On $\mathcal{P}W\pi$-regular rings

Raida D. Mahmood / Khedher J. Khider
Department of Mathematics / College of Computer Science and Mathematics
University of Mosul / Mosul - Iraq
raida.1961@uomosul.edu.iq / khedirjumaa@gmail.com

Abstract.

As a popularization of weakly $\pi$-regular rings, we tender the connotation of $\mathcal{P}W\pi$-regular rings, that is if for each $a \in \mathcal{J}(\mathfrak{R})$, there exist a natural number $n$ such that $a^n \in a^n\mathfrak{R}a^n$ $(a^n \in \mathfrak{R}a^n\mathfrak{R})$. In this treatise, numerous properties of this sort of rings are discussed, some important results are secured. Using the connotation of $\mathcal{P}W\pi$-regular rings. It is show that:

1. Let $\mathfrak{R}$ be a right $\mathcal{P}W\pi$-regular ring and $\mathfrak{M}$-rings with $a^n\mathfrak{R} = \mathfrak{R}a^n$ for every $a \in \mathcal{J}(\mathfrak{R})$ and for at least one of a natural number. Then $\mathcal{J}(\mathfrak{R}) = \mathfrak{N}(\mathfrak{R})$.

2. Let $\mathfrak{R}$ be a right $\mathcal{P}W\pi$-regular ring and $a\mathfrak{R} = \mathfrak{R}a$ for each $a \in \mathcal{J}(\mathfrak{R})$. Then $\mathfrak{R}$ is right $\mathcal{P}$-$\mathcal{I}$-ring.

3. Let $\mathfrak{R}$ be a ring with $r(a) \subseteq l(a)$, for each $a \in \mathcal{J}(\mathfrak{R})$. If any of the next conditions are hold, then $\mathfrak{R}$ is $\mathcal{P}W\pi$-regular rings:

i. Every maximal right ideal of $\mathfrak{R}$ is a right annihilator and right $\mathcal{J}\mathcal{P}\mathcal{P}$-ring.

ii. Any simple singular right $\mathfrak{R}$-module is $\mathcal{J}$-injective and $\mathfrak{R}$ is semi prime.

Keywords: $\mathcal{P}W\pi$-regular ring, $\mathcal{J}$-injective rings, $\mathcal{J}\mathcal{P}\mathcal{P}$-rings, $\mathcal{J}$-regular ring.

1. Introduction.

Over this treatise, $\mathfrak{R}$ refers to an associative ring with identity and each module is unitary $\mathfrak{R}$-module. We write $\mathcal{J}(\mathfrak{R})$, $\mathcal{Y}(\mathfrak{R})$, and $\mathfrak{N}(\mathfrak{R})$ for the Jacobson radical, the right singular ideal and the set of nilpotent elements of $\mathfrak{R}$, respectively. We use the contraction $l(a)$, $r(a)$ for the left, right annihilator of $a$ in $\mathfrak{R}$.

$\mathcal{J}$-injective rings were defined and discussed [5], [10]. A ring $\mathfrak{R}$ is define as a right $\mathcal{J}$-injective [10], whether each $a \in \mathcal{J}(\mathfrak{R})$, $l(r(a)) = \mathfrak{R}a$. Recall that $\mathfrak{R}$ is known as a right (left) weakly $\pi$-regular ($W\pi$-regular) [7], if every $a \in \mathfrak{R}$, there is a natural number $n$ such that $a^n \in a^n\mathfrak{R}a^n$ $(a^n \in \mathfrak{R}a^n\mathfrak{R})$. According to [4] $\mathfrak{R}$ is said to be $n$-weakly regular ring if for any $a \in \mathfrak{N}(\mathfrak{R})$, $a \in a\mathfrak{R}a$. A ring $\mathfrak{R}$ is said to be reduced if $\mathfrak{N}(\mathfrak{R}) = 0$ [3]. $\mathfrak{R}$ is said to be
right ( left ) $\Delta X M$ if for each $0 \neq a \in R$, there is a natural number $n$ such that $a^n \neq 0$, $r(a) = r(a^n)(\lambda(a) = \lambda(a^n))$, every reduced is $\Delta X M$ but convers is not true [8]. A ring is define is semiprime ring if and only if it contains no non-zero nilpotent ideal [2].

