The classical wormhole solution and wormhole wavefunction with a nonlinear Born-Infeld scalar field

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Abstract

On this paper we consider the classical wormhole solution of the Born-Infeld scalar field. The corresponding classical wormhole solution can be obtained analytically for both very small and large \( \dot{\phi} \). At the extreme limits of small \( \dot{\phi} \) the wormhole solution has the same format as one obtained by Giddings and Strominger[10]. At the extreme limits of large \( \dot{\phi} \) the wormhole solution is a new one. The wormhole wavefunctions can also be obtained for both very small and large \( \dot{\phi} \). These wormhole wavefunctions are regarded as solutions of quantum-mechanical Wheeler–Dewitt equation with certain boundary conditions.

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I. INTRODUCTION

The corresponding Lagrangian of Born-Infeld field has been first proposed by Heisenberg[3] in order to describe the process of meson multiple production connected with strong field regime, as a generalization of the Born-Infeld one, \( L_{BJ} = b^2 \left[ \sqrt{1 - \left(1/2b^2\right) F_{ik}^{\mu} F_{ik}^{\mu}} - 1 \right] [4] \), that removes the point–charge singularity that means classical electrodynamics. When the parameter of the field approaches to zero, the corresponding Lagrangian will reduce to linear case[3,4]. Born-Infeld type Lagrangians have also been considered in the theory of strings and branes as well as gravity [5-9]. It shows that the low energy effective field theory on D-branes is of Born-Infeld type[5]. The consistency of the \( \sigma \– \)model for the world sheet of string is shown to require that the brane should be described by Born-Infeld action, just like in the general curved background requiring consistency of string theory leads to the Einstein-Hilbert action.

According to the Euler-Lagrangian equation of motion of Born-Infeld scalar field, we can obtain \( \dot{\varphi} \) at the limit of large and small cosmological scale factors \( R \) respectively. At such limit condition, we found classical wormhole and quantum wormhole solutions. This paper comprises following contents: In section 2 we obtain the classical wormhole solution of Born-Infeld scalar field. In section 3 we found wormhole wavefunction of our nonlinear scalar field model. In last section, we discuss our results and come to conclusions.

II. CLASSICAL WORMHOLE SOLUTION

The Euclidean action of gravitational field interacting with a Born-Infeld type scalar field is given by

\[
S_E = \int \frac{R_c}{16\pi G} \sqrt{g} d^4x + \int L_s \sqrt{g} d^4x
\]  

(1)

Where we have chosen unit so that \( c = 1 \), \( R_c \) is the Ricci scalar curvature and the Lagrangian \( L_s \) of the nonlinear Born-Infeld scalar field is [3]

\[
L_s = \frac{1}{\lambda} \left[ 1 - \sqrt{1 - \lambda \varphi_{\mu} \varphi_{\nu} g^{\mu\nu}} \right]
\]

(2)

When \( \lambda \rightarrow 0 \), based on Taylor expansion (2) approximates to

\[
\lim_{\lambda \rightarrow 0} L_s = \frac{1}{2} \varphi_{\mu} \varphi_{\nu} g^{\mu\nu}
\]

(3)

We choose the standard Euclidean and closed R-W metric

\[
ds^2 = d\tau^2 + R(\tau)^2 \left\{ \frac{dr^2}{1 - r^2} + r^2 \left[ (d\theta^2) + \sin^2 \theta (d\varphi^2) \right] \right\}
\]

(4)

Where \( \tau \) is the Euclidean radial coordinate and \( R(\tau) \) is the radius of curvature of a 3D sphere. According to the “cosmological principle”, \( R \) must only depend on \( \tau \). We write Einstein equations as
\[-3 \frac{\dot{R}^2}{R^2} + 3 \frac{\ddot{R}}{R^2} = 8\pi G T^0_0 \quad (5)\]

\[-\frac{2\dot{R}}{R} - \frac{\dot{R}}{R^2} + 1 \frac{\ddot{R}}{R^2} = 8\pi G T^1_1 = 8\pi G T^2_2 = 8\pi G T^3_3 \quad (6)\]

Where the upper index “·” denotes the derivative with respect to $\tau$. We substitute the Lagrangian (2) into Eule-Lagrange equation

\[
\frac{d}{d\tau} \left( \frac{\partial L_s}{\partial \dot{\varphi}} \right) - \frac{\partial L_s}{\partial \varphi} = 0 \quad (7)
\]

Then we obtain

\[
\frac{R^6 \dot{\varphi}^2}{1 + \lambda \dot{\varphi}^2} = W_0 \quad (8)
\]

and consequently

\[
\dot{\varphi} = \sqrt{\frac{W_0}{R^6 - W_0 \lambda}} \quad (9)
\]

