QUANTUM CRYPTOGRAPHY
VIA PARAMETRIC DOWNCONVERSION

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Abstract

The use of quantum bits (qubits) in cryptography holds the promise of secure cryptographic quantum key distribution schemes. It is based usually on single-photon polarization states. Unfortunately, the implemented “qubits” in the usual weak pulse experiments are not true two-level systems, and quantum key distribution based on these imperfect qubits is totally insecure in the presence of high (realistic) loss rate. In this work, we investigate another potential implementation: qubits generated using a process of parametric downconversion. We find that, to first (two-photon) and second (four-photon) order in the parametric downconversion small parameter, this implementation of quantum key distribution is equivalent to the theoretical version.

Once realistic measurements are taken into account, quantum key distribution based on parametric downconversion suffers also from sensitivity to extremely high (nonrealistic) losses. By choosing the small parameter of the process according to the loss rates, both implementations of quantum key distribution can in principle become secure against the attack studied in this paper. However, adjusting the small parameter to the required levels seems to be impractical in the weak pulse process. On the other hand, this can easily be done in the parametric downconversion process, making it a much more promising implementation.

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INTRODUCTION

Quantum information theory suggests the possibility to accomplish tasks that are beyond the capability of classical computer science, such as information-secure cryptographic key distribution\(^1\). While theoretical quantum key distribution (QKD) schemes are proven secure against very sophisticated attacks\(^2\), the experimental QKD schemes are not yet proven secure even against very simple attacks. In this work, we analyse the effect of losses on the security of experimental quantum key distribution. We investigate a novel implementation, qubits produced by a process of parametric downconversion (PDC), and we compare it to the more common implementation based on weak coherent pulses (WCP).

A protocol is considered secure if the adversary is restricted only by the rules of quantum mechanics, and yet cannot obtain any information on the final key. In the four-state scheme\(^1\) usually referred to as BB84, the sender (Alice) and the receiver (Bob) use two conjugate bases (say, the rectilinear basis, +, and the diagonal basis, \(\times\)) for the polarization of single photons. In basis + (resp. \(\times\)), they use the two orthogonal basis states \(|0+\rangle\) and \(|1+\rangle\) (resp. \(|0\times\rangle\) and \(|1\times\rangle\)) to represent “0” and “1” respectively. The basis is revealed later on, which enables Bob to decode the bit whenever he used the same basis as Alice; otherwise, they throw the bit away. Finally, they use error-correction and privacy amplification to obtain a potentially secure final key\(^3,2\).

All the experiments done so far to demonstrate protocols for secure quantum key distribution use pulses of light containing (on average) much less than one photon. We approximate the state of the modified qubit created by this process to be in single mode, which we call a “weak coherent pulse” (WCP). [For an explanation regarding a description of a pulse, see Blow et al\(^4\).] We analyse the security of WCP-based schemes while paying special attention to the losses. The channel causes huge loss rate (whether a fiber, which causes attenuation, or free space, which causes beam broadening). In the experimental literature, it is usually assumed that the only effect of losses is to reduce the bit rate. We show that there are two different types of losses, channel losses and losses due to the state (“state losses”). The state losses have impact on the bit rate. The channel losses have a vital impact on security, in addition to their impact on the bit rate. A careful analysis of channel losses shows that schemes that were assumed secure are in fact totally insecure even against a simple intercept-resend attack. In intercept-resend attacks, an eavesdropper (Eve) performs a complete measurement on the input qubit, and she prepares and sends to Bob a state of her own, according to the outcome of her measurement. When Alice and Bob are using linearly independent states, Eve can sometimes get full information by performing a “positive operator value measure” (POVM) that conclusively distinguishes such states. This is fatal in presence of high channel losses between Alice and Bob because Eve can recreate the state near Bob and send it to him without loss whenever she measured it conclusively, whereas she forwards nothing to Bob otherwise! We shall refer to this attack as the conclusive-measurement attack. This was discussed when the two-state scheme\(^5\) was invented, and its power against the four-state scheme was realized by Yuen\(^6\).

