Spatially-Coupled Precoded Rateless Codes

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Abstract—Raptor codes are rateless codes that achieve the capacity on the binary erasure channels. However the maximum degree of optimal output degree is unbounded. This leads to a computational complexity problem both at encoders and decoders. Aref and Urbanke investigated the potential advantage of universal achieving-capacity property of proposed spatially-coupled (SC) low-density generator matrix (LDGM) codes. However the decoding error probability of SC-LDGM codes is bounded away from 0. In this paper, we investigate SC-LDGM codes concatenated with SC low-density parity-check codes. The proposed codes can be regarded as SC-HA rateless codes. We derive a lower bound of the asymptotic overhead from stability analysis for successful decoding by density evolution. The numerical calculation reveals that the lower bound is tight. We observe that with a sufficiently large number of information bits, the asymptotic overhead and the decoding error rate approach 0 with bounded maximum degree.

I. INTRODUCTION

Spatially-coupled (SC) low-density parity-check (LDPC) codes attract much attention due to their capacity-achieving performance under low-latency memory-efficient sliding-window belief propagation (BP) decoding. The studies on SC-LDPC codes date back to the invention of convolutional LDPC codes by Felström and Zigangirov [1]. Lentmaier et al. observed that the BP threshold of regular SC-LDPC codes coincides with the maximum a posterior (MAP) threshold of the underlying block LDPC codes with a lot of accuracy by density evolution [2]. Kudekar et al. proved that SC-LDPC codes achieve the MAP threshold of BEC [3] and the binary-input memoryless output-symmetric (BMS) channels [4] under BP decoding.

Rateless codes are a class of erasure-recovering codes which produce limitless sequence of encoded bits from k information bits so that receivers can recover the k information bits from arbitrary (1+\alpha)k/(1-\epsilon) received symbols from BEC(\epsilon). We denote overhead by \alpha. Designing rateless codes with vanishing overhead is desirable, which implies the codes achieve the capacity of BEC(\epsilon). LT codes [5] and raptor codes [6] are rateless codes that achieve vanishing overhead \alpha \rightarrow 0 in the limit of large information size over the BEC. By a nice analogy between the BEC and the packet erasure channel (e.g., Internet), rateless codes have been successfully adopted by several industry standards.

A raptor code can be viewed as concatenation of an outer high-rate LDPC code and infinitely many single parity-check codes of length d, where d is chosen randomly with probability \Omega_d for d \geq 1. Raptor codes need to have unbounded maximum degree d for \Omega_d \neq 0. This leads to a computation complexity problem both at encoders and decoders.

The authors presented empirical results in [7] showing that SC MacKay-Neal (MN) codes and SC Hsu-Anastasopoulos (HA) codes achieve the capacity of BEC with bounded maximum degree. Recently a proof for SC-MN codes are given in [8]. It was observed that the SC-MN codes and SC-HA codes have the BP threshold close to the Shannon limit in [9] over BMS channels.

Aref and Urbanke [10] investigated the potential advantage of universal achieving-capacity property of SC low-density generator matrix (LDGM) codes. They observed that the decoding error probability steeply decreases with overhead \alpha = 0 with bounded maximum degree over various BMS channels. However the decoding error probability was proved to be bounded away from 0 with bounded maximum degree for any \alpha. This is explained from the fact that there are a constant fraction of bit nodes of degree 0.

In this paper, we investigate SC-LDGM codes concatenated with SC-LDPC codes. The proposed codes can be regarded as SC-HA rateless codes. We derive a lower bound of the asymptotic overhead from stability analysis for successful decoding by density evolution. The numerical calculation reveals that the lower bound is tight. We observe that with a sufficiently large number of information bits, the asymptotic overhead and the decoding error rate approach 0 with bounded maximum degree.

