NONLOCAL FINAL VALUE PROBLEM GOVERNED BY SEMILINEAR ANOMALOUS DIFFUSION EQUATIONS

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Abstract. Our goal is to establish some sufficient conditions for the solvability of the nonlocal final value problem involving a class of partial differential equations, which describes the anomalous diffusion phenomenon. Our analysis is based on the theory of completely positive functions, resolvent operators and fixed point arguments in suitable function spaces. Especially, utilizing the regularity of resolvent operators, we are able to deal with non-Lipschitz cases. The obtained results, in particular, extend recent ones proved for fractional diffusion equations.

1. Introduction. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial \Omega$. Consider the following equation

$k \ast \partial_t u + (-\Delta)^\gamma u = f(t, u)$ in $\Omega$, $t \in (0, T]$,  \hspace{1cm} (1)

with the nonlocal final condition

$u(T, \cdot) = g(u)$ in $\Omega$,  \hspace{1cm} (2)

and the Dirichlet boundary condition

$u = 0$ on $\partial \Omega$, $t \geq 0$.  \hspace{1cm} (3)

In equation $1$, $k \in L^1_{loc}(\mathbb{R}^+)$, $\gamma > 0$, and $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a given function. The notation ‘$\ast$’ denotes the Laplace convolution with respect to the time $t$, i.e., $(k \ast v)(t, x) = \int_0^t k(t-s)v(s, x)ds$, and $(-\Delta)^\gamma$ stands for the fractional power operator of the Laplacian.

Equation (1) belongs to a class of nonlocal partial differential equations, which has been employed to describe anomalous diffusion processes, as remarked in [14].

In order to deal with (1), we use the following standing hypothesis.

(\textit{\textit{PC}}) The kernel function $k \in L^1_{loc}(\mathbb{R}^+)$ is nonnegative and nonincreasing, and there exists a function $l \in L^1_{loc}(\mathbb{R}^+)$ such that $k \ast l = 1$ on $(0, \infty)$.

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This hypothesis was employed in some works, see e.g. [3, 8, 14, 15]. The pair \((k, l)\) is called associate Sonine kernels (see [7]), and ones also write \((k, l) \in (PC)\). Let 
\[g_{\alpha}(t) = t^{-\alpha} / \Gamma(\alpha), \quad \alpha > 0 \text{ and } t > 0.\]
We refer to some well-known examples of \((k, l)\):

- **Slow diffusion case**: \(k(t) = g_{1-\alpha}(t)\) and \(l(t) = g_{\alpha}(t), \alpha \in (0, 1).\) Equation (1) is of fractional diffusion type and has been studied extensively.

- **Ultra-slow diffusion case**: 
  \[k(t) = \int_0^1 g_{\beta}(t) d\beta \quad \text{and} \quad l(t) = \int_0^\infty e^{-pt} \frac{1}{1 + p} dp, \quad t > 0.\]

One has an equation of distributed order (see, e.g. [4]).

In fact, if \(k\) is completely monotone, i.e. \(k \in C^\infty(0, \infty)\) such that \((-1)^n k^{(n)}(t) \geq 0\) for all \(t \in (0, \infty)\), then there exists a unique \(l \in L^1_{loc}(\mathbb{R}^+)\) such that \(k * l = 1\) on \((0, \infty)\) (see [2, Theorem 5.4, p.159]). For example, let 
\[k(t) = \sum_{i=1}^m \mu_i g_{1-\alpha_i}(t)\]
with \(\alpha_i \in (0, 1)\) and \(\mu_i > 0.\) Then \(k\) is completely monotone and (1) becomes a multi-term time-fractional equation.

On the other hand, if \(k(t) = t^{\alpha-1} a(t)\) with \(\alpha \in (0, 1),\) \(a(t) = \sum_{n=0}^\infty a_n t^n, a_0 \neq 0,\)
then there exists a unique analytic function 
\[b(t) = \sum_{n=0}^\infty b_n t^n\]
such that \(k\) and \(l\) given by \(l(t) = t^{-\alpha} b(t)\) are associate Sonine kernels (see, e.g. [7]). The kernel 
\[k(t) = g_{\beta}(t) E_{\alpha, \beta}(-\omega t^\alpha)\]
with \(0 < \alpha < \beta < 1, \omega > 0,\) is of this form. Using this kernel, we get the class of weighted time-fractional equations (see [5]).

