Vertices Belonging to All Critical Independent Sets of a Graph

Vadim E. Levit
Ariel University Center of Samaria, Israel
levitv@ariel.ac.il

Eugen Mandrescu
Holon Institute of Technology, Israel
eugen.m@hit.ac.il

Abstract

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is independent if no two vertices from $S$ are adjacent, and by $\text{Ind}(G)$ ($\Omega(G)$) we mean the set of all (maximum) independent sets of $G$, while $\text{core}(G) = \cap \{S : S \in \Omega(G)\}$. [13]. The neighborhood of $A \subseteq V$ is $N(A) = \{v \in V : N(v) \cap A \neq \emptyset\}$. The independence number $\alpha(G)$ is the cardinality of each $S \in \Omega(G)$, and $\mu(G)$ is the size of a maximum matching of $G$.

The number $\text{id}_{c}(G) = \max\{|I| - |N(I)| : I \in \text{Ind}(G)\}$ is called the critical independence difference of $G$, and $A \in \text{Ind}(G)$ is critical if $|A| - |N(A)| = \text{id}_{c}(G)$, [22]. We define $\text{ker}(G) = \cap \{S : S \text{ is a critical independent set}\}$.

In this paper we prove that if a graph $G$ is non-quasi-regularizable (i.e., there exists some $A \in \text{Ind}(G)$, such that $|A| > |N(A)|$), then:

• $\text{ker}(G) \subseteq \text{core}(G)$
• $|\text{ker}(G)| > \text{id}_{c}(G) \geq \alpha(G) - \mu(G) \geq 1$.

Keywords: independent set, critical set, critical difference, maximum matching

1 Introduction

Throughout this paper $G = (V, E)$ is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set $V = V(G)$ and edge set $E = E(G)$. We consider only graphs without isolated vertices.

If $X \subseteq V$, then $G[X]$ is the subgraph of $G$ spanned by $X$. By $G - W$ we mean either the subgraph $G[V - W]$, if $W \subseteq V(G)$, or the partial subgraph $H = (V, E - W)$ of $G$, for $W \subseteq E(G)$. In either case, we use $G - w$, whenever $W = \{w\}$. If $X, Y \subset V$ are non-empty and disjoint, then we denote $(X, Y) = \{xy : xy \in E, x \in X, y \in Y\}$.

The neighborhood of a vertex $v \in V$ is the set $N(v) = \{w : w \in V \text{ and } vw \in E\}$, while the closed neighborhood of $v \in V$ is $N[v] = N(v) \cup \{v\}$; in order to avoid ambiguity, we use also $N_{G}(v)$ instead of $N(v)$. In particular, if $|N(v)| = 1$, then $v$ is a pendant vertex.
of $G$, and $\text{pend}(G) = \{v \in V(G) : v \text{ is a pendant vertex in } G\}$. The neighborhood of $A \subseteq V$ is denoted by $N(A) = N_G(A) = \{v \in V : N(v) \cap A \neq \emptyset\}$, and $N[A] = N(A) \cup A$.

A set $S \subseteq V(G)$ is independent if no two vertices from $S$ are adjacent, and by $\text{Ind}(G)$ we mean the set of all the independent sets of $G$. An independent set of maximum size will be referred to as a maximum independent set of $G$, and the independence number of $G$ is $\alpha(G) = \max\{|S| : S \in \text{Ind}(G)\}$. A graph $G$ is quasi-regularizable if one can replace each edge of $G$ with a non-negative integer number of parallel copies, so as to obtain a regular multigraph of degree $\neq 0$, [2]. For instance, $K_4 - e, e \in E(K_4)$, is quasi-regularizable, while $P_3$ is not quasi-regularizable. It is clear that a quasi-regularizable graph can not have isolated vertices.

**Theorem 1.1** For a graph $G$ the following assertions are equivalent:

(i) quasi-regularizable;

(ii) $|S| \leq |N(S)|$ holds for every $S \in \text{Ind}(G)$;

(iii) $G$ has a perfect 2-matching, i.e., $G$ contains a system of vertex-disjoint odd cycles and edges covering all its vertices.