An element $a$ in the ring $R$ is said to be right (left) $P, \mathcal{I}$-element, if there is an idempotent element $e$ in $R$ such that $a = ae(a = a)$ and $r(a) = r(e)(\lambda(a) = \lambda(e))$. $R$ is known as a right (left) $P, \mathcal{I}$-ring, whether each element in $R$ is right (left) $P, \mathcal{I}$-element [1]. For example $Z_6$ is $P, \mathcal{I}$-ring [1].

In this treatise, we shall popularize the connotation of weakly $\pi$-regular rings to $\mathcal{P}W\pi$-regular, numerous properties of this sort of rings are discussed, little conditions under which $\mathcal{P}W\pi$-regular are $P, \mathcal{I}$-ring, $J$-regular, strongly regular rings will be given.

2. Popularized weakly $\pi$-regular rings.

**Definition 2.1**: $R$ is defined as a right (left) popularized weakly $\pi$-regular ($\mathcal{P}W\pi$-regular) if, for each $a \in \{R\}$, there exist a positive integer $n$ such that $a^n \in a^nRa^nR$ ($a^n \in R^nRa^nR$).

**Example**: Assume that $A$ is division ring. Then the 2 by 2 upper triangle ring $R = \begin{bmatrix} A & A \\ 0 & A \end{bmatrix}$ is $\mathcal{P}W\pi$-regular ring. Clearly $\{I_2(A)\} = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}^n \begin{bmatrix} A & A \\ 0 & A \end{bmatrix}^n = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}^n \begin{bmatrix} A & A \\ 0 & A \end{bmatrix}$.

**Remark**: Every weakly $\pi$-regular ring is $\mathcal{P}W\pi$-regular ring but the converse is not always true: Let $Z$ be the ring of integer. Then $J(Z) = 0$. Then $Z$ is $\mathcal{P}W\pi$-regular ring which is not $W$-$\pi$-regular ring.

**Proposition 2.2**: If $R$ is right $\mathcal{P}W\pi$-regular ring and $r(a) = 0$ for all $0 \neq a \in \{R\}$. Then $R = a^nR$.

**Proof**: Let $R$ be a right $\mathcal{P}W\pi$-regular. Then for all $0 \neq a \in \{R\}$, there is a natural number $n$ such that $a^nR = a^nRa^nR$, this implies $a^nR - Ra^nR = 0$ and hence $(R - Ra^nR) \in r(a^n) = 0$, hence it follows that $R - Ra^nR = 0$. Therefore $R = a^nR$.

**Proposition 2.3**: If $R$ is reduced ring. Then it is a right $\mathcal{P}W\pi$-regular iff $R$ is left $\mathcal{P}W\pi$-regular.

**Proof**: suppose that $R$ is right $\mathcal{P}W\pi$-regular. Then for each $a \in \{R\}$ there is a natural number $n$ and $b, c \in R$ such that $a^n = a^nba^nc$. Now $(a^n - ba^nc)^2 = 0$. Since $R$ is reduced, then $a^n - ba^nc = 0$. Therefore $a^n = ba^nc$. Hence $R$ is left $\mathcal{P}W\pi$-regular. The converse is similar.

Following [2], a ring $R$ is said to be $N|$ if $N(R) \subseteq J(R)$.
**Theorem 2.4**: Let $\mathcal{R}$ be a right $\mathcal{P}W\pi$-regular and $\mathcal{N}$-ring with $a^n\mathcal{R} = \mathcal{R}a^n$ for every $a \in \mathcal{N}(\mathcal{R})$ and for some a natural number $n$. Then $\mathcal{N}(\mathcal{R}) = \mathcal{N}(\mathcal{R})$.

