An Exact Cutting Plane Algorithm to Solve the Selective Graph Coloring Problem in Perfect Graphs

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Graph coloring is the problem of assigning a minimum number of colors to all vertices of a graph in such a way that no two adjacent vertices share the same color. The Selective Graph Coloring Problem is a generalization of the classical graph coloring problem. Given a graph with a partition of its vertex set into clusters, the aim of the selective graph coloring problem is to pick exactly one vertex per cluster so that, among all possible selections, the number of colors necessary to color the vertices in the selection is minimized. This study focuses on an exact cutting plane algorithm for selective coloring in perfect graphs, where the selective coloring problem is known to be NP-hard. Since there exists no suite of perfect graph instances to the best of our knowledge, we also propose an algorithm to randomly generate perfect graphs and provide a large collection of instances available online. We test our method on graphs with different size and densities, present computational results and compare them with an integer programming formulation of the problem solved by CPLEX, and a state-of-the-art algorithm from the literature. Our computational experiments indicate that our approach significantly improves the solution performance.

Key words: selective graph coloring; partition coloring; path coloring; cutting plane algorithm; perfect graph generation

1. Introduction

Graph coloring is the problem of assigning a minimum number of colors to all vertices of a graph in such a way that no two vertices that are linked by an edge share the same color.

*This study is supported by Boğaziçi University Research Fund (grant 11765); and T. Ekim is supported by Turkish Academy of Sciences GEBIP award.
It emerges in a wide range of practical areas including scheduling (Marx 2004), frequency assignment (Hale 1980), register allocation used in compiler optimization (Chaitin 1982), sudoku puzzles (Lewis 2015), and many more. The general framework of a problem that can be modeled by graph coloring is comprised of a set of objects and incompatibilities among them. Each object is represented by a vertex, and each incompatibility by an edge linking the associated two vertices. Coloring of this graph corresponds to dividing the objects into distinct groups subject to the constraint that two incompatible objects cannot lie in the same group.

The selective graph coloring problem, SEL-Col, is a generalization of the classical graph coloring problem. Given a graph and a partition of its vertex set into clusters, the goal in SEL-Col is to pick one vertex per cluster in such a way that, among all possible selections, the number of colors needed to color the vertices in the selection is minimum. When each cluster of a graph contains a single vertex only, SEL-Col becomes equivalent to the classical graph coloring problem (Demange et al. 2014).

The selective graph coloring problem, or partition coloring problem as it is alternatively called in the literature, has emerged as a model to solve the routing and wavelength assignment problem (RWA) in optical networks (Li and Simha 2000). Given a list of source-destination pairs in a network, RWA consists of routing a path between each pair and assigning a wavelength to each of them in such a way that two paths sharing a common link in the network cannot receive the same wavelength. The objective of RWA is to use a minimum number of wavelengths. A well-known approach to solve RWA, called path coloring, works in two phases. The first phase consists of generating a set of paths between the given pairs and the second phase is to choose one path (among the generated ones) for every source-destination pair in such a way that the number of necessary wavelengths is minimized. To model the second phase as SEL-Col, a graph model is constructed where the set of given routes and pairs of routes possessing a common link in the original network correspond to vertices and edges in it, respectively. SEL-Col is then solved on the graph model, where the set of vertices associated with the set of routes connecting a given source-destination pair forms a cluster. Selection of one vertex per cluster achieves the goal of finding a route between each pair of terminals, and coloring of the selected vertices delivers a proper wavelength to each route.
The selective framework is suitable for a wide range of application areas. One typical example provided in (Demange et al. 2015) is timetabling problems, in which each job has its own set of available time slots to be scheduled to. The authors refer to the task of scheduling talks at a conference, where each speaker provides her available time periods to give her talk. The organizer needs to select a time period for each speaker and allocate a minimum number of rooms for the talks such that two talks with overlapping time intervals are not held in the same room. In the graph model, the set of available time slots for a speaker corresponds to a cluster in the graph model, and two vertices are adjacent, i.e., joined by an edge, if the associated time slots intersect. Selection of one vertex from each cluster determines the time slots of the talks for each speaker, and each color used to color the selection corresponds to a distinct room.

In the standard graph coloring problem, assignments (colors) on the objects (vertices) are done without any regard to alternative choices for them. However, as the example applications reveal, there are cases where entities have their own set of feasible options (clusters) and thus should be allocated one among those alternatives. In such cases, where the basic graph coloring framework fails to suffice, SEL-COL bridges the gap by offering the required flexibility.

The selective graph coloring problem is known to be NP-hard, and remains so in many special classes of graphs (Li and Simha 2000, Demange et al. 2014) including several subclasses of perfect graphs motivated by various applications (Demange et al. 2015). Its difficulty is twofold, as Demange et al. (2015) point out. It may be caused by the existence of an exponential number of possible selections and/or by the hardness of optimally coloring the graph induced by the selection, even though a selection yielding an optimal solution can be obtained trivially (for instance because there is only one selection as in the case of classical graph coloring problem). Since SEL-COL is NP-hard in some subclasses of perfect graphs, it remains so in the general class of perfect graphs, too.

In our previous work (Şeker et al. 2018), we investigated SEL-COL in certain perfect graph families, and proposed efficient solution algorithms that exploit special characteristics of the graph families under consideration. In this paper, we generalize our earlier approach to perfect graphs in its general form. We conduct computational experiments on a large suite of randomly generated problem instances with varying size and densities, and compare our results to those of an IP formulation and a branch-and-price algorithm.
by Furini et al. (2018). The results show that our cutting plane algorithm significantly improves the solution performance, and the improvement manifests most noticeably in low-density graphs (see Section 6). Additionally, we compare the performance of our algorithm for perfect graphs in its general form to that of our previous algorithm tailored for three subclasses of perfect graphs, which are permutation, generalized split, and chordal graphs. Our cutting plane algorithm for general perfect graphs results in better performance in permutation graphs, and marked deterioration in the class of chordal graphs. As for generalized split graphs, we observe that, with the algorithm for general perfect graphs, the overall performance is comparable to our specially tailored algorithm in (Şeker et al. 2018).

In this study, we also propose an algorithm to randomly generate perfect graphs, which, to the best of our knowledge, is the first algorithm in the literature that is capable of producing instances from the general class of perfect graphs. Furthermore, we make a large collection of perfect graph instances online accessible. Generation of perfect graphs in its general form, rather than from subclasses of it, has a considerable potential to contribute to the literature by providing a means to test the algorithms specifically designed for perfect graphs. Many problems that are NP-hard in general, including the minimum coloring and maximum stable set problems, become polynomially solvable when confined to the class of perfect graphs. The algorithms to solve these problems in the class of perfect graphs are based on semidefinite programming and the ellipsoid method. Even though these methods are polynomial-time in theory, they are known to demonstrate poor performance in practice (Grötschel et al. 1984). In order to observe how the performance of algorithms designed for perfect graphs manifests in practice, it is important to have a collection of perfect graph instances or a method to generate them. Considering the inherent difficulty of randomly generating perfect graphs, one may resort to producing instances from certain known subclasses of perfect graphs. However, bearing in mind that at least 120 subclasses are known for perfect graphs (Hougardy 2006), such an approach would be quite restrictive, as Yıldırım and Fan-Orzechowski (2006) note, too. In this respect, this study serves as a first step to fill an important gap in the literature.

The rest of this article is organized as follows. In Section 2, we provide some preliminary graph-theoretic definitions and information that relate to perfect graphs and Sel-Col. We then review the exact algorithms from the literature and present an integer programming formulation for Sel-Col in Section 3. In Section 4, we describe our cutting plane algorithm.
for perfect graphs. Then, we introduce our random perfect graph generation method in Section 5, and then present computational results of our cutting-plane approach in comparison to those of the integer programming formulation and a state-of-the-art algorithm by Furini et al. (2018) in Section 6. Finally, Section 7 concludes the paper with a brief summary and presents possible future research directions.

2. Definitions

A graph is an ordered pair $G = (V, E)$ with $V$ being the set of vertices (or nodes) and $E$ being the set of edges, which are pairs of elements of $V$. Two vertices in a graph are called adjacent if they are connected by an edge. A vertex $u$ is called a neighbor of another vertex $v$ if there exists an edge $\{u, v\}$. The neighborhood of a vertex $v$ is the set of all vertices that are adjacent to it, and is denoted by $N(v)$.

The complement of a graph $G = (V, E)$, denoted as $\bar{G}$, is a graph on the same vertex set $V$ and such that two distinct vertices of $\bar{G}$ are adjacent if and only if they are not adjacent in $G$. An induced subgraph is a graph formed by a subset $V'$ of $V(G)$ and all edges connecting the pairs of vertices in $V'$. For a graph $G = (V, E)$ and $V' \subseteq V$, the subgraph induced by $V'$ is shown as $G[V']$.

A (simple) cycle is comprised of a sequence of consecutively adjacent vertices that starts and ends at the same vertex with no repetitions of vertices and edges. An odd cycle is a cycle with an odd number of vertices in it. A clique in a graph is a subset of vertices such that every distinct pair of vertices in the subset is adjacent. In a graph, a given clique is maximal if its size cannot be extended with inclusion of some other vertex; in other words, if it is not a proper subset of another clique. The clique number of a graph $G$ is the size of a largest clique in $G$ and is denoted by $\omega(G)$. A stable set, or equivalently an independent set, is a set of vertices in a given graph in which no two vertices are adjacent. The stability number of a graph $G$, shown by $\alpha(G)$, is the size of a largest stable set in it.

A coloring of a graph using at most $k$ colors is called a (proper) $k$-coloring. A graph is called $k$-colorable if its vertices can be assigned a $k$-coloring. The chromatic number of a graph $G$, denoted by $\chi(G)$, is the minimum number of colors necessary to color all vertices of the graph. Note that a graph $G$ is $k$-colorable for all $k \geq \chi(G)$.

A graph $G$ is perfect if every induced subgraph $G' \subseteq G$ satisfies $\chi(G') = \omega(G')$. The Weak Perfect Graph Theorem (WPGT) (Lovász 1972) states that a graph is perfect if and only if
its complement is perfect. The Strong Perfect Graph Theorem (SPGT) (Chudnovsky et al. 2006) states that a graph $G$ is perfect if and only if neither $G$ nor $\overline{G}$ contains an odd cycle of length at least five as an induced subgraph.

Figure 1  (a) A graph $G$ with a partition of its vertex set into four clusters $\mathcal{V} = \{V_1, \ldots, V_4\}$ shown in dashed ellipses, (b) an optimally colored selection $\{1, 2, 3, 4\}$ in $G$, (c) an optimal selection $\{1, 6, 3, 8\}$ in $G$ with an optimal coloring of it, yielding $\chi_{SEL}(G, \mathcal{V}) = 1$.

Given a graph $G = (V, E)$ with a partition $\mathcal{V} = \{V_1, \ldots, V_P\}$ of its vertex set into $P$ clusters, a selection is a subset $V'$ of vertices of $G$ that contains exactly one vertex from each cluster in the partition; that is, $V' \subseteq V$ such that $|V' \cap V_p| = 1$ for all $p \in \{1, \ldots, P\}$. A selective $k$-coloring of $G$ is defined by a selection $V'$ and a $k$-coloring of $G[V']$. The smallest integer $k$ for which $G$ with vertex partition $\mathcal{V}$ admits a selective $k$-coloring is called the selective chromatic number of $G$ and is denoted by $\chi_{SEL}(G, \mathcal{V})$ (Demange et al. 2015). A selection $V'$ whose optimal coloring yields the selective chromatic number is called an optimal selection (see Figure 1).

