Constructing the Singular Roesser State-Space Model Description of 3D Spatio-Temporal Dynamics From the Polynomial System Matrix

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ABSTRACT This paper considers systems theoretic properties of linear systems defined in terms of spatial and temporal indeterminates. These include physical applications where one of the indeterminates is of finite duration. In some cases, a singular Roesser state-space model representation of the dynamics has found use in characterizing systems theoretic properties. The representation of the dynamics of many linear systems is obtained in terms of transform variables and a polynomial system matrix representation. This paper develops a direct method for constructing the singular Roesser state-space realization from the system matrix description for 3D systems such that relevant properties are retained. Since this method developed relies on basic linear algebra operations, it may be highly effective from the computational standpoint. In particular, spatially interconnected systems of the form of the ladder circuits are considered as the example. This application confirms the usefulness and effectiveness of the proposed method independently of the system spatial order.

INDEX TERMS nD systems, polynomial system matrix, Roesser model, spatio-temporal dynamics.

I. INTRODUCTION

This paper considers linear systems described in terms of spatial and temporal indeterminates, which belong to the general class of nD, n > 1, linear systems. Often the dynamics of the system are described by algebraic equations obtained by applying transforms. The result is a system matrix description of the dynamics where the defining entries are polynomials in more than one indeterminate. Moreover, a transfer-function matrix description can be constructed if required.

In linear systems theory, transformations between model representations are standard in analysis and controller design. This general area is more involved and less well developed for nD linear systems, for which this paper gives new results.

These provide new techniques for onward use in systems theoretic analysis, stability analysis, and controller design.

The particular case of n = 1, sometimes termed 1D systems, there is one indeterminate (time), and primeness of polynomial matrix pairs is critical to the analysis of systems properties, e.g., controllability and observability, together with their preservation under equivalence transformations. Moreover, the construction of a state-space model from the system matrix or transfer-function matrix is well understood. The nD systems case is more complicated due to the underlying ring structure. Specifically, coprimeness is no longer a single property, and for n ≥ 3 zero, factor and minor coprimeness are distinct properties (see e.g., [3], [14] and references therein).

A possible starting point for analyzing nD linear systems is system equivalence, defined in terms of the associated
system matrix. Of significant importance, and especially for applications-related research, is constructing state-space models from the system matrix. Research on this problem can, e.g., be found in [2], [4], [5] for the Roesser model [19], and for the Fornassini-Marchessini model [10] in, e.g., [6], [7].

In most cases, the starting point is the so-called multivariate polynomial system matrix, which can be directly obtained for a given multidimensional system, e.g., the homogeneous, spatially distributed electrical system used to demonstrate the new results in this paper. However, for some problems, there is a need to construct a state-space model. This paper addresses the problem of transforming a system matrix of a 3D linear system to a singular (where in many cases, non-singular models do not exist), 3D Roesser [19] state-space model for examples where there are spatial and temporal indeterminates. It is established that this transformation can be completed by applying a sequence of elementary row and column operations together with an expansion of the original system. Moreover, generalization to the nD case, n > 3, indeterminates is immediate.

The procedure developed relies on basic linear algebra operations and is therefore highly effective from the computational standpoint. Note that the overall computational effort required is determined by the number of states, inputs, and outputs in the system. Consider a system with \( r, l \) and \( m \) entries, respectively, the state, input, and output vectors, respectively. This paper’s new results will produce a singular 3D Roesser model of the dynamics that has order \( 8r + 15m + 22l \). Hence, the overall increase in the order is significant. However, the resulting model matrices are sparse, and this fact could be exploited by using known results for sparse matrices manipulations, see e.g. [8].

A physical circuit area is used to illustrate the application of the results. Note also that the use of elementary operations discussed in this paper is equivalent to applying multivariate polynomial matrix operations. Also, similar results and equivalent approaches can be found, e.g., in [1]. Further, these results could be the basis for the design of iterative learning control laws, see, e.g., [16].

System equivalence has been the subject of considerable attention in the literature for multidimensional systems see, e.g., [9], [11]–[13], [15], [17], but this was mostly for 2D systems. In contrast, this paper considers 3D case systems. Moreover, the example given to highlight this paper’s new results is a physical example from the class of spatially distributed systems. As these results are the first from this approach to 3D systems, comparison with other approaches will have to await future research results.

In this paper, the notation \( O_{e,h} \) and \( I_e \) denotes, respectively, the \( e \times h \) null matrix and the \( e \times e \) identity matrix. In cases where the dimensions are immediate, the subscripts will not be shown. Also, \( \equiv \) denotes matrix equivalence.

II. PRELIMINARIES

Consider a general 3D discrete linear system described in the polynomial form as

\[
P(z_1, z_2, z_3) \begin{bmatrix} x \\ -u \end{bmatrix} = \begin{bmatrix} 0 \\ -y \end{bmatrix},
\]

where \( x \in \mathbb{R}^r \) is the state vector, \( u \in \mathbb{R}^l \) is the input vector, and \( y \in \mathbb{R}^m \) is the output vector, and \( z_i, 1 \leq i \leq 3 \), are the shift operators, which can unlike in the 1D systems case, be temporal or spatial. An example in the 2D case is linear repetitive process [20], where there is one spatial and one temporal variable and the latter is of the finite duration. The stability theory for such processes can be used to design iterative learning control laws, see, e.g., [16]. Moreover, other forms of iterative learning control dynamics can involve 3D (or nD with \( n > 3 \), in general) dynamics [21].

