THE PROMISE POLYNOMIAL HIERARCHY

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Abstract. The polynomial hierarchy is a grading of problems by difficulty, including P, NP and coNP as the best known classes. The promise polynomial hierarchy is similar, but extended to include promise problems. It turns out that the promise polynomial hierarchy is considerably simpler to work with, and many open questions about the polynomial hierarchy can be resolved in the promise polynomial hierarchy.

Goldreich [3] argues that promise problems are a more natural object of study than non-promise problems, and our results would seem to confirm this.

Our main theorem is that, in the world of promise problems, if \( \phi \propto_{T} \text{SAT} \) then \( \phi \propto_{U} \text{VAL}_{2} \), where \( \text{VAL}_{2}(f) \) is the promise problem of finding the unique \( x \) such that \( \forall y, f(x, y) = 1 \). We also give a complete promise problem for the promise problem equivalent of \( \text{UP} \cap \text{coUP} \), and prove the promise problem equivalents of \( \text{P} \cap \text{UP} \cap \text{coUP} = \text{P} \cup \text{UP} \) and \( \text{BPP} \cap \text{UP} \cap \text{coUP} = \text{BPP} \cup \text{NP} \).

Analogous results are known for \( \text{NP} \) and \( \text{coNP} \) [1].

1. Definitions

1.1. The alphabet. We use the alphabet \( \Sigma = \{0, 1\} \). If \( x \in \Sigma^n \), we write \( |x| = n \) and \( x = (x_1, \ldots, x_n) \). If \( 1 \leq i \leq n \), we write \( \pi_i(x) = x_i \). If \( x \in \Sigma^n \), we write \( \neg x = (\neg x_1, \ldots, \neg x_n) \). We regard \( \Sigma^n \) as totally ordered lexicographically, with \( x_1 \) as the most important, so \( (0, 1) < (1, 0) \). We write \( \Sigma^* = \bigcup_{n=1}^{\infty} \Sigma^n \) for the set of finite strings from \( \Sigma \). We write \( P_+ \Sigma = \{\{0\}, \{1\}, \Sigma\} \) for the set of non-empty subsets of \( \Sigma \). We define \( \neg : P_+ \Sigma \to P_+ \Sigma \) by \( \neg\{0\} = \{1\}, \neg\{1\} = \{0\} \) and \( \neg \Sigma = \Sigma \).

1.2. Promise problems. We extend the usual definition of a problem to a promise problem. Promise problems were first introduced by Even, Selman and Yacobi [1]. The concept of a promise problem encompasses two generalizations of a problem simultaneously. Firstly, a promise problem comes with a promise which we may assume is satisfied; if the promise is not satisfied, then either answer is valid. Secondly, a promise problem allows for more than one valid answer to a problem.

Our approach is to reduce non-promise problems to promise problems and then work exclusively with promise problems in the sequel.

A problem (or, for emphasis, a non-promise problem) is a function \( \phi : \Sigma^* \to \Sigma \). A function \( \sigma : \Sigma^* \to \Sigma \) solves \( \phi \) when \( \sigma = \phi \). Let \( \Phi \) be the set of all non-promise problems.

A promise problem is a function \( \phi : \Sigma^* \to P_+ \Sigma \). A function \( \sigma : \Sigma^* \to \Sigma \) solves \( \phi \) when \( \forall x \in \Sigma^*, \sigma(x) \in \phi(x) \). We will treat a problem \( \phi \) as a promise problem \( \hat{\phi} \) by defining \( \hat{\phi}(x) = \{\phi(x)\} \).

There are many other possible equivalent definitions of a promise problem [1, 3]. One is to define a promise problem as a pair \( (\phi, \psi) \), where \( \phi : \Sigma^* \to \Sigma \) is the problem and \( \psi : \Sigma^* \to \{\text{True}, \text{False}\} \) is the promise. Under this definition, a function
We treat such a pair as a promise problem \( \phi \) by defining \( \hat{\phi}(x) = \{ \phi(x) \} \) when \( \psi(x) = \text{True} \) and \( \hat{\phi}(x) = \Sigma \) when \( \psi(x) = \text{False} \).

1.3. **Strong reduction.** If \( \phi_1 \) and \( \phi_2 \) are promise problems, we write \( \phi_1 \supseteq \phi_2 \) to mean that, for all \( x \in \Sigma^* \), \( \phi_1(x) \supseteq \phi_2(x) \). Equivalently, if \( \sigma \) solves \( \phi_2 \) then \( \sigma \) solves \( \phi_1 \). This equivalence requires \( \emptyset \notin \mathbb{P}_+\Sigma \), which is why we defined \( \mathbb{P}_+\Sigma \) as we did. Informally, \( \phi_1 \) is “easier” than (or as easy as) \( \phi_2 \).

If \( \phi_1 \) and \( \phi_2 \) are non-promise problems, we remind the reader that \( \phi_1 \propto \phi_2 \) means that there is a Turing machine \( M \) which runs in time polynomial in \( |x| \) and computes some function \( \mu : \Sigma^* \rightarrow \Sigma^* \) such that \( \forall x \in \Sigma^* , \phi_1(x) = \phi_2(\mu(x)) \). This is sometimes called Karp reduction or many-one reduction.

If \( \phi_1 \) and \( \phi_2 \) are promise problems, we define \( \phi_1 \propto \phi_2 \) to mean that there is a Turing machine \( M \) which runs in time polynomial in \( |x| \) and computes some function \( \mu : \Sigma^* \rightarrow \Sigma^* \) such that, for all \( x \in \Sigma^* \), \( \phi_1(x) \supseteq \phi_2(\mu(x)) \). Note that this generalization of \( \propto \) to promise problems is still transitive.