3. Solution Methods for SEL-Col
To the best of our knowledge, apart from our previous work (Şeker et al. 2018), there are three studies that concentrate on exact solution methods for SEL-Col. The study by Frota et al. (2010) introduce a branch-and-cut algorithm for the partition coloring problem. Hoshino et al. (2011) propose another integer programming model and a branch-and-price algorithm to solve the partition coloring problem. Finally, a recent study by Furini et al. (2018) proposes a new formulation for SEL-Col with an exponential number of variables and designs a branch-and-price algorithm to solve it. The pricing phase of their algorithm is based on a single pricing problem, as opposed to the work by Hoshino et al., in which several pricing problems are solved at each step. Experimental results indicate that the proposed branch-and-price framework improves on the previous state-of-the-art exact approaches.
from the literature. We use the open-source implementation of this algorithm to compare our results to.

Next, we present an integer programming (IP) formulation, Model 1, to solve SEL-COL (Şeker et al. 2018). We assume that we are given a graph $G = (V, E)$ with $V = \{1, \ldots, n\}$ and a partition $\mathcal{V}$ of its vertex set into $P$ clusters $V_1, \ldots, V_P$.

**Model 1:**

$$\text{min } \sum_{k=1}^{P} y_k$$

s.t.

$$x_{ik} + x_{jk} \leq y_k \quad \forall \ (i, j) \in E, \ k \in \{1, \ldots, P\}$$

$$\sum_{i \in V_p} \sum_{k=1}^{P} x_{ik} = 1 \quad \forall p \in \{1, \ldots, P\}$$

$$y_k \in \{0, 1\} \quad \forall k \in \{1, \ldots, P\}$$

$$x_{ik} \in \{0, 1\} \quad \forall (i, k) \in V \times \{1, \ldots, P\},$$

where $x_{ik}$ is a binary variable taking value 1 if vertex $i$ is colored with color $k$ and 0 otherwise, $y_k$ is another binary variable having value 1 if color $k$ is used and 0 otherwise.

Model 1 takes the number of available colors as $P$ because the size of a selection is $P$ and in the worst case each vertex in the selection takes a distinct color. One should note that a feasible $n$-coloring of a selection can choose any $n$-set of the available $P$ colors. Moreover, a feasible $n$-coloring of a selection has $n!$ equivalent alternatives in the solution space that are obtained by simply permuting the indices of the $n$ colors used. In order to reduce the inherent symmetry in this formulation, we add the constraint set (6) to Model 1 (similar to symmetry breaking constraints in (Sherali and Smith 2001)). This way, the program is forced to use the color in increasing order of their indices and clone solutions resulting from alternative combinations of the available $P$ colors are discarded from the solution space.

$$y_k \geq y_{k-1} \quad \forall k \in \{2, \ldots, P\}$$

Model 1 contains $O(|V| \times P)$ binary variables and $O(|E| \times P)$ constraints. Since it is an integer programming formulation, its solution time and memory requirement may increase exponentially with the increase in the size of the input. As a promising alternative to the IP formulation Model 1, we present an exact cutting plane algorithm for perfect graphs next.
4. Cutting Plane Algorithm for Perfect Graphs

Given a graph $G = (V, E)$ with a partition of its vertex set into $P$ clusters as defined in Section 3, an alternative formulation for SEL-COL can be written as follows (Şeker et al. 2018):

Model 2: $\min t$  \hspace{1cm} (7) \\
\text{s.t.} \\
$\sum_{i \in V_p} x_i = 1 \quad \forall p \in \{1, ..., P\}$  \hspace{1cm} (8) \\
t $\geq \chi(G[x])$  \hspace{1cm} (9) \\
t $\geq 0$  \hspace{1cm} (10) \\
x_i \in \{0, 1\} \quad \forall i \in V,$  \hspace{1cm} (11)

where $x_i$ is a binary variable that takes value 1 if vertex $i$ is selected and 0 otherwise, and $G[x]$ is the graph induced by the selection given by the variable vector $x = (x_1, \ldots, x_n)$. The nonnegative variable $t$ is an estimate of the number of colors needed.

The requirement that exactly one vertex is selected from each of the $P$ clusters is met by constraint set (8). The nonnegative variable $t$ is forced to be at least equal to the chromatic number of an optimal selection by constraint set (9). Since the objective is to minimize variable $t$, the optimal objective value of this model is the selective chromatic number $\chi_{SEL}(G, V)$ of the input graph. Because of the $\chi(.)$ operator in constraint set (9), the current form of the model is not well-defined. We need to replace (9) with a set of linear inequalities that will perform the coloring task. Instead of embedding these inequalities to the model all at once, we decompose the problem into two parts to deal with the selection task in one part and the coloring of the given selection in the other.

We first relax constraint set (9) and obtain a linear model that yields a feasible vertex selection for $G$. This relaxed model comprises the initial master problem. At each step, we start with solving the master problem to optimality and obtain a vertex selection. We then feed this selection to a subproblem where the chromatic number of the graph induced by the given selection is computed. If the chromatic number found by the subproblem is higher than the optimal objective value of the master problem, then it means that the current state of the master problem does not fully involve the set of constraints that can
correctly estimate the selective chromatic number of the input graph. In this case, we add a constraint to the master problem, which ensures that the \( t \)-value is at least as large as the chromatic number corresponding to the current selection as long as the same set of vertices is selected. One such constraint can be expressed as follows:

\[
t \geq \chi(G[x^{(j)}]) - \sum_{i \in \{x_i^{(j)} = 1\}} (1 - x_i),
\]

where \( G[x^{(j)}] \) denotes the graph induced by the selection found at iteration \( j \) given by the variable vector \( x^{(j)} \), and \( \chi(G[x^{(j)}]) \) the chromatic number of this induced subgraph.

The rightmost term in inequality (12) is equal to zero only when \( x = x^{(j)} \); i.e., when exactly the same vertices as in \( x^{(j)} \) are picked, and it increases by one with each vertex removed from the selection given by \( x^{(j)} \). This constraint relies on the fact that the chromatic number of a graph induced by a selection can decrease by at most one for each vertex switch, and decreases the lower bound by one for each vertex we replace.

As mentioned before, the chromatic number of a perfect graph is equal to the size of a maximum clique in it and by definition, the property of being perfect is hereditary, i.e., every induced subgraph of a perfect graph is again perfect. Then, each time the subproblem is called, we can equivalently find a maximum clique in it instead of the chromatic number. We can translate this relationship into an inequality as given in (13), and utilize it within our cutting plane algorithm.

\[
t \geq \sum_{i \in K^{(j)}} x_i
\]

where \( K^{(j)} \) is a maximum clique of \( G[x^{(j)}] \).

Given a selection, the cuts as given in (13) enforce the master program’s objective value \( t \) to be at least as large as the size of a maximum clique in it. Moreover, (13) provides positive lower bounds for all other unexplored selections that intersect with the cliques whose cuts have been added before. We prefer constraint (13) over (12) because it is stronger for perfect graphs, which we show in the following.

**Proposition 4.1.** Constraint (13) is stronger than (12) for perfect graphs.
Proof. Given a graph $G$ and a partition $\mathcal{V} = \{V_1, \ldots, V_P\}$ of its vertex set, let us first define the two polyhedra $\mathcal{P}_{12}$ and $\mathcal{P}_{13}$ as follows:

$$\mathcal{P}_{12} := \left\{ x \in [0,1]^{|\mathcal{V}|}, \ t \in \mathbb{R}_{\geq 0}: \sum_{i \in V_p} x_i = 1 \ \forall p \in \{1, \ldots, P\}, \right. $$

$$t \geq \chi(G[\hat{x}]) - \sum_{i \in V[\hat{x}_i = 1]} \sum_{p \in \{1, \ldots, P\}} (1 - x_i) \ \forall \hat{x} \in \{0,1\}^{|\mathcal{V}|} \ \text{s.t.} \ \sum_{i \in V_p} \hat{x}_i = 1 \ \forall p \in \{1, \ldots, P\} $$

$$\left. \forall p \in \{1, \ldots, P\} \right\}$$

$$\mathcal{P}_{13} := \left\{ x \in [0,1]^{|\mathcal{V}|}, \ t \in \mathbb{R}_{\geq 0}: \sum_{i \in V_p} x_i = 1 \ \forall p \in \{1, \ldots, P\}, \right. $$

$$t \geq \sum_{i \in K} x_i \ \forall \hat{x} \in \{0,1\}^{|\mathcal{V}|} \ \text{s.t.} \ \sum_{i \in V_p} \hat{x}_i = 1 \ \forall p \in \{1, \ldots, P\} \ \text{and} $$

$$\hat{K} \text{ is a maximum clique of } G[\hat{x}]$$

In other words, if we let $\mathcal{P}$ be the linear programming (LP) relaxation of the polyhedron defined by the constraint set of our initial master problem, $\mathcal{P}_{12}$ and $\mathcal{P}_{13}$ are constructed by further constraining $\mathcal{P}$ respectively with constraints (12) and (13) defined for each one of all possible vertex selections. We want to prove that $\mathcal{P}_{13} \subseteq \mathcal{P}_{12}$. To do this, we first show that for any $\{\bar{t}, \bar{x}\} \in \mathcal{P}_{13}$, $\{\bar{t}, \bar{x}\} \in \mathcal{P}_{12}$ holds. Since vertex selection constraints are common on both $\mathcal{P}_{12}$ and $\mathcal{P}_{13}$, $\sum_{i \in V_p} \bar{x}_i = 1 \ \forall p \in \{1, \ldots, P\}$ holds by construction. Now, let $\hat{x} \in \{0,1\}^{|\mathcal{V}|}$ such that $\sum_{i \in V_p} \hat{x}_i = 1 \ \forall p \in \{1, \ldots, P\}$ and $\hat{K}$ is a maximum clique of $G[\hat{x}]$. Note that we can write $\sum_{i \in \hat{K}} \bar{x}_i = |\hat{K}| - \sum_{i \in \hat{K}} (1 - \bar{x}_i)$. Since $\hat{K} \subseteq V(G[\hat{x}])$, we have $\sum_{i \in V[\hat{x}_i = 1]} (1 - \bar{x}_i) \geq \sum_{i \in \hat{K}} (1 - \bar{x}_i) \geq 0$. As $\chi(G[\hat{x}]) = |\hat{K}|$ by the perfectness of $G$, we have $\bar{t} \geq \sum_{i \in \hat{K}} \bar{x}_i = |\hat{K}| - \sum_{i \in \hat{K}} (1 - \bar{x}_i) \geq \chi(G[\hat{x}]) - \sum_{i \in V[\hat{x}_i = 1]} (1 - \bar{x}_i)$ and hence $\{\bar{t}, \bar{x}\} \in \mathcal{P}_{12}$.

