On the Barth-Van de Ven-Tyurin-Sato theorem

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Abstract. The Barth-Van de Ven-Tyurin-Sato Theorem states that any finite-rank vector bundle on the complex projective ind-space $\mathbb{P}^\infty$ is isomorphic to a direct sum of line bundles. We establish sufficient conditions on a locally complete linear ind-variety $X$ which ensure that the same result holds on $X$. We then exhibit natural classes of locally complete linear ind-varieties which satisfy these sufficient conditions.

Bibliography: 18 titles.

Keywords: ind-variety, vector bundle.

§1. Introduction

The Barth-Van de Ven-Tyurin-Sato Theorem claims that any finite-rank vector bundle on the complex projective ind-space $\mathbb{P}^\infty$ is isomorphic to a direct sum of line bundles. For rank-two bundles this was established by Barth and Van de Ven in [1], and for finite-rank bundles it was proved by Tyurin in [2] and Sato in [3]. This topic was revived in the more recent papers [4]–[6], where in particular the case of twisted ind-grassmannians was considered.

In the current paper we consider ind-varieties $X = \lim \rightarrow X_m$ given by chains of embeddings of smooth complete algebraic varieties

$$X_1 \xrightarrow{\phi_1} X_2 \xrightarrow{\phi_2} \ldots \xrightarrow{\phi_{m-1}} X_m \xrightarrow{\phi_m} \ldots .$$

We call such ind-varieties locally complete. A locally complete ind-variety $X = \lim \rightarrow X_m$ is linear if the map induced by $\varphi_i$ on Picard groups is a surjection for almost all $i$. Our main objective is to give a reasonably general sufficient condition for the Barth-Van de Ven-Tyurin-Sato Theorem to hold on a locally complete ind-variety $X$.

In the linear case, besides the results from the 1970s and the important results of Sato [7], [8], in which he considers a case when the Barth-Van de Ven-Tyurin-Sato Theorem no longer holds, some more recent results belong to Donin and Penkov [4]. In particular, it was shown in [4] that the Barth-Van de Ven-Tyurin-Sato Theorem holds on any linear direct limit $G(\infty) = \lim \rightarrow G(k_m, \mathbb{C}^{n_m})$, where $G(k_m, \mathbb{C}^{n_m})$

We acknowledge the support and hospitality of the Max Planck Institute for Mathematics in Bonn where the present paper was conceived. We also acknowledge partial support from the DFG through Priority Program “Representation Theory” (SPP 1388) at Jacobs University Bremen.

AMS 2010 Mathematics Subject Classification. Primary 14M15; Secondary 14J60, 32L05.

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denotes the Grassmannian of $k_m$-dimensional subspaces in $\mathbb{C}^{n_m}$, under the assumption that $\lim_{m \to \infty} k_m = \lim_{m \to \infty} (n_m - k_m) = \infty$. It turns out that there is a single isomorphism class of such ind-varieties. Nevertheless, there are other natural homogeneous ind-varieties on which the Barth-Van de Ven-Tyurin-Sato Theorem holds but which have not been considered in the literature. This applies in particular to linear direct limits of isotropic (orthogonal or symplectic) Grassmannians, as well as to direct products of such direct limits.

For this reason we formulate a set of abstract conditions on a linear locally complete ind-variety $X$ which ensure that the Barth-Van de Ven-Tyurin-Sato Theorem (shortly, BVTS Theorem) holds. We then give many examples of ind-varieties $X$ satisfying these sufficient conditions. An interesting new class of such ind-varieties consists of direct limits $Y$ of linear sections $Y_m$ of $G(k_m, \mathbb{C}^{n_m})$, where

$$\lim_{m \to \infty} G(k_m, \mathbb{C}^{n_m}) = G(\infty).$$

Another class of ind-varieties on which the Barth-Van de Ven-Tyurin-Sato Theorem holds are certain ind-varieties of generalized flags: see §6.3.

Probably, there are more general sufficient conditions for the Barth-Van de Ven-Tyurin-Sato Theorem to hold on locally complete ind-varieties. In addition, for nonlinear locally complete ind-varieties nothing seems to be known beyond the results of [6]. Therefore, providing a sufficient condition for the Barth-Van de Ven-Tyurin-Sato Theorem to hold on general locally complete ind-varieties remains a project for the future.

§2. Linear ind-varieties. Statement of the main result

2.1. The ground field is $\mathbb{C}$. We use the term algebraic variety as a synonym for a reduced Noetherian scheme. If $E$ is a vector bundle (or simply a vector space), $E^*$ stands for the dual bundle (or dual space). We use the standard notation $\mathcal{O}_{\mathbb{P}^n}(a)$ for the line bundle $\mathcal{O}_{\mathbb{P}^n}(-1)^{\otimes -a}$, where $\mathcal{O}_{\mathbb{P}^n}(-1)$ is the tautological bundle on the complex $n$-dimensional projective space $\mathbb{P}^n$.

Recall that an ind-variety is the direct limit $X = \lim_{m \to \infty} X_m$ of a chain of morphisms of algebraic varieties

$$X_1 \xrightarrow{\phi_1} X_2 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{m-1}} X_m \xrightarrow{\phi_m} X_{m+1} \xrightarrow{\phi_{m+1}} \cdots.$$ (2.1)

Note that the direct limit of the chain (2.1) does not change if we replace the sequence $\{X_m\}_{m \geq 1}$ by a subsequence $\{X_{i_m}\}_{m \geq 1}$, and the morphisms $\phi_m$ by the compositions $\tilde{\phi}_{i_m} := \phi_{i_m+1} \circ \cdots \circ \phi_{i_m+1} \circ \phi_{i_m}$.

Let $X$ be the direct limit of (2.1) and $X'$ be the direct limit of a chain

$$X'_1 \xrightarrow{\phi'_1} X'_2 \xrightarrow{\phi'_2} \cdots \xrightarrow{\phi'_{m-1}} X'_m \xrightarrow{\phi'_m} X'_{m+1} \xrightarrow{\phi'_{m+1}} \cdots.$$ 

A morphism of ind-varieties $f: X \to X'$ is a map from $X$ to $X'$ induced by a collection of morphisms of algebraic varieties $\{f_m: X_m \to Y_{n_m}\}_{m \geq 1}$ such that $\tilde{\phi}'_{n_m} \circ f_m = f_{m+1} \circ \phi_m$ for all $m \geq 1$. The identity morphism $\text{id}_X$ is a morphism which coincides with the identity as a set-theoretic map from $X$ to $X$. A morphism
f : X → X' is an isomorphism if there exists a morphism g : X' → X such that g ∘ f = id_X and f ∘ g = id_{X'}.

In what follows we only consider chains (2.1) such that the X_m are complete algebraic varieties, \( \lim_{m→∞}(\dim X_m) = ∞ \), and the morphisms \( φ_m \) are embeddings. We call such ind-varieties locally complete. Furthermore, we call a morphism \( f : X = \lim X_m → X' = \lim X'_m \) of locally complete ind-varieties an embedding if all the morphisms \( f_m : X_m → X'_m \) are embeddings.

A vector bundle \( E \) of rank \( r ∈ ℤ_{>0} \) on \( X \) is the inverse limit \( \leftarrow \lim E_m \) of an inverse system of vector bundles \( E_m \) of rank \( r \) on \( X_m \), that is, of a system of vector bundles \( E_m \) with fixed isomorphisms \( ψ_m : E_m \cong φ_m E_{m+1} \); here and below \( φ^* \) stands for the inverse image of vector bundles under a morphism \( φ \). Clearly, \( E|_{X_m} \cong E_m, m \geq 1 \). In particular, the structure sheaf \( Ω_X = \leftarrow \lim Ω_{X_m} \) of an ind-variety \( X \) is well defined. By the Picard group \( \text{Pic}_X \) we understand the group of isomorphism classes of line bundles on \( X \). Clearly, \( \text{Pic}_X \) is the inverse limit \( \leftarrow \lim \text{Pic}_X \) of the system of Picard group homomorphisms \( \{ φ^*_m : \text{Pic}_X \cong φ_m \text{Pic}_{X_m} \}_{m \geq 1} \). In the rest of the paper we automatically assume that all vector bundles considered have finite rank. If \( E \) is a vector bundle on \( X \), \( rE \) stands for the direct sum \( E ⊕ \cdots ⊕ E \) of \( r \) copies of \( E \). A vector bundle \( E \) is trivial if it is isomorphic to \( rφ_X \), \( r = rk E \).

A linear ind-variety is an ind-variety \( X = \lim X_m \) such that, for all large enough \( m \geq 1 \), the induced homomorphisms of Picard groups \( φ^*_m : \text{Pic}_X \cong φ_m \text{Pic}_{X_m} \) are epimorphisms. A typical example of a linear ind-variety is the projective ind-space \( P^∞ \), which is the direct limit of a chain of linear embeddings

\[
\mathbb{P}^{n_1} φ_1 ↪ \mathbb{P}^{n_2} φ_2 ↪ \cdots φ_m ↪ \mathbb{P}^{n_m} φ_m ↪ \cdots
\]

for an arbitrary increasing sequence \( \{ n_m \}_{m \geq 1} \) of nonnegative integers. (It is easy to see that the definition of \( P^∞ \) does not depend, up to isomorphism of ind-varieties, on the choice of the sequence \( \{ n_m \}_{m \geq 1} \) and the embeddings \( φ_m \).) By a projective ind-subspace of an ind-variety \( X \) we understand the image of an embedding \( ψ : P^∞ ↪ X \).

Another example of a linear ind-variety is the ind-grassmannian \( G(∞) \) which is the direct limit of a chain of linear embeddings

\[
G(k_1, \mathbb{C}^{n_1}) φ_1 ↪ G(k_2, \mathbb{C}^{n_2}) φ_2 ↪ \cdots φ_m ↪ G(k_m, \mathbb{C}^{n_m}) φ_m ↪ \cdots
\]

where \( G(k_m, \mathbb{C}^{n_m}) \) is the grassmannian of \( k_m \)-dimensional subspaces in an \( n_m \)-dimensional vector space and \( \lim_{m→∞} k_m = lim_{m→∞}(n_m - k_m) = ∞ \).

2.2. Let \( X = \lim X_m \) be a linear ind-variety such that there is a finite or countable set \( Θ_X \) and a collection \( \{ L_i = \lim L_{im} \}_{i ∈ Θ_X} \) of nontrivial line bundles on \( X \) such that, for any \( m \), \( L_{im} ≅ Θ_{X_m} \) for all but finitely many indices \( i_1(m), \ldots, i_{j(m)}(m) \), and the images of \( L_{i_1(m)m}, \ldots, L_{i_{j(m)}(m)m} \) in \( \text{Pic}_X \) form a basis of \( \text{Pic}_X \), which is assumed to be a free abelian group. It is clear that in this case \( \text{Pic}_X \) is isomorphic to a direct product of infinite cyclic groups with generators the images of the \( L_i \). We let \( \bigotimes_{i∈Θ_X} L_i ⊗ a_i \) denote the linear bundle on \( X \) whose restriction to \( X_m \) is equal to

\[
\bigotimes_i L_{im} ⊗ a_i = L_{i_1(m)m} ⊗ \cdots ⊗ L_{i_{j(m)}(m)m}.
\]
We say that $X$ satisfies property L if, in addition to the above condition, $H^1(X_m, \bigotimes_i L_{i,m}^{\langle a_i \rangle}) = 0$ for any $m \geq 1$ if some $a_i$ is negative.

Let $X$ satisfy property L. For a given $i \in \Theta_X$, a smooth rational curve $C \simeq \mathbb{P}^1$ on $X$ is a projective line of the $i$th family on $X$ (or simply, a line of the $i$th family) if

$$L_j|_C \cong \mathcal{O}_{\mathbb{P}^1}(\delta_{ij}) \quad \text{for } j \in \Theta_X.$$  \hspace{1cm} (2.2)

By $B_i$ we denote the set of all projective lines of the $i$th family on $X$. It has a natural structure of an ind-variety: $B_i = \lim\limits_{\rightarrow} B_{im}$, where $B_{im} := \{ C \in B_i \mid C \subset X_m \}$ for $m \geq 1$. For any point $x \in X$ the subset $B_i(x) = \{ C \in B_i \mid C \ni x \}$ inherits an induced structure of an ind-variety.

Assume that $X$ satisfies property L. Then we say that $X$ satisfies property A if for any $i \in \Theta_X$ there is an ind-variety $\Pi_i$ which is the direct limit of a chain of embeddings

$$\{ \cdots \Pi_{i,m-1} \hookrightarrow \Pi_{i,m} \hookrightarrow \Pi_{i,m+1} \hookrightarrow \cdots \},$$

where points of $\Pi_{i,m}$ are projective subspaces $\mathbb{P}^{nm}$ of $B_{im}$ together with the linear embeddings $\mathbb{P}^{nm} \hookrightarrow \mathbb{P}^{nm+1} = \pi_{im}(\mathbb{P}^{nm})$ induced by embeddings $B_{im} \hookrightarrow B_{i,m+1}$, so that each point in $\Pi_i$ is a projective ind-subspace $\mathbb{P}^\infty = \lim\limits_{\rightarrow} \mathbb{P}^{nm}$ in $B_i$, and, for any point $x \in X$ the following conditions hold:

(A.i) for each $m \geq 1$ such that $x \in X_m$, the sheaf $L_{im}$ defines a morphism $\psi_{im} : X_m \to \mathbb{P}^{r_{im}} := \mathbb{P}(H^0(X_m,L_{im})^*)$ which maps the family of lines $B_{im}(x)$ isomorphically to a subfamily of lines in $\mathbb{P}^{r_{im}}$ passing through the point $\psi_{im}(x)$;

(A.ii) the variety $\Pi_{i,m}(x) := \{ \mathbb{P}^{nm} \in \Pi_{i,m} \mid \mathbb{P}^{nm} \subset B_{im}(x) \}$ is connected for any $m \geq 1$;

(A.iii) the projective ind-subspaces $\mathbb{P}^\infty \subset \Pi_i(x) := \lim\limits_{\rightarrow} \Pi_{i,m}(x)$ fill $B_i(x)$;

(A.iv) for any $d \in \mathbb{Z}_{\geq 1}$ there exists a $m_0(d) \in \mathbb{Z}_{\geq 1}$ such that, for any $d$-dimensional variety $Y$ and any $m \geq m_0(d)$, any morphism $\Pi_{i,m}(x) \to Y$ is a constant map.

In particular, (A.ii) and (A.iii) imply that the varieties $\Pi_{i,m}$, $B_{im}$ and $B_{im}(x)$ are connected.

Let $X$ satisfy properties L and A as above. A vector bundle $E$ on $X$ is called $B_i$-uniform if for any projective line $\mathbb{P}^1 \in B_i$ on $X$, the restricted bundle $E|_{\mathbb{P}^1}$ is isomorphic to $\bigoplus_{j=1}^{r_{i,E}} \mathcal{O}_{\mathbb{P}^1}(k_j)$ for some integers $k_j$ independent of the choice of $\mathbb{P}^1$. If, in addition, all the $k_j = 0$, then $E$ is called $B_i$-linearly trivial. We call $E$ uniform (respectively, linearly trivial) if it is $B_i$-uniform (respectively, $B_i$-linearly trivial) for each $i \in \Theta_X$. Moreover, we say that $X$ satisfies property T if each linearly trivial vector bundle on $X$ is trivial.

Our general version of the BVTS Theorem is the following:

**Theorem 1.** Let $E$ be a vector bundle on a linear ind-variety $X$.

(i) If $X$ has properties L and A for some fixed line bundles $\{L_i\}_{i \in \Theta_X}$ and corresponding families $\{B_i\}_{i \in \Theta_X}$ of projective lines on $X$, then $E$ has a filtration by vector subbundles

$$0 = E_0 \subset E_1 \subset \cdots \subset E_t = E$$

with uniform successive quotients $E_k/E_{k-1}$, $k = 1, \ldots, t$. 


(ii) If, in addition, $X$ has property $T$, then the filtration (2.3) splits and its quotients are of the form

$$E_k/E_{k-1} \cong \text{rk}(E_k/E_{k-1}) \bigotimes_{i \in \Theta_X} L_i^{\otimes a_{ik}}, \quad a_{ik} \in \mathbb{Z}, \quad i \in \Theta_X, \quad 1 \leq k \leq t.$$ 

In particular, $E$ is isomorphic to a direct sum of line bundles.

§ 3. Proof of the main theorem

3.1. Preliminaries on vector bundles. If $C \subset X$ is a smooth irreducible rational curve in an algebraic variety $X$ and $E$ is a vector bundle on $X$, then by a classical theorem often attributed to Grothendieck, $E|_C$ is isomorphic to $\bigoplus \mathcal{O}_C(\delta_i)$ for some $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_{rk E}$. We call the ordered $rk E$-tuple $(\delta_1, \ldots, \delta_{rk E})$ the splitting type of $E|_C$ and denote it by $\text{Split}(E|_C)$. We order the splitting types lexicographically, that is, $(\delta_1, \ldots, \delta_{rk E}) > (\delta'_1, \ldots, \delta'_{rk E})$ if $\delta_1 = \delta'_1, \ldots, \delta_{k-1} = \delta'_{k-1}, \delta_k > \delta'_k$ for some $k$, $1 \leq k \leq rk E$.