Let $\Omega(G) = \{S : S \text{ is a maximum independent set of } G\}$ and $\xi(G) = |\text{core}(G)|$, where $\text{core}(G) = \cap \{S : S \in \Omega(G)\}$, [13].

Similarly, let $\text{corona}(G) = \cup \{S : S \in \Omega(G)\}$, and $\zeta(G) = |\text{corona}(G)|$, [13].

A matching is a set of non-incident edges of $G$; a matching of maximum cardinality $\mu(G)$ is a maximum matching, and a perfect matching is a matching covering all the vertices of $G$.

In the sequel we need the following characterization of a maximum independent set of a graph, due to Berge.

**Theorem 1.2** [2] An independent set $S$ belongs to $\Omega(G)$ if and only if every independent set $A$ of $G$, disjoint from $S$, can be matched into $S$.

$G$ is called a König-Egerváry graph provided $\alpha(G) + \mu(G) = |V(G)|$ [6, 20]. It is known that each bipartite graph satisfies this property.

**Theorem 1.3** [13] If $G$ is a König-Egerváry graph, $M$ is a maximum matching, then $M$ matches $V(G) - S$ into $S$, for every $S \in \Omega(G)$, and $\mu(G) = |V(G) - S|$.

In Boros et al. [3] it has been proved that if $G$ is connected and $\alpha(G) > \mu(G)$, then $\xi(G) = |\text{core}(G)| > \alpha(G) - \mu(G)$. This strengthened the following finding stated in [13]: if $\alpha(G) > (|V(G)| + k - 1)/2$, then $\xi(G) \geq k + 1$; moreover, $\xi(G) \geq k + 2$ is valid, whenever $|V(G)| + k - 1$ is an even number. For $k = 1$, the previous inequality provides us with a generalization of a result of Hammer et al. [8], claiming that if a graph $G$ has $\alpha(G) > |V(G)|/2$, then $\xi(G) \geq 1$. In [12] it was shown that if $G$ is a connected bipartite graph with $|V(G)| \geq 2$, then $\xi(G) \neq 1$. Jamison [9], Zito [29], and Gunther et al. [7] proved independently that $\xi(G) \neq 1$ is true for any tree $T$.

In Chlebík et al. [5] it has been found that if there is some $S \in \text{Ind}(G)$, such that $|S| > |N(S)|$, then $|\text{core}(G)| > \max\{|I| - |N(I)| : I \in \text{Ind}(G)\}$. It strengthens the inequality $|\text{core}(G)| > \alpha(G) - \mu(G)$ [3], since $\max\{|I| - |N(I)| : I \in \text{Ind}(G)\} \geq \alpha(G) - \mu(G)$ [17, 19].
The number $d(X) = |X| - |N(X)|$ is called the difference of the set $X \subseteq V(G)$, and $d_c(G) = \max\{d(X) : X \subseteq V(G)\}$ is the critical difference of $G$. A set $U \subseteq V(G)$ is critical if $d(U) = d_c(G)$ \cite{22}. The number $id_c(G) = \max\{d(I) : I \in \text{Ind}(G)\}$ is called the critical independence difference of $G$. If $A \subseteq V(G)$ is independent and $d(A) = id_c(G)$, then $A$ is called critical independent \cite{22}.

For a graph $G$ let us denote $\ker(G) = \cap \{S : S$ is a critical independent set $\}$ and $\varepsilon(G) = |\ker(G)|$.

For instance, the graph $G_1$ in Figure 1 has $\ker(G_1) = \text{core}(G_1) = \{a, b\}$. The graph $G_2$ from Figure 1 has $X = \{x, y, z, p, q\}$ as a critical non-independent set, because $d(X) = 1 = d_c(G_2)$, while $\ker(G_2) = \{x, y\} \subset \text{core}(G_2) = \{x, y\}$. The graph $G_3$ from Figure 1 has $\{t, u, v\}$ as a critical set, $\ker(G_3) = \{u, v\}$, while $\text{core}(G_3) = \{t, u, v, w\}$ is not a critical set.

