5n Minkowski symmetrizations suffice to arrive at an approximate Euclidean ball

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Abstract

This paper proves that for every convex body in $\mathbb{R}^n$ there exist $5n - 4$ Minkowski symmetrizations, which transform the body into an approximate Euclidean ball. This result complements the sharp $cn \log n$ upper estimate by J. Bourgain, J. Lindenstrauss and V.D. Milman, of the number of random Minkowski symmetrizations sufficient for approaching an approximate Euclidean ball.

1 Introduction

Let $K$ be a compact convex set in $\mathbb{R}^n$ and let $u$ be any vector in $S^{n-1} = \{u; |u| = 1\}$ where $|\cdot|$ denotes the standard Euclidean norm in $\mathbb{R}^n$. Denote by $\pi_u \in O(n)$ the reflection with respect to the hyperplane through the origin orthogonal to $u$, i.e. $\pi_u x = x - 2\langle x, u \rangle u$.

Minkowski symmetrization (often referred to as Blaschke symmetrization) of $K$ with respect to $u$ is defined to be the convex set $\frac{1}{2}(\pi_u K + K)$. Denote by $\|\cdot\|^*$ the dual norm to $K$ (i.e. $\|x\|^* = \sup_{y \in K} \langle x, y \rangle$). Despite the fact that $K$ is not necessarily centrally symmetric and $\|\cdot\|^*$ need not be a norm, this convenient notation will be used for readability. Denote by $M^*(K)$ the half mean width of $K$, defined as $M^*(K) := \int_{S^{n-1}} \|x\|^* d\sigma(x)$, where $\sigma$ is the normalized rotation invariant measure on $S^{n-1}$, and $\|\cdot\|^*$ is the dual norm.

It is easily verified that $M^*(K) = M^*(\frac{1}{2}(\pi_u K + K))$, so the mean width is preserved under Minkowski symmetrizations. Since successive Minkowski symmetrizations make the body more symmetric in some sense, one might expect convergence to a ball of radius $M^*(K)$.

Surprisingly, very few symmetrizations are sufficient for this convergence; In [BLM] it is proven that $cn \log n$ random symmetrizations suffice to obtain from

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any convex body, a new body \( \tilde{K} \), such that \( \frac{1}{2} M^* D \subset \tilde{K} \subset 2M^* D \) with high probability, where \( D = \{ u; |u| \leq 1 \} \) is the standard Euclidean ball in \( \mathbb{R}^n \).

The proof in [BLM] can be slightly refined, and rather than an estimate of \( cn \log n \) for all bodies, in fact \( cn \log \frac{2\text{diam}(K)}{M^*(K)} \) symmetrizations are enough. This quantity is always smaller then \( cn \log n \) but in some cases there is a substantial improvement; For example, the \( n \)-dimensional cube needs only \( cn \) random symmetrizations to be transformed into an almost Euclidean ball.

In [K] it was proven that the aforementioned estimate is very tight and is actually a formula, as follows: For every convex body \( K \) at least \( \tilde{cn} \log \frac{\text{diam}(K)}{M^*(K)} \) random symmetrizations are necessary in order for the body to become close to a Euclidean ball. Hence, bodies such as \( B(l^1_n) \) - the \( n \)-dimensional cross polytope - in fact require at least \( cn \log n \) random symmetrizations.

This paper shows that there exist symmetrizations which are better than random ones. There is a specific choice of \( 5n - 4 \) symmetrizations that transform any convex body into an approximate Euclidean ball. The basic idea underlying the construction is changing the notion of randomness; Rather than symmetrizing with respect to random vectors, symmetrizations with respect to the vectors of a random orthogonal basis will be performed at each iteration.

Six iterations of this kind suffice (totaling \( 6n - 5 \) symmetrizations), however the role of each iteration is slightly different. Precisely, for the first iteration any orthogonal basis is adequate. The remaining five iterations are required to be with respect to random independent orthogonal bases, and the results hold with large probability that tends to 1 when the dimension \( n \) approaches infinity.

There exists a very similar symmetrization process that leads to a slightly better estimate, and consists of \( 5n - 4 \) symmetrizations (only \( 4n - 4 \) symmetrizations, if the body is already unconditional). This process uses symmetrizations with respect to five orthogonal bases, some of which need not be random. An additional basis will be used in this process, and will be referred to here as a Walsh basis. It actually coincides with the regular Walsh basis for dimensions which are powers of two. Let us describe the \( 5n - 4 \) symmetrizations process: The first basis is chosen to be any orthogonal basis, and is used only to create unconditionality. The second basis can be a random basis or a Walsh basis (with respect to the first), and the corresponding symmetrization reduces the diameter of the body to a level of \( \log n \) times its mean width. The third basis is a Walsh basis with respect to the previous, and reduces the diameter further, to a level of \( \log \log n \) times the mean width. The fourth basis must be, in this proof, a random orthogonal basis and the fifth, either a Walsh basis with respect to the fourth, or a random basis. Once the diameter is small enough, the last two bases together transform the body to an approximate Euclidean ball.

