Noise in refrigerating tunnel junctions and in microbolometers

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Microrefrigerators based on normal metal–insulator–superconductor (NIS) junctions represent a very attractive alternative to cool the microbolometers and calorimeters for astrophysical observations in space-borne experiments. The performance in such measurements requires a good knowledge of the noise sources in the detectors. In this paper we present detailed calculations of the thermal fluctuations and of the noise equivalent power due to the heat transfer through the NIS junctions or due to the thermal contact between different subsystems of the detector. The influence of the background radiation will also be evaluated. Analytical approximations, valid at low temperatures, are given.

I. INTRODUCTION

The most sensitive detectors, suitable for observation of cosmic sources in far-infrared and X-ray bands, are the cryogenic bolometers and microcalorimeters. These detectors operate typically at a temperature of about 100 mK. An attractive method for the last stage of detector cooling (from 300 mK to 100 mK) in space-borne experiments is based on normal metal–insulator–superconductor (NIS) refrigerating junctions [1,2]. The sensitivity required by such experiments motivates the study of the noise sources in the detectors.

A thermal detector system, such as a bolometer or a calorimeter, consists of a thermal sensing element (TSE) which is connected to a heat sink. The TSE typically consists of an absorber and a thermometer. The thermometer can be a transition-edge sensor [3–5], or a (SI)NIS tunnel junction thermometer [6,7]. In the rest of this paper we shall concentrate on the junction where the flux of electrons is oriented from the normal metal into the superconductor, where

\[ \theta(\epsilon) = \frac{1}{e^2} \left( \epsilon - \Delta \right)^2 \]

is the normalized density of states in the superconductor, where \( \theta \) is an effective temperature in the superconductor, used to describe the population of the quasiparticle energy levels. V is the voltage across the junction, e is the elementary charge, and \( g(\epsilon) = \theta(\epsilon) \Delta^2 / (\sqrt{\epsilon^2 - \Delta^2}) \) is the normalized density of states in the superconductor, where \( \theta \) is the Heaviside step function. The energy \( \epsilon \) is measured from the Fermi energy in the superconductor and in Eqs. [1] it is always taken in absolute value. Although \( f_{\text{es}} \) may not be Fermi distributions in our case of nonequilibrium [12] we make the assumption that \( 1 - f_{\text{es}}(\epsilon, T_c) = f_{\text{es}}(\epsilon, T_c) \) (which is an identity for a Fermi distribution) to transform the expressions that involved negative \( \epsilon \). In what follows we shall concentrate on the junction where the flux of electrons is oriented from the normal metal into the superconductor, where \( eV \) is positive. Using the definitions given in Eqs. [1], the particle and excitation fluxes, \( \dot{N}_J \) and \( \dot{N}'_J \),

where \( f_{\text{es}}(\epsilon, T_c) \) represent the populations of the electron (in the normal metal) and quasiparticle (in the superconductor) energy levels. \( T_c \) is an effective temperature in the superconductor, used to describe the population of the quasiparticle energy levels. V is the voltage across the junction, e is the elementary charge, and \( g(\epsilon) = \theta(\epsilon) \Delta^2 / (\sqrt{\epsilon^2 - \Delta^2}) \) is the normalized density of states in the superconductor, where \( \theta \) is the Heaviside step function. The energy \( \epsilon \) is measured from the Fermi energy in the superconductor and in Eqs. [1] it is always taken in absolute value. Although \( f_{\text{es}} \) may not be Fermi distributions in our case of nonequilibrium [12] we make the assumption that \( 1 - f_{\text{es}}(\epsilon, T_c) = f_{\text{es}}(\epsilon, T_c) \) (which is an identity for a Fermi distribution) to transform the expressions that involved negative \( \epsilon \). In what follows we shall concentrate on the junction where the flux of electrons is oriented from the normal metal into the superconductor, where \( eV \) is positive. Using the definitions given in Eqs. [1], the particle and excitation fluxes, \( \dot{N}_J \) and \( \dot{N}'_J \),

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respectively, can be written as \[\text{(10)}\]

\[
\hat{N}_j = \frac{1}{e} I_e = \int_{\Delta}^{\infty} (j_1 - j_2 - j_3 + j_4) \, \text{d}\epsilon
\]

\[
= \frac{1}{e^2 R_T} \int_{\Delta}^{\infty} g(\epsilon) [f_e(\epsilon - eV, T_e) - f_e(\epsilon + eV, T_e)] \, \text{d}\epsilon,
\]

\[
\hat{N}'_j = \int_{\Delta}^{\infty} (j_1 + j_2 - j_3 - j_4) \, \text{d}\epsilon
\]

\[
= \frac{1}{e^2 R_T} \int_{\Delta}^{\infty} g(\epsilon) [f_e(\epsilon - eV, T_e) + f_e(\epsilon + eV, T_e) - 2f_e(\epsilon, T_s)] \, \text{d}\epsilon.
\]

If we denote by $\epsilon_F$ the Fermi energy of the ES in the TSE, then the total power extracted through a NIS junction can be written as

\[
\hat{Q}'_j = \int_{\Delta}^{\infty} [(\epsilon - eV + \epsilon_F)(j_1 - j_3) + (\epsilon_F - \epsilon - eV)(j_4 - j_2)] \, \text{d}\epsilon
\]

\[
= \hat{Q}_j + \epsilon_F \hat{N}_j,
\]

where

\[
\hat{Q}_j = \int_{\Delta}^{\infty} [(\epsilon - eV)(j_1 - j_3) - (\epsilon + eV)(j_4 - j_2)] \, \text{d}\epsilon
\]

\[
= \int_{\Delta}^{\infty} [\epsilon(j_1 + j_2 - j_3 - j_4) - eV(j_1 - j_2 - j_3 + j_4)] \, \text{d}\epsilon.
\]

Since the coolers consist usually of pairs of NIS junctions biased in opposite directions \[\text{(1)}\], the total power should be calculated as a sum of terms like the ones given in Eq. \[\text{(6)}\]. In a stationary case the total number of electrons in the TSE should be constant and the terms of the type $\epsilon_F \hat{N}_j$ cancel each other and we are left with the sum over $\hat{Q}_j$’s.

The spectral density of the shot noise in the power extracted through the NIS junction [given by Eq. \[\text{(4)}\]] can be written as:

\[
\langle \delta^2 \hat{Q}'_j \rangle_{\omega} = 2 \int_{\Delta}^{\infty} [(\epsilon - eV + \epsilon_F)^2(j_1 + j_3) + (\epsilon_F - \epsilon - eV)^2(j_4 + j_2)] \, \text{d}\epsilon
\]

\[
= \frac{2}{e^2 R_T} \int_{\Delta}^{\infty} \text{d}g(\epsilon) \left\{ (\epsilon - eV)^2 \right.
\]

\[
\times [f(\epsilon - eV, T_e)(1 - 2f(\epsilon, T_s)) + f(\epsilon, T_s)]
\]

\[
+ (\epsilon + eV)^2[f(\epsilon + eV, T_e)(1 - 2f(\epsilon, T_s)) + f(\epsilon, T_s)]
\]

\[
+ \frac{2}{e^2 R_T} \int_{\Delta}^{\infty} \text{d}g(\epsilon) \left[ f(\epsilon - eV, T_e)(1 - 2f(\epsilon, T_s))
\right.
\]

\[
+ f(\epsilon + eV, T_e)(1 - 2f(\epsilon, T_s)) + 2f(\epsilon, T_s)
\]

\[
\left. + \frac{4}{e^2 R_T \epsilon_F} \int_{\Delta}^{\infty} \text{d}g(\epsilon)(\epsilon - eV)
\times [f(\epsilon - eV, T_e)(1 - 2f(\epsilon, T_s)) + f(\epsilon, T_s)]
\right]
\]

\[
- (\epsilon + eV)[f(\epsilon + eV, T_e)(1 - 2f(\epsilon, T_s)) + f(\epsilon, T_s)]
\right\}
\]

(see \[\text{[13]}, \text{Chap. 1, for a general introduction}\]). In the construction of microbolometers the TSE is sufficiently large and the Coulomb blockade phenomenon cannot influence the tunneling of individual electrons. In such a case, in the calculation of the spectral density of the shot noise in the total power flux, the contributions of the terms \[\text{(6)}, \text{(7)}, \text{and (8)}\], corresponding to different junctions are simply added to each other. As a result, one would obtain a fluctuation in power which is many orders of magnitude larger than what was previously calculated \[\text{[14,15]}\] and which corresponds just to the term labeled as Eq. \[\text{(4)}\]. Nevertheless, we shall show next that the extra terms [(6) and (8)] have no effect on the observable quantities. The shot noise in the output power has no direct connection with the \text{NEP} of the detector, but the fluctuations in the power would produce fluctuations in the electronic temperature,
which depreciate the detection properties of the microbolometer. The NEP is then defined as the minimum power of the input signal that would produce a change in temperature equal to the square root of the mean square fluctuation of temperature.

