DIFFUSION-INDUCED BLOWUP SOLUTIONS FOR THE SHADOW LIMIT MODEL OF A SINGULAR GIERER-MEINHARDT SYSTEM

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Abstract: In the current paper, we provide a thorough investigation of the blowing up behaviour induced via diffusion of the solution of the following non local problem

\[
\begin{align*}
\partial_t u &= \Delta u - u + \frac{u^p}{\left(\int_{\Omega} u^r \, dr\right)^{\gamma}} \quad \text{in } \Omega \times (0, T), \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \Gamma = \partial \Omega \times (0, T), \\
u(0) &= u_0,
\end{align*}
\]

where \(\Omega\) is a bounded domain in \(\mathbb{R}^N\) with smooth boundary \(\partial \Omega\); such problem is derived as the shadow limit of a singular Gierer-Meinhardt system, cf. [KS17, KS18]. Under the Turing type condition

\[
\frac{r}{p - 1} < \frac{N}{2}, \gamma r \neq p - 1,
\]

we construct a solution which blows up in finite time and only at an interior point \(x_0\) of \(\Omega\), i.e.

\[
u(x_0, t) \sim (\theta^*)^{-\frac{1}{r-1}} \left[\kappa(T - t)^{-\frac{1}{p-1}}\right],
\]

where

\[
\theta^* := \lim_{t \to T} \left(\int_{\Omega} u^r \, dr\right)^{-\gamma} \quad \text{and} \quad \kappa = (p - 1)^{-\frac{1}{p-1}}.
\]

More precisely, we also give a description on the final asymptotic profile at the blowup point

\[
u(x, T) \sim (\theta^*)^{-\frac{1}{p-1}} \left[\frac{(p - 1)^2}{8p} \frac{|x - x_0|^2}{\ln |x - x_0|}\right]^{-\frac{1}{r-1}} \quad \text{as } x \to 0,
\]

and thus we unveil the form of the Turing patterns occurring in that case due to driven-diffusion instability.

The applied technique for the construction of the preceding blowing up solution mainly relies on the approach developed in [MZ97] and [DZ19].

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1. Introduction

In as early as 1952, A. Turing in his seminal paper [Tur52] attempted, by using reaction-diffusion systems, to model the phenomenon of morphogenesis, the regeneration of tissue structures in hydra, an animal of a few millimeters in length made up of approximately 100,000 cells. Further observations on the morphogenesis in hydra led to the assumption of the existence of two chemical substances (morphogens), a slowly diffusing (short-range) activator and a rapidly diffusing (long-range) inhibitor. A. Turing, in [Tur52], indicates that although diffusion has a smoothing and trivializing effect on a single chemical, for the case of the interaction of two or more chemicals different diffusion rates could force the uniform steady states of the corresponding reaction-diffusion systems to become unstable and to lead to non-homogeneous distributions of such reactants. Since then, such a phenomenon is known as Turing-type instability or diffusion-driven instability (DDI).

Scrutinizing Turing’s idea further, Gierer and Meinhardt [GM72], proposed in 1972 the following activator–inhibitor system to model the regeneration phenomenon of hydra located in a domain \( \Omega \subset \mathbb{R}^N, N \geq 1 \)

\[
\begin{aligned}
\partial_t u &= \epsilon \Delta u - u + \frac{u^p}{v^q} & \text{in } \Omega \times (0,T), \\
\tau \partial_t v &= D \Delta v - v + \frac{r}{v^s} & \text{in } \Omega \times (0,T), \\
\frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0 & \text{in } \partial \Omega \times (0,T), \\
u(0) &= u_0 > 0 \text{ and } v(0) = v_0 > 0.
\end{aligned}
\]  

(1.1)

Here \( \nu \) stands for the unit outer normal vector to \( \partial \Omega \), whilst \( u \) and \( v \) are the concentrations of the activator and the inhibitor, respectively. Besides, \( \epsilon \) and \( D \) represent the diffusing coefficients and exponents \( p, q, r, s \) measuring the morphogens interactions satisfy

\[ p > 1, q, r > 0 \text{ and } s > -1. \]

A biologically interesting case arises when the activator diffuses much faster compared to the inhibitor. So in the case where \( D \to +\infty \), dividing the second one in (1.1) by \( D \), we take formally that for any \( t \in (0,T) \) and thanks to the Neumann boundary condition, activator’s concentration \( v \) will be spatial homogeneous, i.e. \( v(x,t) = \xi(t) \), cf. [KS17], [KS18] and [KBM19] (a rigorous proof for a version of Gierer-Meinhardt system can be found in [MCM17] and [MCHKS18], whilst for the case of general reaction-diffusion systems the interested reader can check [BK19]).

Next, integrating the second equation in (1.1), we finally derive the shadow system for \( u \) and \( \xi \)

\[
\begin{aligned}
\partial_t u &= \epsilon^2 \Delta u - u + \frac{u^p}{v^q} & \text{in } \Omega \times (0,T), \\
\tau \partial_t \xi &= -\xi + \frac{\int_{\Omega} u^p \, dx}{\xi^r} & \text{in } \Omega \times (0,T), \\
\frac{\partial u}{\partial \nu} &= 0 & \text{in } \partial \Omega \times (0,T), \\
u(0) &= u_0 > 0,
\end{aligned}
\]  

(1.2)

where

\[ \int_{\Omega} u^r \, dx = \frac{1}{|\Omega|} \int_{\Omega} u^r \, dx. \]
We now focus on the case $\tau = 0$, that is when inhibitor’s response rate is quite small against inhibitor’s growth, and thus by the second equation in (1.2) we derive
\begin{equation}
\xi(t) = \left( \int_{\Omega} u^r \, dx \right)^{\frac{1}{1+r}}.
\end{equation}

Next plugging (1.3) into (1.2), we finally obtain the following non-local system
\begin{equation}
\begin{cases}
\partial_t u = \Delta u - u + \frac{u^p}{\left( \int_{\Omega} u^r \, dr \right)} & \text{in } \Omega \times (0, T), \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega \times (0, T) \\
u(0) = u_0 \geq 0,
\end{cases}
\end{equation}
where $\gamma = \frac{1}{s+1}$ and $\epsilon = 1$ for simplicity.

Notably under the Turing condition
\begin{equation}
p - r\gamma < 1
\end{equation}
the spatial homogeneous solution of (1.4) given by
\begin{equation}
\frac{du}{dt} = -u + u^{p-r\gamma}, \quad u(0) = u_0 \geq 0,
\end{equation}
never exhibits blow-up, since the non-linearity is sublinear, and its unique stationary state $u = 1$ is asymptotically stable, cf. [KS17]. On the other hand, under condition (1.5) the solution of (1.4) when domain $\Omega$ is the unit $N$-dimensional sphere, $N \geq 3$, exhibits a Type I finite-time blow-up only at the center of the sphere, cf. [KS17, Theorem 3.7], and thus a Turing type instability (in the form of diffusion-driven blow-up) occurs. Notably, the blowup time $T$ can be done relatively small by reducing properly the size of the initial data, cf. [KS17] and [KS18]. For analogous blowup results for the Gierer-Meinhardt system on an evolving domain and for a non-local Fisher-KPP equation one can see [KBM19] and [KL20] respectively.

The main purpose of the current work is to describe the form of the developing Turing instability (blowup) patterns for the solution of problem (1.4) in a region of any blowup point. To this end we first note that the non-local equation in (1.4) is closely associated to the standard heat equation given in (1.6). Indeed, if we ignore the linear term $-u$ and take $\gamma = 0$, then it turns out the classical nonlinear heat equation
\begin{equation}
\partial_t u = \Delta u + u^p.
\end{equation}
Let us now recall some well known results link with the blowup behaviour of (1.6). Firstly, (1.6) is well-posed in $L^\infty(\Omega)$, hence, for any $u_0 \in L^\infty(\Omega)$, either the solution is global or it blows up in finite time $T = T(u_0)$, i.e
\begin{equation}
\|u(\cdot, t)\|_{L^\infty(\Omega)} \to +\infty \text{ as } t \to T.
\end{equation}

There is also a great amount of works related to the blowing up behaviour of (1.6) and its various generalizations. In particular, the construction of blowup solutions of (1.6) is discussed in [Bre92], [BK94], [MZ97b], [MZ97a] whilst one can find in [IZ19], [DNZ18], [NZ18] analogous constructions for perturbed nonlinear source terms. Besides, in [MZ08] [NZar], [DNZ20] such blowup solutions are build for complex Ginzburg-Landau equations and in [NZ15], [Duo19b], [Duo19a] for complex-valued heat equations which lack variational structure; the case of parabolic systems is examined in [GNZ16]. Additionally, in [DZ19] a singular solution associated with a duality concept to blowup phenomenon, called quenching (or
touch-down in MEMS literature), is constructed. In particular, the authors in [DZ19] developed further the idea of [MZ97a] to describe the quenching behaviour of a non-local problem arising from MEMS industry (see more in [DKN20], [GK12], [GS15], [KLN16] and the references therein).

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2. MAIN RESULTS

In the current paper our main aim is to construct a solution to (1.4) which blows up in finite time $0 < T < 1$ and a point $x_0 \in \Omega$, that is

$$u(x_0, t) \to +\infty, \text{ as } t \to T.$$ 

In that case (1.7) holds, and our result holds true under the Turing condition, i.e. when

$$\frac{r}{p-1} < \frac{N}{2} \quad \text{and} \quad \gamma r \neq p - 1.$$ 

**Theorem 2.1.** Let $\Omega$ be a smooth and bounded domain in $\mathbb{R}^N$ and also assume (2.1) is satisfied. Then, for any arbitrary $x_0 \in \Omega$ there exist initial data $u_0 \geq 0$ such that the solution of (1.4) blows up in finite time $T(u_0) < 1$, only at $x_0$. Moreover, we have the following blowup profiles

(i) The intermediate profile for all $t \in (0, T)$

$$\left\| (T - t)^{\frac{1}{p-r}} u(\cdot, t) - (\theta^*)^{-\frac{1}{p-r}} \varphi_0 \left( \frac{x - x_0}{\sqrt{(T - t) \ln(T - t)}} \right) \right\|_{L^\infty(\Omega)} \leq \frac{C}{1 + \sqrt{\ln(T - t)}}.$$ 

where

$$\theta^* = \lim_{t \to T} \left( \int_\Omega u^r \, dx \right)^{-\gamma} > 0,$$

and

$$\varphi_0(z) = \left( p - 1 + \frac{(p - 1)^2}{4p} |z|^2 \right)^{-\frac{1}{p-r}}.$$ 

(ii) The final profile at $t = T$ is provided by $u(x, t) \to u^*(x) \in C^2(\Omega \setminus \{x_0\})$ uniformly on compact sets of $\Omega \setminus \{x_0\}$. In particular, near $x_0$, solution $u^*$ behaves as

$$u^*(x) \sim (\theta^*)^{-\frac{1}{p-r}} \left[ \frac{(p - 1)^2}{8p} \frac{|x - x_0|^2}{\ln |x - x_0|} \right]^{-\frac{1}{p-r}} \text{ as } x \to x_0.$$ 

Consequently, we also describe the asymptotic of $\|u\|_{L^k(\Omega)}^k$ in the following.

