A Bourgain-like property of Banach spaces with no copies of $c_0$

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Abstract We give a characterization of the existence of copies of $c_0$ in Banach spaces in terms of indexes. As an application, we deduce new proofs of James Distortion theorem and Bessaga-Pełczynski theorem about weakly unconditionally Cauchy series.

Keywords Banach space · Copy of $c_0$ · Index of symmetrization

Mathematics Subject Classification 46B03 · 46B25

1 Introduction

The aim of this paper is to study the existence of copies of $c_0$ in Banach spaces in terms of indexes and by purely geometrical methods. Our motivation for this is the beautiful characterization given by Bourgain [1, Lemma 3.7, p.39] of Banach spaces not containing $\ell^1$, as those satisfying that for every bounded subset $C$ of $X^*$ and each $\epsilon > 0$ there exist relatively weak* -open subsets $U_1, \ldots, U_m$ of $C$ such that $\frac{1}{m}(U_1 + \cdots + U_m)$ has diameter less than $\epsilon$.

Several results concerning this type of spaces follow from this, like the fact that their dual unit ball $(B_{X^*}, \omega^*)$ is convex block compact [1, Proposition 3.11, p. 43].

We prove that a Banach space $X$ does not contain an isomorphic copy of $c_0$ if and only if for every bounded subset $A$ of $X$ and each $\epsilon > 0$ there are $x_1, \ldots, x_m$ in $A$ such that $\bigcap_{j=1}^{m} (A - x_j) \cap (x_j - A)$ has diameter less than $\epsilon$. Actually, we give a quantitative version of this fact. We first associate to any bounded set $A \subset X$ a sequence of indexes $\delta_m(A)$ ($m \geq 0$),

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being $\delta_m(A)$ half of the infimum of all diameters of sets $\bigcap_{j=1}^m (A - x_j) \cap (x_j - A)$ where $x_1, \ldots, x_m \in A$. Then, we prove in Theorem 1 that for each $\epsilon > 0$ we can find a sequence $(x_n)_{n \in \mathbb{N}}$ in the absolute convex hull of $A$ such that

\[
(\delta_2 \cdot \epsilon) \cdot \max_{1 \leq n \leq N} |\lambda_n| \leq \sum_{n=1}^N \lambda_n x_n \leq \delta_0(A) \cdot \max_{1 \leq n \leq N} |\lambda_n|
\]

for every $\lambda_1, \ldots, \lambda_N \in \mathbb{R}$ and $N \in \mathbb{N}$.

From the previous result we deduce the characterization of Banach spaces containing an isomorphic copy of $c_0$ mentioned above (Theorem 2), as well as the known theorems of James (Theorem 3) and Bessaga-Pelczynski (Theorem 4) without using basic sequences.

Our notation is standard and follows [5]. We denote by $X$ a real Banach space with the norm $\| \cdot \|$. Its topological dual will be denoted by $X^*$, and for any $x^* \in X^*$ and $x \in X$ the evaluation of $x^*$ at $x$ is written as $x^*(x) = \langle x^*, x \rangle = \langle x, x^* \rangle$. The closed unit ball (resp. unit sphere) of $X$ is denoted by $B_X$ (resp. $S_X$). If $D \subset X$ then we write $\co(D)$, $\aco(D)$ and span($D$) to denote respectively the convex hull, the absolutely convex hull and the linear hull of $D$. The supremum of $x^* \in X$ on $D$ is denoted by sup($x^*, D$). A slice of $D$ is a set of the form $S(D, x^*, \delta) := \{x \in D : x^*(x) > \sup(x^*, D) - \delta\}$ for some $x^* \in X^*$ and $\delta > 0$. Recall that the diameter of $D$ is defined as $\text{diam}(D) := \sup \{\|x - y\| : x, y \in D\}$.

