ABSTRACT THEORY OF POINTWISE DECAY WITH APPLICATIONS TO
WAVE AND SCHRÖDINGER EQUATIONS

VLADIMIR GEORGESCU\textsuperscript{1}, MANUEL LARENAS\textsuperscript{2}, AND AVY SOFFER\textsuperscript{2}

ABSTRACT. We prove pointwise in time decay estimates via an abstract conjugate operator method. This is then applied to a large class of dispersive equations.

CONTENTS

1. Introduction 2
2. Evanescent states 2
3. Notes on commutators 4
4. Commutators and decay 8
5. Higher order commutators 11
6. Higher order decay 13
7. Applications 16
7.1. Example 1: Laplacian in $\mathbb{R}^n$. 16
7.2. Example 2: $H = -\partial_{xx} + \partial_{yy}$ in $\mathbb{R}^2$.
7.3. Example 3: Electric field in $\mathbb{R}^n$.
7.4. Example 4: $H = -x^{2-\theta} \Delta - \Delta x^{2-\theta}$ in $\mathbb{R}_+$. 16
7.5. Example 5: Fractional laplacian in $\mathbb{R}^n$. 17
7.6. Example 6: Multiplication by $\lambda$ in $L^2(\mathbb{R}_+, d\mu)$. 17
7.7. Example 7: Wave equation in $\mathbb{R}^n$.
7.8. Example 8: Klein-Gordon equation in $\mathbb{R}^n$. 18
8. Appendix 18
References 19

\textsuperscript{1} The second author was partially supported by NSF DMS-1201394.
1. Introduction

In the study of dispersive equations, linear or nonlinear, one is faced with the need to quantitatively estimate the decay rate of the solution in various norms. The known estimates which play a central role in the theory of dispersive equations include local decay estimates, pointwise decay estimates in time, $L^p$ decay estimates and Strichartz estimates. More intricate are microlocal estimates and propagation estimates. The pointwise decay estimates for Schrödinger type operators were proven first in three dimensions and were obtained for short range potentials [JK]. This was later extended by various authors, and unified to arbitrary dimension in [JN1]. These estimates play an important role in proving the more general $L^p$ decay estimates. Such estimates were proven in some generality in [JSS]. Subsequent works have extended the estimates to all dimensions, and general classes of potential perturbations. See e.g. [Ya, Sch, RSch, EGG, DSS, KK] and many more. Common to all these results is the explicit use of the kernel of the (unperturbed) Hamiltonian. Therefore, such methods are difficult to implement on manifolds. In fact, on manifolds most results are of the local decay type and Strichartz estimates [RoT, BS01, BS02, DR, Ta]. The pointwise decay estimates and the $L^p$ estimates are not known or not optimal. An abstract theory to prove decay estimates is based on the Mourre estimate. Mourre’s method and [SS, HSS, Ger] imply decay estimates starting only from an a priory estimate, the Mourre estimate. However, this method does not apply at thresholds. Another way around this problem is the Morawetz type estimates. They apply at thresholds, but limited to nontrapping potentials. The extension to repulsive potentials and low dimension was established as well in some cases. Mourre’s method was extended in many works to include thresholds: [BouG, MR, MT, MRT1, MRT2, BG, GJ1, GJ2, RT1, RT2, RT3, RT4, ? , Sa, BH, So]. However these methods so far could not be versatile enough to include many common systems, mainly due to complicated assumptions or the use of abstract weighted spaces. They only imply local decay estimates. In this work and forthcoming papers, we develop a new abstract theory to prove pointwise decay estimates in weighted spaces, starting only from a general commutator identity that should be satisfied by the Hamiltonian. In this paper we will show that for Schrödinger type equations generated by an abstract Hamiltonian $H$, as well as Klein Gordon and wave equations, the following identity

$$i[H, A] = \theta(H)$$

combined with regularity assumption of the type $H \in C^k(A)$, is sufficient to prove pointwise decay estimates of the Kato-Jensen type. We will then give several examples to show that such estimates follow effortlessly from the general theory. In followup papers we extend this method to include perturbations of $H$ of the types described above.

2. Evanescent states

Let $H$ be a self-adjoint operator on the Hilbert space $\mathcal{H}$ with spectral measure $E$. If $u \in \mathcal{H}$ let $E_u$ be the measure $E_u(J) = \|E(J)u\|^2$ and $\psi_u : \mathbb{R} \to \mathbb{C}$ the function $\psi_u(t) = \langle u | e^{itH}u \rangle = \int \mathbb{R} e^{it\lambda} E_u(d\lambda)$. We are interested in vectors $u$ such that $\psi_u(t) \to 0$ as $t \to \infty$ and in the rapidity of this decay.

Note that $|\psi_u(t)|^2$ is a physically meaningful quantity if we think of $H$ as the Hamiltonian of a system whose state space is $\mathcal{H}$. Indeed, if $u, v$ are vectors of norm one then $|\langle v | e^{itH}u \rangle|^2$ is the probability of finding the system in the state $v$ at moment $t$ if the initial state is $u$,
hence $|\psi_u(t)|^2$ is the probability that at moment $t$ the system be in the same state $u$ as at moment $t = 0$.

**Remark 2.1.** In this paper we are interested in the decay properties of the functions $\psi_u$ for $u$ in the absolute continuity subspace $\mathcal{H}_H^ac$ of $\mathcal{H}$ relatively to $H$. We shall see that $\psi_u \in L^2(\mathbb{R})$ for $u$ in a dense subspace of $\mathcal{H}_H^ac$ but in rather simple cases it may happen that $\psi_u \in L^1(\mathbb{R})$ only for $u = 0$. Formally speaking, the physically interesting quantity $|\psi_u(t)|^2$ generically decays more rapidly than $(t)^{-1}$ but not as rapidly as $(t)^{-2}$. Our results concern mainly the regularity of this decay, for example we give conditions such that $|\psi_u(t)|^2$ is really dominated by $(t)^{-1}$.

Since $\psi_u$ is (modulo a constant factor) the Fourier transform of $E_u$, there is a strong relation between the decay of $\psi_u$ and the smoothness of $E_u$. If $u$ is absolutely continuous with respect to $H$ then $\psi_u \in C_0(\mathbb{R})$ (space of continuous functions which tend to zero at infinity). However, the decay may be quite slow if $E_u$ is not regular enough.

**Example 2.2.** Let $\Lambda$ be a real compact set with empty interior and strictly positive Lebesgue measure and let $H$ be the operator of multiplication by $x$ in $\mathcal{H} = L^2(\Lambda, dx)$. Then the spectrum of $H$ is purely absolutely continuous but $\psi_u \notin L^1(\mathbb{R})$ for all $u \in \mathcal{H}\setminus\{0\}$. Indeed, if $u \in \mathcal{H}$ and we extend it by zero outside $\Lambda$ then $\psi_u(t) = \int e^{ixu} |u(x)|^2 dx$ hence if $\psi_u$ is integrable then $|u|^2$ is the Fourier transform of an integrable function, so it is continuous, so the set where $|u(x)|^2 \neq 0$ is open and contained in $\Lambda$, hence it is empty.

On the other hand, if $H$ has an absolutely continuous component then there are plenty of $u$ such that $\psi_u \in L^2(\mathbb{R})$: indeed, observe that $\psi_u \in L^2(\mathbb{R})$ if and only if $E_u$ is an absolutely continuous measure with derivative $E'_u \in L^2(\mathbb{R})$ and then $\|\psi_u\|_{L^2} = \sqrt{2\pi}\|E'_u\|_{L^2}$.

More generally, if we denote $E_{v,u}$ the complex measure $E_{v,u}(J) = \langle v | E(J) | u \rangle$ then $\langle v | e^{itH} | u \rangle = \int e^{it\lambda} E_{v,u}(d\lambda)$ hence the left hand side belongs to $L^2(\mathbb{R})$ if and only if the measure $E_{v,u}$ is absolutely continuous and has square integrable derivative $E'_{v,u}$ and then we have $\int_{\mathbb{R}} |\langle v | e^{itH} | u \rangle|^2 dt = 2\pi \int |E'_{v,u}(\lambda)|^2 d\lambda$. It is easy to prove the inequality $|E'_{v,u}(\lambda)|^2 \leq E'_{v,u}(\lambda)$, see ([Ro, page 1002] or [BW, Section 3.5]) and this implies $E_{v,u}^{1/2} \leq E_{v,u}^{1/2} + E'_{v,u}^{1/2}$. Thus, if we set for any $u \in \mathcal{H}$

$$[u]_H = \left(\int_{\mathbb{R}} |\psi_u(t)|^2 dt\right)^{1/4} = \left(2\pi \int E'_u(\lambda)^2 d\lambda\right)^{1/4}$$

then $\mathcal{E} = \mathcal{E}(H) = \{u \in \mathcal{H} \mid [u]_H < \infty\}$ is a dense linear subspace of the absolutely continuity subspace of $H$ and $[.]_H$ is a complete norm on it. We mention that the relation $|E'_{v,u}(\lambda)|^2 \leq E'_{v,u}(\lambda)$ implies

$$\left(\int_{\mathbb{R}} |\langle v | e^{itH} | u \rangle|^2 dt\right)^{1/2} \leq [v]_H [u]_H.$$  \hfill (2.1)

**Lemma 2.3.** If $J \in \mathcal{B}(\mathcal{H})$ commutes with $H$ then $J\mathcal{E} \subset \mathcal{E}$ and $[Ju]_H \leq \|J\|[u]_H$. If $J_n = \theta_n(H)$ with $\{\theta_n\}$ a uniformly bounded sequence of Borel functions such that $\lim_n \theta_n(\lambda) = 1$ for all $\lambda \in \mathbb{R}$, then for any $u \in \mathcal{E}$ we have $\lim_n [J_n u]_H = [u]_H$.

**Proof.** For the first part we use $E'_{v,u}(\lambda) \leq \|J\|^2 E'_{v,u}(\lambda)$ (which is obvious) while for the second part $E'_{\theta_n(H)u}(\lambda) \geq \theta_n(\lambda)^2 E'_{u}(\lambda)$ and the dominated convergence theorem. \hfill $\square$
The quantity \( \int_0^t |\psi(u(t)|^2 \, dt \) has a simple physical interpretation in the quantum setting: if \( u,v \) are two state vectors then \( \int_0^t |\langle v^i|H|u \rangle|^2 \, dt \) is the total time spent by the system in the state \( v \) if the initial state is \( u \). We hence may say that \( \int_0^t |\langle u^i|H|u \rangle|^2 \, dt \) is the lifetime of the state \( u \). The elements of \( \mathcal{E}(H) \) are those of finite lifetime, or states in which the system spends a finite total time. We might call them self evanescent states, and they are absolutely continuous with respect to \( H \). Note that there is a Schrödinger Hamiltonian \( H \) and there is a state \( u \) which is singularly continuous with respect to \( H \) such that \( \psi(u)(t) = O(|t|^{-1/2+\varepsilon}) \) for any \( \varepsilon > 0 \) [Si].

Another interesting class \( \mathcal{E}_\infty \equiv \mathcal{E}_\infty(H) \) is that of evanescent states defined by the condition \( \int_0^\infty |\langle v^i|e^{itH}|u \rangle|^2 \, dt < \infty \) for any \( v \): such a state \( u \) spends a finite time in any state \( v \).

The evanescent states disappear (or go to infinity) in a natural quantum mechanical sense, which explains the fundamental role they play in the Rosenblum Lemma [Ro] and later on in the Birman-Kato class tracing theory. A simple argument shows that \( \mathcal{E}_\infty \) is the linear subspace of \( \mathcal{E} \) consisting of vectors \( u \) such that \( E'_u \) is a bounded function. In particular, \( \mathcal{E}_\infty \) is dense in the absolutely continuity subspace associated to \( H \).

**Example 2.4.** If \( H = q = \) operator of multiplication by \( x \) in \( L^2(\mathbb{R}, dx) \) then \( \mathcal{E}(q) = L^4(\mathbb{R}) \) and \( \mathcal{E}_\infty(q) = L^\infty(\mathbb{R}) \). Indeed, \( \langle u|e^{itq}u \rangle = \int_\mathbb{R} e^{itx} |u(x)|^2 \, dx \) is an \( L^2 \) function of \( t \) if and only if \( |u|^2 \in L^2 \) and then \( |u|_q = (2\pi)^{1/4} \|u\|_{L^4} \). On the other hand, \( \langle v|e^{itq}u \rangle = \int_\mathbb{R} e^{itx} \overline{v(x)} u(x) \, dx \) is an \( L^2 \) function of \( t \) for any \( v \in L^2 \) if and only if \( \bar{v}u \in L^2 \) for any \( v \in L^2 \) hence if and only if \( u \in L^\infty \).