II. ENCODER AND DECODER

A. Encoder

Let k denote the number of information bits. We define a (d_l, d_r, d_g, L, w) code for d_l \geq 2, d_r \geq 2, d_g \geq 2 as follows. The (d_l, d_r, d_g, L, w) code are defined on L sections from 0 to L - 1. Each section has M pre-coded bits. Note that, in [3], 2L + 1 sections \{-L, +L\} were considered. Instead, for the sake of simplicity, we consider L sections in [0, L - 1]. First, the k information bits are pre-coded with (d_l, d_r, L, w) codes [3] into LM bits x(0, 0), \ldots, x(L - 1, M - 1). In this paper, we assume that the bits in the i-th section for i \in [0, L - 1] are transmitted and the bits in other sections are shortened. Namely, the shortened bits are set to 0 and are not transmitted. Let R_{pre}(L) denote the design coding rate of (d_l, d_r, L, w) codes. In [3], R_{pre}(L) is given by

\[ R_{pre}(L) = 1 - \frac{d_l}{d_r} \frac{d_l w - 1 - 2 \sum_{i=1}^{w-1} (i/w)^{d_r}}{L} \]
Let \( L = \infty \),
\[
1 - \frac{d_i}{d_e}
\]
It follows that \( k = R_{pre}(L)LM \).
After encoding the \( k \) bits into \( LM \) coded bits by pre-code, the \( LM \) pre-coded bits further will be encoded by an inner code as follows. Repeat the following procedure endlessly for \( t \in \{ 1, \infty \} \).
1. Choose a section \( i(t) \in \{ 0, \ldots , j \} \) uniformly at random from \( L + w - 1 \) sections.
2. Choose \( d_g \) section shifts \( j_1^{(t)}, \ldots , j_{d_g}^{(t)} \in \{ 0, w - 1 \} \) with repetition uniformly at random.
3. Choose \( d_g \) bit-indices \( l_1^{(t)}, \ldots , l_{d_g}^{(t)} \in \{ 0, M - 1 \} \) with repetition uniformly at random.
4. Add \( d_g \) bits and transmit the sum as
\[
x(i^{(t)} - j_1^{(t)}, j_1^{(t)}) + \cdots + x(i^{(t)} - j_{d_g}^{(t)}, j_{d_g}^{(t)})
\]
(1)

### B. Decoder

Assume that transmission takes place over BEC(\( \epsilon \)) and we have \( n \) received symbols \( y^{(1)}, \ldots , y^{(n)} \) each of which is 0, 1 or \( '?' \). Define the **overhead** \( \alpha \) as
\[
\alpha = \frac{n}{k}(1 - \epsilon) - 1.
\]
In this setting, we have \( (1 + \alpha)k = n(1 - \epsilon) \) unerasable received symbols. Independence of the coding scheme ensures that we can assume, without loss of generality, that time indices of \( n \) received symbols are arbitrary. For simplicity, we assume that the receiver receives \( n \) symbols at time \( t = 1, \ldots , n \) without loss of generality.

We assume that the decoder knows \( i(t), d_g \) section shifts \( j_1^{(t)}, \ldots , j_{d_g}^{(t)} \) and bit-indices \( l_1^{(t)}, \ldots , l_{d_g}^{(t)} \) in (1) for each received symbol at time \( t = 1, \ldots , n \). From these information and the knowledge of the precoder, one can construct a factor graph for sum-product decoding [11]. The factor graph consists of \( LM \) variable nodes (bit nodes) \( x(0,0), \ldots , x(L - 1, M - 1) \) and \( (1 - R_{pre}(L))LM \) parity-check factor nodes (check nodes) of pre-code and factor nodes (channel nodes) of factor
\[
1[x(i^{(t)} - j_1^{(t)}, j_1^{(t)}) + \cdots + x(i^{(t)} - j_{d_g}^{(t)}, j_{d_g}^{(t)}) = y^{(t)}]
\]
(2)
for \( t = 1, \ldots , n \), where \( 1[\cdot] \) is defined as 1 if the argument is true and 0 otherwise. We say that the factor node of factor (2) is in the section \( i(t) \).

### III. Performance Analysis

In this section, we investigate the performance of the coupled rateless codes and derive a bound.

#### A. Performance Analysis by Density Evolution

In this subsection, we derive the density evolution update equation. The following lemma clarifies the degree distributions of inner codes.

