Enriched categories of correspondences and characteristic classes of singular varieties

by

Shoji Yokura (Kagoshima)

Abstract. For the category $\mathcal{V}$ of complex algebraic varieties, the Grothendieck group of the commutative monoid of the isomorphism classes of correspondences $X \xleftarrow{f} M \xrightarrow{g} Y$ with a proper morphism $f$ and a smooth morphism $g$ (such a correspondence is called a proper-smooth correspondence) gives rise to an enriched category $\text{Corr}(\mathcal{V})_{\text{pro-sm}}^+$ of proper-smooth correspondences. In this paper we extend the well-known theories of characteristic classes of singular varieties such as Baum–Fulton–MacPherson’s Riemann–Roch transformation (abbr. BFM–RR), MacPherson’s Chern class transformation etc. to this enriched category. In order to deal with local complete intersection ($\ell.c.i.$) morphisms instead of smooth morphisms, in a similar manner we consider an enriched category $\mathcal{Z}\text{igzag}(\mathcal{V})_{\text{pro-\ell.c.i.}}^+$ of proper-\ell.c.i. zigzags and extend BFM–RR to this category. We also consider an enriched category $\mathcal{M}_{*,*}(\mathcal{V})_{\otimes}^+$ of proper-smooth correspondences $(X \xleftarrow{f} M \xrightarrow{g} Y; E)$ equipped with a complex vector bundle $E$ on $M$ (such a correspondence is called a cobordism bicycle of a vector bundle) and we extend BFM–RR to this enriched category as well.

1. Introduction. The algebraic cobordism $\Omega_*(X)$ of Levine and Morel \cite{21} is generated by cobordism cycles. A cobordism cycle is the isomorphism class of 

$$(M \xleftarrow{f} X; L_1, \ldots, L_r)$$

where $M$ is a quasi-projective smooth variety, $f : M \to X$ is a projective morphism and $L_i$’s are line bundles over $M$. In \cite{16} J.-L. González and K. Karu show that the above morphism $f : M \to X$ can be replaced by a proper morphism from a smooth variety $M$. In order to obtain a bivariant-theoretic analogue $\mathbb{B}\Omega(X \xleftarrow{f} Y)$ of the algebraic cobordism $\Omega_*(X)$ in such

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a way that $\mathbb{B}\Omega(X \xrightarrow{f} pt)$ is isomorphic to the algebraic cobordism $\Omega_*(X)$, in [33] we introduce an oriented bivariant theory $\mathbb{OB}(X \xrightarrow{f} Y)$, which is a bivariant theory in the sense of Fulton–MacPherson [13] (see Remark 1.7 below). Note that Fulton–MacPherson’s bivariant theory $\mathbb{B}(X \xrightarrow{f} pt)$ has the property that $\mathbb{B}(X \xrightarrow{id_X} X)$ is a covariant functor like a homology theory and $\mathbb{B}(X \xrightarrow{id_X} X)$ is a contravariant functor like a cohomology theory.

$\mathbb{OB}(X \xrightarrow{f} Y)$ is generated by the cobordism cycles $(M \xrightarrow{h} X; L_1, \ldots, L_r)$ such that

1. $h : M \to X$ is a proper morphism,
2. $f \circ h : M \to Y$ is a smooth morphism.

If $Y = pt$ is a point, then $M$ is smooth and $h : M \to X$ is proper, thus $\mathbb{OB}(X \to pt)$ is generated by the isomorphism classes of proper morphisms from smooth $M$ to $X$, as in González–Karu’s construction. The two morphisms $h : M \to X$ and $f \circ h : M \to Y$ are written as

$$X \xleftarrow{h} M \xrightarrow{f \circ h} Y$$

In general, for a category $\mathcal{C}$ and for three objects $X, Y, M \in \text{Obj}(\mathcal{C})$, $X \xleftarrow{f} M \xrightarrow{g} Y$ is called a correspondence (span or roof) from $X$ to $Y$. The above correspondence $X \xleftarrow{f} M \xrightarrow{g} Y$ with a proper morphism $f$ and a smooth morphism $g$ will be called a proper-smooth correspondence from $X$ to $Y$. In [35] we introduce a cobordism bicycle of a vector bundle $(X \xleftarrow{f} M \xrightarrow{g} Y; E)$, a proper-smooth correspondence carrying a complex vector bundle $E$ on $M$, as a generalization or a bi-variant analogue of a cobordism cycle of a vector bundle $(M \xrightarrow{f} X; E)$ with $f : M \to X$ a proper morphism from a smooth variety $M$ and a complex vector bundle $E$ on $M$, introduced in a recent paper by Lee–Pandharipande [20]. In [35] we discuss bivariant-theoretic properties and aspects of cobordism bicycles of complex vector bundles, but in this paper we will not treat such bivariant-theoretic aspects; instead we consider characteristic classes of singular varieties on proper-smooth correspondences and also on cobordism bicycles of complex vector bundles.

A proper-smooth correspondence $X \xleftarrow{f} M \xrightarrow{g} Y$ can be considered as a morphism from $X$ to $Y$ as follows. Let $\text{Corr}(X, Y)_{\text{pro-sm}}$ be the set of all proper-smooth correspondences from $X$ to $Y$. Then the composition

$$\circ : \text{Corr}(X, Y)_{\text{pro-sm}} \times \text{Corr}(Y, Z)_{\text{pro-sm}} \to \text{Corr}(X, Z)_{\text{pro-sm}}$$

defined by

$$(X \xleftarrow{f} M \xrightarrow{g} Y) \circ (Y \xleftarrow{h} N \xrightarrow{k} Z) := X \xleftarrow{f \circ h} M \times_Y N \xrightarrow{k \circ g} Z$$
is well-defined because the pullback \( \tilde{h} \) of a proper morphism \( h \) is proper and the pullback \( \tilde{g} \) of a smooth morphism \( g \) is smooth, and the composite of proper morphisms is proper and the composite of smooth morphisms is smooth. Here we consider the following commutative diagram where the middle square is the fiber product:

\[
\begin{array}{ccc}
M \times_Y N & \xrightarrow{\tilde{h}} & N \\
\downarrow f & & \downarrow k \\
M & \xrightarrow{g} & Z \\
\uparrow h & & \\
X & \xleftarrow{\cdot} & Y
\end{array}
\]

Then from the category \( \mathcal{V} \) of complex algebraic varieties we get the following category \( \text{Corr}(\mathcal{V})_{\text{pro-sm}} \) of proper-smooth correspondences:

- \( \text{Obj}(\text{Corr}(\mathcal{V})_{\text{pro-sm}}) = \text{Obj}(\mathcal{V}) \),
- for objects \( X \) and \( Y \), \( \text{hom}_{\text{Corr}(\mathcal{V})_{\text{pro-sm}}}(X, Y) = \text{Corr}(X, Y)_{\text{pro-sm}} \).

**Remark 1.2.** For a recent higher-categorical study of correspondences (in derived algebraic geometry), see Gaitsgory–Rozenblyum’s book [14] (cf. [15]), in particular Chapters 5–9.

On the category \( \mathcal{V} \) of complex algebraic varieties, let us consider Baum–Fulton–MacPherson’s Todd class transformation (or Riemann–Roch transformation) \[ td^\text{BFM}_* : G_0(-) \to H_*(-) \otimes \mathbb{Q}, \] which is a unique natural transformation from the covariant functor \( G_0(-) \) of Grothendieck groups of coherent sheaves to the covariant Borel–Moore homology theory \( H_*(-) \otimes \mathbb{Q} \) with rational coefficients, satisfying the “smoothness condition” that if \( X \) is smooth, then the value \( td^\text{BFM}_*(\mathcal{O}_X) \) on the structure sheaf \( \mathcal{O}_X \) is equal to \( \text{td}(TX) \cap [X] \), the Poincaré dual of the total Todd cohomology class \( \text{td}(TX) \) of the tangent bundle \( TX \). Here we remark that the uniqueness of the transformation \( td^\text{BFM}_* : G_0(-) \to H_*(-) \otimes \mathbb{Q} \) is due to the above smoothness condition. The functors \( G_0(-) \) and \( H_*(-) \otimes \mathbb{Q} \) and the the natural transformation \( td^\text{BFM}_* : G_0(-) \to H_*(-) \otimes \mathbb{Q} \) can be naturally extended to the category \( \text{Corr}(\mathcal{V})_{\text{pro-sm}} \) as follows:

**Proposition 1.3.** Define the functors \( G_0 : \text{Corr}(\mathcal{V})_{\text{pro-sm}} \to \mathcal{A}b \) and \( H^\text{Todd}_* : \text{Corr}(\mathcal{V})_{\text{pro-sm}} \to \mathcal{A}b \) as follows:

- for \( X \in \text{Obj}(\text{Corr}(\mathcal{V})_{\text{pro-sm}}) = \text{Obj}(\mathcal{V}) \),
  \[ G_0(X) := G_0(X), \quad H^\text{Todd}_*(X) := H_*(X) \otimes \mathbb{Q}, \]
- for a morphism \( X \xleftarrow{f} M \xrightarrow{g} Y \in \text{hom}_{\text{Corr}(\mathcal{V})_{\text{pro-sm}}}(X, Y) = \text{Corr}(X, Y)_{\text{pro-sm}} \),
Then $G_0 : \text{Corr}(\mathcal{V})_{\text{pro-sm}} \to \mathcal{A}b$ and $H_*^{\text{Todd}} : \text{Corr}(\mathcal{V})_{\text{pro-sm}} \to \mathcal{A}b$ are functors \(^{(1)}\) in the sense that $G_0(\alpha \circ \beta) = G_0(\alpha) \circ G_0(\beta)$ and $H_*^{\text{Todd}}(\alpha \circ \beta) = H_*^{\text{Todd}}(\alpha) \circ H_*^{\text{Todd}}(\beta)$, and Baum–Fulton–MacPherson’s Todd class transformation $\text{td}^*_{\text{BFM}} : G_0(-) \to H_*(-) \otimes \mathbb{Q}$ extends to a natural transformation

$$\text{td}^*_{\text{BFM}} : G_0(-) \to H_*^{\text{Todd}}(-).$$

Furthermore, the set of isomorphism classes of proper-smooth correspondences becomes an Abelian monoid by taking the disjoint union or direct sum $\sqcup$, i.e.,

$$[X \leftarrow f_1 \ M \rightarrow g_1 \ Y] + [X \leftarrow f_2 \ M_2 \rightarrow g_2 \ Y] := [X \leftarrow f_1 \sqcup f_2 \ M_1 \sqcup M_2 \rightarrow g_1 \sqcup g_2 \ Y].$$

Its Grothendieck group will be denoted by $\text{Corr}(X,Y)_{\text{pro-sm}}^+$. Then the above “product” $\circ : \text{Corr}(X,Y)_{\text{pro-sm}}^+ \times \text{Corr}(Y,Z)_{\text{pro-sm}}^+ \to \text{Corr}(X,Z)_{\text{pro-sm}}^+$ extends to

$$\circ : \text{Corr}(X,Y)_{\text{pro-sm}}^+ \times \text{Corr}(Y,Z)_{\text{pro-sm}}^+ \to \text{Corr}(X,Z)_{\text{pro-sm}}^+.$$ Using this we can define the following $\mathcal{A}b$-enriched (or preadditive\(^{(2)}\)) category $\text{Corr}(\mathcal{V})_{\text{pro-sm}}^+$ associated to such correspondences:

- $\text{Obj}(\text{Corr}(\mathcal{V})_{\text{pro-sm}}^+) = \text{Obj}(\mathcal{V})$,
- for objects $X,Y$, $\text{hom}_{\text{Corr}(\mathcal{V})_{\text{pro-sm}}^+}(X,Y) := \text{Corr}(X,Y)_{\text{pro-sm}}^+$.

Then we have the following theorem:

**Theorem 1.4.** Define $G_0, H_*^{\text{Todd}} : \text{Corr}_{\text{pro-sm}}^+ \to \mathcal{A}b$ as follows:

- for $X \in \text{Obj}(\text{Corr}_{\text{pro-sm}}^+)$,
  $$G_0(X) := G_0(X), \quad H_*^{\text{Todd}}(X) := H_*(X) \otimes \mathbb{Q},$$

\(^{(1)}\) Since the correspondence $X \leftarrow f \ M \rightarrow g \ Y$ is considered as a morphism from $X$ to $Y$ and the homomorphism $\mathcal{F}(X \leftarrow f \ M \rightarrow g \ Y) := f_*g^* : \mathcal{F}(Y) \to \mathcal{F}(X)$ is a homomorphism from $\mathcal{F}(Y)$ to $\mathcal{F}(X)$, $\mathcal{F} : \text{Corr}(\mathcal{V}) \to \mathcal{A}b$ should be called a contravariant functor. But since it satisfies $\mathcal{F}(\alpha \circ \beta) = \mathcal{F}(\alpha) \circ \mathcal{F}(\beta)$, we call it a functor, i.e., a covariant functor.

\(^{(2)}\) This cannot be replaced by “additive” because there does not exist a zero object in the category $\text{Corr}(\mathcal{V})_{\text{pro-sm}}^+$. 

• for $\sum_i n_i[X \xleftarrow{f_i} M_i \xrightarrow{g_i} Y] \in \hom_{\corr_{pro-sm}^+} (X, Y) := \corr(X, Y)^{+}_{pro-sm}$,

$\mathcal{G}_0\left(\sum_i n_i[X \xleftarrow{f_i} M_i \xrightarrow{g_i} Y]\right) := \sum_i n_i(f_i)_*(g_i)^* : \mathcal{G}_0(Y) \to \mathcal{G}_0(X)$,

$\mathcal{H}_{\text{Todd}}^*\left(\sum_i n_i[X \xleftarrow{f_i} M_i \xrightarrow{g_i} Y]\right) := \sum_i n_i(f_i)_*(\text{td}(T_{g_i}) \cap (g_i)^*) : \mathcal{H}_{\text{Todd}}^*(Y) \to \mathcal{H}_{\text{Todd}}^*(X)$.

Then $\mathcal{G}_0, \mathcal{H}_{\text{Todd}}^* : \corr_{pro-sm}^+ \to \mathcal{A}b$ are both functors in the sense that $\mathcal{G}_0(\alpha \circ \beta) = \mathcal{G}_0(\alpha) \circ \mathcal{G}_0(\beta)$ and $\mathcal{H}_{\text{Todd}}^*(\alpha \circ \beta) = \mathcal{H}_{\text{Todd}}^*(\alpha) \circ \mathcal{H}_{\text{Todd}}^*(\beta)$ and Baum–Fulton–MacPherson’s Todd class transformation $\text{td}_{BFM}^* : G_0(-) \to H_*(-) \otimes \mathbb{Q}$ extends to a natural transformation

$\text{td}_{BFM}^* : \mathcal{G}_0(-) \to \mathcal{H}_{\text{Todd}}^*(-)$.

Remark 1.5. To avoid possible confusion, we make a remark about notations or symbols concerning the Grothendieck group or the group completion of an Abelian monoid $A$. Let $(A, +)$ be an Abelian monoid. Then its group completion or the Grothendieck group $\hat{K}(A)$ has two constructions; one construction is defined by $\hat{K}(A) := F(A)/E(A)$, where $F(A)$ is the free abelian group generated by the set $A$, i.e.,

$F(A) := \left\{ \sum n_a a \mid a \in A, n_a \in \mathbb{Z}, n_a = 0 \text{ for almost all } a \text{'s} \right\}$

and the group operation $+$ on $F(A)$ is defined by

$\sum n_a a + \sum m_a a := \sum (n_a + m_a) a$

and $E(A)$ is the following subgroup of $F(A)$:

$E(A) := \left\{ \sum n_{a,b} (a + b + (-1)(a + b)) \mid a, b \in A, n_{a,b} \in \mathbb{Z}, n_{a,b} = 0 \text{ for almost all } a \text{'s and } b \text{'s} \right\}$.

The equivalence class $a + E(A) \in \hat{K}(A)$ of $a \in F(A)$ should be denoted by $[a]$ and the group operation on $F(A)$ should be denoted by, say, $\hat{+}$, and $[a] \hat{+} [b] := [a + b]$. Then, since $\hat{K}(A) = F(A)/E(A)$, $[a + b] = [a + b]$ in $\hat{K}(A)$, thus we have $[a] \hat{+} [b] = [a + b]$. In this way, the monoid operation $+$ on $A$ is extended to a group operation $\hat{+}$ on $\hat{K}(A)$. To avoid many symbols or notation, usually $[a] \hat{+} [b]$ is simply denoted by $a + b$ unless confusion is possible. Thus, in the above theorem, an element $\sum_i n_i[X \xleftarrow{f_i} M_i \xrightarrow{g_i} Y] \in \corr(X, Y)^{+}_{pro-sm}$ should be denoted by $\sum_i n_i[[X \xleftarrow{f_i} M_i \xrightarrow{g_i} Y]]$, but we follow the above convention.
In a similar manner we can extend other well-known characteristic classes of singular varieties ([25], [6]) to the enriched category \( \text{Corr}(\mathcal{V})_{\text{pro-sm}}^+ \). Furthermore, similarly, we can define a proper-local complete intersection (abbr. \( \ell.c.i. \)) correspondence. But, in this case, since the pullback of an \( \ell.c.i. \) morphism is not necessarily an \( \ell.c.i. \) morphism, an argument similar to the above does not work. To remedy this, we consider a proper-\( \ell.c.i. \) zigzag, which is a finite sequence of proper-\( \ell.c.i. \) correspondences \( X \xleftarrow{f} M \xrightarrow{g} Y \) with a proper \( f \) and an \( \ell.c.i. \) morphism \( g \). This will be discussed in §4. In §5 we will define the enriched category \( \mathcal{M}_{*,*}(\mathcal{V})_\otimes^+ \) of cobordism bicycles of vector bundles and extend Baum–Fulton–MacPherson’s Riemann–Roch transformation or Todd class transformation to this enriched category:

**Theorem 1.6.**

1. Define \( G_0^\otimes : \mathcal{M}_{*,*}(\mathcal{V})_\otimes^+ \to \mathcal{A}b \) by
   
   (a) for an object \( X, G_0^\otimes(X) := G_0(X) \),
   
   (b) for \( \sum_i n_i[X \xleftarrow{p_i} V_i \xrightarrow{s_i} Y; E_i] \in \text{hom}_{\mathcal{M}_{*,*}(\mathcal{V})_\otimes^+}(X,Y) = \mathcal{M}_{*,*}(X,Y)_\otimes^+ \),
   
   \[
   G_0^\otimes\left(\sum_i n_i[X \xleftarrow{p_i} V_i \xrightarrow{s_i} Y; E_i]\right)
   := \sum_i n_i(p_i)([E_i] \otimes (s_i)^*) : G_0^\otimes(Y) \to G_0^\otimes(X).
   \]

   Then \( G_0^\otimes : \mathcal{M}_{*,*}(\mathcal{V})_\otimes^+ \to \mathcal{A}b \) is a functor in the sense that \( G_0^\otimes(\alpha \otimes \beta) = G_0^\otimes(\alpha) \circ G_0^\otimes(\beta) \).

