Positive definiteness of the asymptotic covariance matrix of OLS estimators in parsimonious regressions

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This version: October 26, 2020
First version: October 13, 2020

Abstract

Recently, Ghysels, Hill, and Motegi (2020) proposed a test for examining whether a large number of coefficients in linear regression models are all zero. The test is called the “max test.” The test statistic is calculated by first running multiple ordinary least squares (OLS) regressions, each including only one of key regressors, whose coefficients are supposed to be zero under the null, and then taking the maximum value of the squared OLS coefficient estimates of those key regressors. They called these regressions “parsimonious regressions.” This paper answers a question raised in their Remark 2.4; whether the asymptotic covariance matrix of the OLS estimators in the parsimonious regressions is generally positive definite. The paper shows that it is generally positive definite, and the result may be utilized to facilitate the calculation of the simulated $p$ value necessary for implementing the max test.

Keywords: Max test; Linear regression; Positive definiteness Many covariates ; Parsimonious regression.
JEL Classification: C12, C22, C51.

*The author would like to thank Kaiji Motegi for his helpful comments. Of course, any remaining errors are the author’s own.
1 Introduction

Several studies have considered the problem of testing a large number of restrictions in linear regression models when the number of regressors is large relative to the number of observations. For instance, Calhoun (2011) showed that the usual F test has an over-rejection tendency, and thus, proposed a modified F test that has correct sizes even in this case. Similarly, Anatolyev (2012) derived modified F, likelihood ratio, and Lagrange multiplier tests under assumptions similar but different from those of Calhoun (2011) (see Richard (2019) for the bootstrapped versions of these tests). These tests are modifications of the well-known classical tests and are derived in the context of cross-sectional regressions with homoscedastic errors. Please refer to Cattaneo, et al. (2018a), Cattaneo, et al. (2018b), Chudik, et al. (2018), and Cattaneo, et al. (2019) for related analyses.

Recently, Ghysels, Hill, and Motegi (2020) (hereafter, GHM) proposed a new test for examining a large set of zero restrictions in linear regression models that possibly include time series variables and heteroscedastic errors. They called this new test the “max test.” Specifically, they considered the following regression model:

$$y_t = z_t' a + x_t' b + \epsilon_t, \quad t = 1, \ldots, n, \quad (1)$$

where $z_t = [z_{1t}, \ldots, z_{pt}]'$ and $x_t = [x_{1t}, \ldots, x_{ht}]'$ are vectors of explanatory variables with dimensions $p$ and $h$, respectively, $a = [a_1, \ldots, a_p]'$, $b = [b_1, \ldots, b_h]'$ are vectors of coefficients, and $\epsilon_t, t = 1, \ldots, n$, are error terms with zero mean and a finite second moment. Dimension $p$ is assumed to be small, while $h$ is large but finite. It is well known that the usual Wald test for the null hypothesis $H_0 : b = 0_{h \times 1}$ and against alternative hypothesis $H_1 : b \neq 0_{h \times 1}$ has a severe over-rejection tendency in small sample due to parameter proliferation, where $0_{a \times b}$ denotes the $a \times b$ vector with all elements being zero. Moreover, bootstrapping the Wald test corrects the finite sample size, reducing the power significantly. To overcome these difficulties, GHM proposed the max test, and their simulation experiments confirmed that the max test does not show the over rejection tendency of the usual Wald test, and is more powerful than the bootstrapped Wald test.

To construct the max test statistic, we first run the following regressions for $i = 1, \ldots, h$, which GHM called “parsimonious regressions”:

$$y_t = z_t' \alpha_i + x_{it}' \beta_i + u_{it}, \quad (2)$$

where $u_{it}$ is an error term that depends on $i$. Let $\hat{\beta}_ni$ be the OLS estimate of $\beta_i$ from the ith parsimonious regression in (2). GHM defined the max test statistic, $\hat{T}_n$, as:

$$\hat{T}_n = \max\{ (\sqrt{n} \hat{\beta}_{n1})^2, \ldots, (\sqrt{n} \hat{\beta}_{nh})^2 \}. \quad (3)$$

When $\hat{T}_n$ is sufficiently large, the null hypothesis is rejected.

