Limits in $n$-categories

Carlos Simpson
CNRS, UMR 5580, Université Paul Sabatier, 31062 Toulouse CEDEX, France.

1. Introduction

One of the main notions in category theory is the notion of limit. Similarly, one of the most commonly used techniques in homotopy theory is the notion of “homotopy limit” commonly called “holim” for short. The purpose of the this paper is to begin to develop the notion of limit for $n$-categories, which should be a bridge between the categorical notion of limit and the homotopical notion of holim.

We treat Tamsamani’s notion of $n$-category [36], but similar arguments and results should hold for the Baez-Dolan approach [3], [5], or the Batanin approach [6], [7].

We define the notions of direct and inverse limits in an arbitrary (fibrant cf [32]) $n$-category $C$. Suppose $A$ is an $n$-category, and suppose $\varphi : A \to C$ is a morphism, which we think of as a family of objects of $C$ indexed by $A$. For any object $U \in C$ we can define the $(n-1)$-category $\operatorname{Hom}(\varphi, U)$ of morphisms from $\varphi$ to $U$. We say that a morphism $\epsilon : \varphi \to U$ (i.e. an object of this $(n-1)$-category) is a direct limit of $\varphi$ (cf 3.2.1 below) if, for every other object $V \in C$ the (weakly defined) composition with $\epsilon$ induces an equivalence of $(n-1)$-categories from $\operatorname{Hom}(U, V)$ to $\operatorname{Hom}(\varphi, V)$.

An analogous definition holds for saying that a morphism $U \to \varphi$ is an inverse limit of $\varphi$ (cf 3.1.1 below).

The main theorems concern the case where $C$ is the $n+1$-category $n\text{CAT}'$ (fibrant replacement of that of) of $n$-categories.

Theorem (4.0.1 5.0.1) The $n+1$-category $n\text{CAT}'$ admits arbitrary inverse and direct limits.

This is the analogue of the classical statement that the category $\text{Sets}$ admits inverse and direct limits—which is the case $n = 0$ of our theorem.

The fact that we work in an $n$-category means that we automatically keep track of “higher homotopies” and the like. This brings the ideas much closer to the relatively simple notion of limits in a category.

I first learned of the notion of “2-limit” from the paper of Deligne and Mumford [13], where it appears at the beginning with very little explanation. Unfortunately at the writing of the present paper I have not been able to investigate the history of the notion of $n$-limits, and I apologize in advance for any references left out.
At the end of the paper we propose many applications of the notion of limit. Most of these are as of yet in an embryonic stage of development and we don’t pretend to give complete proofs.

\textit{Organization}

The paper is organized as follows: we start in §2 with some preliminary remarks recalling the notion of \textit{n}-category from \cite{36} and the closed model structure from \cite{32}. At the end of §2 we define and discuss one of our main technical tools, the \textit{n}+1-precat \( \Upsilon^k(E_1, \ldots, E_k) \) which can be seen as a \textit{k}-simplex with \textit{n}-precats \( E_1, \ldots, E_k \) attached to the principal edges.

In §3 we give the basic definitions of inverse and direct limits, and treat some general properties such as invariance under equivalence, and variation with parameters.

In §4 we start into the main result of the paper which is the existence of inverse limits in \( nCAT' \). Here, the construction is relatively straightforward: if \( \varphi : A \to nCAT' \) is a morphism then the inverse limit of \( \varphi \) is just the \textit{n}-category

\[
\lambda = \text{Hom}_{\text{Hom}(A, C)}(*, \varphi).
\]

This is in perfect accord with the usual situation for inverse limits of families of sets. Our only problem is to prove that this satisfies the definition of being an inverse limit. Thus the reader could read up to here and then skip the proof and move on to direct limits.

We treat direct limits in \( nCAT' \) by a trick in §5: given \( \psi : A \to nCAT' \) we construct an \textit{n}+1-category \( D \) parametrizing all morphisms \( \psi \to B \) to objects of \( nCAT' \), and then construct the direct limit \( U \) as the inverse limit of the functor \( D \to nCAT' \). The main problem here is that, because of set-theoretic considerations, we must restrict to a category \( D_\alpha \) of morphisms to objects \( B \) with cardinality bounded by \( \alpha \). We mimic a possible construction of direct limits in \textit{Sets} and encounter a few of the same difficulties as with inverse limits. Again, the reader might want to just look at the proof for \textit{Sets} and skip the difficulties encountered in extending this to \( nCAT' \).

At the end we discuss some proposed applications:
—First, the notions of homotopy coproduct and fiber product, and their relation to the usual notions which can be calculated using the closed model structure.
—Then we discuss representable functors, give a conjectural criterion for when a functor should be representable, and apply it to the problem of finding internal \textit{Hom}.
—The next subsection concerns \textit{n}-stacks, defined using certain inverse limits.
—We give a very general discussion of the notion of stack in any setting where one knows what limits mean.
—We discuss direct images of families of \textit{n}-categories by functors of the underlying \textit{n}+1-categories, and apply this to give a notion of “realization”.

2
Finally, we use limits to propose a notion of relative Malcev completion of the higher homotopy type.

In all of the above applications except the first, most of the statements which we need are left as conjectures. Thus, this discussion of applications is still only at a highly speculative stage. One recurring theme is that the argument given in §5 should work in a fairly general range of situations.

I would like to thank A. Hirschowitz, for numerous discussions about stacks which contributed to the development of the ideas in this paper. I would like to thank J. Tapia and J. Pradines for a helpful discussion concerning the argument in §5.

2. Preliminary remarks

2.1 $n$-categories

2.1.1 We begin by recalling the correspondence between categories and their nerves. Let $\Delta$ denote the simplicial category whose objects are finite ordered sets $p = \{0, \ldots, p\}$ and morphisms are order-preserving maps. If $C$ is a category then its nerve is the simplicial set (i.e. a functor $A : \Delta^p \to Sets$) defined by setting $A_p$ equal to the set of composable $p$-uples of arrows in $C$. This satisfies the property that the “Segal maps” (cf the discussion of Segal’s delooping machine [27] in [1] for the origin of this terminology)

$$A_p \to A_1 \times_{A_0} \cdots \times_{A_0} A_1$$

are isomorphisms. To be precise this map is given by the $p$-uple of face maps $1 \to p$ which take 0 to $i$ and 1 to $i + 1$ for $i = 0, \ldots, p - 1$. Conversely, given a simplicial set $A$ such that the Segal maps are isomorphisms we obtain a category $C$ by taking

$$\text{Ob}(C) := A_0$$

and

$$\text{Hom}_C(x, y) := A_1(x, y)$$

with the latter defined as the inverse image of $(x, y)$ under the map (given by the pair of face maps) $A_1 \to A_0 \times A_0$. The condition on the Segal maps implies that (with a similar notation)

$$A_2(x, y, z) \to A_1(x, y) \times A_1(y, z)$$

and the third face map $A_2(x, y, z) \to A_1(x, z)$ thus provides the composition of morphisms for $C$. By looking at $A_3(x, y, z, w)$ one sees that the composition is associative and the degeneracy maps in the simplicial set provide the identity elements.
2.1.2 The notion of weak \( n \)-category of Tamsamani \[36\] is a generalization of the above point of view on categories. We present the definition in a highly recursive way, using the notion of \( n-1 \)-category in the definition of \( n \)-category. See \[36\] for a more direct approach. This definition is based on Segal’s delooping machine \[27\] \[1\].

2.1.3 Note that Tamsamani uses the terminology \( n \)-nerve for what we will call “\( n \)-category” since he needed to distinguish this from the notion of strict \( n \)-category. In the present paper we will never speak of strict \( n \)-categories and our terminology “\( n \)-category” means weak \( n \)-category or \( n \)-nerve in the sense of \[36\].

2.1.4 An \( n \)-category \[36\] is a functor \( A \) from \( \Delta^o \) to the category of \( n-1 \)-categories denoted
\[
p \mapsto A_{p/}
\]
such that 0 is mapped to a set \( A_0 \) and such that the Segal maps
\[
A_{p/} \to A_{1/} \times_{A_0} \cdots \times_{A_0} A_{1/}
\]
are equivalences of \( n-1 \)-categories (cf 2.1.8 below).

2.1.5 The category of \( n \)-categories denoted \( n-Cat \) is just the category whose objects are as above and whose morphisms are the morphisms strictly preserving the structure. It is a subcategory of \( Hom(\Delta^o, (n-1)-Cat) \). Working this out inductively we find in the end that \( n-Cat \) is a subcategory of \( Hom((\Delta^n)^o, Sets) \), in other words an \( n \)-category is a certain kind of multisimplicial set. The multisimplicial set is denoted
\[
(p_1, \ldots, p_n) \mapsto A_{p_1, \ldots, p_n}
\]
and the \( (n-1) \)-category \( A_{p/} \) itself considered as a multisimplicial set has the expression
\[
A_{p/} = \left( (q_1, \ldots, q_{n-1}) \mapsto A_{p,q_1,\ldots,q_{n-1}} \right).
\]

2.1.6 The condition that \( A_0 \) be a set yields by induction the condition that if \( p_i = 0 \) then the functor \( A_{p_1,\ldots,p_n} \) is independent of the \( p_{i+1}, \ldots, p_n \). We call this the constancy condition. In \[32\] we introduce the category \( \Theta^n \) which is the quotient of \( \Delta^n \) having the property that functors \( (\Theta^n)^o \to Sets \) correspond to functors on \( \Delta^n \) having the above constancy property. Now \( n-Cat \) is a subcategory of the category of presheaves of sets on \( \Theta^n \).

\(^1\) Recursively an \( n \)-category which is a set is a constant functor where the \( A_{p/} \) are all the same set—considered as \( n-1 \)-categories.
2.1.7 Before discussing the notion of equivalence which enters into the above definition we take note of the relationship with 2.1.1. If $A$ is an $n$-category then its set of objects is the set $A_0$. The face maps give a morphism from $n-1$-categories to sets

$$A_p/ \rightarrow A_0 \times \ldots \times A_0$$

and we denote by $A_p/(x_0, \ldots, x_p)$ the $n-1$-category inverse image of $(x_0, \ldots, x_p)$ under this map. For two objects $x, y \in A_0$ the $n-1$-category $A_1/(x, y)$ is the $n-1$-category of morphisms from $x$ to $y$. This is the essential part of the structure which corresponds, in the case of categories, to the Hom sets. One could adopt the notation

$$\text{Hom}_A(x, y) := A_1/(x, y).$$

The condition that the Segal maps are equivalences of $n-1$-categories says that the $A_p/(x_0, \ldots, x_p)$ are determined up to equivalence by the $A_1/(x, y)$. The role of the higher $A_p/(x_0, \ldots, x_p)$ is to provide the composition (in the case $p = 2$) and to keep track of the higher homotopies of associativity ($p \geq 3$). Contrary to the case of 1-categories, here we need to go beyond $p = 3$.

2.1.8 In order for the recursive definition of $n$-category given in 2.1.4 to make sense, we need to know what an equivalence of $n$-categories is. For this we generalize the usual notion for categories: an equivalence of categories is a morphism which is (1) fully faithful and (2) essentially surjective. We would like to define what it means for a functor between $n$-categories $f : A \rightarrow B$ to be an equivalence. The generalization of the fully faithful condition is immediate: we require that for any objects $x, y \in A_0$ the morphism

$$f : A_1/(x, y) \rightarrow B_1/(f(x), f(y))$$

be an equivalence of $n-1$-categories (and we are supposed to know what that means by recurrence).

2.1.9 The remaining question is how to define the notion of essential surjectivity. Tamasamani does this by defining a truncation operation $T$ from $n$-categories to $n-1$-categories (a generalization of the truncation of topological spaces used in the Postnikov tower). Applying this $n$ times to an $n$-category $A$ we obtain a set $T^n A$ which can also be denoted $\tau_{\leq 0} A$. This set is the set of “objects of $A$ up to equivalence” where equivalence of objects is thought of in the $n$-categorical sense. We say that $f : A \rightarrow B$ is essentially surjective if the induced map

$$\tau_{\leq 0}(f) : \tau_{\leq 0} A \rightarrow \tau_{\leq 0} B$$

is a surjection of sets. One has in fact that if $f$ is an equivalence according to the above definition then $\tau_{\leq 0} f$ is an isomorphism.

\footnote{One might conjecture that it suffices to stop at $p = n + 2.$}
2.1.10 Another way to approach the definition of $\tau_{\leq 0} A$ is by induction in the following way. Suppose we know what $\tau_{\leq 0}$ means for $n-1$-categories. Then for an $n$-category $A$ the simplicial set $p \mapsto \tau_{\leq 0}(A_p)$ satisfies the condition that the Segal maps are isomorphisms, so it is the nerve of a 1-category. This category may be denoted $\tau_{\leq 1} A$. We then define $\tau_{\leq 0} A$ to be the set of isomorphism classes of objects in the 1-category $\tau_{\leq 1} A$.

The above definition is highly recursive. One must check that everything is well defined and available when it is needed. This is done in [36] although the approach there avoids some of the inductive definitions above.

2.2 The closed model structure

An $n$-category is a presheaf of sets on $\Theta^n$ (2.1.6) satisfying certain conditions as described above. Unfortunately $n-Cat$ considered as a subcategory of the category of presheaves, is not closed under pushout or fiber product. This remark is the starting point for [32]. There, one considers the full category of presheaves of sets on $\Theta$ (these presheaves are called $n$-precats) and [32] provides a closed model structure (cf [25] [26] [19]) on the category $nPC$ of $n$-precats, corresponding to the homotopy theory of $n$-categories. In this section we briefly recall how this works.

2.2.1 It is more convenient for the purposes of the closed model structure to work with presheaves over the category $\Theta^n$ (cf 2.1.6 above), defined by the quotient of the cartesian product $\Delta^n$ obtained by identifying all of the objects $(M,0,M')$ for fixed $M = (m_1,\ldots,m_k)$ and variable $M' = (m'_1,\ldots,m'_{n-k-1})$. The object of $\Theta^n$ corresponding to the class of $(M,0,M')$ with all $m_i > 0$ will be denoted $M$. Two morphisms from $M$ to $M'$ in $\Delta^n$ are identified if they both factor through something of the form $(u_1,\ldots,u_i,0,u_{i+2},\ldots,u_n)$ and if their first $i$ components are the same.

2.2.2 An $n$-precat is defined to be a presheaf on the category $\Theta^n$. This corresponds to an $n$-simplicial set $(\Delta^n)^\circ \to Sets$ which satisfies the constancy condition (cf 2.1.6). The category $nPC$ of $n$-precats (with morphisms being the morphisms of presheaves) is to be given a closed model structure.

2.2.3 Note for a start that $nPC$ is closed under arbitrary products and coproducts, what is more (and eventually important for our purposes) it admits an internal $\text{Hom}(A,B)$. These statements come simply from the fact that $nPC$ is a category of presheaves over something.

We denote the coproduct or pushout of $A \to B$ and $A \to C$ by $B \cup_A C$. We denote fiber products by the usual notation.
2.2.4 Cofibrations: A morphism \( A \to B \) of \( n \)-precats is a cofibration if the morphisms \( A_M \to B_M \) are injective whenever \( M \in \Theta^n \) is an object of non-maximal length, i.e. \( M = (m_1, \ldots, m_k, 0, \ldots, 0) \) for \( k < n \). The case of sets \((n = 0)\) shows that we can’t require injectivity at the top level \( n \), nor do we need to.

We often use the notation \( A \hookrightarrow B \) for a cofibration, not meaning to imply injectivity at the top level.

2.2.5 Weak equivalences: In order to say when a morphism \( A \to B \) of \( n \)-precats is a “weak equivalence” we have to do some work. In \([36]\) was defined the notion of equivalence between \( n \)-categories (cf 2.1.8 above), but an \( n \)-precat is not yet an \( n \)-category. We need an operation which specifies the intended relationship between our \( n \)-precats and \( n \)-categories. This is the operation \( A \mapsto \text{Cat}(A) \) which to any \( n \)-precat associates an \( n \)-category together with morphism of precats \( A \to \text{Cat}(A) \), basically by throwing onto \( A \) in a minimal way all of the elements which are needed in order to satisfy the definition of being an \( n \)-category. See \([32]\) §2 for the details of this. Now we say that a morphism

\[
A \to B
\]

of \( n \)-precats is a weak equivalence if the induced morphism of \( n \)-categories

\[
\text{Cat}(A) \to \text{Cat}(B)
\]

is an equivalence as defined in \([36]\)—described in 2.1.8 and 2.1.9 above.

2.2.6 Trivial cofibrations: A morphism \( A \to B \) is said to be a trivial cofibration if it is a cofibration and a weak equivalence.

2.2.7 Fibrations: A morphism \( A \to B \) of \( n \)-precats is said to be a fibration if it satisfies the following lifting property: for every trivial cofibration \( E' \hookrightarrow E \) and every morphism \( E \to B \) provided with a lifting over \( E' \) to a morphism \( E' \to A \), there exists an extension of this to a lifting \( E \to A \).

An \( n \)-precat \( A \) is said to be fibrant if the canonical (unique) morphism \( A \to \ast \) to the constant presheaf with values one point, is a fibration.

A fibrant \( n \)-precat is, in particular, an \( n \)-category. This is because the elements which need to exist to give an \( n \)-category may be obtained as liftings of certain standard trivial cofibrations (those denoted \( \Sigma \to h \) in \([32]\)).
Theorem 2.2.8 ([32] Theorem 3.1) The category $nPC$ of $n$-precats with the above classes of cofibrations, weak equivalences and fibrations, is a closed model category.

The basic “yoga” of the situation is that when we want to look at coproducts, one of the morphisms should be a cofibration; when we want to look at fiber products, one of the morphisms should be a fibration; and when we want look at the space of morphisms from $A$ to $B$, the first object $A$ should be cofibrant (in our case all objects are cofibrant) and the second object $B$ should be fibrant.

2.2.9 We explain more precisely what information is contained in the above theorem, by explaining the axioms for a closed model category structure (CM1–CM5 of [26]). These are proved as such in [32].

CM1—This says that $nPC$ is closed under finite (and in our case, arbitrary) direct and inverse limits (2.2.3).

CM2—Given composable morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

if any two of $f$ or $g$ or $g \circ f$ are weak equivalences then the third is also a weak equivalence.

CM3—The classes of cofibrations, fibrations and weak equivalences are closed under retracts. We don’t explicitly use this condition (however it is the basis for the property 2.2.10 below).

CM4—This says that a pair of a cofibration $E' \to E$ and a fibration $A \to B$ have the lifting property (as in the definition of fibration 2.2.7) if either one of the morphisms is a weak equivalence. Note that the lifting property when $E' \to E$ is a weak equivalence (i.e. trivial cofibration) is just the definition that $A \to B$ be fibrant 2.2.7. The other half, the lifting property for an arbitrary cofibration when $A \to B$ is a weak equivalence, comes from what Jardine calls “Joyal’s trick” [19].

CM5—This says that any morphism $f$ may be factored as a composition $f = p \circ i$ of a cofibration followed by fibration, and either one of $p$ or $i$ may be assumed to be a weak equivalence.

2.2.10 Another axiom in Quillen’s original point of view (Axiom M3 on page 1.1 of [25]) is that if $A \to B$ is a trivial cofibration and $A \to C$ is any morphism then $C \to B \cup^A C$ is again a trivial cofibration; and similarly the dual condition for fibrant weak equivalences and fiber products. In the closed model category setting this becomes a consequence of the axioms CM1–CM5, see [20].

In the proof of [32] (modelled on that of [19]) the main step which is done first ([32] Lemma 3.2) is to prove this property of preservation of trivial cofibrations by coproducts. (On the other hand, note that with our definition 2.2.7 of fibrations, the preservation by fiber products is obvious).
2.2.11 We now try to put these properties in perspective in view of how we will use them. If $A$ is any $n$-precat then applying $CM5$ to the morphism $A \to *$ we obtain a factorization

$$A \to A' \to *$$

with the first morphism a trivial cofibration, and the second morphism a fibration. Thus $A'$ is a fibrant object. We call such a trivial cofibration to a fibrant object $A \hookrightarrow A'$ a fibrant replacement for $A$.

In the constructions of [19], [32] one obtains the fibrant replacement by adding onto $A$ the pushouts by “all possible” trivial cofibrations, making use of 2.2.10. The notion of “all possible” has to be refined in order to avoid set-theoretical problems: actually one looks at $\omega$-bounded cofibrations. The number of them is bounded by the maximum of $2^\omega$ or the cardinality of $A$.

When looking at morphisms into an $n$-category $C$ it is important that $C$ be fibrant, for then we obtain extension properties along trivial cofibrations. In particular, we will only define what it means for limits to exist in $n$-categories $C$ which are fibrant.

When we finally get to our definition of the $n+1$-category $nCAT$ below, it will not be fibrant. Thus one of the main steps is to choose a fibrant replacement $nCAT \hookrightarrow nCAT'$.

2.2.12 There is a nice “interval” in our closed model category (in contrast with the general situation envisioned by Quillen in [25]). Let $I$ denote the 1-category with two objects 0, 1 and with unique morphisms going in either direction between them, whose compositions are the identity. Without changing notation, we can consider $I$ as an $n$-category (pull back by the obvious morphism $\Theta^n \to \Theta^1 = \Delta$).

Claim: Suppose $C$ is a fibrant $n$-category. Then two objects $x, y \in C_0$ are equivalent (i.e. project to the same thing in $\tau_{\leq 0}C$ cf 2.1.10) if and only if there exists a morphism $I \to C$ sending 0 to $x$ and 1 to $y$.

To prove this note that one direction is obvious: if there exists such a morphism then by functoriality of $\tau_{\leq 0}$ $x$ and $y$ are equivalent (because $\tau_{\leq 0}(I) = *$). For the other direction, suppose $x$ and $y$ are equivalent. Use Proposition 6.5 of [32] which says that there is an $n$-category $K$ with objects 0, 1 such that $K \to *$ is a weak equivalence, and there is a morphism $K \to C$ sending 0 to $x$ and 1 to $y$. Applying the factorization statement $CM5$ to the morphism

$$K \cup \{0,1\} I \to *$$

we obtain a cofibration

$$K \cup \{0,1\} I \hookrightarrow A$$

such that $A \to *$ is a weak equivalence. It follows from $CM2$ applied to $K \to A \to *$ that $K \to A$ is a weak equivalence, thus it is a trivial cofibration. Now the fibrant property
of $C$ implies that our morphism $K \to C$ extends to a morphism $A \to C$. This morphism restricted to $T \hookrightarrow A$ provides a morphism $T \to C$ sending 0 to $x$ and 1 to $y$. This proves the other direction of the claimed statement.

2.2.13 As a corollary of the above construction, suppose $f : A \to B$ is a fibrant morphism of fibrant $n$-categories. Suppose that $a \in A_0$ and $b \in B_0$ are objects such that $f(a)$ is equivalent to $b$ (i.e. $f(a)$ is equal to $b$ in $\tau_{\leq 0}B$). Then there is a different object $a' \in A_0$ equivalent to $a$ such that $f(a') = b$ in $B_0$. To prove this, note that the equivalence between $f(a)$ and $b$ corresponds by 2.2.12 to a morphism $I \to B$ sending 0 to $f(a)$ and 1 to $b$. We have a lifting $a$ over $\{0\}$. The inclusion

$$\{0\} \subset T$$

is a trivial cofibration, so the fibrant property of $f$ means that there is a lifting to a morphism $T \to A$. The image of 1 by this map is an object $a'$ equivalent to $a$ and projecting to $b$.

A variant says that if $f : A \to B$ is a fibrant morphism between fibrant $n$-categories and if $f$ is an equivalence then $f$ is surjective on objects. To obtain this note that essential surjectivity of $f$ means that every object $b$ is equivalent to some $f(a)$, then apply the previous statement.

2.2.14 One of the main advantages to using a category of presheaves $nPC$ as underlying category is that we obtain an internal $\text{Hom}(A, B)$ between two $n$-precats. This represents a functor: a map

$$E \to \text{Hom}(A, B)$$

is the same thing as a morphism $A \times E \to B$.

Of course for arbitrary $A$ and $B$, the internal $\text{Hom}(A, B)$ will not have any reasonable properties, for example it will not transform equivalences of the $A$ or $B$ into equivalences. This situation is rectified by imposing the hypothesis that $B$ should be fibrant.

2.2.15 We describe some of the results saying that the internal $\text{Hom}(A, B)$ works nicely when $B$ is fibrant. The following paragraphs are Theorem 7.1 and Lemma 7.2 of [32].

Suppose $A$ is an $n$-precat and $B$ is a fibrant $n$-precat. Then the internal $\text{Hom}(A, B)$ of presheaves over $\Theta^n$ is a fibrant $n$-category. Furthermore if $B' \to B$ is a fibrant morphism then $\text{Hom}(A, B') \to \text{Hom}(A, B)$ is fibrant. Similarly if $A \hookrightarrow A'$ is a cofibration and if $B$ is fibrant then $\text{Hom}(A', B) \to \text{Hom}(A, B)$ is fibrant.
Suppose $A \to A'$ is a weak equivalence, and $B$ fibrant. Then

$$\text{Hom}(A', B) \to \text{Hom}(A, B)$$

is an equivalence of $n$-categories.

If $B \to B'$ is an equivalence of fibrant $n$-precats then $\text{Hom}(A, B) \to \text{Hom}(A, B')$ is an equivalence.

Suppose $A \to B$ and $A \to C$ are cofibrations. Then

$$\text{Hom}(B \cup^A C, D) = \text{Hom}(B, D) \times_{\text{Hom}(A, D)} \text{Hom}(C, D).$$

### 2.2.16

We can relate several different versions of the notion of two morphisms being homotopic. Suppose $A$ and $B$ are $n$-precats with $B$ fibrant. According to Quillen’s definition [25], two maps $f : A \to B$ are homotopic if there is a diagram

$$A \xrightarrow{f} A' \to A$$

such that all morphisms are weak equivalences, the first two morphisms are cofibrations, and such that the compositions are the identity of $A$, plus a morphism $A' \to B$ inducing $f$ and $g$ on the two copies of $A$.

In our situation, if $B$ is fibrant then $\text{Hom}(A, B)$ is a fibrant $n$-category whose objects are the morphisms $A \to B$. Two morphisms are equivalent objects in this $n$-category (cf. 2.1.9 above) if and only if they are homotopic in Quillen’s sense (this is [32] Lemma 7.3).

### 2.2.17

In the above situation apply the claim of 2.2.12. Two objects $f, g$ of the fibrant $n$-category $\text{Hom}(A, B)$ are equivalent if and only if there is a morphism

$$\mathcal{T} \to \text{Hom}(A, B)$$

sending 0 to $f$ and 1 to $g$. Such a morphism corresponds to a map

$$A \times \mathcal{T} \to B;$$

so we can finish up by saying that two morphisms $f, g : A \to B$ are homotopic if and only if there exists a map

$$A \times \mathcal{T} \to B$$

restricting to $f$ on $A \times \{0\}$ and to $g$ on $A \times \{1\}$.
2.2.18 We obtain from CM4 the following characterization of fibrant weak equivalences. A morphism $f : A \to B$ is a fibrant weak equivalence if and only if it satisfies the lifting property for any cofibration $E' \hookrightarrow E$. To prove this, note that CM4 shows that a fibrant weak equivalence has this property. If $f$ has this property then it is fibrant (the case of $E' \hookrightarrow E$ a trivial cofibration). The morphisms of $n-1$-categories

$$A_1/(x, y) \to B_1/(f(x), f(y))$$

also have the same property (one can see this using the construction $\Upsilon(E)$ below) and $f$ is surjective on objects (by the case $\emptyset \hookrightarrow \ast$). Therefore $f$ is an equivalence.

We can give the following variant characterizing when a morphism is an equivalence (not necessarily fibrant). We say that a morphism $f : A \to B$ between fibrant $n$-categories has the **homotopical lifting property** for $E' \hookrightarrow E$ if, given a morphism $v : E \to B$ and a lifting $u' : E' \to A$, there is a homotopy from $v$ to a new morphism $v_1$, a lifting $u_1$ of $v_1$, and a homotopy from $u'$ to $u'_1$ (the restriction of $u_1$ to $E'$) lifting the homotopy from $v'$ to $v'_1$ (restriction of our first homotopy to $E'$). In this definition we can use any of the equivalent notions of homotopy 2.2.16, 2.2.17 above.

