Quantum dynamics of local phase differences between reservoirs of driven interacting bosons separated by simple aperture arrays

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Abstract
We present a derivation of the effective action for the relative phase of driven, aperture-coupled reservoirs of weakly-interacting condensed bosons from a (3 + 1)D microscopic model with local $U(1)$ gauge symmetry. We show that inclusion of local chemical potential and driving velocity fields as a gauge field allows derivation of the hydrodynamic equations of motion for the driven macroscopic phase differences across simple aperture arrays. For a single aperture, the current–phase equation for driven flow contains sinusoidal, linear and current-bias contributions. We compute the renormalization group (RG) beta function of the periodic potential in the effective action for small tunneling amplitudes and use this to analyze the temperature dependence of the low-energy current–phase relation, with application to the transition from linear to sinusoidal current–phase behavior observed in experiments by Hoskinson et al (2006 Nature Phys. 2 23–6) for liquid $^4$He driven through nanoaperture arrays. Extension of the microscopic theory to a two-aperture array shows that interference between the microscopic tunneling contributions for individual apertures leads to an effective coupling between apertures which amplifies the Josephson oscillations in the array. The resulting multiaperture current–phase equations are found to be equivalent to a set of equations for coupled pendula, with microscopically derived couplings.

1. Introduction

1.1. Background

Josephson oscillations of flow between weakly coupled quantum liquids constitute fundamental signatures of macroscopic quantum behavior [2]. Extensively studied for superconductors, such oscillations have also been observed between reservoirs of the cold atomic superfluids $^3$He and $^4$He coupled by arrays composed of nanometer-sized apertures [3, 4, 1]. The Josephson coupling for reservoirs of the liquid $^3$He is analogous to that for high-symmetry superconductors, due to the fermionic nature of the atoms and the angular momentum dependence of the interaction, but occurs only at very low ($\sim$3 mK) temperatures due to the low superfluid transition temperature for $^3$He. In contrast, the recently observed Josephson flow between driven reservoirs of the bosonic superfluid $^4$He separated by an array of nanometer-sized apertures [1] is analogous to Josephson flow between s-wave superconductors, and is observed at much higher temperatures, just below the $^4$He superfluid transition $T_s = 2.17$ K. Several unique features of weakly coupled liquid $^4$He reservoirs subject to driven flow have stimulated research into thermal/quantum fluctuations of the macroscopic phase...
for bosonic quantum liquids in aperture array geometries [5, 6]. These studies provide thermodynamic justification for the transition, observed in [1], between two different current–phase relationships as a function of temperature below the lambda point: the linear regime (occurring at low temperatures, $T_\lambda - T > 5 \, \text{mK}$) in which the current depends linearly on the phase, and the ‘weak-link’ Josephson flow regime in which the current has sinusoidal phase dependence (occurring for $T_\lambda - T < 0.8 \, \text{mK}$). The linear current–phase relationship at low temperatures is thought to be due to independent phase slips occurring at individual apertures in the array in response to external driving by the hydrodynamic resonator. As the temperature is increased toward the lambda point, the coherence length increases and the phase differences across individual apertures appear to become synchronized. It has been proposed that this results in coherent dissipative events, i.e. ‘phase-slip avalanches’, giving way eventually to coherent Josephson flow and a characteristic sinusoidal current–phase relation just below $T_\lambda$ [1, 5, 6].

No microscopic quantum mechanical explanation currently exists for this phenomenon. In order to justify the observed synchronization of the phase differences, Pekker et al postulate an effective long-range interaction between local phase gradients [5] while Chui, et al exploit the analogy between a Josephson junction array and classical coupled pendula to explore thermal phase fluctuations in an aperture array [6]. In this work, we derive an effective theory and equations of motion for the phase difference across a single aperture and a simple two-aperture array, starting from a local $U(1)$ gauge theory coupled to bosonic matter. The gauge field is necessitated by the presence of an external driving velocity which induces a ‘vector potential’ $\nu(r, t)$ and concomitant local chemical potential $\phi(r, t)$, analogous to the electromagnetic gauge field $A_\mu$ in the theory of Bardeen, Cooper and Schrieffer (BCS) for superconductivity.

### 1.2. Theoretical approach

Since our aim is a microscopic derivation of the equations of motion for the local macroscopic phase differences across aperture arrays (section 3) and an examination of the low-energy properties of the resulting current–phase relation across an aperture (section 5), we employ here functional integral techniques rather than well-known mean field or hydrodynamic techniques for bosonic systems (e.g., a gauged Gross–Pitaevskii (GP) equation [7] for weakly-interacting Bose gases or a gauged two-fluid model [8] for $^4$He near the lambda point). The functional integral approach allows the equations of motion to be derived from the microscopic Lagrangian, as was demonstrated for the analogous case of superconducting systems by Ambegaokar et al in [9, 10] (henceforth referred to as AES). Both the stationary phase analyses and the perturbative renormalization group procedure in this work are most convenient to carry out using this formalism. We note that since we utilize an imaginary time path integral, the resulting dynamical equations are all expressed in imaginary time. However the corresponding real time dynamical equations relevant to experimental analysis are readily obtained from these by analytic continuation, specifically by making use of the Wick rotation $\tau \rightarrow i\tau$.

The fundamental variable of our effective theory is a gauge-invariant phase difference across an aperture: $\Delta \gamma(\tau) = \Delta \theta(\tau) + m \int \, dr \cdot \nu(r, \tau)$, where the integral is taken on a short line segment through the aperture. It contains contributions from the background phase texture $\nabla \theta(r, \tau)$ (the irrotational superfluid velocity) and the external driving velocity (the gauge field). We show that the action governing the gauge-invariant phase differences for driven bosonic reservoirs separated by simple aperture arrays provides a qualitative explanation for the experimental observations of $^4$He flow through nanoaperture arrays over a range of temperatures below $T_\lambda$ [1].

As noted above, a related microscopic derivation for a superconducting system appears in AES, in which an effective theory is derived for the dynamics of a superconducting tunnel junction in terms of the macroscopic phase difference across the junction or the magnetic flux threading a superconducting quantum interference device. Like AES, we shall be concerned here only with the dynamics of the low-energy degree of freedom in the system, which is in this case the gauge-invariant phase difference field, $\Delta \gamma(\tau)$, introduced above. In the present work, we focus on incorporating an externally imposed driving velocity into a gauge-invariant description of coupled reservoirs of weakly-interacting bosons, on determining the current–phase relation for this system in different parameter regimes dependent on the energy scale, and on using the results of this analysis to interpret the experimental observations of [1]. We shall not undertake the further analysis of real-time current correlations, dissipation due to quasiparticles, or the effects of noise in the junction that was also made in AES. Explicit comparison between our results for driven, weakly-interacting bosons with the results of AES for superconducting systems will be given where relevant in the subsequent sections.

### 1.3. Summary of results

The microscopic analysis presented in this work shows that the main features of the transition from linear to sinusoidal Josephson flow as a function of temperature are apparent already in the one and two-aperture cases. Starting from a local $U(1)$ gauge-invariant Lagrangian, we derive the effective action for one and two-aperture arrays. We first show that a perturbative expansion of the gauge theory can be used to derive the quantum hydrodynamical equations of motion for the driven superfluid. In particular, we show that the Josephson–Anderson equation for phase evolution [11] in gauge-invariant form, the circulation (superfluid fluxoid) quantization in the presence of a driving velocity field, and the London equation leading to the Hess–Fairbank effect [12] can all be derived from the stationary phase approximation to the effective action. A Legendre transformation of the Euclidean effective action is then used to derive the current–phase relations for one and two-aperture arrays. We show that for a single aperture, the current–phase relation is consistent with a potential composed of sinusoidal, linear,
and quadratic terms, while for a two-aperture array we find that the interference between the microscopic tunneling contributions for individual apertures leads to a coupling of the current–phase equations of the two-aperture system.

For the single aperture case, we then employ a weak-coupling renormalization group calculation to demonstrate the existence of temperature intervals in which the current–phase relation has predominantly linear or predominantly Josephson (sinusoidal) behavior. The critical temperature separating these regions of linear and sinusoidal flow is determined by relating the ratio of two coefficients in the rescaled effective action, each of which we calculate microscopically to one-loop order in perturbation theory, to the finite temperature healing length by making use of the Popov approximation for dilute, finite temperature BECs [13]. We then use the empirical temperature-dependent healing length of liquid $^4$He in this ratio to apply our theory to the experiment in [1] on driven $^4$He flow through arrays of nanometer-sized apertures. This provides a remarkably accurate rationalization for the experimentally observed transition between linear and sinusoidal current–phase relationships as the temperature is increased toward the lambda point.

1.4. Outline

In section 2 we discuss the local $U(1)$ gauge invariant Euclidean action used in the coherent state functional integral and transform this action into a bilinear form in the real density field which can be analyzed using perturbation theory. In section 3 integration over the density field is performed and the resulting perturbation series for the full inverse Green’s function is used to determine the effective action for the phase difference across a single aperture. We show that the stationary phase approximation to the perturbed action allows gauge-invariant forms of several superfluid hydrodynamical equations to be derived, e.g. the Josephson–Anderson equation for phase-difference evolution, the London equation for the gauge-invariant velocity, and circulation (superfluid fluid) quantization. The central result of this paper is the derivation of current–phase relationships for the single aperture and two-aperture array in section 4. We analyze the temperature dependence of the current–phase relation for a single aperture in the limit of small tunneling amplitude by computing the RG beta function of the coupling constant $E_I$ of the periodic potential and use this to analyze the experimental measurements of driven $^4$He flow through arrays of nanometer-sized apertures. We summarize in section 6 and discuss potential directions for future research.

