Deformation Quantization of Odd Dimensional anti-de Sitter Spaces as Contact Manifolds

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Abstract: We quantize odd dimensional anti-de Sitter spaces by applying the method of deforming contact manifolds proposed by Rajeev in [1]. The construction in the present paper consists of the identification of the odd dimensional anti-de Sitter space as a hypersurface of contact type and the subsequent use of 'symplectization' principle. We also show that this construction generalizes to any odd dimensional hypersurface which can be represented as a nonzero level set of a homogenous function.
1 Introduction

Recently Rajeev, in his work on quantization of thermodynamics [1], proposed a method of quantizing contact manifolds. Basic idea behind this method is the symplectization principle of contact manifolds. This principle allows one to set up a 1-1 correspondence between the functions on a contact manifold and a subset \( \mathcal{F} \) of functions on a symplectic manifold. Then one uses the usual deformation of the symplectic manifold to deform \( \mathcal{F} \) and map the result back to the contact manifold. The first order correction to the resulting star product is given by the Legendre bracket which is the analog of Poisson bracket in contact geometry. In [1] this method was applied to quantize odd dimensional spheres.

Considering the central role of AdS/CFT correspondence [2, 3, 4] in M-theory and the important relations between noncommutative geometry [5] and string theory [6] we find it useful to apply Rajeev’s method to quantize odd dimensional anti-de Sitter spaces.

As usual we will regard \( AdS_{2n+1} \) as a hypersurface in \( R^{2n+2} \) defined by the equation

\[
-(x^0)^2 - (x^1)^2 + (x^2)^2 + \ldots + (x^{2n+1})^2 = -1.
\]

We will regard \( R^{2n+2} \) as a symplectic manifold with the canonical symplectic form. Our symplectization will rely on the fact that \( AdS_{2n+1} \) is a hypersurface of contact type. Then we will show how one can lift functions on \( AdS_{2n+1} \) to a subset of homogenous functions on \( R^{2n+2} \). Following the general strategy of [1] we will use the star product on \( R^{2n+2} \) to induce one on \( AdS_{2n+1} \). The homogeneity of the function defining \( AdS_{2n+1} \) is an essential ingredient of the constructions presented in this paper. In fact we will show that our results can be applied to any odd dimensional hypersurface which can be represented as a nonzero level set of a homogenous function.

2 Contact Geometry

In this section we will summarize basic results from contact geometry that will be useful in our constructions. Detailed accounts of the subject can be
Let $M$ be a manifold of dimension $2n+1$ and $\xi$ a $2n$ dimensional subbundle of $TM$. We will assume that there exist a global 1-form $\alpha$ such that $\ker \alpha = \xi$ and $d\alpha$ is nondegenerate on $\xi$. Then $\xi$ is called a contact structure and $\alpha$ is called the contact 1-form. The assumptions on $\alpha$ are equivalent to $\ker \alpha = \xi$ and $\alpha \wedge (d\alpha)^n \neq 0$. Since $d\alpha$ is a 2-form on an odd dimensional manifold, it must be degenerate. However, by assumption $d\alpha$ is nondegenerate on a $2n$ dimensional space $\xi$. Therefore there exist a unique direction along which $d\alpha$ is degenerate. This gives us the definition of the Reeb vector field as the unique vector field $Y$ such that

$$\iota_Y d\alpha = 0, \quad \iota_Y \alpha = 1.$$  \hfill (2)

A vector field $X$ satisfying $\mathcal{L}_X \alpha = g_X \alpha$ where $g_X \in C^\infty(M)$ is called a contact vector field. In particular the Reeb vector field $Y$ is a contact vector field with $g_Y = 0$. There is a one-to-one correspondence between the set of all contact vector fields and $C^\infty(M)$. Given a contact vector field $X$ the corresponding function $H$ is defined as $H = -\iota_X \alpha$. Moreover $g_X = -dH(Y)$. Conversely given a function $H$ the corresponding contact vector field is $X_H = Z - HY$ where $Z \in \xi$ is the unique solution of $\iota_Z d\alpha|_\xi = dH|_\xi$.

