On the Maximum of Random Variables on Product Spaces

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May 5, 2014

Abstract
Let $\xi_i, i = 1, \ldots, n$, and $\eta_j, j = 1, \ldots, m$ be iid $p$-stable respectively $q$-stable random variables, $1 < p < q < 2$. We prove estimates for $\mathbb{E}\Omega_1 \Omega_2 \max_{i,j} |a_{ij} \xi_i(\omega_1) \eta_j(\omega_2)|$ in terms of the $\ell_p^m(\ell_q^n)$-norm of $(a_{ij})_{i,j}$. Additionally, for $p$-stable and standard gaussian random variables we prove estimates in terms of the $\ell_p^m(\ell_{\mathcal{M}_\xi}^n)$-norm, $\mathcal{M}_\xi$ depending on the Gaussians. Furthermore, we show that a sequence $\xi_i, i = 1, \ldots, n$ of iid log-$\gamma(1,p)$ distributed random variables $(p \geq 2)$ generates a truncated $\ell_p$-norm, especially $\mathbb{E} \max_i |a_i \xi_i| \sim \|(a_i)\|_2$ for $p = 2$. As far as we know, the generating distribution for $\ell_p$-norms with $p \geq 2$ has not been known up to now.

Keywords: Random variables, Orlicz norms

1 Introduction and Notation

Let $\xi_i, i = 1, \ldots, n$ be independent copies of a random variable $\xi$ on a probability space $(\Omega_1, \mathcal{A}_1, \mathbb{P}_1)$, whose first moment is finite. Furthermore, let $a_i, i = 1, \ldots, n$ be real numbers. In [4] and [5], the following theorem was shown:

Theorem 1.1. Let

$$M_{\xi}(s) = \int_0^s \frac{1}{t} \mathbb{P}_1(|\xi| \geq \frac{1}{t}) + \int_t^\infty \mathbb{P}_1(|\xi| \geq u)du dt. \quad (1)$$

Then, for all $x \in \mathbb{R}^n$,

$$\mathbb{E} \max_{i=1,\ldots,n} |a_i \xi_i| \sim \|(a_i)_{i=1}^n\|_{\mathcal{M}_\xi}. $$
We recall that a convex function $M : [0, \infty) \to [0, \infty)$ with $M(0) = 0$ is called an Orlicz function. For an Orlicz function $M$ we define the Orlicz norm $\|\cdot\|_M$ on $\mathbb{R}^n$ by

$$
\|x\|_M = \inf \left\{ t > 0 \mid \sum_{i=1}^{n} M \left( \frac{|x_i|}{t} \right) \leq 1 \right\},
$$

and the Orlicz space $\ell^n_M$ to be $\mathbb{R}^n$ equipped with the norm $\|\cdot\|_M$. For references see for example [6].

In the following let also $n_j, j = 1, \ldots, m$ be independent copies of a random variable $\eta$ on a probability space $(\Omega_2, \mathcal{A}_2, \mathbb{P}_2)$, whose first moment is finite and $a_{ij}, i = 1, \ldots, n, j = 1, \ldots, m$ be real numbers. It is a natural question if we can give estimates for

$$
\mathbb{E}_{\Omega_1} \mathbb{E}_{\Omega_2} \max_{i,j} |a_{ij} \eta_j(\omega_2) \xi_i(\omega_1)|.
$$

Since the random variables $(\xi_i \eta_j)_{i,j=1}^{n,m}$ are no longer independent on the product space $(\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2, \mathbb{P}_1 \otimes \mathbb{P}_2)$ the previous result, Theorem 1.1, is not applicable in this case.

We give precise estimates up to absolute constants for a certain class of random variables, namely $p$- and $q$-stable, $p, q \in (1, 2)$, $p < q$, and standard gaussians. This shows in addition that we can treat dependent random variables with a certain structure of dependence and give precise estimates, this has not been feasible at all by now. Considering $p$-stable random variables seems to be natural in this case, since they generate the $\ell_p$-norm, that means the Orlicz function resulting in Theorem 1.1 equals $s \mapsto s^p$ for $p \in (1, 2)$. One would expect, that the standard gaussians generate the $\ell_2$-norm, but in fact, as shown for example in [5], they do not, but we can treat them as well. These estimates can be found in the second section. For applications we refer the reader to [3], [4] and [5].

Furthermore, in this context the question arose which random variables generate the $\ell_2$-norm, since standard gaussians astonishingly do not. We provide the solution together with the solution of the generation of truncated $\ell_p$-norms ($p > 2$) in the third section. Additionally, we give order estimates for (2) for these generating distributions.

In the following we will give order estimates and this will be denoted by $\sim$, since we are not interested in the exact values of the absolute constants. If for example the absolute constants depend on a certain variable $p$ we denote this by $\sim_p$. 

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2 Estimates for $\mathbb{E}_{\Omega_1} \mathbb{E}_{\Omega_2} \max_{i,j} |a_{ij} \eta_j(\omega_2) \xi_i(\omega_1)|$

In the following let $p, q \in (1, 2)$ with $p < q$. We analyze $\mathbb{E}_{\Omega_1} \mathbb{E}_{\Omega_2} \max_{i,j} |a_{ij} \eta_j(\omega_2) \xi_i(\omega_1)|$ under two different assumptions. Let $\eta$ always be a $p$-stable random variable.

