Generalisation of Scott permanent identity

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Abstract

Let $x = \{x_1, \ldots, x_r\}$, $y = \{y_1, \ldots, y_n\}$, $z = \{z_1, \ldots, z_n\}$ be three sets of indeterminates. We give the value of the determinant

$$\prod_{x \in x} (xy - z)^{-1}$$

when specializing $y$ and $z$ to the set of roots of $y^n - 1$ and $z^n - \xi^n$ respectively.

In the case where $r = 2$ and $x = \{1, 1\}$ the determinant $|(y - z)^{-2}|_{y \in y, z \in z}$ factorizes into the determinant of the Cauchy matrix $[(y - z)^{-1}]$ and its permanent. Scott [10, 8] found the value of this permanent when specializing $y$ to the roots of $y^n - 1$ and $z$ to the roots of $z^n + 1$. Han [3] described more generally the case where $z$ is the set of roots of $z^n + az^k + b$ instead of $z^n + 1$.

Instead of restricting to $r = 2$ and specializing $x$, we shall consider the determinant

$$\prod_{x \in x} (xy - z)^{-1}$$

and obtain in Th. 2 its value when specializing $y$ and $z$. The remarkable feature is that this value is a product of sums of monomial functions in $x$ without multiplicities, thus extending the factorized expressions of [10, 3].

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We first need a few generalities about symmetric functions [5].

Given two sets of indeterminates \(x, z\) (we say alphabets), the complete functions \(S_n(x - z)\) are the coefficients of the generating function

\[
\sum_n \gamma^n S_n(x - z) = \prod_{z \in z} (1 - \gamma z) \prod_{x \in x} (1 - \gamma x)^{-1}.
\]

For any \(r\), any \(\lambda \in \mathbb{Z}^r\), \(S_\lambda(x - z) = \det(S_{\lambda+\mu}(x - z))\).

In the case where \(z = 0\), and \(x\) of cardinality \(r\), these functions can be obtained by symmetrisation over the symmetric group \(\mathfrak{S}_r\). Let \(\pi_\omega\) be the following operator on functions in \(x\):

\[
f \to f_{\pi_\omega} := \sum_{\sigma \in \mathfrak{S}_r} \left( f \prod_{1 \leq i < j \leq r} (1 - x_j/x_i)^{-1} \right)^\sigma.
\]

Then, when \(\lambda \geq [1-r, \ldots, -1, 0]\) (i.e. \(\lambda_1 \geq 1-r, \ldots, \lambda_r \geq 0\)), the monomial \(x^\lambda = x_1^{\lambda_1} \cdots x_r^{\lambda_r}\) is sent onto \(S_\lambda(x)\) under \(\pi_\omega\). When \(\lambda\) is a partition (i.e. \(\lambda_1 \leq \cdots \lambda_r \leq 0\)), \(S_\lambda(x)\) is the Schur function of index \(\lambda\).

Let \(n\) be a positive integer, \(x\) and indeterminate and \(z\) be the set of roots of \(z^n - \xi^n\). Equivalently, \(e_i(z) = 0\) for \(1 \leq i \leq n-1\), \(e_n(z) = (-1)^{n-1}\xi^n\). For any integer \(j\), any \(x\), one has

\[
S_j(x - z) = S_j(x) - \xi^n S_{j-n}(x),
\]

and more generally, from the determinantal expression of Schur functions, for any \(\lambda \in \mathbb{N}^r\),

\[
S_\lambda(x - z) = \sum_{u \in \{0, n\}^r} S_{\lambda-u}(x)(-1)^{|u|/n}\xi^{|u|}.
\]

In particular, when \(\lambda = n-1, \ldots, n-1, p\) then the terms with negative last component \(n-p\) vanish, and the set \(\{\lambda - u\}\) to consider is \(\{[v, p] : v \in \{n-1, -1\}^{-1}\}\). Reordering indices, putting \(q = p-r+1\), one rewrites the sum as

\[
S_\lambda(x - z) = (-1)^{r-1} \sum_{u \in \{0, n\}^r} S_{q,u}(x)(-1)^{|\lambda|-|u|-q}/n\xi^{|\lambda|-|u|-q}. \quad (1)
\]

