Remarks on martingale representation theorem for set-valued martingales

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Abstract
Martingale representation theorem for set-valued martingales was proposed by M. Kisielewicz [J. Math. Anal. Appl. 2014]. We shall prove that the result holds only for very special case: the set-valued martingale degenerates to the point-valued one. A revised representation theorem for a special kind of non-degenerate set-valued martingales is presented.

Keywords: Set-Valued Martingale, Interval-Valued Martingale, Martingale Representation Theorem

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1. Introduction
Set-valued function is a natural extension of point-valued one, which has received much attention. A set is called degenerate if it is a singleton and otherwise non-degenerate. A difficulty to handle non-degenerate set-valued functions lies in the fact that the power set of a set is not linear. For example, let \((\Omega, \mathcal{A}, P)\) be a probability space, \(\lambda, \eta, a, b, c, d \in \mathbb{R}\) be real numbers with \(a < b, c < d\) and \(f, g (f \leq g\) a.s.) be two integrable random variables taking values in \(\mathbb{R}\). Define the operations \(\lambda[a, b] := \{\lambda h : a \leq h \leq b\}\) and \([a, b] + [c, d] := [a + c, b + d]\). We then remark that \((\lambda + \eta)[a, b] \neq \lambda[a, b] + \eta[a, b]\) if \(\lambda\) and \(\eta\) have different signs. For the expectation (which will be defined by \(\mathbb{E}(f, g) = \lambda\mathbb{E}([f, g])\) holds, while \(\mathbb{E}(f[a, b]) = \mathbb{E}(f) \times [a, b]\) may not hold. We have to be very careful to deal with set-valued variables.

Martingales are a very important class of stochastic processes with nice properties and wide applications. For set-valued case, Hiai and Umegaki 1977 \cite{HiaiUmegaki} defined set-valued martingale, supermartingale and submartingale. After that, there have been many works studying set-valued martingales. For example, based on the definition of set-valued stochastic integral with respect to Brownian motion given by Jung and Kim \cite{JungKim}, Zhang et al. \cite{ZhangYano1, ZhangYano2} obtained the submartingale property of set-valued stochastic integrals with respect to Brownian motion and to compensated Poisson random variables.
measure. Furthermore, in [11], by using the Hahn decomposition theorem and the properties of compensated Poisson random measure, the authors proved that the non-degenerate set-valued stochastic integral w.r.t. the compensated Poisson measure is not a set-valued martingale but a submartingale. For the non-degenerate case w.r.t. Brownian motion, in a very simple way, we shall show that it is neither a set-valued martingale. The martingale representation theorem for point-valued martingale plays an important role in classical stochastic analysis, see e.g. Theorem 4.3.4 in [7]. For set-valued case, is there a similar representation theorem? It is our task in this short paper to answer the question.

When taking as the underlying space the r-dimensional Euclidean space $\mathbb{R}^r$, Kisielewicz in [4] (2014) proposed a representation theorem for set-valued martingale as follows:

**Theorem 1.1.** For every set-valued $\mathbb{F}$-martingale $F = (F_t)_{t \geq 0}$ defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with the augmented natural filtration $(\mathcal{F}_t)_{t \geq 0}$ of an r-dimensional Brownian motion $B = (B_t)_{t \geq 0}$ defined on $(\Omega, \mathcal{F}, P)$ and such that $F_0 = \{0\}$ there exists a set $G \in \mathcal{P}(\mathcal{L}^2_\mathbb{F})$ such that $F_t = \int_0^t G dB_s$ a.s. for every $t \geq 0$.

Here $\mathcal{L}^2_\mathbb{F}$ denotes the family of all $\mathbb{R}^{rd}$-valued integrable (w.r.t. the Brownian motion $B$) stochastic processes and $\mathcal{P}(\mathcal{L}^2_\mathbb{F})$ denotes the power set of $\mathcal{L}^2_\mathbb{F}$. For $G \in \mathcal{P}(\mathcal{L}^2_\mathbb{F})$, $\int_0^t G dB_s$ is the generalized stochastic integral defined in [5], which is a set-valued random variable for each $t$.

