FULLY SUPERSYMMETRIC HIERARCHIES FROM A ENERGY DEPENDENT SUPER HILL OPERATOR

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Abstract
A super Hill operator with energy dependent potentials is proposed and the associated integrable hierarchy is constructed explicitly. It is shown that in the general case, the resulted hierarchy is multi-Hamiltonian system. The Miura type transformations and modified hierarchies are also presented.

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1 Introduction

Schrödinger equation with energy dependent potential is first studied by Jaulent and Miodek\cite{7} and there in the simplest case, the associated non-linear evolution equations are solved by means of Inverse Scattering Transformation. The problem has been generalized to more general case in\cite{18} and is further shown that the resulted flows are Bi-Hamiltonian system.

The remarkable multi-Hamiltonian structures behind have been explored and Miura type maps are obtained in a series papers of Antonowicz and Fordy\cite{1} - \cite{4}. The Lie algebraic reason for constructing Miura map is provided by Marshall \cite{16} \cite{17} and this subsequently leads to some new results for the Ito’s system\cite{13}. The most recent result for these hierarchies is their relationship with the zero sets of the tau function of the KdV hierarchy\cite{15}.

The generalizations of linear problems with energy potentials are interesting and begins with the third order operator or Lax operator for Boussinesq equation\cite{5}. Unlike the Schrödinger case, one does not have arbitrary polynomial dependent expansions here and to have interesting results, one only obtains four cases(see\cite{5} for details). Similarly, Toda system is generalised this way\cite{5}.

We notice that integrable systems have super counterparts. Indeed there exist two different types of generalizations: fermionic or supersymmetric. For the celebrated KdV system, this corresponds to Kupershmidt’s super KdV\cite{8} and Manin-Radul’s super KdV\cite{14}, respectively. While both generalizations are interesting from the mathematical viewpoint, it is believed that the supersymmetric extensions are physically relevant. In\cite{8}, Kupershmidt’s spectral problem for super KdV is generalized to the context of energy dependent potentials.

The aim of the present paper is to present fully supersymmetric integrable systems resulted from a linear super operator with energy dependent potentials. We will show that like Schrödinger operator case, the resulted systems are multi-Hamiltonian nature and have multi-step modifications. Thus, the remarkable algebraic structures revealed in\cite{1}-\cite{4} are retained for our new supersymmetric systems. The simplest example in this construction includes one of $N = 2$ supersymmetric KdV system\cite{9}.

The paper is organized as follows. In section two, we propose the linear problem and construct the related isospectral flows. We also construct the matrix operators which are our candidates for Hamiltonian operators. In
section three, we proceed to construct the Miura type maps which serve as a simple way to prove some claims made in section two. Section four contains some interesting examples.

2 Linear Problem

We start with the following super linear operator

\[ L = \varepsilon D^3 + uD + \alpha, \quad (1) \]

where \( D = \partial_\vartheta + \vartheta \partial \) is the super derivation with \( \vartheta \) a Grassman odd variable and \( \partial = \frac{\partial}{\partial x} \); \( \varepsilon = \varepsilon(\lambda) \) is a bosonic parameter depending on the spectral parameter \( \lambda \); \( u = u(\lambda; \vartheta, x, t) \) is bosonic variable and \( \alpha = \alpha(\lambda; \vartheta, x, t) \) is fermionic variable.

To obtain isospectral flows associated to \( L \), we consider the linear problem

\[ L \psi_t = 0 \]

with the time evolution of wave function:

\[ \psi_t = P \psi, \quad P = b\partial + \beta D + c, \quad (2) \]

then by simple calculation, we have

\[
L_t - [P, L] = u_t D + \alpha_t - \varepsilon(2\beta - (Db))\vartheta^2 + \varepsilon(b_x + (D\beta))D^3 \\
+ (\varepsilon(Db)_x - \varepsilon\beta_x + \varepsilon(Dc) + u(Db) - 2u\beta)D^2 \\
- (bu_x + \beta(cu) - \varepsilon(D\beta)_x - \varepsilon c_x - u(D\beta))D \\
- (b\alpha_x + \beta(D\alpha) - \varepsilon(Dc)_x - u(Dc)),
\]