Recently, parametric downconversion has been used to generate a polarization singlet state\(^7\) to test Bell’s inequalities, and it is believed that it can be used as a much better single-photon source for quantum key distribution. Here, we explain the potential experiment and we present the modified singlet state resulting from this PDC process. Then, we calculate the state sent to Bob, including two-photon and four-photon terms, assuming dispersion-free devices, no dark counts and perfect detectors. [A different use of a PDC for QKD was previously suggested\(^8\), based on Franson-type
uncertainties, but the polarization encoding we suggest here allows for a much simpler analysis.

We find that PDC-QKD is much more secure than WCP-QKD: The security of WCP-QKD is destroyed in the presence of high channel loss rate due to the linear independence obtained when adding the second-order terms. The crucial advantage of the PDC-QKD is that the second-order terms do not affect the fact that the states in one basis are linearly dependent on the states in the other basis. Thus, the attack that destroys the security of WCP-QKD in the presence of high losses has no impact on PDC-QKD (when second-order calculation and perfect detection are considered).

When imperfections in the process are taken into account, this euphoric picture changes, and the second-order states sent to Bob are not linearly dependent anymore. Fortunately, PDC-QKD becomes totally insecure against the conclusive-measurement attack only in the presence of such extremely high loss rate that more serious practical problems would have already arisen, such as the importance of dark counts, or errors due to various inaccuracies in the devices. As we explain in the discussion, it is probably impossible to make the WCP implementation secure against the conclusive measurement attack, thus we suggest that the experimental effort should be directed towards the implementation of PDC-QKD.

SECURITY OF WCP-BASED QKD

Experimental QKD is mainly based on the use of weak pulses of coherent light. By definition, a pulse consists of a linear superposition of many frequency contributions, but the laser pulse itself can be considered to be in a single, localized mode provided the dispersion is not significant in any of the optical elements.

Using Fock state notation, $|0, 0\rangle$ denotes the vacuum state, and the state $|n, m\rangle$, which describes $n$ photons with vertical polarization and $m$ photons with horizontal polarization, is denoted more simply by $|n, m\rangle$. Ideally, the four BB84 states should be $|\uparrow\rangle = |0, +\rangle = |1, 0\rangle$ and $|\leftrightarrow\rangle = |1, +\rangle = |0, 1\rangle$ in the $+$ basis, and $|0, x\rangle = (1/\sqrt{2})[|1, 0\rangle + |0, 1\rangle]$ and $|1, x\rangle = (1/\sqrt{2})[|1, 0\rangle - |0, 1\rangle]$ in the $\times$ basis.

Consider now a weak coherent pulse with parameter $\alpha$, meaning that a photon would be detected with probability $\alpha^2$ if the pulse were measured by a perfect detector. If this pulse is polarized in the $+$ basis, the two states are simply, to second order in $\alpha$,

$$
|0_{up}^{\alpha}\rangle \approx \left(1 - \frac{\alpha^2}{2}\right)|0, 0\rangle + \alpha|1, 0\rangle + \frac{\sqrt{2}\alpha^2}{2}|2, 0\rangle
$$

$$
|1_{up}^{\alpha}\rangle \approx \left(1 - \frac{\alpha^2}{2}\right)|0, 0\rangle + \alpha|0, 1\rangle + \frac{\sqrt{2}\alpha^2}{2}|0, 2\rangle .
$$

However, the two states in the $\times$ basis, when expressed as Fock states in terms of the $+$ basis, are more complicated:

$$
|0_{x}^{up}^{\alpha}\rangle \approx \left(1 - \frac{\alpha^2}{2}\right)|0, 0\rangle + (\alpha/\sqrt{2})[|1, 0\rangle + |0, 1\rangle] + \frac{\sqrt{2}\alpha^2}{4}[|2, 0\rangle + \sqrt{2}|1, 1\rangle + |0, 2\rangle]
$$

$$
|1_{x}^{up}^{\alpha}\rangle \approx \left(1 - \frac{\alpha^2}{2}\right)|0, 0\rangle + (\alpha/\sqrt{2})[|1, 0\rangle - |0, 1\rangle] + \frac{\sqrt{2}\alpha^2}{4}[|2, 0\rangle - \sqrt{2}|1, 1\rangle + |0, 2\rangle] .
$$

We call those four states the modified qubits. Note that they are not two-level systems anymore but six-level systems, or qu-hexits.

