DOUBLE SOLID TWISTOR SPACES II: GENERAL CASE

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Abstract. In this paper we investigate Moishezon twistor spaces which have a structure of double covering over a very simple rational threefold. These spaces can be regarded as a direct generalization of the twistor spaces studied in the papers [15, 11] to the case of arbitrary signature. In particular, the branch divisor of the double covering is a cut of the rational threefold by a single quartic hypersurface. A defining equation of the hypersurface is determined in an explicit form. We also show that these twistor spaces interpolate LeBrun twistor spaces and the twistor spaces constructed in [8].

1. Introduction

In an influential paper [6], Hitchin initiated a systematic study of compact twistor spaces by algebro-geometric means. In particular, by investigating the half-anticanonical system of twistor spaces, he showed that if a compact twistor space admits a Kähler metric, it must be one of the two standard twistor spaces, the projective space or the flag manifold. This direction of research was succeeded by Poon [14, 15] and Kreussler-Kurke [11] to determine structure of twistor spaces over the connected sum of two or three complex projective planes, by means of the same system on the twistor spaces. Also, LeBrun [12] and Campana-Kreussler [4] utilized the same system to investigate particular Moishezon twistor spaces over the connected sum of any number of complex projective planes.

However, for other twistor spaces, it was evident that the half-anticanonical system no longer brings enough information for analyzing structure of the spaces, because the system is at most a pencil. Therefore multiples of the half-anticanonical system have been used in order to find out and explore new Moishezon twistor spaces. In these analyses the first essential part is always to find a pluri-half-anticanonical system which is not composed with the half-anticanonical system. Once this is established, by investigating the rational map associated to the multiple system, we can make a traditional analysis in algebraic geometry to gain detailed structure of the twistor spaces.

In the paper [7], we pursued such a direction and found a series of Moishezon twistor spaces on \(n\mathbb{CP}^2\), the connected sum of \(n\) complex projective planes where \(n\) being arbitrary with \(n \geq 4\), such that the \((n-2)\)-th power of the half-anticanonical system induces a rational map which is two-to-one over the image. This image is a scroll of 2-planes over a rational normal curve, which is canonically embedded in \(\mathbb{CP}^n\), and the branch divisor is a cut of the scroll by a single quartic hypersurface. Further, a defining equation of the quartic hypersurface was determined.

While these twistor spaces on \(n\mathbb{CP}^2\) can be regarded as a generalization of the twistor spaces on \(3\mathbb{CP}^2\) studied by Poon [15] and Kreussler-Kurke [11], from detail investigation in the case of \(4\mathbb{CP}^2\) [9, 10], it is strongly expected that the twistor spaces in [7] are specialization of more general twistor spaces which also have a double covering structure over the same scroll. A purpose of this paper is to show that this is really the case, and determine the defining equation of the quartic hypersurface which cuts out the branch divisor of the double
covering. However, for the actual analysis totally new method is required, because unlike the ones in [7], the present twistor spaces do not have an effective $\mathbb{C}^*$-action.

In Section 2.1 we construct a rational surface $S$ and investigate its pluri-anticanonical systems. We are concerned with a twistor space on $n\mathbb{CP}^2$ which contains this surface $S$ as a member of the half-anticanonical system $|F|$. We devote most of Section 2.2 to prove that the multiple system $|(n - 2)F|$ of the twistor space is not composed with the system $|F|$ (which is just a pencil). The basic idea for proving this is simple to the effect that we pick up distinct $(n - 2)$ general members of the pencil $|F|$ and look at the restriction of the system $|(n - 2)F|$ to the union of these members. However, this does not work well in this primitive form and we need to blowup the twistor space at the base curve of $|F|$. This makes the $(n - 2)$ divisors disjoint, and further by letting the exceptional divisor of the blowup to be included in the restriction, we obtain a crucial vanishing of cohomology groups (Proposition 2.8). This reduces the computations for the multiple system to those on a divisor of smooth normal crossing. The computations on the last divisor work very effectively, and we can finally show that the system $|(n - 2)F|$ is $n$-dimensional as a linear system (Proposition 2.3). Once this is proved, it is not difficult to show that the image of the rational map associated to the multiple system is a scroll of planes over a rational normal curve in $\mathbb{CP}^{n-2}$, and that the map is two-to-one over the scroll, whose branch divisor is a cut of the scroll by a quartic hypersurface (Proposition 2.10).

Section 3 is employed to show that there exist two special reducible members of the system $|(n - 2)F|$ which consist of two irreducible components. These two divisors play a significant role for obtaining the defining equation of the quartic hypersurface. The Chern classes of the irreducible components of these reducible members are presented in explicit forms (Proposition 3.1). The method for proving this existence is similar to the method in Section 2 but as a restriction we have to take degree-one divisors instead of the divisors in $|F|$, which makes the computations considerably heavier and much more subtle. But again by making the exceptional divisor of the same blowup to be included in the restriction, we are able to obtain a critical vanishing result (Proposition 3.3). Then after long computations over the degree-one divisors and the exceptional divisors, we finally obtain the desired existence of the special reducible members of the system $|(n - 2)F|$.

Like most other non-projective Moishezon twistor spaces, the multiple system $|(n - 2)F|$ has non-empty base locus. In Section 4 we provide a complete elimination of the base locus, by a succession of explicit blowups (Proposition 4.3). While the base locus of the system consists of just strings of smooth rational curves, for a complete elimination a number of blowing-up is required. As a consequence, we obtain information about the image of particular twistor lines and degree-one divisors under the rational map associated to $|(n - 2)F|$.

In the final section, assembling all the results in Sections 2, 3 and 4, we determine a defining equation of the quartic hypersurface which cuts out the branch divisor on the scroll. In Section 5.1 by utilizing the reducible members obtained in Section 3 we find special curves on the branch divisor and prove the existence of a quadratic hypersurface in $\mathbb{CP}^n$ which contains all these special curves. In Section 5.2 we prove the main result which determines the defining equation of the quartic hypersurface (Theorem 5.2). The argument in the proof is mostly algebraic, and is an improvement of the proof given in [9, Theorem 4.5]. Finally in Section 5.3 we first compute the dimension of the moduli space of the present twistor spaces. Next we discuss some global structure of the moduli space. In particular, we see that the present moduli space can be partially compactified (completed)
by attaching some part (a stratum) of the moduli spaces of LeBrun twistor spaces. We also mention that a stratum of of the moduli space of the twistor spaces constructed in [8] is also naturally attached to give a partial compactification of the moduli space of the present twistor spaces.

Notation. The letter $F$ always denotes the canonical square root of the anticanonical line bundle over a twistor space. The degree of a divisor in a twistor space means the intersection number with a twistor line. Linear equivalence between divisors are often denoted by $\sim$. If $Y$ is a subvariety of a complex manifold $X$, and if $\mathcal{S}$ is a sheaf over $X$, we often write $H^q(Y, \mathcal{S})$ to mean $H^q(Y, \mathcal{S}|_Y)$. We also write $h^q(X, \mathcal{S})$ for $\dim H^q(X, \mathcal{S})$. An $(a,b)$-curve on the product surface $\mathbb{CP}^1 \times \mathbb{CP}^1$ means a curve of bidegree $(a,b)$. For a vector space $V$, $S^kV$ denotes the $k$-th symmetric product. We often identify a divisor with the associated line bundle. Usually we use the same letter for an analytic subspace in a complex space and its strict transform into a blowup.

2. Analysis of the pluri-half-anticanonical system

2.1. Construction of a rational surface. First we construct a rational surface $S$ which will be contained in the twistor spaces as a real irreducible member of the system $|F|$.

Let $n \geq 4$ be any fixed integer. First consider the product surface $\mathbb{CP}^1 \times \mathbb{CP}^1$ and define a real structure on it as the product of the complex conjugation and the anti-podal map. Fix a real irreducible $(2,2)$-curve consisting of four irreducible components. We write it as $C_1 + C_2 + C_1 + C_2$, where $C_1$ and $\overline{C}_1$ are $(1,0)$-curves and $C_2$ and $\overline{C}_2$ are $(0,1)$-curves. Next we choose any two points on $C_1$ which are not on $C_2 \cup \overline{C}_2$ (i.e., not on the corners). We also choose any one point on $C_2$ which is not on $C_1 \cup \overline{C}_1$. By taking the conjugation by the real structure of these $3$ points, we obtain $6$ points on the $(2,2)$-curve. Let $S_0 \to \mathbb{CP}^1 \times \mathbb{CP}^1$ be the blowup at these $6$ points. $S_0$ is equipped with a real structure lifted from $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Next we blowup $S_0$ at the two points $C_2 \cap \overline{C}_1$ and $\overline{C}_2 \cap C_1$ $(n-3)$ times respectively, where if $n > 4$ the blowup is always done in the direction of $\overline{C}_1$ and $C_1$ respectively. (This means that blown-up point is always the intersection point of $C_1$ with the exceptional curves of the last blowup, and similar for the conjugate point.) Let $S \to S_0$ be the resulting birational morphism. Since we have blown-up $2n$ times in total, we obtain $K_S^2 = 8 - 2n$. Let

$$C := C_1 + C_2 + \cdots + C_{n-1} + C_1 + \overline{C}_2 + \cdots + \overline{C}_{n-1}$$

be the unique anticanonical curve on $S$ arranged in a cyclic order. From the construction it is immediate to see that the self-intersection numbers of the components $C_1, C_2, \cdots, C_{n-1}$ in $S$ are respectively given by

$$1 - n, -2, -2, \cdots, -2, -1.$$  

These intersection numbers are of fundamental importance throughout this paper. The original real structure on $\mathbb{CP}^1 \times \mathbb{CP}^1$ naturally lifts to the surface $S$, under which the cycle $C$ is real. We note that in this construction of the surface $S$, all freedom is in choosing two points on $C_1$ and one point on $C_2$ in the construction of the surface $S_0$, and there is no freedom in all the remaining blowups $S \to S_0$.

Proposition 2.1. The pluri-anticanonical systems of the surface $S$ enjoy the following properties:

(i) If $0 \leq m < n - 2$, then $h^0(mK_S^{-1}) = 1$. 
(ii) \( h^0((n - 2)K_S^{-1}) = 3 \).

(iii) The fixed components of the system \( |(n - 2)K_S^{-1}| \) is the curve

\[
(2.3) \quad (n - 3)(C_1 + C_1) + \sum_{i=2}^{n-2} (n - 1 - i)(C_i + C_i).
\]

(iv) After removing this curve, the system \( |(n - 2)K_S^{-1}| \) is base point free.

(v) If \( \phi : S \to \mathbb{CP}^2 \) denotes the morphism associated to the last system, \( \phi \) is of degree two and the branch divisor is a quartic curve.

Proof. (i) and (iii) can be proved by computing intersection numbers, and we omit the detail. For (ii), if we define a line bundle \( L \) over \( S \) as \( (n - 2)K_S^{-1} \) minus the curve \((2.3)\), then for any \( i \neq 2 \), the intersection number \( (L, C_i)_S \) can be seen to be zero. Hence we have an exact sequence

\[
0 \to L - (C - C_2 - \overline{C}_2) \to L \to \mathcal{O}_{C - C_2 - \overline{C}_2} \to 0.
\]

If we write \( L' \) for the first non-trivial term of this sequence, by Riemann-Roch formula we readily obtain \( \chi(L') = 1 \). From intersection numbers we also get \( h^0(L') = 1 \). As the line bundle \( L' \) has a non-zero section, we also have \( H^2(L') = 0 \). Hence we obtain \( H^1(L') = 0 \). Therefore noting that the curve \( C - C_2 - \overline{C}_2 \) consists of two connected components, from \((2.4)\), we get \( h^0(L) = 3 \), and we obtain (ii). For (iv) we readily have \( (L, L)_S = 2 \) and \( (L, C_2)_S = 1 \). Also, from the cohomology exact sequence of \((2.4)\) we obtain that \( Bs |L| \) is disjoint from the curve \( C - C_2 - \overline{C}_2 \). These mean that if \( Bs |L| \neq \emptyset \) then \( Bs |L| \) consists of two points, one of which is on \( C_2 \) and the other is on \( \overline{C}_2 \). Moreover these two points are not on the complement \( C - C_2 - \overline{C}_2 \). Let \( \nu : S' \to S \) be the blowup at these two points, \( E \) and \( \overline{E} \) the exceptional curves, and put \( L'' = \nu^*L - E - \overline{E} \). Then we have \( (L'', L'')_{S'} = (L, L)_S - 2 = 0 \). As \( (L, C_2)_S = 1 \) we also obtain that \( Bs |L''| = \emptyset \). Hence the rational map \( \psi : S' \to \mathbb{CP}^2 \) associated to \( |L''| \) is a morphism whose image is a curve. On the other hand, since \( (\nu^*L - E - \overline{E}, E)_{S'} = 1 \), the image \( \psi(E) \) must be a line. Hence \( \psi(S') \) is a line. So \( \phi(S) \) is also a line, which is a contradiction. Therefore we obtain \( Bs |L| = \emptyset \), meaning (iv). For (v), since \( (L, L)_S = 2 \), the morphism \( \phi : S \to \mathbb{CP}^2 \) is of degree two. Also the arithmetic genus of \( L \) is easily seen to be one. Hence the branch curve must be a quartic curve. \( \square \)

We have more detail about the double covering map \( \phi : S \to \mathbb{CP}^2 \). This would be useful for understanding singularities of the branch divisor of the double covering map from the twistor space that will be obtained at the end of this section:

**Proposition 2.2.** Let \( \phi : S \to \mathbb{CP}^2 \) be the degree two morphism as in Proposition 2.1 (v). Then \( \phi \) maps the two curves \( C_2 \) and \( \overline{C}_2 \) to an identical line isomorphically, and maps two connected curves \( C_3 \cup C_4 \cup \cdots \cup C_{n-1} \cup \overline{C}_1 \) and \( \overline{C}_3 \cup \overline{C}_4 \cup \cdots \cup \overline{C}_{n-1} \cup C_1 \) to points on the line. Moreover, the branch quartic curve of \( \phi \) has two ordinary nodes at these two points.

We do not give a proof for this proposition, since it can be shown in a standard way.

### 2.2. Pluri-half-anticanonical systems of the twistor spaces.

We still fix any integer \( n \geq 4 \) and let \( S \) be the surface obtained from \( \mathbb{CP}^1 \times \mathbb{CP}^1 \) by blowing up \( 2n \) times as in Section 2.1. Next let \( Z \) be a twistor space on \( n \mathbb{CP}^2 \) and suppose that \( Z \) contains the surface \( S \) as a real member of the system \( |F| \). As before let \( C \) be the unique anticanonical curve.
(2.1) on $S$. Since $F|_S \simeq K_S^{-1}$, in view of Proposition (2.1) it is tempting to expect that the restriction map

$$H^0(Z, (n-2)F) \to H^0(S, (n-2)F) \simeq H^0((n-2)K_S^{-1})$$

is surjective. However, for any $n > 4$, there is an example of Moishezon twistor space $Z$ on $n\mathbb{CP}^2$, such that $Z$ has a real smooth $S \in |F|$ whose $|(n-2)K_S^{-1}|$ fulfills the properties (i)–(v) (with some minor modification for the explicit form of the fixed component (2.3)) but nevertheless the restriction map (2.5) is not surjective. Thus validity of the above expectation is very subtle.

In spite of this, for the present twistor spaces, we have the following.

**Proposition 2.3.** For the pluri-half-anticanonical systems of the twistor space $Z$ on $n\mathbb{CP}^2$, we have the following.

(i) $h^0(F) = 2$, and $\text{Bs} |F| = C$.

(ii) If $m < n - 2$, then $H^0(mF) = S\cap S^0$, so that $h^0(mF) = m + 1$.

(iii) $h^0((n-2)F) = n + 1$. In particular, the system $|(n-2)F|$ is not composed with the pencil $|F|$.

The assertions (i) and (ii) can be shown in a standard way by using Proposition (2.1) and we omit a proof. For the rest of this section we give a proof of (iii).

For this, we first make it clear about reducible members of the pencil $|F|$. This issue is also now standard and we omit a proof. For each $1 \leq i \leq n - 1$ let $L_i$ be the twistor line through the point $C_i \cap C_{i+1}$ (i.e. a corner of the cycle $C$), where we read $C_n = C_1$. Then there exists a unique reducible member of $|F|$ which contains the twistor line $L_i$. This member consists of two irreducible components, and their intersection is exactly $L_i$. We write this member of $|F|$ as $S_i^+ + S_i^-$, where we make distinction of the two components by promising that $S_i^-$ contains the cycle $C_1$.

