Submartingale property of set-valued stochastic integration associated with Poisson process and related integral equations on Banach spaces

Jinping Zhang\textsuperscript{a}, Itaru Mitoma\textsuperscript{b}, Yoshiaki Okazaki\textsuperscript{c}

\textsuperscript{a}Department of Mathematics and Physics, North China Electric Power University, Beijing, 102206, P.R.China
\textsuperscript{b}Department of Mathematics, Saga University, Saga, 840-8502, Japan
\textsuperscript{c}Department of Systems Design and Informatics, Kyushu Institute of Technology, Iizuka, 820-8502, Japan

Abstract

In an M-type 2 Banach space, firstly we explore some properties of the set-valued stochastic integral associated with the stationary Poisson point process. By using the Hahn decomposition theorem and bounded linear functional, we obtain the main result: the integral of a set-valued stochastic process with respect to the compensated Poisson measure is a set-valued submartingale but not a martingale unless the integrand degenerates into a single-valued process. Secondly we study the strong solution to the set-valued stochastic integral equation, which includes a set-valued drift, a single-valued diffusion driven by a Brownian motion and the set-valued jump driven by a Poisson process.

Keywords: Set-valued stochastic integration, Set-valued submartingale, Poisson process

2000 MSC: Primary 65C30, Secondary 26E25, 54C65

1. Introduction

Set-valued stochastic calculus is the natural extension of single-valued case. Aumann \cite{Aumann}(1965) defined the expectation of set-valued random variables. Hiai and Umegaki \cite{Hiai} (1977) gave the definition of set-valued conditional expectation, set-valued martingale (or super/submartingale). After that research on set-valued stochastic integral and differential equation (or inclusion) has been received much attention.

Kisielewicz \cite{Kisielewicz} (1997) studied the stochastic integral of set-valued stochastic process with respect to Brownian motion in $d$-dimensional Euclidean space $\mathbb{R}^d$, where the integral is defined as a subset of $L^2(\Omega; \mathbb{R}^d)$, which is called the trajectory integral. Kim and Kim \cite{Kim}, Jun and Kim \cite{Jun} defined the set-valued Itô integral (different from the trajectory integral) with respect to Brownian motion by using an indirect method such that the integral is a set-valued stochastic process. After that, there are a lot of research related to set-valued stochastic integrals. For example, Li and Ren \cite{Li} considered the integral as a mapping from the product space $[0, +\infty) \times \Omega$ to the power set of $\mathbb{R}^d$, where the measurability is also considered in the sense of product $\sigma$-algebra generated by $[0, +\infty) \times \Omega$. Michta \cite{Michta} studied both set-valued integral and trajectory integral with respect to semimartingale with finite path variation in $\mathbb{R}^d$. In an M-type Banach space $\mathcal{X}$, Zhang et al. \cite{Zhang1, Zhang2, Zhang3} considered the set-valued integrals with respect to Brownian motion, martingales and Poisson point processes respectively.

There are two ways to extend single-valued stochastic differential equation. One is differential inclusions (also being called multi-valued differential equations in some references). For example, the nice references written by J. Ren and J. Wu \cite{Ren}, J. Ren et al. \cite{Ren2} studied stochastic differential inclusion with single-valued Brownian diffusion in $\mathbb{R}^d$. J. Ren and J. Wu \cite{Ren3} studied the differential inclusion with Brownian diffusion and Poisson jump in $\mathbb{R}^d$ as follows:

\begin{equation}
\begin{aligned}
dX_t &\in -A(X_t)dt + b(X_t)dt + \sigma(X_t)dB_t + \int_{Z_0} f(X_{t-}, z)\tilde{N}(dtdz) + \int_{Z/Z_0} g(X_{t-}, z)N(dtdz),
\end{aligned}
\end{equation}

This a revised version of the manuscript posted in arXiv with Id number 2002.09220. This work is partly supported by Beijing National Science Foundation (1192015), the Construct Program of the Key Discipline in Hunan Province and State Scholarship Fund of CSC.

Email addresses: zhangjinping@ncepu.edu.cn (Jinping Zhang), mitoma@ms.saga-u.ac.jp (Itaru Mitoma), okazaki@fisi.or.jp (Yoshiaki Okazaki)
where $A$ is a set-valued operator. Other mappings are single-valued.

Another way to extend the single-valued stochastic equation is to turn the inclusion ‘$\in$’ into an equality ‘$=’$. Here we call it a set-valued integral (or differential) equation. That is to say, considering the solution $X(t)$ as a set-valued process. In $\mathbb{R}^d$ space, there are some references about set-valued differential equation without jump, e.g. [19, 25, 26]. In [22], the authors studied the set-valued equation driven by martingale, where the set-valued integral is the trajectory integral. In an M-type 2 Banach space, Zhang et al. [30, 31], Mitoma et al. [23, 24] explored the strong solutions to set-valued stochastic differential equations, where the diffusion part is single-valued since the set-valued integral with respect to Brownian motion may be unbounded a.s.

The Poisson point process is a special kind of Lévy process with a wide range of applications. It is important in both random mathematics (see e.g. [6, 11, 16]) and applied fields (see e.g. [19, 34]). For convenience, we consider the stationary Poisson process $\mathbf{p}$ with a finite characteristic measure $\nu$. Both of the Poisson random measure $\mathcal{N}(d\sigma dz)$ (where $\sigma \in \mathbb{Z}$, the state space of $\mathbf{p}$) and the compensated Poisson random measure $\mathcal{N}(dsdz)$ are of finite variation a.s., which is different from Brownian motion. Based on the work [34] and [30], in an M-type 2 Banach space $\mathfrak{X}$, by using the Hahn decomposition theorem of a space and properties of the bounded linear functional, we shall prove that stochastic integrals of set-valued predictable processes with respect to $\mathcal{N}(dsdz)$ and $\mathcal{N}(dsdz)$ are $L^2$-integrably bounded. The integral with respect to the compensated measure is a submartingale. The last theorem (Theorem 3.7) in [34] states that the integral is a set-valued martingale. But unfortunately the integral is not a martingale unless the integrand degenerates into a single-valued stochastic process a.s. See Theorem 3.5 in this paper.

Thanks to the integrable boundeness of set-valued stochastic integral with respect to Poisson point process with finite characteristic measure, based on the work [22, 30], we can study the extended set-valued stochastic integral equations with set-valued Poisson jump and single-valued Brownian motion diffusion as follows:

$$X_t = cl\{X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dB_s + \int_0^{t+} \int Z c(s, z, X_{s-})\mathcal{N}(dzds)\},$$

for $t \in [0, T]$ a.s. where $a(\cdot, \cdot)$ and $c(\cdot, \cdot)$ are set-valued and $b(\cdot, \cdot)$ is single-valued. $\{B_t; t \geq 0\}$ is a real valued Brownian motion. The notation $cl$ stands for the closure in the Banach space $\mathfrak{X}$.

This paper is organized as follows: Section 2 is about basic notations and auxiliary results related to the set-valued theory. In Section 3, firstly we review the stochastic integrals for $\mathfrak{X}$-valued $\mathcal{F}$-predictable processes with respect to $\mathcal{N}(dsdz)$ and $\mathcal{N}(dsdz)$ as required later. Then we study the stochastic integrals for set-valued $\mathcal{F}$-predictable processes with respect to $\mathcal{N}(dsdz)$ and $\mathcal{N}(dsdz)$. Section 4 is about the existence and uniqueness of strong solution to equation (2). Section 5 is a concluding remark.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a filtered complete probability space, in which the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual condition. Let $B(E)$ be the Borel field of a topological space $E$, $(\mathfrak{X}, \|\cdot\|)$ a real separable Banach space equipped with the norm $\|\cdot\|$ and $K(\mathfrak{X})(K_b(\mathfrak{X}), K_c(\mathfrak{X}))$ the family of all nonempty closed (resp. bounded, closed convex) subsets of $\mathfrak{X}$. Let $1 \leq p < +\infty$ and $L^p(\Omega, \mathcal{F}, P; \mathfrak{X})$ (denoted briefly by $L^p(\Omega; \mathfrak{X})$) be the Banach space of equivalence classes of $\mathfrak{X}$-valued $\mathcal{F}$-measurable functions $f : \Omega \to \mathfrak{X}$ such that the norm $\|f\|_p = \left(\int_{\Omega} \|f(\omega)\|^p dP\right)^{1/p}$ is finite. An $\mathfrak{X}$-valued function $f$ is called $L^p$-integrable if $f \in L^p(\Omega; \mathfrak{X})$.

A set-valued function $F : \Omega \to K(\mathfrak{X})$ is said to be measurable if for any open set $O \subset \mathfrak{X}$, the inverse $F^{-1}(O) := \{\omega \in \Omega : F(\omega) \cap O \neq \emptyset\}$ belongs to $\mathcal{F}$. Such a function $F$ is called a set-valued random variable. Let $M(\Omega, \mathcal{F}, P; K(\mathfrak{X}))$ be the family of all set-valued random variables, which is briefly denoted by $M(\Omega; K(\mathfrak{X}))$.

For any open subset $O \subset \mathfrak{X}$, set $Z_0 := \{E \in K(\mathfrak{X}) : E \cap O \neq \emptyset\}$, and $C := \{Z_0 : O \subset \mathfrak{X}, O \text{ is open}\}$, and let $\sigma(C)$ be the $\sigma$-algebra generated by $C$. A set-valued function $F : \Omega \to K(\mathfrak{X})$ is measurable if and only if $F$ is $\mathcal{F}/\sigma(C)$-measurable. By Kuratowski-Ryll-Nardzewski Selection Theorem (see e.g. [4],
page 509), any set-valued random variable \( F : \Omega \rightarrow \mathbf{K}(\mathfrak{X}) \) admits a measurable selection \( f \) such that \( f(\omega) \in F(\omega) \) for each \( \omega \in \Omega \).

For \( A, B \in 2^\mathfrak{X} \) (the power set of \( \mathfrak{X} \)), \( H(A, B) \geq 0 \) is defined by
\[
H(A, B) := \max\{\sup_{x \in A} \inf_{y \in B} ||x - y||, \sup_{y \in B} \inf_{x \in A} ||x - y||\},
\]
which is called the Hausdorff metric. It is well-known that \( \mathbf{K}_b(\mathfrak{X}) \) equipped with the metric \( H \) denoted by \( ((\mathbf{K}_b(\mathfrak{X}), H)) \) is a complete metric space.

For \( F \in \mathcal{M}(\Omega, \mathbf{K}(\mathfrak{X})) \), the family of all \( L^p \)-integrable selections is defined by
\[
S^p_F(\mathcal{F}) := \{ f \in L^p(\Omega, \mathcal{F}, P; \mathfrak{X}) : f(\omega) \in F(\omega) \text{ a.s.} \}.
\]
In the following, \( S^p_F(\mathcal{F}) \) is denoted briefly by \( S^p_F \). If \( S^p_F \) is nonempty, \( F \) is said to be \( L^p \)-integrable. \( F \) is called \( L^p \)-integrably bounded if there exists a function \( h \in L^p(\Omega, \mathcal{F}, P; \mathbb{R}) \) such that \( ||x|| \leq h(\omega) \) for any \( x \) and \( \omega \) with \( x \in F(\omega) \). It is equivalent to that \( ||F||_K \in L^p(\Omega, \mathbb{R}) \), where \( ||F(\omega)||_K := \sup_{a \in F(\omega)} ||a|| \).

