Theory of Games on Quantum Objects

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Effect of replacing the classical game object with a quantum object is analyzed. We find this replacement requires a throughout reformation of the framework of Game Theory. If we use density matrix to represent strategy state of players, they are full-structured density matrices with off-diagonal elements for the new games, while reduced diagonal density matrix will be enough for the traditional games on classical objects. In such formalism, the payoff function of every player becomes Hermitian Operator acting on the density matrix. Therefore, the new game looks really like Quantum Mechanics while the traditional game becomes Classical Mechanics.

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I. INTRODUCTION

The object of Game Theory is a game, a multi-player decision making situation, usually with conflicts between players. For example, in a Penny Flipping Game (PFG), two players play with a coin, say initially with head state. The strategies can be used by players are Non-flip and Flip, which, in the language of Physics, are operators acting on the coin. The payoff is defined such as player 1 wins one dollar for head state after both players applied their strategies, and lose one dollar for tail state. For such static strategy games, Nash Theorem has given a closed conclusion that at least one mixture-strategy Nash Equilibria (NE) exists for any games. Here the NE is defined that under such state no more players will like to change its own strategy state, and mixture strategy is defined as a probability distribution function (PDF) over the strategy space of every player.

Our question is how about we replace the two-side coin here with a $\frac{1}{2}$ quantum spin? What’s the effect of this on Game Theory? It’s still a game-theory question. Players can still choose strategies to act on the spin, although they have much more choices now. Compared with Non-flip and Flip, in Quantum Mechanics, any unitary $2 \times 2$ matrices can be used as operators, and $\{I, X, Y, Z\}$ are the four typical matrices of them. Now the Game Theory must answer how to define the strategy state for this game, how to define NE, and the existence of NE. At last, we have to ask whether such game can be studied within the framework of Traditional Game Theory (TGT), or should we develop a new framework but still with the same spirit of Game Theory? In this work, we will construct a new framework, which can be used both TGT and the game on quantum objects, named Quantum Game Theory (QGT).

II. DENSITY MATRIX: LANGUAGE WE USED

Density Matrix language for Quantum Mechanics is well known. In Schrödinger’s Picture, a state of a quantum object is represented by a density matrix $\rho^q(t)$; the evolution is described by a unitary transformation $U(t) \equiv U(0, t)$ as

$$\rho^q(t) = U(t) \rho^q(0) U^\dagger(t),$$

where generally $U(t)$ is determined by $H$, the Hamiltonian of the quantum object; and any physical quantities should be calculated by

$$A \equiv \left\langle \hat{A} \right\rangle = tr \left( \hat{A} \rho^q \right).$$

Here we want to use density matrix also to describe Classical Mechanics, which originally is described by a PDF, $f(x, \vec{p})$, such as in Liouville Equation. Now we re-express it as a density matrix as

$$\rho^c = \sum_{x \in \Omega} f(x) |x\rangle \langle x|,$$

where $x$ is used to represent all general configuration variables. In fact, even for quantum objects, this kind of states has been used by Von Neumann in his picture of quantum measurement as exclusive mixture states. Its explanation is every sample of this state gives only one realization such as $x^*$ with probability $f(x^*)$. This is exactly the same meaning of the PDF. $f(x)$ normalized within $\Omega = \{x\}$, the set of its all possible states. By assuming

$$\langle x | x' \rangle = \delta(x - x'),$$

the normalizing condition for both quantum and classical density matrix can be written as

$$tr(\rho) = 1.$$
\[ \rho^e(t) = \sum_x p(x) |x \rightarrow x(t)\rangle \langle x \rightarrow x(t)|, \]
\[ = \sum_x p(x) (T|x\rangle \langle x|T^\dagger \rangle) \]
\[ = T \rho^e(0) T^\dagger, \]
and also easy to show \( TT^\dagger = T^\dagger T = I \).