For a promise problem \( \phi_2 \), we define the strong closure \( [\phi_2] = \{ \phi_1 | \phi_1 \propto \phi_2 \} \). For example, \( \text{SAT} \cap \Phi = \text{NP} \). Note that \( [\phi] \) is the downward-closure of an equivalence class under \( \propto \).

1.4. **Weak reduction.** If \( \phi \) is a non-promise problem, we remind the reader that a \( \phi \)-oracle Turing machine is a Turing machine with the extra ability, given \( x \in \Sigma^* \), to compute \( \phi(x) \) in unit time.

If \( \phi \) is a promise problem, we define a \( \phi \)-oracle Turing machine to be a Turing machine \( M \) with the extra ability, given \( x \in \Sigma^* \), to compute some \( y \in \phi(x) \) in unit time. If \( \phi(x) = \Sigma \), \( M \) is non-deterministic as to the value of \( y \). In particular, there is no requirement for two separate calls to the oracle to return the same value of \( y \).

If \( \phi_1 \) and \( \phi_2 \) are non-promise problems, we remind the reader that \( \phi_1 \prec_T \phi_2 \) means that there is a \( \phi_2 \)-oracle Turing machine \( M \) which runs in time polynomial in \( |x| \) and computes \( \phi_1(x) \). This is sometimes called Turing reduction or Cook reduction.

If \( \phi_1 \) and \( \phi_2 \) are promise problems, we define \( \phi_1 \prec_T \phi_2 \) to mean that there is a \( \phi_2 \)-oracle Turing machine \( M \) and a polynomial \( p \) such that, when \( M \) is given \( x \in \Sigma^* \), every possible path of \( M \) runs in time at most \( p(|x|) \) and computes some \( y \in \phi_1(x) \). If \( \phi_1(x) = \Sigma \), there is no need for different paths to return the same value of \( y \). Note that this generalization of \( \prec_T \) to promise problems is still transitive.

Intuitively, \( M \) must compute some \( y \in \phi_1(x) \) in polynomial time even if an adversary is watching the computation and choosing the values returned by the \( \phi_2 \)-oracle when there is a choice.

The condition \( \phi_1 \prec_T \phi_2 \) is weaker than \( \phi \prec \phi_2 \), because \( M \) can call the \( \phi_2 \)-oracle polynomially many times. In contrast, \( \phi \prec \phi_2 \) means that (i) \( M \) may only call the oracle for \( \phi_2 \) once, and (ii) it must return the output from this oracle call unchanged.

For a promise problem \( \phi_2 \), we define the weak closure \( \hat{\phi_2} = \{ \phi_1 | \phi_1 \prec_T \phi_2 \} \). Note that \( \hat{\phi} \) is the downward-closure of an equivalence class under \( \prec_T \). Since strong reduction implies weak reduction, \( \hat{\phi} \) is a union of strong equivalence classes, so we can define \( \hat{\phi} = \hat{\phi} \).
1.5. Duality. If $\phi$ is a promise problem, we define the dual promise problem $\neg\phi$ by $\neg\phi(x) = \neg(\phi(x))$. Note that $\neg\neg\phi = \phi$. We write $\neg\neg[\phi] = [\neg\phi]$. For example, $\neg\text{NP} = \text{coNP}$.

2. The promise polynomial hierarchy

The promise polynomial hierarchy is the partial order on promise problems given by strong reduction. More accurately, it is what we currently know about this partial order; it is of course possible that more strong reduction is true than we can currently prove, for example if $P = \text{NP}$.

Our approach is to introduce promise problems gradually, in what we hope is a natural order, and build the promise polynomial hierarchy as we go. We give frequent diagrams of the partial order as we progress. We start by defining some complexity classes near the bottom of the hierarchy.

2.1. Level 1. We define the following problems and promise problems. In each case, the input is an encoding of some function $f : \Sigma^m \rightarrow \Sigma$, and the suppressed range of $x$ is $\Sigma^m$. The exact encoding is not important, so long as it is not too inefficient. A suitable encoding would be the value of $m$ in binary and a boolean expression in the variables $\{x_1, \ldots, x_m\}$ using the operators $\{\land, \lor, \neg\}$ and parentheses.

Firstly, the non-promise problems. It is traditional to define SAT as taking an expression of the form $\bigwedge (\bigvee x_i \lor \bigvee \neg x_j)$, but allowing general expressions is equivalent. See, for example, [2] LO7 (Satisfiability of Boolean Expressions).

$$\text{SAT}(f) = 1 \iff \exists x : f(x) = 1$$

Now the promise problems. Note that the expression $\{\pi_1 x \mid f(x) = 1\}$ in the definition of VAL($f$) can be equal to $\Sigma$, whereas the same expression in the definition of UVAL($f$) cannot, because of the condition. Our VAL is similar to xSAT of Even, Selman and Yacobi [1] and our USAT is similar to uSAT of Goldreich [3]. [USAT] is also known as promise-UP and [USAT] is also known as promise-coUP.

