. \quad (37) \]

By the same calculation we also obtain
\[
\int_0^R t^{N-1} \int_t^R s^{1-N} \int_0^s \sigma^{N-1} v_\eta(\sigma) d\sigma ds dt \leq \frac{R^2}{2(N-2)(N-\gamma)} \eta^{\delta(N-\gamma)}. \]

As to the term containing \( u_\eta \) and two integrals, we consider the cases of small and slightly larger \( r \) separately. For the sake of a unified form of the explicit computations, we assume \( \gamma \neq 2 \). For \( r \leq r_\eta \) we then have
\[
\int_r^R s^{1-N} \int_0^s \sigma^{N-1} u_\eta(\sigma) d\sigma ds \\
\leq \int_r^R s^{1-N} \int_0^{\min\{s,r_\eta\}} \sigma^{N-1}(\sigma^2 + \eta^2)^{-\frac{1}{2}} \sigma d\sigma ds \\
\leq \int_r^{r_\eta} s^{1-N} \int_0^s \sigma^{N-1-\gamma} d\sigma ds + \int_r^R s^{1-N} \int_0^{r_\eta} \sigma^{N-1-\gamma} d\sigma ds \\
= \frac{1}{N-\gamma} \int_r^{r_\eta} s^{1-\gamma} ds + \frac{1}{N-\gamma} \int_r^{r_\eta} s^{1-N-\gamma} ds \\
= \frac{1}{(N-\gamma)(2-\gamma)} (r_\eta^{2-\gamma} - r^{2-\gamma}) + \frac{1}{(N-\gamma)(N-2)} r_\eta^{N-\gamma} (r_\eta^{2-N} - R^{2-N}), \]

whereas in the case \( r > r_\eta \)
\[
\int_r^R s^{1-N} \int_0^s \sigma^{N-1} u_\eta(\sigma) d\sigma ds \leq \int_r^R s^{1-N} \int_0^{\min\{s,r_\eta\}} \sigma^{N-1}(\sigma^2 + \eta^2)^{-\frac{1}{2}} \sigma d\sigma ds \\
\leq \int_r^R s^{1-N} \int_0^{r_\eta} \sigma^{N-1-\gamma} d\sigma ds \\
= \frac{1}{N-\gamma} r_\eta^{N-\gamma} \int_r^{r_\eta} s^{1-N} ds.
the calculation to small values of $\int R$ would be insufficient.

$v$ term is nonnegative, but since it is this term that has to cause the lower estimate of $\gamma$ we have just obtained enables us to treat the last integral in (35). Namely, as long as $c > c_0$ chosen in the obvious way.

\[ v_\eta(r) \leq C_\eta^{\delta(N-\gamma)} + C r^{2-\gamma} \quad (39) \]

with some $c > 0$.

4.3. Estimates of $v_\eta$ from below. The pointwise upper estimate of $v_\eta$ that we have just obtained enables us to treat the last integral in (35). Namely, as long as $\gamma \neq 4$, we have

\[ \int_r^R s^{1-N} \int_0^s \sigma^{N-1} u_\eta(\sigma) d\sigma ds \]

\[ \leq (c_\eta^{\delta(N-\gamma)} + c r^{2-\gamma}) \int_r^R s^{1-N} \int_0^s \sigma^{N-1} d\sigma ds + c \int_r^R s^{1-N} \int_0^s \sigma^{N-1} \sigma^{2-\gamma} d\sigma ds \]

\[ = C_\eta^{\delta(N-\gamma)} + C r^{2-\gamma} + \frac{c}{N + 2 - \gamma} \int_r^R s^{1-N+2-\gamma} ds \]

\[ \leq C_\eta^{\delta(N-\gamma)} + C r^{2-\gamma} + \frac{c}{(N + 2 - \gamma)(4 - \gamma)} (R^{4-\gamma} - r^{4-\gamma}) \]

\[ \leq C + C_\eta^{\delta(2-\gamma)} + C' r^{4-\gamma}. \]

with $c$ as in (39) and $C > 0$, $C' > 0$ chosen in the obvious way.

The next term to be estimated is $\int_0^{s} \sigma^{N-1} u_\eta(\sigma) d\sigma ds$. Apparently, this term is nonnegative, but since it is this term that has to cause the lower estimate of $v_\eta$ on which we want to rely in having $\int_\Omega u_\eta v_\eta \to \infty$ as $\eta \to \infty$, mere nonnegativity would be insufficient.

