Theorems on Null-Paths and Red-Shift

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ABSTRACT

In the present work, we prove the validity of two theorems on null-paths in a version of absolute parallelism geometry. A version of these theorems has been originally established and proved by Kemack, McCrea and Whittaker (KMW) in the context of Riemannian geometry. The importance of such theorems is their use, in applications, to derive a general formula for the red-shift of spectral lines coming from distant objects. The formula derived in the present work, can be applied for both cosmological and astrophysical red-shifts. It takes into account the shifts resulting from gravitation, different motions of the source of photons, spin of the moving particle (photons) and the direction of the line of sight. It is shown that this formula cannot be derived in the context of Riemannian geometry, but it can be reduced to a formula given by KMW under certain conditions.

Subject headings: Null-G eodesics, Riemannian Geometry, Absolute Parallelism Geometry, Null-Paths, Red-Shift

1. Introduction

In the context of the general theory of relativity (GR), the red-shift in the spectra of distant objects is a metric phenomena. In other words, knowing the metric of space-time, one can calculate the red-shift whether it is astrophysical or cosmological. This depends on the fact that the trajectory of the photons, in a background gravitational field, is assumed to be null-geodesic, while the metric of the space-time is the first integral of this null-geodesic.

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An alternative, and more general, method for calculating the red-shift is by using the results of theorems on null-geodesics, established in Riemannian geometry, by Kermack, McCrea, and Wittaker (KMW) in 1933. It is worth of mention that the applications of the two methods, in the context of GR, give identical results. In both methods, it is assumed that the trajectory of a photon in a gravitational field is a null-geodesic. This implies the neglect of the effect of spin of the photon (spin independent trajectory), on its trajectory, if any. In calculating the red-shift, the first method is more easier and direct than the second; and since the two methods are equivalent in the context of GR, authors usually use the first method neglecting the second one.

Recently, some evidences indicating probable dependence of trajectories of spinning particles on their spin, were reported. One of these evidences, on the laboratory scale, is the discrepancy between theoretical calculations and the results of the COW - experiment (O. Verhaeuser and Colella (1974), Colella et al. (1980) and Wemer et al. (1988). Another evidence, on the galactic scale, concerning the arrival times of photons, neutrinos (and gravitons!) from SN1987A (Schramm and Truran (1990), Weber (1994) and De Rujula (1987)). On the other hand, a new path equation, in the parameterized absolute parallelism (PAP) geometry, giving the trajectory of a spinning particle in a gravitational field (spin dependent trajectory), is suggested (Wanas (1998)). It is worth of mention that the use of this equation has given a satisfactory interpretation of the discrepancy in the COW - experiment Wanas et al. (2000), and can account for the time delay of spinning particles coming from SN1987A (Wanas et al. (2002)). If we use this equation, to describe the trajectory of photons, (spin one massless particle), we encounter a problem concerning red-shift calculations. That is, the metric of space-time is not the first integral of the new path equation; and the red-shift is no longer a metric phenomena. Consequently, one can not calculate the red-shift using any of the above mentioned methods. One way to solve this problem is to develop theorems on the null-path, similar to those of KMW - theorems, in order to apply them for obtaining the red-shift extracted from spinning particles. This is the aim of the present work.

For this aim, we give a brief account on KMW - theorems in section 2. In section 3, we review the main features of the spin dependent path equation. The validity of the KMW - theorems are proved in PAP geometry in section 4. In section 5 we derive a general formula for red-shift, taking into account the spin of the particle from which we extract the red-shift. The work is discussed in section 6.
2. KMW - Theorems

The following theorems on null-geodesics, were established by KMW (Kemak et al. (1933)) in Riemannian space $R^n$ of dimensions (n), whose metric is given by:

$$dS^2 = g \ dx \ dx :$$

(1)

Consider the null-geodesic connecting the two neighboring $C_0$ and $C_1$ in $R^n$. The tangent null-vector (transport vector in (Kemak et al. (1933))) is defined by,

$$\frac{dx}{d} := \frac{dx}{dx} ;$$

(2)

where is a parameter characterizing. The equation for is given by,

$$\frac{d}{d} + \frac{\partial T}{\partial x} = 0;$$

(3)

where is the Christoffel symbol of the second type. Equation (3) follows from the Euler-Lagrange equation,

$$\frac{d}{d} (\frac{\partial T}{\partial x}) - \frac{\partial T}{\partial x} = 0;$$

(4)

upon taking,

$$T := \frac{1}{2} g ;$$

(5)

Let $^0$ be a null-geodesic, parallel to and passing through the point $C^0$ near $C$, and let denotes the vector $CC^0$. Let a scalar $J$ be defined as,

$$J := \frac{1}{2} g ;$$

(6)

where is the covariant form of the vector .