## 2 Indexes of symmetrization

**Definition 1** Given $A \subset X$ bounded, the symmetrized of $A$ with respect to $x_1, \ldots, x_N \in A$ is defined as $\bigcap_{n=1}^N (A - x_n) \cap (x_n - A)$. For each $N \in \mathbb{N}$ we also define:

\[
\Delta_N(A) := \left\{ \bigcap_{n=1}^N (A - x_n) \cap (x_n - A) : x_1, \ldots, x_N \in A \right\}.
\]

\[
\delta_0(A) := \text{diam}(A)/2, \delta_N(A) := \inf \{\delta_0(D) : D \in \Delta_N(A)\}.
\]

It is clear from the definition that $\{\Delta_N(A) : N \in \mathbb{N}\}$ is an increasing sequence of sets, and hence $\{\delta_N(A) : N \in \mathbb{N}\}$ is decreasing. We will write $\delta_\infty(A) := \lim_N \delta_N(A)$. Let us point out that if $x \in A$, then $d \in (A - x) \cap (x - A)$ is equivalent to $x \pm d \in A$. With this in mind, the following (useful) observations are direct:

(I) If $D \in \Delta_N(A)$ is the symmetrized of $A$ with respect to $x_1, \ldots, x_N \in A$, then for every $d \in D$ the set $(D - d) \cap (d - D)$ is the symmetrized of $A$ with respect to $x_1 \pm d, \ldots, x_N \pm d \in A$. In particular, $(D - d) \cap (d - D) \in \Delta_{2N}(A)$.

(II) Given $x^* \in X^*, \delta > 0$ and $x \in S(A, x^*, \delta)$, every $d \in D := (A - x) \cap (x - A)$ satisfies $|x^*(x)| + |x^*(d)| < \sup(x^*, A)$, so that $|x^*(d)| < \delta$. In particular, $x \pm D \subset S(A, x^*, 2\delta)$.

Recall that the Kuratowski measure of non-compactness of a set $S \subset X$ is

\[
\alpha(S) := \inf \{\epsilon > 0 : \text{there are finitely many balls of radius } \epsilon \text{ which cover } S\}.
\]

**Lemma 1** If $A \subset X$ is bounded and $D \in \Delta_N(A)$, then $\alpha(D) \geq \delta_{2N}(A)$.

**Proof** Suppose that $\alpha(D) < \epsilon$, and let $D_1, \ldots, D_n$ be a finite family of subsets of $D$ whose union is equal to $D$ and such that each $D_k$ is contained in a ball of radius less than $\epsilon$. If
\[ D \subset \overline{co}(D_1), \text{ then } \text{diam}(D) < 2\epsilon \text{ and so } \delta_{2N}(A) \leq \delta_N(A) \leq \delta_0(D) < \epsilon. \text{ Otherwise, we can assume that there is } 2 \leq m \leq n \text{ such that} \]
\[ D \subset \overline{co}(D_1 \cup \cdots \cup D_m) \text{ and } D \not\subset \overline{co}(D_1 \cup \cdots \cup D_{m-1}). \quad (2) \]

We can take \( x^*_n \in S_{X^*} \) and \( \delta > 0 \) such that the slice \( S(D, x^*_n, \delta) \) has empty intersection with \( \overline{co}(B_1 \cup \cdots \cup B_{m-1}) \). We claim that for every \( 0 < \eta < 1 \) it holds that
\[ S(D, x^*_n, \eta \delta) \subset D_m + \eta(1 + \text{diam } D)B_X. \quad (3) \]

If \( d \in S(D, x^*_n, \eta \delta) \), then by (2) we can find \( d' := \lambda d_m + (1 - \lambda)c_m \) where \( 0 \leq \lambda \leq 1 \), \( d_m \in D_m \) and \( c_m \in \text{co} (D_1 \cup \cdots \cup D_{m-1}) \) such that \( \|d - d'\| < \eta \) and \( d' \in S(D, x^*_n, \delta \eta) \).