Next, we show that this containment can be strict; i.e., there exists a perfect graph $G$ for which at least one point in $\mathcal{P}_{12}$ is not contained in $\mathcal{P}_{13}$. To this end, consider the graph $G = (V, E)$ with $V = \{V_1, V_2, V_3\}$, where $V = \{1,2,3,4\}$, $E = \emptyset$, $V_1 = \{1,2\}$, $V_2 = \{3\}$, and $V_3 = \{4\}$. There are two possible selections for this graph, which are $\hat{x}^{(1)} = (1,0,1,1)$ and $\hat{x}^{(2)} = (0,1,1,1)$. The constraints of type (12) associated with selections $\hat{x}^{(1)}$ and $\hat{x}^{(2)}$ are respectively $c_1 : t \geq 1 - (3 - (x_1 + x_3 + x_4))$ and $c_2 : t \geq 1 - (3 - (x_2 + x_3 + x_4))$. Now, take the point $\{\bar{t}, \bar{x}_1, \ldots, \bar{x}_4\} = (0.5, 0.5, 0.5, 1, 1)$. This point is contained in $\mathcal{P}_{12}$, because it satisfies the selection constraints as well as $c_1$ and $c_2$. A maximum clique of $G[\hat{x}^{(1)}]$ is $\{3\}$. The corresponding constraint of type (13), $t \geq x_3$ is violated by the given point, as $\bar{t} = 0.5$ and $\bar{x}_3 = 1$. Hence, $(0.5, 0.5, 0.5, 1, 1) \not\in \mathcal{P}_{13}$, and $\mathcal{P}_{13} \subset \mathcal{P}_{12}$. \qed
**Input:** A perfect graph \( G = (V, E) \), and a partition \( \mathcal{V} \) of \( V \)

**Output:** An optimal selection \( x^* \) with \( \chi_{SEL}(G, \mathcal{V}) = z^* \)

\( j \leftarrow 0, \ t^{(j)} \leftarrow 0, \ z_{sp}^{(j)} \leftarrow \infty \)

**while** true **do**

\( j \leftarrow j + 1 \)

Solve the master problem to optimality, find an optimal selection \( x^{(j)} \) having optimal objective value \( t^{(j)} \)

Find a maximum clique \( K^{(j)} \) of \( G[x^{(j)}] \) in the subproblem

\( z_{sp}^{(j)} \leftarrow |K^{(j)}| \)

**if** \( z_{sp}^{(j)} > t^{(j)} \) **then**

Add (13) to the master problem

**else**

break

**end if**

**end while**

\( x^* \leftarrow x^{(j)}, \ z^* \leftarrow t^{(j)} \)

---

**Figure 2** Cutting Plane Algorithm for Perfect Graphs.

Pseudo-code of our cutting plane algorithm for perfect graphs is provided in Figure 2. At each step \( j \) of our cutting plane algorithm, the master problem is solved to optimality yielding a selection \( x^{(j)} \) with a corresponding optimal objective value \( t^{(j)} \), and \( G[x^{(j)}] \) is fed to the subproblem. If the objective value of the subproblem, which is the size of a maximum clique \( K^{(j)} \) in \( G[x^{(j)}] \), turns out to be less than \( t^{(j)} \), we continue iterating because it means the master problem is currently lacking the constraints that will lead to the correct estimate of the optimal value of \( t \). Otherwise, the process is terminated, in which case the incumbent solution \( x^* \) and the associated objective value \( t^* \) are optimal.

In the next two subsections, we discuss the methods that we employ in the subproblem of our cutting plane procedure. It should be noted that even when we make use of polynomial-time algorithms in the subproblem, NP-hardness of \( \text{Sel-Col} \) still persists due to the inherent difficulty of the selection task arising from the existence of exponentially many selections, as mentioned before.
4.1. Semidefinite Programming for the Maximum Clique Problem

In the class of perfect graphs, coloring, or equivalently maximum clique problem is polynomially solvable (up to any desired accuracy) via semidefinite programming (SDP) (Grötschel et al. 1984). Finding the size of a maximum clique in a perfect graph necessitates solving an SDP only once. However, extracting the maximum clique itself involves solving a series of SDP problems on successively smaller graphs for at most \( n \) times, where \( n \) is the number of vertices in the input graph.

Lovász introduced a function \( \vartheta \), known as Lovász’s theta function or theta function, which is polynomial-time computable (Lovász 1979). Given a graph \( G \), Lovász’s theta function \( \vartheta(G) \) satisfies the following inequality:

\[
\alpha(G) \leq \vartheta(G) \leq \chi(\bar{G}),
\]

where \( \alpha(G) \) denotes the size of a maximum stable set in \( G \) and \( \chi(\bar{G}) \) denotes the chromatic number of the complement of \( G \).

One should note that, for any graph, the stability number \( \alpha(G) \) equals the clique number of its complement \( \omega(\bar{G}) \), and \( \chi(\bar{G}) \) is equal to \( \omega(\bar{G}) \) for perfect graphs. Then, \( \vartheta(G) = \omega(\bar{G}) \) holds for perfect graphs. In order to find \( \omega(G) \) of a perfect graph \( G \), we need to use the complement of the graph \( \bar{G} \), which is again perfect by the WPGT (Lovász 1972). The theta function \( \vartheta(G) \) can be computed by several equivalent formulations (Knuth 1994, Grötschel et al. 2012, Yıldırım and Fan-Orzechowski 2006). We provide one of these formulations in (14)–(18), which is an SDP problem.

Let us introduce a few notations first. For two matrices \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times n} \), the trace inner product is denoted by \( A \cdot B = \text{trace}(A^T B) = \text{trace}(BA^T) = \sum_{i,j} A_{ij} B_{ij} \). A symmetric real matrix \( A \) is said to be positive semidefinite if \( zA z^T \geq 0 \) for every \( z \in \mathbb{R}^n \). For an \( n \times n \) real symmetric matrix \( A \), we use \( A \succeq 0 \) to indicate that \( A \) is positive semidefinite. Finally, we use \( S^{n \times n} \) to denote the space of \( n \times n \) symmetric matrices.

Consider the following formulation:

\[
\begin{align*}
\max & \quad J \cdot X \quad \quad \quad \quad \quad (14) \\
\text{s.t.} & \quad I \cdot X = 1 \quad \quad \quad \quad \quad (15) \\
& \quad X_{ij} = 0 \quad \forall \{i,j\} \in E \quad \quad \quad (16)
\end{align*}
\]
\[ X \succeq 0 \quad (17) \]
\[ X \in S^{n \times n}, \quad (18) \]

where \( I \) is the identity matrix, \( J \) is a matrix of all ones, and \( E \) is the edge set of the input graph.

The optimal objective value of the SDP model provided in (14)–(18) gives the stability number \( \alpha(G) \) of a perfect graph \( G \) (Grötschel et al. 1984). However, we cannot directly obtain a maximum stable set itself by solving this model once. A study by Grötschel et al. (1984) proposes a method to extract a maximum stable set in perfect graphs by repeatedly computing the stability number in smaller induced subgraphs of the input graph. The main idea of this method is to remove vertices from the input graph until only the vertices of one maximum stable set remains. It works as follows: First, we find the stability number \( \alpha(G) \) of the original input perfect graph \( G \). Then, we mark all vertices of \( G = (V, E) \) unlabeled. At each step, we select an unlabeled vertex \( v \in V(G) \) and tentatively remove it from \( G \). Note that \( G' = G \setminus \{v\} \) is an induced subgraph of \( G \), and hence is perfect, too. We then calculate \( \alpha(G') \). If \( \alpha(G') = \alpha(G) \), we set \( G = G' \), because it means that \( v \) is not contained in all maximum stable sets of \( G \) and its removal will leave at least one maximum stable set intact. If \( \alpha(G') < \alpha(G) \), then it means that \( v \) intersects with all of the maximum stable sets in the current graph and cannot be eliminated. Therefore, we label \( v \) in this case and keep it in our vertex set. This process continues until there is no unlabeled vertex, in which case the set of all remaining (labeled) vertices form a maximum stable set of the original graph. Since we either label or remove a vertex at each step, each vertex is considered once in this method. It outputs a maximum stable set after \( n \) iterations, with \( n \) being the number of vertices of the original input graph. It is also possible to find other maximum stable sets, if any, by changing the order of vertices to be considered.

This method is the first polynomial-time algorithm for the maximum stable set problem in perfect graphs. Since we are interested in finding a maximum clique, which corresponds to a maximum stable set in the complement of the graph, we simply give the complement of the original graph as input. At each step of this method, we make use of the SDP model provided in (14)–(18) to find a maximum stable set. Note that the original input graph will be a subgraph of a perfect graph induced by a vertex selection. By definition, induced
subgraphs of a perfect graph is again perfect. Therefore, we can safely employ this method in the subroutine of our cutting plane procedure.

We made a minor modification to Grötschel et al.’s algorithm in order to possibly avoid unnecessary computations. As we compute the size of a maximum stable set at the beginning of the algorithm, we continue iterating until the number of labeled vertices in the graph equals maximum stable set size, instead of waiting for all vertices to be considered.

Although SDP is a polynomial-time method in theory, it does not perform very well in practice, as will be revealed by the results of our computational experiments presented in Section 6. As a promising alternative to solve the maximum clique problem in perfect graphs, we consider an algorithm of combinatorial nature from the literature, which we discuss in the sequel.

4.2. A Branch-and-Bound Algorithm for the Maximum Clique Problem

A comprehensive review on both exact and heuristic algorithms for maximum clique problem by Wu and Hao (2015) provides computational performance comparison of ten state-of-the-art exact algorithms on a set of popular DIMACS instances. One of the best-performing algorithms is that of Tomita et al. (2010), which is a branch-and-bound algorithm that the authors call MCS. MCS is based on a previous maximum clique algorithm MCR by Tomita and Kameda (2007) and shows considerably improved performance compared to the previous with the help of newly introduced techniques that reduce the search space.

MCR (Tomita and Kameda 2007) is a branch-and-bound algorithm that begins with a small clique and continues searching for larger and larger cliques until it finds one that can be confirmed to be of maximum size. At every step, it starts from a single vertex and tries to expand it by adding new vertices. In order to avoid unnecessary searching, the algorithm makes use of a greedy coloring of the set $R$ of common neighbors of vertices in the current clique $Q$. Greedy coloring assigns a minimum possible (integer) label to each vertex in $R$, which simply implies that the size of a maximum clique in $R$, $\omega(R)$, can be at most the maximum label used in greedy coloring. Then, current clique $Q$ can be extended by at most $\omega(R)$ vertices. So, if the sum of $|Q|$ and the maximum label from greedy coloring does not exceed the size of a clique of maximum size found so far, $|Q'|$, then there is no need to continue searching for vertices to be included in $Q$ because it is simply not possible to obtain a larger clique on that branch.
In the improved maximum clique algorithm MCS (Tomita et al. 2010), which we utilize in our cutting plane procedure, the authors focus on reducing the search space further by incorporating a recoloring routine. This routine aims to improve the coloring obtained from the greedy coloring procedure by recoloring vertices with the largest color label into a smaller one. One should note that MCS is not specific to perfect graphs; it works on any graph.

5. Data Generation

In order to test the performance of our solution approach, we need random problem instances. A complete problem instance for SEL-COL consists of a graph $G = (V, E)$ and a partition $\mathcal{V}$ of its vertex set $V$. In this section, we first introduce an algorithm to randomly generate perfect graphs and then briefly describe a method to randomly produce vertex set partitions.

The class of perfect graphs has led to a key area of interest in graph theory due to the numerous connections it has to a wide range of fields including linear programming and computational complexity. Perfect graphs have great importance for several reasons. First, many problems that are NP-hard in general, e.g. the maximum clique and the minimum coloring problems, become polynomially solvable when restricted to the class of perfect graphs (Grötschel et al. 1984). Moreover, for many subclasses of perfect graphs, there exist coloring and clique algorithms that are not only polynomial-time but also of combinatorial nature (Golumbic 2004). These subclasses, such as chordal graphs, permutation graphs, and interval graphs have additional importance as they naturally arise in various real-life applications like perfect phylogeny, DNA sequencing, timetabling, and flight altitude assignment (Golumbic 2004, Brandstädt et al. 1999, Spinrad 2003). In this respect, perfect graphs form an umbrella class that unifies the results relating to the complexity of important problems in various graph classes.

