EXPONENTIAL DRIVING FUNCTION FOR THE LÖwner EQUATION

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Abstract. We consider the chordal Löwner differential equation with the model driving function $\sqrt{t}$. Holomorphic and singular solutions are represented by their series. It is shown that a disposition of values of different singular and branching solutions is monotonic, and solutions to the Löwner equation map slit domains onto the upper half-plane. The slit is a $C^1$-curve. We give an asymptotic estimate for the ratio of harmonic measures of the two slit sides.

1. Introduction

The Löwner differential equation introduced by K. Löwner [11] served a source to study properties of univalent functions on the unit disk. Nowadays it is of growing interest in many areas, see, e.g., [12]. The Löwner equation for the upper half-plane $\mathbb{H}$ appeared later (see, e.g., [1]) and became popular during the last decades. Define a function $w = f(z, t)$, $z \in \mathbb{H}$, $t \geq 0$,

$$f(z, t) = z + \frac{2t}{z} + O\left(\frac{1}{z^2}\right), \quad z \to \infty,$$

which maps $\mathbb{H} \setminus K_t$ onto $\mathbb{H}$ and solves the chordal Löwner ordinary differential equation

$$\frac{df(z,t)}{dt} = \frac{2}{f(z, t) - \lambda(t)}, \quad f(z, 0) = z, \quad z \in \mathbb{H},$$

where the driving function $\lambda(t)$ is continuous and real-valued.

The conformal maps $f(z, t)$ are continuously extended onto $z \in \mathbb{R}$ minus the closure of $K_t$ and the extended map also satisfies equation (2). Following [10], we pay attention to an old problem to determine, in terms of $\lambda$, when $K_t$ is a Jordan arc, $K_t = \gamma(t)$, $t \geq 0$, emanating from the real axis $\mathbb{R}$. In this case $f(z, t)$ are continuously extended onto the two sides of $\gamma(t)$,

$$\lambda(t) = f(\gamma(t), t), \quad \gamma(t) = f^{-1}(\lambda(t), t).$$

Points $\gamma(t)$ are treated as prime ends which are different for the two sides of the arc. Note that Kufarev [9] proposed a counterexample of the non-slit mapping for the radial Löwner equation in the disk. For the chordal Löwner equation, Kufarev’s example corresponds to $\lambda(t) = 3\sqrt{2}\sqrt{1-t}$, see [8], [10] for details.

Equation (2) admits integrating in quadratures for partial cases of $\lambda(t)$ studied in [8], [15]. The integrability cases of (2) are invariant under linear and scaling
transformations of \( \lambda(t) \), see, e.g., [10]. Therefore, assume without loss of generality that \( \lambda(0) = 0 \) and, equivalently, \( \gamma(0) = 0 \).

The picture of singularity lines for driving functions \( \lambda(t) \) belonging to the Lipschitz class \( \text{Lip}(1/2) \) with the exponent \( 1/2 \) is well studied, see, e.g., [10] and references therein.

This article is aimed to show that in the case of the cubic root driving function \( \lambda(t) = \sqrt[3]{t} \) in (2), that is,

\[
\frac{df(z,t)}{dt} = 2f(z,t) - \sqrt[3]{t}, \quad f(z,0) = z, \quad \text{Im} \ z \geq 0,
\]

the solution \( w = f(z,t) \) is a slit mapping for \( t > 0 \) small enough, i.e., \( K_t = \gamma(t) \), \( 0 < t < T \).

The driving function \( \lambda(t) = \sqrt[3]{t} \) is chosen as a typical function of the Lipschitz class \( \text{Lip}(1/3) \). We do not try to cover the most general case but hope that the model driving function serves a demonstration for a wider class. By the way, the case when the trace \( \gamma \) is a circular arc meeting the real axis tangentially is studied in [14]. The explicit solution for the inverse function gave a driving term of the form \( \lambda(t) = Ct^{1/3} + \ldots \) which corresponds to the above driving function asymptotically.

The main result of the article is contained in the following theorem which shows that \( f(z,t) \) is a mapping from a slit domain \( D(t) = \mathbb{H} \setminus \gamma(t) \).

**Theorem 1.** Let \( f(z,t) \) be a solution to the Löwner equation (4). Then \( f(\cdot,t) \) maps \( D(t) = \mathbb{H} \setminus \gamma(t) \) onto \( \mathbb{H} \) for \( t > 0 \) small enough where \( \gamma(t) \) is a \( C^1 \)-curve, except probably for the point \( \gamma(0) = 0 \).

Preliminary results of Section 2 in the article concern the theory of differential equations and preparations for the main proof.

Theorem 1 together with helpful lemmas are proved in Section 3.

Section 4 is devoted to estimates for harmonic measures of the two sides of the slit generated by the Löwner equation (1). Theorem 2 in this Section gives the asymptotic relation for the ratio of these harmonic measures as \( t \to 0 \).

In Section 5 we consider holomorphic solutions to (1) represented by power series and propose asymptotic expansions for the radius of convergence of the series.

