An Analytic Expression for the Growth Function in a Flat Universe with a Cosmological Constant

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ABSTRACT

An analytic expression is given for the growth function for linear perturbations in a low-density universe made flat by a cosmological constant. The result involves elliptic integrals but is otherwise straight-forward.

Subject headings: cosmology: theory – large-scale structure of the universe

In linear cosmological perturbation theory, the evolution of perturbations in the absence of pressure is reduced to the superposition of two modes with fixed time dependence and arbitrary spatial dependence (Peebles 1980). The behavior of these modes in time is a function of the background cosmological parameters. The functions of time that describe this behavior are called the growth functions and denoted $D_1$ and $D_2$.

As shown by Heath (1977), the growth functions in cosmologies where pressure fluctuations are negligible can be written as

$$D_1(a) \propto H(a) \int_0^a \frac{da'}{a'^3H(a')^3},$$
$$D_2(a) \propto H(a),$$
$$H(a) = \sqrt{\Omega_0 a^{-3} + \Omega_R a^{-2} + \Omega_\Lambda}.$$ (3)

Here, $a$ is the expansion scale factor of the universe, chosen so that $a = 1$ today; this will be our time coordinate. The redshift $z$ is then $a^{-1} - 1$. $\Omega_0$ is the density of non-relativistic matter in the universe today in units of the critical density. $\Omega_\Lambda = \Lambda/3H_0^2$ is the cosmological constant $\Lambda$ relative to the present-day Hubble constant $H_0$. $\Omega_R$ is the curvature term and is equal to $1 - \Omega_0 - \Omega_\Lambda$. We assume that relativistic matter is a negligible contributor to the density of the universe. Finally, $H(a)$ is proportional to the Hubble constant at epoch $a$, scaled so that $H = 1$ today. We are interested in the behavior of $D_1(a)$, as this is the mode whose amplitude grows in time. However, the overall normalization of $D_1$ is merely a convention.

The analytic form for $D_1$ in the $\Lambda = 0$ case (Weinberg 1972; Groth & Peebles 1975; Edwards & Heath 1976) is widely known, but no analytic solution for the case of $\Lambda \neq 0$ has been presented in the literature. Instead, workers integrate equation (1) numerically or use the approximations given by Lahav et al. (1991) or Carroll et al. (1992). However, the integral can be done in terms of elliptic
integrals, which of course are implemented as special functions in many numerical packages. Here, we restrict ourselves to the case of a flat, low-density universe, i.e. \( \Omega_R = 0, \Omega_0 < 1, \Omega_\Lambda = 1 - \Omega_0 \). This is the case commonly used in practice and the solution can be stated rather cleanly.

We adopt a normalization for \( D_1 \) as

\[
D_1(a) = \frac{5\Omega_0}{2} \frac{da'}{a^2 H(a)'} \int_0^a \frac{du}{(1 + u^3)^{3/2}}.
\]

This is chosen so that \( D_1(a) \rightarrow a \) as \( a \rightarrow 0 \). Defining \( v = a^{-1} \sqrt{\Omega_0/(1 - \Omega_0)} \) and manipulating the integral yields

\[
D_1(a) = \frac{5}{2} a v \sqrt{1 + v^3} \int_v^\infty \frac{du}{(1 + u^3)^{3/2}} - \frac{5}{2} a v \sqrt{1 + v^3} \lim_{v_\infty \rightarrow \infty} \lim_{b \rightarrow 1} \left( -\frac{d}{db} \int_v^{v_\infty} \frac{du}{\sqrt{b + u^3}} \right) = \frac{5}{2} a v \sqrt{1 + v^3} \lim_{v \rightarrow \infty} \left[ -\frac{1}{6} \int_v^{v_\infty} \frac{du}{\sqrt{1 + u^3}} + \frac{1}{3} \left( \frac{v^3}{v_\infty^3} - \frac{1}{3} \right) \right].
\]

