THE GROMOV-WITTEN INVARIANTS OF THE HILBERT SCHEMES OF POINTS ON SURFACES WITH \( p_g > 0 \)

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Abstract. In this paper, we study the Gromov-Witten theory of the Hilbert schemes \( X^{[n]} \) of points on smooth projective surfaces \( X \) with positive geometric genus \( p_g \). Using cosection localization technique due to Y. Kiem and J. Li \([KL1, KL2]\), we prove that if \( X \) is a simply connected surface admitting a holomorphic differential two-form with irreducible zero divisor, then all the Gromov-Witten invariants of \( X^{[n]} \) defined via the moduli space \( \mathcal{M}_{g,r}(X^{[n]}, \beta) \) vanish except possibly when \( \beta = d_0 \beta_{K_X} - d \beta_n \) where \( d \) is an integer, \( d_0 \geq 0 \) is a rational number, and \( \beta_n \) and \( \beta_{K_X} \) are defined in (3.2) and (3.3) respectively. When \( n = 2 \), the exceptional cases can be further reduced to the invariants: \( (1)^{X^{[2]}}_0, \beta_{K_X} - d \beta_2 \) with \( K_X^2 = 1 \) and \( d \leq 3 \), and \( (1)^{X^{[2]}}_1, d \beta_2 \) with \( d \geq 1 \). We show that when \( K_X^2 = 1 \),

\[
\langle 1 \rangle^{X^{[2]}}_0, \beta_{K_X} - 3 \beta_2 = (-1)^{\chi(O_X)} 
\]

which is consistent with a well-known formula of Taubes \([Ta]\). In addition, for an arbitrary smooth projective surface \( X \) and \( d \geq 1 \), we verify that

\[
\langle 1 \rangle^{X^{[2]}}_1, d \beta_2 = \frac{K_X^2}{12d}. 
\]

1. Introduction

Cosection localization via holomorphic two-forms was introduced by Lee and Parker \([LP]\) in symplectic geometry and by Kiem and J. Li \([KL1, KL2]\) in algebraic geometry. It is a localization theorem on virtual cycles such as the virtual fundamental cycles arising from Gromov-Witten theory. Using this technique, Kiem and J. Li \([KL1, KL2]\) studied the Gromov-Witten theory of minimal surfaces of general type, and J. Li and the second author \([LL]\) computed the quantum boundary operator for the Hilbert schemes of points on surfaces. Cosection localization also played a pivotal role in \([LQ2]\) determining the structure of genus-0 extremal Gromov-Witten invariants of these Hilbert schemes and verifying the Cohomological Crepant Resolution Conjecture for the Hilbert-Chow morphisms.

In this paper, we study the Gromov-Witten theory of the Hilbert schemes \( X^{[n]} \) of points on smooth projective surfaces \( X \) with positive geometric genus \( p_g = \)

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Let $h^0(X, \mathcal{O}_X(K_X))$. Let $\mathcal{M}_{g,r}(X^{[n]}, \beta)$ be the moduli space of stable maps $\mu$ from genus-$g$ nodal curves $D$ with $r$-marked points and with $\mu_*[D] = \beta$ to $X^{[n]}$. Let $C$ be a smooth curve in $X$, and fix distinct points $x_1, \ldots, x_{n-1} \in X - C$. Define

$$\beta_n = \{ \xi + x_2 + \ldots + x_{n-1} \in X^{[n]} | \text{Supp}(\xi) = \{ x_1 \} \} ,$$

$$\beta_C = \{ x + x_1 + \ldots + x_{n-1} \in X^{[n]} | x \in C \} .$$

By linearity, extend the notion $\beta_C$ to an arbitrary divisor $C$ (see (3.6) for details).

Using cosection localization technique, we obtain the following vanishing result.

**Theorem 1.1.** Let $X$ be a simply connected surface admitting a holomorphic differential two-form with irreducible zero divisor. If $\beta \neq d_0 \beta_K - d \beta_n$ for some integer $d$ and rational number $d_0 \geq 0$, then all the Gromov-Witten invariants of $X^{[n]}$ defined via the moduli space $\mathcal{M}_{g,r}(X^{[n]}, \beta)$ vanish.

When $n = 2$ and $X$ is further assumed to be a minimal surface of general type, the possible non-vanishing Gromov-Witten invariants $\langle \alpha_1, \ldots, \alpha_r \rangle_{X^{[2]}, \beta}$ (see (2.4) for the precise definition) can be reduced to the 1-point invariants calculated in [LQ1] and the following two types of invariants:

(i) $\langle 1 \rangle_{1, d \beta_2}^{X^{[2]}}$ with $d \geq 1$;

(ii) $\langle 1 \rangle_{0, \beta_K - d \beta_2}^{X^{[2]}}$ with $K_X^2 = 1$, $1 \leq p_g \leq 2$ and $d \leq 3$.

These two types of Gromov-Witten invariants are investigated via detailed analyses of the corresponding virtual fundamental cycles.

**Theorem 1.2.** Let $d \geq 1$. Let $X$ be a smooth projective surface. Then,

$$\langle 1 \rangle_{1, d \beta_2}^{X^{[2]}} = \frac{K_X^2}{12d} .$$

It follows that if $X$ is a simply connected surface admitting a holomorphic differential two-form with irreducible zero divisor and satisfying $K_X^2 > 1$, then all the Gromov-Witten invariants (without descendant insertions) of $X^{[2]}$ can be determined; moreover, the quantum cohomology of $X^{[2]}$ coincides with its quantum corrected cohomology [LQ1, LQ2].

**Theorem 1.3.** Let $X$ be a simply connected minimal surface of general type with $K_X^2 = 1$ and $1 \leq p_g \leq 2$ such that every member in $|K_X|$ is smooth. Then,

(i) $\mathcal{M}_{0,0}(X^{[2]}, \beta_K - 3 \beta_2) \cong |K_X| \cong \mathbb{P}^{p_g-1}$;

(ii) $\langle 1 \rangle_{0, \beta_K - 3 \beta_2}^{X^{[2]}} = (-1)^{\chi(O_X)}$.

We remark that our formula in Theorem 1.3 (ii) is consistent with

$$\langle 1 \rangle_{K_X^2+1, K_X}^{X} = (-1)^{\chi(O_X)}$$

which is a well-known formula of Taubes [Tau] obtained via an interplay between Seiberg-Witten theory and Gromov-Witten theory.

This paper is organized as follows. In §2, we briefly review Gromov-Witten theory. In §3, Theorem 1.1 is proved. In §4, we compute some intersection numbers on certain moduli spaces of genus-1 stable maps. In §5, we study the homology
classes of curves in Hilbert schemes of points on surfaces. In §6, using the results from the previous two sections, we verify Theorem 1.2 and Theorem 1.3.

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2. Stable maps and Gromov-Witten invariants

In this section, we will briefly review the notions of stable maps and Gromov-Witten invariants. We will also recall a result of Behrend from [Beh].

Let $Y$ be a smooth projective variety. An $r$-pointed stable map to $Y$ consists of a complete nodal curve $D$ with $r$ distinct ordered smooth points $p_1, \ldots, p_r$ and a morphism $\mu: D \to Y$ such that the data $(\mu, D, p_1, \ldots, p_r)$ has only finitely many automorphisms. In this case, the stable map is denoted by $[\mu: (D; p_1, \ldots, p_r) \to Y]$. For a fixed homology class $\beta \in H_2(Y, \mathbb{Z})$, let $M_{g,r}(Y, \beta)$ be the coarse moduli space parameterizing all the stable maps $[\mu: (D; p_1, \ldots, p_r) \to Y]$ such that $\mu^*D = \beta$ and the arithmetic genus of $D$ is $g$. Then, we have the $i$-th evaluation map:

$$\text{ev}_i: M_{g,r}(Y, \beta) \to Y$$

defined by $\text{ev}_i([\mu: (D; p_1, \ldots, p_r) \to Y]) = \mu(p_i)$. It is known [LT1, LT2, BF] that the coarse moduli space $M_{g,r}(Y, \beta)$ is projective and has a virtual fundamental class $[\overline{M}_{g,r}(Y, \beta)]^{\text{vir}} \in A_0(\overline{M}_{g,r}(Y, \beta))$ where

$$d = -(K_Y \cdot \beta) + (\dim(Y) - 3)(1 - g) + r$$

is the expected complex dimension of $\overline{M}_{g,r}(Y, \beta)$, and $A_0(\overline{M}_{g,r}(Y, \beta))$ is the Chow group of $d$-dimensional cycles in the moduli space $\overline{M}_{g,r}(Y, \beta)$.

The Gromov-Witten invariants are defined by using the virtual fundamental class $[\overline{M}_{g,r}(Y, \beta)]^{\text{vir}}$. Recall that an element $\alpha \in H^*(Y, \mathbb{C})$ is homogeneous if $\alpha \in H^j(Y, \mathbb{C})$ for some $j$; in this case, we take $|\alpha| = j$. Let $\alpha_1, \ldots, \alpha_r \in H^*(Y, \mathbb{C})$ such that every $\alpha_i$ is homogeneous and

$$\sum_{i=1}^r |\alpha_i| = 2d.$$  

Then, we have the $r$-point Gromov-Witten invariant defined by:

$$\langle \alpha_1, \ldots, \alpha_r \rangle_{g,\beta} = \int_{[\overline{M}_{g,r}(Y, \beta)]^{\text{vir}}} \text{ev}_1^*(\alpha_1) \otimes \ldots \otimes \text{ev}_r^*(\alpha_r).$$

Next, we recall that the excess dimension is the difference between the dimension of $\overline{M}_{g,r}(Y, \beta)$ and the expected dimension $d$ in (2.2). Let $T_Y$ stand for the tangent sheaf of $Y$. For $0 \leq i < r$, we shall use

$$f_{r,i}: \overline{M}_{g,r}(Y, \beta) \to \overline{M}_{g,i}(Y, \beta)$$

to stand for the forgetful map obtained by forgetting the last $(r - i)$ marked points and contracting all the unstable components. It is known that $f_{r,i}$ is flat when $\beta \neq 0$ and $0 \leq i < r$. The following can be found in [Beh].
Proposition 2.1. Let $\beta \in H_2(Y, \mathbb{Z})$ and $\beta \neq 0$. Let $e$ be the excess dimension of the moduli space $\mathcal{M}_{g,r}(Y, \beta)$. If $R^1(f_{r+1,r})_* (\text{ev}_{r+1})^* T_Y$ is a rank-$e$ locally free sheaf over $\mathcal{M}_{g,r}(Y, \beta)$, then $\mathcal{M}_{g,r}(Y, \beta)$ is smooth (as a stack) of dimension
\[
\mathfrak{d} + e = -(K_Y \cdot \beta) + (\dim(Y) - 3)(1 - g) + r + e,
\]
and $[\mathcal{M}_{g,r}(Y, \beta)]^{\text{vir}} = c_e(R^1(f_{r+1,r})_* (\text{ev}_{r+1})^* T_Y) \cap [\mathcal{M}_{g,r}(Y, \beta)]/\mathcal{M}_{g,r}].$

3. Vanishing of Gromov-Witten invariants

In this section, we will recall some basic notations regarding the Hilbert schemes of points on surfaces, and prove Theorem [1].

Let $X$ be a smooth projective complex surface, and $X^{[n]}$ be the Hilbert scheme of points in $X$. An element in $X^{[n]}$ is represented by a length-$n$ 0-dimensional closed subscheme $\xi$ of $X$. For $\xi \in X^{[n]}$, let $I_\xi$ and $\mathcal{O}_\xi$ be the corresponding sheaf of ideals and structure sheaf respectively. It is known from [Fog1, Iar] that $X^{[n]}$ is a smooth irreducible variety of dimension $2n$. The universal codimension-2 subscheme is
\[
Z_n = \{(\xi, x) \in X^{[n]} \times X \mid x \in \text{Supp}(\xi)\} \subset X^{[n]} \times X.
\]
The boundary of $X^{[n]}$ is defined to be the subset
\[
B_n = \{\xi \in X^{[n]} \mid |\text{Supp}(\xi)| < n\}.
\]
Let $C$ be a real-surface in $X$, and fix distinct points $x_1, \ldots, x_{n-1} \in X$ which are not contained in $C$. Define the subsets
\[
\beta_n = \{\xi + x_2 + \ldots + x_{n-1} \in X^{[n]} | \text{Supp}(\xi) = \{x_1\}\}, \quad (3.2)
\]
\[
\beta_C = \{x + x_1 + \ldots + x_{n-1} \in X^{[n]} | x \in C\}, \quad (3.3)
\]
\[
D_C = \{\xi \in X^{[n]} | \text{Supp}(\xi) \cap C \neq \emptyset\}. \quad (3.4)
\]
Note that $\beta_C$ (respectively, $D_C$) is a curve (respectively, a divisor) in $X^{[n]}$ when $C$ is a smooth algebraic curve in $X$. We extend the notions $\beta_C$ and $D_C$ to all the divisors $C$ in $X$ by linearity. For a subset $Y \subset X$, define
\[
M_n(Y) = \{\xi \in X^{[n]} | \text{Supp}(\xi) \text{ is a point in } Y\}.
\]
Nakajima [Nak] and Grojnowski [Gro] geometrically constructed a Heisenberg algebra action on the cohomology of the Hilbert schemes $X^{[n]}$. Denote the Heisenberg operators by $a_m(\alpha)$ where $m \in \mathbb{Z}$ and $\alpha \in H^*(X, \mathbb{C})$. Put
\[
\mathbb{H}_X = \bigoplus_{n=0}^{+\infty} H^*(X^{[n]}, \mathbb{C}).
\]
Then the space $\mathbb{H}_X$ is an irreducible representation of the Heisenberg algebra generated by the operators $a_m(\alpha)$ with the highest weight vector being $|0\rangle = 1 \in H^*(X^{[0]}, \mathbb{C}) = \mathbb{C}$. It follows that the $n$-th component $H^*(X^{[n]}, \mathbb{C})$ in $\mathbb{H}_X$ is linearly spanned by the Heisenberg monomial classes:
\[
a_{-n_1}(\alpha_1) \cdots a_{-n_k}(\alpha_k)|0\rangle
\]
where $k \geq 0$, $n_1, \ldots, n_k > 0$, and $n_1 + \ldots + n_k = n$. We have
\[
\begin{align*}
\beta_n &= \frac{a_2(x)a_{-1}(x)^{n-2}}{n}, \\
\beta_C &= \frac{a_{-1}(C)a_{-1}(x)^{n-1}}{n}, \\
B_n &= \frac{1}{(n-2)!}a_{-1}(1_x)^{n-2}a_{-2}(1_x)|0, \\
D_C &= \frac{1}{(n-1)!}a_{-1}(1_x)^{n-1}a_{-1}(C)|0.
\end{align*}
\]
where $x$ and $1_x$ denote the cohomology classes corresponding to a point $x \in X$ and the surface $X$ respectively. By abusing notations, we also use $C$ to denote the cohomology class corresponding to the real-surface $C$.