We will see in Example 3.2 that Theorem 1.1 imposes so strong assumption $F_0 = \{0\}$ as to exclude non-degenerate set-valued martingales. We will propose in Theorem 3.1 a revised representation theorem.

This paper is organized as follows. In Section 2 we recall several notations and preliminary facts about set-valued processes. In Section 3 we develop our main theorem. We shall prove that the above representation theorem does not hold for non-degenerate set-valued martingale. A revised martingale representation will be given. Except for a special kind of non-degenerate set-valued martingale, the general non-degenerate set-valued martingale has no representation theorem.

2. Notations and Preliminaries

Throughout the paper, let $T$ be a positive number. We consider the complete non-atomic probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ which satisfies the usual condition. We denote by $(\mathcal{X}, \|\cdot\|)$ a separable Banach space equipped with the Borel sigma-algebra $\mathcal{B}(\mathcal{X})$. $K(\mathcal{X})$ denotes the family of all nonempty closed subsets of $\mathcal{X}$. $K_c(\mathcal{X})$ (resp. $K_{nc}(\mathcal{X})$, $K_{bc}(\mathcal{X})$) is the family of all nonempty closed convex (resp. nonempty bounded closed convex, nonempty compact convex) subsets of $\mathcal{X}$. For a set $C \subset \mathcal{X}$, we write $\|C\| := \sup_{x \in \mathcal{X}} \|x\|$. $L^p(\Omega, \mathcal{X}, P; \mathcal{X})$ ($1 \leq p < \infty$) is the set of all $\mathcal{X}$-valued Borel measurable functions $f : \Omega \to \mathcal{X}$ equipped with the norm $\|f\|_p := \left( \int_\Omega \|f(\omega)\|^p dP(\omega) \right)^{\frac{1}{p}} < \infty$.

A mapping $F : \Omega \to K(\mathcal{X})$ is said to be a set-valued random variable (or a random set) if for each open set $O \subset \mathcal{X}$, we have $\{\omega \in \Omega : F(\omega) \cap O \neq \emptyset\} \in \mathcal{F}$. We denote the family of all $K(\mathcal{X})$-valued random variables by $\mathcal{M}(\Omega, \mathcal{F}, P; K(\mathcal{X}))$, or briefly by $\mathcal{M}(\Omega; K(\mathcal{X}))$. Similarly, we also have the notations $\mathcal{M}(\Omega; K_c(\mathcal{X}))$, $\mathcal{M}(\Omega; K_{nc}(\mathcal{X}))$ and
$M(\Omega; K_e(\mathfrak{X}))$. For $F \in M(\Omega, K(\mathfrak{X}))$, the family of all $L^p$-integrable selections is denoted by

$$S^p_F(\mathcal{A}) := \{ f \in L^p(\Omega, \mathcal{A}, P; \mathfrak{X}) : f(\omega) \in F(\omega) \text{ a.s.}, 1 \leq p < \infty \},$$

or briefly by $S^p_F$. A set-valued random variable $F$ is said to be integrable if $S^1_F$ is nonempty. $F$ is called $L^p$-integrably bounded if there exists $h \in L^p(\Omega, \mathcal{A}, P; \mathfrak{X})$ s.t. $\| F(\omega) \| \leq h(\omega)$, i.e. [for all $x \in F(\omega)$ we have $\| x \| \leq h(\omega)$] almost surely. The family of all $K(\mathfrak{X})$-valued $L^p$-integrably bounded random variables is denoted by $L^p(\Omega, \mathcal{A}, P; K(\mathfrak{X}))$, or briefly by $L^p(\Omega, K(\mathfrak{X}))$. Similarly, we have notations $L^p(\Omega; K_e(\mathfrak{X}))$, $L^p(\Omega; K_{ke}(\mathfrak{X}))$ and $L^p(\Omega; K_{ke}(\mathfrak{X}))$.

