1. Introduction

Let $G$ does not an disconnected graph with $m$ vertices and $n$ edges. The $r$-dynamic coloring is an proper vertex coloring such that the coloring of the neighborhood vertices is more than the minimal of $r$ and degree of that vertices (i.e., $|c(N(v))| \geq \min\{r, \deg_G(v)\}$, for each $v \in V(G)$ and it is denoted by $\chi_r(G)$. Snark are bridgeless cubic connected graph in which every vertex has three neighbors. In this paper, we shown the $r$-dynamic coloring for Celmins-swart snark, Double star snark, Loupekine snark, Szekeres snark, Watkins snark and infinite family of flower snark with its special case as Tietze’s graph. This result reinforce that most of the snark are 6-colorable for its maximum degree and also we give the procedures to construct a $r$-dynamic coloring of each snark.

In this study, we contribute some results on several snarks; we found the $r$-dynamic chromatic number of Double-star snark and Loupekine snark to be 6 for higher degree. Also, by using dot product we described the vertex coloring of Watkins snark, Celmins-swart snark and Szekeres snark. Since, to study this operation, some infinite subfamilies of Flower snark are constructed.
by using dot product and the result are obtained for \( r \)-dynamic coloring. In addition, we also constructed the Tietze’s graph which is the special case of flower snark, when \( n = 3 \).

2. Basic Preliminaries
Let \( V = \{v_1, v_2, \cdots \} \) be the vertices of \( G \) and \( c : \{v_1, v_2, \cdots \} \rightarrow \{1, 2, \cdots, k\} \). The \( r \)-dynamic coloring \[3], \[10\] is a proper \( k \)-colorable which satisfies two conditions:

(i) \( c(v_1) \neq c(v_2) \)

(ii) \( |c(N(v))| \geq \min\{r, \deg_G(v)\} \)

The condition (i) is the adjacency condition and (ii) is named as \( r \)-adjacency condition.

Here, we introduced some basic definitions of snarks. Snark is not an disconnected, bridgeless, cubic graph with an chromatic index 4. We also defined it as the connectivity is unnecessary by removing one edge. Since, which would split the graph. The girth is defined as the smallest cycle of \( G \). In this paper we used an nontrivial snark. Snark is trivial, if the girth is less than 5.

The dot product is operated between two snarks \( G_1 \) and \( G_2 \) which is denoted by \( G_1 \cdot G_2 \). The construction are as follows: remove two nonadjacent edges \( ab \) and \( cd \) of \( G_1 \) and two adjacent vertices \( X \) and \( Y \) of \( G_2 \) which must have degree 3. Then, letting \( x_1 \) and \( x_2 \) be two vertices adjacent to \( X \) in \( G_2 \) and \( y_1 \) and \( y_2 \) are adjacent to \( Y \) in \( G_2 \) which includes an \( ax_1, bx_2, cy_1, dy_2 \) edges. Based on the order and size of the snarks \( G_1 \) and \( G_2 \), the dot product \( G_1 \cdot G_2 \) produce different snarks.

3. \( r \)-dynamic coloring on Snark Families
Theorem 1 For \( n \geq 5 \), the \( r \)-dynamic chromatic number of flower snark \( J_n \) is,

\[
\chi_r(J_n) = \begin{cases} 
3, & \text{for } 1 \leq r \leq 2 \\
5, & \text{for } r \geq \Delta, \ n \equiv 0(\text{mod } 3) \text{ and } n \equiv 1(\text{mod } 3) \\
6, & \text{for } r \geq \Delta, \ n \equiv 2(\text{mod } 3)
\end{cases}
\]

Proof: Construct the flower snark through following method: consider \( n \) copies of star graph with 4 nodes. The central vertex of each star graph are represented as \( a_i \) for \( 0 \leq i \leq n - 1 \) and the outer vertices are \( c_i \) and \( c_{ij} \) where, \( 0 \leq i \leq n - 1 \) and \( j = 1, 2 \). To add \( n \) edges, construct \( n \) cycle on \( a_i \) and finally construct \( 2n \) cycle on \( c_{ij} \) so, that adds \( 2n \) edges.