3. Notes on commutators

Let \( A \) be a self-adjoint operator on a Hilbert space \( \mathcal{H} \). If \( S \) is a bounded operator on \( \mathcal{H} \) then we denote \( [A,S]_o \) the sesquilinear form on \( D(A) \) defined by \( [A,S]_o(u,v) = \langle Au|Sv \rangle - \langle u|SAv \rangle \).

As usual, we set \( [S,A]_o = -[A,S]_o, [S,iA]_o = i[S,A]_o \), etc. We say that \( S \) is of class \( C^1(A) \), and we write \( S \in C^1(A) \), if \( [A,S]_o \) is continuous for the topology induced by \( \mathcal{H} \) on \( D(A) \) and then we denote \( [A,S] \) the unique bounded operator on \( \mathcal{H} \) such that \( \langle u|[A,S]_o v = \langle Au|Sv \rangle - \langle u|SAv \rangle \) for all \( u,v \in D(A) \). It is easy to show that \( S \in C^1(A) \) if and only if \( SD(A) \subset D(A) \) and the operator \( SA-AS \) with domain \( D(A) \) extends to a bounded operator \( [A,S] \in B(\mathcal{H}) \). Moreover, \( S \) is of class \( C^1(A) \) if and only if the following equivalent conditions are satisfied

1. the function \( t \mapsto e^{-itA}Se^{itA} \) is Lipschitz in the norm operator topology
2. the function \( t \mapsto e^{-itA}Se^{itA} \) is of class \( C^1 \) in the strong operator topology

and then we have \( [S,iA] = \frac{d}{dt}e^{-itA}Se^{itA}|_{t=0} \).

Clearly \( C^1(A) \) is a \( * \)-subalgebra of \( B(\mathcal{H}) \) and the usual commutator rules hold true: for any \( S,T \in C^1(A) \) we have \( [A,S]^* = -[A,S^*] \) and \( [A,ST] = [A,S]T + [A,T]S \), and if \( S \) is bijective then \( S^{-1} \in C^1(A) \) and \( [A,S^{-1}] = -S^{-1}[A,S]S^{-1} \).

We often abbreviate \( S' = [S,iA] \) if the operator \( A \) is obvious from the context. Then we may write \( (S')^* = (S^*)', (ST)' = ST + TS', \) and \( (S^{-1})' = -S^{-1}S' S^{-1} \).

We consider now the rather subtle case of unbounded operators. Note that we always equip the domain of an operator with its graph topology. If \( H \) is a self-adjoint operator on \( \mathcal{H} \) then \( [A,H]_o \) is the sesquilinear form on \( D(A) \cap D(H) \) defined by \( [A,H]_o(u,v) = \langle Au|Hv \rangle - \langle H u|A v \rangle \). By analogy with the bounded operator case, one would expect that
requiring denseness of $D(A) \cap D(H)$ in $D(H)$ and continuity of $[A, H]_\omega$ for the graph topology of $D(H)$ would give a good $C^1(A)$ notion. For example, this should imply the validity of the virial theorem, nice functions of $H$ (at least the resolvent) should also be of class $C^1$, etc. However this is not true, as the following example from [GG] shows.

**Example 3.1.** In $\mathcal{H} = L^2(\mathbb{R}, dx)$ let $q = \text{operator of multiplication by } x$ and $p = -i \frac{d}{dx}$. Let $A = e^{i p} - p$ and $H = e^{i q}$ with $\omega = \sqrt{2\pi}$. This value of $\omega$ is chosen because $e^{i p}e^{i q} = e^{i q}e^{i p}$ on a very large set although the operators $e^{i p}$ and $e^{i q}$ do not commute. Then $D(A) \cap D(H)$ is dense in both $D(A)$ and $D(H)$ (moreover, $D(H) \cap D(HA)$ is dense in $D(H)$), one has $[H, iA]_\omega = \omega H$ on $D(A) \cap D(H)$, but $(H + i)^{-1} \notin C^1(A)$.

A convenient definition of the $C^1(A)$ class for any self-adjoint operator is as follows. Let $R(z) = (H - z)^{-1}$ for $z$ in the resolvent set $\rho(H)$ of $H$. We say that $H$ is of class $C^1(A)$ if $R(z) \in C^1(A)$ for some (hence for all) $z \in \rho(H)$. In this case the space $R(z)D(A)$ is independent of $z \in \rho(H)$, it is a core of $H$, and is a dense subspace of $D(A) \cap D(H)$ for the intersection topology, i.e. for the norm $\|u\| + \|Au\| + \|Hu\|$. Moreover:

**Proposition 3.2.** Let $A, H$ be self-adjoint operators on a Hilbert space $\mathcal{H}$.

(1) $H$ is of class $C^1(A)$ if and only if the next two conditions are satisfied:
   
   $(a)$ $[A, H]_\omega$ is continuous for the topology induced by $D(H)$ on $D(A) \cap D(H)$,
   
   $(b)$ there is $z \in \rho(H)$ such that $\{u \in D(A) \mid R(z)u \in D(A)\}$ is a core for $A$.

(2) If $H \in C^1(A)$ then $D(A) \cap D(H)$ is dense in $D(H)$ hence $[A, H]_\omega$ extends to a uniquely determined continuous sesquilinear form $[A, H]_\omega$ on $D(H)$. We have:

$$[A, R(z)] = -R(z)[A, H]R(z) \quad \forall z \in \rho(H).$$

(3.2)

This is Theorem 6.2.10 in [ABG]. The condition $(a)$ above is quite easy to check in general but not condition $(b)$ because it involves a certain knowledge of the resolvent of $H$, which is a complicated object. We now describe criteria which allow one to avoid this problem.

We denote $\mathcal{H}^1 = D(H)$ (equipped with the graph topology) and $\mathcal{H}^{-1} = (D(H)^*)^*$ its adjoint space. The identification of the adjoint space $\mathcal{H}^*$ of $\mathcal{H}$ with itself via the Riesz Lemma gives us a scale $\mathcal{H}^1 \subset \mathcal{H} \subset \mathcal{H}^{-1}$ with continuous and dense embeddings. If we define $\mathcal{H}^s := [\mathcal{H}^1, \mathcal{H}^{-1}]_{(1-s)/2}$ for $-1 \leq s \leq 1$ by complex interpolation then $(\mathcal{H}^s)^* = \mathcal{H}^{-s}$ for any $s$ and $\mathcal{H}^{1/2}$ is just $D(|H|^{1/2})$. Finally, we have continuous and dense embeddings

$$\mathcal{H}^1 \subset \mathcal{H}^{1/2} \subset \mathcal{H} \subset \mathcal{H}^{-1/2} \subset \mathcal{H}^{-1}.$$

If $H \in C^1(A)$ the continuous sesquilinear form $[A, H]$ on $D(H)$ is then identified with a linear continuous operator $\mathcal{H}^1 \to \mathcal{H}^{-1}$ and this is useful for example because it gives a simple interpretation to supplementary conditions like $[A, H]\mathcal{H}^1 \subset \mathcal{H}$. Observe that $H' := [H, iA] : \mathcal{H}^1 \to \mathcal{H}^{-1}$ is a continuous symmetric operator. Now the following assertions are consequences of [ABG, Theorem 6.3.4, Lemma 7.5.3] and [GG, Lemma 2].

(1) If $e^{itA}\mathcal{H}^1 \subset \mathcal{H}^1 (\forall t)$ then $H \in C^1(A)$ if and only if condition $(a)$ in part $(1)$ of Proposition 3.2 is satisfied.

(2) If $H \in C^1(A)$ and $H'\mathcal{H}^1 \subset \mathcal{H}$ then $e^{itA}\mathcal{H}^1 \subset \mathcal{H}^1 (\forall t)$ and the restrictions $e^{itA}\mathcal{H}^1$ give a strongly continuous group of operators on the Hilbert space $\mathcal{H}^1$. 


(3) If $e^{itA} \mathcal{H}^1 \subset \mathcal{H}^1(\forall t)$ then $D(A, \mathcal{H}^1) = \{u \in \mathcal{H}^1 : Au \in \mathcal{H}^1\}$ is a dense subspace of $\mathcal{H}^1$ and $H$ is of class $C^1(A)$ with $H' \mathcal{H}^1 \subset \mathcal{H}$ if and only if $\|(Au)Hv - \langle Hu, Av \rangle\| \leq C\|u\|_{\mathcal{H}^1}\|v\|_{\mathcal{H}^1}$ for all $u, v \in D(A, \mathcal{H}^1)$.

(4) Assume $e^{itA} \mathcal{H}^{1/2} \subset \mathcal{H}^{1/2}(\forall t)$. Then $D(A, \mathcal{H}^{1/2}) := \{u \in \mathcal{H}^{1/2} : Au \in \mathcal{H}^1\}$ is dense in $\mathcal{H}^{1/2}$ and if the quadratic form $\langle Hu, Au \rangle - \langle Au, H\rangle$ on $D(A, \mathcal{H}^{1/2})$ is continuous for the topology induced by $\mathcal{H}^{1/2}$ then $H \in C^1(A)$.

We mention that Hypotheses 1, 2′ and 3 on page 62 of [CFKS] imply that $H$ is of class $C^1(A)$, cf. relation (4.10) there.

We now give some “pathological” examples which clarify the notion of $C^1$ regularity.

**Example 3.3.** Let $\mathcal{H} = L^2(\mathbb{R})$ and $A = p$. It is clear that the operator of multiplication by a rational real function is of class $C^1(p)$, in fact of class $C^\infty(p)$ in a natural sense. For example, if $H = q^{-m}$ then $(H + i)^{-1} = q^m(1 + iq^{-m})^{-1}$ is clearly a bounded operator of class $C^1(p)$ if $m \in \mathbb{N}$ and $[q^{-m}, ip] = mq^{-m-1}$ as continuous forms on $D(q^{-m})$.

The worst case is attained when $m = 1$: then $H' = H^2$ hence $H + i : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ is an isomorphism, in particular $H' \mathcal{H}^1$ is not included in any of the smaller spaces $\mathcal{H}^s$ with $s > -1$. If $m \geq 1$ is an odd integer then $H$ is of class $C^1(A)$ and $H' = mH^{1+1/m}$ where $x^{1/m} := |x|^{1/m}$ if $x < 0$; now we have $H' \mathcal{H}^1 \subset \mathcal{H}^{-1/m}$ and this is optimal.

**Remark 3.4.** Example 3.3 shows that if $H \in C^1(A)$ then neither $e^{itA}$ nor $(A + i\lambda)^{-1}$ leave invariant $D(H)$ in general.

If $H \in C^1(A)$ then $D(A) \cap D(H)$ is dense in $D(H)$ but is not dense in $D(A)$ in general.

**Example 3.5.** Let $H = q^{-m}$ with $m \geq 1$ and $A = p$ as in Example 3.3. Then $D(A)$ is the Sobolev space consisting of functions $u \in L^2(\mathbb{R})$ with derivative $u' \in L^2(\mathbb{R})$, so we have $D(A) \subset C_0(\mathbb{R})$ continuously. Thus if $u \in D(A) \cap D(H)$ then $u$ is a continuous function such that $\int |u(x)|^2 x^{-2m}dx < \infty$ which implies $u(0) = 0$. But $\{ u \in D(A) : u(0) = 0 \}$ is a closed hyperplane of codimension one in the Hilbert space $D(A)$.

By taking $m$ large in the preceding example we see that for any $\varepsilon > 0$ there is a self-adjoint operator $H$ of class $C^1(A)$ with $H' \mathcal{H}^1 \subset \mathcal{H}^{-\varepsilon}$ such that $D(A) \cap D(H)$ is not dense in $D(A)$. Thus the next result is optimal.

**Proposition 3.6.** If $H \in C^1(A)$ and $H' \mathcal{H}^1 \subset \mathcal{H}$ then $D(A) \cap D(H)$ is dense in $D(A)$. More precisely, if we set $R_\varepsilon = (1 + i\varepsilon H)^{-1}$ for $\varepsilon > 0$ then $R_\varepsilon D(A) \subset D(A) \cap D(H)$ and $s$-lim $R_\varepsilon = 1$ in the Hilbert space $D(A)$.

