Morphing Planar Graph Drawings Efficiently

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Abstract. A morph between two straight-line planar drawings of the same graph is a continuous transformation from the first to the second drawing such that planarity is preserved at all times. Each step of the morph moves each vertex at constant speed along a straight line. Although the existence of a morph between any two drawings was established several decades ago, only recently it has been proved that a polynomial number of steps suffices to morph any two planar straight-line drawings. Namely, at SODA 2013, Alamdari \textit{et al.} \cite{Alamdari13} proved that any two planar straight-line drawings of a planar graph can be morphed in \(O(n^4)\) steps, while \(O(n^2)\) steps suffice if we restrict to maximal planar graphs.

In this paper, we improve upon such results, by showing an algorithm to morph any two planar straight-line drawings of a planar graph in \(O(n^2)\) steps; further, we show that a morphing with \(O(n)\) steps exists between any two planar straight-line drawings of a series-parallel graph.

1 Introduction

A planar morph between two planar drawings of the same plane graph is a continuous transformation from the first drawing to the second one such that planarity is preserved at all times. The problem of deciding whether a planar morph exists for any two drawings of any graph dates back to 1944, when Cairns \cite{Cairns44} proved that any two straight-line drawings of a maximal planar graph can be morphed one into the other while maintaining planarity. In 1981, Grünbaum and Shephard \cite{GrunbaumShephard81} introduced the concept of linear morph, that is a continuous transformation in which each vertex moves at uniform speed along a straight-line trajectory. With this further requirement, however, planarity cannot always be maintained for any pair of drawings. Hence, the problem has been subsequently studied in terms of the existence of a sequence of linear morphs, also called morphing steps, transforming a drawing into another while maintaining planarity. The first result in this direction is the one of Thomassen \cite{Thomassen85}, who proved that a sequence of morphing steps always exists between any two straight-line drawings of the same plane graph. Further, if the two input drawings are convex, this property is maintained throughout the morph, as well. However, the number of morphing steps used by the algorithm of Thomassen might be exponential in the number of vertices.

Recently, the problem of computing planar morphs gained increasing research attention. The case in which edges are not required to be straight-line segments has been addressed in \cite{Bose11}, while morphs between orthogonal graph drawings preserving planarity and orthogonality have been explored in \cite{Bose12}. Morphs preserving more general edge directions have been considered in \cite{Bose13}. Also, the problem of “topological morphing”, in which the planar embedding is allowed to change, has been addressed in \cite{Brinkmann13}.

In a paper appeared at SODA 2013, Alamdari \textit{et al.} \cite{Alamdari13} tackled again the original setting in which edges are straight-line segments and linear morphing steps are required. Alamdari \textit{et al.} presented the first morphing algorithms with a polynomial number of steps in this setting. Namely, they presented an algorithm to morph straight-line planar drawings of maximal plane graphs with \(O(n^2)\) steps and of general plane graphs with \(O(n^4)\) steps, where \(n\) is the number of vertices of the graph.

In this paper we improve upon the result of Alamdari \textit{et al.} \cite{Alamdari13}, providing a more efficient algorithm to morph general plane graphs. Namely, our algorithms uses \(O(n^2)\) linear morphing steps. Further, we provide a morphing algorithm with a linear number of steps for a non-trivial class of planar graphs, namely series-parallel graphs. These two main results are summarized in the following theorems.

\footnotesize{\textsuperscript{*} Part of the research was conducted in the framework of ESF project 10-EuroGIGA-OP-003 GraDR “Graph Drawings and Representations”\footnote{Part of the research was conducted in the framework of ESF project 10-EuroGIGA-OP-003 GraDR “Graph Drawings and Representations”}.}
Theorem 1. Let \( \Gamma_a \) and \( \Gamma_b \) be two drawings of the same plane series-parallel graph \( G \). There exists a morph \( \langle \Gamma_a, \ldots, \Gamma_b \rangle \) with \( O(n) \) steps transforming \( \Gamma_a \) into \( \Gamma_b \).

Theorem 2. Let \( \Gamma_s \) and \( \Gamma_i \) be two drawings of the same plane graph \( G \). There exists a morph \( \langle \Gamma_s, \ldots, \Gamma_i \rangle \) with \( O(n^2) \) steps transforming \( \Gamma_s \) into \( \Gamma_i \).

The rest of the paper is organized as follows. Section 2 contains preliminaries and basic terminology. Section 3 describes an algorithm to morph series-parallel graphs. Section 4 describes an algorithm to morph plane graphs. Section 5 provides geometric details for the morphs described in Sections 3 and 4. Finally, Section 6 contains conclusions and open problems.

2 Preliminaries

Planar graphs and drawings. A straight-line planar drawing \( \Gamma \) (in the following simply drawing) of a graph \( G(V, E) \) maps vertices in \( V \) to distinct points of the plane and edges in \( E \) to non-intersecting open straight-line segments between their end-vertices. Given a vertex \( v \) of a graph \( G \), we denote by \( \text{deg}(v) \) the degree of \( v \) in \( G \), that is, the number of vertices adjacent to \( v \). A planar drawing \( \Gamma \) partitions the plane into connected regions called faces. The unbounded face is the external face. Also, \( \Gamma \) determines a clockwise order of the edges incident to each vertex. Two planar drawings are equivalent if they determine the same clockwise ordering of the incident edges around each vertex and if they have the same external face. A planar embedding is an equivalence class of planar drawings. A plane graph is a planar graph with a given planar embedding.

Series-parallel graphs and their decomposition. A two-terminal series-parallel graph \( G \) with source \( s \) and target \( t \) can be recursively defined as follows: (i) An edge joining two vertices \( s \) and \( t \) is a two-terminal series-parallel graph. Let \( G' \) and \( G'' \) be two-terminal series-parallel graphs with sources \( s' \) and \( s'' \), and targets \( t' \) and \( t'' \), respectively; (ii) The series composition of \( G' \) and \( G'' \) obtained by identifying \( s'' \) with \( t' \) is a two-terminal series-parallel graph with source \( s' \) and target \( t'' \); and (iii) the parallel composition of \( G' \) and \( G'' \) obtained by identifying \( s' \) with \( s'' \) and \( t' \) with \( t'' \) is a two-terminal series-parallel graph with source \( s' \) and target \( t' \).

A biconnected series-parallel graph is defined as either a single edge or a two-terminal series-parallel graph with the addition of an edge, called root edge, joining \( s \) and \( t \). In the following we deal with biconnected series-parallel graphs not containing multiple edges.

A series-parallel graph is a connected graph whose biconnected components are biconnected series-parallel graphs.

A biconnected series-parallel graph \( G \) with root edge \( e \) is naturally associated with an ordered binary tree \( T^b_e \) rooted at \( e \), called decomposition binary tree. Each node of \( T^b_e \), with the exception of the one associated to \( e \), corresponds to a two-terminal series-parallel graph. Nodes of \( T^b_e \) are of three types, S-nodes, P-nodes, and Q-nodes. Each Q-node represents a single edge. Each S-node represents the series composition of the two-terminal series-parallel graphs associated with its left and right subtrees. Finally, each P-node represents the parallel composition of the two-terminal series-parallel graphs associated with its left and right subtrees.

Observe that, a graph \( G \) may admit more than one binary decomposition tree. Also, since all internal nodes of \( T^b_e \) have degree three, if \( T^b_e \) is rerooted at any other Q-node, corresponding to an edge \( e' \neq e \), the obtained ordered binary tree \( T^b_{e'} \) defines a new set of compositions yielding the same graph \( G \) with root edge \( e' \).