We treat the terms arising from the two summands in (32) separately and restrict the calculation to small values of $r$.

Using that $(\eta^2 + \sigma^2)^{-\frac{2}{N}} \geq (2\eta^2)^{-\frac{2}{N}}$ for any $\sigma \leq \eta$, for $r \leq \eta$ we obtain

\[ \int_r^R s^{1-N} \int_0^{\min(s,r)} \sigma^{N-1} (\sigma^2 + \eta^2)^{-\frac{2}{N}} d\sigma ds \geq 2^{-\frac{2}{N}} \int_r^\eta s^{1-N} \int_0^\eta \sigma^{N-1} \eta^{-\gamma} d\sigma ds \]

\[ \geq \frac{2^{-\frac{2}{N}}}{\eta^{-\gamma}} \int_r^\eta s ds \]

\[ \geq c_1 \eta^{2-\gamma} - c_1 r^2 \eta^{-\gamma}, \]

where $c_1 = 2^{-\frac{2}{N}} N^{-1}$. 

\[ \int_r^R s^{1-N} \int_0^\eta\sigma^{N-1} (\sigma^2 + \eta^2)^{-\frac{2}{N}} d\sigma ds \leq \frac{1}{(N-\gamma)(N-2)} r^{2-\gamma}. \]

Combined, these estimates show that with some $c_1 > 0$

\[ \int_r^R s^{1-N} \int_0^s \sigma^{N-1} u_\eta(\sigma) d\sigma ds \leq c_1 r^{2-\gamma} + c_1 r^{2-\gamma}. \quad (38) \]
Concerning the second term in (32), for \( r < r_\eta \) we have

\[
\int_r^R s^{1-N} \int_0^{\min\{s,r_\eta\}} \sigma^{N-1}(r_\eta^2 + \sigma^2)^{-\frac{\gamma}{2}} d\sigma ds \\
\leq r_\eta^{-\gamma} \int_r^R s^{1-N} \int_0^{\min\{s,r_\eta\}} \sigma^{N-1} d\sigma ds \\
= r_\eta^{-\gamma} \int_r^R s^{1-N} \int_0^s \sigma^{N-1} d\sigma ds + r_\eta^{-\gamma} \int_{r_\eta}^R s^{1-N} \int_0^{r_\eta} \sigma^{N-1} d\sigma ds \\
= \frac{r_\eta^{-\gamma}}{N} \int_r^r s ds + \frac{r_\eta^{-\gamma}}{N} \int_r^{r_\eta} s^{1-N} ds \\
= \frac{r_\eta^{-\gamma}}{2N} (r_\eta^2 - r^2) + \frac{r_\eta^{-\gamma}}{(N-2)N} (r_\eta^2 - N^2 - R^{2-N}) \\
\leq \frac{1}{2N} r_\eta^{2-\gamma} + \frac{1}{(N-2)N} r_\eta^{2-\gamma} = \frac{1}{2(N-2)} r_\eta^{2-\gamma}. 
\]

Combining these two estimates, we see that for \( r < \eta \)

\[
\int_r^R s^{1-N} \int_0^s \sigma^{N-1} u_\eta(s) d\sigma ds \geq \int_r^R s^{1-N} \int_0^{\min\{s,r_\eta\}} \sigma^{N-1}(\sigma^2 + \eta^2)^{-\frac{\gamma}{2}} d\sigma ds \\
- \int_r^R s^{1-N} \int_0^{\min\{r,\eta\}} \sigma^{N-1}(\sigma^2 + \eta^2)^{-\frac{\gamma}{2}} d\sigma ds \\
\geq c_1 \eta^{2-\gamma} - c_1 r^2 \eta^{-\gamma} - \frac{1}{2(N-2)} r_\eta^{2-\gamma}.
\]

