Comparison of metrics obtained with analytic perturbation theory and a numerical code

J. E. Cuchí*, A. Molina† and E. Ruiz*

*Dpto. Física Fundamental. Universidad de Salamanca
†Dpto. de Física Fonamental. Universitat de Barcelona

Abstract. We compare metrics obtained through analytic perturbation theory with their numerical counterparts. The analytic solutions are computed with the CMMR post-Minkowskian and slow rotation approximation due to Cabezas et al. [1] for an asymptotically flat stationary spacetime containing a rotating perfect fluid compact source. The same spacetime is studied with the AKM numerical multi-domain spectral code [2, 3]. We then study their differences inside the source, near the infinity and in the matching surface, or equivalently, the global character of the analytic perturbation scheme.

Keywords: Relativistic astrophysics, post-Minkowskian approximation, Harmonic coordinates, Rotating stars

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INTRODUCTION

Despite the great effort invested, there is still no exact solution of Einstein’s equations able to describe a stellar model, i.e., a singularity-free rotating body that has been matched to an asymptotically flat vacuum exterior. In the last two decades, the attention has moved to the field of approximate solutions. Among the recent ones is the AKM code [2, 3]. It is a multi-domain spectral method, and the difficulties many other codes have on surfaces of discontinuity of some sources due to Gibbs phenomena are solved computing the solution of the different domains and then imposing matching conditions. The number \( n \) of Chebyshev polynomials in the expansions, resolution and equation of state (EOS) can be chosen, and reaches machine accuracy for high enough \( n \). We will use it to check the behaviour of the CMMR post-Minkowskian+slow rotation analytic approximation scheme some of us introduced in [1]. We begin fixing the properties of the problem spacetime while briefing the CMMR basics. Then we study the relative error in the metric functions between schemes and some physical properties of the source.

BUILDING THE METRICS

The spacetime studied \( \mathcal{V} \) is stationary, with timelike Killing vector field \( \xi \), and axisymmetric, being \( \eta \) the associated closed-orbits spacelike Killing vector field that satisfies regularity condition near the axis. It is built from the matching of two spacetimes. The first one, \( \mathcal{V}^- \), is filled with a perfect fluid in circular flow so that its velocity can be written \( u = \psi (\xi + \omega \eta) \), with \( \psi \) adjusted to make \( u^\alpha u_\alpha = -1 \). The function \( \omega \) is constant, making the fluid rigidly rotating. It has constant energy density, \( \mu = \mu_0 \), so integrating...
Euler’s equations gives the pressure $p = \mu_0 \left( \left( \psi / \psi_{\Sigma} \right) - 1 \right)$, with $\psi = \psi_{\Sigma}$ for $p = 0$. The second spacetime, $\mathcal{V}^+$, is asymptotically flat vacuum surrounding $\mathcal{V}^-$. It is not restrictive for us to use global harmonic Cartesian-like coordinates $\{x^\alpha\}$ in $\mathcal{V}[4]$. Working in spherical-like coordinates associated to $\{x^\alpha\}$, the $p = 0$ surface $\Sigma$ on which $\mathcal{V}^-$ and $\mathcal{V}^+$ are matched can be written as an expansion $r_{\Sigma} = r_s \left[ 1 + \sigma \Omega^2 P_2(\cos \theta) \right] + \mathcal{O}(\Omega^4)$ in Legendre polynomials $P_n$, with $\sigma$ a constant. It has been truncated introducing a slow rotation approximation parameter $\Omega$ we have chosen as $\Omega^2 = \omega^2 r_s^3 / m$, where $m \equiv \frac{4}{3} \pi \mu_0 r_s^3$ is the Newtonian mass of a sphere of radius $r_s$.

To solve Einstein’s equations, we use a multipolar post-Minkowskian approximation as follows. Defining a parameter $\lambda = m / r_s$, the exact metric in each spacetime $g^{\pm}$ is decomposed as $g^{\pm}(\lambda, \Omega) = \eta + h^{\pm}(\lambda, \Omega)$, with $\eta$ the flat metric. Then, Einstein’s equations are solved iteratively in $\lambda$. Both $h^{\pm}(\lambda, \Omega)$ are tensor spherical harmonic expansions that are truncated, in this case, to contain $\Omega$ powers lower than $\Omega^4$. This restricts the number of multipole moments $M_i, J_{i+1}$ appearing in the exterior solution. We then match $\mathcal{V}^-$ and $\mathcal{V}^+$ imposing continuity of the metric and its first derivatives. This fixes all coefficients and the stellar model depends then only on $\mu_0$, $\omega$ and $r_s$.
The AKM code computes the matched value of the functions $U, k, W$ and $a$ in the
general line element of a stationary axisymmetric perfect fluid or asymptotically flat
vacuum spacetime in quasi-isotropic coordinates (see, e.g. [2], where $\{\rho, \zeta\}$ are cylin-
drical associated to quasi-isotropic coordinates $\{r, \theta\}$) at each point of a coordinate grid
of user-definable resolution. It also gives a lot of information in terms of physical and
geometric parameters, such as multipolar moments $M_0$ and $J_1$, baryonic mass $M_B$, an-
gular velocity $\omega$, equatorial radius $r_e$ and central pressure $p_c$ among others. Once the
values of two of them and the EOS have been fixed, the code can compute the metric.