$$\text{MaxVAL}(f) = \begin{cases} \{\pi_1 \max \{x \mid f(x) = 1\}\} & (\exists x : f(x) = 1) \\ \Sigma & \text{otherwise} \end{cases}$$

$$\text{MinVAL}(f) = \begin{cases} \{\pi_1 \min \{x \mid f(x) = 1\}\} & (\exists x : f(x) = 1) \\ \Sigma & \text{otherwise} \end{cases}$$

$$\text{VAL}(f) = \begin{cases} \{\pi_1 x \mid f(x) = 1\} & (\exists x : f(x) = 1) \\ \Sigma & \text{otherwise} \end{cases}$$

$$\text{USAT}(f) = \begin{cases} \{\text{SAT}(f)\} & (|\{x \mid f(x) = 1\}| \leq 1) \\ \Sigma & \text{otherwise} \end{cases}$$

$$\text{USAT}(f) = \begin{cases} \{\text{SAT}(f)\} & (|\{x \mid f(x) = 0\}| \leq 1) \\ \Sigma & \text{otherwise} \end{cases}$$

$$\text{UVAL}(f) = \begin{cases} \{\pi_1 x \mid f(x) = 1\} & (\exists! x : f(x) = 1) \\ \Sigma & \text{otherwise} \end{cases}$$

We treat these uniformly as promise problems as discussed earlier. For example, in the proof of the following theorem, the equalities are equalities of elements of $\mathbb{P}^+ \Sigma$. 
Theorem 1.

(i) \(\neg [\text{SAT}] = [\text{SAT}] \).
(ii) \(\neg [\text{MaxVAL}] = [\text{MinVAL}] \).
(iii) \(\neg [\text{VAL}] = [\text{VAL}] \).
(iv) \(\neg [\text{USAT}] = [\text{USAT}] \).
(v) \(\neg [\text{UVAL}] = [\text{UVAL}] \).
(vi) \([\text{SAT}] = [\text{MaxVAL}] \).
(vii) \(\text{UVAL} \propto \text{VAL} \propto \text{MaxVAL} \).
(viii) \(\text{UVAL} \propto \text{USAT} \propto \text{SAT} \).

Proof.

(i) \(\neg \text{SAT}(f(x)) = \overline{\text{SAT}(-f(x))} \).
(ii) \(\neg \text{MaxVAL}(f(x)) = \overline{\text{MinVAL}(f(-x))} \).
(iii) \(\neg \text{VAL}(f(x)) = \overline{\text{VAL}(f(-x))} \).
(iv) \(\neg \text{USAT}(f(x)) = \overline{\text{USAT}(-f(x))} \).
(v) \(\neg \text{UVAL}(f(x)) = \overline{\text{UVAL}(f(-x))} \).
(vi) \(\bullet \ \text{SAT}(f(x)) = \text{MaxVAL}(f(x_2, \ldots, x_{m+1}) \lor \neg x_1) \).
\(\bullet \ \text{MaxVAL}(f(x)) \supseteq \text{SAT}(x_1 \ldots, x_m) \land x_1) \).
(vii) \(\text{UVAL}(f) \supseteq \text{VAL}(f) \supseteq \text{MaxVAL}(f) \).
(viii) \(\bullet \ \text{UVAL}(f) \supseteq \text{USAT}(f(1, x_2, \ldots, x_m)) \).
\(\bullet \ \text{USAT}(f) \supseteq \text{SAT}(f) \). \(\Box\)

This theorem gives a partial order on promise problems by strong reduction, which we summarize in the following diagram. We write \(X \rightarrow Y\) to mean \(X \subseteq Y\). Duality is given by reflection in a horizontal axis.

\[
\begin{align*}
[\text{USAT}] &\rightarrow [\text{SAT}] = [\text{MaxVAL}] \\
[\text{UVAL}] &\rightarrow [\text{VAL}] \\
[\text{USAT}] &\rightarrow [\text{SAT}] = [\text{MinVAL}]
\end{align*}
\]

2.2. Level 2. Level 2 of the hierarchy has a set of problems and promise problems similar to those in level 1. For the moment, however, we wish to define just one of these, so that we can show how levels 1 and 2 are related. We define the following promise problem. The input is an encoding of some \(f: \Sigma^m \times \Sigma^{m'} \rightarrow \Sigma\), which we write as \(f(x, y)\), so the suppressed range of \(x\) is \(\Sigma^m\) and the suppressed range of \(y\) is \(\Sigma^{m'}\).

\[
[\text{USAT}] \rightarrow [\text{SAT}] = [\text{MaxVAL}]
\]

\[
(9)
\]

\[
[\text{UVAL}] \rightarrow [\text{VAL}]
\]

\[
[\text{USAT}] \rightarrow [\text{SAT}] = [\text{MinVAL}]
\]

\[
\text{UVAL}_2(f) = \left\{ \begin{array}{ll}
\{ \pi_1 x \mid \forall y, f(x, y) = 1 \} & (\exists x : \forall y, f(x, y) = 1) \\
\Sigma & \text{otherwise}
\end{array} \right. 
\]

Theorem 2.

(i) \(\neg [\text{UVAL}_2] = [\text{UVAL}_2] \).
(ii) \(\text{MaxVAL} \propto \text{UVAL}_2 \).

Proof. For (i), \(\neg \text{UVAL}_2(f(x, y)) = \text{UVAL}_2(f(-x, y)) \).
For (ii), suppose we are given \( f(x) : \Sigma^m \to \Sigma \). Let \( m' = m(m-1)/2 \) and define \( g(x,y) : \Sigma^m \times \Sigma^{m'} \to \Sigma \) as follows.

\[
g(x,y) = f(x_1, \ldots, x_m) \\
\quad \land (x_1 \lor \neg f(1, y_{1,1}, \ldots, y_{1,m})) \\
\quad \land (x_2 \lor \neg f(x_1, 1, y_{2,1}, \ldots, y_{2,m})) \\
\quad \land \cdots \\
\quad \land (x_{m-1} \lor \neg f(x_1, \ldots, x_{m-2}, 1, y_{m-1,m})) \\
\quad \land (x_m \lor \neg f(x_1, \ldots, x_{m-1}, 1))
\]

(11)

Now \( \text{MaxVAL}(f) \supseteq \text{UVAL}_2(g) \). \( \square \)

We add this information to the partial order (9).

