AUSLANDER-REITEN QUIVER AND REPRESENTATION THEORIES RELATED TO KLR-TYPE SCHUR-WEYL DUALITY

SE-JIN OH

Abstract. We introduce new notions on the sequences of positive roots by using Auslander-Reiten quivers. Then we can prove that the new notions provide interesting information on the representation theories of KLR-algebras, quantum groups and quantum affine algebras including Dorey’s rule, bases theory for quantum groups, and denominator formulas between fundamental representations.

Introduction

The category $\text{Rep}(R)$ consisting of finite dimensional graded modules over KLR algebra $R$ provides the categorification of negative part of $U_q(g)$ for all symmetrizable Kac-Moody algebras $g$ (\cite{25, 39}). When $U_q(g_0)$ is associated with a finite simple Lie algebra $g_0$ of type $A, D$ or $E$, $U_q^-(g_0)$ is also categorified by the categorized the categories $C_Q^{(1)}$ consisting of finite dimensional integrable modules over the quantum affine algebras $U'_q(g)$ ($g = A^{(1)}_n, D^{(1)}_n$ or $E^{(1)}_n$) (\cite{13}). Here the definition of $C_Q^{(1)}$ is closely related to the Auslander-Reiten quiver (AR-quiver) $\Gamma_Q$ of a Dynkin quiver $Q$.

The two categories, $\text{Rep}(R)$ and $C_Q^{(1)}$ for $g = A^{(1)}_n$ or $D^{(1)}_n$, are closely related to each other by the KLR-type Schur-Weyl functor $F_Q^{(1)}$ (\cite{15, 16}) where

(i) $F_Q^{(1)}$ is an exact tensor functor and sends simples to simples,

(ii) $F_Q^{(1)}$ is constructed by the observations on the combinatorial properties of AR-quiver $\Gamma_Q$ and denominator formulas $d_{k,l}(z)$ for fundamental representations over $U'_q(g)$.

The statement (ii) implies that the partial information about $d_{k,l}(z)$ can be observed in the combinatorics of $\Gamma_Q$ (see \cite{34}, Introduction for more detail).

In $U_q^-\left(g_0, (g_0)\right)$, there are distinguished bases, so called (dual) PBW-bases, which are associated to reduced expressions $\tilde{w}_0$ of the longest element $w_0$ of its Weyl group $W_0$ (\cite{31}). Also, in $U^-_a\left(g_0\right)$, there exists a unique basis, the Lusztig/Kashiwara’s canonical/lower global (resp. dual canonical/upper global) basis (\cite{31, 22}).

Interestingly, those distinguished bases for $U_q^-\left(g_0\right)$ are categorified by modules over KLR-algebras $R$ (\cite{6, 24, 32} and \cite{40, 41}) under the suitable assumption for the base field $k$ of $R$. More precisely, (dual) PBW-bases are categorified by the set of all (proper) standard modules over $R$ and the (dual) canonical basis is categorified by the set of all principal indecomposable (simple) modules over $R$. Furthermore, the transition map between the
between the indecomposables \((\tilde{w}_0, (\tilde{w}_0))\), and its convex total order \(<_{\tilde{w}_0}\) on the positive roots \(\Phi^+\) for \(\mathfrak{g}_0\), played an important role.

On the other hand, the dual PBW-basis associated to \(\tilde{w}_0\) adapted to a Dynkin quiver \(Q\) is also categorized by ordered tensor products of fundamental representations with respect to the convex total order \(<_{\tilde{w}_0}\) on \(\Phi^+\). Also, the dual canonical basis is categorized by the set of simple modules in \(C_Q^{(1)}\) ([13]). Furthermore, the modules categorifying PBW-basis associated to \(\tilde{w}_0\) adapted to \(Q\) in both categories can be identified via the functor \(F_Q^{(1)}\) ([16], see also [19] for twisted affine type analogues).

In the aspect of the categorification theory, the combinatorial object \(\Gamma_Q\) is known for the path algebra \(CQ\) in the sense that its vertices can be labeled by the set of all indecomposable modules over \(CQ\) (up to isomorphism), its arrows represent the irreducible morphisms between the indecomposables \((\{1\})\). But also, in the aspect of Lie theory, \(\Gamma_Q\) is quite interesting in the following sense: (i) the set of dimension vectors for all indecomposables coincides with \(\Phi^+\), (ii) each \(\Gamma_Q\) corresponds to the commutation class \(Q\) of reduced expressions \(\tilde{w}_0\) for \(w_0\) adapted to \(Q\), and corresponds to the Coxeter element \(\tau_Q\) determined by \(Q\), (iii) it visualizes the convex partial order \(<_Q\) on \(\Phi^+\) for \(Q\) (see [4, 42]).

In the representation theory of quantum affine algebra \(U_q'(\mathfrak{g})\), Coxeter elements and their twisted analogues play an important role for the Dorey’s rule, which are closely related to the three-point coupling between the quantum particles in Toda field theory [7, 8]. More precisely, for untwisted quantum affine algebras of classical type \(A_n^{(1)}\), \(B_n^{(1)}\), \(C_n^{(1)}\) and \(D_n^{(1)}\), Chari-Pressley used the Coxeter elements and their twisted analogues to prove that the tensor product of two fundamental representations satisfying certain condition has a simple socle as the another fundamental representation:

\[
V^{(1)}(\varpi_k)_z \rightarrow V^{(1)}(\varpi_i)_x \otimes V^{(1)}(\varpi_j)_y \quad \text{if \{(i, x), (j, y), (k, z)\} satisfies the condition.}
\]

To sum up, AR-quivers \(\Gamma_Q\) and hence their Coxeter elements \(\tau_Q\) play central roles in the representation theories of KLR-algebras, quantum affine algebras, quantum group, system of positive roots. The goal of this paper is to understand those representation theories one step further by developing new notions on the sequences of positive roots and by using AR-quivers and commutation classes for reduced expressions \(\tilde{w}\) of \(w \in W_0\).

The most important new notion in this paper is an order

\[
<_{\tilde{w}_0}^{b} \quad \text{on } \mathbb{Z}_{\geq 0}^{f(w_0)} \quad \text{which is far coarser than } <_{\tilde{w}_0}^{b} \quad \text{(see Definition 1.7)}.
\]

Using this new order, we can refine the transition map between a dual PBW-basis associated to any \(\tilde{w}_0\) and the canonical basis by far. Also, we can prove that (dual) PBW-basis of \(U_q'(\mathfrak{g}_0)\) does depend only on the commutation class \([\tilde{w}_0]\) of \(\tilde{w}_0\) (up to \(\mathbb{Z}[q^{-1}]\)) indeed (Theorem 5.10).

With this new order, we define new notions on the sequences of positive roots as follows:

\([\tilde{w}_0]\)-simple sequence \(s\), \([\tilde{w}_0]\)-minimal sequence \(m\) of \(s\),

which depend on the commutation class \([\tilde{w}_0]\) and \(<_{\tilde{w}_0}^{b}\). In particular, when the sequence is a pair (a sequence consisting two distinct positive roots), we define additional new notions

\([\tilde{w}_0]\)-distance of pair, \([\tilde{w}_0]\)-length of a pair, \([\tilde{w}_0]\)-radius of \(\gamma \in \Phi^+\), \([\tilde{w}_0]\)-socle of a pair.
The notions on pairs are motivated by the work of Hernandez in [12] which was concentrated on the tensor product of two simple modules over quantum affine algebras.

Using new notions, we can obtain interesting results on the representation theories of KLR-algebras, quantum affine algebras, quantum group, system of positive roots by concentrating on special commutation classes [Q]. For instances, [Q]-radius of γ ∈ Φ⁺ coincides with the multiplicity of γ when Q is of type An, Dn or E6 (Theorem 4.15) and [Q]-length of a pair tells the composition series of tensor product of two simple modules corresponding to PBW-generators in both categories (Theorem 5.22).

Among the results in this paper, we focus on two results in this introduction.

(A) By the works in [33, 34], the Dorey rules in (0.1) for $U_q'(A_n^{(1)})$ and $U_q'(D_n^{(1)})$ were extended and interpreted in their corresponding categorie s Rep(R) as follows: There exist some Dynkin quiver Q and $\alpha, \beta, \gamma = \alpha + \beta \in \Phi^+$ such that (see Definition 5.6)

$$V(1)(\varpi_k)_{(-q)}^c \mapsto V(1)(\varpi_i)_{(-q)} \otimes V(1)(\varpi_j)_{(-q)}$$

where the coordinates of $\alpha, \beta$ and $\gamma$ in $\Gamma_Q$ are (i, a), (j, b), (l, c), respectively.

By removing the condition that $\alpha + \beta \in \Phi^+$, we prove that the socle of $S_Q(\alpha) \circ S_Q(\beta)$ is isomorphic to the simple module $S_Q(\text{soc}_Q(\alpha, \beta))$ for any $\alpha, \beta \in \Phi^+$ and any Q of type A, D, E, which can be considered as the generalization of KLR-type Dorey’s rule. Here, $\text{soc}_Q(\alpha, \beta)$ denotes the [Q]-socle of a pair $(\alpha, \beta)$ and $S_Q(\text{soc}_Q(\alpha, \beta))$ is a tensor products of simple modules corresponding to PBW-generators (Theorem 5.19). Finding the socle and the head of tensor product of two simple modules has been intensively studied (for instance, see [17, 29]), as the linear combination of the product of two canonical basis elements in terms of canonical basis elements. Interestingly, the result on socles provides characterizations

- for elements in $U_q^-(\mathfrak{g}_0)$ which are contained in a dual PBW-basis and the dual canonical basis simultaneously (Corollary 5.21),
- for a pair of PBW-generators whose product is a linear combination of two distinct canonical basis elements, known as length two property (Corollary 5.23),

when the PBW-basis is associated to [Q] for some Q.

(B) The denominator formulas $d_{k,l}(z)$ between fundamental representations over $U_q'(\mathfrak{g})$ provides crucial information about the representation theory for integrable modules over $U_q'(\mathfrak{g})$ (see, Theorem 6.5). As we mentioned above, one can observe partial information about $d_{k,l}(z)$ from the combinatorics of $\Gamma_Q$. In this paper, we use the new notions to get complete information $d_{k,l}(z)$ from $\Gamma_Q$. In other word, we can read the denominator formulas $d_{k,l}(z)$ in $\Gamma_Q$ completely for $\mathfrak{g} = A_n^{(1)}$ and $D_n^{(1)}$ by defining the distance polynomials $D_{k,l}(z, -q)$ which does not depend on the choice of Q of finite type A, D and E. Thus we can expect naturally that the conjectural formulas for $d_{k,l}(z)$ for $U_q'(E_n^{(1)})$ ($n = 6, 7, 8$) can be read from any $\Gamma_Q$ of finite type E. For example, we give the conjectural denominators for $U_q'(E_6^{(1)})$.

Throughout this paper, we prove our assertions by using the combinatorial properties of $\Gamma_Q$ mainly. Also we usually skip the proofs for finite type $E_n$ (see Convention 4) and give examples in those cases instead, since there are finitely many cases and it is enough to observe one fixed $\Gamma_Q$ (see Convention 3) in Appendix.

Acknowledgements. The author would like to express his sincere gratitude to Professor Masaki Kashiwara, Myungho Kim and Chul-hee Lee for many fruitful discussions.
NOTIONS AND CONVENTIONS

In this preliminary section, we fix the notions and the conventions what we will use in this paper frequently.

Symmetrizable Cartan datum and quantum group. Let I be an index set. A symmetrizable Cartan datum CD is a quintuple \((A, P, \Pi, P^\vee, \Pi^\vee)\) consisting of

(a) a symmetrizable generalized Cartan matrix \(A = (a_{ij})_{i,j \in I}\),
(b) a free abelian group \(P\), called the weight lattice,
(c) \(\Pi = \{\alpha_i \in P \mid i \in I\}\), called the set of simple roots,
(d) \(P^\vee := \text{Hom}(P, \mathbb{Z})\), called the coweight lattice,
(e) \(\Pi^\vee = \{h_i \mid i \in I\} \subset P^\vee\), called the set of simple coroots.

The free abelian group \(Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i\) is also called the root lattice. Set \(Q^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i\).

For \(b = \sum_{i \in I} m_i\alpha_i \in \mathbb{Q}^+\), we set \(\text{ht}(b) = \sum_{i \in I} m_i\).

There exists a positive-definite symmetric bilinear form \(\cdot : Q \times Q \to \mathbb{Z}\) satisfying

\[
A = (a_{ij})_{i,j \in I} = \left(\frac{2\alpha_i \cdot \alpha_j}{\alpha_i \cdot \alpha_i}\right)_{i,j \in I}.
\]

For \(b \in \mathbb{Q}^+\) and \(k \in \mathbb{Z}_{\geq 0}\), we define

\[
\text{supp}(b) := \left\{ i \in I \mid n_i \neq 0 \text{ for } b = \sum_{i \in I} n_i\alpha_i \right\} \quad \text{and} \quad \text{supp}_{\geq k}(b) := \left\{ i \in I \mid n_i \geq k \right\}.
\]

We denote by \(U_q(\mathfrak{g})\) the quantum group associated with a Cartan datum CD.

Dynkin diagrams and Positive roots. In this paper, we mainly deal with the Dynkin diagrams of finite type \(A_n\) and \(D_n\), \(E_6\), \(E_7\) and \(E_8\). In the following, we list Dynkin diagrams \(\Delta\) with an enumeration of vertices by simple roots.

\[
\begin{align*}
A_n &: \circ \quad 1 \quad \cdots \quad n-1 \quad n, \\
D_n &: \circ \quad 1 \quad 2 \quad \cdots \quad n-2 \quad n-1 \quad n, \\
E_6 &: \circ \quad 1 \quad 2 \quad 3 \quad 6 \quad 5 \quad 4, \\
E_7 &: \circ \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7, \\
E_8 &: \circ \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8.
\end{align*}
\]

We denote by \(d(k, l)\) the distance between vertices \(k\) and \(l\) in \(\Delta\).

To denote a positive root, we will use several notations as follows (see [5, PLATE I~IX]):

(a) For \(A_n\) cases, we use a notation \([a, b]\) for the positive root \(\sum_{a \leq k \leq b} \alpha_k\) where \(a \leq b \leq n\). In particular, if \(a = b\), then we use \([a]\) instead of \([a, a]\).
(b) For \(D_n\) cases, we use a notation \(\{a \pm b\}\) for the positive root \(\varepsilon_a \pm \varepsilon_b\) where \(a < b \leq n\). For \(b \in \mathbb{Z}_{<0}\), we sometimes use \(\varepsilon_b\) instead of \(-\varepsilon_{-b}\) and \(\{a|b\}\) \((1 \leq a \leq -b)\) instead of \(\{a|-(b)\}\).
(c) For \(E_6\), \(E_7\) and \(E_8\) cases, we use a notation \((c_1c_2\cdots c_n)\) for the positive root \(\sum_{1 \leq k \leq n} c_k\alpha_k\) \((n = 6, 7, 8)\) where \(c_k \in \mathbb{Z}_{\geq 0}\).
1. Positive roots of finite type

In this section, we briefly recall the system of positive roots of finite types and introduce orders on the system. Then we define new notions on the set of sequences of positive roots with consideration on the orders.

1.1. System of positive roots. In this section and the next section, we choose $A$ as the Cartan matrix of a finite-dimensional simple Lie algebra $\mathfrak{g}_0$ and the index set $I$ as $\{1, 2, \ldots, n\}$. We denote by FCD the finite Cartan datum associated to $\mathfrak{g}_0$. We also denote by $\Phi^+$ the set of positive roots and $\Phi^-$ the set of negative roots associated to $\mathfrak{g}_0$.

Let $W_0$ be the Weyl group associated to $\mathfrak{g}_0$, which is generated by simple reflections $(s_i)_{i \in I}$. We denote by $\ell(w)$ the length of an element $w \in W_0$ and $w_0$ the longest element of $W_0$. We also denote by $\ast$ the involution on $I$ induced by $w_0$; i.e., $w_0(\alpha_i) = -\alpha_i\ast$.

The following proposition is well-known (see, for instance [5]).

Proposition 1.1. For an element $w \in W_0$ and its reduced expression $\tilde{w} = s_{i_1} \cdots s_{i_t}$, the set
\[
\Phi^+_w := \{ s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \mid 1 \leq k \leq t \}
\]
is contained in $\Phi^+$ and has its cardinality as $t = \ell(w)$. Moreover, the set does not depend on the choice of reduced expression of $w$.

Thus we write $\Phi^+_w$ instead of $\Phi^+_w$. Note that $\Phi^+_w$ can be also characterized as follows ([5]):
\[
\Phi^+_w = \{ \beta \in \Phi^+ \mid w^{-1}(\beta) \in \Phi^- \}.
\]

1.2. Total orders on $\Phi^+_w$ and convexity. A fixed reduced expression $\tilde{w} = s_{i_1} \cdots s_{i_t}$ and the characterization $\Phi^+_w$ in (1.1) give a way of defining the total order $<_{\tilde{w}}$ on $\Phi^+_w$ as follows:
\[
\text{(1.3)} \quad \text{Define } \beta_k := s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}), \text{ and } \beta_k <_{\tilde{w}} \beta_j \text{ if and only if } k < j.
\]

The following theorem tells that the order $<_{\tilde{w}}$ on $\Phi^+_w$ induced by $\tilde{w}$ satisfies interesting properties:

Theorem 1.2. [37]
\[
\text{(a) For } \alpha, \beta \in \Phi^+_w \text{ with } \alpha + \beta \in \Phi^+_w, \text{ we have either } \alpha <_{\tilde{w}} \alpha + \beta <_{\tilde{w}} \beta \text{ or } \beta <_{\tilde{w}} \alpha + \beta <_{\tilde{w}} \alpha.
\]
\[
\text{(b) For any total order } < \text{ on } \Phi^+_w \text{ satisfying the above inequality, there exists a unique reduced expression } \tilde{w} \text{ of } w \text{ such that } =<_w.
\]

By abstracting the property of $<_{\tilde{w}}$, we defined a notion of convex order on the subset $L$ of $\Phi^+$ as follows:

Definition 1.3. We say an order $<$ (not necessarily total order) on a subset $L \subset \Phi^+$ convex if $\alpha, \beta \in L$ with $\alpha + \beta \in L$, we have either $\alpha < \alpha + \beta < \beta$ or $\beta < \alpha + \beta < \alpha$.

1.3. Commutation equivalence relation and convex partial orders.

Definition 1.4. We say that two reduced expressions $\tilde{w} = s_{i_1} s_{i_2} \cdots s_{i_{\ell(w)}}$ and $\tilde{w}' = s_{j_1} s_{j_2} \cdots s_{j_{\ell(w)}}$ of $w \in W_0$ are commutation equivalent, denoted by $\tilde{w} \sim \tilde{w}' \in [\tilde{w}]$, if $s_{j_1} s_{j_2} \cdots s_{j_{\ell(w)}}$ is obtained from $s_{i_1} s_{i_2} \cdots s_{i_{\ell(w)}}$ by applying the commutation relations $s_k s_i = s_i s_k$.

By using $[\tilde{w}]$, the convex partial order $<_{[\tilde{w}]}$ is defined on $\Phi^+_w$ in a canonical way:
\[
\text{(1.4) } \alpha <_{[\tilde{w}]} \beta \text{ if and only if } \alpha <_{\tilde{w}'} \beta \text{ for any } \tilde{w}' \in [\tilde{w}].
\]
Remark 1.5. For any \( \tilde{w}' \in [\tilde{w}] \), the convex total order \( \prec_{\tilde{w}_1} \) is compatible with the order \( \prec_{\tilde{w}_2} \) in the following sense:

(i) \( \alpha \prec_{\tilde{w}_1} \beta \) implies \( \alpha \prec_{\tilde{w}_2} \beta \) for any \( \tilde{w}' \in [\tilde{w}] \).

(ii) If \( \{\alpha, \beta\} \subset (\Phi^+_{\tilde{w}})^2 \) is an incomparable pair with respect to the convex partial order \( \prec_{\tilde{w}_1} \), then there exist reduced expressions \( \tilde{w}(1), \tilde{w}(2) \in [\tilde{w}] \) such that

\[ \alpha \prec_{\tilde{w}(1)} \beta \quad \text{and} \quad \beta \prec_{\tilde{w}(2)} \alpha. \]

1.4. Partial orders on the set of sequences of \( \Phi^+_{\tilde{w}} \).

Convention 1. Let us choose a reduced expression \( \tilde{w} = s_{i_1} s_{i_2} \cdots s_{i_{\ell(w)}} \) of \( w \in W_0 \). Fix the convex total order \( \leq_{\tilde{w}} \) induced by \( \tilde{w} \) and the labeling of \( \Phi^+_{\tilde{w}} \) as follows:

\[ \beta_k^\tilde{w} := s_{i_1} \cdots s_{i_{k-1}} \alpha_k \in \Phi^+, \quad \beta_k^\tilde{w} \leq_{\tilde{w}} \beta_l^\tilde{w} \text{ if and only if } k \leq l. \]

(i) We identify a sequence \( \mu_{\tilde{w}} = (m_1, m_2, \ldots, m_{\ell(w)}) \in \mathbb{Z}_{\geq 0}^{\ell(w)} \) with

\[ (m_1 \beta^\tilde{w}_1, m_2 \beta^\tilde{w}_2, \ldots, m_{\ell(w)} \beta^\tilde{w}_{\ell(w)}) \in (Q^+)^{\ell(w)}. \]

(ii) For a sequence \( \mu_{\tilde{w}} \) and another reduced expression \( \tilde{w}' \) of \( w \), \( \mu_{\tilde{w}'_1} \) is a sequence in \( \mathbb{Z}_{\geq 0}^{\ell(w)} \) by considering \( \mu_{\tilde{w}} \) as a sequence of positive roots, rearranging with respect to \( \prec_{\tilde{w}_2} \) and applying (i).

For simplicity of notations, we usually drop the script \( \tilde{w} \) if there is no fear of confusion.

The following order on sequences of positive roots was introduced in [32] when \( w = w_0 \).

Definition 1.6. (cf. [32]) For sequences \( \mu, \mu' \in \mathbb{Z}_{\geq 0}^{\ell(w)} \), we define an order \( \leq^b_{\tilde{w}} \) as follows:

\[ \mu' = (m_1', \ldots, m_{\ell(w)}') \prec_{\tilde{w}}^b \mu = (m_1, \ldots, m_{\ell(w)}) \text{ if and only if there exist integers } k, s \text{ such that } 1 \leq k \leq s \leq \ell(w), m'_k = m_k (t < k), m'_k < m_k, \text{ and } m'_t = m_t (s < t \leq \ell(w)), m'_s < m_s. \]

Note that \( \leq^b_{\tilde{w}} \) is a partial order on the set of sequences of length \( \ell(w) \).

As we define \( \prec_{\tilde{w}}^b \) from \( \prec_{\tilde{w}} \), we define new order \( \prec_{\tilde{w}}^b \) which is far coarser than \( \prec_{\tilde{w}}^b \).

Definition 1.7. For sequences \( \mu, \mu' \in \mathbb{Z}_{\geq 0}^{\ell(w)} \), we define an order \( \prec_{\tilde{w}}^b \) as follows:

\[ \mu' = (m_1', \ldots, m_{\ell(w)}') \prec_{\tilde{w}}^b \mu = (m_1, \ldots, m_{\ell(w)}) \text{ if and only if } \]

\[ \mu'_i <_{\tilde{w}} \mu_i \text{ for all reduced expressions } \tilde{w}' \in [\tilde{w}]. \]

1.5. Sequences and new notions.

Definition 1.8. (i) A sequence \( \mu = (m_1, m_2, \ldots, m_{\ell(w)}) \in \mathbb{Z}_{\geq 0}^{\ell(w)} \) is called a basic if \( m_i \leq 1 \) for all \( i \).

(ii) A sequence \( \mu \) is called a pair if it is basic and \( |\mu| := \sum_{i=1}^{\ell(w)} m_i = 2 \).

(iii) A weight \( \text{wt}(\mu) \) of a sequence \( \mu \) is defined by \( \sum_{i=1}^{\ell(w)} m_i \beta_i \in Q^+ \).

In this paper, we use the notation \( p \) for a pair sequence. For brevity, we write a pair \( p \) as \( (\alpha, \beta) \in (\Phi^+_{\tilde{w}})^2 \) or \( (\underline{p}_1, \underline{p}_2) \) such that \( \beta_{i_1} = \alpha, \beta_{i_2} = \beta \) and \( i_1 < i_2 \).

Definition 1.9. We say a pair \( p \) is \( \tilde{w} \)-simple if there exists no sequence \( \mu \) such that

\[ \mu <_{\tilde{w}}^b (\alpha, \beta) \quad \text{and} \quad \text{wt}(\mu) = \text{wt}(p). \]
The first condition in (1.6) can be re-written as follows:

\[ \alpha < \bar{w} \beta_i < \bar{w} \beta \quad \text{for all } \beta_i \text{ such that } m_i \neq 0. \]

**Definition 1.10.** A sequence \( \bar{w} = (m_1, m_2, \ldots, m_{\ell(w)}) \) is called \( \bar{w} \)-simple if \( |\bar{w}| = 1 \) or all pairs \((p_{i_1}, p_{i_2})\) such that \( m_{i_1}, m_{i_2} > 0 \) are \( \bar{w} \)-simple pairs.

**Example 1.11.** Let us consider the following reduced expression \( \bar{w}_0 \) of \( w_0 \) of \( A_5 \):

\[ s_1 s_3 s_2 s_1 s_4 s_3 s_2 s_1 s_5 s_4. \]

Then the total order on \( \Phi^+ \) of \( A_5 \) can be computed as follows \((< := <_{\bar{w}_0}):\)

\[ [1] < [3] < [1, 3] < [2, 3] < [3, 4] < [1, 4] < [2, 4] < [4] < [3, 5] < [1, 5] < [2, 5] < [4, 5] < [5] < [1, 2]. \]

1. The pair \(([1, 3], [2, 5])\) is not \( \bar{w}_0 \)-simple, since we have \( ([2, 3], [1, 5]) <_{\bar{w}_0} ([1, 3], [2, 5]). \)

2. The pair \(([2, 4], [1, 2])\) is \( \bar{w}_0 \)-simple.

The following definition of a \( \bar{w} \)-minimal sequence \( m \) of a \( \bar{w} \)-simple sequence \( s \) for \(|s| = 1\), \( \bar{w}_0 \) of \( w_0 \) and a pair \( m \) was introduced in \([32]\). We generalize the notion for a general \( \bar{w} \)-simple sequence \( s \) as follows:

**Definition 1.12.** (cf. \([32]\)) For a given \( \bar{w} \)-simple sequence \( s = (s_1, \ldots, s_{\ell(w)}) \in \mathbb{Z}_{\geq 0}^{\ell(w)} \), we say that a sequence \( m \in \mathbb{Z}_{\geq 0}^{\ell(w)} \) is called a \( \bar{w} \)-minimal sequence of \( s \) if

\[ \begin{align*}
(i) & \quad s <_{\bar{w}} m \text{ and } wt(m) = wt(s), \\
(ii) & \quad \text{there exists no pair } m' \text{ such that } m \neq m' \text{ and it satisfies the conditions in (i) and} \\
& \quad s <_{\bar{w}} m' <_{\bar{w}} m.
\end{align*} \]

Now we refine the notions by using \( <_{\bar{w}} \), and introduce new notions.

