AN ANALYTIC PROOF OF THE MATRIX SPECTRAL FACTORIZATION THEOREM

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Abstract. An analytic proof is proposed of Wiener’s theorem on factorization of positive definite matrix-functions.

1. Introduction. The series of papers of Wiener [15], [16], [18], and Helson and Lowdenslager [4], [5] led to the following

Matrix Spectral Factorization Theorem: Let

\( S(z) = \left(S_{jk}(z)\right)_{j,k=1}^r \),

\(|z| = 1\), be a positive definite \( r \times r \) matrix-function with integrable entries, \( S_{jk}(z) \in L_1(\mathbb{T}) \), defined on the unit circle. If

\( \log \det S(z) \in L_1(\mathbb{T}) \),

then \( S(z) \) admits a factorization

\( S(z) = \chi^+(z)(\chi^+(z))^* \),

where \( \chi^+(z), |z| < 1, \) is an analytic \( r \times r \) matrix-function with entries from the Hardy space \( H_2 \), and \( \det \chi^+(z) \) is an outer function.

The equation (3) is assumed to hold a.e. on the unit circle \( \mathbb{T} \) and \( (\chi^+)^* = (\overline{\chi^+})^T \) is the Hermitian conjugate of \( \chi^+ \).

The spectral factorization (3) is unique up to a constant unitary multiplier, and the unique spectral factor with additional requirement that \( \chi^+(0) \) is positive definite is called canonical.

It is well-known that the condition (2) is also necessary for the existence of the factorization (3).

Spectral factorization problem was originally formulated in the scalar case by Wiener [14] and Kolmogorov [9], independently from each other, in connection with developed linear prediction theory of stationary stochastic processes. Wiener [16] used the matrix generalization of the problem in the study of multidimensional stationary time series. Since then

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the spectral factorization has become an important tool in solution of various applied problems in Control Engineering and Communications.

The prediction theory of stationary processes itself, namely the idea derived from the theory of linear least-squares method, is helpful to prove the spectral factorization theorem in the scalar as well in the matrix case and Wiener successfully exploited this idea. Helson and Lowdenslager derived the theorem from the theory of invariant subspaces and this proof, together with its generalizations to positive operator valued functions, is well presented in [3].

However, in the scalar case, the theory of the Hardy spaces $H_p$ provides a simple solution to the spectral factorization problem. Furthermore, the spectral factor can be written in an explicit form (see [10])

\[
\chi^+(z) = \exp \left( \frac{1}{4\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log S(e^{it}) \, dt \right).
\]

No analog of the formula (4) exists in the matrix case since the equation $\exp(A + B) = \exp A \exp B$ breaks down for non-commutative matrices and, in general, the factorization problem becomes more demanding. Nevertheless, one can observe that the formulation of the spectral factorization theorem in the matrix case does not go beyond the theory of $H_p$ spaces and naturally its proof had been searched within this theory. In the present paper we provide such proof on 50 years anniversary of this important theorem.

Together with the existence theorem of the spectral factorization, the challenging problem became the actual approximate computation of the spectral factor $\chi^+(z)$ for a given matrix-function (1) since, as it was mentioned above, this procedure found its immediate applications in solutions of various practical problems. Wiener [17] made first efforts to create a sound computational algorithm of matrix spectral factorization and since then several dozens of papers have addressed to the development of such algorithm. It is surely hopeless to give here a short but still comprehensive account on all these developments (see [7], [13]). Wiener and his followers [17], [11] as well as Kolmogorov’s students in Russia [12], [20] were using methods of Functional Analysis for solution of the problem. A Newton-Raphson type iterative algorithm was constructed by G. Wilson [19]. Kalman’s state-space approach to the Wiener filtering problem [8] became fruitful for spectral factorization as well, and by this way the spectral factorization problem was actually reduced to the solution of algebraic Riccati equation. Until recently this method was
considered as most effective for matrix spectral factorization (see [7], p. 206), though it can be used only for rational matrix functions. Numerous algorithmic type improvements were proposed by the different authors in this direction.

An absolutely new approach to the spectral factorization problem in the matrix case was developed in [6] (see also [1]). Without imposing on matrix-function (1) any additional restriction, apart from the necessary and sufficient condition (2), effective method of approximate computation of $\chi^+$ is proposed. This is the first time that the methods of Complex Analysis and Hardy Spaces were used for construction of the algorithm, which naturally turned out to be very efficient as corresponding software implementation confirms (see demo version at www.rmi.acnet.ge/SpFact). The decisive role of unitary matrix-functions in the factorization process is revealed which, by flexible manipulations, completely absorbs all the technical difficulties of the problem leaving very few and simple procedures for computation. This method of matrix spectral factorization is undoubtedly as simple as possible and in the present paper we would like to demonstrate that in fact it contains an analytic proof of the existence theorem as well, which was formulated in the beginning. No other assumptions beyond the theory of Hardy spaces are used for this purpose.