The associated system matrix for (1) is

\[
P(z_1, z_2, z_3) = \begin{bmatrix} T(z_1, z_2, z_3) & U(z_1, z_2, z_3) \\ -V(z_1, z_2, z_3) & W(z_1, z_2, z_3) \end{bmatrix},
\]

where \( T, U, V \) and \( W \), respectively, are polynomial matrices with entries in \( \mathbb{R}[z_1, z_2, z_3] \) and of dimensions \( r \times r, r \times l, m \times r \) and \( m \times l \). This matrix characterizes the dynamics of the example considered. In many examples, no direct interaction between the system inputs and outputs of the system exists and, in any case, no loss of generality arises from setting \( W(z_1, z_2, z_3) = 0 \).

The following definitions and results are used in the development of the new results in this paper.

Definition 1: Let \( \mathbb{P}(m, l) \) denote the class of \( (r + m) \times (r + l) \) polynomial matrices corresponding to a system with \( l \) inputs and \( m \) outputs, which, by an obvious expansion, can be taken as having the same number of inputs and outputs. Two polynomial system matrices \( P_1(z_1, z_2, z_3) \) and \( P_2(z_1, z_2, z_3) \) are said to be zero coprime equivalent, if there exist polynomial matrices \( S_1(z_1, z_2, z_3) \) and \( S_2(z_1, z_2, z_3) \) of compatible dimensions such that

\[
S_1(z_1, z_2, z_3)P_2(z_1, z_2, z_3) = P_1(z_1, z_2, z_3)S_2(z_1, z_2, z_3),
\]

where \( P_1, S_1 \) are zero left coprime and \( P_2, S_2 \) are zero right coprime. In the case when \( P_1 \) and \( P_2 \) are of the same dimensions and \( S_1 \) and \( S_2 \) are unimodular, \( P_1 \) and \( P_2 \) are said to be unimodular equivalent.

In [18] it was shown that zero coprime equivalence characterizes fundamental algebraic properties of the system amongst its invariants.

A essential transformation developed for the nD system matrices study is zero coprime system equivalence, see, e.g., [11], [13]. This transformation is based on zero coprime equivalence, which is characterized by the following definition.

Definition 2: Two polynomial system matrices \( P_1(z_1, z_2, z_3) \) and \( P_2(z_1, z_2, z_3) \in \mathbb{P}(m, l) \) are said to be zero coprime system
equivalent (ZCSE), if they are related as

\[
\begin{bmatrix}
M(z_1, z_2, z_3) & 0 \\
X(z_1, z_2, z_3) & I_m \\
S_1(z_1, z_2, z_3)
\end{bmatrix}
\begin{bmatrix}
P_2(z_1, z_2, z_3) \\
S_2(z_1, z_2, z_3)
\end{bmatrix}
= \begin{bmatrix}
P_1(z_1, z_2, z_3) \\
0
\end{bmatrix}
\begin{bmatrix}
N(z_1, z_2, z_3) & Y(z_1, z_2, z_3) \\
0 & I_n
\end{bmatrix},
\]

(4)

where \( P_1, S_1 \) are zero left coprime, \( P_2, S_2 \) are zero right coprime and \( M(z_1, z_2, z_3), N(z_1, z_2, z_3), X(z_1, z_2, z_3) \) and \( Y(z_1, z_2, z_3) \) are polynomial matrices of compatible dimensions.

In the case when the system matrices have the same dimensions and \( M(z_1, z_2, z_3) \) and \( N(z_1, z_2, z_3) \) are unimodular, the transformation of ZCSE reduces to that of unimodular system equivalence, were two polynomial system matrices are ZCSE if, and only if, a trivial expansion or deflation of one is unimodular system equivalent to a trivial expansion or deflation of the other. The ZCSE property has a crucial role in several aspects of nD systems theory, see, e.g., [11], [13], [18], where the relevant property for the new developments in this paper is the following.

Lemma 1: [11] The transformation of ZCSE preserves the zero structure of the matrices

\[
T(z_1, z_2, z_3), \quad P(z_1, z_2, z_3),
\begin{bmatrix}
T(z_1, z_2, z_3) & U(z_1, z_2, z_3)
\end{bmatrix},
\begin{bmatrix}
T(z_1, z_2, z_3) \\
V(z_1, z_2, z_3)
\end{bmatrix}.
\]

In the 2D systems case, an equivalent singular Roesser state-space model description of the dynamics provided the route to the characterization of a controllability property that led to stabilizing controller’s characterization and design. Therefore, it is conjectured that a similar role exists for 3D (or nD with \( n > 3 \) in general) linear systems, which is considered in the remainder of this paper.

Let \( y(i, j, k), x(i, j, k) \) and \( u(i, j, k) \) be the output, state and input vectors for a linear system, where the indeterminates \( i, j \) and \( k \) correspond to the directions of information propagation. Then, with the same output, state and input dimensions as (1), the singular 3D Roesser state-space model has the form

\[
\begin{bmatrix}
x_1(i+1, j, k) \\
x_2(i, j+1, k) \\
x_3(i, j, k+1)
\end{bmatrix} = \begin{bmatrix}
x_1(i, j, k) \\
x_2(i, j, k) \\
x_3(i, j, k)
\end{bmatrix} + Bu(i, j, k),
\]

\[
y(i, j, k) = C \begin{bmatrix}
x_1(i, j, k) \\
x_2(i, j, k) \\
x_3(i, j, k)
\end{bmatrix} + Du(i, j, k),
\]

(5)

where the matrix \( E \) is singular. Also define \( z_1, z_2 \) and \( z_3 \), respectively, as the unit forward shift operator acting on \( i, j \) and \( k \). This fact applied to (5) with assumed zero state initial conditions gives

\[
P_{SR}(z_1, z_2, z_3) \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix},
\]

(6)

with associated system matrix:

\[
P_{SR}(z_1, z_2, z_3) = \begin{bmatrix}
z_1E_1 + z_2E_2 + z_3E_3 - A & B \\
-C & D
\end{bmatrix},
\]

(7)

where \( E_1, E_2, E_3 \) are obtained from \( E \) by compatible partitioning. This last matrix is a particular case of (2) and the next section develops an algorithm for constructing the corresponding 3D singular Roesser state-space model for the case when \( W(z_1, z_2, z_3) = 0 \) and therefore \( D = 0 \) in (5).