2. The model

We seek an effective theory for condensed, driven, weakly-interacting bosons separated by an array of one or two apertures in terms of local phase differences across the apertures. While our model includes only a local two-body potential, we will show that the main features of recent experimental results for liquid $^4$He flow through nanoaperture arrays [1] are nevertheless already explained by the current analysis. Our starting point is the Hamiltonian in (equation (1)) for the weakly-interacting Bose gas that is minimally coupled to a local chemical potential field $\phi(r,\tau)$ and a vector field $v(r,\tau)$ which will be interpreted as an external driving velocity. The Hamiltonian (without a tunneling term) is

$$H[\psi,\psi^\dagger] = \frac{1}{2m} \int d^3r \nabla \cdot D \psi^\dagger(r)D\psi(r) + \frac{V_0}{2} \int d^3r \psi^\dagger(r)\psi^2(r) + m \int d^3r \phi(r,\tau)\psi^\dagger(r)\psi(r) + H_{\text{ext}}[v,\phi].$$

The weak interaction is given by the usual delta function two-body potential, with strength $V_0$ (proportional to the s-wave scattering length), and $D = \nabla + imv(r,\tau)$ is the covariant derivative. $H_{\text{ext}}[v,\phi]$ is a classical energy analogous to electromagnetic field energy in superconductors and depends only on external fields.

We then construct the coherent state path integral Lagrangian, equation (2), corresponding to this Hamiltonian and additionally incorporate a single aperture tunneling term $T_{r,r'}$ that couples points $r$ and $r'$ on different sides of the aperture. In the bosonic coherent state path integral, the Lagrangian is given by ($\hbar = k_B = 1):$

$$L[\psi,\psi^*,\Delta,\nu,\phi] = \int d^3r \psi^*(r,\tau)(\partial_\tau + m\phi(r,\tau) - \mu)\psi(r,\tau) + \frac{1}{2m} \int d^3r \nabla \cdot D\psi^*(r,\tau)\nabla \psi(r,\tau) + \frac{1}{2} \int d^3r \nabla^2 \psi^*(r,\tau)T_{r,r'}\psi(r',\tau) + \frac{V_0}{2} \int d^3r \Delta^*(r,\tau)\Delta(r,\tau) - \frac{V_0}{2} \int d^3r \Delta(r,\tau)\nu^*(r,\tau,\nu(r,\tau) - \Delta^*(r,\tau)\psi^*(r,\tau)) + \frac{ml^2}{2} \int d^3r (\nabla \times \nu(r,\tau))^2 + \frac{ml^2}{2} \int d^3r (\nabla \psi(r,\tau) - \nabla\phi(r,\tau))^2. \tag{2}$$

Here, $\Delta(r,\tau)$ and $\Delta^*(r,\tau)$ are Hubbard–Stratonovich fields introduced to decouple the quartic interaction in the weakly-interacting Bose gas. $L$ has dimension of length, and $\tau$ is the imaginary time. In the grand canonical partition function, $Z(\mu,\beta) = \int e^{-\beta L}[v,\phi]$ the functional integration is over the fields $\psi,\psi^*,\Delta$, and $\Delta^*$ and also the gauge field (with the measure defined in the discretized expression for the coherent state path integral [14]). The last two terms are derived from $H_{\text{ext}}[v,\phi]$ in equation (1) and are analogous to the electromagnetic field energy in superconductors; the vorticity (circulation energy density) corresponding to the
magnetic field energy density and an ‘electric’ energy density analogous to the electric field energy density. The fields \( v(r, \tau) \) and \( \phi(r, \tau) \) are analogues of the magnetic vector potential and local voltage of electrodynamics. These will be shown to satisfy stationary phase equations (section 3) and we do not analyze fluctuations of the gauge field configurations.

If the tunneling matrix is multiplied by a \( U(1) \) parallel transporter via:

\[
T_{r', r} \rightarrow T_{r', r} e^{i m \int_{r'}^{r} \nabla \cdot v(r, \tau) \, dr'}
\]  

\( (3) \)

this Lagrangian is clearly invariant under \( \psi(r, \tau) \rightarrow \psi(r, \tau) e^{i \Lambda(r, \tau)} \) (where \( \Lambda(r, \tau) \) is real) as long as \( v(r, \tau) \rightarrow v(r, \tau) - \frac{1}{m} \nabla \Lambda(r, \tau) \) and \( \phi(r, \tau) \rightarrow \phi(r, \tau) - \frac{1}{m} \partial_\tau \Lambda(r, \tau) \).

Put another way, we are analyzing a local \( U(1) \) gauge theory for the superfluid where \( \psi(r, \tau), v(r, \tau) \) is the \( u(1) \) gauge field. The gauge transformation of the \( 0 \)-component is due to working in imaginary time (i.e. the base-space for the \( U(1) \) principal bundle is a Euclidean manifold).

In the analysis to follow, it will lead to e.g. an imaginary Josephson–Anderson equation, which must be Wick rotated to obtain the real-time equation. The mean-field equations of the gauged weakly-interacting Bose gas are the stationary solutions of the Josephson–Anderson equation, which must be Wick rotated to obtain the real-time equation. The mean-field equations of the gauged Gross–Pitaevskii equation [7] for \( \psi \), while \( \frac{\partial L}{\partial \Delta^2} = 0 \) gives a gauged Gross–Pitaevskii equation [7] for \( \psi \), while \( \frac{\partial L}{\partial \Delta^2} = 0 \) gives a steadystate condition. Note that the Hubbard–Stratonovich fields are not complex conjugates! This is a peculiarity of the bosonic Hubbard–Stratonovich transformation. Since the action does not depend on space-time derivatives of \( \Delta(r, \tau) \) or \( \Delta^*(r, \tau) \), they may be taken as real constants at mean-field level. In the following, we choose \( \Delta(r, \tau) = -\Delta \) and \( \Delta^*(r, \tau) = \Delta \) with \( \Delta \) a real constant.

To isolate a local phase field, a polar decomposition can be made on \( \psi \) and \( \psi^* \), e.g. \( \psi \rightarrow \sqrt{\rho(r, \tau)} e^{i \phi(r, \tau)} \). This transformation does not change the measure in the functional integral for the partition function. Physically it means we are considering a restricted ensemble, i.e. we consider only a single condensed mode in the path integral, in particular a coherent state with minimal variance in both amplitude and phase. This is our only explicit reference to Bose–Einstein condensation of the weakly-interacting Bose gas in this work. We always consider the gauge-invariant tunneling amplitude to be nonzero, so that the \( T = 0 \) ground state is never fragmented into spatially separated reservoirs of definite particle number. The action corresponding to the resulting Lagrangian can be brought into bilinear form:

\[
S = \int_{0}^{\beta} d\tau' \int d^3r' \int d^3r \left[ \sqrt{\rho(r, \tau)} G^{-1}(r, \tau; r', \tau') \right. \\
\times \sqrt{\rho(r', \tau')} \left. + \frac{V_0}{2} \Delta^2 \delta(r - r') \delta(\tau - \tau') \right. \\
+ \frac{m t^2}{2} \left( \nabla \times v_g(r, \tau) \right)^2 \delta(r - r') \delta(\tau - \tau') \\
\left. + \frac{m t^2}{2} \left( i \partial_\tau v_g(r, \tau) - \nabla \phi(r, \tau) \right)^2 \delta(r - r') \delta(\tau - \tau') \right]
\]  

\( (4) \)

where \( v_g(r, \tau) := v(r, \tau) + \frac{1}{m} \nabla \phi(r, \tau) \) is the gauge-invariant velocity. The operator \( G^{-1} \) (shown in equations (6)–(7)) is the object of principal computational interest in subsequent sections. Note that besides the field strength contributions and constant offset proportional to \( \Delta^2 \), the complete action can be written as a bilinear form. Using the gauged GP equation [7] and interpreting \( \rho(r, \tau) \) as a local condensate density field, it can be shown that the mean-field hydrodynamic effect of the external driving velocity is a depletion of condensate current [15–17]:

\[
\partial_\tau \rho(r, \tau) - \nabla \cdot \left( \frac{1}{m} \rho(r, \tau) \nabla \phi(r, \tau) \right) = \nabla \cdot (\rho(r, \tau) v(r, \tau)).
\]  

\( (5) \)

In this paper, we take the point of view that \( \phi(r, \tau) \) and \( v(r, \tau) \) comprise the gauge field in the fluid resulting from externally applied driving fields; in particular, the gauge field is not internally generated by fluctuations. In the nanoslit array experiment of Hoskinson et al [1] the oscillations of the hydrodynamic resonator couple to both the condensate atoms and the depletion, like a piston. Thus, the gauge field can be viewed as the externally applied, nonconservative part of the total velocity of the fluid. The response of the phase field to the gauge field is apparent in the Euler and Josephson–Anderson equations that we derive below. An evolving velocity field induces a local chemical potential texture (via the Euler equation) which in turn induces an evolving phase field (via the Josephson–Anderson equation).

