Finite and countable infinite products of
Probabilistic Normed Spaces

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Abstract

In this work we first give for PN spaces results parallel to those obtained by Egbert for the product of PM spaces, and generalize the results by Alsina and Schweizer in [2] in order to study non-trivial products and the product of $m$–transforms of several PN spaces. In addition we present a detailed study of $\alpha$–simple product PN spaces and, finally, the product topologies in PN spaces which are products of countable families of PN spaces.

KEY WORDS: Probabilistic Normed spaces; probabilistic norms; triangle functions; dominates; $t$–norm; $m$–transform; $\tau$–Product and $\Sigma$–Product; strong topology.

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1 INTRODUCTION

We assume that the reader is acquainted with the basic notions of the theory of PN spaces. These, as well as terms and concepts not defined in the body of this paper, may be found in [1, 2, 3, 5, 6].

DEFINITION 1. A probabilistic metric space (henceforth and briefly, a PM space) is a triple $(S, \mathcal{F}, \tau)$ where $S$ is a nonempty set (whose elements are
the points of the space), \( F \) is a function from \( S \times S \) into \( \Delta^+ \), \( \tau \) is a triangle function, and the following conditions are satisfied for all \( p, q, r \) in \( S \):

- \( F(p, p) = \epsilon_0 \)
- \( F(p, q) \neq \epsilon_0 \) if \( p \neq q \)
- \( F(p, q) = F(q, p) \)
- \( F(p, r) \geq \tau(F(p, q), F(p, r)) \).

If \( (S, F, \tau) \) is a PM space, then we also say that \( (S, F) \) is a PM space under \( \tau \).

**DEFINITION 2.** A probabilistic Normed Space, briefly a PN space, is a quadruple \((V, \nu, \tau, \tau^*)\) in which \( V \) is a linear space, \( \tau \) and \( \tau^* \) are continuous triangle functions with \( \tau \leq \tau^* \) and \( \nu \), the probabilistic norm, is a map \( \nu : V \to \Delta^+ \) such that

\[(N1) \quad \nu_p = \epsilon_0 \text{ if, and only if, } p = \theta, \theta \text{ being the null vector in } V; \]
\[(N2) \quad \nu_{-p} = \nu_p \text{ for every } p \in V; \]
\[(N3) \quad \nu_{p+q} \geq \tau(\nu_p, \nu_q) \text{ for all } p, q \in V; \]
\[(N4) \quad \nu_p \leq \tau^*(\nu_{\alpha p}, \nu_{(1-\alpha)p}) \text{ for every } \alpha \in [0, 1] \text{ and for every } p \in V. \]

If, instead of (N1), we only have \( \nu_\theta = \epsilon_0 \), then we shall speak of a **Probabilistic Pseudo Normed Space**, briefly a PPN space. If the inequality (N4) is replaced by the equality \( \nu_p = \tau_M(\nu_{\alpha p}, \nu_{(1-\alpha)p}) \), then the PN space is called a Šerstnev space and, as a consequence, a condition stronger than (N2) holds, namely

\[ \forall \lambda \neq 0 \forall p \in V \quad \nu_{\lambda p} = \nu_p \left( \frac{j}{|\lambda|} \right) \]

Here \( j \) is the identity map on \( \mathbb{R} \), i.e. \( j(x) := x (x \in \mathbb{R}) \).

A Šerstnev space is denoted by \((V, \nu, \tau)\).

There is a natural topology in a PN space \((V, \nu, \tau, \tau^*)\), called the **strong topology**, it is defined, for \( t > 0 \), by the neighbourhoods

\[ \mathcal{I}_p(t) := \{ q \in V : d_S(\nu_{q-p}, \epsilon_0) < t \} = \{ q \in V : \nu_{q-p}(t) > 1 - t \}. \]
By setting \( F \leq G \) whenever \( F(x) \leq G(x) \) for every \( x \in \mathbb{R}^+ \) and \( F, G \in \Delta^+ \), one introduces a natural ordering in \( \Delta^+ \).

**DEFINITION 3.** Let \( (V, \| \cdot \|) \) be a normed space and let \( G \in \Delta^+ \) be different from \( \epsilon_0 \) and \( \epsilon_{+\infty} \); define \( \nu : V \to \Delta^+ \) by \( \nu_0 = \epsilon_0 \) and

\[
\nu_p(t) := G \left( \frac{t}{\| p \|^{\alpha}} \right)
\]

where \( \alpha > 0 \) and \( \alpha \neq 1 \). Then the pair \( (V, \nu) \) will be called the \( \alpha \)-simple space generated by \( (V, \| \cdot \|) \) and by \( G \).

