HOW TO SAMPLE AND WHEN TO STOP SAMPLING: THE GENERALIZED WALD PROBLEM AND MINIMAX POLICIES

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Abstract. Acquiring information is expensive. Experimenters need to carefully choose how many units of each treatment to sample and when to stop sampling. The aim of this paper is to develop techniques for incorporating the cost of information into experimental design. In particular, we study sequential experiments where sampling is costly and a decision-maker aims to determine the best treatment for full scale implementation by (1) adaptively allocating units to two possible treatments, and (2) stopping the experiment when the expected welfare (inclusive of sampling costs) from implementing the chosen treatment is maximized. Working under the diffusion limit, we describe the optimal policies under the minimax regret criterion. Under small cost asymptotics, the same policies are also optimal under parametric and non-parametric distributions of outcomes. The minimax optimal sampling rule is just the Neyman allocation; it is independent of sampling costs and does not adapt to previous outcomes. The decision-maker stops sampling when the average difference between the treatment outcomes, multiplied by the number of observations collected until that point, exceeds a specific threshold. We also suggest methods for inference on the treatment effects using stopping times and discuss their optimality.

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1. INTRODUCTION

It is increasingly common in many fields to run A/B tests, i.e., experiments for determining the best of two treatments. For instance, technology companies like Amazon, Google and Microsoft routinely run hundreds of A/B tests a week to evaluate product changes, such as a tweak to a website layout or an update to a search algorithm. However, experimentation is expensive, especially if the changes being tested are very small and require evaluation on large amounts of data; e.g., state that even hundreds millions of users were considered insufficient at Google to detect the treatment effects they were interested in. Experimenters thus need to carefully choose how many units of each treatment to sample and when to stop sampling. In practice, researchers often have an implicit or explicit stopping time in mind. For instance, in testing the efficacy of vaccines, experimenters stop after a pre-determined number of infections. In other cases, a power analysis may be used to determine sample size. But if the aim is to maximize welfare (or profits), neither of these procedures is optimal.¹

In this paper, we develop techniques for incorporating the cost of information into experimental design. In particular, we study optimal sampling and stopping rules in sequential experiments where sampling is costly and the decision maker (DM) aims to determine the best of two treatments by: (1) adaptively allocating units to one of these treatments, and (2) stopping the experiment when the expected welfare, inclusive of sampling costs, is maximized. We term this the generalized Wald problem, and use minimax regret (Manski, 2021), a natural choice criterion under ambiguity aversion, to determine the optimal decision rule.²

We first derive the optimal decision rule in continuous time, under the diffusion regime (Wager and Xu, 2021; Fan and Glynn, 2021). Then, we show that analogues of this decision rule are also asymptotically optimal under parametric and non-parametric distributions of outcomes. The asymptotics, which appear to be novel, involve taking the sampling costs to 0 at a specific rate. Section 4 motivates these small cost asymptotics and argues for their reasonableness in applications.

¹See, e.g., for a critique on the common use of power analysis for determining the sample size in randomized control trials.
²We do not consider the minimax risk criterion as it leads to a trivial decision: the DM should never experiment and always apply the status quo treatment.
The optimal decision rule has a number of interesting, and perhaps, surprising properties. First, the optimal sampling rule is history independent and also independent of sampling costs. In fact, it is just the Neyman allocation, which is well known in the RCT literature as the (fixed) sampling strategy that minimizes estimation variance; our results state that one cannot better this even when allowing for adaptive strategies. Second, it is optimal to stop when the difference in average outcomes between the treatments, multiplied by the number of observations collected up to that point, exceeds a specific threshold. The threshold depends on sampling costs and the standard deviation of the treatment outcomes. Finally, at the conclusion of the experiment, the DM chooses the treatment with the highest average outcomes.

The decision rule therefore has a simple form that makes it attractive for applications. By allowing for an adaptive stopping time, we save on experimentation costs. Compared to standard, i.e., non-sequential experiments, we show that our decision rules attain the same regret, exclusive of sampling costs, with 40% fewer observations on average; this is independent of model parameters such as sampling costs and outcome variances. Admittedly, due to the nature of the stopping time, point estimation of treatment effects is not straightforward. However, we propose methods for conducting optimal inference using the knowledge of stopping times.

For the most part, this paper focuses on constant sampling costs (i.e., constant per observation). This has been a standard assumption since the classic work of Wald (1947), see also Arrow et al. (1949) and Fudenberg et al. (2018), among others. In fact, many online marketplaces for running experiments, e.g., Amazon Mechanical Turk, charge a fixed cost per query/observation. Note also that the costs may be indirect: for online platforms like Google or Microsoft that routinely run thousands of A/B tests, these could correspond to how much experimentation hurts user experience. Still, one may wonder whether and how our results change under other cost functions and modeling choices, e.g., when data is collected in batches, or, when we measure regret in terms of nonlinear or quantile welfare. We assess this in Section 6. Almost all our results still go through under these variations. We also identify a broader class of cost functions, nesting the constant case, in which the form of the optimal decision stays the same.
1.1. **Related literature.** The question of when to stop sampling has a rich history in economics and statistics. It was first studied by Wald (1947) and Arrow et al. (1949) with the goal being hypothesis testing, specifically, optimizing the trade-off between type I and type II errors, instead of welfare maximization. Still, one can place these results into the present framework by imagining that the distributions of outcomes under both treatments are known, but it is unknown which distribution corresponds to which treatment. This paper generalizes these results by allowing the distributions to be unknown. For this reason, we term the question studied here the generalized Wald problem.

Chernoff (1959) studied the sequential hypothesis testing problem under multiple hypotheses, using large deviation methods. The asymptotics there involve taking the sampling costs to 0, even as there is a fixed reward gap between the treatments. More recently, the stopping rules of Chernoff (1959) were incorporated into the $\delta$-PAC (Probably Approximately Correct) algorithms devised by Garivier and Kaufmann (2016) and Qin et al. (2017) for best arm identification with a fixed confidence. The aim in these studies is to minimize the amount of time needed to attain a pre-specified probability, $1 - \delta$, of selecting the optimal arm. However, these algorithms do not directly minimize a welfare criterion, and the constraint of pre-specifying a $\delta$ could be misplaced, if, e.g., there is very little difference between the first and second best treatments. In fact, under the least favorable prior, our minimax decision rule mis-identifies the best treatment about 23% of the time. Qin and Russo (2022) study the costly sampling problem under fixed reward gap asymptotics using large deviation methods. The present paper differs in using local asymptotics and in appealing to a minimax regret criterion. However, unlike the papers cited above, we only study binary treatments.

A number of papers (Colton, 1963; Lai et al., 1980; Chernoff and Petkau, 1981) have studied sequential trials in which there is a population of $N$ units, and at each period, the DM randomly selects two individuals from this population, and assigns them to the two treatments. The DM is allowed to stop experimenting at any point and apply a single treatment on the remainder of the population. The setup in these papers is intermediate between our own and two-armed bandits: while the aim, as in here, is to minimize regret, acquiring samples is not by itself
expensive and the outcomes in the experimentation phase matter for welfare. This
literature also does not consider optimal sampling rules.

The paper is also closely related to the growing literature on information ac-
quision and design, see, Hébert and Woodford (2017); Fudenberg et al. (2018);
Morris and Strack (2019); Liang et al. (2022), among others. Fudenberg et al.
(2018) study the question of optimal stopping when there are two treatments and
the goal is to maximize Bayes welfare (which is equivalent to minimizing Bayes
regret) under normal priors and costly sampling. While the sampling rule in Fu-
denberg et al. (2018) is exogenously specified, Liang et al. (2022) study a more
general version of this problem that allows for selecting this. In fact, for constant
sampling costs, the setup in Liang et al. (2022) is similar to ours but the welfare
criterion is different. The authors study a Bayesian version of the problem with
normal priors, with the resulting decision rules having very different qualitative
and quantitative properties from ours; see Section 3.2 for a detailed comparison.
These differences arise because the minimax regret criterion corresponds to a least
favorable prior with a specific two-point support. Thus, our results highlight the
important role played by the prior in determining even the qualitative properties
of the optimal decisions. This motivates the need for robust decision rules, and
the minimax regret criterion provides one way to obtain them.

Our results also speak to the literature on drift-diffusion models (DDMs), which
are widely used in neuroscience and psychology to study choice processes (Luce
et al., 1986; Ratcliff and McKoon, 2008; Fehr and Rangel, 2011). The classic
DDM model is based on the binary state hypothesis testing problem of Wald
(1947). Fudenberg et al. (2018) extend this model to allow for continuous states,
using Gaussian priors, and show that the resulting optimal decision rules are very
different, even qualitatively, from the predictions of the DDM model. In this paper,
we show that if the DM is ambiguity averse and uses the minimax regret criterion,
then the predictions of the DDM model are recovered even under continuous states.
In other words, decision making under ignorance brings us back to DDM.

Finally, the results in this paper are unique in regards to all the above strands of
literature in showing that any discrete time parametric and non-parametric version
of the problem can be reduced to the diffusion limit under small cost asymptotics.
Diffusion asymptotics were introduced by Wager and Xu (2021) and Fan and Glynn (2021) to study the properties of Thompson sampling in bandit experiments. The techniques for showing asymptotic equivalence to the limit experiment build on, and extend, previous work on sequential experiments by Adusumilli (2021). Relative to that paper, the technical novelty here is in allowing for stopping times, which makes the length of the experiment endogenous, and also in showing that the proposed decision rule attains the asymptotic minimax lower bound.

2. Setup under diffusion asymptotics

We start by describing the problem under the diffusion regime. There are two treatments 0, 1 corresponding to unknown mean rewards $\mu := (\mu_1, \mu_0)$ and known variances $\sigma_1, \sigma_0$. The aim of the decision maker (DM) is to determine which treatment to implement on the population. To guide her choice, the DM is allowed to conduct a sequential experiment, while paying a flow cost $c$ as long as the experiment is in progress. At each moment in time, the DM chooses which treatment to sample according to the sampling rule $\pi_a(t) \equiv \pi(A = a|F_t), a \in \{0, 1\}$, which specifies the probability of selecting treatment $a$ given some filtration $F_t$. The DM then observes signals, $x_1(t), x_0(t)$ from each of the treatments, as well as the fraction of times, $q_1(t), q_0(t)$ each treatment was sampled so far:

$$dx_a(t) = \mu_a \pi_a(t) dt + \sigma_a \sqrt{\pi_a(t)} dW_a(t), \tag{2.1}$$
$$dq_a(t) = \pi_a(t) dt. \tag{2.2}$$

Here, $W_1(t), W_0(t)$ are independent one-dimensional Weiner processes. The experiment ends in accordance with an $F_\tau$-adapted stopping time, $\tau$. At the conclusion of the experiment, the DM chooses an $F_\tau$ measurable implementation rule, $\delta \in \{0, 1\}$, specifying which treatment to implement on the population. The DM’s decision space thus consists of the triple $d := (\pi, \tau, \delta)$.

Denote $s(t) = (x_1(t), x_0(t), q_1(t), q_0(t))$. We take $F_t \equiv \sigma\{s(u); u \leq t\}$ to be the filtration generated by the state variables $s(\cdot)$ until time $t$.\footnote{As in Liang et al. (2022), we restrict attention to sampling rules $\pi_a$ for which a weak solution to the functional SDEs (2.1), (2.2) exists. This is true if either $\pi_a : \{X_s\}_{s \leq t} \rightarrow [0, 1]$ is continuous, see Karatzas and Shreve (2012, Section 5.4), or, if it is any deterministic function of $t$.} Let $E_{d, \mu}[\cdot]$ denote the expectation under a decision rule $d$, given some value of $\mu$. We evaluate decision
rules under the minimax regret criterion, where the maximum regret is defined as

\[ V_{\text{max}}(d) = \max_{\mu} V(d, \mu), \quad \text{with} \]

\[ V(d, \mu) := \mathbb{E}_{d,\mu}[\max\{\mu_1 - \mu_0, 0\} - (\mu_1 - \mu_0)\delta + c\tau]. \quad (2.3) \]

We refer to \( V(d, \mu) \) as the frequentist regret, i.e., the expected regret of \( d \) given \( \mu \). Recall that regret is the difference in utilities, \( \mu_0 + (\mu_1 - \mu_0)\delta - c\tau \), generated by the oracle decision rule \( \{\tau = 0, \delta = \mathbb{I}\{\mu_1 > \mu_0\}\} \), and a given decision rule \( d \).

2.1. Bayesian formulation. It is convenient to first describe the minimal regret under a Bayesian approach. Suppose the DM places a prior \( p_0 \) on \( \mu \). Bayes regret, \( V(d, p_0) := \int V(d, \mu) dp_0(\mu) \), provides one way to evaluate the decision rules \( d \). In the next section, we characterize minimax regret as Bayes regret under a least-favorable prior.

Let \( p(\mu|s) \) denote the posterior density of \( \mu \) given state \( s \). By standard results in stochastic filtering, (here, and in what follows, \( \propto \) denotes equality up to a normalization constant)

\[ p(\mu|s) \propto p(s|\mu) \cdot p_0(\mu) \]
\[ \propto p_{q_1}(x_1|\mu_1) \cdot p_{q_0}(x_0|\mu_0) \cdot p_0(\mu); \quad p_{q_0}(\cdot|\mu_0) \equiv \mathcal{N}(\cdot|q_0\mu_0, q_0\sigma^2_0) \]

where \( \mathcal{N}(\cdot|\mu, \sigma^2) \) is the normal density with mean \( \mu \) and variance \( \sigma^2 \), and the second proportionality follows from the fact \( W_1(\cdot), W_0(\cdot) \) are independent Weiner processes.

Define \( V^*(s; p_0) \) as the minimal expected Bayes regret, given state \( s \), i.e.,

\[ V^*(s; p_0) = \inf_{d \in D} \mathbb{E}_{\mu|s}[V(d, \mu)], \]

where \( D \) is the set of all decision rules that satisfy the measurability conditions set out above. In principle, one could characterize \( V^*(\cdot; p_0) \) as a HJB Variational Inequality (HJB-VI; Øksendal, 2003, Chapter 10), compute it numerically and characterize the optimal Bayes decision rules. However, this can be computationally expensive, and moreover, does not help us characterize the optimal decisions. Analytical expressions can be obtained under two types of priors.
2.1.1. **Gaussian priors.** In this case, the posterior is also Gaussian and its mean and variance can be computed analytically. Liang et al. (2022) derive the optimal decision rule in this setting. See Section 3.2 for a comparison with our proposals.

2.1.2. **Two-point priors.** Two point priors are closely related to hypothesis testing and the sequential likelihood ratio procedures of Wald (1947) and Arrow et al. (1949). More importantly for us, the least favorable prior for minimax regret, described in the next section, has a two point support. The treatment of two-point priors below is drawn from Adusumilli (2022).

Suppose the prior is supported on the two points \((a_1, b_1), (a_0, b_0)\). Let \(\theta = 1\) denote the state when nature chooses \((a_1, b_1)\), and \(\theta = 0\) the state when nature chooses \((a_0, b_0)\). Also let \((\Omega, \mathbb{P}, \mathcal{F}_t)\) denote the relevant probability space, where \(\mathcal{F}_t\) is the filtration defined previously, and set \(P_0, P_1\) to be the probability measures

\[
\mathbb{P}_0 := \mathbb{P}(A|\theta = 0) \quad \text{and} \quad \mathbb{P}_1 := \mathbb{P}(A|\theta = 1)
\]

for any \(A \in \mathcal{F}_t\).

Clearly, the likelihood ratio process \(\varphi^\pi(t) := \frac{d\mathbb{P}_1}{d\mathbb{P}_0}(\mathcal{F}_t)\) is a sufficient statistic for the DM under the sampling rule \(\pi\). An application of the Girsanov theorem, noting that \(W_1(\cdot), W_0(\cdot)\) are independent of each other, gives (see also Shiryaev, 2007, Section 4.2.1)

\[
\ln \varphi^\pi(t) = \frac{(a_1 - a_0)}{\sigma_1^2} x_1(t) + \frac{(b_1 - b_0)}{\sigma_0^2} x_0(t) - \frac{(a_1^2 - a_0^2)}{2\sigma_1^2} q_1(t) - \frac{(b_1^2 - b_0^2)}{2\sigma_0^2} q_0(t).
\]

(2.4)

Let \(m_0\) denote the prior probability that \(\theta = 1\). Additionally, given a sampling rule \(\pi\), let \(m^\pi(t) = \mathbb{P}(\theta = 1|\mathcal{F}_t)\) denote the belief process describing the posterior probability that \(\theta = 1\). Following Shiryaev (2007, Section 4.2.1), \(m^\pi(t)\) can be related to \(\varphi^\pi(t)\) as

\[
m^\pi(t) = \frac{m_0 \varphi^\pi(t)}{(1 - m_0) + m_0 \varphi^\pi(t)}.
\]

The Bayes optimal implementation rule at the end of the experiment is

\[
\delta^{\pi, \tau} = \mathbb{I} \left\{ a_1 m^\pi(\tau) + a_0 (1 - m^\pi(\tau)) \geq b_1 m^\pi(\tau) + b_0 (1 - m^\pi(\tau)) \right\}
\]

\[
= \mathbb{I} \left\{ \ln \varphi^\pi(\tau) \geq \ln \left( \frac{(b_0 - a_0)(1 - m_0)}{(a_1 - b_1) m_0} \right) \right\}.
\]

(2.5)

The super-script on \(\delta\) highlights that the above implementation rule is conditional on a given choice of \((\pi, \tau)\). Relatedly, the Bayes regret at the end of the experiment
(from employing the optimal implementation rule) is

\[ \varpi^\pi(\tau) := \min \{(a_1 - b_1)m^\pi(\tau), (b_0 - a_0)(1 - m^\pi(\tau))\}. \tag{2.6} \]

Hence, for a given sampling rule \( \pi \), the Bayes optimal stopping time \( \tau^\pi \), can be obtained as the solution to the optimal stopping problem

\[ \tau^\pi = \inf_{\tau \in \mathcal{T}} \mathbb{E}_{\pi}[\varpi^\pi(\tau) + c\tau], \tag{2.7} \]

where \( \mathcal{T} \) is the set of all \( \mathcal{F}_t \) measurable stopping times, and \( \mathbb{E}_{\pi}[\cdot] \) denotes the expectation under the sampling rule \( \pi \).

3. Minimax regret and optimal decision rules

Following Wald (1945), we characterize minimax regret as the value of a zero-sum game played between nature and the DM. Nature’s action consists of choosing a prior, \( p_0 \in \mathcal{P} \), over \( \mu \), while the DM chooses the decision rule \( d \). The minimax regret can then be written as

\[ \inf_{d \in D} V_{\text{max}}(d) = \inf_{d \in D} \sup_{p_0 \in \mathcal{P}} V(d, p_0). \tag{3.1} \]

The equilibrium action of nature is termed the least-favorable prior, and that of the DM, the minimax decision rule.

The following is the main result of this section: Denote \( \gamma^*_0 \approx 0.536357 \), \( \Delta^*_0 \approx 2.19613 \), \( \eta := \left( \frac{2\sigma}{\sigma_1 + \sigma_0} \right)^{1/3} \), \( \gamma^* = \gamma^*_0 / \eta \) and \( \Delta^* = \eta \Delta^*_0 \).

**Theorem 1.** The zero-sum two player game (3.1) has a unique Nash equilibrium. The minimax optimal decision rule is \( d^* := (\pi^*, \tau^*, \delta^*) \), where \( \pi^*_a = \sigma_a / (\sigma_1 + \sigma_0) \) for \( a \in \{0, 1\} \),

\[ \tau^* = \inf \left\{ t : \left| \frac{x_1(t)}{\sigma_1} - \frac{x_0(t)}{\sigma_0} \right| \geq \gamma^* \right\}, \]

and \( \delta^* = \mathbb{I} \left\{ \frac{x_1(\tau^*)}{\sigma_1} - \frac{x_0(\tau^*)}{\sigma_0} \geq 0 \right\} \). Furthermore, the least favorable prior is a symmetric two-point distribution supported on \((\sigma_1 \Delta^*/2, -\sigma_0 \Delta^*/2), (-\sigma_1 \Delta^*/2, \sigma_0 \Delta^*/2)\).

3.1. Proof sketch of Theorem 1. We start by describing the best responses of the DM and nature to specific classes of actions on their opponents’ part. For the actions of nature, we consider the set of indifference priors (Adusumilli, 2022), indexed by \( \Delta \in \mathbb{R} \). These are two-point priors, \( p_\Delta \), supported on \((\sigma_1 \Delta/2, -\sigma_0 \Delta/2), (-\sigma_1 \Delta/2, \sigma_0 \Delta/2)\)
with a prior probability of 0.5 at each support point. For the DM, we consider decision rules of the form \( \mathbf{d}_\gamma = (\pi^*, \tau_\gamma, \delta^*) \), where

\[
\tau_\gamma := \inf \left\{ t : \left| \frac{x_1(t)}{\sigma_1} - \frac{x_0(t)}{\sigma_0} \right| \geq \gamma \right\} ; \quad \gamma \in (0, \infty).
\]

The DM’s response to \( p_\Delta \). The term ‘indifference priors’ indicates that these priors make the DM indifferent between any sampling rule \( \pi \). This was shown in Adusumilli (2022), but let us restate the argument here: Let \( \theta = 1 \) denote the state when \( \mu = (\sigma_1 \Delta/2, -\sigma_0 \Delta/2) \) and \( \theta = 0 \) the state when \( \mu = (-\sigma_1 \Delta/2, \sigma_0 \Delta/2) \). Then, (2.4) implies

\[
\ln \varphi(t) = \Delta \left\{ \frac{x_1(t)}{\sigma_1} - \frac{x_0(t)}{\sigma_0} \right\}.
\]

(3.2)

Suppose \( \theta = 1 \). By (2.1), (2.2)

\[
\frac{dx_1(t)}{\sigma_1} - \frac{dx_0(t)}{\sigma_0} = \frac{\Delta}{2} dt + \sqrt{\pi_1}dW_1(t) - \sqrt{\pi_0}dW_0(t) = \frac{\Delta}{2} dt + d\tilde{W}(t),
\]

(3.3)

where \( \tilde{W}(t) := \sqrt{\pi_1}dW_1(t) - \sqrt{\pi_0}dW_0(t) \) is a one dimensional Weiner process, being a linear combination of two independent Weiner processes with \( \pi_1 + \pi_0 = 1 \). Plugging the above into (3.2) gives

\[
d\ln \varphi(t) = \frac{\Delta^2}{2} dt + \Delta d\tilde{W}(t).
\]

In a similar manner, we can show under \( \theta = 0 \) that \( d\ln \varphi(t) = -\frac{\Delta^2}{2} dt + \Delta d\tilde{W}(t) \).

In either case, the choice of \( \pi \) does not affect the evolution of the likelihood-ratio process \( \varphi(t) \), and consequently, has no bearing on the evolution of the beliefs \( m(t) \).

As the likelihood-ratio and belief processes, \( \varphi(t), m(t) \) are independent of \( \pi \), the Bayes optimal stopping time in (2.7) is also independent of \( \pi \) for indifference priors (standard results in optimal stopping, see e.g., Øksendal, 2003, Chapter 10, imply that the optimal stopping time in (2.7) is a function only of \( m(t) \) which is now independent of \( \pi \)). In fact, it has the same form as the optimal stopping time in the Bayesian hypothesis testing problem of Arrow et al. (1949), analyzed in continuous time by Shiryaev (2007, Section 4.2.1) and Morris and Strack (2019). An adaptation of their results (see, Lemma 1 in Appendix A) shows that the Bayes
optimal stopping time corresponding to \( p_\Delta \) is
\[
\tau_{\gamma(\Delta)} = \inf \left\{ t : \left| \frac{x_1(t)}{\sigma_1} - \frac{x_0(t)}{\sigma_0} \right| \geq \gamma(\Delta) \right\},
\]
where \( \gamma(\Delta) \) is defined in Lemma 1. By (2.5) and (3.2), the corresponding Bayes optimal implementation rule is
\[
\delta^* = \mathbb{I} \left\{ \frac{x_1(t)}{\sigma_1} - \frac{x_0(t)}{\sigma_0} \geq 0 \right\},
\]
and is independent of \( \Delta \). Hence, the decision rule \((\pi^*, \tau_{\gamma(\Delta)}, \delta^*)\) is a best response of the DM to nature’s choice of \( p_\Delta \).

