Abstract: This paper presents the vibration of a transversely cracked rotor supported by anisotropic journal bearings, where the speed-dependent characteristic of bearing is considered. A 3D finite element model and the contact-based approach are employed for the shaft and crack. The governing differential equations of the whole cracked rotor-bearing system were obtained by synthesizing the equations of the cracked shaft, the breathing crack and the journal bearings. In order to solve the computational difficulties caused by the high dimensions of model, the free-interface complex component mode synthesis method (CMS) is employed to reduce the order of the model. On this basis, the eigenvalue and the steady-state forced response of the cracked rotor-bearing system are obtained by the Hill’s method. Finally, the effects of the anisotropic and speed-dependent characteristics of bearings on the vibration of the system are studied. Numerical results show that both the two characteristics can significantly affect the response of the system. The anisotropy in the bearing leads to the split of resonant peaks and influence the amplitudes of the peaks. The speed-dependent characteristic mainly affects the responses at the speeds close to the resonant regions, because the parametric excitation effect of the resonance region is greater than other speeds.

Keywords: cracked rotor; anisotropic journal bearing; speed-dependent characteristic; model order reduction

1. Introduction

Crack damages are common faults in large structures due to the long time of corrosion, creep and high cyclic stresses. As the crack damage is hard to detect in the early stage, it has become severe threats to the operation safety of equipment and attracts the attentions of many researchers. In order to develop effective techniques for detecting the crack damages, a large number of investigations have been published to analyze the nonlinear dynamics of cracked rotors and to gain insights into the dynamic response characteristics [1–3].

A reliable diagnosis method for cracks requires comprehensive insights into the dynamic behaviors of cracked rotor-bearing systems, and an accurate dynamic model is needed. Generally, journal bearings possess the property of anisotropy, which makes the dynamics of rotor-bearing systems much more complex [4]. Besides, the speed-dependent characteristic of bearings [5], which denotes that the stiffness and damping coefficients vary with the rotating speed, is also a common phenomenon in rotor-bearing systems. Current research is mostly concentrated on simply supported cracked rotors or supported by isotropic bearings and the anisotropic rotor-bearing systems without cracks. The
speed-dependent characteristic of journal bearings is also seldom considered in previous studies. With respect to the studies on cracked rotors, the linear fracture mechanics and the strain energy release rate approach were often used to calculate the variation of local stiffness due to cracks [6]. Recently, many crack models, which could provide more reasonable descriptions for the breathing behaviors, were proposed [7,8]. Georgantzinos et al. [9] investigated the time-variant flexibility of cracked shaft due to bending and torsion loads with a 3D finite element model and a nonlinear contact procedure. Lately, they studied the local flexibility of cracked shafts under harmonic type of loadings [10]. Kulesza and Sawicki [11] developed a model for cracked rotating shafts using the rigid finite element method, in which the crack was modeled as two sections connected by several spring-damping elements of variable stiffness. Liong et al. [12] addressed the breathing mechanism of a transverse crack and evaluated the local stiffness loss using the cohesive zone model. Spagnol et al. [13] used non-symmetric bending principles to develop a new crack breathing model. Fu et al. [14,15] investigated the effects of interval uncertain parameters on the dynamic behaviors of a rotor system with crack in the shaft.

Although these investigations provide effective ways for simulating the dynamic behaviors of cracked rotors, the bearings are mostly ignored or simplified as isotropic ones. As the periodically time-variant characteristic caused by the anisotropic bearings can lead to the parametric excitation phenomenon [16,17], it should be appropriately dealt with in order to study the dynamic responses of realistic cracked rotor-bearing systems.

Regarding the anisotropic rotor-bearing systems, a large number of investigations have also been published in the past few decades. In order to quantitatively simulate the vibration of such systems, several accurate models have been developed in recent years. Lazarus et al. [18] addressed a 3D finite element method for the stability analysis of asymmetric rotating rotors basing on the modal theory. Ma et al. [19] suggested a 3D finite element procedure for studying the frequency characteristics of anisotropic rotor-bearing systems. Wang et al. [20] presented a 3D finite element-based model order reduction method for parametric resonance and whirling analysis of anisotropic rotor-bearing systems. Zuo et al. [21] studied the stochastic boundaries of unstable regions of an uncertain asymmetric rotors using a 3D finite element rotor model. Bharti et al. [22] investigated the Sommerfeld effect at forward and backward critical speeds in a rigid rotor shaft system with anisotropic supports. In these investigations, the crack damages in shafts were not taken into account. Although a crack may also result to the asymmetry in shaft, it is quite different from the other kind of structural asymmetry, as the crack involves the breathing behaviors. Besides, few studies are reported to deal with the speed-dependent characteristic. The vibration of cracked rotors supported by bearings with constant coefficients may not accurately reflect that of realistic cases. Thus, comprehensive investigations into the effects of the speed-dependent characteristic are needed to gain deep insights into the dynamic behaviors of cracked rotors.

In this paper, the dynamic response of a transversely cracked rotor supported by anisotropic journal bearings is studied, where the speed-dependent characteristic of bearings has been considered. First, the dynamic models of the cracked shaft, the journal bearings and the crack are established. Model order reduction is conducted using the free-interface complex CMS method. Then, the Hill’s method is employed to compute the eigenvalue and the forced response of the system. Finally, the model is used to study the dynamic characteristics of the cracked rotor-bearing system. This paper is organized as follows. In Section 2, the dynamic modeling procedures are presented. Section 3 contains the formulations of model order reduction. Section 4 presents the numerical results. Discussions are shown in Section 5, and conclusions are given in Section 6.

2. Dynamic Modeling of Cracked Anisotropic Rotor-Bearing System

2.1. Modeling of Cracked Shaft

A realistic crack in rotor is usually gapless. A gapless crack is easy to be modeled in the methods based on the theory of fracture mechanics, which is described by the variation of stiffness. For a 3D
finite element model, it is relatively difficult to make the crack gapless while keeping the nodes on the two crack surfaces independent. With respect to this issue, a two-step approach is developed herein, as is shown by the schematic diagram in Figure 1. Although a transverse crack is illustrated, the proposed method can be used for cracks with arbitrary shapes by adjusting the retaining nodes on the cracked surface in the pretreatment processes.