Assume that the surface $X$ admits a non-trivial holomorphic differential two-form $\theta \in H^0(X, \Omega^2_X) = H^0(X, O_X(K_X))$. By the results of Beauville in [Bea1, Bea2], $\theta$ induces a holomorphic two-form $\theta[n]$ of the Hilbert scheme $X[n]$ which can also be regarded as a map $\theta[n] : T_{X[n]} \rightarrow \Omega_{X[n]}$. For simplicity, put
\[
\overline{M} = \overline{M}_{g,r}(X[n], \beta).
\]
Define the degeneracy locus $\overline{M}(\theta)$ to be the subset of $\overline{M}$ consisting of all the stable maps $u : \Gamma \rightarrow X[n]$ such that the composite
\[
u^*(\theta[n]) \circ du : T_{\Gamma_{\text{reg}}} \rightarrow u^*T_{X[n]}|_{\Gamma_{\text{reg}}} \rightarrow u^*\Omega_{X[n]}|_{\Gamma_{\text{reg}}}
\]
is trivial over the regular locus $\Gamma_{\text{reg}}$ of $\Gamma$. By the results of Kiem-Li [KL1, KL2], $\theta[n]$ defines a regular cosection of the obstruction sheaf of $\overline{M}$:
\[
\eta : \mathcal{O}_{\overline{M}} \rightarrow \mathcal{O}_{\overline{M}}
\]
where $\mathcal{O}_{\overline{M}}$ is the obstruction sheaf and $\mathcal{O}_{\overline{M}}$ is the structure sheaf of $\overline{M}$. Moreover, the cosection $\eta$ is surjective away from the degeneracy locus $\overline{M}(\theta)$, and there exists a localized virtual cycle $\overline{M}^\text{vir}_{\text{loc}} \in A_*(\overline{M}(\theta))$ such that
\[
\overline{M}^\text{vir} = t_*\overline{M}^\text{vir}_{\text{loc}} \in A_*(\overline{M})
\]
where $t : \overline{M}(\theta) \rightarrow \overline{M}$ stands for the inclusion map.

**Lemma 3.1.** Let $C_0$ be the zero divisor of $\theta$. Let $u : \Gamma \rightarrow X[n]$ be a stable map in $\overline{M}(\theta)$, and let $\Gamma_0$ be an irreducible component of $\Gamma$ with non-constant restriction $u|_{\Gamma_0}$. Then there exists $\xi_1 \in X[n_0]$ for some $n_0$ such that $\text{Supp}(\xi_1) \cap C_0 = \emptyset$ and
\[
u(\Gamma_0) \subset \xi_1 + \{\xi_2|\text{Supp}(\xi_2) \subset C_0\}.
\]

**Proof.** For notational convenience, we assume that $\Gamma = \Gamma_0$ is irreducible. Then there exist a nonempty open subset $O \subset \Gamma$ and an integer $n_0 \geq 0$ such that $O$ is smooth and for every element $p \in O$, the image $u(p)$ is of the form
\[
u(p) = \xi_1(p) + \xi_2(p)
\]
where $\xi_1(p) \in X[n_0]$ with $\text{Supp}(\xi_1(p)) \cap C_0 = \emptyset$ and $\text{Supp}(\xi_2(p)) \subset C_0$. This induces a decomposition $u|_O = (u_1, u_2)$ where the morphisms $u_1 : O \rightarrow X[n]$ and...
$u_2 : O \to X^{[n-n_0]}$ are defined by sending $p \in O$ to $\xi_1(p)$ and $\xi_2(p)$ respectively. Since \((3.9)\) is trivial over the regular locus $\Gamma_{\text{reg}}$ of $\Gamma$, the composite

$$u_1^*(\theta^{[n_0]}) \circ du_1 : T_O \to u_1^*T_{X^{[n_0]}}|_O \to u_1^*\Omega_{X^{[n_0]}}|_O$$

\((3.14)\)

is trivial. Note that the holomorphic two-form $\theta^{[n_0]}$ on $X^{[n_0]}$ is non-degenerate at $\xi_1(p), p \in O$ since $\text{Supp}(\xi_1(p)) \cap C_0 = \emptyset$. Thus, $du_1 = 0$ and $u_1$ is a constant morphism. Setting $\xi_1 = \xi_1(p) = u_1(p), p \in O$ proves the lemma. \(\square\)

In the rest of the paper, we will assume that $X$ is simply connected. Then,

$$\text{Pic}(X^{[n]}) \cong \text{Pic}(X) \oplus \mathbb{Z} \cdot (B_n/2)$$

\((3.15)\)

by \([\text{Fog2}]\). Under this isomorphism, the divisor $D_C \in \text{Pic}(X^{[n]})$ corresponds to $C \in \text{Pic}(X)$. Let $\{\alpha_1, \ldots, \alpha_s\}$ be a linear basis of $H^2(X, \mathbb{C})$. Then,

$$\{D_{\alpha_1}, \ldots, D_{\alpha_s}, B_n\}$$

\((3.16)\)

is a linear basis of $H^2(X^{[n]}, \mathbb{C})$. Represent $\alpha_1, \ldots, \alpha_s$ by real-surfaces $C_1, \ldots, C_s \subset X$ respectively. Then a linear basis of $H_2(X^{[n]}, \mathbb{C})$ is given by

$$\{\beta_{C_1}, \ldots, \beta_{C_s}, \beta_n\}.$$  

\((3.17)\)

**Lemma 3.2.** Let the surface $X$ be simply connected. Assume that the zero divisor $C_0$ of $\theta$ is irreducible. If the subset $\mathfrak{M}(\theta)$ of $\mathfrak{M} = \mathfrak{M}_{g,r}(X^{[n]}, \beta)$ is nonempty, then $\beta = d_0\beta_{C_0} - d\beta_n$ for some integer $d$ and some rational number $d_0 \geq 0$. Moreover, if $C_0$ is also reduced, then $d_0$ is an non-negative integer.

**Proof.** Let $u : \Gamma \to X^{[n]}$ be a stable map in $\mathfrak{M}(\theta)$. Restricting $u$ to the irreducible components of $\Gamma$ if necessary, we may assume that $\Gamma$ is irreducible. By Lemma 3.1 there exists $\xi_1 \in X^{[n_0]}$ for some $n_0$ such that $\text{Supp}(\xi_1) \cap C_0 = \emptyset$ and

$$u(\Gamma) \subset \xi_1 + \{\xi_2|\text{Supp}(\xi_2) \subset C_0\}.$$  

\((3.18)\)

We may further assume that $n_0 = 0$ and $C_0$ is reduced. Then for every $p \in \Gamma$,

$$\text{Supp}(u(p)) \subset C_0.$$

Let $C$ be a real-surface in $X$. Assume that $C_0$ and $C$ intersect transversally at $x_{1,1}, \ldots, x_{1,s}, x_{2,1}, \ldots, x_{2,t} \in (C_0)_{\text{reg}}$ such that each $x_{1,i}$ (respectively, $x_{2,i}$) contributes 1 (respectively, $-1$) to the intersection number $C_0 \cdot C$. So $s - t = C_0 \cdot C$. Since $\text{Supp}(u(p)) \subset C_0$ and $C_0$ is irreducible and reduced, there exists an integer $d'$ such that $d'_0$ is independent of $C$ and that each $x_{1,i}$ (respectively, $x_{2,i}$) contributes $d_0$ (respectively, $-d'_0$) to the intersection number $u(\Gamma) \cdot D_C$. Thus,

$$u(\Gamma) \cdot D_C = sd'_0 - td'_0 = (s - t)d'_0 = d'_0(C_0 \cdot C).$$

In view of the bases \((3.17)\) and \((3.16)\), $u(\Gamma) = d'_0\beta_{C_0} - d'\beta_n$ for some integer $d'$. Choosing $C$ to be a very ample curve, we see that $d'_0 \geq 0$. Finally, since $\beta = \deg(u) \cdot u(\Gamma)$, we obtain $\beta = d_0\beta_{C_0} - d\beta_n$ for some integers $d_0 \geq 0$ and $d$. \(\square\)

**Theorem 3.3.** Let $X$ be a simply connected surface admitting a holomorphic differential two-form with irreducible zero divisor. If $\beta \neq d_0\beta_{K_X} - d\beta_n$ for some integer $d$ and rational number $d_0 \geq 0$, then all the Gromov-Witten invariants of $X^{[n]}$ defined via the moduli space $\mathfrak{M}_{g,r}(X^{[n]}, \beta)$ vanish.
Proof. Let $\theta \in H^0(X, \Omega_X^2) = H^0(X, O_X(K_X))$ be the holomorphic differential two-form whose zero divisor $C_0$ is irreducible. By Lemma 3.2, we have $\overline{M}(\theta) = \emptyset$. It follows from (3.11) that $[\overline{M}_{g,r}(X^{[n]}, \beta)]_{\vir} = 0$. Therefore, all the Gromov-Witten invariants defined via the moduli space $\overline{M}_{g,r}(X^{[n]}, \beta)$ vanish.

**Remark 3.4.** From the proof of Lemma 3.2, we see that if $K_X = C_0 = mC'_0$ for some irreducible and reduced curve $C'_0$, then the rational number $d_0$ in Theorem 3.3 is of the form $d_0/m$ for some integer $d_0 \geq 0$.

Recall that $K_{X^{[n]}} = D_{K_X}$. Thus, if $\beta = d_0\beta_{K_X} - d\beta_n$ for some rational number $d_0 \geq 0$ and integer $d$, then the expected dimension of $\overline{M}_{g,r}(X^{[n]}, \beta)$ is

$$d = -K_{X^{[n]}} \cdot \beta + (\dim X^{[n]} - 3)(1 - g) + r$$

$$= -d_0K_X^2 + (2n - 3)(1 - g) + r. \quad (3.19)$$

Our first corollary deals with the case when $X$ is an elliptic surface.

**Corollary 3.5.** Let $X$ be a simply connected (minimal) elliptic surface without multiple fibers and with positive geometric genus. Let $n \geq 2$ and $\beta \neq 0$. Then all the Gromov-Witten invariants without descendant insertions defined via the moduli space $\overline{M}_{g,r}(X^{[n]}, \beta)$ vanish, except possibly when $0 \leq g \leq 1$ and $\beta = d_0\beta_{K_X} - d\beta_n$ for some integer $d$ and rational number $d_0 \geq 0$.

**Proof.** Since $X$ is a simply connected elliptic surface without multiple fibers, $K_X = (p_g - 1)f$ where $p_g \geq 1$ is the geometric genus of $X$ and $f$ denotes a smooth fiber of the elliptic fibration. By Theorem 3.3, it remains to consider the case when $\beta = d_0\beta_{K_X} - d\beta_n$ for some integer $d$ and rational number $d_0 \geq 0$. By (3.19) and $K_X^2 = 0$, the expected dimension of the moduli space $\overline{M}_{g,r}(X^{[n]}, \beta)$ is equal to $d = (2n - 3)(1 - g) + r$. By the Fundamental Class Axiom, all the Gromov-Witten invariants without descendant insertions are equal to zero if $g \geq 2$. \hfill \Box

Our second corollary concentrates on the case when $X$ is of general type.