As mentioned above, some NP-hard problems become polynomially solvable in the class of perfect graphs. The algorithms to solve these problems in the general class of perfect graphs are based on semidefinite programming and the ellipsoid method; they are not combinatorial algorithms. It is known that even though these methods are polynomial in theory, they may perform poorly in practice (Grötschel et al. 1984). In order to observe how the performance of such algorithms manifests in practice, it is important to have a collection of perfect graph instances or a means to generate them.
To the best of our knowledge, there have been only theoretical studies on the generation of perfect graphs in its general form. In his survey, Chvátal (1984) raises the question of whether all perfect graphs are constructible from some “primitive” perfect graphs using perfection-preserving operations, while exemplifying some classes all elements of which can be set up through this idea. To date, only some partial answers have been given to this question. For instance, Burlet and Fonlupt (1984) have proven that all Meyniel graphs are constructible from certain primitive Meyniel graphs by an operation called *amalgam*. Another study by Chudnovsky and Penev (2013) describe the structure of all bull-free perfect graphs, where *bull* is a graph consisting of a triangle and two vertex-disjoint pendant edges. They show that every bull-free perfect graph either belongs to a basic class, or it can be built from smaller bull-free perfect graphs by an operation that preserves the property of being bull-free and perfect.

There exists a polynomial-time recognition algorithm for perfect graphs with $O(|V|^9)$ running time (Chudnovsky et al. 2005). However, generating a random graph and testing for perfectness may not be a viable course of action to obtain test instances, because this recognition algorithm is not practical even for small graphs, as pointed out in (Yıldırım and Fan-Orzechowski 2006). Considering the inherent difficulty of randomly generating perfect graphs in their general form, one may turn to producing instances from certain known subclasses of perfect graphs, like bipartite graphs, line graphs of bipartite graphs, split graphs etc. For instance, two families of perfect graphs for which random generation algorithms are available are chordal graphs (Şeker et al. 2017, Markenzon et al. 2008, Andreou et al. 2005) and generalized split graphs (McDiarmid and Yolov 2016, Şeker et al. 2018). However, as Yıldırım and Fan-Orzechowski (2006) note, this approach would be fairly restrictive in nature since there are at least 120 known subclasses of perfect graphs (Hougardy 2006).

The question of whether all perfect graphs can be built from some primitive perfect graphs still remains to be answered, but there are operations proven to preserve perfection that can seemingly serve well to the purpose of generating perfect graphs. Our algorithm, which we call Algorithm PerfectGen, is based on this idea. We take a diverse set of small-sized perfect graphs and reach an end-graph by combining randomly selected ones via perfection-preserving operations. For this purpose, we made use of the set of all non-isomorphic connected graphs up to nine vertices, offered by McKay (2016). By making use
of the well-known theorems SPGT (Chudnovsky et al. 2006) and WPGT (Lovász 1972), we filtered out the ones that are not perfect, and used the remaining collection to build larger perfect graphs.

In order to check whether a given graph is perfect, we use the well-known characterization of perfect graphs given in SPGT, which states that a graph $G$ is perfect if and only if neither $G$ nor $\bar{G}$ contains an induced odd cycle of length five or more. We check the cycles of the input graph. If we find an odd hole of size five or more; i.e., if we detect an induced cycle with size five or more that is comprised of an odd number of vertices, we conclude that the graph is not perfect. Otherwise (if no odd hole is present in the input graph), we take the complement of the graph and do the same check. If an odd hole of size five or more exists in the complement, we conclude that the original input graph is not perfect; else, it is perfect. This procedure is applied to all connected non-isomorphic graphs having at most nine vertices, which are offered in (McKay 2016), and those passing the check are added to the collection, say $\mathcal{P}$, to be used for perfect graph generation. The number of connected non-isomorphic graphs having one to nine vertices offered by McKay (2016) are respectively 1, 1, 2, 6, 21, 112, 853, 11117, and 261080, and the number of perfect graphs in this collection turned out to be 1, 1, 2, 6, 20, 105, 724, 7805, and 126777, respectively.

Algorithm PerfectGen works as follows: We input a desired number of vertices $n$ and a desired edge density $\rho$ to the algorithm. Initially, we randomly choose a perfect graph from collection $\mathcal{P}$, which is to be extended into a final perfect graph on $n$ vertices. Then, at each step, we first pick a random perfection-preserving operation $op$ among the six such operations we selected from the literature, whose details we are going to provide in the sequel. If the selected operation $op$ necessitates a perfect graph other than the current perfect graph $G$ that is being extended, then we randomly pick a graph $G'$ from $\mathcal{P}$ and combine $G$ and $G'$ via operation $op$. Otherwise, we simply apply operation $op$ to $G$. This routine continues until $G$ has $n$ vertices in total.

The first part of the algorithm explained above has no mechanism to control the number of edges in $G$. In fact, we cannot directly control the number of edges, because the change in the number of edges as well as the number of vertices cannot be foreseen before starting to apply the operation. Moreover, the change in the number of edges is not monotonic throughout the iterations in general; i.e., it can increase, decrease (only possible if we take the complement of the graph), or stay the same. Thus, we first build a perfect graph $G$ on
Input: An integer $n$, two real numbers $\rho$ and $\epsilon$ between 0 and 1

Output: A perfect graph $G$ on $n$ vertices with (approximate) edge density $\rho$

d $\leftarrow 0$

while $d < \rho - \epsilon$ or $d > \rho + \epsilon$ do

Let $G = (V, E)$ be a graph selected randomly from the collection of small-sized perfect graphs $\mathcal{P}$ such that $|V| \leq n$

while $|V| < n$ do

Select a perfection-preserving operation $op$ randomly

if $op$ requires another input graph then

Select a random graph $G' = (V', E')$ from $\mathcal{P}$ with $|V'| \leq n - |V|$

Attach $G'$ to $G$ via operation $op$

else

Modify $G$ with operation $op$

end if

end while

$m \leftarrow |E|$, $d \leftarrow \frac{m}{\frac{n(n-1)}{2}}$

if $\rho - \epsilon < 1 - d < \rho + \epsilon$ then

$G \leftarrow \bar{G}$, where $\bar{G}$ is the complement of $G$, $d \leftarrow 1 - d$

end if

end while

Figure 3 Algorithm PerfectGen.

$n$ vertices and then check its edge density $d$. If $d$ is within some predetermined $\epsilon$-distance from the desired edge density $\rho$, then we accept $G$ and terminate the algorithm. On the other hand, if we can achieve the desired density by taking the complement of $G$, then we deliver $\bar{G}$ as the output graph. Otherwise, we simply discard $G$ and start to construct a new perfect graph from scratch. When generating our instances, we set the value of $\epsilon$ as 0.025. Pseudo-code of the algorithm is provided in Figure 3.

We now present the set of six perfection-preserving operations that we have used in Algorithm PerfectGen.
• **Clique identification** (Berge and Minieka 1973):

Let $G_1$, $G_2$ be disjoint graphs, and $K_i$ be a nonempty clique in $G_i$ satisfying $|K_1| = |K_2|$. Define a one-to-one correspondence between vertices of $K_1$ and $K_2$; i.e., choose a bijective map $f : K_1 \rightarrow K_2$. A graph obtained by unifying each vertex $v$ in $K_1$ with vertex $f(v)$ in $K_2$ is said to arise from $G_1$ and $G_2$ by clique identification. A graph $G$ obtained from two perfect graphs via clique identification is perfect.

![Figure 4](image1.png)  
**Figure 4**  
Two perfect graphs combined by clique identification operation.

• **Substitution** (Lovász 1972):

Let $G_1$, $G_2$ be disjoint graphs, $v$ be a vertex of $G_1$, and $N$ the set of all neighbors of $v$ in $G_1$. Removing $v$ from $G_1$ and linking each vertex in $G_2$ to those in $N$ results in a graph that arises from $G_1$ and $G_2$ by substitution. If $G_1$ and $G_2$ are perfect, a graph $G$ derived via substitution of the two is perfect too. We note that this operation is also known as *Replication Lemma* in the literature and it played an important role in the proof of the WPGT (Lovász 1972).

![Figure 5](image2.png)  
**Figure 5**  
Two perfect graphs combined by substitution operation.

• **“Composition”** (Bixby 1984, Cunningham and Edmonds 1980):

Let $G_1$, $G_2$ be disjoint graphs each with at least three vertices, $v_i$ be a vertex of $G_i$, $N(v_i)$ the set of all neighbors of $v_i$. The composition of $G_1$ and $G_2$ is obtained from $G_1 \setminus \{v_1\}$ and $G_2 \setminus \{v_2\}$ by connecting all vertices in $N(v_1)$ to those in $N(v_2)$. A graph obtained from two perfect graphs via composition operation is again perfect.
Figure 6 Two perfect graphs combined by composition operation.

- **Disjoint union:**
  Let $G_1, G_2$ be two disjoint graphs. The disjoint union of $G_1$ and $G_2$ is simply $G = G_1 \cup G_2$ with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. Disjoint union of two perfect graphs is again perfect (obvious from the definition of perfect graphs).

- **Join:**
  Let $G_1, G_2$ be disjoint graphs. The join of $G_1$ and $G_2$, say $G$, is obtained by connecting all vertices in $G_1$ to all those in $G_2$. A graph obtained from two perfect graphs via join operation is perfect. To show that this operation indeed preserves perfection, assume that $G_1$ and $G_2$ are perfect. Consider $\bar{G}$ which is simply $\bar{G}_1 \cup \bar{G}_2$. $G_1$ and $G_2$ being perfect, $\bar{G}_1$ and $\bar{G}_2$ are so, too, by WPGT. As the disjoint union of two perfect graphs is perfect, $\bar{G} = \bar{G}_1 \cup \bar{G}_2$ and therefore $G$ is perfect.

- **Complement:**
  By WPGT, the complement of a perfect graph is again perfect.

The algorithm we designed to generate a random partition of a given vertex set into clusters takes a pair of integers to be respectively the lower and upper bound on the sizes of clusters as input. The first phase of the algorithm initially creates a random ordering $\sigma$ of vertices. Then, at each step, the size $r$ of the cluster under construction is set uniformly random between the lower and upper bounds input to the algorithm, and a separator is placed $r$-many elements ahead of the previous cluster’s last vertex in $\sigma$. The set of vertices between two consecutive points the separator is placed serves as one cluster. This procedure continues until all vertices in $V$ belong to some cluster.

All of the perfect graph instances and the associated vertex partitions that we have generated with the presented method can be accessed online at http://www.ie.boun.edu.tr/~taskin/data/pg/. Our algorithm for random perfect graph generation and the large collection of randomly generated perfect graph instances we provide online serve as a first step to overcome the difficulty of finding perfect graph instances in their general form.
6. Computational Study

In this section, we present the results of a series of experiments we conducted to evaluate the performance of our cutting plane procedure by comparing it with that of the integer programming formulation Model 1, and the branch-and-price algorithm by Furini et al. (2018).

We implemented the algorithms described in the previous section in C++, and executed them on a computer with 2.00-GHz Intel Xeon CPU. Throughout all the experiments, we used CPLEX version 12.8, and used the callback mechanism of it. To solve the SDP formulations, we used MOSEK version 8.1.0.24. The reason for us to select this SDP solver among several others is that MOSEK turned out to be the best-performing one according to the results of benchmark by Mittelmann (2018) (available at http://plato.asu.edu/ftp/sparse_sdp.html) conducted on a large set of problem instances, both in terms of solution times and the number of instances that are solved optimally.

We randomly generated our test instances for different \( n \) values ranging from 50 to 500, and four different average edge density values 0.1, 0.3, 0.5, and 0.7, where edge density of a graph is defined as \( \frac{m}{\frac{1}{2}(n-1)} \) with \( m \) denoting the number of edges. For each pair of \( n \) and average edge density value, we used five random graph instances.

When an instance could not be solved to optimality by any of the methods we consider, we report the optimality gap percentage, which is calculated as \( \frac{UB-LB}{UB} \times 100 \) with \( UB \) and \( LB \) denoting the upper and lower bounds respectively, to give an indication of how far a feasible solution is away from optimal.