3. Perturbation theory and effective action

The operator \( G^{-1}(r, \tau; r', \tau') \) defining the bilinear form in the action equation (4) is

\[
G^{-1} = G_0^{-1} + G_\theta^{-1} + G_{v_g}^{-1} + G_T^{-1} \equiv G_0^{-1} + \delta G^{-1}
\]  

\( (6) \)

where the individual components of \( \delta G^{-1} \) are:

\[
G_0^{-1} = \left( \partial_\tau - \frac{1}{m} \nabla^2 - \mu + V_0 \Delta \right) \delta(r - r') \delta(\tau - \tau')
\]

\( G_{\theta}^{-1} = [i \partial_\tau \theta(r, \tau) + m \phi(r, \tau)] \delta(r - r') \delta(\tau - \tau') \)

\( G_{v_g}^{-1} = (\frac{1}{2} m v_g(r, \tau)^2) \delta(r - r') \delta(\tau - \tau') \)

\( G_T^{-1} = T_{r', r} e^{i \theta(r', \tau) - \theta(r, \tau)} e^{i m \int_{r'}^{r} \nabla \cdot v(r, \tau) \, dr} \delta(\tau - \tau'). \)

In integration over the density field results in a term \( \frac{1}{2} \text{tr} \log G^{-1} \) in the action. The trace is an integral over all internal positions or momenta and imaginary time \( \tau \in [0, \beta] \) arguments. Details of the perturbation expansion for contributions to the action from each part of \( \delta G^{-1} \) are given in appendix A and the general techniques can be found in [18, 9, 19].

3.1. Self-consistent equation for \( \Delta \)

As mentioned in section 2, since there are no space or time derivatives of the Hubbard–Stratonovich fields \( \Delta(r, \tau) \), \( \Delta^*(r, \tau) \) in the action, we can take them to be constant. From the GP equation, \( |\Delta| \) is equal to the density of condensed
bosons and we take it to have the same value on both sides of the junction for simplicity. To compute the mean-field value of $\Delta$ from equation (4), we require that it extremizes the action: $\frac{\delta S}{\delta G} = 0$. This mean field is only present in $G_{\phi}^{-1}$ and in an additional term of quadratic order, and we use the Matsubara frequency and momentum representation of the free inverse Green’s function to find the extremum [14]. Evaluating the resulting Matsubara sum [18] yields a self-consistent equation for the mean-field $\Delta$ that is analogous to the BCS gap equation:

$$\Delta = \frac{1}{V} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\phi^2(k) - \mu + V_0 \Delta}.$$  

(8)

where $V$ is the volume of the system. In deriving this equation we have assumed that the effect of the gauge field and tunneling across the aperture contribute negligibly to the mean-field value of $\Delta$.

3.2. $G_{\phi}^{-1}$, Josephson–Anderson equation

In superconductors, the dynamical (ac) Josephson effect is expressed by the Josephson–Anderson equation for phase evolution and is dependent on a voltage across the tunnel junction [20]. Although the weakly-interacting Bose gas is not charged, that does not preclude introduction of a 0-component of the gauge field, the fluctuating local chemical potential $\phi(r, \tau)$, that appears in the action with the imaginary-time minimal coupling. The first order contribution of $\delta G^{-1} \equiv G^{-1}_{\phi}$ in equation (A.1) vanishes due to the periodic boundary conditions of $\theta(r, \tau)$ on $[0, \beta]$ and the requirement that the integral over both sides of the aperture (i.e. both reservoirs) is zero. However, a stationary phase equation with respect to $\phi(r, \tau)$ can be derived from the action by finding the extremum of the ‘electric field’ energy density term in equation (4), resulting in:

$$\text{Imaginary time: } i\partial_\tau \psi(r, \tau) = \nabla \phi(r, \tau),$$

$$\text{Real time: } \partial_t \psi(r, t) = \nabla \phi(r, t).$$  

(9)

The real-time version, obtained by a Wick rotation of the imaginary-time equation, is a classical Euler equation relating the acceleration of the driving velocity to a chemical potential difference across the aperture [21].

The second order term in equation (A.1) gives a nonvanishing contribution:

$$-\frac{1}{4} \int_{0}^{\beta} \, d\tau \, \mathcal{G}_{0}(0, \tau; 0, 0, \tau) \nabla \phi(0, \tau) \cdot \nabla \phi(0, \tau).$$  

(10)

where the tilde signifies a move to momentum space. $\mathcal{G}_{0}(0, \tau; 0, 0, \tau) \equiv n$ is the $(\tau$-independent) number of $k = 0$ bosons so it can be pulled out of the integral, resulting in the coefficient $-\frac{n^2}{4}$ (taking into account also the factor of $\frac{1}{4}$ multiplying the perturbation series). The imaginary-time Josephson–Anderson equation can then be derived at this order from $\frac{\delta S}{\delta \phi(r, \tau)} = 0$ together with the global phase and chemical potential configurations for each reservoir, $\theta(r, \tau) = \theta_{R/L}(\tau)$, $\phi(r, \tau) = \phi_{R/L}(\tau)$, yielding:

$$i\partial_\tau \phi_{L}(\tau) = -m \dot{\phi}_{L}(\tau)$$

$$i\partial_\tau \theta_{R}(\tau) = -m \dot{\phi}_{R}(\tau).$$  

(11)

Subtracting these equations gives the usual form of the Josephson–Anderson equation for evolution of the phase difference $\Delta \theta(\tau)$ across a junction:

$$i\partial_\tau \Delta \theta(\tau) = -m \Delta \phi(\tau).$$  

(12)

We can now use the mean-field Euler equation, equation (9), derived from the bare theory to write the Josephson–Anderson equation in gauge-invariant form. Noting that $\phi_{L}(\tau) = \phi_{R}(\tau) + i \int_{r_{L}}^{r_{R}} d\tau \cdot \partial_{\tau} v(r, \tau)$, we define the gauge invariant phase-difference by

$$\Delta \gamma(\tau) \equiv \theta_{L}(\tau) - \theta_{R}(\tau) + m \int_{r_{L}}^{r_{R}} d\tau \cdot v(r, \tau)$$  

(13)

and an analog of the electric field by

$$\xi = \int_{r_{L}}^{r_{R}} d\tau \cdot (-\nabla \phi(r, \tau) + i \partial_\tau v(r, \tau)).$$  

(14)

A rearrangement of equations (12)–(13) then yields the desired gauge-invariant form of the Josephson–Anderson equation as

$$\text{Imaginary time: } i\partial_\tau \Delta \gamma(\tau) = -m \xi(\tau)$$

$$\text{Real time: } \partial_t \Delta \gamma(t) = -m \xi(t).$$  

(15)

where the real-time version is obtained using the Wick rotation.

To assess the contribution to the effective action $S_{\text{eff}}$, equation (10) should be expressed in terms of $\Delta \gamma(\tau)$. $\phi_{R}(\tau)$ can be eliminated from the action using the Euler equation and $\phi_{L}(\tau)$ can be eliminated by a Gaussian integration (or vice versa, see section A.1). The result is a capacitive term in the effective action:

$$S_{C} = \int_{0}^{\beta} \, d\tau \, E_{C}(\partial_{\tau} \Delta \gamma(\tau))^{2}$$  

(16)

where the microscopic expression for $E_{C}$ (from the Gaussian integral) is $\frac{n^2 \xi^2}{8}$. Since in our analysis the Euler equation is considered a hard constraint, the electric energy density vanishes and $S_{C}$ is the only contribution from $G_{\phi}^{-1}$. Here and in other parts of the single aperture calculations, we neglect cross terms of the form $tr \log [G_{0} G_{-1} G_{0} G_{-2} \cdots]$ which are, however, necessary for generating interactions between apertures in the multiaperture case.

3.3. $G_{\psi}^{-1}$, circulation quantization, Hess–Fairbank equation

Using equation (A.1) to expand the contribution of $G_{\psi}^{-1}$ from equation (7) to first order results in a term quadratic in the gauge-invariant velocity field $v_{\psi}(r, \tau)$ (i.e., a massive term for $v_{\psi}$). In superconductors, the physical consequence of a massive vector field is the Meissner effect, a repulsion of magnetic fields from the interior of the superconductor up to a certain penetration depth which is dependent on the
superfluid density [22]. The analogous effect for $^4$He is the Hess–Fairbank effect, in which the superfluid mass density does not respond to rotation of the container due to an energy barrier to vorticity entering the superfluid [12]. The massive term for $v_g$ in the action suppresses fluctuations of the magnitude of the macroscopic phase gradient $\frac{1}{m} \nabla \theta$ from that of the driving velocity $v(r, \tau)$ in the bulk of the system. A stationary phase analysis of the action with respect to $v_g(r, \tau)$ at this order (see section A.2) yields:

$$\frac{n}{2} v_g(r, \tau) - L^2 \nabla^2 v_g(r, \tau) = 0, \quad (17)$$

which is a London equation describing the decay of the gauge-invariant velocity $v_g(r, \tau)$ in the interior of the bosonic system, with penetration depth $\lambda = \sqrt{\frac{2m}{\hbar^2}}$.