Let $f : Z \to \mathbb{CP}^1$ be the rational map associated to the pencil $|F|$. The indeterminacy locus of $f$ is exactly the cycle $C$ by Proposition (2.3) (i). Let $\tilde{Z} \to Z$ be the blowup of $Z$ at $C$, and $E_i$ and $\overline{E}_i$ ($1 \leq i \leq n - 1$) the exceptional divisors over the components $C_i$ and $\overline{C}_i$ respectively. The composition $\tilde{Z} \to Z \to \mathbb{CP}^1$ is a morphism, and has exactly $(n - 1)$ reducible fibers, for which we still write $S_i^+ \cup S_i^-$. From the fact that $C$ constitutes a cycle, it follows that every components $E_i$ and $\overline{E}_i$ are isomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$, and on each of these components the morphism $\tilde{Z} \to \mathbb{CP}^1$ coincides with a projection to one of the two factors. For simplicity of notation we write the total sum of the exceptional divisors as

$$E := \sum_{i=1}^{n-1} E_i + \sum_{i=1}^{n-1} \overline{E}_i.$$

As the curve $C$ forms the cycle, the divisor $E$ constitutes a ‘cylinder’ (see the left picture in Figure [1]). If we denote the strict transform of the twistor line $L_i$ ($1 \leq i \leq n - 1$) into $\tilde{Z}$ by the same letter, the intersection $L_i \cap E$ consists of two points, and these are ordinary double points (ODP-s for short) of the variety $\tilde{Z}$. (So $\tilde{Z}$ has $2(n - 1)$ ODP-s in total.) For each $i$, one of these two ODP-s is shared by the four divisors $S_i^+, S_i^-, E_i$ and $E_{i+1}$, and the other point is shared by the divisors $S_i^+, S_i^-, \overline{E}_i$ and $\overline{E}_{i+1}$, where if $i = n - 1$ we read $E_n = \overline{E}_1$ and $\overline{E}_n = E_1$. We denote by $p_i$ and $\overline{p}_i$ for the former and the latter ODP respectively. (These are indicated in the left picture in Figure [1])
For each of these ODP-s there are two ways of small resolutions. In order for later computations to be transparent, we choose small resolutions for these in the following way. When the index $i$ satisfies $1 \leq i \leq n - 2$, at the point $p_i$, we take the small resolution which blows up the pair $\{S_i^+, E_i\}$. When $i = n - 1$, at the point $p_{n-1}$, we take the small resolution which blows up the alternative pair $\{S_{n-1}^-, E_i\}$. For the conjugate point $\overline{p}_i$, $1 \leq i \leq n - 1$, we take the small resolution which is determined from that of $p_i$ by the real structure.

Let $Z_1 \to \hat{Z}$ be the birational morphism obtained by taking all these small resolutions simultaneously, and $\Delta_i$ and $\Delta_{i-1}$ (1 $\leq i \leq n - 1$) the exceptional curves of the points $p_i$ and $\overline{p}_i$ respectively. We define a morphism $\mu : Z_1 \to Z$ to be the composition $Z_1 \to \hat{Z} \to Z$. Also we write $f_1 : Z_1 \to \mathbb{CP}^1$ for the composition morphism $Z_1 \to \hat{Z} \to \mathbb{CP}^1$. This is precisely the rational map associated with the pencil $|\mu^*F|$. Of course $Z_1$ is non-singular and equipped with a natural real structure. Under the small resolution $Z_1 \to \hat{Z}$, each of the exceptional divisors $E_i$ (and $\overline{E}_i$) receives the following effect:

- the divisor $E_1 \subset \hat{Z}$ is blown up at the two points $p_1$ and $\overline{p}_{n-1}$, and the two curves $\Delta_1$ and $\Delta_{n-1}$ are inserted as the exceptional curves,
- when $1 < i < n - 1$, the divisor $E_i \subset \hat{Z}$ is blown up at one point $p_i$, and the curve $\Delta_i$ is inserted as the exceptional curve,
- the divisor $E_{n-1} \subset \hat{Z}$ remains unchanged, and hence the strict transform $E_{n-1} \subset Z_1$ is still biholomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$.

We illustrate these changes in Figure 1 in the case $n = 7$. The manifold $Z_1$ is the main stage for our computations, in which these exceptional divisors play a principal role.

We use the same letters to mean the strict transforms into $Z_1$ of the above divisors in $\hat{Z}$ and $Z$. From Proposition 2.1 (iii), the pulled-back system $|\mu^*((n - 2)F)|$ has the following divisor as fixed components at least:

\[(2.6) \quad (n - 3)(E_1 + \overline{E}_1) + \sum_{i=2}^{n-2} (n - 1 - i)(E_i + \overline{E}_i).\]

So we define a line bundle $\mathcal{L}_1$ over $Z_1$ by

\[(2.7) \quad \mathcal{L}_1 := \mu^*((n - 2)F) - (n - 3)(E_1 + \overline{E}_1) - \sum_{i=2}^{n-2} (n - 1 - i)(E_i + \overline{E}_i).\]

This is the line bundle we actually need to investigate. Note that this line bundle is also real. Here we outline our method for obtaining $h^0(\mathcal{L}_1)$.

(i) We choose any distinct non-singular members $S_1, S_2, \cdots, S_{n-2}$ of the original pencil $|F|$. We use the same letters to mean the strict transforms of these divisors into $Z_1$.

(ii) We restrict the line bundle $\mathcal{L}_1$ to the divisor

\[(2.8) \quad (S_1 \cup S_2 \cup \cdots \cup S_{n-2}) \cup E,\]

where $E$ is the total sum of the exceptional divisors of the birational morphism $\mu : Z_1 \to Z$. The point here is that unlike the union $S_1 \cup S_2 \cup \cdots \cup S_{n-2}$ in the original space $Z$, the divisor $\{2.8\}$ in $Z_1$ is clearly smooth normal crossing.

(iii) We prove that the restriction map $H^0(Z_1, \mathcal{L}_1) \to H^0((S_1 \cup \cdots \cup S_{n-2}) \cup E, \mathcal{L}_1)$ is surjective, by showing that the cohomology group $H^1$ of the kernel line bundle vanishes.
Figure 1. The exceptional divisor of $\hat{Z} \to Z$ (left) and that of $\mu : Z_1 \to Z$ (right) in the case $n = 7$. In the right picture the intersection with $S_i^+$ ($1 \leq i \leq 6$) is indicated by bold segments. The segment with a small triangle represents a base curve of $|\mathcal{L}_1|$ (Section 4).
(iv) Here comes another point: all the restrictions of $\mathcal{L}_1$ to the divisors $S_i$ and $E$ are explicitly computable. Then thanks to the fact that the divisor (2.8) is smooth normal crossing, it is possible to compute the space $H^0((S_1 \sqcup \cdots \sqcup S_{n-2}) \cup E, \mathcal{L}_1)$.

**Remark 2.4.** Since the above argument might look a bit complicated, it should be explained why we consider the restriction of the line bundle $\mathcal{L}_1$ to the divisor (2.8) for computing $h^0((n-2)F)$. By choosing the divisors $S_1, \cdots, S_{n-2} \in |F|$ as above, we have the exact sequence $0 \to \mathcal{O}_Z \to (n-2)F \to (n-2)F|_{S_1 \sqcup \cdots \sqcup S_{n-2}} \to 0$ on $Z$. Further we have $H^1(\mathcal{O}_Z) = 0$ as $Z$ is simply connected. So we can determine $h^0((n-2)F)$ if we could compute $h^0((n-2)F|_{S_1 \sqcup \cdots \sqcup S_{n-2}})$. As $(n-2)F|_{S_i} \simeq (n-2)K_{S_i}^{-1}$ for each $i$ and as we know $h^0((n-2)K_{S_i}^{-1}) = 3$ by Proposition 2.1, one might think it possible to compute $h^0((n-2)F|_{S_1 \sqcup \cdots \sqcup S_{n-2}})$. However, this seems to be impossible due to the fact that the union $S_1 \sqcup \cdots \sqcup S_{n-2}$ is not smooth normal crossing, when $n > 4$. This situation can be resolved by blowing up the base curve $C$. But it is still impossible to compute $h^0(\mathcal{L}_1)$ if we just restrict the line bundle $\mathcal{L}_1$ to the disjoint union $S_1 \sqcup \cdots \sqcup S_{n-2}$ of the strict transforms, because this time we cannot expect the restriction map to be surjective (at the level of sections). Hence we make the divisor $E$ to be included in the restriction. As we see below, this method works very effectively.

Beginning the actual computations for $h^0(\mathcal{L}_1)$, we first compute the restriction of $\mathcal{L}_1$ to the exceptional divisor $E_i$ for this, we first define curves on $\tilde{Z}_1$ by

$$c_{i,j} := (S_i^+ \cup S_j^-) \cap E_j, \quad i, j \in \{1, 2, \cdots, n-1\}. \quad (2.9)$$

This curve is naturally identified with the rational curve $C_j$ in $Z$ through the birational morphism $\mu$, and the first index $i$ indicates in which degree-one divisor the curve is contained.

We also define other curves on $\tilde{Z}_1$ by

$$\Gamma_i := E_i \cap E_{i+1}, \quad 1 \leq i \leq n-1, \quad (2.10)$$

where we read $E_n = \tilde{E}_1$ when $i = n-1$. See the right picture in Figure 1 for the configuration of the curves $c_{i,j}, \Gamma_i$ and $\Delta_i$. We note that as basis of the cohomology group $H^2(E_i, Z)$ we can take the following curves:

- $C_{1,1}, \Delta_1, \Gamma_1$ and $\Gamma_{n-1}$ when $i = 1$,
- $C_{i,i}, \Delta_i$, and $\Gamma_i$ when $1 < i < n-1$,
- $C_{n-1,n-1}$ and $\Gamma_{n-1}$ when $i = n-1$.

In particular the restriction of the line bundle $\mathcal{L}_1$ to the divisor $E_i$ can be detected from the intersection numbers with these curves.

**Lemma 2.5.** The intersection numbers of $\mathcal{L}_1$ with the above curves are given by

$$\langle \mathcal{L}_1, c_{i,i} \rangle_{\tilde{Z}_1} = \begin{cases} 0, & i = 1, 2, n-1, \\ -1, & 2 < i < n-1, \end{cases} \quad (2.11)$$

$$\langle \mathcal{L}_1, \Delta_i \rangle_{\tilde{Z}_1} = \begin{cases} 0, & i = 1, \\ 1, & 1 < i < n-1, \\ n-3, & i = n-1, \end{cases} \quad (2.12)$$

and

$$\langle \mathcal{L}_1, \Gamma_i \rangle_{\tilde{Z}_1} = \begin{cases} n-2-i, & 1 \leq i \leq n-2, \\ 0, & i = n-1. \end{cases} \quad (2.13)$$
Proof. First noting the relation $\mu^* F \sim f_1^* \mathcal{O}_\Lambda(1) + E$, we have the following useful formula:

\begin{equation}
L_1 \sim f_1^* \mathcal{O}_\Lambda(n-2) + (E_1 + \overline{E}_1) + \sum_{j=2}^{n-1} (j-1)(E_j + \overline{E}_j). \tag{2.14}
\end{equation}

Since the curves $C_{i,j}$ and $\Delta_i$ are contained in a fiber $S_i^+ \cup S_i^-$ of $f_1$ and the curve $\Gamma_i$ is a section of $f_1$, we have

\begin{equation}
(f_1^* \mathcal{O}(1), C_{i,j})Z_i = (f_1^* \mathcal{O}(1), \Delta_i)Z_i = 0, \quad (f_1^* \mathcal{O}(1), \Gamma_i)Z_i = 1. \tag{2.15}
\end{equation}

For the intersection numbers of $E_i$ with the curves in the lemma, if the curve is not contained in $E_i$ but intersects $E_i$, the intersection number is one as they intersect transversally at a point. On the other hand, for a curve which is contained in $E_i$ such as the curve $\Delta_i$ with $1 < i < n - 1$, noting $\Delta_i = S_i^+ \cap E_i$ and $\Delta_i$ is contained in $S_i^+$ as a $-1$-curve, we have

\begin{equation}
(E_i, \Delta_i)Z_i = (\Delta_i, \Delta_i)_{S_i^+} = -1 \quad (1 < i < n - 1). \tag{2.16}
\end{equation}

Similarly, noting that $C_{i,j}$ and $\Gamma_i$ are contained in $E_i$ as $-1$-curves for these $i$, we have

\begin{equation}
(E_i, C_{i,j})Z_i = (E_i, \Gamma_i)Z_i = -1, \quad 1 < i < n - 1. \tag{2.17}
\end{equation}

Also, for the curves in the remaining components $E_1$ and $E_{n-1}$, we have

\begin{equation}
(E_1, C_{1,i})Z_i = 1 - n, \quad (E_1, \Delta_i)Z_i = -1, \quad (E_1, \Gamma_1)Z_i = (E_1, \Gamma_{n-1})Z_i = 0, \tag{2.18}
\end{equation}

\begin{equation}
(E_{n-1}, C_{n-1,n-1})Z_i = (E_i, \Gamma_{n-1})Z_i = -1. \tag{2.19}
\end{equation}

We note that (2.16)–(2.19) uniquely specify the normal bundle $\mathcal{O}_Z(E_i)|_{E_i}$ for any $1 \leq i \leq n - 1$. (These will be frequently used later.)

With these preparatory data, the intersection numbers in the lemma can be computed readily. For example, for proving (2.11), looking the right picture in Figure 1, when $i$ satisfies $1 < i < n - 1$, the curve $C_{i,j}$ intersects only with $E_{i-1}$ and $E_i$. Therefore using (2.14) with the aid of (2.15) and (2.17), when $2 < i < n - 1$, we compute

\begin{align*}
(L_1, C_{i,j})Z_i &= (i - 2)(E_{i-1}, C_{i,j})Z_i + (i - 1)(E_i, C_{i,j})Z_i \\
&= (i - 2) + (i - 1)(-1) = -1.
\end{align*}

(The case $i = 2$ requires an independent treatment because of the form of the R.H.S. of (2.14).) Hence we obtain the second case in (2.11). The first case in (2.11) can be obtained in a similar way by using (2.17)–(2.19). The other two assertions (2.12) and (2.13) can be obtained in a similar way by using (2.14)–(2.19). \qed

Remark 2.6. We have $(L_1, C_{n-1,n-1})Z_i = (L_1, \Gamma_{n-1})Z_i = 0$ by the lemma. It follows that $L_1$ is trivial over the two components $E_{n-1}$ and $\overline{E}_{n-1}$, which will turn out to be useful later. This is a reason why we choose the particular small resolutions $Z_1 \to \hat{Z}$.

By using the lemma, we show the following proposition which will be needed for proving Proposition 2.3 (iii).

Proposition 2.7. For the restriction of the line bundle $L_1$ over $Z_1$, we have

\[ h^0((S_1 \sqcup \cdots \sqcup S_{n-2}) \cup E, L_1) = n. \]
Proof. We first show $h^0(E, \mathcal{L}_1) = 2$. Recalling that $\mathcal{L}_1$ is trivial over $E_{n-1}$ and $\overline{E}_{n-1}$, we prove that the restriction map

\[(2.20) \quad H^0(E, \mathcal{L}_1) \rightarrow H^0(E_{n-1} \cup \overline{E}_{n-1}, \mathcal{L}_1) \simeq \mathbb{C}^2\]

is isomorphic, by verifying that any section over $E_{n-1} \cup \overline{E}_{n-1}$ extends in a unique way to the whole $E$. For this we consider the following three restriction maps:

1. $H^0(E_i, \mathcal{L}_1) \rightarrow H^0(\Gamma_i, \mathcal{L}_1)$, $3 \leq i \leq n-2$,
2. $H^0(E_1, \mathcal{L}_1) \rightarrow H^0(\overline{\Gamma}_{n-1}, \mathcal{L}_1)$,
3. $H^0(E_2, \mathcal{L}_1) \rightarrow H^0(\Gamma_1 \cup \Gamma_2, \mathcal{L}_1)$.

By using Lemma 2.5 it is elementary to see that all of these maps are isomorphic. Then by the isomorphicity of the restriction maps (1) and (2) and their conjugations by the real structure, any section $s \in H^0(E_{n-1}, \mathcal{L}_1)$ (resp. $t \in H^0(\overline{E}_{n-1}, \mathcal{L}_1)$) successively extends in a unique way to a section over the connected union $E_3 \cup E_4 \cup \cdots \cup E_{n-1} \cup \overline{E}_1$ (resp. $\overline{E}_3 \cup \overline{E}_4 \cup \cdots \cup \overline{E}_{n-1} \cup E_1$). In particular, any section $(s, t) \in H^0(E_{n-1} \cup \overline{E}_{n-1}, \mathcal{L}_1)$ uniquely determines a section over the curves $\Gamma_1 \cup \Gamma_2$ and $\overline{\Gamma}_1 \cup \overline{\Gamma}_2$. Hence by the isomorphicity of (3), we conclude that $(s, t) \in H^0(E_{n-1} \cup \overline{E}_{n-1}, \mathcal{L}_1)$ determines a section over the remaining components $E_2$ and $\overline{E}_2$ in a unique way. Therefore the restriction map (2.20) is isomorphic, and we obtain $h^0(E, \mathcal{L}_1) = 2$.

On the other hand, recalling that for any $1 \leq k \leq n-2$ the surface $S_k$ in $Z_1$ is canonically isomorphic to the original surface $S_k$ in $Z$, from the definition of the line bundle $\mathcal{L}_1$, the restriction $\mathcal{L}_1|_{S_k}$ is linearly equivalent to the movable part of the system $|(n-2)K_{S_k}|$. Therefore we have $h^0(S_k, \mathcal{L}_1) = 3$ by Proposition 2.1(ii). We now show that the restriction map

\[(2.21) \quad H^0(S_k, \mathcal{L}_1) \rightarrow H^0(S_k \cap E, \mathcal{L}_1)\]

is surjective for any $1 \leq k \leq n-2$. For this we note that under the above isomorphism, the intersection $S_k \cap E$ is identified with the cycle $C$ on $Z$. Therefore from the self-intersection numbers (2.2) of the components of the cycle $C$, we can readily see that the degree of the line bundle $\mathcal{L}_1|_{S_k}$ over each component of $S_k \cap E$ satisfies

\[(2.22) \quad (\mathcal{L}_1, S_k \cap E)|_{Z_1} = \begin{cases} 0, & i \neq 2, \\ 1, & i = 2. \end{cases}\]

From these it is elementary to show that $h^0(S_k \cap E, \mathcal{L}_1) = 2$ and $h^0(S_k, \mathcal{L}_1|_{S_k} \setminus (S_k \cap E)) = 1$. As the latter space is exactly the kernel of the restriction map (2.21), by dimension counting, we obtain that (2.21) is surjective.

