A Scalable CUR Matrix Decomposition Algorithm: Lower Time Complexity and Tighter Bound*

Shusen Wang and Zhihua Zhang
College of Computer Science & Technology
Zhejiang University
Hangzhou, China 310027
{wss,zhzhang}@zju.edu.cn

Jian Li
Google
Beijing, China 100084
lijian@google.com

May 3, 2014

Abstract

The CUR matrix decomposition is an important extension of Nyström approximation to a general matrix. It approximates any data matrix in terms of a small number of its columns and rows. In this paper we propose a novel randomized CUR algorithm with an expected relative-error bound. The proposed algorithm has the advantages over the existing relative-error CUR algorithm that it possesses tighter theoretical bound and lower time complexity, and that it can avoid maintaining the whole data matrix in main memory. Finally, experiments on several real-world datasets demonstrate significant improvement over the existing relative-error algorithms.

Keywords: Large-scale matrix computations, low-rank matrix approximation, CUR matrix decomposition, randomized algorithms

1. Introduction

Large-scale matrices emerging from stocks, genomes, web documents, web images and videos everyday bring new challenges in modern data analysis. Most efforts have been focused on manipulating, understanding and interpreting large-scale data matrices. In many cases, matrix factorization methods are employed to construct compressed and informative representations to facilitate computation and interpretation. A principled approach is the truncated singular value decomposition (SVD) which finds the best low-rank approximation of a data matrix. Applications of SVD such as eigenface (Sirovich and Kirby, 1987, Turk and Pentland, 1991) and latent semantic analysis (Deerwester et al., 1990) have been illustrated to be very successful.

However, the basis vectors resulting from SVD have little concrete meaning, which makes it very difficult for us to understand and interpret the data in question. An example

* An extended abstract of this paper has been accepted by NIPS 2012.
in (Drineas et al., 2008, Mahoney and Drineas, 2009) has well shown this viewpoint; that is, the vector \([\frac{1}{\sqrt{2}} \text{age} - \frac{1}{\sqrt{2}} \text{height} + \frac{1}{\sqrt{2}} \text{income}]\), the sum of the significant uncorrelated features from a dataset of people’s features, is not particularly informative. Kuruvilla et al. (2002) have also claimed: “it would be interesting to try to find basis vectors for all experiment vectors, using actual experiment vectors and not artificial bases that offer little insight.” Therefore, it is of great interest to represent a data matrix in terms of a small number of actual columns and/or actual rows of the matrix.

The **CUR matrix decomposition** provides such techniques, and it has been shown to be very useful in high dimensional data analysis (Mahoney and Drineas, 2009). Given a matrix \(A\), the CUR technique selects a subset of columns of \(A\) to construct a matrix \(C\) and a subset of rows of \(A\) to construct a matrix \(R\), and computes a matrix \(U\) such that \(\tilde{A} = \text{CUR}\) best approximates \(A\). The typical CUR algorithms (Drineas, 2003, Drineas et al., 2006, 2008) work in a two-stage manner. Stage 1 is a standard column selection procedure, and Stage 2 does row selection from \(A\) and \(C\) simultaneously. Thus, implementing Stage 2 is much more difficult than doing Stage 1.

The CUR matrix decomposition problem is widely studied in the literature (Goreinov et al., 1997a,b, Tyrtyshnikov, 2000, Drineas, 2003, Drineas and Mahoney, 2005, Drineas et al., 2006, 2008, Mahoney and Drineas, 2009, Mackey et al., 2011, Hopcroft and Kannan, 2012). Among the existing work, several recent work are of particular interest. Drineas et al. (2006) proposed a CUR algorithm with additive-error bound. Later on, Drineas et al. (2008) devised randomized CUR algorithms with relative error by sampling sufficiently many columns and rows. Particularly, the algorithm has \((1 + \epsilon)\) relative-error ratio with high probability (w.h.p.). Recently, Mackey et al. (2011) established a divide-and-conquer method which solves the CUR problem in parallel.

Unfortunately, all the existing CUR algorithms require a large number of columns and rows to be chosen. For example, for an \(m \times n\) matrix \(A\) and a target rank \(k \leq \min\{m, n\}\), the state-of-the-art CUR algorithm — the subspace sampling algorithm in Drineas et al. (2008) — requires exactly \(\mathcal{O}(k^3\epsilon^{-6})\) rows or \(\mathcal{O}(k\epsilon^{-4}\log^2 k)\) rows in expectation to achieve \((1 + \epsilon)\) relative-error ratio w.h.p. Moreover, the computational cost of this algorithm is at least the cost of the truncated SVD of \(A\), that is, \(\mathcal{O}(\min\{mn^2, nm^2\})\).\(^1\) The algorithms are therefore impractical for large-scale matrices.

In this paper we develop a CUR algorithm which beats the state-of-the-art algorithm in both theory and experiments. In particular, we show in Theorem 9 a novel randomized CUR algorithm with lower time complexity and tighter theoretical bound in comparison with the state-of-the-art CUR algorithm in Drineas et al. (2008).

The rest of this paper is organized as follows. Section 2 lists some notations that will be used in this paper and Section 3 reviews two classes of CUR algorithms. Section 4 mainly introduces a column selection algorithm to which our work is closely related. Section 5 describes and analyzes our novel CUR algorithm. Section 6 empirically compares our proposed algorithm with the state-of-the-art algorithm. All proofs are deferred to Appendix B.

---

\(^1\) Although some partial SVD algorithms, such as Krylov subspace methods, require only \(\mathcal{O}(mnk)\) time, they are all numerical unstable. See Halko et al. (2011) for more discussions.
2. Notations

For a matrix $A = [a_{ij}] \in \mathbb{R}^{m \times n}$, let $a^{(i)}$ be its $i$-th row and $a_j$ be its $j$-th column. Let $\|A\|_1 = \sum_{i,j} |a_{ij}|$ be the $\ell_1$-norm, $\|A\|_F = (\sum_{i,j} a_{ij}^2)^{1/2}$ be the Frobenius norm, and $\|A\|_2 = \max_{\|x\|_2 = 1} \|Ax\|_2$ be the spectral norm. Moreover, let $I_m$ denote the $m \times m$ identity matrix, and $0$ denotes the zero matrix whose size depends on the context. Let $\rho = \text{rank}(A)$ and $k \leq \rho$, the SVD of $A$ can be written as

$$A = \sum_{i=0}^{\rho} \sigma_{A,i} u_{A,i} v_{A,i}^T = U_A \Sigma_A V_A^T = U_{A,k} \Sigma_{A,k} V_{A,k}^T + U_{A,k\perp} \Sigma_{A,k\perp} V_{A,k\perp}^T,$$

where $U_{A,k}$, $\Sigma_{A,k}$, and $V_{A,k}$ correspond to the top $k$ singular values. We denote $A_k = U_{A,k} \Sigma_{A,k} V_{A,k}^T$. Furthermore, let $A^\dagger = U_{A,\rho} \Sigma_{A,\rho}^{-1} V_{A,\rho}^T$ be the Moore-Penrose inverse of $A$ (Ben-Israel and Greville, 2003).