Let \( G \) be an embedded biconnected series-parallel graph and let \( e \) be an edge incident to the external face of \( G \). Let \( T^b_e \) be one of its binary decomposition trees rooted at \( e \). In order to have a unique decomposition tree \( T^b_e \) of \( G \) rooted at \( e \), we merge together all adjacent P-nodes and all adjacent S-nodes of \( T^b_e \). The order of the children of an S-node of \( T^b_e \) reflects the order of the leaves of the subtree of \( T^b_e \) induced by the merged S-nodes. Observe that, for each P-node \( \mu \) of \( T^b_e \), the embedding of \( G \) induces a circular order on the two-terminal series-parallel graphs corresponding to the children of \( \mu \). We order the children of \( \mu \) according to such an ordering.

Morphs and Pseudo-Morphs. A (linear) morphing step \( \langle \Gamma_1, \Gamma_2 \rangle \), also referred to as linear morph, of two straight-line planar drawings \( \Gamma_1 \) and \( \Gamma_2 \) of a plane graph \( G \) is a continuous transformation of \( \Gamma_1 \) into \( \Gamma_2 \) such that all the vertices simultaneously start moving from their positions in \( \Gamma_1 \) and, moving along a straight-line trajectory, simultaneously stop at their positions in \( \Gamma_2 \) so that no crossing occurs between any two edges during the transformation. A morph \( \langle \Gamma_s, \ldots, \Gamma_i \rangle \) of two straight-line planar drawings \( \Gamma_s \) into \( \Gamma_i \) of a plane graph \( G \) is a finite sequence of morphing steps that transforms \( \Gamma_s \) into \( \Gamma_i \).
Let \( u \) and \( w \) be two vertices of \( G \) such that edge \((u, w)\) belongs to \( G \) and let \( \Gamma \) be a straight-line planar drawing of \( G \). The contraction of \( u \) onto \( w \) results in (i) a graph \( G' = G/(u, w) \) not containing \( u \) and such that each edge \((u, x)\) of \( G \) is replaced by an edge \((w, x)\) in \( G' \), and (ii) a straight-line drawing \( \Gamma' \) of \( G' \) such that each vertex different from \( v \) is mapped to the same point as in \( \Gamma \). In the rest of the paper, the contraction of an edge \((u, w)\) will be only applied if the obtained drawing \( \Gamma' \) is planar. The uncontraction of \( u \) from \( w \) in \( \Gamma' \) yields a straight-line planar drawing \( \Gamma'' \) of \( G \). A morph in which contractions are performed, possibly together with other morphing steps, is a pseudo-morph.

**Kernel of a vertex.** Let \( v \) be a vertex of \( G \) and let \( G' \) be the graph obtained by removing \( v \) and its incident edges from \( G \). Let \( \Gamma'' \) be a planar straight-line drawing of \( G' \). The kernel of \( v \) in \( \Gamma'' \) is the set \( P \) of points such that straight-line segments can be drawn in \( \Gamma'' \) connecting each point \( p \in P \) to each neighbor of \( v \) in \( G \) without intersecting any edge in \( \Gamma'' \).

### 3 Morphing Series-Parallel Graph Drawings in \( \mathcal{O}(n) \) Steps

The aim of this section is to prove the following theorem:

**Theorem 3.** Let \( \Gamma_a \) and \( \Gamma_b \) be two drawings of the same plane series-parallel graph \( G \). There exists a pseudo-morph \( \langle \Gamma_a, \ldots, \Gamma_b \rangle \) with \( \mathcal{O}(n) \) steps transforming \( \Gamma_a \) into \( \Gamma_b \).

We will show in Section 3.1 an algorithm that, given two drawings of the same biconnected plane series-parallel graph \( G \), computes a pseudo-morph transforming one drawing into the other. Then, in Section 3.5 we extend this approach to simply-connected series-parallel graphs, thus proving Theorem 3.

#### 3.1 Biconnected Series-Parallel Graphs

In this section, we show an algorithm to construct a pseudo-morph transforming one drawing of a biconnected plane series-parallel graph into another.

Our approach consists of morphing any drawing \( \Gamma \) of a biconnected plane series-parallel graph \( G \) into a “canonical drawing” \( \Gamma^* \) of \( G \) in a linear number of steps. As a consequence, any two drawings \( \Gamma_1 \) and \( \Gamma_2 \) of \( G \) can be transformed one into the other in a linear number of steps, by morphing \( \Gamma_1 \) to \( \Gamma^* \) and \( \Gamma^* \) to \( \Gamma_2 \).

A canonical drawing \( \Gamma^* \) of a biconnected plane series-parallel graph \( G \) is defined as follows. The decomposition tree \( T_e \) of \( G \) is traversed top-down and a suitable geometric region of the plane is assigned to each node \( \mu \) of \( T_e \); such a region will contain the drawing of the series-parallel graph associated with \( \mu \). The regions assigned to the nodes of \( T_e \) are similar to those used in [43] to construct monotone drawings. Namely, we define three types of regions: Left boomerangs, right boomerangs, and diamonds. A left boomerang is a quadrilateral with vertices \( N, E, S, \) and \( W \) such that \( E \) is inside triangle \( \triangle(N, S, W) \), where \( |NE| = |SE| \) and \( |NW| = |SW| \) (see Fig. 1(a)). A right boomerang is defined symmetrically, with \( E \) playing the role of \( W \), and vice versa (see Fig. 1(b)). A diamond is a convex quadrilateral with vertices \( N, E, S, \) and \( W \), where \( |NW| = |NE| = |SW| = |SE| \). Observe that a diamond contains a left boomerang \( N_l, E_l, S_l, W_l \) and a right boomerang \( N_r, E_r, S_r, W_r \), where \( S = S_l = S_r, N = N_l = N_r, W = W_l, \) and \( E = E_r \) (see Fig. 1(c)).

![Fig. 1](image)

(a) A left boomerang. (b) A right boomerang. (c) A diamond. (d) Diamonds inside a boomerang. (e) Boomerangs (and a diamond) inside a diamond.
We assign boomerangs (either left or right, depending on the embedding of \( G \)) to S-nodes and diamonds to P- and Q-nodes, as follows.

First, consider the Q-node \( \rho \) corresponding to the root edge \( e \) of \( G \). Draw edge \( e \) as a segment between points \((0, 1)\) and \((0, -1)\). Also, if \( \rho \) is adjacent to an S-node \( \mu \), then assign to \( \mu \) the left boomerang \( N = (0, 1), E = (-1, 0), S = (0, -1), W = (-2, 0) \) or the right boomerang \( N = (0, 1), E = (2, 0), S = (0, -1), W = (1, 0) \), depending on the embedding of \( G \); if \( \rho \) is adjacent to a P-node \( \mu \), then associate to \( \mu \) the diamond \( N = (0, 1), E = (+2, 0), S = (0, -1), W = (-2, 0) \).

Then, consider each node \( \mu \) of \( T_e(G) \) according to a top-down traversal.

If \( \mu \) is an S-node (see Fig. 1(d)), let \( N, E, S, W \) be the boomerang associated with it and let \( \alpha \) be the angle \( WNE \). We associate diamonds to the children \( \mu_1, \mu_2, \ldots, \mu_k \) of \( \mu \) as follows. Consider the midpoint \( C \) of segment \( WE \). Subdivide \( NC \) into \( \lfloor \frac{C}{2} \rfloor \) segments with the same length and \( CS \) into \( \lceil \frac{C}{2} \rceil \) segments with the same length. Enclose each of such segments \( NiS_i \), for \( i = 1, \ldots, k \), into a diamond \( N_i, E_i, S_i, W_i \), with \( W_iN_iE_i = \alpha \), and associate it with child \( \mu_i \) of \( \mu \).