For each one of the three methods, we set a time limit of 1200 seconds throughout all the experiments. When an instance could not be solved optimally within the limit, the solution time of that instance is taken as 1200 seconds. In our experiments, the B&P algorithm by Furini et al. failed to report optimality gaps for instances that could not be solved optimally within the time limit. For such cases, we take the optimality gap as 100%.

In our first set of experiments, we test the performance of our cutting plane approach for perfect graphs using SDP in the subproblem versus using the maximum clique algorithm MCS by Tomita et al. (2010). Table 1 summarizes the computational results for perfect graph instances with cluster sizes varying between 2 and 5. The first three columns in this table present the number of vertices ("n"), average edge density ("Avg density"), and
Table 1  Experimental results for perfect graph instances with small clusters to compare SDP with Tomita et al.’s method MCS.

| n  | Avg density | Avg # clust | Cutting Plane w/ SDP | Cutting Plane w/ MCS |
|----|-------------|-------------|----------------------|----------------------|
|    |             |             | # opt | Avg | Avg | Avg | # opt | Avg | Avg | Avg |
|    |             |             | % gap | # clust | time | overall | % gap | # clust | time | overall |
| 50 | 0.110       | 14.4        | 5     | 3.15 | 3.15 | 5     | 0.28  | 0.28 |
|    | 0.203       | 13.8        | 5     | 2.21 | 2.21 | 5     | 0.20  | 0.20 |
|    | 0.495       | 14.2        | 5     | 4.44 | 4.44 | 5     | 0.14  | 0.14 |
|    | 0.710       | 14.0        | 5     | 6.50 | 6.50 | 5     | 0.35  | 0.35 |
| 100| 0.096       | 28.4        | 5     | 94.59| 94.59| 5     | 0.29  | 0.29 |
|    | 0.300       | 29.4        | 5     | 88.51| 88.51| 5     | 0.19  | 0.19 |
|    | 0.488       | 28.0        | 5     | 105.39| 105.39| 5     | 0.34  | 0.34 |
|    | 0.705       | 28.8        | 5     | 314.56| 314.56| 5     | 1.41  | 1.41 |
| 150| 0.098       | 42.6        | 3     | 50.00| 580.73| 828.44| 5     | 0.28  | 0.28 |
|    | 0.298       | 42.2        | 5     | 720.72| 720.72| 5     | 0.20  | 0.20 |
|    | 0.498       | 44.2        | 1     | 34.58| 624.60| 1084.92| 5     | 0.50  | 0.50 |
|    | 0.693       | 43.4        | 0     | 34.29| 1200.00| 5     | 6.12  | 6.12 |
| 200| 0.107       | 56.8        | 0     | 53.33| 1200.00| 5     | 0.21  | 0.21 |
|    | 0.304       | 57.2        | 0     | 48.33| 1200.00| 5     | 0.20  | 0.20 |
|    | 0.496       | 57.2        | 0     | 48.23| 1200.00| 5     | 0.67  | 0.67 |
|    | 0.703       | 57.6        | 0     | 56.20| 1200.00| 5     | 9.09  | 425.64 |
| 250| 0.112       | 71.4        | 0     | 77.33| 1200.00| 5     | 0.19  | 0.19 |
|    | 0.304       | 71.2        | 0     | 82.42| 1200.00| 5     | 0.42  | 0.42 |
|    | 0.497       | 72.2        | 0     | 76.96| 1200.00| 5     | 1.89  | 1.89 |
|    | 0.693       | 70.2        | 0     | 70.12| 1200.00| 2     | 254.25| 821.70 |
| 300| 0.110       | 86.6        | 0     | 1200.00| 5     | 0.23  | 0.23 |
|    | 0.302       | 88.6        | 0     | 1200.00| 5     | 1.04  | 1.04 |
|    | 0.506       | 83.4        | 0     | 92.86| 1200.00| 5     | 4.72  | 4.72 |
|    | 0.691       | 85.6        | 0     | 86.41| 1200.00| 2     | 319.48| 847.79 |
| 350| 0.117       | 102.6       | 0     | 1200.00| 5     | 0.23  | 0.23 |
|    | 0.301       | 100.0       | 0     | 1200.00| 5     | 1.06  | 1.06 |
|    | 0.508       | 98.4        | 0     | 96.91| 1200.00| 4     | 15.55 | 252.44 |
|    | 0.698       | 99.6        | 0     | 96.21| 1200.00| 0     | 18.72 | 1200.00|
| 400| 0.111       | 114.8       | 0     | 1200.00| 5     | 0.34  | 0.34 |
|    | 0.315       | 114.0       | 0     | 1200.00| 5     | 0.65  | 0.65 |
|    | 0.502       | 114.2       | 0     | 1200.00| 4     | 8.33  | 25.32 | 260.25 |
|    | 0.692       | 112.8       | 0     | 97.71| 1200.00| 0     | 24.55 | 1200.00|
| 450| 0.117       | 130.2       | 0     | 1200.00| 5     | 0.39  | 0.39 |
|    | 0.309       | 130.4       | 0     | 1200.00| 5     | 2.02  | 2.02 |
|    | 0.507       | 128.4       | 0     | 1200.00| 4     | 10.00 | 64.26 | 291.41 |
|    | 0.696       | 125.8       | 0     | 1200.00| 0     | 33.30 | 1200.00|
| 500| 0.117       | 142.8       | 0     | 1200.00| 5     | 0.59  | 0.59 |
|    | 0.296       | 143.0       | 0     | 1200.00| 5     | 1.05  | 1.05 |
|    | 0.507       | 144.2       | 0     | 1200.00| 1     | 15.22 | 74.45 | 974.99 |
|    | 0.695       | 141.0       | 0     | 98.36| 1200.00| 0     | 30.15 | 1200.00|

average number of clusters ("Avg # clust") across five random instances. In the next two groups of columns, we report the results of our experiments for the two versions of our algorithm for perfect graphs under “Cutting Plane w/ SDP” and “Cutting Plane w/ MCS” headings, respectively. For the cutting plane method coupled with SDP method, columns 4–7 show the number of instances that could be optimally solved among five ("# opt"), average optimality gap percentages over instances that could not be solved to optimality within the given time limit of 1200 seconds ("Avg % gap in nonopt"), average solution time in seconds over instances that are optimally solved ("Avg time in opt"), and average
solution time over all instances (“Avg overall time”). Columns 8–11 list the same set of results as columns 4–7 for the cutting plane method coupled with MCS. In each row, we report average values across runs on five independent instances. We summarize the performance comparison of the two alternative algorithms in the bottom row by providing the sum for “# opt” column and the averages for other columns.

Our observation from the results listed in Table 1 is that solving the subproblem via Tomita et al.’s MCS algorithm clearly yields superior results in terms of the number of instances solved to optimality, average optimality gap, and average amount of time spent. As \( n \) and edge density increase, the performance of both methods deteriorate as expected; however, coupling of the cutting plane method with MCS outperforms the other in every aspect for all the instances. Out of the 200 instances we experiment with, the version that uses MCS could optimally solve 166 of them, whereas the one using SDP could only solve 49 instances to optimality. Moreover, when we use SDP in the subproblem, we observed that SDP could not even finish solving the maximum clique problem for the first selection the master problem outputs in many instances with 300 or more vertices. In such cases, no optimality gap could be reported, which is revealed by the empty cells in “Avg % gap in nonopt” column in groups of instances for which the number of optimally solved instances shown in the fourth column is zero. In terms of the overall averages shown in the bottom row, when SDP is used in the subproblem, the average % gap and average time spent over all instances are three times higher, and the average time spent in optimally solved instances is seven times higher.

The results in Table 1 show that using SDP in the subproblem of our cutting plane method leads to relatively poor performance in all respects. A study by Yıldırım and Fan-Orzechowski (2006) proposes an SDP-based algorithm to solve the maximum stable set problem in perfect graphs, which shows better performance than the one we utilize in several test instances that they use in their experiments. However, the improvement that their algorithm achieves is far from being comparable to that we achieve by using MCS in the subproblem instead of SDP. Therefore, we did not test the method suggested in (Yıldırım and Fan-Orzechowski 2006), and decided to utilize MCS in the subproblem of our solution procedure for the rest of our computational experiments.

In the remaining portion of this section, we present the experimental results of the IP formulation, B&P algorithm by Furini et al. (2018), and our cutting plane algorithm. Table
summarizes the computational results for perfect graph instances having edge density 0.1 and 0.3 (which are referred to as low-density) with cluster sizes varying between 2 and 5. The first three columns are the same as in Table 1. In the next three groups of columns, we report the results of our experiments for the three algorithms under “IP formulation”, “B&P”, and “Cutting Plane” headings, respectively. For the IP formulation, columns 4–8 show the number of instances that could be optimally solved among five (“# opt”), average optimality gap percentages over instances that could not be solved to optimality within the given time limit of 1200 seconds (“Avg % gap in nonopt”) and over all instances (“Avg % gap overall”), average solution time in seconds over instances that are optimally solved (“Avg time in opt”) and over all instances (“Avg time overall”). Columns 9–13 and 14–18 list the same set of results respectively for B&P of Furini et al. and our cutting plane method. Finally, the rightmost column shows the average time spent in the subproblem of our cutting plane algorithm in seconds across five instances (“Avg time in subpr”). In each row of this table, we report average values across five independent runs. The bottom row provides the totals for columns containing the number of instances solved optimally (“# opt”), and the averages for all other columns.

Next, we present the results of our experiments conducted on instances having edge density 0.5 and 0.7 with cluster sizes ranging between 2 and 5 in Table 7. The structure of this table is the same as Table 6. From the results listed in Tables 6–7, we observe that our approach yields superior results to both of the other two in terms of time and optimality gap. Our method solves all of the low-density instances optimally, while the IP formulation and the B&P method can solve 78% and 53% of low density instances optimally, respectively. The average solution time of our method is considerably lower in general, but the difference becomes more noticeable in optimally solved instances. As \( n \) grows, the performance of all three methods worsen in general. Increasing edge density, however, results in improved performance for B&P method, while deteriorating that of IP formulation and cutting plane method, as previously. Nevertheless, in terms of overall average of percentage optimality gap, the outperformance of our method to the other two persists even in high-density instances.

We conducted two additional sets of experiments on perfect graphs in order to test the effect of cluster sizes. Tables 8–11 report the results obtained on the same set of graphs as before but with cluster sizes between 4–7 and 6–9, respectively. Comparing the
results in Tables 6–7 to those in Tables 8–11, we observe considerable improvement in the performance of all methods. For a given \( n \) value, as average size of clusters increases, the total number of clusters and hence the number of variables and constraints in the IP formulation reduce. This shrinkage in the size leads to improved performance. When cluster sizes vary between 4 and 7, the IP formulation outperforms the B&P method in all respects in graphs with average density 0.1 and 0.3, as we can see from the bottom row of Table 8. For graphs with high density, i.e., those with average density 0.5 and 0.7 and with medium-sized clusters, the B&P algorithm yields the highest number of instances solved optimally. Nevertheless, in terms of overall average of percentage optimality gaps, our cutting plane method performs the best as in the previous set with small clusters. Finally, when we examine the values in Tables 10–11, which present the results when the cluster sizes are large (between 6 and 9 in particular), we observe that our approach yields the best results in all respects regardless of density.