![Figure 1: Non-quasi-regularizable graphs.](image)

Clearly, $d_c(G) \geq id_c(G)$ is true for every graph $G$.

**Theorem 1.4** \cite{22} The equality $d_c(G) = id_c(G)$ holds for every graph $G$.

If $A \in \Omega(G[N[A]])$, then $A$ is called a local maximum independent set of $G$ \cite{14}.

It is easy to see that all pendant vertices are included in every maximum critical independent set. It is known that the problem of finding a critical independent set is polynomially solvable \cite{11,22}.

**Theorem 1.5** (i) \cite{18} Each local maximum independent set is included in a maximum independent set.

(ii) \cite{17} Every critical independent set is a local maximum independent set.

(iii) \cite{4} Each critical independent set is contained in some maximum independent set.

(iv) \cite{11} There is a matching from $N(S)$ into $S$, for every critical independent set $S$.

In this paper we prove that $\ker(G) \subseteq \text{core}(G)$ and $\varepsilon(G) \geq d_c(G) \geq \alpha(G) - \mu(G)$ hold for every graph $G$.

### 2 Results

**Theorem 2.1** Let $A$ be a critical independent set of the graph $G$ and $X = A \cup N(A)$. Then the following assertions are true:

(i) $H = G[X]$ is a Kőnig-Egerváry graph;

(ii) $\alpha(G[V - X]) \leq \mu(G[V - X])$;

(iii) $\mu(G[X]) + \mu(G[V - X]) = \mu(G)$; in particular, each maximum matching of $G[X]$ can be enlarged to a maximum matching of $G$. 


in order to build a maximum matching of $G$. Consequently, we get that
\[ \alpha(H) + \mu(H) = |A \cup N(A)| = |X| = |V(H)|, \]
i.e., $H$ is a König-Egerváry graph.

(ii) According to Theorem [1.5](ii), there exists a maximum independent set $S$ such that $A \subseteq S$. Suppose that $|B| > |N(B)|$ holds for some $B \subseteq S - A$. Then, it follows that
\[ |A| - |N(A)| < (|A| - |N(A)|) + (|B| - |N(B)|) \leq |A \cup B| - |N(A \cup B)|, \]
which contradicts the hypothesis on $A$, namely, the fact that $|A| - |N(A)| = d_e(G)$. Hence $|B| \leq |N(B)|$ is true for every $B \subseteq S - A$. Consequently, by Hall’s Theorem there exists a matching from $S - A$ into $V - S - N(A)$ that implies $|S - A| \leq \mu(G[V - X])$.

It remains to show that $\alpha(G[V - X]) = |S - A|$. By way of contradiction, assume that
\[ \alpha(G[V - X]) = |D| > |S - A| \]
for some independent set $D \subseteq V - X$. Since $D \cap N[A] = \emptyset$, the set $A \cup D$ is independent, and
\[ |A \cup D| = |A| + |D| > |A| + |S - A| = \alpha(G), \]
which is impossible.

(iii) Let $M_1$ be a maximum matching of $H$ and $M_2$ be a maximum matching of $G[V - X]$. We claim that $M_1 \cup M_2$ is a maximum matching of $G$.

![Figure 2: $S \in \Omega(G)$ and $A$ is a critical independent set of $G.$](image)

The only edges that may enlarge $M_1 \cup M_2$ belong to the set $(N(A), V - S - N(A))$. The matching $M_1$ covers all the vertices of $N(A)$ in accordance with Theorem [1.3] and part (i). Therefore, to choose an edge from the set $(N(A), V - S - N(A))$ means to loose an edge from $M_1$. In other words, no matching different from $M_1 \cup M_2$ may overstep $|M_1 \cup M_2|$.

Consequently, each maximum matching of $G[X]$ can find its counterpart in $G[V - X]$ in order to build a maximum matching of $G$. ■

Theorem [2.1] allows us to give an alternative proof of the following inequality due to Lorentzen.
Corollary 2.2 [17], [19] The inequality \( d_e(G) \geq \alpha(G) - \mu(G) \) holds for every graph \( G \).