**Corollary 3.6.** Let $X$ be a simply connected minimal surface of general type admitting a holomorphic differential two-form with irreducible zero divisor. Let $n \geq 2$ and $\beta \neq 0$. Then all the Gromov-Witten invariants without descendant insertions defined via $\overline{M}_{g,r}(X^{[n]}, \beta)$ vanish, except possibly in the following cases

(i) $g = 0$ and $\beta = d\beta_n$ for some integer $d > 0$;

(ii) $g = 1$ and $\beta = d\beta_n$ for some integer $d > 0$;

(iii) $g = 0$ and $\beta = d_0\beta_{K_X} - d\beta_n$ for some integer $d$ and rational number $d_0 > 0$.

**Proof.** In view of Theorem 3.3, it remains to consider the case when $\beta = d_0\beta_{K_X} - d\beta_n$ for some integer $d$ and rational number $d_0 \geq 0$.

When $d_0 = 0$ and $\beta = d\beta_n$ with $d > 0$, we see from (3.19) that the expected dimension of the moduli space $\overline{M}_{g,r}(X^{[n]}, \beta)$ is equal to

$$d = (2n - 3)(1 - g) + r.$$

If $g \geq 2$, then all the Gromov-Witten invariants without descendant insertions defined via $\overline{M}_{g,r}(X^{[n]}, \beta)$ vanish by the Fundamental Class Axiom.
Next, assume that $d_0 > 0$. Since $K_X^2 \geq 1$, we see from (3.19) that
\[ \vartheta < (2n - 3)(1 - g) + r. \]
By the Fundamental Class Axiom, all the Gromov-Witten invariants without descendant insertions vanish except possibly in the case when $g = 0$. \hfill $\square$

4. Intersection numbers on some moduli space of genus-1 stable maps

In this section, we will compute certain intersection numbers on the moduli space of genus-1 stable maps to $\mathbb{P}(V)$ where $V$ is a rank-2 vector bundle over a smooth projective curve $C$. The results will be used in Subsection 6.1.

**Notation 4.1.** Let $V$ be a rank-2 bundle over a smooth projective variety $B$.

(i) $f$ denotes a fiber of the ruling $\pi : \mathbb{P}(V) \to B$ or its cohomology class.

(ii) $H = (f_1, \omega)$ is the rank-1 Hodge bundle over $\overline{M}_{1,0}(\mathbb{P}(V), df)$ where $\omega$ is the relative dualizing sheaf for $f_1 : \overline{M}_{1,1}(\mathbb{P}(V), df) \to \overline{M}_{1,0}(\mathbb{P}(V), df)$.

(iii) $\lambda = c_1(H)$.

Let $d \geq 1$. If $u = [\mu : D \to \mathbb{P}(V)] \in \overline{M}_{1,0}(\mathbb{P}(V), df)$, then $\mu(D)$ is a fiber of the ruling $\pi : \mathbb{P}(V) \to B$. Therefore, there exists a natural morphism
\[ \phi : \overline{M}_{1,0}(\mathbb{P}(V), df) \to B \tag{4.1} \]
whose fiber over $b \in B$ is $\overline{M}_{1,0}(\pi^{-1}(b), d[\pi^{-1}(b)]) \cong \overline{M}_{1,0}(\mathbb{P}^1, [\mathbb{P}^1])$. So the moduli space $\overline{M}_{1,0}(\mathbb{P}(V), df)$ is smooth (as a stack) with dimension\n\[ \dim \overline{M}_{1,0}(\mathbb{P}^1, d[\mathbb{P}^1]) + \dim(B) = 2d + \dim(B). \]

By (2.2), the expected dimension of $\overline{M}_{1,0}(\mathbb{P}(V), df)$ is $2d$. Since $d \geq 1$, the sheaf $R^1(f_1, \omega)_*\mathcal{O}_{\mathbb{P}(V)}(-2)$ on $\overline{M}_{1,0}(\mathbb{P}(V), df)$ is locally free of rank-$2d$. In addition,
\[ \lambda^2 = 0 \tag{4.2} \]
according to Mumford’s theorem in [Mum] regarding the Chern character of the Hodge bundles and the proof of Proposition 1 in [FP].

**Lemma 4.2.** Let $d \geq 1$. Let $V$ be a rank-2 bundle over $B_0 \times C$ where $B_0$ and $C$ are smooth projective curves. Let $V_b = V|_{b \times C}$ for $b \in B_0$. Then,
\[ \int_{\overline{M}_{1,0}(\mathbb{P}(V_b), df)} \lambda \cdot c_{2d}(R^1(f_1, \omega)_*\mathcal{O}_{\mathbb{P}(V_b)}(-2)) \tag{4.3} \]
is independent of the points $b \in B_0$.

**Proof.** This follows from the observation that (4.3) is equal to
\[ \int_{\overline{M}_{1,0}(\mathbb{P}(V), df)} \phi^*[\{b\} \times C] \cdot \lambda \cdot c_{2d}(R^1(f_1, \omega)_*\mathcal{O}_{\mathbb{P}(V)}(-2)) \]
where the morphism $\phi : \overline{M}_{1,0}(\mathbb{P}(V), df) \to B_0 \times C$ is from (4.1). \hfill $\square$

Formula (4.4) below is probably well-known, but we could not find a reference.
Lemma 4.3. Let $d$ be a positive integer. Then, we have

\[ \int_{\mathcal{M}_{1,0}(\mathbb{P}^1,d[\mathbb{P}^1])} c_{2d}(R^1(f_{1,0})_*ev_1^*\mathcal{O}_{\mathbb{P}^1}(-2)) = 0, \quad (4.4) \]

\[ \int_{\mathcal{M}_{1,0}(\mathbb{P}^1,d[\mathbb{P}^1])} \lambda \cdot c_{2d-1}(R^1(f_{1,0})_*ev_1^*\mathcal{O}_{\mathbb{P}^1}(-2)) = -\frac{1}{12d}. \quad (4.5) \]

Proof. We begin with the proof of (4.4). Choose a K3 surface $S$ which contains a smooth rational curve $C$. Then, $C^2 = -2$, $T_S|_C = \mathcal{O}_C(2) \oplus \mathcal{O}_C(-2)$, and $dC$ is the only element in the complete linear system $|dC|$. So we have

\[ \int_{\mathcal{M}_{1,0}(\mathbb{P}^1,d[\mathbb{P}^1])} c_{2d}(R^1(f_{1,0})_*ev_1^*\mathcal{O}_{\mathbb{P}^1}(-2)) \]

\[ = \int_{\mathcal{M}_{1,0}(\mathbb{P}^1,d[\mathbb{P}^1])} c_{2d}(R^1(f_{1,0})_*ev_1^*(T_S|_C)) \]

\[ = \int_{\mathcal{M}_{1,0}(\mathbb{P}^1,d[\mathbb{P}^1])} c_{2d}(R^1(f_{1,0})_*ev_1^*T_S). \]

Note that $R^1(f_{1,0})_*ev_1^*T_S$ is a rank-$2d$ bundle on the $2d$-dimensional moduli space $\mathcal{M}_{1,0}(S,d[C])$ whose virtual dimension is $0$. By Proposition 2.1

\[ \int_{\mathcal{M}_{1,0}(\mathbb{P}^1,d[\mathbb{P}^1])} c_{2d}(R^1(f_{1,0})_*ev_1^*\mathcal{O}_{\mathbb{P}^1}(-2)) = \deg(\mathcal{M}_{1,0}(S,d[C])^{\text{vir}}). \]

Since $\mathcal{M}_{g,r}(S,\beta)^{\text{vir}} = 0$ whenever $\beta \neq 0$, we obtain

\[ \int_{\mathcal{M}_{1,0}(\mathbb{P}^1,d[\mathbb{P}^1])} c_{2d}(R^1(f_{1,0})_*ev_1^*\mathcal{O}_{\mathbb{P}^1}(-2)) = 0. \]

To prove (4.5), we apply $(f_{1,0})_*ev_1^*$ to the exact sequence

\[ 0 \to \mathcal{O}_{\mathbb{P}^1}(-2) \to \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^1} \to 0. \]

Since $(f_{1,0})_*ev_1^*\mathcal{O}_{\mathbb{P}^1} = \mathcal{O}_{\mathcal{M}_{1,0}(\mathbb{P}^1,d[\mathbb{P}^1])}$ and $R^1(f_{1,0})_*ev_1^*\mathcal{O}_{\mathbb{P}^1} = \mathcal{H}^\vee$, we get

\[ 0 \to \mathcal{O}_{\mathcal{M}_{1,0}(\mathbb{P}^1,d[\mathbb{P}^1])} \to R^1(f_{1,0})_*ev_1^*\mathcal{O}_{\mathbb{P}^1}(-2) \to R^1(f_{1,0})_*ev_1^*\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \to \mathcal{H}^\vee \to 0. \]

Calculating the total Chern class and using (4.2), we see that

\[ c(R^1(f_{1,0})_*ev_1^*\mathcal{O}_{\mathbb{P}^1}(-2)) \]

\[ = c(R^1(f_{1,0})_*ev_1^*\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}) / c(\mathcal{H}^\vee) \]

\[ = c(R^1(f_{1,0})_*ev_1^*\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}) \cdot (1 + \lambda). \quad (4.6) \]

Thus, the top Chern class $c_{2d}(R^1(f_{1,0})_*ev_1^*\mathcal{O}_{\mathbb{P}^1}(-2))$ is equal to

\[ \lambda \cdot c_{2d-1}(R^1(f_{1,0})_*ev_1^*\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}). \]

By the Proposition 2 in [GP] and (4.4), we conclude that

\[ \int_{\mathcal{M}_{1,0}(\mathbb{P}^1,d[\mathbb{P}^1])} \lambda \cdot c_{2d-1}(R^1(f_{1,0})_*ev_1^*\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}) = -\frac{1}{12d}. \quad (4.7) \]

By (4.6) again, $\lambda \cdot c_{2d-1}(R^1(f_{1,0})_*ev_1^*\mathcal{O}_{\mathbb{P}^1}(-2)) = \lambda \cdot c_{2d-1}(R^1(f_{1,0})_*ev_1^*\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2})$.

Combining this with (4.7), we obtain our formula (4.5). □
Lemma 4.4. Let \( d \geq 1 \), and \( V \) be a rank-2 bundle over \( \mathbb{P}^1 \). Then,
\[
\int_{[\mathbb{P}^1,0(\mathbb{P}(V),df)]} \lambda \cdot c_{2d}(R^1(f_{1,0})*ev_1^*\mathcal{O}_{\mathbb{P}(V)}(-2)) = \frac{\deg(V)}{12d}.
\]

Proof. First of all, assume that \( \deg(V) = 2k \) for some integer \( k \). Then \( V \) can be deformed to \( \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(k) = (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}) \otimes \mathcal{O}_{\mathbb{P}^1}(k) \). By Lemma 4.2,
\[
\int_{[\mathbb{P}^1,0(\mathbb{P}(V),df)]} \lambda \cdot c_{2d}(R^1(f_{1,0})*ev_1^*\mathcal{O}_{\mathbb{P}(V)}(-2))
= \int_{[\mathbb{P}^1,0(\mathbb{P}^1 \times \mathbb{P}^1,df)]} \lambda \cdot c_{2d}(R^1(f_{1,0})*ev_1^*((\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-2) \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-2k)))).
\]

Note that \( \mathbb{M}_{1,0}(\mathbb{P}^1 \times \mathbb{P}^1, df) \cong \mathbb{P}^1 \times \mathbb{M}_{1,0}(\mathbb{P}^1, d[\mathbb{P}^1]). \) Thus, we obtain
\[
\int_{[\mathbb{P}^1,0(\mathbb{P}(V),df)]} \lambda \cdot c_{2d}(R^1(f_{1,0})*ev_1^*\mathcal{O}_{\mathbb{P}(V)}(-2))
= \int_{[\mathbb{P}^1 \times \mathbb{M}_{1,0}(\mathbb{P}^1, d[\mathbb{P}^1])] \pi_2^* \lambda \cdot c_{2d}(\pi_2^*(R^1(f_{1,0})*ev_1^*\mathcal{O}_{\mathbb{P}^1}(-2)) \otimes \pi_1^*\mathcal{O}_{\mathbb{P}^1}(-2k))
\]
where \( \pi_1 \) and \( \pi_2 \) are the projection on \( \mathbb{P}^1 \times \mathbb{M}_{1,0}(\mathbb{P}^1, d[\mathbb{P}^1]). \) Hence,
\[
\int_{[\mathbb{P}^1,0(\mathbb{P}(V),df)]} \lambda \cdot c_{2d}(R^1(f_{1,0})*ev_1^*\mathcal{O}_{\mathbb{P}(V)}(-2))
= \int_{[\mathbb{P}^1 \times \mathbb{M}_{1,0}(\mathbb{P}^1, d[\mathbb{P}^1])] \pi_2^* \lambda \cdot \pi_2^* c_{2d-1}(R^1(f_{1,0})*ev_1^*\mathcal{O}_{\mathbb{P}^1}(-2)) \cdot \pi_1^*c_1((\mathcal{O}_{\mathbb{P}^1}(-2k)))
= \frac{\deg(V)}{12d}
\]
where we have used formula (4.5) in the last step.

