The number of spanning clusters of the uniform spanning tree in three dimensions

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Abstract

Let $U_δ$ be the uniform spanning tree on $δZ^3$. A spanning cluster of $U_δ$ is a connected component of the restriction of $U_δ$ to the unit cube $[0, 1]^3$ that connects the left face $[0] \times [0, 1]^2$ to the right face $[1] \times [0, 1]^2$. In this note, we will prove that the number of the spanning clusters is tight as $δ \to 0$, which resolves an open question raised by Benjamini in [4].

1 Introduction

Given a finite connected graph $G = (V, E)$, a spanning tree $T$ of $G$ is a subgraph of $G$ that is a tree (i.e. is connected and contains no cycles) with vertex set $V$. A uniform spanning tree (UST) of $G$ is obtained by choosing a spanning tree of $G$ uniformly at random. This is an important model in probability and statistical physics, with beautiful connections to other subjects, such as electrical potential theory, loop-erased random walk and Schramm-Loewner evolution. See [8] for an introduction to various aspects of USTs.

Fix $δ \in (0, 1)$ and $d \in \mathbb{N}$. In [9] it was shown that, by taking the local limit of the uniform spanning trees on an exhaustive sequence of finite subgraphs of $δZ^d$, it is possible to construct a random subgraph $U_δ$ of $δZ^d$. Whilst the resulting graph $U_δ$ is almost-surely a forest consisting on an infinite number of disjoint components that are trees when $d \geq 5$, it is also the case that $U_δ$ is almost-surely a spanning tree of $δZ^d$ with one topological end for $d \leq 4$, see [9]. In the latter low-dimensional case, $U_δ$ is commonly referred to as the UST on $δZ^d$.

In this note, we study a macroscopic scale property of $U_δ$, namely the number of its spanning clusters, as previously studied by Benjamini in [3]. To be more precise, let us proceed to introduce some notation. Write

$$\mathbb{B} = [0, 1]^d = \left\{ (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : 0 \leq x_i \leq 1, i = 1, 2, \ldots, d \right\}$$

for the unit hypercube in $\mathbb{R}^d$. Also, set

$$F = \left\{ (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : x_1 = 0 \right\}$$

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and
\[ G = \left\{ (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d : x_1 = 1 \right\} \]  \hspace{1cm} (3)
for the hyperplanes intersecting the ‘left’ and ‘right’ sides of the hypercube \( \mathbb{B} \). Given a subgraph \( U = (V, E) \) of \( \delta \mathbb{Z}^d \), we write \( U' = (V', E') \) for the restriction of \( U \) to the cube \( \mathbb{B} \), i.e.
we set \( V' = V \cap \mathbb{B} \) and \( E' = \{ \{x, y\} \in E : x, y \in V' \} \). A connected component of \( U' \) is called a cluster of \( U' \). Moreover, following [4], a spanning cluster of \( U \) is a cluster of \( U \) containing vertices \( x \) and \( y \) such that \( \text{dist}(x, F) < \delta \) and \( \text{dist}(y, G) < \delta \), where \( \text{dist}(z, A) := \inf_{w \in A} |z - w| \) is the Euclidean distance between a point \( z \in \mathbb{R}^d \) and subset \( A \subseteq \mathbb{R}^d \). That is, a cluster of \( U \) is called spanning when it connects \( F \) to \( G \) (at the level of discretization being considered).

Concerning the number of spanning clusters of \( U_\delta \), it was proved in [4] that:

- for \( d \geq 4 \), the expected number of spanning clusters of \( U_\delta \) grows to infinity as \( \delta \to 0 \);
- for \( d = 2 \), the number of spanning clusters of \( U_\delta^+ \) is tight as \( \delta \to 0 \), where \( U_\delta^+ \) denotes the uniform spanning tree of the square \( \mathbb{B} \cap \delta \mathbb{Z}^2 \) when all the vertices on the right side of the square are identified to a single point (which is called the right wired uniform spanning tree in [4]). Figure 1 shows the spanning cluster of a realisation of (an approximation to) \( U_\delta \) on \( \delta \mathbb{Z}^2 \).