(12) \[
\begin{aligned}
\text{USAT} & \rightarrow \text{SAT} = \text{MaxVAL} \\
\text{UVAL} & \rightarrow \text{VAL} \quad \rightarrow \quad \text{UVAL}_2 \\
\text{USAT} & \rightarrow \text{SAT} = \text{MinVAL}
\end{aligned}
\]

Later (theorem 7), we will show that, in fact, \( \text{SAT} \subseteq \hat{P}^{\text{SAT}} \subseteq \text{UVAL}_2 \), but we use theorem 2 in the proof of theorem 7, so we must prove it independently.

2.3. Level \( n \). For \( n \geq 1 \), we make the obvious definitions of \( \text{SAT}_n(f) \) and so on, where we may suppress the subscript when \( n = 1 \). The input is an encoding of some function \( f : \Sigma^{m_1} \times \cdots \times \Sigma^{m_n} \to \Sigma \), which we write as \( f(x, y^{(1)}, \ldots, y^{(n-1)}) \), so the suppressed range of \( x \) is \( \Sigma^{m_1} \) and the suppressed range of \( y^{(i)} \) is \( \Sigma^{m_{i+1}} \). The function \( f_x : \Sigma^{m_2} \times \cdots \times \Sigma^{m_n} \to \Sigma \) is the obvious function obtained by fixing the value of \( x \).

Firstly, the non-promise problems.

(13) \( \text{SAT}_n(f) = 1 \iff \exists x : \text{SAT}_{n-1}(f_x) = 1 \)
(14) \( \overline{\text{SAT}}_n(f) = 1 \iff \forall x, \text{SAT}_{n-1}(f_x) = 1 \)

Now the promise problems.

(15) \( \text{MaxVAL}_n(f) = \left\{ \pi_1 \max \{ x \mid \Sigma \text{SAT}_{n-1}(f_x) = 1 \} \right\} \quad (\text{SAT}_n(f) = 1) \)
(16) \( \text{MinVAL}_n(f) = \left\{ \pi_1 \min \{ x \mid \Sigma \text{SAT}_{n-1}(f_x) = 1 \} \right\} \quad (\text{SAT}_n(f) = 1) \)
(17) \( \text{VAL}_n(f) = \left\{ \pi_1 x \mid \Sigma \text{SAT}_{n-1}(f_x) = 1 \right\} \quad (\text{SAT}_n(f) = 1) \)
(18) \( \text{USAT}_n(f) = \left\{ \Sigma \text{SAT}_n(f) \right\} \quad (\left\{ \{ x \mid \Sigma \text{SAT}_{n-1}(f_x) = 1 \} \leq 1 \right\) \)
(19) \( \overline{\text{USAT}}_n(f) = \left\{ \Sigma \text{SAT}_n(f) \right\} \quad (\left\{ \{ x \mid \Sigma \text{SAT}_{n-1}(f_x) = 0 \} \leq 1 \right\) \)
(20) \( \text{UVAL}_n(f) = \left\{ \pi_1 x \mid \Sigma \text{SAT}_{n-1}(f_x) = 1 \right\} \quad (\exists x : \Sigma \text{SAT}_{n-1}(f_x) = 1) \)

otherwise

otherwise

otherwise

otherwise

otherwise
For the following theorem, we hope that it is clear how to generalize the proof from $n = 1$ (theorems 1 and 2) to general $n \geq 1$. We feel that writing it out would require a notation which adds nothing and obscures the ideas.

**Theorem 3.** For $n \geq 1$,

(i) $\neg [\text{SAT}_n] = [\text{SAT}_n]$.

(ii) $\neg [\text{MaxVAL}_n] = [\text{MinVAL}_n]$.

(iii) $\neg [\text{VAL}_n] = [\text{VAL}_n]$.

(iv) $\neg [\text{USAT}_n] = [\text{USAT}_n]$.

(v) $\neg [\text{UVAL}_n] = [\text{UVAL}_n]$.

(vi) $[\text{SAT}_n] = [\text{MaxVAL}_n]$.

(vii) $\text{UVAL}_n \propto \text{VAL}_n \propto \text{MaxVAL}_n \propto \text{UVAL}_{n+1}$.

(viii) $\text{UVAL}_n \propto \text{USAT}_n \propto \text{SAT}_n$.

**Proof.** Omitted. □

We add this information to the partial order (12). To save space, we use the following notation, where $\widehat{\Sigma}_n$ and $\widehat{\Pi}_n$ are chosen to be similar to the standard notation (see section 2.5 below).

\[
\begin{align*}
\widehat{\Sigma}_n &= [\text{SAT}_n] = [\text{MaxVAL}_n] & \text{U}\widehat{\Sigma}_n &= [\text{USAT}_n] \\
\widehat{\Pi}_n &= [\text{SAT}_n] = [\text{MinVAL}_n] & \text{U}\widehat{\Pi}_n &= [\text{USAT}_n] \\
\widehat{V}_n &= [\text{VAL}_n] & \text{U}\widehat{V}_n &= [\text{UVAL}_n] \\
\end{align*}
\]

(21) $\begin{array}{c}
\text{U}\widehat{\Sigma}_1 \rightarrow \widehat{\Sigma}_1 \\
\text{U}\widehat{\Sigma}_2 \rightarrow \widehat{\Sigma}_2 \\
\end{array}$