Since \(x_\nu \pi_\omega = S_\nu(x)\), for any \(\nu \geq [1-r, \ldots, -1, 0]\), one can rewrite (1) as a symmetrisation of monomial functions in \(x - x_1 = \{x_2, \ldots, x_r\}\):

\[
S_\lambda(x - z) = (-1)^{r-1} \sum_{j=0}^{r-1} x_q^{m_{nj}}(x - x_1)(-1)^{r-1-j}\xi^{|\lambda|-j} - 1 - q \pi_\omega. \quad (2)
\]
From the identity
\[ m_{n,j}(x - x_1) = m_{n,j}(x) - x_1^n m_{n,j-1}(x) + x_1^{2n} m_{n,j-2}(x) + \cdots + (-x_1^n)^j, \]
one sees that \[ x_1^n m_{n,j}(x - x_1) \pi_\omega \]
is equal to
\[ S_{q}(x) m_{n,j}(x) - S_{q+n}(x) m_{n,j-1}(x) + \cdots + (-1)^j S_{q+jn}(x). \]

Since on the other hand \( S_{(n-1)r-1_p}(x - z) \) belongs to the linear span of Schur functions, or monomial functions, indexed by partitions \( \mu \) such that \( \mu_1 \leq n-1 \), one can restrict this last sum to the term \((-1)^j S_{q+jn}(x)\).

In summary, one has the following expression for the specialisation of the Schur function that we are considering.

**Proposition 1** Let \( x \) be an alphabet of cardinality \( r \), \( z \) be the set of roots \( z^n - \xi^n = 0 \), \( p \leq n-1 \), \( N = (n-1)(r-1) \). Then
\[ S_{(n-1)r-1_p}(x - z) = \sum_\mu m_\mu(x) \xi^{N+r-p-|\mu|}, \tag{3} \]
sum over all partitions \( \mu \in \mathbb{N}^r \), \( \mu_1 \leq n-1 \).

For example, for \( n = 4 \), \( r = 2 \), one has
\[ S_{30}(x - z) = m_3(x) + m_{21}(x), \quad S_{31}(x - z) = m_{31}(x) + m_{22}(x) + \xi^4, \]
\[ S_{32}(x - z) = m_{32}(x) + \xi^4 m_1(x), \quad S_{33}(x - z) = m_{33}(x) + \xi^4(m_2(x) + m_{11}(x)). \]

Let
\[ D(x, y, z) = \left| \prod_{x \in x} (xy - z)^{-1} \right|_{y \in y, z \in z}. \]

In the case \( r = 2 \), this determinant has been obtained by Izergin and Korepin [4] as the partition function of the Heisenberg XXZ-antiferromagnetic model. Gaudin [2] had previously described the partition function of some other model as the determinant \( |(x - y)^{-1}(x - y + \gamma)^{-1}| \) for some parameter \( \gamma \).

The Izergin-Korepin determinant is used in the enumeration of alternating sign matrices [1]. In that case, one first specializes \( x = \{ e^{2i\pi/3}, e^{4i\pi/3} \} \). Okada [9] evaluates more general partition functions corresponding to similar determinants or Pfaffians, and to other roots of unity (see also [6, Th. 7.2]).

We shall take another point of view, keep \( x \) generic, but specialize instead \( y \) and \( z \). In [7, Formula 4], it is shown that the function
\[ G(x, y, z) = \frac{D(x, y, z)}{\Delta(z)} \prod_{x \in x} \prod_{y \in y} \prod_{z \in z} (xy - z) \]
is equal to the determinant of the matrix

\[
\left[ S_{\square,j}(y_i x - z) \right]_{j=0 \ldots n-1, i=1 \ldots n},
\]

where \( \square = (n-1)^{r-1} \), and \( \Delta(z) = \prod_{i<j} (z_i - z_j) \).