## 2. Preliminary statements

Change variables \( t \to \tau^3, \ g(z,\tau) := f(z,\tau^3) \), and reduce equation (1) to

\[
\frac{dg(z,\tau)}{d\tau} = \frac{6\tau^2}{g(z,\tau) - \tau}, \quad g(z,0) = z, \quad \text{Im} \ z \geq 0.
\]

Note that differential equations

\[
\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}
\]

with holomorphic functions \( P(x,y) \) and \( Q(x,y) \) are well known both for complex and real variables, especially in the case of polynomials \( P \) and \( Q \), see, e.g., [2], [3], [4], [13], [16], [17].
If \( z \neq 0 \), then \( g(z, 0) \neq 0 \), and there exists a regular solution \( g(z, \tau) \) to (5) holomorphic in \( \tau \) for \( |\tau| \) small enough which is unique for every \( z \neq 0 \). We are interested mostly in studying singular solutions to (5), i.e., those which do not satisfy the uniqueness conditions for equation (5). Every point \( (g(z_0, \tau_0), \tau_0) \) such that \( g(z_0, \tau_0) = \tau_0 \) is a singular point for equation (5). If \( \tau_0 \neq 0 \), then \( (g(z_0, \tau_0), \tau_0) \) is an algebraic solution critical point, and corresponding singular solutions to (5) through this point are expanded in series in terms \( (\tau - \tau_0)^{1/m} \), \( m \in \mathbb{N} \). So these singular solutions are different branches of the same analytic function, see [17, Chap.9, §1].

The point \( (g(z_0, \tau_0), \tau_0) = (0, 0) \) is the only singular point of indefinite character for (5). It is determined when the numerator and denominator in the right-hand side of (5) vanish simultaneously. All the singular solutions to (5) which are not branches of the same analytic function pass through this point \( (0, 0) \) [17, Chap.9, §1].

Regular and singular solutions to (5) behave according to the Poincaré-Bendixson theorems [13], [2], [17, Chap.9, §1]. Namely, two integral curves of differential equation (5) intersect only at the singular point \( (0, 0) \). An integral curve of (5) can have multiple points only at \( (0, 0) \). Bendixson [2] considered real integral curves globally and stated that they have endpoints at knots and focuses and have an extension through a saddle. Under these assumptions, the Bendixson theorem [2] makes possible only three cases for equation (5) in a neighborhood of \( (0, 0) \): (a) an integral curve is closed, i.e., it is a cycle; (b) an integral curve is a spiral which tends to a cycle asymptotically; (c) an integral curve has the endpoint at \( (0, 0) \).

Recall the integrability case [8] of the Löwner differential equation (2) with the square root forcing \( \lambda(t) = c \sqrt{t} \). After changing variables \( t \to \tau^2 \), the singular point \( (0, 0) \) in this case is a saddle according to the Poincaré classification [13] for linear differential equations. From the other side, another integrability case [8] with the square root forcing \( \lambda(t) = c \sqrt{1 - t} \), after changing variables \( t \to 1 - \tau^2 \), leads to the focus at \( (0, 0) \).

Going back to equation (5) remark that its solutions are infinitely differentiable with respect to the real variable \( \tau \), see [4, Chap.1, §1], [17, Chap.9, §1]. Hence recurrent evaluations of Taylor coefficients can help to find singular solutions provided that a resulting series will have a positive convergence radius [16, Chap.3, §1]. Apply this method to equation (5). Let

\[
g_s(0, \tau) = \sum_{n=1}^{\infty} a_n \tau^n
\]

be a a formal power series for singular solutions to (5). Note that \( g_s \) is not necessarily unique. It depends on the path along which \( z \) approaches to \( 0, z \notin \mathbb{K}_\tau \). Substitute (6) into (5) and see that

\[
\sum_{n=1}^{\infty} n a_n \tau^{n-1} \left( \sum_{n=1}^{\infty} a_n \tau^n - \tau \right) = 6 \tau^2.
\]

Equating coefficients at the same powers in both sides of (7) obtain that

\[
a_1(a_1 - 1) = 0.
\]
This equation gives two possible values $a_1 = 1$ and $a_1 = 0$ to two singular solutions $g^+(0, \tau)$ and $g^-(0, \tau)$. In both cases equation (7) implies recurrent formulas for coefficients $a^+_n$ and $a^-_n$ of $g^+(0, \tau)$ and $g^-(0, \tau)$ respectively,

\begin{align}
(9) & \quad a^+_1 = 1, \quad a^+_2 = 6, \quad a^+_n = -\sum_{k=2}^{n-1} ka^+_k a^+_{n+1-k}, \quad n \geq 3, \\
(10) & \quad a^-_1 = 0, \quad a^-_2 = -3, \quad a^-_n = \frac{1}{n} \sum_{k=2}^{n-1} ka^-_k a^-_{n+1-k}, \quad n \geq 3,
\end{align}

Show that the series $\sum_{n=1}^{\infty} a^+_n \tau^n$ formally representing $g^+(0, \tau)$ diverges for all $\tau \neq 0$.

**Lemma 1.** For $n \geq 2$, the inequalities

\begin{equation}
6^{n-1}(n-1)! \leq |a^+_n| \leq 12^{n-1}n^{n-3}
\end{equation}

hold.

**Proof.** For $n = 2$, the estimate (11) from below holds with the equality sign. Suppose that these estimates are true for $k = 2, \ldots, n - 1$ and substitute them in (7). For $n \geq 3$, we have

\[
|a^+_n| = \sum_{k=2}^{n-1} k|a^+_k||a^+_n| \geq \sum_{k=2}^{n-1} k6^{k-1}(k-1)!6^{n-k} (n-k)! = 6^{n-1} \sum_{k=2}^{n-1} k!(n-k)! \geq 6^{n-1}(n-1)!
\]

This confirms by induction the estimate (11) from below.