**DEFINITION 4.** Let \( \tau_1, \tau_2 \) be two triangle functions. Then \( \tau_1 \) dominates \( \tau_2 \), and we write \( \tau_1 \gg \tau_2 \), if for all \( F_1, F_2, G_1, G_2 \in \Delta^+ \),

\[
\tau_1(\tau_2(F_1, G_1), \tau_2(F_2, G_2)) \geq \tau_2(\tau_1(F_1, F_2), \tau_1(G_1, G_2))
\]

Notice that since \( \tau_1 \) is associative one has \( \tau_1 \gg \tau_1 \), so that “dominates” is reflexive but its transitivity is still an open question.

**DEFINITION 5.** Given a left-continuous \( t \)-norm \( T \), i.e., a left-continuous binary operation on \([0, 1] \) that is commutative, nondecreasing in each variable and has 1 as identity, a triangle function \( T \) is the function defined via

\[
T(F, G)(x) := T(F(x), G(x)).
\]

**DEFINITION 6.** For \( b \in ]0, +\infty[ \), let \( M_b \) be the set of all continuous and strictly increasing functions \( m \) from \([0, b[ \) onto \( \mathbb{R}^+ = [0, \infty[ \). For any \( F \) in \( \Delta^+ \) and any \( m \) in \( M_b \) let \( Fm \) be the functions on \( \mathbb{R}^+ \) given by

\[
(Fm)(x) = \begin{cases} 
F(m(x)), & x \in [0, b[, \\
\lim_{x \to b^-} F(m(x)), & x = b, \\
1, & x > b,
\end{cases}
\]

if \( b \in ]0, +\infty[ \), and by \( (Fm)(x) = F(m(x)) \) for all \( x \geq 0 \) if \( b = +\infty \). If \( b \in ]0, +\infty[ \), then \( Fm \) is in \( D^+ \) and \( Fm \geq \epsilon_b \). If \( F \) itself is in \( D^+ \), then \( Fm \) is continuous at \( b \). Also, \( \varepsilon_t m = \epsilon_{m^{-1}(t)} \) for any \( t \) in \( \mathbb{R}^+ \).
DEFINITION 7. Let $\tau$ be a triangle function and let $m$ belong to $M_b$. Then $m$ is $\tau$–superadditive if, for all $F, G$ in $\Delta^+$,

$$\tau(F, G)m \geq \tau(Fm, Gm).$$

The function $Fm$ is called the $m$–transform of $F$.

The following results Lemma 1, 2, 3 and Theorem 1 can be seen in [2].

Lemma 1. If $\tau$ is a triangle function such that for all $s, t$ in $\mathbb{R}^+$,

$$\tau(\epsilon_s, \epsilon_t) = \epsilon_{s+t},$$

and if $m$ in $M_b$ is $\tau$– superadditive, then $m$ is superadditive, i.e., for all $x, y \in [0, b]$,

$$m(x + y) \geq m(x) + m(y).$$

Theorem 1. Let $T$ be a continuous $t$-norm and let $m \in M_b$. Then $m$ is $\tau_T$–superadditive if and only if $m$ is superadditive (see [2], Theorem 3).

Lemma 2. The $t$-norm $W$ satisfies the following relationship:

$$W \left( \sum_{i=1}^{\infty} \frac{a_i}{2^i}, \sum_{i=1}^{\infty} \frac{b_i}{2^i} \right) \leq \sum_{i=1}^{\infty} \frac{1}{2^i} W(a_i, b_i),$$

for every $(a_i), (b_i) \in [0, 1]$.

Lemma 3. If $a_i, b_i \in [0, 1]$ for every $i \in \mathbb{N}$, the inequality

$$W^\ast \left( \sum_{i=1}^{\infty} \frac{a_i}{2^i}, \sum_{i=1}^{\infty} \frac{b_i}{2^i} \right) \geq \sum_{i=1}^{\infty} \frac{1}{2^i} W^\ast(a_i, b_i)$$

where $W^\ast(x, y) := M\{x + y, 1\}$, holds.
2 Finite $\tau$–Products of PN spaces

In this section, we give our definition of a $\tau$–product of two probabilistic normed spaces which is a generalization of a parallel result from Egbert about the $\tau$–product of PM spaces. Moreover we show a necessary and sufficient condition for the $\tau$–product of two general PN spaces of Šerstnev to be a Šerstnev space as well as a sufficient condition for the $\tau$–product of two Menger PN spaces to be also a PN space of Menger.

The proof of most theorems is omitted since it is just a matter of straightforward verification of the statements.