**Claim:** A morphism $f : A \to B$ between two fibrant $n$-categories is an equivalence if and only if it satisfies the homotopical lifting property for all $E' \hookrightarrow E$.

To prove this, use CM5 to factor $A \to A' \to B$ with the first morphism a trivial cofibration and the second morphism fibrant. Note that $A'$ is again fibrant. The statement being a homotopical one, the same hypothesis holds for $A' \to B$. If we can prove that $A' \to B$ is an equivalence then the composition with the trivial cofibration $A \to A'$ will be a weak equivalence. Thus we may reduce to the case where $A \to B$ is a fibrant morphism. Now given $E' \hookrightarrow E$ with $E \to B$ lifting to $E' \to A$, choose homotopies

$$E \times I \to B$$

and lifting

$$E' \times I \to A$$

compatible with a lifting $E \times \{1\} \to A$ as in the definition of the homotopical lifting property. These give a lifting

$$E' \times I \cup^{E' \times \{1\}} E \times \{1\} \to A,$$

and the morphism

$$E' \times I \cup^{E' \times \{1\}} E \times \{1\} \to E \times I$$

is a trivial cofibration, so by the fibrant property of $A \to B$ (which we are now assuming) there is a lifting

$$E \times I \to A.$$
The restriction to \( E \times \{0\} \) gives the desired lifting of the original morphism \( E \to B \), coinciding with the given lifting on \( E' \). This proves that \( A \to B \) satisfies the lifting criterion given above so it is a fibrant weak equivalence. This completes the proof of one direction of the claim. A similar argument (using CM2) gives the other direction.

2.3 Families of \( n \)-categories

2.3.1 Using the internal \( \text{Hom}(A, B) \) of 3.2.3 between fibrant \( n \)-categories, we define the \( n+1 \)-category \( n\text{CAT} \) of all fibrant \( n \)-categories (cf [24] §7). This is the “right” category of \( n \)-categories, and is not to be confused with the first approximation \( n-Cat \) as defined in 2.1.5 above.

The objects of \( n\text{CAT} \) are the fibrant \( n \)-categories. Between any two objects we have an \( n \)-category of morphisms \( \text{Hom}(A, B) \). Composition of morphisms gives a morphism of \( n \)-categories

\[
\text{Hom}(A, B) \times \text{Hom}(B, C) \to \text{Hom}(A, C),
\]

which is strictly associative and has a unit element, the identity morphism. Using this we obtain an \( n+1 \)-category \( n\text{CAT} \): to be precise, if \( A_0, \ldots, A_p \) are objects then

\[
n\text{CAT}_p/(A_0, \ldots, A_p) := \text{Hom}(A_0, A_1) \times \ldots \times \text{Hom}(A_{p-1}, A_p)
\]

which organizes into a simplicial collection using the projections or, where necessary, the composition morphisms. The Segal maps are actually isomorphisms here so this is an \( n+1 \)-category.

2.3.2 Unfortunately, \( n\text{CAT} \) is not a fibrant \( n+1 \)-category, although it does have the property that the \( n\text{CAT}_p/( \) are fibrant. Because of this, we must choose a fibrant replacement

\[
n\text{CAT} \hookrightarrow n\text{CAT}' .
\]

2.3.3 Basic to the present paper is the notion of family of \( n \)-categories indexed by an \( n+1 \)-category \( A \), which is defined using our fibrant replacement (2.3.2) to be a morphism \( A \to n\text{CAT}' \).

The \( n+1 \)-category of all families indexed by \( A \) is the \( n+1 \)-category \( \text{Hom}(A, n\text{CAT}') \).
2.3.4 Suppose \( \psi, \psi' : A \to nCAT' \) are families. A morphism from \( \psi \) to \( \psi' \) is an object of the \( n \)-category \( \text{Hom}(A, nCAT')_1/(\psi, \psi') \). Let \( I \) be the category with two objects 0, 1 and a morphism from 0 to 1 (in our notations below this will also be the same as what we will call \( \Upsilon(*) \)). The set of objects of \( \text{Hom}(A, nCAT')_1/(\psi, \psi') \) is equal to the set of morphisms

\[
I \to \text{Hom}(A, nCAT')
\]

sending 0 to \( \psi \) and 1 to \( \psi' \). In view of the definition of internal \( \text{Hom} \) this is the same thing as a morphism

\[
A \times I \to nCAT'
\]

restricting on \( A \times \{0\} \) to \( \psi \) and on \( A \times \{1\} \) to \( \psi' \).

2.4 The construction \( \Upsilon \)

We will now introduce some of our main tools for the present paper. The basic idea is that we often would like to talk about the basic \( n \)-category with two objects (denoted 0 and 1) and with a given \( n-1 \)-category \( E \) of morphisms from 0 to 1 (but no morphisms in the other direction and only identity endomorphisms of 0 and 1). We call this \( \Upsilon(E) \). To be more precise we do this on the level of precats: if \( E \) is an \( n-1 \)-precat then we obtain an \( n \)-precat \( \Upsilon(E) \). The main property of this construction is that if \( A \) is any \( n \)-category then a morphism of \( n \)-precats

\[
f : \Upsilon(E) \to A
\]

corresponds exactly to a choice of two objects \( x = f(0) \) and \( y = f(1) \) together with a morphism of \( n-1 \)-precats \( E \to A_1/(x, y) \).

One can see \( \Upsilon(E) \) as the universal \( n \)-precat \( A \) with two objects \( x, y \) and a map \( E \to A_1/(x, y) \).

2.4.1 We also need more general things of the form \( \Upsilon^2(E, F) \) having objects 0, 1, 2 and similarly a \( \Upsilon^3 \). (These will not have quite so simple an interpretation as universal objects.) Thus we present the definition in a general way.

Suppose \( E_1, \ldots, E_k \) are \( n-1 \)-precats. Then we define the \( n \)-precat

\[
\Upsilon^k(E_1, \ldots, E_k)
\]

in the following way. Its object set is the set with \( k + 1 \) elements denoted

\[
\Upsilon^k(E_1, \ldots, E_k)_0 = \{0, \ldots, k\}.
\]

Then

\[
\Upsilon^k(E_1, \ldots, E_k)_{p/}(y_0, \ldots, y_p)
\]

is defined to be empty if any \( y_i > y_j \) for \( i < j \), equal to \( * \) if \( y_0 = \ldots = y_p \), and otherwise

\[
\Upsilon^k(E_1, \ldots, E_k)_{p/}(y_0, \ldots, y_p) := E_{y_0} \times \ldots \times E_{y_p}.
\]
2.4.2 For example when \( k = 1 \) (and we drop the superscript \( k \) in this case) \( \Upsilon E \) is the \( n \)-precat with two objects 0, 1 and with \( n - 1 \)-precat of morphisms from 0 to 1 equal to \( E \). Similarly \( \Upsilon^2(E, F) \) has objects 0, 1, 2 and morphisms \( E \) from 0 to 1, \( F \) from 1 to 2 and \( E \times F \) from 0 to 2. We picture \( \Upsilon^k(E_1, \ldots, E_k) \) as a \( k \)-gon (an edge for \( k = 1 \), a triangle for \( k = 2 \), a tetrahedron for \( k = 3 \)). The edges are labeled with single \( E_i \), or products \( E_i \times \ldots, E_j \).

2.4.3 There are inclusions of these \( \Upsilon^k \) according to the faces of the \( k \)-gon. The principal faces give inclusions
\[
\Upsilon^{k-1}(E_1, \ldots, E_{k-1}) \hookrightarrow \Upsilon^k(E_1, \ldots, E_k),
\]
\[
\Upsilon^{k-1}(E_2, \ldots, E_k) \hookrightarrow \Upsilon^k(E_1, \ldots, E_k),
\]
and
\[
\Upsilon^{k-1}(E_1, \ldots, E_i \times E_{i+1}, \ldots, E_k) \hookrightarrow \Upsilon^k(E_1, \ldots, E_k).
\]
The inclusions of lower levels are deduced from these by induction. Note that these faces \( \Upsilon^{k-1} \) intersect along appropriate \( \Upsilon^{k-2} \).

2.4.4—Remark: \( \Upsilon(\bullet) = I \) is the category with objects 0, 1 and with a unique morphism from 0 to 1. A map \( \Upsilon(\bullet) \rightarrow A \) is the same thing as a pair of objects \( x, y \) and a 1-morphism from \( x \) to \( y \), i.e. an object of \( A_1/(x,y) \).

Another way of constructing the \( \Upsilon^k \) is given in the following remarks 2.4.5–2.4.7.

2.4.5 For an \( n - 1 \)-precat \( E \), denote by \([p](E)\) the universal \( n \)-precat \( A \) with objects \( x_0, \ldots, x_p \) and with a morphism \( E \rightarrow A_{p}/(x_0, \ldots, x_p) \). This can be described explicitly by saying that \([p](E)\) has objects 0, 1, \ldots, \( p \), and for a sequence of objects \( i_1, \ldots, i_k \) the \( n - 1 \)-precat \([p](E)_{i_1, \ldots, i_k}\) is empty if some \( i_j \leq i_{j+1} \), is equal to \( \ast \) if all \( i_j \) are equal, and is equal to \( E \) if some \( i_j < i_{j+1} \).

2.4.6 One has \([1]E = \Upsilon(E)\). The construction of the higher \( \Upsilon^k \) may be described inductively as follows: we will construct \( \Upsilon^k(E_1, \ldots, E_k) \) together with a morphism
\[
in : [k](E_1 \times \ldots \times E_k) \rightarrow \Upsilon^k(E_1, \ldots, E_k).
\]
Suppose we have constructed these maps up to \( k - 1 \). Note that the first and last face morphisms coupled with the projections onto the first and last \( k - 1 \) factors give a map
\[
[k - 1](E_1 \times \ldots \times E_k) \cup [k-2](E_1 \times \ldots \times E_k) \rightarrow [k - 1](E_1 \times \ldots \times E_k)
\]
\[ \Omega : [k](E_1, \ldots, E_k), \]

but on the other hand the projections onto subsets of factors of the product \( E_1 \times \ldots \times E_k \) together with the maps \( \text{in} \) in our inductive construction for \( k - 1 \) and \( k - 2 \) give a map

\[
[k - 1](E_1 \times \ldots \times E_k) \cup [k - 2](E_1 \times \ldots \times E_k) \to \beta \Upsilon^{k-1}(E_1, \ldots, E_{k-1}) \cup \Upsilon^{k-2}(E_2, \ldots, E_{k-1}) \Upsilon^{k-1}(E_{k-2}, \ldots, E_k).
\]

Finally, \( \Upsilon^k(E_1, \ldots, E_k) \) is the coproduct of the maps \( \alpha \) and \( \beta \).

We can think of this as saying that \( \Upsilon^k(E_1, \ldots, E_k) \) is obtained by adding on the cell \([k](E_1 \times \ldots \times E_k) \cup [k - 1](E_1 \times \ldots \times E_k) \cup [k - 2](E_1 \times \ldots \times E_k) \) to the coproduct

\[
\Upsilon^{k-1}(E_1, \ldots, E_{k-1}) \cup \Upsilon^{k-2}(E_2, \ldots, E_{k-1}) \Upsilon^{k-1}(E_{k-2}, \ldots, E_k)
\]

of the earlier things we have inductively constructed.

### 2.4.7 The case \( k = 2 \) is simpler to write down and is worth mentioning separately. Recall that for \( k = 1 \) we just had \( \Upsilon(E) = [1](E) \). The next step is

\[
\Upsilon^2(E, F) = [2](E \times F) \cup [1](E \times F) \cup [1](E \times F) \cup [1](F) \cup [1](F).
\]

### 2.4.8 One thing which we often will need to know below is when an inclusion from a union of faces, into the whole \( \Upsilon^k \), is a trivial cofibration. For \( k = 2 \) the only inclusion which is a trivial cofibration is

\[
\Upsilon(E_1) \cup [1] \Upsilon(E_2) \hookrightarrow \Upsilon^2(E_1, E_2).
\]

For \( k = 3 \) we denote our inclusions in shorthand notation where 0, 1, 2, 3 refer to the vertices. To fix notations, the above inclusion for \( k = 2 \) would be noted

\[
(01) + (12) \subset (012).
\]

Now for \( k = 3 \) the inclusions which are trivial cofibrations are:

\[
(01) + (12) + (23) \subset (0123)
\]

(which is the standard one, coming basically from the definition of \( n \)-category); and then some others which we obtain from this standard one by adding in triangles on the right, keeping equivalence with \( (01) + (12) + (23) \) according to the result for \( k = 2 \):

\[
(01) + (123) \subset (0123),
\]
Our main examples of inclusions which are not trivial cofibrations are when we leave out the first or the last faces:

\((012) + (023) + (013) \subset (0123)\) not a t.c.;
\((013) + (023) + (123) \subset (0123)\) not a t.c.

We call these the left and right shells. We shall meet both of them and denote the left shell as

\((012) + (023) + (013) = \text{Shell}_3(E_1, E_2, E_3)\),

and the right shell as

\((013) + (023) + (123) = \text{Shell}_r_3(E_1, E_2, E_3)\).

The main parts of our arguments for limits will consist of saying that under certain circumstances we have an extension property for morphisms with respect to these cofibrations which are not trivial.

2.4.9 One of the main technical problems which will be encountered by the reader is deciding when a morphism between \(n\) or \(n+1\)-precats is a trivial cofibration: we use this all the time in order to use the fibrant property of the domain of morphisms we are trying to extend. It is not possible to give all the details each time that this question occurs, as that would be much too long. The general principles at work are: to be aware of the examples given in 2.4.8; to use the fact that the coproduct of a trivial cofibration with something else again yields a trivial cofibration (2.2.10); and to use the fact that if a composable sequence of morphisms

\[ f \Rightarrow g \Rightarrow \]

has composition being a weak equivalence, and one of \(f\) or \(g\) being a weak equivalence, then so is the other (2.2.9 CM2). And of course to use any available hypotheses that are in effect saying that certain morphisms are trivial cofibrations or equivalences. All of the cases where we need to know that something is a trivial cofibration, can be obtained using these principles.
2.4.10 We will often be considering morphisms of the form

\[ f : \Upsilon^k(E_1, \ldots, E_k) \to C. \]

When we would like to restrict this to a face (or higher order face such as an edge) then, denoting the face by \( i_1, \ldots, i_j \) we denote the restriction of \( f \) to the face by

\[ r_{i_1} \ldots i_{j}(f). \]

For example when \( k = 2 \) the restriction of

\[ f : \Upsilon^2(E, F) \to C \]

to the edge (02) (which is a \( \Upsilon(E \times F) \)) would be denoted \( r_{02}(f) \). The object \( f(1) \) could also be denoted \( r_1(f) \).

We make the same convention for restricting maps of the form

\[ A \times \Upsilon^k(E_1, \ldots, E_k) \to C, \]

to maps on \( A \) times some face of the \( \Upsilon^k(E_1, \ldots, E_k) \).

2.5 Inverting equivalences

In preparation for 2.5.8 we need the following result. It says that a morphism which is an equivalence has an inverse which is essentially unique, if the notion of “inverse” is defined in the right way. It is an \( n \)-category version of the theorem of \[29\] which gives a canonical inverse for a homotopy equivalence of spaces.

**Theorem 2.5.1** For any fibrant \( n \)-category \( C \) the morphism restriction from \( I \) to \( I \):

\[ r : \text{Hom}(I, C) \to \text{Hom}(I, C) \]

is fully faithful, so \( \text{Hom}(I, C) \) is equivalent to the full sub-\( n \)-category of invertible elements of \( \text{Hom}(I, C) \).

**Proof:** We first construct some trivial cofibrations.

2.5.2 Recall (2.4.4) that \( I = \Upsilon(*) \). The morphism

\[ \Upsilon(E) \cup^{(1)} I \to \Upsilon^2(E, *) \]

is a trivial cofibration (2.4.8), so by 2.2.10 the coproduct with

\[ \Upsilon(E) \cup^{(1)} I \to \Upsilon(E) \cup^{(1)} \overline{T} \]
gives a trivial cofibration

\[ \Upsilon(E) \cup^{(1)} I \to \Upsilon^2(E, \ast) \cup^I I. \]

The morphism

\[ \Upsilon(E) \to \Upsilon(E) \cup^{(1)} I \]

is a weak equivalence (again by [2.2.10] because it is pushout of the trivial cofibration \( \ast \to I \)). Therefore the composed morphism

\[ i_{01} : \Upsilon(E) \to \Upsilon^2(E, \ast) \cup^I I \]

corresponding to the edge (01) is an equivalence. Thus the projection

\[ \Upsilon^2(E, \ast) \cup^I I \to \Upsilon(E) \]

is an equivalence (by [2.2.9] CM2). This in turn implies that the morphism corresponding to the edge (02)

\[ i_{02} : \Upsilon(E) \to \Upsilon^2(\ast, E) \cup^I I \]

is a trivial cofibration.

**2.5.3** A similar argument shows that

\[ i_{02} : \Upsilon(E) \to \Upsilon^2(\ast, E) \cup^I I \]

is a trivial cofibration.

**2.5.4** Next, note that

\[ \Upsilon(E) \times I = \Upsilon^2(E, \ast) \cup^{\Upsilon(E)} \Upsilon^2(\ast, E) \]

(the square decomposes as a union of two triangles). The morphisms in the coproduct are both \( i_{02} \). Thus if we attach \( I \) to each of the intervals \( I \) on the two opposite sides of this square, the result

\[ \Upsilon(E) \times I \cup^{\{0,1\} \times I} \{0,1\} \times I \]

can be seen as a coproduct of the two objects considered in [2.5.2] and [2.5.3] (we don’t write this coproduct out). Combining with the results of those paragraphs, the morphism from the diagonal

\[ \Upsilon(E) \to \Upsilon(E) \times I \cup^{\{0,1\} \times I} \{0,1\} \times I \]

is an equivalence, which in turn implies that the projection

\[ \Upsilon(E) \times I \cup^{\{0,1\} \times I} \{0,1\} \times I \to \Upsilon(E) \]

is an equivalence.
is an equivalence or, equally well, that the inclusion
\[(\Upsilon(E) \times I) \cup^{(0,1)\times I} \{(0,1) \times T\} \hookrightarrow \Upsilon(E) \times T\]
is a trivial cofibration.

2.5.5 Suppose now that \(E' \subset E\). Let \(G\) denote the pushout of
\[(\Upsilon(E) \times I) \cup^{(0,1)\times I} \{(0,1) \times T\}\]
and \(\Upsilon(E') \times T\) along
\[(\Upsilon(E') \times I) \cup^{(0,1)\times I} \{(0,1) \times T\}.\]

Paragraph 2.5.4 and the usual (2.2.10) and (2.2.9) imply that the morphism
\[G \hookrightarrow \Upsilon(E) \times T\]
is a trivial cofibration. Note, however, the simpler expression
\[G = (\Upsilon(E) \times I) \cup^{\Upsilon(E')\times I} (\Upsilon(E') \times T).\]

2.5.6 We are now ready to prove the theorem. Fix \(u, v\) objects of \(\text{Hom}(T, C)\). Suppose \(E' \hookrightarrow E\) is any cofibration, and suppose given a morphism
\[E \to \text{Hom}(I, C)_{1/(r(u), r(v))}\]
provided with lifting
\[E' \to \text{Hom}(T, C)_{1/(u, v)}\]

These correspond exactly to a morphism
\[G \to C,\]
which since \(C\) is fibrant extends along the trivial cofibration of (2.5.3) to a morphism
\[\Upsilon(E) \times T.\]

This is exactly the lifting to a map
\[E' \to \text{Hom}(T, C)_{1/(u, v)}\]
needed to establish the statement that the morphism induced by \(r\)
\[\text{Hom}(T, C)_{1/(u, v)} \to \text{Hom}(I, C)_{1/(r(u), r(v))}\]
is an equivalence. This proves the theorem.
Corollary 2.5.7 Suppose $f : U \to V$ is a morphism in a fibrant $n$-category $C$. Then the $n$-category of morphisms $\mathcal{T} \to C$ restricting on $I \subset \mathcal{T}$ to $f$ is contractible.

Proof: The $n$-category in question is just the fiber of the morphism in the theorem, over the object $f \in \text{Hom}(I, C)$. 

The next corollary says that equivalences may be inverted with dependence on parameters.

Corollary 2.5.8 Suppose $C$ is a fibrant $n$-category. Suppose $\psi, \psi' : A \to C$ are two morphisms and suppose $f$ is a morphism from $\psi$ to $\psi'$. Suppose that for every object $a \in A$ the induced morphism $f_a : \psi(a) \to \psi'(a)$ is an equivalence in $C$. Then $f$ is an equivalence considered as a 1-morphism in $\text{Hom}(A, C)$.

Proof: The morphism $f$ is a map

$$f : A \times I \to C,$$

which we can think of as a map

$$f_1 : A \to \text{Hom}(I, C).$$

From Theorem 2.5.1 the morphism

$$\text{Hom}(\mathcal{T}, C) \to \text{Hom}(I, C)$$

is a fibrant equivalence onto the full subcategory of invertible objects. The hypothesis of the corollary says exactly that the morphism $f_1$ lands in this full subcategory. Therefore it lifts to a morphism

$$g : A \to \text{Hom}(\mathcal{T}, C),$$

in other words to

$$A \times \mathcal{T} \to C$$

or equally well

$$\mathcal{T} \to \text{Hom}(A, C).$$

This shows that $f$ was an equivalence. 

3. The definitions of direct and inverse limits
One of the most useful tools in homotopy theory is the notion of homotopy limit or “holim”. This can mean either direct or inverse limit and one of the two is called a “colimit” but I don’t know which one! So we’ll call both “limits” and specify which one in context. Our purpose is to define the notions of inverse and direct limit in an $n$-category.

We always suppose that the target category $C$ is fibrant. When this is not the case we first have to take a fibrant replacement (2.2.11).

### 3.1 Inverse limits

Suppose $C$ is a fibrant $n$-category, and suppose $A$ is an $n$-category. Suppose $\varphi : A \to C$ is a morphism. If $U \in C$ is an object then we define

$$\text{Hom}(U, \varphi) := \text{Hom}(A, C)_{1/}(U_A, \varphi)$$

where $U_A$ denotes the constant morphism with value $U$. If $V$ is another object of $C$ then we have a morphism

$$C_{1/}(V, U) \to \text{Hom}(A, C)_{1/}(V_A, U_A)$$

and we use this to define

$$\text{Hom}(V, U, \varphi) := \text{Hom}(A, C)_{2/}(V_A, U_A, \varphi) \times_{\text{Hom}(A, C)_{1/}(V_A, U_A)} C_{1/}(V, U)$$

or more generally if $V^0, \ldots, V^p \in C_0$ we define

$$\text{Hom}(V^0, \ldots, V^p, \varphi) := \text{Hom}(A, C)_{(p+1)/}(V^0_A, \ldots, V^p_A, \varphi) \times_{\text{Hom}(A, C)_{p/}(V^0_A, \ldots, V^p_A)} C_{p/}(V^0, \ldots, V^p).$$

However we won’t need this beyond $p = 2$.

Notice now that since $C$ is fibrant, $\text{Hom}(A, C)$ is fibrant and in particular an $n$-category, thus we get that the morphism

$$\text{Hom}(V, U, \varphi) \to C_{1/}(V, U) \times \text{Hom}(U, \varphi)$$

is an equivalence. On the other hand we have a projection

$$\text{Hom}(V, U, \varphi) \to \text{Hom}(V, \varphi).$$

It is in this sense that we have a “weak morphism” from $C_{1/}(V, U) \times \text{Hom}(U, \varphi)$ to $\text{Hom}(V, \varphi)$. 

22
3.1.1—Definition: We say that an object $U \in C_0$ together with element $f \in \text{Hom}(U, \varphi)_0$ is an inverse limit of $\varphi$ if for any $V \in C_0$ the resulting weak morphism from $C_1/(V, U)$ to $\text{Hom}(V, \varphi)$ is an equivalence. To say this more precisely this means that the morphism

$$\text{Hom}(V, U, \varphi) \times_{\text{Hom}(U, \varphi)} \{f\} \to \text{Hom}(V, \varphi)$$

should be an equivalence. If such an inverse limit exists we say that $\varphi$ admits an inverse limit (we will discuss uniqueness below). If any morphism $\varphi: A \to C$ from any $n$-category $A$ to $C$ admits an inverse limit then we say that $C$ admits inverse limits.

3.1.2 Uniqueness: Suppose $f \in \text{Hom}(U, \varphi)$ and $g \in \text{Hom}(V, \varphi)$ are two different inverse limits of $\varphi$. Then the inverse image of $g$ for the morphism

$$\text{Hom}(V, U, \varphi) \times_{\text{Hom}(U, \varphi)} \{f\} \to \text{Hom}(V, \varphi)$$

is contractible. This gives a contractible $n$-category mapping to $\text{Hom}(V, U)$. We also have a contractible $n$-category mapping to $\text{Hom}(U, V)$. A similar argument with $p = 3$ gives a contractible $n$-category mapping to $\text{Hom}(V, U, V)$ which maps into the contractible things for $V, U$, for $U, V$ and for $V, V$. The image at the end includes the identity. This shows that the composition of the morphisms in the two directions is the identity. The same works in the other direction. This shows that the essentially well defined morphisms $U \to V$ and $V \to U$ are equivalences. (The reader is challenged to find a nicer way of saying this!)

3.1.3 The condition of being an inverse limit may also be interpreted in terms of the construction $\Upsilon$ described in the previous section. To do this, start by noting that for an $n$-precat $E$ a morphism

$$E \to \text{Hom}(U, \varphi)$$

is the same thing as a morphism

$$f: A \times \Upsilon(E) \to C$$

such that $r_0(u) = U_A$ and $r_1(u) = \varphi$.

3.1.4 In view of the discussion 2.4.5–2.4.7, a morphism

$$E \to \text{Hom}(V, U, \varphi)$$
is the same thing as a morphism

\[ g : A \times [2](E) \to C \]

with \( r_0(g) = V_A \), \( r_1(g) = U_A \) and \( r_2(g) = \varphi \) and such that \( r_{01}(g) \) comes from a morphism \( \Upsilon(E) \to C \). To see this, use the definition of \([2](E)\) by universal property 2.4.5.

In a similar way using the description 2.4.7, a morphism

\[ E \to \text{Hom}(V, U, \varphi) \times_{\text{Hom}(U, \varphi)} \{ f \} \]

is the same thing as a morphism

\[ g : A \times \Upsilon^2(E, \ast) \to C \]

such that \( r_2(g) = \varphi \) and \( r_{01}(g) \) comes from a morphism \( g_{01} : \Upsilon(E) \to C \) with \( r_0(g_{01}) = V \) and \( r_1(g_{01}) = U \).

3.1.5 Noting that the morphism

\[ \text{Hom}(V, U, \varphi) \times_{\text{Hom}(U, \varphi)} \{ f \} \to \text{Hom}(V, \varphi) \]

is fibrant, it is an equivalence if and only if it satisfies the lifting property for all cofibrations \( E' \subset E \) (2.2.18).

3.1.6 Using the above descriptions we can describe explicitly the lifting property of the previous paragraph and thus obtain the following characterization. A morphism \( f \in \text{Hom}(U, \varphi) \) is an inverse limit if and only if for every morphism

\[ v : A \times \Upsilon(E) \to C \]

with \( r_0(v) = V_A \) for \( V \in C_0 \) and \( r_1(v) = \varphi \), and for every extension over \( A \times \Upsilon(E') \) to a morphism

\[ w' : A \times \Upsilon^2(E', \ast) \to C \]

with \( r_{12}(w') = f \) and \( r_{01}(w') \) coming from a morphism \( z' : \Upsilon(E') \to C \) with \( r_0(z') = V \) and \( r_1(z') = U \), there exists a common extension of these two: a morphism

\[ w : A \times \Upsilon^2(E, \ast) \to C \]

with \( r_{12}(w) = f \) and \( r_{01}(w) \) coming from a morphism \( z : \Upsilon(E') \to C \) with \( r_0(z) = V \) and \( r_1(z) = U \); such that the restriction of \( w \) to \( A \times \Upsilon^2(E, \ast) \) is equal to \( w' \); and such that \( r_{02}(w) = v \).