2. Notation. $L_p(\mathbb{T})$, $p > 0$, is the class of $p$-integrable complex functions defined on the unit circle, and $H_p$ denotes the Hardy space of analytic functions in the unit disk,

$$H_p = \left\{ f \in \mathcal{A}(D) : \sup_{r < 1} \int_0^{2\pi} |f(re^{it})|^p \, dt < \infty \right\}.$$  

A function $Q(z) \in H_p$ is called outer, we denote $Q(z) \in H_p^O$, if

$$Q(z) = c \cdot \exp \left( \frac{1}{2\pi} \int_0^{2\pi} e^{it} + \frac{z}{e^{it} - z} \log |Q(e^{it})| \, dt \right),$$

where $|c| = 1$.

The $n$th Fourier coefficient of $f \in L_1(\mathbb{T})$ is denoted by $c_n(f)$ and, for $p \geq 1$, $L_p^+(\mathbb{T}) := \{ f \in L_p(\mathbb{T}) : c_n(f) = 0 \text{ for } n < 0 \}$. The spaces $L_p^+(\mathbb{T})$ and $H_p$ are naturally identified, so that we can speak about the value of function $f \in L_p^+(\mathbb{T})$ in $z \in D$.  

Let $L^N_p(\mathbb{T})$, $N > 0$, $p \geq 1$, be the set of functions $f$ from $L^p(\mathbb{T})$ for which $c_n(f) = 0$ whenever $n < -N$, and let $L^+_{[N]}$ be the set of analytic polynomials whose nonzero coefficients range from 0 to $N$.

If $M$ is a matrix (resp. matrix-function), then $\overline{M}$ denotes the matrix (resp. matrix-function) with conjugate entries and $M^* := \overline{M}^T$. The upper-left $m \times m$ submatrix of $M$ is denoted by $M^{[m,m]}$.

We say that matrix-functions have some property, say, belong to $L^p(\mathbb{T})$ or are convergent, etc, if their entries have this property.

A $r \times r$ matrix $U$ is called unitary if $UU^* = I_r$, where $I_r$ is the identity matrix of dimension $r$. Since the rows and columns of $U$ are orthonormal, its entries are bounded by 1. A unitary matrix-function $U(z)$ means that it is unitary for a.a. $z \in \mathbb{T}$.

The notation $\text{diag}(u_1, u_2, \ldots, u_r)$ stands for the diagonal $r \times r$ matrix with corresponding entries on the main diagonal.

3. The uniqueness of spectral factorization. In the proof of the convergence property of the above mentioned algorithm given in [6], at least formally, the existence of spectral factorization is used. However, one can observe that rather the uniqueness than the existence of spectral factorization provides this convergence. So we start with a simple proof of the uniqueness theorem (cf. [2]), emphasizing that it can be obtained without a priori knowledge of the existence of the spectral factorization itself.

We use the following generalization of Smirnov’s theorem concerning functions from the Hardy spaces $H^p$ (see [10], p. 109): Let $f(z) = g(z)/h(z)$, where $g \in H^p_{p_1}$, $p_1 > 0$, and $h \in H^p_{p_2}$, $p_2 > 0$. If the boundary values $f(e^{it}) \in L^p(\mathbb{T})$, $p > 0$, then $f \in H^p$.

Uniqueness Theorem: If

\begin{equation}
S(z) = \chi^+_j(z) (\chi^+_j(z))^*, \quad j = 1, 2,
\end{equation}

are two spectral factorizations of a given spectral density $S(z)$, then

\begin{equation}
\chi^+_1(z) = \chi^+_2(z) \cdot U, \quad |z| < 1,
\end{equation}

for some constant unitary matrix $U$.

Proof. It follows from (5) that $\chi^+_1(z) (\chi^+_1(z))^* = \chi^+_2(z) (\chi^+_2(z))^*$ a.e. on $\mathbb{T}$, so that

\begin{equation}
(\chi^+_2(z))^{-1} \chi^+_1(z) ((\chi^+_2(z))^{-1} \chi^+_1(z))^* = I_r \quad \text{for a.a. } z \in \mathbb{T}.
\end{equation}
Thus the analytic matrix-function
\begin{equation}
U(z) := (\chi_2^+(z))^{-1} \chi_1^+(z), \quad |z| < 1,
\end{equation}
is unitary on the boundary for a.a. \( z \in \mathbb{T} \). Consequently \( U(e^{it}) \in L_\infty(\mathbb{T}) \).