Let $X$ be a locally complete linear ind-variety satisfying properties L and A, and let $x \in X$ and $i \in \Theta_X$. In the notation of (A.i), let $\mathbb{P}^{r_{im}} = \mathbb{P}(H^0(X_m, L_{im})^*)$ and $y = \psi_{im}(x) = Cu$, $0 \neq u \in H^0(X_m, L_{im})^*$, so that

$$B_{im}(x) \subset \mathbb{P}^{r_{im}} := \mathbb{P}(H^0(X_m, L_{im})^*/Cu).$$

Fix a projective subspace $\mathbb{P}^{n_m} \subset B_{im}(x)$, where $\mathbb{P}^{n_m} \in \Pi_{im}(x)$. Then

$$\mathcal{O}_{\mathbb{P}(H^0(X_m, L_{im})^*/Cu)}(1)|_{\mathbb{P}^{n_m}} \simeq \mathcal{O}_{\mathbb{P}^{n_m}}(N(i))$$

for some $N(i) > 0$. Consider the locally closed subvariety

$$Y_m := \{(z, l) \in \mathbb{P}^{r_{im}} \times \mathbb{P}^{n_m} \mid z \in l \setminus \{y\}\}$$

of $\mathbb{P}^{r_{im}} \times \mathbb{P}^{n_m}$, and let $Y_m \hookrightarrow \mathbb{P}^{r_{im}}$ be the embedding induced by the projection $\mathbb{P}^{r_{im}} \times \mathbb{P}^{n_m} \to \mathbb{P}^{r_{im}}$. Then $Y_m$ is isomorphic to the total space of the line bundle $\mathcal{O}_{\mathbb{P}^{n_m}}(N(i))$ (see, for instance, [9], Appendix B) and

$$\mathcal{O}_{\mathbb{P}^{r_{im}}}(1)|_{Y_m} \simeq \tau_m^* \mathcal{O}_{\mathbb{P}^{n_m}}(N(i)), \quad (3.1)$$

where $\tau_m : Y_m \to \mathbb{P}^{n_m}$ is the natural projection. Moreover, by construction we have a commutative diagram of morphisms

$$\begin{array}{c}
\begin{array}{ccc}
\mathbb{P}^{r_{im}} & \xrightarrow{\varphi_y} & \mathbb{P}^{r_{im}} \\
Y_m & \xleftarrow{\tau_m} & Y_m \\
\end{array} \\
\begin{array}{ccc}
\mathbb{P}^{r_{im}} & \xrightarrow{\pi_y} & \mathbb{P}^{r_{im}} \\
\mathbb{P}^{r_{im}} & \xrightarrow{\pi_m} & \mathbb{P}^{n_m} \\
\end{array}
\end{array} \quad (3.2)$$

where $\tau_m : Y_m \hookrightarrow Y_m := Y_m \cup \{y\}$ is the inclusion, $\varphi_y : \mathbb{P}^{r_{im}} \to \mathbb{P}^{r_{im}}$ is the blow-up $\mathbb{P}^{r_{im}}$ with centre at $y$, and $\pi_y$ is the natural projection which is a $\mathbb{P}^1$-bundle. In addition, we have an open embedding

$$\iota_m : Y_m \hookrightarrow \tilde{Y} := \mathbb{P}_{y'}^{r_{im}-1} \times \mathbb{P}^{n_m},$$

and projections \( \mathbf{Y}_m \xrightarrow{\pi} \mathbf{Y}_m \xrightarrow{\tau} \mathbb{P}^{n_m} \) such that
\[
\tau_m = \pi_m \circ l_m, \quad \pi_m = \tau_m \circ l_m. \tag{3.3}
\]
By (A.i), \( \psi_{im} : \psi_{im}^{-1}(\mathbf{Y}_m) \to \overline{\mathbf{Y}}_m \) is an isomorphism. Hence we may regard \( \mathbf{E}|_{\psi_{im}^{-1}(\mathbf{Y}_m)} \) as a vector bundle on \( \overline{\mathbf{Y}}_m \) and denote it by \( \mathbf{E}|_{\overline{\mathbf{Y}}_m} \). We also set \( \mathbf{E}|_{\overline{\mathbf{Y}}_m} := \tau_m^*(\mathbf{E}|_{\mathbf{Y}_m}) \).

For an arbitrary projective line \( \mathbb{P}^1 \subset \mathbb{P}^{n_m} \), we consider the surface \( S = S(x, \mathbb{P}^1) := \pi_y^{-1}(\mathbb{P}^1) \) with the natural projections \( \pi_S := \pi_y|_S : S \to \mathbb{P}^1 \) and \( \sigma_S := \phi_y|_S : S \to \mathbf{X} \). It follows from (3.1) that \( S \) is a surface of type \( F_{N(i)} \).

Let \( \mathbf{E} \) be a vector bundle of rank \( r \) on \( \mathbf{X} \). For any \( i \in \Theta \mathbf{X} \) and \( x \in \mathbf{X} \) we set \( C(i) := c_1(\mathbf{E}|_l) \in \mathbb{Z} \), where \( c_1 \) stands for first Chern class and \( l \in \mathcal{B}_i(x) \). Since \( \mathcal{B}_i(x) \) is connected, \( C(i) \) is well defined. Furthermore, we have \( \delta_1(\mathbf{E}|_l) \geq C(i)/r \geq \delta_B(\mathbf{E}|_l) \).

Hence there are well-defined integers
\[
\delta_1^\min := \min_{l \in \mathcal{B}_i(x)} \delta_1(\mathbf{E}|_l), \quad \delta_1^\max := \max_{l \in \mathcal{B}_i(x)} \delta_1(\mathbf{E}|_l),
\]
and there exist lines \( l_{\min}, l_{\max} \in \mathcal{B}_i(x) \) such that
\[
\delta_1(\mathbf{E}|_{l_{\min}}) = \delta_1^\min, \quad \delta_{rk} E(\mathbf{E}|_{l_{\max}}) = \delta_{rk} E.
\]
The inequality \( C(i) \geq \delta_1^\min + (r - 1) \delta_{rk} E(\mathbf{E}|_{l_{\min}}) \) implies
\[
\delta_1^\min - \delta_{rk} E(\mathbf{E}|_{l_{\min}}) \leq \delta_1^\min - \frac{C(i)}{r - 1}, \quad \delta_1(\mathbf{E}|_{l_{\max}}) - \delta_{rk} E \geq \frac{C(i)}{r - 1} - \delta_{rk} E. \tag{3.4}
\]

Fix \( \mathbf{P}^\infty \in \mathcal{B}_i(x) \) and \( l_{\min} \in \mathbf{P}^\infty \). For an arbitrary point \( l_0 \in \mathbf{P}^\infty \setminus \{l_{\min}\} \) consider the line \( \mathbf{P}^1 = \text{Span}(l_0, l_{\min}) \) in \( \mathbf{P}^\infty \) and the corresponding surface \( S = S(x, \mathbf{P}^1) \) together with the vector bundle \( \mathbf{E}_S := \sigma^*_S \mathbf{E} \) on \( S \). For a general point \( l \in \mathbf{P}^1 \) (\( l \) is a line on \( \mathbf{X} \)), the first inequality in (3.4) implies
\[
\delta_{\text{gen}} := \delta_1(\mathbf{E}|_l) - \delta_{rk} E(\mathbf{E}|_l) \leq \delta_1^\min - \frac{C(i)}{r - 1}. \tag{3.5}
\]

The following lemma is a straightforward consequence of a result of Tyurin.

**Lemma 1.** There exist polynomials \( P_A, P_B \in \mathbb{Q}[x_1, \ldots, x_6] \) such that for any \( l_0 \in \mathbf{P}^\infty \setminus \{l_{\min}\} \)
\[
\delta_1(\mathbf{E}|_{l_0}) \leq P_A(\mathbf{r}, \delta_1^\min, C(i), N(i), c^2_1(E_S), c_2(E_S)) =: P_A(\mathbf{E}, i), \tag{3.6}
\]
\[
\delta_{rk} E(\mathbf{E}|_{l_0}) \geq P_B(\mathbf{r}, \delta_{rk}^\max, C(i), N(i), c^2_1(E_S), c_2(E_S)) =: P_B(\mathbf{E}, i), \tag{3.7}
\]
where \( c^2_1(E_S) \) and \( c_2(E_S) \) are treated as integers.

**Proof.** By construction, \( S \) is a surface of type \( F_{N(i)} \). Hence, repeating the proof of Lemma 5 in [2], Ch. 2, §1 for the vector bundle \( E_S \) we obtain that there exists a polynomial \( f \in \mathbb{Q}[x_1, \ldots, x_6] \) such that
\[
\delta_1(\mathbf{E}|_{l_0}) \leq f(\mathbf{r}, \delta_1^\min, \delta_{\text{gen}}, N(i), c^2_1(E_S), c_2(E_S)).
\]
Thus, in view of (3.5), there exists a polynomial \( P_A \in \mathbb{Q}[x_1, \ldots, x_6] \) satisfying (3.6). The proof of (3.7) is similar.
The next proposition employs in a crucial way results of Sato. Fix $i \in \Theta_X$, $x \in X$ and $\mathbb{P}^m \in \Pi_{im}(x)$ for a large enough $m$. In view of (3.6), there exists a maximal (with respect to lexicographic order) splitting type $S_i(\mathbb{E}, \mathbb{P}^m):=\max_{l \in \mathbb{P}^m} \text{Split}(\mathbb{E}|l)$.

**Proposition 1.** The maximal splitting type $S_i(\mathbb{E}, \mathbb{P}^m)$ depends only on the pair $(\mathbb{E}, i)$, that is, $S_i(\mathbb{E}, \mathbb{P}^m)$ is independent of $x$ and $\mathbb{P}^m \in \Pi_{im}(x)$.

**Proof.** Set

$$M_i(\mathbb{P}^m) := \{l \in \mathbb{P}^m \mid \text{Split}(\mathbb{E}|l) = S_i(\mathbb{E}, \mathbb{P}^m)\}.$$ 

The semicontinuity of $\text{Split}(\mathbb{E}|l)$ implies that $M_i(\mathbb{P}^m)$ is a closed subvariety of $\mathbb{P}^m$. Moreover, Lemma 1, together with [2], Ch. 2, §2, Lemmas 3 and 4, yields the inequality

$$\text{codim}_{\mathbb{P}^m} M_i(\mathbb{P}^m) \leq r(r - 1)(P_A(\mathbb{E}, i) - P_B(\mathbb{E}, i)).$$

Consider the upper row of the diagram (3.2). Since the right-hand side of (3.8) is constant with respect to $m$, for large enough $m$ we have

$$\text{codim}_{\mathbb{P}^m} M_i(\mathbb{P}^m) < \min\left(n_m - r, \frac{n_m - 2r^2}{2}\right).$$

Also, clearly for large enough $m$

$$\text{codim}_{\overline{Y}_m}\{x\} = \text{codim}_{\overline{Y}_m}(\overline{Y}_m \setminus Y_m) > r.$$ 

Inequality (3.10) shows that

$$c_k(\mathbb{E}|Y_m) = \tau^*_m c_k(\mathbb{E}|\overline{Y}_m), \quad 0 \leq k \leq r,$$

where $c_k(\cdot)$ stands for the $k$th Chern class. Moreover, since the Chow group of codimension $k$ of the base of an arbitrary vector bundle pulls back isomorphically to the Chow group of codimension $k$ of the total space of the bundle, we have

$$c_k(\mathbb{E}|Y_m) = \pi^*_m (c_k H^k), \quad 0 \leq k \leq r,$$

where $H$ is the class of a hyperplane divisor on $\mathbb{P}^m$, $c_0 = 1$ and $c_1, \ldots, c_r$ are integers. It is essential to note that the obvious compatibility of the morphisms $\pi_m$ for varying $m$ and the functoriality of Chern classes imply that these integers do not depend on $x$ and on $\mathbb{P}^m \in \Pi_{im}(x)$. Note also that (3.1), (3.3) and equalities (3.11) and (3.12) imply

$$\ell_m^* c_k(\tau_m^* (\mathbb{E} \otimes L_i^{-a}|\overline{Y}_m)) = \tau^*_m c_k(\mathbb{E} \otimes L_i^{-a}|\overline{Y}_m)$$

$$= c_k(\mathbb{E}|Y_m) \otimes \pi^*_m O_{\mathbb{P}^m}(-N(i)aH)), \quad 0 \leq k \leq r, \quad a \in \mathbb{Z}.$$

Next, consider the polynomial

$$h(t) = \sum_{k=0}^r c_k(-t)^{r-k} \in \mathbb{Z}[t],$$

where $c_0 = 1$ and the coefficients $c_k$ for $k \geq 0$ are the integers introduced above. Closely following an idea of Sato, we will now argue that the roots of $h(t)$ constitute a constant multiple of the maximal splitting type $S_i(\mathbb{E}, \mathbb{P}^m)$. More precisely,
let \( a_1 > \ldots > a_\alpha, \alpha \leq r \), be the distinct elements of \( S_i(E, \mathbb{P}^n_m) \) of respective multiplicities \( r_1, \ldots, r_\alpha \) in \( S_i(E, \mathbb{P}^n_m) \). Then we claim that the roots of \( h(t) \) are \( N(i)a_1, \ldots, N(i)a_\alpha \) of respective multiplicities \( r_1, \ldots, r_\alpha \).

The argument in [3], pp. 138–139 shows that in order to prove this claim it suffices to establish the vanishing of

\[
 c_k\left(E|_{\gamma_m} \otimes \pi^*_m \mathcal{O}_{\mathbb{P}^n_m}(-N(i)a_jH)\right)
\]

for \( r - r_j + 1 \leq k \leq r, 1 \leq j \leq \alpha \). By (3.13) it is enough to prove the vanishing of the \( c_k(\tilde{r}_m^* (E \otimes L_i^{-a_j}|_{\gamma_m}) ) \). However, the proof of this fact is practically the same as in [3]. Namely, one defines inductively vector bundles

\[
 F_1 := \tilde{\phi}_m^* E|_{\pi_m^{-1}(M_i(\mathbb{P}^n_m))}, \quad F_2, \ldots, F_\alpha
\]

of ranks \( \text{rk } F_j = \sum_{p=j}^\alpha r_p \) on \( \pi_m^{-1}(M_i(\mathbb{P}^n_m)) \) which fit into the exact triples

\[
 0 \rightarrow r_j \mathcal{O}_{\pi_m^{-1}(M_i(\mathbb{P}^n_m))} \rightarrow F_j \otimes (L_i^{a_j-1-a_j}|_{\pi_m^{-1}(M_i(\mathbb{P}^n_m))}) \rightarrow F_{j+1} \rightarrow 0,
\]

where \( a_0 := 0 \). Using (3.9) and applying the argument in [3], p. 139 to the triples (3.15) we obtain \( c_k(\tilde{r}_m^* (E \otimes L_i^{-a_j}|_{\gamma_m}) ) = 0 \) as desired. Since \( h(t) \) is independent of \( x \) and \( \mathbb{P}^n_m \in \Pi_{im}(x) \), the same applies to \( S_i(E, \mathbb{P}^n_m) \), and the proposition is proved.

### 3.2. Proof of Theorem 1.

**Proof.** According to (3.9), for \( m \gg 0 \) the dimension of \( M_i(\mathbb{P}^n_m) \) is greater than half the dimension of \( \mathbb{P}^n_m \), hence the varieties \( M_i(\mathbb{P}^n_m) \) are connected for large enough \( m \). Consider the variety

\[
 \Gamma_{im}(x) := \{ (l, \mathbb{P}^n_m) \in B_{im}(x) \times \Pi_{im}(x) \mid l \in M_i(\mathbb{P}^n_m) \}
\]

with projections \( B_{im}(x) \xrightarrow{p_1} \Gamma_{im}(x) \xrightarrow{p_2} \Pi_{im}(x) \). Since \( M_i(\mathbb{P}^n_m) = p_2^{-1}(\mathbb{P}^n_m) \) is connected for any \( \mathbb{P}^n_m \in \Pi_{im}(x) \) and \( \Pi_{im}(x) \) is connected by (A.ii), it follows that \( \Gamma_{im}(x) \) is connected. By definition, \( \Gamma_{im}(x) \) is described as

\[
 \Gamma_{im}(x) = \bigsqcup_{\mathbb{P}^n_m \in \Pi_{im}(x)} M_i(\mathbb{P}^n_m), \quad m \geq 1.
\]

(3.16)

In a similar way we consider the varieties

\[
 \Gamma_{im} := \{ (x, l, \mathbb{P}^n_m) \in X_m \times B_{im} \times \Pi_{im} \mid (l, \mathbb{P}^n_m) \in \Gamma_{im}(x) \}.
\]

By construction

\[
 \Gamma_{im} = \bigsqcup_{x \in X_m} \Gamma_{im}(x),
\]

so that each \( \Gamma_{im} \) is connected. Moreover, there is a well-defined ind-variety \( \Gamma_i := \lim_{\longrightarrow} \Gamma_{im} \) and \( \Gamma_i(x) := \lim_{\longrightarrow} \Gamma_{im}(x), x \in X \).
Let \((x, l, \mathbb{P}^{n_m}) \in \Gamma_i\). If \(\delta_i^{\max}\) is the maximal entry of \(\text{Split}(E_{|l}|) = S_i(E, \mathbb{P}^{n_m})\), there is a well-defined subbundle \(E_1(l)\) of \(E_{|l}|\:

\[ E_1(l) := \text{im}(H^0(l, E_{|l}|(-\delta_i^{\max})) \otimes \mathcal{O}_l(\delta_i^{\max})). \]  

(3.17)

Set \(r_1 := \text{rk}\ E_1(l)\) and consider the relative grassmannian \(\rho_1: G(r_1, E) \to X\). According to Proposition 1, \(\delta_i^{\max}\) and \(r_1\) are independent of the point \((x, l, \mathbb{P}^{n_m}) \in \Gamma_i\).