This is the characterization we shall use in our proofs.
3.2 Direct limits

We obtain the notion of direct limit by "reversing the arrows" in the above discussion. Suppose \( C \) is a fibrant \( n \)-category, and suppose \( A \) is an \( n \)-category. Suppose \( \varphi : A \to C \) is a morphism. If \( U \in C_0 \) is an object then we define

\[
\text{Hom}(\varphi, U) := \text{Hom}(A, C)_{1/}(\varphi, U_A)
\]

where again \( U_A \) denotes the constant morphism with value \( U \). If \( V \) is another object of \( C \) then we have a morphism

\[
C_1/(U, V) \to \text{Hom}(A, C)_{1/}(U_A, V_A)
\]

and we use this to define

\[
\text{Hom}(\varphi, U, V) := \text{Hom}(A, C)_{2/}(\varphi, U_A, V_A, \varphi) \times_{\text{Hom}(A, C)_{1/}(U_A, V_A)} C_1/(U, V)
\]

or more generally if \( V^0, \ldots, V^p \in C_0 \) we define

\[
\text{Hom}(\varphi, V^0, \ldots, V^p) := \text{Hom}(A, C)_{(p+1)/}(\varphi, V^0_A, \ldots, V^p_A) \times_{\text{Hom}(A, C)_{p/}(V^0_A, \ldots, V^p_A)} C_{p/}(V^0, \ldots, V^p).
\]

Again we won’t need this beyond \( p = 2 \).

Notice now that since \( C \) is fibrant, \( \text{Hom}(A, C) \) is fibrant and in particular an \( n \)-category, thus we get that the morphism

\[
\text{Hom}(\varphi, U, V) \to C_1/(U, V) \times \text{Hom}(\varphi, U)
\]

is an equivalence. On the other hand we have a projection

\[
\text{Hom}(\varphi, U, V) \to \text{Hom}(V, \varphi).
\]

It is in this sense that we have a “weak morphism” from \( C_1/(U, V) \times \text{Hom}(\varphi, U) \) to \( \text{Hom}(\varphi, V) \).

3.2.1 Definition: We say that an element \( f \in \text{Hom}(U, \varphi)_0 \) is a direct limit of \( \varphi \) if for any \( V \in C_0 \) the resulting weak morphism from \( C_1/(U, V) \) to \( \text{Hom}(\varphi, V) \) is an equivalence. To say this more precisely this means that the morphism

\[
\text{Hom}(\varphi, U, V) \times_{\text{Hom}(\varphi, U)} \{f\} \to \text{Hom}(\varphi, V)
\]

should be an equivalence. If such a direct limit exists we say that \( \varphi \) admits an inverse limit. Exactly the same discussion of uniqueness as above (3.1.2) holds here too. If any morphism \( \varphi : A \to C \) from any \( n \)-category \( A \) to \( C \) admits a direct limit then we say that \( C \) admits direct limits.
3.2.2 We have the following characterization analogue to 3.1.6. Again, this is the characterization which we shall use in the proofs. It comes from considerations identical to 3.1.3–3.1.5 which we omit here.

A morphism \( f \in \text{Hom}(\varphi, U) \) is a direct limit if and only if for every morphism \( v : A \times \Upsilon(E) \to C \) with \( r_0(v) = \varphi \) and \( r_1(v) = V_A \) for \( V \in C_0 \), and for every extension over \( A \times \Upsilon(E') \) to a morphism \( w' : A \times \Upsilon^2(\ast, E') \to C \) with \( r_{01}(w') = f \) and \( r_{12}(w') \) coming from a morphism \( z' : \Upsilon(E') \to C \) with \( r_1(z') = U \) and \( r_2(z') = V \), there exists a common extension of these two: a morphism \( w : A \times \Upsilon^2(\ast, E) \to C \) with \( r_{01}(w) = f \) and \( r_{12}(w) \) coming from a morphism \( z : \Upsilon(E) \to C \) with \( r_1(z) = U \) and \( r_2(z) = V \); such that the restriction of \( w \) to \( A \times \Upsilon^2(\ast, E') \) is equal to \( w' \); and such that \( r_{02}(w) = v \).

3.3 Invariance properties

**Proposition 3.3.1** Suppose \( f : A' \to A \) is an equivalence of \( n \)-categories and suppose \( C \) is a fibrant \( n \)-category. Suppose \( \varphi : A \to C \) is a morphism. Then the inverse (resp. direct) limit of \( \varphi \) exists if and only if the inverse (resp. direct) limit of \( \varphi \circ f \) exists.

Suppose \( \varphi \) and \( \psi \) are morphisms from \( A \) to \( C \), and suppose they are equivalent in \( \text{Hom}(A, C) \). Then the inverse (resp. direct) limit of \( \varphi \) exists if and only if the inverse (resp. direct) limit of \( \psi \) exists.

Finally suppose \( g : C \to C' \) is an equivalence between fibrant \( n \)-categories. Then the inverse (resp. direct) limit of \( \varphi : A \to C \) exists if and only if the inverse (resp. direct) limit of \( g \circ \varphi \) exists. In particular (combining with the previous paragraph) \( C \) admits inverse (resp. direct) limits if and only if \( C' \) does.

**Proof:** There are several statements to prove so we divide the proof into several paragraphs 3.3.2–3.3.11.

3.3.2 Suppose \( f : A' \hookrightarrow A \) is a cofibrant equivalence of \( n \)-categories. Suppose that \( \varphi : A \to C \) is a morphism and that \( u : \varphi \to U \) is a morphism from \( \varphi \) to \( U \in C \) which is a direct limit. This corresponds to a diagram

\[ \epsilon : A \times \Upsilon(\ast) \to C \]
and pullback by $f$ gives a diagram

$$\epsilon' : A' \times \Upsilon(\ast) \to C.$$  

We claim that $\epsilon'$ is a direct limit (note that $\epsilon'$ is a morphism from $\varphi \circ f$ to $U$). Suppose we are given

$$u : A' \times \Upsilon(E) \to C$$

and an extension over $E' \subset E$ to a diagram

$$v_1 : A' \times \Upsilon^2(\ast, E') \to C$$

whose restriction to the edge $(01)$ is $\epsilon'$ and whose restriction to the edge $(02)$ is $u$. Then we can first extend $v_1$ to a diagram

$$A \times \Upsilon(\ast, E') \to C$$

because

$$A' \times \Upsilon^2(\ast, E') \hookrightarrow A \times \Upsilon^2(\ast, E')$$

is a trivial cofibration (and note also that we can assume that the extension satisfies the relevant properties as in the definition of limit); then we can also extend our above morphism $u$ to a diagram

$$A \times \Upsilon(E) \to C$$

compatibly with the extension of $v_1$, because the inclusion from the coproduct of $A \times \Upsilon(E')$ and $A' \times \Upsilon(E)$ over $A' \times \Upsilon(E')$, into $A \times \Upsilon(E)$ is a trivial cofibration. Now we apply the limit property of $\epsilon$ to conclude that there is an extension to a diagram

$$v : A \times \Upsilon^2(\ast, E) \to C.$$  

This restricts over $A'$ to a diagram of the form we would like, showing that $\epsilon'$ is a direct limit.

**3.3.3** Suppose that $f : A' \hookrightarrow A$ is a trivial cofibration and suppose $\varphi : A \to C$ is a morphism to a fibrant $n$-category $C$, and suppose now that we know that $\varphi \circ f$ has a limit

$$\epsilon' : \varphi \circ f \to U$$

for an object $U \in C$. We claim that $\varphi$ has a limit.

The morphism $\epsilon'$ may be considered as a diagram

$$\epsilon' : A' \times \Upsilon(\ast) \to C.$$
This extends along $A' \times \{0\}$ to

$$\varphi : A \times \{0\} \to C$$

and it extends along $A' \times \{1\}$ to

$$U_A : A \times \{1\} \to C.$$ 

Putting these all together we obtain a morphism

$$A \times \{0\} \cup A' \times \{0\} \cup A' \times \Upsilon(\ast) \cup A \times \{1\} \to C.$$ 

Since $A' \subset A$ is a trivial cofibration the morphism

$$A \times \{0\} \cup A' \times \{0\} \cup A' \times \Upsilon(\ast) \cup A \times \{1\} \to A \times \Upsilon(\ast)$$

is a trivial cofibration (applying 2.2.9, first part of CM4, two times), so by the fibrant property of $C$ our morphism extends to a morphism

$$\epsilon : A \times \Upsilon(\ast) \to C$$

with the required properties of being constant along $A \times \{1\}$ and restricting to $\varphi$ along $A \times \{0\}$. Thus we may write $\epsilon : \varphi \to U$.

We claim that this map is a direct limit of $\varphi$. Given a diagram

$$u : A \times \Upsilon(E) \to C$$

going from $\varphi$ to a constant object $B$, the restriction $u'$ to $A' \times \Upsilon(E)$ admits (by the hypothesis that $\epsilon'$ is a direct limit) an extension to

$$v' : A' \times \Upsilon(\ast, E) \to C$$

restricting along the edge (01) to $\epsilon'$ and restricting along the edge (12) to the pullback of a diagram $\Upsilon(E) \to C$. Then using as usual the fibrant property of $C$ and the fact that $A' \to A$ is a trivial cofibration, we can extend $v'$ to a morphism

$$v : A \times \Upsilon(\ast, E) \to C$$

again restricting along the edge (01) to $\epsilon$, restricting along the edge (02) to our given diagram $u$, and restricting along the edge (12) to the pullback of a diagram $\Upsilon(E) \to C$.

If $E' \subset E$ and we are already given an extension $v_{E'}$ over $A \times \Upsilon(\ast, E')$ then (as before, using the fibrant property of $C$ applied to an appropriate cofibration) we can assume that our extension $v$ above restricts to $v_{E'}$. This completes the proof that $\epsilon$ is a direct limit, and hence the proof of the statement claimed for 3.3.3.
3.3.4 Now suppose $p : A' \to A$ is a trivial fibration. Then there exists a section $s : A \to A'$ (with $ps = 1_A$). Note that $s$ is a trivial cofibration. If $\varphi : A \to C$ is a morphism then

$$(\varphi \circ p) \circ s = \varphi$$

so applying the previous two paragraphs 3.3.3 and 3.3.4 to the morphism $s$ we conclude that $\varphi$ admits a limit if and only if $\varphi \circ p$ admits a limit.

3.3.5 Now suppose $f : A' \to A$ is any equivalence between $n$-categories. Decomposing $f = p \circ j$ into a composition of a trivial cofibration followed by a trivial fibration and applying 3.3.2, 3.3.3 to $j$ and 3.3.4 to $p$ we conclude that a functor $\varphi : A \to C$ admits a direct limit if and only if $\varphi \circ f$ admits a direct limit. This proves the first paragraph of Proposition 3.3.1 for direct limits.

3.3.6 The proof of the first paragraph of 3.3.1 for inverse limits is exactly the same as the above.

3.3.7 Now we prove the second paragraph of the proposition. If $f, g : A \to C$ are two morphisms which are equivalent in $\text{Hom}(A, C)$ (i.e. they are homotopic) then there exists a morphism $\varphi : A \times I \to C$ restricting to $f$ on $A \times \{0\}$ and restricting to $g$ on $A \times \{1\}$. Applying the first paragraph of the proposition (for either direct or inverse limits) to the two inclusions $A \to A \times I$ we find that $f$ admits a limit if and only if $\varphi$ admits a limit and similarly $g$ admits a limit if and only if $\varphi$ does—therefore $f$ admits a limit if and only if $g$ admits a limit.

3.3.8 Suppose $C \to C'$ is an equivalence between fibrant $n$-categories. In general if $F' \subset F$ is any cofibration and if $F' \to C$ is a morphism, then there exists an extension to $F \to C$ if and only if the composed morphism $F' \to C'$ extends over $F$. To see this, look at the (exactly commutative) diagram

$$
\begin{array}{ccc}
\text{Hom}(F, C) & \rightarrow & \text{Hom}(F', C) \\
\downarrow & & \downarrow \\
\text{Hom}(F, C') & \rightarrow & \text{Hom}(F', C')
\end{array}
$$

The horizontal arrows are fibrations and the vertical arrows are equivalences. If an element $a \in \text{Hom}(F', C)$ maps to something $b$ which is hit from $c \in \text{Hom}(F, C')$ then there is $d \in \text{Hom}(F, C)$ mapping to something equivalent to $c$; thus the image $e$ of $d$ in $\text{Hom}(F', C)$ maps to something equivalent to $b$. This implies (since the right vertical arrow is an equivalence) that $e$ is equivalent to $a$. Since the top morphism is fibrant, there is another element $d' \in \text{Hom}(F, C)$ which maps directly to $a$. 

29
3.3.9 Suppose still that $C \to C'$ is an equivalence between fibrant $n$-categories. Using the general lifting principle 3.3.8 and the fact that the property of being a limit is expressed in terms of extending morphisms across certain cofibrations $F' \subset F$, we conclude that a functor $A \to C$ has a (direct or inverse) limit if and only if the composition $A \to C'$ does. This proves the first sentence of the last paragraph of the proposition.

3.3.10 If $f : C \to C'$ is an equivalence between fibrant $n$-categories and if $C'$ admits direct (resp. inverse) limits then any functor $A \to C$ admits a direct (resp. inverse) limit by 3.3.9.

3.3.11 Suppose on the other hand that we know that $C$ admits direct (resp. inverse) limits. Suppose that $\varphi : A \to C'$ is a functor. Since $f$ induces an equivalence from $\text{Hom}(A, C)$ to $\text{Hom}(A, C')$ there is a morphism $\psi : A \to C$ such that $f \circ \psi$ is equivalent to $\varphi$ in $\text{Hom}(A, C')$. By the second paragraph of the proposition (proved in 3.3.7 above), $\varphi$ admits direct (resp. inverse) limits if and only if $f \circ \psi$ does. Then by the first part of the last paragraph proved in 3.3.9 above, $f \circ \psi$ admits direct (resp. inverse) limits if and only if $\psi$ does. Now by hypothesis $\psi$ has a direct (resp. inverse) limit, so $\varphi$ does too. This shows that $C'$ admits direct (resp. inverse) limits.

We have now completed the proof of Proposition 3.3.1. ///

3.3.12 We now start to look at variance properties in other situations. Suppose $h : C \to C'$ is a morphism between fibrant $n$-categories, and suppose $\varphi : A \to C$ is a morphism. If $u : \varphi \to U$ is a direct limit then $h(u) : \varphi \circ h \to h(U)$ is a morphism. Suppose that $\varphi \circ h$ admits a direct limit $v : \varphi \circ h \to V$. Then by the limit property there is a factorization i.e. a diagram $[v, w] : \varphi \circ h \to V \to h(U)$ whose third edge (02) is $h(u)$. We say that the morphism $h$ commutes with the direct limit of $\varphi$ if the direct limit of $\varphi \circ h$ exists and if the factorization morphism $w : V \to h(U)$ is an equivalence.

Suppose that $C$ and $C'$ admit direct limits. We say that the morphism $h$ commutes with direct limits if $h$ commutes with the direct limit of any $\varphi : A \to C$ in the previous sense.
3.3.13 We have similar definitions for inverse limits, which we repeat for the record. Suppose again that $h : C \to C'$ is a morphism between fibrant $n$-categories, and suppose $\varphi : A \to C$ is a morphism. If $u : U \to \varphi$ is an inverse limit then $h(u) : h(U) \to \varphi \circ h$ is a morphism. Suppose that $\varphi \circ h$ admits an inverse limit $v : V \to \varphi \circ h$. Then by the limit property there is a factorization i.e. a diagram

$$[w, v] : h(U) \to V \to \varphi \circ h$$

whose third edge (02) is $h(u)$. We say that the morphism $h$ commutes with the inverse limit of $\varphi$ if the inverse limit of $\varphi \circ h$ exists and if the factorization morphism $w : h(U) \to V$ is an equivalence.

Suppose that $C$ and $C'$ admit inverse limits. We say that the morphism $h$ commutes with inverse limits if $h$ commutes with the inverse limit of any $\varphi : A \to C$ in the above sense.

3.4 Behavior under certain precat inverse limits of $C$

We will now study certain situations of what happens when we take fiber products or other inverse limits (here we mean inverse limits in the category of $n$-precats) of the target $n$-category $C$. We study what happens to inverse limits in $C$. We could also say the same things about direct limits in $C$ but the inverse limit case is the one we need, so we state things there and leave it to the reader to make the corresponding statements for direct limits.

**Lemma 3.4.1** Suppose $\{C_i\}_{i \in S}$ is a collection of fibrant $n$-categories indexed by a set $S$. Let $C = \prod_{i \in S} C_i$ and suppose $\varphi = \{\varphi_i\}$ is a morphism from $A$ to $C$. Suppose that the $\varphi_i$ admit inverse limits $u_i : U_i \to \varphi_i$ in $C_i$. Then $U = \{U_i\}$ is an object of $C$ and we have a morphism $u : U \to \varphi$ composed of the factors $u_i$. This morphism is an inverse limit of $\varphi$ in $C$.

**Proof:** The property that $u$ be an inverse limit consists of a collection of extension properties that have to be satisfied. The morphisms $u_i$ admit the corresponding extensions and putting these together we get the required extensions for $u$.  

31
Lemma 3.4.2 Suppose \( f : C \to D \) and \( g : E \to D \) are morphisms of fibrant \( n \)-categories with \( f \) fibrant. Suppose that \( \varphi : A \to C \times_D E \) is a morphism such that the component morphisms \( \varphi_C : A \to C \), \( \varphi_D : A \to D \) and \( \varphi_E : A \to E \) have inverse limits \( \lambda_C \), \( \lambda_D \) and \( \lambda_E \) respectively. Suppose furthermore that \( f \) and \( g \) preserve these inverse limits, which means that the projections of \( \lambda_C \) and \( \lambda_E \) into \( D \) are equivalent (as objects with morphisms to \( \varphi_D \)) to \( \lambda_D \). Then we may (by changing the \( \lambda_C \), \( \lambda_D \), \( \lambda_E \) by equivalences) assume that \( \lambda_C \) and \( \lambda_E \) project to \( \lambda_D \); and the resulting object \( \lambda \in C \times_D E \) is an inverse limit of \( \varphi \).

Proof: Set \( \lambda'_E := \lambda_E \) and let \( \lambda'_D := g(\lambda_E) \) be the projection to \( D \). Note that by hypothesis \( \lambda'_D \) is an inverse limit of \( \varphi_D \). Now \( \lambda_C \) (considered as an object with morphism to \( \varphi_C \)) projects in \( D \) to something equivalent to \( \lambda'_D \) and hence equivalent to \( \lambda'_D \) (equivalence of the diagrams including the morphism to \( \varphi_D \)). Since \( f \) is a fibrant morphism, we can modify \( \lambda_C \) by an equivalence, to obtain \( \lambda'_C \) projecting directly to \( \lambda'_D \). Note that the equivalent \( \lambda'_C \) is again an inverse limit of \( \varphi_C \). Together these give an element \( \lambda \in C \times D E \) with a map

\[
u : \lambda \to \varphi,\]

and we claim that \( \nu \) is an inverse limit. Suppose \( F' \subset F \) is a cofibration of \( n - 1 \)-precats and suppose

\[
v : V \xrightarrow{F} \varphi
\]

is any \( F \)-morphism (i.e. a diagram

\[
A \times \Upsilon(F) \to C \times_D E
\]

restricting on \( A \times \{0\} \) to the constant \( V_A \) and restricting on \( A \times \{1\} \) to \( \varphi \)) provided with an extension over \( F' \) to a diagram

\[
w' : A \times \Upsilon^2(F, \ast) \to C \times_D E
\]

restricting on \( (02) \) to \( v' \) (the restriction of \( v \) to \( F' \)) and on \( (12) \) to \( u \). We look for an extension of \( w' \) to a diagram

\[
w : A \times \Upsilon^2(F, \ast) \to C \times_D E
\]

restricting on \( (02) \) to \( v \) and on \( (12) \) to \( u \). Denoting with subscripts the components in \( C \), \( D \) and \( E \), we have that the pairs \((v_C, w'_C)\) and \((v_E, w'_E)\) admit extensions \( w_C \) and \( w_E \) respectively. The projections of these extensions in \( D \) give diagrams which we denote

\[
w_{C/D}, w_{E/D} : A \times \Upsilon^2(F, \ast) \to D,
\]

both restricting on \( (02) \) to \( v_D \) and on \( (12) \) to \( u_D \), and extending \( w'_D \). Applying again the limit property for \( u_D \) to the cofibration

\[
F \times \{0\} \cup F' \times \{0\} F' \times I \cup F \times \{1\} \hookrightarrow F \times I
\]

32
we find that there is a diagram
\[ z_D : A \times \mathcal{Y}^2(F \times \mathcal{T}, \ast) \to D \]
giving a homotopy between \( w_{C/D} \) and \( w_{E/D} \).

Notice that this is a homotopy in Quillen’s sense \([23]\) because the diagram
\[ \mathcal{Y}^2(F, \ast) \to \mathcal{Y}^2(F \times \mathcal{T}, \ast) \to \mathcal{Y}^2(F, \ast) \]
is of the form used by Quillen \([2.2.10]\). Such a homotopy can be changed into one of the more classical form
\[ A \times \mathcal{Y}^2(F, \ast) \times \mathcal{T} \to D \]
because the relation of homotopy in Quillen’s sense is the same as the relation of equivalence of morphisms, which in turn is the same as existence of a homotopy for the “interval” \( \mathcal{T} \). In fact we don’t use this remark here but we use it with \( D \) replaced by \( C \), below.

Now apply the lifting property for the morphism \( C \to D \), for the above map \( z_D \), with respect to the trivial cofibration
\[ A \times \mathcal{Y}^2(F \times \{0\} \cup F' \times \{0\}, \mathcal{T}, \ast) \]
\[ \to A \times \mathcal{Y}^2(F \times \mathcal{T}, \ast). \]
We get a morphism
\[ z_C : A \times \mathcal{Y}^2(F \times \mathcal{T}, \ast) \to C \]
providing a homotopy between \( w_C \) and a new morphism \( w^n_C \) which projects into \( D \) to \( w_{E/D} \). The new \( w^n_C \) is again a solution of the required extension problem (or to be more precise we can impose conditions on our lifting \( z_C \) to insure that this is the case. The fact that it projects to \( w_{E/D} \) means that the pair \( w = (w^n_C, w_E) \) is a solution of the required extension problem to show that \( u \) is an inverse limit. This completes the proof. \( /// \)

**Lemma 3.4.3** Suppose \( C_i \) is a collection of fibrant \( n \)-categories for \( i = 0, 1, 2, \ldots \) and suppose that \( f_i : C_i \to C_{i-1} \) are fibrant morphisms. Let \( C \) be the inverse limit of this system of \( n \)-precats. Then \( C \) is a fibrant \( n \)-category. Suppose that we have \( \varphi : A \to C \) projecting to the \( \varphi_i : A \to C_i \) and suppose that the \( \varphi_i \) admit inverse limits \( u_i : U_i \to \varphi_i \). Suppose finally that the \( f_i \) commute with the inverse limits of the \( \varphi_i \). Then \( \varphi \) admits an inverse limit and the projections \( C \to C_i \) commute with the inverse limit of \( \varphi \).

**Proof:** The fibrant property of \( C \) may be directly checked by producing liftings of trivial cofibrations.
First we construct a morphism \( u : U \to \phi \) projecting in each \( C_i \) to an inverse limit of \( \varphi_i \). To do this, note by [3.3.1] that it suffices to have \( u \) project to a morphism equivalent to \( u_i \). On the other hand, by the hypothesis that the \( f_i \) commute with the inverse limits \( u_i \), we have that \( f_i(u_i) \) is an inverse limit of \( \varphi_{i-1} \). In particular \( f_i(u_i) \) is equivalent to \( u_{i-1} \) as a diagram from \( A \times \Upsilon(*) \) to \( C_{i-1} \). The morphism from such diagrams in \( C_i \) to such diagrams in \( C_{i-1} \), is fibrant (since it comes from \( f_i \) which is fibrant). Therefore we can change \( u_i \) to an equivalent diagram with \( f_i(u_i) = u_{i-1} \). Do this successively for \( i = 1, 2, \ldots \), yielding a system of morphisms \( u_i \) with \( f_i(u_i) = u_{i-1} \). These now form a morphism

\[
\begin{align*}
u : U &\to \phi.
\end{align*}
\]

We claim that \( u \) is an inverse limit of \( \varphi \). Suppose \( E' \subset E \) is an inclusion of \( n - 1 \)-precats and suppose

\[
w : W \xrightarrow{E} \phi
\]

is an \( E \)-morphism (i.e. a diagram of the form

\[
A \times \Upsilon(E) \to C
\]

being constant equal to \( W \) on \( A \times \{0\} \), provided over \( E' \) with an extension to a diagram

\[
[v', u] : W \xrightarrow{E'} U \to \phi
\]

(i.e. a morphism

\[
A \times \Upsilon^2(E, *) \to C
\]

restricting to \( u \) on the second edge (12) and restricting to \( w|_{E'} \) on the third edge (02)). We would like to extend this to a diagram \([v, u]\) giving \( w \) on the third edge. Let \( w_i \) (resp. \( v'_i \)) be the projections of these diagrams in \( C_i \). These admit extensions \( v_i \). The projection of \( v_i \) to \( C_{i-1} \) is an extension of the desired sort for \( w_{i-1} \) and \( v'_i \). The extensions \( v_i \) are unique up to equivalence—which means a diagram

\[
A \times \Upsilon^2(E, \ast) \times \mathcal{T} \to C_i
\]

satisfying appropriate boundary conditions—and from this and our usual sort of argument constructing a trivial cofibration (this occurs several times below) then making use of the fibrant property of \( f_i \), we conclude that \( v_i \) may be modified by an equivalence so that it projects to \( v_{i-1} \). As before, do this successively for \( i = 1, 2, \ldots \) to obtain a system of extensions \( v_i \) with \( f_i(v_i) = v_{i-1} \). This system corresponds to an extension \( v \) of the desired sort for \( w \) and \( v' \). This shows that the morphism \( u \) is an inverse limit.

Note from our construction the projection of the inverse limit \( u \) to \( C_i \) is an inverse limit \( u_i \) for \( \varphi_i \) so the projections \( C \to C_i \) commute with the inverse limit of \( \varphi \). //

The application of the above results which we have in mind is the following theorem.
Theorem 3.4.4  Suppose \( C \) is a fibrant \( n \)-category and \( B \) is an \( n \)-precat. Then if \( C \) admits inverse (resp. direct) limits, so does the fibrant \( n \)-category \( \text{Hom}(B, C) \). The morphisms of functoriality for \( B' \to B \) commute with inverse (resp. direct) limits.

Proof: Suppose \( \varphi : A \to \text{Hom}(B, C) \). We will construct an inverse limit \( \lambda \in \text{Hom}(B, C) \) such that for any \( b \in B \) the restriction \( \lambda(b) \) is equivalent (via the natural morphism) to the inverse limit of \( \varphi(b) : A \to C \). This condition implies that the restriction morphism for any \( B' \to B \) commutes with the inverse limit. In effect, there is a morphism from the inverse limit over \( B \) (pulled back to \( B' \)) to the inverse limit over \( B' \), and this morphism is an equivalence over every object in \( B' \) by the condition, which implies that it is an equivalence by Lemma 2.5.8.

3.4.5 The first remark is that if \( B \subset B' \) and \( B \subset B'' \) are cofibrations of \( n \)-precats such that \( \text{Hom}(B, C) \), \( \text{Hom}(B', C) \), and \( \text{Hom}(B'', C) \) admit inverse limits complying with the above condition, then \( \text{Hom}(B' \cup^B B'', C) \) admits inverse limits again complying with the above condition. To see this we apply Lemma 3.4.2. The only thing that we need to know is that the restriction of the inverse limits is again equivalent to the inverse limit in \( \text{Hom}(B, C) \). This follows from the fact that there is a morphism which, thanks to the condition given at the start of the proof, is an equivalence for each object of \( B \)—therefore it is an equivalence.

