Twisted Dirac Operators and the noncommutative residue for manifolds with boundary II

Sining Wei\textsuperscript{a}, Yong Wang\textsuperscript{b,}\textsuperscript{*}

\textsuperscript{a}School of Mathematics and Statistics, Northeast Normal University, Changchun, 130024, P.R.China
\textsuperscript{b}School of Mathematics and Statistics, Northeast Normal University, Changchun, 130024, P.R.China

Abstract

In this paper, we establish two kinds of Kastler-Kalau-Walze type theorems for Dirac operators and signature operators twisted by a vector bundle with a non-unitary connection on six-dimensional manifolds with boundary.

Keywords: Twisted Dirac operators; Twisted signature operators; Noncommutative residue; Non-unitary connection.

1. Introduction

The noncommutative residue found in \cite{1,2} plays a prominent role in noncommutative geometry. For one-dimensional manifolds, the noncommutative residue was discovered by Adler \cite{3} in connection with geometric aspects of nonlinear partial differential equations. For arbitrary closed compact $n$-dimensional manifolds, the noncommutative residue was introduced by Wodzicki in \cite{2} using the theory of zeta functions of elliptic pseudodifferential operators. In \cite{4}, Connes used the noncommutative residue to derive a conformal 4-dimensional Polyakov action analogy. Furthermore, Connes made a challenging observation that the noncommutative residue of the square of the inverse of the Dirac operator was proportional to the Einstein-Hilbert action in \cite{4}. In \cite{5}, Kastler gave a brute-force proof of this theorem. In \cite{5}, Kalau and Walze proved this theorem in the normal coordinates system simultaneously. And then, Ackermann proved that the Wodzicki residue of the square of the inverse of the Dirac operator $Wres(D^{-2})$ in turn is essentially the second coefficient of the heat kernel expansion of $D^2$ in \cite{5}.

Recently, Ponge defined lower dimensional volumes of Riemannian manifolds by the Wodzicki residue \cite{4}. Fedosov et al. defined a noncommutative residue on Boutet de Monvel’s algebra and proved that it was a unique continuous trace in \cite{10}. In \cite{11}, Schrohe gave the relation between the Dixmier trace and the noncommutative residue for manifolds with boundary. In \cite{12}, Wang generalized the Kastler-Kalau-Walze type theorem to the cases of 3, 4-dimensional spin manifolds with boundary and proved a Kastler-Kalau-Walze type theorem. In \cite{12,13,14,15}, Y.Wang and his coauthors computed the lower dimensional volumes for 5,6,7-dimensional spin manifolds with boundary and also got some Kastler-Kalau-Walze type theorems. In \cite{15}, authors computed $Wres((\pi^+ D^{-2}) \circ (\pi^+ D^{-n+2}))$ for any-dimensional manifolds with boundary, and proved a general Kastler-Kalau-Walze type theorem.

In \cite{16}, J.Wang and Y.Wang proved the known Lichnerowicz formula for Dirac operators and signature operators twisted by a vector bundle with a non-unitary connection and got two Kastler-Kalau-Walze type theorems for twisted Dirac operators and twisted signature operators on four-dimensional manifolds with boundary.

The motivation of this paper is to establish two Kastler-Kalau-Walze type theorems for twisted Dirac operators and twisted signature operators with non-unitary connections on six-dimensional manifolds with...
boundary.

This paper is organized as follows: In Section 2, we recall the definition of twisted Dirac operators and compute their symbols. In Section 3, we give a Kastler-Kalau-Walze type theorems for twisted Dirac operators on six-dimensional manifolds with boundary. In Section 4 and Section 5, we recall the definition of twisted signature operators and compute their symbols, and we give a Kastler-Kalau-Walze type theorems for twisted signature operators on six-dimensional manifolds with boundary.

2. Twisted Dirac operator and its symbol

In this section we consider a $n$-dimensional oriented Riemannian manifold $(M,g^M)$ equipped with a fixed spin structure. We recall twisted Dirac operators. Let $S(TM)$ be the spinors bundle and $F$ be an additional smooth vector bundle equipped with a non-unitary connection $\nabla_F$. Let $\tilde{\nabla}^{F,*}$ be the dual connection on $F$, and define

$$\nabla_F = \frac{\tilde{\nabla}^F + \tilde{\nabla}^{F,*}}{2}, \quad \Phi = \frac{\tilde{\nabla}^F - \tilde{\nabla}^{F,*}}{2},$$

(2.1)

then $\nabla_F$ is a metric connection and $\Phi$ is an endomorphism of $F$ with a 1-form coefficient. We consider the tensor product vector bundle $S(TM) \otimes F$, which becomes a Clifford module via the definition:

$$c(a) = c(a) \otimes \text{id}_F, \quad a \in TM,$$

(2.2)

and which we equip with the compound connection:

$$\tilde{\nabla}^{S(TM) \otimes F} = \nabla^{S(TM)} \otimes \text{id}_F + \text{id}_{S(TM)} \otimes \tilde{\nabla}^F.$$

(2.3)

Let

$$\nabla^{S(TM) \otimes F} = \nabla^{S(TM)} \otimes \text{id}_F + \text{id}_{S(TM)} \otimes \nabla^F,$$

(2.4)

then the spinor connection $\nabla$ induced by $\nabla^{S(TM) \otimes F}$ is locally given by

$$\nabla^{S(TM) \otimes F} = \nabla^{S(TM)} \otimes \text{id}_F + \text{id}_{S(TM)} \otimes \nabla + \text{id}_{S(TM)} \otimes \Phi.$$

(2.5)

Let $\{e_i\} (1 \leq i,j \leq n) (\{\partial_i\})$ be the orthonormal frames (natural frames respectively ) on $TM$,

$$D_F = \sum_{i,j} g^{ij} c(\partial_j) \nabla^{S(TM) \otimes F}_{\partial_j} = \sum_{j} c(e_j) \nabla^{S(TM) \otimes F}_{e_j},$$

(2.6)

where $\nabla^{S(TM) \otimes F}_{\partial_j} = \partial_j + \sigma^*_j + \sigma_j$ and $\sigma^*_j = \frac{1}{4} \sum_{k} \langle e_j, e_k \rangle c(e_j) c(e_k), \quad \sigma_j$ is the connection matrix of $\nabla^F$, then the twisted Dirac operators $\tilde{D}_F, \tilde{D}_F^*$ associated to the connection $\nabla$ as follows.

For $\psi \otimes \chi \in S(TM) \otimes F$, we have

$$\tilde{D}_F(\psi \otimes \chi) = D_F(\psi \otimes \chi) + c(\Phi)(\psi \otimes \chi),$$

(2.7)

$$\tilde{D}_F^*(\psi \otimes \chi) = D_F(\psi \otimes \chi) - c(\Phi^*)(\psi \otimes \chi),$$

(2.8)

where $c(\Phi) = \sum_{i=1}^n c(e_i) \otimes \Phi(e_i)$ and $c(\Phi^*) = \sum_{i=1}^n c(e_i) \otimes \Phi^*(e_i)$, $\Phi^*(e_i)$ denotes the adjoint of $\Phi(e_i)$.

Then, we have obtain

$$\tilde{D}_F = \sum_{j} c(e_j) \nabla^{S(TM) \otimes F}_{e_j} + c(\Phi),$$

(2.9)

$$\tilde{D}_F^* = \sum_{j} c(e_j) \nabla^{S(TM) \otimes F}_{e_j} - c(\Phi^*).$$

(2.10)
Let $\nabla^{TM}$ denote the Levi-Civita connection about $g^M$. In the local coordinates $\{x_i; 1 \leq i \leq n\}$ and the fixed orthonormal frame $\{e_1, \cdots, e_n\}$, the connection matrix $(\omega_{x,t})$ is defined by

$$\nabla^{TM}(\tilde{e}_1, \cdots, \tilde{e}_n) = (\tilde{e}_1, \cdots, \tilde{e}_n)(\omega_{x,t}).$$ (2.11)

Let $c(\tilde{e}_i)$ denote the Clifford action, $g^{ij} = g(dx_i, dx_j)\nabla^{TM}_{\partial_j} = \sum_k \Gamma^{ik}_{ij}$ and the tangent vector $\xi = \sum \xi_j dx_j$ and $\xi^j = g^{ij} \xi_i$, by Lemma 1 in [13] and Lemma 2.1 in [12], for any fixed point $x_0 \in \partial M$, we can choose the normal coordinates $U$ of $x_0$ in $\partial M$ (not in $M$), by the composition formula and (2.2.11) in [12], we obtain in [10].

**Lemma 2.1.** Let $\tilde{D}_{p}, \tilde{D}_{F}$ be the twisted Dirac operators on $\Gamma(S(TM) \otimes F)$, then

$$\sigma_{-1}(\tilde{D}_{p})^{-1} = \sigma_{-1}(\tilde{D}^{-1}) = \sqrt{-1}e(\xi),$$ (2.12)

$$\sigma_{-2}(\tilde{D}_{p})^{-1} = \frac{c(\xi)c_0(\tilde{D}^{*})c(\xi)}{\|\xi\|^4} + \frac{c(\xi)}{\|\xi\|^6} \sum_j c(dx_j)\left[\partial_{x_j}[c(\xi)]\|\xi\|^2 - c(\xi)\partial_{x_j}[\|\xi\|^2]\right];$$ (2.13)

$$\sigma_{-2}(\tilde{D}_{p}^{-1}) = \frac{c(\xi)c_0(D)c(\xi)}{\|\xi\|^4} + \frac{c(\xi)}{\|\xi\|^6} \sum_j c(dx_j)\left[\partial_{x_j}[c(\xi)]\|\xi\|^2 - c(\xi)\partial_{x_j}[\|\xi\|^2]\right],$$ (2.14)

where

$$\sigma_0(\tilde{D}^{*}) = -\frac{1}{4} \sum_{s,t} \omega_{s,t}(e_i)c(e_t)c(e_s)c(e_t) + \sum_{j=1}^{n} c(e_j)(\sigma_j^{\tilde{p}} - \Phi^{e}(e_j));$$ (2.15)

$$\sigma_0(D) = -\frac{1}{4} \sum_{s,t} \omega_{s,t}(e_i)c(e_t)c(e_s)c(e_t) + \sum_{j=1}^{n} c(e_j)(\sigma_j^{\tilde{p}} + \Phi^{e}(e_j)).$$ (2.16)

Let $\alpha = \sum_{j=1}^{n} c(e_j)(\sigma_j^{\tilde{p}} - \Phi^{e}(e_j)).$ $\beta = \sum_{j=1}^{n} c(e_j)(\sigma_j^{\tilde{p}} + \Phi^{e}(e_j)).$ $\sigma_0(D) = -\frac{1}{4} \sum_{s,t} \omega_{s,t}(e_i)c(e_t)c(e_s)c(e_t), \partial^j = g^{ij}\partial_i, \sigma^j = g^{ij}\sigma_j,$ we note that

$$\sigma_{-2}(\tilde{D}_{p}^{-1}) = \frac{c(\xi)c_0(D)c(\xi)}{\|\xi\|^4} + \frac{c(\xi)}{\|\xi\|^6} \sum_j c(dx_j)\left[\partial_{x_j}[c(\xi)]\|\xi\|^2 - c(\xi)\partial_{x_j}[\|\xi\|^2]\right],$$

$$\sigma_{-2}(\tilde{D}_{p}^{-1}) = \frac{c(\xi)c_0(D)c(\xi)}{\|\xi\|^4} + \frac{c(\xi)}{\|\xi\|^6} \sum_j c(dx_j)\left[\partial_{x_j}[c(\xi)]\|\xi\|^2 - c(\xi)\partial_{x_j}[\|\xi\|^2]\right] + \frac{c(\xi)\beta(e)(\xi)}{\|\xi\|^4},$$ (2.17)

where

$$\sigma_{-2}(\tilde{D}_{p}^{-1}) = \frac{c(\xi)c_0(D)c(\xi)}{\|\xi\|^4} + \frac{c(\xi)}{\|\xi\|^6} \sum_j c(dx_j)\left[\partial_{x_j}[c(\xi)]\|\xi\|^2 - c(\xi)\partial_{x_j}[\|\xi\|^2]\right].$$ (2.18)
By (2.6), (2.9) and (2.10), we have

\[
\bar{D}_F \bar{D}_F^* = D_F^2 - D_F c(\Phi^*) + c(\Phi) D_F - c(\Phi) c(\Phi^*)
\]

\[
= -g^{ij} \partial_i \partial_j - 2 \sigma_{S(TM) \otimes F} \partial_j + \Gamma^k \partial_k + \sum_j \left[ c(\Phi) c(e_j) - c(e_j) c(\Phi^*) \right] e_j + \sum_j c(e_j) (\partial_j S^{(TM) \otimes F}) c(\Phi^*)
\]

\[
- g^{ij} \left[ (\partial_i \sigma_{S(TM) \otimes F})^* + \sigma_{S(TM) \otimes F} (\partial_j \sigma_{S(TM) \otimes F}) - \Gamma_k^{ij} \sigma_k^{S(TM) \otimes F} \right] + \frac{1}{4} s + \frac{1}{2} \sum_{i \neq j} R^F(e_i, e_j) c(e_i) c(e_j)
\]

\[
+ \sum_j \left[ c(\Phi) c(e_j) \right] \sigma_j^{S(TM) \otimes F} - \sum_j c(\Phi) c_j (\Phi^*) - c(\Phi) c(\Phi^*) + \frac{1}{2}
\]

Combining (2.10) and (2.20), we have

\[
\bar{D}_F^* \bar{D}_F^* = \sum_{i=1}^n \left( c(e_i, dxi)(-g^{ij} \partial_i \partial_j) + \sum_{i=1}^n \left( c(e_i, dxi) \right) \left\{ (\partial_i g^{ij}) \partial_j \partial_j - g^{ij} (4 \sigma_{S(TM) \otimes F} \partial_j - 2 \Gamma_k^{ij} \partial_k) \partial_l \right\} \right)
\]

\[
+ \sum_{i=1}^n \left( c(e_i, dxi) \right) \left\{ -2 (\partial_i g^{ij}) \sigma_k^{S(TM) \otimes F} \partial_j + g^{ij} (\partial_j \Gamma_k) \partial_k - 2 g^{ij} (\partial_i \sigma_{S(TM) \otimes F} \partial_j + (\partial g^{ij}) \Gamma_k^{ij} \partial_k \partial_k \right\}
\]

\[
+ \sum_{i=1}^n \left( c(e_i, dxi) \right) \left\{ g^{ij} \left[ (\partial_i \sigma_k^{S(TM) \otimes F}) + \sigma_k^{S(TM) \otimes F} (\partial_j \sigma_{S(TM) \otimes F}) - \Gamma_k^{ij} \sigma_k^{S(TM) \otimes F} \right] + \frac{1}{2} \right\}
\]

\[
+ \sum_{i=1}^n \left( c(e_i, dxi) \right) \left\{ -g^{ij} \left[ \sigma_k^{S(TM) \otimes F} - \Gamma_k^{ij} \sigma_k^{S(TM) \otimes F} \right] + \frac{1}{2} \right\}
\]

\[
+ \left( \sum_{i \neq j} R^F(e_i, e_j) c(e_i) c(e_j) \right) + \left( \sigma_0 (D + \alpha) (-g^{ij} \partial_i \partial_j) + \sum_{i=1}^n \left( c(e_i, dxi) \right) \left\{ 2 \sum_{i=1}^n \left( c(\Phi) c(e_i) - c(e_i) \right) \right\} \right)
\]

\[
+ \left( \sum_{i \neq j} R^F(e_i, e_j) c(e_i) c(e_j) \right) + \left( \sigma_0 (D + \alpha) \left\{ -2 \sigma_k^{S(TM) \otimes F} \partial_j + \Gamma_k \partial_k + \sum_{i=1}^n \left( c(\Phi) c(e_i) - c(e_i) c(\Phi^*) \right) \right\} \right)
\]

\[
+ \left( \sum_{i \neq j} R^F(e_i, e_j) c(e_i) c(e_j) \right) + \left( \sum_{i \neq j} R^F(e_i, e_j) c(e_i) c(e_j) \right)
\]