For any \( k \in \mathbb{N} \), let \( \varphi_k \) be the sum of all monomial functions \( m_\mu(x) \) of degree \( k \), with \( \mu_1 \leq n-1 \) (notice that \( \varphi_k = 0 \) when \( k > (n-1)r \)). From (3), one has that \( S_{\square,j}(y_i x - z) \) specializes, when \( z \) is the set of roots of \( z^n - \xi^n \), into

\[
S_{\square,j}(y_i x - z) = y_i^{N+j} \varphi_{N+j} + \xi^n y_i^{N+j-n} \varphi_{N+j-n} + \xi^{2n} y_i^{N+j-2n} \varphi_{N+j-2n} + \cdots. \tag{5}
\]

Specializing further \( y \) into the roots of \( y^n - 1 \), one sees that the matrix (4) factorizes into the product of the matrix \( \left[ y_i^{(N+j)} \right] \), where \( (k) = k \mod n \), and the diagonal matrix

\[
diag((\varphi_N + \xi^n \varphi_{N-n} + \xi^{2n} \varphi_{N-2n} + \cdots), (\varphi_{N+1} + \xi^n \varphi_{N+1-n} + \xi^{2n} \varphi_{N+1-2n} + \cdots),
\cdots, (\varphi_{N+n-1} + \xi^n \varphi_{N+n-1-n} + \xi^{2n} \varphi_{N+n-1-2n} + \cdots)).
\]

For example, for \( n = 3, r = 3 \),

\[
S_{220}(y_i x - z) = y_i^1 \varphi_4 + y_i \varphi_1 \xi^3, S_{221}(y_i x - z) = y_i^3 \varphi_5 + y_i^2 \varphi_2 \xi^3,
\]

\[
S_{222}(y_i x - z) = y_i^6 \varphi_6 + y_i^3 \varphi_3 \xi^3 + \xi^6,
\]

and the matrix factorizes into

\[
\begin{bmatrix}
y_1 & y_1^2 & 1 \\
y_2 & y_2^2 & 1 \\
y_3 & y_3^2 & 1
\end{bmatrix}
\begin{bmatrix}
\varphi_4 + \varphi_1 \xi^3 & 0 & 0 \\
0 & \varphi_5 + \varphi_2 \xi^3 & 0 \\
0 & 0 & \varphi_6 + \varphi_3 \xi^3 + \xi^6
\end{bmatrix}
\]

Taking into account that \( \prod_{x,y,z} (xy - z) \) specializes into \( \prod_{y,x} (x^n y^n - \xi^n) = \prod_{x \in \mathbb{X}} (x^n - \xi^n)^n \), and that the determinant of powers of the \( y \in \mathbb{Y} \) is a permutation of the Vandermonde in \( \mathbb{Y} \), one obtains the following theorem.

**Theorem 2** Let \( n, r \) be two positive integers, \( N = (n-1)(r-1) \). Let \( x \) be an alphabet of cardinality \( r \), \( y \) be the set of roots of \( y^n - 1 \), \( z \) be the set of roots of \( z^n - \xi^n \). Then

\[
\Delta(y) \Delta(z) \left| \prod_{k=1}^r (x_k y_i - z_j)^{-1} \right|_{i,j=1 \ldots n} = \frac{(-1)^{(n-1)(n/2+r-1)}}{\prod_{x \in \mathbb{X}} (x^n - \xi^n)^n} \prod_{i=0}^{n-1} \left( \sum_{j=0}^{\infty} \varphi_{N+i-nj} \xi^{nj} \right).
\]

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For $x = \{1, 1\}$, this theorem is due to Han[3]. In that case, $\varphi_i = i + 1$ and $\varphi_{n-1+i} = n - i$ for $i = 0, \ldots, n-1$, and the product appearing in the theorem is

$$n (n - 1 + \xi^n) (n - 2 + 2\xi^n) \cdots (1 + (n - 1)\xi^n).$$

For $r = 5, n = 3$, as a further example, the theorem furnishes the expression

$$\prod_{k=1}^{5} (x_k^3 - \xi^3)^{-3} \left( \varphi_8 + \varphi_6\xi^3 + \varphi_2\xi^6 \right) \left( \varphi_9 + \varphi_6\xi^3 + \varphi_3\xi^6 + \xi^9 \right)$$

$$\left( \varphi_{10} + \varphi_7\xi^3 + \varphi_4\xi^6 + \varphi_1\xi^9 \right),$$

which specializes, for $x = \{1, 1, 1, 1, 1\}$, into

$$(1 - \xi^3)^{-15}(15 + 51\xi^3 + 15\xi^6)(5 + 45\xi^3 + 30\xi^6 + \xi^9)(1 + 30\xi^3 + 45\xi^6 + 5\xi^9).$$

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