Since \( \chi_j^+(z), \ j = 1, 2, \) are spectral factors, it is assumed that their entries are from \( H_2 \) and \( \det \chi_j^+(z), \ j = 1, 2, \) are outer analytic functions, so that entries of
\[ U(z) = \frac{1}{\det \chi_2^+(z)} \text{Adj}(\chi_2^+(z)) \chi_1^+(z)\]
can be represented as ratios of two functions from \( H_{2/r} \) and \( H_{2/r}^O \), respectively. Hence, we can use the generalization of Smirnov’s theorem to conclude that \( U(z) \in H_\infty \), i.e. \( U(e^{it}) \in L_\infty^+(\mathbb{T}) \).

By changing the roles of \( \chi_1^+ \) and \( \chi_2^+ \) in this discussion, we get
\[ \left( \chi_1^+(z) \right)^{-1} \chi_2^+(z) \in H_\infty. \]

But \( (U(z))^* = (\chi_1^+(z))^{-1} \chi_2^+(z) \) for a.a. \( z \in \mathbb{T} \), by virtue of (7). Thus, we have
\[ U(e^{it}) \in L_\infty^+(\mathbb{T}) \text{ and } \overline{U(e^{it})} \in L_\infty^+(\mathbb{T}), \]
which implies that (8) is a constant matrix-function and (6) follows. \( \square \)

4. Main Lemmas.

**Lemma 1.** For any \( m \times m \) matrix-function \( F_m(z) \) of the form
\begin{equation}
F_m(z) = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 \\
\varphi_1(z) & \varphi_2(z) & \varphi_3(z) & \cdots & \varphi_{m-1}(z) & f^+(z)
\end{pmatrix},
\end{equation}

\( |z| = 1, \) where \( f^+ \in H_2^O \) and \( \varphi_j \in L_2^N(\mathbb{T}), \ j = 1, 2, \ldots, m-1, \) for some positive integer \( N \), there exists a unitary matrix-function \( U_m(z) \) of the form
\begin{equation}
U_m(z) = \begin{pmatrix}
u_{11}(z) & \nu_{12}(z) & \cdots & \nu_{1m}(z) \\
u_{21}(z) & \nu_{22}(z) & \cdots & \nu_{2m}(z) \\
\vdots & \vdots & \ddots & \vdots \\
u_{m-1,1}(z) & \nu_{m-1,2}(z) & \cdots & \nu_{m-1,m}(z) \\
u_{m1}(z) & \nu_{m2}(z) & \cdots & \nu_{mm}(z)
\end{pmatrix},
\end{equation}
\[ |z| = 1, \text{ where} \]
\[ u_{jk} \in L_{[N]}^+, \quad j, k = 1, 2, \ldots, m, \]
and
\[ (11) \quad \det U_m(z) = 1, \]
\[ |z| = 1, \text{ such that} \]
\[ (12) \quad F_m(z) U_m(z) \in L_2^+(\mathbb{T}). \]

For two dimensional matrices this lemma is proved in [6], and it is generalized for any dimensional matrices in [1]. We emphasize that the lemma can be proved without any reference to the existence theorem of matrix spectral factorization. Furthermore, a system of linear equations which provides the coefficients of functions \( u_{jk}, j, k = 1, 2, \ldots, m, \) whenever \( N \) negative coefficients of \( \varphi_j, j = 1, 2, \ldots, m - 1, \) and \( N \) positive coefficients of \( f^+ \) are given, can be written and solved explicitly (see [1], p. 22).

**Lemma 2.** For any \( m \times m \) matrix-function \( F_m(z), |z| = 1, \) of the form (9), where \( f^+ \in H_2^O \) and \( \varphi_j \in L_2(\mathbb{T}), j = 1, 2, \ldots, m - 1, \) there exists a unitary matrix-function \( U_m(z) \) of the form (10) satisfying (11) a.e. such that
\[ u_{jk}(z) \in L_\infty^+(\mathbb{T}), \quad j, k = 1, 2, \ldots, m, \]
and (12) holds.

