EXTENSIONS, LEVI SUBGROUPS AND CHARACTER FORMULAS

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We dedicate this paper to the memory of Walter Feit

Abstract. This paper consists of three interconnected parts. In Parts I, III we study the relationship between the cohomology of a reductive group \( G \) and that of a Levi subgroup \( H \). For example, we provide a necessary condition, arising from Kazhdan-Lusztig theory, for a natural map \( \text{Ext}^\bullet_G(L, L') \to \text{Ext}^\bullet_H(L_H, L'_H) \) to be surjective, given irreducible \( G \)-modules \( L, L' \) and corresponding irreducible \( H \)-modules \( L_H, L'_H \). In cohomological degree \( n = 1 \), the map is always an isomorphism, under our hypothesis. These results were inspired by recent work of Hemmer [19] obtained for \( G = GL_n \), and both extend and improve upon the latter when our condition is met. Part II obtains results on Lusztig character formulas for reductive groups, obtaining new necessary and sufficient conditions for such formulas to hold. In the special case of \( G = GL_n \), these conditions can be recast in a striking way completely in terms of explicit representation theoretic properties of the symmetric group (and the results improve upon the sufficient cohomological conditions established recently in [25]).

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INTRODUCTION

Let \( H \) be a Levi subgroup of a reductive algebraic group \( G \) defined over an algebraically closed field \( k \) of positive characteristic \( p \). This paper has its origins in the authors’ attempts to understand the relationship between the cohomology of \( H \) and that of \( G \). Usually, we concentrate on comparing \( \text{Ext}^\bullet_G(M, N) \) to \( \text{Ext}^\bullet_H(M_H, N_H) \), where \( M, N \) are rational \( G \)-modules and \( M_H, N_H \) are rational \( H \)-modules obtained from \( M, N \) by a natural “truncation” process.

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Especially important examples include the cases when $M, N$ are taken to be standard (or Weyl) modules, costandard modules, or irreducible modules.

Contributions to the problem have been made by E. Cline [4] and S. Donkin [11]. Their work was the starting point for our discovery in [10] (with E. Cline) of an explicit category equivalence between some highest weight categories associated to the categories of rational modules for $G$ and $H$. Because the methods were generally formal, they also applied to module categories for quantum enveloping algebras and, in type $A$, for $q$-Schur algebras at a root of unity. Furthermore, these results provided natural Ext-transfer results in going from certain $H$-modules to $G$-modules.

Part I of the present paper reviews the category equivalence described above, and gives a number of examples not explicitly considered in [10]. For example, when $H$ decomposes into a direct product of reductive subgroups, the groups $\text{Ext}^*_H(M_H, N_H)$ can be further analyzed by means of an evident application of the Künneth theorem. In type $A$, these results can be stated in an elegant way, using the elementary combinatorics of partitions. In this case, our formulation was inspired by the corresponding degree 0 decomposition given by Lyle-Mathas in a preprint [24]. (More recently, A. Mathas has made a revised version of [24] available to us, in which they also obtain, by different methods, the Künneth decomposition for $q$-Schur algebras.) It should be remarked, however, that the methods of [10] apply for reductive groups or quantum groups in all types. As an illustration, we very briefly consider the case of type $C$, which behaves like the type $A$ case.

The results of Part II improve upon [25, §7] which formulated a vanishing condition for symmetric group cohomology which, if true, would be sufficient to prove the Lusztig character conjecture for $G = SL_n(k)$. In [41] we allow $G$ to be of any type with $p \geq h$, the Coxeter number of $G$. Roughly, we consider finite ideals $\Gamma$ of dominant weights having the following property (†): if $\varpi \in \Gamma$ is regular, then the character of the irreducible $G$-module $L(\varpi)$ of highest weight $\varpi$ is given by the Lusztig character formula; see (3.0.2). Lusztig’s conjecture [23] asserts that the Jantzen region $\Gamma_{\text{Jan}}$ (defined in (3.0.3)) is such an ideal; however, independently of that conjecture, finite ideals $\Gamma$ satisfying property (†) are easily shown to exist. In that case, various homological methods can be brought to bear on the category $C_G[\Gamma]$ of finite dimensional rational $G$-modules which have composition factors $L(\gamma), \gamma \in \Gamma$. For example, [6] establishes that $\Gamma$ satisfies (†) if and only if $\text{Ext}^1_G(L(\lambda), L(\mu)) \neq 0$ whenever $\lambda, \mu \in \Gamma$ are regular weights which are mirror images of each other in adjacent alcoves. This result is recalled in Thm. 4.1 which also presents a useful (and new) method allowing the possible enlargement of (certain ideals) $\Gamma$ by the addition of an alcove adjacent to $\Gamma$. Thm. 4.2 recasts the $\text{Ext}^1$-criterion above in terms of a condition on mappings between $\text{Ext}^1$-groups. The modules involved are no longer irreducible; indeed, all are standard modules or modules with a standard filtration.

In Section 5, Thm. 5.2 translates, in the case $G = SL_n(k)$, the results of [11] into equivalent results stated entirely in symmetric group terms. In particular, we obtain a necessary and sufficient condition for the Lusztig character conjecture to hold for $SL_n(k)$, stated entirely in terms of the representation theory of the symmetric group. This result is in contrast to [25, §7], which only obtained a sufficient condition. Section 5 concludes with a discussion of the relation of Thm. 5.2 to the sufficient condition established in [25, §7]. See Remark 6.5 (b) for a brief comparison between features of our Lusztig conjecture reduction to those
of Erdmann’s general computation \[17\] of Schur algebra decomposition numbers in terms of those of the symmetric group.

The results in Part I for \(q\)-Schur algebras apply to Ext-groups between standard modules, or between costandard modules, or between an irreducible and a standard or costandard module. Part III takes up the question of comparing Ext-groups between irreducible \(G\)-modules with those Ext-groups between similarly parametrized irreducible \(H\)-modules. One result in this direction, for Schur algebras, has already been obtained by Hemmer \[19\], who proves the injectivity of the natural map

\[
\phi^G_H(\varpi, \varpi') : \text{Ext}^1_G(L(\varpi), L(\varpi')) \rightarrow \text{Ext}^1_H(L_H(\varpi), L_H(\varpi'))
\]

when the dominant weights \(\varpi, \varpi'\) lie in the same coset relative to the weight lattice of \(H\). His result is stated in the type \(A\) formalism of row removal in Young diagrams and Schur algebras. However, in §6, Thm. 6.1 proves that Hemmer’s result holds for all types. (A similar result would hold, with the same proof, for quantum enveloping algebras.) Hemmer has asked whether the map \(\phi^G_H(\varpi, \varpi')\) is an isomorphism. In §7, we provide several (related) examples showing that the answer to Hemmer’s question is sometimes negative. The examples involve weights at which the Lusztig character formula fail, and thus suggest a connection between Hemmer’s question and the validity of the Lusztig character formula.

Now assume, for general \(G\), that the Lusztig character formula holds for all irreducible modules \(L(\varpi)\) with regular highest weights \(\varpi\) in a given finite ideal \(\Gamma\) in the poset of dominant weights. Then, as perhaps suggested by the example in \[17\] Thm. 5.3 shows that \(\phi^G_H(\varpi, \varpi')\) is an isomorphism if \(\varpi, \varpi' \in \Gamma\) are regular and lie in the same coset relative to the root lattice of \(H\). Thus, Hemmer’s question does have a positive answer in this case. Moreover, we prove that the restriction map \(\text{Ext}^n_G(L(\varpi), L(\varpi')) \rightarrow \text{Ext}^n_H(L(\varpi), L_H(\varpi'))\) is surjective in all degrees \(n\). A key ingredient in these results is the study in \[17\] of the structure of the algebra \(A^1 = \text{Ext}^1_A(L_0, L_0)\), when \(A\) is a quasi-hereditary algebra such that \(A\)-mod has a Kazhdan-Lusztig theory; here \(L_0\) is the direct sum of the distinct irreducible \(A\)-modules.

Actually, Part III of this paper developed before Part II, following quite naturally upon the investigation in Part I. The relationship with the Lusztig character formula which emerged inspired the revisiting of the latter topic in Part II, though only the results up through Thm. 4.1 are really required in Part III. One interesting by-product of this investigation is a final observation, given in Cor. 5.7 showing that a form of the Lusztig conjecture holds for any Levi subgroup \(H\), once it holds for \(G\).

PART I: A category equivalence and applications

We first review an essential category equivalence involving Levi subgroups of reductive groups. The method has applicability to other situations, e. g., quantum groups, \(q\)-Schur algebras, as we illustrate. In addition, the equivalence will play an important role later in the paper.

1. THE EQUIVALENCE

Fix a reductive group \(G\) over a fixed algebraically closed field \(k\) of positive characteristic \(p\). We will assume that the derived group of \(G\) is simply connected. Let \(B\) be a Borel subgroup of \(G\) and let \(T\) be a maximal torus of \(G\) contained in \(B\). Let \(X = X(T)\) be the character group of \(T\), and set \(X^\vee = \text{Hom}(G_m, T)\), the cocharacter group of \(T\). There is a natural perfect pairing \(\langle , \rangle : X \times X^\vee \rightarrow \mathbb{Z} \cong \text{End}(G_m)\). We let \(\Phi \subset X\) be the root system of \(T\) in
$G$, and $\Phi^+$ (resp., $\Pi$) be the set of positive (resp., simple) roots determined by $B$; thus, if $\alpha \in \Phi^+$, the corresponding one-parameter root subgroup $U_\alpha$ is a subgroup of $B$. For $\alpha \in \Phi$, let $\alpha^\vee \in X^\vee$ be the associated coroot. Now let $P \supseteq B$ be a parabolic subgroup, having Levi factor $H \supseteq T$. We also refer to $H$ as a Levi subgroup. The root system of $T$ in $H$ is denoted $\Phi_H$, while $\Phi_H^+ := \Phi_H \cap \Phi^+$ is a set of positive roots and $\Pi_H = \Pi \cap \Phi_H$ is the corresponding set of simple roots.

We let $X^+ \subset X$ (resp., $X^{(H)+} \subset X$) be the set of dominant (resp., $H$-dominant) weights on $T$. Thus, $\lambda \in X^+$ (resp., $\lambda \in X^{(H)+}$) provided that $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}^+$ for all $\alpha \in \Pi$ (resp., $\alpha \in \Pi_H$). Define poset structures $\leq$ and $\leq_H$ on $X$ by putting $\lambda \leq \mu$ (resp., $\lambda \leq_H \mu$) provided that $\mu - \lambda \in \mathbb{Z}^+ \Phi^+$ (resp., $\mu - \lambda \in \mathbb{Z}^+ \Phi_H^+$). By \cite{10} Lemma 6, $X^+$ is an ideal in the poset $(X^{(H)+}, \leq_H)$, i. e., if $\mu \leq_H \lambda$ and $\mu \in X^{(H)+}$, then $\mu \in X^+$.

Let $\mathcal{C}_G$ (resp., $\mathcal{C}_H$) be the category of finite dimensional rational $G$-modules (resp., $H$-modules). The irreducible objects in $\mathcal{C}$ (resp., $\mathcal{C}_H$) are indexed by the set $X^+$ (resp., $X^{(H)+}$). For $\lambda \in X^+$, let $L(\lambda)$, $\Delta(\lambda)$, and $\nabla(\lambda)$ be the (rational) irreducible, standard, and costandard $G$-modules, respectively, of highest weight $\lambda$. Similarly, for $\sigma \in X^{(H)+}$, $L_H(\sigma)$, $\Delta_H(\sigma)$, $\nabla_H(\sigma)$ be the analogous objects in the category of rational $H$-modules.

Let $\Gamma^+$ be a finite ideal in the poset $(X^+, \leq)$, and let $\mathcal{C}_G[\Gamma^+]$ be the full subcategory of $\mathcal{C}$ consisting of all objects in $\mathcal{C}$ which have composition factors $L(\gamma)$, $\gamma \in \Gamma^+$. Thus, $\mathcal{C}_G[\Gamma^+]$ is a highest weight category with poset $(\Gamma^+, \leq)$.

1. If $\Xi^+ \subseteq \Gamma^+$ is a coideal (i. e., $\Gamma^+ \setminus \Xi^+$ is an ideal), we put $\mathcal{C}_G(\Xi^+)$ equal to the quotient category of $\mathcal{C}_G[\Gamma^+]$ by the Serre subcategory $\mathcal{C}_G[\Gamma^+ \setminus \Xi^+]$. Then $\mathcal{C}_G(\Xi^+)$ is a highest weight category with poset $(\Xi^+, \leq)$. Let $j^\ast : \mathcal{C}_G[\Gamma^+] \rightarrow \mathcal{C}_G(\Xi^+)$ be the natural quotient functor. For $\xi \in \Xi^+$, the coideal $j^\ast \xi$ is the irreducible (resp., standard, costandard) object $L(\xi)$ (resp., $\Delta(\xi), \nabla(\xi)$) in $\mathcal{C}_G[\Gamma^+]$ indexed by $\xi$. If $\xi \in \Xi^+$, then $j^\ast \xi$ annihilates $L(\gamma)$, $\Delta(\gamma)$, $\nabla(\gamma)$. These facts are all well-known general properties of highest weight categories—see, e. g., \cite{10} §2 and the references given there.

Fix $\omega \in X$ and let $\Omega = \omega + \mathbb{Z} \Phi_H$ be the corresponding coset of $\mathbb{Z} \Phi_H$ in $X$. Set $\Omega^+ = \Omega \cap X^+$ and $\Omega^{(H)+} = \Omega \cap X^{(H)+}$. Let $F$ be a finite, nonempty subset of $\Omega^+$.\footnote{If $\Gamma \subset X$, it is convenient to write $\mathcal{C}_G[\Gamma]$ for the full subcategory of $\mathcal{C}_G$ having objects which have composition factors $L(\gamma)$, $\gamma \in \Gamma$. Thus, if $\Gamma^+ := \Gamma \cap X^+$, then $\mathcal{C}_G[\Gamma^+] = \mathcal{C}_G[\Gamma^+]$. Also, if $\Gamma$ is an ideal in $(X, \leq)$, $\Gamma^+$ is an ideal in $(X^+, \leq)$.}

The set $X^+$ in this paragraph may be replaced with the set of highest weights of irreducible modules in any block or union of blocks for the category of rational $G$-modules. The same discussion applies. We will return to this topic later in \cite{10} If will be useful to observe that if $\Gamma$ is a union of blocks in an ideal, then $\mathcal{C}_G[\Gamma]$ is a highest weight category with poset $(\Gamma, \leq)$.

