GLOBAL REGULARITY AND ASYMPTOTIC BEHAVIOR OF MODIFIED NAVIER-STOKES EQUATIONS WITH FRACTIONAL DISSIPATION

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Abstract. This paper is concerned with the Cauchy problem of three-dimensional modified Navier-Stokes equations with fractional dissipation $\nu(-\Delta)^\alpha u$. The results are three-fold. We first prove the global existence of weak solutions for $0 < \alpha \leq 1$ and global smooth solution for $\frac{3}{4} < \alpha \leq 1$. Second, we obtain the optimal decay rates of both weak solutions and the higher-order derivative of the smooth solution. Finally, we investigate the asymptotic stability of the large solution to the system under large initial and external forcing perturbation.

1. Introduction. Mathematical models for fluid dynamics play an important role in theoretical and computational studies in meteorological, oceanographic sciences and petroleum industries, etc. Navier-Stokes equations [22] are generally accepted as proving an accurate model for the incompressible motion of viscous fluids in many practical situations, which presume the derivatives of the components of the velocity are small. Since Leray’s pioneer work [14] in 1930’, however, the question of global regularity or finite-time singularity of three-dimensional Navier-Stokes equations with large initial data is still a big open problem (see [22]). Recently, Caraballo, Real and Kloeden [1] (see also Kloeden, Langa and Real [2, 11]) introduced an interesting and important mathematical model which is so-called the globally modification of the Navier-Stokes equations

\[
\begin{align*}
\partial_t u + F_N(\|\nabla u\|_{L^2})(u \cdot \nabla u) - \nu \Delta u + \nabla p &= 0, \\
\nabla \cdot u &= 0,
\end{align*}
\]

(1.1)

where $F_N$ (for some $N \in \mathbb{R}^+$) is defined by

\[
F_N(r) = \min \left\{ 1, \frac{N}{r} \right\}, \quad r \in \mathbb{R}^+.
\]

As stated by Caraballo, Real and Kloeden [1], the system (1.1) is indeed globally modified — the modifying factor $F_N(\|\nabla u\|_{L^2})$ depends on the norm $\|\nabla u\|_{L^2}$.

2000 Mathematics Subject Classification. 35Q30, 76D05.

Key words and phrases. Modified Navier-Stokes equations, fractional dissipation time decay, asymptotic stability.

This work is partially supported by the NNSF of China (10801001,11071001), NSF of Anhui Province(11040606M02) and is also financed by the 211 Project of Anhui University (KJTD002B, KJJQ005).
Essentially, it prevents large gradients dominating the dynamics and leading to explosions. The system (1.1) may violate the basic laws of mechanics, however, on the viewpoint of mathematics, it is a well defined system of equations, just like the modified versions of the Navier-Stokes equations of Leray and others with other mollifications of the nonlinear term (refer to Constantin [3]). Compared with the 3D Navier-Stokes equations, the system (1.1) in three dimensional case has a unique global smooth solution [1].

Since the presence of the modifying factor $F_N(\|\nabla u\|_{L^2})$ more or less decreases the singularity of the quadratic convection term $u \cdot \nabla u$, it is possible to control the nonlinear term by using the lower dissipation $\nu(-\Delta)^\alpha u$. Once the observation is right, it is an interesting problem to investigate the new feature such as the well-posedness and large time behavior of solutions compared with the classic Navier-Stokes equations. In this study, we consider this sort globally modified Navier-Stokes equations with fractional dissipation in whole space $\mathbb{R}^3$

$$\begin{cases}
\partial_t u + F_N(\|\nabla u\|_{L^2})(u \cdot \nabla u) + \nu(-\Delta)^\alpha u + \nabla p = 0, & 0 < \alpha \leq 1, \\
\nabla \cdot u = 0
\end{cases}$$

(1.3)

with the initial condition

$$u(x, 0) = u_0.$$  \hspace{1cm} (1.4)

The main purpose of this study is to give a complete description on the well-posedness and asymptotic behavior of the solutions to system (1.3)-(1.4). More precisely, on one hand, as regards the existence of solutions, when $0 < \alpha \leq 1$, we first construct a global weak solution $u \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^\alpha(\mathbb{R}^3))$ of system (1.3)-(1.4) with $u_0 \in L^2(\mathbb{R}^3)$ by applying the classic Friedrichs method and Lions-Aubin compactness argument. Second, we prove the local existence of smooth solution $u \in C([0, T^*); H^s(\mathbb{R}^3)) \cap L^2(0, T^*; H^{s+\alpha}(\mathbb{R}^3))$ and obtain a Beale-Kato-Majda type blow-up criterion under $u_0 \in H^s(\mathbb{R}^3), s > 3$. When $\frac{3}{4} < \alpha \leq 1$, we prove the global existence of smooth solution to system (1.3)-(1.4). On the other hand, it is desirable to understand the asymptotic behavior of solutions to system (1.3)-(1.4). As respect to the time decay of solutions, by developing the classic Fourier splitting methods introduced by Schonbek [18], we derive the optimal time decay estimates of the higher-order derivative of smooth smooth, a new analysis method including iterative technique is employed. Furthermore, since the time decay problem of solutions implies that the trivial solution $u = 0$ is asymptotic stable, it is an interesting problem to consider the asymptotic stability for the nontrivial solution of system (1.3)-(1.4) with nonzero force. Another objective of this paper is to investigate the asymptotic stability of large solution to the system, we will show the $L^2$ stability of the difference between the solution of the original system and the perturbed system under large initial data and external forcing perturbation.

To this end, let us introduce the assignment of this paper. In Section 2, we first prove the existence of weak solutions and local smooth solution to system (1.3)-(1.4) with $0 < \alpha \leq 1$, and then further obtain the global smooth solution with $\frac{3}{4} < \alpha \leq 1$. In Section 3, we prove the optimal $L^2$ decay rates for both weak solutions and the higher-order derivative of smooth solution. Finally we study the asymptotic stability of the system under large initial and external forcing perturbation in Section 4.
2. **Global smooth solution.** In this section we will show that system (1.3)-(1.4) has a global weak solution corresponding to any prescribed \(L^2\) initial data and the global smooth solution for \(\frac{3}{4} < \alpha \leq 1\). To do so, let us first recall some basic facts about Littlewood-Paley decomposition (see [4] for more details). Choose two nonnegative radial functions \(\chi, \varphi \in \mathcal{S}(\mathbb{R}^3)\) supported respectively in \(B = \{\xi \in \mathbb{R}^3, |\xi| \leq \frac{\alpha}{4}\}\) and \(C = \{\xi \in \mathbb{R}^3, \frac{3}{4} \leq |\xi| \leq \frac{5}{4}\}\) such that for any \(\xi \in \mathbb{R}^3\),

\[
\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1. \tag{2.1}
\]

Let \(h = \mathcal{F}^{-1} \varphi\) and \(\hat{h} = \mathcal{F}^{-1} \chi\), the frequency localization operator \(\Delta_j\) and \(S_j\) are defined by

\[
\Delta_j f = \varphi(2^{-j}D)f = 2^{3j} \int_{\mathbb{R}^3} h(2^{j}y)f(x - y)\, dy, \quad \text{for } j \geq 0,
\]

\[
S_j f = \chi(2^{-j}D)f = \sum_{-1 \leq k \leq j - 1} \Delta_k f = 2^{3j} \int_{\mathbb{R}^3} \hat{h}(2^{j}y)f(x - y)\, dy, \quad \text{and}
\]

\[
\Delta_{-1} f = S_0 f, \quad \Delta_j f = 0 \quad \text{for } j \leq -2.
\]

For any \(f \in \mathcal{S}'(\mathbb{R}^3)\), we have by (2.1) that

\[
f = S_0(f) + \sum_{j \geq 0} \Delta_j f, \tag{2.2}
\]

which is called Littlewood-Paley decomposition. The norm of Sobolev space \(H^s(\mathbb{R}^3)\) can be characterized in terms of \(\Delta_j\),

\[
\|f\|_{H^s} = \|S_0(f)\|_{L^2} + \left(\sum_{j \geq 0} 2^{2js}\|\Delta_j f\|_{L^2}^2\right)^{\frac{1}{2}}. \tag{2.3}
\]

Before state the main results in this section, we also need some useful lemmas.

**Lemma 2.1.** (Bernstein inequalities [4]) Let \(1 \leq p \leq q \leq \infty\). Assume that \(f \in L^p(\mathbb{R}^3)\), then there hold

\[
\sup \hat{f} \subset \{|\xi| \leq C2^j\} \Rightarrow \|\partial^\alpha f\|_{L^q} \leq C2^{j|\alpha|+3j(\frac{1}{q} - \frac{1}{2})}\|f\|_{L^p}, \tag{2.4}
\]

\[
\sup \hat{f} \subset \{\frac{1}{C}2^j \leq |\xi| \leq C2^j\} \Rightarrow \|f\|_{L^p} \leq C2^{-j|\alpha|} \sup_{|\beta| = |\alpha|} \|\partial^\beta f\|_{L^p}. \tag{2.5}
\]

Here the constant \(C \geq 1\) is independent of \(f\) and \(j\).

**Lemma 2.2.** (Commutator estimates [15]) Let \(s > -1\). Assume that \(u_h \in H^s(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)\) and \(\nabla v \in H^s(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)\) with \(\nabla \cdot v = 0\). Then there holds

\[
\|[\Delta_j, v] \cdot \nabla u\|_{L^2} \leq Cc_j2^{-js}(\|\nabla v\|_{L^\infty}\|u\|_{H^s} + \|\nabla v\|_{H^s}\|u\|_{L^\infty}), \tag{2.6}
\]

where the commutator \([a, b]\) is defined by \([a, b] \triangleq ab - ba\) and \(\{c_j\}\) is a sequence satisfying \(\|\{c_j\}\|_{\ell^2} \leq 1\).