(22) $\begin{array}{c}
\text{U}\widehat{\Pi}_1 \rightarrow \widehat{\Pi}_1 \\
\text{U}\widehat{\Pi}_2 \rightarrow \widehat{\Pi}_2 \\
\end{array}$

(23) $\begin{array}{c}
\text{U}\widehat{V}_1 \rightarrow \widehat{V}_1 \\
\text{U}\widehat{V}_2 \rightarrow \widehat{V}_2 \\
\text{U}\widehat{V}_3 \rightarrow \widehat{V}_3 \\
\end{array}$

2.4. **Level 0.** We have so far ignored the very bottom of the hierarchy. Let $\widehat{P}$ be the set of promise problems $\phi$ such that there is a polynomial-time Turing machine $M$ which computes some function $\mu : \Sigma^* \rightarrow \Sigma$ which solves $\phi$. Alternatively, $\widehat{P} = \widehat{P}^0$, where $0$ is the trivial problem $0(x) = 0$ for all $x \in \Sigma^*$. The next theorem is nearly trivial.

**Theorem 4.** $\widehat{P} \subseteq \text{U}\widehat{V}_1$.

**Proof.** Clearly $\pi_1 \in \widehat{P}$. Let $\phi \in \widehat{P}$ and $\mu$ be as above. Now $\pi_1(\mu(x)) = \mu(x)$, so $\phi \propto \pi_1$, so $\widehat{P} = [\pi_1]$. Now $\pi_1(x) = \text{UVAL}(f_x)$, where $f_x(y) = 1 \iff y = (x_1)$. □

2.5. **The standard hierarchy.** For $n \geq 1$, let $\widehat{\Delta}_{n+1} = \widehat{\Delta}^{\text{SAT}_n}$. We remind the reader that $\Phi$ is the set of non-promise problems. We define $P = \widehat{P} \cap \Phi$, $\Sigma_n = \widehat{\Sigma}_n \cap \Phi$, $\Pi_n = \widehat{\Pi}_n \cap \Phi$, and $\Delta_n = \widehat{\Delta}_n \cap \Phi$ as usual. We have $\Sigma_1 = \text{NP}$ and $\Pi_1 = \text{co-NP}$.
coNP. We remind the reader of the standard polynomial hierarchy \[4, 2\].

\[
\begin{array}{cccc}
\Sigma_1 & \Delta_2 & \Delta_3 \\
\downarrow & \downarrow & \downarrow \\
\Pi_1 & \Pi_2 & \\
\end{array}
\]

We now show that this partial order remains true if we include promise problems, using \(\hat{P}, \hat{\Sigma}_n, \hat{\Pi}_n\) and \(\hat{\Delta}_n\). We already know that \(\hat{P} \subseteq \hat{U} \subseteq \hat{\Sigma}_1\).

**Theorem 5.** \(\hat{\Sigma}_n \subseteq \hat{\Delta}_{n+1}\).

**Proof.** \(\text{SAT}_n \in \hat{P}^{\text{SAT}_n}\). \(\square\)

The proof of the following theorem is the standard argument which shows that \(\Delta_2 \subseteq \Sigma_2\). We write it out in detail, partly so that we can check that it still works for promise problems, but mainly because we will need all the detail anyway when we come to prove theorem 5, and it helps to see it in a more familiar context first.

Incidentally, one might think that the standard argument which shows that \(\Delta_2 \subseteq \Sigma_2\) is just \(P \subseteq \text{NP}\) relative to any oracle, so \(P^{\text{NP}} \subseteq \text{NP}^{\text{NP}}, \) so \(\Delta_2 \subseteq \Sigma_2\). This argument is fine, but then one has to show that \(\text{SAT}_2\) is a complete problem for \(\text{NP}^{\text{NP}}, \) which is essentially the same theorem.

**Theorem 6.** For \(n \geq 2, \hat{\Delta}_n \subseteq \hat{\Sigma}_n\).

**Proof.** We give the proof for \(n = 2\).

Suppose \(\phi \in \hat{\Delta}_2 = \hat{P}^{\text{SAT}_2}\). Then \(\phi\) can be solved by a Turing machine \(M\) which takes \(\alpha \in \Sigma^*\) as input, runs in time at most \(T\), where \(T\) is polynomial in \(m = |\alpha|\), and which is allowed to make calls to a SAT oracle. Conveniently, SAT is a non-promise problem, so \(M\) is deterministic, so we can say that \(M\) calculates \(\sigma\), where \(\sigma\) solves \(\phi\). Let \(X_{i,t}\) be the value on the tape at position \(i\) and time \(t\) and let \(Y_{s,t}\) be 1 if \(M\) is in state \(s\) at time \(t\) and 0 otherwise. The ranges are \(-T \leq i \leq T\) and \(0 \leq t \leq T\), because a Turing machine moves at most 1 cell at each step.

