Perfect Delaunay Polytopes in Low Dimensions

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Abstract

A lattice Delaunay polytope is known as perfect if the only ellipsoid, that can be circumscribed about it, is its Delaunay sphere. Perfect Delaunay polytopes are in one-to-one correspondence with arithmetic equivalence classes of positive quadratic functions on \( \mathbb{Z}^n \), that can be recovered in a unique way from its minimum over \( \mathbb{Z}^n \) and all of its representations. We develop a structural theory of such polytopes and describe all known perfect Delaunay polytopes in dimensions one through eight. We suspect that this list is complete.

1 Introduction

A point lattice is a discrete set of points in \( \mathbb{R}^n \) such that the difference vectors form a subgroup of \( \mathbb{R}^n \). If \( \Lambda \) is a point lattice in \( \mathbb{R}^n (n \geq 0) \), then a convex polytope \( P \subset \mathbb{R}^n \) is called a lattice polytope (or \( \Lambda \)-polytope) if all of its vertices are lattice points. Consider the lattice \( \mathbb{Z}^n \subset \mathbb{R}^n \), and a convex \( \mathbb{Z}^n \)-polytope \( P \). If \( P \) can be circumscribed by an ellipsoid \( \mathcal{E} = \{ x \in \mathbb{R}^n \mid Q_E (x - c) \leq \rho_E^2 \} \) with no interior \( \mathbb{Z}^n \)-elements so that the boundary \( \mathbb{Z}^n \)-elements of \( \mathcal{E} \) are exactly the vertices of \( P \), we will say that \( P \) is a Delaunay polytope with respect to the form \( Q_E \) defined by \( \mathcal{E} \); more informally, we will say that a lattice polytope is Delaunay if it can be circumscribed by an empty ellipsoid. Ellipsoid is commonly used to refer to hypersurfaces, defined by positive definite quadratic forms, as well as solid bodies bounded by such surfaces: the meaning of our usage will be clear from the context. Typically, there is a family of empty ellipsoids that can be circumscribed about a given Delaunay polytope \( P \), but, if there is only one, so that \( \mathcal{E} \) is uniquely determined by \( P \), we will say that \( P \) is a perfect Delaunay polytope in \( \mathbb{Z}^n \). Perfect Delaunay polytopes are also sometimes referred to as extreme. Perfect Delaunay polytopes are fascinating geometrical objects – examples are the six- and seven-dimensional Gosset (1900) polytopes with 27 and 56 vertices respectively, which appear in the Delaunay tilings of the root lattices \( E_6 \) and \( E_7 \) (see e.g. Coxeter (1988) for a description). In this paper we describe, up to an isometry and a dilation, all known perfect Delaunay polytopes in \( \mathbb{R}^n \) for \( n \leq 8 \); we also present a study of the geometry and combinatorics of these polytopes. We suspect that the list of perfect Delaunay polytopes that we give here is complete for \( n \leq 8 \). Erdahl (1975, 1992) proved that 0 and \([0,1] \) are the only perfect Delaunay polytopes for \( n \leq 5 \); Dutour (2002) proved that there is only one perfect Delaunay polytope in \( \mathbb{R}^6 \) – the Gosset 6-polytope (Coxeter’s 2_21), which is described in Section 5. Only two perfect Delaunay polytopes are known in \( \mathbb{R}^7 \): they are
Gosset’s 7-polytope (Coxeter’s 321) and a 35-tope found by Erdahl and Rybnikov (2002), which are described in Section 7. We list 27 8-dimensional perfect Delaunay polytopes, they are identified by numbers, 1 through 27; Section 8 contains a detailed description of these polytopes. There are infinite series of perfect Delaunay polytopes – the first such series was found by Erdahl and Rybnikov in 2001 (Rybnikov, 2001). This series was further generalized in (Erdahl, Ordine, Rybnikov 2004) to a 3-parametric series (where one parameter is the dimension) of perfect Delaunay polytopes. Another infinite series has been found by Dutour (2005). Prior to 2001 only sporadic examples of perfect Delaunay polytopes had been known, besides the cases for \( n \leq 7 \) mentioned above, all of them were found by Deza, Grishukhin and Laurent (1992) and all of their examples were constructed as sections of the Leech and Barnes-Wall lattices.

2 Definitions and Notation

Formally speaking, the subject of this paper is the study of 0-sets of positive \( \mathbb{Q} \)-valued quadratic functions on free \( \mathbb{Z} \)-modules of finite rank. Since any free \( \mathbb{Z} \)-module of finite rank \( \Lambda \) can be realized as a discrete subgroup of \( \mathbb{R}^n \) for any \( n \geq \text{rank} \Lambda \), we can think of \( \Lambda \) geometrically as a discrete set of vectors in vector space \( \mathbb{R}^n \) or as a discrete set of points in affine space \( \mathbb{R}^n \). Whenever we approach a \( \mathbb{Z} \)-module from this point of view, we call it a lattice.

Definition 1. A quadratic lattice is a pair \((\Lambda, \mathbb{R})\), where \( \Lambda \) is a free \( \mathbb{Z} \)-module of finite rank and \( Q : \Lambda \to \mathbb{R} \) is a quadratic form.

A function \( F \) on a module is called quadratic if it can be written as \( \text{QForm} F + A \), where \( \text{QForm} F \) is a quadratic form and \( A \) is an affine function. In general, we denote the quadratic form part of a polynomial \( P \) by \( \text{QForm} P \).

Definition 2. An affine quadratic lattice is a pair \( \text{Aff}(\Lambda, Q) \), where \( \Lambda \) is free \( \mathbb{Z} \)-module of finite rank and \( Q : \Lambda \to E \) is a quadratic function valued in an extension \( E \) of \( \mathbb{Q} \).

Sometimes, to stress that we consider a quadratic lattice, as opposed to an affine quadratic lattice we call the former linear or homogeneous quadratic lattice. \( \text{Aff}(\Lambda, Q) \) (resp. \( (\Lambda, Q) \)) is called degenerate if \( \text{rank} \text{QForm} Q < \text{rank} \Lambda \) (resp. \( \text{rank} Q < \text{rank} \Lambda \)). Quadratic lattices \( (\Lambda, Q) \) and \( (\Lambda_1, Q_1) \) are called isomorphic if there is a \( \mathbb{Z} \)-module isomorphism \( L : \Lambda \to \Lambda_1 \) such that \( Q(x) = Q_1(Lx) \) for any \( x \in \Lambda \). Affine quadratic lattices \( \text{Aff}(\Lambda, Q) \) and \( \text{Aff}(\Lambda_1, Q_1) \) are called isomorphic if there is a \( \mathbb{Z} \)-module isomorphism \( L : \Lambda \to \Lambda_1 \) and some \( z \in \Lambda_1 \) such that \( Q(x) = Q_1(Lx - z) \) for any \( x \in \Lambda \). We allow quadratic forms and functions to take values not only in \( \mathbb{Z} \), the ground ring of the modules under consideration, but also in \( \mathbb{Q} \) and its extensions, unlike the classical case of (linear) quadratic modules, where forms are supposed to be valued in the ground ring of the module (as in, e.g., Serre 1973). Since we restrict ourselves to \( \text{rank} \Lambda < \infty \), any \( \mathbb{Q} \)-valued function on \( \Lambda \) can be rescaled into a \( \mathbb{Z} \)-valued quadratic function.

Suppose \( Q_1 \) and \( Q_2 \) are valued in \( \mathbb{R} \). Two affine quadratic lattices \( \text{Aff}(\Lambda_1, Q_1) \) and \( \text{Aff}(\Lambda_2, Q_2) \) are called equivalent up to scaling if there is a \( \mathbb{Z} \)-module isomorphism \( L : \Lambda_1 \to \Lambda_2 \), some \( z \in \Lambda_2 \) and a real \( c > 0 \) such that \( Q_1(x) = c Q_2(Lx - z) \).
A quadratic form $Q$ induces a symmetric bilinear form on $\Lambda \times \Lambda$ by

$$(x, y) \mapsto \frac{1}{2} \{Q[x + y] - Q[x] - Q[y]\};$$

we will denote this bilinear form also by $Q$—normally, there is no confusion, since a quadratic form has one argument, while the corresponding bilinear form has two. We call a number $a$ positive (resp. negative) if $a \geq 0$ (resp. $a \leq 0$); same terminology is applied to functions. Thus, a form $Q$ is called positive if $Q[x] \geq 0$ for any $x$. A form $Q$ is called positive definite if $Q[x] > 0$ for any $x \neq 0$.

Suppose a quadratic function $Q$ is valued in $\mathbb{R}$. A number $b \in \mathbb{R}$ is called the arithmetic minimum of an affine quadratic lattice $\text{Aff}(\Lambda, Q)$ if $b = \min_{z \in \Lambda} Q(z)$. The vectors of $\Lambda$ on which the minimum of $\text{Aff}(\Lambda, Q)$ is attained are called the minimal vectors of $\text{Aff}(\Lambda, Q)$. The definition of the arithmetic minimum of an affine quadratic lattice is slightly different from that of a homogeneous quadratic lattice: in the case of a homogeneous quadratic lattice the minimum is taken over all non-zero vectors $z \in \Lambda$.

**Definition 3.** Let $Q : \Lambda \to \mathbb{R}$ be a fixed quadratic function with positive $Q$-form $Q$, and let $X$ be a quadratic function with unknown coefficients. Let $b = \min_{z \in \Lambda} Q(z)$. The affine quadratic lattice $\text{Aff}(\Lambda, Q)$ is called perfect if the system of equations

$$\{X(m) = b \mid m \text{ is a minimal vector of } Q\}$$

has the only solution $X = Q$.

When we refer to a function as perfect without specifying $\Lambda$, the meaning of $\Lambda$ is clear from the context. If a concrete formula for the function is given, $\Lambda$ is presumed to be $\mathbb{Z}^n$. We will often use a shorthand such as e.g. $k$-lattice (polytope, cell, etc.), instead of $k$-dimensional lattice (polytope, cell, etc.).

From now on all quadratic functions are valued in $\mathbb{Q}$ or $\mathbb{R}$. In this context a symmetric bilinear form is a scalar product on $\Lambda \otimes \mathbb{Q} \cong \mathbb{Q}^n$ or $\Lambda \otimes \mathbb{R} \cong \mathbb{R}^n$. There are two canonical ways to describe an affine quadratic lattice, one by fixing the lattice to be $\mathbb{Z}^n$ and the other by fixing the quadratic form part of the function to be, say $\sum x_i^2$. The first method is more flexible, since it allows for quadratic forms of arbitrary signature. Furthermore, in any kind of machine computations it is far more convenient to deal with the former representation.

In each dimension there are only finitely many non-isomorphic perfect affine quadratic lattices, up to scale. This follows from Voronoi’s L-type reduction theory (see e.g. Deza and Laurent, 1997). Namely, in each dimension there is a strict L-type reduction domain $D$, which has finitely many extreme rays. The quadratic part of a perfect quadratic function is always arithmetically equivalent to a form lying on an extreme ray of $D$ (but not vice versa). This implies the finiteness.

**Proposition 4.** For $n = 0$ the only perfect affine quadratic lattice is $(\mathbb{Z}^0, 0)$. For $n = 1$ the only perfect affine quadratic lattice, up to isomorphisms and scaling, is $\text{Aff}(\mathbb{Z}^1, (x - \frac{1}{2})^2)$.

Perfect quadratic functions are inhomogeneous analogs of perfect quadratic forms introduced in the middle of 19-th century by Korkine and Zolotareff (1873) and later studied by
Voronoi, Barnes, Conway and Sloane, Stacey, Martinet, and others (see Martinet (2003) for a survey). The interest to perfect forms has been mostly fueled by the theorem, proven by Korkine and Zolotareff (1873), that forms that are extreme points of the ball packing density function are perfect. We prefer not to use the term inhomogeneous form, employed in some number-theoretic literature (e.g. Gruber, Lekkerkerker 1987) since a form is by definition a homogeneous polynomial.

2.1 Delaunay Tilings

The language of Delaunay tilings provides a geometric way of thinking about quadratic functions with positive quadratic part. We denote the vertex set of a polytope $P$ by $\text{vert} P$. A convex $\Lambda$-polytope $D$ is called a Delaunay polytope in a (linear) quadratic lattice $(\Lambda, Q)$, where $Q[x] > 0$ for $x \neq 0$, if there is an ellipsoid $Q(x) \leq 0$, with $Q_{\text{Form}} Q = Q$, whose boundary contains $\text{vert} D$, but no points of $\Lambda \setminus \text{vert} D$. If $\dim D < \text{rank} \Lambda$ such an ellipsoid is not unique; however, the intersection of any such ellipsoid with the affine span of $D$ is unique (for fixed $(\Lambda, Q)$). In particular, such an ellipsoid is unique when $D$ is of maximal dimension – in this case this ellipsoid is called the Delaunay ellipsoid (or empty ellipsoid) of $D$.

For any $S \subset \Lambda$, we denote the affine span of $S$ in $\Lambda \otimes \mathbb{R}$ by $\text{aff} S$. The affine span of $S$ in $\Lambda \otimes \mathbb{Q}$ is denoted by $\text{aff}_Q S$, and the lattice spanned by all vectors $x - y$, where $x, y \in S$, by $\text{aff}_\mathbb{Z} S$. Note that when $0 \in S$, the linear span of $S$, $\text{lin} S$, and the affine span of $S$, $\text{aff} S$, are the same. Often lattices arise as sections of other lattices by affine subspaces of the ambient affine Euclidean space. $\Gamma \subset \mathbb{R}^n$ is called an affine lattice if $\Gamma^* = \{x - y \mid x, y \in \Gamma\}$ is a $\mathbb{Z}$-module of finite rank. In such situations it is convenient to have the notion of isomorphism for affine lattices. Let $\Gamma \subset \mathbb{R}^n$ and $\Gamma' \subset \mathbb{R}^m$ be affine lattices. A map $f : \Gamma \to \Gamma'$ is called an affine isomorphism if there are $0 \in \Gamma$ and $0' \in \Gamma'$ such that $x - o \mapsto fx - o'$ is a $\mathbb{Z}$-module isomorphism from $\Gamma^*$ onto $\Gamma'^*$. Two functions $\varphi : \Gamma \to S$, $\psi : \Gamma' \to S$ on affine lattices $\Gamma$ and $\Gamma'$ are called arithmetically equivalent if there is an affine isomorphism $f : \Gamma \to \Gamma'$ such that $\varphi(x) = \psi(fx)$. A $\Lambda$-polyhedron $P$ can be thought of as the indicator function, which is 1 on $P \cap \Lambda$ and 0 elsewhere on $\Lambda$. Then, arithmetic equivalence of lattice polyhedra is a special case of the arithmetic equivalence between functions. If $\Gamma = \Gamma' = \mathbb{Z}^n$, the arithmetic equivalence is the same as the equivalence with respect to $\text{Aff}(n, \mathbb{Z})$, the group of all transformations of type $z \mapsto Lz + t$, where $L \in GL(n, \mathbb{Z})$ and $t \in \mathbb{Z}^n$.

It is a theorem of Delaunay (1924) that for a positive definite quadratic form $Q : \Lambda \to \mathbb{R}$ the space $\text{aff} \Lambda$ is partitioned into the relative interiors of Delaunay $\Lambda$-polytopes with respect to $Q$; this partition is organized so that the intersection of any family of Delaunay polytopes is again a Delaunay polytope (we add the empty polytope $\emptyset$ to the partition) – in other words the resulting Delaunay tiling is face-to-face. This theorem also says that a Delaunay tiling for $(\Lambda, Q)$ is unique. In studying Delaunay tilings and $L$-types of lattices it is often beneficial not to restrict to positive definite forms, but to use the concept of Delaunay tiling with respect to any positive $Q$-valued quadratic form. Since traditionally, in the geometric context, quadratic forms are valued in $\mathbb{R}$, we will say a few words about the case where $Q$ is $\mathbb{R}$-valued.

**Definition 5.** The rational closure of the cone of positive-definite quadratic forms on $\Lambda$ is the...
set of all positive $\mathbb{R}$-valued forms $Q$ that satisfy the condition \( \text{rank}(\Lambda \cap \text{Re}_\mathbb{R}Q) = \dim \text{Re}_\mathbb{R}Q. \)

It is easy to see that the rational closure of the cone of positive-definite quadratic forms on $\Lambda$ is a convex cone over $\mathbb{R}$. When $\Lambda = \mathbb{Z}^n$ we denote by $\text{Sym}(n, \mathbb{R})$ the space of $\mathbb{R}$-valued quadratic forms on $\mathbb{Z}^n$, and by $\text{Sym}_+(n, \mathbb{R})$ cone of all positive-definite quadratic forms in $\text{Sym}(n, \mathbb{R})$. Then the real closure of $\text{Sym}_+(n, \mathbb{R})$ in $\text{Sym}(n, \mathbb{R})$ is denoted by $\text{Sym}_+^\mathbb{R}(n, \mathbb{R})$, and the rational closure of $\text{Sym}_+(n, \mathbb{R})$ by $\text{Sym}_+^\mathbb{Q}(n, \mathbb{R})$. When we consider an arbitrary lattice $\Lambda$, rather than $\mathbb{Z}^n$, we write $\text{Sym}(\Lambda, \ast)$ instead of $\text{Sym}(n, \ast)$.

