A1-INVARINCE OF NON-STABLE $K_1$-FUNCTORS IN THE EQUICHA RACTERISTIC CASE

ANASTASIA STAVROVA

ABSTRACT. We apply the techniques developed by I. Panin for the proof of the equicharacteristic case of the Serre–Grothendieck conjecture for isotropic reductive groups (I. Panin, A. Stavrova, N. Vavilov, 2015; I. Panin, 2019) to obtain similar injectivity and A1-invariance theorems for non-stable $K_1$-functors associated to isotropic reductive groups. Namely, let $G$ be a reductive group over a commutative ring $R$. We say that $G$ has isotropic rank $\geq n$, if every non-trivial normal semisimple $R$-subgroup of $G$ contains $(G_{m,R})^n$. We show that if $G$ has isotropic rank $\geq 2$ and $R$ is a regular domain containing a field, then $K_1^G(R[x]) = K_1^G(R)$, where $K_1^G(R) = G(R)/E(R)$ is the corresponding non-stable $K_1$-functor, also called the Whitehead group of $G$. If $R$ is, moreover, local, then we show that $K_1^G(R) \to K_1^G(K)$ is injective, where $K$ is the field of fractions of $R$.

1. Introduction

Let $R$ be a commutative ring with 1. Let $G$ be a reductive group scheme over $R$ in the sense of [SGA3]. We say that $G$ has isotropic rank $\geq n$, if every non-trivial normal semisimple $R$-subgroup of $G$ contains $(G_{m,R})^n$.

If $G$ is not a torus, the assumption that $G$ has isotropic rank $\geq 1$ implies that $G$ contains a proper parabolic subgroup [SGA3 Exp. XXVI, Proposition 6.1]. For any reductive group $G$ over $R$ and a parabolic subgroup $P$ of $G$, one defines the elementary subgroup $E_P(R)$ of $G(R)$ as the subgroup generated by the $R$-points of the unipotent radicals of $P$ and of an opposite parabolic subgroup $P^-$, and considers the corresponding non-stable $K_1$-functor $K_1^{G,P}(R) = G(R)/E_P(R)$ [PeSt] [St14]. It does not depend on the choice of $P^-$ by [SGA3 Exp. XXVI Cor. 1.8]. In particular, if $A = k$ is a field and $P = (1)$ is minimal, $E(1)$ is nothing but the group $G(k)^+$ introduced by J. Tits [T], and $K_1^G(k)$ is the subject of the Kneser–Tits problem [G]. If $G = GL_n$ and $P$ is a Borel subgroup, then $K_1^G(R) = GL_n(R)/E_n(R)$, $n \geq 1$, are the usual non-stable $K_1$-functors of algebraic $K$-theory. If $G$ has isotropic rank $\geq 2$, then $K_1^{G,P}(R)$ is independent of $P$ by the main result of [PeSt], and we denote it by $K_1^G(R)$. If $G$ is a torus, we define $K_1^G(R) = G(R)$ for coherence. See §2 for a formal definition and further properties of $K_1^G$.

In [St14] we proved that if $G$ is a reductive group over a field $k$ having isotropic rank $\geq 2$, then

\[ K_1^G(k) = K_1^G(k[x_1, \ldots, x_n]) \quad \text{for any } n \geq 1. \]

This implied the following two statements. First, provided that $k$ is perfect, one has $K_1^G(A) = K_1^G(A[x])$ for any regular ring $A$ containing $k$. Second, provided that $k$ is infinite and perfect, one has $\ker(K_1^G(A) \to K_1^G(K)) = 1$ for any regular local ring $A$ containing $k$, where $K$ is the field of fractions of $A$. Those results generalized several earlier results on split, i.e. Chevalley–Demazure, reductive groups; see [St14] for a historical survey.

In the present text we show how the techniques developed by I. Panin for the proof of the equicharacteristic case of the Serre–Grothendieck conjecture [PeSt] [PeSt1] [Pt1] [Pt2] allow to extend these results to the case where $G$ is defined over a regular ring $A$ containing a field $k$, but not necessarily over $k$ itself, and $k$ is an arbitrary field. The main results are the following.

2010 Mathematics Subject Classification. 19B99, 20G35, 14L15, 20C99.

Key words and phrases. isotropic reductive group, non-stable $K_1$-functor, Whitehead group, Serre–Grothendieck conjecture.

The author is a winner of the contest “Young Russian Mathematics”. The work was supported by RFBR 19-01-00513.
Theorem 1.1. Let $A$ be a regular ring containing a field $k$. Let $G$ be a reductive group scheme over $A$ of isotropic rank $\geq 2$. Then there is a natural isomorphism $K^G_1(A) \cong K^G_1(A[\mathfrak{R}])$.

Theorem 1.2. Let $A$ be a semilocal regular domain containing a field $k$, and let $K$ be the field of fractions of $A$. Let $G$ be a reductive group scheme over $A$ of isotropic rank $\geq 2$. Then the natural homomorphism $K^G_1(A) \to K^G_1(K)$ is injective.

As a corollary of these two theorems, we also obtain some results on the values of non-stable $K_1$-functors on Laurent polynomials and power series rings; see Corollary 3.3 in §3 and §4. In the end of the paper we apply the general results of A. Asok, M. Hoyois and M. Wendt [AHW18] to give an interpretation of $K^G_1$ in terms of the $A^1$-homotopy theory of F. Morel and V. Voevodsky [MV]; see Theorem 5.6.

2. Properties of $K^G_1$ over general commutative rings

Let $R$ be a commutative ring with 1. Let $G$ be a reductive group scheme over $R$, and let $\pi: G \to G^{ad} = G/\text{Cent}(G)$ be the canonical homomorphism of $G$ onto its adjoint group. The correspondence $H \mapsto \pi(H)$ between semisimple normal subgroups of $G$ and $G^{ad}$ is bijective, with the inverse correspondence $H \mapsto \text{der}(\pi^{-1}(T \cdot H))$, where $T$ is any fixed maximal torus of $G^{ad}$. The bijectivity follows by faithfully flat descent from [SGA3, Exp. XXII, Lemme 5.1.5, Lemme 5.2.7, Corollaire 5.3.5]. In particular, $G$ has isotropic rank $\geq n$ if and only if $G^{ad}$ has isotropic rank $\geq n$.