**Proof.** We have $R_\varepsilon D(A) \subset D(A) \cap D(H)$ and $[A, R_\varepsilon] = \varepsilon R_\varepsilon H' R_\varepsilon$ by Proposition 3.2. Then $\varepsilon \|H' R_\varepsilon\| \leq \|H' R_1\| \|(1 + i\varepsilon H)R_\varepsilon\| \leq C$ and $\varepsilon H' R_\varepsilon = \varepsilon H' R_1 R_\varepsilon (1 + iH)u \rightarrow 0$ if $u \in D(H)$. Thus $s$-lim $\varepsilon \rightarrow 0 [A, R_\varepsilon] = 0$ hence $AR \varepsilon u \rightarrow Au$ for any $u \in D(A)$.

This $C^1(A)$ property transfers from $H$ to some functions of $H$: for example, it is easy to prove that $\varphi(H) \in C^1(A)$ if $\varphi \in C^2(\mathbb{R})$ and $|\varphi(\lambda)| + |\varphi'(\lambda)| + |\varphi''(\lambda)| \leq C|\lambda|^{-2}$. But obviously $e^{itH} \notin C^1(A)$ in general.

**Theorem 3.7.** Let $H$ be a self-adjoint operator of class $C^1(A)$ and $t \in \mathbb{R}$. Then the restriction of $[A, e^{itH}]_e$ to $D(A) \cap D(H)$ extends to a continuous form $[A, e^{itH}]$ on $D(H)$ and, in the strong topology of the space of sesquilinear forms on $D(H)$, we have:

$$[e^{itH}, A] = \int_0^t e^{i(t-s)H} H' e^{isH} ds.$$  \hspace{1cm} (3.3)
Proof. Clearly it suffices to assume $t = 1$. For $n \geq 1$ integer let $R_n = (1 - iH/n)^{-1}$. Then $R_n$ has norm $\leq 1$ and $e^{iH} = \text{s-lim}_{n \to \infty} R_n$ in both spaces $\mathcal{H}$ and $D(H)$. Since $H$ is of class $C^1(A)$ we have $R_n \in C^1(A)$ and $[A, R_n] = \frac{1}{n} R_n[A, H] R_n$, so $R_n \in C^1(A)$ and

$$[A, R_n^k] = \sum_{k=0}^{n-1} R_n^k[A, R_n] R_n^{n-1-k} = \frac{1}{n} \sum_{k=1}^{n} R_n^k[A, H] R_n^{n+1-k}.$$  

It is clear that $\langle u| [A, R_n^{k}] v \rangle \to \langle u| [A, e^{iH}] v \rangle$ as $n \to \infty$ for all $u, v \in D(A)$. Thus it remains to be shown that for all $u, v \in D(H)$:

$$\frac{1}{n} \sum_{k=1}^{n} \langle R_n^k u \rangle [A, H] R_n^{n+1-k} v \rangle \to \int_{0}^{1} \langle e^{-isH} u \rangle [A, H] e^{i(1-s)H} v \rangle ds. \quad (3.4)$$

We have

$$\left\| \sum_{k=1}^{n} R_n^k[A, H] R_n^{n+1-k} \right\|_{\mathcal{H} \to \mathcal{H}} \leq n ||[A, H]||_{\mathcal{H} \to \mathcal{H}}^{-1}$$

hence it suffices to prove that $(3.4)$ holds for $u, v$ in a dense subspace of $D(H)$. So we may assume that $u, v$ have compact support with respect to $H$.

Let $a$ be a number such that $|\log(1 + z) - z| \leq a |z|^2$ if $z \in \mathbb{C}$ and $|z| < 1/2$. If $\phi_n(x) = (1 - i\frac{x}{n})^{-1}$ then for $x$ in a real compact set, $1 \leq k \leq n$, and $n$ large, we have

$$|\phi_n(x) e^{i\frac{k}{n}} - e^{i\frac{k}{n}}| = |e^{k \log(1-ix/n)} + i\frac{x}{n}| = Ck |\log(1 - i\frac{x}{n})| \leq C k |\frac{x}{n}|^2$$

where $C$ is a number depending only on the set where varies $x$. Thus the last term above is an $O(x^2/n)$ and so we get $||R_n^k u - e^{i\frac{k}{n}H} u||_{D(H)} = O(\frac{1}{n})$. A similar argument gives $||R_n^{n+1-k} v - e^{i\frac{n+1}{n}H} V||_{D(H)} = O(\frac{1}{n})$. Hence:

$$\frac{1}{n} \sum_{k=1}^{n} \langle R_n^k u \rangle [A, H] R_n^{n+1-k} v \rangle = \frac{1}{n} \sum_{k=1}^{n} \langle e^{-i\frac{k}{n}H} u \rangle [A, H] e^{i\frac{k}{n}H} e^{i\frac{n+1}{n}H} v \rangle + O(\frac{1}{n}).$$

Finally, we have $e^{i\frac{n+1}{n}H} v \to e^{iH} v$ in $D(H)$ and the $(D(H))$-valued functions $s \mapsto e^{-isH} u$ and $s \mapsto e^{iH} v$ are continuous. This proves $(3.3)$. \hfill $\square$

The relation $(3.3)$ also holds in $B(D(H), D(H)^*)$ in the strong topology and then one may easily prove relations like the next one hold in $B(\mathcal{H})$:

$$[A, e^{iH} R(z) \overline{2}] = R(z)[A, e^{iH}] R(z) + [A, R(z)]e^{iH} R(z) + e^{iH} R(z) [A, R(z)]. \quad (3.5)$$

If $H' D(H) \subset \mathcal{H}$ then the right hand side of $(3.3)$ will clearly belong to $B(D(H), \mathcal{H})$ hence we shall also have $[A, e^{iH}] \in B(D(H), \mathcal{H})$ and $(3.3)$ will hold strongly in $B(D(H), \mathcal{H})$.

We say that $H'$, or $[A, H]$, commutes with $H$ if for any $t \in \mathbb{R}$ the relation $H' e^{itH} = e^{itH} H'$ holds in $B(D(H), D(H)^*)$. This is clearly equivalent to $H' \varphi (H) = \varphi (H) H'$ for any bounded Borel function $\varphi : \mathbb{R} \to \mathbb{C}$. Note also that $H'$ commutes with $H$ if and only if there is $z \in \rho(H)$ such that $[A, R(z)]$ commutes with $R(z)$ (this condition is independent of $z$). If we set $R = R(z)$, we then have $R' = -RH'R = -H'R^2$.

If $H'$ commutes with $H$ then Proposition 3.7 can be significantly improved. If $k \in \mathbb{N}$ let $C^k_b(\mathbb{R})$ be the space of functions in $C^k(\mathbb{R})$ whose derivatives of orders $\leq k$ are bounded.
Proposition 3.8. Let $H$ be self-adjoint of class $C^1(A)$ such that $H'$ commutes with $H$ and let $\varphi \in C^1_b(\mathbb{R})$. Then the restriction of $[A, \varphi(H)]_t$ to $D(A) \cap D(H)$ extends to a continuous form $[A, \varphi(H)]$ on $D(H)$ and $[A, \varphi(H)] = [A, H] \varphi'(H) = \varphi'(H)[A, H]$. In other terms:

$$ \varphi(H)' = \varphi'(H)H' = H'\varphi'(H), \text{ in particular } (e^{itH})' = itHe^{itH}. \quad (3.6) $$

Proof. Due to Theorem 3.7 we have $[A, e^{itH}] = it[A, H]e^{itH}$ for any real $t$, hence the proposition is true if $\varphi(\lambda) = e^{i\lambda}$. This clearly implies the proposition if $\varphi$ is the Fourier transform of a bounded measure $\hat{\varphi}$ such that $\int |\hat{\varphi}(x)| dx < \infty$. The general case follows by a standard limiting procedure. \qed

Example 3.9. Consider once again the situation from Example 3.3. Then $[\rho, e^{itH}]_t$ is a restriction of $-mq^{-m-1}e^{itH}$ hence is not a bounded operator but it extends to a continuous form on $D(H)$. In the worst case $m = 1$ we get $[e^{itH}, A] = H^2e^{itH}$, hence the result of Theorem 3.7 is optimal. If $\varphi$ is a $C^1$ function then $\varphi(H)' = H^2\varphi'(H)$ hence $\varphi(H)'$ cannot be bounded unless $|\varphi'(\lambda)| \leq C(\lambda)^{-2}$.

4. Commutators and decay

From Proposition 3.8 we get the following decay result.

Proposition 4.1. Let $H \in C^1(A)$ such that $H' := [H, iA]$ commutes with $H$ and let $u \in D(H) \cap D(A)$. Then $|\langle u|H' e^{itH}u \rangle| \leq 2|t|^{-1} \|A u\| \|u\|$. In particular, if $H' = B^* B$ for some continuous $B : D(H) \to \mathcal{H}$ commuting with $H$, then $|\psi_B u(t)| \leq C_u(t)^{-1}$ for $u \in D(H) \cap D(A)$. If $B$ is bounded on $\mathcal{H}$ then this holds for all $u \in D(A)$.

Proof. The first part is obvious. The fact that $B$ commutes with $H$ means $e^{itH} B = B e^{itH}$ for any $t$ and this clearly implies that $[A, H]$ commutes with $H$. Then $\langle Bu | e^{itH} Bu \rangle = \langle u | [H, iA] e^{itH} u \rangle$ hence the second and the third part are consequences of the first one. \qed

Remark 4.2. Some of the next results are abstract versions of the following estimate: if $H = b(q)$ and $A = -p$ in $L^2(\mathbb{R})$ then $H' = h'(q)$ hence if $|h'| \geq c > 0$ and $h''/h'^2$ is bounded then an integration by parts gives $\left| \int e^{itH(x)} |u(x)|^2 dx \right| \leq C_u(t)^{-1}$ if $u \in D(p)$.

We shall say that a densely defined operator $S$ on $\mathcal{H}$ is boundedly invertible if $S$ is injective, its range is dense, and its inverse extends to a continuous operator on $\mathcal{H}$. If $S$ is symmetric this means that $S$ is essentially self-adjoint and 0 is in the resolvent set of its closure.

Proposition 4.3. Let $H \in C^1(A)$ such that $H'$ commutes with $H$ and $H'D(H) \subset \mathcal{H}$. Assume that $H'$, when considered as operator on $\mathcal{H}$, is boundedly invertible and $H'^{-1}$ extends to a bounded operator of class $C^1(A)$. Then $|\psi_u(t)| \leq C_u(t)^{-1}$ if $u \in D(A)$.

Proof. From Proposition 3.8 we get $[e^{itH}, A] = tH'e^{itH}$ as operators $D(H) \to D(H)^*$ hence $[e^{itH}, A]$ is a bounded operator $D(H) \to \mathcal{H}$ and we have $[e^{itH}, A]H'^{-1} = te^{itH}$ on the range of $H'$. We denote $K$ the continuous extension to $\mathcal{H}$ of $H'^{-1}$ and note that $K$ commutes with $H$ because $H'e^{itH} = e^{itH}H'$ hence $H'^{-1}e^{itH} = e^{itH}H'^{-1}$ for all $t$. If $u \in D(A)$ and $K u \in D(H)$ then $K u \in D(A)$ because $K \in C^1(A)$ hence

$$ t\psi_u(t) = \langle u | e^{itH} A K u \rangle = \langle e^{-itH} u | A Ku \rangle - \langle A u | e^{itH} K u \rangle. $$
This implies \[ |t\psi_u(t)| \leq ||u|| |AKu| + |Au||Ku| \leq ||[A,K]| |u|^2 + 2|K||u||Au|. \]

Now let \( u \) be an arbitrary element of \( D(A) \). Let \( R_e = (1 + i\varepsilon H)^{-1} \) and \( u_e = R_e u \). Then \( u_e \in D(A) \) because \( R_e D(A) \subset D(A) \) and \( Ku_e = R_e Ku \in D(H) \) hence we have
\[
|t\psi_{u_e}(t)| \leq ||[A,K]| |u_e|^2 + 2|K||u_e||Au|. 
\]
Since \([A, R_e] = \varepsilon H'R_e^2\) we get \(|t\psi_u(t)| \leq ||[A,K]| |u|^2 + 2|K||u||Au|\) by making \( \varepsilon \to 0 \) in the preceding inequality. \( \square \)

**Remark 4.4.** We may restate the assumptions of Proposition 4.3 as follows: \( H \) is of class \( C^2(A) \), \( H'D(H) \subset \mathcal{H} \), and \( H' \) when seen as operator on \( \mathcal{H} \) is essentially self-adjoint and 0 is not in its spectrum.