Given matrices $A \in \mathbb{R}^{m \times n}$, $X \in \mathbb{R}^{m \times p}$, and $Y \in \mathbb{R}^{q \times n}$, $XX^\dagger A = U_X U_X^T A \in \mathbb{R}^{m \times n}$ is the projection of $A$ onto the column space of $X$, and $AY^\dagger Y = A V_Y V_Y^T \in \mathbb{R}^{m \times n}$ is the projection of $A$ onto the row space of $Y$. Finally, given an integer $k \leq p$, we define the matrix $\Pi_{X,k}(A) \in \mathbb{R}^{m \times n}$ as the best approximation to $A$ within the column space of $X$ that has rank at most $k$. We have $\Pi_{X,k}(A) = X \hat{Z}$ where $\hat{Z} = \text{argmin}_{\text{rank}(Z) \leq k} \|A - XZ\|_F$. We also have that $\|A - XX^\dagger A\|_F \leq \|A - \Pi_{X,k}(A)\|_F$.

3. Previous Work in CUR Matrix Decomposition

This section discusses two recent developments of the CUR algorithms. Section 3.1 introduces an additive-error CUR algorithm in Drineas et al. (2006), and Section 3.2 describes two relative-error CUR algorithms in Drineas et al. (2008).

3.1 The Linear-Time CUR Algorithm

The linear-time CUR algorithm is proposed by Drineas et al. (2006). It is a highly efficient algorithm. Given a matrix $A$ and a constant $k < \text{rank}(A)$, by sampling $c = 64k\epsilon^{-4}$ columns and $r = 4k\epsilon^{-2}$ rows of $A$ and computing an intersection matrix $U$, the resulting CUR decomposition satisfies the following additive-error bound

$$\mathbb{E}\|A - \text{CUR}\|_F \leq \|A - A_k\|_F + \epsilon\|A\|_F.$$

Furthermore, the decomposition also satisfies $\text{rank}(\text{CUR}) \leq k$. Here we give its main results (Theorem 4 of Drineas et al., 2006) in the following proposition.

**Proposition 1 (The Linear-Time CUR Algorithm)** Given a matrix $A \in \mathbb{R}^{m \times n}$, we let $p_i = \|a^{(i)}\|_2^2/\|A\|_F^2$ and $q_j = \|a_j\|_2^2/\|A\|_F^2$. The linear-time CUR algorithm randomly samples $c$ columns of $A$ with probabilities $\{q_j\}_{j=1}^n$ and $r$ rows of $A$ with probabilities $\{p_i\}_{i=1}^m$. Then

$$\mathbb{E}\|A - \text{CUR}\|_F \leq \|A - A_k\|_F + \left((4k/c)^{1/4} + (k/r)^{1/2}\right)\|A\|_F.$$

The algorithm costs $O(mc^2 + nr + c^2r + c^3)$ time, which is linear in $(m + n)$ by assuming $c$ and $r$ are constants.
3.2 The Subspace Sampling CUR Algorithm

Drineas et al. (2008) proposed a two-stage randomized CUR algorithm which has a relative-error bound w.h.p. In the first stage the algorithm samples \( c \) columns of \( A \) to construct \( C \), and in the second stage it samples \( r \) rows from \( A \) and \( C \) simultaneously to construct \( R \) and \( U^\dagger \). In the first stage the sampling probabilities are proportional to the squared \( \ell_2 \)-norm of the rows of \( V_{A,k} \), in the second stage the sampling probabilities are proportional to the squared \( \ell_2 \)-norm of the rows of \( U_{C,k} \). That is why it is called the “subspace sampling algorithm”. Here we show the main results of the subspace sampling algorithms in the following proposition.

**Proposition 2 (The Subspace Sampling CUR Algorithm)** Given a matrix \( A \in \mathbb{R}^{m \times n} \) and an integer \( k \ll \min\{m, n\} \), the subspace sampling algorithm uses exactly sampling to select exactly \( c = O(k^2\epsilon^{-2}\log(1/\delta)) \) columns of \( A \) to construct \( C \), and then exactly \( r = O(c^2\epsilon^{-2}\log(1/\delta)) \) rows of \( A \) to construct \( R \), or uses expected sampling to select \( c = O(k\epsilon^{-2}\log k\log(1/\delta)) \) columns and \( r = O(c\epsilon^{-2}\log c\log(1/\delta)) \) rows in expectation. Then with probability at least \((1 - \delta)\),

\[
\|A - \text{CUR}\|_F \leq (1 + \epsilon)\|A - A_k\|_F.
\]

Here, the matrix \( U \) is a weighted Moore-Penrose inverse of the intersection between \( C \) and \( R \). The running time of both algorithms is dominated by the truncated SVD of \( A \).

Although the algorithm is \( \epsilon \)-optimal with high probability, it requires too many rows get chosen: at least \( r = O(k\epsilon^{-4}\log^2 k) \) rows in expectation. In this paper we seek to devise an algorithm with mild requirement on column and row numbers.

4. Theoretical Backgrounds

Section 4.1 considers the connections between the column selection problem and the CUR matrix decomposition problem. Section 4.2 introduces a near-optimal relative-error column selection algorithm. Our proposed CUR algorithm is motivated by and partly based on the near-optimal column selection algorithm.

4.1 Connections between Column Selection and CUR Matrix Decomposition

Column selection is a well-established problem which has been widely studied in the literature: (Frieze et al., 2004, Deshpande et al., 2006, Drineas et al., 2008, Deshpande and Rademacher, 2010, Boutsidis et al., 2011b, Guruswami and Sinop, 2012).

Given a matrix \( A \in \mathbb{R}^{m \times n} \), column selection aims to choose \( c \) columns of \( A \) to construct \( C \in \mathbb{R}^{m \times c} \) so that \( \|A - CC^\dagger A\|_F \) achieves the minimum. Since there are \( \binom{n}{c} \) possible choices of constructing \( C \), so selecting the best subset is a hard problem. In recent years, many polynomial-time approximate algorithms have been proposed, among which we are particularly interested in those algorithms with relative-error bounds; that is, with \( c \geq k \) columns selected from \( A \), there is a constant \( \eta \) such that

\[
\|A - CC^\dagger A\|_F \leq \eta\|A - A_k\|_F.
\]
We call $\eta$ the relative-error ratio. For some randomized algorithms, the inequality holds either w.h.p. or in expectation w.r.t. $C$.

The CUR matrix decomposition problem has a close connection with the column selection problem. As aforementioned, the first stage of existing CUR algorithms is simply a column selection procedure. However, the second stage is more complicated. If the second stage is naively solved by a column selection algorithm on $A^T$, then the error ratio will trivially be $2\eta$.

For a relative-error CUR algorithm, the first stage seeks to bound a construction error ratio of $\frac{\|A - CC^\dagger A\|_F}{\|A - A_k\|_F}$, while the section stage seeks to bound $\frac{\|A - CC^\dagger A\|_F}{\|A - CC^\dagger A\|_F} \leq \eta$, given $C$. Actually, the first stage is a special case of the second stage where $C = A_k$. Given a matrix $A$, if an algorithm solving the second stage results in a bound $\frac{\|A - CC^\dagger A\|_F}{\|A - CC^\dagger A\|_F} \leq \eta$, then this algorithm also solves the column selection problem for $A^T$ with an $\eta$ relative-error ratio. Thus the second stage of CUR is a generalization of the column selection problem.

