Saturation of nuclear matter and radii of unstable nuclei

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Abstract

We examine relations among the parameters characterizing the phenomenological equation of state (EOS) of nearly symmetric, uniform nuclear matter near the saturation density by comparing macroscopic calculations of radii and masses of stable nuclei with the experimental data. The EOS parameters of interest here are the symmetry energy \( S_0 \), the symmetry energy density-derivative coefficient \( L \) and the incompressibility \( K_0 \) at the normal nuclear density. We find a constraint on the relation between \( K_0 \) and \( L \) from the empirically allowed values of the slope of the saturation line (the line joining the saturation points of nuclear matter at finite neutron excess), together with a strong correlation between \( S_0 \) and \( L \). In the light of the uncertainties in the values of \( K_0 \) and \( L \), we macroscopically calculate radii of unstable nuclei as expected to be produced in future facilities. We find that the matter radii depend strongly on \( L \) while being almost independent of \( K_0 \), a feature that will help to determine the \( L \) value via systematic measurements of nuclear size.

Key words: Dense matter, Saturation, Unstable nuclei

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1 Introduction

Saturation of density and binding energy, one of the fundamental properties of atomic nuclei, underlies a liquid-drop approach described by a Weizsäcker-Bethe mass formula [1]

\[-E_B = E_{\text{vol}} + E_{\text{sym}} + E_{\text{surf}} + E_{\text{Coul}},\]  

(1)
where $E_B$ is the nuclear binding energy, $E_{\text{vol}}$ is the volume energy, $E_{\text{sym}}$ is the symmetry energy, $E_{\text{surf}}$ is the surface energy, and $E_{\text{Coul}}$ is the Coulomb energy. The sum $E_{\text{vol}} + E_{\text{sym}}$ corresponds to the saturation energy of uniform nuclear matter. Since matter in nuclei is a strongly interacting system, it remains a challenging theoretical problem to understand the nuclear matter equation of state (EOS) through microscopic calculations that utilize a model of the nuclear force duly incorporating low-energy two-nucleon scattering data and properties of light nuclei [2]. Furthermore, it is not straightforward to empirically clarify the EOS, although constraints on the EOS from nuclear masses and radii (e.g., Refs. [3–5]), observables in heavy-ion collision experiments performed at intermediate and relativistic energies (e.g., Refs. [6–9]), the isoscalar giant monopole resonance in nuclei (e.g., Ref. [10]) and even X-ray observations of isolated neutron stars [11] are available. In this work we will consider such constraints from nuclear masses and radii.

The EOS of bulk nuclear matter is a function of nucleon density $n$ and proton fraction $x$, which are related to the neutron and proton number densities $n_n$ and $n_p$ as $n_n = n(1-x)$ and $n_p = nx$. We may generally express the energy per nucleon near the saturation point of symmetric nuclear matter as [12]

$$w = w_0 + \frac{K_0}{18n_0^2}(n - n_0)^2 + \left[ S_0 + \frac{L}{3n_0}(n - n_0) \right] \alpha^2. \tag{2}$$

Here $w_0$, $n_0$ and $K_0$ are the saturation energy, the saturation density and the incompressibility of symmetric nuclear matter, $S_0$ is the symmetry energy $S(n)$ at $n = n_0$, $L = 3n_0(dS/dn)_{n=n_0}$ is the density symmetry coefficient, and $\alpha = 1 - 2x$ is the neutron excess. As the neutron excess increases from zero, the saturation point moves in the density versus energy plane. This movement is determined mainly by the parameters $L$ and $S_0$ associated with the density-dependent symmetry energy $S(n)$ [3]. Up to second order in $\alpha$, the saturation energy $w_s$ and density $n_s$ are given by

$$w_s = w_0 + S_0 \alpha^2 \tag{3}$$

and

$$n_s = n_0 - \frac{3n_0L}{K_0} \alpha^2. \tag{4}$$

The slope, $y$, of the saturation line near $\alpha = 0$ ($x = 1/2$) is thus expressed as

$$y = -\frac{K_0S_0}{3n_0L}. \tag{5}$$
Derivation of $L$ and $S_0$ from nuclear observables is generally obscured by the interfacial and electrostatic properties. Among the observables, the masses and root-mean-square radii of nuclei, which are controlled mainly by the bulk properties, are expected to be good tracers of $L$ and $S_0$. This expectation was stressed by an earlier investigation [3] based on two macroscopic nuclear models. Such macroscopic models are reliable in a range of neutron excess, $\alpha \lesssim 0.3$, and mass number, $A \gtrsim 50$. In this range the neutron separation energy that can be evaluated from a Weizsäcker-Bethe mass formula is greater than 2 MeV, allowing us to preclude the possibility of neutron halo formation expected at large neutron excess (or small separation energy). In constraining the EOS from masses and radii of nuclei of neutron excess $\alpha \lesssim 0.3$ and mass number $A \gtrsim 50$ within the framework of the macroscopic models, systematic study allowing for uncertainties in the EOS parameters is indispensable.

In this paper we thus explore a systematic way of extracting $L$ and $S_0$ from empirical masses and radii of nuclei, together with the parameters, $n_0$, $w_0$ and $K_0$, characterizing the saturation of symmetric nuclear matter. We first set an expression for the energy of uniform nuclear matter, which reduces to the phenomenological form (2) in the limits of $n \to n_0$ and $\alpha \to 0$ ($x \to 1/2$). Using this energy expression within a simplified version of the extended Thomas-Fermi approximation, which permits us to determine the macroscopic features of the nuclear ground state, we calculate charges, charge radii and masses of $\beta$-stable nuclei. Comparing these calculations with empirical values allows us to derive the optimal parameter set for various values of the slope $y$ and the incompressibility $K_0$. We thus find a strong correlation between $L$ and $S_0$. The next step is to calculate root-mean-square charge and matter radii of more neutron-rich nuclei that are expected to be produced in future radioactive ion beam facilities. The results suggest that the density symmetry coefficient $L$ may be constrained from possible systematic data on the matter radii in a way nearly independent of the poorly known $K_0$, while the slope $y$ being deducible as a function of $K_0$. We finally discuss an empirically allowed region in the space of the parameters characterizing the EOS (2).

In Section 2 we construct a macroscopic model of nuclei. Optimization associated with fitting to empirical data for nuclei on the smoothed $\beta$-stability line is illustrated in Section 3. In Section 4 we calculate matter and charge radii of unstable nuclei. Our conclusions are presented in Section 5. Numerical tables are given in the Appendix.
2 Macroscopic description of nuclei

In constructing a macroscopic nuclear model, we begin with the expression for the bulk energy per nucleon [13],

\[ w = \frac{3\hbar^2(3\pi^2)^{2/3}}{10m_n n} (n_n^{5/3} + n_p^{5/3}) + (1 - \alpha^2)v_s(n)/n + \alpha^2 v_n(n)/n, \]  

(6)

where

\[ v_s = a_1 n^2 + \frac{a_2 n^3}{1 + a_3 n} \]  

(7)

and

\[ v_n = b_1 n^2 + \frac{b_2 n^3}{1 + b_3 n} \]  

(8)

are the potential energy densities for symmetric nuclear matter and pure neutron matter, and \( m_n \) is the neutron mass. (Replacement of the proton mass \( m_p \) by \( m_n \) in the proton kinetic energy makes only a negligible difference.) For the later purpose of roughly describing the nucleon distribution in a nucleus, we incorporate into the potential energy densities (7) and (8) a low density behaviour \( \propto n^2 \) as expected from a contact two-nucleon interaction. Note, however, that we will focus on the EOS of nearly symmetric nuclear matter near the saturation density. We will thus determine the parameters included in Eqs. (7) and (8) in such a way that they reproduce data on radii and masses of stable nuclei.