**Definition 1.13.** A pair \( \underline{p} \) is \( \bar{w} \)-simple if there exists no sequence \( m \in \mathbb{Z}_{\geq 0}^{\ell(w)} \) such that

\[ m <_{[\bar{w}]} (\alpha, \beta) \quad \text{and} \quad wt(m) = wt(\underline{p}). \]

**Definition 1.14.** A sequence \( m = (m_1, m_2, \ldots, m_{\ell(w)}) \in \mathbb{Z}_{\geq 0}^{\ell(w)} \) is called \( \bar{w} \)-simple if \( |m| = 1 \) or all pairs \((p_{i_1}, p_{i_2})\) such that \( m_{i_1}, m_{i_2} > 0 \) are \( \bar{w} \)-simple pairs.

**Definition 1.15.** For a given \( \bar{w} \)-simple sequence \( s = (s_1, \ldots, s_{\ell(w)}) \in \mathbb{Z}_{\geq 0}^{\ell(w)} \), a sequence \( m \in \mathbb{Z}_{\geq 0}^{\ell(w)} \) is a \( \bar{w} \)-minimal sequence of \( s \) if

\[ \begin{align*}
(i) & \quad s <_{\bar{w}} m \text{ and } wt(m) = wt(s), \\
(ii) & \quad \text{there exists no sequence } m' \in \mathbb{Z}_{\geq 0}^{\ell(w)} \text{ such that it satisfies the conditions in (i) and} \\
& \quad s <_{\bar{w}} m' <_{\bar{w}} m.
\end{align*} \]

**Definition 1.16.** A pair \( \underline{p} \) has \( \bar{w} \)-distance \( k \) \((k \in \mathbb{Z}_{\geq 0})\), denoted by \( dist_{\bar{w}}(\underline{p}) = k \), if \( \underline{p} \) is not \( \bar{w} \)-simple and

\[ \begin{align*}
(i) & \quad \text{there exists a set of non } \bar{w} \text{-simple pairs } \{p^{(s)} \mid 1 \leq s \leq k\} \text{ such that} \\
& \quad wt(p^{(s)}) = wt(\underline{p}) \quad \text{and} \quad p^{(1)} <_{\bar{w}} \cdots <_{\bar{w}} p^{(k)} = \underline{p}.
\end{align*} \]
(ii) the set of non $[\tilde{w}]$-simple pairs $\{\underline{p}^{(s)}\}$ has maximal cardinality among sets of pairs satisfying (1.8).

If $\underline{p}$ is $[\tilde{w}]$-simple, we define $\text{dist}_{[\tilde{w}]}(\underline{p}) = 0$.

**Definition 1.17.** For pairs $\underline{p}' = (\alpha(1), \beta(1)) \prec_{[\tilde{w}]}^{-b} \underline{p} = (\alpha(2), \beta(2))$ in $\mathbb{Z}_{\geq 0}^{|\ell|}$, we say that they are **good adjacent neighbors** if

(i) there exists $\eta \in \Phi^+_0$ satisfying one of the following conditions:
   (a) $\eta + \beta(2) = \beta(1)$, $\eta + \alpha(1) = \alpha(2)$ and $\text{dist}_{[\tilde{w}]}(\eta, \beta(2)), \text{dist}_{[\tilde{w}]}(\eta, \alpha(1)) < \text{dist}_{[\tilde{w}]}(\underline{p})$,
   (b) $\beta(1) + \eta = \beta(2)$, $\alpha(2) + \eta = \alpha(1)$ and $\text{dist}_{[\tilde{w}]}(\beta(1), \eta), \text{dist}_{[\tilde{w}]}(\alpha(2), \eta) < \text{dist}_{[\tilde{w}]}(\underline{p})$,

(ii) there exists no $\underline{p}'' \prec_{[\tilde{w}]}^{-b} \underline{p}$ such that it satisfies the conditions (i) or

$$
\underline{p}' \prec_{[\tilde{w}]}^{-b} \underline{p}'' \prec_{[\tilde{w}]}^{-b} \underline{p} \quad \text{and} \quad \text{wt}(\underline{p}'') = \text{wt}(\underline{p}).
$$

**Definition 1.18.** For a pair $\underline{p}$, the $[\tilde{w}]$-length of the pair $\underline{p} \in \mathbb{Z}_{\geq 0}^{|\ell|}$, denoted by $\text{len}_{[\tilde{w}]}(\underline{p})$, is the integer which counts the number of non $[\tilde{w}]$-simple pairs $\underline{p}' \in \mathbb{Z}_{\geq 0}^{|\ell|}$ satisfying the following properties:

(i) $\underline{p}' \prec_{[\tilde{w}]}^{-b} \underline{p}$ and $\text{wt}(\underline{p}) = \text{wt}(\underline{p}')$,

(ii) there exists a sequence of pairs

$$
\underline{p}^{(0)} = \underline{p}' \prec_{[\tilde{w}]}^{-b} \underline{p}^{(1)} \prec_{[\tilde{w}]}^{-b} \underline{p}^{(2)} \prec_{[\tilde{w}]}^{-b} \underline{p}^{(k)} \cdots \prec_{[\tilde{w}]}^{-b} \underline{p} \quad (k \in \mathbb{Z}_{\geq 1})
$$

such that $\underline{p}^{(i)}, \underline{p}^{(i+1)}$ are good adjacent neighbor for all $0 \leq i \leq k - 1$.

We call the pairs $\underline{p}', \underline{p}$ **good neighbors**.

The following definition works only for commutation classes of the longest element $w_0$:

**Definition 1.19.** For a non-simple positive root $\gamma \in \Phi^+ \setminus \Pi$, the $[\tilde{w}_0]$-radius of $\gamma$, denoted by $\text{rds}_{[\tilde{w}_0]}(\gamma)$, is the integer defined as follows:

$$
\text{rds}_{[\tilde{w}_0]}(\gamma) = \max(\text{dist}_{[\tilde{w}_0]}(\underline{p}) \mid \gamma \prec_{[\tilde{w}_0]}^{-b} \underline{p} \text{ and } \text{wt}(\underline{p}) = \gamma).
$$

**Example 1.20.** In Example 1.24 below, one can check that

$$
\text{rds}_{[\tilde{w}_0]}(\{1|2\}) = 2 \text{ and } \text{rds}_{[\tilde{w}_0]}(\alpha) = 1 \text{ for all } \alpha \in \Phi^+ \setminus \Pi \sqcup \{\{1|2\}\}.
$$

**Definition 1.21.** For a pair $\underline{p}$, the $[\tilde{w}]$-socle of $\underline{p}$ is a $[\tilde{w}]$-simple sequence $\underline{s}$ such that

$$
\text{wt}(\underline{p}) = \text{wt}(\underline{s}) \quad \text{and} \quad \underline{s} \prec_{[\tilde{w}]}^{-b} \underline{p},
$$

if such $\underline{s}$ exists uniquely.

At this moment, the existence and the uniqueness of $\text{soc}_{[\tilde{w}]}(\underline{p})$ are not guaranteed. In later section, we will prove that $\text{soc}_{[\tilde{w}]}(\underline{p})$ exists uniquely for a certain family of commutation classes $[\tilde{w}]$. For the rest of this subsection, we call the family as the $\langle\langle Q\rangle\rangle$-family.

The following is a consequence of Remark 1.5 (ii) and Definition 1.9:

**Lemma 1.22.** For a pair $(\alpha, \beta)$ which is incomparable with respect to $\prec_{[\tilde{w}]}$,

$$
\text{soc}_{[\tilde{w}]}(\alpha, \beta) = (\alpha, \beta).
$$

**Proposition 1.23.** [6, Lemma 2.6] For $\gamma \in \Phi^+ \setminus \Pi$ and any $\tilde{w}_0$ of $w_0$, a $[\tilde{w}_0]$-minimal sequence of $\gamma$ is indeed a pair $(\alpha, \beta)$ for some $\alpha, \beta \in \Phi^+$ such that $\alpha + \beta = \gamma$. 

Example 1.24.  
(a) Let us consider the reduced expression \( \tilde{w}_0 \) of \( w_0 \) of \( D_4 \) which is given as follows:

\[
\tilde{w}_0 = s_3s_2s_1s_4s_3s_2s_1s_4s_3s_2s_1s_4.
\]

Then one can check the followings:

- \( \{\{2|4\}, \{1|3\}\} \) is \([\tilde{w}_0]\)-simple.
- \( \{(2|4), \{1|2\}\} \) is not \([\tilde{w}_0]\)-simple and
\[
\text{soc}_{[\tilde{w}_0]}(\{(2|4), \{1|2\}\}) = (\{1|4\}, \{2|3\}, \{2|3\}).
\]
- Minimal pairs of the \([\tilde{w}_0]\)-simple sequence \( \{(1|2)\} \) are
\[
\{(\{1|4\}, \{2|4\}),(\{2|3\}, \{1|3\}),(\{2|3\}, \{1|3\})\}.
\]

(b) Let us consider the following reduced expression \( \tilde{w}_0 \) of \( w_0 \) of \( E_6 \):

\[
\tilde{w}_0 = s_1s_2s_6s_3s_5s_4s_6s_3s_2s_6s_5s_6s_8s_6s_3s_2s_6s_3s_5s_6s_4s_1s_3s_2s_6s_3s_5s_6s_4s_1s_3s_2s_6s_3.
\]

Then one can check that, the pair \( p = (110000, 123211) \) is not \([\tilde{w}_0]\)-simple with \( \text{dist}_{[\tilde{w}_0]}(p) = 1 \), and there are three \([\tilde{w}_0]\)-simple pairs

\[
\tilde{s}^1 = (123111, 111100), \quad \tilde{s}^2 = (111000, 122211), \quad \tilde{s}^3 = (111110, 122101)
\]

satisfying \( \text{wt}(\tilde{s}^i) = \text{wt}(p) \) and \( \tilde{s}^i \prec_{[\tilde{w}_0]} p \) for all \( 1 \leq i \leq 3 \). Thus \( \text{soc}_{[\tilde{w}_0]}(p) \) is not well-defined (see Appendix A (2)).

2. Auslander-Reiten quiver

In this section, we briefly review the Auslander-Reiten quiver and its basic properties. For more detail, we refer [2, 3, 13].

2.1. Dynkin quiver \( Q \) of finite type ADE. Let \( Q \) be a Dynkin quiver by orienting edges of a Dynkin diagram \( \text{Delta} \) of type \( ADE \). We say that a vertex \( i \) in \( Q \) is a source (resp. sink) if and only if there are only exiting arrows out of it (resp. entering arrows into it). For a source (resp. sink) \( i \), \( s_iQ \) denotes the quiver obtained by \( Q \) by reversing the arrows incident with \( i \). For a reduced expression \( \tilde{w} = s_{i_1}s_{i_2} \cdots s_{i_\ell(w)} \) of \( w \in W_0 \), it is called adapted to \( Q \) if

\[
(2.1) \quad i_k \text{ is a source of the quiver } s_{i_{k-1}} \cdots s_{i_2}s_{i_1}Q \text{ for all } 1 \leq k \leq \ell(w).
\]

For a reduced expression \( \tilde{w}_0 \) of \( w_0 \) adapted to \( Q \), the following is well-known:

\[
(2.2) \quad \text{If } \tilde{w}'_0 \in [\tilde{w}_0], \text{ then } \tilde{w}'_0 \text{ is adapted to } Q. \quad \text{Conversely, any } \tilde{w}'_0 \text{ adapted to } Q \text{ is contained in } [\tilde{w}_0].
\]

Remark 2.1. The followings are also well-known:

(i) There is a unique Coxeter element \( \tau_Q \in W_0 \) such that all reduced expressions of \( \tau_Q \) are adapted to \( Q \).
(ii) For a sink \( i \) of \( Q \), there exists a reduced expression \( s_{i_1} \cdots s_{i_n} \) of \( \tau_Q \) such that \( i_n = i \).
(iii) The Coxeter element associated with \( s_iQ \) is the same as \( \tau_{s_iQ} = s_{i_n}s_{i_{n-1}} \cdots s_{i_1} \) if \( \tau_Q = s_{i_1} \cdots s_{i_n} \) and \( i = i_n \) is a sink of \( Q \).
(iv) \( s_{i_n}s_{i_{n-1}} \cdots s_{i_1} \) is the Coxeter element of \( Q^{\text{ev}} \) where \( Q^{\text{ev}} \) is the quiver obtained by reversing all arrows of \( Q \).
(v) For any reduced expression of \( \tilde{w}_0 = s_{i_1} \cdots s_{i_n} \) of \( w_0 \) adapted to \( Q \) (\( N := |\Phi^+| \)),
\[ s_{i_1} \cdots s_{i_n} Q = Q^* \] and every \( \tilde{w}_0 \in [\tilde{w}_0] \) is adapted to the \( Q \), also.

Here \( Q^* \) is the quiver obtained from \( Q \) by replacing vertices of \( Q \) from \( i \) to \( i^* \).

2.2. Auslander-Reiten quiver and system of positive roots. A map \( \xi^Q : I \to \mathbb{Z} \) is called a height function if \( \xi^Q_j = \xi^Q_i - 1 \) when there exists an arrow \( i \to j \) in \( Q \). Set
\[ \mathbb{Z}Q := \{(i,p) \in I \times \mathbb{Z} | p - \xi^Q_i \in 2\mathbb{Z}\}. \]

By assigning arrows \((i,p) \to (j,p + 1)\) for indices \( i,j \in I \) with \( d(i,j) = 1 \), we call \( \mathbb{Z}Q \) the repetition quiver. Note that \( \mathbb{Z}Q \) does not depend on \( Q \) but only on \( \Delta \).

For \( i \in I \), we define positive roots \( \gamma^Q_i \) and \( \theta^Q_i \) in the following way: For a reduced expression \( s_{i_1} \cdots s_{i_n} \) of \( \tau_Q \) and \( 1 \leq k \leq n \),
\[ \gamma^Q_{i_k} := s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \quad \text{and} \quad \theta^Q_{i_k} := s_{i_n} \cdots s_{i_{k+1}}(\alpha_{i_k}). \]

The roots \( \gamma_i \) and \( \theta_i \) can be computed in the following way also:
\[ \gamma^Q_i = \sum_{j \in B(i)} \alpha_j \quad \text{and} \quad \theta^Q_i = \sum_{j \in C(i)} \alpha_j \quad \text{where} \]
\begin{itemize}
  \item \( B(i) \) is the set of vertices \( j \) such that there exists a path from \( j \) to \( i \) in \( Q \),
  \item \( C(i) \) is the set of vertices \( j \) such that there exists a path from \( i \) to \( j \) in \( Q \).
\end{itemize}

The following relationship between \( \{\gamma^Q_i \mid i \in I\} \) and \( \{\theta^Q_i \mid i \in I\} \) is known as Nakayama permutation:
\[ \theta^Q_i = \tau^m_Q(\gamma^Q_i), \quad \text{where} \quad m^Q_i := \max(k \geq 0 \mid \tau^k_Q(\gamma^Q_i) \in \Phi^+). \]

Set \( \hat{\Phi}^+ := \Phi^+ \times \mathbb{Z} \). There exists a bijection \( \phi_Q : \mathbb{Z}Q \to \hat{\Phi}^+ \) ([13, §2.2]):

\begin{enumerate}
  \item \( \phi_Q(i, \xi^Q) = (\gamma^Q_i, 0) \),
  \item for a given \( \beta \in \Phi^+ \) with \( \phi_Q(i, p) = (\beta, m) \),
    \begin{itemize}
      \item if \( \tau_Q^m(\beta) \in \Phi^+ \), we set \( \phi_Q(i, p - 2) = (\pm \tau_Q^m(\beta), m) \),
      \item if \( \tau_Q^{-1}(\beta) \in \Phi^+ \), we set \( \phi_Q(i, p + 2) = (\pm \tau_Q^{-1}(\beta), m) \),
    \end{itemize}
\end{enumerate}

For \( \beta \in \Phi^+ \) and \( \phi_Q^{-1}(\beta, 0) = (i, p) \), we denote by
\[ i = \phi_Q^{-1}(\beta) \quad \text{and} \quad p = \phi_Q^{-1}(\beta). \]

We say \( i \) the residue of \( \beta \) with respect to \( Q \).

The Auslander-Reiten quiver (AR-quiver) \( \Gamma_Q \) is the full subquiver of \( \mathbb{Z}Q \) whose set of vertices is \( \phi_Q^{-1}(\Phi^+ \times 0) \). Thus we can label the vertices of \( \Gamma_Q \) by \( \Phi^+ \).

The AR-quiver \( \Gamma_Q \) satisfies the additive property with respect to arrows and \( \tau_Q \) in the following sense: For \( \alpha \in \Phi^+ \) we have
\[ \alpha + \tau^+_{Q}(\alpha) = \sum_{\beta \in \alpha^+} \beta \quad \text{if} \quad \tau^+_{Q}(\alpha) \in \Phi^+, \]
where \( \alpha^- \) (resp. \( \alpha^+ \)) denotes the set of positive roots \( \beta \) such that \( \beta \to \alpha \) (resp. \( \alpha \to \beta \)) in \( \Gamma_Q \).
Furthermore, AR-quiver $\Gamma_Q$ can be characterized in $\mathbb{Z}Q$ as follows:

\[(2.9) \quad \phi_Q^{-1}(\Phi^+, 0) = \{(i, p) \in \mathbb{Z}Q \mid \xi_i^Q - 2m_i^Q \leq p \leq \xi_i^Q\}.\]

The AR-quiver $\Gamma_Q$ also visualizes the convex partial order $\prec_{[\bar{w}_0]}$ on $\Phi^+_w = \Phi^+$ for $\bar{w}_0$ adapted to $Q$, and provides a way of obtaining all reduced expressions in the $[\bar{w}_0]$:

**Theorem 2.2.** [4, 38]

(a) $\alpha \prec_{[\bar{w}_0]} \beta$ if and only if there exists a path from $\beta$ to $\alpha$ in $\Gamma_Q$.

(b) Any reduced expression $\bar{w}_0$ of $w_0$ adapted to $Q$ can be obtained by reading $\Gamma_Q$ in a manner compatible with arrows: If there exists an arrow $\beta \to \alpha$, we read $\alpha$ before $\beta$. Replacing a vertex $\beta$ with $i$ for $\phi_Q^{-1}(\beta, 0) = (i, p)$, we have a sequence $(i_1, i_2, \ldots, i_N)$ which gives a reduced expression $\bar{w}_0$ of $w_0$ adapted to $Q$,

$$\bar{w}_0 = s_{i_1}s_{i_2}\cdots s_{i_N}.$$  

Moreover, if $\beta$ is read at $k$-th step in the reading, $\beta = \beta_{k\bar{w}_0}$.

**Remark 2.3.** In [36], the author and Suh introduced new combinatorial model, combinatorial Auslander-Reiten quiver $\Upsilon_{[\bar{w}]}$ for any reduced expression $\bar{w}$ of $w \in W_0$ of any finite type, which can be understood as a generalization of Auslander-Reiten quiver and visualize the convex partial order $\prec_{[\bar{w}]}$ in terms of paths in the quiver $\Upsilon_{[\bar{w}]}$:

$$\alpha \prec_{[\bar{w}]} \beta \in \Phi^+_w$$ if and only if there exists a path from $\beta$ to $\alpha$ in $\Upsilon_{[\bar{w}]}$.

Furthermore, they introduced the notion of Auslander-Reiten tableau of shape $\Upsilon_{[\bar{w}]}$ which is related to reduced expressions in $[\bar{w}]$.

**Convention 2.** For any $\bar{w}_0$ adapted to $Q$, we write $[Q]$ instead of $[\bar{w}_0]$, to emphasize that the class is determined by the AR-quiver $\Gamma_Q$ and hence $Q$. For brevity, we write $\prec_Q$, $\prec_Q^+$, soc$_Q$, dist$_Q$, len$_Q$ and rds$_Q$ instead of $\prec_{[Q]}$, $\prec_{[Q]}^+$, soc$_{[Q]}$, dist$_{[Q]}$, len$_{[Q]}$ and rds$_{[Q]}$, respectively.

**2.3. Reflection functor.** For a sink $i$ (resp. a source $j$) of $Q$, we denote by $r^+_i$ (resp. $r^-_j$) the reflection functor from the category $\text{Mod}(\mathbb{C}Q)$ to the category $\text{Mod}(\mathbb{C}(s_jQ))$ (resp. $\text{Mod}(\mathbb{C}(s_jQ))$). In this subsection, we briefly review the reflection functors $r^+_i$ in the view points of changing adapted reduced expressions of $w_0$ and changing systems of $\Phi^+$ related to orders.

For a sink $i$ of $Q$, by (1.3), Remark (2.1) and Theorem 2.2, one can check that

(i) there exists a reduced expression $\bar{w}_0 = s_{i_1}\cdots s_{i_N}$ of $w_0$ adapted to $Q$ such that $i_N = i^*$ and $[\bar{w}_0] = [Q]$,

(ii) there exists a convex total order $<_{\bar{w}_0}$ determined by $\{\beta_{k\bar{w}_0}^Q \mid 1 \leq k \leq N := \ell(w_0)\}$ such that $\beta_{k\bar{w}_0}^Q = \theta_i^{\bar{w}_0} = \alpha_i$.

**Remark 2.4.** The reflection functor $r^+_i$ can be interpreted as the way of obtaining $\Gamma_{s_iQ}$ from $\Gamma_Q$ as follows: Let $h^\vee$ be the dual Coxeter number.

1. Remove the vertex $(i^*, p)$ such that $\phi_Q(i^*, p) = \alpha_i$ and the arrows exiting from $(i^*, p)$ in $\Gamma_Q$.

2. Add the vertex $(i, p + h^\vee)$ and the arrows entering into $(i, p + h^\vee)$ in $\mathbb{Z}Q$.

3. Label the vertex $(i, p + h^\vee)$ with $\alpha_i$ and change the labels $\beta$ to $s_i(\beta)$ for all $\beta \in \Gamma_Q \setminus \{\alpha_i\}$. 


Thus taking the reflection functor $r_i^+$ provides

(i') the reduced expression $r_i^+ \cdot \tilde{w}_0 := s_is_1 \cdots s_{iN-1}$ of $w_0$ which is adapted to $s_iQ$ such that $i$ is a source of $s_iQ$ and $[r_i^+ \cdot \tilde{w}_0] = [s_iQ],$

(ii') the convex total order $<_{r_i^+ \cdot \tilde{w}_0}$ determined by $\{\beta_k^{r_i^+ \cdot \tilde{w}_0} \mid 1 \leq k \leq N\}$ such that

$$\beta_1^{r_i^+ \cdot \tilde{w}_0} = \alpha_i \quad \text{and} \quad \beta_{k+1}^{r_i^+ \cdot \tilde{w}_0} = s_i(\beta_k^{r_i^+ \cdot \tilde{w}_0}) \in \Phi^+ \text{ for all } 1 \leq k \leq N - 1.$$

For a sink $i$ of $Q$, $\alpha_i$ is a maximal element with respect to $<_Q$. Thus we can observe the followings:

\begin{align}
\text{(i)} \quad & \alpha <_Q \beta \text{ if and only if } s_i\alpha <_Q s_i\beta \text{ for } \beta \neq \alpha_i.
\text{(ii)} \quad & \text{dist}_Q(\alpha, \beta) = \text{dist}_{s_iQ}(s_i\alpha, s_i\beta) \text{ for } \beta \neq \alpha_i.
\text{(iii)} \quad & \text{For } \gamma \in \Phi^+ \setminus \Pi, \text{ rds}_Q(\gamma) = \text{rds}_{s_iQ}(s_i\gamma) \text{ unless } (\alpha, \alpha_i) \text{ is the unique biggest pair with respect to } <_Q^{s_i} \text{ which determines } \text{rds}_Q(\gamma) \text{ with some } \alpha \in \Phi^+ \setminus \{\alpha_i\}.
\text{(iv)} \quad & \text{Under the assumption that soc}_Q \text{ (resp. soc}_{s_iQ} \text{) is well-defined,}
\text{soc}_Q(\alpha, \beta) = \underline{s} \text{ if and only if } \text{soc}_{s_iQ}(s_i\alpha, s_i\beta) = s_i(\underline{s}) \text{ for } \beta \neq \alpha_i.
\text{Here } s_i(\underline{s}) := (s_i' \mid 1 \leq i \leq N)_{r_i^+ \cdot \tilde{w}_0} \text{ where } \underline{s} = s_i\tilde{w}_0 \text{ such that } \tilde{w}_0 = s_{i_1} \cdots s_{i_{N-1}}s_{i_r} \in [Q], \ s_i' = s_{i+1} \text{ (1} \leq i \leq N - 1 \text{) and } s_1' = s_N = 0.
\end{align}

\textbf{Convention 3.} Note that there are only finitely many Dynkin quivers for finite type $E_n$. In later sections, we will prove our assertions for finite type $E_n$ by observing that each argument holds for one special quiver $Q$ in Appendix. Hence we can check for all other quivers by applying the strategy given in (2.10), since every Dynkin quiver $Q'$ is of a form $Q' = s_{i_1} \cdots s_{i_r}Q$ for some $r \in \mathbb{Z}_{\geq 0}$ and $i_t \in I$ ($1 \leq t \leq r$).

\section{Combinatorial description of $\Gamma_Q$ of finite type $A_n$ and $D_n$}

In this section, we briefly review combinatorial descriptions of $\Gamma_Q$ of finite type $A_n$ and $D_n$ which are studied in [33, 34]. The combinatorial descriptions play an important role in later section for proving an existence of the socle for $\langle(Q)\rangle$-family and well-definedness of distance polynomial, etc.

\textbf{Definition 3.1.} Let $Q$ be the root lattice associated to the Dynkin diagrams of finite type ADE. The \textit{multiplicity} of a non-simple element $\gamma = \sum_{i \in I} n_i\alpha_i \in \Phi^+ \setminus \Pi$ is the integer, denoted by $\text{mul}(\gamma)$, defined as follows:

$$\text{mul}(\gamma) := \max(n_i \mid i \in I).$$

In particular, if $\text{mul}(\gamma) = 1$, then we say that $\gamma$ is \textit{multiplicity free}.

