Near-Optimal Radio Use
For Wireless Network Synchronization

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October 9, 2008
Abstract

In this paper we consider the model of communication where wireless devices can either switch their radios off to save energy (and hence, can neither send nor receive messages), or switch their radios on and engage in communication. The problem has been extensively studied in practice, in the setting such as deployment and clock synchronization of wireless sensor networks – see, for example, the works of McGlynn and Borbash [28], PalChaudhuri and Johnson [30], Moscibroda, Von Rickenbach and Wattenhofer [26], or the survey paper of Sundararaman, Buy and Kshemkalyani [37]. The goal in these works is different from the classic problem of radio broadcast, i.e. avoiding interference. Here, the goal is instead to minimize the use of the radio for both transmitting and receiving, and for most of the time to shut the radio down completely, as the radio even in listening mode consumes a lot of energy. Somewhat surprisingly, in the theoretical community, this model has not been studied.

We distill a clean theoretical formulation of this problem of minimizing radio use and present near-optimal solutions. Our base model ignores issues of communication interference, although we also extend the model to handle this requirement. We assume that nodes intend to communicate periodically, or according to some time-based schedule. Clearly, perfectly synchronized devices could switch their radios on for exactly the minimum periods required by their joint schedules. The main challenge in the deployment of wireless networks is to synchronize the devices’ schedules, given that their initial schedules may be offset relative to one another (even if their clocks run at the same speed). In this paper we study how frequently the devices must switch on their radios in order to both synchronize their clocks and communicate. In this setting, we significantly improve previous results, and show optimal use of the radio for two processors and near-optimal use of the radio for synchronization of an arbitrary number of processors. In particular, for two processors we prove deterministically matching $\Theta(\sqrt{n})$ upper and lower bounds on the number of times the radio has to be on, where $n$ is the discretized uncertainty period of the clock shift between the two processors. (In contrast, all previous results for two processors are randomized, using birthday paradox (e.g. [30, 26]).) For $m = n^\beta$ processors (for any positive $\beta < 1$) we prove $\Omega(n^{(1-\beta)/2})$ is the lower bound on the number of times the radio has to be switched on (per processor), and show a nearly matching (in terms of the radio use) $\tilde{O}(n^{(1-\beta)/2})$ randomized upper bound per processor, (where $\tilde{O}$ notation hides $\text{poly-log}(n)$ multiplicative term) with failure probability exponentially close to 0. For $\beta \geq 1$ our algorithm runs with at most $\text{poly-log}(n)$ radio invocations per processor. Our bounds also hold in a radio-broadcast model where interference must be taken into account. Again, these results are asymptotically superior to all previously proposed algorithms that try to minimize radio use.

While our upper bounds are fairly straightforward (once the problem is properly formulated, which we also consider to be an important contribution) the matching lower bounds (even for the case of two processors) and nearly-matching lower bound (for the case of multiple processors) are nontrivial. We believe the abstract model that we put forward here will be of independent interest and will spark additional theoretical research in this setting.

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1 Introduction

Motivation: Radios are inherently power-hungry. As the power costs of processing, memory, and other computing components drop, the lifetime of a battery-operated wireless network deployment comes to depend largely on how often a node’s radio is left on. System designers therefore try to power down those radios as much as possible. This requires some form of synchronization, since successful communication requires that the sending and receiving nodes have their radios on at the same time. Synchronization is relatively easy to achieve in a wired, powered, and well-administered network, whose nodes can constantly listen for periodic heartbeats from a well-known server. In an ad hoc wireless network or wireless sensor network deployment, the problem becomes much more difficult. Nodes may be far away from any wired infrastructure; deployments are expected to run and even to initialize themselves autonomously (imagine sensors dropped over an area by plane); and environmental factors make sensors prone to failure and clock drift. Indeed there has been a lot of work in this area, see for example: [4, 5, 6, 8, 12, 13, 14, 21, 24, 25, 26, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37]. Many distinct problems are considered in these papers, and it is beyond the scope of this paper to survey all these works, however most if these papers (among other issues) consider the following problem of radio-use consumption:

Informal Problem description:

Consider two (or more) processors that can switch their radios on or off. The processors’ clocks are not synchronized. That is, when a processor wakes up, each clock begins to count up from 0; however, processors may awake at different times. The maximum difference between the time when processors wake up is bounded by some parameter $n$. If processors within radio range have their radios on in the same step, they can hear each other and can synchronize their clocks. When a processor’s radio is off, it saves energy, but can neither receive nor transmit. Initially, processors awaken with clock shifts that differ by at most $n$ time units. The objective for all the processors is to synchronize their clocks while minimizing the use of radio (both transmitting and receiving). We count the maximum number of times any processor’s radio has to be on in order to guarantee synchronization. Indeed, as argued in many papers referenced above, the total time duration during which the radio is on is one of the critical parameters of energy consumption, and operating the radio for considerable time is far costlier than switching radio off and switching it back on. We assume that all the processors that have their radios on at the same time can communicate with each other. The goal of all processors is to synchronize their clocks, i.e. to figure out how much to add to their offset so that all processors wake up at the same time. (We also consider an extension that models radio interference, where if more then one processor is broadcasting at the same time, all receiving processors that have their radio switched on hear only noise.)

For multiple processors, we assume that all processors know the maximum drift $n$, otherwise the adversary can make the delay unbounded, It is also assumed that all processors know the total number of processors $m$, although, we also consider a more general setting where $n$ is known for all processors, but $m$ is not. In this setting, we relax the problem, and instead of requiring synchronization of all $m$ processors, we instead require synchronization of an arbitrarily close to 1 constant fraction of all processors. In this relaxation of our model, we require that the radio usage guarantee holds only for those processors that eventually synchronize.

Furthermore, our model assumes that all processors are within radio range of each other, so that the link graph is complete. Our techniques can be thought of as establishing synchronization within completely connected single-hop regions. Clearly, single-hop synchronization is necessary for multi-hop synchronization. Our single-hop synchronization protocol, with simple changes, can synchronize a connected multi-hop network in the sense that (1) two directly connected nodes know one another’s clock offsets, and (2) given any
two nodes in the network \( v \) and \( w \), there exists a path \( v_0 = v, v_1, \ldots, v_n = w \) where each adjacent pair of nodes is connected and synchronized. Thus our central concern in this paper is on establishing lower bounds and constructing nearly optimal solutions for the single-hop case.

Towards Formalizing the Abstract Model:

To simplify our setting we wish to minimize both transmit and receive cost (i.e., all the times when the radio must be “on” either transmitting or receiving). We discretize time to units of the smallest possible interval that allows a processor to send a message to or receive a message from another processor within radio range. We normalize the cost of transmitting and receiving to 1 unit of energy per time step. (In practice, transmission can be about twice as expensive as receiving. We can easily re-scale our algorithms to accommodate this as well, but for clarity of exposition we make these costs equal.) We ignore the energy consumption needed to power the radio on and to power it off, which is at most comparable but in many cases insignificant compared to the energy consumption of having the radio active. This is the model considered, for example, in [28, 26, 5, 6, 34, 31]. For the purposes of analysis only, we assume that there is global time (mapped to positive integers). All clocks can run at different speeds, but we assume that clock drifts are bounded; i.e., there exists a global constant \( c \), such that for any two clocks their relative speed ratio is bounded by \( c \). Now, we define as a time “unit” the number of steps of the slowest clock, such that if two of the fastest processors’ consecutive awake times overlap by at least a half of their length according to global time, then the number of steps of the slowest clock is sufficient time for any two processors to communicate with each other. (See appendix for further discussion.) We formalize this more precisely in the definition of our model.