The integral (or expectation) of a set-valued random variable \( F \) was defined by Aumann in 1965 (1):
\[
E[F] := \{ E[f] : f \in S^1_F \}.
\]

For \( F_1, F_2 \in \mathcal{M}(\Omega, \mathfrak{X}) \) and \( F(\omega) = cl(F_1(\omega) + F_2(\omega)) \) for all \( \omega \in \Omega \). Then \( F \in \mathcal{M}(\Omega, \mathfrak{X}) \). Moreover if \( S^p_{F_1} \) and \( S^p_{F_2} \) are nonempty where \( 1 \leq p < \infty \), then \( S^p_F = cl(S^p_{F_1} + S^p_{F_2}) \), the closure in \( L^p(\Omega, \mathfrak{X}) \).

Let \( \mathbb{R}_+ \) be the set of all nonnegative real numbers and \( B_+ := B(\mathbb{R}_+) \). \( \mathbb{N} \) denotes the set of natural numbers. An \( \mathfrak{X} \)-valued stochastic process \( f = \{ f_t : t \geq 0 \} \) is called \( \mathcal{F}_t \)-adapted if \( f_t \) is \( \mathcal{F}_t \)-measurable for every \( t \geq 0 \). \( f = \{ f_t : t \geq 0 \} \) is called predictable if it is \( \mathcal{P} \)-measurable, where \( \mathcal{P} \) is the \( \sigma \)-algebra generated by all left continuous and \( \mathcal{F}_t \)-adapted stochastic processes.

In a fashion similar to the \( \mathfrak{X} \)-valued stochastic process, a set-valued stochastic process \( F = \{ F_t : t \geq 0 \} \) is defined as a set-valued function \( F : \mathbb{R}_+ \times \Omega \rightarrow \mathbf{K}(\mathfrak{X}) \) with \( \mathcal{F} \)-measurable section \( F_t \) for \( t \geq 0 \). It is called measurable if it is \( \mathcal{B}_+ \otimes \mathcal{F} \)-measurable, and \( \mathcal{F}_t \)-adapted if for any fixed \( t, F_t(\cdot) \) is \( \mathcal{F}_t \)-measurable. \( F = \{ F_t : t \geq 0 \} \) is called predictable if it is \( \mathcal{P} \)-measurable.

**Definition 2.1.** (see 7) An integrable bounded convex set-valued \( \mathcal{F}_t \)-adapted stochastic process \( \{ F_t, \mathcal{F}_t : t \geq 0 \} \) is called a set-valued \( \mathcal{F}_t \)-martingale if for any \( 0 \leq s \leq t \) it holds that \( E[F_t|\mathcal{F}_s] = F_s \) in the sense of \( S^1_{E[F_t|\mathcal{F}_s]}(\mathcal{F}_s) = S^1_{F_s}(\mathcal{F}_s) \).

It is called a set-valued submartingale (supermartingale) if for any \( 0 \leq s \leq t \), \( E[F_t|\mathcal{F}_s] \supset F_s \) (resp. \( E[F_t|\mathcal{F}_s] \subset F_s \)) in the sense of \( S^1_{E[F_t|\mathcal{F}_s]}(\mathcal{F}_s) \supset S^1_{F_t}(\mathcal{F}_s) \) (resp. \( S^1_{E[F_t|\mathcal{F}_s]}(\mathcal{F}_s) \subset S^1_{F_t}(\mathcal{F}_s) \)).

Note: This is the original definition of set-valued martingale given by Hiai and Umegaki (1977) in 7. There are some references which give the definition without the assumptions of convexity or integrably boundedness (only assume it is integrable and \( \mathcal{F}_t \)-adapted), see e.g. 18. In this paper, we use the original definition.

An \( \mathfrak{X} \)-valued martingale \( f = \{ f_t, \mathcal{F}_t, t \geq 0 \} \) is called an \( L^p \)-martingale selection of the set-valued stochastic process \( F = \{ F_t, \mathcal{F}_t, t \geq 0 \} \) if it is an \( L^p \)-selection of \( F = \{ F_t, \mathcal{F}_t, t \geq 0 \} \). The family of all \( L^p \)-martingale selections of \( F = \{ F_t, \mathcal{F}_t : t \geq 0 \} \) is denoted by \( \mathbf{MS}^p(\mathcal{F}(\cdot)) \). Briefly, write \( \mathbf{MS}(F) = \mathbf{MS}^1(F(\cdot)) \).

For interval-valued martingale, here we list a known result, which will be used later.

**Theorem 2.1.** (1982) Let \( F = \{ F(t), \mathcal{F}_t : t \geq 0 \} \) be an adapted interval-valued stochastic process, and \( F = \{ F(t), \mathcal{F}_t : t \geq 0 \} \subset L^1(\Omega, \mathcal{F}, P; \mathbf{K}(\mathbb{R})) \), then the following two statements are equivalent:

1. \( F = \{ F(t), \mathcal{F}_t : t \geq 0 \} \) is an interval-valued martingale;
2. there exist two real-valued martingale selection \( \xi = \{ \xi(t), \mathcal{F}_t : t \geq 0 \} \) and \( \eta = \{ \eta(t), \mathcal{F}_t : t \geq 0 \} \), s.t. for each \( t, F(t, \omega) = [\xi(t, \omega), \eta(t, \omega)] \) a.s.
3. Properties of set-valued integration associated with Poisson processes

In this section, in an M-type 2 Banach space, at first we will briefly review the stochastic integrals with respect to the Poisson random measure and the compensated Poisson random measure for \( \mathcal{X} \)-valued and \( \mathcal{K}(\mathcal{X}) \)-valued stochastic processes, which are studied in \cite{34}. Then we study some other properties of stochastic integrals for \( \mathcal{K}(\mathcal{X}) \)-valued stochastic processes, such as the \( L^2 \)-integrable boundedness, set-valued submartingale property etc.

3.1. Single-valued stochastic integrals w.r.t. Poisson processes

The following definitions and notations related to Poisson point processes come from \cite{11} and \cite{32}.

Let \( \mathcal{X} \) be a separable Banach space and \( Z \) be another separable Banach space with \( \sigma \)-algebra \( \mathcal{B}(Z) \).

A point function \( p \) on \( Z \) means a mapping \( p : D_p \to Z \), where the domain \( D_p \) is a countable subset of \([0, T]\). \( p \) defines a counting measure \( N_p(dt dz) \) on \([0, T] \times Z \) (with the product \( \sigma \)-algebra \( \mathcal{B}([0, T]) \otimes \mathcal{B}(Z) \)) by

\[
N_p((0, t], U) := \#\{\tau \in D_p : \tau \leq t, p(\tau) \in U\}, \ t \in (0, T], \ U \in \mathcal{B}(Z). \tag{3}
\]

For \( 0 \leq s < t \leq T, N_p((s, t], U) := N_p((0, t], U) - N_p((0, s], U). \) In the following, we also write \( N_p((0, t], U) \) as \( N_p(t, U) \).

A point process (denoted by \( p := (p_{i \geq 0}) \)) is obtained by randomizing the notion of point functions. If there is a continuous \( \mathcal{F}_t \)-adapted increasing process \( N_p \) such that for \( U \in \mathcal{B}(Z) \) and \( t \in [0, T] \), \( N_p(t, U) := N_p(t, U) - N_p(t, U) \) is an \( \mathcal{F}_t \)-martingale, then the random measure \( \{N_p(t, U)\} \) is called the compensator of the point process \( p \) (or \( \{N_p(t, U)\} \)) and the process \( \{\hat{N}_p(t, U)\} \) is called the compensated point process.

A point process \( p \) is called the Poisson Point Process if \( N_p(dt dz) \) is a Poisson random measure on \([0, T] \times Z \). A Poisson point process is stationary if and only if its intensity measure \( \nu_p(dt dz) = E[N_p(dt dz)] \) is of the form \( \nu_p(dt dz) = dt \nu(dz) \) for some measure \( \nu(dz) \) on \((Z, \mathcal{B}(Z))\). \( \nu(dz) \) is called the characteristic measure of \( p \).

Let \( \nu \) be a \( \sigma \)-finite measure on \((Z, \mathcal{B}(Z))\), i.e. there exists \( U_i \in \mathcal{B}(Z), i \in \mathbb{N} \), pairwise disjoint such that \( \nu(U_i) < \infty \) for all \( i \in \mathbb{N} \) and \( Z = \cup_{i=1}^{\infty} U_i \). \( p = (p_{i \geq 0}) \) be the \( \mathcal{F}_t \)-adapted stationary Poisson point process on \( Z \) with the characteristic measure \( \nu \) such that the compensator \( \hat{N}_p(t, U) = E[N_p(t, U)] = t\nu(U) \) (non-random).

For convenience, we will omit the subscript \( p \) in the above notations and assume \( \nu(Z) \) is finite.

For any \( U \in \mathcal{B}(Z) \), both \( \{N(t, U), t \in [0, T]\} \) and \( \{\hat{N}(t, U), t \in [0, T]\} \) are stochastic processes with finite variation a.s. For convenience, from now on, we suppose \( \nu \) is a finite measure in the measurable space \((Z, \mathcal{B}(Z))\).

An \( \mathcal{X} \)-valued mapping \( f \) defined on \([0, T] \times Z \times \Omega \) is called \( \mathcal{S} \)-predictable if the mapping \((t, z, \omega) \to f(t, z, \omega) \) is \( \mathcal{S}/\mathcal{B}(\mathcal{X}) \)-measurable, where \( \mathcal{S} \) is the smallest \( \sigma \)-algebra with respect to which all mappings \( g : [0, T] \times Z \times \Omega \to \mathcal{X} \) satisfying (i) and (ii) below are measurable:

(i) for each \( t \in [0, T] \), the mapping \((z, \omega) \to g(t, z, \omega) \) is \( \mathcal{B}(Z) \otimes \mathcal{F}_t \)-measurable;

(ii) for each \((z, \omega) \in Z \times \Omega \), the mapping \( t \to g(t, z, \omega) \) is left continuous.

**Remark 1.** (see e.g. \cite{32}) \( \mathcal{S} = \mathcal{P} \otimes \mathcal{B}(Z) \), where \( \mathcal{P} \) denotes the \( \sigma \)-field on \([0, T] \times \Omega \) generated by all left continuous and \( \mathcal{F}_t \)-adapted processes.

Set \( \mathcal{L} = \{ f : f \text{ is } \mathcal{S} \text{-predictable and } E\left[\int_0^T \int_Z \|f(t, z, \omega)\|^2 \nu(dz) dt\right] < \infty \} \) equipped with the norm \( \|f\|_\mathcal{L} := \left(E\left[\int_0^T \int_Z \|f(t, z, \omega)\|^2 \nu(dz) dt\right]\right)^{1/2} \).

In the following, when \( f(t, z, \omega) \) (or \( F(t, z, \omega) \)) to appear as the integrand in an integral, for brevity, it will be denoted by \( f_t(z) \) (or \( F_t(z) \) respectively).

In an M-type 2 Banach space (Definition \ref{def:M-type2}), by using the usual method, for any \( f \in \mathcal{L} \), the integrals

\[
J_t(f) = \int_0^{t+} \int_Z f_t(z)N(d\tau dz), \text{ for } t \geq 0
\]

and
Let it possible to study the set-valued stochastic differential equation with set-valued jump.

**Definition 3.1.** A Banach space $(\mathcal{X}, \| \cdot \|)$ is called M-type 2 if and only if there exists a constant $C_X > 0$ such that for any $\mathcal{X}$-valued martingale $\{M_t\}$, it holds that

$$
\sup_k E[\|M_k\|^2] \leq C_X \sum_k E[\|M_k - M_{k-1}\|^2].
$$

About integral processes $(J_t)_{t\in[0,T]}$ and $(I_t)_{t\in[0,T]}$, the following known results will be used to prove some properties of the set-valued case.