\footnote{It seems at first, that six iterations consist of \( 6n \) symmetrizations; However, after the first iteration, the body becomes centrally symmetric. Following that stage, the last vector in each orthogonal basis is unnecessary, because symmetrizing with respect to that vector would not affect the body.}
The proof outlined below is mainly concerned with the first process described (which is purely random). Results for the second process are analogous to those of the first, and may be concluded based on remarks throughout the proof.

The symbols $c, C, c', \tilde{c}$ denote numerical constants which are not necessarily identical throughout this text.

2 First Step: Initial Symmetrizations

Let $K$ be an arbitrary convex body in $\mathbb{R}^n$. For the purpose of normalization, assume $M^*(K) = 1$. Take any orthogonal basis $\{e_1, ..., e_n\}$ and symmetrize $K$ with respect to the vectors $e_1, ..., e_n$ to obtain the new body $\tilde{K}$. Since orthogonal reflections commute, $\tilde{K}$ is invariant under reflection with respect to $e_i$, for $1 \leq i \leq n$. Therefore $\tilde{K}$ is unconditional with respect to the basis $\{e_1, ..., e_n\}$. By Lemma 3.2 from [1] there exists a universal constant $c$ such that,

$$\tilde{K} \subset c\sqrt{n} \text{conv}\{\pm e_i\}_{i=1}^n = c\sqrt{n}B(l^1_n)$$

A specific body will be referred to in this section: $Q = \sqrt{n} \text{conv}\{\pm e_i\}_{i=1}^n$. After a certain symmetrization process its diameter decays from $\sqrt{n}$ to $\tilde{c}\log n$ with high probability. Clearly, applying the same set of symmetrizations to $\tilde{K}$ will reduce its diameter to less than $\tilde{c}\log n$.

**Proposition 2.1** Let $\{e_1,...,e_n\}$ be an orthogonal basis in $\mathbb{R}^n$, and let $Q = \sqrt{n} \text{conv}\{\pm e_i\}_{i=1}^n$. Let $\mu_n$ be the unique rotation invariant probability measure on $O(n)$. Suppose that $\{u_1,...,u_n\} \in O(n)$ is chosen randomly, according to $\mu_n$. After symmetrizing $Q$ with respect to $u_1,...,u_{n-1}$ a new body $\tilde{Q}$ is obtained. Claim:

$$\text{diam}(\tilde{Q}) \leq c\log n$$

with probability greater than $1 - \frac{1}{n^{10}}$.

Remark: The number ‘10’ in the expression $1 - \frac{1}{n^{10}}$ is of course arbitrary, and may be replaced by any other constant. Such a replacement will influence the constant ‘c’ in the concluded inequality “$\text{diam}(\tilde{Q}) \leq c\log n$”.

**Corollary 2.2** For every convex body $K \subset \mathbb{R}^n$ with $M^*(K) = 1$, there exist $2n - 1$ symmetrizations which transform $K$ into $\tilde{K}$, where $\text{diam}(\tilde{K}) < c\log n$ and $\tilde{K}$ is unconditional with respect to some orthogonal basis.

Following is a simple and well-known lemma. For completeness it will be proven at the end of this section.

**Lemma 2.3** Let $\{e_i\}_{i=1}^n$ be any orthogonal basis, and let $\{u_i\}_{i=1}^n$ be a random orthogonal basis. Then for all $1 \leq i, j \leq n$:

$$|\langle u_i, e_j \rangle| \leq c_1 \frac{\sqrt{\log n}}{\sqrt{n}}$$

with probability greater than $1 - \frac{1}{n^{10}}$. 

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Proof of Proposition 2.1: Denote by $\| \cdot \|$ the dual norm of $Q$ (i.e. $\|x\| = \sup_{y \in Q} \langle x, y \rangle$). The dual norm of $Q$ is, by definition (recall that the Minkowski sum of bodies is equivalent to the sum of their dual norms):

$$\|x\| = \frac{1}{2^{n-1}} \sum_{D \subset \{1, \ldots, n\}} \| (\prod_{i \in D} \pi_{u_i}) x \| = \frac{1}{2^{n-1}} \sum_{D} \sqrt{n} \max_{j} \langle \prod_{i \in D} \pi_{u_i} x, e_j \rangle$$

Substitute $x = \sum_{i} \langle x, u_i \rangle u_i$. Since reflecting with respect to $u_i$ means switching the sign of the $i^{th}$ coordinate in $\{u_1, \ldots, u_n\}$ basis,

$$\|x\| = \mathbb{E}_{\varepsilon} \max_{j} \sqrt{n} \sum_{i} \varepsilon_i \langle x, u_i \rangle \langle u_i, e_j \rangle$$

where $\varepsilon = (\varepsilon_i)_{i=1}^n$ is uniformly distributed in $\{-1\}^n$. Therefore, $\|x\|$ is the expectation of a maximum of $n$ random variables. Denote:

$$f^j_x(\varepsilon) = \sqrt{n} \sum_{i} \varepsilon_i \langle x, u_i \rangle \langle u_i, e_j \rangle$$