Our Formal Model and Problem Statement:

Global time is expressed as a positive integer. \( m \) processors start at an arbitrary global time between 1 and \( n \), where each processor starts with a local “clock” counter set to 0. The parameter \( n \) refers to the discretized uncertainty period, or equivalently, to the possible maximal clock difference, i.e., to the maximal offset between clocks; hence, we will use these terms interchangeably. Both global time and each started processor’s clock counter increments by 1 each time unit. The global clock is for analysis only and is not accessible to any of the processors, but an upper bound on \( n \) is known to all processors. Each processor algorithm is synchronous, and can specify, at each time unit, if the processor is “awake” or “sleeping.” (The “awake” period is assumed to be sufficiently long to ensure that the energy consumption of powering the radio on and then shutting it off at each time unit is far less than the energy expenditure to operate the radio even for a single time unit). All processors that are awake at the same time unit can communicate with each other. (Our interference model changes this so that exactly two awake processors can communicate with each other, but if three or more processors are simultaneously awake, none of them can communicate.) The algorithm can specify what information they exchange. The goal is for all \( m \) processors to adjust their local clocks to be equal to each other, at which point they should all terminate. The protocol is correct if this happens either always or if the protocol is randomized with probability of error that is negligible. The objective is to minimize, per processor, the total number of times it’s radio is awake.

We remark that the above model is sufficiently expressive to capture a more general case where clocks at different nodes run at somewhat different speeds, as long as the ratio of different speeds is bounded by a constant. We formally prove this fact in the appendix (see the first section in the appendix).

Our Results:

We develop algorithms for clock synchronization in radio networks that minimize radio use, both with and without modeling of interference. In particular, our results are the following.
1. For two processors we show a $\Omega(\sqrt{n})$ deterministic lower bound and a matching deterministic $O(\sqrt{n})$ upper bound for the number of time intervals a processor must switch its radio on to obtain one-hop synchronization.

2. For arbitrary $m = n^\beta$ processors, we prove $\Omega(n^{1-\frac{\beta}{2}})$ is the lower bound on the number of time intervals the processor must switch its radio for any deterministic protocol and show a nearly-matching (in terms of the number of times the radio is in use) $O(n^{1-\frac{\beta}{2}} \cdot \text{poly-log}(n))$ randomized protocol, which fails to synchronize with probability of failure exponentially (in $n$) close to zero. Furthermore, our upper bound holds even if there is interference, i.e., if more than one processor is broadcasting, listening processors hear noise.

3. It is easy to see that processors could not perform synchronization if $n$ is unknown and unbounded, using standard evasive argument. However, if $n$ is known, we show that $\frac{8}{9}$ (or any other constant fraction) of the processors can synchronize without knowing $m$, yet still using $O(n^{1-\frac{\beta}{2}} \cdot \text{poly-log}(n))$ radio send/receive steps, with probability of failure exponentially close to zero.

We stress that while the upper bound for two processors is simple, the matching lower bound is nontrivial. This (with some additional machinery) holds true for the multi-processor case as well.

**Comparison with Previous (Systems) Work:**

Tiny, inexpensive embedded computers are now powerful enough to run complex software, store significant amounts of information in stable memory, sense wide varieties of environmental phenomena, and communicate with one another over wireless channels. Widespread deployments of such nodes promise to reveal previously unobservable phenomena with significant scientific and technological impact. Energy is a fundamental roadblock to the long-lived deployment of these nodes, however. The size and weight of energy sources like batteries and solar panels have not kept pace with comparable improvements to processors, and long-lived deployments must shepherd their energy resources carefully.

Wireless radio communication is a particularly important energy consumer. Already, communication is expensive in terms of energy usage, and this will only become worse in relative terms: the power cost of radio communication is fundamentally far higher than that of computation. In one example coming from sensor networks, a Mica2 sensor node’s CC1000 radio consumes almost as much current while listening for messages as the node’s CPU consumes in its most active state, and transmitting a message consumes up to 2.5 times more current than active CPU computation \[33\]. In typical wireless sensor networks, transmitting is about two times more expensive than listening, and about 1.5 times more expensive than receiving, but listening or transmitting is about 100 times more expensive as keeping the CPU idle and the radio switched off\(^1\) (i.e., in a “sleep” state).

Network researchers have designed various techniques for minimizing power consumption \[5\] \[6\] \[34\]. For example, Low-Power Listening \[31\] trades more expensive transmission cost for lower listening cost. Every node turns on its radio for listening for a short interval $\tau$ once every interval $n > \tau$. If the channel is quiet, the node returns to sleep for another $n$; otherwise it receives whatever message is being transmitted. To transmit, a node sends a *preamble* of at least $n$ time units long before the actual message. This ensures that no matter how clocks are offset, any node within range will hear some part of the preamble and stay awake for the message. A longer $n$ means a lower relative receive cost (as $\tau/n$ is smaller), but also longer preambles, and therefore higher transmission cost.

\(^1\)Example consumption costs: CPU idle with clock running and radio off (“standby mode”), 0.1–0.2 mA (milliamps); CPU on and radio listening, 10 mA; CPU on and radio receiving, 15 mA; CPU on and radio transmitting, 20–25 mA.
A more efficient solution in terms of radio use was proposed by PalChaudhuri and Johnson [30], and further by Moscibroda, Von Rickenbach and Wattenhofer [26]. The idea is as follows. Notice that in the proposal of [31], the proposal was for a transmitting processor to broadcast continuously for $n$ time units, while receiving processors switch their radios on once every $n$ time units to listen. Even for two processors, this implies that total use of the radio is $n + 1$ time units (i.e., it is linear in $n$). The observation of [30, 26] is that we can do substantially better by using randomization: if both processors wake their radios $O(\sqrt{n})$ time units at random (say both sending and receiving), then by birthday paradox with constant probability they will be awake at the same time and will be able to synchronize their clocks. (As indicated before, we show instead a deterministic solution to this important in practice problem, and a matching lower bound.)

**Comparison with Radio Broadcast:**

Usually, in a broadcast setup, a node is able to receive a message if and only if it does not transmit, and there is one and only one of its neighbors that transmits, at that time. In the case when nodes are not able to detect collision, [3, 2] showed randomized protocols. A deterministic broadcast algorithm, with work time $O(n^{1/6})$, has been given in [10]. The improvements of these algorithms have followed [22]: for undirected radio network graphs, with diameter $D$, for randomized broadcast the expected work time has been $O(D \log(n/D) + \log^2 n)$, while for deterministic broadcast the expected work time has been $\Omega(n \log n \log(\log n) / \log n)$. In [23], a faster algorithm for directed radio network graphs has provided running time $O(n \log n \log D)$. Additionally, other algorithms for broadcast [11, 15, 17, 19, 21] as well as for clock synchronization [29, 32, 12, 4] have been proposed. The work of radio broadcast is different from the problem we address at this paper. However, as we mention in the technical description, once we resolve the problem of meeting times, we can easily combine our solutions with radio broadcast goal to avoid interference.

**High-Level Ideas of Our Constructions and Proofs**

- For two processor upper bound, we prove that two carefully chosen affine functions will overlap no matter what the initial shift is. The only technically delicate part is that the shift is over the reals, and thus the proof must take this into account.

- for the 2-processor lower bound, we show that for any two strings with sufficiently low density (of 1’s) there always exists a small shift such that none of the 1’s overlap. This is done by a combinatorial counting argument.

- for multiple processors, the idea of the lower bound is to extend the previous combinatorial argument, while for the upper bound, the idea is to establish a “connected” graph of pairwise processor synchronization, and then show that this graph is an expander. The next idea is that instead of running global synchronization, we can repeat the same partial synchronization logarithmic number of times (using the same randomness) to yield a communication graph which is an expander. We then use standard synchronization protocol over this “virtual” expander to reach global synchronization.

- for handling interference, we observe that standard “back-off” protocols [11, 9] can be combined with previous machinery to achieve non-interference, costing only poly-logarithmic multiplicative term.