4.2 The Near-Optimal Column Selection Algorithm

Recently, Boutsidis et al. (2011a) proposed a randomized algorithm which selects only $c = 2k(1 + o(1))$ columns to achieve the expected relative-error ratio $(1 + \epsilon)$. Boutsidis et al. (2011a) also proved the lower bound of the column selection problem; that is, at least $c = k(1 + o(1))$ columns are selected to achieve the $(1 + \epsilon)$ ratio. Thus this algorithm is near optimal. Though an optimal algorithm recently proposed by Guruswami and Sinop (2012) achieves the lower bound, the optimal algorithm is quite inefficient compared with the near-optimal algorithm.

The near-optimal algorithm has three steps: the approximate SVD via random projection (Halko et al., 2011), the dual set sparsification algorithm (Boutsidis et al., 2011a), and the adaptive sampling algorithm (Deshpande et al., 2006). Here we present the main results of this algorithm in Lemma 3. To better understand the algorithm, we also give the details of the three steps, respectively.

Lemma 3 (Near-Optimal Column Selection Algorithm) Given a matrix $A \in \mathbb{R}^{m \times n}$ of rank $\rho$, a target rank $k$ ($2 \leq k < \rho$), and $0 < \epsilon < 1$, there exists a randomized algorithm to select at most

$$c = \frac{2k}{\epsilon} \left(1 + o(1)\right)$$

columns of $A$ to form a matrix $C \in \mathbb{R}^{m \times c}$ such that

$$\mathbb{E}^2 \|A - CC^\dagger A\|_F \leq \mathbb{E}\|A - CC^\dagger A\|_F^2 \leq (1 + \epsilon)\|A - A_k\|_F^2,$$

where the expectations are taken w.r.t. $C$. Furthermore, the matrix $C$ can be obtained in $O((mnk + nk^3)\epsilon^{-2/3})$.

The dual set sparsification algorithm requires the top $k$ right singular vectors of $A$ as inputs. Since SVD is time consuming, Boutsidis et al. (2011a) employed an approximation SVD algorithm (Halko et al., 2011) to speedup computation. We give the theoretical analysis of the approximation SVD via random projection in Lemma 4. The resulting matrix $Z$ approximates $V_{A,k}$.
Lemma 4 (Randomized SVD via Random Projection) Given a matrix $A \in \mathbb{R}^{m \times n}$ of rank $\rho$, a target rank $k$ ($k < \rho$), and $0 < \epsilon_0 < 1$, the algorithm computes a factorization $A = BZ^T + E$ with $B = AZ$, $Z^TZ = I_k$, and $EZ = 0$ such that

$$\mathbb{E}\|E\|_F^2 \leq (1 + \epsilon_0)\|A - A_k\|_F^2.$$  

The algorithm runs in $O(mnk\epsilon_0^{-1})$ time.

The second step of the near-optimal column selection algorithm is the dual set sparsification proposed by Boutsidis et al. (2011a). When ones take $A$ and the top $k$ (approximate) right singular vectors of $A$ as inputs, the dual set sparsification algorithm deterministically selects $c_1$ columns of $A$ to construct $C_1$. We present their results in Lemma 5 and attach the concrete algorithm in Appendix A.

Lemma 5 (Column Selection via Dual Set Sparsification Algorithm) Given a matrix $A \in \mathbb{R}^{m \times n}$ of rank $\rho$ and a target rank $k$ ($k < \rho$), the dual set spectral-Frobenius sparsification algorithm deterministically selects $c_1$ ($> k$) columns of $A$ to form a matrix $C_1 \in \mathbb{R}^{m \times c_1}$ such that

$$\|A - \Pi_{C_1,k}(A)\|_F \leq \sqrt{1 + \frac{1}{(1 - k/c_1)^2}} \|A - A_k\|_F.$$  

Moreover, the matrix $C_1$ can be computed in $T_{V_{A,k}} + O(mn + nc_1k^2)$, where $T_{V_{A,k}}$ is the time needed to compute the top $k$ right singular vectors of $A$.

After sampling $c_1$ columns of $A$, the near-optimal column selection algorithm uses the adaptive sampling of Deshpande et al. (2006) to select $c_2$ columns of $A$ to further reduce the construction error. We present Theorem 2.1 in Deshpande et al. (2006) in the following lemma.

Lemma 6 (The Adaptive Sampling Algorithm) Given a matrix $A \in \mathbb{R}^{m \times n}$, we let $C_1 \in \mathbb{R}^{m \times c_1}$ consists of $c_1$ columns of $A$, and define the residual $B = A - C_1C_1^\dagger A$. Additionally, for $i = 1, \cdots, n$, we define

$$p_i = \|b_i\|_2^2/\|B\|_F^2.$$  

We further sample $c_2$ columns i.i.d. from $A$, in each trial of which the $i$-th column is chosen with probability $p_i$. Let $C_2 \in \mathbb{R}^{m \times c_2}$ contain the $c_2$ sampled rows and let $C = [C_1, C_2] \in \mathbb{R}^{m \times (c_1 + c_2)}$. Then, for any integer $k > 0$, the following inequality holds:

$$\mathbb{E}\|A - CC^\dagger A\|_F^2 \leq \|A - A_k\|_F^2 + \frac{k}{r_2^2}\|A - C_1C_1^\dagger A\|_F^2,$$  

where the expectation is taken w.r.t. $C_2$. 

6
Deshpande et al. (2006) is a direct corollary of our following theorem when
original one in Theorem 2.1 of Deshpande et al. (2006). In other words, Theorem 2.1 of
same adaptive sampling algorithm. Interestingly, this new bound is a generalization of the

\[ \mathbf{A} \]

\[ \mathbf{D} \]

\[ \mathbf{C} \times \mathbf{C} \]

Theorem 7 (The Adaptive Sampling Algorithm) Given a matrix \( \mathbf{A} \in \mathbb{R}^{m \times n} \) and a
matrix \( \mathbf{C} \in \mathbb{R}^{m \times c} \) such that \( \text{rank}(\mathbf{C}) = \text{rank}(\mathbf{CC}^\dagger \mathbf{A}) = \rho \ (\rho \leq c \leq n) \), we let \( \mathbf{R}_1 \in \mathbb{R}^{r_1 \times n} \)
consist of \( r_1 \) rows of \( \mathbf{A} \), and define the residual \( \mathbf{B} = \mathbf{A} - \mathbf{A}^\dagger \mathbf{R}_1 \). Additionally, for
\( i = 1, \ldots, m \), we define

\[ p_i = \|b^{(i)}\|_2^2 / \|\mathbf{B}\|_F^2. \]
We further sample \( r_2 \) rows i.i.d. from \( A \), in each trial of which the \( i \)-th row is chosen with probability \( p_i \). Let \( R_2 \in \mathbb{R}^{r_2 \times n} \) contain the \( r_2 \) sampled rows and let \( R = [R_1^T, R_2^T]^T \in \mathbb{R}^{(r_1+r_2) \times n} \). Then the following inequality holds:

\[
\mathbb{E}\|A - CC^\dagger AR^\dagger R\|_F^2 \leq \|A - CC^\dagger A\|_F^2 + \frac{\rho}{r_2}\|A - AR^\dagger R_1\|_F^2,
\]

where the expectation is taken w.r.t. \( R_2 \).

### 5.2 The Fast CUR Algorithm

Based on the randomized SVD algorithm of Lemma 4, the dual set sparsification algorithm of Lemma 5, and the adaptive sampling algorithm of Theorem 7, we develop a randomized algorithm to solve the second stage of the CUR problem. We present the results of the algorithm in the following theorem.