Let $V$ denote the asymptotic covariance matrix of $\sqrt{n} \hat{\beta}_n$, where $\hat{\beta}_n = [\hat{\beta}_{n1}, \ldots, \hat{\beta}_{nh}]'$. The asymptotic distribution of the max test statistic depends on $V$, and is thus not pivotal. To implement the test, GHM utilized simulated $p$ value for $\hat{T}_n$, proposing to compute the simulated $p$ value in the following way. First, generate a sample $\{X_i^{(j)}\}_{i=1}^h$ from the asymptotic
distribution of \( \sqrt{n} \hat{\beta}_n \) with estimated \( V \), or \( \mathcal{N}(0_{h \times 1}, \hat{V}_n) \), where \( \hat{V}_n \) is an estimate of \( V \), then, calculate \( \hat{\gamma}_n^{(j)} \equiv \max \{ \{ \lambda_i^{(j)} \}^2 \}_{i=1}^h \), repeat the procedure \( M \) times to obtain \( \hat{\gamma}_n^{(j)} \), \( j = 1, \ldots, M \), and finally compute the ratio of the number of \( \hat{\gamma}_n^{(j)} \)'s that are greater than \( \hat{\gamma}_n \) to \( M \) as the simulated \( p \) value (see GHM for the computational details). GHM proved that this simulated \( p \) value converges to the true \( p \) value and the test has the correct size asymptotically. They also demonstrated, using simulation experiments, that the test is correctly sized even in finite samples and possesses high powers against various alternatives.

In Remark 2.4, GHM mentioned that \( V \) is positive definite if \( p = 1 \) and \( h = 2 \); however, they did not prove that \( V \) is generally positive definite. GHM emphasized that \( V \) can be estimated consistently even if it is not positive definite, and the test can be implemented regardless of positive definiteness of \( V \).

This paper proves that \( V \) is positive definite in general, which guarantees that the consistently estimated \( V \) is positive definite asymptotically. This implies that it is legitimate, at least asymptotically, to apply a decomposition for positive definite matrices, such as the Cholesky decomposition, to the estimated \( V \). This may facilitate the simulation used for computing the \( p \) value. If \( V \) is not guaranteed to be positive definite, we need to use a decomposition that works even for positive semi definite matrices, such as eigenvalue decomposition, which is much slower to implement than the Cholesky decomposition. The paper also proposes an alternative estimator for \( V \), which is positive definite even in finite sample size. This or a similar positive definiteness property have not been proved for the estimator of \( V \) proposed by GHM.

The paper proceeds as follows. In the next section, it is shown that \( V \) is positive definite for general \( p \) and \( h \). An estimator for \( V \), which is positive definite even in finite sample size, is also proposed. The last section provides concluding remarks. The Appendix presents several proofs.

2 Main results

Let \( X_t = [z'_t, x'_t]' \). Define

\[
\Gamma = E(X_t X'_t), \quad \Gamma_{zz} = E(z_t z'_t), \quad \Gamma_{xx} = E(x_t x'_t), \quad \Gamma_{zx} = E(z_t x'_t), \\
\Lambda = E(x^2_t X_t X'_t), \quad \Lambda_{zz} = E(x^2_t z_t z'_t), \quad \Lambda_{xx} = E(x^2_t x_t x'_t), \quad \Lambda_{zx} = E(x^2_t z_t x'_t), \\
\gamma_{ij} = E(x_i x_j), \quad \gamma_{iz} = E(x_i z_t), \quad \lambda_{ij} = E(x^2_i x_j), \quad \lambda_{iz} = E(x^2_i z_t).
\]

All expected values defined here and hereafter are assumed to exist and be finite. We also make the following assumption throughout the paper.

**Assumption 1.** Assumptions 2.1—2.3 in GHM hold.

The above assumption ensures that \( \Gamma \) is positive definite.