**Lemma 2.3.** (Logarithmic Sobolev inequality [12]) Assume that \(w \in H^s(\mathbb{R}^3), s > 2\). Then there holds

\[
\|w\|_{L^\infty} \leq C(1 + \|w\|_{BMO}) \ln(e + \|w\|_{H^s}). \tag{2.7}
\]

Now our results read as follows.
Theorem 2.1. (i) (Global existence of weak solutions) Let \( u_0 \in L^2(\mathbb{R}^3) \), then the system (1.3)-(1.4) with \( 0 < \alpha \leq 1 \) possesses a global weak solution satisfying

\[
u \in L^\infty(0,T; L^2(\mathbb{R}^3)) \cap L^2(0,T; H^\alpha(\mathbb{R}^3)), \quad \forall \ T > 0. \quad (2.8)
\]

(ii) (Local smooth solution and blow-up criterion) Suppose \( u_0 \in H^s(\mathbb{R}^3), s > 3 \). Then there exists \( T_1 = T_1(\|u_0\|_{H^m}) > 0 \) such that the system (1.3)-(1.4) with \( 0 < \alpha \leq 1 \) has a local smooth solution \( u \) satisfying

\[
u \in C([0,T_1]; H^s(\mathbb{R}^3)) \cap L^2(0,T_1; H^{s+\alpha}(\mathbb{R}^3)). \quad (2.9)
\]

Furthermore, if \( T^* \) is the maximal existence time of the solution, we have the following necessary condition for blow up

\[
u T^* < \infty \Rightarrow \int_0^{T^*} \|\nabla \times u(t)\|_{L^\infty} dt = +\infty. \quad (2.10)
\]

(iii) (Global smooth solution) When \( \frac{3}{4} < \alpha \leq 1 \), the local smooth solution is global, i.e.

\[
u \in C([0,\infty); H^s(\mathbb{R}^3)) \cap L^2(0,\infty; H^{s+\alpha}(\mathbb{R}^3)). \quad (2.11)
\]

Remark 2.1 It is well-known that the global smooth solutions of 3D Navier-Stokes equations with large initial data is a big open problem. The main obstacle lies in the fact that the quadratic convection term \( u \cdot \nabla u \) can not be controlled by the dissipation \( -\nu \Delta u \). Even for 3D Navier-Stokes equations with fractional dissipation

\[
u \partial_t u + (u \cdot \nabla u) + \nu(-\Delta)^\alpha u + \nabla p = 0, \quad (2.12)
\]

Ladyzhenskaya showed the global existence of the smooth solution of (2.12) when \( \alpha \geq \frac{5}{4} \) (see also [9]). The main reason why the smooth solution of the system (1.3)-(1.4) here is global only for \( \alpha > \frac{3}{4} \) is based on our observation that the presence of the modifying factor \( F_N(\|\nabla u\|_{L^2}) \) actually decreases the singularity of the quadratic convection term \( u \cdot \nabla u \).

Proof of Theorem 2.1 The proof is divided into three steps.

Step 1. Existence of weak solutions

We prove the global existence of the weak solution by the classic Friedrichs method which consists of an approximation of (1.3)-(1.4) by a cut-off in the frequency space. Denote \( J_n h = F^{-1}(\chi_{B(0,n)}(\xi)\hat{h}(\xi)) \) for \( n \in \mathbb{N} \) and consider the approximate system of (1.3)-(1.4)

\[
u \partial_t u_n + \nu J_n(-\Delta)^\alpha u_n = -J_n P(F_N(\|\nabla u_n\|_{L^2})) J_n u_n \cdot \nabla J_n u_n, \quad (2.13)
\]

\[
u u_n(x,0) = J_n u_0, \]

where \( P \) is the projection mapping \( L^2 \) onto the subspace \( \{ u \in L^2(\mathbb{R}^3) : \nabla \cdot u = 0 \} \). This is an ODE system on \( L^2 \) and the classic Cauchy-Lipschitz theorem ensures that there exists a unique solution which is continuous in time \([0,T_n]\) with value in \( L^2 \). Furthermore, thanks to \( J_n^2 = J_n \), we claim that \( J_n u_n \) is also a solution of (2.13), so the uniqueness implies that \( J_n u_n = u_n \). Thus \( u_n \) is also a solution of the following system

\[
u \partial_t u_n + \nu(-\Delta)^\alpha u_n = -J_n P(F_N(\|\nabla u_n\|_{L^2})) u_n \cdot \nabla u_n, \quad (2.14)
\]

\[
u u_n(x,0) = J_n u_0, \]

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Noting (1.2), it is easy to verify that the approximate solution \( u_n \) of (2.14) satisfies
\[
\frac{1}{2} \frac{d}{dt} \| u_n(t) \|_{L^2}^2 + \nu \| (-\Delta)^{\frac{3}{2}} u_n(t) \|_{L^2}^2 = 0,
\]
where we have used the fact
\[
\int_{\mathbb{R}^3} J_n(F_N(\| \nabla u_n \|_{L^2}) u_n \cdot \nabla u_n) \, dx = \int_{\mathbb{R}^3} F_N(\| \nabla u_n \|_{L^2}) J_n(u_n \cdot \nabla u_n) \, dx = 0,
\]
due to (1.2) and \( \nabla \cdot u_n = 0 \).

Integrating in time and applying Cauchy inequality gives
\[
\| u_n(t) \|_{L^2}^2 + 2\nu \int_0^t \| (-\Delta)^{\frac{3}{2}} u_n(t) \|_{L^2}^2 \, dt \leq \| u_0 \|_{L^2}^2,
\]
which ensures that \( T_n = T \).

Thus there exists a subsequence (denoted by \( u_n \) again) converges weakly to \( u \) such that \( u \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^\alpha(\mathbb{R}^3)) \). But this weak convergence does not allow us to pass to the limit in the nonlinear term \( J_n P(F_N(\| \nabla u_n \|_{L^2}) u_n \cdot \nabla u_n) \). To do so, we need to show \( \partial_t u_n \) is bounded uniformly in \( L^{\frac{3}{2}}(0, T; H^{-1}(\mathbb{R}^3)) \). Indeed, for any \( \phi \in H^1(\mathbb{R}^3) \)
\[
| < J_n P(F_N(\| \nabla u_n \|_{L^2}) u_n \cdot \nabla u_n), \phi > | \leq C \| \phi \|_{H^1} \| u_n \|_{L^4}^2 \leq C \| \phi \|_{H^1} \| u_n \|_{L^2}^{2-\frac{3}{2\alpha}} \| \nabla u_n \|_{L^2}^{\frac{3}{2\alpha}}
\]
from which we have together with (2.15),
\[
\partial_t u_n = -J_n P(F_N(\| \nabla u_n \|_{L^2}) u_n \cdot \nabla u_n) - \nu (-\Delta)^\alpha u_n + f_n \in L^{\frac{3}{2}}(0, T; H^{-1}(\mathbb{R}^3)).
\]

By using the standard Lions-Aubin compactness theorem [22], which states in our situation that \( L^2(0, T; L^2(\mathbb{R}^3)) \) is compactly imbedded in the space
\[
\left\{ u \mid u \in L^2(0, T; H^\alpha(\mathbb{R}^3)), \partial_t u \in L^{\frac{3}{2}}(0, T; H^{-1}(\mathbb{R}^3)) \right\}.
\]
Thus the strong convergence of \( u_n \in L^2(0, T; L^2(\mathbb{R}^3)) \) will allow us to show that \( u \) is indeed a weak solution of (1.3)-(1.4), which derives the assertion (i) of Theorem 2.1.

**Step 2. Local smooth solution and blow-up criterion**

In order to prove the local existence of the smooth solution, we present the uniform estimate for the approximate solutions \( u_n \) of (2.14) in \( H^2 \). Taking the operator \( \Delta_j \) for \( j \geq 0 \) to both sides of (2.14), we obtain
\[
\begin{cases}
\partial_t \Delta_j u_n + \nu (-\Delta)^\alpha \Delta_j u_n = -J_n P \Delta_j (F_N(\| \nabla u_n \|_{L^2}) u_n \cdot \nabla u_n), \\
\Delta_j u_n(x, 0) = J_n \Delta_j u_0.
\end{cases}
\]

The standard energy method implies that
\[
\frac{d}{dt} \| \Delta_j u_n \|_{L^2}^2 + 2\nu \| (-\Delta)^{\frac{3}{2}} \Delta_j u_n \|_{L^2}^2 \leq -2F_N(\| \nabla u_n \|_{L^2}) \int_{\mathbb{R}^3} \Delta_j(u_n \cdot \nabla u_n) \cdot \Delta_j u_n \, dx = I.
\]