In the limit of \( n \to n_0 \) and \( \alpha \to 0 \) (\( x \to 1/2 \)), expression (6) reduces to the usual form (2) according to

\[ S_0 = \frac{1}{6} \left( \frac{3\pi^2}{2} \right)^{2/3} \frac{\hbar^2}{m_n n_0^{2/3}} + (b_1 - a_1) n_0 + \left( \frac{b_2}{1 + b_3 n_0} - \frac{a_2}{1 + a_3 n_0} \right) n_0^2, \]  

(9)

\[ \frac{1}{3} n_0 L = \frac{1}{9} \left( \frac{3\pi^2}{2} \right)^{2/3} \frac{\hbar^2}{m_n} n_0^{5/3} + (b_1 - a_1) n_0^2 + 2 \left( \frac{b_2}{1 + b_3 n_0} - \frac{a_2}{1 + a_3 n_0} \right) n_0^3 - \left[ \frac{b_2 b_3}{(1 + b_3 n_0)^2} - \frac{a_2 a_3}{(1 + a_3 n_0)^2} \right] n_0^4, \]  

(10)

\[ w_0 = \frac{3}{10} \left( \frac{3\pi^2}{2} \right)^{2/3} \frac{\hbar^2}{m_n} n_0^{2/3} + a_1 n_0 + \frac{a_2 n_0^2}{1 + a_3 n_0}, \]  

(11)
\[ K_0 = -\frac{3}{5} \left( \frac{3\pi^2}{2} \right)^{2/3} \frac{\hbar^2}{m_n} n_0^{2/3} + \frac{18a_2n_0^2}{(1 + a_3n_0)^3}, \] (12)

\[ 0 = \frac{1}{5} \left( \frac{3\pi^2}{2} \right)^{2/3} \frac{\hbar^2}{m_n} n_0^{-1/3} + a_1 + \frac{2a_2n_0}{1 + a_3n_0} - \frac{a_2a_3n_0^2}{(1 + a_3n_0)^2}. \] (13)

Equation (13) comes from the fact that the density derivative of the energy per nucleon at \( n = n_0 \) and \( \alpha = 0 \) (\( x = 1/2 \)) vanishes due to the saturation. Note that the five relations (9)–(13) are not sufficient to determine the six parameters \( a_1, \ldots, b_3 \), and that the parameter \( b_3 \), which controls the EOS of matter at large neutron excess and high density, has little effect on the saturation properties of nearly symmetric nuclear matter. We will thus set \( b_3 \) as a typical value 1.58632 fm\(^3\), which was obtained by one of the authors [13], and determine the other parameters from the empirical radii and masses of stable nuclei.

We now proceed to describe a spherical nucleus of proton number \( Z \) and mass number \( A \) within the framework of a simplified version of the extended Thomas-Fermi theory [13]. We first write the total energy of a nucleus as a function of the density distributions \( n_n(r) \) and \( n_p(r) \) according to

\[ E = E_b + E_g + E_C + Nm_n + Zm_p, \] (14)

where

\[ E_b = \int d^3r n(r) w \left( n_n(r), n_p(r) \right) \] (15)

is the bulk energy,

\[ E_g = F_0 \int d^3r |\nabla n(r)|^2 \] (16)

is the gradient energy with an adjustable constant \( F_0 \),

\[ E_C = \frac{e^2}{2} \int d^3r \int d^3r' \frac{n_p(r)n_p(r')}{|r - r'|} \] (17)

is the Coulomb energy, and \( N = A - Z \) is the neutron number. Here we ignore shell and pairing effects. We also neglect the contribution to \( E_g \) from the gradient of the proton fraction \( x \) [see Eq. (16)]; this contribution makes only a little difference even in the description of extremely neutron-rich nuclei, as clarified in the context of neutron star matter [13].
The energy (14), once optimized, can be mapped onto a Weizsäcker-Bethe mass formula of the form (1) via

\[ E_{\text{surf}} = E_g + (E_b - E_{\text{vol}}) + (E_C - E_{\text{Coul}}), \]  

(18)

where \( E_b - E_{\text{vol}} \) denotes the inhomogeneity contribution to the bulk energy \( E_b \), and \( E_C - E_{\text{Coul}} \) (\( E_{\text{Coul}} = 3Z^2e^2/5R \) with the liquid-drop radius \( R \)) denotes that to the Coulomb energy \( E_C \). These inhomogeneity contributions arise from the fact that matter in a nucleus is compressible. We remark that in equilibrium with respect to nuclear size, \( E_g = E_C \) holds [13]. Combining this relation with the well-known equilibrium condition for the liquid-drop size, \( E_{\text{surf}} = 2E_{\text{Coul}} \), and with \( E_{\text{Coul}} \simeq E_C \), we find a simple relation, \( E_g \simeq E_{\text{surf}}/2 \), for an equilibrium nuclide.

For the present purpose of examining the macroscopic properties of nuclei such as masses and radii, it is sufficient to characterize the neutron and proton distributions for each nucleus by the central densities, radii and surface diffuseness different between neutrons and protons, as in Ref. [13]. We thus assume the nucleon distributions \( n_i(r) \) (\( i = n, p \)), where \( r \) is the distance from the center of the nucleus, as

\[ n_i(r) = \begin{cases} 
  n_i^{\text{in}} \left[ 1 - \left( \frac{r}{R_i} \right)^{t_i} \right]^3, & r < R_i, \\
  0, & r \geq R_i.
\end{cases} \]

(19)

Here \( R_i \) roughly represents the nucleon radius, \( t_i \) the relative surface diffuseness, and \( n_i^{\text{in}} \) the central number density. The proton distribution of the form (19) can fairly well reproduce the experimental data for stable nuclei such as \(^{90}\text{Zr}\) and \(^{208}\text{Pb}\), as we shall see in the next section.