**Theorem 8 (The Fast Row Selection Algorithm)** Given a matrix \( A \in \mathbb{R}^{m \times n} \) and a matrix \( C \in \mathbb{R}^{m \times c} \) such that \( \text{rank}(C) = \text{rank}(CC^\dagger A) = \rho \) \((\rho \leq c \leq n)\), and a target rank \( k \) \((\leq \rho)\), the proposed randomized algorithm selects \( r = \frac{2\rho}{\epsilon}(1 + o(1)) \) rows of \( A \) to construct \( R \in \mathbb{R}^{r \times n} \), such that

\[
\mathbb{E}\|A - CC^\dagger AR^\dagger R\|_F^2 \leq \|A - CC^\dagger A\|_F^2 + \epsilon\|A - A_k\|_F^2,
\]

where the expectation is taken w.r.t. \( R \). Furthermore, the matrix \( R \) can be computed in \( O((mnk + mk^3)\epsilon^{-2/3}) \) time.

Note that Lemma 3, i.e., Theorem 5 of Boutsidis et al. (2011a), is a special case of Theorem 8 when \( C = A_k \). Based on Lemma 3 and Theorem 8, we have the main theorem for the fast CUR algorithm as follows.

**Theorem 9 (The Fast CUR Algorithm)** Given a matrix \( A \in \mathbb{R}^{m \times n} \) and a positive integer \( k \ll \min\{m, n\} \), the fast CUR algorithm described in Algorithm 1 randomly selects \( c = \frac{4k}{\epsilon}(1 + o(1)) \) columns of \( A \) to construct \( C \in \mathbb{R}^{m \times c} \) with the near-optimal column selection algorithm of Lemma 3, and then selects \( r = \frac{2\epsilon}{\epsilon}(1 + o(1)) \) rows of \( A \) to construct \( R \in \mathbb{R}^{r \times n} \) with the fast row selection algorithm of Theorem 8. Then we have

\[
\mathbb{E}\|A - CUR\|_F = \mathbb{E}\|A - C(C^\dagger AR^\dagger)R\|_F \leq (1 + \epsilon)\|A - A_k\|_F.
\]

Moreover, the algorithm runs in time \( O\left(\frac{mnk\epsilon^{-2/3} + (m + n)k^3\epsilon^{-2/3} + mk^2\epsilon^{-2} + nk^2\epsilon^{-4}}{\epsilon}\right) \).

Since \( k, c, r \ll \min\{m, n\} \) by the assumption, so the time complexity of the fast CUR algorithm is lower than that of the SVD of \( A \). This is the main reason why we call it the fast CUR algorithm.

Another advantage of this algorithm is that it can avoid loading the whole \( m \times n \) data matrix \( A \) into main memory. None of three steps — the randomized SVD, the dual set sparsification algorithm, and the adaptive sampling — requires loading the whole of \( A \) into memory. The most memory-expensive operation throughout the fast CUR Algorithm is computing the Moore-Penrose inverses of \( C \) and \( R \), which requires maintaining an \( m \times c \) matrix or an \( r \times n \) matrix in memory. In contrast, the subspace sampling algorithm requires loading the whole matrix into memory to compute its truncated SVD.
6. Empirical Analysis

In this section we conduct empirical comparisons among the relative-error CUR algorithms on several datasets. We report the relative-error ratio and the running time of each algorithm on each data set. The relative-error ratio is defined by

\[ \text{Relative-error ratio} = \frac{\| A - \text{CUR} \|_F}{\| A - A_k \|_F}, \]

where \( k \) is a specified target rank.

6.1 Datasets

We implement experiments on five datasets, including natural images, biology data, and bags of words. Table 1 briefly summarizes some information of the datasets. The Redrock and Edinburgh (Agarwala, 2007) are two large size natural images. Arcene and Dexter are both from the UCI datasets (Frank and Asuncion, 2010). Arcene is a biology dataset with 900 instances and 10000 attributes. Dexter is a bag of words dataset with a 20000-vocabulary and 2600 documents. PicasaWeb image dataset (Wang et al., 2012) contains 6.8 million PicasaWeb images. We use the HMAX features (Serre et al., 2007) and the SIFT features (Lowe, 1999) of the first 50000 images; the features provided by Wang et al. (2012) are all of 3000 dimensions. Each dataset is actually represented as a data matrix, upon which we apply the CUR algorithms.

When the data matrices become very large, e.g., say \( 8K \times 3K \), the truncated SVD and the standard SVD are both infeasible in our experiment environment, and so is the subspace sampling algorithm. Therefore we do not conduct experiments on larger data matrices. In contrast, our fast CUR algorithm actually works well even for \( 30K \times 3K \) matrices.

6.2 Setup

We implement the subspace sampling algorithm and our fast CUR algorithm in MATLAB 7.10.0. We do not compare with the linear-time CUR algorithm for the following reason. There is an implicit projection operation in the linear-time CUR algorithm, so the result satisfies \( \text{rank}(\text{CUR}) \leq k \). However, this inequality does not hold for the subspace sampling algorithm and the fast CUR algorithm. Thus, comparing the construction error among the three CUR algorithms is very unfair for the linear-time CUR algorithm. Actually, the construction error of the linear-time CUR algorithm is much worse than the other two algorithms.
We conduct experiments on a workstation with 12 Intel Xeon 3.47GHz CPUs, 12GB memory, and Ubuntu 10.04 system. According to the analysis in Drineas et al. (2008) and this paper, \( k, c, \) and \( r \) should be integers much less than \( m \) and \( n \). For each data set and each algorithm, we set \( k = 10, 20, \) or \( 50, \) and \( c = \alpha k, r = \alpha c, \) where \( \alpha \) ranges in each set of experiments. We repeat each set of experiments for 20 times and report the average and the standard deviation of the error ratios. The results are depicted in Figures 1, 2, 3, 4, 5, and 6.

### 6.3 Result Analysis

The results show that the fast CUR algorithm has much lower relative-error ratio than the subspace sampling algorithm. The experimental results well match our theoretical analyses in Section 5. As for the running time, the fast CUR algorithm is more efficient when \( c \) and \( r \) are small. When \( c \) and \( r \) become large, the fast CUR algorithm becomes less efficient. This is because the time complexity of the fast CUR algorithm is linear in \( \epsilon^{-4} \) and large \( c \) and \( r \) imply small \( \epsilon \). However, the purpose of CUR is to select a small number of columns and rows from the data matrix, that is, \( c \ll n \) and \( r \ll m \). Thus we are not interested in the cases where \( c \) and \( r \) are large compared with \( m \) and \( n \), e.g., say \( k = 20 \) and \( \alpha = 10 \).
(a) $k = 10$, $c = \alpha k$, and $r = \alpha c$.  
(b) $k = 20$, $c = \alpha k$, and $r = \alpha c$.  
(c) $k = 50$, $c = \alpha k$, and $r = \alpha c$.

Figure 2: Empirical results on the Edinburgh data set.

(a) $k = 10$, $c = \alpha k$, and $r = \alpha c$.  
(b) $k = 20$, $c = \alpha k$, and $r = \alpha c$.  
(c) $k = 50$, $c = \alpha k$, and $r = \alpha c$.

Figure 3: Empirical results on the Arcene data set.
(a) $k = 10$, $c = \alpha k$, and $r = \alpha c$.  
(b) $k = 20$, $c = \alpha k$, and $r = \alpha c$.  
(c) $k = 50$, $c = \alpha k$, and $r = \alpha c$.