Thanks to \( \nabla \cdot u_n = 0 \), we have
\[
I = -2F_N(\| \nabla u_n \|_{L^2}) \int_{\mathbb{R}^3} (\Delta_j(u_n \cdot \nabla u_n) - u_n \cdot \nabla \Delta_j u_n) \cdot \Delta_j u_n \, dx = -2F_N(\| \nabla u_n \|_{L^2}) \int_{\mathbb{R}^3} |\Delta_j(u_n) \cdot \nabla u_n| \cdot \Delta_j u_n \, dx
\]
Thus inserting (2.17) into (2.16) gives
\[\|u\| \leq C\|\Delta_j, u_n\|L^2 \|\Delta_j u_n\|L^2 \leq Cc_j 2^{-2j_{2j}} \|\nabla u_n\|L^\infty \|u_n\|H^s \|u_n\|H^s,\]
(2.17)
Thus inserting (2.17) into (2.16) gives
\[\frac{d}{dt}\|\Delta_j u_n\|L^2 + 2\nu\|(-\Delta)^{\frac{1}{2}} \Delta_j u_n\|L^2 \leq Cc_j 2^{-2j_{2j}} \|\nabla u_n\|L^\infty \|u_n\|H^s,\]
from which, (2.15) and Young’s inequality, it follows that
\[\frac{d}{dt}\|u_n(t)\|H^s + \nu \|u_n(t)\|H^{s+\alpha} \leq C\|\nabla u_n(t)\|L^\infty \|u_n\|H^s,\]
and then taking Gronwall inequality into account implies
\[\|u_n(t)\|H^s + \nu \int_0^t \|u_n(\tau)\|H^{s+\alpha} d\tau \leq \|u_0\|H^s \exp\left\{C \int_0^t \|\nabla u_n(\tau)\|L^\infty\right\}, \]
(2.18)
Denote
\[E_s(u, t) = \|u(t)\|H^s, \quad F(u, t) = \nu \int_0^t \|u(\tau)\|H^{s+\alpha} d\tau,\]
and define \(T_n\) as
\[T_n = \sup\{t \mid \forall t' \leq t, \ E_s(u_n, t') + F(u_n, t') \leq 2E_s(u_0)\}.
\]
From Sobolev embedding inequality, we infer that for \(0 \leq t < T_n\),
\[C \int_0^t \|\nabla u_n(\tau)\|L^\infty\right\}, \]
(2.19)
Choosing \(T_1 > 0\) such that
\[e^{C(T_1^{\frac{1}{2}} + T(1 + E_s(u_0)))} \leq \frac{3}{2},\]
then \(T_n \geq T_1\). Otherwise, we have by (2.18) and (2.19) that
\[E_s(u_n, t) + F(u_n, t) \leq \frac{3}{2}E_s(u_0), \quad \forall n \in N, \quad t \in [0, T_n),\]
which contradicts with the definition of \(T_n\). Thus there holds for any \(t \in [0, T_1]\),
\[\|u_n(t)\|H^s + \nu \int_0^t \|u_n(\tau)\|H^{s+\alpha} d\tau \leq \|u_0\|H^s,\]
(2.20)
Therefore based on the estimate (2.20), a standard compactness argument ensures the existence of the solution \(u\) of the system (1.3)-(1.4) on the interval \([0, T_1]\). Here we omit the details.
Now we prove the Beale-Kate-Majda’s blow-up criterion of the smooth solution. Exactly as in the proof of (2.18), we have
\[\|u_n(t)\|H^s + \nu \int_0^t \|u_n(\tau)\|H^{s+\alpha} d\tau \leq \|u_0\|H^s \exp\left\{C \int_0^t \|\nabla u_n(\tau)\|L^\infty\right\}, \]
or
\[\ln (e + \|u(t)\|H^s) \leq \ln (e + \|u_0\|H^s) + C \int_0^t \|\nabla u(\tau)\|L^\infty\right\}. \]
(2.21)
Thanks to Lemma 2.3,
\[ \int_0^t \|
abla u(\tau)\|_{L^\infty} d\tau \leq C \int_0^t (1 + \|
abla u(\tau)\|_{BMO}) \ln(e + \|u(\tau)\|_{H^s}) d\tau \]
\[ \leq C \int_0^t (1 + \|
abla \times u(\tau)\|_{L^\infty}) \ln(e + \|u(\tau)\|_{H^s}) d\tau. \]
where we used in the last inequality the fact that \( \nabla u = T(\nabla \times u) \) with \( T \) a singular integral operator (Biot-Savart law) and
\[ \|T(\nabla \times u)\|_{BMO} \leq C\|
abla \times u\|_{L^\infty}. \]
Plugging them into (2.21) yields that
\[ \ln(e + \|u(t)\|^2_{H^s}) \leq \ln(e + \|u_0\|^2_{H^s}) + C \int_0^t (1 + \|
abla \times u(\tau)\|_{L^\infty}) \ln(e + \|u(\tau)\|_{H^s}) d\tau, \]
which together with Gronwall’s inequality implies (2.10).

**Step 3. Global smooth solution**

Taking the inner product of (1.3) with \((-\Delta)^m u\) for all \( m > 0 \), we have
\[ \frac{d}{dt} \|
abla^m u\|^2_{L^2} + 2\nu \|
abla^{a+m} u\|^2_{L^2} \leq 2F_N(\|\nabla u\|_{L^2}) \left| \int_{\mathbb{R}^3} (u \cdot \nabla u) \cdot (-\Delta)^m u dx \right|. \] (2.22)
Applying Plancherel theorem, H"older inequality and Young inequality, the right hand side of (2.22) is bounded by
\[ 2F_N(\|\nabla u\|_{L^2}) \left| \int_{\mathbb{R}^3} (u \cdot \nabla u) (-\Delta)^m u dx \right| \]
\[ = 2F_N(\|\nabla u\|_{L^2}) \left| \int_{\mathbb{R}^3} \left( \xi_1 \hat{u}_1 \hat{\xi} + \xi_2 \hat{u}_2 \hat{\xi} + \xi_3 \hat{u}_3 \hat{\xi} \right) \| \xi \|^2 \hat{u} \hat{\xi} \right| \]
\[ \leq 2F_N(\|\nabla u\|_{L^2}) \sum_{i=1}^3 \int_{\mathbb{R}^3} \| \xi \|^{m+1-\alpha}(\hat{u}_i \hat{\xi}) \| \xi \|^{m+\alpha} \hat{u} \hat{\xi} \]
\[ \leq 2F_N(\|\nabla u\|_{L^2}) \sum_{i=1}^3 \|
abla^{m+1-\alpha}(u_i u)\|_{L^2} \|
abla^{m+\alpha} u\|_{L^2}. \] (2.23)

Using the product estimates and embedding theorem of the fractional Sobolev spaces gives
\[ \sum_{i=1}^3 \|
abla^{m+1-\alpha}(u_i u)\|_{L^2} \leq C \|u\|_{L^6} \|
abla^{m+1-\alpha} u\|_{L^3} \leq C \|\nabla u\|_{L^2} \|
abla^{m+1-\alpha + \frac{1}{2}} u\|_{L^2}, \]
from which, (2.22) and (2.23), we have together with (1.2)
\[ \frac{d}{dt} \|
abla^m u\|^2_{L^2} + 2\nu \|
abla^{a+m} u\|^2_{L^2} \leq C \|
abla^{m+1-\alpha + \frac{1}{2}} u\|_{L^2} \|
abla^{m+\alpha} u\|_{L^2}. \] (2.24)
Thanks to \( \frac{3}{4} < \alpha \leq 1 \), *i.e.* \( m < m + 1 - \alpha + \frac{1}{2} < m + \alpha \), applying Gagliardo-Nirenberg inequality and Young inequality yields
\[ \|
abla^{m+1-\alpha + \frac{1}{2}} u\|_{L^2} \|
abla^{m+\alpha} u\|_{L^2} \leq C \|
abla^m u\|^{1-\theta}_{L^2} \|
abla^{m+\alpha} u\|^{1+\theta}_{L^2} \]
\[ \leq C \|
abla^m u\|^2_{L^2} + \frac{\nu}{2C} \|
abla^{m+\alpha} u\|^2_{L^2}. \] (2.25)
Plugging (2.25) into (2.24) and applying Gronwall inequality, we have the uniform estimates
\[
\sup_{0 < \tau < T} \|u(\tau)\|_{H^m}^2 + \nu \int_0^T \|u(\tau)\|_{H^{n+m}}^2 \leq C, \quad \forall \ T > 0. \tag{2.26}
\]
Thus based on the estimate (2.26), the standard continuous method allows us to derive the global smooth solution.
Hence, the proof of Theorem 2.1 is complete.

3. Time decay of solutions.

3.1. $L^2$ decay of weak solutions. In this subsection, we are focused on the optimal $L^2$ decay of weak solutions of the modified system (1.3)-(1.4) with suitable initial data. To do so, we first present some time decay estimates of the linear equations.

**Lemma 3.1.** Suppose the initial data $u_0 \in H^m(\mathbb{R}^3)$ $(m \geq 0)$ and satisfies
\[
\rho(r) = \int_{s^2} |\hat{u}_0(r\omega)|^2 d\omega = cr^{2\gamma - 3} + o(r^{2\gamma - 3}), \text{ for } \gamma > \frac{5}{2} \text{ as } r \to 0 \tag{3.1}
\]
then there exist two positive constants $C_0$ and $C_1$, such that the solution $e^{-\nu(-\Delta)^\alpha t}u_0$ of the linear equations
\[
\begin{cases}
\partial_t u + \nu(-\Delta)^\alpha u + \nabla p = 0, & 0 < \alpha \leq 1 \\
\nabla \cdot u = 0 \\
u(x, 0) = u_0,
\end{cases}
\tag{3.2}
\]
with $0 < \alpha \leq 1$ has the following upper and lower bounds
\[
C_0(1 + t)^{-\frac{m+\gamma}{2m}} \leq \|\nabla^m e^{-\nu(-\Delta)^\alpha t} u_0\|_{L^2} \leq C_1(1 + t)^{-\frac{m+\gamma}{2m}}, \text{ for large } t. \tag{3.3}
\]

**Proof of Lemma 3.1** Applying Plancherel theorem and (3.1), it follows that
\[
\|\nabla^m e^{-\nu(-\Delta)^\alpha t} u_0\|_{L^2}^2 = \|\nabla^m e^{-\nu(-\Delta)^\alpha t} u_0\|_{L^2}^2 = \int_{\mathbb{R}^3} |\xi|^{2m} e^{-2\nu(s^2)\hat{u}_0(\xi)}^2 d\xi
\leq C \int_0^\infty \int_{|\omega| = 1} r^{2m+2} e^{-2\nu(s^2)|\hat{u}_0(\omega)|^2} d\omega dr
\leq C \int_0^t \int_{|\omega| = 1} r^{2m+2} e^{-2\nu(s^2)\rho(r)} dr
\]
\[
+ C e^{-2\sqrt{t}} \int_0^\infty \int_{|\omega| = 1} r^{2m+2} |\hat{u}_0(\omega)|^2 d\omega dr
\leq C e^{-2\sqrt{t}} \int_0^t \frac{2\sqrt{t}}{s^{\frac{m+\gamma}{2}}} e^{-s^{\frac{m+\gamma}{2}}(1 + o(1))ds}
\]
\[
+ C e^{-2\sqrt{t}} \|\nabla^m u_0\|_{L^2}^2 \leq C_1(1 + t)^{-\frac{m+\gamma}{2}}, \text{ for large } t > 0.
\]
this implies the upper bounds in (3.3), and the lower bounds are derived from
\[
\|\nabla^m e^{-\nu(-\Delta)^\alpha t} u_0\|^2_{L^2} = \int_{\mathbb{R}^3} |\xi|^{2m} e^{-\nu|\xi|^{2\alpha} t} |\tilde{u}_0(\xi)|^2 d\xi
\geq C \int_0^\infty \int_{|\xi|=1} r^{2m+2} e^{-2r^{2\alpha} t} |\tilde{u}_0(r\omega)|^2 d\omega dr
= C \int_0^\infty r^{2m+2} e^{-2r^{2\alpha} t} \rho(r) dr
\geq C t^{-\frac{m+\gamma}{\alpha}} \int_0^1 s^{\frac{m+\gamma}{\alpha}-1} e^{-s} ds + o(t^{-\frac{m+\gamma}{\alpha}})
\geq C_0 (1 + t)^{-\frac{m+\gamma}{\alpha}}, \quad \text{for large } t > 0.
\]
Thus we complete the proof of Lemma 3.1.