**Corollary 2.2** (Behavior of $L^k$ norm at blowup time). Let $u$ be the constructed blowup solution given by Theorem 2.1 and $k > 0$. Then, the following hold:

(i) If $\frac{k}{p-1} < \frac{N}{2}$, then

$$\|u\|_{L^k(\Omega)}^k \to C(k) < +\infty, \text{ as } t \to T.$$
\[ \frac{k}{p-1} > \frac{N}{2}, \quad \text{then} \]

\[ \| u(t) \|_{L^k(\Omega)}^k = \left( \theta(t)^{-\frac{k}{p-1}} \int_0^\infty \varphi_0^{k}(r)r^{N-1}dr + o_{t \to T(1)}(1) \right) \left( T-t \right)^{\frac{k}{2} - \frac{1}{p} - \frac{1}{k}} + o_{t \to T(1)}(1). \]

\[ \frac{k}{p-1} - \frac{N}{2} = 0, \quad \text{then} \]

\[ \| u(t) \|_{L^k(\Omega)}^k = \left( \theta(t)^{-\frac{k}{p-1}} \frac{k}{2} + 1 \int_0^\infty \varphi_0^{p-1+k}(r)r^{N-1}dr + o_{t \to T(1)}(1) \right) \left( \ln(T-t) \right)^{k/2 + 1}. \]

**Remark 2.3 (Open problem).** Note that in (2.1), we impose \( \gamma r \neq p-1, \) since otherwise, due to (4.2), our method does not work.

**Remark 2.4.** In Theorem 2.1, a solution blowing up at an arbitrary \( x_0 \) is constructed. However, by the translation \( x - x_0 \) we can always derive a blowing up solution at 0. So, we need to prove Theorem 2.1 only for \( x_0 = 0 \). In addition, we can apply the technique of [Mer92], and we can establish a blowup solution at only \( k \) points \( x_1, \ldots, x_k \) with blowup profiles provided by Theorem 2.1 by replacing \( L^\infty(\Omega) \) with \( L^\infty(|x - x_j| \leq \epsilon_j) \) at each blowup point \( x_j \).

**Remark 2.5 (Stability of the blowup profile).** Let us consider \( \hat{u} \), the constructed solution in Theorem 2.1 with initial data \( \hat{u}_0 \) which blows up at time \( \hat{T} \) and at the point \( \hat{x}_0 \). Then there exists an open neighborhood of \( \hat{u}_0 \) in a sub-space of \( C(\bar{\Omega}) \) and with a suitable topology, named \( \hat{U}_0 \) such that for all \( u_0 \in \hat{U}_0 \), the corresponding solution of (1.4) blows up at \( x(u_0) \) and only at \( T(u_0) \) with blowup profiles given by in Theorem 2.1 by replacing \( x_0 \) to \( x(u_0) \). In particular, we have

\[ (x(u_0), T(u_0), \theta^*(u_0)) \to (\hat{x}_0, \hat{T}, \hat{\theta}^*) \text{ as } u_0 \to \hat{u}_0, \]

where

\[ \theta^*(u_0) = \lim_{t \to T(u_0)} \left( \int_\Omega u^\gamma dx \right)^{-\gamma} \quad \text{and} \quad \hat{\theta}^* = \lim_{t \to T(u_0)} \left( \int_\Omega \hat{u}^\gamma dx \right)^{-\gamma}. \]

The stability result follows by the interpretation of the parameters of the finite-dimensional problem in terms of the blowup time and the blowup point, see more in [MZ97b].

### 3. Formal approach

In this section, we aim at giving a formal approach which explains how the profile in Theorem 2.1 is derived. Firstly, let us denote

\[ \theta(t) = \frac{1}{\left( \int_\Omega u^\gamma(t) dx \right)^{\gamma}}. \]

Henceforth, we rewrite equation (1.4) as follows

\[ \partial_t u = \Delta u - u + \theta(t)u^p. \]

We can see that \( \theta(t) \) strongly affects the blowup dynamic of \( u \). Let us assume that \( u \) blows up in finite time \( T \) and at the origin \( 0 \in \Omega \) (without loss of generality).
Then, there are the following three possibilities:

\[
\theta(t) \to 0 \quad \text{as} \quad t \to T, \tag{3.3}
\]

\[
\theta(t) \to \theta^* > 0 \quad \text{as} \quad t \to T, \tag{3.4}
\]

\[
\theta(t) \to +\infty \quad \text{as} \quad t \to T. \tag{3.5}
\]

In particular, (3.5) is excluded by Theorem 3.1 and Remark 3.2 given in [KS17].

In the context of this work, we aim to handle only the convergent case i.e (3.4), whilst the case (3.3) will be treated in a forthcoming paper. We note that the convergent case (3.4) is quite the same with the situation studied in [DZ19] where such a convergence used to idea to control the behaviour of the non-local term occurred there in towards the quenching time. We rewrite (3.2) as follows

\[
\partial_t u = \Delta u + \theta^* u^p + (\theta(t) - \theta^*)u^p - u
\]

and we formally neglect the term

\[
(\theta(t) - \theta^*)u^p - u,
\]

since, it is relatively small compared to the main non-linear term

\[
\theta^* u^p.
\]

Therefore, it is important to study the limit problem

\[
\partial_t u = \Delta u + \theta^* u^p,
\]

instead of the full model (3.2). In addition to that, using the following re-scaled form

\[
u(x,t) = (\theta^*)^{-\frac{1}{p-1}}U(x,t),
\]

\(U\) then solves

\[
(3.6) \quad \partial_t U = \Delta U + U^p,
\]

i.e. the standard heat equation whose blowup solutions have been studied thoroughly, cf. Section[1]. In particular, the approach developed in [MZ97] to construct a very precise asymptotic profile of blowup solutions to (3.6) is quite related to our work. Indeed, for some positive constants \(K_0\) and \(\epsilon_0\), we now cover \(\Omega\) by

\[
\Omega = P_1(t) \cup P_2(t) \cup P_3(t),
\]

where

\[
P_1(t) = \left\{ x \in \mathbb{R}^N \mid |x| \leq K_0 \sqrt{(T-t)|\ln(T-t)|} \right\}, \tag{3.7}
\]

\[
P_2(t) = \left\{ x \in \mathbb{R}^N \mid \frac{K_0}{4} \sqrt{(T-t)|\ln(T-t)|} \leq |x| \leq \epsilon_0 \right\}, \tag{3.8}
\]

\[
P_3(t) = \left\{ x \in \mathbb{R}^N \mid |x| \geq \frac{\epsilon_0}{4} \right\} \cap \Omega. \tag{3.9}
\]

Here and throughout the proof, we assume that \(T < 1\). Indeed, all the key estimates of our proof (Propositions 4.4, 4.5, etc.) hold for \(T\) small enough. On each domain, we will control the solution with a suitable behavior. In particular, we also name them by \(P_1(t)\)-blowup region; \(P_2(t)\)-intermediate region; and \(P_3(t)\)-regular region.

Asymptotic blowup profile in region \(P_1\):
The region $P_1$ is the region where the blowup phenomenon mainly occurs. Besides, the blowup dynamic is described via the following similarity variable introduced in [GK85]

\begin{equation}
  y = \frac{x}{\sqrt{T - t}}, \quad s = -\ln(T - t) \quad \text{and} \quad W(y, s) = (T - t)^{\frac{1}{p-1}} U(x, t).
\end{equation}

Hence, $W$ solves

\begin{equation}
  \partial_s W = \Delta W - \frac{1}{2} y \cdot \nabla W - \frac{W}{p - 1} + W^p, \quad \forall (y, s) \in \Omega, \times [-\ln(T - t), +\infty),
\end{equation}

where $\Omega_x = e^{\xi} \Omega$. Following [MZ97a] page 149, the generic profile inside this region is given as following

\[ W(y, s) \sim \varphi(y, s) = \left( p - 1 + \frac{(p - 1)^2 |y|^2}{4p} \right)^{-\frac{1}{p-1}} + \frac{\kappa N}{2p^s}. \]

### Asymptotic of the intermediate profile in region $P_2$:

In region $P_2$ we try to control a re-scaled function $U$ instead of $W$. For all $|x|$ small, we can define $t(x)$ as the unique solution of

\begin{equation}
  |x| = \frac{K_0}{4} \sqrt{(T - t(x))|\ln(T - t(x))|} \quad \text{with} \quad t(x) < T.
\end{equation}

Then, we introduce the re-scaled $U$ by

\begin{equation}
  U(x, \xi, \tau) = (T - t(x))^{\frac{1}{p-1}} U \left( x + \xi \sqrt{T - t(x)}, (T - t(x))\tau + t(x) \right),
\end{equation}

where $\xi \in (T - t(x))^{-\frac{1}{2}} (\Omega - x)$ and $\tau \in \left[ -\frac{t(x)}{T - t(x)}, 1 \right]$. Note that, $t(x)$ is well defined as long as $\epsilon_0$ is small enough and we have the following asymptotic behaviour

\[ t(x) \to T, \quad \text{as} \quad x \to 0. \]

For convenience, we introduce

\begin{equation}
  \varrho(x) = T - t(x),
\end{equation}

so, it follows

\[ \varrho(x) \to 0 \quad \text{as} \quad x \to 0, \]

and by virtue of (3.6), we derive that $U$ solves

\[ \partial_\tau U = \Delta U + U^p. \]

Now we recall the main argument in [MZ97a], which demonstrates that is only sufficient to study the dynamic of $U$ on a small region of the local space $(\xi, \tau)$ defined by

\[ |\xi| \leq \alpha_0 \sqrt{|\ln(\varrho(x))|} \quad \text{and} \quad \tau \in \left[ -\frac{t(x)}{\varrho(x)}, 1 \right]. \]

When $\tau = 0$, we are in region $P_1(t(x))$; in fact in that case $P_1(t(x))$ and $P_2(t(x))$ have some overlapping by their definitions. Due to the imposed to constraints in region $P_1(t(x))$, we derive that $U(x, \xi, 0)$ is flat in the sense that

\[ U(x, \xi, 0) \sim \left( p - 1 + \frac{(p - 1)^2 K_0^2}{4p} \right)^{-\frac{1}{p-1}}. \]

The main idea is to show that this flatness is preserved for all $\tau \in [0, 1)$ (that is for all $t \in [t(x), T)$), in the sense that the solution does not depend substantially
on space. For that purpose \( U \) is regarded as a perturbation of \( \hat{U}(\tau) \), where \( \hat{U}(\tau) \) is defined as follows

\[
\begin{aligned}
\partial_{\tau} \hat{U}(\tau) &= \hat{U}^p(\tau), \\
\hat{U}(0) &= \left( p - 1 + \frac{(p - 1)^2 K_n^2}{4p} \right)^{-\frac{1}{p-1}},
\end{aligned}
\]

and is explicitly given by

\[
\hat{U}(\tau) = \left( (p - 1)(1 - \tau) + \frac{(p - 1)^2 K_n^2}{4p} \right)^{-\frac{1}{p-1}}.
\]

Asymptotic profile in the regular region \( P_3 \):

Using the well-posedness of the Cauchy problem for equation (3.6), we derive the asymptotic profile of the solution \( U \) within that region as a perturbation of initial data \( U(0) \).

4. Formulation of the full problem

In the current section, we aim at stating the rigorous steps towards the proof of Theorem 2.1.