II. NOISE IN THE THERMAL SENSING ELEMENT

A. Temperature fluctuations

The detection of radiation using microbolometers is based on the measurement of quantities which depend on the electronic temperature in the detector and, eventually, on the chemical potential of the electrons (as is the case of NIS junctions used as thermometers). Therefore, the noise in the total electronic energy is not directly connected to the figure of merit of the detector, which is the NEP.

At very low temperatures, the chemical potential of a Fermi system is related to the Fermi energy by the relation \( \mu \approx \varepsilon_F \left[ 1 - \left( \frac{\pi^2}{12} \right) \left( k_B T / \varepsilon_F \right)^2 \right] \), where we used some obvious notations. On the other hand, the Fermi energy is determined by the density of particles, \( n = N/V \), in the following way:

\[
 n = \frac{2 m_e}{\hbar^2} \left( \frac{3 \pi^2}{5} \right) \left( \frac{k_B T}{\varepsilon_F} \right)^{3/2}.
\]

Other quantities of interest are the energy per particle \( u \) and the specific heat at constant volume, \( c_V = \left( \frac{\partial u}{\partial T} \right)_{N,V=\text{constant}} \), which, in the low temperature limit have the expressions:

\[
 u = \frac{3}{5} \varepsilon_F \left[ 1 + \left( \frac{5 \pi^2}{12} \right) \left( k_B T / \varepsilon_F \right)^2 \right]
\]

and

\[
 c_V = \frac{\left( \frac{3}{2} \right) \pi^2}{2} \left( k_B T / \varepsilon_F \right). \]

Let us now calculate the change in the temperature of the electron gas when \( \delta N_e \) electrons of energy \( \epsilon + \varepsilon_F \) leave the normal metal. This loss of electrons changes \( u \) by an amount:

\[
 \delta u = \frac{U - \delta N_e (\epsilon + \varepsilon_F)}{N - \delta N_e} - \frac{U}{N} \approx u \frac{\delta N_e}{N} - \epsilon (\epsilon + \varepsilon_F) \frac{\delta N_e}{N}.
\]

which can be expressed as

\[
 \delta u = \frac{\partial u}{\partial T} \delta T + \frac{\partial u}{\partial \varepsilon_F} \delta \varepsilon_F.
\]

The change in the Fermi energy can be written as \( \delta \varepsilon_F = -(2/3) \varepsilon_F \delta N / N \). Using the two equations we have for \( \delta u \), Eqs. [9] and [10], and neglecting the terms of the order \( \alpha^{-2} \equiv (k_B T / \varepsilon_F)^2 \ll 1 \), we end up with an expression for the temperature variation:

\[
 c_V \delta T_e = -\epsilon \frac{\delta N_e}{N}.
\]

Here the subscript \( \epsilon \) in the notation \( \delta T_e \) refers to the fact that the variation in temperature is due to the loss of electrons from the level with energy \( \epsilon + \varepsilon_F \). In the linear approximation, the temperature change is an additive quantity. Taking the time derivative on both sides of Eq. [11], we can write:

\[
 c_V \delta \dot{T}_e = -\epsilon \frac{\delta \dot{N}_e}{N}.
\]

Using the expressions for the chemical potential, the internal energy and Eq. [11], it can be shown that the variation of \( \mu \) with temperature introduces corrections of order \( \alpha^{-2} \) to the first order terms, in all the following equations. Therefore in the rest of this paper we shall consider \( \mu = \varepsilon_F \).

B. Noise Equivalent Power

Having now calculated the variations in temperature due to the variations in the population of the energy levels, we can determine the total temperature fluctuation in the ES of the TSE. The system is represented schematically in Fig. 1. The electrons interact with the lattice, at temperature \( T_1 \), the detector lattice exchanges heat with a heat bath, at temperature \( T_2 \), and the thermal resistance between the lattice and the heat bath is the Kapitza resistance.
where, transforming the summations over the energy levels into integrals, we use the notation $\delta T$ replacing the expression for density of states as being almost constant). Calculating the Fourier transformations of the two equations above and

$\sigma$ heat capacities of the electron gas and of the lattice are for the time variation of the occupation number of the electronic level of energy $\epsilon$ (taken with changed sign sign, as we did in the previous section). The subscripts are used to specify the processes that caused the power transfer or the time variation of the particle population. In this way, $J$ refers to the particle transfer with changed sign sign, as we did in the previous section). The subscripts are used to specify the processes that

We have to explain here the notation. $\dot{Q}$ will be used for the power fluxes (with the directions given by the long arrows in Fig. 1) and $\dot{N}_e$ for the time variation of the occupation number of the electronic level of energy $\epsilon$ (taken with changed sign sign, as we did in the previous section). The subscripts are used to specify the processes that caused the power transfer or the time variation of the particle population. In this way, $J$ refers to the particle transfer through the NIS junction, $\varepsilon$ to the electron-phonon interaction in the TSE, $K$ to the heat transfer through the contact between the lattice of the TSE and the heat bath, while $\omega$ and $\alpha$ refer to the optical input power, due to the external radiation, into the ES and lattice of the TSE, respectively. By $\dot{Q}_b$, we denote the bias power, as shown in the figure, but this contribution will be disregarded until the end of the paper, where it will be discussed in connection with the uncertainty introduced in experiment by the measurement process. Moreover, we shall use the Greek letter $\delta$ in front of the notation to specify the fluctuations of a quantity and the subscript shot for the fluctuations of this quantity, due solely to the shot noise (discrete amounts transported randomly at a constant average rate).

If we add up all the noise terms into the power balance equation, disregarding the fluctuations of $T_2$ and $T_3$, we arrive at the following set of equations:

$$
\begin{align*}
C_{Ve} \delta T_e &= - \epsilon \left( \delta \dot{N}_{e,\text{shoot}} + \delta \dot{N}_{ep,\text{shoot}} + \delta \dot{N}_{con,\text{shoot}} \right) \\
&- \epsilon \frac{\partial (N_{e,\text{shoot}} + N_{ep,\text{shoot}} + N_{con,\text{shoot}})}{\partial T_e} \delta T_e - \epsilon \frac{\partial N_{e,\text{shoot}}}{\partial T_1} \delta T_1, \\
C_{V1} \delta T_1 &= - \delta \dot{Q}_{ep,\text{shoot}} + \delta \dot{Q}_{K,\text{shot}} + \delta \dot{Q}_{op,\text{shot}} \\
&- \frac{\partial (Q_{e,\text{shot}} - Q_{K,\text{shot}} - Q_{op,\text{shot}})}{\partial T_1} \delta T_1 - \frac{\partial Q_{e,\text{shot}}}{\partial T_e} \delta T_e
\end{align*}
$$