**Proof**: Assume that $0 \neq a \in \mathcal{N}(\mathcal{R})$ and let $\mathcal{R}$ be a right $\mathcal{P}W\pi$-regular. Then there is a natural number $n$ and $b,c \in \mathcal{R}$ such that $a^n = a^nba^nc = a^nba^n$, $a^n\mathcal{R} = \mathcal{R}a^n$, $d \in \mathcal{R}$. Then $a^n = a^nba^n$, $(b = bd)$ implies that $a^n(1 - ha^n) = 0$. Since $a \in \mathcal{N}(\mathcal{R})$ then $a^n \in \mathcal{N}(\mathcal{R})$ gives $(1 - ha^n)$ is invertible, so $(1 - ha^n)u = 1$ for some $u \in \mathcal{R}$, implies that $(a^n - a^nha^n)u = a^n = 0$. Thus $\mathcal{R} \in \mathcal{N}(\mathcal{R})$, and hence $\mathcal{N}(\mathcal{R}) \subseteq \mathcal{N}(\mathcal{R})$. But $\mathcal{R}$ is $\mathcal{N}$-ring, therefore $\mathcal{N}(\mathcal{R}) = \mathcal{N}(\mathcal{R})$.

**Theorem 2.5**: If $\mathcal{R}$ is $\mathcal{SX}\mathcal{M}$ ring and right $\mathcal{P}W\pi$-regular, then $\mathcal{N}(\mathcal{R}) \cap \mathcal{N}(\mathcal{R}) = 0$.

**Proof**: Let $\mathcal{N}(\mathcal{R}) \cap \mathcal{N}(\mathcal{R})$ not equal to zero. So there exist $0 \neq a \in \mathcal{N}(\mathcal{R}) \cap \mathcal{N}(\mathcal{R})$. Since $\mathcal{R}$ is right $\mathcal{P}W\pi$-regular, there is a natural number $n$ and $b,c \in \mathcal{R}$ such that $a^n = a^nba^nc$. Thus $a^n(1 - ba^c) = 0$, this implies $(1 - ba^c) \in \mathcal{R}(a^n) = \mathcal{R}(a)$, implies $a(1 - ba^c) = 0$. Since $a \in \mathcal{N}(\mathcal{R})$ then $a^n \in \mathcal{N}(\mathcal{R})$. So $a^n\mathcal{R} \in \mathcal{N}(\mathcal{R})$, gives $(1 - ba^c)$ is invertible, so $(1 - ba^c)u = 1$ for some $u \in \mathcal{R}$, implies that $(a - aba^nc)u = a = 0$. This is contradiction. Hence $\mathcal{N}(\mathcal{R}) \cap \mathcal{N}(\mathcal{R}) = 0$.

**Proposition 2.6**: Let $\mathcal{R}$ be reduced ring. Then $\mathcal{R}$ is right $\mathcal{P}W\pi$-regular iff $\mathcal{R}/\mathcal{R}(a)$ is right $\mathcal{P}W\pi$-regular.

**Proof**: Suppose that $\mathcal{R}/\mathcal{R}(a)$ is right $\mathcal{P}W\pi$-regular, then for every $a \in \mathcal{N}(\mathcal{R})$ there is a natural number $n$ and $b,c \in \mathcal{R}$ such that $(a + f(a))^n = (a + f(a))(b + f(a))(c + f(a))$, implies that $a^n + f(a) = a^nba^nc + f(a)$. Therefore $a^n - a^nba^nc \in f(a)$ and so $aa^n(1 - ba^c) = 0$, implies that $(1 - ba^c) \in f(a^{n+1}) = f(\mathcal{R})$ ($\mathcal{R}$ is reduced). Therefore $a^n(1 - ba^c) = 0$, which yields $a^n = a^nba^nc$. Hence $\mathcal{R}$ is right $\mathcal{P}W\pi$-regular. The conversely is clear.

3. **The relevance among right $\mathcal{P}W\pi$-regular and other rings**

Following [10], $\mathcal{R}$ is called right $\mathcal{J}$-regular ring ($\mathcal{J}$-regular) whether for each $a \in \mathcal{J}(\mathcal{R}), a \in a\mathcal{R}a$.

**Theorem 3.1**: Assume that $\mathcal{R}$ is right $\mathcal{P}W\pi$-regular and $a^n\mathcal{R} = \mathcal{R}a^n$ for each $a \in \mathcal{J}(\mathcal{R})$ and a natural number $n$. Then $\mathcal{R}$ is $\mathcal{J}$-regular.