$\text{Sym}_+^\mathbb{Q}(n, \mathbb{R})$ also can be described as the real cone spanned by rank-one forms in indeterminates $(x_1, \ldots, x_n) = x$ of type $(v \cdot x)^2$ where $v$ runs over $\mathbb{Z}^n$ (see e.g. Dutour, Schuermann, Vallentin, 2006).

Sometimes the condition \( \text{rank}(\mathbb{Z}^n \cap \text{Re}_\mathbb{R}Q) = \dim \text{Re}_\mathbb{R}Q \) is phrased as that $Q$ has rational kernel, although this expression can be somewhat misleading, since $\text{Re}_\mathbb{R}Q \cap \mathbb{Z}^n$ is always a rational subspace of $\mathbb{Q}^n$. Since, $\text{Sym}_+^\mathbb{Q}(n, \mathbb{R}) \cap \text{Sym}(n, \mathbb{Q})$ consists of all positive forms with rational coefficients, perhaps, it would be more elegant to consider only $Q$-valued forms on $\mathbb{Q}^n$, but we decided to follow the tradition and embed the cone of positive $Q$-valued forms into $\text{Sym}_+^\mathbb{Q}(n, \mathbb{R})$. Let us denote by $\text{QP}_+^\mathbb{R}(n, \mathbb{R})$ the cone of all real quadratic polynomials on $\mathbb{R}^n$ whose quadratic form parts belong to $\text{Sym}_+^\mathbb{Q}(n, \mathbb{R})$. It is easy to see that $\text{QP}_+^\mathbb{R}(n, \mathbb{R})$ is a convex cone in the space $\text{QP}(n, \mathbb{R})$ of quadratic polynomials on $\mathbb{R}^n$.

When $Q : \Lambda \to \mathbb{R}$ is positive semidefinite, it defines a tiling of $\Lambda \otimes \mathbb{R}$ only when \( \text{rank}(\text{Re}_\mathbb{R}Q \cap \Lambda) = \dim \text{Re}_\mathbb{R}Q \), i.e. when $Q \in \text{Sym}_+^\mathbb{Q}(\Lambda, \mathbb{R})$ (see e.g. Dutour, Schuermann, Vallentin, 2006). In this case $\text{vert} P$ should be interpreted as $\partial P \cap \Lambda$ rather than as the set of vertices of $P$ in the sense of geometry of $\mathbb{R}^n$; for simplicity, we still call elements of $\text{vert} P$ vertices.

**Definition 6.** Let $(\Lambda, Q)$ be a quadratic lattice with $Q \in \text{Sym}_+^\mathbb{Q}(\Lambda, \mathbb{R})$. A convex polyhedron $P \subset \Lambda \otimes \mathbb{R}$ is a Delaunay polyhedron for $(\Lambda, Q)$ if there is a quadratic polynomial $E_P$ on $\Lambda \otimes \mathbb{R}$, with $\text{QForm} E_P = Q$, such that $E_P(z) = 0$ for any $z \in \Lambda \cap P$ and $E_P(z) > 0$ for any $z \in \Lambda \setminus P$.

In particular, the empty polytope $\emptyset$ and the whole space $\Lambda \otimes \mathbb{R}$ are Delaunay polyhedra in $(\Lambda, 0)$; polynomials $E_\emptyset = 1$ and $E_{\Lambda \otimes \mathbb{R}} = 0$ can serve as witnesses. It is not difficult to show that when $Q : \Lambda \to \mathbb{Q}$ is positive semidefinite, $\text{aff} \Lambda$ is covered by Delaunay polyhedra of various dimensions: some of these polyhedra are polytopes and some are cylinders over Delaunay polytopes of lower dimensions. We often refer to a Delaunay polyhedron as a Delaunay cell. In the semidefinite case the relative interiors of Delaunay cells also form a face-to-face partition of $\Lambda \otimes \mathbb{R}$, but the elements of $\Lambda$ can no longer be considered as 0-cells of the tiling – the tiling in this case does not have any 0-cells unless \( \text{rank} \Lambda = 0 \). We denote the set of all cells of the Delaunay tiling of $\Lambda \otimes \mathbb{R}$ with respect to a semidefinite form $Q$ by $\text{Del}(\Lambda, Q)$. $\text{Del}(\Lambda, Q)$ has a poset structure, namely, $F \preceq C$ if and only if $F \subset \partial C$. Furthermore, since $\emptyset, \Lambda \otimes \mathbb{R} \in \text{Del}(\Lambda, Q)$, it is a lattice. In discussions of concrete Delaunay polytopes it is often more convenient to refer to faces by their vertex sets. The partial order on $\text{Del}(\Lambda, Q)$ induces a partial order on the vertex (in the generalized sense explained above) sets of Delaunay cells of $\text{Del}(\Lambda, Q)$; we denote the resulting poset by $\text{VDel}(\Lambda, Q)$. We will need the notion of Delaunay tiling for degenerate quadratic lattices only in Subsection 3.1,
so, for the exception of that part of the paper, the reader may safely assume that \(Q\) is positive definite and all Delaunay cells are polytopes.

For formal definitions and detailed information on Delaunay tilings of lattices we refer to Deza and Laurent (1997). We only remark that the Delaunay tilings for lattices are classically (Delaunay, 1924) defined with the Euclidean norm \(x_1^2 + \ldots + x_n^2\) (in geometry of numbers the norm of a vector is its squared length), but are most effectively studied by isomorphically mapping the lattice \(\Lambda\) onto \(\mathbb{Z}^n\), and replacing the Euclidean norm by a positive quadratic form \(Q\) that makes \((\mathbb{Z}^n, Q)\) and \((\Lambda, \sum x_i^2)\) equivalent. This allows us to think in terms of Euclidean lattices, i.e. geometrically, but compute in terms of quadratic forms.

**Definition 7.** Let \((\Lambda, Q)\) be a quadratic lattice with \(Q \in \text{Sym}_+^Q(\Lambda, \mathbb{R})\). Suppose \(P \in \text{Del}(\Lambda, Q)\). \(P\) is called perfect if its Delaunay ellipsoid (or elliptic cylinder, if \(\text{rank} \ Q < \text{rank} \ \Lambda\)) with respect to \(Q\) is the only quadric circumscribed about \(P\) in \(\Lambda \times \mathbb{R}\).

Indeed, the notion of perfection, that was introduced in 19th century by the Italian school of algebraic geometry, is independent of the Delaunay property of \(P\) and nature of \(Q\). More generally, let \(F\) be a finite-dimensional linear space of \(\mathbb{R}\)-valued functions on \(\Lambda\). Then a set \(R \subset \Lambda\) is called perfect with respect to \(F\) if the system of linear inhomogeneous equations \(\{f(r) = c \mid r \in R\}\) on the coefficients of \(f\) has a unique solution in \(F\) for any \(c \neq 0\).

We have a natural bijection between perfect affine quadratic lattices \(\text{Aff}(\Lambda, Q)\) and triples \((\Lambda, P, \rho^2)\), where \(P\) is a perfect Delaunay polyhedron and \(\rho^2 \geq 0\) is the squared radius of its Delaunay ellipsoid. Thus, there are only finitely many arithmetically inequivalent perfect Delaunay polyhedra in each dimension, up to rescaling. Furthermore, since for perfect Delaunay polyhedra arithmetic equivalence implies isometry, there are only finitely many nonisometric perfect Delaunay polytopes in each dimension, up to rescaling. For \(n = 0\) there is only one perfect Delaunay and it is \(0\). For \(n = 1\) the polytope \([0, 1]\) is perfect and Delaunay in \((\mathbb{Z}, x^2)\), and it is unique up to arithmetic equivalence.

### 3 Laminar Structure of Delaunay Cells

It is easy to see that a section of the vertex set of a Delaunay polytope by a rational affine subspace is the vertex set of a Delaunay polytope in the induced sublattice. This observation suggests a recursive approach to studying Delaunay polytopes where each newly discovered Delaunay polytope is represented as a union of Delaunay polytopes of smaller dimensions lying in parallel subspaces. Indeed, for \(n > 1\) such a representation is never unique. It has been observed that dealing with a smaller numbers of big laminae is easier than studying a large number of small laminae. In other words, one is usually working with a representation in which the number of laminae is as small as possible.

**Definition 8.** The lamina number \(l(P)\) of a lattice polytope \(P\) in a lattice \(\Lambda\) is the minimal number of disjoint affine subspaces of \(\Lambda \otimes \mathbb{R}\) whose intersections with \(\text{vert} \ P\) form a partition of \(\text{vert} \ P\) into proper subsets.

The natural question is what laminar constructions lead to perfect Delaunay polytopes. In particular, is it possible to construct perfect Delaunay polytopes by using non-trivial (of dimension greater than 1) lower-dimensional perfect Delaunay polytopes as some of the
laminae? It turns out for \( n = 6 - 8 \) such constructions are rather common, although not all of these polytopes have sections that are non-trivial perfect polytopes of smaller dimensions. The only perfect Delaunay 6-polytope, Gosset’s \( G_6 \), does not have non-trivial perfect sections. The only two known perfect 7-polytopes, Gosset’s \( G_7 \) and the 35-tope found by Erdahl and Rybnikov, each have a section isometric to \( G_6 \). Our study showed that of the 27 known perfect Delaunay 8-polytopes 17 have a section which is isometric to \( G_6 \), of which 10 have a section isometric to the 35-tope and one has a section isometric to \( G_7 \). The remaining 10 perfect 8-polytopes do not have non-trivial perfect sections.

The lamina number \( l(P) \) is closely related to the notion of lattice width. Denote by \( \Lambda_\mathcal{Q} \subset \text{aff}\Lambda \) the dual of \( \Lambda \) with respect to the bilinear form \( \mathcal{Q} \). \( \Lambda_\mathcal{Q} \) consists of all vectors of \( \text{aff}\Lambda \) whose \( \mathcal{Q} \)-products with vectors of \( \Lambda \) are integer. If \( B \) is a convex body in \( \text{aff}\Lambda \), then the width \( w(B) \) of \( B \) with respect to \( (\Lambda, \mathcal{Q}) \) is defined as the minimal natural number \( w \) such that \( B \) lies between hyperplanes \( \mathcal{Q}(a^\ast, x) = k \) and \( \mathcal{Q}(a^\ast, x) = k + w \), for some \( a^\ast \in \Lambda^\ast \). It is widely believed (see e.g. Barvinok, 2002) that a body in \( \text{aff}\Lambda \) whose interior is empty of \( \Lambda \)-points cannot have lattice width exceeding \( \text{rank}\Lambda \).

**Proposition 9.** If \( P \) is a Delaunay polytope, then \( l(P) = w(P) + 1 \).

We do not know of any Delaunay polytopes whose lattice width exceeds 2. On the other hand, we have

**Theorem 10.** If \( P \) is a perfect (need not be Delaunay) polytope of dimension \( n > 1 \), then \( w(P) + 1 = l(P) > 2 \).

**Proof.** Let \( (\Lambda, \mathcal{Q}) \) be the lattice in which \( P \) is perfect. Since the partition into laminae must be proper, \( l(P) > 1 \). If \( l(P) = 2 \), then there are affine sublattices \( L_1 \) and \( L_2 \) of codimension 1 such that \( \text{vert} P = (\text{vert} P \cap L_1) \sqcup (\text{vert} P \cap L_2) \). There exists an affine function \( A \) on \( \Lambda \), which is 1 on \( L_1 \) and 0 on \( L_2 \). Then the quadric \( Ax(Ax - 1) = 0 \) is circumscribed about \( P \). If \( Q \) is the quadratic function defined by \( P \) uniquely up to scale, then \( Q + A(A - 1) \) must be proportional to \( Q \), which means that \( \text{QForm}Q \) is of rank 1. Since \( P \) is perfect, \( \dim P = \text{rank}\Lambda \). Since \( P \) is a polytope, \( \text{rank}\Lambda = \text{rank}\ Q\text{Form}Q = 1 \) and \( \dim P = 1 \).

It turns out that all perfect Delaunay polytopes in dimensions \( n = 6 - 8 \) have the lamina number \( l \) equal to 3.

**Theorem 11.** Each perfect Delaunay polytope \( P \) described in Sections 6–8 has \( l(P) = 3 \)

**Proof.** \( \text{vert} G_6 \) has a 3-laminae partition into a vertex, a 5-half-cube, and a 5-cross-polytope (Erdahl, Rybnikov 2002); another partition is into a 5-simplex, a 15-vertex polytope, and another 5 simplex (see Erdahl, 1992). The partition of \( \text{vert} G_7 \) into the union of the vertex sets of a 6-half-cube and two 6-cross-polytopes is given in Lemma 7.1. The partition of \( \Upsilon_7 \) is given in Lemma 7.2 The partitions of the vertex sets of perfect 8-polytopes into layers follow from their coordinate representation given in Section 8.
3.1 Structure of Perfect Affine Lattices

Recall that a pair \((\Lambda, Q)\), where \(\Lambda\) is a lattice and \(Q : \Lambda \to \mathbb{R}\) a quadratic function, is called perfect if the coefficients of \(Q\) are uniquely determined from equations \(X(m) = \min\{Q(z) \mid z \in \Lambda\}\), where \(m\) runs over all minimal vectors of \(Q\) and \(X\) is an unknown quadratic function on \(\Lambda\). In general, for a function \(F\) defined on a lattice \(\Gamma\) denote by \(V(F)\) the variety of \(F\), i.e., the set of lattice points where \(F\) is 0. Perfection is a very natural notion as illustrated by the following theorem of Erdahl (1992). Recall that if \(L\) is an affine sublattice of a lattice \(\Gamma\), then \(\overrightarrow{L} \) stands for the lattice \(L - L\).

Theorem 12. \((\Lambda, Q)\) is perfect if and only if \(V(Q) = \{v + z \mid v \in \text{vert } P, z \in \Gamma\}\), where \(P\) is a perfect Delaunay polytope in \(\text{Aff}(\Lambda \cap \text{aff } P, Q|_{\text{aff } P})\) and \(\Gamma\) is a sublattice of \(\Lambda\) such that \(\Lambda\) is the direct affine sum of \(\Lambda \cap \text{aff } P\) and \(\Gamma\) over \(\mathbb{Z}\), i.e.,

\[
\Lambda = \{(x - x') + z \mid x, x' \in \Lambda \cap \text{aff } P, z \in \Gamma\}.
\]

On the basis of this characterization Erdahl and Rybnikov proved the following theorem (Erdahl, Ordine, Rybnikov, 2004).

Theorem 13. Let \(P\) be a perfect polytope in \(\text{Del}(\Lambda, Q)\) and let \(Q_P\) be its perfect quadratic function, i.e. \(\text{vert } P = V(Q_P)\) and \(Q = Q \circ \text{Form } Q_P\). Suppose \(D \in \text{Del}(\Lambda, Q)\) is another Delaunay cell of full dimension, which is not a \(\Lambda\)-translate of \(P\). If \(e \notin \Lambda\), then there is a positive definite form \(Q'\) on \(\Lambda \oplus \mathbb{Z}e\) and a perfect polytope \(P'\) in \(\text{Del}(\Lambda \oplus \mathbb{Z}e, Q')\) such that \(P' \cap \text{aff } \Lambda = P\) and \(P' \cap \{(\text{aff } \Lambda) + e\} = D + e\).

The delicate part here is the case where \(P\) and \(D\) have identical Delaunay radii. By using the following refinement of Erdahl’s (1992) theorem it is possible to prove that under the assumptions of Theorem 13 a perfect polytope \(P'\) that contains as sections \(P\) and \(D\) cannot be unbounded (Erdahl, Ordine, Rybnikov, 2004). On the basis of Theorem 13 we can make a few useful observations.

- If \(P\) is an antisymmetric perfect polytope in \(\text{Del}(\Lambda, Q)\), then there is a perfect polytope \(P'\) in \(\text{Del}(\Lambda \oplus \mathbb{Z}e, Q')\), for some form \(Q\), with a section isometric to \(P\). For example, for \(P = G_6\) (Gosset’s 6-polytope) there are two perfect 7-polytopes that have \(G_6\) as a section, namely \(G_7\) and the 35-tope. \(G_7\) can be obtained by taking \(D = -P\) in Theorem 13, while the 35-tope cannot be obtained by a direct application of Theorem 13.

- We know only one example of a Delaunay tiling formed by translates of a centrally-symmetric perfect Delaunay polytope: this is the tiling of \(\mathbb{Z}^1\) by unit intervals. Incidentally, we do not know of any Delaunay polytope, except for the \(n\)-cube, that tiles \(\mathbb{R}^n\) by translation. We conjecture that for \(n > 1\) there are no such examples. If this is true, then Theorem 13 gives a construction for a new perfect Delaunay polytope in dimension \(n + 1\) from a perfect Delaunay polytope in dimension \(n\). However, this construction is not uniquely defined, since in Theorem 13 there may be different choices of \(D\).
• When \( P \) is a centrally symmetric perfect \( n \)-polytope in \( \text{Del}(\Lambda, \mathcal{Q}) \) and there is an antisymmetric \( n \)-polytope \( D \) in \( \text{Del}(\Lambda, \mathcal{Q}) \), it may happen that the center of the perfect polytope \( P' \) coincides with the center of \( P \). Then \( P' \) has at least three \( n \)-dimensional layers, which are translates of \(-D, P, \) and \( D \) respectively. The only 8-dimensional polytope from our list that has a section isometric to \( G_7 \) arises from this construction. The role of \( D \) is played by a Delaunay simplex of double volume in the Delaunay tiling defined by \( G_7 \) (see Erdahl, 1992 for a description).