If \( \mu \) is a P-node (see Fig. 1(e)), let \( N, E, S, W \) be the diamond associated with it. Associate boomerangs and diamonds to the children \( \mu_1, \mu_2, \ldots, \mu_k \) of \( \mu \) as follows. If a child \( \mu_i \) of \( \mu \) is a Q-node, then left boomerangs are associated to \( \mu_1, \ldots, \mu_{i-1} \), right boomerangs are associated to \( \mu_{i+1}, \ldots, \mu_k \), and a diamond is associated to \( \mu_i \). Otherwise, boomerangs are associated to all of \( \mu_1, \mu_2, \ldots, \mu_k \). We assume that a child \( \mu_i \) of \( \mu \) that is a Q-node exists, the description for which is present if no child of \( \mu \) is a Q-node. We describe how to associate left boomerangs to the children \( \mu_1, \mu_2, \ldots, \mu_{i-1} \) of \( \mu \). Consider the midpoint \( C \) of segment \( WE \) and consider \( 2l \) equidistant points \( W = p_1, \ldots, p_{2l} = C \) on segment \( WC \). Associate each child \( \mu_i \), with \( i = 1, \ldots, l-1 \), to the quadrilateral \( Ni = N, E_i = p_{2i}, S_i = S, W_i = p_{2i+1} \). Right boomerangs are associated to \( \mu_{i+1}, \mu_{i+2}, \ldots, \mu_k \) in a symmetric way. Finally, associate \( \mu_i \) to any diamond such that \( Ni = N, Si = S, Wi \) is any point between \( C \) and \( E_i \), and \( E_i \) is any point between \( C \) and \( W_{i+1} \).

If \( \mu \) is a Q-node, let \( N, E, S, W \) be the diamond associated with it. Draw the edge corresponding to \( \mu \) as a straight-line segment between \( N \) and \( S \).

Observe that the above described algorithm constructs a drawing of \( G \), that we call the canonical drawing of \( G \). We now argue that no two edges \( e_1 \) and \( e_2 \) intersect in the canonical drawing of \( G \). Consider the lowest common ancestor \( v \) of the Q-nodes \( \tau_i \) and \( \tau_j \) of \( T_e \) representing \( e_1 \) and \( e_2 \), respectively. Also, consider the children \( v_1 \) and \( v_2 \) of \( v \) such that the subtree of \( T_e \) rooted at \( v_i \) contains \( \tau_i \), for \( i = 1, 2 \). Such children are associated with internally-disjoint regions of the plane. Since the subgraphs \( G_1 \) and \( G_2 \) of \( G \) corresponding to \( v_1 \) and \( v_2 \), respectively, are entirely drawn inside such regions, it follows that \( e_1 \) and \( e_2 \) do not intersect except possibly, at common endpoints.

In order to construct a pseudo-morph of a straight-line planar drawing \( \Gamma(G) \) of \( G \) into its canonical drawing \( \Gamma^*(G) \), we do the following: (i) We perform a contraction of a vertex \( v \) of \( G \) into a neighbor of \( v \), hence obtaining a drawing \( \Gamma'(G) \) of a graph \( G' \) with \( n-1 \) vertices; (ii) we inductively construct a pseudo-morph from \( \Gamma'(G) \) to the canonical drawing \( \Gamma^*(G') \) of \( G' \); and (iii) we uncontract \( v \) and perform a sequence of morphing steps to transform \( \Gamma^*(G') \) into the canonical drawing \( \Gamma^*(G) \) of \( G \).

We describe the three steps in more detail.

### 3.2 Step 1: Contract a Vertex \( v \)

Let \( T_e(G) \) be the decomposition tree of \( G \) rooted at some edge \( e \) incident to the outer face of \( G \). Consider a P-node \( \nu \) such that the subtree of \( T_e(G) \) rooted at \( \nu \) does not contain any other P-node. This implies that all the children of \( \nu \), with the exception of at most one Q-node, are S-nodes whose children are Q-nodes. Hence, the series-parallel graph \( G(\nu) \) associated to \( \nu \) is composed of a set of paths connecting its poles \( s \) and \( t \). Let \( p_1 \) and \( p_2 \) be two paths joining \( s \) and \( t \) and such that their union is a cycle \( C \) not containing other vertices in its interior (see Fig. 2(a)). Such paths exist given that the “rest of the graph” with respect to \( \nu \) is in the outer face of \( G(\nu) \), given that the root \( e \) of \( T_e(G) \) is incident to the outer face of \( G \). Internally triangulate \( C \) by adding dummy edges (dashed edges of Fig. 2). Cycle \( C \) and the added dummy edges yield a drawing of a biconnected outerplane graph \( O \) which, hence, has at least two vertices of degree two.

We distinguish two cases depending on the existence of a degree-2 vertex \( v \) different from \( s \) and \( t \).

#### Case 1 (there exists a vertex \( v \) of degree 2 different from \( s \) and \( t \)).

Assume, without loss of generality, that \( v \) belongs to \( p_2 \). Since \( O \) is internally triangulated, both the neighbors \( v_1 \) and \( v_2 \) of \( v \) belong to \( p_2 \), and they are joined
Case 2. Fig. 3. Degree 2. Let \( G \) be a graph with two vertices of degree 2. Let \( T_\Gamma \) be the internally triangulated cycle \( C \) formed by paths \( p_1 \) and \( p_2 \). Dummy edges are drawn as dashed lines. (a–b) Vertex \( v \) of degree 2 can be contracted onto \( v_1 \). (b–c) Vertex \( u_2 \) of degree 3 can be contracted onto \( u_1 \).

by a dummy edge. We obtain \( \Gamma(G') \) from \( \Gamma(G) \) by contracting \( v \) onto one of its neighbors, while preserving planarity (see Figs. 2(a) and 2(b)). Either \( p_2 \) contains more than two edges (Case 1.1) or \( p_2 \) consists of exactly two edges, namely \((v_1, v)\) and \((v, v_2)\). If the latter case holds, either edge \((v_1, v_2)\) exists in \( G \) (Case 1.2) or not (Case 1.3). In the three cases we do the following.

Case 1. Let \( \tau_1 \) and \( \tau_2 \) be the nodes of \( T_\Gamma \) corresponding to paths \( p_1 \) and \( p_2 \). Note that \( \tau_2 \) is an S-node, as \( v \in p_2 \) and \( v \neq s, t \). The two Q-nodes that are children of \( \tau_2 \) and that correspond to edges \((v, v_1)\) and \((v, v_2)\) are removed in \( T_\Gamma \).

Case 2 (the only two vertices of degree 2 in \( G \) are \( s \) and \( t \)). In this case, one of the two vertices \( u_1 \) and \( u_2 \) adjacent to \( s \) has degree 3, say \( u_2 \) (since the removal of \( s \) and its incident edges would yield another biconnected outerplane graph with two vertices of degree 2, namely \( t \) and one of \( u_1 \) and \( u_2 \)). We obtain \( \Gamma(G') \) from \( \Gamma(G) \) by contracting \( u_2 \) onto \( u_1 \). Let \( u_3 \) be the neighbor of \( u_1 \) and \( u_2 \) different from \( s \). Since the edges incident to \( u_2 \) are contained into triangles \( \Delta_{u_1, u_1, u_2} \) and \( \Delta_{u_1, u_2, u_3} \) during the contraction, planarity is preserved (see Figs. 2(b) and 2(c)). Set \( p_2' \) be the path composed of edge \((u_1, u_3)\) and of the subpath of \( p_2 \) between \( u_3 \) and \( t \), and let \( p_1' \) be the subpath of \( p_1 \) between \( u_1 \) and \( t \). Observe that \( G' \) contains edge \((u_1, u_3)\) and does not contain vertex \( u_2 \).

Tree \( T_e(G') \) is obtained from \( T_e(G) \) by performing the local changes described hereunder, with respect to the above cases.

