On the rational symplectic group

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Abstract
We give a short proof of an elementary classical result: any rational symplectic matrix can be put in diagonal form after right and left multiplication by integral symplectic matrices. We also give a new proof for its extension to Chevalley groups due to Steinberg by using the Cartan-Bruhat-Tits decomposition over $p$-adic fields.

1 Introduction
In this expository paper I present a short proof of a classical theorem I needed in [1]: a decomposition of the group $\text{Sp}(n, \mathbb{Q})$ of symplectic matrices with rational coefficients that gives a parametrization of the double quotient $\text{Sp}(n, \mathbb{Z}) \backslash \text{Sp}(n, \mathbb{Q}) / \text{Sp}(n, \mathbb{Z})$ where $\text{Sp}(n, \mathbb{Z})$ is the subgroup of symplectic matrices with integral coefficients.

This decomposition which can already be found in [15] is a symplectic version of the “adapted basis theorem” for $\mathbb{Z}$-modules, or of the “Smith normal form” for integral matrices.

In Section 2 we state precisely this decomposition that we call the “symplectic Smith normal form”.

In Section 3 we explain the analogy with the Cartan-Bruhat-Tits decomposition.

In Section 4 we recall the relevance of Bruhat-Tits buildings in this kind of decomposition.

2020 Math. subject class. Primary 20G30; Secondary 11E57
Key words Symplectic group, Cartan decomposition, Smith normal form.
In Section 5 we give an elementary proof of the symplectic Smith normal form.
In Section 6 we give a non-elementary proof of the symplectic Smith normal form that will be applied to other simply-connected split semisimple algebraic groups $G$ defined over $\mathbb{Q}$ in the last section. Indeed we explain how this symplectic Smith normal form can be deduced from the Cartan-Bruhat-Tits decomposition together with the strong approximation theorem.
In Section 7 we explain the extension due to Steinberg of the Smith normal form to the simply-connected $\mathbb{Q}$-split groups, see Theorem 7.1

The last two sections are a concrete illustration of a classical strategy: if you want to prove a theorem over a global field, prove it first over local fields and then use a local-global principle.

I would like to thank Hee Oh for a very helpful comment on a first draft of this note.

2 The symplectic Smith normal form

For any commutative ring $R$ with unity, we denote by $\text{Sp}(n, R)$ the symplectic group with coefficients in $R$. This group is the stabilizer of the symplectic form $\omega$ on $R^{2n}$ given by, for all $x, y$ in $R^{2n}$,

$$\omega(x, y) = {}^t x J y$$

where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Equivalently, one has

$$\text{Sp}(n, R) := \{ g \in \text{GL}(2n, R) \mid {}^t g J g = J \},$$

If we write the elements of the symplectic group as block matrices with blocks of size $n$, one has

$$\text{Sp}(n, R) = \{ g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mid {}^t \alpha \gamma = {}^t \gamma \alpha, \quad {}^t \beta \delta = {}^t \delta \beta, \quad {}^t \alpha \delta - {}^t \gamma \beta = \mathbf{1}_n \}.$$ 

**Theorem 2.1.** Let $g \in \text{Sp}(n, \mathbb{Q})$. Then there exist two matrices $\sigma$ and $\sigma'$ in $\text{Sp}(n, \mathbb{Z})$ and a positive integral diagonal matrix $d = \text{diag}(d_1, \ldots, d_n)$ with $d_1 | d_2 | \ldots | d_n$, and such that

$$g = \sigma \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} \sigma'.$$
The condition that the coefficients $d_j$ are positive integers with $d_1$ dividing $d_2$, with $d_2$ dividing $d_3$, \ldots, and $d_{n-1}$ dividing $d_n$ ensures that the diagonal matrix $d$ is unique.

I use this precise Theorem 2.1 as a key tool for an apparently completely unrelated problem in my paper [1]. This problem is the construction of functions $f$ on the cyclic group $\mathbb{Z}/d\mathbb{Z}$ of odd order whose convolution square is proportional to their square. Indeed the construction relies on an auxiliary abelian variety endowed with a unitary $\mathbb{Q}$-endomorphism $\nu$, the symplectic form $\omega$ shows up as a polarization of the abelian variety, and the rational symplectic matrix $g$ shows up as the “holonomy” of $\nu$.