III. CONSTRUCTING THE 3D SINGULAR ROESSER MODEL

Consider again (1) and (2). Then, the first stage in obtaining the singular 3D Roesser state-space model with a preserved zero structure (which is the essential requirement), for a system described by these equations is to construct the \([r + m + l + m] \times [(r + m + l + l] \) normalized form, \( P_n, \) of \( P \), i.e.,

\[
P_n(z_1, z_2, z_3) = \begin{bmatrix}
P(z_1, z_2, z_3) & F_{r+m,m} \end{bmatrix} \begin{bmatrix}
0 & l_l \\
0 & -I_m
\end{bmatrix},
\]

(8)

where \( F_{r,m} = [0_{β×(α−β)} I_β] \). Also, it can be easily verified that the normalized system matrix \( P_n \) is ZCSE to the system matrix \( P \).

Consider now \( P_n \) in the form

\[
P_n = \begin{bmatrix}
T & U & 0 & 0 \\
-V & 0 & I_m & 0 \\
0 & -I_l & 0 & I_l \\
0 & 0 & -I_m & 0
\end{bmatrix}
\]

(9)

and apply, in turn, the following block elementary row and column operations, where \( R_e \) and \( C_j \) denote the \( e \)th block row and the \( j \)th column, respectively, of the underlying matrix. In these operations, \( \leftarrow \) denotes the updating the variable on the left-hand side by the result of the operation on the right-hand side, and \( \leftrightarrow \) denotes exchanging the entries on the left and right-hand sides.

\[
\begin{align*}
C_1 & \leftarrow C_1 + C_3V, \\
R_1 & \leftarrow R_1 + UR_3, \\
R_4 & \leftarrow R_4 + R_2, \\
C_4 & \leftarrow C_4 + C_2, \\
C_1 & \leftarrow C_3, \\
R_1 & \leftrightarrow R_2, \\
R_2 & \leftrightarrow R_3, \\
R_2 & \leftarrow -I_lR_2.
\end{align*}
\]

(10)

These operations applied to (9) result in the trivially expanded system matrix, denoted by \( P_r, \) of \( P \) as

\[
P_r = \begin{bmatrix}
I_{m+l} & 0 & 0 \\
0 & T & U \\
0 & -V & 0
\end{bmatrix} = \begin{bmatrix}
I_{m+l} & 0 & 0 \\
0 & T & U \\
0 & -V & 0
\end{bmatrix}
\]

(11)
and it follows that $P_n$ and $P$ are related by the following ZCSE transformation:

$$
\begin{bmatrix}
I_r & 0_{r,m} \\
0_{m,r} & 0_{m,m} \\
0_{l,r} & 0_{l,m} \\
0_{m,r} & I_m
\end{bmatrix}
\begin{pmatrix}
T & U \\
-V & 0
\end{pmatrix}
\begin{bmatrix}
T & U \\
-V & 0
\end{pmatrix} = \begin{bmatrix}
I_{q_1} & 0 & 0 & 0 \\
0 & -I_m & 0 & 0 \\
0 & 0 & I_l & 0 \\
0 & 0 & 0 & -I_l
\end{bmatrix}
$$

(12)

Let $r_1 = r + m + l$, $s_1 = r_1 + m$ and $q_1 = r_1 + l$ and write $P_n$ as

$$
P_n \equiv \begin{bmatrix}
T_{n,0} & U_{n,0} \\
-V_{n,0} & 0
\end{bmatrix} + z_1 \begin{bmatrix}
T_{n,1} & U_{n,1} \\
-V_{n,1} & 0
\end{bmatrix},
$$

(13)

where $P_{n,0}$ and $P_{n,1}$ are $s_1 \times q_1$ polynomial matrices over $\mathbb{R}[z_2, z_3]$. Now let

$$
F_1 = \begin{bmatrix}
I_{q_1} & 0 & 0 & 0 \\
P_{n,1} & I_{q_1} & 0 & 0 \\
0 & 0 & I_l & 0 \\
0 & 0 & 0 & I_l
\end{bmatrix}
$$

and

$$
H_1 = \begin{bmatrix}
I_{q_1} & -z_1 I_{q_1} & 0 & 0 \\
0 & I_{q_1} & 0 & 0 \\
0 & 0 & I_m & 0 \\
0 & 0 & 0 & I_l
\end{bmatrix}
$$

(14)

where it is routine to verify that $F_1$ and $H_1$ are unimodular. Also

$$
F_1 P_{nne} H_1 = Q
$$

$$
= \begin{bmatrix}
I_{r_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_l & 0 & -z_1 I_l & 0 & 0 & 0 & 0 \\
0 & 0 & T_{n,1} & U_{n,1} & 0 & 0 & 0 & 0 \\
0 & 0 & -V_{n,1} & 0 & I_m & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -I_l & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -I_m & 0 & 0
\end{bmatrix}
$$