**COMPARISON RESULTS**

To compare the results of CMMR and AKM for $\mu = \mu_0$, we must first find the change
of coordinates from the spherical-like ones of CMMR to quasi-isotropic. This change
is necessarily approximate, introducing a new source of error in the comparison. This
makes the relative error we compute between the metric functions of each scheme at
a point to be a strict upper bound. For the comparison, CMMR was computed up
to order $O(\lambda^{5/2}, \Omega^3)$, and AKM was set to use 12 Chebyshev polynomials in
each direction. Then, working in dimensionless quantities ($G = c = \mu_0 = 1$) we must
choose which two parameters to fix in both CMMR and AKM. For this work, we have
dealt with two sets, first $\{M_0, \omega\}$ and then $\{r_e, \omega\}$. Once $\omega$ is fixed, CMMR results
depend only on $r_e$. We get its value equating both $M_0$ (alternatively, $r_e$) values. The
$M_0$ adjustment gives better results and is the one we will focus on. Figs. 1-2 show the
relative error in $g_{tt}$ and $g_{t\phi}$ ($g_{ii}$ plots are very similar to $g_{tt}$ ones) on a quadrant of the
plane $\rho - \zeta$ for $\omega = 0.2$ and AKM values of $M_B = 8 \times 10^{-3}, 8 \times 10^{-4}$ and $8 \times 10^{-3}$.
For a typical neutron star density $\mu_0 = 4 \times 10^{17}$ kg m$^{-3}$, they would correspond to a
frequency $\nu \approx 1033$ s$^{-1}$ and $M_0 \approx 0.003M_\odot, 0.03M_\odot$ and $0.3M_\odot$, respectively. Table 1
shows their CMMR values and relative errors of some quantities. The rather extreme
cases of $\omega = 0.7$ are included to check the behaviour of our slow rotation approximation

**FIGURE 2.** Relative error between CMMR and AKM in $g_{tt}$ and $g_{t\phi}$ for $M_B = 8 \times 10^{-3}, \omega = 0.2$ ($\lambda \approx 0.056, \Omega \approx 0.098$). (Continued from Fig. 1)
TABLE 1. CMMR values of some quantities and relative error with AKM $\epsilon = \frac{|CMMR - AKM|}{AKM}$ for two members ($\omega = 0.2$ and $\omega = 0.7$) from each studied sequence: $M_B = 8 \times 10^{-5}$, $8 \times 10^{-4}$ and $8 \times 10^{-3}$

| CMMR $\omega$ | CMMR $\omega$ | CMMR $\omega$ |
|---------------|---------------|---------------|
| 0.2           | 0.7           | 0.2           |
| $M_0$         | 0.00007985    | 0.00007986    |
| $J_1$         | 4.6065620 e-9 | 0.00020       |
| $r_e$         | 0.02674082    | 0.012         |
| $p_c$         | 0.00147636    | 0.02663511    |
| $r_s$         | 0.00079362    | 0.00769003    |
| $J_1$         | 8.10712355 e-7 | 0.018       |
| $r_e$         | 0.05940141    | 0.0065        |
| $p_c$         | 0.00620181    | 0.044         |
| $r_s$         | 0.05667105    | 0.11620612    |

For high values of $\Omega$ ($\Omega \approx 0.49\omega$).

For the three cases studied, relative errors in metric functions increase roughly two orders of magnitude if we make $M_B$ ten times bigger, being higher for $g_{tt\varphi}$ and the interiors. This is expectable since we have fixed $M_0$, i.e. the behaviour of $g_{tt}$ near spatial infinity, what can cause the high values of $\epsilon(p_c)$ as well. Significant error discontinuities are located on equatorial/polar lobes (Fig. 1b) and can be caused by the truncation at $O(\Omega^3)$ of $h^\pm$ and the Legendre expansion of $\Sigma$. This is supported by the smooth plots we get in the static limit with $r_e$ adjustment, and the increased lobular appearance when $M_B$ decreases (giving rise to more oblate configurations for the same $\omega$). We expect this angular dependence of $\epsilon$ to decrease including more terms of the $\Omega$ series.

The error inside the source is systematically bigger than outside it, but comparable. We plan to use $p_c$ and $J_1$ to fix $r_s$ and expect better results in the interior. We will add new physical quantities to the comparison to see how much the general performance improves going further in the approximation as well as other EOS.

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