Nature’s response to \( \tau_{\gamma} \). Next, consider nature’s response to the DM choosing \( \tilde{d}_{\gamma} \). Lemma 2 in Appendix A shows that the frequentist regret \( V(\tilde{d}_{\gamma}, \mu) \), given some \( \mu = (\mu_1, \mu_2) \), depends only on \( |\mu_1 - \mu_2| \). So, \( V(\tilde{d}_{\gamma}, \mu) \) is maximized at \( |\mu_1 - \mu_2| = (\sigma_1 + \sigma_0)\Delta(\gamma)/2 \), where \( \Delta(\gamma) \) is some function of \( \gamma \). The best response of nature to \( \tilde{d}_{\gamma} \) is then to pick any prior that is supported on \( \{ \mu : |\mu_1 - \mu_0| = (\sigma_1 + \sigma_0)\Delta(\gamma)/2 \} \). Therefore, the two-point prior \( p_{\Delta(\gamma)} \) is a best response to \( \tilde{d}_{\gamma} \).

Nash equilibrium. Based on the above observations, we can obtain the Nash equilibrium by numerically solving for the equilibrium values of \( \gamma, \Delta \). This is done in Lemma 3 in Appendix A.

3.2. Discussion.

3.2.1. Sampling rule. Perhaps the most striking aspect of the sampling rule is that it is just the Neyman allocation. It is not adaptive, and is also independent of sampling costs. In fact, both the sampling and implementation rules are the same as in a setting with a pre-determined number of observations - the so called best arm identification problem - see Adusumilli (2022).

The Neyman allocation is also well known as the sampling rule that minimizes the variance for the estimation of treatment effects \( \mu_1 - \mu_0 \). Our results thus imply that practitioners should continue employing the same randomization designs as those employed for standard (i.e., non-sequential) experiments.

By way of comparison, the optimal assignment rule under normal priors is also non-stochastic, but varies deterministically with time (Liang et al., 2022).
3.2.2. Stopping time. The stopping time is adaptive, but it is stationary and has
a simple form: the DM should end the experiment when
\( \rho(t) := \left| \frac{x_1(t) - x_0(t)}{\sigma_1} \right| \)
exceeds \( (\sigma_1 + \sigma_0)^{1/3} \gamma^* \). The threshold is decreasing in \( c \) and increasing in \( \sigma_1 + \sigma_0 \).

Let \( \bar{x}_a(t) := \frac{x_a(t)}{q_a(t)} \) denote the sample average of outcomes from treatment
\( a \) at time \( t \). Since \( q_a(t) = \sigma_a t / (\sigma_1 + \sigma_0) \) under \( \pi^* \), we can rewrite the optimal
stopping rule as \( \tau^* = \inf \{ t : t | \bar{x}_1(t) - \bar{x}_0(t) | \geq (\sigma_1 + \sigma_0) \gamma^* \} \); note that time, \( t \), is
a measure of the number of observations collected so far. From the form of \( \tau^* \), we
can infer that earlier stopping is indicative of larger reward gaps \( \mu_1 - \mu_0 \), with the
average length of the experiment being longest when \( \mu_1 - \mu_0 = 0 \). In Section 3.3,
we exploit this relationship to suggest methods for statistical inference on \( \mu_1 - \mu_0 \).

The stationarity of \( \tau^* \) is in sharp contrast to the properties of the optimal
stopping time under Bayes regret with normal priors. There, the optimal stopping
time is time dependent (Fudenberg et al., 2018; Liang et al., 2022). The following
intuition, adapted from Fudenberg et al. (2018), helps understand the difference:
Suppose that \( \rho(t) \approx 0 \) for some large \( t \). Under a normal prior, this is likely because
\( \mu_1 - \mu_0 \) is close to 0, in which case there is no significant difference between
the treatments and the DM should terminate the experiment straightaway. On
the other hand, the least favorable prior under minimax regret has a two point
support, and under this prior, \( \rho(t) \approx 0 \) would be interpreted as noise, so the DM
should proceed henceforth as if starting the experiment from scratch. Thus, the
qualitative properties of the stopping time are very different depending on the
prior. The above intuition also suggests that the relation between \( \mu_1 - \mu_0 \) and
stopping times is more complicated under normal priors, and not monotone as is
the case under minimax regret.

The stopping time, \( \tau^* \), induces a specific probability of mis-identification of
the optimal treatment under the least favorable prior. By Lemmas 2 and 3, this
probability is
\[
\alpha^* = \frac{1 - e^{-\Delta^* \gamma^*}}{e^{\Delta^* \gamma^*} - e^{-\Delta^* \gamma^*}} = \frac{1 - e^{-\Delta_0^* \gamma^*}}{e^{\Delta_0^* \gamma^*} - e^{-\Delta_0^* \gamma^*}} \approx 0.235. \tag{3.5}
\]
Interestingly, \( \alpha^* \) is independent of the model parameters \( c, \sigma_1, \sigma_0 \). This is because
the least favorable prior adjusts the reward gap in response to these quantities.
Another remarkable property, following from Fudenberg et al. (2018, Theorem 1), is that the probability of mis-identification is independent of the stopping time for any given value of $\mu$, i.e., $P(\delta = 1|\tau, \mu = b) = P(\delta = 1|\mu = b)$. This is again different from the setting with normal priors, where earlier stopping is indicative of higher probability of selecting the best treatment.

### 3.3. Inference on Treatment Effects

Due to the nature of the stopping time, point estimation of the treatment effect $\mu_1 - \mu_0$ is not straightforward. However, statistical inference is possible using information on stopping times. Recall that the optimal stopping time is $\tau^* = \inf \{t:\ |\rho(t)| \geq \gamma^*\}$, where

$$\rho(t) := \frac{x_1(t)}{\sigma_1} - \frac{x_0(t)}{\sigma_0} = \frac{\mu_1 - \mu_0}{\sigma_1 + \sigma_0} t + \tilde{W}(t),$$

with the equality being obtained under the Neyman allocation $\pi^*$ using (2.1), (2.2). It is clear from (3.6) that large values of $\tau^*$ are indicative of smaller values of $|\mu_1 - \mu_0|$. It is also straightforward to derive the distributions of $\tau^*$ under various values of $\Delta\mu := \mu_1 - \mu_0$ using Monte-Carlo simulations or analytic arguments. Figure 3.1, Panel A plots the density of these distributions, $F_{\Delta\mu}(\cdot)$, for a few different values of $\Delta\mu$ under $\sigma_1 = \sigma_0 = 1$ and $\eta = 1/2$. Note that by the symmetry of Brownian motion, $F_{\Delta\mu}(\cdot) = F_{-\Delta\mu}(\cdot)$. Based on the knowledge of these distributions, we can construct $\alpha$-level tests for $H_0 : |\Delta\mu| = b$ vs $H_1 : |\Delta\mu| > b$ as $\phi_b = I\{\tau^* \leq F_b^{-1}(\alpha)\}$.

For the practically important case of $b = 0$, Figure 3.1, Panel B plots $F_0^{-1}(0.05)$ for various values of $\eta$. Unsurprisingly, the critical values are decreasing in $\eta$. 

**Figure 3.1.** Inference using stopping times
For inference on $\Delta \mu$ (as opposed to only its magnitude), we require knowledge of both $\tau^*$ and $\delta^*$. Let $P_b(\cdot)$ denote the probability measure over paths induced by the process $\rho(t)$ when $\Delta \mu = b$. Note that $\delta^* = \mathbb{I}\{\rho(\tau^*) = \gamma^*\}$. Also, as mentioned earlier, $P_b(\delta^* = 1|\tau^* = t)$ is independent of $t$, see, e.g., Fudenberg et al. (2018, Theorem 1). What is more, it is shown in Lemma 2 that

$$
\bar{\phi}_b := P_b(\delta^* = 1) = 1 - \frac{1 - e^{-2b\gamma^*/(\sigma_1 + \sigma_0)}}{e^{2b\gamma^*/(\sigma_1 + \sigma_0)} - e^{-2b\gamma^*/(\sigma_1 + \sigma_0)}}.
$$

Choose $c_{b,\alpha}^+, c_{b,\alpha}^- > 0$ such that $\bar{\phi}_b F^l(c_{b,\alpha}^+) + (1 - \bar{\phi}_b) F^r(c_{b,\alpha}^-) = \alpha$. Then, by the independence of $\tau^*, \delta^*$ given $b$, it is clear that the statistic $\bar{\phi}_b$, defined below, has size $\alpha$ for testing $H_0 : \Delta \mu = b$ vs $H_1 : \Delta \mu \neq b$, when $b \neq 0$:

$$
\{\bar{\phi}_b = 0\} \iff \{\tau^* \geq c_{b,\alpha}^+, \text{sign}(\delta^*) = \text{sign}(b)\} \cup \{\tau^* \geq c_{b,\alpha}^-, \text{sign}(\delta^*) \neq \text{sign}(b)\}.
$$

The critical values $c_{b,\alpha}^+, c_{b,\alpha}^-$ are not uniquely determined; different possibilities correspond to different tests. Setting $c_{b,\alpha}^- > c_{b,\alpha}^+$ provides more power for detecting alternatives $\Delta \mu$ that have the opposite sign as $b$.

Confidence intervals for $|\Delta \mu|$, $\Delta \mu$ can be obtained by inverting $\phi_b, \bar{\phi}_b$. Finite sample counterparts of these tests are described in Section 4.4.

**Optimal tests.** We show in Appendix B.1 that $\bar{\phi}_b$, with some $c_{b,\alpha}^+, c_{b,\alpha}^-$ that depend on $b_1$, is Uniformly Most Powerful (UMP) for testing $H_0 : \Delta \mu = b$ vs $H_1 : \Delta \mu = b_1$ when $b_1 > b$. By varying $b_1$, we can also compute the power envelope for testing $H_0 : \Delta \mu = b$ vs $H_1 : \Delta \mu > b$; however, a UMP test does not exist for the composite alternative as the point-wise optimal tests depend on $b_1$.

### 3.4. Benefit of adaptive experimentation.

In a standard RCT, the number of units of experimentation is specified beforehand. In the diffusion regime, this is equivalent to choosing the duration of the experiment. Now, the Neyman allocation is minimax optimal under both adaptive and non-adaptive experiments. The benefit of our decision rule, however, is that it enables one to stop the experiment early, thus saving on experimental costs. To quantify this benefit, fix some values of $\sigma_1, \sigma_0, c$, and suppose that nature chooses the least favorable prior $p_{\Delta^*}$. Let

$$
R^* := \int \mathbb{E}_{d^*|\mu} [\max\{\mu_1 - \mu_0, 0\} - (\mu_1 - \mu_0)\delta] \, dp_{\Delta^*}.
$$
denote the Bayes regret of the minimax decision rule $d^*$ net of sampling costs. In fact, by symmetry, the above is also the frequentist regret of $d^*$ under both the support points of $p_{\Delta^*}$. Now, let $T_{R^*}$ denote the duration of time required in a non-adaptive experiment to achieve the same Bayes regret $R^*$ (also under the least-favorable prior and net of sampling costs). Then, making use of some results from Shiryaev (2007, Section 4.2.5), we show in Appendix B.2 that

$$\frac{\mathbb{E}[\tau^*]}{T_{R^*}} = \frac{1 - 2\alpha^*}{2(\Phi^{-1}(1 - \alpha^*))^2} \ln \frac{1 - \alpha^*}{\alpha^*} \approx 0.6.$$  

(3.7)

In other words, the use of an adaptive stopping time enables us to attain the same regret with 40% fewer observations on average. Interestingly, the above result is independent of $\sigma_1, \sigma_0, c$, though the values of $\mathbb{E}[\tau^*]$ and $T_{R^*}$ do depend on these quantities (it is only the ratio that is constant). Admittedly, (3.7) does not quantify the welfare gain from using an adaptive experiment - this will depend on the sampling costs - but it is nevertheless useful as an informal measure of how much the amount of experimentation can be reduced.

4. **Parametric regimes and small cost asymptotics**

We now turn to the analysis of parametric models in discrete time. As before, the DM is tasked with selecting a treatment for implementation on the population. To this end, the DM experiments sequentially in periods $j = 1, 2, \ldots$ after paying an ‘effective sampling cost’ $C$ per period. Let $1/n$ denote the time difference between successive time periods. We consider small cost asymptotics, where $C = c/n^{3/2}$ for some $c \in (0, \infty)$, and $n \to \infty$.\(^4\)

Are small cost asymptotics realistic? We contend they are, as $C$ is not the actual cost of experimentation, but rather characterizes the tradeoff between these costs and the benefit accruing from full-scale implementation following the experiment. Indeed, one way to motivate our asymptotic regime is to imagine that there are $n^{3/2}$ population units in the implementation phase (so that the benefit of implementing treatment $a$ on the population is $n^{3/2} \mu_a$), $c$ is the cost of sampling an additional unit of observation, and time, $t$, is measured in units of $n$. This formalizes the

\(^4\)The rationale behind the $n^{3/2}$ normalization is the same as that in time series models with linear drift terms. The author is grateful to Tim Vogelsang for pointing this out.
intuition that, in practice, the cost of sampling is relatively small compared to
the population size; this is particularly true for online platforms (?). The scaling
also suggests that if the population size is $n^{3/2}$, we should aim to experiment on a
sample size of the order $n$ to achieve optimal welfare.

In each period, the DM assigns a treatment to a single unit of observation
according to some sampling rule $\pi_j(\cdot)$. The treatment assignment is a random draw
$A_j \sim \text{Bernoulli}(\pi_j)$. This results in an outcome $Y_j \sim P_{\theta}^{(a)}$, with $P_{\theta}^{(a)}$ denoting the
population distribution of outcomes under treatment $a$. In this section, we assume
that this distribution is known up to some unknown $\theta(a) \in \mathbb{R}^d$. It is without loss of
generality to assume $P_{\theta(1)}^{(1)}, P_{\theta(0)}^{(0)}$ are mutually independent (conditional on $\theta^{(1)}, \theta^{(0)}$)
as we only ever observe the outcomes from one treatment anyway. After observing
the outcome, the DM can decide either to stop sampling, or call up the next unit.
At the end of the experiment, the DM prescribes a treatment to apply on the
population.

Recall that $t$ is the number of periods elapsed divided by $n$. Let $q_a(t) :=
n^{-1} \sum_{j=1}^{\left\lfloor nt \right\rfloor} I(A_j = a)$, and take $\mathcal{F}_t$ to be the $\sigma$-algebra generated by
$$\xi_t \equiv \left\{ \{ A_j \}_{j=1}^{\left\lfloor nt \right\rfloor}, \{ Y_{1i} \}_{i=1}^{\left\lfloor n q_1(t) \right\rfloor}, \{ Y_{0i} \}_{i=1}^{\left\lfloor n q_0(t) \right\rfloor} \right\},$$
the set of all actions and rewards until period $nt$. The sequence of $\sigma$-algebras,
$\{ \mathcal{F}_t \}_{t \in T_n}$, where $T_n := \{ 1/n, 2/n, \ldots \}$, constitutes a filtration. We require $\pi_{nt}(\cdot)$
to be $\mathcal{F}_{t-1/n}$ measurable, the stopping time, $\tau$, to be $\mathcal{F}_{t-1/n}$ measurable, and
the implementation rule, $\delta$, to be $\mathcal{F}_\tau$ measurable. The set of all decision rules
$d \equiv (\{ \pi_{nt} \}_{t \in T_n}, \tau, \delta)$ satisfying these requirements is denoted by $D_n$. As un-
bounded stopping times pose technical challenges, we generally work with $D_{n,T} \equiv
\{ d \in D_n : \tau \leq T \ \text{a.s.} \}$, the set of all decision rules with bounded stopping times.

The mean outcomes under a parameter $\theta$ are denoted by $\mu_a(\theta) := E_{P_{\theta}^{(a)}}[Y_{ai}]$.
Following Hirano and Porter (2009), for each $a \in \{0, 1\}$, we consider local pertur-
bations of the form $\{ \theta^{(a)}_0 + h_a/\sqrt{n}; h_a \in \mathbb{R}^d \}$, with $h_a$ unknown, around a reference
parameter $\theta^{(a)}_0$. As in that paper, $\theta^{(a)}_0$ is chosen such that $\mu_1(\theta^{(1)}_0) = \mu_0(\theta^{(0)}_0) = 0$
for each $a \in \{0, 1\}$; the last equality, which sets the quantities to 0, is not necessary
and is simply a convenient re-centering. This choice of $\theta^{(a)}_0$ defines the hardest in-
stance of the problem, with $\mu_{n,a}(h) := \mu_a(\theta^{(a)}_0 + h/\sqrt{n}) \approx \mu^{(a)}_n h/\sqrt{n}$ for each $h \in \mathbb{R}^d$,16
where $\mu_a := \nabla_\theta \mu_a(\theta^{(0)}_0)$. When $\mu_1(\theta^{(1)}_0) \neq \mu_0(\theta^{(0)}_0)$, determining the best treatment is trivial under large $n$, and many decision rules, including the one we propose here (in Section 4.3), would achieve zero asymptotic regret.

Let $P^{(a)}_h := P^{(a)}_{\theta^{(0)}_0 + h/\sqrt{n}}$ and take $\mathbb{E}^{(a)}_h[\cdot]$ to be its corresponding expectation. We assume $P^{(a)}_\theta$ is differentiable in quadratic mean around $\theta^{(a)}_0$ with score functions $\psi_a(Y_i)$ and information matrices $I_a := \mathbb{E}^{(a)}_0[\psi_a \psi_a^\top]$. To reduce some notational overhead, we set $\theta^{(1)}_0 = \theta^{(0)}_0 = \theta_0$, and also suppose that $\mu_{n,a}(h) = -\mu_{n,a}(-h)$ for all $h$. In fact, the latter is always true asymptotically. Both simplifications can be easily dispensed with (at the expense of some additional notation). We emphasize that our results do not fundamentally require $\theta^{(1)}_0, \theta^{(0)}_0$ to be the same or even have the same dimension.

### 4.1. Bayes and minimax regret under fixed $n$

Let $P^{(a)}_{n,h}$ denote the joint probability over $Y^{(a)}_1, \ldots, Y^{(a)}_n$ - the largest possible (under $\tau \leq T$) iid sequence of outcomes that can be observed from treatment $a$ - when each $Y^{(a)}_i \sim P^{(a)}_{i,n}$. Define $h := (h_1, h_0)$, take $P_{n,h}$ to be the joint probability $P^{(1)}_{n,h_1} \times P^{(0)}_{n,h_0}$, and $\mathbb{E}_{n,h}[\cdot]$ its corresponding expectation. The frequentist regret of decision rule $d$ is defined as

$$V_n(d, h) \equiv V_n(d, (\mu_{n,1}(h_1), \mu_{n,0}(h_0)))$$

$$= \sqrt{n} \mathbb{E}_{n,h} \left[ \max \left\{ \mu_{n,1}(h_1) - \mu_{n,0}(h_0), 0 \right\} - (\mu_{n,1}(h_1) - \mu_{n,0}(h_0)) \delta + \frac{c}{\sqrt{n}^3 T} \right]$$

$$= \sqrt{n} \mathbb{E}_{n,h} \left[ \max \left\{ \mu_{n,1}(h_1) - \mu_{n,0}(h_0), 0 \right\} - (\mu_{n,1}(h_1) - \mu_{n,0}(h_0)) \delta \right] + c \mathbb{E}_{n,h}[\tau],$$

where the multiplication by $\sqrt{n}$ in the second line of the above equation is a normalization ensuring $V_n(d, h)$ converges to a non-trivial quantity.

Let $\nu$ denote a dominating measure over $\{P_\theta : \theta \in \Theta\}$, and define $p_\theta := dP_\theta/d\nu$. Also, take $M_0$ to be some prior over $h$, and $m_0$ its density with respect to some other dominating measure $\nu_1$. By Adusumilli (2021), the posterior density (wrt $\nu_1$), $p(\cdot|F_t)$, of $h$ depends only on $Y^{(a)}_{n\nu_a(t)} = \{Y_{ai}^{(a)}\}_{i=1}^{n\nu_a(t)}$ for $a \in \{0, 1\}$. Hence,

$$p_n(h|F_t) = p_n(h|Y^{(1)}_{n\nu_1(t)}, Y^{(0)}_{n\nu_0(t)})$$

$$\propto \left\{ \prod_{i=1}^{n\nu_1(t)} P^{(1)}_{\nu_1(h_1/\sqrt{n})}(Y_{i1}) \right\} \left\{ \prod_{i=1}^{n\nu_0(t)} P^{(0)}_{\nu_0(h_0/\sqrt{n})}(Y_{i0}) \right\} m_0(h). \quad (4.1)$$

The fixed $n$ Bayes regret of a decision $d$ is given by $V_n(d, m_0) := \int V_n(d, h) dm_0(h)$. 17
Let $\xi_\tau$ denote the terminal state. From the form of $V_n(d, h)$, it is clear that the Bayes optimal implementation rule is $\delta^*(\xi_\tau) = I\{\mu_{n,1}(\xi_\tau) \geq \mu_{n,0}(\xi_\tau)\}$, and the resulting Bayes regret at the terminal state is

$$\varpi_n(\xi_\tau) := \mu_n^{\text{max}}(\xi_\tau) - \max \{\mu_{n,1}(\xi_\tau), \mu_{n,0}(\xi_\tau)\}, \quad (4.2)$$

where $\mu_{n,a}(\xi_\tau) := E_{h|\xi}[\mu_{n,a}(h_a)]$ and $\mu_n^{\text{max}}(\xi_\tau) := E_{h|\xi}[\max \{\mu_{n,1}(h_1), \mu_{n,0}(h_0)\}]$.

We can thus associate each combination, $(\pi, \tau)$, of sampling rules and stopping times with the distribution $P_{\pi,\tau}$ that they induce over $(\varpi_n(\xi_\tau), \tau)$. Thus,

$$V_n(d, m_0) = E_{\pi,\tau} \left[ \sqrt{n \varpi_n(\xi_\tau)} + c \tau \right].$$

For any given $T < \infty$, the minimal Bayes regret in the fixed $n$ setting is therefore

$$V^*_{n,T}(m_0) = \inf_{d \in D_{n,T}} \sup_{h \in J} V_n(d, h),$$

while our interest is in minimax regret, $V^*_{n,T} := \inf_{d \in D_{n,T}} \sup_{h \in J} V_n(d, h)$, the minimal Bayes regret is a useful theoretical device as it provides a lower bound, $V^*_{n,T} \geq V^*_{n,T}(m_0)$ for any prior $m_0$.

4.2. Lower bound on minimax regret. We impose the following assumptions:

**Assumption 1.** (i) The class $\{P^{(a)}_\theta; \theta \in \mathbb{R}\}$ is differentiable in quadratic mean around $\theta_0$ for each $a \in \{0, 1\}$.

(ii) $E^{(a)}_0[\exp |\psi_a(Y_a)|] < \infty$ for $a \in \{0, 1\}$.

(iii) There exist $\mu_1, \mu_0$ and $\epsilon_n \to 0$ s.t. $\sqrt{n} \mu \left( P_{h}^{(a)} \right) \equiv \sqrt{n} \mu_{n,a}(h) = \mu_n^* h + \epsilon_n |h|^2$ for each $a \in \{0, 1\}$ and $h \in \mathbb{R}^d$.

The assumptions are standard, with the only onerous requirement being Assumption 1(ii). This is needed due to the proof techniques, which are adapted from Adusumilli (2021).

Let $V^*$ denote the asymptotic minimax regret, defined as the value of the minimax problem in (3.1).

**Theorem 2.** Suppose Assumptions 1(i)-(iii) hold. Then,

$$\sup_{J} \lim_{T \to \infty} \liminf_{n \to \infty} \inf_{d \in D_{n,T}} V_n(d, h) \geq V^*,$$
where the outer supremum is taken over all finite subsets $\mathcal{J}$ of $\mathbb{R}^d \times \mathbb{R}^d$.