Similarly, for $n = 2, 3$, the estimate (11) from above is easily verified. Suppose that these estimates are true for $k = 2, \ldots, n - 1$ and substitute them in (7). For $n \geq 4$, we have

\[
|a^-_n| = \sum_{k=2}^{n-1} k|a^-_k||a^-_{n+1-k}| \leq \sum_{k=2}^{n-1} k12^{k-1}k^{k-3}12^{n-k}(n+1-k)^{n-2-k} = 12^{n-1} \sum_{k=2}^{n-1} k^{k-2}(n+1-k)^{n-2-k} \leq 12^{n-1} \left( \sum_{k=2}^{n-2} (n-1)^{k-2}(n-1)^{n-2-k} + \frac{(n-1)^{n-3}}{2} \right) < 12^{n-1} n^{n-3}
\]

which completes the proof. □

Evidently, the upper estimates (11) are preserved for $|a^-_n|$, $n \geq 2$.

The lower estimates (11) imply divergence of $\sum_{n=1}^{\infty} a^+_n \tau^n$ for $\tau \neq 0$. Therefore equation (5) does not have any holomorphic solution in a neighborhood of $\tau_0 = 0$. 
There exist some methods to summarize the series \( \sum_{n=1}^{\infty} a_n^+ \tau^n \), the Borel regular method among them \([3], [16, \text{Chap.3, \S1}]\). Let

\[
G(\tau) = \sum_{n=1}^{\infty} \frac{a_n^+}{n!} \tau^n,
\]

this series converges for \(|\tau| < 1/12\) according to Lemma 1. The Borel sum equals

\[
h(\tau) = \int_0^\infty e^{-x}G(\tau x)dx
\]

and solves \([5]\) provided it determines an analytic function. The same approach is applied to \(\sum_{n=1}^{\infty} a_n^- \tau^n\).

In any case solutions \(g_1(0, \tau), g_2(0, \tau)\) to \([5]\) emanating from the singular point \((0, 0)\) satisfy the asymptotic relations

\[
g_1(0, \tau) = \sum_{k=1}^{n} a_k^+ \tau^k + o(\tau^n), \quad g_2(0, \tau) = \sum_{k=1}^{n} a_k^- \tau^k + o(\tau^n), \quad \tau \to 0,
\]

for all \(n \geq 2, o(\tau^n)\) in both representations depend on \(n\).

Let \(f_1(0, t) := g_1(0, \tau^3), f_2(0, t) := g_2(0, \tau^3)\). Since \(f_1(0, t) = 3\sqrt{t} + 6\sqrt{t^2} + o(\sqrt{t^2})\) and \(f_2(0, t) = -3\sqrt{t^2} + o(\sqrt{t^2})\) as \(t \to 0\), the inequality

\[
f_2(0, t) < 3\sqrt{t} < f_1(0, t)
\]

holds for all \(t > 0\) small enough.

Let us find representations for all other singular solutions to equation \([4]\) which appear at \(t > 0\). Suppose there is \(z_0 \in \mathbb{H}\) and \(t_0 > 0\) such that \(f(z_0, t_0) = \sqrt{t}\). Then \((f(z_0, t_0), t_0)\) is a singular point of equation \([4]\), and \(f(z_0, t)\) is expanded in series with powers \((t - t_0)^{n/m}, m \in \mathbb{N},\)

\[
f(z_0, t) = 3\sqrt{t_0} + \sum_{n=1}^{\infty} b_{n/m}(t - t_0)^{n/m}.
\]

Substitute \([12]\) into \([4]\) and see that

\[
\sum_{n=1}^{\infty} \frac{nb_{n/m}(t - t_0)^{n/m-1}}{m} \times
\]

\[
\left(\sum_{n=1}^{\infty} b_{n/m}(t - t_0)^{n/m} \right) \cdot \sum_{n=1}^{\infty} \frac{(-1)^n - 12 \cdots (3n - 4)}{n!} \frac{(t - t_0)^n}{(3t_0)^n} = 2.
\]

Equating coefficients at the same powers in both sides of \([13]\) obtain that \(m = 2\) and

\[
(b_{1/2})^2 = 4.
\]

This equation gives two possible values \(b_{1/2} = 2, b_{1/2} = -2\) to two branches \(f_1(z_0, t)\) and \(f_2(z_0, t)\) of the solution \([12]\). Indeed, we can accept only one of possibilities, for example \(b_{1/2} = 2\), while the second case is obtained by going to another
branch of \((t-t_0)^{n/2}\) when passing through \(t = t_0\). So we have recurrent formulas for coefficients \(b_{n/2}\) of \(f_1(z_0, t)\) and \(f_2(z_0, t)\).

\[
(15) \quad b_{n/2} = 2, \quad b_{n/2} = \frac{1}{n+1}\left(c_{n/2} - \frac{1}{2} \sum_{k=2}^{n-1} k b_{k/2} (b_{(n+1-k)/2} - c_{(n+1-k)/2})\right), \quad n \geq 2,
\]

where

\[
(16) \quad c_{(2k-1)/2} = 0, \quad c_k = \frac{(-1)^{k-1} \cdot 5 \ldots (3k-4)}{3^k k^{k-1/3} k!}, \quad k = 1, 2, \ldots .
\]

Since

\[
f_1(z_0, t) = \sqrt{t} t_0 + 2\sqrt{t-t_0} + o(\sqrt{t-t_0}), \quad f_2(z_0, t) = \sqrt{t} t_0 - 2\sqrt{t-t_0} + o(\sqrt{t-t_0}),
\]

\[
\sqrt{t} = \sqrt{t_0} + \frac{1}{3t_0} (t - t_0) + o(t - t_0), \quad t \to t_0 + 0,
\]

the inequality

\[
f_2(z_0, t) < \sqrt{t} < f_1(z_0, t)
\]

holds for all \(t > t_0\) close to \(t_0\).

