Research article

Some unified bounds for exponentially $tgs$-convex functions governed by conformable fractional operators

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Abstract: In the article, we introduce the concept of the exponentially $tgs$-convex function and discover two new conformable fractional integral identities concerning the first-order differentiable convex mappings. By using these identities, we establish several new right-sided Hermite-Hadamard type inequalities for the exponentially $tgs$-convex functions via conformable fractional integrals. Our outcomes for conformable fractional integral operators are also applied to some special means.

Keywords: integral inequality; exponentially $tgs$-convex function; conformable fractional integral operator; Hermite-Hadamard inequality

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1. Introduction

On different time ranges, fractional calculus has a great impact due to a diversity of applications that have contributed to several fields of technical sciences and engineering [1–12]. One of the principal options behind the popularity of the area is that fractional-order differentiations and integrations are more beneficial tools in expressing real-world matters than the integer-order ones. Various studies in the literature, on distinct fractional operators such as the classical Riemann-Liouville, Caputo, Katugamploa, Hadamard, and Marchaud versions have shown versatility in modeling and control applications across various disciplines. However, such forms of fractional derivatives may not be able to explain the dynamic performance accurately, hence, many authors are
found to be sorting out new fractional differentiations and integrations which have a kernel depending upon a function and this makes the range of definition expanded [13, 14]. Furthermore, models based on these fractional operators provide excellent results to be compared with the integer-order differentiations [15–27].

The derivatives in this calculus seemed complicated and lost some of the basic properties that usual derivatives have such as the product rule and the chain rule. However, the semigroup properties of these operators behave well in some cases. Recently, the authors in [28] defined a new well-behaved simple derivative called “conformable fractional derivative” which depends just on the basic limit definition of the derivative. It will define the derivative of higher-order (i.e., order $\delta > 1$) and also define the integral of order $0 < \delta \leq 1$ only. It will also prove the product rule and the mean value theorem and solve some (conformable) differential equations where the fractional exponential function $e^{\varphi_1}$ plays an important role. Inequalities and their utilities assume a crucial role in the literature of pure and applied mathematics [29–37]. The assortment of distinct kinds of classical variants and their modifications were built up by using the classical fractional operators.

Convexity and its applications exist in almost every field of mathematics due to impermanence in several areas of science, technology and optimization theory. By utilizing the idea of convexity, numerous variants have been derived by researchers, for example, Hardy, Opial, Ostrowski, Jensen and the most distinguished one is the Hermite-Hadamard inequality [38–41].

Let $I \subset \mathbb{R}$ be an interval and $Q : I \to \mathbb{R}$ be a convex function. Then the double inequality

$$\left(l_2 - l_1\right)Q\left(\frac{l_1 + l_2}{2}\right) \leq \int_{l_1}^{l_2} Q(z)dz \leq \left(l_2 - l_1\right)\frac{Q(l_1) + Q(l_2)}{2}, \tag{1.1}$$

holds for all $l_1, l_2 \in I$ with $l_1 \neq l_2$. Clearly, if $Q$ is concave on $I$, then one has the reverse of inequality (1.1). By taking into account fractional integral operators, several lower and upper bounds for the mean value of a convex function can be obtained by utilizing of inequality (1.1).

Exponentially convex functions have emerged as a significant new class of convex functions, which have potential applications in technology, data science, and statistics. In [42], Bernstein introduced the concept of exponentially convex function in covariance formation, then the idea of an exponentially convex function is extended by inserting the condition of $r$-convexity [43]. Following this tendency, Jakšetić and Pečarić introduced various kinds of exponentially convex functions in [44] and have contemplated the applications in Euler-Radau expansions and Stolarsky means. Our aim is to utilize the exponential convexity property of the functions as well as the absolute values of their derivatives in order to establish estimates for conformable fractional integral introduced by Abdeljawed [45] and Jarad et al. [46].

Following the above propensity, we present a novel technique for establishing new generalizations of Hermite-Hadamard inequalities that correlate with exponentially $tgs$-convex functions and conformable fractional operator techniques in this paper. The main purpose is that our consequences, which are more consistent and efficient, are accelerated via the fractional calculus technique. In addition, our consequences also taking into account the estimates for Hermite-Hadamard inequalities for exponentially $tgs$-convex functions. We also investigate the applications of the two proposed conformable fractional operator to exponentially $tgs$-convex functions and fractional calculus. The proposed numerical experiments show that our results are superior to some related results.
2. Preliminaries

Before coming to the main results, we provide some significant definitions, theorems and properties of fractional calculus in order to establish a mathematically sound theory that will serve the purpose of the current article.

Awan et al. [47] proposed a new class of functions called exponentially convex functions.

**Definition 2.1.** (See [47]) A positive real-valued function \( Q : K \subset \mathbb{R} \rightarrow (0, \infty) \) is said to be exponentially convex on \( K \) if the inequality

\[
Q(\vartheta l_1 + (1 - \vartheta)l_2) \leq \vartheta \frac{Q(l_1)}{e^{\vartheta l_1}} + (1 - \vartheta)\frac{Q(l_2)}{e^{\vartheta l_2}},
\]

holds for all \( l_1, l_2 \in \mathbb{R}, \alpha \in \mathbb{R} \) and \( \vartheta \in [0, 1] \).

Now, we introduce a novel concept of convex function which is known as the exponentially tgs-convex function.