Doing this last integral (e.g. using 3.139.8 in Gradshteyn & Ryzhik 1994) yields the final answer

\[
D_1(a) = a \times d_1 \left( \frac{1}{a} \sqrt{\frac{\Omega_0}{1 - \Omega_0}} \right),
\]

where

\[
d_1(v) = \frac{5}{3} v \left\{ 4 \sqrt{3} \sqrt{1 + v^3} \left[ F(\beta, \sin 75^\circ) - \frac{1}{3 + \sqrt{3}} F(\beta, \sin 75^\circ) \right] + \frac{1}{3} \left( \frac{\sqrt{3} + 1}{v + 1 + \sqrt{3}} \right) \right\},
\]

\[
\beta = \arccos \frac{v + 1 - \sqrt{3}}{v + 1 + \sqrt{3}}.
\]

Here, \( F \) and \( E \) are incomplete elliptic integrals of the first and second kind, using the definitions \( F(\phi, k) = \int_0^\phi (1 - k^2 \sin^2 \theta)^{-1/2} d\theta \) and \( E(\phi, k) = \int_0^\phi (1 - k^2 \sin^2 \theta)^{1/2} d\theta \); note that conventions vary on whether \( k \) is squared or not. Also note that \( \beta \) can exceed \( \pi/2 \); some numerical implementations of the elliptic integrals do not handle this properly. In this case one can use the identity \( F(\phi, k) = 2F(\pi/2, k) - F(\pi - \phi, k) \) and likewise for \( E \) to compute for \( \beta > \pi/2 \).

As \( a \rightarrow 0, v \rightarrow +\infty \) and \( d_1 \rightarrow 1 - (2/11)v^{-3} + (16/187)v^{-6} + O(v^{-9}) \). This is not obvious from equation (9); the terms of order \( v^2 \) and \( v \) cancel. This suggests that for numerical evaluation for \( v \gg 1 \), one should be watchful for loss of precision. Alternatively, for \( v \gtrsim 5 \) one can use the large \( v \) expansion given above. As \( a \rightarrow \infty, v \rightarrow 0 \) and \( D_1 = 1.4373 \sqrt{\Omega_0/(1 - \Omega_0)} \).

The approximation of Carroll et al. (1992, adapted from Lahav et al. 1991) that

\[
D_1(1) = \frac{5\Omega_0}{2} \left[ \Omega_0^{4/7} - \Omega_\Lambda + (1 + \Omega_0/2)(1 + \Omega_\Lambda/70) \right]^{-1}
\]

(11)
can be compared to \( d_1(v) \) using \( \Omega_A = 1 - \Omega_0 = (v^3 + 1)^{-1} \). The fit has fractional errors better than 2\% over the range \( z \geq 0 \) and \( \Omega_0 > 0.1 \) (i.e. \( v > 0.48 \)) but deviates for lower \( \Omega_0 \). A plot of \( d_1(v) \) and the Carroll et al. (1992) fit is shown in Figure 1. The fit has similar accuracy for the \( \Lambda = 0 \) case even to much lower \( \Omega_0 \).

For completeness, the solution for the expanding epoch of a flat universe with \( \Omega_0 > 1 \) is

\[
\tilde{D}_1(a) = a \times \tilde{d}_1 \left( \frac{\Omega_0}{\Omega_0 - 1} \right),
\]

\[
\tilde{d}_1(v) = \frac{5}{3} v \left\{ \frac{4}{\sqrt{3}} v^3 - 1 \left[ \frac{1}{3 - \sqrt{3}} F(\gamma, \sin 15^\circ) - E(\gamma, \sin 15^\circ) \right] + \frac{(\sqrt{3} - 1)v^2 + 1}{v - 1 + \sqrt{3}} \right\},
\]

\[
\gamma = \arccos \frac{v - 1 - \sqrt{3}}{v - 1 + \sqrt{3}},
\]

\( v = 1 \) is the time at which the universe reaches maximum expansion. As \( v \to \infty \), \( \tilde{d}_1 \to 1 + (2/11)v^{-3} + (16/187)v^{-6} + O(v^{-9}) \).