Since \( x^*_n(c_m) \leq \text{sup}(x^*_n) - \delta \), we deduce that
\[ \text{sup}(x^*_n, D) - \eta \delta < x^*_n(d') = \lambda x^*_n(d_m) + (1 - \lambda)x^*_n(c_m) \leq \text{sup}(x^*_n, D) - (1 - \lambda)\delta. \]

This yields \( 1 - \lambda < \eta \), and so
\[ \|d - d_m\| \leq \|d - d'\| + \|d' - d_m\| < \eta + (1 - \lambda)\|d_m - c_m\| < \eta(1 + \text{diam } D). \]

This proves the claim. By observations (I) and (II), for every \( d_0 \in S(D, x^*_n, \eta \delta/2) \) the set \( D_0 := (D - d_0) \cap (d_0 - D) \) belongs to \( \Delta_{2N}(A) \) and \( d_0 + D_0 \subset S(D, x^*_n, \eta) \). Hence, we get by (3) that
\[ \delta_{2N}(A) \leq \delta_0(D_0) \leq \frac{1}{2} \text{diam } S(D, x^*_n, \eta \delta) \leq \epsilon + \eta(1 + \text{diam } D). \]

Since \( \eta > 0 \) is arbitrary, we conclude that \( \delta_{2N}(A) < \epsilon). \]

\[ \square \]

**Remark 1** We are thankful to an anonymous referee for pointing out to us that Lemma 1 can be obtained as a corollary of the so-called “Superlemma” of Namioka and Bourgain [3, Chapter IX, p.157]. Indeed, under the assumption (2) we can apply this result to the closed convex hull of \( D \) to obtain a slice \( S = S(D, x^*_n, \delta) \) of \( D \) with diameter smaller than the diameter of \( \overline{co}(D_m) \), which is less than \( 2\epsilon \). Taking \( d_0 \in S(D, x^*_n, \delta/2) \) we can argue as in the last part of the proof of Lemma 1 to conclude the result.

**Lemma 2** Let \( F \subset X \) be a finite-dimensional subspace and \( D \subset X \) bounded. If \( \alpha(D) > \lambda > 0 \), then there exists \( x^*_0 \in S_{F^*} \) such that \( \text{sup}(x^*_0, D) > \lambda. \)

**Proof** Suppose that every \( x^*_0 \in S_{F^*} \) satisfies that \( \text{sup}(x^*_0, D) \leq \lambda \). By Hahn-Banach Theorem we have that \( D \subset F + \lambda B_X \). But then \( D \subset \mu B_F + \lambda B_X \) for some \( \mu > 0 \), which implies that \( \alpha(D) \leq \lambda \) by the compactness of \( B_F \). \[ \square \]

**Theorem 1** Let \( A \subset X \) be bounded. For every \( \epsilon > 0 \) there is a sequence \( (x_n)_{n \in \mathbb{N}} \) in \( aco \ (A) \) such that
\[ (\delta_{2N}(A) - \epsilon) \cdot \max_{1 \leq n \leq N} |\lambda_n| \leq \left\| \sum_{n=1}^{N} \lambda_n x_n \right\| \leq \delta_0(A) \cdot \max_{1 \leq n \leq N} |\lambda_n| \quad (4) \]

for every \( \lambda_1, \ldots, \lambda_N \in \mathbb{R} \) and \( N \in \mathbb{N} \).

**Proof** Write \( \eta = \epsilon/3 \). Fix \( x_0 \in A_0 := A \) and put \( A_1 := (A - x_0) \cap (x_0 - A) \). By Lemma 1 we have that \( \alpha(A_1) > \delta_2(A) - \eta \), so Lemma 2 yields that there are \( x_1 \in A_1 \) and \( x^*_1 \in S_{X^*} \) with \( x^*_1(x_1) > \text{sup}(x^*_1, A_1) - \eta > \delta_2(A) - 2\eta \). Suppose that \( N \geq 1 \) and we have constructed \( (x^*_n)_{n=1}^{N} \in S_{X^*}, (x_n)_{n=1}^{N} \) in \( aco \ (A) \) and \( (A_n)_{n=1}^{N} \) subsets of \( X \) satisfying for each \( 1 \leq n \leq N \):

(a) \( x_{n-1} + A_n \subset A_{n-1} \) and \( A_n \in \Delta_{2n-1}(A) \).
Let \( A_{N+1} := (A_N - x_N) \cap (x_N - A_N) \in \Delta_2 (A) \). By Lemma 1 we have that \( \alpha (A_{N+1}) > \delta_{2N+1} (A) - \eta_{N+1} \), so using Lemma 2 we obtain \( x_{N+1} \in A_{N+1} \) and \( x^*_{N+1} \in S_{X^*} \) such that \( \{ x_k : 1 \leq k \leq N \} \subset \ker x^*_{N+1} \) and \( x^*_{N+1} (x_{N+1}) > \sup (x^*_{N+1}, A_{N+1}) - \eta > \delta_{2N+1} (A) - 2\eta \).