The flower snark are with \( 4n \) vertices and \( 3n \) edges, with the additional condition that \( n \) is odd. At \( n = 3 \), the flower snark becomes an Tietze’s graph so we consider for \( n \geq 5 \). The vertex set are \( \{a_i, c_i, c_{ij}\} \) for \( 0 \leq i \leq n - 1 \), \( j = 1, 2 \). The edge sets are \( \{c_i, c_{i+1}, c_{i+2}\} \cup \{a_i, a_{i+1}, c_{i+1}c_{i+2}\} \cup \{c_{i+2}c_{i+3}, c_{n-1}c_0, c_0c_{n-1}\} \). The maximum and minimum degree of flower snark are \( \delta = \Delta = 3 \). The coloring for flower snark are as follows:

Case : 1 When \( 1 \leq r \leq 2 \), consider the following coloring

Color the vertices \( a_i \) for \( 0 \leq i \leq n - 2 \) with color 1 and 2 cyclically and \( a_{n-1} \) with color 3. Next, color the vertices \( c_i \) for \( 1 \leq i \leq n - 2 \) with color 3, color the vertices \( c_0 \) with color 2 and the vertex \( c_{n-1} \) with color 1. If \( i \) is odd, color the vertices \( c_{i-1} \) and \( c_{i+1} \) with color 1, for \( 1 \leq i \leq n - 2 \) else color the vertices with color 2. Color the left over vertices \( c_0 \) and \( c_2 \) with color 3 and color the vertices \( c_{(n-1)/2} \) with color 2. Hence, \( \chi_r(J_n) \leq 3 \). If \( \chi_r(J_n) < 3 \), the \( r \)-adjacency condition is not fulfilled. Therefore, \( \chi_r(J_n) = 3 \).

Case : 2 When \( r = 3 \), consider the below coloring:

• For \( n \equiv 0(\text{mod } 3) \), color the vertices \( a_i \) for \( 0 \leq i \leq n - 1 \) with color 3, 2 and 1 cyclically.

Next, color the vertices \( c_i \) with color 4 for \( 0 \leq i \leq n - 1 \) and color the vertices \( c_{i+1} \) and \( c_{i+2} \) with color \( 2,1,5,3,3,5 \) cyclically for \( 0 \leq i \leq n - 3 \). Color the vertices \( c_{(n-2)/2} \) and
Figure 1. Flower snark $J_n$

$c_{(n-2)2}$ with color 1 and 3. Finally, color the vertices $c_{(n-1)1}$ and $c_{(n-1)2}$ with color 5 and 2. Therefore, $\chi_3(J_n) \geq 5$. If $\chi_r(J_n) < 5$, the $r$-adjacency condition is not fulfilled. Therefore, $\chi_r(J_n) = 5$

- For $n \equiv 1(\text{mod} \ 3)$, color the vertices $a_i$ for $(1 \leq i \leq n-1)$ with color 3, 2 and 1 in order and color the vertices $a_0$ with 4. Then, color the vertices $c_i$ for $(0 \leq i \leq n-1)$ with color 5 and color the vertices $c_1$ and $c_2$ for $(1 \leq i \leq n-1)$ with color 2,1,3,3,4 orderly. At the end color the last two vertices $c_{01}$ and $c_{02}$ with color 1 and 2. Hence, $\chi_3(J_n) \geq 5$. If $\chi_r(J_n) < 5$, the $r$-adjacency condition is not fulfilled. Therefore, $\chi_r(J_n) = 5$

- For $n \equiv 2(\text{mod} \ 3)$, color the vertices $a_i$ for $(1 \leq i \leq n-2)$ with color 3, 2 and 1 in order, color the vertices $a_0$ with 4 and color the vertices $a_{n-1}$ with color 5. Next, color the vertices $c_i$ for $(0 \leq i \leq n-1)$ with color 6 and color the vertices $c_{11}$ and $c_{12}$ for $(1 \leq i \leq n-2)$ with color 2,1,3,3,2 in order. Atlast, color the vertices $c_{(n-2)1}$ and $c_{(n-2)2}$ with color 2 and 4 and color the vertices $c_{01}$ and $c_{02}$ with color 5 and 3. Thus, $\chi_3(J_n) \geq 6$. If $\chi_r(J_n) < 6$, the $r$-adjacency condition is not fulfilled. Therefore, $\chi_r(J_n) = 6$