The proof proceeds as follows: Let $
abla \sigma_a^2 := \dot{\mu}_a^\top I_a^{-1} \dot{\mu}_a,$

$$h_a^* := \frac{\sigma_a \Delta_a}{2 \mu_a^\top I_a^{-1} \mu_a} I_a^{-1} \dot{\mu}_a,$$

and take $m_0^*$ to be the symmetric two-prior supported on $(h_1^*, -h_0^*)$ and $(-h_1^*, h_0^*)$. This is the parametric counterpart to the least favorable prior described in Theorem 1. Clearly, there exist subsets $\mathcal{J}$ such that

$$\inf_{d \in \mathcal{D}_{n,T}} \sup_{h \in \mathcal{J}} V_n(d, h) \geq \inf_{d \in \mathcal{D}_{n,T}} V_n(d, m_0^*).$$

In Appendix A, we show

$$\lim_{T \to \infty} \lim_{n \to \infty} \inf_{d \in \mathcal{D}_{n,T}} V_n(d, m_0^*) = V^*. \quad (4.3)$$

To prove (4.3), we build on previous work in Adusumilli (2021). Standard techniques, such as asymptotic representation theorems (Van der Vaart, 2000), are not applicable here due to the continuous time nature of the problem. We instead employ a three step approach: First, we replace $P_{n,h}$ with a simpler family of measures whose likelihood ratios (under different values of $h$) are the same as those under Gaussian distributions. Then, for this family, we write down a HJB-Variational Inequality (HJB-VI) to characterize the optimal value function under fixed $n$. PDE approximation arguments then let us approximate the fixed $n$ value function with that under continuous time. The latter is shown to be $V^*$.

The definition of asymptotic minimax risk used in Theorem 1 is standard, see, e.g., Van der Vaart (2000, Theorem 8.11), apart from the $\lim_{T \to \infty}$ operation. The theorem asserts that $V^*$ is a lower bound on minimax regret under any bounded stopping time. The bound $T$ can be arbitrarily large. Our proof techniques require bounded stopping times as our approximation results, e.g., the SLAN property (see, equation (5.2) in Appendix A), are only valid when the experiment is of bounded duration. Nevertheless, we conjecture that in practice there is no loss in setting $T = \infty$.

5 For any given $h$, the dominated convergence theorem implies $\lim_{T \to \infty} \inf_{d \in \mathcal{D}_{n,T}} V_n(d, h) = \inf_{d \in \mathcal{D}_n} V_n(d, h)$. However, to allow $T = \infty$ in Theorem 1, we need to show that this equality holds uniformly over $n$. In specific instances, e.g., when the parametric family is Gaussian, this is indeed the case, but we are not aware of any general results in this direction.
4.3. Attaining the bound. We now describe a decision rule $d_n = (\pi_n, \tau_n, \delta_n)$ that is asymptotically minimax optimal. Let $\sigma_a^2 = \mu_a^I I_a^{-1} \mu_a$ for each $a$ and

$$\rho_n(t) := \frac{x_1(t)}{\sigma_1} - \frac{x_0(t)}{\sigma_0}, \text{ where } x_a(t) := \frac{\mu_a^I I_a^{-1}}{\sqrt{n}} \sum_{i=1}^{[nt]} \psi_a(Y_{ai}).$$

Note that $x_a(t)$ is the efficient influence function process for estimation of $\mu_a(\theta)$. We assume $\mu_a, I_a, \sigma_a$ are known; but in practice, they should be replaced with consistent estimates (from a vanishingly small initial sample) so that they do not require knowledge of the reference parameter $\theta_0$. This can be done without affecting the asymptotic results, see Section 6.3.

Take $\pi_n$ to be any sampling rule such that

$$\left| \frac{q_a(t)}{t} - \frac{\sigma_a}{\sigma_1 + \sigma_0} \right| \leq B |nt|^{-b_0} \text{ uniformly over bounded } t,$$

for some $B < \infty$ and $b_0 > 1/2$. To simplify matters, we suppose that $\pi_n$ is deterministic, e.g., $\pi_{n,1} = \mathbb{I}\{q_{1}(t) \leq t\sigma_1/(\sigma_1 + \sigma_0)\}$. Fully randomized rules, e.g., $\pi_{n,1} = \sigma_1/(\sigma_0 + \sigma_1)$, would also satisfy (4.4) with $b_0 > 1/2$, but we found them to be substantially inferior in practice. We further employ

$$\tau_{n,T} = \inf \{t : |\rho_n(t)| \geq \gamma^*\} \wedge T$$

as the stopping time, and as the implementation rule, set $\delta_{n,T} = \mathbb{I}\{\rho_n(\tau_{n,T}) \geq 0\}$.

Intuitively, $d_{n,T} = (\pi_n, \tau_{n,T}, \delta_{n,T})$ is the finite sample counterpart of the minimax optimal decision rule $d^*$ from Section 3. The following theorem shows that it is asymptotically minimax optimal in that it attains the lower bound of Theorem 2.

**Theorem 3.** Suppose Assumptions 1(i)-(iii) hold. Then,

$$\sup_{J} \lim_{T \to \infty} \lim_{n \to \infty} \inf_{h \in J} V_n(\mathbf{d}_{n,T}, \mathbf{h}) = V^*,$$

where the outer supremum is taken over all finite subsets $J$ of $\mathbb{R}^d \times \mathbb{R}^d$.

An important implication of Theorem 3 is that the minimax optimal decision rule only involves one state variable, $\rho_n(t)$. This is even though the state space in principle includes all the past observations until period $i$, for a total of at least $2i$ variables. The theorem thus provides a major reduction in dimension.
4.4. **Statistical Inference.** Suppose we want to test $H_0 : |\hat{\mu}_1 h_1 - \hat{\mu}_0 h_0| = b$ vs $H_1 : |\hat{\mu}_1 h_1 - \hat{\mu}_0 h_0| > b$. Then, as long as $T \geq F_b^{-1}(\alpha)$, we can employ a finite sample version of the test $\phi_b$ introduced in Section 3.3, given by

$$\hat{\phi}_b = \mathbb{I}\{\tau_{n,T} \leq F_b^{-1}(\alpha)\}.$$ 

Define $\mathcal{H}_b := \{h : |\hat{\mu}_1 h_1 - \hat{\mu}_0 h_0| = b\}$ as the set of all $h$ consistent with the null.

By (A.25) in the proof of Theorem 3, for each $h \in \mathcal{H}_b$, the distribution of $\tau_{n,T}$ under $P_{n,h}$ converges to that of $\tau^* \wedge T$ under $|\mu_1 - \mu_0| = b$ in the diffusion setting. Now, for $T \geq F_b^{-1}(\alpha)$, $\mathbb{I}\{\tau^* \wedge T \leq F_b^{-1}(\alpha)\} = \mathbb{I}\{\tau^* \leq F_b^{-1}(\alpha)\}$. It thereby follows that $\hat{\phi}_b$ has asymptotic size $\alpha$. This is summarized in the following theorem:

**Theorem 4.** Suppose Assumptions 1(i)-(iii) hold. Then, for each $b \geq 0$ and $h \in \mathcal{H}_b$, $\lim_{n \to \infty} P_{n,h} (\hat{\phi}_b = 1) = \alpha$.

Consider the above test for $b = 0$. In Appendix B.3, we show that $\hat{\phi}_0$ has non-trivial power, $F_c(F_b^{-1}(\alpha))$, against local alternatives $(h_1, h_0)$ of the form $|\hat{\mu}_1 h_1 - \hat{\mu}_0 h_0| = c > 0$. But the actual reward gap is $|\hat{\mu}_1 h_1 - \hat{\mu}_0 h_0| / \sqrt{n}$, so this implies $\hat{\phi}_0$ has non-trivial power against local alternatives converging to the null at the $\sqrt{n}$ rate.

The finite sample counterpart, $\hat{\phi}_b$, of $\tilde{\phi}_b$, for testing $H_0 : \hat{\mu}_1 h_1 - \hat{\mu}_0 h_0 = b$ vs $H_1 : \hat{\mu}_1 h_1 - \hat{\mu}_0 h_0 \neq b$, can be constructed in an analogous manner. We omit the details for brevity.

5. **The non-parametric setting**

We now turn to the setting where there is no a-priori information about the distributions $P^{(1)}, P^{(0)}$ of $Y_{0i}$ and $Y_{1i}$. For each $a$, let $\mathcal{P}^{(a)}$ denote a candidate class of probability measures for $P^{(a)}$ with bounded variance, and dominated by some measure $\nu$. Also, let $P_0^{(a)} \in \mathcal{P}^{(a)}$ denote some reference probability distribution. Following Van der Vaart (2000), we consider smooth one-dimensional sub-models of the form $\{P_{s,h}^{(a)} : s \leq \eta\}$ for some $\eta > 0$, where $h(\cdot)$ is a measurable function satisfying

$$\int \left[ \frac{\sqrt{dP_{s,h}^{(a)}}}{s} - \frac{\sqrt{dP_0^{(a)}}}{s} - \frac{1}{2} h(\sqrt{dP_0^{(a)}}) \right]^2 d\nu \to 0 \text{ as } s \to 0.$$ 

(5.1)
By Van der Vaart (2000), (5.1) implies \( \int h dP_0^{(a)} = 0 \) and \( \int h^2 dP_0^{(a)} < \infty \). The set of all such candidate \( h \) is termed the tangent space \( T(P_0^{(a)}) \). This is a subset of the Hilbert space \( L^2(P_0^{(a)}) \), endowed with the inner product \( \langle f, g \rangle_a = \mathbb{E}_{P_0^{(a)}}[fg] \) and norm \( \|f\|_a = \mathbb{E}_{P_0^{(a)}}[f^2]^{1/2} \).

For any \( h_a \in T(P_0^{(a)}) \), let \( P_1^{(a)}_{n,h_a} \) denote the joint probability measure over \( Y_1^{(a)}, \ldots, Y_n^{(a)} \), when each \( Y_i^{(a)} \) is an iid draw from \( P_1^{(a)}_{1/\sqrt{n},h_a} \). Also, denote \( h = (h_1, h_0) \), and take \( P_n,h \) to be the joint probability \( P_1^{(1)}_{n,h_1} \times P_0^{(0)}_{n,h_0} \), with \( \mathbb{E}_{n,h}[] \) being its corresponding expectation. An important implication of (5.1) is the SLAN property that for all \( h \in T(P_0^{(a)}) \),

\[
\sum_{i=1}^{[nq]} \ln \frac{dP_1^{(a)}_{1/\sqrt{n},h}(Y_{ai})}{dP_0^{(a)}(Y_{ai})} = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nq]} h(Y_{ai}) - \frac{q}{2} \|h\|_a^2 + o_{n,0}(1), \text{ uniformly over bounded } q.
\]

(5.2)

See Adusumilli (2021, Lemma 2) for the proof.

The mean rewards under \( P^{(a)} \) are given by \( \mu(P^{(a)}) = \int xdP^{(a)}(x) \). To obtain non-trivial regret bounds, we focus on the case where \( \mu(P^{(a)}) = 0 \) for \( a \in \{0, 1\} \).

Let \( \psi(x) := x \) and \( \sigma_a^2 := \int x^2 dP_0^{(a)}(x) \). Then, \( \psi(\cdot) \) is the efficient influence function corresponding to estimation of \( \mu \), in the sense that under some mild assumptions on \( \{P_{s,h}\} \),

\[
\frac{\mu(P_{s,h}) - \mu(P_0^{(a)})}{s} - \langle \psi, h \rangle_a = \frac{\mu(P_{s,h})}{s} - \langle \psi, h \rangle_a = o(s).
\]

(5.3)

The above implies \( \mu(P_{1/\sqrt{n},h}) \approx \langle \psi, h \rangle_a / \sqrt{n} \). This is just the right scaling for diffusion asymptotics. In what follows, we shall set \( \mu_{n,a}(h) := \mu(P_{1/\sqrt{n},h}) \).

It is possible to select \( \{\phi_{a,1}, \phi_{a,2}, \ldots \} \in T(P_0^{(a)}) \) in such a manner that \( \{\psi/\sigma_a, \phi_{a,1}, \phi_{a,2}, \ldots \} \) is a set of orthonormal basis functions for the closure of \( T(P_0^{(a)}) \); the division by \( \sigma_a \) in the first component ensures \( \|\psi/\sigma_a\|_a^2 = \int x^2/\sigma_a^2 dP_0^{(a)}(x) = 1 \). We can also choose these bases so they lie in \( T(P_0^{(a)}) \), i.e., \( \mathbb{E}_{P_0^{(a)}}[\phi_{a,j}] = 0 \) for all \( j \). By the Hilbert space isometry, each \( h_a \in T(P_0^{(a)}) \) is then associated with an element from the \( l_2 \) space of square integrable sequences, \( (h_a,0/\sigma_a, h_a,1, \ldots) \), where \( h_a,0 = \langle \psi, h_a \rangle_a \) and \( h_a,k = \langle \phi_{a,k}, h_a \rangle_a \) for all \( k \neq 0 \).

As in the previous sections, to derive the properties of minimax regret, it is convenient to first define a notion of Bayes regret. To this end, we follow Adusumilli.
(2021) and define Bayes regret in terms of priors on the tangent space $T(P_0)$, or equivalently, in terms of priors on $l_2$. Let $(\varrho(1), \varrho(2), \ldots)$ denote some permutation of $(1,2,\ldots)$. Define $\mathbf{h} := (h_1, h_0)$, where each $h_a \in T(P_0^{(a)})$. For the purposes of deriving our theoretical results, we may restrict attention to priors, $m_0$, that are supported on a finite dimensional sub-space,

$$\mathcal{H}_I \equiv \left\{ \mathbf{h} \in T(P_0^{(1)}) \times T(P_0^{(0)}) : h_a = \frac{1}{\sigma_a} \langle \psi, h_a \rangle_a + \sum_{k=1}^{I-1} \langle \phi_a, \varrho(k), h_a \rangle_a \phi_a, \varrho(k) \right\}$$

of $T(P_0^{(a)})$, or isometrically, on a subset of $l_2 \times l_2$ of finite dimension $I \times I$. Note that the first component of $h_a \in l_2$ is always included in the prior; this is proportional to $\langle \psi, h_a \rangle_a$, the inner product with the efficient influence function.

In analogy with Section 4, the frequentist expected regret of decision rule $d$ is defined as

$$V_n(d, \mathbf{h}) \equiv \sqrt{n E_{n,\mathbf{h}}} \left[ \max \{ \mu_n(h_1) - \mu_n(h_0), 0 \} - (\mu_n(h_1) - \mu_n(h_0)) \delta + \frac{c}{n^{3/2}} \mu_T \right]$$

$$= \sqrt{n E_{n,\mathbf{h}}} \left[ \max \{ \mu_n(h_1) - \mu_n(h_0), 0 \} - (\mu_n(h_1) - \mu_n(h_0)) \delta \right] + cE_{n,\mathbf{h}}[\tau].$$

The corresponding Bayes regret is

$$V_n(d, m_0) = \int V_n(d, \mathbf{h}) dm_0(\mathbf{h}).$$

5.1. Lower bounds. The following assumptions are similar to Assumption 1:

**Assumption 2.** (i) The sub-models $\{P_{s,h}^{(a)} : h \in T(P_0^{(a)})\}$ satisfy (5.1) for each $a \in \{0,1\}$.

(ii) $\mathbb{E}_{P_0^{(a)}}[\exp |Y_{ai}|] < \infty$ for $a \in \{0,1\}$.

(iii) There exists $\epsilon_n \to 0$ s.t. $\sqrt{n} \mu_{n,a}(h_a) = h_{a,0} + \epsilon_n \|h_a\|^2_a$ for each $a \in \{0,1\}$ and $h_a \in T(P_0^{(a)})$.

We then have the following lower bound:

**Theorem 5.** Suppose Assumptions 2(i)-(iii) hold. Then,

$$\sup_{\mathcal{H}_I} \lim_{T \to \infty} \liminf_{n \to \infty} \inf_{d \in D_n,T} \sup_{h \in \mathcal{H}_I} V_n(d, \mathbf{h}) \geq V^*,$$

where the outer supremum is taken over all possible finite dimensional subspaces, $\mathcal{H}_I$, of $T(P_0^{(1)}) \times T(P_0^{(0)})$. 

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As with Theorem 2, the proof involves lower bounding minimax regret with Bayes regret under a suitable prior. Denote $h^*_{a,0} := \sigma_a \Delta^*/2$ and take $m^*_0$ to be the symmetric two-prior supported on $((h^*_{1,0}, 0, 0, \ldots), (-h^*_{0,0}, 0, 0, \ldots))$ and $((-h^*_{1,0}, 0, 0, \ldots), (h^*_{0,0}, 0, 0, \ldots))$. Note that we are taking $m^*_0$ to be a probability distribution on the space $l_2 \times l_2$. Then, there exist sub-spaces $\mathcal{H}_I$ such that

$$\inf_{d \in \mathcal{D}_{n,T}} \sup_{h \in \mathcal{H}_I} V_n(d, h) \geq \inf_{d \in \mathcal{D}_{n,T}} V_n(d, m^*_0).$$

We can then show

$$\lim_{T \to \infty} \lim_{n \to \infty} \inf_{d \in \mathcal{D}_{n,T}} V_n(d, m^*_0) = V^*.$$

The proof of the above uses the same arguments as that of Theorem 2, and is therefore omitted.

5.2. Attaining the bound. As in Section 4.3, take $\pi_n$ to be any deterministic sampling rule that satisfies (4.4). Let

$$\rho_n(t) := \frac{x_1(t)}{\sigma_1} - \frac{x_0(t)}{\sigma_0}, \quad \text{where} \quad x_a(t) := \frac{1}{\sqrt{n}} \sum_{i=1}^{[nq_a(t)]} Y_{ai}. \quad (5.4)$$

Note that $x_a(t)$, which is the scaled sum of outcomes from each treatment, is again the efficient influence function process for estimation of $\mu(P^{(a)})$ in the non-parametric setting. We choose as the stopping time,

$$\tau_{n,T} = \inf \{ t : |\rho_n(t)| \geq \gamma^* \} \wedge T,$$

and as the implementation rule, set $\delta_{n,T} = I \{ |\rho_n(\tau_{n,T})| \geq 0 \}$.

The following theorem shows that the triple $d_{n,T} = (\pi_n, \tau_{n,T}, \delta_{n,T})$ attains the minimax lower bound in the non-parametric regime.

**Theorem 6.** Suppose Assumptions 2(i)-(iii) hold. Then,

$$\sup_{\mathcal{H}_I} \lim_{T \to \infty} \lim_{n \to \infty} \inf_{h \in \mathcal{H}_I} V_n(d_{n,T}, h) = V^*, \quad \text{where the outer supremum is taken over all possible finite dimensional subspaces,} \quad \mathcal{H}_I, \quad \text{of} \quad T(P^{(1)}_0) \times T(P^{(0)}_0).$$

The proof is similar to that of Theorem 3 and is sketched in Appendix B.4.
6. Variations and Extensions

We now consider various modifications of the basic setup and analyze if, and how, the optimal decisions change.

6.1. Batching. In practice, it may be that data is collected in batches instead of one at a time, and the DM can only make decisions after processing each batch. Let \( B_n \) denote the number of observations considered in each batch. In the context of Section 4, this corresponds to a time duration of \( B_n/n \). An analysis of the proofs of Theorems 2-4 shows that these results continue to hold as long as \( B_n/n \to 0 \). Thus, \( d_{n,T} \) remains asymptotically minimax optimal in this scenario.

Even for \( B_n/n \to m \in (0,1) \), the optimal decision rules remain broadly unchanged. Asymptotically, we have equivalence to Gaussian experiments, so we can analyze batched experiments under the diffusion framework by imagining that the stopping time is only allowed to take on discrete values \( \{0,1/m,2/m,\ldots\} \). It is then clear from the discussion in Section 3.1 that the optimal sampling and implementation rules remain unchanged. The discrete nature of the setting makes determining the optimal stopping rule difficult, but it is easy to show that the decision rule \((\pi^*,\tau^*_m,\delta^*)\), where

\[
\tau^*_m := \inf \left\{ t \in \{0,1/m,2/m,\ldots\} : \left| \frac{x_1(t)}{\sigma_1} - \frac{x_0(t)}{\sigma_0} \right| \geq \gamma^* \right\},
\]

while not being exactly optimal, has a minimax regret that is arbitrarily close to \( V^* \) for large enough \( m \) (note that no batched experiment can attain a minimax regret that is lower than \( V^* \)).

6.2. Alternative cost functions. All our results so far were derived under constant sampling costs. The same techniques apply to other types of flow costs as long as these depend only on \( \rho(t) := \sigma_1^{-1}x_1(t) - \sigma_0^{-1}x_0(t) \). In particular, suppose that the frequentist regret is given by

\[
V(d,\mu) = E_{d|\mu} \left[ \max\{\mu_1 - \mu_0,0\} - (\mu_1 - \mu_0)\delta + \int_0^T c(\rho(t))dt \right],
\]

where \( c(z) \) is the flow cost of experimentation when \( \rho(t) = z \). We require \( c(\cdot) \) to be (i) positive, (ii) bounded away from 0, i.e., \( \inf_z c(z) \geq \xi > 0 \), and (iii) symmetric, i.e., \( c(z) = c(-z) \). By (3.6), \( (\sigma_1 + \sigma_0)\rho(t)/t \) is an estimate of the
treatment effect $\mu_1 - \mu_0$, so the above allows for situations in which sampling costs depend on the magnitude of the estimated treatment effects. While we are not aware of any real world examples of such costs, they could arise if there is feedback between the observations and sampling costs, e.g., if it is harder to find subjects for experimentation when the treatment effect estimates are higher. When there are only two states, the ‘ex-ante’ entropy cost of Sims (2003) is also equivalent to a specific flow cost of the form $c(\cdot)$ above, see Morris and Strack (2019).\(^6\)

For the above class of cost functions, we show in Appendix B.5 that the minimax optimal decision rule, $d^*$, and the least-favorable prior, $p^*_\Delta$, have the same form as in Theorem 1, but the values of $\gamma^*, \Delta^*$ are different and need to be calculated by solving the minimax problem

$$\min_{\gamma} \max_{\Delta} \left\{ \left( \frac{\sigma_1 + \sigma_0}{2} \right) \left( \frac{1 - e^{-\Delta \gamma}}{e^{\Delta \gamma} - e^{-\Delta \gamma}} \right) \right\},$$

where

$$\zeta_\Delta(x) := 2 \int_0^x \int_0^y e^{\Delta(z-y)} c(z) dz dy.$$  

Beyond this class of sampling costs, however, it is easy to conceive of scenarios in which the optimal decision rule differs markedly from the one we obtain here. For instance, Neyman allocation would no longer be the optimal sampling rule if the costs for sampling each treatment were different. Alternatively, if $c(\cdot)$ were to depend on $t$, the optimal stopping time could be non-stationary. The analysis of these cost functions is not covered by the present techniques.

6.3. **Unknown variances.** Replacing unknown variances (and other population quantities) with consistent estimates has no effect on asymptotic regret. We suggest two approaches to attain the minimax lower bounds when the variances are unknown:

The first approach uses ‘forced exploration’ (see, e.g., Lattimore and Szepesvári, 2020, Chapter 33, Note 7): we set $\pi^*_n = 1/2$, for the first $\bar{n} = n^a$ observations, where $a \in (0, 1)$. This corresponds to a time duration of $\bar{t} = n^{a-1}$. We use the data from these periods to obtain consistent estimates, $\hat{\sigma}^2_1, \hat{\sigma}^2_0$ of $\sigma^2_1, \sigma^2_0$. From $\bar{t}$ onwards, we apply the minimax optimal decision $d_{n,T}$ after plugging-in $\hat{\sigma}_1, \hat{\sigma}_0$.

\(^6\)However, we are not aware of any extension of this result to continuous states.
in place of $\sigma_1, \sigma_0$. This strategy is asymptotically minimax optimal for any $a$. Determining the optimal $a$ in finite samples requires going beyond an asymptotic analysis, and is outside the scope of this paper (in fact, this is also an open question in the computer science literature).

Our second suggestion is to place a prior on $\sigma_1, \sigma_0$, and continuously update their values using posterior means. As a default, we suggest employing an inverse-gamma prior and computing the posterior by treating the outcomes as Gaussian (this is of course justified in the limit). This approach has the advantage of not requiring any tuning parameters, but its theoretical properties are as yet unknown.