The KMW - Theorems can be stated as,

Theorem (I)

"The scalar $J$ given by (6) is independent of the choice of the direction $CC'$, and is also independent of the position of $C$ on the null-geodesic." It depends only on the two null-geodesics as a whole and not on any particular point on them. This is the first theorem which was rigorously proved by KMW."
Consider a null-geodesic parallel to , of which there are \(1^{n-1}\). If we form the scalar "\(J\)" corresponding to any of these null-geodesics, we find that for any particular value of "\(J\)" there are \(1^{n-2}\) parallel null-geodesics.

**Theorem (II)**

"If the set of \(1^{n-2}\) null-geodesics lies in a \(R_{n-1}\) which intersects a local at subspace \(E_{n-1}\), at \(C\), in a local at subspace \(E_{n-2}\), then \(E_{n-2}\) is perpendicular to the projection of in \(E_{n-1}\)."

As a consequence of these two theorems KMW were able to derive the following formula for the red-shift extracted from photons, assuming that a photon is moving along null-geodesic of the metric,

\[
Z = \frac{I}{S} = \left[ \begin{array}{ccc} I_1 & S & I_0 \\ S & F_0 \\ I_0 \end{array} \right];
\]

where \(;S\) are the covariant form of the tangents to the geodesics of the observers at \(C_1;C_0\) respectively.

3. Spin-Dependent Path Equation

It is well known that Riemannian geometry possesses two types of the paths. The first is the geodesic path and the second is the null-geodesic. The equation of these two paths can be written in the general form,

\[
\frac{d^2x}{dp^2} + \frac{dx}{dp} \frac{dx}{dp} = 0;
\]

where \(p\) is a parameter characterizing the trajectory of a massive or massless particle (cf. Adler et al. (1975)). This parameter may be related to the parameter \(S\), of (1), by,

\[
dS^2 = E \, dp^2
\]

where \(E\) is a numerical parameter taking the values

\[
E = 0; \text{ for a null geodesic};
E = 1; \text{ for a geodesic}:
\]

Wanas et al. (1995), directed their attention to the absolute parallelism space (AP-space), and by generalizing the method given by Bazanski (1977), (1989), they derived the
The following set of three path equations:

\begin{equation}
\frac{dJ}{dS} + J J = 0; \tag{11}
\end{equation}

\begin{equation}
\frac{dW}{dS^0} + W W = \frac{1}{2} ( \_ \_ \_ ); \tag{12}
\end{equation}

\begin{equation}
\frac{dV}{dS^+} + V V = ( \_ \_ \_ ); \tag{13}
\end{equation}

where \( J, W \) and \( V \) are the tangents to the corresponding curves whose parameters are \( S, S^0 \) and \( S^+ \) respectively, and \( \_ \_ \_ \) is the torsion of the AP-geometry defined by,

\begin{equation}
def \; \; \; \; \; \; \; \; \tag{14}
\end{equation}

where \( \_ \_ \_ \) is the non-symmetric affine connection defined as a consequence of the condition for AP.

Wanas (1998) defined a general expression for a connection formed by taking linear combinations of the available connections in the AP-space. He mentioned that the metricity condition is necessary but not sufficient to define the Christoffel symbol. He generalized the three path equations (11), (12) and (13) in the following equation,

\begin{equation}
\frac{dZ}{d} + Z Z = b ( \_ \_ \_ ); \tag{15}
\end{equation}

where \( b \) is a parameter given by \( b = \frac{n}{2}, \) \( n (= 0; 1; 2; \ldots) \) is a natural number \( \_ \_ \_ \) is the affine structure constant and \( \_ \_ \_ \) is a numerical free parameter, \( Z \) \( \text{def} \; \; \frac{dX}{d}, \) and \( \_ \_ \_ \) is the evolution parameter along the new general path (15) associated with the general connection:

\begin{equation}
r_\cdot = + b \; ; \tag{16}
\end{equation}

The geometric structure characterized by the connection (16) is called the PAP geometry (Wanas (2000)). It is worth mentioning that equation (15), represents a generalization of the three path equations given above. This equation will reduce to the equation of geodesics (null-geodesic upon reparametrization) in the Riemannian geometry, when the parameter \( b = 0. \)

It has been shown that (15), can be used to represent trajectories of spinning test particles, massive or massless, in a gravitational field.
4. Theorems on Null-Paths in PAP-spaces

Let \( \,^0 \) be two neighboring null-paths of the type given by (15) defined in PAP-space of dimensions (\( n \)). Let \( C \) be a point on \( \,^0 \) and \( C^0 \) be a neighboring point on \( C^0 \). Let be the vector \( CC^0 \) connecting the two points. Let us define the following scalar,

\[
J \overset{\text{def}}{=} Z; \quad (17)
\]

where \( Z \) is the null tangent to the path (15), defined at \( C \). Then we can prove the following theorem:

**Theorem (I)**

"The scalar \( J \) is independent of the position of the point \( C \) on the null-path \( \,^0 \) and is also independent of the choice of the direction \( CC^0 \) but depends on the two null-paths them selves."