This means that \(\sigma(\alpha)\) can be written as a circuit \(f\) with \(n\) gates, where \(n\) is polynomial in \(m\), but with the ability to use a special SAT gate, which evaluates \(\text{SAT}(g)\), where \(g\) is (an encoding of) a circuit which is the input to the gate. Let the values at the vertices of \(f\) be \(\alpha = (a_1, \ldots, a_m)\) and \(x = (x_1, \ldots, x_n)\), and label these vertices (conveniently, but rather unusually) such that \(x_1\) is the output vertex. We can write the condition \(v(x)\) that \(x\) is a valid assignment of values to the vertices as follows, where \(a_i(\alpha, x)\) is a function representing the gate which calculates the value at the vertex \(x_i\).

\[
v(\alpha, x) = \bigwedge_{i=1}^{n}(x_i = a_i(\alpha, x))
\]

\[
= \bigwedge_{i=1}^{n}(-x_i \lor a_i(\alpha, x)) \land (x_i \lor -a_i(\alpha, x))
\]

For an ordinary gate, \(a_i(\alpha, x)\) is a polynomial expression. For a SAT gate, we can write \(a_i(\alpha, x) = \exists y : b_i(\alpha, x, y)\), where \(b_i(\alpha, x, y)\) is a polynomial expression. We
Theorem 7. For

\[ v(\alpha, x) = (\exists y : p_1(\alpha, x, y)) \land (\forall z, p_2(\alpha, x, z)) \]

We write \( \sigma(\alpha) = 1 \iff \exists x : v(\alpha, x) \land (x_1 = 1) \), which simplifies to \( \exists(x, y) : \forall z, p_3(\alpha, x, y, z) \) for some polynomial expression \( p_3 \). Now we write \( p_3(\alpha, x, y, z) = h_\alpha(x, y, z) \) for some circuit \( h_\alpha \) of polynomial size which we can construct from \( \alpha \) in polynomial time. We therefore have \( \sigma(\alpha) = 1 \iff \exists(x, y) : \forall z, h_\alpha(x, y, z) \), so \( \sigma \in \Sigma_2 \).

Now \( \phi \in \Sigma_2 \subseteq \hat{\Sigma}_2 \), so \( \phi \in [\sigma] \subseteq \hat{\Sigma}_2 \) (because \( \hat{\Sigma}_2 \) is a union of strong equivalence classes) as required.

We therefore have the promise version of the standard polynomial hierarchy (25).

\[ \hat{\Sigma}_1 \quad \hat{\Sigma}_2 \quad \hat{\Sigma}_3 \]

2.6. Combining the hierarchies. This raises the natural question of how the partial orders (24) and (28) fit together. We already know that \( \hat{P} \subseteq \hat{U}_1 \).

For the next theorem, we need a notation. Suppose \( X \) is a set, suppose \( P : X \to \{\text{True, False}\} \) is some predicate, and suppose \( \exists!x \in X : P(x) \). Then we define \( \exists!x : P(x) \) to be that element. In other words, \( P(\exists!x : P(x)) \) is true. If \( \exists!x \in X : f(x) = 1 \), then \( \text{UVAL}(f) = \{ \pi_1!x : f(x) = 1 \} \).

Theorem 7. For \( n \geq 2 \), \( \hat{\Delta}_n \subseteq \hat{U}_n \).

Proof. We give the proof for \( n = 2 \). We use the notation and most of the proof of theorem 6.

The point is that there is exactly one assignment of \( x \) which satisfies \( v(\alpha, x) \), and \( x_1 \) is the output vertex, so we can write \( \sigma(\alpha) = \pi_1!x : v(\alpha, x) \).

\( \sigma(\alpha) = \pi_1!x : (\exists y : p_1(\alpha, x, y)) \land (\forall z, p_2(\alpha, x, z)) \)

Now theorems 1 and 2 say that \( \text{SAT} \propto \text{UVAL}_2 \) and, by duality, \( \overline{\text{SAT}} \propto \overline{\text{UVAL}_2} \). This means that we can transform the expressions \( \exists y : p_1(\alpha, x, y) \) and \( \forall z, p_2(\alpha, x, z) \) into UVAL form in polynomial time, and therefore polynomially many extra variables.

\( \exists y : p_1(\alpha, x, y) \iff (\pi_1!s : \forall z, p_3(\alpha, s, x, z)) = 1 \)

\( \forall z, p_2(\alpha, x, z) \iff (\pi_1!t : \forall z, p_4(\alpha, t, x, z)) = 1 \)

Note that, for all \( x \in \Sigma^* \), it is true both that \( \exists!s : \forall z, p_3(\alpha, s, x, z) \) and that \( \exists!t : \forall z, p_4(\alpha, t, x, z) \), so the \( !s : \ldots \) notation is justified. We substitute these into \( \sigma(\alpha) \).

\( \sigma(\alpha) = \pi_1!x : (\pi_1!s : \forall z, p_3(\alpha, s, x, z))(\pi_1!t : \forall z, p_4(\alpha, t, x, z)) = 1 \)

We repeatedly simplify this expression for \( \sigma(\alpha) \). We define \( u = (s_t t_1) \) and we define \( v = (u, s, t) \) and \( w = (x, v) \) to be the simple concatenation of previous variables.
Note that the order is important, so that \( v_1 = u_1 = s_1 t_1 \) and \( w_1 = x_1 \). We also define new polynomial expressions \( p_i \) as we go.

\[
\sigma(\alpha) = \ldots
\]

(33)\[ \pi_1!x : (\pi_1!(u, s, t) : (u = s_1 t_1) \land \forall z, p_3(\alpha, s, x, z) \land p_4(\alpha, t, x, z)) = 1 \]

(34)\[ \pi_1!x : (\pi_1!v : \forall z, p_5(\alpha, v, x, z)) = 1 \]

(35)\[ \pi_1!(x, v) : (v_1 = 1) \land \forall z, p_5(\alpha, v, x, z) \]

(36)\[ \pi_1!w : \forall z, p_6(\alpha, w, z) \]

Now we can write \( p_6(\alpha, w, z) = h_\alpha(w, z) \), for some circuit \( h_\alpha \) of polynomial size which we can construct from \( \alpha \) in polynomial time. We therefore have \( \sigma(\alpha) = \pi_1!w : \forall z, h_\alpha(w, z) \), so \( \sigma \in U\hat{V}_2 \).