**Theorem 3.1.** Let $\mathcal{X}$ be of M-type 2 and $(Z,B(Z))$ a separable Banach space with finite measure $\nu$. Let $p$ be a stationary Poisson process with the characteristic measure $\nu$. Taking $f \in \mathcal{L}$, then $(J_t)_{t\in[0,T]}$ and $(I_t)_{t\in[0,T]}$ are uniformly square integrable, right continuous $\mathcal{F}_t$-adapted processes. $(I_t)_{t\in[0,T]}$ is a martingale with mean zero and $E[J_t(f)] = \int_0^t \int_Z E[f_s(z)]d\nu(dz)$. Moreover, there exists a constant $C$ such that

$$
E\left[\sup_{0 \leq s \leq t} \left( \int_0^{s+} \int_Z f_r(z) \tilde{N}(drdz) \right)^2 \right] \leq C \int_0^t \int_Z E[\|f_r(z)\|^2]d\nu(dz),
$$

and

$$
E\left[\sup_{0 \leq s \leq t} \left( \int_0^{s+} \int_Z f_r(z) N(drdz) \right)^2 \right] \leq C \int_0^t \int_Z E[\|f_r(z)\|^2]d\nu(dz),
$$

where $C$ depends on the constant $C_X$ in Definition 3.1.

### 3.2. Set-valued stochastic integrals w.r.t. Poisson processes

For the convenience to read the paper without aiding references, and in order to prove our main results, in this subsection, at first we review the stochastic integral of a set-valued stochastic process with respect to the Poisson point process and list some auxiliary results obtained in [34]. And then we shall study its $L^2$-integrable boundedness, submartingale property and some inequalities, which make it possible to study the set-valued stochastic differential equation with set-valued jump.

A set-valued stochastic process $F = \{F(t)\} : [0,T] \times Z \times \Omega \to \mathcal{K}(\mathcal{X})$ is called $\mathcal{F}$-predictable if $F$ is $\mathcal{F}/\sigma(C)$-measurable.

Set

$$
\mathcal{M} = \left\{ F : [0,T] \times Z \to \mathcal{K}(\mathcal{X}), F \text{ is } \mathcal{F}-\text{predictable and } E\left[\int_0^T \int_Z \|F_t(z)\|^2_2 dt\nu(dz)\right] < \infty \right\}
$$

Given a set-valued stochastic process $\{F(t)\}_{t\in[0,T]}$, the $\mathcal{X}$-valued stochastic process $\{f(t)\}_{t\in[0,T]}$ is called an $\mathcal{F}$-selection if $f(t,z,\omega) \in F(t,z,\omega)$ for all $(t,z,\omega)$ and $f \in \mathcal{F}$. For any $F \in \mathcal{M}$, the $\mathcal{F}$-selection exists and satisfies

$$
E\left[\int_0^T \int_Z \|f_t(z)\|^2 d\nu(dz)\right] \leq E\left[\int_0^T \int_Z \|F_t(z)\|^2_2 d\nu(dz)\right] < \infty,
$$

which means $f \in \mathcal{L}$. The family of all $f$ which belongs to $\mathcal{L}$ and satisfies $f(t,z,\omega) \in F(t,z,\omega)$ for a.e. $(t,z,\omega)$ is denoted by $S(F)$, that is $S(F) = \{ f \in \mathcal{L} : f(t,z,\omega) \in F(t,z,\omega) \text{ for a.e. } (t,z,\omega) \}$. Set

$$
\tilde{\Gamma}_t := \left\{ \int_0^{t+} \int_Z f_s(z) \tilde{N}(dsdz) : (f(t))_{t\in[0,T]} \in S(F) \right\},
$$

$$
\Gamma_t := \left\{ \int_0^t \int_Z f_s(z) N(dsdz) : (f(t))_{t\in[0,T]} \in S(F) \right\}.
$$

Let $\overline{d\tilde{\Gamma}}_t$ (resp. $\overline{d\Gamma}_t$) denote the decomposable closed hull of $\tilde{\Gamma}_t$ (resp. $\Gamma_t$) with respect to $\mathcal{F}_t$, where the closure is taken in $L^1(\Omega,\mathcal{X})$. Then $\overline{d\Gamma}_t$ and $\overline{d\tilde{\Gamma}}_t$ can determine two set-valued random variables respectively, denoted by $I_t(F), J_t(F) \in \mathcal{M}(\Omega,\mathcal{F}_t,P;\mathcal{K}(\mathcal{X}))$ such that $S_{I_t(F)}(\mathcal{F}_t) = \overline{d\tilde{\Gamma}}_t$ and $S_{J_t(F)}(\mathcal{F}_t) = \overline{d\Gamma}_t$. 

$$
I_t(f) = \int_0^{t+} \int_Z f_r(z) \tilde{N}(drdz), \text{ for } t \geq 0
$$

are well defined. See for e.g. [34] and references therein.
Therefore we have the compensated Poisson random measure is not an interval-valued martingale. Particularly, we now decompose it into the difference of two measures. For convenience, let’s review the Hahn decomposition. For a set-valued stochastic process \( \{ I_t : t \in [0, T] \} \) and \( \{ J_t : t \in [0, T] \} \) are of finite variation since both \( I_t \) and \( J_t \) are \( \mathcal{F}_t \)-measurable set.

\[ \int_0^t f_s(z) \tilde{N}(dsdz) = \int_0^t f_s(z) \tilde{N}^+(dsdz) - \int_0^t f_s(z) \tilde{N}^-(dsdz) \]

By additive property of set-valued random variable and Definition 3.2, it is easy to get the proposition below:

**Proposition 3.1.** Assume set-valued stochastic processes \( \{ F_t, \mathcal{F}_t : t \in [0, T] \} \) and \( \{ G_t, \mathcal{F}_t : t \in [0, T] \} \) are integrably bounded and right continuous with respect to \( t \). Moreover, if \( F_t \) is convex for each \( t \), the process \( \{ I_t, t \in [0, T] \} \) is a set-valued submartingale.

Note: The integral \( \{ I_t, t \in [0, T] \} \) is not a set-valued martingale except for special case (the singletons). The counterexample and rigorous proof are given below.

Since the compensated Poisson random measure is a signed measure. In order to give a counterexample, we now decompose it into the difference of two measures. For convenience, let’s review the Hahn decomposition of a space and the Jordan decomposition of a signed measure (see e.g. [21]).

The signed measure \( \tilde{N} \) is defined in the product space \( ([0, t] \times \mathcal{B}([0, t]) \times \mathcal{Z}) \) with finite variation. By the Hahn decomposition theorem, for any fixed \( 0 < t \leq T \), there exists an essential unique \( \mathcal{B}([0, t]) \otimes \mathcal{Z} \)-measurable Hahn decomposition denoted by \( A^+ \) and \( A^- \) such that \( A^+ \cap A^- = \emptyset \), \( A^+ \cup A^- = [0, t] \times \mathcal{Z} \), and for any \( \mathcal{B}([0, t]) \otimes \mathcal{Z} \)-measurable set \( B \subset A^+ \), \( \tilde{N}(B) \geq 0 \), for any \( \mathcal{B}([0, t]) \otimes \mathcal{Z} \)-measurable set \( B \subset A^- \), \( \tilde{N}(B) \leq 0 \). The corresponding unique Jordan decomposition of the signed measure \( \tilde{N} \) is denoted by \( \tilde{N}^+ \) and \( \tilde{N}^- \) such that \( \tilde{N} = \tilde{N}^+ - \tilde{N}^- \). For any \( \mathcal{B}([0, t]) \otimes \mathcal{Z} \)-measurable set \( B \),

\[ \tilde{N}^+(B) := \tilde{N}(B \cap A^+) = \sup_{S \in \mathcal{B}_r \cap \mathcal{B}([0, t]) \otimes \mathcal{Z}} \tilde{N}(S) \]

and

\[ \tilde{N}^-(B) := -\tilde{N}(B \cap A^-) = -\inf_{S \in \mathcal{B}_r \cap \mathcal{B}([0, t]) \otimes \mathcal{Z}} \tilde{N}(S). \]

Particularly,

\[ \tilde{N}^+(A^-) = 0 \text{ and } \tilde{N}^-(A^+) = 0. \]

Therefore we have

\[ \tilde{N}(B) = \tilde{N}^+(B) - \tilde{N}^-(B) = |\tilde{N}||B| = \tilde{N}^+(B) + \tilde{N}^-(B). \]

In addition, the Jordan decomposition is the minimum decomposition, and

\[ \tilde{N}(dsdz) = N(dsdz) - ds\nu(dz) = \tilde{N}^+(dsdz) - \tilde{N}^-(dsdz). \]

Then \( \tilde{N}^+ \) and \( \tilde{N}^- \) are of finite variation since both \( N(dsdz) \) and \( ds\nu(dz) \) are of finite variation. Therefore, for any \( f \in \mathcal{L} \), the integrals \( \int_0^t \int_Z f_s(z) \tilde{N}^+(dsdz) \) and \( \int_0^t \int_Z f_s(z) \tilde{N}^-(dsdz) \) are well defined as a manner similar to \( \int_0^t \int_Z f_s(z) N(dsdz) \). Then we have

\[ \int_0^t \int_Z f_s(z) \tilde{N}(dsdz) = \int_0^t \int_Z f_s(z) \tilde{N}^+(dsdz) - \int_0^t \int_Z f_s(z) \tilde{N}^-(dsdz), \]

\[ \int_0^t \int_Z f_s(z)|\tilde{N}||(dsdz) = \int_0^t \int_Z f_s(z) \tilde{N}^+(dsdz) + \int_0^t \int_Z f_s(z) \tilde{N}^-(dsdz). \]

Now we give an example to show that the interval-valued stochastic integral with respect to the compensated Poisson random measure is not an interval-valued martingale.
Example 3.1. Let $\mathcal{X} = \mathbb{R}$. Take a set-valued stochastic process

$$F(t, z, \omega) \equiv [-1, 1] \text{ for all } (t, z, \omega) \in [0, T] \times Z \times \Omega.$$ 

Then $F = \{F(t, z, \omega), \mathcal{F}_t : t \in [0, T]\} \in \mathcal{M}$. The integral $\int_0^{t+} \int_Z F_s(z) \tilde{N}(dsdz)$ is a closed interval since $F$ is an interval. For any selection $f \in \mathcal{L}$, we have $-1 \leq f(s, z, \omega) \leq 1$. Then for any fixed $t$ ($0 < t \leq T$),

$$\left| \int_0^{t+} \int_Z f_s(z) \tilde{N}(dsdz) \right| \leq \int_0^{t+} \int_Z |f_s(z)||\tilde{N}|(dsdz) \leq \int_0^{t+} \int_Z |\tilde{N}|(dsdz) = |\tilde{N}|([0, t], Z).$$

The extreme point can be attained. In fact, let $A^+$ and $A^-$ be the Hahn decomposition of $(0, t] \times Z$. Taking

$$h(s, z, \omega) = \chi_{A^+} - \chi_{A^-},$$

which is non-random then $\mathcal{B}([0, t]) \otimes \mathcal{B}(Z) \otimes \mathcal{F}_0$-measurable. Furthermore, $h = \{h_t, \mathcal{F}_t : t \in [0, T]\} \in \mathcal{L}$. Then

$$\sup_{\text{all selections } f \in \mathcal{L}} \int_0^{t+} \int_Z f_s(z) \tilde{N}(dsdz) = \int_0^{t+} \int_Z h_s(z) \tilde{N}(dsdz) = \int_{A^+} \tilde{N}(dsdz) - \int_{A^-} \tilde{N}(dsdz) = \tilde{N}(A^+) - \tilde{N}(A^-) = \tilde{N} + (A^+) + \tilde{N}^{-}(A^-) = |\tilde{N}|([0, t], Z)$$

Similarly, taking

$$h(s, z, \omega) = -\chi_{A^+} + \chi_{A^-},$$

we obtain

$$\inf_{\text{all selections } f \in \mathcal{L}} \int_0^{t+} \int_Z f_s(z) \tilde{N}(dsdz) = \int_0^{t+} \int_Z -h_s(z) \tilde{N}(dsdz) = -|\tilde{N}|([0, t], Z).$$

By the convexity and closedness of $\int_0^{t} \int_Z F_s(z) \tilde{N}(dsdz)$, together with (7) and (8), we obtain

$$\int_0^{t+} \int_Z F_s(z) \tilde{N}(dsdz) = [-|\tilde{N}|([0, t], Z), |\tilde{N}|([0, t], Z)].$$

It is obvious that the left end point and the right end point are $\mathcal{F}_t$-supermartingale and submartingale respectively. But not $\mathcal{F}_t$-martingale except for $|\tilde{N}|([0, t], Z) \equiv 0$, a contradiction. Therefore, by Definition 2.1 and Theorem 2.1, the integral process $\{\int_0^{t} \int_Z F_s(z) \tilde{N}(dsdz), \mathcal{F}_t : t \in (0, T]\}$ is an interval-valued submartingale, but not an interval-valued martingale.