3.4.6 The next remark is that weak equivalences of \( n \)-precats \( B \) are turned into equivalences of the \( \text{Hom}(B, C) \). The morphisms

\[
h(1, M') \cup^* \ldots \cup^* h(1, M') \to h(m, M')
\]

are weak equivalences. By the previous remarks if we know the theorem (with the condition of the first paragraph of the proof) for \( h(1, M') \) then we get it for any \( h(m, M') \) (again with the condition of the first paragraph).

As pointed out at the start of the proof, morphisms of restriction between any \( B \)'s for which we know that the limits exist (and satisfying the condition of the first paragraph), commute with the limits. In particular when we apply 3.4.2 and 3.4.3 the hypotheses about commutation with the limits will hold.

Now any \( n \)-precat may be expressed as a direct union of pushouts of the \( h(m, M') \). The pushouts in question may be organized into a countable direct union of pushouts each of which is adding a disjoint direct sum; and the addition will be of a direct sum of things of the form \( h(m, M') \) added along their boundary. Taking \( \text{Hom} \) into \( C \) transforms this expression into an inverse limit indexed by the natural numbers, of fiber products of terms which are direct products of things of the form \( \text{Hom}(h(m, M'), C) \). Applying
Lemmas 3.4.1, 3.4.2 and 3.4.3 we find that if we know the existence of limits for $h(1, M')$ (hence $h(m, M')$)—as always with the additional condition of the first paragraph of the proof—then we get existence of limits for $\text{Hom}(B, C)$ for any $n$-precat $B$.

3.4.7 Therefore it suffices to prove the theorem for $B = h(1, M')$. This is more generally of the form $\Upsilon E$ (in this case $E = h(M')$). Thus it now suffices to prove the theorem for $B = \Upsilon E$.

Suppose we have a morphism $\varphi : A \times \Upsilon E \to C$. Let $\varphi(0)$ (resp. $\varphi(1)$) denote the restriction of $\varphi$ to $A \times \{0\}$ (resp. $A \times \{1\}$). Let $(\lambda(0), \epsilon(0))$ and $(\lambda(1), \epsilon(1))$ denote inverse limits of $\varphi(0)$ and $\varphi(1)$. The morphism $\varphi$ corresponds to a morphism

$$E \to \text{Hom}(A, C)_{1/}(\varphi(0), \varphi(1)).$$

We can lift this together with $\epsilon(0)$ to a morphism

$$E \to \text{Hom}(A, C)_{2/}(\lambda(0), \varphi(0), \varphi(1)).$$

The resulting $E \to \text{Hom}(A, C)_{1/}(\lambda(0), \varphi(1))$ extends to

$$E \to \text{Hom}(A, C)_{2/}(\lambda(0)_A, \lambda(1)_A, \varphi(1))$$

projecting to $\epsilon(1)$ on the second edge, by the limit property for $\epsilon(1)$. The first edge of this comes from a morphism $\lambda : \Upsilon E \to C$. Noting that the product $\Upsilon E \times I$ is a pushout of two triangles the above morphisms glue together to give a morphism

$$\Upsilon E \times I \to \text{Hom}(A, C),$$

in other words we get a morphism $\epsilon$ from $\lambda$ to $\varphi$ considered as families over $A$ with values in $\text{Hom}(\Upsilon E, C)$.

3.4.8 To finish the proof we just have to prove that $\epsilon$ is an inverse limit of $\varphi$. Suppose we have $\mu \in \text{Hom}(\Upsilon E, C)$ and suppose given a morphism $f : \Upsilon F \times A \times \Upsilon E \to C$ restricting over $0^F$ to $\mu_A$ and over $1^F$ to $\varphi$ (here $0^F$ and $1^F$ denote the endpoints of $\Upsilon F$ and we will use similar notation for $E$). This extends to morphisms

$$f'(0^E) : \Upsilon^2(F, *) \times A \to C,$$

$$f'(1^E) : \Upsilon^2(F, *) \times A \to C,$$

by the limit properties of $\lambda(0)$ and $\lambda(1)$ (the above morphisms restricting to $\epsilon(0)$ and $\epsilon(1)$ on the second edges).
Now we try to extend to a morphism on all of $\Upsilon^2(F,*) \times A \times \Upsilon E \to C$. For this we use notation of the form $(i,j)$ for the objects of $\Upsilon^2(F,*) \times \Upsilon E$, where $i = 0,1,2$ (objects in $\Upsilon^2(F,*)$) and $j = 0,1$ (objects in $\Upsilon E$). We are already given maps defined over the triangles $(012,0)$ and $(012,1)$ (these are $f'(0^E)$ and $f'(1^E)$), as well as over the squares $(02,01)$ (our given map $f$) and $(12,01)$ (the map $\epsilon$). First extend using the fibrant property of $C$ to a map on the tetrahedron

$$(0,0)(1,0)(2,0)(2,1).$$

Then extend again using the fibrant property of $C$ to a map on the tetrahedron

$$(0,0)(1,0)(1,1)(2,1).$$

Here note that on the face $(0,0)(1,0)(1,1)$ the map is chosen first as coming from a map $\Upsilon^2(F,E) \to C$. Finally we have to find an extension over the tetrahedron

$$(0,0)(0,1)(1,1)(2,1).$$

Again we require that the map on the first face $(0,0)(0,1)(1,1)$ come from a map $\Upsilon^2(E,F) \to C$.

Our problem at this stage is that the map is already specified on all of the other faces, so we can’t do this using the fibrant property of $C$ (the face that is missing is not of the right kind). Instead we have to use the limit property of $\epsilon(1^E)$.

The limit condition on $\lambda(1^E)$ means that the morphism

$$\text{Hom}^{\epsilon(1^E)}(\mu(0^E),\mu(1^E),\lambda(1^E),\varphi) \to \text{Hom}(\mu(0^E),\mu(1^E),\varphi)$$

is an equivalence. The morphisms

$$\text{Hom}^{\epsilon(1^E)}(\mu(0^E),\lambda(1^E),\varphi) \to \text{Hom}(\mu(0^E),\varphi)$$

and

$$\text{Hom}^{\epsilon(1^E)}(\mu(1^E),\lambda(1^E),\varphi) \to \text{Hom}(\mu(1^E),\varphi)$$

are equivalences too. This implies (in view of the fact that the edges containing $\epsilon(1^E)$ are fixed) that the morphism

$$\text{Hom}^{\epsilon(1^E)}(\mu(0^E),\mu(1^E),\lambda(1^E),\varphi) \to \text{Hom}(\mu(0^E),\varphi) \times \text{Hom}(\mu(0^E),\varphi)$$

37
\[
\text{Hom}(\mu(0^E), \mu(1^E), \varphi) \times \text{Hom}(\mu(1^E), \varphi)
\]
\[
\text{Hom}^{(1^E)}(\mu(1^E), \lambda(1^E), \varphi)
\]
is an equivalence. This exactly implies that the restriction to the shell that we are interested in is an equivalence. The fact that this equivalence is a fibration implies that it is surjective on objects, giving finally the extension that we need.

In the relative case where we are already given an extension over \(F' \subset F\), one can choose our extension in a compatible way (adding on the part concerning \(F'\) in the above argument doesn’t change the properties of the relevant morphisms being trivial cofibrations).

\[\text{Corollary 3.4.9} \] Suppose \(\varphi : A \to C\) is a morphism from an \(n\)-precat to a fibrant \(n\)-category \(C\) admitting inverse limits, and suppose that \(B\) is an \(n\)-precat. Let \(\varphi_B : A \to \text{Hom}(B, C)\) denote the morphism constant along \(B\). Suppose that \(\varphi\) admits an inverse limit \(u : U \to \varphi\). Then \(u_B : U_B \to \varphi_B\) (the pullback of \(u\) along \(B \to *\)) is an inverse limit of \(\varphi_B\).

\[\text{Proof:}\] This is just commutativity for pullbacks for the morphism \(B \to *\). \[\]\\

\[\text{3.4.10} \] We can use the result of the previous theorem to obtain the variation of the limit depending on the family. Suppose \(C\) is a fibrant \(n\)-category in which inverse limits exist, and suppose \(A\) is an \(n\)-precat. Let \(B = \text{Hom}(A, C)\). We have a tautological morphism

\[\zeta : A \to \text{Hom}(B, C)\]

By the previous theorem, limits exist in \(\text{Hom}(B, C)\). Thus we obtain the limit of \(\zeta\) which is an element of \(\text{Hom}(B, C)\): it is a morphism \(\lambda\) from \(B = \text{Hom}(A, C)\) to \(C\), which is the morphism which to \(\varphi \in \text{Hom}(A, C)\) associates \(\lambda(\varphi)\) which is the limit of \(\varphi\).

The same remark holds for direct limits.

\[\text{Theorem 3.4.11} \] Suppose \(A\), \(B\) and \(C\) are \(n\)-categories. Suppose \(F : A \times B \to C\) is a functor. Then letting \(\psi : A \to \text{Hom}(B, C)\) denote the corresponding functor, suppose that \(\psi\) admits an inverse limit \(\lambda \in \text{Hom}(B, C)\). Suppose now that \(\lambda\) (considered as a morphism \(B \to C\)) admits an inverse limit \(\mu \in C\). Then \(\mu\) is an inverse limit of \(F : A \times B \to C\). In particular if the intermediate limits exist going in the other direction then the composed limits are canonically equivalent. Thus if \(C\) admits inverse limits then inverse limits commute with each other.
Proof: The general proof is left to the reader. In the case \( C = n\text{CAT}' \) this will be easy to see from our explicit construction of the limits.

4. Limits in \( n\text{CAT}' \)

Let \( n\text{CAT} \hookrightarrow n\text{CAT}' \) be a trivial cofibration to a fibrant \( n+1 \)-category.

**Theorem 4.0.1** The fibrant \( n+1 \)-category \( n\text{CAT}' \) admits inverse limits.

The rest of this section is devoted to the proof. As a preliminary remark notice that by [3.3.1] the statement doesn’t depend on which choice of \( n\text{CAT}' \) we make. We also remark, in the realm of set-theoretic niceties, that the statement means that \( n\text{CAT}' \) (an \( n+1 \)-category composed of classes) admits inverse limits indexed by any \( n+1 \)-category composed of sets. To be more precise our proof will show that if we restrict to a subcategory of \( n\text{CAT}' \) of \( n \)-categories represented in a certain set of fixed cardinality \( \alpha \), then the inverse limit indexed by \( A \) exists if \( \alpha \) is infinite and at least equal to \( 2^{\# A} \), also at least equal to what is needed for making the fibrant replacement \( n\text{CAT} \subset n\text{CAT}' \).

4.1 Construction of the limit

4.1.1 Suppose \( A \) is an \( n+1 \)-category. If \( B \) is a fibrant \( n \)-category we have denoted by \( B_A \) the constant morphism \( A \to n\text{CAT} \) with value \( B \), considered as a morphism \( A \to n\text{CAT}' \).

4.1.2 We now give the construction of the inverse limit. Suppose \( \varphi : A \to n\text{CAT}' \) is a morphism. We define an \( n \)-category

\[
\lambda := \underbrace{\text{Hom}(A, n\text{CAT}')}_1/\left( \ast_A, \varphi \right).
\]

This has the universal property that for any \( n \)-precat \( B \),

\[
\text{Hom}(B, \lambda) = \text{Hom}_{\ast_A, \varphi}(A \times \Upsilon B, n\text{CAT}').
\]

The notation on the right means the fiber of the map

\[
(r_0, r_1) : \text{Hom}(A \times \Upsilon B, n\text{CAT}') \to \text{Hom}(A, n\text{CAT}') \times \text{Hom}(A, n\text{CAT}')
\]

over \( (\ast_A, \varphi) \).

The problem below will be to prove that \( \lambda \) is an inverse limit of \( \varphi \).

4.2 Diagrams

39
4.2.1 Suppose $C$ is an $n+1$-precat. Then for $n$-precats $E_1, \ldots, E_k$ we define

$$Diag(E_1, \ldots, E_i, \ldots, E_k; C)$$

to be the $n$-precat which represents the functor

$$F \mapsto Hom(\Upsilon^k(E_1, \ldots, E_i \times F, \ldots, E_k), C).$$

We establish some properties.

4.2.2 The first remark is that $Diag(E_1, \ldots, E_i, \ldots, E_k; C)$ decomposes as a disjoint union over all pairs $(a, b)$ where

$$a : \Upsilon^{i-1}(E_1, \ldots, E_{i-1}) \to C$$

and

$$b : \Upsilon^{k-i}(E_{i+1}, \ldots, E_k) \to C$$

are the restrictions to the first and last faces separated by the $i$-th edge. Employ the notation

$$Diag^{a,b}(E_1, \ldots, E_i, \ldots, E_k; C)$$

for the subobject restricting to a given $a$ and $b$. If we don’t wish to specify $b$ for example, then denote this by the superscript $Diag^a$. In particular note that we can decompose into a disjoint union over the $k+1$-tuples of objects which are the images of the vertices $0, \ldots, k$ (these objects are all specified either as a part of $a$ or as a part of $b$).

4.2.3 In case $C = nCAT$ we have

$$Diag^{a,b}(E_1, \ldots, E_i, \ldots, E_k; nCAT) = Hom(U_{i-1} \times E_i, U_i)$$

where $U_j$ are the fibrant $n$-categories which are the images of the vertices

$$j \in \Upsilon^k(E_1, \ldots, E_k)$$

by the maps $a$ (if $j \leq i - 1$) or $b$ (if $j \geq i$).
4.2.4 When we are only interested in the set of objects, it doesn’t matter which $E_i$ is underlined and we denote by

$$\text{Diag}(E_1, \ldots, E_k; C) = \text{Hom}(\Upsilon^k(E_1, \ldots, E_k), C)$$

this set of objects. We can put a superscript $\text{Diag}^{a,b}$ here if we want (with the obvious meaning as above). The edge $i$ dividing between $a$ and $b$ should be understood from the data of $a$ and $b$.

4.2.5 We need a way of understanding the statement that $n\text{CAT}'$ is a fibrant replacement for $n\text{CAT}$. In order to do this we will use the following property of $n\text{CAT}$ which shows that in some sense it is close to being fibrant.

We say that an $n+1$-category $C$ is quasifibrant if for any sequence of objects $x_0, \ldots, x_p$ the morphism

$$C_{p/}(x_0, \ldots, x_p) \to C_{(p-1)/}(x_0, \ldots, x_{p-1}) \times_{C_{(p-1)/}(x_1, \ldots, x_p)} C_{(p-2)/}(x_1, \ldots, x_{p-1})$$

is a fibration of $n$-categories. Note inductively that the morphisms involved in the fiber product here are themselves fibrations, and we get that the projections

$$C_{p/}(x_0, \ldots, x_p) \to C_{(p-1)/}(x_0, \ldots, x_{p-1})$$

and

$$C_{p/}(x_0, \ldots, x_p) \to C_{(p-1)/}(x_1, \ldots, x_p)$$

are fibrations.

4.2.6 The condition that $C$ is an $n+1$-category implies that the morphism in the definition of quasifibrant, is an equivalence whenever $p \geq 2$. Thus if $C$ is quasifibrant, the morphism in question is actually a fibrant equivalence.

4.2.7 If $C'$ is a fibrant $n+1$-category then it is quasifibrant. This is because the morphisms (in the notation of 2.4.3)

$$[p-1](E) \cup [p-2](E) \to [p-1](E) \to [p](E)$$

are trivial cofibrations.
4.2.8 The $n + 1$-category $nCAT$ is easily seen to be quasifibrant: the morphisms in question are actually isomorphisms for $p \geq 2$ and for $p = 1$ they are just projections from the $\underline{Hom}(A_0, A_1)$—which are fibrant—to $\ast$.

We now have two claims which allow us to pass between something quasifibrant such as $nCAT$ and its fibrant completion.

4.2.9 First of all, if $C$ is quasi-fibrant (4.2.5) then $\text{Diag}(E_1, \ldots, E_i, \ldots, E_k; C)$ is fibrant. Furthermore in this case for cofibrations $E'_j \hookrightarrow E_j$ the morphism

$$\text{Diag}(E'_1, \ldots, E'_i, \ldots, E'_k; C) \rightarrow \text{Diag}(E_1, \ldots, E_i, \ldots, E_k; C)$$

is fibrant.

4.2.10 Secondly, if $C$ quasifibrant (4.2.5) and if $C \rightarrow C'$ is an equivalence to a fibrant $C'$ then the morphism

$$\text{Diag}^{a,b}(E_1, \ldots, E_i, \ldots, E_k; C) \rightarrow \text{Diag}^{a',b'}(E_1, \ldots, E_i, \ldots, E_k; C')$$

is an equivalence of fibrant $n$-categories. Here $a, b$ are fixed as in 4.2.2, and $a', b'$ denote the images in $C'$.

Caution: it is essential to restrict to the components for a fixed $a, b$ coming from $C$.

4.2.11 Before getting to the proofs of 4.2.9 and 4.2.10, we discuss diagrams in a quasi-fibrant $C$. A morphism

$$u : \Upsilon^k(E_1, \ldots, E_k) \rightarrow C$$

may be described inductively as triple $u = (\bar{u}, u^-, u^+)$ where

$$u^- : \Upsilon^{k-1}(E_2, \ldots, E_k) \rightarrow C$$

and

$$u^+ : \Upsilon^{k-1}(E_1, \ldots, E_{k-1}) \rightarrow C,$$

are morphisms which agree on $\Upsilon^{k-1}(E_1, \ldots, E_{k-1})$, and where

$$\bar{u} : E_1 \times \ldots \times E_k \rightarrow C_{k/}(x_0, \ldots, x_k)$$

is a lifting of the morphism $(\bar{u}^-, \bar{u}^+)$ (these are the components of $u^-$ and $u^+$ analogous to the component $\bar{u}$ of $u$) along the morphism

$$C_{k/}(x_0, \ldots, x_k) \rightarrow C_{(k-1)/}(x_0, \ldots, x_{k-1}) \times C_{(k-2)/}(x_{k-1}, \ldots, x_k).$$
4.2.12 If $C$ is quasifibrant then the morphisms involved in the previous description are fibrations. We obtain the following result: that if $E'_i \subset E_i$ are trivial cofibrations and $C$ is quasifibrant then any diagram

$$\Upsilon^k(E'_1, \ldots, E'_k) \to C$$

extends to a diagram

$$\Upsilon^k(E'_1, \ldots, E'_k) \to C.$$ 

We can prove this by induction on $k$, and we are reduced exactly to the lifting property for the trivial cofibration

$$E'_1 \times \ldots \times E'_k \hookrightarrow E_1 \times \ldots E_k$$

along the morphism

$$C_{k/}(x_0, \ldots, x_k) \to C_{(k-1)/}(x_0, \ldots, x_{k-1}) \times C_{(k-1)/}(x_1, \ldots, x_k) \to C_{(k-2)/}(x_1, \ldots, x_{k-1}).$$

This morphism being fibrant by hypothesis, the lifting property holds.

4.2.13 Suppose $C$ is quasifibrant. Then for $a, b$ fixed as in 4.2.2 the morphisms

$$Diag^{a,b}(E_1, \ldots, E_i, \ldots, E_k; C) \to Diag^{a(i-1), b(i)}(E_i; C)$$

are fibrant weak equivalences, where $a(i-1)$ and $b(i)$ are the images by $a$ and $b$ of the $i-1$-st and $i$-th vertices.

To prove this we use the description 4.2.11, inductively reducing $k$. The remark 4.2.6 says that for any $k \geq 2$ the choice of lifting $\tilde{u}$ doesn’t change the equivalence type of the $Diag$ n-category. This reduces down to the case $k = 1$, which gives exactly that the restriction to the $i$-th edge is an equivalence (the restrictions to the other edges are fixed and don’t contribute anything because we fix $a, b$).

4.2.14 Proofs of 4.2.9 and 4.2.10: The statements of 4.2.9 are direct consequences of the lifting property 4.2.12.

To prove 4.2.10, in view of 4.2.13 it suffices to consider the case $k = 1$. Now

$$Diag^{U,V}(E; C) = \text{Hom}(E, C_{1/}(U, V)).$$

Therefore if $C \to C'$ is any morphism of quasifibrant $n + 1$-categories which is “fully faithful” i.e. induces equivalences of fibrant $n$-categories

$$C_{1/}(U, V) \to C'_{1/}(U, V),$$
then

\( \text{Diag}^{U,V}(E; C) \to \text{Diag}^{U,V}(E; C') \)

are equivalences by \(6.2.3\).

(Note by \(4.2.7\) that the equivalence \(C \to C'\) to a fibrant \(C'\) that occurs in the hypothesis of \(4.2.10\) is, in particular, a fully faithful morphism of quasifibrant \(n+1\)-categories.)

This proves \(4.2.10\) for \(k = 1\) and hence by \(4.2.13\) for any \(k\).

\(4.2.15\) The hypotheses on \(C \to C'\) used in \(4.2.9\) and \(4.2.10\) are satisfied by \(n\text{CAT} \to n\text{CAT}'\), cf \(4.2.8\). Therefore we may apply the results \(4.2.9\) and \(4.2.10\) to \(n\text{CAT} \to n\text{CAT}'\).

Fix \(a, b\) as in \(1.2.2\) for the following \(\text{Diag}\)'s, and suppose that the restriction of \(a\) to \(\Upsilon(B)\) is equal to \(1_B\). From \(4.2.9\) and \(4.2.10\), the morphism

\[ \text{Diag}^{a,b}(B, E_2, \ldots, E_i, \ldots, E_k; n\text{CAT}) \to \text{Diag}^{a,b}(B, E_2, \ldots, E_i, \ldots, E_k; n\text{CAT}') \]

is an equivalence between fibrant \(n\)-categories.

\(4.3\) Some extensions

\(4.3.1\) In the following preliminary statements we fix \(k \geq 2\). We will only use these statements for \(k = 2, 3\).

\(4.3.2\) We now describe what will be our main technical tool. Suppose \(B\) is a fibrant \(n\)-category. We have a natural morphism

\[ 1_B \in \text{Hom}^{*,B}(\Upsilon B, n\text{CAT}') \]

coming from the identity morphism \(* \times B \to B\) in \(n\text{CAT}\) (which is then considered as a morphism in \(n\text{CAT}'\)).

\(4.3.3\) For any \(E_1, E_2, \ldots, E_k\) let

\[ \text{Shell}\Upsilon^k(E_1, \ldots, E_k) := \]

\[ \Upsilon^{k-1}(E_1, \ldots, E_{k-1}) \cup \bigcup_{i=1}^{k-1} \Upsilon^{k-1}(\ldots, E_i \times E_{i+1}, \ldots) \]

(thus it consists of all of the “faces” except the one \(\Upsilon^{k-1}(E_2, \ldots, E_k)\)). We have a cofibration

\[ \text{Shell}\Upsilon^k(E_1, \ldots, E_k) \to \Upsilon^k(E_1, \ldots, E_k). \]
4.3.4 Now set $E_1 = B$ and let $\text{Hom}^1_B(\Upsilon^k(B, E_2, \ldots, E_k), \text{nCAT}')$ denote the fiber of 
\[ \text{Hom}(\Upsilon^k(1_B, E_2, \ldots, E_k), \text{nCAT}') \to \text{Hom}(\Upsilon B, \text{nCAT}') \]
over $1_B$ and let $\text{Hom}^1_B(\text{Shell}\Upsilon^k(B, E_2, \ldots, E_k), \text{nCAT}')$ denote the fiber of 
\[ \text{Hom}(\text{Shell}\Upsilon^k(1_B, E_2, \ldots, E_k), \text{nCAT}') \to \text{Hom}(\Upsilon B, \text{nCAT}') \]
over $1_B$.

**Lemma 4.3.5** Suppose $B$ is a fibrant $n$-category and $E$ an $n$-precat, and $U$ an object of $\text{nCAT}'$ (it is also an object of $\text{nCAT}$). The morphism 
\[ \text{Diag}^1_B:U_B(\text{B,E}; \text{nCAT}') \to \text{Diag}^U(\text{B \times E}, \text{nCAT}') \]
is an equivalence of $n$-categories.

**Proof:** In view of 4.2.9 and 4.2.10 it suffices to prove the same thing for diagrams in $\text{nCAT}$. In this case, use the calculation of 4.2.3: both sides become equal to $\text{Hom}(\text{B \times E}, U)$. ///

4.3.6 —Remark: Since $\text{nCAT}'$ is a fibrant $n + 1$-category, the morphism
\[ \text{Diag}^1_B:U_B(\text{B,E}; \text{nCAT}') \to \text{Diag}^U(\text{B \times E}, \text{nCAT}') \]
is fibrant. One checks directly the lifting property for a trivial cofibration $F' \hookrightarrow F$, using the fibrant property of $\text{nCAT}'$.

**Corollary 4.3.7** The morphism 
\[ \text{Hom}^1_B(\Upsilon^2(B, E), \text{nCAT}') \to \text{Hom}^U(\Upsilon(B \times E), \text{nCAT}') \]
is surjective.

**Proof:** We can fix an object $U$ for the image of the last vertex. The morphism 
\[ \text{Diag}^1_B:U_B(\text{B,E}; \text{nCAT}') \to \text{Diag}^U(\text{B \times E}, \text{nCAT}') \]
is fibrant by the above remark 4.3.6 and it is an equivalence by 4.3.5. This implies that it is surjective on objects (2.2.13). ///
Corollary 4.3.8 Suppose $E' \subset E$ is a cofibration of $n$-precats. Suppose we are given an object of
\[ \text{Hom}^1_{\mathcal{B}}(\mathcal{T}^2(B, E'), n\text{CAT}'). \]
and an extension over the shell to an object of
\[ \text{Hom}^1_{\mathcal{B}}(\text{Shell}\mathcal{T}^2(B, E), n\text{CAT}'). \]
Then these two have a common extension to an element of
\[ \text{Hom}^1_{\mathcal{B}}(\mathcal{T}^2(B, E), n\text{CAT}'). \]

Proof: Again we can fix $U$. By Lemma 4.3.5 and remark 4.3.6, the morphism
\[ \text{Diag}^1_{\mathcal{B}, U}(B, F; n\text{CAT}') \rightarrow \text{Diag}^*_{\mathcal{B}, U}(B, n\text{CAT}') \]
is a trivial fibration. Therefore it has the lifting property with respect to any cofibration $E' \subset E$. This lifting property gives exactly what we want to show—this is because a morphism
\[ E \rightarrow \text{Diag}^1_{B, U}(B, F; n\text{CAT}') \]
is the same thing as an object of
\[ \text{Diag}^1_{\mathcal{B}, U}(B, E; n\text{CAT}') \]
or equivalently an element of $\text{Hom}^1_{\mathcal{B}}(\mathcal{T}^2(B, E), n\text{CAT}')$. \hfill \Box

Now we treat a similar type of extension problem for shells with $k = 3$.

4.3.9 Now suppose we have an object $b \in \text{Diag}(F; n\text{CAT}')$. Let
\[ \text{Diag}^1_{\text{Shell}}(B, E, F; n\text{CAT}') \]
be the $n$-precat representing the functor
\[ G \mapsto \text{Hom}^1_{B, b}(\text{Shell}\mathcal{T}^3(B, E \times G, F), n\text{CAT}') \]
where the superscript on the $\text{Hom}$ has the obvious meaning that we look only at morphisms restricting to $1_B$ on the edge 01 and to $b$ on the edge 23.

The shell $\text{Shell}\mathcal{T}^3(B, E, F \times G)$ has three faces. We call the faces (013) and (023) the last faces and the face (012) the first face. Restriction to the last faces (which meet along the edge (03)) gives a map
\[ \text{Diag}^1_{\text{Shell}}(B, F; n\text{CAT}') \rightarrow \text{Diag}^1_{\text{Shell}}(B, E \times F; n\text{CAT}') \times_{\text{Diag}^1_{\text{Shell}}(B \times E \times F; n\text{CAT}')} \text{Diag}^1_{\text{Shell}}(B \times E, F; n\text{CAT}'), \]
where $b_3$ denotes the object image of 3 under the map $b$; a similar definition will hold for $b_2$ below—and recall that the image of 0 under the map $1_B$ is $\ast$. 