Figure 4: Empirical results on the Dexter data set.

(a) $k = 10$, $c = \alpha k$, and $r = \alpha c$.  
(b) $k = 20$, $c = \alpha k$, and $r = \alpha c$.  
(c) $k = 50$, $c = \alpha k$, and $r = \alpha c$.

Figure 5: Empirical results on the HMAX features of the PicasaWeb image data set.
In this paper we have proposed a novel randomized algorithm for the CUR matrix decomposition problem. This algorithm is faster, more scalable, and more accurate than the state-of-the-art algorithm, i.e., the subspace sampling algorithm. Our algorithm requires only $c = 2ke^{-1}(1+o(1))$ columns and $r = 2ce^{-1}(1+o(1))$ rows to achieve $(1+\epsilon)$ relative-error ratio. To achieve the same relative-error bound, the subspace sampling algorithm requires $c = \mathcal{O}(ke^{-2}\log k)$ columns and $r = \mathcal{O}(ce^{-2}\log c)$ rows selected from the original matrix. Our algorithm also beats the subspace sampling algorithms in time-complexity. Our algorithm costs $\mathcal{O}(mnke^{-2/3} + (m+n)k^3\epsilon^{-2/3} + m^2k^2\epsilon^{-2} + nk^2\epsilon^{-4})$ time, which is lower than $\mathcal{O}(\min\{mn^2, m^2n\})$ of the subspace sampling algorithms when $k$ is small. Moreover, our algorithm enjoys another advantage of avoiding loading the whole data matrix into main memory, which also makes our algorithm more scalable. Finally, the empirical comparisons have also demonstrated the effectiveness and efficiency of our algorithm.

However, there are several open questions involving the lower bound of the CUR matrix decomposition problem. First, what is the lower bound for the CUR problem? Second, is there any algorithm achieving such a lower bound? Boutsidis et al. (2011b) proved a lower bound for the column selection problem: $\frac{\|A-CC^TA\|_F^2}{\|A-A_k\|_F^2} \geq 1 + \frac{k}{c}$. We thus wonder if there is a similar lower bound on the ratio $\frac{\|A-CC^TAR\|_F^2}{\|A-CC^TA\|_F^2}$, e.g., say $(1 + \frac{\text{rank}(C)}{r})$. We shall address these questions in future work.

Acknowledgments
Algorithm 2 Deterministic Dual Set Spectral-Frobenius Sparsification Algorithm.

1: **Input:** $\mathcal{U} = \{x_i\}_{i=1}^n \subset \mathbb{R}^l$ ($l < n$); $\mathcal{V} = \{v_i\}_{i=1}^n \subset \mathbb{R}^k$, with $\sum_{i=1}^n v_i v_i^T = I_k$ ($k < n$); $k < r < n$;
2: **Initialize:** $s_0 = 0$, $A_0 = 0$;
3: Compute $\|x_i\|^2_2$ for $i = 1, \cdots, n$, and then compute $\delta_U = \frac{\sum_{i=1}^n \|x_i\|^2_2}{1 - \sqrt{k/r}}$;
4: for $\tau = 0$ to $r - 1$ do
5:   Compute the eigenvalue decomposition of $A_\tau$;
6:   Find an index $j$ in $\{1, \cdots, n\}$ and compute a weight $t > 0$ such that
7:   $\delta_U^{-1}\|x_j\|^2_2 \leq t^{-1} \leq \frac{v_j^T (A_\tau - (L_\tau + 1) I_k)^{-2} v_j}{\phi(L_\tau + 1, A_\tau) - \phi(L_\tau, A_\tau) - v_j^T (A_\tau - (L_\tau + 1) I_k)^{-1} v_j};$
8:   Update the $j$-th component of $s_\tau$ and $A_\tau$: $s_{\tau+1}[j] = s_\tau[j] + t$, $A_{\tau+1} = A_\tau + tv_jv_j^T$;
9: end for
10: **return** $s = \frac{1}{r - \sqrt{k/r}} s_r$.

This work has been supported in part by the Natural Science Foundations of China (No. 61070239) and the Google visiting faculty program.

Appendix A. The Dual Set Sparsification Algorithm

For the sake of completeness, we attach the dual set sparsification algorithm here and describe some implementation details. The dual set sparsification algorithms are deterministic algorithms established in Boutsidis et al. (2011a). The fast CUR algorithm calls the dual set spectral-Frobenius sparsification algorithm (Lemma 13 in Boutsidis et al., 2011a) in both stages. We show this algorithm in Algorithm 2 and its bounds in Lemma 10.

Lemma 10 (Dual Set Spectral-Frobenius Sparsification) Let $\mathcal{U} = \{x_1, \cdots, x_n\} \subset \mathbb{R}^l$ ($l < n$) contain the columns of an arbitrary matrix $X \in \mathbb{R}^{l \times n}$. Let $\mathcal{V} = \{v_1, \cdots, v_n\} \subset \mathbb{R}^k$ ($k < n$) be a decompositions of the identity, i.e., $\sum_{i=1}^n v_i v_i^T = I_k$. Given an integer $r$ with $k < r < n$, Algorithm 2 deterministically computes a set of weights $s_i \geq 0$ ($i = 1, \cdots, n$) at most $r$ of which are non-zero, such that

$$\lambda_k \left( \sum_{i=1}^n s_i v_i v_i^T \right) \geq \left( 1 - \sqrt{\frac{k}{r}} \right)^2 \quad \text{and} \quad \text{tr} \left( \sum_{i=1}^n s_i x_i x_i^T \right) \leq \|X\|_F^2.$$

The weights $s_i$ can be computed deterministically in $O(rnk^2 + nl)$ time.

Here we would like to mention the implementation of Algorithm 2, which is not described by Boutsidis et al. (2011a) in details. In each iteration the algorithm performs once eigenvalue decomposition: $A_\tau = WAW^T$. Here $A_\tau$ is guaranteed to be positive semi-definite in each iteration. Since

$$\left( A_\tau - \alpha I_k \right)^q = W \text{Diag} \left( (\lambda_1 - \alpha)^q, \cdots, (\lambda_k - \alpha)^q \right) W^T,$$
we can efficiently compute \((A_\tau - (L_\tau + 1)I_k)^q\) based on the eigenvalue decomposition of \(A_\tau\). With the eigenvalues at hand, \(\phi(L, A_\tau)\) can also be computed directly.

The algorithm runs in \(r\) iterations. In each iteration, the eigenvalue decomposition of \(A_\tau\) requires \(O(k^3)\), and the \(n\) comparisons in Line 6 each requires \(O(k^2)\). Moreover, computing \(\|x_i\|_2^2\) for each \(x_i\) requires \(O(nl)\). Overall, the running time of Algorithm 2 is at most \(O(rk^3) + O(rnk^2) + O(nl) = O(rnk^2 + nl)\).

### Appendix B. Proofs

#### B.1 The Proof of Theorem 7

Theorem 7 can be equivalently expressed in Theorem 11. In order to stick to the column space convention of Boutsidis et al. (2011a), we prove Theorem 11 instead of Theorem 7.