- for the protocol that does not need to know $m$, (recall that $m$ is the total number of processors within radio-reach), we first observe that if $m > n$, by setting $m = n$ our protocol already achieves synchronization with near-optimal radio use. The technical challenge is thus to handle the case where $m < n$ but the value of $m$ is unknown to the protocol. Our first observation is to show that processors can overestimate $m$, in which case the amount of energy needed is much smaller (per processor) than for smaller $m$, and then “check” if the synchronized component of nodes has reached current estimate on
m. If it did not, than our current estimate of m can be reduced (by a constant factor) by all the processors. To assure that estimates are lowered by all the processors at about the same time, we divide the protocol into “epochs” which are big enough not to overlap even with a maximal clock drift (of n). Summing, the energy consumption is essentially dominated by the smallest estimate of m, which is within a constant factor of correct value of m, and all processors that detect it stop running subsequent (more expensive) “epochs”.

2 Mathematical Preliminaries

**Lemma 1 (Two-Color Birthday Problem)** For any absolute constant \( C > \sqrt{1 - \ln 0.1} \approx 1.8173 \) and any positive \( s, t \in (0, 1) \), where \( s + t = 1 \), the following holds: \( r = Cn^s \) identical red balls and \( b = Cn^t \) identical blue balls are thrown independently and uniformly at random into \( n \) bins. Then, for sufficiently big \( n \), probability that there is a bin with both red and blue balls is \( \geq 0.8 \).

**Proof Outline**

We first note that since all balls are thrown independently and uniformly at random, it follows that throwing all of \( r + b \) balls together uniformly at random, is equivalent to the scenario of first throwing \( r \) red balls, then throwing \( b \) blue balls. Thus, we first throw \( r \) red balls, and count the number of unoccupied bins. Let us mention that we keep \( s \) and \( t \) fixed (while \( n \) grows). Using the Theorem 1 in [20] the number of unoccupied bins is \( \Theta(ne^{-r/n}) \) with probability \( \geq 1 - \exp(-n^r) \). Then we throw \( b \) blue balls u.a.r. into \( n \) bins. Given that the number of unoccupied bins after throwing red balls is \( \Theta(ne^{-r/n}) \), with probability aforementioned, it follows that every blue ball independently hits a bin with a red ball with probability \( \geq 1 - \Theta(ne^{-r/n}) \). Now, applying the Chernoff bound on the number the blue balls hitting the bins with the red balls, the proof of the lemma follows. The formal proof is given in the Appendix.

3 Lower Bounds

Recall that \( n \) is the maximum offset between processor starting times and \( m = n^\beta \) is the number of processors. Assume that each processor runs for some time \( L \). Its radio schedule can then be represented as a bit string of length \( L \), where the \( i \)th bit is 1 if and only if the processor turned its radio on during that time unit. We first consider the two-processor case. Recall that in our model maximal assumed offset is at most \( n \). If we take 2 bit strings corresponding to the two processors, the initial clock offset corresponds to a shift of one string against the other by at most \( n \) positions. Note that if we set \( L \geq 4n \), the maximal shift is at most \( n \leq L/4 \).

**Note:** In the next sections, without loss of generality we apply ceiling function to any real number, e.g., \( L^\alpha, L/C^2 \) are treated as \( \lceil L^\alpha \rceil, \lceil L/C^2 \rceil \), respectively.

To prove our lower bound, we need to prove the following: for any two \( L \)-bit strings with at most \( \sqrt{L}/C \) ones in each string (for some constant \( C > 1/\sqrt{2} \)), there always exists a shift \( < L/4 \) of one string against another such that none of the ones after the shift in the first string align with any of the ones in the second string. In this case we say that the strings do not overlap. W.l.o.g., we make both strings (before the shift) identical. To see that this does not limit the generality, we note that if the two strings are not identical, we can make a new string by taking their bitwise OR, what we call the “union” of strings. If the distinct strings overlap at a given offset, then the “union” string will overlap with itself at the same offset.
Lemma 2 (Two Non-Colliding Strings) For any absolute constant $C \geq 1/\sqrt{2}$, and for every $L$-bit string with $\ell \leq \frac{L}{2}$ ones, there is at least one shift within $L/(2C^2)$ such that the string and its shifted copy do not overlap.

Proof For a considered $L$-bit string, let us denote the positions of $\ell$ ones as $1 \leq a_1 < a_2 < \ldots < a_{\ell} \leq L$. Let us consider the set of differences $D = \{a_j - a_i : 1 \leq i < j \leq \ell\}$. The cardinality of that set is at most $|D| \leq \binom{\ell}{2} = \ell(\ell - 1)/2 < L/(2C^2)$ (some of differences may be equal). All differences are greater than zero since $a_i$ are mutually different, and $D \subset \{1,2,\ldots,L-1\}$. Let $\delta \in \{1,2,\ldots,\lfloor \frac{L}{2C^2}\rfloor\}$ be the minimal integer such that $\delta \notin D$. That integer exists since $|D| < \lfloor \frac{L}{2C^2}\rfloor$. For the set of positions of ones $A = \{a_1,a_2,\ldots,a_\ell\}$, let the shifted set be $A + \delta = \{a_i + \delta | a_i \in A\}$. The sets $A$ and $A + \delta$ do not intersect since $\delta \geq 1$, that is $a_i \neq a_i + \delta$, and by construction, since $\delta \notin D$ it follows $a_i \neq a_j + \delta$ for any $a_i,a_j \in A$. This proves the lemma.

Next, we want to prove a general lower bound for multiple strings. The high-level approach of our proof is as follows. We pick one string, and then upper bound the total number of ones possible in the “union” of all the remaining (potentially shifted) strings. If we can prove that assuming the density of all the strings is sufficiently small, and there always exists a shift of the first string that does not overlap the “union” of all the remaining strings, we are done. The “union” string is simply a new string with a higher density.

Lemma 3 (General Two Non-Colliding Strings with Different Densities) Let $s,t > 0$ such that $s+t < 1$, and let $C > 1$. For two $L$-bit strings such that the number of ones in the first string is $a = L^s/C$, and the number of ones in the second string is $b = L^t/C$, there is a shift within $L/C^2 + 1$ such that the first string and the shifted second string do not overlap.

Proof Let the positions of ones in the first string be $P = \{p_1,p_2,\ldots,p_a\}$, and the positions of ones in the second string be $Q = \{q_1,q_2,\ldots,q_b\}$. Let us consider the set of differences $D = \{p - q | p \in P, q \in Q\}$. The cardinality of that set satisfies $|D| \leq |P||Q| = ab = L^{s+t}/C^2 \leq L/C^2 < L$.

Similarly to the proof of the Lemma 2 let us choose $\delta \in \{0,1,2,\ldots,\lfloor L/C^2\rfloor + 1\}$ such that $\delta \notin D$. That integer exists since $|D| \leq L/C^2$. Now, it follows that $P$ and $Q + \delta = \{q + \delta | q \in Q\}$ do not intersect, since by construction $p \neq q + \delta$ for any $p \in P$ and $q \in Q$. This proves the lemma.

Here, w.l.o.g., we considered only “left” shift. If we needed both left and right shift, then we would have an additional factor of 2, and would need both $p - q$ and $q - p$ in $D$. The proof of the previous lemma would follow by the same argument. Using the Lemma 3, the lower bounds immediately follow.

Theorem 4 There exists an absolute constant $C > 1$, such that for any $n^\beta$ strings of length $L$ with at most $n^{(1-\beta)/2}$ ones in each string, there always exists a set of shifts for each string by at most $L/4$ such that no string’s ones overlap any of the ones in all the other strings.

Proof Set $\alpha = (1-\beta)/2$. Add strings sequentially and for each find a shift that does not overlap with (the union of) all the shifted previous strings. The Lemma 3 applies since the smaller string has density $n^\alpha$, and the “union” of all the previous strings has density of at most $n^{\beta} \cdot n^\alpha$. This density is at most $n$, since the sum of the exponents $\alpha + \beta = (1+\beta)/2$ is at most 1, what proves the theorem.
4 Matching Upper Bound for Two Processors

We now show the upper bound. That is, we give the deterministic algorithm for two devices. In particular, for any initial offset of at most \( n \), we show a schedule where two processors meet with probability equal to one inside a “time-window” of length \( W = 2n + 4\sqrt{n} + 2 \).