2. The definition of $\mathcal{C}_G(\Xi^+)$ does not depend on the choice of the finite ideal $\Gamma^+$: Suppose that $\Xi^+$ is any finite subset of $X^+$ which is interval closed, in the sense that, given, $\xi_1 < \xi_2 \in \Xi^+$, any $\xi \in X^+$ satisfying $\xi_1 \leq \xi \leq \xi_2$ automatically belongs to $\Xi^+$. Then $\Xi^+$ is a coideal in the finite ideal $\Gamma^+$ consisting of all $\gamma \in X^+$ satisfying $\gamma \leq \omega$ for some $\xi \in \Xi$. Thus, $\mathcal{C}_G(\Xi^+)$ can be taken to be the highest weight category $\mathcal{C}_G[\Gamma^+] / \mathcal{C}_G[\Gamma^+ \setminus \Xi^+]$. If $\Xi^+$ is a coideal in another finite ideal $\Gamma^+$, it is readily verified that $\mathcal{C}_G(\Xi^+)$ is equivalent to $\mathcal{C}_G[\Gamma^+] / \mathcal{C}_G[\Gamma^+ \setminus \Xi^+]$.

3. If $G$ has simple derived group $G'$, then $\Omega^+$ is itself finite \cite{10} Prop. 9, so we can usually take $F = \Omega^+$. However, even in that case, it might be desirable to consider a proper subset $F$.\footnote{If $G$ has simple derived group $G'$, then $\Omega^+$ is itself finite \cite{10} Prop. 9, so we can usually take $F = \Omega^+$. However, even in that case, it might be desirable to consider a proper subset $F$.}
functor

\[(1.0.1) \quad \pi_\Omega : C_G[\Gamma^+_F] \to C_H[\Omega] = C_H[\Omega^{+(H)}], \quad M \mapsto \pi_\Omega M = \bigoplus_{\tau \in \Omega} M_\tau.\]

Here \(M_\tau, \tau \in \Omega\), denotes the \(\tau\)-weight space of the rational \(G\)-module \(M\). By [11, Prop. 7], \(\pi_\Omega\) maps \(L(\omega), \Delta(\omega), \nabla(\omega), \omega \in \Omega^+_F\) to the corresponding objects \(L_H(\omega), \Delta_H(\omega), \nabla_H(\omega)\) in \(C_H[\Omega^+_F]\). (If \(\gamma \in \Gamma^+_F \setminus \Omega^+_F\), \(\pi_\Omega\) maps these objects to 0.) We have:

**Theorem 1.1.** [10, Thm. 8] The functor \(\pi_\Omega\) factors through the quotient morphism \(j^* : C_G[\Gamma^+_F] \to C_G(\Omega^+_F)\) to produce an equivalence

\[(1.1.1) \quad C_G(\Omega^+_F) \xrightarrow{\cong} C_H[\Omega^+_F]\]

of (highest weight) categories.

This means, given \(\omega \in \Omega^+_F\) and \(M \in C_G[\Delta^+_F]\), there are isomorphisms

\[
\text{Ext}^*_H(\Delta_H(\omega), \pi_\Omega M) \cong \text{Ext}^*_G(\Delta_H(\omega), \pi_\Omega j^* M) \\
\cong \text{Ext}^*_G(\Delta_H(\omega), j^* M) \\
\cong \text{Ext}^*_G(\Delta(\omega), M).
\]

The last isomorphism follows since, at the derived category level, \(C_G[\Gamma^+_F]\) fully embeds into the category of rational \(G\)-modules. The same argument applies for the first isomorphism. See [5, Thm. 3.9]. The second isomorphism is a consequence of Thm. [11] and the fact, noted above, that \(j^* \Delta(\omega)\) is the standard object in the highest weight category \(C_G(\Omega^+_F)\). The third isomorphism follows from [10, Lemma 6]; it is a general fact involving a quotient functor \(j^* : C_G[\Gamma] \to (\Omega^+)\) for any arbitrary coideal in a finite weight poset \(\Gamma^+\).

Similarly, there are isomorphisms

\[
\text{Ext}^*_H(\pi_\Omega M, \nabla_H(\omega)) \cong \text{Ext}^*_G(\pi_\Omega j^* M, \nabla_H(\omega)) \\
\cong \text{Ext}^*_G(j^* M, j^* \nabla(\omega)) \\
\cong \text{Ext}^*_G(M, \nabla(\omega)) \\
\cong \text{Ext}^*_G(M, \nabla(\omega)).
\]

In particular, in both (1.1.2) and (1.1.3), we can take \(M = L(\tau)\), for \(\tau \in \Omega^+_F\), to obtain

\[(1.1.4) \quad \text{Ext}^*_G(\Delta(\omega), L(\tau)) \cong \text{Ext}^*_H(\Delta_H(\omega), L_H(\tau))\]

and

\[(1.1.5) \quad \text{Ext}^*_G(L(\tau), \nabla(\omega)) \cong \text{Ext}^*_H(L(\tau), \nabla(\omega)).\]

See [10] pp. 224–226] for further discussion, including the relation of these Ext-groups with Kazhdan-Lusztig polynomials. In §§67 we will take up the issue of comparing Ext-groups between irreducible modules.
2. Some elementary applications

As noted in \[10\] §6, the proof of Thm. \[11\] relies entirely on properties of highest weight categories, and so it remains valid in other contexts, such as quantum enveloping algebras and \(q\)-Schur algebras at a root of unity. In this section, we present various illustrations. These results will not be used elsewhere in this paper.

First, we introduce some standard notation (which will be important also in \[17\]). If \(\lambda = (\lambda_1, \cdots, \lambda_n)\) is a partition of \(r\) we write \(\lambda \vdash r\); let \(\Lambda^+(r)\) be the set of all partitions of \(r\), and \(\Lambda^+(n, r)\) the set of partitions with at most \(n\) nonzero parts. If \(\lambda \vdash r\), write \(|\lambda| = r\). The set \(\Lambda^+(r)\) is regarded as a poset, using the dominance ordering \(\preceq\): \(\lambda \preceq \mu \iff \sum_{i=1}^j \lambda_i \leq \sum_{i=1}^j \mu_i\) for all \(j\). Then \(\Lambda^+(n, r)\) is a coideal in \(\Lambda^+(r)\). For \(\lambda \in \Lambda^+(r)\), let \(\lambda'\) denote the dual partition; thus, \(\lambda_i' = \#\{\lambda_j \geq i\}\).

Let \(E\) be a vector space over the field of real numbers \(\mathbb{R}\) of dimension \(n\). Assume that \(E\) is a Euclidean space with orthonormal basis \(e_1, \cdots, e_n\). We consider the root system \(\Phi\) of type \(A_{n-1}\) of \(E\) which has roots \(e_i - e_j, 1 \leq i \neq j \leq n\). We label the roots, etc. in the standard way, e. g., as in \[3\] p. 250]. Thus, \(\Phi\) has a simple set \(\Pi = \{\alpha_1, \cdots, \alpha_{n-1}\}\) of roots, putting \(\alpha_i - e_i - e_{i+1}, 1 \leq i < n\). We can identify the \(\mathbb{Z}\)-lattice generated by \(e_1, \cdots, e_n\) with the character group \(X = X(T)\) (written additively) of the \(n\)-dimensional torus \(T = \mathbb{G}_m^\times\), setting \(e_i : T \to \mathbb{G}_m\) to be the projection onto the \(i\)-th factor. With this notation, the “fundamental dominant weights” are defined by putting \(\varpi_i = e_1 + \cdots + e_i, i = 1, \cdots, n-1\). Using the \(\mathbb{Z}\)-basis \(e_1, \cdots, e_n\) for \(X\), we can identify \(X\) as the abelian group of sequences \((\lambda_1, \cdots, \lambda_n)\) of integers. Then \(X^+\) identifies with the set of all \((\lambda_1, \cdots, \lambda_n)\) satisfying \(\lambda_1 \geq \cdots \geq \lambda_n\). In this way, \(\Lambda^+(n, r)\) \(\subset X^+\).

For \(1 \leq d < n\), \(\Pi_d := \Pi \setminus \{\alpha_d\}\) is a base for a subroot system \(\Phi_d\) of type \(A_{d-1} \times A_{n-1-d}\). For \(\lambda, \mu \in \Lambda^+(n, r)\), write \(\lambda =_{d} \mu\) provided that \(\lambda - \mu \in \mathbb{Z}\Phi_d\). Also, put \(\lambda[d] = (\lambda_1, \cdots, \lambda_d)\). The following result is immediately verified.

**Lemma 2.1.** For \(\lambda, \mu \in \Lambda^+(n, r)\), \(\lambda =_{d} \mu\) if and only if \(|\lambda[d]| = |\mu[d]|\).

Now, using Thm. \[11\] we easily obtain expressions for various Ext-groups involving algebraic groups (or quantum groups) of type \(A\). In turn, the answers can be given in terms of Schur algebras \(S(n, r)\) (or \(q\)-Schur algebras \(S_q(n, r)\)). We illustrate these results in several cases, referring the reader to \[10\] for an explanation of the (very standard) notation, and further references. In the first four examples, we work with \(q\)-Schur algebras, letting \(q \in \mathbb{k}\) be a primitive \(\ell\)th root of unity.

**Example 1: Row removal.** Suppose \(\lambda, \mu \in \Lambda^+(n, r)\) satisfy \(\lambda_1 = \mu_1\). Let \(\overline{\lambda} = (\lambda_2, \cdots, \lambda_n), \overline{\mu} = (\mu_2, \cdots, \mu_n) \in \Lambda^+(n-1, r')\), where \(r' = r - \lambda_1\). Then for \((M, N) \in \{(\Delta, L), (L, \nabla), (\Delta, \Delta), (\nabla, \nabla)\}\), we have

\[
\Ext^\bullet_{S_q(n, r)}(M(\lambda), N(\lambda)) \cong \Ext^\bullet_{S_q(n-1, r')}(M(\overline{\lambda}), N(\overline{\mu})).
\]

In fact, this follows immediately from Thm. \[11\] using Lemma \[2\] with \(d = 1\). This result has already been observed in \[10\] §4.6.

**Example 2: Removal of several rows.** Keep the notation of Example 1 above, but replace \(\Phi_1\) by \(\Phi_d\) for some \(d\) satisfying \(1 < d < n\). Now given \(\lambda, \mu \in \Lambda^+(n, r)\), suppose that
|\lambda[d]| = |\mu[d]| = r' and set \( r'' = r - r' \). With \((M, N)\) as before,

\[
\text{Ext}_{S_q(n, r)}^\bullet(M(\lambda), N(\lambda)) \cong \text{Ext}_{S_q(d, r'')}^\bullet(M(\lambda[d]), N(\mu[d])) \otimes \text{Ext}_{S_q(n-d, r'')}^\bullet(M(\lambda[d]), N(\mu[d])),
\]
as graded vectors spaces. We have written \( \lambda[d] \) for \((\lambda_{d+1}, \ldots, \lambda_n)\) and similarly for \( \mu \).

In checking these facts, it is useful to make use of the fact that, for positive integers \( a, b \),

\[
\text{Ext}^\bullet_{S_q(a, b)}(M(\lambda), N(\lambda)) \cong \text{Ext}^\bullet_{S_q(b, a)}(M(\lambda), N(\lambda))
\]
for \( \lambda, \mu \in \Lambda^+(a, b) \). (On the right-hand side, we interpret \( \lambda, \mu \in \Lambda^+(b) \).) Suitably interpreted in the context of \( q \)-Schur algebras, this isomorphism is just that given in (1.1.4) or (1.1.5); see [25, Thm. 6.1].

**Example 3:** Passage to Hecke algebras. Define \( e = \begin{cases} p & \text{if } \ell = 1; \\ \ell & \text{otherwise}, \end{cases} \) where \( k \) has characteristic \( p \). Let \( H_r \) be the Hecke algebra associated to the symmetric group \( \mathfrak{S}_r \) and the parameter \( q \). We use the notation of [25]. For \( \lambda \vdash r \), let \( S_\lambda \) denote the corresponding Specht module for \( H_r \). Let \( m \) be a non-negative integer satisfying \( m \leq e - 3 \). Consider the setting of Example 2 with \( n = d \), and let \( \lambda, \mu \in \Lambda^+(r) \) satisfy \( \lambda[d] = \mu[d] \). Then

\[
\text{Ext}_{H_r}^m(S_\lambda, S_\mu) = \bigoplus_{i=0}^m \text{Ext}_{H_r}^i(S_{\lambda[d]}, S_{\mu[d]}) \otimes \text{Ext}_{H_r}^{m-i}(S_{\lambda[d]}, S_{\mu[d]}).
\]
This result is immediate from [25, Thm. 4.6(ii)].

**Example 4:** Column removal. Fix \( d, 1 \leq d < n \), and suppose that \( \lambda, \mu \in \Lambda^+(n, r) \) satisfy \( \lambda' = d \mu' \). Then we can use Example 2 to give a Künneth factorization of \( \text{Ext}_{S_q(n, r)}^\bullet(\Delta(\lambda), L(\mu)) \), after observing that

\[
(2.1.1) \quad \text{Ext}_{S_q(r, r)}^\bullet(\Delta(\lambda), \Delta(\mu)) \cong \text{Ext}_{S_q(r, r)}^\bullet(\Delta(\mu'), \Delta(\lambda')).
\]

To see this, we use some tilting module results, summarized (with references) in [10, §3]. Let \( Y \) be a complete tilting module for \( S_q(r, r) \mod \). Set \( E_q(r, r) = \text{End}_{S_q(r, r)}(Y) \). Form the contravariant tilting functor \( T : S_q(r, r) \mod \to E_q(r, r) \mod \) given by \( T(M) := \text{Hom}_{S_q(r, r)}(M, Y) \).