\subsection{Type $A_n$}

In this subsection, we assume that $Q$ is of finite type $A_n$. We say that a vertex $i \in Q$ is a right (resp. left) intermediate if

\begin{align*}
\text{(resp. } \includegraphics[width=0.4\textwidth]{right_intermediate.png} \text{)} \text{ in } Q.
\end{align*}

For $\beta = [a, b] \in \Phi^+$, we say $a$ the \textit{first component} of $\beta$ and $b$ the \textit{second component} of $\beta$. 
Example 3.2. Consider the quiver $Q = 1 \to 2 \to 3 \to 4 \to 5$ of finite type $A_5$. We set $\xi^Q_1 = 0$. Then $\Gamma_Q$ inside $\mathbb{Z}Q$ can be drawn as follows:

$$
\begin{array}{ccccccc}
(i, p) & -6 & -5 & -4 & -3 & -2 & -1 & 0 \\
1 & [5] & [4] & [2, 3] & [1, 3] & [1] \\
2 & [4, 5] & [2, 4] & [1, 4] & [3, 4] & [3] \\
3 & [2, 5] & [1, 5] & [3, 5] & [1, 2] \\
4 & [1, 2] & [3, 4] \\
5
\end{array}
$$

There are two canonical readings of $\Gamma_Q$ in Theorem 2.2 as follows:

$[1] < [3] < [1, 3] < [2, 3] < [3, 4] < [1, 4] < [2, 4] < [4] < [3, 5] < [1, 5] < [2, 5] < [4, 5] < [5] < [1, 2] < [2],$

$[3] < [3, 4] < [3, 5] < [1] < [1, 3] < [1, 4] < [1, 5] < [1, 2] < [2, 3] < [2, 4] < [2, 5] < [2] < [4] < [4, 5] < [5].$

Definition 3.3. (a) A connected subquiver $\rho$ in $\Gamma_Q$ is an $S$-sectional (resp. $N$-sectional) path if it a concatenation of arrows whose forms are $(i, p) \to (i + 1, p + 1)$ (resp. $(i, p) \to (i - 1, p + 1)$).

(b) A positive root $\beta \in \Phi^+$ is contained in the subquiver $\rho$ in $\Gamma_Q$ if $\beta$ is an end or a start of some arrow in the subquiver $\rho$.

(c) An $S$-sectional (resp. $N$-sectional) path $\rho$ is maximal if there is no longer $S$-sectional (resp. $N$-sectional) path containing all positive roots in $\rho$.

Theorem 3.4. [33, Theorem 1.11] Every positive root in an $N$-sectional path has the same first component and every positive root in an $S$-sectional path has the same second component. Thus for $1 \leq i \leq n$, $\Gamma_Q$ contains a maximal $N$-sectional path of length $n - i$ once and exactly one. At the same time, $\Gamma_Q$ contains a maximal $S$-sectional path of length $i - 1$ once and exactly one.

Hence we can say that a maximal $N$-sectional path $\rho$ is the $(N, i)$-path if all positive roots contained in $\rho$ have $i$ as a first component. Similarly, the notion $(S, i)$-path is well-defined.

Let $\kappa$ and $\sigma$ be subsets of $\Phi^+$ defined as follows:

$$
\kappa := \{ \beta \in \Phi^+ \mid \varphi^{-1}_{Q,1}(\beta) = 1 \}, \quad \sigma := \{ \beta \in \Phi^+ \mid \varphi^{-1}_{Q,1}(\beta) = n \}.
$$

We enumerate the positive roots in $\kappa = \{\kappa_1, \ldots, \kappa_r\}$ and $\sigma = \{\sigma_1, \ldots, \sigma_s\}$ in the following way:

$$
\varphi^{-1}_{Q,2}(\kappa_{i+1}) + 2 = \varphi^{-1}_{Q,2}(\kappa_i), \quad \varphi^{-1}_{Q,2}(\sigma_{j+1}) - 2 = \varphi^{-1}_{Q,2}(\sigma_j) \text{ for } 1 \leq i < r \text{ and } 1 \leq j < s.
$$

Lemma 3.5. [33, Corollary 1.15]

(a) If $\kappa_i = [a, b]$, then $\kappa_{i+1} = [b + 1, c]$ for some $a \leq b < c$.

(b) If $\sigma_j = [a, b]$, then $\sigma_{j+1} = [b + 1, c]$ for some $a \leq b < c$.

3.2. Type $D_n$. In this subsection, we assume that $Q$ is of finite type $D_n$. The involution $^*$ induced by $w_0 \in W_0$ is given by $^* = i$ for $1 \leq i \leq n - 2$ and $(n - 1)^* = n - 1$, $n^* = n$ if $n$ is even, $(n - 1)^* = n$, $n^* = n - 1$ if $n$ is odd. The $m_i^Q$ in (2.5) is given by $m_i = n - 2$ for
1 ≤ i ≤ n − 2 and ([34, Lemma 1.12])

\[
\begin{cases}
    m_{n-1}^Q = n - 3, & m_n^Q = n - 1 \\
    m_{n-1}^Q = n - 1, & m_n^Q = n - 3 \\
    m_{n-1}^Q = m_n^Q = n - 2
\end{cases}
\]

if \( n \equiv 1 \pmod{2} \) and \( \xi_n^Q = \xi_{n-1}^Q + 2 \),

otherwise.

**Example 3.6.** Let us consider the quiver \( Q = 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \) of finite type \( D_4 \). Then \( \Gamma_Q \) can be drawn as follows:

There are four canonical readings of \( \Gamma_Q \) as follows:

\[
\begin{align*}
\{3 | - 4\} & \prec \{2 | - 4\} \prec \{1 | - 4\} \prec \{2 | 3\} \prec \{2 | - 3\} \prec \{1 | 2\} \prec \{2 | 4\} \prec \{1 | 3\} \prec \{1 | - 3\} \prec \{1 | 4\} \prec \{1 | - 2\} \prec \{3 | 4\}, \\
\{3 | - 4\} & \prec \{2 | - 4\} \prec \{1 | - 4\} \prec \{2 | - 3\} \prec \{2 | 3\} \prec \{1 | 2\} \prec \{2 | 4\} \prec \{1 | 3\} \prec \{1 | - 3\} \prec \{1 | 4\} \prec \{1 | - 2\} \prec \{3 | 4\}, \\
\{3 | - 4\} & \prec \{2 | - 4\} \prec \{2 | - 3\} \prec \{2 | 3\} \prec \{1 | - 4\} \prec \{1 | 2\} \prec \{1 | 3\} \prec \{2 | 4\} \prec \{1 | 4\} \prec \{3 | 4\} \prec \{1 | - 2\}, \\
\{3 | - 4\} & \prec \{2 | - 4\} \prec \{2 | - 3\} \prec \{1 | - 4\} \prec \{1 | 2\} \prec \{1 | 3\} \prec \{1 | - 3\} \prec \{2 | 4\} \prec \{1 | 4\} \prec \{3 | 4\} \prec \{1 | - 2\}.
\end{align*}
\]

**Lemma 3.7.** ([34, Lemma 1.12] For a Dynkin quiver \( Q \), there exists \( t \in \{n - 1, n\} \) such that every positive roots \( \beta \) with \( \phi_Q^{-1}(\beta) \in \{n - 1, n\} \) has \( \varepsilon_t \) or \(- \varepsilon_t\) as its summand. Conversely, every positive roots \( \beta \) having \( \varepsilon_t \) or \(- \varepsilon_t\) as its summand has its residue with respect to \( Q \) as \( n - 1 \) or \( n \). Furthermore, \( t \) and \( t' := \{n - 1, n\} \setminus \{t\} \) are given as follows:

\[
\begin{cases}
    t := \begin{cases}
        n - 1 & \text{if } \xi_{n-1} - \xi_n = \pm 2, \\
        n & \text{if } \xi_{n-1} - \xi_n = 0.
    \end{cases}
\end{cases}
\]

**Definition 3.8.**

(a) A connected subquiver \( \rho \) of \( \Gamma_Q \) is an \( S \)-sectional path if \( \rho \) is a concatenation of arrows whose forms are \((i, p) \rightarrow (i + 1, p + 1)\) for \( 1 \leq i \leq n - 2 \), or \((n - 2, p) \rightarrow (n, p + 1)\).

(b) A connected subquiver \( \rho \) of \( \Gamma_Q \) is an \( N \)-sectional path if \( \rho \) is a concatenation of arrows whose forms are \((i, p) \rightarrow (i - 1, p + 1)\) for \( 2 \leq i \leq n - 1 \), or \((n, p) \rightarrow (n - 2, p + 1)\).

(c) A maximal \( S \)-sectional (resp. \( N \)-sectional) path is shallow if it ends (resp. starts) at level less than \( n - 1 \).

(d) A connected subquiver \( \rho \) in \( \Gamma_Q \) is called a swing if it consists of vertices and arrows in the following way: There exist roots \( \alpha, \beta \in \Phi^+ \) and \( r, s \leq n - 2 \) such that

\[
S_r \rightarrow S_{r+1} \rightarrow \cdots \rightarrow S_{n-2} \rightarrow N_{n-2} \rightarrow N_{n-3} \rightarrow \cdots \rightarrow N_s \text{ where }
\]

\[
\begin{align*}
\text{• } S_r \rightarrow S_{r+1} \rightarrow \cdots \rightarrow S_{n-2} \rightarrow N_{n-2} \rightarrow N_{n-3} \rightarrow \cdots \rightarrow N_s \text{ where }
\end{align*}
\]
Then we have the followings:

- \( \delta \) as their summand and 
- \( \beta \) is located at \((n - 1, u)\) and \( \alpha \) is located at \((n, u)\) for some \( u \in \mathbb{Z} \).

The following lemma tells the information on the positions of positive roots \( \beta \) with \( \text{mul}(\beta) = 2 \) inside of \( \Gamma_Q \):

**Lemma 3.9.** [34, Corollary 1.15] Set

\[
i = \max(\phi_{Q,2}^{-1}(\beta) \mid \phi_{Q,1}^{-1}(\beta) \in \{n - 1, n\}, \ ht(\beta) \geq 2),
\]

\[
j = \min(\phi_{Q,1}^{-1}(\beta) \mid \phi_{Q,1}^{-1}(\beta) \in \{n - 1, n\}, \ ht(\beta) \geq 2).
\]

Then we have the followings:

- (a) \( i - j = 2(n - 3) \).
- (b) Every multiplicity non-free positive root \( \beta \) satisfies the following conditions:

\[
1 < \ell := \phi_{Q,1}^{-1} < n - 1 \text{ and } j - (n - 1 - \ell) \leq \phi_{Q,2}^{-1}(\beta) \leq i - (n - 1 - \ell).
\]

**Theorem 3.10.** [34, Theorem 1.19] For every maximal swing \( \varrho \), there exists \( 1 \leq k \leq n - 2 \) such that all roots in \( \varrho \) contain \( \varepsilon_k \) as their summand. Moreover, \( \varrho \) contains a simple root \( \alpha_k \) and is one of the following two forms:

\[
\theta_k^Q = S_k \longrightarrow S_{k+1} \longrightarrow \cdots \longrightarrow N_{n-2} \begin{array}{c}
\varepsilon_k \pm \varepsilon_t \\
\varepsilon_k \mp \varepsilon_t
\end{array} \longrightarrow N_{n-1},
\]

\[
S_1 \longrightarrow S_2 \longrightarrow \cdots \longrightarrow S_{n-2} \begin{array}{c}
\varepsilon_k \pm \varepsilon_t \\
\varepsilon_k \mp \varepsilon_t
\end{array} \longrightarrow N_{n-2} \longrightarrow N_{n-3} \longrightarrow \cdots \longrightarrow N_k = \gamma_k^Q.
\]

From the above theorem, for \( 1 \leq a \leq n - 2 \), the maximal wing \( \varrho \), containing all positive roots with \( \varepsilon_a \) as its summand, is called by the \( a \)-swing.

**Theorem 3.11.** [34, Theorem 1.22] Let \( \rho \) be a shallow maximal \( S \)-sectional (resp. \( N \)-sectional) path. Then there exists \( k \leq n - 2 + \delta \) such that all positive roots in \( \rho \) contain \( -\varepsilon_k \) as their summand and \( \rho \) starts (resp. ends) at level 1. Here \( \delta = 1 \) if \( \{n - 1, n\} \) are sink or source, \( \delta = 0 \) otherwise.

Similarly, we can define the notion of shallow \((N, -a)\)(resp.\((S, -a)\))-sectional path.

For the rest of this subsection, we record lemmas and notations in [34] which are essential for our assertions in later section.

Let \( \sigma \) be a subset of \( \Phi^+ \) defined as follows:

\[
\sigma := \{ \beta \in \Phi^+ \mid \phi_{Q,1}^{-1}(\beta) = n - 1, \ \beta \notin \{\alpha_{n-1}, \alpha_n\} \}.
\]

Note that \( |\sigma| = n - 2 \). For \( 1 \leq k \leq n - 3 \), We set

- the positive roots in \( \sigma = \{\sigma_1, \ldots, \sigma_{n-2}\} \) as \( \phi_{Q,2}^{-1}(\sigma_{k+1}) + 2 = \phi_{Q,2}^{-1}(\sigma_k) \),
- indices \( \{i_1, i_2, \ldots, i_{n-2}\} = \{1, 2, \ldots, n - 2\} \) such that \( \sigma_k \) is contained in \( i_{\sigma_k}\)-swing.
Lemma 3.12. [34, Corollary 1.25]

(a) A positive root \( \{a|b\} \) with \( \text{mul}(\{a|b\}) = 2 \) is contained in the longer part of the \( b \)-swing.

(b) There exists \( 1 \leq \ell \leq n - 2 \) such that \( i_{\sigma_\ell} = 1 \) and

\[
i_{\sigma_1} > i_{\sigma_2} > \cdots > i_{\sigma_\ell} = 1 < i_{\sigma_{\ell+1}} < \cdots < i_{\sigma_{n-2}},
\]

where the shorter part of \( i_{\sigma_a} \) (\( a < \ell \)) is the \( N \)-part and the shorter part of \( i_{\sigma_b} \) (\( b > \ell \)) is the \( S \)-part.

Let \( \kappa \) be a subset of \( \Phi^+ \) defined as follows:

\[
\kappa := \{ \beta \in \Phi^+ \mid \phi_Q^{-1}(\beta) = 1 \}.
\]

We enumerate the positive roots in \( \kappa = \{ \kappa_1, \ldots, \kappa_{n-1} \} \) in the following way:

\[
\phi_Q^{-1}(\kappa_{i+1}) + 2 = \phi_Q^{-1}(\kappa_i), \quad \text{for } 1 \leq i \leq n - 2.
\]

Note that \( |\kappa| = n - 1 \) and each element in \( \kappa \) is contained in only one shallow maximal path, maximal path sharing \( \varepsilon_{\nu'} \) or maximal path sharing \( -\varepsilon_{\nu'} \). We set, for \( 1 \leq k \leq n - 1 \), indices

\[
\{j_{\kappa_1}, j_{\kappa_2}, \ldots, j_{\kappa_{n-1}}\} = \{-2, \ldots, -n + 2, t', -t'\}
\]

such that \( \kappa_s \) contains \( -\varepsilon_{-j_s} \) or \( \varepsilon' \) as its summand.

The following lemma tells the positions of multiplicity free roots inside of \( \Gamma_Q \).

Lemma 3.13. [34, Corollary 1.26] There exists \( 1 \) such that

\[
|j_{\kappa_{n-1}}| < \cdots < |j_{\kappa_1}| = t' = |j_{\kappa_{n-1}}| > |j_{\kappa_{n-2}}| > \cdots > |j_{\kappa_1}|,
\]

and the maximal sectional path sharing \( -\varepsilon_{j_s} \) is the \( S \)-sectional, if \( s \leq 1 - 1 \), and the maximal sectional path sharing \( -\varepsilon_{j_s} \) is the \( N \)-sectional, otherwise.

4. The Sequences of Positive Roots with Respect to New Notions

In this section, we prove the existence of \([\tilde{w}]\)-socle for \( (Q)\)-family and give a definition of \( (Q)\)-family. Then we investigate the structures of sequences of positive roots by using the new notions in Section 1.

4.1. Directly Q-connected pair. In this subsection, we shall investigate a family of pairs which are directly \( Q \)-connected.

Definition 4.1. (i) We say that a pair \((\alpha, \beta)\) is directly \( Q \)-connected if

\[
d(i, j) = |p - q| \text{ for } \phi_Q^{-1}(\alpha, 0) = (i, p) \text{ and } \phi_Q^{-1}(\beta, 0) = (j, q).
\]

(ii) We say that a full subquiver \( \Omega \) of \( \Gamma_Q \) is sectional if every pair \((\alpha, \beta)\) in \( \Omega \) is directly \( Q \)-connected.

Proposition 4.2. Let us assume that a pair \((\alpha, \beta)\) is directly \( Q \)-connected.

(a) There exists a reduced expression \( \bar{w}_0 \in [Q] \) such that \((\alpha, \beta)\) is \( \bar{w}_0 \)-simple and hence \([Q]\)-simple.
(b) \( \alpha \cdot \beta = 1 \).
(c) \( \alpha - \beta \) or \( \beta - \alpha \in \Phi^+ \).
(d) either \((\alpha, \beta - \alpha)\) or \((\beta, \alpha - \beta)\) is a \([Q]\)-minimal pair of \( \alpha \) or \( \beta \).
Proof. Since \((\alpha, \beta)\) is directly \(Q\)-connected, there exists a unique sectional subquiver \(\Omega\) which starts at \(\beta\) and ends at \(\alpha\). By Theorem 2.2 (b), there exists a reading of \(\Gamma_Q\) which produces a reduced expression \(\bar{w}_0 \in [Q]\), and reads all vertices in \(\Omega\) successively; i.e., \(\beta_{\bar{w}_0} = \alpha\) and \(\beta_{\bar{w}_0}^{[k]} = \beta\) where \(\phi_Q^1(\alpha, 0) = (i, p)\) and \(\phi_Q^1(\beta, 0) = (j, q)\) (see Example 3.2, Example 3.6).

By Theorem 3.4, our assertion for finite type \(A_n\) follows and, by Theorem 3.10 and Theorem 3.11, our assertion for finite type \(D_n\) follows.

For finite type \(E_n\), we observe a special quiver \(Q\) and its AR-quiver \(\Gamma_Q\), and use reflection functors. Assume that our assertions hold for the \(Q\) and \(I\) is a sink of \(Q\). It suffices to prove that the assertions hold for the quiver \(s_iQ\), since all Dynkin quivers can be obtained by applying reflection functors on the \(Q\).

Note that \(\alpha_i\) is a minimal element with respect to \(\prec_{s_iQ}\). If \(\alpha \neq \alpha_i\), then there exist \(\alpha'\) and \(\beta'\) such that \(s_i(\alpha') = \alpha\) and \(s_i(\beta') = \beta\), respectively. Thus our assertions hold by the observations in § 2.3.

Now assume that \(\alpha = \alpha_i\). Then if one of our assertions does not hold, then the assertion does not holds for \((\tau_{s_iQ}(\alpha), \tau_{s_iQ}(\beta))\) neither. However, it contradicts the observations in § 2.3 since \(\tau_{s_iQ}(\alpha) = s_i\alpha''\) and \(\tau_{s_iQ}(\beta) = s_i\beta''\) for some \(\alpha'', \beta'' \in \Phi^+ \setminus \{\alpha_i\}\). Thus our assertions follow by observing the Dynkin quivers \(Q\) and their AR-quivers \(\Gamma_Q\) in Appendix. \(\square\)

4.2. Socle of pairs. This subsection is devoted to prove the following theorem:

**Theorem 4.3.** For a reduced expression \(\bar{w}\) adapted to some Dynkin quiver \(Q\) and a pair \(\underline{p}\), \(\text{soc}_{[\bar{w}]}(\underline{p})\) exists uniquely.

**Proposition 4.4.** For any pair \(\underline{p} = (\alpha, \beta)\) with \(\gamma = \alpha + \beta \in \Phi^+\), \(\text{soc}_Q(\alpha, \beta)\) is well-defined by \(\gamma\). In particular, \(\text{dist}_Q(\underline{p}) = 1\) implies that \(\underline{p}\) is a \([Q]\)-minimal pair of \(\gamma\).

**Proof.** If \((\alpha, \beta)\) is a pair of simple roots, our assertion is trivial. Assume that \((\alpha, \beta)\) is not a pair of simple roots and there exists a \([Q]\)-simple \(s\) such that \(\text{wt}(s) = \alpha + \beta\) and \(s \prec_Q (\alpha, \beta)\). If \(|s| = 2\), then it must be a pair. Then by convexity, \(s\) is not \([Q]\)-simple. Now we can assume that \(|s| > 2\). Then there exist indices \(i \neq j\) such that \(s_i \neq 0, s_j \neq 0\) and \(\beta_i + \beta_j \in \Phi^+\). Thus our first assertion follows from the definition of \([Q]\)-simple sequence and the convexity of \(\prec_Q\). The second assertion follows from Proposition 1.23 and the first assertion. \(\square\)

**Convention 4.** For the rest of this paper, we usually skip the proofs of our assertions for Dynkin quiver \(Q\) of finite types \(E_n\). As we mentioned in Convention 3, there are finitely many Dynkin quivers, and if we give proofs for finite types \(E_n\), the paper would be tedious and long than necessary. Hence, we sometimes give examples for finite types \(E_n\) instead of giving proofs, by using the Dynkin quivers \(Q\) in Appendix.

**Proposition 4.5.** For any pair \(\underline{p}\), \(\text{soc}_Q(\alpha, \beta)\) is well-defined.

**Proof.** By Lemma 1.22 and Proposition 4.2, it suffices to consider a pair \((\alpha, \beta)\) such that

\[(1.1) \quad \text{it is comparable with respect to } \prec_Q \text{ and not directly } Q\text{-connected.}\]

If \(\alpha + \beta \notin \Phi^+\), then our assertion follows from the preceding proposition.

Note that for non-simple positive root \(\beta = \sum_{k=1}^{s \geq 2} \beta_k\), there exist \(1 \leq t \neq l \leq s\) such that \(\beta_l \prec_Q \beta \prec_Q \beta_t\) by the convexity of \(\prec_Q\).

Assume that \(\text{supp}(\alpha) \cap \text{supp}(\beta) = \emptyset\) and \(\alpha + \beta \notin \Phi^+\). If \(\alpha\) and \(\beta\) are simple roots, then there is no sequence \(s \neq (\alpha, \beta)\) with \(\text{wt}(s) = \alpha + \beta\). If \((\alpha, \beta)\) is not a pair of simple roots, then
there exists an index $i$ such that

\begin{equation}
(4.2) \quad s_i \neq 0, \text{ and } \beta_i \prec_Q \alpha \prec_Q \beta \text{ or } \alpha \prec_Q \beta \prec_Q \beta_i,
\end{equation}

by the convexity of $\prec_Q$. Equivalently, $s_i$ cannot be less than $(\alpha, \beta)$ with respect to $\prec_Q^b$. Thus $\text{soc}_Q(\alpha, \beta) = (\alpha, \beta)$. Now it suffices to consider a pair such that $\text{supp}(\alpha) \cap \text{supp}(\beta) \neq \emptyset$ and $\alpha + \beta \notin \Phi^+$. 

(Q of finite type $A_n$) Assume that there exist paths from $\beta$ to $\alpha$ as follows:

Here $\phi_Q^{-1}(\alpha, 0) = (j, t)$ and $\phi_Q^{-1}(\beta, 0) = (i, s)$. Write $\alpha = [a, b]$ and $\beta = [c, d]$. 

(a) In this case, we have $\sigma = [a, d]$ by Theorem 3.4 where $\phi_Q^{-1}(\sigma, 0) = (k, u)$.

(a-1) If $(t - s) - (i + j) + 2 > 2$, then Theorem 3.4 tells that $\eta = [x, b]$ and $\zeta = [c, y]$ where $\phi_Q^{-1}(\eta, 0) = (1, j - (t - 1))$ and $\phi_Q^{-1}(\zeta, 0) = (1, s + (i - 1))$. Moreover, Lemma 3.5 tells that $c - b > 1$. Thus $\text{supp}(\alpha) \cap \text{supp}(\beta) = \emptyset$ and hence $\alpha + \beta \notin \Phi^+$. Thus $\text{soc}_Q(\alpha, \beta) = (\alpha, \beta)$. 

(a-2) If $(t - s) - (i + j) + 2 = 2$, then we have $c - b = 1$ by Lemma 3.5 and hence $\alpha + \beta \in \Phi^+$. Thus $\text{soc}_Q(\alpha, \beta) = (\alpha + \beta)$. The case (b) can be proved by applying similar strategy of (i) by using Theorem 3.4 and Lemma 3.5; i.e.,

\begin{equation}
(4.3) \quad \text{soc}_Q(\alpha, \beta) = \begin{cases} 
(\alpha + \beta) & \text{if } t - s - 2n + i + j = 2, \\
(\alpha, \beta) & \text{otherwise}.
\end{cases}
\end{equation}

(c) In this case, Theorem 3.4 tells that we have

\begin{equation}
(4.4)
\end{equation}

in $\Gamma_Q$. More precisely, there exists paths from $[c, d]$ to $[c, b]$, from $[c, b]$ to $[a, b]$, from $[c, d]$ to $[a, d]$, and from $[a, d]$ to $[a, b]$ in $\Gamma_Q$.

**Strategy** Note that the pair $([a, d], [c, b])$ is $[Q]$-simple since there is no path between $[a, d]$ and $[c, b]$ in $\Gamma_Q$. Assume that there exist a sequence $s \neq ([a, d], [c, b])$ with $\text{wt}(s) = \alpha + \beta$. Then $s$ must have an index $i$ such that $s_i \neq 0$ and

$$
\beta_i = \begin{cases} 
[a, x] & \text{if } \min(a, c) = a, \\
[c, x] & \text{if } \min(a, c) = c,
\end{cases}
$$

for some $x \in I$.

Assume that $\min(a, c) = a$, i.e., $\beta_i = [a, x]$. If $x = b$ or $d$, there exist indices $j, l$ such that $s_j, s_l > 0$ and $\beta_j + \beta_l \in \Phi^+$, since $\text{wt}(s) - \beta_i \in \Phi^+$. If $x \neq b$ and $d$, there exists an index $j$
such that \( s_j > 0 \) and \( \beta_i + \beta_j \in \Phi^+ \), since \( -\varepsilon_{x+1} \) is not a summand of \( \alpha + \beta \). In both cases, \( s \) cannot be \([Q]\)-simple.

One can check that we have the same result when \( \min(a,c) = c \). Thus \( \text{soc}_Q(\alpha, \beta) = ([a,d], [c,b]) \).