By the above composition formulas, then we obtain:

**Lemma 2.2.** Let \( \bar{D}_F, \bar{D}_F^* \) be the twisted Dirac operators on \( \Gamma(S(TM) \otimes F) \),

\[
\sigma_3(\bar{D}_F \bar{D}_F \bar{D}_F^*) = \sqrt{-1} e(\xi) |\xi|^2;
\]

\[
\sigma_2(\bar{D}_F \bar{D}_F^*) = c(dx_n) h'(0) |\xi|^2 + c(\xi) (4 \sigma^k - 2 \Gamma^k) \xi_k + \sigma_0 (D) |\xi|^2 + \alpha |\xi|^2 - 2 \left[ c(\xi) c(\Phi) c(\xi) + |\xi|^2 c(\Phi^*) \right].
\]

Write

\[
\sigma_2(D^3) = c(dx_n) h'(0) |\xi|^2 + c(\xi) (4 \sigma^k - 2 \Gamma^k) \xi_k + \sigma_0 (D) |\xi|^2 = c(\xi) (4 \sigma^k - 2 \Gamma^k) \xi_k - \frac{1}{4} |\xi|^2 h'(0) c(dx_n).
\]

\[
D_x^* = (-\sqrt{-1})^n \partial_x^*; \quad \sigma(\bar{D}_F \bar{D}_F \bar{D}_F^*) = p_3 + p_2 + p_1 + p_0; \quad \sigma((\bar{D}_F \bar{D}_F \bar{D}_F^*)^{-1}) = \sum_{j=3}^\infty q_j.
\]
By the composition formula of pseudodifferential operators, we have

\[
1 = \sigma((\tilde{\Delta}_F^p \tilde{\Delta}_F^q \tilde{\Delta}_F^r)^{-1})
= (p_3 + p_2 + p_1 + p_0)(q_3 + q_4 + q_5 + \cdots) + \sum_j (\partial_{\xi_j} p_3 + \partial_{\xi_j} p_2 + \partial_{\xi_j} p_1 + \partial_{\xi_j} p_0)(D_{x_j} q_3 + D_{x_j} q_4 + D_{x_j} q_5 + \cdots)
= p_3 q_3 + (p_3 q_4 + p_2 q_3 + \sum_j \partial_{\xi_j} p_3 D_{x_j} q_3) + \cdots. \tag{2.25}
\]

Then

\[
q_3 = p_3^{-1}; \quad q_4 = -p_3^{-1} [p_2 p_3^{-1} + \sum_j \partial_{\xi_j} p_3 D_{x_j} (p_3^{-1})]. \tag{2.26}
\]

By Lemma 2.1 in [12] and (2.21)-(2.27), we obtain

**Lemma 2.3.** Let \(\tilde{\Delta}_F^p, \tilde{\Delta}_F^q, \tilde{\Delta}_F^r\) be the twisted Dirac operators on \(\Gamma(S(TM) \otimes F)\), then

\[
\sigma_{-\lambda}((\tilde{\Delta}_F^p \tilde{\Delta}_F^q \tilde{\Delta}_F^r)^{-1}) = \frac{\sqrt{\pi}c(\xi)}{|\xi|^{12}},
\]

\[
\sigma_{-\lambda}((\tilde{\Delta}_F^p \tilde{\Delta}_F^q \tilde{\Delta}_F^r)^{-1}) = \sigma_{-\lambda}(D^{-\lambda}) = -\frac{c(\xi)\sigma_2(D^{\lambda})c(\xi)}{|\xi|^6} + \frac{c(\xi)\sigma_1(D^{\lambda})c(\xi)}{|\xi|^6} + \frac{c(\xi)\sigma_0(D^{\lambda})c(\xi)}{|\xi|^6},
\]

where

\[
\sigma_{-\lambda}(D^{-\lambda}) = \frac{c(\xi)\sigma_2(D^{\lambda})c(\xi)}{|\xi|^6} + \frac{c(\xi)\sigma_1(D^{\lambda})c(\xi)}{|\xi|^6} + \sum_j \left[ c(dx_j) |\xi|^2 + 2\xi_j c(\xi) \right] \left[ \partial_{x_j} [c(\xi)] |\xi|^2 - 2c(\xi) \partial_{x_j} |\xi|^2 \right]. \tag{2.29}
\]

3. A Kastler-Kalau-Walze type theorem for six-dimensional manifolds with boundary associated with twisted Dirac Operators

In this section, we shall prove a Kastler-Kalau-Walze type formula for six-dimensional compact manifolds with boundary. Some basic facts and formulae about Boutet de Monvel’s calculus are recalled as follows.

Let

\[
F : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+); \quad F(u)(v) = \int e^{-ixt} u(t) dt
\]
denote the Fourier transformation and \(\varphi(\mathbb{R}^+)(= r^+ \varphi(\mathbb{R})) \) (similarly define \(\varphi(\mathbb{R}^-)\)), where \(\varphi(\mathbb{R})\) denotes the Schwartz space and

\[
r^+ : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}^+); \quad f \to f(\mathbb{R}^+); \quad \mathbb{R}^+ = \{x \geq 0; \ x \in \mathbb{R}\}. \tag{3.1}
\]

We define \(H^+ = F(\varphi(\mathbb{R}^+)); \ H^- = F(\varphi(\mathbb{R}^-))\) which are orthogonal to each other. We have the following property: \(h \in H^+ (H^-) \) iff \(h \in C^\infty(\mathbb{R})\) which has an analytic extension to the lower (upper) complex half-plane \(\{\text{Im} \xi < 0\} (\{\text{Im} \xi > 0\})\) such that for all nonnegative integer \(l\),

\[
\frac{d^l}{d\xi^l}(\xi) \sim \sum_{k=1}^\infty \frac{d^l}{d\xi^l}(\xi), \tag{3.2}
\]
as \(|\xi| \to +\infty, \\text{Im} \xi \leq 0 (\text{Im} \xi \geq 0)\).

Let \(H^+\) be the space of all polynomials and \(H^- = H_0^- \bigoplus H^+; \ H = H^+ \bigoplus H^-\). Denote by \(\pi^+ (\pi^-) \) respectively the projection on \(H^+ (H^-)\). For calculations, we take \(H = \tilde{\mathbb{H}} = \{\text{rational functions having no poles on the real axis}\} \ (\tilde{\mathbb{H}} \text{ is a dense set in the topology of } \mathbb{H})\). Then on \(\tilde{\mathbb{H}}\),

\[
\pi^+ h(\xi_0) = \frac{1}{2\pi i} \lim_{u \to 0^+} \int_{\xi_0 + iu}^{t_0^+} \frac{h(\xi)}{\xi - \xi \xi_0 + iu} d\xi, \tag{3.3}
\]
where $\Gamma^+$ is a Jordan close curve included $\text{Im} \xi > 0$ surrounding all the singularities of $h$ in the upper half-plane and $\xi_0 \in \mathbb{R}$. Similarly, define $\pi^+$ on $\hat{H}$,

$$\pi^+ h = \frac{1}{2\pi} \int_{\Gamma^+} h(\xi) d\xi.$$  \hspace{1cm} (3.4)

So, $\pi^+(H^-) = 0$. For $h \in H \cap L^1(R)$, $\pi^+ h = \frac{1}{2\pi} \int_{\Gamma^+} h(\nu) d\nu$ and for $h \in H^+ \cap L^1(R)$, $\pi^+ h = 0$. Denote by $\mathcal{B}$ Boutet de Monvel’s algebra (for details, see Section 2 of [14]).

An operator of order $m \in \mathbb{Z}$ and type $d$ is a matrix

$$A = \left( \begin{array}{cc} \pi^+ P + G & K \\ T \\ S \end{array} \right): C^\infty(X, E_1) \oplus C^\infty(\partial X, F_1) \rightarrow C^\infty(X, E_2),$$

where $X$ is a manifold with boundary $\partial X$ and $E_1, E_2$ are vector bundles over $X$ ($\partial X$). Here, $P: C^\infty_0(\Omega, E_1^\dagger) \rightarrow C^\infty(\Omega, E_2^\dagger)$ is a classical pseudodifferential operator of order $m$ on $\Omega$, where $\Omega$ is an open neighborhood of $X$ and $E_i|X = E_i$ ($i = 1, 2$). $P$ has an extension: $E'(\Omega, E_1^\dagger) \rightarrow \mathcal{D}'(\Omega, E_2^\dagger)$, where $E'(\Omega, E_1^\dagger) (\mathcal{D}'(\Omega, E_2^\dagger))$ is the dual space of $C^\infty(\Omega, E_1^\dagger)$ ($C^\infty_0(\Omega, E_2^\dagger)$). Let $e^+ : C^\infty(X, E_1) \rightarrow E'(\Omega, E_1^\dagger)$ denote extension by zero from $X$ to $\Omega$ and $r^+ : D'(\Omega, E_2^\dagger) \rightarrow \mathcal{D}'(\Omega, E_2)$ denote the restriction from $\Omega$ to $X$, then define

$$\pi^+ P = r^+ P e^+ : C^\infty(X, E_1) \rightarrow \mathcal{D}'(\Omega, E_2).$$

In addition, $P$ is supposed to have the transmission property; this means that, for all $j, k, \alpha$, the homogeneous component $p_j$ of order $j$ in the asymptotic expansion of the symbol $p$ of $P$ in local coordinates near the boundary satisfies:

$$\partial^k_x \partial^\alpha_p p_j(x', 0, 0, +1) = (-1)^j \partial^k_x \partial^\alpha_p p_j(x', 0, 0, -1),$$

then $\pi^+ P : C^\infty(X, E_1) \rightarrow C^\infty(X, E_2)$ by Section 2.1 of [14].

In the following, write $\pi^+ D^{-1} = \left( \begin{array}{cc} \pi^+ D^{-1} & 0 \\ 0 & 0 \end{array} \right)$, we will compute

$$\text{Wres}[\pi^+(D_p^{-1}) \circ \pi^+((\tilde{D}_p \tilde{D}_p \tilde{D}_p)^{-1})].$$

Let $M$ be a compact manifold with boundary $\partial M$. We assume that the metric $g^M$ on $M$ has the following form near the boundary

$$g^M = \frac{1}{h(x_n)} g^{\partial M} + dx_n^2,$$  \hspace{1cm} (3.5)

where $g^{\partial M}$ is the metric on $\partial M$. Let $U \subset M$ be a collar neighborhood of $\partial M$ which is diffeomorphic $\partial M \times [0, 1]$. By the definition of $h(x_n) \in C^\infty([0, 1])$ and $h(x_n) > 0$, there exists $\tilde{h} \in C^\infty((-\varepsilon, 1))$ such that $\tilde{h}_{|0, 1)} = h$ and $\tilde{h} > 0$ for some sufficiently small $\varepsilon > 0$. Then there exists a metric $\tilde{g}$ on $\tilde{M} = M \cup_{\partial M} \partial M \times (-\varepsilon, 0]$ which has the form on $U \cup_{\partial M} \partial M \times (-\varepsilon, 0]$

$$\tilde{g} = \frac{1}{h(x_n)} g^{\partial M} + dx_n^2,$$  \hspace{1cm} (3.6)

such that $\tilde{g}|_{\partial M} = g$. We fix a metric $\tilde{g}$ on the $\tilde{M}$ such that $\tilde{g}|_{M} = g$. Note $\tilde{D}_F$ is the twisted Dirac operator on the spinor bundle $S(TM) \otimes F$ corresponding to the connection $\tilde{\nabla}$.

Now we recall the main theorem in [10].

**Theorem 3.1. (Fedosov-Golse-Leichtnam-Schröhe)** Let $X$ and $\partial X$ be connected, dim$X = n \geq 3$, $A = \left( \begin{array}{cc} \pi^+ P + G & K \\ T \\ S \end{array} \right) \in \mathcal{B}$, and denote by $p$, $b$ and $s$ the local symbols of $P, G$ and $S$ respectively. Define:

$$\text{Wres}(A) = \int_X \int_S \text{tr}_E [p_{n-1}(x, \xi)] \sigma(\xi) dx$$

$$+ 2\pi \int_{\partial X} \int_S \{ \text{tr}_E [\text{tr} b_{n-1}(x', \xi')] + \text{tr}_F [s_{1-n}(x', \xi')] \} \sigma(\xi') dx',$$  \hspace{1cm} (3.7)
Then

\[ \text{a) } \text{Wres}(\pi_4(D_F^{-1}) \circ \pi_4((D_F^{*}D_F D_F^{*})^{-1})) = \int_{M} \int_{|\xi|=1} \text{trace}_{S(T(M)) \otimes F}[\sigma_{-n}((D_F^{*}D_F D_F^{*})^{-2})] \sigma(\xi) dx \]

\[ + \int_{\partial M} \Phi. \]  

(3.8)

where

\[ \Phi = \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} (-i)^{|\alpha|+j+k+\ell} \text{trace}_{S(T(M)) \otimes F}[\partial_{\xi}^j \partial_{\xi}^k \sigma_r((D_F^{*}D_F D_F^{*})^{-1})(x',0,\xi',\xi_n) \times \partial_{\xi_n}^\alpha \partial_{\xi_n}^{j+1} \partial_{\xi_n}^{k} \sigma_{\ell}((D_F^{*}D_F D_F^{*})^{-1})(x',0,\xi',\xi_n)] d\xi_n \sigma(\xi') dx'. \]

(3.9)

and the sum is taken over \( r - k + |\alpha| + \ell - j - 1 = -n, r \leq -1, \ell \leq -3. \)

Locally, we can use Theorem 2.4 in [19] to compute the interior term of (3.8), then

\[ \int_{M} \int_{|\xi|=1} \text{trace}_{S(T(M)) \otimes F}[\sigma_{-n}((D_F^{*}D_F D_F^{*})^{-2})] \sigma(\xi) dx \]

\[ = 8 \pi^3 \int_{M} \text{Tr} \left[ \left( -\frac{S}{12} + c(\Phi^*)c(\Phi) - \frac{1}{4} \sum_{i} [c(\Phi^*)c(e_i) - c(e_i)c(\Phi)] \right)^2 + \frac{1}{2} \sum_{j} c(e_j) \nabla_{e_j} c(\Phi) \right] \text{dvol}(M). \]

(3.10)

So we only need to compute \( \int_{\partial M} \Phi. \)

From the formula (3.9) for the definition of \( \Phi, \) now we can compute \( \Phi. \) Since the sum is taken over \( r + \ell - k - j - |\alpha| - 1 = -6, r \leq -1, \ell \leq -3, \) then we have the \( \int_{\partial M} \Phi \) is the sum of the following five cases:

case (a) (I) \( r = -1, l = -3, j = k = 0, |\alpha| = 1. \)

By (3.9), we get

\[ \text{case (a) (I)} = \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha| = 1} \text{trace}_{S(T(M)) \otimes F}[\partial_{\xi}^j \partial_{\xi}^k \sigma_{\ell}((D_F^{*}D_F D_F^{*})^{-1})] (x_0) d\xi_n \sigma(\xi') dx'. \]

(3.11)