The category \( E_q(r, r) \mod \) is equivalent to the category \( S_q(r, r) \mod \), so we identify these two categories. (Caution: The equivalence is not by means of the tilting functor \( T \).) With this identification, \( T\Delta(\lambda) \cong \Delta(\lambda') \). If \( P_\bullet \to \Delta(\lambda) \to 0 \) is a projective resolution, then \( 0 \to \Delta(\lambda') \to TP_\bullet \) is a resolution of \( \Delta(\lambda') \cong T\Delta(\lambda) \) by tilting modules; see [25, Lemmas 1.4&1.5]. Since tilting modules are acyclic for the functor \( \text{Hom}_{E_q(r, r)}(\Delta(\mu'), -) \), (2.1.1) follows from the isomorphism

\[
\text{Hom}_{E_q(r, r)}(\Delta(\mu'), TP_\bullet) \cong \text{Hom}_{S_q(r, r)}(P_\bullet, \text{Hom}_{E_q(r, r)}(\Delta(\mu'), Y))
\]
(see [25, Prop. 1.2]) and the fact that \( \text{Hom}_{E_q(r, r)}(\Delta(\mu'), Y) \cong \Delta(\mu) \). More generally, this argument shows that \( T \) defines a contravariant equivalence from the exact subcategory of \( S_q(r, r) \mod \) with a \( \Delta \)-filtration to the similar category for \( E_q(r, r) \mod \).

In Example 2, the isomorphism

\[
\text{Ext}_{S_q(n, r)}^\bullet(\nabla(\lambda), \nabla(\mu)) \cong \text{Ext}_{S_q(d, r')}^\bullet(\nabla S_q(d, r')(\lambda), \nabla S_q(d, r')(\mu))
\]
is already essentially proved in [12, Formula (17), p. 91], without the explicit interpretation in terms of the decomposition as a tensor product.
The K"unneth results above have only been worked out in type $A$. More generally, in other
types in the case of a reductive algebraic (or quantum) group $G$, a similar such tensor factor-
ization would exist whenever the semisimple part of the Levi subgroup $H$ decomposes into a
product of smaller reductive groups. However, it is not likely that the elegant interpretation
in type $A$ given above holds in all other cases. Our last example below indicates that this
does occur in type $C_n$ at least.

**Example 5:** Type $C$. Suppose that $G \cong Sp_{2n}(k)$ has type $C_n$. We use the notation of
[3] p.254-255 (replacing $l$ there by $n$) as far as listing the set $\Pi$ of simple roots, etc. Thus,
$X = X(T) \cong \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n$, and the fundamental dominant weights $\omega_i, 1 \leq i \leq n$, are
given by $\omega = e_1 + \cdots + e_i$. The set $\Lambda^+(n, \bullet)$ of all partitions $\lambda$ of length at most $n$
indexes the dominant weights $X^+$. Just as for type $A$, for $1 \leq d \leq n$, define $\Phi_d$ as the subroot system
of type $A_{d-1} \times C_{n-d-1}$ of $\Phi$ with simple roots $\Pi_d := \Pi \setminus \{\alpha_d\}$. Put $\lambda = \mu$ if and only if
$\lambda - \mu \in \mathbb{Z}\Phi_d$. Then, by analogy with Lemma 2.1, given $\lambda, \mu \in \Lambda^+(n, \bullet)$, we verify that $\lambda = \mu$
if and only if $|\lambda[d]| = |\mu[d]|$ and $|\lambda| \equiv |\mu| \mod 2$. If $H$ is the Levi subgroup of $G$
corresponding to $\Phi_d$, then $H \cong GL_d(k) \times Sp_{2(n-1-d)}(k)$. Thus, if $\lambda = \mu$, the groups $\text{Ext}^\bullet_{GL_d(k)}(M(\lambda), N(\lambda))$
are isomorphic as a graded vector space to
$$\text{Ext}^\bullet_{GL_d(k)}(M(\lambda[d]), N(\mu[d])) \otimes \text{Ext}^\bullet_{Sp_{2d}(k)}(M(\lambda[d]), N(\mu[d])),$$
where $g = n - d - 1$ and $(M, N) \in \{ (\Delta, L), (L, \nabla), (\Delta, \Delta), (\nabla, \nabla) \}$. We leave further details
to the interested reader.

**PART II: Character formulas**

We study various conditions equivalent to the validity of the Lusztig character formula. We
arrange our formulations, in this section and the next, so that we can discuss cases where
the formula might hold, even though the full Lusztig conjecture might not hold, or might
not be known to hold.

3. The Lusztig character formula

Let $G$ be a simple and simply connected algebraic group over $k$. (The results below and
in Part III easily extend to the case of a general reductive group.) In addition, let $\rho \in X^+$
be defined by $\langle \rho, \alpha^\vee \rangle = 1$ for all simple roots $\alpha \in \Pi$. Let $W = N(T)/T$ be the Weyl group
of $G$ and let $W_p := W_{|\Pi}$ the affine Weyl group, generated by $W$ and the normal subgroup $pZ\Phi$ of
translations by $p$-multiples of roots. It acts naturally (and faithfully) on the space $E = \mathbb{R} \otimes X$.
For $\alpha \in \Phi$, $n \in \mathbb{Z}$, $s_{\alpha, np} \in W_p$ is defined as an operator on $E$ by $s_{\alpha, np}(u) = u - (\langle u, \alpha^\vee \rangle - np)\alpha$, $u \in E$. Set $S_p = \{ s\alpha \mid \alpha \in \Pi \} \cup \{ s_{\alpha, -p} \} \subseteq W_p$, where $\alpha_0 \in \Phi$ is the maximal short root. Then
$(W_p, S_p)$ is a Coxeter system; let $l : W_p \to \mathbb{Z}$ be the length function of $W_p$ defined by $S_p$.
In place of the natural action of $W_p$ on $E$, we prefer the “dot” action given by $w \cdot u = w(u+\rho) - \rho$
for $w \in W_p$, $u \in E$. Given $\alpha \in \Phi$, $n \in \mathbb{Z}$, let $H_{\alpha, np} = \{ u \in E \mid \langle u + \rho, \alpha^\vee \rangle = pn \}$. Thus, $s_{\alpha, np} \in W_p$
is a reflection about the hyperplane $H_{\alpha, np}$. Any connected component $C$ of $E \setminus \bigcup_{\alpha \in \Phi, n \in \mathbb{Z}} H_{\alpha, np}$
is an alcove of $W_p$. The closure $\overline{C}$ a fundamental domain for the dot action of $W_p$ on $E$.

A weight $\lambda \in X$ is called *regular* provided that $\lambda$ belongs to some alcove $C$ of $W_p$, i. e.,
provided $(\lambda + \rho, \alpha^\vee) \neq 0 \mod p$ for all $\alpha \in \Phi$. If $h$ denotes the Coxeter number of $G$, regular
weights exist if and only if $p \geq h$. In particular, if $p \geq h$, $-2\rho = w_0 \cdot 0$ is a regular weight,
where \( w_0 \) denotes the long word in \( W \). If \( \Sigma \subset \mathbb{E} \), let \( \Sigma_{\text{reg}} \) be the set of all regular elements in \( \Sigma \).

For the rest of this section, assume that \( p \geq h \). By definition, the Lusztig character formula holds for \( \lambda \in X^+ \) (in the category \( C_0 \) of finite dimensional rational \( G \)-modules) provided \( \lambda = x \cdot (-2\rho) \) for some \( x \in W_p \) and the irreducible \( G \)-module \( L(\lambda) \) has formal character given by
\[
\chi L(\lambda) = \sum_{y \cdot (-2\rho) \in X^+} (-1)^{l(y)-l(\rho)} P_{y, x}(1) \chi \Delta (y \cdot (-2\rho)).
\]

In this formula, \( P_{y, x} \) is the Kazhdan-Lusztig polynomial\(^4\) associated to the pair \((y, x) \in W_p \times W_p\). In particular, \( P_{y, x} = 0 \) unless \( y \leq x \) in the Bruhat-Chevalley ordering on \( W_p \). For any \( \xi \in X^+ \),
\[
\chi \Delta (\xi) = \frac{\sum_{w \in W} e^{w \cdot \xi}}{\sum_{w \in W} e^{w \cdot 0}}
\]
by Weyl’s character formula. Given a subset \( \Sigma \subseteq W_p \cdot 0 = W_p \cdot (-2\rho) \), we say that the Lusztig character formula holds for \( \Sigma \) if it holds for every \( \lambda \in \Sigma \).

The famous Lusztig conjecture for modular characters\(^{23}\) states that the Lusztig character formula holds for \( W_p \cdot 0 \cap \Gamma_{\text{Jantzen}} \), where
\[
\Gamma_{\text{Jantzen}} = \{ \lambda \in X^+ \mid \langle \lambda + \rho, \alpha^\vee \rangle \leq p(p-h+2), \ \forall \alpha \in \Phi \}
\]
is the Jantzen region. Of course, to test if \( \lambda \in \Gamma_{\text{Jantzen}} \), it suffices to check the inequality in \(3.0.3\) just for \( \alpha = \alpha_o \). For a group \( G \) with a fixed root system \( \Phi \), the conjecture holds provided \( p \) is large enough, through, in general, no sufficient bound on \( p \) is known. See \(22\) and, for a general discussion, \(29\) [§8]. The Lusztig conjecture holds generally in types \( A_n \), \( n \leq 3 \), and \( B_2, G_2 \) \(20\). In addition, it is known to hold in type \( A_4 \) for \( p = 5, 7 \); see the discussion in \(28\).

However, independently of the validity of the Lusztig conjecture, ideals \( \Gamma \) for which the Lusztig character formula holds for \( W_p \cdot 0 \cap \Gamma \) occur naturally, as Thm. \(1.1\) below suggests. A simple (almost trivial) example would be to take \( \Gamma = \{ \lambda \in X^+ \mid \lambda \leq s_{\alpha_p} \cdot 0 \} \).

Let \( C^+ \subset \mathbb{E} \) be the alcove containing \( 0 \), and put \( C^- = w_o \cdot C^+ \) (the alcove containing \(-2\rho\)). Let \( \lambda \in X \cap C^- \) and suppose the Lusztig character formula holds for \( x \cdot (-2\rho), x \in W_p \). Using translation,
\[
\chi L(x \cdot \lambda) = \sum_{y \cdot \lambda \in X^+} (-1)^{l(x)-l(\rho)} P_{y, x}(1) \chi \Delta (y \cdot \lambda).
\]

Thus, we can say that the Lusztig character formula holds for \( x \cdot \lambda \). Suppose that \( \Sigma \subset X^+ \) has the property that \( \Sigma_{\text{reg}} \) is a union of sets of the form \( C \cap X^+ \), \( C \) an alcove. (For example, \( \Gamma_{\text{Jantzen}} \) has this property.) Then, if the Lusztig character formula holds for \( W_p \cdot 0 \cap \Sigma \), it also holds for \( \Sigma_{\text{reg}} \).

\(^4\)The polynomials \( P_{y, x} \) are associated to the generic Hecke algebra \( H \) for the Coxeter system \((W_p, S_p)\). Each \( P_{y, x} \) is, in fact, a polynomial in \( u := t^2 \) and \( P_{x, y}(1) \) means the specialization at \( u = 1 \) (i.e., at \( t = -1 \)). The introduction of the variable \( t = \sqrt{-1} \) is only important in this paper in the proof of Thm. \(3.1.1\) below, involving the theory of enriched Grothendieck groups.

\(^5\)Suppose \( \lambda' \in X \cap C^- \) is not regular. Let \( D_{\lambda'} \) be set of distinguished left coset representatives in \( W_p \) of the stabilizer in \( W_p \) of \( \lambda' \) under the dot action. Suppose that \(3.0.4\) holds for a given \( x \in D_{\lambda'} \). Then, using
Let $X_1^+$ be the set of $p$-restricted dominant weights. A conjecture of Kato [21, Conj. 5.5] asserts that the Lusztig character formula holds for the set $X_1^+ \cap W_p \cdot 0$ if $p \geq h$. Observe that $X_1^+ \subseteq \Gamma_{\text{Jan}}$ if and only if $p \geq 2h - 3$. If his conjecture holds, then (the argument in [21, (5.4)]) also establishes that (3.0.2) holds for any $L(\lambda) = L(\tau) \otimes L(\xi)^{(1)}$, where $\tau \in X_1^+ \cap W_p \cdot 0$ and $\xi \in X^+$ has the property that $\Delta(\xi) \cong L(\xi)^{(1)}$. Here, given a rational $G$-module $V$, $V^{(1)}$ denotes the “twist” of $V$ through the Frobenius morphism $\text{Fr} : G \rightarrow G$.

The following result will be useful in §7. Although the statement of (a) is purely combinatorial, the authors are unaware of a purely combinatorial proof. It would also be interesting to remove the various restrictions in the statement this result.

**Lemma 3.1.** Assume that $p \geq h$ and that the root system of $G$ is simply laced. In addition, if $G$ has type $E_6, E_7, E_8$ assume that $p > h + 1$. Suppose that $x \in W_p$ and $x \cdot (-2\rho) = p\lambda$ for $\lambda \in X^+$.

(a) We have

\[
\text{ch } \Delta(\lambda)^{(1)} = \sum_{y \cdot (-2\rho) \in X^+} (-1)^{l(x) - l(y)} \text{ch } \Delta(y \cdot (-2\rho)).
\]

(b) If $\Delta(\lambda) \neq L(\lambda)$, the Lusztig character formula (3.0.2) does not hold for $p\lambda = x \cdot (-2\rho)$.

**Proof.** Let $U_q$ be the quantum enveloping algebra (integral form) at $q = \sqrt[1]{\tau}$ of the same type as $G$. Let $C_q$ be the category of finite dimensional integral type 1 modules for $U_q$. It is a highest weight category with irreducible (resp. standard) modules $L_\xi(\xi) \otimes L_\eta(\eta)$, $\xi \in X^+$. Furthermore, for any $\xi = x \cdot (-2\rho) \in X^+$, the Lusztig character formula holds for $\xi$ in the category $C_q$; in other words, (3.0.2) remains valid if each “$L$” (resp., “$\Delta$”) is replaced by “$L_q$” (resp., “$\Delta_q$”). This last assertion follows essentially from the validity of Lusztig’s quantum group conjecture: see [29, §7] for a detailed discussion and further references. It is here that the restriction that $p > h + 1$ is required in the exceptional cases—see the (equivalent) case–by–case formulation in [29, Thm. 7.1]. Of course, $\text{ch } \Delta_q(\xi) = \text{ch } \Delta(\xi)$ for any $\xi \in X^+$.