In conclusion, making use of (39) and (40) we obtain a pointwise lower estimate for \( v_\eta(r) \), for any \( r < \eta \):

\[
v_\eta(r) \geq \int_r^R s^{1-N} \int_0^s \sigma^{N-1} u_\eta(s) d\sigma ds \\
- N R^{-N} \int_0^R r^{N-1} \int_t^{r_\eta} s^{1-N} \int_0^s \sigma^{N-1} u_\eta(s) d\sigma ds dt \\
- \int_r^R s^{1-N} \int_0^s \sigma^{N-1} u_\eta(s) d\sigma ds \\
\geq c_1 \eta^{2-\gamma} - c_1 r^2 \eta^{-\gamma} - \frac{1}{2(N-2)} r_\eta^{2-\gamma} \\
- N R^{-N} \left( \frac{R^2}{2(N-2)(N-\gamma)} \eta^{\delta(N-\gamma)} - (C + C\eta^{\delta(2-\gamma)} + C'r^{4-\gamma}) \right) \\
\geq c_1 \eta^{2-\gamma} - c_1 r^2 \eta^{-\gamma} - c_2 r_\eta^{2-\gamma} - c_3 - c_4 r^{4-\gamma}
\]

for suitably chosen constants \( c_1 > 0, c_2 > 0, c_3 > 0, c_4 > 0 \).

4.4. **The estimate for** \( \int_\Omega u_\eta v_\eta \). We choose \( a \in (0, 1) \) such that \( c_5 := \frac{\gamma - \frac{3}{2}}{N} - \frac{\sigma^2}{N+2} > 0 \). Then applying the previously derived estimates we obtain

\[
\frac{1}{\omega_N} \int_\Omega u_\eta v_\eta = \int_0^R r^{N-1} u_\eta(r) v_\eta(r) dr \\
\geq \int_0^{a_\eta} r^{N-1} u_\eta(r) v_\eta(r) dr
\]
Lemma 5.1. Assume the following lemma:

\[ u \]

The only remaining step then is to not just use \( u \), but to approximate any given \( u_0 \) and to adjust arguments where necessary (in particular in (42)). We do this in the following lemma:

**Lemma 5.1.** Assume, \( S \) and \( D \) are such that (6) and \( G \) as defined in (9) is such that (10) is satisfied with some \( \alpha \in \mathbb{R} \) and \( C_G > 0 \). If

\[ -\alpha > \frac{2}{N}, \]

the following holds:

\[
\begin{align*}
&\geq \int_0^{a_0} r^{N-1}(r^2 + \eta^2)^{-\frac{2}{N}} c_1 \eta^{2-\gamma} \rho dr - \int_0^{a_0} r^{N-1}(r^2 + \eta^2)^{-\frac{2}{N}} c_2 \eta^{2-\gamma} \rho dr \\
&- \int_0^{a_0} r^{N-1}(r^2 + \eta^2)^{-\frac{2}{N}} c_2 \eta^{2-\gamma} \rho dr - \int_0^{a_0} r^{N-1}(r^2 + \eta^2)^{-\frac{2}{N}} c_3 \rho dr \\
&- \int_0^{a_0} r^{N-1}(r^2 + \eta^2)^{-\frac{2}{N}} c_4 r^{4-\gamma} \rho - \int_0^{a_0} r^{N-1}(r^2 + \eta^2)^{-\frac{2}{N}} c_1 \eta^{2-\gamma} \rho dr \\
&\geq c_1 \eta^{2-\gamma} 2^{-\frac{2}{N}} \int_0^{a_0} r^{N-1} \eta^{-\gamma} \rho dr \\
&- c_2 \eta^{-\gamma} \int_0^{a_0} r^{N-1} \eta^{-\gamma} \rho dr - c_2 \eta^{-\gamma} \int_0^{a_0} r^{N-1} \eta^{-\gamma} \rho dr \\
&= 2^{-\frac{2}{N}} c_1 \eta^{2-\gamma} \int_0^{a_0} r^{N-1} \eta^{-\gamma} \rho dr - c_2 \eta^{-\gamma} \int_0^{a_0} r^{N-1} \eta^{-\gamma} \rho dr \\
&= \frac{c_1}{N} \eta^{2-\gamma} (\eta N)^N - \frac{c_4}{N+\gamma} (\eta N)^{N+\gamma} - \frac{c_1}{N} \eta^{2-\gamma} (\eta N)^N \\
&\geq c_1 a^N \left( \frac{2^{-\frac{2}{N}}}{N} - \frac{a^2}{N+\gamma} \right) \eta^{2-\gamma} + N \\
&- c_2 \eta^{2-\gamma} (\eta N)^N - c_3 \eta^{2-\gamma} (\eta N)^N - c_4 (N+\gamma)^{N+\gamma} - c_1 (N+\gamma)^{N+\gamma} \\
&\geq 2 - 2\gamma + N = \min\{2 - 2\gamma + N, N - \gamma + (2 - \gamma)\delta, N - \gamma, N + 4 - 2\gamma, N + 2 - \gamma - \gamma \delta\},
\end{align*}
\]

