Crystals arising from Stokes phenomenon

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Abstract

In this paper, we give a complex analysis realization of \(\mathfrak{gl}_n\)-crystals and their tensor products. Motivated by this result, we propose that the WKB approximation of the Stokes matrices, of certain meromorphic linear systems of ordinary differential equations, is characterized by \(\mathfrak{gl}_n\)-crystal structures on the eigenbasis of shift of argument subalgebras of the universal enveloping algebra \(U(\mathfrak{gl}_n)\).

1 Introduction

Crystal bases in representation theory were introduced by Kashiwara [15] and Lusztig [19] in the 1990’s. Since then, they have become ubiquitous in algebra and geometry. We would like to develop an analytic approach to the crystal bases in various settings, via the Stokes phenomenon of differential or difference equations.

Since then, they have become ubiquitous in algebra and geometry. We would like to develop an analytic approach to the crystal bases in various settings, via the Stokes phenomenon of differential or difference equations.

Let us take the Lie algebra \(\mathfrak{gl}_n\) over the field of complex numbers, and its universal enveloping algebra \(U(\mathfrak{gl}_n)\) generated by \(\{E_{ij}\}_{1 \leq i, j \leq n}\) subject to the relation \([E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{kj}\). Let us take the \(n \times n\) matrix \(T = (T_{ij})\) with entries valued in \(U(\mathfrak{gl}_n)\)

\[
T_{ij} = E_{ij}, \quad \text{for } 1 \leq i, j \leq n.
\]

Let \(\mathfrak{h}_{\text{reg}}(\mathbb{R})\) denote the set of \(n \times n\) real diagonal matrices with distinct eigenvalues. Given any highest weight \(\lambda\) and the corresponding representation \(L(\lambda)\) of \(\mathfrak{gl}_n\), let us consider the linear system of differential equation

\[
\frac{dF}{dz} = h\left(\frac{T}{2\pi i z}\right) \cdot F,
\]

for an \(n \times n\) matrix function \(F(z)\) with entries in \(\text{End}(L(\lambda))\). Here \(h = \sqrt{-1}\), \(h\) is a real number, \(u \in \mathfrak{h}_{\text{reg}}(\mathbb{R})\) is seen as a \(n \times n\) matrix with scalar entries in \(U(\mathfrak{gl}_n)\), and the action of the coefficient matrix on \(F(z)\) is given by matrix multiplication and the representation of \(\mathfrak{gl}_n\). The equation (1) is a linear system of rank \(n \times \dim(L(\lambda))\) with a pole of order two at \(z = \infty\). In each Stokes sector \(\text{Sect}_\pm = \{z \in \mathbb{C} \mid \text{ Re}(z) > 0\}\), it has a canonical solution specified by a prescribed asymptotics at \(z = \infty\). The quantum Stokes matrices \(S_{h \pm}(u)\) are then the transition matrices between the two canonical solutions, seen as \(n \times n\) matrices \((s_{i,j}^{\pm}(u))\) with entries in \(\text{End}(L(\lambda))\). Following [25] (see also Section 3), the matrices \(S_{h \pm}(u)\) have canonically regularized limits as some components \(u_i\) of \(u = \text{diag}(u_1, ..., u_n)\) collapse in a comparable speed. The prescription of the possible collapse of the components, therefore the regularized limits of Stokes matrices, is controlled by the geometry of the De Concini-Procesi wonderful space \(\mathfrak{h}_{\text{reg}}(\mathbb{R})\) of the complement \(\mathfrak{h}_{\text{reg}}(\mathbb{R})\) of an arrangement of hyperplanes in \(\mathbb{R}^n\), see [8]. Accordingly, we define Stokes matrices at a boundary point \(u_0 \in \mathfrak{h}_{\text{reg}}(\mathbb{R}) \setminus \mathfrak{h}_{\text{reg}}(\mathbb{R})\) as the corresponding regularized limits. In particular, the 0-dimensional stratum of \(\mathfrak{h}_{\text{reg}}(\mathbb{R})\) contains the caterpillar point \(u_{\text{cat}}\) which corresponds to the limiting point \(u_1 \ll \cdots \ll u_n\), and then the regularized limits \(S_{h \pm}(u_{\text{cat}})\) of \(S_{h \pm}(u)\) as \(u_1 \ll \cdots \ll u_n\) are called the quantum Stokes matrices at \(u_{\text{cat}}\).

The WKB method, named after Wentzel, Kramers, and Brillouin, is for approximating solutions of a differential equation whose highest derivative is multiplied by a small parameter (other names, including Liouville, Green, and Jeffreys are sometimes attached to this method). Accordingly, we will study the asymptotics of \(S_{h \pm}(u)\) as \(h \to -\infty\). For that, we fix a lowest vector \(\xi_0\), and an inner product on \(L(\lambda)\) given by the conditions \(\langle \xi_0, \xi_0 \rangle = 1\) and \(\langle E_{ij}\eta, \zeta \rangle = \langle \eta, E_{ji}\zeta \rangle\) for any \(\eta, \zeta \in L(\lambda)\). For any \(\xi \in L(\lambda)\), we say that a nonzero vector \(\xi'\) is the \(WKB\) \textit{approximation} of the vector valued function \(s_{k,k+1}^{(+)}(u) \cdot \xi\) if there exist constants \(c\) such that (set \(q = e^{\frac{\Delta}{h}}\))

\[
q^{-c} s_{k,k+1}^{(+)}(u) \cdot \xi \to \xi', \text{ as } q \to 0.
\]
1.1 WKB approximation and crystals

Let us first describe the structure of the WKB approximation of \( S_{h\pm}(u_{\text{cat}}) \) at the special point \( u_{\text{cat}} \). Let us take the Gelfand-Tsetlin orthonormal basis \( B(\lambda, u_{\text{cat}}) \) in \( L(\lambda) \), see Section 3.3 Then in Section 5 we prove

**Theorem 1.1.** For each \( k = 1, \ldots, n - 1 \), there exists canonical operators \( \hat{e}_k(u_{\text{cat}}) \) and \( \hat{f}_k(u_{\text{cat}}) \) on the finite set \( B(\lambda, u_{\text{cat}}) \) such that if \( \xi \in B(\lambda, u_{\text{cat}}) \) is a generic element, then the WKB approximation of \( s_{k,k+1}^{(+)}(u_{\text{cat}}) \cdot \xi \) and \( s_{k+1,k}^{(-)}(u_{\text{cat}}) \cdot \xi \) are respectively the actions of \( \hat{e}_k(u_{\text{cat}}) \) and \( \hat{f}_k(u_{\text{cat}}) \) on \( \xi \). Furthermore, the set \( B(\lambda, u_{\text{cat}}) \) equipped with the operators \( \hat{e}_k(u_{\text{cat}}) \) and \( \hat{f}_k(u_{\text{cat}}) \) is a \( \text{gl}_n \)-crystal.

**Remark 1.2.** The meaning of "generic" will become clear in Section 4.3. For a none generic element \( \xi \), the \( q \) leading term of \( s_{k,k+1}^{(+)}(u_{\text{cat}}) \cdot \xi \) has ambiguity and thus is not well defined. The ambiguity in WKB approximation along some codimension one walls is an interesting analysis phenomenon, which is related to cluster mutations in the cluster algebra theory, see our next paper.

The operators \( \hat{e}_k(u_{\text{cat}}) \) and \( \hat{f}_k(u_{\text{cat}}) \) on generic elements of \( B(\lambda, u_{\text{cat}}) \) are determined by WKB approximation, and their extension to none generic elements are uniquely determined by universal property. That is the operators are canonical. Besides, the set \( B(\lambda, u_{\text{cat}}) \) can be obtained via the WKB approximation of Stokes matrices themselves, see Section 4.2 Therefore we call \((B(\lambda, u_{\text{cat}}), \hat{e}_k(u_{\text{cat}}), \hat{f}_k(u_{\text{cat}}))\) the WKB datum at \( u_{\text{cat}} \). Theorem 1.1 states that the WKB datum at \( u_{\text{cat}} \) realizes \( \text{gl}_n \)-crystals. Moreover, the tensor product of two such \( \text{gl}_n \)-crystals can also be realized by the WKB approximation. See Section 5.3 An even stronger statement is that the WKB datum completely characterizes the WKB approximation (including the power of \( q \) in the approximation), see Section 5.4

Let us formulate the conjectural structure of the WKB approximation of \( S_{h\pm}(u) \) at a generic \( u \in \overline{h}_{\text{reg}}(\mathbb{R}) \). Let \( A(u) \) be the shift of argument subalgebras of \( U(\text{gl}_n) \), that are maximal commutative subalgebras parameterized by the de Concini-Procesi space \( u \in \overline{h}_{\text{reg}}(\mathbb{R}) \). See [11][14] for more details. The action of \( A(u) \) on \( L(\lambda) \) has simple spectrum, and thus decomposes \( L(\lambda) \) into eigenlines. Let \( E(\lambda, u) \) denote the set of these eigenlines.

We expect that if \( \xi \in L(\lambda) \) is a generic eigenvector of \( A(u) \), then the WKB approximation of \( s_{k,k+1}^{(+)}(u) \cdot \xi \) and \( s_{k+1,k}^{(-)}(u) \cdot \xi \) are also eigenvectors. Furthermore, the induced operators equip \( E(\lambda, u) \) with a \( \text{gl}_n \)-crystal structure. That is

**Conjecture 1.3.** For any \( u \in \overline{h}_{\text{reg}}(\mathbb{R}) \), the WKB approximation of \( s_{k,k+1}^{(+)}(u) \) and \( s_{k,k+1}^{(-)}(u) \) on \( L(\lambda) \), for each \( k = 1, \ldots, n - 1 \), produce operators \( \hat{e}_k(u) \) and \( \hat{f}_k(u) \) on the set \( E(\lambda, u) \) of eigenlines, such that the WKB datum \((E(\lambda, u), \hat{e}_k(u), \hat{f}_k(u))\) is a \( \text{gl}_n \)-crystal. Moreover, given any two representations \( L(\lambda_1) \) and \( L(\lambda_2) \), the WKB approximation of

\[
\left( s_{k,k}^{(+)} \otimes s_{k,k+1}^{(+)} + s_{k,k}^{(+)} \otimes s_{k,k+1}^{(+)} \right) \quad \text{and} \quad \left( s_{k+1,k}^{(-)} \otimes s_{k,k}^{(-)} + s_{k+1,k+1}^{(-)} \otimes s_{k,k+1}^{(-)} \right)
\]

(2)

produce the crystal operators \( \hat{e}_k(u) \) and \( \hat{f}_k(u) \) on the tensor product \( E(\lambda_1, u) \otimes E(\lambda_2, u) \) of \( \text{gl}_n \)-crystals.

The shift of argument subalgebra \( A(u_{\text{cat}}) \) at \( u_{\text{cat}} \) is the Gelfand-Tsetlin subalgebra, and \( E(\lambda, u_{\text{cat}}) \) is identified with the set of Gelfand-Tsetlin eigenlines. Thus the conjecture includes Theorem 1.1 as a special case.

**Remark 1.4.** The \( \text{gl}_2 \) case can be verified directly, since the quantum Stokes matrices in this case have closed formula, see the Appendix.

Recall that a \( \text{gl}_n \)-crystal is a finite set which models a weight basis for a representation of \( \text{gl}_n \), and crystal operators indicate the leading order behaviour of the simple root vectors on the basis under the crystal limit \( q \to 0 \) in quantum group \( U_q(\text{gl}_n) \). Therefore, from the perspective of quantum groups, the above conjecture and theorem have the following interpretation: in [27] we proved that the quantum Stokes matrices \( S_{h\pm}(u) \) are the \( L_{\pm} \) operators in the FRT realization of the quantum group \( U_q(\text{gl}_n) \) [22] (this is why they are called quantum Stokes matrices). See also the Appendix for more details. Thus the entries \( s_{k,k+1}^{(+)} \), \( s_{k+1,k}^{(-)} \) and \( s_{k,k}^{(+)} \) are certain realization of Chevalley generators of \( U_q(\text{gl}_n) \), and the operators appearing in [22] are just the images of \( s_{k,k+1}^{ (+) } \) and \( s_{k+1,k}^{(-)} \) under the coproduct of \( U_q(\text{gl}_n) \). Conjecture 1.3 and Theorem 1.1 then indicate that the WKB
approximation $h \rightarrow -\infty$ in the differential equation (1) translates to the crystal limit $q = e^{h/2} \rightarrow 0$ in quantum group.

It seems worth pointing out some approaches to the conjecture for general $u$ cases. First note that the equation (1) is equivalent to the (two variables) generalized Knizhnik-Zamolodchikov equations (see [10, Theorem 2.1]) taking values in the tensor product $L(\lambda) \otimes C^n$ of $L(\lambda)$ and the standard representation of $gl_n$. The integral representations for the solutions of (1), obtained by Felder-Markov-Tarasov-Varchenko [10, Theorem 3.1], can be rewritten under Bethe vectors/eigenbasis $B(\lambda, u)$ of the shift of argument subalgebras $A(u)$. Then the Stokes matrices $S_{h+}(u)$, which measure the difference of the asymptotics of these solutions as $z \rightarrow \infty$ within different Stokes sectors, can be expressed under the Bethe vectors $B(\lambda, u)$. The rest is to show the well definedness of the WKB datum $(E(\lambda, u), \tilde{\epsilon}_k(u), \tilde{f}_k(u))$ applying the method of steepest descent, and its continuous dependence on $u \in \hat{h}_{\text{reg}}(\mathbb{R})$. Then by continuity the WKB datum at a generic $u$ is equipped with the discrete combinatorial structure of the WKB datum at $u_{\text{cat}}$, and thus is a $gl_n$-crystal. Some other more general approach is to use the degeneration of isomonodromy equation to isospectral equation. We leave it for our next work.

1.2 The variation of $u$ in $\hat{h}_{\text{reg}}(\mathbb{R})$ and cactus group actions on crystals

The local picture. Now let us discuss the variation of $u$ in the closure $\overline{U_{id}}$ of the connected component $U_{id} = \{ u \in \hat{h}_{\text{reg}}(\mathbb{R}) \mid u_1 < \cdots < u_n \}$ of $\hat{h}_{\text{reg}}(\mathbb{R})$ in the space $\hat{h}_{\text{reg}}(\mathbb{R})$. The WKB datum at $u_{\text{cat}}$, given in Theorem 1.1, should provide an applicable model for general $u \in \overline{U_{id}}$. That is because that the regularized limits $S_{h+}(u_{\text{cat}})$ simply encode the leading terms of $S_{h+}(u)$ as $u \rightarrow u_{\text{cat}}$ from the connected component $U_{id}$ (see Section 2.4 and 3.4 for more details), thus we expect the following commutative diagram.

$$\begin{array}{ccc}
S_{h+}(u) & \xrightarrow{\text{regularized limits as } u \rightarrow u_{\text{cat}} \text{ from } U_{id}} & S_{h+}(u_{\text{cat}}) \\
\downarrow \text{WKB datum} & & \downarrow \text{WKB datum} \\
(E(\lambda, u), \tilde{\epsilon}_k(u), \tilde{f}_k(u)) & \xrightarrow{\text{continuous as } u \rightarrow u_{\text{cat}} \text{ from } U_{id}} & (E(\lambda, u_{\text{cat}}), \tilde{\epsilon}_k(u_{\text{cat}}), \tilde{f}_k(u_{\text{cat}}))
\end{array}$$

In particular, the discrete combinatorial structure encoded in $(E(\lambda, u), \tilde{\epsilon}_k(u), \tilde{f}_k(u))$ should be locally independent of $u \in \overline{U_{id}}$. That is the crystal structure at a generic $u \in U_{id}$ can be understood from the one at $u_{\text{cat}}$. It particularly implies the following construction of the set $E(\lambda, u)$ from Stokes matrices themselves.