4.1. Similarity variable. Let \( u \) be a solution of (1.4) then we introduce

\[
U(x,t) = \theta(t) \left( T - t \right)^{-\frac{1}{p-1}} u(x,t),
\]

which by virtue of (3.1) entails

\[
\theta(t) = \left( \frac{1}{\Omega} \int_{\Omega} U^{r} \, dx \right)^{-\frac{1}{2}}.
\]

Next using equation (1.4), \( U \) reads

\[
\partial_{t} U = \Delta U + U^p + \left( \frac{1}{p - 1} \frac{\theta'(t)}{\theta(t)} - 1 \right) U,
\]

where \( \theta(t) \) is defined as in (4.2). Now, we use the similarity variable introduced in (3.10) to derive

\[
\partial_{s} W = \Delta W - \frac{1}{2} y \cdot \nabla W - \frac{W}{p - 1} + W^p + \left( \frac{1}{p - 1} \frac{\tilde{\theta}(s)}{\theta(s)} - e^{-s} \right) W,
\]

where

\[
\tilde{\theta}(s) = \theta(t(s)), \quad s = -\ln(T - t),
\]

and \( y \in \Omega_s = e^{s} \Omega \).

Note that there are some technical difficulties arising by the evolution of \( \Omega_s \) which we can overcome by using the approach introduced in [MNZ16] (also used in [DZ19]), and resolves this technical issue via the extension of problem on \( \mathbb{R}^n \).

Indeed, let us introduce \( \chi_0 \in C_0^\infty ([0, +\infty)) \), satisfying

\[
\text{supp}(\chi_0) \subset [0, 2], \quad 0 \leq \chi_0(x) \leq 1, \forall x \text{ and } \chi_0(x) = 1, \forall x \in [0, 1].
\]

Then, we define the following function

\[
\psi_{M_0}(y, s) = \chi_0 \left( M_0 ye^{-\frac{s}{2}} \right), \text{ for some } M_0 > 0.
\]
Let us introduce
\begin{equation}
    w(y, s) = \begin{cases}
        W(y, s)\psi_{M_0}(y, s) & \text{if } y \in \Omega_s, \\
        0 & \text{otherwise}.
    \end{cases}
\end{equation}
Using equation (4.4), \( w \) reads
\begin{equation}
    \partial_s w = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{1}{p-1} w + w^p + \left( \frac{1}{p-1} \frac{\bar{\theta}'(s)}{\theta(s)} - e^{-s} \right) w + F(w, W),
\end{equation}
where \( F(w, W) \) is given by
\begin{align}
    F(w, W) &= \begin{cases}
        W \left[ \partial_s \psi_{M_0} - \Delta \psi_{M_0} + \frac{1}{2} y \cdot \nabla \psi_{M_0} \right] - 2\nabla \psi_{M_0} \cdot \nabla W \\
        + \psi_{M_0} (W)^p - w^p, & \text{if } y \in \Omega e^s, \\
        0, & \text{otherwise}.
    \end{cases}
\end{align}
Note that the nonlinear term \( F(w, W) \) is quite the same as the one occurs in \([DZ19]\) and thus can be neglected. More precisely, the growth of following terms
\begin{align}
    \left( \frac{1}{p-1} \frac{\bar{\theta}'(s)}{\theta(s)} - e^{-s} \right) w + F(w, W),
\end{align}
can be controlled and they actually decay exponentially.

Then, using \([BK94]\) and \([MZ97b]\), we get the following blowup profile:
\begin{equation}
    \varphi(y, s) := \left( p - 1 + \frac{(p-1)^2 |y|^2}{4p} \right)^{-\frac{1}{p-1}} + \frac{\kappa N}{2ps}, \quad \kappa = (p - 1)^{-\frac{1}{p-1}}.
\end{equation}
We now linearize around \( \varphi \)
\begin{equation}
    q = w - \varphi,
\end{equation}
hence, \( q \) solves
\begin{equation}
    \partial_s q = (\mathcal{L} + V)q + B(q) + R(y, s) + G(w, W),
\end{equation}
where
\begin{align}
    \mathcal{L} &= \Delta - \frac{1}{2} y \cdot \nabla + Id, \\
    V(y, s) &= p \left( \varphi^{p-1}(y, s) - \frac{1}{p-1} \right), \\
    B(q) &= (q + \varphi)^p - \varphi^p - p\varphi^{p-1}q, \\
    R(y, s) &= -\partial_s \varphi + \Delta \varphi - \frac{1}{2} y \cdot \nabla \varphi - \frac{\varphi^3}{3} + \varphi^p, \\
    G(w, W) &= \left( \frac{1}{p-1} \frac{\bar{\theta}'(s)}{\theta(s)} - e^{-s} \right) (q + \varphi) + F(w, W),
\end{align}
and \( F(w, W) \) is defined by (4.9). Let us remark that equation (4.12) is quite the same as in the classical nonlinear heat equation apart for the extra term \( G \). Let us point out that this term has two important features. On the one hand, it is a novel term, with respect to old literature (\([BK94]\) and \([MZ97b]\) in particular), very delicate to control, which makes our paper completely relevant. On the other hand, we will show in Lemma 9.4 below that \( G \) is exponentially small (in \( s \)), which means that its contribution will not affect the dynamics, which lay in \( s^{-i} \) scales (with possible logarithmic corrections), as one may see from Definition 4.4 of the
shrinking set below, particularly item (i) with the estimates in the blowup region \( P_t(t) \).

In the following, we recall some properties of the linear operator \( \mathcal{L} \) and the potential \( V \).

**Operator \( \mathcal{L} \)**

Operator \( \mathcal{L} \) is self-adjoint in \( \mathcal{D}(\mathcal{L}) \subset L^2_\rho(\mathbb{R}^N) \), where \( L^2_\rho(\mathbb{R}^N) \) defined as follows

\[
L^2_\rho(\mathbb{R}^N) = \left\{ f \in L^2_{\text{loc}}(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} |f(y)|^2 \rho(y)dy < +\infty \right\},
\]

and

\[
\rho(y) := \frac{e^{-|y|^2}}{(4\pi)^{\frac{N}{2}}}.
\]

Besides, its spectrum set is explicitly given by

\[
\text{Spec}(\mathcal{L}) = \left\{ 1 - \frac{m^2}{2} \mid m \in \mathbb{N} \right\}.
\]

Accordingly to the eigenvalue \( \lambda_m = 1 - \frac{m^2}{2} \), the correspond eigen-space is given by

\[
E_m = \left\langle h_{m_1}(y_1), h_{m_2}(y_2), \ldots, h_{m_N}(y_N) \mid m_1 + \ldots + m_N = m \right\rangle,
\]

where \( h_{m_i} \) is the (re-scaled) Hermite polynomial in one dimension.

**Key properties of potential \( V \):**

(i) The potential \( V(\cdot, s) \to 0 \) in \( L^2_\rho(\mathbb{R}^N) \) as \( s \to +\infty \): In particular, in the region \( |y| \leq K_0^\sqrt{s} \) (the singular domain), \( V \) has some weak perturbations on the effect of operator \( \mathcal{L} \).

(ii) \( V(y, s) \) is almost a constant on the region \( |y| \geq K_0^\sqrt{s} \): For all \( \epsilon > 0 \), there exists \( C_\epsilon > 0 \) and \( s_\epsilon \) such that

\[
\sup_{s \geq s_\epsilon, |y| \geq C_\epsilon} \left| V(y, s) - \left( -\frac{p}{p-1} \right) \right| \leq \epsilon.
\]

Note that \( -\frac{p}{p-1} < -1 \) and that the largest eigenvalue of \( \mathcal{L} \) is 1. Hence, roughly speaking, we may assume that \( \mathcal{L} + V \) admits a strictly negative spectrum. Thus, we can easily control our solution in the region \( \{|y| \geq K_0^\sqrt{s} \} \) with \( K_0 \) large enough.

From the preceding properties, it appears that the operator of \( \mathcal{L} + V \) does not share the same behaviour inside and outside of the singular domain \( \{|y| \leq K_0^\sqrt{s} \} \). Therefore, it is natural to decompose every \( r \in L^\infty(\mathbb{R}^N) \) as follows:

\[
r(y) = r_b(y) + r_e(y) = \chi(y, s)r(y) + (1 - \chi(y, s))r(y),
\]

where \( \chi(y, s) \) is defined as follows

\[
\chi(y, s) = \chi_0 \left( \frac{|y|}{K_0^\sqrt{s}} \right),
\]

recalling that \( \chi_0 \) is given by (4.14). From the above decomposition, we immediately have the following:

\[
\text{Supp } (r_b) \subset \{|y| \leq 2K_0^\sqrt{s} \}, \quad \text{Supp } (r_e) \subset \{|y| \geq K_0^\sqrt{s} \}.
\]
In the following we are interested in expanding \( r_b \) in \( L^2_\rho(\mathbb{R}^N) \) according to the basis which is created by the eigenfunctions of operator \( L \):

\[
r_b(y) = r_0 + r_1 \cdot y + y^T \cdot r_2 \cdot y - 2 \operatorname{Tr}(r_2) + r_-(y),
\]

or

\[
r_b(y) = r_0 + r_1 \cdot y + r_-(y),
\]

where

\[
(4.20) \quad r_i = (P_\beta(r_b))_{\beta \in \mathbb{N}^N, |\beta| = i}, \forall i \geq 0,
\]

with \( P_\beta(r_b) \) being the projection of \( r_b \) on the eigenfunction \( h_\beta \) defined as follows:

\[
(4.21) \quad P_\beta(r_b) = \int_{\mathbb{R}^N} r_b \frac{h_\beta}{\|h_\beta\|_{L^2_\rho(\mathbb{R}^N)}} \rho dy, \forall \beta \in \mathbb{N}^N.
\]

Besides that, we also denote

\[
(4.22) \quad r_\perp = P_\perp(r) = \sum_{\beta \in \mathbb{N}^N, |\beta| \geq 2} h_\beta P_\beta(r_b),
\]

and

\[
(4.23) \quad r_- = \sum_{\beta \in \mathbb{R}^N, |\beta| \geq 3} h_\beta P_\beta(r_b).
\]

In other words, \( r_\perp \) is the part of \( r_b \) which is orthogonal to the eigenfunctions corresponding to eigenvalues 0 and 1 and \( r_- \) is orthogonal to the eigenfunctions corresponding to eigenvalues 1, \( \frac{1}{2} \) and 0. We should note that \( r_0 \) is a scalar, \( r_1 \) is a vector and \( r_2 \) is a square matrix of order \( N \); they are all components of \( r_b \) and not \( r \). Finally, we write \( r \) as follows

\[
(4.24) \quad r(y) = r_0 + r_1 \cdot y + y^T \cdot r_2 \cdot y - 2 \operatorname{Tr}(r_2) + r_- + r_e(y).
\]

or

\[
(4.25) \quad r(y) = r_0 + r_1 \cdot y + r_\perp(y) + r_e(y).
\]

4.2. Localization variable. In this part, we will state the rigorous form of problem \( (3.13) \) in region \( P_2 \). Using the equation of (1.4) then \( U \) defined by (3.13) satisfies:

\[
(4.26) \quad \partial_t U = \Delta_2 U + U^p + \left( \frac{1}{p-1} \frac{\dot{\theta}(\tau)}{\theta(\tau)} - \rho(x) \right) U,
\]

where

\[
(4.27) \quad \dot{\theta}(\tau) = \theta(t) = \theta(\tau \rho(x) + t(x)), \quad \text{and } \rho(x) \text{ defined as in } (3.14).
\]

4.3. Shrinking set. Below, we aim to construct a special set where the behaviour of the solution \( U \) of equation (4.3) can be controlled.