Because we have included an external velocity field, it is useful to explore the consequences of this on circulation quantization. In the absence of driving ($v(r, \tau) = 0$), the circulation integral is quantized in values of the circulation flux $\Phi_0 = \frac{2\pi}{m}$, due to the single-valuedness of the phase: $\oint \nabla \theta(r, \tau) = \frac{2\pi\ell}{m}, \ell \in \mathbb{Z}$. It seems clear that some form of the quantization should carry over to the driven case. To this end, we will integrate the London equation over a properly chosen contour. By analogy with Ampère’s law, we interpret a properly chosen contour. To this end, we will integrate the London equation over some form of the quantization should carry over to the driven case.

$$\oint_C \Phi \cdot \nabla \theta(r, \tau) = 2\pi l - \Delta \theta(r, \tau), l \in \mathbb{Z}$$

where $\Delta \theta$ is the line integral of the phase over $C$ (local phase difference across the aperture). The line integral of the driving velocity gives the external circulation $\Phi$

$$\int_C \Phi \cdot v(r, \tau) + \int_C v(r, \tau) \cdot dr = \int_A \omega \cdot dS = \Phi. \quad (19)$$

Combining these two equations gives the generalized circulation quantization condition:

$$\frac{1}{\Phi_0} \left( \Phi + \frac{2L^2}{n} \int_C \Phi \cdot j(r, \tau) \right) = \frac{\Delta \gamma(\tau)}{2\pi} - \ell \quad (20)$$

with $\ell \in \mathbb{Z}$ and $\Delta \gamma(\tau)$ the gauge-invariant phase difference in equation (13). $\Phi_0$ is the circulation quantum, $\Phi_0 = \frac{2\pi}{m}$. At distances into the bulk superfluid greater than the penetration depth, $j(r, \tau) = 0$ so that the above equation reduces to an equation for the quantization of the circulation due to the driving velocity. The general form of equation (20) expresses quantization of the superfluid ‘fluid’ [23] which contains contributions from vorticity due to the driving current in addition to the superfluid circulation.

It remains to determine the contribution of $v_g$ to the effective action for the gauge-invariant phase difference. Inclusion of the vorticity (circulation energy density) in the bare action equation (4) and requiring that the London equation hold results in the cancellation of the first order contribution of $G_{v_g}^{-1}$ by the circulation energy density (see section A.2 for derivation). It should be mentioned that in deriving this cancellation, we ignore a topological surface term $\int_S v_g \wedge \omega$. In fact, had we included in the Higgs action source terms for $v_g(r, \tau)$ and $\omega(r, \tau)$, parametrized the vortex current by an appropriate gauge field, and integrated out $v_g(r, \tau)$, the effective theory for the phase texture and vortex gauge field would be a $BF$ topological field theory [24, 25].

In this work, we do not consider explicitly the dynamics of vortices (but see discussion in section 6).

The second order contribution of $G_{v_g}^{-1}$ gives a non-vanishing contribution to the effective action for $\Delta \gamma$ and simplifies to:

$$- \frac{m^2}{16} \int d^3k d^3k' n_k n_{k'} \int d^3q d^3\xi \frac{\tilde{v}_g(q, \tau) \tilde{v}_g(q', \tau)}{(2\pi)^3}$$

$$\times (k - k' - q, \tau) \tilde{v}_g(\xi, \tau) \tilde{v}_g(k' - k - \xi, \tau). \quad (21)$$

This expression is a convolution in momentum variables, but can be approximated as local in momentum because $n_k$ is exponentially suppressed for $k \neq 0$ at low temperatures. Because the quadratic term in $v_g$ gives rise to a linear term in $\Delta \gamma$ (equation (A.3)) the term quartic in $v_g$ results in a quadratic term for $\Delta \gamma (\tau)$:

$$S_Q = -E_Q \left( \ell - \frac{\Delta \gamma(\tau)}{2\pi} \right)^2 \quad (22)$$

where $E_Q = m^2 n^2 ||\Phi||^2 L^2 \Phi_0^2/16$.

We note here that in the present analysis of driven bosonic flow through an aperture, $G_{v_g}^{-1}$ is strictly second order in the gauge-invariant velocity $v_g$, while the corresponding perturbative contribution for superconducting current flow through a Josephson junction also contains a term linear in $v_g$ [9, 19]. The consequence is that our second order expansion in $G_{v_g}^{-1}$ is quartic in $v_g$. This difference is a result of the polar decomposition of the bosonic fields made here into real components, in contrast to the superconducting case in which one must work with Nambu spinors. Expanding the square in equation (22) shows that the effective action has both quadratic and linear dependence on $\Delta \gamma$. The latter will result in a quantized constant term (a quantized current-bias)
in the current–phase equation while the former will give a term proportional to $\Delta \gamma$ (see section 4).

3.4. $G_T^{-1}$, periodic potential

We require that the tunneling matrix $T_{rr'} = 0$ when $r$ and $r'$ are on the same side of the aperture and, for simplicity, $T_{rr'} = T = \text{const.}$ when $r$ and $r'$ are on opposite sides of the aperture. In the perturbation expansion, we must integrate over all possible positions which give nonzero tunneling matrix elements. The resulting term in the effective action is:

\[ S_I = Tn \int_{0}^{\beta} d\tau \cos \Delta \gamma(\tau) \]  (23)

where $n$ is the zero-momentum occupation. If the perturbation expansion is continued and the imaginary-time integrations are approximated by a single one, higher harmonics of the $\cos(\Delta \gamma(\tau))$ interaction result; we will not include these in our analysis. These interactions can be shown to be of less relevance than the leading interaction (decreasing faster as the high-energy cutoff is lowered) by background field RG methods [26]. However, if one keeps imaginary-time arguments distinct (i.e., preserves time nonlocality) in the second order contribution, the second order term may be included as a dissipative contribution to the effective action (see section A.3). AES use an analogous term of this order to model the effect of quasiparticle-macroscopic phase difference scattering on the current in the Josephson junction.

4. Effective action and current–phase relations

4.1. Effective action

The effective action for the gauge-invariant phase difference is determined from equations (16), (22) and (23) to be $S_{\text{eff}} = S_C + S_Q + S_I$. Explicitly:

\[
S_{\text{eff}}[\Delta \gamma(\tau); l, \beta] = \int_{0}^{\beta} d\tau E_C(\partial_{\tau} \Delta \gamma(\tau))^2 - E_Q \left( \ell - \frac{\Delta \gamma(\tau)}{2\pi} \right)^2 + E_i \cos \Delta \gamma(\tau) \]  (24)

where the microscopic expressions for the coefficients have been derived above: $E_C = \frac{m^2 V^2}{8}$, $E_Q = m^2 n^2 \|v\|^2 L^4/16$, $E_i = Tn$. This effective action describes a particle on a ring with a potential that is a sum of a parabolic and cosine terms, i.e.,

\[
V[\Delta \gamma] = -E_Q(\ell - \Delta \gamma(\tau)/2\pi)^2 + E_i \cos \Delta \gamma(\tau) \]  (25)

(see figure 2). In the partition function involving $S_{\text{eff}}$, the sum over $\ell \in \mathbb{Z}$ counts the winding number of the macroscopic phase, as required by the fluxoid quantization condition equation (20). Changes in $\ell$ correspond to phase slips across the aperture, but we do not consider dynamic transitions $\ell \mapsto \ell \pm 1$ in this work; calculating the amplitude for these events involves an estimate for the tunneling rate for $\Delta \gamma$ between different winding sectors. The behavior of the effective potential for $\ell = 0$ and a range of relative values of the parameters $E_i$, $E_Q$ is shown in figure 2. The generalized circulation quantum condition, equation (20), may be used to further write the effective action solely in terms of the circulation $\Phi$. The Hamiltonian corresponding to this action is formally similar to that used to describe rf SQUIDs and superconducting flux qubits [19, 27, 28] and has been used previously to analyze coherent quantum phase slips [29].

The quadratic contribution of $\Delta \gamma$ in the effective potential differentiates this potential from the sinusoidal-plus-linear or ‘washboard’ form of effective potential found for a current-biased Josephson junction [27]. The effective action derived here for driven bosonic flow through an aperture differs from that derived by AES for superconducting flow through a Josephson junction in two respects. First, for the driven bosonic flow, the gauge field contribution $G_T^{-1}$ to the effective action at second order in perturbation theory is quartic in $\gamma$, resulting in a term quadratic in $\Delta \gamma$ and hence a parabolic contribution to the potential. In contrast, the contribution from the superconducting superfluid velocity to the effective action for a superconducting tunnel junction is linear in the phase difference variable (see equation (31) in [9]) and second order terms arise only from the additional inductive energy. Second, we have neglected the second order tunneling perturbation which is nonlocal in time: inclusion of this would, by analogy with the analysis of AES, give rise to dissipation in the aperture array (see section A.3).

In section 5 below we will analyze the temperature dependence of $E_i$. Because the temperature dependence will enter through the ratio of $E_Q$ to $E_C$ in equation (24), we now show that the latter ratio can be written in terms of the ratio of two characteristic lengths of the system. According to the

\[
\begin{align*}
V(\Delta \gamma) & = -E_Q(\ell - \Delta \gamma(\tau)/2\pi)^2 + E_i \cos \Delta \gamma(\tau) \\
\Delta \gamma & = -E_Q(\ell - \Delta \gamma(\tau)/2\pi)^2 + E_i \cos \Delta \gamma(\tau)
\end{align*}
\]
conjugate to \( 1\gamma(\tau) \), the ‘density difference’ field \( \Delta n(\tau) = n - \langle n \rangle \). This equation is derived in convenient form by first defining the current–phase equation to relate our predicted linear to sinusoidal transition in the strongly interacting \( ^4\text{He} \) system. Although observation of Josephson flow through aperture arrays under external driving has not yet been experimentally observed for dilute atomic BECs, both Josephson oscillations \([30]\) and Persistent flow \([31]\) have been observed for these systems in different geometries (double well and toroidal traps, respectively) so we expect our present approach to be of immediate use in analyzing these dilute systems.