2. For the Todd class \( \text{td} \) and the Chern character \( \text{ch} \), define \( \mathcal{H}_{*,*}^{\text{td, ch}} : \mathcal{M}_{*,*}(\mathcal{V})_\otimes^+ \to \mathcal{A}b \) by
   
   (a) for an object \( X, \mathcal{H}_{*,*}^{\text{td, ch}}(X) := H_*(X) \otimes \mathbb{Q} \),
   
   (b) for \( \sum_i n_i[X \xleftarrow{p_i} V_i \xrightarrow{s_i} Y; E_i] \in \text{hom}_{\mathcal{M}_{*,*}(\mathcal{V})_\otimes^+}(X,Y) = \mathcal{M}_{*,*}(X,Y)_\otimes^+ \),
   
   \[
   \mathcal{H}_{*,*}^{\text{td, ch}}\left(\sum_i n_i[X \xleftarrow{p_i} V_i \xrightarrow{s_i} Y; E_i]\right)
   := \sum_i n_ip_i*\left(\text{td}(T_{s_i}) \cap \text{ch}(E_i) \cap (s_i)^*\right) : \mathcal{H}_{*,*}^{\text{td, ch}}(Y) \to \mathcal{H}_{*,*}^{\text{td, ch}}(X).
   \]

   Then \( \mathcal{H}_{*,*}^{\text{td, ch}} : \mathcal{M}_{*,*}(\mathcal{V})_\otimes^+ \to \mathcal{A}b \) is a functor in the sense that

   \[
   \mathcal{H}_{*,*}^{\text{td, ch}}(\alpha \otimes \beta) = \mathcal{H}_{*,*}^{\text{td, ch}}(\alpha) \circ \mathcal{H}_{*,*}^{\text{td, ch}}(\beta).
   \]

3. Baum–Fulton–MacPherson’s Todd class transformation \( \text{td}_{\text{BFM}}^{\text{td}} \) gives rise to a natural transformation of these two functors \( G_0^\otimes : \mathcal{M}_{*,*}(\mathcal{V})_\otimes^+ \to \mathcal{A}b \) and \( \mathcal{H}_{*,*}^{\text{td, ch}} : \mathcal{M}_{*,*}(\mathcal{V})_\otimes^+ \to \mathcal{A}b \),

\[
\text{td}_{\text{BFM}}^{\text{td}} : G_0^\otimes(-) \to \mathcal{H}_{*,*}^{\text{td, ch}}(-).
\]
Remark 1.7. In [24] P. Lowrey and T. Schürg have constructed a derived algebraic cobordism \( d\Omega_*(X) \) for derived schemes. In [1] T. Annala has obtained a bivariant-theoretic version \( \Omega^*(X \to Y) \) of Levine–Morel’s algebraic cobordism, using the construction of Lowrey and Schürg and the construction of a universal bivariant theory of the author [33] (cf. [34]). Furthermore, in [3] (see also [2]) T. Annala and the author have constructed a bivariant-theoretic version of Lee–Pandharipande’s algebraic cobordism of vector bundles [20].

2. Enriched categories of correspondences

Definition 2.1. Let \( \mathcal{V} \) be the category of complex algebraic varieties. For any pair \((X, Y)\) of complex algebraic varieties \( X \) and \( Y \), the set of all correspondences \( X \leftarrow \overset{f}{V} \overset{g}{\to} Y \) is denoted by \( \text{Corr}(X, Y) \):

\( \text{Corr}(X, Y) := \{ X \leftarrow \overset{f}{V} \overset{g}{\to} Y \mid f \in \text{hom}_\mathcal{V}(V, X), \, g \in \text{hom}_\mathcal{V}(V, Y) \} \).

Definition 2.2. The category \( \text{Corr}(\mathcal{V}) \) of correspondences of \( \mathcal{V} \) is defined by

- \( \text{Obj}(\text{Corr}(\mathcal{V})) = \text{Obj}(\mathcal{V}) \),
- for objects \( X \) and \( Y \), the set of homomorphisms from \( X \) to \( Y \) is defined to be \( \text{Corr}(X, Y) \), i.e., \( \text{hom}_{\text{Corr}(\mathcal{V})}(X, Y) = \text{Corr}(X, Y) \), and the composition \( (2.3) \)

\[ (X \leftarrow \overset{f}{M} \overset{g}{\to} Y) \circ (Y \leftarrow \overset{h}{N} \overset{k}{\to} Z) := X \leftarrow \overset{\tilde{f} \circ h}{M \times_V N} \overset{k \circ g}{\to} Z, \]

where we use the diagram (1.1).

Definition 2.4. Let \( \mathcal{A}b \) be the category of abelian groups. A functor \( E : \mathcal{V} \to \mathcal{A}b \) is called a bifunctor \( (3) \) if it is both a covariant and a contravariant functor.

Lemma 2.5. Let \( \mathcal{A}b \) be the category of abelian groups. A functor \( E : \mathcal{V} \to \mathcal{A}b \) is called a bifunctor \( (3) \) if it is both a covariant and a contravariant functor.

Let \( E : \mathcal{V} \to \mathcal{A}b \) be a bifunctor. Define \( \mathcal{E} : \text{Corr}(\mathcal{V}) \to \mathcal{A}b \) by

- for \( X \in \text{Obj}(\text{Corr}(\mathcal{V})) = \text{Obj}(\mathcal{V}) \), \( \mathcal{E}(X) := E(X) \),
- for \( (X \leftarrow \overset{f}{V} \overset{g}{\to} Y) \in \text{hom}_{\text{Corr}(\mathcal{V})}(X, Y) = \text{Corr}(X, Y) \),

\[ \mathcal{E}(X \leftarrow \overset{f}{V} \overset{g}{\to} Y) := f_*g^* : \mathcal{E}(Y) \to \mathcal{E}(X). \]

(3) Usually a bifunctor is understood to be a functor \( E : \mathcal{C}' \times \mathcal{C}' \to \mathcal{B} \) defined on the Cartesian product of two categories, which is contravariant with respect to the first factor and covariant with respect to the second factor.
If $E$ satisfies the base change formula, i.e., for any fiber square (left) the square on the right commutes:

\[
\begin{array}{c}
A' \xrightarrow{\tilde{g}} A \\
\downarrow \tilde{h} \quad \downarrow h
\end{array}
\quad \Rightarrow 
\begin{array}{c}
E(A') \xleftarrow{(\tilde{g})^*} E(A) \\
\downarrow (\tilde{h})_* \quad \downarrow h_*
\end{array}
\quad \begin{array}{c}
B' \xrightarrow{g} B \\
\downarrow \quad \downarrow
\end{array}
\quad \begin{array}{c}
E(B') \xleftarrow{g^*} E(B)
\end{array}
\]

then $\mathcal{E} : \text{Corr}(\mathcal{V}) \to \mathcal{A}b$ is a functor in the sense that $\mathcal{E}(\alpha \circ \beta) = \mathcal{E}(\alpha) \circ \mathcal{E}(\beta)$ for $\alpha \in \text{hom}_{\text{Corr}(\mathcal{V})}(X, Y) = \text{Corr}(X, Y)$ and $\beta \in \text{hom}_{\text{Corr}(\mathcal{V})}(Y, Z) = \text{Corr}(Y, Z)$.

**Proof.** Let $\alpha = (X \xleftarrow{f} M \xrightarrow{g} Y) \in \text{Corr}(X, Y)$ and $\beta = (Y \xleftarrow{h} N \xrightarrow{k} Z) \in \text{Corr}(Y, Z)$. Then $\alpha \circ \beta = X \xleftarrow{f \circ \tilde{h}} M \times_Y N \xrightarrow{k \circ \tilde{g}} Z$ (see (1.1)). Hence

\[
\mathcal{E}(\alpha \circ \beta) = \mathcal{E}(X \xleftarrow{f \circ \tilde{h}} M \times_Y N \xrightarrow{k \circ \tilde{g}} Z) \\
= (f \circ \tilde{h})_* (k \circ \tilde{g})^* = f_* ((\tilde{h})_* (\tilde{g})^*) k^* \\
= f_* (g_* h_*) k^* \quad \text{(by the above base change formula)} \\
= (f_* g^*) \circ (h_* k^*) = \mathcal{E}(\alpha) \circ \mathcal{E}(\beta).
\]

**Remark 2.6.** If a bifunctor $E : \mathcal{V} \to \mathcal{A}b$ satisfies the above base change formula for a fiber square, then it satisfies the “isomorphism condition” that for an isomorphism $h : M \xrightarrow{\cong} M'$,

\[
h_* h_* = \text{id}_{E(M)}, \quad h_* h_* = \text{id}_{E(M')}
\]

These follow from the fact that the following are fiber squares:

\[
\begin{array}{ccc}
M & \xrightarrow{id_M} & M \\
\downarrow \text{id}_M & \cong & \downarrow h \\
M & \xrightarrow{h} & M'
\end{array}
\quad \begin{array}{ccc}
M & \xrightarrow{h} & M' \\
\downarrow h & \cong & \downarrow \text{id}_{M'} \\
M' & \xrightarrow{\text{id}_{M'}} & M'
\end{array}
\]

Suppose that we have two bifunctors $E_1, E_2 : \mathcal{V} \to \mathcal{A}b$ satisfying the base change formula, and a natural transformation $\tau : E_1 \to E_2$, which means that it is a natural transformation from the covariant functor $E_1$ to the covariant functor $E_2$ and at the same time from the contravariant functor $E_1$ to the contravariant functor $E_2$. Then we get two functors associated to the bifunctors $E_1, E_2$,

\[
\mathcal{E}_1, \mathcal{E}_2 : \text{Corr}(\mathcal{V}) \to \mathcal{A}b.
\]

Since we have the following commutative diagram for a correspondence $X \xleftarrow{f} M \xrightarrow{g} Y$:
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(2.7)

\[ E_1(X) \xleftarrow{f^*} E_1(M) \xleftarrow{g^*} E_1(Y) \]

\[ E_2(X) \xleftarrow{f^*} E_2(M) \xleftarrow{g^*} E_2(Y) \]

the natural transformation \( \tau : E_1 \to E_2 \) gives rise to the associated natural transformation \( \mathcal{T} : E_1 \to E_2 \).

Namely, for an object \( X \), \( \mathcal{T} : E_1(X) \to E_2(X) \) is just the homomorphism \( \tau : \mathcal{E}_1(X) \to \mathcal{E}_2(X) \).

For \( (X \xleftarrow{f} M \xrightarrow{g} Y) \in \text{hom}_{\text{Corr}(\mathcal{V})}(X, Y) = \text{Corr}(X, Y) \), naturality of \( \mathcal{T} : \mathcal{E}_1 \to \mathcal{E}_2 \) means that the following diagram commutes:

\[ \mathcal{E}_1(X) \xleftarrow{f^*g^*} \mathcal{E}_1(Y) \]

\[ \mathcal{E}_2(X) \xleftarrow{f^*g^*} \mathcal{E}_2(Y) \]

which is nothing but the above commutative diagram (2.7).

REMARK 2.8. Define two subcategories of \( \text{Corr}(\mathcal{V}) \):

\[ \text{Corr}(\mathcal{V})_{-\text{id}}(X, Y) := \text{Corr}(X, Y)_{-\text{id}} = \{ X \xleftarrow{f} Y \xrightarrow{id_Y} Y \mid f \in \text{hom}_{\mathcal{V}}(Y, X) \} \cong \text{hom}_{\mathcal{V}}(Y, X), \]

\[ \text{Corr}(\mathcal{V})_{\text{id}, -}(X, Y) := \text{Corr}(X, Y)_{\text{id}, -} = \{ X \xrightarrow{id_X} X \xrightarrow{g} Y \mid g \in \text{hom}_{\mathcal{V}}(X, Y) \} \cong \text{hom}_{\mathcal{V}}(X, Y). \]

If we restrict \( \mathcal{E} : \text{Corr}(\mathcal{V}) \to \mathcal{A} \mathcal{B} \) to these two subcategories, then \( \mathcal{E} : \text{Corr}(\mathcal{V})_{-\text{id}} \to \mathcal{A} \mathcal{B} \) is the same as the covariant functor \( \mathcal{E} : \mathcal{V} \to \mathcal{A} \mathcal{B} \), while \( \mathcal{E} : \text{Corr}(\mathcal{V})_{\text{id}, -} \to \mathcal{A} \mathcal{B} \) is the same as the contravariant functor \( \mathcal{E} : \mathcal{V} \to \mathcal{A} \mathcal{B} \).

Two correspondences \( X \xleftarrow{f_1} M_1 \xrightarrow{g_1} Y \) and \( X \xleftarrow{f_2} M_2 \xrightarrow{g_2} Y \) in the category \( \mathcal{V} \) are called isomorphic if there exists an isomorphism \( h : M_1 \cong M_2 \) such that the following diagram commutes:

The isomorphism class is denoted by \([X \xleftarrow{f} M \xrightarrow{g} Y]\) and the set of such isomorphism classes becomes an Abelian monoid with respect to the dis-
joint union $\sqcup$, i.e.,

$$[X \xleftarrow{f_1} M_1 \xrightarrow{g_1} Y] + [X \xleftarrow{f_2} M_2 \xrightarrow{g_2} Y] := [X \xleftarrow{f_1 \sqcup f_2} M_1 \sqcup M_2 \xrightarrow{g_1 \sqcup g_2} Y].$$

This operation is well-defined, i.e., it does not depend on the choice of representatives. The group completion of this Abelian monoid, i.e., the Grothendieck group of the commutative monoid, is denoted by $\text{Corr}(X,Y)^+$. We observe that the product of correspondences (Definition 2.3)

$$\circ : \text{Corr}(X,Y) \times \text{Corr}(Y,Z) \to \text{Corr}(X,Z)$$

can be extended to the Grothendieck group $\text{Corr}(X,Y)^+$, i.e., we have the following bilinear product:

**Lemma 2.9.** For three varieties $X, Y, Z$ we have a bilinear map

$$\circ : \text{Corr}(X,Y)^+ \times \text{Corr}(Y,Z)^+ \to \text{Corr}(X,Z)^+$$

defined by

$$\left( \sum_i n_i [X \xleftarrow{f_i} M_i \xrightarrow{g_i} Y] \right) \circ \left( \sum_j m_j [Y \xleftarrow{h_j} N_j \xrightarrow{k_j} Z] \right) := \sum_{i,j} n_i m_j [(X \xleftarrow{f_i} M_i \xrightarrow{g_i} Y) \circ (Y \xleftarrow{h_j} N_j \xrightarrow{k_j} Z)].$$

**Proof.** (i) First we show that the product

$$[X \xleftarrow{f} M \xrightarrow{g} Y] \circ [Y \xleftarrow{h} N \xrightarrow{k} Z] := [(X \xleftarrow{f} M \xrightarrow{g} Y) \circ (Y \xleftarrow{h} N \xrightarrow{k} Z)]$$

is well-defined, i.e., it is independent of the choice of representatives. Namely, $(X \xleftarrow{f} M \xrightarrow{g} Y) \cong (X \xleftarrow{f'} M' \xrightarrow{g'} Y)$ and $(Y \xleftarrow{h} N \xrightarrow{k} Z) \cong (Y \xleftarrow{h'} N' \xrightarrow{k'} Z)$ imply that

$$(X \xleftarrow{f} M \xrightarrow{g} Y) \circ (Y \xleftarrow{h} N \xrightarrow{k} Z) \cong (X \xleftarrow{f'} M' \xrightarrow{g'} Y) \circ (Y \xleftarrow{h'} N' \xrightarrow{k'} Z).$$

This isomorphism follows from the universality of the fiber product and from the following commutative diagrams:
(ii) Next, in order to extend the product $\circ$ to the Grothendieck group, we need to show that with respect to each factor, the disjoint union is distributive, i.e., we show the following distributivity (below, $+$ indicates addition defined by the disjoint union as defined above):

\[
([X \leftarrow f_1 M_1 \xrightarrow{g_1} Y] + [X \leftarrow f_2 M_2 \xrightarrow{g_2} Y]) \circ [Y \leftarrow h N \xrightarrow{k} Z]
\]

\[= [X \leftarrow f_1 M_1 \xrightarrow{g_1} Y] \circ [Y \leftarrow h N \xrightarrow{k} Z] + [X \leftarrow f_2 M_2 \xrightarrow{g_2} Y] \circ [Y \leftarrow h N \xrightarrow{k} Z],
\]

\[= [X \leftarrow f M \xrightarrow{g} Y] \circ ([Y \leftarrow h_1 N_1 \xrightarrow{k_1} Z] + [Y \leftarrow h_2 N_2 \xrightarrow{k_2} Z])
\]

\[= [X \leftarrow f M \xrightarrow{g} Y] \circ [Y \leftarrow h_1 N_1 \xrightarrow{k_1} Z] + [X \leftarrow f M \xrightarrow{g} Y] \circ [Y \leftarrow h_2 N_2 \xrightarrow{k_2} Z].
\]

We show only the first one, since the argument for the other one is the same. Namely (see the diagrams below), it suffices to show the following:

\[
[X \leftarrow (f_1 \cup f_2) \circ \tilde{h} (M_1 \sqcup M_2) \times_Y N \xrightarrow{k \circ \tilde{g}_1 \sqcup \tilde{g}_2} Z]
\]

\[= [X \leftarrow f_1 \circ \tilde{h} M_1 \times_Y N \xrightarrow{k \circ \tilde{g}_1} Z] + [X \leftarrow f_2 \circ \tilde{h} M_2 \times_Y N \xrightarrow{k \circ \tilde{g}_2} Z]
\]

\[= [X \leftarrow f_1 \circ \tilde{h} \sqcup f_2 \circ \tilde{h} (M_1 \times_Y N) \sqcup (M_2 \times_Y N) \xrightarrow{k \circ (\tilde{g}_1 \sqcup \tilde{g}_2)} Z]
\]

\[= [X \leftarrow (f_1 \cup f_2) \circ (\tilde{h} \sqcup \tilde{h}) (M_1 \times_Y N) \sqcup (M_2 \times_Y N) \xrightarrow{k \circ (\tilde{g}_1 \sqcup \tilde{g}_2)} Z].
\]
By the universality of the fiber product it follows that \((M_1 \sqcup M_2) \times_Y N \cong (M_1 \times_Y N) \sqcup (M_2 \times_Y N)\), i.e., the fiber product commutes with the disjoint union \(\sqcup\), and \(\overline{h} = \overline{h} \sqcup \overline{h}\) and \(\overline{g_1} \sqcup \overline{g_2} = \overline{g_1} \sqcup \overline{g_2}\), i.e., the pullback commutes with \(\sqcup\). Hence

\[
[X \leftarrow (f_1 \sqcup f_2) \circ \overline{h} (M_1 \sqcup M_2) \times_Y N \twoheadrightarrow \overline{g_1} \sqcup \overline{g_2} Z] = [X \leftarrow (f_1 \sqcup f_2) \circ (\overline{h} \sqcup \overline{h}) (M_1 \times_Y N) \sqcup (M_2 \times_Y N) \twoheadrightarrow \overline{g_1} \sqcup \overline{g_2} Z].
\]

**Corollary 2.10.** The Abelian group \(\text{Corr}(X, X)^+\) becomes a ring with zero \([X \leftarrow \emptyset \rightarrow X]\) and unit \([X \leftarrow \text{id}_X X \rightarrow \text{id}_X X]\) with the above product.

**Remark 2.11.** The ring \(\text{Corr}(X, X)^+\) is called the Hecke ring of \(X\) or the ring of correspondences of \(X\) in Michio Kuga’s article [19, §2] (cf. [18, Appendix, §4]).