(Q of finite type \( D_n \)) Let \( \phi_Q^{-1}(\beta, 0) = (i, s) \) and \( \phi_Q^{-1}(\alpha, 0) = (j, t) \). We assume that \( j \leq i \).

Recall that we have assumed that \( \alpha \prec_Q \beta \) and the pair is not directly \( Q \)-connected,

(The case when \( 1 \leq j \leq i < n - 1 \)) A path between them can be drawn as one of the following forms:

\[
\begin{align*}
(4.5) & \quad (i), (ii), (iii), (iv), (v), (vi).
\end{align*}
\]

We write \( \alpha = \varepsilon_a \pm \varepsilon_b, \beta = \varepsilon_c \pm \varepsilon_d \) and assume that \( \beta \) is in the \( c \)-swing. Recall the index 1 in Lemma 3.13 and the set \( \kappa = \{ \kappa_1, \ldots, \kappa_{n-1} \} \) in (3.8).

Now we shall prove our assertion for \( Q \) of finite type \( D_n \) by dividing cases with respect to the shape of the paths in (4.5). For the cases (i) and (ii), we can assume that,

\[
\alpha \text{ is located at the } N\text{-part of 1-swing; i.e., } \alpha = \{1, b\},
\]

by Theorem 3.10.

(i) By Lemma 3.9, the pair \( (\alpha, \beta) \) can not a pair with \( \text{mul}(\alpha) = \text{mul}(\beta) = 1 \). We write \( \kappa_s \) for \( \phi_Q^{-1}(\kappa_s, 0) = (1, j - (t - 1)) \) and \( \kappa_t \) for \( \phi_Q^{-1}(\kappa_t, 0) = (1, i + (s - 1)) \).

(i-1: \( (t - s) - (i + j) + 2 > 2 \)) (i-1-1: \( s > 1 \)) Since \( s > 1 \), Lemma 3.9 and Lemma 3.13 tell that \( \alpha \) becomes a multiplicity non-free positive root while \( \beta \) is a positive root with \( \text{mul}(\beta) = 1 \) and contained in a shallow maximal path by Theorem 3.11; that is, \( \alpha = \{1, b\}, \beta = \{c, d\} \) where \( 1 < b \leq n - 2, d \in I \setminus \{t\} \) with \( d > c \) and \( \beta \) is located at the \( S\)-part of \( c\)-swing. By Lemma 3.12, we can conclude that \( b \geq c > 0 \).

The multiplicity free root \( \kappa_t \) (\( t > s + 1 \)) is of the form \( \{y| - d\} \). The assumption \( (t - s) - (i + j) + 2 > 2 \) tells that \( \kappa_s + \kappa_t \not\in \Phi^+ \) and hence \( d < b \) by [34, Corollary 1.15, Corollary 1.26].

Thus \( \alpha + \beta = \varepsilon_a + \varepsilon_b + \varepsilon_c - \varepsilon_d \), where \( d > b > c \geq 1 \). If \( c > 1 \), [34, Theorem 1.19, Theorem 1.22] tells that there exist paths from \( \{1| - d\} \) to \( \beta \) and from \( \alpha \) to \( \{b| c\} \). Thus we have

\[
(\alpha, \beta) \prec_Q \{\{c|b\}, \{1| - d\}\}.
\]

If \( 1 = c \), then \( \alpha + \beta = 2\varepsilon_1 + \varepsilon_b - \varepsilon_d \).

One can check that there exists no \([Q]\)-simple sequence \( s \neq (\alpha, \beta) \) satisfying the conditions in Definition 1.21, by using the strategy in Proposition 4.5. Thus \( \text{soc}_Q(\alpha, \beta) = (\alpha, \beta) \).

(i-1-2: \( s \leq 1 \)) Since \( s \leq 1 \), \( \alpha \) becomes multiplicity free. Assume first that \( t > 1 \). Then \( \kappa_t \) becomes \( \{c| - d\} \) where \( \beta = \varepsilon_c - \varepsilon_d \) is multiplicity free.

Since \( (t - s) - (i + j) + 2 > 2 \), \( \kappa_s + \kappa_t \not\in \Phi^+ \) and \( -\varepsilon_b + \varepsilon_c \not\in 0 \). If \( \text{supp}(\alpha) \cap \text{supp}(\beta) = \emptyset \), we do not need to consider since \( \alpha + \beta \not\in \Phi^+ \). If \( \text{supp}(\alpha) \cap \text{supp}(\beta) \neq \emptyset \), there exist paths from \( \{1, -d\} \) to \( \beta \) and from \( \alpha \) to \( \{c| - b\} \). Then \( \text{soc}_Q(\alpha, \beta) = (\alpha, \beta) \) as in (i-1-1).
Now we assume that \( t < 1 \). Then \( \beta \) becomes multiplicity non-free; i.e., \( \beta = \{c|d\} \) such that \( d > c \geq 1 \). In the similar way as before, there exist paths from \( \{1|d\} \) to \( \beta \) and from \( \alpha \) to \( \{c|b\} \), if \( c > 1 \). If \( c = 1 \), \( \alpha + \beta = 2\varepsilon_1 + \varepsilon_d - \varepsilon_b \) where \( d > 1 \) and \( b \in I \setminus \{t'\} \). In both cases, one can check that \( \text{soc}_Q(\alpha, \beta) = (\alpha, \beta) \) by applying the same strategy.

(i-2: \( (t-s) - (i+j) + 2 = 2 \)) Since \( \kappa_s + \kappa_t \in \Phi^+ \), we have

\[
\alpha + \beta = \begin{cases} 
2\varepsilon_1 & \text{if } c = 1, \\
\varepsilon_1 + \varepsilon_c & \text{if } c > 1.
\end{cases}
\]

If \( c = 1 \), the pair \( \{(1|t'), (1|-t)\} \) is only the \([Q]\)-simple pair among the pairs \( \mathfrak{s} \) such that \( \text{wt}(\mathfrak{s}) = 2\varepsilon_1 \) by Theorem 3.10. More precisely, if \( \mathfrak{s} \neq \{(1|t'), (1|-t)\} \), then it must be of the form \( \{(1|k'), (1|-k)\} \) for \( k \in I \setminus \{t\} \). But \( \{(1|t'), (1|-t)\} \not\sim_Q \{(1|k'), (1|-k)\} \) by Theorem 3.10. If there exists a sequence \( \mathfrak{s} \) such that \( |\mathfrak{s}| > 2 \) and \( \text{wt}(\mathfrak{s}) = 2\varepsilon_1 \), then one can check that \( \mathfrak{s} \) can not be \([Q]\)-simple. Thus \( \text{soc}_Q(\alpha, \beta) = \{(1|t'), (1|-t)\} \).

If \( c > 1 \), then our assertion follows from Proposition 4.4.; i.e., \( \text{soc}_Q(\alpha, \beta) = \{(1|c)\} \).

The other pair \( (\alpha', \beta') \) of the form (i) can be obtained by applying \( \tau_Q \) or \( \tau_Q^{-1} \) proper times to \( (\alpha, \beta) \) we already dealt with. Thus we proved.

(ii) We write \( \zeta \) for \( \phi_Q^{-1}(\zeta, 0) = (k, u) \). Recall the assumption (4.6).

(ii-1: \( k' < n - 1 \)) In this case, Theorem 3.10 and Theorem 3.11 tell that

\[
\alpha + \beta = \varepsilon_1 + \varepsilon_b + \varepsilon_c + \varepsilon_d = \eta + \zeta,
\]

where \( \eta, \zeta \) is \([Q]\)-simple. Then there are three positive roots having \( \varepsilon_1 \) as its summand and one of \( \varepsilon_b, \varepsilon_c \) and \( \varepsilon_d \) as its another summand. Thus there are at most three pairs whose weights are the same as \( \alpha + \beta \). Assume that \( \eta' \) and \( \zeta' \) are among the \([Q]\)-simple root pairs among the pairs \( \mathfrak{s} \) such that \( \eta' + \zeta' = \alpha + \beta \). Assume that \( \eta' + \zeta' = \alpha + \beta \). Then \( \eta' + \zeta' \) must be located at the intersection of \( S \)-part of \( 1 \)-swing and the maximal \( N \)-sectional path containing \( \zeta \). Thus there exists a path from \( \zeta' \) to \( \beta \). Similarly, there exists a path from \( \alpha \) to \( \eta' \). Thus we have

\[
(\alpha, \beta) \not\prec_Q (\eta', \zeta').
\]

Thus we can prove that \( \text{soc}_Q(\alpha, \beta) = (\eta, \zeta) \) by applying the strategy in Proposition 4.5.

(ii-2: \( k' = n - 1 \)) In this case, we have

\[
\alpha + \beta = 2\varepsilon_1 + \varepsilon_b + \varepsilon_d = \eta + \eta' + \zeta,
\]

where \( b, d \) are distinct, \( |b|, |d| > 1 \), \( \eta \in \Phi^+ \) for \( \phi_Q^{-1}(\eta, 0) = (n - 1, u') \) and \( \eta' \in \Phi^+ \) for \( \phi_Q^{-1}(\eta', 0) = (n, u) \) such that \( \{\eta, \eta'\} = \{(1|t'), (1|-t)\} \). Then one can check that

\[
\text{soc}_Q(\alpha, \beta) = (\eta, \eta', \zeta).
\]

The other pair \( (\alpha', \beta') \) of form (ii) can be obtained by applying \( \tau_Q \) or \( \tau_Q^{-1} \) proper times to \( (\alpha, \beta) \) we already dealt with. Thus we proved.

For the cases (iii), (iv) and (v), we can assume that \( \beta \) is contained in the \( S \)-part of \( c \)-swing by Theorem 3.10 and Lemma 3.12.

(iii) Write the positive root as \( \zeta \) located in the intersection of the swing containing \( \alpha \) and the maximal \( N \)-sectional path containing \( \beta \), and the positive root as \( \eta \) located in the intersection
of c-swing and the maximal S-sectional path containing α. Then Theorem 3.10 and Theorem 3.11 tell that

\[ \alpha + \beta = \eta + \zeta \text{ and } (\eta, \zeta) \prec c (\alpha, \beta). \]

By (i), we have

\[ \text{soc} Q(\alpha, \beta) = \begin{cases} (\eta, \zeta) & \text{if } (t - s) - (i + j) + 2 > 2, \\ (\alpha + \beta) & \text{if } (t - s) - (i + j) + 2 = 2. \end{cases} \]

(iv) We write \( \kappa_s \) for \( \phi_Q^{-1}(\kappa_s, 0) = (1, j - (t - 1)) \), \( \kappa_s' \) for \( \phi_Q^{-1}(\kappa_s', 0) = (1, s + 2n - 3 - i) \) and \( \kappa_t \) for \( \phi_Q^{-1}(\kappa_t, 0) = (1, i + (s - 1)) (s < s' < t) \). By Lemma 3.13, we have

\[ s < s' < 1 \leq t. \]

More precisely, \( \kappa_s = \{x_1|b\} \), \( \kappa_s = \{c|y_2\} \) and \( \kappa_t = \{x_3|d\} \) where \( 0 < -b < c < |d| \) for \( d \in -I \cup \{t'\} \).

(iv-1: \( s' > s + 1 \)) Since \( \kappa_s + \kappa_s' \notin \Phi^+ \), we have \( 0 < -b < c < |d| \). Thus \( \text{supp}(\alpha) \cap \text{supp}(\beta) \neq 0 \) and \( \alpha + \beta \notin \Phi^+ \). Thus \( \text{soc} Q(\alpha, \beta) = (\alpha, \beta) \).

(iv-2: \( s' = s + 1 \)) In this case, we have \( \kappa_s + \kappa_s' \in \Phi^+ \). Thus \( -b = c \). Hence \( \alpha + \beta \) is a multiplicity free positive root and \( \text{soc} Q(\alpha, \beta) = (\alpha + \beta) \). Moreover, one can check that \( \phi_Q^{-1}(\alpha + \beta, 0) = (k, u) \).

(v) We write \( \alpha' \) for \( \phi_Q^{-1}(\alpha', 0) = (i', s') \), \( \beta' \) for \( \phi_Q^{-1}(\beta', 0) = (j', t') \), \( \eta \) for \( \phi_Q^{-1}(\eta, 0) = (k, u) \) and \( \zeta \) for \( \phi_Q^{-1}(\zeta, 0) = (k', u') \). Then by Theorem 3.10 and Theorem 3.11, we have

\[ \alpha + \beta = \alpha + \beta' = \eta + \zeta. \]

By (ii), we have \( \text{soc} Q(\alpha, \beta) = \text{soc} Q(\alpha', \beta') = (\eta, \zeta) \).

The other pair \( (\alpha', \beta') \) of the form (iii), (iv) or (v) can be obtained by applying \( \tau_Q \) or \( \tau_Q^{-1} \) proper times to \( (\alpha, \beta) \) we already dealt with. Thus we proved.

(The case when \( 1 \leq j < n - 1 \) and \( i \in \{n - 1, n\} \) A path between them can be drawn as one of the following forms:

![Diagram](attachment:image.png)

Write \( \phi_Q^{-1}(\beta, 0) = (n - \delta, s) \) for \( \{\delta, \delta'\} = \{0, 1\} \).

(vi) In this case, Lemma 3.7 and Theorem 3.10 tell that \( \alpha + \beta = \eta + \zeta \) where \( \phi_Q^{-1}(\eta) = (k, u) \) and

\[ \phi_Q^{-1}(\zeta) = \begin{cases} (n - \delta, s + 2l) & \text{if } l \text{ is even}, \\ (n - \delta', s + 2l) & \text{if } l \text{ is odd}. \end{cases} \]

Note that \( (\eta, \zeta) \) is \([Q]\)-simple and there is are three pairs \( p \) such that \( \text{wt}(p) = \alpha + \beta \) and two of them are \( (\alpha, \beta) \) and \( (\eta, \zeta) \). Moreover, we can check that \( (\alpha, \beta) \) and \( (\eta', \zeta') \) are
incomparable with respect to \( \eta' , \zeta' \), where \((\eta', \zeta')\) is the another pair. Then one can check that \( \text{soc}_Q(\alpha, \beta) = (\eta, \zeta) \) by applying the similar arguments of previous cases.

(vii) Applying the similar argument of (iv) and using Lemma 3.7, we have

\[
\text{soc}_Q(\alpha, \beta) = \begin{cases} 
(\alpha, \beta) & \text{if } s - t - n - 3 > 2, \\
(\alpha + \beta) & \text{if } s - t - n - 3 = 2.
\end{cases}
\]

Here, if \( s - t - n - 3 = 2 \), then we have \( \phi_Q^{-1}(\alpha + \beta, 0) = (n - \delta, s + 2l) \) if \( l \) is even, \( \phi_Q^{-1}(\alpha + \beta, 0) = (n - \delta', s + 2l) \) otherwise.

(The case when \( i, j \in \{n - 1, n\} \))

(viii) Applying [34, Proposition 1.14] and Theorem 3.10, one can easily check that

\[
\text{soc}_Q(\alpha, \beta) = \begin{cases} 
(\alpha + \beta) & \text{if } |i - j| \equiv n - k - 1 \text{(mod 2)}, \\
(\alpha, \beta) & \text{otherwise}.
\end{cases}
\]

Here if \( |i - j| \equiv n - k - 1 \text{(mod 2)} \), then we have \( \phi_Q^{-1}(\alpha + \beta, 0) = (k, u) \).

Now we record the socle \( \mathfrak{s} \) of non \([Q]\)-simple pairs \((\alpha, \beta)\) with \( \alpha + \beta \notin \Phi^+ \) of finite type \( D_n \), for convenience of reader and later use.

(4.9)

For the case (6), \( \beta \) is the one of \( \circ \)'s which is determined by (4.8).

For the case when \( j > i \), we can prove by the similar arguments. \( \square \)

In the course of proof of the above proposition, one can notice that the following property holds for type \( A_n \) and \( D_n \):

**Corollary 4.6.** For a pair \( p = (\alpha, \beta) \) of finite type \( A_n \) or \( D_n \) with dist\(_Q(\alpha, \beta) = 1 \), \( p \) is a \([Q]\)-minimal pair of \( \text{soc}_Q(p) \).

**Example 4.7.** In the \( E_6 \) case in Appendix, one can check that the pair

\[
p = (111001, 123212)
\]
has dist$_Q(p) = 1$ with its socle $\underline{s} = (001001,122101,111111)$. But $p$ is not a $[Q]$-minimal sequence of $\underline{s}$. Actually,
$$m = (111101,122111,001001) \quad \text{and} \quad m' = (011001,112101,111111)$$
are less than $p$ with respect to $\prec^b_Q$ and $[Q]$-minimal sequences of $\underline{s}$.

Now we shall generalized the notion of $[\tilde{w}_0]$-distance as follows:

**Definition 4.8.** We say that a sequence $\underline{m}$ has generalized $[\tilde{w}]$-distance $k$ ($k \in \mathbb{Z}_{\geq 0}$), denoted by $\text{gdist}_{[\tilde{w}]}(\underline{m}) = k$, if $\underline{m}$ is not $[\tilde{w}]$-simple and

(i) there exists a set of non $[\tilde{w}]$-simple sequences $\{\underline{m}^{(s)} | 1 \leq s \leq k, \text{wt}(\underline{m}^{(s)}) = \text{wt}(\underline{m})\}$

such that

$$\underline{m}^{(1)} \prec^b_{[\tilde{w}]} \cdots \prec^b_{[\tilde{w}]} \underline{m}^{(k)} = \underline{m.} \tag{4.10}$$

(ii) the set of non $[\tilde{w}]$-simple sequences $\{\underline{m}^{(s)}\}$ has maximal cardinality among sets of sequences satisfying (4.10).

If $\underline{m}$ is $[\tilde{w}]$-simple, we define $\text{gdist}_{[\tilde{w}]}(\underline{m}) = 0$.

**Remark 4.9.** From the proof of Proposition 4.4 and Definition 4.8, we have
$$\text{dist}_Q(\alpha, \beta) = \text{gdist}_Q(\alpha, \beta)$$
for any quiver $Q$ and any pair $(\alpha, \beta)$ of type $A_n$ or $D_n$.

However, the pair $\underline{p}$ in Example 4.7 has its $[Q]$-distance 1 while $\text{gdist}_Q(\underline{p}) = 2$.

**Lemma 4.10.** Assume the followings: For $w \in W_0$, we have

- a reduced expression $\tilde{w}$ of $w \in W$,
- a pair $(\alpha, \beta) \in (\Phi_+^w)^2 \subset (\Phi_+^w)^2$ with $\alpha \prec_{[\tilde{w}]} \beta$,
- a sequence $\underline{s}$ such that $\underline{s} \prec_{[\tilde{w}_0]} (\alpha, \beta)$ for some $\tilde{w}_0$ and $\tilde{w}' = w' \in W_0$ such that $\tilde{w}_0'$ is contained in $\Phi_0^\dagger_{[\tilde{w}]}$.

Then, for any $i$ with $s_i \neq 0$, $\beta_i$ is contained in $\Phi_0^\dagger_{[\tilde{w}]}$.

**Proof.** By the assumptions, it suffices to prove the following argument:

If $\alpha \prec_{[\tilde{w}_0]} \beta_i \prec_{[\tilde{w}_0]} \beta$ and $\alpha \prec_{[\tilde{w}]} \beta$, then we have $\beta_i \in \Phi_0^\dagger_{[\tilde{w}]}$.

If $\beta \notin \Phi_0^\dagger_{[\tilde{w}]}$, then $\beta = \beta_k^\dagger_{[\tilde{w}_0]}$ and $\beta_i = \beta_i^\dagger_{[\tilde{w}_0]}$ such that $k \leq \ell(w) < l$; i.e., $\beta \prec_{[\tilde{w}_0]} \beta_i$, which yields a contradiction to the fact that $\beta_i \prec_{[\tilde{w}_0]} \beta$ if and only if $\beta_i < \tilde{w}_0'$ for all $\tilde{w}_0' \in [\tilde{w}_0]$.

**Proof of Theorem 4.3.** By Proposition 4.5, the $\text{soc}_Q(\alpha, \beta)$ is well-defined for all pairs $(\alpha, \beta)$ and Dynkin quivers $Q$. Then our assertion follows from Lemma 4.10.

Now we can define the notion of $\langle\langle Q\rangle\rangle$-family:

**Definition 4.11.** For $\tilde{w}$ of $w \in W_0$, we say that $\tilde{w}$ is contained in the $\langle\langle Q\rangle\rangle$-family, if there exists a Dynkin quiver $Q'$ such that $\tilde{w}$ is adapted to $Q'$.

**Remark 4.12.** Before we close this subsection, we record the property of $[Q]$-socle of a pair $(\alpha, \beta)$ in this remark: For a given non $Q$-simple pair $p = (\alpha, \beta)$, the $[Q]$-socle $\underline{s}$ of $p$ is (i) a basic sequence and (ii) the number of non-zero parts of $\underline{s}$ is less than or equal to 3.
Example 4.13. For a pair \( p = (11111100, 12233321) \) and the Dynkin quiver \( Q \) of finite type \( E_8 \) in Appendix, the socle of \( p \) is given as follows:

\[
\text{soc}_Q(p) = (1111111, 01111100, 01011100).
\]

4.3. \([Q]\text{-radius and multiplicity of } \gamma \in \Phi^+ \setminus \Pi\). In this subsection, we study the relationship between \([Q]\text{-radius of } (\alpha, \beta) \) and multiplicity of \( \alpha + \beta \) when \( \alpha + \beta \in \Phi^+ \).

Theorem 4.14. [33, Theorem 3.4], [34, Theorem 3.13, Theorem 3.17]

(a) Let \( \gamma \) be a non-simple positive root in \( \Phi^+ \) of finite type \( A_n \) or \( D_n \) with \( \text{mul}(\gamma) = 1 \). For any pair \( (\alpha, \beta) \) with \( \alpha + \beta = \gamma \), we have \( \text{dist}(\alpha, \beta) = 1 \). Hence we have

\[
\text{dist}(\alpha, \beta) = \text{mul}(\gamma) = 1.
\]

(b) Let \( \gamma \) be a non-simple positive root in \( \Phi^+ \) of finite type \( D_n \) with \( \text{mul}(\gamma) = 2 \). Then there are pairs \( (\alpha, \beta) \) such that \( \alpha + \beta = \gamma \) and \( (\alpha, \beta) \) is not \([Q]\)-minimal.

Theorem 4.15. For \( \gamma \in \Phi^+ \setminus \Pi \) and any Dynkin quiver \( Q \) of finite type \( A_n \) \((n \geq 1)\), \( D_n \) \((n \geq 4)\) or \( E_6 \), we have

\[
\text{rds}_Q(\gamma) = \text{mul}(\gamma).
\]

Proof. By Theorem 4.14, our assertion was already proved for a non-simple positive root \( \gamma \) in \( \Phi^+ \) of finite type \( A_n \) and \( D_n \) with \( \text{mul}(\gamma) = 1 \).

Assume that \( \gamma \in \Phi^+ \) of finite type \( D \) with \( \text{mul}(\gamma) = 2 \). By Theorem 3.10, \( \gamma = \{a|b\} \) is located at the intersection of the \( a \)-swing and the \( b \)-swing. Moreover, [34, Theorem 3.17] and Proposition 4.5 \((D \text{ (iii)})\) tell that a non \([Q]\)-minimal pair \( (\alpha, \beta) \) of \( \gamma \) happens in the following form in \( \Gamma_Q \):

\[
\begin{align*}
\text{if } k \text{ is even} & \quad (1,s+(i-1)) & (1,s+(i+1)) \\
\text{if } k \text{ is odd} & \quad (1,s+(i-1)) & (1,s+(i+1)) 
\end{align*}
\]

where \( \phi_Q^{-1}(\beta, 0) = (i, s) \) and \( \phi_Q^{-1}(\alpha, 0) = (j, t) \). Moreover, Theorem 3.10, Theorem 3.11 and [34, Corollary 3.17] tell that

\[
(\alpha', \beta'), (\eta, \zeta) \text{ and } (\eta', \zeta') \text{ are } [Q]\text{-minimal pairs of } \gamma.
\]

Moreover, [34, Proposition 3.15] implies that any non \([Q]\)-minimal pairs \( (\alpha'^{(1)}, \beta'^{(1)}) \) and \( (\alpha'^{(2)}, \beta'^{(2)}) \) of \( \gamma \) are incomparable with respect to \( \prec_Q \). Since

\[
\gamma \prec_Q (\alpha', \beta'), (\eta, \zeta), (\eta', \zeta') \prec_Q (\alpha, \beta),
\]

we can conclude that \( \text{rds}_Q(\gamma) = \text{mul}(\gamma) = 2 \). \( \square \)

In the previous theorem, our choice of \( Q \) of finite type \( A_n \) \((n \geq 1)\), \( D_n \) \((n \geq 4)\) and \( E_6 \) is arbitrary. Thus the value \( \text{rds}_Q(\gamma) \) for \( \gamma \in \Phi_{A_n}, \Phi_{D_n} \) or \( \Phi_{E_6} \) do not depend on the choice of Dynkin quivers indeed:

\[
\begin{align*}
\text{soc}_Q(p) = (1111111, 01111100, 01011100) \\
\end{align*}
\]
Corollary 4.16. For any reduced expressions \( \tilde{w}_0 \) and \( \tilde{w}'_0 \) of \( w_0 \) adapted to some Dynkin quivers \( Q \) and \( Q' \) of finite type \( A_n \) (\( n \geq 1 \)) \( D_n \) (\( n \geq 4 \)) or \( E_6 \), we have
\[
\text{rds}_Q(\gamma) = \text{rds}_{Q'}(\gamma) = \text{mul}(\gamma) \quad \text{for all} \; \gamma \in \Phi^+ \setminus \Pi.
\]

Theorem 4.17. For \( \gamma \in \Phi^+ \setminus \Pi \) and any Dynkin quiver \( Q \) of finite type \( E_7 \) and \( E_8 \), we have
\[
\text{mul}(\gamma) - 1 \leq \text{rds}_Q(\gamma) \leq \text{mul}(\gamma) + 1.
\]

In particular,
- if \( \text{mul}(\gamma) = 1 \), then \( \text{rds}_Q(\gamma) = \text{mul}(\gamma) = 1 \),
- if \( \text{mul}(\gamma) > 1 \), then \( \text{rds}_Q(\gamma) > 1 \).

Example 4.18. (a) For the Dynkin quiver of finite type \( E_7 \) in Appendix, the \([Q]\)-radius of \( \gamma = (1122221) \) is 3 since we have
\[
(1122221) \prec^b_Q (1011110,0111111) \prec^b_Q (1122111,00001110) \prec^b_Q (1122110,0000111),
\]
while \( \text{mul}(\gamma) = 2 \).
(b) For the Dynkin quiver of finite type \( E_8 \) in Appendix, the \([Q]\)-radius of \((23465431)\) is 5 since
\[
(23465432) \prec^b_Q (12233221,11232211) \prec^b_Q (2234321,01121111)
\]
\[
\prec^b_Q (2234321,01122111) \prec^b_Q (22343221,01122111) \prec^b_Q (23454321,00011111)
\]
is the one of the longest sequences for its radius.