The case \( d = 3 \) was left as an open question in [4]. The main purpose of this note is to resolve it by following the following theorem.

**Theorem 1.1.** Let \( d = 3 \). It holds that the number of spanning clusters of \( U_\delta \) is tight as \( \delta \to 0 \).

**Remark 1.2.** In the forthcoming article [3], we establish a scaling limit for the three-dimensional UST in a version of the Gromov-Hausdorff topology, at least along the subsequence \( \delta_n := 2^{-n} \). The corresponding two-dimensional result is also known (along an arbitrary sequence \( \delta \to 0 \)), see [3] and [4, Remark 1.2]. In both cases, we expect that the techniques used to prove such a scaling limit can be used to show that the number of spanning clusters of \( U_\delta \) actually converges in distribution.

The organization of the remainder of the paper is as follows. In Section 2, we introduce some notation that will be used in the paper. The proof of Theorem 1.1 is then given in Section 3.

## 2 Notation

In this section, we introduce the main notation needed for the proof of Theorem 1.1. We write \( |\cdot| \) for the Euclidean norm on \( \mathbb{R}^3 \) and, as in the introduction, \( \text{dist}(\cdot, \cdot) \) for the Euclidean distance between a point and a subset of \( \mathbb{R}^3 \). Given \( \delta \in (0, 1) \), if \( x \in \delta \mathbb{Z}^3 \) and \( r > 0 \), then we write
\[ B(x, r) = \{ y \in \delta \mathbb{Z}^3 : |x - y| < r \} \]
for the lattice ball of centre \( x \) and radius \( r \) (we will commonly omit dependence on \( \delta \) for brevity). Let \( \mathbb{B} \), \( F \) and \( G \) be defined as at [11], [2] and [12] in the case \( d = 3 \).

For \( \delta \in (0, 1) \), a sequence \( \lambda = (\lambda(0), \lambda(1), \cdots, \lambda(m)) \) is said to be a path of length \( m \) if \( \lambda(i) \in \delta \mathbb{Z}^3 \) and \( |\lambda(i) - \lambda(i + 1)| = \delta \) for every \( i \). A path \( \lambda \) is simple if \( \lambda(i) \neq \lambda(j) \) for all \( i \neq j \). For a path \( \lambda = (\lambda(0), \lambda(1), \cdots, \lambda(m)) \), we define its loop-erasure \( \text{LE}(\lambda) \) as follows. Firstly, let
\[ s_0 = \max \{ j \leq m : \lambda(j) = \lambda(0) \}, \]
and for $i \geq 1$, set
\[ s_i = \max \{ j \leq m : \lambda(j) = \lambda(s_{i-1} + 1) \} . \]
Moreover, write $n = \min \{ i : s_i = m \}$. The loop-erasure of $\lambda$ is then given by
\[ \text{LE}(\lambda) = (\lambda(s_0), \lambda(s_1), \ldots, \lambda(s_n)) . \]
We write $\text{LE}(\lambda)(k) = \lambda(s_k)$ for each $0 \leq k \leq n$. Note that the vertices hit by $\text{LE}(\lambda)$ are a subset of those hit by $\lambda$, and that $\text{LE}(\lambda)$ is a simple path such that $\text{LE}(\lambda)(0) = \lambda(0)$ and $\text{LE}(\lambda)(n) = \lambda(m)$. Although the loop-erasure of $\lambda$ has so far only been defined in the case that $\lambda$ has a finite length, it is clear that we can define $\text{LE}(\lambda)$ similarly for an infinite path $\lambda$ if the set $\{ k \geq 0 : \lambda(j) = \lambda(k) \}$ is finite for each $j \geq 0$. Additionally, when the path $\lambda$ is given by a simple random walk, we call $\text{LE}(\lambda)$ a loop-erased random walk (see [7] for an introduction to loop-erased random walks).

Again given $\delta \in (0,1)$, write $U_\delta$ for the uniform spanning tree on $\delta\mathbb{Z}^3$. As noted in the introduction, this object was constructed in [9], and shown to be a tree with a single end, almost-surely. The graph $U_\delta$ can be generated from loop-erased random walks by a procedure now referred to as Wilson’s algorithm (after [11]), which is described as follows.
• Let \((x_i)_{i \geq 1}\) be an arbitrary, but fixed, ordering of \(\delta \mathbb{Z}^3\).