For adjoint groups, semisimple normal subgroups can be described more explicitly. By [SGA3, Exp. XXIV, Proposition 5.10] $G^{ad}$ is isomorphic to a direct product of Weil restrictions $G_i = \prod_{R_i/R} H_i$, where $R_i/R$ is finite étale and $H_i$ is an adjoint simple group over $R_i$. We can assume without loss of generality that every $G_i$ cannot be decomposed further into a product of such factors. Then every non-trivial semisimple normal subgroup of $G^{ad}$ is a direct product of several factors of this decomposition. By adjunction of the Weil restriction and base change, the fact that $G_i$ contains $(G_{m,R})^n$ implies that $H_i$ contains $(G_{m,R_i})^n$. The converse is also true, since $(G_{m,R})^n$ is a subgroup of $\prod_{R'_i/R}(G_{m,R_i})^n$. Therefore, for any ring homomorphism $R \to S$, if $G$ has isotropic rank $\geq n$, $G_S$ also does, since it is a product of Weil restrictions of groups $(H_i)_S$ (cf. [BLR90, §7.6]).

Let $P$ be a parabolic subgroup of $G$ in the sense of [SGA3, Exp. XXVI Cor. 2.3]. There is a unique parabolic subgroup $P^-$ in $G$ which is opposite to $P$ with respect to $L_P$, that is $P^3 \cap P = L_P$, cf. [SGA3, Exp. XXVI Th. 4.3.2]. We denote by $U_P$ and $U_{P^-}$ the unipotent radicals of $P$ and $P^-$ respectively.

Definition 2.1. [PrSi] The elementary subgroup $E_P(R)$ corresponding to $P$ is the subgroup of $G(R)$ generated as an abstract group by $U_P(R)$ and $U_{P^-}(R)$. We denote by $K^{G,P}_1(R) = G(R)/E_P(R)$ the pointed set of cosets $gE_P(R)$, $g \in G(R)$.

Note that if $L_p'$ is another Levi subgroup of $P$, then $L_p'$ and $L_p$ are conjugate by an element $u \in U_P(R)$ [SGA3, Exp. XXVI Cor. 1.8], hence the group $E_P(R)$ and the set $K^{G,P}_1(R)$ do not depend on the choice of a Levi subgroup or an opposite subgroup $P^-$ (and so we do not include $P^-$ in the notation).

Definition 2.2. A parabolic subgroup $P$ in $G$ is called strictly proper, if it intersects every non-trivial normal semisimple subgroup of $G$ properly.

Similarly to the case of semisimple normal subgroups, the correspondence $P \mapsto \pi(P)$ between parabolic subgroups of $G$ and $G^{ad}$ is bijective (cf. [SGA3, Exp. XXVI, §3]). Consequently, $G$ has a strictly proper parabolic subgroup if and only if $G^{ad}$ does.

If $G$ has isotropic rank $\geq 1$, and $G$ is not a torus, then $G$ contains a strictly proper parabolic $R$-subgroup $P$. Indeed, by the above remarks, we can assume that $G = G^{ad}$ is an adjoint reductive group isomorphic to a direct product of Weil restrictions $G_i = \prod_{R_i/R} H_i$, where $R_i/R$ is finite étale and $H_i$ is an adjoint simple group over $R_i$. Let $S \leq G$ be the 1-dimensional split subtorus embedded diagonally into the product of 1-dimensional split subtori in $G_i$ that exist by assumption. By [SGA3, Exp. XXVI, Proposition 6.1] there is a parabolic subgroup $P$ of $G$ such that the centralizer $L = \text{Cent}_G(S)$ of $S$
in $G$ is a Levi subgroup of $P$. Since $S$ acts faithfully on every $G_i$, the subgroup $L$, and hence $P$, intersects properly every factor $G_i$, and hence every semisimple normal subgroup of $G$.

If $G$ has isotropic rank $\geq 2$, then for any strictly parabolic subgroup $P$, the functor $K^G_1$ is group-valued and independent of $P$, as evidenced by the following result.

**Theorem 2.3.** [PeSt, Lemma 12, Theorem 1] Let $G$ be a reductive group over a commutative ring $R$, and let $A$ be a commutative $R$-algebra. If for any maximal ideal $m$ of $R$ the isotropic rank of $G_{R_m}$ is $\geq 2$, then the subgroup $E_P(A)$ of $G(A)$ is the same for any strictly proper parabolic $A$-subgroup $P$ of $G_A$, and is normal in $G(A)$.

**Definition 2.4.** Let $G$ be a reductive group of isotropic rank $\geq 2$ over a commutative ring $R$. If $G$ is not a torus, then for any $R$-algebra $A$, we call the subgroup $E(A) = E_P(A)$, where $P$ is a strictly proper parabolic subgroup of $G$ over $R$, the elementary subgroup of $G(A)$. If $G$ is a torus, we set $E(A) = 1$. The functor $K^G_1$ on the category of commutative $R$-algebras $A$, given by $K^G_1(A) = G(A)/E(A)$, is called the non-stable $K_1$-functor associated to $G$.

We will use the following two properties of $K^G_1$ established in [St14, St15]. The following lemma was established in [Su, Corollary 5.7] for $G = \text{GL}_n$ (the proof goes through for any torus $G$ without any changes).

**Lemma 2.5.** [St15] Lemma 2.7] Let $A$ be a commutative ring, and let $G$ be a reductive group scheme over $A$ of isotropic rank $\geq 1$. Assume moreover that for every maximal ideal $m \subseteq A$, the reductive $A_m$-group $G_{A_m}$ has isotropic rank $\geq 2$. Then for any monic polynomial $f \in A[x]$ the natural homomorphism

$$K^G_1(A[x]) \to K^G_1(A[x]/f)$$

is injective.

The following statement was proved for $G = \text{GL}_n$, $n \geq 3$, by A. Suslin [Su, Th. 3.1]. For the case of split semisimple groups the same result was obtained by E. Abe [A] Th. 1.15.

