Computation of Structure Functions
From a Lattice Hamiltonian

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Abstract
We suggest to compute structure functions in the Hamiltonian formalism on a momentum lattice using a physically motivated regularisation that links the total parton number to the lattice size. We show for the $\phi^4$ theory that our method allows to describe continuum physics. The critical line and the renormalised mass spectrum close to that critical line are computed and scaling behaviour is observed in good agreement with the semi-analytical results of Lüscher and Weisz and with other lattice simulations. We also demonstrate that our method is able to reproduce the $Q^2$ behaviour of deep inelastic structure functions and the typical peak at $x_B = 0$.

1 Introduction

Hadron structure is probed by deep inelastic scattering (DIS). Much effort has been devoted by theorists to compute quark or gluon distribution functions and proton structure functions from QCD with non perturbative methods. E.g., Martinelli et al. have computed the first two moments of the pion structure function. These calculations are notoriously difficult; for the present status of lattice calculations based on the Euclidean action formulation see Ref. In order to compute structure functions it seems natural to use the Hamiltonian formulation that allows to compute wave functions. E.g., scattering wave functions for glueball-like states in compact $QED_{2+1}$ have been computed in a Hamiltonian formulation on a momentum lattice. For a review of Hamiltonian lattice methods see. The usefulness of a momentum lattice to compute physics close to a critical point has been demonstrated in Ref. for the $\phi^4$-model. Unfortunately, Hamiltonian methods are known to pose problems when calculating the huge number of field degrees of freedom due to particle number. Nobody has yet succeeded in seeing scaling laws indicating continuum physics in a $QCD$-Hamiltonian formulation, to the best of the authors’ knowledge. In this work we shall demonstrate that these difficulties can be overcome. We introduce a new type of regularisation which excludes all parton momenta that we deem to be not directly related to deep inelastic
structure measurements. It turns out that such a regularisation scheme provides both an ultraviolet as well as a particle cutoff. We have chosen to test our method on the popular scalar $\phi^4$ theory in four space-time dimensions because this model has been thoroughly investigated numerically and analytically. The principal results are:

1. Using a Hamiltonian approach we observe continuum physics (scaling laws close to the critical line) for the $\phi^4$ and we see
2. Altarelli-Parisi-type behaviour of $\phi^3$ distribution functions in two, three and four dimensions and a peak at $x_B = 0$ similar to the peak occurring in quark distribution functions.

2 DIS and a physical choice of Fock space

Physical observables of relativistic field theories depend continuously on the experimental resolution. Every experiment has a finite resolution and coarse-grains the observed system. This corresponds cum grano salis to a regularisation of the field theory which describes the observed system. The hadron structure functions that we attempt to calculate are a prominent example. The most important experiment in order to probe the structure of hadrons is deep inelastic scattering (DIS). Its simplest form is inclusive scattering of an unpolarised lepton off a hadronic target which we want to build our method upon. For the basics of DIS see, e.g. [8]. Notation: The hadron in its ground state with four momentum $P$ interacts with the probing lepton by the exchange of a virtual photon (our neutrino) with space-like four-momentum $q$. In Feynman’s parton model, the impulse approximation is invoked. It is assumed that the proton consists of constituents, the partons, that are weakly bound when compared to the resolution ability $Q := \sqrt{-q_0 q^\mu q_\mu}$ of the probing photon. In this approximation, the so-called Bjørken scaling variable $x_B := \frac{Q^2}{2P_0 q^0}$ can be interpreted as the momentum fraction of the struck parton in the Breit frame (which is defined by the requirements that the photon energy $q_0$ be zero and that the momentum $q$ point in the opposite direction of the hadron momentum $P$). In this model, structure functions $F(x_B, Q)$ can be interpreted as some linear combinations of parton momentum distribution functions $f(x_B, Q)$. For details see [8].