4.2.1. Current–phase equation for single aperture. The current–phase equation resulting from the effective action \( S_{\text{eff}} \) is obtained as the stationary phase equation \( \delta S_{\text{eff}}/\delta \Delta \gamma(\tau) = 0 \). This equation is derived in convenient form by first defining the ‘density difference’ field \( \Delta n(\tau) \) that is canonically conjugate to \( \Delta \gamma(\tau) \), by Legendre transformation of the Lagrangian in the path integral. Specifically, the kinetic term of equation (24) is changed via:

\[
e^{-\int_0^\tau d\tau E_C(\partial_\tau \Delta \gamma(\tau))^2} \propto \int \mathcal{D}[\Delta n(\tau)] \times e^{-\int_0^\tau d\tau \frac{1}{2E_C} \Delta n(\tau)^2 + i\int_0^\tau d\tau \Delta n(\tau) \partial_\tau \Delta \gamma(\tau)}.
\]

Performing a stationary phase analysis with respect to \( \Delta \gamma(\tau) \) on the resulting Legendre transformed equation (24) then yields the general current–phase equations

\[
\text{Imaginary time:} \quad \imath \partial_\tau \Delta n(t) = E_J \sin(\Delta \gamma(t)) \\Rightarrow \quad \frac{E_Q}{2\pi} \Delta \gamma(t) + \frac{E_Q \ell}{\pi} = 0
\]

\[
\text{Real time:} \quad \partial_\tau \Delta n(t) = E_J \sin(\Delta \gamma(t)) \\Rightarrow \quad \frac{E_Q}{2\pi} \Delta \gamma(t) + \frac{E_Q \ell}{\pi} = 0
\]

where the real-time version is obtained in the same way as the real-time versions of the Euler equation and gauge-invariant Josephson–Anderson equation \((3.2)\). The real-time current is defined by \( I(t) = d\Delta n(t)/dt \). The term linear in \( \Delta \gamma \) confirms that this current–phase relation constitutes an analog for weakly-interacting condensed bosons of the generalized Josephson equation for an \( RF \) SQUID. The current-bias part of the current–phase relation is constant and quantized, proportional to \( \ell \in \mathbb{Z} \). From the form of the effective action, it is clear that \( \ell \) counts the number of phase slips in the system \((\text{i.e. the difference between the number of circulation quanta in the system and the gauge-invariant phase difference})\). If \( \Delta \gamma(0) \in [0, 2\pi) \) then a larger \( |\ell| \) corresponds to a greater absolute number of circulation quanta in the system and hence a greater magnitude of current. We emphasize that equation (29) contains all terms necessary to describe a linear to sinusoidal current–phase transition.

The different forms of the current–phase relation in different physical regimes correspond to specific values of the parameters \( E_Q \) and \( E_J \). For \( E_J = 0 \), the current–phase relationship of equation (29) is linear and corresponds to the small amplitude oscillations of a pendulum \([32, 33]\). However, the effect of an \( \ell \)-dependent current-bias persists. For \( E_Q = 0 \), the real-time version of this equation reduces to the usual Josephson equation \( \partial_\tau \Delta n(t) = E_J \sin(\Delta \gamma(t)) \), with critical number current equal to \( E_J = Tn \).

The classical equation for \( \Delta \gamma \) can be determined (in imaginary time) by requiring the exponent of equation (28) to be stationary with respect to variations in \( \Delta n \). This results in the relation \( \partial_\tau \Delta \gamma = \frac{1}{m\dot{q}} \Delta n \), analogous to \( m\dot{q} = p \) in classical mechanics. Substituting this relation into equation (29), it is then evident that for \( E_Q \ll E_C \) the quantized current-bias and the coefficient of the linear term are negligible; it is in this regime that purely sinusoidal oscillations should be observed.

In this limit, one recovers the imaginary-time version of the classical (fixed length) pendulum equation with amplitude \( E_J/2E_C \), i.e.,

\[
\partial_\tau^2 \Delta \gamma = \frac{E_J}{2E_C} \sin(\Delta \gamma(t)).
\]
Transformation to real-time results in the well-known classical pendulum analogue of the Josephson effect [33].

More generally, the current–phase relation, equation (29), interpolates between two regimes of purely linear and sinusoidal current–phase equations at \( E_I = 0 \) and \( E_Q = 0 \), respectively (plotted in real time in figure 3). These two limiting current–phase behaviors were observed for different temperature intervals in the \(^4\)He nanoaperture array experiments of [1]. In section 5 we derive the temperature dependence of the current–phase relationship.

If we set \( \lambda_L = \lambda_{ap} \) in equation (27), with \( \lambda_{ap} \) the diameter of a single aperture (see section 5 for justification), we may relate the ratio \( E_Q/E_C \) to the ratio \( \lambda_{ap}/2\xi(T) \) of the characteristic aperture size to the temperature-dependent healing length. As the temperature is decreased, the healing length becomes smaller than the aperture size and the ratio \( E_Q/E_C \) grows quartically. Thus if \( E_I \) is considered fixed, the linear term in the current–phase equation equation (29) becomes dominant for low \( T \). In contrast, at higher temperatures, \( \xi(T) \) increases and the ratio becomes larger than the aperture size, the sinusoidal term would become dominant. Whether thermal fluctuations of the gauge-invariant phase difference wash out the sinusoidal part of the current–phase relation as \( T_h \) is approached from below depends on the size of \( E_I \), the scaling of which is derived in terms of \( E_C \) and \( E_Q \) in section 5.

This qualitative analysis shows that as the temperature is increased towards \( T_h \), there can be a transition from a linear current–phase relation at low temperatures to a sinusoidal current–phase relation at higher temperatures (but still below \( T_h \)).

4.2.2. Current–phase relation for two-aperture array. Within the framework of this theoretical analysis, adding an additional aperture is straightforward and results in a substantially richer set of current–phase phenomena. We analyze here just the two-aperture case, leaving the extension to arrays with large numbers of apertures for future investigation. We may assume the cross-sectional areas of the two apertures are identical. There are now two tunneling matrices \( T^{(2)}_{rr'} \); we require that \( T^{(1)}_{rr'} \) is nonzero only when \( r \) and \( r' \) are on opposite sides of aperture 1 and both are in a small vicinity of the aperture (similarly for \( T^{(2)}_{rr'} \)). In addition to the sum of single aperture effective actions for the gauge-invariant phase differences \( \Delta y^{(1)}(\tau) \) and \( \Delta y^{(2)}(\tau) \), which have been derived in section 3.4, there is now also a tunneling cross-term that appears at second order in the perturbation theory. This tunneling cross-term generates an effective aperture interaction that may be expressed in terms of the microscopic phase differences across the individual apertures. In particular, with the tunneling amplitudes assumed to be the same, this term adds an interaction to the effective action for two apertures of the form

\[
S_{int} = -E_I^2 \cos(\Delta y^{(1)}(\tau)) \cos(\Delta y^{(2)}(\tau)).
\] (31)

For small phase differences, expansion of this equation implies that the homogeneous part of degree 2 renormalizes the quadratic parts of the uncoupled contributions to the action and introduces a coupling \( \Delta y^{(1)}(\tau) \Delta y^{(2)}(\tau) \), while the homogeneous part of degree 4 introduces a coupling \( \Delta y^{(1)}(\tau)^2 \Delta y^{(2)}(\tau)^2 \) as well as quartic local potentials for the phase differences. Neglecting these higher order terms, the interaction results in coupled modified Josephson equations which describe classical coupled pendula.

We can use the two-aperture coupling term equation (31) to rationalize the experimentally observed transition from a linear to sinusoidal current–phase relation in a multiaperture array. Because the coefficient of the interaction just derived is the square of \( E_I \) we know that if \( E_I \) is large compared to \( E_Q \), the current–phase relation for each individual aperture is approximately sinusoidal and that the energy cost for having an inter-aperture phase difference of \( \pi \) is \( 2E_I^2 \). This means that for \( E_I \neq 0 \), it is favorable for the difference of the phase differences to be \( \pi \). Hence the amplitude of the oscillation coming from the independent terms is doubled. This is consistent with both the experimental observations of phase difference synchronization as the current–phase relation becomes sinusoidal, i.e. Josephson-like, as well as with the observed linear scaling of the Josephson oscillation amplitude with number of apertures [1].

5. Renormalization group analysis for small \( E_I \) To make contact with experiment and to justify the qualitative argument presented in section 4.2.1, it is desirable to understand how the current–phase relationship of the effective theory, equation (29), and in particular the critical current \( E_I \), depends on temperature. This can be done by employing RG methods in the small \( E_I \) regime and
analyzing the corresponding beta function\(^1\). The sign of this function determines how the coupling constant \(E_1\) behaves (i.e., decreases or increases) at low energies/long length scales.

Full details of the RG calculations are included in appendix B. Here we summarize only the key features of this calculation and the results that are relevant to understanding the temperature dependence of the current–phase relation presented in section 4.2 above. We note that in order for the system to be described by the phase-difference only, we must implicitly assume a high-energy cutoff \(\Lambda\), beyond which energy scale the effective theory is invalid. At the energy scale determined by \(b = \frac{\Lambda}{2}\), with \(\lambda\) a lower energy scale (i.e., \(b \in [1, \infty)\)), \(E_1(b)\) is the critical current of the current–phase relation and its magnitude relative to \(E_Q\) will determine the Josephson character of the current–phase relation.

