Probabilistic analysis of cracked frame structures taking into account the crack trajectory

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Abstract

In this paper a numerical study on the structural behaviour of two-dimensional cracked structures is presented. The stiffness matrix of the cracked element is found as the inverse of the compliance matrix. This matrix is given by the sum of the compliance matrix of the intact element and an additional compliance matrix which contains all the flexibilities given by the presence of the crack. The flexibilities are related to the stress intensity factors. A simple method for obtaining approximate stress intensity factors is applied. It takes into account the elastic crack tip stress singularity while using the elementary beam theory. Moreover, crack depth and location are modelled as random variables in order to take into account the unavoidable uncertainty that always affects damaged structures. A simple and accurate method for the probabilistic characterization of the linear elastic response of cracked structures with uncertain damage is employed. According to this method, the uncertainties are transformed into superimposed deformations depending on the distribution of internal forces and an iterative procedure is established to solve the resultant equations. Numerical tests evidence excellent accuracy for multicracked structures with large fluctuation of damage.

Keywords: probabilistic analysis / cracked frame structures/edge crack/ stress intensity factors.

1. Introduction

Cracked structures have always attracted significant interest by researchers. Analytical, semi-analytical and numerical approaches have been employed in order to model a cracked element in a damaged structure. One of the most common approaches is the finite element method where the key point is to define the stiffness matrix of the cracked element. In this study, the stiffness matrix is found as the inverse of the compliance matrix. The compliance matrix of cracked element is given by adding two terms: the first term is the compliance matrix of the intact beam and the second one is the matrix that contains the additional compliances given by the presence of the crack supposed straight. Stress intensity factors for many configurations are available, but solutions for many structural configurations are not present in the handbooks: simple engineering methods which allow an approximate expression of the stress intensity factors are highly valued to design engineers. In this study, the approach presented by Nobile is followed at the early stage of propagation. This approach is based on elementary beam theory.

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equilibrium condition for internal forces evaluated in the cross-section passing through the crack tip, taking into account the stress singularity at the tip of an elastic crack. The results are in reasonable agreement with other expressions given in the literature. In particular, two dimensional structures with cracked elements are considered. The additional compliances depend on crack trajectory, depth and location and are obtained via energy balance between the external work and fracture work evaluated by means of Strain Energy Density Theory. Moreover, crack depth and crack location are modelled as random variables in order to take into account the unavoidable uncertainty that always affects damaged structures. A computationally efficient and accurate method can be employed for the probabilistic analysis of linear elastic edge-cracked truss and frame structures with uncertain crack features in the two dimensional space. Here, the stochastic method is applied to two-dimensional multicracked frame structures aiming at assessing the overall reliability. The uncertainties affecting cracks are transformed into superimposed strains on a deterministic equivalent structure. For redundant structures, an asymptotic series expansion is obtained. Numerical results show that few terms of the series are enough to accurately characterize the structural response.

2. Probabilistic characterization of structural response

Consider a linear elastic structure subjected to deterministic static loads. The response of such structure can be conveniently computed resorting to a standard finite element discretization procedure. The response of the generic \( a \)-th structural element is governed by the following equations:

\[
e_a = D_a u_a \quad \text{(compatibility)}
\]

\[
D_a^T q_a = S_a \quad \text{(equilibrium)}
\]

\[
e_a = C_a q_a \quad \text{(constitutive law)}
\]

where \( u_a \) is the vector of nodal displacements, \( e_a \) is the vector of element deformations or generalized strains, \( S_a \) is the vector of nodal forces, \( q_a \) is the vector of element internal forces or generalized stresses (work conjugate to \( e_a \)), \( D_a \) and \( D_a^T \) are the compatibility and equilibrium matrices and \( C_a \) is the compliance matrix. For the sake of simplicity, distributed loads over the element are not considered. In the following we refer to Timoshenko beam-type elements. Assume that the structural element is affected by uncertainties, which influence the compliance matrix:

\[
C_a = C_a(\beta_a) \quad (4)
\]

where \( \beta_a \) is a vector of uncertain parameters (crack depth and location) modelled as random variables.