However, although we unified classical and quantum description by density matrix, but those two density matrices are different. The one for classical object is always diagonal, while the one for quantum object has off-diagonal elements and it’s diagonal only under one special basis. This difference roots in the non-commutative relation between quantum operators. The way to use density matrix to describe classical objects is just like to use complex number to reexpress expressions of real numbers. However, when we want to unify expressions of real number and complex number, of course, we need to work in the field of complex number. Here, we are in the same situation: unification of description of states both classical and quantum objects.

Not only states of classical and quantum objects, operators on classical and quantum objects can also be described by Hilbert space and density matrices. We call this the density matrix formalism for Quantum Operators. For the unitary operators on a \( \frac{1}{2} \)-spin, we know they are \( 2 \times 2 \) matrices, and generally can be expanded by \( \{I, X, Y, Z\} \) as
\[ U = e^{i\alpha} \left( \cos \frac{\alpha}{2} \cos \frac{\beta+i\gamma}{2} I + i \sin \frac{\alpha}{2} \sin \frac{\beta+i\gamma}{2} X \right. \]
\[ - i \sin \frac{\alpha}{2} \cos \frac{\beta-i\gamma}{2} Y - i \cos \frac{\alpha}{2} \sin \frac{\beta-i\gamma}{2} Z \right). \]

Now we regard this expansion as a decomposition of a vector \( U \) under the basis of \( \{I, X, Y, Z\} \) of a Hilbert space of operators \( \mathcal{H}^* \equiv \{U\} \). Fortunately, \( \mathcal{H}^* \) is a Hilbert space with a natural defined inner product. The summation and number product of vector is naturally fulfilled by the corresponding usual operation on matrices, the inner product is defined by
\[ (A|B) \equiv (A,B) = \frac{Tr(A^\dagger B)}{Tr(I)}. \]

For simplicity, later on, \( \mathbb{B} \) and \( \mathbb{E} \) is used to denote the basis, \( \mathbb{B}(\mathcal{H}^*) \), and the space expanded by a basis, \( \mathcal{H}^* = \mathbb{E}(\mathcal{H}^*) \) respectively.

Since \( \text{mathcal} H^* \) is also a Hilbert space, we can use density matrix to represent its vectors, which now, physically, is operators. Now, we have prepared everything we will need, the density matrix of any operators, and any probability combinations of operators, can be generally defined as
\[ \rho^{op} = \sum_{\mu, \nu \in \mathbb{B}(\mathcal{H}^*)} \rho^{\mu\nu}_{\mu\nu} |\mu\rangle \langle \nu|. \]

This can be used as strategy state for games on both classical and quantum operators.

### III. DENSITY MATRIX FORMALISM FOR TGT

First, we want to put the TGT into density matrix formalism, which means to put the strategy states as density matrices, payoff function as Hermitian operators, and their relation should obey eqn(4). As we know the general mixture strategy state of a player \( i \) in TGT a PDF over \( i \)'s strategy space, so the density matrix form is
\[ \rho^{c,i} = \sum_{\mu \in \mathbb{B}(\mathcal{H}^*)} \rho^{\mu\mu}_{\mu\mu} |\mu\rangle \langle \mu|, \]
and the density matrix of all players in a non-cooperative game is
\[ \rho^{c,S} = \prod_i \rho^{c,i}. \]

Or if we denote \( |\vec{\mu}\rangle = |\mu_1, \ldots, \mu_i, \ldots, \mu_N\rangle \), then,
\[ \rho^{c,S} = \sum_{\vec{\mu}} \left( \prod_i \rho^{\mu_i\mu_i}_{\mu_i\mu_i} \right) |\vec{\mu}\rangle \langle \vec{\mu}|. \]

The payoff matrix of player \( i \) is defined as
\[ H^i = \sum_{\vec{\mu}} G^i(\vec{\mu}) |\vec{\mu}\rangle \langle \vec{\mu}|, \]
where \( G^i \) is the traditional payoff function in TGT, which give a real number when all the strategies of every player are given, \( G^i(\vec{\mu}) \). It’s easy to check in this abstract form, the payoff is given by
\[ E^i = Tr(\rho^{c,S} H^i), \]
where \( Tr(\cdot) \) is the trace over strategy state space, \( \mathcal{H}^* \).