![Figure 1. The schematic diagram for the modeling of a gapless crack.](image)

In the first step, the cracked rotor is split into two parts according to the position of crack, and the common section is then divided into the connecting surface and the crack surface, respectively, where the shape and size of the crack depend on the problem of interest. As the name implies, the connecting surface is used to connect the two parts, and the crack is defined on the cracked surface. The connecting and the crack surfaces are discretized first and subsequently the left and the right parts. As the shaft and bearings are lack of anisotropy, the governing equations of motion of the system will contain periodically time-variant terms in both the fixed and rotating frames. In order to make it easier in further analysis, the rotating frame is chosen so that the periodical coefficients exist only at the degrees of freedom (DOFs) of the bearing nodes. Then, the governing equations of motion of the two rotating parts in the rotating frame are obtained independently, so that the DOFs of nodes on the connecting and the crack surfaces exist in both the two equations, which can be expressed as

\[
M^{(p)}\ddot{u}^{(p)} + \boldsymbol{\Omega}^{(p)} \mathbf{C}_{\text{cor}}^{(p)} \mathbf{u}^{(p)} + \left( \mathbf{K}^{(p)}_{\text{crk}} - \Omega^2 \mathbf{K}^{(p)}_{\text{d}} \right) \mathbf{u}^{(p)} = \mathbf{f}_{\text{ext}}^{(p)} + \mathbf{f}_{\text{con}}^{(p)} + \mathbf{f}_{\text{crk}}^{(p)}
\]  

where superscript “\(p\)” is a label representing different parts; \(M^{(p)}\) and \(K^{(p)}\) are the mass and elastic stiffness matrices; \(\mathbf{u}^{(p)}\) denotes the vector of nodal displacement; \(\Omega\) is the rotating angular velocity; \(\boldsymbol{\Omega}^{(p)} \mathbf{C}_{\text{cor}}^{(p)}\) is the Coriolis matrix; \(\Omega^2 \mathbf{K}^{(p)}_{\text{d}}\) is the spin softening matrix due to the rotation of rotor; \(\mathbf{f}_{\text{ext}}^{(p)}, \mathbf{f}_{\text{con}}^{(p)}\) and \(\mathbf{f}_{\text{crk}}^{(p)}\) represent the force vectors with respect to the DOFs of nodes on the connecting surface, the crack surface and the external excitation, respectively.

In the second step, the two independent governing equations are synthesized to form that of the whole cracked shaft. The vectors \(\mathbf{u}^{(p)}\) are firstly reorganized as

\[
\mathbf{u}^{(p)} = \begin{cases} 
\mathbf{u}_{\text{crk}}^{(p)}, \\
\mathbf{u}_{\text{con}}^{(p)}, \\
\mathbf{u}_{\text{oth}}^{(p)}
\end{cases}
\]  

where the subscripts “\(\text{crk}\)”, “\(\text{con}\)” and “\(\text{oth}\)” denote the nodal displacement vectors regarding the crack surface, the connecting surface and the other nodes, respectively.

Then, the two parts are connected by applying the displacement compatibility conditions as

\[
\mathbf{u}_{\text{con}}^{(l)} = \mathbf{u}_{\text{con}}^{(r)}
\]  

\(1\)

\(2\)

\(3\)
The compatibility conditions in Equation (3) will lead to a transformation, by which the governing equations of the whole cracked shaft can be obtained

$$
M^{(s)} \ddot{u}^{(s)} + \Omega C^{(s)}_{cor} \dot{u}^{(s)} + \left( K^{(s)}_{y} - \Omega^{2} K^{(s)}_{d} \right) u^{(s)} = f^{(s)}_{ext} + f^{(s)}_{crk}
$$

(4)

where the superscript “$s$” denotes the objects regarding the shaft. As the reaction forces on the connecting surfaces between the two parts are equal but opposite in sign, the force vector $f^{(s)}_{con}$ will no longer exist in Equation (4). Whereas, the nodal displacements of the two crack surfaces are both retained, so the force vector $f^{(s)}_{crk}$ are retained and will be used to apply the penalty contact force in the following part. Although Figure 1 illustrates a transverse crack, this approach can be used for other kinds of cracks, such as the slant and elliptical in shaft, as well as the gapless cracks in other structures.

The breathing behavior is a unique characteristic of cracks and can significantly affect the vibration of cracked structures. Herein, the contact-based approach [23], which is developed for the intermittent contact problem, is employed, combined with the 3D finite element model of shaft. For the sake of generality, the model of breathing crack is developed in the 3D space. First, the two independent crack surfaces are referred to as the contact and the target surfaces, respectively, and contact pairs are defined on the two surfaces. For the convenience of applying penalty forces, local coordinate systems are defined at the contact pairs. Then, the DOFs of the crack nodes and the penalty forces are transformed into the local coordinate systems. In this paper, the crack is assumed to be frictionless, so the penalty forces exist only at the normal direction to penalize the normal penetration.

2.2. Modeling of Anisotropic Speed-Dependent Journal Bearings

The eight-coefficient model [4] is commonly used for the modeling of journal bearing, which considers the cross-coupling interaction between vertical and horizontal motions and can accurately describe the dynamic characteristic of bearings. Therefore, it is used herein to establish the dynamic model for anisotropic journal bearings, where the speed-dependent characteristic is considered. Figure 2 shows the diagram for the model, which contains four stiffness parameters and four damping parameters. As these coefficients are all defined in the fixed reference frame, they should be transformed into the rotating frame. First, the reaction force of the journal bearings in the fixed frame can be expressed as

$$
f_{bf} = \begin{pmatrix}
fx \\
fy
\end{pmatrix} = \begin{pmatrix}
k_{xx}(\Omega) & k_{xy}(\Omega) \\
k_{yx}(\Omega) & k_{yy}(\Omega)
\end{pmatrix} \begin{pmatrix}
u_x \\
u_y
\end{pmatrix} + \begin{pmatrix}
c_{xx}(\Omega) & c_{xy}(\Omega) \\
c_{yx}(\Omega) & c_{yy}(\Omega)
\end{pmatrix} \begin{pmatrix}
u_x \\
u_y
\end{pmatrix} = K_{bf} \mathbf{u}_{bf} + C_{bf} \dot{\mathbf{u}}_{bf}
$$