**Definition 2.12.** The following category \(\mathcal{C}orr(\mathcal{V})^+\) is called the \(\mathcal{A}b\)-enriched (or preadditive) category of correspondences:

- \(\text{Obj}(\mathcal{C}orr(\mathcal{V})^+) = \text{Obj}(\mathcal{V})\),
- for objects \(X\) and \(Y\), \(\text{hom}_{\mathcal{C}orr(\mathcal{V})^+}(X, Y) = \text{Corr}(X, Y)^+\).

Here is an enriched category version of Lemma 2.5.

**Lemma 2.13.** Let \(E : \mathcal{V} \rightarrow \mathcal{A}b\) be a bifunctor such that

(i) it satisfies the base change formula for fiber products, i.e., for a fiber square (left) the right square commutes:

\[
\begin{array}{ccc}
A' & \xrightarrow{\tilde{g}} & A \\
\downarrow{\tilde{h}} & & \downarrow{h} \\
B' & \xrightarrow{g} & B
\end{array}
\quad \implies \quad
\begin{array}{ccc}
E(A') & \xleftarrow{E(\tilde{g})} & E(A) \\
\downarrow{(\tilde{h})_*} & & \downarrow{h_*} \\
E(B') & \xleftarrow{g^*} & E(B)
\end{array}
\]

(ii) it is additive with respect to the disjoint union \(\sqcup\) in the sense that

\[i^* \oplus (i')^* : E(X \sqcup X') \cong E(X) \oplus E(X')\]

where \(i : X \rightarrow X \sqcup X'\) and \(i' : X \rightarrow X \sqcup X'\) are inclusions, and

(iii) it is functorial with respect to pushforward \(f_*\) and pullback \(g^*\).

Define \(\mathcal{E} : \mathcal{C}orr(\mathcal{V})^+ \rightarrow \mathcal{A}b\) by

1. for \(X \in \text{Obj}(\mathcal{C}orr(\mathcal{V})^+) = \text{Obj}(\mathcal{V})\), \(\mathcal{E}(X) := E(X)\),
2. for \(\sum_i n_i[X \xleftarrow{f_i} M_i \xrightarrow{g_i} Y] \in \text{hom}_{\mathcal{C}orr(\mathcal{V})^+}(X, Y) = \text{Corr}(X, Y)^+\),
3. \(\mathcal{E}(\sum_i n_i[X \xleftarrow{f_i} M_i \xrightarrow{g_i} Y]) := \sum_i n_i(f_i)_*(g_i)^* : \mathcal{E}(Y) \rightarrow \mathcal{E}(X)\).
Then $\mathcal{E} : \text{Corr}(\mathcal{V})^+ \to \mathcal{A}b$ is a functor in the sense that $\mathcal{E}(\alpha \circ \beta) = \mathcal{E}(\alpha) \circ \mathcal{E}(\beta)$.

Proof. Before starting the proof, we observe that the definition (2.14) can be interpreted as a group homomorphism

$$\mathcal{E} : \text{Corr}(X, Y)^+ \to \text{Hom}(E(Y), E(X)),$$

where $\text{Hom}(E(Y), E(X))$ is the Abelian group of all homomorphisms from $E(Y)$ to $E(X)$. We also observe that for the inclusions $i_k : M_k \to M_1 \sqcup M_2$ ($k = 1, 2$), we have

$$(i_k)^* (i_k)_* = \text{id}_{E(M_k)} \quad (k = 1, 2), \quad (i_2)^*(i_1)_* = (i_1)^*(i_2)_* = 0$$

by the following fiber squares:

\[
\begin{array}{ccc}
M_k & \xrightarrow{\text{id}_{M_k}} & M_k \\
\downarrow \quad \quad \quad \quad \downarrow & & \downarrow \\
M_k & \xrightarrow{i_k} & M_1 \sqcup M_2 \\
\end{array}
\quad \quad \quad
\begin{array}{ccc}
\emptyset & \rightarrow & M_1 \\
\downarrow & & \downarrow \\
\emptyset & \rightarrow & M_2 \\
\end{array}
\quad \quad \quad
\begin{array}{ccc}
M_2 & \xrightarrow{i_2} & M_1 \sqcup M_2 \\
\downarrow \quad \quad \quad \quad \downarrow & & \downarrow \\
M_1 & \xrightarrow{i_1} & M_1 \sqcup M_2 \\
\end{array}
\]

(i) First we show that $\mathcal{E}([X \xleftarrow{f} M \xrightarrow{g} Y]) := f_* g^*$ is well-defined, namely, it is independent of the choice of a representative, i.e., $(X \xleftarrow{f} M \xrightarrow{g} Y) \cong (X \xleftarrow{f'} M' \xrightarrow{g'} Y)$ implies $f_* g^* = (f')_* (g')^*$. Since there exists an isomorphism $h : M \cong M'$ such that the following diagram commutes:

\[
\begin{array}{ccc}
M & \xrightarrow{f} & M \\
\downarrow \quad \quad \quad \quad \quad \downarrow & & \downarrow \\
X & \xrightarrow{h} & Y \\
\end{array}
\quad \quad \quad
\begin{array}{ccc}
M & \xrightarrow{g} & M \\
\downarrow \quad \quad \quad \quad \quad \downarrow & & \downarrow \\
M' & \xrightarrow{g'} & Y \\
\end{array}
\]

we have

$$f_* g^* = (f' \circ h)_* (g' \circ h)^* = (f')_* (h_* \circ h^*)(g')^* = (f')_* (h_* \circ h^*)(g')^* = (f')_* (g')^* \quad \text{(by the isomorphism condition $h_* h^* = \text{id}_{E(M')}$.)}$$

(ii) Next we show that (2.14) is well-defined on $\text{Corr}(X, Y)^+$. For that it suffices to show that the map

$$\mathcal{E}_0 : \text{Corr}(X, Y) \to \text{Hom}(E(Y), E(X))$$

defined by $\mathcal{E}_0([X \xleftarrow{f} M \xrightarrow{g} Y]) := f_* g^*$ is a homomorphism of monoids, i.e.,

$$\mathcal{E}_0([X \xleftarrow{f_1} M_1 \xrightarrow{g_1} Y] + [X \xleftarrow{f_2} M_2 \xrightarrow{g_2} Y]) = \mathcal{E}_0([X \xleftarrow{f_1} M_1 \xrightarrow{g_1} Y]) + \mathcal{E}_0([X \xleftarrow{f_2} M_2 \xrightarrow{g_2} Y]),$$
namely
\[ \mathcal{E}_0([X \xleftarrow{f_1 \sqcup f_2} M_1 \sqcup M_2 \xrightarrow{g_1 \sqcup g_2} Y]) = \mathcal{E}_0([X \xleftarrow{f_1} M_1 \rightarrow g_1 Y]) + \mathcal{E}_0([X \xleftarrow{f_2} M_2 \rightarrow g_2 Y]), \]
i.e.,
\[ (f_1 \sqcup f_2)_*(g_1 \sqcup g_2)^* = (f_1)^*(g_1)^* + (f_2)^*(g_2)^* \]
—because, as is well-known, it follows from the universality of the Grothen-dieck group that
\[ \mathcal{E} : \text{Corr}(X,Y)^+ \rightarrow \text{Hom}(E(Y), E(X)) \]
is a unique extension of the above monoid homomorphism \( \mathcal{E}_0 : \text{Corr}(X,Y) \rightarrow \text{Hom}(E(Y), E(X)) \):
\[ \begin{array}{ccc}
\text{Corr}(X,Y) & \xrightarrow{\mathcal{E}_0} & \text{Hom}(E(Y), E(X)) \\
\gamma & & \text{Corr}(X,Y)^+
\end{array} \]
where \( \gamma : \text{Corr}(X,Y) \rightarrow \text{Corr}(X,Y)^+ \) maps an element \( x \) to the equivalence class \([x]\).

Here is a proof of (2.15). Consider the commutative diagram
\[ \begin{array}{ccc}
E(X) & \xrightarrow{(f_1 \sqcup f_2)_*} & E(M_1 \sqcup M_2) \\
\downarrow{(f_1)_*pr_1 + (f_2)_*pr_2} & \approx & (i_1)^* \oplus (i_2)^* \\
E(M_1 \oplus E(M_2)) & \downarrow{(g_1)^* \oplus (g_2)^*} & E(Y)
\end{array} \]
Note that
\[ (i_1)^* \oplus (i_2)^* \text{ is } (i_1)_*pr_1 + (i_2)_*pr_2, \]
which implies that
\[ (f_1)^*(g_1)^* + (f_2)^*(g_2)^* \]
\[ = ((f_1)_*pr_1 + (f_2)_*pr_2) \circ ((g_1)^* \oplus (g_2)^*) \]
\[ = ((f_1 \sqcup f_2)_* \circ ((i_1)_*pr_1 + (i_2)_*pr_2)) \circ ((i_1)^* \oplus (i_2)^*) \circ (g_1 \sqcup g_2)_* \]
\[ = (f_1 \sqcup f_2)_* \circ ((i_1)_*pr_1 + (i_2)_*pr_2) \circ ((i_1)^* \oplus (i_2)^*)) \circ (g_1 \sqcup g_2)_* \]
\[ = (f_1 \sqcup f_2)_* \circ (g_1 \sqcup g_2)_* \]
The proof of (2.16) is in two steps:
\begin{itemize}
  \item \(((i_1)^* \oplus (i_2)^*) \circ ((i_1)_*pr_1 + (i_2)_*pr_2)) = \text{id}_{E(M_1) \oplus E(M_2)}.
  
  Indeed, for \((x, y) \in E(M_1) \oplus E(M_2)\),
  \[
  ((i_1)^* \oplus (i_2)^*) \circ ((i_1)_*pr_1 + (i_2)_*pr_2))(x, y) \\
  = ((i_1)^* \oplus (i_2)^*) \circ ((i_1)_*x + (i_2)_*y)) \\
  = ((i_1)^*(i_1)_*x, (i_2)^*(i_2)_*y) \\
  = (x, y) \quad (\text{since } (i_1)^*(i_1)_* = \text{id}_{E(M_1)} \text{ and } (i_2)^*(i_2)_* = \text{id}_{E(M_2)}).
  
  \item \(((i_1)_*pr_1 + (i_2)_*pr_2)) \circ ((i_1)^* \oplus (i_2)^*) = \text{id}_{E(M_1 \sqcup M_2)}.
  
  Indeed, it is clear that
  \[
  ((i_1)_*pr_1 + (i_2)_*pr_2)) \circ ((i_1)^* \oplus (i_2)^*) = (i_1)_*(i_1)^* + (i_2)_*(i_2)^*.
  
  Now \((i_1)_*(i_1)^* + (i_2)_*(i_2)^* = \text{id}_{E(M_1 \sqcup M_2)}\), which can be seen as follows: We can write any element \(a \in E(M_1 \sqcup M_2)\) as \(a = (i_1)_*x + (i_2)_*y\) where \((x, y) \in E(M_1) \oplus E(M_2)\). Then
  \[
  ((i_1)_*(i_1)^* + (i_2)_*(i_2)^*)(a) \\
  = ((i_1)_*(i_1)^* + (i_2)_*(i_2)^*)(a) \\
  = ((i_1)_*(i_1)^* + (i_2)_*(i_2)^*)((i_1)_*x + (i_2)_*y) \\
  = (i_1)_*(i_1)^*((i_1)_*x + (i_2)_*(i_2)^*)x + (i_2)_*(i_2)^*y + (i_1)_*(i_1)^*y) \\
  = (i_1)_*x + (i_2)_*y \\
  \quad (\text{since } (i_1)^*(i_1)_* = \text{id}, (i_2)^*(i_1)_* = (i_1)^*(i_2)_* = 0, (i_2)^*(i_2)_* = \text{id}) \\
  = a. \]
\end{itemize}

Suppose that we have two bifunctors \(E_1, E_2 : \mathcal{V} \to \mathcal{A}b\) satisfying the base change formula for fiber products, and a natural transformation \(\tau : E_1 \to E_2\). Then, in the same way as in the previous section, we get two functors
\[
\mathcal{E}_1, \mathcal{E}_2 : \text{Corr}(\mathcal{V})^+ \to \mathcal{A}b
\]
and a natural transformation \(\mathcal{T} : \mathcal{E}_1 \to \mathcal{E}_2\).

3. Enriched categories of proper-smooth correspondences and characteristic classes of singular varieties. In this section we consider proper-smooth correspondences \(X \xleftarrow{f} M \xrightarrow{g} Y\), i.e., \(f : M \to X\) is a proper morphism and \(g : M \to Y\) is a smooth morphism. The set of all such correspondences from \(X\) to \(Y\) is denoted by \(\text{Corr}(X, Y)_{\text{pro-sm}}\). As explained in the introduction, the following product is well-defined:
\[
\circ : \text{Corr}(X, Y)_{\text{pro-sm}} \times \text{Corr}(Y, Z)_{\text{pro-sm}} \to \text{Corr}(X, Z)_{\text{pro-sm}},
\]
\[
(X \xleftarrow{f} M \xrightarrow{g} Y) \circ (Y \xleftarrow{h} N \xrightarrow{k} Z) := X \xleftarrow{f \circ h} M \times_Y N \xrightarrow{k \circ g} Z,
\]
where we use the diagram \([1.1]\)
Definition 3.1. Let \( \text{Corr}(\mathcal{V})_{\text{pro-sm}} \) be the category of proper-smooth correspondences:

- \( \text{Obj}(\text{Corr}(\mathcal{V})_{\text{pro-sm}}) = \text{Obj}(\mathcal{V}) \),
- for objects \( X, Y \), \( \text{hom}_{\text{Corr}(\mathcal{V})_{\text{pro-sm}}}(X, Y) = \text{Corr}(X, Y)_{\text{pro-sm}} \).

The composition

\[
\circ : \text{hom}_{\text{Corr}(\mathcal{V})_{\text{pro-sm}}}(X, Y) \times \text{hom}_{\text{Corr}(\mathcal{V})_{\text{pro-sm}}}(Y, Z) \to \text{hom}_{\text{Corr}(\mathcal{V})_{\text{pro-sm}}}(X, Z)
\]

is

\[
\circ : \text{Corr}(X, Y)_{\text{pro-sm}} \times \text{Corr}(Y, Z)_{\text{pro-sm}} \to \text{Corr}(X, Z)_{\text{pro-sm}}.
\]

The set of isomorphism classes of proper-smooth correspondences becomes an Abelian monoid with respect to the disjoint union, and its group completion is denoted by \( \text{Corr}(X, Y)^+_{\text{pro-sm}} \). Then the above composition (or product) \( \circ : \text{Corr}(X, Y)_{\text{pro-sm}} \times \text{Corr}(Y, Z)_{\text{pro-sm}} \to \text{Corr}(X, Z)_{\text{pro-sm}} \) extends to

\[
\circ : \text{Corr}(X, Y)^+_{\text{pro-sm}} \times \text{Corr}(Y, Z)^+_{\text{pro-sm}} \to \text{Corr}(X, Z)^+_{\text{pro-sm}}.
\]

Remark 3.2. \( \text{Corr}(X, pt)^+_{\text{pro-sm}} \) is the same as \( \mathcal{M}^+(X) \) defined in \([21, 16]\).

Using this we can define the following \( \mathcal{A} \)-enriched (or preadditive) category:

Definition 3.3.

- \( \text{Obj}(\text{Corr}(\mathcal{V})^+_{\text{pro-sm}}) = \text{Obj}(\mathcal{V}) \),
- for objects \( X, Y \), \( \text{hom}_{\text{Corr}(\mathcal{V})^+_{\text{pro-sm}}}(X, Y) := \text{Corr}(X, Y)^+_{\text{pro-sm}} \).

Baum–Fulton–MacPherson’s Todd class (or the singular Riemann–Roch transformation) \([5]\) is a unique natural transformation \( \text{td}^{\text{BFM}}_*: G_0(-) \to H_*(-) \otimes \mathbb{Q} \) from the covariant functor \( G_0(-) \) of Grothendieck groups of coherent sheaves to the covariant Borel–Moore homology theory \( H_*(-) \otimes \mathbb{Q} \) with rational coefficients such that for a smooth variety \( X \) we have \( \text{td}^{\text{BFM}}_*(\mathcal{O}_X) = \text{td}(TX) \cap [X] \), the Poincaré dual of the total Todd class \( \text{td}(TX) \) of the tangent bundle \( TX \), where \( \mathcal{O}_X \) is the structure sheaf of \( X \).

For a proper map \( f : M \to X \) we have the commutative diagram

\[
\begin{array}{ccc}
G_0(X) & \xleftarrow{f_*} & G_0(M) \\
\downarrow{\text{td}^{\text{BFM}}_*} & & \downarrow{\text{td}^{\text{BFM}}_*} \\
H_*(X) \otimes \mathbb{Q} & \xleftarrow{f_*} & H_*(M) \otimes \mathbb{Q}
\end{array}
\]

For a smooth morphism \( g : M \to Y \) (see Remark 3.13 below), we have the Verdier–Riemann–Roch formula \([29]\) (see \([5\ Conjecture, p. 137]\ and \([12\ Theorem 18.2]\)), i.e., the commutative diagram
Here $td(T_g)$ is the total Todd class of the relative tangent bundle $T_g$ of the smooth morphism $g$. Combining (3.4) and (3.5), for a correspondence $(X \xleftarrow{f} M \xrightarrow{g} Y)$ with $f$ proper and $g$ smooth, we have the commutative diagram

$$
\begin{array}{ccc}
G_0(M) & \xrightarrow{g^*} & G_0(Y) \\
\downarrow td_*^{BFM} & & \downarrow td_*^{BFM} \\
H_*(M) \otimes \mathbb{Q} & \xleftarrow{td(T_g) \cap g^*} & H_*(Y) \otimes \mathbb{Q}
\end{array}
$$

PROPOSITION 3.7. Define $G_0, H_*^{Todd} : \text{Corr}(\mathcal{V})_{pro-sm} \to \mathcal{A}b$ as follows:

- for $X \in \text{Obj}(\text{Corr}(\mathcal{V})_{pro-sm}) = \text{Obj}(\mathcal{V})$, $G_0(X) := G_0(X)$, $H_*^{Todd}(X) := H_*(X) \otimes \mathbb{Q}$,
- for $X \xleftarrow{f} M \xrightarrow{g} Y \in \text{hom}_{\text{Corr}(\mathcal{V})_{pro-sm}}(X,Y) = \text{Corr}(X,Y)_{pro-sm}$, $G_0(X \xleftarrow{f} M \xrightarrow{g} Y) := f_*g^* : G_0(Y) \to G_0(X)$, $H_*^{Todd}(X \xleftarrow{f} M \xrightarrow{g} Y) := f_*(td(T_g) \cap g^*) : H_*^{Todd}(Y) \to H_*^{Todd}(X)$.

Then $G_0, H_*^{Todd} : \text{Corr}(\mathcal{V})_{pro-sm} \to \mathcal{A}b$ are functors in the sense that $G_0(\alpha \circ \beta) = G_0(\alpha) \circ G_0(\beta)$ and $H_*^{Todd}(\alpha \circ \beta) = H_*^{Todd}(\alpha) \circ H_*^{Todd}(\beta)$, and Baum–Fulton–MacPherson’s Todd class transformation $td_*^{BFM} : G_0(-) \to H_*(-) \otimes \mathbb{Q}$ extends to a natural transformation $td_*^{BFM} : G_0(-) \to H_*^{Todd}(-)$.