To evaluate the structural response, internal forces are expressed in terms of nodal displacements from Eqs. (1) and (3):

\[
q_a = G_a(\beta_a) u_a \quad (5)
\]

Substituting Eq. (5) into Eq. (2), the classical relation between nodal forces and nodal displacements is obtained:

\[
S_a = K_a(\beta_a) u_a \quad (6)
\]

where \( G_a(\beta_a) = C_a^{-1}(\beta_a) D_a \), \( K_a(\beta_a) = D_a^T G_a^{-1}(\beta_a) D_a \). Then, according to the standard matrix assembly procedure, equilibrium equations for the whole structure are obtained

\[
K(\beta) u = F \quad (7)
\]

where \( K(\beta) \) is the stochastic structure stiffness matrix, \( u \) is the vector of unknown nodal displacements, \( F \) is the vector of prescribed nodal forces and \( \beta \) is a random vector collecting variables \( \beta_a \).
Natural stresses can be derived from the nodal displacements by the relation

\[ q = G(\beta)u \]  

(8)

To characterize the structural response, nodal displacements should be evaluated as functions of the random variables \( \beta \) by solving Eq. (7). In literature, much effort has been directed towards developing approximate approaches. Here, the stochastic approach \(^7\) can be employed based on splitting the element compliance matrix into a deterministic part \( C_0 \) and an additional part \( C_\beta \) affected by uncertainty.

3. Additional Compliance

The local compliance contributions related to cracked bar element can be determined by Castigliano’s theorem:

\[ \lambda^i_j = \frac{\partial u_i}{\partial P_j} \]  

(9)

where the additional displacement \( u_i \) due to the crack of depth \( a \), in the \( i \)th direction is given by the Paris’ equation:

\[ u_i = \frac{\partial}{\partial P} \left[ \int_{-B/2}^{B/2} \int_0^a U(a)da \right] \]  

(10)

where \( U(a) \) is the total energy due to the crack and \( P_i \) the corresponding load. \( U(a) \) can be related to the Strain energy density factor \(^6\) by:

\[ U(a) = \frac{2\pi(1-v)}{(1-2v)} S da \]  

(11)

where \( S \) is the strain energy density factor, which is a function of the stress intensity factors. In this paper, we refer to the SIFs for an edge-cracked rectangular cross-section determined by the approach presented by Nobile \(^4,5\). In the case of the bar element, the crack trajectory is always straight since only mode-I extension is present. Thus performance of the present procedure can be illustrated through a simple test problem and the results can be compared with those obtained from Monte Carlo simulation \(^7\). The redundant truss structure shown in Fig.1 is considered. A crack with uncertain length is assumed to be present in the fourth bar of the structure.

![Fig.1. The five-bar truss problem](image)

The dimensionless mean depth of the crack is set to \( \xi_0 = 0.5 \) and the depth uncertainty is modelled as a uniformly distributed random variable \( \beta \) in the range \([-0.1, 0.1]\). In this way, the dimensionless crack depth takes values in the interval \([0.4, 0.6]\). Table 1 shows the values obtained for the first moment of axial force \( s \) in the cracked member:
**Table 1. First moment of \( \Delta s \) for the cracked element**

| ELEMENT | \( E[\Delta s]_{\text{Present}} \) (N) | \( E[\Delta s]_{\text{MC}} \) (N) | \( \frac{E[\Delta s]_{\text{MC}} - E[\Delta s]_{\text{Present}}}{E[\Delta s]_{\text{Present}}} \) |
|---------|-----------------------------------|-----------------------------------|---------------------------------|
| 4       | -181.6230                         | -183.4699                         | 1.0169                          |

Consider now a cracked beam with rectangular cross-section \( B \times H \). The beam has an edge curved crack originating from an edge straight crack of depth \( a = \beta H \), with \( \beta \) non dimensional depth, located at \( \alpha l \), with \( \alpha \) non dimensional position (Fig. 2a). Assume that the initial crack propagates slowly with an approximate path depending on mixed mode extension and restoring symmetry to the geometry of fracture, Fig.. Within the framework of brittle fracture, the well-known “Strain energy density factor theory” allows to predict crack initiation in mixed mode. Crack is assumed to run in the direction of \( S_{\text{min}} \), i.e. \( fS/fQ=0 \). Assume that after the first increment \( a \), the crack grows in the direction of the maximum tensile principal stress (Fig. 2b). The procedure by which additional local compliance contributions can be obtained is similar evaluating the approximate energy due to the crack as:

\[
U(3a) = \frac{\pi (1-\nu^2)}{2E} \sum_{i=1}^{2} \int_{\theta_i}^{\theta_i+\pi} S_i(\theta) d\theta
\]

**4. Conclusions**

A simple and accurate method for the probabilistic characterization of the linear response of cracked structures with uncertain damage is presented. The additional damage compliances depend on crack trajectory, depth and location.

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