The analog of the Poisson brackets in contact geometry is the Legendre bracket defined by

$$(F, G) = -\alpha([X_F, X_G]) \hfill (3)$$

$$= dG(Y)F + X_G F \hfill (4)$$

$$= -dF(Y)G - X_F G. \hfill (5)$$

Legendre bracket is bilinear, antisymmetric and satisfies Jacobi identity. However the Leibnitz rule is not satisfied. Instead it satisfies:

$$(f, gh) = g(f, h) + (f, g)h + (1, f)gh \hfill (6)$$

which is sometimes called the generalized Leibnitz rule. Since the Leibnitz rule is not satisfied it follows that the constant function is not in the center of the algebra. In fact, upon quantization \cite{1} the constant function does
not become a constant multiple of the identity. This is the novel feature of quantization on contact manifolds.

Let $Q$ be a hypersurface in a symplectic manifold $(N, \omega)$. If there exist a 1-form $\alpha$ in $Q$ such that $d\alpha = \omega|_Q$ then $\alpha$ is a contact form. In this case $Q$ is called a hypersurface of contact type. A Liouville vector field in a symplectic manifold is defined as a vector field satisfying $\mathcal{L}_T\omega = \omega$. An equivalent condition for $Q$ to be a hypersurface of contact type is the existence of a Liouville vector field $T$ which is transverse to $Q$.

3 Contact Geometry of $AdS_{2n+1}$

Consider the $2n + 1$ dimensional anti-de Sitter space $AdS_{2n+1}$. This is a hypersurface in $\mathbb{R}^{2n+2}$ defined by the equation $R(x) = -1$ where

$$R(x) = -(x_0)^2 - (x_1)^2 + (x_2)^2 + \ldots + (x_{2n+1})^2. \quad (7)$$

It will be useful to call the even coordinates $p$ and odd coordinates $q$. Then

$$R(p, q) = (-p_1)^2 + (q_1)^2 + (-p_2)^2 + (q_2)^2 + \ldots + (-p_{n+1})^2 + (q_{n+1})^2. \quad (8)$$

We will think of $\mathbb{R}^{2n+2}$ as a symplectic manifold with the canonical symplectic form

$$\omega = \sum_k dp^k \wedge dq^k \quad (9)$$

Clearly $\omega = d\alpha_0$ with

$$\alpha_0 = \frac{1}{2} \sum_k p^k dq^k - q^k dp^k. \quad (10)$$

Consider the vector field $T$ in $\mathbb{R}^{2n+2}$ given by

$$T = \frac{1}{2} \sum_k p^k \frac{\partial}{\partial p^k} + q^k \frac{\partial}{\partial q^k}. \quad (11)$$

It follows that $T$ is a Liouville vector field on $\mathbb{R}^{2n+2}$. Notice that $\iota_T \omega = \alpha_0$. Moreover we have

$$\mathcal{L}_T R = R. \quad (12)$$
This is nothing but Euler’s homogenous function theorem applied to $R$, a homogenous function of degree 2. Consequently we see that on $AdS_{2n+1}$ where $R = -1$:

$$\mathcal{L}_TR = -1 \neq 0.$$  \hspace{1cm} (13)

So $T$ is transversal to $AdS_{2n+1}$. Thus odd dimensional $AdS$ spaces are hypersurfaces of contact type. The contact form on $AdS_{2n+1}$ is

$$\alpha = \iota_T \omega|_{AdS} = \alpha_0|_{AdS}$$  \hspace{1cm} (14)

Let us consider the Hamiltonian vector field corresponding to the function $R$

$$\iota_{X_R} \omega = -dR$$  \hspace{1cm} (15)

Explicitly we have

$$X_R = \sum_k -\frac{\partial H}{\partial q^k} \frac{\partial}{\partial p^k} + \frac{\partial H}{\partial p^k} \frac{\partial}{\partial q^k}$$

$$= 2 \left( q^1 \frac{\partial}{\partial p^1} - q^2 \frac{\partial}{\partial p^2} - \ldots \right) + 2 \left( -p^1 \frac{\partial}{\partial q^1} + p^2 \frac{\partial}{\partial q^2} + \ldots \right)$$  \hspace{1cm} (16)