(5)

where $k_{xx}, k_{xy}, k_{yx}, k_{yy}, c_{xx}, c_{xy}, c_{yx}$ and $c_{yy}$ are the stiffness and damping coefficients; $u_x$ and $u_y$ represent the transverse displacements. When the speed-dependent characteristic is considered, the eight coefficients are depending on the rotating speed.
where \( R \) denotes the rotation transformation matrix and \( R^{-1} = R^T \); \( u_{br} \) and \( f_{br} \) represent the displacement and reaction force vectors in the rotating frame, respectively. By substituting Equation (6) into Equation (5) and reasonable simplification, it can be obtained that

\[
\mathbf{f}_{br} = \left[ R^T \mathbf{C}_{bf} \frac{d}{dt} (\mathbf{R}) + R^T \mathbf{K}_{bf} \mathbf{T}_R \right] \mathbf{u}_{br} + \left( R^T \mathbf{C}_{bf} \mathbf{R} \right) \mathbf{u}_{br} = \mathbf{K}_{br} \mathbf{u}_{br} + \mathbf{C}_{br} \ddot{\mathbf{u}}_{br} \tag{7}
\]

As the matrix \( \mathbf{R} \) contains functions of time and the obtained terms cannot be completely turned time-independent, the matrices \( \mathbf{K}_{br} \) and \( \mathbf{C}_{br} \) will contain periodical terms of the sine or cosine functions of \( 2 \Omega t \). By expressing the trigonometric functions as complex exponential functions, the matrices \( \mathbf{K}_{br} \) and \( \mathbf{C}_{br} \) can be expressed as the periodically time-variant form

\[
\mathbf{K}_{br} = \mathbf{K}_{br}(\Omega) + \mathbf{K}_{bn}(\Omega) e^{-2j\Omega t} + \mathbf{K}_{bp}(\Omega) e^{2j\Omega t} \]
\[
\mathbf{C}_{br} = \mathbf{C}_{br}(\Omega) + \mathbf{C}_{bn}(\Omega) e^{-2j\Omega t} + \mathbf{C}_{bp}(\Omega) e^{2j\Omega t} \tag{8}
\]

where \( \mathbf{K}_{br}, \mathbf{K}_{bp}, \mathbf{K}_{bn}, \mathbf{C}_{br}, \mathbf{C}_{bp}, \mathbf{C}_{bn} \) and \( \mathbf{C}_{bn} \) represent the time-independent coefficient matrices.

An isotropic bearing requires the two principle stiffness coefficients, \( k_{xx} \) and \( k_{yy} \), to be identical, as well as the damping coefficients \( c_{xx} \) and \( c_{yy} \). Meanwhile, the two cross-coupling stiffness and damping coefficients need to be opposite in sign. If these two conditions are satisfied, the matrices \( \mathbf{K}_{br} \) and \( \mathbf{C}_{br} \) will no longer contain periodically time-variant terms. However, it is often too strict for realistic bearings to satisfy these constraints, so anisotropic bearings are more common cases. Then, the governing differential equations of the whole cracked rotor-bearing system can be obtained by synthesizing the equations of the cracked shaft, the breathing crack and the journal bearings, which can be expressed as

\[
\mathbf{M} \ddot{\mathbf{u}} + (\Omega \mathbf{C}_{cor} + \mathbf{C}_{b}) \dot{\mathbf{u}} + \left( \mathbf{K}_b - \Omega^2 \mathbf{K}_{d} + \mathbf{K}_b \right) \mathbf{u} = \mathbf{f}_{ext} + \mathbf{f}_{crk}(\mathbf{u}) \tag{9}
\]

where \( \mathbf{K}_b \) and \( \mathbf{C}_b \) denote the periodically time-variant matrices regarding the anisotropic journal bearing, which contain the contributions of several bearings.

**Figure 2.** The diagram for the eight-coefficient model of journal bearings.

Then, the displacements and bearing reaction forces are transformed into the rotating frame

\[
\mathbf{u}_{bf} = \mathbf{R} \mathbf{u}_{br}, \quad \mathbf{f}_{bf} = \mathbf{R}^T \mathbf{f}_{br}, \quad \mathbf{R} = \begin{bmatrix} \cos \Omega t & -\sin \Omega t \\ \sin \Omega t & \cos \Omega t \end{bmatrix} \tag{6}
\]
3. Eigenvalue and Forced Response Analysis

3.1. Model Order Reduction

Directly employing the large order model in conducting nonlinear vibration analysis usually requires huge time in computation, as a large number of iteration steps are needed to reach the converged solutions. In order to solve this issue, model order reduction is conducted before further analysis. In this paper, the free-interface complex component mode synthesis method is employed. First, Equation (9) is converted into the first-order state space form

\[
\begin{pmatrix}
A + A_n e^{-j2\Omega t} + A_p e^{j2\Omega t} \\
A_n + A_p e^{-j2\Omega t} + A_p e^{j2\Omega t}
\end{pmatrix}\dot{X} + \begin{pmatrix}
B + B_n e^{-j2\Omega t} + B_p e^{j2\Omega t} \\
B_n + B_p e^{-j2\Omega t} + B_p e^{j2\Omega t}
\end{pmatrix}X = F_{\text{ext}} + F_{\text{crk}}(X)
\]

where \(A\) and \(B\) are referred to as the state mass and stiffness matrices, respectively;

\[
A = \begin{bmatrix}
0 & M \\
-\Omega C_{\text{cor}} + C_{bc}
\end{bmatrix},
B = \begin{bmatrix}
0 & 0 \\
0 & K_{bc}
\end{bmatrix}.
\]

(11)