**Theorem 11 (Adaptive Sampling Algorithm)** Given a matrix \(A \in \mathbb{R}^{m \times n}\) and a matrix \(R \in \mathbb{R}^{r \times n}\) such that \(\text{rank}(R) = \text{rank}(AR^\dagger R) = \rho \ (\rho \leq r \leq m)\), let \(C_1 \in \mathbb{R}^{m \times c_1}\) consist of \(c_1\) columns of \(A\), and define the residual \(B = A - C_1C_1^\dagger A\). For \(i = 1, \ldots, n\), let

\[
p_i = \frac{\|b_i\|^2}{\|B\|^2_F},
\]

where \(b_i\) is the \(i\)-th column of the matrix \(B\). Sample further \(c_2\) columns from \(A\) in \(c_2\) i.i.d. trials, where in each trial the \(i\)-th column is chosen with probability \(p_i\). Let \(C_2 \in \mathbb{R}^{m \times c_2}\) contain the \(c_2\) sampled columns and \(C = [C_1, C_2] \in \mathbb{R}^{m \times (c_1 + c_2)}\) contain the columns of both \(C_1\) and \(C_2\), all of which are columns of \(A\). Then the following inequality holds:

\[
E\|A - CC^\dagger AR^\dagger R\|^2_F \leq \|A - AR^\dagger R\|^2_F + \frac{\rho}{c_2}\|A - C_1C_1^\dagger A\|^2_F.
\]

where the expectation is taken w.r.t. \(C_2\).

**Proof** With a little abuse of symbols, we use bold uppercase letters to denote matrix random variables and bold lowercase to denote vector random variables, without distinguishing between matrix/vector random variables and constant matrices/vectors.

We denote the \(j\)-th column of \(V_{AR^\dagger R, \rho} \in \mathbb{R}^{r \times \rho}\) as \(v_j\), and the \((i, j)\)-th entry of \(V_{AR^\dagger R, \rho}\) as \(v_{ij}\). Define vector random variables \(x_{j, (l)} \in \mathbb{R}^m\) such that for \(j = 1, \ldots, n\) and \(l = 1, \ldots, c_2\),

\[
x_{j, (l)} = \frac{v_{ij}}{p_i} b_i = \frac{v_{ij}}{p_i} (a_i - C_1C_1^\dagger a_i) \quad \text{with probability } p_i, \quad \text{for } i = 1, \ldots, n,
\]

Note that \(x_{j, (l)}\) is a linear function of a column of \(A\) sampled from the above defined distribution. We have that

\[
E[x_{j, (l)}] = \sum_{i=1}^n p_i \frac{v_{ij}}{p_i} b_i = Bv_j,
\]

\[
E\|x_{j, (l)}\|^2_2 = \sum_{i=1}^n p_i \frac{v_{ij}^2}{p_i} \|b_i\|^2 = \sum_{i=1}^n \frac{v_{ij}^2}{\|b_i\|^2} \|b_i\|^2 = \|B\|^2_F.
\]
Then we let \( x_j = \frac{1}{c_2} \sum_{l=1}^{c_2} x_{j,(l)} \), we have
\[
E[x_j] = E[x_{j,(l)}] = Bv_j,
\]
\[
E\|x_j - Bv_j\|_2^2 = E\|x_j - E[x_j]\|_2^2 = \frac{1}{c_2} E\|x_{j,(l)} - E[x_{j,(l)}]\|_2^2 = \frac{1}{c_2} E\|x_{j,(l)} - Bv_j\|_2^2.
\]
According to the construction of \( x_1, \cdots, x_\rho \), we define the \( c_2 \) columns of \( A \) to be \( C_2 \in \mathbb{R}^{m \times c_2} \). Note that all the random variables \( x_1 \cdots, x_\rho \) lie in the subspace \( \text{span}(C_1) + \text{span}(C_2) \). We define random variables
\[
w_j = C_1 C_1^\dagger A R^\dagger R v_j + x_j = C_1 C_1^\dagger A v_j + x_j, \quad \text{for } j = 1, \cdots, \rho,
\]
where the second equality follows from Lemma 12 that \( AR^\dagger R v_j = A v_j \) if \( v_j \) is one of the top \( \rho \) right singular vectors of \( AR^\dagger R \). Then we have that any set of random variables \( \{w_1, \cdots, w_\rho\} \) lies in \( \text{span}(C) = \text{span}(C_1) + \text{span}(C_2) \). Let \( W = [w_1, \cdots, w_\rho] \) be a matrix random variable, we have that \( \text{span}(W) \subset \text{span}(C) \). The expectation of \( w_j \) is
\[
E[w_j] = C_1 C_1^\dagger A v_j + E[x_j] = C_1 C_1^\dagger A v_j + B v_j = A v_j,
\]
therefore we have that
\[
w_j - A v_j = x_j - B v_j.
\]
The expectation of \( \|w_j - A v_j\|_2^2 \) is
\[
E\|w_j - A v_j\|_2^2 = E\|x_j - B v_j\|_2^2 = \frac{1}{c_2} E\|x_{j,(l)} - B v_j\|_2^2
\]
\[
= \frac{1}{c_2} E\|x_{j,(l)}\|_2^2 - \frac{2}{c_2} (B v_j)^T E[x_{j,(l)}] + \frac{1}{c_2} \|B v_j\|_2^2
\]
\[
= \frac{1}{c_2} E\|x_{j,(l)}\|_2^2 - \frac{1}{c_2} \|B v_j\|_2^2 = \frac{1}{c_2} \|B\|_F^2 - \frac{1}{c_2} \|B v_j\|_2^2
\]
\[
\leq \frac{1}{c_2} \|B\|_F^2.
\]
To complete the proof, we let the matrix variable
\[
F = \left( \sum_{q=1}^{\rho} \sigma_q^{-1} w_q u_q^T \right) AR^\dagger R,
\]
where \( \sigma_q \) is the \( q \)-th largest singular value of \( AR^\dagger R \) and \( u_q \) is the corresponding left singular vector of \( AR^\dagger R \). The column space of \( F \) is contained in \( \text{span}(W) \subset \text{span}(C) \), and thus
\[
\|AR^\dagger R - CC^\dagger AR^\dagger R\|_F^2 \leq \|AR^\dagger R - WW^\dagger AR^\dagger R\|_F^2 \leq \|AR^\dagger R - F\|_F^2.
\]
We use \( F \) to bound the error \( \|AR^\dagger R - CC^\dagger AR^\dagger R\|_F^2 \):
\[
E\|A - CC^\dagger AR^\dagger R\|_F^2 = E\|A - AR^\dagger R + AR^\dagger R - CC^\dagger AR^\dagger R\|_F^2
\]
\[
= E\left[\|A - AR^\dagger R\|_F^2 + \|AR^\dagger R - CC^\dagger AR^\dagger R\|_F^2\right]
\]
\[
\leq \|A - AR^\dagger R\|_F^2 + E\|AR^\dagger R - F\|_F^2.
\]
we decompose \( AR \) projection of \( A \) where (3) follows from Lemma 12 and (4) follows from (1). A similar bound is guaranteed.

Lemma 13 is a direct corollary of Lemma 4 and Lemma 5. If we apply the same algorithm relative-error bound in Lemma 13. This algorithm is described in Line 3 to 6 of Algorithm 1.