Clearly $X_R$ is tangent to $AdS_{2n+1}$. For any vector field $V$ tangent to $AdS_{2n+1}$ we have

$$(\iota_{X_R} d\alpha)(V) = -dR(V) = -\mathcal{L}_V H = 0.$$  \hspace{1cm} (17)

Thus on $AdS_{2n+1}$ we have

$$\iota_{X_R} d\alpha = 0$$  \hspace{1cm} (18)

Moreover by homogeneity of $R$

$$\iota_{X_R} \alpha_0 = \frac{1}{2} \sum_k p^k \frac{\partial R}{\partial p^k} + p^k \frac{\partial R}{\partial p^k} = R$$  \hspace{1cm} (19)

which, when restricted to $AdS_{2n+1}$, gives

$$\iota_{X_R} \alpha_0 = -1$$  \hspace{1cm} (20)

Thus $X_R$ is the negative of the Reeb vector field on $AdS_{2n+1}$. 

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4 Deformation Quantization of $AdS_{2n+1}$

Now let us try to lift functions on $AdS_{2n+1}$ to $\mathbb{R}^{2n+2}$. We will consider the functions that can be expanded in the eigenfunctions of the Laplacian on $AdS_{2n+1}$ \cite{9,10,11,12}. Thus we can expand in e.g.

$$\Phi = e^{-i\nu t}Y_{l,m}(\Omega)F(\sin \rho). \quad (21)$$

Here $F$ is a hypergeometric function as given in \cite{11}, $Y_{l,m}(\Omega)$ are the spherical harmonic on $S^{2n-1}$. The coordinates of a point on $AdS_{2n+1}$ in the ambient space are given in terms of the angular variables as

$$p^1 = \sec \rho \cos \tau \quad (22)$$
$$q^1 = \sec \rho \sin \tau \quad (23)$$
$$x^i = (\tan \rho)\Omega_i. \quad (24)$$

Here $\sum_{i=1}^{n} \Omega_i^2 = 1$, $0 \leq \rho < \frac{\pi}{2}$, $0 \leq \tau < 2\pi$ and we denoted the coordinates $p^2,q^2,\ldots$ by $x^1,x^2,\ldots$, respectively. Expressing $\Phi$’s in terms of the original variables and continuing the result to $\mathbb{R}^{2n+2}$ gives us a function on $\mathbb{R}^{2n+2}$. The effect of this on the coordinates is the substitution:

$$\cos^2 \rho \rightarrow \frac{(p^1)^2 + (q^1)^2 - \sum_i (x^i)^2}{(p^1)^2 + (q^1)^2} \quad (25)$$

$$\tau \rightarrow \tan^{-1} \left( \frac{q^1}{p^1} \right) \quad (26)$$

$$\Omega_i \rightarrow \frac{x^i}{\sqrt{\sum_i (x^i)^2}} \quad (27)$$

The resulting function is defined only on $(p^1)^2 + (q^1)^2 \geq \sum_i (x^i)^2$. Geometrically this can be understood as follows. The transverse vector field $T$ maps a level surface of $R$ onto another, infinitesimally close level set. In such infinitesimal steps one covers the region in question which is thence diffeomorphic to the trivial bundle $AdS_{2n+1} \times \mathbb{R}^+T$. Notice that the resulting function is homogenous of degree 0. In particular it is annihilated by the vector field $T$. Given a function $F$ on $AdS_{2n+1}$ we denote its extension
described above by $F_1$ and define the desired lift by

$$\tilde{F} = -RF_1$$

(28)

This lift is homogenous of degree 2. Notice that $\tilde{I} = -R$. Now we can check that the Poisson bracket of two such lifts gives us a function whose restriction to $AdS_{2n+1}$ gives the Legendre bracket of the original functions. Let $F$ be a function on $AdS_{2n+1}$. We will denote the corresponding contact vector field by $X_F$. The Hamiltonian vector field corresponding to the lift $\tilde{F}$ will be denoted by $\tilde{X}_F$. In order to compare the Poisson bracket with the Legendre bracket we must relate Hamiltonian vectors field to contact vector fields. We will show that the latter are the restrictions of the former on $AdS_{2n+1}$. According to the general theory $X_F = Z_F + FX_R$ where $Z_F$ is the unique vector field in $\xi$ such that

$$d\alpha(Z_F, V) = dF(V) \quad \forall V \in \xi$$

(29)