**Proof.** Let \( F_m^{(N)}(z) \) be the \( L_2 \)-approximation of matrix-function \( F_m(z) \) where the entries \( \varphi_j(z), j = 1, 2, \ldots, m - 1, \) are approximated by their Fourier series
\[ \varphi_j(z) \approx \sum_{n=-N}^{\infty} c_n(\varphi_j) z^n. \]
Then we can use Lemma 1 which provides the existence of unitary matrix-function \( U_m^{(N)}(z), \)
\[ (14) \quad U_m^{(N)}(z)(U_m^{(N)}(z))^* = I_m \text{ (a.e.)}, \]
such that

\[(15)\quad U_m^{(N)}(z) = \begin{pmatrix}
    u_{11}^{(N)}(z) & u_{12}^{(N)}(z) & \cdots & u_{1m}^{(N)}(z) \\
    u_{21}^{(N)}(z) & u_{22}^{(N)}(z) & \cdots & u_{2m}^{(N)}(z) \\
    \vdots & \vdots & \ddots & \vdots \\
    u_{m1}^{(N)}(z) & u_{m2}^{(N)}(z) & \cdots & u_{mm}^{(N)}(z)
\end{pmatrix}, \quad u_{jk}^{(N)} \in L^+_{[N]},\]

\[(16)\quad \det U_m^{(N)}(z) = 1 \quad \text{ (a.e.)},\]

and

\[(17)\quad F_m^{(N)}(z) U_m^{(N)}(z) \in L^+_{2}(\mathbb{T}).\]

Now, a convergent subsequence can be extracted from \(\{U_m^{(N)}\}_{N=1}^{\infty}\). Furthermore, if we require in addition, say, positive definiteness of the matrix \(F_m^{(N)} U_m^{(N)}(0)\) which can be achieved by multiplying, if necessary, \(U_m^{(N)}(z)\) from the right by a constant unitary matrix with determinant 1, then \(U_m^{(N)}(z)\) itself converges at least in measure,

\[(18)\quad U_m^{(N)}(z) \rightrightarrows U_m(z),\]

as \(N \to \infty\). These facts were proved in [6] for two dimensional case and, in the similar way, this can be done for any dimensional matrices as soon as the explicit form of \(U_m^{(N)}(z)\) is obtained. Anyway, the convergence (18), together with boundedness of unitary matrix-functions, guaranties that we can pass to the limit in (14)-(17), so that unitary matrix-function \(U_m(z)\) in (18) satisfies the desired conditions (10)-(13). \(\Box\)

5. Proof of the theorem. First perform the lower-upper triangular factorization of (1) with positive entries on the diagonal, i.e. take

\[(19)\quad S(z) = A(z)(A(z))^* \quad \text{for a.a. } z \in \mathbb{T},\]

where

\[A(z) = \begin{pmatrix}
    f_{11}(z) & 0 & \cdots & 0 \\
    f_{21}(z) & f_{22}(z) & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    f_{r1}(z) & f_{r2}(z) & \cdots & f_{rr}(z)
\end{pmatrix}\]

and \(f_{jj}(z) \geq 0\) for a.a. \(z \in \mathbb{T},\ j = 1, 2, \ldots, r\). This factorization can be achieved pointwise by the well known theorem of Linear Algebra.
Since $S_{jj}(z) \in L_1(\mathbb{T})$, $j = 1, 2, \ldots, r$, by virtue of equations

$$
\sum_{k=1}^{j} |f_{jk}(z)|^2 = S_{jj}(z)
$$

(see (19)), all the entries of $A(z)$ are square integrable,

$$
f_{jk}(z) \in L_2(\mathbb{T}),
$$

$1 \leq j \leq r$, $1 \leq k \leq j$. Furthermore, since

$$
\sum_{j=1}^{r} \log f_{jj}(z) = \log \prod_{j=1}^{r} f_{jj}(z) = \log \det A(z) = \frac{1}{2} \log \det S(z) \in L_1(\mathbb{T})
$$

(see (2)), we have

$$
\log f_{jj}(z) \in L_1(\mathbb{T}), \quad j = 1, 2, \ldots, r,
$$

The condition (20) provides that

$$
f_j^+(z) = \exp \left( \frac{1}{2\pi} \int_{0}^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log f_{jj}(e^{it}) \, dt \right), \quad j = 1, 2, \ldots, r.
$$

are outer analytic functions satisfying $|f_j^+(z)| = f_{jj}(z)$ for a.a. $z \in \mathbb{T}$.