An interval $I$ is called proper if it has infinitely many elements. A convex set $A$ is called non-degenerate if it has infinitely many elements. A set $A$ is called a singleton if it has only one element. In the following, we will show that for any interval-valued integrable stochastic process $F = \{F(t, z, \omega) = \{[f(t, z, \omega), g(t, z, \omega)], \mathcal{F}_t : t \in [0, T]\}$, the integral process $\{I_t(F), \mathcal{F}_t : t \in (0, T]\}$ is not an interval-valued martingale unless the interval process $F$ degenerates into single-valued one.

Theorem 3.2. Assume a proper interval-valued stochastic process $\{F_t = [f_t, g_t], \mathcal{F}_t : t \in [0, T]\}$ is integrable with respect to $N(dsdz)$ and $\tilde{N}(dsdz)$. Then the integral $\{I_t(F), \mathcal{F}_t : t \in (0, T]\}$ is not an interval-valued martingale.

Proof. Let $\mathcal{X} = \mathbb{R}$.

Step 1: At first we consider the symmetric proper interval. Assume $f = \{f_t, \mathcal{F}_t : t \in [0, T]\} \in \mathcal{L}$ and for each $t$, $f_t > 0$ for a.e. $(z, \omega)$. Then the interval stochastic process $\{F_t = [-f_t, f_t], \mathcal{F}_t : t \in [0, T]\} \in \mathcal{M}$. Taking any selection $h \in \mathcal{L}$, we have

$$\int_0^{t+} \int_Z h_s(z) \tilde{N}(dsdz) \leq \int_0^{t+} \int_Z |h_s(z)||\tilde{N}|(dsdz) \leq \int_0^{t+} \int_Z f_s(z)|\tilde{N}|(dsdz).$$
Similarly, by taking

$$h^1(s, z, \omega) = \chi_A + f(s, z, \omega) - \chi_A - f(s, z, \omega),$$

and respectively

$$h^2(s, z, \omega) = -\chi_A + f(s, z, \omega) + \chi_A - f(s, z, \omega),$$

the extreme points \(I^+_0 \int_Z f_s(z) d[N(dsdz)]\) and \(-I^+_0 \int_Z f_s(z) d[N(dsdz)]\) can be attained respectively. 

Therefore, by the closedness and convexity of the integral, we obtain

$$\int_0^{t_+} \int_Z F_s \tilde{N}(dsdz) = [-\int_0^{t_+} \int_Z f_s(z) d\tilde{N}(dsdz), \int_0^{t_+} \int_Z f_s(z) d\tilde{N}(dsdz)],$$

which implies for each \(t\), the integral is a proper interval for a.e. \((z, \omega)\). The \(\mathbb{R}\)-valued stochastic process \(\{\int_0^{t_+} \int_Z f_s(z) d\tilde{N}(dsdz), \mathcal{F}_t : t \in (0, T]\}\) is a submartingale but not a martingale. Indeed, for any \(0 < s < t \leq T\),

$$E\left[\int_s^{t_+} \int_Z f_s(z) d\tilde{N}(dsdz)|\mathcal{F}_s\right] = E\left[\int_s^{t_+} \int_Z f_s(z) d\tilde{N}(dsdz)|\mathcal{F}_s\right] + E\left[\int_s^{t_+} \int_Z f_s(z) d\tilde{N}(dsdz)|\mathcal{F}_s\right] = \int_s^{t_+} \int_Z f_s(z) d\tilde{N}(dsdz) + E\left[\int_s^{t_+} \int_Z f_s(z) d\tilde{N}(dsdz)|\mathcal{F}_s\right] \geq \int_s^{t_+} \int_Z f_s(z) d\tilde{N}(dsdz)$$

but the equality does not always hold for all \(0 < s < t\) since

$$E\left[\int_s^{t_+} \int_Z f_s(z) d\tilde{N}(dsdz)|\mathcal{F}_s\right] = 0$$

does not always hold for all \(s\).

**Step 2.** Let \(0 < f = \{f_t, \mathcal{F}_t : t \in [0, T]\} \in \mathcal{L}\). Setting \(F_t = [0, f_t]\), \(G_1^t = [-f_t, f_t]\) and \(G_2^t = \{f_t\}\) for all \(t \in [0, T]\). Then \(2F_t = G_1^t + G_2^t\), \(F = \{F_t, \mathcal{F}_t : t \in [0, T]\}\), \(G_1^t = \{G_1^t, \mathcal{F}_t : t \in [0, T]\}\) and \(G_2^t = \{G_2^t, \mathcal{F}_t : t \in [0, T]\}\) belong to \(\mathcal{M}\). By Proposition 3.1 for every \(t \in (0, T]\)

$$I_t(2F) = 2I_t(F) = I_t(G_1^1) + I_t(G_2^1) \text{ a.s.}$$

Note: here we need not to take closure since bounded closed set is compact in \(\mathbb{R}\), then the set of sum is closed.

The integral process \(\{I_t(G^2), \mathcal{F}_t : t \in [0, T]\}\) is an \(\mathbb{R}\)-valued martingale. And the process \(\{I_t(G^1), \mathcal{F}_t : t \in [0, T]\}\) is not an interval-valued martingale but an interval-valued submartingale. Then the sum \(\{I_t(2F), \mathcal{F}_t : t \in [0, T]\}\) is an interval-valued submartingale but not an interval-valued martingale, so does \(\{I_t(F), \mathcal{F}_t : t \in [0, T]\}\).

**Step 3.** Assume \(f = \{f_t, \mathcal{F}_t : t \in [0, T]\}\), \(g = \{g_t, \mathcal{F}_t : t \in [0, T]\} \in \mathcal{L}\) and \(f(t, z, \omega) < g(t, z, \omega)\). Setting \(F_t = [f_t, g_t]\) for all \(t\), then \(F = \{F_t, \mathcal{F}_t : t \in [0, T]\} \in \mathcal{M}\) and \(F_t = \{f_t\} + [0, g_t - f_t]\). In a similar way as the proof of Step 2, we obtain \(\{I_t(F), \mathcal{F}_t : t \in [0, T]\}\) is an interval-valued submartingale but not an interval-valued martingale.

From the above proof, we obtain that the integral process \(\{I_t(F), \mathcal{F}_t : t \in [0, T]\}\) is a martingale if and only if the integrand degenerates into a real valued process.

In order to prove the result being also true for M-type 2 Banach space \(\mathcal{X}\), we aid the bounded linear functional \(x^*\), which is defined on \(\mathcal{X}\) and takes values in \(\mathbb{R}\). Let \(\mathcal{X}^*\) be the family of all bounded linear functionals, i.e. the dual space of \(\mathcal{X}\), \(F = \{F_t, \mathcal{F}_t : t \in [0, T]\}\) be a convex set-valued stochastic process. Taking \(x^* \in \mathcal{X}^*\), for any \(t \in [0, T]\), define

$$F_t^{x^*}(\omega) := cl\{< x^*, a > : a \in F_t(\omega)\} \text{ for } \omega \in \Omega,$$

(9)
then $F_t^{x^*}$ is an interval-valued $\mathcal{F}_t$-measurable random variable (Note: for some $t$, the interval $F_t^{x^*}$ may be a singleton for a.e. $(z, \omega)$. For instance, the case $x^*=0$). Indeed, it is convex since the convexity of $F_t(\omega)$ and the linearity of $x^*$. Take any open interval $(c, d) \subset \mathbb{R}$,

$$\{\omega : F_t^{x^*}(\omega) \cap (c, d) \neq \emptyset\} = \Omega \setminus \left(\left\{\omega : \sup_{a \in F_t(\omega)} x^* < a \leq c\right\} \cup \left\{\omega : \inf_{a \in F_t(\omega)} x^* > a > d\right\}\right) \in \mathcal{F}_t,$$

i.e. $F_t^{x^*}$ is $\mathcal{F}_t$-measurable. Further,

$$S^{1}_{F_t^{x^*}}(\mathcal{F}_t) = cl\{< x^*, f_t > : f_t \in S^{1}_{F_t}(\mathcal{F}_t)\}. \tag{10}$$

Therefore, $\{F_t^{x^*} : t \in [0, T]\}$ is an interval-valued stochastic process. Moreover, if $F = \{F_t, \mathcal{F}_t : t \in [0, T]\}$ is convex and belongs to $\mathcal{M}$, then $F^{x^*} = \{F_t^{x^*}, \mathcal{F}_t : t \in [0, T]\}$ is an integrable interval and

$$S(F^{x^*}) = cl\{< x^*, f > : f = (f_s)_{s \in [0, T]} \in S(F)\},$$

where the closure is taken in product space $L^2([0, T] \times \Omega, \mathcal{B}([0, T]) \otimes \mathcal{B}(\Omega) \otimes \mathcal{F}, \lambda \times \nu \times P; \mathbb{X})$, $\lambda$ the Lebesgue measure in $([0, T]; \mathcal{B}([0, T]))$.

In a manner similar to the proof of Theorem 4.5 in [35], we have the following theorems:

**Theorem 3.3.** Assume convex set-valued stochastic process $\{F_t, \mathcal{F}_t : t \in [0, T]\} \in \mathcal{M}$. Then for any $x^* \in \mathbb{X}^*$, $\{I_t^{x^*}(F), \mathcal{F}_t : t \in [0, T]\}$ and $\{J_t^{x^*}(F), \mathcal{F}_t : t \in [0, T]\}$ are interval-valued $\mathcal{F}_t$-adapted processes. For any $t \in [0, T]$,

$$I_t^{x^*}(F)(\omega) = I_t(F^{x^*})(\omega) a.s.$$

$$J_t^{x^*}(F)(\omega) = J_t(F^{x^*})(\omega) a.s.$$

where $I_t^{x^*}(F)(\omega) := (I_t(F))^{x^*}(\omega)$ and $J_t^{x^*}(F)(\omega) := (J_t(F))^{x^*}(\omega)$.