it is easy to see that the usual Lax equation \( L_t = [P, L] \) will not lead to any consistent equation. To have meaningful results, we introduce

\[
Q = ((Db) - 2\beta)D + b_x + (D\beta),
\]

and consider

\[
[P, L] + QL = ( - \varepsilon(Db)_x + \varepsilon\beta_x - \varepsilon(Dc))D^2 + (bu_x + \beta(Du) - \varepsilon(D\beta)_x \\
- \varepsilon c_x - u(D\beta) + (Db)(Du) - 2\beta(Du) - (Db)\alpha + 2\beta\alpha \\
+ b_x u + (D\beta)u)D + b\alpha_x + \beta(D\alpha) - \varepsilon(Dc)_x - u(Dc) \\
+ (Db)(D\alpha) - 2\beta(D\alpha) + b_x \alpha + (D\beta)\alpha,
\]
thus we have to choose \( c = -b_x + (D\beta) \) and then

\[
L_t = [P, L] + QL,
\]
gives us

\[
u_t = (bu)_x - \beta(Du) - 2\varepsilon(D\beta)_x + \varepsilon b_{xx} + (Db)(Du) - (Db)\alpha + 2\beta\alpha,
\]

\[
\alpha_t = (b\alpha)_x + D(\beta\alpha) + \varepsilon(Db)_{xx} - \varepsilon\beta_{xx} + u(Db)_x - u\beta_x + (Db)(D\alpha),
\]

which can be written neatly as

\[
u_t = JR, \quad (3)
\]

where

\[
\begin{align*}
\mathbf{u} &= \begin{pmatrix} u \\ \alpha \end{pmatrix}, \\
\mathbf{R} &= \begin{pmatrix} (Db) - \beta \\ b \end{pmatrix},
\end{align*}
\]

\[
J = \begin{pmatrix}
2\varepsilon D^3 + 2\alpha - (Du) & -\varepsilon\partial^2 - \alpha D + \partial u \\
\varepsilon\partial^2 + u\partial + D\alpha & \alpha\partial + \partial \alpha
\end{pmatrix}. \quad (4)
\]

To obtain the evolution equations, we now specify the \( \varepsilon, u \) and \( \alpha \) in the following way

\[
\varepsilon = \sum_{i=0}^{n} \varepsilon_i \lambda^i, \quad u = \sum_{i=0}^{n} u_i \lambda^i, \quad \alpha = \sum_{i=0}^{n} \alpha_i \lambda^i, \quad (5)
\]

with the above choice(5), the equation (3) is in the form

\[
\sum_{i=0}^{n} \lambda^i \mathbf{u}_{it} = \left( \sum_{i=0}^{n} J_i \lambda^i \right) \mathbf{R}, \quad (6)
\]

where \( \mathbf{u}_i = (u_i, \alpha_i)^T \) and

\[
J_i = \begin{pmatrix}
2\varepsilon_i D^3 + 2\alpha_i - (Du_i) & -\varepsilon_i\partial^2 - \alpha_i D + \partial u_i \\
\varepsilon_i\partial^2 + u_i\partial + D\alpha_i & \alpha_i\partial + \partial \alpha_i
\end{pmatrix}. \quad (7)
\]

We assume that the \( \mathbf{R} \) has following expansion with respect to the spectral parameter \( \lambda \)

\[
\mathbf{R} = \sum_{i=0}^{m} R_{m-i} \lambda^i,
\]
then the coefficients of different powers of $\lambda$ of the equation (8) give us

$$ u_{0t} = J_0 R_m, $$

$$ u_{1t} = J_0 R_{m-1} + J_1 R_m, $$

$$ \vdots $$

$$ u_{nt} = J_0 R_{m-n} + J_1 R_{m-n+1} + \cdots + J_n R_m, $$

(8)

$$ J_0 R_{i-n} + J_1 R_{i-n+1} + \cdots + J_n R_i = 0, \quad i = 0, \ldots, m-1. $$

(9)

From the above systems (8-9), we see that $R_m$ is not determined and we have two basic cases:

- $u_n = -1, \alpha_n = 0$.