The following \(L^p - L^q\) estimates are more or less well-known.

**Lemma 3.2.** Let \(1 \leq p \leq q \leq \infty\), \(m \geq 0\), the following \(L^p - L^q\) estimate of the semigroup \(e^{-\nu(-\Delta)^\alpha t}\)
\[
\|\nabla^m e^{-\nu(-\Delta)^\alpha t} u_0\|_{L^q(\mathbb{R}^3)} \leq C t^{-\frac{m}{\alpha}} - \frac{\nu}{\alpha} \left(\frac{1}{p} - \frac{1}{q}\right) \|u_0\|_{L^p(\mathbb{R}^3)}
\]
(3.4)
is valid for any \(t > 0\).

**Proof of Lemma 3.2** Schonbek [21] proved this result for two-dimensional case and the proof in 3D is parallel to the one in 2D. Here we omit the detail.

Our results read as follows.

**Theorem 3.1.** Suppose \(u_0 \in L^2(\mathbb{R}^3)\) and \(u(x, t)\) is a weak solution of system (1.3)-(1.4) with \(0 < \alpha \leq 1\). Then
\begin{enumerate}
\item \(\lim_{t \to \infty} \|u(t)\|_{L^2} = 0\)
\item Additionally, if \(u_0\) satisfies (3.3), then
\[
\|u(t)\|_{L^2} \leq C (1 + t)^{-\frac{m}{\alpha}}, \quad \text{for large } t.
\]
\end{enumerate}

**Remark 3.1** For the 3D classic Navier-Stokes equations, Schenbek [18, 19] proved that the weak solution decays at \((1 + t)^{-\frac{m}{2}}\), whereas our decay result is \((1 + t)^{-\frac{m}{4}}\) even when \(\alpha = 1\). Our argument here is mainly based on the developing Fourier splitting methods introduced by Schonbek [18] and a new iterative technique. One may also refer to some time decay results of Newtonian flows and non-Newtonian flows [6, 7, 8, 24, 25].

**Remark 3.2** When one considers the decay of system (1.3)-(1.4) with nonzero force \(f(x, t)\), the same results in Theorem 3.1 are also valid if \(f\) satisfies some decay properties: \(\|f(t)\|_{L^2} \leq C(1 + t)^{-\mu}\) for a suitable large \(\mu > 0\). Especially, if \(\mu = 0\), i.e. \(\|f(t)\|_{L^2} \leq C\), then we have not any decay property of the solution.

**Proof of Theorem 3.1** It should be mentioned that the following discussion should be stated rigorously for the smooth approximated solutions firstly and then taken the limits to get the decay results of the relevant weak solutions. For convenience, we directly discuss for weak solutions.

Take the inner product of (1.3) with \(u\), we have
\[
\frac{d}{dt} \|u\|_{L^2}^2 + 2\nu \|\nabla^\alpha u\|_{L^2}^2 = 0,
\]
(3.6)
and Plancherel theorem gives
\[
\frac{d}{dt} \| \hat{u}(t) \|^2_{L^2} + 2 \nu \int_{\mathbb{R}^3} |\xi|^{2\alpha} |\hat{u}(\xi, t)|^2 d\xi = 0. \tag{3.7}
\]
Denote
\[B(t) = \{ \xi \mid |\xi| \leq (g/\nu)^{-\frac{1}{\alpha}}(t) \},\]
and thanks to
\[
2 \nu \int_{\mathbb{R}^3} |\xi|^{2\alpha} |\hat{u}(\xi, t)|^2 d\xi = 2 \nu \int_{B(t)} |\xi|^{2\alpha} |\hat{u}(\xi, t)|^2 d\xi + 2 \nu \int_{B(t)^c} |\xi|^{2\alpha} |\hat{u}(\xi, t)|^2 d\xi \\
\geq 2 \nu \int_{B(t)} |\xi|^{2\alpha} |\hat{u}(\xi, t)|^2 d\xi \\
\geq 2g^{-1}(t) \int_{B(t)^c} |\hat{u}(\xi, t)|^2 d\xi \\
\geq 2g^{-1}(t) \| \hat{u} \|^2_{L^2} - 2g^{-1}(t) \int_{B(t)} |\hat{u}(\xi, t)|^2 d\xi,
\]
the substitution of this inequality into (3.7) gives
\[
\frac{d}{dt} \| \hat{u}(t) \|^2_{L^2} + 2g^{-1}(t) \| \hat{u} \|^2_{L^2} \leq 2g^{-1}(t) \int_{B(t)} |\hat{u}(\xi, t)|^2 d\xi. \tag{3.8}
\]
In order to estimate \( \int_{B(t)} |\hat{u}(\xi, t)|^2 d\xi \), taking the Fourier transform of (1.3) yields
\[
\partial_t \hat{u} + \nu |\xi|^{2\alpha} \hat{u} = - \mathcal{F}[PF_N(\|\nabla u\|_{L^2})(u \cdot \nabla u)]
\]
or
\[
|\hat{u}(\xi, t)| \leq \left| e^{-\nu|\xi|^{2\alpha}t} \hat{u}_0(\xi) \right| + \left| \int_0^t e^{-\nu|\xi|^{2\alpha}(t-s)} (- \mathcal{F}[PF_N(\|\nabla u\|_{L^2})(u \cdot \nabla u)]) ds \right|
\leq \left| e^{-\nu|\xi|^{2\alpha}t} \hat{u}_0(\xi) \right| + C|\xi| \int_0^t \|u\|^2_{L^2} ds. \tag{3.9}
\]
Thus,
\[
\int_{B(t)} |\hat{u}(\xi, t)|^2 d\xi \leq C \int_{B(t)} |e^{-\nu|\xi|^{2\alpha}t} \hat{u}_0(\xi)|^2 d\xi + C \int_{B(t)} |\xi|^2 \left( \int_0^t \|u\|^2_{L^2} ds \right)^2 d\xi \\
\leq C \|e^{-\nu|\xi|^{2\alpha}t} u_0\|^2_{L^2} + C \int_0^{g^{-\frac{1}{\alpha}}(t)} r^4 \left( \int_0^t \|u\|^2_{L^2} ds \right)^2 dr \\
\leq C \|e^{-\nu|\xi|^{2\alpha}t} u_0\|^2_{L^2} + Cg^{-\frac{2}{\alpha}}(t) \left( \int_0^t \|u\|^2_{L^2} ds \right)^2,
\]
from which and Plancherel Theorem, (3.8) becomes
\[
\frac{d}{dt} \|u(t)\|^2 + 2g^{-1}(t) \|u\|^2_{L^2} \leq Cg^{-1}(t) \|e^{-\nu(-\Delta)^{\alpha}t} u_0\|^2_{L^2} + Cg^{-1-\frac{2}{\alpha}}(t) \left( \int_0^t \|u\|^2_{L^2} ds \right)^2.
\]
Now, letting \( g(t) = \frac{2(t+1)}{m} \) with \( m > 0 \) to be chosen sufficient large and multiplying both sides of above inequalities by \((1 + t)^m\), one shows that
\[
\frac{d}{dt} (t+1)^m \|u(t)\|^2_{L^2} \leq C(t+1)^{m-1} \|e^{-\nu(-\Delta)^{\alpha}t} u_0\|^2_{L^2} \\
+ C(t+1)^{m-1-\frac{2}{\alpha}} \left( \int_0^t \|u\|^2_{L^2} ds \right)^2. \tag{3.10}
\]
Theorem 3.2. Suppose solutions of system (1.3)-(1.4). Let us interested in the upper bounds estimates of higher-order derivatives of the smooth solution. Applying Gagliardo-Nirenberg inequality and Young inequality, one shows that upper bounds of higher-order derivatives.