Then $\|x\| = \mathbb{E}_{\varepsilon} \max_{j} f^j_x(\varepsilon)$. For $1 \leq \alpha \leq 2$, and for any measurable $f : \Omega \to \mathbb{R}$ define $\|f\|_{\psi_\alpha} = \inf\{\lambda > 0 : \mathbb{E}_{\varepsilon} e^{\|f\|_{\psi_\alpha}} \leq 2\}$. The equivalent definitions are frequently used:

$$\|f\|_{\psi_\alpha} < c \Leftrightarrow \left(\mathbb{E}|f|^p\right)^{\frac{1}{p}} < c^{\frac{1}{p}} \Leftrightarrow \text{Prob}\{|f| > t\} < e^{-c t^{\alpha}}$$

Khinchine inequality shows that the $\psi_2$ norm of $f^j_x$ is bounded, as follows:

$$\|f^j_x\|_p = \left(\mathbb{E}_{\varepsilon} \left| \sqrt{n} \sum_{i} \varepsilon_i \langle x, u_i \rangle \langle u_i, e_j \rangle \right|^p \right)^{\frac{1}{p}} \leq c \sqrt{p \sqrt{n}} \sqrt{\sum_{i} \langle x, u_i \rangle \langle u_i, e_j \rangle^2}$$

By Lemma 2.3 with large probability, $|\langle u_i, e_j \rangle| \leq c_1 \frac{\log n}{\sqrt{n}}$. Hence, with high probability,

$$\|f^j_x\|_p \leq c \sqrt{p \sqrt{n \log n}} |x| \Rightarrow \|f^j_x\|_{\psi_2} \leq c' \sqrt{\log n} |x|$$

Since $\|x\| = \mathbb{E}_{\varepsilon} \max_{j} f^j_x(\varepsilon)$, the well-known estimate for the expectation of a maximum of $\psi_2$ variables can be used (e.g. [LT] page 79, or the remark after lemma 3.4 in this paper):

$$\forall x \in \mathbb{R}^n \|x\| \leq c \sqrt{\log n (c' \sqrt{\log n} |x|)} = c \log n |x|$$

Thus the proposition is proven. \qed

Remark: For every dimension, there exists an orthogonal basis $\{u_i\}_{i=1}^n$ such that $\forall i, j$:

$$|\langle u_i, e_j \rangle| \leq \frac{2}{\sqrt{n}}$$
Such a basis is called in this paper a “Walsh” basis. Indeed, for dimension $n = 2^k$, the regular Walsh basis is satisfactory, while for other dimensions, an appropriate basis may be constructed using sines and cosines (this basis consists of orthogonal vectors resembling the complex valued characters of the group $\mathbb{Z}/n\mathbb{Z}$). Instead of using Lemma 2.3 in the proof of Proposition 2.1, one can replace the random basis with a Walsh basis, obtaining yet a slightly better result, with “log $n$” replaced by “$\sqrt{\log n}$” in the conclusion of Proposition 2.1.

Proof of Lemma 2.3: Since for every $i$ the vector $u_i$ distributes uniformly over the sphere, by the standard concentration inequality on the sphere (e.g. first pages of [MS]):

$$\text{Prob}\{|\langle u_i, e_j \rangle| > \varepsilon\} \leq \sqrt{\frac{\pi}{2}} e^{-\varepsilon^2 \frac{n}{2}}$$

Select $c_1$ (i.e. $c_1 = 5$) such that for $\varepsilon = c_1 \frac{\sqrt{\log n}}{\sqrt{n}}$, the probability in (1) is less than $\frac{1}{n^2}$. Therefore, the probability that $|\langle u_i, e_j \rangle| < c_1 \frac{\sqrt{\log n}}{\sqrt{n}}$ holds for all $1 \leq i, j \leq n$ is greater than $1 - \frac{1}{n^{10}}$. □

3 Second Step: Logarithmic Decay of the Diameter

In the second step, symmetrizations will be performed with respect to two random orthogonal bases; This section proves that this step reduces the diameter of the body logarithmically: from $c \log n$ to $C \log \log n$, with probability close to 1. Therefore, after the second step (and a total of $4n$ symmetrizations) the diameter is less than $C \log \log n$. This proof extends that of the former section.

Let $K$ be the convex body obtained from the first step of symmetrizations. According to Corollary 2.2, $M^*(K) = 1$, $diam(K) < c \log n$, and $K$ is unconditional with respect to some orthogonal basis (re-denote this basis as $\{e_1, ..., e_n\}$). Once again, by Lemma 3.2 from [K],

$$K \subset c\sqrt{n} \text{conv}\{\pm e_i\}_{i=1}^n = c\sqrt{n}B(l_1^n)$$

Set $t = diam(K) < c \log n$. Clearly, $K \subset \sqrt{n}B(l_1^n) \cap tB(l_2^n)$. As in the first step, rather than working directly with the body $K$, symmetrize $K_i = \sqrt{n}B(l_1^n) \cap tB(l_2^n)$.