**Definition 2.2.** A positive real-valued function \( Q : K \subset \mathbb{R} \rightarrow (0, \infty) \) is said to be exponentially tgs-convex on \( K \) if the inequality

\[
Q(\vartheta l_1 + (1 - \vartheta)l_2) \leq \vartheta(1 - \vartheta)\left[\frac{Q(l_1)}{e^{\vartheta l_1}} + \frac{Q(l_2)}{e^{\vartheta l_2}}\right],
\]

holds for all \( l_1, l_2 \in \mathbb{R}, \alpha \in \mathbb{R} \) and \( \vartheta \in [0, 1] \).

The conformable fractional integral operator was introduced by Abdeljawad [45].

**Definition 2.3.** (See [45]) Let \( \rho \in (n, n + 1] \) and \( \delta = \rho - n \). Then the left and right-sided conformable fractional integrals of order \( \rho > 0 \) is defined by

\[
\mathcal{J}_{l_1}^{\rho} Q(z) = \frac{1}{n!} \int_{l_1}^{z} (z - \vartheta)^{\rho-1} Q(\vartheta) d\vartheta
\]

and

\[
\mathcal{J}_{l_2}^{\rho} Q(z) = \frac{1}{n!} \int_{z}^{l_2} (\vartheta - z)^{\rho-1} Q(\vartheta) d\vartheta.
\]

Next, we demonstrate the following fractional integral operator introduced by Jarad et al. [46].

**Definition 2.4.** (See [46]) Let \( \delta \in \mathbb{C} \) and \( \Re(\delta) > 0 \). Then the left and right-sided fractional conformable integral operators of order \( \rho > 0 \) are stated as:

\[
\mathcal{J}_{l_1}^{\rho, \delta} Q(z) = \frac{1}{\Gamma(\delta)} \int_{l_1}^{z} \left(\frac{(z - l_1)^{\rho} - (\vartheta - l_1)^{\rho}}{\rho}\right)^{\delta-1} \frac{Q(\vartheta)}{(\vartheta - l_1)^{1-\rho}} d\vartheta
\]

and

\[
\mathcal{J}_{l_2}^{\rho, \delta} Q(z) = \frac{1}{\Gamma(\delta)} \int_{z}^{l_2} \left(\frac{(l_2 - z)^{\rho} - (l_2 - \vartheta)^{\rho}}{\rho}\right)^{\delta-1} \frac{Q(\vartheta)}{(l_2 - \vartheta)^{1-\rho}} d\vartheta.
\]
Recalling some special functions which are known as beta and incomplete beta function.

\[ B(l_1, l_2) = \int_0^1 \theta^{l_1-1}(1 - \theta)^{l_2-1} d\theta, \]

\[ B_v(l_1, l_2) = \int_0^v \theta^{l_1-1}(1 - \theta)^{l_2-1} d\theta, \quad v \in [0, 1]. \]

Further, the following relationship holds between classical Beta and incomplete Beta functions:

\[ B(l_1, l_2) = B_v(l_1, l_2) + B_{1-v}(l_1, l_2), \]

\[ B_v(l_1 + 1, l_2) = \frac{l_1 B_v(l_1, l_2) - (\frac{1}{2})^{l_1+l_2}}{l_1 + l_2} \]

and

\[ B_v(l_1, l_2 + 1) = \frac{l_2 B_v(l_1, l_2) - (\frac{1}{2})^{l_1+l_2}}{l_1 + l_2}. \]

3. Hermite-Hadamard type inequality for exponentially tgs-convex functions via conformable fractional integrals

Throughout the article, let \( I = [l_1, l_2] \) be an interval in real line \( \mathbb{R} \). In this section, we shall demonstrate some integral versions of exponentially tgs-convex functions via conformable fractional integrals.

**Theorem 3.1.** For \( \rho \in (n, n+1) \) with \( \rho > 0 \) and let \( Q : I \subset \mathbb{R} \rightarrow \mathbb{R} \) be an exponentially tgs-convex function such that \( Q \in L_1([l_1, l_2]) \), then the following inequalities hold:

\[ \frac{4\Gamma(\rho - n)}{\Gamma(\rho + 1)} Q \left( \frac{l_1 + l_2}{2} \right) \]

\[ \leq \frac{1}{(l_2 - l_1)^\rho} \left[ \mathcal{J}_{l_1}^\rho Q(l_2) e^{al_2} + \mathcal{J}_{l_2}^\rho Q(l_1) e^{al_1} \right] \]

\[ \leq \frac{2(n + 1)!\Gamma(\rho - n + 1)}{\Gamma(\rho + 3)!} \left( \frac{Q(l_1)}{e^{al_1}} + \frac{Q(l_2)}{e^{al_2}} \right). \quad (3.1) \]

**Proof.** By using exponentially tgs-convexity of \( Q \), we have

\[ Q \left( \frac{x + y}{2} \right) \leq \frac{1}{4} \left( \frac{Q(x)}{e^{ax}} + \frac{Q(y)}{e^{ay}} \right). \quad (3.2) \]