Boutsidis et al. (2011a) proposed a randomized algorithm which achieves the expected

\[ E\|A - CC^\dagger AR^\dagger R\|_F^2 \leq \|A - AR^\dagger R\|_F^2 + E\|AR^\dagger R - F\|_F^2, \]

\[ = \|A - AR^\dagger R\|_F^2 + \sum_{j=1}^{\rho} E\|AR^\dagger Rv_j - (\sum_{q=1}^{\rho} \sigma_q^{-1}w_q u_q^T)\sigma_j u_j\|_2^2 \]

\[ = \|A - AR^\dagger R\|_F^2 + \sum_{j=1}^{\rho} E\|AR^\dagger Rv_j - w_j\|_2^2 \]

\[ = \|A - AR^\dagger R\|_F^2 + \sum_{j=1}^{\rho} E\|Av_j - w_j\|_2^2 \]

\[ \leq \|A - AR^\dagger R\|_F^2 + \frac{\rho}{c_2} \|B\|_F^2, \]

where (3) follows from Lemma 12 and (4) follows from (1).

\[ \text{Lemma 12} \quad \text{We are given a matrix } A \in \mathbb{R}^{m \times n} \text{ and a matrix } R \in \mathbb{R}^{r \times n} \text{ such that } \text{rank}(AR^\dagger R) = \text{rank}(R) = \rho (\rho \leq r \leq m). \text{ Letting } v_j \in \mathbb{R}^n \text{ be the } j\text{-th top right singular vector of } AR^\dagger R, \text{ we have that } AR^\dagger Rv_j = Av_j, \quad \text{ for } j = 1, \ldots, \rho. \]

**Proof** First let \( V_{R,\rho} \in \mathbb{R}^{n \times \rho} \) contain the top \( \rho \) right singular vectors of \( R \), then the projection of \( A \) onto the row space of \( R \) is \( AR^\dagger R = AV_{R,\rho} V_{R,\rho}^T \). Let the thin SVD of \( AV_{R,\rho} \in \mathbb{R}^{m \times n} \) be \( \tilde{U} \tilde{\Sigma} \tilde{V}^T \), where \( \tilde{V} \in \mathbb{R}^{\rho \times \rho} \). Then the compact SVD of \( AR^\dagger R \) is

\[ AR^\dagger R = AV_{R,\rho} V_{R,\rho}^T = \tilde{U} \tilde{\Sigma} \tilde{V}^T V_{R,\rho}^T. \]

According to the definition, \( v_j \) is the \( j\)-th column of \( (V_{R,\rho} \tilde{V}) \in \mathbb{R}^{n \times \rho} \), and thus \( v_j \) lies on the column space of \( V_{R,\rho} \), and \( v_j \) is orthogonal to \( V_{R,\rho} \). Finally, since \( A - AR^\dagger R = AV_{R,\rho} V_{R,\rho}^T V_{R,\rho}^T, \) we have that \( v_j \) is orthogonal to \( A - AR^\dagger R \), that is, \( (A - AR^\dagger R)v_j = 0 \), which directly proves the lemma.

**B.2 The Proof of Theorem 8**

Boutsidis et al. (2011a) proposed a randomized algorithm which achieves the expected relative-error bound in Lemma 13. This algorithm is described in Line 3 to 6 of Algorithm 1. Lemma 13 is a direct corollary of Lemma 4 and Lemma 5. If we apply the same algorithm to \( A^T \) to select \( c \) rows of \( A \) to form \( R_1 \), that is, Line 11 to 13 of Algorithm 1, then a very similar bound is guaranteed.
Lemma 13 (Boutsidis et al. (2011a), Theorem 4) \( \) Given a matrix \( A \in \mathbb{R}^{m \times n} \) of rank \( \rho \), a target rank \( 2 \leq k < \rho \), and \( 0 < \epsilon_0 < 1 \), there is a randomized algorithm to select \( c_1 > k \) columns of \( A \) and form a matrix \( C_1 \in \mathbb{R}^{m \times c_1} \) such that

\[
\mathbb{E}\|A - C_1C_1^\dagger A\|_F^2 \leq (1 + \epsilon_0)\left(1 + \frac{1}{(1 - \sqrt{k/c_1})^2}\right)\|A - A_k\|_F^2,
\]

where the expectation is taken w.r.t. \( C_1 \). The matrix \( C_1 \) can be computed in \( \mathcal{O}(mnk\epsilon_0^{-1} + nck_2) \) time.

With Theorem 7 and Lemma 13, we now prove Theorem 8 as follows.

**Proof** \( \) This randomized algorithm has three steps: approximate SVD via randomized projection (Halko et al., 2011), deterministic column selection via dual set sparsification algorithm (Boutsidis et al., 2011a) shown in Lemma 5, and the adaptive sampling algorithm of Theorem 7 proved in this paper. This algorithm is a generalization of the near-optimal algorithm (Boutsidis et al., 2011a) shown in Lemma 5, and the adaptive sampling algorithm (Halko et al., 2011), deterministic column selection via dual set sparsification algorithm taking \( U \) and \( V \) as input, where \( U \) contains all the \( m \) columns of \( (A^T - \hat{A}_k^T) \in \mathbb{R}^{n \times m} \), \( V \) contains all the \( m \) columns of \( \hat{U}_A^T \in \mathbb{R}^{k \times m} \). Lemma 13 shows that

\[
\mathbb{E}\|A - \hat{U}_k \hat{\Sigma}_k \hat{V}_k^T\|_F^2 \leq (1 + \epsilon_0)\|A - A_k\|_F^2.
\]

Step 2 (Line 11 to 13 of Algorithm 1) selects \( r_1 \) rows of \( A \) to construct \( R_1 \) by the dual set sparsification algorithm taking \( U \) and \( V \) as input, where \( U \) contains all the \( m \) columns of \( (A^T - \hat{A}_k^T) \in \mathbb{R}^{n \times m} \), \( V \) contains all the \( m \) columns of \( \hat{U}_A^T \in \mathbb{R}^{k \times m} \). Lemma 13 shows that

\[
\mathbb{E}\|A - AR_1^T R_1\|_F^2 \leq (1 + \epsilon_0)\left(1 + \frac{1}{(1 - \sqrt{k/r_1})^2}\right)\|A - A_k\|_F^2,
\]

where the expectation is taken w.r.t. \( R_1 \). Step 2 costs \( \mathcal{O}(mr_1k^2 + mn) \) time.

Step 3 (Line 14 to 16 of Algorithm 1) samples additional \( r_2 \) rows of \( A \) to construct \( R_2 \in \mathbb{R}^{r_2 \times n} \) by the adaptive sampling algorithm of Theorem 7. Let \( R = [R_1^T, R_2^T]^T \in \mathbb{R}^{(r_1 + r_2) \times n} \). We apply Theorem 7 and have that

\[
\mathbb{E}_R\|A - CC^\dagger AR_1^T R_2\|_F^2 = \mathbb{E}_{R_1}\left[\mathbb{E}_{R_2}\left[\|A - CC^\dagger AR_1^T R_2\|_F^2 | R_1\right]\right]
\]

\[
\leq \mathbb{E}_{R_1}\left[\|A - CC^\dagger A\|_F^2 + \frac{\rho}{r_2}\|A - AR_1^T R_2\|_F^2\right]
\]

\[
\leq \|A - CC^\dagger A\|_F^2 + \frac{\rho}{r_2}(1 + \epsilon_0)\left(1 + \frac{1}{(1 - \sqrt{k/r_1})^2}\right)\|A - A_k\|_F^2.
\]