We want to design a theory where just this picture emerges, describing only the coarse-grained particles that are probed in a DIS experiment. Consequently, we have to devise a regularisation that mimics the coarse-graining due to experiment as close as possible. In order to achieve this goal we have to decide on a convenient Lorentz frame, because firstly the Hamiltonian is only defined with respect to such a frame and, secondly, a parton being a virtual particle is only defined with respect to the specific relativistic frame where experiment performs its coarse-graining of space. Seen from a different Lorentz frame, the parton would emerge as a jet of many partons because the boost generators of the Poincaré group contain vertices that change the parton number! Our candidate is the Breit frame, of course. The struck parton in the Breit frame, however, is collinear to the hadron momentum. Thus, we have to allow for non-collinear partons in more than one space dimension. Therefore we use a ‘generalised Breit frame’ where $q_0 = 0$ but $q$ does not have to be
collinear to $\vec{P}$. Please note that only in such a 'generalised Breit frame' is $Q = |\vec{q}|$ and only there can the wavelength $\lambda = \frac{2\pi}{|\vec{q}|}$ of the virtual photon be interpreted as the resolution ability of the experiment! Please note also that the momentum $\vec{P}$ strongly restricts the domain of all possible photon momenta $\vec{q}$ in the generalised Breit frame. We require therefore that the 'observable' Fock space consist only of partons whose momenta can (but need not) reverse momentum $-\vec{p} \equiv \vec{p} + \vec{q}$ if struck by a virtual photon in the generalised Breit frame. Parton momenta that can not be related by $\vec{p} = -\vec{q}/2$ to some virtual photon with momentum $\vec{q}$ are skipped. Then all 'observable' momenta of the ground state lie inside a sphere

$$ (\vec{p} - \vec{P}/2)^2 \leq (\vec{P}/2)^2 $$

around $\vec{P}/2$ with diameter $|\vec{P}|$ because $x_B = Q^2/2|\vec{q}| = -Q^2/2\vec{p}\cdot\vec{q}$ lies between 0 and 1.

If we go onto a momentum space lattice with momenta $\vec{k} := \vec{n}\Delta k$ where $\vec{n}$ is an integer vector and $\Delta k$ is the momentum lattice spacing, then the constraint (1) on the parton momenta provides an ultraviolet as well as a particle cutoff. This holds because only a finite number of partons with positive longitudinal momentum $p_L := \vec{p}\cdot\vec{e}$ can share the positive total longitudinal momentum $\vec{P}\cdot\vec{e}$, where $\vec{e} := \vec{P}/|\vec{P}|$ denotes the direction of the total hadron. If one performs the continuum limit, the system's velocity $\vec{v} = \vec{P}/E$ grows and one ends up with the velocity of light. For a parton with momentum fraction $x_B = \text{const}$ the continuum limit $a \rightarrow 0$ also implies $Q = \frac{aQ}{a} \rightarrow \infty$ and hence is equivalent to the Bjørken limit. By the way, the transverse momenta in this limit are suppressed by the increasingly dominant kinetic term, at least in asymptotically free and trivial theories.

Since we want to calculate the critical line – which depends on the regularisation – we had to ensure that our regularisation [1] be well comparable to to the lattice regularisation of ref. [9]. To this aim we started on the usual lattice with the position space lattice spacing $a = \frac{\pi}{N\Delta k}$ as ultraviolet regulator, where $2N + 1$ denotes the number of lattice points in each space direction, and put the total momentum $\vec{P}$ on one of the edges $\vec{P} = (1,1,1)\Delta k N$ of the momentum lattice. If we then apply our regularisation [1], every component of the parton momenta is positive, i.e the combination of our regularisation with the lattice cutoff restricts the parton momenta to the first octant of the momentum lattice which further simplifies the set-up of the Fock space. The maximal number of 'partons' equals $N$ in this description. We ought to add that for a small lattice such as $2N + 1 = 7$ the additional lattice spacing cutoff does not exclude states that are allowed by our regularisation – so this seems to be o.k..