6.4. **Other regret measures.** Instead of defining regret, $\max\{\mu(P^{(1)})-\mu(P^{(0)}), 0\} - (\mu(P^{(1)}) - \mu(P^{(0)}))\delta + c\tau$, using the mean values of $P^{(0)}, P^{(1)}$, we can use other functionals of the outcome/welfare distribution in the implementation phase, e.g., $\mu(\cdot)$ could be a quantile function. Note, however, that we still require costs to be linear and additively separable. Let $\psi_a(\cdot)$ denote the efficient influence function corresponding to estimation of $\mu(P^{(a)})$. Then, a straightforward extension of the results in Section 5 shows that Theorems 5 and 6 continue to hold, with $x_a(t)$ in (5.4) replaced with the efficient influence function process $n^{-1/2}\sum_{i=1}^{\left\lfloor np_a(t)\right\rfloor} \psi_a(Y_{ai})$, and $\sigma_a^2$ with $\mathbb{E}_{p^{(a)}}[\psi(Y_{ai})^2]$. See Appendix B.6 for more details.

7. **Numerical illustration**

A/B testing is commonly used in online platforms for optimizing websites. Consequently, to assess the finite sample performance of our proposed policies, we ran a Monte-Carlo simulation calibrated to a realistic example of such an A/B test. Suppose there are two candidate website layouts, with exit rates $p_0, p_1$, and we want to run an A/B test to determine the one with the lowest exit rate.\(^7\) The outcomes are binary, $Y_a \sim \text{Bernoulli}(p_a)$. This is a parametric setting with score functions $\psi_a(Y_{ai}) = Y_{ai}$. We calibrate $p_0 = 0.4$, which is a typical value for an exit rate. The cost of experimentation is normalized to $c = 1$ and we consider various values of $n$, corresponding to different ‘population sizes’ (recall that the benefit during implementation is scaled as $n^{3/2}p_a$). We then set $p_1 = p_0 + \Delta/\sqrt{n}$.

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\(^7\)The exit rate is defined as the fraction of viewers of a webpage who exit from the website it is part of (i.e., without viewing other pages in that website).
and describe the results under varying $\Delta$. We believe local asymptotics provide a good approximation in practice, as the raw performance gains are known to be generally small - typically, $|p_1 - p_0|$ is of the order 0.05 or less (see, e.g., ?) - but they can translate to large profits when applied at scale, i.e., when $n$ is large.

Since $\sigma_\alpha = \sqrt{p_\alpha(1-p_\alpha)}$ is unknown, we employ ‘forced sampling’ with $\bar{n} = \max(50, 0.05n)$, i.e., using about 5% of the sample, to estimate $\sigma_1,\sigma_0$. Note that the asymptotically optimal sampling rule is always $1/2$ in the Bernoulli setting, so forced sampling is in fact asymptotically costless. We also experimented with a beta prior to continuously update $\sigma_\alpha$, but found the results to be somewhat inferior (see Appendix B.7 for details). Figure 7.1, Panel A plots the finite sample frequentist regret profile of our policy rules, $d_n \equiv d_{n,\infty}$ (with $T = \infty$), for various values of $n$, along with that of the minimax optimal policy, $d^*$, under the diffusion regime; the regret profile of the latter is derived analytically in Lemma 3. It is seen that diffusion asymptotics provide a very good approximation to the finite sample properties of $d_n$, even for such relatively small values of $n$ as $n = 1000$. In practice, A/B tests are run with tens, even hundreds, of thousands of observations. We also see that the max-regret of $d_n$ is very close to the asymptotic lower bound $V^*$ (the max-regret of $d^*$).

Figure 7.1, Panel B displays some summary statistics for the Bayes regret of $d_n$ under the least favorable prior, $p_{\Delta^*}$. The regret distribution is positively skewed and heavy tailed. The finite sample Bayes regret is again very close to $V^*$.

Appendix B.7 reports additional simulation results using Gaussian outcomes.

8. Conclusion

This paper proposes a minimax optimal procedure for determining the best treatment when sampling is costly. The optimal sampling rule is just the Neyman allocation, while the optimal stopping rule is time-stationary and advises that the experiment be terminated when the average difference in outcomes multiplies by the number of observations exceeds a specific threshold. While these rules were derived under diffusion asymptotics, it is shown that finite sample counterparts of these rules remain optimal under both parametric and non-parametric regimes. The form of these rules is robust to a number of different variations of the original
problem, e.g., under batching, different cost functions etc. We also propose methods for obtaining inference on treatment effects using the data on stopping times. Given the simple nature of these rules, and the potential for large sample efficiency gains (requiring, on average, 40% fewer observations than standard approaches), we believe they hold a lot of promise for practical use.

The paper also raises a number of avenues for future research. While our results were derived for binary treatments, multiple treatments are common in practice, and it would be useful to derive the optimal decision rules in this setting. We do expect, however, that in this case the optimal sampling rule would no longer be fixed, but history dependent. As noted previously, our setting also does not cover discounting and asymmetric cost functions. It is hoped that the techniques developed in this paper could help answer some of these outstanding questions.

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**Appendix A. Proofs**

**A.1. Proof of Theorem 1.** The proof makes use of the following lemmas:

**Lemma 1.** Suppose nature sets $p_0$ to be a symmetric two-point prior supported on $(\sigma_1 \Delta/2, -\sigma_0 \Delta/2), (\sigma_1 \Delta/2, \sigma_0 \Delta/2)$. Then the decision $d(\Delta) = (\pi^*, \tau_\gamma(\Delta), \delta^*)$, where $\gamma(\Delta)$ is defined in (A.3), is a best response by the DM.

**Proof.** The prior is an indifference-inducing one, so the DM is indifferent between any sampling rule $\pi$. Thus, $\pi^*_a = \sigma_a / (\sigma_1 + \sigma_0)$ is a best-response to this prior. The prior is symmetric with $m_0 = 1/2$, so by (2.5), the Bayes optimal implementation rule is

$$\delta^* = \mathbb{I}\{\ln \varphi(\tau) \geq 0\} = \mathbb{I}\left\{\frac{x_1(t)}{\sigma_1} - \frac{x_0(t)}{\sigma_0} \geq 0\right\}.$$  

It remains to compute the Bayes optimal stopping time. Let $\theta = 1$ denote the state when the prior is $(\sigma_1 \Delta/2, -\sigma_0 \Delta/2)$, with $\theta = 0$ otherwise. The discussion in Section 3.1 implies that, conditional on $\theta$, the likelihood ratio process $\varphi(t)$ does not depend on $\pi$ and evolves as

$$d \ln \varphi(t) = (2\theta - 1) \frac{\Delta^2}{2} dt + \Delta d\tilde{W}(t),$$

where $\tilde{W}(\cdot)$ is one-dimensional Brownian motion. By a similar argument as in Shiryaev (2007, Section 4.2.1), this in turn implies that the posterior probability $m(t) := P(\theta = 1 | \mathcal{F}_t)$ evolves as

$$dm(t) = \Delta m(t)(1 - m(t))d\tilde{W}(t),$$

independent of $\pi$. Therefore, by (2.7) the optimal stopping time also does not depend on $\pi$ and is given by

$$\tau(\Delta) = \inf_{\tau \in T} \mathbb{E}[\varpi(m(\tau)) + c\tau],$$  

where

$$\varpi(m) := \frac{\sigma_1 + \sigma_0}{2} \Delta \min\{m, 1 - m\}.$$  

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Inspection of the objective function in (A.1) shows that this is exactly the same objective as in the Bayesian hypothesis testing problem, analyzed previously by Arrow et al. (1949) and Morris and Strack (2019). We follow the analysis of the latter paper. Morris and Strack (2019) show that instead of choosing the stopping time $\tau$, it is equivalent to imagine that the DM chooses a probability distribution $G$ over the posterior beliefs $m(\tau)$ at an ‘ex-ante’ cost

$$c(G) = \frac{2c}{\Delta^2} \int (2m - 1) \ln \frac{1 - m}{m} dG(m),$$

subject to the constraint $\int m dG(m) = m_0 = 1/2$. Under the distribution $G$, the expected regret, exclusive of sampling costs, for the DM is

$$\int \varpi(m) dG(m) = \frac{(\sigma_1 + \sigma_0)}{2} \Delta \int \min\{m, 1 - m\} dG(m).$$

Hence, the stopping time, $\tau$, that solves (A.1) is the one that induces the distribution $G^*$, defined as

$$G^* = \arg \min_{G: \int m dG(m) = 1/2} \left\{ c(G) + \int \varpi(m) dG(m) \right\} = \arg \min_{G: \int m dG(m) = 1/2} \int f(m) dG(m),$$

where

$$f(m) := \frac{2c}{\Delta^2} (2m - 1) \ln \frac{1 - m}{m} + \frac{(\sigma_1 + \sigma_0)}{2} \Delta \min\{m, 1 - m\}.$$  

Clearly, $f(m) = f(1 - m)$. Hence, setting

$$\alpha(\Delta) := \arg \min_{\alpha} \left\{ \frac{(\sigma_1 + \sigma_0)}{2} \Delta \alpha + \frac{2c}{\Delta^2} (2\alpha - 1) \ln \frac{1 - \alpha}{\alpha} \right\},$$

it is easy to see that $G^*$ is a two-point distribution, supported on $\alpha(\Delta), 1 - \alpha(\Delta)$ with equal probability $1/2$. By Shiryaev (2007, Section 4.2.1), this distribution is induced by the stopping time $\tau_{\gamma(\Delta)}$, where

$$\gamma(\Delta) := \frac{1}{\Delta} \ln \frac{1 - \alpha(\Delta)}{\alpha(\Delta)}.$$  \hspace{1cm} (A.3)

Hence, this stopping time is the best response to nature’s prior. \hfill \Box
Lemma 2. Suppose $\mu$ is such that $|\mu_1 - \mu_0| = \frac{\sigma_1 + \sigma_0}{2} \Delta$. Then, for any $\gamma, \Delta > 0$,

$$V(\tilde{d}_\gamma, \mu) = \frac{(\sigma_1 + \sigma_0)}{2} \Delta \frac{1 - e^{-\Delta \gamma}}{e^{\Delta \gamma} - e^{-\Delta \gamma}} + \frac{2c\gamma e^{\Delta \gamma} + e^{-\Delta \gamma} - 2}{\Delta} \frac{e^{\Delta \gamma} - e^{-\Delta \gamma}}{e^{\Delta \gamma} - e^{-\Delta \gamma}}.$$ 

Thus, the frequentist regret of $\tilde{d}_\gamma$ depends on $\mu$ only through $|\mu_1 - \mu_0|$.

Proof. Suppose that $\mu_1 > \mu_0$. Define

$$\lambda(t) := \Delta \left\{ \frac{x_1(t)}{\sigma_1} - \frac{x_0(t)}{\sigma_0} \right\}.$$ 

Note that under $\tilde{d}_\gamma$ and $\mu$,

$$\frac{x_1(t)}{\sigma_1} - \frac{x_0(t)}{\sigma_0} = \frac{\Delta}{2} t + \tilde{W}(t),$$

where $\tilde{W}(\cdot)$ is one-dimensional Brownian motion. Hence $\lambda(t) = \frac{\Delta^2}{2} t + \Delta \tilde{W}(t)$. We can write the stopping time $\tau_\gamma$ in terms of $\lambda(t)$ as

$$\tau_\gamma = \inf \left\{ t : \left| \frac{x_1(t)}{\sigma_1} - \frac{x_0(t)}{\sigma_0} \right| \geq \gamma \right\} = \inf \left\{ t : |\lambda(t)| \geq \Delta \gamma \right\},$$

and the implementation rule as $\delta^* = I \{ \lambda(\tau) \geq 0 \} = I \{ \lambda(\tau) = \Delta \gamma \}.$

Now, noting the form of $\lambda(t)$, we can apply similar arguments as in Shiryaev (2007, Section 4.2, Lemma 5), to show that

$$\mathbb{E}[\tau_\gamma | \mu] = \frac{2}{\Delta^2} \frac{\Delta \gamma \left( e^{\Delta \gamma} + e^{-\Delta \gamma} - 2 \right)}{e^{\Delta \gamma} - e^{-\Delta \gamma}}.$$ 

Furthermore, following Shiryaev (2007, Section 4.2, Lemma 4), we also have

$$\mathbb{P}(\delta^* = 1 | \mu) = \mathbb{P}(\lambda(\tau) = \Delta \gamma | \mu) = \frac{1 - e^{-\Delta \gamma}}{e^{\Delta \gamma} - e^{-\Delta \gamma}}.$$ 

Hence, the frequentist regret is given by

$$V(\tilde{d}_\gamma, \mu) = \frac{\sigma_1 + \sigma_0}{2} \Delta \mathbb{P}(\delta^* = 1 | \mu) + c \mathbb{E}[\tau_\gamma | \mu]$$

$$= \frac{(\sigma_1 + \sigma_0)}{2} \Delta \frac{1 - e^{-\Delta \gamma}}{e^{\Delta \gamma} - e^{-\Delta \gamma}} + \frac{2c\gamma e^{\Delta \gamma} + e^{-\Delta \gamma} - 2}{\Delta} \frac{e^{\Delta \gamma} - e^{-\Delta \gamma}}{e^{\Delta \gamma} - e^{-\Delta \gamma}}.$$ 

While the above was shown under $\mu_1 > \mu_0$, an analogous argument under $\mu_1 < \mu_0$ gives the same expression for $V(\tilde{d}_\gamma, \mu)$. □
Lemma 3. Consider a two-player zero-sum game in which nature chooses a symmetric two-point prior supported on \((\sigma_1\Delta/2, -\sigma_0\Delta/2)\) and \((-\sigma_1\Delta/2, \sigma_0\Delta/2)\) for some \(\Delta > 0\) and the DM chooses \(d_\gamma = (\pi^*, \tau_\gamma, \delta^*)\) for some \(\gamma > 0\). There exists a unique Nash equilibrium to this game at \(\Delta^* = \eta\Delta_0^*\) and \(\gamma^* = \eta^{-1}\gamma_0^*\), where \(\eta, \Delta_0^*, \gamma_0^*\) are defined in Section 3.

Proof. Let \(p_\Delta\) be the symmetric two-point prior supported on \((\sigma_1\Delta/2, -\sigma_0\Delta/2)\) and \((-\sigma_1\Delta/2, \sigma_0\Delta/2)\). By Lemma 2, the frequentist regret under a given choice of \(\Delta := \frac{2|\mu_1 - \mu_0|}{(\sigma_1 + \sigma_0)}\) and \(\gamma\) is given by \(\frac{(\sigma_1 + \sigma_0)}{2} R(\gamma, \Delta)\), where

\[
R(\gamma, \Delta) := \frac{\Delta}{e^{\Delta \gamma} - e^{-\Delta \gamma}} + \frac{2\eta^3 \gamma e^{\Delta \gamma} - e^{-\Delta \gamma} - 2}{\Delta (e^{\Delta \gamma} - e^{-\Delta \gamma})}.
\]

Lemma 2 further implies that the frequentist regret \(V(d^*, \mu)\) depends on \(\mu\) only through \(\Delta\). Therefore, the frequentist regret under both support points of \(p_\Delta\) must be the same. Hence, the Bayes regret, \(V(d_\gamma, p_\Delta)\), is the same as the frequentist regret at each support point, i.e.,

\[
V(d_\gamma, p_\Delta) = \frac{(\sigma_1 + \sigma_0)}{2} R(\gamma, \Delta).
\]

(A.4)

We aim to find a Nash equilibrium in a two-player game in which natures chooses \(p_\Delta\), equivalently \(\Delta\), to maximize \(R(\gamma, \Delta)\), while the DM chooses \(d_\gamma\), equivalently \(\gamma\), to minimize \(R(\gamma, \Delta)\).

For \(\eta = 1\), it can be verified numerically, using first order conditions on \(R(\gamma, \Delta)\), that the unique Nash equilibrium to this game is given by \(\Delta = \Delta_0^*\) and \(\gamma = \gamma_0^*\). Figure A.1 provides a graphical illustration of the Nash equilibrium.

Now, by the form of \(R(\gamma, \Delta)\), if \(\gamma_0^*\) is a best response to \(\Delta_0^*\) for \(\eta = 1\), then \(\eta^{-1}\gamma_0^*\) is a best response to \(\eta\Delta_0^*\) for general \(\eta\). Similarly, if \(\Delta_0^*\) is a best response to \(\gamma_0^*\) for \(\eta = 1\), then \(\eta\Delta_0^*\) is a best response to \(\eta^{-1}\gamma_0^*\) for general \(\eta\). This proves \(\Delta^* := \eta\Delta_0^*\) and \(\gamma^* := \eta^{-1}\gamma_0^*\) is a Nash equilibrium in the general case. \(\square\)

We now complete the proof of Theorem 1: By Lemma 1, \(d^*\) is the optimal Bayes decision corresponding to \(p_0^*\). We now show

\[
\sup_{\mu} V(d^*, \mu) = V(d^*, p_0^*),
\]

(A.5)
Note: The red curve describes the best response of $\Delta$ to a given $\gamma$, while the blue curve describes the best response of $\gamma$ to a given $\Delta$. The point of intersection is the Nash equilibrium. This is for $\eta = 1$.

**Figure A.1.** Best responses and Nash equilibrium

which implies $d^*$ is minimax optimal according to the verification theorem in Berger (2013, Theorem 17). To this end, recall from Lemma 2 that the frequentist regret $V(d^*, \mu)$ depends on $\mu$ only through $\Delta := 2|\mu_1 - \mu_0|/(\sigma_1 + \sigma_0)$. Furthermore, by Lemma 3, $\Delta^*$ is the best response of nature to $d^*$. These results imply

$$
\sup_{\mu} V(d^*, \mu) = \frac{(\sigma_1 + \sigma_0)}{2} \sup_{\Delta} R(\gamma^*, \Delta) = \frac{(\sigma_1 + \sigma_0)}{2} R(\gamma^*, \Delta^*).
$$

But by (A.4), we also have $V(d^*, p^*_0) = \frac{(\sigma_1 + \sigma_0)}{2} R(\gamma^*, \Delta^*)$. This proves (A.5).

**A.2. Proof of Theorem 2.** Our aim is to show (4.3). The outline of the proof is as follows: First, as in Adusumilli (2021), we use likelihood ratio and posterior approximation arguments to replace the probabilities, $P_{\theta_0 + h/\sqrt{n}}$, with a suitable tilted measure. Next, we apply dynamic programming arguments and viscosity solution techniques to obtain a HJB-variational inequality (HJB-VI) for the value function in the experiment. Finally, the HJB-VI is connected back to the question of optimal stopping time under diffusion asymptotics.

**Step 0 (Definitions and preliminary observations).** Under $m^*_0$, let $\gamma = 1$ denote the state $(h^*_1, -h^*_0)$ and $\gamma = 0$ the state $(-h^*_1, h^*_0)$. Also, let $y^{(a)}_{nq} := \{Y_{ai}\}_{i=1}^{[nq]}$ denote the stacked representation of outcomes $Y_{ai}$ from the first $nq$ observations corresponding to treatment $a$, and take $P_{nq_1,nq_0}$ to be the distribution corresponding to the joint density $p_{n,h(1)}(y^{(1)}_{nq}) \cdot p_{n,h(0)}(y^{(0)}_{nq}) \cdot m^*_0(h)$. Define $\bar{P}_n$ as the marginal of $P_{n,h}$ over $h$, i.e., it is the probability measure whose density, with respect to the dominating
measure $\nu(y^{(1)}_{nT}, y^{(0)}_{nT}) := \prod_{a \in \{0, 1\}} \nu(Y_{a1}) \times \cdots \times \nu(Y_{anT})$, is

$$
p_n \left( y^{(1)}_{nT}, y^{(0)}_{nT} \right) = \int p_{n,h(t)}(y^{(1)}_{nT}) \cdot p_{n,h(0)}(y^{(0)}_{nT}) dm_0(h).
$$

Due to the two-point support of $m_0^*$, the posterior density $p_n(\cdot | \xi_t)$ can be associated with a scalar,

$$
m_n(\xi_t) \equiv m_n \left( y^{(1)}_{nq1(t)}, y^{(0)}_{nq0(t)} \right) := P_n \left( \gamma = 1 | y^{(1)}_{nq1(t)}, y^{(0)}_{nq0(t)} \right).
$$

That the posterior depends on $\xi_t$ only via $y^{(1)}_{nq1}(t), y^{(0)}_{nq0}(t)$ is an immediate consequence of Adusumilli (2021, Lemma 1). Recalling the definition of $\varpi_n(\cdot)$ in (4.2), we have $\varpi_n(\xi_t) = \varpi_n(m_n(\xi_t))$, where

$$
\varpi_n(m) := \min \left\{ \{ \mu_{n,0}(h_0^*) - \mu_{n,1}(h_1^*) \} (1 - m), \{ \mu_{n,1}(h_1^*) - \mu_{n,0}(h_0^*) \} m \right\}
$$

$$
= (\mu_{n,1}(h_1^*) - \mu_{n,0}(h_0^*)) \min \{ m, 1 - m \}.
$$

The first equation above always holds, while the second holds under the simplification $\mu_{n,a}(h) = -\mu_{n,a}(-h)$ described in Section 4.

Let

$$
z_{a,nq} := \frac{I_a^{-1/2}}{\sqrt{n}} \sum_{i=1}^{[nq_a]} \psi_a(Y_{ai}),
$$

(A.6)

denote the (standardized) score process. Under quadratic mean differentiability - Assumption 1(i) - the following SLAN property holds for both treatments:

$$
\sum_{i=1}^{[nq_a]} \ln \frac{dp_{0,h/\sqrt{n}}^{(a)}}{dp_{0}^{(a)}} = h^T I_a^{1/2} z_{a,nq} - \frac{q_a}{2} h^T I_a h + o_{p,a} \left( 1 \right), \text{ uniformly over bounded } q_a.
$$

(A.7)

See Adusumilli (2021, Lemma 2) for the proof.8

We now define the tilted measure, $\lambda_{nq,h}^{(a)}(y^{(a)})$, based on the first two terms of (A.7), as the measure whose density (wrt $\nu$) is

$$
\lambda_{nq,h}^{(a)}(y^{(a)}) = \exp \left\{ h^T I_a^{1/2} z_{a,nq} - \frac{q_a}{2} h^T I_a h \right\} p_{nq,0_h}(y^{(a)}).
$$

(A.8)

8It should be noted that the score process in that paper is defined slightly differently, as $I_a^{-1/2} z_{a,nq}$ under the present notation.
Denote by $\tilde{P}_{nq_1,nq_0}$ the measure whose density is $\lambda_{n,h_1}(y^{(1)}_{nq_1}) \cdot \lambda_{n,h_0}(y^{(0)}_{nq_0}) \cdot m_0^*(h)$, and take $\tilde{P}_{nq_1,nq_0}$ to be its marginal over $h$. The density (wrt $\nu$) of $\tilde{P}_{nq_1,nq_0}$ is

$$
\tilde{p}_{nq_1,nq_0}(y^{(1)}_{nq_1}, y^{(0)}_{nq_0}) = \int \lambda_{n,h_1}(y^{(1)}_{nq_1}) \cdot \lambda_{n,h_0}(y^{(0)}_{nq_0}) \, dm_0^*(h).
$$

(A.9)

Also, let $\hat{\varphi}(t)$ be the likelihood ratio

$$
\hat{\varphi}(t) = \frac{\lambda_{n,h_1}(y^{(1)}_{nq_1}(t)) \cdot \lambda_{n,-h_0}(y^{(0)}_{nq_0}(t))}{\lambda_{n,-h_1}(y^{(1)}_{nq_1}(t)) \cdot \lambda_{n,h_0}(y^{(0)}_{nq_0}(t))} = \exp \{\Delta^* \rho(t)\},
$$

where

$$
\rho(t) := \frac{\hat{\mu}_1^T z_{1,nq_1}(t)}{\sigma_1} - \frac{\hat{\mu}_0^T z_{0,nq_0}(t)}{\sigma_0}.
$$

(A.10)

From the joint measure, $\tilde{P}_{nq_1,nq_0}$, we can obtain the posterior probability of $\gamma = 1$ (i.e., that $h = (h_1^*, -h_0^*)$ as

$$
\frac{\hat{\varphi}(t)}{1 + \hat{\varphi}(t)} = \frac{\exp \{\Delta^* \rho(t)\}}{1 + \exp \{\Delta^* \rho(t)\}} := \tilde{m}(\rho(t)),
$$

where $\tilde{m}(\rho) := \exp(\Delta^* \rho)/(1 + \exp(\Delta^* \rho))$ for $\rho \in \mathbb{R}$. Intuitively, we expect $\tilde{p}_{nq_1,nq_0}(y^{(1)}_{nq_1}, y^{(0)}_{nq_0})$ and $\tilde{m}(\rho(t))$ to be close to $\tilde{p}_n(y^{(1)}_{nT}, y^{(0)}_{nT})$ and $m_n(\xi_t)$. The posterior $\tilde{m}(\rho(t))$ also in turn implies a posterior, $\tilde{p}_n(h|\rho)$, over $h$ that takes the value $(h_1^*, -h_0^*)$ with probability $\tilde{m}(\rho)$ and $(-h_1^*, h_0^*)$ with probability $1 - \tilde{m}(\rho)$.