By Lemma 2.2 in [12], for \( i < n, \) we have

\[ \partial_{\xi} \sigma_{-3}((D_F^{*}D_F D_F^{*})^{-1})(x_0) = \partial_{\xi} \left[ \frac{ic(\xi)}{|\xi|^4} \right] (x_0) = i \partial_{\xi} [c(\xi)] |\xi|^{-4} (x_0) - 2ic(\xi) \partial_{\xi} [|\xi|^2] |\xi|^{-6} (x_0) = 0. \]

(3.12)

so case (a) (I) vanishes.

case (a) (II) \( r = -1, l = -3, |\alpha| = k = 0, j = 1. \)

By (3.9), we have

\[ \text{case (a) (II)} = -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}_{S(T(M)) \otimes F}[\partial_{\xi}^j \partial_{\xi}^k \sigma_{\ell}((D_F^{*}D_F D_F^{*})^{-1})] (x_0) d\xi_n \sigma(\xi') dx'. \]

(3.13)

By (2.23) in [12], we have

\[ \pi^+_\xi \partial_{\xi_n} \sigma_{-1}(D_F^{-1})(x_0)|_{|\xi'|=1} = \frac{\partial_{\xi_n} [c(\xi')(x_0)]}{2(\xi_n - i)} + ih'(0) \left[ \frac{ic(\xi')}{4(\xi_n - i)} + \frac{c(\xi') + ic(dx_n)}{4(\xi_n - i)^2} \right]. \]

(3.14)
By (2.28) and direct calculations, we have
\[
\partial_{\xi_n}^2 \sigma_{-3}((\hat{D}_F^2 \hat{D}_F \hat{D}_F^{-1})^{-1}) = i \left[ \frac{(20 \xi_n^2 - 4)c(\xi') + 12(\xi_n^3 - \xi_n)c(dx_n)}{(1 + \xi_n^4)^4} \right].
\] (3.15)

Since \( n = 6 \), trace_{S(TM)}[\pi_{-1}] = -8\text{dim}F. By the relation of the Clifford action and \( \text{trace}AB = \text{trace}BA \), then
\[
\text{trace}[c(\xi')(\xi dx_n)] = 0; \text{trace}[(\xi dx_n)^2] = -8\text{dim}F; \text{trace}[c(\xi')^2](x_0)|_{\xi'|=1} = -8\text{dim}F;
\]
\[
\text{trace}[\partial_{x_n} c(\xi')(\xi dx_n)] = 0; \text{trace}[\partial_{x_n} c(\xi')^2](x_0)|_{\xi'|=1} = -4h'(0)\text{dim}F.
\] (3.16)

By (3.14), (3.15) and (3.16), we get
\[
\text{trace}\left[ \partial_{x_n} \pi_{-1}^c(\hat{D}_F^{-1}) \times \partial_{\xi_n}^2 \sigma_{-3}((\hat{D}_F^2 \hat{D}_F \hat{D}_F^{-1})^{-1}) \right](x_0) = h'(0)\text{dim}F \frac{-8 - 24 \xi_n i + 40 \xi_n^2 + 24i \xi_n^3}{(\xi_n - i)^6(\xi_n + i)^4}.
\] (3.17)

Then we obtain
\[
\text{case (a) (II)} = \frac{-1}{2} \int_{\xi'|=1}^{+\infty} h'(0)\text{dim}F \frac{-8 - 24 \xi_n i + 40 \xi_n^2 + 24i \xi_n^3}{(\xi_n - i)^6(\xi_n + i)^4} d\xi_n \sigma(\xi') dx'
\]
\[
= h'(0)\text{dim}F \Omega_4 \int_{\hat{TM}^+} \frac{4 + 12 \xi_n i - 20 \xi_n^2 - 12i \xi_n^3}{(\xi_n - i)^6(\xi_n + i)^4} d\xi_n dx'
\]
\[
= h'(0)\text{dim}F \Omega_4 \frac{\pi i}{5!} \left[ \frac{8 + 24 \xi_n i - 40 \xi_n^2 - 24i \xi_n^3}{(\xi_n + i)^4} \right] |_{\xi_n=0} dx'
\]
\[
= \frac{-15}{16} \frac{\pi h'(0)\text{dim}F dx'}.
\] (3.18)

where \( \Omega_4 \) is the canonical volume of \( S_4 \).

\text{case (a) (III)} \( r = -1, l = -3, |\alpha| = j = 0, k = 1. \) By (3.9), we have
\[
\text{case (a) (III)} = \frac{-1}{2} \int_{\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}\left[ \partial_{x_n} \pi_{-1}^c(\hat{D}_F^{-1}) \times \partial_{\xi_n} \partial_{x_n} \sigma_{-3}((\hat{D}_F^2 \hat{D}_F \hat{D}_F^{-1})^{-1}) \right](x_0) d\xi_n \sigma(\xi') dx'.
\] (3.19)

By (2.2.29) in [12], we have
\[
\partial_{\xi_n} \pi_{-1}^c(\hat{D}_F^{-1})(x_0)|_{\xi'|=1} = -\frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)^2}. \] (3.20)

By (2.28) and direct calculations, we have
\[
\partial_{\xi_n} \partial_{x_n} \sigma_{-3}((\hat{D}_F^2 \hat{D}_F \hat{D}_F^{-1})^{-1}) = -\frac{4i \xi_n \partial_{x_n} c(\xi')(x_0)}{(1 + \xi_n^4)^3} + i \frac{12h'(0)\xi_n c(\xi')}{(1 + \xi_n^4)^4} - i \frac{(2 - 10 \xi_n^2) h'(0) c(dx_n)}{(1 + \xi_n^4)^4}.
\] (3.21)

Combining (3.16), (3.20) and (3.21), we have
\[
\text{trace}\left[ \partial_{\xi_n} \pi_{-1}^c(\hat{D}_F^{-1}) \times \partial_{\xi_n} \partial_{x_n} \sigma_{-3}((\hat{D}_F^2 \hat{D}_F \hat{D}_F^{-1})^{-1}) \right](x_0)|_{\xi'|=1} = h'(0)\text{dim}F \frac{8i - 32 \xi_n - 8i \xi_n^2}{(\xi_n - i)^6(\xi + i)^4}.
\] (3.22)
Then
\[
\text{case (a) III) } = -\frac{1}{2} \int_{|\xi'|=1}^{+\infty} h'(0) \text{dimK} \frac{8i - 32\xi_n - 8i\xi_n^2}{(\xi_n - i)^{\delta}} d\xi_n \sigma(\xi') dx' \\
= -\frac{1}{2} h'(0) \text{dimF} \frac{8i - 32\xi_n - 8i\xi_n^2}{(\xi_n - i)^{\delta}} d\xi_n dx' \\
= -h'(0) \text{dimF} \frac{8i - 32\xi_n - 8i\xi_n^2}{(\xi_n - i)^{\delta}} |_{\xi_n = -1} dx' \\
= -\frac{25}{16} \pi h'(0) \text{dimF} dx',
\]
where \( \Omega_4 \) is the canonical volume of \( S_4 \).

\textbf{case (b)} \( r = -1, l = -4, |\alpha| = j = k = 0. \)

By (3.9), we have
\[
\text{case (b) } = -i \int_{|\xi'|=1}^{+\infty} \text{trace} \left[ \pi^+_{\xi_n} \sigma_{a-1}(\tilde{D}_F^{-1}) \times \partial_{\xi_n} \sigma_{-4}((\tilde{D}_F^{\alpha} \tilde{D}_F^{-1})) \right] (x_0) d\xi_n \sigma(\xi') dx' \\
= -i \int_{|\xi'|=1}^{+\infty} \text{trace} \left[ \pi^+_{\xi_n} \sigma_{a-1}(\tilde{D}_F^{-1}) \times \partial_{\xi_n} \left( \sigma_{-4}(D^{-3}) + \frac{c(\xi) c(\Phi)}{|\xi|^6} \right) - \frac{2c(\xi) c(\Phi)}{|\xi|^4} \right] (x_0) d\xi_n \sigma(\xi') dx' \\
:= D_1 + D_2 + D_3 + D_4, \tag{3.24}
\]
where
\[
D_1 = -i \int_{|\xi'|=1}^{+\infty} \text{trace} \left[ \pi^+_{\xi_n} \sigma_{a-1}(\tilde{D}_F^{-1}) \times \partial_{\xi_n} \sigma_{-4}(D^{-3}) \right] (x_0) d\xi_n \sigma(\xi') dx'; \tag{3.25}
\]
\[
D_2 = -i \int_{|\xi'|=1}^{+\infty} \text{trace} \left[ \pi^+_{\xi_n} \sigma_{a-1}(\tilde{D}_F^{-1}) \times \partial_{\xi_n} \left( \frac{c(\xi) c(\Phi)}{|\xi|^6} \right) \right] (x_0) d\xi_n \sigma(\xi') dx'; \tag{3.26}
\]
\[
D_3 = 2i \int_{|\xi'|=1}^{+\infty} \text{trace} \left[ \pi^+_{\xi_n} \sigma_{a-1}(\tilde{D}_F^{-1}) \times \partial_{\xi_n} \left( \frac{c(\xi) c(\Phi)}{|\xi|^6} \right) \right] (x_0) d\xi_n \sigma(\xi') dx'; \tag{3.27}
\]
\[
D_4 = 2i \int_{|\xi'|=1}^{+\infty} \text{trace} \left[ \pi^+_{\xi_n} \sigma_{a-1}(\tilde{D}_F^{-1}) \times \partial_{\xi_n} \left( \frac{c(\Phi)}{|\xi|^4} \right) \right] (x_0) d\xi_n \sigma(\xi') dx'. \tag{3.28}
\]

By (2.2.44) in [12], we have
\[
\pi^+_{\xi_n} \sigma_{a-1}(\tilde{D}_F^{-1}) = \frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)}. \tag{3.29}
\]
In the normal coordinate, \( g^j(x_0) = \delta^j_j \) and \( \partial_{x_j} (g^{\alpha\beta}) (x_0) = 0 \), if \( j < n \); \( \partial_{x_j} (g^{\alpha\beta}) (x_0) = h'(0) \delta^j_j \), if \( j = n \). So by Lemma A.2 in [12], we have \( \Gamma^\alpha (x_0) = \frac{1}{2} h'(0) \Gamma^k (x_0) = 0 \) for \( k < n \). By the definition of \( \delta^k \) and Lemma 2.3 in [12], we have \( \delta^m (x_0) = 0 \) and \( \delta^k = \frac{1}{2} h'(0) c^k \left( \tilde{e}_k \right) c(\tilde{e}_n) \) for \( k < n \). We obtain
\[ \sigma_{-4}(D^{-3})(x_0) = \frac{1}{|\xi|^8} c(\xi) \left( h'(0)c(\xi) \sum_{k \neq n} \xi_k c(\xi_k) c(\xi_n) - 5h'(0)\xi_n c(\xi) - \frac{5}{4} h'(0)|\xi|^2 c(dx_n) \right) c(\xi) \\
+ \frac{c(\xi)}{|\xi|^10} \left( |\xi|^4 c(dx_n) \partial_{x_n} [c(\xi)](x_0) - 2h'(0)|\xi|^2 c(dx_n) c(\xi) + 2\xi_n |\xi|^2 c(\xi) \partial_{x_n} [c(\xi)](x_0) \right) \\
+ 4\xi_n h'(0)c(\xi) c(\xi) \left( \frac{c(\xi) c(dx_n) c(\xi)}{|\xi|^6} \right) + h'(0) \left( \frac{c(\xi) c(dx_n) c(\xi)}{|\xi|^2} \right) \\
= \frac{-17 - 9\xi_n^2}{4(1 + \xi_n^2)^4} h'(0)c(\xi) c(dx_n) c(\xi') + \frac{33\xi_n + 17\xi_n^3}{2(1 + \xi_n^2)^4} h'(0)c(\xi) + \frac{49\xi_n^2 + 25\xi_n^4}{2(1 + \xi_n^2)^4} h'(0)c(dx_n) \\
+ \frac{1}{(1 + \xi_n^2)^3} c(\xi') c(dx_n) \partial_{x_n} [c(\xi)](x_0) - \frac{3\xi_n}{(1 + \xi_n^2)^4} \partial_{x_n} [c(\xi')] c(\xi)(x_0) \\
+ \frac{3}{(1 + \xi_n^2)^3} h'(0)c(\xi) c(\xi'). \] (3.30)

Then

\[ \partial_{\xi_n} \sigma_{-4}(D^{-3})(x_0) = \frac{59\xi_n + 27\xi_n^3}{2(1 + \xi_n^2)^4} h'(0)c(\xi) c(dx_n) c(\xi') + \frac{33 - 180\xi_n^2 - 85\xi_n^4}{2(1 + \xi_n^2)^4} h'(0)c(\xi) \\
+ \frac{49\xi_n - 97\xi_n^3 - 50\xi_n^5}{2(1 + \xi_n^2)^4} h'(0)c(dx_n) - \frac{6\xi_n}{(1 + \xi_n^2)^4} c(\xi) c(dx_n) \partial_{x_n} [c(\xi')](x_0) \\
- \frac{3 - 15\xi_n^2}{(1 + \xi_n^2)^4} \partial_{x_n} [c(\xi')](x_0) + \frac{4\xi_n^3 - 8\xi_n^5}{(1 + \xi_n^2)^4} h'(0)c(dx_n) + \frac{2 - 10\xi_n^2}{(1 + \xi_n^2)^4} h'(0)c(\xi'). \] (3.31)

By (3.16),(3.29) and (3.31), we obtain

\[ \text{trace} \left[ \pi_\xi^+ \sigma_{-4}(D^{-3}) \right] (x_0) |_{\xi = 1} = h'(0)dimF \frac{4i(-17 - 42i\xi_n + 50\xi_n^2 - 16i\xi_n^2 + 29\xi_n^4)}{(\xi_n - i)^5(\xi_n + i)^3}. \] (3.32)

By (3.25) and (3.32), we have

\[ D_1 = 4h'(0)dimF \frac{2\pi i}{4!} \left[ -17 - 42i\xi_n + 50\xi_n^2 - 16i\xi_n^2 + 29\xi_n^4 \right] \mid_{\xi = 1} = \frac{-129}{16} \pi h'(0)dimF \Omega_4 dx'. \] (3.33)

Since

\[ \partial_{\xi_n} \left( \frac{c(\xi) c(\xi)}{|\xi|^6} \right) = c(dx_n) \alpha c(\xi') + c(\xi') c(dx_n) + 2\xi_n c(dx_n) \alpha c(dx_n) - \frac{6\xi_n c(\xi) c(\xi)}{(1 + \xi_n^2)^4}, \] (3.34)

then

\[ \text{trace} \left[ \pi_\xi^+ \sigma_{-4}(D^{-3}) \right] \times \partial_{\xi_n} \left( \frac{c(\xi) c(\xi)}{|\xi|^6} \right) (x_0) \]
\[ = \frac{4(\xi_n + i) + 2}{2(\xi_n + i)(1 + \xi_n^2)^3} \text{trace}[c(\xi') \alpha] + \frac{4\xi_n i + 2}{2(\xi_n + i)(1 + \xi_n^2)^3} \text{trace}[c(dx_n) \alpha]. \] (3.35)

By the relation of the Clifford action and \text{trace}AB = \text{trace}BA, we then have the equalities

\[ \text{trace} \left[ c(dx_n) \sum_{j=1}^n c(e_j)(\sigma_j^F - \Phi^*(e_j)) \right] = \text{trace} \left[ - |\boldsymbol{d}| \otimes (\sigma_j^F - \Phi^*(e_j)) \right]; \] (3.36)

\[ \text{trace} \left[ c(\xi') \sum_{j=1}^n c(e_j)(\sigma_j^F - \Phi^*(e_j)) \right] = \text{trace} \left[ - \sum_{j=1}^{n-1} \xi_j (\sigma_j^F - \Phi^*(e_j)) \right]. \] (3.37)
We note that \( i < n \), \( \int_{|\xi'|=1} \xi \sigma(\xi') = 0 \), so \( \text{trace}[c(\xi')\alpha] \) has no contribution for computing case (b).

