Dark matter stability at fixed points in a modular $A_4$ symmetry

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Abstract

We propose a new mechanism of stabilization for dark matter under fixed points of modular $A_4$ symmetry, thanks to recovering a residual symmetry for each of fixed points at $\tau = \omega, i, i \times \infty$. We find that a model at $\tau = i$ successfully reproduces the neutrino oscillation data without conflict of the DM stability.

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I. INTRODUCTION

Modular flavor symmetries are powerful tools not only to obtain predictions for lepton and quark masses and mixings but also to induce dark matter (DM) stability. One of the most attractive points is that one does not need to introduce so many bosons such as flavons anymore other than the fields contents of the standard model (SM) and right-handed neutrinos. Therefore one finds sharper predictions, which could be obtained without any assumptions such as alignment of Higgs of vacuum expectations (VEVs). It has originally been proposed by the paper [1]. (See also for useful subgroups of the modular group Ref. [2].) Recently, a vast literature has arisen along the idea, e.g., $A_4$ [1, 3–41], $S_3$ [42–47], $S_4$ [48–60], $A_5$ [61–62], double covering of $A_4$ [63–65], $S_4$ [66–67], $A_5$ [68–71], multiple modular symmetries [72] and other types of groups [73–79], which have nicely predicted masses, mixings, and CP phases for the SM fermions.1 Interesting ideas on DM have been proposed [6, 10, 13, 52], in which modular weights assure the stability of DM. A systematic approach to understanding the origin of CP transformations has been discussed in Ref. [88], CP or flavor violation with modular symmetry has been discussed in Refs. [89–91] 2, the electric dipole moment originated from modulus has been discussed in Ref. [93], and a possible correction from Kähler potential has been discussed in Ref. [94]. Furthermore, systematic analysis of the fixed points, which is called ”stabilizers”, has been discussed in Ref. [95], and moduli stabilization has also been investigated by three-form flux background [96] and Fayet-Iliopoulos terms [97]. These analyses show that the modulus $\tau$ is stabilized at fixed points, $\tau = \omega$ and $i$, with a certain probability, where $\omega \equiv e^{2\pi i/3}$.

In this letter, we propose a new mechanism for stabilization of DM, applying a residual symmetry, which is recovered at each of three fixed points at $\tau = i, \omega, i \times \infty$. Since the DM stability is very stringent, we consider $\tau = i, \omega$ only.3 In fact it is known that $Z_3$ is recovered at $\tau = \omega$, and $Z_2$ is recovered at $\tau = i$, as we will explain later in details. Also, we show how to get collect relic density of DM, introducing a gauged $U(1)_{B-L}$ symmetry as a simple example. Then, we discuss the neutrino oscillation data for each of the cases. Even

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1 As for any readers who need to know about these groups more, we give several review papers, which are helpful to understand the non-Abelian groups and their applications to flavor structures [80–87].

2 See also for spontaneous CP violation Ref. [92].

3 If one would consider the stability of DM at nearby $\tau = i \times \infty$, the required $\tau$ be more than $10i$. 
though a model at $\tau = \omega$ needs more parameters to reproduce the neutrino oscillation data, we find that a model at $\tau = i$ successfully reproduces the neutrino oscillation data.

This paper is organized as follows. In Sec. II, we explain our model setup and how to stabilize DM without conflict. Then, we show a simple example to induce the relic density and estimate how stringent for the stability of DM. After that, we discuss the lepton sector and show our successful model at $\tau = i$, demonstrating the benchmark point for normal and inverted hierarchies. Finally, we conclude and discuss in Sec. III. In Appendix, we show various modular forms.

II. MODEL

In this section, we show a successful model that has a stable DM candidate at two fixed points $\tau = \omega, i$, introducing three heavy Majorana fermions where the lightest one is supposed to be DM. There are only 2 inequivalent finite points except $\tau = i\infty$ in the fundamental domain of $\Gamma$, namely, $\tau = i$ and $\tau = \omega = -1/2 + i\sqrt{3}/2$. $\tau = i$ is invariant under the $S$ transformation: $\tau = -1/\tau$. The subgroup of $A_4$: $\mathbb{Z}^S_2 = \{I, S\}$ is preserved at the fixed point. $\tau = \omega$ is the left cusp in the fundamental domain of the modular group, which is invariant under the $ST$ transformation: $\tau = -1/(\tau + 1)$. Indeed, $\mathbb{Z}^{ST}_3 = \{I, ST, (ST)^2\}$ is one of subgroups of $A_4$ group. The right cusp at $\tau = -\omega^2 = 1/2 + i\sqrt{3}/2$ is related to $\tau = \omega$ by the $T$ transformation: $\tau = \tau + 1$. There is also infinite point $\tau = i\infty$, in which the subgroup of $A_4$: $\mathbb{Z}^T_3 = \{I, T, T^2\}$ is preserved. It is possible to calculate the values of the $A_4$ singlet and triplet modular forms of weights 2, 4, 6, 8 and 10 at $\tau = i$, $\tau = \omega$ and $\tau = i\infty$. The results are summarized in Table I. We use the modular forms in Appendix A.