In the first case let $\xi$ be a $q$-stable random variable, we prove the following:

$$\mathbb{E}_{\Omega_1} \mathbb{E}_{\Omega_2} \max_{i,j} |a_{ij} \eta_j(\omega_2) \xi_i(\omega_1)| \sim \left( \left\| \left( \sum_{i=1}^{n} \left| a_{ij} \right| \right)^{\frac{n}{q}} \right\|_{p} \right)^{m}.$$

In the second case let $\xi$ be a standard gaussian random variable, we prove under this assumption

$$\mathbb{E}_{\Omega_1} \mathbb{E}_{\Omega_2} \max_{i,j} |a_{ij} \eta_j(\omega_2) \xi_i(\omega_1)| \sim \left( \left\| \left( \sum_{i=1}^{n} \left| a_{ij} \right| \right)^{\frac{n}{M_{\xi}}} \right\|_{p} \right)^{m},$$

where $\| \cdot \|_{M_{\xi}}$ denotes the Orlicz norm given by the Orlicz function

$$M_{\xi}(s) = \begin{cases} 0, & \text{if } s = 0 \\ e^{-\frac{s^2}{2}}, & \text{if } s \in (0, 1) \\ e^{-\frac{s}{2}(3s - 2)}, & \text{if } s \geq 1. \end{cases}$$

The idea to prove these two results is using the triangle inequality and Jensen’s inequality for getting a lower and an upper bound. Afterwards we show that the resulting expressions are equal up to constants depending only on $p$ and $q$ using Theorem 1.1. Furthermore, we show that we can express this resulting object in terms of a product norm, as above. This also allows us, in these cases, to express a result due to S. Kwapien and C. Schütt, [8] (Example 1.6), in terms of random variables and in a very handy form.

Applying the results from [3], combined with the first steps of the proof of Theorem 2.1 one obtains

$$c_1 \alpha^{-1} \left\| \max_{1 \leq l \leq n} \left( \frac{n + 1 - j}{\sum_{i=1}^{n} \frac{1}{a_{ij}}} \right)^{n} \right\|_{p} \leq \mathbb{E}_{\Omega_1} \mathbb{E}_{\Omega_2} \max_{i,j} |a_{ij} \eta_j(\omega_2) \xi_i(\omega_1)| \leq c_p \beta^{-1} \ln(n + 1) \left\| \max_{1 \leq l \leq n} \left( \frac{n + 1 - j}{\sum_{i=1}^{n} \frac{1}{a_{ij}}} \right)^{n} \right\|_{p},$$

where $\eta$ is $p$-stable and $\xi$ is a standard gaussian. Since there is a logarithmic factor in the upper bound, this obviously does not give the correct order.
With our method we give the correct order up to absolute constants in a very handy form.

**Theorem 2.1.** Let \( p, q \in (1, 2) \) with \( p < q \). Additionally, let \( \xi_i, i = 1, \ldots, n \) be independent copies of a \( q \)-stable random variable \( \xi \) on \((\Omega_1, \mathcal{A}_1, \mathbb{P}_1)\) and let \( \eta_j, j = 1, \ldots, m \) be independent \( p \)-stable copies of a random variable \( \eta \) on \((\Omega_2, \mathcal{A}_2, \mathbb{P}_2)\). Then, for all \((a_{ij})_{i,j} \in \mathbb{R}^{n \times m}\),

\[
\mathbb{E}_{\Omega_1} \mathbb{E}_{\Omega_2} \max_{1 \leq i,j \leq m} |a_{ij} \eta_j(\omega_2) \xi_i(\omega_1)| \sim_{p \cdot q} \left\| \left( \mathbb{E}_{\Omega_1} \max_{1 \leq i \leq n} |a_{ij} \xi_i(\omega_1)| \right)^m \right\|_p.
\]

**Proof.** Let \( a_j, j = 1, \ldots, m \) be real numbers. In [5] it was shown that

\[
\mathbb{E}_{\Omega_2} \max_{1 \leq j \leq m} |a_j \eta_j(\omega_2)| \sim \left\| (a_j)^m \right\|_p. \tag{4}
\]

Applying this, we get

\[
\mathbb{E}_{\Omega_1} \mathbb{E}_{\Omega_2} \max_{i,j} |a_{ij} \eta_j(\omega_2) \xi_i(\omega_1)| \sim \mathbb{E}_{\Omega_1} \left( \max_{1 \leq i \leq n} |a_{ij} \xi_i(\omega_1)| \right)^m \left\| \left( (a_j)^n \right)^m \right\|_p.
\]

Using the triangle inequality and (4) for the \( q \)-stable \( \xi_i, i = 1, \ldots, n \), we get

\[
\mathbb{E}_{\Omega_1} \mathbb{E}_{\Omega_2} \max_{i,j} |a_{ij} \eta_j(\omega_2) \xi_i(\omega_1)| \geq \left( \mathbb{E}_{\Omega_1} \max_{1 \leq i \leq n} |a_{ij} \xi_i(\omega_1)| \right)^m \left\| \left( (a_j)^n \right)^m \right\|_p.
\]

For the upper bound we apply Jensen’s inequality and obtain

\[
\mathbb{E}_{\Omega_1} \mathbb{E}_{\Omega_2} \max_{i,j} |a_{ij} \eta_j(\omega_2) \xi_i(\omega_1)| \leq \left( \mathbb{E}_{\Omega_1} \max_{1 \leq i \leq n} \left| a_{ij} \xi_i^p(\omega_1) \right|^p \right)^{\frac{1}{p}} \left\| (a_j)^m \right\|_p.
\]