Next, we provide a brief synopsis of our results in Table 2. Out of the 600 instances in total, the cutting plane algorithm was able to solve about 92% of them to optimality, whereas IP and B&P could solve 80% and 84%, respectively. In terms of the average optimality gap, our algorithm yields an order of magnitude better optimality gaps on the average as compared to the B&P algorithm, and the average solution time is about 30% and 27% of those of IP and B&P, respectively. Although the B&P method is able to yield a higher number of optimally solved instances for high-density graphs in the case of small and medium-sized clusters, the cutting plane method still delivers the best average optimality gaps and average solution times both in high-density graphs and in general.

| Sizes of clusters | Density | IP formulation | | B&P | | Cutting plane |
|-------------------|---------|----------------|----------------|----------------|----------------|
|                   |         | # opt | Avg % gap | Avg time | # opt | Avg % gap | Avg time | # opt | Avg % gap | Avg time |
| small             | low     | 78    | 19.37 | 372.23 | 53    | 47.00 | 731.35 | 100   | 0.00 | 0.48     |
|                   | high    | 48    | 44.03 | 707.97 | 88    | 12.00 | 469.88 | 66    | 7.26 | 434.51   |
|                   | all     | 126   | 31.70 | 540.10 | 141   | 29.50 | 600.62 | 166   | 3.63 | 217.50   |
| medium            | low     | 97    | 2.75  | 175.36 | 80    | 20.00 | 339.75 | 100   | 0.00 | 0.96     |
|                   | high    | 73    | 24.17 | 448.30 | 92    | 8.00  | 365.42 | 88    | 3.57 | 177.81   |
|                   | all     | 170   | 13.46 | 311.83 | 172   | 14.00 | 352.50 | 188   | 1.78 | 89.39    |
| large             | low     | 100   | 0.00  | 72.54  | 100   | 0.00  | 9.34   | 100   | 0.00 | 0.47     |
|                   | high    | 85    | 15.00 | 302.46 | 92    | 8.00  | 331.86 | 100   | 0.00 | 8.42     |
|                   | all     | 185   | 7.50  | 187.50 | 192   | 4.00  | 170.60 | 200   | 0.00 | 4.44     |
|                   |         | 481   | 17.55 | 346.48 | 505   | 15.83 | 374.60 | 554   | 1.80 | 103.77   |

| Table 2 Summary of experimental results for all perfect graph instances. |
Now, let us compare the performance of our algorithm for general perfect graphs with that for the subclasses of perfect graphs investigated in (Şeker et al. 2018); namely, permutation, generalized split, and chordal graphs. The subproblems in these three graph classes were solved via specialized combinatorial algorithms that are polynomial-time, whereas the maximum clique algorithm MCS is not so, though it runs quite efficiently in practice. Using the same experimental environment as we did in (Şeker et al. 2018), we run our cutting plane algorithm for general perfect graphs on the test instances from the three subclasses of perfect graphs. The number of vertices of these instances range from 100 to 500 for permutation and generalized split graphs, and from 100 to 1000 for chordal graphs. The average edge densities are the same as here; namely, 0.1, 0.3, 0.5, and 0.7. The total number of instances tested are 1200 for chordal graphs, and 600 for the other two classes.

Table 3 summarizes the results for permutation graphs. The structure of this table is the same as Table 2, except that the two sets of columns list the results for our algorithms for the general and special cases, respectively. Our first observation from this table is that the general algorithm surprisingly yields better results than the one tailored for permutation graphs. The improvement is particularly evident in high-density instances in terms of all three measures we list here. The average % gap value in high-density instances with small clusters drops from 23.32% to 12.98%, and the overall average of optimality gap improves by 5%. Table 4 contains the summary of the results we obtained for generalized split graphs and has the same format as the previous one. As opposed to the case of permutation graphs, there is no monotonicity in the change of the performance between the cutting plane and the decomposition algorithm, not even within a given density or cluster size.
category. The overall performance citep deteriorates when we use our algorithm for general perfect graphs, but it is still comparable to that of the algorithm tailored for generalized split graphs.

We finally present the computational results we obtained for chordal graphs in Table 5, which has the same structure as the previous two. The algorithm we present in (Şeker et al. 2018) yields the best results in the class of chordal graphs by solving all of the instances to optimality in approximately 0.16 seconds. In this case, the difference between the two methods is clear; the one custom-tailored for chordal graphs clearly outperforms in all respects.

7. Conclusions and Future Research

In this paper, we presented an exact cutting plane algorithm for the selective graph coloring problem in perfect graphs, which is a generalization of the method presented in (Şeker et al. 2018). We also introduced an algorithm to generate random perfect graphs, which, to the best of our knowledge is the first algorithm for this purpose in the literature. Given an input
graph with a partition of its vertex set into clusters, the master problem of our cutting plane procedure seeks an optimal selection, and the subproblem seeks a maximum clique in the graph induced by that selection by making use of two alternative methods. One of these is based on semidefinite programming and works in polynomial time (up to a selected accuracy) in the class of perfect graphs. The other one is a general-purpose maximum clique algorithm from the literature and performs quite efficiently in practice. We tested the performance of our algorithm on a large suite of randomly generated problem instances, and compared the results to those of an IP formulation and a branch-and-price algorithm from the literature. The computational results show that the cutting plane algorithm significantly improved the solution performance in general and the improvement manifests most evidently in low-density graphs. We also compared the performance of our cutting plane algorithm for perfect graphs in its general form to that of our previous algorithm tailored for three subclasses of perfect graphs; namely, permutation, generalized split, and chordal graphs. The use of our cutting plane algorithm for general perfect graphs resulted in better performance in permutation graphs, and marked deterioration in chordal graphs regardless of edge density. In the class of generalized split graphs, the overall performance became citep worse with the algorithm for general perfect graphs.

As future research, the presented solution strategy can be adapted to graph classes where the size of a maximum clique is not necessarily equal to the chromatic number. In that case, the clique cuts used here will not be sufficient, different cuts will be needed additionally. Such cuts would require the use of a coloring algorithm in the subproblem of our solution procedure. Even though both the maximum clique and minimum coloring problems are NP-hard in general, coloring problem usually turns out to be more difficult in practice. To facilitate the solution procedure, alternative cuts can be designed and incorporated into the presented cutting plane method. As for perfect graph generation, our proposed algorithm can be enriched by including different perfection-preserving methods.

Acknowledgement
We are grateful to Pınar Heggernes for her helpful suggestions in the perfect graph generation algorithm when she was on her sabbatical leave at Boğaziçi University and when the first author was a visiting scholar at the University of Bergen.
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**Appendix**

The Appendix can be moved to Online Supplement if needed.
Table 6  Experimental results for low-density perfect graph instances with small clusters.

| n   | Avg density | Avg # clust | IP formulation | B&P | Cutting Plane |
|-----|-------------|-------------|----------------|-----|---------------|
|     | Avg # opt | Avg % gap in nonopt | Avg % gap overall | Avg time in opt | Avg % gap in nonopt | Avg % gap overall | Avg time in opt | Avg time overall | Avg % gap in nonopt | Avg % gap overall | Avg time in opt | Avg time overall | Avg % gap overall |
| 50  | 0.110     | 14.4        | 5              | 0.00 | 0.13          | 0.13 | 5              | 0.00           | 0.14          | 0.14 | 5              | 0.00           | 0.28          | 0.28           | 45.43          |
|     | 0.293     | 13.8        | 5              | 0.00 | 0.21          | 0.21 | 5              | 0.00           | 2.02          | 2.02 | 5              | 0.00           | 0.20          | 0.20           | 37.60          |
| 100 | 0.096     | 28.4        | 5              | 0.00 | 0.90          | 0.90 | 5              | 0.00           | 23.07         | 23.07 | 5              | 0.00           | 0.29          | 0.29           | 44.59          |
|     | 0.300     | 29.4        | 5              | 0.00 | 3.24          | 3.24 | 5              | 0.00           | 32.02         | 32.02 | 5              | 0.00           | 0.19          | 0.19           | 45.24          |
| 150 | 0.098     | 42.6        | 5              | 0.00 | 2.97          | 2.97 | 5              | 0.00           | 146.58        | 146.58 | 5              | 0.00           | 0.28          | 0.28           | 48.99          |
|     | 0.298     | 42.2        | 5              | 0.00 | 8.70          | 8.70 | 4              | 100.00         | 140.79        | 352.63 | 5              | 0.00           | 0.20          | 0.20           | 44.94          |
| 200 | 0.107     | 56.8        | 5              | 0.00 | 7.93          | 7.93 | 5              | 0.00           | 429.52        | 429.52 | 5              | 0.00           | 0.21          | 0.21           | 54.51          |
|     | 0.304     | 57.2        | 5              | 0.00 | 36.53         | 36.53 | 5              | 0.00           | 293.05        | 293.05 | 5              | 0.00           | 0.20          | 0.20           | 54.27          |
| 250 | 0.112     | 71.4        | 5              | 0.00 | 30.25         | 30.25 | 5              | 0.00           | 1023.62       | 1023.62 | 5              | 0.00           | 0.19          | 0.19           | 49.82          |
|     | 0.304     | 71.2        | 5              | 0.00 | 85.98         | 85.98 | 5              | 0.00           | 712.18        | 712.18 | 5              | 0.00           | 0.42          | 0.42           | 51.83          |
| 300 | 0.110     | 86.6        | 5              | 0.00 | 74.87         | 74.87 | 5              | 0.00           | 100.00        | 100.00 | 1200.00        | 5              | 0.23          | 0.23           | 57.38          |
|     | 0.302     | 88.6        | 5              | 0.00 | 340.37        | 340.37 | 4              | 100.00         | 726.59        | 821.27 | 5              | 0.00           | 1.04          | 1.04           | 62.28          |
| 350 | 0.117     | 102.6       | 5              | 0.00 | 204.47        | 204.47 | 0              | 100.00         | 1200.00       | 1200.00 | 5              | 0.00           | 0.23          | 0.23           | 63.60          |
|     | 0.301     | 100.0       | 4              | 94.52 | 189.00     | 689.68 | 791.74 | 100.00         | 1200.00       | 1200.00 | 5              | 0.00           | 0.51          | 0.51           | 69.82          |
| 400 | 0.111     | 114.8       | 5              | 0.00 | 265.87        | 265.87 | 0              | 100.00         | 1200.00       | 1200.00 | 5              | 0.00           | 0.34          | 0.34           | 70.10          |
|     | 0.315     | 114.0       | 0              | 51.02 | 51.02       | 1200.00 | 0              | 100.00         | 1200.00       | 1200.00 | 5              | 0.00           | 0.65          | 0.65           | 68.71          |
| 450 | 0.117     | 130.2       | 3              | 98.42 | 39.37       | 523.80 | 794.28 | 0              | 100.00         | 1200.00       | 1200.00 | 5              | 0.00           | 0.39          | 0.39           | 69.22          |
|     | 0.309     | 130.4       | 0              | 99.17 | 99.17       | 1200.00 | 0              | 100.00         | 1200.00       | 1200.00 | 5              | 0.00           | 2.02          | 2.02           | 76.74          |
| 500 | 0.117     | 142.8       | 1              | 98.59 | 78.87       | 1180.90 | 1196.18 | 0              | 100.00         | 1200.00       | 1200.00 | 5              | 0.00           | 0.59          | 0.59           | 71.36          |
|     | 0.296     | 143.0       | 0              | 100.00 | 100.00     | 1200.00 | 0              | 100.00         | 1200.00       | 1200.00 | 5              | 0.00           | 1.05          | 1.05           | 71.71          |
|     |           |             |                | 78   | 90.29        | 19.37  | 203.34 | 372.23         | 53              | 100.00        | 47.00          | 320.05        | 731.35        | 100   | -               | 0.00           | 0.48          | 0.48           | 57.91          |
Table 7  Experimental results for high-density perfect graph instances with small clusters.