**Theorem 5** For any \( n \), there exists a string of length \( W = 2n + 4\sqrt{n} + 2 \) with at most \( 2\sqrt{n} + 2 \) ones such that this string will overlap itself for all shifts from 1 to \( n \).

**Proof** Let us define the string \( S \) of the length \( W \), that has ones at the following positions (from the perspective of its local clock): Set the bits at positions \((i\sqrt{n} + i)\) and \((i\sqrt{n})\) to 1, for \( i \in \{1, \ldots, \lfloor 2\sqrt{n} \rfloor \} \); Set the remaining bits to 0.

For the analysis, we consider the “global” clock. Further, we consider two strings \( A \) and \( B \), being the shifted versions of the string \( S \), shifted by \( a_0 \) and \( b_0 \), respectively. (Both \( a_0 \) and \( b_0 \) are \( \leq n \), by the conditions of the Theorem 5.) Since the string \( S \) is deterministically defined, we know the exact appearances of ones in the strings \( A \) and \( B \). Thus, from the global clock point of view, in the strings \( A \) and \( B \), respectively, ones appear during the following time intervals:

\[
\lfloor a_{i1} \rfloor, \lfloor a_{i1} \rfloor + 1, \lfloor a_{i2} \rfloor, \lfloor a_{i2} \rfloor + 1 \quad \text{and} \quad \lfloor b_{i1} \rfloor, \lfloor b_{i1} \rfloor + 1, \lfloor b_{i2} \rfloor, \lfloor b_{i2} \rfloor + 1,
\]

where the values \( a_{i1}, a_{i2}, b_{i1}, b_{i2} \) are given by

\[
a_{i1} = a_{0} + i\sqrt{n} + i,
\]
\[
a_{i2} = a_{0} + i\sqrt{n}
\]
\[
b_{i1} = b_{0} + i\sqrt{n} + i,
\]
\[
b_{i2} = b_{0} + i\sqrt{n},
\]

for \( i \in \{1, \ldots, \lfloor 2\sqrt{n} \rfloor \} \). The initial values of strings are: \( a_{01} = a_{02} = a_{0} \) and \( b_{01} = b_{02} = b_{0} \). We show that there exist integers \( i, j \in \{0, 1, \ldots, \lfloor 2\sqrt{n} \rfloor \} \), such that for some \( s_1, s_2 \in \{1, 2\} \) the following is satisfied

\[
\delta = |a_{is_1} - b_{js_2}| < 1. \tag{1}
\]

Although the schedule we propose may look simple, in general \( \sqrt{n} \) is not an integer, thus we have to perform the precise analysis below.

We will widely use the following property of the floor function and the fractional part for the reals. For any \( x \in \mathbb{R}, x = \lfloor x \rfloor + \{x\} \), where \( \{x\} \in [0, 1) \) is the fractional part, and \( \lfloor x \rfloor \in \mathbb{Z} \) is the floor function, the following is satisfied

\[
\lfloor x \rfloor - 1 \leq x - 1 < \{x\} \leq x < \lfloor x \rfloor + 1 \leq x + 1.
\]

Now, we explicitly construct \( i, j \) such that Eq. (1) is satisfied, that is, \( a_{is_1} = b_{js_2} \pm \delta \) for some fractional part \( \delta \in [0, 1) \) and for some \( s_1, s_2 \in \{1, 2\} \). Let us call the absolute difference between initial values of the strings \( \Delta = |a_{0} - b_{0}| \). Since the protocol is symmetric in \( A \) and \( B \), w.l.o.g., let us first consider \( a_{0} - b_{0} \geq 0 \). From \( a_{0}, b_{0} \in \{0, 1, \ldots, n\} \), it follows \( \Delta \leq n \). We consider the following three cases.

**Case:** \( a_{0} - b_{0} = 0 \). It follows \( a_{0} = b_{0} \) and Eq. (1) is satisfied for \( i = j = 0 \).
Case: $1 \leq a_0 - b_0 \leq n$. Let $q = \lfloor \Delta / \sqrt{n} \rfloor$. Then we have:

\[
q = \left\lfloor \frac{\Delta}{\sqrt{n}} \right\rfloor \leq \frac{\Delta}{\sqrt{n}} \leq \frac{n}{\sqrt{n}} = \sqrt{n},
\]

\[
\left\lfloor \frac{\Delta}{\sqrt{n}} \right\rfloor \leq \frac{\Delta}{\sqrt{n}} < \left\lfloor \frac{\Delta}{\sqrt{n}} \right\rfloor + 1,
\]

\[
q \leq \frac{\Delta}{\sqrt{n}} < q + 1,
\]

\[
q \sqrt{n} \leq \Delta < q \sqrt{n} + \sqrt{n}.
\]

Now we give the exact $i$ and $j$ where the strings meet. Consider $i$ and $j$ given by

\[
i = \lfloor (q + 1) \sqrt{n} \rfloor - \Delta,
\]

\[
j = i + q + 1.
\]

Then we have:

\[
i = \lfloor (q + 1) \sqrt{n} \rfloor - \Delta \leq (q + 1) \sqrt{n} - \Delta = q \sqrt{n} - \Delta + \sqrt{n} \leq \sqrt{n},
\]

\[
j = q + 1 + i \leq \sqrt{n} + 1 + \sqrt{n} = 2 \sqrt{n} + 1.
\]

From above, it follows that since $\Delta$ is an integer, $\Delta \leq \lfloor (q + 1) \sqrt{n} \rfloor \Rightarrow i \geq 0 \Rightarrow j > 0$. Substituting the values for $i$ and $j$, we get $\delta = |a_{i,1} - b_{j,2}| = (q+1)\sqrt{n} \in [0,1)$.

Case: $-n \leq a_0 - b_0 \leq -1$. In this case, for the previously defined $i, j, \Delta$, it follows $|a_{j,2} - b_{i,1}| \in [0,1)$.

Finally, for any $a_0, b_0 \in \{0, 1, \ldots, n\}$ there are $i, j$ such that the shifted strings meet. Since $\max_{i,j} \{a_i, b_j\} \leq n + (2\sqrt{n} + 2)\sqrt{n} + (2\sqrt{n} + 2) = 3n + 4\sqrt{n} + 2$, and subtracting $n$, the length of the strings, it follows that the strings meet with the probability equal to 1 inside the time-window of the length $W = 2n + 4\sqrt{n} + 2$. This completes the proof.

5 Upper Bound for $m$ Processors

In this setting we have $m = n^\beta$ processors (and as before the maximum shift is at most $n$). We first state our theorem:

**Theorem 6** There exists a randomized protocol for $n^\beta$ processors (which fails with probability at most $1/2^\Theta(n)$) such that:

1. if $\beta < 1$ the protocol is using at most $O\left( n^{1-\beta} \cdot \text{poly-log}(n) \right)$ radio steps per processor, and

2. if $\beta \geq 1$ using at most $O(\text{poly-log}(n))$ radio steps per processor.

Furthermore, the same bounds hold for the synchronization in the radio communication model, where a processor can hear a message if one (and strictly one) message is broadcasted.
First, we give a high-level outline of the construction of our algorithm for $\beta \in [0, 1)$. For the case of $\beta \geq 1$ we only need Steps 4 and 5, see below. The formal analysis and proofs of the Main Algorithm are given in the Appendix.

**Outline of the Main Algorithm:**

**Step 1.** We let each processor run for $L = 4n$ steps, waking up during this time $O \left( n^{\frac{1-\beta}{2}} \right)$ times uniformly at random. It is important to point out that each processor uses independent randomness. We view it as an $m$-row and $L (L \geq W + n)$ column matrix $A$ (taking into account all the shifts), where $W = 2n + 4\sqrt{n} + 2$ is defined in Theorem 5. Fix any row of this matrix (say the first one). We say that this row “meets” some other row, if 1 in the first row also appears (after the shifts) in some other row. If this happens, the first processor can “communicate” with another processor. We show that for a fixed row, this happens with a constant probability.