**Remark 4.5.** The good decay \( \psi_u(t) = O(t^{-1}) \) obtained in Proposition 4.3 depends on a quite strong condition on \( H' \) which in particular forces \( H' \) to be an essentially self-adjoint operator on \( \mathcal{H} \) whose spectrum does not contain zero. In the “classical” case mentioned in the Remark 4.2 this means \( |h'(x)| \geq c > 0 \) which is rather natural when one has to estimate an integral like \( \psi(t) = \int e^{ih(x)} f(x)dx \) for large positive \( t \): points of stationary phase should be avoided, otherwise we cannot expect more than \( \psi(t) = O(t^{-1/2}) \).

We now consider operators satisfying some special commutation relations but allow \( H' \) to have zeros, e.g. we treat the simplest case \( H, iA = cH \). Note that Example 3.1 shows that requiring only an algebraic relation like \( [H, iA] = cH \) is highly ambiguous; the property \( H \in C^1(A) \) is then necessary and is not automatically satisfied.

In many of the applications of the conjugate operator method, see for example Section 7, the operator \( A \) is unbounded in energy space. However, it is possible to introduce an energy cut-off for \( A \) that does not alter the \( C^1(A) \) condition for \( H \) and preserves the behavior of the commutation relation at thresholds. For instance, consider \( H \in C^1(A) \) bounded from below and such that \( H' = cH \). Define the operators \( g(H) = (H + c)^{-1/4} \) and \( \tilde{A} = g(H)Ag(H) \). Then, it is easy to see that \( H \in C^1(\tilde{A}) \) and \([H, i\tilde{A}] = H'g(H)^2\).

**Remark 4.6.** The subsequent results will hold for self evanescent states \( u \in \mathcal{E} \), but they also rely on the condition \( Au \in \mathcal{E} \). The latter assumption is not satisfied in general, in fact, it is implied by a stronger localization condition for \( u \). To elude this, it will be assumed that there is a projection \( P \) which commutes with \( H \), and such that \( u \in \text{Ran} P \). Then the condition \( Au \in \mathcal{E} \) can be replaced by \( PAu \in \mathcal{E} \), which is easier to satisfy. This idea will be explored further in forthcoming work. For instance, one can choose \( P \) as the projection on the continuous spectrum of \( H \) and the proofs presented below can be slightly modified to obtain the same decay estimates.

**Proposition 4.7.** Let \( H \in C^1(A) \) such that \( H' = cH \) with \( c \neq 0 \) and let \( u \in D(A) \) such that \( u, Au \in \mathcal{E} \). Then \( |\psi_u(t)| \leq C_u(t)^{-1/2} \).

**Proof.** We have \( \psi_u \in L^2(\mathbb{R}) \) because \( u \in \mathcal{E} \) hence, according to Corollary 8.2, it suffices to show that the function \( (\delta \psi)(t) = t\psi'_u(t) \) also belongs to \( L^2(\mathbb{R}) \). If \( u \in D(|H|^{1/2}) \) then \( t\psi'_u(t) = \langle u | ih u | e^{itH} \rangle = \langle u | h e^{itH} | u \rangle \) so that by using Proposition 3.8 we get:
\[
ict\psi'_u(t) = \langle u | t e^{itH} u \rangle = \langle u | [A, itH] e^{itH} u \rangle = \langle u | [A, e^{itH}] u \rangle.
\]
Then, if \( u \in D(A) \) we get \( i ct \psi'_u(t) = \langle Au|e^{itH}u \rangle - \langle u|e^{itH}Au \rangle \) hence (2.1) implies:

\[
c^2||\delta \psi_u||^2 \leq 2||\psi_u||_{L^2}||\psi_{Au}||_{L^2} = 2[u]_H^2 [Au]_H^2.
\]

(4.7)

So the proposition is proved under the supplementary condition \( u \in D(H) \) and the estimate (4.7) depends only on \( c \). Now consider an arbitrary \( u \in D(A) \) such that \( u, Au \in \mathcal{E} \) and for \( \varepsilon > 0 \) let \( R_\varepsilon = (1 + i\varepsilon H)^{-1} \). Then from Proposition 3.2 we get \( R_\varepsilon u \in D(A) \) and \( [A, R_\varepsilon] = R_\varepsilon i[\varepsilon H, A]R_\varepsilon = \varepsilon HR_\varepsilon^2 \). If we set \( u_\varepsilon = R_\varepsilon u \) then the estimate (4.7) gives \( c^2||\delta \psi_{u_\varepsilon}||^2 \leq 2[u_\varepsilon]_H^2 [Au_\varepsilon]_H^2 \). Finally, let \( \varepsilon \to 0 \) and use Fatou’s lemma in the left hand side and Lemma 2.3 on the right hand side to get (4.7) without the condition \( u \in D(H) \).

\[ \square \]

**Proposition 4.8.** Let \( H \in C^1(A) \) such that \( H \geq 0 \) and \( H' = H(1 + H)^{-1} \). If \( u \in D(A) \) and \( u, Au \in \mathcal{E} \) then \( |\psi_u(t)| \leq C_u(t)^{-1/2} \).

**Proof.** Choose \( \varphi \in C_0^\infty(\mathbb{R}) \) with \( 0 \leq \varphi \leq 1 \) and equal to one on a neighbourhood of zero and set \( \phi = \varphi(H) \), \( \phi^+ = 1 - \phi^2 \). Then \( \psi_u(t) = \psi_{\phi u}(t) + \langle u|\phi^+ e^{itH}u \rangle \) and we have \( \langle |\phi^+ e^{itH}u| \rangle \leq C(t)^{-1} \) by an argument almost identical to that of the proof of Proposition 4.7 (note that \( \xi(H)D(A) \subset D(A) \) if \( \xi \) is a smooth function constant near infinity).

Hence it suffices to prove the proposition under the supplementary assumption that the \( H \)-support of \( u \) (i.e. \( \text{supp} E_u \)) is compact. As before, we have \( tv'_u(t) = \langle u|itHe^{itH}u \rangle \) so that by using Proposition 3.8 and with the notation \( \Lambda = \sqrt{H} + 1 \) we get:

\[ -i\psi'_u(t) = \langle u|H e^{itH}u \rangle = \langle Au|tH e^{itH}u \rangle = \langle \Lambda u|e^{itH}, A\Lambda u \rangle. \]

We have \( \Lambda u \in D(A) \cap \mathcal{E} \) because \( \Lambda u = \theta(H)u \) for some smooth function with compact support \( \theta \) and \( u \in D(A) \cap \mathcal{E} \). Thus

\[ |tv'_u(t)| = \langle u|tH e^{itH}u \rangle | \leq |(Au|e^{itH}\Lambda u) + |(AAu|e^{itH}\Lambda u). \]

Finally, the relation

\[ A\Lambda u = A\theta(H)u = [A, \theta(H)]u + \theta(H)Au = i\theta'(H)H(H + 1)^{-1}u + \theta(H)Au \]

gives \( A\Lambda u \in \mathcal{E} \). Thus \( \delta \psi_u \in L^2(\mathbb{R}) \) hence Corollary 8.2 implies \( |\psi_u(t)| \leq C_u(t)^{-1/2} \).

\[ \square \]

The next result is of the same nature but more general; the proof is essentially the same.

**Theorem 4.9.** Let \( H \in C^1(A) \) such that \( H' = \theta(H) \) with \( \theta \) real of class \( C^1 \) with bounded derivative and such that: (1) if \( |\lambda| \geq \varepsilon > 0 \) then \( |\theta(\lambda)| \geq \varepsilon > 0 \), (2) \( \lambda \theta(\lambda) \) extends to a \( C^1 \) function on \( \mathbb{R} \). If \( u \in D(A) \) and \( u, Au \in \mathcal{E} \) then \( |\psi_u(t)| \leq C_u(t)^{-1/2} \).

**Proof.** Let \( \varphi \in C_0^\infty(\mathbb{R}) \) real and equal to one on a neighbourhood of zero and let us set \( \phi = \varphi(H), \phi^+ = 1 - \phi^2 \), so that \( \psi_u(t) = \psi_{\phi u}(t) + \langle u|\phi^+ e^{itH}u \rangle \). We first show that the second term is \( O(t^{-1}) \). We have \( \phi^+ H^{-1} = \xi(H) \) with \( \xi(\lambda) = (1 - \varphi^2(\lambda))/\theta(\lambda) \) hence

\[ t\langle u|\phi^+ e^{itH}u \rangle = \langle \phi^+ H^{-1}u|tH e^{itH}u \rangle = \langle \xi(H)u|tH e^{itH}u \rangle = \langle \xi(H)u|e^{itH}, A|u \rangle. \]

Until here \( u \) was an arbitrary element of \( \mathcal{H} \). If \( u \in D(H) \cap D(A) \) then we can expand the commutator and get

\[ |t\langle u|\phi^+ e^{itH}u \rangle| = |(e^{-itH}\xi(H)u|Au) - \langle A\xi(H)u|e^{itH}u \rangle| \leq ||Au||||\xi(H)u|| + ||A\xi(H)u||. \]

Since \( \xi \) is a bounded function of class \( C^1 \) with bounded derivative we can use Proposition 3.6 and get \( \xi(H)' = \xi(H)' = (\xi'\theta)(H) \). We have \( \xi'\theta = -\theta'/\theta \) outside a compact
neighbourhood of zero, hence \( \xi(H)' \) is a bounded operator, so \( \xi(H) \) is of class \( C^1(A) \), hence \( \xi(H)D(A) \subset D(A) \). Then, since \( H'D(H) \subset H \), this estimate remains true for any \( u \in D(A) \) by Proposition 3.6. Thus \( |\langle u|e^{itH} u \rangle| \leq C_u(t)^{-1} \) for any \( u \in D(A) \).

From now on we change notations: \( \phi_u \) will be denoted \( u \). So we may assume \( \text{supp}E_u \subset [-1,1] \) and \( u,Au \in \mathcal{E} \), cf. Lemma 2.3, and we want to prove that the function \( t\psi_u'(t) \) belongs to \( L^2(\mathbb{R}) \). Let \( \eta \) be the \( C^1 \) function on \( \mathbb{R} \) which extends \( \lambda/\theta(\lambda) \), let \( \zeta \in C_{c}^{\infty}(\mathbb{R}) \) such that \( \zeta(H)u = u \), and let us set \( \tilde{\eta} = \eta\zeta \). Then \( \tilde{\eta}(H)u \in D(A) \cap \mathcal{E} \) and \( A\tilde{\eta}(H)u = [A, \tilde{\eta}(H)]u + \tilde{\eta}(H)Au \in \mathcal{E} \). Finally

\[
-\ii t\psi_u'(t) = \langle u|tHe^{itH}u \rangle = \langle u|\tilde{\eta}(H)tH'e^{itH}u \rangle = \langle \tilde{\eta}(H)u|[e^{itH}, A]\zeta(H)u \rangle \\
= \langle e^{-\ii tH}\tilde{\eta}(H)u|A\zeta(H)u \rangle - \langle A\tilde{\eta}(H)u|e^{itH}\zeta(H)u \rangle
\]

and from (2.1) we get the square integrability of \( t\psi_u'(t) \). \( \square \)

5. Higher order commutators

The decay estimates obtained so far on \( \psi_u(t) \) are at most of order \( O(t^{-1}) \) and it is clear that to obtain \( O(t^{-k}) \) for some integer \( k > 1 \) we need conditions of the form \( u \in D(A^k) \) and assumptions on the higher order commutators of \( A \) with \( H \). We recall here the necessary formalism.

Let \( A \) be a self-adjoint operator on a Hilbert space \( \mathcal{H} \) and \( k \in \mathbb{N} \). We say that \( S \) is of class \( C^k(A) \), and we write \( S \in C^k(A) \), if the map \( \mathbb{R} \ni t \mapsto e^{-\ii tA}Se^{\ii tA} \in B(\mathcal{H}) \) is of class \( C^k \) in the strong operator topology. It is clear that \( S \in C^{k+1}(A) \) if and only if \( S \in C^1(A) \) and \( S' \in C^k(A) \). If \( S \in C^k(A) \) we set \( (S')' = S'' = S^{(2)} \), etc. Clearly \( C^k(A) \) is a \(*\)-subalgebra of \( B(\mathcal{H}) \) and if \( S \in B(\mathcal{H}) \) is bijective and \( S \in C^k(A) \) then \( S^{-1} \in C^k(A) \).