**Proposition 3.1** Let $K_t = \sqrt{n}B(l_1^n) \cap tB(l_2^n)$. Assume that $\{u_1, ..., u_n\} \in O(n)$ and $\{v_1, ..., v_n\} \in O(n)$ are chosen uniformly and independently. After symmetrizations with respect to $u_1, ..., u_{n-1}$ and $v_1, ..., v_{n-1}$, a new body $\tilde{K}_t$ is obtained such that:

$$\tilde{K}_t \subset C \log tB(l_2^n)$$

with probability greater than $1 - e^{-c\sqrt{n}}$ of choosing the orthogonal bases.
**Corollary 3.2** For every convex body \( K \subset \mathbb{R}^n \) with \( M^*(K) = 1 \), there exist \( 4n-3 \) symmetrizations which transform \( K \) into \( \tilde{K} \), where \( \text{diam}(\tilde{K}) < c \log \log n \).

Begin by describing the body \( K_t \) through its dual norm. Denote by \( \| \cdot \|'_t \) the norm:
\[
\|x\|'_t = \inf \{ \|x'\|_2 + t\|x''\|_\infty : x = x' + x'' \}
\]
The dual norm of \( K_t \) is exactly \( t\| \cdot \|'_\sqrt{n} \), as can be verified. Put \((a^*_i)_{i=1}^n\) for the non-increasing rearrangement of the absolute values of \((a_i)_{i=1}^n\). The following two lemmas are well-known. The first lemma essentially appears in [BL], but for lack of concise references, attached here are the short elementary proofs.

**Lemma 3.3**

\[
\forall x \in \mathbb{R}^n \quad \|x\|'_k \approx \sqrt{\sum_{i=1}^{k^2} (x^*_i)^2}
\]
(and the equivalence constant is not more than \( \sqrt{2} \)).

**Proof:** For \( i \) where \( |x_i| \geq x^*_k \), set \( x'_i = (x_i - \text{sgn}(x_i)x^*_k) \). For other \( i \)'s, set \( x'_i = 0 \). Let \( x'' = x - x' \). Then:
\[
\|x\|'_k \leq \|x'\|_2 + k\|x''\|_\infty
\]
\[
= \sqrt{\sum_{i=1}^{k^2} (x^*_i - x^*_k)^2 + kx^*_k}
\]
\[
\leq \sqrt{2} \sqrt{\sum_{i=1}^{k^2} [(x^*_i - x^*_k)^2 + (x^*_k)^2]}
\]
\[
\leq \sqrt{2} \sqrt{\sum_{i=1}^{k^2} (x^*_i)^2}
\]
On the other hand, assume \( x = x' + x'' \). Surely \( x^*_i \leq x'^*_i + x''^*_i \), so:
\[
\sqrt{\sum_{i=1}^{k^2} (x^*_i)^2} \leq \sqrt{\sum_{i=1}^{k^2} (x'^*_i)^2} + \sqrt{\sum_{i=1}^{k^2} (x''^*_i)^2}
\]
\[
\leq \|x'\|_2 + k\|x''\|_\infty
\]

**Lemma 3.4** Let \((X_i)_{i=1}^n\) be \( \psi_1 \) random variables (i.e. random variables that satisfy: \( \mathbb{E} e^{X_i} \leq C \)), and let \((X^*_i)_{i=1}^n\) be the non-increasing rearrangement of the \( X_i \)'s. Then:
\[
\mathbb{E} \sqrt{\frac{1}{k} \sum_{i=1}^{k} (X^*_i)^2} \leq c_2 \log \frac{2n}{k}
\]
Proof: Since the $X_i$’s are $\psi_1$ variables,
\[ E \frac{1}{k} \sum_{i=1}^{k} e^{X_i} \leq E \frac{1}{k} \sum_{i=1}^{n} e^{X_i} \leq C_n \frac{n}{k} \]  \hfill (2)

Let $(a_i)_{i=1}^{k}$ be any real numbers such that $\forall i \ a_i \geq 1$. Since the function $e^x$ is convex on $[1, \infty)$, by Jensen inequality:
\[ e^{\sqrt{\frac{1}{k} \sum_{i=1}^{k} (a_i)^2}} \leq \frac{1}{k} \sum_{i=1}^{k} e^{a_i} \]  \hfill (3)

Replace $X_i$ by $\max(X_i, 1)$, and combine inequalities (2) and (3):
\[ E e^{\sqrt{\frac{1}{k} \sum_{i=1}^{k} (X_i^*)^2}} \leq \frac{C_n}{k} \]  \hfill (4)

Another application of Jensen inequality ($E \log X \leq \log E X$) yields:
\[ E \left[ \frac{1}{k} \sum_{i=1}^{k} (X_i^*)^2 \right] \leq \log \frac{C_n}{k} \]
which concludes the proof. \hfill \qed

Remark: If $X_i$ are $\psi_2$ variables, then it can be simply verified that:
\[ E \left[ \frac{1}{k} \sum_{i=1}^{k} (X_i^*)^2 \right] \leq c \sqrt{\log \frac{2n}{k}} \]