Thus there is a morphism of ind-varieties

\[ f_{i1} : \Gamma_i \to G(r_1, E), \quad (x, l) \mapsto E_1(l)|_x, \quad x \in X. \]  

(3.18)

Since \(f_{i1}(\Gamma_{im}(x)) \subset \rho_1^{-1}(x) = G(r_1, E|_x)\), by (3.16) we have

\[ f_{i1}(M_i(\mathbb{P}^{n_m})) \subset G(r_1, E|_x), \quad \mathbb{P}^{n_m} \in \Pi_{im}(x). \]

According to (3.8), the codim\(\mathbb{P}^{n_m} M_i(\mathbb{P}^{n_m})\) are bounded as \(m \to \infty\). This means that, for large enough \(m\) the morphism \(f_{i1}: M_i(\mathbb{P}^{n_m}) \to G(r_1, E|_x)\) satisfies the assumptions of Proposition 3.2 in [3], in which we set \(n = n_m, X = M_i(\mathbb{P}^{n_m}), Y = G(r_1, E|_x)\) and \(f = f_{i1}\). By this proposition \(f_{i1}|_{M_i(x)}\) is a constant map, hence it induces a morphism

\[ \phi_{i1}(x) : \Pi_{im}(x) \to G(r_1, E|_x), \quad x \in X. \]

Now (A.iv) implies that there exists a positive integer \(m_1\) such that the morphism \(\phi_{i1}(x)\) is a constant map for any \(m \geq m_1\). We thus obtain that (3.18) induces a constant morphism

\[ \phi_{i1}(x) : \Sigma_i(x) \to G(r_1, E|_x). \]

Consider the ind-variety \(\Sigma_i = \lim \Sigma_{im},\) where

\[ \Sigma_{im} := \{(x, \mathbb{P}^{n_m}) \in X_m \times \Pi_{im} | \mathbb{P}^{n_m} \in \Pi_{im}(x)\}, \]

and let \(p_i : \Sigma_i \to X\) be the natural projection with fibre \(\Pi_i(x), x \in X\). The above constant morphisms \(\phi_{i1}(x)\) extend to a morphism \(\Phi_{i1} : \Sigma_i \to G(r_1, E),\) which is constant on the fibres of \(p_i\). In addition, the morphism \(f_{i1} : M_i(x)\) is a constant map. We thus obtain a well-defined morphism

\[ \Phi_{i1} : X \to G(r_1, E), \quad x \mapsto f_{i1}(\Gamma_i(x)). \]  

(3.19)

Let \(\mathcal{S}\) be the tautological bundle of rank \(r_1\) on \(G(r_1, E)\). Set \(E_{1i} := \Phi_{i1}^+\mathcal{S}\). It follows now from (3.17)–(3.19) that \(E_{1i}\) is a subbundle of \(E\) such that

\[ E_{1i}|_l = E_1(l) \simeq r_1 \mathcal{O}_l(\delta_i^{\max}), \quad l \in q_i(\Gamma_i), \]

where \(q_i : \Gamma_i \to B_i : (x, l, \mathbb{P}^{n_m}) \mapsto l\) is the natural projection. Using the semi-continuity of \(\dim H^0(l, E_{1i}(-\delta_1^{\max})|_l)\) one checks immediately that the last equality is true for any \(l \in B_i\).

Applying the above argument to the quotient \(E' = E/E_{1i}\) etc., we obtain a filtration of the bundle \(E\)

\[ 0 \subset E_{1i} \subset E_{2i} \subset \cdots \subset E_{\alpha_i} = E_{1i} \]

with \(B_i\)-uniform successive quotients \(F_{ki} = E_{ki}/E_{k-1,i}\).
Now fix \( j \in \Theta_X, i \neq j \). Applying the same procedure to all bundles \( F_{ki} \) we obtain a bundle filtration of \( E \) whose quotients are \( B_i \)-uniform and \( B_j \)-uniform. After finitely many iterations we finally obtain a filtration

\[
0 = E_0 \subset E_1 \subset \cdots \subset E_s = E
\]

(3.20)
of \( E \) with uniform successive quotients. This yields (i).

Note that any uniform vector bundle on \( X \) becomes linearly trivial after twisting by an appropriate line bundle. This means that each successive quotient \( E_k/E_{k-1} \) is isomorphic to \( M_k \otimes F_k \), where \( M_k \) is a line bundle and \( F_k \) is linearly trivial. In addition, assume that property T is satisfied. Then the bundles \( F_k \) are trivial, that is,

\[
E_k/E_{k-1} \cong \text{rk}(E_k/E_{k-1})M_k, \quad 1 \leq k \leq s.
\]

Furthermore, for \( p < k \)

\[
\text{Ext}^1(M_k, M_p) = H^1(X, M_k^* \otimes M_p),
\]

and according to a well-known fact ([10], Theorem 4.5; see also [11], Proposition 10.3)

\[
H^1(X, M_k^* \otimes M_p) = \lim H^1(X_m, (M_k^* \otimes M_p)|_{X_m}).
\]

However, the above construction shows that

\[
(M_k^* \otimes M_p)|_{X_m} \cong \bigotimes_i L_{im_i}^{a_i}
\]

with some \( a_i \) negative. Therefore the vanishing part of property L yields

\[
H^1(X_m, (M_k^* \otimes M_p)|_{X_m}) = 0
\]

for \( m \geq 1 \), and hence \( \text{Ext}^1(M_k, M_p) = 0 \) for \( 1 \leq p \leq k \leq s \). This is sufficient to conclude that the filtration (3.20) splits, that is, (ii) follows.

The rest of the paper (with the exception of the appendix) is devoted to examples of linear ind-varieties satisfying properties L, A and T.

§ 4. Linear ind-grassmannians satisfying properties L, A and T

4.1. Finite-dimensional orthogonal and symplectic grassmannians. Let \( V \) be a finite-dimensional vector space. In what follows we will consider both symmetric and symplectic quadratic forms \( \Phi \) on \( V \). Under the assumption that \( \Phi \) is fixed, for any subspace \( W \subset V \) we set \( W^\perp := \{ v \in V \mid \Phi(v, w) = 0 \text{ for each } w \in W \} \). Recall that \( W \) is isotropic (or \( \Phi \)-isotropic) if \( W \subset W^\perp \).

Let \( \Phi \in S^2V^* \) be a nondegenerate symmetric form on \( V \). For \( \dim V \geq 3 \) and \( 1 \leq k \leq \left[ \frac{\dim V}{2} \right] \), the orthogonal grassmannian \( GO(k, V) \) is defined as the subvariety of \( G(k, V) \) consisting of all \( \Phi \)-isotropic \( k \)-dimensional subspaces of \( V \). Unless \( \dim V = 2n, k = n, GO(k, V) \) is a smooth irreducible variety. For \( \dim V = 2n, k = n, GO(k, V) \) is smooth and has two irreducible components, both of which are isomorphic to \( GO(n-1, \tilde{V}) \) for \( \dim \tilde{V} = 2n - 1 \).

If \( \Phi \in \Lambda^2 V^* \) is a nondegenerate symplectic form on \( V \), \( \dim V = 2n \), we recall that the symplectic grassmannian \( GS(k, V) \) is a smooth irreducible subvariety of \( G(k, V) \) consisting of all \( \Phi \)-isotropic \( k \)-dimensional subspaces of \( V \).
4.2. Definition of linear ind-grassmannians. We start by recalling the definition of standard extensions of grassmannians [12].

By a standard extension of grassmannians we understand an embedding of grassmannians \( f : G(k, V) \to G(k, V') \) for \( \dim V \geq \dim V, k' \geq k \), given by the formula
\[
    f : V_k \hookrightarrow V_k \oplus W,
\]
for some fixed isomorphism \( V' \cong V \oplus \hat{W} \) and a fixed subspace \( W \subset \hat{W} \) of dimension \( k' - k \). Correspondingly, by a standard extension of orthogonal (respectively, symplectic) grassmannians \( f : GO(k, V) \to GO(k, V') \) (respectively, \( f : GS(k, V) \to GS(k, V') \)) given by (4.1) for some fixed orthogonal (respectively, symplectic) isomorphism \( V' \cong V \oplus \hat{W} \) and a fixed isotropic subspace \( W \subset \hat{W} \) of dimension \( k' - k \); cf. [12], Definitions 3.2 and 3.5. Note that standard extensions are linear morphisms.

Next we recall the definition of a standard ind-grassmannian [12].

**Definition 1.** Fix an infinite chain of vector spaces
\[
    V_{n_1} \subset V_{n_2} \subset \cdots \subset V_{n_m} \subset V_{n_{m+1}} \subset \cdots
\]
(4.2)
of dimensions \( n_m, n_m < n_{m+1} \).

a) For an integer \( k, 1 \leq k < n_1 \), set \( G(k) := \lim_{\to} G(k, V_{n_m}) \), where
\[
    G(k, V_{n_1}) \hookrightarrow G(k, V_{n_2}) \hookrightarrow \cdots \hookrightarrow G(k, V_{n_m}) \hookrightarrow G(k, V_{n_{m+1}}) \hookrightarrow \cdots
\]
is the chain of canonical inclusions of grassmannians induced by (4.2).

b) For a sequence of integers \( 1 \leq k_1 < k_2 < \cdots \) such that \( k_m < n_m \), \( \lim_{m \to \infty} (n_m - k_m) = \infty \), set \( G(\infty) := \lim_{\to} G(k_m, V_{n_m}) \), where
\[
    G(k_1, V_{n_1}) \hookrightarrow G(k_2, V_{n_2}) \hookrightarrow \cdots \hookrightarrow G(k_m, V_{n_m}) \hookrightarrow G(k_{m+1}, V_{n_{m+1}}) \hookrightarrow \cdots
\]
is an arbitrary chain of standard extensions of grassmannians.

c) Assume that the \( V_{n_m} \) are endowed with compatible nondegenerate symmetric (respectively, symplectic) forms \( \Phi_m \). In the symmetric case \( n_m/2 \in \mathbb{Z}_+ \). For an integer \( k, 1 \leq k \leq [n_1/2] \), set \( GO(k, \infty) := \lim_{\to} GO(k, V_{n_m}) \) (respectively, \( GS(k, \infty) := \lim_{\to} GS(k, V_{n_m}) \)), where
\[
    GO(k, V_{n_1}) \hookrightarrow GO(k, V_{n_2}) \hookrightarrow \cdots \hookrightarrow GO(k, V_{n_m}) \hookrightarrow GO(k, V_{n_{m+1}}) \hookrightarrow \cdots
\]
(respectively,
\[
    GS(k_1, V_{n_1}) \hookrightarrow GS(k_2, V_{n_2}) \hookrightarrow \cdots \hookrightarrow GS(k_m, V_{n_m}) \hookrightarrow GS(k_{m+1}, V_{n_{m+1}}) \hookrightarrow \cdots
\]
(4.3)
is the chain of inclusions of isotropic grassmannians induced by (4.2).

d) For a sequence of integers \( 1 \leq k_1 < k_2 < \cdots \) such that \( k_m < [n_m/2] \), \( \lim_{m \to \infty} ([n_m/2] - k_m) = \infty \) set \( GO(\infty, \infty) := \lim_{\to} GO(k_m, V_{n_m}) \) (respectively, \( GS(\infty, \infty) := \lim_{\to} GS(k_m, V_{n_m}) \)), where
\[
    GO(k_1, V_{n_1}) \hookrightarrow GO(k_2, V_{n_2}) \hookrightarrow \cdots \hookrightarrow GO(k_m, V_{n_m}) \hookrightarrow GO(k_{m+1}, V_{n_{m+1}}) \hookrightarrow \cdots
\]
(4.3)
(respectively,
\[
    GS(k_1, V_{n_1}) \hookrightarrow GS(k_2, V_{n_2}) \hookrightarrow \cdots \hookrightarrow GS(k_m, V_{n_m}) \hookrightarrow GS(k_{m+1}, V_{n_{m+1}}) \hookrightarrow \cdots
\]
(4.4)
is an arbitrary chain of standard extensions of isotropic grassmannians.
e) In the symplectic case, consider a sequence of integers \(1 \leq k_1 < k_2 < \cdots\) such that \(k_m \leq n_m/2\), \(\lim_{m \to \infty} (n_m/2 - k_m) = k \in \mathbb{N}\), and set \(\text{GS}(\infty, k) := \lim \text{GS}(k_m, V_{n_m})\) for any chain of standard extensions (4.4). In the orthogonal case, assume first that the \(\dim V_{n_m}\) are even. Then set \(\text{GO}^0(\infty, k) := \lim \text{GO}(k_m, V_{n_m})\) for a chain (4.3) where \(k_m < n_m/2\), \(\lim_{m \to \infty} (n_m/2 - k_m) = k \in \mathbb{N}\), \(k \geq 2\). Finally, consider the orthogonal case under the assumption that the \(\dim V_{n_m}\) are odd. Then set \(\text{GO}^1(\infty, k) := \lim \text{GO}(k_m, V_{n_m})\) for a chain (4.3) where \(k_m \leq [n_m/2]\), \(\lim_{m \to \infty} ([n_m/2] - k_m) = k \in \mathbb{N}\).

In particular, \(\mathbf{P}^\infty = \mathbf{G}(1) \simeq \text{GS}(1)\). Note that the above standard ind-grassmannians are well-defined, that is, a standard ind-grassmannian does not depend, up to an isomorphism of ind-varieties, on the specific chain of standard embeddings used in its definition. Furthermore, the main result of [12] claims that, with the exception of the isomorphism \(\mathbf{P}^\infty \simeq \text{GS}(1)\), the standard ind-grassmannians are pairwise nonisomorphic as ind-varieties.

In all cases the maximal exterior power of a tautological bundle generated by its global sections yields an ample line bundle \(\mathcal{O}_{X_m}(1)\), where \(X_m = \mathbf{G}(k_m, V_{n_m})\), \(\text{GO}(k_m, V_{n_m})\), \(\text{GS}(k_m, V_{n_m})\). It is well known that \(\mathcal{O}_{X_m}(1)\) generates Pic \(X_m\). Moreover, if \(i_m : X_m \hookrightarrow X_{m+1}\) is one of the embeddings in Definition 1, there is an isomorphism \(i_m^* \mathcal{O}_{X_{m+1}}(1) \simeq \mathcal{O}_{X_m}(1)\). This allows us to conclude that \(X = \lim \mathcal{O}_{X_m}(1)\)

4.3. BVTS Theorem for \(\mathbf{G}(\infty), \text{GO}(\infty, \infty), \text{GS}(\infty, \infty), \text{GO}^1(\infty, 0)\) and \(\text{GS}(\infty, 0)\). We first note that if \(X = \mathbf{G}(k), \text{GO}(k, \infty), \text{GS}(k, \infty)\), there is a tautological rank-\(k\) bundle \(\mathbf{S}\) on \(X\). If \(k \geq 2\), this bundle is not isomorphic to a direct sum of line bundles, hence the BVTS Theorem does not hold for these ind-grassmannians. Moreover, it is known [7] that, for \(X = \mathbf{G}(k), \text{GO}(k, \infty), X = \text{GS}(k, \infty)\), any simple vector bundle of finite rank on \(X\), that is, a vector bundle which does not have a nontrivial proper subbundle, is a direct summand in a tensor power of \(\mathbf{S}\).

**Theorem 2.** Any vector bundle \(E\) on \(X \simeq \mathbf{G}(\infty), \text{GO}(\infty, \infty), \text{GS}(\infty, \infty), \text{GO}^1(\infty, 0), \text{GS}(\infty, 0)\) is isomorphic to \(\bigoplus_i \mathcal{O}_X(k_i)\) for some \(k_i \in \mathbb{Z}\).

For \(X = \mathbf{G}(\infty)\) this was proved in [4] (see also [5], §4). For the remaining standard ind-grassmannians the claim of Theorem 2 follows from Theorem 1 and the following theorem.

**Theorem 3.** Let \(X \simeq \text{GO}(\infty, \infty), \text{GS}(\infty, \infty), \text{GO}^1(\infty, 0)\) or \(\text{GS}(\infty, 0)\). Then \(X\) satisfies properties L, A and T.\(^1\)

**Proof.** \(X\) satisfies property L as \(\mathcal{O}_X(1)\) generates Pic \(X\) and \(H^1(X_m, \mathcal{O}_{X_m}(a))\) vanishes for all \(a\) and sufficiently large \(m\) by the Borel-Weil-Bott Theorem. Furthermore, property T follows from Proposition 9 below.

It remains to establish property A. Part (A.i) holds here simply because \(\mathcal{O}_X(1)\) is very ample. We therefore discuss parts (A.ii)–(A.iv).

Let \(X = \text{GO}(\infty, \infty) = \lim \text{GO}(k_m, V_{n_m})\). For \(m \geq 1\), the base \(B_m\) of the family of projective lines on \(\text{GO}(k_m, V_{n_m})\) coincides with the variety of isotropic flags of

\(^1\)The reader can check that \(\mathbf{G}(\infty)\) also satisfies properties L, A and T.
type \((k_m - 1, k_m + 1)\) in \(V_{n_m}^\ast\):

\[
B_m = \{(V_{k_m - 1}, V_{k_m + 1}) \in GO(k_m - 1, V_{n_m}) \times GO(k_m + 1, V_{n_m}) \mid V_{k_m - 1} \subset V_{k_m + 1}\}
\]

(see [12], Lemma 2.2, (i)). Furthermore, set

\[
\Pi_m := \{(V_{k_m}, V_{k_m + 1}) \in GO(k_m, V_{n_m}) \times GO(k_m + 1, V_{n_m}) \mid V_{k_m} \subset V_{k_m + 1}\}.\]

(4.6)

A point \(y = (V_{k_m}, V_{k_m + 1}) \in \Pi_m\) corresponds to the projective subspace

\[
G(k_m - 1, V_{k_m}) \times \{V_{k_m + 1}\} \subset B_m.
\]

It is easy to see that \(B := \lim \Pi_m\) is a well-defined ind-variety and that a point of \(\Pi\) represents a projective ind-subspace of \(B := \lim B_m\).

Next, (4.5), together with [12], Lemma 2.2, (iv), implies that for any points \(x = \{V_{k_m}\} \in GO(k_m, V_{n_m})\)

\[
B_\bar{m}(x) := \{\mathbb{P}^1 \in B_\bar{m} \mid \mathbb{P}^1 \ni x\} \simeq \mathbb{P}((\phi_{\bar{m}-1} \circ \cdots \circ \phi_m)(V_{k_m})^\ast)
\]

\[
\times GO(1, (\phi_{\bar{m}-1} \circ \cdots \circ \phi_m)(V_{k_m})^\perp / (\phi_{\bar{m}-1} \circ \cdots \circ \phi_m)(V_{k_m})), \quad \bar{m} \geq m,
\]

(4.7)

and

\[
\Pi_\bar{m}(x) := \{(\phi_{\bar{m}-1} \circ \cdots \circ \phi_m)(V_{k_m}), V_{k_m + 1}\} \in \Pi_\bar{m}
\]

\[
\simeq GO(1, V_{k_m}^\perp / V_{k_m}), \quad \bar{m} \geq m.
\]

(4.8)

Since the quadrics \(GO(1, (\phi_{\bar{m}-1} \circ \cdots \circ \phi_m)(V_{k_m})^\perp / (\phi_{\bar{m}-1} \circ \cdots \circ \phi_m)(V_{k_m}))\) are connected, (A.ii) follows from (4.5). Furthermore, as for each \(\bar{m} \geq 1\) the variety \(GO(1, V_{k_m}^\perp / V_{k_m})\) is a smooth quadric hypersurface in the projective space \(\mathbb{P}^n_{\bar{m} - 2k_m - 1}\), (4.7) and (4.8) imply (A.iii) and (A.iv) directly.