For completing the proof of the proposition, let

\[(2.23) \quad H^0(E, \mathcal{L}_1) \oplus \left( \bigoplus_{k=1}^{n-2} H^0(S_k, \mathcal{L}_1) \right) \xrightarrow{d} \bigoplus_{k=1}^{n-2} H^0(E \cap S_k, \mathcal{L}_1)\]

be the linear map which takes differences on all connected components of the intersection $E \cap (S_1 \cup \cdots \cup S_{n-1})$. Then since the divisor (2.8) is smooth normal crossing, we have

\[\text{Ker } d \simeq H^0((S_1 \cup \cdots \cup S_{n-2}) \cup E, \mathcal{L}_1).\]

The map $d$ is surjective since all the maps (2.21) are surjective. Therefore, again by dimension counting, we finally obtain

\[h^0((S_1 \cup \cdots \cup S_{n-2}) \cup E, \mathcal{L}_1) = 2 + 3(n-2) - 2(n-2) = n,\]
and we obtain Proposition 2.7.

Continuing a proof of Proposition 2.3 (iii), we define another line bundle over $Z_1$ by

$$(2.24) \quad \mathcal{L}'_1 := \mathcal{L}_1 - \sum_{k=1}^{n-2} S_k - E.$$ 

This is the kernel from the restriction of $\mathcal{L}'_1$ to the key divisor (2.8). This line bundle is still real. For this line bundle we have the following critical vanishing result:

**Proposition 2.8.** Let $\mathcal{L}'_1$ be the line bundle (2.24) over $Z_1$ as above. Then we have $h^0(Z_1, \mathcal{L}'_1) = 1$ and $H^q(Z_1, \mathcal{L}'_1) = 0$ for any $q > 0$.

**Proof.** First, summing up the relation $S_k \sim \mu^* F - E$ on $Z_1$ for each $1 \leq k \leq n - 2$, we have $\sum_{k=1}^{n-2} S_k \sim \mu^*((n - 2)F) - (n - 2)E$. Hence from (2.7) we readily obtain

$$(2.25) \quad \mathcal{L}'_1 \sim \sum_{i=3}^{n-1} (i - 2)(E_i + \overline{E}_i).$$ 

We note that thanks to Lemma 2.5 and the intersection numbers (2.16)–(2.19), all the restrictions $\mathcal{O}_{Z_1}(E_i + \overline{E}_i)|_{E_i \cup \overline{E}_i}$ are explicitly computable for any $i$ and $j$. By using these, it is possible to decrease the coefficients in (2.25) one by one without changing arbitrary cohomology groups, by subsequently considering the restrictions to the exceptional divisors in the following order:

$$E_3 \sqcup \overline{E}_3 \quad E_4 \sqcup \overline{E}_4 \quad E_5 \sqcup \overline{E}_5 \quad E_6 \sqcup \overline{E}_6 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10$$

Here for simplicity we are displaying the order in the case $n = 7$. This means that we first restrict $\mathcal{L}'_1$ to $E_6 \cup \overline{E}_6$, and second restrict $\mathcal{L}'_1 - (E_6 + \overline{E}_6)$ to $E_5 \cup \overline{E}_5$, and third restrict $\mathcal{L}'_1 - (E_6 + \overline{E}_6 + E_5 + \overline{E}_5)$ to $E_4 \cup \overline{E}_4$, and so on. Then if we write $\mathcal{O}_{E_i}(a, b)$ for the pullback of the line bundle $\mathcal{O}(a, b)$ over the original $E_i \simeq \mathbb{CP}^1 \times \mathbb{CP}^1 \subset \mathcal{Z}$ by the small resolution $Z_1 \to \mathcal{Z}$, then the bundles over $E_i$ appearing in the above restriction process are always of the form $\mathcal{O}_{E_i}(-1, d)$ for some $d \in \mathbb{Z}$. (Here, more precisely, $\mathcal{O}(0, 1)$ represents the fiber class of the projection $E_i \to \Lambda$.) By the real structure we have the same result for the restrictions to $\overline{E}_i$. Therefore any cohomology group vanishes, and we finally obtain

$$H^q(Z_1, \mathcal{L}'_1) \simeq H^q(Z_1, \mathcal{O}_{Z_1}), \quad q \geq 0.$$ 

Further since the morphism $\mu : Z_1 \to Z$ is birational, we have $H^q(\mathcal{O}_{Z_1}) \simeq H^q(\mathcal{O}_Z)$ for any $q \geq 0$. Therefore we have $H^q(\mathcal{L}'_1) \simeq H^q(\mathcal{O}_Z)$ for any $q \geq 0$, which easily implies the claim of the proposition. \qed

**Proof of Proposition 2.3 (iii).** From the definition of the line bundles $\mathcal{L}'_1$ and $\mathcal{L}'_1$, we have an isomorphism $H^0(Z, (n - 2)F) \simeq H^0(Z_1, \mathcal{L}_1)$ and an exact sequence

$$(2.26) \quad 0 \to \mathcal{L}'_1 \to \mathcal{L}_1 \to \mathcal{L}_1|_{(S_1 \sqcup \ldots \sqcup S_{n-2}) \cup E} \to 0.$$ 

Hence by making use of Propositions 2.7 and 2.8 we obtain

$$h^0(Z, (n - 2)F) = h^0(\mathcal{L}_1|_{(S_1 \sqcup \ldots \sqcup S_{n-2}) \cup E}) + 1 = n + 1.$$
This completes a proof.

This readily implies the following.

**Proposition 2.9.** For any non-singular member \( S \in |F| \), the restriction map \( H^0(Z, (n-2)F) \to H^0((n-2)K_S^{-1}) \) (see (2.5)) is surjective.

**Proof.** From the standard exact sequence

\[
0 \to (n-3)F \to (n-2)F \to (n-2)K_S^{-1} \to 0
\]

and Proposition 2.3 (ii), we obtain an exact sequence

\[
0 \to \mathbb{C}^{n-2} \to H^0((n-2)F) \to H^0((n-2)K_S^{-1}).
\] (2.27)

Moreover we have \( h^0((n-2)F) = n+1 \) by Proposition 2.3 (iii), and also \( h^0((n-2)K_S^{-1}) = 3 \) by Proposition 2.1 (ii). Therefore from the exact sequence (2.27) we obtain that the restriction map is surjective. \( \square \)

For the rational map associated to the system \( |(n-2)F| \), we have the following

**Proposition 2.10.** Let \( \Phi : Z \to \mathbb{C}^n \) be the rational map associated to the linear system \( |(n-2)F| \). Then we have:

(i) The image \( \Phi(Z) \) is a scroll of 2-planes over a rational normal curve in \( \mathbb{C}^{n-2} \).

(ii) The map \( \Phi \) is two to one over the scroll.

(iii) The branch divisor of \( \Phi \) is a cut of the scroll by a single quartic hypersurface.

**Proof.** We just write an outline of the proof, since these can be proved in a similar way to [9, Propositions 3.2 and 3.4]. From the subspace \( S^{n-2}H^0(F) \subset H^0((n-2)F) \) and the rational maps associated to these linear systems, we obtain the following commutative diagram of rational maps:

\[
\begin{array}{ccc}
Z & \xrightarrow{\Phi} & \mathbb{C}^n \\
\Phi|_{(n-2)F} \downarrow & & \downarrow \pi \\
\mathbb{C}^1 & \xrightarrow{\iota} & \mathbb{C}^{n-2},
\end{array}
\] (2.28)

where \( \pi \) is the linear projection induced from the above inclusion of the subspace, and \( \iota \) is an embedding as a rational normal curve. Moreover by Proposition 2.9 the restriction map (2.5) is surjective. Hence the restriction of \( \Phi \) to any non-singular member \( S \in |F| \) is exactly the rational map \( \phi : S \to \mathbb{C}^2 \) associated to the net \( |(n-2)K_S^{-1}| \). Hence by the commutativity of the diagram (2.28) we obtain the claim (i). Also, Proposition 2.1 (v) means the assertions (ii) and (iii). \( \square \)

### 3. Finding reducible members

Our final goal is to determine a defining equation of the quartic hypersurface which cuts out the branch divisor of the map \( \Phi : Z \to Y \) (see Proposition 2.10 (iii)). For this purpose, in this section, we find two reducible members of the system \( |(n-2)F| \), each of which consists of two irreducible components. As in the case of \( 4\mathbb{C}^2 \) studied in [9, 10], existence of these reducible members brings a strong constraint for a defining equation of the quartic hypersurface. But in contrast with the case of \( 4\mathbb{C}^2 \), for many reasons, finding these divisors in the present case is incomparably difficult.
Let $S$ be the rational surface constructed in Section 2.1 which is contained in the twistor space $Z$ as a real member of $|F|$ by our assumption. Let $\epsilon : S \to \mathbb{CP}^1 \times \mathbb{CP}^1$ be the composition of the explicit blowups give in Section 2.1. (So $\epsilon$ is the composition of $S \to S_0$ and $S_0 \to \mathbb{CP}^1 \times \mathbb{CP}^1$.) Let $e_1, \cdots, e_n$ and $\overline{e}_1, \cdots, \overline{e}_n$ be the elements of $H^2(S, \mathbb{Z})$ which are represented by the exceptional curves of $\epsilon$, named after the following natural rule: $e_1$ and $e_2$ are represented by the exceptional curves of the two blowup points on $C_1$ for obtaining the surface $S_0$, and $e_3$ is represented by the exceptional curve of the blowup point on $C_2$ for obtaining $S_0$. The remaining classes $e_4, e_5, \cdots, e_n$ are chosen in a standard way from the iterated blowup $S \to S_0$. In particular, the classes $e_4, e_5, \cdots, e_{n-1}$ are not represented by an irreducible curve, and $e_n$ is exactly the class of the curve $C_{n-1}$. From the choice these classes satisfy $(e_i, e_j)_S = -\delta_{ij}$. As a basis of the cohomology group $H^2(S, \mathbb{Z})$ we can take the following $(2n+2)$ classes:

\begin{equation}
(3.1) \quad e_1, \cdots, e_n, \overline{e}_1, \cdots, \overline{e}_n, e^*O(1,0), e^*O(0,1).
\end{equation}

Note that the roles of the first two classes $e_1$ and $e_2$ are in some sense ‘symmetric’.

Let $\varpi : Z \to n\mathbb{CP}^2$ be the twistor fibration, and let $\alpha_1, \cdots, \alpha_n$ be elements of $H^2(n\mathbb{CP}^2, \mathbb{Z})$ which are uniquely determined from the condition $(\varpi^*\alpha_i)|_S = e_i - \overline{e}_i$ (see [13] for the structure of the restriction $\varpi|_S : S \to n\mathbb{CP}^2$). As the pullback $\varpi^* : H^2(n\mathbb{CP}^2, \mathbb{Z}) \to H^2(Z, \mathbb{Z})$ is injective, in the following we just write $\alpha_i$ to mean $\varpi^*\alpha_i$. Then the purpose of this section is to prove the following existence result:

**Proposition 3.1.** Let $\mathcal{M}$ be the holomorphic line bundle over $Z$ whose cohomology class is given by

\begin{equation}
(3.2) \quad \frac{n-2}{2}F - \frac{1}{2}\left( (n-2)\alpha_1 + (n-4)\sum_{i=2}^n \alpha_i \right).
\end{equation}

Then the linear system $|\mathcal{M}|$ consists of a single member, and it is irreducible. Also, the same conclusion holds for another cohomology class

\begin{equation}
(3.3) \quad \frac{n-2}{2}F - \frac{1}{2}\left( (n-2)\alpha_2 + (n-4)\sum_{i \neq 2} \alpha_i \right).
\end{equation}

We note that the latter class (3.3) is obtained from (3.2) by just exchanging the role of $\alpha_1$ and $\alpha_2$. (This reflects the above ‘symmetric’ property of $e_1$ and $e_2$.) We also note that the line bundle $\mathcal{M}$ and the other one are not real, and satisfy the relation

$$\mathcal{M} + \sigma^*\mathcal{M} \simeq (n-2)F.$$ 

Therefore the single member of $|\mathcal{M}|$ (and also the single member of another system) gives a reducible member of the system $|(n-2)F|$ consisting of two irreducible components.

Our proof of Proposition 3.1 broadly proceeds in a similar way to Proposition 2.3 (iii). Namely we pullback the line bundle $\mathcal{M}$ to the same blowup space $Z_1$, subtract obvious fixed components from the pullback, and then restrict the resulting bundle to some divisors of smooth normal crossing. But the choice of the last divisor is much more subtle than we did in Section 2 as we see below.

We begin with determining fixed components of the linear system $|\mathcal{M}|$ on the surface $S$: 

\[ \text{\ldots} \]
**Proposition 3.2.** Let $S \in |F|$ be any non-singular member of the pencil $|F|$. Then the linear system $|\mathcal{M}|_S$ contains the following curve as fixed components at least:

\begin{equation}
(n - 3)C_1 + \sum_{i=2}^{n-2} (n - 1 - i)C_i.
\end{equation}

In other words, any section of the line bundle $\mathcal{M}|_S$ vanishes along the curve $C_i$ by the order indicated by the coefficient at least.

**Proof.** Though this is not immediate to see, it can be proved in an elementary way, so we just give an outline. From the explicit form of the line bundle $\mathcal{M}$ and the relation $\alpha_i|_S = e_i - \tau_i$ for $1 \leq i \leq n$, we can concretely write down the cohomology class of the line bundle $\mathcal{M}|_S$, in terms of the basis (3.1). Also, the cohomology class of the curve $C_i$ can be expressed in terms of the same basis. Therefore we can compute the intersection numbers of the line bundle $\mathcal{M}|_S$ with the curve $C_i$. From this, by checking negativity or vanishing of the intersection numbers successively, we can show that any section of $\mathcal{M}|_S$ has to vanish along the curve (3.4) with multiplicities indicated by the coefficients. \hfill \Box

Let $\mu : Z_1 \to Z$ be the birational morphism given in Section 2.2. By Proposition 3.2, if we define a line bundle $\mathcal{M}_1$ over $Z_1$ by

\begin{equation}
\mathcal{M}_1 := \mu^* \mathcal{M} - (n - 3)E_1 - \sum_{i=2}^{n-2} (n - 1 - i)E_i,
\end{equation}

then we have an isomorphism

\begin{equation}
H^0(Z, \mathcal{M}) \simeq H^0(Z_1, \mathcal{M}_1).
\end{equation}

We are going to compute the right-hand-side by restricting $\mathcal{M}_1$ to the divisor

\begin{equation}
E + \sum_{i=1}^{n-2} S_i^-,
\end{equation}

where as in Section 2.2, $S_i^-$ is the strict transform of an irreducible component of a reducible member of the pencil $|F|$, and $E$ is the total sum of the exceptional divisors of the birational morphism $\mu$. We note that the divisor (3.7) is again smooth normal crossing, and that the degree of the divisor (3.7) is equal to that of $\mathcal{M}$. (Note that in (3.7) the divisor $S_{n-1}^-$ is not included. The reason why we restrict to this particular divisor among numerous possible choices would become evident in the course of the proof of Proposition 3.3 below.)

We define another line bundle $\mathcal{M}_1'$ over $Z_1$ by

\begin{equation}
\mathcal{M}_1' := \mathcal{M}_1 - \left( E + \sum_{i=1}^{n-2} S_i^- \right),
\end{equation}

which is the kernel of the restriction of the line bundle $\mathcal{M}_1$ to the divisor (3.7). Then one of the keys for computing $H^0(\mathcal{M}_1)$ is the following critical vanishing result:

**Proposition 3.3.** For any $q \geq 0$, we have $H^q(Z_1, \mathcal{M}_1') = 0$.

**Proof.** The strategy is the same as a similar result Proposition 2.8 in the last section, but the required computations are more involved. In this proof for distinguish divisor $S_i^-$ in $Z$ and its strict transform into $Z_1$, we write $\mathcal{O}_Z(S_i^-)$ and $\mathcal{O}_{Z_1}(S_i^-)$ respectively.
First by using the fact that the restriction map $H^2(Z, \mathbb{Z}) \to H^2(S, \mathbb{Z})$ is injective, and also from a concrete form of the divisor $S_i^-|_S$ which is exactly a half of the cycle $C$, it is possible to write down the Chern classes of the original divisors $S_i^-$ in $Z$, in terms of $F$ and the classes $\alpha_1, \cdots, \alpha_n$. The result is as follows:

$$\mathcal{O}_Z(S_i^-) \simeq \frac{1}{2} F - \frac{1}{2} \sum_{j=1}^{n} \epsilon_{ij} \alpha_j, \quad \text{for } 1 \leq i \leq n - 2,$$

where $\epsilon_{ij} = 1$ except $j = n - i + 1$ while $\epsilon_{ij} = -1$ if $j = n - i + 1$, and for $i = n - 1$

$$\mathcal{O}_Z(S_{n-1}^-) \simeq \frac{1}{2} F - \frac{1}{2} \sum_{j=1}^{n} \alpha_j.$$(3.10)

(The formula (3.10) will be needed later.) Summing up (3.9) for $1 \leq i \leq n - 2$, we easily obtain

$$\sum_{i=1}^{n-2} \mathcal{O}_Z(S_i^-) \simeq \frac{n-2}{2} F - \frac{1}{2} \left\{ (n-2)(\alpha_1 + \alpha_2) + (n-4) \sum_{i=3}^{n} \alpha_i \right\}.$$

(3.11)

Next as the divisor $S_1^-$ in $Z$ contains the $n$ curves $\overline{C}_2, \overline{C}_3, \cdots, \overline{C}_{n-1}$ and $C_1$ by multiplicity one, we have $\mathcal{O}_{Z_1}(S_1^-) \simeq \mu^* \mathcal{O}_Z(S_i^-) - \sum_{i=2}^{n-2} \overline{E}_i - E_1$. We have a similar isomorphism for the line bundle $\mathcal{O}_{Z_1}(S_i^-)$ for any $i$. Summing these up for $1 \leq i \leq n - 2$, we obtain

$$\sum_{i=1}^{n-2} \mathcal{O}_{Z_1}(S_i^-) \simeq \mu^* \left( \sum_{i=1}^{n-2} \mathcal{O}_Z(S_i^-) \right) - \left( \sum_{i=1}^{n-1} (n-1-i)E_i + \sum_{i=1}^{n-1} (i-1)\overline{E}_i \right).$$

(3.12)

After substituting (3.11) into (3.12), we deduce

$$\mathcal{M}' = \mu^* \mathcal{M} - (n-3)E_1 - \sum_{i=2}^{n-2} (n-1-i)E_i - E - (\text{R.H.S. of (3.12)})$$

$$= \mu^* \mathcal{O}_Z(\alpha_2) - \sum_{i=2}^{n-1} E_i + \sum_{i=1}^{n-1} (i-2)\overline{E}_i.$$

(3.13)

(In the equality (3.13) almost all terms in the pullback term canceled out, and this is the reason why we choose the particular divisor (3.7) for the restriction. The ‘smallness’ of the pullback term is crucial as we see in the following argument.)