By (3.26) and (3.35), then

\[
D_2 = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{4\xi_n i + 2}{2(\xi_n + i)|\xi_n - i|} \text{trace}\left[ -\mathbb{I} \otimes (\sigma_n^\xi - \Phi^*(\epsilon_n)) \right] d\xi_n \sigma(\xi') dx' \\
= \frac{3}{2} \pi \text{dim} F \text{trace}\left[ \sigma_n^\xi - \Phi^*(\epsilon_n) \right] \Omega_4 dx'.
\]

(3.38)

Since

\[
\partial_{\xi_n} \left( \frac{c(\xi) c(\Phi^*) c(\xi)}{\xi^6} \right) = \frac{c(dx_n) c(\Phi^*) c(\xi') + c(\xi') c(\Phi^*) c(dx_n) + 2\xi_n c(dx_n) c(\Phi^*) c(dx_n) - 6\xi_n c(\xi) c(\Phi^*) c(\xi)}{(1 + \xi_n^4)^4},
\]

then

\[
\text{trace}\left[ \pi_{+}^\xi \sigma_{-1} (\tilde{D}_F^{-1}) \times \partial_{\xi_n} \left( \frac{c(\xi) c(\Phi^*) c(\xi)}{\xi^6} \right) \right] (x_0) \\
= \frac{(4\xi_n i + 2)i}{2(\xi_n + i)(1 + \xi_n^4)^4} \text{trace}[c(\xi') c(\Phi^*)] + \frac{4\xi_n i + 2}{2(\xi_n + i)(1 + \xi_n^4)^4} \text{trace}[c(dx_n) c(\Phi^*)].
\]

(3.40)

By the relation of the Clifford action and \( \text{trace} AB = \text{trace} BA \), then we have the equalities

\[
\text{trace}\left[ c(dx_n) \sum_{j=1}^{n} c(e_j) \otimes \Phi^*(e_j) \right] = \text{trace}\left[ -\mathbb{I} \otimes \Phi^*(\epsilon_n) \right], \quad (3.41)
\]

\[
\text{trace}\left[ c(\xi') \sum_{j=1}^{n} c(e_j) \otimes \Phi^*(e_j) \right] = \text{trace}\left[ -\sum_{j=1}^{n-1} \xi_j \Phi^*(e_j) \right], \quad (3.42)
\]

We note that \( i < n \), \( \int_{|\xi'|=1} \xi \sigma(\xi') = 0 \), so \( \text{trace}[c(\xi') c(\Phi^*)] \) has no contribution for computing case (b).

By (3.27) and (3.40), then

\[
D_3 = 2i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{4\xi_n i + 2}{2(\xi_n + i)(1 + \xi_n^4)} \text{trace}\left[ -\mathbb{I} \otimes \Phi^*(\epsilon_n) \right] d\xi_n \sigma(\xi') dx' \\
= -3\pi \text{dim} F \text{trace}\left[ \Phi^*(\epsilon_n) \right] \Omega_4 dx'.
\]

(3.43)

Since

\[
\partial_{\xi_n} \left( \frac{c(\Phi)}{\xi^4} \right) = -\frac{2\xi_n c(\Phi)}{(1 + \xi_n^4)^4}, \quad (3.44)
\]

then

\[
\text{trace}\left[ \pi_{+}^\xi \sigma_{-1} (\tilde{D}_F^{-1}) \times \partial_{\xi_n} \left( \frac{c(\Phi)}{\xi^4} \right) \right] (x_0) \\
= \frac{-2\xi_n}{2(\xi_n - i)(1 + \xi_n^4)^4} \text{trace}[c(\xi') c(\Phi)] + \frac{-2\xi_n i}{2(\xi_n - i)(1 + \xi_n^4)^4} \text{trace}[c(dx_n) c(\Phi)].
\]

(3.45)
By the relation of the Clifford action and \( \text{trace} AB = \text{trace} BA \), then we have the equalities
\[
\text{trace} \left[ c(dx_n) \sum_{j=1}^n c(e_j) \otimes \Phi(e_j) \right] = \text{trace} \left[ -id \otimes \Phi(e_n) \right], \tag{3.46}
\]
\[
\text{trace} \left[ c(\xi') \sum_{j=1}^n c(e_j) \otimes \Phi(e_j) \right] = \text{trace} \left[ -\sum_{j=1}^{n-1} \xi_j \Phi(e_j) \right]. \tag{3.47}
\]

We note that \( i < n \), \( \int_{|\xi'|=1} \xi_i \sigma(\xi') = 0 \), so \( \text{trace}[c(\xi')c(\Phi)] \) has no contribution for computing case (b).

By (3.28) and (3.45), then
\[
D_4 = 2i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{-2\xi_n i}{2(\xi_n + i)^3(\xi_n - i)^2} \text{trace} \left[ -id \otimes \Phi(e_n) \right] d\xi_n \sigma(\xi') d\xi'.
\]

By (3.24), then
\[
\text{case (b)} = -\frac{129}{16} \pi h'(0) \text{dim} F \Omega_4 dx' + \frac{3}{2} \pi \text{dim} F \text{trace} \left[ \sigma_n - \sigma_n F \phi(e_n) \right] \Omega_4 dx' - 3\pi \text{dim} F \text{trace} \left[ \phi^*(e_n) \right] \Omega_4 dx' - \pi \text{dim} F \text{trace} \left[ \phi(e_n) \right] \Omega_4 dx'. \tag{3.49}
\]

**case (c)** \( r = -2, l = -3, |\alpha| = j = k = 0 \).

By (3.9), we have
\[
\text{case (c)} = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[ \pi_{\xi_n}^+ \sigma_{-2}(\tilde{D}_F^{-1}) \times \partial_{\xi_n} \sigma_{-3}(\tilde{D}_F \tilde{D} \tilde{D}_I^{-1}) \right] (x_0) d\xi_n \sigma(\xi') d\xi'. \tag{3.50}
\]

By (2.18), we have
\[
\pi_{\xi_n}^+ \sigma_{-2}(\tilde{D}_F^{-1}) = \pi_{\xi_n}^+ \left( \sigma_{-2}(D^{-1}) + \frac{c(\xi)\beta_c(\xi)}{|\xi|^4} \right). \tag{3.51}
\]

By (2.19), we have
\[
\pi_{\xi_n}^+ \left( \sigma_{-2}(D^{-1}) \right)(x_n)|_{|\xi'|=1} = \pi_{\xi_n}^+ \left[ \frac{c(\xi)\sigma_0(D)(x_0)c(\xi) + c(\xi)c(dx_n)\partial_{\xi_n}c(\xi')(x_0)}{(1 + \xi_n^2)^2} \right] - h'(0) \pi_{\xi_n}^+ \left[ \frac{c(\xi)c(dx_n)c(\xi)}{1 + \xi_n^2} \right] := A_1 - A_2, \tag{3.52}
\]

where
\[
A_1 = -\frac{1}{4(\xi_n - i)^2} \left[ (2 + i\xi_n)c(\xi')c_0(D)c(\xi') + i\xi_n c(dx_n)\sigma_0(D)c(dx_n) \right. \nonumber
\]
\[
+ (2 + i\xi_n)c(\xi')c(dx_n)\partial_{\xi_n}[c(\xi')] + ic(dx_n)\sigma_0(D)c(\xi') + ic(\xi')\sigma_0(D)c(dx_n) - i\partial_{\xi_n}[c(\xi')] \right] \nonumber
\]
\[
= \frac{1}{4(\xi_n - i)^2} \left[ 5h'(0)c(dx_n) - \frac{5i}{2} h'(0)c(\xi') - (2 + i\xi_n)c(\xi')c(dx_n)\partial_{\xi_n}[c(\xi')] + i\partial_{\xi_n}[c(\xi')] \right]; \tag{3.53}
\]
\[
A_2 = \frac{h'(0)}{2} \left[ c(dx_n) \right] + \frac{c(dx_n) - ic(\xi')}{8(\xi_n - i)^2} + \frac{3\xi_n - 7i}{8(\xi_n - i)^3} \left( ic(\xi') - c(dx_n) \right). \tag{3.54}
\]
On the other hand,

$$\pi^+_{\xi_0} \left( \frac{c(\xi)\beta c(\xi)}{|\xi|^4} \right)(x_0)|_{|\xi'|=1} = \frac{(-i\xi_n - 2)c(\xi')\beta c(\xi') - i\left[c(dx_n)\beta c(\xi') + c(\xi')\beta c(dx_n)\right]}{4(\xi_n - i)^2}. \quad (3.55)$$

By (2.28), we obtain

$$\partial_{\xi_n} \sigma_3((\tilde{\mathcal{D}}^*_{\xi_0} \tilde{\mathcal{D}}^*_{\xi_0} \mathcal{D}^*_{\xi_0})^{-1}) = -\frac{4i\xi_n c(\xi') + (i - 3i\xi_n^2)c(dx_n)}{(1 + \xi_n)^3}. \quad (3.56)$$

By (3.53) and (3.56), we have

$$\text{tr}[A_1 \times \partial_{\xi_n} \sigma_3((\tilde{\mathcal{D}}^*_{\xi_0} \tilde{\mathcal{D}}^*_{\xi_0} \mathcal{D}^*_{\xi_0})^{-1})]|_{|\xi'|=1} = \frac{h'(0) \text{dim} \mathcal{F} \left[ \frac{3 + 12i\xi_n + 3\xi_n^2}{(\xi_n - i)^3(\xi_n + i)^3} \right]}{(3.57)}$$

Similarly, we have

$$\text{trace}[A_2 \times \partial_{\xi_n} \sigma_3((\tilde{\mathcal{D}}^*_{\xi_0} \tilde{\mathcal{D}}^*_{\xi_0} \mathcal{D}^*_{\xi_0})^{-1})]|_{|\xi'|=1} = \frac{h'(0) \text{dim} \mathcal{F} \left[ \frac{4i - 11\xi_n - 6i\xi_n^2 + 3\xi_n^3}{(\xi_n - i)^3(\xi_n + i)^3} \right]}{(3.58)}$$

By (3.57) and (3.58), we obtain

$$-i \int_{|\xi'|=1}^{+\infty} \text{trace}\left[ \pi^+_{\xi_0} \sigma_2(D^{-1}) \times \partial_{\xi_n} \sigma_3((\tilde{\mathcal{D}}^*_{\xi_0} \tilde{\mathcal{D}}^*_{\xi_0} \mathcal{D}^*_{\xi_0})^{-1}) \right] (x_0) d\xi_n \sigma(\xi') d\xi'$$

$$= -i \text{dim} \mathcal{F} h'(0) \int_{|\xi'|=1}^{+\infty} \frac{7i + 26\xi_n + 15i\xi_n^2}{(\xi_n - i)^3(\xi_n + i)^3} d\xi_n \sigma(\xi') d\xi'$$

$$= -i \text{dim} \mathcal{F} h'(0) \frac{2\pi i}{4i} \left[ \frac{-7i + 26\xi_n + 15i\xi_n^2}{(\xi_n + i)^3} \right]_{|\xi_n|=\Omega_4} d\xi'$$

$$= \frac{55}{16} \text{dim} \mathcal{F} h'(0) \Omega_4 d\xi'. \quad (3.59)$$

By (3.55) and (3.56), we have

$$\text{trace}\left[ \pi^+_{\xi_0} \left( \frac{c(\xi)\beta c(\xi)}{|\xi|^4} \right) \times \partial_{\xi_n} \sigma_3((\tilde{\mathcal{D}}^*_{\xi_0} \tilde{\mathcal{D}}^*_{\xi_0} \mathcal{D}^*_{\xi_0})^{-1}) \right] (x_0)$$

$$= \frac{(3\xi_n - i)}{2(\xi_n - i)(1 + \xi_n^2)} \text{trace}[c(dx_n)\beta] + \frac{3\xi_n - i}{2(\xi_n - i)(1 + \xi_n^2)} \text{trace}[c(\xi')\beta]. \quad (3.60)$$
By the relation of the Clifford action and trace $AB = \text{trace}BA$, then we have the equalities
\begin{equation}
\text{trace}\left[ c(dx_n) \sum_{j=1}^{n} c(e_j)(\sigma_j F + \Phi(e_j)) \right] = \text{trace}\left[ -\mathbf{id} \otimes (\sigma_n F + \Phi(e_n)) \right]; \tag{3.61}
\end{equation}

\begin{equation}
\text{trace}\left[ c(\xi') \sum_{j=1}^{n} c(e_j)(\sigma_j F + \Phi(e_j)) \right] = \text{trace}\left[ -\sum_{j=1}^{n-1} \xi_j(\sigma_j F + \Phi(e_j)) \right]. \tag{3.62}
\end{equation}

We note that $i < n$, \( \int_{|\xi'|=1} \xi_i \sigma(\xi') = 0 \), so trace\([c(\xi')\beta]\) has no contribution for computing case (c). Then, we obtain
\begin{align*}
-i \int_{|\xi'|=1} & \int_{-\infty}^{+\infty} \text{trace}\left[ \pi_+ \left( \frac{c(\xi')\beta c(\xi)}{|\xi'|^4} \right) \times \partial_{\xi_n} \sigma_{-1} \left( (\tilde{D}_F \tilde{D}_F^{-1})^{-1} \right) \right] (x_0) d\xi_n \sigma(\xi') dx' \\
= -i \int_{|\xi'|=1} & \int_{-\infty}^{+\infty} \frac{(3\xi_n - i)\pi_{-1}}{2(\xi_n - i)(1 + \xi_n^2)^2} \text{trace}[c(dx_n)\beta] d\xi_n \sigma(\xi') dx' \\
= -2\pi \text{dim} F \text{trace}[\sigma_n F + \Phi(e_n)] \Omega_4 dx'. \tag{3.63}
\end{align*}

Then
\begin{equation*}
\text{case (c)} = \frac{55}{16} \text{dim} F \pi h'(0) \Omega_4 dx' - 2\pi \text{dim} F h'(0) \text{trace}[\sigma_n F + \Phi(e_n)] \Omega_4 dx'. \tag{3.64}
\end{equation*}

Now \( \Phi \) is the sum of the cases (a), (b) and (c), then
\begin{equation*}
\Phi = \left[ 4h'(0) - \text{trace}\left( \Phi(e_n) \right) - 3\text{trace}\left( \Phi^*(e_n) \right) + \frac{3}{2} \text{trace}\left( \sigma_n F - \Phi^*(e_n) \right) - 2\text{trace}\left( \sigma_n F + \Phi(e_n) \right) \right] \pi \text{dim} F \Omega_4 dx'. \tag{3.65}
\end{equation*}

By (4.2) in \cite{12}, we have
\begin{equation*}
K = \sum_{1 \leq i,j \leq n-1} K_{i,j} \delta_{ij}^M; K_{i,j} = -\Gamma_{i,j},
\end{equation*}
and \( K_{i,j} \) is the second fundamental form, or extrinsic curvature. For \( n = 6 \), then
\begin{equation*}
K(x_0) = \sum_{1 \leq i,j \leq n-1} K_{i,j}(x_0) \delta_{ij}^M(x_0) = \sum_{i=1}^{5} K_{i,i}(x_0) = -\frac{5}{2} h'(0). \tag{3.66}
\end{equation*}
Hence we conclude that