Since we are concerned here with the scaling of \(E_1(b)\), we neglect the scaling of \(E_C\) and \(E_Q\). If \(E_1(b)\) decreases (increases) as we consider low-energy scales, we infer that the low-energy current–phase relation equation (29) does not contain (does contain) a sinusoidal term. The resulting beta function is then given by

\[
\beta(E_1) \equiv \left. \frac{d\beta}{db} \right|_{b=1} = \left( 1 + \frac{2\pi \Lambda}{4\pi^2 E_C^2 - E_Q} \right) E_1. \tag{32}
\]

Integrating this differential equation by separating variables and transforming to dimensionless parameters (using the naive scaling dimension of each) \(E_Q = E_Q' = E_C = E_C\Lambda\), yields the following scaling field for \(E_1\):

\[
E_1(b) \propto E_1(b^1+g), \quad g = \frac{1}{2\pi(E_1'^2 - E_Q')} \tag{33}
\]

We have confirmed the validity of this scaling field with a background RG calculation, similar to the analysis of the noncompact clock model in \[26\]. When the exponent \(1 + g\) is negative, \(E_1(b)\) will be irrelevant and disappear at low energies, while when the exponent is positive \(E_1(b)\) is relevant and grows at low energies. Figure 4 shows the resulting RG flow diagram for \(E_1\) in the positive \((E_Q', E_C')\) quadrant.

There are two important features in this RG diagram for \(E_1\). First, the singular line defined by \(E_Q' = 4\pi^2 E_C'\) (where the denominator of \(g\) goes to zero) and second, the marginal line at \(E_Q' = 2\pi(2\pi E_C' + 1)\) (where \(1 + g = 0\)). We can analyze the singular line in terms of the ratio \(E_Q'/E_C'\) considered at the beginning of section 4 (see equation (27)). In order to evaluate this ratio as a function of the renormalization scaling \(b\), we must choose a value for the high-energy cutoff, \(\Lambda\). In the low temperature helium nanoperture array experiments of [1], the largest energy scale is the kinetic energy of the driving velocity. We therefore employ an energy cutoff value \(\Lambda = \frac{\hbar v}{2\xi(T)}\). Returning to equation (27), we see that the condition for the singularity will then occur at a temperature \(T_1\) such that

\[
\frac{E_Q'}{E_C'} = \left( \frac{\lambda_L}{2\xi(T_1)} \right)^4 = 4\pi^2. \tag{34}
\]

\(^1\) This result can also be obtained in a background field RG calculation by considering fluctuations around a low-energy \(\Delta\gamma\) configuration.
and sinusoidal contributions as a function of temperature within this regime. At low temperature when the number of condensed bosons is large, or whenever the tunneling amplitude is very large or very small, the $E_3 \cos \Delta \gamma$ part of the action can be treated using the Villain approximation [37] which would renormalize $S_0$ and lead to a purely linear current/phase equation. In support of this argument is the fact that for $E_C' > 2\pi$ and constant, the value of $E_C'$ is lower in region I than in region III. A low value of $E_C'$ implies a high energetic cost for density difference fluctuations (see equation (28)). Since the density difference is canonically conjugate to the gauge-invariant phase difference, we expect that a low variance in the value of the former quantity allows for a high variance in $\Delta \gamma$ and hence for the Josephson flow contribution to the current–phase relation to be washed out.

At temperatures above $T_1$, the periodic potential is relevant (region III). The current–phase relationship, equation (29) will always have a nonvanishing contribution from sinusoidal flow in this regime (while the system remains below $T_2$, although it may be mixed with linear flow. For $E_Q$ small and $E_C'$ large, nearly pure Josephson oscillations should be observed.

To support the validity of this analysis of the small $E_3$ current–phase relation, we place two results from the experiments of [1] that exhibit different current–phase behaviors into the context of the RG diagram, figure 4. For example, at $T_0 = T = 27$ mK a linear current–phase relation is observed. Employing the experimental formula for the healing length [36] and an aperture width $\lambda_{ap} = 40$ nm, yields the ray $E_Q/E_C' \approx 94$ for this temperature. Since the experimental current–phase relation has linear character at this temperature, we expect that this point lies in region II below the $E_Q = 2\pi(2\pi E_C + 1)$ line. The second point we analyze is $T_1 - T = 0.8$ mK. Here the experiment shows nearly pure Josephson oscillations and experimental estimates for healing length and aperture width yield the ray $E_Q/E_C' \approx 8.0 \times 10^{-3}$. Consequently this higher temperature point lies in region III, far below the $E_Q = E_C'$ line and in a region where $E_Q$ is negligible.

It is interesting to consider what the present RG calculation predicts for the effect of phase slips on the transition from linear to sinusoidal current–phase relation. While we do not explicitly study the dynamics of phase slips, we can nevertheless provide a qualitative analysis of their role since the integer $\ell$ characterizing the phase slips appears in the current–phase relationship equation (29) as a background current whose magnitude is also proportional to $E_Q$. The RG analysis above shows that the $E_Q$ terms in the current–phase relation are most important at low $T$ and we therefore expect that the $\ell \rightarrow \ell \pm 1$ phase slips are most important in this ‘strong coupling’ regime where Josephson flow is irrelevant. This is consistent with the observations and conclusions of the experiment in [1].

We emphasize that pure sinusoidal Josephson oscillations (without the modified dynamics due to parabolic potential) should be found in region III of figure 4) only. This is a regime of considerable interest for applications of Josephson phenomena in liquid $^4$He to metrology [38] and for development of circulation analogues of superconducting flux qubits [39, 40]. The experimental challenge in accessing this regime lies in the fabrication of small enough nanoaperture arrays in order for the $E_C' = 4\pi^2 E_C$ line to be reached deep in the condensed phase and not near the critical point.

5.1. Single versus multiple apertures

The above analysis is made for a single aperture. Observing a Josephson current for a bosonic superfluid across a single driven nanoaperture is known to be a challenging task, due to the small amplitude of oscillation compared to the amplitude of oscillations of the driving device. Our analysis shows that if the healing length of an interacting Bose gas can be made over twice the characteristic aperture diameter, nearly pure Josephson oscillations would be observable even with a single aperture. Unfortunately, for driven liquid $^4$He in aperture arrays of $\lambda_{ap} \sim 40$ nm, which is the one bosonic superfluid for which such experiments have been performed to date, this regime is nearly precluded by the lambda transition. In superfluids with larger zero-temperature coherence lengths (e.g., the paired fermion superfluids, including type-II superconductors and in particular $^3$He, which has a zero-temperature coherence length on the order of 100 times that of $^4$He) robust Josephson oscillations across a single aperture with $\lambda_{ap} \sim 40$ nm can be observed deep in the superfluid phase [3].

For bosonic superfluids such as liquid $^4$He and trapped dilute Bose gases, it is of interest to consider what changes to the present analysis are required by having multiple apertures. If tunneling amplitudes at each aperture are the same and each aperture has the same size and shape, even the particularities of the weak $E_3$ coupling RG calculation should carry over. The most important change in going from one aperture to multiple apertures is the presence of the interaction between phase differences across spatially separated apertures and the independent tunneling terms as mentioned in section 4.2.2. If the phase-difference interaction favors a uniform value, the classical configurations will be phase-locked, independent tunneling terms will add up and the overall tunneling amplitude will be scaled by $M$, with $M$ the number of apertures in the array. Consequently, the amplitude of the Josephson oscillation is multiplied by $M$ and it is easier to observe. It should be noted that the presence of multiple apertures introduces new, higher-order operators in the effective action. In general, their anomalous scaling dimensions (and hence their operator relevance) are different from that of the $\cos(\Delta \gamma)$ potential. Finally, we note that for the paired fermion superfluid $^3$He, more complicated current–phase relations across an aperture array are observed due to the higher symmetry of the $^3$He order parameter [41].

6. Summary and conclusions

In this work we have derived and analyzed an effective theory of gauge-invariant phase differences for interacting bosons driven across simple aperture arrays, starting from a local $U(1)$ gauge theory. The stationary phase approximation to the local $U(1)$ gauge theory at first and second order
The expansion of the one-loop contribution to the action was shown to reproduce many well-known equations of motion, e.g., the Josephson–Anderson equation, the Euler equation, the London equation, the equation of superfluid fluxoid quantization, and the dc Josephson equation. We have shown that the general current–phase relationship is consistent with the phase dynamics in a potential formally analogous to that of a rf SQUID, consisting of quadratic, linear and sinusoidal terms whose relative strength is determined by the magnitudes of the charging and Josephson couplings, \( E_0 \) and \( E_I \), respectively. The effective action leading to this current–phase relationship differs from that derived by AES in the context of superconductive tunneling [9] due to the explicit presence of the parabolic potential in the phase-difference action, as well as to the locality in time assumed in our analysis. Analysis of dissipation in the aperture array deriving from the second order time nonlocal contribution of \( G_T^{-1} \) will be addressed in future work.