Step 1 (Posterior and probability approximations). Set $V_{n,T}^* = \inf_{d \in D_{n,T}} V_n^*(d, m_0^*)$. Using dynamic programming arguments, it is straightforward to show that there exists a non-randomized sampling rule and stopping time that minimizes $V_{n,T}^*(d, m_0)$ for any prior $m_0$. We therefore restrict $D_{n,T}$ to the set of all deterministic rules, $\tilde{D}_{n,T}$. Under deterministic policies, the sampling rules $\pi_{nt}$, states $\xi_t$ and stopping times $\tau$ are all deterministic functions of $y^{(1)}_{nT}, y^{(0)}_{nT}$. Recall that $y^{(1)}_{nT}, y^{(0)}_{nT}$ are the vector of outcomes under $nT$ observations of each treatment. It is useful to think of $\{\pi_{nt}\}_{t=1/n}$ as quantities mapping $(y^{(1)}_{nT}, y^{(0)}_{nT})$ to realizations of regret. Taking $\bar{E}_n[\cdot]$ to be the expectation under $\tilde{P}_n$, we then have

$$
V_{n,T}^*(d, m_0^*) = \bar{E}_n \left[ \sqrt{n \omega_n(m_n(\xi_\tau))} + cT \right],
$$

9Formally, this follows by the disintegration of measure, see, e.g., Adusumilli (2021, p.17).

10Note that $\pi, \tau$ still need to satisfy the measurability restrictions, and some components of $y^{(o)}_{nT}$ may not be observed as both treatments cannot be sampled $nT$ times.
for any deterministic \(d\).

Now, take \(\tilde{E}[]\) to be the expectation under \(\tilde{P}_n\), and define

\[
\tilde{V}_n(d, m_0^*) = \tilde{E}_n \left[ \sqrt{n} \omega_n (\tilde{m}_n (\rho(t))) + ct \right].
\]

(A.11)

Then by similar likelihood ratio and posterior approximation arguments as in Steps 1-3 of the proof of Adusumilli (2021, Theorem 5), we can show

\[
\lim_{n \to \infty} \sup_{d \in \mathcal{D}_{n,T}} \left| V_n^*(d, m_0^*) - \tilde{V}_n(d, m_0^*) \right| = 0.
\]

This in turn implies \(\lim_{n \to \infty} |V_{n,T}^* - \tilde{V}_{n,T}^*| = 0\), where \(\tilde{V}_{n,T}^* := \inf_{d \in \mathcal{D}_{n,T}} \tilde{V}_n^*(d, m_0^*)\).

**Step 2 (Recursive formula for \(\tilde{V}_{n,T}^*\)).** We now employ dynamic programming arguments to obtain a recursion for \(\tilde{V}_{n,T}^*\). This requires a bit of care since \(\tilde{P}_n\) is not a probability, even though it does integrate to 1 asymptotically.

Recall that \(\tilde{p}_n(h | \rho)\) is the probability measure on \(h\) that assigns probability \(\tilde{m}(\rho)\) to \((h_1^*, -h_0^*)\) and probability \(1 - \tilde{m}(\rho)\) to \((-h_1^*, h_0^*)\). Next, define

\[
\tilde{p}_n(Y_a | \rho) = \tilde{p}_n^{(a)}(Y_a) \cdot \int \exp \left\{ \frac{1}{\sqrt{n}} h_1^* \psi_a(Y_a) - \frac{1}{2n} h_1^* I_a h_a \right\} d\tilde{p}_n(h | \rho),
\]

\[
\tilde{\rho}_n(y_{-nq}^{(1)}, y_{-nq}^{(0)} | \rho, q_1, q_0) = \int \frac{\lambda_{n,h(1)}^{(1)}(y_{nT}^{(1)}) \cdot \lambda_{n,h(0)}^{(0)}(y_{nT}^{(0)})}{\lambda_{n,h(1)}^{(1)}(y_{nq}^{(1)}) \cdot \lambda_{n,h(0)}^{(0)}(y_{nq}^{(0)})} d\tilde{p}_n(h | \rho), \quad \text{and}
\]

\[
\eta(\rho, q_1, q_0) = \int d\tilde{p}_n \left( y_{-nq}^{(1)}, y_{-nq}^{(0)} | \rho, q_1, q_0 \right),
\]

(A.12)

where \(y_{-nq}^{(a)} := \{Y_{a(nq+1)}, \ldots, Y_{a(nT)}\}\). Note that, \(\eta(\rho, q_1, q_0)\) is the normalization constant of \(\tilde{p}_n(y_{-nq}^{(1)}, y_{-nq}^{(0)} | \rho, q_1, q_0)\).

In Lemma 5 in Appendix B.8, we show that \(\tilde{V}_{n,T}^* = \tilde{V}_{n,T}^*(0,0,0,0,0)\), where \(\tilde{V}_{n,T}^*(\cdot)\) solves the recursion

\[
\tilde{V}_{n,T}^*(\rho, q_1, q_0, t) = \min \left\{ \sqrt{n} \eta(\rho, q_1, q_0) \omega_n(\tilde{m}(\rho)), \right.
\]

\[
\frac{\eta(\rho, q_1, q_0)c}{n} + \min_{a \in \{0,1\}} \int \tilde{V}_{n,T}^* \left( \rho - \frac{(2a - 1) \mu_1^T I_a^{-1} \psi_a(Y_a)}{\sqrt{n} \sigma_a}, q_1 + \frac{a}{n}, q_0 + \frac{1 - a}{n}, t + \frac{1}{n} \right) d\tilde{p}_n(Y_a | \rho) \right\},
\]

(A.13)

for \(t \leq T\), and

\[
\tilde{V}_{n,T}^*(\rho, q_1, q_0, T) = \sqrt{n} \eta(\rho, q_1, q_0) \omega_n(\tilde{m}(\rho)).
\]
The function $\eta(\cdot)$ accounts for the fact $\tilde{P}_n$ is not a probability.

Now, Lemma 6 in Appendix B.8 shows that

$$\sup_{\rho,q_1,q_0} |\eta(\rho,q_1,q_0) - 1| \leq M n^{-\vartheta}$$

for some $M < \infty$ and any $\vartheta \in (0, 1/2)$. Furthermore, by Assumption 1(iii),

$$\lim_{n \to \infty} \sup_{m \in [0,1]} |\sqrt{n} \varpi_n(m) - \varpi(m)| = 0,$$

where $\varpi(m) := \frac{\sigma_1 + \sigma_0}{2} \Delta^* \min\{m, 1 - m\}$. Since $\varpi(\cdot)$ is uniformly bounded, it follows from (A.15) that $\sqrt{n} \varpi_n(\cdot)$ is also uniformly bounded. Then, (A.14) and (A.15) imply

$$\lim_{n \to \infty} \left| \tilde{V}_{n,T}^*(0) - \tilde{V}_{n,T}^*(0) \right| = 0,$$

where $\tilde{V}_{n,T}(\rho,t)$ is defined as the solution to the recursion

$$\tilde{V}_{n,T}^* (\rho, t) = \min \left\{ \varpi(\tilde{m}(\rho)), \frac{c}{n} + \min_{a \in \{0,1\}} \int \tilde{V}_{n,T}^* \left( \rho + \frac{(2a - 1)\mu_a I_a^{-1} \psi_a(Y_a)}{\sqrt{n} \sigma_a}, t + \frac{1}{n} \right) d\tilde{p}_n(Y_a | \rho) \right\}$$

for $t \leq T$,

$$\tilde{V}_{n,T}^* (\rho, T) = \varpi(\tilde{m}(\rho)).$$

We can drop the state variables $q_1, q_0$ in $\tilde{V}_{n,T}^* (\cdot)$ as they enter the definition of $\tilde{V}_{n,T}^* (\rho, q_1, q_0, t)$ only via $\eta(\rho, q_1, q_0)$, which was shown in (A.14) to be uniformly close to 1.

**Step 3 (PDE approximation and relationship to optimal stopping).** Let

$$\varpi(\rho) := \varpi(\tilde{m}(\rho)) = \frac{(\sigma_1 + \sigma_0) \Delta^*}{2} \min \left\{ \frac{\exp(\Delta^* \rho)}{1 + \exp(\Delta^* \rho)} \right\}.$$

Lemma 7 in Appendix B.8 shows that $\tilde{V}_{n,T}^* (\cdot)$ converges locally uniformly to $V_T^*(\cdot)$, the unique viscosity solution of the HJB-VI

$$\min \left\{ \varpi(\rho) - V_T^*(\rho, t), c + \partial_t V_T^* + \frac{\Delta^*}{2} (2\tilde{m}(\rho) - 1) \partial_\rho V_T^* + \frac{1}{2} \partial^2_\rho V_T^* \right\} = 0 \text{ for } t \leq T,$$

$$V_T^*(\rho, T) = \varpi(\rho).$$

(A.17)
Note that the sampling rule does not enter the HJB-VI. This is a consequence of the choice of the prior, \( m_0^* \).

There is a well known connection between HJB-VIs and the problem of optimal stopping that goes by the name of smooth-pasting or the high contact principle, see Øksendal (2003, Chapter 10) for an overview. In the present context, letting \( W(t) \) denote one-dimensional Brownian motion, it follows by Reikvam (1998) that

\[
V_T^*(0,0) = \inf_{\tau \leq T} \mathbb{E} [\varpi(\rho_\tau) + c\tau], \text{ where }
\]
\[
d\rho_t = \frac{\Delta^*}{2} (2\tilde{m}(\rho_t) - 1)dt + dW(t); \quad \rho_0 = 0,
\]
and \( \tau \) is the set of all stopping times adapted to the filtration \( \mathcal{F}_t \) generated by \( \rho_t \).

**Step 4 (Taking \( T \to \infty \)).** Through steps 1-3, we have shown

\[
\lim_{n \to \infty} \inf_{d \in D_{n,T}} \sup_h V_n(d, h) \geq \lim_{n \to \infty} \inf_{d \in D_{n,T}} V_n(d, m_0^*) = V_T^*(0,0).
\]

We now argue that

\[
\lim_{T \to \infty} V_T^*(0,0) = V_\infty^* := \inf_{\tau} \mathbb{E} [\varpi(\rho_\tau) + c\tau].
\]

Suppose not: Then, there exists \( \epsilon > 0 \), and some stopping time \( \bar{\tau} \) such that \( V(\bar{\tau}) := \mathbb{E} [\varpi(\rho_{\bar{\tau}}) + c\bar{\tau}] < V_T^*(0,0) - \epsilon \) for all \( T \) (note that we always have \( V_T^*(0,0) \geq V_\infty^* \) by definition). Now, \( \varpi(\cdot) \) is uniformly bounded, so by the dominated convergence theorem, \( \lim_{T \to \infty} \mathbb{E} [\varpi(\rho_{T+T})] = \mathbb{E} [\varpi(\rho_{\bar{\tau}})] \). Hence,

\[
\lim_{T \to \infty} V_T^*(0,0) \leq \lim_{T \to \infty} \mathbb{E} [\varpi(\rho_{T+T}) + c(T \wedge T)] = \mathbb{E} [\varpi(\rho_{\bar{\tau}})] + \lim_{T \to \infty} c\mathbb{E} [(T \wedge T)] \leq V(\bar{\tau}).
\]

This is a contradiction.

It remains to show \( V_\infty^* \) is the same as \( V^* \), the value of the two-player game in Theorem 1. Define

\[
m_t = \frac{\exp(\Delta^* \rho_t)}{1 + \exp(\Delta^* \rho_t)}.
\]

By a change of variables from \( \rho_t \) to \( m_t \), we can write \( V_\infty^* := \inf_{\tau} \mathbb{E} [\varpi(m_\tau) + c\tau] , \) where \( dm_t = \Delta^* m_t (1 - m_t) dW_t \) by Ito’s lemma. But by way of the proof of Lemma 1, see (A.1), this is just \( V^* \). The theorem can therefore be considered proved.
A.3. Proof of Theorem 3. For any \( h = (h_1, h_0) \), let \( P_{n,h} \) denote the joint distribution with density \( P_{h_1+h_0/\sqrt{n}}(y^{(1)}_{nT}) \cdot P_{\hat{\theta}_0+h_0/\sqrt{n}}(y^{(0)}_{nT}) \). Take \( E_{n,h}[\cdot] \) to be the corresponding expectation. We can write \( V_n(d_{n,T}, h) \) as

\[
V_n(d_{n,T}, h) = E_{n,h} \left[ \sqrt{n} (\mu_{n,1}(h_1) - \mu_{n,0}(h_0)) \mathbb{I}\{\delta_{n,T} \geq 0\} + c_{n,T} \right].
\]

Define \( \mu(h) = (\mu_1^1 h_1, \mu_0^1 h_0) \), \( \Delta \mu(h) = \mu_1^1 h_1 - \mu_0^1 h_0 \) and \( \Delta_n \mu(h) = \mu_{n,1}(h_1) - \mu_{n,0}(h_0) \). In addition, we also define \( \bar{q}_a(t) := \sigma_a t / (\sigma_1 + \sigma_0) \).

**Step 1 (Weak convergence of \( \rho_n(t) \)).** Denote \( P_{n,0} = P_{n,(0,0)} \). By the SLAN property (A.7), independence of \( y^{(1)}_{nT}, y^{(0)}_{nT} \) given \( h \), and the central limit theorem,

\[
\ln \frac{dP_{n,h}}{dP_{n,0}}(y^{(1)}_{nT}, y^{(0)}_{nT}) = \sum_{a \in \{0,1\}} \left\{ h_a I_a^{1/2} z_{a,nT} - \frac{T}{2} h_a^2 I_a h_a \right\} + o_{P_{n,0}}(1) \quad (A.18)
\]

\[
\frac{d}{P_{n,0}} \rightarrow \mathcal{N} \left( \frac{-T}{2} \sum_{a \in \{0,1\}} h_a^2 I_a h_a, T \sum_{a \in \{0,1\}} h_a^4 I_a h_a \right). \quad (A.19)
\]

Therefore, by Le Cam’s first lemma, \( P_{n,h} \) and \( P_{n,0} \) are mutually contiguous.

We now determine the distribution of \( \rho_n(t) \). We start by showing

\[
\left| \frac{\hat{\mu}_a I_a^{-1} \left[ \psi_{a}(Y_{ai}) \right]}{\sigma_a \sqrt{n}} \sum_{i=1}^{[n q_a(t) \rho]} - \frac{\hat{\mu}_a I_a^{-1} \left[ \psi_{a}(Y_{ai}) \right]}{\sigma_a \sqrt{n}} \sum_{i=1}^{[n \bar{q}_a(t) \rho]} \psi_{a}(Y_{ai}) \right| = o_{P_{n,0}}(1), \quad (A.20)
\]

uniformly over \( t \leq T \). Choose any \( b \in (1/2, 1) \). For \( t \leq n^{-b} \), we must have \( q_a(t), \bar{q}_a(t) \leq n^{-b} \), so (A.20) follows from Assumption 1(ii), which implies

\[
\sup_{1 \leq i \leq nT} |\psi_{a}(Y_{ai})| = O_{P_{n,0}}(n^{1/r}), \text{ for any } r > 0. \quad (A.21)
\]

As for the other values of \( t \), by (4.4) and (A.21),

\[
\frac{\hat{\mu}_a I_a^{-1} \left[ \psi_{a}(Y_{ai}) \right]}{\sigma_a \sqrt{n}} \sum_{i=1}^{[n q_a(t) \rho]} - \sum_{i=1}^{[n \bar{q}_a(t) \rho]} \psi_{a}(Y_{ai}) \lesssim \sqrt{n} |q_a(t) - \bar{q}_a(t)| \sup_{i} |\psi_{a}(Y_{ai})| = o_{P_{n,0}}(1),
\]

uniformly over \( t \in (n^{-b}, T] \).

Now, (A.20) implies

\[
\rho_n(t) = \frac{\hat{\mu}_1 I_1^{-1} \left[ \psi_{1}(Y_{i}) \right]}{\sigma_1 \sqrt{n}} \sum_{i=1}^{[n q_1(t) \rho]} \psi_{1}(Y_{i}) - \frac{\hat{\mu}_0 I_0^{-1} \left[ \psi_{0}(Y_{i}) \right]}{\sigma_0 \sqrt{n}} \sum_{i=1}^{[n \bar{q}_0(t) \rho]} \psi_{0}(Y_{i}) + o_{P_{n,0}}(1) \text{ uniformly over } t \leq T.
\]

(A.22)
By Donsker’s theorem, and recalling that \( \bar{q}_a(t) = \sigma_a t / (\sigma_1 + \sigma_0) \),
\[
\frac{\mu_a^T I_a^{-1} [\bar{q}_a(t) \psi_a]}{\sigma_a \sqrt{n}} \sum_{i=1} \psi_a(Y_{ai}) \xrightarrow{d,P_{n,0}} \frac{\sigma_a}{\sigma_1 + \sigma_0} W_a(\cdot),
\]
where \( W_1(\cdot), W_0(\cdot) \) can be taken to be independent Weiner processes due to the independence of \( y_{nT}^{(1)}, y_{nT}^{(0)} \) under \( P_{n,0} \). Combined with (A.22), we conclude
\[
\rho_a(\cdot) \xrightarrow{d,P_{n,0}} \bar{W}(\cdot),
\]
where \( \bar{W}(\cdot) = \sqrt{\frac{\sigma_1}{\sigma_1 + \sigma_0}} W_1(\cdot) - \sqrt{\frac{\sigma_0}{\sigma_1 + \sigma_0}} W_0(\cdot) \) is another Weiner process.

Let \( Z \) denote the normal random variable in (A.19). Equations (A.19) and (A.23) imply that \( \rho_a(\cdot), \ln \left( \frac{dP_{n,h}}{dP_{n,0}} \right) \) are asymptotically tight, and therefore, the joint \( \rho_a(\cdot), \ln \left( \frac{dP_{n,h}}{dP_{n,0}} \right) \) is also asymptotically tight under \( P_{n,0} \). Furthermore, for any \( t \in [0, T] \), it can be shown using (A.22) and (A.18) that
\[
\left( \begin{array}{c}
\rho_n(t) \\
\ln \frac{dP_{n,h}}{dP_{n,0}}
\end{array} \right) \xrightarrow{d,P_{n,h}} \left( \begin{array}{c}
\bar{W}(t) \\
Z
\end{array} \right) \sim \mathcal{N} \left( \begin{array}{c}
0 \\
\frac{-T}{2} \sum_a h_a^T I_a h_a
\end{array} \right),
\left[ t \frac{\Delta \mu(h)}{\sigma_1 + \sigma_0} t \right]
\]
Based on the above, an application of Le Cam’s third lemma as in Van Der Vaart and Wellner (1996, Theorem 3.10.12) then gives
\[
\rho_n(\cdot) \xrightarrow{d,P_{n,h}} \rho(\cdot) \quad \text{where} \quad \rho(t) := \frac{\Delta \mu(h)}{\sigma_1 + \sigma_0} t + \bar{W}(t).
\]

**Step 2 (Weak convergence of \( \delta_{n,T}, \tau_{n,T} \).)** Let \( \mathbb{D}[0, T] \) denote the metric space of all functions from \([0, T]\) to \( \mathbb{R} \) equipped with the sup norm. For any element \( z(\cdot) \in \mathbb{D}[0, T] \), define \( \tau_T(z) = T \wedge \inf \{ t : |z(t)| \geq \gamma \} \) and \( \delta_T(z) = \mathbb{I} \{ z(\tau_T(z)) > 0 \} \).

Now, under \( h = (0, 0), \rho(\cdot) \) is the Weiner process, whose sample paths take values (with probability 1) in \( \overline{C}[0, T] \), the set of all continuous functions such that \( \gamma, -\gamma \) are regular points (i.e., if \( z(t) = \gamma, z(\cdot) - \gamma \) changes sign infinitely often in any time interval \([t, t + \epsilon]\), \( \epsilon > 0 \); a similar property holds under \( z(t) = -\gamma \). The latter is a well known property of Brownian motion, see Karatzas and Shreve (2012, Problem 2.7.18), and it implies \( z(\cdot) \in \overline{C}[0, T] \) must ‘cross’ the boundary within an arbitrarily small time interval after hitting \( \gamma \) or \( -\gamma \). It is then easy to verify that if \( z_n \to z \) with \( z_n \in \mathbb{D}[0, T] \) for all \( n \) and \( z \in \overline{C}[0, T] \), then \( \tau_T(z_n) \to \tau_T(z) \) and \( \delta_T(z_n) \to \delta_T(z) \). By construction, \( \tau_{n,T} = \tau_T(\rho_n) \) and \( \delta_{n,T} = \delta_T(\rho_n) \), so by (A.23) and the extended continuous mapping theorem (Van Der Vaart and Wellner, 1996,
Theorem 1.11.1)

\[ (\tau_{n,T}, \delta_{n,T}) \xrightarrow{d_{P_n,0}} (\tau^*_T, \delta^*_T), \]

where \( \tau^*_T := \tau_T(\rho) \) and \( \delta^*_T := \delta_T(\rho) \).