For any \( S \in B(\mathcal{H}) \) let \( \mathcal{A}(S) = [S,iA] \) considered as a sesquilinear form on \( D(A) \). We may iterate this and define a sesquilinear form on \( D(A^k) \) by:

\[
S^{(k)} \equiv \mathcal{A}^k(S) = i^k \sum_{i+j=k} \frac{k!}{i!j!} (-1)^i S A^j.
\]

Then \( S \in C^k(A) \) if and only if this form is continuous for the topology induced by \( \mathcal{H} \) on \( D(A^k) \). We keep the notation \( \mathcal{A}^k(S) \) or \( S^{(k)} \) for the bounded operator associated to its continuous extension to \( \mathcal{H} \).

Strictly speaking, the operator \( \mathcal{A} \) acting in \( B(\mathcal{H}) \) must be defined as the infinitesimal generator of the group of automorphisms \( \mathcal{U} = \{ \mathcal{U}_t \}_{t \in \mathbb{R}} \) of \( B(\mathcal{H}) \) given by \( \mathcal{U}_t(S) = e^{tA}S e^{-tA} \). This group is not of class \( C_0 \) and so \( \mathcal{A} \) is not densely defined. Then \( C^k(A) \) is just the domain of \( \mathcal{A}^k \). One may also define \( C^\alpha(A) \) if \( \alpha \) is not an integer as the Besov space of order \( (\alpha, \infty) \) associated to \( \mathcal{U} \).

We denote \( B_1(\mathcal{H}) \) the Banach algebra of trace class operators on \( \mathcal{H} \). Its dual is identified with the space \( B(\mathcal{H}) \) of all bounded operators on \( \mathcal{H} \) with the help of the bilinear form \( \text{Tr}(Sp) \). It is clear that the restrictions of the \( \mathcal{U}_t \) to \( B_1(\mathcal{H}) \subset B(\mathcal{H}) \) give a group of automorphisms of \( B_1(\mathcal{H}) \) and that this group is of class \( C_0 \). We do not distinguish in notation between \( \mathcal{U} \) and \( \mathcal{A} \) and their restrictions to \( B_1(\mathcal{H}) \) but note that for example the domain of \( \mathcal{A} \) in \( B_1(\mathcal{H}) \) is the set of \( S \in C^1(A) \cap B_1(\mathcal{H}) \) such that \( \mathcal{A}(S) \in B_1(\mathcal{H}) \). Moreover, if \( S = \langle u|v \rangle \) and \( u,v \in D(A^k) \) then \( S \) belongs to the domain of \( \mathcal{A}^k \) in \( B_1(\mathcal{H}) \).
Now let $H$ be a self-adjoint operator on $\mathcal{H}$ and $R(z) = (H - z)^{-1}$ for $z$ in the resolvent set $\rho(H)$ of $H$. We say that $H$ is of class $C^k(A)$ if $R(z_0) \in C^k(A)$ for some $z_0 \in \rho(H)$; then we shall have $R(z) \in C^k(A)$ for all $z \in \rho(H)$ and more generally $\varphi(H) \in C^k(A)$ for a large class of functions $\varphi$ (e.g., rational and bounded on the spectrum of $H$).

For each real $m$ let $S^m(\mathbb{R})$ be the set of symbols of class $m$ on $\mathbb{R}$, i.e., the set of functions $\varphi : \mathbb{R} \to \mathbb{C}$ of class $C^\infty$ such that $|\varphi^{(k)}(\lambda)| \leq C_k|\lambda|^{m-k}$ for all $k \in \mathbb{N}$. Note that $S^m \cdot S^n \subset S^{m+n}$ and $\varphi(j) \in S^{m-j}$ if $\varphi \in S^m$ and $j \in \mathbb{N}$.

**Proposition 5.1.** Let $H$ be a self-adjoint operator of class $C^1(A)$ with $H' = \theta(H)$ for some $\theta \in S^2(\mathbb{R})$. Then $H$ is of class $C^\infty(A)$. Let $\delta_\theta$ be the first order differential operator given by $\delta_\theta = \theta(\lambda) \frac{d}{d\lambda}$. If $\theta \in S^1(\mathbb{R})$ and $\varphi \in S^0(\mathbb{R})$ then $\varphi(H)$ is of class $C^\infty(A)$ and

$$A^k(\varphi(H)) = (\delta_{\theta}^k \varphi)(H) \quad \forall k \in \mathbb{N}. \quad (5.8)$$

**Proof.** We begin with a general remark. By using Proposition 3.8 we see that if $H$ is of class $C^1(A)$ and $H' = \theta(H)$ for some real Borel function $\theta$, and if $\varphi \in C^1_0(\mathbb{R})$, then $\varphi'(H) = A(\varphi(H)) = \theta(H)\varphi'(H) = (\theta \varphi)'(H)$. In particular, if $\delta_\theta \varphi = \theta \varphi'$ is a bounded function then $\varphi(H)$ is of class $C^1(A)$.

If we take $\theta \in S^2$ and $\varphi(\lambda) = (\lambda + i)^{-1}$ then $\varphi \in S^1$ hence $H \varphi' \in S^0$. Thus the operator $R = (H + i)^{-1} = \varphi(H)$ satisfies $R' = \psi(H)$ with $\psi \in S^0$. Now we may apply the preceding argument with $\varphi$ replaced by $\psi$ and get $\psi \in C^1(A)$, so $R' \in C^1(A)$, etc. This proves that $H$ is of class $C^\infty(A)$.

In the preceding argument we clearly may take any $\varphi \in S^{-1}$. If $\theta \in S^1$ then the same argument works for any $\varphi \in S^0$ and gives the last assertion of the proposition.

**Remark 5.2.** If $\theta \in S^m$ and $\varphi \in S^{-(m-1)}$ with $1 \leq m \leq 2$ the last assertion of the proposition remains true (with the same proof).

We finish this section with some comments in connection with relation (5.8). At a formal level (5.8) means

$$e^{-itA} \varphi(H) e^{itA} \equiv e^{itA} (\varphi(H)) = (e^{it\delta_{\theta}} \varphi)(H). \quad (5.9)$$

We shall explain without going into details how one may rigorously interpret this relation and how one may use it to get decay estimates.

Let $\xi_t$ be the flow of diffeomorphisms of the real line defined by the vector field $\delta_\theta = \theta(\lambda) \frac{d}{d\lambda}$. This means that $\frac{d}{dt} \xi_t(\lambda) = \theta(\xi_t(\lambda))$ and $\xi_0(\lambda) = \lambda$ for all $\lambda \in \mathbb{R}$ (we assume that such a global flow exists). Then if $\varphi : \mathbb{R} \to \mathbb{C}$ is a smooth function we have $\frac{d}{dt} \varphi \circ \xi_t = (\delta_\theta \varphi) \circ \xi_t$ or $\varphi \circ \xi_t = e^{d\theta \psi} \varphi$. Hence (5.9) may be written $e^{-it\delta_{\theta}} \varphi(H) e^{itA} = (\varphi \circ \xi_t)(H)$. This can be easily checked independently of what we have done before.

Let $M(\mathbb{R})$ be the space of all bounded Borel measures on $\mathbb{R}$. We associate to $H$ a continuous linear map $\Phi : B_1(\mathcal{H}) \to M(\mathbb{R})$ defined as follows: if $\rho \in B_1(\mathcal{H})$ then $\int \varphi \Phi(\rho) = \text{Tr}(\varphi(H) \rho)$ for any bounded Borel function $\varphi$. Then

$$\text{Tr}(\varphi(H) \mathcal{U}_{-t}(\rho)) = \text{Tr}(e^{-itA} \varphi(H) e^{itA} \rho) = \text{Tr}((\varphi \circ \xi_t)(H) \rho)$$

which means that the measure $\Phi(\mathcal{U}_{-t}(\rho))$ is equal to the image of the measure $\Phi(\rho)$ through the map $\xi_t$. Or, if we denote $V_t$ the map $M(\mathbb{R}) \to M(\mathbb{R})$ which sends a measure $\mu$ into its image $\xi_t(\mu)$ through $\xi_t$, we have $\Phi \circ \mathcal{U}_{-t} = V_t \circ \Phi$. 


Thus, if \( \rho \) belongs to the Besov space \( B_1(\mathcal{H})_{s,p} \) associated to the group of automorphisms \( \mathcal{U}_t \) of \( B_1(\mathcal{H}) \) then \( \Phi(\rho) \) belongs to the Besov space \( M(\mathbb{R})_{s,p} \) associated to the group of automorphisms \( \mathcal{V}_t \) of \( M(\mathbb{R}) \) (notations as in [ABG]). This gives smoothness properties of the measure \( \Phi(\rho) \) with respect to the differential operator \( \delta_0 \) in terms of smoothness properties of \( \rho \) with respect to the operator \( A \). In particular, since \( \text{Tr}(e^{itH}\rho) = \int e^{it\lambda}\Phi(\rho)(d\lambda) \) is just the Fourier transform of the measure \( \Phi_\rho \equiv \Phi(\rho) \), this allows us to control the decay as \( t \to \infty \) of \( t \mapsto \text{Tr}(e^{itH}\rho) \) in terms of the local behaviour of the measure \( \Phi_\rho \).

The operators \( \mathcal{V}_t \) can be explicitly computed in many situations and the preceding strategy gives optimal results. For example, in the simplest case \( [H, iA] = 1 \) we get for any \( s > 0 \)

\[
\| \langle A \rangle^{-s} e^{itH} \langle A \rangle^{-s} \| \leq C_s \langle t \rangle^{-s} \tag{5.10}
\]

If \( [H, iA] = H \) then such a good decay is impossible because zero is a threshold (see remark 6.2) but if \( \eta \) is a smooth function equal to zero near zero and to one near infinity then

\[
\| \langle A \rangle^{-s} e^{itH} \eta(H) \langle A \rangle^{-s} \| \leq C_s \langle t \rangle^{-s}. \tag{5.11}
\]

This may be extended to a large class of functions \( \theta \).

6. HIGHER ORDER DECAY

The expressions \( \psi_u(t) = \langle u|e^{itH}u \rangle \) that we considered until now are quadratic in \( u \) and this complicates the higher order computations. To elude this we note that \( \psi_u(t) = \text{Tr}(e^{itH}\rho) \) with \( \rho = |u\rangle \langle u| \), expression which makes sense for any \( \rho \in B_1(\mathcal{H}) \) and is linear in \( \rho \).

We begin with an extension to higher orders of Proposition 4.3.

**Theorem 6.1.** Let \( k \in \mathbb{N} \) and \( s \in [0, k] \) real. Assume that \( H \) is of class \( C^{k+1}(A) \) and \( H' \) commutes with \( H \), satisfies \( H'D(H) \subset \mathcal{H} \), and is boundedly invertible. Then for each vector \( u \in D(|A|^s) \) we have \( \psi_u(t) = O(t^{-s}) \).

**Proof.** By an interpolation argument it suffices to prove \( |\psi_u(t)| \leq C_k(\|u\| + \|A^k u\|)^2 \langle t \rangle^{-k} \) for \( u \) in a dense subspace of \( D(A^k) \). Formally this is quite straightforward starting with the formula \( \langle i t H' \rangle^{-1} A(e^{i t H}) = e^{i t H} \) and then iterating it \( k \) times; we next sketch the rigorous proof. We change slightly the notations from the proof of Proposition 4.3 and denote \( K \) the continuous extension to \( \mathcal{H} \) of \( -i H'^{-1} \). Then \( K \) commutes with \( H \), is of class \( C^k(A) \), and we have \( KA(e^{i t H}) = A(e^{i t H})K = t e^{i t H} \). Let \( u \in D(A^k) \) and \( \rho = |u\rangle \langle u| \) or a more general trace class operator. Let \( L_K \) and \( R_K \) be the operators of right and left multiplication by \( K \), which act both in \( B(\mathcal{H}) \) and in \( B_1(\mathcal{H}) \). Then \( R_K A(e^{i t H}) = t e^{i t H} \) hence

\[
t \psi_u(t) = \text{Tr} ((R_K A)(e^{i t H})\rho) = \text{Tr} (A(e^{i t H})(K \rho)) = -\text{Tr} (e^{i t H} A(K \rho)) = -\text{Tr} (e^{i t H} (A L_K) \rho)
\]

This is easy to justify since \( K u \in D(A^k) \) because \( K \) is of class \( C^k(A) \). In exactly the same way, starting with \( (R_K A)^k (e^{i t H}) \rho = t^k e^{i t H} \) we get

\[
t^k \psi_u(t) = \text{Tr} ((R_K A)^k (e^{i t H}) \rho) = (-1)^k \text{Tr} (e^{i t H} (A L_K)^k \rho).
\]

Finally, it remains to note that \( \|(A L_K)^k \rho\|_{B_1(\mathcal{H})} \leq C_k (\|u\| + \|A^k u\|)^2 \). \( \square \)
Remark 6.2. The following example shows that such a good decay as in Theorem 6.1 cannot be expected if \( H' \) is not boundedly invertible. In the Hilbert space \( \mathcal{H} = L^2(0, \infty) \) let \( H \) be the operator of multiplication by the independent variable \( x \) and let \( A \) be the self-adjoint realization of \( \frac{d}{dx} \left( \frac{x}{2} \frac{d}{dx} + \frac{d}{dx} x \right) \). Then \( H \) is of class \( C^\infty(A) \) and \( H' = [H, iA] = H \).