**Step 2.** Each processor repeats Step 1 (using independent randomness) $O(\log m)$ times. Here, we show that a fixed row has at least $O(\log m)$ connections to other rows (not necessarily distinct) with probability greater than $1 - 1/\text{poly}(m)$.

**Step 3.** From Step 2, we conclude that the first row meets at least a constant number of distinct other rows with probability greater than $1 - 1/2m$.

**Step 4.** We use the union bound to conclude that every row meets at least constant number of distinct other rows with probability greater than $1/2$. If we repeat this process a logarithmic number of times, we show that we get an expander graph with overwhelming probability (for the definition of an expander see [27]). Thus, considering every row (i.e., every processor) as a node, this represents a random graph with degree of at least at least constant number for each node, which is an expander with high probability.

**Step 5.** During the synchronization period, a particular processor will synchronize with some other processor, without collision, by attempting to communicate whenever it has a 1 in its row. (In case of interference, the processor can communicate if only one other processor is up at this column, which we can achieve as well, using standard "back-off" protocols [1, 9], costing only poly-logarithmic multiplicative term.)

**Step 6.** The processors can now communicate along the edges of the formed expander (which has logarithmic diameter) as follows. The main insight that we prove below is that if processors repeat the same random choices of Step 1 through Step 5, the communication pattern of the expander graph is preserved. Hence, the structure developed in Step 2 can be reused to establish a logarithmic-diameter (in $m$) spanning tree and synchronize nodes with poly-logarithmic overhead (using known machinery over this “virtual” graph). We show in the appendix, using standard methods, that communicating over the implicit expander graph to synchronize all nodes can be done in $D + 2$ steps, where $D$ is the diameter of the expander.

**6 Protocol That Does Not Need to Know the Number of Processors**

Suppose our processors know the offset $n$ but not the number of all processors in the system, that is, $m$. The main observation here is that once we make a spanning tree of the graph, each node can also compute the number of nodes in its spanning tree. Hence, we can make an estimate of $m$ and then check to see if this estimate is too big. Thus, until the right (within a constant factor) estimate is reached, all nodes will reject...
the estimate and continue. Adjusting constants appropriately, we can guarantee that an arbitrary constant fraction of the processors will terminate with the right estimate of \( m \) (within some fixed constant fraction). The algorithm for the estimation of \( m \) is as follows.

**Algorithm: Estimation of \( m \)**

1. Set \( i = 0 \).
2. Build a spanning tree using the Main Algorithm (from the previous section) for \( m_i = n/2^i \) and count the number of nodes in the tree. If the number of the nodes in the tree is less than \( m_i \) then set \( i := i + 1 \) and go to step 2.
3. Output \( m_i \).

**End of Algorithm: Estimation of \( m \)**

**Theorem 7** Any constant fraction of the processors can synchronize without knowing \( m \), yet still use \( O(n^{1-\beta} \text{poly-log}(n)) \) radio send/receive steps (with probability of failure exponentially close to zero). The bound on the radio use holds only for processors that synchronize.

**Proof** We showed that the used power is \( O(n^{1-\beta} \text{poly-log}(n)) \) for a particular number of processors \( m_i = n^{\beta} \). Let us consider \( m_i = n/2^i \). Since \( \alpha_i = (1 - \beta_i)/2 \), it follows \( n^{\alpha_i} = \sqrt{n/m_i} = 2^{i/2} \). Let \( i_{\max} = \lceil \log(n/m) \rceil \). Then the total power, used in the protocol that does not know \( m \), is

\[
\sum_{i=0}^{i_{\max}} O(n^{\alpha_i} \text{poly-log}(n)) = \sum_{i=0}^{i_{\max}} O(2^{i/2} \text{poly-log}(n)) = \sum_{i=0}^{i_{\max}} 2^{i/2} O(\text{poly-log}(n)) = O(2^{i_{\max}/2} O(\text{poly-log}(n)) = O(n^{(1-\beta)/2} \text{poly-log}(n)),
\]

since \( 2^{i_{\max}/2} = O(\sqrt{n/m}) = O(n^{1-\beta/2}) \).

**7 Conclusions and Open Problems**

In this paper, we have studied an important problem of power consumption in radio networks and completely resolved the deterministic case for two processors, showing matching upper and lower bound. For multiple processors, we were able to show a poly-logarithmic gap between our randomized protocol and our deterministic lower bound. However, this is not completely satisfactory. Our lower bound holds only for deterministic protocols, while our upper bound in multi-processor case is probabilistic (unlike the two-processor case, where our upper bound is deterministic as well). Closing this gap remains an interesting open problem.

Another interesting question is the following. It is important to note that in radio communication, conservation of power can be achieved in two different ways: one approach is to always broadcast the signal with the same intensity (or to power down radios completely in order to save energy); this is what we explored in this paper. The second approach is the ability for a radio to broadcast and receive signals at different intensity, the stronger the signal the further it reaches. In the case where all processors are at the same distance from each other, this is a non-issue (i.e., our single-hop networks, the main focus of this paper). However, for multi-hop networks the question of optimal power-consumption strategies with varying signal strength is completely open.
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Appendix

A Proof of Lemma

Proof We first note that since all balls are thrown independently and uniformly at random, it follows that throwing all \( r + b \) balls together uniformly at random, is equivalent to the scenario of first throwing \( r \) red balls, then throwing \( b \) blue balls. Thus, we first throw \( r \) red balls, and count the number of unoccupied bins. Let \( Z \) be a random variable, denoting the number of empty bins, after throwing \( r = Cn^r \) balls into \( n \) bins, u.a.r. The expectation of \( Z \) is

\[
\mathbb{E}[Z] = n(1 - \frac{1}{n})^r.
\]

By the occupancy bound, Theorem 1 in [20], for any \( \theta > 0 \), the tail of \( Z \) is given by

\[
\Pr[|Z - \mathbb{E}[Z]| \geq \theta \mathbb{E}[Z]] \leq 2\exp \left( -\frac{\theta^2 \mathbb{E}[Z]^2 (n - 1/2)}{n^2 - \mathbb{E}[Z]^2} \right).
\]

Now, let us throw \( b \) blue balls into \( n \) bins, u.a.r. We have that some of these \( n \) bins have been previously occupied with red balls. Let us denote the event \( \mathcal{H} \) that exists at least one bin with both red and blue balls. (The goal is to show \( \Pr[\mathcal{H}] > 0.8 \) for the assumptions given in the lemma.) Given \( Z = z \), the probability that one blue ball does not hit a bin with a red ball is \( z/n \). Then it follows \( \Pr[\mathcal{H}|Z = z] = 1 - (z/n)^b \), and furthermore,

\[
\Pr[\mathcal{H}|Z \leq (1 + \theta)\mathbb{E}[Z]] \geq 1 - ((1 + \theta)\mathbb{E}[Z]/n)^b,
\]

where we will appropriately chose \( \theta = o(1) \), later. We now use the formula of the total probability, and the bound on \( Z \), to obtain a lower bound on the probability of \( \mathcal{H} \)

\[
\Pr[\mathcal{H}] = \Pr[\mathcal{H}|Z \leq (1 + \theta)\mathbb{E}[Z]] \Pr[Z \leq (1 + \theta)\mathbb{E}[Z]].
\]
\[ \text{Pr}[\mathcal{H} | Z > (1 + \theta)\mathbb{E}[Z]] \text{Pr}[Z > (1 + \theta)\mathbb{E}[Z]] \]
\[ > \text{Pr}[\mathcal{H} | Z \leq (1 + \theta)\mathbb{E}[Z]] \text{Pr}[Z \leq (1 + \theta)\mathbb{E}[Z]] \]
\[ \geq \left(1 - ((1 + \theta)\mathbb{E}[Z]/n)^b\right) \left(1 - 2\exp\left(-\frac{\theta^2\mathbb{E}[Z]^2(n - 1/2)}{n^2 - \mathbb{E}[Z]^2}\right)\right) \]
\[ = \left(1 - (1 + \theta)^b\left(1 - \frac{1}{n}\right)^{rb}\right) \left(1 - 2\exp\left(-\frac{\theta^2\mathbb{E}[Z]^2(n - 1/2)}{n^2 - \mathbb{E}[Z]^2}\right)\right) \]