(13)

where, transforming the summations over the energy levels into integrals, we use the notation $\delta T_e \equiv \int d\epsilon \sigma_0 \delta T_e$, in which $\sigma_0$ is the density of the energy levels of the electrons in the normal metal, at Fermi energy (we make here the usual assumption that the energy range of interest is much smaller than the Fermi energy, so we can consider the density of states as being almost constant). Calculating the Fourier transformations of the two equations above and replacing the expression for $\delta T_1(\omega)$ obtained from the second equation into the first equation, we get

$$
\begin{align*}
\dot{\omega} C_{Ve} \delta T_e(\omega) + \delta T_e(\omega) \left[ \epsilon \frac{\partial N_{e,\text{shoot}}}{\partial T_1} \frac{\partial Q_{ep,\text{shot}}}{\partial T_e} \frac{1}{\dot{\omega} C_{V1} + \frac{\partial Q_{ep,\text{shot}} - Q_{op,\text{shot}}}{\partial T_1}} \right] &= \\
= - \epsilon \left[ \delta \dot{N}_{e,\text{shoot}}(\omega) + \delta \dot{N}_{ep,\text{shoot}}(\omega) + \delta \dot{N}_{con,\text{shoot}}(\omega) \right] - \epsilon \frac{\partial N_{e,\text{shoot}}}{\partial T_e} \frac{\partial \dot{Q}_{ep,\text{shoot}}}{\partial N} \frac{\delta \dot{N}_{e,\text{shoot}}(\omega)}{\dot{\omega}}
\end{align*}
$$
We introduce the notation:

\[ \Delta T_c(\omega) \equiv i\omega C_{Vc} \delta T_e(\omega) + \delta T_e(\omega) \left[ \frac{\partial (N_e + \delta N_{ep} + \delta N_{em})}{\partial T_e} \right] - \frac{\partial \dot{N}_{em}}{\partial T_1} - \frac{\partial \dot{Q}_{ep}}{\partial T_c} \left( \frac{1}{i\omega C_{Vc}} \right) \left( \frac{\partial \dot{Q}_{ep}}{\partial T_e} \right) \].

(14)

We can now integrate both sides of Eq. (13) and write

\[ \Delta T(\omega) = \delta T_e(\omega) \left[ i\omega C_{Vc} + \frac{\partial}{\partial T_e} \left( \dot{Q}_F - \dot{Q}_{ep} - \dot{Q}_{oe} \right) \right] + \frac{1}{i\omega C_{Vc} + \frac{\partial (Q_{ep} - Q_K - Q_{op})}{\partial T_1}} \left( \frac{\partial \dot{Q}_{ep}}{\partial T_e} \right). \]

(16)

Integrating the r.h.s. of Eq. (14) we obtain another expression for \( \Delta_T(\omega) \):

\[ \Delta_T(\omega) = -\delta \dot{Q}_{1,\text{shot}}(\omega) + \delta \dot{Q}_{ep,\text{shot}}(\omega) + \delta \dot{Q}_{oe,\text{shot}}(\omega) \]

\[ \frac{\partial \dot{Q}_{ep}}{\partial T_1} - \frac{\partial \dot{Q}_{ep,\text{shot}}(\omega) + \delta \dot{Q}_{K,\text{shot}}(\omega) + \delta \dot{Q}_{op,\text{shot}}(\omega)}{i\omega C_{Vc} + \frac{\partial (Q_{ep} - Q_K - Q_{op})}{\partial T_1}}. \]

(17)

Using Eqs. (16) and (17) we obtain the following expressions for the mean square value of \( \Delta_T(\omega) \):

\[ \langle \Delta_T^2(\omega) \rangle = \langle \delta^2 T_e(\omega) \rangle \left[ i\omega C_{Vc} + \frac{\partial}{\partial T_e} \left( \dot{Q}_F - \dot{Q}_{ep} - \dot{Q}_{oe} \right) \right]^2 \]

\[ + \frac{1}{i\omega C_{Vc} + \frac{\partial (Q_{ep} - Q_K - Q_{op})}{\partial T_1}} \left( \frac{\partial \dot{Q}_{ep}}{\partial T_e} \right)^2 \]

\[ = \langle \delta^2 \dot{Q}_{ep,\text{shot}}(\omega) \rangle \left[ 1 + \frac{\partial \dot{Q}_{ep}}{\partial T_1} \right] \left( \frac{1}{i\omega C_{Vc} + \frac{\partial (Q_{ep} - Q_K - Q_{op})}{\partial T_1}} \right)^2 \]

\[ + \langle \delta^2 \dot{Q}_{K,\text{shot}}(\omega) + \delta^2 \dot{Q}_{op,\text{shot}}(\omega) \rangle \left( \frac{\partial \dot{Q}_{ep}}{\partial T_1} \right)^2 \]

\[ \times \left\{ \omega^2 C_{Vc} \right\}^{-1} \left[ \frac{\partial (Q_{ep} - Q_K - Q_{op})}{\partial T_1} \right]^2 \]

\[ + \langle \delta^2 \dot{Q}_{oe,\text{shot}}(\omega) \rangle \]

\[ + \left( \langle \delta \dot{Q}_{1,\text{shot}}(\omega) \rangle + \frac{1}{\omega^2} \left( \frac{\partial \dot{Q}_F}{\partial N} \frac{\partial \dot{Q}_1}{\partial E} \right) \right) \left( \langle \delta^2 \dot{N}_{1,\text{shot}}(\omega) \rangle \right)^2. \]

(19)

The last term of Eq. (13), say \( \Upsilon(\omega) \), can be calculated. The correlation function between \( \delta \dot{Q}_{1,\text{shot}}(\omega) \) and \( \delta \dot{N}_{1,\text{shot}}(\omega)/i\omega \) gives no contribution due to the \( \pi/2 \) phase difference, and the final result is:

\[ \Upsilon(\omega) = \langle \delta^2 \dot{Q}_{1,\text{shot}}(\omega) \rangle + \frac{1}{\omega^2} \left( \frac{\partial \dot{Q}_F}{\partial N} \frac{\partial \dot{Q}_1}{\partial E} \right)^2 \langle \delta^2 \dot{N}_{1,\text{shot}}(\omega) \rangle. \]

(20)

From the expressions (18) and (14), using the result (20), one can calculate the spectral density of the temperature noise.
To determine the amplitude of the signal power that has to be introduced into the detector, to produce changes in the temperature of the ES equal to the fluctuations calculated above, we write a set of equations similar to (13):

\[ C_v \delta T_e = \dot{Q}_s - \dot{Q}_{J0} + \dot{Q}_{ep0} + \dot{Q}_{oe0} \]

\[ - \frac{\partial(\dot{Q}_J - \dot{Q}_{ep} - \dot{Q}_{oe})}{\partial T_e} \delta T_e + \frac{\partial \dot{Q}_{ep}}{\partial T_1} \delta T_1, \]

\[ C_V \delta T_1 = \dot{Q}_{K0} + \dot{Q}_{op0} - \dot{Q}_{ep0} + \frac{\partial(\dot{Q}_K + \dot{Q}_{op0} - \dot{Q}_{ep})}{\partial T_1} \delta T_1 \]

\[ - \frac{\partial \dot{Q}_{ep}}{\partial T_e} \delta T_e. \]  

(22)

Here \( \dot{Q}_s \) is the power of the input signal into the TSE. The subscript 0 is added to the usual subscripts to denote the equilibrium values of the powers defined in Fig. 1. Therefore, \(-\dot{Q}_{J0} + \dot{Q}_{ep0} + \dot{Q}_{oe0} = \dot{Q}_{K0} + \dot{Q}_{op0} - \dot{Q}_{ep} = 0\). Calculating the Fourier transformations of Eqs. (21) and (22) and replacing \( \delta T_1(\omega) \) from Eq. (22) into Eq. (21), we obtain the Noise Equivalent Power, as defined in the end of Section I:

\[ NEP^2 \equiv \langle |\dot{Q}_s(\omega)|^2\rangle = |\Delta_5^2(\omega)|. \]  

(23)

Therefore, \( NEP \) can be calculated directly from Eq. (19), with the use of Eq. (20). In this way we observe that when the detector is cooled indirectly (\( \dot{Q}_J \equiv 0 \) and \( T_2 \approx T_1 \approx T_e \), \( Y \equiv 0 \) and \( NEP \) is lower. We shall evaluate the total \( NEP \) in the next sections.