Let $\Gamma$ be a set of measurable functions $f : \Omega \to \mathfrak{X}$. We call $\Gamma$ decomposable with respect to the $\sigma$-algebra $\mathcal{A}$ if for any finite $\mathcal{A}$-measurable partition $A_1, ..., A_n$ of $\Omega$ and for any $f_1, ..., f_n \in \Gamma$ it follows that $1_{A_1} f_1 + ... + 1_{A_n} f_n \in \Gamma$, where $1_A$ is the indicator function of $A$. From (3), we know that a nonempty set $\Gamma \subset L^p(\Omega, \mathcal{A}, P; \mathfrak{X})$ determines a $p$-integrable set-valued random variable $F$ such that $G = S^p_F$ if and only if $\Gamma$ is decomposable with respect to $\mathcal{A}$. Note also that $F(\omega) = G(\omega)$ a.s. iff $S^p_F = S^p_G$.

Therefore, in order to study $F$, we have only to study $S^p_F$.

There is the Castaing representation for a $p$-integrable set-valued random variable.

Lemma 2.1 (3). For a $p$-integrable set-valued random variable $F \in M(\Omega, \mathcal{A}, P; K(\mathfrak{X}))$, there exists a sequence $\{ f^i : i \in \mathbb{N} \} \subset S^p_F$ such that $F(\omega) = \text{cl} \{ f^i(\omega) : i \in \mathbb{N} \}$ for all $\omega \in \Omega$, where the closure is taken in $\mathfrak{X}$. In addition, $S^p_F = \overline{\text{de}} \{ f^i : i = 1, 2, ... \}$, where the $\overline{\text{de}}$ denotes the decomposable closure of the sequence $\{ f^i : i = 1, 2, ... \}$ in the space $L^p(\Omega; \mathfrak{X})$.

The integral (or expectation) of a set-valued random variable $F$ was defined by Aumann (1) as

$$E(F) := \left\{ E(f) : f \in S^1_F \right\},$$

where $E(f) = \int_{\Omega} f dP$ is the Bochner integral. If the probability space is non-atomic, we have $\text{cl} \{ E(F) \}$ is convex. In general, the expectation $E(F)$ is not closed. But under some conditions, it is closed. For example, if the Banach space $\mathfrak{X}$ has the Radon Nikodym property, and if $F \in L^1(\Omega, K_e(\mathfrak{X}))$ then $E(F)$ is closed in $\mathfrak{X}$. Moreover, if the space $\mathfrak{X}$ is reflexive, and if $F \in L^1(\Omega, K_e(\mathfrak{X}))$, then the expectation $E(F)$ is closed in $\mathfrak{X}$ (7). It is well-known that finite dimensional spaces and $L^p(\Omega; \mathfrak{X}) (1 < p < \infty)$ are reflexive.

A set-valued stochastic process $F = \{ F_t : 0 \leq t \leq T \}$ is called uniformly integrable if the real-valued stochastic process $\| F \| = \{ \| F_t \| : 0 \leq t \leq T \}$ is uniformly integrable.

$F = \{ F_t : 0 \leq t \leq T \}$ is called an $\mathcal{F}$-adapted set-valued stochastic process if $F_t \in M(\Omega, \mathcal{A}_t, P; K(\mathfrak{X}))$ for each $t$. $f = \{ f_t : 0 \leq t \leq T \}$ is called a martingale selection of $F$ if $f$ is an $\mathfrak{X}$-valued $\mathcal{F}$-martingale and $f_t(\omega) \in F_t(\omega)$ a.s. for each $t$. The family of all martingale selections of $F$ is denoted by $MS(F)$. The set-valued conditional expectation $E[F_t|\mathcal{A}_s]$ is defined by $S^1_{E[F_t|\mathcal{A}_s]}(\mathcal{A}_s) := \text{cl} \{ E[f_t|\mathcal{A}_s] : f_t \in S^1_{F_t}(\mathcal{A}_s) \}$ for $0 \leq s \leq t$, where the closure is taken in $L^1(\Omega; \mathfrak{X})$. 

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**Definition 2.1.** An integrable convex set-valued $\mathcal{A}$-adapted stochastic process $F = \{F_t : 0 \leq t \leq T\}$ is called a set-valued $\mathcal{A}$-martingale if for any $0 \leq s \leq t$ it holds that $E(F_t|\mathcal{A}_s) = F_s$ in the sense of $S^1_{E(F_s|\mathcal{A}_s)}(\mathcal{A}_s) = S^1_{F_s}(\mathcal{A}_s)$. It is called a set-valued submartingale (resp. supermartingale) if for any $0 \leq s \leq t$, $E(F_t|\mathcal{A}_s) \geq F_s$ (resp. $E(F_t|\mathcal{A}_s) \leq F_s$) in the sense of $S^1_{E(F_s|\mathcal{A}_s)}(\mathcal{A}_s) \geq S^1_{F_s}(\mathcal{A}_s)$ (resp. $S^1_{E(F_s|\mathcal{A}_s)}(\mathcal{A}_s) \leq S^1_{F_s}(\mathcal{A}_s)$).