A further feature of the proposed ansatz is remarkable: the connection between boosts and renormalisation group transformations. A boost that reduces the total momentum $|\vec{P}|$ equally reduces the number of observable Fock states in our approach and hence constitutes a coarse-graining procedure on a momentum grid with fixed infrared regularisation – in contrast to that, a boost without our regularisation would increase the (unobservable) particle number. Since the hadron momentum $|\vec{P}| = \frac{\pi}{a}$ is the cutoff in our approach, the scale $1/a$ is reduced and the bare parameters have to be re-adjusted. This means that such a boost acts in a way similar to a renormalisation...
group transformation in the Kadanoff/Wilson sense. Such a transformation is not invertible. Intuitively this corresponds to the fact that decreasing the momentum of the probing lepton relative to the target hadron corresponds to a lesser structure resolution. If $|\vec{P}|$ gets too small we have coarse-grained too much and the system breaks down, of course. In the momentum-sector with $\vec{P} = \vec{0}$ only one state, the vacuum, remains. Within our method we cannot describe a hadron in its rest frame, just as one cannot perform structure measurements without a particle beam moving relative to the target. This means that we cannot give a Fock space expansion of the physical vacuum either. This is no draw-back, it is a desirable feature. It makes simply no sense to calculate structure functions for the vacuum. We treat it as a background field with the same quantum numbers as the perturbative vacuum because only vacuum expectation values of fields or of the energy (Casimir effect) originate from the vacuum. The fact that the perturbative vacuum is annihilated by our Hamiltonian does not mean that it is the physical vacuum – the latter one is far from being trivial. It only means that per constructionem our theory is designed for deep inelastic scattering and breaks down if we set $\vec{P}$ equal zero. If we, however, boost the momentum grid also, the transformation is an invertible scale transformation making the physical quantities change according to the Callan-Symanzik equation. Three little remarks at the end of this section:

1. Theoretically, our regularisation admits partons with zero momentum. An infinite number of these can be present because they do not take away longitudinal momentum. It is these that are responsible for vacuum expectation values. Fortunately, for a system with spontaneous symmetry breaking as the $\phi^4$ theory, we can treat them on the classical level since at least in the infinite volume limit, quantum mechanical fluctuations of the vacuum expectation values must vanish.

2. It is obvious that our approach sheds some light on the relation of the so-called front- and instant forms in two dimensions. A discussion of this must be reserved for another paper.

3. Our method would, of course, permit to calculate the hadronic tensor and thus the structure functions directly from the electromagnetic current operators. But the calculation of distribution functions is much easier.

### 3 Application on the scalar $\phi^4_4$ theory

Now we apply our method to the $\phi^4_4$ theory as a testing ground. We shall compare it to some existing results in the lattice regularisation. The Hamiltonian of the $\phi^4$ theory is

$$H = \frac{1}{2} \int d^4x \left( \frac{1}{2} \left( \frac{\partial \phi}{\partial t} \right)^2 + \left( \frac{\partial \phi}{\partial x} \right)^2 + m_0^2 \phi^2 + \frac{g_0}{4!} \phi^4 \right)$$

where $\phi(x)$ is the scalar field, $m_0$ and $g_0$ are the bare mass and coupling constant respectively.

We set the scale $\Delta k$ to one and insert the discretised and quantised solution

$$\phi(x) = \sum_{\vec{k}} \frac{1}{(2\pi)^3\sqrt{2\omega(\vec{k})}} \left( a(\vec{k}) e^{-ik_\mu x^\mu} + a^\dagger(\vec{k}) e^{+ik_\mu x^\mu} \right)$$
of the kinetic field equations of motion back into the Hamiltonian, where \( a, a^\dagger \) are the usual annihilation and creation operators \([a^\dagger(\vec{k}), a(\vec{l})] = \delta_{\vec{k},\vec{l}} \) respectively and \( \omega(\vec{k}) = k_0 = \sqrt{m_0^2 + \vec{k}^2} \) is the kinetic energy of a parton. This yields the discretised Hamiltonian that we diagonalised.