The effect of the sinusoidal term in the current–phase relation was further analyzed using finite temperature renormalization group methods, allowing us to make contact with the temperature-dependent transition from low-\( T \) linear to high-\( T \) sinusoidal current observed in the \( ^4\text{He} \) experiment of [1]. We have shown that the sinusoidal part of the current–phase relationship is expected to become significant in two different regimes, but that it is most important when the coherence length \( \xi(T) \) is larger than the characteristic size of the aperture, \( \lambda_{ap} \). By exploiting the relationship between \( E_0/E_C \) and the ratio of the aperture size to the temperature-dependent healing length within the Popov approximation for dilute BECs, we were able to examine the temperature scaling of \( E_I \) with respect to this ratio. This analysis identified regions II and III, separated by a singular line in the RG diagram, that are respectively consistent with the linear and sinusoidal current–phase relations that were observed experimentally for \(^4\text{He} \) in [1]. Using the empirical temperature-dependent scaling of the liquid \(^4\text{He} \) healing length and experimental aperture dimensions, we have shown that the singular line separating these regions, \( E_0 = 4\pi^2E_C \), occurs about 20 mK below the lambda transition for \(^4\text{He} \). The qualitative agreement of this value with the experimentally observed transition for \(^4\text{He} \) at \( \sim5 \) mK below \( T_\lambda \) in [1] provides evidence for the validity of this effective theory even beyond its expected limitation to weakly-interacting Bose condensed systems. We consider this wide applicability of the theory to be due to the fact that \( E_0/E_C \) depends only on the ratio of a length scale characterizing the aperture to the healing length, the scaling of which determines the universality class of the system. In addition, generalization of the effective action derived in this theory from one to two apertures shows that phase-difference coupling between multiple apertures leads to phase-difference synchronization and to a doubled amplitude of Josephson oscillation in the array. Our analysis indicates that for \( M \) parallel apertures in an array, we may expect the amplitude of Josephson oscillations to behave as \( \mathcal{O}(M) \).

In this paper we have considered neither the dynamics of phase slips and the vortices by which they are carried, nor the details of their role in the transition from linear to sinusoidal current–phase relationship (see section 3.3). Thus we cannot make direct contact with the detailed analysis of synchronicity that was made in [42, 43], although the renormalization group analysis does allow an identification of the temperature regime for which phase slips should play the most important role, namely at low temperatures and strong coupling, when Josephson flow is irrelevant. This is in agreement with the interpretation of the experiments on nanoaperture arrays, where the low-temperature linear current–phase characteristic was concluded to be due to independent nucleation and subsequent slippage of vortices at individual apertures [1]. These events dissipate the kinetic energy of the hydrodynamic resonator slowly, as opposed to the large scale coherent phase slips occurring at higher temperatures. In the high-temperature regime just below the lambda point, the diameters of vortex cores are nearly as large as the apertures themselves. In this regime, coherent phase slips across the aperture array could not be energy-conserving so a qualitatively different type of flow must occur. The coincidence of temperature-dependent changes in flow and temperature-dependent changes in vortex properties suggests a physical picture of vortex proliferation at the nanoaperture array leading to Josephson oscillations in the high-temperature regime. This picture is consistent with our requirements that (i) \( E_I\cos(\Delta \gamma) \) be relevant in order to observe Josephson oscillations (i.e. the temperature be lower than, but sufficiently close to \( T_\lambda \)), and (ii) \( E_Q \) be small so that the gauge-invariant phase difference is not pinned to an integer multiple of \( 2\pi \ell \) (i.e. quantized phase-slips not the only allowed mode of flow). The latter requirement is equivalent to destruction of off-diagonal long range order in the vortex cores filling the aperture allowing \( \Delta \gamma \) to fluctuate away from \( 2\pi \ell \).

We note that inclusion of the nonlocal interaction between \( \omega(r) \) and \( \omega(r') \) is expected to lead to the hydrodynamic equations for vortices first presented in [44]. In future work, the dynamics of superfluid vortices in an aperture array may be studied by deriving their effective theory using boson–vortex duality [45]. Such a study would be useful both to confirm in the dual picture the features of the phase diagram derived here, and to investigate the properties of a vortex condensate, for which the bosonic field operator used in this work no longer describes particles above the vacuum, in single apertures and aperture arrays.

In utilizing the current approach to interpret experiments on liquid \(^4\text{He} \), we have neglected the strongly-interacting nature of superfluid helium, i.e., we do not consider a realistic two-body potential. Realistic studies of Josephson effects in liquid helium driven through nanoscale aperture arrays may be undertaken with path integral Monte Carlo methods [46]. To our knowledge there has so far been no observation of Josephson oscillations between driven reservoirs of weakly-interacting condensed bosons separated by nanoaperture arrays, nor indeed of any Josephson effects under driving flow conditions for weakly-interacting Bose condensate systems. However, the Josephson effect has been observed for weakly coupled Bose–Einstein condensates [30, 47], and persistent currents have been observed in toroidally

\[ T \ J \ Volkoff \ and \ K \ B \ Whaley \]
trapped condensates [31]. Taken together with the recently demonstrated ability to make arbitrary potentials in such geometries [48], the rapid progress in experimental study and manipulation of rotating BECs in toroidal traps holds out the prospect of future realization of Josephson phenomena in confined atomic BECs.

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Appendix A. Perturbative expansion of $G^{-1}$

For convenience and clarity, the perturbative expansion of $G^{-1}$ is included in this appendix. We use the following perturbative series to analyze the action equation (4):

$$
\text{tr log} [G_0^{-1} + \delta G^{-1}] = \text{tr log} [G_0^{-1}] + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \text{tr} [(G_0 \delta G^{-1})^k]. \quad (A.1)
$$

The first term in this series is a constant which cancels due to the normalization of the partition function. The second term, a series in powers of $\delta G^{-1}$, gives important contributions to the effective action. The free Green’s function of the action is found by inverting the $G_0^{-1}$ operator [14]

$$
\tilde{G}_0(k, \tau; k', \tau') = (2\pi)^3 \delta(k-k') \exp[-\mathcal{E}_k(\tau - \tau')] \times \left( \Theta(\tau - \tau')(1+n_k) + \Theta(\tau' - \tau) n_k \right) \quad (A.2)
$$

where $\mathcal{E}_k = \frac{k^2}{2m} - \mu + V_0 \Delta$ and $n_k = \frac{1}{e^{2\mathcal{E}_k} - 1}$ is the Bose–Einstein distribution. When $\tau = \tau'$, the time-ordered correlation function is the normal ordered correlation function and so $\tilde{G}_0(k, \tau; k', \tau') = n_k \delta(k-k')$.

In evaluating the integrations over internal momenta, we frequently use the fact that $G_0(k, \tau; k', \tau') \propto \delta(k-k')$. Treating the perturbation series exactly results in nonlocal contributions to the action. We assume when needed that the imaginary time arguments of the higher order terms are the same, by appealing to the fact that the free Green’s function is exponentially suppressed as distance in imaginary time increases. Momentum integrals over the free Green’s function are restricted to $k = 0$ because we are considering the low-energy dynamics of the condensed mode.

A.1. $G^{-1}_g$ contribution

The $\text{tr log}$ expansion with respect to this perturbation is outlined in the text (section 3.2). To convert equation (10) to a functional of $\Delta \mathcal{Y}$ in the effective action, equation (10) is split (in position space) into left and right parts as

$$
-\frac{1}{4L^2} \int_0^\beta \text{d} \tau \left( \partial_\tau \theta_L + m \phi_L \right)^2 + (L \to R). \quad (A.3)
$$

where $\partial_\tau \theta_L = \partial_\tau \theta_R$.

The Euler equation, $\phi_R$ is eliminated from the action. We then perform the Gaussian integral over $\phi_R$ to arrive at an effective term involving $\Delta \mathcal{Y}(\tau)$. The Gaussian integral is:

$$
\left[ \int \mathcal{D}[\phi_R] \exp \left[ -\frac{1}{2} \int_0^\beta \text{d} \tau \ V_2 n^2 m^2 \phi_R^2 \\
- \frac{1}{2} \int_0^\beta \text{d} \tau \ \left( \partial_\tau \theta_L + \partial_\tau \theta_R \right) \\
+ m \int_{\tau_L}^{\tau_R} \text{d} \tau \cdot \partial_\tau \nu(r, \tau) \phi_R \right] \\
\propto \exp \left[ -\frac{n^2 V^2}{8} \int_0^\beta \text{d} \tau \ \left( \partial_\tau \theta_L + \partial_\tau \theta_R \right) \\
+ m \int_{\tau_L}^{\tau_R} \text{d} \tau \cdot \partial_\tau \nu(r, \tau) \right]^2 \right].
$$

The $\phi_R$-independent part of the contribution is added into the exponent and the square expanded, yielding $S_C$ after simplification. Elimination of the 0-component of the gauge field from the action is reasonable because it is not a dynamical field.

A.2. $G^{-1}_g$ contribution

Analyzing the contribution of the vorticity energy density and the first order term in $G^{-1}_g$ to the perturbation expansion in equation (A.1) shows that the gauge-invariant velocity satisfies a London equation. We will use this equation to show that the first order contribution of $G^{-1}_g$ to the effective action for $\Delta \mathcal{Y}$ is canceled by the vorticity energy density term. Specifically, the first order contribution to the action is

$$
\frac{1}{2} \int \mathcal{D}[\tilde{G}_0(k; \tau; k, \tau)] \cdot \frac{m}{2} \vec{v}_g(q, \tau) \vec{v}_g(-q, \tau)
$$

where $n$ is the number of condensed bosons. We have restricted the sum over momenta in the free Green’s function to $k = 0$ because contributions from $k \neq 0$ are exponentially suppressed at low temperatures.