3 Optimization

For fixed mass number \( A \), we then optimize the energy (14) with respect to the parameters \( R_i, t_i \) and \( n_i^{\text{in}} \). (In a general Thomas-Fermi approach, such optimization is carried out without assuming a particular form of the distributions.) The resultant optimal values of charge number, nuclear mass and root-mean-square charge radius

\[ R_c = \left[ Z^{-1} \int d^3r r^2 \rho_c(r) \right]^{1/2}, \]  

(20)
Fig. 1. Various optimal relations among the parameters $S_0$, $n_0$, $w_0$, $L$ and $K_0$ characterizing the EOS of nearly symmetric nuclear matter. In addition to the present results (crosses), the Skyrme-Hartree-Fock predictions [dots except for SIII (square)] and the TM1 prediction (triangle) are plotted. In (c), the thin lines are lines of constant $y$.

where

$$\rho_c(r) = (\pi^{1/2}a_p)^{-3} \int d^3r' \exp \left(-|r-r'|^2/a_p^2\right) n_p(r')$$

(21)

with $a_p = 0.65$ fm is the charge distribution folded with the proton form factor [14], are functions of $a_1-b_3$ and $F_0$. These optimal values are in turn compared with the empirical values for nuclei on the smoothed $\beta$-stability line ranging $25 \leq A \leq 245$ (see Table A.1 in Ref. [13], which is based on Refs. [15,16]). For fixed slope $y$ and incompressibility $K_0$, such a comparison can be made by a usual least squares fitting, which gives rise to an optimal set of the parameters $a_1-b_3$ and $F_0$. Here, we set $y$ and $K_0$ as $-1800$ MeV fm$^3 \leq y \leq -200$ MeV fm$^3$ and $180$ MeV $\leq K_0 \leq 360$ MeV; the numerical results for $n_0$, $w_0$, $S_0$, $L$ and $F_0$ are tabulated in the Appendix. When the slope is very gentle ($0 > y \gtrsim -200$
Fig. 2. The energy per nucleon as a function of nucleon density. The dotted lines denote the saturation lines.

MeV fm$^3$), the optimal set is unavailable for large $K_0$; when the slope is very steep, the optimal parameters converge on the values close to those obtained for $y = -1800$ MeV fm$^3$. Nuclear masses that can be calculated from the optimal parameter sets agree well with the experimental data of 1962 nuclides ($A \geq 2$) [17]; for all the combinations of $y$ and $K_0$, the root-mean-square deviations of the masses are $\sim 3$–$5$ MeV, which are roughly as large as those obtained from a Weizsäcker-Bethe type formula. We likewise evaluated the root-mean-square charge radii of various stable nuclei; the root-mean-square deviations from the experimental data of 92 nuclides ($A \geq 50$) [16] are about 0.06 fm.

The optimal results for $S_0$, $L$, $n_0$ and $w_0$ are plotted in Fig. 1. From Fig. 1a, we obtain a relation nearly independent of $K_0$:

$$S_0 \approx B + CL,$$  \hspace{1cm} (22)
where $B \approx 28$ MeV and $C \approx 0.075$. We find from Figs. 1b and 1d that the saturation energy and density of symmetric nuclear matter always take on a value of $-16.0 \pm 0.5$ MeV and $0.155 \pm 0.015$ fm$^{-3}$. Several data in Fig. 1d having $n_0$ smaller than the standard values correspond to the case of $y = -200$ MeV fm$^3$, where the fitting is no longer effective. In Fig. 1c, we see a band on which the optimal values of $L$ and $K_0$ are scattered; in this band $L$ increases with increasing $y$ for fixed $K_0$. For comparison we also plot the predictions from various Skyrme-Hartree-Fock schemes (references in Ref. [4]; Refs. [18–20]) and a relativistic mean field model (TM1 in Ref. [21]); the values for $n_0$, $w_0$, $K_0$, $S_0$, $L$ and $y$ are tabulated in Table 1. These predictions are distributed over the band, among which only two (TM1 and SIII) were considered in the previous analysis [3]. We remark in passing that the optimal values of $F_0$ are confined to $66 \pm 6$ MeV fm$^5$, consistent with the result of Ref. [13].

In Fig. 2 we display the EOS (6) for various sets of $y$ and $K_0$. Whereas $y$ affects the slope of the saturation line, $K_0$ controls $n_0$ as well as the curvature of the line of constant $\alpha$. The neutron and proton distributions in $^{208}$Pb and $^{90}$Zr modelled via Eq. (19) are plotted in Fig. 3. The question we consider in the next section is how such differences in the saturation properties as shown in Fig. 2 affect matter and charge radii of unstable nuclei that can be evaluated from the distributions of the form illustrated in Fig. 3.

### 4 Matter and charge radii

For neutron-rich nuclides we now obtain the root-mean-square charge radii $R_c$ and matter radii $R_m$, defined as

$$ R_m = \left[A^{-1} \int d^3r r^2 \rho_m(r)\right]^{1/2}, \quad (23) $$

where

$$ \rho_m(r) = \left(\pi^{1/2}a_p\right)^{-3} \int d^3r' \exp\left(-|r - r'|^2/a_p^2\right) n(r') \quad (24) $$

is the matter distribution folded with the proton charge form factor equally for neutrons and protons. We evaluated the radii $R_c$ and $R_m$ of Ni and Sn isotopes for combinations of $y = -220, -350, -1800$ MeV fm$^3$ and $K_0 = 180, 230, 360$ MeV. The results are shown at neutron excess of up to $\alpha \sim 0.4$ in Fig. 4. We find from the upper panels of Fig. 4 (see also the upper panels of Fig. 5 for further clarity) that at $\alpha \sim 0.3$ ($x \sim 0.35$) a difference of order 0.05–0.1 fm occurs in the matter radii due to variation in the slope $y$. This is because as the slope becomes gentler, the saturation density difference $n_0 - n_s$ becomes
Fig. 3. The neutron and proton density distributions in $^{208}$Pb and $^{90}$Zr. Experimental data on the charge distributions (crosses) are taken from Ref. [16].

larger [see Eq. (4)]. We also find that the charge radii depend only weakly on $y$, a feature consistent with the prediction made in Ref. [3]. We can thus expect that forthcoming empirical data on matter radii of unstable neutron-rich nuclei with accuracy down to order $\pm 0.01$ fm will at least answer the question of whether the slope $y$ is steep or gentle.

Several remarks, however, are needed here. First, it is to be noted that the experimental matter radii that can be derived from measurements of interaction cross sections [22] and elastic scattering of protons and alpha particles [23] depend strongly on treatment of the optical potential, in contrast to the charge radii that can be determined from elastic electron scattering [16,24], muonic X-ray experiments [16,24] and isotope-shift measurements [25,24]. This dependence contributes to intrinsic uncertainties in the derived matter radii, which are typically of order or greater than $\pm 0.05$ fm [23,26]. Second, we recall that the present calculations of the radii $R_m$ and $R_c$ ignore the tails of the nucleon distributions arising from quantum-mechanical effects; the absence
of such tails tends to reduce the radii. Third, there are uncertainties in the calculated radii due to the absence of shell and pairing effects in the present macroscopic models. These models, which are fitted to the charge radii of nuclei on the smoothed $\beta$ stability line, provide the proton-closed-shell nuclei, Ni and Sn, with larger charge radii than the empirical values, as shown in Fig. 4. Fourth, we can see from the lower panels of Fig. 4 that with $K_0$ increased and $y$ fixed, the matter radii $R_m$ increase, a feature that prevents a clear derivation of $y$ from $R_m$.