\(A_n, A_p, B_n\) and \(B_p\) denotes the matrices regarding the anisotropic journal bearings;

\[
A_n = \begin{bmatrix}
0 & 0 \\
0 & C_{bn}
\end{bmatrix},
A_p = \begin{bmatrix}
0 & 0 \\
0 & C_{bp}
\end{bmatrix},
B_n = \begin{bmatrix}
0 & 0 \\
0 & K_{bn}
\end{bmatrix},
B_p = \begin{bmatrix}
0 & 0 \\
0 & K_{bp}
\end{bmatrix}.
\]

(12)

\(X\) denotes the state displacement vector; \(F_{\text{ext}}\) and \(F_{\text{crk}}(X)\) are the state force vectors;

\[
X = \begin{bmatrix}
\dot{u} \\
u
\end{bmatrix},
F_{\text{ext}} = \begin{bmatrix}
0 \\
f_{\text{ext}}
\end{bmatrix},
F_{\text{crk}}(X) = \begin{bmatrix}
0 \\
f_{\text{crk}}(u)
\end{bmatrix}.
\]

(13)

In order to simulate the breathing behaviors of the crack, the DOFs of the nodes on the crack surfaces are retained in the reduced order model. Besides, the DOFs of the nodes for applying external forces and connecting journal bearings are also retained to enhance the analysis accuracy. The other DOFs are represented by the generalized coordinates to reduce the order of the model. The coordinate transformation of the free-interface complex component mode synthesis method can be written as

\[
X = \begin{bmatrix}
\Phi_d & \Phi_n
\end{bmatrix} \begin{bmatrix}
q_d \\
q_n
\end{bmatrix} = \Phi_{\text{red}} q
\]

(14)

where \(\Phi_d\) is referred to as the attachment mode matrix; \(\Phi_n\) denotes the truncated eigenvector matrix; \(q_d\) and \(q_n\) are the generalized coordinates. The attachment mode matrix \(\Phi_d\) and the eigenvector matrix \(\Phi_n\) are generated by the time-invariant terms of Equation (10).

Then, the governing equations of the reduced order model can be expressed as

\[
(\bar{A} + \bar{A}_n e^{-j2\Omega t} + \bar{A}_p e^{j2\Omega t})\ddot{q} + (\bar{B} + \bar{B}_n e^{-j2\Omega t} + \bar{B}_p e^{j2\Omega t})q = \bar{F}_{\text{ext}} + \bar{F}_{\text{crk}}(q)
\]

(15)

where

\[
\begin{align*}
\bar{A} &= \Psi \text{red}^T A \Phi_{\text{red}} \\
\bar{B} &= \Psi \text{red}^T B \Phi_{\text{red}} \\
\bar{A}_n &= \Psi \text{red}^T A_n \Phi_{\text{red}} \\
\bar{B}_n &= \Psi \text{red}^T B_n \Phi_{\text{red}} \\
\bar{A}_p &= \Psi \text{red}^T A_p \Phi_{\text{red}} \\
\bar{B}_p &= \Psi \text{red}^T B_p \Phi_{\text{red}} \\
\bar{F}_{\text{ext}} &= \Psi \text{red}^T F_{\text{ext}} \\
\bar{F}_{\text{crk}}(q) &= \Psi \text{red}^T F_{\text{crk}}(X)
\end{align*}
\]

(16)
where \( \Psi_{red} \) denotes the left coordinate transformation matrix, which is similar to \( \Phi_{red} \) and is computed by the adjoint equation of the time-invariant terms of Equation (10).

### 3.2. Eigenvalue Analysis

Based on the obtained reduced order model, the eigenvalues and the force dynamic response of the cracked rotor-bearing system can be computed efficiently. The eigenvalues can be used to generate the Campbell diagram, which is an important tool in designing rotating machinery. Besides, the real parts of the eigenvalues can reflect the stability of the system. Different from the analysis of conventional structures, the governing equation of the anisotropic cracked rotor-bearing system contains periodically time-variant terms. With respect to this issue, the Hill’s method is employed to obtain the eigenvalue equation of Equation (15). First, the generalized coordinates \( q \) are expressed as the product of a periodical function \( \tilde{\Phi}(t) \) and an exponential function

\[
q = \tilde{\Phi}(t)e^{\lambda t}, \quad \tilde{\Phi}(t) = \sum_{h=-\infty}^{+\infty} \tilde{\phi}_h e^{j2ht} \quad (17)
\]

where \( \lambda \) represents the eigenvalue variable; \( \tilde{\phi}_h \) denotes the mode shape of the \( h \)th harmonic.

By ignoring the force vectors at the right part of Equation (15) and substituting Equation (17) into Equation (15), the eigenvalue equation can be expressed as

\[
\lambda \left( \tilde{A} + \tilde{A}_n e^{-j2ht} + \tilde{A}_p e^{j2ht} \right) \sum_{h=-\infty}^{+\infty} \tilde{\phi}_h e^{j2ht} + \sum_{h=-\infty}^{+\infty} \left[ \tilde{B}_h + \tilde{B}_n e^{-j2ht} + \tilde{B}_p e^{j2ht} \right] \tilde{\phi}_h e^{j2ht} = 0 \quad (18)
\]

where

\[
\begin{align*}
\tilde{B}_h &= j2ht \tilde{A}\tilde{B} \\
\tilde{B}_n,h &= j2ht \tilde{A}_n \tilde{B} \\
\tilde{B}_p,h &= j2ht \tilde{A}_p \tilde{B}
\end{align*} \quad (19)
\]

Equation (18) consists of a series of algebraic equations regarding the different harmonics. By balancing the terms in Equation (18), the equation for the \( h \)th harmonic can be expressed as

\[
\lambda \left( \tilde{A}\tilde{\phi}_h + \tilde{A}_n \tilde{\phi}_{h+1} + \tilde{A}_p \tilde{\phi}_{h-1} \right) + \left( \tilde{B}_h \tilde{\phi}_h + \tilde{B}_n,h+1 \tilde{\phi}_{h+1} + \tilde{B}_p,h-1 \tilde{\phi}_{h-1} \right) = 0 \quad (20)
\]