So for the proof of the proposition it suffices to show that the cohomology group $H^q$ of the line bundle (3.13) vanishes for any $q$. We first see that the first summation in (3.13) can be entirely removed without changing any cohomology group. By adding $E_2$, we obtain the standard exact sequence

$$0 \rightarrow \mathcal{M}' \rightarrow \mathcal{M}' + E_2 \rightarrow \mathcal{M}' + E_2|_{E_2} \rightarrow 0.$$

(3.14)

For the restricted term, from (3.13) we have $\mathcal{M}' + E_2|_{E_2} \simeq \mu^* \mathcal{O}_Z(\alpha_2) - E_3|_{E_2}$. Further since the curves $e_2 \cup \overline{E}_2$ and $C_2$ (in $S$) are disjoint, this is isomorphic to $-E_3|_{E_2}$. It is easy to see that all cohomology groups vanish for the last class, and hence we obtain isomorphisms $H^q(\mathcal{M}'_1) \simeq H^q(\mathcal{M}'_1 + E_2)$ for any $q \geq 0$. Repeating this process by adding $E_3, E_4, \cdots, E_{n-1}$
one by one, we finally obtain an isomorphism

\[ H^q(\mathcal{M}_1') \cong H^q \left( \mu^* \mathcal{O}_Z(\alpha_2) + \sum_{i=1}^{n-1} (i-2)\overline{E}_i \right), \quad q \geq 0. \tag{3.15} \]

(We remark that all the line bundles over \( E_i \)'s appearing in this restriction process are mutually isomorphic, which considerably decreases the computations.)

Note that a negative term \(-\overline{E}_1\) is still included in the ingredient of the R.H.S. of (3.15). We next show that this term can also be removed. Recalling that in the surface \( \mathcal{S} \) the two curves \( \overline{C}_1 \) and \( e_2 \) are disjoint and that \( \overline{C}_1 \) and \( \overline{C}_2 \) intersect transversally at a point, we have

\[ (\mu^* \mathcal{O}_Z(\alpha_2))|_{\overline{C}_1} \cong \mu^* ( \mathcal{O}_Z(\alpha_2)|_{\overline{C}_1} ) \cong \mu^* ( \mathcal{O}_S(\alpha_2)|_{\overline{C}_1} ) \]

\[ \cong \mu^* ( \mathcal{O}_S(e_2 - \overline{C}_2)|_{\overline{C}_1} ) \cong \mu^* ( \mathcal{O}_{\overline{C}_1}(-1) ). \tag{3.16} \]

Hence noting that the component \( \overline{E}_3 \) is not included in the R.H.S. of (3.15), we obtain that the restriction of \( \mathcal{O}(\alpha_2) \) plus \( \overline{E}_1 \) to the divisor \( \overline{C}_1 \) is isomorphic to just \( \mu^* ( \mathcal{O}_{\overline{C}_1}(-1) ) \), whose all cohomology groups can be easily seen to vanish. Hence by the exact sequence similar to (3.14), we can remove the negative term \(-\overline{E}_1\) without changing any cohomology group.

Thus for completing the proof of Proposition 3.3 we are reduced to show

\[ H^q \left( \mu^* \mathcal{O}_Z(\alpha_2) + \sum_{i=3}^{n-1} (i-2)\overline{E}_i \right) = 0, \quad q \geq 0. \tag{3.17} \]

Here we note that the summation in (3.17) is exactly the one included in the line bundle \( \mathcal{L}_1' \) (see (2.25)) in the last section. Therefore the computations in the proof of Proposition 2.8 perfectly work in order to decrease the coefficients of \( \overline{E}_i \)'s one by one, and finally we obtain an isomorphism

\[ H^q \left( \mu^* \mathcal{O}_Z(\alpha_2) + \sum_{i=3}^{n-1} (i-2)\overline{E}_i \right) \cong H^q \left( \mu^* \mathcal{O}_Z(\alpha_2) \right), \quad q \geq 0. \tag{3.18} \]

The R.H.S. of (3.18) is of course isomorphic to \( H^q(\mathcal{Z}, \mathcal{O}_Z(\alpha_2)) \). For \( q = 0 \) and \( q = 3 \), this is zero by obvious reasons. For \( q = 2 \), this also vanishes by the vanishing theorem of Hitchin [7]. On the other hand, the Riemann-Roch formula gives

\[ \chi(\mathcal{O}_Z(\alpha_2)) = \frac{1}{6} \alpha_2^3 + \frac{1}{4} \alpha_2^2 c_1 + \frac{1}{12} \alpha_2 (c_1^2 + c_2) + \frac{1}{24} c_1 c_2, \tag{3.19} \]

where \( c_i \) denotes the Chern class of \( Z \). We have \( \alpha_2^3 = 0 \) since \( \alpha_2 \) is a lift from \( n\mathbb{CP}^2 \). We have \( \alpha_2^2 \cdot c_1 = -4 \) because \( K_Z \) is of degree 4 over a twistor line. On the other hand both \( c_1^2 \) and \( c_2 \) are lifts from \( n\mathbb{CP}^2 \) (see [7]), and therefore their product with \( \alpha_2 \) is zero. Finally \( c_1 c_2 \) is 24. Hence we obtain \( \chi(\mathcal{O}_Z(\alpha_2)) = -1 + 1 = 0 \). Thus we get \( H^1(\mathcal{O}_Z(\alpha_2)) = 0 \). Therefore we obtain \( H^q(\mathcal{O}_Z(\alpha_2)) = 0 \) for any \( q \geq 0 \), and finally obtain \( H^q(\mathcal{M}_1') = 0 \) for any \( q \geq 0 \). \[ \square \]

The following result is also indispensable for proving Proposition 3.1.

**Proposition 3.4.** For the restriction of the line bundle \( \mathcal{M}_1 \) over \( Z_1 \), we have

\[ h^0 \left( (S_1^- \cup \cdots \cup S_{n-2}^-) \cup E, \mathcal{M}_1 \right) = 1. \]

For the proof, we first show the following

**Proposition 3.5.** We have \( h^0(E, \mathcal{M}_1) = 1 \).
Proof. The idea is similar to the first half of the proof of Proposition \[2.7\] but since the line bundle \( \mathcal{M} \) possesses terms coming from \( n\mathbb{CP}^2 \), the computations are much more involved. As in the case of the line bundle \( \mathcal{L}_i \), we exhibit the restrictions of \( \mathcal{M}_i \) to the components of \( E \) in terms of the basis of \( H^2(E_i,\mathbb{Z}) \) given just before Lemma \[2.5\].

First for obtaining the restrictions of the pulled-back term \( \mu^*\mathcal{M} \), we first compute the intersection numbers \( (\mathcal{M},C_i)_Z \) and \( (\mathcal{M},\overline{C}_i)_Z \). (Note that since \( \mathcal{M} \) is non-real, these are not necessarily equal.) Putting \( \alpha := (n-2)\alpha_1 + (n-4)(\alpha_2 + \alpha_3 + \cdots + \alpha_n) \) so that \( 2\mathcal{M} = (n-2)F - \alpha \), and taking a non-singular member \( S \in |F| \), we have

\[
2(\mathcal{M},C_i)_Z = 2(\mathcal{M}|_S,C_i)_S = ((n-2)K^{-1}_S,C_i)_S - (\alpha|_S,C_i)_S,
\]

and a similar equality for \( (\mathcal{M},\overline{C}_i)_Z \). As \( C_i \) is a rational curve, we have \( (K^{-1}_S,C_i)_S = (C_i,C_i)_S + 2 \). For computing another term \( (\alpha|_S,C_i)_S \), it suffices to compute \( (\alpha|_S,C_i)_S \) for each \( 1 \leq j \leq n \), and this is equal to \( (e_j - \overline{\tau}_j,C_i)_S \) by our definition of the class \( \alpha_j \). From the choice of the classes \( e_1, \cdots, e_n \) given at the beginning of this section, it is not difficult to deduce the relations

\[
e_j = \sum_{k=n-j+2}^{n-1} \overline{C}_k, \quad 3 \leq j \leq n-1,
\]

and similar relations for \( \tau_j \) and \( C_k \). From these, by using the self-intersection numbers \[2.2\], we can quickly compute the intersection numbers \( (e_j,C_i)_S \) and \( (\tau_j,C_i)_S \) for any \( 1 \leq i \leq n-1 \) and \( 1 \leq j \leq n \). By using these and \[3.20\], after long but elementary computations, we obtain that

\[
(\mathcal{M},C_i)_Z = \begin{cases} 
-(n-2)(n-3) & i = 1, \\
0 & 1 < i < n-1, \\
1 & i = n-1,
\end{cases}
\]

and

\[
(\mathcal{M},\overline{C}_i)_Z = \begin{cases} 
0 & 1 \leq i < n-1, \\
n-3 & i = n-1.
\end{cases}
\]

On the other hand, since \( E_i \) (resp. \( \overline{E}_i \)) is an exceptional divisor over the curve \( C_i \) (resp. \( \overline{C}_i \)) in \( Z \), we have

\[
(\mu^*\mathcal{M})|_{E_i} \simeq \mu^*(\mathcal{M}|_{C_i}) \quad \text{and} \quad (\mu^*\mathcal{M})|_{\overline{E}_i} \simeq \mu^*(\mathcal{M}|_{\overline{C}_i}).
\]

Further, recalling the concrete form of the restriction of the birational morphism \( \mu \) to the divisor \( E_i \) (see Section \[2.2\]), we have, for any \( d \in \mathbb{Z} \),

\[
(\mu^*\mathcal{O}_{C_i}(d))|_{C_i,i} \simeq \mathcal{O}_{C_i,i}(d), \quad 1 \leq i \leq n-1,
\]

and

\[
(\mu^*\mathcal{O}_{\Delta_i})|_{\Delta_i} \simeq \mathcal{O}_{\Delta_i}, \quad (\mu^*\mathcal{O}_{\Gamma_i})|_{\Gamma_i} \simeq \mathcal{O}_{\Gamma_i}.
\]

Hence \( (\mu^*\mathcal{M},C_{i,i})_{Z_1} \) and \( (\mu^*\mathcal{M},\overline{C}_{i,i})_{Z_1} \) are exactly given by the R.H.S-s of \(3.22\) and \(3.23\) respectively, and \( (\mu^*\mathcal{M},\Gamma_i)_{Z_1} = (\mu^*\mathcal{M},\Delta_i)_{Z_1} = 0 \) for any \( i \).

On the other hand, the intersection numbers of the subtraction terms (see \[3.5\]) with the above curves in \( E_i \) and \( \overline{E}_i \) can be readily computed by using the intersection numbers \(2.16\) - \(2.19\).
Combining these, by elementary calculations, we can deduce that the intersection numbers of \( \mathcal{M} \) with the above curves are given by

\[
(\mathcal{M}, C_{i,i})_{Z_1} = \begin{cases} 
0 & i \in \{1, 2, n-1\}, \\
-1 & i \notin \{1, 2, n-1\}, 
\end{cases}
\]

\[
(\mathcal{M}, C_{i,i})_{Z_1} = 0 \quad 1 \leq i \leq n-1,
\]

\[
(\mathcal{M}, \Delta_i)_{Z_1} = \begin{cases} 
0 & i \in \{1, n-1\}, \\
1 & i \notin \{1, n-1\}, 
\end{cases}
\]

\[
(\mathcal{M}, \Delta_i)_{Z_1} = \begin{cases} 
0 & i \neq n-1, \\
n-3 & i = n-1, 
\end{cases}
\]

\[
(\mathcal{M}, \Gamma_i)_{Z_1} = \begin{cases} 
n-i-2 & i \neq n-1, \\
0 & i = n-1, 
\end{cases}
\]

\[
(\mathcal{M}, \Gamma_i)_{Z_1} = 0 \quad 1 \leq i \leq n-1.
\]

In particular, we get that \( \mathcal{M} \) is trivial over the \( n \) components \( E_{n-1}, E_1, E_2, \ldots, E_{n-1} \). Therefore by connectedness (see Figure 1), we have \( h^0(E_{n-1} \cup E_1 \cup E_2 \cup \cdots \cup E_{n-1}, \mathcal{M}) = 1 \).

For completing a proof of Proposition 3.5, we show that the restriction map

\[
H^0(E, \mathcal{M}) \to H^0(E_{n-1} \cup E_1 \cup E_2 \cup \cdots \cup E_{n-1}, \mathcal{M}) \simeq \mathbb{C}
\]

is isomorphic, by showing that any element of the R.H.S. uniquely extends to the whole \( E \). Similarly to the first part of the proof of Proposition 2.7, we consider the following three restriction maps:

1. \( H^0(E_i, \mathcal{M}) \to H^0(\Gamma_i, \mathcal{M}), \quad 3 \leq i \leq n-2 \),
2. \( H^0(E_1, \mathcal{M}) \to H^0(\Gamma_{n-1}, \mathcal{M}) \),
3. \( H^0(E_2, \mathcal{M}) \to H^0(\Gamma_1 \cup \Gamma_2, \mathcal{M}) \).

By using (3.27)–(3.29), it is elementary to see that all these are isomorphisms. Then by the argument in the proof of Proposition 2.7 we conclude that the restriction map (3.30) is isomorphic. This means the claim of Proposition 3.5.

Next for the proof of Proposition 3.6 we further need to show

**Proposition 3.6.** For the line bundle \( \mathcal{M} \) over the original twistor space \( Z \), we have

\[
h^0(S_i^-, \mathcal{M}) = 1, \quad 1 \leq i \leq n-2.
\]

Before proceeding to the proof, we note that on each divisor \( S_i^- \) (resp. \( S_i^+ \)) there exist \((-1\)-curves \( e'_1 \) and \( e'_2 \) (resp. \( \overline{e}'_1 \) and \( \overline{e}'_2 \)) such that \( \alpha_j|_{S_i^- \cup S_i^+} = e'_j - \overline{e}'_j \) for \( j = 1, 2 \). The existence of these \((-1\)-curves) can be derived from the self-intersection numbers of the components of the cycle \( C \) inside \( S_i^- \) and \( S_i^+ \), and also from the fact that the intersection \( S_i^- \cap S_i^+ \) is a twistor line, which is contained in \( S_i^+ \) and \( S_i^- \) as a \((+1\)-curve). The curves \( e'_j \) and \( \overline{e}'_j \) are respectively homologous to the exceptional curves \( e_j \) and \( \overline{e}_j \) contained in each non-singular member \( S \in |F| \). Just like \( e_1 \) and \( e_2 \) in \( S \), each of \( e'_1 \) and \( e'_2 \) (resp. \( \overline{e}'_1 \) and \( \overline{e}'_2 \)) intersects \( C_1 \) (resp. \( \overline{C}_1 \)) transversally at a unique point respectively.