| n   | Avg density | Avg # clust | IP formulation | B&P | Cutting Plane |
|-----|-------------|-------------|----------------|-----|---------------|
|     |             |             | Avg # opt | Avg % gap in nonopt | Avg % gap overall | Avg time in opt | Avg time overall | # opt | Avg % gap in nonopt | Avg % gap overall | Avg time in opt | Avg time overall | Avg % time in subpr |
| 50  | 0.495       | 14.2        | 5         | 0.00                    | 0.62               | 0.62            | 5               | 0.00              | 2.33              | 2.33            | 5               | 0.00              | 0.14              | 0.14              | 35.20 |
|     | 0.710       | 14.0        | 5         | 0.00                    | 0.79               | 0.79            | 5               | 0.00              | 1.91              | 1.91            | 5               | 0.00              | 0.35              | 0.35              | 50.82 |
| 100 | 0.488       | 28.0        | 5         | 0.00                    | 3.02               | 3.02            | 5               | 0.00              | 17.38             | 17.38           | 5               | 0.00              | 0.34              | 0.34              | 55.38 |
|     | 0.705       | 28.8        | 5         | 0.00                    | 5.65               | 5.65            | 5               | 0.00              | 14.91             | 14.91           | 5               | 0.00              | 1.41              | 1.41              | 31.38 |
| 150 | 0.498       | 44.2        | 5         | 0.00                    | 23.10              | 23.10           | 5               | 0.00              | 68.55             | 68.55           | 5               | 0.00              | 0.50              | 0.50              | 53.36 |
|     | 0.693       | 43.4        | 5         | 0.00                    | 38.99              | 38.99           | 5               | 0.00              | 61.09             | 61.09           | 5               | 0.00              | 6.12              | 6.12              | 19.41 |
| 200 | 0.496       | 57.2        | 5         | 0.00                    | 84.10              | 84.10           | 5               | 0.00              | 152.09            | 152.09          | 5               | 0.00              | 0.67              | 0.67              | 55.49 |
|     | 0.703       | 57.6        | 4         | 9.09                    | 1.82               | 637.24          | 749.79          | 5               | 0.00              | 102.61            | 102.61          | 4               | 9.09              | 1.82              | 232.05            | 425.64 | 3.11 |
| 250 | 0.497       | 72.2        | 5         | 0.00                    | 480.98             | 480.98          | 5               | 0.00              | 253.53            | 253.53          | 5               | 0.00              | 1.89              | 1.89              | 48.41 |
|     | 0.693       | 70.2        | 2         | 23.89                   | 14.33              | 830.33          | 1052.13         | 5               | 0.00              | 215.79            | 215.79          | 2               | 13.33             | 8.00              | 254.25            | 821.70 | 1.94 |
| 300 | 0.506       | 83.4        | 2         | 24.68                   | 14.81              | 500.43          | 920.17          | 5               | 0.00              | 507.82            | 507.82          | 5               | 18.51             | 11.10             | 319.48            | 847.79 | 4.03 |
|     | 0.691       | 85.6        | 0         | 83.96                   | 89.96              | 1200.00         | 0.00            | 5               | 0.00              | 358.41            | 358.41          | 0               | 18.72             | 18.72             | 1200.00           | 3.26   |
| 350 | 0.508       | 98.4        | 0         | 94.75                   | 94.75              | 1200.00         | 5               | 0.00              | 657.70            | 657.70          | 4               | 8.33              | 1.67              | 15.55             | 252.44            | 20.48 |
|     | 0.698       | 99.6        | 0         | 89.50                   | 89.50              | 1200.00         | 5               | 0.00              | 526.98            | 526.98          | 0               | 24.55             | 24.55             | 1200.00           | 4.76   |
| 400 | 0.502       | 114.2       | 0         | 86.91                   | 86.91              | 1200.00         | 4               | 100.00             | 1092.62           | 1114.09         | 4               | 8.33              | 1.67              | 25.32             | 260.25            | 31.24 |
|     | 0.692       | 112.8       | 0         | 97.81                   | 97.81              | 1200.00         | 5               | 0.00              | 742.60            | 742.60          | 0               | 24.55             | 24.55             | 1200.00           | 4.76   |
| 450 | 0.507       | 128.4       | 0         | 98.72                   | 98.72              | 1200.00         | 2               | 100.00             | 975.35            | 1110.14         | 3               | 10.00             | 2.00              | 64.26             | 291.41            | 33.84 |
|     | 0.696       | 125.8       | 0         | 98.32                   | 98.32              | 1200.00         | 2               | 100.00             | 1059.82           | 1059.82         | 0               | 30.15             | 30.15             | 1200.00           | 7.89   |
| 500 | 0.507       | 144.2       | 0         | 99.59                   | 99.59              | 1200.00         | 2               | 100.00             | 1193.47           | 1197.39         | 1               | 15.22             | 12.18             | 74.45             | 974.89            | 9.07   |
|     | 0.695       | 141.0       | 0         | 100.00                  | 100.00             | 1200.00         | 5               | 0.00              | 1059.82           | 1059.82         | 0               | 30.15             | 30.15             | 1200.00           | 7.89   |

48  75.60  44.03  236.84  707.97  88  100.00  12.00  423.02  469.88  66  17.23  7.26  62.59  434.51  25.07
| n  | Avg density | Avg # clust | IP formulation | B&P | Cutting Plane |
|----|-------------|-------------|----------------|-----|---------------|
|    |             |             | # opt | Avg % gap in nonopt | Avg time in opt | # opt | Avg % gap in nonopt | Avg time in opt | # opt | Avg % gap in nonopt | Avg time in opt |
| 50 | 0.110       | 9.0         | 5     | 0.00 | 0.08 | 0.08 | 5     | 0.00 | 0.01 | 0.01 | 5     | 0.00 | 0.10 | 0.10 | 5     | 0.00 | 0.14 | 0.14 | 35.13 |
| 100| 0.096       | 17.8        | 5     | 0.00 | 0.20 | 0.20 | 5     | 0.00 | 0.02 | 0.02 | 5     | 0.00 | 0.24 | 0.24 | 42.40 |
| 150| 0.098       | 27.6        | 5     | 0.00 | 1.04 | 1.04 | 5     | 0.00 | 0.06 | 0.06 | 5     | 0.00 | 0.26 | 0.26 | 55.77 |
| 200| 0.107       | 36.8        | 5     | 0.00 | 1.71 | 1.71 | 5     | 0.00 | 0.08 | 0.08 | 5     | 0.00 | 0.38 | 0.38 | 57.27 |
| 250| 0.112       | 45.0        | 5     | 0.00 | 16.56 | 16.56 | 5     | 0.00 | 1.29 | 1.29 | 5     | 0.00 | 0.43 | 0.43 | 62.99 |
| 300| 0.110       | 54.4        | 5     | 0.00 | 10.13 | 10.13 | 5     | 0.00 | 14.05 | 14.05 | 5     | 0.00 | 0.46 | 0.46 | 73.36 |
| 350| 0.117       | 63.6        | 5     | 0.00 | 22.88 | 22.88 | 5     | 0.00 | 163.62 | 163.62 | 5     | 0.00 | 1.21 | 1.21 | 84.68 |
| 400| 0.111       | 71.4        | 5     | 0.00 | 33.68 | 33.68 | 5     | 0.00 | 221.57 | 221.57 | 5     | 0.00 | 1.06 | 1.06 | 81.53 |
| 450| 0.117       | 82.2        | 5     | 0.00 | 493.34 | 493.34 | 5     | 0.00 | 221.57 | 221.57 | 5     | 0.00 | 1.06 | 1.06 | 83.63 |
| 500| 0.117       | 90.6        | 5     | 0.00 | 449.23 | 449.23 | 5     | 0.00 | 190.43 | 190.43 | 5     | 0.00 | 1.06 | 1.06 | 87.44 |

Table 8 Experimental results for low-density perfect graph instances with medium-sized clusters.

| n  | Avg density | Avg # clust | IP formulation | B&P | Cutting Plane |
|----|-------------|-------------|----------------|-----|---------------|
|    |             |             | # opt | Avg % gap in nonopt | Avg time in opt | # opt | Avg % gap in nonopt | Avg time in opt | # opt | Avg % gap in nonopt | Avg time in opt |
| 50 | 0.110       | 9.0         | 5     | 0.00 | 0.08 | 0.08 | 5     | 0.00 | 0.01 | 0.01 | 5     | 0.00 | 0.10 | 0.10 | 5     | 0.00 | 0.14 | 0.14 | 35.13 |
| 100| 0.096       | 17.8        | 5     | 0.00 | 0.20 | 0.20 | 5     | 0.00 | 0.02 | 0.02 | 5     | 0.00 | 0.24 | 0.24 | 42.40 |
| 150| 0.098       | 27.6        | 5     | 0.00 | 1.04 | 1.04 | 5     | 0.00 | 0.06 | 0.06 | 5     | 0.00 | 0.26 | 0.26 | 55.77 |
| 200| 0.107       | 36.8        | 5     | 0.00 | 1.71 | 1.71 | 5     | 0.00 | 0.08 | 0.08 | 5     | 0.00 | 0.38 | 0.38 | 57.27 |
| 250| 0.112       | 45.0        | 5     | 0.00 | 16.56 | 16.56 | 5     | 0.00 | 1.29 | 1.29 | 5     | 0.00 | 0.43 | 0.43 | 62.99 |
| 300| 0.110       | 54.4        | 5     | 0.00 | 10.13 | 10.13 | 5     | 0.00 | 14.05 | 14.05 | 5     | 0.00 | 0.46 | 0.46 | 73.36 |
| 350| 0.117       | 63.6        | 5     | 0.00 | 22.88 | 22.88 | 5     | 0.00 | 163.62 | 163.62 | 5     | 0.00 | 1.21 | 1.21 | 84.68 |
| 400| 0.111       | 71.4        | 5     | 0.00 | 33.68 | 33.68 | 5     | 0.00 | 221.57 | 221.57 | 5     | 0.00 | 1.06 | 1.06 | 81.53 |
| 450| 0.117       | 82.2        | 5     | 0.00 | 493.34 | 493.34 | 5     | 0.00 | 221.57 | 221.57 | 5     | 0.00 | 1.06 | 1.06 | 83.63 |
| 500| 0.117       | 90.6        | 5     | 0.00 | 449.23 | 449.23 | 5     | 0.00 | 190.43 | 190.43 | 5     | 0.00 | 1.06 | 1.06 | 87.44 |

97  93.75  2.75  165.91  175.36  80  100.00  20.00  119.80  339.75  100  -  0.00  0.96  0.96  68.16
Table 9 Experimental results for high-density perfect graph instances with medium-sized clusters.

| n  | Avg density | Avg # clust | IP formulation | B&P | Cutting Plane |
|----|-------------|------------|----------------|-----|---------------|
|    | Avg # opt | Avg % gap in | Avg % gap | Avg time in | Avg time in | Avg % gap in | Avg % gap | Avg time in | Avg time in | Avg % gap in | Avg % gap | Avg time in | Avg time in | Avg % time in |
|    |    | nonopt | overall | opt | overall | nonopt | overall | opt | overall | nonopt | overall | subpr |
| 50 | 0.495 | 9.0 | 0.00 | 0.15 | 0.15 | 5 | 0.00 | 0.04 | 0.04 | 5 | 0.00 | 0.22 | 0.22 | 45.21 |
| 100 | 0.488 | 17.6 | 5 | 0.00 | 3.20 | 3.20 | 5 | 0.00 | 3.44 | 3.44 | 5 | 0.00 | 0.47 | 0.47 | 66.76 |
| 150 | 0.498 | 26.6 | 5 | 0.00 | 9.84 | 9.84 | 5 | 0.00 | 20.0 | 20.1 | 5 | 0.00 | 1.64 | 1.64 | 60.08 |
| 200 | 0.496 | 35.6 | 5 | 0.00 | 19.81 | 19.81 | 5 | 0.00 | 98.02 | 98.02 | 5 | 0.00 | 1.18 | 1.08 | 70.71 |
| 250 | 0.497 | 45.2 | 5 | 0.00 | 58.95 | 58.95 | 5 | 0.00 | 209.77 | 209.77 | 5 | 0.00 | 0.65 | 0.65 | 70.86 |
| 300 | 0.506 | 55.0 | 5 | 0.00 | 195.94 | 195.94 | 5 | 0.00 | 450.78 | 450.78 | 5 | 0.00 | 0.98 | 0.98 | 74.14 |
| 350 | 0.508 | 63.8 | 4 | 0.00 | 200.47 | 200.47 | 5 | 0.00 | 734.28 | 734.28 | 5 | 0.00 | 1.71 | 1.71 | 67.28 |
| 400 | 0.502 | 73.0 | 5 | 0.00 | 713.59 | 713.59 | 4 | 0.00 | 200.47 | 200.47 | 5 | 0.00 | 734.28 | 734.28 | 67.28 |
| 450 | 0.507 | 82.8 | 2 | 0.00 | 200.47 | 200.47 | 5 | 0.00 | 200.47 | 200.47 | 5 | 0.00 | 1.18 | 1.08 | 70.71 |
| 500 | 0.507 | 91.2 | 0 | 0.00 | 895.48 | 895.48 | 1 | 0.00 | 895.48 | 895.48 | 5 | 0.00 | 6.02 | 6.02 | 54.75 |