**Theorem 3.4.** Assume convex set-valued stochastic process $\{F_t, \mathcal{F}_t : t \in [0, T]\} \in \mathcal{M}$, then for any $x^* \in \mathbb{X}^*$ and $s < t \in [0, T]$,

$$E[I_t^{x^*}(F)|\mathcal{F}_s](\omega) = E^{x^*}[I_t(F)|\mathcal{F}_s](\omega) \text{ a.s.}$$

$$E[J_t^{x^*}(F)|\mathcal{F}_s](\omega) = E^{x^*}[J_t(F)|\mathcal{F}_s](\omega) \text{ a.s.}$$

where $E^{x^*}[I_t(F)|\mathcal{F}_s]$ is the $K(\mathbb{X})$-valued random variable determined by

$$S^{1}_{E^{x^*}[I_t(F)|\mathcal{F}_s]}(\mathcal{F}_s) = cl\{< x^*, g > : g \in S^{1}_{E[I_t(F)|\mathcal{F}_s]}(\mathcal{F}_s)\} = cl\{< x^*, E[g_t|\mathcal{F}_s] > : g_t \in S^{1}_{I_t(F)}(\mathcal{F}_t)\}$$

and $E^{x^*}[J_t(F)|\mathcal{F}_s]$ is the $K(\mathbb{X})$-valued random variable determined by

$$S^{1}_{E^{x^*}[J_t(F)|\mathcal{F}_s]}(\mathcal{F}_s) = cl\{< x^*, g > : g \in S^{1}_{E[\mathcal{F}_t]}(\mathcal{F}_s)\} = cl\{< x^*, E[g_t|\mathcal{F}_s] > : g_t \in S^{1}_{J_t(F)}(\mathcal{F}_t)\}.$$

**Theorem 3.5.** Assume a non-degenerate convex set-valued stochastic process $F = \{F_t : t \in [0, T]\} \in \mathcal{M}$. The integral process $\{I_t(F) : t \in [0, T]\}$ is not a set-valued martingale.

**Proof.** Since for each $t$, $F_t$ is a non-degenerate closed convex subset of $\mathbb{X}$ for a.e. $(z, \omega)$ then so is the integral $I_t(F)$ a.s. Moreover, the expectation $E[I_t(F)]$ is a non-degenerate convex subset of $\mathbb{X}$. Taking $x, y \in E[I_t(F)]$ and $x \neq y$, by using the Hahn-Banach extension theorem of functionals, there exists an $x^* \in \mathbb{X}^*$ (independent of $t, z, \omega$), such that $x^*(x) \neq x^*(y)$. Indeed, set $A(x, y) = \{a(x - y) : a \in \mathbb{R}\}$ and define $< \phi, a(x - y) > = a\|x - y\|$ for every $a(x - y) \in A(x, y)$. It is observed that $\phi$ is a bounded linear functional with operator norm $\|\phi\| = 1$ defined in the subspace $A(x, y)$. Then by the Hahn-Banach
extension theorem (c.f. [21]), there exists a bounded linear functional \( x^* : \mathcal{X} \to \mathbb{R} \) such that \( x^* \) restricted to \( A(x, y) \) is equal to \( \phi \) and \( \|x^*\| = 1 \).

On the other hand, for the linearity of \( x^* \),

\[
< x^*, E \left[ \int_0^{T_+} \int_Z F_s(z) \tilde{N}(dsdz) \right] >= E \left[ \int_0^{T_+} \int_Z < x^*, F_s(z) > \tilde{N}(dsdz) \right].
\]

Since \( E \left[ \int_0^{T_+} \int_Z F(s, z, \omega) \tilde{N}(dsdz) \right] \) is a non-degenerate convex subset of \( \mathcal{X} \), by the choice of \( x^* \), \( < x^*, E \left[ \int_0^{T_+} \int_Z F_s(z) \tilde{N}(dsdz) \right] > \) is a proper interval, which implies \( E \left[ \int_0^{T_+} \int_Z < x^*, F(s, z, \omega) > \tilde{N}(dsdz) \right] \) is a proper interval. Furthermore, by the convexity of \( F(s, z, \omega) \), \( < x^*, F(s, z, \omega) > \) is a proper interval for a.e. \( (s, z, \omega) \). That means the stochastic processes \( F^{x^*} \) is a proper interval-valued process.

By Theorem [32] the interval-valued stochastic process \( \{I_i(F^{x^*}), \mathcal{F}_t : t \in (0, T] \} \) is not a proper interval-valued martingale. As a further result, we will show that \( \{I_i(F), \mathcal{F}_t : t \in (0, T] \} \) is not a \( \mathbf{K}(\mathcal{X}) \)-valued martingale.

Otherwise, suppose \( \{I_i(F), \mathcal{F}_t : t \in (0, T] \} \) is a \( \mathbf{K}(\mathcal{X}) \)-valued martingale. Taking the same functional \( x^* \) as above, for each \( t \), both \( I_i(F^{x^*}) \) and \( I_i(F) \) are non-degenerate convex sets a.s. Then \( \{I_i(F^{x^*}), \mathcal{F}_t : t \in (0, T] \} \) is a proper interval stochastic process. By the property of set-valued martingale, we have

\[
S^1_{I_i(F)}(\mathcal{F}_s) = S^1_{E[I_i(F)|\mathcal{F}_s]}(\mathcal{F}_s). \tag{11}
\]

According to Theorem [3.3] for any \( 0 < s < t \leq T \), we obtain

\[
S^1_{I_i(F^{x^*})}(\mathcal{F}_s) = \text{cl}\{< x^*, g_s > : g_s \in S^1_{I_i(F)}(\mathcal{F}_s)\}
\]

\[
= \text{cl}\{< x^*, g_s > : g_s \in S^1_{E[I_i(F)|\mathcal{F}_s]}(\mathcal{F}_s)\} \quad \text{(by (11))}
\]

\[
= S^1_{E[x^*I_i(F)|\mathcal{F}_s]}(\mathcal{F}_s) \quad \text{(by Theorem [3.3])}
\]

\[
= S^1_{E[I_i(F^{x^*})|\mathcal{F}_s]}(\mathcal{F}_s) \quad \text{(by Theorem [3.3])},
\]

which implies that \( \{I_i(F^{x^*}), \mathcal{F}_t : t \in (0, T] \} \) is a proper interval-valued martingale according to the equivalent conditions (see e.g. Theorem 3.1. in [33]), a contradiction to Theorem 3.2. \( \square \)

**Remark 2.** Theorem 3.7 in our previous paper [34] states that the integral \( \{I_i(F), t = 0, \ldots, T\} \) is a set-valued martingale with a very simple proof. In fact at that time we carelessly misused the martingale equivalent condition of Theorem 3.1. in [33]. The martingale equivalent condition is for each \( t \),

\[
S^1_{I_i(F)}(\mathcal{F}_t) = \text{cl}\{g_s : (g_s)_{s \in [0, T]} \in \mathbf{MS}(F)\},
\]

where \( g \) is an \( \mathcal{X} \)-valued martingale. The condition is different from the following Castaing representation

\[
I_i(F)(\omega) = \text{cl}\{\int_0^{t_+} \int_Z f_s(z) \tilde{N}(dsdz)(\omega) : i = 1, 2, \ldots \} \text{ a.s.}
\]

The latter is weaker. So we can not get the martingale property of \( \{I_i(F), t \in [0, T]\} \) from the Castaing representation even though for each \( i \), \( \int_0^{t_+} \int_Z f_s(z) \tilde{N}(dsdz), t \in [0, T] \) is an-\( \mathcal{X} \)-valued martingale.

In [33], Theorem 3.3 shows that \( \{I_i(F)\} \) and \( \{J_i(F)\} \) are \( L^1 \)-integrably bounded. We now show the \( L^2 \)-integrable boundedness. Set

\[
S^2_{I_i(F)}(\mathcal{F}_t) := \text{cl}_{L^2}\{\int_0^{t_+} \int_Z f_s(z) \tilde{N}(dsdz) : (f_s)_{s \in [0, T]} \in S(F)\},
\]

\[
S^2_{J_i(F)}(\mathcal{F}_t) := \text{cl}_{L^2}\{\int_0^{t_+} \int_Z f_s(z) \tilde{N}(dsdz) : (f_s)_{s \in [0, T]} \in S(F)\},
\]

where the closure is taken in \( L^2 \). We have the following result:
Lemma 3.1. Assume a set-valued stochastic process \( \{ F_t : t \in [0, T] \} \in \mathcal{M} \). Then for every \( t \in [0, T] \)
\( S_{t(F)}^1 (\mathcal{F}_t) = S_{t(F)}^2 (\mathcal{F}_t) \), and \( S_{t\downarrow(F)}^1 (\mathcal{F}_t) = S_{t\downarrow(F)}^2 (\mathcal{F}_t) \).

Proof. Obviously, \( S_{t(F)}^1 (\mathcal{F}_t) \supseteq S_{t(F)}^2 (\mathcal{F}_t) \), and \( S_{t\downarrow(F)}^1 (\mathcal{F}_t) \supseteq S_{t\downarrow(F)}^2 (\mathcal{F}_t) \). It is sufficient to prove the converse inclusions.

Step 1: At first we shall show that \( d \Gamma_t \) and \( d \tilde{\Gamma}_t \) are bounded in \( L^2(\Omega; \mathfrak{X}) \).

For any finite \( \mathcal{F}_t \)-measurable partition \( \{ A_1, ..., A_m \} \) of \( \Omega \) and a finite sequence \( \{ f^1, ..., f^m \} \subset S(F) \),

\[
E[\sum_{i=1}^m \chi_{A_i} \int_0^t \int_Z f_i^*(z) N(dsdz)]^2
\]

\[
= \sum_{i=1}^m E[\chi_{A_i} \int_0^t \int_Z f_i^*(z) N(dsdz)]^2
\]

\[
\leq \sum_{i=1}^m E[\chi_{A_i} (\int_0^t \int_Z f_i^*(z) N(dsdz)]^2]
\]

\[
\leq \sum_{i=1}^m E[\chi_{A_i} (\int_0^t \int_Z f_i^*(z) N(dsdz)]^2]
\]

\[
= E[(\int_0^t \int_Z f_i^*(z) N(dsdz)]^2].
\]

The process \( \{ \| F(t) \|_k : t \in (0, T) \} \) is a real valued predictable (with three parameters \( t, z, \omega \)) process since the set-valued stochastic process \( \{ F(t), t \in (0, T) \} \) is \( \mathcal{F} \)-predictable. Then by Theorem 3.1 we have

\[
E[(\int_0^t \int_Z \| F_s(z) \|_k N(dsdz)]^2] \leq C E[(\int_0^t \int_Z \| F_s(z) \|^2_k N(dsdz)] < \infty,
\]

where \( C \) is the constant that depends on \( C_X \). The inequality (12) implies that \( d \Gamma_t \) is bounded in \( L^2(\Omega; \mathfrak{X}) \).

\[
E[\sum_{i=1}^m \chi_{A_i} \int_0^t \int_Z f_i^*(z) \tilde{N}(dsdz)]^2
\]

\[
= \sum_{i=1}^m E[\chi_{A_i} \int_0^t \int_Z f_i^*(z) \tilde{N}(dsdz)]^2
\]

\[
\leq 2 \sum_{i=1}^m E[\chi_{A_i} \int_0^t \int_Z f_i^*(z) N(dsdz)]^2]
\]

\[
\leq 2 \sum_{i=1}^m E[\chi_{A_i} \int_0^t \int_Z f_i^*(z) N(dsdz)]^2]
\]

\[
\leq 2 \sum_{i=1}^m E[\chi_{A_i} \int_0^t \int_Z f_i^*(z) N(dsdz)]^2 + 2 \sum_{i=1}^m E[\chi_{A_i} \int_0^t \int_Z f_i^*(z) d\nu(Z)]^2
\]

\[
\leq 2 \sum_{i=1}^m E[\chi_{A_i} \int_0^t \int_Z f_i^*(z) N(dsdz)]^2 + 2 \sum_{i=1}^m E[\chi_{A_i} \int_0^t \int_Z f_i^*(z) d\nu(Z)]^2
\]

\[
= 2 E[(\int_0^t \int_Z f_i^*(z) N(dsdz)]^2 + 2 E[(\int_0^t \int_Z f_i^*(z) d\nu(Z)]^2
\]

\[
\leq 2 C E[\int_0^t \int_Z f_i^*(z) N(dsdz)]^2 + 2 T \nu(Z) E[\int_0^t \int_Z f_i^*(z) d\nu(Z)]^2
\]

\[
= 2(C + T \nu(Z)) E[\int_0^t \int_Z f_i^*(z) d\nu(Z)]^2 < \infty,
\]

which yields that \( d \tilde{\Gamma}_t \) is bounded in \( L^2(\Omega; \mathfrak{X}) \).