Where $W_0$ is a constant of integration. We write components of energy-momentum tensor of Born-Infeld scalar field as

\[
T^{\mu}_{\nu} = \frac{g^{\mu\rho} \varphi_{,\rho} \varphi_{,\nu}}{\sqrt{1 - \lambda \varphi_{,\mu} \varphi_{,\nu} g^{\mu\nu}}} - \delta^\mu_\nu L_s \quad (10)
\]

Substitute equations (9) and (2) into (10), we obtain

\[
T^0_0 = \frac{1}{\lambda} \left[ \sqrt{R^6 - \lambda W_0 / R^3} - 1 \right] \quad (11)
\]

\[
T^1_1 = T^2_2 = T^3_3 = -\frac{1}{\lambda} \left[ 1 - R^3 / \sqrt{R^6 - \lambda W_0} \right] \quad (12)
\]

Substitute (11) into Einstein equations (5), we can obtain

\[
-3 \frac{\dot{R}^2}{R^2} + 3 \frac{\ddot{R}}{R^2} = \frac{8\pi G}{\lambda} \left[ \sqrt{R^6 - \lambda W_0 / R^3} - 1 \right] \quad (13)
\]

\[
\dot{R}^2 = 1 - \frac{8\pi G}{3\lambda} \left[ R^2 \sqrt{1 - \lambda W_0 R^{-6} - R^2} \right] \quad (14)
\]

From equation (9), we can find that $R$ is very small or very large when $\dot{\varphi}$ is very large or very small respectively. Assuming that $\dot{\varphi}$ is very small (i.e. $R$ is very large), equation (14) becomes
When $W_0 < 0$, the wormhole solution of equation (15) is
\[
\frac{\tau}{R_0} = \sqrt{\frac{1}{2} F \left[ \cos^{-1} \left( \frac{R_0}{R} \right) \right] - \sqrt{2} E \left[ \cos^{-1} \left( \frac{R_0}{R} \right) \right] + \frac{\sqrt{R^4 - R_0^4}}{R_0 R}}
\] (16)

Where $R_0 = \sqrt[4]{-\frac{4 \pi G W_0}{3}}$. We note that this wormhole solution has the same format as one obtained by Giddings and Strominger[10]. When $\dot{\phi}$ is very large (i.e., $R$ is very small), we can obtain from equation (14)
\[
\dot{R}^2 = 1 - \frac{8 \pi G \sqrt{-W_0}}{9a}
\] (17)

We restrict $\lambda > 0$, integrating (17) we can obtain wormhole solution of equation (17), that is
\[
R \sqrt{1 - \frac{N}{R}} + N \log \left[ \frac{\sqrt{\frac{R}{N}} - 1 + \sqrt{\frac{R}{N}} - 1}{\sqrt{\frac{R}{N}} - 1 - \sqrt{\frac{R}{N}} + 1} \right] = \tau
\] (18)

Where $N = 8 \pi G \sqrt{-W_0} > 0$. From equation (18) we can find that $\lim_{\tau \to \infty} R(\tau) = \infty$. Using $\dot{R}(0) = 0$ from equation (17) we can obtain the size of wormhole throat: $R(0) = N$ and $\ddot{R}(0) = \frac{1}{2N}$. Thus we obtain a new wormhole solution.

### III. WORMHOLE WAVEFUNCTION

It is possible that the wormholes are regarded as solutions of quantum-mechanical Wheeler—Dewitt (WD) equation. These wavefunctions have to obey certain boundary conditions in order that they represent wormholes. The wavefunction will be damped at large radius $R$, i.e., such wavefunction tends to zero as $R \to \infty$, and when $R$ nears 0 it should be oscillatory[11]. Wavefunction should tend to a constant as $R \to 0$[12]. The Lorentz action of the gravitational field interacting with a Born–Infeld type scalar field is given by
\[
S = \int \frac{R_c}{16 \pi G} \sqrt{-g} d^4x + \int L_s \sqrt{-g} d^4x
\] (19)

Where $R_c$ is the Ricci scalar curvature and $L_s$ is equation (2). However, in equation (19), $g_{\mu\nu}$ is decided by equation (20). The closed R–W spacetime metric is
\[
d s^2 = -dt^2 + R^2(t) \left\{ \frac{dr^2}{1 - r^2} + r^2 \left[ d\theta^2 + \sin^2 \theta \left( d\varphi^2 \right) \right] \right\}
\] (20)

Using equation (20) and integrating space–components, the action (19) becomes (The upper-dot means the derivative with respect to the time $t$):
\[ S = \int \frac{3\pi}{4G} (1 - \dot{R}^2) R dt + \int 2\pi^2 R^3 \left[ \frac{1}{\lambda} \left( 1 - \sqrt{1 - \lambda \varphi, \varphi, g} \right) \right] dt \equiv \int \mathcal{L}_g dt + \int \mathcal{L}_s dt \quad (21) \]