Now, as mentioned in the previous section, the equation of the null paths in the PAP Geometry is given by (15) and the equation of null geodesic in the Riemannian Geometry is given by (8). It is clear that equation (15) tends to equation (8), if \( b = 0 \). Keeping in mind the fact that for every absolute parallelism space there exists an associated Riemannian one, then we can relate the objects in the first equation (15) to the objects of the second (8) by the following relations

\[
Z = (1 + g(b)); \quad (18)
\]

\[
= p \ (1 + f(b)); \quad \text{and} \quad (19)
\]

\[
= (1 + l(b)); \quad (20)
\]

where \( g(b); f(b) \) and \( l(b) \) are positive functions of the parameter \( b \) such that these functions tend to zero when \( b \) goes to zero. Now let us evaluate the scalar \( J \overset{\text{def}}{=} Z \). By using equation (19), we get

\[
\frac{dp}{d} = \frac{1}{(1 + f(b))} \quad (21)
\]

If we use (18), (20) and (21), we have

\[
\frac{d}{d} (Z) = \frac{1}{(1 + f(b))} \frac{dp}{d} (Z)
\]
\[
\begin{align*}
\text{(7)} & \quad = \frac{1}{(1 + f(\phi))} \left( \frac{d}{dp} (1 + g(\phi))(1 + l(\phi)) \right) \\
& \quad = \frac{1}{(1 + f(\phi))} (1 + g(\phi))(1 + l(\phi)) \left( \frac{d}{dp} (\text{ )} \right)
\end{align*}
\]

Now by recalling equation (6) and theorem (I) of KMW, we get

\[
\frac{d}{d (2)} Z = 0;
\]

\[i.e. \quad Z = \text{constant};\]  \( 23 \)

This result proves theorem (I) on the null-path of the PAP-space. Now, as an extension of the idea of null-path passing through a given point C in the PAP-space \( T_n \), one can find \( 1 \ n \) of null-paths, in the neighborhood of the point C, parallel to the null-path. The second theorem can be stated as follows:

Theorem (II)

For a definite value of the scalar \( J \), we have \( 1 \ n \) of parallel null-paths lying in a subspace \( T_{(n-1)} \), which intersects a local at subspace \( E_{(n-1)} \) of \((n-1)\) dimensions at the point C in a local \( (n-2)\)-dimensions at subspace \( E_{(n-2)} \). This \( E_{(n-2)} \) is perpendicular to the projection of the null-path in \( E_{(n-1)} \).

To prove this theorem, we shall assume that at the point C, or at any point in its neighborhood, there is no singularity. If we assume any rectangular axes at C, such that the direction ratios along the null-path are \( (1; 2; \ldots; n) \), then:

\[
1^2 + 2^2 + \ldots + n^2 = 0;
\]

So, the tangent null-vector \( Z \), to the null-path at C, has the components, \( (m; m; m; \ldots; m) \), where \( m \) is some constant. If the coordinates of the point C are \( (x^1; x^2; \ldots; x^n) \), and if we take the vector \( Z \), which is defined above, as \( (x^1; x^2; \ldots; x^n) \), then the scalar \( J \) takes the form:

\[
J = (x^1 + x^2 + \ldots + x^n) m;
\]

\[i.e.\]

\[
(x^1 + x^2 + \ldots + x^n) = \frac{J}{m};
\]  \( 26 \)
Due to theorem (I), $J$ remains constant wherever $(x^1; x^2; \ldots; x^n)$ lies in the hyperplane, i.e., for any other point $C^0$ on null path $\phi^0$, parallel to the null-path $\phi^0$, $J$ remains constant. So, if we take the components of $Z$ at point $C^0$ on the null-path $\phi^0$, to be $(x^1 + k^1_1; x^2 + k^2_2; \ldots; x^n + k^n_n)$, and substitute in equation (28), we have:

$$J = m \left( x^1_1 + k^1_1 + x^2_2 + k^2_2 + \cdots + x^n_n + k^n_n \right).$$

Then, using (24), we get the same equation (25). This means that any one of the set of the null-paths parallel to the null-path $\phi^0$ must lie in the hyperplane given by (26), in order to keep $J$ constant. Now if we put $x^1 = 0$, it follows directly that all the points in which these null-paths cut any local subspace $E_{(n, 1)}$ lie in a local subspace $E_{(n, 2)}$ given by:

$$(x^2_2 + \cdots + x^n_n) = \frac{J}{m};$$

This $E_{(n, 2)}$ is perpendicular to the null-path whose projection in $E_{(n, 1)}$ is given by:

$$\frac{x^2}{2} = \cdots = \frac{x^n}{n}; \quad (27)$$

Hence, the second theorem is proved.