Case 1. Let \( \tau_1 \) and \( \tau_2 \) be the nodes of \( T_e(G') \) corresponding to paths \( p_1 \) and \( p_2 \). Note that \( \tau_2 \) is an S-node, as \( v \in p_2 \) and \( v \neq s, t \). The two Q-nodes that are children of \( \tau_2 \) and that correspond to edges \((v, v_1)\) and \((v, v_2)\) are removed in \( T_e(G') \).

Case 2 (the only two vertices of degree 2 in \( G \) are \( s \) and \( t \)). In this case, one of the two vertices \( u_1 \) and \( u_2 \) adjacent to \( s \) has degree 3, say \( u_2 \) (since the removal of \( s \) and its incident edges would yield another biconnected outerplane graph with two vertices of degree 2, namely \( t \) and one of \( u_1 \) and \( u_2 \)). We obtain \( \Gamma(G') \) from \( \Gamma(G) \) by contracting \( u_2 \) onto \( u_1 \). Let \( u_3 \) be the neighbor of \( u_1 \) and \( u_2 \) different from \( s \). Since the edges incident to \( u_2 \) are contained into triangles \( \Delta_{u_1, u_1, u_2} \) and \( \Delta_{u_1, u_2, u_3} \) during the contraction, planarity is preserved (see Figs. 2(b) and 2(c)). Set \( p_2' \) be the path composed of edge \((u_1, u_3)\) and of the subpath of \( p_2 \) between \( u_3 \) and \( t \), and let \( p_1' \) be the subpath of \( p_1 \) between \( u_1 \) and \( t \). Observe that \( G' \) contains edge \((u_1, u_3)\) and does not contain vertex \( u_2 \).

Tree \( T_e(G') \) is obtained from \( T_e(G) \) by performing the local changes described hereunder, with respect to the above cases.

**Case 1.** Let \( \tau_1 \) and \( \tau_2 \) be the nodes of \( T_e(G') \) corresponding to paths \( p_1 \) and \( p_2 \). Note that \( \tau_2 \) is an S-node, as \( v \in p_2 \) and \( v \neq s, t \). The two Q-nodes that are children of \( \tau_2 \) and that correspond to edges \((v, v_1)\) and \((v, v_2)\) are removed in \( T_e(G') \).

**Case 2.** Let \( \tau_1 \) and \( \tau_2 \) be the nodes of \( T_e(G') \) corresponding to paths \( p_1 \) and \( p_2 \). Note that \( \tau_2 \) is an S-node, as \( v \in p_2 \) and \( v \neq s, t \). The two Q-nodes that are children of \( \tau_2 \) and that correspond to edges \((v, v_1)\) and \((v, v_2)\) are removed in \( T_e(G') \).
cases in which \( \nu \) has more than two children in \( T_e(G) \) (Case 2.1) and when \( \nu \) has exactly two children in \( T_e(G) \) (Case 2.2).

**Case 2.1** An S-node \( \nu_S \) and a P-node \( \nu_P \) are introduced in \( T_e(G') \), in such a way that (i) \( \nu_S \) is a child of \( \nu \), (ii) the Q-node corresponding to \( (s, u_1) \) and \( \nu_P \) are children of \( \nu_S \), (iii) \( \tau_1 \) and \( \tau_2 \) are children of \( \nu_P \), and (iv) \( \nu_Q \) is a child of \( \tau_2 \). See Figs. 4(a) and 4(b).

![Fig. 4. Construction of \( T_e(G') \) starting from \( T_e(G) \) in Case 2. (a–b) \( T_e(G) \) and \( T_e(G') \), respectively, in Case 2.1. (c–d) \( T_e(G) \) and \( T_e(G') \), respectively, in Case 2.2.](image)

**Case 2.2** Node \( \nu \) is removed from the children of \( \mu \), and a P-node \( \nu_P \) is introduced in \( T_e(G') \) in such a way that (i) the Q-node corresponding to \( (s, u_1) \) and \( \nu_P \) are children of \( \mu \), (ii) \( \tau_1 \) and \( \tau_2 \) are children of \( \nu_P \), and (iii) \( \nu_Q \) is a child of \( \tau_2 \). See Figs. 4(c) and 4(d).

### 3.3 Step 2: Recursive Call

Let \( \Gamma(G') \) be the drawing of the graph \( G' = G \setminus \{v\} \) obtained after the contraction of vertex \( v \) performed in Case 1 or in Case 2.

Inductively construct a morphing from \( \Gamma(G') \) to the canonical drawing \( \Gamma^*(G') \) of \( G' \) in \( c \cdot (n - 1) \) steps, where \( c \) is a constant.

### 3.4 Step 3: Uncontract Vertex \( v \) and Construct a Canonical Drawing of \( G \)

We describe how to obtain \( \Gamma^*(G) \) from \( \Gamma^*(G') \) by uncontracting \( v \) and performing a constant number of morphing steps. The description follows the cases discussed in Appendix 3.2.

**Case 1** (there exists a vertex \( v \) of degree 2 different from \( s \) and \( t \)).

**Case 1.1** This case is discussed in Section 3.1.

**Case 1.2 and Case 1.3** Note that \( \Gamma^*(G') \) and \( \Gamma^*(G) \) coincide, except for the fact that: (i) \( \Gamma^*(G) \) contains one boomerang more than \( \Gamma^*(G') \) (the one associated to \( \tau_2 \)) inside the diamond associated to \( \nu \), (ii) \( \Gamma^*(G) \) might not contain the diamond associated to the Q-node corresponding to edge \( (s, t) \) (in Case 1.3), and (iii) the boomerangs inside the diamond associated to \( \nu \) have a different drawing in \( \Gamma^*(G') \) and \( \Gamma^*(G) \). Drawing \( \Gamma^*(G') \) is illustrated in Fig. 5(a), drawing \( \Gamma^*(G) \) in Case 1.2 is illustrated in Fig. 5(b), drawing \( \Gamma^*(G) \) in Case 1.3 is illustrated in Fig. 5(d).

Since edge \( (v_1, v_2) \) exists in \( G' \), its drawing in \( \Gamma^*(G') \) is the straight-line segment between the points \( N' \) and \( S' \) of a diamond \( N', E', S', W' \). Also, the drawing \( \Gamma^*(p_2) \) of \( p_2 \) in \( \Gamma^*(G) \) lies inside a boomerang \( N, E, S, W \) with \( N = N' \) and \( S = S' \).

In order to construct a pseudo-morph from \( \Gamma^*(G') \) to \( \Gamma^*(G) \), initially place points \( E \) and \( W \) on segment \( EW' \), on the same side with respect to segment \( N'S' \) (in Case 1.2, the side depends on the order of the children of \( \nu \) in \( T_e(G) \)). With one morphing step, move \( v \) to the midpoint of segment \( EW \) (see Fig. 5(b)).
are white. (b) Vertices \( u \) from diamonds associated to (see Fig. 6. (a)) edge \( S = v \) \( v \) is moved to the midpoint of \( EW \). (c) \( \Gamma^* (G) \) in Case 1.2, where edge \((v_1, v_2)\) exists in \( G \). (d) \( \Gamma^* (G) \) in Case 1.3.

Consider the children \( \tau_i \) of \( \nu \) in \( T_v (G) \) that are not Q-nodes, with \( i = 1, \ldots, q \), and note that the drawing of each \( \tau_i \) is composed of two straight-line segments \( NC_i \) and \( SC_i \). With a second morphing step, move the vertex \( w_i \) of \( \tau_i \) lying on \( C_i \), for each \( i = 1, \ldots, q \), and vertex \( v \) along the line through \( EW \) till reaching their positions in \( \Gamma^* (G) \). In the same morphing step, for each \( i = 1, \ldots, q \), the vertices on the path between \( s \) and \( w_i \) are moved as convex combination of the movements of \( s \) and \( w_i \), and the vertices on the path between \( t \) and \( w_i \) are moved as linear combination of the movements of \( t \) and \( w_i \). Hence, at the end of the morphing step, also such vertices reach their positions in \( \Gamma^* (G) \) (see Figs. 5(c) and 5(d)).