• For many a perfect 8-polytope the Delaunay tiling has a significant number of arithmetically inequivalent 8-cells. This suggests that starting from \( n = 9 \) the number of perfect Delaunay \( n \)-polytopes explodes. (see http://www.liga.ens.fr/~dutour for the enumeration) It is likely that \( n = 8 \) is the highest dimension in which a complete classification is within reach.

4 Symmetries of Perfect Delaunay Polytopes

Recall that the group \( O(\mathbb{Z}^n, \mathcal{Q}) \) of linear automorphisms of a quadratic lattice \((\mathbb{Z}^n, \mathcal{Q})\) is defined as the full subgroup of \( O(\mathbb{R}^n, \mathcal{Q}) \) that maps \( \mathbb{Z}^n \) onto itself, in other words, the set-stabilizer of \( \mathbb{Z}^n \) in \( O(\mathbb{R}^n, \mathcal{Q}) \):

\[
O(\mathbb{Z}^n, \mathcal{Q}) = \{ \tau \in O(\mathbb{R}^n, \mathcal{Q}) \mid \tau(\mathbb{Z}^n) = \mathbb{Z}^n \}.
\]

The group \( O(\mathbb{Z}^n, \mathcal{Q}) \) can also be seen as the subgroup of \( GL_n(\mathbb{Z}) \) that consists of transformations preserving \( \mathcal{Q} \):

\[
O(\mathbb{Z}^n, \mathcal{Q}) = \{ \tau \in GL(n, \mathbb{Z}) \mid \forall z \in \mathbb{Z}^n : \mathcal{Q}(\tau z) = \mathcal{Q}[z] \}.
\]

Denote by \( Iso(\mathbb{R}^n, \mathcal{Q}) \) the group of affine automorphisms of \( \mathbb{R}^n \) which preserve \( \mathcal{Q} \). If \( D \) is a \( \mathbb{Z}^n \)-polytope, then \( Iso(D, \mathcal{Q}) \) denotes the group of all transformations from \( Iso(\mathbb{R}^n, \mathcal{Q}) \) that map \( D \) onto itself. Denote by \( LatIso(D, \mathcal{Q}) \) the group of all transformations from \( O(\mathbb{Z}^n, \mathcal{Q}) \) that map \( D \) to itself. Clearly \( LatIso(D, \mathcal{Q}) \leq Iso(D, \mathcal{Q}) \). When \( \{ x - y \mid x, y \in \text{vert} D \} = \mathbb{Z}^n \), the polytope \( D \) is called generating. All known perfect Delaunay polytopes are generating. Obviously, for generating polytopes \( Iso(D, \mathcal{Q}) = LatIso(D, \mathcal{Q}) \).

An important invariant of an extreme Delaunay polytope \( D \) in \((\mathbb{Z}^n, \mathcal{Q})\) is the dimension of the subspace of quadratic forms in \( n \) variables preserved by \( O(\mathbb{Z}^n, \mathcal{Q}) \). We denote this space by \( \text{QuadInv}[D] \).

The metric geometry of a Delaunay polytope \( D \) is reflected in the norm spectrum of \( D \), which is just the set of all possible value for \( \mathcal{Q}[x - y] \), where \( x, y \in \text{vert} D \). We denote the norm spectrum by \( \text{Spec}(D) \).

We have classified the isometry groups of all known perfect Delaunay polytopes for \( n \leq 8 \). The isometry groups of six- and seven-dimensional perfect polytopes are distinct. Among the isometry groups of the 27 8-polytopes there are 21 non-isomorphic. Polytopes in the following five groups have isomorphic groups: \#2 and \#5; \#3 and \#13; \#12 and \#21; \#14, \#19, and \#25; \#24 and \#27. The most interesting is the case of \#2 and \#5: both polytopes have 72 vertices in two orbits of size 56. Their group contains \( S_8 \) as a subgroup of index 2.
5 Perfect Affine Quadratic Lattices for $n < 6$

Aff($0,0$) is a perfect affine lattice of rank 0. All perfect affine lattices of rank 1 are obviously equivalent, up to scaling, to Aff($\mathbb{Z}^1, (x-\frac{1}{2})^2$). The inequality $(x-\frac{1}{2})^2 \leq \frac{1}{4}$ describes the Delaunay ellipsoid for the Delaunay polytope $[0,1]$. The Delaunay tiling for $(\mathbb{Z}^1, \text{QForm}(x-\frac{1}{2})) = (\mathbb{Z}^1, x^2)$ consists of points of $\mathbb{Z}^1$, which are 0-dimensional polytopes, and segments $[k,k+1]$, where $k \in \mathbb{Z}$, which are 1-dimensional polytopes of the tiling. The symmetry group of $[0,1]$ consists of 2 elements and is generated by reflection about $\frac{1}{2}$. Surprisingly, there are no perfect affine modules of ranks 2, 3, 4, and 5. This was first proven by Erdahl in 1975 (see also Erdahl, 1992).

6 Perfect affine quadratic lattices of rank 6

The affine lattice Aff($\mathbb{Z}^6, E_6[x-c]$), where $E_6$ is given by

$$E_6(x) = x^t\begin{array}{cccccc} 4 & 3 & 3 & 3 & 3 & 5 \\ 3 & 4 & 3 & 3 & 3 & 5 \\ 3 & 3 & 4 & 3 & 3 & 5 \\ 3 & 3 & 3 & 4 & 3 & 5 \\ 5 & 5 & 5 & 5 & 5 & 8 \end{array} x \quad \text{and} \quad c = \frac{1}{3} \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -2 \end{array},$$

turns out to be perfect, which was first observed by Erdahl (1975). Quadratic form $E_6$ is of type $E_6$, that is $(\mathbb{Z}^6, E_6[x])$ is equivalent, up to scaling, to $(E_6, \sum x_i^2)$, where $E_6$ is a well-known root lattice in $\mathbb{R}^6$ (see, e.g. Coxeter, 1991). Lattice $(E_6, \sum x_i^2)$ has first been constructed by Korkin and Zolotarev (1873), to which they referred as the fifth perfect form in 6 variables and denoted it by $X$.

Inequality $E_6[x-c] \leq \frac{1}{2}$ defines the Delaunay ellipsoid for a Delaunay polytope in $(\mathbb{Z}^6, E_6)$. The set of 27 vertices of this polytope is given by the following table ($[-1,0^4;-1]$ means the entry 0 is repeated 4 times and all permutations of the first 5 positions are taken – the last entry is separated by semicolon and is not permuted; the other records of the table are interpreted similarly).

| $x_6$  | $x_6 = -3$ | $x_6 = -2$ | $x_6 = -1$ | $x_6 = 0$  | $x_6 = 1$  |
|--------|-------------|-------------|-------------|-------------|-------------|
| $[1^5; -3] \times 1$ | $[0, 1^4; -2] \times 5$ | $[1^2, 0^3; -1] \times 10$ | $[0^6] \times 1$ | $[1, 0^4; -1] \times 5$ |

This polytope is known as Gosset’s (1900) 6-dimensional semiregular polytope, which we denote by $G_6$. It is a two distance set and $\text{Spec}(G_6) = \{2, 4\}$. Typically, $G_6$ is described as a Delaunay polytope for $(E_6, \sum x_i^2)$, in which case it is commonly denoted by 221 – the notation going back to Coxeter (see his 1988 paper for history). $G_6$ has two orbits of facets, regular simplices and regular cross-polytopes. It does not have interior diagonals and all segments joining its vertices are either edges, or diagonals of its facets. The 1-skeleton of $G_6$ is a strongly regular graph known as Schlafli graph. Its isometry group $\text{Iso}(G_6, E_6)$,
of order 51820, is the famous group of automorphisms of the 27 lines on a general cubic surface. \( \text{Iso}(G_6, E_6) \) is isomorphic to the semidirect product of a 2-element group generated by a reflection and a reflection-free normal that consists of \( 2^5 \times 3^4 \times 5 = 25920 \) elements; the latter group is simple and has a number of descriptions as a group of Lie type (see ATLAS, 1986, for more details). The isometry group of \( G_6 \) is transitive on its vertex set. A remarkable property of \( G_6 \) is that the convex hull of any subset of \( \text{vert } G_6 \) is a Delaunay polytope for \( (Z^6, Q) \) for some positive definite form \( Q \) (Erdahl and Rybnikov, 2002). For more details see Coxeter (1988) and Erdahl & Rybnikov (2002).

6.0.1 Laminar Structure of Gosset’s \( G_6 \)

Let us denote by \( J(n,s) \) the polytope formed by all \( \{0,1\} \)-vectors in \( (Z^n, \sum_{i=1}^{n} x_i) \) with the sum of the coordinates equal to \( s \). It is known that for each \( s \), such that \( 0 \leq s < n \), the polytope \( J(n,s) \) is isometric to a Delaunay polytopes in \( (A_{n-1}, \sum_{i=1}^{n-1} x_i) \), where \( A_{n-1} \) is a root lattice of type \( A \) of rank \( n - 1 \). \( G_6 \) can be represented as the union of 3 laminae that are isometric to \( J(6,1) \), \( J(6,2) \), and \( J(6,1) \), where the regular 6-simplexes ( \( J(6,1) \)) are parallel (see Erdahl, 1992).

It was long suspected that \( \text{Aff}(Z^6, E_6[x - c]) \) is the only perfect affine lattice of rank 6 up to scaling. Finally, Dutour (2004), using his \( \text{EXT-HYP7} \) program, created in 2002, proved that this is the case.

7 Perfect Affine Quadratic Lattices of rank 7

7.1 Gosset Polytope in Lattice \( E_7 \)

The affine lattice \( \text{Aff}(Z^7, E_7[x - c]) \), where \( E_7 \) is given by

\[
\begin{align*}
\mathcal{E}_7(x) &= x^t \begin{bmatrix} 4 & 3 & 3 & 3 & 3 & 5 & 4 \\
3 & 4 & 3 & 3 & 3 & 5 & 4 \\
3 & 3 & 4 & 3 & 3 & 5 & 4 \\
3 & 3 & 3 & 4 & 3 & 5 & 4 \\
5 & 5 & 5 & 5 & 5 & 8 & 6 \\
4 & 4 & 4 & 4 & 4 & 6 & 6 \\
\end{bmatrix} x \\
\begin{bmatrix} 0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
1 \\
\end{bmatrix} \\
\begin{align*}
c &= \frac{1}{2}
\end{align*}
\]

turns out to be perfect, which was first observed by Erdahl (1975). Quadratic form \( E_7 \) is of type \( E_7 \), that is \( (Z^7, E_7) \) is equivalent, up to scaling, to \( (E_7, \sum x_i^2) \), where \( E_7 \) is a well-known root lattice in \( \mathbb{R}^7 \) (see, e.g. Coxeter, 1991). Lattice \( (E_7, \sum x_i^2) \) has first been constructed by Korkine and Zolotareff (1873), to which they referred as the sixth perfect form in 7 variables and denoted it by \( Y \).

Inequality \( E_7[x - c] \leq 3 \) defines the Delaunay ellipsoid for a polytope in \( \text{Del}(Z^7, E_7) \), whose vertex set is given below.
\[
\begin{array}{|c|c|c|c|}
\hline
x_7 = -1 & x_7 = 0 & x_7 = 1 & x_7 = 2 \\
\hline
[1^5; -2; -1] \times 1 & [0^7] \times 1 & [0^6; 1] \times 1 & [-1^5; 2; 2] \times 1 \\
\hline
\end{array}
\]

This centrally-symmetric polytope has 56 vertices and is known as Gossel’s (1900) 7-dimensional semiregular polytope, which we denote by \(G_7\). It is a 3-distance set and \(\text{Spec}(G_7) = \{2, 4, 6\}\). Typically, \(G_7\) is described as a Delaunay polytope for \((E_7, \sum x_i^2)\), in which case it is commonly denoted by \(3_{21}\) after Coxeter (see Coxeter (1988) for history). The 1-skeleton of \(G_7\) is known as Gossel graph, which is a strongly-regular graph. \(G_7\) has 28 interior diagonals passing through its center; in fact, Patrick du Val discovered that \(G_7\) can be thought of as the convex hull of seven congruent 3-dimensional cubes in \(\mathbb{R}^7\) with common center (see Coxeter, 1998). The isometry group of \(G_7\) is transitive on vertices and is isomorphic to the semidirect product of a 2-element group generated by a reflection and a reflection-free normal subgroup of \(4 \times 9! = 1451520\) elements; the latter group is simple and isomorphic, among other groups of Lie type, to \(O_7(2)\) (see ATLAS, 1986).

**Lemma 7.1.** \(l(G_7) = 3\) and \(\text{vert}(G_7)\) is the union of the vertex sets of two 6-cross-polytopes and a 6-half-cube.

**Proof.** The fact \(l(G_7) = 3\) follows from the representation of \(G_7\) as the following subset of the 7-cube \([-1, +1]^7\). Consider all cyclic permutations of 8 vectors \((\pm 1, \pm 1; 0; \pm 1; 0, 0, 0)\). This defines a 56-element subset \(V\) of \([-1, +1]^7\). P. Du Val (attributed to du Val by Coxeter, 1988) had shown that the convex hull of these 56 points is the Gosset 7-polytope in the lattice \(\text{aff}_Z V = \mathbb{Z}^7\) with respect to the usual metric \(\sum x_i^2\). Note that \(\text{conv} V \notin \text{Del}(\mathbb{Z}^7, \sum x_i^2)\), since \(\text{conv} V\) obviously contains the origin.

In each of the coordinate directions \(\text{conv} V\) has three layers defined by inequalities \(x_i = -1, 0, +1\) and thus \(l(\text{conv} V) = l(G_7) = 3\). It is easy to see that the sections \(x_i = -1\) and \(x_i = +1\) are 6-cross-polytopes and the section \(x_i = 0\) is a 6-half-cube. □

### 7.1.1 Laminar Structure of Gosset’s \(G_7\)

\(G_7\) can be represented as the union of 4 laminae that are isometric to \(J(7, 1), J(7, 2), J(7, 2), \) and \(J(7, 1)\) (obviously, \(J(7, 1)\) is regular simplex). Another lamination of \(\text{vert} G_7\) is into 3 layers that consist of vertex sets of a 6-cross-polytope, 6-halfcube, and another 6-cross-polytope. Yet another lamination is given in the above table, where the layers are a 0-simplex, a copy of \(G_6\), another copy of \(G_6\), and another 0-simplex. All these laminations correspond to the unique subdiagrams of types \(A_6, D_6, \) and \(E_6\) of the Coxeter diagram \(E_7\), which represents the isometry group of \(G_7\) (see Humphreys, 1990).
7.2 The 35-tope

The only known perfect affine lattice of rank 7, that is not equivalent to \( \text{Aff}(\mathbb{Z}^7, \mathcal{E}_7[\mathbf{x} - \mathbf{c}]) \), was constructed by Erdahl and Rybnikov in 2000 (see Erdahl and Rybnikov (2002) and Rybnikov (2001)). It is \( \text{Aff}(\mathbb{Z}^7, \mathcal{E}_7 \mathcal{R}_7[\mathbf{x} - \mathbf{c}]) \), where \( \mathcal{E}_7 \mathcal{R}_7 \) is given by

\[
\mathcal{E}_7 \mathcal{R}_7(\mathbf{x}) = \mathbf{x}^t \begin{pmatrix}
8 & 6 & 6 & 6 & 6 & 6 & 9 \\
6 & 8 & 6 & 6 & 6 & 6 & 9 \\
6 & 6 & 8 & 6 & 6 & 6 & 9 \\
6 & 6 & 6 & 8 & 6 & 6 & 9 \\
6 & 6 & 6 & 6 & 8 & 6 & 9 \\
9 & 9 & 9 & 9 & 9 & 9 & 13 \\
\end{pmatrix} \quad \text{and} \quad \mathbf{c} = \frac{1}{16}
\]

The lattice \( (\Lambda, \mathcal{E}_7 \mathcal{R}_7) \) has 12 shortest vectors and \( \det \Lambda_{\mathcal{E}_7 \mathcal{R}_7} = 256 \). The order of \( O(\Lambda, \mathcal{E}_7 \mathcal{R}_7) \) is 2880 and the dimension of the space of invariant forms is 3. \( (\Lambda, \mathcal{E}_7 \mathcal{R}_7) \) is not perfect, but the lattice obtained from \( (\Lambda, \mathcal{E}_7 \mathcal{R}_7) \) by adding the centers of perfect ellipsoids is perfect with 70 shortest vectors.