By Theorem 1.1 we get

\[
\mathbb{E}_{\Omega_1} \max_{1 \leq i \leq n} \left| a_{ij}^p \xi_i^p(\omega_1) \right| \sim \left\| (a_{ij})^m \right\|_{M_{\xi^p}},
\]

where

\[
M_{\xi^p}(s) = \int_0^s \left( \frac{1}{t} \mathbb{P}_1 \left( |\xi|^p \geq \frac{1}{t} \right) + \int_s^\infty \mathbb{P}_1 \left( |\xi|^p \geq u \right) du \right) dt. \tag{5}
\]

To prove that the upper and lower bound are equal up to constants, we show that \( M_{\xi^p}(s) \sim s^{q/p} \). This is equivalent to \( M_{\xi^p}(s^p) \sim s^q \) and hence we get

\[
\left\| (a_{ij})^n \right\|_{M_{\xi^p}} = \left\| (a_{ij})^n \right\|_{M_{\xi^{p^2}}} \sim \left\| (a_{ij})^n \right\|_q.
\]

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since the function $s \mapsto s^q$ generates the $l_q$-norm. To do so, we use the fact that if a random variable $\xi$ is $q$-stable for all $t > 0$ it holds true that

$$P(|\xi| \geq t) \lesssim t^{-q},$$

for references see for example [2]. Combining (3) and (4) we get, since $q > p$,

$$M_{\xi^p}(s) = \int_0^s \left( \frac{1}{t} \mathbb{P}_1(|\xi|^p \geq \frac{1}{t}) \right) dt + \int_\frac{1}{t}^\infty \mathbb{P}_1(|\xi|^p \geq u) du \lesssim \int_0^s \left( \frac{1}{t} \left( \frac{1}{t} \right)^{-\frac{q}{p}} + \int_\frac{1}{t}^\infty u^{-\frac{q}{p}} du \right) dt \leq 2 \int_0^s \left( t^{-1+\frac{q}{p}} + \left[ \frac{-u^{-\frac{q}{p}}}{t} \right]_t^{\infty} \right) dt = 2 \int_0^s t^{-1+\frac{q}{p}} dt \sim s^\frac{q}{p},$$

which yields the desired result.

**Theorem 2.2.** Let $p \in (1, 2)$, let $\xi_i, i = 1, ..., n$ be independent copies of a $p$-stable random variable $\xi$ on $(\Omega_1, \mathcal{A}_1, \mathbb{P}_1)$ and let $\eta_j, j = 1, ..., m$ be independent copies of a standard gaussian random variable $\eta$ on $(\Omega_2, \mathcal{A}_2, \mathbb{P}_2)$. Then, for all $(a_{ij})_{i,j} \in \mathbb{R}^{n \times m}$,

$$\mathbb{E}_{\Omega_1} \mathbb{E}_{\Omega_2} \max_{i,j} |a_{ij}\eta_j(\omega_2)\xi_i(\omega_1)| \sim_p \left\| \left\| (a_{ij})_{i=1}^n \right\|_{\mathcal{M}_p} \right\|_p \mathcal{M}_p,$$

where $\| \cdot \|_{\mathcal{M}_p}$ denotes the Orlicz norm given by the Orlicz function

$$M_\xi(s) = \begin{cases} 0 & \text{if } s = 0 \\ e^{-\frac{3}{2}s^2} & \text{if } s \in (0, 1) \\ e^{-\frac{3}{2}(3s - 2)} & \text{if } s \geq 1. \end{cases} \hspace{1cm} (7)$$

Before giving the proof, we need the following observation concerning standard gaussian random variables:

**Observation 2.3.** Let $\xi$ be a standard gaussian random variable, then the following holds for all $t > 0$, since the distribution of $\xi$ is symmetric

$$\mathbb{P}(|\xi| \geq t) = 2\mathbb{P}(\xi \geq t) = \sqrt{\frac{2}{\pi}} \int_\frac{t}{\sqrt{2}}^\infty e^{-\frac{x^2}{2}} dx.$$
Now, applying the results from [3], we get

\[ P(|ξ| ≥ t) = \sqrt{\frac{2}{\pi}} \int_t^{\infty} e^{-\frac{x^2}{2}} \, dx \sim \frac{1}{t} e^{-\frac{t^2}{2}}. \]  

\( (8) \)

Proof. (Theorem 2.2) Let \((a_i)_{i=1}^n \in \mathbb{R}^n\). Applying Theorem 1.1, we get

\[ E \max_{i=1,\ldots,n} |a_i ξ_i| \sim \| (a_i)_{i=1}^n \|_{M_ξ}, \]

where, as shown in [5], the following holds

\[ M_ξ(s) = 0 \left\{ \begin{array}{ll}
0 & \text{, if } s = 0 \\
\frac{t}{2} & \text{, if } s \in (0, 1) \\
\frac{1}{2} (3s - 2) & \text{, if } s \geq 1.
\end{array} \right. \]

In accordance with the ideas from the proof of Theorem 2.1, we get

\[ E_{Ω_1} E_{Ω_2} \max_{i,j} |a_{ij} η_j(ω_2) ξ_i(ω_1)| \geq \left\| \left( \left\| \left( a_{ij} \right)_{i=1}^n \|_{M_ξ} \right\|_{M_ξ(Ω)} \right)_{j=1}^m \right\|_p 
\]

and

\[ E_{Ω_1} E_{Ω_2} \max_{i,j} |a_{ij} η_j(ω_2) ξ_i(ω_1)| \leq \left\| \left( \left\| \left( a_{ij} \right)_{i=1}^n \|_{M_ξ(Ω)} \right\|_{M_ξ} \right)_{j=1}^m \right\|_p. \]