**Proof.** Let \( A \) be a critical independent set of \( G \), and \( X = A \cup N(A) \).

By Theorem 2.1(ii), we get \( \alpha(G[V - X]) - \mu(G[V - X]) \leq 0 \). Hence it follows that

\[
\alpha(G[X]) - \mu(G[X]) \geq (\alpha(G[X]) + \alpha(G[V - X])) - (\mu(G[X]) + \mu(G[V - X])).
\]

Theorem 2.1(iii) claims that \( \mu(G[X]) + \mu(G[V - X]) = \mu(G) \).

Since \( A \) is a critical independent set, there exists some \( S \in \Omega(G) \) such that \( A \subseteq S \), and \( \alpha(G[X]) = |A| \), by Theorem 1.4(ii). Hence we have

\[
\alpha(G[X]) + \alpha(G[V - X]) = |A| + |S - A| = \alpha(G).
\]

In addition, Theorem 2.1(i) and Theorem 1.3 imply that \( \mu(G[X]) = |N(A)| \).

Finally, we obtain

\[
d_e(G) = \max \{|I| - |N(I)| : I \in \text{Ind}(G)\} = |A| - |N(A)| = \\
\alpha(G[X]) - \mu(G[X]) \geq \alpha(G) - \mu(G),
\]

and this completes the proof. ■

Applying Theorem 2.1 and Theorem 1.5(iii) we get the following.

Corollary 2.3 [17] Let \( J \) be a maximum critical independent set of \( G \), and \( X = J \cup N(J) \). Then the following assertions are true:

(i) \( \alpha(G) = \alpha(G[X]) + \alpha(G[V - X]) \);
(ii) \( \alpha(G) = \alpha_c(G) + \alpha(G[V - X]) \);
(iii) \( G[X] \) is a König-Egerváry graph.

The graph \( G \) from Figure 3 has \( \ker(G) = \{a, b, c\} \). Notice that \( \ker(G) \subseteq \text{core}(G) \); \( S = \{a, b, c, v\} \) is a largest critical independent set, and neither \( S \subseteq \text{core}(G) \) nor \( \text{core}(G) \subseteq S \). In addition, \( \text{core}(G) \) is not a critical independent set of \( G \).

![Figure 3](image)

**Figure 3:** \( G \) is a non-quasi-regularizable graph with \( \text{core}(G) = \{a, b, c, u\} \).

Theorem 2.4 For a graph \( G = (V, E) \) of order \( n \), the following assertions are true:

(i) the function \( d \) is supermodular, i.e., \( d(A \cup B) + d(A \cap B) \geq d(A) + d(B) \) for every \( A, B \subseteq V(G) \);
(ii) if \( A \) and \( B \) are critical in \( G \), then \( A \cup B \) and \( A \cap B \) are critical as well;
(iii) \( \ker(G) = \cap \{B : B \text{ is a critical set of } G\} \).
Proof. (i) Let us notice that $N(A \cup B) = N(A) \cup N(B)$ and $N(A \cap B) \subseteq N(A) \cap N(B)$. Further, we obtain

$$d(A \cup B) = |A \cup B| - |N(A \cup B)| = |A \cup B| - |N(A) \cup N(B)| =$$

$$= |A| + |B| - |A \cap B| - |N(A)| - |N(B)| + |N(A) \cap N(B)| =$$

$$= (|A| - |N(A)|) + (|B| - |N(B)|) + |N(A) \cap N(B)| - |A \cap B| =$$

$$= d(A) + d(B) - (|A \cap B| - |N(A \cap B)|) + |N(A) \cap N(B)| - |N(A \cap B)| =$$

$$= d(A) + d(B) - d(A \cap B) + |N(A) \cap N(B)| - |N(A \cap B)| \geq$$

$$\geq d(A) + d(B) - d(A \cap B).$$

(ii) By part (i), we have that

$$d(A \cup B) + d(A \cap B) \geq d(A) + d(B) = 2d_c(G).$$

Consequently, we get that $d(A \cup B) = d(A \cap B) = d_c(G)$, i.e., both $A \cup B$ and $A \cap B$ are critical sets.