Let \( x = \theta l_1 + (1 - \theta)l_2 \) and \( y = (1 - \theta)l_1 + \theta l_2 \), we get

\[ 4Q \left( \frac{l_1 + l_2}{2} \right) \leq \frac{Q(\theta l_1 + (1 - \theta)l_2)}{e^{al_{\theta l_1 + (1 - \theta)l_2}}} + \frac{Q(\theta l_2 + (1 - \theta)l_1)}{e^{al_{\theta l_2 + (1 - \theta)l_1}}}. \quad (3.3) \]
If we multiply (3.3) by \( \frac{1}{n!} \theta^n (1 - \theta)^{n-1} \) with \( \theta \in (0, 1), \rho > 0 \) and then integrating the resulting estimate with respect to \( \theta \) over \([0, 1]\), we find

\[
\frac{4}{n!} Q\left( \frac{l_1 + l_2}{2} \right) \int_0^1 \theta^n (1 - \theta)^{n-1} d\theta 
\]

\[
\leq \frac{1}{n!} \int_0^1 \theta^n (1 - \theta)^{n-1} \frac{Q(\theta l_1 + (1 - \theta) l_2) \rho l_1}{\rho (1 - \theta) l_1 + \theta l_2} d\theta 
\]

\[
+ \frac{1}{n!} \int_0^1 \theta^n (1 - \theta)^{n-1} \frac{Q(\theta l_2 + (1 - \theta) l_1) \rho l_2}{\rho (1 - \theta) l_1 + \theta l_2} d\theta 
\]

\[
= I_1 + I_2 \quad \tag{3.4}
\]

By setting \( u = \theta l_1 + (1 - \theta) l_2 \), we have

\[
I_1 = \frac{1}{n!} \int_0^1 \theta^n (1 - \theta)^{n-1} \frac{Q(\theta l_1 + (1 - \theta) l_2) \rho l_1}{\rho (1 - \theta) l_1 + \theta l_2} d\theta 
\]

\[
= \frac{1}{n!(l_2 - l_1)^\rho} \int_{l_1}^{l_2} (l_2 - 1)^\rho (u - l_1)^\rho - 1 Q(u) \frac{1}{e^{\rho u}} du 
\]

\[
= \frac{1}{(l_2 - l_1)^\rho} \int_{l_1}^{l_2} Q(l_2) e^{\rho l_1}. \quad \tag{3.5}
\]

Analogously, by setting \( v = \theta l_2 + (1 - \theta) l_1 \), we have

\[
I_2 = \frac{1}{n!} \int_0^1 \theta^n (1 - \theta)^{n-1} \frac{Q(\theta l_2 + (1 - \theta) l_1) \rho l_2}{\rho (1 - \theta) l_1 + \theta l_2} d\theta 
\]

\[
= \frac{1}{n!(l_2 - l_1)^\rho} \int_{l_1}^{l_2} (v - l_1)^\rho (l_2 - v)^\rho - 1 Q(v) \frac{1}{e^{\rho v}} dv 
\]

\[
= \frac{1}{(l_2 - l_1)^\rho} \int_{l_1}^{l_2} Q(l_1) e^{\rho l_2}. \quad \tag{3.6}
\]

Thus by using (3.5) and (3.6) in (3.4), we get the first inequality of (3.1).

Consider

\[
Q(\theta l_1 + (1 - \theta) l_2) \leq \theta (1 - \theta) \left( \frac{Q(l_1)}{e^{\rho l_1}} + \frac{Q(l_2)}{e^{\rho l_2}} \right)
\]

and

\[
Q(\theta l_2 + (1 - \theta) l_1) \leq \theta (1 - \theta) \left( \frac{Q(l_1)}{e^{\rho l_1}} + \frac{Q(l_2)}{e^{\rho l_2}} \right).
\]

By adding

\[
Q(\theta l_1 + (1 - \theta) l_2) + Q(\theta l_2 + (1 - \theta) l_1) \leq 2 \theta (1 - \theta) \left( \frac{Q(l_1)}{e^{\rho l_1}} + \frac{Q(l_2)}{e^{\rho l_2}} \right). \quad \tag{3.7}
\]

If we multiply (3.7) by \( \frac{1}{n!} \theta^n (1 - \theta)^{n-1} \) with \( \theta \in (0, 1), \rho > 0 \) and then integrating the resulting inequality with respect to \( \theta \) over \([0, 1]\), we get

\[
\frac{1}{(l_2 - l_1)^\rho} \left[ \int_{l_1}^{l_2} Q(l_2) e^{\rho l_1} + \int_{l_1}^{l_2} Q(l_1) e^{\rho l_2} \right]
\]
≤ \frac{2(n+1)\Gamma(\rho-n+1)}{\Gamma(\rho+3)} \left( \frac{Q(l_1)}{e^{\alpha l_1}} + \frac{Q(l_2)}{e^{\alpha l_2}} \right), \quad (3.8)

which is the required result. □

Some special cases of above theorem are stated as follows:

**Corollary 3.1.** Choosing \( \alpha = 0 \), then Theorem 3.1 reduces to a new result

\[
\frac{4\Gamma(\rho-n)}{\Gamma(\rho+1)} \left( \frac{Q(l_1+l_2)}{2} \right)
\]

\[
\leq \frac{1}{(l_2-l_1)^\rho} \left[ J_{l_1}^\rho Q(l_2) + J_{l_2}^\rho Q(l_1) \right]
\]

\[
\leq \frac{2(n+1)\Gamma(\rho-n+1)}{\Gamma(\rho+3)} (Q(l_1) + Q(l_2)).
\]

**Remark 3.1.** Choosing \( \rho = n+1 \) and \( \alpha = 0 \), then Theorem 3.1 reduces to Theorem 3.1 in [19].

4. Hermite-Hadamard type inequality for differentiable exponentially tgs-convex functions via conformable fractional integrals

Our next result is the following lemma which plays a dominating role in proving our coming results.