A. Model in case of $\tau = \omega$

We show an example how to get a DM candidate in case of $\tau = \omega$. The field contents and their assignments are summarized in Table II. Under this symmetry, the renormalizable

\footnote{Since $\tau = i\infty$ would be too difficult to deal with a model, we will not consider the case hereafter. $S$ and $T$ are generators defined satisfying the following algebras $S^2 = T^3 = (ST)^3 = 1$.}
| $k$ | $r$ | $\tau = i$ | $\tau = \omega$ | $\tau = i\infty$ |
|-----|-----|-------------|-----------------|----------------|
| 2   | $\mathbf{3}$ | $(1, 1 - \sqrt{3}, -2 + \sqrt{3})$ | $(1, \omega, -\frac{1}{2}\omega^2)$ | $(1, 0, 0)$ |
| 4   | $\mathbf{3}$ | $(1, 1, 1)$ | $(1, -\frac{1}{2}\omega, \omega^2)$ | $(1, 0, 0)$ |
|     | $\{1, 1'\}$ | $\{1, 1\}$ | $\{0, 1\}$ | $\{1, 0\}$ |
| 6   | $\mathbf{3}_1$ | $(1, 1 - \sqrt{3}, -2 + \sqrt{3})$ | $(0, 0, 0)$ | $(1, 0, 0)$ |
|     | $\mathbf{3}_2$ | $(1, 1 - \sqrt{3}, 1 + \sqrt{3})$ | $(1, \omega, -\frac{1}{2}\omega^2)$ | $(0, 0, 0)$ |
|     | $1$ | $0$ | $1$ | $1$ |
| 8   | $\mathbf{3}_1$ | $(1, 1, 1)$ | $(0, 0, 0)$ | $(1, 0, 0)$ |
|     | $\mathbf{3}_2$ | $(1, 1, 1)$ | $(1, -\frac{1}{2}\omega, \omega^2)$ | $(0, 0, 0)$ |
|     | $\{1, 1', 1''\}$ | $\{1, 1, 1\}$ | $\{0, 0, 1\}$ | $\{1, 0, 0\}$ |
| 10  | $\mathbf{3}_1$ | $(1, 1 - \sqrt{3}, -2 + \sqrt{3})$ | $(0, 0, 0)$ | $(1, 0, 0)$ |
|     | $\mathbf{3}_2$ | $(1, 1 - \sqrt{3}, 1 + \sqrt{3})$ | $(0, 0, 0)$ | $(0, 0, 0)$ |
|     | $\mathbf{3}_3$ | $(1, 1 - \sqrt{3}, -2 + \sqrt{3})$ | $(1, \omega, -\frac{1}{2}\omega^2)$ | $(0, 0, 0)$ |
|     | $\{1, 1'\}$ | $\{0, 0\}$ | $\{0, 1\}$ | $\{1, 0\}$ |

**TABLE I:** Modular forms of weight $k = 2, k = 4, k = 6, k = 8$ and $k = 10$ at fixed points of $\tau$, where we ignore overall factors.

| $k$ | $r$ | $\tau = i$ | $\tau = \omega$ | $\tau = i\infty$ |
|-----|-----|-------------|-----------------|----------------|
| $-k$ | $\mathbf{2}$ | $\mathbf{1}$ | $\mathbf{1}$ | $\mathbf{2}$ |
| $\frac{1}{2}$ | $\mathbf{1}$ | $\mathbf{1}$ | $\mathbf{1}$ | $\mathbf{1}$ |
| $\mathbf{1}'$ | $\mathbf{1}'$ | $\mathbf{1}'$ | $\mathbf{1}'$ | $\mathbf{1}'$ |
| $\mathbf{1}''$ | $\mathbf{1}''$ | $\mathbf{1}''$ | $\mathbf{1}''$ | $\mathbf{1}''$ |

**TABLE II:** Field contents of fermions and bosons and their charge assignments in case of $\tau = \omega$ under $SU(2)_L \times U(1)_Y \times A_4$ in the lepton and boson sector, where $-k$ is the number of modular weight and the quark sector is the same as the SM.
Lagrangian is given by

\[-\mathcal{L} = a_\ell Y_1^{(6)} L_{\ell e} H e_R + b_\ell Y_1^{(6)} L_{\ell \mu} H \mu_R + c_\ell Y_1^{(6)} L_{\ell \tau} H \tau_R + d_\ell Y_1^{(6)} L_{\ell e} H \tau_R + e_\ell Y_1^{(6)} L_{\ell \mu} H \mu_R
+ a_D Y_1^{(4)} L_{\ell \mu} \tilde{H} N_{R_1} + b_D Y_1^{(4)} L_{\ell e} \tilde{H} N_{R_2} + c_D Y_1^{(4)} L_{\ell \tau} \tilde{H} N_{R_3}
+ a'_D Y_4^{(4)} L_{\ell e} \tilde{H} N_{R_3} + b'_D Y_4^{(4)} L_{\ell \mu} \tilde{H} N_{R_2} + c'_D Y_4^{(4)} L_{\ell \tau} \tilde{H} N_{R_2}
+ M_2 Y_1^{(4)} N_{R_2} N_{R_2} + M_1 Y_1^{(4)} N_{R_3} N_{R_3} + M_{23} Y_1^{(4)} N_{R_2} N_{R_3} + \text{h.c.},\]

(II.1)

where \( \tilde{H} \equiv i\sigma_2 H^* \), \( \sigma_2 \) being the second Pauli matrix. Here, we denote \( H \equiv (h^+, (v_H + h + i z)/\sqrt{2})^T \), where \( h^+ \) and \( z \) are respectively eaten by gauge bosons \( W^+ \) and \( Z \) in the SM.