Remark 4.19. Consider the following reduced expression \( \tilde{w}_0 \) of \( w_0 \) of finite type \( A_5 \) which is not adapted to any Dynkin quiver \( Q \):
\[
\tilde{w}_0 = s_1 s_2 s_3 s_5 s_4 s_3 s_2 s_3 s_5 s_3 s_1 s_2 s_3.
\]
The convex partial order \( \prec_{[\tilde{w}_0]} \) can be visualized the following quiver \( \Upsilon_{[\tilde{w}_0]} \) by [36].

The distances of pairs.

Theorem 4.20. Let \( \mathfrak{m} = \max(\text{mul}(\gamma) \mid \gamma \in \Phi^+ \setminus \Pi) \). For any pair \( (\alpha, \beta) \), we have
\[
\text{dist}_Q(\alpha, \beta) \leq \mathfrak{m}.
\]
Proof. It suffices to consider a pair \((\alpha, \beta)\) such that \(\text{soc}_Q(\alpha, \beta) \neq (\alpha, \beta)\).

(Q of finite type \(A_n\)) Note that \(m = 1\) for \(\Phi^+\) of finite type \(A_n\). If \(\alpha + \beta \in \Phi^+\), we have \(\text{dist}_Q(\alpha, \beta) = 1\) by [33, Theorem 3.4]. Thus the remained pairs we have to deal with satisfy the following property:

\[
\text{supp } \alpha \cap \text{supp } \beta \neq \emptyset \quad \text{and} \quad \alpha + \beta \notin \Phi^+.
\]

Equivalently \(c \leq b\) or \(a \leq d\) when we write \(\alpha = [a, b]\) and \(\beta = [c, d]\). Then there exists only one pair \((\eta, \zeta)\) such that \((\eta, \zeta) \neq (\alpha, \beta)\) and \(\alpha + \beta = \eta + \zeta\). By (A-iii) in Proposition 4.5, \(\text{dist}_Q(\alpha, \beta) \leq 1\).

(Q of finite type \(D\)) Note that \(m = 2\) for \(\Phi^+\) of finite type \(D_n\). If \(\alpha + \beta \in \Phi^+\), we have \(\text{dist}_Q(\alpha, \beta) \leq 2\) by Theorem 4.15. Thus the remained pairs satisfy the following properties:

\[
\text{supp } \alpha \cap \text{supp } \beta \neq \emptyset \quad \text{and} \quad \alpha + \beta \notin \Phi^+.
\]

By recalling the cases in Proposition 4.5, such pairs happen in the cases (D-ii), (D-v) or (D-vi).

Write \(\alpha = [a, b]\) and \(\beta = [c, d]\) and set \(m = \min(a, c) \in I\). Assume first that \(a \neq c\). Then there are three positive roots having \(\varepsilon_a\) as its summand and one of \(\varepsilon_b, \varepsilon_c\) and \(\varepsilon_d\) as its another summand. Thus there are at most three pairs whose weights are the same as \(\alpha + \beta\). Moreover, we proved that one of them is \([Q]\)-simple. Thus \(\text{dist}_Q(\alpha, \beta) \leq 2\).

Now we assume that \(a = b\). Then such pairs happen only in the case (D-ii) with \(k' = n - 1\). The pair \((\alpha, \beta)\) is the only the pair with its weight \(\alpha + \beta\) and hence \(\text{dist}_Q(\alpha, \beta) = 1\) since \(\text{soc}_Q(\alpha, \beta) \neq (\alpha, \beta)\). Thus our assertion follows. \(\Box\)

By the above theorem, one can notice that every pair \((\alpha, \beta)\) of finite type \(A_n\) is \([Q]\)-minimal or simple.

**Proposition 4.21.** Let \(Q\) be a Dynkin quiver of finite type \(A_n, D_n\) and \(E_6\). For a pair \(\mathcal{P} = (\alpha, \beta)\) with \(\text{dist}_Q(\alpha, \beta) \geq 2\) and \(\alpha + \beta = \gamma \in \Phi^+\), there are at least two \([Q]\)-minimal pairs \(\mathcal{P}^{(1)}\) and \(\mathcal{P}^{(2)}\) such that \(\mathcal{P}^{(1)} = (\alpha^{(1)}, \beta^{(1)})\) and \(\mathcal{P}^{(2)} = (\alpha^{(2)}, \beta^{(2)})\) are good neighbors.

**Proof.** (Q of finite type \(D\)) By Theorem 4.15, we can assume that \((\alpha, \beta)\) is a pair such that \(\text{dist}_Q(\alpha, \beta) = 2\). Then we know that \(\gamma\) is a multiplicity non-free root and the relative positions of \(\alpha, \beta\) and \(\gamma\) in \(\Gamma_Q\) can be described as in (4.11). Take any pair among the minimal pairs \((\alpha', \beta'), (\eta, \zeta), (\eta', \zeta')\) of \(\gamma\) in (4.13) and write it as \((\alpha^{(1)}, \beta^{(1)})\). By Proposition 4.2 and (4.12), we have either

(i) \(\beta - \beta^{(1)} \in \Phi^+\) and \(\alpha^{(1)} - \alpha \in \Phi^+\), or

(ii) \(\alpha - \alpha^{(1)} \in \Phi^+\) and \(\beta^{(1)} - \beta \in \Phi^+\).

Assume the case (i) first. Write \(\eta = \beta - \beta^{(1)} \in \Phi^+\). Then we have \(\beta \prec_Q \eta, \alpha^{(1)} \prec_Q \eta\) and \(\alpha^{(1)} + \eta = \alpha\) by the convexity of \(\prec_Q\) and (4.12). Since the pairs \((\beta^{(1)}, \beta)\) and \((\alpha, \alpha^{(1)})\) are directly \(Q\)-connected, Proposition 4.2 tells that the pairs \((\beta^{(1)}, \eta)\) and \((\alpha, \eta)\) are minimal pairs for \(\beta\) and \(\alpha^{(1)}\) respectively; i.e.,

\[
\text{dist}_Q(\beta^{(1)}, \eta) = \text{dist}_Q(\alpha, \eta) = 1.
\]

Thus our assertion follows since \(\text{dist}_Q(\alpha, \beta) = 2\). The case (ii) can be proved by applying the similar argument. \(\Box\)

Now we record the following observation on \(E_n\)-types which can be obtained by applying the strategy in (2.10):
Proposition 4.22. For any pairs \( \rho = (\alpha, \beta) \) of \( \gamma = \alpha + \beta \in \Phi^+ \) and \( \text{dist}_Q(\rho) \geq 3 \), there are at least two \( [Q] \)-minimal pairs \( \rho^{(1)} \) and \( \rho^{(2)} \) of \( \gamma \) such that \( \rho^{(1)} \) and \( \rho \) are good neighbors \( (i = 1, 2) \).

Example 4.23. For a pair \( \rho = (111100, 011221) \) and the Dynkin quiver \( Q \) of finite type \( E_7 \) in Appendix, the minimal pairs \( \rho^{(1)} = (011111, 1112210) \) and \( \rho^{(2)} = (1122211, 0101110) \) are good neighbors of \( \rho \) since we have sequences of pairs

\[
\begin{align*}
\rho^{(1)} &\ll_{Q} (111110, 011221) \ll_{Q} \rho, \\
\rho^{(2)} &\ll_{Q} (011100, 111221) \ll_{Q} \rho,
\end{align*}
\]

which satisfy the conditions in Definition 1.18.

4.5. **The number of pairs** \( (\alpha, \beta) \) of \( \gamma \in \Phi^+ \). In [33, 34], the following propositions are already observed for finite types \( A_n \) and \( D_n \). We record the extended results here for finite types \( A_n, D_n \) and \( E_n \) without their proofs.

Proposition 4.24. For \( \gamma \in \Phi^+ \), there exists \( h(\gamma) \)-many sequences \( m \bar{w}_0 \) (for any \( \bar{w}_0 \) of \( w_0 \)) such that \( \text{wt}(m) \leq 2 \) and \( \text{wt}(m) = \gamma \).

Proposition 4.25. For \( \gamma \in \Phi^+ \setminus \Pi \) of finite type \( A_n, D_n \) or \( E_6 \) and any Dynkin quiver \( Q \),

\[
\left| \{ \rho \mid \rho \text{ is a pair, } \text{wt}(\rho) = \gamma \text{ and } k = \text{dist}_Q(\rho) \geq 1 \} \right| = \text{supp}_{k}(\gamma) - \delta_{k,1}.
\]

5. **Application to KLR algebras**

In this section, we shall apply our results in the previous sections to the representation theory of KLR-algebras.

5.1. **KLR algebras and categorifications.** In this subsection, we briefly recall the representation theories on KLR algebras which were introduced in [25, 26, 39].

Let \( k \) be a field. For a given symmetrizable Cartan datum \( CD := (\Lambda, P, \Pi, P^\vee, \Pi^\vee) \), we choose a polynomial \( Q_{ij}(u, v) \in k[u, v] \) for \( i, j \in I \) which is of the form

\[
Q_{ij}(u, v) = \delta(i \neq j) \sum_{(p, q) \in \mathbb{Z}_{\geq 0}^2} t_{i, j; p, q} u^p v^q
\]

with \( t_{i, j; p, q} \in k \), \( t_{i, j; p, q} = t_{j, i; q, p} \) and \( t_{i, j; -a_{ij}, 0} \in k^\times \). Thus we have \( Q_{i, j}(u, v) = Q_{j, i}(v, u) \).

The symmetric group \( S_m = \langle s_1, s_2, \ldots, s_{m-1} \rangle \) acts on \( I^m \) by place permutations. For \( n \in \mathbb{Z}_{\geq 0} \) and \( b \in Q^+ \) such that \( \text{ht}(b) = n \), we set

\[
I^b = \{ \nu = (\nu_1, \ldots, \nu_n) \in I^n \mid \alpha_{\nu_1} + \cdots + \alpha_{\nu_n} = b \}.
\]

For \( b \in Q^+ \), we denote by \( R(b) \) the KLR algebra at \( b \) associated with \( CD \) and \((Q_{i, j})_{i, j \in I} \). It is a \( Z \)-graded \( k \)-algebra generated by the generators \( \{ e(\nu) \}_{\nu \in P}, \{ x_k \}_{1 \leq k \leq \text{ht}(b)}, \{ \tau_m \}_{1 \leq m < \text{ht}(b)} \) with the certain defining relations (see [33, Definition 2.7] for the relations).

For a graded \( R(b) \)-module \( M = \bigoplus_{k \in \mathbb{Z}} M_k \), we define \( qM = \bigoplus_{k \in \mathbb{Z}} (qM)_k \), where

\[
(qM)_k = M_{k-1} \quad (k \in \mathbb{Z}).
\]

We call \( q \) the grading shift functor on the category of graded \( R(b) \)-modules.

Let \( \text{Rep}(R(b)) \) be the category consisting of finite dimensional graded \( R(b) \)-modules and \([\text{Rep}(R(b))] \) be the Grothendieck group of \( \text{Rep}(R(b)) \). Then \([\text{Rep}(R(b))] \) has a natural
Theorem 5.1 ([25, 39]). For a given symmetrizable Cartan datum \( \text{CD} \), let \( U^-_k(g)^\vee \) (\( A = \mathbb{Z}[q^{\pm 1}] \)) the dual of the integral form of the negative part of quantum groups \( U_q(g) \) related to \( \text{CD} \) and \( R \) be the KLR algebra corresponding to \( \text{CD} \) and \( (Q_{ij}(u,v))_{i,j \in I} \). Then we have
\[
(5.2) \quad U^-_k(g)^\vee \simeq [\text{Rep}(R)].
\]

Definition 5.2. We say that the KLR-algebra \( R \) is symmetric if \( A \) is symmetric and \( Q_{i,j}(u,v) \) is a polynomial in \( u - v \) for all \( i, j \in I \).

Theorem 5.3. [40, 41] Assume that the KLR-algebra \( R \) is symmetric and \( k \) is of characteristic zero. Then under the isomorphism (5.2) in Theorem 5.1, the dual canonical/upper global basis of \( U^-_k(g)^\vee \) corresponds to the set of the isomorphism classes of self-dual simple \( R \)-modules.

5.2. Simple heads and socles. In this subsection, we collect the results on the convolution product of the real simple module and simple module over symmetric KLR algebra for our purpose ([17, 20, 21]). Throughout this subsection, we assume that all KLR algebras are symmetric.

Theorem 5.4. ([17, Theorem 3.2], [20, Theorem 3.1], [21, Proposition 4.5]) Let \( M \) be a real simple \( R(a) \)-module and \( N \) be a simple \( R(b) \)-module. Then there exist unique non-zero homomorphisms
\[
r_{M,N} : M \circ N \longrightarrow N \circ M \quad \text{and} \quad r_{N,M} : N \circ M \longrightarrow M \circ N,
\]
satisfying the following properties:
(i) \( M \circ N \) has a simple socle and a simple head.
(ii) \( \text{Hom}_R(M \circ N, N \circ M) = k \cdot r_{M,N} \).
(iii) The socle and head of \( M \circ N \) are distinct and appear once in the composition series of \( M \circ N \) unless \( M \circ N \simeq N \circ M \) is simple.

The similar result holds for \( N \circ M \) and we have
\[
\text{Im}(r_{N,M}) \simeq \text{soc}(M \circ N) \simeq \text{hd}(N \circ M) \quad \text{and} \quad \text{Im}(r_{M,N}) \simeq \text{soc}(N \circ M) \simeq \text{hd}(M \circ N).
\]

Proposition 5.5. [17, Corollary 3.11] Let \( M_k \) be a finite dimensional graded \( R \)-module \((k = 1, 2, 3)\), and let \( \varphi_1 : L \longrightarrow M_1 \circ M_2 \) and \( \varphi_2 : M_2 \circ M_3 \longrightarrow L' \) be non-zero homomorphisms. Assume further that \( M_2 \) is simple. Then the composition
\[
L \circ M_3 \xrightarrow{\varphi_1 \circ M_3} M_1 \circ M_2 \circ M_3 \xrightarrow{M_1 \circ \varphi_2} M_1 \circ L'
\]
does not vanish. Similarly, for non-zero homomorphisms \( \varphi_1 : L \rightarrow M_2 \circ M_3, \varphi_2 : M_1 \circ M_2 \rightarrow L' \) and an assumption that \( M_2 \) is simple, we have a non-zero composition

\[
M_1 \circ L \xrightarrow{M_1 \circ \varphi_1} M_1 \circ M_2 \circ M_3 \xrightarrow{\varphi_2 \circ M_3} L' \circ M_3.
\]

Let \( a, b \in Q^+ \). For a simple \( R(a) \)-module \( M \) and a simple \( R(b) \)-module \( N \), let us denote by \( M \circ N \) the head of \( M \circ N \).

5.3. Rep(\( R \)) associated to FCD and \( \prec_\beta^{\tilde{\omega}_0} \). In this subsection, we first review the results on Rep(\( R \)) associated to finite Cartan datum FCD which were investigated in [6, 24, 27, 32], briefly. Then we shall refine the results by using the order \( \prec_\beta^{\tilde{\omega}_0} \) which is far coarser than \( \prec_\beta^{\tilde{\omega}_0} \). We would like to emphasize that the results in this subsection work for all reduced expressions \( \tilde{\omega}_0 \) of all finite types.

**Theorem 5.6.** [6, 32] (see also [24, 27]) For a finite simple Lie algebra \( \mathfrak{g}_0 \), we fix a convex total order \( \leq^{\tilde{\omega}_0} \) on \( \Phi^+ \) induced from the reduced expression \( \tilde{\omega}_0 \) (see Convention 1). Let \( R \) be the KLR algebra corresponding to \( \mathfrak{g}_0 \). For each positive root \( \beta \in \Phi^+ \), there exists a simple module \( S_{\tilde{\omega}_0}(\beta) \) satisfying the following properties:

(a) \( S_{\tilde{\omega}_0}(\beta) \circ_m \) is a real simple \( R(m\beta) \)-module.

(b) For every \( m_{\tilde{\omega}_0} \in \mathbb{Z}_{\geq 0}^{\ell(\tilde{\omega}_0)} \), there exists a non-zero \( R \)-module homomorphism

\[
\rho_m : \tilde{S}_{\tilde{\omega}_0}(m) := S_{\tilde{\omega}_0}(\beta_1)^{\circ m_1} \circ \cdots \circ S_{\tilde{\omega}_0}(\beta_N)^{\circ m_N} \rightarrow \tilde{S}_{\tilde{\omega}_0}(m) := S_{\tilde{\omega}_0}(\beta_N)^{\circ m_N} \circ \cdots \circ S_{\tilde{\omega}_0}(\beta_1)^{\circ m_1}
\]

such that

(i) \( \text{Hom}_{R(\text{wt}(m))}(\tilde{S}_{\tilde{\omega}_0}(m), \tilde{S}_{\tilde{\omega}_0}(m)) = k \cdot \rho_m \).

(ii) \( \text{Im}(\rho_m) \cong \text{hd}(\tilde{S}_{\tilde{\omega}_0}(m)) \cong \text{soc}(\tilde{S}_{\tilde{\omega}_0}(m)) \) is simple.

(c) For any sequence \( m_{\tilde{\omega}_0} \in \mathbb{Z}_{\geq 0}^{\ell(\tilde{\omega}_0)} \), we have

\[
[S_{\tilde{\omega}_0}(m)] \in [\text{Im}(\rho_m)] + \sum_{\text{wt}(m') = \text{wt}(m)} \mathbb{Z}_{\geq 0}[\text{Im}(\rho_{m'})].
\]

(d) For any sequence \( m_{\tilde{\omega}_0} \in \mathbb{Z}_{\geq 0}^{\ell(\tilde{\omega}_0)} \), \( \tilde{S}_{\tilde{\omega}_0}(m) \) has a unique simple head \( \text{hd}(\tilde{S}_{\tilde{\omega}_0}(m)) \).

and \( \tilde{S}_{\tilde{\omega}_0}(m) \neq \tilde{S}_{\tilde{\omega}_0}(m') \) if \( m \neq m' \).

(e) For every simple \( R \)-module \( M \), there exists a unique sequence \( m \in \mathbb{Z}_{\geq 0}^{\ell(\tilde{\omega}_0)} \) such that \( M \cong \text{Im}(\rho_m) \cong \text{hd}(\tilde{S}_{\tilde{\omega}_0}(m)) \).

(f) For any \( \tilde{\omega}_0 \)-minimal pair \((\beta_k^{\tilde{\omega}_0}, \beta_l^{\tilde{\omega}_0})\) of \( \beta_j^{\tilde{\omega}_0} = \beta_k^{\tilde{\omega}_0} + \beta_l^{\tilde{\omega}_0} \), there exists an exact sequence

\[
0 \rightarrow S_{\tilde{\omega}_0}(\beta_j) \rightarrow S_{\tilde{\omega}_0}(\beta_k) \circ S_{\tilde{\omega}_0}(\beta_l) \xrightarrow{\rho_m} S_{\tilde{\omega}_0}(\beta_l) \circ S_{\tilde{\omega}_0}(\beta_k) \rightarrow S_{\tilde{\omega}_0}(\beta_j) \rightarrow 0,
\]

where \( m_{\tilde{\omega}_0} \in \mathbb{Z}_{\geq 0}^{\ell(\tilde{\omega}_0)} \) such that \( m_k = m_l = 1 \) and \( m_i = 0 \) for all \( i \neq k, l \).
Remark 5.7. The property in (f) of Theorem 5.6 is called a length two property of minimal pair, since the composition series of $S_{\bar{w}_0} (\beta_l) \circ S_{\bar{w}_0} (\beta_k)$ has its length as 2 and consists of their distinct head and socle. The length two property sometimes holds for some $\bar{w}_0$ and non $\bar{w}_0$-minimal pair $(\alpha, \beta)$ of $\gamma$. For the $\bar{w}_0$ in Remark 4.19 and the non $\bar{w}_0$-minimal pair $([1], [2, 5])$, one can compute that $S_{\bar{w}_0} ([1], [2, 5])$ has its composition length 2 where its composition series consists of $S_{\bar{w}_0} ([1]) \circ S_{\bar{w}_0} ([2, 5])$ as its head and $S_{\bar{w}_0} ([1, 3]) \circ S_{\bar{w}_0} ([4, 5]) \simeq S_{\bar{w}_0} ([2, 5]) \circ S_{\bar{w}_0} ([1])$ as its socle.

Remark 5.8. For any $\bar{w}_0, \bar{w}_0' \in [\bar{w}_0]$, we have 

$$S_{\bar{w}_0} (\beta) \simeq S_{\bar{w}_0'} (\beta) \quad \text{for all } \beta \in \Phi^+,$$

by Remark 1.5 and Theorem 5.6 (f). Thus we denote by $S_{[\bar{w}_0]} (\beta)$ the simple module $S_{\bar{w}_0'} (\beta)$ for any $\bar{w}_0' \in [\bar{w}_0]$. However, at this moment, the definition of $S_{\bar{w}_0} (m)$ and (5.6) depend on $\bar{w}_0$.

Proposition 5.9. Let $(\alpha, \beta)$ be an incomparable pair with respect to the order $\prec_{[\bar{w}_0]}$. Then we have 

$$S_{[\bar{w}_0]} (\alpha) \circ S_{[\bar{w}_0]} (\beta) \simeq S_{[\bar{w}_0]} (\beta) \circ S_{[\bar{w}_0]} (\alpha) \quad \text{is simple.}$$

Proof. By the assumption and Remark 1.5, there exist reduced expressions $\bar{w}_0 (1), \bar{w}_0 (2) \in [\bar{w}_0]$ such that 

(i) $\alpha <_{\bar{w}_0 (1)} \beta$ and (ii) $\beta <_{\bar{w}_0 (2)} \alpha$.

By Theorem 5.6, we have 

(i) \[
[\overrightarrow{S_{\bar{w}_0}} (\alpha, \beta)] := [S_{[\bar{w}_0]} (\alpha) \circ S_{[\bar{w}_0]} (\beta)] \in \Im (r_{(\alpha, \beta)}) + \sum_{m'_{\bar{w}_0} (1) \prec_{\bar{w}_0 (1)} (\alpha, \beta)} \mathbb{Z}_{\geq 0} \Im (r_{m'_{\bar{w}_0} (1)}),
\]

where $r_{(\alpha, \beta)} : S_{[\bar{w}_0]} (\alpha) \circ S_{[\bar{w}_0]} (\beta) \rightarrow S_{[\bar{w}_0]} (\beta) \circ S_{[\bar{w}_0]} (\alpha)$ is the unique non-zero homomorphism (up to constant).

(ii) \[
[\overrightarrow{S_{\bar{w}_0}} (\beta, \alpha)] := [S_{[\bar{w}_0]} (\beta) \circ S_{[\bar{w}_0]} (\alpha)] \in \Im (r_{(\beta, \alpha)}) + \sum_{m'_{\bar{w}_0} (2) \prec_{\bar{w}_0 (2)} (\beta, \alpha)} \mathbb{Z}_{\geq 0} \Im (r_{m'_{\bar{w}_0} (2)}),
\]

where $r_{(\beta, \alpha)} : S_{[\bar{w}_0]} (\beta) \circ S_{[\bar{w}_0]} (\alpha) \rightarrow S_{[\bar{w}_0]} (\alpha) \circ S_{[\bar{w}_0]} (\beta)$ is the unique non-zero homomorphism (up to constant).

Assume that $\Im (r_{(\alpha, \beta)}) \not\simeq \Im (r_{(\beta, \alpha)})$. Then there exists one of $r_{m'_{\bar{w}_0} (2)}$ for some $m'_{\bar{w}_0} (2)$ which is less than $(\beta, \alpha)$ with respect to the order $\prec_{\bar{w}_0 (2)}$ by Theorem 5.6 (c) and (e). However, it can not happen and hence our assertion follows.

By the above proposition, the isomorphism class of the module $\overrightarrow{S_{\bar{w}_0}} (m)$ and the homomorphism $r_{m_{\bar{w}_0}}$ do not depend on the reduced expression $\bar{w}_0' \in [\bar{w}_0]$ but on $[\bar{w}_0]$. Thus we can denote it by $\overrightarrow{S_{[\bar{w}_0]}} (m)$ even though $m = m_{\bar{w}_0}'$ for $\bar{w}_0' \in [\bar{w}_0]$. Furthermore, (5.6) can be refined:

Theorem 5.10. \hspace{1cm} (a) $\overrightarrow{S_{[\bar{w}_0]}} (m)$ is well-defined; that is, 

$$\overrightarrow{S_{\bar{w}_0}} (m_{\bar{w}_0}) \simeq \overrightarrow{S_{\bar{w}_0}} (m_{\bar{w}_0'}) \quad \text{for all } \bar{w}_0, \bar{w}_0' \in [\bar{w}_0].$$
(b) For any $\bar{w}_0$ of $w_0$ and any sequence $m$, we have

\[
\overrightarrow{S_{\bar{w}_0}(m)} \in \text{Im}(\overrightarrow{r_m}) + \sum_{\substack{m' \in \bar{w}_0 \cdot m \\text{wt}(m') = \text{wt}(m)}} \mathbb{Z}_{\geq 0}[\text{Im}(\overrightarrow{r_m})].
\]

Corollary 5.11. If a pair $p$ is $[\bar{w}_0]$-simple, then $\overrightarrow{S_{\bar{w}_0}(p)} \simeq \overrightarrow{S_{\bar{w}_0}(p)}$ is real simple.

Proof. By the definition of $[\bar{w}_0]$-simple sequence, the $[\bar{w}_0]$-simple pair $p$ is a minimal element with respect to $\preceq_{[\bar{w}_0]}$ among the sequences $m$ such that $\text{wt}(m) = \text{wt}(p)$. Thus our assertion follows from Theorem 5.10. \hfill \Box

Corollary 5.12. For a $[\bar{w}_0]$-simple sequence $s$, the module $\overrightarrow{S_{\bar{w}_0}(s)} \simeq \overrightarrow{S_{\bar{w}_0}(s)}$ is real simple.

Proof. For any $i, j$ such that $s_i, s_j \neq 0$, the pair $\overrightarrow{S_{\bar{w}_0}(\beta_i, \beta_j)} \simeq \overrightarrow{S_{\bar{w}_0}(\beta_i, \beta_j)}$ by Corollary 5.11. Thus there exists an isomorphism between $\overrightarrow{S_{\bar{w}_0}(s)}$ and $\overrightarrow{S_{\bar{w}_0}(s)}$. Then our assertion follows from Theorem 5.6 (b). \hfill \Box

For the simplicity of notations, we denote by $S_{[\bar{w}_0]}(g)$ instead of $\overrightarrow{S_{\bar{w}_0}(g)}$ when the KLR-algebra $R$ is symmetric and $g$ is a $[\bar{w}_0]$-simple sequence.