**Lemma 4.1.** For \( \rho \in (n, n+1) \) with \( \rho > 0 \) and let \( Q : I \subset \mathbb{R} \to \mathbb{R} \) be differentiable function on \( \mathcal{I}^\circ \) (interior of \( \mathcal{I} \)) with \( l_1 < l_2 \) such that \( Q' \in L_1([l_1, l_2]) \), then the following inequality holds:

\[
\mathbb{B}(n+1, \rho-n) \left( \frac{Q(l_1) + Q(l_2)}{2} \right) - \frac{n!}{2(l_2-l_1)^\rho} \left[ J_{l_1}^\rho Q(l_2) + J_{l_2}^\rho Q(l_1) \right]
\]

\[
= \int_0^1 (\mathbb{B}_{1-u}(n+1, \rho-n) - \mathbb{B}_u(n+1, \rho-n)) Q'(\vartheta l_1 + (1-\vartheta)l_2) d\vartheta. \quad (4.1)
\]

**Proof.** It suffices that

\[
\int_0^1 (\mathbb{B}_{1-u}(n+1, \rho-n) - \mathbb{B}_u(n+1, \rho-n)) Q'(\vartheta l_1 + (1-\vartheta)l_2) d\vartheta
\]

\[
= \int_0^1 \mathbb{B}_{1-u}(n+1, \rho-n) Q'(\vartheta l_1 + (1-\vartheta)l_2) d\vartheta
\]

\[
- \int_0^1 \mathbb{B}_u(n+1, \rho-n) Q'(\vartheta l_1 + (1-\vartheta)l_2) d\vartheta
\]

\[
= S_1 - S_2 \quad (4.2)
\]

Then by integration by parts, we have

\[
S_1 = \int_0^1 \mathbb{B}_{1-u}(n+1, \rho-n) Q'(\vartheta l_1 + (1-\vartheta)l_2) d\vartheta
\]
Analogously, the following inequality holds:

$$\text{Theorem 4.2.}$$

For $1 < \Upsilon < 2$, let $Q = Q(\partial l_1 + (1 - \theta)l_2)\in (Q_1(l_1), Q_2(l_1))\subset \mathbb{R}^n$ be a differentiable function on $I^\circ$. Let $Q' = Q' : I^\circ \subset \mathbb{R}^n \to \mathbb{R}$ be a differentiable function on $I^\circ$ with $l_1 < l_2$ such that $Q' \in L_1([l_1, l_2])$. If $|Q'|r$, with $r \geq 1$, is an exponentially tgs-convex function, then the following inequality holds:

$$\|T_Q(\rho; B; n; l_1, l_2)\|_{L^1(I^\circ)} \leq \frac{l_2 - l_1}{2} \left( \mathbb{E}(n + 1, \rho - n + 1) - \mathbb{E}(n + 1, \rho - n) + \mathbb{E}(n + 2, \rho - n) \right)^{1/2} \times \left( \frac{e^{\text{ar}l_1|Q'(l_1)|} + e^{\text{ar}l_2|Q'(l_2)|}}{6e^{\text{ar}l_1}e^{\text{ar}l_2}} \right)^{1/2}) \cdot (4.5)$$

For the sake of simplicity, we use the following notation:

$$T_Q(\rho; B; n; l_1, l_2) = \mathbb{E}(n + 1, \rho - n) \left( \frac{Q(l_1) + Q(l_2)}{2} \right) - \frac{n!}{2(l_2 - l_1)^{r+1}} \left[ J_Q^Q(l_2) + J_Q^Q(l_1) \right].$$

Theorem 4.2. For $\rho \in (n, n+1)$ with $\rho > 0$ and let $Q : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $I^\circ$ with $l_1 < l_2$ such that $Q' \in L_1([l_1, l_2])$. If $|Q'|r$, with $r \geq 1$, is an exponentially tgs-convex function, then the following inequality holds:

$$|T_Q(\rho; B; n; l_1, l_2)| \leq \frac{l_2 - l_1}{2} \left( \mathbb{E}(n + 1, \rho - n + 1) - \mathbb{E}(n + 1, \rho - n) + \mathbb{E}(n + 2, \rho - n) \right)^{1/2} \times \left( \frac{e^{\text{ar}l_1|Q'(l_1)|} + e^{\text{ar}l_2|Q'(l_2)|}}{6e^{\text{ar}l_1}e^{\text{ar}l_2}} \right)^{1/2}) \cdot (4.5)$$
Proof. Utilizing exponentially tgs-convex function of \(|Q'|\), Lemma 4.1 and Hölder's inequality, one obtains

\[
| \gamma_{Q}(\rho; B; n; l_1, l_2)|
\]

\[
= \left| \frac{l_2 - l_1}{2} \int_{0}^{1} (B_{1-u}(n + 1, \rho - n) - B_u(n + 1, \rho - n))Q'(\partial l_1 + (1 - \partial)l_2)d\partial \right|
\]

\[
\leq \frac{l_2 - l_1}{2} \left( \int_{0}^{1} (B_{1-u}(n + 1, \rho - n) - B_u(n + 1, \rho - n))d\partial \right)^{1 - \frac{1}{r}}
\]

\[
\times \left( \int_{0}^{1} |Q'(\partial l_1 + (1 - \partial)l_2)|^r d\partial \right)^{\frac{1}{r}}
\]

\[
\leq \frac{l_2 - l_1}{2} \left( |B(n + 1, \rho - n + 1) - B(n + 1, \rho - n) + B(n + 2, \rho - n)|^{1 - \frac{1}{r}}
\]

\[
\times \left( \frac{e^{\alpha l_1} |Q'(l_1)|^r + e^{\alpha l_2} |Q'(l_2)|^r}{6e^{\alpha l_1} e^{\alpha l_2}} \right)^{\frac{1}{r}},
\]