When we focus on \( \tau = \omega \); i.e., \( Y_1^{(4)} = 0 \), the above Lagrangian is reduced to be

\[-\mathcal{L} = a_\ell Y_1^{(6)} (\omega) L_{\ell e} H e_R + b_\ell Y_1^{(6)} (\omega) L_{\ell \mu} H \mu_R + c_\ell Y_1^{(6)} (\omega) L_{\ell \tau} H \tau_R
+ d_\ell Y_1^{(6)} (\omega) L_{\ell e} H \tau_R + e_\ell Y_1^{(6)} (\omega) L_{\ell \mu} H \mu_R
+ a'_D Y_1^{(4)} (\omega) L_{\ell e} \tilde{H} N_{R_3} + b'_D Y_1^{(4)} (\omega) L_{\ell \mu} \tilde{H} N_{R_2} + c'_D Y_1^{(4)} (\omega) L_{\ell \tau} \tilde{H} N_{R_2}
+ M_1 Y_1^{(4)} (\omega) N_{R_2} N_{R_3} + \text{h.c.},\]

(II.2)

It implies that \( N_{R_i} \) cannot decay into SM particles any more due to the vanishing Dirac terms which are proportional to \( Y_1^{(4)} \). Therefore, the \( Z_3 \) is restored at \( \tau = \omega \). This is why \( N_{R_1} \) is stable.

**B. Model in case of \( \tau = i \)**

In case of \( \tau = i \), we find rather large degrees of freedom on models that have a stable DM candidate at \( \tau = i \). Thus, we restrict ourselves that isospin doublet leptons \( L_L \equiv (1_{-4}, 1'^{-4}, 1'^{-4}) \), \( N_R \equiv (1_{-4}, 1'^{-4}, 1'^{-4}) \), and \( (e_R, \mu_R, \tau_R) \equiv (1, 1', 1')_{-k_R} \) under the \( A_4 \) symmetry and modular weight. Notice here that the number of modular weight for the right-handed charged-leptons, which is denoted by \( -k_R \), is allowed to be even numbers including zero; \( 0, -2, -4, \cdots \). The field contents and their assignments are summarized in Table [III] The degrees of freedom in the charged-lepton sector is straightforwardly found as follows: In case of \( -k_R = 0 \), there exist six free parameters. In case of \( -k_R = -2 \), there exist three free parameters but \( Y_1^{(6)} \) is needed to be invariant. Thus, all the charged-lepton masses
TABLE III: Field contents of fermions and bosons and their charge assignments in case of \( \tau = i \) under \( SU(2)_L \times U(1)_Y \times A_4 \) in the lepton and boson sector, where \( -k \) is the number of modular weight and the quark sector is the same as the SM. \( -k_R \) is allowed to be even numbers including zero: 0, \(-2\), \(-4\), \cdots.

are vanishing at \( \tau = i \), and the model spoils. In case of \( -k_R = -4 \), there exist nine free parameters, since there exist three types of non-vanishing singlets with modular weight 8; \( Y_{1,1',1''}^{(8)} \neq 0 \). If one assigns \( -k_R = -6 \), the right-handed leptons have to be triplet under the \( A_4 \) symmetry, since there exists one degree of freedom \( Y_{10}^{(10)} \) only, while three types of triplets for modular weight 10. These degrees of freedom would be useful to build a lepton mass model. However since it is beyond our scope, we just select the simplest case, i.e., \( -k_R = 0 \) hereafter. Under this symmetry, the renormalizable Lagrangian is given by

\[
-L = a_\ell Y_1^{(4)} \bar{L}_{Le} e_R + b_\ell Y_1^{(4)} \bar{L}_{L\mu} H \mu_R + c_\ell Y_1^{(4)} \bar{L}_{L\tau} H \tau_R \\
+ a'_\ell Y_1^{(4)} \bar{L}_{Le} H e_R + b'_\ell Y_1^{(4)} \bar{L}_{L\mu} H \mu_R + c'_\ell Y_1^{(4)} \bar{L}_{L\tau} H \mu_R \\
+ a_D Y_1^{(6)} \bar{L}_{Le} \tilde{H} N_{R1} + b_D Y_1^{(8)} \bar{L}_{L\mu} \tilde{H} N_{R2} + c_D Y_1^{(8)} \bar{L}_{L\tau} \tilde{H} N_{R3} \\
+ a'_D Y_1^{(8)} \bar{L}_{Le} \tilde{H} N_{R2} + b'_D Y_1^{(8)} \bar{L}_{L\mu} \tilde{H} N_{R3} + a''_D Y_1^{(8)} \bar{L}_{L\tau} \tilde{H} N_{R3} + c''_D Y_1^{(8)} \bar{L}_{Le} \tilde{H} N_{R3} \\
+ M_1 Y_1^{(4)} \tilde{N}_{R1}^C N_{R1} + M_2 Y_1^{(8)} \tilde{N}_{R2}^C N_{R3} + M_3 Y_1^{(8)} \tilde{N}_{R3}^C N_{R3} + h.c. \quad (II.3)
\]

When we focus on \( \tau = i \); i.e., \( Y_1^{(6)} = 0 \), \( N_{R1} \) cannot decay into SM particles due to absence of \( a_D Y_1^{(6)} \bar{L}_{Le} \tilde{H} N_{R1} \). Therefore, the \( Z_2 \) is restored at \( \tau = i \), and \( N_{R1} \) is stable.