(15)

where $P_{nne}$ is given by:

$$
P_{nne} = \begin{bmatrix}
I_{q_1} & 0 & 0 & 0 \\
0 & P_n & F_{s_1,m} & 0 \\
0 & 0 & -F_{q_1,l}^T & 0 \\
0 & 0 & 0 & I_l
\end{bmatrix}
$$

(16)

To show that $P_{nne}$ is the unimodular system equivalent trivial expansion of $P_n$, apply the following elementary row and column operations (similar to (10)) to $P_{nne}$

$$
R_5 \leftarrow R_5 + R_2, \quad C_2 \leftarrow C_4, \\
C_2 \leftarrow C_3.
$$

(17)

to obtain

$$
\begin{bmatrix}
I_{q_1} & 0 & 0 & 0 \\
0 & -I_m & 0 & 0 \\
0 & 0 & I_l & 0 \\
0 & 0 & -F_{s_1,m}^T & -F_{q_1,l}^T
\end{bmatrix} P_n
$$

(18)

Expand, next $F_{s_1,m}$ and $F_{q_1,l}$ to obtain

$$
\begin{bmatrix}
I_{q_1} & 0 & 0 & 0 \\
0 & -I_m & 0 & 0 \\
0 & 0 & I_l & 0 \\
0 & 0 & 0 & -I_l
\end{bmatrix}
$$

(19)

and then apply the elementary operations

$$
R_5 \leftarrow R_5 + R_2, \quad C_5 \leftarrow C_3 + C_5, \\
R_2 \leftarrow -I_m R_2.
$$

(20)

These steps result in

$$
P_{nne} = \begin{bmatrix}
I_{q_1 + m + l} & 0 \\
0 & P_n
\end{bmatrix}
$$

(21)

and it follows in that $Q$ and $P_n$ are related by the ZCSE transformation

$$
\begin{bmatrix}
0_{r_1,r_1} & 0_{r_1,m} \\
0_{l,r_1} & 0_{l,m} \\
I_{r_1} & 0_{r_1} \\
0_{m,r_1} & 0_{m,m} \\
0_{l,r_1} & 0_{l,l} \\
0_{m,r_1} & 0_{m,m}
\end{bmatrix}
\begin{bmatrix}
T_n & U_n \\
-V_n & 0
\end{bmatrix} = Q
$$

(22)

where

$$
Q S_4 = \begin{bmatrix}
0_{r_1,r_1} & 0_{r_1,l} \\
0_{l,r_1} & 0_{l,l} \\
T_n & U_n \\
0_{m,r_1} & 0_{r_1} \\
0_{l,r_1} & 0_{l,l} \\
0_{m,r_1} & 0_{m,m}
\end{bmatrix}
$$

(23)

and therefore $Q$ is ZCSE to $P_n$ and $P$. Also

$$
Q = \begin{bmatrix}
T_Q & U_Q \\
-V_Q & 0
\end{bmatrix} + z_2 \begin{bmatrix}
T_{Q,1} & U_{Q,1} \\
-V_{Q,1} & 0
\end{bmatrix},
$$

(24)

where $Q_0$ and $Q_1$ are $s_2 \times q_2$ polynomial matrices over $\mathbb{R}[z_1, z_3]$ and $s_2 = 2r_1 + m + 2l$ and $q_2 = 2r_1 + m + 3l$.

Let

$$
F_2 = \begin{bmatrix}
I_{q_1} & 0 & 0 & 0 \\
0 & I_l & 0 & 0 \\
0 & 0 & I_m & 0 \\
0 & 0 & 0 & I_l
\end{bmatrix}
$$

(25)

and introduce $r_2 = 2r_1 + m + 2l$. Also the matrices $F_2$ and $H_2$ are unimodular and hence

$$
F_2 Q_{nne} H_2
$$
\[ R = \begin{bmatrix}
I_{q_2} & -z_2I_{q_2} & 0 & 0 \\
Q_1 & Q_0 & F_{s_2,m} & 0 \\
0 & -F_{q_2,l}^T & I_l & 0 \\
0 & 0 & -I_m & 0
\end{bmatrix} \]

where \( R_0 \) and \( R_1 \) are \( s_3 \times q_3 \) polynomial matrices over \( \mathbb{R}[z_1, z_2] \). Also introduce

\[
F_3 = \begin{bmatrix}
I_{q_3} & 0 & 0 & 0 \\
R_1 & I_{s_3} & 0 & 0 \\
0 & 0 & I_l & I_m \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
H_3 = \begin{bmatrix}
I_{q_3} & -z_3I_{q_3} & 0 & 0 \\
0 & I_{s_3} & 0 & 0 \\
0 & 0 & I_l & I_m \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

which are unimodular and

\[
F_3R_{ne}H_3 = \tilde{P}
\]

\[
\begin{bmatrix}
I_{r_3} & 0 & -z_3I_{r_3} & 0 & 0 & 0 \\
0 & I_l & 0 & -z_3I_l & 0 & 0 \\
T_{R,1} & U_{R,1} & T_{R,0} & U_{R,0} & 0 & 0 \\
0 & 0 & -V_{R,1} & 0 & 0 & I_m \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]

where

\[
\begin{bmatrix}
I_{r_3} & 0 & -z_3I_{r_3} & 0 & 0 & 0 \\
0 & I_l & 0 & -z_3I_l & 0 & 0 \\
T_{R,1} & U_{R,1} & T_{R,0} & U_{R,0} & 0 & 0 \\
0 & 0 & -V_{R,1} & 0 & 0 & I_m \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Applying block elementary row and column operations defined by (17) to \( R_{ne} \) gives the unimodular system equivalent trivial expansion of \( R \) as

\[
R_e = \begin{bmatrix}
I_{q_3} & 0 & 0 & 0 \\
0 & R & F_{s_3,m} & 0 \\
0 & 0 & -F_{q_3,l}^T & 0 \\
0 & 0 & 0 & -I_m
\end{bmatrix},
\]

where

\[
\begin{bmatrix}
I_{q_3} & 0 & 0 & 0 \\
0 & R & F_{s_3,m} & 0 \\
0 & 0 & -F_{q_3,l}^T & 0 \\
0 & 0 & 0 & -I_m
\end{bmatrix}
\]