• Write \(R^{x_1}\) for a simple random walk on \(\delta \mathbb{Z}^3\) started at \(x_1\). Let \(\gamma_{x_1} = \text{LE}(R^{x_1})\) be the loop-erasure of \(R^{x_1}\) – this is well-defined since \(R^{x_1}\) is transient. Set \(U^1 = \gamma_{x_1}\).

• Given \(U^i\) for \(i \geq 1\), let \(R^{x_{i+1}}\) be a simple random walk (independent of \(U^i\)) started at \(x_{i+1}\) and stopped on hitting \(U^i\). We let \(U^{i+1} = U^i \cup \text{LE}(R^{x_{i+1}})\).

It is then the case that the output random tree \(\bigcup_{i=1}^{\infty} U^i\) has the same distribution as \(\mathcal{U}_\delta\). In particular, the distribution of the output tree does not depend on the ordering of points \((x_i)_{i \geq 1}\).

Similarly to above, for \(z \in \delta \mathbb{Z}^3\), we will write \(\gamma_z\) for the infinite simple path in \(\mathcal{U}_\delta\) starting from \(z\). Given a point \(z \in \delta \mathbb{Z}^3\), it follows from the construction of \(\mathcal{U}_\delta\) explained hitherto that the distribution of \(\gamma_z\) coincides with that of \(\text{LE}(R^z)\), where \(R^z\) is a simple random walk on \(\delta \mathbb{Z}^3\) started at \(z\).

Furthermore, as we explained in the introduction, we will write \(U^i_\delta\) for the restriction of \(U^i\) to the cube \(B\). A connected component of \(U^i_\delta\) is called a cluster. Also, as we defined previously, a spanning cluster is a cluster connecting \(F\) to \(G\). We let \(N_\delta\) be the number of spanning clusters of \(U^i_\delta\).

Finally, we will use \(c, C, c_0\), etc. to denote universal positive constants which may change from line to line.

3 Proof of the main result

In this section, we will prove the following theorem, which incorporates Theorem

**Theorem 3.1.** There exists a universal constant \(C\) such that: for all \(M < \infty\) and \(\delta > 0\),

\[
\mathbb{P}(N_\delta \geq M) \leq CM^{-1}.
\]

In particular, the laws of \((N_\delta)_{\delta \in (0,1)}\) form a tight sequence of probability measures on \(\mathbb{Z}_+\).

**Remark 3.2.** In [7], Aizenman proved that for critical percolation in two dimensions, the probability of seeing \(M\) distinct spanning clusters is bounded above by \(Ce^{-cM^2}\). We do not expect that the polynomial bound in [7] is sharp, but leave it as an open problem to determine the correct tail behaviour for number of spanning clusters of the UST in three dimensions, and, in particular, ascertain whether it also exhibits Gaussian decay.

**Proof.** Let \(\delta \in (0,1)\), and suppose \(M \geq 1\) is such that \(\delta < M^{-1}\). Define

\[
A = [-1,2]^3, \\
A' = F \cap B, \\
A'' = \{(x_1,x_2,x_3) \in B : x_1 = 2/3\}, \\
B' = \{(x_1,x_2,x_3) \in B : x_1 \leq 2/3\}.
\]

Moreover, let \((z_i)_{i=1}^L\) be a sequence of points in \(A \cap \delta \mathbb{Z}^3\) such that \(A \subseteq \bigcup_{i=1}^{L} B(z_i,1/M)\) and \(L \leq 10^6 M^3\).