Proof. Since the naturality of $td_*^{BFM} : G_0(-) \to H_*^{Todd}(-)$ follows from (3.6), we only have to show the covariance of the functors $G_0$ and $H_*^{Todd}$.

1) Functoriality of $G_0(-)$: It is well-known (e.g., see [28 Lemma 29.5.2]) that the covariant functor $G_0(-)$ of Grothendieck groups of coherent sheaves satisfies the base change formula. Hence, as in the proof of Lemma 2.5, we see that $G_0 : \text{Corr}(\mathcal{V})_{pro-sm} \to \mathcal{A}b$ is a functor in the sense that $G_0(\alpha \circ \beta) = G_0(\alpha) \circ G_0(\beta)$.

2) Functoriality of $H_*^{Todd}(-)$: First we remark that the Borel–Moore homology theory (\(^4\)) also satisfies the base change formula (cf. [8]).
For $\alpha = (X \xleftarrow{f} M \xrightarrow{g} Y)$ and $\beta = (Y \xleftarrow{h} N \xrightarrow{k} Z)$, we have
\[ \alpha \circ \beta = X \xleftarrow{f \circ h} M \times_Y N \xrightarrow{k \circ g} Z \]
(see Definition 1.1). Hence
\[
\mathcal{H}^{\text{Todd}}_*(\alpha \circ \beta) = \mathcal{H}^{\text{Todd}}_*(X \xleftarrow{f \circ h} M \times_Y N \xrightarrow{k \circ g} Z)
\]
\[ = (f \circ \tilde{h})_* (T_{k \circ g})(k \circ \tilde{g})^* \]
\[ = f_* (\tilde{h})_* ((T_{\tilde{g}} \cup T((\tilde{g})^* T_k)) \cap (\tilde{g})^* k^*) \]
(since $T$ is multiplicative, thus $T(T_{k \circ g}) = T(T_{\tilde{g}} \cup T((\tilde{g})^* T_k))$)
\[ = f_* (\tilde{h})_* (T_{\tilde{g}}) \cap (\tilde{g})^* T_k \cap (\tilde{g})^* k^*) \]
\[ = f_* (h)_* (T_{T_{\tilde{g}}}) \cap (\tilde{g})^* (T_k) \cap (\tilde{g})^* k^*) \]
(by $[12]$ Theorem 3.2(d)(Pull-back))
\[ = f_* (\tilde{h})_*(T_{\tilde{g}}) \cap (\tilde{g})^* (T_k) \cap (\tilde{g})^* k^*) \]
(since $T_{\tilde{g}} = (\tilde{h})^* T_g$)
\[ = f_* (\tilde{h})_* ((\tilde{h})^* T_{\tilde{g}}) \cap (\tilde{g})^* (T_k) \cap (\tilde{g})^* k^*) \]
(by the projection formula)
\[ = f_* (T_{\tilde{g}}) \cap (\tilde{h})_* (\tilde{g})^* (T_k) \cap (\tilde{g})^* k^*) \]
(by the base change formula $(\tilde{h})_* (\tilde{g})^* = g^* h_*$)
\[ = f_* (T_{\tilde{g}}) \cap g^* (h_* (T_k) \cap (\tilde{g})^* k^*)) \]
\[ = (f_* (T_{\tilde{g}}) \cap g^*) (h_* (T_k) \cap k^*)) \]
\[ = \mathcal{H}^{\text{Todd}}_* (\alpha) \circ \mathcal{H}^{\text{Todd}}_* (\beta). \blacklozenge \]

Remark 3.8. The proof above only uses the fact that the Todd class $c_\ell = T$ is a cohomological characteristic class of isomorphism classes of complex algebraic vector bundles, which is functorial for pullbacks $g^*$ and multiplicative, i.e., $c_\ell(F) = c_\ell(F') \cup c_\ell(F'')$ for a short exact sequence $0 \to F' \to F \to F'' \to 0$ of complex algebraic vector bundles.\(^{(5)}\) In the proof above we have a short exact sequence $0 \to T_{\tilde{g}} \to T_{k \circ g} \to (\tilde{g})^* T_k \to 0$.

Theorem 3.9. Define $\mathcal{G}_0, \mathcal{H}^{\text{Todd}}_* : \mathcal{C}^*_\text{pro-sm} \to \mathcal{A}b$ as follows:

- for $X \in \text{Obj}(\mathcal{C}^*_\text{pro-sm})$,
  \[
  \mathcal{G}_0(X) := G_0(X), \quad \mathcal{H}^{\text{Todd}}_*(X) := H_*(X) \otimes \mathbb{Q},
  \]

\(^{(5)}\) Note that if we consider this short exact sequence $0 \to F' \to F \to F'' \to 0$ as one of topological complex vector bundles, forgetting the algebraic structure, then it splits, i.e., $F \cong F' \oplus F''$. 

• for $\sum_i n_i [X \xleftarrow{f_i} M_i \xrightarrow{g_i} Y] \in \text{hom}_{\text{Corr}_{\text{pro-sm}}}(X, Y) := \text{Corr}(X, Y)_{\text{pro-sm}}$, 

$$\mathcal{G}_0(\sum_i n_i [X \xleftarrow{f_i} M_i \xrightarrow{g_i} Y]) := \sum_i n_i (f_i)_*(g_i)^*: \mathcal{G}_0(Y) \to \mathcal{G}_0(X),$$

$$\mathcal{H}_*^{\text{Todd}}(\sum_i n_i [X \xleftarrow{f_i} M_i \xrightarrow{g_i} Y]) := \sum_i n_i (f_i)_*(\text{td}(g_i) \cap (g_i)^*): \mathcal{H}_*^{\text{Todd}}(Y) \to \mathcal{H}_*^{\text{Todd}}(X).$$

Then $\mathcal{G}_0, \mathcal{H}_*^{\text{Todd}}: \text{Corr}_{\text{pro-sm}}^+ \to \mathcal{A}b$ are functors in the sense that $\mathcal{G}_0(\alpha \circ \beta) = \mathcal{G}_0(\alpha) \circ \mathcal{G}_0(\beta)$ and $\mathcal{H}_*^{\text{Todd}}(\alpha \circ \beta) = \mathcal{H}_*^{\text{Todd}}(\alpha) \circ \mathcal{H}_*^{\text{Todd}}(\beta)$, and Baum–Fulton–MacPherson’s Todd class transformation $\text{td}_{\text{BFM}}^*: \text{G}_0(-) \to \text{H}_*(-) \otimes \mathbb{Q}$ extends to a natural transformation

$$\text{td}_{\text{BFM}}^*: \mathcal{G}_0(-) \to \mathcal{H}_*^{\text{Todd}}(-).$$

**Proof.** We note that $\text{G}_0(-)$ and $\text{H}_*(-) \otimes \mathbb{Q}$ are additive bifunctors with respect to pushforward $f_*$ for proper morphisms $f$ and pullbacks $g^*$ for smooth morphisms $g$. Thus it suffices to show the following equalities:

$$\mathcal{G}_0([X \xleftarrow{f_1 \sqcup f_2} M_1 \sqcup M_2 \xrightarrow{g_1 \sqcup g_2} Y]) = \mathcal{G}_0([X \xleftarrow{f_1} M_1 \xrightarrow{g_1} Y]) + \mathcal{G}_0([X \xleftarrow{f_2} M_2 \xrightarrow{g_2} Y]),$$

$$\mathcal{H}_*^{\text{Todd}}([X \xleftarrow{f_1 \sqcup f_2} M_1 \sqcup M_2 \xrightarrow{g_1 \sqcup g_2} Y]) = \mathcal{H}_*^{\text{Todd}}([X \xleftarrow{f_1} M_1 \xrightarrow{g_1} Y]) + \mathcal{H}_*^{\text{Todd}}([X \xleftarrow{f_2} M_2 \xrightarrow{g_2} Y]),$$

that is,

(3.10) \( (f_1 \sqcup f_2)_*(g_1 \sqcup g_2)^* = (f_1)_*(g_1)^* + (f_2)_*(g_2)^* \),

(3.11) \( f_1 \sqcup f_2)_*(\text{td}(T_{g_1 \sqcup g_2}) \cap (g_1 \sqcup g_2)^*) = (f_1)_*(\text{td}(T_{g_1}) \cap (g_1)^*) + (f_2)_*(\text{td}(T_{g_2}) \cap (g_2)^*). \)

The proof is the same as that of (2.15). Here we only show (3.11):

$$ (f_1)_*(\text{td}(T_{g_1}) \cap (g_1)^*) + (f_2)_*(\text{td}(T_{g_2}) \cap (g_2)^*) = ((f_1)_* \circ pr_1 + (f_2)_* \circ pr_2) \circ (\text{td}(T_{g_1}) \cap (g_1)^*) + (f_2)_*(\text{td}(T_{g_2}) \cap (g_2)^*) \circ (\text{td}(T_{g_1 \sqcup g_2}) \cap (g_1 \sqcup g_2)^*) \circ ((i_1)_* \oplus (i_2)_*) \circ (\text{td}(T_{g_1 \sqcup g_2}) \cap (g_1 \sqcup g_2)^*)) \circ (\text{td}(T_{g_1 \sqcup g_2}) \cap (g_1 \sqcup g_2)^*)) = (f_1 \sqcup f_2)_* \circ (((i_1)_* \circ pr_1 + (i_2)_* \circ pr_2) \circ ((i_1)_* \oplus (i_2)_*)) \circ (\text{td}(T_{g_1 \sqcup g_2}) \cap (g_1 \sqcup g_2)^*)) \circ ((i_1)_* \oplus (i_2)_*) \circ (\text{td}(T_{g_1 \sqcup g_2}) \cap (g_1 \sqcup g_2)^*)) = (f_1 \sqcup f_2)_*(\text{td}(T_{g_1 \sqcup g_2}) \cap (g_1 \sqcup g_2)^*). \)
Remark 3.12. We define the following subcategories of $\text{Corr}(\mathcal{V})_{\text{pro-sm}}$ and $\text{Corr}(\mathcal{V})_{\text{pro-sm}}^+$:

\[
\text{Corr}(\mathcal{V})_{\text{pro-id}}(X,Y) := \{ X \xleftarrow{f} Y \xrightarrow{\text{id}_Y} Y \mid f \in \text{hom}_\mathcal{V}(Y,X) \text{ proper}\},
\]

\[
\text{Corr}(\mathcal{V})_{\text{id-sm}}(X,Y) := \{ X \xleftarrow{\text{id}_X} X \xrightarrow{g} Y \mid g \in \text{hom}_\mathcal{V}(X,Y) \text{ smooth}\},
\]

\[
\text{Corr}(\mathcal{V})_{\text{pro-id}}^+(X,Y) := \{ [X \xleftarrow{f} Y \xrightarrow{\text{id}_Y} Y] \mid f \in \text{hom}_\mathcal{V}(Y,X) \text{ proper}\},
\]

\[
\text{Corr}(\mathcal{V})_{\text{id-sm}}^+(X,Y) := \{ [X \xleftarrow{\text{id}_X} X \xrightarrow{g} Y] \mid g \in \text{hom}_\mathcal{V}(X,Y) \text{ smooth}\}.
\]

Note that

\[
\text{Corr}(\mathcal{V})_{\text{pro-id}}^+(X,Y) = \text{Corr}(\mathcal{V})_{\text{id-sm}}^+(X,Y) = \{ [X \xleftarrow{f} Y] \mid f \in \text{hom}_\mathcal{V}(Y,X) \text{ proper}\}.
\]

Then $\mathcal{G}_0, \mathcal{H}_*^{\text{Todd}} : \text{Corr}(\mathcal{V})_{\text{pro-id}}^+ \rightarrow \mathcal{A}\mathcal{B}$ are covariant functors (in the usual sense) for proper morphisms and $td_*^{\text{BFM}} : \mathcal{G}_0(-) \rightarrow \mathcal{H}_*^{\text{Todd}}(-)$ is Baum–Fulton–MacPherson’s Todd class transformation. On the other hand, $\mathcal{G}_0, \mathcal{H}_*^{\text{Todd}} : \text{Corr}(\mathcal{V})_{\text{id-sm}}^+ \rightarrow \mathcal{A}\mathcal{B}$ are contravariant functors for smooth morphisms and $td_*^{\text{BFM}} : \mathcal{G}_0(-) \rightarrow \mathcal{H}_*^{\text{Todd}}(-)$ yields the Verdier–Riemann–Roch formula for Baum–Fulton–MacPherson’s Todd class transformation.

Remark 3.13. The Verdier–Riemann–Roch formula (3.5) holds for any \(\ell.c.i\.) morphism $g : M \rightarrow Y$ instead of a smooth morphism $g : M \rightarrow Y$. The main reason why we restrict ourselves to smooth morphisms $g : M \rightarrow Y$, i.e., considering proper-smooth correspondences instead of proper-\(\ell.c.i\.) correspondences $X \xleftarrow{f} M \xrightarrow{g} Y$ with a proper morphism $f$ and an \(\ell.c.i\.) morphism $g$, is that a local complete intersection morphism is not necessarily stable under a base change, i.e., in taking the product (see the diagram (1.1)) the pullbacked morphism $\tilde{g} : M \times_Y N \rightarrow N$ is not necessarily an \(\ell.c.i\.) morphism even if $g : M \rightarrow Y$ is. Thus the composite $k \circ \tilde{g} : M \times_Y N \rightarrow Z$ is not necessarily an \(\ell.c.i\.) morphism even if $g : M \rightarrow Y$ and $k : N \rightarrow Z$ are \(\ell.c.i\.). We can remedy this also by considering zigzags instead of correspondences, which we will discuss later.

Remark 3.14. Given any two classes of morphisms, we can consider correspondences similar to proper-smooth correspondences. Let $\mathcal{M}_1, \mathcal{M}_2$ be classes of morphisms that contain all identity morphisms and are stable by base change and closed under composition, i.e.,
• if $f : X \to Y$ is in the class $\mathcal{M}_i$, then for any fiber square

\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow{f'} & & \downarrow{f} \\
Y' & \xrightarrow{g} & Y
\end{array}
\]

$f'$ is in $\mathcal{M}_i$;

• if $f : X \to Y$ and $g : Y \to Z$ are in $\mathcal{M}_i$, then $g \circ f$ is also in $\mathcal{M}_i$.

Then a correspondence $X \xleftarrow{f} V \xrightarrow{g} Y$ with $f \in \mathcal{M}_1$ and $g \in \mathcal{M}_2$ will be called an $(\mathcal{M}_1, \mathcal{M}_2)$-correspondence from $X$ to $Y$. In a similar manner we can get the category $\text{Corr}(\mathcal{V})(\mathcal{M}_1, \mathcal{M}_2)$ and the $\mathcal{A}b$-enriched category $\text{Corr}(\mathcal{V})^+(\mathcal{M}_1, \mathcal{M}_2)$ of $(\mathcal{M}_1, \mathcal{M}_2)$-correspondences. If a functor $E : \mathcal{V} \to \mathcal{A}b$ is

• covariant for morphisms in $\mathcal{M}_1$,
• contravariant for morphisms in $\mathcal{M}_2$,

then it will be called a partial bifunctor with respect to $(\mathcal{M}_1, \mathcal{M}_2)$. Suppose that a partial bifunctor $E : \mathcal{V} \to \mathcal{A}b$ with respect to $(\mathcal{M}_1, \mathcal{M}_2)$ satisfies (the base change formula)

\[
\begin{array}{ccc}
A' & \xrightarrow{\tilde{g}} & A \\
\downarrow{\tilde{h}} & & \downarrow{h} \\
B' & \xrightarrow{g} & B
\end{array}
\]

\[
\begin{array}{ccc}
E(A') & \xleftarrow{(\tilde{g})^*} & E(A) \\
\downarrow{h_*} & & \downarrow{h_*} \\
E(B') & \xleftarrow{g^*} & E(B)
\end{array}
\]

where $h \in \mathcal{M}_1$ and $g \in \mathcal{M}_2$. Then $E$ will be called nice. Let $E_1, E_2 : \mathcal{V} \to \mathcal{A}b$ be two nice partial bifunctors with respect to $(\mathcal{M}_1, \mathcal{M}_2)$ and let $\tau : E_1 \to E_2$ be a natural transformation. Then in the same way as above we get the following:

Define $\mathcal{E}_1, \mathcal{E}_2 : \text{Corr}(\mathcal{V})(\mathcal{M}_1, \mathcal{M}_2) \to \mathcal{A}b$ as follows:

• for $X \in \text{Obj}(\text{Corr}(\mathcal{V})(\mathcal{M}_1, \mathcal{M}_2)) = \text{Obj}(\mathcal{V})$,

\[
\mathcal{E}_1(X) := E_1(X), \quad \mathcal{E}_2(X) := E_2(X),
\]

• for $X \xleftarrow{f} M \xrightarrow{g} Y \in \text{hom}_{\text{Corr}(\mathcal{V})(\mathcal{M}_1, \mathcal{M}_2)}(X, Y) = \text{Corr}(X, Y)(\mathcal{M}_1, \mathcal{M}_2)$,

\[
\begin{array}{c}
\mathcal{E}_1(X \xleftarrow{f} M \xrightarrow{g} Y) := f_*g^* : \mathcal{E}_1(Y) \to \mathcal{E}_1(X), \\
\mathcal{E}_2(X \xleftarrow{f} M \xrightarrow{g} Y) := f_*g^* : \mathcal{E}_2(Y) \to \mathcal{E}_2(X).
\end{array}
\]

Then $\mathcal{E}_1, \mathcal{E}_2 : \text{Corr}(\mathcal{V})(\mathcal{M}_1, \mathcal{M}_2) \to \mathcal{A}b$ are functors and the natural transformation $\tau : E_1 \to E_2$ extends to a natural transformation $\tau : \mathcal{E}_1(-) \to \mathcal{E}_2(-)$.

If $E_i$ is additive, then similarly we define $\mathcal{E}_1, \mathcal{E}_2 : \text{Corr}(\mathcal{V})^+(\mathcal{M}_1, \mathcal{M}_2) \to \mathcal{A}b$ as follows:
• for $X \in \text{Obj}(\mathcal{C}orr(\mathcal{V})^+_{(\mathcal{A}_1, \mathcal{A}_2)})$,
  $\mathcal{E}_1(X) := E_1(X), \quad \mathcal{E}_2(X) := E_2(X),$

• for $\sum_i n_i [X \xleftarrow{f_i} M_i \xrightarrow{g_i} Y] \in \text{hom}_ {\mathcal{C}orr(\mathcal{V})^+_{(\mathcal{A}_1, \mathcal{A}_2)}}(X, Y) := \text{Corr}(X, Y)^+_{(\mathcal{A}_1, \mathcal{A}_2)}$, 
  $\mathcal{E}_1 \left( \sum_i n_i [X \xleftarrow{f_i} M_i \xrightarrow{g_i} Y] \right) := \sum_i n_i (f_i)_* (g_i)^* : \mathcal{E}_1(Y) \to \mathcal{E}_1(X)$,
  $\mathcal{E}_2 \left( \sum_i n_i [X \xleftarrow{f_i} M_i \xrightarrow{g_i} Y] \right) := \sum_i n_i (f_i)_* (g_i)^* : \mathcal{E}_2(Y) \to \mathcal{E}_2(X)$.