5.4. $[Q]$-socle. In this subsection, we only deal with the equivalent class $[Q]$ for some Dynkin quiver $Q$. We shall study the representation theoretical meanings of $[Q]$-socle and $[Q]$-length in the category $\text{Rep}(R)$, where $R$ is a symmetric KLR algebra associated to simply laced finite simple Lie algebra $\mathfrak{g}_0$.

For the simplicity of notation, we denote by $\overrightarrow{S_Q(m)}$ instead of $\overrightarrow{S_{[Q]}(m)}$.

Proposition 5.13. Let $M_i$ and $N_i$ $(i = 1, 2)$ be simple $R$-modules such that one of them is real and $M_i \circ N_i$ has a composition length 2. For a real simple $R$-modules $M$ and a simple module $N$, we assume that

(a) there exists a non-zero $R$-homomorphism $\varphi_i: M_i \circ N_i \to M \circ N$ $(i = 1, 2)$,

(b) $\text{soc}(M_i, N_i) \simeq \text{soc}(M_2, N_2)$ and the pairs $(M_i, N_i)$ are distinct.

Then we have

\[
\text{soc}(M_1, N_1) \simeq \text{soc}(M_2, N_2) \simeq \text{soc}(M, N) \quad \text{and} \quad \varphi_i: M_i \circ N_i \to M \circ N.
\]

Proof. By the assumptions and Theorem 5.4, the simple socle of $M \circ N$ is contained in $\text{Im}\varphi_1 \cap \text{Im}\varphi_2$. However, if one of $\varphi_1$ and $\varphi_2$ is not injective, $\text{Im}\varphi_1 \cap \text{Im}\varphi_2$ becomes empty since $\text{hd}(M_k \circ N_k)$ is not isomorphic to $\text{hd}(M_i \circ N_i)$ and $\text{soc}(M_1, N_1) \simeq \text{soc}(M_2, N_2)$ for $\{k,l\} = \{1, 2\}$. Thus our assertion follows. \hfill \Box

Since $\overrightarrow{S_Q(\alpha, \beta)}$ has a composition length 2 for a $[Q]$-minimal pair $(\alpha, \beta)$ of $\gamma$, we have the following lemma:

Corollary 5.14. Let $(\alpha_i, \beta_i)$ $(i = 1, 2, 3)$ be distinct pairs for $\gamma = \alpha_i + \beta_i \in \Phi^+$ such that $(\alpha_j, \beta_j)$ $(j = 1, 2)$ is $[Q]$-minimal and $(\alpha_3, \beta_3)$ is not. If there exists a non-zero $R(\gamma)$-homomorphism

\[
\overrightarrow{S_Q(\alpha_j, \beta_j)} \to \overrightarrow{S(\alpha_3, \beta_3)} \quad (j = 1, 2),
\]

then $\text{soc}(\overrightarrow{S_Q(\alpha_i, \beta_i)}, \gamma)$ has a composition length 2 for $\gamma = \alpha_i + \beta_i \in \Phi^+$ such that $(\alpha_j, \beta_j)$ $(j = 1, 2)$ is $[Q]$-minimal and $(\alpha_3, \beta_3)$ is not.
then we have
\[ \text{soc}(\overrightarrow{S}(\alpha_3, \beta_3)) \simeq S_Q(\gamma) \quad \text{and} \quad \overrightarrow{S}(\alpha_j, \beta_j) \twoheadrightarrow \overrightarrow{S}(\alpha_3, \beta_3) \quad (j = 1, 2). \]

**Proposition 5.15.** We assume that
(a) \( M_i \) and \( N_i \) \( (i = 1, 2) \) be simple \( R \)-modules such that one of them is real,
(b) there exists an injective \( R \)-homomorphism \( M_1 \circ N_1 \hookrightarrow M_2 \circ N_2 \).

Then we have
\[ \text{soc}(M_1 \circ N_1) \simeq \text{soc}(M_2 \circ N_2). \]

**Proof.** The proof follows from the definition of socle and Theorem 5.4. \( \square \)

**Corollary 5.16.** Let \( (\alpha_i, \beta_i) \) \( (i = 1, 2) \) be pairs for \( \gamma = \alpha_i + \beta_i \in \Phi^+ \) such that \( \text{soc}(\overrightarrow{S}(\alpha_1, \beta_1)) \simeq S_Q(\gamma) \). Assume there exists an injective \( R \)-homomorphism
\[ \overrightarrow{S}(\alpha_i, \beta_i) \hookrightarrow M \circ N \quad (i = 1, 2) \]
for simple \( R \)-modules \( M \) and \( N \) such that one of them is real. Then we have
\[ \text{soc}(\overrightarrow{S}(\alpha_2, \beta_2)) \simeq S_Q(\gamma). \]

**Example 5.17.** Consider the positive root \( \gamma = (0112111) \) of \( \Phi_{E_7}^+ \). From the quiver \( Q \) in Appendix B, one can check that the pair \( (0112100, 0000011) \) of \( \gamma \) has \([Q] \)-distance 2 since
\[ \gamma' := (0111111, 0001000) \twoheadrightarrow_b p := (0112100, 0000011). \]

Note that \( \gamma', p \) are not good neighbor. Furthermore, there is no more pair \( (\alpha, \beta) \) of \( \gamma \) such that \( (\alpha, \beta) \twoheadrightarrow_b p \). Then we have
\[
\begin{array}{ccc}
S_Q(0112100) \circ S_Q(0000011) \\
\downarrow S_Q(0112111) \\
S_Q(0111111) \circ S_Q(0001000)
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
S_Q(0111100) \circ S_Q(0000011) \\
\downarrow S_Q(0111100) \circ S_Q(0000011)
\end{array}
\end{array}
\]
by (f) of Theorem 5.6 and the \([Q] \)-minimality of \((0111100, 0001000), (0111100, 0000011) \) and \((0111111, 0001000) \). Note that the isomorphism in the second column follows from the fact that the pair \((0001000, 0000011) \) is \( Q \)-simple. Then Corollary 5.11 tells that \( S_Q(0001000) \circ S_Q(0000011) \) is real simple. By setting \( M = S_Q(0111100) \) and \( N = S_Q(0001000) \circ S_Q(0000011) \), we can apply the above corollary.

The following theorem is already proved for finite types \( A_n \) and \( D_n \) in [33, 34] by using Dorey’s rule and KLR-type Schur-Weyl duality functor. Now, we provide an alternative proof including finite \( E_n \)-type.

**Theorem 5.18.** For a pair \( p = (\alpha, \beta) \) with \( \alpha + \beta = \gamma \in \Phi^+ \), we have
\[ \text{soc}(\overrightarrow{S}(\alpha, \beta)) \simeq S_Q(\gamma). \]

**Proof.** For the case when \( (\alpha, \beta) \) is a \([Q] \)-minimal pair of \( \gamma \), i.e., \( \text{dist}_Q(\alpha, \beta) = 1 \), our assertion follows from (f) of Theorem 5.6. Since each pair \( (\alpha, \beta) \) of \( \gamma \) of finite type \( A_n \) is \([Q] \)-minimal ([33, Theorem 3.4]), we can first assume that \( \text{dist}_Q(\alpha, \beta) = 2 \) and \( Q \) is of finite type \( D_n \) or \( E_n \). From the Proposition 4.21, there exist good adjacent neighbors \( \gamma^{(1)} \) and \( \gamma^{(2)} \) of \( \gamma \) which are also minimal pairs of \( \gamma \); that is, there exists \( \eta^{(i)} \in \Phi^+ \) satisfying either
(a1) \( \eta^{(i)} + \beta = \beta^{(i)} \), \( \eta + \alpha^{(i)} = \alpha \) and \( 1 = \text{dist}_Q(\eta, \beta) = \text{dist}_Q(\eta, \alpha^{(i)}) < \text{dist}_Q(\underline{p}) = 2 \),

(b1) \( \beta^{(i)} + \eta^{(i)} = \beta \), \( \alpha + \eta^{(i)} = \alpha^{(i)} \) and \( 1 = \text{dist}_Q(\beta^{(i)}, \eta) = \text{dist}_Q(\alpha, \eta) < \text{dist}_Q(\underline{p}) = 2 \),

except when \( Q \) is of finite type \( E_7 \) or \( E_8 \). In both cases, we have homomorphisms

(a2) \( S_Q(\beta^{(i)}) \to S_Q(\eta^{(i)}) \circ S_Q(\beta), \) \( S_Q(\alpha^{(i)}) \circ S_Q(\eta^{(i)}) \to S_Q(\alpha) \),

(b2) \( S_Q(\eta^{(i)}) \circ S_Q(\beta^{(i)}) \to S_Q(\beta), \) \( S_Q(\alpha^{(i)}) \to S_Q(\alpha) \circ S_Q(\eta^{(i)}) \).

Thus we have non-zero compositions

(a3) \( S_Q(\alpha^{(i)}) \circ S_Q(\beta^{(i)}) \to S_Q(\alpha^{(i)}) \circ S_Q(\eta^{(i)}) \circ S_Q(\beta) \to S_Q(\alpha) \circ S_Q(\beta) \),

(b3) \( S_Q(\alpha^{(i)}) \circ S_Q(\beta^{(i)}) \to S_Q(\alpha) \circ S_Q(\eta^{(i)}) \circ S_Q(\beta^{(i)}) \to S_Q(\alpha) \circ S_Q(\beta) \),

by Proposition 5.5. Then our assertion follows from 5.14.

For the cases when \( Q \) is of finite type \( E_7 \) or \( E_8 \), there are several \( \gamma \in \Phi^+ \) which has a pair \( p = (\alpha, \beta) \) of \( \gamma \) (see Example 5.17) such that \( \text{dist}_Q(\alpha, \beta) = 2 \) and

- \( p \) does not have good neighbor at all,
- there exists a unique minimal pair \( p' = (\alpha', \beta') \) of \( \gamma \) such that \( p' \prec Q \underline{p} \) and \( \text{wt}(p') = \text{wt}(p) \).

However, we can apply the same argument of Example 5.17 in those cases. More precisely, there exists \( \eta \in \Phi^+ \) such that one of the following conditions holds:

(i) \( (\eta, \beta') \) is a \([Q]-\text{minimal pair of } \alpha \), \( (\eta, \beta) \) is a \([Q]-\text{minimal pair of } \alpha' \) and \( (\beta, \beta') \) are \([Q]-\text{simple} \),

(ii) \( (\alpha', \eta) \) is a \([Q]-\text{minimal pair of } \beta \), \( (\alpha, \eta) \) is a \([Q]-\text{minimal pair of } \beta' \) and \( (\alpha, \alpha') \) are \([Q]-\text{simple} \).

Thus we have

\[
\begin{array}{ccc}
S_Q(\alpha) \circ S_Q(\beta) & \cong & S_Q(\eta) \circ S_Q(\beta') \circ S_Q(\beta) \\
& & \text{(resp. } S_Q(\alpha) \circ S_Q(\alpha') \circ S_Q(\eta)) \\
& \cong & S_Q(\eta) \circ S_Q(\beta) \circ S_Q(\beta') \\
S_Q(\alpha') \circ S_Q(\beta') & & \text{(resp. } S_Q(\alpha') \circ S_Q(\alpha) \circ S_Q(\eta))
\end{array}
\]

Thus our assertion holds for the pairs \( (\alpha, \beta) \) with \( \text{dist}_Q(\alpha, \beta) \leq 2 \).

For the pairs \( (\alpha, \beta) \) with \( \text{dist}_Q(\alpha, \beta) \geq 3 \), we can apply the induction. More precisely, by Proposition 4.21 and Proposition 4.22, there are good neighbors \( p^{(i)} = (\alpha^{(i)}, \beta^{(i)}) \) \( (i = 1, 2) \) of \( \underline{p} \), which are also \([Q]-\text{minimal} \). By the induction hypothesis, we have non-zero compositions

(a) \( S_Q(\alpha^{(i)}) \circ S_Q(\beta^{(i)}) \to S_Q(\alpha^{(i)}) \circ S_Q(\eta^{(i)}) \circ S_Q(\beta) \to S_Q(\alpha) \circ S_Q(\beta) \) or

(b) \( S_Q(\alpha^{(i)}) \circ S_Q(\beta^{(i)}) \to S_Q(\alpha) \circ S_Q(\eta^{(i)}) \circ S_Q(\beta^{(i)}) \to S_Q(\alpha) \circ S_Q(\beta) \),

since

\[
\text{dist}_Q(\eta, \beta), \text{dist}_Q(\eta, \alpha^{(i)}) < \text{dist}_Q(\underline{p}) \quad \text{or} \quad \text{dist}_Q(\beta^{(i)}, \eta), \text{dist}_Q(\alpha, \eta) < \text{dist}_Q(\underline{p}).
\]

Thus our assertion follows from Corollary 5.14.

Now we shall generalize Theorem 5.18 to all pairs \( (\alpha, \beta) \) by using the assumptions and the arguments of Proposition 4.5.

**Theorem 5.19.** For a pair \( (\alpha, \beta) \), we have

\[
S_Q(\text{soc}_Q(\alpha, \beta)) \simeq \text{soc}(\overrightarrow{S_Q(\alpha, \beta)}).
\]
Proof. The case when \( \alpha + \beta \in \Phi^+ \) is already covered in Theorem 5.18. Thus we suffice to consider the case when

\[
\alpha + \beta \notin \Phi^+ \quad \text{and} \quad \underline{p} = (\alpha, \beta) \text{ is not } [Q]-\text{simple.}
\]

(Q of finite type \( A_n \)) In this case, \( (5.7) \) can happen only in \( (4.4) \); i.e.,

\[
\begin{align*}
\beta & \qquad \alpha \qquad \beta \\
\alpha & \qquad \beta
\end{align*}
\]

Then \( \underline{s} \) and \( \underline{p} \) are good adjacent neighbor such that \( \underline{s} \prec_Q \underline{p} \) by Proposition 4.2 and Theorem 4.20. Thus our assertion holds from the existence of non-zero homomorphism \( \overrightarrow{S_Q(\underline{s})} \to \overrightarrow{S_Q(\underline{p})} \), the simplicity of \( \overrightarrow{S_Q(\underline{s})} \) and Theorem 5.4.

(Q of finite type \( D \)) In this case, \( (5.7) \) can happen only in \( (4.9) \) which we have recorded; i.e.,

\[
\begin{align*}
\beta & \qquad \alpha \qquad \beta \\
\alpha & \qquad \beta
\end{align*}
\]

(Recall that we have assumed \( j \leq i \) where \( \phi_Q^{-1}(\alpha, 0) = (j, t) \) and \( \phi_Q^{-1}(\beta, 0) = (i, s) \).)

In the cases (1), (2), (4) and (6), one can easily check that \( \underline{s} \) and \( \underline{p} \) form good adjacent neighbors by Proposition 4.5. Hence our assertion follows from the same argument of finite type \( A_n \). In the case (3), \( \underline{s} \prec_Q \underline{p}' = (\alpha', \beta') \prec_Q \underline{p} \) is a sequence of good neighbors such that \( (\underline{s}, \underline{p'}) \) and \( (\underline{p'}, \underline{p}) \) are pairs of good adjacent neighbors. Thus our assertion follows.

In the case (5), the positive roots should be of the following forms:

\[
\alpha = \varepsilon_a - \varepsilon_c, \quad \beta = \varepsilon_a + \varepsilon_b, \quad s_3 = \varepsilon_b - \varepsilon_c, \quad \{s_1, s_2\} = \{\varepsilon_a - \varepsilon_t, \varepsilon_a + \varepsilon_t\}.
\]

Thus there exists \( \alpha' = \varepsilon_a - \varepsilon_b \) such that \( \alpha' \prec_Q \alpha \) and the pair \( (\alpha', \beta) \) is of the case (4) and has its socle as \( (s_1, s_2) \). Thus we have a non-zero composition

\[
(5.8) \quad S_Q(\beta) \circ S_Q(\alpha) \rightarrow S_Q(\beta) \circ S_Q(s_3) \circ S_Q(\alpha') \simeq S_Q(s_3) \circ S_Q(\beta) \circ S_Q(\alpha') \rightarrow S_Q(s_3) \circ S_Q(s_1) \circ S_Q(s_2) \simeq S_Q(\underline{s})
\]

by Proposition 5.5. Thus our assertion can be obtained by taking dual on \( (5.8) \).

Recall Remark 4.12. For \( Q \) of finite type \( E_n \), we can apply the same strategy of finite type \( D_n \) and we shall show a non-trivial example below. \( \square \)

Example 5.20. (a) Consider

the pair \( \underline{p} = (111001, 123212) \) and its socle \( \underline{s} = (111111, 122101, 001001) \)
in Example 4.7. Since \((110000,001001)\) is a pair for \((111001)\), we have
\[
S_Q(123212) \circ S_Q(111001) \to S_Q(122101) \circ S_Q(111111) \circ S_Q(001001).
\]

Here the second surjection can be obtained by the following way: Since
\[
\bullet \quad (122101,001111) \text{ is a pair for } (123212),
\bullet \quad (110000,001111) \text{ is a pair for } (111111),
\]
we have a non-zero composition
\[
S_Q(123212) \circ S_Q(110000) \to S_Q(122101) \circ S_Q(001111) \circ S_Q(110000) \to S_Q(122101) \circ S_Q(111111).
\]

Since \(S_Q(122101) \circ S_Q(111111)\) is \([Q]\)-simple, the non-zero composition is surjective.

By taking dual on \((5.9)\), we have
\[
\to S_Q(s) \to S_Q(p).
\]

(b) In the Dynkin quiver \(Q\) of Example 3.6, the socle of the module \(S_Q(\{2\mid -4\}) \circ S_Q(\{1\mid 2\})\) has its socle as \(S_Q(\{1\mid -4\}) \circ S_Q(\{2\mid -3\}) \circ S_Q(\{2\mid 3\})\); i.e.,
\[
S_Q(\{1\mid -4\}) \circ S_Q(\{2\mid -3\}) \circ S_Q(\{2\mid 3\}) \to S_Q(\{2\mid -4\}) \circ S_Q(\{1\mid 2\}).
\]

**Corollary 5.21.** For \(m \in \mathbb{Z}^{(w_0)}_{\geq 0}\),
\[
\to S_Q(m) \simeq \to S_Q(m) \text{ is simple if and only if } m \text{ is } [Q]-simple.
\]

**Proof.** Our assertion is an immediate consequence of Corollary 5.12 and Theorem 5.19. \(\Box\)

**Theorem 5.22.** For non \([Q]\)-simple pairs \(p = (\alpha, \beta)\) and \(p' = (\alpha', \beta')\) such that
\[
p' \preceq_Q p \quad \text{and they are good neighbors,}
\]
we have an injective homomorphism
\[
\to S_Q(p') \to \to S_Q(p).
\]

Hence the composition length of \(\to S_Q(p)\) for \(\text{dist}_Q(p) \geq 1\) is larger than or equal to \(\text{len}_Q(p) + 2\).

**Proof.** If \(p\) and \(p'\) are good adjacent neighbors, we have a non-zero homomorphism \(\to S_Q(p') \to \to S_Q(p)\) by the same argument of the preceding theorem. Since their socles are isomorphic to each other and the socle appears once in their composition series, the homomorphism should be injective. Then our first assertion follows from the definition of good neighbors. The second assertion follows from the first assertion and (d) in Theorem 5.6. \(\Box\)

**Corollary 5.23.** For a pair \(p = (\alpha, \beta)\) with \(\text{gdist}_Q(p) = 1\), the module \(\to S_Q(p)\) has a composition length 2 such that
\[
[\to S_Q(p)] = [S_Q(\alpha) \circ S_Q(\beta)] + [S_Q(\beta) \circ S_Q(\alpha)].
\]

**Proof.** Our assertion follows from (iii) of Theorem 5.4, (c) of Theorem 5.6, Theorem 5.19 and the assumption that \(\text{gdist}_Q(p) = 1\). \(\Box\)
Example 5.24. Recall that for the pair $p = (111001, 123212)$ in Example 4.7, we have $\text{dist}_Q(p) = 1$ and $\text{gdist}_Q(p) = 2$. The module $\tilde{S}_Q(p)$ has its composition length larger than or equal to 4 by the following argument: For sequences
\[
m = (111101, 122111, 001001) \quad \text{and} \quad m' = (011001, 112101, 111111),\]
we already observe that
\[
\mathfrak{s} \prec_Q m \quad \text{and} \quad m' \prec_Q p.
\]
Furthermore $(122111, 001001)$ and $(12101, 11111)$ are $[Q]$-simple pairs. Thus one can check that $\tilde{S}_Q(m)$ and $\tilde{S}_Q(m')$ have their socles as $\tilde{S}_Q(\mathfrak{s})$ and there exist non-zero homomorphisms $\tilde{S}_Q(m) \rightarrow \tilde{S}_Q(p)$ and $\tilde{S}_Q(m') \rightarrow \tilde{S}_Q(p)$. Since the socles of $\tilde{S}_Q(m)$, $\tilde{S}_Q(m')$ and $\tilde{S}_Q(p)$ are the same as $\tilde{S}_Q(\mathfrak{s})$, the non-zero homomorphisms are injective indeed. Thus the composition length of $\tilde{S}_Q(p)$ is larger than or equal to 4 and $\tilde{S}_Q(m)$ and $\tilde{S}_Q(m')$ have their composition length as 2.

6. Application to Quantum affine algebras

In this section, we first prove that our results in Section 5 hold also for the representation theory for quantum affine algebras through the KLR-type Schur-Weyl duality functor. Then we prove the denominator formulas for $U_q'(A_n^{(1)})$ and $U_q'(D_n^{(1)})$ can be read from $\Gamma_Q$ for any Dynkin quiver $Q$ of finite type $A_n$ and $D_n$. In the last part of section, we shall propose several conjectures on $U_q'(E_n^{(1)})$.

6.1. Quantum affine algebras and categorifications. Let $A$ be a generalized Cartan matrix of affine type; i.e., $A$ is positive semi-definite of corank 1. We choose $0 \in I := \{0, 1, \ldots, n\}$ as the leftmost vertices in the tables in [14, pages 54, 55] except $A_{2n}^{(2)}$-case in which we take the longest simple root as $\alpha_0$. We set $I_0 := I \setminus \{0\}$. We denote by ACD the affine Cartan datum associated to the affine Cartan matrix $A$. Then the weight lattice $P$ is given as follows:
\[
P = \left( \bigoplus_{i \in I} \mathbb{Z} \Lambda_i \right) \oplus \mathbb{Z} \delta,
\]
where $\delta := \sum_{i \in I} d_i \alpha_i$ denotes the imaginary root. We also denote by $c := \sum_{i \in I} c_i h_i$ the center.

We denote by
\begin{itemize}
  \item $g$ the affine Kac-Moody algebra associated to ACD,
  \item $g_0$ the subalgebra of $g$ generated by $e_i, f_i$ and $h_i$ for $i \in I_0$,
  \item $U_q'(g)$ the quantum group associated to the affine Cartan datum ACD,
  \item $U_q'(g)$ the subalgebra of $U_q'(g)$ generated by $e_i, f_i$ and $K_i^{\pm 1}$ for all $i \in I$.
\end{itemize}

Note that $g_0$ is a finite simple Lie algebra. We call $U_q'(g)$ the quantum affine algebra.

For the rest of this paper, we take the field $k$, the algebraic closure of $\mathbb{C}(q)$ in $\cup_{m>0} \mathbb{C}((q^{1/m}))$, as the base field of $U_q'(g)$-modules.

Set $P_{cl} := P/\mathbb{Z} \delta$ and call it the classical weight lattice. We say that a $U_q'(g)$-module $M$ is integrable if
\begin{itemize}
  \item[(i)] it is $P_{cl}$-graded; i.e.,
  \[
  M = \bigoplus_{\lambda \in P_{cl}} M_\lambda \quad \text{where} \quad M_\lambda = \{ u \in M \mid K_i u = q_i^{(h_i, \lambda)} u \quad \text{for all} \quad i \in I \},
  \]
\end{itemize}
(ii) for all \( i \in I \), \( e_i \) and \( f_i \) act on \( M \) locally nilpotently.

We denote by \( \mathcal{C}_g \) the category of finite-dimensional integrable \( U'_q(\mathfrak{g}) \)-modules. A simple module \( M \) in \( \mathcal{C}_g \) contains a non-zero vector \( u \) of weight \( \lambda \in \mathcal{P}_\text{cl} \) such that

- \( \langle c, \lambda \rangle = 0 \) and \( \langle h_i, \lambda \rangle \geq 0 \) for all \( i \in I_0 \),
- all the weights of \( M \) are contained in \( \lambda - \sum_{i \in I_0} Z \geq 0 \mathcal{P}_\text{cl}(\alpha_i) \).

Such a \( \lambda \) is unique and \( u \) is unique up to a constant multiple. We call \( \lambda \) the dominant extremal weight of \( M \) and \( u \) the dominant extremal weight vector of \( M \).

For \( M \in \mathcal{C}_g \) and \( x \), let \( M_x \) be the \( U'_q(\mathfrak{g}) \)-module with the actions of \( e_i, f_i, K_i \) replaced with \( x^\delta e_i, x^{-\delta} f_i, K_i \), respectively. Then \( M_x \) is contained in \( \mathcal{C}_g \) when \( x \in k \) and isomorphic to \( k[x, x^{-1}] \otimes M \) when \( x \) is an indeterminate.

For each \( i \in I_0 \), we set
\[
\varpi_i := \gcd(c_0, c_i)^{-1} \text{cl}(c_0 \Lambda_i - c_i \Lambda_0) \in \mathcal{P}_\text{cl}.
\]

Then there exists a unique simple \( U'_q(\mathfrak{g}) \)-module \( V(\varpi_i) \) in \( \mathcal{C}_g \) with its dominant extremal weight \( \varpi_i \) and its dominant extremal weight vector \( u_{\varpi_i} \), called the fundamental representation of weight \( \varpi_i \), satisfying the certain conditions (see [1, §1.3] for more detail). Moreover, there exist the left dual \( V(\varpi_i)^* \) and the right dual \( V(\varpi_i) \) of \( V(\varpi_i) \) with the following \( U'_q(\mathfrak{g}) \)-homomorphisms

\[
(6.1) \quad V(\varpi_i)^* \otimes V(\varpi_i) \xrightarrow{\text{tr}} k \quad \text{and} \quad V(\varpi_i) \otimes V(\varpi_i) \xrightarrow{\text{tr}} k
\]

where
\[
(6.2) \quad V(\varpi_i)^* := V(\varpi_i^\rho)^*_{\rho^*}, \quad V(\varpi_i)^* := V(\varpi_i^\rho)_{\rho^*} \quad \text{and} \quad p^* := (-1)^{\langle \rho^\vee, \delta \rangle} q^{\langle \rho, \delta \rangle}.
\]

Here \( \rho \) is defined by \( \langle h_i, \rho \rangle = 1 \), \( \rho^\vee \) is defined by \( \langle \rho^\vee, \alpha_i \rangle = 1 \) and \( * \) is the involution on \( I_0 \).