(4.6)

which is the required result. \(\Box\)

**Theorem 4.3.** For \(\rho \in (n, n + 1]\) with \(\rho > 0\) and let \(Q : I \subset \mathbb{R} \to \mathbb{R}\) be a differentiable function on \(I^n\) with \(l_1 < l_2\) such that \(Q' \in L_1([l_1, l_2])\). If \(|Q'|^r\), with \(r > 1\) such that \(\frac{1}{r} + \frac{1}{s} = 1\), is exponentially tgs-convex function, then the following inequality holds:

\[
| \gamma_{Q}(\rho; B; n; l_1, l_2)| \leq \frac{l_2 - l_1}{2} \left( 2 \int_{0}^{1} \left( \int_{u}^{1-u} v^r(1 - v)^{r-n-1} dv \right)^{\frac{1}{r}} \right)^{\frac{1}{r}}
\]

\[
\times \left( \frac{e^{\alpha l_1} |Q'(l_1)|^r + e^{\alpha l_2} |Q'(l_2)|^r}{6e^{\alpha l_1} e^{\alpha l_2}} \right)^{\frac{1}{r}}.
\]

(4.7)

**Proof.** Utilizing exponentially tgs-convex function of \(|Q'|^r\) and well-known Hölder inequality, one obtains

\[
| \gamma_{Q}(\rho; B; n; l_1, l_2)|
\]

\[
= \left| \frac{l_2 - l_1}{2} \int_{0}^{1} (B_{1-u}(n + 1, \rho - n) - B_u(n + 1, \rho - n))Q'(\partial l_1 + (1 - \partial)l_2)d\partial \right|
\]

\[
\leq \frac{l_2 - l_1}{2} \left( \int_{0}^{1} |B_{1-u}(n + 1, \rho - n) - B_u(n + 1, \rho - n)|^{1 - \frac{1}{r}}d\partial \right)^{\frac{1}{r}}
\]

\[
\times \left( \int_{0}^{1} |Q'(\partial l_1 + (1 - \partial)l_2)|^r d\partial \right)^{\frac{1}{r}}
\]

\[
\leq \frac{l_2 - l_1}{2} \left( \int_{0}^{1} (B_{1-u}(n + 1, \rho - n) - B_u(n + 1, \rho - n))^r du \right)^{\frac{1}{r}}
\]
Proof. Taking into account (3.3) and conducting product of (3.3) by \( \frac{u(1 - \vartheta)}{e^{\alpha r_1}} \) within the generalized conformable integral operator.

5. Hermite-Hadamard inequality within the generalized conformable integral operator

This section is devoted to proving some new generalizations for exponentially tgs-convex functions within the generalized conformable integral operator.

**Theorem 5.1.** For \( \rho > 0 \) and let \( Q : [l_1, l_2] \subset \mathbb{R} \to \mathbb{R} \) be an exponentially tgs-convex function such that \( Q \in L_{1}[l_1, l_2] \), then the following inequality holds:

\[
\frac{4}{\delta \rho^\delta} Q\left(\frac{l_1 + l_2}{2}\right) \leq \frac{\Gamma(\delta)}{(l_2 - l_1)^{2\delta}} \left[ J_{l_1}^{\rho, \delta} Q(l_2) + J_{l_2}^{\rho, \delta} Q(l_1) \right] + \frac{1}{\rho} \left[ B\left(\frac{\rho + 1}{\rho}, \delta\right) + B\left(\frac{\rho + 2}{\rho}, \delta\right) \right] \left( \frac{Q(l_1)}{e^{\alpha r_1}} + \frac{Q(l_2)}{e^{\alpha r_2}} \right)
\]

(5.1)

**Proof.** Taking into account (3.3) and conducting product of (3.3) by \( \frac{u(1 - \vartheta)}{e^{\alpha r_1}} \) with \( \vartheta \in (0, 1), \rho > 0 \) and then integrating the resulting estimate with respect to \( \vartheta \) over \([0, 1]\), we find

\[
4Q\left(\frac{l_1 + l_2}{2}\right) \int_0^1 \left( \frac{1 - \vartheta^\rho}{\rho} \right)^{\delta - 1} \vartheta^{\rho - 1} d\vartheta
\]

\[
\leq \int_0^1 \left( \frac{1 - \vartheta^\rho}{\rho} \right)^{\delta - 1} \vartheta^{\rho - 1} Q(\vartheta l_1 + (1 - \vartheta)l_2) \frac{e^{\alpha \vartheta l_1 + (1 - \vartheta) l_2}}{e^{\alpha \Omega l_1 + (1 - \vartheta) l_2}} \ d\vartheta
\]

\[
+ \int_0^1 \left( \frac{1 - \vartheta^\rho}{\rho} \right)^{\delta - 1} \vartheta^{\rho - 1} Q(\vartheta l_2 + (1 - \vartheta) l_1) \frac{e^{\alpha \vartheta l_2 + (1 - \vartheta) l_1}}{e^{\alpha \Omega l_2 + (1 - \vartheta) l_1}} \ d\vartheta
\]

\[
= R_1 + R_2.
\]