(iii) Let $\Gamma_{ci}$ be the family of all critical independent sets of $G$, while $\Gamma_c$ denotes the family $\{B : B$ is a critical set in $G\}$.

By part (ii), both sets

$$\ker(G) = \cap \{S \in \Gamma_{ci}\} \quad \text{and} \quad Q_c = \cap \{B \in \Gamma_c\}$$

are critical. Theorem 1.4 implies that $\Gamma_{ci} \subseteq \Gamma_c$, and therefore, $Q_c \subseteq \ker(G)$. On the other hand, $Q_c$ is independent, because by Theorem 1.4, one of the critical sets from $\Gamma_c$ is independent. Consequently, we obtain $\ker(G) \subseteq Q_c$, and this completes the proof. ■

Theorem 2.5 For a graph $G = (V, E)$ of order $n$, the following assertions are true:

(i) $V \supseteq \text{corona}(G) \supseteq S \supseteq \text{core}(G) \supseteq \ker(G)$, for every $S \in \Omega(G)$;

(ii) $n \geq \xi(G) \geq \alpha(G) \geq \xi(G) \geq \varepsilon(G) \geq d_c(G) \geq \alpha(G) - \mu(G)$;

(iii) $\xi(G) \geq \alpha(G) - \mu(G) + \varepsilon(G) - d_c(G)$.

Proof. (i) Clearly, $\text{core}(G) \subseteq S \subseteq \text{corona}(G) \subseteq V$ hold for each $S \in \Omega(G)$. The set $\ker(G)$ is independent by definition. According to Theorem 1.4(ii), $\ker(G)$ is critical. Consequently, by Theorem 1.4(iv), there exists a matching $M_L$ from $N(\ker(G))$ into $\ker(G)$. Figure 3 will accompany us all the way to the end of the proof.

Let $S \in \Omega(G)$, and $A_1 = \ker(G) \cap S$. Since $\ker(G) - A_1$ is stable and disjoint from $S$, Theorem 1.2 ensures that there is a matching $M_B$ from $\ker(G) - A_1$ into $S$, covering some subset $A_2$ of $S - A_1$. Let $S \in \Omega(G)$, and $A_1 = \ker(G) \cap S$. Since $\ker(G) - A_1$ is stable and disjoint from $S$, Theorem 1.2 ensures that there is a matching $M_B$ from $\ker(G) - A_1$ into $S$, covering some subset $A_2$ of $S - A_1$. Clearly, we have

$$|\ker(G) - A_1| = |A_2|, \quad A_1 \cap A_2 = \emptyset, \quad \text{and} \quad A_2 \subseteq N(\ker(G) - A_1) \cap S.$$

Assume that there is some $v \in (N(\ker(G) - A_1) \cap S) - A_2$. The vertex $v$ must be matched with some vertex from $\ker(G) - A_1$ by $M_L$, because $\{v\} \cup A_1 \subseteq S$. Hence $M_L$ matches the set $N(\ker(G) - A_1) \cap S$ into $\ker(G) - A_1$, which is impossible, since

$$|N(\ker(G) - A_1) \cap S| \geq |\{v\} \cup A_2| > |A_2| = |\ker(G) - A_1|.$$
Corollary 2.7

If there is some $S \in \text{Ind}(G)$ with $|S| \geq |N(S)|$, then $\xi(G) > d_c(G)$.

**Proof.** According to Theorem 2.5, $G$ is non-quasi-regularizable if and only if $\ker(G) \neq \emptyset$, i.e., $|\ker(G)| \geq 2$. The fact that $G$ has no isolated vertices implies $N(\ker(G)) \neq \emptyset$, and consequently, it follows $\varepsilon(G) = |\ker(G)| > |\ker(G)| - |N(\ker(G))| = d_c(G)$. Further, using Theorem 2.5, we get both (i) and (ii). ■

Corollary 2.6

If $d_c(G) > 0$ or, equivalently, $G$ is a non-quasi-regularizable graph, then

(i) $\eta(G) \geq \alpha(G) \geq \xi(G) \geq \varepsilon(G) > d_c(G) \geq \alpha(G) - \mu(G) \geq 1$;

(ii) $\xi(G) \geq \alpha(G) - \mu(G) + \varepsilon(G) - d_c(G)$.