Step 2. We shall show that the closure of \( d \Gamma_t \) (\( d \tilde{\Gamma}_t \)) in \( L^1 \) is also a subset of \( L^2(\Omega; \mathfrak{X}) \).
Taking any \( h \in S_{J_t(F)}^1(\mathcal{F}_t) \), there exists a sequence
\[
\{h^k : k = 1, 2, \ldots \} \subset d\Gamma_t,
\]
such that
\[
E\|h^k - h\| \to 0 \quad \text{as} \quad k \to +\infty.
\]
Then there exists a subsequence \( \{h^{k_i} : i = 1, 2, \ldots \} \) of \( \{h^k : k = 1, 2, \ldots \} \) such that
\[
\|h^{k_i} - h\| \to 0 \quad \text{as} \quad i \to +\infty \quad \text{a.s.}
\]
For any \( h^k \), we have \( h^k \leq \int_0^{t+} \int_Z F_s(z) N(dsdz) \) a.s. In addition, \( \int_0^{t+} \int_Z \|F_s\|_{KL}(dsdz) \) is \( L^2 \)-integrable. Therefore, by the Lebesgue dominated convergence theorem, we obtain
\[
E\|h^{k_i} - h\|^2 \to 0 \quad \text{as} \quad i \to +\infty
\]
By the inequality,
\[
\|h(\omega)\|^2 \leq 2\|h(\omega) - h^{k_i}(\omega)\|^2 + 2\|h^{k_i}(\omega)\|^2, \quad \text{a.s.}
\]
immediately, we obtain \( h \in S_{J_t(F)}^2(\mathcal{F}_t) \), which implies \( S_{J_t(F)}^1(\mathcal{F}_t) \subset S_{J_t(F)}^2(\mathcal{F}_t) \). Similarly, we have \( S_{I_t(F)}^1(\mathcal{F}_t) \subset S_{I_t(F)}^2(\mathcal{F}_t) \).

By Lemma 3.1 and its proof, we get Theorem 3.6 below, which is necessary to guarantee the availability to study the set-valued stochastic differential equation with set-valued jump part.

**Theorem 3.6.** Assume a set-valued stochastic process \( \{F_t, \mathcal{F}_t : t \in [0, T]\} \in \mathcal{M} \). Then both \( \{J_t(F)\} \) and \( \{I_t(F)\} \) are \( L^2 \)-integrably bounded.

If \( \mathcal{F} \) is separable, by Theorem 3.5 in [34], for stochastic processes \( \{I_t, \mathcal{F}_t : t \in (0, T]\} \) and \( \{J_t, \mathcal{F}_t : t \in (0, T]\} \), there exist \( \mathcal{F} \otimes \mathcal{B}([0, T]) \)-measurable and \( \mathcal{F}_t \)-adapted versions. From now on, we always take the measurable versions.

**Lemma 3.2.** Assume \( \mathcal{F} \) is separable with respect to \( P \). For set-valued stochastic processes \( \{F_t\}_{t \in [0, T]}, \{G_t\}_{t \in [0, T]} \in \mathcal{M} \), and for all \( t \), we have
\[
H\left(\int_0^{t+} \int_Z F_s(z) N(dsdz), \int_0^{t+} \int_Z G(s, z, \omega) N(dsdz)\right) 
\leq \int_0^{t+} \int_Z H(F_s(z), G_s(z)) N(dsdz) \quad \text{a.s.}
\]
(13)

**Proof.** By Theorem 3.5 in [34], there exists a sequence \( \{f^i : i \in \mathbb{N}\} \subset S(F) \), such that
\[
F(t, z, \omega) = \text{cl}\{f^i(t, z, \omega) : i \in \mathbb{N}\} \quad \text{a.e.} \quad (t, z, \omega)
\]
and, for each \( t \in [0, T] \),
\[
\int_0^{t+} \int_Z F_s(z) N(dsdz) = \text{cl}\left\{\int_0^{t+} \int_Z f^i_s(z) N(dsdz) : i \in \mathbb{N}\right\}.
\]
For each \( i \geq 1 \), we can choose a sequence \( \{g^{ij} : j \in \mathbb{N}\} \subset S(G) \) (this sequence depends on \( i \)), such that
\[
\|f^i - g^{ij}\|_{\mathcal{L}^1} \downarrow d(f^i, S(G)) \quad (j \to +\infty),
\]
where
\[
\|f^i - g^{ij}\|_{\mathcal{L}^1} = \int_0^T \int_Z \|f^i_s(z) - g^{ij}_s(z)\| d\nu(dz) dp,
\]
and
\[
d(f^i, S(G)) = \inf_{g \in S(G)}\|f^i - g\|_{\mathcal{L}^1}.
\]
In fact,
\[
\int_{\Omega} \int_{0}^{T} \int_{Z} \|f_{s}^{i}(z) - g_{s}^{ij}(z)\| N(dsdz)dp = \int_{\Omega} \int_{0}^{T} \int_{Z} \|f_{s}^{i}(z) - g_{s}^{ij}(z)\| dsdv(dz)dp < +\infty
\]

since \(F, G \in \mathcal{M}\).

By (14) and Theorem 2.2 in [7], we have
\[
d\left(f^{i}, S(G)\right) = \inf_{g \in S(F)} \|f^{i} - g\|_{\mathcal{X}}
\]
\[
= \inf_{g \in S(G)} \int_{\Omega} \int_{0}^{T} \int_{Z} \|f_{s}^{i}(z) - g_{s}(z)\| dsdv(dz)dp
\]
\[
= \inf_{g \in S(G)} \int_{\Omega} \int_{0}^{T} \int_{Z} \|f_{s}^{i}(z) - g_{s}(z)\| N(dsdz)dp
\]
\[
= \inf_{g \in S(G)} \int_{\Omega} \int_{0}^{T} \int_{Z} \|f_{s}^{i}(z) - g_{s}(z)\| N(dsdz)dp
\]
\[
= \int_{\Omega} \int_{0}^{T} \int_{Z} \inf_{y \in G_{s}(z)} \|f_{s}^{i}(z) - y\| N(dsdz)dp
\]
\[
= \int_{\Omega} \int_{0}^{T} \int_{Z} d(f_{s}^{i}(z), G_{s}(z)) N(dsdz)dp.
\]
Namely, noticing that \(\|f^{i} - g^{ij}\|_{\mathcal{X}} \geq d\left(f^{i}, S(G)\right)\) and \(\|f^{i}(s, z, \omega) - g^{ij}(s, z, \omega)\| \geq d(f^{i}(s, z, \omega), G(s, z, \omega))\) for a.e. \((s, z, \omega)\), then for any \(\varepsilon > 0\), there exists a natural number \(M\) such that for any \(j \geq M\),
\[
\varepsilon > \int_{\Omega} \int_{0}^{T} \int_{Z} \|f_{s}^{i}(z) - g_{s}^{ij}(z)\| N(dsdz)dp
\]
\[
- \int_{\Omega} \int_{0}^{T} \int_{Z} d(f_{s}^{i}(z), G_{s}(z)) N(dsdz)dp
\]
\[
= \int_{\Omega} \int_{0}^{T} \int_{Z} \|f_{s}^{i}(z) - g_{s}^{ij}(z)\| N(dsdz)dp
\]
\[
- \int_{\Omega} \int_{0}^{T} \int_{Z} d(f_{s}^{i}(z), G_{s}(z)) N(dsdz)dp
\]
\[
= \int_{\Omega} \int_{0}^{T} \int_{Z} (\|f_{s}^{i}(z) - g_{s}^{ij}(z)\| - d(f_{s}^{i}(z), G_{s}(z))) N(dsdz)dp
\]
\[
= \int_{\Omega} \int_{0}^{T} \int_{Z} (\|f_{s}^{i}(z) - g_{s}^{ij}(z)\| - d(f_{s}^{i}(z), G_{s}(z))) N(dsdz)dp.
\]
Hence there exists a subsequence of \(\{g^{ij} : j \in \mathbb{N}\}\), denoted as \(\{g^{ij_{k}} : k \in \mathbb{N}\}\) such that
\[
\|f^{i}(s, z, \omega) - g^{ij_{k}}(s, z, \omega)\| \to d(f^{i}(s, z, \omega), G(s, z, \omega)) (k \to +\infty) \text{ a.e. } (s, z, \omega).
\]
Because \(\{F_{t}\}_{t \in [0, T]}\) and \(\{G_{t}\}_{t \in [0, T]}\) are in \(\mathcal{M}\), we have
\[
\int_{\Omega} \int_{0}^{T} \int_{Z} (\|F_{s}(z)\|_{K} + \|G_{s}(z)\|_{K}) N(dsdz)dp < \infty,
\]
which yields
\[
\int_{0}^{T} \int_{Z} (\|F_{s}(z)\|_{K} + \|G_{s}(z)\|_{K}) N(dsdz) < \infty \text{ a.s.,}
\]
Since 
\[ \| f^t(s, z, \omega) - g^{ijk}(s, z, \omega) \| \leq \| F(s, z, \omega) \| K + \| G(s, z, \omega) \| K \text{ for a.e.}(s, z, \omega) \]

and almost sure 
\[ \| f^t(s, z, \omega) - g^{ijk}(s, z, \omega) \| \leq \| F(s, z, \omega) \| K + \| G(s, z, \omega) \| K \text{ for a.e.}(s, z, \omega) \]

then together with (16), by the Lebesgue dominated convergence theorem, for all \( t \) and almost sure \( \omega \), we obtain that 
\[ \int_0^{t+} \int_Z f^t_s(z) - g^{ijk}_s(z) \| N(\text{dsd}z) \rightarrow \int_0^{t+} \int_Z d(f^t_s(z), G_s(z)) N(\text{dsd}z) \]

when \( k \rightarrow +\infty \). Therefore, for all \( t \) and almost sure \( \omega \)
\[ \inf_k \int_0^{t+} \int_Z \| f^t_s(z) - g^{ijk}_s(z) \| N(\text{dsd}z) \leq \int_0^{t+} \int_Z d(f^t_s(z), G_s(z)) N(\text{dsd}z). \]