Let \( u \) be a \( C^\infty \) function on \((0, \infty)\) which is zero for \( x > 2 \) and equal to \( x^{-\theta} \) for \( x < 1 \) with \( 0 < \theta < 1 / 2 \). Then \( u \in D(|A|^s) \) for all \( s > 0 \) but \( \psi_u(t) \sim \int_0^1 e^{itx} x^{-2\theta} dx \sim t^{2\theta - 1} \) for \( t \rightarrow \infty \), hence the decay can be made as bad as possible. On the other hand, Example 2.4 explains why the space \( \mathcal{E} \) helps to improve the behaviour.

We now give a higher order version of Theorem 4.9. Recall that \( \theta \in S^m(\mathbb{R}) \) is an elliptic symbol if there is \( c > 0 \) such that \( |\theta(\lambda)| \geq c|\lambda|^m \) near infinity. Then \( \eta/\theta \in S^{-m}(\mathbb{R}) \) for any \( C^\infty \) function \( \eta \) with support in the region where \( \theta \neq 0 \) and equal to one near infinity.

Theorem 6.3. Let \( H \in C^1(A) \) such that \( H' = \theta(H) \) for some elliptic symbol \( \theta \in S^m(\mathbb{R}) \) with \( 0 \leq m \leq 1 \). Assume: (1) \( \theta(\lambda) \neq 0 \) if \( \lambda \neq 0 \) and (2) \( \lambda \theta(\lambda) \) extends to a \( C^\infty \) function on \( \mathbb{R} \). Let \( k \) be an odd integer and let \( u \in \mathcal{H} \) be of the form \( |H|^{(k-1)/4} v \) for some \( v \in D(A^k) \) such that \( A^j v \in \mathcal{E} \) for \( 0 \leq j \leq k \). Then \( |\psi_u(t)| \leq C_u(t)^{-k/2} \).

Proof. Denote \( S^0(\mathbb{R}) \) the set of \( a \in S^0(\mathbb{R}) \) such that \( a(\lambda) = 0 \) near zero. We first prove the following: if \( n \in \mathbb{N} \) and \( a \in S^0(\mathbb{R}) \) then there are \( a_0, a_1, \ldots, a_n \in S^0(\mathbb{R}) \) such that:

\[
 t^n a(H)e^{itH} = \sum_{j=0}^n A^j \left( a_j(H)e^{itH} \right). \tag{6.12}
\]

Of course, the \( a_j \) also depend on \( n \). If \( n = 1 \) we write (see also the proof of Theorem 4.9):

\[
 ta(H)e^{itH} = -\frac{a(H)}{\theta(H)} A(e^{itH}) = A \left( a_1(H)e^{itH} \right) + a_0(H)e^{itH} \tag{6.13}
\]

where \( a_1 = \frac{a}{\theta} \) and \( a_0 = -\theta a_1' \) (use Proposition 5.1). We mention that we use without comment the relation \( A(ST) = A(S)T + SA(T) \) with the further simplification that in our context \( S \) and \( T \) are functions of \( H \) hence commute. Now assume (6.12) is true and let us prove it with \( n \) replaced by \( n + 1 \). Let \( b \in C^\infty \) equal to zero near zero and to 1 near infinity and such that \( a_j = a_j b \) for all \( j \). Then

\[
 t^{n+1} a(H)e^{itH} = \sum_{j=0}^n A^j \left( a_j(H)tb(H)e^{itH} \right)
\]

Now we use (6.13) and replace \( tb(H)e^{itH} = A \left( b_1(H)e^{itH} \right) + b_0(H)e^{itH} \). Thus

\[
 a_j(H)tb(H)e^{itH} = a_j(H)A \left( b_1(H)e^{itH} \right) + a_j(H)b_0(H)e^{itH}
\]

\[
 = A \left( a_j(H)b_1(H)e^{itH} \right) + (a_j(H)b_0(H) - A(a_j(H))b_1(H))e^{itH}
\]

which clearly gives the required result.

Now we begin the proof of the theorem. As in the proof of Theorem 4.9 we consider separately the case when \( u \) is zero near energy zero and that when \( u = \varphi(H)u \) for some \( \varphi \in C^\infty_{\text{c}} \). The first case is an immediate consequence of (6.12) because there is \( a \in S^0(\mathbb{R}) \) such that \( a(H)u = u \) hence (recall the notation \( \rho = |u\rangle\langle u| \))

\[
 t^k \langle u|e^{itH}u \rangle = \text{Tr} \left( t^k a(H)e^{itH} \rho \right) = \sum_{j=0}^k (-1)^j \text{Tr} \left( a_j(H)e^{itH} A^j(\rho) \right) \tag{6.14}
\]
which implies \( \psi_u(t) = O(t^{-k}) \) because obviously \( \rho \in D(A^k) \) if \( u \in D(A^k) \).

Note that the facts established above hold for an arbitrary \( u \in \mathcal{H} \). The condition involving \( v \) is needed to have some control on the behaviour of \( u \) at zero energy, which cannot be arbitrary as explained in Remark 6.2. When we localize near zero energy we replace \( u \) by \( \varphi(H)u \) with \( \varphi \in C_c^\infty \) equal to one on a neighbourhood of zero. If \( u = |H|^m v \) with \( m = (k - 1)/4 \) and \( v \in D(A^k) \) such that \( A^j v \in \mathcal{E} \) for \( 0 \leq j \leq k \) then \( \varphi(H)u = |H|^m \varphi(H)v \). By Proposition 5.1 \( H \) is of class \( C^\infty(A) \) so \( \varphi(H)D(A^j) \subset D(A^j) \) for any \( j \) and \( A^j \varphi(H)v \in \mathcal{E} \) by Lemma 2.3.

Thus for the rest of the proof we may assume that the support of \( u \) in a spectral representation of \( H \) is included in \([-1, 1]\) and \( u = |H|^m v \) with \( v \in D(A^k) \) such that \( A^j v \in \mathcal{E} \) for \( 0 \leq j \leq k \). It is clear that \( v \) has the same \( H \)-support as \( u \). Our purpose is to check the assumptions of the Corollary 8.3 for \( \psi = \psi_u \). There are two conditions to be verified: the functions \( t \frac{d}{dt} \psi_u(t) \) and \( t \frac{d}{dt} \psi_u(t) \) should be in \( L^2(\mathbb{R}) \). We treat only the second one, the first is treated similarly. If \( \ell = 2m + 1 = (k + 1)/2 \) then

\[
\left( t^\ell \frac{d}{dt} \psi_u(t) \right) = \langle u | i^{\ell} t^\ell H e^{iT} | v \rangle = \langle |H|^m v | i^{\ell} t^\ell H e^{iT} | |H|^m v \rangle = \langle v | i^{\ell} t^\ell H sgn^{\ell + 1}(H) e^{i\ell T} | v \rangle.
\]

Let \( \eta \) be a \( C^\infty \) function with compact support such that \( \eta(\lambda) = \lambda/\theta(\lambda) \) on \([-1, 1]\). Then \( \lambda = \eta(\lambda) \theta(\lambda) \) on \([-1, 1]\) \( \setminus \{0\} \) hence on \([-1, 1]\) so we have

\[
i^{\ell - 1} t^\ell \psi_u(t) = \eta(\ell) e^{i \ell \theta(\ell)} e^{i \ell T} v = \langle v | i^{\ell} t^\ell H \theta(e^{i \ell T}) e^{i \ell T} | v \rangle.
\]

Recall that we have \( \mathcal{A} e^{iH} = iH e^{iH} \) in a sense described in Proposition 3.8. But under the present conditions we have much more because \( H'D(H) \subset \mathcal{H} \) hence \( e^{iA} \) leaves invariant the domain of \( H \) and induces there a \( C_0 \)-group (see the assertion (2) page 5). In particular, the set of \( u \in D(H) \cap D(A^j) \) such that \( A^j u \in D(H) \) for any \( j \in \mathbb{N} \) is dense in \( D(H) \) (and is a core for \( A \)). Moreover, the \( A^j(H) \) are bounded operators if \( j \geq 2 \). This allows us to compute \( \mathcal{A} e^{iH} \) inductively as usual. Our next computations look slightly formal but it is straightforward, although a little tedious, to rigorously justify each step.

Above we fixed \( \ell \) to the value \((k + 1)/2\) but now we allow it to take any value smaller than this one. For the case \( \ell = 1 \) see the proof of Theorem 4.9. For \( \ell = 2 \) we write

\[
\mathcal{A}^2 e^{iH} = \mathcal{A} iH e^{iH} = \frac{H''}{H'} iH e^{iH} + (iH')^2 e^{iH} = \frac{H''}{H'} \mathcal{A} e^{iH} + (iH')^2 e^{iH}.
\]

By “localising” Proposition 5.1 we get \( H'' = \mathcal{A} (H') = \mathcal{A} (\theta(H)) = \theta(H) \theta'(H) = H' \theta'(H) \) hence \( \frac{H''}{H'} = \theta'(H) \). Thus

\[
(iH')^2 e^{iH} = \mathcal{A}^2 e^{iH} - \theta'(H) \mathcal{A} e^{iH}.
\]

Then if we set \( \rho = \eta^2(H) v \) we get

\[
i^2 \psi_u(t) = \text{Tr} \left( (iH')^2 e^{iH} \rho \right) = \text{Tr} \left( \mathcal{A}^2 (e^{iH}) \rho \right) - \text{Tr} \left( \theta'(H) \mathcal{A} (e^{iH}) \rho \right) = \text{Tr} \left( e^{iH} \mathcal{A}^2 (\rho) \right) - \text{Tr} \left( e^{iH} \mathcal{A} (\rho \theta'(H)) \right).
\]

The right hand side belongs to \( L^2(\mathbb{R}) \) by the argument from Theorem 4.9, which finishes the proof in the case \( \ell = 2 \). The general case does not involve any new idea: by writing conveniently \( \frac{H''}{H'} \) one may express \( (iH')^2 e^{iH} \) as a linear combination of functions of \( H \) times commutators \( \mathcal{A}^j (e^{iH}) \) and one may proceed as above. \( \square \)
7. Applications

We will use the previous results to obtain decay estimates for \( \psi_u(t) = \langle u | e^{itH} u \rangle \) in several situations. Note that Example 3.1 and Proposition 3.2 show that the commutation relation is not enough to prove the \( C^1(A) \) condition for \( H \). For instance, in addition to the continuity of \([A, H]\) on \( D(A) \cap D(H)\), it suffices to verify the invariance of domain \( R(z)D(A) \subset D(A) \). In other cases it is convenient to verify the simplified assumptions of Mourre [Mo], which are stronger than the \( C^1(A) \) property [ABG]:

(a) \( e^{i\theta A}D(H) \subset D(H) \)

(b) There is a subspace \( \mathcal{S} \subset D(A) \cap D(H) \) which is core for \( H \) such that \( e^{i\theta A} \mathcal{S} \subset \mathcal{S} \) and the form \([H, iA]\) on \( \mathcal{S} \) extends to a continuous operator \( D(H) \to \mathcal{H} \).

Recall the subspace \( \mathcal{E} = \{ u \in \mathcal{H} | [u]_H < \infty \} \), where \([u]_H = (\int_\mathbb{R} |\psi_u(t)|^2 dt)^{1/4} \).