By assembling the equations of each harmonic, the overall eigenvalue equation can be written as

\[
\left( \lambda \tilde{A} + \tilde{B} \right) \tilde{\Phi} = 0 \quad (21)
\]
where

\[
\hat{\mathbf{A}} = \begin{bmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots \\
\ldots & \tilde{\mathbf{A}} & \mathbf{A}_n & 0 & 0 & \ldots \\
\ldots & \tilde{\mathbf{A}}_p & \tilde{\mathbf{A}} & \mathbf{A}_n & 0 & \ldots \\
\ldots & 0 & \tilde{\mathbf{A}}_p & \tilde{\mathbf{A}} & \mathbf{A}_n & \ldots \\
\ldots & 0 & 0 & \tilde{\mathbf{A}}_p & \tilde{\mathbf{A}} & \mathbf{A}_n & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{bmatrix},
\]

\[
\hat{\mathbf{B}} = \begin{bmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots \\
\ldots & \tilde{\mathbf{B}}_{-2} & \tilde{\mathbf{B}}_{n-1} & 0 & 0 & \ldots \\
\ldots & \tilde{\mathbf{B}}_{p-2} & \tilde{\mathbf{B}}_{-1} & \tilde{\mathbf{B}}_{n,0} & 0 & \ldots \\
\ldots & 0 & \tilde{\mathbf{B}}_{p-1} & \tilde{\mathbf{B}}_{n,0} & \tilde{\mathbf{B}}_{n,1} & 0 & \ldots \\
\ldots & 0 & 0 & \tilde{\mathbf{B}}_{p,0} & \tilde{\mathbf{B}}_1 & \tilde{\mathbf{B}}_{n,2} & \ldots \\
\ldots & 0 & 0 & 0 & \tilde{\mathbf{B}}_{p,1} & \tilde{\mathbf{B}}_2 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{bmatrix},
\]

\[
\hat{\Phi} = \begin{bmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots \\
\ldots & \tilde{\phi}_{-2} & \tilde{\phi}_{-1} & \tilde{\phi}_0 & \tilde{\phi}_1 & \tilde{\phi}_2 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{bmatrix}
\]

In order to solve the infinitely dimensional eigenvalue equation, necessary truncation to the equation is needed. In real applications, a satisfactory approximation for the eigenvalues can be reached by retaining only several harmonics of low orders. After the truncation, Equation (21) becomes finitely dimensional and the eigenvalues can be obtained. As the nonlinear breathing effects of crack are not considered in the eigenvalue equation, the obtained eigenvalues are actually that of the anisotropic rotor-bearing system with an open crack. Nevertheless, they can also be used to reflect the effects of the anisotropy in journal bearings on the system’s natural vibration.

### 3.3. Forced Response Analysis

Similar to the deduction for eigenvalue analysis, the nonlinear steady-state forced response of the system can also be obtained using the Hill’s method. The time-domain response \(\mathbf{q}\) and the force vectors \(\tilde{\mathbf{F}}_{\text{ext}}\) and \(\tilde{\mathbf{F}}_{\text{crk}}\) are all expressed as the Fourier series

\[
\mathbf{q} = \sum_{h=-\infty}^{\infty} \phi_h e^{jh\Omega t},
\]

\[
\tilde{\mathbf{F}}_{\text{ext}} = \sum_{h=0}^{\infty} \mathbf{F}_{\text{ext},h} e^{jh\Omega t},
\]

\[
\tilde{\mathbf{F}}_{\text{crk}}(\mathbf{q}) = \sum_{h=0}^{\infty} \mathbf{F}_{\text{crk},h} e^{jh\Omega t},
\]

where \(\phi_h, \mathbf{F}_{\text{ext},h}\) and \(\mathbf{F}_{\text{crk},h}\) denote the coefficient vectors of the \(h\)th harmonic. By substituting Equation (23) into Equation (15), conducting differential manipulation and merging similar terms, Equation (15) becomes

\[
\sum_{h=-\infty}^{\infty} \left( A_h \phi_h + B_h \phi_{h+2} e^{-2j\Omega t} + C_h \phi_{h+2} e^{2j\Omega t} \right) e^{jh\Omega t} = \sum_{h=0}^{\infty} \left( \mathbf{F}_{\text{ext},h} + \mathbf{F}_{\text{crk},h} \right) e^{jh\Omega t}
\]

where

\[
A_h = jh\Omega \tilde{\mathbf{A}} + \tilde{\mathbf{B}}
\]

\[
B_h = jh\Omega \mathbf{A}_n + \tilde{\mathbf{B}}_n
\]

\[
C_h = jh\Omega \mathbf{A}_p + \tilde{\mathbf{B}}_p
\]

Similarly, the algebraic equation for the \(h\)th harmonic can be expressed as

\[
A_h \phi_h + B_h \phi_{h+2} + C_h \phi_{h+2} = \mathbf{F}_{\text{ext},h} + \mathbf{F}_{\text{crk},h}
\]
By assembling the balancing equations of each harmonic, the overall equation can be expressed as

$$\hat{\Phi}_{nl} = \hat{F}_{nl}^{ext} + \hat{F}_{nl}^{crk}$$  \hspace{1cm} (27)

where

$$\hat{\Phi}_{nl} = \begin{bmatrix} \phi_0 & \phi_1 & \phi_2 & \cdots & \phi_n \end{bmatrix}^T, \quad \hat{F}_{nl}^{ext} = \begin{bmatrix} F_{nl}^{ext,0} & F_{nl}^{ext,1} & F_{nl}^{ext,2} & \cdots & F_{nl}^{ext,n} \end{bmatrix}^T, \quad \hat{F}_{nl}^{crk} = \begin{bmatrix} F_{nl}^{crk,0} & F_{nl}^{crk,1} & F_{nl}^{crk,2} & \cdots & F_{nl}^{crk,n} \end{bmatrix}^T$$  \hspace{1cm} (28)

The obtained equation should be truncated before further computation. As nonlinear terms are included in Equation (27), the solving process involves several iteration steps to reach converged solution. The solution is deemed to be converged when the difference in \( \Phi_{nl} \) between two sequential steps is smaller than a prescribed tolerance. Then, the time-domain response of the system can be obtained.