C. Relic density

The remaining task is to get a correct relic density to provide appropriate interactions of DM. One of the simplest ways is to introduce a gauged \( U(1)_{B-L} \) symmetry, under which
we introduce an isospin singlet $\phi$ with $+2 B - L$ charge and zero modular weight in order to break the spontaneous $B - L$ symmetry. Therefore, $\phi$ has nonzero vacuum expectation value that is denoted by $\phi \equiv (v' + r + i z')/\sqrt{2}$, where $z'$ is absorbed by $Z'$ vector gauge boson. Then, we have two promising channels via $Z'$ and $\phi$ with s-channels. However since the channels via $Z'$ is severely restricted by LEP and LHC, it would not be so easy to get sizable relic density of DM. In fact, sizable relic density; $\Omega h^2 \approx 0.12$, is induced at s-channel via $r$; $N_{R_1}^{C} N_{R_1} \rightarrow r \rightarrow 2h$, where $N_{R}$'s are assigned to be $-1$ under $U(1)_{B-L}$ symmetry.  

Then, only the Majorana terms are changed into as follows:

$$\tau = \omega :$$

$$y_{M_2} \phi Y^{(4)}_1(\omega)N_{R_2}^C N_{R_2} + y_{M_{13}} \phi Y^{(4)}_1(\omega)N_{R_1}^C N_{R_3} + y_{M_1} \phi Y^{(4)}_1'(\omega)N_{R_1}^C N_{R_1} + y_{M_{23}} \phi Y^{(4)}_1'(\omega)N_{R_2}^C N_{R_3} + \text{h.c.}.$$ (II.4)

$$\tau = i :$$

$$y_{M_1} \phi Y^{(4)}_1(i)N_{R_1}^C N_{R_1} + y_{M_{13}} \phi Y^{(4)}_1(i)N_{R_2}^C N_{R_3} + y_{M_2} \phi Y^{(8)}_1(i)N_{R_2}^C N_{R_2} + y_{M_{33}} \phi Y^{(8)}_1'(i)N_{R_3}^C N_{R_3} + \text{h.c.}.$$ (II.5)

The structure of lepton Yukawa sector does not change at all even after breaking the $B - L$ symmetry, and the DM candidate, which is denoted by $X_R \equiv N_{R_1}$ hereafter, does not mix with the other two Majorana particles. Thus, the stability can never be broken.

Then, we write the valid Lagrangian to induce the cross section that can explain the sizable relic density as follows:

$$\mathcal{L}_{\text{int}} = -\frac{Y}{\sqrt{2}} \bar{X} P_R X r + \mu_{hh r} h h r,$$ (II.6)

where $Y$ corresponds to $y_{M_1} Y^{(4)}_1(\omega(i))$ for $\tau = \omega(i)$, and $\mu_{hh r}$ is trilinear terms on $h h r$ interaction. We assume to be $m_h \lesssim M_X$ where $M_X$ is DM mass. The relic density of DM is formulated by \[98, 99\]

$$\Omega h^2 \approx \frac{1.07 \times 10^9}{\sqrt{g^*(x_f) M_{Pl} J(x_f)} \text{[GeV]}},$$ (II.7)

where $g^*(x_f \approx 25) \approx 100$ is the degrees of freedom for relativistic particles at freeze out.
temperature $T_f = M_X/x_f$, $M_{Pl} \approx 1.22 \times 10^{19}$ GeV, and

$$J(x_f) = \int_{x_f}^{\infty} d\bar{x} \frac{\int_{4M_X^2}^{\infty} ds \sqrt{s - 4M_X^2 W^h(s) K_1 \left( \frac{\bar{x}}{M_X} \right)}}{16M_X^5 x[K_2(x)]^2}, \quad (\text{II.8})$$

$$W^h(s) = \frac{1}{32\pi} \left| \frac{Y \mu_{hhr}}{s - m_r^2 + m_r \Gamma_r} \right|^2 (s - 2M_X^2) \sqrt{1 - \frac{4m_r^2}{s}}. \quad (\text{II.9})$$

The decay width of $r$ is given by

$$\Gamma_r = \frac{1}{4\pi} \left[ \frac{\mu_{hhr}^2}{m_r} \sqrt{1 - \frac{4m_r^2}{m_r^2}} + \frac{Y^2}{8m_r} \left( 1 - 2 \frac{M_X^2}{m_r^2} \right) \sqrt{1 - \frac{4m_r^2}{s}} \right]. \quad (\text{II.10})$$

In Fig. 1 we show the relation between $M_X$ and $\Omega h^2$, where blue line represents $\mu_{hhr} = 5$ GeV, $Y = 0.07$, green one $\mu_{hhr} = 10$ GeV, $Y = 0.1$, and red one $\mu_{hhr} = 50$ GeV, $Y = 0.3$. One finds that there is a solution at each of the resonant points.