**DEFINITION 8.** Let $(V_1, \nu_1, \tau, \tau^*)$ and $(V_2, \nu_2, \tau, \tau^*)$ be two PN spaces under the same triangle functions $\tau$ and $\tau^*$. Let $\tau_1$ be a triangle function. Their $\tau_1$–product is the quadruple

$$(V_1 \times V_2, \nu^{\tau_1}, \tau, \tau^*)$$

where

$$\nu^{\tau_1} : V_1 \times V_2 \rightarrow \Delta^+$$

is a probabilistic seminorm defined by

$$\nu^{\tau_1}(p, q) := \tau_1(\nu_1(p), \nu_2(q))$$

for any $(p, q) \in V_1 \times V_2$.

**Theorem 2.** Let $(V_1, \nu_1, \tau, \tau^*)$, $(V_2, \nu_2, \tau, \tau^*)$ and $\tau_1$ be two PN spaces under the same triangle functions and a triangle function $\tau_1$ respectively. Assume that $\tau^* \gg \tau_1$ and $\tau_1 \gg \tau$, then the $\tau_1$–product $(V_1 \times V_2, \nu^{\tau_1})$ is a PN space under $\tau$ and $\tau^*$.

**Example 1.** The $T$–product $(V_1 \times V_2, \nu^T)$ of the two PN spaces $(V_1, \nu_1, \tau_T, M)$ and $(V_2, \nu_2, \tau_T, M)$ is a PN space under $\tau_T$ and $M$.

**Example 2.** Let $(V_1, F, M)$ and $(V_2, G, M)$ be two equilateral PN spaces with distribution functions $F$, $G$ respectively. Then, their $M$–product is an equilateral PN space with a d.d.f. given by $M(F, G)$. In particular, if $F \equiv G$, the $M$–product is an equilateral PN space with the same distribution function $F$. 

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One may wonder whether the PM space associated with the $\tau_1$–product of two PN spaces characterized in Theorem 2 coincides with the $\tau_1$–product of corresponding PM spaces. The following theorem gives an answer in the affirmative to this question.

**Theorem 3.** Let $(V_1, \nu_1, \tau, \tau^*)$ and $(V_2, \nu_2, \tau, \tau^*)$ be two PN spaces under the same triangle functions $\tau$ and $\tau^*$ and let the triangle function $\tau_1$ be such that $\tau^* \gg \tau_1$ and $\tau_1 \gg \tau$. Let $(V_1, F_1, \tau)$ and $(V_2, F_2, \tau)$ be the same spaces regarded as PM spaces. Then, the $\tau_1$–product $(V_1 \times V_2, \nu^{\tau_1}, \tau, \tau^*)$ regarded as a PM space coincides with the $\tau_1$–product $(V_1 \times V_2, F^{\tau_1})$.

It is known that if $(S_1, F_1)$ and $(S_2, F_2)$ are the simple spaces $(S_1, d_1, G)$ and $(S_2, d_2, G)$, respectively, and if $d_{Max}$ is the metric on $S_1 \times S_2$ defined by

$$d_{Max}((p_1, p_2), (q_1, q_2)) = \text{Max}(d_1(p_1, q_1), d_2(p_2, q_2))$$

then the $M$–product of $(S_1, F_1)$ and $(S_2, F_2)$ is the simple space $(S_1 \times S_2, d_{Max}, G)$ (see [12], p. 211, Theorem 12.7.8). But in principle, if one has two simple PN spaces $(V_1, (\| \cdot \|)_1, G, M)$ and $(V_2, (\| \cdot \|)_2, G, M)$ with the same d.d.f. $G$ its $M$–product is not necessarily a PN space because of the assumption $\tau_{M^*} \gg M$ of Theorem 1 fails here.

Now then, when one replaces $\tau_{M^*}$ by $M$ in these simple PN spaces we obtain the PN spaces $(V_1, \| \cdot \|_1, G, M)$ and $(V_2, \| \cdot \|_2, G, M)$ respectively, as is easily checked. Both of them are Šerstnev. Can the $M$–product of these be a simple PN space?

The following theorem answers that question in the affirmative.

**Theorem 4.** Let $(V_1, \| \cdot \|_1, G, \tau_M, M)$ and $(V_2, \| \cdot \|_2, G, \tau_M, M)$ and $\| \cdot \|_3$ be the two above mentioned PN spaces and the norm defined on $V_1 \times V_2$ by

$$\|\bar{p}\|_3 := \|p_1\|_1 \vee \|p_2\|_2,$$

with $\bar{p} = (p_1, p_2) \in V_1 \times V_2$. Then $(V_1 \times V_2, \| \cdot \|_3, G, \tau_M, M)$ is a simple PN space that coincides with the $M$–product of the given simple spaces. Furthermore it is Šerstnev.
Can a \( \tau \)-product of two simple PN spaces with the same generator function \( G \) be a simple PN space also with \( G \) as generator? The following theorem answers this question in the affirmative.

