Deformation of Homogeneous structures and Homotopy of symplectomorphisms groups.

Introduction.

Let \((N, \omega)\) a symplectic manifold, the group \(\text{Symp}(N)\) of symplectomorphisms of \((N, \omega)\) acts transitively on it. Moreover, Banyaga has shown that \(\text{Sym}(N)\) determines completely the symplectic structure of \((N, \omega)\) when \(N\) is compact. This motivates the study of the properties of \(\text{Symp}(N)\) which must enable to understand the geometric properties of \((N, \omega)\).

The beginning of the study of the homotopy properties of \(\text{Symp}(N)\) has its origin in the theory of pseudo-holomorphic curves defined by Gromov, using this theory, Gromov has shown that the group of compactly supported symplectomorphisms in the interior of the symplectic 4-dimensional standard ball is contractible. Let \((S^2 \times S^2, \omega_0)\) be the product of two copies of the 2-dimensional sphere \(S^2\), endowed with the symplectic structure which is the product of the canonical volume form of \(S^2\). Gromov has also shown that \(\text{Symp}(S^2 \times S^2)\) has the same homotopy type than \(SO(3) \times SO(3)\).

The work of Gromov has been generalized by many authors, remark that if we endow \(S^2 \times S^2\) with a symplectic form \(\omega\) such that the volume of the fiber of the fibration \(S^2 \times S^2 \to S^2\) is not the volume of its base, the homotopy of \(\text{Symp}(S^2 \times S^2, \omega)\) can be different than the homotopy of a finite dimensional Lie group. This phenomenon has been firstly observed by Abreu, and completely studied by McDuff and Abreu. Their study use the theory of pseudo-holomorphic curves, and stratification of the space of pseudo-convex structures.

In his thesis Pinsouault has studied the homotopy type of the one point blow-up \(N\) of \(S^2 \times S^2\), if the volume of the fiber and base of the fibration \(S^2 \times S^2\) are equal, he has shown that \(\text{Symp}(N)\) has the homotopy type of a 2-dimensional torus. Joseph Coffey has used the decomposition of a symplectic 4-dimensional manifold to relate the homotopy type of the group of symplectomorphisms \(\text{Symp}(N)\) of a 4-dimensional manifold to a configuration space, more precisely Paul Biran has shown that we can remove a simplex \(D\) on a 4-dimensional symplectic manifold in such a way that the resulting space is a disc bundle over a symplectic surface, the group \(\text{Symp}(N)\) is then an extension of its subgroup which preserves \(D\) by the orbit of \(D\) by the group \(\text{Symp}(N)\). This approach must be related to the work of Lalonde and Pinsouault who
have studied the relations of the space of embedding of a symplectic ball with a given radius in a symplectic 4-dimensional manifold and $\text{Symp}(N)$.

As remarked Pinsonnault in his thesis, the calculation of the homotopy type of the group of symplectomorphisms of $S^2 \times S^2$ and its blow-up is possible because the topology of these manifolds is very simple, and there exists a good knowledge of the theory of pseudo-holomorphic curves of these manifolds. Another simple example of symplectic 4-dimensional manifold is the 4-dimensional torus $N^4$, the study of $\text{Symp}(N^4)$ is hardly tractable using the theory of pseudo-holomorphic curves since $H^2(N^4, \mathbb{R})$ is a 6-dimensional real vector space.

The purpose of this paper is to describe the homotopy type of $\text{Symp}(N^4)$ using new ideas. The torus $N^4$ is the quotient of the affine space $\mathbb{R}^4$ by 4-translations which preserve the standard symplectic form. Let $D'$ be the boundary of a fundamental domain of this action. It projects to $N^4$ to define a union $D$ of 3-dimensional torus. Our approach may be related to the work of Joseph Coffey and Lalonde-Pinsonnault, we remark that the group $\text{Symp}(N^4)$ is the total space of a fibration whose base space is the orbit of $D$ under $\text{Symp}(N^4)$ and fiber the group of symplectomorphisms which preserve $D$.