In the remaining cases the same argument goes through if one makes the following modifications.

If \(X = GS(\infty, \infty)\), the formulae for \(B_m\), \(\Pi_m\) and \(\Pi_\bar{m}(x)\) are the same as (4.5), (4.6), (4.7) and (4.8), respectively, with \(GO\) substituted by \(GS\) (use [15], Lemma 2.5). Note also that \(GS(1, V_{k_m}^\perp / V_{k_m})\) is isomorphic to the projective space \(\mathbb{P}(V_{k_m}^\perp / V_{k_m})\).

For

\[
X = GO^\ast(\infty, 0) = \lim GO(k_m, V_{2k_m + 1})
\]

we first identify \(GO(k_m, V_{2k_m + 1})\) with an irreducible component \(GO(k_m + 1, V_{2k_m + 2})^\ast\) of \(GO(k_m + 1, V_{2k_m + 2})\) (see [12], § 2.3). Consequently, \(X \simeq \lim GO(k_m, V_{2k_m})^\ast\).

Next, instead of (4.5) and (4.6) one has

\[
B_m \simeq GO(k_m - 2, V_{2k_m}), \quad \Pi_m \simeq GO(k_m - 1, V_{2k_m}), \quad m \geq 1.
\]

Correspondingly, instead of (4.7) and (4.8) one has

\[
B_\bar{m}(x) \simeq G(k_m - 2, (\phi_{\bar{m}-1} \circ \cdots \circ \phi_m)(V_{k_m}))
\]
for \( x = \{V_{k_m}\} \). The latter fact can be proved by an argument similar to that of [12], Lemma 2.2. In addition, \( \Pi_{\tilde{m}}(x) \simeq \mathbb{P}((\phi_{m-1} \circ \cdots \circ \phi_m)(V_{k_m})^*) \), \( \tilde{m} \geq m \).

For \( X = GS(\infty, 0) = \lim_{\rightarrow} GS(k_m, V_{2k_m}) \) one can show that (4.5) and (4.6) can be replaced by

\[
B_m \simeq GS(k_m - 1, V_{2k_m}), \quad \Pi_m \simeq GS(k_m, V_{2k_m}).
\]

Correspondingly, (4.7) and (4.8) for \( x = \{V_{k_m}\} \in GS(k_m, V_{2k_m}) \) can be replaced by

\[
B_{\tilde{m}}(x) \simeq G(k_{\tilde{m}} - 1, (\phi_{m-1} \circ \cdots \circ \phi_m)(V_{k_m})),
\]

and \( \Pi_{\tilde{m}}(x) \simeq \{\mathbb{P}((\phi_{m-1} \circ \cdots \circ \phi_m)(V_{k_m})^*)\} \) is a point for \( \tilde{m} \geq m \).

§ 5. Linear sections of \( G(\infty), GO(\infty, \infty), GS(\infty, \infty) \)

5.1. Linear sections of finite-dimensional grassmannians. Let \( G = G(k, V), \)
\( GO(k, V), \) and \( GS(k, V). \) Assume that \( 1 \leq k < \dim V - 1 \) for \( G = G(k, V) \) and \( 1 \leq k < [\dim V/2] \) for \( G = GO(k, V), \) \( GS(k, V). \) Put \( N := \dim H^0(\Theta_G(1)) \) and \( V_N := H^0(\Theta_G(1))^*. \) We regard \( G \) as a subvariety of \( \mathbb{P}(V_N) \) via the Plücker embedding. For a given integer \( c, 1 \leq c \leq k - 1, \) set

\[
X := G \cap \mathbb{P}(U),
\]

where \( U \subset V_N \) is a subspace of codimension \( c. \) We call \( X \) a linear section of \( G \) of codimension \( c. \)

Note that there is a single family of maximal projective spaces of dimension \( k \) on \( G \) with base \( \tilde{G}, \) where \( \tilde{G} = G(k+1, V) \) if \( G = G(k, V), \) respectively, \( \tilde{G} = GO(k+1, V) \) if \( G = GO(k, V), \) and \( \tilde{G} = GS(k+1, V) \) if \( G = GS(k, V) \) (see [12], Lemmas 2.2, (i) and 2.5, (i)). Consider the graph of incidence \( \Sigma := \{(V_k, V_{k+1}) \in G \times \tilde{G} \mid V_k \subset V_{k+1}\} \)

with projections \( \tilde{G} \xrightarrow{p} \Sigma \xrightarrow{q} G \) and set \( \pi := p|_{q^{-1}(X)} : q^{-1}(X) \to \tilde{G}. \) The condition \( 1 \leq c \leq k - 1 \) implies that \( \pi \) is a surjective projective morphism.

Proposition 2. For a subspace \( U \subset V_N \) of codimension \( c \) in general position the following statements hold.

(i) The varieties \( X \) and \( q^{-1}(X) \) are smooth and

\[
\pi_* \Theta_{q^{-1}(X)} = \Theta_{\tilde{G}}. \tag{5.1}
\]

(ii) \( Z(U) := \{x \in \tilde{G} \mid \dim \pi^{-1}(x) > k - c\} \) is a proper closed subset of \( \tilde{G} \)

\[
\text{codim}_{\tilde{G}} Z(U) \geq 3, \quad \text{codim}_{q^{-1}(X)} \pi^{-1}(Z(U)) \geq 2. \tag{5.2}
\]

(iii) The projection \( \pi : q^{-1}(X)\backslash\pi^{-1}(Z(U)) \to \tilde{G}\backslash Z(U) \) is a projective \( \mathbb{P}^{k-c} \)-bundle.

Proof. We give the proof for the case \( G = GO(k, V). \) The other cases are very similar and we leave them to the reader.

(i) Since the projective subspace \( \mathbb{P}(U) \) is in general position in \( \mathbb{P}(V_N), \) we have \( \text{codim}_G X = c \) and hence there is a Koszul resolution of the \( \Theta_G \)-sheaf \( \Theta_X \)

\[
0 \to \Theta_G(-c) \to \cdots \to \left(\begin{array}{c} c \\ i \end{array}\right) \Theta_G(-i) \to \cdots \to c \Theta_G(-i) \to \Theta_G \to \Theta_X \to 0. \tag{5.3}
\]
The pullback of (5.3) under the projection \( q \) is a \( \mathcal{O}_\Sigma \)-resolution of the sheaf \( \mathcal{O}_{q^{-1}(X)} = \pi^* \mathcal{O}_{\widetilde{G}} \) of the form
\[
0 \to \mathcal{L}_c \to \cdots \to \mathcal{L}_1 \to \mathcal{O}_\Sigma \to \pi^* \mathcal{O}_{\widetilde{G}} \to 0, \tag{5.4}
\]
where \( \mathcal{L}_i := q^*((i)_! \mathcal{O}_G(-i)), i = 1, \ldots, c. \)

For any \( x \in \widetilde{G} \) we have \( p^{-1}(x) \simeq \mathbb{P}^k \), so the condition \( c \leq k - 1 \) implies
\[
H^j(p^{-1}(x), \mathcal{L}_i|_{p^{-1}(x)}) \simeq H^j(\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(-i)) = 0
\]
for \( j \geq 0, i = 1, \ldots, c. \) Hence the Base-change Theorem ([13], Ch. III, Theorem 12.11) for the flat projective morphism \( p \) shows that \( R^j p_* \mathcal{L}_i = 0 \). In addition, for the same reason, \( R^j p_* \mathcal{O}_\Sigma = 0, j > 0 \), and clearly \( p_* \mathcal{O}_\Sigma = \mathcal{O}_{\widetilde{G}} \). Therefore, applying the functor \( R^j p_* \) to (5.4), we obtain (5.1).

(ii) We now prove (5.2). Fix an arbitrary point \( V_{k+1} \in \widetilde{G} \). Since \( p^{-1}(x) = \mathbb{P}(V_{k+1}^*) \) (see [12], Lemma 2.2, (i)), there is an induced monomorphism
\[
0 \to V_{k+1}^* \to V_N. \tag{5.5}
\]
Consider the varieties
\[
\Gamma_i := \{(W, V_{k+1}) \in G(N-c, V_N) \times \widetilde{G} \mid \dim(W \cap V_{k+1}^*) \geq k-c+i+2, 0 \leq i \leq c-1, \}
\]

Together with the natural projections
\[
G(N-c, V_N) \xrightarrow{p_i} \Gamma_i \xrightarrow{q_i} \widetilde{G},
\]

For an arbitrary \( U \in G(N-c, V_N) \) denote
\[
Z_i(U) := q_i(p_i^{-1}(U)), \quad 0 \leq i \leq c-1.
\]

By construction, \( Z_0(U) = Z(U) \), the \( Z_i(U) \) are closed subvarieties of \( \widetilde{G} \), and we have a filtration
\[
\emptyset =: Z_c(U) \subset Z_{c-1}(U) \subset \cdots \subset Z_0(U) = Z(U) \tag{5.6}
\]
such that the \( Z_i(U)' := Z_i(U) \setminus Z_{i+1}(U) \) are locally closed subvarieties of \( \widetilde{G} \). Consequently, the \( B_i(U)' := \pi^{-1}(Z_i(U)) \) are locally closed subvarieties of \( \pi^{-1}(X) \). Moreover,
\[
\pi|_{B_i(U)'} : B_i(U)' \to Z_i(U)'
\]
is a \( \mathbb{P}^{k+1-c+i} \)-bundle, so that
\[
\dim B_i(U) = \dim Z_i(U) + k+1-c+i.
\]
Equivalently,
\[
\operatorname{codim}_{\pi^{-1}(X)} B_i(U)' = \operatorname{codim}_{\widetilde{G}} Z_i(U)' - (i+1). \tag{5.7}
\]
Note also that \( Z(U) = \bigcup_{i=0}^{c-1} Z_i(U)' \), hence
\[
\pi^{-1}(Z(U)) = \pi^{-1}\left(\bigcup_{i=0}^{c-1} Z_i(U)'ight) = \bigcup_{i=0}^{c-1} B_i(U)'.
\] (5.8)

We now calculate the dimensions of \( Z_i(U) \) under the assumption that \( U \) is in general position. For this, let \( Y := q_i^{-1}(x) \) be the fibre of the projection \( q_i \) over a point \( x = V_{k+1} \in Z_i(U) \). Consider the variety
\[
\tilde{Y} = \{(W, V_{k-c+i+2}) \in G(N-c, V_N) \times G(k-c+i+2, V_{k+1}^*) \mid W \supset V_{k-c+i+2} \subset V_{k+1}^*\}.
\]
The natural projection \( \tilde{Y} \to G(k-c+i+2, V_{k+1}^*) \) is a fibration with the Grassmannian \( G(N-k-i-2, \mathbb{C}^{N-2+c}) \) as a fibre. On the other hand, one has a birational surjective morphism \( \tilde{Y} \to Y, (W, V_{k-c+i+2}) \mapsto W \). Therefore, in view of (5.5),
\[
\dim Y = \dim \tilde{Y} = \dim G(k-c+i+2, V_{k+1}^*) + \dim G(N-k-i-2, \mathbb{C}^{N-2+c})
= cN - c^2 + (i+1)(c-k-i-2).
\]
As \( q_i \) is surjective, this yields
\[
\dim \Gamma_i = \dim \tilde{G} + \dim Y = \dim \tilde{G} + cN - c^2 + (i+1)(c-k-i-2).
\]
Since \( p_i \) is also surjective, for a point \( U \in G(N-c, V_N) \) in general position we have
\[
\dim Z_i(U) = \dim \Gamma_i - \dim G(N-c, V_N) = \dim \tilde{G} - (i+1)(k+i+c),
\]
that is,
\[
\text{codim}_{\tilde{G}} Z_i(U) = (i+1)(k+i+c), \quad 0 \leq i \leq c-1.
\] (5.9)
This, together with (5.7), implies \( \text{codim}_{q^{-1}(X)} B_i(U)' = (i+1)(k+i+1-c), \) \( 0 \leq i \leq c-1 \). Therefore, in view of (5.8) and the assumption \( c \leq k-1 \), we obtain
\[
\text{codim}_{q^{-1}(X)} \pi^{-1}(Z(U)) = \min_{0 \leq i \leq c-1} \text{codim}_{q^{-1}(X)} B_i(U)' = k + 1 - c \geq 2.
\] (5.10)
The inequality \( \text{codim}_{\tilde{G}} Z(U) \geq 3 \) follows now from (5.7), and the proposition is proved.

Corollary 1. Under the assumptions of Proposition 2 let \( \mathcal{E} \) be a vector bundle on \( q^{-1}(X) \) that is trivial along the fibres of the morphism \( \pi : q^{-1}(X) \to \tilde{G} \). Then there is an isomorphism \( \text{ev} : \pi^* \pi_* \mathcal{E} \cong \mathcal{E} \).

Proof. Apply Proposition 7 from the appendix to the morphism \( \pi : q^{-1}(X) \to \tilde{G} \), the subvariety \( Z(U) \) in \( \tilde{G} \) and the vector bundle \( \mathcal{E} \) on \( q^{-1}(X) \).

Lemma 2. Let \( X = G \cap \mathbb{P}(U) \) be a linear section of \( G \) of codimension \( c \) for \( 1 \leq c \leq (k-1)/2 \), and let \( \mathbb{P}^1 \) be a projective line on \( \tilde{G} \). Then there exists a rational curve \( C \subset q^{-1}(X) \) such that \( \pi|_C \) is an isomorphism of \( C \) with \( \mathbb{P}^1 \), and \( q|_C \) is either an isomorphism or a constant map.
Proof. We only consider the case $G = GO(k, V)$. It is clear that
$$\mathbb{P}^1 = \{ V_{k+1} \in \tilde{G} \mid V_k \subset V_{k+1} \subset V_{k+2} \}$$
for a unique isotropic flag $V_k \subset V_{k+2}$ in $V$. If $V_k \in X$, we set
$$C := \{ (V_k, V_{k+1}) \in \Sigma \mid V_k \subset V_{k+1} \subset V_{k+2} \}.$$ 
Then $\pi|_C : C \to \mathbb{P}^1$ is an isomorphism and $q(C)$ equals the point $\{ V_k \} \in G$.
Assume that $V_k \notin X$. It is straightforward to check that the intersection $q(p^{-1}(\mathbb{P}^1)) \cap \mathbb{P}^{N-2}$, for a hyperplane $\mathbb{P}^{N-2} \subset \mathbb{P}(V_N)$ such that $V_k \notin \mathbb{P}^{N-2}$, is isomorphic to the direct product $\mathbb{P}(V_k^*) \times \mathbb{P}(V_{k+2}/V_k)$ embedded by Segre in $\mathbb{P}^{N-2}$. Let $\mathbb{P}^{k-1}_a$ and $\mathbb{P}^{k-1}_b$ be the fibres in $\mathbb{P}(V_k^*) \times \mathbb{P}(V_{k+2}/V_k)$ over two points $a, b \in \mathbb{P}(V_{k+2}/V_k)$. The projection
$$\text{pr}_1 : \mathbb{P}(V_k^*) \times \mathbb{P}(V_{k+2}/V_k) \to \mathbb{P}(V_k^*)$$
induces an isomorphism $f : \mathbb{P}^{k-1}_a \sim \mathbb{P}^{k-1}_b$.
Set
$$\mathbb{P}^{k-c-1}_a := \mathbb{P}^{k-1}_a \cap \mathbb{P}(U), \quad \mathbb{P}^{k-c-1}_b := \mathbb{P}^{k-1}_b \cap \mathbb{P}(U).$$
Since $1 \leq c \leq (k - 1)/2$, the intersection $\mathbb{P}^{k-c-1}_b \cap f(\mathbb{P}^{k-c-1}_a) \subset \mathbb{P}^{k-1}_b$ is nonempty. Consider a point $x$ in this latter intersection. By construction, the fibre $\mathbb{P}_x^1 := \pi^{-1}_x(x)$ lies in $q(p^{-1}(\mathbb{P}^1)) \cap X$. Finally, the preimage of $\mathbb{P}_x^1$ in $p^{-1}(\mathbb{P}^1)$ is a rational curve $C$ as desired.

**Proposition 3.** Let $X$ be a linear section of $G$ of codimension $c$ for $1 \leq c \leq (k - 1)/2$. Then a linearly trivial vector bundle $E$ on $X$ is trivial.

Proof. Consider the vector bundle $\mathcal{E} := q^* E$ on $q^{-1}(X)$. Since $E$ is linearly trivial, for any $x \in \tilde{G}$, $\mathcal{E}|_{\pi^{-1}(x)}$ is a linearly trivial bundle on the projective space $\pi^{-1}(x)$. A well-known theorem ([14], Ch. I, Theorem 3.2.1) implies that $\mathcal{E}|_{\pi^{-1}(x)}$ is trivial. Therefore, ev: $\pi^* \pi_* \mathcal{E} \to \mathcal{E}$ is an isomorphism by Corollary 1.

Next, Lemma 2 allows us to conclude that $\pi_* \mathcal{E}$ is linearly trivial. Indeed, if $\mathbb{P}^1 \subset \tilde{G}$ is a projective line and $C \subset q^{-1}(X)$ is a rational curve as in Lemma 2, then $\pi_* \mathcal{E}|_{\mathbb{P}^1} \simeq \mathcal{E}|_C$, and hence $\mathcal{E}|_C$ is trivial because of the linear triviality of $E$.

Consequently, $\pi_* \mathcal{E}$ is trivial by Proposition 9 below. Then $\mathcal{E} \simeq \pi^* \pi_* \mathcal{E}$ is also trivial. Finally, since $q : q^{-1}(X) \to X$ is a flat projective morphism with irreducible fibres, $E = q_* \mathcal{E}$ is trivial by Proposition 6 below.