Proof of Theorem 3.2

Exactly as in the proof of (2.24), we have

\[
\frac{d}{dt}\|u(t)\|_{L^2}^2 + 2\nu\|\nabla^{\alpha+m}u\|_{L^2}^2 \leq C\|\nabla^{m+1-\alpha+\frac{1}{2}}u\|_{L^2}\|\nabla^{m+\alpha}u\|_{L^2}.
\]  

(3.14)

Thanks to \(\frac{3}{4} < \alpha \leq 1\), i.e. \(m < m + 1 - \alpha + \frac{1}{2} < m + \alpha\), letting \(0 \leq k \leq m\) and applying Gagliardo-Nirenberg inequality and Young inequality, one shows that

\[
\|\nabla^{m+1-\alpha+\frac{1}{2}}u\|_{L^2}\|\nabla^{m+\alpha}u\|_{L^2} \leq C\|\nabla^{k}u\|_{L^2}^{1-\theta}\|\nabla^{m+\alpha}u\|_{L^2}^{1+\theta}
\]

\[
\leq C\|\nabla^{k}u\|_{L^2}^2 + \frac{\nu}{2C}\|\nabla^{m+\alpha}u\|_{L^2}^2.
\]  

(3.15)
Plugging (3.15) into (3.14) becomes

\[
\frac{d}{dt}\|\nabla^m u\|_{L^2}^2 + \nu \|\nabla^{\alpha+m} u\|_{L^2}^2 \leq C\|\nabla^k u\|_{L^2}^2. \tag{3.16}
\]

Similar to the proof of the case (i) in Theorem 3.1, we have noting \(B(t) = \{\xi \mid |\xi| \leq (g/\nu)^{-\frac{m}{\nu}}(t)\}\),

\[
\nu \|\nabla^{\alpha+m} u\|_{L^2}^2 = \nu \int_{\mathbb{R}^3} |\xi|^{2(m+\alpha)}|\widehat{u}(\xi,t)|^2 \, d\xi
\]

\[
\geq \nu \int_{B(t)^c} |\xi|^{2(m+\alpha)}|\widehat{u}(\xi,t)|^2 \, d\xi
\]

\[
\geq g^{-1}(t) \int_{B(t)^c} |\xi|^{2m}|\widehat{u}(\xi,t)|^2 \, d\xi
\]

\[
\geq g^{-1}(t) \left( \|\nabla^m u\|_{L^2}^2 - \int_{B(t)} |\xi|^{2m} |\widehat{u}(\xi,t)|^2 \, d\xi \right)
\]

\[
\geq g^{-1}(t) \left( \|\nabla^m u\|_{L^2}^2 - (g/\nu)^{-\frac{m}{\nu}}(t)\|u\|_{L^2}^2 \right).
\]

Inserting the above inequality into (3.16) and applying (3.5) produce

\[
\frac{d}{dt}\|\nabla^m u\|_{L^2}^2 + g^{-1}(t)\|\nabla^m u\|_{L^2}^2 \leq Cg^{-1-\frac{m}{\nu}}(t)(1+t)^{-\frac{2}{\nu}} + C\|\nabla^k u\|_{L^2}^2. \tag{3.17}
\]

Now, letting \(g(t) = \frac{t+1}{2}\) with a suitable large integer \(\beta > 0\), then multiplying both sides of (3.17) by \((1+t)^\beta\) and integrating with respect to \(t\), it follows that

\[
\|\nabla^m u\|_{L^2}^2 \leq C(t+1)^{-\frac{m}{\nu} - \frac{2}{\nu}} + C(t+1)^{-\beta} \int_0^t (s+1)^\beta \|\nabla^k u\|_{L^2}^2 \, ds. \tag{3.18}
\]

Employing (3.5) to the right hand of (3.18) with \(k = 0\), we get

\[
\|\nabla^m u\|_{L^2}^2 \leq C(t+1)^{-\left(\frac{m}{\nu} - \frac{1}{2}\right)}, \quad \text{for } \forall \ m > 0. \tag{3.19}
\]

Consider the integral equation of (1.3)

\[
u u(t) = e^{-\nu(\nabla^\alpha)^t} u_0 + \int_0^t e^{-\nu(\nabla^\alpha)^{t-s}} P(F_N(\nabla u)(u \cdot \nabla u)) \, ds, \tag{3.20}
\]

Taking \(\nabla^k\) to both sides of equation (3.20) and applying Lemma 3.1, one shows that

\[
\|\nabla^k u(t)\|_{L^2} \leq \|\nabla^k e^{-\nu(\nabla^\alpha)^t} u_0\|_{L^2} + \left\| \int_0^t \nabla^k e^{-\nu(\nabla^\alpha)^{t-s}} P(u \cdot \nabla u) \, ds \right\|_{L^2}
\]

\[
\leq C(1+t)^{-\frac{k+s}{\nu}} + C \int_0^t \|\nabla^k e^{-\nu(\nabla^\alpha)^{t-s}} (u \cdot \nabla u)\|_{L^2} \, ds
\]

\[
+ C \int_0^t \|\nabla^k e^{-\nu(\nabla^\alpha)^{t-s}} (u \cdot \nabla u)\|_{L^2} \, ds = C(1+t)^{-\frac{k+s}{\nu}} + I + II. \tag{3.21}
\]
For $I$, using Lemma 3.2 and Theorem 3.1, we have

\[
I \leq \sum_{i=1}^{3} \int_{\frac{t}{2}}^{t} \| \nabla^{k+1} e^{-\nu(-\Delta)^{\alpha}(t-s)}(u_i u) \|_{L^2} ds \\
\leq C \sum_{i=1}^{3} \int_{0}^{\frac{t}{2}} (t-s)^{-\frac{k+1}{2\alpha} - \frac{3}{4\alpha}} \| u_i u \|_{L^2} ds \\
\leq C \int_{0}^{\frac{t}{2}} (t-s)^{-\frac{k+1}{2\alpha} - \frac{3}{4\alpha}} \| u_i u \|_{L^2} ds \\
\leq C \int_{0}^{\frac{t}{2}} (t-s)^{-\frac{k+1}{2\alpha} - \frac{3}{4\alpha}} (1+s)^{-\frac{3}{4\alpha}} ds \leq C(1+t)^{-\frac{2\alpha}{3} - \frac{5}{4\alpha}},
\]

(3.22)

and for $II$

\[
II \leq C \sum_{i=1}^{3} \int_{\frac{t}{2}}^{t} \| e^{-\nu(-\Delta)^{\alpha}(t-s)} \nabla^{k+1}(u_i u) \|_{L^2} ds \\
\leq C \sum_{i=1}^{3} \int_{0}^{\frac{t}{2}} (t-s)^{-\frac{k+1}{2\alpha} + \frac{3}{4\alpha}} \| \nabla^{k+1} u_i u \|_{L^2} ds \\
\leq C \int_{0}^{\frac{t}{2}} (t-s)^{-\frac{k+1}{2\alpha} + \frac{3}{4\alpha}} \| \nabla^{k+1} u_i u \|_{L^2} ds \\
\leq C \int_{0}^{\frac{t}{2}} (t-s)^{-\frac{k+1}{2\alpha} + \frac{3}{4\alpha}} (1+s)^{-\frac{3}{4\alpha}} ds \leq C(1+t)^{-\frac{2\alpha}{3} + \frac{3}{4\alpha}}.
\]

(3.23)

Plugging (3.21)-(3.22) into the right hand side of (3.21), we derive

\[
\| \nabla^{k} u(t) \|_{L^2} \leq C(1+t)^{-\left(\frac{3}{4\alpha} - \frac{2}{3} - \frac{k}{2} - \frac{3}{2}\right)} \text{ for } 0 \leq k \leq m.
\]

In particular, we derive an improved decay rate with respect to (3.19)

\[
\| \nabla^{m} u(t) \|_{L^2} \leq C(1+t)^{-\left(\frac{3}{4\alpha} - \frac{2}{3} - \frac{3}{2}\right)} \text{ for } m \geq 0.
\]

(3.24)

Repeating this process $N$ times till

\[
\left(\frac{5}{4\alpha} - \frac{1}{2}\right) + N\left(\frac{2}{\alpha} - \frac{3}{2}\right) \geq \frac{k}{2\alpha} + \frac{5}{4\alpha},
\]

then we reach the desired decay results

\[
\| \nabla^{m} u(t) \|_{L^2} \leq C(1+t)^{-\frac{m}{4\alpha} - \frac{3}{4\alpha}}, \text{ for large } t.
\]

Hence the proof of Theorem 3.2 is complete.

4. Asymptotic stability. The time decay results in Section 3 imply the trivial solution $u = 0$ of zero-force system (1.3)-(1.4) is asymptotic stable, it is an interesting problem to consider the asymptotic stability for the nontrivial solution of system (1.3)-(1.4) with nonzero force $f(x,t)$. More precisely, consider the original system

\[
\left\{ \begin{array}{l}
\partial_t u + F_N(|\nabla u|_{L^2})(u \cdot \nabla u) + \nu(-\Delta)^{\alpha} u + \nabla p = f, \\
\nabla \cdot u = 0,
\end{array} \right.
\]

(4.1)
and the perturbed system
\[
\begin{aligned}
\partial_t v + F_N(\|\nabla v\|_{L^2})(v \cdot \nabla v) + \nu(-\Delta)^\alpha v + \nabla p & = f + g, \\
\nabla \cdot v & = 0 \\
v(x,0) & = u_0 + w_0,
\end{aligned}
\]
(4.2)

with the initial data and external forcing perturbation \(w_0(x)\) and \(g(x,t)\), we are focused on the asymptotic stability of system (4.1) with external forcing \(f(x,t)\) under the large initial data and external forcing perturbation \(w_0\) and \(g\). Our result reads as follows.

**Theorem 4.1.** Let \(\frac{3}{4} < \alpha \leq 1, u_0 \in H^1(\mathbb{R}^3), w_0 \in L^2(\mathbb{R}^3)\) and \(f(x,t) \in L^2(0,T; L^2(\mathbb{R}^3))\), \(g(x,t) \in L^1(0,\infty; L^2(\mathbb{R}^3))\). Suppose \(u(x,t)\) is a global solution of the original system (4.1), then the asymptotic stability property
\[
\|u(t) - v(t)\|_{L^2} \to 0 \text{ as } t \to +\infty
\]
(4.3)
holds true for every weak solution \(v(x,t)\) of the perturbed equations (4.2).

**Remark 4.1** The proofs of the global existence of the solution for both the original system (4.1) and the perturbed system (4.2) in Theorem 4.1 are parallel to the ones in Theorem 2.1 in Section 2. It should be mentioned that since nonzero force \(f(x,t) \in L^2(0,T; L^2(\mathbb{R}^3))\), the original system (4.1) has a nontrivial solution \(u \neq 0\) which has not any decay property according to Remark 3.2 in Section 3. The same assertion is also valid for the perturbed system (4.2).

**Remark 4.2** Compared with the asymptotic stability results of the 3D Navier-Stokes equations by Kawanago [10], Ponce et al [17], Kozono [13], Zhou [23], neither additional assumption on the solutions nor small assumption on initial and external forcing perturbation is added in Theorem 4.1. The key in the proof of Theorem 4.1 is to prove the global estimate of the difference \(u(t) - v(t)\) and an auxiliary decay estimate by choosing a suitable test function which was first introduced by Masuda [16] (see also Dong and Chen [5]).