Let us choose \( \theta = 1/b = 1/(Cn^t) \). The goal is to obtain an upper bound on \( \text{Pr}[\mathcal{H}] \), sufficiently close to 1, thus we discuss the following terms. First,

\[
1 - (1 + \theta)^b\left(1 - \frac{1}{n}\right)^{rb} = 1 - (1 + \frac{1}{b})^b\left(1 - \frac{1}{n}\right)^{C^2n^{t+1}}
\]
\[
= 1 - (1 + \frac{1}{Cn^t})^{C^2n^t}\left(1 - \frac{1}{n}\right)^{C^2n}
\]
\[
\geq 1 - e^{1-C^2}
\]

where we have used \((1 + 1/n)^n \leq e\) and \((1 - 1/n)^n \leq 1/e\), for every \( n \). (Furthermore, the sequences \((1 + 1/n)^n\) and \((1 - 1/n)^n\) are both increasing, with the limits \( e \) and \( 1/e \), respectively.) Second, let us consider the term \( \frac{\theta^2\mathbb{E}[Z]^2(n - 1/2)}{n^2 - \mathbb{E}[Z]^2} \). We will use the following: (i) \( n - 1/2 \geq n/2 \), for any positive integer \( n \); (ii) Bernoulli’s inequality \((1 + x)^r \geq 1 + rx\) for \( x > -1 \) and \( r \geq 1 \). Thus,

\[
\frac{\theta^2\mathbb{E}[Z]^2(n - 1/2)}{n^2 - \mathbb{E}[Z]^2} \geq \frac{\theta^2n/2}{(n/\mathbb{E}[Z])^2 - 1}
\]
\[
= \frac{\theta^2n}{2} \frac{(1 - 1/n)^{2r}}{1 - (1 - 1/n)^{2r}} \quad \text{using the expression for } \mathbb{E}[Z]
\]
\[
\geq \frac{\theta^2n}{2} \frac{(1 - 1/n)^{2r}}{2r/n} \quad \text{by Bernoulli’s inequality}
\]
\[
= \frac{n}{2b^2} \frac{(1 - 1/n)^{2r}}{2r/n} \quad \text{we already chose } \theta = \frac{1}{b}
\]
\[
= \frac{r}{4C^4} (1 - 1/n)^{2r} \quad \text{by definition for } r \text{ and } b, \text{ it follows } rb = C^2n
\]

In order to find an upper bound on the last expression, let us consider the logarithm of it.

\[
\ln \frac{r}{4C^4} (1 - 1/n)^{2r} = -\ln(4C^4) + \ln r + 2r \ln(1 - 1/n)
\]
\[
\geq -\ln(4C^4) + \ln r + 2r \left(-\frac{2}{n}\right) \quad \text{by the fact } \ln(1 - \frac{1}{n}) \geq -\frac{2}{n}, \text{ for } n \geq 2
\]
\[
= -\ln(4C^4) + \ln r - \frac{4r}{n}
\]
\[
= -\ln(4C^4) + s \ln C + s \ln n - \frac{4C}{n^{1-s}} \quad \text{from definition } r = Cn^s
\]
\[
\geq \frac{s}{2} \ln n \quad \text{for sufficiently large } n
\]

That is,

\[
1 - 2\exp\left(-\frac{\theta^2\mathbb{E}[Z]^2(n - 1/2)}{n^2 - \mathbb{E}[Z]^2}\right) \geq 1 - \exp(-\frac{r}{4C^4} (1 - 1/n)^{2r})
\]
\[
\geq 1 - \exp(-n^{s/2}). \quad (2)
\]
Thus, for sufficiently large \( n \), we obtain the lower bound on the probability of \( \mathcal{H} \)

\[
\Pr[\mathcal{H}] \geq (1 - e^{1-C^2})(1 - 2\exp(-n^{s/2})) = 1 - e^{1-C^2} - 2\exp(-n^{s/2})\exp(-n^{s/2}).
\]

(3)

For given \( \varepsilon \in (0,1) \), if both \( \varepsilon / 2 \geq e^{1-C^2} \) and \( \varepsilon / 4 \geq \exp(-n^{s/2}) \), then Eq. (3) implies \( \Pr[\mathcal{H}] \geq 1 - \varepsilon \). These two conditions are equivalent to

\[
c \geq \sqrt{1 - \ln \frac{\varepsilon}{2}}
\]

and

\[
n \geq \left( -\ln \left( \frac{\varepsilon}{4} \right) \right)^{2/s}.
\]

Finally, we obtain

\[
\Pr[\mathcal{H}] \geq 1 - \varepsilon.
\]

Specifically, let \( C > \sqrt{1 - \ln 9.1} \approx 1.8173 \), that is \( 1 - e^{1-C^2} > 0.8 \). Then for sufficiently big \( n \), it follows \( \Pr[\mathcal{H}] \geq 0.8 \), which completes the proof of lemma.

### B Proof of Correctness of the Main Algorithm

**Proof** Analysis of Step 1. We generate a random matrix \( A \) as follows.

**Definition 8 (Generation of the Random Matrix \( A \))** For each row of the \( n^\beta \times L \) random matrix \( A \), independent of the content of other rows, we uniformly and independently generate \( CL^\alpha \) integers \( t_1, t_2, \ldots, t_{CL^\alpha} \in \{1,2,\ldots,L\} \), where \( \alpha, \beta \in [0,1] \). Each \( t_i \) corresponds to one energy unit for the unit time \( t_i \) of that row (note, \( t_i \)’s are not necessarily different, and the sum of each row is \( CL^\alpha \)).

**Lemma 9** Let \( A \) be the matrix (given by the Definition 8), such that \( 2\alpha + \beta = 1 \). Let us consider one particular row from \( A \). That row “meets” with some other row with the probability > 0.8.

**Proof** In the Section 3 we used \( L = 4n \). Each row in the matrix \( A \) has \( \hat{C}L^\alpha = \hat{C}n^\alpha \) ones, denoting the constant \( \hat{C} = 4^\alpha C \).

Let us consider a particular row of the matrix \( A \), w.l.o.g. let it be the first row. The first row has \( \hat{C}n^\alpha \) ones, what we call the blue balls. Let all of the remaining \( n^\beta - 1 \) rows be collapsed into one row, what we call the “collapsed row”, and each entry of this collapsed row will represent one of \( n \) bins. The number of ones in the collapsed row is

\[
(n^\beta - 1)\hat{C}n^\alpha = n^{\alpha+\beta} \hat{C} \left( 1 - \frac{1}{n^\beta} \right).
\]

The ones in the collapsed row, we call the red balls.

Since the positions of the balls in each row are generated randomly, independently and uniformly, also row by row independently, it follows that the balls in the collapsed row are generated independently, uniformly at random. That is, the process of throwing the first \( \hat{C}n^\alpha \) red balls into “collapsed bins”, then the second
Now we show that other positive constant, and taking in

At this step, we specifically choose the number 10, to be a constant number of different rows that one row

With the random variable, such that,

We see this experiment as the throwing of

of "meetings" is chosen independently with replacement.) Again, w.o.l.g. we consider the first row. We have that the number

For the sake of brevity, let us call 

where

Analysis of Step 3. Here, we prove that each row in the matrix A "meets" with at least a constant number of different rows with the probability > 1/2. We specify this constant later. (Furthermore, the meetings are chosen independently with replacement.) Again, w.o.l.g. we consider the first row. We have that the number of "meetings" is 

We see this experiment as the throwing of 

balls, one by one independently, into 

bins. Let 

be a binary random variable, such that, 

if and only if the number of already occupied bins is increased by one, with the 

th thrown ball, otherwise 

. Note that the variables 

are not independent, so that we cannot apply the Chernoff bound. We have 

and 

. The following is satisfied 

, and 

for every 

. Let the value 

be the number of occupied bins after throwing all 

balls.