**5.2. Linear sections of $G(\infty)$, $GO(\infty, \infty)$, $GS(\infty, \infty)$ of small codimension.** Let $G = G(\infty)$, $GO(\infty, \infty)$, $GS(\infty, \infty)$, in particular, $G = \lim_{\longrightarrow} G(k_m, V_{n_m})$, $\lim_{\longrightarrow} GO(k_m, V_{n_m})$, $\lim_{\longrightarrow} GS(k_m, V_{n_m})$; see Definition 1. Fix a nondecreasing sequence $\{ c_m \}_{m \geq 1}$ of integers satisfying the condition
$$1 \leq c_m \leq \frac{k_m - 1}{2}. \quad (5.11)$$
Consider an ind-variety $X = \lim_{\longrightarrow} X_m$ such that, for each $m \geq 1$, $X_m$ is a smooth linear section of codimension $c_m$ of $G_m = G(k_m, V_{n_m})$, $GO(k_m, V_{n_m})$, $GS(k_m, V_{n_m})$. 

and the embedding $\phi_m: X_m \hookrightarrow X_{m+1}$ is induced by the embedding $G_m \hookrightarrow G_{m+1}$. In what follows we call such ind-varieties linear sections of $G$ of small codimension.

The existence of linear sections $X$ of $G$ of small codimension is a consequence of the Bertini Theorem. Moreover, such a linear section $X$ is a linear ind-variety and $\text{Pic} X$ is generated by $\mathcal{O}_X(1) := \lim \mathcal{O}_{X_m}(1)$. This follows from the observation that $\mathcal{O}_{X_m}(1)$ generates $\text{Pic} X_m$ by the Lefschetz Theorem, and from the linearity of the embeddings $G_m \hookrightarrow G_{m+1}$.

**Proposition 4.** Let $G = G(\infty), \text{GO}(\infty, \infty), \text{GS}(\infty, \infty)$. There exists a linear section $X$ of $G$ of small codimension which is not isomorphic to either of the ind-varieties $G(\infty), \text{GO}(\infty, \infty)$ or $\text{GS}(\infty, \infty)$.

**Proof.** Let $G = \varinjlim G_m$, where $G_m = G(k_m, V_{n_m}), \text{GO}(k_m, V_{2m+1}), \text{GS}(k_m, V_{2m+1})$. Fix $m \geq 1$ and let $P^1$ be a projective line on $G_m$. Then there exist unique maximal projective subspaces $P^k \subset G_m$ and $P^s \subset G_m$ which intersect in $P^1$. For $G_m = G(k_m, V_{n_m})$ one has $s_m = n_m - k_m$, and for $G_m = \text{GO}(k_m, V_{n_m}), \text{GS}(k_m, V_{2m+1})$ one has $s_m = [n_m/2] - k_m$: see [12], Lemmas 2.3, (iii) and 2.6. (ii).

For $\tilde{m} \geq m$ the embeddings $\phi_{\tilde{m}}: G_{\tilde{m}} \hookrightarrow G_{m+1}$ in the direct limit $\varinjlim G_m$ are given by (4.1), which makes it easy to verify that $P^k \subset G_{\tilde{m}}$ and $P^s \subset G_{\tilde{m}}$ such that $P^k \subset P^{k_{\tilde{m}}} \subset P^s \subset P^{s_{\tilde{m}}}$ and

$$P^1 = P^{k_{\tilde{m}}} \cap P^{s_{\tilde{m}}}, \quad \tilde{m} \geq m.$$  

Denoting $P^\infty_\alpha := \varinjlim P^{k_{\tilde{m}}}$ and $P^\infty_\beta := \varinjlim P^{s_{\tilde{m}}}$ we have

$$P^1 = P^\infty_\alpha \cap P^\infty_\beta. \tag{5.12}$$

We now choose $n_{\tilde{m}}$ and $k_{\tilde{m}}$ in a specific way. Namely, we assume that $n_{\tilde{m}} = 3t_{\tilde{m}}$ for $G_m = G(k_m, V_{n_m})$ and $[n_{\tilde{m}}/2] = 3t_{\tilde{m}}$ for $G_m = \text{GO}(k_m, V_{n_m}), \text{GS}(k_m, V_{2m+1})$, $k_{\tilde{m}} = 2t_{\tilde{m}}, t_{\tilde{m}} \in \mathbb{Z}_{\geq 1}$. Set $c_{\tilde{m}} := t_{\tilde{m}} - 1$. Then

$$s_{\tilde{m}} - c_{\tilde{m}} = 1 \tag{5.13}$$

and inequality (5.11) is satisfied, together with the conditions

$$\lim_{\tilde{m} \to \infty} k_{\tilde{m}} = \lim_{\tilde{m} \to \infty} s_{\tilde{m}} = \infty.$$  

Next, using (5.13) and the Bertini Theorem we choose a tower of projective subspaces $P^{N_{\tilde{m}} - c_{\tilde{m}} - 1} \subset P(V_{N_{\tilde{m}}})$ for $\tilde{m} \geq m$ in general position so that $P^{N_{\tilde{m}} - c_{\tilde{m}} - 1} \cap P^{s_{\tilde{m}}} = P^1$ and $P^{N_{\tilde{m}} - c_{\tilde{m}} - 1} \cap P^{k_{\tilde{m}}} = P^{(k_{\tilde{m}}/2) + 1}$ for some projective subspaces $P^{(k_{\tilde{m}}/2) + 1} \subset G_{\tilde{m}}$. As a result, we obtain a linear section $X := \varinjlim (P^{N_{\tilde{m}} - c_{\tilde{m}} - 1} \cap G_{\tilde{m}})$ of $G$ of small codimension and a projective line $P^1 \subset X$ such that $P^1$ is contained in a unique linear ind-projective subspace $P^\infty$ of $X$, namely $P^\infty := \varinjlim P^{(k_{\tilde{m}}/2) + 1}$. Were $X$ isomorphic to $G(\infty), \text{GO}(\infty, \infty)$ or $\text{GS}(\infty, \infty)$, this would contradict (5.12).

Now we show that a linear section $X$ of $G$ of small codimension satisfies properties L, A and T. Property L is clear as Pic $X$ is generated by the class of $\mathcal{O}_X(1)$, and $H^1(X_m, \mathcal{O}_{X_m}(a))$ vanishes for $\dim X_m > 1$ and $a < 0$ by Kodaira’s Theorem. Property $T$ is established in Proposition 3. It remains to establish property A.
Part (A.i) is clear as $\mathcal{O}_{X_m}(1)$ is very ample. For parts (A.ii)–(A.iv) we consider in detail only the case when $X$ is a linear section of $GO(\infty, \infty)$.

Let $B(G_m)$ be the family of all projective lines in $G_m = GO(k_m, V_{n_m})$ and $B_m$ be its subfamily consisting of those projective lines which lie in $X_m$. By definition, $X_m$ is the intersection of $G_m$ with a subspace $\mathbb{P}(U_m)$ of $\mathbb{P}(V_{N_m})$ for a fixed $U_m \in G(N_m - c_m, V_{N_m})$ in general position. The grassmannians $G(2, V_{N_m})$ and $G(2, U_m)$ can be thought of as the grassmannians of projective lines in $\mathbb{P}(V_{N_m})$ and $\mathbb{P}(U_m)$, respectively. Then $B_m = B(G_m) \cap G(2, U_m)$, where the intersection is taken in $G(2, V_{N_m})$. We show next that $B_m$ is irreducible.

Let $B$ be an irreducible component of $B_m$. Since $G(2, V_{N_m})$ is smooth, the subadditivity of codimensions ([15], Theorem 17.24) yields

$$\text{codim}_{B(G_m)} B \leq \text{codim}_{G(2, V_{N_m})} G(2, U_m) = 2c_m. \quad (5.14)$$

Consider the graph of incidence $\Sigma_m := \{(x, \mathbb{P}^{k_m}) \in G_m \times \widetilde{G}_m \mid x \in \mathbb{P}^{k_m}\}$ with its projections $\tilde{G}_m \xrightarrow{p_m} \Sigma_m \xrightarrow{q_m} G_m$, where $\widetilde{G}_m := GO(k_m + 1, V_{n_m})$. Let $B(\Sigma_m)$ be the family of all projective lines in $\Sigma_m$ lying in the fibres of the projection $p_m$. Denote by $B(q_m^{-1}(X_m))$ the subfamily of $B(\Sigma_m)$ consisting of those projective lines which lie in $q_m^{-1}(X_m)$. The projection $q_m : \Sigma_m \to G_m$ induces a morphism $r_{G_m} : B(\Sigma_m) \to B(G_m)$, which is bijective since any projective line on $G_m$ lies in a unique maximal projective space $\mathbb{P}^{k_m}$ (see [12], §2). The space $\mathbb{P}^{k_m}$ is an isomorphic image via $q_m$ of some fibre of $p_m$. Consequently, the restricted morphism

$$r_{X_m} := r_{G_m} \mid_{B(q_m^{-1}(X_m))} : B(q_m^{-1}(X_m)) \to B_m$$

is a bijection. Hence, for any irreducible component $B'$ of $B(q_m^{-1}(X_m))$, (5.14) yields the inequality

$$\text{codim}_{B(\Sigma_m)} B' \leq \text{codim}_{G(2, V_{N_m})} G(2, U_m) = 2c_m. \quad (5.15)$$

The projection $p_m$ induces a projection $\rho_m : B(\Sigma_m) \to \widetilde{G}_m$. Let

$$\emptyset = Z_{c_m}(U_m) \subset Z_{c_m-1}(U_m) \subset \cdots \subset Z_0(U_m) \subset \widetilde{G}_m$$

be the filtration (5.6) of $\widetilde{G}_m$ by closed subvarieties $Z_i(U_m)$ of codimensions in $\widetilde{G}_m$ given by (5.9), where we put $c = c_m$ and $k = k_m$. This filtration yields a decomposition

$$B(q_m^{-1}(X_m)) = \bigsqcup_{0 \leq i \leq c_m} B_i,$$

where

$$B_0 := B(q_m^{-1}(X_m)) \cap \rho_m^{-1}(\widetilde{G}_m \setminus Z_0(U_m));$$

$$B_i := B(q_m^{-1}(X_m)) \cap \rho_m^{-1}(Z_{i-1}(U_m) \setminus Z_i(U_m)), \quad i = 1, \ldots, c_m.$$

Formula (5.9) implies that $\text{codim}_{B(\Sigma_m)} B_0 = 2c_m$, $\text{codim}_{B(\Sigma_m)} B_i > 2c_m$ for $i = 1, \ldots, c_m$. This, together with (5.15), yields the irreducibility of $B(q_m^{-1}(X_m))$, hence of $B_m$ as well.
Now let $\Pi(G_m)$ be the family of projective spaces $\mathbb{P}^{k_m-1}$ lying in $B(G_m)$ and defined by the right-hand side of (4.6). Set
\[ \Pi := \lim_{\to} \Pi_m, \]
where
\[ \Pi_m := \{ \mathbb{P}^{k_m-c_m-1} \subset B_m | \mathbb{P}^{k_m-c_m-1} \text{ is a linear subspace of some } \mathbb{P}^{k_m-1} \in \Pi(G_m) \}. \]

Fix $m$ and let $\pi := p_m|_{q_m^{-1}(X_m)} : q_m^{-1}(X_m) \to \tilde{G}_m$ be the projection. Consider the relative grassmannian
\[ G_\pi := \{ \mathbb{P}^1 \subset q_m^{-1}(X_m) | \mathbb{P}^1 \text{ lies linearly in a fibre of the projection } \pi \} \]
with induced projections $\rho : G_\pi \to \tilde{G}_m$ and $q_\pi : G_\pi \to B_m$. By definition, the fibre $\rho^{-1}(x)$ over an arbitrary point $x \in \tilde{G}_m$ is the grassmannian of projective lines in the projective space $\pi^{-1}(x)$. (Note that for a point $x \in \tilde{G}_m$ in general position the fibre $\pi^{-1}(x)$ is a projective space $\mathbb{P}^{k_m-c_m}$, hence $\rho^{-1}(x) \simeq G(2, \mathbb{C}^{k_m-c_m+1})$.) Furthermore, the projection $q_\pi$ is birational.

By construction, the projective spaces $\mathbb{P}^{k_m-c_m-1} \in \Pi_m$ are isomorphic images under $q_\pi$ of projective spaces $\mathbb{P}^{k_m-c_m-1}$ lying linearly in the fibres of $\rho : G_\pi \to \tilde{G}_m$. Considering the set
\[ G_\rho := \{ \mathbb{P}^{k_m-c_m} \subset q_m^{-1}(X_m) | \mathbb{P}^{k_m-c_m} \text{ lies linearly in a fibre of } \pi : q_m^{-1}(X_m) \to \tilde{G}_m \}, \]
we obtain a $\mathbb{P}^{k_m-c_m}$-fibration $q_\rho : \Pi_m \to G_\rho$ with fibre $q_\rho^{-1}(y) = \mathbb{P}^{k_m-c_m}$ over a given point $y = \{ \mathbb{P}^{k_m-c_m} \} \in G_\rho$. Therefore, to check the irreducibility of $\Pi_m$ it suffices to check the irreducibility of $G_\rho$. Note that the projection $\pi$ induces a projection $\tau : G_\rho \to \tilde{G}_m$ such that the fibre of $\tau$ over a point $V_{k_m+1} \in \tilde{G}_m$ coincides with the grassmannian $G(k_m-c_m+1,W)$, where $W \subset V_{k_m+1}$ is the subspace defined by the condition $P(W) = \pi^{-1}(x)$. As above, (5.9) implies that, for $i \geq 1$, the locally closed subsets $\tau^{-1}(Z_i(W))$ of $G_\rho$ have dimensions strictly less than that of the open subset $\tau^{-1}(Z(W) \setminus Z_i(W))$. This proves the irreducibility of $G_\rho$, hence of $\Pi_m$.

Next, (4.7) implies that, for any point $x = V_{k_m} \in X_m$ and $\tilde{m} \geq m$, the base $B_{\tilde{m}}(x)$ of the family of projective lines on $X_{\tilde{m}}$ passing through $x$ is a linear section of the variety
\[ \mathbb{P}((\phi_{\tilde{m}-1} \circ \cdots \circ \phi_m)(V_{k_m})^*) \times GO(1,(\phi_{\tilde{m}-1} \circ \cdots \circ \phi_m)(V_{k_m})^\perp/(\phi_{\tilde{m}-1} \circ \cdots \circ \phi_m)(V_{k_m})) \]
in $\mathbb{P}(V_{k_m}^* \otimes V_{k_m}^\perp/V_{k_m})$ by a projective subspace of codimension $c_{\tilde{m}}$ in
\[ \mathbb{P}(V_{k_m}^* \otimes V_{k_m}^\perp/V_{k_m}). \]

Let
\[ b_{\tilde{m}}(x) : B_{\tilde{m}}(x) \to Q_{(\tilde{m})}(x) := GO(1,V_{k_m}^\perp/V_{k_m}) \]
be the natural projection. Note that the fibres of $b_m(x)$ are projective spaces of dimension at least $k_{\tilde{m}} - c_{\tilde{m}} - 1$, and, for points $x$ of $X_m$ and $z \in Q_{\tilde{m}}(x)$ in general position the fibre $b_m(x)^{-1}(z)$ is a projective space $\mathbb{P}^{k_{\tilde{m}} - c_{\tilde{m}} - 1}$ by the Bertini Theorem. Moreover, we have an ind-variety $B(x) = \lim B_{\tilde{m}}(x)$, $\tilde{m} \geq m$.

In a similar way we obtain that $\Pi(x) := \{ \mathbb{P}^\infty \in \Pi \mid \mathbb{P}^\infty \supseteq x \}$ is the ind-variety $\lim_{\longrightarrow} \Pi_{\tilde{m}}(x)$, $\tilde{m} \geq m$, where

$$
\Pi_{\tilde{m}}(x) := \{ \mathbb{P}^{k_{\tilde{m}} - c_{\tilde{m}} - 1} \subset B_m \mid \mathbb{P}^{k_{\tilde{m}} - c_{\tilde{m}} - 1} \text{ lies as a linear projective subspace in a fibre of the projection } b_m(x) \}.
$$

Let $p(x) : \Pi_{\tilde{m}}(x) \rightarrow Q_{\tilde{m}}(x)$ be the induced projection. By construction, for any point $z \in Q_{\tilde{m}}(x)$ the fibre $p(x)^{-1}(z)$ is the grassmannian $G(k_{\tilde{m}} - c_{\tilde{m}}, \mathbb{C}^{\dim(b_m(x)^{-1}(z)) + 1})$.

(In particular, this grassmannian is just a point for $x \in X_{\tilde{m}}$ and $z \in Q_{\tilde{m}}(x)$ in general position.) This implies property (A.ii) since $Q_{\tilde{m}}(x)$ is an irreducible quadric hypersurface.

Property (A.iii) is evident. As for (A.iv), let $Y$ be a fixed variety and $f : \Pi_{\tilde{m}}(x) \rightarrow Y$ be a morphism. For $\tilde{m} \rightarrow \infty$ the fibres of $p(x)$ are either points or are grassmannians whose dimensions tend to infinity. Therefore $f$ maps each fibre of $p(x)$ to a point, that is, $f$ factors through the induced morphism $g : Q_{\tilde{m}}(x) \rightarrow Y$. As $Q_{\tilde{m}}(x)$ is a smooth quadric hypersurface whose dimension tends to infinity as $\tilde{m} \rightarrow \infty$, it follows that for large enough $\tilde{m}$ the morphism $g$ is a constant map. Hence, $f$ is constant too, and (A.iv) is proved.

Theorem 1 now yields the following.

**Theorem 4.** A vector bundle on a linear section $X$ of small codimension of $G(\infty)$, $GO(\infty, \infty)$, $GS(\infty, \infty)$ is isomorphic to a direct sum of line bundles $\mathcal{O}(a_i)$ for $a_i \in \mathbb{Z}$.