**Lemma 2.6.** [PeSt, Lemma 17] Let $A$ be a commutative ring, and let $G$ be a reductive group scheme over $A$ of isotropic rank $\geq 1$. Assume moreover that for every maximal ideal $m \subseteq A$, the reductive $A_m$-group $G_{A_m}$ has isotropic rank $\geq 2$. Then for any $g(X) \in G(A[X])$ such that $g(0) \in E(A)$ and $F_m(g(X)) \in E(A_m[X])$ for all maximal ideals $m$ of $A$, one has $g(X) \in E(A[X])$.

**Proof.** If $G$ is not a torus, the claim is proved as in the proof of [PeSt, Lemma 17]. If $G$ is a torus, then the claim follows from the fact that $G$ is a sheaf for Zariski topology.

The following lemma is a straightforward extension of [V] Lemma 2.4 for $G = \text{GL}_n$ and [A] Lemma 3.7 for split reductive groups.

**Lemma 2.7.** Let $G$ be a reductive group of isotropic rank $\geq 2$ over a Noetherian commutative ring $B$. Let $\phi : B \to A$ be a homomorphism of commutative rings, and $h \subseteq B$ be such that $\phi : B/hB \to A/\phi(h)A$ is an isomorphism and the restriction of $h$ to every connected component of $B$ is non-nilpotent. Assume moreover that the commutative square

$$
\begin{array}{ccc}
\text{Spec } A_{\phi(h)} & \xrightarrow{F_{\phi(h)}} & \text{Spec } A \\
\downarrow & & \downarrow \\
\text{Spec } B_h & \xrightarrow{F_h} & \text{Spec } B
\end{array}
$$

is a distinguished Nisnevich square in the sense of [MV, Def. 3.1.3]. Then the sequence of pointed sets

$$K^G_1(B) \xrightarrow{(F_h, \phi)} K^G_1(B_h) \times K^G_1(A) \xrightarrow{(g_{\phi(h)}, \phi(g_{\phi(h)}))F_1^{-1}} K^G_1(A_{\phi(h)})$$

is exact.

**Proof.** If $G$ is a torus, then $K^G_1(-) = G(-)$ by definition, and the claim follows immediately from the fact that $G$ is a sheaf for the fpqc topology [FGA05, Theorem 2.55]. If $G$ is not a torus, we use [St14, Corollary 3.5]. In order to use this result, we only need to check that $G$ has a strictly
proper parabolic subgroup $P$ such that the system of relative roots $\Phi_P$ in the sense of [St14] is has trivial kernel. Since $B$ is Noetherian, $h$ is non-nilpotent on every connected component of $B$, and $K^G_1$ commutes with finite direct products of rings, we can check the claim of the lemma individually for every connected component of $B$. In other words, we can assume that $B$ is connected. Let $G^a = G/\text{Cent}(G)$ be the adjoint group corresponding to $G$. The construction of $\Phi_P$ in [St14] Lemma 3.6 and Definition 3.7 implies that $\Phi_P$ the same for $P$ and $P/\text{Cent}(G)$. Consequently, we can assume that $G = G^a$ is an adjoint reductive group. Then $G$ is isomorphic to a direct product of Weil restrictions $G_i = \prod_{B_i/B} H_i$, where $B_i/B$ is finite étale and $H_i$ is an adjoint simple group over $B_i$, and the factors $G_i$ cannot be decomposed further. Let $S \leq G$ be the 2-dimensional split subtorus embedded diagonally into the product of 2-dimensional split subtori in $G_i$ that exist by assumption. By [SGA3] Exp. XXVI, Proposition 6.1 there is a parabolic $B$-subgroup $P$ of $G$ such that the centralizer $L = \text{Cent}_G(S)$ of $S$ in $G$ is a Levi subgroup of $P$. Since $S$ is contained in the center of $L$, and acts faithfully on every $G_i$, by [St15] Lemma 3.6 the system of relative roots $\Phi_P$ has rank $\geq 2$ over $B$.

### 3. Proof of the main results

The following theorem proved in [P19] extends [PSIV] Theorem 7.1 to arbitrary reductive groups $G$ and arbitrary fields $k$.

**Theorem 3.1.** [P19] Theorem 2.5]

(i) Let $X$ be an affine $k$-smooth irreducible $k$-variety, let $U = \text{Spec}(O_X(x_1, x_2, \ldots, x_n))$ be the spectrum of the semilocal ring of $X$ at a finite set of closed points $x_1, x_2, \ldots, x_n$, $n \geq 1$, and let $0 \neq f \in O_X(X)$ be such that $x_1, \ldots, x_n \notin X_f$. Then, possibly after replacing $X$ by a smaller affine open neighborhood of $U$, one can find a $U$-scheme $Y$ which is finite étale neighborhood

![Diagram](image)

of the $U$-point $0 \in A^1_U$, and a morphism $p : Y \to X$, such that $Y \xrightarrow{\tau} A^1_U$ together with a principal open $(A^1_U)_h \to A^1_U$, where $h \in O_X(x_1, x_2, \ldots, x_n)[t]$ is a monic polynomial with $h(1) \in (O_X(x_1, x_2, \ldots, x_n))^\times$, form an elementary distinguished Nisnevich square of $U$-schemes in the sense of [MV] Def. 3.1.3], and the following diagram commutes.

(1)

![Diagram](image)

(ii) Let $G$ be a reductive group scheme over $X$, and denote by $G_U$ its restriction to $U$. Then one can choose $Y$ in (i) in such a way that, moreover, the base change $p^*(G)$ of $G$ to $Y$ is $Y$-isomorphic to the restriction $\tau^*(H)$ to $Y$ of the “constant” group scheme $H = G_U \times_U A^1_U$ over $A^1_U$.