Proof of Proposition 3.4: Let $\|x\| = t\|x\|_{\psi_1}$, the dual norm of $K_t$. Take two random bases $\{u_i\}_{i=1}^{n}$ and $\{v_i\}_{i=1}^{n}$. The symmetrized norm $\|\cdot\|$ is:
\[ \|x\| = E_{\varepsilon, \varepsilon'} \left[ \sum_{j,k} \varepsilon_j \varepsilon_k' \langle x, v_j \rangle \langle v_j, u_k \rangle \langle u_k, e_i \rangle \right] \]
where $\varepsilon, \varepsilon'$ are independent and uniformly distributed in $\{\pm 1\}^n$. For $1 \leq i \leq n$ and $\varepsilon, \varepsilon' \in \{\pm 1\}^n$ define:
\[ \phi^i_x(\varepsilon, \varepsilon') = \sum_{j,k} \varepsilon_j \varepsilon_k' \langle x, v_j \rangle \langle v_j, u_k \rangle \langle u_k, e_i \rangle \]
Then:
\[ \|x\| = tE_{\varepsilon, \varepsilon'} \left[ \|\phi^1_x(\varepsilon, \varepsilon'), \ldots, \phi^n_x(\varepsilon, \varepsilon')\|_{\psi_1} \right] \]
By Lemma 3.3
\[ \|x\| \leq \sqrt{2tE_{\varepsilon, \varepsilon'} \left[ \sum_{i=1}^{\left\lfloor \frac{n}{k} \right\rfloor + 1} \phi^i_x(\varepsilon, \varepsilon')^2 \right]} \]
The following lemma, estimating the $\psi_1$ norm of those variables, will be proved later.
Lemma 3.5

\[ \| \phi^i \|_{\psi_1} < \frac{c_3}{\sqrt{n}} |x| \]

with probability greater than \( 1 - e^{-c \sqrt{n}} \) of choosing the orthogonal bases.

Lemma 3.4 may now be used (for \( k = \left\lfloor \frac{n}{2} \right\rfloor + 1 \)). It shows that:

\[ \|x\| \leq \sqrt{2t} \left( \frac{c_3}{\sqrt{n}} |x| \right) \cdot c_2 \left( \frac{\sqrt{n}}{t} + 1 \right) \log \frac{2n}{t} \leq c \log t |x| \]

with probability greater than \( 1 - ne^{-c \sqrt{n}} \) of choosing the bases. \( \square \)

Before turning to the proof of lemma 3.5, prove another lemma, which is believed to be known to experts:

**Lemma 3.6** Let \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) be two random independent vectors in \( S^{n-1} \). Then with probability greater than \( 1 - e^{-c \sqrt{n}} \),

\[ \sum_i x_i^2 y_i^2 \leq \frac{c}{n} \]

**Proof of Lemma 3.6.** Let \( \{\gamma_i\}_{i=1}^n \) and \( \{\eta_i\}_{i=1}^n \) be independent standard Gaussian variables. Since the measure on the sphere is the radial projection of the standard Gaussian measure in \( \mathbb{R}^n \), then:

\[ \text{Prob}\left\{ \sum_i x_i^2 y_i^2 > t \right\} = \text{Prob}\left\{ \frac{1}{\sum_j \gamma_j^2 \sum_j \eta_j^2} \sum_i \gamma_i^2 \eta_i^2 > t \right\} \]

To prove the lemma, it is sufficient to bound from below \( \sum_j \gamma_j^2 \sum_j \eta_j^2 \) and bound from above \( \sum_i \gamma_i^2 \eta_i^2 \). Begin with the second expression. Note that \( \gamma_i^2 \eta_i^2 \) is a \( \psi_1^2 \) variable:

\[ (\mathbb{E} \gamma_i^{2p} \eta_i^{2p})^{1/2} \leq (c \sqrt{p})^{1/2} = c^{1/2} \]

for \( \alpha = \frac{1}{2} \). Therefore, \( \sum_i \gamma_i^2 \eta_i^2 \) is a sum of independent copies of a \( \psi_1^2 \) random variable. By a deviation inequality for sums of i.i.d \( \psi_{\alpha} \) random variables (see [3]),

\[ \text{Prob}\left\{ \sum_i \gamma_i^2 \eta_i^2 > cn \right\} < \exp(-c' \sqrt{n}) \]

The fact that \( \text{Prob}\{ \sum_j \gamma_j^2 < \frac{n}{2} \} < e^{-cn} \) follows from Large Deviations technique (e.g. Cramér’s Theorem, [5]). To conclude, with probability greater than \( 1 - e^{-c \sqrt{n}} \),

\[ \frac{1}{\sum_j \gamma_j^2 \sum_j \eta_j^2} \sum_i \gamma_i^2 \eta_i^2 < \frac{cn}{n} \cdot \frac{n}{2} = \frac{c'}{n} \]
Proof of Lemma 3.5: Let $\phi^1_x(\varepsilon, \varepsilon') = |\sum_{j,k} \varepsilon_j \varepsilon'_k \langle x, v_j \rangle \langle v_j, u_k \rangle \langle u_k, e_i \rangle|$. This random variable is a particular case of a Rademacher Chaos variable. It is well known (e.g. see [LT]), that a $\psi_1$ estimate holds true for such variables:

$$
\|\phi^1_x\|_{\psi_1} \leq c \|\phi^1_x\|_2 = c \sqrt{\sum_j \langle x, v_j \rangle^2 \sum_k \langle v_j, u_k \rangle^2 \langle u_k, e_i \rangle^2}
$$