For any integers $i, j$ with $1 \leq n-i \leq j \leq n-1$, we introduce the monomial $Q_{ij} = s_{j,j+1}^{(n)} s_{j-1,j}^{(n)} \cdots s_{n-i,n-i+1}^{(n)}$. For any tuple of nonnegative integers $d = (d_{ij})_{1 \leq i \leq j \leq n}$, we define the operator

$$Q(u, h)^d = \prod_{i+j \geq n} (Q_{ij})^{d_{ij}} \in \text{End}(L(\lambda)), \tag{3}$$

where the product is taken with $Q_{ij}$ to the right of $Q_{i'j'}$ if $j > j'$; or $j = j'$ and $i > i'$. We say that a nonzero vector $\xi_d$ is the WKB approximation of $Q(u, h)^d \cdot \xi_0$ if there exists a constant $c$ such that

$$q^c Q(u, h)^d \cdot \xi_0 \rightarrow \xi_d \text{ as } q \rightarrow 0.$$  

Then for any $u \in U_{id}$, the set of one dimensional subspaces $[\xi_d] := \mathbb{C} \xi_d$ for all possible $d$ coincides with $E(\lambda, u)$, and equips $E(\lambda, u)$ with a natural parametrization by the integer points in the Gelfand-Tsetlin polytope. See Section 1.2 for more details. Furthermore, the WKB operators $\tilde{\epsilon}_k(u), \tilde{f}_k(u)$ are explicitly given, see the formula in Theorem 5.3. It gives a realization of the set $E(\lambda, u)$ via the WKB approximation of polynomial combinations of Stokes matrices themselves.

Remark 1.5. The polynomials $Q(u, h)^d$ in (3) simply label the elements in $E(\lambda, u)$ by encoding the ordering of actions of crystal operators on the lowest vector. However, for $u$ in some other component which is not $U_{id}$, the WKB approximation of $Q(u, h)^d \cdot \xi_0$ may not produce a basis anymore. There is a natural candidate resolving this problem. That is the canonical basis of quantum groups. Since $S_{h+}(u)$ is a realization of the plus part of $U_q(gl_n)$, we can replace the generators of the plus part of $U_q(gl_n)$ by the corresponding entries of $S_{h+}(u)$, and thus consider the WKB approximation of the canonical basis. Given the above result, it is natural to expect that the WKB approximation gives a one-to-one correspondence between the canonical bases and the integer points in the Gelfand-Tsetlin cone. A more interesting question is that if the canonical basis can be characterized by the analytic data.
The global picture. For any $\sigma \in S_n$, let $\overline{U_\sigma}$ be the closure of the connected component $U_\sigma = \{ u \in h_{\text{reg}}(\mathbb{R}) \mid u_{\sigma(1)} < \cdots < u_{\sigma(n)} \}$ of $h_{\text{reg}}(\mathbb{R})$. The finite set $E(\lambda, u)$ over $u$ in different chambers can glue together along the intersection $\overline{U_\sigma} \cap \overline{U_{\tau_i}}$, according to the “wall-crossing formula” of the regularized limits of Stokes matrices at a boundary point, see [25]. In this way, we get a nature cover of the space $h_{\text{reg}}(\mathbb{R})$, whose monodromy representation on the fibre $E(\lambda, u_{\text{cat}})$ at $u_{\text{cat}}$ can be computed via the explicit “wall-crossing formula” of the regularized limits of Stokes matrices, as $u \rightarrow u_{\text{cat}}$ from different chambers $U_{\tau_i}$ with $\tau_i \in S_n$ being the permutation reversing the segment $[1, \ldots, i]$ in $[1, \ldots, n]$. In [14] Halacheva, Kamnitzer, Rybnikov and Weekes (HKRW) defined a covering of the space $h_{\text{reg}}(\mathbb{R})$, whose fibre at $u$ is the set $E(\lambda, u)$ of eigenlines, and studied its monodromy representation. In particular, the known $\mathfrak{gl}_n$-crystal structure on the eigenlines $(E(\lambda, u_{\text{cat}}), \hat{e}_k(u_{\text{cat}}), \hat{f}_k(u_{\text{cat}}))$ of the Gelfand-Tsetlin subalgebra induces crystal structures on other discrete fibre $E(\lambda, u)$ by continuity, and globally the monodromy representation gives rise to the cactus group actions on $\mathfrak{gl}_n$-crystals [4]. It is rather similar to the above consideration. Therefore we expect that the variation of the WKB datum of quantum Stokes matrices over $h_{\text{reg}}(\mathbb{R})$ gives a complex analysis realization of the HKRW cover, as well as the the crystal structure on each fibre constructed in [14]. Since the quantum Stokes matrices and their WKB datum can be seen as the monodromy and spectral datum of the equation (1) respectively, it is then interesting to realize the cactus group actions on crystals via an isospectral deformation over $h_{\text{reg}}(\mathbb{R})$.

The construction of [14] works for any simple Lie algebra $\mathfrak{g}$. One can ask the similar relation between the $\mathfrak{g}$-crystals and the WKB approximation of quantum Stokes matrices associated to $\mathfrak{g}$. Furthermore, using the same idea, we expect that various wall crossing type formula in representation theory can be interpreted via Stokes phenomenon. For example, motivated by [14] Conjecture 1.14, we expect that the cactus group action on the Weyl group constructed by Losev [18], via perverse equivalences coming from wall crossing functors in category $\mathcal{O}$ (a more elementary definition was given shortly afterwards by Bonnafé [6]), can be realized via the stokes phenomenon of the affine KZ equations introduced by Cherednik [7].

The organization of the paper is as follows. Section [2] gives the preliminaries of Stokes matrices of meromorphic linear systems, isomonodromy deformation, and recalls the regularized limits of Stokes matrices. Section [3] recalls the quantum Stokes matrices and their regularized limits at $u_{\text{cat}}$. Section [4] computes the WKB approximation of the quantum Stokes matrices at $u_{\text{cat}}$, and introduces the notion of WKB datum. The last section shows that the WKB datum of quantum Stokes matrices are $\mathfrak{gl}_n$-crystals.

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2 Stokes matrices and their regularized limits at caterpillar points

This section gives an introduction to Stokes matrices of meromorphic linear systems of ordinary differential equations, and gives a brief recall of the closure of Stokes matrices developed in our previous paper [25]. Section [2.1] recalls the Stokes matrices of certain meromorphic linear systems. Section [2.2] recalls the regularized limits of Stokes matrices and introduces the Stokes matrices at a caterpillar point. Section [2.3] gives the expression of the leading terms of Stokes matrices in terms of the Gelfand-Tsetlin systems.

2.1 Stokes matrices

Let $\mathfrak{h}(\mathbb{R})$ (resp. $h_{\text{reg}}(\mathbb{R})$) denote the set of diagonal matrices with (resp. distinct) real eigenvalues. Let us consider the meromorphic linear system of differential equation

$$
\frac{dF}{dz} = \left( iu - \frac{1}{2\pi i} A \right) \cdot F,
$$

where $F(z)$ is valued in $\mathbb{C}^n$, $u \in \mathfrak{h}(\mathbb{R})$ and $A \in \text{Herm}(n)$. The system has an order two pole at $\infty$ and (if $A \neq 0$) a first order pole at $0$. 

4
Definition 2.1. The two Stokes sectors $\text{Sect}_\pm$ of the system are the right/left half planes $\text{Sect}_\pm = \{ z \in \mathbb{C} \mid \pm \Re(z) > 0 \}$.

Let us choose the branch of $\log(z)$, which is real on the positive real axis, with a cut along the nonnegative imaginary axis $i \mathbb{R}_{\geq 0}$. For any two real numbers $a, b$, an open sector and a closed sector with opening angle $b - a > 0$ are respectively denoted by

$$S(a, b) := \{ z \in \mathbb{C} \mid a < \arg(z) < b \}, \quad \overline{S}(a, b) := \{ z \in \mathbb{C} \mid a \leq \arg(z) \leq b \}.$$

The following result are known, see e.g., [2, Chapter 9] or [25].

Theorem 2.2. For any $u \in \mathfrak{h}(\mathbb{R})$, on $\text{Sect}_\pm$ there is a unique fundamental solution $F_\pm : \text{Sect}_\pm \to \text{GL}(n, \mathbb{C})$ of equation (4) such that $F_+ \cdot e^{-iu z} \cdot z^\frac{\delta(A)}{2\pi i}$ and $F_- \cdot e^{-iu z} \cdot z^\frac{\delta(A)}{2\pi i}$ can be analytically continued to $S(-\pi, \pi)$ and $S(0, 2\pi)$ respectively, and for every small $\varepsilon > 0$,

$$\lim_{z \to \infty} F_+(z; u) \cdot e^{-iu z} \cdot z^\frac{\delta(A)}{2\pi i} = \text{Id}_n, \quad \text{as} \quad z \in \text{S}(\pi + \varepsilon, \pi - \varepsilon),$$

$$\lim_{z \to \infty} F_-(z; u) \cdot e^{-iu z} \cdot z^\frac{\delta(A)}{2\pi i} = \text{Id}_n, \quad \text{as} \quad z \in \overline{\text{S}}(\varepsilon, 2\pi - \varepsilon),$$

Here $\text{Id}_n$ is the rank $n$ identity matrix, and $\delta(A)$ is the projection of $A$ to the centralizer of $u$ in $\mathfrak{gl}_n$. In particular, if $u$ has distinct eigenvalues, $\delta(A)$ is the diagonal part of $A$.

Definition 2.3. The Stokes matrices of the system (4) (with respect to $\text{Sect}_+$ and the branch of $\log(z)$) are the elements $S_+(u, A) \in \text{GL}_n$ determined by

$$F_+(z) = F_-(z) \cdot e^{-\frac{\delta(A)}{2} S_+(u, A)}, \quad F_-(z) = F_+(z) \cdot S_-(u, A) e^{-\frac{\delta(A)}{2}},$$

where the first (resp. second) identity is understood to hold in $\text{Sect}_\pm$ (resp. $\text{Sect}_\pm$) after $F_+$ (resp. $F_-$) has been analytically continued clockwise.

The following lemma follows from the fact that if $F(z)$ is a solution, so is $F(-z)^\dagger$, see [5].

Lemma 2.4. Let $S_+(u, A)^\dagger$ denote the conjugation transpose of $S_+(u, A)$, then $S_-(u, A) = S_+(u, A)^\dagger$.

2.2 Connection matrices

Since the system (4) is non-resonant, i.e., no two eigenvalues of $\frac{\delta(A)}{2\pi i}$ for $A \in \text{Herm}(n)$ are differed by a positive integer, we have (see e.g. [24] Chapter 2).

Lemma 2.5. There is a unique holomorphic fundamental solution $F_0(z) \in \text{GL}(n)$ of the system (4) on a neighbourhood of $\infty$ slit along $i \mathbb{R}_{\geq 0}$, such that $F_0 \cdot z^\frac{\delta(A)}{2\pi i} \to \text{Id}_n$ as $z \to 0$.

Definition 2.6. The connection matrix $C(u, A) \in \text{GL}_n(\mathbb{C})$ of the system (4) (with respect to $\text{Sect}_+$) is determined by $F_0(z) = F_+(z) \cdot C(u, A)$, as $F_0(z)$ is extended to the domain of definition of $F_+(z)$.

Due to the symmetry imposed on the coefficient of the equation (4), the connection matrix $C(u, A)$ is valued in $\text{U}(n)$ (see e.g., [5] Lemma 29).

2.3 Regularized limits at caterpillar points and closure of Stokes matrices

It is well known that for a generic $A \in \text{Herm}(n)$, the limit of the Stokes matrices $S_\pm(u, A)$, as some components $u_i$ of $u = \text{diag}(u_1, \ldots, u_n) \in \text{h}_{\text{reg}}(\mathbb{R})$ collapse, do not exist. However, in [25] we prove that $S_\pm(u, A)$ have canonically regularized limit as $u_1, \ldots, u_n$ collapse in a comparable speed. In the following, we will only focus on the special case that all $u_1, \ldots, u_n$ collapse in the speed that $u_i$ approaches to $u_1$ much faster than $u_{i+1}$ approaches to $u_1$. We refer the reader to [25] for more general cases.

For any $n \times n$ matrix $A$ and $k = 0, \ldots, n - 1$, we denote by $\delta_k(A)$ the matrix

$$\delta_k(A)_{ij} = \begin{cases} A_{ij}, & \text{if} \quad 1 \leq i, j \leq k, \text{ or } i = j \\ 0, & \text{otherwise}. \end{cases} \quad (5)$$

Then we have
Theorem 2.7. \([25]\) For any fixed \(A \in \text{Herm}(n)\), the limit of the matrix valued function

\[
\left( \prod_{k=0,\ldots,n-1}^{\rightarrow} \frac{u_k}{u_{k+1}} \right)^{\log \left( \delta_k(S_-^{(k)} \delta_k(S_+^{(k+1)}) \right)} \cdot S_-^L(u, A) S_+^R(u, A) \cdot \left( \prod_{k=0,\ldots,n-1}^{\leftarrow} \frac{u_k}{u_{k+1}} \right)^{\log \left( \delta_k(S_-^{(k)} \delta_k(S_+^{(k+1)}) \right)}^{-1},
\]

as \(0 < u_1 \ll u_2 \ll \ldots \ll u_n\) exist. Here \(u_0 := 1\) and \(\log(\delta_k(S_-^{(k)} \delta_k(S_+^{(k+1)})))\) is the logarithm of the positive definite Hermitian matrix \(\delta_k(S_-^{(k)} \delta_k(S_+^{(k+1)}))\). Furthermore, if we use the Gauss decomposition to decompose the corresponding limit of \([6]\) into the product \(S_-^L(u, A) S_+^R(u, A)\) of upper and lower triangular matrices \(S_-^L(u, A)\) (with the same diagonal part), then

- the sub-diagonals of \(S_-^L(u, A)\) are given by

\[
{\left( S_-^L \right)}_{k+1,k} = e^{-\frac{(k+1)-2(k)}{4}} \prod_{i=1}^{k} \left( \frac{1}{1-i} \Gamma \left( 1 + \frac{\lambda_{k+1} - 2(k)}{2(k)} \right) \right) \prod_{i=1}^{k} \left( \frac{1}{1-i} \Gamma \left( 1 + \frac{\lambda_{k+1} - 2(k)}{2(k)} \right) \right) \cdot m_{k}^{(k)},
\]

where \(k = 1, \ldots, n-1\), \(\lambda_{k+1,\ldots,n} = \lambda_{k,\ldots,n-1,\ldots,1} - A\), and

\[
m_{k}^{(k)}(A) = \sum_{j=1}^{k} (-1)^{k-j} \Delta_{1,\ldots,k-1}^{(k)}(\lambda_{k}^{(k)} - A) A_{j,k+1}.
\]

Here \(\Delta_{1,\ldots,k-1}^{(k)}(\lambda_{k}^{(k)} - A)\) is the \((k-1) \times (k-1)\) minor of the matrix \(\lambda_{k}^{(k)} - A\) (the \(j\) means that the row index \(j\) is omitted).

- the other entries of \(S_-^L(u, A)\) are given by explicit algebraic combinations of the sub-diagonal ones. The algebraic combinations are determined by the classical RLL formulation, see Section 5.4 for more details.

Through the Gauss decomposition, Theorem 2.7 states that \(S_-^L(u, A)\) have properly regularized limits \(S_-^L(u, A)\) as \(u_1 \ll \cdots \ll u_n\). In Section 2.4 we will see that the regularized limits completely encode the leading terms of \(S_-^L(u, A)\).

More generally, the properly regularized limit of the Stokes matrices \(S_-^L(u, A)\), as some components \(u_j\) of \(u = \text{diag}(u_1, \ldots, u_n)\) collapse in a comparable speed, was studied in [25]. The prescription of all the possible collapse of the components \(u_j\), therefore the properly regularized limit, is controlled by the geometry of the De Concini-Procesi space \(\mathfrak{h}_{\text{reg}}(\mathbb{R})\). In [25] we define the Stokes matrices at the boundary \(\mathfrak{h}_{\text{reg}}(\mathbb{R}) \setminus \mathfrak{h}_{\text{reg}}(\mathbb{R})\) as the corresponding regularized limits, and call them the closure of Stokes matrices. In particular, the "infinite" point \(u = \text{diag}(u_1, \ldots, u_n)\), with \(u_1 \ll \cdots \ll u_n\), is a point in the 0-dimensional stratum of \(\mathfrak{h}_{\text{reg}}(\mathbb{R})\), called a caterpillar point \(u_{\text{cat}}\) (see [23], page 16). Accordingly,\end{quote}

Definition 2.8. \([25]\) For any \(A \in \text{Herm}(n)\), the regularized limits \(S_-^L(u_{\text{cat}}, A)\) of \(S_-^L(u, A)\), as \(0 < u_1 \ll u_2 \ll \cdots \ll u_n\), are called the Stokes matrices at the caterpillar point \(u_{\text{cat}}\).