For general \( h \), \( \rho(\cdot) \) is distributed as in (A.24). By the Girsanov theorem, the probability law induced on \( D[0, T] \) by the process \( \frac{\Delta \mu(h)_{\sigma_1 + \sigma_0}}{\sigma_1} + \tilde{W}(t) \) is absolutely continuous with respect to the probability law induced by \( \tilde{W}(t) \). Hence, with probability 1, the sample paths of \( \rho(\cdot) \) again lie in \( \tilde{C}[0, T] \). Then, by similar arguments as in the case with \( h = (0, 0) \), but now using (A.24), we conclude

\[ (\tau_{n,T}, \delta_{n,T}) \xrightarrow{d_{P_n,0}} (\tau^*_T, \delta^*_T). \]  

(A.25)

Step 3 (Convergence of \( V_n(d_{n,T}, h) \)). From (3.6) and the discussion in Section 3.1, it is clear that the distribution of \( \rho(t) \) is the same as that of \( \sigma_1^{-1} x_1(t) - \sigma_0^{-1} x_0(t) \) in the diffusion regime. Thus, the joint distribution, \( \mathbb{P} \), of \((\tau^*_T, \delta^*_T)\), defined in Step 2, is the same as the joint distribution of

\[ \left( \tau^*_T \equiv \tau^* \land T, \delta^*_T \equiv 1 \left\{ \frac{x_1(\tau^* \land T)}{\sigma_1} - \frac{x_0(\tau^* \land T)}{\sigma_0} \geq 0 \right\} \right) \]

in the diffusion regime, when the optimal sampling rule \( \pi^* \) is used. Therefore, defining \( d^*_T \equiv (\pi^*, \tau^*_T, \delta^*_T) \) and \( \mathbb{E}[\cdot] \) to be the expectation under \( \mathbb{P} \), we obtain

\[ V(d^*_T, \mu(h)) = \mathbb{E}[\Delta \mu(h) \delta^*_T + c \tau^*_T], \]

where \( V(d, \mu) \) denotes the frequentist regret of \( d \) in the diffusion regime. Now, recall that by the definitions stated early on in this proof,

\[ V_n(d_{n,T}, h) = \mathbb{E}_{n,h} \left[ \sqrt{n} \Delta_n \mu(h) \delta_{n,T} + c \tau_{n,T} \right]. \]

Since \( \delta_n, \tau_n \) are bounded and \( \sqrt{n} \Delta_n \mu(h) \to \Delta \mu(h) \) by Assumption 1(iii), it follows from (A.25) that for each \( h \),

\[ \lim_{n \to \infty} V_n(d_{n,T}, h) = V(d^*_T, \mu(h)). \]  

(A.26)

For any given \( h \) and \( \epsilon > 0 \), a dominated convergence argument as in Step 4 of the proof of Theorem 2 shows that there exists \( T_h \) large enough such that

\[ V(d^*_T, \mu(h)) \leq V(d^*, \mu(h)) + \epsilon \]  

(A.27)

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for all $T \geq \bar{T}_h$. Fix a finite subset $\mathcal{J}$ of $\mathbb{R}$ and define $\bar{T}_\mathcal{J} = \sup_{h \in \mathcal{J}} T_h$. Then, (A.26) and (A.27) imply

$$\liminf_{n \to \infty} \sup_{h \in \mathcal{J}} V_n(d_{n,T}, h) \leq \sup_{h \in \mathcal{J}} V(d^*_T, \mu(h)) \leq \sup_{h \in \mathcal{J}} V(d^*, \mu(h)) + \epsilon,$$

for all $T \geq \bar{T}_\mathcal{J}$. Since the above is true for any $\mathcal{J}$ and $\epsilon > 0$,

$$\sup_{\mathcal{J}} \lim_{T \to \infty} \liminf_{n \to \infty} \sup_{h \in \mathcal{J}} V_n(d_{n,T}, h) \leq \sup_{\mathcal{J}} \sup_{h \in \mathcal{J}} V(d^*, \mu(h))$$

$$\leq \sup_{\mu} V(d^*, \mu) = V^*.$$

The inequality can be made an equality due to Theorem 2. We have thereby proved Theorem 3.
B.1. Optimal tests. We start by deriving the UMP test in a setting with two simple hypotheses and show that the resulting test is also UMP more generally. Fix some $b, b_1$ with $b_1 > b$, and consider testing $H_0 : \mu = (\sigma_1 \Delta_0 + a, -\sigma_0 \Delta_0 + a)$ vs $H_1 : \mu = (\sigma_1 \Delta_1 + a, -\sigma_0 \Delta_1 + a)$, where $\Delta_0 = b / (\sigma_1 + \sigma_0)$, $\Delta_1 = b_1 / (\sigma_1 + \sigma_0)$ and $a \in \mathbb{R}$ is arbitrary. By the Neyman-Pearson lemma and (2.4), the optimal test at the $\alpha$-significance level is $\phi^* : = \mathbb{I}\{\ln \varphi^*(\tau^*) \geq c_\alpha\}$, where (after some algebra)

$$\ln \varphi^*(\tau^*) = (\Delta_1 - \Delta_0) \left(\frac{x_1(\tau^*)}{\sigma_1} - \frac{x_0(\tau^*)}{\sigma_0}\right) - \frac{\Delta^2_1 - \Delta^2_0}{2} \tau^*$$

$$= (\Delta_1 - \Delta_0) \gamma^*(2\delta^* - 1) - \frac{\Delta^2_1 - \Delta^2_0}{2} \tau^*.$$

Note that the above does not depend on $a$. Consequently,

$$\phi^* = \mathbb{I}\left\{\tau^* \leq \frac{2\gamma^*(2\delta^* - 1)}{\Delta_1 + \Delta_0} - \frac{2c_\alpha}{\Delta^2_1 - \Delta^2_0}\right\}, \quad (B.1)$$

with $c_\alpha$ being determined by the requirement that $P_b(\phi^* = 1) = \alpha$. Here, $P_b(\cdot)$ denotes the probability measure over paths induced by the process $\rho(t) := \frac{x_1(t)}{\sigma_1} - \frac{x_0(t)}{\sigma_0}$ when $\Delta \mu = b$. As noted in Section 3.3, $\tau^*$ is independent of $\delta^*$ under $P_b$, and $P_b(\delta^* = 1) = \varepsilon_b$. Hence, $c_\alpha$ is the value, always negative, such that

$$\varepsilon_b P_b \left(\tau^* \leq \frac{2\gamma^*}{\Delta_1 + \Delta_0} - \frac{2c_\alpha}{\Delta^2_1 - \Delta^2_0}\right) + (1 - \varepsilon_b) P_b \left(\tau^* \leq -\frac{2\gamma^*}{\Delta_1 + \Delta_0} - \frac{2c_\alpha}{\Delta^2_1 - \Delta^2_0}\right) = \alpha.$$

In this manner, we have determined that the UMP test of $H_0 : \mu = (\sigma_1 \Delta_0, -\sigma_0 \Delta_0)$ vs $H_1 : \mu = (\sigma_1 \Delta_1, -\sigma_0 \Delta_1)$ is of the form $\phi_b$ from Section 3.3, with

$$c^+_b, a = \frac{2\gamma^*}{\Delta_1 + \Delta_0} - \frac{2c_\alpha}{\Delta^2_1 - \Delta^2_0}, \quad c^-_b, a = \frac{-2\gamma^*}{\Delta_1 + \Delta_0} - \frac{2c_\alpha}{\Delta^2_1 - \Delta^2_0}.$$

Now consider testing $H_0 : \mu_1 - \mu_0 = b$ vs $H_1 : \mu_1 - \mu_0 = b_1$. The previous null and alternative hypotheses are special cases of the present ones. It is clear from Section 3.3 that the distribution of $(\tau^*, \delta^*)$ depends only on $\mu_1 - \mu_0$. Since $\phi^*$ is a function only of $(\tau^*, \delta^*)$, see (B.1), it follows that it has size $\alpha$ for all $\{\mu : \mu_1 - \mu_0 = b\}$. To prove $\phi^*$ is also UMP in the general testing problem, we can argue as follows: Suppose, to the contrary, there is an $\alpha$-level test, $\tilde{\phi}$, of $H_0 : \mu_1 - \mu_0 = b$ vs $H_1 : \mu_1 - \mu_0 = b_1$ that has higher power than $\phi^*$ at some $(\tilde{\mu_1}, \tilde{\mu_0})$
with $\bar{\mu}_1 - \bar{\mu}_0 = b_1$. Set $a = \bar{\mu}_1 - \sigma_1 \Delta_1$. Then, $(\sigma_1 \Delta_1 + a, -\sigma_0 \Delta_1 + a) = (\bar{\mu}_1, \bar{\mu}_0)$ and $(\sigma_1 \Delta_0 + a) - (-\sigma_0 \Delta_0 + a) = b$; hence, $\tilde{\phi}$ must also be more powerful than $\phi^*$ for testing $H_0 : \mu = (\sigma_1 \Delta_0 + a, -\sigma_0 \Delta_0 + a)$ vs $H_1 : \mu = (\sigma_1 \Delta_1 + a, -\sigma_0 \Delta_1 + a)$.
But by the previous step, this is a contradiction. We conclude $\phi^*$ is UMP.

B.2. Proof of equation (3.7). We exploit the fact that the least favorable prior has a two point support, and that the reward gap is the same under both support points. Fix some values of $c, \sigma_1, \sigma_0$. Recall the definition of $\alpha^*$ as the probability of mis-identification error from (3.5), and observe that $R^* = (\sigma_1 + \sigma_0) \Delta^* \alpha^*/2$. Furthermore, by Lemma 2,
\[
\mathbb{E}[\tau^*] = \frac{2}{\Delta^2} \frac{\Delta^* \gamma^* (e^{\Delta^* \gamma^*} + e^{-\Delta^* \gamma^*} - 2)}{e^{\Delta^* \gamma^*} - e^{-\Delta^* \gamma^*}} = \frac{2}{\Delta^2} (1 - 2\alpha^*) \ln \frac{1 - \alpha^*}{\alpha^*},
\]
where the second equality follows from the expression for $\alpha^*$ in (3.5).

Let $\theta = 1$ denote the state when $\mu = (\sigma_1 \Delta^*/2, -\sigma_0 \Delta^*/2)$ and $\theta = 0$ the state when $\mu = (-\sigma_1 \Delta^*/2, \sigma_0 \Delta^*/2)$. Because of the nature of the prior, we can think of a non-sequential experiment as choosing a set of mis-identification probabilities $\alpha_s, \beta_s$ under the two states (e.g., $\alpha_s$ is the probability of choosing treatment 0 under $\theta = 1$), along with a duration (i.e., a sample size), $T_{R^*}$. To achieve a Bayes regret of $R^*$, we would need $\alpha_s + \beta_s = 2\alpha^*$. For any $\alpha_s, \beta_s$, let $T(\alpha_s, \beta_s)$ denote the minimum duration of time needed to achieve these mis-identification probabilities. Following Shiryaev (2007, Section 4.2.5), we have
\[
T(\alpha_s, \beta_s) = \frac{\Phi^{-1}(1 - \alpha_s) + \Phi^{-1}(1 - \beta_s))^2}{\Delta^2}.
\]
Hence,
\[
T_{R^*} = \min_{\alpha_s + \beta_s = 2\alpha^*} \frac{(\Phi^{-1}(1 - \alpha_s) + \Phi^{-1}(1 - \beta_s))^2}{\Delta^2}.
\]
It can be seen that the minimum is reached when $\alpha_s = \beta_s = \alpha^*$, and we thus obtain
\[
T_{R^*} = \frac{4(\Phi^{-1}(1 - \alpha^*))^2}{\Delta^2}.
\]
Therefore,
\[
\frac{\mathbb{E}[\tau^*]}{T_{R^*}} = \frac{(1 - 2\alpha^*) \ln \frac{1 - \alpha^*}{\alpha^*}}{2(\Phi^{-1}(1 - \alpha^*))^2} \approx 0.6.
\]
B.3. **Power properties of** \( \hat{\phi}_0 \). Consider alternatives \( h = (h_1, h_0) \) such that 
\[ |\mu_1^0 h_1 - \hat{\mu}_1^0 h_0| = c > 0. \]
As described in Section 3.3, the distribution of \( \tau_{n,T} \) under \( P_{n,h} \) converges to that of \( \tau^* \) under \( |\mu_1 - \mu_0| = c \) in the diffusion setting. Now, as long as we choose \( T \geq F_0^{-1}(\alpha) \), 
\[ \mathbb{I} \{ \tau^* \wedge T \leq F_0^{-1}(\alpha) \} = \mathbb{I} \{ \tau^* \leq F_0^{-1}(\alpha) \}. \]
This leads to the following power envelope for \( \hat{\phi}_0 \):

**Lemma 4.** Suppose Assumptions 1(i)-(iii) hold. Then, for each \( h \) such that 
\[ |\mu_1^0 h_1 - \hat{\mu}_1^0 h_0| = c, \lim_{n \to \infty} P_{n,h} (\hat{\phi}_0 = 1) = F_c(F_0^{-1}(\alpha)). \]

Note that the test is unbiased (i.e., its power is always greater than \( \alpha \)) since \( F_0(\cdot) \) first order stochastically dominates \( F_c(\cdot) \).

B.4. **Proof sketch of Theorem 6.** For any \( h = (h_1, h_0) \), \( h_a \in T(P_0^{(a)}) \), let \( P_{n,h} \) denote the joint distribution \( P_{1/\sqrt{n},h_1}(y_{n,T}^{(1)}) \cdot P_{1/\sqrt{n},h_0}(y_{n,T}^{(0)}) \). Take \( \mathbb{E}_{n,h}[\cdot] \) to be the corresponding expectation. As in Section 5, we can associate each \( h_a \in T(P_0^{(a)}) \) with an element from the \( l_2 \) space of square integrable sequences \( \{h_{a,0}/\sigma_a, h_{a,1}, \ldots\} \). In what follows, we write \( \mu_a := h_{a,0} \), and define \( \mu = (\mu_1, \mu_0) \) and \( \Delta \mu = \mu_1 - \mu_0 \).

We only rework the first step of the proof of Theorem 3 as the remaining steps can be applied with minor changes.

Denote \( P_{n,0} = P_0^{(1)}(y_{n,T}^{(1)}) \cdot P_0^{(0)}(y_{n,T}^{(0)}) \). By the SLAN property (5.2), independence of \( y_{n,T}^{(1)}, y_{n,T}^{(0)} \) given \( h \), and the central limit theorem,
\[
\ln \frac{dP_{n,h}}{dP_{n,0}} (y_{n,T}^{(1)}, y_{n,T}^{(0)}) = \sum_{a \in \{0,1\}} \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{nT} h_a(Y_{ai}) - \frac{T}{2} \|h_a\|^2 \right\} + o_{P_{n,0}}(1)
\]
\[
\xrightarrow{P_{n,0}} \mathcal{N} \left( -\frac{T}{2} \sum_{a \in \{0,1\}} \|h_a\|^2, T \sum_{a \in \{0,1\}} \|h_a\|^2 \right). \tag{B.2}
\]

Therefore, by Le Cam’s first lemma, \( P_{n,h} \) and \( P_{n,0} \) are mutually contiguous. Next, define
\[ \rho_n(t) = \frac{x_1(t)}{\sigma_1} - \frac{x_0(t)}{\sigma_0}. \]

By similar arguments as in the proof of Theorem 3,
\[ \rho_n(t) = \frac{1}{\sigma_1 \sqrt{n}} \sum_{i=1}^{[n\rho_1(t)]} Y_{1i} - \frac{1}{\sigma_0 \sqrt{n}} \sum_{i=1}^{[n\rho_0(t)]} Y_{0i} + o_{P_{n,0}}(1) \text{ uniformly over } t \leq T. \tag{B.3} \]
Then, by Donsker's theorem, and recalling that \( \tilde{q}_a(t) = \sigma_a t / (\sigma_1 + \sigma_0) \), we obtain

\[
\frac{1}{\sigma_a \sqrt{n}} \sum_{i=1}^{[n\tilde{q}_a(t)]} Y_{ai} \overset{d}{\to} \sqrt{\frac{\sigma_a}{\sigma_1 + \sigma_0}} W_a(\cdot),
\]

where \( W_1(\cdot), W_0(\cdot) \) can be taken to be independent Weiner processes due to the independence of \( Y_{nT}, Y_{0T} \) under \( P_{n,0} \). Combined with (B.3), we conclude

\[
\rho_n(\cdot) \overset{d}{\to} \tilde{W}(\cdot),
\]

(B.4)

where \( \tilde{W}(\cdot) = \sqrt{\frac{\sigma_1}{\sigma_1 + \sigma_0}} W_1(\cdot) - \sqrt{\frac{\sigma_0}{\sigma_1 + \sigma_0}} W_0(\cdot) \) is another Weiner process.

Equations (B.2) and (B.4) imply that \( \rho_n(\cdot), \ln (dP_{n,h}/dP_{n,0}) \) are asymptotically tight, and therefore, the joint \( (\rho_n(\cdot), \ln (dP_{n,h}/dP_{n,0})) \) is also asymptotically tight under \( P_{n,0} \). It remains to determine the point-wise distributional limit of \( (\rho_n(\cdot), \ln (dP_{n,h}/dP_{n,0})) \) for each \( t \). By our \( l_2 \) representation of \( h_a \), we have \( h_a = (\mu_a / \sigma_a)(\psi / \sigma_a) + h_{a,-1} \), where \( h_{a,-1} \) is orthogonal to the influence function \( \psi(Y_{ai}) := Y_{ai} \). This implies \( \mathbb{E}_{n,0}[h_a(Y_{ai})Y_{ai}] = \mu_a \), and therefore, after some straightforward algebra exploiting the fact that \( Y_{nT}^{(1)}, Y_{nT}^{(0)} \) are independent iid sequences, we obtain

\[
\mathbb{E}_{n,0} \left[ \sum_a \frac{(2a - 1)}{\sigma_a \sqrt{n}} \sum_{i=1}^{[n\tilde{q}_a(t)]} Y_{ai} \right] \cdot \left[ \sum_a \frac{1}{\sqrt{n}} \sum_{i=1}^{[n\tilde{q}_a(t)]} h_a(Y_{ai}) \right] = \frac{\Delta \mu}{\sigma_1 + \sigma_0} t.
\]

Combining the above with (B.3) and the first line of (B.2), we find

\[
\begin{pmatrix}
\rho_n(t) \\
\ln \frac{dP_{n,h}}{dP_{n,0}}
\end{pmatrix}
= \begin{pmatrix} 0 \\
- \frac{T}{2} \sum_a \|h_a\|^2_a
\end{pmatrix}
+ \begin{pmatrix}
\sum_a \frac{(2a - 1)}{\sigma_a \sqrt{n}} \sum_{i=1}^{[n\tilde{q}_a(t)]} Y_{ai} \\
\sum_a \frac{1}{\sqrt{n}} \sum_{i=1}^{[n\tilde{q}_a(t)]} h_a(Y_{ai})
\end{pmatrix}
+ \ldots
\begin{pmatrix}
0 \\
- \frac{T}{2} \sum_a \|h_a\|^2_a
\end{pmatrix}
+ o_{P_{n,0}}(1)
\]

\[
\frac{d}{P_{n,0}} \begin{pmatrix} \tilde{W}(t) \\ Z \end{pmatrix} \sim \mathcal{N} \left( \begin{pmatrix} 0 \\ - \frac{T}{2} \sum_a \|h_a\|^2_a \end{pmatrix}, \begin{pmatrix} t & \frac{\Delta \mu}{\sigma_1 + \sigma_0} t \\ \frac{\Delta \mu}{\sigma_1 + \sigma_0} t & T \sum_a \|h_a\|^2_a \end{pmatrix} \right)
\]

for each \( t \), where the last step makes use of the independence of \( (Y_{n\tilde{q}_a(t)}, Y_{0\tilde{q}_a(t)}) \) and \( (Y_{-n\tilde{q}_a(t)}, Y_{-0\tilde{q}_a(t)}) \). Based on the above, an application of Le Cam’s third lemma as in Van Der Vaart and Wellner (1996, Theorem 3.10.12) then gives

\[
\rho_n(\cdot) \overset{d}{\to} \rho(\cdot) \quad \text{where} \quad \rho(t) := \frac{\Delta \mu}{\sigma_1 + \sigma_0} t + \tilde{W}(t).
\]

(B.5)
B.5. **Alternative cost functions.** We follow the basic outline of Section 3.1 and Lemmas 1-3. Our ansatz is that the least favorable prior should be within the class of indifference priors, $p_\Delta$, and the minimax decision rule should lie within the class $\tilde{d}_\gamma = (\pi^*, \tau_\gamma, \delta^*)$.

*The DM’s response to $p_\Delta.* Suppose nature employs the indifference prior $p_\Delta$. Then it is clear from the discussion in Section 3.1, and the symmetry of the sampling costs $c(\cdot)$ that the DM is indifferent between any sampling rule, and the Bayes optimal implementation rule is $\delta^* = \mathbb{1}\{\rho(t) \geq 0\}$. To determine the Bayes optimal stopping rule, we employ a similar analysis as in Lemma 1. Define

$$\tilde{c}(m) := c\left(\frac{1}{\Delta} \ln \frac{m}{1-m}\right),$$
$$\phi_c(m) := \int_{1/2}^{m} \int_{1/2}^{x} \frac{\tilde{c}(z)}{2(z(1-z))^2}dzdx.$$

Note that $\tilde{c}(\cdot)$ is the sampling cost in terms of the posterior probability $m(t)$, as $\rho(t) = \Delta^{-1} \ln \left(\frac{m(t)}{1-m(t)}\right)$. Let $\mathbb{E}[\cdot]$ denote the expectation over $\tau$ given the prior $p_\Delta$ and sampling rule $\pi$. By Morris and Strack (2019, Proposition 2),

$$\mathbb{E}\left[\int_0^\tau c(\rho(t))dt\right] = \mathbb{E}\left[\int_0^\tau \tilde{c}(m(t))dt\right] = \int_0^1 \phi_c(m)dG_\tau(m),$$

where $G_\tau(\cdot)$ is the distribution induced over $m(\tau)$ by the stopping time $\tau$. Hence, as in Lemma 1, we can suppose that instead of choosing $\tau$, the DM chooses a probability distribution $G$ over the posterior beliefs $m(\tau)$ at an ‘ex-ante’ cost

$$c(G) = \int_0^1 \phi_c(m)dG(m),$$

subject to the constraint $\int m dG(m) = m_0 = 1/2$. Hence, the Bayes optimal stopping time is the one that induces the distribution $G^*$, defined as

$$G^* = \arg\min_{G: \int m dG(m) = \frac{1}{2}} \int f(m)dG(m), \text{ where}$$

$$f(m) := \phi_c(m) + \frac{(\sigma_1 + \sigma_0)\Delta}{2} \min\{m, 1-m\}.$$

As $\phi_c'(1/2) = 0$, $f(m)$ cannot be minimized at $1/2$. Consider, then, $f(m)$ for $m \in [0, 1/2)$. In this region, $f(m) = \phi_c(m) + \frac{(\sigma_1 + \sigma_0)\Delta}{2}m$, where $\phi''(m) > 0$ by the assumption $\tilde{c}(m) > 0$. This proves $f(m)$ is convex in $[0, 1/2)$. Also,
\( \phi_c(1/2) = 0 \), and under the assumption \( c(\cdot) \geq \zeta_c \), it is easy to see that \( \phi_c(m) \to \infty \) as \( m \to 0 \), with \( \phi_c(m) \) monotonically decreasing on \( (0, 1/2) \). Taken together, these results imply \( f(m) \) has a unique minimum in \( (0, 1/2) \). Denote

\[
\alpha(\Delta) := \arg \min_{m \in (0, 1/2)} f(m).
\]

By the symmetry of sampling costs, \( f(m) = f(1 - m) \), and so the global minima of \( f(\cdot) \) are \( \alpha(\Delta), 1 - \alpha(\Delta) \). Given the constraint \( m \, d G^*(m) = 1/2 \), we conclude that \( G^* \) is a two-point distribution, supported on \( \alpha(\Delta), 1 - \alpha(\Delta) \) with equal probability \( 1/2 \). By Shiryaev (2007, Section 4.2.1), this distribution is induced by the stopping time \( \tau_{\gamma(\Delta)} \), where

\[
\gamma(\Delta) := \frac{1}{\Delta} \ln \frac{1 - \alpha(\Delta)}{\alpha(\Delta)}.
\]

This stopping time is the best response to nature’s prior \( p_\Delta \).

**Nature’s response to \( \tau_{\gamma} \).** We will determine nature’s best response to the DM choosing \( \tilde{d}_\tau \) by obtaining a formula for the frequentist regret \( V(\tilde{d}_\tau, \mu) \). Denote \( \Delta = 2(\mu_1 - \mu_0)/(\sigma_1 + \sigma_0) \), and take \( \zeta_{\Delta}(x) \) to be the solution of the ODE

\[
\frac{1}{2} \zeta''_{\Delta}(x) + \frac{\Delta}{2} \zeta'_{\Delta}(x) = c(x); \quad \zeta_{\Delta}(0) = \zeta'_{\Delta}(0) = 0.
\]

It is easy to show that the solution is

\[
\zeta_{\Delta}(x) = 2 \int_0^x e^{-\Delta y} \int_0^y e^{\Delta z} c(z) dz dy.
\]

In what follows we write \( \rho_t = \rho(t) \).

We now claim that for any stopping time, \( \tau \),

\[
\mathbb{E}_{d, \mu} \left[ \int_0^\tau c(\rho_t) dt \right] = \mathbb{E}_{d, \mu} [\zeta_{\Delta}(\rho_\tau)]. \tag{B.6}
\]

To prove the above, we start by recalling from (3.6) that

\[
\rho_t = \frac{\Delta}{2} t + \tilde{W}(t),
\]

where \( \tilde{W}(\cdot) \) is a one-dimensional Weiner process. Then, for any bounded stopping time \( \tau \), Ito’s lemma implies

\[
\zeta_{\Delta}(\rho_\tau) = \zeta_{\Delta}(\rho_0) + \frac{\Delta}{2} \int_0^\tau \zeta''_{\Delta}(\rho_t) dt + \frac{1}{2} \int_0^\tau \zeta'_{\Delta}(\rho_t) dt + \int_0^\tau \zeta'_{\Delta}(\rho_t) d\tilde{W}(t)
\]

\[
= \int_0^\tau c(\rho_t) dt + \int_0^\tau \zeta_{\Delta}(\rho_t) d\tilde{W}(t),
\]

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where the last step follows from the definition of $\zeta_\Delta(\cdot)$. This proves (B.6) for bounded stopping times. The extension to unbounded stopping times follows by a similar argument as in the proof of Proposition 2 in Morris and Strack (2019).