At this step, we specifically choose the number 10, to be a constant number of different rows that one row in A "meets". We do that wlog, since the proofs below apply, by simple changing the number 10 with any other positive constant, and taking 

To be that constant plus 2.

Now we show that 



Analysis of Step 2. We repeat 

times the procedure “Generation of the random matrix A.” Such constructed matrices concatenated to each other form the matrix 

of the dimension 

.

For the sake of brevity, let us call 

. Let us generate 

random matrices 

, where 

is a constant to be determined later in the analysis of the Step 4. That is, let us repeat 

times the procedure “Generation of the random matrix A.” We prove that each row in the matrix 

has 

“meetings” with the probability > 1/2. Again, w.o.l.g. let us consider the first row. Let 

be a random zero-one variable, indicating that the first row “meets” with some other row in the matrix 

, for 

. The variables 

are independent Bernoulli trials, since matrices 

are generated independently. By the Lemma 

it follows 

Again, w.o.l.g. let us consider the first row. Let 

be the number of these “meetings”. The following is satisfied

Applying the Chernoff Bound on 

it follows,

Taking 

we have the following bound on 



Analysis of Step 3. Here, we prove that each row in the matrix A “meets” with at least a constant number of different rows with the probability > 1/2. We specify this constant later. (Furthermore, the meetings are chosen independently with replacement.) Again, w.o.l.g. we consider the first row. We have that the number of “meetings” is 

with probability at least 

.
For \( T > 0.4K \log M > 12 \) it follows
\[
e T^T / M^{T-10} = \exp(1 + T \log T - (T - 10) \log M) < \exp(-2 \log M) = M^{-2},
\]
what completes the proof of the Step 3.

**Analysis of Step 4.** Taking \( K \) such that \( 0.1K > 1 \) and \( K \log M > 30 \), by the union bound, from the analysis of the previous steps, it follows that
\[
\Pr[\text{the first row has at least 10 different “meetings”}] > 1 - \left( 1 / M^{0.1K} + 1 / M^2 \right)
\]
\[
> 1 - \frac{1}{2(M+1)}
\]
\[
= 1 - \frac{1}{2m}.
\]
Finally, by the union bound applied over all of \( m \) rows, it follows that every row has at least 10 different “meetings” with probability \( > 1/2 \), that is,
\[
\Pr[\text{every row has at least 10 different “meetings”}] > 1/2.
\]

The matrix \( \hat{A} \) uniquely defines a random graph \( \hat{G} \), for which we show that has a minimum degree is at least 10, with the probability \( > 1/2 \). Here we define the undirected graph \( G = (V, E) \) that corresponds to the random matrix \( A \).

**Definition 10** For a graph \( G = (V, E) \), the set of nodes \( V = \{v_1, v_2, \ldots, v_m\} \) corresponds to the set of rows of the matrix \( A \) (i.e., to the set of devices). For \( 1 \leq i < j \leq m \) the edge \( (i, j) \in E \), if and only if there is a column \( t \) in the matrix \( A \) such that \( A_{i,t} \neq 0 \) and \( A_{j,t} \neq 0 \) (\( 1 \leq t \leq n \)).

Let \( \hat{G} = (V, \hat{E}) \) be the graph with the set of nodes \( V \) and the set of edges \( \hat{E} = \cup_{i}^{l} E^{(i)} \) obtained as the union of edges corresponding to the matrices \( A^{(i)} \), for \( i = 1, \ldots, l \). We have just proven that with probability \( > 1/2 \) every vertex in the graph \( \hat{G} \) has degree \( \geq 10 \). Finally, we note that we can repeat this entire process another \( \text{poly-log}(n) \) times to guarantee a success probability exponentially close to 1.

**Analysis of Step 5.** A fixed row in a matrix \( \hat{A} \) will meet with some other row, without collision, with probability \( > 1 - 0.4^{\Theta(log m)} = 1 - m^{-\Theta(1)} \). In case of interference, use standard back-off protocol analysis, with \( O(\log^2 m) \) multiplicative overhead.

**Analysis of Step 6.** We recall that if we have a random graph with node-degree at least a constant, then we can use the following theorem:

**Theorem 11 (Bollobas, de la Vega [7])** A random \( d \)-regular graph on \( m \) nodes has diameter \( (\log m + \log \log m) / \log(d - 1) + c \), for some small constant \( c < 10 \). This is the best possible since any \( d \)-regular graph has diameter at least \( \log m / \log(d - 1) \).

In the graph \( \hat{G} \) each node has a degree of at least \( d \) (we have specifically chosen, wlog, \( d \) to be 10). Furthermore, by our construction, the edges are independent. It follows that the \( \text{diam}(\hat{G}) \) is at most the diameter of a random \( d \)-regular subgraph. That is, \( \text{diam}(\hat{G}) \leq O(\log m) \) with high probability.
**Definition 12** We generally say that an $m \times n$ zero-one matrix $B = (b_{i,j})$ is associated to an undirected graph $G = (V,E)$ if and only if: the set of nodes is $V = [m]$, and between two nodes $i \neq j$ there is an edge $(i,j) \in E$ if and only if there is a column $t \in [n]$ in the matrix $B$ such that $b_{i,t} = b_{j,t} = 1$, and $b_{k,t} = 0$ for all $k \in [n] \setminus \{i,j\}$. We also say that the graph $G$ is associated to the matrix $B$.

Every processor $i \in [m]$ generates $O(n \log^2 m)$ random variables $C_i^1, C_i^2, \ldots, C_i^{O(n \log^2 m)}$, repeating $O(\log^2 m)$ times the procedure (Generation of the Random Matrix $A$) (see Definition 8). That is, $i$ randomly generates a string of the length $n$, with exactly $n^a = n^{(1-\beta)/2}$ ones, while the rest of the entries are zeros. That string is mapped onto $C_i^1, C_i^2, \ldots, C_i^n$. Then $i$, independent of the previous outcomes, repeats (Generation of the Random Matrix $A$) for the next $n$ variables $C_i^{n+1}, C_i^{n+2}, \ldots, C_i^{2n}$, and so on; totally repeating $O(\log^2 m)$ times the procedure (Generation of the Random Matrix $A$).

We define the zero-one matrix $\hat{A}$, such that $\hat{A}_{i,j} = C_i^j$, for $1 \leq i \leq m$, $1 \leq j \leq O(n \log^2 m)$; that is, the $i$th row corresponds to the coin outcomes of the $i$th processor. According to the way the random matrix $\hat{A}$ of the size $m \times O(n \log^2 m)$ is created, it can be divided into $O(n \log m)$ blocks of the matrices $\hat{A}$’s, each of the size $m \times O(n \log m)$. Finally, each of these $\hat{A}$’s matrices, can be subdivided into $O(n \log m)$ blocks of the matrices $A$’s, each of the size $m \times n$. For matrices $A, \hat{A}, \hat{A}$, let the associated graphs be $G, \hat{G}, \hat{G}$, respectively.

In the Analysis of Step 2, we have proven that a particular row in $\hat{A}$ has at least 10 meetings with probability $> 1/(2m)$. That is, every node in $\hat{G}$, has a degree of at least 10 with probability $> 1/2$. Then it follows that every row in $A$ has at least 10 meetings with probability close to $1 - m^{-\Theta(1)}$, i.e., every node in $G$ has a degree of at least 10 with probability $1 - m^{-\Theta(1)}$. Finally, let us define $CommGraph$.