**Theorem 3.2.** Let \( M \) be a 6-dimensional compact spin manifolds with the boundary \( \partial M \). Then
\begin{equation*}
\text{Wres}[\pi^+((\tilde{D}_F^{-1}) \circ \pi^+((\tilde{D}_F^{-1})) \right] = 8\pi^3 \int_{M} \text{Tr} \left[ \left[ -\frac{8}{12} + c(\Phi^*) c(\Phi) - \frac{1}{4} \sum_{i} [c(\Phi^*) c(e_i) - c(e_i) c(\Phi)]^2 - \frac{1}{2} \sum_{j} \nabla_{e_j}^F (c(\Phi^*)) c(e_j) \right) \right] \text{vol}_{M} + \int_{\partial M} \left[ -\frac{8}{3} K - \text{trace}\left( \Phi(e_6) \right) - 3\text{trace}\left( \Phi^*(e_6) \right) + \frac{3}{2} \text{trace}\left( \sigma_6 F - \Phi^*(e_6) \right) - 2\text{trace}\left( \sigma_6 F + \Phi(e_6) \right) \right] \pi \text{dim} F \Omega_4 dx'. \tag{3.67}
\end{equation*}
where \( s \) is the scalar curvature.
4. Twisted signature operator and its symbol

Let us recall the definition of twisted signature operators. We consider a $n$-dimensional oriented Riemannian manifold $(M,g^M)$. Let $F$ be a real vector bundle over $M$. Let $g^F$ be an Euclidean metric on $F$. Let

\[ \wedge^* (T^*M) = \bigoplus_{i=0}^{n} \wedge^i (T^*M) \]  

be the real exterior algebra bundle of $T^*M$. Let

\[ \Omega^*(M,F) = \bigoplus_{i=0}^{n} \Omega^i(M,F) = \bigoplus_{i=0}^{n} C^\infty (M, \wedge^i (T^*M) \otimes F) \]

be the set of smooth sections of $\wedge^* (T^*M) \otimes F$. Let $\ast$ be the Hodge star operator of $g^{TM}$. It extends on $\wedge^* (T^*M) \otimes F$ by acting on $F$ as identity. Then $\Omega^*(M,F)$ inherits the following standardly induced inner product

\[ \langle \zeta, \eta \rangle = \int_M \langle \zeta \wedge \ast \eta \rangle_F, \quad \zeta, \eta \in \Omega^*(M,F). \]

Let $\nabla^F$ be the non-Euclidean connection on $F$. Let $d^F$ be the obvious extension of $\nabla^F$ on $\Omega^*(M,F)$. Let $\delta^F = d^F \ast$ be the formal adjoint operator of $d^F$ with respect to the inner product. Let $\bar{D}^F$ be the differential operator acting on $\Omega^*(M,F)$ defined by

\[ \bar{D}^F = d^F + \delta^F. \]

Then $\nabla^{F,e}$ is an Euclidean connection on $(F,g^F)$. Let $\nabla^{\wedge^* (T^*M)}$ be the Euclidean connection on $\wedge^* (T^*M)$ induced canonically by the Levi-Civita connection $\nabla^{TM}$ of $g^{TM}$. Let $\nabla^e$ be the Euclidean connection on $\wedge^* (T^*M) \otimes F$ obtained from the tensor product of $\nabla^{\wedge^* (T^*M)}$ and $\nabla^{F,e}$. Let $\{e_1, \ldots, e_n\}$ be an oriented (local) orthonormal basis of $T^M$. The following result was proved by Proposition in [20].

The following identity holds

\[ d^F + \delta^F = \sum_{i=1}^{n} c(e_i) \nabla^e_{e_i} - \frac{1}{2} \sum_{i=1}^{n} \bar{c}(e_i) \omega(F,g^F)(e_i). \]

Let $D_{\bar{F}} = \sum_{i=1}^{n} c(e_i) \nabla^e_{e_i}$ and $\omega(F,g^F)$ be any element in $\Omega(M,EndF)$, then we define the generalized twisted signature operators $\bar{D}_F$, $\bar{D}_{\bar{F}}$ as follows.

For sections $\psi \otimes \chi \in \wedge^* (T^*M) \otimes F$,

\[ \bar{D}_F(\psi \otimes \chi) = D_{\bar{F}}(\psi \otimes \chi) - \frac{1}{2} \sum_{i=1}^{n} \bar{c}(e_i) \omega(F,g^F)(e_i) (\psi \otimes \chi), \]

\[ \bar{D}_{\bar{F}}(\psi \otimes \chi) = D_{\bar{F}}(\psi \otimes \chi) - \frac{1}{2} \sum_{i=1}^{n} \bar{c}(e_i) \omega^*(F,g^F)(e_i) (\psi \otimes \chi). \]

Here $\omega^*(F,g^F)(e_i)$ denotes the adjoint of $\omega(F,g^F)(e_i)$.

In the local coordinates $\{x_1; 1 \leq i \leq n\}$ and the fixed orthonormal frame $\{\tilde{e}_1, \ldots, \tilde{e}_n\}$, the connection matrix $(\omega_{s,t})$ is defined by

\[ \tilde{\nabla}(\tilde{e}_1, \ldots, \tilde{e}_n) = (\tilde{e}_1, \ldots, \tilde{e}_n)(\omega_{s,t}). \]

Let $M$ be a 6-dimensional compact oriented Riemannian manifold with boundary $\partial M$. We define that $\bar{D}_F: C^\infty (M, \wedge^* (T^*M) \otimes F) \rightarrow C^\infty (M, \wedge^* (T^*M) \otimes F)$ is the generalized twisted signature operator. Take
the coordinates and the orthonormal frame as in Section 3. Let \( \epsilon(e_j^*) \), \( \iota(e_j^*) \) be the exterior and interior multiplications respectively. Write

\[
\epsilon(\hat{e}_j) = \epsilon(e_j^*) - \iota(e_j^*); \quad \hat{\epsilon}(\hat{e}_j) = \epsilon(e_j^*) + \iota(e_j^*),
\]

(4.10)

We’ll compute \( \text{tr}_{\Lambda^r(T^*M) \otimes F} \) in the frame \( \{ e_{i_1}^* \wedge \cdots \wedge e_{i_k}^* \mid 1 \leq i_1 < \cdots < i_k \leq 6 \} \). By (3.2) and (4.8) in [12], we have

\[
\hat{D}_F = \sum_{i=1}^{n} c(e_i) \nabla e_i - \frac{1}{2} \sum_{i=1}^{n} \hat{c}(e_i) \omega(F, g^F)(e_i)
\]

\[
= \sum_{i=1}^{n} c(e_i) \left( \nabla^{\Lambda^n(T^*M)} \otimes id_F + \text{id}_{\Lambda^r(T^*M) \otimes \nabla^{F^c}} \right) - \frac{1}{2} \sum_{i=1}^{n} \hat{c}(e_i) \omega(F, g^F)(e_i)
\]

\[
\hat{D}_F^* = \sum_{i=1}^{n} c(e_i) \left[ \hat{e}_i + \frac{1}{4} \sum_{s,t} \omega_{s,t}(\hat{e}_i)[\hat{c}(\hat{e}_s)\hat{c}(\hat{e}_t) - c(\hat{e}_s)c(\hat{e}_t)] \otimes \text{id}_F + \text{id}_{\Lambda^r(T^*M) \otimes \sigma_i} \right]
\]

\[
- \frac{1}{2} \sum_{i=1}^{n} \hat{c}(e_i) \omega(F, g^F)(e_i),
\]

(4.11)

\[
\sigma_1(\hat{D}_F) = \sigma_1(\hat{D}_F^*) = \sqrt{-1}c(\xi);
\]

(4.13)

\[
\sigma_0(\hat{D}_F) = \sum_{i=1}^{n} c(e_i) \left[ \frac{1}{4} \sum_{s,t} \omega_{s,t}(\hat{e}_i)[\hat{c}(\hat{e}_s)\hat{c}(\hat{e}_t) - c(\hat{e}_s)c(\hat{e}_t)] \otimes \text{id}_F + \text{id}_{\Lambda^r(T^*M) \otimes \sigma_i} \right]
\]

\[
- \frac{1}{2} \sum_{i=1}^{n} \hat{c}(e_i) \omega(F, g^F)(e_i);
\]

(4.14)

\[
\sigma_0(\hat{D}_F^* = \sum_{i=1}^{n} c(e_i) \left[ \frac{1}{4} \sum_{s,t} \omega_{s,t}(\hat{e}_i)[\hat{c}(\hat{e}_s)\hat{c}(\hat{e}_t) - c(\hat{e}_s)c(\hat{e}_t)] \otimes \text{id}_F + \text{id}_{\Lambda^r(T^*M) \otimes \sigma_i} \right]
\]

\[
- \frac{1}{2} \sum_{i=1}^{n} \hat{c}(e_i) \omega(F, g^F)(e_i).
\]

(4.15)

By the composition formula and (2.2.11) in [12], we obtain in [19],

**Lemma 4.1.** Let \( \hat{D}_F, \hat{D}_F^* \) be the twisted signature operators on \( \Gamma(\Lambda^r(T^*M) \otimes F) \), then

By the composition formula of pseudodifferential operators in Section 2.2.1 of [12], we have

**Lemma 4.2.** The symbol of the twisted signature operators \( \hat{D}_F, \hat{D}_F^* \) as follows:

\[
\sigma_{-1}(\hat{D}_F^{-1}) = \sigma_{-1}(\hat{D}_F^*)^{-1} = \frac{\sqrt{-1}c(\xi)}{|\xi|^2};
\]

(4.16)

\[
\sigma_{-2}(\hat{D}_F^{-1}) = \frac{c(\xi)\sigma_0(\hat{D}_F)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} \sum_j c(\text{d}x_j) \left[ \partial_{x_j}(c(\xi))|\xi|^2 - c(\xi)\partial_{x_j}(|\xi|^2) \right];
\]

(4.17)

\[
\sigma_{-2}(\hat{D}_F^*)^{-1} = \frac{c(\xi)\sigma_0(\hat{D}_F^*)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} \sum_j c(\text{d}x_j) \left[ \partial_{x_j}(c(\xi))|\xi|^2 - c(\xi)\partial_{x_j}(|\xi|^2) \right].
\]

(4.18)
Since $\Psi$ is a global form on $\partial M$, so for any fixed point $x_0 \in \partial M$, we can choose the normal coordinates $U$ of $x_0$ in $\partial M$ (not in $M$) and compute $\Psi(x_0)$ in the coordinates $\tilde{U} = U \times [0, 1)$ and the metric $\frac{1}{h(x_n)}g_{\partial M} + dx_n^2$.

The dual metric of $g_{\partial M}$ on $\tilde{U}$ is $\frac{1}{h(x_n)}g_{\partial M} + dx_n^2$. Write $g^M_{ij} = g^M(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$; $\tilde{g}^M_{ij} = g^M(dx_i, dx_j)$, then
\[
\begin{bmatrix}
\frac{1}{h(x_n)}g^M_{ij} & 0 \\
0 & 1
\end{bmatrix}; \quad \begin{bmatrix}
\frac{1}{h(x_n)}g^M_{ij} & 0 \\
0 & 1
\end{bmatrix},
\]
(4.19)
and
\[
\partial x_i, g^M_{ij}(x_0) = 0, \quad 1 \leq i, j \leq n - 1; \quad g^M_{ij}(x_0) = \delta_{ij}.
\]
(4.20)

Let $\{e_1, \ldots, e_{n-1}\}$ be an orthonormal field in $U$ about $g_{\partial M}$ which is parallel along geodesics and
\[
e_i = \frac{\partial}{\partial x_i}(x_0),
\]
then $\tilde{e}_1 = \sqrt{h(x_n)}e_1, \ldots, \tilde{e}_{n-1} = \sqrt{h(x_n)}e_{n-1}, \tilde{e}_n = dx_n$ is the orthonormal frame field in $\tilde{U}$ about $g_{\partial M}$. Locally $\Lambda^\ast(T^*M)|\tilde{U} \cong \tilde{U} \times \Lambda^\ast_{\tilde{\mathbb{C}}}(\mathbb{T})$. Let $\{f_1, \ldots, f_n\}$ be the orthonormal basis of $\Lambda^\ast_{\tilde{\mathbb{C}}}(\mathbb{T})$.

Take a spin frame field $\sigma : \tilde{U} \to \text{Spin}(M)$ such that $\pi\sigma = \{\tilde{e}_1, \ldots, \tilde{e}_n\}$ where $\pi : \text{Spin}(M) \to O(M)$ is a double covering, then $\{[\sigma, f_i], 1 \leq i \leq 4\}$ is an orthonormal frame of $\Lambda^\ast(T^*M)|\tilde{U}$. In the following, since the global form $\Psi$ is independent of the choice of the local frame, so we can compute $\text{tr}_{\Lambda^\ast(\text{Spin}(M))}$ in the frame $\{[\sigma, f_i], 1 \leq i \leq 4\}$. Let $\{E_1, \ldots, E_n\}$ be the canonical basis of $\mathbb{R}^n$ and $c(E_i) \in \mathfrak{d}_{C}(n) \cong \text{Hom}(\Lambda^\ast_{C}(\mathbb{T}), \Lambda^\ast_{\tilde{\mathbb{C}}}(\mathbb{T}))$ be the Clifford action. By [12], then
\[
c(\tilde{e}_i) = [(\sigma, c(E_i))]; \quad c(\tilde{e}_i)(\sigma, f_i) = [\sigma, (c(E_i))f_i]; \quad \frac{\partial}{\partial x_i} = [(\sigma, \partial_{\tilde{\mathbb{C}}})],
\]
(4.21)
then we have $\frac{\partial}{\partial x_i}c(\tilde{e}_i) = 0$ in the above frame. By Lemma 2.2 in [12], we have

\textbf{Lemma 4.3.}

\[
\begin{aligned}
\partial_{x_i}(|\xi|^2_{g_{\partial M}})(x_0) &= \begin{cases} 0, & \text{if } j < n; \\ h'(0)|\xi|^2_{g_{\partial M},} & \text{if } j = n. \end{cases} \\
\partial_{x_j}[c(\xi)](x_0) &= \begin{cases} 0, & \text{if } j < n; \\ \partial x_i(c(\xi'))(x_0), & \text{if } j = n. \end{cases}
\end{aligned}
\]
(4.22)
(4.23)

where $\xi = \xi' + \xi_n dx_n$.