Recall that $\tau_\gamma := \inf\{t: |\rho_t| \geq \gamma\}$. By Lemma 2,
$$P(\rho_{\tau_\gamma} = \gamma | \mu) = \frac{1 - e^{-\Delta \gamma}}{e^{\Delta \gamma} - e^{-\Delta \gamma}}.$$  
This implies
$$\mathbb{E}_{d,\mu}[\zeta_\Delta(\rho_{\tau_\gamma})] = \frac{1 - e^{-\Delta \gamma}}{e^{\Delta \gamma} - e^{-\Delta \gamma}} \zeta_\Delta(\gamma) + \frac{e^{\Delta \gamma} - 1}{e^{\Delta \gamma} - e^{-\Delta \gamma}} \zeta_\Delta(-\gamma).$$  
(B.8)

Combining (B.6)-(B.8), we obtain
$$V(\tilde{d}_\gamma, \mu) = \frac{\sigma_1 + \sigma_0}{2} \Delta \mathbb{P}(\delta^* = 1 | \mu) + \mathbb{E}_{d,\mu}\left[\int_0^{\tau_\gamma} c(\rho_t) dt\right]$$
$$= \frac{\sigma_1 + \sigma_0}{2} \Delta \frac{1 - e^{-\Delta \gamma}}{e^{\Delta \gamma} - e^{-\Delta \gamma}} + \frac{(1 - e^{-\Delta \gamma}) \zeta_\Delta(\gamma) + (e^{\Delta \gamma} - 1) \zeta_\Delta(-\gamma)}{e^{\Delta \gamma} - e^{-\Delta \gamma}}.$$  

Thus, the best response of nature to $\tilde{d}_\gamma$ is to pick any prior supported on
$$\left\{\mu: |\mu_1 - \mu_0| = \frac{\sigma_1 + \sigma_0}{2} \Delta(\gamma)\right\},$$
where
$$\Delta(\gamma) := \arg \max_{\Delta} \left\{\frac{(\sigma_1 + \sigma_0)}{2} \frac{(1 - e^{-\Delta \gamma}) \Delta}{e^{\Delta \gamma} - e^{-\Delta \gamma}} + \frac{(1 - e^{-\Delta \gamma}) \zeta_\Delta(\gamma) + (e^{\Delta \gamma} - 1) \zeta_\Delta(-\gamma)}{e^{\Delta \gamma} - e^{-\Delta \gamma}}\right\}.$$  

Therefore, the two-point prior $p_{\Delta(\gamma)}$ is a best response to $\tilde{d}_\gamma$.

_Nash equilibrium._ By similar arguments as in the proof of Theorem 1, the Nash equilibrium is given by $(p_{\Delta^*}, \tilde{d}_{\gamma^*})$ where $(\Delta^*, \gamma^*)$ is the solution to the minimax problem
$$\min_{\gamma} \max_{\Delta} \left\{\left(\frac{\sigma_1 + \sigma_0}{2}\right) \frac{(1 - e^{-\Delta \gamma}) \Delta}{e^{\Delta \gamma} - e^{-\Delta \gamma}} + \frac{(1 - e^{-\Delta \gamma}) \zeta_\Delta(\gamma) + (e^{\Delta \gamma} - 1) \zeta_\Delta(-\gamma)}{e^{\Delta \gamma} - e^{-\Delta \gamma}}\right\}.$$  

_B.6. Analysis of other regret measures._ Following the discussion in Section 6.4, suppose that we measure regret in the implementation phase using some non-linear functional $\mu(\cdot)$ of the outcome distributions $P^{(0)}, P^{(1)}$. We assume that $\mu(\cdot)$
is a regular functional of the data, i.e., for each \( a \in \{0, 1\} \), there is a \( \psi_a \in L^2(P_0^{(a)}) \) such that

\[
\frac{\mu(P_{t,h}^{(a)}) - \mu(P_0^{(a)})}{t} - \langle \psi_a, h \rangle_a = o(t),
\]

for each of the sub-models \( \{P_{t,h}^{(a)} : t \leq \eta\} \) introduced in Section 5.\(^{11}\) The function \( \psi_a(\cdot) \) is termed the efficient influence function.

Define \( \sigma_a^2 := \mathbb{E}_{P_0^{(a)}}[\psi_a(Y_{ai})^2] \). It is possible to select \( \{\phi_{a,1}, \phi_{a,2}, \ldots\} \in T(P_0^{(a)}) \) in such a manner that \( \{\psi_a/\sigma_a, \phi_{a,1}, \phi_{a,2}, \ldots\} \) is a set of orthonormal basis functions for the closure of \( T(P_0^{(a)}) \). We can also choose these bases so they lie in \( T(P_0^{(a)}) \), i.e., \( \mathbb{E}_{P_0^{(a)}}[\phi_{a,j}] = 0 \) for all \( j \). By the Hilbert space isometry, each \( h_a \in T(P_0^{(a)}) \) is then associated with an element from the \( l_2 \) space of square integrable sequences, \( (h_{a,0}/\sigma_a, h_{a,1}, \ldots) \), where \( h_{a,0} = \langle \psi_a, h_a \rangle_a \) and \( h_{a,k} = \langle \phi_{a,k}, h_a \rangle_a \) for all \( k \neq 0 \).

Note that the above setup closely mirrors the discussion in Section 5. Indeed, when \( \mu(\cdot) \) is the mean functional, the efficient influence function is just \( \psi(Y) := Y \), as defined in that section. It is then easy to verify that the derivation of the minimax lower bound in Theorem 5, and the discussion preceding it, goes through unchanged even for general functionals.

For decision rules that attain the lower bound, consider \( d_{n,T} = (\pi_n, \tau_{n,T}, \delta_{n,T}) \), as defined in Section 4.3, but with \( x_a(t) \) in (5.4) now representing the efficient influence function process for treatment \( a \), i.e.,

\[
x_a(t) := \frac{1}{\sqrt{n}} \sum_{i=1}^{[nq_a(t)]} \psi_a(Y_{ai}),
\]

and \( \sigma_a^2 := \mathbb{E}_{P_0^{(a)}}[\psi(Y_{ai})^2] \). By the same method of proof as in Section B.4, it is easy to see that \( d_{n,T} \) attains the lower bound \( V^* \) and Theorem 6 thereby applies to general functionals as well.

### B.7. Additional simulations.

#### B.7.1. Updating \( \sigma_1, \sigma_0 \) using a prior

As noted in Section 6.3, instead of using forced sampling to estimate the values of \( \sigma_1, \sigma_0 \), we could instead employ a prior \(^{11}\)

\(^{11}\)Following Van der Vaart (2000, Section 25.3), we may restrict attention to those \( h \in T(P_0^{(a)}) \) for which the Hadamard derivative of \( \mu(P_{t,h}^{(a)}) \), as given by (B.9), exists.
and continuously update these parameter values. Here, we report simulation re-
sults from using this approach in the context of the numerical illustration from
Section 7.

Since the outcome model is Bernoulli, we employ a beta prior over the unknown
quantities $p_0, p_1$. The prior parameters are taken to be $\alpha_0 = 2, \beta_0 = 3$; these imply
that the prior is centered around $p_0(= 0.4)$. We then apply our proposed policies
while continuously updating the values of $\sigma_1, \sigma_0$ using the posterior means of $p_1, p_0$.
We experimented with alternative prior parameters, but they did not change the
results substantially.

Figure B.2, Panel A plots the finite sample frequentist regret profile of $d_n :=
d_{n,\infty}$ (i.e., $d_{n,T}$ with $T = \infty$) for various values of $n$, along with that of $d^*$ un-
der diffusion asymptotics. The approach of using the prior performs substantially
worse than forced sampling (as can be seen by comparing the figure to Figure 7.1),
with the performance being worse for higher values of $\Delta$. It appears that contin-
uously updating the prior results in more variability, leading to higher expected
regret than that under the minimax policy (for large $\Delta$). However, the minimax
regret itself is actually close to the asymptotic value, as can be seen by comparing
the maximum values of the regret profiles; in particular, the difference is less than
3%. Nevertheless, we recommend employing forced sampling in practice, at-least
for Bernoulli outcomes, as it has a strong theoretical justification.
A: Frequentist regret profiles

B: Performance under least-favorable prior

Note: The solid curve in Panel A is the regret profile of $d^*$; the vertical red line denotes $\Delta^*$. We only plot the results for $\Delta > 0$ as the values are close to symmetric. The dashed red line in Panel B is $V^*$, the asymptotic minimax regret. Black lines within the bars denote the Bayes regret in finite samples, under the least favorable prior. The bars describe the interquartile range of regret. Parameter values are $c = 1, \sigma_0 = \sigma_1 = 1$.

**Figure B.2.** Finite sample performance of $d_n$

**B.7.2. Simulations using Gaussian outcomes.** To assess the finite sample performance of the proposed policies under continuous outcomes, we ran additional Monte-Carlo simulations assuming Gaussian outcomes $Y_{ai} \sim \mathcal{N}(\mu_a/\sqrt{n}, \sigma^2_a)$ for each treatment. This is a parametric setting in which $\rho_n(t)$ has the form

$$
\rho_n(t) = \frac{1}{\sqrt{n}\sigma_1} \sum_{i=1}^{\lfloor n q_1(t) \rfloor} Y_{1i} - \frac{1}{\sqrt{n}\sigma_0} \sum_{i=1}^{\lfloor n q_0(t) \rfloor} Y_{0i}.
$$

Figure B.2, Panel A plots the finite sample frequentist regret profile of $d_n := d_{n,\infty}$ (i.e., $d_{n,T}$ with $T = \infty$) for various values of $n$, along with that of $d^*$ under diffusion asymptotics. The parameter values are $c = 1$ and $\sigma_0^2 = \sigma_1^2 = 1$. Given these parameter values, each $n$ corresponds to a sampling cost of $C = n^{-3/2}$. It is seen that diffusion asymptotics provide a very good approximation to the finite sample properties of $d_n$, even for such relatively small values of $n$ as $n = 200$. Furthermore, $d_n$ can be seen to attain the lower bound for minimax regret. Panel B of the same figure displays some summary statistics for Bayes regret under $d_n$ when nature chooses the least favorable prior, $p_{\Delta^*}$. The finite sample expected regret is very close to $V^*$, the value of minimax regret under diffusion asymptotics. We can also infer that the distribution of regret under $p_{\Delta^*}$ is positively skewed and heavy tailed.
Note: Panel A plots the size of \( \hat{T}_0 \) at the nominal 5% level. Panel B plots the finite sample power envelopes for different \( n \). The reward gap is defined as \( \Delta = |\mu_1 - \mu_0| \). Parameter values are \( c = 1 \), \( \sigma_0 = \sigma_1 = 1 \).

**Figure B.3.** Finite sample performance of \( \hat{T}_0 \)

We also assess the finite sample performance of the test \( \hat{T}_0 \), described in 4.4, for testing \( H_0 : \mu_1 - \mu_0 = 0 \) against \( H_1 : \mu_1 \neq \mu_0 \). Figure B.3, Panel A plots the size of the test for different values of \( n \) under the nominal 5% significance level. Even for relatively small values of \( n \), the size is close to nominal. Panel B of the same figure plots the finite sample power functions for this test under different values of \( n \). Note that power here is computed against local alternatives; the reward gap in that figure is the scaled one, \( \Delta = |\mu_1 - \mu_0| \). But for any given \( n \), the actual difference in mean outcomes is \( \Delta/\sqrt{n} \). Thus our test has non-trivial power against alternatives converging to 0 at the rate \( 1/\sqrt{n} \).

B.8. Supporting lemmas for the proof of Theorem 2. We suppose that Assumption 1 holds for all the results in this section.

**Lemma 5.** The function \( \tilde{V}_{n,T}^* := \inf_{d \in \tilde{D}_{n,T}} \tilde{V}_n(d, m_0) \), where \( \tilde{V}_n(d, m_0) \) is defined in (A.11), is the solution at \((0, 0, 0, 0)\) of the recursive equation (A.13).

**Proof.** In what follows, we define \( \tilde{\omega}_n(\rho) := \omega_n(\tilde{m}_n(\rho)) \).

**Step 1 (Disintegration of \( \tilde{P}_n \).** We start by presenting a disintegration result for \( \tilde{P}_n \); this will turn out to be convenient when applying a dynamic programming argument on \( \tilde{V}_{n,T}^* \). Let \( \tilde{p}_{nq_1,nq_0} \left( y_{nq_1}^{(1)}, y_{nq_0}^{(0)}, h \right) \) denote the probability density (wrt \( \nu \times \nu_1 \)) of \( \tilde{P}_{nq_1,nq_0} \), defined in Step 0 of the proof of Theorem 2, and recall the definition of \( \tilde{p}_{nq_1,nq_0} \left( y_{nq_1}^{(1)}, y_{nq_0}^{(0)} \right) \) from (A.9). By the disintegration theorem and
the definition of \( \tilde{p}_n(h|\rho) \), we have
\[
\hat{p}_{nq_1, nq_0}(y_{nq_1}^{(1)}, y_{nq_0}^{(0)}, h) = \tilde{p}_{nq_1, nq_0}(y_{nq_1}^{(1)}, y_{nq_0}^{(0)}) \cdot \tilde{p}_n(h|\rho). \tag{B.10}
\]

Note that in the above \( \rho \) is a function of \( y_{nq_1}^{(1)}, y_{nq_0}^{(0)} \) - it is defined in (A.10) - but we have elected to suppress this dependence.

Now, \( \lambda_{nT,h}(y_{nT}^{(a)}) \) can be written as
\[
\lambda_{nT,h}(y_{nT}^{(a)}) = \prod_{i=1}^{nT} \exp \left\{ \frac{h^\top}{\sqrt{n}} \psi(Y_{ai}) - \frac{1}{2n} h^\top I_a h \right\} p^{(a)}_0(Y_{ai}). \tag{B.11}
\]

Hence, for any \( q_1, q_0 \),
\[
\hat{p}_{nT,nT}(y_{nT}^{(1)}, y_{nT}^{(0)}, h) = \hat{p}_{nq_1, nq_0}(y_{nq_1}^{(1)}, y_{nq_0}^{(0)}, h) \frac{\lambda_{n,h}^{(1)}(y_{nT}^{(1)}) \cdot \lambda_{n,h}^{(0)}(y_{nT}^{(0)})}{\lambda_{n,h}^{(1)}(y_{nq_1}^{(1)}) \cdot \lambda_{n,h}^{(0)}(y_{nq_0}^{(0)})} \\
= \tilde{p}_{nq_1, nq_0}(y_{nq_1}^{(1)}, y_{nq_0}^{(0)}) \cdot \tilde{p}_n(h|\rho) \frac{\lambda_{n,h}^{(1)}(y_{nT}^{(1)}) \cdot \lambda_{n,h}^{(0)}(y_{nT}^{(0)})}{\lambda_{n,h}^{(1)}(y_{nq_1}^{(1)}) \cdot \lambda_{n,h}^{(0)}(y_{nq_0}^{(0)})},
\]

where the first equality is a consequence of (B.11) and the definition of \( \hat{p}_{nT,nT}(y_{nT}^{(1)}, y_{nT}^{(0)}, h) \), and the second equality follows from (B.10). Integrating with respect to the dominating measure, \( \nu_1(h) \), on both sides of the expression then gives (the quantity \( \tilde{p}_n(y_{-nq_1}^{(1)}, y_{-nq_0}^{(0)}|\rho, q_1, q_0) \) is defined in A.12) \(^{12}\)
\[
\hat{p}_{nT,nT}(y_{nT}^{(1)}, y_{nT}^{(0)}) = \tilde{p}_{nq_1, nq_0}(y_{nq_1}^{(1)}, y_{nq_0}^{(0)}) \cdot \tilde{p}_n(y_{-nq_1}^{(1)}, y_{-nq_0}^{(0)}|\rho, q_1, q_0). \tag{B.12}
\]

Step 2 (Relating successive values of \( \hat{p}_n(Y^{(a)} | \cdot, \cdot | \rho, q_1, q_0) \)). The quantity \( \hat{p}_n(y_{-nq_1}^{(1)}, y_{-nq_0}^{(0)}|\rho, q_1, q_0) \) specifies the density of the unobserved elements, \( y_{-nq_1}^{(1)}, y_{-nq_0}^{(0)} \), of \( y_{nT}^{(1)}, y_{nT}^{(0)} \) when the current state is \( \rho, q_1, q_0 \). In this step, we aim to characterize the density of the remaining elements of \( y_{-nq_a}^{(a)} \), if starting from the state \( \rho, q_1, q_0 \), we assign treatment \( a \) and observe the first element, \( Y_{a(nq_a+1)} \), of \( y_{-nq_a}^{(a)} \).

We start by noting that (B.10), (B.12) are valid for any \( \rho, q_1, q_0 \) as long as \( q_1, q_0 < T \). Suppose treatment 1 is employed when the current state \( y_{nq_1}^{(1)}, y_{nq_0}^{(0)} \).

---

\(^{12}\)Recall that \( \nu_1(h) \) is some dominating measure for the prior \( m_0 \). Here, it can be taken to be the counting measure on \( (-h_1^*, h_0^*) \) and \( (h_1^*, h_0^*) \).
Then, it is easily verified that
\[
\tilde{p}_{nq_1+nq_0} \left( y_{nq_1+1}, y_{nq_0}, h \right) \\
= \tilde{p}_{nq_1,nq_0} \left( y_{nq_1}, y_{nq_0}, h \right) \exp \left\{ \frac{1}{\sqrt{n}} h_1^\top \psi \left( Y_{1(nq_1+1)} \right) - \frac{1}{2n} h_1^\top I_1 h_1 \right\} \tilde{p}_0^{(1)} \left( Y_{1(nq_1+1)} \right) \\
= \tilde{p}_{nq_1,nq_0} \left( y_{nq_1}, y_{nq_0}, h \right) \tilde{p}_n(h|\rho) \exp \left\{ \frac{1}{\sqrt{n}} h_1^\top \psi \left( Y_{1(nq_1+1)} \right) - \frac{1}{2n} h_1^\top I_1 h_1 \right\} \tilde{p}_0^{(1)} \left( Y_{1(nq_1+1)} \right),
\]
where the last equality follows from (B.10). Integrating with respect to \( \nu_1(h) \) on both sides then gives\(^{13}\)
\[
\tilde{p}_{nq_1+nq_0} \left( y_{nq_1+1}, y_{nq_0} \right) = \tilde{p}_{nq_1,nq_0} \left( y_{nq_1}, y_{nq_0} \right) \tilde{p}_n \left( Y_{1(nq_1+1)} | \rho \right).
\]
(B.13)

Applying (B.12) twice, with the values \( \rho, q_1, q_0 \) and \( \rho', q_1 + \frac{1}{n}, q_0 \), and making use of (B.13) together with the definition of \( \rho \) from (A.10), we conclude
\[
\tilde{p}_n \left( y_{-nq_1}, y_{-nq_0} | \rho, q_1, q_0 \right) = \tilde{p}_n \left( y_{-nq_1-1}, y_{-nq_0} | \rho', q_1 + \frac{1}{n}, q_0 \right) \cdot \tilde{p}_n \left( Y_{1(nq_1+1)} | \rho \right),
\]
(B.14)
where \( \rho' := \rho + n^{-1/2} I_1^{-1} \psi_1 \left( Y_{1(nq_1+1)} \right) \). Analogously,
\[
\tilde{p}_n \left( y_{-nq_1}, y_{-nq_0} | \rho, q_1, q_0 \right) = \tilde{p}_n \left( y_{-nq_1-1}, y_{-nq_0-1} | \rho', q_1, q_0 + \frac{1}{n} \right) \cdot \tilde{p}_n \left( Y_{0(nq_0+1)} | \rho \right),
\]
(B.15)
with \( \rho' \) now being \( \rho - n^{-1/2} I_0^{-1} \psi_0 \left( Y_{0(nq_0+1)} \right) \).

**Step 3 (Recursive expression for \( V_{n,T}^* \)).** Suppose that at period \( j \) of the experiment, the state is \( \xi_j = (y_{nq_1}, y_{nq_0}) \) with the value of \( \rho \) being \( \rho_j \). The posterior, \( \tilde{p}_n(\cdot | \rho_j, q_1, q_0) \) provides the density of the remaining elements \( y_{nq_1}, y_{nq_0} \) of the vector \( y_{nT}, y_{nT}^0 \). By extension, we may define \( \tilde{p}_n, y_{nT}, y_{nT}^0 | \rho_j, q_1, q_0 \) as the density induced over paths \( y_{nT}, y_{nT}^0 \) given the knowledge of \( \xi_j \). This density consists of a point mass for \( y_{nq_1}, y_{nq_0} \), with the rest distributed as \( \tilde{p}_n \left( y_{nq_1}, y_{nq_0} | \rho_j, q_1, q_0 \right) \).

Let \( \mathcal{T}_j = \{ j/n, (j+1)/n, \ldots, 1 \} \) and take \( \mathcal{D}_{n,j,T} \) to be the set of all possible decision rules \( \{ \pi_n \} \) starting from period \( j \) with the usual measurability restrictions, i.e., \( \pi_n(\cdot) \in \mathcal{F}_{t-1/n} \) measurable, the stopping time \( \tau \) is sequentially \( \mathcal{F}_{t-1/n} \) measurable, and the implementation rule \( \delta \) is \( \mathcal{F}_\tau \) measurable. Recalling

\(^{13}\)The quantity \( \tilde{p}_n(Y_a | \rho) \) is defined in Step 2 of the proof of Theorem 2.
that \( \varpi_n(\rho) : = \varpi_n(\tilde{m}_n(\rho)) \), define

\[
\tilde{V}^*_n,T(\xi) = \inf_{d \in D_{n,T}} \int \left\{ \sqrt{n} \varpi_n(\rho(\tau)) + c \left( \tau - \frac{j}{n} \right) \right\} d\tilde{p}_{n,j}(\mathbf{y}^{(1)}_{nT}, \mathbf{y}^{(0)}_{nT}|\rho_j, q_1, q_0),
\]

with the convention that at \( j = 0 \),

\[
\tilde{V}^*_n,T(\xi_0) = \inf_{d \in D_{n,T}} \int \left\{ \sqrt{n} \varpi_n(\rho(\tau)) + c \tau \right\} d\tilde{p}_n(\mathbf{y}^{(1)}_{nT}, \mathbf{y}^{(0)}_{nT}).
\]

The quantity \( \tilde{V}^*_n,T(\xi_j) \) is akin to the value function at period \( j \). Note also that the quantities \( \rho(\tau), \tau \) in (B.16) are functions of \( \mathbf{y}^{(1)}_{nT}, \mathbf{y}^{(0)}_{nT} \).

Clearly, \( \tilde{V}^*_n,T = \tilde{V}^*_n,T(\xi_0) \) by definition, so the claim follows if we show: (i) \( \tilde{V}^*_n,T(\xi_j) = \tilde{V}^*_n,T(\rho_j, q_1, q_0, j/n) \), i.e., it is function only of \( (\rho_j, q_1, q_0, t = j/n) \); and (ii) it satisfies the recursion (A.13). To show this, we adopt the usual approach in dynamic programming of using backward induction.

First, we argue that the induction hypothesis holds at \( j = nT \) (corresponding to \( t = T \)). Indeed,

\[
\tilde{V}^*_n,T(\xi_{nT}) := \int \sqrt{n} \varpi_n(\rho_{nT}) d\tilde{p}_{n,nT}(\mathbf{y}^{(1)}_{nT}, \mathbf{y}^{(0)}_{nT}|\rho_{nT}, q_1, q_0)
\]

\[
= \int \sqrt{n} \varpi_n(\rho_{nT}) d\tilde{p}_n(\mathbf{y}^{(1)}_{-nq_1}, \mathbf{y}^{(0)}_{-nq_0}|\rho_{nT}, q_1, q_0) = \sqrt{n}\eta(\rho_{nT}, q_1, q_0) \varpi_n(\rho_{nT})
\]

and we can therefore write \( \tilde{V}^*_n,T(\xi_n) = \tilde{V}^*_n,T(\rho_n, q_1, q_0, T) \) as a function only of \( \rho_n, q_1, q_0, T \).