Let $\sigma : U_q \rightarrow U$ be the Frobenius morphism from $U_q$ to the universal enveloping algebra over $\mathbb{C}$ of the complex Lie algebra $g$ having the same type as $G$. Then $\Delta_q(p\lambda) = \sigma^* L(\lambda)$, the pull-back through $\sigma$ of the complex irreducible $g$-module of highest weight $\lambda$. Hence, $\text{ch } \Delta_q(p\lambda) = \text{ch } \Delta(\lambda)^{(1)}$, and (a) follows.

Since $L(p\lambda) \cong L(\lambda)^{(1)}$, (b) also follows. \qed

**Remark 3.2.** See [29] for a discussion of the status of the Lusztig quantum group conjecture used in the above proof in the non-simply laced case. Published results giving a character formula require $p \geq r_0$, where $r_0$ is sufficiently large, depending on the root system, but unknown; see [29, Thm. 7.1] and remarks following it. However, [29] also notes an improvement to $p > h$ in all cases using unpublished work of Soergel. A similar improvement in all cases might be based on the category equivalences given in [1], though specific details leading from the final result in [1] to a character formula are not included. In our paper, the above lemma Jantzen translation, $\text{ch } L(x \cdot \lambda) = \sum_{y \cdot \xi \in X^+} (-1)^{l(x) - l(y)} \text{ch } \Delta(y \cdot \lambda)$. In this way, it would be possible to speak of the Lusztig character formula holding for an ideal $\Gamma$, not just $\Gamma_{\text{reg}}$, though we will not use this more general terminology.

In general, such weights $\tau + p\xi$ need not always lie in $\Gamma_{\text{Jan}}$. 

---

1. The authors of the original paper note that the Lusztig character formula (3.0.2) remains valid if each “$L$” (resp., “$\Delta$”) is replaced by “$L_q$” (resp., “$\Delta_q$”). This last assertion follows essentially from the validity of Lusztig’s quantum group conjecture.
2. In general, such weights $\tau + \xi$ need not always lie in $\Gamma_{\text{Jan}}$. This is due to the fact that the Lusztig character formula holds for the set $X_1^+ \cap W_p \cdot 0$, where $p \geq h$. If his conjecture holds, then the result is established for $p \geq 2h - 3$. If his conjecture holds, then (the argument in [21, (5.4)]) also establishes that (3.0.2) holds for any $L(\lambda) = L(\tau) \otimes L(\xi)^{(1)}$, where $\tau \in X_1^+ \cap W_p \cdot 0$ and $\xi \in X^+$ has the property that $\Delta(\xi) \cong L(\xi)^{(1)}$. Here, given a rational $G$-module $V$, $V^{(1)}$ denotes the “twist” of $V$ through the Frobenius morphism $\text{Fr} : G \rightarrow G$. The following result will be useful in §7. Although the statement of (a) is purely combinatorial, the authors are unaware of a purely combinatorial proof. It would also be interesting to remove the various restrictions in the statement this result.

**Lemma 3.1.** Assume that $p \geq h$ and that the root system of $G$ is simply laced. In addition, if $G$ has type $E_6, E_7, E_8$ assume that $p > h + 1$. Suppose that $x \in W_p$ and $x \cdot (-2\rho) = p\lambda$ for $\lambda \in X^+$.

(a) We have

\[
\text{ch } \Delta(\lambda)^{(1)} = \sum_{y \cdot (-2\rho) \in X^+} (-1)^{l(x) - l(y)} \text{ch } \Delta(y \cdot (-2\rho)).
\]

(b) If $\Delta(\lambda) \neq L(\lambda)$, the Lusztig character formula (3.0.2) does not hold for $p\lambda = x \cdot (-2\rho)$.

**Proof.** Let $U_q$ be the quantum enveloping algebra (integral form) at $q = \sqrt[1]{\tau}$ of the same type as $G$. Let $C_q$ be the category of finite dimensional integral type 1 modules for $U_q$. It is a highest weight category with irreducible (resp. standard) modules $L_\xi(\xi) \otimes L_\eta(\eta)$, $\xi \in X^+$. Furthermore, for any $\xi = x \cdot (-2\rho) \in X^+$, the Lusztig character formula holds for $\xi$ in the category $C_q$; in other words, (3.0.2) remains valid if each “$L$” (resp., “$\Delta$”) is replaced by “$L_q$” (resp., “$\Delta_q$”). This last assertion follows essentially from the validity of Lusztig’s quantum group conjecture: see [29, §7] for a detailed discussion and further references. It is here that the restriction that $p > h + 1$ is required in the exceptional cases—see the (equivalent) case–by–case formulation in [29, Thm. 7.1]. Of course, $\text{ch } \Delta_q(\xi) = \text{ch } \Delta(\xi)$ for any $\xi \in X^+$.

Let $\sigma : U_q \rightarrow U$ be the Frobenius morphism from $U_q$ to the universal enveloping algebra over $\mathbb{C}$ of the complex Lie algebra $g$ having the same type as $G$. Then $\Delta_q(p\lambda) = \sigma^* L(\lambda)$, the pull-back through $\sigma$ of the complex irreducible $g$-module of highest weight $\lambda$. Hence, $\text{ch } \Delta_q(p\lambda) = \text{ch } \Delta(\lambda)^{(1)}$, and (a) follows.

Since $L(p\lambda) \cong L(\lambda)^{(1)}$, (b) also follows. \qed

**Remark 3.2.** See [29] for a discussion of the status of the Lusztig quantum group conjecture used in the above proof in the non-simply laced case. Published results giving a character formula require $p \geq r_0$, where $r_0$ is sufficiently large, depending on the root system, but unknown; see [29, Thm. 7.1] and remarks following it. However, [29] also notes an improvement to $p > h$ in all cases using unpublished work of Soergel. A similar improvement in all cases might be based on the category equivalences given in [1], though specific details leading from the final result in [1] to a character formula are not included. In our paper, the above lemma.
is only used in type $A$. In fact, in this case, as noted in [29], the equivalence between representations of the quantum enveloping algebra at an \( \ell \)th root of unity and the corresponding representations for affine Lie algebras holds for all \( \ell \), without any \( \ell \geq h \) restriction.

4. Equivalent conditions

Assume \( G \) is simple (any type), simply connected and \( p \geq h \). The faces of an alcove \( C \) are naturally labelled by elements of \( S_p \): if \( C = C^- \), the faces are already labelled by \( S_p \) by definition; otherwise, a face \( F \) is \( W_p \) dot-conjugate to a unique face of \( C^- \), and we assign to \( F \) the corresponding element \( s \in S_p \). Thus, for \( s \in S_p \), we speak of an \( s \)-face of \( C \). For \( \lambda, \mu \in X^+_W \) and \( s \in S_p \), write \( \mu = \lambda s \) provided: (1) \( \lambda, \mu \) lie in adjacent alcoves \( C, C' \) separated by an \( s \)-face, and (2) \( \lambda \) is the reflection of \( \mu \) through that \( s \)-face. In other words, \( \lambda = s_{\beta, np} \cdot \mu \), where the common \( s \)-face of \( C, C' \) lies in the hyperplane \( H_{\beta, np} \). Then \( \lambda \) and \( \lambda s \) are called adjacent. If \( \lambda = w \cdot (-2\rho), w \in W_p \) then \( \lambda s = w s \cdot (-2\rho) \). Also, \( \lambda s > \lambda \iff l(ws) = l(w) + 1 \), and \( \lambda = w \cdot (-2\rho) \in X^+ \iff w = w_{\rho y} \), with \( y \) a “distinguished” right coset representative of \( W \) in \( W_p \), i.e., \( l(w) = l(w_o) + l(y) \). In particular, if \( y \) is distinguished and \( y = s_1 \cdots s_n \), \( s_i \in S_p \), then each product \( s_1 \cdots s_m \), \( m \leq n \), is also a distinguished distinguished right coset representative. Thus,

\[
0 = w_o \cdot (-2\rho) < w_o s_1 \cdot (-2\rho) < \cdots < w_o s_1 \cdots s_n \cdot (-2\rho)
\]

provides a “path” of adjacent dominant weights from 0 to \( \lambda \).

In the following theorem, part (a) plays a basic role in the sequel. Part (b) gives a way to construct ideals \( \Gamma \) for which the Lusztig character formula holds. However, it and part (c) will not be used in the paper\(^7\) so that the proofs are rather brief, and rely heavily on some of the machinery developed in [9], [8].

**Theorem 4.1.** Let \( \Gamma \) be a finite ideal in the poset \( (X^+, \leq) \).

(a) The Lusztig character formula holds for \( \Gamma \cap W_p \cdot 0 \) if and only if

\[
\text{Ext}^1_{\mathcal{C}}(L(\lambda), L(\lambda s)) \neq 0
\]

whenever \( \lambda < \lambda s \) both belong to \( \Gamma \cap W_p \cdot 0 \).

(b) Suppose that there exists \( \tau \in X^+ \cap W_p \cdot 0 \) such that \( \Gamma = \{ \xi \in X^+ | \xi \leq \tau \} \) and form the ideal \( \tilde{\Gamma} = \{ \xi \in \Gamma | \xi < \tau \} \) of \( \Gamma \). Suppose that \( s \in S_p \) and \( \tau s \in \tilde{\Gamma} \). Assume that the Lusztig character formula holds for \( \tilde{\Gamma} \cap W_p \cdot 0 \). Then the Lusztig character formula holds for \( \Gamma \cap W_p \cdot 0 \) if and only if \( \text{Ext}^1_{\mathcal{C}}(L(\tau s), L(\tau)) \neq 0 \).\(^8\)

(c) Given \( \xi \in X^+ \cap W_p \cdot 0 \), the Lusztig character formula holds for \( \xi \) provided there is a path \( 0 = \xi_0 < \xi_1 \cdots < \xi_m = \xi \) in \( X^+ \cap W_p \cdot 0 \) with \( \xi_{i-1} \) adjacent to \( \xi_i \) for \( 0 < i \leq m \) and with each \( \text{Ext}^1_{\mathcal{C}}(L(\xi_{i-1}), L(\xi_i)) \neq 0 \).

**Proof.** Part (a) is proved in [9] Thm. 5.3 for \( \Gamma = \Gamma_{\text{Jan}} \), but the argument holds equally well for any ideal \( \Gamma \) in \( X^+ \), making use of the discussion in the paragraph immediately above the statement of the theorem.

Although (c) implies (b), we prove (b) first, then augment the argument to obtain (c).

In the proof of (b), we will need to use the theory (and notation) of pre-Hecke operators

\(^7\) More precisely, (c) makes a brief cameo appearance in [27].

\(^8\) In the statement of (b) and (c), we could replace \( \tau \) and \( \xi \) be regular weights, using a standard translation functor argument.
discussed in §5 and §4 (the latter reference contains many more details). Let \( C_{G,0} \) be the category of all finite dimensional rational \( G \)-modules with composition factors \( L(\lambda) \) for some \( \lambda \in W_p \cdot 0 \). Then \( C_{G,0} \) is a highest weight category with poset \((W_p \cdot 0, \uparrow)\), letting \( \uparrow \) be the Jantzen “up arrow” poset structure. Form the exact, additive functor \( \Theta_s : C_{G,0} \to C_{G,0} \) (the composite of two Jantzen translation functors) [8, p. 88]. If \( M \) has the form \( \Delta(\lambda), \nabla(\lambda) \), or \( L(\lambda) \) for some \( \lambda \in W_p \cdot 0 \), then let \( \beta_s M \) equal to the complex

\[
0 \to M \to \beta_s M \to M \to 0,
\]

defined in § (4.8.2). It is concentrated in degrees \(-1, 0, \) and \(+1\). When \( \lambda < \lambda_s \), \( \beta_s L(\lambda) \) is isomorphic in the bounded derived category \( D^b(C_{G,0}) \) to its cohomology, which is concentrated in degree 0. (We will take \( \lambda_s = \tau \) in the notation of (b).) From the discussion in § pp. 89–90, if \( \lambda \in \Gamma \), \( \beta_s L(\lambda) \in \mathcal{E}^L \cap \mathcal{E}^R \). See also § Remark 2.2. Here \( \mathcal{E}^L \cap \mathcal{E}^R \) is a subset of objects in \( D^b(C_{G,0}) \) defined by an even-odd vanishing property similar to that described for irreducible modules in (S.0.2) below. The condition \( \operatorname{Ext}_G^1(L(\tau s), L(\tau)) \neq 0 \) implies that \( L(\tau) \) is a direct summand of \( \beta_s L(\tau s) \); see the end of the proof of “(c) \implies (b)” of § Thm. 5.5. Therefore, \( L(\tau) \in \mathcal{E}^L \cap \mathcal{E}^R \). This implies that \( C_0[\Gamma] \) has a Kazhdan-Lusztig theory (again, see § S.0.2 below), and so the Lusztig character formula holds for \( \Gamma \), as required.

The proof of (c) requires the use of the “enriched Grothendieck groups” \( K^L_0 := K^L_0(C_{G,0}, l) \) and \( K^R_0 := K^R_0(C_{G,0}, l) \) introduced in §2 (see also § §2). In fact, these groups can be associated to any highest weight category \( C \), with weight poset \( \Lambda \) and length function \( l : \Lambda \to \mathbb{Z} \). As discussed in § Prop. 2.3, \( K^L_0 \) and \( K^R_0 \) are related by a non-degenerate sesquilinear pairing \( K^L_0 \times K^R_0 \to \mathbb{Z}[u, u^{-1}] \) (Laurent polynomials). When \( C_{G,0}[\Gamma \cap W_p \cdot 0] \) has a Kazhdan-Lusztig theory § Defn. 3.3 (see also § S.0.2 below), the group \( K^L_0 \) (resp., \( K^R_0 \)) keeps track of the dimensions of the groups \( \operatorname{Ext}^*_{\mathcal{C}_{G,0}}(L(\lambda), \nabla(\mu)) \) (resp., \( \operatorname{Ext}^*_{\mathcal{C}_{G,0}}(\Delta(\mu), L(\lambda)) \)) for \( \lambda, \mu \in \Gamma \cap W_p \cdot 0 \).