**Theorem 3.2.** Let $k$ be a field, let $A$ be a semilocal ring of several closed points on a smooth irreducible $k$-variety, and let $K$ be the field of fractions of $A$. Let $G$ be a reductive group over $A$ such that every semisimple normal subgroup of $G$ contains $(G_m)^2$. Then for any commutative $k$-algebra $B$, the natural map

$$K^G_1(A \otimes_k B) \to K^G_1(K \otimes_k B)$$

has trivial kernel.
Proof. Let \( g \in \ker(K^G_1(A) \otimes_k B) \to K^G_1(K \otimes_k B) \). By \cite[Tag 01ZU]{stacks-project} there are a smooth irreducible affine \( k \)-variety \( X = \text{Spec}(C) \) and \( f \in C \) such that \( A \) is a semilocal ring of several closed points on \( X \). Since \( G \) is finitely presented, and \( K^G_1 \) commutes with filtered direct limits of rings, we can assume that \( B \) is a finitely generated \( k \)-algebra, \( G \) is defined and has isotropic rank \( \geq 2 \) over \( C \), the element \( g \in K^G_1(A) \otimes_k B \) is the image of an element \( g_0 \in \ker(K^G_1(C) \otimes_k B) \to K^G_1(C) \otimes_k B) \). Apply Theorem \ref{thm:main} (i). Replace all schemes in the diagram \ref{diagram} by their fiber products with \( \text{Spec} \) over \( \text{Spec} k \). This gives an element
\[
p^*(g_0) \in \ker(K^G_1(O_Y(Y) \otimes_k B) \to K^G_1(O_Y(Y)_{\tau(h)} \otimes_k B) ,
\]
where we write \( G \) instead of \( p^*(G) \) thanks to Theorem \ref{thm:main} (ii). Note that the polynomial \( h \) is nilpotent on every connected component of the Noetherian ring \( A[x] \otimes_k B = (A \otimes_k B)[x] \). By Lemma \ref{lem:embedding} there is an element
\[
\tilde{g} \in \ker(K^G_1(A[x] \otimes_k B) \to K^G_1(A[x] \otimes_k B) ,
\]
such that \( \tau^*(\tilde{g}) = p^*(g_0) \). Since \( h \) is a monic polynomial, by Lemma \ref{lem:monic} we have \( \tilde{g} = 1 \). By the commutativity of the diagram \ref{diagram} we have \( \tilde{g}|_{x = 0} = \delta(p^*(g_0)) = g \). Hence \( g \geq 1 \).

Proof of Theorem \ref{thm:main}. Clearly, we can assume that \( k \) is a finite field or \( \mathbb{Q} \) without loss of generality. The embedding \( k \to A \) is geometrically regular, since \( k \) is perfect \cite[28.M]{ma} \cite[28.N]{swasser}. Then by Popescu’s theorem \cite{popescu} \cite{swasser} \( A \) is a filtered direct limit of smooth \( k \)-algebras \( R \). Since the group scheme \( G \) and the unipotent radicals of its parabolic subgroups are finitely presented over \( A \), the functors \( G(-) \) and \( E(-) = E_p(-) \) commute with filtered direct limits. Hence we can replace \( A \) by a smooth \( k \)-algebra \( R \). By the local-global principle Lemma \ref{lem:local-global} to show that \( K^G_1(R) = K^G_1(R[x]) \), it is enough to show that for every maximal localization \( R_m \) of \( R \). Let \( K \) be the field of fractions of \( R_m \). By Theorem \ref{thm:structure} the maps \( K^G_1(R_m) \to K^G_1(K) \) and \( K^G_1(R_m[x]) \to K^G_1(K[x]) \) are injective. By \cite[Theorem 1.2]{saito} \( K^G_1(K) = K^G_1(K[x]) \). Hence \( K^G_1(R_m) = K^G_1(R_m[x]) \), and we are done.

Proof of Theorem \ref{thm:simple}. As in the proof of Theorem \ref{thm:main} we are reduced to the case where \( A \) is a filtered direct limit of smooth \( k \)-algebras \( R \), and \( G \) is defined and has isotropic rank \( \geq 2 \) over \( R \). Also, since \( A \) is a semilocal domain, we can assume that \( R \) is a domain with a fraction field \( L \), and an element \( g \in \ker(K^G_1(A) \to K^G_1(K) \)) comes from an element \( g' \in K^G_1(R) \) that vanishes in \( K^G_1(L) \). By Theorem \ref{thm:structure} \( g' \) vanishes in every semilocalization of \( R \) at a finite set of maximal ideals, and hence in every semilocalization of \( R \) at a finite set of prime ideals. Since the map \( K^G_1(R) \to K^G_1(A) \) factors through such a semilocalization of \( R \), it follows that \( g = 1 \).

Corollary 3.3. Let \( A \) be a semilocal regular domain containing a field \( k \). Let \( G \) be a simply connected semisimple group scheme over \( A \) of isotropic rank \( \geq 2 \). Then \( K^G_1(A) = K^G_1(A[x_{1 \pm 1}, \ldots, x_{n \pm 1}]) \).

Proof. Recall that for any field \( K \) and any simply connected semisimple \( G \) of isotropic rank \( \geq 2 \) over \( K \) one has \( K^G_1(K[x_{1 \pm 1}, \ldots, x_{n \pm 1}]) \cong K^G_1(K) \) by \cite{st14}. Let \( K \) be the field of fractions of \( A \). Then the claim follows from Theorem \ref{thm:structure} exactly as Theorem \ref{thm:simple} via the following diagram:
\[
\begin{array}{ccc}
K^G_1(A[x_{1 \pm 1}, \ldots, x_{n \pm 1}]) & \xrightarrow{x_1 = \ldots = x_n = 1} & K^G_1(A) \\
\downarrow & & \downarrow \\
K^G_1(K[x_{1 \pm 1}, \ldots, x_{n \pm 1}]) & \xrightarrow{x_1 = \ldots = x_n = 1} & K^G_1(K).
\end{array}
\]

Remark 3.4. It is clear that Theorem \ref{thm:main} can be applied to deduce analogs of Theorems \ref{thm:main} and \ref{thm:simple} for any reasonably good functor defined in terms of a reductive group scheme and satisfying properties similar to Lemmas \ref{lem:embedding} \ref{lem:local-global} \ref{lem:monic}. The paradigm example is the funtor \( H^1_1(-, G) \), in which case Theorem \ref{thm:simple} corresponds to the Serre–Grothendieck conjecture (the non-isotropic cases involve additional modification of the counterpart of Lemma \ref{lem:embedding}). One can axiomatize this approach similarly to the “constant” case \cite[Théorème 1.1]{cto}, however, the axioms are, naturally, more cumbersome.
We will need the following analog of a well-known theorem of Quillen on projective modules [Q Theorem 3].