GHM showed that the asymptotic covariance matrix of \( \sqrt{n} \hat{\beta}_n \), namely, \( V \), is expressed under the null hypothesis as:

\[
V = RSR',
\]
where

\[
R = \begin{bmatrix}
0_{1 \times p} & 0_{1 \times p} & \cdots & 0_{1 \times p} \\
0_{1 \times p} & 0_{1 \times p} & \cdots & 0_{1 \times p} \\
\vdots & \vdots & \ddots & \vdots \\
0_{1 \times p} & 0_{1 \times p} & \cdots & 0_{1 \times p}
\end{bmatrix}, \quad S = \begin{bmatrix}
\Sigma_{11} & \cdots & \Sigma_{1h} \\
\vdots & \ddots & \vdots \\
\Sigma_{h1} & \cdots & \Sigma_{hh}
\end{bmatrix},
\]

\[
\Sigma_{ij} = \Gamma_{ii}^{-1} A_{ij} \Gamma_{jj}^{-1}, \quad \Gamma_{ij} = E[X_{it} X'_{jt}], \quad A_{ij} = E[\epsilon_i^2 X_{it} X'_{jt}], \quad X_{it} = [z_t', x_{it}]', \quad i, j \in 1, \ldots, h.
\]

From this expression, it is unclear whether \( V \) is generally positive definite. In Remark 2.4, GHM mentioned that they proved that \( V \) is positive definite only in the case that \( p = 1 \) and \( h = 2 \). The following proposition shows that \( V \) is generally positive definite for any \( p \) and \( h \).

**Proposition 1** The asymptotic covariance matrix \( V \) is expressed as:

\[
V = D^{-1} (\Omega'_{xx} \Omega_{xx} \Omega_{xx} + \Lambda_{xx} - \Lambda'_{xx} \Lambda_{xx}^{-1} \Lambda_{xx}) D^{-1},
\]

where \( D \) is an \( h \times h \) diagonal matrix whose \( i \)-th diagonal element \( d_{ii} \) is \( d_{ii} \equiv \gamma_{ii} - \gamma_{iz} \Gamma_{zz}^{-1} \gamma_{iz} \), and matrices \( \Omega_{xx} \) and \( \Omega_{zz} \) are:

\[
\Omega_{xx} = [\Gamma'_{xx}, \Lambda'_{xx}], \quad \text{and} \quad \Omega_{zz} = \begin{bmatrix}
\Gamma_{zz}^{-1} & \Gamma_{zz}^{-1} & -\Gamma_{zz}^{-1} & -\Gamma_{zz}^{-1} \\
\Gamma_{zz}^{-1} & -\Gamma_{zz}^{-1} & \Gamma_{zz}^{-1} & -\Gamma_{zz}^{-1} \\
-\Gamma_{zz}^{-1} & \Gamma_{zz}^{-1} & -\Gamma_{zz}^{-1} & \Gamma_{zz}^{-1} \\
-\Gamma_{zz}^{-1} & \Gamma_{zz}^{-1} & -\Gamma_{zz}^{-1} & \Gamma_{zz}^{-1}
\end{bmatrix}.
\]

Furthermore, \( V \) is positive definite if \( \Lambda \) is positive definite.

**Proof.** See the Appendix.

Note that \( d_{ii} \) is equal to the conditional variance of \( x_{it} \), conditional on \( z_t \), when \( X_{it} \) is jointly normally distributed.

An immediate corollary of Proposition \((\text{1})\) is as follows (cf. Remark 2.2. in GHM).