**Proof of Proposition 3.6.** Again we first compute the restriction of the line bundle \( \mathcal{M} \) to the divisor \( S_i^- \) in a concrete form. For this we need to compute the restriction of the class \( \alpha_j \) to \( S_i^- \) for any \( j \), which is quite difficult in contrast with their restriction to \( S \) (except the cases \( j = 1, 2 \)). To avoid this, we make use of the remaining component \( S_{n-1}^- \). Let \( \alpha \) be as in the proof of Proposition 3.5, so that \( \mathcal{M} = \{(n-2)F - \alpha\}/2 \). By using the concrete
form of $\alpha$ and the Chern class formula (3.10) of $S_{n-1}^-$, we rewrite $\mathcal{M}$ as
\[
\frac{n-2}{2}F - \frac{\alpha}{2} = F + (n - 4) \left( \frac{1}{2}F - \frac{1}{2} \sum_{i=1}^{n} \alpha_i \right) - \alpha_1
\]
\[
= F + (n - 4)S_{n-1}^- - \alpha_1.
\]
(3.31)
Since $\alpha_1|_{S_i^-} = e'_1$ for any $i$ as above, we obtain from (3.31) that
\[
\mathcal{M}|_{S_i^-} = F|_{S_i^-} + (n - 4)S_{n-1}^-|_{S_i^-} - e'_1.
\]
(3.32)
The first term $F|_{S_i^-}$ is exactly a half of the cycle $C$ contained in $S_i^-$, and the restriction $S_{n-1}^-|_{S_i^-}$ in the second term is also a part of the cycle $C$, which can be immediately written down. From these we obtain
\[
\mathcal{M}|_{S_i^-} = \sum_{j=i+1}^{n-1} C_{i,j} + (n - 3) \sum_{j=1}^{i} C_j - e'_1.
\]
(3.33)
By computing intersection numbers, it is immediate to see that the second term $(n - 3) \sum_{j=1}^{i} C_j$ is a fixed component of this system, and that the system $| \sum_{j=i+1}^{n-1} C_{i,j} |$ is a base point free pencil. From the latter we obtain that the system $| \sum_{j=i+1}^{n-1} C_{i,j} - e'_1 |$ has a unique (effective) member, and that it is disjoint from the cycle $C$. These in particular mean the claim of the proposition. □

By using Propositions 3.5 and 3.6 we show Proposition 3.4:

Proof of Proposition 3.4. First we compute the restriction of the pulled-back bundle $\mu^* \mathcal{M}$ to the divisor $S_i^- \subset Z_1$ for $1 \leq i \leq n - 2$. For this we recall from Section 2.2 that due to the small resolution $Z_1 \to \hat{Z}$, the restriction of the birational morphism $\mu : Z_1 \to Z$ to the divisor $S_i^- \subset Z_1$ is not isomorphic but identified with the blowing-up at the point $\overline{C}_i \cap \overline{C}_{i+1}$, and the curve $\Delta_i$ is inserted as the exceptional curve. Then noting that, among the curves in the R.H.S. of (3.33), only $C_{i+1}$ contains the blown-up point $\overline{C}_{i+1} \cap \overline{C}_i$, and that the coefficient of $\overline{C}_{i+1}$ is one, we have
\[
(\mu^* \mathcal{M})|_{S_i^-} = \sum_{j=i+1}^{n-1} \overline{C}_{i,j} + (n - 3) \sum_{j=1}^{i} C_{i,j} - e'_1 + \Delta_i,
\]
(3.34)
where we are using the curves $C_{i,j}$ defined in Section 2.2. On the other hand, the subtraction term in $\mathcal{M}_1$ is (see (3.31))
\[
(n - 3)E_1 + \sum_{j=2}^{n-2} (n - 1 - j)E_j.
\]
(3.35)
When $j > i$, the intersection $S_i^- \cap E_j$ is at most a point. Hence by disposing them we obtain that the restriction of (3.35) to the divisor $S_i^- \ (1 \leq i \leq n - 2)$ is given by
\[
(n - 3)C_{i,1} + \sum_{j=2}^{i} (n - 1 - j)C_{i,j}.
\]
Subtracting this from (3.34) we get

\[ \mathcal{M}_1|_{S_i^1} = \sum_{j=1}^{i} (j-2)C_{i,j} + \Delta_i + \sum_{j=i+1}^{n-1} C_{i,j} - e'_1. \]  

By computing intersection numbers, the first summation in (3.36) can easily seen to be fixed components of $|\mathcal{M}|_{S_i^1}$. On the other hand from self-intersection numbers the system $|\Delta_i + \sum_{j=i+1}^{n-1} C_{i,j}|$ is again a base point free pencil, and it follows that the system $|\Delta_i + \sum_{j=i+1}^{n-1} C_{i,j} - e'_1|$ consists of a single member. In particular we obtain $h^0(S_i^1, \mathcal{M}_1) = 1$. Moreover when $1 \leq i < n - 1$, we have

\[ \left( C_{i,n-1}, \sum_{j=i+1}^{n-1} C_{i,j} + \Delta_i - e'_1 \right)_{S_i^1} = \left( C_{i,n-1}, C_{i,n-1} + C_{i,n_2} - e'_1 \right)_{S_i^1} = (-1) + 1 - 0 = 0. \]

When $i = n - 1$, by replacing $C_{i,n-2}$ with $\Delta_{n-1}$, we obtain the same conclusion. Therefore the unique member of the system $|\mathcal{M}_1|_{S_i^1}$ is disjoint from the curve $C_{i,n-1}$ for any $1 \leq i \leq n - 1$.

For completing the proof of Proposition 3.4 by Proposition 3.5 it is enough to show that any element of $H^0(E, \mathcal{M}_1)$ extends to $S_i^1$ for any $1 \leq i \leq n - 2$ in a unique way. For this we first recall from the proof of Proposition 3.5 that $\mathcal{M}_1$ is trivial over the component $E_{n-1}$, and the restriction map $H^0(E, \mathcal{M}_1) \to H^0(E_{n-1}, \mathcal{M}_1)$ is isomorphic. Further as above the unique member of the system $|\mathcal{M}_1|_{S_i^1}$ is disjoint from the curve $C_{i,n-1}$. In particular we obtain that both of the two restriction maps

1. $H^0(E, \mathcal{M}_1) (\simeq \mathbb{C}) \to H^0(C_{i,n-1}, \mathcal{M}_1) (\simeq \mathbb{C})$,
2. $H^0(S_i^1, \mathcal{M}_1) (\simeq \mathbb{C}) \to H^0(C_{i,n-1}, \mathcal{M}_1) (\simeq \mathbb{C})$,

are isomorphic. As the divisor (3.7) is smooth normal crossing, this immediately means that any element of $H^0(E, \mathcal{M}_1)$ uniquely extends to $S_i^1$ for any $1 \leq i \leq n - 2$. \[\square\]

Now we are able to prove the proposition presented in the beginning of this section.

**Proof of Proposition 3.1** By the isomorphism (3.6), in order to prove $h^0(Z, \mathcal{M}) = 1$, it suffices to show $h^0(Z_1, \mathcal{M}_1) = 1$. But now this is an immediate consequence of the standard exact sequence

\[ 0 \to \mathcal{M}' \to \mathcal{M}_1 \to \mathcal{M}_1|_{(S_i^1 \cup \cdots \cup S_{n-2}) \cup E} \to 0 \]

and Propositions 3.3 and 3.4.

Next let $D$ be the unique member of the system $|\mathcal{M}|$ and show that $D$ is irreducible. Suppose that $D$ is reducible, and let $D_1$ be any irreducible component of $D$. Then we have $D_1 + D_1 \in |kF|$ for some $k$ with $0 < k < n - 2$. But by Proposition 2.3 (i), (ii), we have $|kF| = S^k H^0(F)$ for these $k$. This means that $D_1$ is a degree-one divisor, or otherwise $D_1 \in |F|$. Thus if $D$ is reducible, all irreducible components must be some $S \in |F|$, $S^+_i$ or $S^-_i$. In order to show that this cannot happen, we first notice that the coefficients of $\alpha_1$ and $\alpha_2$ of the cohomology class $\mathcal{M}$ (see (3.5)) do not coincide (namely $(n - 2)$ and $(n - 4)$ respectively). On the other hand the class $F$ does not contribute for the pullback term (i.e. $\alpha_2$-terms). Furthermore, most importantly, the Chern class formulae (3.10) and (3.10) for degree-one divisors imply that the coefficients of $\alpha_1$ and $\alpha_2$ coincide for any $S^+_i$ and $S^-_i$, $1 \leq i \leq n - 1$. Therefore, $D$ cannot be a sum of $S^+_i$, $S^-_i$ $(1 \leq i \leq n - 1)$ and $S \in |F|$. 

The claim for another line bundle (3.3) follows by exchanging the role of the two classes \(\alpha_1\) and \(\alpha_2\) in all the arguments throughout this section.

4. Elimination of the base locus of the pluri-half-anticanonical system

In Section 2 we proved that the linear system \(|(n - 2)F|\) of the present twistor space induces a rational map \(\Phi: Z \to Y \subset \mathbb{CP}^n\), where \(Y\) is the scroll of planes over the rational normal curve \(\Lambda \subset \mathbb{CP}^{n-2}\), and that \(\Phi\) is of degree two over \(Y\) (Proposition 2.10). The restriction of the line bundle \((n - 2)F\) to a smooth member \(S \in |F|\) is isomorphic to \((n - 2)K_S^{-1}\), and this line bundle has base points along some components of the cycle \(C\) (Proposition 2.1 (iii)). Then from the surjectivity of the restriction map (2.5) (see Proposition 2.9), we have a coincidence \(B_s |(n - 2)F| = B_s |(n - 2)K_S^{-1}|\). In this section, we give a complete elimination of this base locus, via the space \(Z_1\) we have used throughout Sections 2 and 3. While the elimination requires some complicated calculations, this process seems to be indispensable for reaching our final goal. We use the notations from Sections 2 and 3, and continue to use the same letters to mean subsets of \(Z\) and their strict transforms into \(Z_1\). Also, since all the operations preserve the real structure, we often omit to mention the counterpart by the real structure.

From the construction in Section 2, in order to eliminate the base locus of \(|(n - 2)F|\), it is enough to eliminate the base locus of the system \(|L_1|\) on \(Z_1\), where \(L_1\) is the line bundle (2.7). We have the morphism \(f_1: Z_1 \to \Lambda \cong \mathbb{CP}^1\) whose fibers are strict transforms of the members of the pencil \(|F|\). Also recall that the restriction of \(L_1\) to a general fiber \(S\) of \(f_1\) is isomorphic to the line bundle \((n - 2)K_S^{-1}\) with the fixed components (2.3) removed. Let \(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}\) be points on the rational normal curve \(\Lambda\) which correspond to the reducible members \(S_i^+ + S_i^- \in |F|\), \(1 \leq i \leq n - 1\), respectively. Of course, the collection \(f_1^{-1}(\lambda_i)\), \(1 \leq i \leq n - 1\), are all reducible fibers of \(f_1\). Recall that as defined in (2.9), on the divisors \(S_i^+\) and \(S_i^-\) in \(Z_1\), there is a curve \(C_{i,j}\) which is identified with the curve \(C_j\) in \(Z\) under the birational morphism \(\mu: Z_1 \to Z\). The following property of the base locus of the system \(|L_1|\) follows immediately from Lemma 2.5 (See also the right picture in Figure 1 where the base curves lying on \(S_{n-2}^+ \cup S_{n-2}^-\) are written as segments with small triangles in the case \(n = 7\).)

**Proposition 4.1.** The base locus of \(|L_1|\) contains the following curves:

\[
\bigsqcup_{3 \leq i \leq n-2} \left( \bigsqcup_{3 \leq j \leq i} C_{i,j} \right) \quad \text{and} \quad \bigsqcup_{3 \leq i \leq n-2} \left( \bigsqcup_{3 \leq j \leq i} \overline{C}_{i,j} \right),
\]

and

\[
C_{n-1,1} \cup \overline{C}_{n-1,1}.
\]

Note that for each \(i\) with \(3 \leq i \leq n - 2\) the curves in the parentheses of (4.2) are connected. So each of the two curves (4.2) consists of \((n - 4)\) connected components. In particular they are empty when \(n = 4\).

**Remark 4.2.** As we shall see below, the two base curves (4.2) and (4.3) have quite different nature.
Figure 2. The sequence of blowups in the case $n=7$, which eliminates the base locus of the system $|\mathcal{L}_{n-2}|$ lying on $S_5^+ \cup S_5^-$. 
Figure 3. The sequence of blowups in the case $n = 6$, which eliminates the base locus of the system $|\mathcal{L}_{n-2}|$ lying on $S^+_5 \cup S^-_5$. 
Let $\mu_2 : Z_2 \to Z_1$ be the blowup of $Z_1$ at all the curves (1.2) and (1.3), and $D_{i,j}^{(2)}$ the exceptional divisor over $C_{i,j}$. Each $D_{i,j}^{(2)}$ is isomorphic to a ruled surface over $C_{i,j}$ (which is not isomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$ in general). When $n > 5$, as each connected component of the center $\text{(4.2)}$ is reducible for $4 \leq i \leq n - 2$, the variety $Z_2$ has an ODP at the common point of the four divisors

$$E_j, E_{j+1}, D_{i,j}^{(2)}, D_{i,j+1}^{(2)},$$

where $4 \leq i \leq n - 2$ and $3 \leq j \leq i - 1$. (In Figure 1 these are indicated by dotted points in the case $(n, i) = (7, 5)$.) In particular, these ODP-s are not lying on (the strict transforms of) the divisors $S_i^+$ and $S_i^-$. Define a line bundle $\mathcal{L}_2$ over $Z_2$ by

$$\mathcal{L}_2 := \mu_2^*\mathcal{L}_1 - \sum_{i=3}^{n-2} \sum_{j=3}^i (D_{i,j}^{(2)} + \bar{D}_{i,j}^{(2)}) - D_{n-1,1}^{(2)} - \bar{D}_{n-1,1}^{(2)}.$$  

Namely we are just subtracting all the exceptional divisors from the pulled-back bundle. The bundle $\mathcal{L}_2$ is clearly real. Further, for each $i$ and $j$ with $3 \leq i \leq n - 2$ and $3 \leq j \leq i$, we define curves on $Z_2$ by

$$C_{i,j}^{(2)} := D_{i,j}^{(2)} \cap S_i^- \quad \text{and} \quad \bar{C}_{i,j}^{(2)} := \bar{D}_{i,j}^{(2)} \cap S_i^+.$$  

Since the variety $Z_2$ is non-singular on these curves, their intersection numbers with the line bundle $\mathcal{L}_2$ make sense. These can be computed in a similar way to the proof of Lemma 2.5, and we obtain

$$(\mathcal{L}_2, C_{i,j}^{(2)})_{Z_2} = \begin{cases} 1, & j = 3, \\ 0, & 3 < j < i, \\ -1, & j = i. \end{cases}$$  

Hence disposing the curves $C_{i,3}^{(2)}$ and $\bar{C}_{i,3}^{(2)}$ for each $i$, the curves

$$\bigcup_{4 \leq i \leq n-2} \left( \bigcup_{4 \leq j \leq i} C_{i,j}^{(2)} \right) \quad \text{and} \quad \bigcup_{4 \leq i \leq n-2} \left( \bigcup_{4 \leq j \leq i} \bar{C}_{i,j}^{(2)} \right)$$  

are base curves of the system $|\mathcal{L}_2|$. (In Figure 2 these curves are indicated by bold segments in the case $(n, i) = (7, 5)$.) In particular, not only the number of the base curves lying on $S_i^-$ for fixed $i$, but also the number of the connected components of the base curves decrease by one as an effect of the blowup $\mu_2$, and (4.7) is empty when $n = 5$.

Next, in order to inspect the base locus on the isolated exceptional divisor $D_{n-1,1}^{(2)}$, we put

$$C_{n-1,1}^{(2)} := D_{n-2,1}^{(2)} \cap E_1 \quad \text{and} \quad \bar{C}_{n-1,1}^{(2)} := \bar{D}_{n-2,1}^{(2)} \cap \bar{E}_1.$$  

(In Figure 3 these curves are indicated by bold segments in the case $n = 6$.) Here note that unlike the above curves (4.5), we are taking intersection with the exceptional divisors $E_1$ and $\bar{E}_1$, and hence these curves are not lying on $S_i^-$ nor $S_i^+$. Then $Z_2$ is non-singular on these curves, and we can compute as

$$(\mathcal{L}_2, C_{n-1,1}^{(2)})_{Z_2} = 4 - n.$$  

In view of Lemma 2.5, this increases from $(\mathcal{L}_1, C_{n-1,1})_{Z_1}$ by one. In particular, the curves (4.8) are base curves when $n > 4$. 

When $n = 4$, we finish the operation here (i.e., at $Z_2$). If $n > 4$, let $\mu_3 : Z_3 \to Z_2$ be the blowup at the curves $\mathcal{L}_3$ and $\mathcal{L}_4$. Let $D_{i,j}^{(3)} \subset Z_3$ be the exceptional divisor over the curve $C_{i,j}^{(2)}$, and define a line bundle over $Z_3$ by

\begin{equation}
\mathcal{L}_3 := \mu_2^* \mathcal{O}_2 - \sum_{i=4}^{n-2} \sum_{j=4}^{i} (D_{i,j}^{(3)} + \overline{D}_{i,j}^{(3)}) - D_{n-1,1}^{(3)} - \overline{D}_{n-1,1}^{(3)},
\end{equation}

which is still real. Then by the same reason for the blowup $\mu_2 : Z_2 \to Z_1$, if $n > 6$, the variety $Z_3$ has new ordinary double points over the singular points of the curves $\mathcal{L}_3$. But again they are not lying on (the strict transform of) the divisors $S_{i}^\pm$ and $S_{1}^\pm$. If we put

\begin{equation}
C_{i,j}^{(3)} := D_{i,j}^{(3)} \cap S_i^- \quad \text{and} \quad \overline{C}_{i,j}^{(3)} := \overline{D}_{i,j}^{(3)} \cap S_i^+,
\end{equation}

then again by computing intersection numbers with $\mathcal{L}_3$, we deduce that these are base curves of $|\mathcal{L}_3|$ as long as $5 \leq i \leq n - 2$ and $5 \leq j \leq i$. (In Figure 2 these curves are indicated by bold segments in the case $(n, i) = (7, 5)$.) In particular, the number of the base curves lying on $S_i^-$ for fixed $i$ and the number of the connected components of the base curves again decrease by one by the effect of the blowup $\mu_3$. Also, the intersection numbers of $\mathcal{L}_3$ with the curves

\begin{equation}
C_{n-1,1}^{(3)} := D_{n-1,1}^{(3)} \cap E_1 \quad \text{and} \quad \overline{C}_{n-1,1}^{(3)} := \overline{D}_{n-1,1}^{(3)} \cap E_1
\end{equation}

increase by one from (4.9), and the curves (4.12) are base curves if $n > 5$. (In Figure 3 these curves are indicated by bold segments in the case $n = 6$.) When $n = 5$, we stop the operation here. If $n > 5$, we blowup $Z_3$ at the base curves (4.11) and (4.12), and repeat the same operation above.