**Theorem 4.1.** [PSIV Theorem 1.3] Let $B$ be a commutative ring, and let $G$ be a simply connected semisimple group over $B$ of isotropic rank $\geq 1$. Let $f \in B[x]$ be a monic polynomial. Then the natural map of étale cohomology sets $H^1_{\acute{e}t}(B[x], G) \rightarrow H^1_{\acute{e}t}(B[x]_f, G)$ has trivial kernel.

**Proof.** Assume first that $B$ is a local ring. Since $G$ is simply connected, similarly to the adjoint case, by [SGA3 Exp. XXIV, Proposition 5.10] implies that $G$ is isomorphic to a direct product of Weil restrictions $G_i = \prod_{B'_i/R} H_i$, where $R'_i/R$ is finite étale and $H_i$ is a simply connected group over $B_i$, i.e. the Dynkin diagram of $H_i$ over every geometric point of $\text{Spec}(B_i)$ is irreducible. Again as in the adjoint case (see §2), the assumption that $G$ has isotropic rank $\geq 1$ implies that $H_i$ contains $G_{m,B_i}$. Then $H^1_{\acute{e}t}(B[x], G) \rightarrow H^1_{\acute{e}t}(B[x]_f, H_i)$ has trivial kernel by [Ce20 Proposition 5.2.2 (i)]. Then the same claim for $G$ follows from a version of Faddeev–Shapiro’s lemma [PSIV Exp. XXIV, Proposition 8.4].

Now assume that $B$ is arbitrary. Since $G$ is finitely presented, and $H^1_{\acute{e}t}(-, G)$ commutes with filtered direct limits [Mar07], we can assume that $B$ is Noetherian. Since $G$ is semisimple, it is $B$-linear by [ThS7 Corollary 3.2]. Then Lemma [4.2] below finishes the proof. $\square$

**Lemma 4.2.** Let $B$ be a Noetherian ring, and let $G$ be a $B$-linear flat group scheme. Let $f \in B[x]$ be a monic polynomial. Assume that for every maximal ideal $m$ of $B$ the natural map of étale cohomology sets $H^1_{\acute{e}t}(B_m[x], G) \rightarrow H^1_{\acute{e}t}(B_m[x]_f, G)$ and $H^1_{\acute{e}t}(B_m[x], G) \rightarrow H^1_{\acute{e}t}(B_m[x]_x, G)$ have trivial kernels. Then the map $H^1_{\acute{e}t}(B[x], G) \rightarrow H^1_{\acute{e}t}(B[x]_f, G)$ has trivial kernel.

**Proof.** Let $\xi \in H^1_{\acute{e}t}(B[x], G)$ be in the kernel. Set $y = x^{-1}$ and choose $g(y) \in B[y]$ so that $x^{\deg(f)}g(y) = f(x)$. Then $g(0) = 1 \in B^\times$, and $B[x]_f = B[y]_{|g}$. Hence $\text{Spec}(B[y]) = \text{Spec}(B[y]_{|g}) \cup \text{Spec}(B[y]_{|g})$. Extend $\xi|_{\text{Spec}(B[y]_{|g})} = \xi|_{\text{Spec}(B[y]_{|g})}$ to a bundle $\eta$ on $\text{Spec}(B[y])$ by gluing it to the trivial bundle on $\text{Spec}(B[y]_{|g})$ (this can be done e.g. by [CTO Proposition 2.6 (iv)]). By assumption, for any maximal ideal $m$ of $B$ the $G$-bundle $\xi|_{B_m[x]}$ is trivial, hence $\eta|_{\text{Spec}(B_m[y]_{|g})}$ is trivial. Then, again by assumption, $\eta|_{B_m[y]}$ is trivial. Since $G$ is $B$-linear, by [AHW18 Theorem 3.2.5] the fact that for any maximal ideal $m$ of $B$ the $G$-bundle $\eta|_{B_m[y]}$ is trivial implies that $\eta$ is extended from $B$. However, $\eta$ is trivial at $y = 0$ by construction, so $\eta$ is trivial. Hence $\xi$ is trivial at $x = y = 1$. Since $\xi|_{B_m[x]}$ is also trivial for any $m$, by [AHW18 Theorem 3.2.5] $\xi$ is also extended from $B$. Hence $\xi$ is trivial. $\square$

**Remark 4.3.** Theorem [4.4] was first established in [PSIV Theorem 1.3] with the following additional assumptions. First, $B$ was assumed to be a Noetherian $k$-algebra, where $k$ was an arbitrary field. Second, $G$ was assumed to be simple (that is, the root system of $G$ over any geometric point of $\text{Spec}(B)$ is irreducible). Third, it was assumed that $f(1) \in B^\times$. The assumptions that $B$ is Noetherian and $G$ is simple are unsubstantial, as explained in our proof of Theorem [4.4]. The fact that $B$ is a $k$-algebra was used in the proof of [PSIV Theorem 1.3] at one point only. Namely, in [PSIV Lemma A.1], we referred to the classical results of W. J. Haboush [H] and M. Nagata [N] that a quotient of a split reductive group over a field $k$ by a reductive subgroup is representable by an affine scheme. It was observed in [Ce20] that this result was generalized to arbitrary reductive group schemes over Noetherian rings by C. S. Seshadri [S77] and J. Alper [Al14 9.4.1 and 9.7.5]. The condition $f(1) \in B^\times$ was used in the proof of [PSIV Theorem 1.3] in order to deduce the case of a not necessarily local ring $B$ from the local ring case, the latter being established without this assumption. This argument is replaced by the new Lemma [4.2].