To determine the homotopy type of $\text{Symp}(N^4)$, one needs to determine the homotopy type of the space $N(D)$ of orbits of $D$ under $\text{Symp}(N)$. This is performed using the theory of deformation of homogeneous structures as it is described by W. Goldman. We show that $N(D)$ is contractible.

**Decomposition of symplectic manifolds and group of symplectomorphism.**

The purpose of this paragraph is to define a notion of canonical decomposition of a compact symplectic 4-dimensional manifold, and to show that the connected component of the group of Hamiltonian diffeomorphisms which preserve such a configuration is contractible.

**Definition**

Let $(N, \omega)$ be a compact 4-dimensional symplectic manifold, a configuration $D$ of $(N, \omega)$ is a finite union $D$ of compact 3-dimensional submanifolds $D_1, \ldots, D_p$ which satisfy the following conditions:

1. The manifold $N - D$ is symplectomorphic to the 4-dimensional symplectic standard ball or to a polydisc endowed with the restriction of the standard symplectic form of $\mathbb{R}^4$, $D$ is connected.
2. The restriction of $\omega$ defines on $D_i$ a contact structure such that the fibers of the characteristic foliation are circles, $D_i$ is the total space of a bundle whose leaves are the characteristic leaves, and which base space is an intersection $D_i \cap D_j$. The fibers of the characteristic foliation of $D_i$ intersects at least two different cells.
3. The intersection $D_{ij} = D_i \cap D_j$ is either a symplectic 2-dimensional submanifold of $(N, \omega)$, or a Lagrangian 2-dimensional submanifold stable by the characteristic foliations.
Suppose that $D_{ij}$ is a symplectic submanifold and consider $h_t$ a path of Hamiltonian diffeomorphisms of $D_{ij}$ such that the restriction of $h_t$ to $D_{ij}$ is the identity if $D_{ij}$ is a circle, then we can extend $h_t$ to a path of Hamiltonian diffeomorphisms $h'_t$ of $N$ in such a way that it preserves the configuration, and its restriction to the cells $D_t$ such that $D_t$ is either a circle, or is empty is the identity. The restriction of $h'_t$ to a symplectic surface $D_{ij}$ different from $D_{ij}$ is the identity.

The configuration defines on $N$ the structure of a CW-complex with one 4-dimensional cell.

An example of a configuration is the 4-dimensional torus $N^4$. It is the product of 2-dimensional torus $N_2$ and $N'_2$. $N^4$ is endowed with the symplectic form which is the product of the standard symplectic forms of $N_1$ and $N_2$. Let $l_1$, $l_2$, be the two curves of $N_2$ parallel in respect to the flat structure which generate $\pi_1(N_2)$, and $l_3$, $l_4$, the curves of $N'_2$ which satisfy the same properties. The configuration is the union of $N_2 \times l_3, N_2 \times l_4, N'_2 \times l_1, N'_2 \times l_2$. Each function $H : N_2 \to \mathbb{R}$, which is constant on $l_1$ and $l_2$ can be extended to $N_2 \times N'_2$ by the projection $N_2 \times N'_2 \to N_2$, and it preserves the configuration.

We denote by $\text{Symp}(N, D)$ the group of symplectomorphisms of $(N, \omega)$ which preserve $D$, $\text{Ham}(N, D)$ its subgroup of Hamiltonian symplectomorphisms, $\text{Symp}_{Id}(N, D)$ is the subgroup of $\text{Symp}(N, D)$ whose restriction to $D$ is the identity, and $\text{Ham}_{Id}(N, D)$ is the subgroup of Hamiltonian diffeomorphisms of $\text{Symp}_{Id}(N, D)$.