Example 2.9 (2 by 2 cases). Let us consider the rank two case, that is

\[
\frac{dF}{dz} = \left( \begin{array}{cc} t_1 & 0 \\ 0 & t_2 \end{array} \right) - \frac{1}{2\pi iz} \left( \begin{array}{cc} 0 & a \\ \bar{a} & 0 \end{array} \right) \cdot F.
\]

Following [3] Proposition 8, the Stokes matrices (with respect to the chosen branch of \(\log(z)\)) are

\[
S_-^L(u, A) = \begin{pmatrix} e^{t_2} & 0 \\ \frac{a e^{t_2} (u_2 - u_1) + t_2}{\Gamma(1 + \frac{\lambda_1 - t_2}{2\pi i})} & e^{t_2} \end{pmatrix},
\]

\[
S_+^R(u, A) = \begin{pmatrix} 0 & e^{t_2} \\ \frac{\bar{a} e^{t_2} (u_2 - u_1) + t_2}{\Gamma(1 + \frac{\lambda_1 - t_2}{2\pi i})} & 0 \end{pmatrix}
\]

\[
F.\]
By the definition of $\delta_k$ given in (5), we have that $\delta_k(S_\pm(u, A)) = \delta_1(S_\pm(u, A))$ is the diagonal part of $S_\pm(u, A)$. Then \(\frac{\log(\delta_1(S_-)(\delta_1(S_+)))}{2\pi t} = \text{diag}(\frac{t_1}{2\pi t}, \frac{t_1}{2\pi t})\), and we have
\[
\left(\begin{array}{cc}
-u_2 \frac{t_1}{2\pi t} & 0 \\
 0 & -u_2 \frac{t_1}{2\pi t}
\end{array}\right)S_-(u, A)S_+(u, A) \left(\begin{array}{cc}
u_2 & 0 \\
 0 & u_2 \frac{t_1}{2\pi t}
\end{array}\right) \rightarrow S_-(u_{\text{cat}}, A)S_+(u_{\text{cat}}, A), \text{ as } u_1 \ll u_2,
\]
where
\[
S_-(u_{\text{cat}}, A)^\dagger = S_+(u_{\text{cat}}, A) = \left(\begin{array}{ccc}
e^{\frac{t_1}{\pi\sqrt{4 - t_1^2}}} & \frac{e^{\frac{t_1}{\pi\sqrt{4 - t_1^2}}}}{\Gamma(1 + \frac{t_1}{\pi\sqrt{4 - t_1^2}})} & 0 \\
 0 & e^{\frac{t_1}{\pi\sqrt{4 - t_1^2}}} & 0 \\
 0 & 0 & 1
\end{array}\right).
\]

2.4 The leading terms of Stokes matrices via the Gelfand-Tsetlin systems

Theorem [27] was proved in [25] by the method of isomonodromy deformation. In particular, the asymptotics of the Tsetlin map. Its image of
\[
\log(t)
\]
by the definition of $\delta_k$ are strict. Let $\delta_k$ be the corresponding dense open subset of $\text{Herm}(n)$. For any fixed $A \in \text{Herm}(n)$, let us introduce the unitary matrix
\[
g(u; A) = \prod_{k=0,\ldots,n-1} (u_k/u_{k+1})^{\frac{h_k(A)}{2\pi t}}.
\]

Proposition 2.10. [25] For any fixed $A \in \text{Herm}(n)$, we have, as $0 < u_1 \ll u_2 \ll \ldots \ll u_n$,
\[
S_\pm(u, A) = S_\pm(u_{\text{cat}}, g(u; A) \cdot A \circ g(u; A)^{-1} + O((u_2 - u_1)^{-1})).
\]

As a corollary, we can describe explicitly the leading terms of $S_\pm(u, A)$ as $u \rightarrow u_{\text{cat}}$.

Corollary 2.11. For any fixed $A \in \text{Herm}(n)$, the subdiagonal entries of Stokes matrices $S_\pm(u, A)$ satisfy
\[
(S_+(u, A))_{k,k+1} = l_k^{(\pm)}(u, A) + o(l_k^{(\pm)}(u, A)), \text{ as } 0 < u_1 \ll u_2 \ll \ldots \ll u_n,
\]
where $l_k^{(\pm)}(u)$ coincides with the $(S_+(u_{\text{cat}}, A))_{k,k+1}$ given by (7), provided we replace the term $m_k^{(\pm)}(A)$ by
\[
m_k^{(\pm)}(A) \cdot u_k \frac{\Gamma(1 + \frac{t_1}{\pi\sqrt{4 - t_1^2}})}{\Gamma(1 + \frac{t_1}{\pi\sqrt{4 - t_1^2}})} \frac{\lambda_k^{(\pm)}(A)}{2\pi t},
\]
where $\lambda_k^{(\pm)}(A)$ for $1 \leq i \leq k \leq n$ is the Gelfand-Tsetlin cone, cut out by the following inequalities,
\[
\lambda_i^{(k+1)} \leq \lambda_i^{(k)} \leq \lambda_{i+1}^{(k+1)}, \quad 1 \leq i \leq k \leq n - 1.
\]

Thimm torus actions. Let $C_0(n) \subset C(n)$ denote the subset where all of the eigenvalue inequalities are strict. Let $\text{Herm}_0(n) := \text{Im}(C_0(n))$ be the corresponding dense open subset of $\text{Herm}(n)$. The $k$-torus $T(k) \subset U(k)$ of diagonal matrices acts on $\text{Herm}_0(n)$ as follows,
\[
t \cdot A = \text{Ad}_{U(k)} A, \quad t \in T(k), A \in \text{Herm}_0(n).
\]
Here $U \subset U(k) \subset U(n)$ is a unitary matrix such that $\text{Ad}_{U} A^{(k)}$ is diagonal, with entries $\lambda_1^{(k)}, \ldots, \lambda_k^{(k)}$. The action is well-defined since $U^{-1} t U$ does not depend on the choice of $U$, and preserves the Gelfand-Tsetlin
map \((13)\). The actions of the various \(T(k)\)'s commute, hence they define an action of the Gelfand-Tsetlin torus \(T(1) \times \cdots \times T(n-1) \cong U(1)^{(n-2)n}\). Here the torus \(T(n)\) is excluded, since the action \((15)\) is trivial for \(k = n\).

**Action-angle coordinates.** Let \(A \in \text{Herm}_0(n)\). There exists a unique unitary matrix \(P_k(A) \in U(k) \subset U(n)\), whose entries in the \(k\)-th row are positive and real, such that the upper left \(k\)-th submatrix of \(P_k(A)^{-1}AP_k(A)\) is the diagonal matrix \(\text{diag}(\lambda_1^{(k)}, \ldots, \lambda_k^{(k)})\), i.e.,

\[
P_k(A)^{-1}AP_k(A) = \begin{pmatrix}
\lambda_1^{(k)} & a_1^{(k)} & \cdots \\
\vdots & \ddots & \vdots \\
a_k^{(k)} & \cdots & \lambda_k^{(k)} \\
1 & \cdots & \cdots
\end{pmatrix},
\]

(16)

The \((i, k + 1)\) entry \(a_i^{(k)}(A)\) are seen as functions on \(\text{Herm}_0(n)\). Then the functions \(\{\lambda_i^{(k)}\}_{1 \leq i \leq k \leq n}\) and \(\{\psi_i^{(k)} = \text{Arg}(a_i^{(k)})\}_{1 \leq i \leq k \leq n-1}\) on \(\text{Herm}_0(n)\) are called the Gelfand-Tsetlin action and angle coordinates. Under the coordinates, the Thimm action of

\[
\text{diag}(e^{i\theta_1^{(k)}}, \ldots, e^{i\theta_k^{(k)}}) \in T(k)
\]

is given by

\[
\lambda_j^{(i)} \mapsto \lambda_j^{(i)}, \quad \psi_j^{(i)} = \psi_j^{(i)} + \delta_k \delta_{j+1} \theta_i^{(k)}.
\]

**Proof of Corollary 2.11.** First, for any \(A\) and \(u\), let us introduce an element in the torus

\[
\prod_{k=1,\ldots,n} (u_{k-1}/u_k)^{\frac{\lambda^{(k-1)}(A)}{2\pi i}} \in T(1) \times \cdots \times T(n-1)
\]

where \(\lambda^{(k-1)}(A) := \text{diag}(\lambda_1^{(k-1)}, \ldots, \lambda_{k-1}^{(k-1)})\). Let us introduce the diagonal matrix

\[
D(u; A) = \text{diag}(u_1^{A_{11}/2\pi i}, \ldots, u_n^{A_{nn}/2\pi i}) \in T(n).
\]

Then one checks

\[
g(u; A)Ag(u; A)^{-1} = D(u; A)^{-1} \cdot \left( \prod_{k=1,\ldots,n} (u_{k-1}/u_k)^{\frac{\lambda^{(k-1)}(A)}{2\pi i}} \cdot A \right) \cdot D(u; A).
\]

(17)

Therefore, we have

\[
\lambda_i^{(k)}(g(u; A)Ag(u; A)^{-1}) = \lambda_i^{(k)}(A),
\]

\[
a_i^{(k)}(g(u; A)Ag(u; A)^{-1}) = a_i^{(k)}(A) \cdot u_i^{\frac{A_{ii}}{2\pi i}} (u_k/u_{k+1})^{\frac{\lambda^{(k)}(A)}{2\pi i}},
\]

where the \(u_i^{A_{ii}/2\pi i}\) term in the second identity comes from the action of \(D(u; A)\) in \((17)\). Furthermore, using the identity of functions \(a_i^{(k)}\) and \(m_i^{(k)}\) on \(\text{Herm}_0(n)\),

\[
a_i^{(k)} = m_i^{(k)} \cdot \sqrt{\frac{\prod_{k=1,\ldots,n-1}^{k+1} (\lambda_i^{(k+1)} - \lambda_i^{(k)}) \prod_{k=1}^{n-1} (\lambda_i^{(k)} - \lambda_i^{(k+1)})}{\prod_{k=1}^{n} (\lambda_i^{(k)} - \lambda_i^{(k+1)})}},
\]

we get that the values of the function \(m_i^{(k)}\) at the two points \(A\) and \(g(u; A)Ag(u; A)^{-1}\) are related by

\[
m_i^{(k)}(g(u; A)Ag(u; A)^{-1}) = m_i^{(k)}(A) \cdot u_k^{\frac{A_{kk}}{2\pi i}} (u_k/u_{k+1})^{\frac{\lambda_i^{(k)}(A)}{2\pi i}},
\]

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that is the identity \((12)\). In the end, the corollary follows from the real analyticity of \(S_\pm (u_{\text{cat}}, A)\) with respect to \(A\).

One can prove similar results for the other entries of \(S_\pm (u, A)\). Since the functions \(m_i^{(k)}\) are expressed by \(a_i^{(k)} = e^{\psi_i^{(k)}} , \lambda_i^{(k)}\), and \(\lambda_i^{(k+1)}\), Corollary 2.11 gives the expression of the leading terms of Stokes matrices as \(u \to u_{\text{cat}}\), in terms of the Gelfand-Tsetlin action and angle coordinates. Furthermore, we see that the leading terms include a fast spin on the corresponding Liouville torus of the integrable systems, which is cancelled out as far as the regularized limit is considered.

2.5 Degenerate irregular terms

The diagonal matrix \(\nu u\) is called the irregular term of the meromorphic linear system of differential equation \((4)\). Although the results in the previous sections are obtained under the assumption that \(u \in h_{\text{reg}}(\mathbb{R})\), a generalization of the construction in \([25]\) to the case of degenerate irregular terms, i.e., \(u \in h(\mathbb{R}) \setminus h_{\text{reg}}(\mathbb{R})\) is direct. For example, given any partition \(d\) of \(n\), i.e., a set of integers \(\{d_j\}_{j=1,\ldots,m}\) such that \(n = d_1 + \cdots + d_m\), let us consider the subspace of \(h(\mathbb{R})\)

\[
U_d = \{ u = \text{diag}(u_1, \ldots, u_{d_1}, u_{d_1+1}, \ldots, u_{d_1+d_2}, \ldots, u_m, \ldots, u_m) \in h(\mathbb{R}) \mid u_i \neq u_j \text{ if } i \neq j \}.
\]

Any \(n \times n\) matrix \(A\) can be seen as a blocked matrix \(A = (A_{ij})_{1 \leq i, j \leq m}\) according to the tuple \(d\), with each \(A_{ij}\) a \(d_i \times d_j\) block. Then for each \(k = 0, \ldots, m - 1\), we denote by \(\delta_k(A)\) the blocked matrix

\[
\delta_k(A)_{ij} = \begin{cases} A_{ij}, & \text{if } 1 \leq i, j \leq k, \text{ or } i = j, \\ 0, & \text{otherwise.} \end{cases} \tag{18}
\]

Remark 2.12. The identity \((18)\) depends on the partition \(d_1, \ldots, d_m\), which generalizes the definition of \(\delta_k(A)\) given in \((5)\) corresponding to the case \(d_1 = \cdots = d_n = 1\). Although this more general notation can lead to confusion with \((5)\), the context typically eliminates any ambiguity.

Then the analytic branching rule in \([25]\) states

**Theorem 2.13.** \([25]\) For any fixed \(A \in \text{Herm}(n)\), the limit of the matrix valued function of \(u \in U_d\)

\[
\left( \prod_{k=0,\ldots,m-1}^u \frac{u_k}{u_{k+1}} \right)^{\log\left(\frac{\delta_k(S_-)\delta_k(S_+)}{2\pi i}\right)} \cdot S_-(u, A)S_+(u, A) \cdot \left( \prod_{k=0,\ldots,m-1}^u \frac{u_k}{u_{k+1}} \right)^{\log\left(\frac{\delta_k(S_-)\delta_k(S_+)}{2\pi i}\right)}^{-1}, \tag{19}
\]

as \(0 < u_1 \ll u_2 \ll \ldots \ll u_m\), equals to (here \(u_0 := 1\))

\[
\left( \prod_{k=1,\ldots,m}^u C^{(k)}(A) \right) \cdot e^A \cdot \left( \prod_{k=1,\ldots,m}^u C^{(k)}(A) \right)^{-1}. \tag{20}
\]

Here the product \(\prod\) is taken with the index \(i\) to the right of \(j\) if \(j < i\), and each \(C^{(k)}(A)\) denotes the connection matrix of the system

\[
\frac{dF}{dz} = \left( iE(d)_k - \frac{1}{2\pi i} \frac{\delta_k(A)}{z} \right) F, \tag{21}
\]

where \(\delta_k(A)\) is given in \((18)\), and \(E(d)_k = \text{diag}(0, \ldots, 0, \text{Id}_{d_k}, 0, \ldots, 0)\) is the blocked diagonal matrix whose \(k\)-th blocked diagonal entry is the rank \(d_k\) identity matrix \(\text{Id}_{d_k}\).

**Remark 2.14.** If the partition \(d = (1, 1, \ldots, 1)\), we arrive at the case of nondegenerate irregular term \(u \in h_{\text{reg}}(\mathbb{R})\) studied in Section 2.3–2.4. In this case, the system \((21)\) is a confluent hypergeometric system, and the associated connection matrices and Stokes matrices have closed formula. This is how we derive the explicit formula in Theorem 2.7.
3 Quantum Stokes matrices and their regularized limits at caterpillar points

This section gives the quantum analog of the results in the previous section, i.e., the expression of the regularized limits, as well as the leading terms, of quantum Stokes matrices as $u \to u_{cat}$, in terms of the Gelfand-Tsetlin basis. In Section 3.1, we introduce the quantum Stokes matrices of the linear system (1). In Section 3.2 we recall the regularized limits of quantum Stokes matrices at a caterpillar point, and their explicit expressions. In Section 3.3 and 3.4 we introduce the Gelfand-Tsetlin basis, and describe the leading terms of quantum Stokes matrices as $u \to u_{cat}$ in terms of the basis.