It is sufficient to show that the inequality $\sum_k \langle v_j, u_k \rangle^2 \langle u_k, e_i \rangle^2 \leq \frac{n}{\sqrt{n}}$ holds with high probability, since in that case, with the same probability:

$$
\sqrt{\sum_j \langle x, v_j \rangle^2 \sum_k \langle v_j, u_k \rangle^2 \langle u_k, e_i \rangle^2} \geq \frac{\sqrt{c}}{\sqrt{n}} |x|
$$

The fact that $\sum_k \langle v_j, u_k \rangle^2 \langle u_k, e_i \rangle^2 \leq \frac{n}{\sqrt{n}}$ holds with probability greater than $1 - e^{-cn}$ follows directly from Lemma 3.4. Take $U \in O(n)$ such that $U(u_k) = e_k$. $U$ is distributed uniformly over $O(n)$.

$$
\sum_k \langle v_j, u_k \rangle^2 \langle u_k, e_i \rangle^2 = \sum_k \langle Uv_j, e_k \rangle^2 \langle e_k, Ue_i \rangle^2
$$

Since $UV_j$ and $Ue_i$ are independent and distributed uniformly over the sphere - the claim is proven, by Lemma 3.6. $\square$

Remark: Proposition 3.1 may be adapted to suit Walsh-type symmetrizations. If $K_t = \sqrt{n}B(l^n) \bigcap tB(l^n)$ is symmetrized with respect to Walsh vectors $w_1, \ldots, w_{n-1}$, a slightly better conclusion than that in Proposition 3.1 is obtained; In this setting, it is true that:

$$
\tilde{K}_t \subset C \sqrt{\log t} B(l^n)
$$

The differences between the proofs are minor. Lemma 3.3 becomes much easier as it follows immediately from Khinchine inequality, even with a $\psi_2$ estimate rather than $\psi_1$. To take advantage of this improvement, use the remark after Lemma 3.4, to obtain the better conclusion.

Re-iteration of this proposition, where each iteration uses a Walsh basis with respect to the previous, would result in a rapid decay of the body’s diameter. After $\log^* n$ iterations, a body whose $\frac{\text{diam}(K)}{M^*(K)}$ ratio is bounded by a universal constant is obtained. Note that this specific choice of symmetrizations decreases the diameter of all possible convex bodies in $\mathbb{R}^n$, to be a constant times their mean width. Of course, once the $\frac{\text{diam}(K)}{M^*(K)}$ ratio is bounded, $cn$ random independent Minkowski symmetrizations suffice for transforming the body into an approximate Euclidean ball.
4 Third step: Concentration Techniques

Take any convex body $K$ in $\mathbb{R}^n$. According to Corollary 3.2 from the previous steps (which consist of $4n$ symmetrizations) a new body is obtained, with $M^* = 1$ and with diameter less than $c \log \log n$. As before, the third step involves symmetrizing with respect to two random orthogonal bases. A total of $2n$ symmetrizations will make the body very close to Euclidean.

Let $\|\cdot\|$ be the dual norm of the body obtained after the previous steps. Since $M^*(K) = 1$, then $M(\|\cdot\|) \equiv \int_{S^{n-1}} |x| d\sigma(x) = 1$, and $b(\|\cdot\|) \equiv \sup_{x \in S^{n-1}} \|x\| \leq c \log \log n$. Let $\{u_i\}_{i=1}^n, \{v_i\}_{i=1}^n$ be random orthogonal bases and denote for $x \in \mathbb{R}^n$ a set:

$$\mathcal{F}(x) = \{ \sum_{i,j} \varepsilon_i \varepsilon'_j \langle x, v_i \rangle \langle v_i, u_j \rangle u_j : \varepsilon, \varepsilon' \in \{\pm\}^n \}$$

The symmetrized norm $\|\cdot\|$ satisfies $|||x||| = \frac{1}{2n} \sum_{v \in \mathcal{F}(x)} \|v\|$. This section will prove that for the new norm:

$$\forall x \in \mathbb{R}^n \quad \frac{1}{2} |x| \leq |||x||| \leq 2|x|$$

with large probability of choosing $\{u_i\}_{i=1}^n, \{v_i\}_{i=1}^n \in O(n)$. In fact, a somewhat stronger theorem is proved, where instead of $\frac{1}{2}$ and $2$, better estimates are given.

Useful remark: Let $|||x||| = \frac{1}{2n} \sum_{v \in \mathcal{F}(x)} \|v\|$ be the norm obtained after symmetrizing with respect to $\{u_i\}$ and $\{v_i\}$. Take $U \in O(n)$, and let $\|\cdot\|_U$ be the norm obtained after symmetrizing with respect to $\{Uu_i\}$ and $\{Uv_i\}$. Then $|||Ux|||_U = \frac{1}{\sqrt{n}} \sum_{v \in \mathcal{F}(x)} \|Uv\|$. Therefore, due to the rotation invariance of the measure $\mu_n$ in $O(n)$, it is possible to fix an orthonormal system $\{u_i\}$, and prove the following:

**Theorem 4.1** With the above definitions,

$$\forall x \in S^{n-1} \quad (1 - c \frac{(\log \log n)^{\frac{3}{2}}}{\sqrt{\log n}}) \leq |||x|||_U \leq (1 + c \frac{(\log \log n)^{\frac{3}{2}}}{\sqrt{\log n}})$$

with probability greater than $1 - e^{-Cn}$ of choosing $U \in O(n)$, and probability greater than $1 - \frac{1}{n^{10}}$ of choosing $\{v_i\}$.