The first reference to Theorem 2.1 that I know is Shimura’s paper [15, Prop. 1.6]. Moreover in [16], Shimura points out the relevance of this theorem to show the commutativity of a Hecke algebra and hence to better understand the modular forms on Siegel upper halfspace. This theorem is also in [9, p.232] and is also used by Clozel, Oh and Ullmo in [8, p.23].

As we have seen, there is a version of Theorem 2.1 for the linear group $\text{SL}(n, \mathbb{Q})$, see for instance Proposition 5.1. More generally, there is also a version of Theorem 2.1 for any simply-connected split semisimple algebraic group $G$ defined over $\mathbb{Q}$, if one chooses suitably the $\mathbb{Z}$-form, see Section 7.

3 The symplectic group over local fields

Before going on I would like to emphasize the analogy of this theorem with two classical theorems. These two classical theorems are valid for all algebraic semisimple groups $G$ and are due respectively to E. Cartan and to F. Bruhat and J. Tits. I will not quote here their general formulation that can be found respectively in [12] and in [5] but only the special case where $G$ is the symplectic group.

The first theorem is a decomposition theorem over the real field $\mathbb{R}$ due to E. Cartan which is called either the “polar decomposition” or the “Cartan decomposition”. We set

$$\text{SO}(2n) := \{ g \in \text{GL}(2n, \mathbb{R}) \mid ^tgg = 1_{2n} \} \text{ and } \text{Sp}(n) := \text{Sp}(n, \mathbb{R}) \cap \text{SO}(2n).$$

Note that the group $\text{Sp}(n)$ is a maximal compact subgroup of the group $\text{Sp}(n, \mathbb{R})$. 

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**Theorem 3.1. (Cartan)** Let $g \in \text{Sp}(n, \mathbb{R})$. Then there exist two matrices $\sigma$ and $\sigma'$ in $\text{Sp}(n)$ and a positive real diagonal matrix $d = \text{diag}(d_1, \ldots, d_n)$ with $d_1 \leq d_2 \leq \ldots \leq d_n \leq 1$ such that

$$g = \sigma \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} \sigma'.$$

The second theorem is a decomposition theorem over a local non-archimedean field $k$ due to F. Bruhat and J. Tits. We denote by $\mathcal{O}_k$ the ring of integers of $k$ and choose a uniformizer $\pi$ in $k$, i.e. a generator of the maximal ideal of $\mathcal{O}_k$.

Note again that the group $\text{Sp}(n, \mathcal{O}_k)$ is a maximal compact subgroup of the group $\text{Sp}(n, k)$.

**Theorem 3.2. (Bruhat, Tits)** Let $g \in \text{Sp}(n, k)$. Then there exist two matrices $\sigma$ and $\sigma'$ in $\text{Sp}(n, \mathcal{O}_k)$ and a diagonal matrix $d = \text{diag}(\pi^{p_1}, \ldots, \pi^{p_n})$ with $p_1 \geq p_2 \geq \ldots \geq p_n \geq 0$ integers such that

$$g = \sigma \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} \sigma'.$$

The analogy between these three theorems is striking. It extends the analogy between the Smith normal form of an integral matrix and the singular value decomposition of a real matrix.

In this analogy the group of integers points of a group defined over the rational should be handled as the maximal compact subgroup of a group defined over the real. This rough analogy is an equality when dealing with non-archimedean local field. Indeed, when $k$ is a non-archimedean local field, the group of integer points is an open compact subgroup.

### 4 Bruhat-Tits buildings

F. Bruhat and J. Tits have described the analog of the Cartan decomposition for semisimple groups over non-archimedean local fields, in [4], [5], [6] and [7], by introducing new geometric spaces that are nowadays called Bruhat-Tits buildings and that extend the space of $p$-adic norms studied by Goldman and Iwahori in [10].
As explained in the book [13], these Bruhat-Tits buildings are very useful. One of the reasons is that they are $K(\pi, 1)$-spaces for the lattices in semisimple $p$-adic groups.