**Remark:** As $n = 3$ flower snark becomes an Tietze’s graph $T_z$. The degrees are are same as flower snark. The coloring for $1 \leq r \leq 2$ are same as flower snark. At $r = 3$, color the vertices $a_1$, $a_2$ and $a_3$ with color 1,2,3 respectively. Color the vertices $c_i$ for $(0 \leq i \leq 2)$ with color 4. Finally, color the left over vertices $a_{ij}$ for $(0 \leq i \leq 2)$ and $j = 1, 2$. with colors 1, 2, 3, 4, 5, 6 and 7 except 4. Since, it is colored in $c_i$. Therefore, $\chi_3(T_z) = 7$.

**Theorem 2** The $r$-dynamic chromatic number of double star snark $DS$ are given by,

$$\chi_r(DS) = \begin{cases} 3, & \text{for } 1 \leq r \leq 2 \\ 6, & \text{for } r \geq 3 \end{cases}$$

**Proof:** Double star snark has 30 vertices and 45 edges, with a girth 6. The maximum and minimum degree of Double star snark are $\delta = \Delta = 3$. Let $\{a_i, b_i, a_{ij}, b_{ij}\}$ be the vertices for $(1 \leq i \leq 5)$ and $(1 \leq j \leq 2)$. The inner cycle of double star are with vertices $a_i$ and $a_{ij}$ and the outer cycle are with $b_i$ and $b_{ij}$. The vertices in both the cycles are connected by the edges. The edge sets of Double star snark are $\{a_1a_{11}, a_1a_{22}, a_2b_1 : 1 \leq i \leq 5\} \cup \{a_1a_{(i+2)2}, a_1a_{(i+3)2} : 1 \leq i \leq 2\} \cup \{a_1a_{(i-2)2}, a_1a_{(i+3)2} : i = 4, 5\} \cup \{a_1a_{(i+2)2}, a_1a_{(i-2)2} : i = 3\} \cup \{a_2a_{(i+2)1}, a_2a_{(i+3)1} : i = 1, 2\} \cup \{a_2a_{(i+2)1}, a_2a_{(i-2)1} : i = 3\} \cup \{a_1a_{(i-2)1}, a_2a_{(i-3)1} : i = 4, 5\} \cup \{b_1b_{11}, b_1b_{12} : 1 \leq i \leq 5\} \cup \{b_1b_{21}, b_1b_{22} : 1 \leq i \leq 5\}$.
Therefore, \( \chi = 3 \). The coloring are as follows: [\( \chi = 5 \)] \( \cup \{ b_1 b_{(i+1)} : 2 \leq i \leq 4 \} \cup \{ b_1 b_{(i+1)} : 2 \leq i \leq 4 \} \cup \{ b_1 b_{(i+1)} : 2 \leq i \leq 4 \} \). The coloring for Double star snark are given below:

**Case : 1** When \( 1 \leq r \leq 2 \) the \( r \)-coloring are as follows:

- Color the vertices \( a_1, a_0, a_2 \) with color 3, 1 and 2 in cyclic order for \( (1 \leq i \leq 5) \). Thus, the inner cycle had been color. To color the outer cycle, color the vertices \( b_i \) for \( (1 \leq i \leq 5) \) with color 2, then the vertices \( b_{i+1} \) with color 3 and color the vertices \( b_{i+2} \) for \( (1 \leq i \leq 5) \) with color 1. Hence, \( \chi_{1 \leq r \leq 2}(DS) \leq 3 \). If \( \chi_{1 \leq r \leq 2}(DS) < 3 \), the \( r \)-adjacency condition is not fulfilled. Therefore, \( \chi_{1 \leq r \leq 2}(DS) = 3 \)