If we considered only the first order in $\alpha$, as is usually done, the four states would behave very much like the ideal BB84 states leading us to the wrong conclusion that the
CREATING A MODIFIED SINGLET STATE IN THE PROCESS OF PDC

In this section, we present the parametric downconversion process and we give the output state to second order in the PDC parameter. The PDC process provides a source of photons for Bob and Alice with important advantages over the weak coherent pulse discussed in the previous section. A classical pump field with vertical polarization drives a PDC crystal below threshold, thereby producing photon pairs from a two-mode vacuum state input field $|0, 0, 0, 0\rangle$. The two output fields from the parametric downconverter are correlated in time of emission as well as polarization, and conservation laws apply to the sum of energies and momenta of the photons in the two fields. The quantum field input to the parametric downconverter is assumed to be in the vacuum state. We consider the field emitted by the PDC process and channeled through a polarization rotator and a beam splitter, which creates entanglement between them. One arm of the resulting output goes to Alice and the other arm goes to Bob.

We denote by $|k, a, n, b, m\rangle$, or more simply $|k, l, n, m\rangle$, the state in which there are $k$ photons with vertical polarization and $l$ photons with horizontal polarization going into Alice’s arm “a”, and $n$ photons with vertical polarization and $m$ photons with horizontal polarization going into Bob’s arm “b”. The PDC small parameter $\chi$, which is proportional to the strength of the pump field, the interaction time between the field and the crystal and the nonlinearity of the medium, is so that a photon pair would be detected with probability $\chi^2$ if the output of the interaction were measured by perfect detectors. The state created by this process is an entangled state, and it is usually assumed to be a singlet $|\psi_-\rangle = (1/\sqrt{2})[|0, 1, 1, 0\rangle - |1, 0, 0, 1\rangle]$, but we show in the final paper how to calculate it more precisely, to obtain the modified singlet $|\chi\rangle = |\psi^{(mod)}_-\rangle$ to second order in $\chi$:

$$|\chi\rangle = \left(1 - \frac{\chi^2}{2}\right)|0, 0, 0, 0\rangle + \frac{\chi}{2}[|0, 1, 1, 0\rangle + |1, 1, 0, 0\rangle] - |0, 0, 1, 1\rangle - |1, 0, 0, 1\rangle$$

$$+ \frac{\chi^2}{2}(|0, 0, 2, 0\rangle + |2, 2, 0, 0\rangle + |0, 0, 2, 2\rangle + |2, 0, 0, 2\rangle - 2|1, 1, 1, 1\rangle$$

$$+ \sqrt{2}|1, 0, 1, 2\rangle - \sqrt{2}|0, 1, 2, 1\rangle + \sqrt{2}|1, 2, 1, 0\rangle - \sqrt{2}|2, 1, 0, 1\rangle \right) . \quad (1)$$
CREATING A MODIFIED QUBIT IN A PDC PROCESS

In order to use PDC for performing the BB84 four-state scheme, we need to consider the state sent from Alice to Bob. This is produced by Alice measuring her arm in a basis (+ or ×) of her choice and letting the other arm, which is the modified qubit, go to Bob through the quantum channel. More precisely, Alice directs her arm to an adjustable rotator (to choose the basis of measurement: angle 0 for + and π/4 for ×) followed by a polarization-dependent beam splitter that sends the horizontal mode to one direction and the vertical mode to another spatial direction. Each of these spatial modes is now subjected to a measurement, which in the limit of perfect efficiency provides an exact count of the number of photons that reached each detector. In this section, we analyse to orders χ and χ² the modified qubit thus sent to Bob resulting from the modified singlet state.

Considering |χ⟩ to order χ and perfect detectors (used by Alice), the modified singlet is projected to yield a perfect qubit that is sent towards Bob in one of the BB84 states. With imperfect detection, but not allowing dark counts, Alice might send the vacuum, while she thinks she sent a single photon, but this causes only state losses and it has no effect on security as far as we could see.