3. Proof of the main results

The theory of differential equations claims that integral curves of equation (4) intersect only at the singular point \((0, 0)\) \([17, \text{Chap.9, §1}]\). In particular, this implies the local inequalities \(f_2(0, t) < f_2(z_0, t) < \sqrt{t} < f_1(z_0, t) < f_1(0, t)\) where \((f(z_0, t_0), t_0)\) is an algebraic solution critical point for equation (4). We will give an independent proof of these inequalities which can be useful for more general driving functions.

**Lemma 2.** For \(t > 0\) small enough and a singular point \((f(z_0, t_0), t_0)\) for equation (4), \(0 < t_0 < t\), the following inequalities

\[
f_2(0, t) < f_2(z_0, t) < \sqrt{t} < f_1(z_0, t) < f_1(0, t)
\]

hold.

**Proof.** To show that \(f_1(z_0, t) < f_1(0, t)\) let us subtract equations

\[
\frac{df_1(0, t)}{dt} = \frac{2}{f_1(0, t) - \sqrt{t}}, \quad f_1(0, 0) = 0,
\]

\[
\frac{df_1(z_0, t)}{dt} = \frac{2}{f_1(z_0, t) - \sqrt{t}}, \quad f_1(z_0, t_0) = \sqrt{t_0},
\]

and obtain

\[
\frac{d(f_1(0, t) - f_1(z_0, t))}{dt} = \frac{2(f_1(z_0, t) - f_1(0, t))}{(f_1(0, t) - \sqrt{t})(f_1(z_0, t) - \sqrt{t})},
\]

which can be written in the form

\[
\frac{d\log(f_1(0, t) - f_1(z_0, t))}{dt} = \frac{-2}{(f_1(0, t) - \sqrt{t})(f_1(z_0, t) - \sqrt{t})}.
\]
Suppose that $T > t_0$ is the smallest number for which $f_1(0, T) = f_1(z_0, T)$. This implies that

\[
\int_{t_0}^{T} \frac{dt}{(f_1(0, t) - \sqrt{t})(f_1(z_0, t) - \sqrt{t})} = \infty.
\]

To evaluate the integral in (17) we should study the behavior of $f_1(z_0, t) - \sqrt{t}$ with the help of differential equation

\[
\frac{d(f_1(z_0, t) - \sqrt{t})}{dt} = \frac{2}{f_1(z_0, t) - \sqrt{t}} - \frac{1}{3\sqrt{t}^2} = \frac{\sqrt{t} + 6\sqrt{t}^2 - f_1(z_0, t)}{3\sqrt{t}^2(f_1(z_0, t) - \sqrt{t})}.
\]

Calculate that $a_3^+ = -72$ and write the asymptotic relation

\[f_1(0, t) = \sqrt{t} + 6\sqrt{t}^2 - 72t + o(t), \ t \to +0.\]

There exists a number $T' > 0$ such that for $0 < t < T'$, $\sqrt{t} + 6\sqrt{t}^2 > f_1(0, t)$. Consequently, the right-hand side in (18) is positive for $0 < t < T'$. Note that $T'$ does not depend on $t_0$. The condition ”$t > 0$ small enough” in Lemma 2 is understood from now as $0 < t < T'$. We see from (18) that for such $t$, $f_1(z_0, t) - \sqrt{t}$ is increasing with $t$, $t_0 < t < T < T'$. Therefore, the integral in the left-hand side of (17) is finite. The contradiction against equality (17) denies the existence of $T$ with the prescribed properties which proves the third and the fourth inequalities in Lemma 2.

The rest of inequalities in Lemma 2 are proved similarly and even easier. To show that $f_2(z_0, t) > f_2(0, t)$ let us subtract equations

\[
\frac{df_2(0, t)}{dt} = \frac{2}{f_2(0, t) - \sqrt{t}}, \ f_2(0, 0) = 0,
\]

\[
\frac{df_2(z_0, t)}{dt} = \frac{2}{f_2(z_0, t) - \sqrt{t}}, \ f_2(z_0, t_0) = \sqrt{t_0},
\]

and obtain

\[
\frac{d(f_2(0, t) - f_2(z_0, t))}{dt} = \frac{2(f_2(z_0, t) - f_2(0, t))}{(f_2(0, t) - \sqrt{t})(f_2(z_0, t) - \sqrt{t})},
\]

which can be written in the form

\[
\frac{d\log(f_2(z_0, t) - f_2(0, t))}{dt} = \frac{-2}{(f_2(0, t) - \sqrt{t})(f_2(z_0, t) - \sqrt{t})}.
\]

Suppose that $T > t_0$ is the smallest number for which $f_2(z_0, T) = f_2(0, T)$. This implies that

\[
\int_{t_0}^{T} \frac{dt}{(f_2(0, t) - \sqrt{t})(f_2(z_0, t) - \sqrt{t})} = \infty.
\]
To evaluate the integral in (19) we should study the behavior of \( f_2(z_0, t) - \sqrt[3]{t} \) with the help of differential equation

\[ (20) \quad \frac{d(f_2(z_0, t) - \sqrt[3]{t})}{dt} = \frac{2}{f_2(z_0, t) - \sqrt[3]{t}} - \frac{1}{3\sqrt[3]{t^2}} = \frac{\sqrt[3]{t} + 6\sqrt[3]{t^2} - f_2(z_0, t)}{3\sqrt[3]{t^2}(f_2(z_0, t) - \sqrt[3]{t})} \]

Since \( f_2(0, t) = -3\sqrt[3]{t^2} + o(\sqrt[3]{t^2}), \ t \rightarrow +0, \) there exists a number \( T'' > 0 \) such that for \( 0 < t < T'' \), \( \sqrt[3]{t} + 6\sqrt[3]{t^2} > f_2(0, t) \). Consequently, the right-hand side in (20) is positive for \( 0 < t < T'' \). We see from (20) that for such \( t \), \( f_2(0, t) - \sqrt[3]{t} \) is decreasing with \( t \), \( t_0 < t < T < T'' \). Therefore, the integral in the left-hand side of (19) is finite. The contradiction against equality (20) denies the existence of \( T \) with the prescribed properties which completes the proof.