**Proposition** The connected component $\text{Ham}_{Id}(N, D)_0$ of the group $\text{Ham}_{Id}(N, D)$ is contractible.

A theorem of Weinstein asserts that there exists a neighborhood $U_{Id}$ of the identity in $\text{Symp}(N)$ which is contractible. Without restricting the generality, we can suppose that the image of $U_{Id}$ by the flux map is a contractible open subset. Thus we can suppose that $\text{Ham}(N) \cap U_{Id}$ is contractible, and the existence of a continuous map $\text{Ham}(N) \cap U_{Id}$ to the space of exact time dependent 1-forms, $h \to dH^h$, where $H^h : N \to \mathbb{R}$ is a time dependent differentiable function such that $h$ is the value at 1 of the flow generated by $dH$.

Without restricting the generality, we can suppose that for every element $h \in \text{Ham}_{Id}(N, D)_0$, there exists a neighborhood $V$ of $D$ such that the restriction of $h$ to $V$ is the restriction of an element of $U_{Id}$, $h'$ to $V$ which coincide with $h$ on an open neighborhood $U$ which contains $V$, and $h(V) \subset U_1 \subset U$, where $U_1$ is open. The existence of $h'$ can be shown using a cut function. We can also suppose that and $N - V, N - U$ are polydiscs. Let $f$ be a cut function such that the restriction of $f$ to $U_1$ is 1 and the restriction of $f$ to $N - U$ is zero, we define a map $\Phi_t, h \to \psi_t^{-1} \circ h$, where $\psi_t$ is the flow generated by $fH^h$, $\Phi_0(h) = h$, $\Phi_1(h)$ is an Hamiltonian map whose support is contained in the interior of $N - V$. Let $\text{Ham}_{Id}(N, V, U_1, U, D)$ be the subset of $\text{Ham}_{Id}(N, D)_0$ such that the restriction of $h$ to $V$ is the restriction of an element $h'$ of $U_{Id}$, and $h(V) \subset U_1$, the restrictions of $h$ and $h'$ to $U$ coincide. A result of Gromov says that the group of symplectomorphisms whose supports are in the interior
of $N - V$ is contractible. The map $\Phi_t$ implies that $\operatorname{Ham}_{td}(N, V, U_1, U, D)$ is contractible.

Let $(V_n)_{n \in \mathbb{N}}$, be a family of open subsets which contain $D$ such that $V_{n+1}$ is contained in $V_n$, $\cap_{n \in \mathbb{N}} V_n = D$. Consider the families of open subsets $U_n$ and $U_1^n$, such that $V_n \subset U_1^n \subset U_n$, we suppose that $\cap_{n \in \mathbb{N}} U_n = \cap_{n \in \mathbb{N}} U_1^n = D$, $U_1^{n+1} \subset U_1^n$, $U_{n+1} \subset U_n$. Then $\operatorname{Ham}_{td}(N, D)_0$ is the limit of $\operatorname{Ham}_{td}(N, V_{n+1}, U_1^{n+1}, U_n, D)$ when the previous expression has a sense. We suppose that $N - V_0$ is a polydisc.

Since $V_{n+1}$ is contained in the interior of $V_n$, we deduce that $\operatorname{Ham}_{td}(N, V_{n+1}, U_1^{n+1}, U_n, D)$ is contained in the interior of $\operatorname{Ham}_{td}(N, V_n, U_1^n, U_n, D)$. This implies that $\operatorname{Ham}_{td}(N, D)_0$ is contractible.

Proposition. The group $\operatorname{Ham}(N, D)_0$ is contractible.

Proof. Let $L_D$ be the restriction to $D$ of the subgroup $\operatorname{Ham}'(N, D)$ of $\operatorname{Ham}(N, D)_0$ whose restriction to $D_{i_1 i_2 i_3}$ is the identity if $D_{i_1 i_2 i_3}$ is a curve. If $D_{ij}$ is a symplectic submanifold, we denote by $L_{ij}$ the group of Hamiltonian diffeomorphisms of $D_{ij}$ whose restriction to $D_{i_1 i_2 i_3}$ is the identity, if $D_{i_1 i_2 i_3}$ is a circle, we denote by $L_D'$ the product of the groups $L_{ij}$. The extension property verified by the configuration implies that the restriction map $L_D \to L_D'$ is a fibration. The fiber of this fibration is contractible, since the gauge group of the fiber induced by the characteristic foliation is contractible, since each fiber intersects two different cells, and a connected subgroup of diffeomorphisms of an interval is contractible. Since $L_D'$ is contractible, we deduce that $L_D$ is contractible.