Then an application of Lemma 2.3 in [12] shows

\textbf{Lemma 4.4.} The symbol of the twisted signature operators $\tilde{D}_F, \tilde{D}_F$ as follows:

\[
\sigma_0(\tilde{D}_F) = -\frac{5}{4} h'(0)c(dx_n) + \frac{1}{4} h'(0) \sum_{i=1}^{n-1} c(\tilde{e}_i)c(\tilde{c}_n)c(\tilde{c}_i)(x_0) \otimes \text{id}_F
\]
\[+ \sum_{i=1}^{n} c(\tilde{e}_i)\sigma_i^{F, c} - \frac{1}{2} \sum_{i=1}^{n} \tilde{c}(e_i)\omega(f, g^F)(e_i);
\]
(4.24)
\[
\sigma_0(\tilde{D}_F) = -\frac{5}{4} h'(0)c(dx_n) + \frac{1}{4} h'(0) \sum_{i=1}^{n-1} c(\tilde{e}_i)c(\tilde{c}_n)c(\tilde{c}_i)(x_0) \otimes \text{id}_F
\]
\[+ \sum_{i=1}^{n} c(\tilde{e}_i)\sigma_i^{F, c} - \frac{1}{2} \sum_{i=1}^{n} \tilde{c}(e_i)\omega(f, g^F)(e_i).
\]
(4.25)
We write

\[
\theta := -\frac{5}{4} h'(0) c(dx_n) + \frac{1}{4} h'(0) \sum_{i=1}^{n-1} c(\tilde{e}_i) \tilde{c}(\tilde{e}_i) (x_0) \otimes i d_F := -\frac{5}{4} h'(0) c(dx_n) + m;
\]

\[
\vartheta^* := \sum_{i=1}^n c(\tilde{e}_i) \sigma^{F,e}_i - \frac{1}{2} \sum_{i=1}^n \tilde{c}(e_i) \omega^*(F, g^F)(e_i);
\]

\[
\vartheta := \sum_{i=1}^n c(\tilde{e}_i) \sigma^{F,e}_i - \frac{1}{2} \sum_{i=1}^n \tilde{c}(e_i) \omega(F, g^F)(e_i).
\]

Let \( \tilde{c}(\omega) = \sum_i c(e_i) \omega(F, g^F)(e_i) \) and \( \tilde{c}(\omega^*) = \sum_i c(e_i) \omega^*(F, g^F)(e_i) \), then similar to (2.20), we have

\[
\hat{D}_F \hat{D}^*_F = -g^{ij} \partial_i \partial_j - 2 \sigma^{j,(T^*M)\otimes F}_i \partial_j + \Gamma^k \partial_k - \frac{1}{2} \sum_j \left( \tilde{c}(\omega) c(e_j) + c(e_j) \tilde{c}(\omega^*) \right) e_j
\]

\[
- g^{ij} \left[ (\partial_i \sigma^{\lambda,(T^*M)\otimes F}_j) + \sigma^{\lambda,(T^*M)\otimes F} \sigma^{j,(T^*M)\otimes F,e} - \Gamma^k_{ij} \sigma^{\lambda,(T^*M)\otimes F} \right]
\]

\[
- \frac{1}{2} \sum_j \tilde{c}(\omega) c(e_j) \sigma^{\lambda,(T^*M)\otimes F,e}_j - \frac{1}{2} \sum_j c(e_j) e_j (\tilde{c}(\omega^*)) + \frac{1}{4} s + \frac{1}{4} \tilde{c}(\omega) \tilde{c}(\omega^*)
\]

\[
- \frac{1}{2} \sum_j c(e_j) \sigma^{\lambda,(T^*M)\otimes F,e}_j (\omega^*) + \frac{1}{2} \sum_{i \neq j} R^{F,e}(e_i, e_j) c(e_i) c(e_j).
\]

where \( s \) is the scalar curvature, \( R^{F,e} \) denotes the curvature-tensor on \( F \).
Combining (4.9) and (4.10), we have

\[
\hat{D}_F^*D_FD_F = \sum_{i=1}^{n} c(e_i)(e_i, dx_i)(-g^{ij}\partial_i\partial_j) + \sum_{i=1}^{n} c(e_i)(e_i, dx_i) \left\{ - (\partial_i g^{ij})\partial_i\partial_j - g^{ij}(4\sigma_i^{\Lambda^*(T^*M)\otimes F}\partial_j - 2\Gamma_{ij}^{k}\partial_k) \partial_i \right\} \\
+\left(\theta + \vartheta^*\right)(-g^{ij}\partial_i\partial_j) - \frac{1}{2} \sum_{i=1}^{n} c(e_i)(e_i, dx_i) \left\{ 2 \sum_{j,k} \left[ \hat{(c)}(w)c(e_j) + (c(e_j)\hat{(c)}(w^*)) \right](e_j, dx^k) \right\} \times \partial_k \\
+\frac{n}{2} \sum_{i=1}^{n} c(e_i)(e_i, dx_i)\partial_i \left\{ - g^{ij} \left[ (\partial_i \sigma_i^{\Lambda^*(T^*M)\otimes F}) + \sigma_i^{\Lambda^*(T^*M)\otimes F} \sigma_i^{\Lambda^*(T^*M)\otimes F} + \Gamma_{ij}^{k} \sigma_i^{\Lambda^*(T^*M)\otimes F} \right] + \frac{1}{4} \hat{c}(\omega) \hat{c}(\omega^*) \right\} \\
- \frac{1}{2} \sum_{i=1}^{n} \hat{c}(\omega)c(e_j)\sigma_j^{\Lambda^*(T^*M)\otimes F,e} - \frac{1}{2} \sum_{i=j} c(e_j)(e_j, \hat{(c)}(\omega^*)) + \frac{1}{2} \sum_{i,j} R_{F,e}(e_i, e_j) c(e_i)c(e_j) + \frac{1}{4} \hat{c}(\omega) \hat{c}(\omega^*) \\
- \frac{1}{2} \sum_{i,j} c(e_j)\sigma_j^{\Lambda^*(T^*M)\otimes F,e} \hat{(c)}(\omega^*) \left\{ - 2 \sigma_i^{\Lambda^*(T^*M)\otimes F} \partial_j + \Gamma_{ik}^{k}\partial_k - \frac{1}{2} \sum_{j} \hat{c}(\omega)c(e_j) + c(e_j) \right\} \\
\times \hat{c}(\omega^*) - g^{ij} \left[ (\partial_i \sigma_i^{\Lambda^*(T^*M)\otimes F}) + \sigma_i^{\Lambda^*(T^*M)\otimes F} \sigma_i^{\Lambda^*(T^*M)\otimes F} + \Gamma_{ij}^{k} \sigma_i^{\Lambda^*(T^*M)\otimes F} \right] + \frac{1}{4} \hat{c}(\omega) \hat{c}(\omega^*) \\
+ \frac{1}{2} \sum_{j} c(e_j)(e_i, dx_i)\partial_i \left\{ - g^{ij} \left[ (\partial_i \sigma_i^{\Lambda^*(T^*M)\otimes F}) + \sigma_i^{\Lambda^*(T^*M)\otimes F} \sigma_i^{\Lambda^*(T^*M)\otimes F} + \Gamma_{ij}^{k} \sigma_i^{\Lambda^*(T^*M)\otimes F} \right] + \frac{1}{4} \hat{c}(\omega) \hat{c}(\omega^*) \right\} \\
\times \hat{c}(\omega^*) - \frac{1}{2} \sum_{j} c(e_j)\sigma_j^{\Lambda^*(T^*M)\otimes F,e} \hat{c}(\omega^*) + \frac{1}{2} \sum_{i,j} R_{F,e}(e_i, e_j) c(e_i)c(e_j) \right\} \times \partial_k \\
\times (\hat{(c)}(w)c(e_j) + (c(e_j)\hat{(c)}(w^*)) \left\{ \partial_k(e_j, dx^k) \right\} \right\}. \quad (4.28)
\]

By the above composition formulas, then we obtain:

**Lemma 4.5.** Let \(\hat{D}_F^*, D_F^*\) be the twisted signature operators on \(\Gamma(\Lambda^*(T^*M) \otimes F)\), then

\[
\sigma_3(\hat{D}_F^* D_F D_F^*) = \sqrt{-1} c(\xi)|\xi|^2; \quad (4.29)
\]

\[
\sigma_2(\hat{D}_F^* D_F D_F^*) = \sigma_2(D^3) + |\xi|^2 |\theta^*| + |c(\xi)\hat{c}(w)c(\xi) - |\xi|^2\hat{c}(w^*)|, \quad (4.30)
\]

where,

\[
\sigma_2(D^3) = c(\xi)(4\sigma^k - 2\Gamma^k)\xi_k - \frac{1}{4} |\xi|^2 h'(0)c(dx_n). \quad (4.31)
\]

\[
p = \frac{1}{4} h'(0) \sum_{i=1}^{5} \hat{(c)}(\xi_i)\hat{(c)}(\xi_i)(\xi_i(x_0)). \quad (4.32)
\]

Write

\[
D_x^* = (-\sqrt{-1})^{a_j}e_j^*; \quad \sigma(\hat{D}_F^* D_F D_F^*) = p_3 + p_2 + p_1 + p_0; \quad \sigma((\hat{D}_F^* D_F D_F^*)^{-1}) = \sum_{j=3}^{\infty} q_{-j}. \quad (4.33)
\]
By the composition formula of pseudodifferential operators, we have

\[
1 = \sigma((\hat{D}_F^\ast \hat{D}_F \hat{D}_F)^{-1}) = (p_3 + p_2 + p_1 + p_0)(q_{-3} + q_{-4} + q_{-5} + \cdots)
+ \sum_j (\partial_{\xi_j} p_3 + \partial_{\xi_j} p_2 + \partial_{\xi_j} p_1 + \partial_{\xi_j} p_0) (D_{x_j} q_{-3} + D_{x_j} q_{-4} + D_{x_j} q_{-5} + \cdots)
= p_3 q_{-3} + (p_3 q_{-4} + p_2 q_{-4} + \sum_j \partial_{\xi_j} p_3 D_{x_j} q_{-3}) + \cdots.
\]

(4.34)

Then

\[
q_{-3} = p_3^{-1}; \quad q_{-4} = -p_3^{-1}[p_2 p_3^{-1} + \sum_j \partial_{\xi_j} p_3 D_{x_j} (p_3^{-1})].
\]

(4.35)

By Lemma 2.1 in [12] and (4.30)-(4.36), we obtain

\[\tag{Lemma 4.6}
\]

Let \( \hat{D}_F^\ast \hat{D}_F \) be the generalized twisted signature operators on \( \Gamma(\wedge^*(T^*M) \otimes F) \), then

\[
\sigma_{-3}((\hat{D}_F^\ast \hat{D}_F \hat{D}_F)^{-1}) = \frac{\sqrt{-1} c(\xi)}{|\xi|^4}; \quad (4.36)
\]

\[
\sigma_{-4}((\hat{D}_F^\ast \hat{D}_F \hat{D}_F)^{-1}) = \sigma_{-4}(D^{-3}) + \frac{c(\xi) p c(\xi)}{|\xi|^6} + \frac{c(\xi) \tilde{c} c(\xi)}{|\xi|^6} - \frac{c(\xi) \tilde{c}(\omega^*) c(\xi)}{|\xi|^6} + \frac{\tilde{c}(\omega)}{|\xi|^4}. \quad (4.37)
\]

where

\[
\sigma_{-4}(D^{-3}) = \frac{c(\xi) \sigma_2(D^3) c(\xi)}{|\xi|^8} + \frac{c(\xi)}{|\xi|^{10}} \sum \left[ c(dx_j)|\xi|^2 + 2\tilde{c}(\xi) \left[ \partial_{x_j} c(\xi) \right] |\xi|^2 - 2c(\xi) \partial_{x_j} (|\xi|^2) \right]. \quad (4.38)
\]

Hence we conclude that

\[\tag{Theorem 4.7}
\]

For even \( n \)-dimensional oriented compact Riemannian manifolds without boundary, the following equality holds:

\[
Wres(\hat{D}_F^\ast \hat{D}_F) = \frac{(2\pi)^{\frac{n+2}{2}}}{(4\pi^2 - 2)!} \int_M \text{Tr} \left[ -\frac{s}{12} + \frac{n}{16} \left[ \delta(\omega^*) - \delta(\omega) \right]^2 + \frac{1}{4} \delta(\omega^*) \delta(\omega) - \frac{1}{4} \sum_j \nabla^F_{x_j} (\delta(\omega^*) c(e_j) + \frac{1}{4} \sum_j c(e_j) \nabla^F_{x_j} \delta(\omega)) \right] \text{vol} M. \quad (4.39)
\]

5. A Kastler-Kalau-Walze theorem for six-dimensional Riemannian manifolds with boundary associated to twisted Signature Operators

In this section, we shall prove a Kastler-Kalau-Walze type formula for \( \hat{D}_F^\ast \hat{D}_F \). An application of (2.1.4) in [14] shows that

\[
\widetilde{Wres}[\pi^+(\hat{D}_F^{-1}) \circ \pi^+((\hat{D}_F^\ast \hat{D}_F \hat{D}_F)^{-1})] = \int_M \int_{|\xi|=1} \text{Tr} \left[ \sigma_{-n}(\sigma_{-n}((\hat{D}_F^\ast \hat{D}_F \hat{D}_F)^{-1})) \sigma(\xi) dx + \int_{\partial M} \Psi \right]. \quad (5.1)
\]

where

\[
\Psi = \int_{|\xi|=1} \int_{|\xi|=1}^{+\infty} \sum_{j,k=0}^\infty \frac{(-i)^{\alpha+j+k+\ell}}{\alpha! (j+k+1)!} \text{Tr} \left[ \partial_{\xi_j}^{\ell} \partial_{\xi_k}^{l} \sigma_{-n}((\hat{D}_F^{-1})(x', 0, \xi', \ell)) \right] d\xi_n \sigma(\xi') dx', \quad (5.2)
\]
and the sum is taken over \( r - k + \vert \alpha \vert + \ell - j - 1 = -n, r \leq -1, \ell \leq -1 \).

Locally we can use Theorem 4.3 [19] to compute the interior term of (5.1), then

\[
\int_M \int_{\vert \xi \vert = 1} \text{trace}_{\pi_\ast \tilde{T} \ast \tilde{M}} |(\sigma_{-4}(\hat{D}_{\kappa}^\ast \tilde{D}_{\kappa}^F)^{-2})| \sigma(\xi) d\xi
\]

\[
= 8\pi^3 \int_M \text{Tr} \left[ \frac{8}{12} + \frac{3}{8} [\hat{c}(\omega^*) - \hat{c}(\omega)]^2 - \frac{1}{4} \hat{c}(\omega^*) \hat{c}(\omega) - \frac{1}{2} \sum_j \nabla_{\xi_j} (\hat{c}(\omega^*)) \alpha(e_j) + \frac{1}{4} \sum_j \alpha(e_j) \nabla^{\tilde{F}}_{\xi_j} (\hat{c}(\omega^*)) \right] d\text{vol}_M. \tag{5.3}
\]

So we only need to compute \( \int_{D_M} \Psi \). From the remark above, now we can compute \( \Psi \) (see formula (3.6) for the definition of \( \Psi \)). Since the sum is taken over \( r + \ell - k - j - \vert \alpha \vert - 1 = -6, r \leq -1, \ell \leq -3 \), then we have the \( \int_{D_M} \Psi \) is the sum of the following five cases:

**case a) I)** \( r = -1, l = -3, j = k = 0, \vert \alpha \vert = 1 \).

By (5.2), we get

**case a) I)** \( - \int_{\vert \xi \vert = 1} \int_{-\infty}^{+\infty} \sum_{\vert \alpha \vert = 1} \text{trace} \left[ \partial_{n_\xi} \pi_{\ast \xi} \sigma_{-1}(\hat{D}_{\kappa}^\ast)^\dagger \times \partial_{n_\xi} \partial_{n_\xi} \sigma_{-3}(\hat{D}_{\kappa}^\ast \tilde{D}_{\kappa}^F \tilde{D}_{\kappa}^\ast)^{-1}) \right] (\xi_0) d\xi_n \sigma(\xi') d\xi'. \tag{5.4}
\]

By Lemma 2.2 in [12], for \( i < n \) we have

\[
\partial_{n_\xi} \sigma_{-3}(\hat{D}_{\kappa}^\ast \tilde{D}_{\kappa}^F \tilde{D}_{\kappa}^\ast)^{-1}) (\xi_0) = \partial_{n_\xi} \left[ \frac{ic(\xi')}{\vert \xi \vert} \right] (\xi_0) = i \left[ \partial_{n_\xi} [c(\xi)] \vert \xi \vert^{-4}(\xi_0) - 2c(\xi) \partial_{n_\xi} [\vert \xi \vert^2] \vert \xi \vert^{-6}(\xi_0) \right] = 0. \tag{5.5}
\]

so **case a) I)** vanishes.