Another reason is that they played the role of a model to follow in order to understand other finitely generated groups, like Coxeter groups, Artin groups, Baumslag-Solitar groups or Mapping class groups.

The relevance of the Bruhat-Tits buildings became even clearer to me when I used them with Hee Oh to prove a general polar decomposition for $p$-adic symmetric spaces in [2]. This polar decomposition was a key ingredient in our proof of equidistribution of $S$-integral points on rational symmetric spaces in [3].

5 The symplectic adapted basis

In this section we come back to elementary consideration and we discuss the structure of the rational symplectic group $\text{Sp}(n, \mathbb{Q})$, and its relation with the integral symplectic group $\text{Sp}(n, \mathbb{Z})$.

We first recall the well-known undergraduate “adapted basis theorem” for $\mathbb{Z}$-module or, equivalently, the “Smith normal form” for integral matrices. We denote by $\mathcal{M}(n, \mathbb{Z})$ the ring of $n \times n$ integral matrices.

**Proposition 5.1. (Smith)** Let $g \in M(n, \mathbb{Z})$. Then there exist $\sigma$ and $\sigma'$ in $\text{SL}(n, \mathbb{Z})$ and an integral diagonal matrix $a = \text{diag}(a_1, \ldots, a_n)$ with $a_1 | a_2 | \ldots | a_n$, and such that

$$g = \sigma a \sigma'.$$

Theorem 2.1 follows from the following proposition which is a variation of the “adapted basis theorem” which takes into account the existence of a symplectic form. We introduce the set $\mathcal{M}p(n, \mathbb{Z})$ of nonzero integral matrices which are proportional to elements of $\text{Sp}(n, \mathbb{R})$,

$$\mathcal{M}p(n, \mathbb{Z}) := \{ g \in M(2n, \mathbb{Z}) \mid {}^t gJg = \lambda^2 J \text{ for some } \lambda \text{ in } \mathbb{R}^* \}.$$

**Proposition 5.2.** Let $g \in \mathcal{M}p(n, \mathbb{Z})$. Then there exist two matrices $\sigma$ and $\sigma'$ in $\text{Sp}(n, \mathbb{Z})$ and a positive integral diagonal matrix $a = \text{diag}(a_1, \ldots, a_{2n})$ with $a_1 | a_2 | \ldots | a_n$, with $a_n | a_{2n}$ and such that

$$g = \sigma a \sigma'.$$
Note that the matrix $a$ is also in $Mp(n, \mathbb{Z})$ and hence the products $a_j a_{n+j}$ do not depend on the positive integer $j \leq n$. Indeed it is equal to $\lambda^2$. In particular, one has $a_{2n} | a_{2n-1} \cdots | a_{n+1}$.

For the proof of Proposition 5.2 we need the following lemma. We recall that a nonzero vector $v$ of $\mathbb{Z}^k$ is primitive if it spans the $\mathbb{Z}$-module $\mathbb{R}v \cap \mathbb{Z}^k$.

**Lemma 5.3.** The group $Sp(n, \mathbb{Z})$ acts transitively on the set of primitive vectors in $\mathbb{Z}^{2n}$.

Denote by $e_1, \ldots, e_n, f_1, \ldots, f_n$ the canonical basis of $\mathbb{Z}^{2n}$ so that our symplectic form is $\omega = e_1^* \wedge f_1^* + \cdots + e_n^* \wedge f_n^*$.

**Proof of Lemma 5.3.** Let $v = (x_1, \ldots, x_{2n})$ be a primitive vector in $\mathbb{Z}^{2n}$. We want to find $\sigma \in Sp(n, \mathbb{Z})$ such that $\sigma v = e_1$.

This is true for $n = 1$. Using the subgroups $Sp(1, \mathbb{Z})$ for the planes $Ze_j \oplus Zf_j$, with $j = 1, \ldots, n$, we can assume that

$$x_{n+1} = \cdots = x_{2n} = 0.$$ 

In this case the vector $(x_1, \ldots, x_n)$ is primitive in $\mathbb{Z}^n$.