Let $L_D^1$ be the restriction of $\operatorname{Ham}(N, D)_0$ to $D$, and $L_D^2$ the restriction of $\operatorname{Ham}(N, D)_0$ to the circles $D_{i_1 i_2 i_3}$, we have a fibration $L_D^1 \to L_D^2$ whose fiber is $L_D$, since $L_D$ is contractible, and $L_D^2$ is contractible since a connected group of diffeomorphism of an interval is contractible, we deduce that $L_D^2$ is a contractible. The fiber of the fibration $\operatorname{Ham}(N, D)_0 \to L_D^1$ is $\operatorname{Ham}_{td}(N, D)_0$ which is contractible. This implies that $\operatorname{Ham}(N, D)_0$ is contractible.

Isotopy of configurations and homogeneous structures.

To determine the homotopy of $\operatorname{Ham}(N)$, we have to determine the homotopy of the orbit of $D$ under the action of $\operatorname{Ham}(N)$. To solve this problem, we consider the situation when $N$ is a locally homogeneous manifold.

Definition Let $V$ be a differentiable manifold, and $H$ a Lie subgroup acting on $V$, the action of $H$ satisfies the unique extension property, if and only if two elements $h_1, h_2$ of $H$ which coincide on an open subset of $V$, coincide on $V$.

Definition Let $(V, H)$ be a $n$-dimensional manifold endowed with the action of a Lie group which satisfies the unique extension property. A manifold $N$ is endowed with a $(V, H)$ (homogeneous) structure, if and only if there exists a $V$-atlas $(U_i, \phi_i)_{i \in I}$ of $N$, such that $\phi_i : U_i \to V$, and $\phi_j \circ \phi_i^{-1}$ is the restriction of the action of an element $h_{ij}$ of $H$ on $\phi_i(U_i \cap U_j)$.

Let $N$ be a manifold endowed with a $(V, H)$ homogeneous structure, and $\hat{N}$ be the universal cover $\hat{N}$, $\hat{N}$ inherits from $\hat{N}$ a structure (the pull-back)
of a \((V,H)\) manifold defined by a local diffeomorphism \(D_N: \hat{N} \to V\) called the developing map. This developing map gives rise to a representation \(h_N: \pi_1(N) \to H\) defined by the following commutative diagram:

\[
\begin{array}{ccc}
\hat{N} & \xrightarrow{d} & \hat{N} \\
\downarrow D_N & & \downarrow D_N \\
V & \xrightarrow{h_N(d)} & V
\end{array}
\]

called the holonomy representation.

Examples of homogeneous manifolds are:

Affine manifolds, here \(V\) is the affine space \(\mathbb{R}^n\), and \(H = \text{Aff}(\mathbb{R}^n)\) the group of affine transformations.

Projective manifolds, here \(V\) is either \(P^n(\mathbb{R})\), or \(P^n(\mathbb{C})\), and \(H\) is the group of projective transformations.

Let \(N\) be a manifold endowed with a \((V,H)\) structure, we denote by \(D_N(N,V,H)\) the space of \((H,N)\) structures defined on \(N\). To each element \([h_N]\) of \(D_N(N,V,H)\), we can associate the following flat bundle \(V_{h_N}\): let \((U_i,\phi_i)\) be the atlas which defines the \((V,H)\)-structure of \(N\). The total space of \(V_{h_N}\) is the quotient of \(\hat{N} \times V\) by the action of \(\pi_1(N)\) defined on the first factor by the Deck transformations, and on the second factor by the holonomy \(h_N\) of the \((V,H)\) structure.

The following result is shown in Goldman:

**Theorem.**

Let \(N\) be a compact manifold endowed with a \((V,H)\) structure, then the map \(D(N,V,H) \to \text{Rep}(\pi_1(N),H)\) which assigns to each \((V,H)\) structure its holonomy representation is an open map, two nearby \((V,H)\) structures with the same holonomy are isotopic.