Add and complete the inequalities of Lemma 2 by the following statements demonstrating a monotonic disposition of values for different singular solutions.

**Lemma 3.** For \( t > 0 \) small enough and singular points \( (f(z_1, t_1), t_1), (f(z_0, t_0), t_0) \) for equation (7), \( 0 < t_1 < t_0 < t \), the following inequalities

\[ f_2(z_1, t) < f_2(z_0, t), \ f_1(z_0, t) < f_1(z_1, t) \]

hold.

**Proof.** Similarly to Lemma 2, subtract equations

\[ \frac{df_1(z_1, t)}{dt} = \frac{2}{f_1(z_1, t) - \sqrt[3]{t}}, \ f_1(z_1, t_1) = \sqrt[3]{t_1}, \]

\[ \frac{df_1(z_0, t)}{dt} = \frac{2}{f_1(z_0, t) - \sqrt[3]{t}}, \ f_1(z_0, t_0) = \sqrt[3]{t_0}, \]

and obtain

\[ \frac{d(f_1(z_1, t) - f_1(z_0, t))}{dt} = \frac{2(f_1(z_0, t) - f_1(z_1, t))}{(f_1(z_1, t) - \sqrt[3]{t})(f_1(z_0, t) - \sqrt[3]{t})}, \]

which can be written in the form

\[ \frac{d \log(f_1(z_1, t) - f_1(z_0, t))}{dt} = \frac{-2}{(f_1(z_1, t) - \sqrt[3]{t})(f_1(z_0, t) - \sqrt[3]{t})}. \]

Suppose that \( T > t_0 \) is the smallest number for which \( f_1(z_1, T) = f_1(z_0, T) \). This implies that

\[ (21) \quad \int_{t_0}^{T} \frac{dt}{(f_1(z_1, t) - \sqrt[3]{t})(f_1(z_0, t) - \sqrt[3]{t})} = \infty. \]

To evaluate the integral in (21) apply to (18) and obtain that there exists a number \( T'' > 0 \) such that for \( 0 < t < T'' \), \( f_1(z_0, t) - \sqrt[3]{t} \) is increasing with \( t \), \( t_0 < t < T < T'' \). Therefore, the integral in the left-hand side of (21) is finite. The contradiction
against equality (21) denies the existence of \( T \) with the prescribed properties which proves the second inequality of Lemma 3.

To prove the first inequality of Lemma 3 subtract equations

\[
\frac{df_2(z_1,t)}{dt} = \frac{2}{f_2(z_1,t) - \sqrt{t}}, \quad f_2(z_1,t_1) = \sqrt{t_1},
\]

\[
\frac{df_2(z_0,t)}{dt} = \frac{2}{f_2(z_0,t) - \sqrt{t}}, \quad f_2(z_0,t_0) = \sqrt{t_0},
\]

and obtain after dividing by \( f_2(z_1,t) - f_2(z_0,t) \)

\[
\frac{d \log(f_2(z_0,t) - f_2(z_1,t))}{dt} = \frac{-2}{(f_2(z_1,t) - \sqrt{t})(f_2(z_0,t) - \sqrt{t})}.
\]

Suppose that \( T > t_0 \) is the smallest number for which \( f_2(z_0,T) = f_2(z_1,T) \). This implies that

\[
(22) \quad \int_{t_0}^{T} \frac{dt}{(f_2(z_1,t) - \sqrt{t})(f_2(z_0,t) - \sqrt{t})} = \infty.
\]

To evaluate the integral in (22) apply to (20) and obtain that \( \sqrt{t} + 6\sqrt{t^2} > \sqrt{t} > f_2(0,t) \). Consequently, the right-hand side in (20) is positive and we see that \( f_2(0,t) - \sqrt{t} \) is decreasing with \( t, t_0 < t < T \). Therefore, the integral in the left-hand side of (22) is finite. The contradiction against equality (22) denies the existence of \( T \) with the prescribed properties which completes the proof.

**Proof of Theorem 1.**

For \( t_0 > 0 \), there is a hull \( K_{t_0} \subset \mathbb{H} \) such that \( f(\cdot,t_0) \) maps \( \mathbb{H} \setminus K_{t_0} \) onto \( \mathbb{H} \). We refer to [10] for definitions and more details. The hull \( K_{t_0} \) is driven by \( \sqrt{t} \). The function \( f(\cdot,t_0) \) is extended continuously onto the set of prime ends on \( \partial(\mathbb{H} \setminus K_{t_0}) \) and maps this set onto \( \mathbb{R} \). One of the prime ends is mapped on \( \sqrt{t}_0 \). Let \( z_0 = z_0(t_0) \) represent this prime end.