46
4.3.10 Claim: that the map at the end of the previous paragraph is a fibrant equivalence. Call the object on the right in this morphism $D$. Restriction to the edge $(02)$ is a map

$$D \to \text{Diag}^{*b_2}(B \times E; n\text{CAT}').$$

We have an isomorphism

$$\text{Diag}^{1_{B,b}}_\text{shell}(B, E, F; n\text{CAT}') \cong D \times_{\text{Diag}^{*b_2}(B \times E; n\text{CAT}')} \text{Diag}^{1_{B,b}}(B, E; n\text{CAT}').$$

However, the second morphism in this fiber product is

$$\text{Diag}^{1_{B,b}}(B, E; n\text{CAT}') \to \text{Diag}^{*b_2}(B \times E; n\text{CAT}')$$

which is a fibrant equivalence by Lemma 4.3.5. It follows that the morphism

$$\text{Diag}^{1_{B,b}}_\text{shell}(B, E, F; n\text{CAT}') \to D$$

is a weak equivalence (note also that it is fibrant). This proves the claim.

**Corollary 4.3.11** The morphism

$$\text{Diag}^{1_{B,b}}(B, E, F; n\text{CAT}') \to \text{Diag}^{1_{B,b}}_\text{shell}(B, E, F; n\text{CAT}')$$

is a fibrant equivalence.

**Proof:** It is fibrant because $n\text{CAT}'$ is fibrant. In view of the claim 4.3.10 it suffices to note that the map

$$\text{Diag}^{1_{B,b}}(B, E, F; n\text{CAT}') \to D$$

is an equivalence, and this by the fibrant property of $n\text{CAT}'$ (the union of the faces $(013)$ and $(023)$ is one of the admissible ones in our list of 2.4.8). ///

**Corollary 4.3.12** Suppose $E' \subset E$ is a cofibration of $n$-precats. Then for any morphism

$$\Upsilon^3(B, E', F) \to n\text{CAT}'$$

sending the edge $(01)$ to $1_B$, and any extension of this over the shell to a morphism

$$\text{Shell}\Upsilon^3(B, E, F) \to n\text{CAT}'$$

again restricting to $1_B$ on the edge $(01)$, there exists a common extension to a morphism

$$\Upsilon^3(B, E, F) \to n\text{CAT}'. $$
Proof: By the previous Corollary 4.3.11 the morphism
\[ \text{Diag}^{1_{B,b}}(B, F; nCAT') \to \text{Diag}_{\text{Shell}}^{1_{B,b}}(B, F; nCAT'), \]
is a fibrant equivalence. Therefore it satisfies the lifting property for any cofibration \( E' \hookrightarrow E \), and as before (4.3.8) a map from \( E \) into \( \text{Diag}^{1_{B,b}}(B, F; nCAT') \) is the same thing as an object of \( \text{Diag}_{\text{Shell}}^{1_{B,b}}(B, F; nCAT') \) (and the same things for \( E' \) and for \( \text{Diag}_{\text{Shell}}^{1_{B,b}} \)). This gives the required statement. ///

4.4 EXTENSION PROPERTIES FOR INTERNAL \( \text{Hom} \)

Now we take the above extension properties and recast them in terms of internal \( \text{Hom} \). This is because we will need them for products of our precats \( \Upsilon \) with an arbitrary \( A \). Note that there is a difference between the internal \( \text{Hom} \) referred to in this section (which are \( n + 1 \)-categories) and the \( \text{Diag} n \)-categories above.

We state the following lemma for any value of \( k \) but we will only use \( k = 2 \) and \( k = 3 \); and we give the proofs only in these cases, leaving it to the reader to fill in the combinatorial details for arbitrary \( k \).

Lemma 4.4.1 For any \( n \)-precats \( E_2, \ldots, E_k \), the morphism
\[ \text{Hom}^{1_B}(\Upsilon^k(B, E_2, \ldots, E_k), nCAT') \to \text{Hom}^{1_B}(\text{Shell}\Upsilon^k(B, E_2, \ldots, E_k), nCAT') \]
is an equivalence of \( n + 1 \)-categories.

Proof: The morphism in question is fibrant—cf 6.2.3.

The proof is divided into several paragraphs. In 4.4.2–4.4.4 we give the proof for \( k = 2 \). Then in 4.4.5 we give the proof for \( k = 3 \).

4.4.2 We begin the proof for \( k = 2 \). Corollary 4.3.7 implies that the morphism in question
\[ \text{Hom}^{1_B}(\Upsilon^2(B, E_2), nCAT') \to \text{Hom}^*(\Upsilon(B \times E_2), nCAT') \]
is surjective on objects.

4.4.3 Now we have to prove that our morphism induces equivalences between the morphism \( n \)-categories. Suppose
\[ f, g : \Upsilon^2(B, E_2) \to nCAT' \]
are two morphisms (with the appropriate behavior on (01)). Then the \( n \)-category of morphisms between them represents the functor
\[ F \mapsto \text{Hom}^{f,g}^{1_B}(\Upsilon F \times \Upsilon^2(B, E_2), nCAT') \]
where the superscript means morphisms restricting to $f$ and $g$ over $0, 1 \in \Upsilon F$ and restricting to $1_B$ over $\Upsilon F \times \Upsilon B$. This maps (by restricting to the edge 02) to the functor
\[ F \mapsto \text{Hom}_{\text{g}^*}(\Upsilon F \times \Upsilon (B \times E_2), n\text{CAT}'). \]

We would like to prove that this restriction map of functors is an equivalence. In order to prove this it suffices to prove that it has the lifting property for any cofibrations $F' \subset F$. Thus we suppose that we have a morphism
\[ \eta : \Upsilon F \times \Upsilon (B \times E_2) \to n\text{CAT}' \]
(restricting appropriately to $f$ and $g$ and to $*$), as well as a morphism
\[ \zeta' : \Upsilon F' \times \Upsilon^2(B, E_2) \to n\text{CAT}', \]
restricting appropriately to $f$, $g$ and $1_B$. We would like to extend this latter to a map defined on $F$ and compatible with the previous one. This extension will complete the proof for $k = 2$.

4.4.4 We now prove the extension statement claimed above. As in 3.4.8 we consider the diagram as the product of an interval $(01)$ and a triangle $(012)$ and we denote the points by $(i,j)$ for $i = 0, 1$ and $j = 0, 1, 2$. More generally for example $(ab, cd)$ denotes the square which is the edge $(ab)$ crossed with the edge $(cd)$. We are provided with maps on the end triangles $f$ on $(0, 012)$ and $g$ on $(1, 012)$ as well as $\eta$ on the top square $(01, 02)$. We fix the map on the square $(01, 01)$ (which is $\Upsilon(F) \times \Upsilon(B)$ pullback of $1_B$, and call this again $1_B$). We are also provided with a map $\zeta'$ defined on the whole diagram with respect to $F'$ and we would like to extend this all to $\zeta$ defined on the whole diagram.

Note that we can write $\Upsilon(F) \times \Upsilon^2(B, E)$ as the coproduct of three tetrahedra which we denote
\[ (0, 0) (1, 0) (1, 1) (1, 2) \quad \text{i.e.} \quad \Upsilon^3(F, B, E), \]
\[ (0, 0) (0, 1) (0, 2) (1, 2) \quad \text{i.e.} \quad \Upsilon^3(B, E, F), \]
\[ (0, 0) (0, 1) (1, 1) (1, 2) \quad \text{i.e.} \quad \Upsilon^3(B, F, E). \]

The first step is to use the fibrant property of $n\text{CAT}'$ to extend our given morphisms $g$, the restriction of $1_B$ to the triangle $(0, 0), (1, 0), (1, 1)$, and the restriction of $\eta$ to the triangle $(0, 0), (1, 0), (1, 2)$, to a map on the tetrahedron
\[ (0, 0) (1, 0) (1, 1) (1, 2). \]

We can do this in a way which extends the map $\zeta'$.

49
Next we again use the fibrant property of $nCAT'$ to extend across the tetrahedron

$$(0, 0) \ (0, 1) \ (0, 2) \ (1, 2).$$

Note that we are provided with the map $f$ on the triangle $(0, 012)$, and the restriction of the $\eta$ on the triangle $(0, 0), (0, 2), (1, 2)$. We can find our extension again in a way extending the given map $\zeta'$.

Finally we come to the tetrahedron

$$(0, 0) \ (0, 1) \ (1, 1) \ (1, 2),$$

which is of the form $\Upsilon^3(B, F, E)$. Here we are given maps on all of the faces except the last one, i.e. on the shell of this tetrahedron, and we would like to extend it. The given maps are the pullback of $1_B$ on the first face, and the maps coming from the two previous paragraphs on the other two faces. Furthermore we already have a map $\zeta'$ over the tetrahedron $\Upsilon^3(B, F', E)$. The given map on the shell restricts on the first edge to $1_B$, so this is an extension problem of the type which we have already treated in Corollary 4.3.12 above. (N.B. the notations $E$ and $F$ are interchanged between 4.3.12 and the present situation.) Thus Corollary 4.3.12 provides the extension we are looking for, and we have finished making our extension across the three tetrahedra. This completes the proof of “fully faithfulness” so the morphism in the lemma is an equivalence in the case $k = 2$.

4.4.5 Here is the proof for $k = 3$. First of all, the morphism

$$\text{Hom}^1_B(\Upsilon^3(B, E_2, E_3), nCAT') \to \text{Hom}^1_B(\Upsilon^2(B, E_2), nCAT') \times_{\text{Hom}(\Upsilon(B \times E_2), nCAT')} \text{Hom}(\Upsilon^2(B \times E_2, E_3), nCAT')$$

is an equivalence, by the fibrant property for $nCAT'$.

By the case $k = 2$ (4.4.2 4.4.3) applied to the face 012, the morphism

$$\text{Hom}^1_B(\text{Shell}\Upsilon^3(B, E_2, E_3), nCAT') \to \text{Hom}^1_B(\Upsilon^2(B, E_2), nCAT') \times_{\text{Hom}(\Upsilon(B \times E_2), nCAT')} \text{Hom}(\Upsilon^2(B \times E_2, E_3), nCAT')$$

is an equivalence. This implies that the morphism

$$\text{Hom}^1_B(\Upsilon^3(B, E_2, E_3), nCAT') \to \text{Hom}^1_B(\text{Shell}\Upsilon^3(B, E_2, E_3), nCAT')$$

is an equivalence. This completes the case $k = 3$. 

50
This completes the proof of the lemma (as far as we are going).

4.4.6 — Remark: One might think that we have a simple argument for the case \( k = 3 \), and the only difficult part of the argument for \( k = 2 \) was the part where we used \( k = 3 \). However one cannot simplify the proof: the simple argument for \( k = 3 \) is based upon the use of internal \( \text{Hom} \) and to get \( k = 2 \) for internal \( \text{Hom} \) we need a statement like the case of \( k = 3 \)—the statement which in the above proof is provided by Corollary 4.3.12. This is why we were obliged to do all of the stuff in the previous subsection.

We will only use the subsequent corollaries in the cases \( k = 2 \) and \( k = 3 \), so the proof we have given of 4.4.1 is sufficient. Again the reader is invited to treat the case of any \( k \).

**Corollary 4.4.7** Suppose \( A \) is an \( n + 1 \)-precat and suppose \( E_i \) are \( n \)-precats for \( i = 2, \ldots, k \). Suppose we are given a morphism

\[
A \times \text{Shell} \Upsilon^k(B, E_2, \ldots, E_k) \to n \text{CAT}'
\]

restricting to \( 1_B \) on \( A \times \Upsilon B \). Then there is an extension to a morphism

\[
A \times \Upsilon^k(B, E_2, \ldots, E_k) \to n \text{CAT}'.
\]

**Proof:** The restriction morphism on the \( \text{Hom} \) is fibrant and an equivalence by the previous lemma, therefore it is surjective on objects.

What we really need to know is a relative version of this for cofibrations \( E'_i \subset E_i \).

**Corollary 4.4.8** Suppose \( A \) is an \( n \)-category and suppose \( E'_i \subset E_i \) are cofibrations of \( n \)-precats for \( i = 2, \ldots, k \). Suppose we are given a morphism

\[
A \times \text{Shell} \Upsilon^k(B, E_2, \ldots, E_k) \to n \text{CAT}'
\]

restricting to \( 1_B \) on \( A \times \Upsilon B \), together with a filling-in

\[
A \times \Upsilon^k(B, E'_2, \ldots, E'_k) \to n \text{CAT}',
\]

then there is an extension of all of this to a morphism

\[
A \times \Upsilon^k(B, E_2, \ldots, E_k) \to n \text{CAT}'.
\]
Proof: Let $H_E$ denote the $\text{Hom}$ for the full $\mathcal{Y}$ and let $H_{E}^{\text{Sh}}$ denote the $\text{Hom}$ for $\text{Shell} \mathcal{Y}$. The morphism

$$H_E \to H_{E}^{\text{Sh}} \times_{H_{E}^{\text{Sh}}} H_{E}'$$

is an equivalence (as is seen by applying the lemma for both $E$ and $E'$) and it is fibrant (since it comes from $\text{Hom}$ applied to a cofibration). Therefore it is surjective on objects, which exactly means that we have the above extension property. ///

4.5 Proof of Theorem 4.0.1

4.5.1 Recall that $\lambda$ was defined in 4.1.2. We first apply the above statements to find our morphism $\epsilon : \lambda_A \to \varphi$. The universal property of $\lambda$ (4.1.2) applied to the identity map $\lambda \to \lambda$ gives a morphism

$$\eta : A \times \mathcal{Y}(\lambda) \to n\text{CAT}'$$

sending $A \times \{0\}$ to $*_A$ and sending $A \times \{1\}$ to $\varphi$. By Corollary 4.4.7 (for $k = 2$) there is a morphism

$$\epsilon^{(2)} : A \times \mathcal{Y}^2(\lambda, *) \to n\text{CAT}'$$

such that

$$r_{02}(\epsilon^{(2)}) = \eta$$

and

$$r_{01}(\epsilon^{(2)}) = 1_{\lambda}.$$ 

Note that $r_{12}(\epsilon^{(2)})$ is a morphism from $A \times \mathcal{Y} = A \times I$ into $n\text{CAT}'$ restricting to $\lambda_A$ and $\varphi$, which by definition means a morphism $\lambda \to \varphi$. Call this morphism $\epsilon$.

4.5.2 Claim: that for any fibrant $n$-category $B$ and any morphism

$$f : A \times \mathcal{Y}E \to n\text{CAT}'$$

with

$$r_0(f) = B_A, \quad r_1(f) = \varphi$$

there is a morphism

$$f' : \mathcal{Y}^2(E, *) \to n\text{CAT}'$$

with

$$r_{02}(f') = f, \quad r_{12}(f') = \epsilon.$$ 

This almost gives the required property to show that $\lambda \Rightarrow \varphi$ is an inverse limit. Technically speaking we also will have to show the above claim in the relative situation of $E' \subset E$. This we will do below (4.5.8–4.5.13) after first going through the argument in the absolute case (4.5.3–4.5.7).
4.5.3 The basic idea is to use what we know up until now to construct a morphism

\[ F : A \times \Upsilon^3(B, E, *) \to nCAT' \]

with

\[ r_{01}(F) = 1_B, \quad r_{12}(F) = f, \quad r_{23}(F) = \epsilon. \]

Setting \( f' = r_{123}(F) \) we will obtain the morphism asked for in the previous paragraph. In order to follow the construction the reader is urged to draw a tetrahedron with vertices labeled 0, 1, 2, 3, putting respectively \( B, E, *, B \times E, E, B \times E \) along the edges 01, 12, 23, 02, 13, 03; then putting in \( *_A, B_A, \lambda_A \) and \( \varphi \) at the vertices 0, 1, 2, 3 respectively. And finally putting in \( 1_B \) along edge 01, \( f \) along edge 13 and \( \epsilon \) along edge 23.

Our strategy is to fill in all of the faces except 123, then call upon Corollary 4.4.7 to fill in the tetrahedron thus getting face 123.

4.5.4 The first step is the face 013. This we fill in using simply the fact that \( nCAT' \) is a fibrant \( n + 1 \)-category. The edges 01 and 13 are specified so we can fill in to a morphism \( A \times \Upsilon^2(B, E) \to nCAT' \) (restricting to \( 1_B \) and \( f \) on the edges 01 and 13). Now the restriction of this morphism to edge 03 provides a morphism \( g : A \times \Upsilon(B \times E) \to nCAT' \) restricting to \( *_A \) and \( \varphi \).

4.5.5 The next step is to notice that by the universal property 4.1.2 of \( \lambda \) there is a morphism \( B \times E \to \lambda \) such that \( g \) is deduced from \( \eta \) by pullback via \( \Upsilon(E \times B) \to \Upsilon(\lambda) \).

This same morphism yields

\[ \Upsilon^2(E \times B, *) \to \Upsilon^2(\lambda, *), \]

and we can use this to pull back the above morphism \( \epsilon^{(2)} \). This gives a morphism

\[ h : A \times \Upsilon^2(E \times B, *) \to nCAT' \]

where (adopting exceptionally for obvious reasons here the notations 0, 2 and 3 for the vertices of this \( \Upsilon^2 \))

\[ r_{03}(h) = g, \quad r_{23}(h) = \epsilon. \]

This treats the face 023.
Finally, for the face 012 we have a morphism

\[ r_{02}(h) : A \times \Upsilon(E \times B) \to nCAT' \]

restricting to \(*_A\) and \(\lambda_A\). By Corollary \[4.4.7\] applied with \(k = 2\) (for the map

\[ A \times Shell\Upsilon^2(B, E) \to nCAT' \]

given by \(1_B\) and \(h\)) we get a morphism

\[ m : \Upsilon^2(B, E) \to nCAT' \]

with \(r_{01}(m) = 1_B\) and \(r_{02}(m) = r_{02}(h)\).

Putting all of these together we obtain a morphism

\[ F' : A \times Shell\Upsilon^3(B, E, \ast) \to nCAT' \]

restricting to \(1_B\) on edge 01 and restricting to \(f\) on edge 13 and \(\epsilon\) on edge 23. Corollary \[4.4.7\] applied with \(k = 3\) gives an extension over the tetrahedron to a morphism

\[ F : A \times \Upsilon^3(B, E, \ast) \to nCAT'' \]

again restricting to \(1_B\) on edge 01 and restricting to \(f\) on edge 13 and \(\epsilon\) on edge 23. The restriction to the last face \(r_{123}\) yields the filling-in desired.

We now treat the case where \(E' \subset E\) is a cofibration and where we already have a filling-in of the face 123 for \(E'\). We would like to obtain a filling-in of this face for \(E\). Basically the only difficulty is that we don’t yet know that the filling-in of face 123 for \(E'\) comes from a filling-in of the whole tetrahedron compatible with the above process. In particular this causes a problem at the step where we fill in face 023.

Before getting started we use the fibrant property of \(nCAT'\) to obtain a morphism

\[ A \times \Upsilon^3(B, E', \ast) \to nCAT' \]

restricting to our given morphism on the face 123, and restricting to \(1_B\) on the edge 01. Actually we would like to insure that the restriction to the face 012 comes from a morphism

\[ \Upsilon^2(B, E') \to nCAT' \]
by pulling back along the projection $A \to \ast$. In order to do this notice that the restriction of the given map to the edge 12 comes from $\Upsilon(E') \to nCAT'$. Thus we can first extend this map combined with $1_B$ to a morphism $\Upsilon^2(B, E') \to nCAT'$. Now the morphism

$$(A \times \Upsilon^2(B, E')) \cup^{A \times \Upsilon E'} A \times \Upsilon^2(E', \ast) \to \Upsilon^3(B, E', \ast)$$

is a trivial cofibration so we can extend from here to obtain $A \times \Upsilon^3(B, E', \ast) \to nCAT'$ with restriction to the face 012 coming from $\Upsilon^2(B, E') \to nCAT'$. This is our point of departure for the rest of the argument.

4.5.10 The first step following the previous outline is to fill in the face 013. We note that the morphism $\Upsilon B \cup^{\{1\}} \Upsilon E \to \Upsilon^2(B, E)$ is a trivial cofibration. Thus also the morphism

$$(\Upsilon B \cup^{\{1\}} \Upsilon E) \cup^{\Upsilon B \cup^{\{1\}} \Upsilon E'} \Upsilon^2(B, E') \to \Upsilon^2(B, E)$$

is a trivial cofibration, so given the edges 01 and 13 (for $E'$) with filling-in over the face 013 with respect to $E'$, we can fill in 013 with respect to $E$.

4.5.11 The face 023

Now we treat the face 023. Let

$$g : A \times \Upsilon(E \times B) \to nCAT'$$

be the restriction of the map obtained in 4.5.10 to the edge 03. Let $g'$ denote its restriction to $A \times \Upsilon(E' \times B)$. The map given in 4.5.9 restricts on (023) to a morphism

$$h' : A \times \Upsilon^2(E' \times B, \ast) \to nCAT'$$

where (using as above the notations 0, 2 and 3 for the vertices of this $\Upsilon^2$)

$$r_{03}(h') = g', \quad r_{23}(h') = \epsilon.$$ 

Let $a' = r_{02}(h')$. It is a morphism

$$a'_A : A \times \Upsilon(E' \times B) \to nCAT'$$

with $r_0(a'_A) = \ast_A$ and $r_2(a'_A) = \lambda_A$. By hypothesis on our map over the full tetrahedron for $E'$ (cf 4.5.3), $a'_A$ comes from a map

$$a' : \Upsilon(E' \times B) \to nCAT'$$
again with values $*$ and $\lambda$ on the endpoints. This map corresponds to

$$E' \times B \to nCAT'_{1/}(\ast, \lambda).$$

The morphism $nCAT' \to nCAT''$ is an equivalence so our morphism is equivalent to a different morphism $b': E' \times B \to nCAT'_{1/}(\ast, \lambda)$. These two resulting morphisms $\Upsilon(E' \times B) \to nCAT''$ are equivalent so by 2.2.16, 2.2.17 there is a morphism

$$T \times \Upsilon(E' \times B) \to nCAT'$$

sending the endpoints $0, 1 \in T$ to $a'$ and $b'$.

Using this different morphism $b'$ (which is now the same thing as a map $E' \times B \to \lambda$) we pull back our standard $\eta \in \text{Hom}(A \times \Upsilon^2(\lambda, \ast), nCAT'')$ to get a morphism

$$A \times \Upsilon^2(E' \times B, \ast) \to nCAT'$$

restricting on the edges to $b'$ and $\epsilon$ respectively.

Now we have a map from

$$\left( A \times \Upsilon^2(E' \times B, \ast) \right) \cup \left( A \times T \times \left[ \Upsilon(E' \times B) \cup \Upsilon(\ast) \right] \right) \cup \left( A \times \Upsilon^2(E' \times B, \ast) \right)$$

to $nCAT'$, where the first term is glued to the second term along $1 \in T$ and the last term is glued to the second term along $0 \in T$ (we have omitted in the notation the $n+1$-precats along which the glueing takes place, the reader may fill them in as an exercise!).

The morphism from the above domain to

$$A \times T \times \Upsilon^2(E' \times B, \ast)$$

is a trivial cofibration, so since $nCAT'$ is fibrant there exists an extension of the above to a morphism

$$A \times T \times \Upsilon^2(E' \times B, \ast) \to nCAT''.$$

This morphism is a standard one coming from $b': E' \times B \to \lambda$ on the end $1 \in T$, and it is our given $h'$ on the end $0 \in T$.

We now go to the edge 03 of the triangle 023. We are also given an extension of $g'$ to $g: A \times \Upsilon(E \times B) \to nCAT''$ along the edge 03 of the triangle and 0 of the interval $T$. Thus, using the face (03) $\times T$, we have a morphism

$$(A \times \Upsilon(E \times B)) \cup A \times \Upsilon(E' \times B) \to nCAT''.$$
Fill this in along the trivial cofibration
\[ (A \times \Upsilon(E \times B)) \cup^{A \times \Upsilon(E' \times B)} (A \times T \times \Upsilon(E' \times B)) \hookrightarrow A \times T \times \Upsilon(E \times B), \]
to give on the whole a morphism
\[ A \times T \times \Upsilon(E \times B) \cup^{A \times T \times \Upsilon(E' \times B)} (A \times T \times \Upsilon^2(E' \times B, \ast) \to nCAT', \]
where the morphism \( \Upsilon(E' \times B) \to \Upsilon^2(E' \times B, \ast) \) in question is the one coming from the edge 03.

Next we extend down along the triangle 023 times the end \( 1 \in T \). To do this, notice that our extension from the previous paragraph gives an extension of the morphism \( b' : \Upsilon(E' \times B) \to nCAT' \) to a morphism \( b : \Upsilon(E \times B) \to nCAT' \). By the universal property of \( \lambda \) this corresponds to an extension \( E \times B \to \lambda \). Now the morphism that we already have on the end \( 1 \in T \) comes by pulling back the standard \( \eta : A \times \Upsilon^2(\lambda, \ast) \to nCAT' \) via the map \( E' \times B \to \lambda \) so our extension allows us to pull back \( \eta \) to get a map \( b : A \times \Upsilon^2(E \times B, \ast) \to nCAT' \) extending the previous \( b' \).

Now we have our map
\[ A \times T \times \Upsilon^2(E' \times B, \ast) \to nCAT' \]
which is provided with an extension from \( E' \) to \( E \), over the faces \( (03) \times T \) and \( 023 \times 1 \) of the product of the triangle with the interval. Another small step is to notice that along the face \( (02) \times T \) the morphism is pulled back along \( A \to \ast \) from a morphism \( T \times \Upsilon(E' \times B) \to nCAT' \). On the other hand at the edge \( (02) \times \{1\} \) the extension from \( E' \) to \( E \) again comes from a morphism \( \Upsilon(E \times B) \to nCAT' \). We get a morphism
\[ T \times \Upsilon(E' \times B) \times^{\{1\} \times \Upsilon(E' \times B)} \Upsilon(E \times B) \to nCAT', \]
which can be extended along the trivial cofibration
\[ T \times \Upsilon(E' \times B) \times^{\{1\} \times \Upsilon(E' \times B)} \Upsilon(E \times B) \hookrightarrow T \times \Upsilon(E \times B) \]
to give a map
\[ T \times \Upsilon(E \times B) \to nCAT'. \]
Similarly we note that the map on the face \( (23) \times T \) is pulled back from our map \( \epsilon : \Upsilon(\ast) \to nCAT' \).

All together on the triangular icosahedron \( (023) \times T \) we have a morphism defined for \( E' \) plus, along the faces
\[ (03) \times T, \ (02) \times T, \ (23) \times T, \ (023) \times \{1\} \]
extensions from $E'$ to $E$ (all compatible on intersections of the faces and having the required properties along 02 and 23). The inclusion of this $n+1$-precat (which we will call $G$ for “gory” instead of writing it out) into

$$A \times T \times \Upsilon^2(E \times B, \ast)$$

is a trivial cofibration. Indeed $G$ comes by attaching to the end

$$A \times \{1\} \times \Upsilon^2(E \times B, \ast),$$

something of the form

$$A \times \partial \Upsilon^2(E \times B, \ast) \cup A \times \partial \Upsilon^2(E' \times B, \ast) \times A \times \Upsilon^2(E' \times B, \ast)$$

where $\partial \Upsilon^2(E \times B, \ast)$ denotes the coproduct of the three “edges” $\Upsilon(E \times B)$ (two times) and $T(\ast)$. The inclusion of the end $A \times \{1\} \times \Upsilon^2(E \times B, \ast)$ into $G$ is an equivalence, as is the inclusion of this end into the full product

$$A \times T \times \Upsilon^2(E \times B, \ast),$$

which proves that the map in question (from $G$ to the above full product) is a weak equivalence (and it is obviously a cofibration).

Now we again make use of the fibrant property to extend our map from $G$ to a morphism

$$A \times T \times \Upsilon^2(E \times B, \ast) \to n\text{CAT}'$$

When restricted to $A \times \{0\} \times \Upsilon^2(E \times B, \ast)$ this gives the extension $h$ desired in order to complete our treatment of the face 023.