**Case 2 (the only two vertices of degree 2 in \( O \) are \( s \) and \( t \)).**

**Case 2.1** Note that \( \Gamma^* (G') \) and \( \Gamma^* (G) \) coincide, except for the drawing of \( p_1, p_2, p'_1, \) and \( p'_2 \).

Namely, \( p_1 \) and \( p_2 \) are drawn in \( \Gamma^* (G) \) in two boomerangs \( N_1, E_1, S_1, W_1 \) and \( N_2 = N_1, E_2, S_2 = S_1, W_2 \) lying inside the diamond associated to \( \nu \) (see Fig. 6(d)), while \( p'_1 \) and \( p'_2 \) are drawn in \( \Gamma^* (G') \) in two boomerangs \( N'_1, E'_1, S'_1 = S_1, W'_1 \) and \( N'_2 = N_1, E'_2, S'_2 = S'_1 = S_1, W'_2 \) lying inside a diamond associated to \( \nu_p \), that lies inside a boomerang \( N_S, E_S, S_S, W_S \) associated to \( \nu_S \) (with \( S_S = S'_2 = S'_1 = S_1 \)), that lies inside the diamond associated to \( \nu \) (see Fig. 6(a)).

Note that, since \( \nu_S \) has two children in \( T_v (G') \), vertex \( u_1 \) is placed on the midpoint \( S_S \) of segment \( E_S W_S \), that is, \( C_S = N'_1 = N'_2 \).

Let \( w_1 \) and \( w_2 \) be the vertices of \( p'_1 \) and \( p'_2 \), respectively, placed on the midpoints \( C'_1 \) and \( C'_2 \) of segments \( E'_1 W'_1 \) and \( E'_2 W'_2 \).

Fig. 6. Construction of \( \Gamma^* (G) \) from \( \Gamma^* (G') \) when Case 2.1 applied. (a) \( \Gamma^* (G') \). The boomerang associated to \( \nu_S \) is light-grey, the diamonds associated to \( \nu_F \) and to the Q-node corresponding to \( (s, u_2) \) are dark-grey, and the boomerangs associated to \( \tau_1 \) and \( \tau_2 \) are white. (b) Vertices \( w_1 \) and \( w_2 \) are moved to points \( C'_2 \) and \( C'_3 \), and \( u_1 \) is moved to a point of \( N_S C'_3 \). (c) Vertex \( v \) is uncontracted from \( u_2 \) and moved to a point of \( N_S C'_2 \). (d) \( \Gamma^* (G) \). The boomerangs associated to \( \tau_1 \) and \( \tau_2 \) are light-grey.
With one morphing step, move \( w_1 \) to any point \( p_1 \) on \( ES^[_S] \), move \( w_2 \) to any point \( p_2 \) on \( WS^[_S] \), and move \( u_1 \) to any point on segment \( NS^[_S] \) (see Fig. 7(b)). In the same morphing step, for each two vertices in \( w_1, w_2, u_1, \) and \( t \), say \( w_1 \) and \( t \), the vertices lying on segment \( w_1t \) are moved as linear combination of the movements of \( w_1 \) and \( t \). Hence, at the end of the morphing step, all these vertices still lie on \( w_1t \).

Next, with one morphing step, uncontract \( u_2 \) from \( u_1 \) and move it to any internal point on segment \( NS^[_S] \) (see Fig. 7(c)). In the same morphing step, the vertices lying on segment \( w_1t \) are moved as linear combination of the movements of \( w_2 \) and \( w_1 \).

Further, perform the same operation as in Case 1.1 to redistribute the vertices of \( p_1 \) on \( NS^[_S] \) and \( S_W^[_S] \), and the vertices of \( p_2 \) on \( NS^[_S] \) and \( S_W^[_S] \). After this step, for each child \( \tau_i \) of \( \nu \), the vertex \( w_i \) of \( \tau_i \) lying on segment \( w_1w_2 \), lying in \( \Gamma^*(G) \) lies on \( w_1w_2 \) also in the current drawing.

Finally, perform the same operation as in Case 1.2 to move the vertex \( w_i \) of each child \( \tau_i \) of \( \nu \) to its final position (on segment \( w_1w_2 \)) in \( \Gamma^*(G) \). In the same morphing step, the vertices on the path between \( s \) and \( w_i \) are moved as linear combination of the movements of \( s \) and \( w_i \), and the vertices on the path between \( t \) and \( w_i \) are moved as linear combination of the movements of \( t \) and \( w_i \). Hence, at the end of the morphing step, also such vertices reach their positions in \( \Gamma^*(G) \).

**Case 2.2** Note that \( \Gamma^*(G') \) and \( \Gamma^*(G) \) coincide, except for the drawing of \( p_1, p_2, p'_1, \) and \( p'_2 \).

Namely, \( p_1 \) and \( p_2 \) are drawn in \( \Gamma^*(G) \) in two boomerangs (associated to \( \tau_1 \) and \( \tau_2 \)) lying inside the diamond associated to \( \nu \) (see Fig. 7(b)), that lies inside the boomerang associated to \( \mu \). Also \( p'_1 \) and \( p'_2 \) are drawn in \( \Gamma^*(G') \) in two boomerangs lying inside a diamond (associated to \( \nu_p \)) that lies inside the boomerang associated to \( \mu \) (see Fig. 7(a)). However, the boomerang associated to \( \mu \) in \( \Gamma^*(G) \) has one diamond less than in \( \Gamma^*(G') \), since in \( \Gamma^*(G') \) it also contains the diamond associated to edge \((s, u_1)\). Also, the vertices in the boomerangs associated to \( \tau_1 \) and \( \tau_2 \) have different positions in \( \Gamma^*(G') \) and in \( \Gamma^*(G) \), since vertex \( u_2 \) is not present in \( \Gamma^*(G') \).

With three morphing steps analogous to those performed in Case 1.1, we redistribute the vertices inside the boomerang \( N, E, S, W \) associated to \( \mu \) in such a way that the vertex lying on the midpoint \( C \) of \( EW \) is the same in \( \Gamma^*(G') \) and in \( \Gamma^*(G) \). Note that, after these steps, the diamonds associated to \( \nu_p \) and to edge \((s, u_1)\) lie on the same segment, either \( NC \) or \( SC \), say \( SC \), and that the vertices lying on segment \( NC \) already are at their final position in \( \Gamma^*(G) \). Then, with three morphing steps analogous to those performed in Case 2.1, we uncontract \( u_2 \) and collapse the two diamonds associated to \( \nu_p \) and to \((s, u_1)\) into a single diamond. Then, with one morphing step (analogous to one of the steps performed in Case 1.1), we move the vertices lying on segment \( SC \) till they reach their final position in \( \Gamma^*(G) \).

![Fig. 7](image_url)

**Fig. 7.** Construction of \( \Gamma^*(G) \) from \( \Gamma^*(G') \) when Case 2.2 applied. (a) \( \Gamma^*(G') \). The boomerang associated to \( \mu \) is light-grey, the diamonds associated to the children of \( \mu \), including \( \nu_p \) and the Q-node corresponding to \((s, u_1)\) are dark-grey, and the boomerangs associated to \( \tau_1 \) and \( \tau_2 \) are white. (b) \( \Gamma^*(G) \).
3.5 Simply-Connected Series-Parallel Graphs

In this section we show how, by preprocessing the input drawings $\Gamma_a$ and $\Gamma_b$ of any series-parallel graph $G$, the algorithm presented in Section 3.1 can be used to compute a pseudo-morph $M = \langle \Gamma_a, \ldots, \Gamma_b \rangle$. The idea is to augment both $\Gamma_a$ and $\Gamma_b$ to two drawings $\Gamma'_a$ and $\Gamma'_b$ of a biconnected series-parallel graph $G'$, compute the morph $M' = \langle \Gamma'_a, \ldots, \Gamma'_b \rangle$, and obtain $M$ by restricting $M'$ to the vertices and edges of $G$.