**Proof of Theorem 4.1** The proof is divided into three steps.

**Step 1 An auxiliary identity**

This section is devoted to the derivation of the following auxiliary identity
\[
\int_0^t \{2\nu(\Lambda^\alpha v, \Lambda^\alpha u) + F_N(\|\nabla u\|_{L^2})(u \cdot \nabla u, v) + F_N(\|\nabla v\|_{L^2})(v \cdot \nabla v, u)\} \, d\tau
= \int_0^t (f, v) + (f + g, u) \, d\tau - (u(t), v(t)) + (u_0, v_0)
\]
(4.4)
the proof of (4.4) is based on a special choice of test functions. The idea of choosing test functions is developed from the argument of Masuda [16]. We introduce the test functions
\[
u_\epsilon(t) \equiv \int_0^t \eta_\epsilon(\tau - \sigma) u(\sigma) \, d\sigma,
\]
\[
u_\epsilon(t) \equiv \int_0^t \eta_\epsilon(\tau - \sigma) v(\sigma) \, d\sigma \quad (0 \leq \tau < t)
\]
for the solutions $u$ and $v$. We define the mollifier function $\eta_\varepsilon$ ($\varepsilon > 0$) expressed as

$$\eta_\varepsilon = \frac{1}{\varepsilon} \eta\left(\frac{t}{\varepsilon}\right)$$

with $\eta \in C_0^\infty(-1, 1)$, $\eta(t) \geq 0$, $\eta(-t) = \eta(t)$, $\int_{-1}^{1} \eta(t) dt = 1$ \hspace{1cm} (4.5)

It follows from the definition of the weak solutions with the choosing test functions $v_\varepsilon$ and $u_\varepsilon$ that

$$\int_{0}^{t} \{ -(u, \partial_\tau v_\varepsilon) + \nu(\Lambda^\alpha u, \Lambda^\alpha v_\varepsilon) + F_N(\|\nabla u\|_{L^2})(u \cdot \nabla u, v_\varepsilon) \} d\tau$$

$$= \int_{0}^{t} (f, v_\varepsilon) d\tau - (u(t), v_\varepsilon(t)) + (u_0, v_\varepsilon(0)), \hspace{1cm} (4.6)$$

and

$$\int_{0}^{t} \{ -(v, \partial_\tau u_\varepsilon) + \nu(\Lambda^\alpha v, \Lambda^\alpha u_\varepsilon) + F_N(\|\nabla v\|_{L^2})(v \cdot \nabla v, u_\varepsilon) \} d\tau$$

$$= \int_{0}^{t} (f + g, u_\varepsilon) d\tau - (v(t), u_\varepsilon(t)) + (v_0, u_\varepsilon(0)). \hspace{1cm} (4.7)$$

Combination of these two equations (4.6) and (4.7) yields

$$\int_{0}^{t} \{ \nu(\Lambda^\alpha u, \Lambda^\alpha v_\varepsilon) + \nu(\Lambda^\alpha v, \Lambda^\alpha u_\varepsilon) + F_N(\|\nabla u\|_{L^2})(u \cdot \nabla u, v_\varepsilon)$$

$$+ F_N(\|\nabla v\|_{L^2})(v \cdot \nabla v, u_\varepsilon) \} d\tau$$

$$= \int_{0}^{t} (f, v_\varepsilon) + (f + g, u_\varepsilon) d\tau - (u(t), v_\varepsilon(t)) - (v(t), u_\varepsilon(t))$$

$$+ (u_0, v_\varepsilon(0)) + (v_0, u_\varepsilon(0)) \hspace{1cm} (4.8)$$

where we have used the following observation

$$\int_{0}^{t} \{ -(u, \partial_\tau v_\varepsilon) - (v, \partial_\tau u_\varepsilon) \} d\tau = 0,$$

due to the Fubini theorem and the symmetry of $\eta_\varepsilon$.

We now prove the convergence of the equation (4.8) to the desired identity (4.4) as $\varepsilon \to 0$. Firstly, for the right-hand side of (4.8), due to the properties of the mollifier function $\eta_\varepsilon$, therefore, it is readily deduced that

$$\lim_{\varepsilon \to 0} \int_{0}^{t} \{ (f, v_\varepsilon) + (f + g, u_\varepsilon) \} d\tau = \int_{0}^{t} \{ (f, v) + (f + g, u) \} d\tau$$

With the aid of the weak continuity of $u$ and $v$ in $L^2(\mathbb{R}^3)$ and the definition of the function $\eta_\varepsilon$, one shows that

$$\lim_{\varepsilon \to 0} \left| (u(t), v_\varepsilon(t)) - \frac{1}{2}(u(t), v(t)) \right|$$

$$= \lim_{\varepsilon \to 0} \left| \int_{0}^{t} \eta_\varepsilon(\sigma)(u(t), v(t - \sigma) - v(t)) d\sigma \right|$$

$$\leq \frac{1}{2} \lim_{\varepsilon \to 0} \sup_{0 < \sigma < \varepsilon} |(u(t), v(t - \sigma) - v(t))| = 0.$$

Similarly, we have
\[
\lim_{\varepsilon \to 0} \{-(v(t), u_\varepsilon(t)) + (u_0, v_\varepsilon(0))\} = \frac{1}{2} \{-(v(t), u(t)) + (u_0, v_0) + (v_0, u_0)\}.
\]

For the left-hand side of (4.8), since
\[
\lim_{\varepsilon \to 0} \{\|\Lambda^\alpha u_\varepsilon - \Lambda^\alpha u\|_{L^2(0,t;L^2)} + \|\Lambda^\alpha v_\varepsilon - \Lambda^\alpha v\|_{L^2(0,t;L^2)}\} = 0,
\]
due to the properties of the mollifier function \(\eta_\varepsilon\), therefore, it is readily deduced that
\[
\lim_{\varepsilon \to 0} \int_0^t \{\nu(\Lambda^\alpha u_\varepsilon, \Lambda^\alpha v_\varepsilon) + \nu(\Lambda^\alpha v_\varepsilon, \Lambda^\alpha v_\varepsilon)\} \, d\tau = \int_0^t \{2\nu(\Lambda^\alpha u, \Lambda^\alpha v)\} \, d\tau
\]
Now it remains to prove the validity of the following convergence results
\[
\lim_{\varepsilon \to 0} \int_0^t (u \cdot \nabla u, v_\varepsilon) \, d\tau = \int_0^t (u \cdot \nabla u, v) \, d\tau
\]
and
\[
\lim_{\varepsilon \to 0} \int_0^t (v \cdot \nabla v, u_\varepsilon) \, d\tau = \int_0^t (v \cdot \nabla v, u) \, d\tau
\]
By using Hölder inequality, Gagliardo-Nirenberg inequality, we have
\[
\int_0^t (u \cdot \nabla u, v_\varepsilon - v) \, d\tau \\
\leq \int_0^t \left| \int_{\mathbb{R}^3} (\xi_1 \tilde{u}_1 u + \xi_2 \tilde{u}_2 u + \xi_3 \tilde{u}_3 u)(v_\varepsilon - v)(\xi) \, d\xi \right| \, d\tau \\
\leq \int_0^t \sum_{i=1}^3 \left| \int_{\mathbb{R}^3} |\xi|^{1-\alpha} |\tilde{u}_i u| |\xi|^{\alpha} |(v_\varepsilon - v)(\xi)| \, d\xi \right| \, d\tau \\
\leq \int_0^t \sum_{i=1}^3 \|\nabla^{1-\alpha}(u_i u)\|_{L^2} \|\nabla^\alpha(v_\varepsilon - v)\|_{L^2} \, d\tau \\
\leq C \int_0^t \|u\|_{L^2(0,t;H^{\alpha-1})} \|\nabla^{1-\alpha} u\|_{L^2(0,t;H^{\alpha})} \|\nabla^\alpha(v_\varepsilon - v)\|_{L^2} \, d\tau \\
\leq C\|u\|_{L^\infty(0,t;L^2)} \|u\|_{L^2(0,t;H^{\alpha+1})} \|v_\varepsilon - v\|_{L^2(0,t;H^{\alpha})} \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]
Similarly, since
\[
\int_0^t (v \cdot \nabla v, u_\varepsilon) \, d\tau = -\int_0^t (v \cdot \nabla u_\varepsilon, v) \, d\tau
\]
due to the divergence free condition on $v$, we have
\[
\int_0^t (v \cdot \nabla v, u \varepsilon - u) d\tau
\leq C \int_0^t \|v\|^2_{L^{\frac{2(\frac{2}{3} - \alpha)}{2(\frac{2}{3} - \alpha) - 1}}(\Omega)} \|\nabla (u \varepsilon - u)\|_{L^{\frac{2(\frac{2}{3} - \alpha)}{2(\frac{2}{3} - \alpha) - 1}}} d\tau
\leq C \int_0^t \|v\|^2_{L^{2}(\Omega)} \|\nabla^\alpha v\|_{L^{2}}^{\frac{2}{3} - \frac{1}{2}} \|u \varepsilon - u\|_{L^{2}(\Omega)}^{\frac{2}{3} - \frac{1}{2}} \|\nabla^\alpha (u \varepsilon - u)\|_{L^{2}(\Omega)} d\tau
\leq C \|v\|_{L^{\infty}(0, T; L^{2}(\Omega))} \|v\|_{L^{2}(0, T; H^\alpha)} \|u \varepsilon - u\|_{L^{2}(0, T; H^{\alpha + 1})} \rightarrow 0
\]

Hence, letting $\varepsilon \rightarrow 0$ in (4.8), we have the auxiliary identity.

**Step 2 Global estimates of $u(x, t) - v(x, t)$**

Denote by $w(x, t) = u(x, t) - v(x, t)$ the difference between the solution $u(x, t)$ of the original system (4.1) and the solution $v(x, t)$ of the perturbed equations (4.2).