Let \( H \) denote the generic Hecke algebra of the Coxeter system \((W_p, S_p)\) over \( \mathbb{Z}[t, t^{-1}] \). Then \( H \) acts on the groups \( K^L_0 \) and \( K^R_0 \), using the \( \beta_s \), \( s \in S_p \), mentioned above § Prop. 5.6. The argument for (b) shows, recursively, that all the \( L(\xi_i) \)'s may be represented in both \( K^L_0 \) and \( K^R_0 \). Also, the discussion in § §4 shows the resulting representations are completely determined by Kazhdan-Lusztig polynomials. Using the “Euler characteristic map” \( K^L_0 \to K_0(C_{G,0}) \) (the Grothendieck group of \( \mathcal{G} \)-mod) obtained by specializing \( u \mapsto -1 \) (and hence \( t \mapsto 1 \); cf. footnote 4) yields (3.0.2) for \( L(\xi) \).

Let \( \lambda \in X^\text{reg} \) and suppose that \( \lambda < \lambda_s \in X^+ \) for some \( s \in S_p \). By § II, Prop. 7.21], \( \dim \operatorname{Ext}_G^1(\Delta(\lambda), \Delta(\lambda s)) = 1 \). In simple terms, this means there is a non-split short exact sequence

\[
(4.1.2) \quad 0 \to \Delta(\lambda s) \to E \to \Delta(\lambda) \to 0
\]

of rational \( G \)-modules, and that, given any other non-split short exact sequence \( 0 \to \Delta(\lambda s) \to E' \to \Delta(\lambda) \to 0 \), it is “scalar equivalent” to (4.1.2) in the sense that there is a commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & \Delta(\lambda s) & \longrightarrow & E & \longrightarrow & \Delta(\lambda) & \longrightarrow & 0 \\
\alpha \id & & & \downarrow & & & \downarrow \id & & \\
0 & \longrightarrow & \Delta(\lambda s) & \longrightarrow & E' & \longrightarrow & \Delta(\lambda) & \longrightarrow & 0
\end{array}
\]
Thus, the Lusztig character formula holds for \( \Gamma \). We will make use of elementary recollement results; these are described in \([10, \text{Proof.} \])

\[
\dim \text{Ext}^1_G(L(\lambda), L(\lambda_s)) \leq 1.
\]

If a non-trivial extension of \( L(\lambda) \) by \( L(\lambda_s) \) exists, it must come, up to scalar equivalence, from \((4.1.2)\) by factoring out \( \text{rad}(\Delta(\lambda_s)) \). That is, if the resulting extension of \( \Delta(\lambda) \) by \( L(\lambda_s) \) must split on \( \text{rad}(\Delta(\lambda)) \). If it does, we get a non-split extension of \( L(\lambda) \) by \( L(\lambda_s) \); if it does not, there is no such nontrivial extension. Observe that

\[
\text{Ext}^1_G(\Delta(\lambda), \Delta(\lambda_s)) \cong \text{Ext}^1_G(\Delta(\lambda), L(\lambda_s)).
\]

These statements can be easily verified, using the long exact sequence of \( \text{Ext}^\bullet \). Note that \( \text{Ext}^n_G(\Delta(\lambda), L(\nu)) = 0 \) unless \( \nu > \lambda \).

For each \( \mu \in \Gamma \cap W_p \cdot 0 \), let \( P_\mu \) be a fixed projective in the category \( C_G[\Gamma] \) such that the projective cover \( P(\mu) \) of \( L(\mu) \) is a direct summand of \( P_\mu \), and all other indecomposable summands \( P(\nu) \) satisfy \( \mu < \nu \in \Gamma \).\(^9\) Explicitly, \( P_\mu \in C_G[\Gamma] \), while \( P(\mu) \in C_G[\Gamma \cap W_p \cdot 0] \), the category of all finite dimensional rational \( G \)-modules with composition factors of the form \( L(\nu), \nu \in \Gamma \cap W_p \cdot 0 \). For \( \gamma \in \Gamma \), let \( P_{\mu, < \gamma} \) be the largest quotient module of \( P_\mu \) with all composition factors \( L(\nu), \nu < \gamma \).

The module \( P_{\mu, < \gamma} \) has a filtration by submodules with successive sections standard modules \( \Delta(\tau), \tau < \gamma \).

**Theorem 4.2.** Let \( \Gamma \) be a finite ideal as above. For any given \( \lambda, \lambda_s \in \Gamma \cap W_p \cdot 0 \) with \( \lambda < \lambda_s \), \( s \in S_p \), condition \((4.1.1)\) holds if and only if, for each \( \mu < \lambda \) in \( \Gamma \cap W_p \cdot 0 \), the following condition holds:

\[
\begin{cases}
\text{For any map } P_{\mu, < \lambda_s} \rightarrow \Delta(\lambda) \text{ which is not a split surjection, the induced map } \text{Ext}^1_G(\Delta(\lambda), \Delta(\lambda_s)) \rightarrow \\
\text{Ext}^1_G(P_{\mu, < \lambda_s}, \Delta(\lambda_s)) \text{ is zero.}
\end{cases}
\]

Thus, the Lusztig character formula holds for \( \Gamma \cap W_p \cdot 0 \) if and only if \((4.2.1)\) holds for all \( \mu, \lambda, \lambda_s \in \Gamma \cap W_p \cdot 0 \).

**Proof.** We will make use of elementary recollement results; these are described in \([10, \text{§2}]\)

\[^9\text{Such projective modules } P_\mu \text{ arise naturally in many contexts, where } P(\mu) \text{ itself may not be explicitly known. We allow summands } P(\nu) \text{ for } \nu \notin W_p \cdot 0, \text{ though they will be irrelevant, since then } L(\nu) \text{ and } L(\mu) \text{ lie in different blocks.}
\]

\[^{10}\text{We only use that the other summands } P(\nu) \text{ of } P_\mu \text{ satisfy } \mu \leq \nu \text{ in this paper, though the natural examples have } \mu < \nu. \text{ That is, we assume that } P(\mu) \text{ is a summand of } P_\mu \text{ with multiplicity one.}
\]
Suppose that condition (4.2.1) fails for some \( \mu < \lambda \). Proposition 4.3. Let this failure occurs, then Ext \(_G^1(\Delta(\lambda), L(\lambda)) \) is zero. However, the long exact sequence of Ext shows that the map Ext \(_G^1(\Delta(\lambda), L(\lambda)) \) is zero for each component \( \psi \mid \Gamma \). Obvi-
ously, the component maps \( \mu, \lambda \in \Gamma \) do not give split surjections when composed with the inclusion \( \mathrm{rad}(\Delta(\lambda)) \subseteq \Delta(\lambda) \). Thus, the composites Ext \(_G^1(\Delta(\lambda), L(\lambda)) \) are zero by (4.2.2). Using the above diagram, we obtain that the map

\[
\text{Ext}_G^1(\Delta(\lambda), L(\lambda)) \to \text{Ext}_G^1(\mathrm{rad}(\Delta(\lambda)), L(\mu)) \to \text{Ext}_G^1(\mu, L(\lambda)) \to \text{Ext}_G^1(\mu, L(\lambda))
\]

is zero for each component \( \mu, L(\lambda) \) of \( P_{<ls} \). Thus, the map

\[
\text{Ext}_G^1(\Delta(\lambda), L(\lambda)) \to \text{Ext}_G^1(\mathrm{rad}(\Delta(\lambda)), L(\lambda)) \to \text{Ext}_G^1(P_{<ls}, L(\lambda))
\]

is also zero. However, the long exact sequence of Ext shows that the map Ext \(_G^1(\Delta(\lambda), L(\lambda)) \) is injective, since \( \text{Hom}_G(N, L(\lambda)) \) for the kernel \( N \) of \( P_{<ls} \) to \( \text{rad}(\Delta(\lambda)) \). This proves that the validity of (4.2.1) for all possible \( \mu, \lambda, \lambda_s \) as above, implies the validity of (4.2.2).

The following results ties the above equivalence in with the even–odd varnishing equivalence (as a sufficient condition) for the Lusztig conjecture, see, e. g. [25, §7]. See also [18] below.

**Proposition 4.3.** Let \( \Gamma \) be a finite ideal in \( X^+ \). Fix \( \lambda, \lambda_s \in \Gamma \) with \( s \in S_p \), such that \( \lambda < \lambda_s \). Suppose that condition (4.2.1) fails for some \( \mu < \lambda \). If \( \mu < \lambda \) is chosen maximal for which this failure occurs, then Ext \(_G^1(\Delta(\mu), L(\lambda)) \) \( \neq 0 \) and Ext \(_G^1(\Delta(\mu), L(\lambda)) \) \( \neq 0 \).

In particular, condition (4.2.1) holds if Ext \(_G^1(\Delta(\mu), L(\gamma)) \) \( = 0 \) always whenever \( \gamma \in \Gamma \cap W_p \), \( 0 \), and \( \mu, \gamma \) have lengths of the same parity with respect to some “length” function \( l : \Gamma \cap W_p \to \mathbb{Z} \) for which \( l(\lambda_s) \equiv l(\lambda) + 1 \mod 2 \).

**Proof.** If (4.2.1) fails for some \( \mu \), then the argument for Thm. 4.2 shows that the homomorphism \( \psi : \text{Ext}_G^1(\Delta(\lambda), L(\lambda)) \to \text{Ext}_G^1(\text{rad}(\Delta(\lambda)), L(\lambda)) \) is not zero. Moreover, if \( E \) is a nonsplit extension \( 0 \to L(\lambda) \to E \to \Delta(\lambda) \to 0 \), and if \( F \) is the largest submodule of \( E \) without \( L(\lambda) \) as a composition factor, then \( \psi \) factors through \( \text{Ext}_G^1(E, L(\lambda)) \), where \( F' \) is the image of \( F \) in \( \Delta(\lambda) \) and \( F' = \Delta(\lambda)/F' \). Also, (4.2.1) fails for \( \mu \) if and only if \( L(\mu) \) is a composition factor of \( E' = \text{rad}(\Delta(\lambda))/F' \). Taking \( \mu \) maximal among such weights, there are nonzero homomorphisms \( \Delta(\mu) \to E' \) and \( \Delta(\mu) \to E'^* \), where \( E'^* \) is the dual of \( E' \). (As
well-known, the category $C_G[\Gamma]$ has a duality $M \mapsto M^*$ which fixes irreducible modules.) Pulling back to $\Delta(\mu)$ the evident extensions of $E'$ (resp., $E'^*$) by $L(\lambda s)$ (resp., $L(\lambda)$) gives the required nonsplit extensions of $\Delta(\mu)$ by $L(\lambda s)$ (resp., $L(\lambda)$).

\[\square\]

The papers [9, 7] contain many other conditions which are equivalent to the validity of the Lusztig character formula. We will return to the even-odd vanishing result mentioned in the proof of Thm. 4.1 later in §8.

5. Type A

We now consider the case $G = SL_n(k)$. For technical reasons, we will assume throughout this section that $p > 3$. See the discussion below concerning (5.0.3).

For positive integers $n, r$, let (as before) $\Lambda^+(n, r)$ be the set of all partitions of $r$ of length at most $n$. We regard $\Lambda^+(n, r)$ as a poset, using the dominance ordering $\preceq$. We will need to make use of a well-known method of associating to any partition with at most $n$ nonzero parts a dominant weight in $X^+$. For this, we will use a local notational convention (i. e., it will be used only in this section!): We will denote partitions by symbols $\Lambda, \mu$, etc. Thus, if $\bar{\lambda} \in \Lambda^+(n, r)$, then $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$. Label the fundamental dominant weights $\varpi_1, \cdots, \varpi_{n-1}$ for $SL_n(k)$ as in [8] p. 250. We have a mapping $\Lambda^+(n, r) \to X^+$ which associates to $\bar{\lambda} \in \Lambda^+(n, r)$ the dominant weight $\lambda = \sum_{i=1}^{n-1} a_i \varpi_i$, where $a_i = \lambda_i - \lambda_{i+1}$.

If a partition $\bar{\lambda} \in \Lambda^+(n, r)$ corresponds to a dominant weight $\lambda \in X^+$ as above, then $\lambda$ is regular (in the sense of [8]) if and only if

\[\lambda_i - \lambda_j \neq i - j \mod p, \forall i < j \leq n\]

(5.0.1)

(This above geometry notion of regularity is different than the Young diagram notion of $p$-regularity, which requires that no row be repeated $p$ times.)

For a fixed pair $(n, r)$ of positive integers, let $S(n, r)$ be the Schur algebra over $k$ of bidegree $(n, r)$. Our notation will largely be consistent with that in [10] §4, where the reader can find for more details on Schur algebras in the spirit of this paper. (Also, in the notation of [2] in Part I, $S(n, r) = S_q(n, r)$ for $q = 1$.) The category $S(n, r)$–mod identifies with the category of rational $GL_n(k)$-modules which are polynomial of homogeneous degree $r$. The category $S(n, r)$–mod is a highest weight category with poset $(\Lambda^+(n, r), \preceq)$.

Now fix a finite ideal $\Gamma \subset X^+$. Choose a sufficiently large $r \equiv 0 \mod n$ such that every weight in $\Gamma \cap W_p \cdot 0$ corresponds (as described above) to a partition $\bar{\lambda} \in \Lambda^+(n, r)$. Any $S(n, r)$-module $M$ is naturally a $GL_n(k)$-module and hence an $SL_n(k)$-module, and $C_G[\Gamma \cap W_p \cdot 0]$ is a full subcategory of the image of $S(n, r)$–mod $\to SL_n(k)$–mod under this identification functor, with standard, costandard, and irreducible modules going to standard, costandard, and irreducible modules, respectively. In addition, if $M, N \in S(n, r)$–mod, then

\[\text{Ext}^\bullet_{S(n, r)}(M, N) \cong \text{Ext}^\bullet_{GL_n(k)}(M, N) \cong \text{Ext}^\bullet_{SL_n(k)}(M, N),\]

(5.0.2)

The first isomorphism is well-known (see [10] (6)) and the references there) and the second one follows immediately from an elementary Hochschild-Serre spectral sequence argument, using the normal subgroup $G = SL_n(k)$ of $GL_n(k)$.