Applying the block elementary row and column operations defined by (17) to \( R_{ne} \) gives the unimodular system equivalent trivial expansion of \( R \) as

\[
R_e = \begin{bmatrix}
I_{q_3} & 0 & 0 & 0 \\
0 & R & F_{s_3,m} & 0 \\
0 & 0 & -F_{q_3,l}^T & 0 \\
0 & 0 & 0 & -I_m
\end{bmatrix},
\]

and it follows that \( \tilde{P} \) and \( R \) are related by the ZCSE transformation

\[
\begin{bmatrix}
0_{r_2,r_2} & 0_{r_2,m} & 0_r & 0 \\
0_{l_r,l_r} & 0_{l_m} & 0_l & 0 \\
I_r & 0 & I_l & 0 \\
0_{m_r,m_r} & 0_{m_m} & 0_m & 0 \\
0_{l_r,l_m} & I_m & 0 & 0 \\
\end{bmatrix}
\]

where \( \tilde{P} \) given in (34) is the \( s_4 \times q_4 \) polynomial matrix over \( \mathbb{R}[z_1, z_2, z_3] \). (s4 = s3 + q3 + l + m, s4 = q3 + m + l) and

\[
S_8 = \begin{bmatrix}
z_2I_{r_2} & 0 & z_2I_l & 0 \\
0 & I_r & 0 & I_l \\
V_Q & 0 & 0 & 0 \\
0 & 0 & I_l & 0 \\
\end{bmatrix}
\]

Therefore \( S_8 \) is ZCSE to \( \tilde{P} \) and \( P_n \) and \( P \).

Let \( r_3 = q_2 + s_2 + l, s_3 = r_3 + m \) and \( q_3 = r_3 + l \) and write \( R \) as

\[
R = \begin{bmatrix}
T_R & U_R \\
-V_R & 0
\end{bmatrix} = R_0 + z_3R_1
\]

\[
R = \begin{bmatrix}
T_{R,0} & U_{R,0} \\
-V_{R,0} & 0
\end{bmatrix} + z_3 \begin{bmatrix}
T_{R,1} & U_{R,1} \\
-V_{R,1} & 0
\end{bmatrix},
\]

where

\[
\begin{bmatrix}
T_{R,0} & U_{R,0} \\
-V_{R,0} & 0
\end{bmatrix} + z_3 \begin{bmatrix}
T_{R,1} & U_{R,1} \\
-V_{R,1} & 0
\end{bmatrix}
\]

and it follows in that \( Q \) and \( R \) are related by the following ZCSE transformation

\[
Q_{ne} = \begin{bmatrix}
I_{q_2} & 0 & 0 & 0 \\
0 & Q^T & F_{s_2,m} & 0 \\
0 & 0 & -F_{q_2,l}^T & 0 \\
0 & 0 & 0 & -I_m
\end{bmatrix}
\]

and it follows in that \( Q \) and \( R \) are related by the following ZCSE transformation

\[
Q_{ne} = \begin{bmatrix}
I_{q_2+m+l} & 0 \\
0 & Q
\end{bmatrix}
\]

where \( Q_e = \begin{bmatrix}
I_{q_2+m+l} & 0 \\
0 & Q
\end{bmatrix} \]

Applying block elementary row and column operations defined by (17) to \( Q_{ne} \) gives the unimodular system equivalent trivial expansion of \( Q \) as

\[
Q_e = \begin{bmatrix}
I_{q_2+m+l} & 0 \\
0 & Q
\end{bmatrix}
\]
Moreover,
\[
\tilde{P}S_8 = \begin{bmatrix}
0_{r_3, r_3} & 0_{r_3, l} \\
0_{l, r_3} & 0_{l, l} \\
T_R & U_R \\
o_{r, r_3} & 0_{r, r_3} \\
o_{r, l} & 0_{r, l} \\
o_{r_3, r_3} & 0_{r_3, l} \\
-V_R & 0
\end{bmatrix}
\]  
(39)
and it follows that \( \tilde{P} \) is ZCSE to \( R \) and \( P_e \) and \( P \) and the polynomial matrix \( \tilde{P} \) corresponds to the 3D singular Roesser state-space model given by (5). Hence, the following theorem has therefore been constructively established.

**Theorem 1:** The matrix \( \tilde{P} \) is unimodular system equivalent to the trivial expansion matrix of \( P \)
\[
P_{ee} = \begin{bmatrix}
I_{3N} & 0 \\
0 & P
\end{bmatrix}
\]  
(40)
Also the matrices \( \tilde{P} \) and \( P \) are related by the ZCSE transformation:
\[
\tilde{S}_1 P = \tilde{P} \tilde{S}_2 = \begin{bmatrix}
T & U \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
-V & 0
\end{bmatrix}
\]  
(41)
where
\[
\tilde{S}_1 = S_7 S_5 S_3 S_1 = \begin{bmatrix}
I_t & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & I_m
\end{bmatrix}
\]  
(42)
\[
\tilde{S}_2 = S_8 S_6 S_4 S_2 = \begin{bmatrix}
I_t & 0 \\
0 & I_t \\
V & 0 \\
0 & I_l \\
0 & V \\
0 & 0
\end{bmatrix}
\]