Inequality \( \mathcal{E}_7 \mathcal{R}_7(\mathbf{x} - \mathbf{c}) \leq \frac{43}{16} \) defines the Delaunay ellipsoid for a perfect Delaunay \( \Upsilon^7 \) in \( \text{Del}(\mathbb{Z}^7, \mathcal{E}_7 \mathcal{R}_7) \), whose vertex set is given below.

\[
\begin{array}{c}
x_7 = -4 \\
[1^6; -4] \times 1
\end{array}, \quad
\begin{array}{c}
x_7 = -3 \\
[0, 1^5; -3] \times 6
\end{array}, \quad
\begin{array}{c}
x_7 = -1 \\
[1^2, 0^4; -1] \times 15
\end{array}, \quad
\begin{array}{c}
x_7 = 0 \\
[0^7] \times 1
\end{array}, \quad
\begin{array}{c}
x_7 = 1 \\
[-1, 0^5; 1] \times 6
\end{array}
\]

\[
\text{Spec}(\Upsilon^7) = \{3, 4, 5, 7, 8, 9\}. \quad \text{Polytope } \Upsilon^7 \text{ generalizes to an infinite series of perfect Delaunay polytopes } \Upsilon^n \quad (n \geq 7) \text{ with } \frac{n(n+3)}{2} \text{ vertices (see Erdahl, Rybnikov, 2002):}
\]

\[
\begin{array}{c}
x_n = -(n - 3) \\
[1^{n-1}; -(n - 3)] \times 1
\end{array}, \quad
\begin{array}{c}
x_n = -(n - 4) \\
[0, 1^{n-2}; -(n - 4)] \times (n - 1)
\end{array}, \quad
\begin{array}{c}
x_n = -1 \\
[1^2, 0^{n-3}; -1] \times \frac{(n-1)(n-2)}{2}
\end{array}
\]

\[
\begin{array}{c}
x_n = 0 \\
[0^n] \times 1
\end{array}, \quad
\begin{array}{c}
x_n = 1 \\
[-1, 0^{n-2}; 1] \times (n - 1)
\end{array}
\]

Polytope \( \Upsilon^7 \) has lamina number \( l(\Upsilon^7) = 3 \) and can be represented as the union of Gosset polytope \( G_6 \) and regular simplices of dimensions 2 and 4 lying in parallel subspaces of \( \mathbb{R}^7 \):

**Lemma 7.2.** The vertex set of \( \Upsilon^7 \) is, up to scaling, isometric to the disjoint union of the vertex sets of a Gosset polytope \( G_6 \), a regular 5-simplex, and a 1-simplex.
\textit{Proof.} Consider the following partition of
\[
\text{vert } \mathcal{Y}^7 = S_1 \bigsqcup S_2 \bigsqcup S_3 = \\
\{[-1, 0^5; 1] \times 6, [0, 1^5; -3] \times 6, [1^2, 0^4; -1] \times 15\} \bigsqcup \{[0^5, 1; 0] \times 6\} \bigsqcup \{[0^7, 1^6; -4]\}.
\]
Let us show that the affine rank of the first subset is 6. The first subset \(S_1\) can be represented as \(S_1 \bigsqcup S_{12} \bigsqcup S_{13}\), where \(\{S_{11} = [-1, 0^5; 1] \times 6\}, \{S_{12} = [0, 1^5; -3] \times 6\}, \{S_{13} = [1^2, 0^4; -1] \times 15\}\).

Notice that
\[
[-1, 0, 0, 0, 0, 0, 0, 1] = [0, 1, 1, 0, 0, 0, -1] + ([0, 0, 0, 1, 1, 0, -1] - [1, 1, 1, 1, 1, 0, -3])
\]
Applying cyclic permutations of the first six characters to the above identity, we see that each element of \(S_{11}\) can be written as \(p_1 + (p_2 - p)\), where \(p_1, p_2 \in S_3\) and \(p \in S_2\). Therefore, \(\text{aff}(S_{11} \cup S_{12} \cup S_{13}) = \text{aff}(S_{12} \cup S_{13})\). Since both \(S_{12}\) and \(S_{13}\) lie on the hyperplane \(2(x_1 + x_2 + x_3 + x_4 + x_5 + x_6) + 3x_7 = 1\), \(\dim \text{aff}(S_1 \cup S_2 \cup S_3) = 6\). Notice that this argument does not work if we replace \(\mathcal{Y}^7\) with \(\mathcal{Y}^n\) for \(n > 7\). It turns out that for \(n > 7\) we have \(\text{aff}(S_{11} \cup S_{12} \cup S_{13}) \neq \text{aff}(S_{12} \cup S_{13})\).

It is clear that the affine subspaces generated by \(S_2\) and \(S_3\) are parallel to that generated by the first subset. Computing squared distances between the elements of the first subset with metric \(\mathcal{E}\mathcal{R}_7\) shows that it is a two-distance set isometric to \(G_6\) (dilated by a factor of 2). The second and the third sets are obviously regular simplices. \(\square\)

8 \textbf{Perfect Affine Quadratic Lattices of Rank 8}

We denote the perfect quadratic functions by \(Q_8^s[x - c]\), where \(c \in \mathbb{Q}^n\), and the corresponding perfect Delaunay polytopes by \(D_8^s\). For each \(Q_8^s[x - c]\) we give
\begin{itemize}
  \item an integer Gram matrix,
  \item the center \(c\) of the perfect ellipsoid,
  \item the order of the group \(O(\mathbb{Z}^8, Q_8^s)\) (the group’s generators are available from the first author upon request), together with the size of the maximal symmetric subgroup,
  \item the number \(s(\mathbb{Z}^8, Q_8^s)\) of shortest vectors,
  \item the dimension of the subspace of \(\text{Sym}(8, \mathbb{R})\) that consists of forms \(Q\) such that \(Q[Tz] = Q[z]\) for every \(T \in O(\mathbb{Z}^8, Q_8^s)\), we denote this subspace by \(\text{QuadInv}[O(\mathbb{Z}^8, Q_8^s)]\).
\end{itemize}

For each \(D_8^s\) we give
\begin{itemize}
  \item the coordinates of the vertices,
  \item \(|\text{Iso}(D_8^s, Q_8^s)| = \text{Order of the isometry group of } D_8^s|\),
\end{itemize}
• Whether $D^8_i$ is Centrally-symmetric or Antisymmetric,

• Maximal non-trivial perfect polytope of smaller dimension (i.e. $G_6$, $G_7$, or the 35-tope) contained in $D^8_i$,

• Information on certain types of Delaunay polytopes contained in $D^8_i$. If $X$ is an arithmetic type of a lattice Delaunay polytope such as, e.g., $J(n, s)$, and $D^8_i$ contains a copy of $J(n, s)$, which is not a proper subpolytope of a $J(n', s') \subset D^8_i$, then we state that $J(n, s)$ is maximally included into $D^8_i$.

• the norm spectrum $\text{Spec}(D^8_i)$.

• the lamina number $l(D^8_i)$ (always 3).

The coordinatization of all polytopes is chosen so that the three laminae structure is transparent. For some of the polytopes we give additional geometric information.

8.1 Delaunay Tilings of Lattices $A_n$ and $D_n$

The geometric structure of 8-dimensional perfect Delaunay polytopes can be analyzed by relating the geometry of these polytopes to the geometry of Delaunay tilings of lattices $A_n$ and $D_n$, which is explicit in our 8-dimensional data. In this section let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{Z}^n \subset \mathbb{E}^n$, and let $I = \{x \in \mathbb{R}^n \mid \forall i : 0 \leq x \cdot e_i \leq 1\}$ denote the standard unit cube.

$A_n$ can be defined as $(\mathbb{Z}^n, \sum_{i=1}^n x_i^2 + \sum_{1 \leq i < j \leq n} x_i x_j)$ or, in terms of the Euclidean space $\mathbb{E}^n$, as the lattice based on a regular $n$-simplex. Lattice $D_n$ can be defined as $(\mathbb{Z}^n, \sum_{i=1}^n x_i^2 + x_1 x_3 + \sum_{2 \leq i < j \leq n} x_i x_j)$ or, in terms of the Euclidean space $\mathbb{E}^n$, as the sublattice of $\mathbb{Z}^n$ that consists of all points with even sum of the coordinates; another Euclidean construction of $D_n$ is obtained by taking $\mathbb{Z}^n$ and adding to it the centers of all facets of the unit $n$-cubes with integral vertices – this is an $n$-dimensional generalization of what is known in crystallography as the face-centered cubic lattice, or fcc. Note that for $n = 3$, $A_n$ and $D_n$ coincide.

Delaunay tilings of $A_n$ have been described by Barnes (1959) and Delaunay tilings of $D_n$ have been described by Ryshkov and Shushbaev (1981). Below we give a brief description of these tilings borrowed from Baranovski (1991).

Let $d = \sum e_i$. Consider the sections of the standard unit cube by hyperplanes perpendicular to $d$ and passing through the points $\frac{2}{n}d$ for $q = 1, \ldots, n$. These hyperplanes induce a partition of $I$ into $n$ $n$-polytopes $P(q)$, where each $P(q)$ is squeezed between hyperplanes $\sum x_i = q - 1$ and $\sum x_i = q$. It can be shown (see Barnes (1959) or Baranovski (1991)) that with respect to quadratic form $\sum_{i=1}^n x_i^2 + \sum_{1 \leq i < j \leq n} x_i x_j$ these polytopes are Delaunay. Thus, any Delaunay $n$-polytope in $A_n$ is a translate of one these polytopes. The 1-skeletons of the faces of $P(q)$ that are defined by $P(q) \cap \{x \in \mathbb{R}^n \mid \sum x_i = q - 1\}$ and $P(q) \cap \{x \in \mathbb{R}^n \mid \sum x_i = q\}$ are Johnson graphs $J(n, q - 1)$ and $J(n, q)$. We also use $J(n, q - 1)$ and $J(n, q)$ to refer to the arithmetic classes of these polytopes. Note that $J(n, q)$ and $J(n, n - q)$ are isometric with respect to any quadratic form on $\mathbb{Z}^n$, since one of them can be obtained from the other by a combination of a lattice translation and an inversion with respect to a lattice point; in the terminology of geometry of numbers such polytopes are known as homologous.
Let us consider $D_n$ as the sublattice of $\mathbb{Z}^n \subset \mathbb{E}^n$ that consists of all points with even sum of the coordinates. Then any Delaunay $n$-polytopes is homologous to one of the following:

1. a cross-polytope, with vertices in $D_n$, centered at $x \in \mathbb{Z}^n$, where $\sum x_i \equiv 1 \mod 2$,
2. the convex hull of points of $D_n$ that belong to the standard cube $I$,
3. the convex hull of points of $D_n$ that belong to the shifted cube $I + e_n$.

The polytopes in 2) and 3) are known as $n$-semicubes (or halfcubes). Note that for $n = 3$ a semicube is a tetrahedron and for $n = 4$ a semicube is a cross-polytope; the latter fact explains why the Delaunay tiling of $D_4$ is formed by three homology classes of cross-polytopes.
8-dimensional perfect Delaunay polytopes

Perfect affine quadratic lattice \( \text{Aff}(\mathbb{Z}^8, Q_1^8[x - c]) \), where \( Q_1^8 \) is given by

\[
Q_1^8(x) = x^T \begin{pmatrix}
4 & 1 & 1 & 1 & 1 & 1 & 1 & -5 \\
1 & 4 & 1 & 1 & 1 & 1 & 1 & -5 \\
1 & 1 & 4 & 1 & 1 & 1 & 1 & -5 \\
1 & 1 & 1 & 4 & 1 & 1 & 1 & -5 \\
1 & 1 & 1 & 1 & 4 & 1 & 1 & -5 \\
1 & 1 & 1 & 1 & 1 & 4 & 1 & -5 \\
1 & 1 & 1 & 1 & 1 & 1 & 4 & -5 \\
-5 & -5 & -5 & -5 & -5 & -5 & -5 & 19
\end{pmatrix}
\]

and \( c = \frac{1}{10} \).

- \(|O(\mathbb{Z}^8, Q_1^8)| = 20160; S_7 < O(\mathbb{Z}^8, Q_1^8)\)
- \( s(\mathbb{Z}^8, Q_1^8) = 14 \)
- \( \dim \quad \text{QuadInv}[\mathbb{Z}^8, Q_1^8] = 3 \)

Inequality \( Q_1^8[x - c] \leq \frac{43}{10} \) defines the Delaunay ellipsoid for a perfect polytope \( D_1^8 \in \text{Del}(\mathbb{Z}^8, Q_1^8) \), whose vertex set (\( |\text{vert} D_1^8| = 44 \)) is given below.

| \( x_8 = 0 \) | \( x_8 = 1 \) | \( x_8 = 2 \) |
|---|---|---|
| \([0^8] \times 1\) | \([0^2, 1^5; 1] \times 21\) | \([1^7; 2] \times 1\) |
| \([0^6, 1; 0] \times 7\) | \([0, 1^6; 1] \times 7\) | \([1^6, 2; 2] \times 7\) |

- \( \text{Spec}(D_1^8) = 4, 6, 7, 9, 10, 12, 13, 15 \)
- \( |\text{Iso}(D_1^8, Q_1^8)| = 10080 \)
- \( l(D_1^8) = 3 \)
- Antisymmetric
- Maximally contained subpolytopes: \( H(2), \frac{1}{2}H(4), J(8, 6) \)
Perfect affine quadratic lattice $\text{Aff}(\mathbb{Z}^8, \mathcal{Q}_2^8[x - c])$, where $\mathcal{Q}_2^8$ is given by

$$
\begin{align*}
\mathcal{Q}_2^8(x) &= x^t \\
\begin{array}{cccccccc}
8 & 6 & 6 & 6 & 6 & 10 & 8 & 4 \\
6 & 8 & 6 & 6 & 6 & 10 & 8 & 5 \\
6 & 6 & 8 & 6 & 6 & 10 & 8 & 6 \\
6 & 6 & 6 & 8 & 6 & 10 & 8 & 4 \\
6 & 6 & 6 & 6 & 8 & 10 & 8 & 4 \\
10 & 10 & 10 & 10 & 10 & 16 & 12 & 7 \\
8 & 8 & 8 & 8 & 8 & 12 & 12 & 7 \\
4 & 5 & 6 & 4 & 4 & 7 & 7 & 7 \\
\end{array}
\end{align*}
$$

- $|O(\mathbb{Z}^8, \mathcal{Q}_2^8)| = 80640; S_8 < O(\mathbb{Z}^8, \mathcal{Q}_2^8)$
- $s(\mathbb{Z}^8, \mathcal{Q}_2^8) = 16$
- $\dim \text{QuadInv}[\mathbb{Z}^8, \mathcal{Q}_2^8] = 2$

Inequality $\mathcal{Q}_2^8[x-c] \leq 3$ defines the Delaunay ellipsoid for a perfect polytope $D_2^8 \in \text{Del}(\mathbb{Z}^8, \mathcal{Q}_2^8)$, whose vertex set (|vert $D_2^8| = 72$) is given below.

| $x_8 = -1$ | $x_8 = 0$ | $x_8 = 1$ |
| --- | --- | --- |
| $[-1^2;0;-1^2;e^2;1;1]\times 1$ | $[-1^5;2^2;0]\times 1$ | $[0;-1^2;0^2;1;0;1]\times 1$ |
| $[-1;0^2;-1^2;1;2;-1]\times 1$ | $[-1^5;3;1;0]\times 1$ | $[0^2;-1;0;1;0^2;1]\times 2$ |
| $[0^6;1;0;1;0^3;1;-1]\times 1$ | $[0^6;1;0^3;1;-1]\times 10$ | $[1;0;-1;0^4;1]\times 1$ |
| $[0^2;1;-1;0;0;1;-1]\times 2$ | $[1;0^4;1;0^2]\times 5$ | $[1;0^2;1^2;-1^2;1]\times 1$ |
| $[0;1^2;0^2;-1;1;-1]\times 1$ | $[0^8]\times 1$ | $[1^2;0;1^2;-2;-1;1]\times 1$ |
| $[0^6;1;0]\times 1$ | $[0^4;1;-1;1;0]\times 1$ | $[0^4;1;0^3]\times 5$ |
| $[0^3;1^2;-1;0^2]\times 10$ | $[0;1^4;-2;0^2]\times 5$ | $[1^5;-3;0^2]\times 1$ |
| $[1^5;-2;-1;0]\times 1$ | | |

- $\text{Spec}(D_2^8) = 3, 4, 5, 7, 8, 9, 12$
- $|\text{Iso}(D_2^8, \mathcal{Q}_2^8)| = 80640$
- $l(D_2^8) = 3$
- Centrally-symmetric
- Maximally contained subpolytopes: $35 - \text{tope}, H(3), \frac{1}{2}H(6), J(9,7)$
Perfect affine quadratic lattice \( \text{Aff}(\mathbb{Z}^8, Q_3^8[x - \mathbf{c}]) \), where \( Q_3^8 \) is given by

\[
\begin{array}{cccccccc}
11 & 8 & 8 & 8 & -3 & 4 & -20 & 4 \\
8 & 11 & 8 & 8 & -3 & 4 & -20 & 4 \\
8 & 8 & 11 & 8 & -3 & 4 & -20 & 4 \\
8 & 8 & 8 & 11 & -3 & 4 & -20 & 4 \\
-3 & -3 & -3 & -3 & 3 & -1 & 6 & -1 \\
4 & 4 & 4 & 4 & -1 & 4 & -10 & 1 \\
-20 & -20 & -20 & -20 & 6 & -10 & 48 & -10 \\
-20 & -20 & -20 & -20 & 6 & -10 & 48 & -10 \\
4 & 4 & 4 & 4 & -1 & 1 & -10 & 4 \\
\end{array}
\]

\( Q_3^8(x) = x' \quad \text{and} \quad \mathbf{c} = \frac{1}{92} \).