Let $\text{mod}$-$k\mathcal{S}_r$ denote the category of all finite dimensional modules for the symmetric group $\mathcal{S}_r$ of degree $r$ over $k$. We will make use of Schur-Weyl duality between $S(n, r)$–mod and $\text{mod}$-$k\mathcal{S}_r$. In particular, let $S(n, r)$–mod($\Delta$) be the full, exact subcategory of finite
dimensional $S(n,r)$-modules which have a $\Delta$-filtration, i.e., a filtration with sections of the form $\Delta(\lambda)$, $\lambda \in \Lambda^+(n,r)$. Similarly, let $\text{mod--}\mathfrak{S}_r(S_{n,r})$ be the full, exact subcategory of all finite dimensional (right) $k\mathfrak{S}_r$-modules which have a filtration with sections of the form $S_{\lambda}$, $\lambda \in \Lambda^+(n,r)$, where $S_{\lambda}$ is the Specht module for $k\mathfrak{S}_r$ corresponding to the partition $\lambda$. Now assume that $p \geq n = h$ and recall the blanket assumption that $p > 3$. By [9, (3.8.3.2)], there is a contravariant equivalence

$$\tag{5.0.3} S(n,r)\text{-mod}(\Delta) \cong \text{mod--}k\mathfrak{S}_r(S_{n,r}).$$

If $\mu \in \Lambda^+(n,r)$, let $\mathfrak{S}_\mu$ be the associated Young (or parabolic) subgroup of $\mathfrak{S}_r$. Let $T_\mu = \text{Ind}_{\mathfrak{S}_\mu} k$ be the corresponding permutation module obtained by inducing the trivial module $k$ from $\mathfrak{S}_\mu$ to $\mathfrak{S}_r$. If $V$ is the standard $n$-dimensional module of column vectors for $GL_n(k)$, then $\mathfrak{S}_r$ acts on tensor space $V^\otimes r$ by place permutation. As a right $k\mathfrak{S}_r$-module, $V^\otimes r$ decomposes into a direct sum of permutation modules $T_{\lambda}$, $\lambda \in \Lambda^+(n,r)$ (each summand occurring with some positive multiplicity). Then $\text{Hom}_{k\mathfrak{S}_r}(V^\otimes r, V^\otimes r) \cong S(n,r)$. The contravariant equivalence (5.0.3) is given explicitly by restricting to the exact subcategory $S(n,r)\text{-mod}(\Delta)$ the functor

$$\tag{5.0.4} S(n,r)\text{-mod} \to \text{mod--}k\mathfrak{S}_r,$$

$$M \mapsto M^\circ := \text{Hom}_{S(n,r)}(M, V^\otimes r).$$

The equivalence (5.0.3) takes the standard module $\Delta(\lambda) \in S(n,r)\text{-mod}$ to the Specht module $S_{\lambda}$. Also, the permutation module $T_{\lambda}$ corresponds to a projective module $P_{\lambda} := \text{Hom}_{k\mathfrak{S}_r}(T_{\lambda}, V^\otimes r)$ of the type discussed in [41, viz., it is a direct sum of the projective indecomposable module $P(\tau)$ (with multiplicity one), together with various $P(\sigma)$, for $\tau \triangleright \lambda$, letting $\triangleright$ denote the dominance order on $\Lambda^+(n,r)$. For more details, see [29, 13], and [15].

As a consequence of (5.0.3) and the uniqueness of the short exact sequence (4.1.2), we have the following result. (Recall that $p > 3$ throughout this section.)

**Lemma 5.1.** Let $\Gamma$ be a finite ideal in $X^+$ as above. Choose $r \equiv 0 \mod n$ so that every $\lambda \in \Gamma \cap W_p \cdot 0$ corresponds to a partition $\lambda \in \Lambda^+(n,r)$. Suppose $\lambda < \lambda_s$ both belong to $\Gamma \cap W_p \cdot 0$ (where $s \in S_p$). Then, up to scalar equivalence, there is a unique non-split short exact sequence

$$\tag{5.1.1} 0 \to S_{\lambda} \to F \to S_{\lambda_s} \to 0$$

in $k\mathfrak{S}_r$-mod.

Each weight $\nu \in \Gamma \cap W_p \cdot 0$ is represented by a partition $\nu \in \Lambda^+(n,r)$. The weight $0$ is represented $0 = (r/n, \cdots, r/n)$. The partitions $\nu$ with $\nu \in \Gamma \cap W_p \cdot 0$ may be recursively defined as follows: Suppose this notation is understood for all small partitions with at most $n$ parts. Then $\nu \in \Gamma \cap W_p \cdot 0$ if and only if there is a partition $\omega < \nu$, with $\omega \in \Gamma \cap W_p \cdot 0$, such that, for some $0 \leq i < j < n$ and $0 < m$, we have:

$$\text{(a)} \quad \nu_i = \omega_i + m \quad \text{and} \quad \nu_j = \omega_j - m;$$

$^{10}$The argument given in [4] is incomplete when $n = p$, unless $n = r$. However, because of the assumption that $p > 3$, the argument is easily repaired using [13] (3.3.4) or the $i = 1$ case of [23, Thm. 4.5]. In fact, these more recent results imply that the assumption that $p \geq n$ may even be dropped in the equivalence (5.0.3), when $p > 3$. A related version of (5.0.3) for $p > n$ is essentially a result in [16, p. 124]; restated in [9, ftm. 15].
(b) \( \nu_i - \nu_j + j - i \equiv m \mod p \).

The above conditions are just a translation of conditions for \( \omega \) to be related to \( \nu \) by a certain kind of reflection. Notice that \( \nu_i - \nu_j + j - i < np < \omega_i - \omega_j + j - i \) for some integer \( n \). If the pair \( i, j \) are unique with this property, and only \( m < p \) works in (a) above, then \( \omega = \nu s \) (and conversely, if \( \omega > \nu \)).

Given \( \mu \in \Lambda^+(n, r) \), \( \mu \in \Gamma_{Jan} \) if and only if

\[
\mu_i - \mu_n + n - 1 \leq p(p - n + 2).
\]

As before, we will consider if the Lusztig character formula holds for \( \Gamma \cap W_p \cdot 0 \) for any ideal \( \Gamma \) in \( X^+ \).

There is a natural and unique submodule \( T_{\mu_\gamma, < \lambda_s} \) for any \( \gamma \in \Lambda^+(r) \), which has a filtration by Specht modules \( S_\lambda \) with \( \lambda \leq \gamma \), and with \( T_{\mu_\gamma, < \lambda_s} \) filtered by \( S_\lambda \) with \( \lambda \) not smaller than \( \gamma \).

The existence and uniqueness of \( T_{\mu_\gamma, < \lambda_s} \) follow from (5.0.3) and the existence and uniqueness of the module \( P_{\nu_\gamma, < \lambda_s} \) from §4 (which follows from well-known quasi-hereditary algebra theory).

**Theorem 5.2.** Assume that \( p \geq n \) and let \( \Gamma \) be a finite ideal in \( X^+ \). Then the Lusztig character formula holds for \( \Gamma \cap W_p \cdot 0 \) if and only if for each map \( S_\lambda \rightarrow T_{\mu_\gamma, < \lambda_s} \), which is not a split injection, the induced map

\[
\text{Ext}_{kS_r}^1(S_{\lambda_s}, S_\lambda) \rightarrow \text{Ext}_{kS_r}^1(S_{\lambda_s}, T_{\mu_\gamma, < \lambda_s})
\]

on \( \text{Ext}^1 \)-groups is zero, whenever \( \mu < \lambda < \lambda_s \) in \( \Gamma \cap W_p \cdot 0 \) (for some \( s \in S_p \)).

**Proof.** This follows from Thm. 4.2 and (5.0.2) using the contravariant equivalence of exact categories (5.0.3).

Using Lemma 5.1, we can restate the above result in a slightly different, but suggestive way.

**Theorem 5.3.** Assume that \( p \geq n \) and let \( \Gamma \) be a finite ideal in \( X^+ \). Then the Lusztig character formula holds for \( \Gamma \cap W_p \cdot 0 \) if and only if, whenever \( \mu < \lambda < \lambda_s \) belong to \( \Gamma \cap W_p \cdot 0 \) (for some \( s \in S_p \)), any morphism \( S_\lambda \rightarrow T_{\mu_\gamma, < \lambda_s} \) which is not a split injection, fits into a commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & S_\lambda \\
& & E \\
&T_{\mu_\gamma, < \lambda_s} & \rightarrow & S_{\lambda_s} & \rightarrow & 0
\end{array}
\]

of \( kS_r \)-modules, in which the horizontal row is the unique (up to scalar equivalence) short exact sequence defined in (5.1.1).

\[\text{11} \text{There is always a natural construction of } T_{\mu_\gamma, < \lambda_s} \text{, obtained by reduction mod } p \text{ from a similar module defined over a principal ideal domain } Z \text{ of characteristic zero. See } [9, \text{Thm. 5.2.1}] \text{ for the existence of the relevant filtrations over } Z \text{, which uses no algebraic group theory. The uniqueness of the analogue of } T_{\mu_\gamma, < \lambda_s} \text{ over } Z \text{ is easy. Indeed, over } Z, T_{\mu_\gamma, < \lambda_s} \text{ may be described as the intersection of the (Z-version) of } T_{\mu_\gamma} \text{ with an evident canonical module defined over the quotient field of } Z.\]
Taking $\Gamma = \Gamma_{\text{Jan}}$, the above theorems present a specific necessary and sufficient condition in terms of symmetric group modules for the validity of the Lusztig conjecture for $SL_n(k)$. This result should be compared to [25, Prop. 7.1], where the authors proved a sufficient condition for the validity of the Lusztig conjecture for $SL_n(k)$. The condition involved only the cohomology of symmetric groups, stating that the Lusztig conjecture was true for certain groups $\text{Ext}^1_{k\mathfrak{S}_\lambda}(S_{\lambda'}, D_{\lambda'})$ were equal to 0. Here the (usually large) integer $s$ is defined in terms of $n$ and the weights $\Gamma_{\text{Jan}} \cap \text{W}_p \cdot 0$. Also, $\tau, \sigma$ are various (regular) partitions of $s$ indexed by the set $\Gamma_{\text{Jan}} \cap \text{W}_p \cdot 0$, which are defined using the Erdmann function $d$ in [25 (6.1.2)]. We also have the following result which is in the spirit of [25].

**Proposition 5.4.** Assume that $p \geq n$. Let $\Gamma \subset X^+$ be a finite ideal as before. Suppose, for fixed $\lambda < \lambda_s$ as above, that (5.2.1) fails for some $\mu < \lambda$. Let $\mu < \lambda$ be maximal so that (5.2.1) fails. Assume that $\mu, \lambda$ and $\lambda_s$ are all $p$-restricted. Then $\text{Ext}^1_{k\mathfrak{S}_\mu}(S_{\mu'}, D_{(\lambda_s')}) \neq 0$ and $\text{Ext}^1_{k\mathfrak{S}_\lambda}(S_{\lambda'}, D_{\lambda'}) \neq 0$.

**Proof.** By [13, Thm. (4.3)] (or [25, Cor. 5.3]),

(5.4.1) $\dim \text{Ext}^1_{S(r,r)}(\Delta(\lambda), L(\mu)) \leq \dim \text{Ext}^1_{k\mathfrak{S}_\lambda}(S_{\lambda'}, D_{\lambda'})$,

whenever the partitions $\lambda'$ and $\mu'$ are $p$-regular. For any partition $\lambda$, the weight $\lambda \in X^+$ is restricted if and only if the dual partition $\lambda'$ is $p$-regular. Now the result follows from (5.4.1), together with Thm. 4.2 and (5.0.2). \qed

**Remarks 5.5.** (a) Assume that $\Gamma = \Gamma_{\text{Jan}}$ and that $p \geq 3p - 2$ so that all restricted weights lie in $\Gamma_{\text{Jan}}$. By [21] (see [13], to see that the Lusztig character formula holds for $\Gamma_{\text{Jan}} \cap \text{W}_p \cdot 0$, it is enough to check it for restricted regular weights. Then the condition that $\lambda, \lambda_s$ be restricted in the statement of Prop. 5.4 is not so severe: Let $\tau \in X^+ \cap \text{W}_p \cdot 0$, and suppose $\tau = \lambda \tau_s$ for some $\lambda \in X^+, \lambda < \tau$. Further, if $\tau$ is restricted, then $\lambda$ is also restricted. This follows since $\lambda$ and $\tau$ are separated by only one affine hyperplane $H_{a,mp}$, and $H_{a,mp}$ separates $\tau$ from 0. So $H_{a,mp}$ is not among those hyperplanes defining the restricted region. If the Lusztig character formula holds for each weight in $X^+ \cap \text{W}_p \cdot 0$ smaller than $\tau$, then Thm. 4.1(b) shows that it holds for $\tau$ if and only the inequality (4.1.1) holds. Thus, in Prop. 5.4 we can assume to start that both $\lambda$ and $\lambda_s$ restricted. (However, it still may be true that $\mu$ is not restricted. The Erdmann function $d$ [25 (6.1.2)] in offers some way around this problem, but it would seem best to first check if the conclusion of Prop. 5.4 holds when all three weights involved are restricted.)

(b) The results of this section avoid the use of the function $d$ altogether. Thus, they give a way of using a smaller symmetric group than in Erdmann’s paper [17] to determine, say, the decomposition numbers for the “principal” block (containing the determinant module) of the Schur algebra $S(n, r)$ in terms of $\mathfrak{S}_r$, without increasing $r$. However, it has been long known that the decomposition numbers of the endomorphism ring of a permutation module can be equated with multiplicities of ordinary characters inside those of $p$-adic indecomposable components of the permutation module. See [27]. So, Schur algebra decomposition numbers can be computed in terms of symmetric group permutation module theory in this way, though this method would require knowledge the indecomposable components of permutation modules. Erdmann’s reduction notably uses (albeit after a big increase in $r$) just the projective indecomposable components, whose multiplicities are actual symmetric group decomposition
numbers. Perhaps the notable aspect of our reduction, aside from the fact that no increase in \( r \) is required, is that no indecomposable components of permutation modules are required at all, only certain submodules of permutation modules whose definition is transparent in terms of ordinary character theory.

**PART III: Levi subgroups and irreducible modules**

Again, \( G \) is a simple, simply connected algebraic group over \( k \). When \( p \geq h \), the discussion will often make use of the notion, defined in \[ 13 \] of what it means for the Lusztig character formula to hold for a subset \( \Sigma \subseteq W_p \).