3.1 Quantum Stokes matrices

In this subsection, we recall the Stokes matrices of the linear system (1) associated to a representation $L(\lambda)$

$$\frac{dF}{dz} = h\left(\left(u + \frac{1}{2\pi i} T\right) \cdot F\right).$$

(22)

Under a choice of basis in $L(\lambda)$, it becomes a special case of the equation (4) with rank $n \times \dim(L(\lambda))$. But it is more conveniently seen as a $n \times n$ system with coefficients valued in a non-commutative space $\text{End}(L(\lambda))$. The two Stokes sectors are still the right/left half planes $\text{Sect}_\pm = \{ z \in \mathbb{C} \mid \pm \text{Re}(z) > 0 \}$. And again, let us choose the branch of $\log(z)$, which is real on the positive real axis, with a cut along $i\mathbb{R}_{\geq 0}$. As a special case of Theorem 2.2 we have that

Corollary 3.1. For any $u \in h_{reg}(\mathbb{R})$, on $\text{Sect}_\pm$ there is a unique (therefore canonical) holomorphic fundamental solution $F_{h\pm}(z; u) \in \text{End}(\mathbb{C}^n) \otimes \text{End}(L(\lambda))$ of (22) such that $F_{h\pm} \cdot e^{-ihuzz - \frac{b\|T\|}{2\pi \varepsilon}}$ can be analytically continued to $S(-\pi, \pi)$ and $S(0, 2\pi)$ respectively, and for every small $\varepsilon > 0$,

$$\lim_{z \to \infty} F_{h\pm}(z; u) \cdot e^{-ihuzz - \frac{b\|T\|}{2\pi \varepsilon}} = 1, \quad \text{as } z \in \mathbb{S}(-\pi + \varepsilon, \pi - \varepsilon),$$

$$\lim_{z \to \infty} F_{h\pm}(z; u) \cdot e^{-ihuzz - \frac{b\|T\|}{2\pi \varepsilon}} = 1, \quad \text{as } z \in \mathbb{S}(\varepsilon, 2\pi - \varepsilon),$$

where $[T] = \text{diag}(E_{11}, ..., E_{nn}) \in \text{End}(\mathbb{C}^n) \otimes \text{End}(L(\lambda))$.

Definition 3.2. The quantum Stokes matrices of (22) (with respect to $\text{Sect}_+$ and the chosen branch of $\log(z)$) are the elements $S_{h\pm}(u) \in \text{End}(\mathbb{C}^n) \otimes \text{End}(L(\lambda))$ determined by

$$F_{h\pm}(z) = F_{h\pm}(z) \cdot S_{h\pm}(u), \quad F_{h\pm}(z) = F_{h\pm}(z) \cdot S_{h\pm}(u) e^{-\frac{b\|T\|}{2\pi \varepsilon}},$$

where the first (resp. second) identity is understood to hold in $\text{Sect}_-$ (resp. $\text{Sect}_+$) after $F_{h\pm}$ (resp. $F_{h\pm}$) has been analytically continued clockwise.

Let us assume $u \in U_{\text{id}} \subset h_{reg}(\mathbb{R})$, then the asymptotics in Theorem 3.1 ensures that $S_{h\pm} = (s_{ij}^{(\pm)})$ is a upper triangular matrix, and $S_{h\pm} = (s_{ij}^{(-)})$ is lower triangular with entries $s_{ij}^{(\pm)} \in \text{End}(L(\lambda))$, see e.g., [24 Chapter 9.1].

3.2 Quantum Stokes matrices at a caterpillar point

Since (22) can be seen as a special form of the general system (4), we have

$$S_{h\pm}(u) = S_{\pm}(hu, -hT),$$

where the right hand side are the (classical) Stokes matrices of the equation (4) with $u$ and $A$ replaced by $hu$ and $-hT$. Here the irregular term $ithu = \text{diag}(ithu_{11}, ..., ithu_{nn}) \in \text{End}(\mathbb{C}^n) \otimes \text{End}(L(\lambda))$ is degenerate, i.e., has repeated eigenvalues. One can still study the regularized limits of $S_{h\pm}(u)$ as the components $u_i$ of $u \in h_{reg}(\mathbb{R})$ collapse in a comparable speed, and introduce the quantum Stokes matrices for any boundary point in $h_{reg}(\mathbb{R}) \setminus h_{reg}(\mathbb{R})$. In particular, the construction in Section 2.5 enables us to define the quantum Stokes matrices at $u_{cat}$ as follows.

First, the system (22) has rank $n \times \dim(L(\lambda))$. In terms of the notation in Section 2.5 let us take the partition $d$ of $n \times \dim(L(\lambda))$ with $d_1 = \cdots = d_n = \dim(L(\lambda))$, then the irregular term in the Stokes matrices $S_{\pm}(hu, -hT)$ of (22) lives in $U_{\pm}$. Following Theorem 2.13 the matrix $S_{\pm}(hu, -hT) \cdot S_{\pm}(hu, -hT)$ has a regularized limit.

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Definition 3.3. The quantum Stokes matrices at $u_{\mathrm{cat}}$, are the upper and lower $n \times n$ triangular matrices $S_{h \pm}(u_{\mathrm{cat}})$ (having the same diagonal part), with entries valued in $\text{End}(L(\lambda))$, such that (the blocked Gauss decomposition)

$$S_{h -}(u_{\mathrm{cat}})S_{h +}(u_{\mathrm{cat}})$$

equals to the regularized limit of the function $S_{-}(hu, -hT) \cdot S_{+}(hu, -hT)$ on $U_{\Delta}$, i.e., the limit of

$$\left( h^{-\frac{\delta_0(T)}{2\pi n}} \prod_{k=0,\ldots,n-1} u_k^{-1} \log \left( \frac{\delta_k(S_{-})}{\delta_k(S_{+})} \right) \right) \cdot S_{-} \cdot \left( h^{-\frac{\delta_0(T)}{2\pi n}} \prod_{k=0,\ldots,n-1} u_k^{-1} \log \left( \frac{\delta_k(S_{-})}{\delta_k(S_{+})} \right) \right)^{-1}$$

(23)

as $0 < u_1 \ll u_2 \ll \ldots \ll u_n$, where $\delta_k(S_{\pm}(hu, -hT))$ takes the $\text{End}(\lambda)$ valued entries of the matrices $S_{\pm}(hu, -hT)$ as in (18) (Here the extra term $h^{-\frac{\delta_0(T)}{2\pi n}}$ comes from the convention that $u_0 = 1$ in the product $\prod$).

To write down the explicit expression of $S_{h \pm}(u_{\mathrm{cat}})$, as a non-commutative version of the formula in Theorem 2.7, let us define $T(\zeta) := \zeta \text{Id} - T$. Then for any $1 \leq m \leq n$, elements $(a_1, \ldots, a_m)$ and $(b_1, \ldots, b_m)$ of $\{1, \ldots, n\}$, let us consider the equation $\delta_{E(k)}$ as in (18) (Here the extra term $h^{-\frac{\delta_0(T)}{2\pi n}}$ comes from the convention that $u_0 = 1$ in the product $\prod$).

Theorem 3.4. For any $1 \leq k \leq n - 1$, the $(k, k + 1)$-entry of $S_{h \pm}(u_{\mathrm{cat}})$, as an element in $\text{End}(L(\lambda))$ is given by

$$s_{k,k+1}^{(\pm)} = -\left( ih \right) \frac{h(E_{k,k} - E_{k+1,k+1+1})}{2\pi i} \sum_{i=1}^{k} \left( \prod_{l=1,\not= l \not= i}^{k} \Gamma(1 + h\frac{\delta_{E(k)}}{2\pi i}) \prod_{l=1,\not= l \not= i}^{k+1} \Gamma(1 + h\frac{\delta_{E(k)}}{2\pi i}) \right) \cdot h\alpha_{i}^{(k)},$$

$$s_{k+1,k}^{(\pm)} = -\left( ih \right) \frac{h(E_{k,k} - E_{k+1,k+1+1})}{2\pi i} \sum_{i=1}^{k+1} \left( \prod_{l=1,\not= l \not= i}^{k} \Gamma(1 - h\frac{\delta_{E(k)}}{2\pi i}) \prod_{l=1,\not= l \not= i}^{k+1} \Gamma(1 - h\frac{\delta_{E(k)}}{2\pi i}) \right) \cdot h\beta_{i}^{(k)},$$

with

$$\alpha_{i}^{(k)} = \sum_{j=1}^{k} \left( -1 \right)^{k-j} \frac{\delta_{j,j+1,\ldots,k}^{(\pm)}}{\Gamma(1 + h\frac{\delta_{E(k)}}{2\pi i})} \cdot E_{j,k+1}$$

(25)

$$\beta_{i}^{(k)} = \sum_{j=1}^{k} \left( -1 \right)^{k-j} \frac{\delta_{j,j+1,\ldots,k}^{(\pm)}}{\Gamma(1 - h\frac{\delta_{E(k)}}{2\pi i})} \cdot E_{k+1,j}$$

(26)

Remark 3.5. A universal version of the formula of quantum Stokes matrices at $u_{\mathrm{cat}}$ is given in [27], and is used to derive an explicit Drinfeld isomorphism.
3.3 Gelfand-Tsetlin basis for representations of $\mathfrak{gl}_n$

The expression of quantum Stokes matrices at the caterpillar point is closely related to the Gelfand-Tsetlin basis. Let us introduce the Gelfand-Tsetlin orthonormal basis in this subsection.

Recall that $\{ E_{ij} \}_{i,j=1,...,n}$ is the standard basis of the Lie algebra $\mathfrak{gl}_n$. Denote by $\mathfrak{gl}_{n-1}$ the subalgebra spanned by the elements $\{ E_{ij} \}_{i,j=1,...,n-1}$. Finite dimensional irreducible representations of $\mathfrak{gl}_n$ are parameterized by the highest weight, i.e., $n$-tuples of numbers $\lambda = (\lambda_1, ..., \lambda_n)$ with

$$\lambda_i - \lambda_{i+1} \in \mathbb{Z}_+, \quad \forall i = 1, ..., n - 1.$$

We denote by $L(\lambda)$ the corresponding representation. It has a lowest vector $\xi_0$ such that $E_{ii} \xi = \lambda_i \xi$ for $i = 1, ..., n$, and $E_{ij} \xi = 0$ for $1 \leq j < i \leq n$. Then the simple branching rule for the reduction from $\mathfrak{gl}_n$ to $\mathfrak{gl}_{n-1}$ states that the restriction of $L(\lambda)$ to the subalgebra $\mathfrak{gl}_{n-1}$ is isomorphic to the direct sum of pairwise inequivalent irreducible representations

$$L(\lambda)|_{\mathfrak{gl}_{n-1}} \cong \bigoplus_{\lambda^{(n-1)}} L'(\lambda^{(n-1)}),$$

where the summation is over the highest weights $\lambda^{(n-1)}$ satisfying the interlacing conditions

$$\lambda_i^{(n)} - \lambda_i^{(n-1)} \in \mathbb{Z}_{\geq 0}, \quad \lambda_i^{(n-1)} - \lambda_{i+1}^{(n)} \in \mathbb{Z}_{\geq 0}, \quad \forall i = 1, ..., n - 1. \quad (27)$$

Thus a chain of subalgebras

$$\mathfrak{gl}_1 \subset \cdots \subset \mathfrak{gl}_{n-1} \subset \mathfrak{gl}_n$$

produces a basis in $L(\lambda)$, parameterized by the Gelfand-Tsetlin patterns. Such a pattern $\Lambda$ (for fixed $\lambda^{(n)}$) is a collection of numbers $\{ \lambda_{ij}^{(1)} \}_{1 \leq i \leq j \leq n-1}$ satisfying the interlacing conditions

$$\lambda_{ij}^{(1)} - \lambda_{ij}^{(1-1)} \in \mathbb{Z}_{\geq 0}, \quad \lambda_{ij}^{(1-1)} - \lambda_{ij+1}^{(1)} \in \mathbb{Z}_{\geq 0}, \quad \forall i = 1, ..., n - 1.$$

**Definition 3.6.** We denote by $P_{GZ}(\lambda; \mathbb{Z})$ the set of Gelfand-Tsetlin patterns in $L(\lambda)$, seen as the set of integer points in the real Gelfand-Tsetlin polytope $P_{GZ}(\lambda; \mathbb{R})$.

The structure of the basis obtained in this way is summarized in the following proposition.

**Proposition 3.7.** There exists an orthonormal basis $\xi_\Lambda$ of $L(\lambda)$, called the Gelfand-Tsetlin basis, parameterized by all patterns $\Lambda \in P_{GZ}(\lambda; \mathbb{Z})$, such that for any $1 \leq i \leq k \leq n$, the actions of $E_{kk}$, $\zeta_i^{(k)}$ and $\alpha_i^{(k)}$ on the basis $\xi_\Lambda$ of $L(\lambda)$ are given by

$$E_{kk} \cdot \xi_\Lambda = \left( \sum_{i=1}^{k} \lambda_i^{(k)} - \sum_{i=1}^{k-1} \lambda_i^{(k-1)} \right) \xi_\Lambda, \quad (28)$$

$$\zeta_i^{(k)} \cdot \xi_\Lambda = \left( \lambda_i^{(k)}(\Lambda) - i + 1 + (k-1)/2 \right) \xi_\Lambda, \quad (29)$$

$$\alpha_i^{(k)} \cdot \xi_\Lambda = \left[ -\frac{\Pi_{l=1,l \neq i}^{k}(\zeta_l^{(k)} - \zeta_i^{(k)})}{\Pi_{l=1,l \neq i}^{k+1}(\zeta_l^{(k+1)} - \zeta_i^{(k+1)})} - \frac{\Pi_{l=1,l \neq i}^{k}(\zeta_l^{(k)} - \zeta_i^{(k)})}{\Pi_{l=1,l \neq i}^{k-1}(\zeta_l^{(k-1)} - \zeta_i^{(k-1)})} \right] \xi_\Lambda + \delta_i^{(k)} \xi_\Lambda, \quad (30)$$

where the pattern $\Lambda + \delta_i^{(k)}$ is obtained from $\Lambda$ by replacing $\lambda_i^{(k)}$ by $\lambda_i^{(k)} + 1$. It is supposed that $\xi_\Lambda$ is zero if $\Lambda$ is not a pattern.

**Proof.** We refer to [20] Section 2] for the existence of the orthogonal Gelfand-Tsetlin basis, and the norms of the elements in the basis [20] Proposition 2.4]. We then perform the explicit normalization to get an orthonormal basis. The actions of $E_{kk}$ and $\zeta_i^{(k)}$ on the orthogonal basis given in [20] Theorem 2.3] imply the identities (28) and (29).