**Case : 2** When \( r = 3 \), the coloring are as follows:

- Color the inner cycle \( a_i \) for \( (1 \leq i \leq 3) \) with color 5 and 3 sequencily, color the vertex \( a_4 \) with color 5 and color the vertex \( a_5 \) with color 3. Next, the remaining vertices \( a_6, a_7 \) and \( a_8 \), the vertices \( b_1, b_2 \) are connected to the vertices \( a_{11}, a_{12} \), the vertices \( b_3, b_4 \) are connected to the vertices \( a_{21}, a_{22} \) and the leftover vertices \( b_5, b_6 \) are connected to the vertices \( a_{31}, a_{32} \). The vertices of \( a_5 \) are connected to \( a_{23}, a_{24} \) and \( a_{25}, a_{26} \) to \( a_{13} \). Similarly for \( a_{25} \). And also an edges are formed between the vertex \( b_2, b_3 \) and between \( b_4, b_5 \). In addition, \( [LK_1] \) snark forms an edge \( \{ b_1 b_3, a_{15} a_{22} \} \) whereas in \( [LK_2] \) snark forms an edge \( \{ b_1 a_{22}, b_3 a_{15} \} \). The \( r \)-dynamic coloring for \( [LK_1] \) and \( [LK_2] \) are explained in the following case

**Case 1:** When \( 1 \leq r \leq 2 \), the coloring are as follows:

- The center vertex \( b_0 \) with color 2, the remaining vertices are \( b_2, b_3, b_5 \) with color 2, the vertices \( b_1, b_4 \) with color 3 and the vertex \( b_6 \) are with color 1.
- Next, color the vertices \( a_{1j} \) for \( (1 \leq j \leq 5) \) with color 1,2,3,3 orderly, the vertices \( a_{2j} \) for \( (1 \leq j \leq 5) \) with color 1,2,3,3,1 cyclically and the vertices \( a_{3j} \) for \( (1 \leq j \leq 5) \) with color 1,1,3,3,2 orderly, in order to obtain the coloring. Hence \( \chi_{r}[LK_1] = \chi_{r}[LK_2] \leq 3 \). If \( \chi_{r}[LK_1] = \chi_{r}[LK_2] < 3 \), the \( r \)-adjacency condition is not fulfilled. Therefore, \( \chi_{r}[LK_1] = \chi_{r}[LK_2] = 3 \).

**Case 2** The \( r \)-dynamic coloring for \( r = 3 \) are as follows:

In order to obtain the coloring, color the vertices \( a_{ij} \) for \( (1 \leq i \leq 3) \) and \( (1 \leq j \leq 5) \) with five different colors. Thus, the vertices \( a_{ij} \) are colored with 1,5,4,3,2 colors, the vertices \( a_{2j} \) are colored with 3,2,4,1,5 colors and finally, color the vertices \( a_{3j} \) for are colored with 4,3,1,5,2 colors. Next, the remaining vertices \( b_1, b_5 \) are colored with color 2, the vertices \( b_3, b_6 \) are with
color 3 and then color the vertices $b_2, b_4$ with color 4 and color 6. At last, color the center vertex $b_0$ with color 6. Therefore, $\chi_r = 3[LK_1] = 3[LK_2] = 6$. If $\chi_r [LK_1] = \chi_r [LK_2] < 6$, the r-adjacency condition is not fulfilled. Therefore, $\chi_r [LK_1] = \chi_r [LK_2] = 6$.

**Theorem 4** If $1 \leq r \leq 2$ the r-dynamic chromatic number of Szekeres snark $[S]$ is 3-colorable else 6-colorable.

**Proof**: Szekeres snark is a snark with 50 vertices and 75 edges. The maximum and minimum degree of Szekeres snark are $\delta = \Delta = 3$. The vertices of $[S]$ are $\{a_i, a_{ij} : 0 \leq i < 4, 1 \leq j \leq 9\}$. The vertices $a_i$ are the center vertices of $S_i$ that are connected to $a_{i2}, a_{i5}$ and $a_{i8}$. The edge sets are $\{a_{ij} a_{ij+1} : 0 \leq i \leq 4, 1 \leq j \leq 8\} \cup \{a_{i1} a_{i6}, a_{i4} a_{i9} : 0 \leq i \leq 4\} \cup \{a_{i0} a_{i9}, a_{i1} a_{(i-1)9} : 1 \leq i \leq 4\} \cup \{a_{33} a_{07}, a_{43} a_{17}, a_{33} a_{(i+2)7} : 0 \leq i \leq 2\}$. The r-dynamic coloring are as follows:

**Case 1**: The coloring for $1 \leq r \leq 2$ are explained below:

- The center vertices $a_i$ for $(0 \leq i \leq 4)$ with color 3. Next color the vertices $a_{00}$ with color 3, $a_{02}$ with color 2 and the vertex $a_{03}$ with color 1. The remaining vertices of $a_{0j}$ for $(4 \leq j \leq 9)$ are with color 3, 1, 2 sequentially. Color the vertices $a_{ij}$ for $(1 \leq j \leq 6)$ with color 1, 2, 3 orderly and color the remaining vertices $a_{17}, a_{18}$ and $a_{19}$ with color 2, 1, 3.

- Color the vertices $a_{21}$ and $a_{22}$ with color 2, the vertices $a_{23}$ with color 1 and color the remaining vertices $a_{2j}$ for $(4 \leq j \leq 9)$ with color 3, 2 and 1 orderly. Next color the vertices $a_{31}, a_{32}$ and $a_{33}$ as same as the vertices $a_{21}, a_{22}, a_{23}$ and color the remaining vertices $a_{3j}$ for $(4 \leq j \leq 9)$ with color 1, 2 and 3 cyclically. At last, color the vertices $a_{4j}$ for $(1 \leq j \leq 6)$ with color 2, 1 and 3 alternatively and color the remaining vertices $a_{47}$ and $a_{49}$ with color 1 and the vertex $a_{48}$ with color 2. Therefore, $\chi_{1 \leq r \leq 2} [S] = 3$. If $\chi_r [S] < 3$, the r-adjacency condition is not fulfilled. Therefore, $\chi_r [S] = 3$.

**Case 2**: When $r = 3$, the coloring of Szekeres snark are as follows:

- Color the center vertices $a_i$ for $(0 \leq i \leq 4)$ with color 6. Color the vertices $a_{01}$ and $a_{04}$ with color 2, the vertices $a_{02}$ and $a_{09}$ are with color 4, the vertices $a_{03}$ and $a_{08}$ are with color 3. Then the vertices $a_{06}$ and $a_{09}$ are with color 5 and the last vertex $a_{05}$ are colored with color 1. Next, we consider the other set of vertices $a_{11}$ and $a_{14}$ are with color 1, the vertices $a_{12}$ and $a_{17}$ are with color 2, next the vertices $a_{13}$ and $a_{18}$ are with color 4. Then the vertices $a_{16}$ and $a_{19}$ are with color 3 and the last vertex $a_{15}$ are colored with color 5.

- Next color the vertices $a_{21}$ and $a_{24}$ are with color 2, the vertices $a_{22}$ and $a_{27}$ are with color 1, next the vertices $a_{23}$ and $a_{28}$ are with color 5. Then the vertices $a_{26}$ and $a_{29}$ are with color 4 and the last vertex $a_{25}$ are colored with color 3. Color the vertices $a_{31}$ and $a_{34}$ are with color 3, the vertices $a_{32}$ and $a_{37}$ are with color 5, next the vertices $a_{33}$ and $a_{38}$ are with color 2. Then the vertices $a_{36}$ and $a_{39}$ are with color 6 and the last vertex $a_{35}$ are colored with color 4. Finally we color the vertices $a_{41}$ and $a_{44}$ are with color 5, the vertices $a_{42}$ and $a_{47}$ are with color 4, next the vertices $a_{43}$ and $a_{48}$ are with color 1. Then the vertices $a_{46}$ and $a_{49}$ are with color 3 and the last vertex $a_{45}$ are colored with color 2. Therefore, $\chi_{r=3} [S] = 6$. If $\chi_r [S] < 6$, the r-adjacency condition is not fulfilled. Therefore, $\chi_r [S] = 6$.

**Theorem 5** If $1 \leq r \leq 2$ the r-dynamic chromatic number of Watkins snark $[WK]$ are 3-colorable else 6-colorable.