**Proof.** According to Theorem 2.4, if $G$ is non-quasi-regularizable and $\ker(G) \neq \emptyset$, then $\eta(G) \geq \alpha(G) \geq \xi(G) = \varepsilon(G) > d_c(G) \geq \alpha(G) - \mu(G) \geq 1$. Further, using Theorem 2.5, we get both (i) and (ii). ■

Consequently, we get that $N(\ker(G) - A_1) \cap S = A_2$. Thus $M_L$ matches the set $N(\ker(G) - A_1) \cap S$ onto $\ker(G) - A_1$, and $N(A_1)$ into $A_1$. Clearly, we have

$$|\ker(G) - A_1| = |A_2|, A_1 \cap A_2 = \emptyset, \text{ and } A_2 \subseteq N(\ker(G) - A_1) \cap S.$$ 

Assume that there is some $v \in (N(\ker(G) - A_1) \cap S) - A_2$. The vertex $v$ must be matched with some vertex from $\ker(G) - A_1$ by $M_L$, because $\{v\} \cup A_1 \subseteq S$. Hence $M_L$ matches the set $N(\ker(G) - A_1) \cap S$ into $\ker(G) - A_1$, which is impossible, since

$$|N(\ker(G) - A_1) \cap S| \geq |\{v\} \cup A_2| > |A_2| = |\ker(G) - A_1|.$$ 

Consequently, we get that $N(\ker(G) - A_1) \cap S = A_2$. Thus $M_L$ matches the set $N(\ker(G) - A_1) \cap S$ onto $\ker(G) - A_1$, and $N(A_1)$ into $A_1$.

In conclusion, we may assert that $|\ker(G)| - |N(\ker(G))| = |A_1| - |N(A_1)|$. Hence, we infer that $\ker(G) - A_1 = \emptyset$, otherwise we have that $A_1$ is a critical independent set of $G$ with $|A_1| < |\ker(G)|$, in contradiction with the hypothesis on minimality of $\ker(G)$.

This ensures that $\ker(G) \subseteq S$ for every $S \in \Omega(G)$, which means that $\ker(G) \subseteq \text{core}(G)$.

(ii) Using part (i), Theorem 2.4(iii), and Corollary 2.2, we deduce that

$$n \geq \zeta(G) \geq \alpha(G) \geq \xi(G) \geq \varepsilon(G) = |\ker(G)| \geq |\ker(G)| - |N(\ker(G))| = d_c(G) \geq \alpha(G) - \mu(G),$$

which completes the proof.

(iii) It follows immediately from part (ii). ■

Notice that $\xi(K_{2,3}) = \varepsilon(K_{2,3}) > d_c(K_{2,3}) = 1 = \alpha(K_{2,3}) - \mu(K_{2,3})$, while the graph $G_2$ is from Figure 4 satisfies $\xi(G_2) > \varepsilon(G_2) = d(G_2) = 1$.

Figure 4: $S \in \Omega(G)$, $\ker(G)$, and $A_1 = S \cap \ker(G)$.
3 Conclusions

Writing this paper we have been motivated by the inequality

$$\xi(G) = |\text{core}(G)| > \alpha(G) - \mu(G),$$

which is true for every graph $G$ without isolated vertices, such that $\alpha(G) > \mu(G)$ [3].

What we have found is that there exists a subset of $\text{core}(G)$, which is a real obstacle to its nonemptiness. The cardinality of this subset, namely, $\varepsilon(G) = |\ker(G)|$ stands out above $\alpha(G) - \mu(G)$ on its own.

The problem of whether there are vertices in a given graph $G$ belonging to $\text{core}(G)$ is NP-hard [3]. On the other hand, it has been noticed that for some families of graphs $\text{core}(G)$ may be computed in polynomial time.

We conclude with the following question.

**Problem 3.1** Is it true that for any fixed positive integer $k$, to decide if $\varepsilon(G) > k$ is NP-complete?

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