Next, assume that \( \deg(V) = 2k + 1 \) for some integer \( k \). Then \( V \) can be deformed to \( (\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \otimes \mathcal{O}_{\mathbb{P}^1}(k) \). As in the previous paragraph, we have
\[
\int_{[\mathbb{P}^1,0(\mathbb{P}(V),df)]} \lambda \cdot c_{2d}(R^1(f_{1,0})*ev_1^*\mathcal{O}_{\mathbb{P}(V)}(-2))
= \int_{[\mathbb{M}_{1,0}(S,df)]} \lambda \cdot c_{2d}(R^1(f_{1,0})*ev_1^*(\mathcal{O}_S(-2) \otimes \mathcal{O}_S(-2k)))
= \int_{[\mathbb{M}_{1,0}(S,df)]} \lambda \cdot c_{2d}(R^1(f_{1,0})*ev_1^*\mathcal{O}_S(-2)) + \frac{2k}{12d}
\]
where \( S = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \). Let \( \mathbb{F}_1 \) be the blown-up of \( \mathbb{P}^2 \) at a point \( p \), and \( \sigma \) be the exceptional curve. Then, \( T_{\mathbb{F}_1}|_{\sigma} \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \). So

\[
\int_{[\mathbb{M}_{1,0}(\mathbb{P}(V), df)]} \lambda \cdot c_{2d}(R^1(f_{1,0})_*ev^*_1\mathcal{O}_{\mathbb{P}(V)}(-2)) = \int_{[\mathbb{M}_{1,0}(\mathbb{P}(T_{\mathbb{F}_1}|_{\sigma}), df)]} \lambda \cdot c_{2d}(R^1(f_{1,0})_*ev^*_1\mathcal{O}_{\mathbb{P}(T_{\mathbb{F}_1}|_{\sigma})}(-2)) + \frac{2k}{12d}
\]

where the morphism \( \phi : \mathbb{M}_{1,0}(\mathbb{P}(T_{\mathbb{F}_1}), df) \to \mathbb{F}_1 \) is from \((4.1)\). Let \( f_0 \) be a fiber of the ruling \( \mathbb{F}_1 \to \mathbb{P}^1 \), and \( C \) be a smooth conic in \( \mathbb{P}^2 \) such that \( p \notin C \). We use \( C \) to denote its strict transform in \( \mathbb{F}_1 \). Then, \( [\sigma] = [C]/2 - [f_0] \). By \((4.10)\),

\[
\int_{[\mathbb{M}_{1,0}(\mathbb{P}(V), df)]} \lambda \cdot c_{2d}(R^1(f_{1,0})_*ev^*_1\mathcal{O}_{\mathbb{P}(V)}(-2)) = \frac{1}{2} \cdot \int_{[\mathbb{M}_{1,0}(\mathbb{P}(T_{\mathbb{F}_1})), df)]} \phi^*[C] \cdot \lambda \cdot c_{2d}(R^1(f_{1,0})_*ev^*_1\mathcal{O}_{\mathbb{P}(T_{\mathbb{F}_1})}(-2))
\]

\[
- \int_{[\mathbb{M}_{1,0}(\mathbb{P}(T_{\mathbb{F}_1}), df)]} \phi^*[f_0] \cdot \lambda \cdot c_{2d}(R^1(f_{1,0})_*ev^*_1\mathcal{O}_{\mathbb{P}(T_{\mathbb{F}_1})}(-2)) + \frac{2k}{12d}
\]

\[
= \frac{1}{2} \cdot \int_{[\mathbb{M}_{1,0}(\mathbb{P}(T_{\mathbb{F}_1}|_C), df)]} \lambda \cdot c_{2d}(R^1(f_{1,0})_*ev^*_1\mathcal{O}_{\mathbb{P}(T_{\mathbb{F}_1}|_C)}(-2))
\]

\[
- \int_{[\mathbb{M}_{1,0}(\mathbb{P}(T_{\mathbb{F}_1}|_{f_0}), df)]} \lambda \cdot c_{2d}(R^1(f_{1,0})_*ev^*_1\mathcal{O}_{\mathbb{P}(T_{\mathbb{F}_1}|_{f_0})}(-2)) + \frac{2k}{12d}
\]

Note that \( \deg(T_{\mathbb{F}_1}|_C) = 6 \) and \( \deg(T_{\mathbb{F}_1}|_{f_0}) = 2 \). By \((4.9)\),

\[
\int_{[\mathbb{M}_{1,0}(\mathbb{P}(V), df)]} \lambda \cdot c_{2d}(R^1(f_{1,0})_*ev^*_1\mathcal{O}_{\mathbb{P}(V)}(-2)) = \frac{1}{2} \cdot \frac{6}{12d} - \frac{2}{12d} + \frac{2k}{12d} = \frac{\deg(V)}{12d}. \quad \square
\]

**Proposition 4.5.** Let \( d \) be a positive integer. Assume that \( V \) is a rank-2 vector bundle over a smooth projective curve \( C \). Then, we have

\[
\int_{[\mathbb{M}_{1,0}(\mathbb{P}(V), df)]} \lambda \cdot c_{2d}(R^1(f_{1,0})_*ev^*_1\mathcal{O}_{\mathbb{P}(V)}(-2)) = \frac{\deg(V)}{12d}. \quad (4.11)
\]

**Proof.** It is well-known that there exist a rank-2 bundle \( \mathcal{V} \) over \( \mathbb{P}^1 \times C \) and two points \( b_1, b_2 \in \mathbb{P}^1 \) such that \( \mathcal{V}|_{b_1 \times C} = V \) and \( \mathcal{V}|_{b_2 \times C} = (\mathcal{O}_C \oplus M) \otimes N \) where \( M \) and \( N \) are line bundles on \( C \) with \( M \) being very ample. As in the proof of Lemma \((4.4)\), we conclude from Lemma \((4.2)\) that

\[
\int_{[\mathbb{M}_{1,0}(\mathbb{P}(V), df)]} \lambda \cdot c_{2d}(R^1(f_{1,0})_*ev^*_1\mathcal{O}_{\mathbb{P}(V)}(-2)) = \int_{[\mathbb{M}_{1,0}(\mathbb{P}(\mathcal{O}_C \oplus M), df)]} \lambda \cdot c_{2d}(R^1(f_{1,0})_*ev^*_1\mathcal{O}_{\mathbb{P}(\mathcal{O}_C \oplus M)}(-2)) + \frac{2\deg(N)}{12d}. \quad (4.12)
\]
Since \( M \) is very ample, there exists a morphism \( \alpha : C \to \mathbb{P}^1 \) such that \( M = \alpha^* \mathcal{O}_{\mathbb{P}^1}(1) \). Then, \( \mathcal{O}_C \oplus M = \alpha^* (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \). This induces an isomorphism

\[
\overline{M}_{1,0}(\mathbb{P}(\mathcal{O}_C \oplus M), df) \cong C \times_{\mathbb{P}^1} \overline{M}_{1,0}(\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)), df).
\]

Let \( \tilde{\alpha} : \overline{M}_{1,0}(\mathbb{P}(\mathcal{O}_C \oplus M), df) \to \overline{M}_{1,0}(\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)), df) \) be the projection. Then

\[
\begin{align*}
\int_{\overline{M}_{1,0}(\mathbb{P}(\mathcal{O}_C \oplus M), df)} & \lambda \cdot c_{2d}(R^1(f_{1,0})*_{\mathcal{O}_{\mathbb{P}(C \oplus M)}}(-2)) \\
= \int_{\overline{M}_{1,0}(\mathbb{P}(\mathcal{O}_C \oplus M), df)} & \tilde{\alpha}^* \lambda \cdot c_{2d}\left(\tilde{\alpha}^*(R^1(f_{1,0})*_{\mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))}}(-2))\right) \\
= \deg(\tilde{\alpha}) \cdot \int_{\overline{M}_{1,0}(\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)), df)} & \lambda \cdot c_{2d}(R^1(f_{1,0})*_{\mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))}}(-2)).
\end{align*}
\]

By Lemma 4.4 and noticing \( \deg(\tilde{\alpha}) = \deg(\alpha) = \deg(M) \), we get

\[
\int_{\overline{M}_{1,0}(\mathbb{P}(\mathcal{O}_C \oplus M), df)} \lambda \cdot c_{2d}(R^1(f_{1,0})*_{\mathcal{O}_{\mathbb{P}(\mathcal{O}_C \oplus M)}}(-2)) = \frac{\deg(M)}{12d}. \quad (4.13)
\]

Now our formula \((4.11)\) follows immediately from \((4.12)\) and \((4.13)\). \(\square\)

5. The homology classes of curves in Hilbert schemes

This section contains some technical lemmas which will be used in Subsection 6.2. These lemmas deal with the homology classes of curves in Hilbert schemes.

**Lemma 5.1.** Let \( n \geq 2 \) and \( X \) be a simply connected surface. Let \( \Gamma \) be an irreducible curve in the Hilbert scheme \( X^{[n]} \). Then,

\[
\Gamma \sim \beta_C + d\beta_n \in H_2(X^{[n]}, \mathbb{C})
\]

for some effective curve class \( C \) (possibly zero) and some integer \( d \).

**Proof.** Let \( \pi_1, \pi_2 \) be the two projections of \( X^{[n]} \times X \), and recall the universal codimension-2 subscheme \( \mathcal{Z}_n \) from (3.1). Define

\[
\mathcal{Z}_\Gamma = \Gamma \times_{X^{[n]}} \mathcal{Z}_n.
\]

Let \( \tilde{\pi}_1 = \pi_1|_{\mathcal{Z}_\Gamma} : \mathcal{Z}_\Gamma \to \Gamma \) and \( \tilde{\pi}_2 = \pi_2|_{\mathcal{Z}_\Gamma} : \mathcal{Z}_\Gamma \to X \). Define

\[
C_\Gamma = \tilde{\pi}_2(\mathcal{Z}_\Gamma) \subset X.
\]

Let \( C_1, \ldots, C_t \) (possibly \( t = 0 \)) be the irreducible components of \( C_\Gamma \) such that \( \dim C_i = 1 \) for all \( 1 \leq i \leq t \). Let \( m_1, \ldots, m_t \) be the degrees of the restrictions of \( \pi_2|_{\mathcal{Z}_\Gamma} \) to the reduced curves \( (\tilde{\pi}_2^{-1}(C_1))_{\text{red}}, \ldots, (\tilde{\pi}_2^{-1}(C_t))_{\text{red}} \) respectively.

Fix a very ample curve \( H \) in \( X \). Let \( d_i = C_i \cdot H \) for \( 1 \leq i \leq t \). Choose the curve \( H \) such that the following conditions are satisfied:

- for each \( 1 \leq i \leq t \), \( H \) intersects \( C_i \) transversally at \( d_i \) distinct smooth points \( \tilde{x}_{i,1}, \ldots, \tilde{x}_{i,d_i} \);
For $1 \leq i \leq t$ and $1 \leq j \leq d_i$, \( \tilde{\pi}_2^{-1}(\tilde{x}_{i,j}) \) consists of distinct smooth points

\[
(\xi_{i,j;1}, \tilde{x}_{i,j}), \ldots, (\xi_{i,j;m_i}, \tilde{x}_{i,j}) \in (\tilde{\pi}_2^{-1}(C_i))_{\text{red}}
\]

at which the restriction of \( \tilde{\pi}_1 \) to \((\tilde{\pi}_2^{-1}(C_i))_{\text{red}}\) is also unramified.

For $1 \leq i \leq t$, $1 \leq j \leq d_i$ and $1 \leq k \leq m_i$, let \( \tilde{m}_{i,j,k} \) be the multiplicity of the unique irreducible component of \( Z \) containing the smooth point \((\xi_{i,j;k}, \tilde{x}_{i,j})\). Then the contribution of \((\xi_{i,j;k}, \tilde{x}_{i,j})\) to \( \Gamma \cdot D_H \) is exactly \( \tilde{m}_{i,j,k} \). Therefore,

\[
\Gamma \cdot D_H = \sum_{i=1}^{t} \sum_{j=1}^{d_i} \sum_{k=1}^{m_i} \tilde{m}_{i,j,k}.
\]  

(5.3)

Note that \( \sum_{k=1}^{m_i} \tilde{m}_{i,j,k} \) is independent of \( j \) and \( H \). Put \( \sum_{k=1}^{m_i} \tilde{m}_{i,j,k} = e_i \). By (5.3),

\[
\Gamma \cdot D_H = \sum_{i=1}^{t} \sum_{j=1}^{d_i} e_i = \sum_{i=1}^{t} e_i d_i = \sum_{i=1}^{t} e_i (C_i \cdot H) = \left( \sum_{i=1}^{t} e_i C_i \right) \cdot H. \]  

(5.4)

Since \( X \) is simply connected, \( \text{Pic}(X^{[n]}) \cong \text{Pic}(X) \oplus \mathbb{Z} \cdot (B_n/2) \) by [3,13]. By the duality between divisor classes and curve classes, \( \Gamma \sim \beta_C + d \beta_n \) for some integer \( d \) and some class \( C \in A_1(X^{[n]}) \). Combining with (5.4), we get

\[
C \cdot H = (\beta_C + d \beta_n) \cdot D_H = \Gamma \cdot D_H = \left( \sum_{i=1}^{t} e_i C_i \right) \cdot H.
\]

So \( C \) and \( \sum_{i=1}^{t} e_i C_i \) are numerically equivalent divisors on the surface \( X \). Since \( X \) is simply connected, we see that \( C = \sum_{i=1}^{t} e_i C_i \) as divisors.