**Proof** : Watkins snark $[WK]$ is with 50 vertices and 75 edges. The maximum and minimum degree are $\delta = \Delta = 3$. The vertices are $\{a_i, a_{ij} : 0 \leq i < 4, 1 \leq j \leq 9\}$. Watkins snark has same number of vertices and edges as szekeres snark with a different that the vertices $a_{ij}$ forms a cycle
in Watkins snark but it is not possible in Szekeres snark. The vertices $a_i$ are the center vertices of $a_{ij}$ that is connected to three vertices namely $a_{i2}, a_{i5}$ and $a_{i8}$. The edge sets of Watkins snark are \{\{a_{04}a_{49}, a_{i4}a_{(i-1)9} : 1 \leq i \leq 4\} \cup \{a_{i1}a_{(i+3)6} : i = 0, 1\} \cup \{a_{i1}a_{(i-2)6} : 2 \leq i \leq 4\} \cup \{a_{33}a_{7} : 0 \leq i \leq 4\}\}. Thus, the r-dynamic coloring are as follows:

**Case 1:** When $1 \leq r \leq 2$, the coloring are as follows:

- Color the vertices $a_i$ for $(0 \leq i \leq 4)$ with color 3.
- Color the vertices $a_{01}$ and $a_{07}$ with color 3, color the vertices $a_{0j}$ for $(2 \leq i \leq 6)$ with color 1 and 2 alternatively and color the left over vertices $a_{08}$ and $a_{09}$ with color 1 and 2. Next, color the vertices $a_{i1}$ for $(1 \leq i \leq 3)$ with color 3, 2, 1. Then, color the vertices $a_{14}$ with color 1, $a_{15}$ with color 3, $a_{16}$ with color 2 and finally, color the vertices $a_{1j}$ for $(7 \leq i \leq 9)$ with color 3, 2, 1.
- Color the vertices $a_{2j}$ for $(1 \leq i \leq 6)$ with color 2, 1, 3 orderly. Color the remaining vertices $a_{27}$ with color 1, $a_{28}$ with color 2 and $a_{29}$ with color 3. Next, color the vertices $a_{31}$ with color 1, $a_{32}$ with color 2 and $a_{33}$ with color 3. Color the other vertices $a_{3j}$ for $(4 \leq i \leq 8)$ with color 1 and 2 alternatively and color the last vertex $a_{39}$ with color 3.
- Color the vertices $a_{41}$ with color 2, $a_{42}$ with color 1 and $a_{43}$ with color 3. Color the leftover vertices $a_{4j}$ for $(4 \leq i \leq 8)$ with color 2 and 1 alternatively. Atlast, color the last vertex $a_{49}$ with color 3. Thus, $\chi_r[WK] \leq 3$. If $\chi_r[WK] < 3$, the r-adjacency condition is not fulfilled. Therefore, $\chi_r[WK] = 3$.

**Case 2:** when $r = 3$ the coloring are as follows:

Color the vertices $a_i$ for $(0 \leq i \leq 4)$ with color 6. Since the vertices $a_{ij}$ forms a cycle, the method for coloring are as follows: color the vertices $a_{01}$ and $a_{07}$ with color 4, color the vertices $a_{02}$ and $a_{06}$ with color 5, the vertices $a_{03}$ and $a_{09}$ are colored with color 3, color the vertices $a_{04}$