- \( |O(\mathbb{Z}^8, Q_3^8)| = 96; S_4 < O(\mathbb{Z}^8, Q_3^8) \)
- \( s(\mathbb{Z}^8, Q_3^8) = 2 \)
- \( \text{dim QuadInv}[\mathbb{Z}^8, Q_3^8] = 12 \)

Inequality \( Q_3^8[x - \mathbf{c}] \leq \frac{189}{46} \) defines the Delaunay ellipsoid for a perfect polytope \( D_3^8 \in \text{Del}(\mathbb{Z}^8, Q_3^8) \), whose vertex set (\( |\text{vert} D_3^8| = 47 \)) is given below.

| \( x_5 = -1 \) | \( x_5 = 0 \) | \( x_5 = 1 \) |
|----------------|----------------|----------------|
| \( [0^4; -1; 0^3] \times 1 \) | \( [-1, 0^3; 0; -1^3] \times 4 \) | \( [0^4; 1; -2; -1^2] \times 1 \) |
| \( [0^5; -2; -1; -2] \times 1 \) | \( [0^4; 1; -1^2; -2] \times 1 \) | \( [0^4; 1; -1^3] \times 1 \) |
| \( [0^5; -2; -1^2] \times 1 \) | \( [0^4; 1; -1^3] \times 1 \) | \( [0^3; 1; 1; 0^3] \times 4 \) |
| \( [0^6] \times 1 \) | \( [0^2; 1^2; 1; -1; 0; -1] \times 6 \) | \( [0^7; 1] \times 1 \) |
| \( [0^7; 1] \times 1 \) | \( [0; 1^3; 1; 0; 1; 0] \times 4 \) | \( [1^5; -1; 1; 1] \times 1 \) |
| \( [0^5; 1; 0^2] \times 1 \) | \( [1^5; -1; 1; -1] \times 1 \) | \( [0^3; 1; 0^3; -1] \times 4 \) |
| \( [0^3; 1; 0^3; -1] \times 4 \) | \( [1^5; -1; 1; 0] \times 1 \) | \( [0^3; 1; 0^3; -1] \times 4 \) |
| \( [0^3; 1; 0^4] \times 4 \) | \( [1^5; 0; 1; -1] \times 1 \) | \( [0^3; 1; 0^4] \times 4 \) |
| \( [0, 1^3; 0^2; 1; 0] \times 4 \) |

- \( \text{Spec}(D_3^8) = 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15 \)
- \( |\text{Iso}(D_3^8, Q_3^8)| = 48 \)
- \( l(D_3^8) = 3 \)
- Antisymmetric
- Maximally contained subpolytopes: \( G_6, H(3), \frac{1}{2}H(5), J(7, 5) \)
Perfect affine quadratic lattice \( \text{Aff}(\mathbb{Z}^8, Q^8_5[x - c]) \), where \( Q^8_5 \) is given by

\[
Q^8_5(x) = x^t \begin{bmatrix}
15 & 11 & 11 & 11 & 11 & 11 & 11 & -3 \\
11 & 9 & 8 & 8 & 8 & 8 & 8 & -3 \\
11 & 8 & 9 & 8 & 8 & 8 & 8 & -3 \\
11 & 8 & 8 & 9 & 8 & 8 & 8 & -3 \\
11 & 8 & 8 & 8 & 9 & 8 & 8 & -3 \\
11 & 8 & 8 & 8 & 8 & 9 & 8 & -3 \\
11 & 8 & 8 & 8 & 8 & 8 & 9 & -3 \\
11 & 8 & 8 & 8 & 8 & 8 & 8 & -3 \\
\end{bmatrix}
\]

\( x \) and \( c = \frac{1}{8} \).

- \( |O(\mathbb{Z}^8, Q^8_5)| = 161280; \) \( S_8 < O(\mathbb{Z}^8, Q^8_5) \)
- \( s(\mathbb{Z}^8, Q^8_5) = 56 \)
- \( \dim \text{QuadInv}[\mathbb{Z}^8, Q^8_5] = 2 \)

Inequality \( Q^8_5[x - c] \leq \frac{15}{8} \) defines the Delaunay ellipsoid for a perfect polytope \( D^8_5 \in \text{Del}(\mathbb{Z}^8, Q^8_5) \), whose vertex set (\( |\text{vert} D^8_5| = 72 \)) is given below.

| \( \sum x_i = -4 \) | \( \sum x_i = -3 \) | \( \sum x_i = -2 \) |
|-----------------|-----------------|-----------------|
| \([2; -1^6, 0] \times 7\) | \([-1^3, 0^5] \times 56\) | \([-3; 0^6, 1] \times 7\) |
| \([3; -1^7] \times 1\) | \([2; 0^7] \times 1\) |

- \( \text{Spec}(D^8_5) = 2, 3, 4, 5, 6 \)
- \( |\text{Iso}(D^8_5, Q^8_5)| = 80640 \)
- \( l(D^8_5) = 3 \)
- Antisymmetric
- Maximally contained subpolytopes: \( H(3), \frac{1}{2}H(5), J(8,5) \)
Perfect affine quadratic lattice \( \text{Aff}(\mathbb{Z}^8, Q_6^8[x - c]) \), where \( Q_6^8 \) is given by

\[
Q_6^8(x) = x' \quad \text{and} \quad c = \frac{1}{8}
\]

- \(|O(\mathbb{Z}^8, Q_6^8)| = 768; S_4 < O(\mathbb{Z}^8, Q_6^8)\)
- \(s(\mathbb{Z}^8, Q_6^8) = 2\)
- \(\dim \text{QuadInv}[\mathbb{Z}^8, Q_6^8] = 5\)

Inequality \( Q_6^8[x - c] \leq \frac{11}{4} \) defines the Delaunay ellipsoid for a perfect polytope \( D_6^8 \in \text{Del}(\mathbb{Z}^8, Q_6^8) \), whose vertex set (\( |\text{vert} \ D_6^8| = 54 \)) is given below.

| \( x_8 = -1 \) | \( x_8 = 0 \) | \( x_8 = 1 \) |
|----------------|----------------|----------------|
| \([0^3; -1; 1; 0; 1; -1] \times 1\) | \([-1, 0^5; 1; 0] \times 6\) | \([0^7; 1] \times 1\) |
| \([0^6; 1; -1] \times 1\) | \([0^8] \times 1\) | \([0^3; 1; -1; 0^2; 1] \times 1\) |
| \([0^4; 1; 0^2; -1] \times 1\) | \([0^5; 1; 0^2] \times 6\) | \([0^3; 1; 0^2; -1; 1] \times 1\) |
| \([0^2; 1; 0; 1^2; -1^2] \times 3\) | \([0^4; 1^2; -1; 0] \times 15\) | \([0; 1^2; 0; 1^2; -1^2] \times 3\) |
| \([0; 1^2; 0; 1; 0; -1^2] \times 3\) | \([0; 1^5; -3; 0] \times 6\) | \([0; 1^2; 0; 1^2; -2; -1] \times 3\) |
| \([1^3; 0; 1; 0; -2; -1] \times 1\) | \([1^6; -4; 0] \times 1\) | \([1^3; 0; 2; 1; -3; -1] \times 1\) |
| \([1^6; -3; -1] \times 1\) | | \([1^4; 2; 1; -4; -1] \times 1\) |

- \(\text{Spec}(D_6^8) = 2, 3, 4, 5, 6, 7, 8, 9, 10\)
- \(|\text{Iso}(D_6^8, Q_6^8)| = 384\)
- \(l(D_6^8) = 3\)
- Antisymmetric
- Maximally contained subpolytopes: 35 – tope, \(H(3)\), \(\frac{1}{2}H(5)\), \(J(7, 5)\)
Perfect affine quadratic lattice $\text{Aff}(\mathbb{Z}^8, Q^8_7[x - c])$, where $Q^8_7$ is given by

$$Q^8_7(x) = x^t 12 9 9 9 9 -15 9 8$$

and $c = \frac{1}{66}$.

- $|O(\mathbb{Z}^8, Q^8_7)| = 72$; $S_3 < O(\mathbb{Z}^8, Q^8_7)$
- $s(\mathbb{Z}^8, Q^8_7) = 2$
- $\dim \text{QuadInv}[\mathbb{Z}^8, Q^8_7] = 12$

Inequality $Q^8_7[x - c] \leq \frac{91}{22}$ defines the Delaunay ellipsoid for a perfect polytope $D^8_7 \in \text{Del}(\mathbb{Z}^8, Q^8_7)$, whose vertex set ($|\text{vert } D^8_7| = 46$) is given below.

| $x_8 = -1$ | $x_8 = 0$ | $x_8 = 1$ |
|-----------|-----------|-----------|
| $[0; 1^5, -1^2] \times 1$ | $[-1, 0^4; -1, 0^2] \times 5$ | $[0; -1, -1^2, 0, -1; 1^2] \times 3$ |
| $[0^5, -1^2; 0] \times 1$ | $[0; -1; 0^2; -1; 0; 1] \times 3$ | $[0; -1; 0^4; 1^2] \times 1$ |
| $[0^6; 1; 0] \times 1$ | $[0^7; 1] \times 1$ | $[0^5; 1^3] \times 1$ |
| $[0^4, 1; 0^3] \times 5$ | $[0^3, 1^2; 1; 0^2] \times 10$ | $[1; -1; 0^5; 1] \times 1$ |
| $[0^3, 1^4; 2; 0^2] \times 5$ | $[1; -1; 0^3; 1^3] \times 1$ | $[1; -1; 0^3; 1^3] \times 1$ |
| $[1^5, 2; -1; 0] \times 1$ | $[1; 0^4; 1; 0; 1] \times 1$ | $[1; 0^4; 1; 0; 1] \times 1$ |
| $[1^5; 3; 0^2] \times 1$ | $[1^5; 3; 0^2] \times 1$ | $[0; 1^4; 1^2; 0] \times 1$ |
| $[0^2, 1^4; -1; 0] \times 1$ | $[0; 1; 0^3; 1^2; 0] \times 1$ | $[1; 0^4; 1^2; 0] \times 1$ |

- $\text{Spec}(D^8_7) = 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15$
- $|\text{Iso}(D^8_7, Q^8_7)| = 36$
- $l(D^8_7) = 3$
- Antisymmetric
- Maximally contained subpolytopes: $G_6, H(3), \frac{1}{2}H(5), J(7, 5)$
Perfect affine quadratic lattice $\text{Aff}(\mathbb{Z}^8, Q_8^8[x - c])$, where $Q_8^8$ is given by

$$Q_8^8(x) = x^t 6 6 6 6 6 9 5$$

and $c = \frac{1}{20}$.

- $|O(\mathbb{Z}^8, Q_8^8)| = 384$; $S_4 < O(\mathbb{Z}^8, Q_8^8)$
- $s(\mathbb{Z}^8, Q_8^8) = 2$
- $\dim \text{QuadInv}[\mathbb{Z}^8, Q_8^8] = 7$

Inequality $Q_8^8[x - c] \leq \frac{14}{5}$ defines the Delaunay ellipsoid for a perfect polytope $D_8^8 \in \text{Del}(\mathbb{Z}^8, Q_8^8)$, whose vertex set ($|\text{vert} D_8^8| = 52$) is given below.

| $x_8 = -1$ | $x_8 = 0$ | $x_8 = 1$ |
|-----------|-----------|-----------|
| $[0; 1^2; 0^2; 1; -1^2] \times 1$ | $[-1; 0^5; 1; 0] \times 6$ | $[0; -1^2; 0^2; -1; 2; 1] \times 1$ |
| $[0^8] \times 1$ | $[0; -1; 0^3; -1; 1^2] \times 1$ |
| $[0^5; 1; 0^2] \times 6$ | $[0^2; -1; 0^2; -1; 1^2] \times 1$ |
| $[0^4; 1^2; -1; 0] \times 15$ | $[0^7; 1] \times 1$ |
| $[0; 1^5; -3; 0] \times 6$ | $[0^3; 0; 1; -1; 0; 1] \times 2$ |
| $[1^6; -4; 0] \times 1$ | $[0^3; 0; 1; 0; -1; 1] \times 2$ |
| $[0^3; 1^2; -1^2; 1] \times 1$ |
| $[1; 0^4; -1; 0; 1] \times 1$ |
| $[1; 0^5; -1; 1] \times 1$ |
| $[1; 0^2; 0; 1; -1^2; 1] \times 2$ |
| $[1; 0^2; 1^2; 0; -2; 1] \times 1$ |
| $[1; 0; 1^3; 0; -3; 1] \times 1$ |
| $[1^2; 0; 1^2; 0; -3; 1] \times 1$ |

- $\text{Spec}(D_8^8) = 2, 3, 4, 5, 6, 7, 8, 9, 10$
- $|\text{Iso}(D_8^8, Q_8^8)| = 192$
- $l(D_8^8) = 3$
- Antisymmetric
- Maximally contained subpolytopes: $35 - \text{tope}, H(3), \frac{1}{2}H(5), J(7, 5)$

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Perfect affine quadratic lattice $\text{Aff}(\mathbb{Z}^8, \mathcal{Q}_0^8[x - c])$, where $\mathcal{Q}_0^8$ is given by

$$
\mathcal{Q}_0^8(x) = x'^t \begin{array}{cccccccc}
6 & 4 & 4 & 4 & 4 & 4 & -5 & 6 \\
4 & 6 & 4 & 4 & 4 & 4 & -5 & 6 \\
4 & 4 & 6 & 4 & 4 & 4 & -5 & 6 \\
4 & 4 & 4 & 6 & 4 & 4 & -5 & 6 \\
4 & 4 & 4 & 4 & 6 & 4 & -5 & 6 \\
-5 & -5 & -5 & -5 & -5 & 9 & -6 & 6 \\
6 & 6 & 6 & 6 & 6 & 6 & -6 & 9 \\
\end{array} x \quad \text{and} \quad c = \frac{1}{8}.
$$

- $|O(\mathbb{Z}^8, \mathcal{Q}_0^8)| = 2880$; $S_6 < O(\mathbb{Z}^8, \mathcal{Q}_0^8)$
- $s(\mathbb{Z}^8, \mathcal{Q}_0^8) = 12$
- $\dim \text{QuadInv}[\mathbb{Z}^8, \mathcal{Q}_0^8] = 5$

Inequality $\mathcal{Q}_0^8[x - c] \leq \frac{27}{8}$ defines the Delaunay ellipsoid for a perfect polytope $D_9^8 \in \text{Del}(\mathbb{Z}^8, \mathcal{Q}_0^8)$, whose vertex set ($|\text{vert } D_9^8| = 58$) is given below.

| $x_7 = 0$ | $x_7 = 1$ | $x_7 = 2$ |
|------------|------------|------------|
| $[-1, 0^5; 0; 1] \times 6$ | $[0^6; 1^2] \times 1$ | $[1^6; 2; -2] \times 1$ |
| $[0^8] \times 1$ | $[0^4, 1^2; 1; 0] \times 15$ | |
| $[0^7; 1] \times 1$ | $[0^3, 1^3; 1; -1] \times 20$ | |
| $[0^5, 1; 0^2] \times 6$ | $[0, 1^5; 1; -2] \times 6$ | $[1^7; -3] \times 1$ |

- $\text{Spec}(D_9^8) = 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$
- $|\text{Iso}(D_9^8, \mathcal{Q}_0^8)| = 1440$
- $l(D_9^8) = 3$
- Antisymmetric
- Maximally contained subpolytopes: $H(3), \frac{1}{2}H(5), J(7, 5)$
Perfect affine quadratic lattice \( \text{Aff}(\mathbb{Z}^8, \mathcal{Q}_{10}^8[x - c]) \), where \( \mathcal{Q}_{10}^8 \) is given by

\[
\begin{array}{cccccc}
8 & 6 & 6 & 6 & 6 & 9 \\
6 & 8 & 6 & 6 & 6 & 9 \\
6 & 6 & 8 & 6 & 6 & 9 \\
6 & 6 & 8 & 6 & 6 & 9 \\
6 & 6 & 8 & 6 & 6 & 9 \\
9 & 9 & 9 & 9 & 9 & 13 \\
4 & 4 & 3 & 3 & 2 & 4 \\
\end{array}
\]

\( x \) and \( c = \frac{1}{26} \).