This augmentation is performed on $G$ by repeatedly applying the following lemma.

Lemma 1. Let $v$ be a cut-vertex of a plane series-parallel graph $G$ with $n_b$ blocks. Let $e_1 = (u, v)$ and $e_2 = (w, v)$ be two consecutive edges in the circular order around $v$ such that $e_1$ belongs to block $b_1$ of $G$ and $e_2$ belongs to block $b_2 \neq b_1$ of $G$. The graph $G^*$ obtained from $G$ by adding a vertex $z$ and edges $(u, z)$ and $(w, z)$ is a plane series-parallel graph with $n_b - 1$ blocks.

Proof: First, observe that by adding $z$, $(u, z)$, and $(w, z)$ to $G$, blocks $b_1$ and $b_2$ are merged together into a single block $b_{1,2}$ of $G^*$ (see Figs. 8(a) and 8(b)). Hence, the number of blocks of $G^*$ is $n_b - 1$. It remains to show that $G^*$ is a series-parallel graph.

Assume for a contradiction that $G^*$ is not a series-parallel graph. It follows that $G^*$ contains a subdivision of the complete graph on four vertices $K_4$, i.e., there is a set $V_{K_4}$ of four vertices of $G^*$ such that any two of them are joined by three vertex-disjoint paths. Observe that the vertices in $V_{K_4}$ cannot belong to different blocks of $G^*$. Further, since $G$ is a series-parallel graph, the vertices in $V_{K_4}$ belong to $b_{1,2}$. Since $z$ has degree two, $z \notin V_{K_4}$; hence, the vertices in $V_{K_4}$ are also vertices of $G$. This gives a contradiction since: (i) The vertices in $V_{K_4}$ cannot all belong to $b_1$, as otherwise $G$ would not be series-parallel, contradicting the hypothesis; (ii) the vertices in $V_{K_4}$ cannot all belong to $b_2$, for the same reason; and (iii) the vertices in $V_{K_4}$ cannot belong both to $b_1$ and $b_2$, as otherwise there could not exist three vertex-disjoint paths joining them in $G^*$, contradicting the hypothesis that $G^*$ contains a subdivision of $K_4$. \[\square\]

Observe that, when augmenting $G$ to $G^*$, both $\Gamma_a$ and $\Gamma_b$ can be augmented to two planar straight-line drawings $\Gamma^*_a$ and $\Gamma^*_b$ of $G^*$ by placing vertex $z$ suitably close to $v$ and with direct visibility to vertices $u$ and $w$, as in the proof of Fáry’s Theorem [3]. By repeatedly applying such an augmentation we obtain a biconnected series-parallel graph $G'$ and its drawings $\Gamma'_a$ and $\Gamma'_b$, whose number of vertices and edges is linear in the size of $G$. Hence, the algorithm described in Section 3.1 can be applied to obtain a pseudo-morph $\langle \Gamma'_a, \ldots, \Gamma'_b \rangle$, thus proving Theorem 3. We will show in Section 5 how to obtain a morph starting from the pseudo-morph computed in this section.

4 Morphing Plane Graph Drawings in $O(n^2)$ Steps

In this section we prove the following theorem.

Theorem 4. Let $\Gamma_s$ and $\Gamma_t$ be two drawings of the same plane graph $G$. There exists a pseudo-morph $\langle \Gamma_s, \ldots, \Gamma_t \rangle$ with $O(n^2)$ steps transforming $\Gamma_s$ into $\Gamma_t$.  

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Preliminary definitions Let \( \Gamma \) be a planar straight-line drawing of a plane graph \( G \). A face \( f \) of \( G \) is empty in \( \Gamma \) if it is delimited by a simple cycle. Consider a vertex \( v \) of \( G \) and let \( v_1 \) and \( v_2 \) be two of its neighbors. Vertices \( v_1 \) and \( v_2 \) are consecutive neighbors of \( v \) if no edge appears between edges \((v, v_1)\) and \((v, v_2)\) in the circular order of the edges around \( v \) in \( \Gamma \). Let \( v \) be a vertex with \( \deg(v) \leq 5 \) such that each face containing \( v \) on its boundary is empty. We say that \( v \) is contractible \( \Box \) if, for each two neighbors \( u_1 \) and \( u_2 \) of \( v \), edge \((u_1, u_2)\) exists in \( G \) if and only if \( u_1 \) and \( u_2 \) are consecutive neighbors of \( v \). We say that \( v \) is quasi-contractible if, for each two neighbors \( u_1 \) and \( u_2 \) of \( v \), edge \((u_1, u_2)\) exists in \( G \) only if \( u_1 \) and \( u_2 \) are consecutive neighbors of \( v \). In other words, no edge exists between non-consecutive neighbors of a contractible or quasi-contractible vertex; also, each face incident to a contractible vertex \( v \) is delimited by a 3-cycle, while a face incident to a quasi-contractible vertex might have more than three incident vertices. We have the following.

**Lemma 2.** Every planar graph contains a quasi-contractible vertex.

**Proof:** Let \( \Gamma \) be a planar drawing of a graph \( G \). Add vertices \( a, b, \) and \( c \) so that the triangle composed by these vertices completely encloses \( \Gamma \), and augment the obtained drawing to the drawing \( \Gamma' \) of a maximal planar graph \( G' \) by adding dummy edges. Since \( G' \) is maximal planar, it contains a contractible vertex \( v \) (different from \( a, b, \) and \( c \)), as shown in \( \Box \). Since \( v \) is contractible in \( G' \), it is either contractible or quasi-contractible in \( G \), as the edges incident to a vertex in \( G \cap G' \) are at most those of \( G' \).

Further, given a neighbor \( x \) of \( v \), we say that \( v \) is \( x \)-contractible onto \( x \) in \( \Gamma \) if: (i) \( v \) is quasi-contractible, and (ii) the contraction of \( v \) onto \( x \) in \( \Gamma \) results in a straight-line planar drawing \( \Gamma'' \) of \( G' = G/(v, x) \).

**The algorithm** We describe the main steps of our algorithm to pseudo-morph a drawing \( \Gamma_s \) of a plane graph \( G \) into another drawing \( \Gamma_t \) of \( G \).

First, we consider a quasi-contractible vertex \( v \) of \( G \), that exists by Lemma \( \Box \). Second, we compute a pseudo-morph with \( O(n) \) steps of \( \Gamma_s \) into a drawing \( \Gamma'_s \) of \( G \) and a pseudo-morph with \( O(n) \) steps of \( \Gamma_t \) into a drawing \( \Gamma'_t \) of \( G \), such that \( v \) is \( x \)-contractible onto the same neighbor \( x \) both in \( \Gamma'_s \) and in \( \Gamma'_t \). We will describe later how to perform these pseudo-morphs. Third, we contract \( v \) onto \( x \) both in \( \Gamma'_s \) and in \( \Gamma'_t \), hence obtaining two drawings \( \Gamma'_s \) and \( \Gamma'_t \) of a graph \( G' = G/(v, x) \) with \( n-1 \) vertices. Fourth, we recursively compute a pseudo-morph transforming \( \Gamma'_s \) into \( \Gamma'_t \). This completes the description of the algorithm for constructing a pseudo-morphing transforming \( \Gamma_s \) into \( \Gamma_t \).