**Lemma 1:** Let \( \Lambda_d \) be the probability that a bit node has \( d \) neighboring channel nodes. Let \( \beta \) be the average number of channel nodes adjacent to a bit node. In the limit of large \( M \), we have
\[
\beta = \frac{d_g}{1 - \epsilon} \frac{LR_{pre}(L)(1 + \alpha)}{L + w - 1},
\]
\[
\sum_{d \geq 0} \Lambda_d x^d = e^{-\beta(1-x)} = \sum_{d \geq 0} \beta^d \frac{e^{-\beta}}{d!} x^d.
\]
**Proof:** Let \( \bar{N} \) denote the average number of channel nodes per section. There are \( L + w - 1 \) sections containing channel nodes. We have \( n \) channel nodes in total.
\[
N = \frac{n}{L + w - 1} = \frac{1}{1 - \epsilon} \frac{(1 + \alpha)k}{L + w - 1} = \frac{1}{1 - \epsilon} \frac{(1 + \alpha)R_{pre}(L)LM}{L + w - 1},
\]
where we used \( k = R_{pre}(L)LM \). Recalling that \( \beta \) is the average number of channel nodes adjacent to a bit node, we have
\[
\beta = \frac{d_g N}{M}.
\]
Equation (3) immediately follows from this. Each section has \( N \) channel nodes of degree \( d_g \), in other words, we have \( d_g N \) edges in each section. Let \( \Lambda_d \) denote the probability that a bit node in the \( i \)-th section has \( d \) channel nodes within sections from \( i \) to \( i + w - 1 \). Since each channel node is generated independently, the probability \( \Lambda_d \) follows a binomial distribution as follows.
\[
\Lambda_d = \binom{d_g N}{d} \left( \frac{1}{M} \right)^d \left( 1 - \frac{1}{M} \right)^{d_g N - d}
\]
The probability generating function of \( \Lambda_d \) is given as follows.
\[
\Lambda(x) := \sum_{d \geq 0} \Lambda_d x^d = \left( \frac{x}{M} + 1 - \frac{1}{M} \right)^{d_g N}
\]
\[
M \equiv \infty \exp[-\beta(1-x)] = \sum_{d \geq 0} \beta^d \frac{e^{-\beta}}{d!} x^d.
\]
This implies \( \Lambda_d = \frac{\beta^d e^{-\beta}}{d!} \) in the limit of \( M \rightarrow \infty \). In other words, the degree \( d \) follows the Poisson distribution of average \( \beta \). □

Let us describe density evolution update equations. Let \( p_i^{(t)} \) and \( s_i^{(t)} \) be the erasure probability of messages sent from bit nodes in the \( i \)-th section to check nodes and channel nodes, respectively, at the \( t \)-th iteration of BP decoding of \((d_1, d_2, d_g, L, w)\) codes in the limit of large \( M \). The density evolution [12] gives update equations for \( p_i^{(t)} \) and \( s_i^{(t)} \) as follows. For \( i \notin [0, L - 1] \), \( p_i^{(0)} = s_i^{(0)} = 0 \). For \( i \in [0, L - 1] \), \( p_i^{(0)} = s_i^{(0)} = 1 \), and for \( t \geq 0 \),
\[
p_i^{(t+1)} = \left( \frac{1}{w} \sum_{j=0}^{w-1} (1 - (1 - \frac{1}{w} \sum_{k=0}^{w-1} p^{(t)}_{i+j-k})^{d_i-1}) \right)^{d_i-1}
\]
\[
\cdot \Lambda \left( \frac{1}{w} \sum_{j=0}^{w-1} (1 - (1 - \epsilon)(1 - \frac{1}{w} \sum_{k=0}^{w-1} s^{(t)}_{i+j-k})^{d_g-1}) \right)
\]
\[s_i^{(\ell+1)} = \left(\frac{1}{w} \sum_{j=0}^{w-1} (1 - \frac{1}{w} \sum_{k=0}^{w-1} P_{i+j-k}^{(\ell)} d_{i-k}) \right) d_i \]
\[\cdot \lambda \left(\frac{1}{w} \sum_{j=0}^{w-1} (1 - (1 - \epsilon)(1 - \frac{1}{w} \sum_{k=0}^{w-1} P_{i+j-k}^{(\ell)} d_{i-k}) \right),\]
where \(\lambda(x) = \frac{x}{\Lambda(x)} = \exp[-\beta(1-x)] = \Lambda(x).\)