- \(|O(\mathbb{Z}^8, \mathcal{Q}_{10}^8)| = 576; S_3 < O(\mathbb{Z}^8, \mathcal{Q}_{10}^8)\)
- \(s(\mathbb{Z}^8, \mathcal{Q}_{10}^8) = 4\)
- \(\dim \text{QuadInv}[\mathbb{Z}^8, \mathcal{Q}_{10}^8] = 4\)

Inequality \( \mathcal{Q}_{10}^8[x - c] \leq \frac{35}{15} \) defines the Delaunay ellipsoid for a perfect polytope \( D_{10}^8 \in \text{Del}(\mathbb{Z}^8, \mathcal{Q}_{10}^8) \), whose vertex set (\(|\text{vert } D_{10}^8| = 55\)) is given below.

| \( x_8 = -1 \) | \( x_8 = 0 \) | \( x_8 = 1 \) |
|-----------------|----------------|----------------|
| \([1^2; 0; 1; 0; 1; -2; -1]\) \times 1 | \([-1; 0^6; 1; 0]\) \times 6 | \([-1^4; 0^2; 3; 1]\) \times 1 |
| \([1^3; 0^2; 1; -2; -1]\) \times 1 | \([0^8]\) \times 1 | \([-1^3; 0^2; -1; 3; 1]\) \times 1 |
| \([1^4; 0^2; -2; -1]\) \times 1 | \([0^6; 1; 0^2]\) \times 6 | \([-1^3; 0^3; 2; 1]\) \times 1 |
| \([1^4; 0; 1; -3; -1]\) \times 1 | \([0^4; 1^2; -1; 0]\) \times 15 | \([-1^2; 0; -1; 0; -1; 3; 1]\) \times 1 |
| \([1; 2; 1^4; -4; -1]\) \times 2 | \([0; 1^5; -3; 0]\) \times 6 | \([-1^2; 0; -1; 0^2; 2; 1]\) \times 1 |
| \([2^2; 1^2; 0; 1; -4; -1]\) \times 1 | \([1^6; -4; 0]\) \times 1 | \([-1^2; 0^3; -1; 2; 1]\) \times 1 |
| \([2^2; 1^4; -5; -1]\) \times 1 | | \([-1^2; 0^2; 1; 0; 1^2]\) \times 1 |
| | | \([-1; 0; 0^4; 1^2]\) \times 2 |
| | | \([-1; 0; 0^2; 1; 0^2; 1]\) \times 2 |
| | | \([0^7; 1]\) \times 1 |

- \(\text{Spec}(D_{10}^8) = 2, 3, 4, 5, 6, 7, 8, 9\)
- \(|\text{Iso}(D_{10}^8, \mathcal{Q}_{10}^8)| = 288\)
- \(l(D_{10}^8) = 3\)
- Antisymmetric
- Maximally contained subpolytopes: 35 – tope, \( H(3), \frac{1}{2}H(5), J(7, 5) \)
Perfect affine quadratic lattice \( \text{Aff}(\mathbb{Z}^8, Q_{11}^8[x - c]) \), where \( Q_{11}^8 \) is given by

\[
Q_{11}^8(x) = x^t 24 18 18 18 18 27 15 36
18 24 18 18 18 27 15 36
18 24 18 18 18 27 15 36
18 24 18 18 18 27 15 36
18 24 18 18 18 27 15 36
18 24 18 18 18 27 15 36
18 24 18 18 18 27 15 36
18 24 18 18 18 27 15 36
15 15 15 15 15 15 11 20 17 36
\]

\( x \) and \( c = \frac{1}{90} \). 

- \(|O(\mathbb{Z}^8, Q_{11}^8)| = 480; S_5 < O(\mathbb{Z}^8, Q_{11}^8)\)
- \(s(\mathbb{Z}^8, Q_{11}^8) = 4\)
- \(\dim \text{QuadInv}[\mathbb{Z}^8, Q_{11}^8] = 8\)

Inequality \( Q_{11}^8[x - c] \leq \frac{124}{13} \) defines the Delaunay ellipsoid for a perfect polytope \( D_{11}^8 \in \text{Del}(\mathbb{Z}^8, Q_{11}^8) \), whose vertex set \(|\text{vert } D_{11}^8| = 44\) is given below.

| \( x_8 = -1 \) | \( x_8 = 0 \) | \( x_8 = 1 \) |
|----------------|----------------|----------------|
| \([0^6; 1; -1] \times 1 \) | \([-1, 0^5; 1; 0] \times 6 \) | \([0^7; 1] \times 1 \) |
| \([0, 1^4; 0; -2; -1] \times 5 \) | \([0^8] \times 1 \) | \([0^5; 1; -1; 1] \times 1 \) |
| \([1^6; -3; -1] \times 1 \) | \([0^5; 1; 0^2] \times 6 \) | \([0^4; 1^2; -1; 0] \times 15 \) |
| \([0; 1^5; -3; 0] \times 6 \) | \([1^6; -4; 0] \times 1 \) |

- \(\text{Spec}(D_{11}^8) = 8, 9, 11, 12, 13, 15, 16, 17, 19, 20, 21, 23, 24, 27, 28, 31\)
- \(|Iso(D_{11}^8, Q_{11}^8)| = 240\)
- \(|l(D_{11}^8)| = 3\)
- Antisymmetric
- Maximally contained subpolytopes: 35 – tope, \( H(2) \), \( \frac{1}{2}H(5) \), \( J(7, 5) \)
Perfect affine quadratic lattice \( \text{Aff}(\mathbb{Z}^8, \mathcal{Q}_{12}^8[x - c]) \), where \( \mathcal{Q}_{12}^8 \) is given by

\[
\begin{array}{cccccccc}
12 & 9 & 9 & 9 & 9 & -15 & 7 & 6 \\
9 & 12 & 9 & 9 & 9 & -15 & 10 & 6 \\
9 & 9 & 12 & 9 & 9 & -15 & 8 & 5 \\
9 & 9 & 9 & 12 & 9 & -15 & 8 & 5 \\
-15 & -15 & -15 & -15 & -15 & 24 & -12 & -9 \\
7 & 10 & 8 & 8 & 8 & -12 & 12 & 5 \\
6 & 6 & 5 & 5 & 5 & -9 & 5 & 6 \\
\end{array}
\]

\[
x \quad \text{and} \quad c = \frac{1}{31}
\]

- \( |O(\mathbb{Z}^8, \mathcal{Q}_{12}^8)| = 96; S_3 < O(\mathbb{Z}^8, \mathcal{Q}_{12}^8) \)
- \( s(\mathbb{Z}^8, \mathcal{Q}_{12}^8) = 14 \)
- \( \dim \text{QuadInv}[\mathbb{Z}^8, \mathcal{Q}_{12}^8] = 8 \)

Inequality \( \mathcal{Q}_{12}^8[x - c] \leq \frac{126}{31} \) defines the Delaunay ellipsoid for a perfect polytope \( D_{12}^8 \in \text{Del}(\mathbb{Z}^8, \mathcal{Q}_{12}^8) \), whose vertex set \( \{ \text{vert} D_{12}^8 \} = 45 \) is given below.

| \( x_8 = -1 \) | \( x_8 = 0 \) | \( x_8 = 1 \) |
|----------------|----------------|----------------|
| \([0; -1^4; -3; 1; -1] \times 1 \) | \([-1; 0^4; -1; 0^2] \times 5 \) | \([0^7; 1] \times 1 \) |
| \([0; -1; -1^2; 0; -2; 1; -1] \times 3 \) | \([0^8] \times 1 \) | \([0^2; 1^3; 2; 0; 1] \times 1 \) |
| \([0; -1; 0^3; -1; 1; -1] \times 1 \) | \([0^6; 1; 0] \times 1 \) | \([0; 1^4; 2; -1; 1] \times 1 \) |
| \([0^5; -1; 0; -1] \times 1 \) | \([0^4; 1; 0^3] \times 5 \) | \([1^5; 3; 0; 1] \times 1 \) |
| \([0; 1; 0^5; -1] \times 1 \) | \([0^9; 1^2; 1; 0^2] \times 10 \) | \([0; 1^4; 2; 0^2] \times 5 \) |
| \([1; 0^6; -1] \times 1 \) | \([1^5; 3; 0^2] \times 1 \) | \([0; -1^2; 0^2; -1; 1; 0] \times 1 \) |
| \([0; -1; 0; -1; 0; -1; 1; 0] \times 1 \) | \([0; -1; 0^2; -1^2; 1; 0] \times 1 \) | \([0; -1; 0^4; 1; 0] \times 1 \) |
| \([1; -1; 0^4; 1; 0] \times 1 \) | \([1; 2; 1^3; 3; -1; 0] \times 1 \) |

- \( \text{Spec}(D_{12}^8) = 2, 3, 4, 5, 6, 7 \)
- \( |\text{Iso}(D_{12}^8, \mathcal{Q}_{12}^8)| = 48 \)
- \( l(D_{12}^8) = 3 \)
- Antisymmetric
- Maximally contained subpolytopes: \( G_6, H(2), \frac{1}{2}H(5), J(6, 4) \)
Perfect affine quadratic lattice \( \text{Aff}(\mathbb{Z}^8, \mathcal{Q}_{13}^8[x - c]) \), where \( \mathcal{Q}_{13}^8 \) is given by

\[
\mathcal{Q}_{13}^8(x) = x' \quad \text{and} \quad c = \frac{1}{74}.
\]

\[
\begin{array}{cccccccccc}
24 & 18 & 18 & 18 & 18 & 27 & 6 & 18 & 24 & 18 & 18 & 18 & 27 & 10 & 18 & 18 & 24 & 18 & 18 & 27 & 10 & 18 & 18 & 18 & 18 & 24 & 18 & 27 & 10 & 27 & 27 & 27 & 27 & 27 & 39 & 12 & 18 & 18 & 18 & 24 & 18 & 18 & 18 & 27 & 10 & 6 & 10 & 10 & 10 & 10 & 12 & 12 & 16 & 16 & 16 & 13 & 16 & -40 & 9 & 22 & 16 & 16 & 16
\end{array}
\]

- \( |O(\mathbb{Z}^8, \mathcal{Q}_{13}^8)| = 96; S_4 < O(\mathbb{Z}^8, \mathcal{Q}_{13}^8) \)
- \( s(\mathbb{Z}^8, \mathcal{Q}_{13}^8) = 2 \)
- \( \dim \text{QuadInv}[\mathbb{Z}^8, \mathcal{Q}_{13}^8] = 12 \)

Inequality \( \mathcal{Q}_{13}^8[x - c] \leq \frac{300}{37} \) defines the Delaunay ellipsoid for a perfect polytope \( D_{13}^8 \in \text{Del}(\mathbb{Z}^8, \mathcal{Q}_{13}^8) \), whose vertex set \( \{|\text{vert} D_{13}^8| = 44\} \) is given below.

\[
\begin{array}{ccc}
x_8 = -1 & x_8 = 0 & x_8 = 1 \\
[0; 1^5, -3; -1] \times 1 & [-1, 0^5; 1; 0] \times 6 & [0; -1^3, 0; -1; 3; 1] \times 4 \\
[1^5; 2; -4; -1] \times 1 & [0^8] \times 1 & [0^5; -1; 1^2] \times 1 \\
& [0^5, 1; 0^2] \times 6 & [0^7; 1] \times 1 \\
& [0^4, 1^2; -1; 0] \times 15 & [1; 0^4; -1; 0; 1] \times 1 \\
& [0, 1^5; -3; 0] \times 6 & \\
& [1^6; -4; 0] \times 1 & \\
\end{array}
\]

- \( \text{Spec}(D_{13}^8) = 5, 8, 9, 11, 12, 13, 15, 16, 17, 19, 20, 21, 23, 24, 25, 27, 28 \)
- \( |\text{Iso}(D_{13}^8, \mathcal{Q}_{13}^8)| = 48 \)
- \( l(D_{13}^8) = 3 \)
- Antisymmetric
- Maximally contained subpolytopes: 35 – tope, \( H(2), \frac{1}{2}H(5) \), \( J(7, 5) \)
Perfect affine quadratic lattice $\text{Aff}(\mathbb{Z}^8, \mathcal{Q}_8^{\text{f}}[x - c])$, where $\mathcal{Q}_8^{\text{f}}$ is given by

$$\mathcal{Q}_8^{\text{f}}(x) = x' \cdot \begin{pmatrix} 9 & 5 & 5 & -15 & 0 & 3 & -12 \\ 5 & 9 & 5 & 5 & -15 & 0 & 3 & -12 \\ 5 & 5 & 9 & 5 & -15 & 0 & 3 & -12 \\ -15 & -15 & -15 & -15 & 42 & 3 & -9 & 33 \\ 0 & 0 & 0 & 0 & 3 & 6 & 0 & 3 \\ 3 & 3 & 3 & 3 & -9 & 0 & 4 & -8 \\ -12 & -12 & -12 & -12 & 33 & 3 & -8 & 28 \end{pmatrix} \cdot x \quad \text{and} \quad c = \frac{1}{16}$.$$

- $|O(\mathbb{Z}^8, \mathcal{Q}_8^{\text{f}})| = 576; S_4 < O(\mathbb{Z}^8, \mathcal{Q}_8^{\text{f}})$
- $s(\mathbb{Z}^8, \mathcal{Q}_8^{\text{f}}) = 6$
- $\dim \text{QuadInv}[\mathbb{Z}^8, \mathcal{Q}_8^{\text{f}}] = 6$

Inequality $\mathcal{Q}_8^{\text{f}}[x - c] \leq \frac{41}{8}$ defines the Delaunay ellipsoid for a perfect polytope $D_8^{\text{f}} \in \text{Del}(\mathbb{Z}^8, \mathcal{Q}_8^{\text{f}})$, whose vertex set ($|\text{vert } D_8^{\text{f}}| = 46$) is given below.

| $x_6 = -1$ | $x_6 = 0$ | $x_6 = 1$ |
|------------|------------|------------|
| $[1^4; 0; -1; 1; 2] \times 1$ | $[0^4; -1; 0^2; 1] \times 1$ | $[-1, 0^3; 0; 1; -1^2] \times 4$ |
| $[1^4; 0; -1; 2^2] \times 1$ | $[0^8] \times 1$ | $[0^4; -1; 1; 0; 1] \times 1$ |
| $[1^5; -1; 2; 1] \times 1$ | $[0^6; 1; 0] \times 1$ | $[0^5; 1; -2; -1] \times 1$ |
| $[0^3; 1; 0^4] \times 4$ | $[0^5; 1; -1^2] \times 1$ | $[0^2; 1^2; 0^2; 1^2] \times 6$ |
| $[0^2; 1^2; 0^2; 1^2] \times 6$ | $[0^5; 1; 0^2] \times 1$ | $[0^2; 1^2; 0^2; 1^2] \times 6$ |
| $[0, 1^3; 0^3; 1] \times 4$ | $[0^5; 1^2; 0] \times 1$ | $[0, 1^3; 1; 0; 1; 0] \times 4$ |
| $[0, 1^3; 1; 0; 1; 0] \times 4$ | $[0^5; 1^2; 0] \times 1$ | $[0, 1^3; 0; 1; 0; 0^2] \times 4$ |
| $[1^4; 0^2; 1; 2] \times 1$ | $[0^3; 1; 0; 1; 0^2] \times 4$ | $[1^4; 0^2; 1; 2] \times 1$ |
| $[1^4; 0^2; 2^2] \times 1$ | $[1^5; 0^3] \times 1$ | $[1^5; 0^3] \times 1$ |
| $[1^5; 0^3] \times 1$ | $[1^5; 0^3] \times 1$ |

- $\text{Spec}(D_8^{\text{f}}) = 4, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 18$
- $|\text{Iso}(D_8^{\text{f}}, \mathcal{Q}_8^{\text{f}})| = 288$
- $l(D_8^{\text{f}}) = 3$
- Antisymmetric
- Maximally contained subpolytopes: $H(3), \frac{1}{2}H(5), J(7, 5)$
Perfect affine quadratic lattice $\text{Aff}(\mathbb{Z}^8, Q_{15}^8[x - c])$, where $Q_{15}^8$ is given by

$$Q_{15}^8(x) = x' \cdot \begin{bmatrix} 8 & 6 & 6 & 6 & 6 & 6 & 9 & 2 \\ 6 & 8 & 6 & 6 & 6 & 6 & 9 & 3 \\ 6 & 6 & 8 & 6 & 6 & 6 & 9 & 3 \\ 6 & 6 & 6 & 8 & 6 & 6 & 9 & 3 \\ 6 & 6 & 6 & 6 & 8 & 6 & 9 & 4 \\ 9 & 9 & 9 & 9 & 9 & 9 & 13 & 4 \\ 2 & 3 & 3 & 3 & 3 & 4 & 4 & 4 \end{bmatrix} x \quad \text{and} \quad c = \frac{1}{20}.$$

- $|O(\mathbb{Z}^8, Q_{15}^8)| = 384$; $S_4 < O(\mathbb{Z}^8, Q_{15}^8)$
- $s(\mathbb{Z}^8, Q_{15}^8) = 24$
- $\dim \text{QuadInv}[\mathbb{Z}^8, Q_{15}^8] = 6$