The proof shall use three lemmas:

**Lemma 4.2** $\forall x, y \in S^{n-1} \quad \frac{1}{\sqrt{n}} \sum_{v \in \mathcal{F}(x)} |\langle v, y \rangle| \leq 2c_1 \frac{\sqrt{\log n}}{\sqrt{n}}$

for any $\{u_i\}$, with probability of choosing $\{v_i\}$ greater than $1 - \frac{1}{n^{10}}$.

**Proof:** According to Lemma 2.3, with probability greater than $1 - \frac{1}{n^{10}}$, for all $i, j$ the inequality $|\langle v_i, u_j \rangle| \leq c_1 \frac{\sqrt{\log n}}{\sqrt{n}}$ holds. Thus:

$$\frac{1}{4n} \sum_{v \in \mathcal{F}(x)} |\langle v, y \rangle| = \mathbb{E}_{\varepsilon, \varepsilon'} \left| \sum_{i,j} \varepsilon_i \varepsilon'_j \langle x, v_i \rangle \langle v_i, u_j \rangle \langle u_j, y \rangle \right|$$

10
\[
\left[ \sum_{i,j} \epsilon_i \epsilon_j \langle x, v_i \rangle \langle v_j, u_j \rangle \right]^2 \leq \sqrt{E_{e,e'} \left[ \sum_{i,j} \epsilon_i \epsilon_j \langle x, v_i \rangle \langle v_j, u_j \rangle \right]^2}
\]

\[
= \sqrt{\sum_{i,j} \langle x, v_i \rangle^2 \langle v_i, u_j \rangle^2 \langle u_j, y \rangle^2} \leq c_1 \frac{\sqrt{\log n}}{\sqrt{n}}
\]

since \(x \) and \(y\) are sphere vectors.

The next lemma is copied from [BLM], where it is proven.

Lemma 4.3 Assume that \(\{w_\alpha\}_{\alpha \in A} \subset S^{n-1}\), and for some \(\delta > 0\),

\[
\sup_{y \in S^{n-1}} \frac{1}{\#A} \sum_{\alpha \in A} |\langle w_\alpha, y \rangle| \leq \delta
\]

Let \(0 < \lambda < 1\) and let \(k \leq n\) be an integer. Then the set \(A\) can be partitioned into families \(\mathcal{F}_\beta = \{\beta_i\}_{i=1}^k \subset A, \, \beta \in B\), so that \(#(\bigcup_{\beta \in B} \mathcal{F}_\beta) > (1 - \lambda) \#A - k\) and so that for every \(\beta \in B\) there is an orthonormal set of vectors \(\{v_\beta_i\}_{i=1}^k\) satisfying

\[
|v_\beta_i - w_\beta_i| \leq \frac{\delta^4 k}{\lambda}
\]

Concentration on the orthogonal group shall be used in the proof of Theorem 4.1, due to [GM] (see [MS], page 29):

Lemma 4.4 Let \(\| \cdot \|\) be a norm on \(\mathbb{R}^n\) such that \(\|x\| \leq b|x| \forall x \in S^{n-1}\). Let \(k \leq n\) be a positive integer, and \(\{x_i\}_{i=1}^k\) be orthonormal vectors. Denote \(M = \int_{S^{n-1}} \|x\| d\sigma_n(x)\). Then:

\[
\mu_n \{U \in O(n) ; \left| \frac{1}{k} \sum_{i=1}^k \|U x_i\| - M \right| \geq \varepsilon \} \leq \exp(-c_4 \frac{\varepsilon^2 nk}{b^2})
\]

Proof of Theorem 4.1: Fix \(x \in S^{n-1}\). Let \(\varepsilon = c_5 \frac{(\log \log n)^2}{\sqrt{\log n}}\), \(\lambda = \frac{\varepsilon}{b}\) and \(k = \log n\). According to Lemma 4.2, the collection of vectors \(\mathcal{F}(x)\) satisfies the requirement of Lemma 4.3 for \(\delta = 2c_1 \frac{\sqrt{\log n}}{\sqrt{n}}\), with large probability of choosing \(\{v_i\}\) (and of course independently of \(U\)). As a result, \(\mathcal{F}(x)\) can be decomposed into disjoint almost orthogonal families \(\{\mathcal{F}_\beta\}_{\beta \in B}\), which cover all but a \(\lambda\) fraction of \(\mathcal{F}(x)\).