When we consider |χ⟩ to order χ² and perfect detectors, this process yields a modification of the four BB84 states, but surprisingly still results in a perfect BB84 scheme! With perfect measurements, only the terms with exactly one photon at Alice’s site will not be discarded, so that we need only consider the terms $\frac{X}{2}[[0,1,1,0] − [1,0,0,1]]$ and $\frac{X}{2\sqrt{2}}[[1,0,1,2] − [0,1,2,1]]$. In case Alice decides to rotate her mode by angle π/4 in order to send Bob a qubit in the × basis, the above terms change to $\frac{X}{2\sqrt{2}}[[0,1,1,0] + [1,0,1,0] − [1,0,0,1] + [0,1,0,1]]$ and $\frac{X^2}{2}[[1,0,1,2] − [0,1,1,2] − [0,1,2,1] − [1,0,2,1]]$.

With ideal detectors and Alice measuring without rotation, the state of Eq. 1 is projected onto |0,1⟩ or |1,0⟩ (in Alice’s arm), yielding respectively

$$|0^\text{pdc}_{+}\rangle \approx \frac{X}{2}|0,1\rangle - \frac{X^2}{2\sqrt{2}}|2,1\rangle$$

$$|1^\text{pdc}_{+}\rangle \approx -\frac{X}{2}|0,1\rangle + \frac{X^2}{2\sqrt{2}}|1,2\rangle$$

(since Alice used the + basis). When Alice uses the × basis, the rotated terms (calculated as before) provide the relevant contribution, yielding

$$|0^\text{pdc}_{\times}\rangle \approx \frac{X}{2\sqrt{2}}[[1,0] + |0,1\rangle] - \frac{X^2}{4}[[1,2] + |2,1\rangle]$$

$$|1^\text{pdc}_{\times}\rangle \approx \frac{X}{2\sqrt{2}}[[1,0] − |0,1\rangle] + \frac{X^2}{4}[[1,2] − |2,1\rangle] .$$

The modified qubit is not a two-level system but a four-level system. Yet, all four states lie in a two-level system spanned by any two of them. Furthermore, they satisfy the same conditions as the theoretical BB84 states; each one in the × basis is an equal superposition of the states in the + basis. Thus, all theoretical security analyses apply to these states.

DISCUSSION

We have seen that PDC-QKD has a crucial advantage over WCP-QKD due to the fact that the four states created in the PDC process are equivalent to the theoretical BB84 states.
states. However, the calculation so far assumed that Alice uses perfect measuring devices. A calculation taking account of realistic measurements will contain also other corrections. Then the states will be linearly independent, so that Eve can find a POVM to distinguish between them conclusively.

Nevertheless, let us show a vital advantage of the more realistic PDC-QKD over WCP-QKD. Even though both schemes are insecure in principle in the presence of high channel losses, the use of PDC as a source of qubits is potentially much preferable: For PDC qubits, the controlled parameter $\chi$ is usually smaller than $10^{-3}$, thus the probability of having more than one photon is $10^{-6}$, conditional to having at least one photon, and seems to be negligible when the channel losses are 99% or even much more. Furthermore, the small parameter can be easily further decreased according to the loss rate to potentially solve the problem, perhaps while increasing the pulse frequency to keep the same bit rate. In WCP, the corresponding parameter $\alpha$ is usually around 0.3. Unfortunately, this parameter cannot be adjusted so easily because it plays a dual role. Decreasing it immediately increases the state losses, which are $1 - \alpha^2$. Although these are state losses and not channel losses—hence we didn’t see any effect of these losses on security—they are crucial in this implementation: with much smaller $\alpha$ it is impossible to achieve any reasonable bit rate since the state loss rate is $1 - \alpha^2$. Increasing the number of pulses to overcome this problem is not an appropriate solution since Alice needs to write down the polarization of the states in all pulses, and change the polarization for each one.

Another important advantage of PDC-QKD is that it solves a problem usually left unnoticed: Eve can attack WCP-QKD by eavesdropping into Alice’s lab; this can be done by finding the setting of Alice’s polarizers using a strong pulse sent to, and reflected from the polarizers in between Alice’s pulses. We are not aware of any such attack that can be used against the PDC-QKD implementation.

Our work is only an initial step. Analysis of more realistic scenarios and of other attacks might show that PDC-QKD is not as superior to WCP-QKD as this preliminary study indicates.

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