**Proof**: Assume that $a \notin \mathcal{J}(\mathcal{R})$, and let $\mathcal{R}$ is right $\mathcal{P}W\pi$-regular, then there is a natural number $n$ such that $a^n\mathcal{R} = a^n\mathcal{R}a^n\mathcal{R}$, since $a^n\mathcal{R} = \mathcal{R}a^n$, then $a \in \mathcal{R}a^n = a^n\mathcal{R}a^n = a\mathcal{R}a$, implies that $a \in a\mathcal{R}a$ for every $a \in \mathcal{J}(\mathcal{R})$. Hence $\mathcal{R}$ is $\mathcal{J}$-regular.

**Proposition 3.2**: Suppose that $\mathcal{R}$ is $\mathcal{N}$-ring with $a^n\mathcal{R} = \mathcal{R}a^n$ for each $a \in \mathcal{J}(\mathcal{R})$ and a natural number $n$. Then $\mathcal{R}$ is $\mathcal{N}$-weakly regular ring iff $\mathcal{R}$ is $\mathcal{P}W\pi$-regular.

**Proof**: $\mathcal{J}(\mathcal{R}) = \mathcal{N}(\mathcal{R})$ (Theorem 2.4). So $\mathcal{R}$ is $\mathcal{N}$-weakly regular iff $\mathcal{R}$ is $\mathcal{P}W\pi$-regular.

**Theorem 3.3**: Suppose that $\mathcal{R}$ is right $\mathcal{P}W\pi$-regular and $a\mathcal{R} = \mathcal{R}a$ for each $a \in \mathcal{J}(\mathcal{R})$. Then $\mathcal{R}$ is right $\mathcal{P}, \mathcal{I}$-ring.
Proof: Since $\mathcal{R}$ is right $\mathcal{P}\mathcal{W}\pi$-regular ring. Then for any $a \in \mathcal{J}(\mathcal{R})$, there is a natural number $n$ and $b, d \in \mathcal{R}$ such that $a^n = a^n ba^n d = a^n bca^n = a^n w a^n$, when $w = b c$, if we take $f = wa^n$, then $f^2 = wa^n wa^n = wa^n = f$, then $f$ is idempotent element and $a^n = a^n f$. Now let $b \in \mathfrak{r}(f)$, implies $fb = 0$, and $wa^n b = 0$, implies that $a^n wa^n b = 0$, and hence $a^n b = 0$. Therefore $b \in \mathfrak{r}(a^n)$ and we get $\mathfrak{r}(f) \subseteq \mathfrak{r}(a^n)$ $(1)$. Now let $a \in \mathfrak{r}(a^n)$, implies $a^n z = 0$ and $wa^n z = 0$, implies that $f z = 0$. Therefore $z \in \mathfrak{r}(f)$ and we get $\mathfrak{r}(a^n) \subseteq \mathfrak{r}(f)$ $(2)$. From $(1)$ and $(2)$ we get $\mathfrak{r}(a^n) = \mathfrak{r}(f)$. Hence $\mathcal{R}$ is $\mathcal{P}\mathcal{W}\pi$-ring.

Following [10] , $\mathcal{R}$ is said to be right $\mathcal{J}\mathcal{P}\mathcal{P}$-ring. If $a\mathcal{R}$ is projective for each $\in \mathcal{J}(\mathcal{R})$. In [10] we give the following lemma:

Lemma 3.4: Let $\mathcal{R}$ be a ring. Then it is right $\mathcal{J}\mathcal{P}\mathcal{P}$-ring iff $\mathfrak{r}(a) = \mathfrak{e}\mathcal{R}$, $\mathfrak{e}$ is some idempotent element in $\mathcal{R}, a \in \mathcal{J}(\mathcal{R})$.

Proposition 3.5: If $\mathcal{R}$ is right $\mathcal{J}\mathcal{P}\mathcal{P}$-ring, then $Y(\mathcal{R}) = 0$.