Next, we study the homology classes of curves in \( C^{(n)} \subset X^{[n]} \) where \( C \) denotes a smooth curve in \( X \). The case when \( g_C = 0 \) has been settled in [LQZ]. So we will assume \( g_C \geq 1 \). We recall some standard facts about \( C^{(n)} \) from [ACGH] [BT]. For a fixed point \( p \in C \), let \( \Xi \) denote the divisor \( p + C^{(n-1)} \subset C^{(n)} \). Let

\[
\text{AJ} : C^{(n)} \to \text{Jac}_n(C)
\]

be the Abel-Jacobi map sending an element \( \xi \in C^{(n)} \) to the corresponding degree-\( n \) divisor class in \( \text{Jac}_n(C) \). For an element \( \delta \in \text{Jac}_n(C) \), the fiber \( \text{AJ}^{-1}(\delta) \) is the complete line system \( |\delta| \). Let \( \mathcal{Z}_n(C) \subset C^{(n)} \times C \) be the universal divisor, and let \( \tilde{\pi}_1, \tilde{\pi}_2 \) be the two projections on \( C^{(n)} \times C \). By the Lemma 2.5 on p.340 of [ACGH] and the Proposition 2.1 (iv) of [BT], we have

\[
c_1(\tilde{\pi}_1^* \mathcal{O}_{\mathcal{Z}_n(C)}) = (1 - g_C - n) \Xi + \Theta \]  

(5.5)

where \( \Theta \) is the pull-back via \( \text{AJ} \) of a Theta divisor on \( \text{Jac}_n(C) \).

**Lemma 5.2.** Let \( n \geq 2 \) and \( X \) be a simply connected surface. Let \( C \) be a smooth curve in \( X \), and \( \Gamma \subset C^{(n)} \) be a curve. Then,

\[
\Gamma \sim (\Xi \cdot \Gamma) \beta_C + \left( -(n + g_C - 1)(\Xi \cdot \Gamma) + (\Theta \cdot \Gamma) \right) \beta_n \in H_2(X^{[n]}, \mathbb{C}).
\]  

(5.6)

In addition, for every line \( \Gamma_0 \) in a positive-dimensional fiber \( \text{AJ}^{-1}(\delta) \), we have

\[
\Gamma_0 \sim \beta_C - (n + g_C - 1) \beta_n.
\]  

(5.7)
admitting a holomorphic differential two-form with irreducible zero divisor. Let

\[ \beta \neq 0. \]

Proposition 6.1. Let \( \beta \neq 0 \). Then all the Gromov-Witten invariants without descendant insertions defined via \( \mathfrak{M}_{n,r}(X^2, \beta) \) vanish, except possibly in the following cases

(i) \( g = 0 \) and \( \beta = d \beta_2 \) for some integer \( d > 0 \);
(ii) \( g = 1 \) and \( \beta = d \beta_2 \) for some integer \( d > 0 \);
(iii) \( K_X^2 = 1, g = 0 \) and \( \beta = \beta_{K_X} - d \beta_2 \) for some integer \( d \).

Proof. Cases (i) and (ii) follow from Corollary 3.6 (i) and (ii) respectively. In the case of Corollary 3.6 (iii), we have \( g = 0 \) and \( \beta = d_0 \beta_{K_X} - d \beta_2 \) for some rational number \( d_0 > 0 \) and some integer \( d \). We see from (3.19) that the expected dimension of the moduli space \( \mathfrak{M}_{0,r}(X^2, \beta) \) is equal to

\[ d = -d_0 K_X^2 + 1 + r. \]

Since \( d_0 K_X^2 \) must be a positive integer, we conclude from the Fundamental Class Axiom that all the Gromov-Witten invariants without descendant insertions defined via \( \mathfrak{M}_{n,r}(X^2, \beta) \) vanish except possibly when \( d_0 K_X^2 = 1 \). Now write \( K_X = m C_0' \) where \( C_0' \) is an irreducible and reduced curve, and \( m \geq 1 \) is an integer. By Remark 3.4 \( d_0 = d'_0 / m \) for some integer \( d'_0 \geq 1 \). Therefore, we obtain

\[ 1 = d_0 K_X^2 = d'_0 m (C_0')^2. \]

It follows that \( d'_0 = m = (C_0')^2 = 1 \). Hence \( K_X^2 = 1 \) and \( d_0 = 1 \). □
The invariants (6.1) and (6.2) will be studied in the next two subsections. Similarly, Case (iii) in Proposition 6.1 can be reduced to the invariant Witten theory and the results in [LQ1]. By the Divisor Axiom, Case (ii) in Proposition 6.1 can be handled via the Divisor Axiom of Gromov-Witten theory and the results in [LQ1].

**Proof.** (i) Recall the evaluation map \( \text{ev} \) sending an element \( u \) to the curve \( \mu \) in \( X \). By (3.19), the expected dimension of \( 0 \) is locally free of rank 2d + 2.

(ii) \( \overline{M}_{1,0,d} \) is locally free of rank 2d + 2.

Since \( d \geq 1 \), \( \mu(D) = M_2(x) \) for some point \( x \in X \). By the results in [LQZ],

\[
T_{X[2]}|_{\mu(D)} = \mathcal{O}(\mu(D))(-2) \oplus \mathcal{O}(\mu(D)) \oplus \mathcal{O}(\mu(D)).
\]

Since the curve \( D \) is of genus-1, \( h^1(D, \mu^*(T_{X[2]})|_{\mu(D)}) = 2d + 2 \). It follows that the sheaf \( R^1(f_{1,0})_* (ev^*_1 T_{X[2]}) \) is locally free of rank 2d + 2.

(ii) First of all, note that there exists a natural morphism

\[
\phi : \overline{M}_{1,0,d} \to X
\]

sending an element \( u = [\mu : D \to X] \in \overline{M}_{1,0,d} \) to \( x \in X \) if \( \mu(D) = M_2(x) \). The fiber \( \phi^{-1}(x) \) over \( x \in X \) is \( \overline{M}_{1,0}(M_2(x), d|M_2(x)|) \cong \overline{M}_{1,0}(P^1, d|P^1|) \). So the moduli space \( \overline{M}_{1,0,d} \) is smooth (as a stack) with dimension

\[
\dim \overline{M}_{1,0,d} = \dim \overline{M}_{1,0}(P^1, d|P^1|) + 2 = 2d + 2.
\]

By (3.19), the expected dimension of \( \overline{M}_{1,0,d} \) is 0. Thus, the excess dimension of \( \overline{M}_{1,0,d} \) is 2d + 2. By (i) and Proposition 2.1, \( \overline{M}_{1,0,d}[\text{vir}] = c_{2d+2}(Ob) \cap [\overline{M}_{1,0,d}] \).

Via the inclusion map \( B_2 \hookrightarrow X \), the evaluation map \( ev_1 : \overline{M}_{1,1,d} \to X \) factors through a morphism \( \overline{ev}_1 : \overline{M}_{1,1,d} \to B_2 \). Also, \( B_2 \cong \mathbb{P}(T_X^*) \). Let \( \rho : B_2 \to X \) be the canonical projection. Then, there exists a commutative diagram of morphisms:

\[
\begin{array}{ccc}
\overline{M}_{1,1,d} & \xrightarrow{\overline{ev}_1} & B_2 \\
\downarrow_{f_{1,0}} & & \downarrow_{\rho} \\
\overline{M}_{1,0,d} & \xrightarrow{\phi} & X
\end{array}
\]
Lemma 6.3.  
(i) Let \( \mathcal{H} \) be the Hodge bundle over \( \mathfrak{M}_{1,0,d} \). Then,
\[
R^1(f_{1,0})_*\tilde{\ev}_1^*T_{B_2} \cong \mathcal{H}^\vee \otimes \phi^*T_X;
\]
(ii) There exists an exact sequence of locally free sheaves:
\[
0 \to R^1(f_{1,0})_*\tilde{\ev}_1^*T_{B_2} \to \mathcal{O} \to R^1(f_{1,0})_*\tilde{\ev}_1^*\mathcal{O}_{B_2}(-2) \to 0.
\]

Proof. (i) Let \( T_{B_2/X} \) be the relative tangent sheaf for the projection \( \rho : B_2 \to X \). Applying the functors \( \tilde{\ev}_1 \) and \( (f_{1,0})_* \) to the exact sequence
\[
0 \to T_{B_2/X} \to T_{B_2} \to \rho^*T_X \to 0
\]
of locally free sheaves, we obtain an exact sequence
\[
R^1(f_{1,0})_*\tilde{\ev}_1^*T_{B_2/X} \to R^1(f_{1,0})_*\tilde{\ev}_1^*T_{B_2} \to R^1(f_{1,0})_*\tilde{\ev}_1^*(\rho^*T_X) \to 0.
\]
where we have used \( R^2(f_{1,0})_*\tilde{\ev}_1^*T_{B_2/X} = 0 \) since \( f_{1,0} \) is of relative dimension 1.

We claim that \( R^1(f_{1,0})_*\tilde{\ev}_1^*T_{B_2/X} = 0 \). Indeed, let \( u = [\mu : D \to X^{[2]}] \in \mathfrak{M}_{1,0,d} \), and assume that \( \mu(D) = M_2(x) \). Since \( T_{B_2/X}|_{M_2(x)} = T_{M_2(x)} = \mathcal{O}_{M_2(x)}(2) \),
\[
H^1(f_{1,0}^{-1}(u), \tilde{\ev}_1^*T_{B_2/X}|_{f_{1,0}^{-1}(u)}) \cong H^1(D, \mu^*\mathcal{O}_{M_2(x)}(2)) = 0.
\]

By (6.6), \( R^1(f_{1,0})_*\tilde{\ev}_1^*T_{B_2} \cong R^1(f_{1,0})_*\tilde{\ev}_1^*(\rho^*T_X) \). Since \( \rho \circ \tilde{\ev}_1 = \phi \circ f_{1,0} \), we get
\[
R^1(f_{1,0})_*\tilde{\ev}_1^*T_{B_2} \cong R^1(f_{1,0})_*(f_{1,0}^*(\rho^*T_X))
\]
\[
\cong R^1(f_{1,0})_*\mathcal{O}_{\mathfrak{M}_{1,1,d}} \otimes \phi^*T_X
\]
\[
\cong \mathcal{H}^\vee \otimes \phi^*T_X.
\]

(ii) Since \( \tilde{\ev}_1 \) factors through \( \tilde{\ev}_1 \), we see from Lemma 6.2 (i) that
\[
\mathcal{O} = R^1(f_{1,0})_*((\tilde{\ev}_1^*T_{X^{[2]}}) = R^1(f_{1,0})_*((\tilde{\ev}_1^*(T_{X^{[2]}}|_{B_2})).
\]

Since \( B_2 \) is a smooth divisor in \( X^{[2]} \) and \( \mathcal{O}_{B_2}(B_2) = \mathcal{O}_{B_2}(-2) \), we have
\[
0 \to T_{B_2} \to T_{X^{[2]}}|_{B_2} \to \mathcal{O}_{B_2}(-2) \to 0.
\]

Applying the functors \( \tilde{\ev}_1^* \) and \( (f_{1,0})_* \), we obtain an exact sequence
\[
(f_{1,0})_*\tilde{\ev}_1^*\mathcal{O}_{B_2}(-2) \to R^1(f_{1,0})_*\tilde{\ev}_1^*T_{B_2} \to \mathcal{O} \to R^1(f_{1,0})_*\tilde{\ev}_1^*\mathcal{O}_{B_2}(-2) \to 0.
\]

We claim that \( (f_{1,0})_*\tilde{\ev}_1^*\mathcal{O}_{B_2}(-2) = 0 \). Indeed, let \( u = [\mu : D \to X^{[2]}] \in \mathfrak{M}_{1,0,d} \), and assume that \( \mu(D) = M_2(x) \). Then, since \( \mathcal{O}_{B_2}(-2)|_{M_2(x)} = \mathcal{O}_{M_2(x)}(-2) \),
\[
H^0(f_{1,0}^{-1}(u), \tilde{\ev}_1^*\mathcal{O}_{B_2}(-2)|_{f_{1,0}^{-1}(u)}) \cong H^0(D, \mu^*\mathcal{O}_{M_2(x)}(-2)) = 0.
\]

So \( (f_{1,0})_*\tilde{\ev}_1^*\mathcal{O}_{B_2}(B_2) = 0 \), and (6.7) is simplified to the exact sequence (6.5). \( \square \)