Suppose now that the \((V,H)\)-manifold \(N\) is endowed with a symplectic form \(\omega\), and there exists a configuration \(D\) of \(N\) such that \(N - D\) is symplectomorphic to the standard ball, or the standard polydisc. The idea that will use is to study the deformation space of \(D\) under the action of the group of symplectomorphisms, using the deformation theory of locally homogeneous structures developed in Goldman. Let \(\phi\) be a symplectomorphism of \((N,\omega)\) the symplectomorphism \(\phi\) acts on \(D(N,V,H)\) as follows: let \((U_i,\phi_i)\) be an atlas which defines the \((V,H)\)-structure of \(N\). The total space of \(V_{h_N}\) is the quotient of \(\hat{N} \times V\) by the action of \(\pi_1(N)\) defined on the first factor by the Deck transformations, and on the second factor by the holonomy \(h_N\) of the \((V,H)\) structure. The following result is shown in Goldman:

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Suppose that \((V, H) = (\mathbb{R}^n, \text{Aff}(\mathbb{R}^n))\), in this situation, the bundle \(V_{h,N}\) is the affine vector bundle defined on \(N\), it is associated to a principal bundle defined on \(N\), which is the bundle of affine frames. Recall that an affine manifold is also defined by a connection whose curvature and torsion forms vanish identically, this connection induces on the bundle of affine frames a connection à la Erhesmann, the horizontal distribution which defines the Erhesmann connection induced the horizontal foliation of \(h_{h,N}\).

**The action of \(\text{Symp}(N)_0\) on \(V_{h,N}\).**

We are going to define an action of the connected component of the universal cover \(\text{Symp}(N)_0\) of the group of symplectomorphisms \(\text{Symp}(N)\) of \((N, \omega)\) on \(V_{h,N}\). We suppose that our \((V, H)\) structure is complete, or more generally, we suppose that the developing map is a covering map, this is equivalent to saying that \(N\) is the quotient of an open subset \(U_N\) of \(V\) by a discrete subgroup of \(H\).

We define \(V^1_{h,N}\) to be the \(U_N\)-bundle defined on \(N\), by making the quotient of \(\hat{N} \times U_N\) by the action of \(\pi_1(N)\) defined on the first factor by the deck transformations, and on the second factor by the action of the holonomy.

Let \(h'\) be an element of \(\text{Symp}(N)_0\), we can lift \(h'\) to an element \(h''\) of \(U_N\), since \(h'\) is an element of \(\text{Symp}(N)_0\), \(h''\) commutes with the action of \(\pi_1(N)\) on \(U_N\). We denote by \(\text{Symp}^1(N)_0\), the group of symplectomorphisms of \((U_N, \omega)\) (where \(\omega\) is the symplectic form of \(U_N\) lifted to \(U_N\) by the covering map \(\pi' : U_N \rightarrow N\) which commutes with the holonomy: this group is a cover of \(\text{Symp}(N)_0\). Since the element \(h''\) commutes with \(\pi_1(N)\), it induces a gauge transformation on \(V^1_{h,N}\) which covers the identity of \(N\). The covering map \(\text{Symp}(N)_0 \rightarrow \text{Symp}^1(N)_0\) induces an action of \(\text{Symp}(N)_0\) on \(V^1_{h,N}\).

To study the action of \(\text{Symp}(N)\) on the configuration \(D\), we make the following assumption: let \(N_U\) be a fundamental domain of the action of \(\pi_1(N)\) on \(U_N\), we denote by \(D\) the boundary of \(U_N\), we suppose that the image of \(D\) by \(\pi'\) is \(D\). We obtain the following proposition:

**Proposition.** The group \(\text{Symp}(N)_0\) acts naturally on \(V^1_{h,N}\). The stabilizer of the image of \(\hat{N} \times N_U - D\) by the covering map \(\rho^\circ : \hat{N} \times U_N \rightarrow V_{h,N}\) is \(\text{Symp}(N, N_U)_0\), the connected component of \(\text{Symp}(N)_0\) which fixes \(N_U\).