4.5.12 For the face 012 the argument is the same as in the previous case but we apply Corollary 4.4.8 rather than 4.4.7 in view of our relative situation $E' \subset E$.

4.5.13 End of the proof of 4.0.1

We have constructed a morphism

$$F' : A \times \text{Shell} \Upsilon^3(B, E, \ast) \to n\text{CAT}'$$

restricting to $1_B$ on edge 01 and restricting to $f$ on edge 13 and $\epsilon$ on edge 23. Furthermore, by construction it restricts to our already-given morphism over $E'$. Corollary 4.4.8 applied with $k = 3$ gives an extension over the tetrahedron to a morphism

$$F : A \times \Upsilon^3(B, E, \ast) \to n\text{CAT}'$$
again restricting to $1_B$ on edge 01 and restricting to $f$ on edge 13 and $\epsilon$ on edge 23, and restricting to the already-given morphism over $E'$. The restriction to the last face $r_{123}(F)$ yields the filling-in desired. This completes the proof that $\lambda \rightarrow \varphi$ is an inverse limit, finishing the proof of Theorem 4.0.1.

Corollary 4.5.14 If $F : A \times B \to nCAT'$ is a functor from the product of two $n + 1$-categories, then taking the inverse limits first in one direction and then in the other, is independent of which direction is chosen first.

Proof: This is a consequence of Theorem 3.4.11 but can also be seen directly from the construction 4.1.2 of the limit.

5. Direct limits

Theorem 5.0.1 The $n + 1$-category $nCAT'$ admits direct limits.

One should probably be able to construct these direct limits in much the same way as in the topological case, roughly speaking by replacing a family by an equivalent one in which the morphisms are cofibrations (some type of telescope construction) and then taking the direct limit of $n$-precats in the usual sense. This seems a bit complicated to put into practice so we will avoid doing so by a trick.

5.0.2 The argument for a 1-category: Consider the following argument which shows that if $C$ is a category in which all inverse limits exist and in which projectors are effective, then $C$ admits direct limits. For a functor $\psi : A \to C$ let $D$ be the category whose objects are pairs $(c, u)$ where $c$ is an object of $C$ and $u : \psi \to c$ is a morphism. There is a forgetful functor $f : D \to C$. Let $\delta \in C$ be the inverse limit of $f$. Then for any $a \in A$ there is a unique morphism $\psi(a) \to f$. By the inverse limit property this yields a morphism $\psi(a) \to \delta$ and uniqueness implies that it is functorial in $a$. Thus we get a morphism $v : \psi \to \delta$ and $(\delta, v)$ is in $D$. As an object of $D$, $\delta$ has a morphism

$$p := v(\delta) : \delta \to \delta.$$ 

This is itself a morphism in $D$, so we get $p \circ p = p$ from naturality of $v$. Thus $p$ is a projector. Let $t$ be the direct factor of $\delta$ given by $p$. Composition $\psi \to \delta \to t$ gives a map $\psi \to t$ and we get a factorization $\psi \to t \to \delta$. Now $t$ is seen to be an initial object of $D$, hence $\psi \to t$ is a direct limit.
5.0.3 The only problem with this argument is a set-theoretic one. Namely, when one speaks of “limits” it is presupposed that the indexing category $A$ is small, i.e. is a set of some cardinality rather than a class. However our category $C$ is likely to be a class. Thus, in the above argument, $D$ is not small and we are not allowed to take the inverse limit over $D$.

5.0.4 Let’s see how to fix this up in the case $C = \text{Set}$ is the category of sets. Suppose we have a functor $\psi : A \to \text{Set}$ from a small category $A$. Let $\alpha$ be a cardinal number bigger than $|A|$ and bigger than the cardinal of any set in the image of $\psi$. Let $D_\alpha$ be the category of pairs $(c, u)$ as above where $c$ is contained in a fixed set of cardinality $\alpha$. Note that $D_\alpha$ has cardinality $\leq 2^\alpha$. Let $(\delta, v)$ be as above. The only hitch is that (since we know an expression of $\delta$ as a subset of certain types of functions on $D_\alpha$ with values in the parametrizing sets which themselves have cardinality $\leq \alpha$) the cardinality of $\delta$ seems a priori only to be bounded by $2^{2^\alpha}$. Let $\delta' \subset \delta$ be the smallest subset through which the map $v : \psi \to \delta$ factors. Note that the cardinality of $\delta'$ cannot be bigger than the sum of the cardinals of the $\psi(a)$ over $a \in A$, in particular $\delta'$ has cardinal $\leq \alpha$. But now the universal property of $\delta$ implies that $\delta = \delta'$, for it is easy to see that $\delta' \to f$ is again an inverse limit. Thus by actually counting we see that the cardinality of $\delta$ is really $\leq \alpha$ and up to isomorphism we may assume that $(\delta, v) \in D_\alpha$. This argument actually shows that the cardinality of $\delta$ is bounded independently of the choice of $\alpha$. Thus $\delta$ satisfies the universal property of a direct limit for morphisms to a set of any cardinality, so $\delta$ is the direct limit of $A$.

More generally, in the situation of 5.0.2 if we can define the $D_\alpha$ and if we know for some reason that every object $B \in D$ admits a map $B' \to D$ from an object $D' \in D_\alpha$ then we can fix up the argument.

We would like to do the same thing for limits in $n\text{CAT}'$, namely show that direct limits exist just using a general argument working from the existence of inverse limits. In order to do this we first need to discuss cardinality questions for $n$-categories.

5.1 Cardinality

Suppose $A$ is an $n$-category. We define the cardinal of $A$, denoted $\# A$ in the following way. Choose for every $y \in \pi_0(A) = \tau_{\leq 0}(A)$ (the set of equivalence classes of objects) a lifting to an object $\bar{y} \in A_0$. Then

$$\# A := \sum_{y, z \in \pi_0(A)} \# A_1/(\bar{y}, \bar{z}).$$

The sum of cardinals is of course the cardinal of the disjoint union of representing sets. This definition is recursive, as what goes into the formula is the cardinal of the $n - 1$-category $A_1/(\bar{y}, \bar{z})$. At the start we define the cardinal of a 0-category (i.e. a set) in the usual way.
Lemma 5.1.1 The above definition of $\#A$ doesn't depend on choice of representatives. If $A \to B$ is an equivalence of $n$-categories then $\#A = \#B$.

Proof: Left to the reader.  

An easier and more obvious notion is the precardinality of $A$. If $A$ is any $n$-precat we define (with the notations of [12])

$$\#^{\text{pre}}A := \sum_{M \in \Theta^n} \#(A_M).$$

For infinite cardinalities the precardinal of $A$ is also the maximum of the cardinalities of the sets $A_M$. In any case note that the precardinality is infinite unless $A$ is empty.

5.1.2 — Remark: Let $A \mapsto \text{Cat}(A)$ denote the operation of replacing an $n$-precat by the associated $n$-category. Then the precardinal of $\text{Cat}(A)$ is bounded by the maximum of $\omega$ and the precardinal of $A$. Similarly by the argument of ([12] §6, proof of CM5(1)), for any $n$-precat $A$ there is a replacement by a fibrant $n$-category $A \hookrightarrow A'$ with

$$\#^{\text{pre}}A' \leq \max(\omega, \#^{\text{pre}}A).$$

Actually since $\#^{\text{pre}}A \geq \omega$ we can write more simply that $\#^{\text{pre}}A' = \#^{\text{pre}}A$.

Note trivially that

$$\#A \leq \#^{\text{pre}}A.$$

The following lemma gives a converse up to equivalence.

Lemma 5.1.3 Suppose $A$ is an $n$-category with $\#A \leq \alpha$ for an infinite cardinal $\alpha$. Then $A$ is equivalent to an $n$-category $A'$ of precardinality $\leq \alpha$.

Proof: Left to the reader.  

5.2 A CRITERION FOR DIRECT LIMITS IN $n\text{CAT}'$

Before getting to the application of the theory of cardinality we give a criterion which simplifies the problem of finding direct limits in $n\text{CAT}'$.

5.2.1 For this section we need another type of universal morphism. Suppose $E$ and $B$ are $n$-precats, with $B$ fibrant. Then $\underline{\text{Hom}}(B, E)$ is fibrant and we have a canonical morphism

$$\underline{\text{Hom}}(B, E) \times E \to B.$$
This may be interpreted as an object
\[ \nu \in \text{Diag}_{\text{Hom}(B,E),B}(E;nCAT) \]
which yields by composition with \( nCAT \to nCAT' \) the element which we denote by the same symbol
\[ \nu \in \text{Diag}_{\text{Hom}(B,E),B}(E;nCAT') \]

**5.2.2** The element \( \nu \) has the following universal property: for any \( n \)-precat \( F \) the morphism
\[ \text{Diag}^{U,\nu}(F,E;nCAT') \to \text{Diag}^{U,B}(F \times E;nCAT') \]
is a fibrant equivalence of fibrant \( n \)-categories.

To prove this note that the fibrant property comes from the fact that \( nCAT' \) is fibrant. Note that both sides are fibrant by 4.2.9. The fact that it is an equivalence may be checked using diagrams in \( nCAT \) rather than diagrams in \( nCAT' \), according to 4.2.10. Using 4.2.3 for diagrams in \( nCAT \), both sides are equal to
\[ \text{Hom}(U \times F \times E,B) \]
where \( U \) is the image of the first object \( 0 \in \Upsilon^2(F,E) \). This shows that the morphism is an equivalence.

**5.2.3** As a corollary of the above, given a morphism
\[ f : \Upsilon(F \times E) \to nCAT' \]
with image of the last vertex equal to \( B \), there is an extension to a morphism
\[ g : \Upsilon^2(F,E) \to nCAT' \]
such that \( r_{02}(g) = f \) and \( r_{12}(f) = \nu \). Similarly if \( E' \subset E \) and we are already given the extension \( g' \) for \( E' \) then we can assume that \( g \) is compatible with \( g' \).

**5.2.4** We also have a version of this universal property for shell-extension in higher degree. This concerns the right shell \( \text{Shelr} \Upsilon^k \) (cf 4.3.3). Suppose we are given a morphism
\[ f : \text{Shelr} \Upsilon^k(F_1,\ldots,F_{k-1},E) \to nCAT' \]
such that \( f \) restricts on the last edge to \( \nu \). Then there is a filling-in to a morphism
\[ g : \Upsilon^k(F_1,\ldots,F_{k-1},E) \to nCAT' \]
5.2.5 The above property also works in a family. Given a morphism
\[ f : A \times \Upsilon(F \times E) \to n\text{CAT}' \]
sending the last vertex to the constant object \( B_A \), there is an extension to a morphism
\[ g : A \times \Upsilon^2(F, E) \to n\text{CAT}' \]
such that \( r_{02}(g) = f \) and \( r_{12}(f) = \nu_A \) is the morphism pulled back from \( \nu \). Again if an extension \( g' \) is already given on \( E' \subset E \) then \( g \) may be chosen compatibly with \( g' \).

Similarly there is a shell-extension property as in 5.2.4 in a family.

For the proof one has to go through a procedure analogous to the passage from diagrams to internal \( \text{Hom} \) in 4.4.2–4.4.8. This discussion of the universal morphism \( \nu \) is parallel to the discussion of the discussion of the universal \( 1_B \), but with “arrows reversed”.

We now come to our simplified criterion for limits in \( n\text{CAT}' \).

5.2.6 Caution: Note that the following lemma only applies as such to limits taken in \( n\text{CAT}' \) and not in general to limits in an arbitrary \( n+1 \)-category \( \mathcal{C} \). The proof uses in an essential way the fact that the morphism objects for the “category” \( n\text{CAT}' \) are \( n \)-categories which are also basically the same thing as the objects of \( n\text{CAT}' \). Of course it is possible that the same techniques of proof might work in a limited other range of circumstances which are closely related to these.

**Lemma 5.2.7** Suppose \( A \) is an \( n+1 \)-precat and \( \psi : A \to n\text{CAT}' \) is a morphism. Suppose that \( \epsilon : \psi \to \delta \) is a morphism to an object \( \delta \in n\text{CAT}' \) having the following weak limit-like property: for any other morphism \( f : \psi \to \mu \) there exists a morphism \( g : \delta \to \mu \) such that the composition \( g \epsilon \) (well defined up to homotopy) is homotopic to \( f \); and furthermore that such a factorization is unique up to a (not necessarily unique) homotopy of the factorization. Then \( \psi \xrightarrow{\epsilon} \delta \) is a direct limit.

**Proof:** First we explain more precisely what the existence and uniqueness of the factorization mean. Given an element \( f \in \text{Hom}(\psi, \mu) \) there exists an element \( g' \in \text{Hom}'(\psi, \delta, \mu) \) projecting via \( r_{02} \) to a morphism equivalent to \( f \). This equivalence may be measured in the \( n \)-category \( \text{Hom}(\psi, \mu) \). Note that since \( n\text{CAT}' \) is fibrant the projection
\[ \text{Hom}'(\psi, \delta, \mu) \to \text{Hom}(\psi, \mu) \]
is fibrant, so if an object equivalent to \( f \) is in the image then \( f \) is in the image. Thus we can restate the criterion as saying simply that there exists an element \( g' \) projecting via \( r_{02} \) to \( f \).

63
Suppose given two such factorizations $g'_1$ and $g'_2$. By “homotopy of the factorization” we mean a homotopy between $r_{12}(g'_1)$ and $r_{12}(g'_2)$ such that the resulting homotopy between $f$ and itself (this homotopy being well defined up to 2-homotopy) is 2-homotopic to the identity $1_f$. Again using the fibrant condition of $nCAT'$ we obtain that this condition implies the simpler statement that there exists a morphism

$$A \times \Upsilon^2(\ast, \ast) \times \mathcal{T} \rightarrow nCAT'$$

restricting to $g'_1$ and $g'_2$ on the two endpoints 0, 1 $\in \mathcal{T}$; restricting to the pullback $\epsilon$ on the edge (01) of the $\Upsilon^2$, this edge being $A \times \Upsilon(\ast) \times \mathcal{T}$; and restricting to the pullback of $f$ on the edge (02) which is $A \times \Upsilon(\ast) \times \mathcal{T}$.

5.2.8 Simple factorization We start by showing the simple version of the factorization property necessary to show that $\epsilon$ is an inverse limit; we will treat the relative case for $E' \subset E$ below. So for now, suppose that we are given a morphism

$$u : A \times \Upsilon(E) \rightarrow nCAT'$$

restricting to $\psi$ on $A \times \{0\}$ and restricting to a constant object $B \in nCAT'$ (i.e. to the pullback $B_A$) on $A \times \{1\}$. We would like to extend this to a morphism

$$v : A \times \Upsilon^2(\ast, E) \rightarrow nCAT'$$

restricting to our given morphism on the edge (02), and restricting to $\epsilon$ on the edge (01). Our given morphism corresponds by 5.2.5 to a morphism $w : A \times \Upsilon(\ast) \rightarrow nCAT'$ restricting to $\psi$ on $A \times \{0\}$ and restricting to the constant object $\text{Hom}(E, B)$ (pulled back to $A$) on $A \times \{1\}$. More precisely there is a morphism

$$w' : A \times \Upsilon^2(\ast, E) \rightarrow nCAT'$$

restricting to $u$ over the edge (02), and restricting to the universal morphism $\nu$ (cf 5.2.1) over the edge (12). The restriction to the edge (01) is the morphism $w$.

Now $w$ is an element of $\text{Hom}(\psi, B)$, so by hypothesis there is a diagram

$$g : A \times \Upsilon^2(\ast, \ast) \rightarrow nCAT'$$

sending the edge (01) to $\epsilon$ and sending the edge (02) to $w$. Putting this together with the diagram $w'$ and using the fibrant property of $nCAT'$ (i.e. composing these together) we obtain existence of a diagram

$$A \times \Upsilon^3(\ast, \ast, E) \rightarrow nCAT'$$
restricting to \( g \) on the face (012) and restricting to \( w' \) on the face (023). The face (013) yields a diagram

\[
A \times \Upsilon^2(\ast, E) \to nCAT'
\]

restricting to \( \epsilon \) on the first edge and restricting to our original morphism \( u \) on the edge (03): this is the morphism \( v \) we are looking for.

5.2.9 Uniqueness of these factorizations The homotopy uniqueness property for factorization of morphisms implies a similar property for the factorizations of \( E \)-morphisms obtained in the previous paragraph. Suppose that we are given a morphism

\[
u : A \times \Upsilon(E) \to nCAT'
\]
as above, and suppose that we are given two extensions

\[
v_1, v_2 : A \times \Upsilon^2(\ast, E) \to nCAT'
\]

restricting to our given morphism on the edge (02), and restricting to \( \epsilon \) on the edge (01). We can complete the \( v_i \) to diagrams

\[
z_i : A \times \Upsilon^3(\ast, \ast, E) \to nCAT'
\]

restricting to \( v_i \) on the faces (013) and restricting to the universal morphism \( \nu \) of \( \Upsilon^2 \) on the edge (23). To do this, use the universal property of \( \nu \) (cf. 5.2.1) to fill in the faces (023) and (123); then we have a map defined on the shell and by the universal property of \( \nu \) which gives shell extension (5.2.4, 5.2.5) we can extend to the whole tetrahedron.

Note furthermore that we can assume that the restrictions to the faces (023) are the same for \( z_1 \) and \( z_2 \) (because we have chosen these faces using only the map \( u \) and not referring to the \( v_i \)). Call these \( r_{023}(z) \). In particular the restrictions to (02) give the same map \( w : \psi \to \text{Hom}(E, B) \). Now the restrictions of the above diagrams \( z_i \) to the faces (012) give two different factorizations of this map \( w \) so by hypothesis there is a homotopy between these factorizations: it is a morphism

\[
A \times \Upsilon^2(\ast, \ast) \times \mathcal{T} \to nCAT'
\]

restricting to \( r_{012}(z_i) \) on the endpoints \( i = 0, 1 \) of \( \mathcal{T} \), restricting to the pullback of \( \epsilon \) along (01) \( \times \mathcal{T} \) and restricting to the pullback of our morphism \( w \) along (02) \( \times \mathcal{T} \). We can attach this homotopy to the constant homotopy which is the pullback of \( r_{023}(z) \) from \( A \times \Upsilon^2(\ast, E) \) to \( A \times \Upsilon^2(\ast, E) \times \mathcal{T} \). We obtain a homotopy defined on the union of the faces (012) and (023) and going between \( z_0 \) and \( z_1 \). Using the fact that the inclusion of this union of faces into the tetrahedron is a trivial cofibration (see the list 2.4.8 above)
we get that the inclusion (written in an obvious shorthand notation where \((0123)\) stands for \(A \times \Upsilon^3(*,*,E)\) and \((012 + 023)\) for the union of the two faces)
\[
(0123) \times \{0\} \cup^{(012+023)\times\{0\}} (012 + 023) \times \Upsilon \cup^{(012+023)\times\{1\}} (0123) \times \{1\} \hookrightarrow (0123) \times \Upsilon
\]
is a trivial cofibration. We have a map from the left side into \(nCAT'\) so it extends to a map
\[
A \times \Upsilon^3(*,*,E) \times \Upsilon \rightarrow nCAT'.
\]
The restriction of this map to the face \((013)\) is a homotopy
\[
A \times \Upsilon^2(*,E) \times \Upsilon \rightarrow nCAT'
\]
between our factorizations \(v_1\) and \(v_2\).

5.2.10 The relative case To actually prove the lemma, we need to obtain a factorization property as above in the relative situation \(E' \subset E\) where we already have the factorization over \(E'\) and we would like to extend to \(E\). This is where we use the homotopy uniqueness of factorization which was in the hypothesis of the lemma (we use it in the form given in the previous paragraph 5.2.9). Suppose we are given
\[
v' : A \times \Upsilon^2(*,E') \rightarrow nCAT'
\]
restricting to \(\epsilon\) on the first edge, and suppose we are given
\[
u : A \times \Upsilon(E) \rightarrow nCAT'
\]
restricting to \(\psi\) on \(A \times \{0\}\) and restricting to a constant object \(B \in nCAT'\) (i.e. to the pullback \(B_A\)) on \(A \times \{1\}\). Suppose that the restriction of \(u\) to \(A \times \Upsilon(E')\) is equal to the restriction of \(v'\) to the edge \((02)\). By 5.2.8 there exists an extension
\[
v_0 : A \times \Upsilon^2(*,E) \rightarrow nCAT'
\]
which restricts to \(\epsilon\) on the first edge and to \(u\) on the edge \((02)\). Let \(v'_0\) denote the restriction of \(v_0\) to \(A \times \Upsilon^2(*,E')\). By the uniqueness statement 5.2.9 there exists a homotopy
\[
A \rightarrow \Upsilon^2(*,E') \times \Upsilon \rightarrow nCAT'
\]
between \(v'_0\) and \(v'\), constant along edges \((01)\) and \((02)\). Let \(D\) be the coproduct of \(\Upsilon^2(*,E')\) and \(\Upsilon(E)\) with the latter attached along the edge \((02)\) (i.e. the coproduct is taken over the copy of \(\Upsilon(E') \subset \Upsilon^2(*,E')\) corresponding to the edge \((02)\)). Our homotopy glues with the constant map \(u\) to give a morphism
\[
A \times D \times \Upsilon \rightarrow nCAT',
\]
and this glues with \( u \) to obtain

\[
A \times \Upsilon^2(*, E) \times \{0\} \cup^{A \times D \times \{0\}} A \times D \times I \to n\text{CAT}'.
\]

The inclusion

\[
A \times D \times \{0\} \hookrightarrow A \times D \times I
\]

is a trivial cofibration so the inclusion

\[
A \times \Upsilon^2(*, E) \times \{0\} \cup^{A \times D \times \{0\}} A \times D \times I \hookrightarrow A \times \Upsilon^2(*, E) \times I
\]

is a trivial cofibration and by the fibrant property of \( n\text{CAT}' \) there exists an extension of the above morphism to a morphism

\[
A \times \Upsilon^2(*, E) \times I \to n\text{CAT}'.
\]

The value of this over \( 1 \in I \) is the extension

\[
v : A \times \Upsilon^2(*, E) \to n\text{CAT}'
\]

we are looking for: it restricts to \( \epsilon \) on the edge \((01)\), it restricts to \( u \) on the edge \((02)\), and it restricts to \( v' \) over \( A \times \Upsilon^2(*, E') \). This completes the proof of Lemma 5.2.7. ///

5.2.11—Remark: One also obtains a criterion similar to 5.2.7 for inverse limits in \( n\text{CAT}' \). The proof is the same as above but using the universal diagram

\[
B \xrightarrow{E} B \times E
\]

in the place of \( \text{Hom}(E, B) \xrightarrow{E} B \). We did not choose to use this in the proof of 4.0.1 because it didn’t seem to make any substantial savings (and in fact probably would have complicated the notation in many places).

5.2.12 We now improve the above criterion with a view toward applying this in the argument [5.0.3] given above for the case of sets. Fix a functor \( \psi : A \to n\text{CAT}' \) of \( n+1 \)-categories. Suppose \( \alpha \) is an infinite cardinal number such that

\[
\#A \leq \alpha
\]

and

\[
\#\psi(a) \leq \alpha
\]

for all \( a \in A \).
5.2.13 Suppose $B \in nCAT'$ and suppose $u : \psi \to B$ is a morphism. Then we claim that there is $B' \in nCAT'$ with $\#^{pre}B' \leq \alpha$, and with a factorization $\psi \to B' \to B$.

To prove this notice that $u$ is a morphism

$$u : A \times I \to nCAT',$$

and the image of $u$ is contained in an $\alpha$-bounded set of additions of trivial cofibrations to $nCAT$ (recall that $nCAT'$ was constructed by adding pushouts along trivial cofibrations to $nCAT$ [2.2.11]).

We can take $B' \subset B$ to be a sub-precat containing all of the objects necessary for the morphisms involved in the trivial cofibrations which are added in the previous paragraph, as well as the morphisms involved in $u$ given that we fix $\psi$, all of the objects necessary for the structural morphisms of a precat, and finally add on what is necessary to get $B'$ fibrant. This has cardinality $\#^{pre}B' \leq \alpha$.

5.3 A CONSTRUCTION

5.3.1 Hypothesis— With the above notations, suppose we have an object $U \in nCAT'$ together with a morphism $a : \psi \to U$ provided with the following data:

(A)—for every morphism $\psi \to V$ where $\#^{pre}V \leq \alpha$, a factorization which we call the official factorization

$$\psi \xrightarrow{a} U \xrightarrow{} V$$

(in other words a diagram

$$A \times Y^2(\ast, \ast) \to nCAT'$$

restricting to $a$ on the edge (01) and restricting to our given morphism on the edge (02));

(B)—for every diagram

$$\psi \to V \to V',$$

a completion of this and the official factorization diagrams

$$\psi \to U \to V, \; \psi \to U \to V'$$

to a diagram (the official commutativity diagram)

$$\psi \to U \to V \to V'$$

(which again means a morphism

$$A \times Y^3(\ast, \ast, \ast) \to nCAT''$$

restricting to our given diagrams on the faces (023), (012), and (013)).
5.3.2 Keep the above hypothesis [5.3.1]. Let 

\[ [b, i] : \psi \rightarrow U' \rightarrow U \]

be a factorization of the morphism \( a \) as above [5.2.13] with \( \#U' \leq \alpha \). This means a diagram whose third edge (02) is equal to \( a \).

Unfortunately at this point we have no control over the choice of \( U' \), so the “real” \( U' \) which we would like to choose to satisfy the criterion of 5.2.7 may be a direct factor of this \( U' \). To explain this notice that by hypothesis [5.3.1 (A)] there is a morphism 

\[ q : U \rightarrow U' \]

giving a factorization 

\[ [a, q] : \psi \rightarrow U \rightarrow U'. \]

Let \( b \) be the edge (02) of this diagram, so we can write \( b \sim qa \).

Using the fibrant property of \( nCAT' \) we can glue the diagrams \([b, i]\) and \([a, q]\) together to give a diagram 

\[ [b, i, q] : \psi \rightarrow U' \rightarrow U \rightarrow U', \]

in other words a morphism 

\[ A \times Y^3(\ast, \ast, \ast) \rightarrow nCAT' \]

restricting to \([b, i]\) on the face (012) and restricting to \([a, q]\) on the face (023) (and satisfying the usual condition that the restriction to the face (123) be constant in the \( A \) direction). Denote by \( p \) the restriction to the edge (13), and denote by \([b, p]\) the restriction to the face (013). Thus 

\[ [b, p] : \psi \rightarrow U' \rightarrow U' \]

is a diagram whose restrictions to the edges (01) and (02) are both equal to the morphism \( b \).

Restriction to the face (123) is a diagram \([i, q]\) with third edge equal to \( p \), in other words we can write \( p \sim q \circ i \).

The official commutativity diagram for \([b, p]\) is a diagram of the form 

\[ [a, q, p] : \psi \rightarrow U \rightarrow U' \rightarrow U'. \]

The restriction of this diagram to the face (023) is the diagram \([b, p]\). The restrictions to (012) and (013) are both equal (by hypothesis (B)) to the official factorization diagram \([a, q]\). In particular, the face (123) gives a diagram 

\[ [q, p] : U \rightarrow U' \rightarrow U' \]
whose third edge (which we should here denote (13)) is again the morphism $p$. Homotopically we get an equation

$$p \circ q \sim q.$$ 

In view of the fact that $p \sim q \circ i$ we get

$$p \circ p \sim p.$$ 

This equation says that, up to homotopy, $p$ is a projector. It is the projector onto the answer that we are looking for.

### 5.3.3 Construction — Continuing with hypothesis 5.3.1 and the notations of 5.3.2, we will construct the object corresponding to the “image” of the homotopy projector $p$. To do this we will take the “mapping telescope” of the sequence

$$U' \xrightarrow{p} U' \xrightarrow{p} U' \xrightarrow{p} \ldots$$

In the present setting of $n$-categories we do this as follows (which is basically just the mapping telescope in the closed model category structure of [32]). Recall that $\overline{T}$ is the 1-category with two objects 0, 1 and two morphisms inverse to each other between the objects. We consider it as an $n$-category. Glue together the $n$-precats $U' \times \overline{T}$, one for each natural number, by attaching $U' \times \{1\}$ in the $i-1$-st copy to $U' \times \{0\}$ in the $i$-th copy via the map $p : U' \times \{1\} \to U' \times \{0\}$. Denote by $T'$ the resulting $n$-precat and by $T' \hookrightarrow T$ a fibrant replacement. Inclusion of $U' \times \{0\}$ in the first copy gives a morphism

$$j : U' \to T.$$ 

On the other hand, using the projection $p$ in each variable and the homotopy $p \circ p \sim p$ gives a morphism

$$r : T \to U'$$

(which comes by extension from a map $r' : T' \to U'$) and we have $rj = p$.