**Figure 2.** Watkins snark [$WK$]
and \(a_{08}\) with color 1 and atlast the vertex \(a_{05}\) are with color 2. Next, color the vertices \(a_{11}\) and \(a_{17}\) with color 2, color the vertices \(a_{12}\) and \(a_{16}\) with color 3, the vertices \(a_{13}\) and \(a_{19}\) are colored with color 4, color the vertices \(a_{14}\) and \(a_{18}\) with color 5 and finally, the vertex \(a_{15}\) are with color 1. Then, color the vertices \(a_{21}\) and \(a_{27}\) with color 1, color the vertices \(a_{22}\) and \(a_{26}\) with color 4, the vertices \(a_{23}\) and \(a_{29}\) are colored with color 2, color the vertices \(a_{24}\) and \(a_{28}\) with color 3 and atlast the vertex \(a_{25}\) are with color 5. Color the vertices \(a_{31}\) and \(a_{37}\) with color 5, color the vertices \(a_{32}\) and \(a_{36}\) with color 2, the vertices \(a_{33}\) and \(a_{39}\) are colored with color 1, color the vertices \(a_{34}\) and \(a_{38}\) with color 4 and atlast the vertex \(a_{35}\) are with color 3. Finally, color the vertices \(a_{41}\) and \(a_{47}\) with color 3, color the vertices \(a_{42}\) and \(a_{46}\) with color 1, the vertices \(a_{43}\) and \(a_{49}\) are colored with color 5, color the vertices \(a_{44}\) and \(a_{48}\) with color 2 and atlast the vertex \(a_{45}\) are with color 4. Therefore, \(\chi_r[WK]\) \(\leq 6\). If \(\chi_r[WK] \leq 6\), the r-adjacency condition is not fulfilled. Therefore, \(\chi_r[WK] = 6\).

**Theorem 6** If \(1 \leq r \leq 2\) the r-dynamic chromatic number of first celmins swart snark \([CS_0]\) are 3-colorable else 6-colorable.

**Proof** : The 1-celmins swart snark are with 26 vertices and 39 edges. The maximum and minimum degree are \(\delta = \Delta = 3\). The vertices are \(\{a_{ij}, b_k : 1 \leq i \leq 3, 1 \leq j \leq 5, 1 \leq k \leq 11\}\). The vertices \(a_{ij}\) are connected to \(b_k\) by the edges so it forms an connected graph. The coloring for first celmins swart snark \([CS_0]\) are as follows:

**Case : 1** When \(1 \leq r \leq 2\) the r-dynamic coloring are given below:

Color the vertices \(b_k\) for \((2 \leq k \leq 10)\) with 3,1,2 colors cyclically. Then, color the vertices \(b_1\) with color 2 and \(b_{11}\) with color 1. Next, color the vertices \(a_{1j}\) for \((1 \leq j \leq 5)\) with color 3,2,1,1,3 cyclically. Then, color the vertices \(a_{2j}\) for \((1 \leq i \leq 5)\) with color 1,3,3,2,1 orderly. Finally color the vertices \(a_{3j}\) for \((1 \leq i \leq 5)\) with color 1,1,2,2,3. Hence \(\chi_r[CS_0] \leq 3\). If \(\chi_r[CS_0] < 3\), the r-adjacency condition is not fulfilled. Therefore, \(\chi_r[CS_0] = 3\).

**Case : 2** To exhibit the result for \(r = 3\) and to satisfy the r-dynamic coloring, color the vertices \(a_{ij}\) for \((1 \leq i \leq 3), (1 \leq j \leq 5)\) with different colors i.e., 1,2,3,4,5. With the condition that the vertices of \(a_{ij}\) that are adjacent to the vertices of \(a_{2j}\) should receives the same color. Similarly, for \(a_{2j}\) and \(a_{3j}\). Next, color the vertices \(b_1\) and \(b_7\) with color 3, the vertices \(b_2\) and \(b_6\) with color 5, color the vertices \(b_3\) with color 4, the vertices \(b_4\) and \(b_9\) are with color 2 and the vertices \(b_5, b_8\) and \(b_{11}\) are with color 6. Atlast color the vertices \(b_{10}\) with color 1. Hence, \(\chi_r[CS_0] \leq 6\). If \(\chi_r[CS_0] < 6\), the r-adjacency condition is not fulfilled. Therefore, \(\chi_r[CS_0] = 6\).

**Theorem 7** The r-dynamic chromatic number of second celmins swart snark \([CS_1]\) are given by,

\[
\chi_r(CS_1) = \begin{cases} 
3, & \text{for } r = 1 \\
4, & \text{for } r = 2 \\
6, & \text{for } r = 3 = \delta \\
7, & \text{for } r = 4 = \Delta 
\end{cases}
\]

**Proof** : The 2-celmins swart snark is same as the 1-celmins swart snark with same number of vertices and edges. But with the slight change (i.e.,) the minimum degree of \(\delta(CS_1) = 3\) and the maximum degree \(\Delta(CS_1) = 4\). The vertices of \(V[CS_1] = \{a_{ij}, b_k, c_t, 1 \leq i \leq 3, 1 \leq j \leq 5, 1 \leq k \leq 9, t = 0, 1\}\).