Such $K_0$ dependence of the matter radii $R_m$ is due mainly to the $K_0$ dependence of the saturation density, $n_s$, given by Eq. (4). For fixed $y$, the $K_0$ dependence of $n_s$ is dominated by $n_0$, which increases with decreasing $K_0$ as in Fig. 1d. Generally, this increase in $n_0$ tends to increase the inner nucleon densities (see the lower panels of Fig. 3) and hence reduce the radii $R_m$. Note,
Fig. 5. Differences of the root-mean-square matter radii of Ni and Sn isotopes, \( \Delta R_m \), from those calculated for \( y = -350 \) MeV fm\(^3\) and \( K_0 = 230 \) MeV (upper panels) and for \( L = 50 \) MeV and \( K_0 = 230 \) MeV (lower panels). The thin curves in the lower panels are from formulas (26) and (27).

However, that there is an opposite effect of \( K_0 \) on \( R_m \). This effect comes from the fact that a significant part of the nucleons are present in the deepest region of the nuclear surface where \( r^2 n_n(r) \) and \( r^2 n_p(r) \) are peaked. In this region, the nucleon densities begin to drop in such a way that with decreasing \( K_0 \), the surface diffuses further away. This diffuseness, which can also be seen from the lower panels of Fig. 3, is consistent with the results from microscopic nuclear models [27].

A cancellation between those counteracting effects is favoured for the purpose of deriving information about the saturation properties in a way independent of \( K_0 \). It turns out that better cancellation can be achieved if we calculate, as in the lower panels of Fig. 6, the matter radii of Ni and Sn isotopes for the EOS parameters optimized under fixed \( L \) rather than \( y \). The charge radii likewise calculated are also nearly independent of \( K_0 \). On the
other hand, under fixed $K_0$, the dependence of the matter and charge radii on $L$, as depicted in the upper panels of Fig. 6, is the same as that on $y$. These results from Fig. 6 may open an opportunity to empirically determine the density symmetry coefficient $L$ from the isotopic dependence of matter radii.

For the purpose of parametrizing the matter and charge radii as functions of $A$, $\alpha$ and $L$, it is useful to first obtain fitting formulas for the root-mean-square neutron and proton radii, defined as

$$R_n = \left[N^{-1} \int d^3r r^2 n_n(r)\right]^{1/2}, \quad R_p = \left[Z^{-1} \int d^3r r^2 n_p(r)\right]^{1/2}.$$  \hspace{1cm} (25)

To be fitted to are the radii $R_n$ and $R_p$ of Ni, Sn and Pb isotopes ranging $0 \leq \alpha \leq 0.3 \ (0.35 \leq x \leq 0.5)$ as calculated from the present macroscopic model for the EOS parameter sets tabulated in Table A.1. In Fig. 7 the results for $^{56}$Ni, $^{80}$Ni, $^{116}$Sn and $^{142}$Ni are plotted as functions of $L$. We observe from
Fig. 7. The root-mean-square neutron (dots) and proton (crosses) radii of $^{58}$Ni, $^{80}$Ni, $^{116}$Sn and $^{142}$Sn for the EOS parameter sets tabulated in Table A.1. The solid and dotted lines are from formulas (27) and (26), respectively.

This figure shows that no uncertainties in $R_p$ more than $\pm 0.03$ fm arise from the various values of $K_0$ and $L$, while the $K_0$ dependence of $R_n$ can be ignored as compared with the $L$ dependence, which is stronger at larger neutron excess. These features, consistent with the dependence of $R_m$ and $R_c$ on $L$ and $K_0$ as mentioned above, lead us to the formulas,

$$R_p = c_1 A^{1/3} + c_2 + c_3 (\alpha - \alpha_0)^2,$$

(26)

with $c_1 = 0.914961$ fm, $c_2 = -0.102372$ fm, $c_3 = 0.388905$ fm and $\alpha_0 = 0.879704$, and

$$R_n = c_4 A^{1/3} (1 + c_5 L \alpha^2 + c_6 L^2 \alpha^4) + c_7 + c_8 \alpha,$$

(27)

with $c_4 = 0.880489$ fm, $c_5 = 0.00635080$ MeV$^{-1}$, $c_6 = -0.000172275$ MeV$^{-2}$, $c_7 = 0.301616$ fm and $c_8 = 0.193326$ fm. Expressions (26) and (27) reproduce
the original values with the root-mean-square deviations of 0.010 fm and 0.013 fm, respectively, and roughly obey a usual $A^{1/3}$ law for fixed $L$ and $\alpha$. Finally, the parametrization of the matter and charge radii can be constructed from Eqs. (26) and (27) via the relations $R_{m}^{2} = (Z/A)R_{p}^{2} + (N/A)R_{n}^{2} + 3a_{p}^{2}/2$ and $R_{c}^{2} = R_{p}^{2} + 3a_{p}^{2}/2$ that can be derived from Eqs. (20), (23) and (25).

The lower panels of Fig. 5 depict the matter radii of Ni and Sn isotopes calculated for various combinations of $L$ and $K_{0}$ relative to those calculated for $L = 50$ MeV and $K_{0} = 230$ MeV. As can be seen from these panels, $K_{0}$-induced uncertainties in the matter radii at constant $L$ are limited to the order of $\pm 0.015$ fm in the neutron-rich side, small enough for the $L$ dependence of the matter radii to be seen clearly. This may play a role in determining $L$ from measurements of the matter radii. Note, however, that possible data on matter radii for isotopes of a specific element would be insufficient to determine the density symmetry coefficient $L$, as inferred from the differences between the empirical and calculated charge radii shown for Ni and Sn in Fig. 6. In view of such differences as well as intrinsic uncertainties in the matter radii that can be deduced from proton and alpha-particle elastic scattering data, systematics of the isotopic dependence of matter radii would be required.

It is interesting to note that the difference between the neutron and proton radii increases with increasing $L$ in a way dependent on neutron excess but almost independent of $K_{0}$ (see Fig. 7). This leads us to study the isotopic dependence of neutron skin thickness in order to derive the density symmetry coefficient $L$. The previous investigation [3] has already suggested the possibility that the EOS of asymmetric nuclear matter will be probed by future detection of neutron skin thickness in unstable neutron-enriched nuclei. However, the neutron skin structure depends strongly on the EOS of nuclear matter at large neutron excess and low density [4], which is empirically hard to determine. Moreover, the fact [28] that the neutron skin thickness is thermodynamically relevant to the dependence of the surface tension on neutron excess complicates an extraction of the bulk properties of asymmetric nuclear matter; the connection between the surface and bulk properties remains to be clarified. We will elsewhere reinvestigate how the neutron skin thickness is phenomenologically related to the EOS of nearly symmetric nuclear matter [29].

5 Conclusion

In this paper we have derived the relations between the parameters characterizing the EOS of nearly symmetric nuclear matter from experimental data on radii and masses of stable nuclei. We have found the linear relation (22) between the parameters $L$ and $S_{0}$ associated with the symmetry energy. It is
interesting to regard the band structure on the $K_0$ versus $L$ plane (see Fig. 1c) as an empirically allowed region of $K_0$ and $L$. Future systematic measurements of the root-mean-square matter radii of unstable neutron-rich nuclei may narrow the allowed region in such a way as to fix $L$ almost independently of $K_0$. Once $L$ is determined, the slope $y$ would be given as a function of $K_0$. It is instructive to note that the incompressibility was estimated to be $K_0 = 231 \pm 5$ MeV from the observations of the giant monopole resonances [10]. This estimate can stringently limit the allowed region, although it is model-dependent in the sense that it involves microscopic calculations of the resonance energies from a specific model of the effective nucleon-nucleon interactions [30].