We suppose that the manifold \(V\) is endowed with a symplectic form \(\omega_V\) such that the action of \(H\) on \(V\) is symplectic, and the maps \(\phi_i : U_i \rightarrow V\) which define the \((V, H)\)-structure of \(N\) are symplectic maps. This enables to define on \(V^1_{h,N}\) the following symplectic structure: on the trivialization \(U_i \times V\), we define the form \(\Omega_i\) to be the product of the restriction of \(\omega\) to \(U_i\) with \(-\omega_V\). Since the action of \(H\) on \(V\) is symplectic, we deduce that the forms \(\Omega_i\) glue together to define a form \(\Omega\) on \(V^1_{h,N}\). Since we can assume that the section \(l_{h,N}\) takes it values in \(V^1_{h,N}\), we have the following proposition:

**Proposition** The section \(l_{h,N}\) which defines the \((V, H)\)-structure is a Lagrangian submanifold. The orbits of \(l_{h,N}\) under \(\text{Symp}(N)_0\) are also Lagrangian submanifolds, conversely every Lagrangian submanifold in the connected com-
ponent of \( l_{hN} \) in the space of Lagrangian submanifolds transverse to the vertical foliation of \( V^1_{hN} \) is the image of \( l_{hN} \) by an element of \( \text{Symp}(N)_0 \).

**proof.**

The section \( l_{hN} \) is defined on \( U_i \) by \( l_{hN}^i(u) = (u, \phi_i(u)) \). Since the maps \( \phi_i \) are symplectic maps, we deduce that the section \( l_{hN}^i \) is a Lagrangian submanifold as is the section \( \phi(l_{hN}^i) \) defined by \( \phi(l_{hN}^i)(u) = (u, \phi \circ \phi_i(u)) \).

Let \( d' \) be a Lagrangian submanifolds of \( V^1_{hN} \) transverse to the vertical foliation of \( V^1_{hN} \), \( d' \) is the image of the section \( d \) of \( V^1_{hN} \). Consider a trivialization \( (U_i \times \phi_i(U_i), \phi_j \circ \phi_i^{-1}) \) of \( V^1_{hN} \), we define the symplectomorphism \( \phi_d \) of \( N \) whose restriction \( \phi_d^U \) to \( U_i \) is defined for every element \( u \) of \( U_i \) by the projection of \( d_i(u) \) by the covering map \( p^N : U_N \to N \), where \( d_i \) is the restriction of \( d \) to \( U_i \). The morphism \( \phi_d \) is well-defined indeed, suppose that \( u \in U_i \cap U_j \), then \( d_j(u) = \phi_j \circ \phi_i^{-1} d_i(u) \). This implies that \( \phi^U_d(u) = \phi^U_d(u) \). The section \( d \) is the image of \( l_{hN} \) by an element \( \phi_d \) of \( \text{Symp}(N) \) above \( \phi_d \). If \( d' \) is in the connected component of \( l_{hN} \) in the space of Lagrangian submanifolds transverse to the vertical foliation of \( V^1_{hN} \), we can suppose that \( \phi_d \) is an element of \( \text{Symp}(N)_0 \).

Suppose that there exists a pseudo-complex structure \( J \) adapted to the symplectic form of \( V^1_{hN} \) such that the induced differentiable metric is complete, \( J \) enables to define an isomorphism of bundle between the cotangent bundle of \( l_{hN} \), \( T^*l_{hN} \) and \( Nl_{hN} \) the orthogonal of the tangent bundle \( Tl_{hN} \) of \( l_{hN} \) in \( V^1_{hN} \). We consider the differentiable map \( P_N : T^*l_{hN} \to V^1_{hN} \) defined as follows: identify \( T^*l_{hN} \) with the normal bundle of \( l_{hN} \) in \( V^1_{hN} \) by the map which assigns to an element \( n_u \) of the fiber of the normal bundle at \( u \) the 1-form \( i_{n_u} \Omega \). Let \( v_u \) be an element of the fiber of \( u \) in \( T^*l_{hN} \), \( P_N(v_u) = \exp_u(v_u) \), where the exponential map is defined by the differentiable metric \( \Omega(J,.) \).