We denote the n-th eigenvalue of the Hamiltonian Matrix as \( E_n \) and the n-th physical mass as \( M_n := \sqrt{E_n^2 - \vec{P}^2} \). We did not normal order – which would not have been allowed for an interacting Hamiltonian – but we subtracted the vacuum energy. The latter is equivalent to the requirement that the perturbative vacuum be annihilated by the Hamiltonian. Only if the vacuum-energy \( E_0 \) is gauged to be zero can we use the mass-shell condition \( M_n := \sqrt{E_n^2 - \vec{P}^2} \) to calculate the physical mass spectrum from the energy spectrum. We do not have to take the field renormalisation constant \( Z_\phi \) into account, but one problem arises. The critical line lies entirely in the region where \( m_0^2 \) is negative and we cannot build up the Fock-space in terms of partons with imaginary masses. Instead, we have chosen to build it up in terms of partons with mass \( m \) by splitting the \( m_0^2 \phi^2/2 \) mass term into a kinetic \( m_K^2 \phi^2/2 \) and a dynamic \( m^2_{\text{int}} \phi^2/2 \) part where \( m_K^2 = m_0^2 - m^2_{\text{int}} > 0 \) is the kinetic energy of the partons and \( m^2_{\text{int}} \) is a sufficiently large negative number that takes the rôle of a coupling constant. We have presented our results for the choice \( m_K^2 = 0 \). We have found, however, that the critical behaviour was almost independent of a specific choice of \( m_K \) in the \( N = 4 \)-sector, if only \( m_K^2 \) is reasonably small.

4 Critical behaviour of the scalar \( \phi^4 \)-theory

We diagonalised the described Hamiltonian on a \( 7^3 \) and a \( 9^3 \) lattice. To do this, Fock spaces of only 6 resp. 21 states are needed(and even less if the \( Z_2 \) symmetry is removed in the symmetric phase). In order to compare our results to the work of Lüscher and Weisz we have to introduce the parameters \( \lambda \) and \( \kappa \) which are related to \( m_0 \) and \( \lambda \) by \( m_0^2 = (1 - 2\lambda)/\kappa - 8 \) and \( g_0 = 6\lambda/\kappa^2 \). In Fig.1 the first of the three data columns of the renormalised masses \( m_R \) in terms of \( \kappa \) as given by Lüscher and Weisz is shown and compared to our ground state masses \( M_1 \). We have diagonalised the Hamiltonian for \( 50 \times 30 \) different bare parameters \( m_0, g_0 \) and fitted the results to obtain the energy spectra in terms of \( \lambda \) and \( \kappa \). It can be seen in Fig. 1 that the shape of the function is well described even on the \( 7^3 \)-lattice. For the \( 9^3 \)-lattice we are able to describe most of the scaling region \( M_1 \leq \frac{a}{2} \). The ground state mass \( M_1(\kappa) \) inside the validity domain \( 1/(2N) << aM_1 << 1 \) of our model is seen to obey the scaling law \( M_1(\kappa) \sim C_4 T^{1/2} |\ln \tau|^{-1/6} \), where \( C_4 \) is a constant and \( \tau := 1 - \kappa/\kappa_{\text{crit}} \) denotes the reduced temperature, meaning that we have correctly described the critical exponent \( \nu = 1/2 \) plus the logarithmic correction \( |\ln \tau|^{-1/6} \) in our Hamiltonian approach. It can also be seen that the critical point is well reproduced. This is true also for curves of different \( \lambda \) close to the Gaussian fixed point as it is shown in Table 1. where we listed the quotients \( \alpha_N := \kappa_{\text{crit}}/\kappa_{(N)}^{\text{crit}} \), the quantity \( \kappa_{(N)}^{\text{crit}} \) denotes the point where the renormalised mass \( M_1 \) becomes imaginary(It becomes real again in the asymmetric phase if we break the \( Z_2 \) symmetry by choosing a non zero field expectation value.).

Table 1 shows that the quantity \( \kappa_{(N)}^{\text{crit}} \) is an excellent estimator for the critical line. In Fig.2 finally,
the mass spectrum related to the first excited state, $M_n/M_1$, is depicted for fixed $\lambda$ in terms of $\kappa$. It clearly demonstrates the fact that the partons of the $\phi^4_4$ theory interact by repulsive forces. The first state above the vacuum is a one particle state with negative parity and the positive parity state above consists of two particles.