As a convenience to the reader, we collect other exact equations relevant to low-density, flat universes from the literature (Edwards 1972, 1973; Dabrowski & Stelmach 1986, 1987; Weinberg 1972, 1989). The time-redshift relation may be written as

\[
t(z) = \frac{2}{3H_0} \frac{1}{\sqrt{1 - \Omega_0}} \sinh^{-1} \left( (1 + z)^{-3/2} \sqrt{\frac{1 - \Omega_0}{\Omega_0}} \right).
\]

The conformal time \( \eta \equiv \int a^{-1} dt \) is

\[
\eta(z) = \frac{1}{H_0} \int_0^{(1+z)^{-1}} \frac{da}{a^2 H(a)} = \frac{1}{H_0 \sqrt{3}} \left[ \Omega_0^3 (1 - \Omega_0) \right]^{-1/6} F[\beta(z), \sin 75^\circ]
\]

\[
\beta(z) = \arccos \frac{1 + z + (1 - \sqrt{3})^{3/\sqrt{3} \Omega_0^{-1} - 1}}{1 + z + (1 + \sqrt{3})^{3/\sqrt{3} \Omega_0^{-1} - 1}}
\]

where \( H(a) \) is defined in equation (3). The coordinate radius out to redshift \( z \) is \( r_1(z) = c[\eta(0) - \eta(z)] \), where \( c \) is the speed of light and the flatness of the universe has been used. Then the luminosity distance \( d_L \) is \( (1 + z)r_1(z) \), the angular diameter distance \( d_A \) to an object of given physical size is \( (1 + z)^{-1} r_1(z) \), and the comoving volume per unit redshift per unit solid angle is

\[
dV_c/dz d\Omega = \frac{c}{H_0 \sqrt{1 - \Omega_0 + \Omega_0 (1 + z)^3}}.
\]

A quantity related to the growth function is \( f \equiv (a/D_1)(dD_1/da) \), used to relate density perturbations to velocity perturbations. Assuming the normalization in equation (4), one finds
\begin{equation}
    f(a) = \frac{\Omega_0}{(1-\Omega_0)a^3 + \Omega_0} \left( \frac{5a}{2D_1(a)} - \frac{3}{2} \right).
\end{equation}

In summary, the expression presented here for the growth function is marginally more complicated than the \( \Lambda = 0 \) case, but if one evaluates the elliptic integrals using available numerical packages, the exact function is quite tractable.

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REFERENCES

Carlson, B.C. 1977, SIAM Journal on Mathematical Analysis, 8, 231
Carroll, S.M., Press, W.H., Turner, E.L. 1992, ARA&A, 30, 499
Dabrowski, M.P., & Stelmach, J. 1986, AJ, 92, 1272
Dabrowski, M.P., & Stelmach, J. 1987, AJ, 94, 1373
Edwards, D. 1972, MNRAS, 159, 51
Edwards, D. 1973, Ap Space Sci, 24, 563
Edwards, D., & Heath, D. 1976, Ap Space Sci, 41, 183
Gradshteyn, I.S., & Ryzhik, I.M. 1994, Table of Integrals, Series, and Products, 5th ed. (Boston: Academic Press)
Groth, E.J., & Peebles, P.J.E. 1975, A&A, 41, 143
Heath, D.J. 1977, MNRAS, 179, 351
Lahav, O., Rees, M.J., Lilje, P.B., & Primack, J.R. 1991, MNRAS, 251, 128
Peebles, P.J.E. 1980, Large-Scale Structure of the Universe (Princeton: Princeton Univ. Press)
Peebles, P.J.E. 1984, ApJ, 284, 439
Weinberg, S. 1972, Gravitation and Cosmology (New York: Wiley)
Weinberg, S. 1989, Rev Mod Phys, 61, 1

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Fig. 1.— The function $d_1(v)$ from equation (9) (solid line). The growth function $D_1(a)$ is $a d_1(a^{-1} \sqrt[3]{\Omega_0/(1 - \Omega_0)}$. The approximation of Carroll et al. (1992) (dashed line), using $1 - \Omega_0 = (v^3 + 1)^{-1}$, fits quite well in the observationally relevant portion of parameter space.