4. Numerical Results

Based on the proposed method, the effects of the anisotropic and speed-dependent characteristics of bearings on the eigenvalues and the forced response of the cracked rotor-bearing system are presented. The model shown in Figure 3 is employed as an example, where the rotor model contains 13,368 elements and 56,413 nodes. The dimensions of the shaft are as follows. The length and radius of the shaft equal 1 and 0.05 m, respectively. The thickness and outer radius of the disk are 0.075 and 0.15 m, respectively. The length of the center of the disk to the left side of the shaft is 0.6125 m. The material of the rotor is steel with Young’s modulus \( E = 210 \text{ GPa} \), density \( \rho = 7850 \text{ Kg/m}^3 \) and Poisson’s ratio \( \nu = 0.3 \).

![Figure 3. The finite element model of a cracked rotor-bearing system.](image)

The first 60 eigenvectors are retained during model order reduction, as the 60th natural frequency has been much larger than the excitation frequencies. The retained harmonics in truncating Equations (21) and (27) is \( h \in [-7, 7] \). In this paper, the anisotropy in the stiffness coefficients of bearing is considered, and two dimensionless parameters are defined to quantify the anisotropy patterns

$$a_k = \frac{k_{xx}}{k_{yy}}, \quad \beta_k = \frac{k_{xy}}{k_{yy}}$$  \hspace{1cm} (29)

where \( a_k \) and \( \beta_k \) are referred to as the anisotropic parameters of principal and cross-coupling stiffness coefficients, respectively.

First, the proposed method is validated. Although the full order model may not be perfectly consistent with the realistic rotor, it can be modified by employing the model updating method, which
The external excitations consist of gravity and unbalance force. In order to reveal the effects of the level of unbalance and get the response characteristics of the cracked rotor at different rotating speeds, the unbalance force is defined by its ratio to the gravity of rotor, where the unbalance force is set to be 10% of gravity and is along the crack opening direction. The crack with a depth of 20 mm, which is 20% of the diameter, is employed as a representative. It can be seen in the figures that the amplitudes obtained from the reduced order model agree well with that by the full order model, where the maximum relative error of the real parts is less than 0.5% and that of the imaginary parts is less than 0.1%. Although the real parts possess larger relative errors, the absolute values of errors of the real parts are actually quite small as the real parts are much smaller that the imaginary parts.

![Figure 4](image-url)

**Figure 4.** The relative errors of the first 50 damped eigenvalues of the anisotropic rotor-bearing system with a 20-mm crack at the speeds of 3800 and 5000 r/min: (a) the real parts, (b) the imaginary parts.

![Figure 5](image-url)

**Figure 5.** The response amplitudes of nodes on shaft axis of the rotor-bearing system with a 20-mm crack at the speeds of 3800 and 5000 r/min: (a) along the horizontal direction; (b) along the vertical direction. The labels “FEM” and “ROM” denote the full order model and the reduced order model, respectively.

Then, the forced responses obtained by the two models are compared, which is shown in Figure 5. The external excitations consist of gravity and unbalance force. In order to reveal the effects of the level of unbalance and get the response characteristics of the cracked rotor at different rotating speeds, the unbalance force is defined by its ratio to the gravity of rotor, where the unbalance force is set to be 10% of gravity and is along the crack opening direction. The crack with a depth of 20 mm, which is 20% of the diameter, is employed as a representative. It can be seen in the figures that the amplitudes obtained by the reduced order model agree well with that by the full order model, where the maximum relative error is less than 1%. Then, the reduced order model is employed in the following sections.
4.1. Effects of the Anisotropy in Bearings

The effects of the anisotropy in journal bearings are presented in this section. The stiffness coefficient $k_{yy}$ is set to be $2 \times 10^5 \text{ N/m}$. The anisotropy in the damping coefficients is ignored, and $c_{xx} = c_{yy} = 1 \times 10^3 \text{ Ns/m}$. For the sake of brevity, the node on the axis of the shaft at the crack section is called the central node in the following parts.

Figure 6 shows the response amplitudes of the central node versus the rotating speed. Two kinds of models are considered, including an uncracked rotor supported by isotropic bearings and three cracked models. The crack depth is 20 mm. As can be seen in the figures, the existence of crack leads to significant changes in the forced responses of the rotor system. The most obvious difference of the cracked rotor and uncracked rotor is that the breathing behavior of the crack leads to the 1/2 and 1/3 subharmonic resonance of the system. This is easy to understand and is reported in a vast of investigations. Besides, the anisotropy in the principle and the cross-coupling stiffness coefficients significantly affects the response of the system, in both the amplitudes and the resonant frequencies. As is reported in previous paper [18], the amplitude curves of the isotropy case in the frequency regions near the 1, 1/2 and 1/3 critical speeds contain only one resonant peak. The introduction of anisotropy in the cross-coupling stiffness coefficient leads to the split of the isolated resonant peaks, and the anisotropy in the principle stiffness coefficients results in more obvious split. It can be easily understood that the decrease of stiffness in the horizontal direction reduces the corresponding natural frequency. Moreover, the amplitudes of the resonant peaks are also affected by the anisotropy of bearings, which are the results of the complex interactions of the anisotropy of bearings and the nonlinear breathing behavior of the cracked rotor.

![Figure 6](image)

**Figure 6.** The response amplitudes of the central node of the anisotropic rotor-bearings system with a 20-mm crack versus the rotating speed: (a) along the horizontal direction; (b) along the vertical direction.

Figure 7 shows the evolution of the whirling orbits of the central node versus the parameters of $\alpha_k$ at the speed of 3800 r/min, where $\beta_k$ is set to be 0.2 and the unbalance force is 10% of gravity. The rotating speed is selected to be 3800 as it is in or near the resonant region of the 1/2 critical speed, at which the response is more sensitive to the variation of system parameters. As can be seen in the figures, the variation of $\alpha_k$ significantly affects the whirling orbits, especially when $\alpha_k$ is smaller than 1.0. The orbits gradually change from petals to twisty heart shapes as $\alpha_k$ increases. These interesting evolution processes of whirling orbits may come from two aspects of reasons. First, the variation of $\alpha_k$ changes the resonant regions, then the rotating speed is within the resonant region at some cases while is not at the other cases. Second, the different level of anisotropy affects the weight of each harmonic component, resulting in the different shapes in orbits.
Therefore, only the central eigenvalue $\lambda$ systems, especially when the responses in the high frequency ranges are of interest. It should be well dealt with in order to gain accurate insights into the dynamic behaviors of such the system's parameters.