6. A THEOREM OF HEMMER AND A GENERALIZATION

It is natural to ask for a version of \[ 14.1.4 \] relating the groups \( \text{Ext}^\bullet_G(L(\lambda),L(\mu)) \) to the groups \( \text{Ext}^\bullet_H(L_H(\lambda),L_H(\mu)) \). This section considers this question. Thm. \[ 6.1 \] below establishes an inequality for \( \text{Ext}^1 \)-groups which is in the spirit of \[ 14.1.3 \] and \[ 14.1.5 \]. A version of this result for quantum enveloping algebras at a root of unity is easily obtained along the same lines.

In \[ 19 \] Thm. 2.3, Hemmer proves that if \( \lambda,\mu \in \Lambda^+(n,r) \) satisfy \( \lambda_1 = \mu_1 = m \) then there is an injection

\[
(6.0.1) \quad \text{Ext}^1_{S(n,r)}(L(\lambda),L(\mu)) \hookrightarrow \text{Ext}^1_{S(n-1,r-m)}(L(\bar{\lambda}),L(\bar{\mu})),
\]

where \( \bar{\lambda} \) is the partition of \( r-m \) obtained by removing \( \lambda_1 \) from \( \lambda \), and \( \bar{\mu} \) is defined similarly. This result is then interpreted in terms of comparing the Ext-quiver of \( S(n,r) \) with that of \( S(n-1,r-m) \).

We can generalize Hemmer’s theorem to other types:

Let \( H \) be a Levi subgroup of \( G \) and \( \Omega = \omega + \mathbb{Z}\Phi_H \) is a coset of \( \mathbb{Z}\Phi_H \) in \( \mathbb{Z}\Phi \). Put \( \Omega^+ = \Omega \cap X^+ \).

**Theorem 6.1.** Let \( \lambda,\mu \in \Omega^+ \). Then there is a natural injection

\[
(6.1.1) \quad \text{Ext}^1_G(L(\lambda),L(\mu)) \hookrightarrow \text{Ext}^1_H(L_H(\lambda),L_H(\mu)).
\]

**Proof.** We will argue that the required injection is induced by the truncation functor \( \pi_\Omega \) (10.1). We can assume that \( \lambda \leq \mu \). There is a natural short exact sequence \( 0 \to Q(\lambda) \to \Delta(\lambda) \to L(\lambda) \to 0 \). The long exact sequence of \( \text{Ext}^\bullet_G \) with respect to the functor \( \text{Hom}_G(-,L(\mu)) \) gives an exact sequence

\[
\text{Hom}_G(Q(\lambda),L(\mu)) \to \text{Ext}^1_G(L(\lambda),L(\mu)) \to \text{Ext}^1_G(\Delta(\lambda),L(\mu)).
\]

All the composition factors \( L(\tau) \) of \( Q(\lambda) \) satisfy \( \tau < \lambda < \mu \); so \( \text{Hom}_G(Q(\lambda),L(\mu)) = 0 \). A similar conclusion holds for the analogous exact sequence \( 0 \to Q_H(\lambda) \to \Delta_H(\lambda) \to L_H(\lambda) \to 0 \). Therefore, the functor \( \pi_\Omega \) provides a commutative diagram

\[
0 \to \text{Ext}^1_G(L(\lambda),L(\mu)) \to \text{Ext}^1_G(\Delta(\lambda),L(\mu)) \quad \pi_\Omega
\]

\[
0 \to \text{Ext}^1_H(L_H(\lambda),L_H(\mu)) \to \text{Ext}^1_H(\Delta_H(\lambda),L_H(\mu)) \quad \pi_\Omega
\]
with exact rows. By (1.1.2) for \( M = L(\mu) \) (or [10 Cor. 10]), the right hand vertical map is
an isomorphism. Hence, the left hand vertical map is an injection, as required. \( \square \)

7. An example

In general, the inclusion map \( \text{Ext}^1_G(L(\lambda), L(\mu)) \hookrightarrow \text{Ext}^1_H(L_H(\lambda), L_H(\mu)) \) given in Thm. 6.1 need not be an isomorphism.\(^{12}\)

Let \( G = \text{SL}_3(k) \), with \( \text{char} k = p = 3 \). We list the dominant weights as in [3, p. 250]. The
Lusztig conjecture is known to be true for \( G \). In particular, by the result of Kato discussed
in [3] the Lusztig character formula holds for \( 4\varpi_1 + \varpi_2 = (\varpi_1 + \varpi_2) + 3\varpi_1 \), since \( \varpi_1 + \varpi_2 \)
is a restricted weight in \( W_p \cdot 0 \) and \( \varpi_1 \in \mathbb{C}^\times \). Similarly, it holds for \( 3\varpi_1 \). Thus, the Lusztig
character formula holds for each weight in the sequence \( 0 < \varpi_1 \varpi_2 < 3\varpi_1 < 4\varpi_1 + \varpi_2 \) of
adjacent dominant weights. On the other hand, Lemma 3.1 (or an unpleasant computation)
shows that the Lusztig character formula fails for \( 3\varpi_1 + \varpi_2 = (4\varpi_1 + \varpi_2)_{\alpha_2} \). By Thm.
4.1(c), \( \text{Ext}^1_G(L(4\varpi_1 + \varpi_2), L(3\varpi_1 + 3\varpi_2)) = 0 \). However, let \( H \) be the Levi subgroup with
\( \Phi_H = \{\alpha_2\} \) (i.e., defined by \( \Pi_1 \) in the notation of [2]), then, directly or using Thm. 4.1(c)
again, we have that \( \text{Ext}^1_H(L_H(4\varpi_1 + \varpi_2), L_H(3\varpi_1 + 3\varpi_2)) \neq 0 \).

In terms of the group \( GL_3(k) \), let \( \lambda = (6, 2, 1), \mu = (6, 3, 0) \in \Lambda^+(3, 9) \). In the notation of
[2] Example 1,

\[
\text{Ext}^1_{GL_3(k)}(L(\lambda), L(\mu)) = 0,
\]

while

\[
\text{Ext}^1_H(L(\lambda), L_H(\mu)) \neq 0.
\]

Therefore, the map (6.0.1) is not an isomorphism in this case.

Another, similar but interesting example, is provided by \( GL_5(k) \) when \( p = 5 \), taking
\( \lambda = (10, 5^2), (10, 5^2, 4, 1) \in \Lambda^+(5, 25) \). (From computer results discussed in [28], and the
translation principle, \( \text{Ext}^1(L(\lambda), L(\mu)) \) holds in the category \( \mathcal{C}_G \) for any regular restricted weight \( \nu \).) In
fact, this last example let us to the “smaller” one above.

Despite these examples, [28] below shows that (6.1.1) is an isomorphism for regular weights
lying in a finite ideal \( \Gamma \) with the property that the Lusztig character formula holds for \( \Gamma \cap W_p \cdot 0 \).

8. An \( \text{Ext}^1 \)-isomorphism result and higher \( \text{Ext}^\bullet \)-groups

In this section, we take up, for arbitrary cohomological degree \( n \), the study of the map
\( \text{Ext}^n_G(L(\lambda), L(\mu)) \to \text{Ext}^n_H(L_H(\lambda), L_H(\mu)) \) induced by the truncation functor \( \pi_\Omega \). We will
show that if \( \lambda, \mu \) are regular weights, lying in a finite ideal \( \Gamma \) such that the Lusztig character
formula holds for \( \Gamma_{\text{reg}} \), then \( \pi_\Omega \) always induces an \textit{surjection} provided \( \lambda, \mu \) belong to the same
\( \mathbb{Z} \Phi_H \)-coset. In particular, (6.1.1) is an isomorphism in this case. To obtain such a result, it
will be necessary to work in the context of the homological dual of a quasi-hereditary algebra,
and so we begin by recalling some general results.

\(^{12}\)The examples in this section provide a (negative) answer to a question raised by Hemmer in his lecture
at the AMS Conference “Representations of algebraic groups, quantum groups, and Lie algebras,” Snowbird,
Utah, July 2004. The form of the examples suggest a positive answer to Hemmer’s question is almost equivalent
to the validity of the Lusztig conjecture.
In \([7]\), the homological dual \(A^!\) of a quasi-hereditary algebra \(A\) was defined to be the Yoneda Ext-algebra
\[
A^! = \text{Ext}^*_A(L_0, L_0)
\]
where \(L_0\) is the direct sum of the distinct irreducible \(A\)-modules. If \(C = A\text{-mod}\), write \(C^!\) for \(A^!\text{-mod}\).

The rest of this section will make use of properties of a highest weight category \(C \cong A\text{-mod}\) having a Kazhdan-Lusztig theory with respect to a length function \(l : \Lambda \to \mathbb{Z}^+\) on its weight poset \(\Lambda\). This means that, for any \(\lambda, \mu \in \Lambda\),
\[
\begin{align*}
\text{Ext}_C^\lambda(\lambda, \nabla(\mu)) &\neq 0 \\
\text{or} \\
\text{Ext}_C^\lambda(\Delta(\mu), \lambda) &\neq 0
\end{align*}
\tag{8.0.2}
\]
implies \(n \equiv l(\lambda) - l(\mu) \mod 2\).

See \([5]\) or \([7]\), §1.3] for further discussion. We will always assume below that \(C\) has finite poset \(\Lambda\), so that \(C \cong A\text{-mod}\) for some finite dimensional algebra \(A\). The following result, proved in \([7]\), summarizes some basic properties of the homological dual.

**Theorem 8.1.** Assume that \(C \cong A\text{-mod}\) has a Kazhdan-Lusztig theory with respect to a length function \(l : \Lambda \to \mathbb{Z}^+\). Let \(\Xi\) be a coideal in \(\Lambda\). Then:

(a) \(A^!\) is a quasi-hereditary algebra; more precisely, \(C^! = A^!\text{-mod}\) is a highest weight category with weight poset \(\Lambda^{op}\) (the poset opposite to \(\Lambda\)).

(b) \(C(\Xi)\) has a Kazhdan-Lusztig theory with respect to \(l|_{\Xi}\).

(c) \(C(\Xi)^! \cong C^!(\Xi^{op})\) as highest weight categories.

**Proof.** (a) is proved in \([7]\) Thm. 2.1. (b) is well-known, but follows directly in the spirit of this paper as follows: Let \(j^* : C \to C(\Xi)\) be the natural quotient functor. By \([10]\) Lemma 6, given \(\lambda, \mu \in \Xi\), \(\text{Ext}^*_C(L(\lambda), \nabla(\mu)) \cong \text{Ext}^\lambda_C(L(\lambda), \nabla(\mu))\) and \(\text{Ext}^*_C(\Delta(\mu), L(\lambda)) \cong \text{Ext}^\lambda_C(\Delta(\mu), L(\lambda))\), where we have written \(L(\lambda)\), etc. for the irreducible, etc. object in \(C(\Xi)\) indexed by \(\lambda\). Now (b) follows from the definition \(8.0.2\). For (c), see \([7]\) (2.3)]. \(\square\)

In (c) above, \(C(\Xi)^! \cong (A_{\Xi})^!\text{-mod}\), where \(A_{\Xi}\) is a quasi-hereditary algebra such that \(A_{\Xi}\text{-mod} \cong C(\Xi)\). The proof given in \([7]\) p. 308] is more precise, showing that the natural algebra homomorphism
\[
\Phi : A^! = \text{Ext}_C^*(L_0, L_0) \to \text{Ext}_{C(\Xi)}^*(j^*L_0, j^*L_0) = (A_{\Xi})^!;
\]
induced by the quotient functor \(j^* : C \to C(\Xi)\), defines an equivalence
\[
\Phi^* : ((A_{\Xi})^!)^\text{-mod} \to A^!/J\text{-mod}
\]
of highest weight categories, for some (idempotent) ideal \(J\) (denoted \(A^!\text{st}A^!\) in \([7]\)) which is part of a defining sequence for \(A^!\). Thus, \(A^!/J \cong (A_{\Xi})^!\). In particular, we have:

**Corollary 8.2.** Assume that \(A\text{-mod}\) has a Kazhdan-Lusztig theory. Then the map \(\Phi\) defined in \(8.1.1\) is surjective.

We now return to the setting of a simple, simply connected algebraic group \(G\) over \(k\), assuming that \(p \geq h\). Fix a finite ideal \(\Gamma \subset X^+\), and let \(C_{G,0}[\Gamma]\) denote the highest weight category consisting of all finite dimensional \(G\)-modules with composition factors \(L(\xi)\) with
\( \xi \in \Gamma \cap W_p \cdot 0. \) Define a length function \( l : \Gamma \cap W_p \cdot 0 \to \mathbb{N} \) by \( l(w \cdot 0) = l(w) \). By the argument in [3, Thm. 5.3] (see the discussion of Thm. 1111), \( C_{G,0}[\Gamma] \) has a Kazhdan-Lusztig theory if and only if the Lusztig character formula holds for \( \Gamma \cap W_p \cdot 0 \).

Let \( H \) be a Levi subgroup of \( G \) as in [11]. In Thm. 1144, \( \Gamma_F^+ \) can be replaced by a “block analog”. That is, let \( B \) be any union of \( W_p \)-orbits in the dot action on \( X \). Put \( \Gamma_{F,B}^+ = \Gamma_F^+ \cap B \). Then clearly \( \Omega_{F,B}^+ := \Gamma_{F,B}^+ \cap \Omega \) is a union of dominant weights in orbits of the affine Weyl group \( W_{H,p} \) of \( H \). (Use the same weight \( \rho \) in defining the dot action of \( W_{H,p} \) as is used for \( W_p \).) Thm. 1 remains true with \( \Gamma_{F,B}^+ = \Gamma_F^+ \), and \( \Gamma_{F,B}^+ \) replaced by \( \Gamma_{F,B}^+ \) and \( \Omega_{F,B}^+ \), respectively.