**case a) II)** \( r = -1, l = -3, \vert \alpha \vert = k = 0, j = 1 \).

By (5.2), we have

**case a) II)** \( - \int_{\vert \xi \vert = 1} \int_{-\infty}^{+\infty} \text{trace} \left[ \partial_{n_\xi} \pi_{\ast \xi} \sigma_{-1}(\hat{D}_{\kappa}^\ast)^\dagger \times \partial_{n_\xi} \partial_{n_\xi} \sigma_{-3}(\hat{D}_{\kappa}^\ast \tilde{D}_{\kappa}^F \tilde{D}_{\kappa}^\ast)^{-1}) \right] (\xi_0) d\xi_n \sigma(\xi') d\xi'. \tag{5.6}
\]

By Lemma 2.2 in [12], we have

\[
\pi_{\ast \xi} \partial_{n_\xi} \sigma_{-1}(\hat{D}_{\kappa}^\ast)^\dagger (\xi_0) \vert \xi \vert = 1 = \frac{\partial_{n_\xi} [c(\xi') \vert \xi \vert]}{2(\xi_n - i)} + \frac{ic(\xi')}{4(\xi_n - i)^2}. \tag{5.7}
\]

By direct calculations we have

\[
\partial_{n_\xi} \sigma_{-3}(\hat{D}_{\kappa}^\ast \tilde{D}_{\kappa}^F \tilde{D}_{\kappa}^\ast)^{-1}) \right] (\xi_0) = i \left[ \frac{(20\xi_n^2 - 4)c(\xi') + 12(\xi_n^2 - \xi_n)c(\xi_n)}{(1 + \xi_n^2)} \right]. \tag{5.8}
\]

By (5.7) and (5.8), we obtain

\[
\text{trace} \left[ \partial_{n_\xi} \pi_{\ast \xi} \sigma_{-1}(\hat{D}_{\kappa}^\ast)^\dagger \times \partial_{n_\xi} \sigma_{-3}(\hat{D}_{\kappa}^\ast \tilde{D}_{\kappa}^F \tilde{D}_{\kappa}^\ast)^{-1}) \right] (\xi_0) = 8h'(0) dim F \frac{8 - 24\xi_n i + 40\xi_n^2 + 24i\xi_n^3}{(\xi_n - i)^6(\xi_n + i)^4}. \tag{5.9}
\]

Then we obtain

**case a) II)** \( - \frac{1}{2} \int_{\vert \xi \vert = 1} \int_{-\infty}^{+\infty} 8h'(0) dim F \frac{8 - 24\xi_n i + 40\xi_n^2 + 24i\xi_n^3}{(\xi_n - i)^6(\xi_n + i)^4} d\xi_n \sigma(\xi') d\xi'. \)

\[
= 8h'(0) dim F \frac{\pi i}{5!} \left[ \frac{8 + 24\xi_n i - 40\xi_n^2 - 24i\xi_n^3}{(\xi_n + i)^4} \right] \bigg|_{\xi_n = i} d\xi' \tag{5.10}
\]

\[
= \frac{15}{2} \pi h'(0) \Omega_4 dim F d\xi'.
\]
where $\Omega_4$ is the canonical volume of $S_4$.

**case a) III**) $r = -1, l = -3, |\alpha| = j = k = 1.$

By (5.2) and an integration by parts, we have

$$
\text{case a) III} = -\frac{1}{2} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\hat{D}_F^{-1}) \times \partial_{\xi_n} \partial_{\xi_n} \sigma_{-3}((\hat{D}_F^* \hat{D}_F \hat{D}_F^*)^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx'.
$$

By Lemma 2.2 in [12], we have

$$
\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\hat{D}_F^{-1})(x_0)||_{|\xi'|=1} = \frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)^2}.
$$

By (4.37) and direct calculations, we have

$$
\partial_{\xi_n} \partial_{\xi_n} \sigma_{-3}((\hat{D}_F^* \hat{D}_F \hat{D}_F^*)^{-1}) = -\frac{4\xi_0 \partial_{\xi_n} c(\xi')(x_0) + i(12\xi_0^2 + 2(2 - 10\xi_0^2)h(0)c(dx_n))}{(1 + \xi_n^2)^4}.
$$

Combining (5.12) and (5.13), we have

$$
\text{trace} \left[ \partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\hat{D}_F^{-1}) \times \partial_{\xi_n} \partial_{\xi_n} \sigma_{-3}((\hat{D}_F^* \hat{D}_F \hat{D}_F^*)^{-1}) \right] (x_0)||_{|\xi'|=1} = 8h'(0)dimF \frac{8i - 32\xi_n^2 - 8i\xi_n^2}{(\xi_n - i)^4(\xi + i)^4}.
$$

Then

$$
\text{case a) III} = -\frac{1}{2} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} 8h'(0)dimF \frac{8i - 32\xi_n^2 - 8i\xi_n^2}{(\xi_n - i)^4(\xi + i)^4} d\xi_n \sigma(\xi') dx'
$$

$$
= -8h'(0)dimF \Omega_4 \pi \left[ \frac{8i - 32\xi_n^2 - 8i\xi_n^2}{(\xi + i)^4} \right]_{|\xi'|=1}^{(4)} dx'
$$

$$
= \frac{25}{2} \pi h'(0) \Omega_4 \text{dim} F dx'
$$

(5.15)

where $\Omega_4$ is the canonical volume of $S_4$.

**case b) $r = -2, l = -3, |\alpha| = j = k = 0.$**

By (5.2) and an integration by parts, we have

$$
\text{case c)} = -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \pi_{\xi_n}^+ \sigma_{-2}(\hat{D}_F^{-1}) \times \partial_{\xi_n} \sigma_{-3}((\hat{D}_F^* \hat{D}_F \hat{D}_F^*)^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx'.
$$

Then an application of Lemma 4.3 shows

$$
\sigma_{-2}(\hat{D}_F^{-1})(x_0) = \frac{c(\xi)\sigma_0(\hat{D}_F)(x_0)c(\xi)}{|\xi'|^4} + \frac{c(\xi)}{|\xi'|^6} \sum_j c(dx_j) \left[ \partial_{\xi_j}(c(\xi))(\xi)|^2 - c(\xi)\partial_{\xi_j}(\xi)|^2 \right] (x_0)
$$

$$
= \frac{c(\xi)\sigma_0(\hat{D}_F)(x_0)c(\xi)}{|\xi'|^4} + \frac{c(\xi)}{|\xi'|^6} c(dx_n) \left[ \partial_{\xi_n}(c(\xi))(x_0) - c(\xi)h'(0)|\xi'|^2 \right]_{|\xi'|=0}.
$$

(5.17)

Hence,

$$
\pi_{\xi_n}^+ \sigma_{-2}((\hat{D}_F)^{-1})(x_0) := B_1 + B_2 + B_3 + B_4,
$$

(5.18)
where

\[
B_1 = -\frac{1}{4(\xi_n - i)^2} \left[ (2 + i\xi_n)c(\xi')(-\frac{5}{4}h'(0)c(dx_n))c(\xi') + i\xi_n c(dx_n)(-\frac{5}{4}h'(0)c(dx_n))c(dx_n) + (2 + i\xi_n)c(\xi')c(dx_n)\partial_{x_n} c(\xi') + ic(dx_n)(-\frac{5}{4}h'(0)c(dx_n))c(\xi') - i\partial_{x_n} c(\xi') \right]
\]

\[
B_2 = -\frac{1}{4(\xi_n - i)^2} \left[ \frac{5}{2}h'(0)c(dx_n) - \frac{5i}{2}h'(0)c(\xi') - (2 + i\xi_n)c(\xi')c(dx_n)\partial_{x_n} c(\xi') + i\partial_{x_n} c(\xi') \right];
\]

\[
B_3 = -\frac{1}{4(\xi_n - i)^2} \left[ (2 + i\xi_n)c(\xi')c(\xi') + i\xi_n c(dx_n)pc(dx_n) + (2 + i\xi_n)c(\xi')c(dx_n)\partial_{x_n} c(\xi') + ic(dx_n)pc(\xi') + ic(\xi')pc(dx_n) - i\partial_{x_n} c(\xi') \right]
\]

\[
B_4 = -\frac{1}{4(\xi_n - i)^2} \left[ (2 + i\xi_n)c(\xi')pc(\xi') + i\xi_n c(dx_n)pc(dx_n) + ic(dx_n)pc(\xi') + ic(\xi')pc(dx_n) \right]
\]

On the other hand,

\[
\partial_{\xi_n} \sigma_{-3}((\tilde{D}^\nu_F \tilde{D} F \tilde{D}^\nu_F)^{-1}) = -\frac{4i\xi_n c(\xi')}{(1 + \xi_n^2)^3} + \frac{i(1 - 3\xi_n^2)}{(1 + \xi_n^2)^3}.
\]

From (5.19) and (5.24), we have

\[
\operatorname{tr}[B_1 \times \partial_{\xi_n} \sigma_{-3}((\tilde{D}^\nu_F \tilde{D} F \tilde{D}^\nu_F)^{-1})(x_0)]|_{\xi'|=1}
= \operatorname{tr}\left\{ \frac{1}{4(\xi_n - i)^2} \left[ \frac{5}{2}h'(0)c(dx_n) - \frac{5i}{2}h'(0)c(\xi') - (2 + i\xi_n)c(\xi')c(dx_n)\partial_{x_n} c(\xi') + i\partial_{x_n} c(\xi') \right] \right\}
\times \frac{8h'(0)}{(\xi_n - i)^2}\frac{3 + 12i\xi_n + 3\xi_n^2}{(\xi_n + i)^3}.
\]

Similarly, we obtain

\[
\operatorname{tr}[B_2 \times \partial_{\xi_n} \sigma_{-3}((\tilde{D}^\nu_F \tilde{D} F \tilde{D}^\nu_F)^{-1})(x_0)]|_{\xi'|=1}
= \operatorname{tr}\left\{ -\frac{h'(0)}{2} \left[ \frac{c(dx_n)}{4i(\xi_n - i)} + \frac{c(dx_n) - ic(\xi')}{8(\xi_n - i)^2} + \frac{3\xi_n - 7i}{8(\xi_n - i)^3}[ic(\xi') - c(dx_n)] \right] \right\}
\times \frac{4i\xi_n c(\xi') + (i - 3i\xi_n^2) c(dx_n)}{(1 + \xi_n^2)^3}
\]

\[
= -8h'(0) \frac{4i - 11\xi_n - 6i\xi_n^2 + 3\xi_n^3}{(\xi_n - i)^2(\xi_n + i)^3}.
\]

For the signature operator case,

\[
\operatorname{tr}[c(\xi')pc(\xi')c(dx_n)](x_0) = \operatorname{tr}[pc(\xi')c(dx_n)c(\xi')](x_0) = |\xi'|^2 \operatorname{tr}[pc(dx_n)],
\]

and

\[
c(dx_n)p(x_0) = -\frac{1}{4}h'(0) \sum_{i=1}^{n-1} c(\bar{e}_i) \bar{c}(\bar{e}_i) c(\bar{e}_n) \bar{c}(\bar{e}_n)
= -\frac{1}{4}h'(0) \sum_{i=1}^{n-1} [c(\bar{e}_i) \bar{c}(\bar{e}_i) - \bar{c}(\bar{e}_i) c(\bar{e}_i)][c(\bar{e}_n) \bar{c}(\bar{e}_n) - \bar{c}(\bar{e}_n) c(\bar{e}_n)].
\]
By Section 3 in [12], then

$$\text{tr}_{\Lambda \cdot (T^*M)} \{ [e(e_i*)u(e_i*) - u(e_i*)e(e_i*)][e(e_n*)u(e_n*) - u(e_n*)e(e_n*)] \}$$

$$= a_{n,m} \langle e_i*, e_n* \rangle^2 + b_{n,m} |e_i*|^2 |e_n*|^2 = b_{n,m}. \quad (5.28)$$

where $b_{n,m} = \left( \frac{4}{m - 2} \right) + \left( \frac{4}{m} \right) - 2 \left( \frac{4}{m - 1} \right)$.

Then

$$\text{tr}_{\Lambda \cdot (T^*M)} \{ [e(e_i*)u(e_i*) - u(e_i*)e(e_i*)][e(e_n*)u(e_n*) - u(e_n*)e(e_n*)] \} = \sum_{m=0}^{6} b_{6,m} = 0. \quad (5.29)$$

Hence in this case,

$$\text{tr}_{\Lambda \cdot (T^*M)} [e(dx_n) p(x_0)] = 0. \quad (5.30)$$

We note that $\int_{|\xi|=1} \xi_1 \cdots \xi_{2q+1} \sigma(\xi') = 0$, then $\text{tr}_{\Lambda \cdot (T^*M)} [e(\xi') p(x_0)]$ has no contribution for computing Case (b).

So, we obtain

$$\text{tr} [B_3 \times \partial_{\xi_n} \sigma_{-3} ((\hat{D}_F \hat{D}_F \hat{D}_F)^{-1}) (x_0)]_{|\xi|=1}$$

$$= \text{tr} \left\{ -\frac{1}{4(\xi_n - i)^2} \left[ (2 + i\xi_n)c(\xi') P_1 c(\xi') + i\xi_n c(dx_n) P_1 c(dx_n) + (2 + i\xi_n)c(\xi') P_1 c(dx_n) \right] \right\} \times \frac{-4i\xi_n c(\xi') + (i - 3i\xi_n^2)c(dx_n)}{(1 + \xi_n^2)^3}$$

$$= 8h'(0) \text{dim} F \frac{3\xi_n^2 - 3i\xi_n - 2}{(\xi_n - i)^2(\xi_n + i)^3}. \quad (5.31)$$

Then, we have

$$\text{trace} [B_3 + B_2 + B_3] \times \partial_{\xi_n} \sigma_{-3} ((\hat{D}_F \hat{D}_F \hat{D}_F)^{-1}) (x_0) dx_n \sigma(\xi') dx'$$

$$= 8h'(0) \text{dim} F \frac{3\xi_n^2 + 9\xi_n^2 + 21\xi_n - 5i}{(\xi_n - i)^2(\xi_n + i)^3}. \quad (5.32)$$

By the relation of the Clifford action and $\text{tr} AB = \text{tr} BA$, then we have the equalities

$$\text{tr} [c(\xi') c(dx_n)] = 0, i < n; \text{tr} [c(\xi') c(dx_n)] = -64 \text{dim} F, i = n; \text{tr} [c(\xi') c(dx_n)] = \text{tr} [c(\xi') c(dx_n)] = 0. \quad (5.33)$$

Then $\text{tr} [\partial c(\xi')]$ has no contribution for computing Case b.