By the Laplace expansion, the $\alpha_i^{(k)}$ in (25) equals to the normalized quantum minor

$$\alpha_i^{(k)} = -\frac{1}{\Pi_{l=1,l \neq i}^{k}(\zeta_l^{(k)} - \zeta_i^{(k)})} \Delta_{1,...,k-1,k+1}^{1,...,k}(T(\zeta - \frac{1}{2}(k-1))).$$

Then the identity (30) follows from the action of $\Delta_{1,...,k-1,k+1}^{1,...,k}$ on the orthogonal basis given in [20] Section 2.5], provided the normalization, from the orthogonal to the orthonormal basis, is accounted for.
Given any $L(\lambda)$, let us think of the associated quantum Stokes matrices as blocked matrices. Let us take a proper norm on the vector space $\text{End}(L(\lambda))$, then up to a slight modification, Proposition 2.10 and Corollary 2.11 can be applied to the quantum/blocked cases. That is if we introduce the unitary matrix
\[
g(u; T) = \prod_{k=0, \ldots, n-1} (u_k / u_{k+1}) \frac{\delta_k(T)}{2\pi i},
\]
then

**Corollary 3.8.** For any fixed none zero real number $h$, we have, as $0 < u_1 \ll u_2 \ll \ldots \ll u_n$,\[
S_{h \pm}(u) = S_{\pm}(u_{\text{cat}}, \ g(hu; hT) \cdot hT \cdot g(hu; hT)^{-1} + O((u_2 - u_1)^{-1})).
\] (31)

Furthermore, the subdiagonal entries of $S_{h \pm}(u)$ satisfy
\[
s_{k,k+1}^{(\pm)}(u) = t_{k,k+1}^{(\pm)}(u) + o(t_{k,k+1}^{(\pm)}(u)), \quad \text{as } 0 < u_1 \ll u_2 \ll \ldots \ll u_n,
\] (32)

where the leading terms $t_{k,k+1}^{(\pm)}(u)$ is given by the expressions $s_{k,k+1}^{(\pm)}(u_{\text{cat}})$ of in Theorem 3.4 provided we replace respectively $\alpha_i^{(k)}$ and $\beta_i^{(k)}$ by
\[
\frac{e_{kk}^{-1}}{2\pi i} (u_k / u_{k+1}) \frac{h_i^{(k)}}{2\pi i} \cdot \alpha_{i}^{(k)} \quad \text{and} \quad \frac{e_{kk}^{-1}}{2\pi i} (u_k / u_{k+1}) \frac{h_i^{(k)}}{2\pi i} \cdot \beta_{i}^{(k)}.
\] (33)

Here $\frac{k-1}{2}$ in $(\zeta_{i}^{(k)} - \frac{k-1}{2})$ comes from the Capelli correction in the quantum minors, see the computation in Appendix.

In particular, by Proposition 3.7 and Corollary 3.8 the leading terms of entries of quantum Stokes matrices, as elements in $\text{End}(L(\lambda))$, can be written down explicitly in terms of the Gelfand-Tsetlin basis.

**Remark 3.9.** In the next section, we will study the WKB approximation of $S_{h \pm}(u_{\text{cat}})$. By the definition given in the introduction and the replacement of $\alpha_i^{(k)}$ and $\beta_i^{(k)}$ given in (33), we see that $t_{k,k+1}^{(\pm)}(u)$ and $s_{k,k+1}^{(\pm)}(u_{\text{cat}})$ have the same WKB leading term as $h \to -\infty$. Note that the relation in (32) is for any fixed $h > 0$ (the proof of Proposition 2.10 given in [25] makes the assumption that the norm of $A$ is fixed), therefore, the WKB datum at $u_{\text{cat}}$ encodes the "repeated limit" of $S_{h \pm}(u)$ as $u \to u_{\text{cat}}$ and then $h \to -\infty$.

## 4 WKB approximation of quantum Stokes matrices at caterpillar points

In the last two sections, we introduce the relations between the regularized limits of Stokes matrices at $u_{\text{cat}}$, and Gelfand-Tsetlin systems in classical and quantum settings respectively. In this section, we deepen the relation between Stokes phenomenon and representation theory by showing that the WKB approximation of quantum Stokes matrices at a caterpillar point gives rise to $\mathfrak{gl}_n$-crystals. In particular, Section 4.1 computes explicitly the WKB approximation. Section 4.2 unveils that how to recover the Gelfand-Tsetlin basis from the WKB approximation. Then Section 4.3 introduces the notion of WKB datum of quantum Stokes matrices.

### 4.1 WKB approximation of quantum Stokes matrices at caterpillar points

Since the derivative in equation (11) is multiplied by a small parameter $1/h$, we will call the leading term, as $h \to -\infty$, of Stokes matrices of (11) as the WKB approximation. In this subsection, we study the WKB approximation of Stokes matrices given in (34).

First, for any Gelfand-Tsetlin pattern $\Lambda$ of $L(\lambda)$ given in Section 3.3 set
\[
x_j^{(k)}(\Lambda) := -\lambda_j^{(k)} + \lambda_{j-1}^{(k-1)} - \lambda_{j+1}^{(k)} + \lambda_j^{(k+1)}, \quad 1 \leq j \leq k + 1,
\] (34)
\[
y_j^{(k)}(\Lambda) := \lambda_j^{(k)} - \lambda_{j-1}^{(k-1)} + \lambda_{j+1}^{(k)} - \lambda_j^{(k+1)}, \quad 0 \leq j \leq k,
\] (35)
and
\[
X_j^{(k)}(\Lambda) := \sum_{i=1}^j \nu_i^{(k)}(\Lambda), \quad 1 \leq j \leq k + 1,
\]
\[
Y_j^{(k)}(\Lambda) := \sum_{i=j}^k \nu_i^{(k)}(\Lambda), \quad 0 \leq j \leq k.
\]

Note that \(Y_0^{(k)}(\Lambda) = Y_1^{(k)}(\Lambda) - X_1^{(k)}(\Lambda)\) for any \(j = 0, \ldots, k + 1\). Furthermore, we define
\[
\varepsilon_k(\Lambda) = \max\{X_1^{(k)}(\Lambda), X_2^{(k)}(\Lambda), \ldots, X_k^{(k)}(\Lambda)\},
\]
\[
\phi_i(\Lambda) = \max\{Y_1^{(k)}(\Lambda), Y_2^{(k)}(\Lambda), \ldots, Y_k^{(k)}(\Lambda)\},
\]
and define the functions \(l_1(\Lambda) < \cdots < l_{m_k}(\Lambda)\) of \(\Lambda \in P_{GZ}(\lambda; Z)\) be those ordered labels such that
\[
X_{l_j}^{(k)}(\Lambda) = \varepsilon_k(\Lambda).
\]

Let us denote by
\[
P^{k}_{GZ}(\lambda; Z) := \{\Lambda \in P_{GZ}(\lambda; Z) \mid \Lambda + \delta_i^{(k)} \in P_{GZ}(\lambda; Z) \text{ and } \Lambda + \delta_i^{(k)} \notin P_{GZ}(\lambda; Z) \text{ for } i = 2, \ldots, m_k\}.
\]
In particular, all \(\Lambda\) satisfying \(m_k(\Lambda) = 1\) and \(\Lambda + \delta_i^{(k)} \notin P_{GZ}(\lambda; Z)\) are in \(P^{k}_{GZ}(\lambda; Z)\). From (34)–(41), the function \(m_k(\Lambda)\) is defined for any point \(\Lambda\) in the real polytope \(P_{GZ}(\lambda; \mathbb{R})\). Thus \(P^{k}_{GZ}(\lambda; \mathbb{R}) := P^{k}_{GZ}(\lambda; Z) \otimes \mathbb{R}\) is an open dense part of \(P_{GZ}(\lambda; \mathbb{R})\), whose complements are cut out by various equalities between \(\Lambda_i^{(j)}(\Lambda)\) for \(j = k - 1, k, k + 1\). By this reason, the elements in the subset \(P^{k}_{GZ}(\lambda; Z)\) are called generic (it gives precise meaning of the word "generic" in Theorem 1.1).

Let \(\xi_{\Lambda}\) be the Gelfand-Tsetlin orthonormal basis given in Section 3.3. Then we have

**Proposition 4.1.** For \(k = 1, \ldots, n - 1\) and for any \(\Lambda \in P^{k}_{GZ}(\lambda; Z)\), there exists a real number \(\theta_k \in [0, 2\pi)\) independent of \(q = e^{h/2}\) such that
\[
s^{(\pm)}_{k,k+1} : \xi_{\Lambda} \sim q^{-\varepsilon_k(\Lambda + \delta_i^{(k)}) + \theta_k} \xi_{\Lambda + \delta_i^{(k)}}, \quad \text{as } q \to 0.
\]

Here note that \(\varepsilon_k(\Lambda + \delta_i^{(k)}(\Lambda)) = \varepsilon_k(\Lambda) + 1\).

**Proof.** By the asymptotics of gamma function
\[
\ln(\Gamma(1 + z)) \sim z \ln(z) - z + \frac{1}{2} \ln(z) + \frac{1}{2} \ln(2\pi) + \frac{1}{12z} + o\left(\frac{1}{z}\right), \quad \text{as } |z| \to \infty \text{ at constant } |\arg(z)| < \pi,
\]
For \(r\) a real number and \(h \to -\infty\), we have
\[
\ln(\Gamma(1 + \frac{rh}{2\pi i})) \sim \left\{ \begin{array}{ll}
\frac{r h \ln(-h)}{2\pi i} + \frac{rh}{2\pi i} \ln(\frac{r}{h}) + \frac{rh}{4} - \frac{rh}{4} + \frac{1}{2} \ln(\frac{r}{h}) & , \quad r > 0,

\frac{r h \ln(-h)}{2\pi i} + \frac{rh}{2\pi i} \ln(\frac{-r}{h}) - \frac{rh}{4} - \frac{rh}{4} + \frac{1}{2} \ln(\frac{-r}{h}) & , \quad r < 0.
\end{array} \right.
\]
Here we use \(\ln(\frac{a}{b}) = \mp \frac{\pi}{2}\) to separate the real and imaginary part of the \(h\) linear terms. By (29),
\[
\zeta_i^{(k)}, \xi_{\Lambda} = (\lambda_i^{(k)}(\Lambda) - i + 1 + (k - 1)/2) \cdot \xi_{\Lambda},
\]
by abuse of notation, we will take \(\zeta_i^{(k)}\) as the number \(\lambda_i^{(k)}(\Lambda) - i + 1 + (k - 1)/2\) when the vector \(\xi_{\Lambda}\) is specified. Then by (43) and the interlacing inequalities between \(\lambda_j^{(i)}\) for \(i = k - 1, k, k + 1\),
\[
\ln \left( \prod_{i=1, i \neq i}^{\Lambda} \Gamma(1 + h \frac{\zeta_i^{(k)}}{2\pi i}) \prod_{i=1, i \neq i}^{\Lambda} \Gamma(1 + h \frac{\zeta_i^{(k-1)}}{2\pi i}) \right) \sim \frac{h \ln(-h)}{2\pi i} A_i^{(k)} + i h \theta_i^{(k)} + \frac{h}{4} B_i^{(k)} + C_i^{(k)}, \quad \text{as } h \to -\infty,
\]
where
\[ A_i^{(k)} = \sum_{l=1}^{k-1} \zeta_l^{(k-1)} + \sum_{l=1}^{k+1} \zeta_l^{(k+1)} - 2 \sum_{l=1}^{k} \zeta_l^{(k)} + 1, \]
\[ B_i^{(k)} = -2 \sum_{l=1}^{i-1} (\zeta_l^{(k)} - \zeta_l^{(k)}) + 2 \sum_{l=i+1}^{k} (\zeta_l^{(k)} - \zeta_l^{(k)}) + \sum_{l=1}^{i-1} (\zeta_l^{(k)} - \zeta_l^{(k-1)}) \]
\[ - \sum_{l=i}^{k-1} (\zeta_l^{(k)} - \zeta_l^{(k-1)}) + \sum_{l=i}^{k} (\zeta_l^{(k)} - \zeta_l^{(k+1)}) - \sum_{l=i+1}^{k} (\zeta_l^{(k)} - \zeta_l^{(k+1)}), \]
\[ C_i^{(k)} = \frac{1}{2} \text{ln} \left( \frac{1}{h^2} \prod_{l=1}^{k} (\zeta_l^{(k)} - \zeta_l^{(k)}) \right), \]
\[ \text{and} \]
\[ \theta_i^{(k)} = \frac{h}{2\pi} A_i^{(k)} - \sum_{l=1}^{i-1} \frac{\zeta_l^{(k)} - \zeta_l^{(k)}}{2\pi} \text{ln} \left( \frac{\zeta_l^{(k)} - \zeta_l^{(k)}}{2\pi} \right) + \sum_{l=i+1}^{k} \frac{\zeta_l^{(k)} - \zeta_l^{(k)}}{2\pi} \text{ln} \left( \frac{\zeta_l^{(k)} - \zeta_l^{(k)}}{2\pi} \right) \]
\[ - \sum_{l=1}^{i-1} \frac{\zeta_l^{(k-1)} - \zeta_l^{(k-1)}}{2\pi} \text{ln} \left( \frac{\zeta_l^{(k-1)} - \zeta_l^{(k-1)}}{2\pi} \right) + \sum_{l=i+1}^{k} \frac{\zeta_l^{(k-1)} - \zeta_l^{(k-1)}}{2\pi} \text{ln} \left( \frac{\zeta_l^{(k-1)} - \zeta_l^{(k-1)}}{2\pi} \right) \]
\[ + \sum_{l=1}^{i-1} \frac{\zeta_l^{(k+1)} - \zeta_l^{(k+1)}}{2\pi} \text{ln} \left( \frac{\zeta_l^{(k+1)} - \zeta_l^{(k+1)}}{2\pi} \right) - \sum_{l=1}^{i-1} \frac{\zeta_l^{(k+1)} - \zeta_l^{(k+1)}}{2\pi} \text{ln} \left( \frac{\zeta_l^{(k+1)} - \zeta_l^{(k+1)}}{2\pi} \right). \]

Finally, the action of \( s_{k,k+1}^{(+)} \) on \( \xi_\Lambda \) is given by
\[ \langle h \rangle^{(E_{k,k} - E_{k+1,k+1} - 1)} \sum_{i=1}^{k} \left( \prod_{l=1,l \neq i}^{k+1} \text{ln} \left( 1 + h \frac{\zeta_l^{(k)} - \zeta_l^{(k)}}{2\pi} \right) \right) \left( \prod_{l=1}^{k} \text{ln} \left( 1 + h \frac{\zeta_l^{(k+1)} - \zeta_l^{(k+1)}}{2\pi} \right) \right) \cdot (\text{ln} h \alpha_i^{(k)}). \]
\[ \sim \langle -h \rangle^{(E_{k,k} - E_{k+1,k+1} - 1)} \left( -i \right) \left( \text{ln} \left( 1 + h \frac{\zeta_l^{(k)} - \zeta_l^{(k)}}{2\pi} \right) \right) \left( \text{ln} \left( 1 + h \frac{\zeta_l^{(k+1)} - \zeta_l^{(k+1)}}{2\pi} \right) \right) \cdot (\text{ln} h \alpha_i^{(k)}). \]
\[ = \sum_{i=1}^{k} \left( -h \right)^{\left( E_{k,k} - E_{k+1,k+1} - 1 \right)} \text{ln} \left( 1 + h \frac{\zeta_l^{(k)} - \zeta_l^{(k)}}{2\pi} \right) \left( \text{ln} \left( 1 + h \frac{\zeta_l^{(k+1)} - \zeta_l^{(k+1)}}{2\pi} \right) \right) \left( -h \alpha_i^{(k)} \right). \]
\[ = \sum_{i=1}^{k} q^{-X_i^{(k)}(\Lambda+\delta_i^{(k)})} \cdot \xi_\Lambda + \delta_i^{(k)}. \]

Here in the last equality, we use the identities \( (28), (29), (34) \) and \( (35) \) to get
\[ \left( -h \right)^{\left( E_{k,k} - E_{k+1,k+1} - 1 \right)} \text{ln} \left( 1 + h \frac{\zeta_l^{(k)} - \zeta_l^{(k)}}{2\pi} \right) = 1, \]
\[ e^{-\frac{h}{2\pi \text{ln} h} A_i^{(k)}} = 1, \]
\[ e^{-\frac{h}{2\pi \text{ln} h} A_i^{(k)}} = 1, \]
\[ q^{-X_i^{(k)}(\Lambda+\delta_i^{(k)})} \cdot \xi_\Lambda + \delta_i^{(k)} = q^{-X_i^{(k)}(\Lambda+\delta_i^{(k)})} \cdot \xi_\Lambda + \delta_i^{(k)}. \]

and use the expression \( (39) \) of \( \alpha_i^{(k)} \cdot \xi_\Lambda \) and the expression \( (44) \) of \( C_i^{(k)} \) to get
\[ -h \alpha_i^{(k)} \cdot \xi_\Lambda = \xi_\Lambda + \delta_i^{(k)}. \]