§ 6. Ind-products and their subvarieties satisfying properties L, A and T

6.1. Finite or countable ind-products satisfying properties L, A and T. Let $X^\xi = \lim_{\longrightarrow} X^\xi_m$, $\xi \in \Xi$, be a countable collection of ind-varieties. We assume that for each $\xi \in \Xi$ and each $m \geq 1$ we have a fixed inclusion $X^\xi_m \subset X^\xi_{m+1}$. On every $X^\xi$ we fix a point $x_0^\xi$. Without loss of generality we assume that $x_0^\xi \in X^\xi_1$. Fix a bijection $\nu : \mathbb{N} \rightarrow \Xi$ and denote $\underline{m} := \{ 1, 2, \ldots, m \}$. Set

$$
\nu X_m := \times_{\xi \in \nu(\underline{m})} X^\xi_m
$$

and consider the embeddings

$$
\nu X_m \hookrightarrow \nu X_{m+1} = \nu X_m \times X^{\nu(m+1)}_m, \quad x \mapsto (x, x^{\nu(m+1)}_0), \quad m \geq 1.
$$

We call the ind-variety $X := \lim_{\longrightarrow} \nu X_m$ an ind-product of the ind-varieties $\{ X^\xi \}_{\xi \in \Xi}$ and denote it as

$$
X = \times_{\xi \in \Xi} X^\xi. \quad (6.1)
$$
Note that $X$ does not depend, up to isomorphism of ind-varieties, on the choice of the bijection $\nu : \mathbb{N} \to \Xi$, and thus the notation (6.1) is consistent. Indeed, let $\nu' : \mathbb{N} \to \Xi$ be another bijection and let $\psi := \nu' \circ \nu : \mathbb{N} \to \mathbb{N}$ be the induced bijection. An isomorphism

$$f : \lim_{\nu} X_m \sim \lim_{\nu'} X_m, \quad f = \{f_m : \nu X_m \to \nu' X_{\tilde{m}(m)}\},$$

and its inverse

$$g = f^{-1} : \lim_{\nu'} X_m \sim \lim_{\nu} X_m, \quad g = \{g_m : \nu' X_m \to \nu X_{\tilde{m}(m)}\},$$

for $\tilde{m}(m) := \min_{m' > m}\{m' \mid \psi(m) \subset m'\}$, are given by the formulae

$$f_m : \nu X_m = \times_{\xi \in \nu(m)} X_{\xi} \to \nu' X_{\tilde{m}(m)} = (\times_{\xi \in \nu'} X_{\xi})_{\tilde{m}(m)}$$

$$\times (\times_{\xi \in \nu'(\tilde{m}(m)) \setminus \nu(m)} X_{\xi}), \quad x \mapsto (x, \{x_{\xi}^0\}_{\xi \in \nu'(\tilde{m}(m)) \setminus \nu(m)}),$$

$$g_m : \nu' X_m = \times_{\xi \in \nu'(m)} X_{\xi} \to \nu X_{\tilde{m}(m)} = (\times_{\xi \in \nu(m)} X_{\xi})_{\tilde{m}(m)}$$

$$\times (\times_{\xi \in \nu(\tilde{m}(m)) \setminus \nu(m)} X_{\xi}), \quad x \mapsto (x, \{x_{\xi}^0\}_{\xi \in \nu(\tilde{m}(m)) \setminus \nu(m)}).$$

Note in addition that in principle $X$ depends on the choice of points $x_{\xi}^0$, however we suppress this dependence in the notation (6.1).

The reason we call $X$ an ind-product rather than a product is that $X$ is not a direct product in the category of ind-varieties. Of course, there are well-defined projections of ind-varieties $p_\xi : X \to X_\xi$, $p_\xi = \lim p_{\xi m}$, where $p_{\xi m} : \nu X_m \to X_\xi$ is the projection onto the $\xi$-factor for $\xi \in \nu(m)$, and the constant map $p_{\xi m} : \nu X_m \to x_0^{\nu(m)}$ for $\xi \notin \nu(m)$. However, $X$ fails to satisfy the universality property of a product.

If $\Xi$ is finite, we define the ind-product $\times_{\xi \in \Xi} X_\xi$ as the set-theoretic direct product of the $X_\xi$’s. Then $\times_{\xi \in \Xi} X_\xi = \lim_{\Xi} (\times_{\xi \in \Xi} X_\xi)$ is an ind-variety, and $\times_{\xi \in \Xi} X_\xi$ clearly satisfies the universality property of a direct product in the category of ind-varieties.

Now let $\Xi$ be finite or countable and let the ind-varieties $X_\xi = \lim_{\Xi} X_\xi$ satisfy properties $L$, $A$ and $T$. This means that on each $X_\xi$ there exists a collection $L_\xi := \{L_\theta^\xi \mid \theta \in \Theta X_\xi\}$ of line bundles and a collection $B_\xi := \{B_\theta^\xi = \lim_{\Xi} B_\theta^{\xi m} \mid \theta \in \Theta X_\xi\}$ of ind-varieties of projective lines such that $X_\xi$ satisfies properties $L$, $A$ and $T$. The collections $\{L_\xi^\Xi\}_{\xi \in \Xi}$ yield the following countable collection of line bundles on $X$:

$$L = \{p_\xi^* L_\theta^\xi \mid \theta \in \Theta X_\xi, \xi \in \Xi\}. \quad (6.2)$$

Moreover, any projective line $\mathbb{P}^1$ on $X_\xi$ determines a projective line $\mathbb{P}^1 X$ on $X$ such that $p_\xi(\mathbb{P}^1 X) = \mathbb{P}^1$ and $p_{\xi'}(\mathbb{P}^1 X) = \{x_{\xi'}^0\}$ for $\xi' \neq \xi$. Therefore, each ind-variety $B_\theta^\xi$ of projective lines on $X_\xi$ ‘lifts’ to an ind-variety of projective lines on $X$. In this way we obtain a collection $B$ of ind-varieties of projective lines on $X$. Since each $X_\xi$ satisfies properties $L$, $A$ and $T$, it is easy to check that $X$ satisfies the same properties with respect to the collections $L$ and $B$.

This, together with Theorem 1, leads to the following theorem.

**Theorem 5.** A vector bundle on $X = \times_{\xi \in \Xi} X_\xi$, where each ind-variety $X_\xi$ satisfies properties $L$, $A$ and $T$, is isomorphic to a direct sum of line bundles.
6.2. Linear sections of ind-products. In this subsection we assume that \( \Xi \) is finite, \( \Xi = \{1, 2, \ldots, l\} \), and that the ind-varieties \( X^i = \lim X^i_m \), \( i \in \Xi \), are copies of the standard ind-grassmannians \( G(\infty), GO(\infty, \infty) \) or \( G^S(\infty, \infty) \). By the above, \( \times_{i \in \Xi} X^i \) is a direct limit \( \lim_{m \to \infty} \times_{i \in \Xi} X^i_m \). Each \( X^i_m \) is a (possibly isotropic) grassmannian which we consider as lying via the Plücker embedding in \( \mathbb{P}^{N_i-1} = \mathbb{P}(V_{N_i}) \), and the embeddings

\[
\times_{i \in \Xi} X^i_m \hookrightarrow \times_{i \in \Xi} X^i_{m+1}
\]

are induced by the standard extensions \( X^i_m \hookrightarrow X^i_{m+1} \). We also assume that \( \times_{i \in \Xi} X^i_m \) lies in \( \mathbb{P}^{N_i-1} \) via the Segre embedding \( \times_{i \in \Xi} \mathbb{P}^{N_i-1} \hookrightarrow \mathbb{P}^{N_i-1} \).

For each \( m \geq 1 \) and \( i \in \Xi \) we set

\[
\hat{X}^i_m := X^1_m \times \cdots \times X^{i-1}_m \times X^i_m \times \cdots \times X^l_m,
\]

and for \( X^i_m = G(k_{im}, V_{n_{im}}) \) (respectively, \( X^i_m = GO(k_{im}, V_{n_{im}}) \) or \( X^i_m = GS(k_{im}, V_{n_{im}}) \)) set \( X^i_m : = G(k_{im}+1, V_{n_{im}}) \) (respectively, \( X^i_m : = GO(k_{im}+1, V_{n_{im}}) \) or \( X^i_m : = GS(k_{im}+1, V_{n_{im}}) \)). Consider the flag variety

\[
\Sigma_{im} := \{(V_{k_{im}}, V_{k_{im}+1}) \in X^i_m \times X^{i+1}_m | V_{k_{im}} \subset V_{k_{im}+1}\}
\]

with the natural projections \( X^i_m : \Sigma_{im} \to X^i_m \). There are induced projections

\[
X^i_m \times \hat{X}^i_m \overset{p_{im}}{\longrightarrow} \Sigma_{im} \times \hat{X}^i_m \overset{q_{im}}{\longrightarrow} X^i_m \times \hat{X}^i_m \simeq \hat{X}^i_m, \quad i \in \Xi;
\]

and we put

\[
\pi_{im} := p_{im}|_{q_{im}^{-1}(X^i_m)}: q_{im}^{-1}(X^i_m) \to X^i_m \times \hat{X}^i_m, \quad i \in \Xi.
\]

Now let \( \{c_m\}_{m \geq 1} \) be a nondecreasing sequence of integers satisfying the conditions

\[
1 \leq c_m \leq \min\left\{ \frac{k_{im} - 1}{2} \middle| i \in \Xi \right\}, \quad (6.5)
\]

where \( X^i_m = G(k_{im}, V_{n_{im}}) \), \( GO(k_{im}, V_{n_{im}}) \), \( GS(k_{im}, V_{n_{im}}) \). For each \( m \geq 1 \) consider a linear section \( X^i_m \) of

\[
X^i_m := (\times_{i \in \Xi} X^i_m) \cap \mathbb{P}^{N_i-c_m-1},
\]

where \( \mathbb{P}^{N_i-c_m-1} = \mathbb{P}(U_m) \) for \( U_m \in G(N_m-c_m, V_{N_m}) \). We call \( X^i_m \) a linear section of \( \times_{i \in \Xi} X^i_m \) of codimension \( c_m \).

**Proposition 5.** For a given \( m \geq 1 \) such that \( k_{im} \geq 2 \) for all \( i \in \Xi \) fix an integer \( c_m \) satisfying (6.5). Then for a projective subspace \( \mathbb{P}^{N_i-c_m-1} = \mathbb{P}(U_m) \) of general position in \( \mathbb{P}^{N_i-1} \), \( U_m \in G(N_m-c_m, V_{N_m}) \), and any \( i \in \Xi \) the following statements hold:

1) the varieties \( X^i_m \) and \( q_{im}^{-1}(X^i_m) \) are smooth;

2) \( \text{codim}_{X^i_m \times \hat{X}^i_m} X^i_m = \text{codim}_{\Sigma_{im} \times \hat{X}^i_m} q_{im}^{-1}(X^i_m) = c_m \), \( \pi_{im} \circ q_{im}^{-1}(X^i_m) = \pi_{im} \circ q_{im}^{-1}(X^i_m) = q_{im}^{-1}(X^i_m) \times \hat{X}^i_m \), and for a point \( x = (x_1, x_2) \in X^i_m \times \hat{X}^i_m \) in general position, the projective subspace \( \mathbb{P}^{k_{im}} = q_{im}p_{im}^{-1}(x) \) of \( X^i_m \times \{x_2\} \) satisfies the condition

\[
\text{codim}_{\mathbb{P}^{k_{im}}} (\mathbb{P}^{k_{im}} \cap \mathbb{P}^{N_i-c_m-1}) = c_m;
\]
so that $Z_m^i(U_m) := \{x \in X_m^i \times \hat{X}_m^i \mid \dim \pi_{im}^{-1}(x) > k_{im} - c_m\}$ is a proper closed subset of $X_m^i \times \hat{X}_m^i$;

3) let $B_m^i(U_m) := \pi_{im}^{-1}(Z_m^i(U_m));$ then
\[\operatorname{codim}_{\pi_{im}(X_m)} B_m^i(U_m) \geq 2, \quad \operatorname{codim}_{X_m^i \times \hat{X}_m^i} Z_m^i(U_m) \geq 3;\]

4) the projection
\[\pi_{im}: q_{im}^{-1}(X_m) \setminus B_m^i(U_m) \to X_m^i \times \hat{X}_m^i \setminus Z_m^i(U_m)\]
is a projective $\mathbb{P}^{k_{im} - c_m}$-bundle.

The proof is similar to the proof of Proposition 2.

**Corollary 2.** Under the assumptions of Proposition 5, let $i \in \Xi$ and let $\mathcal{E}$ be a vector bundle on $q_{im}^{-1}(X_m)$ that is trivial along the fibres of the morphism
\[\pi_{im}: q_{im}^{-1}(X_m) \to X_m^i \times \hat{X}_m^i.\]

Then the sheaf $\pi_{im*}\mathcal{E}$ is locally free and the canonical morphism $\operatorname{ev}: \pi_{im*}\pi_{im*}\mathcal{E} \to \mathcal{E}$ is an isomorphism.

**Proof.** Proposition 5 implies that $X = X_m^i \times \hat{X}_m^i, Y = q_{im}^{-1}(X_m), B = B_m^i(U_m), Z = Z_m^i(U_m)$ and $E = \mathcal{E}$ satisfy the conditions of Proposition 7 in the appendix. Therefore this latter proposition yields the corollary.

Below we will need the following lemma, the proof of which is similar to that of Lemma 2.

**Lemma 3.** For $i = 1, 2$ let $X_i = G(k_i, V_i), \ GO(k_i, V_i), \ GS(k_i, V_i),$ and let $X$ be a linear section of codimension $c$ of $X_1 \times X_2,$ where $1 \leq c \leq \min\{(k_1+1)/2, (k_2+1)/2\}$. Then for any projective line $\mathbb{P}_1 \subset X_i$ there exists a projective line $\mathbb{P}^1$ such that $\operatorname{pr}_i|_{\mathbb{P}_1}$ is an isomorphism of $\mathbb{P}^1$ and $\mathbb{P}^1_i$. Here $\operatorname{pr}_i$ stands for the natural projection $X_1 \times X_2 \to X_i$.

Consider an ind-variety $X = \varinjlim X_m$ such that, for each $m \geq 1$, $X_m$ is a smooth linear section of codimension $c_m$ of $\times_{i \in \Xi} X_{im}$, where $c_m$ satisfies (6.5), and the embeddings $\phi_m: X_m \hookrightarrow X_{m+1}$ are induced by the corresponding embeddings $\times_{i \in \Xi} X_m^i \hookrightarrow \times_{i \in \Xi} X_{m+1}^i$. In what follows we call $X$ a linear section of $\times_{i \in \Xi} X^i$ of small codimension. The existence of linear sections $X$ of $\times_{i \in \Xi} X^i$ of small codimension follows immediately from Bertini’s Theorem.

**Theorem 6.** Let $X$ be a linear section of $\times_{i \in \Xi} X^i$ of small codimension. Then a vector bundle on $X$ is a direct sum of line bundles.

**Proof.** We give a proof for the case when all $X^i$ are isomorphic to $\operatorname{GO}(\infty, \infty)$. The case when some $X^i$ are isomorphic to $G(\infty)$ or $\operatorname{GS}(\infty, \infty)$ is treated similarly.

First we construct families $B_i, i \in \Xi,$ on $X$. For each $m \geq 1$ and each $i$, $1 \leq i \leq l,$ consider the natural projection $p_{im}: X_m \to \hat{X}_m^i$ and for a variable point $y \in \hat{X}_m^i$ set $X_{mi}(y) := p_{im}^{-1}(y)$. By definition, $X_{mi}(y)$ is a linear section of the grassmannian $\operatorname{GO}(k_{im}, V_{n_{im}})$. Let $B_i(m)(y)$ be the family of projective
lines in $GO(k_{im}, V_{a_{im}})$ lying on $X^i_m(y)$ and set $B_i(m) := \bigcup_{y \in X^i_m} B_i(m)(y)$. Then $B_i := \lim_{\rightarrow} B_i(m), \ i \in \Xi$. Furthermore, the ind-varieties $\Pi_i, \ i \in \Xi$, parametrizing certain families of ind-projective spaces $\mathbb{P}^\infty$ in $B_i$, are defined in the same way as the ind-variety $\Pi$ (see (5.16)).

Next, recall the collection of line bundles (6.2) on $\times_{i \in \Xi} \mathbb{X}^i$. In our case $\Theta_{\mathbb{X}^\Xi}$ consists of a single point for each $i$, hence we can write simply $L = \{L_i\}_{i \in \Xi}$. We now define a family of line bundles $L_{\mathbb{X}}$ by putting $L_{\mathbb{X}} := \{L_i|_{\mathbb{X}}\}_{i \in \Xi}$. Then by the Lefschetz Theorem $L_{\mathbb{X}}$ freely generates $\text{Pic} \mathbb{X}$; in addition, (2.2) is clearly satisfied. To see that $\mathbb{X}$ satisfies property L, it remains to notice that $H^1(X_m, \times_{i \in \Xi} L_i^{\otimes a_i}|_{X_m}) = 0$ for all $a_i$. Indeed, the vanishing of $H^1(\times_{i \in \Xi} X^i_m, \times_{i \in \Xi} L_i^{\otimes a_i}|_{X_m})$ follows from Kunneth’s and Bott’s formulae. Since $\times_{i \in \Xi} L_i^{\otimes a_i}|_{X_m}$ admits a Koszul resolution similar to (5.3), this is sufficient to conclude that $H^1(X_m, \times_{i \in \Xi} L_i^{\otimes a_i}|_{X_m}) = 0$.

Using Proposition 5 and repeating the argument from §5.2 it is easy to check that $\mathbb{X}$ satisfies property A. Let us show that $\mathbb{X}$ satisfies property T. The case $|\Xi| = 1$ was treated in the proof of Theorem 4. It is enough to give the proof in the case $|\Xi| = 2$; the proof for $|\Xi| = 3$ goes along the same lines. We thus assume that $X_m$ is a linear section of $X^1_m \times X^2_m$. According to (6.3) and (6.4) we have a commutative diagram

$$ q^{-1}_{1m}(X_m) \xrightarrow{\pi_{1m}} X_m \xrightarrow{\rho} X^1_m \times X^2_m \xrightarrow{\lambda} X^2_m, $$

where $l = 2$ and $i = 1$, so that $\tilde{X}^1_m = X^2_m$, and $\lambda$ and $\rho$ are the natural projections.