Now \( \phi \propto \sigma \in U\hat{V}_2 \), so \( \phi \in [\sigma] \subseteq U\hat{V}_2 \) as required. \( \square \)

We can now combine the partial orders (24) and (28).

(37) \[ \hat{P} \rightarrow U\hat{V}_1 \rightarrow \hat{V}_1 \rightarrow \hat{\Delta}_2 \rightarrow U\hat{V}_2 \rightarrow \hat{V}_2 \rightarrow \hat{\Delta}_3 \rightarrow U\hat{\Pi}_1 \rightarrow \hat{\Pi}_1 \rightarrow U\hat{\Pi}_2 \rightarrow \hat{\Pi}_2 \]

2.7. The last class. The partial order (37) is clearly missing something, so we make the following definition.

(38) \[ U\hat{\Delta}_{n+1} = \hat{P}^{U\hat{\Sigma}_n} = \hat{P}^{U\hat{\Pi}_n} \]

Theorem 8. \( U\hat{\Sigma}_n \subseteq U\hat{\Delta}_{n+1} \subseteq \hat{\Delta}_{n+1} \).

Proof. \( \text{USAT}_n \in \hat{P}^{\text{USAT}_n} \) and \( U\hat{\Sigma}_n \subseteq \hat{\Sigma}_n \). \( \square \)

We add this information to the partial order (37) to get our final diagram. Note that the periodic “bulge” is a cube.

Diagram (39) summarizes our current knowledge about the partial order in the promise polynomial hierarchy. However, the promise polynomial hierarchy is easier to work with than the non-promise polynomial hierarchy, and there are interesting questions which we can answer in the promise polynomial hierarchy that are open in the non-promise polynomial hierarchy.
3.1. Intersections. There are two intersections in diagram \( (29) \) that we would like to identify, and it turns out that in the promise polynomial hierarchy we can do so. We know that \( \widehat{V}_n \subseteq \widehat{\Sigma}_n \cap \widehat{\Pi}_n \), and \( U \widehat{V}_n \subseteq U \widehat{\Sigma}_n \cap U \widehat{\Pi}_n \). The next two theorems show that these inclusions are, in fact, equalities.

**Theorem 9.** \( \widehat{V}_n = \widehat{\Sigma}_n \cap \widehat{\Pi}_n \).

**Proof.** We give the proof for \( n = 1 \).

Suppose \( \phi \in \widehat{\Sigma}_1 \cap \widehat{\Pi}_1 \). Since \( \phi \in \widehat{\Sigma}_1 = [\text{SAT}] \), there is a polynomial expression \( g(x,y) \) so that \( \exists y : g(x,y) = 1 \) implies \( 1 \in \phi(x) \) and \( \forall y, g(x,y) = 0 \) implies \( 0 \in \phi(x) \), and if neither of these holds then \( \phi(x) = \Sigma \). Since \( \phi \in \widehat{\Pi}_1 = [\text{USAT}] \), there is also a polynomial expression \( h(x,y) \) so that \( \forall y, h(x,y) = 1 \) implies \( 1 \in \phi(x) \), \( \exists y : h(x,y) = 0 \) implies \( 0 \in \phi(x) \), and if neither of these holds then \( \phi(x) = \Sigma \).

Now define \( f_x(i,y) \) (where \( |i| = 1 \)) by \( f_x(1,y) = g(x,y) \) and \( f_x(0,y) = \neg h(x,y) \).

By construction, if \( f_x(i,y) = 1 \) then \( i \in \phi(x) \).

Suppose that there is no \( (i,y) \) such that \( f_x(i,y) = 1 \). Then \( \forall y, g(x,y) = 0 \) and \( \forall y, h(x,y) = 1 \), so \( 0 \in \phi(x) \) and \( 1 \in \phi(x) \), so \( \phi(x) = \Sigma \).

Therefore \( \phi(x) \supseteq \text{VAL}(f_x) \), so \( \phi \in [\text{VAL}] = \widehat{V}_1 \). \( \square \)

**Theorem 10.** \( U \widehat{V}_n = U \widehat{\Sigma}_n \cap U \widehat{\Pi}_n \).

**Proof.** We give the proof for \( n = 1 \). The proof is similar to the proof of theorem \( 9 \).

Suppose \( \phi \in U \widehat{\Sigma}_1 \cap U \widehat{\Pi}_1 \). Since \( \phi \in U \widehat{\Sigma}_1 = [\text{SAT}] \), there is a polynomial expression \( g(x,y) \) so that \( \exists y : g(x,y) = 1 \) implies \( 1 \in \phi(x) \), \( \forall y, g(x,y) = 0 \) implies \( 0 \in \phi(x) \), and if neither of these holds then \( \phi(x) = \Sigma \). Since \( \phi \in U \widehat{\Pi}_1 = [\text{USAT}] \), there is also a polynomial expression \( h(x,y) \) so that \( \forall y, h(x,y) = 1 \) implies \( 1 \in \phi(x) \), \( \exists y : h(x,y) = 0 \) implies \( 0 \in \phi(x) \), and if neither of these holds then \( \phi(x) = \Sigma \).

Now define \( f_x(i,y) \) (where \( |i| = 1 \)) by \( f_x(1,y) = g(x,y) \) and \( f_x(0,y) = \neg h(x,y) \).

By construction, \( \exists y : f_x(1,y) = 1 \) implies \( 1 \in \phi(x) \) and \( \exists y : f_x(0,y) = 1 \) implies \( 0 \in \phi(x) \), so \( \exists(i,y) : f_x(i,y) = 1 \) implies \( i \in \phi(x) \) for that value of \( i \).