Then the proposition follows from the definitions \( (39) \) and \( (41) \) of \( \varepsilon_i^{(k)}(\Lambda) \) and \( l_i(\Lambda) \) with \( i = 1, ..., m_k \), and the assumption \( m_{k}^{(i)}(\Lambda) = 1 \). Particularly the constant \( \theta_k^{(i)} \) in \( (42) \) is just \( 2\theta_i^{(k)}. \)
4.2 Gelfand-Tsetlin basis form WKB approximation

Recall that for any tuple \( \underline{d} = (d_{ijk}^{(k)})_{1 \leq j, k \leq n-1} \in \mathbb{N}^{n(n-1)/2} \), we have defined the operator in the introduction

\[
Q(\text{ucat})\underline{d} = \prod_{i+j \geq n} (Q_{ij}(\text{ucat}))^{d_{ij}^{(k)}} \in \text{End}(L(\lambda)),
\]

where \( Q_{ij}(\text{ucat}) = s_{i,j}^{(1)} s_{j-1,i}^{(2)} \cdots s_{n-1,n-1}^{(n)} \) are combinations of the entries of Stokes matrix at \( \text{ucat} \). Let \( \xi_{\lambda_0} \) be the lowest vector in \( L(\lambda) \), which corresponds to the pattern \( \Lambda_0 \) satisfying \( \lambda_i^{(k)}(\Lambda_0) = \lambda_{i+1}^{(k+1)}(\Lambda_0) \) for all \( i, k \). Following the definition of \( Q\underline{d} \) and the proof of Proposition 4.1, we have

**Proposition 4.2.** For any \( \underline{d} \), there exists a constant \( c(\underline{d}) \) independent of \( q \) such that

\[
q^{c(\underline{d})}Q(\text{ucat})\underline{d} : \xi_{\lambda_0} \rightarrow \xi_{\lambda}, \quad \text{as} \quad q \rightarrow 0,
\]

where \( \Lambda \in P_{GZ}(\lambda; \mathbb{Z}) \) is uniquely determined by the linear equations

\[
\lambda_i^{(j)}(\Lambda) - \lambda_{i+1}^{(j+1)}(\Lambda) = d_{ij}^{(k)}, \quad \text{for all} \ 1 \leq j \leq i \leq n-1. \quad (44)
\]

**Proof.** The ordering in the definition of \( Q\underline{d} \), together with the formula (30) and the expressions in Theorem 3.4 enforces

\[
Q(\text{ucat})\underline{d} : \xi_{\lambda_0} = f(q)\xi_{\lambda},
\]

for certain function of \( q \), where \( \Lambda \) is determined by (44). Take \( Q\underline{d} = s_{34}(s_{34}s_{23})^2(s_{34}s_{23}s_{12})^2 \) as an example, by (3.4) and (30) we have

\[
\begin{align*}
&\quad s_{12} : \xi_{\lambda_0} = f_1 \xi_{\lambda_0 + \delta_1^{(1)}}, \quad s_{23}s_{12} : \xi_{\lambda_0} = f_2 \xi_{\lambda_0 + \delta_2^{(1)} + \delta_2^{(2)}}, \quad s_{34}s_{23}s_{12} : \xi_{\lambda_0} = f_3 \xi_{\lambda_0 + \delta_3^{(1)} + \delta_3^{(2)}}, \\
&\quad s_{12}s_{34}s_{23}s_{12} : \xi_{\lambda_0} = f_4 \xi_{\lambda_0 + \delta_4^{(1)} + \delta_4^{(2)} + \delta_4^{(3)}}, \quad s_{23}s_{34}s_{23}s_{12} : \xi_{\lambda_0} = f_5 \xi_{\lambda_0 + \delta_5^{(1)} + \delta_5^{(2)} + \delta_5^{(3)}}, \\
&\quad (s_{34}s_{34}s_{23}s_{12})^2 : \xi_{\lambda_0} = f_6 \xi_{\lambda_0 + \delta_6^{(1)} + \delta_6^{(2)} + \delta_6^{(3)}}, \quad s_{34}s_{34}s_{23}s_{12} : \xi_{\lambda_0} = f_7 \xi_{\lambda_0 + \delta_7^{(1)} + \delta_7^{(2)} + \delta_7^{(3)}}, \\
&\quad (s_{34}s_{34}s_{23}s_{12})^2 : \xi_{\lambda_0} = f_8 \xi_{\lambda_0 + \delta_8^{(1)} + \delta_8^{(2)}}, \quad (s_{34}s_{34}s_{23}s_{12})^2 : \xi_{\lambda_0} = f_9 \xi_{\lambda_0 + \delta_9^{(1)} + \delta_9^{(2)} + \delta_9^{(3)}},
\end{align*}
\]

for certain coefficients \( \{f_i(q)\}_{i=1}^{9} \). Thus \( Q(\text{ucat})\underline{d} : \xi_{\lambda_0} \) corresponds to a chain of vectors \( f_i(q)\eta_i \) as shown in the example, and the ordering in the product \( Q\underline{d} \) guarantees that in each immediate step the assumption in Proposition 4.1 is satisfied. By Proposition 4.1, for the \( i \)-th vector in the chain, there exists a constant \( c_i \) such that \( q^{c_i}f_i(q) \rightarrow 1 \) as \( q \rightarrow 0 \). Here we assume \( f_0(q) = 1 \). Since \( f(q) = \prod_{i=1}^{9} \frac{f_i(q)}{f_{i-1}(q)} \), it follows from that there exists constants \( c \) such that \( q^{c}f(q) \rightarrow 1 \) as \( q \rightarrow 0 \). It proves the proposition.

**Corollary 4.3.** The set \( B(\lambda, \text{ucat}) \) of the WKB approximation of all nonzero vectors \( Q(\text{ucat}, h)\underline{d} : \xi_0 \) is a base of \( L(\lambda) \), with a natural parameterization: the element corresponding to \( Q(\text{ucat}, h)\underline{d} : \xi_0 \) is labelled by the integer point \( \Lambda \in P_{GZ}(\lambda; \mathbb{Z}) \) satisfying (44). Actually \( B(\lambda, \text{ucat}) \) is isomorphic to the set of Gelfand-Tsetlin basis.

4.3 WKB operators

In this subsection, we introduce a combinatorial structure to encode the WKB leading terms (of entries) of quantum Stokes matrices at a catpeller point.

As we have seen from the above sections, the \( q \) leading term of the action of monomials \( Q^{d} \) for all possible \( d \) on the lowest vector \( \xi_0 \) produces a basis \( B(\lambda, \text{ucat}) \) of \( L(\lambda) \), parametrized by \( P_{GZ}(\lambda; \mathbb{Z}) \). It coincides with the Gelfand-Tsetlin basis. If we denote by \( B^k(\lambda, \text{ucat}) \subset B(\lambda, \text{ucat}) \) the subset consisting of the elements parametrized by \( P_{GZ}^k(\lambda; \mathbb{Z}) \subset P_{GZ}(\lambda; \mathbb{Z}) \), then the formula (42) shows that taking the \( q \) leading term of \( s_{k,k+1}^{(+)\underline{d}} \) naturally induces a map

\[
\hat{e}_k : B^k(\lambda, \text{ucat}) \rightarrow B(\lambda, \text{ucat}) ; \quad \hat{e}_k(\xi_\Lambda) = \xi_{\Lambda + \delta_1^{(k)}}.
\]

Equivalently, as we identify \( B(\lambda, \text{ucat}) \) with \( P_{GZ}(\lambda; \mathbb{Z}) \) by mapping \( \xi_\Lambda \) to \( \Lambda, \hat{e}_k \) can be seen as a map

\[
\hat{e}_k : P_{GZ}^k(\lambda; \mathbb{Z}) \rightarrow P_{GZ}(\lambda; \mathbb{Z}) ; \quad \hat{e}_k(\Lambda) = \Lambda + \delta_1^{(k)}.
\]
The map has a canonical extension to the whole $P_{GZ}(\lambda; \mathbb{Z})$ as follows.

First, the expression of $\tilde{e}_k$ is universal, i.e., doesn’t depend on the choice of $\lambda$. Thus for any positive integer $N$, if we set $N\lambda = (N\lambda_1^{(n)},...,N\lambda_n^{(n)})$, then $\tilde{e}_k$ can be equivalently seen as a map from $P_{GZ}^k(\lambda; \mathbb{Z}) := \{ \lambda \in P_{GZ}(\lambda; \mathbb{Z}) | \lambda_j^{(i)}(\Lambda) \in \mathbb{Z} \}$ to $P_{GZ}(\lambda; \mathbb{Z}) := \{ \Lambda \in P_{GZ}(\lambda; \mathbb{Z}) | \lambda_j^{(i)}(\Lambda) \in \mathbb{Z} \}$

$$\tilde{e}_k(\Lambda) := \Lambda + \frac{1}{N} \delta_{1}(k),$$

(45)

where $\lambda_j^{(i)}(\Lambda + \frac{1}{N} \delta_{1}) = N \lambda_j^{(i)} + \frac{\delta_{1} \delta_{1}}{N} \delta_{1}$. Now as $N \to \infty$, the discrete "dynamical system" (45) approaches to a unique continuous system in the inner part of the whole real polytope $P_{GZ}(\lambda; \mathbb{Z})$:

$$\tilde{e}_k(\Lambda) := \Lambda + t_{1} \delta_{1}^{(k)} + \cdots + t_{m} \delta_{1}^{(k)},$$

(46)

where the time

$$t \in \left[ 0, \sum_{i=1}^{k} \left( \min(\lambda_i^{(k+1)}(\Lambda), \lambda_{i-1}^{(k+1)}(\Lambda)) \right) \right],$$

and $t_1, ..., t_m, r_1, ..., r_m$ are determined by

$$r_1 = l_1(\Lambda + s \delta_{1}^{(k)}), \text{ for all } 0 \leq s < t_1,$$

$$r_i = l_1(\Lambda + t_{i-1} \delta_{1}^{(k)} + \cdots + r_{i-1} \delta_{1}^{(k)} + s \delta_{1}^{(k)}), \text{ for all } 0 \leq s < t_i, \text{ for } i = 2, ..., m,$$

$$t = t_{1} + \cdots + t_{m}.$$

Here the function $l_1(\Lambda)$ is defined for any point $\Lambda$ in $P_{GZ}(\lambda; \mathbb{R})$ just as (34)–(41).

This continuation is unique, thus canonically determines an extension of $\tilde{e}_k$ to the complements of the "generic part" $P_{GZ}^k(\lambda; \mathbb{Z})$ in $P_{GZ}(\lambda; \mathbb{Z})$. It gives rise to

**Definition 4.4.** For each $k$, the WKB operator $\tilde{e}_k$ from $B(\lambda, u_{\text{cat}}) \cong P_{GZ}(\lambda; \mathbb{Z})$ to $B(\lambda, u_{\text{cat}}) \cup \{ 0 \}$ is given by

$$\tilde{e}_k \cdot \xi_{\Lambda} := \xi_{\Lambda + \delta_{1}(k)}, \quad \forall \Lambda \in P_{GZ}(\lambda; \mathbb{Z})$$

(47)

Here recall that $l_1(\Lambda)$ is the integer given by (41), and it is supposed that $\tilde{e}_k \cdot \xi_{\Lambda}$ is zero if $\Lambda + \delta_{1}(k)$ doesn’t belong to $P_{GZ}(\lambda; \mathbb{Z})$.

The above computation and discussion carry to the WKB approximation of the entries $s_{k+1,k}^{(-)}$ of lower triangular Stokes matrix $S_{h-}(u_{\text{cat}})$. In particular, it produces the same set $B(\lambda, u_{\text{cat}})$, and induces operators $\tilde{f}_k$ on $B(\lambda, u_{\text{cat}})$

$$\tilde{f}_k \cdot \xi_{\Lambda} = \xi_{\Lambda - \delta_{1}(k)}, \quad \forall \Lambda \in P_{GZ}(\lambda; \mathbb{Z}).$$

(48)

One checks that $\tilde{e}_k$ and $\tilde{f}_k$ satisfy that for all $\Lambda, \Lambda' \in P_{GZ}(\lambda; \mathbb{Z})$,

$$\varepsilon_{k}(\Lambda) = \max \{ j : e_{k}^{j}(\xi_{\Lambda}) \neq 0 \},$$

$$\phi_{k}(\Lambda) = \max \{ j : f_{k}^{j}(\xi_{\Lambda}) \neq 0 \},$$

and

$$\tilde{e}_k \cdot \xi_{\Lambda} = \xi_{\Lambda} \text{ if and only if } \tilde{f}_k \cdot \xi_{\Lambda} = \xi_{\Lambda}.$$  

**Definition 4.5.** We call $(B(\lambda, u_{\text{cat}}), \tilde{e}_k, \tilde{f}_k, \varepsilon_k, \phi_k)$ the WKB datum of the Stokes matrices $S_{h-}(u_{\text{cat}})$ at the caterpillar point associated to the representation $L(\lambda)$.

## 5 WKB datum are crystals

In Section 5.1 we recall the notion of crystals and their tensor products. In Section 5.2 we prove that the WKB datum of quantum Stokes matrices at $u_{\text{cat}}$ are $\mathfrak{gl}_n$-crystals. Furthermore, in Section 5.3 we also realize the tensor products of $\mathfrak{gl}_n$-crystals by WKB analysis.
5.1 Crystals and tensor products

Let $\mathfrak{g}$ be a semisimple Lie algebra with a Cartan datum $(A, \Delta_+ = \{\alpha_i\}_{i \in I}, \Delta_-^\vee = \{\alpha_i^\vee\}_{i \in I}, P, P^\vee)$ be a Cartan datum, where $P \subset \mathfrak{h}^*$ denotes the weight lattice, $I$ denotes the set of vertices of its Dynkin diagram, $\alpha_i \in I$ denote its simple roots, and $\alpha_i^\vee$ the simple coroots.

**Definition 5.1.** A $\mathfrak{g}$-crystal is a finite set $B$ along with maps

$$wt : B \to P,$$

$$\tilde{e}_i, \tilde{f}_i : B \to B \cup \{0\}, \quad i \in I,$$

$$\varepsilon_i, \phi_i : B \to \mathbb{Z} \cup \{-\infty\}, \quad i \in I,$$

satisfying for all $b, b' \in B$, and $i \in I$,

- $\tilde{f}_i(b) = b'$ if and only if $b = \tilde{e}_i(b')$, in which case
  $$wt(b') = wt(b) - \alpha_i, \quad \varepsilon_i(b') = \varepsilon_i(b) + 1, \quad \phi_i(b') = \phi_i(b) + 1.$$

- $\phi_i(b) = \varepsilon_i(b) + (wt(b), \alpha_i^\vee)$, and if $\phi_i(b) = -\infty$, then $\tilde{e}_i(b) = \tilde{f}_i(b) = 0$.

The map $wt$ is called the weight map, $\tilde{e}_i$ and $\tilde{f}_i$ are called Kashiwara operators or crystal operators.

**Definition 5.2.** Let $B_1$ and $B_2$ be two crystals. The tensor product $B_1 \otimes B_2$ is the crystal with the underlying set $B_1 \times B_2$ (the Cartesian product) and structure maps

$$wt(b_1, b_2) = wt(b_1) + wt(b_2),$$

$$\tilde{e}_i(b_1, b_2) = \begin{cases} (\tilde{e}_i(b_1), b_2), & \text{if } \varepsilon_i(b_1) > \phi(b_2) \\ (b_1, \tilde{e}_i(b_2)), & \text{otherwise} \end{cases}$$

$$\tilde{f}_i(b_1, b_2) = \begin{cases} (\tilde{f}_i(b_1), b_2), & \text{if } \varepsilon_i(b_1) \geq \phi(b_2) \\ (b_1, \tilde{f}_i(b_2)), & \text{otherwise}. \end{cases}$$

5.2 $\mathfrak{g}l_n$-crystals from WKB approximation

Let us take the Cartan datum of type $A_{n-1}$, where $I = \{1, 2, ..., n - 1\}$, the weight lattice $P = \mathbb{C}\{v_1, ..., v_n\}$, and $\alpha_i = v_i - v_{i+1}$, $\alpha_i^\vee$ is given by $\langle v_i, \alpha_j^\vee \rangle = \delta_{ij} - \delta_{i,j+1}$. Let us consider the WKB datum, given in Definition 5.1 of the quantum Stokes matrices at the caterpillar point $u_{\text{cat}}$ associated to the representation $L(\lambda)$. Let us define a weight map

$$wt : B(\lambda, u_{\text{cat}}) \to P ; \xi_\lambda \to \sum_{k=1}^n wt_k(\lambda)v_k = \sum_{k=1}^n \left( \sum_{i=1}^k \lambda_i^{(k)}(\lambda) - \sum_{i=1}^{k-1} \lambda_i^{(k-1)}(\lambda) \right)v_k.$$  

The following theorem implies Theorem 1.1 in the introduction.