One important character should be noticed that both above \( \rho^S \) and \( G^i \) have only diagonal terms, which is a character of classical systems. It’s easy to check that every classical game can be re-expressed in this language of density matrix and Hermitian Hamiltonian. The only difference between this TGT and Classical Mechanics is that here every player has its own Hamiltonian, while in Physics, we only have a common one for the whole system. This reflects the conflict of interests between players. For example, PFG, which in TGT notation is
\[ G^{1,2} = \begin{bmatrix} 1, -1 & -1, 1 \\ -1, 1 & 1, -1 \end{bmatrix}, \]
can be redefined as
\[ H^{1,2} = \begin{bmatrix} 1, -1 & 0 & 0 & 0 \\ 0 & -1, 1 & 0 & 0 \\ 0 & 0 & -1, 1 & 0 \\ 0 & 0 & 0 & 1, -1 \end{bmatrix}, \]
and
\[ \rho^{c,S} = \rho^{c,1} \otimes \rho^{c,2} = \begin{bmatrix} \rho^{c,1}_{\mu_1\mu_1} & 0 & 0 \\ 0 & \rho^{c,1}_{\mu_2\mu_2} & 0 \\ 0 & 0 & \rho^{c,2}_{\mu_2\mu_2} \end{bmatrix}. \]
IV. DENSITY MATRIX FORMALISM FOR QGT

In order to put QGT into density matrix form, first, we need to define the traditional payoff function \( G^i (\vec{\mu}) \), and then similarly define \( H^i \) from \( G^i \). However, the definition of \( G^i \) is not trivial, because there are infinite number of strategies (unitary operators) even for the 1/2-spin SFG. Fortunately, the inherent relation between quantum operators, such as eq(\ref{eq:4}), will save us out of this mud. The idea is choose a basis, \( \mathbb{B} (\mathcal{H}^*) \) for Hilbert space of operators \( \mathcal{H}^* \), then define \( G^i \) as a function over \( \mathbb{B} (\mathcal{H}^*) \) first, and by eq(\ref{eq:4}), at last \( G^i \) will be defined on the whole \( \mathcal{H}^* \). Before we go into the detail, there is a mine making our life not so easy: like operators in Quantum Mechanics, \( G^i \) will also be matrix operator over \( \mathbb{B} (\mathcal{H}^*) \) with off-diagonal elements. This means usually, \( G^i (\vec{\mu}, \vec{\nu}) \neq 0 \), while in TGT, \( G^i \) has only diagonal elements. Let’s demonstrate it by one example. In SFG, \( \{I, X, Y, Z\} \) is used as the operator basis, and the payoff is still defined such that player 1 gets payoff \( p^1 = \rho^0_{I\uparrow I\uparrow} - \rho^0_{I\uparrow I\downarrow} = -p^2 \), where the states are measured in z-direction. This can be written in a matrix form that

\[
P^1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = -P^2 \text{ and } p^i = tr \left( P^i \rho^{spin} \right). \tag{18}
\]

And the state of the spin changes according to the strategies of players by

\[
\rho^{spin}_{end} = (U_2 U_1) \rho^{spin}_{initial} (U_2 U_1)^\dagger. \tag{19}
\]