Then, we have

$$\text{trace} [B_3 + \partial_{\xi_n} \sigma_{-3} ((\hat{D}_F \hat{D}_F \hat{D}_F)^{-1})]_{|\xi|=1}$$

$$= \text{trace} \left\{ -\frac{1}{4(\xi_n - i)^2} \left[ (2 + i\xi_n)c(\xi') \beta_1 c(\xi') + i\xi_n c(dx_n) \beta_1 c(dx_n) + ic(dx_n) \beta_1 c(\xi') + ic(\xi') \beta_1 c(dx_n) \right] \right\} \times \frac{-4i\xi_n c(\xi') + (i - 3i\xi_n^2)c(dx_n)}{(1 + \xi_n^2)^3}$$

$$= \frac{i(3\xi_n - i)}{2(\xi_n - i)^2(\xi_n + i)^3} \text{trace}[c(dx_n) \beta_1]$$

$$= -32 \text{dim} F \frac{1 + 3i\xi_n}{(\xi_n - i)^2(\xi_n + i)^3} \text{trace}[\sigma_{F,n}] \quad (5.34)$$
From (5.33), we obtain
\[
-i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}\left[ (B_1 + B_2 + B_3) \times \partial_{\xi_n} \sigma_{-3}((\hat{D}_F^* \hat{D}_F \hat{D}_F^*)^{-1}) \right](x_0) d\xi_n \sigma(\xi') dx'
\]
\[
= -8 \dim F h'(0) \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \frac{3\xi_n^3 + 9\xi_n^2 i + 21\xi_n - 5i}{(\xi_n - i)^4 (\xi_n + i)^3} d\xi_n \sigma(\xi') dx'
\]
(5.35)

From (5.35), we obtain
\[
-i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}\left( B_4 \times \partial_{\xi_n} \sigma_{-3}((\hat{D}_F^* \hat{D}_F \hat{D}_F^*)^{-1}) \right)(x_0) d\xi_n \sigma(\xi') dx'
\]
\[
= 32 \dim F \text{trace}[\sigma_n^{F,e}] \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \frac{1 + 3\xi_n}{(\xi_n - i)^4 (\xi_n + i)^3} d\xi_n \sigma(\xi') dx'
\]
\[
= -16 \dim F \text{trace}[\sigma_n^{F,e}][\Omega_4 dx']
\]
(5.36)

Then

\[
\text{case b)} = \left[ \frac{45}{2} h'(0) - 16 \text{trace}[\sigma_n^{F,e}] \right] \pi \dim F \Omega_4 dx'
\]
(5.37)

\text{case c)} r = -1, l = -4, |\alpha| = j = k = 0.

By (5.2) and an integration by parts, we have
\[
\text{case b)} = -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}\left( \pi_n^{+} \sigma_{-1}(\hat{D}_F^*)^{-1} \times \partial_{\xi_n} \sigma_{-4}((\hat{D}_F^* \hat{D}_F \hat{D}_F^*)^{-1}) \right)(x_0) d\xi_n \sigma(\xi') dx'
\]
\[
= -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}\left( \pi_n^{+} \sigma_{-1}(\hat{D}_F^*)^{-1} \times \partial_{\xi_n} \left( \sigma_{-4}(D^{-3}) + \frac{c(\xi)p(x_0)c(\xi) + c(\xi)\partial_n(x_0)c(\xi)}{|\xi|^6} \right) \right)(x_0) d\xi_n \sigma(\xi') dx'.
\]
(5.38)

By (3.12) in [19], we have
\[
\pi_n^{+} \sigma_{-1}(\hat{D}_F^*)^{-1} = -\frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)}
\]
(5.39)

In the normal coordinate, \( g^{ij}(x_0) = \delta_i^j \) and \( \partial_{x_j}(g^{ij})(x_0) = 0 \), if \( j < n \); \( \partial_{x_j}(g^{0j})(x_0) = h'(0) \delta_0^j \), if \( j = n \). So by Lemma A.2 in [12], we have \( \Gamma^m(x_0) = \frac{1}{2} h'(0) \) and \( \Gamma^k(x_0) = 0 \) for \( k < n \). By the definition of \( \delta^k \) and Lemma 2.3 in [12], we have \( \delta^m(x_0) = 0 \) and \( \delta^k = \frac{1}{4} h'(0) c(\xi) c(\xi_0) \) for \( k < n \). By (3.15) in [19], we obtain
\[
\sigma_{-4}(D^{-3})(x_0) = \frac{1}{|\xi|^6} e(\xi)(h'(0)c(\xi) \sum_{k<n} \xi_k e(\xi_k) c(\xi_n) - 5h'(0)\xi_n c(\xi) - \frac{5}{4} h'(0)|\xi|^2 c(dx_n))c(\xi) \\
+ \frac{c(\xi)}{|\xi|^{10}} (|\xi|^4 c(dx_n) \partial_{x_n} [c(\xi')])(x_0) - 2h'(0)|\xi|^2 c(dx_n) c(\xi) + 2\xi_n |\xi|^2 c(\xi) \partial_{x_n} [c(\xi')](x_0) \\
+ 4\xi_n h'(0)c(\xi)c(\xi) + h'(0) \frac{c(\xi)c(dx_n)c(\xi)}{|\xi|^6}.
\]

Then

\[
\partial_{\xi_n} \sigma_{-4}(D^{-3})(x_0) = \frac{59\xi_n + 27\xi_n^3 h'(0)c(\xi')c(dx_n) c(\xi') + \frac{33 - 180\xi_n^2 - 85\xi_n^4}{2(1 + \xi_n^2)^5} h'(0)c(\xi')
\]

\[
+ \frac{49\xi_n - 97\xi_n^2 - 50\xi_n^5}{2(1 + \xi_n^2)^5} h'(0)c(dx_n) c(\xi') - \frac{3}{(1 + \xi_n^2)^4} \partial_{x_n} [c(\xi')](x_0) + \frac{4\xi_n^3 - 8\xi_n^4}{(1 + \xi_n^2)^4} h'(0)c(dx_n) c(\xi') + \frac{2 - 10\xi_n^2}{(1 + \xi_n^2)^4} h'(0)c(\xi').
\]

Combining (5.40) and (5.42), we obtain

\[
\text{trace} \left[ \pi_+^+ \sigma_{-1}(\hat{D}^{-1}_F) \times \partial_{\xi_n} \sigma_{-4}(D^{-3}) \right](x_0)|_{\xi |^1 = 32h'(0)dimF \frac{(-17 - 42i\xi_n + 50\xi_n^2 - 16i\xi_n^3 + 29\xi_n^4)}{\left(\xi_n - i\right)^5 (\xi + i)^5}}.
\]

Then

\[
-i \int_{|\xi| = 1}^{\infty} \int_{-\infty}^{\infty} \text{trace} \left[ \pi_+^+ \sigma_{-1}(\hat{D}^{-1}_F) \times \partial_{\xi_n} \sigma_{-4}(D^{-3}) \right](x_0) d\xi_n c(\xi') dx'
\]

\[
= 32h'(0)dimF \frac{2\pi i}{4!} \left[ \frac{(-17 - 42i\xi_n + 50\xi_n^2 - 16i\xi_n^3 + 29\xi_n^4)}{(\xi + i)^5} \right]^{(4)}|_{\xi_n = -i} \Omega_4 dx'
\]

\[
= \frac{129}{2} \pi h'(0)dimF \Omega_4 dx'.
\]

Since

\[
\partial_{\xi_n} \left( \frac{c(\xi)p(x_0)c(\xi)}{|\xi|^6} \right) = \frac{c(dx_n)p(x_0)c(\xi') + c(\xi')p(x_0)c(dx_n) + 2\xi_n c(dx_n)p(x_0)c(dx_n) - 6\xi_n c(\xi)p(x_0)c(\xi)}{(1 + \xi_n^2)^4}
\]

from (5.40) and (5.45), then we have

\[
\text{trace} \left[ \pi_+^+ \sigma_{-1}(\hat{D}^{-1}_F) \times \partial_{\xi_n} \left( \frac{c(\xi)p(x_0)c(\xi)}{|\xi|^6} \right) \right](x_0)
\]

\[
= \frac{(4\xi_n i + 2)i}{2(\xi_n + i)(1 + \xi_n^2)^3} \text{trace}[c(\xi')p(x_0)] + \frac{4\xi_n i + 2}{2(\xi_n + i)(1 + \xi_n^2)^3} \text{trace}[c(dx_n)p(x_0)].
\]
We have (5.46) has no contribution for computing case b).
Similarly, we have

\[
\text{trace} \left[ \pi^{+}_{\xi_n} \sigma_{-1}(\hat{D}^{-1}_F) \times \partial_{\xi_n} \left( \frac{c(\xi) \partial^* c(\xi)}{\lvert \xi \rvert^6} \right) \right](x_0)
\]

\[
= \frac{(4\xi_n i + 2)i}{2(\xi_n + i)(1 + \xi_n^2)^2} \text{trace}[c(\xi') \partial^*] + \frac{4\xi_n i + 2}{2(\xi_n + i)(1 + \xi_n^2)^2} \text{trace}[c(dx_n) \partial^*].
\] (5.46)

Then

\[
-i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{\infty} \text{trace} \left[ \pi^{+}_{\xi_n} \sigma_{-1}(\hat{D}^{-1}_F) \times \partial_{\xi_n} \left( \frac{c(\xi) \partial^* c(\xi)}{\lvert \xi \rvert^6} \right) \right](x_0) d\xi_n \sigma(\xi') dx'
\]

\[
= -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{\infty} \frac{4\xi_n i + 2}{2(\xi_n + i)^2(\xi_n - i)^2} \text{trace} \left[ -\mathbf{i} \sigma \otimes \sigma_{n}^{F,c} \right] d\xi_n \sigma(\xi') dx'.
\]

\[
= 12\pi \text{dim} F \text{trace} \left[ \sigma_{n}^{F,c} \right] \Omega_4 dx'.
\] (5.47)

Similarly,

\[
-i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{\infty} \text{trace} \left[ \pi^{+}_{\xi_n} \sigma_{-1}(\hat{D}^{-1}_F) \times \partial_{\xi_n} \left( \frac{c(\xi) \partial^* c(\xi)}{\lvert \xi \rvert^6} \right) \right](x_0) d\xi_n \sigma(\xi') dx'
\]

\[
= 4\pi \text{dim} F \text{trace} \left[ w(F, g^F)(e_n) \right] \Omega_4 dx' - 12\pi \text{dim} F \text{trace} \left[ w^*(F, g^F)(e_n) \right] \Omega_4 dx'.
\] (5.48)

Then

\[
\text{case c) } = -\frac{129}{2} \pi h'(0) \text{dim} F \Omega_4 dx' + 12\pi \text{dim} F \text{trace} \left[ \sigma_{n}^{F,c} \right] \Omega_4 dx'
\]

\[
+ 4\pi \text{dim} F \text{trace} \left[ w(F, g^F)(e_n) \right] \Omega_4 dx' - 12\pi \text{dim} F \text{trace} \left[ w^*(F, g^F)(e_n) \right] \Omega_4 dx'.
\] (5.49)

Now \( \Psi \) is the sum of the cases a), b) and c), then

\[
\Psi = 23\pi h'(0) \text{dim} F \Omega_4 dx' - 4\pi \text{dim} F \text{trace} \left[ \sigma_{n}^{F,c} \right] \Omega_4 dx'
\]

\[
+ 4\pi \text{dim} F \text{trace} \left[ w(F, g^F)(e_n) \right] \Omega_4 dx' - 12\pi \text{dim} F \text{trace} \left[ w^*(F, g^F)(e_n) \right] \Omega_4 dx'.
\] (5.50)

By (4.2) in [12], we have

\[
K = \sum_{1 \leq i, j \leq n-1} K_{i,j} g_{ij}^{i,j}, \quad K_{i,j} = -\Gamma_{i,j},
\]

and \( K_{i,j} \) is the second fundamental form, or extrinsic curvature. For \( n = 6 \), then

\[
K(x_0) = \sum_{1 \leq i, j \leq n-1} K_{i,j}(x_0) g_{ij}^{i,j}(x_0) = \sum_{i=1}^{5} K_{i,i}(x_0) = -\frac{5}{2} h'(0).
\] (5.51)

Hence we conclude that
Theorem 5.1. Let $M$ be a 6-dimensional compact manifolds with the boundary $\partial M$. Then

$$\widetilde{\text{Wres}}[\pi^+(\hat{D}_F^{-1}) \circ \pi^+((\hat{D}_F^* \hat{D}_F \hat{D}_F^*)^{-1})]$$

$$= 8\pi^3 \int_M \text{Tr} \left[ -\frac{s}{12} + \frac{3}{8} [\hat{c}(\omega^*) - \hat{c}(\omega)]^2 - \frac{1}{4} \hat{c}(\omega^*) \hat{c}(\omega) - \frac{1}{4} \sum_j \nabla_{e_j}^F (\hat{c}(\omega^*)) c(e_j) \right. $$

$$+ \frac{1}{4} \sum_j c(e_j) \nabla_{e_j}^F (\hat{c}(\omega)) \bigg] d\text{vol}_M + \int_{\partial M} \left[ -\frac{46}{8} \pi \text{dim} F \text{dim} \text{trace} (\sigma_{6,F}^F, e) \right.$$  

$$+ 4\pi \text{dim} F \text{trace} \left( w^*(F, g^F)(e_6) \right) - 12\pi \text{dim} F \text{trace} \left( w(F, g^F)(e_6) \right) \bigg] \Omega_4 d\nu'.$$

(5.52)

where $s$ is the scalar curvature.

Acknowledgements

This work was supported by NSFC. 11771070 . The authors thank the referee for his (or her) careful reading and helpful comments.

References

References

[1] V. W. Guillemin: A new proof of Weyl's formula on the asymptotic distribution of eigenvalues. Adv. Math. 55, no. 2, 131-160, (1985). 

[2] M. Wodzicki: local invariants of spectral asymmetry. Invent. Math. 75(1), 143-178, (1995). 

[3] M. Adler: On a trace functional for formal pseudo-differential operators and the symplectic structure of Korteweg-de Vries type equations, Invent. Math. 50, 219-248, (1979).

[4] A. Connes: Quantized calculus and applications. Xth International Congress of Mathematical Physics(Paris,1994), Inter- 

[5] A. Connes: The action functional in Noncommutative geometry. Comm. Math. Phys. 117, 673-683, (1998).

[6] D. Kastler: The Dirac Operator and Gravitation. Comm. Math. Phys. 166, 633-643, (1995).

[7] W. Kalau and M. Walze: Gravity, Noncommutative geometry and the Wodzicki residue. J. Geom. Physics. 16, 327-344, (1995).

[8] T. Ackermann: A note on the Wodzicki residue. J. Geom. Phys. 20, 404-406, (1996).

[9] R. Ponge.: Noncommutative Geometry and lower dimensional volumes in Riemannian geometry, Lett. Math. Phys. 83, 1-19 (2008).

[10] B. V. Fedosov, F. Golse, E. Leichtnam, E. Schrohe: The noncommutative residue for manifolds with boundary. J. Funct. 

[11] E. Schrohe: Noncommutative residue, Dixmier's trace, and heat trace expansions on manifolds with boundary. Contemp. Math. 242, 161-186, (1999).

[12] Y. Wang: Gravity and the Noncommutative Residue for Manifolds with Boundary. Letters in Mathematical Physics. 80, 

[13] Y. Wang: Lower-Dimensional Volumes and Kastler-kalau-Walze Type Theorem for Manifolds with Boundary. Commun. 

[14] Y. Wang, Differential forms and the Wodzicki residue for manifolds with boundary, J. Geom. Phys. 56 731-753,(2006).

[15] J. Wang, Y. Wang. : A Kastler-Kalau-Walze Type Theorem for 7-Dimensional Manifolds with Boundary. Abstract and 

[16] J. Wang, Y. Wang. : A Kastler-Kalau-Walze Type Theorem for five-dimensional manifolds with boundary[1]. International 

[17] J. Wang and Y. Wang: The Kastler-Kalau-Walze type theory for 6-dimensional manifolds with boundary.Journal of 

[18] J. Wang and Y. Wang:A general A Kastler-Kalau-Walze type theorem for manifolds with boundary.International Journal 

[19] J. Wang and Y. Wang:Twisted Dirac operators and the noncommutative residue for manifolds with boundary.J.Pseudo-

[20] J. M. Bismut and W. Zhang.: An Extension of a theorem by Cheeger and Müller, Astérisque, No. 205, paris, (1992).