**Theorem 5.3.** The WKB datum $(B(\lambda, u_{\text{cat}}), \tilde{e}_k, \tilde{f}_k, \varepsilon_k, \phi_k)$ with the weight map $wt$ is a $\mathfrak{g}l_n$-crystal.

**Proof.** By the formula (47) and (48), the WKB operators on the finite set $B(\lambda, u_{\text{cat}})$ are:

$$\tilde{e}_k : \xi_\lambda = \begin{cases} \xi_{\lambda + \delta_i^{(k)}}, & \text{if } \varepsilon_k(\lambda) > 0, \\ 0, & \text{if } \varepsilon_k(\lambda) = 0, \end{cases} \quad \text{where } l = \min\{j = 1, ..., k \mid X_j^{(k)}(\lambda) = \varepsilon_k(\lambda)\},$$

$$\tilde{f}_k : \xi_\lambda = \begin{cases} \xi_{\lambda - \delta_i^{(k)}}, & \text{if } \phi_k(\lambda) > 0, \\ 0, & \text{if } \phi_k(\lambda) = 0, \end{cases} \quad \text{where } l = \max\{j = 1, ..., k \mid Y_j^{(k)}(\lambda) = \phi_k(\lambda)\},$$

The explicit realization (49) and (50) of the WKB operators $\tilde{e}_k, \tilde{f}_k$ coincides with the known crystal operators of $\mathfrak{g}l_n$-crystal realized on the Gelfand-Tsetlin basis, see e.g., [21].
5.3 Tensor products from WKB approximation

Recall that given any representation $L(\lambda)$, the quantum Stokes matrices at $u_{\text{cat}}$ produce operators $s_{ij}^{(\pm)} \in \text{End}(L(\lambda))$. Now given two representations $L(\lambda_1)$ and $L(\lambda_2)$, let us consider the actions of

$$
s_{k,k}^{(+)} \otimes s_{k+1,k+1}^{(+)} + s_{k,k+1}^{(+)} \otimes s_{k+1,k}^{(+)} \quad \text{and} \quad s_{k,k}^{(+)} \otimes s_{k,k}^{(-)} + s_{k,k+1}^{(+)} \otimes s_{k+1,k}^{(-)} \quad k = 1, \ldots, n-1,
$$

(51)
on the tensor product $L(\lambda_1) \otimes L(\lambda_2)$. Let $\{\xi_{\lambda_1}\}$ and $\{\xi_{\lambda_2}\}$ be the basis of $L(\lambda_1)$ and $L(\lambda_2)$ respectively. Similar to Proposition 4.1, one can compute the WKB leading term of the operators in (51) under the basis $\xi_{\lambda_1} \otimes \xi_{\lambda_2}$. (Since the diagonal elements $s_{k,k}^{(\pm)}$ of quantum Stokes matrices have a rather simple expression, the computation is direct and is omitted here.) Then following the same argument as previous sections, one verifies

Proposition 5.4. The WKB approximation of the operators in (51) induces the crystal operators $\tilde{e}_k$ and $\tilde{f}_k$ on the tensor product $E(\lambda_1, u_{\text{cat}}) \otimes E(\lambda_2, u_{\text{cat}})$ of $\mathfrak{gl}_n$-crystals.

5.4 The structure theorem of the WKB approximation

In a summary, the WKB approximation of quantum Stokes matrices at caterpillar points naturally produces Gelfand-Tsetlin basis in any $L(\lambda)$: the WKB approximation of monomials $Q^d$ of entries of Stokes matrices produces the basis $B(\lambda, u_{\text{cat}})$ of $L(\lambda)$. Then for any generic $v \in B(\lambda, u_{\text{cat}})$, the WKB approximation of the vector $s_{k,k+1} \cdot v$ takes the form

$$
q^{-\epsilon_k(v) + \partial_k(v)} \tilde{e}_k(v),
$$

(52)
where $\tilde{e}_k$ is the WKB/crystal operator on $B(\lambda, u_{\text{cat}})$. The expression (52) can be understood as a structure theorem of quantum Stokes matrices at $u_{\text{cat}}$ under the Gelfand-Tsetlin basis. In a next paper, we would like to use the method of isomonodromy deformation to obtain the same expression (52) for quantum Stokes matrices at general $u$, with $B(\lambda, u_{\text{cat}})$ replaced by the eigenbasis $B(\lambda, u)$ of the shift of argument subalgebra $A(u)$ of $U(\mathfrak{gl}_n)$.

Appendix

A. The RLL relations and semiclassical limits

This appendix explains the reason why the WKB approximation in differential equations should be related to the theory of crystal bases in quantum groups. It recalls the quantum and classical RLL relations satisfied by quantum and classical Stokes matrices respectively.

In [27], we showed that $S_{h^+}(u) = (s_{ij}^{(\pm)})$ satisfy the RLL relation, i.e., are $L_{\pm}$-operators, in the Faddeev-Reshetikhin-Takhtajan realization [22] of the quantum group $U_q(\mathfrak{gl}_n)$ of $\mathfrak{gl}_n$. To be more precise, let us take the R-matrix $R \in \text{End}(\mathbb{C}^n) \otimes \text{End}(\mathbb{C}^n)$ given by

$$
R = \sum_{i \neq j, i,j=1}^n E_{ii} \otimes E_{jj} + e^{\frac{1}{2}} \sum_{i=1}^n E_{ii} \otimes E_{ii} + (e^{\frac{1}{2}} - e^{-\frac{1}{2}}) \sum_{1 \leq j < i \leq n} E_{ij} \otimes E_{ji}.
$$

(53)
Set $S_{h^+}(u) := S_{h^+}(u)$ and $S_{h^-}(u) := S_{h^+}(u)^{-1} \in \text{End}(\mathbb{C}^n) \otimes \text{End}(L(\lambda))$. The following theorem can be found in [26,27] in various generalities.

Theorem 5.5. For any $u \in \mathfrak{h}_{\text{reg}}(\mathbb{R})$ and $L(\lambda)$, the quantum Stokes matrices $S_{h^\pm}(u)$ satisfy

$$
R^{12} S_{h^+}^{13} S_{h^+}^{23} = S_{h^+}^{23} S_{h^+}^{13} R^{12} \in \text{End}(\mathbb{C}^n) \otimes \text{End}(\mathbb{C}^n) \otimes \text{End}(\lambda),
$$

(54)
$$
R^{12} S_{h^+}^{13} S_{h^-}^{23} = S_{h^+}^{23} S_{h^-}^{13} R^{12} \in \text{End}(\mathbb{C}^n) \otimes \text{End}(\mathbb{C}^n) \otimes \text{End}(\lambda),
$$

(55)
and

$$(S_{h^+})_{ii} \cdot (S_{h^-})_{ii} = (S_{h^-})_{ii} \cdot (S_{h^+})_{ii} = 1, \quad \text{for } i = 1, \ldots, n.$$
Thus the Stokes matrices $S_{h\pm}(u_{\text{cat}})$ can be understood as the $L_{\pm}$ operators in the FRT realization of the quantum group $U_q(\mathfrak{gl}_n)$ \cite{22} in a representation. The associated coproduct takes the form

$$
\Delta(s^{(\pm)}_{ij}) = \sum_{k=1}^{n} s^{(\pm)}_{ik} \otimes s^{(\pm)}_{kj}, \ i, j = 1, \ldots, n.
$$

Thus the operators in \cite{51} are just the coproduct of the sub-diagonal elements of quantum Stokes matrices.

One checks that the "gauge transformation" appearing in \cite{23} preserves the RLL relations, that is if we write the term in \cite{23} as $G_{h^-}(u)^{-1}G_{h^+}(u)$ via the Gauss decomposition, then $G_{\pm}(u)$ satisfy the identities \cite{54} and \cite{55}. Since $S_{h\pm}(u_{\text{cat}})$ are the limits of $G_{h\pm}(u)$, we get

**Corollary 5.6.** The quantum Stokes matrices $S_{h\pm}(u_{\text{cat}})$ at $u_{\text{cat}}$ satisfy the RLL relations \cite{54} and \cite{55}.

Similarly, one can show that the RLL relations are preserved under the regularized limits of quantum Stokes matrices at a general boundary point in $h_{\text{reg}}(\mathbb{R})$. We refer to \cite{25} for the classical analog, where the classical RLL relation \cite{56} is shown to be preserved under the closure of (classical) Stokes matrices. We see that the Stokes matrices at a caterpillar point share many properties with the generic ones, thus we can first study these properties using the closed formula at $u_{\text{cat}}$, and then use the method of isomonodromy/isospectral deformation to understand them at generic $u$.

**Semiclassical limits.** Let us consider the Lie algebra $u(n)$ of the unitary group $U(n)$, consisting of skew-Hermitian matrices, and identify $\text{Herm}(n) \cong u(n)^*$ via the pairing $\langle A, \xi \rangle = 2\text{Im}(trA\xi)$. Thus $\text{Herm}(n)$ inherits a Poisson structure from the canonical linear (Kostant-Kirillov-Souriau) Poisson structure on $u(n)^*$.

For any fixed $u \in h_{\text{reg}}(\mathbb{R})$, by varying $A \in \text{Herm}(n)$, we can think of the Stokes matrices $S_{\pm}(u; A)$ of equation \cite{4} as matrix valued functions $S_{\pm}(u)$ on $\text{Herm}(n) \cong u(n)^*$. Let us introduce the function $M(u) = S_{\pm}(u)^{-1}S_{\pm}(u)$, and let $r_{\pm} \in \text{End}(\mathbb{C}^n) \otimes \text{End}(\mathbb{C}^n)$ be the standard classical $r$-matrices, that is the semiclassical limit of the $R$-matrix in \cite{53}, see e.g., \cite{1} Formula (212) and (213)]. Then a reformulation of Boalch’s remarkable theorem states that

**Theorem 5.7.** \cite{57} For any fixed $u \in h_{\text{reg}}(\mathbb{R})$, the matrix valued function $M(u)$ satisfies

$$
\{M^1, M^2\} = r_+ M^1 M^2 + M^1 M^2 r_+ - M^1 r_+ M^2 - M^2 r_- M^1. \quad (56)
$$

Here the tensor notation is used: both sides are $\text{End}(\mathbb{C}^n) \otimes \text{End}(\mathbb{C}^n)$ valued functions on $\text{Herm}(n)$, $M^1 := M \otimes 1$, $M^2 := 1 \otimes M$, and the $ij, kl$ coefficient of the matrix $\{M^1, M^2\}$ is defined as the Poisson bracket $\{M_{ij}, M_{kl}\}$ of the functions on $\text{Herm}(n) \cong u(n)^*$.

The right hand side of \cite{56} coincides with the classical RLL formulation of the Poisson brackets on the standard dual Poisson Lie group (see \cite{1} Formula (235)). In particular, the relation \cite{56} can be seen as a semiclassical limit of the relations \cite{54}, \cite{55}. This is why the Stokes matrices of the equation \cite{22} are called quantum.

**B. The proof of Theorem 5.4.**

The explicit expressions in Theorem \cite{34} can be derived in the same way as the ones in Theorem \cite{27} except that in the non-commutative setting we should use Proposition \cite{58} to exchange the orders of the $\text{End}(L(\lambda))$ valued functions $\zeta_1^{(k)}$ and $\alpha_1^{(k)}$.

Following Definition \cite{33} and Theorem \cite{213} we have

$$
S_{h-}(u_{\text{cat}})S_{h+}(u_{\text{cat}}) = \left( \prod_{k=1,\ldots,n} C^{(k)}_h(T) \right) e^{hT} \cdot \left( \prod_{k=1,\ldots,n} C^{(k)}_h(T) \right)^{-1}, \quad (57)
$$

where each $C^{(k+1)}_h(T)$ denotes the connection matrix of

$$
\frac{dF}{dz} = h \left( iE_{k+1,k+1} + \frac{1}{2\pi i} \frac{\delta_{k+1}(T)}{z} \right) \cdot F. \quad (58)
$$
Here \( E_{k+1} \in \text{End}(C^n) \otimes \text{End}(L(\lambda)) \) is the matrix whose \((k + 1, k + 1)\) entry is 1 \( \in \text{End}(L(\lambda)) \) and other entries are zero. Therefore, to get the explicit expressions of \( S_h(u_{\text{cat}}) \), we need to compute each \( C_h^{(k+1)}(T) \).

Diagonalization in stages. To simplify the equation \((53)\), let us diagonalize the upper left \( k \)-th submatrix of its coefficient matrix. Recall that \( \zeta_1^{(k)}, \ldots, \zeta_k^{(k)} \) of \( M_k(\zeta) \) denote the roots of the quantum minor \( \Delta_{1,\ldots,k}^{1,\ldots,k}(T(\zeta - \frac{h}{2}(k - 1))) \), which act diagonally on the Gelfand-Zeitlin basis in \( L(\lambda) \).

**Proposition 5.8.** (a) For any 2 \( \leq k \leq n \), the \( n \times n \) matrix \( P_k \) with entries in \( \text{End}(L(\lambda)) \),

\[
(P_k)_{ij} := \frac{(-1)^{k-j}}{\prod_{l=1, l \neq i}^{k}(\zeta_i^{(k)} - \zeta_l^{(k)})} \Delta_{1,\ldots,k}^{1,\ldots,k}(T(\zeta_i^{(k)} - \frac{1}{2}(k - 1))) \tag{59}
\]

\[
(P_k)_{ii} := 1, \text{ if } i > k, \quad (P_k)_{ij} := 0, \text{ otherwise.}
\]
diagonalizes the upper left \( k \)-th principal submatrix \( T^{(k)} \) of \( T \), and is such that \( T_k := P_k \cdot T \cdot P_k^{-1} \) takes the form

\[
T_k = \begin{pmatrix}
\zeta_1^{(k)} + \frac{1}{2}(k - 1) & \alpha_1^{(k)} & \cdots \\
& \ddots & \cdots & \cdots \\
& & \zeta_k^{(k)} + \frac{1}{2}(k - 1) & \alpha_k^{(k)} \\
& & & E_{k+1,k+1}
\end{pmatrix}
\]

(b) Furthermore, for any 1 \( \leq k \leq n - 1 \), and 1 \( \leq i, j \leq k \), we have the commutators

\[
[\zeta_i^{(k)}, \alpha_j^{(k)}] = \delta_{ij}\alpha_j^{(k)}.
\]

**Proof.** The Laplace expansion, of the quantum minor \( M_k(\zeta) \) as a weighted sum of \( k \) quantum minors of size \( k - 1 \), shows that for any tuples \((a_1, \ldots, a_k)\) and \((b_1, \ldots, b_k)\),

\[
\Delta_{b_1,\ldots,b_k}^{a_1,\ldots,a_k}(T(\zeta)) = \sum_{j=1}^{k} (-1)^{k-j} \Delta_{b_1,\ldots,b_{k-1}}^{a_1,\ldots,a_j\ldots,a_k}(T(\zeta)) \cdot T_{a_j,b_k}(\zeta + (k - 1)).
\]

In the following, let us fix \((a_1, \ldots, a_k) = (1, \ldots, k)\) and \((b_1, \ldots, b_{k-1}) = (1, \ldots, k - 1)\). If we take \( b_k = k \) and for each \( i = 1, \ldots, k \) plug in \( \zeta = \zeta_i^{(k)} - \frac{1}{2}(k - 1) \), we get the identity

\[
0 = \Delta_{1,\ldots,k}^{a_1,\ldots,a_k}(T(\zeta_i^{(k)} - \frac{1}{2}(k - 1))) = \sum_{j=1}^{k} (-1)^{k-j} \Delta_{1,\ldots,k-1}^{a_1,\ldots,a_j\ldots,a_k}(T(\zeta_i^{(k)} - \frac{1}{2}(k - 1))) \cdot T_{j,k}(\zeta_i^{(k)} + \frac{1}{2}(k - 1)).
\]

Similarly, taking \( b_k = 1, \ldots, k - 1 \) and \( \zeta = \zeta_i^{(k)} - \frac{1}{2}(k - 1) \) eventually implies that for each \( i \), the row vector

\[
v_i = \frac{1}{\prod_{l=1, l \neq i}^{k}(\zeta_i^{(k)} - \zeta_l^{(k)})} ((P_k)_{i1}, \ldots, (P_k)_{ik})
\]

satisfies

\[
v_i \cdot T^{(k)} = \left(\zeta_i^{(k)} + \frac{h}{2}(k - 1)\right) v_i.
\]

Here we denote by \( T^{(k)} \) the upper left \( k \)-th principal submatrix of \( T \). It verifies (a).