**Theorem.**

Suppose that \( P_N \) is a symplectomorphism which induces a one to one map between the Lagrangian submanifolds of \( T^*l_{hN} \) transverse to the vertical foliation of \( T^*l_{hN} \), and the Lagrangian submanifolds of \( V^1_{hN} \) transverse to the vertical foliation of \( V^1_{hN} \), then the orbit of \( l_{hN} \) under \( \text{Symp}(N)_0 \) is contractible.

**proof.**

Consider the cotangent bundle \( T^*l_{hN} \) \( l_{hN} \) endowed with the differential of the Liouville form. The map \( P_N \) is a symplectomorphism which send Lagrangian submanifolds of \( T^*l_{hN} \) transverse to the vertical foliation to Lagrangian submanifolds of \( V^1_{hN} \) transverse to the vertical foliation. Since the Lagrangian submanifolds of \( T^*l_{hN} \) transverse to the vertical foliation are one to one with closed 1-forms defined on \( l_{hN} \), we deduce that the orbit of \( l_{hN} \) under \( \text{Symp}(N)_0 \) is contractible.

**Symplectomorphisms group of symplectic affine manifolds.**

Let \((N, \nabla_N)\) be a complete compact \( n \)-dimensional affine manifold endowed with the parallel symplectic form \( \omega \). The bundle \( V_{hN} \) is the quotient of \( IR^n \times
\( \mathbb{R}^n \) by the action of \( \pi_1(N) \) which acts on the both factors by the holonomy representation. The principal bundle associated to \( V_{hN} \) is the bundle of affine frames.

**Theorem** Let \( N \) be an \( n \)-dimensional compact affine manifold, suppose that the affine structure of \( N \) is defined by a flat differentiable metric, and \( N \) is endowed with a symplectic form parallel in respect to the flat connection, then if \( \text{Ham}(N,D)_0 \) is contractible, then \( \text{Ham}(N)_0 \) is also contractible.

**Proof.**

The manifold \( N \) is the quotient of \( \mathbb{R}^n \) by a subgroup \( \pi_1(N) \) whose linear part is contained in \( O(n) \). The parallel symplectic form \( \omega_0 \) is invariant by the holonomy and gives rise to the form \( \omega \) of \( N \). The parallel complex structure \( J_0 \) adapted to the flat metric of \( \mathbb{R}^n \), and to \( \omega_0 \) gives rise to the pseudo-complex structure \( J \) of \( (N,\omega) \). In this situation, \( P_N : T^*\mathfrak{l}_{hN} \rightarrow V_{hN} \) is a symplectomorphism.

Let \( \alpha \) be a closed 1-form defined on \( N \), suppose that \( P_N(\alpha) \) is transverse to the vertical foliation, then it defines a symplectomorphism \( \hat{\phi}_\alpha \) of \( V_{hN} \) which gives rise to a symplectomorphism \( \phi_\alpha \) of \( N \), we can define \( \phi_\alpha(D) \) the image of \( D \) by \( \phi_\alpha \). Suppose that \( P_N(\alpha) \) is not transverse to the vertical foliation, then the infinite dimensional Sard lemma implies the existence of a sequence of closed 1-forms \( (\alpha_n)_{n \in \mathbb{N}} \), such that \( \alpha_n \) is transverse to the vertical foliation, and \( \alpha_n \) converges towards \( \alpha \). We define \( \phi_\alpha(D) \) to be the limit of \( \phi_{\alpha_n}(D) \). This make sense because the group of symplectomorphisms is \( C^0 \)-closed in the group of diffeomorphisms.