**Theorem 5.** The \( \tau_M \)-product of two simple PN spaces \((V_1, \| \cdot \|_1, G, M)\) and \((V_2, \| \cdot \|_2, G, M)\) is the simple space under \( M \) generated by \((V_1 \times V_2, \| \cdot \|_s, G, M)\) and the same d.d.f. \( G \), namely, \((V_1 \times V_2, \| \cdot \|_s, G, M)\), where \( \| \cdot \|_s \) is the classic norm defined via

\[
\| \cdot \|_s := \| \cdot \|_1 + \| \cdot \|_2.
\]

\(\Box\)

**Theorem 6.** Let \((V_1, \nu_1, \tau)\) and \((V_2, \nu_2, \tau)\) be two Šerstnev spaces under the same triangle function \( \tau \). Let us assume that \( \tau_1 \) is a triangle function such that \( \tau_1 \gg \tau \), then, their \( \tau_1 \)-product is also a Šerstnev space if, and only if \( \tau_1 \gg \tau_M \) and \( \tau_M \gg \tau_1 \).

**Proof:** By Theorem 1 the \( \tau_1 \)-product of the two Šerstnev spaces exists. Now since both PN spaces are Šerstnev one has, for all \( \alpha \in [0,1] \) and for all \((p, q) \in V_1 \times V_2,\)

\[
\nu^{\tau_1}(p, q) = \tau_1(\nu_1(p), \nu_2(q))
= \tau_1[\tau_M(\nu_1(\alpha p), \nu_1((1-\alpha)p)), \tau_M(\nu_2(\alpha q), \nu_2((1-\alpha)q))]
= \tau_M[\nu^{\tau_1}(\alpha(p, q)), \nu^{\tau_1}((1-\alpha)(p, q))]
= \tau_M[\tau_1(\nu_1(\alpha p), \nu_2(\alpha q)), \tau_1(\nu_1((1-\alpha)p), \nu_2((1-\alpha)q))];
\]
this implies \( \tau_1 \gg \tau_M \) and \( \tau_M \gg \tau_1 \).

Conversely, let \( \tau_1 \gg \tau_M \) and \( \tau_M \gg \tau_1 \); then

\[
\tau_1(\nu_1(p), \nu_2(q))
= \tau_1[\tau_M(\nu_1(\alpha p), \nu_1((1-\alpha)p)), \tau_M(\nu_2(\alpha q), \nu_2((1-\alpha)q))]
= \tau_M[\tau_1(\nu_1(\alpha p), \nu_2(\alpha q)), \tau_1(\nu_1((1-\alpha)p), \nu_2((1-\alpha)q))]
= \tau_M[\nu^{\tau_1}(\alpha(p, q)), \nu^{\tau_1}((1-\alpha)(p, q))],
\]

hence the assertion. \(\Box\)
Corollary 1. Under the same assumptions as in Theorem 5, if \( \tau \equiv \tau_T \) then the \( \tau_1 \)-product \((V_1 \times V_2, \nu_1, \tau_T)\) is a Menger space.

Now, the \( \tau_1 \)-product of two Menger spaces can again be a Menger space. A sufficient condition is provided by the following theorem.

Theorem 7. Let \((V_1, \nu_1, T)\) and \((V_2, \nu_2, T)\) be two Menger PN spaces. If \( T_0 \) is a left-continuous \( t \)-norm that satisfies the conditions \( T^* \gg T_0 \) and \( T_0 \gg T \), then the \( \tau_{T_0} \)-product

\[
(V_1 \times V_2, \nu_1, \nu_2, \tau_T, \tau_{T^*})
\]

is a Menger PN space under \( T \).

Proof: It suffices to apply Lemma 12.7.3 in [12] and Theorem 1. \( \Box \)

Example 3. It is known that for every \( t \)-norm \( T \) one has \( M \gg T \) (this is a result due to R. Tardiff) and it is easily checked that \( T^* \gg M^* \). Then the \( \tau_M \)-product of two Menger PN spaces is also a Menger PN space.

In the next section we study non trivial products.

3 Countable \( \tau \)-Products of PN spaces

We shall need some preliminaries before stating the main results of this section.

Let \((V, \| \cdot \|)\) be a normed space, \( G \in \mathcal{D}^+ \) a strictly increasing continuous d.d.f., \( T \) a strict \( t \)-norm with additive generator \( f \). It is known (see [8]: Lemma 3.2) that when \( \alpha \in ]1, +\infty[ \), then \((V, \| \cdot \|, G; \alpha)\) is a Menger PN space under \( T \) if, and only if, for \( s, t \in ]0, +\infty[ \) and for all \( p, q \in V \) such that \( p \neq \theta, q \neq \theta, p + q \neq \theta \), the inequality

\[
\| p + q \| ^\alpha (f \circ G)^{-1}(s + t) \leq \| p \| ^\alpha (f \circ G)^{-1}(s) + \| q \| ^\alpha (f \circ G)^{-1}(t)
\]

holds.
Let \((V, \| \cdot \|)\) be a normed space and let \(\alpha > 1\). If the d.f. \(G \in \mathcal{D}^+\) is continuous and strictly increasing, then (\[6\]; Section 3) \((V, \| \cdot \|, G; \alpha)\) is a Menger PN space under the strict t-norm defined for all \(x, y\) in \([0, +\infty]\) by

\[
T_G(x, y) := G \left( \left\{ [G^{-1}(x)]^{\frac{1}{1-\alpha}} + [G^{-1}(y)]^{\frac{1}{1-\alpha}} \right\}^{1-\alpha} \right)
\]