Since $SL(n, \mathbb{Z})$ acts transitively on the set of primitive vectors in $\mathbb{Z}^n$, we can find a block diagonal matrix $\sigma = \text{diag}(\sigma_0, \cdot \sigma_0^{-1})$, with $\sigma_0 \in SL(n, \mathbb{Z})$ such that $\sigma v = e_1$. This matrix $\sigma$ belongs to $Sp(n, \mathbb{Z})$. \qed

**Proof of Proposition 5.2.** Set $\Gamma := Sp(n, \mathbb{Z})$. The proof is by induction on $n$. It relies on a succession of steps, in the spirit of the Smith normal form, in which one multiplies on the right or on the left the matrix $g$ by an “elementary” matrix to obtain a simpler matrix $g' \in \Gamma g \Gamma$. We have to pay attention that at each step the elementary matrix is symplectic.

We can assume that the gcd of the coefficients of $g$ is equal to 1. We denote by $\lambda$ the positive real factor such that $g/\lambda$ belongs to $Sp(n, \mathbb{R})$. Note that $\lambda^2$ is a positive integer. At the end of the proof we will see that $a_1 = 1$ and $a_{n+1} = \lambda^2$.

**1st step:** We find $g' \in \Gamma g \Gamma$ such that $g' e_1 = e_1$.

Since the coefficients of the integral matrix $g$ are relatively prime, by Proposition 5.1 there exists a primitive vector $v$ in $\mathbb{Z}^{2n}$ such that $gv$ is also primitive. According to lemma 5.3 there exists $\sigma, \sigma'$ in $\Gamma$ such that $\sigma gv = e_1$ and $\sigma' e_1 = v$. Then the matrix $g' := \sigma g \sigma'$ satisfies $g' e_1 = e_1$. 

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2nd step: We find $g' \in \Gamma g \Gamma$ with $g' e_1 = e_1$ and $\omega(g' e_j, f_1) = 0$ for $j > 1$.

By the first step, we can assume that

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

with $\alpha e_1 = e_1$ and $\gamma e_1 = 0$.

In particular the first column of the integral matrix $\alpha$ is $(1, 0, \ldots, 0)$. We would like the first row of $\alpha$ to be also of the form $(1, 0, \ldots, 0)$. For that we choose $g' = g\sigma'$ where $\sigma'$ is the symplectic transformation

$$\sigma' = 1_n + \sum_{1 < j \leq n} \alpha_{1,j}(f_j \otimes f_1^* - e_1 \otimes e_j^*) \in \text{Sp}(n, \mathbb{Z}),$$

where $\alpha_{1,j}$ are the coefficients of the first row of the matrix $\alpha$.

3rd step: We find $g' \in \Gamma g \Gamma$ such that $g' e_1 = e_1$ and $g' f_1 = \lambda^2 f_1$.

By the second step, we can assume, writing $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ that both the first row and first column of $\alpha$ are $(1, 0, \ldots, 0)$, and the first column of $\gamma$ is $(0, \ldots, 0)$. We would also like the first row of $\beta$ to be $(0, \ldots, 0)$. For that we choose $g' = g\sigma'$ where $\sigma'$ is the symplectic transformation

$$\sigma' = 1_n - \beta_{1,1} e_1 \otimes f_1^* - \sum_{1 < j \leq n} \beta_{1,j}(e_j \otimes f_1^* + e_1 \otimes f_j^*) \in \text{Sp}(n, \mathbb{Z}).$$

Now by construction one has

$$\omega(g' e_j, f_1) = 0 \quad \text{for} \quad j < n,$$

$$\omega(g' e_1, f_1) = 1 \quad \text{and}$$

$$\omega(g' f_j, f_1) = 0 \quad \text{for} \quad j \leq n.$$  

Since $g' / \lambda$ is symplectic, this implies that $g'^{-1} f_1 = \lambda^{-2} f_1$, or equivalently, $g' f_1 = \lambda^2 f_1$ as required.