Definition 4.1 (Definition of \( S(t) \)). Let us consider positive constants \( K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0 \) and take \( t \in [0, T) \) for some \( T > 0 \). Then, we introduce the following set

\[
S(K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0, t) \quad (S(t) \text{ in short}),
\]

as the subset of \( C^2(\Omega) \cap C(\bar{\Omega}) \), containing all functions \( U \) satisfying the following conditions:
In particular, we obtain the following improved estimate in the blowup region:

\[ |r_i| \leq \frac{A}{s^2}(i = 0, 1), \text{ and } |r_2| \leq \frac{A^2 \ln s}{s^2}, \]

\[ |r_-(y)| \leq \frac{A^2}{s^2}(1 + |y|^3) \text{ and } |(\nabla r_\perp)| \leq \frac{A}{s^2}(1 + |y|^3), \forall y \in \mathbb{R}^N, \]

\[ \|r_e\|_{L^\infty(\mathbb{R}^N)} \leq \frac{A^2}{\sqrt{s}}, \]

where the definitions of \(r_i, r_-, (\nabla r)_\perp\) and \(r_e\) are given by (4.20), (4.22), (4.23), and (4.18), respectively, depending on the similarity variables defined by (4.10).

(ii) **Estimates in \(P_2(t)\):** For all \(|x| \in \left[\frac{K_0}{\sqrt{t}}, \sqrt{(T-t)}][\ln(T-t)], c_0\right]\), \(\tau(x,t) = \frac{t-t(x)}{g(x)}\) and \(|\xi| \leq \alpha_0 \sqrt{\ln g(x)}\), we have the following

\[ |U(x,\xi,\tau(x,t)) - \hat{U}(\tau(x,t))| \leq \delta_0, \]

\[ |\nabla_\xi U(x,\xi,\tau(x,t))| \leq \frac{C_0}{\sqrt{\ln g(x)}} \]

where \(U, \hat{U}\) and \(g(x)\) defined as in (3.13), (3.14) and (3.15), respectively.

(iii) **Estimates in \(P_3(t)\):** For all \(|x| \in \{ |x| \geq \frac{A}{2} \} \cap \Omega\), we have

\[ |U(x,t) - U(x,0)| \leq \eta_0, \]

\[ |\nabla U(x,t) - \nabla e^{\Lambda t} U(x,0)| \leq \eta_0. \]

Using the definition of \(S(t)\), we have the following

**Lemma 4.2** (Growth estimates for \(q\) belonging to \(V_A(s)\)). Let us consider \(K_0 \geq 1\) and \(A \geq 1\). Then, there exists \(s_1 = s_1(A, K_0)\) such that for all \(s \geq s_1\) and \(q \in V_A(s)\), we have the following estimates:

\[ |q(y, s)| \leq \frac{C(K_0) A^2}{\sqrt{s}} \text{ and } |q(y, s)| \leq \frac{C(K_0) A^2 \ln s}{s^2}(1 + |y|^3). \]

In particular, we obtain the following improved estimate in the blowup region

\[ \|q\|_{L^\infty(\{|y| \leq K_0 \sqrt{s}\})} \leq \frac{C(K_0) A}{\sqrt{s}}. \]

**Proof.** The proof immediately arises by adding the given bounds in Definition 4.1.

**Remark 4.3** (Universality constant). In our proof, we introduce a lot of parameters, then, the universality upper bound will depend on these parameters. For more convenience, from now on, we denote the universality constant by \(C\) as long as it only depends on \(K_0, \Omega, r, \gamma, \mu\) and \(N\), intrinsic constants. However, once it depends more on extra-parameters, example \(C_2\) as in Proposition 4.4 below, we will write \(C(C_2)\).

Next we derive the dynamic of \(\theta\) defined as in (4.2):
4.4. Constructing initial data. In this part, we want to build initial data $U_0 \in S(0)$ for equation (4.3). To this end we follow a similar approach with one developed in [DZ19]. Firstly, we introduce the following cut-off function

$$
\chi_1(x) = \chi_0 \left( \frac{|x|}{\sqrt{T} \ln |T|} \right),
$$

where $\chi_0$ defined in (4.6). Next, we introduce $H^*$ as a suitable modification of the final asymptotic profile in the intermediate region:

$$
H^*(x) = \begin{cases}
\left[ \frac{(p-1)^2}{8p} \frac{|x|^2}{\ln |x|} \right]^{-\frac{1}{p-1}}, & \forall|x| \leq \min\left( \frac{1}{4}d(0, \partial \Omega), \frac{1}{2} \right), x \neq 0, \\
1, & \forall|x| \geq \frac{1}{4}d(0, \partial \Omega).
\end{cases}
$$

Now for any $(d_0, d_1) \in \mathbb{R}^{1+N}$, we define initial data

$$
U_{d_0, d_1}(x, 0) = H^*(x) (1 - \chi_1(x)),
$$

where $z_0 = \frac{x}{\sqrt{T} \ln |T|}$, $s_0 = -\ln T$; $\varphi, \chi_0, \chi_1$ and $H^*$ are defined as in (4.10), (4.6), (4.31) and (4.32), respectively.

**Proposition 4.4** (Dynamic of $\theta$). Let us consider (2.1) with $\Omega$ a bounded domain with smooth boundary, and $C_2 > 0$. Then, there exists $K_2 > 0$ such that for all $K_0 \geq K_2, \delta_0 > 0$, we can find $\epsilon_2(K_0, \delta_0, C_2) > 0$ such that for all $\epsilon_0 \leq \epsilon_2$ and $A \geq 1, C_0 > 0, \eta_0 > 0$, and $\alpha_0 > 0$, there exists $T_2 > 0$ such that for all $T \leq T_2$ the following holds: Assuming $U$ is a non-negative solution of equation (4.3) on $[0, t_1]$, for some $t_1 < T$ and $U(t) \in S(K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0, t) = S(t)$ for all $t \in [0, t_1]$ with initial data $U(0) = U_0$ satisfying the following estimate

$$
\frac{1}{C_2} \leq \int_{\Omega} U(t)^r dx \leq C_2,
$$

then there hold:

(i) For all $t \in [0, t_1]$, we have $\theta(t) > 0$ and

$$
\frac{1}{C(C_2)} \leq \theta(t) \leq C(C_2).
$$

Moreover, there exists $\varepsilon := \varepsilon(N, r, p) > 0$ such that for all $t \in (0, t_1)$, we have

$$
|\theta'(t)| \leq (T - t)^{-1+\varepsilon}.
$$

(ii) In particular, if $U \in S(t)$ for all $t \in [0, T)$, then there exists a constant $\theta^*(\Omega, r, p, N, \gamma, U(0)) > 0$ such that

$$
\theta(t) \to \theta^* \text{ as } t \to T.
$$

**Proof.** We kindly refer the readers to see the details of the proof in Appendix S.1. □
Note that we can also write the initial data in the variable similarity (4.10) corresponding \( y_0 = \frac{v}{\sqrt{T}} \in \Omega_{y_0} \) and \( s_0 = -\ln T \).

- For \( W(y_0, s_0) \):
  \[
  W(y_0, s_0) = \left\{ \begin{array}{l}
  \varphi(y_0, s_0) + (d_0 + z_0 \cdot d_1)\chi_0 \left( \frac{|z_0|}{T} \right) \\
  + \frac{T}{\pi} H^*(y_0 \sqrt{T}) \left( 1 - \chi_1(y_0 \sqrt{T}) \right)
  \end{array} \right.
  \]

- For \( w(y_0, s_0) \):
  \[
  w(y_0, s_0) = \left\{ \begin{array}{l}
  \varphi(y_0, s_0) + (d_0 + z_0 \cdot d_1)\chi_0 \left( \frac{|z_0|}{T} \right) \\
  + \frac{T}{\pi} H^*(y_0 \sqrt{T}) \left( 1 - \chi_1(y_0 \sqrt{T}) \right)
  \end{array} \right\} \psi_{M_0}(y_0, s_0),
  \]
  for all \( y_0 \in \Omega_{y_0} \). We should point out that \( \psi_{M_0} \) defined as in (4.37) and \( w \) vanishes outside \( \Omega_{y_0} \).

- For \( q(y_0, s_0) \):
  \[
  q(y_0, s_0) = w(y_0, s_0) - \varphi(y_0, s_0).
  \]

In the following, we construct appropriate initial data of the form (4.33), i.e. initial data which belong to the shrinking set \( S(0) \).

**Proposition 4.5** (Construction of initial data). There exists \( K_3 > 0 \) such that for all \( K_0 \geq K_3 \) and \( \delta_3 > 0 \), there exist \( \alpha_3(K_0, \delta_3) > 0 \) and \( C_3(K_0) > 0 \) such that for every \( \epsilon_0 \in (0, \epsilon_3] \), there exists \( \epsilon_3(K_0, \delta_3, \alpha_0) > 0 \) such that for every \( \epsilon_0 \in (0, \epsilon_3] \) and \( A \geq 1 \), there exists \( T_3(K_0, \delta_3, \epsilon_0, A, C_3) > 0 \) such that for all \( T \leq T_3 \) and \( s_0 = -\ln T \), then the following hold:

(I) We can find a set \( \mathcal{D}_A \subset [-2, 2] \times [-2, 2]^N \) such that if we define the following mapping

\[
\Gamma : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \times \mathbb{R}^N
\]

\[
(d_0, d_1) \mapsto (q_0, q_1)(s_0),
\]

then, \( \Gamma \) is affine, one to one from \( \mathcal{D}_A \) to \( \hat{\mathcal{V}}_A(s_0) \), where \( \hat{\mathcal{V}}_A(s) \) is defined as follows

\[
\hat{\mathcal{V}}_A(s) = \left[ \begin{array}{cc}
A & A \\
\frac{A}{s^2} & \frac{A}{s^2}
\end{array} \right]^{1+N}.
\]

Moreover, we have

\[
\Gamma |_{\partial \mathcal{D}_A} \subset \partial \hat{\mathcal{V}}_A(s_0),
\]

and

\[
\deg(\Gamma |_{\partial \mathcal{D}_A}) \neq 0,
\]

where \( q_0, q_1 \) are defined as in (1.21) and considered as the components of \( q_{d_0, d_1}(s_0) \) given by (4.30).

(II) For all \( (d_0, d_1) \in \mathcal{D}_A \) then, initial data \( U_{d_0, d_1} \) defined in (4.33) belongs to \( S(K_0, \epsilon_0, \alpha_0, A, \delta_3, C_3, 0, 0) = S(0) \),

where \( S(0) \) is defined in Definition 4.1. Moreover, the following estimates hold

(i) Estimates in \( P_1(0) \): We have \( q_{d_0, d_1}(s_0) \in \mathcal{V}_{K_0, A}(s_0) \) and

\[
|q_0(s_0)| \leq \frac{A}{s_0}, \quad |q_{1,j}(s_0)| \leq \frac{A}{s_0^2}, \quad |q_{2,i,j}(s_0)| \leq \frac{\ln s_0}{s_0^2}, \forall i, j \in \{1, ..., N\},
\]
Proposition 5.1
Proof of Theorem 2.1 will be given in the end of the current section. Though is quite technical, and it requires the following auxiliary result. Then the proof of the current lemma we kindly refer the reader to check [MZ97a, Lemma 2.4]. □

Notably, the shrinking set and initial data are the same as in [MZ97a], Proof.

where $U$ solution

4.4]. In particular, the argument consists of the following two main steps: [BK94], [MZ97b], [MZ97a]. In particular, we refer the readers to [DZ19, Proposition 4.4]. In particular, the argument consists of the following two main steps:

- Step 1: We reduce our problem to a finite dimensional one. More precisely, we prove that the task of controlling the asymptotic behaviour of $U(t) \in S(t)$ for all $t \in [0, T)$ is reduced to governing the behaviour of $(q_0, q_1)(s)$ in $V_A(s)$ (see Proposition 7.4 below).