From Lemma 4.3, for each family \(\mathcal{F} = \{x_1, ..., x_k\} \subset \mathcal{F}(x)\), there exist orthonormal vectors \(\{t_i\}_{i=1}^k\) such that \(|t_i - x_i| \leq \frac{\delta^4 k}{\lambda}\). Since \(\{t_i\}_{i=1}^k\) are orthonormal, then by Lemma 4.4

\[
\mu_n \{U \in O(n) ; \left| \frac{1}{k} \sum_{i=1}^k \|U t_i\| - 1 \right| \geq \varepsilon \} \leq \exp(-c_4 \frac{n k \varepsilon^2}{b^2})
\]
where \( b = \sup_{x \in S^{n-1}} \|x\| \). Since \( \|Ut_i - Ux_i\| \leq b \frac{\delta 4^k}{\lambda} \), then:

\[
\left| \frac{1}{K} \sum_{i=1}^{k} \|Ux_i\| - 1 \right| \leq \varepsilon + b \frac{\delta 4^k}{\lambda} \tag{4}
\]

with probability (of choosing \( U \in O(n) \)) of at least \( 1 - \exp(-c_4 \frac{nk^2}{\delta}) \).

This holds for a single family \( F \). The number of families is less than \( 4^n \), so inequality (4) holds for all families \( \{F_{\beta}\}_{\beta \in B} \) together, with probability greater than \( 1 - 4^n \exp(-c_4 \frac{nk^2}{\delta}) = 1 - \exp(-c_4 n(\frac{k^2}{\delta} - \log 4)) \).

There still remains a \( \lambda \) fraction of the collection \( F(x) \), not covered by the disjoint families \( \{F_{\beta}\}_{\beta \in B} \). Their contribution to the relevant expression, which is \( \frac{1}{4^n} \sum_{v \in F(x)} \|Uv\| - 1 \), can be bounded by \( \lambda b \). Hence:

\[
\left| \|Ux\|_U - 1 \right| = \frac{1}{4^n} \sum_{v \in F(x)} \|Uv\| - 1 \leq \left( \frac{1}{4^n} \sum_{v \in F(x)} \|Uv\| - 1 \right) + \frac{\lambda b}{4^n} + b \leq (1 - \lambda) (\varepsilon + \frac{\delta 4^k}{\lambda}) + \lambda b
\]

In summary: choose \( \{v_i\} \) by random. With probability of at least \( 1 - \frac{1}{n^{6/5}} \), the following holds: the set of \( U \in O(n) \) for which

\[
\left| \|Ux\|_U - 1 \right| \leq \varepsilon + \frac{\delta 4^k}{\lambda} + \lambda b \tag{5}
\]

has measure of at least \( 1 - \exp(-c_4 n(\frac{k^2}{\delta} - \log 4)) \). From substituting the values of the variables \( k, \varepsilon, \lambda \), it follows that \( \lambda b \leq \varepsilon \), and also \( \frac{b \delta 4^k}{\lambda} < \varepsilon \), for \( n > c_6 \). Therefore, the quantity discussed in (5) is less than \( 3\varepsilon \), for \( n > c_6 \).

The inequality \( \left| \|Ux\|_U - 1 \right| \leq 3\varepsilon \) holds with probability (with respect to \( U \)) of at least \( 1 - \exp(-c_4 n(\frac{k^2}{\delta} - \log 4)) \). With a suitable universal constant \( c_5 \) this probability would be greater than \( 1 - \exp(-10n \log \log n) \).

This analysis considered a fixed \( x \in S^{n-1} \). Now, take an \( \varepsilon \)-net on the sphere denoted by \( N \). There exists such a net with \( \#N \leq (\frac{3}{4})^n \). For each \( x \in N \), \( \left| \|Ux\|_U - 1 \right| \leq 3\varepsilon \) with probability greater than \( 1 - \exp(-10n \log \log n) \).

Since \( (\frac{3}{4})^n \leq \exp(\log \log n) \) for \( n > c_6 \), then \( \left| \|Ux\|_U - 1 \right| \leq 3\varepsilon \) holds for all \( x \in N \), with more than exponentially close to 1 probability.

For a general \( x \in S^{n-1} \), write \( x = \sum_{i=0}^{\infty} \theta_i x_i \), where \( Ux_i \in N \), and \( \theta_0 = 1, 0 \leq \theta_i \leq \varepsilon^i \). Then \( \|x\|_U \leq \sum_{i=0}^{\infty} (1 + 3\varepsilon) \varepsilon^i = \frac{1 + 3\varepsilon}{1 - \varepsilon} \leq 1 + 5\varepsilon \). Finally, \( \|x\|_U \geq \|x_0\|_U - \sum_{i=1}^{\infty} |\theta_i| \cdot \|x_i\|_U \geq 1 - 5\varepsilon \).

Hence, with slightly better than exponentially close to 1 probability, the new norm \( \|\cdot\| \) satisfies

\[
\forall x \in \mathbb{R}^n \quad (1 - \varepsilon) |x| \leq \|x\| \leq (1 + \varepsilon) |x|
\]
where \( \varepsilon < c \frac{\log \log n}{\sqrt{\log n}} \), and the theorem is proven, for \( n > c_6 \).

\[ \square \]

**Remark:** Using a Walsh-type symmetrization in the second step, the theorem can be proven with \( \varepsilon < c \frac{\log \log n}{\sqrt{\log n}} \), an improvement of a mere \( \sqrt{\log \log n} \) factor.

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