## 5 Distribution functions

Since the $\phi^4$ theory describes partons with repulsive two particle exchange forces and since it has a trivial, i.e. free continuum limit, its distribution functions are not comparable to the QCD structure functions. In the $\phi^4$ theory, a renormalisation group transformation(RGT) decreases the lattice size as well as the marginal coupling constant. In an asymptotically free theory such as QCD, the coupling constant increases under such a transformation. This means a RGT enforces particle creation for QCD, whereas particle creation is weakened for the scalar theory. Since the renormalisation group in the Wilson/Kadanoff sense is used to make a complicated system numerically tractable by coarse-graining it into a theory with equivalent long distance behaviour, this means for the $\phi^4$-theory that the coarse-grained system has a strongly reduced particle content so that a particle cutoff may be applied after a few RGTs. This can not be expected to hold for QCD where a RGT actually increases the interaction. This effect slows down or may even invert the net particle number decimation of a RGT due to a reduction of momentum space points and this is the problem we designed our method for. The distribution function $f(x_B, Q)$ of finding some parton with momentum fraction $x_B$ inside the hadron is determined by the parton momentum distribution function $\tilde{f}(\vec{p}, \vec{P})$ for finding a parton with momentum $\vec{p}$ inside the hadron with momentum $\vec{P}$.

Since $Q$ is a dimensionful quantity, its scale is set by the lattice spacing $Q \sim \frac{1}{a(m_0, g_0)}$ and thus depends on the bare parameters if one keeps the renormalised mass fixed. The continuum limit $a \to 0$ corresponds to the the limit towards arbitrarily high resolution ability. If one keeps the renormalised mass and the renormalised coupling constant fixed, then $Q$ is a function of the bare coupling constant $g_0$ and vice versa – invertibility of $Q(g_0)$ assumed. Hence $f(x_B, Q)$ is related by the function $Q(g_0)$ to the distribution function $\tilde{f}(x_B, g_0)$ that only depends on dimensionless parameters. Consequently, a calculation of the distribution function along a renormalisation group trajectory can be used to compute some peculiarities of the $Q$-dependence of the quark structure functions such as the increasing of the peak at $x_B = 0$ when the resolution $Q$ is increased. In order to demonstrate that the suggested method is capable of describing them, we calculate the distribution functions of the $\phi^3$ theory whose forces are attractive one particle exchange forces. Although this theory is known to suffer from an unstable vacuum, it serves well for our purpose of demonstrating qualitatively that the singularity at $x_B = 0$ and its change in terms of the resolution $Q$ of experimentally measured structure functions can be modelled in our approach (The unstable vacuum of the $\phi^3$ theory does not allow us to calculate meaningful ground state masses that are needed to specify renormalisation group trajectories and consequently the exact relation between the resolution $Q$ and the bare coupling constant $g_0$. But the distribution function $\tilde{f}(x_B, g_0)$ suffices
for our purposes). To increase the number of particles that can be modeled on our tiny workstation, we have chosen to present the results of the two dimensional $\phi^3$ theory. The results, however, are qualitatively alike in higher dimensions except for the smaller particle number. In Fig.3 we depict the function $\bar{f}(x_B, g_0)$ to illustrate that our method is – at least in principle – powerful enough to describe the known features of the proton structure function. This can *by no means* be taken for granted. Fock space methods that have to apply a low particle number cutoff in addition to the phase space cutoff are in principle unable to show this feature[10]. This is so because a system of $n$ identical *observable* particles must have an expectation value of $x_B$ around $<x_B> = 1/n$ for symmetry reasons. Furthermore, the behaviour of the structure functions in terms of the resolution $Q$ is dominated not by the power law rest effects of the interaction between the partons but rather by the fact that a higher resolution power unveils *more* particles to the observer. With a low particle number cutoff, the splitting functions appearing in the Altarelli-Parisi equations would be meaningless, anyway. For the $\phi^4$ theory, however, an additional particle cutoff is perfectly feasible for describing the lowest-lying eigenstates of the Hamiltonian.

