Fundamental Nature of Viscous Incompressible Flow and Minimization of Fluid Dynamic Drag

Yasuhiko AIHARA†

Professor Emeritus, Department of Aeronautics and Astronautics, The University of Tokyo, Tokyo 113–8656, Japan

The fundamental nature of viscous incompressible flow is investigated by integrating the first and second laws of thermodynamics. The free energy of fluid, the fundamental thermodynamic function for system equilibrium, is shown to be a synthetic combination of the entropy production rate, flow steadiness and Lagrange function (kinetic energy – potential energy). When the flow is stationary and there is no energy exchange at the boundary, the principle of minimum entropy production rate in the thermodynamics of irreversible processes is extended to the space integration of the flow field. The results for minimizing the dynamic drag of fluid flow are discussed.

Key Words: Viscous Flows, Fundamental Nature of Viscous Incompressible Flow, Minimization of Fluid Dynamic Drag

1. Introduction

Fluid motions are natural phenomena that have brought about many practical benefits as well as a better understanding of nature. Despite the long history of fluid science and technology, the study of fluid physics has not remained a classical field of dynamics, but has continued to receive attention as a new area of interest in nonlinear sciences.

A major subject in fluid sciences that remains unresolved even now is the principle that governs flow phenomena. Natural principles have been discovered in several other fields, such as the Lagrange functions, which express the difference between kinetic energy and potential energy, that work to realize extremes in analytical dynamics. However, up to now, a satisfactory answer has not been found in fluid flows. Because of the complexity of fluid phenomena and the non-linearity of the governing equations, the calculation of variations that has been used to reveal other natural principles, has not succeeded in this case. To the best of the author’s knowledge, a Lagrange function for the Navier-Stokes equations of a steady incompressible flow has been shown to be nonexistent by Serrin3) and Millikan,2) and detailed discussions have been reviewed by Serrin.3)

As an alternative approach, the thermodynamics of irreversible processes,2) one of the new sciences developed in the 20th century, have resulted in a new viewpoint that phenomena in a wide range of dissipative systems follow the principle of minimizing the rate of entropy production. This has been demonstrated to be appropriate in a number of cases including linear problems of flow instability.

The former is based on the fluid dynamics, that is, the first law of thermodynamics, while the latter is based on the second law.

In this paper, based on the above, the natural principle of flow is investigated in more general by integrating the first and second laws of thermodynamics. The results are extended to the design policy for the optimum body geometry of minimum fluid dynamic drag.

2. Equilibrium of Flow

Thermodynamics are the basis of natural science and applied to viscous flow dynamics introducing the characteristic features of the field. The total energy of an element including kinetic energy is considered to take into account the motion of fluid. Viscous dissipation is analyzed to determine entropy production. The flow field is generally unsteady and three-dimensional, and the substantial derivatives are used for analyses.

From the viewpoint of the subject, fundamental equations must be analyzed rigorously. The use of terminologies and understanding the results should be done carefully so as to avoid confusion due to the insufficient experience in the intermediate field of research.

From the equations that describe the motion of an incompressible viscous fluid (Navier-Stokes equations) and the equation of continuity in a Cartesian coordinate system, the change of total energy $p_0$ (kinetic energy $(1/2)\rho V^2 +$ static pressure $p$) per unit volume is expressed as:

$$\frac{Dp_0}{Dt} = \frac{\partial p}{\partial t} + \nabla^2 p_0 - \mu \omega^2. \quad (1)$$

Here, $V$ is the velocity vector, $D/Dt$ is the substantial derivative, and $t$ is time. $\rho$, $\mu$, and $\nabla$ are correspondingly the density, viscosity and kinematic viscosity of the fluid. $\nabla^2$ is the Laplace operator, and $\omega = \nabla \times V$ expresses the vorticity of the flow.

The total energy changes with the unsteadiness of the static pressure and spatial distribution of total energy, and decays with the square of vorticity. Equation (1) is the first
law of thermodynamics of viscous flow. When the flow is steady and inviscid, \( p_0 \) is constant along the stream line; that is, Bernoulli’s theorem is obtained. \( p \) is seen to work as the potential energy of the flow.

Thermodynamically, \( p = \text{enthalpy} \rho c_p T - \text{internal energy} \rho c_v T \) per unit volume, where, \( T, \rho c_p, \) and \( c_v \) are absolute temperature, specific heat at constant pressure, and specific heat at constant volume, respectively.\(^5\)

In the present case, \( p \gg (1/2) \rho V^2 \). The left-hand side of Eq. (1) shows the change (total enthalpy – internal energy) in the viscous flow element and the right-hand side terms are obtained using flow analysis under given conditions.

In the same way, the change in entropy \( S \) per unit volume, the second law of thermodynamics of viscous flow,\(^6\) is expressed as follows:

\[
\rho T \frac{DS}{Dt} = -2\nu \nabla^2 p + \mu \omega^2, \tag{2}
\]

where, \( T \) is a constant absolute temperature. The dissipation function is transformed as shown on the right-hand side of the equation. The entropy changes with the spatial distribution of static pressure and increases with the square of vorticity.

The effective energy of the flowing element, free energy, is obtained as \( G = p_0 - \rho T S \), and the following equation is obtained for \( G \) from Eq. (1) and Eq. (2).

\[
\frac{DG}{Dt} = -2\rho T \frac{DS}{Dt} + \frac{\partial p}{\partial t} + \nu \nabla^2 \left( \frac{1}{2} \rho V^2 - p \right). \tag{3}
\]

Thermodynamics show that the increase in free energy is always less than the external work, and the equilibrium condition of the system at constant pressure is obtained at minimum free energy. Equation (3) shows that equilibrium in a fluid element is achieved, in general, through synthetic minimization of the entropy production rate, unsteady pressure, and Lagrange function distribution. Not only the mechanical balance of the element, but also the environmental relations are necessary to achieve equilibrium, as a feature of fluid dynamics.