C. Calculation of the shot noise terms

a. Electron-phonon shot noise. The heat power transferred between the ES and the lattice of the TSE, \( \dot{Q}_{ep} \), due to the electron-phonon interaction, can be calculated using the formula

\[ \dot{Q}_{ep} = \Sigma_{ep}\Omega(T_e^5 - T_c^5), \]  

(24)

where \( \Omega \) is the volume of the TSE and \( \Sigma_{ep} \) is the electron-phonon coupling constant [10]. If, for example, the TSE is made of copper, then \( \Sigma_{ep} \approx 4 \text{nW/K}^2\mu\text{m}^3 \) [1]. Using the same model as in Ref. [16], the shot noise of \( \dot{Q}_{ep} \) has been evaluated in Ref. [14]. An approximative expression for this power shot noise, namely

\[ \langle \delta^2 \dot{Q}_{ep,\text{shot}}\rangle_\omega \approx 5\Sigma_{ep}\Omega(T_e^6 + T_1^6), \]  

(25)

has been used in Ref. [13] (although a factor of two was introduced by mistake there). The formula above is identical to the exact expression for \( T_e = T_1 \) and deviates from it with less than 2% for any \( T_e \leq T_1 \). Therefore we shall use Eq. (25) in the rest of this paper, since its accuracy is good enough for our purposes.

b. Kapitza shot noise. We assume that the dynamics of the detector lattice is well described by a three-dimensional distribution of acoustic phonons, with sound velocity \( v \) (the same for all three phonon modes). If the heat bath consists of a dielectric membrane, with the sound velocity \( v_m \), then the phonon flux that penetrates through the separation surface, of area \( S \), from the detector into the membrane has the expression:

\[ \dot{N}_K = \begin{cases} \frac{S}{4\pi^2} \left( \frac{v}{v_m} \right)^2 \left( \frac{k_B T}{\hbar v} \right)^3 \zeta(3)t, & \text{for } v \leq v_m, \\ \frac{S}{4\pi} \left( \frac{k_B T}{\hbar v} \right)^3 \zeta(3)t, & \text{for } v > v_m, \end{cases} \]  

(26)

where \( t \) is the transmission coefficient, related to the acoustic impedances of the detector lattice, \( Z \), and of the membrane, \( Z_m \), by the relation \( t = 4ZZ_m(Z + Z_m)^{-2} \) [17]. The energy flux from the heat bath into the detector is

\[ \dot{Q}_K = \frac{S}{4\pi^2} \left( \frac{v}{v_m} \right)^2 k_B T (T_2^4 - T_1^4) \equiv \Sigma_K S(T_2^4 - T_1^4), \]  

(27)

and the power shot noise due to the quantisation of the phonon energy is

\[ \langle \delta^2 \dot{Q}_{K,\text{shot}}\rangle_\omega = 8 \zeta(5) \zeta(4) \Sigma_K S(T_2^4 + T_1^4), \]  

(28)

which, when \( T_1 = T_2 \), assumes the form \( \langle \delta^2 \dot{Q}_{K,\text{shot}}\rangle_\omega = \left( \zeta(5)/\zeta(4) \right)4k_B T^2 G_K \approx 4k_B T^2 G_K \), where \( G_K \equiv |\partial \dot{Q}_K/\partial T_1|_{T_1=T_2} \) is the Kapitza conductance.
c. Optical input power shot noise. Let us suppose that the detector admits a frequency band \( \Delta f_o \) from the electromagnetic spectrum within which the quantum efficiency is unity. If we denote by \( n(\omega_o) \) the density of phonons with the angular frequency \( \omega_o \) in a spatial region around the detector, then the energy flux and the number flux of photons on the detector are \( \dot{Q}_o(\omega_o) = \hbar \omega_o cn(\omega_o) \Delta \omega_o / 4 \) and \( \dot{N}_o(\omega_o) = Scn(\omega_o) \Delta \omega_o / 4 \), respectively, where \( \omega_o \equiv 2 \pi f_o \), as usual, \( \epsilon \) is the velocity of light, and we assume that \( \Delta \omega_o \ll \omega_o \). The electromagnetic radiation interacts with the ES, therefore we shall take \( \dot{Q}_o(\omega_o) = \dot{Q}_{oN}(\omega_o) \) and \( \dot{Q}_{opt}(\omega_o) = 0 \). Since \( n(\omega_o) = (1/\pi^2 e^2) \omega_o^2 (exp (h \omega_o / k_B T) - 1)^{-1} \) (see for example Chap. 10 in Ref. [18]), where \( T \) is the temperature of the background radiation, and if we assume that each photon produces a \( \delta \) peak of power into the ES, we obtain the following expressions for the input power and the shot noise into the ES of the TSE, respectively:

\[
\dot{Q}_{oe}(\omega_o) = \frac{S}{4} \frac{\hbar}{\pi^2 e^2} \frac{\omega_o^3 \Delta \omega_o}{\exp (\hbar \omega_o / k_B T) - 1}
\]

and

\[
\langle \delta^2 \dot{Q}_{oe,\text{shot}}(\omega_o) \rangle = \frac{S}{2} \left( \frac{\hbar}{\pi e} \right)^2 \frac{\omega_o^4 \Delta \omega_o}{\exp (\hbar \omega_o / k_B T) - 1}.
\]

The radiation that influences the astrophysical measurements is the cosmic background radiation, which corresponds to \( T \equiv T_b \approx 3 \) K.

d. Shot noise in the junction. As anticipated in Section II, the shot noise fluctuation of the total power extracted through the NIS junction has no relevance to our problem. The calculations in Sections II A and II B showed in a rigorous manner that the quantity of interest is \( \dot{Q}_{J} \), given by Eq. (3). The shot noise fluctuation of this quantity, which enters directly into the calculation of NEP, has the expression

\[
\langle \delta^2 \dot{Q}_{J,\text{shot}} \rangle = \frac{2}{e^2 R_T} \int_{\Delta}^{\infty} d\epsilon \left( \epsilon - eV \right)^2 f(\epsilon - eV, T_s) \psi_{\epsilon}(\epsilon - eV, T_s)
\]

\[
\times f(\epsilon, T_s)(1 - 2f(\epsilon, T_s)) + f(\epsilon, T_s)]
\]

\[
+ (\epsilon + eV)^2 f(\epsilon + eV, T_s) \psi_{\epsilon}(\epsilon + eV, T_s)
\]

\[
\times f(\epsilon, T_s)(1 - 2f(\epsilon, T_s)) + f(\epsilon, T_s)]
\]

which is the term labeled by (3) in Section II. The fluctuation of the particle flux, \( \langle \delta^2 \dot{N}_{J,\text{shot}} \rangle \), that enters the expression of \( Y(\omega) \), can be calculated in a similar way as \( \langle \delta^2 \dot{Q}_{J,\text{shot}} \rangle \), with the result

\[
\langle \delta^2 \dot{N}_{J,\text{shot}} \rangle = \frac{2}{e^2 R_T} \int_{\Delta}^{\infty} d\epsilon \left[ f(\epsilon - eV, T_s) + f(\epsilon + eV, T_s) \right]
\]

\[
\times \left[ f(\epsilon + eV, T_s) - f(\epsilon, T_s) \right] + 2f(\epsilon, T_s)]
\]

We have written down explicitly all the equations needed for the calculation of NEP. We are now left with the task of calculating the parameters of the refrigerating junctions for specific working conditions. Since the formulae involved are rather complicated and the microbolometers work at low temperatures, we give below some useful analytical approximations, valid for this range of temperatures.