This finishes the inductive construction. Notice that conditions (a) and (c) imply that

\[ |x^*_n (z)| < \eta \text{ whenever } z \in A_{N+1}. \]

Given \( N \in \mathbb{N} \), we show now that the sequence \( (x_n)_{n \in \mathbb{N}} \) satisfies (4). For every \( 0 \neq (\lambda_n)_{n=1}^{N} \in \mathbb{R}^N \) we can write

\[
\left\| \sum_{n=1}^{N} \lambda_n x_n \right\| = |\lambda_m| \cdot \left\| \sum_{n=1}^{N} \frac{\lambda_n}{\lambda_m} x_n \right\| \leq |\lambda_m| \cdot \delta_0 (A)
\]

being \( m \) such that \( |\lambda_m| = \max \{|\lambda_n| : 1 \leq n \leq N\} \), since \( x_0 + \sum_{n=1}^{N} \pm x_n \in A \). Furthermore

\[
\left\| \sum_{n=1}^{N} \frac{\lambda_n}{\lambda_m} x_n \right\| \geq \langle x^*_m, x_m \rangle + \sum_{m<n \leq N} \lambda_n \frac{\lambda_m}{\lambda_m} x_n \geq \delta_{2m} (A) - 3\eta \geq \delta_{2N} (A) - 3\eta,
\]

where we have used (b), (c), (d) and the fact that

\[
\sum_{m<n \leq N} \frac{\lambda_n}{\lambda_m} x_n \in \text{co} (A_{m+1}),
\]

which is a consequence of (a).

\[ \square \]

**Corollary 1** Let \( A \subset X \) be bounded. For every \( \epsilon > 0 \) there is a sequence in \( (x_n)_{n \in \mathbb{N}} \) in \( \text{aco} (A) \) such that

\[
(\delta_{\infty} (A) - \epsilon) \max_{n \in \mathbb{N}} |\lambda_n| \leq \left\| \sum_{n=1}^{\infty} \lambda_n x_n \right\| \leq \delta_0 (A) \cdot \max_{n \in \mathbb{N}} |\lambda_n|.
\]

for every finitely supported sequence \( (\lambda_n)_{n \in \mathbb{N}} \in \mathbb{R} \).

## 3 Copies of \( c_0 \) in Banach spaces

**Theorem 2** Let \( X \) be a Banach space. The following assertions are equivalent:

(i) \( c_0 \) is not isomorphic to a subspace of \( X \).

(ii) \( \delta_{\infty} (C) = 0 \) for every bounded set \( C \subset X \).

(iii) \( \delta_{\infty} (C) = 0 \) for every bounded, convex and closed set \( C \subset X \).

**Proof** Implication (i)\( \Rightarrow \) (ii) is a consequence of Corollary 1, while (ii)\( \Rightarrow \) (iii) is obvious. We just have to check that (iii)\( \Rightarrow \) (i). Let \( T : c_0 \rightarrow X \) be an isomorphism, and consider \( A := T (B_{c_0}) \). Given \( a_1, \ldots, a_N \in A \) and \( 0 < \epsilon < 1 \) we can find \( m \in \mathbb{N} \) such that \( a_n \pm (1-\epsilon) T(e_m) \in A \) for every \( 1 \leq n \leq N \). This shows that \( \delta_N (A) \geq (1-\epsilon) / \| T^{-1} \| \) for each \( N \in \mathbb{N} \).