Then $\mathcal{E}_1, \mathcal{E}_2 : \mathcal{C}orr(\mathcal{V})^+_{(\mathcal{A}_1, \mathcal{A}_2)} \to \mathcal{A}b$ are functors and the natural transformation $\tau : E_1 \to E_2$ extends to a natural transformation $\tau : \mathcal{E}_1(-) \to \mathcal{E}_2(-)$.

Baum–Fulton–MacPherson’s Riemann–Roch transformation was motivated by MacPherson’s Chern class transformation [25], which is the unique natural transformation $c^\text{Mac}_* : F(-) \to H_*(-)$ from the covariant functor $F(-)$ of constructible functions to the covariant Borel–Moore homology theory $H_*(-)$ satisfying the “smoothness condition” that for a smooth variety $X$, $c^\text{Mac}_*(1_X) = c(TX) \cap [X]$, the Poincaré dual of the total Chern class $c(TX)$ of the tangent bundle $TX$ (see [12]). Here $1_X$ is the characteristic function on $X$.

For a proper morphism $f : M \to X$ we have the commutative diagram

$$
\begin{array}{ccc}
F(X) & \xleftarrow{f_*} & F(M) \\
\downarrow c^\text{Mac}_* & & \downarrow c^\text{Mac}_* \\
H_*(X) & \xleftarrow{f_*} & H_*(M)
\end{array}
$$

(3.15)

For a smooth morphism $g : M \to Y$ we have the following Verdier–Riemann–Roch formula for MacPherson’s Chern class transformation [32]:

$$
\begin{array}{ccc}
F(M) & \xleftarrow{g^*} & F(Y) \\
\downarrow c^\text{Mac}_* & & \downarrow c^\text{Mac}_* \\
H_*(M) & \xleftarrow{c(T_g) \cap g^*} & H_*(Y)
\end{array}
$$

(3.16)

Here $c(T_g)$ is the total Chern class of the relative tangent bundle $T_g$ of the smooth morphism $g$. The pullback $g^* : F(Y) \to F(M)$ is simply the pullback of functions, i.e., for a constructible function $\gamma : Y \to \mathbb{Z}$, $g^*(\gamma)$ is defined by $(g^*(\gamma))(m) := \gamma(g(m))$ for $m \in M$. Combining (3.15) and (3.16), for a

(6) If $g : M \to Y$ is a local complete intersection morphism, this Verdier–Riemann–Roch formula (3.16) does not hold in general, but there is some defect as proved by J. Schürmann [27].
correspondence \((X \leftarrow f \rightarrow M \rightarrow g \rightarrow Y)\) with proper \(f\) and smooth \(g\), we have the commutative diagram

\[
\begin{array}{c}
F(X) & \xleftarrow{f^*} & F(M) & \xleftarrow{g^*} & F(Y) \\
c_{*}^{Mac} & \downarrow & c_{*}^{Mac} & \downarrow & c_{*}^{Mac} \\
H_*(X) & \xleftarrow{f^*} & H_*(M) & \xleftarrow{c(T) \cap g^*} & H_*(Y)
\end{array}
\]  
(3.17)

As in the case of Baum–Fulton–MacPherson’s Riemann–Roch transformation \(td_{*}^{BFM}: G_0(X) \rightarrow H_*(X) \otimes \mathbb{Q}\), we obtain the following:

**Proposition 3.18.** Define \(\mathcal{F}, \mathcal{H}_{*}^{Chern} : Corr(\mathcal{V})_{pro-sm} \rightarrow \mathcal{A}b\) as follows:
- for \(X \in \text{Obj}(Corr(\mathcal{V})_{pro-sm}) = \text{Obj}(\mathcal{V})\),
  \[
  \mathcal{F}(X) := F(X), \quad \mathcal{H}_{*}^{Chern}(X) := H_*(X).
  \]
- for \(X \leftarrow f \rightarrow M \rightarrow g \rightarrow Y \in \text{hom}_{Corr(\mathcal{V})_{pro-sm}}(X,Y) = Corr(X,Y)_{pro-sm}\),
  \[
  \mathcal{F}(X \leftarrow f \rightarrow M \rightarrow g \rightarrow Y) := f^*g^* : F(Y) \rightarrow F(X),
  \]
  \[
  \mathcal{H}_{*}^{Chern}(X \leftarrow f \rightarrow M \rightarrow g \rightarrow Y) := f^*(c(T) \cap g^*) : \mathcal{H}_{*}^{Chern}(Y) \rightarrow \mathcal{H}_{*}^{Chern}(X).
  \]

Then \(\mathcal{F}, \mathcal{H}_{*}^{Chern} : Corr(\mathcal{V})_{pro-sm} \rightarrow \mathcal{A}b\) are functors in the sense that \(\mathcal{F}(\alpha \circ \beta) = \mathcal{F}(\alpha) \circ \mathcal{F}(\beta)\) and \(\mathcal{H}_{*}^{Chern}(\alpha \circ \beta) = \mathcal{H}_{*}^{Chern}(\alpha) \circ \mathcal{H}_{*}^{Chern}(\beta)\), and MacPherson’s Chern class transformation \(c_{*}^{Mac} : F(-) \rightarrow H_*(-)\) extends to a natural transformation

\[
c_{*}^{Mac} : \mathcal{F}(-) \rightarrow \mathcal{H}_{*}^{Chern}(-).
\]

**Theorem 3.19.** Define \(\mathcal{F}, \mathcal{H}_{*}^{Chern} : Corr(\mathcal{V})^+_{pro-sm} \rightarrow \mathcal{A}b\) as follows:
- for \(X \in \text{Obj}(Corr(\mathcal{V})^+_{pro-sm})\),
  \[
  \mathcal{F}(X) := F(X), \quad \mathcal{H}_{*}^{Chern}(X) := H_*(X),
  \]
- for \(\sum_i n_i[X \leftarrow f_i \rightarrow M_i \rightarrow g_i \rightarrow Y] \in \text{hom}_{Corr(\mathcal{V})^+_{pro-sm}}(X,Y) := Corr(X,Y)^+_{pro-sm}\),
  \[
  \mathcal{F} \left( \sum_i n_i[X \leftarrow f_i \rightarrow M_i \rightarrow g_i \rightarrow Y] \right) := \sum_i n_i(f_i)^*(g_i)^* : \mathcal{F}(Y) \rightarrow \mathcal{F}(X),
  \]
  \[
  \mathcal{H}_{*}^{Chern} \left( \sum_i n_i[X \leftarrow f_i \rightarrow M_i \rightarrow g_i \rightarrow Y] \right)
  := \sum_i n_i(f_i)^*(c(T) \cap (g_i)^*) : \mathcal{H}_{*}^{Chern}(Y) \rightarrow \mathcal{H}_{*}^{Chern}(X).
  \]

Then \(\mathcal{F}, \mathcal{H}_{*}^{Chern} : Corr(\mathcal{V})^+_{pro-sm} \rightarrow \mathcal{A}b\) are functors in the sense that \(\mathcal{F}(\alpha \circ \beta) = \mathcal{F}(\alpha) \circ \mathcal{F}(\beta)\) and \(\mathcal{H}_{*}^{Chern}(\alpha \circ \beta) = \mathcal{H}_{*}^{Chern}(\alpha) \circ \mathcal{H}_{*}^{Chern}(\beta)\), and MacPherson’s Chern class transformation \(c_{*}^{Mac} : F(-) \rightarrow H_*(-)\) extends to a natural transformation

\[
c_{*}^{Mac} : \mathcal{F}(-) \rightarrow \mathcal{H}_{*}^{Chern}(-).
\]
Remark 3.20. $\mathcal{F}, \mathcal{H}^{\text{Chern}}_*: \text{Corr}(\mathcal{V})^+_{\text{pro-id}} \to \mathcal{A}b$ are covariant functors (in the usual sense) for proper morphisms and $c^{\text{Mac}}_* : \mathcal{F}(-) \to \mathcal{H}^{\text{Chern}}_*(-)$ is MacPherson’s Chern class transformation. On the other hand, $\mathcal{F}, \mathcal{H}^{\text{Chern}}_*: \text{Corr}(\mathcal{V})^+_{\text{id-sm}} \to \mathcal{A}b$ are contravariant functors for smooth morphisms and $\mathcal{H}^{\text{Chern}}_*: \mathcal{F}(-) \to \mathcal{H}^{\text{Chern}}_*(-)$ yields the Verdier–Riemann–Roch formula for MacPherson’s Chern class transformation.

Remark 3.21. $\mathcal{F}(X \leftarrow^f M \rightarrow^g Y) := f^*g^*: F(Y) \to F(X)$ is called the topological Radon transformation of constructible functions [10, 11, 26]. The base change formula for constructible functions is proved in [10, Proposition 3.5] (cf. [11, Lemma 2.4]). $\mathcal{H}^{\text{Chern}}_*(X \leftarrow^f M \rightarrow^g Y) := f_*(c(T_g) \cap g^*) : H_*(Y) \to H_*(X)$ for compact smooth manifolds $X, Y, M$ is called the Verdier–Radon transformation in [11] (cf. [31]). Here $f$ and $g$ can be any morphisms and $T_g := TM - g^*TY$ is the virtual relative tangent bundle.

In [23] E. Looijenga defines the relative Grothendieck group $K_0(\mathcal{V}/X)$ as the free abelian group generated by the isomorphism classes $[V \xrightarrow{h} X]$ of morphisms $h: V \to X$ modulo the relation $[V \xrightarrow{h} X] = [W \xrightarrow{f \circ h} X] + [V \setminus W \xrightarrow{h|V\setminus W} X]$ for a closed subvariety $W \subset V$. For a morphism $f: X \to Y$ the pushforward $f_*: K_0(\mathcal{V}/X) \to K_0(\mathcal{V}/Y)$ is defined by $f_*([V \xrightarrow{h} X]) := [V \xrightarrow{f \circ h} Y]$ and clearly $(g \circ f)_* = g_* \circ f_*$, so $K_0(\mathcal{V}/-)$ is a covariant functor. For a morphism $g: X' \to X$, the pullback $g^*: K_0(\mathcal{V}/X) \to K_0(\mathcal{V}/X')$ is defined by $g^*([V \xrightarrow{h} X]) := [V' \xrightarrow{h'} X']$ where we use the fiber square

\[
\begin{array}{ccc}
V' & \xrightarrow{g'} & V \\
\downarrow h' & & \downarrow h \\
X' & \xrightarrow{g} & X
\end{array}
\]

Then it is clear that for morphisms $g: X' \to X$ and $f: X'' \to X$ we have $(g \circ f)^* = f^* \circ g^*$, thus it is a contravariant functor. We observe that the functor $K_0(\mathcal{V}/-)$ satisfies the base change formula:

\[
\begin{array}{c}
\begin{array}{ccc}
A' & \xrightarrow{\bar{g}} & A \\
\downarrow \bar{h} & & \downarrow h \\
B' & \xrightarrow{g} & B
\end{array} & \implies & \begin{array}{ccc}
K_0(\mathcal{V}/A') & \xrightarrow{(\bar{g})^*} & K_0(\mathcal{V}/A) \\
\downarrow (\bar{h})_* & & \downarrow h_* \\
K_0(\mathcal{V}/B') & \xrightarrow{g^*} & K_0(\mathcal{V}/B)
\end{array}
\end{array}
\]
This follows from considering the following fiber squares for $[V \xrightarrow{f} A] \in K_0(\mathcal{V}/A)$:

\[
\begin{align*}
V' &\xrightarrow{\tilde{g}'} V \\
\downarrow f' &\quad & \downarrow f \\
A' &\xrightarrow{\tilde{g}} A \\
\downarrow \tilde{h} &\quad & \downarrow h \\
B' &\xrightarrow{g} B
\end{align*}
\]

In [6] we showed that there exists a unique natural transformation

$$T_y_* : K_0(\mathcal{V}/X) \to H_*(X) \otimes \mathbb{Q}[y]$$

such that for a nonsingular variety $X$, $T_{y_*}([X \xrightarrow{id_X} X]) = T_{y}(TX) \cap [X]$. Here $T_y(TX)$ is the Hirzebruch class of the tangent bundle $TX$. Recall that the Hirzebruch class $T_y(E)$ of a complex vector bundle is

$$T_y(E) := \prod_{i=1}^{\dim X} \left( \frac{\alpha_i(1+y)}{1-e^{-\alpha_i(1+y)}} - \alpha_i y \right)$$

where $\alpha_i$ are the Chern roots of the bundle $E$.

**Remark 3.22.** The Hirzebruch class $T_y(E)$ unifies the following three important characteristic cohomology classes of $TX$:

1. ($y = -1$): $c(TX) = \prod_{i=1}^{\dim X} (1 + \alpha)$, the total Chern class,
2. ($y = 0$): $td(TX) = \prod_{i=1}^{\dim X} \frac{\alpha}{1-e^{-\alpha}}$, the total Todd class,
3. ($y = 1$): $L(TX) = \prod_{i=1}^{\dim X} \frac{\alpha}{\tanh \alpha}$, the total Thom–Hirzebruch $L$-class.

The natural transformation $T_{y_*} : K_0(\mathcal{V}/X) \to H_*(X) \otimes \mathbb{Q}[y]$ is called the *motivic Hirzebruch class*. We also have the Verdier–Riemann–Roch formula for the motivic Hirzebruch class for a smooth morphism ([6]). Thus for a correspondence $(X \leftarrow^f M \rightarrow^g Y)$ with $f$ proper and $g$ smooth, we have the commutative diagram

\[
\begin{array}{ccccc}
K_0(\mathcal{V}/X) &\xrightarrow{f_*} & K_0(\mathcal{V}/M) &\xleftarrow{g^*} & K_0(\mathcal{V}/Y) \\
T_{y_*} &\downarrow & T_{y_*} &\downarrow & T_{y_*} \\
H_*(X) \otimes \mathbb{Q}[y] &\xleftarrow{f_*} & H_*(M) \otimes \mathbb{Q}[y] &\xleftarrow{T_y(T_g) \cap g^*} & H_*(Y) \otimes \mathbb{Q}[y]
\end{array}
\]

Here $T_y(T_g)$ is the Hirzebruch class of the relative tangent bundle of the smooth morphism $g : M \to Y$. 
Thus, as in the above discussion, we obtain the following:

**Proposition 3.24.** Define $K_0(\mathcal{V} / -)$, $\mathcal{H}_*^{Hirz} : \text{Corr}(\mathcal{V})_{\text{pro-sm}} \to \text{Ab}$ as follows:

- for $X \in \text{Obj}(\text{Corr}(\mathcal{V})_{\text{pro-sm}}) = \text{Obj}(\mathcal{V})$, 
  $$K_0(\mathcal{V} / -)(X) := K_0(\mathcal{V} / X), \quad \mathcal{H}_*^{Hirz}(X) := H_*(X) \otimes \mathbb{Q}[y],$$

- for $X \xleftarrow{f} M \xrightarrow{g} Y \in \text{hom}_{\text{Corr}(\mathcal{V})_{\text{pro-sm}}}(X, Y) = \text{Corr}(X, Y)_{\text{pro-sm}}$, 
  $$K_0(\mathcal{V} -)(X \xleftarrow{f} M \xrightarrow{g} Y) := f_*g^* : K_0(\mathcal{V} / -)(Y) \to K_0(\mathcal{V} / -)(X),$$
  $$\mathcal{H}_*^{Hirz}(X \xleftarrow{f} M \xrightarrow{g} Y) := f_*(T_y(T_g) \cap g^*) : \mathcal{H}_*^{Hirz}(Y) \to \mathcal{H}_*^{Hirz}(X).$$

Then $K_0(\mathcal{V} -), \mathcal{H}_*^{Hirz} : \text{Corr}(\mathcal{V})_{\text{pro-sm}} \to \text{Ab}$ are functors and the motivic Hirzebruch class transformation $T_{y*} : K_0(\mathcal{V} / X) \to H_*(X) \otimes \mathbb{Q}[y]$ extends to a natural transformation

$$T_{y*} : K_0(\mathcal{V} -) \to \mathcal{H}_*^{Hirz}(-).$$

**Theorem 3.25.** Define $\mathcal{K}_0(\mathcal{V} / -)$, $\mathcal{H}_*^{Hirz} : \text{Corr}(\mathcal{V})^+_{\text{pro-sm}} \to \text{Ab}$ as follows:

- for $X \in \text{Obj}(\text{Corr}(\mathcal{V})^+_{\text{pro-sm}})$, 
  $$\mathcal{K}_0(\mathcal{V} / -)(X) := F(X), \quad \mathcal{H}_*^{Hirz}(X) := H_*(X) \otimes \mathbb{Q}[y],$$

- for $\sum_i n_i[X \xleftarrow{f_i} M_i \xrightarrow{g_i} Y] \in \text{hom}_{\text{Corr}(\mathcal{V})^+_{\text{pro-sm}}}(X, Y) := \text{Corr}(X, Y)^+_{\text{pro-sm}}$, 
  $$\mathcal{K}_0(\mathcal{V} / -)\left(\sum_i n_i[X \xleftarrow{f_i} M_i \xrightarrow{g_i} Y]\right)$$
  $$:= \sum_i n_i(f_i)_*(g_i)^* : \mathcal{K}_0(\mathcal{V} / -)(Y) \to \mathcal{K}_0(\mathcal{V} / -)(X),$$
  $$\mathcal{H}_*^{Hirz}\left(\sum_i n_i[X \xleftarrow{f_i} M_i \xrightarrow{g_i} Y]\right)$$
  $$:= \sum_i n_i(f_i)_*(T_y(T_{g_i}) \cap (g_i)^*) : \mathcal{H}_*^{Hirz}(Y) \to \mathcal{H}_*^{Hirz}(X).$$

Then $\mathcal{K}_0(\mathcal{V} / -), \mathcal{H}_*^{Hirz} : \text{Corr}(\mathcal{V})^+_{\text{pro-sm}} \to \text{Ab}$ are functors and the motivic Hirzebruch class transformation $T_{y*} : \mathcal{K}_0(\mathcal{V} / X) \to H_*(X) \otimes \mathbb{Q}[y]$ extends to a natural transformation

$$T_{y*} : \mathcal{K}_0(\mathcal{V} / -) \to \mathcal{H}_*^{Hirz}(-).$$

**Remark 3.26.** The functors $\mathcal{K}_0(\mathcal{V} / -), \mathcal{H}_*^{Hirz} : \text{Corr}(\mathcal{V})^+_{\text{pro-id}} \to \text{Ab}$ are covariant (in the usual sense) for proper morphisms and $T_{y*} : \mathcal{K}_0(-) \to \mathcal{H}_*^{Todd}(-)$ is the motivic Hirzebruch class transformation. On the other hand, $\mathcal{K}_0(\mathcal{V} / -), \mathcal{H}_*^{Hirz} : \text{Corr}(\mathcal{V})^+_{\text{id-sm}} \to \text{Ab}$ are contravariant for smooth
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morphisms and $T_y : \mathcal{K}_0(\mathcal{V}/-) \to \mathcal{H}_{\operatorname{Todd}}(-)$ yields the Verdier–Riemann–Roch formula for the motivic Hirzebruch class transformation.