Theorem 6.4. Let \( d \geq 1 \). Let \( X \) be a smooth projective surface. Then,
\[
(1)^{X^{[2]}}_{1,d'} = \frac{K_X^2}{12d'}
\]
Proof. By Lemma 6.2 (ii), \(1\,\chi[2]_{1,d\beta_2} = \deg(\overline{\mathfrak{M}}_{1,0,d})^{\text{vir}} = \deg(c_{2d+2}(\mathcal{O}_b \cap \overline{\mathfrak{M}}_{1,0,d}))\). The Hodge bundle \(\mathcal{H}\) and \(R^1(f_{1,0})_*\overline{\mathcal{V}}_1^*\mathcal{O}_{\mathcal{B}_2}(-2)\) on the moduli space \(\overline{\mathfrak{M}}_{1,0,d}\) are of ranks 1 and 2 respectively. Therefore, by Lemma 6.3 and (4.2),

\[
\langle 1\,\chi[2]_{1,d\beta_2} = c_2(\mathcal{H}^* \otimes \phi^*T_X) \cdot c_{2d}(R^1(f_{1,0})_*\overline{\mathcal{V}}_1^*\mathcal{O}_{\mathcal{B}_2}(-2))
\]

\[
= (\lambda^2 + \phi^*K_X \cdot \lambda + \phi^*c_2(T_X)) \cdot c_{2d}(R^1(f_{1,0})_*\overline{\mathcal{V}}_1^*\mathcal{O}_{\mathcal{B}_2}(-2))
\]

\[
= (\phi^*K_X \cdot \lambda + \phi^*c_2(T_X)) \cdot c_{2d}(R^1(f_{1,0})_*\overline{\mathcal{V}}_1^*\mathcal{O}_{\mathcal{B}_2}(-2)).
\]

Let \(\chi(X)\) be the Euler characteristic of \(X\). By (4.4), we obtain

\[
\phi^*c_2(T_X) \cdot c_{2d}(R^1(f_{1,0})_*\overline{\mathcal{V}}_1^*\mathcal{O}_{\mathcal{B}_2}(-2)) = \chi(X) \cdot \int_{[\mathfrak{M}_{1,0}(\mathbb{P}^1,d\mathbb{P}^1)]} c_{2d}(R^1(f_{1,0})_*\overline{\mathcal{V}}_1^*\mathcal{O}_{\mathbb{P}^1}(-2))
\]

\[
= 0.
\]

Hence, \(1\,\chi[2]_{1,d\beta_2} = \phi^*K_X \cdot \lambda \cdot c_{2d}(R^1(f_{1,0})_*\overline{\mathcal{V}}_1^*\mathcal{O}_{\mathcal{B}_2}(-2))\). Choose two smooth irreducible curves \(C_1\) and \(C_2\) satisfying \([C_1] - [C_2] = K_X\). Since \(\mathcal{B}_2 \cong \mathbb{P}(T_X^\vee)\),

\[
\langle 1\,\chi[2]_{1,d\beta_2} = \phi^*([C_1] - [C_2]) \cdot \lambda \cdot c_{2d}(R^1(f_{1,0})_*\overline{\mathcal{V}}_1^*\mathcal{O}_{\mathcal{B}_2}(-2))
\]

\[
= \int_{[\mathfrak{M}_{1,0}(\mathbb{P}(T_X^\vee)\widetilde{\mathbb{P}}_{T_X}])} \lambda \cdot c_{2d}(R^1(f_{1,0})_*\overline{\mathcal{V}}_1^*\mathcal{O}_{\mathbb{P}(T_X^\vee)\widetilde{\mathbb{P}}_{T_X}}(-2))
\]

\[
- \int_{[\mathfrak{M}_{1,0}(\mathbb{P}(T_X^\vee)\widetilde{\mathbb{P}}_{T_X}])} \lambda \cdot c_{2d}(R^1(f_{1,0})_*\overline{\mathcal{V}}_1^*\mathcal{O}_{\mathbb{P}(T_X^\vee)\widetilde{\mathbb{P}}_{T_X}}(-2))
\]

\[
= \frac{\deg(T_X^\vee|C_1)}{12d} - \frac{\deg(T_X^\vee|C_2)}{12d}
\]

\[
= \frac{K_X^2}{12d}
\]

where we have used Proposition 4.5 in the third step. \(\square\)

6.2. Calculation of \((6.2)\).

Let \(X\) be a minimal surface of general type with \(K_X^2 = 1\) and \(p_g \geq 1\). By Noether’s inequality, \(p_g \leq 2\). Thus, \(p_g = 1\) or \(2\). If \(p_g = 2\), then we see from the proof of Proposition (8.1) in [BPV] that \(|K_X|\) is a pencil without fixed part and with one base point, and the general canonical curve is a genus-2 smooth irreducible curve. If \(p_g = 1\), then \(|K_X|\) consists of a single element which is a connected curve of arithmetic genus-2. In this subsection, we will study \((6.2)\) by assuming that \(X\) is a simply connected minimal surface of general type with \(K_X^2 = 1\) and \(1 \leq p_g \leq 2\), and that every member in \(|K_X|\) is a smooth irreducible curve (of genus-2).

Our first lemma asserts that the lower bound of \(d\) for the class \(\beta_X + d\beta_2\) to be effective is equal to \(-3\), and classifies all the curves homologous to \(\beta_X - 3\beta_2\).

Lemma 6.5. Let \(X\) be a simply connected minimal surface of general type with \(K_X^2 = 1\) and \(1 \leq p_g \leq 2\) such that every member in \(|K_X|\) is a smooth curve. Let \(\Gamma\) be a curve in \(X^{[2]}\) such that \(\Gamma \sim \beta_X + d\beta_2\) for some integer \(d\). Then,

(i) \(d \geq -3\).
(ii) \( d = -3 \) if and only if \( \Gamma = g_2^1(C) \subset C^{(2)} \), where \( C \in \mid K_X \mid \) and \( g_2^1(C) \) is the unique linear system of dimension 1 and degree 2 on the genus-2 curve \( C \).

**Proof.** Let \( \Gamma_1, \ldots, \Gamma_t \) be the irreducible components of \( \Gamma \). By Lemma 5.1, there exists \( t_1 \) with \( 1 \leq t_1 \leq t \) such that for \( 1 \leq i \leq t_1 \), \( \Gamma_i \sim \beta_{C_i} + d_i \beta_2 \) for some curve \( C_i \) and integer \( d_i \), and that for \( t_1 < j \leq t \), \( \Gamma_j \sim d_j \beta_2 \) for some integer \( d_j > 0 \). Then,

\[
\beta_{K_X} + d \beta_2 \sim \Gamma = \sum_{i=1}^{t_1} \Gamma_i \sim \sum_{i=1}^{t_1} (\beta_{C_i} + d_i \beta_2) + \sum_{j=t_1+1}^{t} (d_j \beta_2). \tag{6.8}
\]

So \( K_X \sim \sum_{i=1}^{t_1} C_i \). Since \( X \) is simply connected, \( K_X = \sum_{i=1}^{t_1} C_i \) as divisors. Since every member in \( \mid K_X \mid \) is a smooth irreducible curve, \( t_1 = 1 \) and \( C_1 \in \mid K_X \mid \). By (6.8), \( d = d_1 + \sum_{j=2}^{t} d_j \geq d_1 \). To prove the lemma, it suffices to prove \( d_1 \geq -3 \), i.e., we will assume in the rest of the proof that \( \Gamma = \Gamma_1 \) is irreducible.

We claim that there exists a non-empty open subset \( U \) of \( \Gamma \) such that every \( \xi \in U \) consists of two distinct points in \( X \). Indeed, assume that this is not true. Then \( \Gamma \subset B_2 \). Let \( \alpha \in H^2(X, \mathbb{C}) \). By abusing notations, we also use \( \alpha \) to denote a real surface in \( X \) representing the cohomology class \( \alpha \). Note that \( D_{\alpha | B_2} = D_{\alpha | M_2(X)} = 2M_2(\alpha) \). Thus, we obtain

\[
K_X \cdot \alpha = (\beta_{K_X} + d \beta_2) \cdot D_\alpha = \Gamma \cdot D_\alpha = \Gamma \cdot D_{\alpha | B_2} = 2\Gamma \cdot M_2(\alpha).
\]

So \( K_X \cdot \alpha \) is even for every \( \alpha \in H^2(X, \mathbb{C}) \). This contradicts to \( K_X^2 = 1 \).

Next, we claim that either \( \Gamma \sim \beta_{K_X} \), or there exists an irreducible curve \( C \subset X \) such that \( \text{Supp}(\xi) \in C \) for every \( \xi \in \Gamma \). Assume that there is no irreducible curve \( C \subset X \) such that \( \text{Supp}(\xi) \in C \) for every \( \xi \in \Gamma \). Then by the previous paragraph, we conclude that there exist a non-empty open subset \( U \) of \( \Gamma \) and two distinct irreducible curves \( \widetilde{C}_1, \widetilde{C}_2 \subset X \) such that every \( \xi \in U \) is of the form \( x_1 + x_2 \) with \( x_1 \in \widetilde{C}_1, x_2 \in \widetilde{C}_2 \) and \( x_1 \neq x_2 \). This leads to two (possibly constant) rational maps \( f_1 : \Gamma \to \widetilde{C}_1 \) and \( f_2 : \Gamma \to \widetilde{C}_2 \). Choose the real surface \( \alpha \subset X \) in the previous paragraph such that \( \alpha, \widetilde{C}_1 \) and \( \widetilde{C}_2 \) are in general position. Then,

\[
K_X \cdot \alpha = \Gamma \cdot D_\alpha = \deg(f_1) \widetilde{C}_1 \cdot \alpha + \deg(f_2) \widetilde{C}_2 \cdot \alpha.
\]

So \( K_X = \deg(f_1) \widetilde{C}_1 + \deg(f_2) \widetilde{C}_2 \) in \( H^2(X, \mathbb{C}) \). Since \( X \) is simply-connected, \( K_X = \deg(f_1) \widetilde{C}_1 + \deg(f_2) \widetilde{C}_2 \) as divisors. Since every member in \( \mid K_X \mid \) is a smooth irreducible curve and \( K_X^2 = 1 \), we must have an equality of sets:

\[
\{ \deg(f_1), \deg(f_2) \} = \{ 1, 0 \}.
\]

For simplicity, let \( \deg(f_1) = 1 \) and \( \deg(f_2) = 0 \). Then, \( x_2 \in \text{Supp}(\xi) \) for every \( \xi \in \Gamma \), and \( \widetilde{C}_1 \in \mid K_X \mid \). By our assumption, \( x_2 \notin \widetilde{C}_1 \) (otherwise, \( \text{Supp}(\xi) \in \widetilde{C}_1 \) for every \( \xi \in \Gamma \)). Thus, \( \Gamma = \widetilde{C}_1 + x_2 \). So \( \Gamma \sim \beta_{\widetilde{C}_1} \sim \beta_{K_X} \).

Finally, we assume that there exists an irreducible curve \( C \subset X \) such that \( \text{Supp}(\xi) \in C \) for every \( \xi \in \Gamma \). We claim that \( C \in \mid K_X \mid \). Note that there exists a positive integer \( s \) such that for a general point \( x \in C \), there exist \( s \) distinct
elements $\xi_1, \ldots, \xi_s \in \Gamma$ such that $x \in \text{Supp}(\xi_i)$ for every $i$. Choose the real surface $\alpha \subset X$ such that $\alpha$ and $C$ are in general position. Then, we have

$$K_X \cdot \alpha = sC \cdot \alpha.$$  

So $K_X = sC$ in $H^2(X, \mathbb{C})$. Since $X$ is simply connected, $K_X = sC$ as divisors.