Let $E_m$ be a $B_i$-trivial vector bundle on $X = \lim_{\rightarrow} X_m$ for $i = 1, 2$. This means that each vector bundle $E_m$ is a $B_i(m)$-trivial bundle on $X_m$; that is, $E_m|_{\mathbb{P}^1}$ is trivial for any $\mathbb{P}^1 \in B_i(m), \ i = 1, 2$. Consider the vector bundle

$$ \tilde{E}_m := q_{1m}^* E_m. $$

Since $E_m$ is $B_1(m)$-trivial, that is, linearly trivial on the fibres of $\rho$, $E_m$ is trivial by Proposition 3. It follows that $\tilde{E}_m$ is trivial along the fibres of $\pi_{1m}$. Therefore, by Corollary 2 there is an isomorphism

$$ \text{ev}: \pi_{1m}^* \pi_{1m*} \tilde{E}_m \simeq \tilde{E}_m. $$

Moreover, as in the proof of Proposition 3, we obtain that the bundle $\pi_{1m*} \tilde{E}_m|_{\lambda^{-1}(y)}$ is trivial for any $y \in X^2_m$.

Thus Proposition 6 below and diagram (6.6) imply that $E_m^\lambda := \lambda_* \pi_{1m*} \tilde{E}_m$ is a vector bundle on $X^2_m$ such that

$$ \tilde{E}_m \simeq \pi_{1m*} \tilde{E}_m \simeq q_{1m}^* \rho^* E_m^\lambda. $$

Applying Proposition 6 again we obtain

$$ E_m \simeq q_{1m*} \tilde{E}_m \simeq q_{1m}^* q_{1m*} \rho^* E_m^\lambda \simeq \rho^* E_m^\lambda. $$

(6.7)
Since $E_m$ is $B_2(m)$-trivial, it follows from (6.7) and Lemma 3 that $E_m^\lambda$ is a linearly trivial bundle on $X_{2m}^\lambda$. Hence, $E_m^\lambda$ is trivial by Proposition 9 below, and $E_m$ is trivial as well. Thus, we have shown that $X$ satisfies property T.

### 6.3. Ind-varieties of generalized flags.

We first recall some basic definitions concerning generalized flags in a vector space: see [16], §§3–5. Let $V$ be a countable-dimensional vector space. A set $\mathcal{C}$ of pairwise distinct subspaces of $V$ is called a chain if it is linearly ordered by inclusion. A chain $\mathcal{F}$ of subspaces of $V$ is a generalized flag in $V$ if it satisfies the following conditions:

(i) each $F \in \mathcal{F}$ has an immediate successor or an immediate predecessor, that is, $\mathcal{F} = \mathcal{F}' \cup \mathcal{F}''$, where $\mathcal{F}' \subset \mathcal{F}$ (respectively, $\mathcal{F}'' \subset \mathcal{F}$) is the set of subspaces in $\mathcal{F}$ having an immediate successor (respectively, predecessor);

(ii) $V \setminus \{0\} = \bigsqcup_{F' \in \mathcal{F}'} (F'' \setminus F')$, where $F'' \in \mathcal{F}''$ is the immediate successor of $F' \in \mathcal{F}'$.

We define a flag in $V$ to be a generalized flag in $V$ which is isomorphic as an ordered set to a subset of $\mathbb{Z}$. A flag can be defined equivalently as a chain of subspaces of $V$ such that

$$\bigcap_{F \in \mathcal{F}} F = 0, \quad \bigcup_{F \in \mathcal{F}} F = V$$

and there exists a strictly monotonic map of ordered sets

$$\phi : \mathcal{F} \to \mathbb{Z}.$$ 

If $\mathcal{F}$ is a generalized flag in $V$ and $\{e_\alpha\}_{\alpha \in A}$ is a basis of $V$ ($A$ being a countable set), we say that $\mathcal{F}$ and $\{e_\alpha\}_{\alpha \in A}$ are compatible if there exists a strict partial order $\prec$ on $A$ such that, for any $F' \in \mathcal{F}'$, $F' = \text{Span}\{e_\beta \mid \beta \prec \alpha\}$ for a certain $\alpha \in A$, and $F'' = F' \oplus \text{Span}\{e_\gamma \mid \gamma \text{ is not } \prec\text{-comparable with } \alpha\}$.

For the rest of this section we fix a basis $E = \{e_n\}$ of $V$. We call a generalized flag $\mathcal{F}$ weakly compatible with $E$ if $\mathcal{F}$ is compatible with a basis $L$ of $V$ such that $E \setminus (E \cap L)$ is a finite set. Furthermore, we define two generalized flags $\mathcal{F}$ and $\mathcal{G}$ in $V$ to be $E$-commensurable if both $\mathcal{F}$ and $\mathcal{G}$ are weakly compatible with $E$ and there exists an inclusion-preserving bijection $\phi : \mathcal{F} \to \mathcal{G}$ and a finite-dimensional subspace $U$ in $V$ such that, for every $F \in \mathcal{F}$

(i) $F \subset \phi(F) + U$, $\phi(F) \subset F + U$,

(ii) $\dim(F \cap U) = \dim(\phi(F) \cap U)$.

Let $\mathcal{F}l(\mathcal{F}, E)$ be the set of all generalized flags in $V$ that are $E$-commensurable with $\mathcal{F}$. Following [16], Proposition 5.2 we endow $\mathcal{F}l(\mathcal{F}, E)$ with the structure of an ind-variety in the following way. For each $m \geq 1$ denote $E_m := \{e_\alpha\}_{\alpha \leq m}$, $V_m := \text{Span}(E_m)$, $E_m^c := \{e_\alpha\}_{\alpha > m}$ and $V_m^c := \text{Span}(E_m^c)$. Next, for each $\mathcal{G} \in \mathcal{F}l(\mathcal{F}, E)$ choose a positive integer $m_\mathcal{G}$ such that $\mathcal{F}$ and $\mathcal{G}$ are compatible with bases containing $E_{m_\mathcal{G}}^c$, and $V_{m_\mathcal{G}}$ contains a finite-dimensional subspace $U$ which, together with the corresponding inclusion-preserving bijection $\phi_{m_\mathcal{G}} : \mathcal{F} \rightarrow \mathcal{G}$, makes $\mathcal{F}$ and $\mathcal{G}$ $E$-commensurable. We can pick $n_{\mathcal{F}}$ so that $n_{\mathcal{F}} \leq m_\mathcal{G}$ for every $\mathcal{G} \in \mathcal{F}l(\mathcal{F}, E)$. Set

$$\mathcal{G}_m := \{G \cap V_m \mid G \in \mathcal{G}\}, \quad m \geq m_\mathcal{G}.$$
The type of the flag $F_m$ yields a sequence of integers

$$0 < d_{m,1} < \cdots < d_{m,s_m-1} < d_{m,s_m} = m.$$  

Let $\mathcal{F}l(d_m, V_m)$ be an ordinary flag variety of type $d_m = (d_{m,1}, \ldots, d_{m,s_m-1})$ in $V_m$. Notice that $s_{m+1} = s_m$ or $s_{m+1} = s_m + 1$. Furthermore, in both cases an integer $j_m$ is determined as follows: in the former case $d_{m+1,i} = d_{m,i}$ for $0 \leq i < j_m$ and $d_{m+1,i} = d_{m,i} + 1$ for $j_m \leq i < s_m$, and in the latter case $d_{m+1,i} = d_{m,i}$ for $0 \leq i < j_m$ and $d_{m+1,i} = d_{m,i-1} + 1$ for $j_m \leq i < s_m$. 

Now we define a map $\iota_m: \mathcal{F}l(d_m, V_m) \to \mathcal{F}l(d_{m+1}, V_{m+1})$ for every $m \geq m_F$. Given a flag

$$G_m = \{0 = G_0^m \subset G_1^m \subset \cdots \subset G_{s_m}^m = V_m\} \in \mathcal{F}l(d_m, V_m)$$

put

$$\iota_m(G_m) = G_{m+1} := \{0 = G_0^{m+1} \subset G_1^{m+1} \subset \cdots \subset G_1^{s_m+1} = V_{m+1}\},$$

where

$$G_i^{m+1} = \begin{cases} G_i^m & \text{if } 0 \leq i < j_m, \\ G_i^m \oplus ke_{m+1} & \text{if } j_m \leq i \leq s_m + 1 \text{ and } s_{m+1} = s_m, \\ G_{i-1}^m \oplus ke_{m+1} & \text{if } j_m \leq i \leq s_{m+1} \text{ and } s_{m+1} = s_m + 1. \end{cases}$$

The maps $\iota_m$ are closed embeddings of algebraic varieties, and hence $\lim \mathcal{F}l(d_m, V_m)$ is an ind-variety. A bijection between $\mathcal{F}l(\mathcal{F}, E)$ and $\lim \mathcal{F}l(d_m, V_m)$ is given by

$$\tau: \mathcal{F}l(\mathcal{F}, E) \cong \lim \mathcal{F}l(d_m, V_m), \quad G \mapsto \lim G_m$$

(see [16], Proposition 5.2). Assume now that $\mathcal{F}$ is a flag of subspaces in $V$. Then $\mathcal{F} = \{\cdots \subset F_i \subset F_{i+1} \subset \cdots\}_{i \in \Theta}$, where $\Theta$ is one of the four linearly ordered sets \{1, \ldots, n\}, $\mathbb{Z}$, $\mathbb{Z}_{\geq 0}$, $\mathbb{Z}_{\leq 0}$. Assume, in addition, that

$$\dim(F''/F') = \infty \quad (6.8)$$

for all $i \in \Theta$ such that $i + 1 \in \Theta$. Denote by $\widehat{\mathcal{F}}(i)$ the flag

$$\widehat{\mathcal{F}} \setminus \{F_i\} = \{\cdots \subset F_{i-1} \subset F_{i+1} \subset \cdots\}.$$  

There is a natural projection $\pi_i: \mathcal{F}l(\mathcal{F}, E) \to \mathcal{F}l(\widehat{\mathcal{F}}(i), E)$. Let

$$\widehat{G} = \{\cdots \subset G_{i-1} \subset G_{i+1} \subset \cdots\} \in \mathcal{F}l(\widehat{\mathcal{F}}(i), E),$$

$$G = \{\cdots \subset G_{i-1} \subset G_i \subset G_{i+1} \subset \cdots\} \in \pi_i^{-1}(\widehat{G}).$$

Then the fibre $\pi_i^{-1}(\widehat{G})$ equals $\mathcal{F}l(G_i/G_{i-1}, E(i))$, where

$$E(i) = (E \cap G_{i+1}) \setminus (E \cap G_i).$$

Note that the ind-variety $\mathcal{F}l(G_i/G_{i-1}, E(i))$ is isomorphic to the ind-grassmannian $G(\infty)$. 
Moreover, there is a well-defined line bundle $L_i := \mathcal{O}_{\pi_i(1)} := \varinjlim \mathcal{O}_{\pi_{im}}(1)$ on $\mathcal{F}(\mathcal{F}, E)$, where $\pi_{im} : \mathcal{F}(d_m, V_m) \to \mathcal{F}(\mathcal{d}_m(i), V_m)$ is the natural projection and $\mathcal{d}_m(i)$ is defined in the same way as $d_m$ using the flag $\mathcal{F}(i)$ instead of $\mathcal{F}$. The fact that the line bundles $\mathcal{O}_{\pi_{im}}(1)$ yield a well-defined bundle $\mathcal{O}_{\pi_i(1)}$ is established by a straightforward check using the explicit form of the embeddings $\iota_m$.

By $B_i(G)$ we denote the ind-variety of projective lines on $\mathcal{F}(\mathcal{F}, E)$ passing through a point $G \in \mathcal{F}(\mathcal{F}, E)$ and lying in the fibre of $\pi_i$ which contains $G$. Finally, we define the ind-variety $\Pi_i(G)$ as the ind-variety $\Pi(G)$ for the ind-grassmannian $\mathcal{F}(G_i/G_{i-1}, E(i)) \simeq G(\infty)$ as defined in §4.3.

It is easy to check that $\mathcal{F}(\mathcal{F}, E)$ satisfies properties L, A and T with respect to the data

$$\Theta_{\mathcal{F}(\mathcal{F}, E)} := \Theta, \quad L_i, \quad B_i := \bigcup_{G \in \mathcal{F}(\mathcal{F}, E)} B_i(G), \quad \Pi_i := \bigcup_{G \in \mathcal{F}(\mathcal{F}, E)} \Pi_i(G).$$

As a result, Theorem 1 implies the following theorem.

**Theorem 7.** Let $V$ be a countable-dimensional vector space with basis $E$. Let $\mathcal{F}$ be a flag in $V$ satisfying (6.8) and weakly compatible with $E$. Then any vector bundle on $\mathcal{F}(\mathcal{F}, E)$ is isomorphic to a direct sum of line bundles.

It is an interesting question whether the BVTS Theorem holds on each ind-variety of generalized flags $\mathcal{F}(\mathcal{F}, E)$ under the assumption that the generalized flag $\mathcal{F}$ satisfies (6.8) for all $F' \in \mathcal{F}'$ and their respective successors $F''$.

### §7. Appendix

In this appendix we collect some general facts about coherent sheaves on projective varieties and their behaviour under flat projective morphisms, which are used throughout the paper.

**Proposition 6.** Let $p : Y \to X$ be a smooth flat projective morphism of projective varieties with irreducible fibres.

1) If $E$ is a vector bundle on $Y$, trivial on the fibres of $p$, then the evaluation morphism $ev : p^*p_*E \to E$ is an isomorphism.

2) If $F$ is a vector bundle on $X$, then the canonical morphism $F \cong p_*p^*F$ is an isomorphism.

**Proof.** 1) This follows easily from the Base-change Theorem ([13], Ch. III, Corollary 12.9).

2) Consider the Stein factorization $p : Y \xrightarrow{p'} X' \xrightarrow{g} X$ of $p$, where $X' = \text{Spec}(p_*\mathcal{O}_Y)$ and $p'_*\mathcal{O}_Y = \mathcal{O}_{X'}$ (see [13], Ch. III, Corollary 12.9). Since $p_*\mathcal{O}_Y$ is an invertible sheaf by 1), it follows that $g$ is an isomorphism. Therefore $f_*\mathcal{O}_Y = \mathcal{O}_{X'}$. This, together with the projection formula ([13], Ch. III, Exercise 8.3), gives the desired assertion.

**Proposition 7.** Let $\pi : Y \to X$ be a surjective morphism of smooth irreducible projective varieties such that:

(i) the fibres of $\pi$ are projective spaces;
(ii) the variety $Z := \{ x \in X \mid \dim \pi^{-1}(x) > \dim Y - \dim X \}$ has codimension at least 3 in $X$, and $B := \pi^{-1}(Z)$ has codimension at least 2 in $Y$;

(iii) there exists a vector bundle $F$ on $X \setminus Z$ such that $\pi: Y \setminus B \cong \mathbb{P}(F) \to X \setminus Z$ is the structure map of the projectivized vector bundle $F$.

Next, let $E$ be a vector bundle on $Y$, trivial along the fibres of $\pi$. Then the $\mathcal{O}_X$-sheaf $\pi_*E$ is locally free and the evaluation morphism $ev: \pi^*\pi_*E \to E$ is an isomorphism.

Proof. We show first that $ev: \pi^*\pi_*E \to E$ is an isomorphism. For this, consider an arbitrary open subvariety $U \subset X$ and its closed subvariety $A \subset U$ such that

$$\text{codim}_U A \geq 2. \quad (7.1)$$

Since $X$ is smooth and $Z$ has codimension $\geq 3$ in $X$, it follows that $\text{codim}_U (Z \cap A) \geq 3$ and $\text{codim}_{\pi^{-1}(U)} \pi^{-1}(Z \cap A) \geq 2$. Next, (iii) and (7.1) imply that $\text{codim}_{\pi^{-1}(U)} \pi^{-1}(Z \setminus Z \cap A) \geq 2$. Hence

$$\text{codim}_{\pi^{-1}(U)} \pi^{-1}(A) \geq 2. \quad (7.2)$$

Let $s \in H^0(U \setminus A, \pi_*E|_{U \setminus A})$ and let $\tilde{s} := \phi(s)$, where

$$\phi: H^0(U \setminus A, \pi_*E|_{U \setminus A}) \cong H^0(\pi^{-1}(U \setminus A), E|_{\pi^{-1}(U \setminus A)})$$

is the canonical isomorphism. Since $E$ is a locally free sheaf on a smooth variety $Y$, $E$ is normal by [17], Proposition 1.6, (ii), so that (7.2) implies that $\tilde{s}$ extends uniquely to a section $\tilde{s}' \in H^0(\pi^{-1}(U), E|_{\pi^{-1}(U)})$. Then $s$ extends to the section $s' := \psi(\tilde{s}') \in H^0(U, \pi_*E|_U)$, where

$$\psi: H^0(\pi^{-1}(U), E|_{\pi^{-1}(U)}) \cong H^0(U, \pi_*E|_U)$$

is the canonical isomorphism. In view of (7.1), this means that the sheaf $\pi_*E$ is normal.

Note that $\pi_*E$ is torsion-free. Indeed, if the torsion subsheaf $\text{Tors}(\pi_*E)$ were nonzero, then since $E$ is locally free, by (iii) any section $0 \neq s \in H^0(Y, \text{Tors}(\pi_*E))$ would be supported in $Z$. Then the section $0 \neq \tilde{s} := \psi^{-1}(s)$ would be supported in $B$, that is, $\text{Tors}(E) \neq 0$. This contradicts the assumptions that $E$ is locally free and $Y$ is smooth and irreducible.