Remark 3.27. The motivic Hirzebruch class $T_y : K_0(\mathcal{V}/X) \to H_*(X) \otimes \mathbb{Q}[y] \sqcup$ unifies” the above MacPherson’s Chern class transformation $c_{\operatorname{Mac}}^*$, Baum–Fulton–MacPherson’s Todd class transformation $td_{\operatorname{BFM}}^*$ and Cappell–Shaneson’s $L$-class transformation $L_{\operatorname{CS}}^* : \Omega(-) \to H_*(-) \otimes \mathbb{Q}$ (see below) in the sense that we have the following commutative diagrams:

\[
\begin{array}{ccc}
K_0(\mathcal{V}/X) & \xrightarrow{\epsilon} & F(X) \\
\downarrow & & \downarrow \text{c}_{\operatorname{Mac}}^* \otimes \mathbb{Q} \\
H_*(X) \otimes \mathbb{Q} & \xrightarrow{T_{-1}^*} & H_*(X) \otimes \mathbb{Q}
\end{array}
\]

\[
\begin{array}{ccc}
K_0(\mathcal{V}/X) & \xrightarrow{\Gamma} & G_0(X) \\
\downarrow & & \downarrow \text{td}_{\operatorname{BFM}}^* \\
H_*(X) \otimes \mathbb{Q} & \xrightarrow{T_{0*}} & H_*(X) \otimes \mathbb{Q}
\end{array}
\]

\[
\begin{array}{ccc}
K_0(\mathcal{V}/X) & \xrightarrow{\omega} & \Omega(X) \\
\downarrow & & \downarrow L_{\operatorname{CS}} \\
H_*(X) \otimes \mathbb{Q} & \xrightarrow{T_{1*}} & H_*(X) \otimes \mathbb{Q}
\end{array}
\]

Goresky–MacPherson’s homology $L$-class [17] extends to a natural transformation by S. Cappell and J. Shaneson [7] (see also [30]): There exists a unique natural transformation $L_{\operatorname{CS}}^* : \Omega(X) \to H_*(X) \otimes \mathbb{Q}$ such that for a nonsingular compact variety $X$,

\[
L_*(\mathbb{Q}_X[2 \dim X]) = L(TX) \cap [X].
\]

Here $\Omega$ is the covariant functor assigning to $X$ the cobordism group $\Omega(X)$ of self-dual constructible sheaf complexes on $X$. For the case of Cappell–Shaneson’s $L$-class $L_{\operatorname{CS}}^* : \Omega(-) \to H_*(-) \otimes \mathbb{Q}$ we do not know whether the Verdier–Riemann–Roch formula holds for smooth morphisms. Thus, unlike the cases of MacPherson’s Chern class, Baum–Fulton–MacPherson’s Todd class and the motivic Hirzebruch class, at the moment we cannot define $L_{\operatorname{CS}}^* : \Omega(-) \to \mathcal{H}_{L\text{-class}}(-)$ in a similar manner to $c_{\operatorname{Mac}}^* : \mathcal{F}(-) \to \mathcal{H}_{\operatorname{Chern}}(-)$, $td_{\operatorname{BFM}}^* : G_0(-) \to \mathcal{H}_{\operatorname{Todd}}(-)$ and $T_y : \mathcal{K}_0(\mathcal{V}/-) \to \mathcal{H}_{\operatorname{Hirz}}(-)$.

Remark 3.28. The motivic Chern class transformation $mC_y : K_0(\mathcal{V}/-) \to G_0(-) \otimes \mathbb{Z}[y]$ and the motivic Hodge Chern class transformation $MHC_y :
$K_0(MHM(-)) \to G_0(-) \otimes \mathbb{Z}[y, y^{-1}]$ also satisfy the Verdier–Riemann–Roch formula for smooth morphisms (see [6]). Thus we can get the same formulations for these transformations. Here $K_0(MHM(X))$ is the Grothendieck group of the derived category of mixed Hodge modules on $X$.

Remark 3.29. In [22] Levine and Pandharipande show that Levine–Morel’s algebraic cobordism $\Omega_*(X)$ can be obtained as a quotient group of the Grothendieck group $\mathcal{M}^+(X) = \text{Corr}(X, pt)^+_{\text{pro-sm}}$ modulo the double-point degeneration relation. If we could get some quotient group $B\Omega_*(X, Y) := \text{Corr}(X, Y)^+_{\text{pro-sm}}$ of $\text{Corr}(X, Y)^+_{\text{pro-sm}}$ in an analogous manner to [22], in such a way that

- $B\Omega_*(X, pt) \cong \Omega_*(X)$,
- $\circ : \text{Corr}(X, Y)^+_{\text{pro-sm}} \times \text{Corr}(Y, Z)^+_{\text{pro-sm}} \to \text{Corr}(X, Z)^+_{\text{pro-sm}}$ extends to $\circ : B\Omega_*(X, Y) \times B\Omega_*(Y, Z) \to B\Omega_*(X, Z)$,

then we would call $B\Omega_*(X, Y)$ a bi-variant algebraic cobordism of bicycles, which is treated in [35]. Then we would get an $Ab$-enriched category $\mathcal{B}\Omega_*(\mathcal{V})$ of algebraic cobordisms of bicycles such that

- $\text{Obj}(\mathcal{B}\Omega_*(\mathcal{V})) = \text{Obj}(\mathcal{V})$,
- for objects $X$ and $Y$, $\text{hom}_{\mathcal{B}\Omega_*(\mathcal{V})}(X, Y) = B\Omega_*(X, Y)$,

and we could also consider whether one can extend characteristic classes of singular varieties to the enriched categories of algebraic cobordisms of bicycles.

4. Enriched categories of zigzags and characteristic classes of singular varieties. An $\ell.c.i.$ morphism $f : X \to Y$ is a regular embedding $r : X \to P$ followed by a smooth morphism $p : P \to Y$ (see [12]). In the context of $\ell.c.i.$ morphisms we assume as in [12] that all varieties have a closed embedding into a smooth variety (like e.g. quasi-projective varieties), so that the composition of $\ell.c.i.$ morphisms is again an $\ell.c.i.$ morphism. The virtual tangent bundle $T_f$ of an $\ell.c.i.$ morphism $f : X \to Y$ is defined to be $r^*T_p - N_X P \in K^0(X)$, which has similar properties to the fiber tangent bundle $T_f$ of a smooth morphism $f$ (for more details see [12]).

The Verdier–Riemann–Roch formula for Baum–Fulton–MacPherson’s Todd class transformation holds for $\ell.c.i.$ morphisms (see [5]). A smooth morphism is also an $\ell.c.i.$ morphism. Hence, similarly we can consider a proper-$\ell.c.i.$ correspondence $X \xleftarrow{f} M \xrightarrow{g} Y$ with a proper morphism $f$ and an $\ell.c.i.$ morphism $g$. Unfortunately, the pullback of an $\ell.c.i.$ morphism in a fiber square is not necessarily an $\ell.c.i.$ morphism. Hence, we cannot do the same argument for proper-$\ell.c.i.$ correspondences. To remedy this, we use zigzags instead of correspondences.
Definition 4.1. The following finite sequence of correspondences is called a \( k \)-zigzag or a \( k \)-correspondence of complex algebraic varieties:

\[
\begin{align*}
X = X_0 & \xleftarrow{f_1} M_1 & \xrightarrow{g_1} X_1 \xleftarrow{f_2} M_2 & \xrightarrow{g_2} X_2 \cdots \xleftarrow{f_k} M_k & \xrightarrow{g_k} X_k = Y
\end{align*}
\]

The set of all zigzags of finite length from \( X \) to \( Y \) is denoted by \( \text{Zigzag}(X, Y) \).

Lemma 4.2. For two zigzags

\[
\alpha = (X \xleftarrow{f_1} M_1 \xrightarrow{g_1} X_1 \cdots \xleftarrow{f_i} M_i \xrightarrow{g_i} Y) \in \text{Zigzag}(X, Y),
\]

\[
\beta = (Y \xleftarrow{h_1} N_1 \xrightarrow{k_1} Y_1 \cdots \xleftarrow{h_j} N_j \xrightarrow{k_j} Z) \in \text{Zigzag}(Y, Z),
\]

we define the product \( \alpha \land \beta \) by juxtaposition:

\[
\alpha \land \beta := (X \xleftarrow{f_1} M_1 \xrightarrow{g_1} X_1 \cdots \xleftarrow{f_i} M_i \xrightarrow{g_i} Y \xleftarrow{h_1} N_1 \xrightarrow{k_1} Y_1 \cdots \xleftarrow{h_j} N_j \xrightarrow{k_j} Z).
\]

Then the juxtaposition \( \land \) is well-defined:

\[
\land : \text{Zigzag}(X, Y) \times \text{Zigzag}(Y, Z) \to \text{Zigzag}(X, Z).
\]

If a zigzag consists of proper-\( \ell \).c.i. correspondences, i.e., each \( X_{i-1} \xleftarrow{f_i} M_i \xrightarrow{g_i} X_i \) is a proper-\( \ell \).c.i. correspondence, such a zigzag is called a proper-\( \ell \).c.i. zigzag, and the set of all proper-\( \ell \).c.i. zigzags from \( X \) to \( Y \) is denoted by \( \text{Zigzag}_{\text{pro-\( \ell \).c.i.}}(X, Y) \).

Then we define the category \( \text{Zigzag}_{\text{pro-\( \ell \).c.i.}}(\mathcal{Y}) \) of proper-\( \ell \).c.i. zigzags:

- \( \text{Obj}(\text{Zigzag}_{\text{pro-\( \ell \).c.i.}}(\mathcal{Y})) = \text{Obj}(\mathcal{Y}) \),
- \( \text{hom}_{\text{Zigzag}_{\text{pro-\( \ell \).c.i.}}(\mathcal{Y})}(X, Y) = \text{Zigzag}_{\text{pro-\( \ell \).c.i.}}(X, Y) \).

Two proper-\( \ell \).c.i. zigzags (of the same length)

\[
(X \xleftarrow{f_1} M_1 \xrightarrow{g_1} X_1 \xleftarrow{f_2} M_2 \xrightarrow{g_2} X_2 \cdots \xleftarrow{f_k} M_k \xrightarrow{g_k} Y)
\]

and

\[
(X = X_0 \xleftarrow{f'_1} M'_1 \xrightarrow{g'_1} X'_1 \xleftarrow{f'_2} M'_2 \xrightarrow{g'_2} X'_2 \cdots \xleftarrow{f'_k} M'_k \xrightarrow{g'_k} X_k = Y)
\]

are called isomorphic if there exist isomorphisms \( h_i : M_i \to M'_i \) for \( 1 \leq i \leq k \) and \( \phi_j : X_j \to X'_j \) for \( 1 \leq j \leq k - 1 \) such that the following diagrams commute:
The set of isomorphism classes of proper-ℓ.c.i. zigzags (of length $k$) becomes an Abelian monoid by taking the disjoint union

$$[X \xleftarrow{f_1} M_1 \xrightarrow{g_1} X_1 \xleftarrow{f_2} \ldots \xleftarrow{f_{k-1}} X_{k-1} \xrightarrow{f_k} M_k \xrightarrow{g_k} Y]$$

$$+ [X \xleftarrow{f_1'} M_1' \xrightarrow{g_1'} X_1' \xleftarrow{f_2'} \ldots \xleftarrow{f_{k-1}'} X_{k-1}' \xrightarrow{f_k'} M_k' \xrightarrow{g_k'} Y]$$

$$:= [X \xleftarrow{f_1+f_1'} M_1 \sqcup M_1' \xrightarrow{g_1+g_1'} X_1 \sqcup X_1' \xleftarrow{f_2} \ldots \xleftarrow{f_{k-1}} X_{k-1} \sqcup X_{k-1}' \xrightarrow{f_k} M_k \sqcup M_k' \xrightarrow{g_k+g_k'} Y].$$

Then its group completion is denoted by $\text{Zigzag}_k(X,Y)^{+}_{\text{pro-ℓ.c.i.}}$ and

$$\text{Zigzag}(X,Y)^{+}_{\text{pro-ℓ.c.i.}} := \bigoplus_k \text{Zigzag}_k(X,Y)^{+}_{\text{pro-ℓ.c.i.}}.$$

The juxtaposition product

$$\wedge : \text{Zigzag}_{\text{pro-ℓ.c.i.}}(X,Y) \times \text{Zigzag}_{\text{pro-ℓ.c.i.}}(Y,Z) \to \text{Zigzag}_{\text{pro-ℓ.c.i.}}(X,Z)$$

extends to $\text{Zigzag}_{\text{pro-ℓ.c.i.}}(-,-)^{+}$:

$$\wedge : \text{Zigzag}(X,Y)^{+}_{\text{pro-sm}} \times \text{Zigzag}(Y,Z)^{+}_{\text{pro-ℓ.c.i.}} \to \text{Zigzag}(X,Z)^{+}_{\text{pro-ℓ.c.i.}}.$$

Then the $\mathcal{A}b$-enriched category $\text{Zigzag}(\mathcal{V})^{+}_{\text{pro-ℓ.c.i.}}$ of zigzags is defined by

- $\text{Obj}(\text{Zigzag}(\mathcal{V})^{+}_{\text{pro-ℓ.c.i.}}) = \text{Obj}(\mathcal{V})$,
- $\text{hom}_{\text{Zigzag}(\mathcal{V})^{+}_{\text{pro-ℓ.c.i.}}}(X,Y) = \text{Zigzag}(X,Y)^{+}_{\text{pro-ℓ.c.i.}}$.

Now Proposition 3.7 and Theorem 3.9 become:

**Proposition 4.3.** Define $G_0, H^{\text{Todd}}_\ast : \text{Zigzag}_{\text{pro-ℓ.c.i.}}(\mathcal{V}) \to \mathcal{A}b$ as follows:

- for $X \in \text{Obj}(\text{Zigzag}_{\text{pro-ℓ.c.i.}}(\mathcal{V})) = \text{Obj}(\mathcal{V})$,
  $$G_0(X) := G_0(X),$$
  $$H^{\text{Todd}}_\ast(X) := H_\ast(X) \otimes \mathbb{Q}, \text{ the Borel–Moore homology theory},$$
for

\[ \alpha = (X \xleftarrow{f_1} M_1 \xrightarrow{g_1} X_1 \ldots \ldots X_{i-1} \xleftarrow{f_i} M_i \xrightarrow{g_i} Y) \in \text{hom}_{\text{zigzag}_{\text{pro-\ell.c.i.}}}(Y)(X,Y) = \text{zigzag}_{\text{pro-\ell.c.i.}}(X,Y) \]

set

\[ G_0(\alpha) := (f_1)_*(g_1)^* \circ \cdots \circ (f_i)_*(g_i)^* : G_0(Y) \to G_0(X), \]
\[ H_{\text{Todd}}^*(\alpha) := (f_1)_*(\text{td}(T_{g_1})(g_1)^*) \circ \cdots \circ (f_i)_*(\text{td}(T_{g_i})(g_i)^*)) : \]
\[ H_{\text{Todd}}^*(Y) \to H_{\text{Todd}}^*(X). \]

Then \( G_0, H_{\text{Todd}}^* : \text{zigzag}_{\text{pro-\ell.c.i.}}(Y) \to \mathcal{A}b \) are functors in the sense that \( G_0(\alpha \land \beta) = G_0(\alpha) \circ G_0(\beta) \) and \( H_{\text{Todd}}^*(\alpha \land \beta) = H_{\text{Todd}}^*(\alpha) \circ H_{\text{Todd}}^*(\beta) \), and Baum–Fulton–MacPherson’s Todd class transformation \( \text{td}_{\text{BFM}}^* : G_0(-) \to H_{\text{Todd}}^*(-) \otimes \mathbb{Q} \) extends to a natural transformation

\[ \text{td}_{\text{BFM}}^* : G_0(-) \to H_{\text{Todd}}^*(-). \]

**Theorem 4.4.** Define \( G_0, H_{\text{Todd}}^* : \text{zigzag}(Y)^{\text{pro-\ell.c.i.}} \to \mathcal{A}b \) as follows:

- for \( X \in \text{Obj}(\text{zigzag}(Y)^{\text{pro-\ell.c.i.}}) \),
  \[ G_0(X) := F(X), \quad H_{\text{Todd}}^*(X) := H_{\ell.c.i.}(X) \otimes \mathbb{Q}, \]

- for \( \sum_i n_i[\alpha_i] \in \text{hom}_{\text{zigzag}(Y)^{\text{pro-\ell.c.i.}}}(X,Y) := \text{zigzag}(X,Y)^{\text{pro-\ell.c.i.}} \)
  \[ G_0\left( \sum_i n_i[\alpha_i] \right) := \sum_i n_i G_0(\alpha_i) : G_0(Y) \to G_0(X), \]
  \[ H_{\text{Todd}}^*(\sum_i[n_i \alpha_i]) := \sum_i n_i H_{\text{Todd}}^*(\alpha_i) : H_{\text{Todd}}^*(Y) \to H_{\text{Todd}}^*(X). \]

Then \( G_0, H_{\text{Todd}}^* : \text{zigzag}(Y)^{\text{pro-\ell.c.i.}} \to \mathcal{A}b \) are functors in the sense that \( G_0(\alpha \land \beta) = G_0(\alpha) \circ G_0(\beta) \) and \( H_{\text{Todd}}^*(\alpha \land \beta) = H_{\text{Todd}}^*(\alpha) \circ H_{\text{Todd}}^*(\beta) \), and MacPherson’s Chern class transformation \( \text{td}_{\text{BFM}}^* : G_0(-) \to H_{\text{Todd}}^*(-) \otimes \mathbb{Q} \) extends to a natural transformation

\[ \text{td}_{\text{BFM}}^* : G_0(-) \to H_{\text{Todd}}^*(-). \]

**Remark 4.5.** We note the following:

1. A zigzag of proper-identity correspondences is the same as a proper-identity correspondence \( X \xleftarrow{\text{id}_Y} Y \xrightarrow{\text{id}_Y} Y \) with a proper \( f \).
2. A zigzag of identity-smooth correspondences is the same as an identity-smooth correspondence \( X \xleftarrow{\text{id}_X} X \xrightarrow{g} Y \) with a smooth \( g \).
3. A zigzag of identity-\( \ell.c.i. \) correspondences is the same as an identity-\( \ell.c.i. \) correspondence \( X \xleftarrow{\text{id}_X} X \xrightarrow{g} Y \) with an \( \ell.c.i. \) morphism \( g \).
5. Cobordism bicycles of vector bundles and Baum–Fulton–MacPherson’s Todd classes. In this section we consider extending the notion of algebraic cobordism of vector bundles due to Lee and Pandharipande [20] to correspondences.

Definition 5.1. Let $X \overset{p}{\leftarrow} V \overset{s}{\rightarrow} Y$ be a proper-smooth correspondence and let $E$ be a complex algebraic vector bundle on $V$. Then the pair

$$(X \overset{p}{\leftarrow} V \overset{s}{\rightarrow} Y; E)$$

is called a cobordism bicycle of a vector bundle.

Remark 5.2. The above notion is just a proper-smooth correspondence equipped with a complex vector bundle, but we mimick the terminology of [4], [21] and [20]. A similar object is used in the so-called KK-theory (e.g., see [9]). KK-theoretic issues, i.e., bivariant-theoretic aspects, are treated in [35].