As in the previous proof it remains to show that

\[ \left\| \left( \left( a_{ij} \right)_{i=1}^n \|_{M_ξ} \right)_{j=1}^m \right\|_p \sim \left\| \left( a_{ij} \right)_{i=1}^n \right\|_{M_ξ}. \]

Therefore, we prove again that \( M_{ξ^p}(s) \sim_p M_ξ \left( s^\frac{1}{p} \right) \), since this is equivalent to \( M_{ξ^p}(s^p) \sim_p M_ξ (s) \) and yields

\[ \left\| \left( a_{ij} \right)_{i=1}^n \right\|_{M_ξ(Ω)} \sim \left\| \left( a_{ij} \right)_{i=1}^n \right\|_{M_ξ(Ω^p)} \sim \left\| \left( a_{ij} \right)_{i=1}^n \right\|_{M_ξ}. \]

First we show \( M_{ξ^p}(s) \leq_p M_ξ \left( s^\frac{1}{p} \right) \) and afterwards we prove the reverse inequality. To do so, we distinguish between \( s \leq 1 \) and \( s > 1 \).

Upper bound \( M_{ξ^p}(s) \leq_p M_ξ \left( s^\frac{1}{p} \right): \)
Case 1: Let $s \leq 1$.

$$M_{\xi^p}(s) = \int_0^s \left( \frac{1}{t} P_1(|\xi|^p \geq \frac{1}{t}) + \int_0^\infty P_1(|\xi|^p \geq u)du \right) dt$$

$$\sim_p \int_0^s \left( \frac{1}{x} P_1(|\xi| \geq \frac{1}{x}) + \int_{\frac{1}{x}}^\infty P_1(|\xi|^p \geq y) dy \right) x^{p-1} dx.$$  

By (8) we get

$$(I) \sim \int_\frac{1}{x}^\infty e^{-\frac{y^2}{2}} y^{p-2} dy.$$  

Since $p - 2 < 0$ and $y \geq \frac{1}{x} \geq 1$, we have $y^{p-2} \leq 1$ and so we get again by (8)

$$(I) \lesssim \int_\frac{1}{x}^\infty e^{-\frac{y^2}{2}} dy \sim x e^{-\frac{1}{2x^2}}.$$  

Altogether

$$M_{\xi^p}(s) \lesssim_p \int_0^s \left( \frac{1}{x} P_1(|\xi| \geq \frac{1}{x}) + x e^{-\frac{1}{2x^2}} \right) x^{p-1} dx.$$  

To estimate $P\left(|\xi| \geq \frac{1}{x}\right)$, we apply (8) and then take into account that for all $x \in (0, s^{1/p})$ it holds that $x \leq 1$ and so $e^{-\frac{1}{2x^2}} + x e^{-\frac{1}{2x^2}} \leq 2 e^{-\frac{1}{2x^2}}$. Using this, we get

$$M_{\xi^p}(s) \lesssim_p \int_0^s x^{p-1} \left( e^{-\frac{1}{2x^2}} + x e^{-\frac{1}{2x^2}} \right) dx$$

$$= \int_0^s x^{p-1} \left( e^{-\frac{1}{2x^2}} + x e^{-\frac{1}{2x^2}} \right) dx$$

$$\lesssim \int_0^s x^{p-1}e^{-\frac{1}{2x^2}} dx$$

$$= \int_0^\infty t^{-p-1}e^{-\frac{t^2}{2}} dt.$$  

Since $-p - 1 < -2$ and $t \geq s^{-\frac{1}{p}} \geq 1$, it holds that $t^{-p-1} \leq 1$. Applying this and (8), we get

$$M_{\xi^p}(s) \lesssim_p \int_0^s \int_0^\infty e^{-\frac{t^2}{2}} dt \sim s^{\frac{1}{p}} e^{-\frac{1}{2s^{\frac{1}{p}}}}.$$  

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Finally, as \( s \leq 1 \), we get

\[
M_{\xi^p}(s) \lesssim e^{-\frac{1}{2s^2}} \sim M_{\xi}\left(s^{\frac{1}{2}}\right).
\]

**Case 2:** Let \( s > 1 \).

\[
M_{\xi^p}(s) = \int_0^1 \left( \frac{1}{t} \mathbb{P}_1 \left( |\xi|^p \geq \frac{1}{t} \right) + \int_{\frac{1}{t}}^\infty \mathbb{P}_1 \left( |\xi|^p \geq u \right) du \right) dt.
\]

\[
= \int_0^1 \left( \frac{1}{t} \mathbb{P}_1 \left( |\xi|^p \geq \frac{1}{t} \right) + \int_{\frac{1}{t}}^\infty \mathbb{P}_1 \left( |\xi|^p \geq u \right) du \right) dt
+ \int_{\frac{1}{s}}^s \left( \frac{1}{t} \mathbb{P}_1 \left( |\xi|^p \geq \frac{1}{t} \right) + \int_{\frac{1}{t}}^\infty \mathbb{P}_1 \left( |\xi|^p \geq u \right) du \right) dt.
\]

(a) can be estimated by case 1 and so yields

\[
\int_0^1 \left( \frac{1}{t} \mathbb{P}_1 \left( |\xi|^p \geq \frac{1}{t} \right) + \int_{\frac{1}{t}}^\infty \mathbb{P}_1 \left( |\xi|^p \geq u \right) du \right) dt \lesssim e^{-\frac{1}{4}}.
\]