One is now ready to state the main results of this section. The following one is the analogue of Theorem 12.7.9 in [12] and shows the relevance of the t-norm \(T_G\) in countable \(\tau\)-products.

But in order to state it one gives a previous definition.

**DEFINITION 9.** The mapping \(\| \cdot \|_\beta\colon V_1 \times V_2 \to \mathbb{R}^+\) is defined for all \(\bar{p} = (p_1, p_2) \in V_1 \times V_2\) via

\[
\| \bar{p} \|_\beta := \left( \| p_1 \|_1^\beta + \| p_2 \|_2^\beta \right)^{\frac{1}{\beta}}.
\]

**Theorem 8.** Let \((V_1, \| \cdot \|_1, G; \alpha)\) and \((V_2, \| \cdot \|_2, G; \alpha)\) be two Menger PN spaces under \(T_G\) and let \(\alpha > 1\). If the d.f. \(G \in \mathcal{D}^+\) is continuous and strictly increasing, then, the \(T_G\)-product \((V_1 \times V_2, \| \cdot \|_\beta, G; \alpha)\) of the given PN spaces is a Menger PN space under \(T_G\).

**Proof:** It suffices to notice that the mapping in definition 9 is a norm for all \(\beta \in [0, +\infty[\). \(\square\)

The space \((V_1 \times V_2, \| \cdot \|_\beta, G; \alpha)\) with \(\beta = \frac{\alpha}{\alpha-1}\) is \(\alpha\)-simple: it suffices to show that

\[
\nu^{T_G}(\bar{p})(j) = T_G \left( G \left( \frac{j}{\| p_1 \|_1^\alpha} \right), G \left( \frac{j}{\| p_2 \|_2^\alpha} \right) \right) =
\]

\[
G \left( \frac{j}{\left( \frac{\| p_1 \|_1^\alpha}{\| p_1 \|_1^\alpha + \| p_2 \|_2^\alpha} \right)^{\alpha-1}} \right) = G \left( \frac{j}{\| \bar{p} \|_\beta^\alpha} \right).
\]

where \(j\) is the identity map on \(\mathbb{R}\)

The analogous theorem for the case \(\alpha \in ]0, 1[\) is an open problem (see \[6\]; Section 3).
We know that if $\alpha > 1$ there exist normed spaces, $(V, \| \cdot \|)$ with the following property: “if $G \in \Delta^+$ is continuous and strictly increasing, then the t-norm $T_G$ is the strongest continuous t-norm under which $(V, \| \cdot \|, G; \alpha)$ is a Menger PN space ” (see [6]; Theorem 3.3). However, a new phenomenon arises in the case of product PN spaces, for contrary to the above, in this case the $t$–norm $T_G$ is not the strongest continuous $t$–norm under which $(V_1 \times V_2, \| \cdot \|_{\beta}, G; \alpha)$ is a Menger PN space, as is easily checked. Such a phenomenon is today an open problem.

In the sequel we study a special kind of probabilistic norms on the countable product of a family of PN spaces.

**Theorem 9.** Let $(V, \nu, \tau_T, T)$ be a PN space and suppose $m$ in $M_b$ is $\tau_T$–superadditive. Let $\nu_m$ be the map defined for any $p$ in $V$ by

$$\nu_m(p) := \nu_pm,$$

and $\nu_p = \nu(p)$. Then the m-transform $(V, \nu_m, \tau_T, T)$ is a PN space.

**Proof:** (N1) and (N2) are evident.

(N3) Since $(V, \nu, \tau_T, T)$ is a PN space and $m$ is superadditive, for any $p$ in $V$, one has

$$\nu_{p+q}m \geq \tau_T(\nu_p, \nu_q)m \geq \tau_T(\nu_pm, \nu_qm).$$

(N4)

$$\nu_pm(x) \leq T(\nu_{\alpha p}, \nu_{(1-\alpha)p}m)(x) = T(\nu_{\alpha p}m(x), \nu_{(1-\alpha)p}m(x)) = T(\nu_{\alpha p}m, \nu_{(1-\alpha)p}m)(x),$$

for every $\alpha \in [0, 1]$, for every $p \in V$ and for every $x \in \mathbb{R}^+$, whence

$$\nu_pm \leq T(\nu_{\alpha p}m, \nu_{(1-\alpha)p}m),$$

for every $\alpha \in [0, 1]$ and for every $p \in V$.