We will take \( B = \mathcal{X}_{\text{reg}}^+ \), the set of regular dominant weights, writing \( \Gamma_{F,\text{reg}}^+ \) and \( \Omega_{F,\text{reg}}^+ \) for \( \Gamma_{F,B}^+ \) and \( \Omega_{F,B}^+ \), respectively,

Fix a finite ideal \( \Sigma \) in \( \mathcal{X}^+ \) such that the Lusztig character formula holds for \( \Sigma_{\text{reg}} \neq \emptyset \). Let \( \Omega = \omega + \mathbb{Z}\Phi_H \) be a coset of \( \mathbb{Z}\Phi_H \) in \( X \), with \( \omega \in \Sigma \). Put \( F = \Sigma \cap \Omega \) and, as in Thm. 1144, form \( \Gamma_F^+ \) and \( \Omega_F^+ \). Thus, \( \Omega_F^+ \) is a coideal in \( \Gamma_F^+ \) and an ideal in \( \mathcal{X}(\mathcal{H})^+ \). Set \( \Lambda_0 = \Gamma_{F,\text{reg}}^+ \) and \( \Omega_0 = \Omega_{F,\text{reg}}^+ \); and the highest weight categories \( C_{G,0} = C_{G,0}[\Lambda_0] \) and \( C_{H,0} = C_{H,0}[\Omega_0] \). Let \( L_0 \) be the direct sum of the distinct irreducible \( G \)-modules \( L(\lambda) \) for \( \lambda \in \Lambda_0 \). Similarly, let \( L_{H,0} \) be the direct sum of the distinct irreducible \( H \)-modules \( L_H(\lambda) \) for \( \lambda \in \Omega_0 \).

**Theorem 8.3.** Assume the notation of the previous paragraph. Then:

(a) The category \( C_{H,0} \) has a Kazhdan-Lusztig theory. In particular, the algebra \( A_H^1 := \operatorname{Ext}_H(\mathcal{X}_{\text{reg}}^+, L_{0,H}, H_{0,H}) \) is quasi-hereditary.

(b) The natural map

\[
A_H^1 = \operatorname{Ext}_G^*(L_0, L_0) \to \operatorname{Ext}_H^*(L_{0,H}, H_{0,H}) = A_H^1
\]

induced by \( \pi_\Omega \) is surjective, and its kernel is a defining ideal \( J \) of the quasi-hereditary algebra \( A^1 \).

(c) We have \( A^1/J\cdot\mathcal{X} \cong C_{G,0}[\Omega_0] \).

**Proof.** (a) follows from Thm. 8.3 using Thm. 1144 which shows that \( C_{H,0} \cong C_{G,0}[\Omega_0] \). Then (b) and (c) follow from Cor. 8.2. \( \square \)

**Remarks 8.4.** (a) A somewhat less precise way to state part (c) is, in view of part (b),

\[
C_{G,0}[\Omega_0] \cong C_H[\Omega_0]^{-1}.
\]

(b) As remarked earlier, for semisimple algebraic groups having a fixed root system, the Lusztig conjecture is known to hold provided that the characteristic \( p \) is large enough, though lower bound is known on the size of \( p \) in general. Thus, in those cases, we can in Thm. 8.3 take \( \Sigma = \Gamma_{\text{Jan}} \), the Jantzen region.

**Corollary 8.5.** Assume the set-up in the paragraph immediately before Thm. 8.3. Let \( \lambda, \mu \in \Sigma_{\text{reg}} \cap \omega + \mathbb{Z}\Phi_H \). Then

(a) \( \operatorname{Ext}_G^1(L(\lambda), L(\mu)) \cong \operatorname{Ext}_H^1(L_{H}(\lambda), L_{H}(\mu)) \).

(b) The map \( \operatorname{Ext}_G^n(L(\lambda), L(\mu)) \to \operatorname{Ext}_H^n(L_{H}(\lambda), L_{H}(\mu)) \) which is induced by \( \pi_\Omega \) is surjective for all \( n \geq 0 \). In particular,

\[
\dim \operatorname{Ext}_G^1(L(\lambda), L(\mu)) \geq \dim \operatorname{Ext}_H^1(L_{H}(\lambda), L_{H}(\mu)).
\]

---

13 If we replace the \( \leq \) partial ordering on \( \mathcal{X}^+ \) by the Jantzen \( \uparrow \) ordering, then \( \Gamma \cap W_p \cdot 0 \) becomes an ideal in \( (\mathcal{X}^+, \uparrow) \).
Proof. Part (b) follows directly from Thm. 6.3. Part (a) is then a consequence of (b) for $n = 1$ and Thm. 6.1. □

Remarks 8.6. (a) An alternative proof of the inequality in Cor. 8.5(b) can be based on the Ext-formula proved in [6, Thm. 3.5], which states that

$$\dim \text{Ext}^n_G(L(\lambda), L(\mu)) = \sum_{m=0}^{n} \sum_{\nu} \dim \text{Ext}^m_G(L(\lambda), \nabla(\nu)) \cdot \dim \text{Ext}^{n-m}_G(\Delta(\nu), L(\mu))$$

in any highest weight category $C$ with a Kazhdan-Lusztig theory. This formula shows just what part of $\dim \text{Ext}^n_G(L(\lambda), L(\mu))$ contributes of $\text{Ext}^n_H(L_H(\lambda), L_H(\mu))$—namely, the weight $\nu$ must lie in $\Omega$.

(b) Now we consider Hemmer’s injective map (6.0.1) in the case when the Lusztig conjecture is true for $SL_n$. Assume that $\lambda, \mu \in \Lambda_r^+(n, r)$ satisfy $\lambda_1 = \mu_1 = m$, and let $\lambda, \mu$ be as before. Assume that $\lambda_i - \lambda_j \neq i - j$ and $\mu_i - \mu_j \neq i - j$ mod $p$ for all $i < j \leq n$ (see 5.1.1) and that $\lambda_1 - \lambda_n + n - 1 < p(p - n + 2)$ and $\mu_1 - \mu_n + n - 1 < p(p - n + 2)$ (see 5.1.2). Then the map (6.0.1) is an isomorphism. Moreover, for higher degree Ext-groups, there is a corresponding surjection, a somewhat surprising turnabout.

(c) Observe that an alternative proof of Cor. 8.5(a) can be based on the commutative diagram in the proof of Thm. 6.1 once it is observed that [6, Thm. 4.3] implies that the horizontal map is surjective.

Corollary 8.7. Let $\Sigma_{\text{reg}}$ be as in the set-up in the paragraph immediately before Thm. 8.3 and put $X := \Sigma_{\text{reg}}$. (In particular, the Lusztig character formula holds for irreducible modules $L(\lambda)$, $\lambda \in X$.) Let $\mathcal{C}_H[X]$ denote the category of all finite dimensional rational $H$-modules which have composition factors $L_H(\xi)$, $\xi \in X$. Then $\mathcal{C}_H[X]$ is a highest weight category with respect to the poset $(X, \leq_H)$, and, further, $\mathcal{C}_H[X]$ has a Kazhdan-Lusztig theory in the sense of [8, 1.2].

Proof. First note, that if $\Delta_H(\nu)$ is a standard module for $\nu \in X$, and $L_H(\varpi)$ is a composition factor, then $\varpi \in X^+$ since $X^+$ is an ideal in $(X^H, +, \leq_H)$. Also, $L(\omega)$ is a composition factor of $\Delta(\nu)$—this is a result of Donkin [11]—see also [10, Cor. 12]. In particular, $\nu$ and $\varpi$ belong to the same block of $G$, and so $\nu$ and $\varpi$ are conjugate under the dot action of $W_p$. Hence, $\varpi \in X$. Now consider the projective cover $P_{\Gamma_H}(\varpi)$ of $L_H(\varpi)$ in the category $\mathcal{C}_H[\Gamma]$, where $\Gamma$ is the set of dominant weights (for $G$) regarded as an ideal in $(X^H, +, \leq_H)$. If $\Delta_H(\xi)$ is a $\Delta_H$-section, then $L_H(\varpi)$ is a composition factor of $\nabla_H(\xi)$ by Brauer-Humphreys reciprocity [3, Thm. 3.11]. Hence, $L_H(\varpi)$ is a composition factor of $\Delta_H(\xi)$. As before, this forces $\xi \in X$. Thus, $\mathcal{C}_H[X]$ is a highest weight category.

The final assertion follows from Thm. 8.3(a), since $\text{Ext}_H^1$ vanishes between modules with respective composition factors in different cosets of the root lattice of $H$. □

Finally, we have the following general (and elementary) result which shows that the $\text{Ext}^1$-isomorphism property behaves well with respect to Frobenius twisting.

Theorem 8.8. Assume that if $G$ or $H$ has a component of type $C_m$, then $p > 2$. Suppose that $\lambda, \mu$ lie in the same $\mathbb{Z}\Phi$-coset $\Omega = \omega + \mathbb{Z}\Phi_H$ and that $\pi_\Omega$ induces an isomorphism $\text{Ext}^1_G(L(\lambda), L(\mu)) \cong \text{Ext}^1_H(L_H(\lambda), L_H(\mu))$. For any positive integer $r$, put $\Omega(r) = p^r \omega + \mathbb{Z}\Phi_H$. Then $\pi_{\Omega(r)}$ induces an isomorphism $\text{Ext}^1_G(L(p^r \lambda), L(p^r \mu)) \cong \text{Ext}^1_H(L_H(p^r \lambda), L_H(p^r \mu))$. 

□
Proof. It suffices to consider the \( r = 1 \) case. The hypothesis on \( p \) guarantees that, if \( G_1 \) denotes the first Frobenius kernel of \( G \), then \( H^1(G_1, k) = 0 \) \[\text{Prop. 12.9}\]. Hence, by a Hochschild-Serre spectral sequence argument for \( G_1 \triangleleft G \), \( \Ext^1_G(L(p\lambda), L(p\mu)) \cong \Ext^1_G(L(\lambda), L(\mu)) \). Similar remarks apply to \( \Ext^1 \).

Thus, in general, the injection \( \text{(6.1.1)} \) is an isomorphism for some regular weights well outside \( \Gamma_{\text{Jan}} \), where the Lusztig character formula fails.\[14\]

References

[1] S. Arkhipov, R. Bezrukavnikov, and V. Ginzburg, Quantum groups, the loop Grassmannian, and the Springer resolution, *J. Amer. Math. Soc.* 17 (2004), 595–678.

[2] H. Andersen, J. Jantzen, W. Soergel, *Representations of quantum groups at a \( p \)th root of unity and of semisimple groups in characteristic \( p \)*, Astérique 220 (1994).

[3] N. Bourbaki, *Groupes et algèbres de Lie*, IV, V, VI, Hermann (1968).

[4] E. Cline, On injective modules for infinitesimal algebraic groups, II, *J. Algebra* 134 (1990), 271–297.

[5] E. Cline, B. Parshall, and L. Scott, Finite dimensional algebras and highest weight categories, *J. reine angew. Math.* 391 (1988), 85–99.

[6] E. Cline, B. Parshall, and L. Scott, Abstract Kazhdan-Lusztig theories, *Tohoku Math. J.* 45 (1993), 511–534.

[7] E. Cline, B. Parshall, and L. Scott, The homological dual of a highest weight category, *Proc. London Math. Soc.* 68 (1994), 294–316.

[8] E. Cline, B. Parshall, and L. Scott, Simulating perverse sheaves in modular representation theory, *Proc. of Symposia in Pure Math.* 56(1) (1994), 63–104.

[9] E. Cline, B. Parshall, and L. Scott, *Stratifying endomorphism algebras*, Memoirs Amer. Math. Soc. 591 (1996).

[10] E. Cline, B. Parshall, and L. Scott, An Ext-transfer theorem for algebraic groups, *J. Transformation Groups* 9 (2004), 213–236.

[11] S. Donkin, A note on decomposition numbers for reductive algebraic groups, *J. Algebra* 80 (1983), 226–234.

[12] S. Donkin, *The \( q \)-Schur algebra*, Cambridge (1998).

[13] S. Doty, K. Erdmann, and D. Nakano, Extensions of modules over Schur algebras, symmetric groups, and Hecke algebras, *Algebras and Representation Theory* 7 (2004), 67–99.

[14] J. Du, B. Parshall, and L. Scott, Quantum Weyl reciprocity and tilting modules, *Comm. Math. Physics* 195 (1998), 321–352.

[15] J. Du, B. Parshall, and L. Scott, Cells and \( q \)-Schur algebras, *J. Transformation Groups* 3 (1998), 33–49.

[16] K. Erdmann, Symmetric groups and quasi-hereditary algebras, in *Finite dimensional algebras and related topics*, ed. V. Dlab and L. Scott, NATO ASI Series 424 (1994), 123–162.

[17] K. Erdmann, Decomposition numbers for symmetric groups and composition factors of Weyl modules, *J. Algebra* 180 (1996), 316–320.

[18] D. Hemmer and D. Nakano, Specht filtrations for Hecke algebras of type \( A \), *J. London Math. Soc.* 69 (2004), 623–638.

[19] D. Hemmer, A row removal theorem for the \( \Ext^1 \) quiver of symmetric groups and Schur algebras, *Proc. Amer. Math. Soc.* 133 (2005), no. 2, 403–414 (electronic).

[20] J.C. Jantzen, *Representations of algebraic groups*, 2nd ed. American Mathematical Society (2003).

[21] S. I. Kato, On Kazhdan-Lusztig polynomials for affine Weyl groups, *Adv. Math.* 55 (1985), 103–130.

\[14\] Assume \( p \geq h \). Lemma \[51\] provides a way to construct \( \lambda \in W_p \cdot 0 \cap X^+ \) such that the Lusztig character formula holds for \( \lambda \) but fails for \( p\lambda \).
[22] D. Kazhdan and G. Lusztig, Tensor structures arising from affine Lie algebras, I–III; IV–VI, *J. Amer. Math. Soc.* 6 (1993), 905-1011; 7 (1994), 335–453.

[23] G. Lusztig, Some problems in the representation theory of finite Chevalley groups, Proc. Symp. Pure Math. 37 (1980), 313–317.

[24] S. Lyle and A. Mathas, Row and column removal theorems for homomorphisms of Specht modules and Weyl modules, preprint (2004).

[25] B. Parshall and L. Scott, Quantum Weyl reciprocity for cohomology, *Proc. London Math. Soc.*, in press.

[26] B. Parshall and J.-p. Wang, Quantum Linear Groups, *Mem. Amer. Math. Soc.* 439 (1991).

[27] L. Scott, Modular permutation representations, *Trans. Amer. Math. Soc.* 175 (1973), 101–123.

[28] L. Scott, Some new examples in 1-cohomology, *J. Algebra* 260 (2003), 416–425.

[29] T. Tanisaki, Character formulas of Kazhdan-Lusztig type, *Representations of finite dimensional algebras and related topics in Lie theory and geometry*, Fields Inst. Commun. 40, Amer. Math. Soc., Providence, RI (2004) 261–276.

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