As far as the problem (1)-(3) is concerned, the objective is to detect the previous state of process from its present one, in the situation that, the process cannot be observed at the time \(t = 0\) and then the initial data is not known. When the measured final value depends on some information, e.g. the energy, of the system, the problem involves a nonlocal final condition like (2). Unlike the problem with nonlocal initial condition (i.e. \(u(0) = g(u)\), forward problem), the nonlocal final value problem is of backward type and much more complicated. The main reason is the smoothing effect of the forward problem, i.e. \(u(t), \text{ for } t > 0,\) is more regular than \(u(0)\). Consequently, \(t = 0\) may become a singular point of \(u\) if the given final value is not regular enough (see Remark 2).

In the case \(k(t) = g_{1-\alpha}(t)\) (fractional/slow diffusion case), there has been a number of works devoted to (1)-(3). The linear problem \((f \text{ and } g \text{ are independent of } u)\) was considered in [10, 11, 17, 18, 19], where the problem was proved to be ill-posed in the sense that, the solution is unstable with respect to final data, and some regularization methods were proposed to obtain the solution. The semilinear problem with final value condition \((g \text{ is independent of } u)\) was addressed in [13]. Recently, the semilinear problem with nonlocal final value condition was solved in [12]. It should be mentioned that, the analysis in these works is based on the formulation of solution utilizing the Mittag-Leffler functions. Taking other cases into account, for instant, ultra-slow diffusion or multi-term fractional diffusion case, such a nice representation is unavailable. The aim of our work is to give a comprehensive analysis for solving the problem in a general case, i.e. when \(k\) is a Sonine kernel function. From the technical point of view, our approach employs the theory of completely positive functions (see [2]) and resolvent operators (see, e.g. [9]), combining with fixed point principles. Particularly, using the parabolicity of the anomalous diffusion equation, we get a regularity of resolvent operators, which enable us to solve the problem in non-Lipschitz cases. In comparison with the results made for the
fractional diffusion case, we deal with a more general class of nonlinearity functions and give more concrete conditions.

Our paper is organized as follows. In the next section, we give a presentation of mild solution to (1)-(3) in linear case using resolvent operators. Section 3 is devoted to the existence results under regular setting, that is, $f$ is defined on $[0,T] \times L^2(\Omega)$ and $g$ is defined on $C([0,T]; L^2(\Omega))$. With this setting, we obtain mild solutions in $C([0,T]; L^2(\Omega))$. In Section 4, we prove the solvability of (1)-(3) in the space $C_1((0,T]; L^2(\Omega))$, consisting of functions possibly discontinuous at $t = 0$. In the last section, we show two examples demonstrating the obtained results.

2. Preliminaries. Consider the following scalar integral equations

$$s(t) + \mu(l * s)(t) = 1, \quad t \geq 0,$$

$$r(t) + \mu(l * r)(t) = l(t), \quad t > 0.$$

The existence and uniqueness of $s$ and $r$ were analyzed in [6]. Recall that the function $l$ is called a completely positive kernel iff $s(\cdot)$ and $r(\cdot)$ take nonnegative values for every $\mu > 0$. The complete positivity of $l$ is equivalent to that (see [1]), there exist $\alpha \geq 0$ and $k \in L_{\text{loc}}^1(\mathbb{R}^+)$ nonnegative and nonincreasing which satisfy $\alpha l + l * k = 1$. So the kernel function $l$ taken from the hypothesis (PC) is completely positive. In the case $l(t) = g_\mu(t)$, by using the Laplace transform, one can see that $s(t) = E_{\alpha,1}(-\mu t^\alpha)$ and $r(t) = t^{\alpha-1} E_{\alpha,1}(-\mu t^\alpha)$, where

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, z \in \mathbb{C},$$

is the Mittag-Leffler function.