7.1. Example 1: Laplacian in \( \mathbb{R}^n \).

Let \( H = -\Delta \) in \( L^2(\mathbb{R}^n) \) with domain the Sobolev space \( \mathcal{H}^2(\mathbb{R}^n) \) and \( A = -\frac{1}{2}(x \cdot \nabla + \nabla \cdot x) \) the generator of dilations which is essentially self-adjoint on the Schwartz space \( \mathcal{S}(\mathbb{R}^n) \). Condition (a) is a consequence of the formula \( e^{i\theta A}(H + i)^{-1} = (e^{-2\theta H} + i)^{-1}e^{-i\theta A} \), and (b) is satisfied since \( \mathcal{S} \) is a core for \( H \) which is trivially invariant under the dilatation group. Integration by parts on \( \mathcal{S} \) shows that \([H, iA] = 2H\). We conclude from Proposition 4.7 that for \( u \in D(A) \) such that \( u, Au \in \mathcal{E} \), \( \psi_u \) satisfies the decay estimate \( |\psi_u(t)| \leq C_u(t)^{-1/2} \). Higher-order decay estimates follow from Theorem 6.3.

7.2. Example 2: \( H = -\partial_{xx} + \partial_{yy} \) in \( \mathbb{R}^2 \).

Let \( H = -\partial_{xx} + \partial_{yy} \) and \( A = -\frac{1}{2}(x \cdot \nabla + \nabla \cdot x) \) in \( L^2(\mathbb{R}^2) \). With the help of a Fourier transformation we see that \( H \) is essentially self-adjoint on \( S(\mathbb{R}^2) \). Clearly \([H, iA] = 2H\), hence the estimate of Example 1 holds. One may treat similarly the case when the operator \( H \) in \( L^2(\mathbb{R}^n) \) is an arbitrary homogeneous polynomial of order \( m \) in the derivatives \( i\partial_1, \ldots, i\partial_n \) with constant coefficients: then \([H, iA] = mH\).

7.3. Example 3: Electric field in \( \mathbb{R}^n \).

Here we study the case \( H = -\Delta + \vec{h} \cdot x \) and \( A = i\vec{h} \cdot \nabla \) in \( \mathbb{R}^n \), where \( \vec{h} \) is a fixed unitary vector. We take again \( \mathcal{S} = S(\mathbb{R}^n) \) as a core for \( H \) and then it is easy to check the commutation relation \([H, iA] = 1\). Therefore Proposition 4.1 provides the estimate \( |\psi_u(t)| \leq C_u(t)^{-1} \) for \( u \in D(A) \), where \( C_u = 2\|u\|\|Au\| \). Further estimates follow from Theorem 6.1.

7.4. Example 4: \( H = -x^{2-\theta} \Delta - \Delta x^{2-\theta} \) in \( \mathbb{R}_+ \).

For \( 0 < \theta < 2 \) consider \( H = -x^{2-\theta} \Delta - \Delta x^{2-\theta} \) and \( A = -\frac{1}{2}(x \cdot \nabla + \nabla \cdot x) \) in \( \mathbb{R}_+ \). Then \( \mathcal{S} = C_0(\mathbb{R}_+) \) is a core for \( H \) and the domain conditions follow from the formula \( e^{-i\alpha A}H e^{i\alpha A} = e^{i\alpha H} \). The commutation relation is \([H, iA] = \theta H\), which yields the estimate of Example 1.
7.5. **Example 5**: Fractional laplacian in $\mathbb{R}^n$.

For $0 < s < 2$, let $H = (-\Delta)^{s/2}$ with domain the Sobolev space $\mathcal{H}^s(\mathbb{R}^n)$ and consider $A = -\frac{1}{2}(x \cdot \nabla + \nabla \cdot x)$. Then $\mathcal{S} = C_c^\infty(\mathbb{R}_+)$ is a core for $H$ and homogeneity of $H$ with respect to $A$ implies that $[H, iA] = sH$. The estimate of Example 1 follows.

7.6. **Example 6**: Multiplication by $\lambda$ in $L^2(\mathbb{R}_+, d\mu)$.

Let $H = \lambda$ and $A = -\frac{1}{2}(\lambda \partial_\lambda + g(\lambda))$ on $L^2(\mathbb{R}_+, d\mu)$, where $g$ is to be determined. Assume that $d\mu = h(\lambda)d\lambda$, for some non-vanishing function $h$ of class $C^1(\mathbb{R}_+)$. It can be shown that if $g$ satisfies the relation $g(\lambda) = \lambda^{N-1} + 1$, then $A$ is self-adjoint in $L^2(\mathbb{R}_+, d\mu)$. For instance, if $h(\lambda) = \lambda^N$ then choose $g(\lambda) = N + 1$. If $g$ is a bounded function, $\mathcal{S} = C_c^\infty(\mathbb{R}_+)$ is a core for $A$ and the commutation relation is $[H, iA] = 2H$. For $z \in \rho(H)$ the function $(\lambda - z)^{-1}$ is smooth and has bounded derivative on $\mathbb{R}_+$, hence the domain invariance $R(z)D(A) \subset D(A)$ can be easily checked. Therefore $H$ is of class $C^1(A)$, which gives the estimate of Example 1.

7.7. **Example 7**: Wave equation in $\mathbb{R}^n$.

For $H > 0$ consider the equation

\[
\begin{aligned}
\partial_t u + H^2 u &= 0 \\
u(0) &= f \\
\partial_t u(0) &= g.
\end{aligned}
\]

Assume $\mathcal{H} = L^2(\mathbb{R}^n)$. Define $u_1(t) := \cos(tH)$, $u_2(t) := \frac{\sin(tH)}{H}$. Then $u(t) := u_1(t)f + u_2(t)g$ is a solution to (WE).

For $f$, $g \in \mathcal{H}$ define the function $\psi_{f,g}(t) := \langle f|u_1(t)f \rangle + \langle f|u_2(t)g \rangle$ and the subspace $\mathcal{E} = \{u \in \mathcal{H} | |u|_H < \infty\}$, where $|h|_H = \|\langle h|u_1(t)h \rangle\|_{L^2_t}^{1/2} + \|\langle h|u_2(t)h \rangle\|_{L^2_t}^{1/2}$.

**Proposition 7.1.** Let $H$ and $A$ be self-adjoint operators, assume $H \in C^1(A)$ and the commutation relation $[H, iA] = cH$, with $c \neq 0$. Then for $f, g \in D(A) \cap \mathcal{E}$ such that $Af, Ag \in \mathcal{E}$, one has the estimate $|\psi_{f,g}(t)| \leq C_{f,g}(t)^{-1/2}$.

**Proof.** Similarly to Proposition 3.8, the following two sesquilinear forms restricted to $D(A) \cap D(H)$ extend to continuous forms on $D(H)$ satisfying the identities

\[
\begin{bmatrix}
\cos(tH), iA \\
\sin(tH), iA
\end{bmatrix}
= -ctH \sin(tH) \\
= ct \cos(tH) - c \frac{\sin(tH)}{H}.
\]

We will use Corollary 8.2 for $f, g \in D(|H|^{1/2})$. Clearly $\psi_{f,g} \in L^2(\mathbb{R})$. Now we calculate

\[
sct \psi_{f,g}(t) = -\langle f|ctH \sin(tH)f \rangle + \langle f|c \cos(tH)g \rangle
= \langle f|[u_1, iA]f \rangle + \langle f|[u_2, iA]g \rangle + c\langle f, u_2g \rangle
= \langle u_1f, iAf \rangle + \langle u_2f, iAg \rangle + \langle u_2f, iAg \rangle + \langle u_2f, iAg \rangle + c\langle f, u_2g \rangle.
\]

Thus $c\|\psi_{f,g}\|_{L^2} \leq C(f, g) + |Ag|_H + |Ag|_H + |Af|_H$. For $f, g$ not necessarily in $D(H)$ we can proceed analogously to Proposition 4.7 using $u_\epsilon = R_\epsilon u$ and letting $\epsilon \to 0$. \(\square\)
7.8. Example 8: Klein-Gordon equation in $\mathbb{R}^n$.

Now we draw our attention to (WE) in the case $H = \sqrt{-\Delta + m^2}$, for $m > 0$. The vector space is again defined as $E = \{u \in H | \|u\|_H < \infty\}$, where $\|h\|_H = \|(h|e^{itH}h)\|_{L^2}$. Let $A$ be the generator of dilations, then $H$ is of class $C^1(A)$ and it can be formally shown that $[H, iA] = H - m^2H^{-1}$.

Let $u_1, u_2$ be as in (WE), define $\psi_{f,g}^1(t) = \langle f|u_1(t)f \rangle$ and $\psi_{f,g}^2(t) = \langle f|u_2(t)g \rangle$. We are interested in the decay rate of $\psi_{f,g}(t) := \psi_{f,g}^1(t) + \psi_{f,g}^2(t)$.

**Proposition 7.2.** For $H$ and $A$ defined as above and $f, g \in D(A) \cap E$ such that $Af, Ag \in E$, then $|\psi_{f,g}(t)| \leq C_{f,g}^{-1/2}$.

**Proof.** Note that this result is a direct consequence of Proposition 4.9. Higher-order decay estimates follow from Proposition 6.3. Here we present a direct proof.

As continuous forms on $D(H)$

$$[e^{it(H-m)}A] = t(H-m)(1+mH^{-1})e^{it(H-m)}. \quad (7.15)$$

Define the auxiliary function $\psi(t) := \langle f|e^{it(H-m)}g \rangle$ and we now prove the conditions of Corollary 8.2. It is clear that $\psi \in L^2(\mathbb{R})$ since $f, g \in E$. Assume $g \in D(H)$ and we estimate

$$t\psi^2(t) = \langle f|t(H-m)e^{it(H-m)}g \rangle$$

$$= \langle f|([e^{it(H-m)}A]H(H+m)^{-1}g) \rangle$$

$$= \langle e^{-it(H-m)}fAH(H+m)^{-1}g + (Af)e^{it(H-m)}(H(H+m)^{-1}g).$$

By Lemma 2.3 we conclude that $\|\psi\|_{L^2} \leq C([f]_{H[|g]_H} + [f]_{H[Ag]} + [Af]_{H[Ag]}).$ For general $g \in D(A)$, replace it by $g_\epsilon = R_\epsilon g$ and let $\epsilon \to 0$.

We conclude that $|\psi(t)| \leq C_{f,g}(t)^{-1/2}$. Notice that $|\psi|_{L^2} = |\psi(t)|$ satisfies the same inequality.

Now we prove the desired estimate. Observe that $\psi_{f,g}^1(t) = \frac{1}{2} (\langle f|e^{itH}f \rangle + \langle f|e^{-itH}f \rangle$), therefore $|\psi_{f,g}^1(t)| \leq C_{f,g}(t)^{-1/2}$.

For the second term, we write $\psi_{f,g}^2(t) = \frac{1}{2} (\langle f|H^{-1}e^{itH}g \rangle + \langle f|H^{-1}e^{-itH}g \rangle$) and redefine the auxiliary function $\psi(t) := \langle f|H^{-1}e^{it(H-m)}g \rangle$, which is in $L^2(\mathbb{R})$ by the spectral theorem. Now

$$t\psi^2(t) = \langle f|tH^{-1}(H-m)e^{it(H-m)}g \rangle$$

$$= \langle f|([e^{it(H-m)}A]H(H+m)^{-1}g) \rangle$$

$$= \langle e^{-it(H-m)}fAH(H+m)^{-1}g + (Af)e^{it(H-m)}(H(H+m)^{-1}g),$$

which again yields the estimate $|\psi(t)| \leq C_{f,g}(t)^{-1/2}$, concluding the proof. □

8. Appendix

We prove here an auxiliary estimate. We consider functions $g$ defined on $\mathbb{R}_+ = (0, \infty)$ and denote $\|g\|_p$ their $L^p$ norms. Let $\delta$ the operator $(\delta g) = zg'(x)$ acting in the sense of distributions and set $\tilde{g}(t) = \int_0^\infty e^{itx}g(x)dx$ for $t > 0$ (improper integral).
Lemma 8.1. \(|\tilde{g}(t)| \leq |t|^{-1/2}2^{3/2}(p-1)^{-1/2p}|g|_p^{1/2}||\delta g||_q^{1/2}\) if \(1 < p < \infty\) and \(\frac{1}{p} + \frac{1}{q} = 1\).