Lemmas 2 and 3 describe the structure of the pre-image of \( \mathbb{H} \) under \( f(\cdot,t) \). All the singular solutions \( f_1(0,t), f_2(0,t), f_1(z_0,t), f_2(z_0,t), 0 < t_0 < t < T', \) are real-valued and satisfy the inequalities of Lemmas 2 and 3. So the segment \( I = [f_2(0,t), f_1(0,t)] \) is the union of the segments \( I_2 = [f_2(0,t), \sqrt{t}] \) and \( I_1 = [\sqrt{t}, f_1(0,t)] \). The segment \( I_2 \) consists of points \( f_2(z(\tau),t), 0 \leq \tau < t \), and the segment \( I_1 \) consists of points \( f_1(z(\tau),t), 0 \leq \tau < t \). All these points belong to the boundary \( \mathbb{R} = \partial \mathbb{H} \). This means that all the points \( z(\tau), 0 \leq \tau < t, \) belong to the boundary \( \partial(\mathbb{H} \setminus K_t) \) of \( \mathbb{H} \setminus K_t \). Moreover, every point \( z(\tau) \) except for the tip determines exactly two prime ends corresponding to \( f_1(z(\tau),t) \) and \( f_2(z(\tau),t) \). Evidently, \( z(\tau) \) is continuous on \( [0,t] \). This proves that \( z(\tau) := \gamma(\tau) \) represents a curve \( \gamma := K_t \) with prime ends corresponding to points on different sides of \( \gamma \). This proves that \( f^{-1}(w,t) \) maps \( \mathbb{H} \) onto the slit domain \( \mathbb{H} \setminus \gamma(t) \) for \( t > 0 \) small enough.

It remains to show that \( \gamma(t) \) is a \( C^1 \)-curve. Fix \( t_0 > 0 \) from a neighborhood of \( t = 0 \). Denote \( g(w,t) = f^{-1}(w,t) \) an inverse of \( f(z,t) \), and \( h(w,t) := f(g(w,t_0),t), t \geq t_0 \). The arc \( \gamma[t_0,t] := K_t \setminus K_{t_0} \) is mapped by \( f(z,t_0) \) onto a curve \( \gamma_1(t) \) in \( \mathbb{H} \).
emanating from $\sqrt[3]{t_0} \in \mathbb{R}$. So the function $h(w, t)$ is well defined on $\mathbb{H} \setminus \gamma_1(t_0)$, $t \geq t_0$. Expand $h(w, t)$ near infinity,

$$h(w, t) = g(w, t_0) + \frac{2t}{g(w, t_0)} + O \left( \frac{1}{g^2(w, t_0)} \right) = w + \frac{2(t - t_0)}{w} + O \left( \frac{1}{w^2} \right).$$

Such expansion satisfies (11) after changing variables $t \to t - t_0$. The function $h(w, t)$ satisfies the differential equation

$$\frac{dh(w, t)}{dt} = \frac{2}{h(w, t) - \sqrt{t}}, \quad h(w, t_0) = w, \quad w \in \mathbb{H}.$$

This equation becomes the Löwner differential equation if $t_1 := t - t_0$, $h_1(w, t_1) := h(w, t_0 + t_1)$,

$$\frac{dh_1(w, t_1)}{dt_1} = \frac{2}{h_1(w, t_1) - \sqrt{t_1} + t_0}, \quad h_1(w, 0) = w, \quad w \in \mathbb{H}.$$ (23)

The driving function $\lambda(t_1) = \sqrt{t_1 + t_0}$ in (23) is analytic for $t_1 \geq 0$. It is known [1, p.59] that under this condition $h_1(w, t_1)$ maps $\mathbb{H} \setminus \gamma_1$ onto $\mathbb{H}$ where $\gamma_1$ is a $C^1$-curve in $\mathbb{H}$ emanating from $\lambda(0) = \sqrt{t_0}$. The same does the function $h(w, t)$.

Go back to $f(z, t) = h(f(z, t_0), t)$ and see that $f(z, t)$ maps $\mathbb{H} \setminus \gamma(t)$ onto $\mathbb{H}$, $\gamma(t) = \gamma[0, t_0] \cup \gamma[t_0, t]$, and $\gamma[t_0, t]$ is a $C^1$-curve. Tending $t_0$ to 0 we prove that $\gamma(t)$ is a $C^1$-curve, except probably for the point $\gamma(0) = 0$. This completes the proof.

4. Harmonic measures of the slit sides

The function $f(z, t)$ solving [11] maps $\mathbb{H} \setminus \gamma(t)$ onto $\mathbb{H}$. The curve $\gamma(t)$ has two sides. Denote $\gamma_1 = \gamma_1(t)$ the side of $\gamma$ which is mapped by the extended function $f(z, t)$ onto $I_1 = [\sqrt{t}, f_1(0, t)]$. Similarly, $\gamma_2 = \gamma_2(t)$ is the side of $\gamma$ which is the pre-image of $I_2 = [f_2(0, t), \sqrt{t}]$ under $f(z, t)$.