(5.2)

By making change of variable \( u = \vartheta l_1 + (1 - \vartheta) l_2 \), we have

\[
R_1 = \int_0^1 \left( \frac{1 - \vartheta^\rho}{\rho} \right)^{\delta - 1} \vartheta^{\rho - 1} Q(\vartheta l_1 + (1 - \vartheta) l_2) \frac{e^{\alpha \vartheta l_1 + (1 - \vartheta) l_2}}{e^{\alpha u l_1 + (1 - \vartheta) l_2}} \ d\vartheta
\]

\[
= \int_{l_1}^{l_2} \left( \frac{1 - (\frac{u-l_1}{l_2-l_1})^\rho}{\rho} \right)^{\delta - 1} \left( \frac{u-l_2}{l_1-l_2} \right)^{\rho - 1} Q(u) \frac{du}{l_1-l_2}
\]

\[
R_2 = \int_0^1 \left( \frac{1 - \vartheta^\rho}{\rho} \right)^{\delta - 1} \vartheta^{\rho - 1} Q(\vartheta l_2 + (1 - \vartheta) l_1) \frac{e^{\alpha \vartheta l_2 + (1 - \vartheta) l_1}}{e^{\alpha u l_2 + (1 - \vartheta) l_1}} \ d\vartheta
\]

\[
= \int_{l_1}^{l_2} \left( \frac{1 - (\frac{u-l_2}{l_2-l_1})^\rho}{\rho} \right)^{\delta - 1} \left( \frac{u-l_1}{l_2-l_1} \right)^{\rho - 1} Q(u) \frac{du}{l_2-l_1}
\]
that the desired inequality is the right hand side of (5.1).

Lemma 5.2. For $Q : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $(l_1, l_2)$ with $l_1 < l_2$ such that $Q' \in L_1[l_1, l_2]$, then the following identity holds:

$$\left( \frac{Q(l_1) + Q(l_2)}{2} \right) - \frac{\rho^\delta \Gamma(\delta + 1)}{2(l_2 - l_1)^\delta} \left[ \mathcal{I}_{l_1}^{\rho,\delta} Q(l_2) + \mathcal{I}_{l_2}^{\rho,\delta} Q(l_1) \right] = \frac{(l_2 - l_1)^\delta}{2} \int_0^1 \left[ \left( \frac{1 - \vartheta^\rho}{\rho} \right)^\delta - \left( \frac{1 - (1 - \vartheta)^\rho}{\rho} \right)^\delta \right] Q(\vartheta l_1 + (1 - \vartheta)l_2) d\vartheta. \tag{5.7}$$

Our main results depend on the following identity.

$$\lim_{\rho \to 0} \Gamma(\delta) \mathcal{I}_{l_1}^{\rho,\delta} Q(l_2) \mathcal{I}_{l_2}^{\rho,\delta} Q(l_1) \mathcal{I}_{l_1}^{\rho,\delta} Q(l_1) = \mathcal{I}_{l_1}^{\rho,\delta} Q(l_2) \mathcal{I}_{l_2}^{\rho,\delta} Q(l_1) \mathcal{I}_{l_1}^{\rho,\delta} Q(l_1). \tag{5.8}$$

Thus by using (5.2) and (5.3) in (5.4), we get the first inequality of (5.1).

Consider

$$Q(\vartheta l_1 + (1 - \vartheta)l_2) \leq \vartheta(1 - \vartheta) \left( \frac{Q(l_1)}{e^{\alpha l_1}} + \frac{Q(l_2)}{e^{\alpha l_2}} \right)$$

and

$$Q(\vartheta l_2 + (1 - \vartheta)l_1) \leq \vartheta(1 - \vartheta) \left( \frac{Q(l_1)}{e^{\alpha l_1}} + \frac{Q(l_2)}{e^{\alpha l_2}} \right).$$

By adding

$$Q(\vartheta l_1 + (1 - \vartheta)l_2) + Q(\vartheta l_2 + (1 - \vartheta)l_1) \leq 2\vartheta(1 - \vartheta) \left( \frac{Q(l_1)}{e^{\alpha l_1}} + \frac{Q(l_2)}{e^{\alpha l_2}} \right). \tag{5.5}$$

If we multiply (5.5) by $(\frac{1 - \vartheta^\rho}{\rho})^{\delta - 1} \rho^\rho$ with $\vartheta \in (0, 1), \rho > 0$ and then integrating the resulting estimate with respect to $\vartheta$ over $[0, 1]$, we get

$$\int_0^1 \left( \frac{1 - \vartheta^\rho}{\rho} \right)^{\delta - 1} \vartheta(1 - \vartheta) \left( \frac{Q(l_1)}{e^{\alpha l_1}} + \frac{Q(l_2)}{e^{\alpha l_2}} \right) d\vartheta \leq 2\vartheta(1 - \vartheta) \left( \frac{Q(l_1)}{e^{\alpha l_1}} + \frac{Q(l_2)}{e^{\alpha l_2}} \right). \tag{5.6}$$

the desired inequality is the right hand side of (5.1). \qed
Proof. It suffices that
\[
\int_0^1 \left( \frac{1 - \vartheta^\rho}{\rho} \right)^\delta \left( 1 - \frac{(1 - \vartheta)^\rho}{\rho} \right)^\delta Q'(\vartheta l_1 + (1 - \vartheta)l_2) d\vartheta \\
= \int_0^1 \left( \frac{1 - \vartheta^\rho}{\rho} \right)^\delta Q'(\vartheta l_1 + (1 - \vartheta)l_2) d\vartheta - \left( \frac{1 - (1 - \vartheta)^\rho}{\rho} \right)^\delta Q'(\vartheta l_1 + (1 - \vartheta)l_2) d\vartheta \\
= M_1 - M_2.
\]
(5.8)