  In this case, the last equation of (8) takes the same form of (9) with $i = m$, which enables us to determine $R_m$ in principal. This case is refereed as KdV case.

- $u_0 =$constant, $\alpha_0 =$ (fermionic) constant

  This case leads to $R_m = 0$ for compatibility and is refereed as Harry Dym case.

Since the second case can be studied similarly, next we will only consider the first case in detail.

The evolution equation (8) can be reformed as

$$ \begin{pmatrix} u_0 \\ \vdots \\ u_{n-1} \end{pmatrix}_{t_m} = \begin{pmatrix} 0 & J_0 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ J_0 & \cdots & J_{n-1} & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & J_0 & \cdots & \cdots & 0 \end{pmatrix} \begin{pmatrix} R_{m-n+1} \\ \vdots \\ \vdots \\ R_{m} \end{pmatrix}, $$

(10)

and the recursion relation (9) can be written as

$$ B_i R^{(k)} = B_{i-1} R^{(k+1)}, \quad i = 1, \ldots, n, $$

(11)

where

$$ B_i = \begin{pmatrix} 0 & J_0 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ J_0 & \cdots & J_{i-1} & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & 0 & \cdots & \cdots & J_0 & \cdots & \cdots & 0 \end{pmatrix}, $$

(12)
and
\[ \mathbf{R}^{(k)} = (R_{k-n+1}, \ldots, R_k)^T, \]
these operators \( B_i \) are our candidates of Hamiltonian operators. In order to obtain Hamiltonian description of the evolution system (10), we need to prove

(i) \( B_i \) are Hamiltonian operators;
(ii) \( J\mathcal{R} = 0 \) admits the formal power series solution \( \mathcal{R} = \sum_{i=0}^{\infty} R_i \lambda^{-i}; \)
(iii) \( \mathbf{R}^{(i)} \) can be written as variational derivatives of some functionals \( \mathcal{H}_i \).

With the assumption that above statements are proved, we now has
\[ \mathbf{U}_{t_m} = B_{n-k} \delta \mathcal{H}_{m+k}, \quad k = 0, \ldots, n, \tag{13} \]
where \( \mathbf{U} = (u_0, \ldots, u_{n-1})^T \) and \( \delta \) denotes the variational derivative with respect to \( \mathbf{U} \). Thus, our system is a \((n + 1)\) Hamiltonian system.

Now we prove the statement (ii-iii). To this end, we introduce \( \eta = D(\ln \psi) \), then \( L\psi = 0 \) becomes
\[ \varepsilon (\eta_x + \eta(D\eta)) + u\eta + \alpha = 0, \tag{14} \]
it is easy to see that (14) has the following solution \( \eta = \sum_{s}^{\infty} \eta^{-j} \lambda^j \) for certain \( s \). It is also ready to see that each \( \eta \) provides us a conserved quantity in principal. Next we show that the solution of (14) will supply a set solution for \( J\mathcal{R} = 0 \). For clarity, we formulate it as

**Proposition 1** For each solution \( \eta \) of the equation (14), its variational derivative, with respect to \((u, \alpha)\), provides us a solution for \( J\mathcal{R} = 0 \).

**Proof:** We introduce an additional variable \( y \) so that the equation (14) is written as
\[ u = \varepsilon((D\eta) - y), \]
\[ \alpha = \varepsilon(\eta_x - 2\eta(D\eta) + \eta y), \tag{15} \]
notice that the equation (14) serves as a map between \((u, \alpha)\) and \((y, \eta)\). Thus, we have the following formula
\[ \begin{pmatrix} \delta y \\ \delta \eta \end{pmatrix} = F^\dagger \begin{pmatrix} \delta u \\ \delta \alpha \end{pmatrix}, \tag{16} \]
where \( \delta_v = \frac{\delta}{\delta v} \) and

\[
F = \varepsilon \left( \begin{array}{cc}
-1 & D \\
\eta & -\partial - 2\eta D - 2(D\eta) + y
\end{array} \right),
\]

is the Fréchet derivative of (13) and \( \dagger \) denotes adjoint.