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Appendix: Evaluation of the parameters in Eq. (2)

In this appendix we tabulate (Table A.1) the optimal values of various EOS parameters in Eq. (2).

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| Force | \( n_0 \) | \(-w_0\) | \( K_0 \) | \( S_0 \) | \( L \) | \(-y\) |
|-------|--------|--------|--------|--------|--------|--------|
| SI    | 0.155  | 16.0   | 371    | 29.2   | 1.18   | 19700  |
| SII   | 0.148  | 16.0   | 342    | 34.2   | 50.0   | 524    |
| SIII  | 0.145  | 15.9   | 356    | 28.2   | 9.87   | 2330   |
| SIV   | 0.151  | 16.0   | 325    | 31.2   | 63.5   | 353    |
| SV    | 0.155  | 16.1   | 306    | 32.8   | 96.1   | 224    |
| SVI   | 0.144  | 15.8   | 364    | 26.9   | -7.38  | -3080  |
| Ska   | 0.155  | 16.0   | 263    | 32.9   | 74.6   | 249    |
| Skb   | 0.155  | 16.0   | 263    | 23.9   | 47.5   | 284    |
| SG-0  | 0.168  | 16.7   | 253    | 35.6   | 41.6   | 430    |
| SGI   | 0.154  | 15.8   | 261    | 28.3   | 64.1   | 250    |
| SGII  | 0.158  | 15.6   | 215    | 26.8   | 37.6   | 322    |
| SkM   | 0.160  | 15.8   | 217    | 30.7   | 49.3   | 281    |
| SkM*  | 0.160  | 15.8   | 217    | 30.0   | 45.8   | 296    |
| \( E \) | 0.159  | 16.1   | 334    | 27.6   | -31.3  | -617   |
| \( E_\sigma \) | 0.163  | 16.0   | 249    | 26.4   | -36.9  | -364   |
| \( Z \) | 0.159  | 16.0   | 330    | 26.8   | -49.8  | -373   |
| \( Z_\sigma \) | 0.163  | 15.9   | 233    | 26.7   | -29.4  | -432   |
| \( Z^*_\sigma \) | 0.163  | 16.0   | 235    | 28.8   | -4.58  | -3030  |
| \( R_\sigma \) | 0.158  | 15.6   | 238    | 30.6   | 85.7   | 179    |
| \( G_\sigma \) | 0.158  | 15.6   | 237    | 31.4   | 94.0   | 167    |
| MSkA  | 0.154  | 16.0   | 314    | 30.4   | 57.2   | 361    |
| SkT6  | 0.161  | 16.0   | 236    | 30.0   | 30.8   | 475    |
| SkP   | 0.163  | 16.0   | 201    | 30.0   | 19.5   | 632    |
| SkSC4 | 0.161  | 15.9   | 235    | 28.8   | -2.17  | -6460  |
| SkX   | 0.155  | 16.1   | 271    | 31.1   | 33.2   | 545    |
| MSk7  | 0.158  | 15.8   | 231    | 27.9   | 9.36   | 1460   |
| BSk1  | 0.157  | 15.8   | 231    | 27.8   | 7.15   | 1908   |
| SLY4  | 0.160  | 16.0   | 230    | 32.0   | 45.9   | 335    |
| SLY7  | 0.158  | 15.9   | 230    | 32.0   | 47.2   | 328    |
| TM1   | 0.145  | 16.3   | 281    | 37.9   | 114    | 215    |

Table 1
The saturation properties of nuclear matter obtained from various effective forces
Table A.1
Optimal values of the parameters $n_0$, $w_0$, $S_0$, $L$ and $F_0$ for fixed $y$ and $K_0$