Using integration by parts and making change of variable technique, we have
\[
M_1 = \int_0^1 \left( \frac{1 - \vartheta^\rho}{\rho} \right)^\delta Q'(\vartheta l_1 + (1 - \vartheta)l_2) d\vartheta \\
= \frac{1}{l_1 - l_2} \left( \frac{1 - \vartheta^\rho}{\rho} \right)^\delta Q(\vartheta l_1 + (1 - \vartheta)l_2) d\vartheta|_0^1 \\
+ \frac{\delta}{l_1 - l_2} \int_0^1 \left( \frac{1 - \vartheta^\rho}{\rho} \right)^{\delta-1} \vartheta^\rho Q'(\vartheta l_1 + (1 - \vartheta)l_2) d\vartheta \\
= \frac{Q(l_2)}{(l_2 - l_1)\rho^\delta} - \frac{\delta}{l_2 - l_1} \int_0^1 \left( \frac{1 - \vartheta^\rho}{\rho} \right)^{\delta-1} \vartheta^\rho Q'(\vartheta l_1 + (1 - \vartheta)l_2) d\vartheta \\
= \frac{Q(l_2)}{(l_2 - l_1)\rho^\delta} - \frac{\delta \Gamma(\delta)}{(l_2 - l_1)\rho^{\delta+1} J_{l_2}^{\rho,\delta} Q(l_1)}.
\]

Analogously
\[
M_2 = \int_0^1 \left( \frac{1}{\rho} - \frac{(1 - \vartheta)^\rho}{\rho} \right)^\delta Q'(\vartheta l_1 + (1 - \vartheta)l_2) d\vartheta \\
= \frac{1}{l_1 - l_2} \left( \frac{1}{\rho} - \frac{(1 - \vartheta)^\rho}{\rho} \right)^\delta Q(\vartheta l_1 + (1 - \vartheta)l_2) d\vartheta|_0^1 \\
- \frac{1}{l_1 - l_2} \int_0^1 Q \left( \frac{1 - (1 - \vartheta)^\rho}{\rho} \right)^{\delta-1} (1 - \vartheta)^\rho Q'(\vartheta l_1 + (1 - \vartheta)l_2) d\vartheta \\
= \frac{-Q(l_1)}{(l_2 - l_1)\rho^\delta} + \frac{\delta}{l_2 - l_1} \int_0^1 \left( \frac{1}{\rho} - \frac{(1 - \vartheta)^\rho}{\rho} \right)^{\delta-1} (1 - \vartheta)^\rho Q'(\vartheta l_1 + (1 - \vartheta)l_2) d\vartheta \\
= \frac{-Q(l_1)}{(l_2 - l_1)\rho^\delta} + \frac{\delta \Gamma(\delta)}{(l_2 - l_1)\rho^{\delta+1} J_{l_1}^{\rho,\delta} Q(l_2)}.
\]
(5.9)

By substituting values of \(M_1\) and \(M_2\) in (5.8) and then conducting product on both sides by \(\frac{(l_2 - l_1)\rho^\delta}{2}\), we get the desired result. \(\square\)

**Theorem 5.3.** For \(\rho > 0\) and let \(Q : I \subset \mathbb{R} \rightarrow \mathbb{R}\) be a differentiable function on \(I^r\) with \(l_1 < l_2\) such that \(Q' \in L_1([l_1, l_2])\). If \(|Q'|\), with \(r \geq 1\), is an exponentially tgs-convex function, then the following inequality holds
\[
\left( Q(l_1) + Q(l_2) \right) - \frac{\rho^{\delta} \Gamma(\delta + 1)}{2(l_2 - l_1)\rho^\delta} \left[ J_{l_1}^{\rho,\delta} Q(l_2) + J_{l_2}^{\rho,\delta} Q(l_1) \right] \\
\leq \frac{(l_2 - l_1)\rho^\delta}{2} \left( \frac{1}{\rho^{\delta+1}} \zeta(\rho, \delta + 1) + \frac{1}{\rho^{\delta+2}} \zeta(\rho, \delta + 1) \right)^{1/2} \left( \frac{Q(l_1)}{Q(l_2)} \right)^{1/2}.
\]
(5.10)