Now it is easy to check \( G^i (\{I, I\}, \{I, I\}) = 1 \) as usual as in TGT, but we have new elements such as \( G^i (\{Y, Y\}, \{Y, Y\}) = 1 \), and even off-diagonal elements \( G^i (\{X, X\}, \{I, I\}) = 1 \). This shows \( G^i (\vec{\mu}, \vec{\nu}) \neq 0 \) so that later on,

\[
H^i = \sum_{\vec{\mu}, \vec{\nu}} G^i (\vec{\mu}, \vec{\nu}) |\vec{\mu}\rangle \langle \vec{\nu}| \tag{20}
\]

will also have off-diagonal elements. Not only the payoff matrix, but also the density matrix of strategy state has off-diagonal elements. Considering a player chooses strategy \( U = \frac{1}{\sqrt{2}} (X + Y) \), expressed under the basis, it’s

\[
\rho^{op} = \frac{1}{2} (|X \rangle \langle X| + |X \rangle \langle Y| + |Y \rangle \langle X| + |Y \rangle \langle Y|), \tag{21}
\]

which obviously has non-zero off-diagonal elements such as \( \frac{1}{2} |X \rangle \langle Y| \). Generally, the state of players in a non-cooperative QGT is

\[
\rho^{i,S} = \prod_i \rho^{i,S}, \tag{22}
\]

where

\[
\rho^{i,S} = \sum_{\vec{\mu}, \vec{\nu} \in \mathbb{B}(\mathcal{H}^* \otimes \mathcal{H}^*)} \rho^{i,S}_{\vec{\mu}, \vec{\nu}} |\vec{\mu}\rangle \langle \vec{\nu}|. \tag{23}
\]

Or put in another way,

\[
\rho^{i,S} = \sum_{\vec{\mu}, \vec{\nu}} \left( \prod_i \rho^{i,S}_{\vec{\mu}, \vec{\nu}} \right) |\vec{\mu}\rangle \langle \vec{\nu}|. \tag{24}
\]

And then the payoff value is given by

\[
E^i = Tr \left( \rho^{i,S} H^i \right). \tag{25}
\]

Compare eq(\ref{eq:24}) and eq(\ref{eq:20}) with eq(\ref{eq:13}) and eq(\ref{eq:12}), we notice that the existence of off-diagonal elements is the difference between QGT and TGT.

Eq(\ref{eq:13}), the scale matrix used to assign payoff value to each player according to the state of object and eq(\ref{eq:19}), the evolution of state of object, can also be generalized to games on any classical and quantum objects. This has been done in \cite{4}, where it is named as “Manipulative Definition” of game, the payoff of player \( i \) is given by a physical process changing the state of the object and a scale to readout the end state into payoff value,

\[
E^i (S) = tr \left( P^i \mathcal{L} (S) \rho^{object}_{initial} \mathcal{L}^i (S) \right), \tag{26}
\]

where \( S = (s^1, s^2, \ldots, s^N) \) is an ordered sequence of the strategies used by all players.

The manipulative definitions of PFG and SFG is given respectively as followings,

\[
\rho_0^0 = |+1\rangle \langle +1| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad S^1 = S^2 = \{I, X\}
\]

\[
\mathcal{L} (s^1, s^2) = s^2 s^1, \quad P^1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = -P^2, \tag{27}
\]

and

\[
\rho_0^0 = |+1\rangle \langle +1| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad S^1 = S^2 = \mathbb{E} \{I, X, Y, Z\}
\]

\[
\mathcal{L} (s^1, s^2) = s^2 s^1, \quad P^1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = -P^2. \tag{28}
\]

From these manipulative definition, it’s easy to see that the only difference coming from the strategy space, that \( \{I, X\} \) for classical object and \( \mathbb{E} \{I, X, Y, Z\} \) for quantum object. And this difference requires the off-diagonal terms in both density matrix of strategy state and Hermitian payoff matrices. The explicit form of \( H^1 \), payoff
matrix of player 1 in SFG is