### 5.3.4 Claim: The morphism $jr : T \to T$ is homotopic to the identity, via a homotopy compatible with the homotopy $p \circ p \sim p$.

This is by a classical construction that works in any closed model category with “interval object” such as $\overline{T}$. As a sketch of proof, let $T^m$ denote the subobject of $T'$ obtained by taking only the first $m$ copies of $U' \times \overline{T}$. Then $T^m$ retracts to the last copy of $U'$, so the restriction of $r'$ to $T^m$ is homotopic (via this retraction) to $p$. On the other hand, the inclusion $T^m \hookrightarrow T^{m+1}$ is also homotopic to $p$ (via the retractions to the end copies of $U'$).
Thus we may choose a homotopy (in Quillen’s sense cf 2.2.16) between the restriction of \( r' \) to \( T^m \), and the inclusion \( T^m \hookrightarrow T^{m+1} \). We can make this into a homotopy between

\[
r', 1_{T^m} : T^m \xrightarrow{\sim} T,
\]

and since \( T \) is fibrant we can do this with a homotopy using the interval \( I \). Again because \( T \) is fibrant we can assume that these homotopies are compatible for all \( m \), so they glue together to give a homotopy between the two maps

\[
r', 1_{T'} : T' \xrightarrow{\sim} T.
\]

Then extend from \( T' \) to \( T \).

5.3.5 We wrap things up by pointing out how \( T \) fits in with the situation of 5.3.2. Consider the sequence of morphisms

\[
\psi \to U' \to T \to U' \to T.
\]

The composition of the first two gives a morphism \( jb : \psi \to T \). The composition of the first three morphisms is equal to \( rjb \sim pb \) which has a homotopy to the usual morphism \( b : \psi \to U' \). Thus the morphism \( b \) factors through \( T \). Finally from our claim 5.3.4 the composition of the last two arrows is homotopic to the identity on \( T \).

Our original morphism \( \psi \to U \) factors through \( U' \) hence it factors through \( T \): the composition

\[
\psi \to T \to U' \to U
\]

is equal to the original morphism \( a : \psi \to U \).

We have the morphism

\[
jq : U \to T
\]

providing a factorization

\[
\psi \to U \to T.
\]

The composition \( T \to U \to T \) is homotopic to the identity on \( T \) by claim 5.3.4.

Lemma 5.3.6 Under hypothesis 5.3.1 and with the above notations, the morphism \( \psi \to T \) has the unique homotopy factorization property of 5.2.1 with respect to any morphism \( \psi \to B \) (without bound on the cardinality of \( B \)).
Proof: This is really only a statement about 1-categories. We can consider the 1-category \( M \) which is the truncation of the \( n + 1 \)-category of objects under \( \psi \) (cf. 5.4.1 below). Our objects \( U, U', T \) and so on together with maps from \( \psi \) may be considered as objects in the category \( M \). The result of 5.3.2 says that \( p : U' \to U' \) is a projector in the category \( M \), and in 5.3.3, 5.3.4 and 5.3.5 we show that the object \( T \) corresponding to this projector exists. The criterion of 5.2.7 asks simply that \( T \) be an initial object in \( M \).

What we know from hypothesis 5.3.1 is that \( T \) is provided with a collection of morphisms \( T \to B' \) to every \( \alpha \)-bounded object of \( M \), in such a way that these form a natural transformation from the constant functor \( T \) to the identity functor \( M_\alpha \to M \) (where \( M_\alpha \) is the full subcategory of objects having cardinality bounded by \( \alpha \)).

The fact that \( T \) is the object corresponding to the projector \( p \) (and that \( p \) was the projector defined by the natural transformation for \( U' \)) means that the value of this natural transformation on \( T \) itself is the identity.

Suppose \( \psi \to B \) is an object of \( M \). Then there is a factorization through \( \psi \to B' \to B \) with \#pre \( B' \leq \alpha \). This just says that every object of \( M \) has a morphism from an object in \( M_\alpha \). It is worth mentioning that if \( B \in M \) and if \( B' \to B \) and \( B'' \to B \) are two morphisms from objects in \( M_\alpha \) then they both factor through a common morphism \( B'' \to B \) from an object in \( M_\alpha \).

Using the above formal properties, we show that \( T \) is an initial object of \( M \) to prove the lemma. Suppose \( B \) is an object of \( M \). There exists a morphism \( B' \to B \) from an object of \( M_\alpha \) so applying our natural transformation, there exists a morphism \( T \to B' \) and hence a morphism \( T \to B \). Suppose that \( T \to B \) is a pair of morphisms. These factor through a common object of \( M_\alpha \)

\[
T \to B' \to B,
\]

and applying our natural transformation we obtain that the compositions of the two morphisms

\[
T \to T \to B'
\]

are equal to the given morphism \( T \to B' \); however, since the natural transformation \( T \to T \) is the identity, this implies that our two morphisms \( T \to B' \) were equal and hence that the two original morphisms \( T \to B \) were equal. 

\[\text{///}\]

**Corollary 5.3.7** In the situation of Lemma 5.3.6 the map \( \psi \to T \) is a direct limit.

**Proof:** By 5.3.6 it satisfies the condition of 5.2.7 so by the latter, it is a direct limit. 

\[\text{///}\]

**5.4 Proof of Theorem 5.0.1**
5.4.1 Objects under $\psi$: In order to replicate the proof that was given above for the category of sets, we need to know what the category of "objects under $\psi$" is. Suppose $C$ is an $n+1$-category and $A$ another $n+1$-category and suppose $\psi : A \to C$ is a morphism. We define the $n+1$-category $\psi/C$ of objects under $\psi$ to be the category of morphisms

$$(A \times I) \cup^{A \times \{1\}} \{1\} \to C$$

restricting to $\psi$ on $A \times \{0\}$. In other words, it is the fiber of the morphism

$$\text{Hom}((A \times I) \cup^{A \times \{1\}} \{1\}, C) \to \text{Hom}(A, C)$$

over $\psi$.

5.4.2 Let $(\psi/C)_\alpha$ denote the category of objects under $\psi$ which are (set-theoretically speaking) contained in a given fixed set of cardinality $\alpha$. It has cardinality $\leq 2^\alpha$. It is a full subcategory of $\psi/C$.

5.4.3 We can restate the criterion of 5.2.7 in terms of the above definition. Let $\tau_{\leq 1}(\psi/C)$ denote the 1-truncation of the category of objects under $\psi$ defined in 5.4.1. It is a 1-category. The criterion says that if $u : \psi \to U$ is an initial object in this category then it is (the image under the truncation operation of) a direct limit of $\psi$. Definition 5.4.1 and the present remark were used in the proof of 5.3.6 already, where we denoted $\psi/C$ by $M$.

Proof of Theorem 5.0.1: Suppose $\psi : A \to n\text{CAT}'$. Fix a cardinal $\alpha$ bounding $(A, \psi)$ as above. Let $M_\alpha := (\psi/C)_\alpha$ denote the $n+1$-category of objects under $\psi$, of cardinality bounded by $\alpha$. Let $U$ be the inverse limit of the forgetful functor $f : M_\alpha \to n\text{CAT}'$, given by Theorem 4.0.1. By Corollary 3.4.9, the pullback of $U$ to a constant family $U_A$ over $A$ is again an inverse limit of the functor

$$f_A : A \times M_\alpha \to n\text{CAT}'$$

($f$ pulled back along the second projection to $M_\alpha$).

We have a morphism of families over $A \times M_\alpha$, from $\psi$ to $f_A$, which thus factorizes into

$$\psi \to U_A \to f_A.$$ 

The morphism $\psi \to U_A$ is automatically provided with the data required for Hypothesis 5.3.1.

Apply the above construction 5.3.2 to 5.3.5 to obtain $\psi \to T_A$, and Lemma 5.3.6 and Corollary 5.3.7 show that $\psi \to T_A$ is a direct limit of $\psi$. ///
6. Applications

We will discuss several different possible applications for the notions of inverse and direct limit in $n$-categories in general, and of the existence of limits in $nCAT'$ in particular. Many of these applications are only proposed as conjectural. Only in the first section do we give full proofs.

The conjectures are for the most part supposed to be possible to do with the present techniques, except possibly 6.7.6.

6.1 Coproducts and fiber products

6.1.1 Taking $A$ to be the category with three objects $a, b$ and $c$ and morphisms $a \to b$ and $c \to b$, a functor $A \to nCAT$ is just a triple of $n$-categories $X, Y, Z$ with maps $u : X \to Y$ and $v : Z \to Y$. The inverse limit of the projection into $nCAT'$ is the homotopy fiber product denoted $X \times^h_Y Z$.

**Lemma 6.1.2** Suppose $A$ is as above and $\varphi : A \to nCAT$ is a morphism corresponding to a pair of maps $u : X \to Y$ and $v : Z \to Y$ of $n$-categories such that $u$ is fibrant. Then the usual fiber product $X \times_Y Z$ is a limit of $\varphi$ so we can write

$$X \times^h_Y Z = X \times^h_Y Z.$$

**Proof:** One way to prove this is to use our explicit construction of the inverse limit (4.1.2). The second way is to show that $U := X \times_Y Z$ satisfies the required universal property as follows. First of all note that the commutative square

$$
\begin{array}{ccc}
U & \to & X \\
\downarrow & & \downarrow \\
Z & \to & Y
\end{array}
$$

corresponds to a map $I \times I \to nCAT$ which we can project into $I \times I \to nCAT'$. Then combine this with the projection

$$A \times I \to I \times I$$

which sends $A \times \{0\}$ to $(0, 0)$ and sends $A \times \{1\}$ to the copy of $A \subset I \times I$ corresponding to the sides $(1, 01)$ and $(01, 1)$ of the square. We get a map $A \times I \to nCAT'$ having the required constancy property to give an element $\epsilon \in Hom(U, \varphi)$. This is the map which we claim is an inverse limit.
In passing note that since $X, Y, Z$ are elements of $n\text{CAT}$ they are by definition fibrant, and since by hypothesis the map $X \rightarrow Y$ is fibrant, the map $U \rightarrow Z$ is fibrant too, and so $U$ is fibrant.

We now fix a fibrant $n$-category $V$ and study the functor which to an $n$-precat $F$ associates the set of morphisms

$$g : A \times \Upsilon(F) \rightarrow n\text{CAT}'$$

with $r_0(g) = V_A$ and $r_1(g) = \varphi$. This is of course just the functor represented by $\text{Hom}(V, \varphi)$. Recalling that $\text{Hom}(V, U)$ is the morphism set in $n\text{CAT}$, we obtain by composition with $\epsilon$ a morphism

$$C_\epsilon : \text{Hom}(V, U) \rightarrow \text{Hom}(V, \varphi).$$

In this case, since $\epsilon$ comes from $n\text{CAT}$ in which the composition at the first stage is strict, the morphism $C_\epsilon$ is strictly well defined rather than being a weak morphism as usual in the notion of limit. We would like to show that $C_\epsilon$ is an equivalence (which would prove the lemma).

A morphism $g : A \times \Upsilon(F) \rightarrow n\text{CAT}'$ decomposes as a pair of morphisms $(g_1, g_2)$ with

$$g_i : I \times \Upsilon(F) \rightarrow n\text{CAT}';$$

in turn these decompose as pairs $g_i^+$ and $g_i^-$ where

$$g_i^+ : \Upsilon(\ast, F) \rightarrow n\text{CAT}',$$

$$g_i^- : \Upsilon(F, \ast) \rightarrow n\text{CAT}'.

(Decompose the square $I \times \Upsilon(F)$ into two triangles, drawing the edge $I$ vertically with vertex 0 on top.) The conditions on everything to correspond to a morphism $g$ are that

$$r_{12}(g_i^+) = r_{01}(g_i^-)$$

and

$$r_{02}(g_1^-) = r_{02}(g_2^-).$$

The endpoint conditions on $g$ correspond to the conditions

$$r_{12}(g_1^-) = u, \quad r_{12}(g_2^-) = v,$$

and

$$r_{01}(g_i^+) = 1_V.$$
Putting these all together we see that our functor of $F$ is of the form a fiber product of four diagram $n$-categories \[4.2.1\]. More precisely, put

$$M_u := \text{Diag}^{V} (\star, \star; n\text{CAT}') \times_{\text{Diag}^{V} (\star, \star; n\text{CAT}') \times \text{Diag}^{V} (\star, \star; n\text{CAT}')} \text{Diag}^{V} (\star, \star; n\text{CAT}')$$

where the morphisms in the fiber product are $r_{12}$ then $r_{01}$; and define $M_v$ similarly. Then

$$\text{Hom}(V, \varphi) = M_u \times_{\text{Diag}^{V} (\star, \star; n\text{CAT}')} M_v,$$

where here the morphisms in the fiber product are the restrictions $r_{02}$ on the second factors of the $M$.

Refer now to the calculation of \[4.2.3\] in view of the comparison result \[4.2.10\] (applied to $n\text{CAT} \rightarrow n\text{CAT}'$).

By this calculation the restriction morphism

$$r_{12} : \text{Diag}^{V} (\star, \star; n\text{CAT}') \rightarrow \text{Diag}^{V} (\star, \star; n\text{CAT}')$$

is a fibrant equivalence. Therefore the second projections are equivalences

$$M_u \rightarrow \text{Diag}^{V} (\star, \star; n\text{CAT}')$$

and similarly for $v$. Using these second projections in each of the factors $M$ we get an equivalence

$$\text{Hom}(V, \varphi) \rightarrow$$

$$\text{Diag}^{V} (\star, \star; n\text{CAT}') \times_{\text{Diag}^{V} (\star, \star; n\text{CAT}') \times \text{Diag}^{V} (\star, \star; n\text{CAT}')} \text{Diag}^{V} (\star, \star; n\text{CAT}')$$

where the morphisms in the fiber product are $r_{02}$. There is a morphism from the same fiber product taken with respect to $n\text{CAT}$, into here. In the case of the fiber product taken with respect to $n\text{CAT}$ the calculation of \[4.2.3\] gives directly that it is equal to

$$\text{Hom}(V, X) \times_{\text{Hom}(V, Y)} \text{Hom}(V, Z)$$

which is just $\text{Hom}(V, U)$. The morphism

$$\text{Hom}(V, X) = \text{Diag}^{V} (\star, \star; n\text{CAT}') \rightarrow \text{Diag}^{V} (\star, \star; n\text{CAT}')$$

is an equivalence by \[4.2.10\], and similarly for the other factors in the fiber product.

Now we are in the general situation that we have equivalences of fibrant $n$-precats $P \rightarrow P'$, $Q \rightarrow Q'$ and $R \rightarrow R'$ compatible with diagrams

$$P \rightarrow Q \leftarrow R, \quad P' \rightarrow Q' \leftarrow R'.$$
If we know that the morphisms $P \to Q$ and $P' \to Q'$ are fibrant then we can conclude that these induce an equivalence

$$P \times_Q R \to P' \times_{Q'} R'.$$

Prove this in several steps using 2.2.10 and [32] Theorem 6.7:

$$P' \times_{Q'} R' \simeq P' \times_{Q'} R = (P' \times_{Q'} Q) \times_Q R$$

and

$$P' \times_{Q'} Q \simeq P'$$

so

$$P \simeq P' \times_{Q'} Q$$

giving finally

$$P \times_{Q} R \simeq (P' \times_{Q'} Q) \times_Q R;$$

then apply (2.2.9, CM2).

Applying this general fact to the previous situation gives that the morphism

$$\text{Hom}(V, X) \times_{\text{Hom}(V, Y)} \text{Hom}(V, Z) \to \text{Diag}^{V, u}(\ast, \ast; nCAT') \times_{\text{Diag}^{V, Y}(\ast; nCAT')} \text{Diag}^{V, u}(\ast, \ast; nCAT')$$

is an equivalence. By (2.2.9, CM2) this implies that

$$C_\epsilon : \text{Hom}(U, V) \to \text{Hom}(V, \varphi)$$

is an equivalence. ///

Lemma 6.1.2 basically says that for calculating homotopy fiber products we can forget about the whole limit machinery and go back to our usual way of assuming that one of the morphisms is fibrant.

6.1.3 Taking $A$ to be the opposite of the category in the previous paragraph, a functor $A \to nCAT$ is a triple $U, V, W$ with morphisms $f : V \to U$ and $g : V \to W$. The direct limit is the homotopy pushout of $U$ and $W$ over $V$, denoted $U \cup^V W$.

**Lemma 6.1.4** Suppose that $f$ is cofibrant. Let $P$ denote a fibrant replacement

$$U \cup^V W \hookrightarrow P.$$
Then there is a natural morphism \( u : \varphi \to P \) which is a direct limit. Thus we can say that the morphism
\[
U \cup^V W \to U \cup^V_{\text{ho}} W
\]
is a weak equivalence, or equivalently that the morphism of \( n \)-categories
\[
\text{Cat}(U \cup^V W) \to U \cup^V_{\text{ho}} W
\]
is an equivalence.

The proof is similar to the proof of 6.1.2 and is left as an exercise.

This lemma provides justification \textit{a posteriori} for having said in [32] that \( \text{Cat}(U \cup^V W) \) is the “categorical pushout of \( U \) and \( W \) over \( V \).” It also shows that this pushout, which occurs in the generalized Seifert-Van Kampen theorem of [32], is the same as the homotopy pushout.

\section{Representable functors and internal Hom}

Suppose \( A \) is an \( n+1 \)-category. Recall that \( A^\text{o} \) is the first opposed category obtained by switching the directions of the 1-arrows but not the rest (this comes from the inversion automorphism on the first simplicial factor of \( \Delta^{n+1} \)). The “arrow family” is a family
\[
\text{Arr}(A) : A^\text{o} \times A \to \text{nCAT}',
\]
associating to \( X \in A^\text{o} \) and \( Y \in A \) the \( n \)-category \( A_1/(X,Y) \). We will not discuss here the existence and uniqueness of this family (there is not actually a natural way to define this family in Tamsamani’s point of view on \( n \)-categories, so it must be done by constructing the family by hand making choices of various morphisms when necessary).

The arrow family gives two functors
\[
\alpha : A \to \text{Hom}(A^\text{o},\text{nCAT}')
\]
and
\[
\beta : A^\text{o} \to \text{Hom}(A,\text{nCAT}').
\]

\textbf{Conjecture 6.2.1} That \( \alpha \) and \( \beta \) are fully faithful (as is the case for \( n = 0 \)).

We say that an object of \( \text{Hom}(A^\text{o},\text{nCAT}') \) (resp. \( \text{Hom}(A,\text{nCAT}') \)) is representable if it comes from an object of \( A \) (resp. \( A^\text{o} \)). Note that such objects are themselves functors \( A \to \text{nCAT}' \) or \( A^\text{o} \to \text{nCAT}' \), and we call them \textit{representable functors}.

\footnote{And therefore running a certain risk of being circular...}
Conjecture 6.2.2 Suppose that an $n + 1$-category $A$ admits arbitrary direct and inverse limits. Then a functor $h : A^o \to nCAT'$ is representable by an object of $A$ if and only it transforms direct limits into inverse limits. A functor $g : A \to nCAT'$ is representable by an object of $A^o$ if and only if it transforms inverse limits into inverse limits.

6.2.3 If this conjecture is true we would obtain the following corollary: that if an $n + 1$-category $A$ admits arbitrary direct and inverse limits, then $A$ has an internal $Hom$. To see this, fix objects $x, y \in A$. Denote by $\times$ the functor $A \times A \to A$ which associates to $(u, v)$ the direct product of $u$ and $v$ (considered as an inverse limit). This functor comes from Theorem 3.4.4 as described in (3.4.10).

Now the functor $u \mapsto Arr(A)(x \times u, y)$ from $A^o$ to $nCAT$ transforms direct limits to inverse limits (this uses one direction of Conjecture 6.2.2, and I suppose without proof that the functor $u \mapsto x \times u$ is known to preserve direct limits).

Therefore by (the other direction of) Conjecture 6.2.2, the functor $u \mapsto Arr(A)(x \times u, y)$ is representable by an object $Hom_A(x, y)$.

6.2.4 We are obviously going toward some sort of theory of $n$-topoi: an $n$-topos would be an $n$-category admitting arbitrary direct and inverse limits (indexed by small $n$-categories). There may be some other conditions that one would have to impose...

6.3 $n$-stacks

Suppose $\mathcal{X}$ is a site. Consider the underlying category as an $n+1$-category. An $n$-stack over $\mathcal{X}$ is a morphism $F : \mathcal{X} \to nCAT'$ such that for every object $X \in \mathcal{X}$ and every sieve $\mathcal{B} \subset \mathcal{X}/X$ the morphism

$$\Gamma(\mathcal{X}/X, F|_{\mathcal{X}/X}) \to \Gamma(\mathcal{B}, F|_{\mathcal{B}})$$

is an equivalence of $n$-categories, where $\Gamma(\mathcal{B}, F|_{\mathcal{B}})$ denotes the inverse limit of $F|_{\mathcal{B}}$ and the same for $\Gamma(\mathcal{X}/X, F|_{\mathcal{X}/X})$. We define $nSTACK/\mathcal{X}$ to be the full subcategory of the (already fibrant) $n + 1$-category $Hom(\mathcal{X}, nCAT')$ whose objects are the morphisms $F$ satisfying the above criterion.

The $n+1$-category $nSTACK/\mathcal{X}$ admits inverse limits—since the only thing involved in its definition is an inverse limit and inverse limits commute with each other. In particular we may speak of homotopy fiber products of $n$-stacks.

Conjecture 6.3.1 Homotopy projectors are effective for $n$-stacks, in other words given an $n$-stack $U'$ with endomorphism $p$ such that $p \circ p \sim p$, the “telescope construction” $T$ of §5 is again an $n$-stack.
Assuming this conjecture, the same argument as in §5 would work to show that \( n\text{STACK}/\mathcal{X} \) admits direct limits.

A \( n \)-prestack over \( \mathcal{X} \) is just a morphism \( F : \mathcal{X} \to n\text{CAT}' \) without any other condition (this makes sense for any category \( \mathcal{X} \) and in fact for any \( n+1 \)-category, it is just our notion of family of \( n \)-categories indexed by \( \mathcal{X} \)). We can adopt the notation

\[
n\text{PRESTACK}/\mathcal{X} := \text{Hom}(\mathcal{X}, n\text{CAT}').
\]

Suppose \( F \) is an \( n \)-prestack. We define the associated stack denoted \( \text{st}(F) \) to be the universal \( n \)-stack to which \( F \) maps. Assuming Conjecture 6.3.1, the associated stack \( \text{st}(F) \) exists again by copying the argument of §5 above.

6.3.2—Remark: The inverse limit of a family of stacks is the same as the inverse limit of the underlying family of prestacks. However this is not true for direct limits.

6.3.3 By 6.2.3 which is based on Conjecture 6.2.2, the \( n+1 \)-category \( n\text{STACK}/\mathcal{X} \) admits internal \( \text{Hom} \). Using this (or alternatively using a direct construction which associates to any \( X \in \mathcal{X} \) the \( n+1 \)-category \( n\text{STACK}/(\mathcal{X}/X) \)) we should be able to construct the \( n+1 \)-stack \( n\text{STACK}/\mathcal{X} \).

6.3.4 Now that we have a notion of \( n \)-stack not necessarily of groupoids, one can ask how to generalize the definition of geometricity given in [31], to the case where the values may not be groupoids.

If \( A \) is an \( n \)-category and \( X, Y \) are sets with maps \( a : X \to A \) and \( b : Y \to A \) then the pullback

\[
(a^\circ, b)^*(\text{Arr}(A)) =: X \times Y \to (n-1)\text{CAT}'
\]

may be considered as an \( (n-1) \)-category (taking the union over all of the points of \( X \times Y \)) which we denote by \( \text{Hom}_A(a, b) \). However, it is no longer the same thing as the fiber product \( X \times_A Y \). Both of these still satisfy the recurrence-enabling fact that they are \( n-1 \)-categories. Thus we can still employ the same type of definition as in [31]. However, as many common examples quickly show, the smoothness condition should only be imposed on the fiber product, not the arrows. Thus we say that \( A \) is geometric (resp. locally geometric) if:

(GS1) for any two morphisms from schemes \( a : X \to A \) and \( b : Y \to A \), the arrow \( n-1 \)-stack \( \text{Hom}_A(a, b) \to X \times Y \) and the product \( X \times_A Y \) are both geometric (resp. locally geometric); and

(GS2) there exists a smooth morphism from a finite type scheme (resp. locally finite type scheme) \( X \to A \) surjective on the truncations to 0-stacks;
where the morphism $X \to A$ is said to be smooth if for any morphism from a scheme of finite type $Y \to A$, the locally geometric $n-1$-stack $X \times_A Y$ is actually geometric and is smooth, this latter condition meaning that the smooth surjection to it from GS2 comes from a smooth scheme of finite type.

6.3.5 Here is an example to show what we are thinking of (this type of example—even if relatively unknown on “alg-geom”—apparently comes up very often on “q-alg”). The stack of vector bundles on a given variety, for example, is locally geometric. It has an additional operation, tensor product, which allows it to be considered as a monoidal (or braided or symmetric) monoidal 1-stack, thus allowing us to consider it as a 2, 3 or 4-stack. In these cases there are only one object (locally speaking) and in the 3 and 4 cases, only one morphism (in the 4 case only one 2-morphism). The original stack comes back as an arrow stack (possibly after iterating). In this example, if we want a tensor product we are forced to consider things not of finite type, so the arrow stacks should often be allowed to be only locally geometric (also one readily sees that the arrow stacks will not necessarily be smooth). On the other hand the finite type and smoothness conditions in GS2 correspond in this example to the smoothness and finite type conditions for the Picard scheme.

6.3.6 Suppose $\mathcal{P}$ is a property of $n$-stacks of groupoids. Then we say that an $n$-stack $A$ is locally $\mathcal{P}$ (and we call this property $\text{loc}\mathcal{P}$) if $F = \tau_{\leq 0} A$ is an filtered inductive limit of open subsheaves $F_i \subset F$ (the openness condition means that for any scheme $X \to F$, $X \times_F F_i$ is an open subset of $X$) such that $A \times_F F_i$ has property $\mathcal{P}$.

In particular we obtain notions of locally presentable and locally very presentable $n$-stacks of groupoids.

We claim that for $\mathcal{P} =$ “geometric” the above definition gives the same definition as the previous definition of locally geometric. Suppose that $A$ is locally $\mathcal{P}$. Let $A_i := A \times_F F_i$. This is an open substack of $A_i$. Let $X_i \to A_i$ be the smooth surjections from schemes of finite type. Then $X_i \to A$ is smooth (for example by the formal criterion for smoothness). Thus the morphism from the disjoint union of the $X_i$ to $A$ is a smooth surjection proving that $A$ is locally geometric according to the old definition.

Suppose now that $A$ is locally geometric for the old definition, and let $X_i \to A$ be the smooth morphisms from schemes of finite type which together cover $A$. Let $F = \tau_{\leq 0} A$ and let $F_i \subset F$ be the images of $X_i$. Let $A_i = A \times_F F_i$. It is clear that $X_i$ maps to $A_i$ by a map which is, on the one hand, smooth by the formal criterion, and on the other hand surjective on the level of $\pi_0$ by definition. Thus the $A_i$ are geometric, i.e. have property $\mathcal{P}$. It is clear that the union of the $A_i$ is $A$. Finally, the $F_i$ are open subsheaves of $F$, using smoothness of $X_i \to A$ plus Artin approximation.
6.3.7 — **Definition:** If \( P \) is a property of \( n \)-stacks of groupoids (say, independent of \( n \)... then we can extend \( P \) to a property of \( n \)-stacks in a minimal way such that the following conditions hold:

(A) If \( A \) has property \( P \) then so does the interior groupoid \( A^{\text{int}} \);

(B) If \( A \) has property \( P \) and \( a : X \to A \) and \( b : Y \to A \) are morphisms from schemes of finite type then \( \text{Hom}_A(a, b) \) has property \( P \).