Continuing this process, for the twistor space on $n\mathbb{CP}^2$ with $n$ being arbitrary, we obtain a sequence of the explicit blowups

\begin{equation}
Z_{n-2} \xrightarrow{\mu_{n-2}} Z_{n-3} \xrightarrow{\mu_{n-3}} \cdots \xrightarrow{\mu_3} Z_2 \xrightarrow{\mu_2} Z_1,
\end{equation}

together with a real line bundle $\mathcal{L}_m$ over each $Z_m$. Let $\nu : Z_{n-2} \to Z_1$ be the composition of all these blowups. (We note that the number of times of blowups are $(n - 4)$ for the base curves (4.2), and $(n - 3)$ for the base curves (4.3). The final blowup $\mu_{n-2} : Z_{n-2} \to Z_{n-3}$ changes only the latter base curves (4.3) and does not change the former base curve (4.2).)

Then the exceptional divisor of $\nu$ consists of

\begin{equation}
\nu^{-1}(C_{i,j}) = \bigcup_{k=2}^{j-1} D_{i,j}^{(k)}, \quad \text{where} \quad 3 \leq i \leq n - 2 \quad \text{and} \quad 3 \leq j \leq i,
\end{equation}

\begin{equation}
\nu^{-1}(C_{n-1,1}) = \bigcup_{k=2}^{n-2} D_{n-1,1}^{(k)}
\end{equation}

and the images of these divisors by the real structure. From the explicit construction, for divisors in (4.14) we readily have

\begin{equation}
D_{n-1,1}^{(k)} \simeq \Sigma_{n-k-1}
\end{equation}

where $\Sigma_d$ denotes the ruled surface of degree $d$ over $\mathbb{CP}^1$. In particular, the divisor $D_{n-1,1}^{(n-2)}$ (obtained from the final blowup) is isomorphic to one point blown-up of $\mathbb{CP}^2$. Further, the intersection of the adjacent components $D_{n-1,1}^{(k)}$ and $D_{n-1,1}^{(k+1)}$ is always a section of the ruling.
Furthermore, these sections are mapped isomorphically to the curve $C_{n-1,1}$ by $\nu$. Thus it would be possible to say that the divisor $D$ has a structure of a ladder over $C_{n-1,1}$ by $\nu$. Similarly, for each $i$ and $j$, the divisor $D_{i,j}$ forms a ladder over $C_{i,j}$, but this ladder is growing up in the reverse direction with the ladder $D$.

Then we have the following

**Proposition 4.3.** The linear system $|L_{n-2}|$ on the variety $Z_{n-2}$ obtained above is base point free.

**Proof.** Let $Z'_{n-2} \rightarrow Z_{n-2}$ be any small deformation of all ODP-s on $Z_{n-2}$ which preserves the real structure, and let $L'_{n-2}$ be the pullback of $L_{n-2}$ to $Z_{n-2}$. It suffices to show that $|L'_{n-2}|$ is base point free. For this purpose, we compute the restrictions of the line bundle $L'_{n-2}$ to the exceptional divisors of $\nu$ and $\mu$ (namely $D_{i,j}$ and $E_{j}$). In the sequel we obtain surjective morphisms from each of these divisors to $\mathbb{C}P^1$ (including the above ruling map for $D_{n-1,1} \simeq \Sigma_{n-1}$), for which we use the common letter $p$. (In Figures 2 and 3, fibers of these morphisms are indicated by gray curves in the cases $(n, i) = (7, 5)$ and $(n, i) = (6, 5)$ respectively.)

First for the exceptional divisor $D_{i,j}$ in (4.13), by computing the intersection numbers of $L'_{n-2}$ with curves in $D_{i,j}$ which are obtained as an intersection of other exceptional divisors, it is possible to show that the line bundle $L'_{n-2}$ is trivial over the divisors

$$
D_{i,j} \; 3 \leq i \leq n - 2, \; 3 \leq j \leq i, \; 2 \leq k \leq j - 2.
$$

Among all the divisors (4.13), these are characterized by disjointness with the divisor $S_{i}^{-}$ (i.e. we are just excluding the case $k = j - 1$ from the divisors (4.13).) Note that for any fixed $i$, the union of all the divisors (4.16) is connected. On the other hand, over the remaining exceptional divisors

$$
D_{i,j}^{(j-1)} \; 3 \leq i \leq n - 2, \; 3 \leq j \leq i,
$$

the line bundle $L'_{n-2}$ is not trivial but of the form $p^* D(1)$, where $p$ is a surjective morphism $D_{i,j}^{(j-1)} \rightarrow \mathbb{C}P^1$ which has the intersection curve $D_{i,j}^{(j-1)} \cap S_{i}^{-}$ as a smooth fiber.

Next for the restriction to the divisor $D_{n-1,1}$ in (4.14) (or (4.15)), for any $k$ with $2 \leq k < n - 2$, over $D_{n-1,1}^{(k)}$, the line bundle $L'_{n-2}$ is isomorphic to $p^* E(1)$, where $p : D_{n-1,1}^{(k)} \rightarrow \mathbb{C}P^1$ is the ruling map. Over the remaining divisor $D_{n-1,1}^{(n-2)}$ (which is placed at an end of the ladder), $L'_{n-2}$ is of the form $\varepsilon^* E(1)$, where $\varepsilon : D_{n-1,1}^{(n-2)} \simeq \Sigma_{1} \rightarrow \mathbb{C}P^2$ is a blow-down.

For the restriction to the exceptional divisor $E_{i}$ ($1 \leq i \leq n - 1$), we recall that the line bundle $L_{1}$ over $Z_{1}$ is trivial over $E_{n-1}$ by Remark 2.6. On the other hand, $L_{1}$ is non-trivial over the remaining divisors $E_{i}, i < n - 1$. However, as an effect of the blowups in $\nu$, the final line bundle $L'_{n-2}$ is trivial over any divisors $E_{i}$, so far as $i \neq 2$. On the other hand, over the component $E_{2}$, $L'_{n-2}$ is of the form $p^* E(1)$, where $p : E_{2} \rightarrow \mathbb{C}P^1$ is a surjective morphism for which the intersection $S \cap E_{2}$ is a section, with $S$ being any fiber of the composition

$$
Z'_{n-2} \rightarrow Z_{n-2} \overset{\nu}{\rightarrow} Z_{1} \overset{f_{1}}{\rightarrow} \mathbb{C}P^1.
$$

We also need to know the restriction of $L'_{n-2}$ to the divisor $S_{n-1}^{-}$. For this we again notice that there is a surjective morphism $p : S_{n-1}^{-} \rightarrow \mathbb{C}P^1$ for which the two intersection
curves \( S_{-1}^{-1} \cap D^{(2)}_{n-1,1} \) and \( S_{-1}^{-1} \cap D^{(n-2)}_{n-1,1} \) are (mutually disjoint) sections of the morphism. (The morphism is induced by the linear system \( |\sum_{i=2}^{n-1} C_{n-1,i} + \mathcal{C}_{n-1,1}| \) on \( S_{-1}^{-1} \).) Then the restriction of \( \mathcal{L}_n' \) to \( S_{-1}^{-1} \) is again of the form \( p^* \mathcal{O}(1) \).

Utilizing all these restriction data, we show that \( |\mathcal{L}_n'| \) is base point free. First it is clear from the beginning that \( Bs|\mathcal{L}_n'| \) is contained in the exceptional divisors \((4.13), (4.14), E_i \) (\( 1 \leq i \leq n-1 \)), or their conjugations by the real structure. If there is a base point on the divisors \((4.16)\) for some \( i \) (so that \( 3 \leq j \leq i \leq n-2 \)), then by the triviality of \( \mathcal{L}_n' \) over these divisors, all the divisors \((4.16)\) must be fixed components of \( |\mathcal{L}_n'| \) for the above \( i \). Then since the component \( D_{i,j}^{(2)} \) intersects \( E_i \) and \( \mathcal{L}_n' \) is trivial over \( \bigcup_{j \neq 2} (E_j \cup \mathcal{E}_j) \) as above, whole of the last union must also be fixed components of \( |\mathcal{L}_n'| \). This means that the original system \( |\mathcal{L}_1| \) on the space \( Z_1 \) has \( E_i \) (\( i \neq 2 \)) as a fixed component. But this contradicts the fact that \( |\mathcal{L}_1| \) does not have a base point on smooth fibers of \( f_1 : Z_1 \to \mathbb{CP}^1 \). Therefore \( Bs|\mathcal{L}_n'| \) is disjoint from the divisors \((4.16)\) and also from their conjugations. Moreover if there is a base point on the divisor \( D_{i,j}^{(j-1)} \) in \((4.17)\), then through the morphism \( p : D_{i,j}^{(j-1)} \to \mathbb{CP}^1 \) (where we decrease \( j \) one by one until it becomes 3), we finally obtain that there must be a base point on \( E_2 \). Hence \( Bs|\mathcal{L}_n'| \cap E_2 \) has to be a fiber of the above morphism \( p : E_2 \to \mathbb{CP}^1 \). But since fibers of the last morphism intersect any fiber of the morphism \((4.18)\), this again contradicts the fact that \( |\mathcal{L}_1| \) does not have a base point on smooth fibers of \( f_1 \). So we conclude \( Bs|\mathcal{L}_n'| \cap D_{i,j}^{(j-1)} = \emptyset \) also for the divisors in \((4.17)\).

Furthermore, these argument clearly imply that \( Bs|\mathcal{L}_n'| \cap E_i = \emptyset \) for any \( 1 \leq i \leq n-1 \).

It remains to see that \( |\mathcal{L}_n'| \) does not have a base point on the ladder \((4.14)\). If the system \( |\mathcal{L}_n'| \) has a base point on the divisor \( D_{n-1,1}^{(k)} \) for some \( k < n-2 \), then via the ruling map \( D_{n-1,1}^{(k)} \to \mathbb{CP}^1 \) (where we decrease \( k \) until it becomes 2), the system has a base point on the intersection curve \( D_{n-1,1}^{(2)} \cap S_{-1}^{-1} \), which is a section of the morphism \( p : S_{-1}^{-1} \to \mathbb{CP}^1 \). Hence \( |\mathcal{L}_n'| \) has a base point along a fiber of the morphism \( p : S_{-1}^{-1} \to \mathbb{CP}^1 \). But this contradicts the fact that \( Bs|\mathcal{L}_n'| \) is contained in the divisor \((4.14)\) (which is already proved). Hence if \( k < n-2 \) we have \( D_{n-1,1}^{(k)} \cap Bs|\mathcal{L}_n'| = \emptyset \). Finally we show \( D_{n-1,1}^{(n-2)} \cap Bs|\mathcal{L}_n'| = \emptyset \). Let \( \Phi_{n-2} : Z_{n-2} \to \mathbb{CP}^m \) be the rational map associated to \( |\mathcal{L}_n'| \). Then the restriction of \( \mathcal{L}_n' \) to the divisor \( D_{n-1,1}^{(k)} \) is of the form \( p^* \mathcal{O}(1) \) and also that \( Bs|\mathcal{L}_n'| \cap D_{n-1,1}^{(k)} = \emptyset \) for \( 2 \leq k < n-2 \), the image \( \Phi_{n-2}^{(k)}(D_{n-1,1}^{(k)}) \) must be a line for this range of \( k \). Moreover this line is independent of \( k \), because of the ladder structure of the divisor \((4.14)\). The ladder structure also implies that the curve \( S_{-1}^{-1} \cap D_{n-1,1}^{(2)} \) (which is a section of the morphism \( p : S_{-1}^{-1} \to \mathbb{CP}^1 \)) and the curve \( D_{n-1,1}^{(n-2)} \cap D_{n-1,1}^{(n-3)} \) are also mapped to the same line by \( \Phi_{n-2} \). Hence, via the morphism \( p : S_{-1}^{-1} \to \mathbb{CP}^1 \), the other section \( S_{-1}^{-1} \cap D_{n-1,1}^{(n-2)} \) is also mapped to the same line by \( \Phi_{n-2} \). Therefore, by the real structure, the image \( \Phi_{n-2}(S_{n-1}^{-1} \cap D_{n-1,1}^{(n-2)}) \) of the conjugate curve is a line. Hence \( \Phi_{n-2} \) maps the two curves \( D_{n-1,1}^{(n-2)} \cap D_{n-1,1}^{(n-3)} \) and \( S_{n-1}^{-1} \cap D_{n-1,1}^{(n-2)} \) on the surface \( D_{n-1,1}^{(n-2)} \cong \Sigma_1 \) to lines. However, these lines cannot be identical, since the former curve belongs to \( |\varepsilon^* \mathcal{O}(1)| \) on \( D_{n-1,1}^{(n-2)} \), while the latter does not. Hence we conclude that the image \( \Phi_{n-2}^{(n-2)}(D_{n-1,1}^{(n-2)}) \) contains two lines. Therefore \( \Phi_{n-2}^{(n-2)}(D_{n-1,1}^{(n-2)}) \) has to
be two-dimensional, meaning that $\Phi_{n-2}'(D_{n-1,1}^{(n-2)}) = \pi^{-1}(\lambda_{n-1})$. Hence recalling that $\mathcal{L}_{n-2}'$ is isomorphic to $\mathcal{E}^*\mathcal{O}_{\mathbb{P}^n}(1)$ over $D_{n-1,1}^{(n-2)}$, we obtain $D_{n-1,1}^{(n-2)} \cap \text{Bs} |\mathcal{L}_{n-2}'| = \emptyset$.

Thus we have completed a proof of Proposition 4.3. \hfill \Box

By using this elimination, we prove the following result concerning the behavior of the rational map $\Phi$ on the degree-one divisors, which will be needed in the next section. Recall that for $1 \leq i \leq n - 1$, $L_i$ means the twistor line $S_i^+ \cap S_i^-$. 

**Proposition 4.4.** Let $\Phi : Z \to Y \subset \mathbb{P}^n$ be the rational map associated to the system $|(n - 2)F|$ as before.

(i) If $1 \leq i < n - 1$, both of the restrictions $\Phi|_{S_i^+}$ and $\Phi|_{S_i^-}$ are birational over the plane $\pi^{-1}(\lambda_i)$. Moreover, the image $\Phi(L_i)$ is a conic in the plane.

(ii) Both of the images $\Phi(S_{n-1}^+)$ and $\Phi(S_{n-1}^-)$ are lines in the plane $\pi^{-1}(\lambda_{n-1})$. Further, these lines are distinct.

**Proof.** From our construction, it is enough to show the same claim for the morphism $\Phi_{n-2}'$ associated to $|\mathcal{L}_{n-2}'|$ on $Z_{n-2}'$. For the birationality in (i), as we know that $\Phi_{n-2}'$ is of degree two preserving the real structure, it is enough to show that $\Phi_{n-2}'|_{S_i^-}$ is surjective over the plane $\pi^{-1}(\lambda_i)$. As in the above proof of Proposition 4.3, the line bundle $\mathcal{L}_{n-2}'$ over $Z_{n-2}'$ is trivial over the exceptional divisors (4.16), and therefore their images by $\Phi_{n-2}'$ must be a point. Similarly, since the restriction of $\mathcal{L}_{n-2}'$ to the divisor $D_{i,j}^{(j-1)}$ in (4.17) is of the form $p^*\mathcal{O}(1)$ where $p : D_{i,j}^{(j-1)} \to \mathbb{P}^1$ is the surjective morphism, the image $\Phi_{n-2}'(D_{i,j}^{(j-1)})$ must be a line. Therefore, since we already know that $|\mathcal{L}_{n-2}'|$ is base point free, the remaining divisor $S_i^-$ has to be mapped surjectively to the plane $\pi^{-1}(\lambda_{n-1})$, as claimed.

For the assertion about the images of the twistor lines in (i), since $\text{Bs} |\mathcal{L}_{n-2}'| = \emptyset$ and $\Phi_{n-2}'$ is degree one over $S_i^-$ as above, it suffices to show that $(\mathcal{L}_{n-2}', L_i)|_{Z_{n-2}'} = 2$ for $1 \leq i < n - 1$. From the formula (2.14), on the manifold $Z_1$ we have

$$(\mathcal{L}_1, L_i)|_{Z_1} = 2(i - 1),$$

since $(f_1^*\mathcal{O}(1), L_i)|_{Z_1} = 0$ (as $L_i \subset f_1^{-1}(\lambda_i)$) and, among the divisors $E_j$ and $F_j$, $1 \leq j \leq n - 1$, only $E_i$ and $F_i$ intersect $L_i$ and the intersections are transversal. Then for each of the blowup $\mu_m : Z_m \to Z_{m-1}$, since we are removing the exceptional divisors of $\mu_m$ by multiplicity one (see (4.4) and (4.10)), if the center of the blowup intersects $L_i$, the intersection number satisfies

$$(\mathcal{L}_m, L_i)|_{Z_m} = (\mathcal{L}_{m-1}, L_i)|_{Z_{m-1}} - 2.$$

Further from the explicit centers of the blowups $\mu_m$, this actually happens exactly when $2 \leq m \leq i - 1$. Hence the decreasing (4.19) happens precisely $(i - 2)$ times. Therefore we have

$$(\mathcal{L}_{n-2}', L_i)|_{Z_{n-2}'} = (\mathcal{L}_{n-2}, L_i)|_{Z_{n-2}} = (\mathcal{L}_1, L_i)|_{Z_1} - 2(i - 2) = 2,$$

as claimed.