\[
\begin{pmatrix}
1 & 1 & 1 & -i & i & 1 & 1 & 1 \\
-1 & i & -1 & -1 & i & i & 1 & -i \\
-1 & i & -1 & -i & -1 & -1 & i & 1 \\
1 & 1 & 1 & -i & i & 1 & 1 & 1 \\
1 & 1 & 1 & -i & -1 & i & 1 & -i \\
-1 & i & -1 & -i & -1 & -1 & i & 1 \\
-1 & -i & -i & -1 & i & 1 & -i & -1 \\
1 & 1 & 1 & -i & i & 1 & 1 & 1
\end{pmatrix}
\]

while the general strategy state when there is no cooperation between players is

\[
\rho_{S} = \left[ \begin{array}{cccc}
\rho_{11}^{S} & \rho_{12}^{S} & \rho_{13}^{S} & \rho_{14}^{S} \\
\rho_{21}^{S} & \rho_{22}^{S} & \rho_{23}^{S} & \rho_{24}^{S} \\
\rho_{31}^{S} & \rho_{32}^{S} & \rho_{33}^{S} & \rho_{34}^{S} \\
\rho_{41}^{S} & \rho_{42}^{S} & \rho_{43}^{S} & \rho_{44}^{S}
\end{array} \right] \otimes \left[ \begin{array}{cccc}
\rho_{11}^{S} & \rho_{12}^{S} & \rho_{13}^{S} & \rho_{14}^{S} \\
\rho_{21}^{S} & \rho_{22}^{S} & \rho_{23}^{S} & \rho_{24}^{S} \\
\rho_{31}^{S} & \rho_{32}^{S} & \rho_{33}^{S} & \rho_{34}^{S} \\
\rho_{41}^{S} & \rho_{42}^{S} & \rho_{43}^{S} & \rho_{44}^{S}
\end{array} \right].
\]

(29)

Compare eqn (29) and eqn (30) with eqn (16) and eqn (17), 16 x 16 full-structure matrices are used to replace 4 x 4 diagonal matrices, first, because we have four base vectors of strategy space in SFG other than two in PFG; second, because of the non-zero off-diagonal elements. Such elements have no corresponding meaning in classical game: what’s the meaning of $\langle XX | H \rangle$? From left side, it looks like both players choose $X$, while from right side, both players choose $I$.

In fact, this remind us the meaning of the off-diagonal terms in density matrix and Hamiltonian of a quantum object. There such terms also have no classical correspondence, and they are a distinguishable character of quantum system compared with classical system. Therefore, the relation between TGT and QGT looks exactly like the relation between Classical Mechanics and Quantum Mechanics.

**V. DISCUSSION**

In fact, the idea of Quantum Game Theory has long been proposed in [4] and developed in [3, 6], and currently in an active development stage [5]. However, in all the former general prescription of Quantum Game Theory, although they do consider games on quantum objects, the strategy state is always treated as a probability distribution function over $\mathcal{H}^*$, the whole operator space, not a density matrix expanded on $\mathcal{B}(\mathcal{H}^*)$, a basis of $\mathcal{H}^*$. Obviously, those two descriptions of state are different. In [4], we did a detailed comparison between them and gave an argument that why our density matrix representation should be used instead of the probability distribution function.

In this paper, we first related abstract strategies in Game Theory with operators acting on physical objects, named game objects. Then, operators are treated as vectors in Hilbert space. Because Traditional Game Theory use probability distribution functions over the classical operators Hilbert space to act as general mixture strategy, the density matrix form of a probability distribution function is diagonal. However, in Quantum Game Theory, when we replace the classical game object with a quantum object, a strategy state must be a full-structured density matrix over Hilbert space of quantum operators, not the diagonal density matrix coming from probability distribution function. This is just like the relation between Classical Mechanics and Quantum Mechanics.

Besides the non-cooperative game, this new framework of Game Theory can also be used to discuss Coalitional Game Theory (CGT). When the system-level density matrix, $\rho^S \neq \prod_i \rho^i$, not a direct-product state, it naturally leads to correlation between players. This implies cooperation between players. The possibility to link this new framework with CGT will be an interesting topic [3].

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