Definition 5.3. Let $(X \overset{p}{\leftarrow} V \overset{s}{\rightarrow} Y; E)$ and $(X \overset{p'}{\leftarrow} V' \overset{s'}{\rightarrow} Y; E')$ be cobordism bicycles of vector bundles of the same rank. If there exists an isomorphism $h : V \cong V'$ such that $(X \overset{p}{\leftarrow} V \overset{s}{\rightarrow} Y) \cong (X \overset{p'}{\leftarrow} V' \overset{s'}{\rightarrow} Y)$ as correspondences and $E \cong h^*E'$ as well, then the two cobordism bicycles are called isomorphic and we write

$$(X \overset{p}{\leftarrow} V \overset{s}{\rightarrow} Y; E) \cong (X \overset{p'}{\leftarrow} V' \overset{s'}{\rightarrow} Y; E').$$

The isomorphism class of $(X \overset{p}{\leftarrow} V \overset{s}{\rightarrow} Y; E)$ is denoted by $[X \overset{p}{\leftarrow} V \overset{s}{\rightarrow} Y; E]$, and still called a cobordism bicycle of a vector bundle. For a fixed rank $r$ of vector bundles, the set of isomorphism classes of cobordism bicycles for a pair $(X,Y)$ becomes a commutative monoid with respect to the disjoint union:

$$[X \overset{p_1}{\leftarrow} V_1 \overset{s_1}{\rightarrow} Y; E_1] + [X \overset{p_2}{\leftarrow} V_2 \overset{s_2}{\rightarrow} Y; E_2] := [X \overset{p_1 + p_2}{\leftarrow} V_1 \sqcup V_2 \overset{s_1 + s_2}{\rightarrow} Y; E_1 + E_2],$$

where $E_1 + E_2$ is a vector bundle with the property that $(E_1 + E_2)|_{V_1} = E_1$ and $(E_1 + E_2)|_{V_2} = E_2$. This monoid is denoted by $\mathcal{M}_r(X,Y)$, and another grading of $[X \overset{p}{\leftarrow} V \overset{s}{\rightarrow} Y; E]$ is defined by the relative dimension $\dim s$ of the smooth map $s$; thus with double grading, $[X \overset{p}{\leftarrow} V \overset{s}{\rightarrow} Y; E] \in \mathcal{M}_{n,r}(X,Y)$ means that $n = \dim s$ and $r = \rank E$. The group completion of this monoid, i.e., the Grothendieck group, is denoted by $\mathcal{M}_{n,r}(X,Y)^+$. This notation mimicks [21] [20] (cf. Remark 3.2).

Remark 5.4. For a fixed rank $r$, $\mathcal{M}_{*,r}(X,Y)^+ = \bigoplus \mathcal{M}_{n,r}(X,Y)^+$ is a graded Abelian group.
Remark 5.5. \( \mathcal{M}_{n,r}(X, pt)^+ \) is nothing but \( \mathcal{M}_{n,r}(X)^+ \) considered by Lee–Pandharipande \[20\]. Thus, \( \mathcal{M}_{n,r}(pt, Y)^+ \) is a new object to investigate.

Definition 5.6 (product of cobordism bicycles). For three varieties \( X, Y, Z \), we define the following two kinds of product:

(1) (by the Whitney sum \( \oplus \))

\[ \circ \oplus : \mathcal{M}_{m,r}(X, Y)^+ \times \mathcal{M}_{n,k}(Y, Z)^+ \to \mathcal{M}_{m+n,r+k}(X, Z)^+ , \]

\[ [X \xleftarrow{p} V \xrightarrow{s} Y; E] \circ \oplus [Y \xleftarrow{q} W \xrightarrow{t} Z; F] := [(X \xleftarrow{p} V \xrightarrow{s} Y) \circ (Y \xleftarrow{q} W \xrightarrow{t} Z); \tilde{q}^* E \oplus \tilde{s}^* F] , \]

(2) (by the tensor product \( \otimes \))

\[ \circ \otimes : \mathcal{M}_{m,r}(X, Y)^+ \times \mathcal{M}_{n,k}(Y, Z)^+ \to \mathcal{M}_{m+n,rk}(X, Z)^+ , \]

\[ [X \xleftarrow{p} V \xrightarrow{s} Y; E] \circ \otimes [Y \xleftarrow{q} W \xrightarrow{t} Z; F] := [(X \xleftarrow{p} V \xrightarrow{s} Y) \circ (Y \xleftarrow{q} W \xrightarrow{t} Z); \tilde{q}^* E \otimes \tilde{s}^* F] , \]

where we consider the commutative diagram

\[ \tilde{q}^* E \oplus \tilde{s}^* F \text{ or } \tilde{q}^* E \otimes \tilde{s}^* F \]

\[ \begin{array}{ccc}
E & \xrightarrow{q} & V \times_Y W \\
\downarrow & & \downarrow \tilde{s} \\
V & \xrightarrow{s} & W \\
\downarrow p & & \downarrow q \\
X & \xrightarrow{\bar{s}} & Y \\
\end{array} \]

\[ \begin{array}{ccc}
F & & \xleftarrow{t} Z \\
\downarrow & & \\
W & \xleftarrow{t} Z \\
\end{array} \]

Lemma 5.7. The products \( \circ \oplus \) and \( \circ \otimes \) are both bilinear.

Remark 5.8. \( \mathcal{M}_{n,*}(X, X)^+ \) is a double graded commutative ring with respect to both products \( \circ \oplus \) and \( \circ \otimes \).

Remark 5.9. Since \( \mathcal{M}_{n,r}(X, pt)^+ = \mathcal{M}_{n,r}(X)^+ \) and \( \mathcal{M}_{n,r}(pt, pt)^+ = \mathcal{M}_{n,r}(pt)^+ \), we have

\[ \circ \oplus : \mathcal{M}_{m,r}(X)^+ \times \mathcal{M}_{n,k}(pt)^+ \to \mathcal{M}_{m+n,r+k}(X)^+ , \]

\[ [X \xleftarrow{p} V \xrightarrow{s} pt; E] \circ \oplus [pt \xleftarrow{q} W \xrightarrow{t} pt; F] = [(X \xleftarrow{p} V \xrightarrow{s} pt) \circ (pt \xleftarrow{q} W \xrightarrow{t} pt); (pr_1)^* E \oplus (pr_2)^* F] , \]

which, in the notations of \[20\], can be rewritten as follows:

\[ [V \xrightarrow{p} X, E] \circ \oplus [W; F] = [V \times W \xrightarrow{pr_1 \circ pr_2} X; (pr_1)^* E \oplus (pr_2)^* F] . \]
Then \[ \langle X \xleftarrow{p} V \xrightarrow{s} Y; E \rangle \in \mathcal{M}_{m,r}(X,Y)_{+}, \]
and let
\[ \langle X \xleftarrow{p} V \xrightarrow{id} V \rangle \in \mathcal{M}_{0,0}(X,V)_{+}, \]
\[ \langle V; E \rangle := \langle V \xleftarrow{id} V \xrightarrow{id} V; E \rangle \in \mathcal{M}_{0,r}(V,V)_{+}, \]
\[ \langle V \xrightarrow{s} Y \rangle := \langle V \xleftarrow{id} V \xrightarrow{\alpha} Y \rangle \in \mathcal{M}_{m,0}(V,Y)_{+}. \]

Then \[ \langle X \xleftarrow{p} V \xrightarrow{s} Y; E \rangle = \langle X \xleftarrow{p} V \rangle \circ_{+} \langle V; E \rangle \circ_{+} \langle V \xrightarrow{s} Y \rangle. \]

**Remark 5.11.** As “bicycle” suggests, we can discuss bivariant-theoretic aspects of cobordism bicycles, but this will not be done in this paper. For more details, see [35].

**Definition 5.12.** Define the enriched categories \( \mathcal{M}_{*,*}(Y)_{+}, \mathcal{M}_{*,*}(Y)_{\otimes} \) of cobordism bicycles of vector bundles as follows:

- \( \text{Obj}(\mathcal{M}_{*,*}(Y)_{+}) = \text{Obj}(\mathcal{M}_{*,*}(Y)_{\otimes}) = \text{Obj}(Y), \)
- for objects \( X \) and \( Y \),
  \[ \hom_{\mathcal{M}_{*,*}(Y)_{+}}(X,Y) := \mathcal{M}_{*,*}(X,Y)_{+}, \]
  \[ \hom_{\mathcal{M}_{*,*}(Y)_{\otimes}}(X,Y) := \mathcal{M}_{*,*}(X,Y)_{\otimes}. \]

**Proposition 5.13.** Let \( \mathcal{c} \ell \) be a multiplicative characteristic class of complex vector bundles (hence in particular \( \mathcal{c} \ell(F_1 \oplus F_2) = \mathcal{c} \ell(F_1) \cup \mathcal{c} \ell(F_2) \)) with \( \mathcal{c} \ell(E) \in H^*(X) \otimes \Lambda \) for some ring \( \Lambda \). Define \( \mathcal{H}_{*}^{\mathcal{c} \ell} : \mathcal{M}_{*,*}(Y)_{+} \to \mathcal{A}b \) by

- for an object \( X \), \( \mathcal{H}_{*}^{\mathcal{c} \ell}(X) = H_{*}(X) \otimes \Lambda \), the Borel–Moore homology theory with coefficients in \( \Lambda \),
- for \( \sum_{i} n_{i}[X \xleftarrow{p_i} V_i \xrightarrow{s_i} Y; E_i] \in \hom_{\mathcal{M}_{*,*}(Y)_{+}}(X,Y), \)
  \[ \mathcal{H}_{*}^{\mathcal{c} \ell}\left( \sum_{i} n_{i}[X \xleftarrow{p_i} V_i \xrightarrow{s_i} Y; E_i] \right) := \sum_{i} n_{i}(p_i)_{*}(\mathcal{c} \ell(E_i) \cap (s_i)_{*}) : \mathcal{H}_{*}^{\mathcal{c} \ell}(Y) \to \mathcal{H}_{*}^{\mathcal{c} \ell}(X). \]

Then \( \mathcal{H}_{*}^{\mathcal{c} \ell} : \mathcal{M}_{*,*}(Y)_{+} \to \mathcal{A}b \) is a functor in the sense that \( \mathcal{H}_{*}^{\mathcal{c} \ell}(\alpha \circ_{\otimes} \beta) = \mathcal{H}_{*}^{\mathcal{c} \ell}(\alpha) \circ_{\otimes} \mathcal{H}_{*}^{\mathcal{c} \ell}(\beta). \)
Proof. It suffices to show
\[ \mathcal{H}^c_{*}(\{X \xleftarrow{p} V \xrightarrow{s} Y; E\} \circ \bowtie \{Y \xleftarrow{q} W \xrightarrow{t} Z; F\}) \]
\[ = \mathcal{H}^c_{*}(\{X \xleftarrow{p} V \xrightarrow{s} Y; E\}) \circ \mathcal{H}^c_{*}(\{Y \xleftarrow{q} W \xrightarrow{t} Z; F\}). \]
The proof is the same as that of Proposition 3.7, but for the reader’s convenience we write it down:

\[ \mathcal{H}^c_{*}(\{X \xleftarrow{p} V \xrightarrow{s} Y; E\} \circ \bowtie \{Y \xleftarrow{q} W \xrightarrow{t} Z; F\}) \]
\[ = \mathcal{H}^c_{*}(\{X \xleftarrow{p \circ q} V \times_Y W \xrightarrow{to \tilde{s}} Z; \tilde{q}^* E \oplus \tilde{s}^* F\}) \]
\[ = (p \circ \tilde{q})_* (\tilde{c}^* E \oplus \tilde{s}^* F) \cap (t \circ \tilde{s})^* \]
\[ = p_* \tilde{q}_* ((\tilde{q}^* \tilde{c}^* (E) \cup \tilde{s}^* \tilde{c}^* (F)) \cap (\tilde{s}^* \circ t^*)) \]
\[ = p_* \tilde{q}_* (\tilde{q}^* \tilde{c}^* (E) \cap (\tilde{s}^* \tilde{c}^* (F) \cap (\tilde{s}^* \circ t^*)) \]
\[ = p_* (\tilde{c}^* (E) \cap \tilde{q}_* (\tilde{s}^* (\tilde{c}^* (F) \cap t^*)) \]) \quad \text{(by the projection formula)} \]
\[ = p_* (\tilde{c}^* (E) \cap \tilde{q}_* (\tilde{s}^* (\tilde{c}^* (F) \cap (\tilde{s}^* \circ t^*)) \]
\[ = p_* (\tilde{c}^* (E) \cap \tilde{q}_* (\tilde{s}^* (\tilde{c}^* (F) \cap t^*)) \]
\[ = p_* (\tilde{c}^* (E) \cap \tilde{s}^* (q_* (\tilde{c}^* (F) \cap t^*)) \]
\[ = (\mathcal{H}^c_{*}(\{X \xleftarrow{p} V \xrightarrow{s} Y; E\}) \circ \mathcal{H}^c_{*}(\{Y \xleftarrow{q} W \xrightarrow{t} Z; F\}). \]

As a corollary of the above proof we get the following for \( \mathcal{M}_{*,*}(\mathcal{Y})^+ \), since \( ch(E \otimes F) = ch(E) \cup ch(F) \) for the Chern character \( ch \):

**Corollary 5.14.** Let \( ch \) be the Chern character. Define
\[ \mathcal{H}^c_{*} : \mathcal{M}_{*,*}(X,Y)^+ \to \mathcal{A}b \]
by

- for an object \( X \), \( \mathcal{H}^c_{*}(X) = H_*(X) \otimes \mathbb{Q} \), the Borel–Moore homology with rational coefficients,
- for \( \sum_i n_i [X \xleftarrow{p_i} V_i \xrightarrow{s_i} Y; E_i] \in hom_{\mathcal{M}_{*,*}(\mathcal{Y})^+}(X,Y), \)
\[ \mathcal{H}^c_{*} \left( \sum_i n_i [X \xleftarrow{p_i} V_i \xrightarrow{s_i} Y; E_i] \right) \]
\[ := \sum_i n_i (p_i)_* (ch(E_i) \cap (s_i)^*) : \mathcal{H}^c_{*}(Y) \to \mathcal{H}^c_{*}(X). \]

Then \( \mathcal{H}^c_{*} : \mathcal{M}_{*,*}(\mathcal{Y})^+ \to \mathcal{A}b \) is a functor in the sense that \( \mathcal{H}^c_{*}(\alpha \circ \otimes \beta) = \mathcal{H}^c_{*}(\alpha) \circ \mathcal{H}^c_{*}(\beta). \)

Since a smooth map \( s : V \to Y \) has the relative tangent bundle \( T_s \), we can make another functor as follows.
Proposition 5.15. For multiplicative characteristic classes \( c_1, c_2 \) (with coefficients in a ring \( \Lambda \)) of complex algebraic vector bundles, define \( \mathcal{H}^{c_1, c_2}_* : \mathcal{M}^{c_1, c_2}_*(\mathcal{Y})^+ \rightarrow \mathcal{A}b \) by

- for an object \( X, \mathcal{H}^{c_1, c_2}_*(X) := H_*(X) \otimes \Lambda \), the Borel–Moore homology with coefficients in \( \Lambda \),
- for \( \sum_i n_i [X \xrightarrow{p_i} V_i \xrightarrow{s_i} Y; E_i] \in \text{hom}_{\mathcal{M}^{c_1, c_2}_*(\mathcal{Y})^+}(X, Y) \),

\[
\mathcal{H}^{c_1, c_2}_*(\sum_i n_i [X \xrightarrow{p_i} V_i \xrightarrow{s_i} Y; E_i]) := \sum_i n_i (p_i)_*(c_1(T_{s_i}) \cap c_2(E_i) \cap (s_i)^*) : \mathcal{H}^{c_1, c_2}_*(Y) \rightarrow \mathcal{H}^{c_1, c_2}_*(X).
\]

Then \( \mathcal{H}^{c_1, c_2}_* : \mathcal{M}^{c_1, c_2}_*(\mathcal{Y})^+ \rightarrow \mathcal{A}b \) is a functor in the sense that

\[
\mathcal{H}^{c_1, c_2}_*(\alpha \circ \beta) = \mathcal{H}^{c_1, c_2}_*(\alpha) \circ \mathcal{H}^{c_1, c_2}_*(\beta).
\]

Proof. It suffices to show that

\[
\mathcal{H}^{c_1, c_2}_*[X \xrightarrow{p} V \xrightarrow{s} Y; E] \circ \mathcal{H}^{c_1, c_2}_*[Y \xrightarrow{q} W \xrightarrow{t} Z; F] = \mathcal{H}^{c_1, c_2}_*[X \xrightarrow{p \circ q} V \times Y W \xrightarrow{t \circ s} Z; \widetilde{F} \oplus \widetilde{F}]
\]

where \( \widetilde{F} \) is multiplicative, thus

\[
\mathcal{H}^{c_1, c_2}_*[X \xrightarrow{p \circ q} V \times Y W \xrightarrow{t \circ s} Z; \widetilde{F} \oplus \widetilde{F}] = \mathcal{H}^{c_1, c_2}_*[X \xrightarrow{p \circ q} V \times Y W \xrightarrow{t \circ s} Z; T]
\]

and \( T \) is also multiplicative.