Proof: Assume that $0 \neq a \in Y(\mathcal{R})$, $a^2 = 0$, it is clear that $a\mathcal{R}$ is projective, then $\mathfrak{r}(a)$ must be direct summand of $\mathcal{R}$. But $\in Y(\mathcal{R})$, $\mathfrak{r}(a)$ it then essential in $\mathcal{R}$, but this is contradiction. Therefore $Y(\mathcal{R}) = 0$.

Lemma 3.6: Assume that $\mathcal{R}$ is right $\mathcal{J}\mathcal{P}\mathcal{P}$-ring, $\mathfrak{r}(a) \subseteq \mathfrak{l}(a)$, for each $a \in \mathcal{J}(\mathcal{R})$. Then $\mathcal{R}$ is reduced.

Proof: Trivial.

Theorem 3.7: Let $\mathcal{R}$ is right $\mathcal{J}\mathcal{P}\mathcal{P}$-ring, $\mathfrak{r}(a) \subseteq \mathfrak{l}(a)$, for each $a \in \mathcal{J}(\mathcal{R})$, and any right maximal ideal of $\mathcal{R}$ is a right annihilator. Then $\mathcal{R}$ is $\mathcal{P}\mathcal{W}\pi$-regular.

Proof: Suppose that $a \in \mathcal{J}(\mathcal{R})$, we must show that $\mathcal{R}a^n + \mathfrak{r}(a^n) = \mathcal{R}$. If it is not hold, then there is a right maximal ideal $\mathcal{N}$ containing $\mathcal{R}a^n + \mathfrak{r}(a^n)$. If $= \mathfrak{r}(b)$, for some $0 \neq b \in \mathcal{J}(\mathcal{R})$, we have $b \in \mathcal{L}(\mathcal{R}a^n + \mathfrak{r}(a^n)) \subseteq \mathcal{L}(a^n) = \mathfrak{r}(a^n) \subseteq \mathcal{N} = \mathfrak{r}(b)$, which implies $b \in \mathfrak{r}(b)$. Then $b^2 = 0, b = 0$, a contradiction. Therefore $\mathcal{R}a^n + \mathfrak{r}(a^n) = \mathcal{R}$. In particular $xa^n y + d = 1$, with $x, y \in \mathcal{R}$, and $d \in \mathfrak{r}(a^n)$, hence $a^n xa^n y = a^n$ which proves $\mathcal{R}$ is right $\mathcal{P}\mathcal{W}\pi$-regular.

Following [3], $\mathcal{R}$ is called strongly regular ring, if for each $a \in \mathcal{R}$, there is $b \in \mathcal{R}, a = a^2 b$.

Theorem 3.8: Assume that $\mathcal{R}$ is right $\mathcal{P}\mathcal{W}\pi$-regular, $\mathcal{J}(\mathcal{R})$ is reduced and $a^n \mathcal{R} = \mathcal{R}a$, for each $a \in \mathcal{J}(\mathcal{R})$. Then $\mathcal{J}(\mathcal{R})$ is strongly regular ideal.

Proof: Assume that $\mathcal{J}(\mathcal{R})$ be a reduced of $\mathcal{R}$ and let $a \in \mathcal{J}(\mathcal{R})$. Since $\mathcal{R}$ is right $\mathcal{P}\mathcal{W}\pi$-regular, there is a natural number $n$ and $c, b \in \mathcal{R}$ such that $a^n = a^n ba^nc$, which implies $a^n(1 - ba^nc) = 0$ and $(1 - ba^nc) \in \mathfrak{r}(a^n) = \mathfrak{r}(a)$, gives $a = aba^nc = aha$ ( $a^n \mathcal{R} = \mathcal{R}a$ ). Consider $(a - a^2 h)^2 = a^2 - a^2 h - a^2 ha + a^2 ha^2 h = a^2 - a^2 h - a^2 h - a(aha) + a(aha)ah = a^2 - a^2 h - a^2 + a^2 h = 0$. But $\mathcal{J}(\mathcal{R})$ is reduced, then $a - a^2 h = 0$, implies that $a = a^2 h$. Hence $\mathcal{J}(\mathcal{R})$ is strongly regular ideal.
Theorem 3.9: Assume that \( R \) is semi prime and any singular simple right \( R \)-module is \( J \)-injective with \( r(a) \subseteq l(a) \), for each \( a \in l(R) \). Then \( R \) is right \( PW\pi \)-regular.