III. APPROXIMATIVE ANALYTICAL EXPRESSIONS FOR THE CALCULATION OF THE JUNCTIONS PARAMETERS

In order to calculate the efficiency of the microbolometer we have to know the parameters of the refrigerating junctions for given working conditions. The physical phenomena that take place in NIS microrefrigerators have been presented in a series of publications (see for example Refs. [20,1]). Yet, the formulae involved are rather complicated and, in the general case, have to be calculated numerically. The metal that had been used extensively as a superconductor in the construction of the NIS refrigerating junctions is Al, which has an energy gap \( \Delta \approx 200 \) \( \mu \)eV at temperatures much lower than the critical one. Fortunately, the temperature range of interest for us \( (T \leq 0.3 \) K) is well below the critical temperature and since in this range \( \Delta / k_B T \gg 1 \), we shall give here a general method to calculate analytical approximations for all the formulae presented in connection with the NIS junctions.
A. General formulae

In the previous sections we saw that all the formulae needed for the quantitative evaluation of the processes in NIS junctions [see Eqs. (2), (3), (5), (6), (7), (8), (31), and (32)] have the general form:

\[
(r) F_l^{(s)} = \frac{1 + \theta(s)}{e^{2RT}} \int_{\Delta}^\infty \frac{\epsilon}{\sqrt{\epsilon^2 - \Delta^2}} \left\{ (\epsilon - E)^l \left[ f(\epsilon - E, T_e)(1 - f(\epsilon, T_s)) + s(1 - f(\epsilon - E, T_e)f(\epsilon, T_s)) + r(\epsilon + E)^l \cdot f(\epsilon + E, T_e)f(\epsilon, T_s) \right] \right\} \]

\[
= \frac{1 + \theta(s)}{e^{2RT}} \int_{\Delta}^\infty \frac{\epsilon}{\sqrt{\epsilon^2 - \Delta^2}} \left\{ (\epsilon - E)^l \left[ f(\epsilon - E, T_e)f(\epsilon, T_s) - (1 + s)f(\epsilon - E, T_e)f(\epsilon, T_s) \right] + r(\epsilon + E)^l \cdot f(\epsilon + E, T_e)f(\epsilon, T_s) \right\} .
\]

The parameter \( l \) takes one of the values 0, 1, or 2, while \( r \) and \( s \) can be +1 or -1. The function \( \theta(s) \) is the Heaviside step function. For example, \((+)^l F_l^{(s)} \equiv (\delta^2 \hat{Q}_{1,\text{shot}})^l, (+) F_0^{(s)} \equiv (\delta^2 \hat{N}_{1,\text{shot}})^l, \) and \((+) F_1^{(s)} \equiv \hat{Q}_1. \) The integral in Eq. (34) splits up in an obvious way into six terms. We shall evaluate them one by one in the approximation

\[
A_e \equiv \frac{\Delta}{k_B T_e} \geq A_s \equiv \frac{\Delta}{k_B T_s} \gg 1.
\]

Other notations that we shall use are: \( \beta \equiv 1/k_B T, \) where any subscripts attached to \( T \) will be transferred to \( \beta, \delta \equiv \Delta - E, a_e \equiv \beta \delta, a_s \equiv \beta \delta, B_e \equiv \beta_\omega(\Delta + E) \gg 1, \) and \( B_s \equiv \beta_\omega(\Delta + E) \gg 1. \)

The analytical approximations are possible due to the exponential dependence of the Fermi functions on energy, and are based on the Taylor expansion of the function \( 1/\sqrt{1 + x}, \) around \( x = 0. \) If we also write \( (a + b)^l = a^l + la^{l-1}b + [l(l-1)/2]a^{l-2}b^2 \) (which is exact for \( l = 0, 1, 2), \) we can calculate general analytical expressions for the six integrals in Eq. (34). In what follows we shall be interested just in the two highest order terms in \( A_e, A_s, B_e, \) and \( B_s \) in each of the integrals mentioned. For each \( l = 0, 1, 2 \) these highest order terms can be extracted rigorously from the general formulae given below, but in some cases the remaining lower order terms do not have the complete form (therefore we advise the reader to recalculate these lower orders, if they are necessary).

Let us now start to calculate each of the integrals from Eq. (34). In these calculations, the values of the parameters \( r \) and \( s \) have no relevance, so they will be omitted. They should be introduced just in the end, when the expression of \((r) F_l^{(s)} \) is to be reconstructed. The first term is

\[
I_{11} \equiv \int_{\Delta}^\infty \frac{\epsilon}{\sqrt{\epsilon^2 - \Delta^2}} \left\{ A_e \left[ \Gamma(l + 1/2)g_{l+1/2}(a_e) \right] + \Gamma(l - 1/2)a_e g_{l-1/2}(a_e) + \frac{1}{2} \left[ a_e \cdot g_{l+1/2}(a_e) + a_e \cdot g_{l-1/2}(a_e) \right] \right\} ,
\]

where the functions \( g_l(\alpha) \) are the \( l \)th order polylogarithmic functions of argument \(-e^{-\alpha}, g_l(\alpha) \equiv [1/\Gamma(l)] \int_0^\infty t^{l-1}(e^{t+\alpha} - 1)^{-1} dt \) (see, for example, Refs. [21] and references therein for more details). In the second term:

\[
I_{21} \equiv \int_{\Delta}^\infty \frac{\epsilon}{\sqrt{\epsilon^2 - \Delta^2}} \left\{ A_s \left[ \Gamma(l + 1/2)g_{l+1/2}(A_s) \right] \right\} ,
\]
$$+ l \Gamma(l - 1/2) a_s g_{l-1/2}(A_s) + \frac{l(l-1)}{2} \Gamma(l - 3/2) a_s^2 g_{l-3/2}(A_s)$$

$$+ \frac{3}{4} \left\{ \Gamma(l + 3/2) g_{l+3/2}(A_s) + l \Gamma(l + 1/2) a_s g_{l+1/2}(A_s) + \frac{l(l-1)}{2} \Gamma(l - 1/2) a_s^2 g_{l-1/2}(A_s) \right\} ,$$

since $g_l(\alpha) = e^{-\alpha} [1 + O(e^{-\alpha})]$, for any $l$, if $\alpha \gg 1$, we shall use the approximation $g_l(A_s) = e^{-A_s}$, so:

$$I_{2i} = \frac{(k_B T_e)^{l+1}}{\sqrt{2 A_s}} e^{-A_s} \left\{ A_s \left[ \Gamma(l + 1/2) e^{-l} \right] + \frac{l(l-1)}{2} \Gamma(l - 3/2) a_s^2 \right\} .$$

Within the same approximation, the third term,

$$I_{3i} = 2 \int_\Delta \frac{d\epsilon}{\sqrt{\epsilon^2 - \Delta^2}} \frac{(\epsilon - E)^l}{\exp [\beta_c(\epsilon - E)] + 1} \exp (\beta_c \epsilon) + 1 ,$$

can be written in the form:

$$I_{3i} = 2 e^{-A_s} \frac{(k_B T_e)^{l+1}}{\sqrt{2 A_e}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(T_e)}{T_s^k} \left\{ A_e \left[ \Gamma(l + k + 1/2) g_{l+k+1/2}(A_e) \right] + \frac{l(l-1)}{2} \Gamma(l + k - 3/2) a_e^2 g_{l+k-3/2}(A_e) \right\} .$$

If $T_s \gg T_e$ we may just take the first term in the summation and obtain

$$I_{3i} \approx 2 e^{-\beta_c \Delta} I_{1i} .$$

This is a reasonable approximation, since for a typical case of $T_e \approx 0.1 \text{ K}$ and $T_s \approx 0.4 \text{ K} > T_2$ (9) $A_s \approx 6$ and $T_s/T_e \approx 4$. In such a case $I_{3i} \ll I_{1i}$ so there is no point in taking higher order terms in Eq. (36). If $T_s = T_e$, the infinite summations in Eq. (36) do not converge. In such a case we can make use of the general inequality $I_{3i} < 2 e^{-\beta_s \Delta} I_{1i}$, valid for any $T_s$ and eventually neglect this term, since in the cases of practical interest $A_e > 20$.