**Corollary 4.4.** Let $B$ be a commutative ring, and let $G$ be a simply connected semisimple group over $B$ of isotropic rank $\geq 1$. Then $G(B((x))) = G(B[[x]])G(B[[x^{-1}]]).$ If, moreover, $G$ has isotropic rank $\geq 2$, then the sequence of pointed sets

$$1 \rightarrow K^G_1(B[x]) \xrightarrow{g \mapsto (g, 0)} K^G_1(B[[x]]) \times K^G_1(B[x, x^{-1}]) \xrightarrow{(g_1, g_2) \mapsto g_1g_2^{-1}} K^G_1(B((x))) \rightarrow 1$$

is exact.
Proof. Take an element $g \in G(B((x)))$. The schemes $\text{Spec}(B[x]), \text{Spec}(B((x))), \text{Spec}(B[[x]])$ and $\text{Spec}(B[x^{\pm 1}])$ together form a patching square for $G$-torsors [BC Lemma 2.2.11]. Then one can patch $G_B[[x]]$ and $G_{B[[x]]}$ by means of a shift by $g$ to obtain a $G$-torsor over $B[x]$. Since $G$ is smooth, this torsor is étale-locally trivial [Groth. 11.7]. By Theorem 4.11 this torsor is trivial, since its restriction to $B[x^{\pm 1}]$ is trivial. Therefore, $g \in G(B[[x]])G(B[x^{\pm 1}])$.

The second claim follows from the first and [SL ε] Corollary 3.2] \[ \]

The following result, in particular, generalizes a theorem of Ph. Gille [G] Théorème 5.8] that $K^G_1(k) = K^G_1(k((x)))$ for any isotropic simply connected semisimple group $G$ over a field $k$.

**Theorem 4.5.** Let $B$ be a semilocal regular ring containing a field, and let $G$ be a simply connected semisimple group over $B$ of isotropic rank $\geq 2$. Then

$$K^G_1(B) = K^G_1(B((x)))) = K^G_1(B[[x]]) = K^G_1(B[x]) = K^G_1(B[x^{\pm 1}]).$$

**Proof.** Since $K^G_1$ commutes with finite products, we can assume that $B$ is a domain. It follows from Theorem 1.1 and Corollary 3.3 that $K^G_1(B) = K^G_1(B[x]) = K^G_1(B[x^{\pm 1}])$.

Let $K$ be the field of fractions of $B$. Since $B[[x]]$ and $K[[x]]$ are semilocal regular rings containing a field, by Theorem 1.12 the maps $K^G_1(B[[x]]) \rightarrow K^G_1(K((x)))$ and $K^G_1(K[[x]]) \rightarrow K^G_1(K((x)))$ are injective. Consequently, also the map $K^G_1(B[[x]]) \rightarrow K^G_1(K[[x]])$ is injective. By [G] Lemme 4.5, p. 983–15 one has

$$\ker(G(K[[x]]) \xrightarrow{x=0} G(K)) \subseteq E(K((x))),$$

and hence $K^G_1(K[[x]]) = K^G_1(K)$. It follows that $K^G_1(B[[x]]) \xrightarrow{x=0} K^G_1(B)$ is injective, and hence $K^G_1(B[[x]]) = K^G_1(B)$. Finally, since $K^G_1(B) \rightarrow K^G_1(B[[x]]) \rightarrow K^G_1(K((x)))$ is injective, it follows that $K^G_1(B) \rightarrow K^G_1(B((x)))$ is injective.

By Corollary 4.4, we have $G(B((x))) = G(B[x^{\pm 1}])G(B[[x]])$, therefore, $K^G_1(B[[x]]) = K^G_1(B)$ implies that the map $K^G_1(B) \rightarrow K^G_1(B((x)))$ is surjective. \[ \]

5. $\mathbb{A}^1$-HOMOGENEOUS INTERPRETATION OF $K^G_1$

Let $A$ be any commutative ring, $Sm_A$ be the category of finitely presented smooth $A$-schemes, and $Sm_A^{\text{aff}}$ be the full subcategory of affine schemes.

For any presheaf $F$ on $Sm_A$, we denote by $\text{Sing}_{\mathbb{A}^1}(F)$ the simplicial presheaf $U \mapsto F(\Delta^* \times U)$, where $\Delta^*$ is the standard cosimplicial object made of affine spaces (see e.g. [MV] p. 88).

**Definition 5.1.** Let $G$ be a group-valued presheaf on $Sm_A$. For any $n \geq 1$, define the $n$-th Karoubi–Villamayor $K$-theory functor associated to $G$ to be the presheaf on $Sm_A$ given by

$$KV_n^G(U) = \pi_{n-1}(\text{Sing}_{\mathbb{A}^1}(G)(U)).$$

This definition goes back to J. F. Jardine [J] who defined the non-stable Karoubi–Villamayor $K$-theory associated to a group valued functor on a category of $k$-algebras, where $k$ is any commutative unitary ring, similarly to Gersten’s definition of the usual Karoubi–Villamayor $K$-theory. For any commutative $A$-algebra $R$, one can explicitly compute $KV_n^G(R)$ as follows:

$$KV_n^G(R) = G(R)/\{g \in G(R) \mid \exists h(x) \in G(R[x]) : h(0) = 1, h(1) = g\}.$$

Let $G$ be a reductive group scheme over $A$, and let $P$ be a proper parabolic subgroup of $G$, and let $U_P$ be the unipotent radical of $P$. By SGA3 Exp. XXVI Corollaire 2.5 $U_P$ is $A$-schematically isomorphic to the canonical scheme of a projective $A$-module $V$, with the point $0 \in V$ corresponding to $1 \in U_P(A)$. Consequently, every $g \in U_P(R)$ corresponds to an element $v \in V \otimes_A R$, and then $vx \in V \otimes_A R[x]$ corresponds to $h(x) \in U_P(R[x])$ such that $h(0) = 1, h(x) = g$. This implies that there is a natural map of pointed sets

$$K_{1,P}^G(R) = G(R)/E_P(R) \rightarrow KV_1^G(R).$$

If $G$ has isotropic rank $\geq 2$, this map becomes a group homomorphism

$$K_1^G(R) \rightarrow KV_1^G(R).$$
A. Asok, M. Hoyois and M. Wendt proved the following “affine representability” result for $KV^G_1$ in the Morel–Voevodsky $\mathbb{A}^1$-homotopy category $\mathcal{H}^A_{\mathbb{A}^1}$ over $A$, which is the homotopy category of the category of simplicial sheaves in the Nisnevich topology on $Sm_A$ with respect to the $\mathbb{A}^1$-model structure [MV p. 109]. Note that the definition of $\mathcal{H}^A_{\mathbb{A}^1}$ used in [AHW18] and its prequel [AHW17] is different from the definition of [MV], however, the two categories are equivalent under the assumption that $A$ is Noetherian and has finite Krull dimension [AHW17, Remarks 3.1.4 and 5.1.1].