We say that a \( U'_q(\mathfrak{g}) \)-module \( M \) is good if it has a bar involution, a crystal basis with simple crystal graph, and a global basis (see [23] for precise definitions). For instance, \( V(\varpi_i)_x \) is a good module for every \( i \in I \) and \( x \in k^\times \). Note that every good module is a simple \( U'_q(\mathfrak{g}) \)-module and real, i.e., \( M \otimes M \) is simple again.

**Definition 6.1.** [13] Let \( Q \) be a Dynkin quiver of finite type \( A_n, D_n \) or \( E_n \), and \( U'_q(\mathfrak{g}) \) be the quantum affine algebra of type \( A_n^{(1)}, D_n^{(1)} \) or \( E_n^{(1)} \), respectively. For any positive root \( \beta \in \Phi^+ \) associated to \( g_0 \), we define the \( U'_q(\mathfrak{g}) \)-module \( V_Q^{(1)}(\beta) \) in \( \mathcal{C}_g \) as follows:

\[
(6.3) \quad V_Q^{(1)}(\beta) := V(\varpi_i)_{(-q)p} \quad \text{where} \quad \phi_Q^{(1)}(\beta, 0) = (i, p).
\]

We define the smallest abelian full subcategory \( \mathcal{C}_Q^{(1)} \) inside \( \mathcal{C}_g \) such that

- (a) it is stable by taking subquotient, tensor product and extension,
- (b) it contains \( V_Q^{(1)}(\beta) \) for all \( \beta \in \Phi^+ \).

**Theorem 6.2.** [13] Keeping the notations in Definition 6.1, we have a ring isomorphism given as follows:

\[
(6.4) \quad \left[ \mathcal{C}_Q^{(1)} \right] \simeq U_A^{-}(g_0)_{q=1}^\vee.
\]

Here \( \left[ \mathcal{C}_Q^{(1)} \right] \) denotes the Grothendieck ring of \( \mathcal{C}_Q^{(1)} \).

Now we recall one of the main results of [19] which can be understood as the twisted version of Theorem 6.2 for types \( A_n \) and \( D_n \).
**Definition 6.3.** [19] Let $Q$ be a Dynkin quiver of finite type $A_n$ or $D_n$, and $U'_q(g)$ be the quantum affine algebra of type $A^{(2)}_n$ or $D^{(2)}_n$, respectively. For any positive root $\beta$ contained in $\Phi^+$ associated to $g_0$ of the finite type, we set the $U'_q(g)$-module $V_Q(\beta)$ defined as follows:

For $\phi^{-1}_Q(\beta,0) = (i,p)$, we define

\[(6.5)\]

\[V_Q^{(2)}(\beta) := V(\varpi_{i*})(-q)^p,\]

where $i^*$ and $(-q)^p$ are given as follows:

\[(6.6)\]

\[(i^*,(-q)^p)^{\ast} := \begin{cases} 
(i,(-q)^p) & \text{if } g = A_n \text{ and } 1 \leq i \leq \left\lfloor \frac{n+1}{2} \right\rfloor, \\
(n+1-i,(-1)^n(-q)^p) & \text{if } g = A_n \text{ and } \left\lceil \frac{n+1}{2} \right\rceil \leq i \leq n, \\
(i,\sqrt{-1})^{n-i}(-q)^p & \text{if } g = D_n \text{ and } 1 \leq i \leq n-2, \\
(n-1,(-1)^i(-q)^p) & \text{if } g = D_n \text{ and } n-1 \leq i \leq n.
\end{cases}\]

We define the smallest abelian full subcategory $C^{(2)}_Q$ inside $C_g$ such that

(a) it is stable by taking subquotient, tensor product and extension,

(b) it contains $V_Q^{(2)}(\beta)$ for all $\beta \in \Phi^+$.

**Theorem 6.4.** [19] There exist a ring isomorphism

\[(6.7)\]

\[U_q^{-}(g_0)^{\mathcal{V}}|_{q=1} \simeq \left[ C^{(2)}_Q \right].\]

Hence we have

\[(6.8)\]

\[\left[ C^{(1)}_Q \right] \simeq U_q^{-}(g_0)^{\mathcal{V}}|_{q=1} \simeq \left[ C^{(2)}_Q \right].\]

6.2. Denominators and KLR-type Schur-Weyl duality. For a good module $M$ and $N$, there exist a $U'_q(g)$-homomorphism

\[R_{M,N}^{\text{norm}} : M_{z_M} \otimes M_{z_N} \to k(z_M,z_N) \otimes k[z^\pm 1, z_M^\pm 1] N_{z_N} \otimes M_{z_M}\]

such that

\[R_{M,N}^{\text{norm}} \circ z_M = z_M \circ R_{M,N}^{\text{norm}}, \quad R_{M,N}^{\text{norm}} \circ z_N = z_N \circ R_{M,N}^{\text{norm}} \text{ and } R_{M,N}^{\text{norm}}(u_M \otimes u_N) = u_N \otimes u_M,\]

where $u_M$ (resp. $u_N$) is the dominant extremal weight vector of $M$ (resp, $N$).

The denominator $d_{M,N}$ of $R_{M,N}^{\text{norm}}$ is the unique non-zero monic polynomial $d(u) \in k[u]$ of smallest degree such that

\[(6.9)\]

\[d_{M,N}(z_{N}/z_{M})R_{M,N}^{\text{norm}}(M_{z_M} \otimes N_{z_N}) \subset (N_{z_N} \otimes M_{z_M}).\]

**Theorem 6.5.** [1, ?, 23]

1. For good modules $M_1$ and $M_2$, the zeros of $d_{M_1,M_2}(z)$ belong to $\mathbb{C}[[q^{1/m}]] q^{1/m}$ for some $m \in \mathbb{Z}_{>0}$.

2. $V(\varpi_i)_{a_i} \otimes V(\varpi_j)_{a_j}$ is simple if and only if

\[d_{i,j}(z) := d_{V(\varpi_i),V(\varpi_j)}(z)\]

does not have zeros at $z = a_i/a_j$ or $a_j/a_i$. 
(3) Let \( M \) be a finite dimensional simple integrable \( U_q'(\mathfrak{g}) \)-module. Then, there exists a finite sequence

\[
((i_1, a_1), \ldots, (i_l, a_l)) \quad \text{in} \quad (I_0 \times \mathbb{K}^\times)^l
\]

such that \( d_{i_k, i_{k'}}(a_k/a_{k'}) \neq 0 \) for \( 1 \leq k < k' \leq l \) and \( M \) is isomorphic to the head of \( \bigotimes_{k=1}^l V(w_{i_k})a_{i_k} \). Moreover, such a sequence \( ((i_1, a_1), \ldots, (i_l, a_l)) \) is unique up to permutation.

(4) \( d_{i_k, l}(z) = d_{l, k}(z) = d_{k^*, l^*}(z) = d_{l^*, k^*}(z) \) for \( k, l \in I_0 \).

From the above theorem, one can notice that the denominator formulas between fundamental representations provides crucial information of the representation theory on \( U \) consisting of (a) a quantum affine algebra \( \mathcal{A} \), (b) a symmetric Cartan matrix \( \mathbf{A} \), and (c) a family of good \( U_q'(\mathfrak{g}) \)-modules \( \{V_s\} \). For a given \( \mathcal{A} \), we define a quiver \( \Gamma^{SWD} \) in the following way:

(1) \( \Gamma_0^{SWD} = J \).

(2) For \( i, j \in J \), we assign \( d_{ij} \) many arrows from \( i \) to \( j \), where \( d_{ij} \) is the order of the zero of \( d_{V_{ij}(i, V_{ij})}((z_2/z_1)^{i, j}) \) at \( X(j)/X(i) \).

We call \( \Gamma^{SWD} \) the (KLR-type) Schur-Weyl quiver associated to \( \mathcal{A} \).

For a Schur-Weyl datum \( \mathcal{A} \), we have

- \( \mathcal{A}^{SWD} = (a_{ij}^{SWD})_{i, j \in J} \) by

\[(6.10)\]

\[
a_{ij}^{SWD} = 2 \quad \text{if} \quad i = j, \quad a_{ij}^{SWD} = -d_{ij} - d_{ji} \quad \text{if} \quad i \neq j,
\]

- the set of polynomials \( (q_{ij}^{SWD}(u, v))_{i, j \in J} \)

\[q_{ij}^{SWD}(u, v) = (u - v)^{d_{ij}}(v - u)^{d_{ji}} \quad \text{if} \quad i \neq j.
\]

We denote by \( R^{SWD} \) the symmetric quiver Hecke algebra associated with \( (q_{ij}^{SWD}(u, v)) \).

**Theorem 6.6.** [15] For a given \( \mathcal{A} \), there exists a functor

\[
\mathcal{F} : \text{Rep}(R^{SWD}) \to \mathcal{C}_q,
\]

Moreover, \( \mathcal{F} \) satisfies the following properties:

(a) \( \mathcal{F} \) is a tensor functor; that is, there exist \( U_q'(\mathfrak{g}) \)-module isomorphisms

\[
\mathcal{F}(R^{SWD}(0)) \simeq \mathbb{K} \quad \text{and} \quad \mathcal{F}(M_1 \circ M_2) \simeq \mathcal{F}(M_1) \otimes \mathcal{F}(M_2)
\]

for any \( M_1, M_2 \in \text{Rep}(R^{SWD}) \).

(b) If the Schur-Weyl quiver \( \Gamma^{SWD} \) is a Dynkin quiver of finite type \( A_n, D_n \) or \( E_n \), then \( \mathcal{F} \) is exact.

We call the functor \( \mathcal{F} \) the generalized quantum affine Schur-Weyl duality functor.
Theorem 6.7. [16, 19] Let $U_q'(\mathfrak{g})$ be a quantum affine algebra of type $A_n^{(1)}$, $A_{2n-1}^{(2)}$, or $A_{2n}^{(2)}$ (resp. $D_n^{(1)}$ or $D_{n+1}^{(2)}$) and $Q$ be a Dynkin quiver of finite type $A_n$, $A_{2n-1}$ or $A_{2n}$ (resp. $D_n$ or $D_{n+1}$). Take $J$ and $S$ as the set of simple roots $\Pi$ associated to $Q$. Set $J$ as the set of simple roots $\Pi$ associated to $Q$. We define two maps
\[ s : \Pi \to \{V(\varpi_i) \mid i \in \{1, \ldots, n\}\} \quad \text{and} \quad X : \Pi \to \mathbb{k}^x \]
as follows: For $\alpha \in \Pi$ with $\phi_Q^{-1}(\alpha, 0) = (i, p)$, we define
\[ s(\alpha) = \begin{cases} V(\varpi_i) & \text{if } g = A_n^{(1)} \text{ or } D_n^{(1)}; \\ V(\varpi_i^*) & \text{otherwise}, \end{cases} \]
and
\[ X(\alpha) = \begin{cases} (-q)^p & \text{if } g = A_n^{(1)} \text{ or } D_n^{(1)}; \\ (-q)^p & \text{otherwise}, \end{cases} \]
which are induced by the maps $\phi_Q$ and $\ast$.

(a) The quiver $\Gamma^{\text{SWD}}$ is isomorphic to $Q^{\text{rev}}$. Hence the functor
\[ F_Q^{(i)} : \text{Rep}(R^{\text{SWD}}) \to C_Q^{(i)} \quad (i = 1, 2) \]
in Theorem 6.6 is exact.

(b) The functor $F_Q^{(i)}$ sends simples to simples, bijectively. In particular, $F_Q^{(i)}$ sends
\[ S_Q(\beta) \text{ to } V_Q^{(i)}(\beta), \]
for $\beta, \alpha \in \Pi$.

(c) The functors $F_Q^{(1)}$ and $F_Q^{(2)}$ induce the ring isomorphisms in (6.8). Moreover, the induced functor between $C_Q^{(1)}$ and $C_Q^{(2)}$ preserves dimensions and sends simples to simples, bijectively.

By applying the exact functors in Theorem 6.7, the results in previous section holds for $C_Q^{(i)}$ of affine types $A_n^{(1)}$ and $D_n^{(1)}$ also.

Theorem 6.8. Let $Q$ be a Dynkin quiver of finite type $A_n$ or $D_n$, $i \in \{1, 2\}$ and $\underline{m}$, $\underline{n}$ for some $\bar{\omega} \in [Q]$.

1. $\overrightarrow{V_Q^{(i)}(\underline{m})} := V_Q^{(i)}(\beta_1^{\underline{m}} \otimes \cdots \otimes \partial_1^{\underline{n}}) \otimes \cdots \otimes V_Q^{(i)}(\beta_{\underline{n}}^{\underline{m}}) \otimes \cdots \otimes V_Q^{(i)}(\partial_{\underline{n}}^{\underline{m}})$ is well-defined.

2. $[V_Q^{(i)}(\underline{m})] [\in \mathcal{F}_Q^{(i)}(\text{Im}(\text{r}_m))] + \sum_{w \in \mathbb{Z}^+} [\mathcal{F}_Q^{(i)}(\text{Im}(\text{r}_m))]$.

3. For any pair $\underline{p} = (\alpha, \beta)$, $\underline{m} \simeq \overrightarrow{V_Q^{(i)}(\underline{p})}$ is simple if and only if $\underline{m}$ is $[Q]$-simple.

4. $\overrightarrow{V_Q^{(i)}(\underline{m})} \simeq \overrightarrow{V_Q^{(i)}(\underline{m})}$ is simple if and only if $\underline{m}$ is $[Q]$-simple.

5. For any pair $\underline{p} = (\alpha, \beta)$ with $\text{dist}_Q(\underline{p}) \geq 1$, the composition length of $\overrightarrow{V_Q^{(i)}(\underline{p})}$ is larger than or equal to $\text{len}_Q(\underline{p}) + 2$. In particular, if $\text{gdist}_Q(\underline{p}) = 1$, the composition length of $\overrightarrow{V_Q^{(i)}(\underline{p})}$ is equal to 2.

Example 6.9. From Theorem 6.8 and (b) of Example 5.20, we have the following injective $U_q'(D^{(1)}_5)$-homomorphism:
\[ V(\varpi_1)(-q) \otimes V(\varpi_4) \otimes V(\varpi_5) \Rightarrow V(\varpi_3)(-q)\otimes V(\varpi_2)(-q)^2. \]
Moreover, the composition length of $V(\varpi_3)(-q)\otimes V(\varpi_2)(-q)^2$ is equal to 2.
Remark 6.10. From Theorem 6.5, 6.6 and 6.7, we can observe the following:

For a Dynkin quiver $Q$ of finite type $A_n$ or $D_n$,

the pair $(\alpha, \beta)$ is not $[Q]$-simple $\iff$ $S_Q(\alpha) \circ S_Q(\beta)$ is reducible

$\iff \begin{cases} d_{k,l}(z) \text{ has a root at one of } \frac{(-q)^a}{(-q)^b} \text{ and } \frac{(-q)^b}{(-q)^a} \\
\quad \text{ if } i = 1, \\
\quad \frac{((-q)^a)^*}{((-q)^b)^*} \text{ and } \frac{((-q)^b)^*}{((-q)^a)^*} \text{ if } i = 2,
\end{cases}$

where $\phi_Q^{-1}(\alpha, 0) = (k, a)$ and $\phi_Q^{-1}(\beta, 0) = (l, b)$.

Note that the Schur-Weyl datum in Theorem 6.7 and the functors $\mathcal{F}_Q^{(1)}$ (resp. $\mathcal{F}_Q^{(2)}$) are determined by $(U'_q(g), \phi_Q^{-1})$ (resp. $(U'_q(g), \phi_Q^{-1})$) indeed.

Conjecture 1. [16, Conjecture 4.3.2] Let $Q$ be a Dynkin quiver of finite type $E_n$. Then the functor $\mathcal{F}_Q^{(1)}$ determined by $(U'_q(E_n^{(1)}), \phi_Q^{-1})$ enjoys the properties in Theorem 6.7.

By the lack of denominator formulas for the quantum affine algebra of type $E_n^{(1)}$, we do not know whether Conjecture 1 holds or not.

6.3. Distance polynomials on AR-quivers. In this subsection, we denote by $U'_q(g)$ the quantum affine algebra of untwisted type $A_n^{(1)}$, $D_n^{(1)}$ or $E_n^{(1)}$, and $Q$ the Dynkin quiver of finite type $A_n$, $D_n$ or $E_n$, respectively. Thus we take index sets $I = \{0, 1, \ldots, n\}$ and $I_0 = I \setminus \{0\}$, and the set of positive roots $\Phi^+$ associated to $Q$.

Now we shall develop the observation in Remark 6.10; that is, under the situation in Remark 6.10, we shall interpret the meaning of $\text{gdist}_Q(\alpha, \beta)$ in the denominator formula $d_{k,l}(z)$.

Conversely, we shall check the existence of a pair $(\alpha, \beta)$ and its generalized $[Q]$-distance having $\{\phi_Q^{-1}(\alpha, 0), \phi_Q^{-1}(\beta, 0)\} = \{(k, a), (l, b)\}$ with $|a - b| = t \in \mathbb{N}$, when $d_{k,l}(z)$ has an order $m$ root at $(-q)^t$.

Definition 6.11. For a Dynkin quiver $Q$, indices $k, l \in I_0$ and an integer $t \in \mathbb{N}$, we define the subset $\Phi_Q(k, l)[t]$ of $\Phi^+ \times \Phi^+$ as follows:

A pair $(\alpha, \beta)$ is contained in $\Phi_Q(k, l)[t]$ if $\alpha \prec_Q \beta$ or $\beta \prec_Q \alpha$ and

$\{\phi_Q^{-1}(\alpha, 0), \phi_Q^{-1}(\beta, 0)\} = \{(k, a), (l, b)\}$ such that $|a - b| = t$.

Lemma 6.12. For any $(\alpha^{(1)}, \beta^{(1)})$ and $(\alpha^{(2)}, \beta^{(2)})$ in $\Phi_Q(k, l)[t]$, we have

$\phi^t_Q(k, l) := \text{gdist}_Q(\alpha^{(1)}, \beta^{(1)}) = \text{gdist}_Q(\alpha^{(2)}, \beta^{(2)})$.

Proof. Note that for $\alpha \prec_Q \gamma \prec_Q \beta$,

$\tau_Q(\beta) \in \Phi^+$ implies $\tau_Q(\gamma) \in \Phi^+$ and $\tau_Q^{-1}(\alpha) \in \Phi^+$ implies $\tau_Q^{-1}(\gamma) \in \Phi^+$.

Since, for triples $(\alpha^{(1)}, \beta^{(1)}; Q), (\alpha^{(2)}, \beta^{(2)}; Q) \in \Phi_Q(k, l)[t]$, there exists $k \in \mathbb{Z}$ such that $\tau_Q^k(\alpha^{(1)}) = \alpha^{(2)}$ and $\tau_Q^k(\beta^{(1)}) = \beta^{(2)}$, we have

$\text{gdist}_Q(\alpha^{(1)}, \beta^{(1)}) = \text{gdist}_Q(\alpha^{(2)}, \beta^{(2)})$. 

$\square$
From the definition, we have \( \alpha^Q_t(k, l) = \alpha^Q_t(l, k) \).

**Lemma 6.13.** For \( k, \ell \in I_0 \) and \( t \in \mathbb{Z} \), we have
\[
\alpha^Q_t(k, l) = \alpha^Q_{t^*}(l^*, k^*) \quad \text{and} \quad \alpha^{Q^*}_t(k, l) = \alpha^{Q^*}_{t^*}(l^*, k^*).
\]

**Proof.** Note that for \( \alpha <_Q \beta \), we have
\[
\beta <_{Q^{rev}} \alpha \quad \text{and} \quad (\phi^{-1}_{Q,1}(\alpha))^* = \phi^{-1}_{Q^{rev},1}(\alpha).
\]
Thus our assertion follows. \( \square \)

**Lemma 6.14.** For \( k, \ell \in I_0 \) and \( t \in \mathbb{Z} \), we have
\[
\alpha^Q_t(k, l) = \alpha^Q_{t^*}(l, k) \quad \text{and} \quad \alpha^{Q^*}_t(k, l) = \alpha^{Q^*}_{t^*}(l, k).
\]

**Proof.** Note that for \( \alpha <_Q \beta \), we have
\[
-w_0(\beta) <_{Q^{rev}} -w_0(\alpha) \quad \text{and} \quad \phi^{-1}_{Q,1}(\alpha) = \phi^{-1}_{Q^{rev},1}(-w_0(\alpha)).
\]
Thus our assertion follows. \( \square \)

From the above lemmas, we have also \( \alpha^Q_t(k, l) = \alpha^{Q^*}_t(k^*, l^*) \).

**Definition 6.15.** For \( k, l \in I_0 \) and a Dynkin quiver \( Q \), we define a polynomial \( D^Q_{k, l}(z; -q) \in k[z] \) as follows:
\[
D^Q_{k, l}(z; -q) := \prod_{t \in \mathbb{Z}_{\geq 0}} (z - (-q)^t)^{\phi^Q_t(k, l)},
\]
where \( \phi^Q_t(k, l) := \max(\alpha^Q_t(k, l), \alpha^{Q^*}_t(k, l)) \).

Note that
\[
\tag{6.11}
\begin{align*}
(a) & \quad \phi^Q_t(k, l) = 0 \quad \text{for} \quad 0 \leq t < d(k, l) + 2, \\
(b) & \quad D^Q_{k, l}(z; -q) = D^Q_{l, k}(z; -q) = D^{Q^*}_{l^*, k^*}(z; -q) = D^{Q^*}_{k^*, l^*}(z; -q),
\end{align*}
\]
by the definition of \( \phi^Q_t(k, l) \) and (a) of Proposition 4.2.

**Proposition 6.16.** For \( k, l \in I_0 \) and any Dynkin quivers \( Q \) and \( Q' \), we have
\[
D^Q_{k, l}(z; -q) = D^{Q'}_{k, l}(z; -q).
\]

**Proof.** By (2.1) and (2.8), for Dynkin quivers \( Q \) and \( Q' \), we have
\[
\alpha^Q_t(k, l) = \alpha^{Q'}_t(k, l) \quad \text{if} \quad \Phi_Q(k, l)[t], \Phi_{Q'}(k, l)[t] > 0.
\]
To prove our assertion, it suffices to prove that
\[
\phi^Q_t(k, l) = \phi^{Q'}_{Q,1}(k, l) \quad \text{for every source} \quad i \quad \text{of} \quad Q,
\]
since every Dynkin quiver \( Q' \) can be obtained by applying sequence of the reflection functors \( r^- \) properly.

Assume that \( a = \phi^Q_t(k, l) > 0 \); that is \( (\alpha, \beta) \in \Phi_Q(k, l)[t] \) or \( \Phi_{Q^*}(k, l)[t] \) such that \( \text{gdist}_Q(\alpha, \beta) = a \) or \( \text{gdist}_{Q^{rev}}(\alpha, \beta) = a \).

Without loss of generality, set \( (\alpha, \beta) \in \Phi_Q(k, l)[t], \phi^{-1}_{Q,1}(\alpha) = k, \phi^{-1}_{Q,1}(\beta) = l \) and \( \alpha <_Q \beta \).
If $\alpha \neq \alpha_k$, then $s_i(\alpha), s_i(\beta) \in \Phi^+$ for every source $i$ of $Q$ and hence $\text{gdist}_Q(\alpha, \beta) = \text{gdist}_{k, l}^+(s_i(\alpha), s_i(\beta)) = a$. Thus $\phi_i^+(k, l) = \phi_i^+ \circ s_i^+(Q)(k, l)$ for every source $i$ of $Q$.

If $|\Phi_Q(k, l)[t]| > 1$, then we have $\phi_i^+(k, l) = \phi_i^+ \circ s_i(\Phi_Q(k, l))$ for every source $i$ of $Q$ by the same reason.

Now assume that

(6.12) (1) $k$ is a source of $Q$ and $\alpha = \alpha_k$, (2) $\Phi_Q(k, l)[t] = \{(\alpha_k, \beta)\}$ and $\phi_i^+(k, l) > 0$.

Since $(\alpha, \beta)$ are not directly $Q$-connected, we have

(6.13)

\begin{itemize}
  \item $\tau_Q(\alpha_k) \in \Phi^+$ and $\beta = \theta_{1, l}^Q$;
  \item $\phi_{Q, l}^{-1}(\alpha) - \phi_{Q, l}^{-1}(\beta) = |k - l| + 2s = t$ for some $1 \leq s \leq m_l^Q$.
\end{itemize}

$(Q$ is of finite type $A_n$, $k \neq l^*)$ In this case, the Dynkin quiver $Q$ is the one of the following forms:

(6.14) $Q: (a) \quad \circ \quad k \quad \circ \quad k+1 \quad \cdots \quad \circ \quad l^* \quad \circ \quad (b) \quad \circ \quad k-1 \quad \circ \quad k \quad \circ \quad k+1 \quad \cdots$.

by (2.4) and (2.5). Hence we have $\gamma = \alpha_k + \theta_{l^*}^Q \in \Phi^+$, $\text{mul}(\gamma) = 1$ and $\text{dist}_Q(\alpha_k, \beta) = 1$.

Otherwise,

$$\text{supp}(\theta_{l^*}^Q) \cap k = \emptyset \quad \text{and} \quad \theta_{l^*}^Q + \alpha_k \notin \Phi^+$$

and we have a contradiction to the assumption that $gdist_Q(\alpha, \beta) > 0$.

By Proposition 4.5, $\alpha$ and $\beta$ are located at one of the following form:

(6.15)

Then the Dynkin quivers $s_k(Q)$ and $(s_k(Q))^{rev}$ are give as follows:

$s_k(Q): (a) \quad \circ \quad k-1 \quad \circ \quad k \quad \circ \quad k+1 \quad \cdots \quad \circ \quad l^* \quad \circ \quad (b) \quad \circ \quad k-1 \quad \circ \quad k \quad \circ \quad k+1 \quad \cdots$.

$(s_k(Q))^{rev}: (a) \quad \circ \quad k \quad \circ \quad k+1 \quad \cdots \quad \circ \quad l^* \quad \circ \quad (b) \quad \circ \quad k-1 \quad \circ \quad k \quad \circ \quad k+1 \quad \cdots$.