When \( c_0 \) is isomorphic to a subspace of \( X \), it is also said that \( X \) has a copy of \( c_0 \). It turns out that these spaces have indeed almost isometric copies of \( c_0 \), which means that for every \( \epsilon > 0 \) we can find a closed subspace \( Y \subset X \) and an isomorphism \( T : c_0 \rightarrow Y \) such that \( \| T \| \| T^{-1} \| \leq 1 + \epsilon. \)
Theorem 3 (James) If $X$ has a copy of $c_0$, then it has almost isometric copies of $c_0$.

Proof If $c_0$ embeds into $X$ then there exists a bounded set $A \subset X$ with $\delta_\infty(A) > 0$ by Theorem 2. It follows from the definition of $\delta_\infty(A)$ that for every $\epsilon > 0$ there exists an element $D \in \bigcup_{N \in \mathbb{N}} \Delta_N(A)$ such that

$$\delta_\infty(A) \leq \delta_0(D) \leq (1 + \epsilon)\delta_\infty(A).$$

Since $\delta_\infty(A) \leq \delta_\infty(D)$, we deduce that $\delta_0(D) \leq (1 + \epsilon)\delta_\infty(D)$, so an application of Corollary 1 with $D$ leads to the desired copy of $c_0$. \qed

Another easy consequence is the Bessaga-Pełczynski criterion for the existence of copies of $c_0$. Recall that a series $\sum_n x_n$ in a Banach space $X$ is said to be wuC if $\sum_n |x^*(x_n)|$ converges for every $x^* \in X^*$, which by the Uniform Boundedness Principle implies that $\sum_n |x^*(x_n)|$ is uniformly bounded for $x^* \in B_{X^*}$.

Theorem 4 (Bessaga-Pełczynski) If $c_0 \not\subseteq X$ and $\sum_n x_n$ is wuC, then the series is unconditionally convergent.

Proof Consider the uniformly bounded sets given by

$$A_m = \left\{ \sum_{n=1}^m \theta_n x_n : \theta_n \in \{-1, 1\} \text{ for each } 1 \leq n \leq m \right\}, \quad A = \bigcup_{m \in \mathbb{N}} A_m.$$

If $X$ does not contain a copy of $c_0$, then $\delta_\infty(A) = 0$, so given $\epsilon > 0$ we can find $x_1, \ldots, x_N \in A$ with

$$\text{diam} \left( \bigcap_{j=1}^N (A - a_j) \cap (a_j - A) \right) < \epsilon.$$

There is $M \in \mathbb{N}$ such that $\{a_n : 1 \leq n \leq N\} \subset \bigcup_{m \leq M} A_m$, so $\sum_{n=M}^{M'} \theta_n x_n \leq \epsilon$ for every $\theta_n \in \{-1, 1\}$ and $M' \geq M$. \qed

We finish with a non-symmetrized characterization of Banach spaces with no copies of $c_0$.

Proposition 1 A Banach space $X$ does not contain an isomorphic copy of $c_0$ if and only if for every bounded set $A \subset X$ and each $\epsilon > 0$ there are $x_1, \ldots, x_N \in A$ such that

$$\text{diam} \left( \bigcap_{j=1}^N (A - x_j) \right) < \epsilon.$$

Proof The sufficiency of the condition is consequence of Theorem 2. To see the converse, assume that there exists $A \subset X$ and $\epsilon > 0$ such that any intersection like in the statement has diameter greater or equal than $\epsilon$. Fix an arbitrary $x_0 \in A$ and then pick $x_1 \in (A - x_0)$ such that $\|x_1\| \geq \epsilon$. Consider the set $A_1 := \{x_0, x_0 + x_1\} \subset A$. Now we take

$$x_2 \in \bigcap_{x \in A_1} (A - x) \text{ with } \|x_2\| \geq \epsilon \text{ and } A_2 := A_1 \cup (A_1 + x_2).$$
Following in this way, we will have a sequence \((x_n)_{n \in \mathbb{N}}\) of vectors of norm greater or equal to \(\epsilon\) for \(n \geq 1\) and sets \(A_n \subset A\) of cardinality \(2^n\). Then consider

\[
x_{n+1} \in \bigcap_{x \in A_n} (A - x) \quad \text{with} \quad \|x_{n+1}\| \geq \epsilon \quad \text{and} \quad A_{n+1} := A_n \cup (x_n + A_n).
\]

Notice that the sums \(\sum_{n=1}^{N} \theta_n x_n\) are uniformly bounded independently of \(N\) and the choice of \(\theta_n \in \{-1, 1\}\), since they are difference of two elements of \(A_N \subset A\). Now Theorem 4 implies that \(X\) contains a copy of \(c_0\).