Proof: Assume that \( Ra^nR + r(a^n) = R \), for every \( a \in l(R) \). If \( a^nR + r(a^n) \neq R \), then there is a right maximal ideal \( N \) of \( R \) such that \( Ra^nR + r(a^n) \subseteq N \) and if \( N \) is not essential of \( R \). Then \( N \) is a direct summand. And then there exists \( 0 \neq \varphi = \varphi^2 \in R \) such that \( = r(\varphi) \). Now, \( Ra^n\varphi \subseteq Ra^nR \subseteq N = r(\varphi) \), implies that \( \varphi Ra^n\varphi = 0 \) and \( (a^n\varphi R)^2 = a^n\varphi Ra^n\varphi = 0 \). So \( a^n\varphi R = 0 \) (\( R \) is semi prime ) and \( a^n\varphi = 0 \), \( \varphi \in r(a^n) \subseteq N = r(\varphi) \), and \( \varphi^2 = 0 \), a contradiction. So \( N \) is maximal essential right ideal of \( R \). Since \( R/N \) is \( J \)-injective, then for any right \( R \)-homomorphism, \( f : a^nR \rightarrow R/N \), known as \( f(a^nz) = z + N \), for every \( z \in R \). Note \( f \) is well define and it will be extended from \( R \) into \( R/N \). So \( 1 + N = f(a^n) = ca^n + N \), where \( c \in R \), and \( (1 - ca^n) \in N \). Since \( a^n \in Ra^nR \subseteq N \). So that \( 1 \in N \), and this is contradiction, hence \( Ra^nR + r(a^n) = R \). In specific \( ax^n + a^n = 1 \), \( a^nxa^n + a^n = a^n \). Therefore \( nxa^n = a^n \), and \( R \) is right \( PW\pi \)-regular.

From Theorem 3.9 and Lemma 3.6 we get:

Corollary 3.10: If every simple singular right \( R \)-module is \( J \)-injective and \( R \) is right \( JJP \)-ring, \( r(a) \subseteq l(a) \), for each \( a \in l(R) \). Then \( R \) is \( PW\pi \)-regular.

Acknowledgment.
The authors are very grateful to the university of Mosul College of Computer Science and Mathematics for their provided facilities, which helped to improve the quality of this work.

References
[1] A. Ab. Bilal, On periodic rings, Ms. C. Uni. Thesis, Mosul university (2002).
[2] Ch. I. Lee and S. Y. Park, When nilpotents are contained in Jacobson radical, J. Korean Math., Soc. 55, No. 5 (2018), P. P. 1193–1205.
[3] J. Luh, A note on strongly regular rings, Proc. J. Acad., Vol. 40 (1964), P. P. 74–75.
[4] R. D. Mahmood and M. T. Yunis, On n-weakly regular rings, Raf. J. Com. And Math., Vol. 9 No. 2 (2012), P. P. 53–59.
[5] R. D. Mahmood, On almost J-injectivity and J-regularity of rings, Tikrit J. Of Pure Sci., No. 18 (2012), P. P. 206–210.
[6] S. B. Nam, N. K. Kim and J. Y. Kim, On simple GP-injective modules, Comm. Algebra, Vol. 23 No. 14 (1995), P. P. 5437–5444.
[7] V. S. Ramamurhi, Weakly regular rings, Cana. Math. Bull., Vol. 16 No. 3 (1973), P. P. 317–321.
[8] J. C. Wei, On simple singular YJ-injective modules, Son. Asian Bull. Of Math., Vol. 31 (2007), P. P. 1–10.
[9] R. Yue Chi Ming, On Von Neumann regular rings, Proc. Edinburgh Math., Soc. 19, P. P. 89–91.
[10] Z. Yue and Z. Shujuam, On JPP rings, JPF rings and J-regular rings, Inte. Math. Four., Vol. 6 No. 34 (2011), P. P. 1691–1696.