A set-valued martingale has the property: $cl\{E(F_t)\} = cl\{E(F_0)\}$ for every $0 \leq t \leq T$.

In the case $\mathfrak{X} = \mathbb{R}$, it is known (Theorem 3.1.1 of [4]) that $\{F_t = [f_t, g_t] : 0 \leq t \leq T\}$ is an interval-valued $\mathcal{A}$-martingale iff both the endpoints are $\mathbb{R}$-valued $\mathcal{A}$-martingales. For discrete time, the result holds too.

**3. Main result**

Let $(B_t)_{0 \leq t \leq T}$ be a real-valued standard Brownian motion. Let $\mathcal{A} = (\mathcal{A}_t)_{0 \leq t \leq T}$ denote the augmented natural filtration of $(B_t)_{0 \leq t \leq T}$. We adopt the notations of Section 2.

**Theorem 3.1.** Suppose $(\mathfrak{X}, \|\cdot\|)$ is a separable reflexive $M$-type 2 Banach space. Let $\{M_t, 0 \leq t \leq T\}$ be a $K_r(\mathfrak{X})$-valued $\mathcal{A}$-martingale. Then the following statements are equivalent:

(i) There exists a set-valued stochastic process $(G_t)_{0 \leq t \leq T}$ such that

$$M_t = E(M_0) + \int_0^t G_s dB_s \text{ a.s. for each } t. \quad (2)$$

(ii) There exists an $\mathfrak{X}$-valued stochastic process $g = \{g_t : t \in [0, T]\}$ and a bounded, closed and convex subset $C \subset \mathfrak{X}$ such that

$$M_t = C + \left\{ \int_0^t g_s dB_s \right\} \text{ a.s. for each } t.$$

(iii) There exists a sequence $\{f^1, \cdots, f^n, \cdots\}$ of $\mathfrak{X}$-valued $\mathcal{A}$-martingales such that

$$M_t = cl\{f^1, \cdots, f^n, \cdots\} \text{ a.s. for each } t$$

and $f^i_t - f^j_t$ is non-random and independent of $t$ for any $i, j \geq 1$.

**Remark 1.** For the $M$-type 2 Banach space, the set-valued stochastic integral w.r.t. Brownian motion $I_t(G) = \int_0^t G_s dB_s$ is defined in Definition 4.2 in [10], which is a $K(\mathfrak{X})$-valued random variable for each $t$. $I_t(G)$ is determined by

$$S^1_{I_t(G)}(\mathcal{A}_t) = \overline{de}\left\{ \int_0^t g_s dB_s : g \text{ is the integrable selection of } G \right\}.$$ 

If $\mathfrak{X}$ is not reflexive, replacing (2) by $M(t) = cl\{E(M_0)\} + \int_0^t G_s dB_s$, the result still holds. It is also valid for $\mathfrak{X} = \mathbb{R}^d$ with $d$-dimensional Brownian motion and corresponding $K(\mathbb{R}^{rd})$-valued integrable stochastic process $G$.

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In the following, the family of all \( \mathbb{X} \)-valued square-integrable (w.r.t Brownian motion) stochastic processes is denoted by \( \mathcal{L}^2(\mathbb{X}) \). The family of all \( K(\mathbb{X}) \)-valued integrable (w.r.t. Brownian motion) is denoted by \( \mathcal{L}^2(K(\mathbb{X})) \). Before prove Theorem 3.1 firstly, we give a result about expectation of set-valued random variable.

**Lemma 3.1.** For any set-valued random variable \( F \in \mathcal{L}^1(\Omega, K(\mathbb{X})) \) and any deterministic element \( a \in \mathbb{X} \), the expectation \( E(F) = \{a\} \) iff \( F \) degenerates to a random singleton \( \{f\} \) with \( E(f) = a \).