- Step 2: In this step, we aim at proving the existence of a $(d_0, d_1) \in \mathbb{R}^{1+N}$ such that the solution of the solution $U$ of equation (4.3) and with initial data $U_{d_0, d_1}$ given in (4.33) belongs to $S(t)$ for all $t \in [0, T)$, where $S(t)$ introduced in Definition 4.1.

Proof. This proposition plays a vital role in proving Theorem 2.1. However, the conclusion is very classical and stems from a robustness argument is encountered in [BK94], [MZ97b], [MZ97a]. In particular, we refer the readers to [DZ19, Proposition 4.4]. In particular, the argument consists of the following two main steps:

- Step 1: We reduce our problem to a finite dimensional one. More precisely, we prove that the task of controlling the asymptotic behaviour of $U(t) \in S(t)$ for all $t \in [0, T)$ is reduced to governing the behaviour of $(q_0, q_1)(s)$ in $V_A(s)$ (see Proposition 7.4 below).

- Step 2: In this step, we aim at proving the existence of a $(d_0, d_1) \in \mathbb{R}^{1+N}$ such that the solution of the solution $U$ of equation (4.3) and with initial data $U_{d_0, d_1}$ given in (4.33) belongs to $S(t)$ with suitable parameters. Then, the conclusion follows from a topological argument based on Index theory. This completes the proof of the proposition.

□

Conclusion of Theorem 2.1
Let us fix positive parameters $T, K_0, \epsilon_0, A, \delta_0, C_0, \eta_0$ such that Propositions 4.4 and 5.1 hold true. Then, we obtain

$U(t) \in S(t) \quad \forall t \in [0, T).$

Using item (ii) of Proposition 4.4 we derive

$\theta(t) \to \theta^*$, as $t \to T,$
for some $\theta^* = \theta^*(\Omega, p, \gamma, r) > 0$. In particular, we also obtain the fact that the constructed solution satisfied

$$|\theta'(t)| \leq (T - t)^{-1 + \varepsilon},$$

which yields

(5.1) $$|\theta(t) - \theta^*| \leq (T - t)^{\varepsilon}.$$ 

Now we are ready to proceed with the proof of Theorem 2.1 which follows the same lines as in [DZ19, pages 1306-1310]. However, for readers convenience we present in the following the key estimates, whilst further details can be found in the aforementioned work.

- Proof of item (i): By virtue of in Definition 4.1 (i), (iii) and equality (4.8), we obtain

$$\|W(y, s) - (p - 1 + \frac{(p-1)^2}{4p} \frac{|y|^2}{s})\|^{\frac{1}{p-1}}_{L^\infty(\Omega_s)} \leq \frac{C}{1 + \sqrt{s}},$$

which via (3.10) yields

(5.2) $$\|U(., t) - \left(p - 1 + \frac{(p-1)^2}{4p} \frac{|.|^2}{(T-t)\ln(T-t)}\right)\|^{\frac{1}{p-1}}_{L^\infty(\Omega)} \leq \frac{C}{1 + \sqrt{\ln(T-t)}}.$$ 

From (4.1), (5.1) and (5.2), we directly derive (2.2). The uniqueness of the blowup point follows by the result [GK89, Theorem 2.1] in applying (5.2), and for the complete argument for this point, the readers can see the same one in . Finally the existence of the blowup profile $u^*$ is quite the same as in [DZ19, Proposition 3.5]. This concludes the proof of Theorem 2.1.

6. Behavior of $L^k$ norm at blowup time

This part is devoted to the proof of Corollary 2.2. Let us show some useful estimates for proof.

Lemma 6.1. Let us consider $t(x)$, defined as in (3.12) for all $|x| \leq \epsilon_0$, and $\rho(x) = T - t(x)$. Then, we have

$$\rho(x) \sim \frac{8}{K_0^2} \frac{|x|^2}{\ln |x|},$$

and

$$\ln \rho(x) \sim 2 \ln |x|,$$

as $x \to 0$.

Proof. The proof is straightforward by definition 2.2 of $t(x)$.

Now, we aim to give some estimate on $U$, trapped in $S(t)$.

Lemma 6.2. Let us consider $U(t) \in S(K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0, t)$ for all $t \in [0, T)$, defined as in Definition 4.4 where $\eta_0 \ll 1$, and $K'_0 \geq K_0$. Then, the following hold:
Proof. The proof is mainly based on estimates in Definition 4.1.

By definition of $P_K$, that, the second one is proved by the same technique.

(6.1) 

Then, from Lemma 4.2, we get

Using (3.10), we derive

For all $x,t \in \rho(x)$, we have

provided that $K_0 \geq K_6$ and $\epsilon_0 \leq \epsilon_0(K_0)$.

(iii) For all $|x| \geq \epsilon_0$, we have

and

provided that $\eta_0 \ll 1$.

Proof. The proof is mainly based on estimates in Definition 4.1.

- The proof of item (i): Let us consider $|x| \leq \frac{K_6}{4} \sqrt{(T-t)\ln(T-t)}$, where $K_6 \geq K_0$. Once $\epsilon_0 \leq \frac{1}{M_0}$, where $M_0$ introduced in (4.7), we have

Using (3.10), we derive

Then, from Lemma 4.2, we get

By definition of $\varphi$, defined in (4.10), we conclude the first estimate. In addition to that, the second one is proved by the same technique.

- The proof of item (ii): We now consider $|x| \in \left[\frac{K_6}{4} \sqrt{(T-t)\ln(T-t)}\right], \epsilon_0 \subset P_2(t)$, recall item (ii) in Definition 4.1, we have

and

(6.1) 

\[ |\nabla \xi U(x_0, \tau(x,t))| \leq \frac{C_0}{|\ln(\rho(x))|}. \]
We also have the fact that
\[ 1 - \tau(x, t) = \frac{t - t(x)}{T - t(x)} \in [0, 1]. \]

Then, it follows
\[ \left( (p - 1)(1 - \tau(x, t)) + \frac{(p - 1)^2}{4p} \frac{K_0^2}{16} \right)^{-\frac{1}{p - 1}} \leq \left( \frac{(p - 1)^2}{4p} \frac{K_0^2}{16} \right)^{-\frac{1}{p - 1}}. \]

Taking \( \delta_0 \leq \left( \frac{(p - 1)^2}{4p} \frac{K_0^2}{16} \right)^{-\frac{1}{p - 1}} \), we derive
\[ |U(x, 0, \tau(x, t))| \leq 2 \left( \frac{(p - 1)^2}{4p} \frac{K_0^2}{16} \right)^{-\frac{1}{p - 1}}. \]

Using the fact that,
\[ U(x, t) = (T - t(x))^{-\frac{1}{p - 1}} U(x, 0, \tau(x, t)), \]
then, we have
\[ |U(x, t)| \leq 2 \left( \frac{(p - 1)^2 K_0^2}{4p} \right)^{-\frac{1}{p - 1}} (T - t(x))^{-\frac{1}{p - 1}}. \]

Apply Lemma 6.1 with \( \epsilon_0 \leq \epsilon_6(K_0) \), we obtain
\[ |U(x, t)| \leq 4 \left( \frac{(p - 1)^2}{8p} \frac{|x|^2}{|\ln |x||} \right)^{-\frac{1}{p - 1}}. \]

We conclude the first one, and the second one follows (6.1) and a same technique. In particular, item (iii) directly follows from the third one in Definition 4.1. This completes the proof of Lemma 2.2. \( \square \)

Now, we produce the proof of Corollary 2.2.

\textbf{Proof of Corollary 2.2} We will consider the two cases mentioned in the statement.

- The case where \( \frac{k}{p - 1} - \frac{N}{2} < 0 \), we decompose as follows

\[ \|U\|_{L^k}^k(\Omega) = \int_\Omega U^k \, dx = \int_{|x| \leq \frac{K_0}{\sqrt{(T - t)|\ln(T - t)|}}} U^k \, dx + \int_{\frac{K_0}{\sqrt{(T - t)|\ln(T - t)|}} \leq |x| \leq \epsilon_0} U^k \, dx + \int_{|x| \geq \epsilon_0} U^k \, dx. \]

And, we also have a fundamental integral: for all \( k > 0 \) and \( K'_0 \)

\[ \int_{\frac{K'_0}{\sqrt{(T - t)|\ln(T - t)|}} \leq |x| \leq \epsilon_0} \left( \frac{|x|^2}{|\ln |x||} \right)^{-\frac{k}{p - 1}} \, dx \]
\[ = \left( \frac{K'_0}{4} \right)^{-\frac{2k}{p - 1}} \left( \frac{N}{2} - \frac{k}{p - 1} \frac{|\ln(T - t)|}{\frac{2}{N} \left( 1 + o_{t \rightarrow T}(1) \right)} \right) \]
\[ + f(\epsilon_0), \]

where \( f \) is some regular function.
In addition to that, we also have

\[
(6.4) \quad (T-t)^{-\frac{k}{p-1}} \int_{|x| \leq \frac{k'}{p-1} \sqrt{(T-t)|\ln (T-t)|}} \varphi_0 \left( \frac{|x|}{\sqrt{(T-t)|\ln (T-t)|}} \right) \, dx
\]

\[
= (T-t)^{\frac{N}{2} - \frac{k'}{p-1}} \ln (T-t) \int_{0}^{\frac{k'}{p-1}} \varphi_0 (r) r^{N-1} \, dr.
\]

Then, using (6.2), (6.3), and (6.4), with \( K_0' = K_0 \), we derive

\[ \|U(t)\|_{L^k(\Omega)}^k \leq C(A, \epsilon_0, \eta_0), \forall t \in [0, T). \]

We now aim to prove that

\[ \lim_{t \to T} \|U(t)\|_{L^k(\Omega)}^k < +\infty. \]

It is sufficient to prove

\[ \left| \partial_t \|U(t)\|_{L^k(\Omega)}^k \right| \leq C(A, \epsilon_0, \eta_0)(T-t)^{-1+\epsilon}, \]

with a small positive \( \epsilon \in (0, 1) \). Indeed, using (143), we derive

\[ \int \Delta U U^{k-1} \, dx = \int \nabla U^2 U^{k-2} + \int \nabla U \nabla U U^{k-1}. \]

By (143), we obtain,

\[ \left| \left( \frac{1}{p-1} \frac{\theta'(t)}{\theta(t)} - 1 \right) \int \nabla U \right| \leq C(A, \epsilon_0, \eta_0)(T-t)^{-1+\epsilon_1}, \]

with \( \epsilon_1 \in (0, 1) \).