4th step: Conclusion.

By the third step, we can assume that $g e_1 = e_1$ and $g f_1 = \lambda^2 f_1$. Therefore $g$ preserves the symplectic $\mathbb{Z}$-submodule of $\mathbb{Z}^{2n}$ orthogonal of $\mathbb{Z} e_1 \oplus \mathbb{Z} f_1$, which admits $e_2, \ldots, e_n, f_2, \ldots, f_n$ as $\mathbb{Z}$-basis. We conclude by applying the induction hypothesis to the restriction $g' \in \text{Mp}(n-1, \mathbb{Z})$ of $g$ to this $\mathbb{Z}$-module. \qed
6 The strong approximation theorem

In this section, we give a non elementary proof of the decomposition theorem 2.1 for Sp(n, \Q). We will deduce this theorem from the Bruhat-Tits decomposition theorem 3.2 for Sp(n, \Q_p) thanks to the strong approximation theorem.

First, I recall the strong approximation theorem. I will not quote here the general formulation for a simply-connected isotropic \Q-simple algebraic group defined over \Q that can be found in [14] but only the special case where \G is the symplectic group.

For \( p = 2, 3, 5, \ldots \) a prime number, we denote by \( \Q_p \) the \( p \)-adic local field and by \( \Z_p \) its ring of integers.

We denote by \( \Q = \prod_p \Q_p \) the locally compact ring of finite adèles which is the restricted product of the \( \Q_p \) with respect to the open subrings \( \Z_p \). The product \( \hat{\Z} := \prod_p \Z_p \) is then a maximal compact open subring of \( \Q \).

Note that, thanks to the diagonal embedding, \( \Q \) is a dense subring in \( \Q \). This means that \( \Q = \Q + \hat{\Z} \) and that \( \Z \) is dense in \( \hat{\Z} \).

By construction the symplectic group Sp(n, \Q) is a locally compact group that contains Sp(n, \hat{\Z}) as a maximal compact open subgroup. It also contains the group Sp(n, \Q).

Here is the strong approximation theorem for the symplectic group.

**Theorem 6.1.** The group Sp(n, \Q) is dense in Sp(n, \hat{\Q}).

This means that,

\[
\text{Sp}(n, \hat{\Q}) = \text{Sp}(n, \Q) \text{Sp}(n, \hat{\Z})
\]

and that

\[
\text{Sp}(n, \Z) \text{ is dense in } \text{Sp}(n, \hat{\Z}).
\]

If we collect together the Bruhat-Tits decomposition in Theorem 3.2 for all \( p \)-adic fields \( k = \Q_p \), one gets

**Theorem 6.2.** Let \( g \in \text{Sp}(n, \hat{\Q}) \). Then there exist two matrices \( \sigma \) and \( \sigma' \) in Sp(n, \hat{\Z}) and a positive integral diagonal matrix \( \mathbf{d} = \text{diag}(d_1, \ldots, d_n) \) with
such that
\[ g = \sigma \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} \sigma'. \]

We can now give the non-elementary proof of the symplectic Smith normal form.

**Proof of Theorem 2.1.** Let \( g \in \text{Sp}(n, \mathbb{Q}) \).

According to the combined Bruhat-Tits decomposition theorem 6.2, one can write
\[ g = \sigma a \sigma' \]
with \( \sigma, \sigma' \) in \( \text{Sp}(n, \hat{\mathbb{Z}}) \) and with \( a = \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} \) where \( d = \text{diag}(d_1, \ldots, d_n) \) is a positive integral diagonal matrix with \( d_1 | d_2 | \ldots | d_n \).