![Fig. 1: Relic density in terms of DM mass, where the black line represents the observed relic density. Blue line represents $\mu_{hhr} = 5$ GeV, $Y = 0.07$, green one $\mu_{hhr} = 10$ GeV, $Y = 0.1$, and red one $\mu_{hhr} = 50$ GeV, $Y = 0.3$.](image)

1. **The exactness of fixed points**

If $\tau$ is slightly deviated from fixed points, $X_R$ has to decay into SM Higgs and neutrino. Here, we simply estimate the decay rate via $\frac{Y}{\sqrt{2}} \bar{\nu} P_R X h$, assuming the vanishing neutrino mass and massive Higgs mass (to be $m_h \approx 125.5$ GeV):

$$\Gamma_X \approx \frac{|Y|^2 m_X}{16\pi} \left( 1 - \frac{m_h^2}{m_X^2} \right)^2. \quad (\text{II.11})$$
The lifetime is then estimated as follows:

\[
\tau_X = \Gamma_X^{-1} \simeq 3.31 \times 10^{-26} \times \frac{1}{|Y|^2} \left( \frac{1 \text{ TeV}}{m_X} \right) \left( 1 - \frac{m_\tau^2}{m_X^2} \right)^2 \text{ sec.} \tag{II.12}
\]

Once \( m_X = 1 \text{ TeV} \), the upper limit of \( Y \) should be less than the order \( 10^{-21} \) in order \( X \) to be quasi-stable particle, where the age of universe is about \( 10^{17} \) sec. If the the mass of DM is extremely degenerate to the mass of Higgs, the limit would be relaxed.

**D. Evaluations of lepton mass matrix**

In this subsection, we will briefly discuss lepton mass matrix, depending on two fix points; \( \tau = \omega, \ i \).

After spontaneous symmetry breakings, the charged-lepton mass matrix is given by

\[
\tau = \omega : \quad M_e = \frac{v_H}{\sqrt{2}} \begin{pmatrix} 0 & y_{12}(\omega) & y_{13}(\omega) \\ y_{21}(\omega) & 0 & y_{23}(\omega) \\ y_{31}(\omega) & y_{32}(\omega) & 0 \end{pmatrix},
\]

\[
\tau = i : \quad M_e = \frac{v_H}{\sqrt{2}} \begin{pmatrix} y_{11}(i) & 0 & y_{13}(i) \\ y_{21}(i) & y_{22}(i) & 0 \\ y_{31}(i) & y_{32}(i) & y_{33}(i) \end{pmatrix},
\]

where \( y_{11}(\omega) \equiv a_\ell Y_1^{(4)}, y_{22}(\omega) \equiv b_\ell Y_1^{(4)}, y_{33}(\omega) \equiv c_\ell Y_1^{(4)}, y_{23}(\omega) \equiv d_\ell Y_1^{(4)}, y_{32}(\omega) \equiv e_\ell Y_1^{(4)}, y_{12}(i) \equiv a_\ell' Y_1^{(4)}, y_{13}(i) \equiv b_\ell' Y_1^{(4)}, y_{21}(i) \equiv c_\ell' Y_1^{(4)}, y_{31}(i) \equiv d_\ell' Y_1^{(4)}, y_{32}(i) \equiv e_\ell' Y_1^{(4)}. M_e \) is diagonalized by bi-unitary mixing matrix as follows: \( D_\ell \equiv \text{diag}(m_e, m_\mu, m_\tau) = V_{eL}^\dagger M_e V_{eR} \). We numerically solve three diagonal elements; \( y_{11}^\ell, y_{22}^\ell, y_{33}^\ell \) applying the following three relations, where we input the experimental masses of charged-lepton masses and off-diagonal parameters \([y_{23}^\ell(\omega), y_{32}^\ell(\omega)]\), and \([y_{13}(i), y_{21}(i), y_{32}(i)]\),

\[
\text{Tr}[M_e M_e^\dagger] = |m_e|^2 + |m_\mu|^2 + |m_\tau|^2, \quad \text{Det}[M_e M_e^\dagger] = |m_e|^2|m_\mu|^2|m_\tau|^2,
\]

\[
(\text{Tr}[M_e M_e^\dagger])^2 - \text{Tr}[M_e M_e^\dagger]^2 = 2(|m_e|^2|m_\mu|^2 + |m_\mu|^2|m_\tau|^2 + |m_e|^2|m_\tau|^2). \tag{II.15}
\]
The Dirac mass and Majorana mass terms at $\tau = \omega$ are found as

$$
\tau = \omega : \\
\begin{pmatrix}
0 & 0 & y_{13}^D(\omega) \\
y_{22}^D(\omega) & 0 & 0 \\
y_{32}^D(\omega) & 0 & 0
\end{pmatrix}
\equiv \frac{v_H}{\sqrt{2}} \tilde{m}_D, \\
\begin{pmatrix}
y_{11}^M(\omega) & 0 & 0 \\
0 & 0 & y_{23}^M(\omega) \\
0 & y_{23}^M(\omega) & 0
\end{pmatrix}
\equiv \frac{v'}{\sqrt{2}} \tilde{M}_N,
$$