To quantize the model, we first find the canonical moment

\[ P_R = \left( \frac{\partial \mathcal{L}_g}{\partial \dot{R}} \right) = -\left( \frac{3\pi}{2G} \right) R \dot{R} \hspace{1cm} P_\varphi = \left( \frac{\partial \mathcal{L}_s}{\partial \dot{\varphi}} \right) = \left( 2\pi^2 R^3 \dot{\varphi}/\sqrt{1 + \lambda \varphi^2} \right) \]

and the Hamiltonian

\[ H = -\frac{G}{3\pi R} P_R^2 - \frac{3\pi}{4G} R + \frac{P_\varphi^2}{4\pi^2 R^3} - \frac{\lambda P_\varphi^4}{64\pi^6 R^9} \quad (22) \]

For small \( \dot{\varphi} \), the Hamiltonian (22) can be simplified by using the Taylor expansion

\[ H = -\frac{G}{3\pi R} P_R^2 - \frac{3\pi}{4G} R + \frac{2\pi^2 R^3}{\lambda} \left( 1 - \frac{\lambda P_\varphi^2}{4\pi^4 R^6} \right) \]

If \( \dot{\varphi} \) is large, then equation (22) becomes

\[ H = -\frac{G}{3\pi R} P_R^2 - \frac{3\pi}{4G} R + \frac{2\pi^2 R^3}{\lambda} \]

The WD equation is obtained from \( \dot{H} \psi = 0 \) and equations (23) as well as (24) by replacing \( P_R \rightarrow -i \left( \frac{\partial}{\partial R} \right) \) and \( P_\varphi \rightarrow -i \left( \frac{\partial}{\partial \varphi} \right) \). Then we obtain

\[ \left[ \frac{\partial^2}{\partial R^2} + \frac{P}{R} \frac{\partial}{\partial R} - \frac{1}{R^2} \frac{\partial}{\partial \Phi^2} - \frac{\lambda}{16\pi^4 R^8} \frac{\partial^4}{\partial \Phi^4} - U(R) \right] \psi = 0 \quad (25) \]

and

\[ \left[ \frac{\partial^2}{\partial R^2} + \frac{P}{R} \frac{\partial}{\partial R} - u(R) \right] \psi = 0 \quad (26) \]

Where \( \Phi^2 = 4\pi G \varphi^2/3 \) and the parameter \( P \) represents the ambiguity in the ordering of factors \( R \) and \( \frac{\partial}{\partial R} \) in the first term of equations (23) and (24). We have also denoted

\[ U(R) = \left( \frac{3\pi}{2G} \right)^2 R^2 \]

\[ u(R) = \left( \frac{3\pi}{2G} \right)^2 R^2 \left[ 1 - \frac{8\pi G}{3\lambda} R^2 \right] \]

Equations (25) and (26) are the WD equations corresponding to action (19) in the cases of small and large \( \dot{\varphi} \) respectively. Together we can obtain the equation of motion of Born–Infeld scalar field when we substitute the Lagrangian \( L_s \) into the Euler–Lagrangian equation

\[ \frac{d}{dt} \left( \frac{\partial L_s}{\partial \dot{\varphi}} \right) - \frac{\partial L_s}{\partial \varphi} = 0 \quad (27) \]

Then we obtain

\[ \dot{\varphi} = \sqrt{\frac{C}{R^6 + C\lambda}} \quad (28) \]
The upper–dot means the derivative with respect to the $t$. Where $C$ is a constant of integration. From equation (28) we find that $R$ is very small or very large when $\dot{\varphi}$ is very large or very small respectively. In other word, equations (25) and (26) are the WD equations corresponding to action (19) in the cases of large and small $R$ respectively. When $R$ is very large, we take the ambiguity of ordering factor $P = -1$ and set transformation $(R/R_0)^2 = \sigma$, with $R_0$ the Planck’s length. Choosing appropriate units makes the Planck constant $\hbar = 1$, speed of light $c = 1$, and $R_0 \sim \sqrt{\frac{4G}{3\pi}}$. Then equation (25) becomes

$$\frac{\partial^2 \psi}{\partial \sigma^2} - \frac{1}{\sigma^2} \frac{\partial^2 \psi}{\partial \Phi^2} - \frac{\lambda}{16\pi^4 R_0^6 \sigma^5} \frac{\partial^4 \psi}{\partial \Phi^4} - \tilde{U} \psi = 0$$  \hspace{1cm} (29)

Where $\tilde{U} = (3\pi/4G)^2 R_0^4$. Assuming $\psi (\sigma, \Phi) = Q (\sigma) e^{-K\Phi}$, with $K$ an arbitrary constant, equation (29) takes the form:

$$\frac{d^2 Q}{d\sigma^2} - \left( \frac{K^2}{\sigma^2} + \frac{\mu K^4}{\sigma^5} + \tilde{U} \right) Q = 0$$  \hspace{1cm} (30)