To derive the London equation equation (17) we set $\frac{\delta S}{\delta \vec{v}_g} = 0$ at first order in the expansion in $G^{-1}$ and make use of the Euler equation, the identity $\nabla \times (\nabla \times \vec{v}_g(r, \tau)) = \nabla(\nabla \cdot \vec{v}_g(r, \tau)) - \nabla^2 \vec{v}_g(r, \tau)$, and the physical requirement that $\nu(r, \tau)$ be divergence-free. Owczarek has exploited a similar ‘Higgs’-type argument to rationalize the expulsion of circulation by a superfluid, noting that if the source of $\vec{v}_g(r, \tau)$ is a roton, the penetration depth is roughly the same as experimentally observed vortex core diameters [49].

In section 3.3, we stated that the first order contribution of $G^{-1}_g$ is canceled by the circulation energy density. This can be seen as follows: consider integrating the first order contribution along an integral curve of $\nu(r, \tau)$ as experimentally observed vortex core diameters [49].

In section 3.3, we stated that the first order contribution of $G^{-1}_g$ is canceled by the circulation energy density. This can be seen as follows: consider integrating the first order contribution along an integral curve of $\nu(r, \tau)$ as experimentally observed vortex core diameters [49].
where the latter integral is with respect to arc-length and $L^2$ is the area factor multiplying the vorticity energy density in the microscopic Lagrangian. This integral can be converted to a line integral by identifying the tangent vector to $\Gamma$ with the driving velocity at each point. This is justifiable because (i) $\Gamma$ is an integral curve of the superfluid velocity and it is physically reasonable to assume that for low $T$, $\nabla \theta(r, \tau)$ is parallel to $v(r, \tau)$ at each point in space–time, and also since (ii) $\|\nabla \theta(r, \tau)\| \ll \|v(r, \tau)\|$. For a constant magnitude driving velocity, the first order contribution becomes:

$$\frac{nmL^2}{4} \int_\Gamma d\tau v_g(r, \tau)^2 = \frac{nm\|v\|^2}{4} \int_\Gamma d\tau \cdot v_g(r, \tau)$$

$$= \frac{nm\|v\|^2}{4} \Phi_0 \left( \frac{\Delta \gamma(\tau)}{2\pi} \right). \quad \text{(A.5)}$$

This contribution is canceled by the circulation energy density, which can be rewritten:

$$\frac{mL^2}{2} \int d^3r v_g(r, \tau) v_g(r, \tau). \quad \text{(A.6)}$$

The vector identity $a \cdot (\nabla \times b) = b \cdot (\nabla \times a) - \nabla \cdot (a \times b)$ has been used in deriving this formula. Substituting into equation (A.6) the London equation in the form $j(r, \tau) = \frac{\pi n}{2\hbar} v_g(r, \tau)$, one obtains the perturbation contribution in equation (A.5) but multiplied by a factor of $-1$. In using the vector identity above, we have neglected a topological contribution to the effective action. This is discussed in section 3.3.

### A.3. $G_T^{-1}$ contribution

In this calculation, as in previous ones, we specialize to the left/right reservoir phase configuration $\theta(r, \tau) = \theta_{\text{LR}}(r, \tau)$. In the multiaperture case, these become local left/right macroscopic phases in the vicinity of each aperture. Employing the convention that the left-to-right gauge-invariant phase difference is defined to be $-\Delta \gamma(\tau)$, the first order contribution is

$$\frac{1}{4} [G_0(r_L, \tau; r_R, \tau) T_{rgG} e^{-i\Delta \gamma(\tau)}] + \frac{1}{2} [G_0(r'_L, \tau; r'_R, \tau) T_{rgG} e^{i\Delta \gamma(\tau)}].$$

The corresponding first order contribution $S_{1}$ to the effective action results from using the fact that $\int d^3r d^3r' G_0(r, \tau; r', \tau) = G_0(k = k' = 0, \tau)$.

In our analysis of the effective theory for the gauge-invariant phase difference, nonlocal imaginary time terms in the perturbation expansion have been neglected. Here we derive one of these nonlocal terms arising from the second order contribution of $G_T^{-1}$; diagrams corresponding to this contribution are shown in (figure A.1).

The resulting contribution is:

$$\exp[-\frac{1}{4} [tr\{G_0(r_L, \tau; r_R, \tau) T_{rgG} e^{-i\Delta \gamma(\tau)}]$$

$$\times G_0(r'_L, \tau'; r'_R, \tau) T_{rgG} e^{i\Delta \gamma(\tau')} + (R \leftrightarrow L)]$$

$$+ tr\{G_0(r_L, \tau; r'_R, \tau') T_{rgG} e^{i\Delta \gamma(\tau')}$$

$$\times G_0(r'_L, \tau; r_R, \tau) T_{rgG} e^{-i\Delta \gamma(\tau)} + (R \leftrightarrow L)]]. \quad \text{(A.7)}$$

Transforming to momentum space and taking the tunneling amplitude to be a constant, $T$, yields:

$$-T^2 \int_0^\beta d\tau d\tau' \int d^3k d^3k' e^{i\frac{1}{2}(k-k')^2(\tau-\tau') + i\omega_0 k} \times \cos(\Delta \gamma(\tau)) \cos(\Delta \gamma(\tau')). \quad \text{(A.8)}$$

Our expression for the nonlocal contribution for this driven bosonic flow differs from that of AES because we do not have a particle-hole symmetry.

### Appendix B. RG for periodic potential

We start by expressing the effective action equation (24) in terms of Matsubara frequencies (we use the $S_{\text{eff}}$ label for both the imaginary time and Matsubara representations of the action):

$$S_{\text{eff}}[\tilde{\Delta} \gamma; l, \beta] = 2 \sum_{n=0}^{\infty} \left( \frac{E_C \omega_n^2}{4\pi^2} \right) \tilde{\Delta} \gamma(\omega_n) \tilde{\Delta} \gamma(\omega_{-n})$$

$$+ \int_0^\beta d\tau E_3 \cos \Delta \gamma(\tau) + \frac{\beta E_C \ell}{2\pi^2} \Delta \gamma(0). \quad \text{(B.1)}$$

Choosing a high-energy cutoff $\Lambda$, $\Delta \gamma(\tau)$ can then be split into low-frequency (slow, $s$) and high-frequency ($f$, $f'$) terms, $\Delta \gamma(\tau) = \int_{|\omega| > \beta} \frac{d\omega}{2\pi} e^{-i\omega \tau} \Delta \gamma(\omega)$. The renormalization scaling. The effective action is then split into slow ($s$), fast ($f$) and combination ($U$) components:

$$S_{\text{eff}}[\tilde{\Delta} \gamma; l, \beta] = S_l[\tilde{\Delta} \gamma] + S_U[\tilde{\Delta} \gamma]$$

$$+ S_f[\Delta \gamma_s + \Delta \gamma_f]. \quad \text{(B.2)}$$

We note that the slow part gets an additional contribution from the zero mode in equation (B.1). Assuming that $T \ll \Lambda$, so

![Figure A.1. Diagrammatic representation of the two second order contributions from $G_T^{-1}$ in equation (A.7) which result in an imaginary time nonlocal contribution to the effective action. The convention $-\Delta \gamma$ is used for left-to-right hopping. We omit the delta function vertices as they are omitted in equation (A.7).](image-url)
that the Matsubara sums become integrals, it is then possible to integrate over the fast components by making use of a small $E_{J}$ approximation [18]
\[
e^{-\text{low energy} \{\Delta \gamma \}} = e^{-S_{\gamma} \{\Delta \gamma \}} e^{-\sum_{f} S_{\gamma} \{\Delta \gamma_{f}, \Delta \gamma \} + \ldots} \approx e^{-\sum_{f} S_{\gamma} \{\Delta \gamma_{f}, \Delta \gamma \}} e^{-S_{\gamma} \{\Delta \gamma_{f}, \Delta \gamma \}}.
\]
to obtain an effective low-energy action $S_{\text{low energy}} \{\Delta \gamma \}$. Here the $f$ subscript denotes an expectation value using $S_{\gamma}$ as the action.

An explicit evaluation of $\{S_{\gamma} \{\Delta \gamma_{f}, \Delta \gamma \} \}_{f}$ results in a $b$-dependent multiplicative renormalization of $E_{J}$, which we call $E_{J}(b)$. This integration over fast modes is given explicitly by:
\[
\{S_{\gamma} \{\Delta \gamma_{f}, \Delta \gamma \} \}_{f} = E_{J} \int_{\beta}^{1} d\epsilon \ e^{-2\int_{0}^{\beta} \frac{d\omega}{\pi} \left( \epsilon \cos^{2} \frac{\omega}{\pi} \right) \Delta \gamma^{2} / Q^{2}} \left( \epsilon \cos^{2} \frac{\omega}{\pi} \right) \Delta \gamma^{2} / Q^{2} \left( \epsilon \cos^{2} \frac{\omega}{\pi} \right) \left( \epsilon \cos^{2} \frac{\omega}{\pi} \right).
\]

Carrying out the Gaussian integration (and neglecting the divergent contributions) gives
\[
\{S_{\gamma} \}_{f} = E_{J} \int_{0}^{\beta} d\epsilon \ e^{-\frac{1}{4} \int_{0}^{\beta} d\omega \ \frac{d\omega}{\pi} \left( \epsilon \cos^{2} \frac{\omega}{\pi} \right) \cos(\Delta \gamma_{f}(\tau))}.
\]

Rescaling $\tau \rightarrow \frac{\tau}{\epsilon}$ to ensure that the Matsubara frequency still lies within the positive interval $[0, \infty)$ results in (equation (32)) in the main text and we see that the periodic potential is multiplicatively renormalized. We have verified our result for the $\beta$-function using a background field RG analysis according to the procedure outlined in [26].

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