73 81.76 24.17 217.16 448.30 92 100.00 8.00 310.98 365.42 88 27.41 3.57 56.94 177.81 46.09
Table 10  Experimental results for low-density perfect graph instances with large clusters.

| n   | Avg density | Avg # clust | IP formulation | B&P | Cutting Plane |
|-----|-------------|-------------|----------------|-----|---------------|
|     |             |             | # opt | Avg % gap in nonopt | Avg % gap overall | Avg time in opt | Avg time overall | # opt | Avg % gap in nonopt | Avg % gap overall | Avg time in opt | Avg time overall | # opt | Avg % gap in nonopt | Avg % gap overall | Avg time in opt | Avg time overall | Avg % time in subpr |
| 50  | 0.110       | 6.6         | 5    | 0.00 | 0.06  | 0.06 | 5    | 0.00 | 0.01  | 0.01 | 5    | 0.00 | 0.01  | 0.01 | 5    | 0.00 | 0.17  | 0.17 | 19.72             |
|     | 0.293       | 6.6         | 5    | 0.00 | 0.08  | 0.08 | 5    | 0.00 | 0.01  | 0.01 | 5    | 0.00 | 0.01  | 0.01 | 5    | 0.00 | 0.19  | 0.19 | 29.79             |
| 100 | 0.096       | 13.4        | 5    | 0.00 | 0.16  | 0.16 | 5    | 0.00 | 0.02  | 0.02 | 5    | 0.00 | 0.02  | 0.02 | 5    | 0.00 | 0.10  | 0.10 | 30.79             |
|     | 0.300       | 13.2        | 5    | 0.00 | 0.72  | 0.72 | 5    | 0.00 | 0.03  | 0.03 | 5    | 0.00 | 0.03  | 0.03 | 5    | 0.00 | 0.16  | 0.16 | 43.86             |
| 150 | 0.098       | 19.4        | 5    | 0.00 | 0.53  | 0.53 | 5    | 0.00 | 0.08  | 0.08 | 5    | 0.00 | 0.08  | 0.08 | 5    | 0.00 | 0.23  | 0.23 | 38.11             |
|     | 0.298       | 19.8        | 5    | 0.00 | 1.01  | 1.01 | 5    | 0.00 | 0.05  | 0.05 | 5    | 0.00 | 0.05  | 0.05 | 5    | 0.00 | 0.38  | 0.38 | 53.32             |
| 200 | 0.107       | 26.8        | 5    | 0.00 | 1.03  | 1.03 | 5    | 0.00 | 0.11  | 0.11 | 5    | 0.00 | 0.11  | 0.11 | 5    | 0.00 | 0.38  | 0.38 | 53.28             |
|     | 0.304       | 26.6        | 5    | 0.00 | 3.37  | 3.37 | 5    | 0.00 | 0.14  | 0.14 | 5    | 0.00 | 0.14  | 0.14 | 5    | 0.00 | 0.50  | 0.50 | 64.61             |
| 250 | 0.112       | 33.2        | 5    | 0.00 | 2.67  | 2.67 | 5    | 0.00 | 0.24  | 0.24 | 5    | 0.00 | 0.24  | 0.24 | 5    | 0.00 | 0.23  | 0.23 | 50.76             |
|     | 0.304       | 33.4        | 5    | 0.00 | 9.18  | 9.18 | 5    | 0.00 | 0.36  | 0.36 | 5    | 0.00 | 0.36  | 0.36 | 5    | 0.00 | 0.37  | 0.37 | 69.05             |
| 300 | 0.110       | 40.2        | 5    | 0.00 | 6.15  | 6.15 | 5    | 0.00 | 0.52  | 0.52 | 5    | 0.00 | 0.52  | 0.52 | 5    | 0.00 | 0.30  | 0.30 | 53.70             |
|     | 0.302       | 40.2        | 5    | 0.00 | 22.06 | 22.06 | 5    | 0.00 | 4.85  | 4.85 | 5    | 0.00 | 4.85  | 4.85 | 5    | 0.00 | 0.63  | 0.63 | 71.75             |
| 350 | 0.117       | 46.2        | 5    | 0.00 | 14.54 | 14.54 | 5    | 0.00 | 0.75  | 0.75 | 5    | 0.00 | 0.75  | 0.75 | 5    | 0.00 | 0.29  | 0.29 | 60.45             |
|     | 0.301       | 47.6        | 5    | 0.00 | 52.97 | 52.97 | 5    | 0.00 | 85.89 | 85.89 | 5    | 0.00 | 85.89 | 85.89 | 5    | 0.00 | 0.72  | 0.72 | 70.16             |
| 400 | 0.111       | 53.2        | 5    | 0.00 | 22.06 | 22.06 | 5    | 0.00 | 1.24  | 1.24 | 5    | 0.00 | 1.24  | 1.24 | 5    | 0.00 | 0.43  | 0.43 | 66.04             |
|     | 0.315       | 52.6        | 5    | 0.00 | 137.30 | 137.30 | 5    | 0.00 | 1.51  | 1.51 | 5    | 0.00 | 1.51  | 1.51 | 5    | 0.00 | 0.90  | 0.90 | 81.08             |
| 450 | 0.117       | 59          | 5    | 0.00 | 137.30 | 137.30 | 5    | 0.00 | 1.90  | 1.90 | 5    | 0.00 | 1.90  | 1.90 | 5    | 0.00 | 0.47  | 0.47 | 69.24             |
|     | 0.309       | 59.4        | 5    | 0.00 | 439.23 | 439.23 | 5    | 0.00 | 44.07 | 44.07 | 5    | 0.00 | 44.07 | 44.07 | 5    | 0.00 | 1.24  | 1.24 | 84.12             |
| 500 | 0.117       | 65.8        | 5    | 0.00 | 143.72 | 143.72 | 5    | 0.00 | 2.71  | 2.71 | 5    | 0.00 | 2.71  | 2.71 | 5    | 0.00 | 0.73  | 0.73 | 76.06             |
|     | 0.296       | 66.4        | 5    | 0.00 | 480.92 | 480.92 | 5    | 0.00 | 42.40 | 42.40 | 5    | 0.00 | 42.40 | 42.40 | 5    | 0.00 | 0.94  | 0.94 | 82.08             |

100 - 0.00 | 72.54 | 72.54 | 100 - 0.00 | 9.34 | 9.34 | 100 - 0.00 | 0.47 | 0.47 | 58.40
Table 11  Experimental results for high-density perfect graph instances with large clusters.

| n  | Avg density | Avg # clust | IP formulation | B&P | Cutting Plane |
|----|-------------|-------------|----------------|-----|---------------|
|    |             |             | Avg % gap in nonopt | Avg % gap overall | Avg time in opt | Avg % gap in nonopt | Avg % gap overall | Avg time in opt | Avg % gap in nonopt | Avg % gap overall | Avg time in opt | Avg % gap in nonopt | Avg % gap overall | Avg time in opt | Avg % time in subpr |
| 50 | 0.495       | 6.4         | 0.00 | 0.11 | 0.00 | 0.01 | 0.00 | 0.00 | 0.23 | 0.23 | 33.81 |
|    | 0.710       | 7           | 0.00 | 0.25 | 0.25 | 0.00 | 0.38 | 0.38 | 0.26 | 0.26 | 59.92 |
| 100| 0.488       | 13.2        | 0.00 | 2.13 | 2.13 | 0.00 | 0.05 | 0.05 | 0.22 | 0.22 | 53.08 |
|    | 0.705       | 13.4        | 0.00 | 4.12 | 4.12 | 0.00 | 5.83 | 5.83 | 0.77 | 0.77 | 69.83 |
| 150| 0.498       | 19          | 0.00 | 1.94 | 1.94 | 0.00 | 0.62 | 0.62 | 0.35 | 0.35 | 64.90 |
|    | 0.693       | 20.2        | 0.00 | 4.65 | 4.65 | 0.00 | 26.65 | 26.65 | 0.93 | 0.93 | 61.71 |
| 200| 0.496       | 26.6        | 0.00 | 7.13 | 7.13 | 0.00 | 12.97 | 12.97 | 0.98 | 0.98 | 58.60 |
|    | 0.703       | 26.8        | 0.00 | 15.67 | 15.67 | 0.00 | 55.31 | 55.31 | 4.13 | 4.13 | 51.20 |
| 250| 0.497       | 33.2        | 0.00 | 19.66 | 19.66 | 0.00 | 62.03 | 62.03 | 0.79 | 0.79 | 77.09 |
|    | 0.693       | 33.4        | 0.00 | 42.21 | 42.21 | 0.00 | 129.70 | 129.70 | 4.05 | 4.05 | 51.07 |
| 300| 0.506       | 40          | 0.00 | 47.66 | 47.66 | 0.00 | 123.92 | 123.92 | 1.36 | 1.36 | 80.21 |
|    | 0.691       | 40.6        | 0.00 | 118.47 | 118.47 | 0.00 | 244.15 | 244.15 | 5.04 | 5.04 | 44.38 |
| 350| 0.508       | 46.6        | 0.00 | 274.99 | 274.99 | 0.00 | 394.37 | 394.37 | 2.25 | 2.25 | 82.15 |
|    | 0.698       | 46.8        | 0.00 | 268.45 | 406.76 | 0.00 | 407.26 | 407.26 | 8.06 | 8.06 | 40.88 |
| 400| 0.502       | 52.8        | 0.00 | 298.38 | 298.38 | 0.00 | 728.67 | 728.67 | 2.58 | 2.58 | 64.96 |
|    | 0.692       | 52.2        | 0.00 | 354.49 | 692.69 | 0.00 | 658.65 | 658.65 | 4.23 | 4.23 | 46.87 |
| 450| 0.507       | 59.6        | 0.00 | 538.64 | 803.18 | 0.00 | 372.79 | 869.11 | 1.92 | 1.92 | 81.49 |
|    | 0.696       | 60.6        | 0.00 | 896.27 | 957.01 | 0.00 | 733.06 | 733.06 | 85.12 | 85.12 | 12.77 |
| 500| 0.507       | 66          | 0.00 | 960.58 | 1152.12 | 0.00 | 903.36 | 1081.34 | 4.65 | 4.65 | 69.60 |
|    | 0.695       | 65          | 0.00 | 1200.00 | 1200.00 | 0.00 | 1038.45 | 1103.07 | 40.48 | 40.48 | 25.01 |
| 85 | 100.00      | 15.00       | 199.78 | 302.46 | 92 | 100.00 | 8.00 | 294.91 | 331.86 | 0.00 | 8.42 | 8.42 | 56.48 |