The second assertion (ii) is already shown in the final part of the above proof of Proposition 4.3. \hfill \Box

**Remark 4.5.** As showed in the final part of the proof of Proposition 4.3, the final exceptional divisors $D_{n-1,1}^{(n-2)}$ and $\mathcal{D}_{n-1,1}^{(n-2)}$ are mapped birationally onto the plane $\pi^{-1}(\lambda_{n-1})$. 


5. DEFINING EQUATION OF THE QUARTIC HYPERSURFACE

In this section, assembling all the results obtained so far, we shall obtain defining equation of a quartic hypersurface in $\mathbb{CP}^n$ which cut out the branch divisor of the pluri-half-anticanonical map $\Phi : Z \to Y \subset \mathbb{CP}^n$.

5.1. Double curves on the branch divisor, and a quadratic hypersurface containing them. In the study of plane quartic curves, the notion of bitangent has been played a significant role. Similarly, for quartic surfaces in a projective space, a plane which touches the surface along a curve is meaningful, because such a plane brings much information about a defining equation of the surface. This is also the case for the study of the branch divisor for the present twistor spaces.

As before let $Z$ be the twistor space on $n\mathbb{CP}^2$ which has the surface $S$ constructed in Section 2.1 as a real member of the system $|F|$, and $\Phi : Z \to \mathbb{CP}^n$ be the rational map associated to the system $|(n-2)F|$. If $\pi : \mathbb{CP}^n \to \mathbb{CP}^{n-2}$ denotes the natural linear projection corresponding to the subspace $S^{n-2}H^0(F) \subset H^0((n-2)F)$ as before, the image $\Phi(Z)$ is the scroll $Y = \pi^{-1}(\Lambda)$, where $\Lambda$ is a rational normal curve in $\mathbb{CP}^{n-2}$ (Proposition 2.10). We also know that the branch divisor $B$ of $\Phi : Z \to Y$ is of the form $Y \cap \mathcal{B}$, where $\mathcal{B}$ is a quartic hypersurface in $\mathbb{CP}^n$. In order to determine a defining equation of $\mathcal{B}$, we call a curve on the branch divisor $B$ to be a double curve if there exists a hyperplane $H \subset \mathbb{CP}^n$ such that $H \cap B$ is a non-reduced curve on the surface $B$. Geometrically, this means that the hyperplane section $H \cap Y$ is tangent to $B$ along the curve.

As before let $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ be points on the curve $\Lambda \subset \mathbb{CP}^{n-2}$ which correspond to the reducible divisors $S^+_i + S^-_i \in |F|$ respectively, where we are identifying the curve $\Lambda$ with the space $\mathbb{P}^*H^0(F) \simeq \mathbb{CP}^1$ through the diagram (2.28). By Proposition 4.4 (i), for $1 \leq i \leq n-2$, we have $\Phi(S^+_i) = \Phi(S^-_i) = \pi^{-1}(\lambda_i)$ and the image $\Phi(L_i)$ is a conic in the plane $\pi^{-1}(\lambda_i)$. We put

$$C_i := \Phi(L_i), \quad 1 \leq i \leq n-2$$

for these conics. Since $\Phi$ preserves the real structure, these conics are real. Then since $\Phi^{-1}(\pi^{-1}(\lambda_i))$ splits into the union $S^+_i \cup S^-_i$, the plane $\pi^{-1}(\lambda_i)$ must touch the divisor $B$ along the conic $C_i$. Hence for $1 \leq i \leq n-2$, the conic $C_i$ is a double curve in the above sense. We call these curves as double conics. For the case $i = n-1$, by Proposition 4.4 (ii), the image $\Phi(S^+_i)$ and $\Phi(S^-_i)$ are mutually distinct lines. We denote the union of these two lines by $C_{n-1}$. Namely, we put

$$C_{n-1} := \Phi(S^+_n \cup S^-_{n-1}).$$

As in the case $i < n-1$, this also has to be a double curve on $B$, and we call it as a splitting double conic.

By Proposition 4.1, there exist two special reducible members of the system $|(n-2)F|$, both of which consist of two irreducible components. Let $H_n$ and $H_{n+1}$ be the hyperplanes in $\mathbb{CP}^n$ which correspond to these two reducible members. Then since the preimage $\Phi^{-1}(H_n)$ (resp. $\Phi^{-1}(H_{n+1})$) splits into the two irreducible components, if we denote $C_n$ (resp. $C_{n+1}$) for the image of the intersection of the two irreducible components of $\Phi^{-1}(H_n)$ (resp. $\Phi^{-1}(H_{n+1})$), these are also double curves on $B$. If we write $l$ for the line which is the indeterminacy locus (the center) of the linear projection $\pi$, the hyperplane section $H_i \cap Y$ ($i \in \{n, n+1\}$) is a cone over the curve $\Lambda$ whose vertex is the point $l \cap H_i$. Moreover since the twice of the double curve $C_i$ ($i \in \{n, n+1\}$) is an element of the system $|\mathcal{O}(4)|$ on the
cone, the curve $\mathcal{C}_t$ belongs to the system $|\mathcal{O}(2)|$ on the cone. In particular the degree of $\mathcal{C}_n$ and $\mathcal{C}_{n+1}$ in $\mathbb{CP}^n$ is $2(n-2)$.

Thus we have $(n-1)$ double conics $\mathcal{C}_1, \ldots, \mathcal{C}_{n-1}$ on the planes and two double curves $\mathcal{C}_n, \mathcal{C}_{n+1}$ on the cones. These double curves play an essential role for obtaining a defining equation of the quartic hypersurface.

**Proposition 5.1.** Let $\mathcal{C}_1, \ldots, \mathcal{C}_{n+1}$ be the double curves on the branch divisor $B \subset Y$ as above. Then there exists a quadratic hypersurface $Q$ in $\mathbb{CP}^n$ which contains all these double curves, and which is different from the scroll $Y$.

**Proof.** For $1 \leq i \leq n-1$ we denote $P_i$ for the plane $\pi^{-1}(\lambda_i)$. We also write $D_n := Y \cap H_n$ and $D_{n+1} := Y \cap H_{n+1}$ for the cones, on which $\mathcal{C}_n$ and $\mathcal{C}_{n+1}$ lie respectively. Let $\delta : \tilde{Y} \to Y$ be the blowup at the line (ridge) $l$, and $\Sigma$ the exceptional divisor. $\Sigma$ is birational to $\mathbb{CP}^1 \times \mathbb{CP}^1$. Let $\tilde{\pi} : \tilde{Y} \to \mathbb{CP}^1$ be the composition $\pi \circ \delta$, which is clearly a morphism.

We again use the same letters for the strict transforms into $\tilde{Y}$ of the divisors and curves in $Y$. On the resolved space $\tilde{Y}$, the divisors $D_n$ and $D_{n+1}$ are smooth and birational to $\Sigma_{n-2}$, the ruled surface of degree $(n-2)$. Moreover, on $\tilde{Y}$, the union

$$D := \left( \bigcup_{i=1}^{n-1} P_i \right) \cup D_n \cup D_{n+1} \cup \Sigma$$

is not only smooth normal crossing but also simply connected. (Note that the fundamental group of the union $D_n \cup D_{n+1} \cup \Sigma$ is $\mathbb{Z}$, but the generator becomes homotopic to the identity after adding the plane $P_1$.)

As the degree of $\Lambda$ in $\mathbb{CP}^{n-2}$ is $(n-2)$, we have

$$\delta^* \mathcal{O}_Y(1) \sim \Sigma + \tilde{\pi}^* \mathcal{O}_\Lambda(n-2).$$

Therefore, noting $P_i \in |\tilde{\pi}^* \mathcal{O}_\Lambda(1)|$ and $D_j \in |\delta^* \mathcal{O}_Y(1)|$ on $\tilde{Y}$, we obtain that the divisor (5.2) (with all components counted just once) belongs to the linear system

$$|3\Sigma + \tilde{\pi}^* \mathcal{O}_\Lambda(3n-5)|.$$

Hence again by using the relation (5.3) we obtain an exact sequence

$$0 \to -\Sigma + \tilde{\pi}^* \mathcal{O}_\Lambda(1-n) \to \delta^* \mathcal{O}_Y(2) \to \delta^* \mathcal{O}_Y(2)|_D \to 0.$$

Now a defining quartic polynomial of the hypersurface $\mathcal{B}$ gives a section of the line bundle $\mathcal{O}_Y(4)$ as a restriction. Let $s \in H^0(\mathcal{O}_Y(4))$ be this section. Then because all $\mathcal{C}_t$-s are double curves of $B$, for each $1 \leq i \leq n-1$ and $j \in \{n, n+1\}$ we can take sections

$$t_i \in H^0(P_i, \mathcal{O}_{P_i}(2)) \quad \text{and} \quad t_j \in H^0(D_j, \mathcal{O}_{D_j}(2))$$

such that $s|_{P_i} = t_i^2$ and $s|_{D_j} = t_j^2$. Also, since the intersection $B \cap l$ consists of two points (see Proposition 2.2), there also exists an element $t_0 \in H^0(l, \mathcal{O}(2))$ such that $t_0^2 = s|_l$. All these sections $t_i$, $t_j$ and $t_0$ are determined only up to sign. Let $\tilde{t}_i$, $\tilde{t}_j$ and $\tilde{t}_0$ be the natural lifts of these sections by the blowup $\delta$. Then from the simply connectedness of the divisor $D$, we can choose the signs in a way that any two agree on the intersections of the components of $D$. Hence since $D$ in $\tilde{Y}$ is smooth normal crossing, we obtain an element $t_j \in H^0(D, \delta^* \mathcal{O}_Y(2))$ such that it defines the double curves on the plane $P_i$ and the cone $D_j$. 
On the other hand, by restricting the line bundle $\pi^*G_A(1)$ to the divisor $D_n$, it is easy to see that $H^1(Y, -\Sigma + \pi^*G_A(1-n)) = 0$. Hence from the exact sequence (5.4), the restriction map $H^0(\delta^*G_Y(2)) \rightarrow H^0(\delta^*G_Y(2)|_D)$ is surjective. Hence we obtain an element $\tilde{t} \in H^0(\delta^*G_Y(2))$ such that $\tilde{t}|_D = t_D$. Letting $t \in H^0(Y, G_Y(2))$ be the element corresponding to $\tilde{t}$, the divisor $(t)$ on $Y$ contains all the $(n+1)$ double curves. Then since the restriction $H^0(\mathbb{CP}^n, G(2)) \rightarrow H^0(Y, G(2))$ is surjective, there is an element $q \in H^0(\mathbb{CP}^n, G(2))$ such that $q|_Y = t$. Putting $Q := (q)$, we obtain the required quadratic hypersurface $Q$.

5.2. Defining equation of the quartic hypersurface. Now we are ready for determining defining equation of the branch divisor. Let all the notations be as in the last subsection, and we choose homogeneous coordinates $(z_0, z_1, \cdots, z_{n-2})$ on $\mathbb{CP}^{n-2}$ such that the rational normal curve $\Lambda$ is realized as an image of the standard holomorphic map

\[(5.5)\quad \mathbb{CP}^1 \ni (u_0, u_1) \mapsto (u_0^{n-2}, u_0^{n-3}u_1, \cdots, u_1^{n-2}) \in \mathbb{CP}^{n-2}.
\]

We further normalize the coordinates in such a way that the last point $\lambda_{n-1}$ is the point $(0, 0, \cdots, 0, 1) \in \mathbb{CP}^{n-2}$. Then the hyperplane $\{z_0 = 0\} \subset \mathbb{CP}^{n-2}$ intersects $\Lambda$ at $\lambda_{n-1}$ by the highest multiplicity $(n-2)$. Under these normalizations of the coordinates, our main result is stated as follows:

**Theorem 5.2.** Let $\Phi : Z \rightarrow Y \subset \mathbb{CP}^n$, $\mathcal{B}$, $\Lambda$ and $(z_0, z_1, \cdots, z_{n-2})$ be as above. Then for appropriate homogeneous coordinates $(z_0, z_1, \cdots, z_n)$ on $\mathbb{CP}^n$ which are obtained as an extension of the above ones on $\mathbb{CP}^{n-2}$, the quartic hypersurface $\mathcal{B}$ is defined by the following polynomial:

\[(5.6)\quad z_0z_{n-1}z_nf(z_0, z_1, \cdots, z_{n-2}) = Q(z_0, z_1, \cdots, z_n)^2,
\]

where $f(z_0, z_1, \cdots, z_{n-2})$ is a linear form (not on $\mathbb{CP}^n$ but) on $\mathbb{CP}^{n-2}$, and $Q(z_0, z_1, \cdots, z_n)$ is a quadratic form on $\mathbb{CP}^n$.

**Remark 5.3.** As the proof below shows, the quadratic form $Q(z_0, \cdots, z_n)$ is exactly the defining equation of the hyperquadric in Proposition 5.1. Therefore, since the restriction of the hyperquadric $Q$ to the plane $\pi^{-1}((\lambda_{n-1}) = \{z_0 = z_1 = \cdots = z_{n-3} = 0\}$ is the splitting double conic $\mathcal{C}_{n-1}$, we have a constraint that the conic defined by the equation

\[(5.7)\quad Q(0, 0, \cdots, 0, z_{n-2}, z_{n-1}, z_n) = 0
\]

is reducible under the above normalization of the coordinates.

**Remark 5.4.** As the proof below shows, up to a non-zero constant, the linear form $f$ in the equation (5.6) is uniquely determined from the $(n-2)$ points $\lambda_1, \cdots, \lambda_{n-2} \in \Lambda$.

**Remark 5.5.** At first sight one might think that when $n = 4$ the equation (5.6) coincides with the equation (1.2) in [9] of the quartic hypersurface in $\mathbb{CP}^4$. But this is not correct, since the linear polynomial $f$ in [9] (1.2)] includes not only $z_0, z_1, \cdots, z_{n-2}$ but also $z_{n-1}$ and $z_n$. (This is not a minor difference, as the type of the singularities of the branch divisor becomes quite different.) This means that the twistor spaces in this paper is not a direct generalization of the twistor spaces studied in [9]. Rather, they are a direct generalization of one type of the twistor spaces studied in [10], which we call ‘type II’ there.

**Proof of Theorem 5.2.** First let $z_{n-1}$ and $z_n$ be linear forms on $\mathbb{CP}^n$ such that $(z_{n-1}) = H_n$ and $(z_n) = H_{n+1}$, where as before $H_n$ and $H_{n+1}$ are the hyperplanes corresponding to the two reducible members of $[(n-2)F]$. Then $(z_0, z_1, \cdots, z_n)$ provides homogeneous coordinates on $\mathbb{CP}^n$, with respect to which the line $l$ is defined by $z_0 = z_1 = \cdots = z_{n-2} = 0$. 

For an algebraic variety \( X \subset \mathbb{CP}^n \), we denote by \( I_X \subset \mathbb{C}[z_0, \ldots, z_n] \) for the homogeneous ideal of \( X \). Let \( F = F(z_0, \ldots, z_n) \) be a defining quartic polynomial of \( \mathcal{B} \). Obviously \( F \) is defined only up to an ideal \( I_Y \subset \mathbb{C}[z_0, \ldots, z_n] \). Let \( Q(z_0, \ldots, z_n) \) be a defining polynomial of the hyperquadric \( Q \) whose existence was proved in Proposition [5,1]. \( Q \) contains all the double curves \( \mathcal{E}_i \), \( 1 \leq i \leq n + 1 \). For each \( i \) with \( 1 \leq i \leq n - 1 \) let \( P_i \subset Y \) be the plane \( \pi^{-1}(\lambda_i) \) as before. Then as \( (F|_{P_i}) = 2\mathcal{E}_i = (Q|_{P_i}) \) as divisors on the plane \( P_i \), there exists a constant \( c_i \) such that \( F - c_i Q^2 \in I_{P_i} \subset \mathbb{C}[z_0, \ldots, z_n] \). If \( c_i \neq c_j \) for some \( i \neq j \), we obtain \( Q^2 \in I_{P_i} + I_{P_j} \). Further the last ideal is readily seen to be equal to \( I_{P_i \cap P_j} \), and therefore equals to \( I_l = (z_0, z_1, \ldots, z_{n-2}) \subset \mathbb{C}[z_0, \ldots, z_n] \). Hence \( Q \in (z_0, z_1, \ldots, z_{n-2}) \). This means that the divisor \( (Q|_{P_i}) \) contains the line \( l \), which contradicts irreducibility of the double conic \( \mathcal{E}_i \) for \( i < n - 1 \), and non-reality of each lines of the splitting double conic \( \mathcal{E}_{n-1} \) (see Section [5,1]). Therefore \( c_i = c_j \) for any \( i, j \in \{1, 2, \ldots, n - 1\} \).