**Theorem 5.2.** [AHW18 Theorem 2.4.2] Let $A$ be a Noetherian scheme of finite Krull dimension. Let $G$ be finitely presented smooth $A$-group scheme such that the natural map $H^1_{\text{Nis}}(X, G) \to H^1_{\text{Nis}}(\mathbb{A}^1_X, G)$ is bijective for all $X \in Sm^\text{aff}_A$. Then for all $X \in Sm^\text{aff}_A$ and $n \geq 0$ the canonical map $KV^G_1(X) \to \text{Hom}_{H^A_{\mathbb{A}^1}}(X, G)$ is bijective.

If $A$ is a regular ring containing a field, then the condition on Nisnevich cohomology in Theorem 5.2 is satisfied for reductive $A$-groups of isotropic rank $\geq 1$. This follows from the Serre–Grothendieck conjecture [P20] and the extended version of the Serre–Grothendieck conjecture for simply connected isotropic groups [P19] Theorem 2.6). The next three statements provide details of this implication.

**Lemma 5.3.** Let $A$ be a regular semilocal domain, let $K$ be the fraction field of $A$, and let $G$ be a reductive group scheme over $A$. Let $\text{der}(G)$ be the derived group of $G$ in the sense of [SGA3], and let $G^{\text{sc}}$ be the simply connected semisimple group corresponding to $\text{der}(G)$. Assume that the three maps

$$H^1_{\text{et}}(A, G) \to H^1_{\text{et}}(K, G),$$

$$H^1_{\text{et}}(A, \text{der}(G)) \to H^1_{\text{et}}(K, \text{der}(G)),$$

$$H^1_{\text{et}}(A[x_1, \ldots, x_n], G^{\text{sc}}) \to H^1_{\text{et}}(K[x_1, \ldots, x_n], G^{\text{sc}})$$

have trivial kernels. Then the map

$$H^1_{\text{et}}(A[x_1, \ldots, x_n], G) \to H^1_{\text{et}}(K[x_1, \ldots, x_n], G)$$

has trivial kernel.

**Proof.** There are two short exact sequences of reductive $A$-groups

$$1 \to \text{der}(G) \to G \to \text{corad}(G) \to 1$$

and

$$1 \to C \to G^{\text{sc}} \to \text{der}(G) \to 1,$$

where $\text{corad}(G)$ and $C$ are $A$-groups of multiplicative type [SGA3 Exp. XXII]. By [S19 Lemma 4.1] the second sequence and the fact that (5) and (6) have trivial kernels imply that

$$H^1_{\text{et}}(A[x_1, \ldots, x_n], \text{der}(G)) \to H^1_{\text{et}}(K[x_1, \ldots, x_n], \text{der}(G))$$

has trivial kernel. Also by [S19 Lemma 4.1] the first sequence and the fact that (2) and (6) have trivial kernels imply that (5) has trivial kernel. □

**Lemma 5.4.** Let $A$ be a regular semilocal domain containing a field $k$, let $K$ be the fraction field of $A$, and let $G$ be a reductive group scheme over $A$ of isotropic rank $\geq 1$. Then the natural map $H^1_{\text{et}}(A[x], G) \to H^1_{\text{et}}(K[x], G)$ has trivial kernel.

**Proof.** By Lemma 5.3 it is enough to check that (2), (5), (6) have trivial kernels. By the Serre–Grothendieck conjecture over $A$ [P20 Theorem 1.1] the maps (2) and (3) have trivial kernels. Since $G^{\text{sc}}$ has isotropic rank $\geq 1$, by [P19 Theorem 2.6] the map (4) has trivial kernel, under the additional assumption that $A$ is a semilocal ring of several closed points on a $k$-smooth irreducible affine variety $X$. Using Popescu’s theorem as in the proof of Theorem 1.1, and the fact that $H^1_{\text{et}}(-, G)$ commutes with filtered direct limits [Mar07], we conclude that (4) has trivial kernel for any regular semilocal domain $A$ containing a field. □

**Corollary 5.5.** Let $A$ be a regular ring containing a field $k$, and let $G$ be a reductive group scheme over $A$ of isotropic rank $\geq 1$. Then the natural map $H^1_{\text{Nis}}(A, G) \to H^1_{\text{Nis}}(A[x], G)$ is bijective.
Proof. It is enough to show that any Nisnevich $G$-torsor over $A[x]$ is extended from $A$. By [Th87, Corollary 3.2] $G$ is linear, hence by the local-global principle for torsors [AHW18, Theorem 3.2.5] (see also [Ko, Korrall 3.5.2]) it is enough to prove that every $G$-torsor over $A_A[x]$ is extended from $A_m$, for every maximal localization $A_m$ of $A$. Thus, we can assume from the start that $A$ is regular local.

Next, for every regular local ring $A$ containing a field, we show that every Nisnevich $G$-torsor over $A[x]$ is in fact trivial, and hence extended. By Lemma 5.4 $H^1_{Nis}(A[x], G) \rightarrow H^1_{G}(K[x], G)$ has trivial kernel, where $K$ is the fraction field of $A$. By [CTO, Proposition 2.2] $H^1_{G}(K[x], G) \rightarrow H^1_{G}(K(x), G)$ has trivial kernel. Since $H^1_{Nis}(K(x), G) = 1$, every Nisnevich $G$-torsor over $A[x]$ is trivial.

Theorem 5.6. Let $A$ be a regular ring of finite Krull dimension and containing a field $k$. Let $G$ be a reductive group scheme over $A$ of isotropic rank $\geq 2$. Then the canonical map $K^G(A) \rightarrow \text{Hom}_{H^1_A}(\text{Spec}(A), G)$ is bijective.

Proof. The map $K^G(A) \rightarrow KV^G(A)$ is bijective by [St14, Lemma 3.3], given that $K^G(A[x]) \cong K^G(A)$ by Theorem 5.1. The map $KV^G(A) \rightarrow \text{Hom}_{H^1_A}(\text{Spec}(A), G)$ is bijective by Theorem 5.2 and Corollary 5.5.

References

[A] E. Abe, Whitehead groups of Chevalley groups over polynomial rings, Comm. Algebra 11 (1983), 1271–1307.