To construct \(\mathcal{U}_\delta\), we first perform Wilson’s algorithm for \((z_i)_{i=1}^L\) (see Section 2). Namely, we consider

\[
U^1 := \bigcup_{i=1}^{L} \gamma_{z_i},
\]
which is the subtree of $\mathcal{U}_i$ spanned by $(z_i)_{i=1}^L$. (Recall that for $z \in \delta \mathbb{Z}^3$ we denote the infinite simple path in $\mathcal{U}_i$ starting from $z$ by $\gamma_z$.) The idea of the proof is then as follows. Crucially, each branch of $\mathcal{U}^1$ is a ‘hittable’ set, in the sense that for a simple random walk $R$ whose starting point is close to $\mathcal{U}^1$, it is likely that $R$ hits $\mathcal{U}^1$ before moving far away. As a result, Wilson’s algorithm guarantees that, with high probability, the spanning clusters of $\mathcal{U}_i$ correspond to those of $\mathcal{U}^1$ when $M$ is sufficiently large. So, the problem boils down to the tightness of the number of spanning clusters of $\mathcal{U}^1$, which is not difficult to prove.

To make the above argument rigorous, we introduce the following two “good” events for $\mathcal{U}^1$:

$$H_i = H_i(\xi) := \left\{ \text{For any } x \in B(0,4) \cap \delta \mathbb{Z}^3 \text{ with } \text{dist}(x, \gamma_{z_i}) \leq 1/M, \right. \left. P^x_R(R[0,T] \cap \gamma_{z_i} = \emptyset) \leq M^{-\xi} \right\},$$

$$I_i := \left\{ \text{The number of crossings of } \gamma_{z_i} \text{ between } A' \text{ and } A'' \right\},$$

for $1 \leq i \leq L$, where

- $R$ is a simple random walk which is independent of $\gamma_{z_i}$, the law of which is denoted by $P^x_R$ when we assume $R(0) = x$;
- $T$ is the first time that $R$ exits $B(x,1/\sqrt{M})$;
- a crossing of $\gamma_{z_i}$ between $A'$ and $A''$ in $\mathbb{B}'$ is a connected component of the restriction of $\gamma_{z_i}$ to $\mathbb{B}'$ that connects $A'$ to $A''$.

Namely, the event $H_i$ guarantees that the branch $\gamma_{z_i}$ is a hittable set, and the event $I_i$ controls the number of crossings of $\gamma_{z_i}$.

Now, [10, Theorem 3.1] ensures that there exist universal constants $\xi_0, C > 0$ such that

$$\mathbb{P} \left( \bigcap_{i=1}^L H_i(\xi_0) \right) \geq 1 - CM^{-10}.$$ 

Thus, with high probability (for $\mathcal{U}^1$), each branch of $\mathcal{U}^1$ is a hittable set.

The probability of the event $I_i$ is easy to estimate. Indeed, suppose that the event $I_i$ does not occur. This implies that the number of “traversals” of $S_{z_i}$ from $A'$ to $A''$ or vice versa must be bigger than $M$, where $S_{z_i}$ stands for a simple random walk starting from $z_i$. Notice that there exists a universal constant $c_0 > 0$ such that for any point $w \in A'$ (respectively $w \in A''$), the probability that $S''_w$ hits $A''$ (respectively $A'$) is smaller than $1 - c_0$ (see [8, Proposition 1.5.10], for example). Thus, the probability of the event $I_i$ is bounded below by $1 - (1 - c_0)^M = 1 - e^{-aM}$, where $a > 0$. Letting $b = a/2$ and taking sum over $1 \leq i \leq L$, we find that

$$\mathbb{P} \left( \bigcap_{i=1}^L I_i \right) \geq 1 - Ce^{-bM}.$$ 

To put the above together, let

$$J = \bigcap_{i=1}^L H_i(\xi_0) \cap I_i.$$ 

For $1 \leq i \leq L$, set $\mathcal{U}^1_i = \cup_{j=1}^L \gamma_{z_j}$ so that $\mathcal{U}^1 = \bigcup_{i=1}^L \mathcal{U}^1_i$. As above, by a spanning cluster of $\mathcal{U}^1_i$ between $A'$ and $A''$ in $\mathbb{B}'$ we mean a connected component of the restriction of $\mathcal{U}^1_i$ to $\mathbb{B}'$ which
connects $A'$ to $A''$. We write $n_i$ for the number of spanning clusters of $U^1_i$ between $A'$ and $A''$ in $B'$. On the event $J$, we have that

$$n_i \leq iM + i - 1,$$

for all $1 \leq i \leq L$, since $n_{i+1} - n_i$ is at most $M + 1$ for each $i \geq 1$. In particular, we see that the number of spanning clusters of $U^1_i$ between $A'$ and $A''$ in $B'$ is bounded above by $L(M + 1)$, which is comparable to $M^4$.