Hence, $\pi_*E$ is reflexive by [17], Proposition 1.6. Set

$$\tilde{E} := \pi^*\pi_*E / \text{Tors}(\pi^*\pi_*E).$$

Proposition 6, together with (iii), implies the existence of an isomorphism

$$\alpha: \tilde{E}|_{Y \setminus B} \cong E|_{Y \setminus B}. \quad (7.3)$$

Now by [17], Proposition 1.1, $\pi_*E$ can, locally on $X$, be included in an exact sequence

$$0 \to \pi_*E \to L_1 \to L_2 \quad (7.4)$$
with locally free sheaves $L_1$ and $L_2$. Applying the functor $\pi^*$ to (7.4) we obtain the sequence
\[ 0 \to \widetilde{E} \to \pi^*L_1 \to \pi^*L_2, \]
which is exact when restricted onto $Y \setminus B$. Hence, this sequence is itself exact as $\widetilde{E}$ is torsion free and the sheaves $\pi^*L_1$ and $\pi^*L_2$ are locally free. By [17], Proposition 1.1 this implies that $\widetilde{E}$ is reflexive. Therefore, denoting by $i$ the inclusion $Y \setminus B \hookrightarrow Y$ and using the isomorphism (7.3) and [17], Proposition 1.6,(iii) we obtain an isomorphism
\[ \pi^*\pi_*E = \widetilde{E} \cong i_*(\widetilde{E}|_{Y \setminus B}) \overset{i_*\pi_*}{\cong} i_*(E|_{Y \setminus B}) \cong E. \]
This isomorphism is nothing but the evaluation morphism $\text{ev}$.

It remains to show that $\pi_*E$ is locally free. The isomorphism $\text{ev}$ implies that $\pi^*\pi_*E$ is locally free. Therefore, for any $y \in Y$, $r := \dim_{\mathbb{C}(y)}(\pi^*\pi_*E \otimes \mathbb{C}(y))$ does not depend on $y$, and consequently, $\dim_{\mathbb{C}(x)}(\pi_*E \otimes \mathbb{C}(x)) = r$. According to [18], §5, Lemma 1, since $X$ is smooth, the fact that $\dim_{\mathbb{C}(x)}(\pi_*E \otimes \mathbb{C}(x))$ does not depend on $x \in X$ implies that $\pi_*E$ is locally free.

**Proposition 8.** Let $Q_n$ be a nonsingular $n$-dimensional quadric in $\mathbb{P}^{n+1}$ for $n \geq 2$, and let $E$ be a linearly trivial vector bundle on $Q_n$. Then $E$ is trivial.

**Proof.** We argue by induction on $n$. For $n = 2$ the proof is easy and is the same as for a projective space as given in [14], Ch. 1, Theorem 3.2.1. Thus we may assume that $n \geq 3$. Consider a codimension-2 subspace $\mathbb{P}^{n-1}$ in $\mathbb{P}^{n+1}$ such that $Q_{n-2} := Q_n \cap \mathbb{P}^{n-1}$ is a smooth quadric of dimension $n - 2$. If $n \geq 4$ then $E|_l$ is trivial for any projective line $l \subset Q_{n-2}$, hence $E|_{Q_{n-2}}$ is trivial by the induction assumption. For $n = 3$ the quadric $Q_{n-2}$ is a smooth conic $C$. By Bertini’s Theorem there exists a smooth quadric surface $Q_2$ on $Q_3$ passing through the conic $C$. Since $E|_{Q_2}$ is trivial (being linearly trivial), $E|_C$ is also trivial, that is, our claim holds for $n = 3$.

We will now use the triviality of $E|_{Q_{n-2}}$ for $n \geq 4$ to show that $E$ is trivial. Let $\sigma_Q : \tilde{Q}_n \to Q_n$ be the blow-up of $Q_n$ with centre at $Q_{n-2}$, and let $D := \sigma_Q^{-1}(Q_{n-2})$ be the exceptional divisor. Clearly, $D \cong Q_{n-2} \times \mathbb{P}^1$ and there is a flat surjective morphism $\pi : \tilde{Q}_n \to \mathbb{P}^1$ fitting in the commutative diagram

\[ \begin{array}{ccc}
D = Q_{n-2} \times \mathbb{P}^1 & \xrightarrow{i} & \tilde{Q}_n \\
\downarrow \text{pr}_2 & & \downarrow \pi \\
\mathbb{P}^1, & & 
\end{array} \]

(7.5)

where $i$ is the embedding of the exceptional divisor. By construction, there exist two distinct points $t_1, t_2 \in \mathbb{P}^1$ such that the fibre $Q_{n-1}(t) = \pi^{-1}(t)$ is a smooth quadric for $t \in U := \mathbb{P}^1 \setminus \{t_1 \cup t_2\}$, and $Q_{n-1}(t_j) := \pi^{-1}(t_j)$ for $j = 1, 2$ are quadratic cones whose vertices are points.

Consider the vector bundle $\widehat{E} := \sigma_Q E$ on $\tilde{Q}_n$. By construction, $\widehat{E}$ is trivial on any projective line $l \subset Q_{n-1}(t)$, $t \in U$. Hence, by the induction assumption,
This, together with the Base-change Theorem for $f_j$, shows that $R^i f_j \ast \widetilde{K}_j = 0$ for $i \geq 1$ and $R^i f_j \ast (\widetilde{K}_j (-\sigma^* D)) = 0$ for $i \geq 0$. Hence the Leray spectral sequence for the projection $f_j$ yields

$$H^i(K_j, \widetilde{K}_j) = 0, \quad i \geq 1, \quad H^i(K_j, \widetilde{K}_j (-\sigma^* D)) = 0, \quad i \geq 0, \quad j = 1, 2. \tag{7.7}$$

Next, one uses the blow-up of the cone $Q_{n-1}(t_j)$ embedded in $\mathbb{P}^n$ to show that $\sigma_* \mathcal{O}_{K_j} = \mathcal{O}_{Q_{n-1}(t_j)}$ and $R^i \sigma_* \mathcal{O}_{K_j} = 0, \ i \geq 1$. Therefore, setting $\tilde{E}_{t_j} := \tilde{E}|_{Q_{n-1}(t_j)}$ we have by the projection formula $\sigma_* \tilde{E}_{K_j} = \tilde{E}_{t_j}$, $R^i \sigma_* \tilde{E}_{K_j} = 0, \ i \geq 1$, and

$$\sigma_* (\tilde{E}_{K_j} (-\sigma^* D)) = \tilde{E}_{t_j} (-D), \quad R^i \sigma_* (\tilde{E}_{K_j} (-\sigma^* D)) = 0, \quad i \geq 1.$$

Now the Leray spectral sequence applied to $\sigma$ shows, in view of (7.7), that

$$H^i(Q_{n-1}(t_j), \tilde{E}_{t_j}) = H^i(K_j, \tilde{E}_{K_j}) = 0, \quad i \geq 1, \tag{7.8}$$

$$H^i(Q_{n-1}(t_j), \tilde{E}_{t_j} (-D)) = H^i(K_j, \tilde{E}_{K_j} (-\sigma^* D)) = 0, \quad i \geq 0, \quad j = 1, 2.$$

Equalities (7.6) and (7.8) yield via base change for the flat morphism $\pi$

$$R^i \pi_* \tilde{E} = 0, \quad i \geq 1, \quad R^i \pi_* (\tilde{E} (-D)) = 0, \quad i \geq 0. \tag{7.9}$$

The same argument yields base change isomorphisms

$$b_t : \pi_* \tilde{E} \otimes \mathbb{C}_t \xrightarrow{\simeq} H^0(Q_{n-1}(t), \tilde{E}|_{Q_{n-1}(t)}), \quad t \in \mathbb{P}^1. \tag{7.10}$$

Consider the divisor $D = Q_{n-2} \times \mathbb{P}^1$ on $\tilde{Q}$ (see diagram (7.5)) and the projections $Q_{n-2} \xrightarrow{pr_1} Q_{n-2} \times \mathbb{P}^1 \xrightarrow{pr_2} \mathbb{P}^1$. By definition, $\tilde{E}|_D = pr_1^*(\tilde{E}|_{Q_{n-2}})$, hence, since $\tilde{E}|_{Q_{n-2}}$ is trivial, the base change for the flat morphism $pr_2$ gives the isomorphisms

$$b' : pr_2^*(\tilde{E}|_D) \xrightarrow{\simeq} H^0(Q_{n-2}, E|_{Q_{n-2}}) \otimes \mathcal{O}_{\mathbb{P}^1} \simeq \mathbb{C}^r \otimes \mathcal{O}_{\mathbb{P}^1}, \tag{7.11}$$

$$b'_t : pr_2^*(\tilde{E}|_D) \otimes \mathbb{C}_t \xrightarrow{\simeq} H^0(Q_{n-2}, E|_{Q_{n-2}}) \simeq \mathbb{C}^r, \quad t \in \mathbb{P}^1. \tag{7.12}$$
Now consider the exact triple
\[ 0 \to \tilde{E}(-D) \to \tilde{E} \to E|_D \to 0 \]
and its restriction
\[ 0 \to \tilde{E}(-D)|_{Q_{n-1}(t)} \to \tilde{E}|_{Q_{n-1}(t)} \to (E|_{Q_{n-2}}) \otimes C_t \to 0 \]
to a fibre \(Q_{n-1}(t)\) of the projection \(\pi\) over an arbitrary point \(t \in \mathbb{P}^1\). Applying the functor \(R^i\pi_*\) to (7.13) and using (7.9) and (7.11) we obtain the isomorphism of sheaves
\[ r_D: \pi_*\tilde{E} \simeq \text{pr}_2^*(\tilde{E}|_D) \simeq \mathbb{C}^r \otimes \mathcal{O}_{\mathbb{P}^1}. \]
In particular, \(\pi_*\tilde{E}\) is a trivial bundle. Respectively, passing to cohomology of the exact sequence (7.14) and using (7.6), (7.8) and (7.12) we obtain the isomorphisms
\[ \text{res}_t: H^0(Q_{n-1}(t), \tilde{E}|_{Q_{n-1}(t)}) \simeq H^0(Q_{n-2}, E|_{Q_{n-2}}), \quad t \in \mathbb{P}^1. \]
By construction, the isomorphisms (7.10), (7.12), (7.15) and (7.16) constitute the commutative diagram
\[
\begin{array}{ccc}
\pi_*\tilde{E} \otimes C_t & \xrightarrow{r_D \otimes C_t} & \text{pr}_2^*(\tilde{E}|_D) \otimes C_t \\
\text{res}_t & \simeq & \text{res}_t \\
H^0(Q_{n-1}(t), \tilde{E}|_{Q_{n-1}(t)}) & \xrightarrow{\text{res}_t} & H^0(Q_{n-2}, E|_{Q_{n-2}}) = \mathbb{C}^r
\end{array}
\]
for \(t \in \mathbb{P}^1\). Next, since \(E|_{Q_{n-2}}\) is trivial, the evaluation map \(H^0(Q_{n-2}, E|_{Q_{n-2}}) \otimes \mathcal{O}_{Q_{n-2}} \to E|_{Q_{n-2}}\) is an isomorphism, so that its composition \(e_t\) with the restriction
\[ H^0(Q_{n-2}, E|_{Q_{n-2}}) \otimes \mathcal{O}_{Q_{n-2}} \to H^0(Q_{n-2}, E|_{Q_{n-2}}) \otimes \mathcal{O}_{Q_{n-1}(t)} \]
is an epimorphism for any \(t \in \mathbb{P}^1\) and fits into the commutative diagram
\[
\begin{array}{ccc}
H^0(Q_{n-1}(t), \tilde{E}|_{Q_{n-1}(t)}) \otimes \mathcal{O}_{Q_{n-2}} & \xrightarrow{\pi^*\text{res}_t} & H^0(Q_{n-2}, E|_{Q_{n-2}}) \otimes \mathcal{O}_{Q_{n-1}(t)} \\
\text{ev}_t & \simeq & \text{ev}_t \\
\tilde{E}|_{Q_{n-1}(t)} & \xrightarrow{\text{res}_{Q_{n-2}}} & E|_{Q_{n-2}}.
\end{array}
\]
Here we understand \(Q_{n-2}\) as lying in \(Q_{n-1}(t)\) as a divisor. In particular, through any point of \(Q_{n-1}(t) \setminus Q_{n-2}\) there passes a line, say \(l\), intersecting \(Q_{n-2}\) at a point, say \(y\). Therefore, since \(\tilde{E}|_l\) is trivial, we have a commutative diagram of restriction maps
\[
\begin{array}{ccc}
H^0(Q_{n-1}(t), \tilde{E}|_{Q_{n-1}(t)}) & \xrightarrow{\text{res}_t} & H^0(Q_{n-2}, E|_{Q_{n-2}}) \\
\rho & \simeq & \rho \\
H^0(l, \tilde{E}|_l) & \xrightarrow{\simeq} & H^0(y, \tilde{E}|_y).
\end{array}
\]
Hence, \( \rho \) is an isomorphism, and therefore the evaluation morphism \( \ev_t \) in (7.18) is an isomorphism of sheaves. Composing it with the isomorphism

\[
\pi^*b_t: \pi^*\pi_*\widetilde{E}|_{Q_{n-1}(t)} \xrightarrow{\cong} H^0(Q_{n-1}(t), \widetilde{E}|_{Q_{n-1}(t)}) \otimes \mathcal{O}_{Q_{n-1}(t)},
\]

arising from the left vertical isomorphism in (7.17) we obtain the (evaluation) isomorphism \( \ev|_{Q_{n-1}(t)}: \pi^*\pi_*\widetilde{E}|_{Q_{n-1}(t)} \xrightarrow{\cong} \widetilde{E}|_{Q_{n-1}(t)} \). Since this is true for any \( t \in \mathbb{P}^1 \), we obtain the isomorphism \( \ev: \pi^*\pi_*\widetilde{E} \xrightarrow{\cong} \widetilde{E} \), which together with (7.15) leads to the triviality of \( \widetilde{E} \). Since clearly \( \sigma_Q \mathcal{O}_Q = \mathcal{O}_Q \), it follows that \( E = \sigma_Q \widetilde{E} = \sigma_Q(r \partial_{\widetilde{Q}}) = r \partial_Q \), that is, we obtain the statement of the proposition.

**Proposition 9.** Let \( E \) be a linearly trivial vector bundle on \( GO(k, V) \) or \( GS(k, V) \). Then \( E \) is trivial.

**Proof.** Consider the case \( GO(k, V) \). We give a proof by induction under the assumption that \( n := \dim V/2 \in \mathbb{Z}_{>0} \). The case when \( \dim V \) is odd can be treated similarly.

For \( n = 2 \) we have \( GO(1, V) \simeq \mathbb{P}^1 \times \mathbb{P}^1 \), \( GO(2, V) \simeq \mathbb{P}^1 \), and for these varieties our claim clearly holds. Therefore we assume that \( n \geq 3 \) and argue by induction on \( k \). If \( k = 1 \), \( GO(k, V) \) is a \( (2n-2) \)-dimensional quadric in \( \mathbb{P}^{2n-1} \), so our statement holds by Proposition 8. Now let \( 1 \leq k \leq n-2 \) and recall the graph of incidence \( \Sigma \) with natural projections

\[
GO(k, V) \xrightarrow{q} \Sigma \xrightarrow{p} GO(k + 1, V)
\]

(see § 5.1).

Let \( E \) be a linearly trivial vector bundle on \( GO(k + 1, V) \). Then the bundle \( p^*E \) is linearly trivial on the fibres of \( q \). Since these fibres are quadrics, Proposition 8 implies that \( p^*E \) is trivial on the fibres of \( q \). Furthermore, Proposition 6 yields an isomorphism \( q^*q_*p^*E \xrightarrow{\cong} p^*E \). Hence, since \( p^*E \) is trivial along the fibres of \( p \) which are mapped by \( q \) isomorphically to projective spaces \( \mathbb{P}^k \) on \( GO(k, V) \), it follows that \( q_*p^*E \) is trivial along these projective subspaces \( \mathbb{P}^k \) of \( GO(k, V) \). Consequently, \( q_*p^*E \) is linearly trivial on \( GO(k, V) \). Thus, by the induction assumption, \( q_*p^*E \) is trivial. Hence \( p^*E \) and \( E = p_*p^*E \) are trivial.

It remains to consider \( GO(n, V) \). Here we employ induction on \( n \). For \( n = 3 \) \( GO(n, V) \simeq \mathbb{P}^3 \), hence the statement holds in this case. For \( n \geq 4 \) consider the graph of incidence

\[
\Pi_n := \{(V_1, V_n) \in Q_{2n-2} \times GO(n, V) \mid V_1 \subset V_n\}
\]

with natural projections

\[
Q_{2n-2} \xleftarrow{p} \Pi_n \xrightarrow{q} GO(n, V).
\]

Let \( E \) be a linearly trivial vector bundle on \( GO(n, V) \). Then \( q^*E \) is trivial on lines lying in the fibres of \( p \) which are isomorphic to \( GO(n-1, \mathbb{C}^{2n-2}) \). Hence \( q^*E \) is trivial along the fibres of \( p \) by the induction assumption. Next, Proposition 6 yields an isomorphism \( p^*p_*q^*E \xrightarrow{\cong} q^*E \). Since \( q^*E \) is trivial on the fibres of \( p \), it follows that \( p_*q^*E \) is trivial on the projective subspaces \( \mathbb{P}^{n-1} \) of the quadric \( Q_{2n-2} \).
Therefore $p_* q^* E$ is trivial on the lines in $Q_{2n-2}$, so it is trivial by Proposition 8. Finally, $q^* E \simeq p_* p_* q^* E$ and $E = q_* q^* E$ are trivial as well.

We proceed to the case of $G\ell(k, V)$. Substituting $G\ell$ by $G\ell$ in diagram (7.19) we obtain a diagram $G\ell(k, V) \xrightarrow{q'} \Sigma \xrightarrow{p} G\ell(k+1, V)$, where $p$ is a $P^k$-bundle and $q$ is a $P^{2n-2k-1}$-bundle. Correspondingly, substituting $G\ell$ by $G\ell$ and $Q_{2n-2}$ by $P(V_n)$ in diagram (7.20) we obtain a diagram $P(V_n) \xrightarrow{p} \Pi_n \xrightarrow{q} G\ell_n$. This enables us to carry out an argument very similar to the one for $G\ell(k, V)$.

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Received 8/MAY/14 and 4/FEB/15  
Translated by A. TIKHOMIROV