[A14] J. Alper, Adequate moduli spaces and geometrically reductive group schemes, Algebr. Geom. 1 (2014), no. 4, 489–531.

[AHW17] A. Asok, M. Hoyois, and M. Wendt, Affine representability results in $\mathbb{A}^1$-homotopy theory, I: vector bundles, Duke Math. J. 166 (2017), no. 10, 1923–1953.

[AHW18] A. Asok, M. Hoyois, and M. Wendt, Affine representability results in $\mathbb{A}^1$-homotopy theory, II: Principal bundles and homogeneous spaces, Geom. Topol. 22 (2018), no. 2, 1181–1225.

[BLR90] S. Bosch, W. Lütkebohmert, and M. Raynaud, Néron models, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 21, Springer-Verlag, Berlin, 1990.

[BC] A. Bouthier, K. Česnavičius, Torsors on loop groups and the Hitchin fibration, to appear in Ann. Sci. ÉNS.

[Cv20] K. Česnavičius, Grothendieck-Serre in the split unramified case, arXiv preprint arXiv:2009.05299 (2020).

[CTO] J.-L. Colliot-Thélène, M. Ojanguren, Espaces Principaux Homogènes Localement Triviaux, Publ. Math. IHÉS 75, no. 2 (1992), 97–122.

[SAG3] M. Demazure, A. Grothendieck, Schémas en groupes, Lecture Notes in Mathematics, vol. 151–153, Springer-Verlag, Berlin-Heidelberg-New York, 1970.

[FAG05] B. Fantechi, L. Göttsche, I. Illusie, N. Nitsure, and A. Vistoli, Fundamental algebraic geometry, Grothendieck’s FGA explained, Mathematical Surveys and Monographs, vol. 123, American Mathematical Society, Providence, RI, 2005.

[H] W.J. Haboush, Reductive groups are geometrically reductive, Ann. Math. 102 (1975), no. 1, 67–83.

[G] Ph. Gille, Le problème de Kneser-Tits, Sém. Bourbaki 983 (2007), 983-01–983-39.

[Gro68b] A. Grothendieck, Le groupe de Brauer. III. Exemples et compléments, Dix exposés sur la cohomologie des schémas, Adv. Stud. Pure Math., vol. 3, North-Holland, Amsterdam, 1968, pp. 88–188.

[J] J.-F. Jardine, On the homotopy groups of algebraic groups, J. Algebra 81 (1983), 180–201.

[Mar07] B. Margaux, Passage to the limit in non-abelian Čech cohomology, J. Lie Theory 17 (2007), no. 3, 591–596.

[Ma] H. Matsumura, Commutative algebra, second ed., Math. Lect. Note Series 56, Benjamin/Cummings Publishing Co., Inc., Reading, Massachusetts, 1980.

[MV] F. Morel, V. Voevodsky, $\mathbb{A}^1$-homotopy theory of schemes, Publ. Math. I.H.É.S. 90 (1999), 45–143.

[Mo] L.-F. Moser, Rational triviale Torsoren und die Serre-Grothendiecksche Vermutung, Diplomarbeit, 2008, http://www.mathematik.uni-muenchen.de/~lfmoser/da.pdf

[N] M. Nagata, Invariants of a group in an affine ring, J. Math. Kyoto Univ. 3 (1964), no. 3, 369–377.

[PSV] I. Panin, A. Stavrova, and N. Vavilov, On Grothendieck-Serre’s conjecture concerning principal $G$-bundles over reductive group schemes: I, Compos. Math. 151 (2015), no. 3, 535–567.

[P18] I. Panin, On Grothendieck-Serre’s conjecture concerning principal bundles, Proc. Int. Cong. of Math. 2018, Rio de Janeiro, vol. 1, 201–222.

[P19] I. Panin, Nice triples and the Grothendieck-Serre conjecture concerning principal $G$-bundles over reductive group schemes, Duke Math. J. 168 (2019), no. 2, 351–375.

[P20] I. A. Panin, Proof of the Grothendieck-Serre conjecture on principal bundles over regular local rings containing a field, Izv. Ross. Akad. Nauk Ser. Mat. 84 (2020), no. 4, 169–186.

[PeSt] V. Petrov, A. Stavrova, Elementary subgroups of isotropic reductive groups, St. Petersburg Math. J. 20 (2009), 625–644.

[Po] D. Popescu, Letter to the Editor: General Néron desingularization and approximation, Nagoya Math. J. 118 (1990), 45–53.

[Q] D. Quillen, Projective modules over polynomial rings, Invent. Math. 36 (1976), 167–171.
A¹-invariance of non-stable $K_1$-functors in the equicharacteristic case

C. S. Seshadri, Geometric reductivity over arbitrary base, Advances in Math. 26 (1977), no. 3, 225–274.

[Stacks] The Stacks project, https://stacks.math.columbia.edu

A. Stavrova, Homotopy invariance of non-stable $K_1$-functors, J. K-Theory 13 (2014), 199–248.

A. Stavrova, Non-stable $K_1$-functors of multiloop groups, Canad. J. Math. 68 (2016), 150–178.

A. Stavrova, Isotropic reductive groups over discrete Hodge algebras, J. Homotopy Relat. Str. 14 (2019), 509–524.

A. A. Suslin, On the structure of the special linear group over polynomial rings. Math. USSR Izv. 11 (1977), 221–238.

R. G. Swan, Néron-Popescu desingularization, in Algebra and Geometry (Taipei, 1995), Lect. Alg. Geom. 2 (1998), 135–198. Int. Press, Cambridge, MA.

R. W. Thomason, Equivariant resolution, linearization, and Hilbert’s fourteenth problem over arbitrary base schemes, Adv. in Math. 65 (1987), no. 1, 16–34.

J. Tits, Algebraic and abstract simple groups, Ann. of Math. 80 (1964), 313–329.

T. Vorst, The general linear group of polynomial rings over regular rings, Comm. Algebra 9 (1981), 499–509.

Chebyshev Laboratory, Department of Mathematics and Computer Science, St. Petersburg State University, 14th Line V.O. 29B, 199178 Saint Petersburg, Russia

Email address: anastasia.stavrova@gmail.com