That such a minimal extension exists is obvious by induction.

6.3.8 Taking the property \( P \) in the above definition 6.3.7 to be “locally presentable” or “locally very presentable” or “locally geometric” we obtain reasonable properties for \( n \)-stacks not necessarily of groupoids. The use of the locality properties is natural here since the composition operation will often be something like tensor product, which does not preserve any substack of finite type.

6.4 **The notion of stack, in general**

We give here a very general discussion of the notion of “stack”. This was called “homotopy-sheaf” in [28] (cf also [29] which predates [28] but which was made available much later), however that was not the first time that such objects were encountered—the condition of being a homotopy sheaf is the essential part of the condition of being a fibrant (or “flasque”) simplicial presheaf [11] [19] [21].

Suppose \( C \) is some type of category-like object (such as an \( n \)-category or \( \infty \)-category or other such thing). Suppose that we have a notion of inverse limit of a family of objects of \( C \) indexed by a category \( B \). If we call the family \( F : \mathcal{B}^\circ \to C \) (contravariant on \( B \), for our purposes) then we denote this limit—if it exists—by \( \Gamma(\mathcal{B}, F) \in C \). This should be sufficiently functorial in that if we have a functor \( \mathcal{B} \to \mathcal{B}' \) and \( F \) is the pullback of a family \( F' \) on \( \mathcal{B}' \) (denoted \( F = F'|_\mathcal{B} \)) then we should obtain a morphism of functoriality (i.e. an arrow in \( C \))

\[
\Gamma(\mathcal{B}', F') \to \Gamma(\mathcal{B}, F),
\]

possibly only well-defined up to some type of homotopy in \( C \). Similarly if \( \mathcal{B} \) has a final object \( b \) (initial for our functoriality which is contravariant) then the morphism (obtained from above for the inclusion \( \{b\} \to \mathcal{B} \))

\[
\Gamma(\mathcal{B}, F) \to F(b)
\]

should be an “equivalence” in \( C \) (one has to know what that means).

With all this in hand (and note that we do not assume the existence of arbitrary limits, only existence of limits indexed by categories with final objects) we can define the notion of **stack over a site \( \mathcal{X} \) with coefficients in \( C \)**. This is to be a family \( F \) of objects
of \( \mathcal{C} \) indexed by \( \mathcal{X} \) (i.e. a morphism \( \mathcal{X}^o \to \mathcal{C} \)) which satisfies the following property: for every object \( X \in \mathcal{X} \) and every sieve \( \mathcal{B} \subset \mathcal{X}/X \) the morphism

\[
\Gamma(\mathcal{X}/X, F|_{\mathcal{X}/X}) \to \Gamma(\mathcal{B}, F|_{\mathcal{B}})
\]

is an equivalence in \( \mathcal{C} \), meaning that the limit on the right exists. (Note that since \( \mathcal{X}/X \) has a final object \( X \), the morphism

\[
\Gamma(\mathcal{X}/X, F|_{\mathcal{X}/X}) \to F(X)
\]

is assumed to exist and to be an equivalence.)

Taking the inverse limit of a family of stacks will again be a family of stacks because inverse limits should (when that notion is defined) commute with each other. If \( \mathcal{C} \) admits arbitrary (set-theoretically reasonable) inverse limits then taking the inverse limit of a family of stacks gives again a stack. Using this we can define the stack associated to a prestack. A prestack is just any family \( F : \mathcal{X}^o \to \mathcal{C} \) not necessarily satisfying the stack condition. The associated stack is defined to be the inverse limit of all stacks \( G \) to which \( F \) maps. Of course this needs to be investigated some more in any specific case, in order to get useful information.

When \( \mathcal{C} \) is the 2-category of categories we obtain the classical notion of stack. When \( \mathcal{C} \) is the \( \infty \)-category of simplicial sets we obtain the notion of “homotopy sheaf” which is equivalent in Jardine’s terminology to a simplicial presheaf which is flasque with respect to each object of the underlying site. In particular, fibrant simplicial presheaves satisfy this condition, and the condition is just that of being object-by-object weak equivalent to a fibrant simplicial presheaf. The process of going from a prestack to the associated stack is basically the process of going from a simplicial presheaf to an equivalent fibrant simplicial presheaf.

The case where \( \mathcal{C} \) is the \( n+1 \)-category \( n\text{CAT}' \) of \( n \)-categories yields the notion of \( n \text{-stack} \) described above.

6.5 Localization

6.5.1 Universal morphisms with certain properties

We often encounter the following situation. Suppose \( X \in n\text{CAT}' \) and suppose and suppose \( \mathcal{P} \) is a property of morphisms \( X \to B \) in \( n\text{CAT}' \). Then we can look for a universal morphism \( \nu : X \to U \) with property \( \mathcal{P} \).

The “universal” property can be written out in terms of our construction \( \Upsilon \): it means that for any cofibration of \( n \)-precats \( E' \hookrightarrow E \) and any morphism (from an edge labeled (02))

\[
f : \Upsilon(E) \to n\text{CAT}'
\]
with $f(0) = X$, $f(2) = B$ sending $\Upsilon(E_0)$ to a collection of morphisms having property $\mathcal{P}$, together with an extension along $E'$ to a morphism

$$g' : \Upsilon^2(\ast, E') \to n\text{CAT}^r$$

with $r_{01}(g') = \nu$ and $g'(2) = B$, there exists

$$g : \Upsilon^2(\ast, E) \to n\text{CAT}^r$$

extending $g'$ and with $r_{01}(g) = \nu$ and $r_{02}(g) = f$.

### 6.5.2

Suppose $\psi : A \to n\text{CAT}^r$ is a functor and $\mathcal{P}$ a property of morphisms $\psi \to B$ to objects $B \in n\text{CAT}^r$. Then we can make a similar definition of “universal morphism” $\nu : \psi \to U$ having property $\mathcal{P}$.

In this case, it also makes sense to ask for a morphism $\nu : \psi \to U$ to an object of $n\text{CAT}^r$, “universal for morphisms with property $\mathcal{P}$” (the definition is the same as above but we don’t require $\nu$ to have property $\mathcal{P}$). Note that this definition in the case of one object $X$ is vacuous: the answer would just be the identity morphism $1_X : X \to X$.

### 6.5.3

To construct $\nu$ we can try to follow the argument of §5, taking the full subcategory $M(\mathcal{P}) \subset \psi/n\text{CAT}^r$ of objects under $\psi$ having property $\mathcal{P}$. As before we consider the subcategory $M(\mathcal{P})_\alpha$ of objects of $\alpha$-bounded cardinality, and let $U$ be the inverse limit of the forgetful functor $M(\mathcal{P})_\alpha \to n\text{CAT}^r$.

We now need to know four things:

(i) that the morphism $\psi \to U$ again has property $\mathcal{P}$ (preservation of $\mathcal{P}$ by inverse limits);

(ii) that there is a factorization $\psi \to U' \to U$ with $\psi \to U'$ again having property $\mathcal{P}$ and $\#^{pre} U' \leq \alpha$ (for $\alpha$ chosen appropriately);

(iii) that the “telescope” construction of (5.3.3) preserves property $\mathcal{P}$; and

(iv) that if $f : \psi \to \text{Hom}(E, B)$ is a morphism which, when restricted to every object of $E_0$ gives a morphism $\psi \to B$ with property $\mathcal{P}$, then $f$ has property $\mathcal{P}$ (this is so that a criterion analogue to 5.2.7 applies).

**Conjecture 6.5.4** If we know these four things then the argument of §5 works to construct a universal $\nu : \psi \to T$ with property $\mathcal{P}$.
6.5.5 Localization:

If $X$ is an $n$-category then we denote $Fl^i(X)$ the set of $i$-morphisms, which is the same as $X_{1,\ldots,1}$. Suppose we are given a collection of subsets $S = \{ S^i \subset Fl^i(X) \}$. Then we can define $S^{-1}X$ to be the universal $n$-category with map $X \to S^{-1}X$ sending the elements of $S^i$ to $i$-morphisms in $S^{-1}X$ which are invertible up to equivalence (i.e. morphisms which are invertible in $\tau_{\leq i}(S^{-1}X)$). To construct $S^{-1}X$, let $P$ be the property of a map $X \to B$ that the arrows in $S^i$ become invertible in $B$. One has to verify the properties 6.5.3(i)--6.5.3(iv), and then apply Conjecture 6.5.4. To verify the properties (i)--(iv) use Theorem 2.5.1.

This is the $n$-categorical analogue of [14].

Caution: If $A$ is an $m$-category considered as an $n$-category then $S^{-1}A$ may not be an $m$-category. In particular, note that by taking the group completion (see below) of 1-categories one gets all homotopy types of $n$-groupoids. (This fact, which seems to be due to Quillen, was discussed at length in [17]...).

6.5.6 Group completion:

The theory of $n$-categories which are not groupoids actually has a long history in homotopy theory, in the form of the study of topological monoids. In Adams’ book [1] the chapter after the one on loop-space machinery, concerns the notion of “group completion”, namely how to go from a topological monoid to a homotopy-theoretic group ($H$-space). This is a special example of going from an $n$-category to an $n$-groupoid by “formally inverting all arrows”.

Taking $S$ to be all of the arrows in a fibrant $n$-category $X$, the localization $S^{-1}X$ is the group completion of $X$ denoted $X^{gc}$. It is the universal $n$-groupoid to which $X$ maps. This may also be constructed by a topological approach (which has the merit of not depending on Conjecture 6.5.4), as

$$X^{gc} = \Pi_n(|X|),$$

using Tamsamani’s realization $|X|$ and Poincaré $n$-groupoid $\Pi_n$ constructions [30].

As an example, 2.5.1 allows us to describe the group completion of $I$ which is contractible, as one might expect.

Corollary 6.5.7 The morphism $I \to \overline{I}$ is the group completion in the context of $n$-categories.

Proof: This follows immediately from Theorem 2.5.1. ///
**Lemma 6.5.8** Group completion commutes with coproduct. More precisely, suppose $B \leftarrow A \rightarrow C$ are morphisms of $n$-precats. Then the morphism

$$(B \cup^A C)^{gp} \to B^{gp} \cup^{A^{gp}} C^{gp}$$

is an equivalence.

*Proof:* This can be seen directly from the topological definition $X^{gc} = \Pi_n(|X|)$ using the results of [32] §9.

---

**6.5.9 Interior groupoid** We can do a similar type of definition as 6.5.1 for universal maps from $B$ to $X$ having certain properties. Applying this again to the property that all $i$-morphisms become invertible, we get the following definition. If $X$ is a fibrant $n$-category then its interior groupoid $X^{int}$ is the universal map $X^{int} \to X$ for this property. It is an $n$-groupoid, and may be seen as the “largest $n$-groupoid inside $X$”.

Without referring to conjectures, we can construct $X^{int} \subset X$ explicitly as follows. Assume that $X$ is an $n$-category. First we define $X^{1-int} \subset X$ with the same objects as $X$, by setting

$$X^{1-int}_{p/}(x_0, \ldots, x_p) := X_{p/}(x_0, \ldots, x_p)^{int}$$

(note that we use inductively the definition of $Y^{int} \subset Y$ for $n-1$-categories as well as the fact that this construction takes equivalences to equivalences).

Now let

$$X^{1-int}_{1/}(x, y) \subset X^{1-int}_{1/}(x)$$

be the full sub-$(n-1)$-category of objects corresponding to morphisms which are invertible up to equivalence. Let $X^{1-int}_{p/}(x_0, \ldots, x_p)$ be the full sub-$(n-1)$-category of $X^{1-int}_{p/}(x_0, \ldots, x_p)$ consisting of objects which project to elements of $X^{1-int}_{1/}(x_{i-1}, x_i)$ on the principal edges.

Another way of saying this is to note that there is a morphism

$$X^{1-int} \to \tau_{\leq 1}(X)$$

(cf the notation of 2.1.10). Then define the “interior 1-groupoid” of the 1-category $\tau_{\leq 1}(X)$ to be the subcategory consisting only of invertible morphisms, and set $X^{int}$ to be the fiber product of $X^{1-int} \to \tau_{\leq 1}(X)$ and interior 1-groupoid of $\tau_{\leq 1}(X)$, over $\tau_{\leq 1}(X)$.

**6.5.10 $k$-groupic completion and interior**

More generally we say that an $n$-category $B$ is $k$-groupic for $0 \leq k \leq n$ if the $n-k$-categories $B_{m_1, \ldots, m_k}$ are groupoids. In other words this says that the $n-k$-category whose objects are the $k$-morphisms of $B$ should be an $n-k$-groupoid. Note that being $O$-groupic means that $B$ is an $n$-groupoid, and the condition of being $n$-groupic is void of content.
We can define the $k$-groupic completion $X^{k-\text{gp}}$ as the universal $k$-groupic $n$-category to which $X$ maps. We can define the $k$-groupic interior $X^{k-\text{int}} \subset X$ to be the universal $k$-groupic $n$-category mapping to $X$. For $k = 0$ these reduce to the group completion and interior groupoid. For the $k$-groupic interior, we have the following formula whenever $k \geq 1$:

$$X^{k-\text{int}}(x_0, \ldots, x_p) = X_p/(x_0, \ldots, x_p)^{(k-1)-\text{int}},$$

which gives an inductive construction.

6.6 Direct images and realizations

Suppose $F : A \to B$ is a morphism of $n+1$-categories and suppose $\varphi : A \to \text{nCAT}^r$ is a family of $n$-categories over $A$. Then we can look for a universal family $\psi : B \to \text{nCAT}^r$ together with morphism $\varphi \to F^*(\psi)$. If it exists, we call $\psi$ the direct image and denote it by $F_*(\varphi)$.

Conjecture 6.6.1 The direct image $F_*(\varphi)$ always exists, and is essentially unique.

Again, the argument of §5 should work to give the construction of $F_*(\varphi)$, with several things to verify analogous to 6.5.3(i-iv).

6.6.2 Caution: the notations “direct image” $F_*$ and “inverse image” $F^*$ are switched from the usual notations for functoriality for “morphisms of sites”.

6.6.3 Realization:

Suppose $A$ is an $n+1$-category and suppose

$$\varphi : A \to \text{nCAT}^r$$

is a family of $n$-categories, and

$$\psi : A^o \to \text{nCAT}^r$$

is a contravariant family of $n$-categories. Then we define the realization of this pair, denoted $\langle \varphi, \psi \rangle$, as follows. The arrow family for $A$ corresponds to a morphism

$$\alpha : A \to \text{Hom}(A^o, \text{nCAT}^r).$$

The direct image $\alpha_*(\varphi)$ is therefore a morphism

$$\alpha_*(\varphi)\text{Hom}(A^o, \text{nCAT}^r) \to \text{nCAT}^r.$$

Put

$$\langle \varphi, \psi \rangle := \alpha_*(\varphi)(\psi).$$
6.6.4 An example of this is when $A = \mathcal{X}$ is a site, and when $\varphi$ and $\psi$ are families of $n$-groupoids. Then $\langle \varphi, \psi \rangle$ is an $n$-groupoid, and we conjecture that it corresponds to the topological space given as realization of the two functors as defined in [30].

6.6.5 In the main example of [30] one took $\mathcal{X}$ to be the site of schemes over $\text{Spec}(\mathcal{C})$ and one took $\varphi$ to be the functor associating to each scheme the $n$-truncation of the homotopy type of the underlying topological space. Then for any presheaf $\psi$ of $n$-truncated topological spaces one obtained the “topological realization” of $\psi$.

6.6.6 One can do the operation of 6.6.3 in the other order, using the arrow family considered as a morphism

$$\beta : A^o \to \mathbb{Hom}(A, n\text{CAT}'')$$

and looking at $\beta_\ast(\psi)(\varphi)$.

*Conjecture*—that these two ways of defining $\langle \varphi, \psi \rangle$ give the same answer.

6.6.7 The above construction is a special case of the more general phenomenon which we call “triple combination”. Suppose $A$ and $B$ are $(n+1)$-categories and suppose that we have functors

$$F : A \to n\text{CAT}'',
G : B \to n\text{CAT}'',$$

and

$$H : A \times B \to n\text{CAT}''.$$

Then we can consider $H$ as a functor

$$H : A \to \mathbb{Hom}(B, n\text{CAT}'')$$

and define

$$H(F,G) := H_\ast(F)(G).$$

As above, one conjectures that $H(F,G) = H_\ast(G,F)$ (applying the symmetry $\sigma : A \times B \cong B \times A$). The previous construction is just

$$\langle \varphi, \psi \rangle = \text{Arr}(A)(\varphi, \psi).$$

The same definition of triple combination works for functors $F, G, H$ in any fibrant $n$-category $C$ which admits limits as does $n\text{CAT}''$. 

88
6.7 Relative Malcev completion

An example which gets more to the point of my motivation for doing all of this type of thing is the following generalization of relative Malcev completion \[18\] to higher homotopy.

6.7.1 Fix a $\mathbb{Q}$-algebraic group $G$. Fix an $n$-groupoid $X$ with base-object $x$ (which is the same thing as an $n$-truncated pointed homotopy type). Fix a representation $\rho : \pi_1(X, x) \to G$. Let $\mathcal{C}$ be the $n+1$-category of quadruples $(R, r, p, f)$ where $R$ is a connected $n$-groupoid, $r$ is an object, $p : R \to BG$ is a morphism sending $r$ to the base-object $o$, and $f : X \to R$ is a morphism sending $x$ to $r$ such that the induced morphism $\pi_1(X, x) \to G$ is equal to $\rho$. Let $\mathcal{C}^\uni$ denote the subset of objects satisfying the following properties: that $\pi_1(R)$ is a $\mathbb{Q}$-algebraic group and $p : \pi_1(R) \to G$ is a surjection with unipotent kernel; and that $\pi_1(R)$ acts algebraically on the higher homotopy groups $\pi_1(R)$ which are themselves assumed to be finite dimensional $\mathbb{Q}$-vector spaces.

6.7.2 Inverse limits exist in $\mathcal{C}$. To see this, note that $\mathcal{C}$ is an $n+1$-category of morphisms $V \to nCAT'$ where $V$ is the category with objects $v_R, v_r, v_{BG}, v_X$ and morphisms $v_r \to v_R, v_R \to v_{BG}, v_X \to v_R, v_r \to v_X$. The $n+1$-category $\mathcal{C}$ is the subcategory of morphisms $V \to nCAT'$ which send $v_r$ to $\ast$, send $v_{BG}$ to $BG$ and send $v_X$ to $X$, and which send the maps $v_r \to v_X$ to the basepoint $\ast \to X$, similarly for the map $v_r \to v_{BG}$, and which send $v_X \to v_{BG}$ to the map induced by $\rho$. Our Theorem \[4.0.1\] as well as \[3.4.4\] and Lemma \[3.4.2\] imply that $\mathcal{C}$ admits inverse limits. Of course $\mathcal{C}^\uni$ is not closed under inverse limits. However we can still take the inverse limit in $\mathcal{C}$ of all the objects in $\mathcal{C}^\uni$. We call this the relative Malcev completion of the homotopy type of $X$ at $\rho$, and denote it by $\text{Malc}(X, \rho)$ (technically this is the notation for the underlying $n$-groupoid which is the inverse limit of the $R$’s).

6.7.3 We have, for example, that $\pi_1(\text{Malc}(X, \rho), \ast)$ is equal to the relative Malcev completion of the fundamental group $\pi_1(X)$ at $\rho$. For this statement we fall back into the realm of 1-categories, where our Malcev completion coincides with the usual notion \[18\].

6.7.4 We can do the same thing with stacks. For a field $k$ (of characteristic zero, say) an algebraic group $G$ over $k$ and a representation $\rho : \pi_1(X, x) \to G$, let $\mathcal{C}(X, \rho)/k$ be the $n+1$-category of quadruples $(R, r, p, f)$ where $R$ is a connected $n$-stack of groupoids on $\text{Sch}/k$, $r$ is a basepoint, $p : R \to BG$, and $f : X \to R$ are as above. Here $X$ is the constant stack with values $X$. Let $\mathcal{C}^\uni(X, \rho)/k$ be the subcategory of objects such that $\pi_1(R)$ is an algebraic group surjecting onto $G$ and where the $\pi_1(R)$ are linear finite dimensional representations of $\pi_1(R)$. Again inverse limits will exist in $\mathcal{C}(X, \rho)/k$ and we can take the inverse limit here of the objects of $\mathcal{C}^\uni(X, \rho)/k$. Call this $\text{Malc}(X, \rho)/k$.

Note that $\text{Malc}(X, \rho)/\mathbb{Q}$ is an $n$-stack on $\text{Sch}/\mathbb{Q}$ whose $n$-groupoid of global sections is $\text{Malc}(X, \rho)$. 89
6.7.5 Suppose $X$ is a variety and let $X_B$ be the $n$-groupoid truncation of the homotopy type of $X^{\text{top}}$. Fix a representation $\rho$. Then we obtain the “Betti” Malcev completion $\text{Malc}(X_B, \rho)/\mathbb{C}$. On the other hand suppose $P$ is the principal $G$-bundle with integrable connection (with regular singularities at infinity) corresponding to $\rho$, then we can define in a similar way $\text{Malc}(X_{\text{DR}}, P)/\mathbb{C}$. The GAGA results imply that these two are naturally equivalent:

$$\text{Malc}(X_B, \rho)/\mathbb{C} \cong \text{Malc}(X_{\text{DR}}, P)/\mathbb{C}.$$ 

Similarly we can define, for a principal Higgs bundle $Q$ with vanishing Chern classes, $\text{Malc}(X_{\text{Dol}}, Q)/\mathbb{C}$, and (in the case $X$ smooth projective) if $Q$ corresponds to $\rho$ then

$$\text{Malc}(X_B, \rho)/\mathbb{C} \cong \text{Malc}(X_{\text{Dol}}, Q)/\mathbb{C}.$$ 

Finally, suppose $\rho$ is an $\mathbb{R}$-variation of Hodge structure and $Q$ the corresponding system of Hodge bundles. Then $\mathbb{C}^*$ acts on $\text{Malc}(X_{\text{Dol}}, Q)/\mathbb{C}$ giving rise to a “weight filtration” and “Hodge filtration”. We conjecture that these (together with the $\mathbb{R}$-rational structure $\text{Malc}(X_B, \rho)/\mathbb{R}$) define a “mixed Hodge structure” on $\text{Malc}(X_B, \rho)/\mathbb{R}$. (One has to give this definition, specially in view of the infinite size of $\text{Malc}(X_B, \rho)/\mathbb{R}$).

More generally we have the following conjecture.

**Conjecture 6.7.6** Suppose $\rho$ is a reductive representation of the fundamental group of a projective variety $X$ (we assume it is reductive when restricted to the fundamental group of the normalization). Then the relative Malcev completion of the higher homotopy type $\text{Malc}(X_B, \rho)/\mathbb{C}$ defined above carries a natural mixed twistor structure (cf [33]).

There should also be a statement for quasiprojective varieties, but in this case one probably needs some additional hypotheses on the behavior of $\rho$ at infinity.

**References**

[1] J. Adams. *Infinite Loop Spaces*, Princeton University Press *Annals of Math. Studies* **90** (1978).

[2] M. Artin. Versal deformations and algebraic stacks, *Inventiones Math.* **27** (1974), 165-189.

[3] J. Baez, J. Dolan. $n$-Categories, sketch of a definition. Letter to R. Street, 29 Nov. and 3 Dec. 1995, available at [http://math.ucr.edu/home/baez/ncat.def.html](http://math.ucr.edu/home/baez/ncat.def.html)
[4] J. Baez, J. Dolan. Higher-dimensional algebra and topological quantum field theory. *Jour. Math. Phys* 36 (1995), 6073-6105 (preprint dating from q-alg 95-03).

[5] J. Baez, J. Dolan. Higher dimensional algebra III: n-categories and the algebra of opetopes. Preprint available on q-alg (97-02).

[6] M. Batanin. On the definition of weak ω-category. Macquarie mathematics report number 96/207, Macquarie University, NSW Australia.

[7] M. Batanin. Monoidal globular categories as a natural environment for the theory of weak n-categories. Preprint, April 1997.

[8] J. Bénabou. *Introduction to Bicategories*, Lect. Notes in Math. 47, Springer-Verlag (1967).

[9] A. Bousfield, D. Kan. *Homotopy limits, completions and localizations*. Springer Lecture Notes in Mathematics 304 (1972).

[10] L. Breen. On the classification of 2-gerbs and 2-stacks. *Astérisque* 225, Soc. Math. de France (1994).

[11] K. Brown. Abstract homotopy theory and generalized sheaf cohomology. *Trans. A.M.S.* 186 (1973), 419-458.

[12] K. Brown, S. Gersten. *Algebraic K-theory as generalize sheaf cohomology* Springer Lecture Notes in Math. 341 (1973), 266-292.

[13] P. Deligne, D. Mumford. On the irreducibility of the space of curves of a given genus. *Publ. Math. I.H.E.S.* 36 (1969), 75-109.

[14] P. Gabriel, M. Zisman. *Calculus of fractions and homotopy theory*. Springer, New York (1967).

[15] J. Giraud. *Cohomologie nonabélienne*, Grundlehren der Wissenschaften in Einzel-darstellung 179 Springer-Verlag (1971).

[16] R. Gordon, A.J. Power, R. Street. Coherence for tricategories *Memoirs A.M.S.* 117 (1995), 558 ff.

[17] A. Grothendieck. *Pursuing Stacks*, unpublished manuscript.

[18] R. Hain. Completions of mapping class groups and the cycle $C - C^-$.

[19] J.F. Jardine. Simplicial presheaves, *J. Pure and Appl. Algebra* 47 (1987), 35-87.
[20] M. Johnson. The combinatorics of $n$-categorical pasting. *J. Pure and Appl. Algebra* **62** (1989), 211-225.

[21] A. Joyal. Letter to A. Grothendieck (referred to in Jardine’s paper).

[22] G. Laumon, L. Moret-Bailly. Champs algébriques. Preprint, Orsay **42** (1992).

[23] S. MacLane. Categories for the working mathematician. Springer (1971).

[24] J. P. May. *Simplicial objects in algebraic topology*. Van Nostrand (1967).

[25] D. Quillen. *Homotopical algebra* Springer *L.N.M.* **43** (1967).

[26] D. Quillen. Rational Homotopy Theory. *Ann. Math.* **90** (1969), 205-295.

[27] G. Segal. Homotopy everything $H$-spaces. Preprint.

[28] C. Simpson. Homotopy over the complex numbers and generalized de Rham cohomology. *Moduli of Vector Bundles*, M. Maruyama (Ed.) *Lecture Notes in Pure and Applied Math.* **179**, Marcel Dekker (1996), 229-263.

[29] C. Simpson. Flexible sheaves. Preprint available on q-alg (96-08).

[30] C. Simpson. The topological realization of a simplicial presheaf. Preprint, available on q-alg 96-09.

[31] C. Simpson. Algebraic (geometric) $n$-stacks. Preprint, available on alg-geom 96-09.

[32] C. Simpson. A closed model structure for $n$-categories, internal $Hom$, $n$-stacks and generalized Seifert-Van Kampen. Preprint, available on alg-geom 97-04.

[33] C. Simpson. Mixed twistor structures. Preprint, available on alg-geom 97-05.

[34] J. Stasheff. Homotopy associativity of $H$-spaces. *Trans. Amer. Math. Soc.* **108** (1963), 275-312.

[35] R. Street. The algebra of oriented simplexes. *Jour. Pure and Appl. Algebra* **49** (1987), 283-335.

[36] Z. Tamsamani. Sur des notions de $n$-categorie et $n$-groupeoide non-strictes via des ensembles multi-simpliciaux. Thesis, Université Paul Sabatier, Toulouse (1996) available on alg-geom (95-12 and 96-07).

[37] D. Tanré. *Homotopie Rationnelle: modèles de Chen, Quillen, Sullivan*. Springer *Lecture Notes in Mathematics* **1025** (1983).