Let $\{V_a\}$ be a basis for $\xi$ and recall that $d\alpha = \omega|_{AdS}$. Expanding $Z_F = Z^aV_a$ we get

$$Z^a\omega_{ab} = dF(V_b).$$

(30)

Let us also express $\tilde{X}_F$ in the basis $\{T, X_H, V_a\}$ as

$$\tilde{X}_F = A^aV_a + BT + CX_R.$$  

(31)

Then

$$\left\{ \tilde{F}, \tilde{G} \right\} = -\tilde{X}_F\tilde{G} =$$

$$= (\tilde{X}_F R)G_1 + R\tilde{X}_F G_1$$

(32)

(33)

Now

$$\tilde{X}_F R\big|_{AdS} = -B\big|_{AdS}$$

(34)

and

$$\tilde{X}_F G_1\big|_{AdS} = (A^aV_a G_1 + CX_R G_1)|_{AdS}$$

(35)
So
\[ \left\{ \tilde{F}, \tilde{G} \right\}_{AdS} = -B|_{AdS} G - A^a|_{AdS} V_a G - C|_{AdS} X_R G. \]  
(36)

On the other hand we have
\[ t_{\tilde{X}_k} \omega = A^a \omega(V_a, ) + B \omega(T, ) + C \omega(X_R, ) \] 
(37)
\[ = A^a \omega(V_a, ) + B \alpha( ) + C \omega(X_R, ) = d(-RF_1) = -dRF_1 - RdF_1. \]  
(38)

Contracting this with \( V_b \) we get
\[ A^a \omega_{ab} = -FdR(V_b) + dF(V_b) = dF(V_b) \]  
(39)
Comparing this with (30) we see that \( A^a = Z^a \). Similarly contraction with \( T \) gives
\[ C = F, \]  
(40)
and finally contracting by \( X_R \) we get
\[ B = -dF(X_R) = -\mathcal{L}_{X_R} F \]  
(41)

Substituting these into (35) we get the desired result
\[ \left\{ \tilde{F}, \tilde{G} \right\}_{AdS} = -dF(Y)G - X_F G = (F, G) \]  
(42)

Now, following Rajeev [11] we can define the star product on \( AdS_{2n+1} \) as
\[ F \ast G = \tilde{F} \exp \left[ -\frac{i\hbar}{2} \left( \frac{\partial}{\partial q_k} \frac{\partial}{\partial p_k} - \frac{\partial}{\partial p_k} \frac{\partial}{\partial q_k} \right) \right] \tilde{G} \bigg|_{AdS} \]  
(43)
\[ = FG - \frac{i\hbar}{2} (F, G) + \ldots. \]  
(44)

This provides the deformation of \( AdS_{2n+1} \) as a contact manifold.
5 Generalizations and Conclusion

Our results can be generalized as follows. Let $Q$ be a hypersurface defined as a nonzero level set of a homogenous function on an even dimensional space. If $Q$ is defined by an equation of the form $R = a \neq 0$ where $R$ is a homogenous function of degree $r$ the transverse Liouville vector field can be chosen as

$$T = \frac{1}{r} \sum_k p^k \frac{\partial}{\partial p^k} + q^k \frac{\partial}{\partial q^k}. \quad (45)$$

This turns $Q$ into a hypersurface of contact type. Again thanks to the homogeneity of $R$ the Hamiltonian vector field $X_R$ gives the Reeb field on the hypersurface. The crucial point in the problem is the construction of the lift. This can be accomplished if one can find a suitable basis for the functions on the hypersurface which can be lifted to homogenous functions of degree $r$ in the ambient space. In particular this construction works for odd dimensional spheres $\Pi$, for de Sitter spaces and, as we have shown explicitly, for anti-de Sitter spaces.

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