**Corollary 1** (conditional homoscedasticity) When \( \Lambda = \sigma^2 \Gamma \), \( V \) reduces to:

\[
V = \sigma^2 D^{-1} (\Gamma_{xx} - \Gamma_{xx} \Gamma_{zz}^{-1} \Gamma_{xx}) D^{-1}.
\]

**Proof.** When \( \Lambda = \sigma^2 \Gamma \), matrices \( \Lambda_{zz} \), \( \Lambda_{xx} \), and \( \Lambda_{xx} \) are expressed as \( \Lambda_{zz} = \sigma^2 \Gamma_{zz} \), \( \Lambda_{xx} = \sigma^2 \Gamma_{xx} \), and \( \Lambda_{xx} = \sigma^2 \Gamma_{xx} \), respectively, and we have:

\[
\Omega'_{xx} \Omega_{xx} \Omega_{xx} = \begin{bmatrix}
\Gamma'_{xx} & \sigma^2 \Gamma'_{xx} \\
\sigma^2 \Gamma_{xx} & \Gamma_{xx}
\end{bmatrix} \begin{bmatrix}
\sigma^2 \Gamma_{zz}^{-1} & -\Gamma_{zz}^{-1} \\
-\Gamma_{zz}^{-1} & \sigma^2 \Gamma_{zz}^{-1}
\end{bmatrix} \begin{bmatrix}
\Gamma_{xx} \\
\sigma^2 \Gamma_{xx}
\end{bmatrix} = \begin{bmatrix}
0_{h \times p} & 0_{h \times p}
\end{bmatrix} \begin{bmatrix}
\Gamma_{xx} \\
\sigma^2 \Gamma_{xx}
\end{bmatrix} = 0_{h \times h}.
\]

\(^1S\) is positive semidefinite for general \( p \) and \( h \). See Remark 2.3 in GHM.
From (8) and (9), we obtain the result in (7). □

The expression of $V$ in (7) is, in fact, different from the asymptotic covariance matrix of $\sqrt{n}\hat{\theta}_n$ in the case of conditional homoscedasticity, namely, $\sigma^2(\Gamma_{xx} - \Gamma'_{xx}\Gamma_{xx}^{-1}\Gamma_{xx})^{-1}$, where $\hat{b}_n$ is the OLS estimator of $b$ in the full regression model in (1).

Next, we consider the estimation of $V$. Define $\theta_i = [\alpha_i', \beta_i]'$. Let $\hat{\theta}_{ni}$ denote the OLS estimator for $\theta_i$ in the $i$th parsimonious regression: $y_t = X'_it\theta_i + u_{it}$. To estimate $V$, GHM proposed the following estimator:

$$\hat{V}_n = R\hat{S}\hat{R}'$$

where $\hat{S} = [\hat{\Sigma}_{ij}]_{i,j}, \hat{\Sigma}_{ij} = \hat{\Gamma}_{ii}^{-1}\hat{\Lambda}_{ij}\hat{\Gamma}_{jj}^{-1}, \hat{\Gamma}_{ij} = n^{-1}\sum_{t=1}^n X_{it}X'_{jt}$, $\hat{\Lambda}_{ij} = n^{-1}\sum_{t=1}^n \hat{u}_{it}^2X_{it}X'_{jt}$, and $\hat{u}_{it} = y_t - X'_it\hat{\theta}_{ni}$. Under Assumption 1, $\hat{V}_n$ is a consistent estimator for $V$ under $H_0$, and converges in probability to a matrix under $H_1$ (Theorem 2.3 in GHM). Because it is a consistent estimator for $V$ under $H_0$ and $V$ is positive definite, as shown in Proposition 1 above, $\hat{V}_n$ is asymptotically positive definite under $H_0$. However, the positive definiteness may not be guaranteed in finite sample because it uses different estimates of the error term in constructing $\hat{\Lambda}_{ij}$, depending on $i$.

This paper proposes a slightly different consistent estimator for $V$ as:

$$\tilde{V}_n = R\tilde{S}\tilde{R}'$$

where $\tilde{S} = [\tilde{\Sigma}_{ij}]_{i,j}, \tilde{\Sigma}_{ij} = \tilde{\Gamma}_{ii}^{-1}\tilde{\Lambda}_{ij}\tilde{\Gamma}_{jj}^{-1}, \tilde{\Gamma}_{ij} = n^{-1}\sum_{t=1}^n \tilde{u}_{it}^2X_{it}X'_{jt}$, $\tilde{\Lambda}_{ij} = n^{-1}\sum_{t=1}^n \tilde{u}_{it}^2X_{it}X'_{jt}$, $\tilde{u}_{it} = y_t - z_i\tilde{\alpha}_n$, and $\tilde{\alpha}_n$ is the OLS estimator of $\alpha$ in the regression: $y_t = z'_i\alpha + \epsilon_t$, which is the regression obtained under $H_0$. The estimator $\tilde{V}_n$ is almost always positive definite in practice, even in finite sample size. To see this, note that $\tilde{u}_{it}$, which is the estimate of the error term $\epsilon_t$ under $H_0$, does not depend on $i$, and is common in all $\tilde{\Lambda}_{ij}$. Then, it follows from the proof of Proposition 1 that $\tilde{V}_n$ can be expressed as:

$$\tilde{V}_n = \tilde{D}^{-1}(\tilde{\Omega}_{xx}\tilde{\Omega}_{zz}\tilde{\Omega}_{zx} + \tilde{\Lambda}_{xx} - \tilde{\Lambda}'_{xx}\tilde{\Lambda}_{zz})\tilde{D}^{-1},$$

where $\tilde{\Lambda}_{xx} = n^{-1}\sum_{t=1}^n \tilde{u}_{it}^2x'_ix'_t$, $\tilde{\Lambda}_{xx} = n^{-1}\sum_{t=1}^n \tilde{u}_{it}^2z'_iz'_t$, $\tilde{\Lambda}_{zz} = n^{-1}\sum_{t=1}^n \tilde{u}_{it}^2z'_i\tilde{\alpha}_n$, and $\tilde{D}, \tilde{\Omega}_{xx}, \tilde{\Omega}_{zz},$ are defined similarly to $D, \Omega_{xx}, \Omega_{zz}$, respectively, where $\Lambda_{xx}$ and $\Lambda_{zz}$ are replaced with $\tilde{\Lambda}_{xx}$, and $\tilde{\Lambda}_{zz}$, respectively, and $\Gamma_{xx}, \Gamma_{zz}, \gamma_{ii},$ and $\gamma_{iz}$ are replaced with their corresponding sample moments, $\Gamma_{xx}, \tilde{\Gamma}_{zz}, \tilde{\gamma}_{ii},$ and $\tilde{\gamma}_{iz}$; here, for example, $\Gamma_{xx}$ is defined as $n^{-1}\sum_{t=1}^n z'_ix'_t$. The proof of Proposition 1 implies that $\tilde{V}_n$ is positive definite whenever $\Lambda = n^{-1}\sum_{t=1}^n \tilde{u}_{it}^2X'_{it}X'_{it}$ is positive definite. This holds true, for example, when $\tilde{u}_{it} \neq 0$ for all $t = 1, ..., n$, which is practically always satisfied. Moreover, we obtain the following

2. The matrix $\tilde{\Lambda}$ is expressed as $\tilde{\Lambda} = n^{-1}X'D_u(X'D_u)'$, where $X \equiv [X_1, \cdots, X_n]'$ and $D_u$ is the $n \times n$ diagonal matrix whose $ith$ diagonal element is equal to $\tilde{u}_i$. When $\tilde{u}_t \neq 0$ for all $t = 1, ..., n$, $D_u$ is full rank and thus $\text{rank}(\Lambda) = \text{rank}(X'D_u) = \text{rank}(X')$. Then, the positive definiteness of $\Lambda$ follows from Assumption 2.2. in GHM, namely, $X$ is of full column rank $p + h$ almost surely.

3. Actually, up to $n - (p + h)$ values of $\tilde{u}_i$ among the $n$ values of $\tilde{u}_i, t = 1, ..., n$, can be zero, depending on $X$. For example, any $n - (p + h)$ diagonal elements of $D_u$ can be zero as long as $p + h$ column vectors of $X'D_u$ are linearly independent.
proposition on the consistency property of \( \tilde{V}_n \), which is similar to Theorem 2.3 in GHM.