Now suppose that the induction hypothesis holds for the periods \( j + 1, \ldots, nT \). Consider the various possibilities at period \( j \). If the experiment is stopped right away, the continuation value of this choice is

\[
\tilde{V}^*_n,T(\xi_j | \tau = j) := \int \sqrt{n} \varpi_n(\rho_j) d\tilde{p}_{n,j}(\mathbf{y}^{(1)}_{nT}, \mathbf{y}^{(0)}_{nT}|\rho_j, q_1, q_0)
\]

\[
= \int \sqrt{n} \varpi_n(\rho_j) d\tilde{p}_n(\mathbf{y}^{(1)}_{-nq_1}, \mathbf{y}^{(0)}_{-nq_0}|\rho_j, q_1, q_0) = \sqrt{n}\eta(\rho_j, q_1, q_0) \varpi_n(\rho_j).
\]
On the other hand, if the experiment is continued and treatment 1 is sampled, the resulting continuation value is

\[ \hat{V}_{n,T}^{*}(\xi_j | \pi_j = 1) \]

\[ := \inf_{\{\pi_j = 1\}} \int \left\{ \sqrt{n} \omega_n (\rho(\tau)) + c \left( \tau - \frac{j}{n} \right) \right\} d\tilde{p}_{n,j} \left( Y_{nT}, Y_{nT}^{(0)} | \rho_j, q_1, q_0 \right) \]

\[ = \frac{c}{n} \int d\tilde{p}_{n,j} (Y_{nT}, Y_{nT}^{(0)} | \rho_j, q_1, q_0) + \ldots \]

\[ + \frac{n}{d \in D_{n,j+1:T}} \int \int \left\{ \sqrt{n} \omega_n (\rho(\tau)) + c \left( \tau - \frac{j + 1}{n} \right) \right\} d\tilde{p}_{n,j+1} \left( Y_{nT}, Y_{nT}^{(0)} | \rho_{j+1}, q_1 + \frac{1}{n}, q_0 \right) d\tilde{p}_n (Y_1 | \rho_j) \]

\[ = \frac{n}{\eta(\rho_j, q_1, q_0)c} + \int \hat{V}_{n,T}^{*} (\xi_{j+1}) d\tilde{p}_n (Y_1 | \rho_j) \]

\[ = \frac{n}{\eta(\rho_j, q_1, q_0)c} + \int \hat{V}_{n,T}^{*} (\rho_{j+1}, q_1 + 1, q_0, j + 1 \frac{1}{n}) d\tilde{p}_n (Y_1 | \rho_j), \]

where \( \rho_{j+1} := \rho_j + n^{-1/2} I_i^{-1} \psi_1 (Y_1) \) and \( \xi_{j+1} = \xi_j \cup \{ y_{l(nm+1)} = Y_1 \} \). The first equality follows from (B.14), the second follows from a suitable measurable selection theorem (see, e.g., Bertsekas, 2012, Proposition A.5), the third from the definition of \( \hat{V}_{n,T}^{*}(\xi_{j+1}) \), and the last equality from the induction hypothesis. In a similar vein, if treatment 0 were sampled, we would have

\[ \hat{V}_{n,T}^{*}(\xi_j | \pi_j = 0) = \frac{n}{\eta(\rho_j, q_1, q_0)c} + \int \hat{V}_{n,T}^{*}(\rho', q_1, q_0 + 1, j + 1 \frac{1}{n}) d\tilde{p}_n (Y_0 | \rho_j). \]

Now,

\[ \hat{V}_{n,T}^{*}(\xi_j) = \min \left\{ \hat{V}_{n,T}^{*}(\xi_j | \tau = j), \hat{V}_{n,T}^{*}(\xi_j | \pi_j = 1), \hat{V}_{n,T}^{*}(\xi_j | \pi_j = 0) \right\}. \quad (B.17) \]

We have shown above that each of the three terms within the minimum in (B.17) are functions only of \( \rho, q_1, q_0, j/n \). Hence, \( \hat{V}_{n,T}^{*}(\xi_j) = \hat{V}_{n,T}^{*}(\rho_j, q_1, q_0, j/n) \). Furthermore, by the expressions for these quantities, it is clear that (B.17) is none other than (A.13). This proves the induction hypothesis for period \( j \). The claim follows. \( \square \)

**Lemma 6.** There exist non-random constants, \( M < \infty \) and \( \theta \in (0, 1/2) \) such that

\[ \sup_{\rho, q_1, q_0} |\eta(\rho, q_1, q_0) - 1| \leq Mn^{-\theta}. \]
Proof. By (B.11),
\[
\frac{\lambda_{n,h}(1)^{(1)}}{\lambda_{n,h}(1)^{(0)}} \cdot \frac{\chi_{n,T}^{(1)}(Y_{nT})}{\chi_{n,T}^{(0)}(Y_{nT})} = \left\{ \prod_{i=nq_1+1}^{nT} \exp \{ h_i^T \psi_1(Y_{1i}) - \frac{1}{2} h_i^T I_1 h_i \} p_{\theta_0}^{(1)}(Y_{1i}) \right\} \cdot \left\{ \prod_{i=nq_0+1}^{nT} \exp \{ h_i^T \psi_0(Y_{0i}) - \frac{1}{2} h_i^T I_0 h_i \} p_{\theta_0}^{(0)}(Y_{0i}) \right\}.
\]

Making use of the above in the definition of \( \bar{p}_n(\cdot, \cdot; \rho, q_1, q_0) \) and applying Fubini’s theorem gives
\[
\eta(\rho, q_1, q_0) = \int \prod_{a \in \{0, 1\}} \prod_{i=nq_a+1}^{nT} \left\{ \int \exp \left( h_a^T \psi_a(Y_{ai}) - \frac{1}{2} h_a^T I_a h_a \right) p_{\theta_0}^{(a)}(Y_{ai}) dY_{ai} \right\} d\bar{p}_n(h|\rho).
\]

(B.18)

Denote
\[
g_{an}(h, Y) = \frac{1}{\sqrt{n}} h_a^T \psi_a(Y) - \frac{1}{2n} h_a^T I_a h,
\]
\[
\delta_{an}(h, Y) = \exp \{ g_{an}(h, Y) \} - \{ 1 + g_{an}(h, Y) + g_{an}(h, Y)^2/2 \},
\]
and taken \( \mathbb{E}_{p_{\theta_0}^{(a)}[\cdot]} \) to be the expectation corresponding to \( p_{\theta_0}^{(a)}(Y_{ai}) \). Then, writing the inner integral (within the \( \{ \} \) brackets) in (B.18) as \( b_a(h_a) \), we find
\[
b_a(h_a) = \mathbb{E}_{p_{\theta_0}^{(a)}} \left[ \exp \left\{ \frac{1}{\sqrt{n}} h_a^T \psi_a(Y_a) - \frac{1}{2n} h_a^T I_a h_a \right\} \right]
\]
\[
= \mathbb{E}_{p_{\theta_0}^{(a)}} \left[ 1 + g_{an}(h_a, Y_a) + \frac{1}{2} g_{an}^2(h_a, Y_a) \right] + \mathbb{E}_{p_{\theta_0}^{(a)}} [\delta_{an}(h_a, Y_a)]
\]
\[
:= Q_{n1}(h_a) + Q_{n2}(h_a).
\]

(B.19)

Since \( \psi(\cdot) \) is the score function at \( \theta_0 \), \( \mathbb{E}_{p_{\theta_0}^{(a)}[\psi_a(Y_a)]} = 0 \) and \( \mathbb{E}_{p_{\theta_0}^{(a)}[\psi_a(Y_a)\psi_a(Y_a)^T]} = I_a \). Using these results, and noting that the support of \( h_a \) is only \( \{ h_a^+, h_a^- \} \) with \( \| h_a^+ \| := \Gamma < \infty \) due to the form of the prior, some straightforward algebra implies
\[
Q_{n1}(h_a) = 1 + b_n, \quad \text{where } b_n \leq \Gamma^4/(8n^2 \text{eig}(I_a)).
\]

Here, \( \text{eig}(I_a) \) denotes the minimum eigenvalue of \( I_a \). Next, we can expand \( Q_{n2} \) as:
\[
Q_{n2}(h_a) = \mathbb{E}_{p_{\theta_0}^{(a)}} \left[ \mathbb{I}_\|\psi_a(Y_a)\| \leq k \delta_n(h_a, Y_a) \right] + \mathbb{E}_{p_{\theta_0}^{(a)}} \left[ \mathbb{I}_\|\psi_a(Y_a)\| > k \delta_n(h_a, Y_a) \right].
\]

(B.20)

Since \( \| h_a^+ \| = \Gamma \) and \( e^x - (1+x+x^2/2) = O(|x|^3) \), the first term in (B.20) is bounded by \( K^3 \Gamma^2 n^{-3/2} \) over \( h_a \in \{ h_a^+ - h_a^- \} \). Furthermore, for large enough \( n \), the second
term in (B.20) is bounded by $E_{\tilde{p}_n^{(\omega)}}[\exp \| \psi_a(Y_a) \|] / \exp(bK)$ for any $b < 1$. Hence, setting $K = (3/2b) \ln n$ gives $\sup_{h_a \in \{h^-_a, h^+_a\}} Q_n(h_a) = O \left( \ln^3 n/n^{3/2} \right)$. Combining the above, we conclude there exists some non-random $L < \infty$ such that

$$\sup_{h_a \in \{h^-_a, h^+_a\}} |b_a(h_a) - 1| \leq L n^{-c} \text{ for any } c < 3/2.$$

Substituting the above bound on $b_a(h_a)$ into (B.18) gives

$$\eta(\rho, q_1, q_0) \leq \prod_{a \in \{0,1\}} \prod_{i = nq_a + 1}^{nT} (1 + L n^{-c}) \leq (1 + L n^{-c})^{2nT} \leq 1 + M n^{-(c-1)},$$

for some $M < \infty$. Since we can choose any $c \in (0, 3/2)$, it follows $\vartheta := c - 1 \in (0, 1/2)$ and the claim follows. □

**Lemma 7.** The solution $\hat{V}_{n,T}(\rho, t)$ of (A.16) converges locally uniformly to the unique viscosity solution of the HJB-VI (A.17).

**Proof.** The proof consists of two steps. In the first step, we derive some preliminary results for expectations under the posterior $\tilde{p}_n(Y_a|\rho)$. Then, we use the abstract convergence result of Barles and Souganidis (1991) to show that $\hat{V}_{n,T}(\rho, t)$ converges locally uniformly to the viscosity solution of (A.17).

**Step 1 (Some results on moments of $\tilde{p}_n(\cdot|\rho)$).** Let $\tilde{E}^\rho[\cdot]$ denote the expectation under $\tilde{p}_n(\cdot|\rho)$. In this step, we show that there exists $\xi_n \to 0$ independent of $\rho$ and $a \in \{0,1\}$ such that

$$n \tilde{E}^\rho \left[ \frac{2a - 1}{\sqrt{n} \sigma_a} \tilde{m}_a(Y_a) \right] = \frac{\Delta^*}{2} (2\bar{m}(\rho) - 1) + \xi_n, \quad \text{and} \quad (B.21)$$

$$\tilde{E}^\rho \left[ \frac{(\hat{\mu}_a I_a^{-1} \psi_a(Y_a))^2}{\sigma_a} \right] = \frac{1}{2} + \xi_n. \quad \text{(B.22)}$$

Furthermore,

$$\tilde{E}^\rho \left[ \left| \frac{\hat{\mu}_a I_a^{-1} \psi_a(Y_a)}{\sqrt{n} \sigma_a} \right|^3 \right] < \infty. \quad \text{(B.23)}$$

Start with (B.21). Suppose $a = 1$. By the definition of $\tilde{p}_n(\cdot|\rho)$,

$$\tilde{p}_n(Y_1|\rho) = p_{b_1}^{\langle 1 \rangle}(Y_1) \left[ \bar{m}(\rho) \exp \left\{ \frac{1}{\sqrt{n}} \frac{h_1^* \psi(Y_1)}{2n} - \frac{1}{2n} h_1^* I_1 h_1^* \right\} + (1 - \bar{m}(\rho)) \exp \left\{ -\frac{1}{\sqrt{n}} h_1^* \psi(Y_1) - \frac{1}{2n} h_1^* I_1 h_1^* \right\} \right].$$
Hence,
\[
\tilde{E}^p \left[ \frac{\hat{\mu}_1 I_1^{-1} \psi_1(Y_1)}{\sigma_1} \right] = \tilde{m}(\rho) \frac{\hat{\mu}_1 I_1^{-1}}{\sigma_1} \int \psi_1(Y_1) \exp \left\{ \frac{1}{\sqrt{n}} h_1^\ast \psi_1(Y_1) - \frac{1}{2n} h_1^\ast I_1 h_1 \right\} d\rho_0(Y_1) + (1 - \tilde{m}(\rho)) \frac{\hat{\mu}_1 I_1^{-1}}{\sigma_1} \int \psi_1(Y_1) \exp \left\{ -\frac{1}{\sqrt{n}} h_1^\ast \psi_1(Y_1) - \frac{1}{2n} h_1^\ast I_1 h_1 \right\} d\rho_0(Y_1).
\]

Now, for each \( h_1 \in \{h_1^+, -h_1^+\} \), define
\[
g_{\infty}(h_1, Y) = \frac{1}{\sqrt{n}} h_1^\ast \psi_1(Y) - \frac{1}{2n} h_1^\ast I_1 h_1, \quad \text{and} \quad \delta_{\infty}(h_1, Y) = \exp\{g_{\infty}(h_1, Y)\} - \{1 + g_{\infty}(h_1, Y)\}.
\]

Then,
\[
\int \psi(Y_1) \exp \left\{ \frac{1}{\sqrt{n}} h_1^\ast \psi_1(Y_1) - \frac{1}{2n} h_1^\ast I_1 h_1 \right\} d\rho_0(Y_1) = \mathbb{E}_{\rho_0}^{(1)} \left[ \psi_1(Y_1) \exp \left\{ \frac{1}{\sqrt{n}} h_1^\ast \psi_1(Y_1) - \frac{1}{2n} h_1^\ast I_1 h_1 \right\} \right] = \mathbb{E}_{\rho_0}^{(1)} \left[ \psi_1(Y_1) \left\{ 1 + \frac{1}{\sqrt{n}} h_1^\ast \psi_1(Y_1) - \frac{1}{2n} h_1^\ast I_1 h_1 \right\} \right] + \mathbb{E}_{\rho_0}^{(1)} \left[ \psi_1(Y_1) \delta_{\infty}(h_1, Y_1) \right].
\]

Now, \( \mathbb{E}_{\rho_0}^{(1)}[\psi_1(Y_1)] = 0 \) and \( \mathbb{E}_{\rho_0}^{(1)}[\psi_1(Y_1)\psi_1(Y_1)^\top] = I_1 \). Hence, the first term in the above expression equals \( I_1 h \). As for the second term,
\[
\mathbb{E}_{\rho_0}^{(1)}[\psi_1(Y_1)\delta_{\infty}(h_1, Y_1)] = \mathbb{E}_{\rho_0}^{(1)} \left[ \mathbb{E}_{\rho_0}^{(1)} \left[ \psi_1(Y_1) \delta_{\infty}(h_1, Y_1) \right] \right]
\]
\[
+ \mathbb{E}_{\rho_0}^{(1)} \left[ \mathbb{E}_{\rho_0}^{(1)} \left[ \psi_1(Y_1) \delta_{\infty}(h_1, Y_1) \right] \right].
\]

Since \( h_1 \in \{h_1^+, -h_1^+\} \) with \( \|h_1^+\|^2 = \Gamma \), and \( e^x - (1 + x) = o(x^2) \), the first term in (B.24) is bounded by \( K^3 \Gamma^2 n^{-1} \). The second term in (B.24) is bounded by \( \mathbb{E}_{\rho_0}^{(1)}[\exp\|\psi_1(Y_1)\|]/\exp(aK) \) for any \( a < 1 \). Hence, setting \( K = (1/a) \ln n \) gives
\[
\max_{h_1 \in \{h_1^+, -h_1^+\}} \|\mathbb{E}_{\rho_0}^{(1)}[\psi_1(Y_1)\delta_{\infty}(h_1, Y_1)]\| = O(\ln^3 n/n).
\]

Combining the above results, we obtain
\[
\sqrt{n} \tilde{E}^p \left[ \frac{\hat{\mu}_1 I_1^{-1} \psi_1(Y_1)}{\sigma_1} \right] = \tilde{m}(\rho) \frac{\hat{\mu}_1 h_1^\ast}{\sigma_1} - (1 - \tilde{m}(\rho)) \frac{\hat{\mu}_1 h_1^\ast}{\sigma_1} + \xi_n
\]
\[
= (2\tilde{m}(\rho) - 1) \frac{\hat{\mu}_1 h_1^\ast}{\sigma_1} + \xi_n = (2\tilde{m}(\rho) - 1) \frac{\Delta^*}{2} + \xi_n,
\]

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where $\xi_n \approx \ln^3 n/\sqrt{n}$, and the last equality follows from the definition of $h^*_1$. In a similar manner, we can show for $a = 0$ that
\[
\sqrt{n}E^\rho \left[ \frac{\hat{m}_0 I_0^{-1} \psi_0(Y_0)}{\sigma_0} \right] = -(2\hat{m}(\rho) - 1)\frac{\Delta^*}{2} + \xi_n.
\]
This proves (B.21).

The proofs of (B.22) and (B.23) are analogous.

\textbf{Step 2 (Convergence to the HJB-VI).} We now make the time change $\tau := T - t$.

Let $\mathbb{I}_n = \{ \tau < 1/n \}$ and $\mathbb{I}_n^c = \{ \tau \geq 1/n \}$. Also, denote the state variables by $s := (\rho, \tau)$ and take $S$ to the domain of $s$. Finally, let $C^\infty(S)$ denote the set of all infinitely differentiable functions $\phi : S \to \mathbb{R}$ such that $\sup_{q \geq 0} |D^q \phi| \leq M$ for some $M < \infty$ (these are also known as test functions).

Following the time change, we can alternatively represent the solution, $\hat{V}^*_n(T)(\cdot)$, to (A.16) as solving the approximation scheme
\[
S_n(s, \phi(s), [\phi]) = 0 \quad \text{for } \tau > 0; \quad \phi(\rho, 0) = 0,
\]
where for any $u \in \mathbb{R}$ and $\phi : S \to \mathbb{R}$,
\[
S_n(s, u, [\phi]) := -\mathbb{I}_n^c \min \left\{ \frac{\varpi(\hat{m}(\rho)) - u}{n}, \frac{c}{n} + \min_{a \in \{0, 1\}} E^\rho \left[ \phi \left( \rho + \frac{(2a - 1)\hat{m}_a I_a^{-1} \psi_a(Y_a)}{\sqrt{2n}\sigma_a}, \tau - \frac{1}{n} \right) - u \right] \right\} + \\
- \mathbb{I}_n \frac{\varpi(\hat{m}(\rho)) - u}{n}.
\]

Here, $[\phi]$ refers to the fact that it is a functional argument. Define
\[
F(D^2 \phi, D\phi, \phi, s) = -\min \left\{ \varpi(\hat{m}(\rho)) - \phi, -\partial_\tau \phi + c + \frac{\Delta^*}{2}(2\hat{m}(\rho) - 1)\partial_\rho \phi + \frac{1}{2} \partial_\rho^2 \phi \right\},
\]
as the left-hand side of HJB-VI (A.17) after the time change. By Barles and Souganidis (1991), the solution, $\hat{V}^*_n(T)(\cdot)$, of (B.25) converges to the solution, $V^*_T(\cdot)$, of $F(D^2 \phi, D\phi, \phi, s) = 0$ with the boundary condition $\phi(\rho, 0) = 0$ if the scheme $S_n(\cdot)$ satisfies the properties of monotonicity, stability and consistency.

\footnote{This alternative representation does not follow from an algebraic manipulation, but can be verified by checking that the relevant inequalities hold, e.g., $\varpi(\rho) - V^*_T(\rho, t) > 0$ implies $c + \partial_\tau V^*_T + \frac{\Delta^*}{2}(2\hat{m}(\rho) - 1)\partial_\rho V^*_T + \frac{1}{2} \partial_\rho^2 V^*_T = 0$, etc.}
Monotonicity requires \( S_n(s, u, [\phi_1]) \leq S_n(s, u, [\phi_2]) \) for all \( s \in \mathcal{S} \), \( u \in \mathbb{R} \) and \( \phi_1 \geq \phi_2 \). This is clearly satisfied.

Stability requires (B.25) to have a unique solution, \( \hat{V}_{n,T}^*(\cdot) \), that is uniformly bounded. That a unique solution exists follows from backward induction. An upper bound on \( \hat{V}_{n,T}^*(\cdot) \) is \( \sup_\rho \varpi(\hat{m}(\rho)) = (\sigma_1 + \sigma_0)\Delta^*/2 \).

Finally, the consistency requirement has two parts: for all \( \phi \in C^\infty(\mathcal{S}) \), and \( s \equiv (\rho, \tau) \in \mathcal{S} \) such that \( \tau > 0 \), we require

\[
\limsup_{n \to \infty} \limsup_{\gamma \to 0} n S_n(z, \phi(z) + \gamma, [\phi + \gamma]) \leq F(D^2 \phi(s), D\phi(s), \phi(s), s), \text{ and} \quad (B.26)
\]

\[
\liminf_{n \to \infty} \liminf_{\gamma \to 0} n S_n(z, \phi(z) + \gamma, [\phi + \gamma]) \geq F(D^2 \phi(s), D\phi(s), \phi(s), s). \quad (B.27)
\]

For boundary values, \( s \in \partial \mathcal{S} \equiv \{(\rho, 0) : \rho \in \mathbb{R}\} \), the consistency requirements are (see, Barles and Souganidis, 1991)

\[
\limsup_{n \to \infty} \limsup_{\gamma \to 0} n S_n(z, \phi(z) + \gamma, [\phi + \gamma]) \leq \max \left\{ F(D^2 \phi(s), D\phi(s), \phi(s), s), \phi(s) - \varpi(\hat{m}(\rho)) \right\}, \quad (B.28)
\]

\[
\liminf_{n \to \infty} \liminf_{\gamma \to 0} n S_n(z, \phi(z) + \gamma, [\phi + \gamma]) \geq \min \left\{ F(D^2 \phi(s), D\phi(s), \phi(s), s), \phi(s) - \varpi(\hat{m}(\rho)) \right\}. \quad (B.29)
\]

Using (B.21)-(B.23), it is straightforward to show (B.26) and (B.27) by a third order Taylor expansion, see Adusumilli (2021) for an illustration. For the boundary values, we can show (B.28) as follows (the proof of (B.29) is similar): Let \( z \equiv (\rho_z, \tau) \) denote some sequence converging to \( s \equiv (\rho, 0) \in \partial \mathcal{S} \). By the definition of \( S_n(\cdot) \), for every sequence \( (n \to \infty, \gamma \to 0, z \to s) \), there exists a sub-sequence such that either \( n S_n(z, \phi(z) + \gamma, [\phi + \gamma]) = -(\varpi(\hat{m}(\rho_z)) - \phi(z)) \) or

\[
n S_n(z, \phi(z) + \gamma, [\phi + \gamma]) = -\min \left\{ \frac{\varpi(\hat{m}(\rho)) - u}{n}, \frac{c}{n} + \min_{a \in \{0, 1\}} \frac{\hat{E}_p}{\sqrt{n\sigma_a}} \left[ \phi \left( \rho + \frac{(2a - 1)\hat{\mu}_a I_a^{-1} \psi_a(Y_a)}{\sqrt{n\sigma_a}}, \tau - \frac{1}{n} \right) - u \right] \right\}.
\]

In the first instance, \( n S_n(z, \phi(z) + \gamma, [\phi + \gamma]) \to -(\varpi(\hat{m}(\rho)) - \phi(s)) \) by the continuity of \( \varpi(\hat{m}(\cdot)) \), while the second instance gives rise to the same expression for \( S_n(\cdot) \) as
being in the interior, so that $nS_n(z, \phi(z) + \gamma, [\phi + \gamma]) \to F(D^2\phi(s), D\phi(s), \phi(s), s)$ by similar arguments as that used to show (B.26). Thus, in all cases, the limit along subsequences is smaller than the right hand side of (B.28). □