So it suffices to estimate

\[
(b) = \int_{\frac{1}{s}}^s \frac{1}{t} \mathbb{P}_1 \left( |\xi|^p \geq \frac{1}{t} \right) dt + \int_{\frac{1}{t}}^\infty \mathbb{P}_1 \left( |\xi|^p \geq u \right) du dt
\]

At first, we estimate (I). Using Markov’s inequality, we get

\[
(I) = \int_1^s \frac{1}{t} \mathbb{P}_1 \left( |\xi| \geq \frac{1}{t} \right) dt \leq \int_1^s \frac{1}{t} \mathbb{E} |\xi| \left\{ \left( \frac{1}{t} \right)^{\frac{p}{2}} \right\} dt \sim \int_1^s t^{-1+\frac{1}{p}} dt = \left[ t^{\frac{1}{p}} \right]_1^{s^{\frac{1}{p}}} = s^{\frac{1}{p}} - 1 \lesssim \frac{1}{2s^{\frac{1}{2}}}
\]

To estimate (II), we use (8) and get

\[
(II) \sim \int_1^s \int_1^\infty u^{-\frac{1}{2}} e^{-\frac{u^2}{2}} du dt = \int_1^s \int_1^\infty y^{-1} e^{-\frac{y^2}{2}} p y^{p-1} dy dt \sim \int_1^s \int_1^\infty y^{p-2} e^{-\frac{y^2}{2}} dy dt
\]

\[
\sim \int_1^s \left[ -\Gamma \left( \frac{p-1}{2}, \frac{y^2}{2} \right) \right]_{\frac{1}{t^{\frac{1}{p}}}}^\infty dt = \int_1^s \Gamma \left( \frac{p-1}{2}, \frac{1}{2t^{\frac{1}{p}}} \right) dt.
\]

In general, we have

\[
\int x^{b-1} \Gamma(t, x) dx = \left[ \frac{1}{b} \left( x^b \Gamma(t, x) - \Gamma(t + b, x) \right) \right]
\]
see for example [1]. With $b = 1$ this provides

$$
(II) \sim \int_1^s \Gamma \left( \frac{p-1}{2}, \frac{1}{2t^p} \right) dt = \left[ \frac{1}{2t^p} \Gamma \left( \frac{p-1}{2}, \frac{1}{2t^p} \right) - \Gamma \left( \frac{p-1}{2} + 1, \frac{1}{2t^p} \right) \right]_1^s
$$

$$
= \frac{1}{2s^p} \Gamma \left( \frac{p-1}{2}, \frac{1}{2s^p} \right) - \Gamma \left( \frac{p-1}{2} + 1, \frac{1}{2s^p} \right) + c_p
$$

$$
\text{for } s \geq 1.
$$

Generally by integration by parts we have

$$
\Gamma(t, x) = (t-1)\Gamma(t-1, x) + xe^{t-1}e^{-x}.
$$

We apply this for $t = \frac{p-1}{2} + 1$ and $x = \frac{1}{2s^p}$. Since $1 < s < \infty$ and $1 < p < 2$,

$$
0 \leq xe^{t-1}e^{-x} = \frac{1}{2s^p} e^{-\frac{1}{2} + \frac{2}{s^p}} \leq \frac{1}{2},
$$

$$
\text{for } s \geq 1.
$$

Altogether, this yields

$$
\Gamma \left( \frac{p-1}{2} + 1, \frac{1}{2s^p} \right) \sim_p \Gamma \left( \frac{p-1}{2}, \frac{1}{2s^p} \right) + \tilde{c}_p.
$$

Overall, we have

$$
(II) \lesssim_p \tilde{c}_p.
$$

Combining the previous, we get

$$
M_{\xi^p}(s) = (a) + (b) \lesssim_p e^{-\frac{s}{2}} + s^{\frac{1}{p}} - 1 + \tilde{c}_p \lesssim_p \sim M_{\xi} \left( \frac{s}{p} \right).
$$

Subsumed, we proved for all $s$

$$
M_{\xi^p}(s) \lesssim_p M_{\xi} \left( \frac{s}{p} \right).
$$

Lower bound $M_{\xi^p}(s) \gtrsim_p M_{\xi} \left( \frac{s}{p} \right)$:
Case 1: Let \( s \leq 1 \).

\[
M_{\xi^p}(s) = \int_0^s \left( \frac{1}{t} \mathbb{P}_1 \left( |\xi|^p \geq \frac{1}{t} \right) + \int_0^\infty \mathbb{P}_1(|\xi|^p \geq u) du \right) dt
\]

\[
\sim_p \int_0^{s^{\frac{1}{p}}} \left( \frac{1}{x} \mathbb{P}_1 \left( |\xi| \geq \frac{1}{x} \right) + \int_{\frac{1}{x}}^\infty \mathbb{P}_1(|\xi| \geq y) y^{p-1} dy \right) x^{p-1} dx
\]

\[
\geq \int_0^{s^{\frac{1}{p}}} x^{p-2} \mathbb{P}_1 \left( |\xi| \geq \frac{1}{x} \right) dx.
\]

We have \(-1 < p - 2 < 0\) and \( x \leq 1 \), so \( 1 \leq x^{p-2} \leq x^{-1} \) holds and therefore

\[
M_{\xi^p}(s) \geq_p \int_0^{s^{\frac{1}{p}}} x^{p-2} \mathbb{P}_1 \left( |\xi| \geq \frac{1}{x} \right) dx \geq \int_0^{s^{\frac{1}{p}}} \mathbb{P}_1 \left( |\xi| \geq \frac{1}{x} \right) dx.
\]