The regions adjacent to the bearing nodes are more likely to participate in the deflection in shapes of the system tend to be bowed, whereas the high-order ones often involve complex bending influenced. These phenomena depend on the characteristics of the mode shapes. The first-order mode differences between the two cases are relatively obvious. Therefore, the high-order critical speeds will be obviously una

Figure 7. The evolution of orbits of the central node versus the anisotropic parameters of principle stiffness coefficient at the speed of 3800 r/min: (a) $\alpha_k = 0.5$; (b) $\alpha_k = 0.6$; (c) $\alpha_k = 0.7$; (d) $\alpha_k = 0.8$; (e) $\alpha_k = 0.9$; (f) $\alpha_k = 1.0$; (g) $\alpha_k = 1.5$; (h) $\alpha_k = 2.0$.

From the results shown above, it can be seen that the periodically time-variant characteristic caused by the anisotropy of bearings has significant influence on the vibration of the cracked rotor-bearing system. Then, its effects on the system’s eigenvalues are discussed. The eigenvalues of periodically time-variant systems obtained by the Hill’s method are often appeared in groups. If $\lambda_1$ is one eigenvalue of the system, then $\lambda_1 = \pm 2h\Omega$, $h = 1, 2, 3, \cdots N_k$ will be also its eigenvalues, where $N_k$ denotes the retained number of harmonics. If the eigenvalues are all illustrated in one figure, it will be disordered. Therefore, only the central eigenvalue $\lambda_1$ is included to avoid the clusters of eigenvalues. Figure 8 shows the Campbell diagram of the system with a 20-mm crack, where an isotropic case and an anisotropic case $\alpha_k = 0.5, \beta_k = 0.2$ are included. The natural frequencies are often coming in pairs, which are referred to as the forward and backward frequencies, respectively.

As is illustrated in the figure, the anisotropy of bearings mainly affects the eigenvalues of relatively high orders, and the first pair of natural frequencies and the torsional natural frequencies is almost unaffected. With respect to the second and the higher order natural frequencies, the differences between the two cases are relatively obvious. Therefore, the high-order critical speeds will be obviously influenced. These phenomena depend on the characteristics of the mode shapes. The first-order mode shapes of the system tend to be bowed, whereas the high-order ones often involve complex bending deflections. The regions adjacent to the bearing nodes are more likely to participate in the deflection in the high-order modes, and then the high-order natural frequencies are more sensitive to the change of the system’s parameters.

It can be concluded that the periodically time-variant characteristic caused by the anisotropy of journal bearings makes the dynamic response of the cracked rotor-bearing system more complicated. It should be well dealt with in order to gain accurate insights into the dynamic behaviors of such systems, especially when the responses in the high frequency ranges are of interest.
variation of the stiffness of lubrication oil is VG46, and the load is set to be 1125 N during the simulations. It can be seen that different, not only in the shape but also in the size. After passing through the resonant region, the coefficients has some impacts, whereas the parametric excitation effect caused by the anisotropy of 1/2 critical speed, such as the speed of 3800, 4000, 4125 and 4300 rpm, the orbits become obviously speed ranges far from the resonant regions, such as in the ranges with the speed smaller than the 1/2 speed ranges.

4.2. Effects of the Speed-Dependent Characteristic of Journal Bearings

The speed-dependent characteristic is a widely existed phenomenon in journal bearings, which is caused by the complex fluid dynamic behaviors of lubricating oil. Nevertheless, this characteristic has not been paid enough attention in the dynamic analysis of cracked rotors. In previous sections, the stiffness and damping coefficients are all assumed to be fixed. In this section, the effects of the speed-dependent characteristic on the vibration of the system are discussed. Figure 9 illustrates the variation of the stiffness and damping coefficients of a journal bearing. The tilting-pad type bearings are used, and the stiffness and damping coefficients are obtained by the MADYN2000 software. The type of lubrication oil is VG46, and the load is set to be 1125 N during the simulations. It can be seen that the stiffness and damping coefficients change remarkably versus rotating speed. A fixed case is also employed, where \( k_{yy} = 2 \times 10^8 \text{N/m}, c_{xx} = c_{yy} = 1 \times 10^3 \text{Ns/m} \) and \( c_{xy} = c_{yx} = 0 \). The coefficients \( k_{yy} \) and \( c_{xx} \) of the fixed case are selected according to the mean values of the speed-dependent ones in the speed range [0, 10,000], where \( k_{xx} \) and \( k_{xy} \) are determined by \( \alpha_k \) and \( \beta_k \).

![Figure 8](image_url)

**Figure 8.** The Campbell diagram of the rotor-bearings system with a 20-mm crack. ▼ and ◆ denote the results of \( \alpha_k = 1.0, \beta_k = 0 \) and \( \alpha_k = 0.5, \beta_k = 0.2 \), respectively.

![Figure 9](image_url)

**Figure 9.** The speed-dependent coefficients of a journal bearings versus the rotating speed: (a) stiffness coefficients, (b) damping coefficients.
Figure 10 shows the response amplitudes of the cracked rotor-bearing system with fixed and speed-dependent coefficients. The depth of the crack and the unbalance force are same as those in previous section. The parameter \(a_k\) of both cases equals 1.0, and the parameter \(b_k\) of the fixed case is 0.2. As can be seen in the figures, although the stiffness and damping coefficients of the fixed and speed-dependent cases are obviously different, the amplitudes of the two cases agree well in the speed ranges far from the resonant regions, such as in the ranges with the speed smaller than the 1/2 critical speed. Nevertheless, the differences in the amplitudes of the two cases become quite obvious when the speed is close to the resonant regions. Of course, the difference in the stiffness and damping coefficients has some impacts, whereas the parametric excitation effect caused by the anisotropy of bearings plays a more important role. For the speeds close to the resonant regions, the parametric excitation effect is much stronger than those far away from the resonant regions, so that the dynamic responses are more sensitive to the changes of the system’s parameters.