**Proposition 2** Under Assumption 1, it holds that, as \( n \to \infty \), \( \tilde{V}_n \overset{p}{\to} V \) under \( H_0 \), and \( \tilde{V}_n \overset{p}{\to} \bar{V} \) under \( H_1 \), where \( \bar{V} \) is a matrix of positive definite.

**Proof.** The proof is similar to the proof for Theorem 2.3 in GHM, and is thus omitted here.

One may concern that the performance of the max test can be affected by the choice of the estimator for \( V \). Therefore, we conducted a simulation experiment to check whether there are any differences in the performances of the max tests with \( \tilde{V}_n \) and \( \hat{V}_n \). Due to space considerations, we do not report detailed simulation results, but the unreported simulation experiment confirmed that the performance of the max test is not affected when replacing \( \hat{V}_n \) with \( \tilde{V}_n \) in terms of both size and power. Using an estimator for \( V \) that is positive definite even for finite sample size may facilitate the computation of the simulated \( p \) value. For example, we can use the estimate \( \bar{V}_n \) when the estimate \( \tilde{V}_n \) is not positive definite.

3 Concluding remarks

This paper showed that the asymptotic covariance matrix of the OLS estimators in the parsimonious regressions considered in Ghysels, Hill, and Motegi (2020) is generally positive definite. This result may facilitate the computation of the simulated \( p \) value needed to implement the max test. One may consider applying a bootstrap method to the max test. However, to achieve an asymptotic refinement by bootstrap, it is crucial that the test statistic is (asymptotically) pivotal (see Hill and Motegi (2020) on the asymptotic validity of a bootstrap method for the max test in a different context). Although the max test statistic in Ghysels, Hill, and Motegi (2020) is not even asymptotically pivotal, the results in this paper may be utilized to construct a pivotal version of the max test.
Appendix: Proofs

First, we state the following Lemma, which is used in the proof of Proposition 1 below.

**Lemma 1** If $A$ is positive definite, then $\begin{bmatrix} B'AB & B' \\ B & A^{-1} \end{bmatrix}$ is positive semidefinite.

**Proof.** For any vector $x = [x_1', x_2']'$, we have

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} \begin{bmatrix} B'AB & B' \\ B & A^{-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1'B'ABx_1 + x_1'B'x_2 + x_2'Bx_1 + x_2'A^{-1}x_2$$

$$= (x_1'B'A + x_2')(Bx_1 + A^{-1}x_2)$$

$$= (Bx_1 + A^{-1}x_2)'A(Bx_1 + A^{-1}x_2) \geq 0.$$

Because $A$ is positive definite, the above quadratic form is zero if and only if $Bx_1 + A^{-1}x_2 = 0$. Thus $x' \begin{bmatrix} B'AB & B' \\ B & A^{-1} \end{bmatrix} x = 0$, when $x = [x_1', -x_1'B'A]'$, which shows that the matrix is positive semidefinite. □

**Proof of Proposition 1.**

Let $g_i$ be the $(p + 1)$th row of $\Gamma_{ii}^{-1}$. By the matrix inversion formula, we obtain:

$$g_i = \begin{bmatrix} -d_{ii}^{-1}\gamma_{i}^2\Gamma_{zz}^{-1} \\ d_{ii}^{-1} \end{bmatrix}, \tag{9}$$

and $V$ is rewritten using $g_i$ as:

$$V = RSR' = \begin{bmatrix} g_1A_{11}g_1' & g_1A_{12}g_2' & \cdots & g_1A_{1h}g_h' \\ g_2A_{21}g_1' & g_2A_{22}g_2' & \cdots & g_2A_{2h}g_h' \\ \vdots & \vdots & \ddots & \vdots \\ g_hA_{h1}g_1' & g_hA_{h2}g_2' & \cdots & g_hA_{hh}g_h' \end{bmatrix}. \tag{10}$$