**Corollary 2.** Let $T_1, T_2$ be two $t$-norms such that $T_1 \leq T_2$, then the $m$–transform of any one of PN spaces $(V, \nu, \tau_{T_1}, T_2)$ is a PN space under $\tau_{T_1}$ and $T_2$. 

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Proof: If $T_1 \leq T_2$ one has $\tau_{T_1} \leq \tau_{T_2}$, and now it suffices to apply to the axiom (N4) the well-known inequality $\tau_T \leq T$ for any $t$–norm $T$ (see [12]).

Now let us recall some conventions and results about infinite $\tau$-products. Since the $\tau_T$ operations are associative, for any sequence $F_i \in \Delta^+$, the $n$–fold $\tau_T$–product $\tau^n_T(F_1, \ldots, F_{n+1})$ is well defined for each $n$ as the serial iterates of $\tau_T$, defined recursively via

$$Dom \tau^n_T = (\Delta^+)^{n+1}, \quad \tau^n_T = \tau_T$$

and

$$\tau^{n+1}_T(F_1, \ldots, F_{n+1}, F_{n+2}) = \tau_T(\tau^n_T(F_1, \ldots, F_{n+1}), F_{n+2}).$$

If $T$ is a continuous $t$–norm, then it is well known that

$$\lim_{n \to +\infty} \tau^n_T(F_1, \ldots, F_{n+1})(x) = \sup\{\lim_{n \to +\infty} T^n(F_1(x_1), \ldots, F_{n+1}(x_{n+1}))\},$$

where the supremum is taken with respect to all sequences $\{x_n\}$ of positive numbers such that $\sum_{i=1}^{\infty} x_n = x$.

Furthermore $\tau^\infty_T$ is defined on sequences $\{F_n\}$ in $\Delta^+$ by

$$\tau^\infty_T F_n(x) = t^- \left( \lim_{n \to +\infty} \tau^n_T(F_1, \ldots, F_{n+1})(x) \right).$$

DEFINITION 10. Let $\{(V_i, \nu^i, \tau_T, T)| i \in \mathbb{N}\}$ be a countable family of proper PN spaces, i.e., PN spaces with a continuous triangle function $\tau$ that satisfies $\tau(\epsilon_s, \epsilon_t) \geq \epsilon_{s+t}$. The function $\tau_T$ is one of these. Let $b_i$ be an infinite sequence of positive numbers such that the series $\sum_{i=1}^{\infty} b_i$ converges. For each $i \in \mathbb{N}$ one chooses a function $m_i \in M_{b_i}$ that is $\tau_T$–superadditive. Let $\{(V_i, G^i, \nu^i m_i, \tau_T, T)| i \in \mathbb{N}\}$ be the $m_i$–transform of the family given before, in which the map $G^i : V_i \to \mathbb{R}^+$ is defined by $G^i_{p_i} := \nu^i m_i$. We adopt the convention $V = \prod_{i=1}^{\infty} V_i$. Now, for any sequence $\bar{p} = (p_i) \in \prod_{i=1}^{\infty} V_i$ for all $i \in \mathbb{N}$, we set $G_{\bar{p}} = \tau^\infty_T G^i_{p_i}$.

Lemma 4. The function $G_{\bar{p}}$ defined as in Definition 7 is in $\mathcal{D}^+$. 11
Proof: The proof in [2] only needs to be supplemented by the new notation of PN spaces with respect to the PM spaces as follows:
For any positive integer \( n \), let \( \sigma_n = \sum_{i=1}^{n} b_i \) and let \( \sigma = \sum_{i=1}^{\infty} b_i \). Then, one has
\[
\tau_T^n(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \geq \epsilon_{\sigma_n} > \epsilon_{\sigma},
\]
whence \( \tau_T^\infty \epsilon_{b_i} \geq \epsilon_{\sigma} \). Consequently, since \( G_{p_i} = G_{\nu_i} \geq \epsilon_{b_i} \) for all \( i \in \mathbb{N} \), one has \( G_p = \tau_T^\infty G_{p_i} \geq \tau_T^\infty \epsilon_{b_i} \geq \epsilon_{\sigma} \). Thus \( G_p(x) = 1 \) for \( x > \sigma \). \( \Box \)

According to the result in Lemma 4 about the non trivial limit of the infinite \( \tau_T \)-product, one has that \( G_{\alpha p} = G_{(1-\alpha) p} \) are in \( D^+ \) for every \( \alpha \in [0,1] \).