The abovementioned characteristics will also be reflected in the whirling orbits. Figure 11 shows the orbits of the central node of the fixed and the speed-dependent cases at ten typical rotating speeds. It can be seen in the figures that the evolution characteristics of the orbits are similar to the variation of response amplitudes. At low rotating speeds, the orbits of the two cases are very similar in shape and size, as the first three figures show. When the rotating speed is within the resonant region of the 1/2 critical speed, such as the speed of 3800, 4000, 4125 and 4300 rpm, the orbits become obviously different, not only in the shape but also in the size. After passing through the resonant region, the orbits are both gradually turning into circular shape, until the speed reaches the resonant region of the 1× critical speed, at which the orbits become elliptical shapes.
Then, the effects of the speed-dependent characteristic on the eigenvalues are discussed. Figure 12 illustrates the evolution of the natural frequencies versus the rotating speed, where both the fixed and the speed-dependent cases have been included. As is shown in the figure, the evolution lines of natural frequencies of the speed-dependent case are remarkably different from that of the fixed case, except the first natural frequency. The reason of this phenomenon is similar to the result shown in Figure 8. Besides, the most conspicuous feature in the diagram is that the lines of the fixed case are all straight, whereas that of the speed-dependent case become curved lines. This is caused by the variation of the stiffness and damping coefficients. In the range of low speed, the coefficients decrease rapidly but with decreased gradients. As a result, the second and higher order natural frequencies are also decreased with decreased gradients. In the range of higher speed, the variations of the coefficients tend to be slow and smooth, so that the evolution lines gradually get straight. As the differences between the stiffness coefficients of the two cases are almost invariant in this range, the evolution lines are almost parallel. Nevertheless, the torsional natural frequencies are not affected by the speed-dependent characteristic.

Figure 13 shows the Campbell diagrams of the speed-dependent cases with $a_k = 0.5$ and $a_k = 1.0$, respectively. This figure is illustrated to show the effects of the anisotropy in the stiffness coefficient. It can be seen that the anisotropy of bearings leads to relatively obvious changes in the natural frequencies of higher orders. Consequently, the $2\times$, $3\times$ and higher order critical speeds will be affected. Therefore, it can be concluded that the speed-dependent characteristic of bearings can also significantly affect the dynamic response of a cracked rotor-bearing system, and it should be carefully taken into consideration to accurately investigate the vibration of cracked rotor-bearing systems.
Figure 12. The Campbell diagram of the rotor-bearings system with a 20-mm crack. ▼ and ★ denote the results of the fixed and speed-dependent cases, respectively. The parameter $\alpha_k$ of both cases equals 1.0, and the parameter $\beta_k$ of the fixed case is 0.2.

Figure 13. The Campbell diagram of the rotor-bearings system with a 20-mm crack. ▼ and ★ denote the results of the speed-dependent cases with $\alpha_k = 0.5$ and $\alpha_k = 1.0$, respectively.
5. Discussions

From the numerical results in previous sections, it can be seen that the anisotropic and speed-dependent characteristics of bearings could remarkably affect the vibrations of cracked rotors and make the dynamic responses and the evolutions of eigenvalues be more complex. Although the rotors of practical rotating machinery are much more complicated than the model employed in this paper, the obtained results could reflect the influence characteristics of bearings. In available investigations, including those for dynamic and stability analysis and those for crack diagnosis, these two factors are seldom addressed. This may be one of the reasons why available techniques could not thoroughly solve the crack diagnosis issues. In reality, the anisotropic and speed-dependent characteristics are inevitable in journal bearings, due to the fluid dynamic mechanics. Thus, these two factors should be well considered in order to accurately obtain the vibration characteristics of cracked rotors and to develop effective techniques for crack diagnosis, especially for those model-based methods.

The main objectives of this paper were to gain insights into the influences of the anisotropic and speed-dependent characteristics of bearings. Nevertheless, developing effective techniques for crack identification and preventing severe accidents are still the ultimate goal. From the point of crack diagnosis, data-driven approaches are more practical ways. Nevertheless, a reliable method for crack identification requires deep insights into the dynamic characteristics caused by the crack. Therefore, the presented method in this paper may help to find interesting dynamic features for crack identification in rotors supported by anisotropic and speed-dependent journal bearings in the future.

6. Conclusions

In this paper, the vibration of a transversely cracked rotor supported by anisotropic journal bearings is presented, where the speed-dependent characteristic of bearing is taken into account. First, the 3D finite element model, the contact-based approach and the eight-coefficient model are used for the accurate modeling of shafts, crack and bearings, respectively. Model order reduction is conducted on the model to increase the efficiency of analysis. The Hill's method is employed to compute the eigenvalue and the forced response for the nonlinear periodically time-variant system. Finally, the effects of the anisotropy and speed-dependent characteristic on the system's vibration are studied.

Numerical results show that both the characteristics of anisotropy and speed-dependence can significantly affect the dynamic responses of cracked anisotropic rotor-bearing systems. The anisotropy in bearing leads to the split of resonant peaks and affects the amplitudes of the peaks due to the complex interactions of the anisotropic journal bearings and the nonlinear breathing crack. The effects of the speed-dependent characteristic present a different feature. Specifically, the responses at the speeds far from the resonant regions are rarely affected, whereas the effects become much more obvious at the speed close to the resonant regions. This is because the parametric excitation effect is much stronger at the resonant regions than the other ranges of speed, which makes the responses be much more sensitive to the variation of the system's parameters. Regarding the natural frequencies, the anisotropic characteristic mainly affects the values of the high-order ones, whereas the speed-dependent characteristic makes their evolution lines in Campbell diagram be curved. Moreover, the gradients of the curves change rapidly at low rotating speeds and tends to be constants at high speeds. Consequently, these two factors should be appropriately dealt with in order to accurately study the dynamic behaviors of cracked rotors and to develop quantitative diagnosis techniques for crack faults.

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