Observe that the algorithm has \( p(n) \in O(n^2) \) steps, thus proving Theorem \( \Box \). Namely, it will be described later, \( O(n) \) steps suffice to construct pseudo-morphings of \( \Gamma_s \) and \( \Gamma_t \) into drawings \( \Gamma'_s \) and \( \Gamma'_t \) of \( G \), respectively, such that \( v \) is \( x \)-contractible onto the same neighbor \( x \) both in \( \Gamma'_s \) and in \( \Gamma'_t \). Further, two steps are sufficient to contract \( v \) onto \( x \) in both \( \Gamma'_s \) and \( \Gamma'_t \), obtaining drawings \( \Gamma''_s \) and \( \Gamma''_t \), respectively. Finally, the recursion on \( \Gamma''_s \) and \( \Gamma''_t \) takes \( p(n-1) \) steps. Thus, \( p(n) = p(n-1) + O(n) \in O(n^2) \). We will show in Section \( \Box \) how to obtain a morph starting from the pseudo-morph computed in this section.

We remark that our approach is similar to the one proposed by Alamdari et al. \( \Box \). In \( \Box \), \( \Gamma_s \) and \( \Gamma_t \) are augmented to drawings of the same maximal planar graph with \( m \in O(n^2) \) vertices; then, Alamdari et al. \( \Box \) show how to construct a morphing in \( O(m^2) \) steps between two drawings of the same \( m \)-vertex maximal planar graph. This results in a morphing between \( \Gamma_s \) and \( \Gamma_t \) with \( O(n^4) \) steps. Here, we also augment \( \Gamma_s \) and \( \Gamma_t \) to drawings of maximal planar graphs. However, we only require that the two maximal planar graphs coincide in the subgraph induced by the neighbors of \( v \).

Since this can be achieved by adding a constant number of vertices to \( \Gamma_s \) and \( \Gamma_t \), namely one for each of the at most five faces \( v \) is incident to, our morphing algorithm has \( O(n^2) \) steps.

**Making \( v \) \( x \)-contractible** Let \( v \) be a quasi-contractible vertex of \( G \). We show an algorithm to construct a pseudo-morph with \( O(n) \) steps transforming any straight-line planar drawing \( \Gamma \) of \( G \) into a straight-line planar drawing \( \Gamma' \) of \( G \) such that \( v \) is \( x \)-contractible onto any neighbor \( x \). If \( v \) has degree 1, then it is contractible into its unique neighbor in \( \Gamma \), and there is nothing to prove.

In order to transform \( \Gamma \) into \( \Gamma' \), we use a support graph \( S \) and its drawing \( \Sigma \), initially set equal to \( G \) and \( \Gamma \), respectively. The goal is to augment \( S \) and \( \Sigma \) so that \( v \) becomes a contractible vertex of \( S \). In order to do this, we have to add to \( S \) an edge between any two consecutive neighbors of \( v \). However, the insertion of these edges might
not be possible in $\Sigma$, as it might lead to a crossing or to enclose some vertex inside a cycle delimited by $v$ and by two consecutive neighbors of $v$ (see Fig. 9(b)).

Let $a$ and $b$ be two consecutive neighbors of $v$. If the closed triangle $\langle a, b, v \rangle$ does not contain any vertex other than $a$, $b$, and $v$, then add edge $\langle a, b \rangle$ to $S$ and to $\Sigma$ as a straight-line segment. Otherwise, proceed as follows. Let $\Sigma_u$ be the drawing of a plane graph $S_u$ obtained by adding a vertex $u$ and the edges $\langle u, v \rangle$, $\langle u, a \rangle$, and $\langle u, b \rangle$ to $S$ and to $\Sigma$, in such a way that the resulting drawing is straight-line planar and each face containing $u$ do not have direct visibility and the triangle $\langle a, b, v \rangle$ is not empty. As in the proof of Fáry’s Theorem [8], a position for $u$ with such properties can be found in $\Sigma$, suitably close to $v$. See Fig. 9(b) for an example.

Augment $\Sigma_u$ to the drawing $\Theta$ of a maximal plane graph $T$ by first adding three vertices $p, q$, and $r$ to $\Sigma_u$, so that triangle $\langle p, q, r \rangle$ completely encloses the rest of the drawing, and then adding dummy edges [7] till a maximal plane graph is obtained. If edge $\langle a, b \rangle$ has been added in this augmentation (this can happen if $a$ and $b$ share a face not having $v$ on its boundary), subdivide $\langle a, b \rangle$ in $\Theta$ (namely, replace edge $\langle a, b \rangle$ with edges $\langle a, w \rangle$ and $\langle w, b \rangle$, placing $w$ along the straight-line segment connecting $a$ and $b$) and triangulate the two faces vertex $w$ is incident to.

Next, apply the algorithm described in [1], that we call $\text{CONVEXIFIER}$, to construct a morphing of $\Theta$ into a drawing $\Theta'$ of $T$ in which polygon $\langle a, v, b, u \rangle$ is convex. The input of algorithm $\text{CONVEXIFIER}$ consists of a planar straight-line drawing $G^*$ of a plane graph $G$ and of a set of at most five vertices of $G^*$ inducing a biconnected outerplane graph not containing any other vertex in its interior in $G^*$. The output of algorithm $\text{CONVEXIFIER}$ is a sequence of $O(n)$ linear morphing steps transforming $G^*$ into a drawing of $G^*$ in which the at most five input vertices bound a convex polygon. Since, by construction, vertices $a, v, b, u$ satisfy all such requirements, we can apply algorithm $\text{CONVEXIFIER}$ to $\Theta$ and to $a, v, b, u$, hence obtaining a morphing with $O(n)$ steps transforming $\Theta$ into the desired drawing $\Theta'$ (see Fig. 9(c)).

Let $\Sigma''_u$ be the drawing of $S_u$ obtained by restricting $\Theta'$ to vertices and edges of $S_u$. Since $\langle a, v, b, u \rangle$ is a convex polygon containing no vertex of $S_u$ in its interior, edge $\langle u, v \rangle$ can be removed from $\Sigma''_u$ and an edge $\langle a, b \rangle$ can be introduced in $\Sigma''_u$, so that the resulting drawing $\Sigma''$ is planar and cycle $\langle a, b, v \rangle$ does not contain any vertex in its interior (see Fig. 9(d)).

Once edge $\langle a, b \rangle$ has been added to $S$ (either in $\Sigma$ or after the described procedure transforming $\Sigma$ into $\Sigma''$), if $\deg(v) = 2$ then $v$ is both $a$-contractible and $b$-contractible. Otherwise, consider a new pair of consecutive vertices of $v$ not creating an empty triangular face with $v$, if any, and apply the same operations described before.

Once every pair of consecutive vertices has been handled, vertex $v$ is contractible in $S$. Let $\Sigma_v$ be the current drawing of $S$. Augment $\Sigma_v$ to the drawing $\Theta_v$ of a triangulation $T_v$ (by adding three vertices and a set of edges), contract $v$ onto a neighbor $w$ such that $v$ is $w$-contractible (one of such neighbors always exists, given that $v$ is contractible), and apply $\text{CONVEXIFIER}$ to the resulting drawing $\Theta'_v$ and to the neighbors of $v$ to construct a morphing $\Theta''_v$ to a drawing $\Sigma''_v$, in which the polygon defined by such vertices is convex. Drawing $G'$ of $G$ in which $v$ is $x$-contractible for any neighbor $x$ of $v$ is obtained by restricting $\Sigma''_v$ to the vertices and the edges of $G$. We can now contract $v$ onto $x$ in $G'$ and recur on the obtained graph (with $n - 1$ vertices) and drawing.