Multiplying (4.1) by $u$, and integrate over $\mathbb{R}^3$, we obtain
\[
\|u(t)\|_{L^2}^2 + 2\nu \int_0^t \|\Lambda^\alpha u\|^2_{L^2} d\tau \leq \|u_0\|_{L^2}^2 + 2 \int_0^t (f, u) d\tau
\]
and, likewise, for global weak solution $v(x, t)$,
\[
\|v(t)\|_{L^2}^2 + 2\nu \int_0^t \|\Lambda^\alpha v\|^2_{L^2} d\tau \leq \|v_0\|_{L^2}^2 + 2 \int_0^t (f + g, v) d\tau.
\]

the summation of (4.9-4.10) gives that
\[
\|u(t)\|_{L^2}^2 + \|v(t)\|_{L^2}^2 \leq \|u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2 + 2 \int_0^t \{(f, u) + (f + g, v)\} d\tau. \tag{4.11}
\]

On the other hand, the difference $w(x, t) = u(x, t) - v(x, t)$ also satisfies the energy-type inequality
\[
\|u - v\|_{L^2}^2 + 2\nu \int_0^t \|\Lambda^\alpha (u - v)\|^2_{L^2} d\tau
= \|u(t)\|_{L^2}^2 + \|v(t)\|_{L^2}^2 + 2\nu \int_0^t \{\|\Lambda^\alpha u\|^2_{L^2} + \|\Lambda^\alpha v\|^2_{L^2}\} d\tau
-2(u(t), v(t)) - 4\nu \int_0^t (\Lambda^\alpha u, \Lambda^\alpha v) d\tau. \tag{4.12}
\]

Therefore the substitution of (4.11) into (4.12) produces
\[
\|u - v\|_{L^2}^2 + 2\nu \int_0^t \|\Lambda^\alpha (u - v)\|^2_{L^2} d\tau
\leq \|u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2 + 2 \int_0^t \{(f, u) + (f + g, v)\} d\tau
-2(u(t), v(t)) - 4\nu \int_0^t (\Lambda^\alpha u, \Lambda^\alpha v) d\tau. \tag{4.13}
\]
With the use of the auxiliary identity (4.4), one shows that

\[
\|u - v\|_{L^2}^2 + 2\nu \int_0^t \|\Lambda^\alpha (u - v)\|_{L^2}^2 d\tau
\]

\[
\leq \|w_0\|_{L^2}^2 + 2\nu \int_0^t \{F_N(\|\nabla u\|_{L^2}) (u \cdot \nabla v, v) + F_N(\|\nabla v\|_{L^2}) (v \cdot \nabla u, u)\} d\tau
\]

\[+ 2 \int_0^t (g, v - u) d\tau\]

or

\[
\|w\|_{L^2}^2 + 2\nu \int_0^t \|\Lambda^\alpha w\|_{L^2}^2 d\tau
\]

\[
\leq \|w_0\|_{L^2}^2 + 2\nu \int_0^t (u \cdot \nabla v, v) d\tau + 2 \int_0^t (v \cdot \nabla u, u) d\tau + 2 \int_0^t (g, w) d\tau
\]

\[
\leq \|w_0\|_{L^2}^2 + 4\nu \int_0^t (u \cdot \nabla v, w) d\tau + 2 \int_0^t (w \cdot \nabla u, w) d\tau + 2 \int_0^t (g, w) d\tau.
\]

(4.14)

By using Hölder inequality and Gagliardo-Nirenberg inequality, one shows that for the right hand side of (4.14) one by one

\[
2 \int_0^t (w \cdot \nabla u, w) d\tau \leq 2 \int_0^t \|\nabla u\|_{L^2}^2 \|w\|_{L^{4\alpha/(2\alpha - 1)}}^2 \|w\|_{L^{8\alpha/(4\alpha - 1)}}^2 d\tau
\]

\[
\leq C \int_0^t \|u\|_{L^{2\alpha/(2\alpha + 4\alpha - 3)}} \|\nabla^{\alpha + 1} u\|_{L^2}^{2(\frac{2}{3} - \alpha)} \|w\|_{L^2}^{2(\frac{2}{3} - \alpha)} \|\nabla^\alpha w\|_{L^2}^{2\alpha - \frac{2}{3}} d\tau
\]

\[
\leq C \int_0^t \|\nabla^{\alpha + 1} u\|_{L^2}^{2(\frac{2}{3} - \alpha)} \|w\|_{L^2}^{2(\frac{2}{3} - \alpha)} \|\nabla^\alpha w\|_{L^2}^{2\alpha - \frac{2}{3}} d\tau
\]

\[
\leq C \int_0^t \|u\|_{H^{\alpha + 1}}^2 d\tau + \nu \int_0^t \|\nabla^\alpha w\|_{L^2}^2 d\tau.  
\]

(4.15)

and

\[
2 \int_0^t (g, w) d\tau \leq 2 \int_0^t \|g\|_{L^2} \|w\|_{L^2} d\tau \leq \int_0^t \|g\|_{L^2}^2 d\tau + \int_0^t \|g\|_{L^2} \|w\|_{L^2}^2 d\tau.  
\]

(4.17)
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Inserting (4.15-4.17) into (4.14) to deduce
\[
\|w\|_{L^2}^2 + \nu \int_0^t \|\nabla^\alpha w\|_{L^2}^2 d\tau \leq \|a\|_{L^2}^2 + C \int_0^t \|u\|_{H^{\alpha+1}}^2 d\tau + \int_0^t \|g\|_{L^2}^2 d\tau + C \int_0^t (\|u\|_{H^{\alpha+1}}^2 + \|g\|_{L^2}) \|w\|_{L^2}^2 d\tau.
\] (4.18)

Taking Gronwall inequality into consideration gives
\[
\|w\|_{L^2}^2 + \nu \int_0^t \|\nabla^\alpha w\|_{L^2}^2 d\tau \leq (\|a\|_{L^2}^2 + C \int_0^\infty (\|u\|_{H^{\alpha+1}}^2 + \|g\|_{L^2}) d\tau) \exp \left\{ C \int_0^\infty (\|g\|_{L^2} + \|u\|_{H^{\alpha+1}}) d\tau \right\},
\] (4.19)
or
\[
\|w\|_{L^2}^2 + \nu \int_0^t \|\nabla^\alpha w\|_{L^2}^2 d\tau \leq C
\] (4.20)
for the constant C independent of \(t > 0\), where we have used the bounds of the solution \(u\) with initial data \(u_0 \in H^1\)
\[
u \in L^2(0, \infty, H^{1+\alpha}(\mathbb{R}^3)).\] (4.21)

Furthermore, the derivation of (4.18) and (4.20) also implies the estimate, for \(0 \leq s < t < \infty\)
\[
\|w\|_{L^2}^2 + \nu \int_s^t \|\nabla^\alpha w\|_{L^2}^2 d\tau \leq \|w(s)\|_{L^2}^2 + C \int_s^t (\|u\|_{H^{\alpha+1}}^2 + \|g\|_{L^2}) d\tau + C \int_s^t (\|g\|_{L^2} + \|u\|_{H^{\alpha+1}}) \|w\|_{L^2}^2 d\tau \leq \|w(s)\|_{L^2}^2 + C \int_s^t (\|g\|_{L^2} + \|u\|_{H^{\alpha+1}}) d\tau.
\] (4.22)

Step 3 Proof of (4.3)

In order to prove the asymptotic stability
\[
\|w(t)\|_{L^2} \to 0 \quad \text{as} \quad t \to \infty,
\]
we need the following auxiliary decay estimate

Lemma 4.1. Under the same assumption of Theorem 4.1, the following auxiliary decay
\[
\lim_{t \to \infty} \frac{1}{t} \int_{t/2}^t \|w(\tau)\|_{L^2}^2 d\tau = 0 \quad (4.23)
\]
holds true.
By Lemma 4.1, \((4.21)\) and the assumption on \(g\), the integration of \((4.22)\) with respect to \(s\) on the interval \(\left(\frac{t}{2}, t\right)\) yields

\[
\|w(t)\|_{L^2}^2 \leq \frac{2}{t} \int_{\frac{t}{2}}^{t} \|w(s)\|_{L^2}^2 ds + C \int_{\frac{t}{2}}^{t} (\|u\|_{H^{\alpha+1}}^2 + \|g\|_{L^2}) d\tau ds \\
\leq \frac{2}{t} \int_{\frac{t}{2}}^{t} \|w(s)\|_{L^2}^2 ds + C \int_{\frac{t}{2}}^{t} (\tau - \frac{t}{2}) (\|u\|_{H^{\alpha+1}}^2 + \|g\|_{L^2}) d\tau \\
\leq \frac{2}{t} \int_{\frac{t}{2}}^{t} \|w(s)\|_{L^2}^2 ds + C \int_{\frac{t}{2}}^{t} (\|u\|_{H^{\alpha+1}}^2 + \|g\|_{L^2}) d\tau \to 0 \quad \text{as} \quad t \to \infty,
\]

which derive the desired assertion \((4.3)\) in Theorem 4.1.

Hence it remains to prove Lemma 4.1. 

**Proof of Lemma 4.1** According to the definition of the solutions \(u\) and \(v\), the difference \(w = u - v\) satisfies the following identity

\[
(w(t), \varphi_e(t)) - (w(s), \varphi_e(s)) + \int_{s}^{t} \{-(w(\tau), \partial \varphi_e(\tau)) + (\nu(-\Delta)^{3/2} w, (-\Delta)^{3/2} \varphi_e)\} d\tau \\
= \int_{s}^{t} (g, \varphi_e) d\tau - \int_{s}^{t} \{F_N(\|\nabla v\|_{L^2})(v \cdot \nabla w, \varphi_e) + F_N(\|\nabla v\|_{L^2})(w \cdot \nabla u, \varphi_e)\} d\tau \\
+ \int_{s}^{t} (F_N(\|\nabla u\|_{L^2}) - F_N(\|\nabla v\|_{L^2})(u \cdot \nabla u, \varphi_e)) d\tau \tag{4.24}
\]

for \(0 \leq s \leq t < \infty\) and \(\epsilon > 0\), where the test function \(\varphi_e\) is defined as

\[
\varphi_e(\tau) = U(\tau) \int_{s}^{t} \eta_\epsilon(\tau - \sigma) U(\sigma) w(\sigma) d\sigma \quad \text{with} \quad U(\tau) = (1 - \Delta)^{-\frac{3\alpha}{2}} e^{-\nu(-\Delta)^{\alpha}(t-\tau)}
\]

for \(s \leq \tau \leq t\) and the function \(\eta_\epsilon\) given by \((4.5)\).