**Case : 1** When \(r = 1\), consider the following coloring:

Color the vertices \(b_k\) for \((1 \leq k \leq 6)\) with color 3 and 1 alternatively and color the remaining \(b_k\) for \((7 \leq k \leq 9)\) with color 2 and 1 orderly. Next, color the vertices \(c_9\) and \(c_1\) with color 2 and 3. Then, color the vertices \(a_{ij}\) for \((1 \leq j \leq 5)\) with colors 3,2,2,1,1 and color the vertices \(a_{2j}\) for \((1 \leq j \leq 5)\) with colors 2,2,1,1,3. Finally, color the vertices \(a_{3j}\) for \((1 \leq j \leq 5)\) with
colors 2,2,3,1,1. Hence, \(\chi_1[CS_1] \leq 3\). If \(\chi_r[CS_1] < 3\), the r-adjacency condition is not fulfilled. Therefore, \(\chi_r[CS_1] = 3\).

**Case : 2** When \(r = 2\), follow the below coloring:

Color the vertices \(c_0\) and \(c_1\) with color 2 and 4. Color the vertices \(b_k\) for \((1 \leq k \leq 6)\) with colors 3,1,2,1,3,1 and the vertices \(b_7\) and \(b_8\) are with color 1 and 3 and color the vertices \(b_9\) with color 2. Next color the vertices \(a_{1j}\) for \((1 \leq j \leq 5)\) with colors 1,1,3,2,2, and color the vertices \(a_{2j}\) for \((1 \leq j \leq 5)\) with colors 1,2,2,3,3. Finally, color the vertices \(a_{3j}\) for \((1 \leq j \leq 5)\) with colors 2,3,3,1,1. Hence, \(\chi_2[CS_1] \leq 4\). If \(\chi_r[CS_1] < 4\), the r-adjacency condition is not fulfilled. Therefore, \(\chi_r[CS_1] = 4\).

**Case : 3** When \(r = 3\), consider the following coloring:

Color the vertices \(c_0\) and \(c_1\) with color 1 and 6. Color the vertices \(b_1\) and \(b_4\) with color 6. The vertices \(b_2, b_5\) and \(b_8\) are colored with color 3, color the vertices \(b_3\) and \(b_7\) with color 2 and color the vertices \(b_9\) and \(b_6\) with color 5. Color the vertices \(a_{ij}\) for \((1 \leq i \leq 3)\) and \((1 \leq j \leq 5)\) with five different colors as 1,2,3,4,5, to satisfy the r-dynamic coloring. The coloring for the vertices \(a_{ij}\) are given with the condition that the vertices \(a_{1j}\) that are adjacent to \(a_{2j}\) should not have the same color. Similarly, for the vertices \(a_{2j}\) and \(a_{3j}\). Therefore, \(\chi_3[CS_1] \leq 6\). If \(\chi_r[CS_1] < 6\), the r-adjacency condition is not fulfilled. Therefore, \(\chi_r[CS_1] = 6\).

**Case : 4** When \(r = 4\), consider the following coloring:

Color the vertices \(c_0\) and \(c_1\) with color 7 and 6. Color the vertices \(a_{ij}\) for \((1 \leq i \leq 3, 1 \leq j \leq 5)\) as in case 3. Next, color the vertices \(b_1\) and \(b_4\) with color 6. Color the vertices \(b_2\) and \(b_7\) with color 3, the vertices \(b_3\) and \(b_6\) are colored with color 5 and then color the vertices \(b_5\) and \(b_9\) with color 4. Finally, color the vertex \(b_8\) with color 2. Hence, \(\chi_4[CS_1] \leq 7\). If \(\chi_r[CS_1] < 7\), the r-adjacency condition is not fulfilled. Therefore, \(\chi_r[CS_1] = 7\).

4. Conclusion
we obtained the exact values of r-dynamic chromatic number of some snark families. Further studies of distance graphs may well give an additional insight to the r-dynamic coloring problem.

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