Acting the equation (16) on \( \eta \) and denoting \( \xi = \delta_u \eta, \ p = \delta_\alpha \eta, \) we obtain

\[
\xi - \eta p = 0,
\]

\[
\varepsilon \left[ (D\xi) + p_x + 2\eta(Dp) - 4(D\eta)p + yp \right] = 1. \tag{17}
\]

We claim that the solution \((\xi, p)\) of the system (17) provides us a solution of \( J\mathcal{R} = 0. \) To see the validity of this claim, we eliminate the variable \( \xi \) in (17) and have

\[
\varepsilon \left[ p_x + \eta(Dp) + yp - 3(D\eta)p \right] = 1. \tag{18}
\]

Differentiating above equation leads to

\[
\varepsilon \left[ (Dp)_x + (y - 2(D\eta))(Dp) + (Dy - 3\eta_x + \eta y - 3\eta(D\eta)p \right] = \eta, \tag{19}
\]

\[
\varepsilon \left[ p_{xx} + (\eta_x - 2\eta y + 5\eta(D\eta))(Dp) + (y_x - 3(D\eta_x) - \eta(Dy) + 3\eta\eta_x - y^2 + 6y(D\eta) - 9(D\eta)^2)p \right] + y - 3(D\eta) = 0, \tag{20}
\]

now using above formulae (19)-(20) and meanwhile keeping in mind the mapping (13), one can easily show that

\[
2\varepsilon(D\xi_x) + 2\alpha \xi - (Du)\xi - \varepsilon p_{xx} - \alpha(Dp) + (up)_x = 0.
\]

Similarly, we can check the

\[
\varepsilon \xi_{xx} + u\xi_x + D(\alpha \xi) + 2\alpha p_x + \alpha_x p = 0,
\]

is an identity. Last two equation is nothing but \( J\mathcal{R} = 0 \) and the proposition is proved.

Remark: Solvability is justified by supplying a set of solutions as above. So the strategy used here is different from bosonic case, where one is able to prove this fact directly (cf. [3]).
3 Miura Maps and Modifications

To construct the Miura type map for the systems presented in last section, we first consider the basic case: $u \rightarrow u - \lambda, \alpha \rightarrow \alpha$.

By the following factorization

$$L = (D + \theta_1)(D + \theta_1 + \theta_2)(D + \theta_2),$$

we have

$$u = w_x + (D\theta) + \theta(Dw),$$
$$\alpha = (Dw)_x + (D\theta)(Dw),$$

(21)

where we made redefinitions of coordinates $\theta_1 = \theta$ and $\theta_2 = Dw$ for convenience.

The Fréchet derivation of the map (21) and its adjoint are

$$m = \left( \begin{array}{cc} \partial + \theta D & D - (Dw) \\ D\partial + (D\theta)D & (Dw)D \end{array} \right),$$
$$m^\dagger = \left( \begin{array}{cc} -\partial + D\theta & D\partial - D(D\theta) \\ D + (Dw) & D(Dw) \end{array} \right),$$

and we can verify the following identity holds

$$mKm^\dagger = J,$$

where $K = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$ and $J$ is given by (4).

Modifying the map (21) with a parameter $\gamma$ so that we have

$$u = \gamma w_x + (D\theta) + \theta(Dw), \quad \alpha = \gamma(Dw)_x + (D\theta)(Dw).$$

(22)

With the preparation, we now follow the method presented in [3] and construct the Miura maps. Since the construction following closely from one presented in [3]-[4], we just present final results here. Factorizing $J = \sum_{i=0}^n \lambda_i J_i$ in the following way

$$J = (m_0, m_1, \ldots, m_n)K\Lambda_k(m_0^\dagger, m_1^\dagger, \ldots, m_n^\dagger)^T,$$