**IV. CASE STUDY**
Consider a particular case of spatially interconnected systems in the form of the ladder circuit of Fig. 1. To obtain the 3D model take the state vector as the inductor current and capacitor voltage in each internal node \( p,f, \) \( 0 < p < N, \) \( 0 < q < M, \) i.e.,
\[
x(p, q, t) = \begin{bmatrix}
U_C(p, q, t) \\
i_l(p, q, t)
\end{bmatrix}.
\]  
(43)
Also, as one option, take the system output to be the state vector and the sources to be controlled as
\[
i(p, q, t) := \gamma i_e(p + 1, q, t) = \gamma U_C(p + 1, q, t).
\]  
(44)
By Kirchoff laws, the dynamics of this circuit are described applying the shift operators
\[
\begin{align*}
\hat{A}_{111} & = \frac{0}{0} \cdot \hat{A}_{101} = \frac{0}{0} \cdot \hat{A}_{011} = \frac{-C}{0} \cdot \hat{A}_{001} = \frac{0}{0}, \\
\hat{A}_{110} & = \frac{0}{-1-R_1} \cdot \hat{A}_{100} = \frac{0}{0}, \\
\hat{A}_{010} & = \frac{-1-R_2}{1+\gamma} \cdot \hat{A}_{100} = \frac{0}{0}, \\
\hat{A}_{000} & = \frac{-1-R_2}{1-\gamma} \cdot \hat{B}_{110} = \frac{0}{0}.
\end{align*}
\]  
(46)
In this case both state variables, i.e., \( U_C(p, q, t) \) and \( i_l(p, q, t) \), are selected as outputs and hence \( \tilde{C} = \tilde{I}_2 \).

To obtain a discrete representation of the dynamics of (45), as shown at the bottom of the next page, the forward Euler algorithm with discretion period \( h \) is employed, i.e.,
\[
\begin{align*}
x(p, q, t) & \rightarrow x(p, q, k + 1) - x(p, q, k) \\
x(p, q, t) & \rightarrow x(p, q, k)
\end{align*}
\]  
(46)
The resulting discrete state-space model is given in (47), as shown at the bottom of the next page, below with
\[
\begin{align*}
\tilde{A}_{111} & = \frac{\hat{A}_{111}}{h}, \quad \tilde{A}_{101} = \frac{\hat{A}_{101}}{h}, \quad \tilde{A}_{011} = \frac{\hat{A}_{011}}{h} \\
\tilde{A}_{110} & = \frac{\hat{A}_{110}}{h}, \quad \tilde{A}_{100} = \frac{\hat{A}_{100}}{h}, \\
\tilde{A}_{010} & = \frac{\hat{A}_{010}}{h}, \quad \tilde{A}_{000} = \frac{\hat{A}_{000}}{h}, \quad \tilde{B}_{110} = \frac{\hat{B}_{110}}{h}, \quad \tilde{B}_{100} = \frac{\hat{B}_{100}}{h}.
\end{align*}
\]
The related polynomial system matrix (1) is obtained by applying the shift operators
\[
z_1 g(p, q, k) = g(p + 1, q, k), \quad z_2 g(p, q, k) = g(p, q + 1, k), \quad z_3 g(p, q, k) = g(p, q, k + 1),
\]
where \( z_1 \) is a temporal shift operator and \( z_2 \) and \( z_3 \) spatial. In the resulting matrix
\[
T(z_1, z_2, z_3) = z_1 z_2 z_3 \tilde{A}_{111} - z_1 z_2 \tilde{A}_{110} \]
\[
- z_1 z_3 \tilde{A}_{101} - z_2 z_3 \tilde{A}_{011} - z_1 \tilde{A}_{100}
\]
As a numerical example, the data $L = 0.1\,H$, $C = 20 \times 10^{-6}\,F$, $R_1 = 2000\,\Omega$, $R_2 = 1000\,\Omega$ and $h = 0.01\,\text{secs}$ is used. For ease of presentation the following notation is used:

$X(a, b)$ denotes the entry in row $a$ and column $b$ of the matrix $X$ and $X(c : d, e : f)$ denotes the matrix formed by rows $c$ through to $d$ and columns $e$ through to $f$.

For this example, $P_n$ defined in (13) is given by (49), as shown at the bottom of the next page. Also, $T_{n,0}$ and $T_{n,1}$ are given in (50), as shown at the bottom of the next page, and

$$
U(z_1, z_2, z_3) = z_1 z_2 B_{110} + z_1 B_{100},
V(z_1, z_2, z_3) = I_m, \quad W(z_1, z_2, z_3) = 0.
$$ (48)

In what follows, $Q \in \mathbb{R}(z_1, z_2, z_3)^{16 \times 15}$ of (15) has zero entries except for

- $Q(9 : 16, 7 : 14) = -I_8$
- $Q(7, 2) = -1$
- $Q(1 : 6, 1 : 6) = I_6$
- $Q(9 : 14, 10 : 15) = I_6$
- $Q(1 : 6, 7 : 12) = -z_1 I_6$
- $Q(8, 1) = 100 z_2$
- $Q(8, 3) = z_2 - 1$
- $Q(7, 8) = 1.01 - 1.01 z_2$
- $Q(7, 7) = 0.002 z_2 - 0.001 z_2 + 0.002 z_2 z_3$
- $Q(8, 2) = 200000 z_2 - 10 z_3 + 10 z_2 * z_3$
- $Q(8, 2) = -0.003$
- $Q(8, 2) = -1990.$