We next consider a sequence of subsets of $A$ as follows. Let $a^* > 0$ be the positive constant such that

$$a^* \sum_{k=1}^{\infty} k^{-2} = 10^{-1}. \tag{5}$$

Set $\eta_1 = 0$, and $\eta_k = a^* \sum_{j=1}^{k-1} j^{-2}$ for $k \geq 2$. Finally, for $k \geq 1$, let

$$A_k = [-1 + \eta_k, 2 - \eta_k]^3.$$ 

Notice that $A_{k+1} \subseteq A_k$ and $[-1/2, 3/2]^3 \subseteq A_k$ for all $k \geq 1$, and moreover dist$(\partial A_k, \partial A_{k+1}) = a^*k^{-2}$. We further introduce sequences $(z^k_i)_{i=1}^{L_k}$ consisting of points in $A_k \cap \delta Z^3$ such that

$$A_k \subseteq \bigcup_{i=1}^{L_k} B\left(z^k_i, \delta_k\right),$$

and $L_k \leq 10^5 \delta_k^{-3}$, where $\delta_k := M^{-1}2^{-(k-1)}$. Note that we may assume that $L_1 = L$ and $(z^1_i)_{i=1}^{L_1} = (z_i)_{i=1}^{L}$.

We set

$$H_k^i = H_k^i(\xi) := \left\{ \text{For any } x \in B(0, 4) \cap \delta Z^3 \text{ with dist}\left(x, \gamma_{z^k_i}\right) \leq \delta_k, \quad P^R_{\xi}\left(R[0, T^k] \cap \gamma_{z^k_i} = \emptyset\right) \leq \delta_k^\xi \right\},$$

where $R$ is a simple random walk that is independent of $\gamma_{z^k_i}$, with law denoted by $P^R_{\xi}$ when we assume $R(0) = x$, and $T^k$ is the first time that $R$ exits $B(x, \sqrt{k})$. By [11] Theorem 3.1] again, there exist universal constants $\xi_1, C > 0$ (which do not depend on $k$) such that

$$P\left(\bigcap_{i=1}^{L_k} H_k^i(\xi_1)\right) \geq 1 - C\delta_k^{10},$$

for all $k = 1, 2, \cdots, k_0$, where $k_0$ is the smallest integer $k$ such that $\delta_k < \delta$. Thus if we write

$$H^k = \bigcap_{i=1}^{L_k} H_k^i(\xi_1)$$

and

$$J' = J \cap \bigcap_{k=1}^{k_0} H^k,$$

we have

$$P(J') \geq 1 - CM^{-10}.$$

Given the above setup, we perform Wilson’s algorithm as follows:
• recall that $\mathcal{U}^1$ is the tree spanned by $(z_i^{L_1})_{i=1}^{L_1} = (z_i)_{i=1}^{L_1};$

• next perform Wilson’s algorithm for $(z_i^{L_2})_{i=1}^{L_2}$ for each $z_i^2$, run a simple random walk $Rz_i^2$ from $z_i^2$ until it hits the part of the tree that has already been constructed, and adding its loop-erasure as a new branch – the output tree is denoted by $\mathcal{U}^2;

• repeat the previous step for $(z_i^{L_k})_{i=1}^{L_k}$ to construct $\mathcal{U}^k$ for $k = 1, 2, \cdots, k_0$.