This definition implies that

\[
\partial \tau \varphi_e(\tau) + \nu(-\Delta)^{3/2} \varphi_e(\tau) = U(\tau) \int_{s}^{t} \partial \tau \eta_\epsilon(\tau - \sigma) U(\sigma) w(\sigma) d\sigma \tag{4.25}
\]

and by \((4.20)\)

\[
\sup_{s \leq \tau \leq t} \|\varphi_e(\tau)\|_{L^2} \leq C, \tag{4.26}
\]

furthermore, by Sobolev embedding inequality and \((4.20)\)

\[
\|\nabla \varphi_e(\tau)\|_{L^2} \leq C \left\|\nabla\frac{1}{2} U(\tau) \int_{s}^{t} \eta_\epsilon(\tau - \sigma) U(\sigma) w(\sigma) d\sigma\right\|_{L^2} \\
\leq C \left\|\int_{s}^{t} \eta_\epsilon(\tau - \sigma) e^{-\nu(-\Delta)^{\alpha}(t-\sigma)} w(\sigma) d\sigma\right\|_{L^2} \leq C. \tag{4.27}
\]
We now estimate (4.24) one by one. Firstly, for the first two terms of the left hand side of (4.24), by the definition of $\eta$

\[
(w(t), \varphi_\varepsilon(t)) - (w(s), \varphi_\varepsilon(s))
\]

\[
= \int_s^t \eta_\varepsilon(t - \sigma)(U(t)w(t), U(\sigma)w(\sigma))d\sigma - \int_s^t \eta_\varepsilon(s - \sigma)(U(s)w(s), U(\sigma)w(\sigma))d\sigma
\]

\[
= \frac{1}{2}\|U(t)w(t)\|_{L^2}^2 - \frac{1}{2}\|U(s)w(s)\|_{L^2}^2
\]

\[
= \frac{1}{2}\|\epsilon^{-\Delta}^{-\frac{2-n}{4}}w(t)\|_{L^2}^2 - \frac{1}{2}\|\epsilon^{-\Delta}^{-\frac{2-n}{4}}w(s)\|_{L^2}^2 \quad (\epsilon \to 0).
\]

(4.28)

Secondly, for others of the left hand side of (4.24), we obtain that

\[
\int_s^t \{-\langle w(\tau), \partial_\tau \varphi_\varepsilon(\tau)\rangle + \langle w, \nabla \varphi_\varepsilon \rangle\}d\tau
\]

\[
= \int_s^t \{-\langle w(\tau), \partial_\tau \varphi_\varepsilon(\tau)\rangle + \langle w, \nabla (-\Delta)^{\alpha_\varepsilon} \varphi_\varepsilon \rangle\}d\tau
\]

\[
= \int_s^t \langle w(\tau), \nabla (-\Delta)^{\alpha_\varepsilon} \varphi_\varepsilon - \partial_\tau \varphi_\varepsilon(\tau)\rangle d\tau
\]

\[
= - \int_s^t \int_s^t \partial_{\tau}\eta_\varepsilon(\tau - \sigma)U(\tau)w(\sigma)\langle U(\sigma)w(\sigma)\rangle d\tau = 0.
\]

(4.29)

Moreover, by using Hölder inequality and Sobolev embedding inequality and combining (4.20),(4.26) and (4.27), the last three terms of the right hand side of (4.24) is estimated as follows:

\[
- \int_s^t \{F_N(\|\nabla v\|_{L^2})(v \cdot \nabla w, \varphi_\varepsilon) + F_N(\|\nabla v\|_{L^2})(w \cdot \nabla u, \varphi_\varepsilon)\}d\tau
\]

\[
\leq \int_s^t \|v\|_{L^\frac{n}{n-\alpha}} \|\nabla \varphi_\varepsilon\|_{L^\frac{2}{\alpha}} \|w\|_{L^\frac{n}{n-\alpha}} d\tau + \int_s^t \|u\|_{L^\frac{n}{n-\alpha}} \|\nabla \varphi_\varepsilon\|_{L^\frac{2}{\alpha}} \|w\|_{L^\frac{n}{n-\alpha}} d\tau
\]

\[
\leq C \left(\sup_{s \leq \tau \leq t} \|\nabla \varphi_\varepsilon(\tau)\|_{L^\frac{2}{\alpha}}\right) \left(\int_s^t \|v\|_{L^\frac{n}{n-\alpha}} \|w\|_{L^\frac{n}{n-\alpha}} d\tau \right.
\]

\[
+ \int_s^t \|u\|_{L^\frac{n}{n-\alpha}} \|w\|_{L^\frac{n}{n-\alpha}} d\tau \right)
\]

\[
\leq C \int_s^t \|\nabla^\alpha v\|_{L^2} \|\nabla^\alpha w\|_{L^2} d\tau + C \int_s^t \|\nabla^\alpha u\|_{L^2} \|\nabla^\alpha w\|_{L^2} d\tau
\]

\[
\leq C \left(\int_s^t \|\nabla^\alpha v\|_{L^2}^2 d\tau \right) \frac{1}{2} \left(\int_s^t \|\nabla^\alpha w\|_{L^2}^2 d\tau \right) \frac{1}{2}
\]

\[
+ C \left(\int_s^t \|\nabla^\alpha u\|_{L^2}^2 d\tau \right) \frac{1}{2} \left(\int_s^t \|\nabla^\alpha w\|_{L^2}^2 d\tau \right) \frac{1}{2}
\]

\[
\leq C \left(\int_s^t \|\nabla^\alpha v\|_{L^2}^2 d\tau \right)^\frac{1}{2} + \left(\int_s^t \|\nabla^\alpha u\|_{L^2}^2 d\tau \right)^\frac{1}{2}, \quad (4.30)
\]
and
\[ -\int_s^t (F_N(\|\nabla u\|_{L^2}) - F_N(\|\nabla v\|_{L^2}))(u \cdot \nabla u, \varphi) dt \]
\[ \leq C \int_s^t \|u\|_{L^\infty}^2 \|\nabla \varphi\|_{L^\frac{3}{2}} dt \leq C \left( \sup_{s \leq \tau \leq t} \|\nabla \varphi(\tau)\|_{L^{\frac{3}{2}}} \right) \int_s^t \|u\|_{L^\infty}^2 \|u\|_{L^{\infty}} \|u\|_{L^2} \frac{1}{
abla} d\tau \]
\[ \leq \int_s^t \|\nabla u\|_{L^2}^2 dt. \] (4.31)

Finally, we have for the last term
\[ \int_s^t (g, \varphi) dt \leq \int_s^t \|g\|_{L^2} \|\varphi\|_{L^2} dt \leq C \int_s^t \|g\|_{L^2} dt. \] (4.32)

Thus, inserting (4.28-4.32) into (4.24) and letting \( \epsilon \to 0 \), we obtain
\[ \frac{1}{2} \|(1 - \Delta)^{-\frac{2}{4}} w(t)\|_{L^2}^2 - \frac{1}{2} \int e^{-\nu(-\Delta)^{\alpha}(t-s)}(1 - \Delta)^{-\frac{2}{4}} w(s)\|_{L^2}^2 \]
\[ \leq C \left\{ \int_s^t \|\nabla^\alpha v\|_{L^2}^2 dt \right\}^\frac{1}{2} + \left( \int_s^t \|\nabla u\|_{L^2}^2 dt \right)^\frac{1}{2} + \int_s^t \|\nabla u\|_{L^2}^2 dt + \int_s^t \|g\|_{L^2} dt \}
\]
which implies the desired auxiliary decay estimate
\[ \limsup_{t \to \infty} \|(1 - \Delta)^{-\frac{2}{4}} w\|_{L^2}^2 \to 0 \] (4.33)

after letting \( t \to \infty \) and then \( s \to \infty \).

Using Gagliardo-Nirenberg inequality yields
\[ \|w\|_{L^2}^2 \leq C \|(1 - \Delta)^{-\frac{2}{4}} w(t)\|_{L^2}^\frac{4}{\alpha} \|(1 - \Delta)^{\frac{3}{2}} w\|_{L^4}^\frac{4-2\alpha}{\alpha} \]
which implies by combining (4.20)-(4.33)
\[ \frac{1}{t} \int_s^t \|w\|_{L^2}^2 dt \leq C \int_s^t \|(1 - \Delta)^{-\frac{2}{4}} w\|_{L^2}^\frac{4}{\alpha} \|w\|_{L^2}^\frac{4-2\alpha}{\alpha} d\tau \]
\[ \leq C \int_s^t \|(1 - \Delta)^{-\frac{2}{4}} w\|_{L^2}^\frac{4}{\alpha} \left( \|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 \right) \frac{2-\alpha}{2+\alpha} d\tau \]
\[ \leq C \left( \frac{1}{t} \int_s^t \|(1 - \Delta)^{-\frac{2}{4}} w\|_{L^2}^2 d\tau \right)^\frac{2-\alpha}{2+\alpha} \times \left( \frac{1}{t} \int_s^t \left( \|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 \right) d\tau \right)^\frac{2-\alpha}{2+\alpha} \]
\[ \leq C \left( \frac{1}{t} \int_s^t \|(1 - \Delta)^{-\frac{2}{4}} w\|_{L^2}^2 d\tau \right) \frac{2-\alpha}{2+\alpha} \left( 1 + \frac{1}{t} \right)^\frac{2-\alpha}{2+\alpha} \to 0 \]
as \( t \to \infty \). Hence we complete the proof of Lemma 4.1.

Acknowledgments. The authors express their sincere thanks to the editor and the referee for their invaluable comments and suggestions.
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