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Proof. Using exponentially tgs-convexity of $|Q|'$, Lemma 5.2, and the well-known Hölder inequality, we have

$$
\left| \frac{(Q(l_1) + Q(l_2))}{2} - \frac{\rho \Gamma(\delta + 1)}{2(l_2 - l_1)\rho^\delta} \left[ J_{l_1}^{\rho,\delta} Q(l_2) + J_{l_1}^{\rho,\delta} Q(l_1) \right] \right|
\leq \frac{(l_2 - l_1)\rho^\delta}{2} \left( \int_0^1 \left[ \frac{1 - \theta^\rho}{\rho} \right]^{\frac{1}{\rho}} d\theta \right)^{\frac{1}{\rho}} \left( \int_0^1 \left[ \frac{1 -(1 - \theta)^\rho}{\rho} \right]^{\frac{1}{\rho}} d\theta \right)^{\frac{1}{\rho}}
\leq \frac{(l_2 - l_1)\rho^\delta}{2} \left( \int_0^1 \frac{1 - \theta^\rho}{\rho} d\theta - \int_0^1 \frac{1 -(1 - \theta)^\rho}{\rho} d\theta \right)^{\frac{1}{\rho}}
\times \left( \int_0^1 \theta(1 - \theta)\left( \frac{1}{\rho^{\delta+1}} + \frac{1}{\rho^{\delta+2}} \right) d\theta \right)^{\frac{1}{\rho}}
= \frac{(l_2 - l_1)\rho^\delta}{2} \left( \frac{1}{\rho^{\delta+1}} + \frac{1}{\rho^{\delta+2}} \right) \left( \frac{1}{\rho^{\delta+1}} + \frac{1}{\rho^{\delta+2}} \right)^{\frac{1}{\rho}}
\leq \left( \frac{1}{\rho^{\delta+1}} + \frac{1}{\rho^{\delta+2}} \right)^{\frac{1}{\rho}}
= \left( \frac{1}{\rho^{\delta+1}} + \frac{1}{\rho^{\delta+2}} \right)^{\frac{1}{\rho}}
\leq \left( \frac{1}{\rho^{\delta+1}} + \frac{1}{\rho^{\delta+2}} \right)^{\frac{1}{\rho}}.
$$

the required result. \(\square\)

6. Applications

Let $l_1, l_2 > 0$ with $l_1 \neq l_2$. Then the arithmetic mean $A(l_1, l_2)$, harmonic mean $H(l_1, l_2)$, logarithmic mean $L(l_1, l_2)$ and $n$-th generalized logarithmic mean $L_n(l_1, l_2)$ are defined by

$$
A(l_1, l_2) = \frac{l_1 + l_2}{2},
$$

$$
H(l_1, l_2) = \sqrt{l_1 l_2},
$$

$$
L(l_1, l_2) = \frac{l_2 - l_1}{\ln l_2 - \ln l_1}
$$

and

$$
L_n(l_1, l_2) = \left[ \frac{n + 1}{l_2 - l_1} \right]^{\frac{1}{n}}
$$

respectively. Recently, the bivariate means have attracted the attention of many researchers [47–58] due to their are closely related to the special functions.

In this section, we use our obtained results in section 5 to provide several novel inequalities involving the special bivariate means mentioned above.

Proposition 6.1. Let $l_1, l_2 > 0$ with $l_2 > l_1$. Then

$$
\left| A(l_1^2, l_2^2) - \frac{1}{2} L_3^3(l_1, l_2) \right| \leq \frac{l_2 - l_1}{6} \left( e^{\alpha_2 l_2} + e^{\alpha_1 l_1} \right)^{\frac{1}{6}}.
$$
Proof. Let $\rho = \delta = 1$ and $Q(z) = z^2$. Then the desired result follows from Theorem 5.3. \qed

Proposition 6.2. Let $l_1, l_2 > 0$ with $l_2 > l_1$. Then

$$
\left| \mathcal{H}^{-1}(l_1, l_2) - \frac{1}{2} \mathcal{L}^{-1}(l_1, l_2) \right| \leq \frac{l_2 - l_1}{2(6)^{\frac{n}{2}} e^{\alpha l_1 l_2}} \left( \frac{e^{\alpha l_2^2} + e^{\alpha l_1^2}}{(l_1 l_2)^{2n}} \right)^{\frac{1}{n}}.
$$

Proof. Let $\rho = \delta = 1$ and $Q(z) = \frac{1}{z}$. Then the desired result follows from Theorem 5.3. \qed

Proposition 6.3. Let $l_1, l_2 > 0$ with $l_2 > l_1$. Then

$$
\left| \mathcal{A}(l_1, l_2) - \frac{1}{2} \mathcal{L}_n(l_1, l_2) \right| \leq \frac{l_2 - l_1}{2} \left| n! \left( \frac{e^{\alpha l_2^2} + e^{\alpha l_1^2}}{6e^{\alpha l_1 l_2}} \right) \right|.
$$

Proof. Let $\rho = \delta = 1$ and $Q(z) = z^n$. Then the desired result follows from Theorem 5.3. \qed

7. Conclusions

In this paper, we proposed a novel technique with two different approaches for deriving several generalizations for an exponentially tgs-convex function that accelerates with a conformable integral operator. We have generalized the Hermite-Hadamard type inequalities for exponentially tgs-convex functions. By choosing different parametric values $\rho$ and $\delta$, we analyzed the convergence behavior of our proposed methods in form of corollaries. Another aspect is that to show the effectiveness of our novel generalizations, our results have potential applications in fractional integro-differential and fractional Schrödinger equations. Numerical applications show that our findings are consistent and efficient. Finally, we remark that the framework of the conformable fractional integral operator, it is of interest to further our results to the framework of Riemann-Liouville, Hadamard and Katugampola fractional integral operators. Our ideas and the approach may lead to a lot of follow-up research.

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Conflict of interest

The authors declare no conflict of interest.

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