It remains to estimate for the first and second ones. Estimate for \( \int \Delta U U^{k-1} \):

Using the integration by parts, we have

\[ \int \Delta U U^{k-1} = (1-k) \int |\nabla U|^2 U^{k-2}. \]

By using Lemma 6.2, 6.3, 6.4, and a similar decomposition as in (6.2), we estimate

\[ \int |\nabla U|^2 U^{k-2} \leq C(A, \epsilon_0, \eta_0)(1 + (T-t)^{\frac{N}{2} - \frac{k'}{p-1}}), \]

Similarly, we obtain

\[ \int U^{p-1+k} \leq C(A, \epsilon_0, \eta_0)(1 + (T-t)^{\frac{N}{2} - \frac{k'}{p-1}}). \]

Finally, we derive

\[ |\partial_t \|U\|_{L^k(\Omega)}^k| \leq C(A, \epsilon_0, \eta_0)(T-t)^{-1+\epsilon}, \]

with a small \( \epsilon \in (0, 1) \). Thus, we conclude the result for the case \( \frac{k}{p-1} - \frac{N}{2} < 0 \).

Now, we start to the case where \( \frac{k}{p-1} - \frac{N}{2} > 0 \): Let us consider an arbitrary \( K_0' \geq K_0 \), and we take (6.2) by replacing \( K_0 \) to \( K_0' \). Then, we repeat the process for the case sub-critical \( \frac{k}{p-1} - \frac{N}{2} < 0 \) by using Lemma 6.2, 6.3, 6.4, we can write

\[ \int_{|x| \leq \frac{k'}{p-1} \sqrt{(T-t)|\ln (T-t)|}} U \, dx = (T-t)^{\frac{N}{2} - \frac{k'}{p-1}} \ln (T-t)^{\frac{k'}{p-1}} \int_{0}^{\frac{k'}{p-1}} \varphi_0 (r) r^{N-1} \, dr \]

\[ + o_{t \to T}((T-t)^{\frac{N}{2} - \frac{k'}{p-1}} \ln (T-t)^{\frac{k'}{p-1}}), \]
and
\[
\left| \int_{\Omega} \frac{U^k}{\sqrt{(T-t)\ln(T-t)}} \right| \leq \left( K_0' \right)^{1/2} \cdot C(p). (T-t)^{\frac{k-1}{2}} \ln(T-t) \left| \frac{T-t}{T} \right|^{\frac{k}{2}} + o_{t \to T} ((T-t)^{\frac{k-1}{2}} \ln(T-t) \left| \frac{T-t}{T} \right|^{\frac{k}{2}}),
\]
and
\[
\int_{\{|x| \geq \epsilon_0\} \cap \Omega} U^k \leq C(\epsilon_0, \eta_0).
\]
Let us define
\[
v(t) = (T-t)^{\frac{k}{2}} \cdot \frac{T-t}{T} \ln(T-t) \left| \frac{T-t}{T} \right|^{\frac{k}{2}}.
\]
We see that $K'_0$ is free, then, taking $K'_0 \to +\infty$, we derive
\[
\|U\|_{L^k(\Omega)}^k = \left( \int_0^\infty \varphi_0(r) r^{N-1} dr \right) v(t) + o_{t \to T}(v(t)).
\]
- The case $\frac{k}{p-1} - \frac{N}{2} = 0$: This case arises a different situation. Indeed, we use again
\[
\partial_t \|U\|_{L^k(\Omega)} = k \int \nabla U \cdot \partial_t U dx = k \int \Delta U U^{k-1} + k \int \partial_t U^{p-1+k} + k \left( \frac{1}{p} \frac{\theta'(t)}{\theta(t)} - 1 \right) \int U^k,
\]
We now use the result in item (ii) with $k' = p - 1 + k$, and we derive
\[
\int \Delta U U^{k-1} = (1 - k) \int |\nabla U|^2 U^{k-2} = o_{t \to T} \left( (T-t)^{-1} \ln(T-t) \left| \frac{T-t}{T} \right|^{\frac{k}{2}} \right).
\]
In addition to that, repeating the process in the proof of item (ii) by using Lemma 6.3, 6.3, and 6.3, we derive
\[
\int \Delta U U^{k-1} = (1 - k) \int |\nabla U|^2 U^{k-2} = o_{t \to T} \left( (T-t)^{-1} \ln(T-t) \left| \frac{T-t}{T} \right|^{\frac{k}{2}} \right),
\]
and use more
\[
\left( \frac{1}{p} \frac{\theta'(t)}{\theta(t)} - 1 \right) \int U^k = o_{t \to T} \left( (T-t)^{-1} \ln(T-t) \left| \frac{T-t}{T} \right|^{\frac{k}{2}} \right).
\]
Then, we derive
\[
\partial_t \|U\|_{L^k(\Omega)} = \left( k \int_{0}^{\infty} \varphi_0^{p-1+k} r^{N-1} + o_{t \to T}(1) \right) (T-t)^{-1} \ln(T-t) \left| \frac{T-t}{T} \right|^{\frac{k}{2}}.
\]
Thus, we derive
\[
\|U\|_{L^k(\Omega)}^k = \left( \frac{k}{\frac{k}{2}+1} \int_{0}^{\infty} \varphi_0^{p-1+k} r^{N-1} + o_{t \to T}(1) \right) \ln(T-t) \left| \frac{T-t}{T} \right|^{\frac{k}{2}+1}.
\]
\[\square\]
7. Finite dimensional reduction

In the current section, we try to reduce the infinite dimensional problem of controlling $U(t) \in S(t)$ to the finite dimensional problem of controlling the two positive spectrum modes $q_0$ and $q_1$ in $\hat{V}_A(s)$. More precisely, this reduction concludes Step 1 of Proposition 5.1.

Proposition 7.1 (Reduction to a finite dimensional problem). There exist positive $T, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0$ and $\eta_0$ such that the following holds: Assume that $(d_0, d_1) \in D$ and the solution $U$ of equation (4.33) exists on $[0, t_1]$, for some $t_1 < T$, with initial data $U_{d_0, d_1}$ as defined in (4.33). Furthermore, we assume that we have $U \in S(t)$ for all $\forall t \in [0, t_1]$ and $U(t_1) \in \partial S(t_1)$ (see the definition of $S(t) = S(K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0, t)$ in Definition 4.1 and the set $D_A$ given in Proposition 4.5). Then, the following statements hold:

(i) We have $(q_0, q_1)(s_1) \in \partial \hat{V}_A(s_1)$, where $q_0, q_1$ are the components of $q(s)$ given by (4.23) and $s_1 = \ln(T - t_1)$.

(ii) There exists $\nu_0 > 0$ such that for all $\nu \in (0, \nu_0)$, we have

$$(q_0, q_1)(s_1 + \nu) \notin \hat{V}_A(s_1 + \nu).$$

Consequently, there exists $\nu_1 > 0$ such that

$$U \notin S(t_1 + \nu), \forall \nu \in (0, \nu_1).$$

Proof. The proof is quite the same as [DZ19 Proposition 3.4] that mainly relies on a priori estimates in regions $P_1, P_2$ and $P_3$ of $S(t)$ introduced in Definition 4.1. For that reason, we ignore the detailed arguments and we only provide below the required a priori estimates to conclude the proof. \qed

7.1. A priori estimates on $P_1$. We aim to prove the following Lemma:

Lemma 7.2. There exists $K_4 > 0, A_4 > 0$ such that for all $K_0 \geq K_4, A \geq A_4$ and $l^* > 0$ there exists $T_3(K_0, A, l^*)$ such that for all positive $\epsilon_0, \alpha_0, \delta_0, \eta_0, C_0$ and for $T \leq T_4, l \in [0, l^*]$, the following holds:

Assume that we have the following conditions:

- We consider initial data $U_{d_0, d_1}$, defined as in (4.33) and $(d_0, d_1) \in D_A$, given in Proposition 4.5 such that $(q_0, q_1)(s_0)$ belongs to $\hat{V}_A(s_0)$ (defined in (4.37)), where $s_0 = -\ln T$, $\hat{V}_A(s)$ is and $q_0, q_1$ are components of $q_{d_0, d_1}(s_0)$.

- We have $U(t) \in S(T, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0, t)$ for all $t \in [T - e^{-\sigma}, T - e^{-(\sigma+1)}]$, for some $\sigma \geq s_0$ and $l \in [0, l^*]$.

Then, the following estimates hold:

(i) For all $s \in [\sigma, \sigma + l]$, we have

$$|q_0(s) - q_0(s)| + \left| q_{1,i}(s) - \frac{1}{2} q_{1,i}(s) \right| \leq \frac{C}{s^2}, \forall i \in \{1, ..., N\},$$

and

$$\left| q_{2,i,i}(s) + \frac{2}{s} q_{2,i,j}(s) \right| \leq \frac{CA}{s^3}, \forall i, j \in \{1, ..., N\},$$

where $q_1 = (q_{1,i})_{1 \leq i \leq N}, q_2 = (q_{2,i,j})_{1 \leq i,j \leq N}$ and $q_1, q_2$ are defined in (4.20).

(ii) Control of $q_-(s)$: For all $s \in [\sigma, \sigma + l]$ and $y \in \mathbb{R}^N$, we have the following two cases:
- When $\sigma > s_0$ then:

\begin{equation}
|q_-(y, s)| \leq C \left( Ae^{-\frac{y}{2s^2}} + A^2 e^{-\frac{(s-\sigma)^2}{s^2}} + (s - \sigma) \right) \frac{(1 + |y|^{3})}{s^2},
\end{equation}

- When $\sigma = s_0$ then:

\begin{equation}
|q_-(y, s)| \leq C(1 + (s - \sigma)) \frac{(1 + |y|^{3})}{s^2}.
\end{equation}

(iii) Control of the gradient term of $q$: For all $s \in [\sigma, \sigma + l], y \in \mathbb{R}^N$, we have the two following cases:

- When $\sigma > s_0$ then:

\begin{equation}
|\nabla q_\perp(y, s)| \leq C \left( Ae^{-\frac{y}{2s^2}} + e^{-\frac{(s-\sigma)^2}{s^2}} + (s - \sigma) + \sqrt{s - \sigma} \right) \frac{(1 + |y|^{3})}{s^2},
\end{equation}

- When $\sigma = s_0$ then:

\begin{equation}
|\nabla q_\perp(y, s)| \leq C \left( 1 + (s - \sigma) + \sqrt{s - \sigma} \right) \frac{(1 + |y|^{3})}{s^2}.
\end{equation}

(iii) Control of the outside part $q_e$: For all $s \in [\sigma, \sigma + \lambda]$, we have the two following cases:

- When $\sigma > s_0$ then:

\begin{equation}
\|q_e(\cdot, s)\|_{L^\infty(\mathbb{R}^N)} \leq C \left( A^2 e^{-\frac{y}{2s^2}} + Ae^{(s-\sigma)} + 1 + (s - \sigma) \right) \frac{1}{\sqrt{s}},
\end{equation}

- When $\sigma = s_0$ then:

\begin{equation}
\|q_e(\cdot, s)\|_{L^\infty(\mathbb{R}^N)} \leq C \left( 1 + (s - \sigma) \right) \frac{1}{\sqrt{s}}.
\end{equation}

Proof. Let us remark that the proof is similar to [MZ97a, Lemma 3.2] (see also [MZ97b]). Indeed, our situation corresponds to equation (24) in [MZ97a] Lemma 3.2 with $a = 0$ and with a small perturbation $G(u, W)$. In particular, the structure of our shrinking set $S(t)$ in exactly the same as [MZ97a] Lemma 3.2 (i)]. For those reasons, we kindly refer the readers to see the technical estimates in this work. \qed

The preceding Lemma directly implies the following:

**Proposition 7.3** (A priori estimates in $P_1(t)$). There exist $K_5, A_5 \geq 1$ such that for all $K_0 \geq K_5, A \geq A_5, \epsilon_0 > 0, \alpha_0 > 0, \delta_0 \leq \frac{1}{2}U(0), C_0 > 0, \eta_0 > 0$, there exists $T_5(K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0)$ such that for all $T \leq T_5$, the following holds: If $U$ is a non negative solution of equation (4.3) satisfying $U(t) \in S(T; K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0, t)$ for all $t \in [0, t_5]$ for some $t_5 \in [0, T]$, and with initial data $U_{d_0, d_1}$ given in (4.33) for some $d_0, d_1 \in D_A$ (cf. Proposition 4.3), then, for all $s \in [-\ln T, -\ln(T - t_5)]$, we have the following:

\[
\forall i, j \in \{1, \cdots, n\}, \quad |q_{2i, j}(s)| \leq \frac{A^2 \ln s}{2s^2},
\]

\[
\|q_-(\cdot, s)\|_{L^\infty(\mathbb{R}^N)} \leq A + \|q_-(\cdot, s)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{A}{2s^2},
\]

\[
\|q_e(s)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{A^2}{2\sqrt{s}}.
\]
7.2. **A priori estimates in** $P_2$: In order to derive the required a priori estimates within the region $P_2$ we first need the auxiliary result:

**Lemma 7.4** (A priori estimates in the intermediate region). There exist $K_6, A_6 > 0$, such that for all $K_0 \geq K_6, A \geq A_6, \delta_0 > 0$, there exists $\alpha_0(K_0, \delta_0), C_6(K_0, A) > 0$ such that for all $\alpha \leq \alpha_0, C_0 > 0$, there exists $\epsilon_6(\alpha, \delta_0, C_0) > 0$ such that for all $\epsilon_0 \leq \epsilon_6$, there exists $T_6(\epsilon_0, \alpha, \delta_0, C_0)$ and $\eta_6(\epsilon_0, \alpha, \delta_0, C_0) > 0$ such that for all $T \leq T_6, \eta_0 \leq \eta_6, \delta_0 \leq \frac{1}{2} \left( p - 1 + \frac{(p-1)^2}{4p} K_0^2 \right) \frac{1}{T^{p-1}}$, the following holds: if $U \in S(T, K_0, \epsilon_0, \alpha, \delta_0, C_0, \eta_0, t)$ for all $t \in [0, t_s]$, for some $t_s \in [0, T)$, then, for all $|x| \in \left[ T^{p-1} \sqrt{(T-t_s)} \right] \ln(T-t_s), \epsilon_0 \right]$, we have:

1. For all $|\xi| \leq \frac{2}{\alpha_6} \sqrt{\ln \varrho(x)}$ and $\tau \in \left[ \max \left( 0, -\frac{t(x)}{\varrho(x)} \right), \frac{T-t(x)}{\varrho(x)} \right]$, the transformed function $U(x, \xi, \tau)$ defined in (7.13) satisfies the following:

$$\nabla_x U(x, \xi, \tau) \leq \frac{2C_0}{\sqrt{\ln \varrho(x)}},$$

(7.9)

$$U(x, \xi, \tau) \geq \frac{1}{4} \left( 1 + \frac{(p-1)^2}{4p} K_0^2 \right) \frac{1}{T^{p-1}},$$

(7.10)

$$|U(x, \xi, \tau)| \leq 4.$$  

(7.11)

2. For all $|\xi| \leq 2\alpha_0 \sqrt{\ln \varrho(x)}$ and $\tau_0 = \max \left( 0, -\frac{t(x)}{\varrho(x)} \right)$, we have

$$\left| U(x, \xi, \tau_0) - U(\tau_0) \right| \leq \delta_0 \text{ and } |\nabla_x U(x, \xi, \tau_0)| \leq \frac{C_0}{\sqrt{\ln \varrho(x)}},$$

Proof. For the proof see [DZ19, Lemma 4.2], or [MZ97a, Lemma 2.6].

Next by following the same reasoning as in the proof of Proposition 4.2 in [DZ19] and taking into account Lemma 7.3, we derive the following result:

**Proposition 7.5** (A priori estimates in $P_2(t)$). There exist $K_7, A_7 > 0$ such that for all $K_0 \geq K_7, A \geq A_7$, there exists $\delta_7 \leq \frac{1}{2} \hat{U}(0)$ and $C_7(K_0, A) > 0$ such that for all $\delta_0 \leq \delta_7, C_0 \geq C_7$ there exists $\alpha_7(K_0, \delta_0) > 0$ such that for all $\alpha \leq \alpha_7$, there exist $\epsilon_7(K_0, \delta_0, C_0) > 0$ such that for all $\epsilon_0 \leq \epsilon_7$, there exists $T_7(\epsilon_0, \delta, C_0) > 0$ such that for all $T \leq T_7$ the following holds: If $U \in S(T, K_0, \epsilon_0, C_0, \alpha, \delta_0, \eta_0, t)$ for all $t \in [0, t_s]$ for some $t_s \in [0, T)$, then, for all $|x| \in \left[ T^{p-1} \sqrt{(T-t_s)} \ln(T-t_s), \epsilon_0 \right]$, $|\xi| \leq \alpha_0 \sqrt{\ln \varrho(x)}$ and $\tau \in \left[ \max \left( -\frac{t(x)}{\varrho(x)}, 0 \right), \frac{T-t(x)}{\varrho(x)} \right]$, we have

$$\left| U(x, \xi, \tau_s) - \hat{U}(x, \xi, \tau_s) \right| \leq \frac{\delta_0}{2} \text{ and } |\nabla_x U(x, \xi, \tau_s)| \leq \frac{C_0}{2 \sqrt{\ln \varrho(x)}},$$

where $\varrho(x) = T - t(x)$.

7.3. **Priori estimates in** $P_3$: In that region we claim that the following holds:

**Proposition 7.6** (A priori estimates in $P_3$). Let us consider positive $K_0, \epsilon_0, \alpha_0, A, C_0, \eta_0$ and $\delta_0 \in [\frac{1}{2} \hat{U}(0)]$. Then, there exists $T_8(\eta_0) > 0$ such that for all $T \leq T_8$, the following holds: We assume that $U$ is a non negative solution of (7.13) on $[0, t_8]$ for some $t_8 < T$, and $U \in S(K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0, t)$ for all $t \in [0, t_8]$ with initial
data \( U(0) = U_{d_0, d_1} \) given in (4.33) for \(|d_0|, |d_1| \leq 2\). Then, for all \(|x| \geq \frac{\alpha}{4}\) and \(t \in (0, t_8]\), we get the a priori estimates:

\[
\begin{align*}
|U(x, t) - U(x, 0)| &\leq \frac{\eta_0}{2}, \\
|\nabla U(x, t) - \nabla e^{t\Delta} U(x, 0)| &\leq \frac{\eta_0}{2}.
\end{align*}
\]

Proof. The proof relies on the well-posedness property of problem (4.3). More precisely, we directly derive by (4.3) the following integral equation

\[
U(t) = e^{t\Delta} U_0 + \int_0^t e^{(t-s)\Delta} \left[ U^p(s) + \left( \frac{1}{p-1} \frac{\theta'(s)}{\theta(s)} - 1 \right) U(s) \right] ds,
\]

where \( e^{t\Delta} \) stands for the semi-group of the laplacian \( \Delta \) associated with Neuman boundary conditions. Using now the rough estimates stem from the definition of \( S(t) \), then by virtue of (7.14) we derive the desired a priori estimates (7.12) and (7.13).

8. Appendix

8.1. Proof of Proposition 4.4

We only consider the sub-critical case

\[
1 - \frac{r\gamma}{p-1} > 0
\]

together with super-critical one

\[
1 - \frac{r\gamma}{p-1} < 0,
\]

whilst the critical case \( 1 - \frac{r\gamma}{p-1} = 0 \) is not considered in the current work.

Using now Definition 4.2 we write

\[
\theta(t) = |\Omega|^{-\frac{\gamma}{p-1}} \left( \int_{P_{\beta}(t)} U^r + \int \left\{ K_0 \sqrt{(T-t)|\ln(T-t)| \leq |x| \leq \frac{\alpha t}{4} \} \right\} U^r + \int_{P_3(t)} U^r \right)^{-\frac{\gamma}{p-1}}.
\]

- The proof of item (i): Since the goal is to prove the integrals in the above line to be bounded, the arguments of (8.1) and (8.2) are the same. For that reason, we only give the proof involving to (8.1). Next using that (8.1) entails

\[
- \frac{\gamma}{1 - \frac{r\gamma}{p-1}} < 0,
\]

then in conjunction with hypothesis (4.28) and via Definition 4.1 (iii) we deduce

\[
\theta(t) \leq |\Omega|^{-\frac{\gamma}{p-1}} \left( \int_{P_3} U^r \right)^{-\frac{\gamma}{p-1}} \leq C(\eta_0, C_2).
\]

Using Lemma 6.2, item (i), we have

\[
\int_{P_3} U^r \leq C(T-t)^{\frac{\gamma}{p-1}} |\ln(T-t)| \leq \int_{\Omega} U_0^r,
\]
provided that $T \leq T'(K_0, C_2)$ and $\frac{N}{2} - \frac{\gamma}{p-1} > 0$. Then, we use again item (ii) in the Lemma to derive
\[
\int_{\{K_0 \sqrt{(T-t) \ln(T-t)} \leq |x| \leq \frac{\delta_0}{|x|}\}} U_t \leq C \int_{|x| \leq \epsilon_0} \left[ \frac{|x|^2}{|\ln |x||} \right]^{-\frac{1}{p-1}} dx
\leq \int_{\Omega} U_0^r,
\]
provided that $\epsilon \leq \epsilon'(K_0, C_2)$ and $\frac{N}{2} - \frac{\gamma}{p-1} > 0$. Similarly, we also get
\[
\int_{P_3(t)} U_t \leq C(\eta_0) \left( \int_{\Omega} U_0^r + 1 \right).
\]
Thus, we obtain
\[
(8.3) \quad \theta(t) \geq \frac{1}{C(C_2, \eta_0)},
\]
which concludes (4.29).

Next, we proceed with the proof of (4.30): To this end, we express below the formula of $\theta'(t)$
\[
\theta'(t) = -\frac{\gamma}{1 - \frac{1}{p-1}} \left( \int_{\Omega} U_t^r \ dx \right)^{-\frac{1}{p-1}} \times \frac{1}{|\Omega|} \int_{\Omega} rU_t U_t^{-1} dx,
\]
by which it is evident that it is sufficient to estimate the following integral
\[
\int_{\Omega} U_t^{-1} U_t dx.
\]
Note that by virtue of equation (4.13) we can write
\[
(8.4) \int_{\Omega} U_t^{-1} U_t dx = \int_{\Omega} \Delta U U_t^{r-1} + \int_{\Omega} U_t^{p-1+r} + \left( \frac{1}{p-1} \frac{\theta(t)}{\theta(t)} - 1 \right) \int_{\Omega} U_t^r.
\]
Then, we derive
\[
(8.5) \quad \theta'(t) = -\frac{r\gamma}{p-1} \left\{ \int_{\Omega} \Delta U U_t^{r-1} + \int_{\Omega} U_t^{p-1+r} - \int_{\Omega} U_t^r \right\}.
\]

Next we recall some necessary material for the proof: From Lemma 6.2 we roughly estimate
\[
|\nabla_i U(x, t)| \leq C(T-t)^{-\frac{1}{p-1}} \quad \text{for all } x \in P_1(t),
\]
\[
(8.6) \quad |\nabla_i^2 U(x, t)| \leq C(\delta_0, C_0) \left[ \frac{|x|^2}{|\ln |x||} \right]^{-\frac{1}{p-1}} \quad \text{for all } x \in P_2(t),
\]
\[
|\nabla_i^3 U(x, t)| \leq |\nabla_i^3 U_0| + \eta_0 \quad \text{for all } x \in P_3(t),
\]
for all $i \in \{0, 1\}$.