Remind that the harmonic measures $\omega(f^{-1}(i, t); \gamma_k, \mathbb{H} \setminus \gamma(t), t)$ of $\gamma_k$ at $f^{-1}(i, t)$ with respect to $\mathbb{H} \setminus \gamma(t)$ are defined by the functions $\omega_k$ which are harmonic on $\mathbb{H} \setminus \gamma(t)$ and continuously extended on its closure except for the endpoints of $\gamma$, $\omega_k|_{\gamma(t)} = 1$, $\omega_k|_{\mathbb{R} \cup \gamma(t)} = 0$, $k = 1, 2$, see, e.g., [6, Chap.3, §3.6]. Denote

$$m_k(t) := \omega(f^{-1}(i, t); \gamma_k, \mathbb{H} \setminus \gamma(t), t), \quad k = 1, 2.$$

**Theorem 2.** Let $f(z, t)$ be a solution to the Löwner equation (4). Then

$$\lim_{t \to +0} \frac{m_1(t)}{m_2(t)} = 6\pi.$$ (24)

**Proof.** The harmonic measure is invariant under conformal transformations. So

$$\omega(f^{-1}(i, t); \gamma_k, \mathbb{H} \setminus \gamma(t), t) = \Omega(i; f(\gamma_k, t), \mathbb{H}, t)$$

are defined by the harmonic functions $\Omega_k$ which are harmonic on $\mathbb{H}$ and continuously extended on $\mathbb{R}$ except for the endpoints of $f(\gamma_k, t)$, $\Omega_k|_{f(\gamma_k, t)} = 1$, $\Omega_k|_{\mathbb{R} \setminus f(\gamma_k, t)} = 0$, $k = 1, 2$. The solution of the problem is known, see, e.g., [5, p.334]. Namely,

$$m_k(t) = \frac{\alpha_k(t)}{\pi}$$
where $\alpha_k(t)$ is the angle under which the segment $I_k = I_k(t)$ is observed from the point $w = i$, $k = 1, 2$. It remains to find asymptotic expansions for $\alpha_k(t)$.

Since

$$f_1(0, t) = \sqrt[3]{t} + 6\sqrt[3]{t^2} + O(t), \quad f_2(0, t) = -3\sqrt[3]{t^2} + O(t), \quad t \to +0,$$

after elementary geometrical considerations we have

$$\alpha_1(t) = \arctan f_1(0, t) - \arctan \sqrt[3]{t} = 6\sqrt[3]{t^2} + O(t), \quad t \to +0,$$

$$\alpha_2(t) = \arctan \sqrt[3]{t} - \arctan f_2(0, t) = \sqrt[3]{t} + 3\sqrt[3]{t^2} + O(t), \quad t \to +0.$$ 

This implies that

$$\frac{m_1(t)}{m_2(t)} = \pi \frac{6\sqrt[3]{t^2} + O(t)}{\left(\sqrt[3]{t} + 3\sqrt[3]{t^2} + O(t)\right)^2} = 6\pi \left(1 + O(\sqrt[3]{t})\right), \quad t \to +0,$$

which leads to (24) and completes the proof. □

Remark 1. The relation similar to (24) follows from [14] for the two sides of the circular slit $\gamma(t)$ in $\mathbb{H}$ such that $\gamma(t)$ is tangential to $\mathbb{R}$ at $z = 0$.

5. Representation of holomorphic solutions

Holomorphic solutions to (4) or, equivalently, to (5) appear in a neighborhood of every non-singular point $(z_0, 0)$. We will be interested in real solutions corresponding to $z_0 \in \mathbb{R}$.

Put $z_0 = \epsilon > 0$ and let

$$f(\epsilon, t) = \epsilon + \sum_{n=1}^{\infty} a_n(\epsilon) t^{n/3}$$

be a solution of equation (4) holomorphic with respect to $\tau = \sqrt[3]{t}$. Change $\sqrt[3]{t}$ by $\tau$ and substitute (25) in (5) to get that

$$\sum_{n=1}^{\infty} n a_n(\epsilon) \tau^{n-1} \left[ \epsilon - \tau + \sum_{n=1}^{\infty} a_n(\epsilon) \tau^n \right] = 6\tau^2.$$ 

Equate coefficients at the same powers in both sides of (26) and obtain equations

$$a_1(\epsilon) = 0, \quad a_2(\epsilon) = 0, \quad a_k(\epsilon) = \frac{6}{k\epsilon^{k-2}}, \quad k = 3, 4, 5,$$

and

$$a_n(\epsilon) = \frac{1}{n\epsilon} \left[ (n - 1)a_{n-1}(\epsilon) - \sum_{k=3}^{n-3} (n - k)a_{n-k}(\epsilon)a_k(\epsilon) \right], \quad n \geq 6.$$ 

The series in (25) converges for $|\tau| = |\sqrt[3]{t}| < R(\epsilon)$.

Theorem 3. The series in (27) converges for

$$|t| < \epsilon^3 + o(\epsilon^3), \quad \epsilon \to +0.$$

Theorem 4. The series in (25) converges for

$$|t| < \epsilon^3 + o(\epsilon^3), \quad \epsilon \to +0.$$
Proof. Estimate the convergence radius $R(\epsilon)$ following the Cauchy majorant method, see, e.g., [4, Chap. 1, §§2-3], [16, Chap. 3, §1]. The Cauchy theorem states: if the right-hand side in (5) is holomorphic on a product of the closed disks $|g - \epsilon| \leq \rho_1$ and $|\tau| \leq r_1$ and is bounded there by $M$, then the series $\sum_{n=1}^{\infty} a_n(\epsilon)\tau^n$ converges in the disk

$$|\tau| < R(\epsilon) = r_1 \left(1 - \exp\left\{-\frac{\rho_1}{2Mr_1}\right\}\right).$$

In the case of equation (5) we have

$$\rho_1 + r_1 < \epsilon, \quad \text{and} \quad M = \frac{6r_1^2}{\epsilon - (\rho_1 + r_1)}.$$

This implies that for $\rho_1 + r_1 = \epsilon - \delta$, $\delta > 0$,

$$R(\epsilon) = r_1 \left(1 - \exp\left\{-\frac{\delta - r_1}{12r_1^2}\right\}\right).$$