Example 4. (A particular countable, but finite, \( \tau \)-product) Let \((V, G)\) be the product of the Definition 8 for \( i = 1, 2 \), then \((V_1 \times V_2, \tau_T(\nu_{p_1}^1 m_1, \nu_{p_2}^2 m_2), T, T)\) is a PN space.
To prove this it suffices to check axiom (iv): By Theorem 9 and since \( T \gg \tau_T \) for every \( t \)-norm \( T \) (see [12]) one has
\[
G_p = \tau_T(\nu_{m_1}^1, \nu_{m_2}^2) \leq \tau_T(T(\nu_{(1-\alpha)p_1}^1 m_1, \nu_{(1-\alpha)p_2}^1 m_1), T(\nu_{(1-\alpha)p_2}^2 m_2)) \leq T(\tau_T(\nu_{(1-\alpha)p_1}^1 m_1, \nu_{(1-\alpha)p_2}^1 m_1), \tau_T(\nu_{p_1}^1 m_1, \nu_{p_2}^2 m_2)) = T(G_{\alpha p}, G_{(1-\alpha) p}). \quad \Box
\]

DEFINITION 11. Let \( \{(V_i, \nu_i^{m_i}, T, T)\} \) be a countable family of proper PN spaces. Then the \( T \)-product of this family is the pair \((V, G)\), where \( V \) and \( G \) are the same as in previous definition.

Just now one has the following question: Is the product \((V, G)\) a PN space under \( \tau_T \) and \( \tau_{T^*}\) ?
First of all, can any member of the family \((V_i, \nu_i^{m_i}, \tau_T, \tau_{T^*})\) be a PN space? We answer that question in the negative because of the axiom (iv). Axiom (iii) works with \( m \) superadditive, but axiom (iv) needs \( m \) to be subadditive. The same function \( m \) must to appear in second and third terms of the following chain of inequalities
\[
\nu_p m \leq \tau_{T^*}(\nu_{op}, \nu_{(1-\alpha)p}) m \leq \tau_{T^*}(\nu_{op} m, \nu_{(1-\alpha)p} m),
\]
and it would be meaningful that \( m \) is \( \tau_{T^*} \)-subadditive, whatever convergence factor one introduces in \( \tau_{T^*} \), which is absurd.
In order to make the reader’s task easier the following result in PN spaces will be needed, for the last section of this paper.

**Theorem 10.** Let \( \{ (V_i, \nu^{i m_i}, T, T) \} \) be a countable family of proper PN spaces. Then the pair \((V, G)\) is a PN space under \(T\).

**Proof:** Clearly, for every \(\bar{p}, \bar{q}\) in \(V\), \(G\bar{p} = \epsilon_0\) if, and only if \(\bar{p} = \theta\); and \(G\bar{p} = G_{\bar{p}}\).

(N3) By Lemma 4

\[
G_{\bar{p}+\bar{q}} = T^\infty G_{p_i+q_i} \geq T^\infty (T(G_{p_i}, G_{q_i})) = T(T^\infty G^i_{p_i}, T^\infty G^i_{q_i}) = T(G_{\bar{p}}, G_{\bar{q}})
\]

(N4) By Lemma 4

\[
G_{\bar{p}} \geq \epsilon_{\sigma}
\]

for every \(\bar{p} \in V\). Moreover,

\[
G_{\bar{p}} = T^\infty \nu^i_{p_i} m_i \leq T^\infty (T(\nu^i_{\alpha p_i}, \nu^i_{(1-\alpha)p_i}) m_i) = T(T^\infty \nu^i_{\alpha p_i} m_i, T^\infty \nu^i_{(1-\alpha)p_i} m_i) = T(G_{\alpha \bar{p}}, G_{(1-\alpha)\bar{p}}).
\]

\(\square\)

**Example 5.** Every member of the family \(\{ (V_i, \nu_i, \Pi, \Pi) \}\), with the map \(\nu_{p_i}\) defined via \(\nu_{p_i} := \exp (- \| p_i \|)\) is a PN space as it is easy to check. Then, the family \(\{ (V_i, \nu^{i m_i}, \Pi, \Pi) \}\) of \(m_i\)-transforms of the members of the above family is also a family of PN spaces, and his \(\Pi\)-product \((V, G)\) is a PN space under \(\Pi\), whence there exist countable infinite products of PN spaces.

**DEFINITION 12.** Let \(\{ (V_i, \nu^i, \tau_i, \tau^+_i) \mid i \in \mathbb{N} \}\) be a countable family of PN spaces. The \(\Sigma\)-product of this family is the pair \((\prod_{i=1}^{\infty} V_i, \nu^\Sigma)\) where \(\nu^\Sigma : \prod_{i=1}^{\infty} V_i \to \Delta^+\) is a map given by

\[
\nu^\Sigma_{\bar{p}} := \sum_{i=1}^{\infty} 2^{-i} \nu^i_{p_i},
\]

for every sequence \((p_i) = p \in \prod_{i=1}^{\infty} V_i\).