As shown in (49),

$$
\begin{align*}
U_{n,0} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T, \\
U_{n,1} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \\
V_{n,0} &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \\
V_{n,1} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}.
\end{align*}
$$

As shown in (49),

$$
\begin{align*}
&\hat{A}_{111} x(p + 1, q + 1, t) = \hat{A}_{101} x(p + 1, q + 1, t) + \hat{A}_{011} x(p + 1, q + 1, t) + \hat{A}_{110} x(p + 1, q + 1, t) + \hat{A}_{001} x(p, q, t) \\
&\quad + \hat{B}_{010} x(p + 1, q + 1, t) + \hat{A}_{100} x(p + 1, q, t) + \hat{A}_{000} x(p, q, t) + \hat{B}_{110} E(p + 1, q + 1, t) \\
&\quad + \hat{B}_{100} E(p + 1, q, t), \\
y(p, q, t) = \hat{C} x(p, q, t)
\end{align*}
$$ (45)

$$
\begin{align*}
&\hat{A}_{111} x(p + 1, q + 1, k + 1) = \hat{A}_{101} x(p + 1, q + 1, k + 1) + \hat{A}_{011} x(p + 1, q + 1, k + 1) + \hat{A}_{110} x(p + 1, q + 1, k) \\
&\quad + \hat{A}_{001} x(p, q + 1, k + 1) + \hat{A}_{100} x(p + 1, q + 1, k) + \hat{A}_{100} x(p + 1, q, k) + \hat{A}_{100} x(p + 1, q, k) \\
&\quad + \hat{B}_{110} E(p + 1, q + 1, k) + \hat{B}_{100} E(p + 1, q, k), \\
y(p, q, k) = C x(p, q, k)
\end{align*}
$$ (47)
Hence, the entries in $T_{Q,0}$ and $T_{Q,1}$ are zero except for

$$
T_{Q,0}(9 : 14, 7 : 12) = -I_6
$$
$$
T_{Q,0}(7 : 8, 2 : 3) = -I_2
$$
$$
T_{Q,0}(1 : 6, 1 : 6) = I_6
$$
$$
T_{Q,0}(9 : 13, 10 : 14) = I_5
$$
$$
T_{Q,0}(1 : 6, 7 : 12) = -z_1 I_6
$$
$$
T_{Q,0}(7, 8) = 1.01
$$
$$
T_{Q,0}(8, 2) = -10z_3 - 1990
$$
$$
T_{Q,0}(7, 7) = 0.002z_3 - 0.003
$$
$$
T_{Q,1}(8, 1) = 100
$$
$$
T_{Q,1}(8, 2) = 10z_3 + 200000
$$
$$
T_{Q,1}(8, 3) = 1
$$
$$
T_{Q,1}(7, 7) = 0.002z_3 - 0.001
$$
$$
T_{Q,1}(7, 7) = -1.01
$$

and

$$
U_{Q,0} = \begin{bmatrix} 0_{13 \times 1} \\ 1 \end{bmatrix}.
$$
$$
U_{Q,1} = \begin{bmatrix} 0_{14 \times 1} \end{bmatrix}.
$$
$$
V_{Q,0} = \begin{bmatrix} 0_{2 \times 12} & I_2 \end{bmatrix}.
$$
$$
V_{Q,1} = \begin{bmatrix} 0_{2 \times 14} \end{bmatrix}.
$$

Also, $R \in \mathbb{R}(z_1, z_2, z_3)^{34 \times 33}$ of (26) has zero entries except for

$$
R(22 : 23, 17 : 18) = -I_2
$$
$$
R(24 : 32, 22 : 32) = -I_{11}
$$
$$
R(1 : 21, 1 : 21) = I_{21}
$$
$$
R(24 : 32, 25 : 33) = I_9,
$$

and hence $T_{R,0}$ has zero entries except for

$$
T_{R,0}(23, 17) = -1990
$$
$$
T_{R,0}(22 : 23, 17 : 18) = -I_2
$$
$$
T_{R,0}(24 : 32, 22 : 32) = -I_{11}
$$
$$
T_{R,0}(1 : 21, 1 : 21) = I_{21}
$$
$$
T_{R,0}(24 : 31, 25 : 32) = I_8, T_{R,0}(23, 3) = 1
$$
$$
T_{R,0}(23 : 23, 1) = 100
$$
$$
T_{R,0}(23 : 23, 2) = 200000
$$
$$
T_{R,0}(16 : 21, 22 : 27) = -z_1 I_6
$$
$$
T_{R,0}(1 : 15, 16 : 30) = -z_2 I_{15}
$$
$$
T_{R,0}(22 : 23, 1) = 1.01
$$
$$
T_{R,0}(22 : 23, 8) = -1.01
$$
$$
T_{R,0}(22 : 23, 7) = -0.001
$$
$$
T_{R,0}(22 : 22, 22) = -0.003.
$$

Moreover, the $32 \times 32$ matrix $T_{R,1}$ has zero entries except for

$$
T_{R,1}(2, 23) = 10
$$

$$
P_n = \begin{bmatrix} 0.002z_3 - 0.001z_2 + 0.002z_2 z_3 - 0.003 & 1.01 - 1.01z_2 - z_1 & 0 & 0 & 0 & 0 \\
100z_1 z_2 & 200000 z_1 z_2 - 1990 z_1 - 10 z_1 z_3 + 10 z_1 z_2 z_3 z_1 z_2 - z_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

(49)

$$
T_{n,0} = \begin{bmatrix} 0.002z_3 - 0.001z_2 + 0.002z_2 z_3 - 0.003 & 1.01 - 1.01z_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