The fourth term is

$$I_{4i} = \int_\Delta \frac{d\epsilon}{\sqrt{\epsilon^2 - \Delta^2}} \frac{(\epsilon + E)^l}{\exp [\beta_c(\epsilon + E)] + 1}$$

$$= \frac{(k_B T_e)^{l+1}}{\sqrt{2 A_e}} \left\{ A_e \left[ \Gamma(l + 1/2) g_{l+1/2}(B_e) \right] + \frac{l(l-1)}{2} \Gamma(l - 3/2) B_e^2 g_{l-3/2}(B_e) \right\} .$$

$$\approx e^{-B_e} \frac{(k_B T_e)^{l+1}}{\sqrt{2 A_e}}$$

$$\times \left\{ A_e \left[ \Gamma(l + 1/2) + l \Gamma(l - 1/2) B_e + \frac{l(l-1)}{2} \Gamma(l - 3/2) B_e^2 \right] + \frac{3}{4} \left[ \Gamma(l + 3/2) + l \Gamma(l + 1/2) B_e + \frac{l(l-1)}{2} \Gamma(l - 1/2) B_e^2 \right] \right\} .$$
The fifth term is

\[
I_{5j} = \int_{\Delta} e_{\Delta}^{(\epsilon + E)^l} \frac{1}{\sqrt{\epsilon^2 - \Delta^2}} \exp(\beta \epsilon) + 1
\]

\[
= \frac{(k_B T)^{l+1}}{\sqrt{2A_e}} \left\{ A_e \left[ \Gamma(l + 1/2)g_{l+1/2}(A_e) + \frac{l(l-1)}{2} \Gamma(l - 3/2)B_e^2 g_{l-3/2}(A_e) \right] \\
+ 3 \frac{\Gamma(l + 3/2)g_{l+3/2}(A_e) + l \Gamma(l + 1/2)B_e g_{l+1/2}(A_e) + l(l-1) \Gamma(l - 1/2)B_e^2 g_{l-1/2}(A_e)}{2} \right\}
\]

\[
\approx \frac{(k_B T)^{l+1}}{\sqrt{2A_e}} e^{-A_e} \left\{ A_e \left[ \Gamma(l + 1/2) \\
+ l \Gamma(l - 1/2)B_e + \frac{l(l-1)}{2} \Gamma(l - 3/2)B_e^2 \right] \\
+ 3 \frac{\Gamma(l + 3/2) + l \Gamma(l + 1/2)B_e + l(l-1) \Gamma(l - 1/2)B_e^2}{2} \right\}
\]

and the last term is

\[
I_{6j} = 2 \int_{\Delta} e_{\Delta}^{(\epsilon + E)^l} \frac{1}{\sqrt{\epsilon^2 - \Delta^2}} \exp(\beta \epsilon) + 1 \exp(\beta \epsilon) + 1
\]

\[
\approx 2e^{-(A_e + B_e)} \frac{(k_B T_e)^{l+1}}{\sqrt{2A_e}} \left\{ A_e \left[ \frac{\Gamma(l + 1/2)}{(1 + T_e/T_s)^{(l+1)/2}} + \frac{l \Gamma(l - 1/2)}{(1 + T_e/T_s)^{(l-1)/2} B_e} \right] \\
+ \frac{l(l-1)}{2} \frac{\Gamma(l + 3/2)}{(1 + T_e/T_s)^{(l+3)/2} B_e} \right\}
\]

One way to derive the last expression is to use the identity \( \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(k+r)}{\Gamma(k+1)} \epsilon^k = (1 + \epsilon)^{-r+1} \), valid for any \( c \in (-1, 1) \). For \( c = 1 \) the left hand side of the identity is not convergent, but we can still define it as \( \Gamma(r)/2^r \), using the continuity property of the right hand side.

Using the above analytical expressions, we can calculate easily the quantity \((\pm) F_l(\pm)\) for any \( l \). As mentioned in the beginning of this section, in the concrete calculations we refer to the case in which the superconductor is Al, with \( \Delta \approx 200 \mu eV \). Then for a temperature \( T_e = 0.1 \) K, \( B_e \approx 46 \), so we shall neglect the terms \( I_{1j} \) and \( I_{6j} \), since they are proportional to \( \exp(-B_e) \). Moreover, it should be noted that three of the other terms, \( I_{2j}, I_{3j}, \) and \( I_{5j} \), are all proportional to \( \exp(-A_e) \). Among these three, \( I_{5j} \) contains the highest orders for \( l = 1 \) and \( 2 \), while for \( l = 0 \) all of them are of the same order. Therefore care should be taken to calculate consistently the highest order of the terms proportional to \( \exp(-A_e) \).

B. Power flux through the NIS junctions

In order to study the power flux through the NIS junctions, in the limit of low temperatures, let us take just the leading orders in \( A_e, A_s, B_e, \) and \( B_s \) from the formulae given for \( I_{ji}, j = 1, \ldots, 6 \) and \( l = 1 \). After dropping the irrelevant terms we arrive at the following expression [13]:

\[
\dot{Q}_j = \frac{\Delta^2}{e^2 R_T} \sqrt{\frac{\pi}{2A_e^3}} \left[ \frac{1}{2} \frac{\Gamma(3/2)}{\Gamma(1/2)} g_{3/2}(a_e) + a_e g_{1/2}(a_e) \right] - \frac{\Delta^2}{e^2 R_T} \sqrt{\frac{2\pi}{A_e}} e^{-A_e}.
\]

A problem that occurs in all the cooling experiments by NIS junctions is the heating of the superconductor. Due to the poor heat transport properties of the superconductor, the temperature \( T_e \) increases too much and \( \dot{Q}_j \) becomes negative, or very close to zero, for any bias voltage. In such a case the cooler is practically ineffective. The heating can be controlled to some extent by placing normal metal traps on top of the superconductor. This has been discussed.
in Refs. [2, 19]. To avoid getting into too many details, here we shall consider $T_e$ as an external parameter which is independent of the applied bias voltage (for any specific design more detailed calculations can be made). In case of an ideal quasiparticle trap, we set $T_e = T_2$. For a constant $T_e$, $\dot{Q}_J$ is just a function of $a_e$. This function reaches its maximum value at $a_e = a_{opt,0} \approx 0.66$ [12], which corresponds to what we shall call the optimum bias voltage, $V_{opt} \equiv \Delta - k_B T_e a_{opt,0}$. From here we can draw an important conclusion: in the limit of low temperatures, the difference between the optimum bias voltage, $V_{opt}$, and the energy gap $\Delta$ in the superconductor is independent of $\Delta$ and scales with the temperature of the ES in the normal metal. This means that $\delta_{opt} \equiv \Delta - V_{opt} \rightarrow 0$ as $T_e \rightarrow 0$, and $a_{opt}(T_e) \rightarrow a_{opt,0}$ as $T_e \rightarrow 0$, where $a_{opt}(T_e)$ is the exact value of $a_e$ at the maximum cooling power, at finite $T_e$. The ratio between $a_{opt}$ and $a_{opt,0}$ as a function of $A_e$, is shown in Fig. 2. We have to note that the formulae containing $g_k(a_{opt,0})$ can be simplified by replacing this function neither by $g_k(0) \equiv |(1 - 2^{-k})\zeta(k)|$ nor by $\exp(-a_{opt,0})$ in the limit of low temperature. Replacing $a_e$ by $a_{opt,0}$ in Eq. (38), we obtain the optimum cooling power [4]:

$$\dot{Q}_{J,\text{opt}} \approx 0.59 \frac{\Delta^2}{e^2 R_T} \left(\frac{k_B T_e}{\Delta}\right)^{3/2} - \frac{\Delta^2}{e^2 R_T} \sqrt{\frac{2\pi k_B T_e}{\Delta}} e^{-\Delta/k_B T_e}. \quad (39)$$