Now, condition $\mathcal{U}^1$ on the event $J$ above. We will show that, with high (conditional) probability, every new branch in $\mathcal{U}^2 \setminus \mathcal{U}^1$ has diameter smaller than $M^{-1/4}$. To this end, for $1 \leq i \leq L_2$, we write $d_i^2$ for the Euclidean diameter of the path from $z_i^2$ to $\mathcal{U}^1$ in $\mathcal{U}^2$, and define the event $W_i^2$ by setting

$$W_i^2 = \{d_i^2 \geq M^{-1/4}\}.$$ 

Suppose that the event $W_i^2$ occurs. By Wilson’s algorithm, the simple random walk $Rz_i^2$ must not hit $\mathcal{U}^1$ until it exits $B(z_i^2, M^{-1/4})$. Since $\text{dist}(z_i^2, \partial A) \geq a^*$ (for the constant $a^*$ defined at (4)), it holds that $B(z_i^2, M^{-1/4}) \subseteq A$. With this in mind, we set $u_0 = 0$, and

$$u_m = \inf \left\{j \geq u_{m-1} : \left|Rz_i^2(j) - Rz_i^2(u_{m-1})\right| \geq M^{-1/2}\right\}$$

for $m \geq 1$. We then have that

$$Rz_i^2[u_{m-1}, u_m] \cap \mathcal{U}^1 = \emptyset$$

for all $1 \leq m \leq M^{1/4}$. Since $A \subseteq \bigcup_{i=1}^{L_2} B(z_i, 1/M)$, it follows that for each $1 \leq m \leq M^{1/4}$, there exists a $z_i$ such that $Rz_i^2(u_{m-1}) \in B(z_i, 1/M)$. Thus the event $H_i(\xi_0)$ guarantees that

$$\mathbb{P}\left(Rz_i^2[u_{m-1}, u_m] \cap \mathcal{U}^1 = \emptyset \text{ for all } 1 \leq m \leq M^{1/4}\right) \leq M^{-\xi_0 M^{1/4}}.$$

Consequently, the conditional probability of $\bigcup_{i=1}^{L_2} W_i^2$ is bounded above by $L_2 M^{-\xi_0 M^{1/4}}$, which is smaller than $Ce^{-cM^{1/4}}$. This implies that, with probability at least $1 - Ce^{-cM^{1/4}}$, every new branch in $\mathcal{U}^2 \setminus \mathcal{U}^1$ has diameter smaller than $M^{-1/4}$. Notice that once each new branch has such a small diameter, the event $J$ guarantees that the number of spanning clusters of $\mathcal{U}^2$ between $A'$ and $A_2''$ in $\mathbb{B}$ is bounded above by $L(M + 1) \leq 10^6 M^{4}$, where $A_2''$ is defined by setting

$$A_2'' = \mathbb{B} \cap \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = 2/3 + M^{-1/4}\right\}.$$ 

Essentially the same argument is valid for $\mathcal{U}^k$. Indeed, conditioning $\mathcal{U}^k$ on the good event $J \cap \bigcap_{i=1}^{k} H_i'$ as above, it holds that, with probability at least $1 - Ce^{-c\xi_k^{1/4}}$ every new branch in $\mathcal{U}^{k+1} \setminus \mathcal{U}^k$ has diameter smaller than $\xi_k^{1/4}$. Notice that $\sum_k \xi_k^{1/4} \leq 10M^{-1/4} < 10^{-2}$ when $M$ is large. Therefore, with probability at least $1 - CM^{-10}$, the number of spanning clusters of $\mathcal{U}^{k_0}$ between $A'$ and $A''$ in $\mathbb{B}$ is bounded above by $L(M + 1) \leq CM^4$ for some universal constant $C$, where $A''$ is defined by setting

$$A'' = \mathbb{B} \cap \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = 3/4\right\}.$$ 

Finally, we perform Wilson’s algorithm for all of the remaining points in $\delta \mathbb{Z}^3$ to construct $\mathcal{U}_\delta$. Since the mesh size of $(z_i^{L_k})_{i=1}^{L_k}$ is smaller than $\delta$, it follows that the restriction of $\mathcal{U}_\delta$ to $\mathbb{B}$ coincides with that of $\mathcal{U}^{k_0}$. Thus we conclude that there exists a universal constant $C$ such that: for all $M < \infty$ and $\delta \in (0, M^{-1})$,

$$\mathbb{P}(N_\delta \geq M) \leq CM^{-2}.$$ 

For the case that $\delta > M^{-1}$, it is clear that $N_\delta < 100M^2$. Combining these two bounds, we readily obtain the bound at (4).
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