| $-y$ | $K_0$ | $n_0$  | $-w_0$ | $S_0$  | $L$    | $F_0$  |
|------|-------|--------|--------|--------|--------|--------|
| 200  | 180   | 0.16931| 16.257 | 32.899 | 58.296 | 71.300 |
| 200  | 190   | 0.16695| 16.239 | 33.154 | 62.887 | 70.892 |
| 200  | 200   | 0.16491| 16.227 | 33.430 | 67.572 | 70.534 |
| 200  | 210   | 0.16298| 16.214 | 33.723 | 72.421 | 70.221 |
| 200  | 220   | 0.16125| 16.204 | 34.037 | 77.398 | 69.914 |
| 200  | 230   | 0.15965| 16.193 | 34.365 | 82.511 | 69.539 |
| 200  | 240   | 0.15793| 16.120 | 34.905 | 99.328 | 63.190 |
| 200  | 250   | 0.15685| 16.186 | 35.113 | 93.278 | 69.320 |
| 200  | 260   | 0.15228| 15.932 | 34.905 | 99.328 | 63.190 |
| 200  | 270   | 0.15200| 15.870 | 34.933 | 103.42 | 60.072 |
| 200  | 280   | 0.15298| 16.157 | 36.349 | 110.88 | 68.403 |
| 200  | 290   | 0.15184| 16.153 | 36.856 | 117.32 | 68.297 |
| 200  | 300   | 0.14880| 15.858 | 36.503 | 122.66 | 59.622 |
| 200  | 310   | 0.14972| 16.144 | 37.939 | 130.93 | 68.070 |
| 200  | 320   | 0.14867| 16.139 | 38.540 | 138.26 | 68.006 |
| 200  | 330   | 0.14772| 16.135 | 39.190 | 145.92 | 67.944 |
| 200  | 340   | 0.14673| 16.131 | 39.904 | 154.10 | 67.911 |
| 200  | 350   | 0.14582| 16.130 | 40.684 | 162.75 | 67.969 |
| 200  | 360   | 0.14444| 16.056 | 41.232 | 171.28 | 65.851 |
| 220  | 180   | 0.16925| 16.250 | 32.350 | 52.129 | 71.133 |
| 220  | 190   | 0.16698| 16.234 | 32.564 | 56.141 | 70.709 |
| 220  | 200   | 0.16489| 16.218 | 32.791 | 60.264 | 70.320 |
| 220  | 210   | 0.16303| 16.207 | 33.031 | 64.467 | 69.959 |
| 220  | 220   | 0.16131| 16.196 | 33.286 | 68.784 | 69.625 |
| 220  | 230   | 0.15969| 16.184 | 33.550 | 73.214 | 69.276 |
| 220  | 240   | 0.15821| 16.175 | 33.835 | 77.767 | 68.969 |
| 220  | 250   | 0.15748| 16.184 | 34.183 | 82.224 | 68.520 |
| 220  | 260   | 0.15412| 16.053 | 34.231 | 87.496 | 65.686 |
| 220  | 270   | 0.15438| 16.179 | 34.850 | 92.349 | 69.206 |
| $-y$ | $K_0$ | $n_0$ | $-w_0$ | $S_0$ | $L$ | $F_0$ |
|------|-------|-------|-------|-------|-----|------|
| 220  | 280   | 0.15265 | 16.003 | 34.543 | 96.002 | 63.371 |
| 220  | 290   | 0.15128 | 16.019 | 35.173 | 102.16 | 63.971 |
| 220  | 300   | 0.15110 | 16.133 | 35.917 | 108.05 | 67.413 |
| 220  | 310   | 0.15019 | 16.135 | 36.354 | 113.69 | 67.388 |
| 220  | 320   | 0.14856 | 16.000 | 36.340 | 118.60 | 63.115 |
| 220  | 330   | 0.14832 | 16.126 | 37.295 | 125.72 | 67.102 |
| 220  | 340   | 0.14742 | 16.123 | 37.813 | 132.14 | 67.039 |
| 220  | 350   | 0.14651 | 16.128 | 38.395 | 138.97 | 67.357 |
| 220  | 360   | 0.14570 | 16.114 | 38.963 | 145.86 | 66.841 |
| 250  | 180   | 0.16876 | 16.191 | 31.643 | 45.001 | 69.337 |
| 250  | 190   | 0.16694 | 16.225 | 31.893 | 48.399 | 70.488 |
| 250  | 200   | 0.16490 | 16.210 | 32.071 | 48.399 | 70.075 |
| 250  | 210   | 0.16305 | 16.197 | 32.251 | 51.862 | 70.075 |
| 250  | 220   | 0.16136 | 16.184 | 32.449 | 58.988 | 69.240 |
| 250  | 230   | 0.15978 | 16.174 | 32.655 | 62.675 | 68.890 |
| 250  | 240   | 0.15829 | 16.154 | 32.851 | 66.410 | 68.193 |
| 250  | 250   | 0.15707 | 16.155 | 33.098 | 70.239 | 68.083 |
| 250  | 260   | 0.15590 | 16.159 | 33.388 | 74.246 | 68.228 |
| 250  | 270   | 0.15479 | 16.129 | 33.574 | 78.083 | 66.874 |
| 250  | 280   | 0.15364 | 16.142 | 33.882 | 82.331 | 67.477 |
| 250  | 290   | 0.15087 | 15.950 | 33.612 | 86.147 | 62.109 |
| 250  | 300   | 0.15173 | 16.132 | 34.449 | 90.818 | 66.855 |
| 250  | 310   | 0.14900 | 16.081 | 34.601 | 95.985 | 67.159 |
| 250  | 320   | 0.14894 | 16.028 | 34.781 | 99.635 | 63.947 |
| 250  | 330   | 0.14889 | 16.113 | 35.403 | 104.62 | 66.202 |
| 250  | 340   | 0.14818 | 16.074 | 35.654 | 109.08 | 64.564 |
| 250  | 350   | 0.14727 | 16.105 | 36.144 | 114.54 | 65.893 |
| 250  | 360   | 0.14653 | 16.103 | 36.548 | 119.72 | 65.769 |
| $-y$ | $K_0$ | $n_0$ | $-w_0$ | $S_0$ | $L$ | $F_0$ |
|------|------|------|------|------|------|------|
| 300  | 180  | 0.16915 | 16.232 | 31.027 | 36.685 | 70.687 |
| 300  | 190  | 0.16679 | 16.206 | 31.127 | 39.398 | 69.997 |
| 300  | 200  | 0.16492 | 16.201 | 31.261 | 42.123 | 69.789 |
| 300  | 210  | 0.16304 | 16.185 | 31.387 | 44.919 | 69.318 |
| 300  | 220  | 0.16138 | 16.172 | 31.522 | 47.747 | 68.875 |
| 300  | 230  | 0.15991 | 16.164 | 31.669 | 50.612 | 68.492 |
| 300  | 240  | 0.15845 | 16.152 | 31.816 | 53.547 | 68.116 |
| 300  | 250  | 0.15716 | 16.144 | 31.977 | 56.518 | 67.747 |
| 300  | 260  | 0.15594 | 16.135 | 32.136 | 59.533 | 67.382 |
| 300  | 270  | 0.15487 | 16.129 | 32.312 | 62.595 | 67.030 |
| 300  | 280  | 0.15386 | 16.128 | 32.498 | 65.712 | 66.662 |
| 300  | 290  | 0.15301 | 16.134 | 32.722 | 68.907 | 66.850 |
| 300  | 300  | 0.15309 | 16.206 | 33.118 | 72.109 | 68.333 |
| 300  | 310  | 0.15101 | 16.103 | 33.093 | 75.485 | 65.682 |
| 300  | 320  | 0.14987 | 16.058 | 33.149 | 78.644 | 64.466 |
| 300  | 330  | 0.14883 | 16.071 | 33.480 | 82.484 | 65.001 |
| 300  | 340  | 0.14699 | 15.926 | 33.240 | 85.428 | 60.873 |
| 300  | 350  | 0.14680 | 15.964 | 33.630 | 89.092 | 61.549 |