We claim that if it is not true that \( \exists(i,y) : f_x(i,y) = 1 \), then \( \phi(x) = \Sigma \). Firstly, suppose that there is no \( (i,y) \) such that \( f_x(i,y) = 1 \). Then \( \forall y, g(x,y) = 0 \) and \( \forall y, h(x,y) = 1 \), so \( 0 \in \phi(x) \) and \( 1 \in \phi(x) \), so \( \phi(x) = \Sigma \). Now suppose that \( f_x(i_1,y_1) = f_x(i_0,y_0) = 1 \) with \( (i_1,y_1) \neq (i_0,y_0) \). We may suppose that \( \phi(x) \neq \Sigma \), so there is at most one \( y \) such that \( g(x,y) = 1 \) and there is at most one \( y \) such that \( h(x,y) = 0 \). Therefore \( i_1 \neq i_0 \), and we may assume \( i_1 = 1 \) and \( i_0 = 0 \). Now \( \exists y : g(x,y) = 1 \) and \( \exists y : h(x,y) = 0 \), so \( 1 \in \phi(x) \) and \( 0 \in \phi(x) \), so \( \phi(x) = \Sigma \).

Therefore \( \phi(x) \supseteq \text{UVAL}(f_x) \), so \( \phi \in [\text{UVAL}] = U \widehat{V}_1 \). \( \square \)

3.2. Weak closures. There are two weak closures in diagram \( (39) \) that we would like to identify, and it turns out that in the promise polynomial hierarchy we can do so. We know that \( \overline{\text{P}}V_n \subseteq \overline{\text{P}} \Sigma_n = \Delta_{n+1} \), and \( \overline{\text{P}}U \overline{\text{V}}_n \subseteq \overline{\text{P}} \Sigma_n = U \Delta_{n+1} \). The next two theorems show that these inclusions are, in fact, equalities.

**Theorem 11.** \( \overline{\text{P}} \overline{\text{V}}_n = \Delta_{n+1} \).

**Proof.** We give the proof for \( n = 1 \). The proof is standard \( 1 \).

We show that \( \text{SAT} \in \overline{\text{P}}[\text{VAL}] \), which suffices. Given a polynomial expression \( f_1(x_1,\ldots,x_m) \), we call the oracle for \( \text{VAL}(f_1) \) to get the first bit \( x'_1 \) of a putative solution. We then define \( f_2(x_2,\ldots,x_m) = f_1(x'_1, x_2,\ldots,x_m) \) and call \( \text{VAL}(f_2) \) to get
the second bit \( x'_2 \) of a putative solution. We continue until we have \( x' = (x'_1, \ldots, x'_m) \) and then we evaluate \( f_1(x'_1, \ldots, x'_m) \).

\[ \square \]

**Theorem 12.** \( \widehat{\text{P}}^U \widehat{\Delta}_n = U \widehat{\Delta}_{n+1} \).

**Proof.** We give the proof for \( n = 1 \). The proof is very similar to the proof of theorem 11.

We show that USAT \( \in \widehat{\text{P}}^\text{[UVAL]} \), which suffices. Given a polynomial expression \( f_1(x_1, \ldots, x_m) \) with at most one solution, we call the oracle for UVAL\((f_1)\) to get the first bit \( x'_1 \) of the putative solution. We then define \( f_2(x_2, \ldots, x_m) = f_1(x'_1, x_2, \ldots, x_m) \) and call UVAL\((f_2)\) to get the second bit \( x'_2 \) of the putative solution. We continue until we have \( x' = (x'_1, \ldots, x'_m) \) and then we evaluate \( f_1(x'_1, \ldots, x'_m) \).

\[ \square \]

3.3. **Randomized reduction.** We can further weaken weak reduction to random reduction. If \( \phi_1 \) and \( \phi_2 \) are promise problems, we define \( \phi_1 \propto_R \phi_2 \) to mean that there is a \( \phi_2 \)-oracle probabilistic Turing machine \( M \) and a polynomial \( p \) such that, when \( M \) is given \( x \in \Sigma^* \), every possible path \( \pi \) of \( M \) runs in time at most \( p(|x|) \) and computes some \( y_\pi \), and \( y_\pi \in \phi_1(x) \) with probability at least \( 2/3 \). We define \( \text{BPP}^{\phi_2} = \{ \phi_1 \mid \phi_1 \propto_R \phi_2 \} \). If \( X = [\phi] \) or \( X = \widehat{\text{P}}^\phi \), we define \( \text{BPP}^X = \text{BPP}^{\phi} \).

We could similarly generalize RP and coRP, which allow a probability of error for one answer but not the other, but we do not pursue that possibility here.

Under randomized reduction, it turns out that the entire “bulge” from \( U \widehat{\Delta}_n \) to \( \widehat{\Delta}_{n+1} \) is equivalent. Our proof is a simple corollary of theorem 12 and a result of Valiant and Vazirani [5].

**Theorem 13.** \( \text{BPP}^{U \widehat{\Delta}_n} = \text{BPP}^{\widehat{\Delta}_{n+1}} \).

**Proof.** We rely on the result that SAT \( \propto_R \) USAT [5,3], and the relativization of this result which states that SAT\(_n \propto_R \) USAT\(_n \). Now SAT\(_n \in \text{BPP}^{\sim \text{USAT}_n} = \text{BPP}^{U \text{VAL}_n} \) by theorem 12 and the fact that weak reduction implies random reduction.

\[ \square \]

4. **Bibliography**

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