Hence there exist no intersection between (a) the $N$-sectional (resp. (b) $S$-sectional) path containing $\theta_{l^*}^{s_k(Q^{rev})}$ and (a) the $S$-sectional (resp. (b) $N$-sectional) path containing $\alpha_k$. Thus there exists $x \in \mathbb{Z}_{>0}$ such that $\tau_{s_k(Q^{rev})}^{-x}(\theta_{l^*}^{(Q^{rev})})$ and $\alpha_k$ are of the form (6.15).

$(Q$ is of finite type $A_n$, $k = l^*)$ In this case, by (6.13), we have

$$\left(\gamma_{s_k(Q)}^{s_k(Q)}, \alpha_k\right) \in \Phi_{s_k(Q)}(k, l)[t]$$

and hence our assertion follows.

$(Q$ is of finite type $D_n$, $1 \leq k, l \leq n - 2$) In this case, the Dynkin diagram $Q$ is one of the forms in (6.14), $k^* = k$ and $l^* = l$. The case when $k = l$ is trivial. Thus we assume that
44 SE-JIN OH

\( k \neq l \). Moreover, \( \alpha = \varepsilon_k - \varepsilon_{k+1} \beta = \theta_l^Q, \alpha + \beta \in \Phi^+ \) and \( \text{mul}(\alpha + \beta) = 1 \). Thus \( \Theta_l^Q(k, l) = 1 \) and \( \Gamma_Q \) can be depicted as follows:

\[
(6.16)
\]

By Theorem 3.11, all positive roots in the shallow \( N \)-sectional path containing \( \alpha \) shares \( -\varepsilon_{k+1} \) as their summand, and all positive roots in the shallow \( S \)-sectional path containing \( \beta \) are contained in \( \Gamma_Q \) indeed. As in the \( A_n \)-case, \( k \) becomes a sink of \( s_k(Q) \) and \( \Gamma(s_k(Q)) \) can be depicted as follows:

\[
(6.17)
\]

by Theorem 3.10. Thus there exist \( \beta' \in \Phi^+ \) such that \( (\beta', \alpha) \in \Phi_{s_k(Q)}(k, l)[t] \).

(Q is of finite type \( D_n \), \(|\{k, l\} \cap \{n - 1, n\}| = 1\)). We shall prove only for \( 1 \leq k \leq n - 2 \) and \( l \in \{n - 1, n\} \). For the case when \( 1 \leq l \leq n - 2 \) and \( k \in \{n - 1, n\} \), we can apply the similar strategy. In this case, the Dynkin diagram \( Q \) is of the following form:

Furthermore, \( \Gamma_Q \) can be depicted as follows:

where \( \beta \) is one of the \( \circ \)'s. Then \( \Gamma(s_k(Q)) \) can be depicted as follows:
Thus there exist \( \beta' \in \Phi^+ \) such that \((\beta', \alpha) \in \Phi_{s_k(Q)}(k, l)[t]\) where \( \beta' \) is one of the \( \circ \)'s.

\( Q \) is of finite type \( D_n, \{k, l\} \subset \{n-1, n\} \) Note that \( k \) (resp. \( l \)) is a sink or a source. Thus the Dynkin quiver is of the following form:

![Dynkin quiver diagram]

Then \( t = h^\vee - 2 \) and \( |\Phi_Q(k, l)[t]| = 2 \) by \( (3.2) \). Then it contradicts our assumption \( (6.12) \). \( \square \)

From the above proposition, we can define \( D_{k,l}(z) \) in a natural way and call it the distance polynomial at \( k \) and \( l \).

Now, we recall the denominator formulas for \( U'_q(g) \) here.

**Theorem 6.17.** \([1, 16] \)

(a) \( d_{k,l}^{A_n^{(1)}}(z) = \prod_{x=1}^{\min(k,l,n+1-k,n+1-l)} (z - (-q)^{k-l+2x}) \).

(b) \( d_{k,l}^{D_n^{(1)}}(z) = \begin{cases} \prod_{x=1}^{\min(k,l)} (z - (-q)^{k-l+2x})(z - (-q)^{2n-2-k-l+2x}) & 1 \leq k, l \leq n-2, \\ \prod_{x=1}^{k} (z - (-q)^{n-k-2x}) & 1 \leq k \leq n-2, \ l \in \{n-1, n\}, \\ \prod_{x=1}^{l} (z - (-q)^{4x-2}) & k = l \in \{n-1, n\}. \end{cases} \)

In particular, \( d_{k,l}^{D_n^{(1)}}(z) \) has a zero of multiplicity 2 at \( z = (-q)^s \) when

\[(6.18) \quad 2 \leq k, l \leq n-2, \ k + l > n - 1, \ 2n - k - l \leq s \leq k + l \text{ and } s \equiv k + l \mod 2.\]

**Theorem 6.18.** For any Dynkin quiver \( Q \) of finite type \( A_n \) (resp. \( D_n \)), the denominator formulas for \( U'_q(A_n^{(1)}) \) (resp. \( U'_q(D_n^{(1)}) \)) can be read from \( \Gamma_Q \) and \( \Gamma_{Q^{rev}} \) as follows:

\[d_{k,l}(z) = D_{k,l}(z; -q) \times (z - (-q)^{h^\vee})^{\delta_{k,l}}\]

where \( h^\vee \) is the dual Coxeter number of \( A_n \) (resp. \( D_n \)).

**Proof.** By Proposition 6.16, It is enough to show our assertion for a fixed Dynkin quiver \( Q \). Among \( 2^{n-1} \)-many Dynkin quivers, we choose a canonical one whose height function \( \xi^Q \) is given by \( \xi^Q_i = n - d(1, i) \); i.e.,

\[
(6.19) Q : \begin{array}{cccccccccc}
1 & 2 & \cdots & n-1 & n \\
\_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ \end{array}
\]

if \( Q \) is of finite type \( A_n \),

\[
(6.19) Q : \begin{array}{cccccccccc}
1 & 2 & \cdots & n-2 & n \\
\_ & \_ & \_ & \_ & \_ \end{array}
\]

if \( Q \) is of finite type \( D_n \).
(Q of finite type $A_n$) The Auslander-Reiten quivers $\Gamma_Q$ and $\Gamma_{Q^{rev}}$ are given as follows:

\begin{equation}
\begin{array}{c}
\text{[n]} \quad \ldots \quad [n-1, n] \quad \llap{\ldots} \quad [2, 3] \quad [2, 1, 2, [1, n-1, 1, n] \quad [1, n] \quad [2, n] \quad [2, 1, 2, [1, n-1, 1, n] \quad [1, n] \quad [2, n] \\
\Gamma_Q \quad \llap{\ldots} \quad [2, n] \quad [1, n-1] \quad [1, 2] \quad [2, n] \quad [2, n] \quad [1, n-1] \quad [1, n] \quad [2, n]
\end{array}
\end{equation}

Thus $m_k^Q = n - k$ and $m_k^{Q^{rev}} = k - 1$ for $1 \leq k \leq n$.

Recall that, for any Dynkin quiver $Q'$ of finite type $A_n$, we have $0 \leq \text{gdist}_{Q'}(\alpha, \beta) \leq 1$. In particular, if $\text{gdist}_{Q'}(\alpha, \beta) = 1$, $\alpha, \beta$ is located in $Q'$ satisfying one of the following forms:

\begin{equation}
\begin{array}{c}
\text{[1]} \quad \ldots \quad 2 \quad \llap{\ldots} \quad (k, p) \quad \alpha + \beta \quad (k, p) \quad \beta \quad \alpha \\
\text{(a)} \quad \llap{\ldots} \quad \alpha + \beta \quad (l, q) \quad (l, q) \quad (l, q) \quad (l, q)
\end{array}
\end{equation}

for $\phi_{Q'}^{-1}(\alpha, 0) = (k, p)$ and $\phi_{Q'}^{-1}(\beta, 0) = (l, q)$. Then one can see that

- (a) and (b) are contained in $Q$,
- (b) and (c) are contained in $Q^{rev}$.

Furthermore, there exists $1 \leq s \leq \min(k, l, n + 1 - k, n + 1 - l)$ such that the length between $\alpha$ and $\beta$, that is $|p - q|$, is equal to $|k - l| + 2s$. Conversely, for $1 \leq s \leq \min(k, l, n + 1 - k, n + 1 - l)$, we can draw one of (a) \sim (c) in $\Gamma_Q$ or $\Gamma_{Q^{rev}}$ satisfying whose length between $\alpha$ and $\beta$ is $|a - b| = |k - l| + 2s$, except when $k = l$ and $|a - b| = h'$ (see Remark 2.4).

Thus our assertion follows.

(Q of finite type $D$) The Auslander-Reiten quivers $\Gamma_Q$ and $\Gamma_{Q^{rev}}$ are given as follows:

\begin{equation}
\begin{array}{c}
\text{(Q of finite type $D$) The Auslander-Reiten quivers $\Gamma_Q$ and $\Gamma_{Q^{rev}}$ are given as follows:}
\end{array}
\end{equation}

Thus $m_k^Q = m_k^{Q^{rev}} = 2n - 4$ for $1 \leq k \leq n$.

Recall that, for any Dynkin quiver $Q'$ of finite type $D$, we have $0 \leq \text{gdist}_{Q'}(\alpha, \beta) \leq 2$. 

(1) If $\operatorname{gdist}_{Q'}(\alpha, \beta) = 1$, $\alpha$ and $\beta$ are located in $Q'$ satisfying one of the following forms:

![Diagram](image_url)

where $\phi_{Q'}^{-1}(\alpha, a) = k$, $\phi_{Q'}^{-1}(\beta) = (l, b)$ and $k \leq l$.

(2) If $\operatorname{gdist}_{Q'}(\alpha, \beta) = 2$, $\alpha$ and $\beta$ are located in $Q'$ satisfying one of the following forms:

![Diagram](image_url)

where $\phi_{Q'}^{-1}(\alpha) = (k, a)$, $\phi_{Q'}^{-1}(\beta) = (l, b)$ and $k \leq l$. 
By choosing \( \alpha = \gamma_Q^Q \), one can check that \((\alpha, \beta)\) in (i) \(\sim\) (ix) is contained in \(\Phi_Q(k, l)[[a - b]]\) when \(k \leq l\). For the case when \(k \geq l\), one can easily check that \((\alpha, \beta)\) in (i) \(\sim\) (ix) is contained in \(\Phi_{Q^\text{rev}}(k, l)[[a - b]]\), by choosing \(\beta = \theta_Q^Q\).

Furthermore, the length of a path between \(\alpha\) and \(\beta\), \(s:=|a-b|\), satisfies the condition in (6.18) only when \((\alpha, \beta)\) is one of the forms (viii) \(\sim\) (ix).

Conversely, we can draw one of the diagrams (i) \(\sim\) (ix) in \(\Gamma_Q\) or \(\Gamma_{Q^\text{rev}}\) satisfying whose length \(|a - b|\) between \(\alpha\) and \(\beta\), \((\phi_Q^{-1}(\alpha) = (k, a), \phi_Q^{-1}(\beta) = (l, b)\) or \(\phi_{Q^\text{rev}}^{-1}(\alpha) = (k, a), \phi_{Q^\text{rev}}^{-1}(\beta) = (l, b)\)) is given as

\[
|a - b| = \begin{cases} 
|k - l| + 2x & \text{if } 1 \leq k, l \leq n - 2 \text{ and } 1 \leq x \leq \min(k, l), \\
2n - 2 - k - l + 2x & \text{if } 1 \leq k \leq n - 2, l \in \{n - 1, n\} \text{ and } 1 \leq k \leq x, \\
n - k - 1 + 2x & \text{if } \{k, l\} = \{n, n - 1\} \text{ and } 1 \leq x \leq \left\lfloor \frac{a - 1}{2} \right\rfloor, \\
4x & \text{if } \{k, l\} = \{n, n - 1\} \text{ and } 1 \leq x \leq \left\lfloor \frac{a}{2} \right\rfloor,
\end{cases}
\]

except when \(k = l^*\) and \(|a - b| = h^y\) (see Remark 2.4). Thus our assertion follows. □

**Theorem 6.19.** Let \(g\) be a quantum affine algebra of type \(A_n^{(1)}\) or \(D_n^{(1)}\). For any \(i, j \in I_0\) and \(x, y \in k\), we can compute the simple socle or the simple head of the tensor product of two fundamental representations

\[ V := V(\varpi_i)_x \otimes V(\varpi_j)_y \]

as a \(U'_q(g)\)-module. Moreover, it is a tensor product of fundamental representations.

**Proof.** By Theorem 6.5 and Theorem 6.17, we suffice to consider that \(x/y := (-q)^a \in \mathbb{Z}[q, q^{-1}]^\times\) for some \(a \in \mathbb{Z} \setminus \{0\}\) and hence can assume that \(V\) is reducible. Theorem 6.18 tells that there exist a Dynkin quiver \(Q\) and positive roots \(\alpha, \beta \in \Phi^+\) such that \(V(1)(\varpi_i)_x = V_Q(\alpha)\) and \(V(1)(\varpi_j)_y = V_Q(\beta)\) (up to parameter shift) unless \(i = j^*\) and \(|a| = h\). By (6.1) and Theorem 6.8, we have

\[
\begin{align*}
\text{hd}(V) &\simeq V_Q(\mathfrak{a}) & \text{if } a < 0 \text{ and } a \neq -h, \\
\text{soc}(V) &\simeq V_Q(\mathfrak{a}) & \text{if } a > 0 \text{ and } a \neq h, \\
\text{hd}(V) &\simeq k & \text{if } i = j^* \text{ and } a = -h, \\
\text{soc}(V) &\simeq k & \text{if } i = j^* \text{ and } a = h,
\end{align*}
\]

where \(\mathfrak{a}\) is equal to \(\text{soc}_Q(\alpha, \beta)\) if \(a > 0\) and \(a \neq h\), and \(\text{soc}_Q(\beta, \alpha)\) otherwise.

Then our assertion follows from the fact that \(V_Q(\mathfrak{a})\) is of the form of tensor product of fundamental representations. □

**Remark 6.20.** By the similarities between untwisted and twisted affine types ([11, 21]), one can also read the denominator formulas for \(A_n^{(2)}\) and \(D_n^{(2)}\), which were computed in [35], by folding \(\Gamma_Q\) with respect to the map \(*\). Furthermore, we have the same result of Theorem 6.19 for twisted affine types \(A_n^{(2)}\) and \(D_n^{(2)}\).
6.4. Conjectures on $E_n^{(1)}$. From Theorem 6.18, we have a conjecture given as follows:

**Conjecture 2.** For any Dynkin quiver $Q$ of finite type $E_n$, the denominator formulas for $U_q'(E_n^{(1)})$ can be read from $\Gamma_Q$ and $\Gamma_{Q^{\text{rev}}}$ as follows:

$$d_{k,l}(z) = D_{k,l}(z) \times (z - (-q)^{h^\vee})^{d_i,k^*}$$

where $h^\vee$ is the dual Coxeter number of $E_n$.

Since $\text{gdist}_Q(\alpha_i, \alpha_j) = 1$ for $d(i,j) = 1$ and $\text{gdist}_Q(\alpha_i, \alpha_j) = 0$ for $d(i,j) > 1$, the Schur-Weyl quiver $\Gamma_{Q^{\text{SW}}}^\text{rev}$ determined by $(U_q'(E_n^{(1)}), \phi_Q^{-1})$ is isomorphic to $Q^\text{rev}$ under the assumption that Conjecture 2 holds. Moreover, one can prove

1. the functor $\mathcal{F}_Q^{(1)}$ satisfies the properties (a) and (b) in Theorem 6.7 by using the same arguments in [16],

2. the $E_n^{(1)}$-analogues of Theorem 6.8 by using the results in this paper, under the same assumption.

Now we give a conjectural denominator formulas $d_{k,l}(z)$ for $E_6^{(1)}$ here ($p^* = q^{12}$):

$$d_{1,1}(z) = d_{5,5}(z) = (z - q^2)(z - q^8)$$
$$d_{1,2}(z) = d_{2,1}(z) = d_{4,5}(z) = d_{5,4}(z) = (z + q^3)(z + q^7)(z + q^9)$$
$$d_{1,3}(z) = d_{3,1}(z) = d_{3,5}(z) = d_{5,3}(z) = (z - q^4)(z - q^6)(z - q^8)(z - q^{10})$$
$$d_{1,4}(z) = d_{4,1}(z) = d_{2,5}(z) = d_{5,2}(z) = (z + q^5)(z + q^7)(z + q^9)(z + q^{11})$$
$$d_{1,5}(z) = d_{5,1}(z) = (z - q^6)(z - q^{12})$$
$$d_{1,6}(z) = d_{6,1}(z) = d_{5,6}(z) = d_{6,5}(z) = (z + q^5)(z + q^9)$$
$$d_{2,2}(z) = d_{4,4}(z) = (z - q^2)(z - q^4)(z - q^6)(z - q^8)^2(z - q^{10})$$
$$d_{2,3}(z) = d_{3,2}(z) = d_{3,4}(z) = d_{4,3}(z) = (z + q^3)(z + q^5)^2(z + q^7)^2(z + q^9)^2(z + q^{11})$$
$$d_{2,4}(z) = d_{4,2}(z) = (z - q^4)(z - q^6)^2(z - q^8)(z - q^{10})(z - q^{12})$$
$$d_{2,6}(z) = d_{6,2}(z) = d_{4,6}(z) = d_{6,4}(z) = (z - q^4)(z - q^6)(z - q^8)(z - q^{10})$$
$$d_{3,2}(z) = (z - q^2)(z - q^4)^2(z - q^6)^2(z - q^8)^3(z - q^{10})^2(z - q^{12})$$
$$d_{3,6}(z) = d_{6,3}(z) = (z + q^3)(z + q^5)(z + q^7)^2(z + q^9)(z + q^{11})$$
$$d_{6,6}(z) = (z - q^2)(z - q^6)(z - q^8)(z - q^{12}).$$
Appendix A. Dynkin Quiver $Q$ of Type $E_6$ and Its AR-Quiver $\Gamma_Q$

(1) Let us consider the Dynkin quiver $Q$ given as follows:

$$E_6: \begin{array}{cccccc}
& & & \circ & \circ & \circ \\
\circ & & & & & \\
\circ & & & & & \\
\circ & & & & & \\
\circ & & & & & \\
& & & & & \\
\end{array}$$

Note that $1^* = 5$, $2^* = 4$, $3^* = 3$ and $6^* = 6$. Its AR-quiver $\Gamma_Q$ can be drawn as follows $\left(\begin{array}{c}
\begin{array}{c}
(010) \\
(000) \\
(001) \\
(011) \\
(111) \\
(110) \\
\end{array}
\begin{array}{c}
(012) \\
(010) \\
(011) \\
(112) \\
(110) \\
(110) \\
\end{array}
\begin{array}{c}
(122) \\
(121) \\
(121) \\
(122) \\
(110) \\
(110) \\
\end{array}
\begin{array}{c}
(112) \\
(101) \\
(101) \\
(111) \\
(010) \\
(000) \\
\end{array}
\begin{array}{c}
(011) \\
(001) \\
(011) \\
(111) \\
(111) \\
(111) \\
\end{array}
\begin{array}{c}
(001) \\
(000) \\
(110) \\
(110) \\
(110) \\
(110) \\
\end{array}
\end{array}\right)$.

(2) The convex order $\prec_{[\tilde{a}_6]}$ in (b) of Example 1.24 can be visualized by the result of [36] as follows:
Appendix B. Dynkin quiver $Q$ of type $E_7$ and its AR-quiver $\Gamma_Q$

Let us consider the Dynkin quiver $Q$ given as follows:

$$Q :$$

Note that $i^* = i$ for all $i \in I$. Its AR-quiver $\Gamma_Q$ can be drawn as follows:

Here \(a_1a_2a_3a_4a_5a_6a_7\) denotes \(a_1a_2a_3a_4a_5a_6a_7\).
APPENDIX C. DYNKIN QUIVER $Q$ OF TYPE $E_8$ AND ITS AR-QUIVER $\Gamma_Q$

Let us consider the Dynkin quiver $Q$ given as follows:

$$Q : \begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{array}$$

Note that $i^* = i$ for all $i \in I$. Its AR-quiver $\Gamma_Q$ can be drawn as follows:

Here $\begin{pmatrix} a_1 a_2 \\ a_3 a_4 \\ a_5 a_6 \\ a_7 a_8 \end{pmatrix}$ denotes $(a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8)$. 
References

[1] T. Akasaka and M. Kashiwara, *Finite-dimensional representations of quantum affine algebras*, Publ. Res. Inst. Math. Sci., 33 (1997), 839-867.

[2] M. Auslander, I. Reiten and S. Smalø, *Representation theory of Artin algebras*, Cambridge studies in advanced mathematics 36. Cambridge 1995.

[3] I. Assem, D. Simson and A. Skowroński, *Elements of the representation theory of associative algebras. Vol. I*, London Math. Soc. Student Texts 65, Cambridge 2006.

[4] R. Bedard, *On commutation classes of reduced words in Weyl groups*, European J. Combin. 20 (1999), 483–505.

[5] N. Bourbaki. *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitres IV–VI*. Actualités Scientifiques et Industrielles, No. 1337. Hermann, Paris, 1968.

[6] J. Brundan, A. Kleshchev and P. J. McNamara, *Homological properties of finite Khovanov-Lauda-Rouquier algebras*, Duke Math. J., 163 (2014), 1353–1404.

[7] V. Chari and A. Pressley, *Yangians, integrable quantum systems and Dorey’s rule*, Comm. Math. Phys. 181 (1996), no. 2, 265-302.

[8] E. Date and M. Okado, *Calculation of excitation spectra of the spin model related with the vector representation of the quantized affine algebra of type $A_{1}^{(1)}$*. Internat. J. Modern Phys. A 9 (3) (1994), 399–417.

[9] P.E. Dorey, *Root systems and purely elastic S-matrices*, Nucl. Phys. B358, 654-676 (1991)

[10] P. Gabriel, *Auslander-Reiten sequences and Representation-finite algebras*, Lecture notes in Math., vol. 831, Springer-Verlag, Berlin and New York, 1980, 1–71.

[11] D. Hernandez, *Kirillov-Reshetikhin conjecture: the general case*, Int. Math. Res. Not. 2010 no.7, 149–193.

[12] , *Simple tensor products*, Invent. Math. 181 (2010), 649-675.

[13] D. Hernandez and B. Leclerc, *Quantum Grothendieck rings and derived Hall algebras*, J. Reine Angew. Math. 701 (2015), 77–126.

[14] V. Kac, *Infinite dimensional Lie algebras*, 3rd ed., Cambridge University Press, Cambridge, 1990.

[15] S.-J. Kang, M. Kashiwara and M. Kim, *Symmetric quiver Hecke algebras and R-matrices of quantum affine algebras*, arXiv:1304.0323 [math.RT].

[16] S.-J. Kang, M. Kashiwara and M. Kim, *Symmetric quiver Hecke algebras and R-matrices of quantum affine algebras II*, Duke Math. J. 164 (2015), no. 8, 1549–1602.

[17] S.-J. Kang, M. Kashiwara, M. Kim and S.-J. Oh, *Simplicity of heads and socles of tensor products*, Compos. Math. 151 (2015), no. 2, 377–396.

[18] , *Symmetric quiver Hecke algebras and R-matrices of quantum affine algebras III*, Proc. Lond. Math. Soc. 111 (2015), no. 2, 420–444.

[19] , *Symmetric quiver Hecke algebras and R-matrices of quantum affine algebras IV*, arXiv:1502.07415, to appear in Selecta Math.

[20] , *Monoidal categorification of cluster algebras*, arXiv:1412.8106v1.

[21] , *Monoidal categorification of cluster algebras II*, arXiv:1502.06714v1.

[22] M. Kashiwara, *Global crystal bases of quantum groups*, Duke Math. J. 69 (1993), no. 2, 455–485.

[23] , *On level zero representations of quantum affine algebras*, Duke. Math. J. 112 (2002), 117–175.

[24] S. Kato, *Poincaré-Birkhoff-Witt bases and Khovanov-Lauda-Rouquier algebras*, Duke Math. J. 163, 3 (2014), 619–663.

[25] M. Khovanov and A. D. Lauda, *A diagrammatic approach to categorification of quantum groups I*, Represent. Theory 13 (2009), 309–347.

[26] , *A diagrammatic approach to categorification of quantum groups II*, Trans. Amer. Math. Soc. 363 (2011), no. 5, 2685–2700.

[27] A. Kleshchev and A. Ram, *Representations of Khovanov-Lauda-Rouquier algebras and combinatorics of Lyndon words*, Math. Ann. 349 (2011), no. 4, 943-975.

[28] A. Lauda and M. Vazirani, *Crystals from categorified quantum groups*, Adv. Math., 228 (2011), 803–861.

[29] B. Leclerc, *Imaginary vectors in the dual canonical basis of $U_q(n)$*, Transform. Groups 8 (2003), no. 1, 95–104.

[30] G. Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. 3 (1990), 447–498.
[31] ______, Introduction to Quantum Groups, Birkhäuser, 1993.
[32] P. McNamara, Finite dimensional representations of Khovanov-Lauda-Rouquier algebras I: finite type, arXiv:1207.5860 [math.RT], to appear in J. Reine Angew. Math.
[33] S.-j. Oh, Auslander-Reiten quiver of type A and generalized quantum affine Schur-Weyl duality, arXiv:1405.3336v3 [math.RT], to appear in Trans. Amer. Math. Soc.
[34] ______, Auslander-Reiten quiver of type D and generalized quantum affine Schur-Weyl duality, arXiv:1406.4555v3 [math.RT].
[35] ______. The Denominators of normalized R-matrices of types $A_{2n-1}^{(2)}$, $A_{2n}^{(2)}$, $B_n^{(1)}$, and $D_{n+1}^{(2)}$, Publ. Res. Inst. Math. Sci. 51 (2015), 709–744.
[36] S.-j. Oh and U. Suh, Combinatorial Auslander-Reiten quivers and reduced expressions, arXiv:1509.04820.
[37] P. Papi, A characterization of a special ordering in a root system, Proc. Amer. Math. 120 (1994), 661–665.
[38] C. Ringel, PBW-bases of quantum groups, J. Reine Angew. Math. 470 (1996), pp. 51–88.
[39] R. Rouquier, 2 Kac-Moody algebras, arXiv:0812.5023 (2008).
[40] ______. Quiver Hecke algebras and 2-Lie algebras, Algebra Colloq. 19 (2012), no. 2, 359–410.
[41] M. Varagnolo and E. Vasserot, Canonical bases and KLR algebras, J. reine angew. Math. 659 (2011), 67–100.
[42] S. Zelikson, Auslander-Reiten quivers and the Coxeter complex, Algebr. Represent. Theory 8 (2005), no. 1, 35–55.

Department of Mathematics Ewha Womans University Seoul 120-750, Korea
E-mail address: sejin092@gmail.com