In order to simplify the notation we replace henceforth \(\nu^\Sigma_{\bar{p}}\) by \(\nu_{\bar{p}}\).
Theorem 11. Let \( \{(V_i, \nu_i, \tau_i, \tau^*_i) \mid \in \mathbb{N}\} \) be a countable family of PN spaces and let \( \tau_i \geq \tau_W \) and \( \tau^*_i \leq \tau^*_W \) for all \( i \in \mathbb{N} \), then the \( \Sigma \)-product of this family denoted by

\[
\left( \prod_{i=1}^{\infty} V_i, \nu^\Sigma, \tau_W, \tau_W^* \right)
\]

is a Menger space under \( W \).

The proof is similar to the one in Alsina [2]. It suffices to apply Lemma 1 and Lemma 2.

4 Product topology for countable \( \tau \)-products

In this section we want to show that the product topology and the strong topology in countable infinite \( \tau \)-products are not equal. Let us recall that this is what happened with the same type of \( \tau \)-products of PM spaces.

Theorem 12. Let each of the PN spaces \( (V_i, \nu^i, \tau_i, \tau^*_i) \) be endowed with the strong topology corresponding to \( \nu^i, i \in \mathbb{N}, \) and \( \Delta^+ \) with the topology of weak convergence. Then the product topology is weaker than the strong topology in \( (V, G) \).

Proof: Let \( U = \prod_{i=m}^{n} N_{p_i}(\epsilon_i) \times \prod_{i=m+1}^{\infty} V_i \) be a standard neighborhood in the product topology. Choose \( \epsilon = \min\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\} \) and let \( \bar{q} \in N_{p}(\epsilon) \). Then, since \( G_{\bar{p} \bar{q}} \leq \nu_{q_i - p_i}^i \) for all \( i \in \mathbb{N}, \) one has

\[
1 - \epsilon_i \leq 1 - \epsilon < G_{\bar{p} \bar{q}}(\epsilon) \leq \nu_{q_i - p_i}^i(\epsilon) \leq \nu_{q_i - p_i}^i(\epsilon_i),
\]

whence

\[
N_{p}(\epsilon) \subset \prod_{i=1}^{\infty} N_{p_i}(\epsilon) \subset U.
\]

In general, the two topologies are not equal. For, if this were the case, given \( N_{p}(\epsilon) \) there would exist a product neighborhood \( U = \prod_{i=m}^{n} N_{p_i}(\epsilon_i) \times \prod_{i=m+1}^{\infty} V_i \) such that \( U \subset N_{p}(\epsilon) \subset \prod_{i=1}^{\infty} N_{p_i}(\epsilon) \), which implies that \( V_i = N_{p_i}(\epsilon) \) for all \( i > m \), a very strong condition.
5 Product topology for $\Sigma$–products

Contrary to what happens with countable infinite $\tau$–products, in $\Sigma$–products the two topologies are equal.

Let us recall that if $(V, \nu, \tau_W)$ is a PM space with $\tau_M$ uniformly continuous, then for the strong neighborhoods $N_p(t)$, $p \in V$, $t > 0$, the following statements hold:

- If $q \in N_p(p)$, there exists a $t' > 0$ such that $N_q(t') \subset N_p(t)$
- If $p \neq q$, there exists a $t > 0$ such that $N_p(t) \cap N_q(t) = \emptyset$

The proof of the following theorem is similar to that of Theorem 1.4 in [3]: only small changes in the notations are needed.

**Theorem 13.** Let $\{(V_i, \nu^i, \tau_i, \tau_{W^i})|i \in \mathbb{N}\}$ and $V, \nu^\Sigma$ be as in theorem 8. Let each $V_i$ be endowed with the strong topology induced by $\nu^i$. Then the strong topology on $V$ induced by $\nu^\Sigma$ is the product topology.

The reason for the difference between Theorem 11 and 12 is easily understood if one pays attention to the probabilistic interpretation of the $\epsilon$–neighborhood in the respective products spaces:

If $N_p(\epsilon)$ is a neighborhood in the $\tau$–product, and $\bar{q} \in N_p$ then, with probability greater than $1 - \epsilon$, all the components $p_i$ of $p$ are at a distance (the one associated to the norm in $V_i$) less than $\epsilon$ from the corresponding $q_i$. On the other hand, if $N_p(\epsilon)$ is a neighborhood in the $\Sigma$–product then $\bar{q} \in N_p$ implies that, with probability greater than $1 - \epsilon$, at least one of the components $p_i$ of $p$ is at a distance less than $\epsilon$ from the corresponding $q_i$.

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