(50)

$$
T_{n,1} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
100 z_2 & 200000 z_2 - 10 z_3 + 10 z_2 z_3 - 1990 z_2 - 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

(51)
Moreover, \( \tilde{P} \) corresponds to (6) with

\[
\begin{align*}
T_{R,1}(7, 22) &= 0.002 \\
T_{R,1}(17, 23) &= -10 \\
T_{R,1}(22, 22) &= 0.002
\end{align*}
\]

and

\[
\begin{align*}
U_{R,0} &= \begin{bmatrix} 0_{31 \times 1} \\ 1 \end{bmatrix} \\
U_{R,1} &= \begin{bmatrix} 0_{32 \times 1} \\ 1 \end{bmatrix} \\
V_{R,0} &= \begin{bmatrix} 0_{2 \times 30} \\ I_2 \end{bmatrix} \\
V_{R,1} &= \begin{bmatrix} 0_{2 \times 32} \end{bmatrix}.
\end{align*}
\]

In the next step \( \tilde{P} \in \mathbb{R}^{(z_1, z_2, z_3)^{70 \times 69}} \) of (38) is obtained. Again, this matrix has zero entries except for

\[
\begin{align*}
\tilde{P}(55:56, 50:51) &= -I_2 \\
\tilde{P}(57:70, 55:68) &= -I_{14} \\
\tilde{P}(56, 17) &= -10 \\
\tilde{P}(56, 50) &= -1990 \\
\tilde{P}(1:54, 1:54) &= I_{54} \\
\tilde{P}(57:68, 58:69) &= I_{12} \\
\tilde{P}(56, 36) &= 1 \\
\tilde{P}(56, 2) &= 10 \\
\tilde{P}(56, 34) &= 100 \\
\tilde{P}(56, 35) &= 200000 \\
\tilde{P}(49:54, 55:60) &= -z_1 I_6 \\
\tilde{P}(34:48, 49:63) &= -z_2 I_{15} \\
\tilde{P}(1:33, 34:66) &= -z_3 I_{33} \\
\tilde{P}(55, 56) &= 1.01 \\
\tilde{P}(55, 41) &= -1.01 \\
\tilde{P}(55, 40) &= -0.001 \\
\tilde{P}(55, 55) &= -0.003 \\
\tilde{P}(55, 7) &= 0.002 \\
\tilde{P}(55, 22) &= -0.002.
\end{align*}
\]

Moreover, \( \tilde{P} \) retains the structure defined in (34) and hence corresponds to (6) with

\[
\begin{align*}
E_1 &= \begin{bmatrix} 0_{48 \times 54} & 0_{48 \times 6} & 0_{48 \times 8} \\ 0_{6 \times 54} & I_6 & 0_{6 \times 6} \\ 0_{14 \times 54} & 0_{14 \times 6} & 0_{14 \times 8} \end{bmatrix} \\
E_2 &= \begin{bmatrix} 0_{33 \times 48} & 0_{33 \times 15} & 0_{33 \times 5} \\ 0_{15 \times 48} & -I_{15} & 0_{15 \times 5} \\ 0_{20 \times 48} & 0_{20 \times 15} & 0_{20 \times 5} \end{bmatrix} \\
E_3 &= \begin{bmatrix} 0_{33 \times 33} & -I_{33} & 0_{33 \times 2} \\ 0_{35 \times 33} & 0_{35 \times 33} & 0_{35 \times 2} \end{bmatrix},
\end{align*}
\]

Also, \( A \in \mathbb{R}^{68 \times 68} \) has zero entries except for

\[
\begin{align*}
A(1:54, 1:54) &= -I_{54} \\
A(57:67, 58:68) &= -I_{11} \\
A(57:68, 55:66) &= I_{12} \\
A(55:56, 50:51) &= I_2 \\
A(56, 36) &= -1
\end{align*}
\]

and

\[
C = \begin{bmatrix} 0_{2 \times 66} & I_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0_{1 \times 67} \\ 1 \end{bmatrix}.
\]

and these matrices form the singular 3D Roesser state-space model of (5) in this case.

To summarize the case study presented, the model of 3D spatially interconnected system with respectively: 2 states, 1 input, and 2 outputs, has resulted in a singular 3D Roesser model of 68th order. A significant increase in the order will eventually appear for more complicated systems but, due to the sparse model matrices, this should not be computationally prohibitive.

V. CONCLUSION

A constructive method has been developed that constructs a singular Roesser state-space model of a 3D linear system with spatial and temporal indeterminates from its system matrix. The form of transformation used in the reduction procedure is generated by a finite sequence of elementary row and column operations and a trivial expansion of the original polynomial system. An advantage of the developed method is the possible application to the analysis and synthesis of 3D spatially interconnected systems. Since there are no results referring directly applying the original models, the transformation into the known 3D Roesser model and the theory developed for this class of systems is promising for future research. Similar structure nD models appear in modeling other classes of interconnected systems, e.g., mechanical (connections of masses and dampers) heat exchangers and electromechanical systems. The results in this paper are also applicable.

REFERENCES

[1] E. Antoniou and S. Vologiannidis, “On the reduction of 2-D polynomial systems into first order equivalent models,” Multidimensional Syst. Signal Process., vol. 31, no. 1, pp. 249–268, Jan. 2020.

[2] J. A. Ball, “Multidimensional circuit synthesis and multivariable dilation theory,” Multidimensional Syst. Signal Process., vol. 22, nos. 1–3, pp. 27–44, Mar. 2011.

[3] N. K. Bose, B. Buchberger, and J. Guiver, Multidimensional Systems Theory and Applications. Dordrecht, The Netherlands: Kluwer, 2003.
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