Applying (8), we get

\[
M_{\xi^p}(s) \geq_p \int_0^{s^{\frac{1}{p}}} x e^{-\frac{x^2}{2}} dx = \int_{s^{-\frac{1}{p}}}^{\infty} t^{-3} e^{-\frac{t^2}{2}} dt = \int_{s^{-\frac{1}{p}}}^{\infty} e^{-\frac{t^2}{2}-3 \ln(t)} dt \geq \int_{s^{-\frac{1}{p}}}^{\infty} e^{-\frac{t^2}{2}} dt \sim s^{\frac{1}{p}} e^{-\frac{s}{2p}}.
\]

Finally, we proved

\[
M_{\xi^p}(s) \geq_p M_{\xi^p} (s^{\frac{1}{p}}).
\]
Case 2: Let $s > 1$.

$$M_{\xi^p}(s) = \int_0^s \left( \frac{1}{t} \mathbb{P}_1 \left( |\xi|^p \geq \frac{1}{t} \right) + \int_{\frac{1}{t}}^\infty \mathbb{P}_1 (|\xi|^p \geq u) du \right) dt$$

$$\sim_p \int_0^{s^\frac{1}{p}} \left( \frac{1}{x} \mathbb{P}_1 \left( |\xi| \geq \frac{1}{x} \right) + \int_{\frac{1}{x}}^\infty \mathbb{P}_1 \left( |\xi| \geq y \right) y^{p-1} dy \right) x^{p-1} dx$$

$$= \int_0^{s^\frac{1}{p}} \frac{1}{x} \mathbb{P}_1 \left( |\xi| \geq \frac{1}{x} \right) x^{p-1} dx + \int_0^{s^\frac{1}{p}} \int_{\frac{1}{x}}^\infty \mathbb{P}_1 \left( |\xi| \geq y \right) y^{p-1} dy x^{p-1} dx$$

$$\geq \int_0^{s^\frac{1}{p}} \int_{\frac{1}{x}}^\infty \mathbb{P}_1 \left( |\xi| \geq y \right) y^{p-1} dy x^{p-1} dx$$

$$\geq \int_0^{s^\frac{1}{p}} \int_{\frac{1}{x}}^\infty \mathbb{P}_1 \left( |\xi| \geq y \right) x^{-p+1} dy x^{p-1} dx$$

$$= \int_0^{s^\frac{1}{p}} \int_{\frac{1}{x}}^\infty \mathbb{P}_1 \left( |\xi| \geq y \right) dy dx$$

$$\geq \int_1^{s^\frac{1}{p}} \int_{\frac{1}{x}}^\infty \mathbb{P}_1 \left( |\xi| \geq y \right) dy dx$$

$$\geq \int_1^{s^\frac{1}{p}} \int_1^\infty \mathbb{P}_1 \left( |\xi| \geq y \right) dy dx.$$

By (8), we get

$$\int_1^\infty \mathbb{P}_1 \left( |\xi| \geq y \right) dy \sim \int_1^\infty \frac{1}{y} e^{-\frac{y^2}{4}} dy = \int_1^\infty e^{-\frac{y^2}{4}-\ln(y)} dy \sim \int_1^\infty e^{-\frac{y^2}{4}} dy \sim e^{-1},$$

which yields

$$M_{\xi}(s) \gtrsim_p \int_1^{s^\frac{1}{p}} e^{-1} dx \sim s^{\frac{1}{p}} - 1 \sim s^{\frac{1}{p}} \sim M_{\xi^p}(s^{\frac{1}{p}}).$$

altogether, we proved that for all $s$

$$M_{\xi^p}(s) \gtrsim_p M_{\xi^p} \left( s^{\frac{1}{p}} \right).$$

With regard to the previous, we proved for all $s$

$$M_{\xi^p}(s) \sim_p M_{\xi^p} \left( s^{\frac{1}{p}} \right),$$

which concludes the proof.
3 Generation of truncated $\ell_p$-norms ($p > 1$)

Since standard gaussian random variables do not generate the $\ell_2$-norm, the question arises what distribution does. We prove that $\log - \gamma_{1,p}$ distributed random variables generate more or less the $\ell_p$-norm and especially $\log - \gamma_{1,2}$ distributed random variables generate exactly the $\ell_2$-norm.

We remind the reader that the density of a $\log - \gamma_{q,p}$ distributed random variable $\xi$ with parameters $q,p > 0$ is given by

$$f_\xi(x) = \begin{cases} \frac{p^q}{\Gamma(q)} x^{-p-1} (\ln(x))^{q-1}, & x \geq 1, \\ 0, & x < 1. \end{cases}$$

We prove the following theorem.