For small values of \( \eta \), the first of these terms dominates the others if \( 2 - 2\gamma + N \) is negative and

\[ 2 - 2\gamma + N = \min\{2 - 2\gamma + N, N - \gamma + (2 - \gamma)\delta, N - \gamma, N + 4 - 2\gamma, N + 2 - \gamma - \gamma \delta\}, \]

which is ensured if \( \gamma > 2 \), since \( \delta < 1 \).

We can therefore summarize the result of subsections 4.2 – 4.4 as follows:

There are \( \eta_0 > 0 \) and \( c_0 > 0 \) such that for all \( \eta \in (0, \eta_0) \):

\[
\int_0^{\Omega} u_{\eta} v_\eta \geq c_0 \eta^{2-2\gamma+N} \tag{41}
\]

5. **An upper bound for the positive contribution to \( \mathcal{F}(u_{\eta}, v_\eta) \).** Proof of Theorem 1.4. Under the assumption (10),

\[
\int_\Omega G(u_\eta) \leq C_G|\Omega| + C_G \omega N \int_0^{\Omega} r^{N-1}(r^2 + \eta^2)^{-\frac{2}{N}(2+\alpha)} dr \\
\leq c_1 + c_1 \eta^{-\gamma(2+\alpha)}r_\eta^N = c_1 + c_1 \eta^{-\gamma(2+\alpha)}r_\eta^N. \tag{42}
\]

If we want the term in (41) to dominate that of (42), we have to ensure that

\[ 2 - 2\gamma + N < N\delta - \gamma(2 + \alpha). \]

The only remaining step then is to not just use \( u_\eta \), but to approximate any given \( u_0 \) and to adjust arguments where necessary (in particular in (42)). We do this in the following lemma:
Let \( p \in [1, \frac{2N}{N+2}] \) if \( \alpha \leq -\frac{4}{N+2} \) and \( p \in [1, -\frac{\alpha N}{2}) \) if \( \alpha > -\frac{4}{N+2} \). Given radially symmetric \( u_0 \in C^\beta(\Omega) \) for some \( \beta > 0 \), there are radially symmetric functions \( u_\eta, v_\eta \) such that \( 0 = \Delta v_\eta - v_\eta + u_\eta, \partial_\nu v_\eta \big|_{\partial \Omega} = 0 \) for any \( \eta \in (0, 1) \), and
\[
\mathcal{F}(u_\eta, v_\eta) \to -\infty \quad \text{as } \eta \searrow 0
\]
and
\[
\|u_\eta - u_0\|_{L^p(\Omega)} \to 0 \quad \text{as } \eta \searrow 0.
\]

Proof. Since \( 2 < -N\alpha \) by (43), we can choose
\[
\gamma \in \left(\frac{N+2}{2}, N\right) \setminus \{2, 4\}
\]
such that
\[
2 < -\gamma\alpha.
\]
We can, moreover, make this choice in such a way that
\[
\frac{N}{\gamma} > p,
\]
because \( \frac{N}{\gamma} > \max\{\frac{N+2}{2}, \frac{2}{\gamma}\} \) by the conditions on \( p \). In light of (45), it is possible to furthermore choose \( \delta \in (0, 1) \) satisfying
\[
2 + (1 - \delta)N < -\gamma\alpha
\]
so that, finally,
\[
2 - 2\gamma + N < N\delta - \gamma(\alpha + 2)
\]
holds.

With \( \gamma \) and \( \delta \) as chosen here, we now define \( u_\eta \) according to (32) and \( v_\eta \) by (33). We then pick a small number \( q > 0 \) such that
\[
2 - 2\gamma + N < (\alpha + 2)q
\]
and define
\[
\hat{u}_\eta := u_0 + u_\eta + \eta^q.
\]
(The last summand will only be needed if \( \alpha + 2 < 0 \).) We let \( v_0 \) be the corresponding solution to the Neumann problem of \(-\Delta v_0 + v_0 = u_0 \) and define \( \hat{v}_\eta = v_0 + v_\eta + \eta^q \).