**Proof.** We may assume \( a = 0 \) without loss of generality thanks to translation. The sufficiency is obvious. For the converse, assume \( F \in \mathcal{L}^1(\Omega, K(\mathbb{X})) \) is non-degenerate. By the Castaing representation Lemma 2.1, there exist at least two different \( \mathbb{X} \)-valued functions \( f_1, f_2 \in \mathcal{F} \). If one of \( f_1, f_2 \) is not mean zero, then the result is obtained. We assume \( E(f_1) = E(f_2) = 0 \) and \( f_1 \neq f_2 \) with a positive probability. Since \( (\Omega, \mathcal{A}, P) \) is non-atomic, there exists a measurable partition \( \{A, B\} \) of \( \Omega \) such that \( 0 < P(A) < 1 \) and

\[
E(F) \ni E(1_A f_1 + 1_B f_2) = E(1_A f_1) + E(1_B f_2)
\]

\[
= E(1_A f_1) - E(1_A f_2) \quad \text{(since } E(f_2) = E(1_A f_2 + 1_B f_2) = 0)\]

\[
= E(1_A (f_1 - f_2)) \neq 0.
\]

\( \square \)

By using Lemma 3.1 it is convenient to judge some set-valued stochastic processes are not set-valued martingales. There are some examples.

**Example 3.1.** Let \( G \in \mathcal{L}^2(K(\mathbb{X})) \) be non-degenerate. Then the set-valued stochastic integral \( \left\{ \int_0^t G_s dB_s : t \in [0, T]\right\} \) is not a set-valued martingale.

In fact, by the definition of \( I_t(G) = \int_0^t G_s dB_s \), we know \( I_0(G) = \{0\} \) a.s. Then \( E(I_0(G)) = \{0\} \). But there exists \( t > 0 \) such that \( I_t(G) \) is non-degenerate. By Lemma 3.1 \( E(I_t(G)) \neq E(I_0(G)) \), then \( \left\{ \int_0^t G_s dB_s : t \in [0, T]\right\} \) is not a set-valued martingale.

In [10], we know that it is a set-valued submartingale.

**Example 3.2.** Now let us concentrate on the martingale representation Theorem 1.1.

By using Lemma 3.1 it is easy to see that there is no non-degenerate set-valued martingale such that its expectation is a singleton. Particularly, there is no non-degenerate set-valued martingale with expectation zero. From this point of view, the martingale representation Theorem 1.1 given by Kisielewicz in [4] does not hold for any non-degenerate set-valued martingale since \( F_0 = \{0\} \) a.s. In fact, in Theorem 1.1 if the subset \( \mathbb{G} \) is not a singleton, then the integral process \( \left\{ I_t(\mathbb{G}) := \int_0^t \mathbb{G} dB_s : 0 \leq t \leq T\right\} \) is non-degenerate. \( \{I_t(\mathbb{G}) : 0 \leq t \leq T\} \) is not a non-degenerate set-valued martingale since \( I_0(\mathbb{G}) = \{0\} \) a.s.

If the set \( \mathbb{G} \) only includes one \( \mathbb{X} \)-valued integrable stochastic process, Theorem 1.1 becomes the classical martingale representation theorem.

**Example 3.3.** Take two different stochastic processes \( f, g \in \mathcal{L}^2(\mathbb{R}) \) and set

\[
\xi_t = \int_0^t f_s dB_s, \quad \eta_t = \int_0^t g_s dB_s.
\]
For each $t \in [0, T]$, define

$$M_t = cl \{ \lambda \xi_t + (1 - \lambda) \eta_t : \lambda \in [0, 1] \cap \mathbb{Q} \},$$

i.e., $M_t$ is the segment of $\int_0^t f_s dB_s$ and $\int_0^t g_s dB_s$. It is clear that $\{M_t : t \in [0, T]\}$ is a non-degenerate interval-valued stochastic process. But it is not an interval-valued martingale since $M_0 = [0] a.s.$

We know that, $M_t = \{ \min \{ \xi_t, \eta_t \}, \max \{ \xi_t, \eta_t \} \}$, and that it is a non-degenerate interval-valued submartingale. In fact, the process $\min \{ \xi_t, \eta_t \}$ is a real-valued supermartingale and the process $\max \{ \xi_t, \eta_t \}$ is a real-valued submartingale. So that $E(M_{t\wedge s}) \supset M_s$ for $0 \leq s \leq t \leq T$.