**Theorem 3.1.** Let $p > 1$ and $\xi_1, \ldots, \xi_n$ be i.i.d. copies of a $\log - \gamma_{1,p}$ distributed random variable $\xi$. Then for all $x \in \mathbb{R}^n$

$$\mathbb{E} \max_{1 \leq i \leq n} |x_i \xi_i| \sim \|x\|_{M_\xi},$$

and $M_\xi(s) = \begin{cases} \frac{1}{p-1}s^p, & s \leq 1; \\ \frac{1}{p-1}s - 1, & s > 1. \end{cases}$

**Proof.** By Theorem 1.1, we have

$$\mathbb{E} \max_{1 \leq i \leq n} |x_i \xi_i| \sim \|x\|_{M_\xi},$$

where

$$M_\xi(s) = \int_0^s \frac{1}{t} \mathbb{P}\left(|\xi| \geq \frac{1}{t}\right) dt + \int_t^\infty \mathbb{P}(|\xi| \geq u) du dt. \quad (9)$$

**Case 1:** Let $s \leq 1$. Since we have integration limits 0 and $s$, $\frac{1}{s} \geq 1$ holds. For all $y \geq 1$

$$\mathbb{P}(|\xi| \geq y) = \int_y^\infty f_\xi(x) dx = \int_y^\infty px^{-p-1} dx = [-x^{-p}]^\infty_y = y^{-p}. \quad (10)$$

Therefore, by (10)

$$\int_0^s \frac{1}{t} \mathbb{P}\left(|\xi| \geq \frac{1}{t}\right) dt = \int_0^s \frac{1}{t} t^p dt = \frac{s^p}{p}.$$ 

Furthermore, by (10) and because $p > 1$

$$\int_\frac{1}{t}^\infty \mathbb{P}(|\xi| \geq u) du = \int_\frac{1}{t}^\infty u^{-p} du = \frac{1}{p-1}t^{p-1}$$
and hence
\[ \int_0^s \frac{1}{p-1} t^{p-1} dt = \frac{1}{p(p-1)} s^p. \]

Using the representation (9), we obtain
\[ M_{\xi}(s) = \frac{1}{p} s^p + \frac{1}{p(p-1)} s^p = \frac{1}{p-1} s^p. \]

**Case 2:** Let \( s > 1 \). We first calculate
\[ \int_0^s \frac{1}{t} \mathbb{P} \left( |\xi| \geq \frac{1}{t} \right) dt. \]

We have
\[ \int_0^s \frac{1}{t} \mathbb{P} \left( |\xi| \geq \frac{1}{t} \right) dt = \int_0^1 \frac{1}{t} \mathbb{P} \left( |\xi| \geq \frac{1}{t} \right) dt + \int_1^s \frac{1}{t} \mathbb{P} \left( |\xi| \geq \frac{1}{t} \right) dt. \]

In (I), we have \( \frac{1}{t} \geq 1 \) and therefore (10) applies and we obtain
\[ \mathbb{P} \left( |\xi| \geq \frac{1}{t} \right) = t^p. \]

Hence
\[ (I) = \int_0^1 \frac{1}{t} t^p dt = \left[ \frac{1}{p} t^p \right]_0^1 = \frac{1}{p}. \]

In (II), we have \( \frac{1}{t} \leq 1 \) and therefore
\[ \mathbb{P} \left( |\xi| \geq \frac{1}{t} \right) = \int_{1/t}^\infty f_{\xi}(x) dx = \int_1^{\infty} f_{\xi}(x) dx = \int_1^{\infty} px^{-p-1} dx = [-x^{-p}]_1^\infty = 1. \]

So we obtain
\[ (II) = \int_1^s \frac{1}{t} dt = \ln(s). \]

Therefore
\[ \int_0^s \frac{1}{t} \mathbb{P} \left( |\xi| \geq \frac{1}{t} \right) dt = \ln(s) + \frac{1}{p}. \]

We now calculate
\[ \int_0^s \int_{1/t}^{\infty} \mathbb{P}(|\xi| \geq u) du \ dt. \]

Again, we have
\[ \int_0^s \int_{1/t}^{\infty} \mathbb{P}(|\xi| \geq u) du \ dt = \int_0^1 \int_{1/t}^{\infty} \mathbb{P}(|\xi| \geq u) du \ dt + \int_1^s \int_{1/t}^{\infty} \mathbb{P}(|\xi| \geq u) du \ dt. \]
In part \((III)\), we have \(\frac{1}{t} \geq 1\) and hence by \((10)\)
\[
P(|\xi| \geq u) = u^{-p}.
\]
So we obtain
\[
\int_{\frac{1}{t}}^{\infty} u^{-p} = \frac{1}{p-1} t^{p-1}.
\]
Therefore
\[
(III) = \int_0^1 \int_{1/t}^{\infty} P(|\xi| \geq u) du \; dt = \int_0^1 \frac{1}{p-1} t^{p-1} dt = \frac{1}{p(p-1)}.
\]
In part \((IV)\), we have \(\frac{1}{t} \leq 1\), so we get
\[
\int_{\frac{1}{t}}^{\infty} P(|\xi| \geq u) du = \int_{\frac{1}{t}}^{1} P(|\xi| \geq u) du + \int_{1}^{\infty} P(|\xi| \geq u) du.
\]
Since in \((IV.1)\) we have \(u \leq 1\), we obtain
\[
(IV.1) = \int_{\frac{1}{t}}^{1} \int_{u}^{\infty} f_{\xi}(x) dx \; du = \int_{\frac{1}{t}}^{1} \int_{1}^{\infty} f_{\xi}(x) dx \; du = \int_{1}^{1} 1 du = 1 - \frac{1}{t}.
\]
In \((IV.2)\), we have \(u \geq 1\) and therefore by \((10)\)
\[
P(|\xi| \geq u) = u^{-p}.
\]
Hence
\[
(IV.2) = \int_{1}^{\infty} u^{-p} du = \frac{1}{p-1}.
\]
So
\[
(IV) = \int_{1}^{s} \int_{1/t}^{\infty} P(|\xi| \geq u) du \; dt = \int_{1}^{s} 1 - \frac{1}{t} + \frac{1}{p-1} dt = \frac{p}{p-1} (s - 1) - \ln(s).
\]
Altogether, we have
\[
M_\xi(s) = \ln(s) + \frac{1}{p} + \frac{1}{p(p-1)} + \frac{p}{p-1} (s - 1) - \ln(s),
\]
i.e. for \(s > 1\) we have \(M_\xi(s) = \frac{p}{p-1} s - 1\).