Hence, for all \( t \) and almost sure \( \omega \), we have
\[
\begin{align*}
&\sup_{x \in f^t_s G_s(z) N(\text{dsd}z)} d(x, \int_0^{t+} \int_Z f^t_s(z) N(\text{dsd}z)) \\
&\leq \sup_i \inf_j \int_0^{t+} \int_Z f^t_s(z) N(\text{dsd}z) - \int_0^{t+} \int_Z g^{ij}_s(z) N(\text{dsd}z) \| \\
&\leq \sup_i \inf_k \int_0^{t+} \int_Z f^t_s(z) N(\text{dsd}z) - \int_0^{t+} \int_Z g^{ijk}_s(z) N(\text{dsd}z) \\
&\leq \sup_i \int_0^{t+} \int_Z d(f^t_s(z), G_s(z)) N(\text{dsd}z) \\
&\leq \int_0^{t+} \int_Z \sup_i d(f^t_s(z), G_s(z)) N(\text{dsd}z). 
\end{align*}
\]

Similarly, by Theorem 3.5 in [34], there exists a sequence \( \{ g^m : m \in \mathbb{N} \} \subset S(G) \) such that
\[ G(t, z, \omega) = \text{cl} \{ g^m(t, z, \omega) : m \in \mathbb{N} \} \text{ a.e. } (t, z, \omega) \]

and, for each \( t \in [0, T] \),
\[
\int_0^{t+} \int_Z G_s(z) N(\text{dsd}z) = \text{cl} \{ \int_0^{t+} \int_Z g^m_s(z) N(\text{dsd}z) : m \in \mathbb{N} \}.
\]

In the same way as above, we obtain that for all \( t \) and almost sure \( \omega \),
\[
\begin{align*}
&\sup_{y \in f^t_s G_s(z) N(\text{dsd}z)} d(y, \int_0^{t+} \int_Z F_s(z) N(\text{dsd}z)) \\
&\leq \int_0^{t+} \int_Z \sup_m d(g^m_s(z), F_s(z)) N(\text{dsd}z). 
\end{align*}
\]

Therefore, the inequality
\[
\begin{align*}
H \left( \int_0^{t+} \int_Z F_s(z) N(\text{dsd}z), \int_0^{t+} \int_Z G_s(z) N(\text{dsd}z) \right) \\
\leq \int_0^{t+} \int_Z H(F_s(z), G_s(z)) N(\text{dsd}z)
\end{align*}
\]

holds for all \( t \) and almost sure \( \omega \). 

\[\square\]
Theorem 3.7. Assume \( \mathcal{F} \) is separable with respect to \( P \). Let \( \{F_t\}_{t \in [0,T]} \) and \( \{G_t\}_{t \in [0,T]} \) be set-valued stochastic processes in \( \mathcal{M} \). Then for all \( t \), it follows that

\[
E \left[ H \left( \int_0^{t+} \int_Z F_s(z)N(ds,dz), \int_0^{t+} \int_Z G_s(z)N(ds,dz) \right) \right] \\
\leq E \left[ \int_0^{t+} \int_Z H(F_s(z), G_s(z))N(ds,dz) \right] \\
= E \left[ \int_0^{t+} \int_Z H(F_s(z), G_s(z))ds \right] \\
\tag{17}
\]

and

\[
E \left[ H^2 \left( \int_0^{t+} \int_Z F_s(z)N(ds,dz), \int_0^{t+} \int_Z G_s(z)N(ds,dz) \right) \right] \\
\leq CE \left[ \int_0^{t+} \int_Z H^2(F_s(z), G_s(z))N(ds,dz) \right] \\
= CE \left[ \int_0^{t+} \int_Z H^2(F_s(z), G_s(z))ds \right] \\
\tag{18}
\]

where \( C \) is the constant appearing in Theorem 3.6.

Proof. Since

\[
H(F(s,z,\omega), G(s,z,\omega)) \leq H(F(s,z,\omega), \{0\}) + H(G(s,z,\omega), \{0\}) \\
= \|F(s,z,\omega)\|_k + \|G(s,z,\omega)\|_k,
\]

\[
H^2(F(s,z,\omega), G(s,z,\omega)) \leq (H(F(s,z,\omega), \{0\}) + H(G(s,z,\omega), \{0\}))^2 \\
\leq 2\|F(s,z,\omega)\|_k^2 + 2\|G(s,z,\omega)\|_k^2,
\]

and \( F, G \in \mathcal{M} \), therefore both \( E \left[ \int_0^{T+} \int_Z H(F(s,z,\omega), G(s,z,\omega))N(ds,dz) \right] \) and \( E \left[ \int_0^{T+} \int_Z H^2(F(s,z,\omega), G(s,z,\omega))N(ds,dz) \right] \) are finite. By taking expectation on both sides of \( (13) \), immediately we obtain that

\[
E \left[ H \left( \int_0^{T+} \int_Z F_s(z)N(ds,dz), \int_0^{T+} \int_Z G_s(z)N(ds,dz) \right) \right] < \infty
\]

and \( (17) \) holds. By Theorem 3.6 we have that

\[
E \left[ H^2 \left( \int_0^{T+} \int_Z F_s(z)N(ds,dz), \int_0^{T+} \int_Z G_s(z)N(ds,dz) \right) \right]
\]

is finite. Then by \( (13) \) and Theorem 3.6 we have

\[
E \left[ H^2 \left( \int_0^{T+} \int_Z F_s(z)N(ds,dz), \int_0^{T+} \int_Z G_s(z)N(ds,dz) \right) \right] \leq E \left[ \left( \int_0^{T+} \int_Z H(F_s(z), G_s(z))N(ds,dz) \right)^2 \right] \\
\leq CE \left[ \int_0^{T+} \int_Z H^2(F_s(z), G_s(z))ds \right] \\
= CE \left[ \int_0^{T+} \int_Z H^2(F_s(z), G_s(z))N(ds,dz) \right],
\]

which implies \( (18) \).
4. Set-valued stochastic integral equation

In this section, we study the strong solution to a set-valued stochastic integral equation. Assume $X$ is a separable $M$-type 2 Banach space, $\mathcal{F}$ is separable with respect to $P$. $(Z, \mathcal{B}(Z))$ is a separable Banach space with finite measure $\mu$. Let the functions $\sigma : [0, T] \times [0, T] \times \Omega \rightarrow K(\mathbb{X})$ be measurable functions $a, b, c$ also satisfy the following conditions:

$$
\|a(t, X)\|_K + \|b(t, X)\| + \int_Z \|c(t, z, X)\| K\mu(dz) \leq C_1 (1 + \|X\|_K),
$$

for $X \in K(\mathbb{X}), t \in [0, T]$ and some constant $C_1$ and

$$
H^2 (a(t, X), a(t, Y)) + \|b(t, X) - b(t, Y)\|^2 + \int_Z H^2 (c(t, z, X), c(t, z, Y)) \nu(dz) \leq C_2 H^2 (X, Y),
$$

for $X, Y \in K(\mathbb{X}), t \in [0, T]$ and some constant $C_2$.

Let $X_0$ be an $L^2$-integrably bounded set-valued random variable, $\{B_t : t \in [0, T]\}$ a real valued Brownian motion and $N_t$ a stationary Poisson point process with characteristic measure $\nu$. It is reasonable to define the set-valued stochastic integral equation as follows:

**Definition 4.1.**

$$
X_t = c \left\{ X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dB_s + \int_0^{t+} \int_Z c(s, z, X_s-)N(dzds) \right\},
$$

for $t \in [0, T]$ a.s.

Suppose that $\{X_t : t \in [0, T]\}$ is an $\mathcal{F}_t$-adapted and measurable set-valued process, which is right continuous in $t$ with respect to $H$ almost surely. Then it is called a strong solution if it satisfies the equation (21).

**Remark 3.** There are four terms on the right hand side of equation (21). Every term is measurable and bounded a.s. Then the closure of the sum is measurable and bounded a.s. Thus the right hand side of formula (21) makes sense.

If the initial value is not only $L^2$-integrably bounded but also weakly compact in $\mathbb{X}$, then it is not necessary to take the closure in the right hand side in (21) (cf. (4.3) in [36]).

**Theorem 4.1.** Assume that $\mathcal{F}$ is separable with respect to $P$. Let $T > 0$, and let $a(\cdot, \cdot) : [0, T] \times K(\mathbb{X}) \rightarrow K(\mathbb{X})$, $b(\cdot, \cdot) : [0, T] \times K(\mathbb{X}) \rightarrow \mathbb{X}$ and $c(\cdot, \cdot, \cdot) : [0, T] \times Z \times K(\mathbb{X}) \rightarrow K(\mathbb{X})$ be measurable functions satisfying conditions (19) and (20). Then for any given $L^2$-integrably bounded initial value $X_0$, there exists a unique strong solution to (21). The unique strong solution is right continuous in $t$ with respect to the Hausdorff metric. In the above, the uniqueness means $P \left( H(X_t, Y_t) = 0 \text{ for all } t \in [0, T] \right) = 1$ for any strong solutions $X_t$ and $Y_t$ to (21).

**Proof.** As a manner similar to that of solving single-valued stochastic differential equation, we use the successive approximation method to construct a solution of equation (21).

Define $Y^0_t = X_0$, and $Y^k_t = Y^k_t(\omega)$ for $k \in \mathbb{N}$ inductively as follows:

$$
Y^{k+1}_t = c \left\{ X_0 + \int_0^t a(s, Y^k_s)ds + \int_0^t b(s, Y^k_s)dB_s + \int_0^{t+} \int_Z c(s, z, Y^k_{s-})N(dzds) \right\}.
$$