Next for the double curves \( \mathcal{E}_n \) and \( \mathcal{E}_{n+1} \), since \( (F|_{H_k \cap Y}) = 2\mathcal{E}_k = (Q^2|_{H_k \cap Y}) \) for \( k \in \{n, n+1\} \) on the cone, there exists a constant \( c_k \) such that \( F - c_k Q^2 \in I_{H_k \cap Y} = (z_{k-1}) + I_Y \). So taking a difference with \( F - c_1 Q^2 \in I_{P_1} \), we obtain that \( (c_1 - c_k)Q^2 \in (z_{k-1}) + I_Y + I_{P_1} \). But since \( P_1 \subset Y \), we have \( I_{P_1} \supset I_Y \), and therefore \( (c_1 - c_k)Q^2 \in (z_{k-1}) + I_{P_1} \). Hence if \( c_1 \neq c_k \) we have \( Q^2 \in (z_{k-1}) + I_{P_1} \), which means \( Q^2|_{P_1} \in (z_{k-1}|_{P_1}) \). This implies that the divisor \( (Q^2)|_{P_1} \) contains a line \( (z_{k-1}) \) on the plane \( P_1 \) as an irreducible component, which again contradicts the irreducibility of the double conic \( \mathcal{E}_1 \). Therefore we have \( c_1 = c_k \) for \( k \in \{n, n+1\} \). By rescaling we can suppose \( c_1 = 1 \) for any \( 1 \leq i \leq n + 1 \). Thus we have

\[
(5.8) \quad F - Q^2 \in \left( \bigcap_{1 \leq i \leq n-1} I_{P_i} \right) \cap ((z_{n-1}) + I_Y) \cap ((z_n) + I_Y).
\]

Let \( \Pi \) be the linear subspace of \( \mathbb{CP}^{n-2} \) spanned by the \( (n - 2) \) points \( \lambda_1, \ldots, \lambda_{n-2} \) on \( \Lambda \). Since \( \Lambda \) is a rational normal curve, \( \Pi \) is \( (n - 3) \)-dimensional. Let \( f \in \mathbb{C}[z_0, \ldots, z_{n-2}] \) be a defining linear polynomial of the hyperplane \( \Pi \). Then we have

\[
(5.9) \quad \pi^{-1}(\Pi) \cap Y = \bigcup_{1 \leq i \leq n-2} P_i,
\]

and therefore

\[
\bigcap_{1 \leq i \leq n-2} I_{P_i} = I_{P_1 \cup \cdots \cup P_{n-2}} = I_{\pi^{-1}(\Pi) \cap Y} = I_{\pi^{-1}(\Pi)} + I_Y.
\]

Hence (5.8) can be rewritten as

\[
(5.10) \quad F - Q^2 \in ((f) + I_Y) \cap ((z_{n-1}) + I_Y) \cap ((z_n) + I_Y) \cap I_{P_{n-1}}.
\]

Further, by an elementary argument, it is easy to see that

\[
((z_{n-1}) + I_Y) \cap ((z_n) + I_Y) = (z_{n-1}z_n + I_Y
\]

and

\[
((f) + I_Y) \cap ((z_{n-1}z_n) + I_Y) = (fz_{n-1}z_n + I_Y).
\]

Therefore we can write

\[
(5.11) \quad F - Q^2 = z_{n-1}z_nf + g, \quad y \in I_Y.
\]

Then recalling that this is also in the ideal \( I_{P_{n-1}} \), by restricting both hand sides to the plane \( P_{n-1} \), we obtain \( g \in I_{P_{n-1}} \). From our normalization of the homogeneous coordinates given just before Theorem 5.2, we have \( I_{P_{n-1}} = (z_0, z_1, \ldots, z_{n-3}) \). Therefore \( g \in (z_0, z_1, \ldots, z_{n-3}) \). As \( g \) is linear, this means \( g \in \mathbb{C}[z_0, z_1, \ldots, z_{n-3}] \). Then if we regard the divisor \( (fg) \) as a
sum of two hyperplanes in $\mathbb{CP}^{n-2}$, since $S_i^+ + S_i^-$, $1 \leq i \leq n-1$, are all reducible members of the pencil $|F|$, we have, as sets,

\[(fg) \cap \Lambda = \{\lambda_1, \cdots, \lambda_{n-1}\}.\]  

(5.12)

Furthermore, from the definition of the hyperplane $\Pi$, we have $(f) \cap \Lambda = \{\lambda_1, \cdots, \lambda_{n-2}\}$, where all points are included by multiplicity one. Now if some $\lambda_i$, $1 \leq i \leq n-2$, are contained in the intersection $(g) \cap \Lambda$, then from (5.11), in a neighborhood of the double conic $C_i$, the defining equation on $Y$ of the branch divisor $B$ is of the form

$$q^2 + (\lambda - \lambda_i)^k, \quad k \geq 2,$$

where $\lambda$ is a local coordinate on a neighborhood of $\lambda_i$ in the curve $\Lambda$, and $q$ is a non-homogeneous representative of $Q$. This means that the divisor $B$ is singular along the double conic $C_i$, $1 \leq i \leq n-2$. But if this is actually the case, the double cover of $Y$ with branch divisor being $B$ would have singularity along the (inverse image of) the real irreducible curve $C_i$. (Note that here we have used the assumption $i \neq n-1$.) Therefore the morphism $\Phi'_n : Z'_n \to Y$ obtained in Section 4 must contract a real divisor. But from our explicit elimination, there is no real divisor over the plane $P_i$, $1 \leq i \leq n-2$. This is a contradiction, and hence we obtain that $(g) \cap \Lambda = \{\lambda_{n-1}\}$. This means that $g = cz_0$ for some $c \neq 0$. Hence, from (5.11), we finally obtain

$$F - Q^2 = z_0 z_{n-1} z_n f + y, \quad y \in I_Y.$$

Therefore by disposing $y$, we obtain the claim of the theorem. \[\square\]

5.3. Dimension of the moduli space. Finally in this subsection we first compute dimension of the moduli space of the present twistor spaces, and then explain relationship between other twistor spaces.

Let $Z$ be any one of the relevant twistor spaces on $n\mathbb{CP}^2$ and $S$ a real irreducible member of the pencil $|F|$ as before. Then by a similar argument to [9, Proposition 5.1], for the tangent sheaf of $Z$ we have

\[(5.13) \quad H^i(\Theta_Z) = 0 \text{ for } i \neq 1, \quad h^1(\Theta_Z) = 7n - 15.\]

Also, it is easy to show

\[(5.14) \quad H^i(\Theta_S) = 0 \text{ for } i \neq 1, \quad h^1(\Theta_S) = 4n - 6,\]

\[(5.15) \quad h^0(K_S^{-1}) = 1, \quad h^1(K_S^{-1}) = 2n - 8.\]

Let $\Theta_{Z,S}$ denote the subsheaf of $\Theta_Z$ consisting of germs of vector fields which are tangent to $S$, and write $\Theta_Z(-S) := \Theta_Z \otimes \mathcal{O}_Z(-S)$. Then since $Z$ is Moishezon, we have $H^2(\Theta_Z(-S)) = 0$ by [3, Lemma 1.9]. Hence from the cohomology exact sequence $0 \to \Theta_Z(-S) \to \Theta_{Z,S} \to \Theta_S \to 0$, by using (5.11), we obtain $H^2(\Theta_{Z,S}) = 0$. Therefore by several standard exact sequences of sheaves including this one, and noting $N_{S/Z} \simeq K_S^{-1}$, we obtain the following
From the middle column of this diagram we obtain \( h^1(\Theta_{Z,S}) = 5n - 6 \) by (5.13) and (5.15), which means \( h^1(\Theta_Z(-S)) = n \) from the middle row and (5.14).

In order to compute the dimension of the moduli space, we recall that our twistor spaces can be characterized by the property that they have one of the rational surface \( S \) constructed in Section 2.1 as a member of the system \(|F|\). From the construction, the surface \( S \) is determined from 6 points on the anticanonical cycle \( C_1 + C_2 + \overline{C}_1 + \overline{C}_2 \) on \( \mathbb{CP}^1 \times \mathbb{CP}^1 \) (which give the intermediate surface \( S_0 \)), and therefore by taking automorphisms of \( \mathbb{CP}^1 \times \mathbb{CP}^1 \) which preserve the cycle into account, they determine a 4-dimensional subspace of \( H^1(\Theta_S) \). We denote this subspace by \( V \). We have \( \dim \alpha^{-1}(V) = \dim V + h^1(\Theta_Z(-S)) = n + 4 \). The tangent space of the moduli space of the present twistor spaces can be considered as the space \( \beta(\alpha^{-1}(V)) \subset H^1(\Theta_Z) \). The image of the map \( \gamma \) in the diagram corresponds to deformations of the pair \((Z,S)\) that can be obtained by moving \( S \) in \( Z \), and of course they do not give a non-trivial deformation of \( Z \). Further, from the characterization of \( Z \) by the presence of \( S \), even if we move \( S \) in \( Z \), the deformed \( S \) must still be the one discussed in Section 2.1. This means that the image of \( \gamma \) is contained in \( \alpha^{-1}(V) \). Thus the tangent space of the moduli space of the present twistor spaces can be identified with the quotient space

\[
\alpha^{-1}(V)/\gamma H^0(K_S^{-1}),
\]

and this is \((n + 3)\)-dimensional by (5.15). Thus the dimension of the moduli space is strictly larger than that of the twistor spaces in [7], which was \( n \)-dimensional.

Recall that the twistor spaces studied in [7] also have a structure of a branched double covering over the same scroll \( Y \) under the same multiple system \(|(n - 2)F|\), and the branch divisor is a cut of the scroll by a quartic hypersurface. (In [7] the twistor spaces are presented rather as a double cover over the resolved space \( \tilde{Y} \), but it is not difficult to rewrite it as a double cover over the scroll \( Y \).) Hence the structure of the two kinds of the twistor spaces is very similar. Looking defining equations of the quartic hypersurfaces, or inspecting structure of the surface \( S \) in the pencil \(|F|\), it is easy to see that the twistor spaces in [7] are obtained as a limit (under a deformation) of the present twistor spaces.
Next we explain a relationship between the present twistor spaces (and also those in [7]), and LeBrun twistor spaces [12], from a viewpoint of moduli. For this, we recall that LeBrun’s self-dual conformal classes on \(n\mathbb{CP}^2\) are determined from distinct \(n\) points on the hyperbolic space \(\mathcal{H}^3\). So for each \(n \geq 3\), let \(H^{[n]}\) be the space of configurations of distinct \(n\) points on \(\mathcal{H}^3\). This is a dense open subset of the symmetric product of \(n\) copies of \(\mathcal{H}^3\). For each \(k\) with \(2 \leq k \leq n\), we denote by \(H^{[n]}_k\) for the subset of \(H^{[n]}\) consisting of configurations for which the maximal number of points lying on a common geodesic is exactly \(k\). These provide \(H^{[n]}\) with a natural stratification
\[
H^{[n]} = H^{[n]}_2 \supset H^{[n]}_3 \supset H^{[n]}_4 \supset \cdots \supset H^{[n]}_{n-1} \supset H^{[n]}_n.
\]
The space \(H^{[n]}\) is \(3n\)-dimensional, and we clearly have
\[
\dim H^{[n]}_{k+1} = \dim H^{[n]}_k - 2, \quad 2 \leq k < n.
\]
In particular we have \(\dim H^{[n]}_k = 3n - 2(k - 2)\). The isometric action of the group \(\text{PSL}(2, \mathbb{C})\) on the hyperbolic space \(\mathcal{H}^3\) naturally induces an action on the space \(H^{[n]}\) by the same group, and it clearly preserves the stratification (5.17). So we can define the quotients of the strata by
\[
\mathcal{LB}^{[n]}_k := H^{[n]}_k / \text{PSL}(2, \mathbb{C}).
\]
The largest space \(\mathcal{LB}^{[n]} := \mathcal{LB}^{[n]}_{2n}\) is exactly the moduli space of LeBrun’s self-dual conformal classes on \(n\mathbb{CP}^2\), and from (5.17) it is equipped with a natural stratification
\[
\mathcal{LB}^{[n]} = \mathcal{LB}^{[n]}_2 \supset \mathcal{LB}^{[n]}_3 \supset \mathcal{LB}^{[n]}_4 \supset \cdots \supset \mathcal{LB}^{[n]}_{n-1} \supset \mathcal{LB}^{[n]}_n.
\]
We note that the smallest strata \(\mathcal{LB}^{[n]}_n\) is precisely the moduli space of toric LeBrun metrics on \(n\mathbb{CP}^2\). For any \(k \neq n\), the \(\text{PSL}(2, \mathbb{C})\)-action on \(H^{[n]}_k\) is effective and therefore we have
\[
\dim \mathcal{LB}^{[n]}_k = (3n - 2k + 4) - 6 = 3n - 2k - 2.
\]
On the other hand, for the case \(k = n\), a \(U(1)\)-subgroup of \(\text{PSL}(2, \mathbb{C})\) which fixes the geodesic acts trivially on the smallest stratum \(H^{[n]}_n\), and hence we have
\[
\dim \mathcal{LB}^{[n]}_n = (n + 4) - 5 = n - 1.
\]
This stratification on the moduli space of LeBrun metrics is closely related to the moduli space of the present twistor spaces and also that of the twistor spaces in [7] in the following way. As mentioned in the final portion of [7], the twistor spaces in [7] can be obtained as a small (\(\mathbb{C}^*\)-equivariant) deformation of the twistor spaces of toric LeBrun metrics, where a \(\mathbb{C}^*\)-subgroup of the torus \(\mathbb{C}^* \times \mathbb{C}^*\) is chosen in an (explicit) appropriate way. Since the moduli space of toric LeBrun metrics on \(n\mathbb{CP}^2\) is \((n - 1)\)-dimensional as in (5.21) and the moduli space of the twistor spaces on \(n\mathbb{CP}^2\) studied in [7] is \(n\)-dimensional as computed in the paper, we can conclude that the former moduli space is contained in the closure of the latter moduli space as a hypersurface. In other words, the moduli space of the twistor spaces in [7] can be partially compactified by attaching the moduli space of toric LeBrun metrics, and the last moduli space is a hypersurface in the partial compactification.

In order to explain a similar relationship between the present twistor spaces and LeBrun twistor spaces on \(n\mathbb{CP}^2\), we look at the stratum
\[
\mathcal{LB}^{[n]}_{n-2}
\]
in the moduli space of LeBrun metrics on \( \mathbb{CP}^2 \). This is \((n+2)\)-dimensional by (5.20).

Now from the construction in Section 2.1, it is immediate to see that a member \( S \in |F| \) in the present twistor spaces can be obtained as a small deformation of a member in \(|F|\) on a LeBrun twistor space \( Z \in \mathcal{LB}_{n-2} \). From this, by using a standard argument in deformation theory, we can show that the present twistor spaces are obtained as a small deformation of a twistor space belonging to \( \mathcal{LB}_{n-2} \). Therefore, recalling that the moduli space of the present twistor space is \((n+3)\)-dimensional as seen above, we again conclude that the moduli space of the twistor spaces studied in this paper can be partially compactified by attaching the stratum \((5.22)\) in the moduli space of LeBrun metrics on \( n \mathbb{CP}^2 \), and the last moduli space is a hypersurface in the partial compactification.

One might wonder if a similar relationship between the moduli spaces carries over to other strata in the stratification \((5.19)\) and twistor spaces of double solid type. In this respect, it seems quite certain that the real situation is summarized as in the following diagram:

\[
\begin{array}{cccccc}
\mathcal{LB}_n & \leftarrow & \mathcal{LB}_{n-1} & \leftarrow & \mathcal{LB}_{n-2} & \leftarrow & \mathcal{LB}_{n-3} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathcal{DS}_{IV} & \leftarrow & \mathcal{DS}_{III} & \leftarrow & \mathcal{DS}_{II} & \leftarrow & \mathcal{DS}_{I},
\end{array}
\]

(5.23)

where \( \mathcal{DS}_{IV} \) and \( \mathcal{DS}_{II} \) are respectively the moduli spaces of the twistor spaces in \([7]\) and the ones in the present paper, and \( \mathcal{DS}_{III} \) and \( \mathcal{DS}_{I} \) are moduli spaces of some unknown twistor spaces of double solid type. Also, in the diagram (5.23), the notation \( A \to B \) means that the twistor spaces belonging to the moduli space \( B \) are obtained as a limit (specialization) of twistor spaces belonging to the moduli space \( A \). Furthermore, the upper four moduli spaces should always be a hypersurface in the closure of the lower moduli spaces. We note that the results in \([9, 10]\) rigorously shows this is actually the case when \( n = 4 \). Actually, the above notations I, II, III and IV for the double solid twistor spaces are taken from the paper \([10]\).

We leave the investigation of the ‘unknown’ twistor spaces belonging to \( \mathcal{DS}_{I} \) and \( \mathcal{DS}_{III} \) in a future paper; here we just mention that unlike the ones belonging to \( \mathcal{DS}_{II} \) and \( \mathcal{DS}_{IV} \), the multiple system \(|(n-2)F|\) of twistor spaces belonging to \( \mathcal{DS}_{I} \) is composed with a pencil \(|F|\), and therefore for analysis of the spaces we need to consider a linear system of higher degree. On the other hand, concerning the remaining strata \( \mathcal{LB}_k \), \( k < n - 3 \), although we can consider similar small deformations of LeBrun twistor spaces in these strata, they are not Moishezon anymore.

The above discussion concerns relations between the twistor spaces of double solid type and LeBrun twistor spaces. Now recall that in \([8]\) we have constructed a family of Moishezon twistor spaces over \( n \mathbb{CP}^2 \) which share many properties with LeBrun twistor spaces. If we call these as LeBrun-like twistor spaces, analogous relations to (5.23) hold between the twistor spaces of double solid type and LeBrun-like twistor spaces. Namely, we can define a stratification on the moduli space of LeBrun-like twistor spaces in a similar way to \((5.17)\) in terms of the structure of a member of the system \(|F|\), and for the smallest four strata among them, the relations \((5.23)\) carry over, including the dimension of the moduli spaces. (We recall that the full moduli space of LeBrun-like twistor spaces on \( n \mathbb{CP}^2 \) is \((3n-6)\)-dimensional as computed in \([8\) Section 5.2], which is exactly the same as the original LeBrun metrics.)
Hence it might be possible to say that, while the moduli spaces of LeBrun twistor spaces and LeBrun-like twistor spaces are not adjacent, they are interpolated by the moduli space of the twistor spaces of double solid type.

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