![FIG. 2. The ratio between $a_{opt}$, the value of $a_e$ at maximum cooling power and finite temperature $T_e$, and its limit value at zero temperature, $a_{opt,0}$, as a function of $A_e \equiv \Delta/k_B T_e$. Note that a range interesting for applications would be $A_e > 20$.](image)

**IV. RESULTS**

Based on Eqs. (24) and (19) and using the formulae given in Sections II C and III, we can make concrete calculations of $NEP$ in the TSE of the microbolometer. In Ref. [22] the direct and indirect cooling methods were compared, for the case of zero bias and optical input power. The conclusion was that $NEP$ was more than one order of magnitude lower in the case of indirect cooling than in the case of direct cooling, clearly recommending the first method for applications. When the optical input power is not zero the indirect cooling of the TSE might not be enough, due to the poor electron-phonon coupling at low temperatures [see Eq. (24)]. Therefore another set of junctions should be attached directly to the TSE to compensate for the optical power. To evaluate the situation, let us suppose that the resistances of these junctions are calibrated in such a way that $\dot{Q}_{J,\text{opt}} = \dot{Q}_b + \dot{Q}_{oe}$ (we take $\dot{Q}_{op} \equiv 0$), so $T_e = T_1$. We consider that $T_1 = T_2 = 0.1$ K and that the superconductor is in contact with an ideal quasiparticle trap ($T_e = T_2$). The optical power incident on the detector is due to the cosmic background radiation and corresponds to a temperature $T_1 = 3$ K. As seen from Eqs. (29) and (30), both $\dot{Q}_{oe}(\omega_o)$ and $(\delta^2 \dot{Q}_{oe,\text{shot}}(\omega_o))_\omega$ depend on the detector dimensions, emissivity, bandwidth and frequency of the absorbed radiation. To eliminate the first three of these parameters, we shall use the formula $(\delta^2 \dot{Q}_{oe,\text{shot}}(\omega_o))_\omega = 2\hbar \omega_o \dot{Q}_{oe}(\omega_o)$ and treat $\dot{Q}_{oe}(\omega_o)$ as a parameter obtainable from experiment. On the other hand, making use of the formulae given for the functions $I_{F_1}, p = 1, \ldots, 6$
and keeping only the highest orders in $A_e = A_s$, we obtain the following expression for the fluctuations:

$$
\langle \delta^2 \dot{Q}_{1,\text{shot}} \rangle_{\omega} \approx 2.05 \frac{3}{\epsilon^2 R_T} \left( \frac{k_B T_e}{\Delta} \right)^{5/2} + 4 \sqrt{2} \pi \frac{2}{\epsilon^2 R_T} \left( \frac{k_B T_e}{\Delta} \right)^{1/2} e^{-\Delta/k_B T_e} .
$$

(40)

Now we can compare Eqs. (39) and (40). If we neglect the terms proportional to $e^{-\Delta/k_B T_e}$, we can write in general

$$
\langle \delta^2 \dot{Q}_{1,\text{shot}} \rangle_{\omega} \approx 3.47 k_B T_e \dot{Q}_{1,\text{opt}}(\omega_o) = 3.47 k_B T_e \dot{Q}_{\text{oe}}(\omega_o) .
$$

(41)

Comparing the above equation with the equation for $\langle \delta^2 \dot{Q}_{\text{oe,shot}}(\omega_o) \rangle_{\omega}$, we obtain $\langle \delta^2 \dot{Q}_{1,\text{shot}} \rangle_{\omega} \approx 1.74(k_B T_e/\hbar \omega)(\delta^2 \dot{Q}_{\text{oe,shot}}(\omega_o))_{\omega}$. If we write $\omega_o \equiv \omega_{o,\text{max}}$, where $\omega_{o,\text{max}}$ is the angular frequency corresponding to the maximum of the energy density of the cosmic background radiation, $\hbar \omega_{o,\text{max}} = y k_B T_b$ ($y \approx 2.82$), then we can write

$$
\langle \delta^2 \dot{Q}_{1,\text{shot}} \rangle_{\omega} \approx 0.62 \frac{T_e}{x T_b} \langle \delta^2 \dot{Q}_{\text{oe,shot}}(x) \rangle_{\omega} .
$$

(42)

Since $\partial Q_1/\partial V = 0$ at the optimum bias voltage, Eq. (20) simplifies and we obtain $T(\omega) = \langle \delta^2 \dot{Q}_{1,\text{shot}} \rangle_{\omega}$. In experiments it is expected that $T_b/T_e \geq 30$, so, for $x$ of the order of 1 [Infra Red (IR) radiation] the shot noise in the junction, in the case of the combined cooling method, is very small as compared to the shot noise due to the background radiation. On the other hand it should be noted that the optical power of the IR radiation is absorbed by an antenna and is transformed into heat by the Joule effect in the TSE. In such a case, the noise contribution due to the input optical power may not be well described by Eq. (30) (here further investigation is needed). Therefore we shall write in general $\langle \delta^2 \dot{Q}_{\text{oe}}(\omega_o) \rangle_{\omega} = 2 \phi \hbar \omega_o \dot{Q}_{\text{oe}}(\omega_o)$, where $\phi$ is a parameter of value between 0 and 1.

Taking again the two limit situations from Ref. [22] and using Eq. (42) we say that if the Kapitza resistance is much smaller than the thermal resistance between the electrons and phonons, then

$$
NEP^2 \approx \langle \delta^2 \dot{Q}_{\text{ep,shot}} \rangle_{\omega} + \langle \delta^2 \dot{Q}_{K,\text{shot}} \rangle_{\omega} \left( \frac{\partial \dot{Q}_{\text{ep}}}{\partial T_1} \right)^2 \left( \frac{\partial \dot{Q}_K}{\partial T_1} \right)^{-2} + \left( \phi + 0.62 \frac{T_e}{x T_b} \right) k_B T_b \dot{Q}_{\text{oe}}(x) .
$$

(43)

where the last approximation holds if $(T_2/T_1)^5$ is much smaller than the ratio between the electron-phonon and the Kapitza resistance $[(T_2/T_1)^5(\Sigma T_1/\Sigma K)S] \ll 1$, which is certainly the case for the indirect and combined cooling, under the assumptions made. In the other limiting situation, when the Kapitza resistance is much higher than the thermal resistance between the electrons and phonons, we can write

$$
NEP^2 \approx 4 \langle \delta^2 \dot{Q}_{\text{ep,shot}} \rangle_{\omega} + \left( \phi + 0.62 \frac{T_e}{x T_b} \right) k_B T_b \dot{Q}_{\text{oe}}(x) .
$$

(44)

Note that the coupling between the phonon and electron systems introduces a factor 4 in front of $\langle \delta^2 \dot{Q}_{\text{ep,shot}} \rangle_{\omega}$ in this situation. As in [22] we assume that the volume of the TSE is $\Omega = 1 \mu m^3$ and if we take $T_1 = T_e = 0.1$ K, we obtain $\langle \delta^2 \dot{Q}_{\text{ep,shot}} \rangle_{\omega}^{1/2} \approx 7.43 \times 10^{-19}$ W/\(\sqrt{Hz}\). Let us suppose that we detect IR radiation and suppose further that in such a case $\phi \approx 0$. From Eqs. (43) and (44) we can calculate two critical input optical powers, $\dot{Q}_{\text{oe,1}}(x)$ and $\dot{Q}_{\text{oe,2}}(x)$, respectively, by the equation $\dot{Q}_{\text{oe,1}}(x) = \dot{Q}_{\text{oe,2}}(x)/4 = \langle \delta^2 \dot{Q}_{\text{ep,shot}} \rangle_{\omega}/(0.62 y k_B T_e)$. For our choice of temperatures we obtain $\dot{Q}_{\text{oe,1}}(x) \approx 2.29 \times 10^{-13}$ W and $\dot{Q}_{\text{oe,2}}(x) \approx 9.15 \times 10^{-13}$ W. Therefore, for optical input power smaller than $\dot{Q}_{\text{oe,1}}(x)$ (depending whether we are in the first or in the second case), the noise contribution comes essentially from the electron-phonon coupling. For higher input power, the noise due to the electron tunneling through the NIS junctions dominates. As seen from Eq. (42), in such a case the $NEP$ has a weaker dependence on temperature, being proportional to $T_e^{1/2}$, for constant optical input power.