By setting \( r_1 = \mathcal{O}(k\epsilon^{-2/3}) \), \( r_2 \approx \frac{2\rho}{\epsilon} \), and \( \epsilon_0 = \epsilon^{2/3} \), we conclude that

\[
\mathbb{E}\|A - CC^\dagger AR_1^T R_2\|_F^2 \leq \|A - CC^\dagger A\|_F^2 + \epsilon\|A - A_k\|_F^2.
\]

The total computation time of the three steps is \( \mathcal{O}(mnk/\epsilon_0 + mr_1k^2 + mn) = \mathcal{O}(mnk/mk^3\epsilon^{-2/3}) \). \( \square \)
B.3 The Proof of Theorem 9

Proof Since $C$ is constructed by columns of $A$, the column space of $C$ is contained in the column space of $A$, so $\text{rank}(CC^\dagger A) = \text{rank}(C) = \rho \leq c$, and thus the assumptions of Theorem 8 are satisfied. Lemma 3 and Theorem 8 together prove Theorem 9:

$$E^2\|A - \text{CUR}\|_F \leq E\|A - \text{CUR}\|^2_F = E_{C,R}\|A - CC^\dagger AR^\dagger R\|^2_F$$

$$= E_C[E_R[\|A - CC^\dagger AR^\dagger R\|^2_F | C]]$$

$$\leq E_C[\|A - CC^\dagger A\|^2_F + \epsilon\|A - A_k\|^2_F]$$

$$\leq (1 + 2\epsilon)\|A - A_k\|^2_k.$$

Finally we have $E\|A - \text{CUR}\|_F \leq (1 + \epsilon)\|A - A_k\|_k$ because $1 + 2\epsilon \leq (1 + \epsilon)^2$.

The time cost of the fast CUR algorithm is the sum of Stage 1, Stage 2, and the Moore-Penrose inverse of $C$ and $R$, i.e. $O((mnk + nk^3)\epsilon^{-2/3}) + O((mnk + mk^3)\epsilon^{-2/3}) + O(m\epsilon^2) + O(nr^2) = O(mnk\epsilon^{-2/3} + (m + n)k^3\epsilon^{-2/3} + mk^2\epsilon^{-2} + nk^2\epsilon^{-4})$. \hfill \blacksquare

References

Aseem Agarwala. Efficient gradient-domain compositing using quadtrees. In SIGGRAPH 2007, 2007.

Adi Ben-Israel and Thomas N.E. Greville. Generalized Inverses: Theory and Applications. Second Edition. Springer, 2003.

Christos Boutsidis, Petros Drineas, and Malik Magdon-Ismail. Near-optimal column-based matrix reconstruction. CoRR, abs/1103.0995, 2011a.

Christos Boutsidis, Petros Drineas, and Malik Magdon-Ismail. Near optimal column-based matrix reconstruction. In Proceedings of the 2011 IEEE 52nd Annual Symposium on Foundations of Computer Science, FOCS ’11, pages 305–314, 2011b.

Scott Deerwester, Susan T. Dumais, George W. Furnas, Thomas K. Landauer, and Richard Harshman. Indexing by latent semantic analysis. Journal of The American Society for Information Science, 41(6):391–407, 1990.

Amit Deshpande and Luis Rademacher. Efficient volume sampling for row/column subset selection. In Proceedings of the 2010 IEEE 51st Annual Symposium on Foundations of Computer Science, FOCS ’10, pages 329–338, 2010.

Amit Deshpande, Luis Rademacher, Santosh Vempala, and Grant Wang. Matrix approximation and projective clustering via volume sampling. Theory of Computing, 2(2006):225–247, 2006.

Petros Drineas. Pass-efficient algorithms for approximating large matrices. In In Proceeding of the 14th Annual ACM-SIAM Symposium on Discrete Algorithms, pages 223–232, 2003.
Petros Drineas and Michael W. Mahoney. On the Nyström method for approximating a
gram matrix for improved kernel-based learning. *Journal of Machine Learning Research*,
6:2153–2175, 2005.

Petros Drineas, Ravi Kannan, and Michael W. Mahoney. Fast monte carlo algorithms for
matrices iii: Computing a compressed approximate matrix decomposition. *SIAM Journal
on Computing*, 36(1):184–206, 2006.

Petros Drineas, Michael W. Mahoney, and S. Muthukrishnan. Relative-error CUR matrix
decompositions. *SIAM Journal on Matrix Analysis and Applications*, 30(2):844–881,
September 2008.

A. Frank and A. Asuncion. UCI machine learning repository, 2010. URL
http://archive.ics.uci.edu/ml.

Alan Frieze, Ravi Kannan, and Santosh Vempala. Fast monte-carlo algorithms for finding
low-rank approximations. *Journal of the ACM*, 51(6):1025–1041, November 2004. ISSN
0004-5411.

S. A. Goreinov, E. E. Tyrtyshnikov, and N. L. Zamarashkin. A theory of pseudoskeleton
approximations. *Linear Algebra and Its Applications*, 261:1–21, 1997a.

S. A. Goreinov, N. L. Zamarashkin, and E. E. Tyrtyshnikov. Pseudo-skeleton approxima-
tions by matrices of maximal volume. *Mathematical Notes*, 62(4):619–623, 1997b.

Venkatesan Guruswami and Ali Kemal Sinop. Optimal column-based low-rank matrix re-
construction. In *Proceedings of the Twenty-Third Annual ACM-SIAM Symposium on
Discrete Algorithms*, SODA ’12, pages 1207–1214. SIAM, 2012.

Nathan Halko, Per-Gunnar Martinsson, and Joel A. Tropp. Finding structure with random-
ness: Probabilistic algorithms for constructing approximate matrix decompositions.
*SIAM Review*, 53(2):217–288, 2011.

John Hopcroft and Ravi Kannan. *Computer Science Theory for the Information Age*. 2012.

Finny G. Kuruvilla, Peter J. Park, and Stuart L. Schreiber. Vector algebra in the analysis
of genome-wide expression data. *Genome Biology*, 3:research0011–research0011.1, 2002.

David G. Lowe. Object recognition from local scale-invariant features. In *Proceedings of
the International Conference on Computer Vision*, ICCV 99, 1999.

Lester Mackey, Ameet Talwalkar, and Michael I. Jordan. Divide-and-conquer matrix fac-
tORIZATION. In *Advances in Neural Information Processing Systems 24*. 2011.

Michael W. Mahoney and Petros Drineas. CUR matrix decompositions for improved data
analysis. *Proceedings of the National Academy of Sciences*, 106(3):697–702, 2009. URL
http://www.pnas.org/content/106/3/697.abstract.

Thomas Serre, Lior Wolf, Stanley Bileschi, Maximilian Riesenhuber, and Tomaso Poggio.
Robust object recognition with cortex-like mechanisms. *IEEE Transactions on Pattern
Analysis and Machine Intelligence*, 29:411–426, 2007.

20
L. Sirovich and M. Kirby. Low-dimensional procedure for the characterization of human faces. *Journal of the Optical Society of America A*, 4(3):519–524, Mar 1987.

Matthew Turk and Alex Pentland. Eigenfaces for recognition. *Journal of Cognitive Neuroscience*, 3(1):71–86, 1991.

Eugene E. Tyrtyshnikov. Incomplete cross approximation in the mosaic-skeleton method. *Computing*, 64:367–380, 2000.

Zhiyu Wang, Fangtao Li, Edward Y. Chang, and Shiqiang Yang. A data-driven study on image feature extraction and fusion. *Manuscript*, 2012.