If we denote the ratio $f_j^+(z)/f_{jj}(z)$ by $u_j(z)$, then $|u_j(z)| = 1$ for a.a. $z \in \mathbb{T}$, $j = 1, 2, \ldots, r$, and $U(z) = \text{diag}(u_1(z), u_2(z), \ldots, u_r(z))$ is unitary matrix-function. Thus, for matrix-function $M(z) = A(z)U(z)$, we have

$$
S(z) = M(z)(M(z))^* \quad \text{for a.a. } z \in \mathbb{T},
$$

where

$$
M(z) = \begin{pmatrix}
    f_1^+(z) & 0 & \cdots & 0 & 0 \\
    \varphi_{21}(z) & f_2^+(z) & \cdots & 0 & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    \varphi_{r-1,1}(z) & \varphi_{r-1,2}(z) & \cdots & f_{r-1,1}^+(z) & 0 \\
    \varphi_{r,1}(z) & \varphi_{r,2}(z) & \cdots & \varphi_{r,r-1}(z) & f_r^+(z)
\end{pmatrix},
$$

$f_j^+(z) \in H_2^0$, $\varphi_{jk} \in L_2(\mathbb{T})$, $1 \leq j \leq r$, $1 \leq k \leq j$. Define

$$
M_1(z) = M(z) \quad \text{and} \quad M_m(z) = M_{m-1}(z)V_m(z), \quad m = 2, 3, \ldots, r,
$$

where $V_m(z)$, $m = 2, 3, \ldots, r$, are unitary matrix-functions

$$
V_m(z)V_m^*(z) = I_m, \quad \text{for a.a. } z \in \mathbb{T},
$$

constructed recurrently as follows: let $F_m(z)$ be the $m \times m$ matrix of the form (9), where its last row coincides with the last row of $M_{m-1}^{[m,m]}(z)$, and
let $U_m(z)$ be the corresponding unitary matrix-function which existence is proved in Lemma 2, so that (11) and (12) hold. Define $V_m(z)$ as the block matrix-function

$$V_m(z) = \begin{pmatrix} U_m(z) & 0 \\ 0 & I_{r-m} \end{pmatrix},$$

$m = 2, 3, \ldots, r$. It is assumed that $V_r(z) = U_r(z)$. Obviously $V_m(z)$ is unitary matrix-function and (see (11))

$$\det V_m(z) = \det U_m(z) = 1 \text{ for a.a. } z \in \mathbb{T}.$$

Pay attention that the following equation holds

$$M_m \left[ m^{-1}, m^{-1} \right] (z) \cdot F_m(z) = M_m \left[ m, 1 \right] (z),$$

while

$$M_m^\left[ m^{-1}, m^{-1} \right] (z) \in L^+_2(\mathbb{T})$$

for $m = 2$ (see (23), (22)) and for $m > 2$ as well, because of the construction process (see (30) below). Indeed, by virtue of (23) and (25), we have

$$M_m \left[ m, m \right] (z) = M_m \left[ m^{-1}, m^{-1} \right] (z) U_m(z), \quad m = 2, 3, \ldots, r.$$

Thus, it follows from (29), (27), (28), and (12) that

$$M_m^\left[ m, m \right] (z) = \begin{pmatrix} M_m^\left[ m^{-1}, m^{-1} \right] (z) & 0 \\ 0 & 1 \end{pmatrix} F_m(z) U_m(z) \in L^+_2(\mathbb{T}).$$

For $m = r$, we have

$$M_r(z) \in L^+_2(\mathbb{T}),$$

and we conclude that $M_r(z)$ is a spectral factor of $S(z)$,

$$\chi^+(z) = M_r(z).$$

Indeed, since (21), (23), and (24) hold, we have

$$S(z) = M_r(z)(M_r(z))^* \text{ for a.a. } z \in \mathbb{T},$$

and it remains to show that $\det M_r(z)$ is outer, where $M_r(z)$ is assumed extended in the unit disk $D$ in this case, by virtue of (31).

The equations in (23) imply that

$$M_r(z) = M(z) U_2(z) U_3(z) \cdots U_r(z) \text{ for a.a. } z \in \mathbb{T}.$$
Hence, taking into account (33), (22), and (29), we have
\[
\det M_r(z) = f_1^+(z)f_2^+(z)\ldots f_r^+(z) \text{ for a.a. } z \in \mathbb{T},
\]
(34)

The both sides of (34) are functions from $H_{2/r}$ and they coincide on the boundary almost everywhere. Thus the equation in (34) is valid inside the unit circle for each $z \in D$, and since each $f_j^+(z)$ is outer, $j = 1, 2, \ldots, r$, their product $\det M_r(z)$ is outer as well.

The proof of the relation (32) is completed.

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