Throughout this paper we did not refer to the measurement of the temperature of the ES of the TSE. As mentioned in the beginning, there exist different methods to do that and we want to keep the arguments here as general as possible. Therefore, let us suppose that in an experiment the temperature is calculated from the measured value of a quantity $M(T_e)$ (for example $M$ can be the voltage across the NIS junction biased at constant current, or the current in a voltage biased transition-edge sensor). If the inaccuracy in reading the quantity $M$ is $\langle \delta_1 M \rangle_{\omega}$ (this should be
significant just in badly designed experiments) and the total mean square fluctuations of $M$ (for example the shot noise added quadratically to the ampifier noise) are $\langle \delta^2 M_t \rangle$, then the uncertainty in the measurement of $M$ is:

$$\langle \delta^2 M \rangle^{1/2} = \sqrt{\langle \delta^2 M_{\text{shot}} \rangle + \frac{2 \partial M}{\partial T_e} \text{Re} \left( \langle \delta M \delta T_e \rangle \right) + \left( \frac{\partial M}{\partial T_e} \right)^2 \langle \delta^2 T_e \rangle + \langle \delta_1 M \rangle}, \quad (45)$$

where we assumed again that the changes in the chemical potential $\mu$, due to temperature fluctuations, are negligible and $\text{Re} (\langle \delta M \delta T_e \rangle)$ represent the real part of the correlations between the fluctuations of $M$ and of $T_e$. To evaluate Eq. (45) we have to introduce in Eqs. (13) the noise contribution due to the bias power. In this way we obtain extra terms in the expressions for $\Delta_T (\omega)$ given by Eqs. (13) and (17). Denoting the new function, influenced by the measurement process, by $\Delta_{mT} (\omega)$, and using in an obvious manner the two expressions for $\Delta_T (\omega)$, we write the equivalent of Eqs. (13) and (17) as

$$\Delta_{mT} (\omega) \equiv \delta T_e (\omega) Z_{\text{NEP} \omega} = \Delta_T (\omega) + \delta T_e (\omega) \frac{\partial \dot{Q}_b}{\partial T_e} \quad (46)$$

and

$$\Delta_{mT} (\omega) = \Delta_T (\omega) + \delta \dot{Q}_{b,\text{shot}} (\omega) - \frac{\partial \epsilon_F}{\partial N} \frac{\partial \dot{Q}_b}{\partial N} \frac{\dot{N}_{b,\text{shot}} (\omega)}{i \omega} \quad (47)$$

respectively, where $E_b$ is the bias voltage of the thermometer and $\dot{N}_{b,\text{shot}}$ is the shot noise of the particle current due to the thermometer. The presence of the last term in Eq. (17) depends on the temperature measurement setup. It should also be noted that when $\dot{Q}_b$ is due to a flux of particles, so when $\dot{N}_b$ is not zero, the $\pi/2$ angle between the phases of the noise spectra of these two quantities annihilates their correlations. Taking this into account, we can write in general

$$\langle \Delta_{mT}^2 (\omega) \rangle = \langle \Delta_T^2 (\omega) \rangle + \langle \delta^2 \dot{Q}_{b,\text{shot}} (\omega) \rangle + \frac{1}{\omega^2} \left( \frac{\partial \epsilon_F}{\partial N} \frac{\partial \dot{Q}_b}{\partial N} \right)^2 \langle \delta^2 \dot{N}_{b,\text{shot}} (\omega) \rangle$$

In this situation, the total Noise Equivalent Power, $\text{NEP}_\omega$, should be defined as the input power that would produce the same change in the quantity $M$ as the one in Eq. (15). Simple calculations lead us to the general result

$$\text{NEP}_\omega = \sqrt{\langle \Delta_{mT}^2 (\omega) \rangle + 2 \left( \frac{\partial M}{\partial T_e} \right)^{-1} |Z_{\text{NEP} \omega}| \text{Re} \left( \langle \delta M_t \delta T_e \rangle \right) + \left( \frac{\partial M}{\partial T_e} \right)^{-2} |Z_{\text{NEP} \omega}|^2 \langle \delta^2 M_t \rangle} \quad (48)$$

where $Z_{\text{NEP} \omega}$ was defined in Eq. (15). Using Eq. (17) we can write

$$\text{Re} \left( \langle \delta M_t \delta T_e \rangle \right) \equiv \text{Re} \left[ \left( \delta M_t \left( \delta \dot{Q}_{b,\text{shot}} - \frac{\partial \epsilon_F}{\partial N} \frac{\partial \dot{Q}_b}{\partial N} \frac{\dot{N}_{b,\text{shot}} (\omega)}{i \omega} \right) \right) \right] \omega |Z_{\text{NEP} \omega}| .$$

For the case when the thermometer is a NIS junction, current or voltage biased, the $\text{NEP}$ was calculated in Refs. [14,15,23] in the limit of zero Kapitza resistance, infinitely good reading accuracy of quantity $M$, and when the only refrigerating junction in the system is the thermometer junction.

**V. CONCLUSIONS**

In this paper we calculated the temperature fluctuations and the Noise Equivalent Power $\text{NEP}$ in the thermal sensing element TSE of a microbolometer cooled by NIS junctions. From the general result we extracted simple expressions for two limiting cases. In the first case we considered that the Kapitza resistance is much smaller than the thermal resistance between the electrons and phonons, while in the second case we considered the opposite situation. In both cases the important role in the calculation of the $\text{NEP}$ was played by the electron-phonon shot noise and the noise due to the cooling NIS junction, coupled directly to the TSE, and to the input optical power [see Eqs. (13) and (14)].
We gave analytical expressions for the quantitative evaluation of the processes that take place in NIS junctions, which are valid in the range of low temperatures. The validity of these formulae can be checked by the evaluation of the lower order terms in the quantities $A_s/e$ (see the definitions in Section III A). Using the analytical expressions we showed that in the limit of low temperature of the electron gas in the normal metal ($T_e$) the difference between the gap energy in the superconductor, $\Delta$, and the bias voltage corresponding to the maximum cooling power, $V_{\text{opt}}$ (supposing that the temperature of the superconductor does not vary with the bias voltage), is independent of $\Delta$ and scales with $T_e$, like $\Delta - V_{\text{opt}} = 0.66k_B T_e$. This is a very useful result since in such a case, in the quantitative evaluation of the processes that take place in NIS junctions, we can use neither the limit $(\Delta - V_{\text{opt}})/k_B T_e \gg 1$, nor the limit $(\Delta - V_{\text{opt}})/k_B T_e \ll 1$, for $T_e \to 0$.

In the end we took into account in a general way the effect of the measurement process on the Noise Equivalent Power and we gave general expressions for its calculation. As expected, the measurement increases the uncertainty in the detected input optical power [see Eq. (48)].

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