Inequality $Q_{15}^8[x - c] \leq \frac{27}{10}$ defines the Delaunay ellipsoid for a perfect polytope $D_{15}^8 \in \text{Del}(\mathbb{Z}^8, Q_{15}^8)$, whose vertex set ($|\text{vert} D_{15}^8| = 45$) is given below.

| $x_8 = -1$ | $x_8 = 0$ | $x_8 = 1$ |
|---|---|---|
| $[0^5; 1; 0; -1] \times 1$ | $[-1, 0^5; 1; 0] \times 6$ | $[0^5; -1; 1^2] \times 1$ |
| $[0; 0, 1^3; 1; -2; -1] \times 4$ | $[0^8] \times 1$ | $[0^7; 1] \times 1$ |
| $[0; 1^5; -3; -1] \times 1$ | $[0^5, 1; 0^2] \times 6$ | $[1; 0^4, -1; 0; 1] \times 1$ |
| $[1^5; 2; -4; -1] \times 1$ | $[0^4, 1^2, -1; 0] \times 15$ | $[0^4, 1^5; -3; 0] \times 6$ |
| | | $[1^6, -4; 0] \times 1$ |

- $\text{Spec}(D_{15}^8) = 3, 4, 5, 6, 7, 8, 9$
- $|\text{Iso}(D_{15}^8, Q_{15}^8)| = 192$
- $l(D_{15}^8) = 3$
- Antisymmetric
- Maximally contained subpolytopes: 35 — tope, $H(2), \frac{1}{2}H(5), J(7, 5)$
Perfect affine quadratic lattice $\text{Aff}(\mathbb{Z}^8, Q_{16}^8[x - c])$, where $Q_{16}^8$ is given by

$$Q_{16}^8(x) = x^t Q_{16}^8 x$$

and $c = \frac{1}{14}$.

\begin{align*}
8 & 6 & 6 & 6 & 6 & 6 & 9 & 6 \\
6 & 8 & 6 & 6 & 6 & 6 & 9 & 5 \\
6 & 6 & 8 & 6 & 6 & 6 & 9 & 5 \\
6 & 6 & 6 & 8 & 6 & 6 & 6 & 9 \\
6 & 6 & 6 & 6 & 8 & 6 & 9 & 5 \\
9 & 9 & 9 & 9 & 9 & 9 & 13 & 8 \\
6 & 5 & 5 & 6 & 6 & 5 & 8 & 7
\end{align*}

• $|O(\mathbb{Z}^8, Q_{16}^8)| = 288$; $S_3 < O(\mathbb{Z}^8, Q_{16}^8)$

• $s(\mathbb{Z}^8, Q_{16}^8) = 24$

• $\text{dim QuadInv}[\mathbb{Z}^8, Q_{16}^8] = 5$

Inequality $Q_{16}^8[x - c] \leq \frac{19}{7}$ defines the Delaunay ellipsoid for a perfect polytope $D_{16}^8 \in \text{Del}(\mathbb{Z}^8, Q_{16}^8)$, whose vertex set ($|\text{vert } D_{16}^8| = 45$) is given below.

\begin{align*}
|0^6; 1; -1| & \times 1 \\
|1; 0^2; 1^2; 0; -1^2| & \times 1 \\
|0^8| & \times 1 \\
|0^6; 1; 0^2| & \times 6 \\
|0^4; 1^2; -1; 0| & \times 15 \\
|0; 1^5; -3; 0| & \times 6 \\
|1^6; -4; 0| & \times 1 \\
|0; 1^3; 0; 1; -3| & \times 1 \\
|1^3; 0^2; 1; -3| & \times 1
\end{align*}

• $\text{Spec}(D_{16}^8) = 3, 4, 5, 6, 7, 8, 9$

• $|\text{Iso}(D_{16}^8, Q_{16}^8)| = 144$

• $l(D_{16}^8) = 3$

• Antisymmetric

• Maximally contained subpolytopes: $35 - \text{tope}, H(2), \frac{1}{2}H(5), J(7, 5)$
Perfect affine quadratic lattice \( \text{Aff}(\mathbb{Z}^8, \mathbb{Q}^8_{17}[x - c]) \), where \( \mathbb{Q}^8_{17} \) is given by

\[
\mathbb{Q}^8_{17}(x) = x' - c
\]

\[
x = \begin{bmatrix}
8 & 6 & 6 & 6 & 6 & 9 & 4 \\
6 & 8 & 6 & 6 & 6 & 9 & 4 \\
6 & 6 & 8 & 6 & 6 & 6 & 9 \\
6 & 6 & 8 & 6 & 6 & 9 & 6 \\
6 & 6 & 6 & 6 & 6 & 9 & 4 \\
9 & 9 & 9 & 9 & 9 & 13 & 6 \\
4 & 4 & 4 & 6 & 4 & 4 & 6 \\
\end{bmatrix}
\]

\[
c = \frac{1}{10}
\]

- \(|O(\mathbb{Z}^8, \mathbb{Q}^8_{17})| = 5760; S_6 < O(\mathbb{Z}^8, \mathbb{Q}^8_{17})
- \(s(\mathbb{Z}^8, \mathbb{Q}^8_{17}) = 2
- \dim \text{QuadInv}[\mathbb{Z}^8, \mathbb{Q}^8_{17}] = 4

Inequality \( \mathbb{Q}^8_{17}[x - c] \leq \frac{27}{10} \) defines the Delaunay ellipsoid for a perfect polytope \( D^8_{17} \in \text{Del}(\mathbb{Z}^8, \mathbb{Q}^8_{17}) \), whose vertex set (\(|\text{vert} D^8_{17}| = 44\) is given below.

| \( x_8 = -1 \) | \( x_8 = 0 \) | \( x_8 = 1 \) |
|---|---|---|
| \([0^3; 1; 0^2; -1]\times 1\) | \([-1, 0^5; 1; 0]\times 6\) | \([0^3; -1; 0^2; 1^2]\times 1\) |
| \([1^5; 2; 1^2; -4; -1]\times 1\) | \([0^8]\times 1\) | \([0^3; -1; 0; 1; 0; 1]\times 1\) |
| \([0^5; 1; 0^2]\times 6\) | \([0^3; -1; 0; 1; 0; 1]\times 1\) |
| \([0^4; 1^2; -1; 0]\times 15\) | \([0^7; 1]\times 1\) |
| \([0, 1^5; -3; 0]\times 6\) | \([0^2, 1; -1; 0^3; 1]\times 3\) |
| \([1^6; -4; 0]\times 1\) |

- \(\text{Spec}(D^8_{17}) = 2, 3, 4, 5, 6, 7, 8, 9\)
- \(|\text{Iso}(D^8_{17}, \mathbb{Q}^8_{17})| = 2880\)
- \(l(D^8_{17}) = 3\)
- Antisymmetric
- Maximally contained subpolytopes: 35 – tope, \(H(2), \frac{1}{2}H(5), J(8, 6)\)
Perfect affine quadratic lattice \( \text{Aff}(\mathbb{Z}^8, \mathcal{Q}_{18}^8[x - c]) \), where \( \mathcal{Q}_{18}^8 \) is given by

\[
\mathcal{Q}_{18}^8(x) = x^t \begin{bmatrix}
11 & 6 & 6 & 6 & -7 & 11 & 4 & 3 \\
6 & 11 & 6 & 6 & -7 & 11 & 4 & 3 \\
6 & 6 & 11 & 6 & -7 & 11 & 4 & 3 \\
6 & 6 & 6 & 11 & -7 & 11 & 4 & 3 \\
-7 & -7 & -7 & -7 & 12 & -8 & -4 & 4 \\
11 & 11 & 11 & 11 & -8 & 20 & 4 & 8 \\
4 & 4 & 4 & 4 & -4 & 4 & 8 & 0 \\
3 & 3 & 3 & 3 & 4 & 8 & 0 & 13 
\end{bmatrix} x \quad \text{and} \quad c = \frac{1}{76}.
\]

- \(|O(\mathbb{Z}^8, \mathcal{Q}_{18}^8)| = 288; S_4 < O(\mathbb{Z}^8, \mathcal{Q}_{18}^8)
- s(\mathbb{Z}^8, \mathcal{Q}_{18}^8) = 6
- \dim \text{QuadInv}[\mathbb{Z}^8, \mathcal{Q}_{18}^8] = 8

Inequality \( \mathcal{Q}_{18}^8[x - c] \leq \frac{501}{76} \) defines the Delaunay ellipsoid for a perfect polytope \( D_{18}^8 \in \text{Del}(\mathbb{Z}^8, \mathcal{Q}_{18}^8) \), whose vertex set \(|\text{vert } D_{18}^8| = 44\) is given below.

| \( x_6 = -1 \) | \( x_6 = 0 \) | \( x_6 = 1 \) | \( x_6 = 2 \) |
|----------------|----------------|----------------|----------------|
| \[0, 1^3; 1; -1; 0^2\] | \[0^4; -1; 0^2; 1\] \times 1 | \[-1^4; -2; 1^3\] \times 1 | \[-1^5; 2; 1; 0\] \times 1 |
| \[1^5; -1^2; 0\] \times 1 | \[0^8\] \times 1 | \[-1^3; 0; -1; 1^3\] \times 4 | \[-1^5; 2; 1^2\] \times 1 |
| \[1^5; -1; 0^2\] \times 1 | \[0^7; 1\] \times 1 | \[-1; 0^3; 0; 1^2; 0\] \times 4 | |
| \[1^4; 2; -1; 0^2\] \times 1 | \[0^6; 1; 0\] \times 1 | \[-1; 0^3; 0; 1^2; 0\] \times 4 | |
| | \[0^6; 1^2\] \times 1 | \[0^5; 1; 0^2\] \times 1 | |
| | \[0^5; 1; 0^4\] \times 4 | \[0^4; 1^2; 0^2\] \times 1 | |
| | \[0^2; 1^2; 1; 0^3\] \times 6 | \[0^4; 1^3; 0\] \times 1 | |
| | \[0; 1^3; 2; 0^2; -1\] \times 4 | | |

- \( \text{Spec}(D_{18}^8) = 5, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 24 \)
- \(|\text{Iso}(D_{18}^8, \mathcal{Q}_{18}^8)| = 144 \)
- \( l(D_{18}^8) = 3 \)
- Antisymmetric
- Maximally contained subpolytopes: \( H(2), \frac{1}{2}H(5), J(7, 5) \)
Perfect affine quadratic lattice $\text{Aff}(\mathbb{Z}^8, Q_{19}^8[x - c])$, where $Q_{19}^8$ is given by

$$Q_{19}^8(x) = x^t \begin{pmatrix} 8 & 5 & 5 & 5 & 1 & 1 & 8 & 4 \\ 5 & 8 & 5 & 5 & 1 & 1 & 8 & 4 \\ 5 & 5 & 8 & 5 & 1 & 1 & 8 & 4 \\ 5 & 5 & 5 & 8 & 1 & 1 & 8 & 4 \\ 1 & 1 & 1 & 1 & 7 & 3 & 2 & 0 \\ 1 & 1 & 1 & 1 & 3 & 7 & 6 & 4 \\ 8 & 8 & 8 & 8 & 2 & 6 & 15 & 8 \\ 4 & 4 & 4 & 4 & 0 & 4 & 8 & 8 \end{pmatrix} x$$

and $c = \frac{1}{46} \begin{pmatrix} -11 \\ -11 \\ -11 \\ -11 \\ -11 \\ -29 \\ -32 \\ 49 \\ 12 \end{pmatrix}$.

- $|O(\mathbb{Z}^8, Q_{19}^8)| = 576$; $S_4 < O(\mathbb{Z}^8, Q_{19}^8)$
- $s(\mathbb{Z}^8, Q_{19}^8) = 8$
- $\dim \text{QuadInv}[\mathbb{Z}^8, Q_{19}^8] = 6$

Inequality $Q_{19}^8[x - c] \leq \frac{229}{46}$ defines the Delaunay ellipsoid for a perfect polytope $D_{19}^8 \in \text{Del}(\mathbb{Z}^8, Q_{19}^8)$, whose vertex set ($|\text{vert} D_{19}^8| = 49$) is given below.

| $x_8 = -1$ | $x_8 = 0$ | $x_8 = 1$ |
|---|---|---|
| $[0^6; 1; -1] \times 1$ | $[-1^4; 1; -2; 3; 0] \times 1$ | $[-1^4; 1; -3; 3; 1] \times 1$ |
| $[-1^3; 0; 1; -2; 3; 0] \times 4$ | $[-1^4; 1; -2; 3; 1] \times 1$ | $[-1^4; 1; -2; 3; 1] \times 1$ |
| $[-1^2; 0^2; 1; -1; 2; 0] \times 6$ | $[-1^4; 2; -3; 3; 1] \times 1$ | $[-1^3; 0; 1; -2; 2; 1] \times 4$ |
| $[-1; 0^3; 0^2; 1; 0] \times 4$ | $[-1^3; 0; 1; -2; 2; 1] \times 4$ | $[-1; 0^3; 1; -1; 1^2] \times 4$ |
| $[0^8] \times 1$ | $[0^7; 1] \times 1$ | |
| $[0^6; 1; 0] \times 1$ | $[0^4; 1; -1; 0; 1] \times 1$ | |
| $[0^5; 1; 0^2] \times 1$ | $[0^4; 1; 0^2; 1] \times 1$ | |
| $[0^4; 1; -1; 1; 0] \times 1$ | $[0^4; 1; 0^3] \times 1$ | |
| $[0^3; 1; 0^4] \times 4$ | $[0^2; 1^2; 0; 1; -1; 0] \times 6$ | |

- $\text{Spec}(D_{19}^8) = 4, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 18$
- $|\text{Iso}(D_{19}^8, Q_{19}^8)| = 288$
- $l(D_{19}^8) = 3$
- Antisymmetric
- Maximally contained subpolytopes: $H(3), \frac{1}{2}H(5), J(7,5)$
Perfect affine quadratic lattice $\text{Aff}(\mathbb{Z}^8, Q_{20}^8[x - c])$, where $Q_{20}^8$ is given by

$$Q_{20}^8(x) = x^t \begin{pmatrix} 12 & 9 & 9 & 9 & 9 & -15 & 3 & 3 \\ 9 & 12 & 9 & 9 & 9 & -15 & 4 & 5 \\ 9 & 9 & 12 & 9 & 9 & -15 & 4 & 3 \\ 9 & 9 & 9 & 12 & 9 & -15 & 4 & 5 \\ -15 & -15 & -15 & -15 & -12 & 24 & -6 & -6 \\ 3 & 4 & 4 & 4 & 4 & -6 & 4 & 3 \\ 3 & 5 & 3 & 5 & 4 & -6 & 3 & 5 \end{pmatrix} x \quad \text{and} \quad c = \frac{1}{20}. $$

- $|O(\mathbb{Z}^8, Q_{20}^8)| = 48, S_3 < O(\mathbb{Z}^8, Q_{20}^8)$
- $s(\mathbb{Z}^8, Q_{20}^8) = 4$
- $\dim \text{QuadInv}[\mathbb{Z}^8, Q_{20}^8] = 11$

Inequality $Q_{20}^8[x - c] \leq \frac{81}{20}$ defines the Delaunay ellipsoid for a perfect polytope $D_{20}^8 \in \text{Del}(\mathbb{Z}^8, Q_{20}^8)$, whose vertex set ($\text{vert } D_{20}^8 = 47$) is given below.

| $x_8 = -1$ | $x_8 = 0$ | $x_8 = 1$ |
|-------------|-------------|-------------|
| $[0^3; 1; 0^3; -1] \times 1$ | $[-1; 0^4; -1; 0^2] \times 5$ | $[0; -1; 0^3; -1^2; 1^1] \times 1$ |
| $[0; 1; 0^5; -1] \times 1$ | $[0^8] \times 1$ | $[0^3; -1; 0; -1^2; 1^1] \times 1$ |
| $[0; 1; 0; 1; 0; 1^2; -1] \times 1$ | $[0^6; 1; 0] \times 1$ | $[0^7; 1] \times 1$ |
| $[0; 1; 0; 1^3; 0; -1] \times 1$ | $[0^4; 1; 0^3] \times 5$ | $[0^2; 1; 0^3; -1; 1] \times 1$ |
| $[0; 1^4; 2; 0; -1] \times 1$ | $[0^3; 1^2; 1; 0^2] \times 10$ | $[0^2; 1; 0; 1^2; -1; 1] \times 1$ |
| $[1^2; 0; 1^2; 2; 0; -1] \times 1$ | $[0; 1^4; 2; 0^2] \times 5$ | $[1; 0; 1; 0^2; 1; -1; 1] \times 1$ |
| $[1^2; 0; 1^2; 2; 1; -1] \times 1$ | $[1^5; 3; 0^2] \times 1$ | $[1; 0; 1; 0^2; 1; 0; 1] \times 1$ |
| $[0^4; 1; 0; -1; 0] \times 1$ | $[0^3; 1; 0^2; -1; 0] \times 1$ | $[0^2; 1; 0^3; -1; 0] \times 1$ |
| $[0; 1; 0^4; -1; 0] \times 1$ | $[0; 1^4; 2; -1; 0] \times 1$ |