Since $K_X^2 = 1$, $s = 1$ and $C \in |K_X|$. Since $\Gamma \in C^{(2)}$ and $g_C = 2$,

$$\beta_{K_X} + d\beta_2 \sim \Gamma \sim (\Xi \cdot \Gamma)\beta_{K_X} + (-3(\Xi \cdot \Gamma) + (\Theta \cdot \Gamma))\beta_2$$

by (5.6). Thus, $\Xi \cdot \Gamma = 1$ and $d = -3 + (\Theta \cdot \Gamma)$. Since $\Theta$ is a nef divisor of $C^{(2)}$, we have $d \geq -3$. In addition, $d = -3$ if and only if $\Theta \cdot \Gamma = 0$. Since $\Theta$ is the pull-back of a Theta divisor on $\text{Jac}_2(C)$ via the map $AJ : C^{(2)} \to \text{Jac}_2(C)$, $d = -3$ if and only if $\Gamma$ is contracted to a point by $AJ$. Note that the only positive-dimensional fiber of the map $AJ$ is $g_2^1(C) \cong \mathbb{P}^1$. Hence $d = -3$ if and only if $\Gamma = g_2^1(C) \subset C^{(2)}$. \qed

Fix $C \in |K_X|$ and an isomorphism $\mu_0 : \mathbb{P}^1 \to g_2^1(C)$. Via the inclusion $g_2^1(C) \subset C^{(2)} \subset X^{[2]}$, regard $\mu_0$ as a morphism from $\mathbb{P}^1$ to $X^{[2]}$. Then, we have

$$[\mu_0 : \mathbb{P}^1 \to X^{[2]}] \in \mathcal{W}_{0,0}(X^{[2]}, \beta_{K_X} - 3\beta_2).$$

(6.9)

**Lemma 6.6.** $h^1(\mathbb{P}^1, \mu_0^* T_{X^{[2]}}) = p_g - 1$.

**Proof.** From the exact sequence $0 \to \mathcal{O}_X \to \mathcal{O}_X(K_X) \to \mathcal{O}_C(K_X) \to 0$, we get

$$0 \to H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X(K_X)) \to H^0(C, \mathcal{O}_C(K_X)) \to H^1(X, \mathcal{O}_X).$$

Since $X$ is a simply connected surface, $h^1(X, \mathcal{O}_X) = 0$. So

$$h^0(C, \mathcal{O}_C(K_X)) = p_g - 1. \quad (6.10)$$

Put $\Gamma = g_2^1(C) \subset C^{(2)} \subset X^{[2]}$. Since the smooth rational curve $\Gamma$ is the only positive-dimensional fiber of $AJ : C^{(2)} \to \text{Jac}_2(C)$, $\Gamma$ is a $(-1)$-curve contracted by $AJ$. So the normal bundle $N_{\Gamma \subset C^{(2)}}$ of $\Gamma$ in $C^{(2)}$ is given by:

$$N_{\Gamma \subset C^{(2)}} = \mathcal{O}_\Gamma(-1). \quad (6.11)$$

Since $T_\Gamma = \mathcal{O}_\Gamma(2)$, we see from $0 \to T_\Gamma \to T_{C^{(2)}}|_\Gamma \to N_{\Gamma \subset C^{(2)}} \to 0$ that

$$T_{C^{(2)}}|_\Gamma = \mathcal{O}_\Gamma(2) \oplus \mathcal{O}_\Gamma(-1). \quad (6.12)$$

Since $K_X^{[2]} : \Gamma = D_{K_X} \cdot (\beta_{K_X} - 3\beta_2) = K_X^2 = 1$, we have

$$\deg N_{\Gamma \subset X^{[2]}} = -K_X^{[2]} : \Gamma - \deg T_\Gamma = -3.$$

So $\deg (N_{C^{(2)} \subset X^{[2]}}|_\Gamma) = -2$ in view of (6.11) and the exact sequence

$$0 \to N_{\Gamma \subset C^{(2)}} \to N_{\Gamma \subset X^{[2]}} \to N_{C^{(2)} \subset X^{[2]}}|_\Gamma \to 0. \quad (6.13)$$

We claim that $N_{C^{(2)} \subset X^{[2]}}|_\Gamma = \mathcal{O}_\Gamma(-p_g) \oplus \mathcal{O}_\Gamma(p_g - 2)$. Since $N_{C^{(2)} \subset X^{[2]}}|_\Gamma$ is a degree-$(-2)$ rank-2 bundle on $\Gamma \cong \mathbb{P}^1$, it suffices to prove

$$h^0(\Gamma, N_{C^{(2)} \subset X^{[2]}}|_\Gamma) = p_g - 1. \quad (6.14)$$

It is known from [AIK] that $N_{C^{(2)} \subset X^{[2]}} = \pi_1^* \pi_2^* \mathcal{O}_X(C)|_{C^{(2)}} = \pi_1^* \pi_2^* \mathcal{O}_X(K_X)|_{C^{(2)}}$ where $\pi_1 : Z_2 \to X^{[2]}$ and $\pi_2 : Z_2 \to X$ are the natural projections. Let $Z_\Gamma = \pi_1^{-1}(\Gamma) \subset Z_2$. Note that $\pi_2(Z_\Gamma) = C$. Put $\bar{\pi}_1 = \pi_1|_{Z_\Gamma} : Z_\Gamma \to \Gamma$ and $\bar{\pi}_2 = \pi_2|_{Z_\Gamma} : Z_\Gamma \to \Gamma$. 

\[ \]
\( \mathcal{Z}_T \rightarrow C \). Then, \( \tilde{\pi}_2 \) is an isomorphism. Up to an isomorphism, \( \tilde{\pi}_1 \) is the double cover \( C \rightarrow \mathbb{P}^1 \) corresponding to the linear system \( g_2^1(C) \). We have
\[
N_{C(2) \subset X[2]}|_{\Gamma} = \pi_1*\pi_2^*O_X(K_X)|_{\Gamma} = \tilde{\pi}_1*\tilde{\pi}_2^*(O_X(K_X)|_{C}) = \tilde{\pi}_1*\tilde{\pi}_2^*O_C(K_X). \quad (6.15)
\]
Thus \( H^0(\Gamma, N_{C(2) \subset X[2]}|_{\Gamma}) = H^0(\Gamma, \tilde{\pi}_1*\tilde{\pi}_2^*O_C(K_X)) = H^0(\mathcal{Z}_T, \tilde{\pi}_2^*O_C(K_X)) \). Since \( \tilde{\pi}_2 \) is an isomorphism, we conclude from \((6.10)\) that
\[
H^0(\Gamma, N_{C(2) \subset X[2]}|_{\Gamma}) \cong H^0(C, O_C(K_X)) = p_g - 1.
\]
This proves \((6.14)\). Therefore, we obtain
\[
N_{C(2) \subset X[2]}|_{\Gamma} = \mathcal{O}_T(-p_g) \oplus \mathcal{O}_T(p_g - 2). \quad (6.16)
\]
Now the exact sequence \((6.13)\) becomes
\[
0 \rightarrow \mathcal{O}_T(-1) \rightarrow N_{T \subset X[2]} \rightarrow \mathcal{O}_T(-p_g) \oplus \mathcal{O}_T(p_g - 2) \rightarrow 0
\]
which splits since \( 1 \leq p_g \leq 2 \). So \( N_{T \subset X[2]} = \mathcal{O}_T(-1) \oplus \mathcal{O}_T(-p_g) \oplus \mathcal{O}_T(p_g - 2) \). From \( T_{\Gamma} = \mathcal{O}_T(2) \) and the exact sequence \( 0 \rightarrow T_{\Gamma} \rightarrow T_X|_{\Gamma} \rightarrow N_{T \subset X[2]} \rightarrow 0 \), we see that \( T_X|_{\Gamma} = \mathcal{O}_T(2) \oplus \mathcal{O}_T(-1) \oplus \mathcal{O}_T(-p_g) \oplus \mathcal{O}_T(p_g - 2) \). Finally,
\[
h^1(\mathbb{P}^1, \mu_0^*T_{X[2]}) = h^1(\mathbb{P}^1, \mu_0^*(T_X|_{\Gamma})) = p_g - 1. \tag*{\Box}
\]

**Theorem 6.7.** Let \( X \) be a simply connected minimal surface of general type with \( K_X^2 = 1 \) and \( 1 \leq p_g \leq 2 \) such that every member in \( |K_X| \) is smooth. Then,

(i) \( \overline{M}_{0,0}(X[2], \beta_{K_X} - 3\beta_2) \cong |K_X| \cong \mathbb{P}^{p_g-1} \);

(ii) \( \langle 1 \rangle_{\overline{M}_{0,0}(X[2], \beta_{K_X} - 3\beta_2)} = (-1)^{\chi(O_X)} \).

**Proof.** For simplicity, we denote \( \overline{M}_{0,0}(X[2], \beta_{K_X} - 3\beta_2) \) by \( \overline{M} \).

(i) Let \([\mu : D \rightarrow X[2]] \in \overline{M}\). Put \( \Gamma = \mu(D) \). As in the first paragraph in the proof of Lemma \((6.5)\) let \( \Gamma_1, \ldots, \Gamma_t \) be the irreducible components of \( \Gamma \). Let \( m_i \) be the degree of the restriction \( \mu|_{\mu^{-1}(\Gamma_i)} : \mu^{-1}(\Gamma_i) \rightarrow \Gamma_i \). Then,
\[
\beta_{K_X} - 3\beta_2 = \mu_*[D] = \sum_{i=1}^t m_i[\Gamma_i]. \quad (6.17)
\]
By Lemma \((5.1)\) there exists some \( t_1 \) with \( 1 \leq t_1 \leq t \) such that for \( 1 \leq i \leq t_1 \), \( \Gamma_i \sim \beta_{C_i} + d_i\beta_2 \) for some curve \( C_i \) and integer \( d_i \), and that for \( t_1 < j \leq t \), \( \Gamma_j \sim d_j\beta_2 \) for some integer \( d_j > 0 \). Combining with \((6.17)\), we obtain
\[
\beta_{K_X} - 3\beta_2 = \sum_{i=1}^{t_1} m_i(\beta_{C_i} + d_i\beta_2) + \sum_{j=t_1+1}^t m_j(d_j\beta_2).
\]
So \( K_X \sim \sum_{i=1}^{t_1} m_iC_i \). Since \( X \) is simply connected, \( K_X = \sum_{i=1}^{t_1} m_iC_i \) as divisors. Since every member in \( |K_X| \) is smooth, \( t_1 = 1 \), \( m_1 = 1 \) and \( C_1 \in |K_X| \). By Lemma \((6.5)\) \( t = 1 \) and \( \Gamma_1 = g_2^1(C_1) \). So \( \mu(D) = g_2^1(C_1) \). Since \( m_1 = 1 \) and there is no marked points, \( D = \mathbb{P}^1 \) and \( \mu \) is an isomorphism from \( D = \mathbb{P}^1 \) to \( g_2^1(C_1) \). Therefore, \( \overline{M} = |K_X| \) as sets. Since the stable map \( \mu : D \rightarrow X[2] \) has the trivial automorphism group, \( \overline{M} \) is a fine moduli space.
Next, we construct a morphism $\psi : |K_X| \to \overline{\mathcal{M}}$. Let $\mathcal{C} \subset |K_X| \times X$ be the family of curves parametrized by $|K_X|$. Then we have the relative Hilbert schemes $(\mathcal{C}/|K_X|)^{(2)} \subset (|K_X| \times X/|K_X|)^{(2)}$, i.e., $(\mathcal{C}/|K_X|)^{(2)} \subset |K_X| \times X^{[2]}$. Let

$$\tilde{\Psi} : (\mathcal{C}/|K_X|)^{(2)} \to X^{[2]}$$

be the composition of the inclusion $(\mathcal{C}/|K_X|)^{(2)} \subset |K_X| \times X^{[2]}$ and the projection $|K_X| \times X^{[2]} \to X^{[2]}$. In the relative Jacobian $\text{Jac}_2(\mathcal{C}/|K_X|)$, let $\Sigma$ be the section to the natural projection $\text{Jac}_2(\mathcal{C}/|K_X|) \to |K_X|$ such that for $C \in |K_X|$, the point $\Sigma(C) = K_C \in \text{Jac}_2(C)$. Then the natural map $(\mathcal{C}/|K_X|)^{(2)} \to \text{Jac}_2(\mathcal{C}/|K_X|)$ is the blowing-up of $\text{Jac}_2(\mathcal{C}/|K_X|)$ along $\Sigma$. Let $\mathcal{E} \subset (\mathcal{C}/|K_X|)^{(2)}$ be the exceptional divisor of this blowing-up. Put

$$\Psi = \tilde{\Psi}|_\mathcal{E} : \mathcal{E} \to X^{[2]}.$$ 

Then $\Psi$ is a family of stable maps in $\overline{\mathcal{M}}$ parametrized by $|K_X|$. By the universality of the moduli space $\overline{\mathcal{M}}$, $\Psi$ induces a morphism $\psi : |K_X| \to \overline{\mathcal{M}}$. By the discussion in the previous paragraph, we conclude that $\psi$ is bijective.

By (3.19), the expected dimension of $\overline{\mathcal{M}}$ is 0. Since $\dim \overline{\mathcal{M}} = \dim |K_X| = p_g - 1$, the excess dimension of $\overline{\mathcal{M}}$ is equal to $p_g - 1$. By Lemma 6.6, $R^1(f_{1,0})_*\text{ev}_1^*T_{X^{[2]}}$ is a rank-$(p_g - 1)$ locally free sheaf, where $f_{1,0} : \overline{\mathcal{M}}_{0,1}(X^{[2]}, \beta_{K_X} - 3\beta_2) \to \overline{\mathcal{M}}$ is the forgetful map and $\text{ev}_1 : \overline{\mathcal{M}}_{0,1}(X^{[2]}, \beta_{K_X} - 3\beta_2) \to X^{[2]}$ is the evaluation map. By Proposition 2.1, $\overline{\mathcal{M}}$ is smooth (as a scheme since it is a fine moduli space). By Zariski’s Main Theorem, the bijective morphism $\psi : |K_X| \to \overline{\mathcal{M}}$ is an isomorphism. So $\overline{\mathcal{M}} \cong |K_X| \cong \mathbb{P}^{p_g - 1}$. Note also from Proposition 2.1 that

$$\overline{\mathcal{M}}^{\text{vir}} = c_{p_g - 1}(R^1(f_{1,0})_*\text{ev}_1^*T_{X^{[2]}}) \cap [\overline{\mathcal{M}}]. \quad (6.18)$$

(ii) Since $X$ is simply connected, we have $\chi(\mathcal{O}_X) = 1 + p_g$. When $p_g = 1$, $\overline{\mathcal{M}}_{0,1}(X^{[2]}, \beta_{K_X} - 3\beta_2)$ is a smooth point; so $\langle 1 \rangle^{X^{[2]}}_{0, \beta_{K_X} - 3\beta_2} = 1$, and our formula holds. In the rest of the proof, let $p_g = 2$. We will prove that $\langle 1 \rangle^{X^{[2]}}_{0, \beta_{K_X} - 3\beta_2} = -1$.