According to the strong approximation theorem 6.1 one can write
\[ \sigma = \sigma_0 \eta \]
with \( \sigma_0 \) in \( \text{Sp}(n, \mathbb{Z}) \) and with \( \eta \) in an arbitrarily small neighborhood of 1 in \( \text{Sp}(n, \hat{\mathbb{Z}}) \). More precisely we choose \( \eta \) such that the element \( \sigma'_0 := a^{-1} \eta a \sigma' \) belongs to \( \text{Sp}(n, \hat{\mathbb{Z}}) \). Then one has the equality
\[ g = \sigma_0 a \sigma'_0 \]
where both \( \sigma_0 \) and \( \sigma'_0 = a^{-1} \sigma_0^{-1} g \) belong to \( \text{Sp}(n, \mathbb{Z}) \). \( \square \)

### 7 Chevalley groups

Let \( \mathbf{G} \) be a simply-connected Chevalley group. See [17] for a concrete presentation of the group \( \mathbf{G}(\mathbb{Z}) \), and see [11] for other nice examples of \( \mathbb{Z} \)-models of simple algebraic groups over \( \mathbb{Q} \). This \( \mathbf{G} \) is a reductive scheme-group over \( \mathbb{Z} \) such that as a \( \mathbb{Q} \)-group \( \mathbf{G} \) is a \( \mathbb{Q} \)-split simply connected quasi-simple algebraic group. By construction, this algebraic group contains a \( \mathbb{Q} \)-split maximal torus \( \mathbf{A} \) such that the group of integral points \( \mathbf{N}(\mathbb{Z}) \) of the normalizer of \( \mathbf{A} \) surjects onto the Weyl group of \( (\mathbf{A}(\mathbb{C}), \mathbf{G}(\mathbb{C})) \).

Since \( \mathbf{G} \) is simply connected and \( \mathbb{R} \)-isotropic, by strong approximation, the group \( \mathbf{G}(\mathbb{Q}) \) is dense in \( \mathbf{G}(\hat{\mathbb{Q}}) \). On the other hand, for all prime integer \( p \), one can consider the simply connected simple \( p \)-adic Lie group \( \mathbf{G} := \mathbf{G}(\mathbb{Q}_p) \),
its split maximal torus $A := A(\mathbb{Q}_p)$ and its normalizer $N := N(\mathbb{Q}_p)$. The maximal compact subgroup $K := G(\mathbb{Z}_p)$ is a good compact subgroup in the sense that one has the equality $N = (N \cap K)A$. Hence, according to Bruhat-Tits, one has the decomposition $G(\mathbb{Q}_p) = G(\mathbb{Z}_p)A(\mathbb{Q}_p)G(\mathbb{Z}_p)$.

Therefore the same proof as in Chapter 6 gives the following theorem due to Steinberg in [17, Theorem 21]

**Theorem 7.1.** Let $G$ be a simply connected Chevalley group and $g \in G(\mathbb{Q})$. Then there exist two elements $\sigma$ and $\sigma'$ in $G(\mathbb{Z})$ and an element $a$ in $A(\mathbb{Q})$ such that

$$g = \sigma a \sigma'.$$

**Remark.** Such a decomposition is not true when we replace $\mathbb{Q}$ by a number field $\mathbb{K}$ whose ring of integer $\mathcal{O}$ is not principal. Here is an example with

$$G(\mathbb{K}) := \text{SL}(2, \mathbb{K}), \quad G(\mathcal{O}) := \text{SL}(2, \mathcal{O}),$$

$$A(\mathbb{K}) := \left\{ a = \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} \mid d \in K^* \right\}.$$

In this case the product $G(\mathcal{O})A(\mathbb{K})G(\mathcal{O})$ is not equal to $G(\mathbb{K})$. For instance, when $K = \mathbb{Q}[i\sqrt{5}]$ and $\mathcal{O} = \mathbb{Z}[i\sqrt{5}]$, this product does not contain the matrix

$$g = \begin{pmatrix} (1-i\sqrt{5})/2 & i\sqrt{5} \\ -1 & 2 \end{pmatrix}.$$

Indeed the element $d \in K^*$ should be a unit in all completion $K_p$ except for the prime ideal $p_0 = 2\mathbb{Z} \oplus (1+i\sqrt{5})\mathbb{Z}$ in which case it should be a uniformizer. Such an element $d$ would be a generator of the ideal $p_0$. This is a contradiction, since this ideal $p_0$ is not principal.

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