So $R(\epsilon)$ depends on $\delta$ and $r_1$. Maximum of $R$ with respect to $\delta$ is obtained for $\delta = (\epsilon - r_1)/2$. Hence, this maximum is equal to

$$(30) \quad R_1(\epsilon) = r_1 \left(1 - \exp\left\{-\frac{(\epsilon - r_1)^2}{48r_1^2}\right\}\right),$$

where $R_1(\epsilon)$ depends on $r_1$. Let us find a maximum of $R_1$ with respect to $r_1 \in (0, \epsilon)$. Notice that $R_1$ vanishes for $r_1 = 0$ and $r_1 = \epsilon$. Therefore the maximum of $R_1$ is attained for a certain root $r_1 = r_1(\epsilon) \in (0, \epsilon)$ of the derivative of $R_1$ with respect to $r_1$. To simplify the calculations we put $r_1(\epsilon) = \epsilon c(\epsilon)$, $0 < c(\epsilon) < 1$. Now the derivative of $R_1$ vanishes for $c = c(\epsilon)$ satisfying

$$(31) \quad 1 - \exp\left\{-\frac{(1 - c)^2}{48\epsilon c^3}\right\} \left(1 + \frac{(1 - c)(3 - c)}{48\epsilon c^3}\right) = 0.$$

Choose a sequence $\{\epsilon_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n \to \infty} \epsilon_n = 0$, such that $c(\epsilon_n)$ converge to $c_0$ as $n \to \infty$. Suppose that $c_0 < 1$. Then

$$\exp\left\{-\frac{(1 - c(\epsilon_n))^2}{48\epsilon_n c^3(\epsilon_n)}\right\} \left(1 + \frac{(1 - c(\epsilon_n))(3 - c(\epsilon_n))}{48\epsilon_n c^3(\epsilon_n)}\right) < 1$$

for $n$ large enough. Therefore $c(\epsilon_n)$ is not a root of equation (31) for $\epsilon = \epsilon_n$ and $n$ large enough. This contradiction claims that $c_0 = 1$ for every sequence $\{\epsilon_n > 0\}_{n=1}^{\infty}$ tending to 0 with $\lim_{n \to \infty} c(\epsilon_n) = c_0$. So we proved that $c(\epsilon) \to 1$ as $\epsilon \to +0$.

Consequently, the maximum of $R_1$ with respect to $r_1$ is attained for $r_1(\epsilon) = \epsilon c(\epsilon) = \epsilon(1 + o(1))$ as $\epsilon \to +0$. Let $R_2 = R_2(\epsilon)$ denote the maximum of $R_1$ with respect to $r_1$. It follows from (30) that

$$(32) \quad R_2(\epsilon) = r_1(\epsilon) \left(1 - \exp\left\{-\frac{(\epsilon - r_1(\epsilon))^2}{48r_1^2(\epsilon)}\right\}\right) = \epsilon c(\epsilon) \left(1 - \exp\left\{-\frac{(1 - c(\epsilon))^2}{48\epsilon c^3(\epsilon)}\right\}\right).$$

Examine how fast does $c(\epsilon)$ tends to 1 as $\epsilon \to +0$. Choose a sequence $\{\epsilon_n > 0\}_{n=1}^{\infty}$, $\lim_{n \to \infty} \epsilon_n = 0$, such that the sequence $(1 - c(\epsilon_n))^2/\epsilon_n$ converges to a non-negative
number or to $\infty$. Denote

$$l := \lim_{n \to \infty} \frac{(1 - c(\epsilon_n))^2}{\epsilon_n}, \quad 0 \leq l \leq \infty.$$ 

If $0 < l < \infty$, then $(1 - c(\epsilon_n))/\epsilon_n$ tends to $\infty$, and equation (31) with $\epsilon = \epsilon_n$ has no roots for $n$ large enough.

If $l = 0$, then, according to (31), $\lim_{n \to \infty} (1 - c(\epsilon_n))/\epsilon_n = 0$, and

$$\exp \left\{ -\frac{(1 - c(\epsilon_n))^2}{48\epsilon_n^3(\epsilon)} \right\} \left( 1 + \frac{(1 - c(\epsilon_n))(3 - c(\epsilon_n))}{48\epsilon_n^3(\epsilon)} \right) = \left( 1 - \frac{(1 - c(\epsilon_n))^2}{48\epsilon_n^3(\epsilon)} + o\left( \frac{(1 - c(\epsilon_n))^2}{\epsilon_n} \right) \right) \left( 1 + \frac{(1 - c(\epsilon_n))(3 - c(\epsilon_n))}{48\epsilon_n^3(\epsilon)} + o\left( \frac{(1 - c(\epsilon_n))^2}{\epsilon_n} \right) \right) = 1 + \frac{1 - c(\epsilon_n)}{24\epsilon_n} + o\left( \frac{1 - c(\epsilon_n)}{\epsilon_n} \right), \quad n \to \infty.$$ 

This implies again that equation (31) with $\epsilon = \epsilon_n$ has no roots for $n$ large enough.

Thus the only possible case is $l = \infty$ for all sequences $\{\epsilon_n > 0\}_{n=1}^\infty$ converging to 0. It follows from (32) that

$$(33) \quad R_2(\epsilon) = \max_{0 < r_1(\epsilon) < \epsilon} R_1(\epsilon) = \epsilon + o(\epsilon), \quad \epsilon \to 0.$$ 

In other words, the series in (25) converges for $|t| < (\epsilon + o(\epsilon))^3$, $\epsilon \to 0$, which implies the statement of Theorem 3 and completes the proof. □

Remark 2. Evidently, a similar conclusion with the same formulas (27) and (28) is true for $\epsilon < 0$.

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