Where $\mu = \lambda/16\pi^4 R_0^6$. When $R$ (and consequently $\sigma$) is very large, equation (30) approximates to

$$\frac{d^2 Q}{d\sigma^2} - \beta^2 Q = 0$$  \hspace{1cm} (31)

Where $\beta = \left( \frac{3\pi}{4G} \right) R_0^2$. The solution of equation (31) is

$$Q = \exp (-\beta \sigma)$$  \hspace{1cm} (32)

From (32) we can find that wavefunction $\psi \to 0$ when $R \to \infty$ (and consequently $\sigma \to \infty$). If $R$ is very small, we take the ambiguity of ordering factor $P = -1$ and set the transformation $(R/R_0)^2 = \sigma$ ,with $R_0$ the Planck length. Choosing appropriate units makes the Planck constant $\hbar = 1$, the speed of light $c = 1$ and $R_0 \sim \sqrt{\frac{4G}{3\pi}}$. Then equation (26) becomes

$$\frac{d^2 \psi}{d\sigma^2} - \left( \frac{3\pi}{4G} \right)^2 R_0^4 \left( 1 - \frac{8\pi G}{\lambda} \sigma R_0^2 \right) \psi = 0$$  \hspace{1cm} (33)

When $R \gg \sqrt{\frac{3\lambda}{8\pi G}}$ (and consequently $\sigma \gg \frac{3\lambda}{8\pi G R_0}$), equation (33) can be approximated as

$$\frac{d^2 \psi}{d\sigma^2} + \gamma^2 \sigma \psi = 0$$  \hspace{1cm} (34)

Where $\gamma = \left( \frac{3\pi^3}{2G\lambda} \right)^{1/2} R_0^2$. Equation (34) has the solution

$$\psi = \sqrt{\sigma} Z_{\gamma} \left( \frac{2\gamma}{3} \sigma^{3/2} \right)$$  \hspace{1cm} (35)

Now $\psi$ is an oscillatory function. When $\sigma \to 0$ (and consequently $R \to 0$), equation (33) can be approximated as
\[ \frac{d^2 \psi}{d\sigma^2} - \left( \frac{3\pi}{4G} \right)^2 R_0^4 \psi = 0 \] (36)

Equation (36) has the solution

\[ \psi = N e^{-\frac{3\pi}{4G} R^2} \] (37)

In the geometry described by the R-W metric, the probability of wormhole situated between \( R \to R + dR \) is

\[ \omega (R) \propto \psi^2 R^2 dR \] (38)

The probability density is \( \psi^2 R^2 \). The position of the maximum probability can be determined by

\[ \frac{d}{dR} (\psi^2 R^2) = 0 \] (39)

From (39) we can obtain

\[ R = \sqrt{\frac{4G}{6\pi}} \] (40)

Equation (40) implies that most probable radius of wormhole is of the Planck scale, namely the quantum effect can make a wormhole survive gravitational collapse.

**IV. CONCLUSION**

At the extreme limits of small \( \dot{\varphi} \), the classical wormhole solution of the Born–Infeld scalar field has the same format as one obtained by Giddigs and Strominger. If \( \dot{\varphi} \) is very large, a new wormhole solution can be obtained. From the Euler–Lagrange equation of the Born–Infeld scalar field, we find that cosmological scale factors is very large or very small when \( \dot{\varphi} \) is very small or very large respectively. We obtain the wormhole wavefunction. It is the solution of quantum–mechanical Wheeler–Dewitt equation with certain boundary conditions. The wavefunction is exponentially damped for large three geometries and the wavefunction tends to a zero as cosmological scale factors tend to a infinity. They oscillate near zero radius, it tends to a constant as cosmological radius tends to a zero.

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REFERENCES

[1] Hawking S W 1987 *Phys. Lett.* **195B** 337
[2] Hawking S W 1988 *Phys. Rev.* D **37** 904
[3] Heisenberg W 1952 Z. *Phys.* **133** 79
[4] Born M and Infeld Z 1934 *Proc. Roy. Soc.* A **144** 425
[5] Tseytlin A 1986 *Nucl. Phys.* B **276** 391
[6] Palatnik D 1998 *Phys. Lett.* **432B** 287
[7] Feigenbaum J A 1998 *Phys. Rev.* D **58** 124023
[8] Boillat G and Strumia A 1998 *J Math. Phys.* **40** 1
[9] Deser S and Gibbons G W 1998 *Class. Quantum Grav.* **15** 135
[10] Giddings S B and Strominger A 1988 *Nucl. Phys.* B **306** 890
[11] Hawking S W and Page D N 1990 *Phys Rev.* D **42** 2655
[12] Coule D H 1992 *Class. Quantum Grav.* **9** 2353-2360