The proof is the same as that of Proposition 5.13, but we write it down:

\[
\mathcal{H}^{c_1, c_2}_*[X \xrightarrow{p} V \xrightarrow{s} Y; E] \circ \mathcal{H}^{c_1, c_2}_*[Y \xrightarrow{q} W \xrightarrow{t} Z; F] = \mathcal{H}^{c_1, c_2}_*[X \xrightarrow{p \circ q} V \times Y W \xrightarrow{t \circ s} Z; T]
\]

(by the projection formula)

\[
= p_* ((c_1(T_s) \cup \tilde{s}^* c_1(T_i)) \cap (\tilde{q}^* c_2(E) \cup \tilde{s}^* c_2(F)) \cap (\tilde{s}^* \circ t^*)
\]

(since \( T_s = \tilde{q}^* T_s \) and \( c_2 \) is also multiplicative)

\[
= p_* (\tilde{q}^* c_1(T_s) \cap \tilde{s}^* c_1(T_i) \cap \tilde{q}^* c_2(E) \cap \tilde{s}^* c_2(F) \cap (\tilde{s}^* \circ t^*)
\]

= \( \tilde{q}_* (\tilde{q}^* c_1(T_s) \cup \tilde{s}^* c_1(T_i)) \cap (\tilde{q}^* c_2(E) \cup \tilde{s}^* c_2(F)) \cap (\tilde{s}^* \circ t^*)
\]

(by the projection formula)

\[
= \tilde{q}_* (\tilde{q}^* (c_1(T_s) \cup c_2(E)) \cap (\tilde{q}^* c_2(E) \cup \tilde{s}^* c_2(F)) \cap (\tilde{s}^* \circ t^*)
\]

(since \( \tilde{q}_* \tilde{s}^* = s^* q_* \))
For the Todd class

• define

\[ H \]

Then

\[ X \] for an object

\[ \sum \] for \( c_\ell, c \), \( H \)

Corollary 5.16

As a corollary of the above proof we get the following for \( M_{m,r}(X,Y)_\otimes^+ \):

**Corollary 5.16.** For a multiplicative characteristic class \( c \) (with rational coefficients) of complex vector bundles and the Chern character \( ch \), define \( H_{*}^{c,f,ch} : M_{*,*}(V)_\otimes^+ \to Ab \) by

- for an object \( X \), \( H_{*}^{c,f,ch}(X) := H_*(X) \otimes \mathbb{Q} \), the Borel–Moore homology with rational coefficients,
- for \( \sum_i n_i[X \xrightarrow{p_i} V_i \xrightarrow{s_i} Y ; E_i] \in \text{hom}_{M_{*,*}(V)_\otimes^+}(X,Y) \),

\[ H_{*}^{c,f,ch} \left( \sum_i n_i[X \xrightarrow{p_i} V_i \xrightarrow{s_i} Y ; E_i] \right) \]

\[ := \sum_i n_i(p_i)_*(c(T_{s_i}) \cap ch(E_i) \cap (s_i)^*) : H_{*}^{c,f,ch}(Y) \to H_{*}^{c,f,ch}(X). \]

Then \( H_{*}^{c,f,ch} : M_{*,*}(V)^+_\otimes \to Ab \) is a functor in the sense that

\[ H_{*}^{c,f,ch}(\alpha \circ \beta) = H_{*}^{c,f,ch}(\alpha) \circ H_{*}^{c,f,ch}(\beta). \]

**Theorem 5.17.**

1. Define \( G^\otimes_0 : M_{*,*}(V)_\otimes^+ \to Ab \) by

   - for an object \( X \), \( G^\otimes_0(X) := G_0(X) \),
   - for \( \sum_i n_i[X \xrightarrow{p_i} V_i \xrightarrow{s_i} Y ; E_i] \in \text{hom}_{M_{*,*}(V)_\otimes^+}(X,Y) := M_{*,*}(X,Y)_\otimes^+ \),

\[ G^\otimes_0 \left( \sum_i n_i[X \xrightarrow{p_i} V_i \xrightarrow{s_i} Y ; E_i] \right) \]

\[ := \sum_i n_i(p_i)_*([E_i] \otimes (s_i)^*) : G^\otimes_0(Y) \to G^\otimes_0(X). \]

Then \( G^\otimes_0 : M_{*,*}(V)_\otimes^+ \to Ab \) is a functor in the sense that

\[ G^\otimes_0(\alpha \circ \beta) = G^\otimes_0(\alpha) \circ G^\otimes_0(\beta). \]

2. For the Todd class \( td \) and the Chern character \( ch \), define \( H_{*}^{td,ch} : M_{*,*}(V)^+_\otimes \to Ab \) by

   - for an object \( X \), \( H_{*}^{td,ch}(X) := H_*(X) \otimes \mathbb{Q} \),
   - for \( \sum_i n_i[X \xleftarrow{p_i} V_i \xrightarrow{s_i} Y ; E_i] \in \text{hom}_{M_{*,*}(V)^+_\otimes}(X,Y) = M_{*,*}(X,Y)^+_\otimes \),

\[ H_{*}^{td,ch} \left( \sum_i n_i[X \xleftarrow{p_i} V_i \xrightarrow{s_i} Y ; E_i] \right) \]

\[ := \sum_i n_i p_i_*(td(T_{s_i}) \cap ch(E_i) \cap (s_i)^*) : H_{*}^{td,ch}(Y) \to H_{*}^{td,ch}(X). \]
Then $\mathcal{H}^{td, ch}_* : \mathcal{M}_*(\mathcal{V})^+ \to \mathcal{A}b$ is a functor in the sense that

$$\mathcal{H}^{td, ch}_*(\alpha \circ \otimes \beta) = \mathcal{H}_{id, ch}(\alpha) \circ \mathcal{H}_{id, ch}(\beta).$$

(3) Baum–Fulton–MacPherson’s Todd class transformation $td_{BFM}^*$ gives rise to a natural transformation of the functors $G^* \otimes, \mathcal{H}^{td, ch}_* : \mathcal{M}_*(\mathcal{V})^+ \to \mathcal{A}b$:

$$td_{BFM}^* : G^* \otimes (-) \to \mathcal{H}^{td, ch}_*(-).$$

**Proof.** It suffices to show that $td_{BFM}^* : G^* \to \mathcal{H}^{td, ch}_*$ is a natural transformation, i.e., the following diagram commutes for $(X \xleftarrow{p} V \xrightarrow{s} Y; E)$:

$$\begin{array}{ccc}
G_0(X) & \xleftarrow{p_*} & G_0(V) & \xleftarrow{[E] \otimes s^*} & G_0(Y) \\
H_*(X) \otimes \mathbb{Q} & \xleftarrow{p_*} & H_*(V) \otimes \mathbb{Q} & \xleftarrow{td(T_s) \cap ch(E) \cap s^*} & H_*(Y) \otimes \mathbb{Q}
\end{array}$$

In other words, it suffices to show the commutativity of the right square, i.e., for $\theta \in G_0(Y)$,

$$td_{BFM}^*([E] \otimes s^*(\theta)) = td(T_s) \cap ch(E) \cap s^*(td_{BFM}^*(\theta)).$$

It follows from [5, Theorem, p. 119] (see also [12, Theorem 18.2, (2) Module]) that for any class $\beta \in K^0(V)$, the $K$-theory of complex algebraic vector bundles, and any $\alpha \in G_0(V)$, we have

$$(5.18) \quad td_{BFM}^*(\beta \otimes \alpha) = ch(\beta) \cap td_{BFM}^*(\alpha).$$

Hence

$$td_{BFM}^*([E] \otimes s^*(\theta))$$

$$= ch(E) \cap td_{BFM}^*(s^*(\theta)) \quad \text{(by (5.18))}$$

$$= ch(E) \cap td(T_s) \cap s^*(td_{BFM}^*(\theta)) \quad \text{(by the Verdier–Riemann–Roch formula)}$$

$$= td(T_s) \cap ch(E) \cap s^*(td_{BFM}^*(\theta)).$$

**Remark 5.19.** The above natural transformation $td_{BFM}^* : G^* \to \mathcal{H}^{td, ch}_*$ is an $\mathcal{M}_*(\mathcal{V})^+$-version of the natural transformation $td_{BFM}^* : G^*(-) \to \mathcal{H}^{Todd}_*(-)$ of the functors $G^*, \mathcal{H}^{Todd}_* : Corr^{+}_\text{pro-sm} \to \mathcal{A}b$.

We define the operations of pushforward and pullback of cobordism bicycles of vector bundles for later use. We can of course discuss plausible or natural relations among the operations of product, pushforward and pullback of cobordism bicycles of complex algebraic vector bundles, but they are treated in [35].
**Definition 5.20.**

(1) (Pushforward)

(a) For a **proper** map $f : X \to X'$, $f_* : \mathcal{M}_{m,r}(X,Y)^+ \to \mathcal{M}_{m,r}(X',Y)^+$ is defined by

$$f_*([X \xleftarrow{p} V \xrightarrow{s} Y; E]) := [X' \xleftarrow{f_{\text{op}}} V \xrightarrow{s} Y; E].$$

(b) For a **smooth** map $g : Y \to Y'$,

$$g_* : \mathcal{M}_{m,r}(X,Y)^+ \to \mathcal{M}_{m+\dim g,r}(X,Y')^+$$

is defined by

$$g_*([X \xleftarrow{p} V \xrightarrow{s} Y; E]) := [X \xleftarrow{p} V \xrightarrow{g \circ s} Y'; E].$$

(Note that $m = \dim s$ and $\dim(g \circ s) = \dim s + \dim g = m + \dim g$.)

(2) (Pullback)

(a) For a **smooth** map $f : X' \to X$,

$$f^* : \mathcal{M}_{m,r}(X,Y)^+ \to \mathcal{M}_{m+\dim f,r}(X',Y)^+$$

is defined by

$$f^*([X \xleftarrow{p} V \xrightarrow{s} Y; E]) := [X' \xleftarrow{p'} X' \times_X V \xrightarrow{s \circ f'} Y; (f')^* E].$$

Here we consider the following commutative diagram:

$$\begin{array}{ccc}
(f')^* E & \longrightarrow & E \\
\downarrow & & \downarrow \\
X' \times_X V & \longrightarrow & V \\
\uparrow_{p'} & & \uparrow_s \\
X' & \longrightarrow & X \\
\downarrow_f & & \downarrow_p \\
X & \longrightarrow & Y
\end{array}$$

(Note that the left diamond is a fiber square, thus $f' : X' \times_X V \to V$ is smooth and $p' : X' \times_X V \to X'$ is proper. Note that $\dim f' = \dim f$ and $\dim(s \circ f') = \dim s + \dim f' = m + \dim f$.)

(b) For a **proper** map $g : Y' \to Y$,

$$g^* : \mathcal{M}_{m,r}(X,Y)^+ \to \mathcal{M}_{m,r}(X,Y')^+$$

is defined by

$$g^*([X \xleftarrow{p} V \xrightarrow{s} Y; E]) := [X \xleftarrow{p \circ g'} V \times_Y Y' \xrightarrow{s'} Y'; (g')^* E].$$
Here we consider the following commutative diagram:

\[ E \leftarrow (g')^*E \]

\[ \downarrow \quad \downarrow \]

\[ V \leftarrow V \times_Y Y' \]

\[ \begin{array}{c}
\xymatrix{
X & V \\
& Y \ar[l]_p \ar[r]^s & Y' \\
& X \ar[l]^{g'} & (g')^*E \\
& V \ar[l]_g & (g')^*E \\
}
\end{array} \]

(Note that the right diamond is a fiber square, thus \( s' : V \times_Y Y' \to Y' \) is smooth and \( g' : V \times_Y Y' \to V \) is proper, and \( \dim s = \dim s' \).)

**Remark 5.21.** When we deal with a smooth map \( f \) or \( g \), both in pushforward and pullback, the first grading is incremented by the relative dimension \( \dim f \) or \( \dim g \), but when we deal with proper maps, the first grading is unchanged. In both pushforward and pullback, the second grading (referring to the dimension of vector bundle) is unchanged.

Let \( f : X \to Y \) be a proper and smooth map. Then we have the pushforward \( f_* : \mathcal{M}_{m,r}(X, X)^+ \to \mathcal{M}_{m,r}(Y, X)^+ \) by properness and the pushforward \( f_* : \mathcal{M}_{m,r}(Y, X)^+ \to \mathcal{M}_{m+\dim f, r}(Y, Y)^+ \) by smoothness. The composition \( f_* \circ f_* : \mathcal{M}_{m,r}(X, X)^+ \to \mathcal{M}_{m+\dim f, r}(Y, Y)^+ \) is a pushforward, denoted by \( f_* \); we have \( f_*([X \xleftarrow{p} V \xrightarrow{s} X; E]) = [Y \xleftarrow{f_* \circ p} V \xrightarrow{f_* \circ s} Y; E] \). This is clearly covariantly functorial for proper and smooth maps.

Similarly, by properness we have the pullback \( f^* : \mathcal{M}_{m,r}(Y, Y)^+ \to \mathcal{M}_{m,r}(Y, X)^+ \) and by smoothness we have the pullback \( f^* : \mathcal{M}_{m,r}(Y, X)^+ \to \mathcal{M}_{m+\dim f, r}(Y, Y)^+ \). The composition of these two pullbacks \( f^* \circ f^* : \mathcal{M}_{m,r}(X, X)^+ \to \mathcal{M}_{m,r}(Y, Y)^+ \) is a pullback, denoted by \( f^{**} \); we have \( f^{**}([X \xleftarrow{p} V \xrightarrow{s} X; E]) = [X \xleftarrow{f^* \circ f} V' \times_Y V'' \xrightarrow{s' \circ \tilde{f}} X; (\tilde{f} \times_Y f')^*E] \). Here we consider the following fiber squares:
Note that $X \times_Y V$ on the left is $X \times_{f=p} V$, and $V \times_Y X$ on the right is $X \times_{s=f} V$, so they are different. Hence $(X \times_Y V) \times_Y (V \times_Y X)$ is $(X \times_{f=p} V) \times_Y (X \times_{s=f} V)$.

**Proposition 5.22.** Let $f : X \to Y$ be a proper and smooth morphism and let $c^{\ell_1}, c^{\ell_2}$ be multiplicative characteristic classes of complex algebraic vector bundles. Then we have the following commutative diagrams:

\begin{align*}
\mathcal{M}_{m,r}(X, X)^+ & \xrightarrow{f^*} \mathcal{M}_{m+\dim f, r}(Y, Y)^+ \\
\mathcal{H}_{c^{\ell_1}, c^{\ell_2}} & \xrightarrow{H^*} \mathcal{H}_{c^{\ell_1}, c^{\ell_2}}
\end{align*}

where $f^* : \text{Hom}(H_*(X), H_*(X)) \to \text{Hom}(H_*(Y), H_*(Y))$ is defined by

\[ f^*(H) := f_* \circ H \circ (c^{\ell_1}(T_f) \cap f^*) \]

and

\begin{align*}
\mathcal{M}_{m,r}(Y, Y)^+ & \xrightarrow{f^{**}} \mathcal{M}_{m+\dim f, r}(X, X)^+ \\
\mathcal{H}_{c^{\ell_1}, c^{\ell_2}} & \xrightarrow{H^*} \mathcal{H}_{c^{\ell_1}, c^{\ell_2}}
\end{align*}

where $f^{**} : \text{Hom}(H_*(Y), H_*(Y)) \to \text{Hom}(H_*(X), H_*(X))$ is defined by

\[ f^{**}(H) := (c^{\ell_1}(T_f) \cap f^*) \circ H \circ f_* \]

**Proof.** We just show (2). Let $[Y \xrightarrow{p} V \xrightarrow{s} Y; E] \in \mathcal{M}_{m,r}(Y, Y)^+$. Then

\[ \mathcal{H}_{c^{\ell_1}, c^{\ell_2}}(f^{**}([Y \xrightarrow{p} V \xrightarrow{s} Y; E])) \]

\[ = \mathcal{H}_{c^{\ell_1}, c^{\ell_2}}(X \xleftarrow{p' \circ f''} V' \times_Y V'' \xrightarrow{s' \circ \tilde{f}} X; (\tilde{f} \times_Y f')^* E) \]

\[ = (p' \circ f'')_* (c^{\ell_1}(T_{s' \circ \tilde{f}}) \cap c^{\ell_2}((\tilde{f} \times_Y f')^* E) \cap (s' \circ \tilde{f})^*) \]

\[ = (p')_* (f'')_* (c^{\ell_1}((f'')^* T_{s' \circ \tilde{f}}) \cap c^{\ell_2}((\tilde{f} \times_Y f')^* E) \cap (s' \circ \tilde{f})^*) \]

\[ = (p')_* (f'')_* (c^{\ell_1}(T_{s' \circ \tilde{f}}) \cap c^{\ell_2}(f' \circ \tilde{f})^* E) \cap (s' \circ \tilde{f})^*) \]

\[ = (p')_* (f'')_* (c^{\ell_1}(T_{s' \circ \tilde{f}}) \cap (\tilde{f})^* (f')^* c^{\ell_2}(E) \cap \tilde{f}^* (s')^*) \]

\[ = (p')_* (c^{\ell_1}(T_{s' \circ \tilde{f}}) \cap (\tilde{f})^* (f')^* c^{\ell_2}(E) \cap (s')^*) \]

(by the projection formula)

\[ = (p')_* (c^{\ell_1}(T_{s' \circ \tilde{f}}) \cap (\tilde{f})^* (f')^* c^{\ell_2}(E) \cap (s')^*) \]
Therefore

\[
\begin{align*}
\text{(since } c\ell_1 \text{ is multiplicative, thus } &c\ell_1(T_{s_0 f}) = c\ell_1(T_f) \cup c\ell_1((\hat{f})^* T_s)) \\
= (p')_*((\hat{f})^* (c\ell_2(E) \cap f'_*(s')^*)) \\
&\text{(since } T_\hat{f} = (p')^* T_f) \\
= c\ell_1(T_f) \cap (p')_*(\hat{f})^* (c\ell_1(T_s) \cap c\ell_2(E) \cap f'_*(s')^*) \\
&\text{(by the projection formula)}
\end{align*}
\]

\[
= c\ell_1(T_f) \cap f^* p_* (c\ell_1(T_s) \cap c\ell_2(E) \cap s^* f_*)
\]

\[
= c\ell_1(T_f) \cap f^* (p_*(c\ell_1(T_s) \cap c\ell_2(E) \cap s^*)) f_*
\]

\[
= f^{**}(\mathcal{H}_{c\ell_1, c\ell_2}([Y \overset{\hat{s}}{\leadsto} V \wedge Y; E])).
\]

Therefore \(\mathcal{H}^{c\ell_1, c\ell_2} \circ f^{**} = f^{**} \circ \mathcal{H}^{c\ell_1, c\ell_2}\).

**Remark 5.23.** Finally, we remark that given a cobordism bicycle of a vector bundle \([X \overset{fb^*}{\leadsto} V \overset{s^*}{\leadsto} Y; E]\), we can consider a canonical functor of Fourier–Mukai type on derived categories of coherent sheaves. Let \(D^b \text{Coh}(X)\) denote the derived category of bounded complexes of coherent sheaves on \(X\). Then we have the following functor of Fourier–Mukai:

\[
\mathcal{H}([X \overset{fb^*}{\leadsto} V \overset{s^*}{\leadsto} Y; E]) := p_*([T_s] \otimes [E] \otimes s^*) : D^b \text{Coh}(Y) \to D^b \text{Coh}(X).
\]

Here a vector bundle is considered as a locally free sheaf, thus a coherent sheaf. We will treat this aspect in another paper. Here we just remark that the \(D^b \text{Coh}\)-analogue of Proposition 5.22 is as follows:

\[
\begin{array}{ccc}
\mathcal{M}_{m,r}(X, X)^+ & \xrightarrow{f^{**}} & \mathcal{M}_{m+\dim f, r}(Y, Y)^+ \\
\mathcal{H} & \downarrow & \mathcal{H} \\
\text{Functor}(D^b \text{Coh}(X), D^b \text{Coh}(X)) & \xrightarrow{f^{**}} & \text{Functor}(D^b \text{Coh}(Y), D^b \text{Coh}(Y))
\end{array}
\]

where\( f^{**} : \text{Functor}(D^b \text{Coh}(X), D^b \text{Coh}(X)) \to \text{Functor}(D^b \text{Coh}(Y), D^b \text{Coh}(Y))\) is defined by

\[
f^{**}(\mathcal{H}) := f_* \circ \mathcal{H} \circ ([T_f] \otimes f^*), \quad \mathcal{H} \in \text{Functor}(D^b \text{Coh}(X), D^b \text{Coh}(X));
\]

and

\[
\begin{array}{ccc}
\mathcal{M}_{m,r}(Y, Y)^+ & \xrightarrow{f^{**}} & \mathcal{M}_{m+\dim f, r}(X, X)^+ \\
\mathcal{H} & \downarrow & \mathcal{H} \\
\text{Functor}(D^b \text{Coh}(Y), D^b \text{Coh}(Y)) & \xrightarrow{f^{**}} & \text{Functor}(D^b \text{Coh}(X), D^b \text{Coh}(X))
\end{array}
\]
where
\( f^* : \text{Functor}(\mathcal{D} \mathcal{b} \text{Coh}(Y), \mathcal{D} \mathcal{b} \text{Coh}(Y)) \to \text{Functor}(\mathcal{D} \mathcal{b} \text{Coh}(X), \mathcal{D} \mathcal{b} \text{Coh}(X)) \)
is defined by
\[
 f^*(\mathcal{H}) := ([T_f] \otimes f^*) \circ \mathcal{H} \circ f_* , \quad \mathcal{H} \in \text{Functor}(\mathcal{D} \mathcal{b} \text{Coh}(Y), \mathcal{D} \mathcal{b} \text{Coh}(Y)).
\]

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Shoji Yokura
Graduate School of Science and Engineering
Kagoshima University
21-35 Korimoto 1-chome
Kagoshima 890-0065, Japan
E-mail: yokura@sci.kagoshima-u.ac.jp