Besides that, we use the Green formula to derive
\[
(8.7) \quad \int_{\Omega} U_t^{-1} \Delta U = (1 - r) \int_{\Omega} U_t^{-2}|\nabla U|^2,
\]
for all $U \in W^{2,\infty}$ and $\frac{\partial U}{\partial v} = 0$. 

In particular, we have the following fundamental integral for $a < b \ll 1, n > 0, m < 0$ and $m \neq -1$:

$$
\int_a^b (- \ln s)^n s^m ds \leq C(n, m) \left( (- \ln b)^n b^{1+m} + (- \ln a)^n a^{1+m} \right).
$$

- Integrals in $P_1$: Using the first estimate in (8.6) we can obtain

$$
\int_{P_1} U^{p-1+r} \leq C(T-t)^{-1+\frac{n}{p-1} - \frac{2}{p-1}} |\ln(T-t)|^{\frac{n}{p-1}},
$$

$$
\int_{P_1} U^{r-2} |\nabla U|^2 \leq C(T-t)^{-1+\frac{n}{p-1} - \frac{2}{p-1}} |\ln(T-t)|^{\frac{n}{p-1}}.
$$

- Integrals in $P_2$: In fact, we use the second estimate in (8.6) to obtain the following

$$
|U(x, t)|^{p-1+r} \leq C(\delta_0) \left[ \frac{|x|^2}{|\ln |x||} \right]^{-1 - \frac{2}{p-1}},
$$

$$
|U(x, t)|^{r-2} |\nabla U(x, t)|^2 \leq C(C_0) \left[ \frac{|x|^2}{|\ln |x||} \right]^{-1 - \frac{2}{p-1}}.
$$

Besides

$$
\int_{K_0 \sqrt{(t-t)|\ln(T-t)|}} |x| \leq \epsilon_0 \left[ \frac{|x|^2}{|\ln |x||} \right]^{-1 - \frac{2}{p-1}} z^{N-1} dz
$$

$$
= \int_{K_0 \sqrt{(t-t)|\ln(T-t)|}} \left( - \ln z \right)^{1+\frac{2}{p-1}} z^{-3-\frac{2}{p-1}+N} dz := I,
$$

and thus by virtue of (8.8) we deduce

$$
I \leq C \left( \epsilon_0^{-1+\frac{2}{p-1} - \frac{2}{p-1}} |\ln \epsilon_0|^{1+\frac{2}{p-1}} + (T-t)^{-1+\frac{n}{p-1} - \frac{2}{p-1}} |\ln(T-t)|^{\frac{n}{p-1}} \right),
$$

which finally entails

$$
\int_{P_2} [U^{p-1+r} + U^{r-2} |\nabla U|^2]
\leq C(\delta, C_0) \left( \epsilon_0^{-1+\frac{2}{p-1} - \frac{2}{p-1}} |\ln \epsilon_0|^{1+\frac{2}{p-1}} + (T-t)^{-1+\frac{n}{p-1} - \frac{2}{p-1}} |\ln(T-t)|^{\frac{n}{p-1}} \right),
$$

Additionally, since in $P_3$ the gradient $\nabla U$ is considered as a small perturbation of initial data we have

$$
\int_{P_3} [U^{p-1+r} + U^{r-2} |\nabla U|^2] \leq C(\Omega, \eta_0).
$$

Finally, choose $\varepsilon = \frac{1}{2} \left( \frac{n}{2} - \frac{2}{p-1} \right)$, then, there exists $T \leq T'(K_0, \delta_0, C_0, \eta_0)$ and there holds

$$
\int_{\Omega} [U^{p-1+r} + U^{r-2} |\nabla U|^2] \leq (T-t)^{-1+\varepsilon}.
$$

Using (4.25), we have

$$
\int_{\Omega} U^r \leq C(\eta_0, C_2).
$$
Regarding to (8.5), it concludes (1.30). Thus, item (i) follows. Clearly, (ii) is derived by item (i).

9. Parabolic estimates with Neumann condition

In this section we provide some useful parabolic estimates by using well known growth conditions of the Neumann heat kernel. Let us consider the following equation

\[
\begin{aligned}
\frac{\partial U}{\partial t} &= \Delta U \text{ in } \Omega \times (0, T) \\
\frac{\partial U}{\partial \nu} &= 0 \text{ on } \partial\Omega \times (0, T) \\
U(t = 0) &= U_0 \text{ on } \Omega.
\end{aligned}
\]

Problem (9.1) generates the associated semi-group \(e^{t\Delta}\) and its solution is given by

\[U(t) = e^{t\Delta}(U_0) = \int_{\Omega} G(x, y, t)U_0(y)dy,\]

and thus

\[\frac{\partial e^{t\Delta}U_0}{\partial \nu} = 0 \text{ for all } t > 0 \text{ and } x \in \partial\Omega.\]

In particular, the Neumann heat kernel \(G(x, y, t)\) satisfies the following growth estimates

\[|\nabla_i G(x, y, t)| \leq Ct^{-\frac{N+i}{2}}\exp \left( -C|\Omega| \frac{|x-y|^2}{t} \right), C(|\Omega|) > 0, \forall i = 0, 1,\]

for all \(x, y \in \Omega\) and \(t \neq 0\), cf. [YZ13, Lemma 3.3].

Now we can prove the following:

**Lemma 9.1.** Let us consider initial data \(U_0(d_0, d_1)\) defined as in (4.33) and define

\[L(t) := e^{t\Delta}(U_0(d_1, d_2)).\]

Then,

\[\|L(t)\|_{L^\infty(\Omega \cap \{x| x| \geq \frac{d_0}{2}\})} + \|\nabla L(t)\|_{L^\infty(\Omega \cap \{x| x| \geq \frac{d_0}{2}\})} \leq C(\epsilon_0)\]

**Proof.** We observe that the first desired estimate for \(L\) follows directly from (9.2) with \(i = 0\). It remains to prove the second one. To this end the technique from [DZ19, Lemma 4.3] can be applied thanks to estimate (9.2). So, we kindly refer the readers to check the proof of this lemma. □

**Lemma 9.2 (Parabolic estimates).** Let us consider positive parameters \(T, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0 \) and \(\eta_0\) such that Proposition 4.4 holds true and we assume furthermore that

\[U(t) \in S(K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0, t), \forall t \in [0, t_1).\]

Then we have the following

\[|\nabla_i U(x, t)| \leq C(T - t)^{-\frac{1}{2}}\]

for all \(i = 0, 1\) and for all \((x, t) \in \Omega \times [0, t_1)\). In particular, for all \(x \in \Omega\), there exists \(R_x > 0\) such that

\[|\partial_t U(x, t)| \leq C(A, T, R_x).\]

**Proof.** Estimate (9.3) is directly derived by Definition 4.1. Besides, the proof of (9.4) is quite the same as the proof estimate (F.4) in [DZ19, Lemma F.1]. □
9.1. Some estimates of terms in equation \((4.12)\). In this part, we aim to estimate the terms \(V, B, R\) and \(G\) in equation \((4.12)\).

**Lemma 9.3.** For all \(A, K_0 \geq 1\), there exists \(s_7(A, K_0) \geq 1\), such that for all \(s \geq s_7\) and \(q \in V_A(s)\), then the following hold:

(i) Estimate for \(V\), defined as in \((4.14)\):

\[
|V(y, s)| \leq C \quad \text{and} \quad \left| \frac{|y|^2 - 2N}{4s} \right| \leq \frac{C(1 + |y|^4)}{s^2}, \forall y \in \mathbb{R}^N.
\]

(ii) Estimate for \(B(q)\), defined as in \((4.15)\):

\[
|\chi(y, s)B(q)| \leq C|q|^2 \quad \text{and} \quad |B(q)| \leq C|q|^\bar{p}, \quad \text{where} \quad \bar{p} = \min(p, 2),
\]

for all \(y \in \mathbb{R}^N\).

(iii) Estimate for \(R(y, s)\):

\[
|R(y, s)| \leq \frac{C}{s},
\]

\[
\left| R(y, s) - \frac{c_1}{s^2} \right| \leq \frac{C(1 + |y|^4)}{s^3},
\]

\[
|\nabla R(y, s)| \leq \frac{C(|y| + |y|^3)}{\bar{p} + 1},
\]

for all \(y \in \mathbb{R}^N\).

**Proof.** The proof of item (i) arises from a simple Taylor expansion. The proofs of (ii) and (iii) can be found in of [MZ97b, Lemma 3.15] and [MZ97a, Lemma B.5]. \(\square\)

Next, Proposition \(4.4\) ensures that \(G\), defined \((4.17)\), decays exponentially. In particular there holds:

**Lemma 9.4** (Size of \(G\)). There exists \(K_7 \geq 1\) such that for all \(K_0 \geq K_7, \delta_0 > 0\), there exists \(\alpha_7(K_0, \delta_0) > 0\) such that for all \(\alpha_0 \leq \alpha_7, M_0 > 0\) there exists \(\epsilon_7(K_0, \delta_0, \alpha_0, M_0) > 0\) such that for all \(\epsilon_0 \leq \epsilon_7\) and \(A \geq 1, C_0 > 0, \eta_0 > 0\), there exists \(T_7 > 0\) such that for all \(T \leq T_7\) the following holds: Assuming \(U\) is a non negative solution of equation \((4.3)\) on \([0, t_1]\), for some \(t_1 < T\) and \(U \in \mathcal{S}(K_0, \epsilon_0, \alpha, A, \delta_0, C_0, \eta_0, t) = S(t)\) for all \(t \in \mathbb{R}^N\) with initial data \(U_0\) introduced as in \((4.13)\) for \(|d_1|, |d_2| \leq 2\). Then, \(G\), defined as in \((4.17)\) satisfies

\[
\|G\|_{L^\infty(\mathbb{R}^N)} \leq e^{-\eta s}, \forall s \in [-\ln T, \ln(T - t_1)],
\]

for some \(\eta > 0\) and small.

**Proof.** We note that the constants in the hypothesis mainly satisfy the assumptions of Proposition \(4.4\) thus

\[
|\theta^*_s(t)| \leq C(T - t)^{-1+\epsilon},
\]

and so we obtain

\[
|\tilde{\theta}^*_s(s)| \leq e^{-\epsilon s}.
\]

Now, we rewrite \(G\) as follows

\[
G = \left( \frac{1}{p - 1} \tilde{\theta}^*_s(s) - e^\epsilon \right) w + F(w, W),
\]
and thus it directly follows
\[
\left| \left( \frac{1}{p-1} \frac{\bar{\theta}^\prime(s)}{\bar{\theta}(s)} - e^s \right) w \right| \leq Ce^{-\min(1, \epsilon)s}.
\]

In particular, the nonlinear term \( F \) is similar to the one treated in [DZ19] and thus one derives
\[
\| F \|_{L^\infty} \leq Ce^{-\epsilon_1 s}.
\]

Finally we arrive at the desired estimate for \( G \) for \( \eta = \min(1, \epsilon, \epsilon_1) \). \( \square \)

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