From (9) and noting that $A_{ij} = \begin{bmatrix} \Lambda_{zz} & \lambda_{jz} \\ \lambda_{iz} & \lambda_{ij} \end{bmatrix}$, we have:

$$g_iA_{ij}g_j$$

$$= \begin{bmatrix} -d_{ii}^{-1}\gamma_{i}^2\Gamma_{zz}^{-1} \\ d_{ii}^{-1} \end{bmatrix} \begin{bmatrix} \Lambda_{zz} & \lambda_{jz} \\ \lambda_{iz} & \lambda_{ij} \end{bmatrix} \begin{bmatrix} \frac{-d_{jj}^{-1}\Gamma_{zz}^{-1}\gamma_{jz}}{d_{jj}^{-1}} \\ \frac{-d_{jj}^{-1}\Gamma_{zz}^{-1}\gamma_{jz}}{d_{jj}^{-1}} \end{bmatrix}$$

$$= d_{ii}^{-1}d_{jj}^{-1} \begin{bmatrix} \gamma_{i}^2\Gamma_{zz}^{-1}\Lambda_{zz} & \gamma_{i}^2\Gamma_{zz}^{-1}\lambda_{jz} \\ \gamma_{i}^2\lambda_{iz} & \gamma_{i}^2\lambda_{ij} \end{bmatrix} \begin{bmatrix} \Gamma_{zz}^{-1}\Lambda_{zz} & \Gamma_{zz}^{-1}\lambda_{jz} \\ \Gamma_{zz}^{-1}\lambda_{iz} & \Gamma_{zz}^{-1}\lambda_{ij} \end{bmatrix}$$

$$= d_{ii}^{-1}d_{jj}^{-1} \begin{bmatrix} \gamma_{i}^2\Lambda_{zz} & \gamma_{i}^2\lambda_{jz} \\ \gamma_{i}^2\lambda_{iz} & \gamma_{i}^2\lambda_{ij} \end{bmatrix} + \lambda_{ij} - \lambda_{iz}^2\Lambda_{zz}^{-1}\lambda_{jz}$$

$$= d_{ii}^{-1}d_{jj}^{-1} \{ \omega_{i}^2\Omega_{zz}\omega_{iz} + \lambda_{ij} - \lambda_{iz}^2\Lambda_{zz}^{-1}\lambda_{jz} \},$$

where $\omega_{i}$ and $\Omega_{zz}$ are defined as in the text.
where \( \omega_{iz} = \begin{bmatrix} \gamma'_{iz} & \chi'_{iz} \end{bmatrix} \). From (10) and (11), we obtain:

\[
V = D^{-1} \left\{ \begin{bmatrix} \omega_{iz} \Omega_{zz} \omega_{1z} & \omega_{iz} \Omega_{zz} \omega_{hz} \\ \vdots & \vdots \\ \omega_{hz} \Omega_{zz} \omega_{1z} & \omega_{hz} \Omega_{zz} \omega_{hz} \end{bmatrix} + \begin{bmatrix} \lambda_{11} & \cdots & \lambda_{1h} \\ \vdots & \ddots & \vdots \\ \lambda_{h1} & \cdots & \lambda_{hh} \end{bmatrix} \right\} D^{-1}
\]

\[
= D^{-1} \left\{ \begin{bmatrix} \omega_{iz} \\ \vdots \\ \omega_{hz} \end{bmatrix} \Omega_{zz} \begin{bmatrix} \omega_{1z} & \cdots & \omega_{hz} \end{bmatrix} + \Lambda_{xx} - \begin{bmatrix} \lambda'_{iz} \\ \vdots \\ \lambda'_{hz} \end{bmatrix} \Lambda_{zz}^{-1} \begin{bmatrix} \lambda_{1z} & \cdots & \lambda_{hz} \end{bmatrix} \right\} D^{-1}
\]

which completes the proof of the first part of Proposition 1. Lemma 1 implies that, if \( \Lambda \) is positive definite, then \( \Omega_{zz} \) is positive semidefinite. Similarly, matrix \( \Lambda_{xx} - \Lambda_{xx} \Lambda_{zz}^{-1} \Lambda_{xx} \) is positive definite when \( \Lambda \) is positive definite because it is the Schur complement of \( \Lambda \) with respect to \( \Lambda_{zz} \). These results imply that \( V \) is positive definite. \( \square \)
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