(II.16)

where $y_{13}^D(\omega) \equiv d_{13}' Y_1^{(4)}$, $y_{22}^D(\omega) \equiv b_{12}' Y_1^{(4)}$, $y_{32}^D(\omega) \equiv c_{13}' Y_1^{(4)}$, $y_{11}^M(\omega) \equiv y_{M1} Y_1^{(4)}$, $y_{23}^M(\omega) \equiv y_{M2} Y_1^{(4)}$. Then, the neutrino mass matrix is given by

$$
m_\nu \approx -m_D M_N^{-1} m_D^T = \frac{v_H^2}{\sqrt{2} v'} \tilde{m}_D \tilde{M}_N^{-1} \tilde{m}_D^T \equiv \kappa \bar{m}_D \bar{M}_N^{-1} \bar{m}_D^T.
$$

(II.17)

The neutrino mass eigenvalues are obtained as follows: $D_\nu = \kappa \bar{D}_\nu = U_\nu^T m_\nu U_\nu = \kappa U_\nu^T \bar{m}_\nu U_\nu$, where $U_\nu$ is a unitary matrix. The Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix is given by $U_{PMNS} \equiv V_{el}^\dagger U_\nu$. One straightforwardly finds that the neutrino masses are 0, $-m$ and $m$ because $\det[m_\nu] = \text{Tr}[m_\nu] = 0$. The degenerate mass matrix is excluded by neutrino oscillation experiments. Therefore, the model at $\tau = \omega$ has to be modified in order to reproduce neutrino oscillation data.

On the other hand, the Dirac mass and Majorana mass terms at $\tau = i$ are found as

$$
\tau = i : \\
\begin{pmatrix}
0 & y_{12}^D(i) & y_{13}^D(i) \\
y_{22}^D(i) & 0 & y_{23}^D(i) \\
y_{32}^D(i) & y_{33}^D(i) & 0
\end{pmatrix}
\equiv \frac{v_H}{\sqrt{2}} \tilde{m}_D, \\
\begin{pmatrix}
y_{11}^M(i) & 0 & 0 \\
0 & y_{22}^M(i) & y_{23}^M(i) \\
0 & y_{23}^M(i) & y_{33}^M(i)
\end{pmatrix}
\equiv \frac{v'}{\sqrt{2}} \tilde{M}_N,
$$

(II.18)

where $y_{12}^D(i) \equiv b_{12} Y_1^{(8)}$, $y_{13}^D(i) \equiv c_D Y_1^{(8)}$, $y_{22}^D(i) \equiv d_{22}' Y_1^{(4)}$, $y_{13}^D(i) \equiv d_{13}' Y_1^{(4)}$, $y_{23}^D(i) \equiv d_{23}' Y_1^{(4)}$, $y_{12}^D(i) \equiv b_{12}' Y_1^{(8)}$, $y_{11}^M(i) \equiv y_{M1} Y_1^{(4)}$, $y_{22}^M(i) \equiv y_{M2} Y_1^{(4)}$, $y_{23}^M(i) \equiv y_{M3} Y_1^{(4)}$, $y_{23}^M(i) \equiv y_{M3} Y_1^{(4)}$. Here, the lightest neutrino mass is zero because of the rank two neutrino mass matrix, and $\kappa$ is described by one experimental values and dimensionless neutrino mass eigenstates as follows:

$$
(NH) : \; \kappa^2 = \frac{|\Delta m^2_{\text{ atm}}|}{\bar{D}_{\nu s}^2}, \\
(\text{IH}) : \; \kappa^2 = \frac{|\Delta m^2_{\text{ atm}}|}{\bar{D}_{\nu e}^2},
$$

(II.19)
where $\Delta m^2_{\text{atm}}$ is the atmospheric neutrino mass-squared difference and NH(IH) stands for normal(inverted) hierarchy. Then, the solar mass-squared difference is given in terms of $\kappa$ and dimensionless neutrino mass eigenvalues:

\begin{align}
(\text{NH}) & : \Delta m^2_{\text{sol}} = \kappa^2 \tilde{D}_{\nu_2}^2, \\
(\text{IH}) & : \Delta m^2_{\text{sol}} = \kappa^2 (\tilde{D}_{\nu_2}^2 - \tilde{D}_{\nu_1}^2),
\end{align}

(II.20)

Due to enough free parameters, we reproduce the neutrino oscillation data where we adopt Nufit 5.0 [100]. We show a benchmark point to satisfy the experimental data within 3 $\sigma$ for NH in Table IV and IH in Table V.

|              | NH |
|--------------|----|
| $\frac{\sqrt{2}}{v_H} M_e$ | $\begin{pmatrix}
0.0586 & 0 & 0.00385 + 0.00943i \\
-0.00718 + 0.00193i & 0.999 & 0 \\
0 & -0.00187 + 0.0347i & 0.000253
\end{pmatrix}$ |
| $\tilde{m}_D$ | $\begin{pmatrix}
0 & 0.0161 & 0.0148 - 0.589i \\
0 & -0.324 & 0.00542 + 0.00912i \\
0 & 0.158 - 0.0659i & 0.173
\end{pmatrix}$ |
| $\tilde{M}_N$ | $\begin{pmatrix}
-0.00370 & 0 & 0 \\
0 & -0.00700 - 0.0962i & 0.0808 - 0.137i \\
0 & 0.0808 - 0.137i & 0.224 + 0.00388i
\end{pmatrix}$ |
| $\frac{\kappa^2}{\text{GeV}^2}$ | $1.25 \times 10^{-22}$ |
| $\Delta m^2_{\text{atm}}$ | $2.47 \times 10^{-3}\text{eV}^2$ |
| $\Delta m^2_{\text{sol}}$ | $7.94 \times 10^{-5}\text{eV}^2$ |
| $\sin \theta_{12}$ | 0.520 |
| $\sin \theta_{23}$ | 0.759 |
| $\sin \theta_{13}$ | 0.146 |
| $[\delta_C^{\ell}, \alpha_{21}]$ | $[211^\circ, 128^\circ]$ |
| $\sum m_i$ | 58.6 meV |
| $\langle m_{ee} \rangle$ | 3.17 meV |

TABLE IV: Numerical benchmark point of our input parameters and observables in NH within 3$\sigma$. 