In order to investigate the total energy balance and equilibrium of the flow system, space integrals of those relations are discussed.

Assume there is a flow field around an arbitrary solid body. If stationary, that is, when the flow is steady or the fluctuation is small and the time mean of the flow is steady, the integration of Eq. (3) in a sufficiently large space \( V \), leads to the total free energy balance in the space, showing an open dissipative system.

The third term on the right-hand side of the equation becomes the surface integral of the perpendicular gradient at the boundary. The flow velocity is zero on the surface. As a normal pressure gradient is caused by the change in stream-wise skin friction, stream-wise integration from the upstream stagnation point to the separation point is zero. Therefore, energy transfer between the flow and body at the boundary does not occur, and integration ultimately expresses the relationship between free energy and entropy as

\[
\int_V \frac{DG}{Dt} dV = -2\rho T \int_V \frac{DS}{Dt} dV. \tag{4}
\]

Equation (4) does not show the so-called detailed balance between \( G \) and \( S \), but expresses the total balance within the space \( V \) for dissipative systems that may be turbulent, where the free energy changes with the generation of vorticity due to the existence of a solid body in the flow.

Bringing Eq. (4) closer to a static state where \( p_0 \) becomes \( p \), the well-known equilibrium condition of thermodynamics, that is, where the free energy becomes minimum with minimum entropy production, is attained. Equation (4) can be regarded as an extension of the rule to fluid systems with the addition of kinetic energy of the flow to the energy. In other words, it extends the theorem of minimum entropy production rate in non-equilibrium thermodynamics to a more general flow field.

The same space integration of Eq. (2) under the conditions for Eq. (4) shows that the minimum entropy production rate is achieved by the minimum volume integration of

\[
\rho T \int_V \frac{DS}{Dt} dV = \mu \int_V \omega^2 dV. \tag{5}
\]

In the same way, from Eq. (1), the total energy loss of the flow in the space is also seen to be the integration of

\[
\int_V \frac{Dp_0}{Dt} dV = -\mu \int_V \omega^2 dV. \tag{6}
\]

3. Minimization of Fluid Dynamic Drag

The total energy loss of the flow is equivalent to the work done between the flow and the body; that is, the drag of the body times the relative velocity. As can be seen from Eq. (4) to Eq. (6), minimum free energy production rate means minimizing drag. In other words, in the stationary state, the flow around any body is established so as to minimize the drag of the body.

This result means, thermodynamically, that a peculiar flow field is established for a respective equilibrium state. The flow field is obtained so as to minimize the volume integration of the right-hand side of Eq. (6). The calculation of variation\(^7\) in the three-dimensional velocity field shows that the corresponding flow satisfies the following equation.

\[
\nabla^2 V = 0. \tag{7}
\]

The fundamental equation of the viscous incompressible flow, Navier-Stokes equations and the equation of continuity, is expressed in terms of \( \omega \) as follows,

\[
\frac{\partial \omega}{\partial t} + V \cdot \nabla \omega = \omega \cdot \nabla V + \nu \nabla^2 \omega. \tag{8}
\]

The flow field is obtained by analyzing Eqs. (7) and (8) with boundary conditions.

As the total fluid dynamic drag is due to the surface integration of skin friction and static pressure distributions, the corresponding flow field minimizing the total drag is an elab-
orrate control system.

As another point of view, the following approach may be considered.

As the basic flow is steady and uniform, the vorticities in Eq. (8) are caused in the flow due to the existence of a body. The following relation is noticed

\[ \omega = KV \quad (k: \text{constant}) \]  

(9)

in view of satisfying Eqs. (7) and (8) simultaneously. Equation (9) gives spiral flows. Vorticities are formed with the axes aligned to local flow direction, or flows are induced along the axes of local vorticities. Longitudinal vortices have been widely noticed as a typical example of the organized motion of fluid.

For an axisymmetric body flying in the axial direction, reducing drag by inducing rotation around the axis has been known empirically. In American football games, the long touchdown pass from the quarterback to the receiver is a familiar example.

For general body geometries, organized motion in longitudinal vortices is observed in various flows, and the causes of generation in the boundary layer have been discussed. A row of counter-rotating longitudinal vortices, Taylor-Goertler vortices, is known to appear in dynamically unstable velocity distributions. This is to restrict skin friction by lifting the row from the wall in the downstream nonlinear region.

The behaviors of longitudinal vortices to reduce base drag have also been investigated. The near wake is an important flow field linking the upstream flow over the body with the wake where substantial energy dissipation takes place.

The approach based on Eq. (9) is to recognize the organized motion of longitudinal vortices for controlling the local flow field. The role of the motion in the whole flow field is unknown.

Further work in the field of fluid science is required in order to understand the increase in drag associated with unsteady flows.

4. Conclusions

1. The fundamental nature of viscous incompressible flow is discussed by integrating the first and second laws of thermodynamics. The change in flow free energy is shown to be due to the entropy production rate, flow steadiness and Lagrange function. The volume integration shows that, in the stationary flow without energy exchange at the boundary, the principle of minimum entropy production rate for the equilibrium discussed in non-equilibrium thermodynamics is extended to the entire flow field.

2. The results are applied to investigate the flow field of minimum fluid dynamic drag, and fundamental equations were derived.

3. The generation of longitudinal vortices was observed as local organized motion in relation to minimum entropy production.

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Yuichi Matsuo
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