(22)
By property of Hausdorff metric, we have
\[
H(Y_t^{k+1}, Y_{t}^{k}) \leq H \left( \int_0^t a(s, Y_s^k)ds, \int_0^t a(s, Y_s^{k-1})ds \right) + \left\| \int_0^t (b(s, Y_s^k) - b(s, Y_s^{k-1}))dB_s \right\|
\]
\[
+ H \left( \int_0^{t+} \int_Z c(s, z, Y^k_s)N(dzds), \int_0^{t+} \int_Z c(c, s, Y_s^{k-1})N(dzds) \right).
\]
\[
E \left[ \sup_{s \in [0, t]} H^2(Y_{s}^{k+1}, Y_{s}^{k}) \right]
\]
\[
\leq 3E \left[ \sup_{s \in [0, t]} H^2 \left( \int_0^s a(\tau, Y^k_{\tau})d\tau, \int_0^s a(\tau, Y^{k-1}_{\tau})d\tau \right) + \sup_{0 \leq s \leq t} \left\| \int_0^s b(\tau, Y^k_{\tau})dB_\tau - \int_0^s b(\tau, Y^{k-1}_{\tau})dB_\tau \right\|^2
\]
\[
+ \sup_{0 < s \leq t} H \left( \int_0^{t+} \int_Z c(s, z, Y^k_s)N(dzds), \int_0^{t+} \int_Z c(c, s, Y_s^{k-1})N(dzds) \right) \right].
\]
By condition (20) and Doob maximal martingale inequality, we have
\[
E \left[ \sup_{0 \leq s \leq t} H^2 \left( \int_0^s a(\tau, Y^k_{\tau})d\tau, \int_0^s a(\tau, Y^{k-1}_{\tau})d\tau \right) \right] \leq TC_2 E \left[ \int_0^t H^2(Y^k_{\tau}, Y^{k-1}_{\tau})d\tau \right]
\]
(23)
\[
E \left[ \sup_{0 \leq s \leq t} \left\| \int_0^s b(\tau, Y^k_{\tau})dB_\tau - \int_0^s b(\tau, Y^{k-1}_{\tau})dB_\tau \right\|^2 \right] \leq 4C_1 C_2 E \left[ \int_0^t H^2(Y^k_{\tau}, Y^{k-1}_{\tau})d\tau \right].
\]
(24)
By Theorem 3.1 Lemma 3.2 Theorem 3.7 and condition (20),
\[
E \left[ \sup_{0 \leq s \leq t} H^2 \left( \int_0^{t+} \int_Z c(s, z, Y^k_s)N(dzds), \int_0^{t+} \int_Z c(c, s, Y_s^{k-1})N(dzds) \right) \right]
\]
\[
\leq E \left[ \sup_{0 \leq s \leq t} \left( \int_0^{t+} \int_Z H \left( c(s, z, Y^k_s), c(c, s, Y_s^{k-1}) \right)N(dzds) \right)^2 \right]
\]
\[
\leq CE \left[ \int_0^{t+} \int_Z H^2(c(s, z, Y^k_s), c(c, s, Y_s^{k-1}))ds\nu(dz) \right]
\]
\[
= CE \left[ \int_0^{t+} \left( \int_Z H^2(c(s, z, Y^k_s), c(c, s, Y_s^{k-1}))\nu(dz) \right)ds \right] \leq CC_2 E \left[ \int_0^t H^2(Y^k_{\tau}, Y^{k-1}_{\tau})d\tau \right].
\]
Therefore, we obtain
\[
E \left[ \sup_{s \in [0, t]} H^2(Y_{s}^{k+1}, Y_{s}^{k}) \right] \leq (3TC_2 + 12C_1 C_2 + 3CC_2) E \left[ \int_0^t H^2(Y^k_{\tau}, Y^{k-1}_{\tau})d\tau \right]
\]
Setting \( c := 9C_2(T \vee 4C_1 \vee C) \) and \( \Delta_k(t) := E \left[ \sup_{s \in [0, t]} H^2(Y_{s}^{k+1}, Y_{s}^{k}) \right] \), then by induction, we have
\[
\Delta_k(T) = E \left[ \sup_{s \in [0, T]} H^2(Y_{s}^{k+1}, Y_{s}^{k}) \right] \leq c \int_0^T \Delta_{k-1}(\tau)d\tau
\]
\[
\leq c^k \int_0^T \int_{\tau_{k-1}}^{\tau_k} \int_{\tau_{k-2}}^{\tau_{k-1}} \cdots \int_{\tau_0}^{\tau_1} \Delta_0(\tau_0)d\tau_0 \cdots d\tau_{k-2} \leq c^k \Delta_0(T) \int_0^T \int_{\tau_{k-1}}^{\tau_k} \cdots \int_{\tau_0}^{\tau_1} \Delta_0(d\tau_0 \cdots d\tau_{k-2}).
\]
Hence, we obtain \( \Delta_k(T) \leq \frac{(cT)^k}{k!} \Delta_0(T) \). Therefore, the series \( \sum_{k=1}^{\infty} \Delta_k(T) \) converges. Then
\[
\sum_{k=1}^{\infty} \sup_{t \in [0, T]} H^2(Y^k_{t}, Y^{k-1}_{t}) < +\infty \ a.s.,
\]
which implies the sequence \( \{ Y^k : k \in \mathbb{N} \} \) uniformly (with respect to \( t \)) converges to a set-valued stochastic process denoted by \( \{ Y_t : t \in [0, T] \} \) by the completeness of the space \( L^2(\Omega; (\mathcal{K}_b(\mathbb{X}), H)) \). Since both the
integral of set-valued stochastic processes with respect to Lebesgue measure $t$ and the integral with respect to Brownian motion are continuous in $t$, together with Theorem 3.7 we obtain that the process \{Y_t\} is right continuous in $t$ with respect to the Hausdorff metric $H$ and satisfies (21).

Now we show the uniqueness of solutions. Assume there are two solutions \{X_t: t \in [0, T]\} and \{Y_t: t \in [0, T]\} with the same initial value $X_0$. Denote $\triangle(t) = E\left[ \sup_{s \in [0,t]} H^2(X_s, Y_s) \right]$. Then through the same way as above, we have $\triangle(T) \leq \left(\frac{cT}{k^2}\right) \triangle(T)$. Letting $k \to \infty$, we obtain $\triangle(T) = 0$, which implies $P\left( H(X_t, Y_t) = 0 \text{ for all } t \in [0, T] \right) = 1$.

5. Concluding remark

The main result of this paper is that the set-valued integral with respect to the compensated Poisson measure is not a martingale unless the integrand degenerates into a single-valued process. The proof uses the Hahn decomposition of a Banach space and bounded linear functionals. Since integrals with respect to Poisson point process are integrably bounded, the differential equation with set-valued jump makes sense. Due to the complexity in real world, set-valued random variable is a good tool to model the uncertainty including both randomness and imprecision. We expect that the model (21) has potential applications to practical fields. For instance, single-valued stochastic calculus has surprising applications in mathematical finance and dynamics [16]. It is also reasonable to consider the price of finance derivative as an interval-valued random variable due to high frequency fluctuations and unseen events. Ogura [25] studied the set-valued Black-Scholes equation. Sometimes there is a big change of price since some unusual and unpredictable causes. A possible model for this situation is set-valued stochastic differential equation with jump, which is a natural extension of the equation in [25]. Another example of potential application is on detection of echo signal of a sea clutter, which is very important in the defense and civilian business. Due to the fluid dynamics, classical stochastic differential equation is used to modeling the echo signal’s phase and amplitude ([31]). The sea surface may have a big change during a very short period since the complex fluid dynamics or the sudden strong wind. It is reasonable to consider the amplitude of sea clutter as a set-valued process. The sharp change of sea surface can be described as a Poisson jump.

References

[1] R. Aumann, Integrals of set-valued functions, J.Math.Anal.Appl. 12 (1965) 1-12.
[2] J.K. Brooks and N. Dinculeanu, Weak compactness in spaces of Bochner integrable functions and applications, Advances in Mathematics 24 (1977) 172-188.
[3] Z. Brzeźniak, A. Carroll, Approximations of the Wong-Zakai differential equations in M-type 2 Banach spaces with applications to loop spaces, Séminaire de Probabilités, XXXVII (2003) 251-289.
[4] D.A. Charalambos and C.B. Kim, Infinite Dimensional Analysis, Springer-Verlag, Berlin, 1994.
[5] C. Castaing and M. Valadier, Convex Analysis and Measurable Multifunctions, Lecture Notes in Math 580 Springer-Verlag, Berlin, 1977.
[6] E. Dettweiler, A characterization of the Banach spaces of type $p$ by Lévy measures, Math.Z. 157 (1977) 121-130.
[7] F. Hiai and H. Umegaki, Integrals, conditional expectations and martingales of multivalued functions, Jour. Multivar. Anal. 7 (1977) 149-182.
[8] F. Hiai, Convergence of conditional expectations and strong laws of large numbers for multivalued random variables, Trans. A.M.S. 291 (1985) 613-627.
[9] A. Honda and Y. Okazaki, Theory of inclusion-exclusion integral, Information Sciences, 376 (2017): 136-147.
[10] A. Honda, Y. Okazaki and Y. Takahashi, A generalization of the Hanner’s inequality and the type 2 (cotype 2) constant of a Banach space, *Bulletin of the Kyushu Institute of Technology, Pure and Applied Mathematics* 44 (1995) 29-34.

[11] N. Ikeda and S. Watanabe, Stochastic Differential Equations and Diffusion Processes, North-Holland publishing company, 1981.

[12] E.J. Jung and J.H. Kim, On set-valued stochastic integrals, *Stoch Anal Appl* 21 (2) (2003) 401-418.

[13] B.K. Kim and J.H. Kim, Stochastic integrals of set-valued processes and fuzzy processes, *J.Math.Anal.Appl.* 236 (1999) 480-502.

[14] M. Kisielewicz, Set-valued stochastic integrals and stochastic inclusions, *Stoch Anal Appl* 15 (1997) 780-800.

[15] H. Kunita, Stochastic Differential Equations and Stochastic Flows of Diffeomorphisms, *Lect Notes in Math* 1097, Springer, 1984.

[16] H. Kunita, Itô’s stochastic calculus: Its surprising power for applications, *Stochastic Processes and Their Applications* 120 (2010) 622-652.

[17] S. Li and A. Ren, Representation theorems, set-valued and fuzzy set-valued Itô Integral, *Fuzzy Sets and Systems* 158 (2007) 949-962.

[18] S. Li, Y. Ogura and V. Kreinovich, Limit Theorems and Applications of Set-Valued and Fuzzy Set-Valued Random Variables, Kluwer Academic Publishers, 2002.

[19] J. Li, S. Li and Y. Ogura, Strong solution of Itô type set-valued stochastic differential equation, *Acta Mathematica Sinica, English Series* 26(9) (2010) 1739-1748.

[20] M.Malinowski and M.Michta, Set-valued stochastic integral equations driven by martingales, *J.Math.Anal.Appl.*, 394 (12) (2012) 30-47.

[21] R.E. Megginson, An Introduction to Banach Space Theory, Springer, New York, 1998.

[22] M. Michta, On set-valued stochastic integrals and fuzzy stochastic equations, *Fuzzy Sets and Systems* 177 (2011) 1-19.

[23] I. Mitoma, Y. Okazaki and J. Zhang, Set-valued stochastic differential equations in M-type 2 Banach space, *Communications on Stochastic Analysis* 4(2) (2010) 215-237.

[24] I. Molchanov, Theory of Random Sets, Springer-Verlag, London, 2005.

[25] Y. Ogura, On stochastic differential equations with set coefficients and the Black-Scholes model, in: *Proceedings of the Eighth International Conference on Intelligent Technologies*, 2008, pp: 263-270.

[26] P. Terán, Distributions of random closed sets via containment functional, *Nonlinear Convex Anal.* 15(5) (2014) 907-917.

[27] J. Ren, J. Wu and X. Zhang, Exponential ergodicity of non-Lipschitz multivalued stochastic differential equations, *Bull. Sci. math.* Vol 134 (2010) 391-404.

[28] J. Ren and S. Xu, A transfer principle for multivalued stochastic differential equations, *Journal of Functional Analysis* Vol 256 (2009) 2780-2814.

[29] J. Ren and J. Wu, Multi-valued Stochastic Differential Equations Driven by Poisson Point Processes, *Progress in Probability* Vol 65 (2011) 191-205.

[30] K.I. Sato, Lévy Processes and Infinitely Divisible Distributions, Cambridge University Press, 1999.

[31] K.D.Ward, R.J.A.Tough and S.Watts, Sea Clutter: Scattering, the K Distribution and Radar Performance, The institute of engineering and technology, London, 2006.
[32] S. Watanabe, Itô’s theory of excursion point processes and its developments, *Stochastic Processes and their Applications* **120** (2010) 653-677.

[33] J. Zhang, Set-valued stochastic integrals with respect to a real valued martingale. In: *Soft Method for Handling Variability and Imprecision ASC 48*, Springer-Verlag, Berlin Heidelberg, 2008, pp: 253-259.

[34] J. Zhang, I. Mitoma and Y. Okazaki, Set-valued stochastic integral with respect to Poisson process in a Banach space, *International Journal of Approximate Reasoning* **54** (3) (2013) 404-417.

[35] J. Zhang, S. Li, I. Mitoma and Y. Okazaki, On Set-Valued Stochastic Integrals in an M-type 2 Banach Space, *J. Math. Anal. Appl.* **350** (2009) 216-233.

[36] J. Zhang, S. Li, I. Mitoma and Y. Okazaki, On the solution of set-valued stochastic differential equations in M-type 2 Banach space, *Tohoku Mathematical Journal* **61** (2009) 417-440.