**Remark 2.** For discrete time set-valued martingales, Ezzaki and Tahri (Corollary 3.8 of [2]) proposed that in a separable RNP (Radon Nikodym Property) Banach space $\mathfrak{X}$, a sequence $\{F_n, n = 1, 2, \cdots \}$ is a uniformly integrable $K_{bc}(\mathfrak{X})$-valued martingale if and only if it admits a Castaing representation of regular martingale selections such that $\lim inf_{n \rightarrow \infty} \int_\Omega \|F_n\|dP < \infty$. Unfortunately, the ‘if’ part may not hold. For example, let $\{f_n, n = 1, \cdots \}$ be a uniformly integrable $\mathbb{R}$-valued martingale. For each $n$, we assume that the value of $f_n$ changes its signs with positive probability. Set $M_n = cl \{ \lambda f_n + 2(1 - \lambda)f_n; \lambda \in [0, 1] \cap \mathbb{Q} \}$. Then $M_n = \{ \min \{f_n, 2f_n\}, \max \{f_n, 2f_n\} \}$. As a manner similar to Example 3.3 we know that $\{M_n; n = 1, \cdots \}$ is not an interval-valued martingale. But $cl \{ \lambda f_n + 2(1 - \lambda)f_n; \lambda \in [0, 1] \cap \mathbb{Q} \}$ is a Castaing representation of regular martingale selections with

$$\lim inf_{n \rightarrow \infty} \int_\Omega \|M_n\|dP \leq 2\lim inf_{n \rightarrow \infty} \int_\Omega |f_n|dP < \infty.$$

**Proof of Theorem 3.1:**

**Proof.** (i) $\Rightarrow$ (ii): Assume there exists $G \in L^2(K(\mathfrak{X}))$ such that $M_t = E(M_0) + \int_0^t G_s dB_s$ a.s. for each $t$. Then $E(M_t) = cl \{E(M_0) + \int_0^t G_s dB_s\} = E(M_0)$, which implies $E\left(\int_0^t G_s dB_s\right) = [0]$. By Lemma 3.1 we obtain that $G$ is a degenerate set-valued stochastic process, which is denoted by $\{g\} (g \in L^2(\mathfrak{X}))$. Hence (ii) holds with $C = E(M_0)$.

(ii) $\Rightarrow$ (iii): Since $E(M_0)$ is a closed subset of the separable Banach space $\mathfrak{X}$, there exists a sequence $\{x_1, \cdots, x_n, \cdots \} \subset \mathfrak{X}$ such that $E(M_0) = cl \{x_1, \cdots, x_n, \cdots \}$. Then we have

$$M_t = cl \{x_1, \cdots, x_n, \cdots \} + \left\{ \int_0^t g_s dB_s \right\}$$

$$= cl \left\{x_1 + \int_0^t g_s dB_s, \cdots, x_n + \int_0^t g_s dB_s, \cdots \right\} =: cl \{f_1, \cdots, f_n, \cdots \}.$$

Apparently, such a sequence $\{f_1, \cdots, f_n, \cdots \}$ satisfies the requirement.

(iii) $\Rightarrow$ (i): Assume there exists an $\mathfrak{X}$-valued $\mathbb{A}$-martingale sequence $\{f_1, \cdots, f_n, \cdots \}$ such that $M_t = cl \{f_1, \cdots, f_n, \cdots \}$ a.s. for all $t$. Then by the classical martingale rep-
presentation theorem, there exists \( g \in L^2(X) \) such that \( f^i_t = E(f^i_0) + \int_0^t g_s dB_s \). Notice that for any \( i, j \), the difference \( f^i - f^j \) is non-random and independent of \( t \), then \( f^n_t = E(f^n_0) + \int_0^t g_s dB_s \) for each \( n \). Thus

\[
M_t = \text{cl} \left\{ E(f^n_0) + \int_0^t g_s dB_s, \cdots, E(f^n_0) + \int_0^t g_s dB_s, \cdots \right\} \\
= \text{cl} \left\{ E(f^n_0), \cdots, E(f^n_0), \cdots \right\} + \left\{ \int_0^t g_s dB_s \right\} = E(M_0) + \int_0^t G_s dB_s,
\]

where \( G = \{g\} \). In fact, each \( f^n_0 \) is constant a.s. since \( \mathcal{A}_0 \) is trivial.  \( \square \)

Interval-valued martingale is concrete and easy to deal with. It is sufficient to consider the endpoints. Now we give the representation theorem of interval-valued martingale and some examples when taking \( X = \mathbb{R} \).