Fig. 9. Vertex $v$ and its neighbors. (a) Vertices $a$ and $b$ do not have direct visibility and the triangle $\langle a, b, v \rangle$ is not empty. (b) A vertex $u$ is added suitably close to $v$ and connected to $v$, $a$, and $b$. (c) The output of $\text{CONVEXIFIER}$ on the quadrilateral $\langle a, b, v, u \rangle$. (d) Vertex $u$ and its incident edges can be removed in order to insert edge $\langle a, b \rangle$.

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It remains to observe that, given a quasi-contractible vertex $v$, the procedure to construct a pseudo-morph of $\Gamma$ into $\Gamma'$ consists of at most $\deg(v) + 1$ executions of CONVEXIFIER, each requiring a linear number of steps \[\Pi\]. As $\deg(v) \leq 5$, the procedure to pseudo-morph $\Gamma$ into $\Gamma'$ has $O(n)$ steps. This concludes the proof of Theorem 4.

5 Transforming a Pseudo-Morph into a Morph

In this section we show how to obtain an actual morph $M$ from a given pseudo-morph $\mathcal{M}$, by describing how to compute the placement and the motion of any vertex $v$ that has been contracted during $\mathcal{M}$. By applying this procedure to Theorems 3 and 4, we obtain a proof of Theorems 1 and 2.

Let $\Gamma'$ be a drawing of a graph $G$ and let $\mathcal{M} = (\Gamma', \ldots, \Gamma^*)$ be a pseudo-morph that consists of the contraction of a vertex $v$ of $G$ onto one of its neighbors $x$, followed by a pseudo-morph $\mathcal{M}'$ of the graph $G' = G/(v, x)$, and then of the uncontraction of $v$.

The idea of how to compute $M$ from $\mathcal{M}$ is the same as in \[\Pi\]. Namely, morph $M$ is obtained by (i) recursively converting $\mathcal{M}'$ into a morph $M'$; (ii) modifying $M'$ to a morph $M''$ obtained by adding vertex $v$ (and its incident edges) to each drawing of $\mathcal{M}'$, in a suitable position; (iii) replacing the contraction of $v$ onto $x$, performed in $\mathcal{M}$, with a linear morph that moves $v$ from its initial position in $\Gamma'$ to its position in the first drawing of $\mathcal{M}'$; and (iv) replacing the uncontraction of $v$, performed in $\mathcal{M}$, with a linear morph that moves $v$ from its position in the last drawing of $\mathcal{M}'$ to its final position in $\Gamma^*$. Note that, in order to guarantee the planarity of $M$ when adding $v$ to any drawing of $\mathcal{M}'$ in order to obtain $\mathcal{M}'$, vertex $v$ must lie inside its kernel. Since vertex $x$ lies in the kernel of $v$ (as $x$ is adjacent to all the neighbors of $v$ in $G'$), we achieve this property by placing $v$ suitably close to $x$, as follows.

At any time instant $t$ during $\mathcal{M}'$, there exists an $\epsilon_t > 0$ such that the disk $D$ centered at $x$ with radius $\epsilon_t$ does not contain any vertex other than $x$. Let $\epsilon$ be the minimum among the $\epsilon_t$ during $\mathcal{M}'$. We place vertex $v$ at a suitable point of a sector $S$ of $D$ according to the following cases.

Case (a) $v$ has degree 1 in $G$. Sector $S$ is defined as the intersection of $D$ with the face containing $v$ in $G$. See Fig. 10(a).

Case (b) $v$ has degree 2 in $G$. Sector $S$ is defined as the intersection of $D$ with the face containing $v$ in $G$ and with the halfplane defined by the straight-line passing through $x$ and $r$, and containing $v$ in $\Gamma$. See Fig. 10(b).

Otherwise, $\deg(v) \geq 3$ in $G'$. Let $(r, v)$ and $(l, v)$ be the two edges such that $(r, v)$, $(x, v)$, and $(l, v)$ are clockwise consecutive around $v$ in $G$. Observe that edges $(r, x)$ and $(l, x)$ exist in $G'$. Assume that $x, r$, and $l$ are not collinear in any drawing of $\mathcal{M}'$, as otherwise we can slightly perturb such a drawing without compromising the planarity of $\mathcal{M}'$. Let $\alpha_i$ be the angle $\hat{xrv}$ in any intermediate drawing of $\mathcal{M}'$.

Case (c) $\alpha_i < \pi$. Sector $S$ is defined as the intersection of $D$ with the wedge delimited by edges $(x, r)$ and $(x, l)$. See Fig. 10(c).

Case (d) $\alpha_i > \pi$. Sector $S$ is defined as the intersection of $D$ with the wedge delimited by the elongations of $(x, r)$ and $(x, l)$ emanating from $x$. See Fig. 10(d).

Fig. 10. Sector $S$ (in grey) when: (a) $\deg(v) = 1$, (b) $\deg(v) = 2$, and (c)-(d) $\deg(v) \geq 3$.

By exploiting the techniques shown in \[\Pi\], the motion of $v$ can be computed according to the evolution of $S$ over $\mathcal{M}'$, thus obtaining a planar morph $\mathcal{M}_\mathcal{M}'$. For convenience, we report hereunder the statement of Lemma 5.2 of \[\Pi\].
showing that a contracted vertex can be placed and moved according to the evolution of a sector defined on one of its neighbors lying in the kernel.

Lemma 3. Let \(\Gamma_1, \ldots, \Gamma_k\) be straight-line planar drawings of a 5-gon \(C\) on vertices \(a, b, c, d, e\) in clockwise order such that the morph \(\langle \Gamma_1, \ldots, \Gamma_k \rangle\) is planar and vertex \(a\) is inside the kernel of the polygon \(C\) at all times during the morph. Then we can augment each drawing \(\Gamma_i\) to a drawing \(\Gamma_i^p\) by adding vertex \(p\) at some point \(p_i\) inside the kernel of the polygon \(C\) in \(\Gamma_i\), and adding straight line edges from \(p\) to each of \(a, b, c, d, e\) in such a way that the morph \(\langle \Gamma_1, \ldots, \Gamma_k \rangle\) is planar.

Observe that, in the algorithm described in Section 4, the vertex \(x\) onto which \(v\) has been contracted might be not adjacent to \(v\) in \(G\). However, since a contraction has been performed, \(x\) is adjacent to \(v\) in one of the graphs obtained when augmenting \(G\) during the algorithm. Hence, a morph of \(G\) can be obtained by applying the above procedure to the pseudo-morph computed on this augmented graph and by restricting it to the vertices and edges of \(G\).

6 Conclusions and Open Problems

In this paper we studied the problem of designing efficient algorithms for morphing two planar straight-line drawings of the same graph. We proved that any two planar straight-line drawings of a series-parallel graph can be morphed with \(O(n)\) linear morphing steps, and that a planar morph with \(O(n^2)\) linear morphing steps exists between any two planar straight-line drawings of any planar graph.

It is a natural open question whether the bounds we presented are optimal or not. We suspect that planar straight-line drawings exist requiring a linear number of steps to be morphed one into the other. However, no super-constant lower bound for the number of morphing steps required to morph planar straight-line drawings is known.

It would be interesting to understand whether our techniques can be extended to compute morphs between any two drawings of a partial planar 3-tree with a linear number of steps. We recall that, as observed in [1], a linear number of morphing steps suffices to morph any two drawings of a maximal planar 3-tree.

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for each non-triangular face $v$ is incident to

$v$ quasi-contractible in $S$ and $G$

$v$ contractible in $S$

$v$ collapsible in $\Theta_v'$ and $\Gamma_v'$. subphase 1

subphase 2

subphase 3