| 300  | 360  | 0.14746 | 16.025 | 34.068 | 92.411 | 62.205 |
| 350  | 180  | 0.16905 | 16.224 | 30.543 | 30.974 | 70.513 |
| 350  | 190  | 0.16677 | 16.201 | 30.617 | 33.221 | 69.869 |
| 350  | 200  | 0.16472 | 16.180 | 30.696 | 35.495 | 69.261 |
| 350  | 210  | 0.16297 | 16.174 | 30.811 | 37.813 | 69.037 |
| 350  | 220  | 0.16086 | 16.131 | 30.837 | 40.166 | 67.933 |
| 350  | 230  | 0.15979 | 16.145 | 31.002 | 42.498 | 67.975 |
| 350  | 240  | 0.15877 | 16.167 | 31.189 | 44.902 | 68.422 |
| 350  | 250  | 0.15729 | 16.138 | 31.258 | 47.315 | 67.474 |
| 350  | 260  | 0.15608 | 16.128 | 31.374 | 49.775 | 67.057 |
| 350  | 270  | 0.15500 | 16.121 | 31.503 | 52.263 | 66.690 |
| $-y$ | $K_0$ | $n_0$ | $-w_0$ | $S_0$ | $L$  | $F_0$ |
|------|------|------|-------|------|------|------|
| 350  | 280  | 0.15412 | 16.143 | 31.701 | 54.852 | 67.309 |
| 350  | 290  | 0.15305 | 16.108 | 31.777 | 57.343 | 65.982 |
| 350  | 300  | 0.15211 | 16.101 | 31.917 | 59.953 | 65.635 |
| 350  | 310  | 0.15133 | 16.099 | 32.071 | 62.569 | 65.373 |
| 350  | 320  | 0.15117 | 16.227 | 32.539 | 65.599 | 69.348 |
| 350  | 330  | 0.14978 | 16.088 | 32.384 | 67.950 | 64.685 |
| 350  | 340  | 0.14959 | 16.073 | 32.489 | 70.326 | 63.401 |
| 350  | 350  | 0.14785 | 16.073 | 32.714 | 73.757 | 64.408 |
| 350  | 360  | 0.14741 | 16.025 | 32.734 | 76.137 | 62.538 |
| 400  | 180  | 0.16902 | 16.219 | 30.206 | 26.807 | 70.423 |
| 400  | 190  | 0.16675 | 16.200 | 30.263 | 28.736 | 69.895 |
| 400  | 200  | 0.16446 | 16.164 | 30.287 | 30.693 | 69.005 |
| 400  | 210  | 0.16303 | 16.173 | 30.412 | 32.645 | 68.967 |
| 400  | 220  | 0.16138 | 16.158 | 30.481 | 34.628 | 68.483 |
| 400  | 230  | 0.15988 | 16.148 | 30.566 | 36.642 | 68.057 |
| 400  | 240  | 0.15855 | 16.138 | 30.654 | 38.668 | 67.603 |
| 400  | 250  | 0.15728 | 16.130 | 30.746 | 40.727 | 67.250 |
| 400  | 260  | 0.15615 | 16.122 | 30.845 | 42.801 | 66.850 |
| 400  | 270  | 0.15505 | 16.113 | 30.942 | 44.900 | 66.429 |
| 400  | 280  | 0.15406 | 16.107 | 31.043 | 47.016 | 66.044 |
| 400  | 290  | 0.15315 | 16.100 | 31.156 | 49.162 | 65.654 |
| 400  | 300  | 0.15227 | 16.095 | 31.265 | 51.332 | 65.344 |
| 400  | 310  | 0.15146 | 16.090 | 31.381 | 53.526 | 65.004 |
| 400  | 320  | 0.15070 | 16.087 | 31.508 | 55.754 | 64.721 |
| 400  | 330  | 0.14996 | 16.081 | 31.627 | 58.000 | 64.364 |
| 400  | 340  | 0.14938 | 16.164 | 31.976 | 60.648 | 67.165 |
| 400  | 350  | 0.14854 | 16.071 | 31.895 | 62.628 | 63.808 |
| 400  | 360  | 0.14701 | 16.095 | 32.082 | 65.471 | 65.588 |
| $y$ | $K_0$ | $n_0$ | $-w_0$ | $S_0$ | $L$ | $F_0$ |
|-----|-------|-------|--------|-------|-----|-------|
| 500 | 180   | 0.16895 | 16.213 | 29.750 | 21.131 | 70.289 |
| 500 | 190   | 0.16674 | 16.194 | 29.786 | 22.628 | 69.745 |
| 500 | 200   | 0.16506 | 16.201 | 29.868 | 24.127 | 69.843 |
| 500 | 210   | 0.16324 | 16.179 | 29.901 | 25.644 | 69.044 |
| 500 | 220   | 0.16136 | 16.149 | 29.914 | 27.191 | 68.228 |
| 500 | 230   | 0.15991 | 16.141 | 29.972 | 28.739 | 67.851 |
| 500 | 240   | 0.15856 | 16.130 | 30.028 | 30.302 | 67.377 |
| 500 | 250   | 0.15731 | 16.120 | 30.085 | 31.875 | 66.940 |
| 500 | 260   | 0.15620 | 16.112 | 30.152 | 33.459 | 66.504 |
| 500 | 270   | 0.15513 | 16.104 | 30.216 | 35.060 | 66.079 |
| 500 | 280   | 0.15421 | 16.098 | 30.288 | 36.663 | 65.685 |
| 500 | 290   | 0.15330 | 16.092 | 30.359 | 38.286 | 65.293 |
| 500 | 300   | 0.15243 | 16.086 | 30.437 | 39.934 | 64.956 |
| 500 | 310   | 0.15165 | 16.080 | 30.511 | 41.581 | 64.550 |
| 500 | 320   | 0.15091 | 16.076 | 30.594 | 43.249 | 64.240 |
| 500 | 330   | 0.15022 | 16.073 | 30.678 | 44.930 | 63.950 |
| 500 | 340   | 0.14957 | 16.069 | 30.765 | 46.624 | 63.597 |
| 500 | 350   | 0.14892 | 16.063 | 30.852 | 48.341 | 63.250 |
| 500 | 360   | 0.14912 | 16.222 | 31.314 | 50.399 | 68.034 |
| 600 | 180   | 0.16890 | 16.208 | 29.461 | 17.443 | 70.159 |
| 600 | 190   | 0.16699 | 16.188 | 29.475 | 18.664 | 69.602 |
| 600 | 200   | 0.16470 | 16.172 | 29.496 | 19.899 | 69.132 |
| 600 | 210   | 0.16393 | 16.245 | 29.692 | 21.132 | 70.991 |
| 600 | 220   | 0.16137 | 16.148 | 29.560 | 22.389 | 68.204 |
| 600 | 230   | 0.15990 | 16.135 | 29.596 | 23.651 | 67.684 |
| 600 | 240   | 0.15856 | 16.123 | 29.634 | 24.920 | 67.189 |
| 600 | 250   | 0.15735 | 16.114 | 29.675 | 26.194 | 66.714 |
| 600 | 260   | 0.15626 | 16.108 | 29.724 | 27.476 | 66.344 |
| 600 | 270   | 0.15519 | 16.097 | 29.766 | 28.770 | 65.856 |
| $-y$ | $K_0$ | $n_0$ | $-w_0$ | $S_0$ | $L$ | $F_0$ |
|------|-------|-------|-------|------|-----|-------|
| 600  | 280   | 0.15422 | 16.091 | 29.818 | 30.077 | 65.479 |
| 600  | 290   | 0.15336 | 16.085 | 29.871 | 31.381 | 65.049 |
| 600  | 300   | 0.15254 | 16.081 | 29.930 | 32.701 | 64.714 |
| 600  | 310   | 0.15176 | 16.075 | 29.988 | 34.031 | 64.339 |
| 600  | 320   | 0.15106 | 16.070 | 30.047 | 35.362 | 63.953 |
| 600  | 330   | 0.15042 | 16.072 | 30.118 | 36.708 | 63.752 |
| 600  | 340   | 0.14968 | 16.061 | 30.168 | 38.071 | 63.287 |
| 600  | 350   | 0.15232 | 16.340 | 30.965 | 39.528 | 69.329 |