By linearity of the elliptic equation, \( \hat{v}_\eta \) then solves \(-\Delta \hat{v}_\eta + \hat{v}_\eta = \hat{u}_\eta \) and furthermore obeys \( \partial_\nu \hat{v}_\eta \big|_{\partial \Omega} = 0 \).

We note that
\[
\|\hat{u}_\eta - u_0\|_{L^p(\Omega)}^p = \|u_\eta + \eta^q\|_{L^p(\Omega)}^p \leq 2^p\omega_N \int_0^R r^{N-1}u_\eta^p(r)dr + 2^p|\Omega|\eta^{pq}
\]
\[
\leq 2^p\omega_N \int_0^\rho r^{N-1}(r^2 + \eta^2)^{-\frac{2p}{2}}dr + 2^p|\Omega|\eta^{pq}
\]
\[
\leq 2^p\omega_N \int_0^{N-1}\gamma^pdr + 2^p|\Omega|\eta^{pq}
\]
\[
= \frac{2^p\omega_N}{N - \gamma p}r^{N-\gamma p} + 2^p|\Omega|\eta^{pq} = \frac{2^p\omega_N}{N - \gamma p}q^{(N - \gamma p)} + 2^p|\Omega|\eta^{pq} \to 0
\]
as \( \eta \to 0 \), due to \( q > 0 \) and (46).

If \( 2 + \alpha < 0 \), then \( \hat{u}_\eta \geq \eta^q \) together with (10) ensures that
\[
\int_\Omega G(\hat{u}_\eta) \leq C_G \int_\Omega \left(1 + \eta^{q(2+\alpha)}\right) \leq C_G|\Omega| \left(1 + \eta^{q(2+\alpha)}\right).
\]
If \(2 + \alpha \geq 0\), then we use that with some constant \(c_1 > 0\), \(u_0(x) + \eta^q \leq c_1\) for all \(x \in \Omega\) and \(\eta \in (0, 1)\) and employ the estimate

\[G(u_0 + u_\eta + \eta^q) \leq C_G(u_0 + \eta^q)^{2+\alpha} + 2^{2+\alpha}C_Gu_\eta^{2+\alpha} \leq c_2 + c_2u_\eta^{2+\alpha}\]

for suitable \(c_2 > 0\), yielding

\[
\begin{align*}
\int_\Omega G(u_0 + u_\eta + \eta^q) &\leq c_2|\Omega| + c_2 \int_\Omega u_\eta^{2+\alpha} \\
&\leq c_2|\Omega| + c_2\omega_N \int_0^{r_\eta} r^{N-1}(r^2 + \eta^q)^{-\frac{2}{2}}(2+\alpha)dr \\
&\leq c_2|\Omega| + c_2\omega_N \eta^{-\gamma(2+\alpha)} \int_0^{r_\eta} r^{N-1}dr \\
&\leq c_3 + c_3\eta^{N\delta-\gamma(2+\alpha)}
\end{align*}
\]

with some \(c_3 > 0\).

Moreover, \(\hat{u}_\eta \geq u_\eta\) and \(\hat{v}_\eta \geq v_\eta\) and hence

\[
\int_\Omega \hat{u}_\eta \hat{v}_\eta \geq \int_\Omega u_\eta v_\eta \geq c_0\eta^{2-2\gamma+N}
\]

by (41). Therefore

\[
\mathcal{F}(\hat{u}_\eta, \hat{v}_\eta) = \int_\Omega G(u_0 + u_\eta + \eta^q) - \frac{1}{2} \int_\Omega \hat{u}_\eta \hat{v}_\eta \\
\leq c_4\eta^{(2+\alpha)} + c_5 + c_3\eta^{N\delta-\gamma(2+\alpha)} - \frac{c_0}{2}\eta^{2-2\gamma+N},
\]

where \(c_4 = C_G|\Omega|\) and \(c_5 = \max\{c_3, c_4\}\). Due to (44) and (49), the exponent in the last of these terms is negative and according to (48), it is also the smallest exponent. From negativity of its coefficient, we may immediately conclude

\[
\mathcal{F}(\hat{u}_\eta, \hat{v}_\eta) \rightarrow -\infty \quad \text{as } \eta \rightarrow 0. \tag{50}
\]

Theorem 1.4 now becomes a straightforward consequence:

**Proof of Theorem 1.4.** We combine Lemma 5.1 and Lemma 3.5.

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