Let \(P_b^{(\ell)}\) be the decoding error probability at the \(\ell\)-th iteration of BP decoding given as follows.

\[P_b^{(\ell)} := \frac{1}{L} \sum_{i=1}^{L} P_i^{(\ell)}.\]

Definition 1: One can easily check \(P_b^{(\ell)}\) has its limit \(P_b^{(\infty)}(L) = \lim_{\ell \to \infty} P_b^{(\ell)}(L)\) since \(P_b^{(\ell)}\) is decreasing in \(\ell\).

We define the \textit{overhead threshold} \(\alpha_L^*\) and its corresponding \(\beta_L^*\) as follows.

\[\alpha_L^* := \inf \{\alpha > 0 \mid P_b^{(\infty)}(L) = 0\},\]
\[\beta_L^* := \inf \{\beta > 0 \mid P_b^{(\infty)}(L) = 0\}.\]

We say \((d_1, d_r, d_g, L, w)\) codes \textit{achieve the capacity of BEC(\epsilon)} if

\[\limsup_{L \to \infty} \alpha_L^* = 0.\]

Discussion 1: We will explain why we exclude the case \(d_g = 1\). Assume \(d_g = 1\). The density evolution update equations can be reduced as follows.

\[p_i^{(\ell+1)} = \begin{cases} 
\Lambda(\epsilon) \left(\frac{1}{w} \sum_{j=0}^{w-1} (1 - (1 - \frac{1}{w} \sum_{k=0}^{w-1} P_{i+j-k}^{(\ell)} d_{i-k}) \right) d_i^{-1} & (i \in [0, L-1]), \\
0 & (i \notin [0, L-1]).
\end{cases}\]

This is equivalent to the density evolution update equation of the precoder that is a \((d_1, d_r, d_g, w, L)\) code transmitted over BEC(\(\epsilon\)) [3]. If the error probability goes to 0, \(\Lambda(\epsilon)\) has to be less than the Shannon limit \(\Lambda(\epsilon) = e^{-\beta_L^*(1-\epsilon)} < 1 - R_{\text{pre}}(L)\).

It follows that \(\beta_L^*\) is bounded as follows.

\[\beta_L^* > \frac{1}{1 - \epsilon} \ln \frac{1}{1 - R_{\text{pre}}(L)}.\]

From (3), we have

\[\alpha_L^* > \frac{L + w - 1}{LR_{\text{pre}}(L)} \ln \frac{1}{1 - R_{\text{pre}}(L)} - 1.\]

This implies the \((d_1, d_r, d_g = 1, L, w)\) codes do not achieve the capacity of BEC(\(\epsilon\)). This is the reason why we exclude the case \(d_g = 1\) in this paper.

Lemma 2: The \((d_1, d_r, d_g, L, w)\) codes achieve the capacity of BEC(\(\epsilon\)) if and only if

\[\limsup_{L \to \infty} \beta_L^* = \frac{d_g}{1 - \epsilon} \left(1 - \frac{d_i}{d_r} \right).\]

Proof: This is straightforward from (3), we have

\[\beta_L^* = \frac{d_g}{1 - \epsilon} R_{\text{pre}}(L) \frac{L}{L + w - 1} (1 + \alpha_L^*)\]
\[= \frac{d_g}{1 - \epsilon} \left(1 - \frac{d_i}{d_r}\right) (L \to \infty).\]

\[QED\]

B. Performance Bound by Stability Analysis

In the following theorem, we derive a lower bound of overhead threshold \(\alpha_L^*\).