| 600  | 360   | 0.14849 | 16.053 | 30.297 | 40.808 | 62.629 |
| 800  | 180   | 0.16884 | 16.203 | 29.105 | 12.929 | 70.077 |
| 800  | 190   | 0.16688 | 16.200 | 29.137 | 13.823 | 69.926 |
| 800  | 200   | 0.16491 | 16.200 | 29.142 | 14.726 | 69.926 |
| 800  | 210   | 0.16455 | 16.283 | 29.351 | 15.607 | 71.847 |
| 800  | 220   | 0.16132 | 16.135 | 29.140 | 16.558 | 67.801 |
| 800  | 230   | 0.15982 | 16.127 | 29.147 | 17.478 | 67.534 |
| 800  | 240   | 0.15858 | 16.118 | 29.174 | 18.397 | 67.030 |
| 800  | 250   | 0.15734 | 16.107 | 29.193 | 19.327 | 66.550 |
| 800  | 260   | 0.15624 | 16.099 | 29.217 | 20.258 | 66.079 |
| 800  | 270   | 0.15527 | 16.092 | 29.249 | 21.193 | 65.645 |
| 800  | 280   | 0.15432 | 16.085 | 29.276 | 22.133 | 65.237 |
| 800  | 290   | 0.15341 | 16.076 | 29.305 | 23.083 | 64.764 |
| 800  | 300   | 0.15265 | 16.073 | 29.342 | 24.028 | 64.400 |
| 800  | 310   | 0.15186 | 16.067 | 29.373 | 24.983 | 64.001 |
| 800  | 320   | 0.15117 | 16.063 | 29.414 | 25.943 | 63.644 |
| 800  | 330   | 0.15047 | 16.057 | 29.448 | 26.909 | 63.268 |
| 800  | 340   | 0.14983 | 16.053 | 29.489 | 27.881 | 62.933 |
| 800  | 350   | 0.14925 | 16.049 | 29.529 | 28.852 | 62.575 |
| 800  | 360   | 0.14866 | 16.041 | 29.572 | 29.839 | 62.127 |
| $-y$ | $K_0$ | $n_0$ | $-w_0$ | $S_0$ | $L$ | $F_0$ |
|------|------|------|------|------|------|------|
| 1000 | 180  | 0.16891 | 16.213 | 28.929 | 10.276 | 70.446 |
| 1000 | 190  | 0.16656 | 16.179 | 28.890 | 10.985 | 69.460 |
| 1000 | 200  | 0.16460 | 16.164 | 28.888 | 11.700 | 68.969 |
| 1000 | 210  | 0.16411 | 16.266 | 29.106 | 12.415 | 71.720 |
| 1000 | 220  | 0.16127 | 16.133 | 28.888 | 13.136 | 67.828 |
| 1000 | 230  | 0.15989 | 16.124 | 28.890 | 13.853 | 67.378 |
| 1000 | 240  | 0.15857 | 16.114 | 28.907 | 14.584 | 66.911 |
| 1000 | 250  | 0.15738 | 16.103 | 28.920 | 15.314 | 66.392 |
| 1000 | 260  | 0.15626 | 16.095 | 28.933 | 16.047 | 65.958 |
| 1000 | 270  | 0.15525 | 16.086 | 28.950 | 16.783 | 65.496 |
| 1000 | 280  | 0.15433 | 16.080 | 28.969 | 17.520 | 65.071 |
| 1000 | 290  | 0.15345 | 16.074 | 28.988 | 18.261 | 64.677 |
| 1000 | 300  | 0.15268 | 16.069 | 29.012 | 19.001 | 64.238 |
| 1000 | 310  | 0.15191 | 16.063 | 29.037 | 19.752 | 63.864 |
| 1000 | 320  | 0.15121 | 16.059 | 29.060 | 20.499 | 63.503 |
| 1000 | 330  | 0.15053 | 16.052 | 29.085 | 21.253 | 63.069 |
| 1000 | 340  | 0.14991 | 16.048 | 29.110 | 22.008 | 62.729 |
| 1000 | 350  | 0.14935 | 16.045 | 29.141 | 22.763 | 62.395 |
| 1000 | 360  | 0.14880 | 16.042 | 29.171 | 23.525 | 62.056 |
| 1400 | 180  | 0.16872 | 16.195 | 28.677 | 7.2841 | 69.908 |
| 1400 | 190  | 0.16657 | 16.178 | 28.657 | 7.7828 | 69.419 |
| 1400 | 200  | 0.16460 | 16.160 | 28.641 | 8.2861 | 68.842 |
| 1400 | 210  | 0.16291 | 16.146 | 28.633 | 8.7877 | 68.295 |
| 1400 | 220  | 0.16130 | 16.132 | 28.624 | 9.2955 | 67.785 |
| 1400 | 230  | 0.15979 | 16.115 | 28.608 | 9.8043 | 67.148 |
| 1400 | 240  | 0.15867 | 16.117 | 28.633 | 10.311 | 66.953 |
| 1400 | 250  | 0.15735 | 16.097 | 28.615 | 10.825 | 66.220 |
| 1400 | 260  | 0.15632 | 16.093 | 28.625 | 11.336 | 65.868 |
| 1400 | 270  | 0.15527 | 16.084 | 28.627 | 11.852 | 65.403 |
| $-y$ | $K_0$ | $n_0$ | $-w_0$ | $S_0$ | $L$ | $F_0$ |
|------|------|------|------|------|----|------|
| 1400 | 280  | 0.15433 | 16.075 | 28.634 | 12.369 | 64.909 |
| 1400 | 290  | 0.15352 | 16.071 | 28.645 | 12.883 | 64.546 |
| 1400 | 300  | 0.15269 | 16.063 | 28.655 | 13.405 | 64.064 |
| 1400 | 310  | 0.15196 | 16.058 | 28.667 | 13.924 | 63.663 |
| 1400 | 320  | 0.15126 | 16.053 | 28.681 | 14.447 | 63.317 |
| 1400 | 330  | 0.15065 | 16.048 | 28.693 | 14.965 | 62.884 |
| 1400 | 340  | 0.15003 | 16.044 | 28.708 | 15.490 | 62.512 |
| 1400 | 350  | 0.14941 | 16.039 | 28.713 | 16.015 | 62.182 |
| 1400 | 360  | 0.14889 | 16.035 | 28.738 | 16.544 | 61.778 |
| 1800 | 180  | 0.16868 | 16.193 | 28.552 | 5.643 | 69.913 |
| 1800 | 190  | 0.16652 | 16.174 | 28.525 | 6.027 | 69.339 |
| 1800 | 200  | 0.16460 | 16.158 | 28.508 | 6.415 | 68.779 |
| 1800 | 210  | 0.16281 | 16.142 | 28.488 | 6.805 | 68.254 |
| 1800 | 220  | 0.16126 | 16.129 | 28.475 | 7.190 | 67.714 |
| 1800 | 230  | 0.15983 | 16.116 | 28.465 | 7.586 | 67.193 |
| 1800 | 240  | 0.15846 | 16.104 | 28.446 | 7.978 | 66.736 |
| 1800 | 250  | 0.15738 | 16.098 | 28.459 | 8.371 | 66.256 |
| 1800 | 260  | 0.15629 | 16.090 | 28.457 | 8.766 | 65.790 |
| 1800 | 270  | 0.15530 | 16.082 | 28.457 | 9.161 | 65.332 |
| 1800 | 280  | 0.15437 | 16.074 | 28.459 | 9.559 | 64.869 |
| 1800 | 290  | 0.15349 | 16.065 | 28.457 | 9.957 | 64.391 |
| 1800 | 300  | 0.15271 | 16.058 | 28.463 | 10.354 | 63.912 |
| 1800 | 310  | 0.15199 | 16.056 | 28.471 | 10.754 | 63.584 |
| 1800 | 320  | 0.15129 | 16.050 | 28.478 | 11.155 | 63.167 |
| 1800 | 330  | 0.15068 | 16.046 | 28.485 | 11.553 | 62.780 |
| 1800 | 340  | 0.15009 | 16.043 | 28.498 | 11.955 | 62.431 |
| 1800 | 350  | 0.14948 | 16.037 | 28.507 | 12.361 | 62.062 |
| 1800 | 360  | 0.14895 | 16.032 | 28.515 | 12.763 | 61.648 |