(23)
where

\[ m_i = \begin{pmatrix} \gamma_i \partial + \theta_i D & D - (Dw_i) \\ \gamma_i D \partial + (D \theta_i) D & (Dw_i) D \end{pmatrix}, \]

\[ m_i^\dagger = \begin{pmatrix} -\gamma_i \partial + D \theta_i & \gamma_i D \partial - D(D \theta_i) \\ D + (Dw_i) & D(Dw_i) \end{pmatrix}, \]

and

\[ \Lambda_k = \begin{pmatrix} 1 & \ldots & \lambda^{k-1} \\ \vdots & \ddots & \vdots \\ \lambda^{k-1} & & 0 \\ 0 & \cdots & \lambda^{k} \end{pmatrix}, \]

and comparing the coefficients of \( \lambda \) of the equation (23) we have

\[ \varepsilon_k = \sum_{i=0}^{k} \gamma_i, \quad k = 0, \ldots, r - 1, \quad (24) \]

\[ \varepsilon_k = \sum_{i=k}^{n} \gamma_i, \quad k = r, \ldots, n, \]

\[ u_k = \frac{1}{2} \sum_{i=0}^{k} W_{i,k-i}, \quad \alpha_k = \frac{1}{2} \sum_{i=0}^{k} \Omega_{i,k-i}, \quad k = 0, \ldots, r - 1, \quad (25) \]

\[ u_k = \frac{1}{2} \sum_{i=0}^{n-k} W_{k+i,n-i}, \quad \alpha_k = \frac{1}{2} \sum_{i=0}^{n-k} \Omega_{k+i,n-i}, \quad k = r, \ldots, n, \quad (26) \]

where

\[ W_{i,j} = (D \theta_i) + (D \theta_j) + \gamma_i w_{j,x} + \gamma_j w_{i,x} + \theta_i (D w_j) + \theta_j (D w_i), \]

\[ \Omega_{i,j} = (Dw_i)(D \theta_j) + (Dw_j)(D \theta_i) + \gamma_i (Dw_j)_x + \gamma_j (Dw_i)_x. \]

To have the reduction to the KdV case, we specify

\[ u_n = -1, \quad \alpha_n = 0, \]
and we only need to choose $\theta_n = -\vartheta, w_n = 0$ for consistence. Thus, the formulae (26) become

$$u_k = -1 + (D\theta_k) + \eta_n w_{k,x} + \frac{1}{2} \sum_{i=1}^{n-k-1} W_{k+i,n-i},$$
$$\alpha_k = -(Dw_k) + \eta_n (Dw_k)_x + \frac{1}{2} \sum_{i=1}^{n-k-1} \Omega_{k+i,n-i}. \tag{27}$$

Having the maps constructed, we now obtain

**Proposition 2** Solving the equations (24) for $\kappa_i$, the operators $B_k(12)$ is related to a constant coefficient Hamiltonian operator in

$$B_k = M_k \hat{B}_k(M_k)^\dagger,$$

where $M_k$ is the Fréchet derivative of (25)(27) and

$$\hat{B}_k = \begin{pmatrix} 0 & -K & \cdots & 0 \\ -K & 0 & \cdots & -K \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & K \end{pmatrix},$$

where $\hat{B}_k$ has the same block structures as $B_k(12)$.

**Proof:** Direct computation.

**Remarks:**

- The Hamiltonian nature of our operators $B_k$ is proved as a simple corollary of above proposition for the generic case. The general case can be proved as taking limits as in [3].

- The general Miura map (25)(27) can be regarded as decomposition of $n$ step elementary maps as in [3]. In this way, the remarkable picture of [3](cf. Fig 1 of [3]) reappears here.
4 Examples

In this section, we present some interesting examples. We will concentrate on the simplest cases, that is $n = 2$ and $n = 4$ case.