11
III. CONCLUSION AND DISCUSSION

We have proposed a new type of DM stability at fixed points at \( \tau = \omega, i \) under a modular \( A_4 \) symmetry, introducing three Majorana neutral fermions where the lightest one is assumed to be DM. We have shown that the observed relic density is simply induced at boson mediation via s-channel. Then, we have demonstrated a benchmark points satisfying the current neutrino oscillation data that can also be reproduced in case of \( \tau = i \) for both the cases of NH and IH. As a future work, we would try to build a model that generates predictions for lepton sector (as well as quark sector), maintaining the stable DM candidate at fixed points.
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Appendix A: Modular forms

Modular forms $Y_r^{(k)}$ are given by $Y_1^{(4)}$, $Y_1^{(4)}$, $Y_3^{(2)}$ and $Y_3^{(4)}$ where
\begin{align}
Y_3^{(2)} &\equiv (y_1,y_2,y_3)^T, \\
Y_3^{(4)} &\equiv (y_1^2 - y_2y_3, y_3^2 - y_1y_2, y_2^2 - y_1y_3)^T, \\
Y_1^{(4)} &\equiv y_1^2 + 2y_2y_3, \\
Y_1^{(6)} &\equiv y_3^2 + 2y_1y_2, \\
Y_1^{(4)'} &\equiv y_2^2 + 2y_1y_3,
\end{align}
(A.1)

and
\begin{align}
y_1(\tau) &\equiv \frac{i}{2\pi} \left( \frac{\eta'((\tau + 1)/3)}{\eta((\tau + 1)/3)} + \frac{\eta'((\tau + 2)/3)}{\eta((\tau + 2)/3)} - \frac{27\eta'(3\tau)}{\eta(3\tau)} \right), \\
y_2(\tau) &\equiv -\frac{i}{\pi} \left( \frac{\eta'((\tau + 1)/3)}{\eta((\tau + 1)/3)} + \omega^2 \frac{\eta'((\tau + 2)/3)}{\eta((\tau + 2)/3)} \right), \\
y_3(\tau) &\equiv -\frac{i}{\pi} \left( \frac{\eta'((\tau + 1)/3)}{\eta((\tau + 1)/3)} + \omega \frac{\eta'((\tau + 2)/3)}{\eta((\tau + 2)/3)} \right),
\end{align}
(A.3)

where $\eta(\tau)$ is Dedekind eta function. $Y_1^{(4)'}$ vanishes at all $\tau$ however it is useful to find identities of modular forms.

$Y_1^{(k)}$ is obtained by $Y_1^{(4)}$ and $Y_1^{(6)}$:
\begin{align}
Y_1^{(k)} &\equiv \left(Y_1^{(4)}\right)^i \left(Y_1^{(6)}\right)^j, \\
\end{align}
(A.4)

where $k = 4i + 6j$. In the same way, $Y_1^{(k)'}$ and $Y_1^{(k)''}$ are obtained by $Y_1^{(4)}$, $Y_1^{(6)}$ and $Y_1^{(4)}$,
\begin{align}
Y_1^{(k)'} &\equiv Y_1^{(4)} \left(Y_1^{(4)}\right)^i \left(Y_1^{(6)}\right)^j, \\
Y_1^{(k)''} &\equiv \left(Y_1^{(4)}\right)^2 \left(Y_1^{(4)}\right)^i \left(Y_1^{(6)}\right)^j\end{align}
(A.5)
where \( k' = 4 + 4i' + 6j' \) and \( k'' = 8 + 4i'' + 6j'' \). There are three \( Y_1^{(12)} \) however the independent forms are only two. Using the following identity,
\[
\left( Y_1^{(4)} \right)^3 + \left( Y_1^{(4)}' \right)^3 + \left( Y_1^{(4)}'' \right)^3 - \left( Y_1^{(6)} \right)^2 = 3Y_1^{(4)}Y_1^{(4)}'Y_1^{(4)}'',
\] (A.7)
and \( Y_1^{(4)}'' = 0 \), we can obtain
\[
\left( Y_1^{(4)} \right)^3 = \left( Y_1^{(6)} \right)^2 - \left( Y_1^{(4)} \right)^3.
\] (A.8)
Therefore (A.5) and (A.6) have only these forms.

\( Y_3^{(8)} \) has three different forms: \( Y_1^{(6)}Y_3^{(2)} \), \( Y_1^{(4)}Y_1^{(4)}' \) and \( Y_1^{(4)}'TY_3^{(4)} \), where
\[
T = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}.
\] (A.9)
We can find two independent forms using the following identity:
\[
Y_1^{(6)}Y_3^{(2)} = \left( Y_1^{(4)}I_3 + Y_1^{(4)}'T + Y_1^{(4)}'T^2 \right)Y_3^{(4)}
= \left( Y_1^{(4)}I_3 + Y_1^{(4)}'T \right)Y_3^{(4)},
\] (A.10)
where \( I_3 \) is \( 3 \times 3 \) identity matrix and \( Y_1^{(4)}'' = 0 \).