An Orlicz norm \(\| \cdot \|_M\) is uniquely determined on the interval \([0, s_0]\) where \(M(s_0) = 1\). Therefore, we obtain the following interesting corollary.
Corollary 3.2. Let $\xi_1, \ldots, \xi_n$ be i.i.d. copies of a log $-\gamma_{1,2}$ distributed random variable $\xi$. Then, for all $x \in \mathbb{R}^n$,

$$E \max_{1 \leq i \leq n} |x_i \xi_i| \sim \|x\|_2.$$ 

In fact, this is interesting since one would assume standard gaussians to generate the $\ell_2$-norm. In fact, the norm generated by Gaussians is far from being the $\ell_2$-norm.

Naturally now the question arises, can we prove Theorem 2.1 and Theorem 2.2 also in case that $p = 2$, this means in the case that the random variables $\xi_i$, $i = 1, \ldots, n$, are independent log $-\gamma_{1,2}$ distributed. We can do so, as provided in the following.

Theorem 3.3. Let $p \in (1, 2)$, let $\xi_i$, $i = 1, \ldots, n$ be independent copies of a log $-\gamma_{1,2}$ distributed random variable $\xi$ on $(\Omega_1, \mathcal{A}_1, \mathbb{P}_1)$ and let $\eta_j$, $j = 1, \ldots, m$ be independent $p$-stable copies of a random variable $\eta$ on $(\Omega_2, \mathcal{A}_2, \mathbb{P}_2)$. Then, for all $(a_{ij})_{i,j} \in \mathbb{R}^{n \times m}$,

$$E_{\Omega_1} E_{\Omega_2} \max_{i,j} |a_{ij} \xi_i(\omega_1) \eta_j(\omega_2)| \sim_p \left( \| (a_{ij})_{i=1}^n \|_2 \right)_{j=1}^m.$$

Proof. We follow the proof of Theorem 2.1. Therefore we have

$$E_{\Omega_1} E_{\Omega_2} \max_{1 \leq i \leq n, 1 \leq j \leq n} |a_{ij} \xi_i(\omega_1) \eta_j(\omega_2)| \sim E_{\Omega_1} \left( \max_{1 \leq i \leq n} |a_{ij} \xi_i(\omega_1)| \right)^n_{j=1}.$$

As before, we have to show that $M_{\xi}(s) \sim M_{\xi}(s^{1/p}) = s^{2/p}$. We calculate $M_{\xi}(s)$ and start with $s \leq 1$. Since for all $y \geq 1$

$$\mathbb{P}(|\xi| \geq y) = \int_y^\infty f_\xi(x)dx = y^{-2},$$

we obtain

$$M_{\xi}(s) = \int_0^1 \frac{1}{t} \mathbb{P}(|\xi| \geq t^{-1/p}) + \int_t^\infty \mathbb{P}(|\xi| \geq u^{1/p})du \, dt$$

$$= \int_0^s t^{2/p-1} + \left[ -\frac{1}{2/p-1} u^{-2/p+1} \right]_t^\infty \, dt$$

$$= \frac{2}{2-p} \int_0^s t^{2/p-1} \, dt$$

$$= \frac{p}{2-p} s^{2/p}.$$
So for all $s \leq 1$ we have $M_{\xi p}(s) = \frac{p}{2-p}s^{2/p} = \frac{p}{2-p}M_{\xi}(s^{1/p})$. Since $\frac{p}{2-p} > 1$, the case $0 \leq s \leq 1$ suffices because $M_{\xi p}(1) > 1$ and therefore the Orlicz norm $\|\cdot\|_{M_{\xi p}}$ is uniquely determined on this interval.

**Theorem 3.4.** Let $\xi_i, i = 1, ..., n$ be independent copies of a log $-\gamma_{1,2}$ distributed random variable $\xi$ on $(\Omega_1, \mathcal{A}_1, \mathbb{P}_1)$ and let $\eta_j, j = 1, ..., m$ be independent copies of a standard gaussian random variable $\eta$ on $(\Omega_2, \mathcal{A}_2, \mathbb{P}_2)$. Then, for all $(a_{ij}) \in \mathbb{R}^{n \times m}$,

$$
\mathbb{E}_{\Omega_1, \Omega_2} \max_{i,j} |a_{ij}\eta_j(\omega_2)\xi_i(\omega_1)| \sim \left(\left\|\left\|(a_{ij})_{i=1}^n\right\|_{M_{\xi}}\right\|_{j=1}^m\right)^2,
$$

where $\|\cdot\|_{M_{\xi}}$ denotes again the Orlicz norm given by the Orlicz function

$$
M_{\xi}(s) = \begin{cases} 
0, & \text{if } s = 0 \\
e^{-\frac{3}{2}s^2}, & \text{if } s \in (0, 1) \\
e^{-\frac{3}{2}(3s-2)}, & \text{if } s \geq 1.
\end{cases}
$$

(11)

The proof works exactly in the same way as the proof of Theorem 2.2.

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