Let \( N'(D) \) be the image of the space of closed 1-forms \( C^1(\alpha) \) by the map \( \Phi \) defined by \( \Phi(\alpha) = \phi_\alpha(D) \), \( N'(D) \) contains an open and dense subset of \( N(D) \). Indeed, consider the subspace of image of closed 1-forms \( \alpha \) such that \( \alpha \) is transverse to the vertical foliation. This image is open, since the neighborhood of a symplectomorphism can be identified with a neighborhood of the zero section in the cotangent bundle, and the image of the set \( N''(D) \) of closed 1-forms such that the image of their graph by \( P_N \) are transverse to the vertical foliation is dense in the space of graphs of symplectomorphisms this implies also that \( N'(D) \) is closed in \( N(D) \), thus \( N'(D) = N(D) \).

The space of closed 1-forms \( C^1(N) \) which induces symplectomorphisms which fix \( D \) is a vector subspace since the components of \( D \) are affine submanifolds. To see this we can use a generating functions-like theory. Indeed, let \( \alpha \) be a closed 1-form, the image of the lift of the graph to the universal cover of \( N \) by \( P_N \) is the space of elements of \( \mathbb{R}^n \times \mathbb{R}^n \) of the form \((u_1 + \alpha_1, ..., u_n + \alpha_n, u_1 - \alpha_1, ..., u_n - \alpha_n)\) where \((\alpha_1, ..., \alpha_n)\) are the component of the lift \( \hat{\alpha} \) of \( \alpha \) to \( \mathbb{R}^n \). The fact that \( \hat{\phi}_\alpha \) preserves \( D \) implies that if \((u_1 + \alpha_1, ..., u_n + \alpha_n)\) is an element of the subspace \( \tilde{D} \) of \( \mathbb{R}^n \) over \( D \), then \((u_1 - \alpha_1, ..., u_n - \alpha_n)\) is also an element of \( \tilde{D} \). Let \( \alpha, \alpha' \) be closed 1-forms which define symplectomorphisms of \( N \) which preserves \( D \), then if \( U \) is a connected component of \( \tilde{D} \) in \( \mathbb{R}^n \), then the element \((u_1 - \alpha_1, ..., u_n - \alpha_n)\) is tangent to \( U \) if \((u_1 + \alpha_1, ..., u_n + \alpha_n)\) in this situation, \((u_1, ..., u_n)\) and \((\alpha_1(u), ..., \alpha_n(u))\) are tangent to \( U \). This implies that \((\alpha_1 + \alpha'_1, ..., \alpha_n + \alpha'_n)\) is tangent to \( U \), and henceforth that if \( \alpha + \alpha' \) induces a sym-
plectomorphism then the fact that $(u_1-(\alpha_1+\alpha_1'),\ldots,u_n-(\alpha_n+\alpha_n'))$ is an element of $U$, if $(u_1+\alpha_1+\alpha_1',\ldots,u_n+\alpha_n+\alpha_n')$ is an element of $U$ implies that this symplectomorphism preserves $D$. We have a fibration $C^1(N) \to C^1(N) \to N(D)$. Since $C^1(N)$ and $C^1(N)$ are contractible, we deduce that $N(D)$ is contractible.

**Corollary.**

The universal cover $\hat{\text{Symp}}(N)$ of the connected component of the group of symplectomorphisms $\text{Symp}(N)$ of a 4-dimensional compact manifold endowed with a flat metric, and a parallel symplectic structure $\omega$ contractible.

**Proof.** $N$ is the quotient of $\mathbb{R}^4$ by $\pi_1(N)$, the boundary $D$ of this action is a configuration. We have a fibration $\text{Symp}(N,D)_0 \to \text{Symp}(N)_0 \to N(D)$, since $N(D)$ is contractible, we deduce that $\text{Symp}(N)_0$ has the homotopy type of $\text{Symp}(N,D)_0$. We have an exact sequence $\text{Ham}(N,D) \to \hat{\text{Symp}}(N,D) \to L$, where $L$ is the image of $\hat{\text{Symp}}(N,D)$ by the flux. Since $\text{Ham}(N,D)$ is contractible, we deduce that $\text{Symp}(N,D)$ is contractible.

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