\( Y_3^{(10)} \) has four different forms: \( \left( Y_1^{(4)} \right)^2 Y_3^{(2)} \), \( Y_1^{(4)}Y_1^{(4)}'TY_3^{(2)} \), \( \left( Y_1^{(4)}' \right)^2 Y_3^{(2)} \) and \( Y_1^{(6)}Y_3^{(4)} \).
We can find three independent forms using the following identity:
\[
Y_1^{(6)}Y_3^{(4)} = \left( \left( Y_1^{(4)} \right)^2 - Y_1^{(4)}Y_1^{(4)}' \right)I_3 + \left( \left( Y_1^{(4)}' \right)^2 - Y_1^{(4)}Y_1^{(4)}'' \right)T + \left( \left( Y_1^{(4)}' \right)^2 - Y_1^{(4)}Y_1^{(4)}'' \right)T^2 \right)Y_3^{(2)}
= \left( \left( Y_1^{(4)} \right)^2 I_3 - Y_1^{(4)}Y_1^{(4)}' \right)T + \left( \left( Y_1^{(4)}' \right)^2 T^2 \right)Y_3^{(2)}.
\] (A.11)
The basic modular forms have the following values at fixed points,
\[
Y_3^{(2)}(i) = Y_0(i) (1, 1 - \sqrt{3}, -2 + \sqrt{3})^T,
\] (A.12)
\[
Y_3^{(2)}(\omega) = Y_0(\omega) (1, \omega, -\frac{1}{2}\omega^2)^T,
\] (A.13)
\[
Y_3^{(2)}(i\infty) = Y_0(i\infty)(1, 0, 0)^T,
\] (A.14)
\[
Y_3^{(4)}(i) = 3(2 - \sqrt{3})Y_0(i)^2 (1, 1, 1)^T,
\] (A.16)
\[
Y_3^{(4)}(\omega) = \frac{3}{2}Y_0(\omega)^2 (1, -\frac{1}{2}\omega^2, \omega)^T,
\] (A.17)
\[
Y_3^{(4)}(i\infty) = Y_0(i\infty)^2 (1, 0, 0)^T.
\] (A.18)
\[ Y_1^{(4)}(i) = 3 \left( -3 + 2\sqrt{3} \right) Y_0(i)^2, \]  
\[ Y_1^{(4)}(\omega) = 0, \]  
\[ Y_1^{(4)}(i\infty) = Y_0(i\infty)^2, \]  
\[ Y_1^{(6)}(i) = 0, \]  
\[ Y_1^{(6)}(\omega) = \frac{3}{2} Y_0(\omega)^3, \]  
\[ Y_1^{(6)}(i\infty) = Y_0(i\infty)^3, \]  
\[ Y_1^{(4)\prime}(i) = 3 \left( 3 - 2\sqrt{3} \right) Y_0(i)^2, \]  
\[ Y_1^{(4)\prime}(\omega) = 2 Y_0(\omega)^2, \]  
\[ Y_1^{(4)\prime}(i\infty) = 0, \]  
\[ Y_0(i) = 1.0225 \cdots, \]  
\[ Y_0(\omega) = 0.9486 \cdots, \]  
\[ Y_0(i\infty) = 1. \]

Modular forms and zero at fixed points are listed in Tab. VI

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| $k$ | $r$ | Modular form | Zero point |
|-----|-----|--------------|------------|
| 2   | 3   | $Y_3^{(2)}$  | $-$        |
| 4   | 3   | $Y_3^{(4)}$  | $-$        |
|     | 1   | $Y_1^{(4)}$  | $\omega$  |
|     | 1'  | $Y_1^{(4)}$  | $i\infty$ |
| 6   | 3   | $Y_1^{(4)}Y_3^{(2)}$ | $\omega$  |
|     | 3   | $Y_1^{(4)}TY_3^{(2)}$ | $i\infty$ |
|     | 1   | $Y_1^{(6)}$  | $i$        |
| 8   | 3   | $Y_1^{(4)}Y_3^{(4)}$ | $\omega$  |
|     | 3   | $Y_1^{(4)}TY_3^{(4)}$ | $i\infty$ |
|     | 1   | $(Y_1^{(4)})^2$ | $\omega$  |
|     | 1'  | $Y_1^{(4)}Y_1^{(4)}$ | $\omega, i\infty$ |
|     | 1'' | $(Y_1^{(4)})^2$ | $i\infty$ |
| 10  | 3   | $(Y_1^{(4)})^2 Y_3^{(2)}$ | $\omega$  |
|     | 3   | $Y_1^{(4)}Y_1^{(4)}TY_3^{(2)}$ | $\omega, i\infty$ |
|     | 3   | $(Y_1^{(4)})^2 Y_3^{(4)}$ | $i\infty$ |
|     | 1   | $Y_1^{(4)}Y_1^{(6)}$ | $\omega, i$ |
|     | 1'  | $Y_1^{(4)}Y_1^{(6)}$ | $i, i\infty$ |

TABLE VI: Modular forms

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