We adopt the notations from the proof of (i). For simplicity, put $\overline{\mathcal{M}} = |K_X|$. Then we have identifications $\overline{\mathcal{M}}_{0,1}(X^{[2]}, \beta_{K_X} - 3\beta_2) = \mathcal{E}$ and $\text{ev}_1 = \Psi$. Moreover, the forgetful map $f_{1,0} : \overline{\mathcal{M}}_{0,1}(X^{[2]}, \beta_{K_X} - 3\beta_2) \to \overline{\mathcal{M}}$ is identified with the natural projection $f : \mathcal{E} \to |K_X|$. In view of (6.18), we have

$$\overline{\mathcal{M}}^{\text{vir}} = c_1(R^1f_*\Psi^*T_{X^{[2]}}) \cap [\overline{\mathcal{M}}]. \quad (6.19)$$

To understand the line bundle $R^1f_*\Psi^*T_{X^{[2]}}$, we take the exact sequence of relative tangent bundles associated to the pair $(\mathcal{C}/|K_X|)^{(2)} \subset |K_X| \times X^{[2]}$:

$$0 \to T_{(\mathcal{C}/|K_X|)^{(2)}/|K_X|} \to T_{|K_X| \times X^{[2]}/|K_X|}|_{(\mathcal{C}/|K_X|)^{(2)}} \to N_{(\mathcal{C}/|K_X|)^{(2)} \subset |K_X| \times X^{[2]}} \to 0.$$ 

Note that $T_{|K_X| \times X^{[2]}/|K_X|}|_{(\mathcal{C}/|K_X|)^{(2)}} = \tilde{\Psi}^*T_{X^{[2]}}$. In addition, we have

$$N_{(\mathcal{C}/|K_X|)^{(2)} \subset |K_X| \times X^{[2]}} = \left(p_1^*p_2^*\mathcal{O}_{|K_X| \times X(\mathcal{C})}|_{(\mathcal{C}/|K_X|)^{(2)}}\right)$$

(6.20)
by [AIK], where \( p_1 : |K_X| \times Z_2 \to |K_X| \times X^{[2]} \) and \( p_2 : |K_X| \times Z_2 \to |K_X| \times X \) are the natural projections. Therefore, the above exact sequence becomes

\[
0 \to T_{(|K_X|^{[2]}/|K_X|)} \to \Psi^* T_{X^{[2]}} \to (p_1*p_2^* O_{|K_X| \times X(C)})_{|C/|K_X|^{[2]} / |K_X|} \to 0.
\]

Restricting it to \( \mathcal{E} \subset (C/|K_X|^{[2]}) \), we obtain the exact sequence

\[
0 \to T_{(\mathcal{E}/|K_X|^{[2]}/|K_X|)} \mathcal{E} \to \Psi^* T_{X^{[2]}} \mathcal{E} \to (p_1*p_2^* O_{|K_X| \times X(C)})_{|\mathcal{E}|} \to 0.
\]  

(6.21)

Let \( Z_{\mathcal{E}} = p_1^{-1}(\mathcal{E}). \) Then \( p_2(Z_{\mathcal{E}}) = \mathcal{C}. \) Put \( \tilde{p}_1 = p_1|_{Z_{\mathcal{E}}} : Z_{\mathcal{E}} \to \mathcal{E} \) and \( \tilde{p}_2 = p_2|_{Z_{\mathcal{E}}} : Z_{\mathcal{E}} \to \mathcal{C}. \) Then the exact sequence (6.21) can be rewritten as

\[
0 \to T_{(\mathcal{C}/|K_X|^{[2]}/|K_X|)} \mathcal{E} \to \Psi^* T_{X^{[2]}} \mathcal{E} \to \tilde{p}_1*p_2^* O_{\mathcal{C}}(\mathcal{C}) \to 0.
\]  

(6.22)

By (6.12), \( R^1 f_* (T_{(\mathcal{C}/|K_X|^{[2]}/|K_X|)} \mathcal{E}) = 0. \) Thus applying \( f_* \) to (6.22) yields

\[
R^1 f_* \Psi^* T_{X^{[2]}} \cong R^1 f_* (\tilde{p}_1*p_2^* O_{\mathcal{C}}(\mathcal{C})).
\]  

(6.23)

Since \( K_X^2 = 1 \), the complete linear system \( |K_X| \) has a unique base point, denoted by \( x_0. \) The blowing-up morphism \( \tilde{q}_2 : \tilde{X} \to X \) of \( X \) at \( x_0 \) resolves the rational map \( X \dashrightarrow |K_X|, \) and leads to a morphism \( \tilde{q}_1 : \tilde{X} \to |K_X|. \) In addition, \( \tilde{X} = \mathcal{C}, \) the inclusion \( \mathcal{C} \subset |K_X| \times X \) is given by the map \( (\tilde{q}_1, \tilde{q}_2) : \tilde{X} \to |K_X| \times X, \) and there exists a commutative diagrams of morphisms:

\[
\begin{array}{ccc}
\tilde{X} = \mathcal{C} & \xrightarrow{\tilde{q}_1} & \mathcal{E} \\
\downarrow \mathcal{X}_1 & & \downarrow f \\
|K_X| & \xrightarrow{f} & Z_{\mathcal{E}} \\
\end{array}
\]

From the exact sequence \( 0 \to T_{\mathcal{C}} \to T_{|K_X| \times X}|_{|K_X|} \to O_{\mathcal{C}}(\mathcal{C}) \to 0, \) we get

\[
O_{\mathcal{C}}(\mathcal{C}) \cong O_{\tilde{X}}(K_{\tilde{X}}) \otimes \tilde{q}_1^* O_{|K_X|}(2) \otimes \tilde{q}_2^* O_X(-K_X) \cong O_{\tilde{X}}(E) \otimes \tilde{q}_1^* O_{|K_X|}(2)
\]  

(6.24)

where \( E \subset \tilde{X} \) is the exceptional curve. Combining with (6.23), we obtain

\[
R^1 f_* \Psi^* T_{X^{[2]}} \cong O_{|K_X|}(2) \otimes R^1 f_* (\tilde{p}_* O_{\tilde{X}}(E)).
\]  

(6.25)

Next, we determine the rational ruled surface \( \mathcal{E}. \) Let \( \sigma = \tilde{p}(E). \) Since \( E \) is a section to \( \tilde{q}_1, \) \( \sigma \) is a section to \( f. \) Let \( \mathcal{C} \) be a fiber of \( \tilde{q}_1. \) Then \( \Gamma = \tilde{q}_1^{-1}(C) \) is the fiber of \( f \) over the point \( \tilde{q}_1(C), \) and \( K_{\tilde{X}} = \tilde{q}_2^* K_X + E = C + 2E. \) By the adjunction formula, \( K_C = (K_{\tilde{X}} + C)|_{C} = 2(E)|_{C} = 2(E \cap C). \) So the double cover \( C \to \Gamma \) is ramified at the intersection point \( E \cap C. \) It follows that the double cover \( \tilde{p} \) is ramified along \( E. \) Thus \( \tilde{p}^* \sigma = 2E. \) By the projection formula, \( \sigma^2 = (\tilde{p}^* \sigma)^2/2 = 2E^2 = -2. \) Thus, \( \mathcal{E} \) is the Hirzebruch surface \( \mathbb{F}_2 = \mathbb{P}(O_{|K_X|} \oplus O_{|K_X|}(-2)) \) with \( O_{\mathcal{E}}(1) = O_{\mathcal{E}}(\sigma). \) It follows that \( K_{\mathcal{E}} = -2\sigma - 4\Gamma. \)

Let \( B \subset \mathcal{E} \) be the branch locus of the double cover \( \tilde{p}, \) and \( \tilde{B} = \tilde{p}^{-1}(B). \) Then \( B \) and \( \tilde{B} \) are smooth, \( \sigma \subset B, E \subset \tilde{B}, \) and \( B = 2L \) for some divisor \( L \) on \( \mathcal{E}. \) Also,

\[
C + 2E = K_{\tilde{X}} = \tilde{p}^*(K_{\mathcal{E}}) + \tilde{B} = \tilde{p}^*(-2\sigma - 4\Gamma) + \tilde{B} = -4E - 4C + \tilde{B}.
\]
So \( \tilde{B} = 6E + 5C \), and \( B = \tilde{p}_* \tilde{B} = 6\sigma + 10\Gamma \) (thus \( B \) is the disjoint union of \( \sigma \) and a smooth curve in \(|5\sigma + 10\Gamma|\)). Since \( \tilde{p}_* \mathcal{O}_X = \mathcal{O}_X \oplus \mathcal{O}_X(-L) \),

\[
\begin{align*}
\chi (\tilde{p}_* \mathcal{O}_X (E)) &= ch (\tilde{p}_* (\mathcal{O}_X (E))) = \{ \tilde{p}_* (ch (\mathcal{O}_X (E)) \cdot \text{td}(T_{\tilde{p}})) \}_1 \\
&= \tilde{p}_* E + 1 \cdot (\tilde{p}_* \mathcal{O}_X) = \sigma + (-L) = -2\sigma - 5\Gamma
\end{align*}
\]
(6.26)
by the Grothendieck-Riemann-Roch Theorem, where \( T_{\tilde{p}} \) is the relative tangent sheaf of \( \tilde{p} \). Since \( h^0 (\mathcal{E}, \tilde{p}_* \mathcal{O}_X (E)) = h^0 (\tilde{X}, \mathcal{O}_X (E)) = 1 \), there exists an injection \( \mathcal{O}_E \to \tilde{p}_* \mathcal{O}_X (E) \) which in turn induces an injection \( \mathcal{O}_E (a\sigma) \to \tilde{p}_* \mathcal{O}_X (E) \) with \( a \geq 0 \) and with torsion-free quotient \( \tilde{p}_* \mathcal{O}_X (E) / \mathcal{O}_E (a\sigma) \). So we have an exact sequence

\[
0 \to \mathcal{O}_E (a\sigma) \to \tilde{p}_* \mathcal{O}_X (E) \to \mathcal{O}_E ((-a - 2)\sigma - 5\Gamma) \otimes I_{\eta} \to 0
\]
(6.27)
where \( \eta \) is a 0-cycle on \( E \). By (6.16), (6.20) and (6.24), we get

\[
\begin{align*}
\mathcal{O}_E \oplus \mathcal{O}_R (-2) &= \mathcal{N}_{C^{(2)} \subset X^{[2]}} | I = (\mathcal{N}_{C^{(2)} / K_X^{[(2)]}} | \mathcal{O}_{K_X^{[(2)]}}) | I \\
&= (p_1^* p_2^* \mathcal{O}_{K_X^{[(2)]} \times X^{[2]}} (C)) | I = (\tilde{p}_1^* \tilde{p}_2^* \mathcal{O}_{C^{[(2)]}}) | I \\
&= (\tilde{p}_*(\mathcal{O}_X (E) \otimes \mathfrak{q}_1 \mathcal{O}_{K_X^{[(2)]}})) | I = (\tilde{p}_* \mathcal{O}_X (E)) | I.
\end{align*}
\]

Since this holds for every fiber \( \Gamma \) of \( f \), we conclude from (6.27) that \( a = 0 \) and \( \eta = 0 \). So the exact sequence (6.27) is simplified to

\[
0 \to \mathcal{O}_E \to \tilde{p}_* \mathcal{O}_X (E) \to \mathcal{O}_E (-2\sigma - 5\Gamma) \to 0.
\]

Applying the functor \( f_* \) to the above exact sequence yields

\[
R^1 f_* (\tilde{p}_* \mathcal{O}_X (E)) \cong R^1 f_* \mathcal{O}_E (-2\sigma - 5\Gamma) \cong \mathcal{O}_E (1) \cong \mathcal{O}_E (\sigma) \oplus \mathcal{O}_E (\sigma).
\]

Since \( E = \mathbb{P} (\mathcal{O}_{K_X} \oplus \mathcal{O}_{K_X} (-2)) \) with \( \mathcal{O}_E (1) = \mathcal{O}_E (\sigma) \) and \( f_* \mathcal{O}_E = \mathcal{O}_{K_X} \),

\[
R^1 f_* (\tilde{p}_* \mathcal{O}_X (E)) \cong \mathcal{O}_{K_X} (-5) \otimes (f_*(\mathcal{O}_E) \otimes \mathcal{O}_{K_X} (-2)) \cong \mathcal{O}_{K_X} (-3).
\]

By (6.25), \( R^1 f_* \Psi^* T_{X^{[2]}} \cong \mathcal{O}_{K_X} (-1) \). Finally, by (6.19), we obtain

\[
\begin{align*}
\langle 1 \rangle_{X^{[2]}}, 3 = \text{deg} [\overline{\mathcal{M}}]_{\text{vir}} &= \text{deg} c_1 (R^1 f_* \Psi^* T_{X^{[2]}}) = -1.
\end{align*}
\]

\( \square \)

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