\(\square\)

4 Remarks

Let \(A\) be a subset of \(X\). Recall that an \(\epsilon\)-tree in \(A\) is a a sequence \(\{x_n : n \in \mathbb{N}\}\) such that \(x_n = (x_{2n} + x_{2n+1})/2\) and \(\|x_{2n} - x_{2n+1}\| \geq \epsilon\) for every \(n \in \mathbb{N}\). The index \(\delta_1(A)\) is directly related to existence of \(\epsilon\)-trees inside \(A\). In fact, if \(\delta_1(A) > \epsilon\), then we can construct a \(2\epsilon\)-tree inside of \(A\) in the following way: fix any \(x_1 \in A\). Since \(\text{diam}((A - x_1) \cap (x_1 - A)) > 2\epsilon\), we can find \(a_1 \in A\) such that \(\|a_1\| \geq \epsilon\) and \(x_1 \pm a_1 \in A\). Put \(x_2 := x_1 - a_1\) and \(x_3 := x_1 + a_1\). Repeating this process with \(x_2, x_3\) and the subsequent constructed points, we obtain the desired \(2\epsilon\)-tree. On the other hand, it is clear that every \(\epsilon\)-tree \(A'\) satisfies that \(\delta_1(A') \geq \epsilon/2\).

As a consequence, we can conclude that a set \(A \subset X\) contains no \(\epsilon\)-trees (for any \(\epsilon > 0\)) if and only if \(\delta_1(A') = 0\) for every \(A' \subset A\). In particular, if \(C\) is a closed and convex set having the Radon-Nikodým Property (RNP), then \(\delta_1(A) = 0\) for every \(A \subset C\).

We say that \(x_0 \in A\) is an \(\epsilon\)-extreme point of \(A\) if \(\text{diam}((A - x_0) \cap (x_0 - A))\) is less than \(2\epsilon\). It is not difficult to see that \(x_0\) is an extreme point of \(A\) if and only if it is \(\epsilon\)-extreme for every \(\epsilon > 0\). As a consequence, if \(K \subset X\) is a bounded, closed and convex set having the Krein-Milman Property (KMP), then \(\delta_1(C) = 0\) for every closed and convex set \(C \subset K\).

The previous notion reminds of the following concept introduced by Kunen and Rosenthal [6]: \(x_0 \in A\) is an \(\epsilon\)-strong extreme point of \(A\) if there is \(\delta > 0\) such that whenever \(a_1, a_2 \in A\) and there exists a point \(u = \lambda a_1 + (1 - \lambda)a_2\) \((0 < \lambda < 1)\) with \(\|x_0 - u\| < \delta\), then \(\|u - a_1\| < \epsilon\) or \(\|u - a_2\| < \epsilon\). If \(x_0\) is \(\epsilon\)-strong extreme for every \(\epsilon > 0\), then we simply say that it is a strong extreme point. It is not difficult to see that every \(\epsilon\)-strong extreme point of \(A\) is an \(\epsilon\)-extreme point of the same set. The converse is not true, since as it is pointed out in [6, Remark 3, p. 173] every strong extreme point of a bounded, closed and convex set is also an extreme point of its \(\sigma(X^{**}, X^*)\)-closure (in the terminology of [4] we might say that these are preserved extreme points), while there are, for instance, Banach spaces where \(B_X\) has extreme points that are not extreme points of \(B_X^{**}\) (see [4]). With this formulation we have that if \(K\) is a bounded, closed and convex set such that every \(A \subset K\) has \(\epsilon\)-extreme points for every \(\epsilon > 0\) (i.e. \(\delta_1(A) = 0\)), then each closed and convex set \(C \subset K\) has an \(\epsilon\)-strong extreme point for every \(\epsilon > 0\) (see [6, Proposition 3.2, p. 170]).

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