By using the two previous theorems and the same idea of KMW, we can write a general expression of the red-shift. This expression depends essentially on the idea of the wave fronts, which are represented by a set of null-paths passing through the points of the wave front. Their projections in the local subspace are perpendicular to this wave front. So, the actual wavelength is determined by the perpendicular distance between the wave fronts corresponding to two parallel sets of null-paths. In other words, it is the interval between the points of intersection of the two local subspaces defined by the two sets of parallel null-paths and the observer's world line.

5. General Expression of Red-Shift in PAP-Space

In the previous section, we have shown that the two theorems on null-geodesics, proved by KMW, are also applicable to the null-paths (15). Now, we are going to assume that the trajectory of a photon (spin-1 particle) in a gravitational field is spin dependent and given by equation (15). So, we can easily establish a general formula for the red-shift of spectral lines similar to that given by KMW.

Consider a null path of the form (15) connecting the two points $C_1$, $C_0$ at which two observers A and B are located, respectively. The null path belongs to the same
wave front observed by A and B. Let $1$ and $0$ be the components of the transport null tangent to $C_1$ and $C_0$, respectively. Let $0$ be a null path parallel to , belonging to the succeeding wave front, intersecting the world lines of A and B at $C_1^0$ and $C_0^0$, respectively. If $1$, $0$ are the wavelengths of the same spectral line as observed by A and B, respectively, then the components of the vectors $C_1C_1^0$ and $C_0C_0^0$ are ; which represent the components of the unit tangents to the world lines of A and B, respectively. These unit tangents are solutions of (15), for A and B, upon taking $b = 0$. The vectors and $0$ are the values of the vectors $1$, of theorem (D), evaluated at $C_1$, $C_0$ respectively, while $1$, $0$ represent the values of the vector $Z$, of the same theorem, evaluated at $C_1$, $C_0$ as stated above. Now applying theorem (I) and equating the values of $J$ at $C_1$, and at $C_0$ we get

$$1 1 = 0 0$$

(28)

$$1 1 = 0 0$$

(29)

thus,

$$1 1 = 0 1 = 1 0$$

(30)

This gives a general formula for the red-shift of spectral lines coming from a distant object.

6. Concluding Remarks

In the present work, we have investigated the validity of two important theorems on null-geodesics in the context of the PAP-geometry. These two theorems are important to establish a general formula for the red-shift of spectral lines especially when the trajectories of massless particles are spin dependent. Equation (15) is the equation representing such trajectories. This equation can be used as an equation of motion in the context of any field theory written in the AP-geometry including GR (cf. Wanas (1990)).

In conclusion we can write the following general remarks

(1) In the present work, we tried, as far as we could, to use the same notations, as those used in the original work of KMW in order to facilitate comparison.

(2) The path equation (15) can be used to represent the trajectory of a test particle or the trajectory of a massless particle, in a background gravitational field, upon adjusting the parameter $b$. The right hand side of this equation is suggested to represent a type of interaction between the spin of the moving particle and the torsion of the background gravitational field. The parameter $(b)$ is a spin dependent parameter (Wanas (1998)).
(3) The two theorems on (15), when it represents null paths, can be reduced to the original KMW theorems, reviewed in section 2, upon taking \( b = 0 \).

(4) Equation (30) gives the red-shift taking into account the spin-torsion interaction. This equation appears to be the same as that given by KMW, but the main difference is that the effect of the spin of the moving particle will appear in the values of the null-vectors \( 0 \) and \( 1 \) which are solutions of equation (15) and not of the equation of null-geodesic.

(5) In KMW-paper, the authors used formula (7) to get Doppler shift. In an attempt to interpret the solar limb effect (Mikhail, et al. (2001)), the use of (7) have been widened to account, not only for the relative radial velocity between A and B (Doppler-shift), but also for:

(i) The effect of gravity (gravitational red-shift).

(ii) The effect of the direction of the null-geodesic.

In addition to these effects, the formula (30) will account also for the effect of the spin-torsion interaction on the value of the red-shift.

(6) Formula (30) can be used to get the red-shift whether it is treated as a metric phenomena or not.

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