- $\text{Spec}(D_{20}^8) = 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14$
- $|\text{Iso}(D_{20}^8, Q_{20}^8)| = 24$
- $l(D_{20}^8) = 3$
- Antisymmetric
- Maximally contained subpolytopes: $G_6, H(3), \frac{1}{2}H(5), J(6, 4)$
Perfect affine quadratic lattice \( \text{Aff}(\mathbb{Z}^8, \mathcal{Q}^8_{21}[x - c]) \), where \( \mathcal{Q}^8_{21} \) is given by

\[
\mathcal{Q}^8_{21}(x) = x^t \begin{bmatrix}
12 & 9 & 9 & 9 & 9 & -15 & 6 & 5 \\
9 & 12 & 9 & 9 & 9 & -15 & 3 & 4 \\
9 & 9 & 12 & 9 & 9 & -15 & 4 & 3 \\
9 & 9 & 9 & 12 & 9 & -15 & 4 & 3 \\
9 & 9 & 9 & 9 & 12 & -15 & 4 & 3 \\
-15 & -15 & -15 & -15 & -15 & 24 & -6 & -5 \\
6 & 3 & 4 & 4 & 4 & -6 & 6 & 4 \\
5 & 4 & 3 & 3 & 3 & -5 & 4 & 5 \\
\end{bmatrix} \begin{bmatrix}
x \\
c = \frac{1}{50} \\
\end{bmatrix}.
\]

- \( |O(\mathbb{Z}^8, \mathcal{Q}^8_{21})| = 96; S_3 < O(\mathbb{Z}^8, \mathcal{Q}^8_{21}) \)
- \( s(\mathbb{Z}^8, \mathcal{Q}^8_{21}) = 4 \)
- \( \dim \text{QuadInv}[\mathbb{Z}^8, \mathcal{Q}^8_{21}] = 8 \)

Inequality \( \mathcal{Q}^8_{21}[x - c] \leq \frac{201}{50} \) defines the Delaunay ellipsoid for a perfect polytope \( D^8_{21} \in \text{Del}(\mathbb{Z}^8, \mathcal{Q}^8_{21}), \) whose vertex set (\( |\text{vert} \ D^8_{21}| = 47 \)) is given below.

| \( x_8 = -1 \) | \( x_8 = 0 \) | \( x_8 = 1 \) |
|---|---|---|
| \([0; 1; 0^4; 1; -1] \times 1\) | \([-1; 0^4; -1; 0^2] \times 5\) | \([-1; 0^4; -1; 0; 1] \times 1\) |
| \([0; 1; 0^2; 1; 1^2; -1] \times 3\) | \([0^8] \times 1\) | \([0; -1; 0^3; -1^2; 1] \times 1\) |
| \([1; 0^6; -1] \times 1\) | \([0^6; 1; 0] \times 1\) | \([0^7; 1] \times 1\) |
| \([1^2; 0^3; 1^2; -1] \times 1\) | \([0^4; 1; 0^3] \times 5\) | \([1; 0^3; 2; -1; 1] \times 1\) |
| \([1^2; 0; 1^2; 2; 0] \times 3\) | \([0^3; 1^2; 1; 0^2] \times 10\) | \([0; 1^4; 2; 0^2] \times 5\) |
| \([2; 1^4; 3; 0] \times 1\) | \([1^5; 3; 0^2] \times 1\) | \([-1; 0^4; -1; 1; 0] \times 1\) |
| \([-1; 1; 0^4; 1; 0] \times 1\) | \([1; 0^5; -1; 0] \times 1\) | \([1; 0; 1^3; 2; -1; 0] \times 1\) |
| \([2; 1^4; 3; -1; 0] \times 1\) | \[
- \text{Spec}(D^8_{21}) = 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13 \\
- |\text{Iso}(D^8_{21}, \mathcal{Q}^8_{21})| = 48 \\
- \ell(D^8_{21}) = 3 \\
- \text{Antisymmetric} \\
- \text{Maximally contained subpolytopes: } G_6, H(3), \frac{1}{2}H(5), J(6, 4) \\
\]
Perfect affine quadratic lattice \( \text{Aff}(\mathbb{Z}^8, \mathcal{Q}_{22}^8[x - c]) \), where \( \mathcal{Q}_{22}^8 \) is given by

\[
\mathcal{Q}_{22}^8(x) = x^t \begin{bmatrix}
3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 4 & 2 & 2 & 2 & 2 & 2 & 2 \\
0 & 2 & 2 & 1 & 1 & 1 & 1 & 1 \\
0 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 2 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix} x \quad \text{and} \quad c = \frac{1}{10}
\]

- \( |O(\mathbb{Z}^8, \mathcal{Q}_{22}^8)| = 645120 \); \( S_7 < O(\mathbb{Z}^8, \mathcal{Q}_{22}^8) \)
- \( s(\mathbb{Z}^8, \mathcal{Q}_{22}^8) = 84 \)
- \( \dim \text{QuadInv}[\mathbb{Z}^8, \mathcal{Q}_{22}^8] = 2 \)

Inequality \( \mathcal{Q}_{22}^8[x - c] \leq \frac{9}{5} \) defines the Delaunay ellipsoid for a perfect polytope \( D_{22}^8 \in \text{Del}(\mathbb{Z}^8, \mathcal{Q}_{22}^8) \), whose vertex set (\( |\text{vert} D_{22}^8| = 79 \)) is given below.

| \( x_1 = 0 \) | \( x_1 = 1 \) | \( x_1 = 2 \) |
|---|---|---|
| \( 0^8 \times 1 \) | \( 1; -3; 1^6 \times 1 \) | \( 2; -3; 1^6 \times 1 \) |
| \( 0^2; 0^5; 1 \times 6 \) | \( 1; -2; 0^2; 1^4 \times 15 \) |   |
| \( 0; 1; -1; 0^6 \times 6 \) | \( 1; -2; 0; 1^5 \times 6 \) |   |
| \( 0; 1; 0^6 \times 1 \) | \( 1; -1; 0^4; 1^2 \times 15 \) |   |
|   | \( 1; -1; 0^3; 1^3 \times 20 \) |   |
|   | \( 1; 0^7 \times 1 \) |   |
|   | \( 1; 0^6; 0; 1 \times 6 \) |   |

- \( \text{Spec}(D_{22}^8) = 2, 3, 4, 5, 6 \)
- \( |\text{Iso}(D_{22}^8, \mathcal{Q}_{22}^8)| = 322560 \)
- \( l(D_{22}^8) = 3 \)
- \( \text{Antisymmetric} \)
- Maximally contained subpolytopes: \( H(3), \frac{1}{2}H(7), J(8, 6) \)
Perfect affine quadratic lattice \( \text{Aff}(\mathbb{Z}^8, Q^8_{23}[x - c]) \), where \( Q^8_{23} \) is given by

\[
Q^8_{23}(x) = x' \quad \text{and} \quad c = \frac{1}{20} - 7 - 7 - 7 - 7 - 7 - 7 - 7 - 7 - 7 - 7 - 14.
\]

- \( |O(\mathbb{Z}^8, Q^8_{23})| = 1920; S_5 < O(\mathbb{Z}^8, Q^8_{23}) \)
- \( s(\mathbb{Z}^8, Q^8_{23}) = 10 \)
- \( \dim \text{QuadInv}[\mathbb{Z}^8, Q^8_{23}] = 4 \)

Inequality \( Q^8_{23}[x - c] \leq \frac{49}{10} \) defines the Delaunay ellipsoid for a perfect polytope \( D^8_{23} \in \text{Del}(\mathbb{Z}^8, Q^8_{23}) \), whose vertex set \( |\text{vert } D^8_{23}| = 49 \) is given below.

\[
\begin{array}{cccc}
7 & 4 & 4 & 4 & 4 & 4 & 2 & 7 & 4 & -7 \\
4 & 7 & 4 & 4 & 4 & 2 & 7 & 4 & -7 \\
4 & 4 & 7 & 4 & 4 & 2 & 7 & 4 & -7 \\
4 & 4 & 4 & 7 & 4 & 2 & 7 & 4 & -7 \\
2 & 2 & 2 & 2 & 7 & 7 & 0 & -7 \\
7 & 7 & 7 & 7 & 7 & 7 & 14 & 4 \\
4 & 4 & 4 & 4 & 0 & 4 & 8 & 14 \\
\end{array}
\]

- \( \text{Spec}(D^8_{23}) = 4, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16 \)
- \( |\text{Iso}(D^8_{23}, Q^8_{23})| = 960 \)
- \( l(D^8_{23}) = 3 \)
- Antisymmetric
- Maximally contained subpolytopes: \( H(3), \frac{1}{2}H(4), J(7, 5) \)
Perfect affine quadratic lattice $\text{Aff}(\mathbb{Z}^8, Q^8_{24}[x - c])$, where $Q^8_{24}$ is given by

$$Q^8_{24}(x) = x'$$

$x$ and $c = \frac{1}{36}$.

- $|O(\mathbb{Z}^8, Q^8_{24})| = 144$; $S_3 < O(\mathbb{Z}^8, Q^8_{24})$
- $s(\mathbb{Z}^8, Q^8_{24}) = 2$
- $\dim \text{QuadInv}[^8\mathbb{Z}, Q^8_{24}] = 9$

Inequality $Q^8_{24}[x - c] \leq \frac{97}{12}$ defines the Delaunay ellipsoid for a perfect polytope $D^8_{24} \in \text{Del}(\mathbb{Z}^8, Q^8_{24})$, whose vertex set ($|\text{vert } D^8_{24}| = 44$) is given below.

| $x_8 = -1$ | $x_8 = 0$ | $x_8 = 1$ |
|-----------|-----------|-----------|
| $[0^3; \uparrow 1; 0^3; \downarrow 1] \times 1$ | $[-1, 0^5; \uparrow 1; 0] \times 6$ | $[0^7; \uparrow 1] \times 1$ |
| $[0^3; \uparrow 1^2; \downarrow 1^2] \times 1$ | $[0^8] \times 1$ | $[0, 1; 0; -1; 0^3; 1] \times 2$ |
| $[0^2; \uparrow 1^2; 0; \downarrow 1^2] \times 2$ | $[0^5, 1; 0^2] \times 6$ | |
| $[0^2; \uparrow 1^4; -2; \downarrow \uparrow 1] \times 1$ | $[0^4, 1^2; -1; 0] \times 15$ | |
| $[1^3; \uparrow 2; 1^2; -4; \downarrow -1] \times 1$ | $[0, 1^5; -3; 0] \times 6$ | |
| | | $[1^6; -4; 0] \times 1$ |

- $\text{Spec}(D^8_{24}) = 5, 8, 9, 11, 12, 13, 15, 16, 17, 19, 20, 21, 23, 24, 25, 27$
- $|\text{Iso}(D^8_{24}, Q^8_{24})| = 72$
- $l(D^8_{24}) = 3$
- Antisymmetric
- Maximally contained subpolytopes: 35 - tope, $H(2)$, $\frac{1}{2}H(5)$, $J(7, 5)$
Perfect affine quadratic lattice $\text{Aff}(\mathbb{Z}^8, \mathcal{Q}^8_{26}[x - c])$, where $\mathcal{Q}^8_{26}$ is given by

$$
\begin{array}{cccccc}
12 & 9 & 9 & 9 & -15 & 6 & 3 \\
9 & 12 & 9 & 9 & -15 & 4 & 5 \\
9 & 9 & 12 & 9 & -15 & 7 & 5 \\
9 & 9 & 9 & 12 & 9 & -15 & 6 & 3 \\
9 & 9 & 9 & 9 & 12 & 9 & -15 & 7 & 5 \\
-15 & -15 & -15 & -15 & 24 & -9 & -6 \\
6 & 4 & 7 & 6 & 7 & -9 & 8 & 4 \\
3 & 5 & 5 & 3 & 5 & -6 & 4 & 6 \\
\end{array}
$$

$x$ and $c = \frac{1}{3}$.

- $|O(\mathbb{Z}^8, \mathcal{Q}^8_{26})| = 2592$; $S_3 < O(\mathbb{Z}^8, \mathcal{Q}^8_{26})$
- $s(\mathbb{Z}^8, \mathcal{Q}^8_{26}) = 18$
- $\dim \text{QuadInv}[\mathbb{Z}^8, \mathcal{Q}^8_{26}] = 2$

Inequality $\mathcal{Q}^8_{26}[x - c] \leq 4$ defines the Delaunay ellipsoid for a perfect polytope $D^8_{26} \in \text{Del}(\mathbb{Z}^8, \mathcal{Q}^8_{26})$, whose vertex set $|\text{vert} D^8_{26}| = 45$ is given below.

| $x_8 = -1$ | $x_8 = 0$ | $x_8 = 1$ |
|---|---|---|
| $[0; 1; 0^2; 1^3; -1] \times 1$ | $[-1; 0^4; -1; 0^2] \times 5$ | $[0; -1^2; 0; -1; -2; 0; 1] \times 1$ |
| $[0; 1^2; 0^2; 1^2; -1] \times 1$ | $[0^3] \times 1$ | $[0; -1; 0^3; -1^2; 1] \times 1$ |
| $[0; 1^2; 0; 1^2; 0; -1] \times 1$ | $[0^6; 1; 0] \times 1$ | $[0^7; 1] \times 1$ |
| $[0; 1^4; 2; 0; -1] \times 1$ | $[0^4; 1; 0^3] \times 5$ | $[1; -1; 0; 1; 0^2; -1; 1] \times 1$ |
| $[0; 2; 0^3; 1^2; -1] \times 1$ | $[0^3; 1^2; 1; 0^2] \times 10$ | $[1; 0^2; 1; 0; 1; 0; 1] \times 1$ |
| $[1^3; 0; 1; 2; 0; -1] \times 1$ | $[0; 1^4; 2; 0^2] \times 5$ | $[1; 0; 1^3; 2; -1; 1] \times 1$ |

- $\text{Spec}(D^8_{26}) = 2, 3, 4, 5, 6$
- $|\text{Iso}(D^8_{26}, Q^8_{26})| = 1296$
- $l(D^8_{26}) = 3$
- Antisymmetric
- Maximally contained subpolytopes: $G_6$, $H(2)$, $\frac{1}{2}H(5)$, $J(7, 5)$
Perfect affine quadratic lattice $\text{Aff}(\mathbb{Z}^8, \mathcal{Q}_{27}^8[x - c])$, where $\mathcal{Q}_{27}^8$ is given by

$$
\begin{bmatrix}
12 & 9 & 9 & 9 & 9 & -15 & 7 & 3 \\
9 & 12 & 9 & 9 & 9 & -15 & 6 & 2 \\
9 & 9 & 12 & 9 & 9 & -15 & 6 & 3 \\
9 & 9 & 9 & 12 & 9 & -15 & 8 & 5 \\
9 & 9 & 9 & 9 & 12 & -15 & 8 & 5 \\
-15 & -15 & -15 & -15 & -15 & 24 & -9 & -5 \\
7 & 6 & 6 & 6 & 8 & -9 & 11 & 3 \\
3 & 2 & 3 & 3 & 5 & -5 & 3 & 5
\end{bmatrix}
\begin{bmatrix}
x \\
c = \frac{1}{12}
\end{bmatrix}
$$

- $|O(\mathbb{Z}^8, \mathcal{Q}_{27}^8)| = 144$; $S_3 < O(\mathbb{Z}^8, \mathcal{Q}_{27}^8)$
- $s(\mathbb{Z}^8, \mathcal{Q}_{27}^8) = 24$
- $\dim \text{QuadInv}[\mathbb{Z}^8, \mathcal{Q}_{27}^8] = 8$

Inequality $\mathcal{Q}_{27}^8[x - c] \leq \frac{17}{4}$ defines the Delaunay ellipsoid for a perfect polytope $D_{27}^8 \in Del(\mathbb{Z}^8, \mathcal{Q}_{27}^8)$, whose vertex set ($\vert \text{vert} D_{27}^8 \vert = 44$) is given below.

| $x_8 = -1$ | $x_8 = 0$ | $x_8 = 1$ |
|-----------|-----------|-----------|
| $[0^4; 1; 0^2; -1] \times 1$ | $[1^5; -3; 1; 0] \times 1$ | $[-1; 0; -1^2; -2; -3; 1^2] \times 1$ |
| $[1^7; 0^2; -1; -2; -1; 0] \times 1$ | $[0^6; 1; 0] \times 1$ | $[0; 1; 0^2; -1; 0^2; 1] \times 1$ |
| $[0^4; 1; 0^3] \times 5$ | $[0^7; 1^2; 0] \times 1$ | $[0; 1; 0^2; 1; 0; 1] \times 2$ |
| $[0^8; 1^2; 0^2] \times 10$ | $[1^2; 0^3; 1; 0; 1] \times 1$ |
| $[1^5; 3; 0^2] \times 1$ |

- $\text{Spec}(D_{27}^8) = 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15$
- $|\text{Iso}(D_{27}^8, \mathcal{Q}_{27}^8)| = 72$
- $l(D_{27}^8) = 3$
- Antisymmetric
- Maximally contained subpolytopes: $G_6, H(2), \frac{1}{2}H(5), J(7, 5)$
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