**Definition 13** $CommGraph$ is obtained by concatenating, one by one in time of $D + 1$ identical copies of $\hat{A}$.

The nodes will be able to communicate over $CommGraph$ in time and synchronize their clocks’ drifts. The synchronization scheme and the proof is given by the following ‘Synchronization Algorithm.’ ■

**C Our Model Can Handle Different Clocks’ Speeds with Bounded Ratio**

Here are the technical details that explain why our model is realistic even if processors have somewhat different clock speeds. For $m$ processors, let their clock ticks be $\{\tau_1, \tau_2, \ldots, \tau_m\}$. Let $\tau_{\text{min}}, \tau_{\text{max}}$ be minimum, maximum of the set $\{\tau_1, \tau_2, \ldots, \tau_m\}$, respectively. The clock ticks are in general different, but the ratio $\tau_{\text{max}}/\tau_{\text{min}} \leq c$ bounded by some constant $c$, and each processor knows that upper bound $c$. Let $\tau_{\text{trans}}$ be the lower bound on the time necessary for the transmission, i.e., on the time necessary for communication and synchronization between two processors. It is also assumed that the lower bound on $\tau_{\text{trans}}$ is known to all processors. Now, knowing $c$ and $\tau_{\text{trans}}$, each processor $i$ counts $k_i$ clock ticks as a single time step $s_i$ such that $k_i$ is defined by $s_i = k_i \tau_i \geq 2 \tau_{\text{trans}}$. In other words, each processor enables the condition necessary for the communication, making its time step $s_i \geq 2 \tau_{\text{trans}}$. It follows that if two processors $i$ and $j$ overlap for a period of time $\geq \frac{1}{4} \min\{s_i, s_j\}$, then they can communicate.

For the purposes of analysis only, we assume that there is a global time axis, and time is mapped to the set of non-negative reals. Note that there is no a real global time, i.e., neither processors know nor need a real global time clock. Now we define a “time unit” on the global time axis to be $5s_{\text{max}}$, and we call $5s_{\text{max}}$ a single unit of the “global time.”
Fact 14 For every processor $i$ that works within a single global “time unit” there are at least three complete “time steps” that this processor’s radio is awake.

Proof For the processor $i$ that starts working at some time $v_i \in [0,s_i)$ within a single unit time $u$, the following is satisfied $v_i + 3 \leq 5$, since we had previously defined the global time unit to be $5s_{\text{max}}$. ■

Claim 15 If two processors $i, j$ work within the same global time unit, then they can communicate and can synchronize.

Proof Let us consider one time unit $u$ (with the length of $5s_{\text{max}}$). Let $v_i \in [0,s_i)$ be the time where the processor $i$ starts working within the unit $u$, and analogously let $v_j \in [0,s_j)$ be the time where the processor $j$ starts working within the unit $u$. We argue that if they both happen to be awake in the same time unit, there is an overlap of time $\geq \frac{1}{5} \min(s_i, s_j)$ when they both work and hence can communicate and can synchronize.

The processors $i, j$ certainly start working at times $v_i \in [0,s_i)$, $v_j \in [0,s_j)$, respectively, and then continue working over the period $u$. Let $s_i \geq s_j$ (the other case is symmetric). By the Claim 15 both processors work at least for three full-time steps. Then it follows that there exist time instances $a > b$ within the unit $u$ such that: $i$ works over periods $[a - \tau, a]$ and $[a, a + \tau]$; and $b \in [a - s_i, a]$ and $j$ works over periods $[b - s_j, b]$ and $[b, b + s_j]$. We purse the analysis as follows.

If $a - b \geq s_j$ then $j$ is entirely covered by $i$, otherwise let us consider the case $a - b < s_j$. Further, let us consider two time intervals $[a - s_i, a]$ and $[a, a + s_i]$ when $i$ works, as well as two intervals $[b - s_j, b]$ and $[b, b + s_j]$ when $j$ works. Let us consider the differences $b + s_j - a = t_j - (a - b)$ and $(a + s_i) - (b + s_j) = (s_i - s_j) + (a - b)$. If $(a - b) \leq s_j / 2$ then the first interval is $\geq s_j / 2$, otherwise the other interval is $\geq s_j / 2$, what completes the proof of the claim. ■

D Synchronization

Every processor $i$ has its own identification $ID_i$, which is a random number. Let the number of random bits, representing an $ID_i$, be much larger then $\log m$. Then it follows that all $ID_i$’s are different, with probability arbitrarily close to one. Further, we will use the terms node and processor interchangeably.

Every node knows $m$, so that it can compute $D \leq O(\log m)$. Also, every node $i$ keeps the following variables: $\text{Max}(i)$, a set of neighbors $\text{Neighbors}(i)$, and $\text{RootTime}(i)$. Besides $\text{RootTime}(i)$, a node keeps its own local-clock time, $\text{OwnTime}(i)$. Here we explain the variables.

At the beginning, the initialization for any node $i$ is the following. Every node assumes that it has the maximum $ID$, that is $\text{Max}(i) = ID_i$, at the beginning. The set of neighbors $\text{Neighbors}(i)$ is the set of neighbors $i$ in the $\text{CommGraph}$, $\hat{G}$, and the set of neighbors $\text{Neighbors}(i)$ is known to $i$. The $\text{RootTime}(i)$ is set to the node’s current time, $\text{RootTime}(i) = \text{OwnTime}(i)$. Every node $i$ performs the following Synchronization Algorithm, defined below.

Synchronization Algorithm

1. DO ONCE Subroutine $T$. 


**Subroutine T:** Transmitting only
SEND ITS $\text{Max}(i)$ AND $\text{RootTime}(i)$ TO THE SET OF ITS NEIGHBORS $\text{Neighbors}(i)$.

**End of Subroutine T**

2. REPEAT $D + 1$ TIMES Subroutine LT.

**Subroutine LT:** Listening and Transmitting
IF THE NODE $i$ HEARS, FROM A NODE $j$, AN $\text{Max}(j)$ HIGHER THEN $\text{Max}(i)$ THEN:

LT.1. SET $\text{Max}(i) := \text{Max}(j)$.
Explanation: Propagate $\text{Max}(ID)$.

LT.2. SET THE NEW $\text{RootTime}(i) := \text{RootTime}(j) + \Delta_{tr}$.
Explanation: $i$ must update the time, $\text{RootTime}(i)$, of the node with ‘the maximal ID,’ $\text{Max}(i)$. Furthermore, $\Delta_{tr}$ is a transmission time of the message $\text{RootTime}(j)$, sent from $j$ to $i$. Furthermore, we assume that $\Delta_{tr}$ is the fixed transmission time for any $i \neq j$, and the message $\text{RootTime}(j)$ is transmitted during that period of time.

LT.3. PROPAGATE THE NEW $\text{Max}(i)$ AND $\text{RootTime}(i)$ TO $\text{Neighbors}(i) \setminus \{j\}$.
Explanation: Let all other nodes, but $j$, know about the recent updates $\text{RootTime}(i)$ and $\text{Max}(i)$.

ELSE, IF THE NODE $i$ HEARS FROM A NODE $j$, AN $\text{ID} \leq \text{Max}(i)$ THEN DO NOTHING.

**End of Subroutine LT**

3. DO ONCE Subroutine C.

**Subroutine C:** Set the Clock
SET OWN CLOCK TO THE TIME OF THE NODE WITH THE MAXIMAL $\text{ID}$, I.E., $\text{OwnTime}(i) = \text{RootTime}(i)$.

**End of Subroutine C**

**END OF SYNCHRONIZATION ALGORITHM**

The communication over the graph $\text{CommGraph}$ is possible for every node, since $\text{CommGraph}$ is built as a concatenation in time of $D + 1$ identical copies of $G$.

Let us now prove the correctness of the algorithm. With high probability all $\text{ID}_i$’s are different. There is the unique node, let us call it $\text{root} = \max_{i \in [m]} \text{ID}_i$. We have to show that all nodes in the network, after the synchronization algorithm, have the same time, synchronized to the time of the node $\text{root}$. Let us consider any node $i$ in the network. Since the graph distance between the $\text{root}$ and $i$ is less than equal to the diameter $D$, it follows that the entire synchronization procedure will be done in $D + 1$ steps, and all nodes will know the time of the $\text{root}$. After Subroutine C (Set the Clock), all nodes will set their own clocks, $\text{OwnTime}(i) = \text{RootTime}(i)$, all being equal to $\text{OwnTime}(\text{root})$ with high probability. This proves the correctness of the Synchronization Algorithm.