Proof. We may assume that \(g \in L^p\) and \(\delta g \in L^q\). For any \(s > 0\) we have

\[
\left| \int_0^s e^{itx} g(x) dx \right| \leq s^{1/q}||g||_p. \tag{8.16}
\]

Since \(g \in L^p\) with \(p < \infty\), there is a sequence \(a_n \to \infty\) such that \(g(a_n) \to 0\) (otherwise \(|g(x)| \geq c > 0\) on a neighbourhood of infinity, so \(|g|^p\) cannot be integrable). Since \(p > 1\), after integrating over \((s, a_n)\) and then making \(n \to \infty\), we also obtain

\[
|g(s)| \leq \int_s^\infty |g'(x)| dx \leq (p - 1)^{-1/p} s^{-1/q} ||\delta g||_q \tag{8.17}
\]

by Hölder inequality. Then

\[
\int_s^\infty e^{itx} g(x) dx = \lim_{a \to \infty} \int_s^a \left( \frac{d}{dx} \frac{1}{it} e^{itx} \right) g(x) dx \\
= \lim_{a \to \infty} \left[ e^{itx} g(a) - e^{itx} g(s) \right] - \frac{1}{it} \int_s^a e^{itx} g'(x) dx.
\]

We take here \(a = a_n\) and make \(n \to \infty\) to get

\[-it \int_s^\infty e^{itx} g(x) dx = e^{itx} g(s) + \int_s^\infty e^{itx} g'(x) dx\]

and then by using (8.17) two times we obtain

\[
\left| \int_s^\infty e^{itx} g(x) dx \right| \leq 2(p - 1)^{-1/p} s^{-1/q} t^{-1} ||\delta g||_q.
\]

Let \(\varepsilon > 0\) and \(s = \varepsilon^q/t\). Then (8.16) and the last inequality give

\[
|\tilde{g}(t)| \leq \varepsilon t^{-1/q} ||g||_p + 2(p - 1)^{-1/p} s^{-1/q} t^{-1} ||\delta g||_q.
\]

The infimum over \(\varepsilon > 0\) of an expression \(\varepsilon a + e^{-1}b\) is \(2\sqrt{ab}\). This finishes the proof. \(\square\)

Corollary 8.2. If \(\psi \in L^2(\mathbb{R})\) and \(t^\psi(t) \in L^2(\mathbb{R})\) then \(|\psi(t)| \leq C_\psi |t|^{-1/2}\) for \(t \in \mathbb{R}\setminus \{0\}\).

Proof. We use Lemma 8.1 with \(p = 2\) and \(g\) equal to the Fourier transform of \(\psi\). \(\square\)

Corollary 8.3. If a function \(\psi\) is such that \(t^{\frac{k+1}{2}} \psi(t)\) and \(t^{\frac{k+1}{4}} \psi'(t)\) belong to \(L^2(\mathbb{R})\) for some \(k \geq 1\) then \(|\psi(t)| \leq C_\psi |t|^{-k/2}\) for all \(t \in \mathbb{R}\setminus \{0\}\).

References

[ABG] W. Amrein, A. Boutet de Monvel, V. Georgescu: \textit{C\(_0\)-groups, commutator methods and spectral theory of N-body Hamiltonians}, Birkhäuser, Basel-Boston-Berlin, (1996).

[BouG] A. Boutet de Monvel and V. Georgescu: Some developments and applications of the abstract Mourre theory. In “\textit{Méthodes semi-classiques}”, vol. 2 (Nantes,1991). Astérisque 210, 27-48, 1992.

[BG] N. Bousaid and S. Golénia: Limiting absorption principle for some long range perturbations of Dirac systems at threshold energies. \textit{Comm. Math. Phys.} 299(3):677-708, 2010.

[BH] J.-F. Bony and and D. Hafner: The semilinear wave equation on asymptotically euclidean manifolds. \textit{Comm. Partial Differential Equations} 35(1):23-67, 2010.

[BS01] P. Blue and A. Soffer: Semilinear wave equations on the Schwarzschild manifold I: Local decay estimates. \textit{Advances in Dif. Eqs.} 8(5):595-614, 2003.

[BS02] P. Blue and A. Soffer: Phase space analysis on some black hole manifolds. \textit{J. Funct. Anal.} 256(1):1-90, 2009.

[BW] H. Baumgärtel, M. Wollenberg: \textit{Mathematical scattering theory}, Springer, 1983.
[CFKS] H.L. Cycon, R. Froese, W. Kirsch, B. Simon: Schrödinger operators, with applications to quantum mechanics and global geometry, second edition, Springer (2008).

[DR] M. Dafermos and I. Rodnianski: The black hole stability problem for linear scalar perturbations, in Proceedings of the Twelfth Marcel Grossmann Meeting on General Relativity, Singapore, 2011, 132-189, arXiv: 1010.5137

[DSS] R. Donninger, W. Schlag and A. Soffer: A proof of Price’s Law on Schwarzschild blackhole manifolds for all angular momenta. Advances in Mathematics 226:484-540, 2011.

[EGG] B. Erdogan, M. Goldberg and W. Green: Dispersive estimates for four dimensional Schrödinger and wave equations with obstructions at zero energy. Comm. PDE. 39(10):1936-1964, 2014.

[FRT] C. Fernández, S. Richard and R. Tiedra de Aldecoa: Commutator methods for unitary operators. J. Spectr. Theory 3:271-292, 2013.

[Ger] C. Gérard: A proof of the abstract limiting absorption principle by energy estimates. J. Funct. Anal. 254:2070-2704, 2008.

[Go] S. Golénia: Positive commutators, Fermi golden rule and the spectrum of the zero temperature Pauli-Fierz hamiltonians. Journal of Functional Analysis 256(8):2587-2620, 2009.

[GG] V. Georgescu and C. Gérard: On the virial theorem in quantum mechanics. Communications in Mathematical Physics 208(2):275-281, 1999.

[GJ1] S. Golénia and T. Jecko: A new look at Mourre’s commutator theory. Complex Analysis and Operator Theory 1(3): 399-422, 2007.

[GJ2] S. Golénia and T. Jecko: Rescaled Mourre’s commutator theory, application to Schrödinger operators with oscillating potential. Journal of Operator Theory 70(1):109-144, 2013.

[GL] V. Georgescu and S. Golénia: Isometries, Fock spaces and spectral analysis of Schrödinger operators on trees. J. Funct. Anal. 227: 389-429, 2005.

[GM1] S. Golénia and S. Moroianu: Spectral analysis of magnetic Laplacians on conformally cusp manifolds. Ann. Henri Poincaré 9(1):131-179, 2008.

[GM2] S. Golénia and S. Moroianu: The spectrum of Schrödinger operators and Hodge Laplacians on conformally cusp manifolds. Trans. Amer. Math. Soc. 364(1):1-29, 2012.

[HSS] J.-L. Journé, A. Soffer and C. D. Sogge: Decay estimates for Schrödinger operators. Comm. Pure Appl. Math. 44:573-604, 1991.

[JK] A. Jensen and T. Kato: Spectral properties of Schrödinger operators and time-decay of the wave functions. Duke Math. J. 46(3):583-611, 1979.

[JV1] A. Jensen and G. Nenciu: A unified approach to resolvent expansions at thresholds. Rev. Math. Phys. 13(6):717-754, 2001.

[JV2] A. Jensen and G. Nenciu: Erratum: “A unified approach to resolvent expansions at thresholds”. Rev. Math. Phys. 16(5):675-677, 2004. [Rev. Math. Phys. 13(6):717-754, 2001]

[JSS] J.-L. Journé, A. Soffer and C. D. Sogge: Decay estimates for Schrödinger operators. Comm. Pure Appl. Math. 44:573-604, 1991.

[KK] A. Komech and E. Kopylova: Dispersion Decay and Scattering Theory. John Wiley & Sons, Hoboken, NJ, 2012.

[LS] H. Leinfelder and C. Simader: Schrödinger operators with singular magnetic vector potentials. Mathematische Zeitschrift 176(1):1-19, 1981.

[Mo] E. Mourre: Absence of singular continuous spectrum for certain self-adjoint operators. Communications in Mathematical Physics 78(3):519-567, 1981.

[MR] M. Măntoiu and S. Richard: Absence of singular spectrum for Schrödinger operators with anisotropic potentials and magnetic fields. J. Math. Phys. 41: 2732-2740, 2000.

[MT] M. Măntoiu and R. Tiedra de Aldecoa. Spectral analysis for convolution operators on locally compact groups. J. of Funct. Anal. 253(2):675-691, 2007.

[MRT1] M. Măntoiu, S. Richard and R. Tiedra de Aldecoa: Spectral Analysis for Adjacency Operators on Graphs. Ann. Henri Poincaré 8:1401-1423, 2007.

[MRT2] M. Măntoiu, S. Richard and R. Tiedra de Aldecoa: The Method of the Weakly Conjugate Operator: Extensions and Applications to Operators on Graphs and Groups. Petroleum - Gas University of Ploiesti Bulletin, Mathematics - informatics - physics Series LXI, LXI: 1-12, 2009.

[Ro] M. Rosenblum: Perturbation of the continuous spectrum and unitary equivalence. Pacific Journal of Mathematics 7: 997-1010, 1957.

[RS] M. Reed, B. Simon: Methods of modern mathematical physics, 4 volumes, Academic Press.
ABSTRACT THEORY OF POINTWISE DECAY

[RSch] W. Schlag and I. Rodnianski: Time decay for solutions of Schrödinger equations with rough and time-dependent potentials. Invent. Math. 155(3):451-513, 2004.

[RoT] I. Rodnianski and T. Tao: Long time decay estimates for the Schrödinger equation on manifolds. Mathematical aspects of nonlinear dispersive equations, Ann. of Math. Stud. 1:223-253, 2007.

[RT1] S. Richard, R. Tiedra de Aldecoa: On perturbations of Dirac operators with variable magnetic field of constant direction. J. Math. Phys. 45: 4164-4173, 2004.

[RT2] S. Richard, R. Tiedra de Aldecoa: On the spectrum of magnetic Dirac operators with Coulomb-type perturbations. J. Funct. Ana. 250:625-641, 2007

[RT3] S. Richard, R. Tiedra de Aldecoa: A few results on Mourre theory in a two-Hilbert spaces setting. Anal. Math. Phys. 3:183-200, 2013.

[RT4] S. Richard, R. Tiedra de Aldecoa: Spectral analysis and time-dependent scattering theory on manifolds with asymptotically cylindrical ends. Rev. Math. Phys. 25: 1350003-1 - 1350003-40, 2013.

[Sa] J. Sahbani: The conjugate operator method for locally regular Hamiltonians. J. Op. Theory., vol. 38, no. 2, 297-322, 1997.

[Si] K.B. Sinha: On the absolutely and singularly continuous subspaces in scattering theory. Annales de l'I.H.P., section A 26(3): 263-277, 1977.

[Sch] W. Schlag: Dispersive estimates for Schrödinger operators: a survey. Mathematical aspects of nonlinear dispersive equations, Ann. of Math. Stud. 1:255-285, 2007.

[So] A. Soffer: Monotonic local decay estimates. arXiv:1110.6549, 2011.

[SS] I. M. Sigal and A. Soffer: Local decay and velocity bounds for time-independent and time-dependent Hamiltonians. Preprint, Princeton, 1987.

[Ta] D. Tataru: Local decay of waves on asymptotically flat stationary space-times. American Journal of Mathematics 135(2):361-401, 2013.

[Ti] R. Tiedra de Aldecoa: Commutator methods for the spectral analysis of uniquely ergodic dynamical systems. Ergod. Th. & Dynam. Sys., First view, 1-24, 2014.

[Ya] K. Yajima: Dispersive estimates for Schrödinger equations with threshold resonance and eigenvalue. Commun. Math. Phys. 259:475-509, 2005.

(1) V. GEORGESCU, DÉPARTEMENT DE MATHEMATIQUES, UNIVERSITÉ DE CERGY-PONTOISE, 95000 CERGY-PONTOISE, FRANCE

E-mail address: vladimir.georgescu@math.cnrs.fr

(2) RUTGERS UNIVERSITY, DEPARTMENT OF MATHEMATICS, 110 FREYLINGHUYSEN ROAD, PISCATAWAY, NJ 08854, U.S.A.

E-mail address: mlarenas@math.rutgers.edu, soffer@math.rutgers.edu