Theorem 1: For \((d_1 = 2, d_r, d_g, L, w)\) codes, if \(P_b^{(\infty)}(L) = 0\) then there exist \(\alpha_L^*\) and \(\beta_L^*\) such that

\[\alpha_L^* \geq \alpha_L^*,\]
\[\beta_L^* \geq \beta_L^*;\]
\[\lim_{L \to \infty} \alpha_L^* = \max \left\{ \frac{\ln(d_r - 1)}{d_g(1 - 2/d_r)}, 1, 0 \right\},\]
\[\lim_{L \to \infty} \beta_L^* = \max \left\{ \frac{\ln(d_r - 1)}{1 - \epsilon} \frac{d_g}{1 - \epsilon} \left(1 - \frac{d_i}{d_r}\right) \right\}.\]

Proof: Let \(P_L\) denote an \(L \times L\) matrix whose \((i, j)\) entry is \(\partial p^{(\ell)} / \partial p^{(\ell)}_{ij}\). As we will see, this does not depend on \(\ell\). Let \(\rho(P_L)\) denote the spectral radius of \(P_L\). We will derive a lower bound of \(\rho(P_L)\).

Some calculation reveals that at \(p^{(\ell)} = s^{(\ell)} = 0\) for \(d_i = 2\).

\[\frac{\partial p_i^{(\ell+1)}}{\partial p_j^{(\ell)}} = \left\{ \begin{array}{ll}
\frac{w - (i - j)}{w} (d_r - 1) \lambda(\epsilon) & (|i - j| \leq w) \\
0 & (|i - j| > w)
\end{array} \right.\]

and \(\frac{\partial p_i^{(\ell+1)}}{\partial p_j^{(\ell)}} = 0\) for \(d_i > 2\). It holds that for \(d_i \geq 2\),

\[\frac{\partial s_i^{(\ell+1)}}{\partial s_j^{(\ell)}} = \frac{\partial s_i^{(\ell+1)}}{\partial p_j^{(\ell)}} = \frac{\partial s_i^{(\ell+1)}}{\partial p_j^{(\ell)}} = 0.\]

at \(p^{(\ell)} = s^{(\ell)} = 0\). We drop \(\ell\) since (5) is independent of \(\ell\).

From (3), we can see that \(P_L\) is a \textit{positive band matrix} of width \(w\), which is defined in Definition 4 in Appendix. Since \(P_L\) is a positive band matrix of width \(w\), one can see that \(P_L\) is irreducible from Lemma 4 in Appendix. Let \(\lambda_1, \ldots, \lambda_L\) be the eigenvalues of \(P_L\); recall that \(\rho(P_L)\) is the spectral radius of \(P_L\). We have

\[\rho(P_L) := \max_i (|\lambda_i|).\]

Since \(P_L\) is symmetric, the eigenvalues are real.

Let \(\lambda_1 > \ldots > \lambda_L\) be the eigenvalues of \(P_L\). Perron-Frobenius theorem [13] asserts that the eigenvalue that gives the spectral radius of a non-negative irreducible matrix is positive. Since \(P_L\) is non-negative symmetric irreducible matrix, the eigenvalue that gives spectral radius of \(P_L\) is positive. Then we have

\[\rho(P_L) = \lambda_1.\]
For $\delta > 0$, we define $\beta := \beta_0^r + \delta$. Since $\beta > \beta_0^r$, it follows $\mathbb{P}_b(\infty) = 0$. From (6), we have for $\forall x \in \mathbb{R}^L \setminus \{0\}$,

$$1 > \rho(P_L) \overset{(a)}{=} \max_{x \in \mathbb{R}^L \setminus \{0\}} \frac{x^T P_L x}{x^T x} \geq \frac{1^T P_L 1}{1^T 1} = (d_r - 1)e^{-\beta(w-1)} = (d_r - 1)e^{-\beta_0^r(w-1)} + \frac{1}{w^2L} - \frac{w(w+1)/3}{w^2L}$$

where we used [14, Theorem 4.2.2] for (a). Solving $\lim_{L \to \infty} (d_r - 1)e^{-\beta_0^r(w-1)}$, we obtain

$$\beta > \frac{1}{1 - \varepsilon} \ln \left( \frac{(d_r - 1)(1 - (w-1)(w+1))}{3wL} \right).$$

**Discussion 2:** For $L \geq 2w-1$, $P_L$ has entries taking value from 1 to $w$. From [14, Lemma 5.6.10], we can bound $\rho(P_L)$ as follows.