Denote by $s(\cdot, \mu)$ and $r(\cdot, \mu)$ the solutions of (4) and (5), respectively. As mentioned in [15], the functions $s(\cdot, \mu)$ and $r(\cdot, \mu)$ take nonnegative values for all $\mu \in \mathbb{R}$. Some additional properties of these functions are collected in the following proposition.

**Proposition 1.** Let the hypothesis (PC) hold. Then for every $\mu > 0$, $s(\cdot, \mu), r(\cdot, \mu) \in L_{\text{loc}}^1(\mathbb{R}^+)$. In addition, we have:

(a) The function $s(\cdot, \mu)$ is nonincreasing. Moreover,

$$\frac{1}{1 + \mu k(t)^{-1}} \leq s(t, \mu) \leq \frac{1}{1 + \mu (1 + l)(t)}, \quad \forall t \geq 0. \quad (6)$$

(b) The functions $s(\cdot, \mu)$ and $r(\cdot, \mu)$ satisfy

$$s(t, \mu) = 1 - \mu \int_0^t r(\tau, \mu) d\tau = k * r(\cdot, \mu)(t), \quad t \geq 0. \quad (7)$$

(c) For each $t > 0$, the functions $\mu \mapsto s(t, \mu)$ and $\mu \mapsto r(t, \mu)$ are nonincreasing.

(d) Equation (4) is equivalent to the problem

$$\frac{d}{dt} [k * (s - 1)] + \mu s = 0, \quad s(0) = 1.$$

(e) Let $v(t) = s(t, \mu)v_0 + (r(\cdot, \mu) * g)(t)$, here $g \in C(\mathbb{R}^+)$. Then $v$ solves the problem

$$\frac{d}{dt} [k * (v - v_0)](t) + \mu v(t) = g(t), \quad v(0) = v_0.$$
Proof. The proofs for (a) and (d) can be found in [14]. The justification for (b) is given in [15], while (c) is proved in [8]. We show a proof for (e). By formulation, we have

\[ v - v_0 = (s - 1)v_0 + r \ast g. \]

Using the relation \( k \ast r = s \), one gets

\[ k \ast (v - v_0) = k \ast (s - 1)v_0 + k \ast r \ast g. \]

So

\[
\frac{d}{dt}[k \ast (v - v_0)] = \frac{d}{dt}[k \ast (s - 1)v_0 + s(0, \mu)g + \frac{d}{dt}s(\cdot, \mu) \ast g] = -\mu s(\cdot, \mu)v_0 + s(0, \mu)g + \frac{d}{dt}s(\cdot, \mu) \ast g = -\mu v_0 + g,
\]

thanks to the fact that \( s(0, \mu) = 1 \) and \( \frac{d}{dt}s(t, \mu) = -\mu r(t, \mu), t > 0 \). The proof is complete.

Let \( \{e_n\} \) be the orthonormal basis of \( L^2(\Omega) \) consisting of eigenfunctions of the operator \(-\Delta\) with homogeneous Dirichlet boundary condition, i.e.,

\[ -\Delta e_n = \lambda_n e_n \text{ in } \Omega, e_n = 0 \text{ on } \partial \Omega, \]

where we can assume that \( \{\lambda_n\} \) is nondecreasing with \( \lambda_1 > 0 \). Then the operator of fractional power can be defined by

\[ (-\Delta)^{\beta}v = \sum_{n=1}^{\infty} \lambda_n^{2\beta} (v, e_n) e_n, \]

\[ D((-\Delta)^{\beta}) = \{ v \in L^2(\Omega) : \sum_{n=1}^{\infty} \lambda_n^{2\beta} (v, e_n)^2 < \infty \}, \]

for \( \beta \in \mathbb{R} \), here the notation \((\cdot, \cdot)\) denotes the inner product in \( L^2(\Omega) \).

Let \( V_\beta = D((-\Delta)^{\beta}) \), then \( V_\beta \) is a Hilbert space with the inner product

\[ (u, v)_{V_\beta} = \sum_{n=1}^{\infty} \lambda_n^{2\beta} (u, e_n)(v, e_n). \]

Furthermore, for any \( \beta > 0 \), the embedding \( V_\beta \subset L^2(\Omega) \) is compact.