**Two Component Case** In this case, we take $\varepsilon_0 = 1$, $\varepsilon_1 = 0$ and $u(x, t; \lambda) = u(x, t) - \lambda$ and $\alpha(x, t; \lambda) = \alpha(x, t)$. Then, we seek the formal solution $\eta = \sum_{i=1}^{\infty} \eta_i \lambda^{-i}$ of the equation

$$
\eta_x + \eta(D\eta) + u\eta - \lambda \eta + \alpha = 0,
$$

the first a few solutions, denoted as, are

$$
\mathcal{H}_1 = \alpha, \quad \mathcal{H}_2 = u\alpha, \\
\mathcal{H}_3 = u\alpha_x + \alpha(D\alpha + u^2),
$$

which serve as the first Hamiltonians. The corresponding first non trivial system $(t_2$ flow)$ is

$$
u_t = -u_{xx} + 2uu_x + 2(D\alpha)_x, \\
\alpha_t = \alpha_{xx} + 2(u\alpha)_x, \\
(28)
$$

we note that the above system reduces to Burgers equation when $\alpha = 0$, so it can be regarded as a supersymmetric Burgers equation. We also remark that the next flow $(t_3$-flow)$ can be transformed to one of $N = 2$ supersymmetric KdV equation by an invertable change of coordinates

The Miura map in the present case is the basic one $(21)$ and the modified system for $(28)$ is

$$
\begin{pmatrix}
    w \\
    \theta
\end{pmatrix}_t = 
\begin{pmatrix}
    0 & 1 \\
    -1 & 0
\end{pmatrix}
\begin{pmatrix}
    \delta_w \\
    \delta_\theta
\end{pmatrix}
\hat{\mathcal{H}}_2,
$$

where $\hat{\mathcal{H}}_2 = w_x(D\theta)(Dw) + (D\theta)^2(Dw) + (Dw)(Dw)_x\theta + (D\theta)(Dw)_x$.

To have new example, we choose $\varepsilon$ as before and $u = u_0 + \lambda u_1$ and $\alpha = \lambda \alpha_1$ with $u_0$ is a constant. The Hamiltonian operators are

$$
J_0 = \begin{pmatrix}
    2D\partial \\
    \partial^2 + u_0\partial
\end{pmatrix}, \\
J_1 = -\begin{pmatrix}
    2\alpha_1 - (Du_1) \\
    u_1\partial + D\alpha_1 \\
    -\alpha_1 D + \partial u_1
\end{pmatrix}.
$$
In this case we seek the formal solution of the form \( \eta = \sum_{i=0}^{\infty} \eta_i \lambda^{-i} \) of the equation (14) and the first two are listed as follows

\[
\mathcal{H}_0 = -\frac{\alpha_1}{u_1},
\]

\[
\mathcal{H}_1 = u_1^{-1} \left( \left( \frac{\alpha_1}{u_1} \right)_x - \left( \frac{\alpha_1}{u_1} \right) D \left( \frac{\alpha_1}{u_1} \right) + \frac{u_0 \alpha_1}{u_1} \right).
\]

With \( u_0 = c(\text{constant}) \), we have

\[
u_{1,t} = 2D \left( \frac{\alpha_1}{u_1^2} \right)_x + \frac{1}{u_1} \left( \frac{1}{u_1} \right)_{xx} - c \left( \frac{1}{u_1} \right)_x,
\]

\[
\alpha_{1,t} = \left( \frac{\alpha_1}{u_1^2} \right)_{xx} + \frac{1}{u_1} \left( \frac{\alpha_1}{u_1^2} \right)_x,
\]

interestingly, above system admits the reduction \( \alpha_1 = 0 \), which reads as

\[
v_t = -v^2 (v_{xx} - cv_x), \quad (29)
\]

with \( v = u_1^{-1} \). The system (29) passes Painleve test as shown in [3]. We also note that when \( c = 0 \), the system (29) is the one discussed in [13] and is in the list of evolution equations classified in [20] by symmetry approach.

**Four Component Case**  Now we present the last example - system with four component case: \( \varepsilon = 1, \ u = u_0 + \lambda u_1 - \lambda^2 \) and \( \alpha = \alpha_0 + \lambda \alpha_1 \). Similarly, we have the Hamiltonians

\[
\mathcal{H}_1 = \alpha_1, \quad \mathcal{H}_2 = \alpha_0 + u_1 \alpha_1, \quad \mathcal{H}_3 = u_0 \alpha_1 + u_1^2 \alpha_1 + u_1 \alpha_0,
\]