$$\rho(P_L) \leq \|P_L\|_1 := \max_{1 \leq i \leq L} \sum_{j=1}^L \|P_L\|_{i,j}$$

$$= (d_r - 1)e^{-\beta_0^r(w-1)} = (d_r - 1)e^{-\beta_0^r(w-1)} = \frac{d_g}{d_g L R_{\text{pre}}(L)} - 1 =: \alpha^*_L$$

In the limit of large $L$, we have

$$\lim_{L \to \infty} \beta^*_L = \max \left[ \frac{\ln(d_r - 1)}{1 - \varepsilon}, \frac{d_g}{d_g R_{\text{pre}}(L)} \right],$$

$$\lim_{L \to \infty} \alpha^*_L = \max \left[ \frac{d_r \ln(d_r - 1)}{d_g (d_r - 2)} - 1, 0 \right].$$

This concludes Theorem [1] and Theorem [2].

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In the limit of large $L$, we have

$$\lim_{L \to \infty} \beta^*_L = \max \left[ \frac{\ln(d_r - 1)}{1 - \varepsilon}, \frac{d_g}{d_g R_{\text{pre}}(L)} \right],$$

$$\lim_{L \to \infty} \alpha^*_L = \max \left[ \frac{d_r \ln(d_r - 1)}{d_g (d_r - 2)} - 1, 0 \right].$$

This condition is not satisfied for $d_r = 2$ or $d_g = 2$.

**Proof:** From Definition [1] capacity-achieving codes satisfy $\alpha^*_L$ goes to 0 in the limit of large $L$. To be precise,

$$\lim_{L \to \infty} \alpha^*_L = \max \left[ \frac{d_r \ln(d_r - 1)}{d_g (d_r - 2)} - 1, 0 \right] = 0.$$

The inequality (9) immediately follows from this.}

**Fig. 1.** The asymptotic overhead $\alpha^*_L$ and the average degree of $\beta^*_L$ and their lower bounds $\alpha^*_L$ and $\beta^*_L$ of $(d_l = 2, d_r = 3, d_g = 2, L, w = 2)$ codes over BEC($\varepsilon = 0.5$). The asymptotic overhead threshold $\alpha^*_L$ does not converge to 0 since the codes do not satisfy the condition of Corollary [1]. Figure suggests the lower bounds are tight for large $L$.

**IV. Decoding Performance**

In this section, we demonstrate the decoding performance of the $(d_l, d_r, d_g, L, w)$ codes.

**Discussion 2:** For $L \geq 2w-1$, $P_L$ has entries taking value from 1 to $w$. From [14, Lemma 5.6.10], we can bound $\rho(P_L)$ as follows.

$$\rho(P_L) \leq \|P_L\|_1 := \max_{1 \leq i \leq L} \sum_{j=1}^L \|P_L\|_{i,j}$$

$$= (d_r - 1)e^{-\beta_0^r(w-1)} = (d_r - 1)e^{-\beta_0^r(w-1)} = \frac{d_g}{d_g L R_{\text{pre}}(L)} - 1 =: \alpha^*_L$$

In the limit of large $L$, we have

$$\lim_{L \to \infty} \beta^*_L = \max \left[ \frac{\ln(d_r - 1)}{1 - \varepsilon}, \frac{d_g}{d_g R_{\text{pre}}(L)} \right],$$

$$\lim_{L \to \infty} \alpha^*_L = \max \left[ \frac{d_r \ln(d_r - 1)}{d_g (d_r - 2)} - 1, 0 \right].$$

This condition is not satisfied for $d_r = 2$ or $d_g = 2$.

**Proof:** From Definition [1] capacity-achieving codes satisfy $\alpha^*_L$ goes to 0 in the limit of large $L$. To be precise,

$$\lim_{L \to \infty} \alpha^*_L = \max \left[ \frac{d_r \ln(d_r - 1)}{d_g (d_r - 2)} - 1, 0 \right] = 0.$$

The inequality (9) immediately follows from this.
is an extension to BMS channels and a proof for capacity-achievability.

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