The relation in (b) follows from Proposition \([57]\). ■

One can define \( P_k \) recursively. That is for any 1 \( \leq k \leq n - 1 \), we introduce the \( n \times n \) matrix \( L^{(k+1)} \) with entries in \( \text{End}(L(\lambda)) \)

\[
L_{ij}^{(k+1)} = \alpha_i^{(k)} \cdot \frac{1}{\zeta_j^{(k+1)} - \zeta_i^{(k+1)}} 1 \leq i \leq k, \ 1 \leq j \leq k + 1,
\]

\[
L_{k+1,j}^{(k+1)} = 1, \quad 1 \leq j \leq k + 1.
\]

\[
L_{ii}^{(k+1)} = 1, \quad if \ i > k + 1, \quad L_{ij}^{(k+1)} = 0, otherwise.
\]
Then $L^{(k+1)} \cdot T_k \cdot L^{(k+1)} = T_{k+1}$, and $P_k = L^{(k+1)} P_{k+1}$.

**Confluent hypergeometric functions in representation spaces.** Diagonalizing the upper left $k$-th submatrix of the coefficient of (58), we are left with the equation

$$\frac{dF}{dz} = h \left( t E_{k+1,k+1} + \frac{1}{2\pi i} \frac{\delta_{k+1}(T_k)}{z} \right) \cdot F. \quad (63)$$

A solution of this equation, as a function valued in $\text{End}(\mathbb{C}^n) \otimes \text{End}(L(\lambda))$, is described using the generalized confluent hypergeometric functions as follows. First recall that the confluent hypergeometric functions associated to any $\alpha_j \in \mathbb{C}$ and $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}, 1 \leq j \leq m$, are

$$kF_k(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_k)_n z^n}{(\beta_1)_n \cdots (\beta_k)_n n!}, \quad (64)$$

where $(\alpha)_0 = 1$, $(\alpha)_n = \alpha \cdot \cdots \cdot (\alpha + n - 1)$, $n \geq 1$. Since $\{\xi_i^{(k)}\}_{1 \leq i \leq k \leq n}$ are commutative elements in $\text{End}(L(\lambda))$, we can introduce the following confluent hypergeometric functions valued in $\text{End}(L(\lambda))$,

$$H(z)_{ij} = \frac{1}{h(\xi_i - \xi_j)} \cdot kF_k(\alpha_{ij,1}, \ldots, \alpha_{ij,k}, \beta_{ij,1}, \ldots, \beta_{ij,k+1}; tz), \quad 1 \leq i \leq k, 1 \leq j \leq k + 1,$$

$$H(z)_{k+1,j} = F_k(\alpha_{k+1,j,1}, \ldots, \alpha_{k+1,j,k}, \beta_{k+1,j,1}, \ldots, \beta_{k+1,j,k+1}; t z), \quad i = k + 1, 1 \leq j \leq k + 1,$$

with the variables $\{\alpha_{ij,l}\}$ and $\{\beta_{ij,l}\}$ given by

$$\alpha_{ij,l} = h \left( \xi_{ij} - \xi_l - \frac{1}{2} \right), \quad 1 \leq i \leq k, 1 \leq j \leq k + 1,$$

$$\alpha_{(k+1),j,l} = 1 + h \left( \xi_{ij} - \xi_l - \frac{1}{2} \right), \quad 1 \leq j \leq k + 1,$$

$$\alpha_{ij,l} = 1 - h \left( \xi_{ij} - \xi_l - \frac{1}{2} \right), \quad l \neq i, 1 \leq l \leq k, 1 \leq i, j \leq k + 1,$$

$$\beta_{ij,l} = 1 + h \left( \xi_{ij} - \xi_l^{(k+1)} \right), \quad l \neq j, 1 \leq l \leq k + 1, 1 \leq i, j \leq k + 1.$$

**Proposition 5.9.** The equation (63) has a fundamental solution $F(z) \in \text{End}(\mathbb{C}^n) \otimes \text{End}(L(\lambda))$ taking the form

$$F(z) = \left( \begin{array}{cc} \text{diag}(h a_1^{(k)} \cdots h a_k^{(k)}, 1) & 0 \\ 0 & \text{Id}_{n-k-1} \end{array} \right) \left( \begin{array}{cc} (H_{ij}(z))_{k+1,1} & 0 \\ 0 & \text{Id}_{n-k-1} \end{array} \right) \cdot \frac{h(T_{k+1})}{2\pi i}, \quad (65)$$

where $[T_{k+1}]$ is the diagonal part of $T_{k+1}$, i.e.,

$$[T_{k+1}] = \left( \begin{array}{cc} \text{diag}(\xi_1^{(k+1)} + \frac{k}{2}, \ldots, \xi_{k+1}^{(k+1)} + \frac{k}{2}) & 0 \\ 0 & \text{diag}(E_{k+2,k+2}, \ldots, E_{n,n}) \end{array} \right).$$

**Proof.** We only need to verify the identities

$$z \frac{d}{dz} (F(z))_{ij} = h \left( \xi_i^{(k)} + \frac{1}{2} (k-1) \right) F(z)_{ij} + h \frac{1}{2\pi i} a_i^{(k)} F(z)_{k+1,j}, \quad \text{for } 1 \leq i \leq k, \quad (66)$$

$$z \frac{d}{dz} (F(z))_{k+1,j} = h t z F(z)_{k+1,j} + h \frac{1}{2\pi i} \sum_{i=1}^{k} \beta_i^{(k)} F(z)_{ij} + h \frac{1}{2\pi i} \xi_i^{(k+1)} F(z)_{k+1,j}, \quad (67)$$

for the functions

$$F(z)_{ij} = h a_i^{(k)} \cdot H_{ij}(z) \cdot z^{\frac{h}{2\pi i} (\xi_i^{(k+1)} + \frac{k}{2})}, \quad \text{for } 1 \leq i \leq k, 1 \leq j \leq k + 1,$$

$$F(z)_{k+1,j} = H_{k+1,j}(z) \cdot z^{\frac{h}{2\pi i} (\xi_{k+1}^{(k+1)} + \frac{k}{2})}, \quad \text{for } 1 \leq j \leq k + 1.$$
Since $\alpha_i^{(k)}$ and $\zeta_{ij}^{(k)}$ do not commute, to compare the left and right hand sides of (66), we need to keep the terms $\alpha_i^{(k)}$ in the front. That is to pull the term $\alpha_i^{(k)}$ in

$$\left(\zeta_{ij}^{(k)} + \frac{1}{2}(k-1)\right) F(z)_{ij} = \left(\zeta_{ij}^{(k)} + \frac{1}{2}(k-1)\right) \cdot \alpha_i^{(k)} H_{ij}(z) z^{\frac{1}{2}\ell_1} z^{\frac{1}{2}(k+1) + \frac{1}{2}}$$

in front, via the commutator relation

$$\left(\zeta_{ij}^{(k)} + \frac{1}{2}(k-1)\right) \cdot \alpha_i^{(k)} = \alpha_i^{(k)} \cdot \left(\zeta_{ij}^{(k)} + \frac{1}{2}(k+1)\right).$$

Then the equation (66) is equivalent to the differential equation

$$\frac{dH_{ij}}{dz} = \frac{h(\zeta_{ij}^{(k)} - \zeta_{ij}^{(k+1)} + \frac{1}{2})}{2\pi i} \cdot \alpha_i^{(k)} H_{ij} - \frac{1}{2\pi i} H_{k+1,ij}, \text{ for } 1 \leq i \leq k,$$

which in turn can be verified by the definition (64) of the confluent hypergeometric functions. Using the similar differential equation of confluent hypergeometric functions, as well as the characteristic polynomials of $\delta_{k+1}(T)$, one can verify (66).

We denote by $C_{h}^{(k+1)}(T_k)$ the connection matrix of the equation (63). Then the Stokes matrices $S_{h}^{(k+1)}(T)$ and the renormalized connection matrix $\tilde{C}_{h}^{(k+1)}(T_k) := C_{h}^{(k+1)}(T_k) \cdot L^{(k+1)}$ of the linear system (63) is described by

**Proposition 5.10.** (1) The entries $s_{ij}$ of $S_{h}^{(k+1)}(T)$ are

$$s_{j,k+1} = \left(-\frac{h}{\pi}\frac{\zeta_{ij}^{(k)} - \zeta_{j,k+1}^{(k+1)} - 1}{2\pi i} e^{-\frac{h\zeta_{ij}^{(k)}}{2\pi i}} \prod_{l=1, l \neq j}^{k+1} \Gamma(1 + h\zeta_{ij}^{(k)} - \zeta_{ij}^{(k+1)} - \frac{1}{2}) \prod_{l=1}^{k+1} \Gamma(1 + h\zeta_{ij}^{(k)} - \zeta_{ij}^{(k+1)} - \frac{1}{2}) \cdot h\alpha_j^{(k)} \right), \text{ for } j = 1, ..., k;$$

and

$$s_{ii} = e^{\frac{F_k}{2}}, \quad i = 1, ..., n; \quad s_{ij} = 0, \text{ otherwise.}$$

(2) The entries $c_{ij}$ of the matrix $\tilde{C}_{h}^{(k+1)}(T)$ are

$$c_{ij} = -\left(\frac{h}{\pi} \frac{\zeta_{ij}^{(k+1)} - \zeta_{ij}^{(k)}}{2\pi i} \prod_{l=1}^{k+1} \Gamma(1 + h\zeta_{ij}^{(k+1)} - \zeta_{ij}^{(k+1)} - \frac{1}{2}) \prod_{l=1}^{k+1} \Gamma(1 + h\zeta_{ij}^{(k+1)} - \zeta_{ij}^{(k+1)} - \frac{1}{2}) \right) \cdot h\alpha_i^{(k)},$$

for $1 \leq j \leq k + 1, 1 \leq i \leq k$;

$$c_{k+1,j} = \left(\frac{h}{\pi} \frac{E_{K,k+1} - h\zeta_{ij}^{(k+1)}}{2\pi i} \prod_{l=1}^{k+1} \Gamma(1 + h\zeta_{ij}^{(k+1)} - \zeta_{ij}^{(k+1)} - \frac{1}{2}) \prod_{l=1}^{k+1} \Gamma(1 + h\zeta_{ij}^{(k+1)} - \zeta_{ij}^{(k+1)} - \frac{1}{2}) \right), \text{ for } 1 \leq j \leq k + 1;$$

and

$$c_{ii} = 1 \text{ for } k + 2 \leq i \leq n; \quad c_{ij} = 0 \text{ otherwise.}$$

**Proof.** It follows from Proposition 5.9 and the known asymptotics of the functions $k F_k(\alpha_1, ..., \alpha_k, \beta_1, ..., \beta_k; z)$:

$$\prod_{l=1}^{k} \Gamma(\alpha_l) \cdot k F_k(\alpha_1, ..., \alpha_k, \beta_1, ..., \beta_k; z) \sim \sum_{m=1}^{k} \Gamma(\alpha_m) \prod_{l=1, l \neq m}^{k} \Gamma(\alpha_l - \alpha_m) (\mp z)^{-\alpha_m} (1 + O(\frac{1}{z})) + e^z z^k \sum_{l=1}^{k} (\alpha_l - \beta_l) (1 + O(\frac{1}{z})). \tag{69}$$
where upper or lower signs are chosen according as \( z \) lies in the upper or lower half-plane. To be more precise, if we take the solution \( \hat{F}(z) \) in Proposition [5.9] then

\[
F(z) \sim \hat{F}(z) \cdot Y \cdot U_+, \quad \text{as } z \to \infty \text{ in } \mathbb{S}(-\pi + \varepsilon, \pi - \varepsilon)
\]

\[
F(z) \sim \hat{F}(z) \cdot Y \cdot U_-, \quad \text{as } z \to \infty \text{ in } \mathbb{S}(\varepsilon, 2\pi - \varepsilon),
\]

for small \( \varepsilon \), where

\[
\hat{F}(z) = (1 + O(z^{-1})) e^{\varepsilon z} E_{k+1,k+1} z^{\frac{k}{\pi \varepsilon}} [T_k]
\]

is the unique formal solution, and \( U_\pm \) are explicit matrices determined by the asymptotics (69). Note that the matrices \( U_\pm \) are certain functions of the commutative variables \( \{ \zeta_j^{(k)} \}_{j=1, \ldots, k} \) and \( \{ \zeta_j^{(k+1)} \}_{j=1, \ldots, k+1} \). Then by the uniqueness in Theorem [2.2] we know that the canonical solutions

\[
F(z) = F_+(z) Y U_+ \quad \text{in } \text{Sect}_+,
\]

\[
F(z) = F_-(z) Y U_- \quad \text{in } \text{Sect}_-.
\]

Then by definition, the Stokes matrices are given by (here to derive the second formula, the change in choice of \( \log(z) \) is accounted for)

\[
S^{(k+1)}_{h+}(T) = Y U_- U_+^{-1} Y^{-1}, \quad S^{(k+1)}_{h-}(T) = Y U_- e^{h[T_{k+1}]} U_+^{-1} Y^{-1}.
\]

Then to get the expression of \( \delta^{(k+1)}_j \) in the proposition, we only need to use the commutative relation (60) to commute \( a_j^{(k)} \) in the first factor of \( Y \) in \( Y U_- U_+^{-1} Y^{-1} \) with the arguments \( \zeta_j^{(k)} \) in \( U_- U_+^{-1} \). Similarly, one can get the expression of the renormalized connection matrix \( \tilde{C}^{(k+1)}(T) \). The involved computation using the asymptotics of \( _k F_k \) is straightforward but lengthy. Here we only outline the computation, and refer the reader to [17] for the full details including the explicit expressions of \( U_\pm \).

**Subdiagonal entries of quantum Stokes matrices.** The connection matrices of the equations (58) and (63) are related by

\[
C^{(k+1)}_h(T) = P_{k-1}^{-1} \cdot C^{(k+1)}_h(T_k) \cdot P_k,
\]

where \( P_k \) is given in Proposition [5.8] By the relation \( \tilde{C}^{(k+1)}(T) = C^{(k+1)}_h(T_k) \cdot L^{(k+1)} \) and \( P_k = L^{(k+1)} P_{k+1} \),

we get

\[
C^{(k+1)}_h(T) = P_{k-1}^{-1} \cdot \tilde{C}^{(k+1)}_h(T_k) \cdot P_{k+1}.
\]

Plugging (70) for all \( k = 1, \ldots, n - 1 \) into (57) gives

\[
S_{h-}(u_{\text{cat}}) S_{h+}(u_{\text{cat}}) = \left( \prod_{k=1, \ldots, n} \tilde{C}^{(k+1)}_h(T_k) \right) \cdot e^{h[T]} \cdot \left( \prod_{k=1, \ldots, n} \tilde{C}^{(k+1)}_h(T_k) \right)^{-1}.
\]

A manipulation of (blocked) Gauss decomposition, as well as the explicit formula of \( \tilde{C}^{(k+1)}(T) \) and \( S^{(k+1)}_{h+}(T) \) given in Proposition [5.10] gives rise to the formula of \( s^{(k+1)}_{h,k+1} \) in Theorem [3.4]. We refer the reader to [17] for the full details.

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