and the system are tri-Hamiltonian with

\[
B_0 = \begin{pmatrix} J_1 & -J_2 \\ -J_2 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} J_0 & 0 \\ 0 & J_2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & J_0 \\ J_0 & J_1 \end{pmatrix},
\]

where

\[
J_0 = \begin{pmatrix} 2D \partial + 2\alpha_0 - (Du_0) & -\partial^2 - \alpha_0 D + \partial u_0 \\ \partial^2 + u_0 \partial + D\alpha_0 & \alpha_0 \partial + \partial \alpha_0 \end{pmatrix},
\]

\[
J_1 = \begin{pmatrix} 2\alpha_1 - (Du_1) & \alpha_1 D + \partial u_1 \\ u_1 \partial + D\alpha_1 & \alpha_1 \partial + \partial \alpha_1 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & -\partial \\ -\partial & 0 \end{pmatrix},
\]

12
the first interesting flow is

\[ u_{0,t} = 2(D\alpha_1)_x + 2\alpha_0\alpha_1 - (Du_0)\alpha_1 - u_{1,xx} - \alpha_0(Du_1) + (u_0u_1)_x, \]
\[ \alpha_{0,t} = \alpha_{1,xx} + u_0\alpha_{1,x} + D(\alpha_0\alpha_1) + 2\alpha_0u_{1,x} + \alpha_{0,x}u_1, \]
\[ u_{1,t} = u_{0,x} + 2u_1u_{1,xx}, \]
\[ \alpha_{1,t} = \alpha_{0,x} + 2(u_1\alpha_1)_x. \]

The Miura map here reads as

\[ u_0 = w_{0,x} + (D\theta_0) + \theta_0(Dw_0), \]
\[ \alpha_0 = (Dw_0)_x + (D\theta_0)(Dw_0), \]
\[ u_1 = (D\theta_0) + (D\theta_1) - w_{0,x} + w_{1,x} + \theta_0(Dw_1) + \theta_1(Dw_0), \]
\[ \alpha_1 = (Dw_0)(D\theta_1) + (Dw_1)(D\theta_0) - (Dw_0)_x + (Dw_1)_x, \]

above Miura map (30) can be decomposed as follows

\[ u_0 = v_{0,x} + (D\mu_0) + \mu_0(Dv_0), \]
\[ \alpha_0 = (Dv_0)_x + (D\mu_0)(Dv_0), \]
\[ u_1 = v_1, \]
\[ \alpha_1 = \mu_1, \]

and

\[ v_0 = w_0, \]
\[ \mu_0 = \theta_0 \]
\[ v_1 = (D\theta_0) + (D\theta_1) - w_{0,x} + w_{1,x} + \theta_0(Dw_1) + \theta_1(Dw_0), \]
\[ \mu_1 = (Dw_0)(D\theta_1) + (Dw_1)(D\theta_0) - (Dw_0)_x + (Dw_1)_x. \]

Thus, we have two step modifications here. The modified systems under all these Miura maps can be easily calculated and we will not present them here.

Remark: The Miura map (30) is resulted from the general construction of the section 3. It is possible to rederive it by linearization of the basic one. Indeed, linearizing the basic map (21), we have

\[ u_0 = \hat{w}_{0,x} + (D\hat{\theta}_0) + \hat{\theta}_0(D\hat{w}_0), \]
\[ \alpha_0 = (D\hat{w}_0)_x + (D\hat{\theta}_0)(D\hat{w}_0), \]
\[ u_1 = \hat{w}_{1,x} + (D\hat{\theta}_1) + \hat{\theta}_0(D\hat{w}_1) + \hat{\theta}_1(D\hat{w}_0), \]
\[ \alpha_1 = (D\hat{w}_1)_x + (D\hat{\theta}_1)(D\hat{w}_0) + (D\hat{\theta}_0)(D\hat{w}_1), \]
the above map is equivalent to (30) by a simple transformation, namely,
\[
\hat{w}_0 = w_0, \quad \hat{\theta}_0 = \theta_0, \quad \hat{w}_1 = w_1 - w_0, \quad \hat{\theta}_1 = \theta_0 + \theta_1.
\]

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