**Theorem 3.2.** Let \( M = \{M_t = [a_t, b_t] : 0 \leq t \leq T\} \) be a \( K_1(\mathbb{R}) \)-valued \( \mathcal{A} \)-martingale. Then there exists an interval-valued stochastic process \( G \in L^2(K(\mathbb{R})) \) such that

\[
M_t = [E(a_0), E(b_0)] + \int_0^t G_s dB_s, \quad \text{for every } 0 \leq t \leq T \tag{3}
\]

if and only if \( b_t - a_t \) is a constant a.s. for each \( t \) and \( G \) degenerates to a singleton process \( \{g\} \) with \( g \in L^2(\mathbb{R}) \).

**Proof.** This is a direct consequence of Theorem 5.1. \( \square \)

**Example 3.4.** Let \( B = \{B_t : 0 \leq t \leq T\} \) be the real-valued Brownian motion as above. For \( t \in [0, T] \), set \( M_t = \left[ e^{B_t-\frac{t}{2}}, 1 + e^{B_t-\frac{t}{2}} \right] \), then \( \{M_t : t \in [0, T]\} \) is an interval-valued martingale with the following martingale representation

\[
M_t = [1, 2] + \left\{ \int_0^t e^{B_t-\frac{t}{2}} dB_s \right\} \text{ a.s. for all } t.
\]

In fact, \( E \left( \int_0^T (e^{B_t-\frac{t}{2}})^2 dt \right) = e^T - 1 < \infty \), then \( \left\{ e^{B_t-\frac{t}{2}} : t \in [0, T] \right\} \in L^2(\mathbb{R}) \). By Ito’s formula,

\[
d(e^{B_t-\frac{t}{2}}) = -\frac{1}{2} e^{B_t-\frac{t}{2}} dt + e^{B_t-\frac{t}{2}} dB_t + \frac{1}{2} e^{B_t-\frac{t}{2}} dt = e^{B_t-\frac{t}{2}} dB_t.
\]

Then \( e^{B_t-\frac{t}{2}} = 1 + \int_0^t e^{B_s-\frac{s}{2}} dB_s > 0 \) and \( E \left( e^{B_t-\frac{t}{2}} \right) = 1. \) Thus \( E(M_t) = [1, 2] \).

**Example 3.5.** Setting \( M_t = \left[ e^{B_t-\frac{t}{2}}, 2e^{B_t-\frac{t}{2}} \right] \), then \( \{M_t : t \in [0, T]\} \) is an interval-valued martingale since both the endpoints are real-valued martingales and \( E(M_t) = [1, 2] \). But \( M_t \neq [1, 2] + \int_0^t \left[ e^{B_s-\frac{s}{2}}, 2e^{B_s-\frac{s}{2}} \right] dB_s \). \( M_t \) does not have the martingale representation since the difference between endpoints of \( M_t \) is \( e^{B_t-\frac{t}{2}} \), not a constant.
Interval-valued martingale is a concrete example of set-valued martingales with wide applications in statistical modelling and practical fields such as econometrics, mathematical finance etc. For example, Sun et al. [8] proposed a threshold autoregressive model based on interval-valued data. In the model, \( \{u_t : t \in [0, T]\} \) is an interval-valued martingale difference sequence (the difference between two intervals is the Hukuhara difference which guarantees the difference of two identical sets is zero). There is an assumption in the model: \( E(u_t|I_{t-1}) = [0, 0] \) where \( I_{t-1} \) is the information set up to time \( t-1 \). Unfortunately, the assumption is not appropriate for non-degenerate interval-valued stochastic process.

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