6 Discussion

We have shown that the suggested principle that only measurable quantities be described works well for the scalar theory. Even a $6 \times 6$ Hamiltonian matrix is able to correctly describe important features of the critical behaviour of the $\phi^4$-theory. We are, however, aware of the fact that there is a huge difference between the scalar $\phi^4$-theory which does not even possess a nontrivial continuum limit and nonabelian gauge theories which do possess a continuum limit with strongly bound particles and also an additional topological structure. But we have reasons for being optimistic. Our method has been designed to sooth one major nuisance of highly relativistic systems, the uncontrolled particle production. This feature has not even been necessary for the $\phi^4$ theory where we could introduce a *modest* particle cutoff without mutilating the results. This is encouraging since we did not aim at the scalar theory in the first place. The aim is the calculation of structure functions, one of the outstanding problems of particle physics, directly from QCD where high particle numbers are inevitable. We hope that our results can be extended to gauge theories and stress that the suggested method has the power to significantly simplify the Kogut-Susskind Hamiltonian because it also works on a position space lattice. This is important since there is no QCD Hamiltonian for momentum space available yet. The Kogut-Susskind Hamiltonian being a position space method is plagued by mirror fermions. Our approach, taking only parton momenta of the first octant in the Brillouin zone, *formally* eliminates them from the set of 'observable' partons, which seems to be a remedy to this problem (We break the mirror-symmetry by the choice of a specific hadron momentum $\vec{P}$ different from $\vec{0}$). If this suffices to describe the axial anomaly remains to be seen. We have to be very cautious here: the Nielsen-Ninomiya no-go theorem[12] has yet always resisted attempts to circumnavigate it.
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\footnote{Sic. Authors. improved version of this paper to be submitted as a letter.}
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Figure Caption

**Fig. 1** The ground state mass $m_R(\lambda, \kappa)$ in lattice units ($a \equiv 1$) is depicted as a function of the lattice parameter $\kappa$ for the semi-analytical data of ref. [9] (points) and for our data obtained on the $7^3$-lattice (dashed line) and on the $9^3$-lattice (solid line). The parameter $\lambda$ is taken to be $\lambda = 0.00345739$.

**Fig. 2** The lowest lying mass spectrum on the $9^3$ lattice is depicted in terms of $\kappa$ for $\lambda = 0.00345739$. The ground state is set to one.

**Fig. 3** The distribution function $\bar{f}(x_B, g_0)$ of $\phi_2$ in terms of the momentum fraction $x_B$ and the coupling constant $g_0$. The mass $m_0$ is set to be $m_0 = 3\Delta k$. The maximum particle number $\bar{N}$ is $\bar{N} = 11$. 
Table Caption

| $\lambda$ | .001 | .005 | .01  | .02  | .03  | .04  | .07  |
|-----------|------|------|------|------|------|------|------|
| $\kappa_{\text{crit}}$ | 0.125202 | 0.125991 | 0.126968 | 0.128604 | 0.130096 | 0.133096 | 0.133825 |
| $\alpha_3$ | 1.01327 | 1.01194 | 1.01192 | 1.01718 | 1.0227 | 1.02664 | 1.03075 |
| $\alpha_4$ | 1.00949 | 1.00518 | 1.0043 | 1.00262 | 1.00079 | 1.00068 | 0.95589 |

The semi-analytically determined critical points $\kappa_{\text{crit}}$ in terms of the lattice parameter $\lambda$ and the quotients $\alpha_N := \kappa_{\text{crit}}/\kappa_{\text{crit}}^{(N)}$. The quantity $\kappa_{\text{crit}}^{(N)}$ denotes the points we found where the renormalised mass $M_1$ becomes imaginary. A tiny systematical error of about 1% can be expected to be present due to effects of our having interpolated the function $M_1(m_0,g_0)$ on a finite (but large) number of parameter points.
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