We now define the following operators

\[ S(t)v = \sum_{n=1}^{\infty} s(t, \lambda_n^\gamma) (v, e_n) e_n, \quad t \geq 0, v \in L^2(\Omega), \quad (8) \]

\[ R(t)v = \sum_{n=1}^{\infty} r(t, \lambda_n^\gamma) (v, e_n) e_n, \quad t > 0, v \in L^2(\Omega). \quad (9) \]

It is easily seen that \( S(t) \) and \( R(t) \) are linear. We show some basic properties of these operators in the following lemma.

**Lemma 2.1.** Let \( \{S(t)\}_{t \geq 0} \) and \( \{R(t)\}_{t > 0} \) be the families of linear operators defined by (8) and (9), respectively. Then
(a) For each \( v \in L^2(\Omega) \) and \( T > 0 \), \( S(\cdot)v \in C([0,T]; L^2(\Omega)) \) and \( (-\Delta)^\gamma S(\cdot)v \in C((0,T]; L^2(\Omega)) \). Moreover,
\[
\|S(t)v\| \leq s(t, \lambda_n^\gamma)v, \ t \in [0,T],
\]
\[
\|S(t)v\|_{V^1_h} \leq \frac{\|v\|}{(1 * l)(t)}, \ t \in (0,T].
\]  
(b) Let \( v \in L^2(\Omega), T > 0 \) and \( f \in C([0,T]; L^2(\Omega)) \). Then \( R(\cdot)v \in C((0,T]; L^2(\Omega)) \) and \( R * f \in C([0,T]; V^2) \). Furthermore,
\[
\|R(t)v\| \leq r(t, \lambda_n^\gamma)v, \ t \in (0,T],
\]
\[
\| (R * f)(t) \|_{V^2} \leq \left( \int_0^t r(t - \tau, \lambda_n^\gamma) \|f(\tau)\|^2 d\tau \right)^{\frac{1}{2}}, \ t \in [0,T].
\]

Proof. (a) By formulation, we have
\[
\|S(t)v\|^2 = \sum_{n=1}^\infty \lambda_n^{2\gamma} s(t, \lambda_n^\gamma)^2(v, e_n)^2
\leq s(t, \lambda_n^\gamma)^2\|v\|^2, \ t \geq 0, v \in L^2(\Omega),
\]
thanks to Proposition 1(c). So (10) follows. In addition, one sees that the last series is uniformly convergent on \([0,T]\), which ensures that \( S(\cdot)v \in C([0,T]; L^2(\Omega)) \).

For \( t > 0 \), we see that
\[
\|(-\Delta)^\gamma S(t)v\|^2 = \sum_{n=1}^\infty \lambda_n^{2\gamma} s(t, \lambda_n^\gamma)^2(v, e_n)^2
\leq \sum_{n=1}^\infty \left( \frac{\lambda_n^\gamma}{1 + \lambda_n^\gamma(1 * l)(t)} \right)^2 (v, e_n)^2
\leq \frac{\|v\|^2}{[(1 * l)(t)]^2},
\]
which proves (11). Moreover, the series (14) is uniformly convergent on \([\epsilon, T]\) for each \( \epsilon > 0 \). So \( (-\Delta)^\gamma S(\cdot)v \in C((0,T]; L^2(\Omega)) \).

(b) We get (12) from the estimate
\[
\|R(t)v\|^2 = \sum_{n=1}^\infty r(t, \lambda_n^\gamma)^2(v, e_n)^2
\leq r(t, \lambda_n^\gamma)^2\|v\|^2, \ \forall t > 0, v \in L^2(\Omega),
\]
thanks to Proposition 1(c). Concerning (13), let \( f_n(t) = (f(t), e_n) \), then
\[
\|(-\Delta)^\gamma (R * f)(t)\|^2 = \sum_{n=1}^\infty \lambda_n^{2\gamma} \left( \int_0^t r(t - \tau, \lambda_n^\gamma)f_n(\tau)d\tau \right)^2
\leq \sum_{n=1}^\infty \lambda_n^{2\gamma} \left( \int_0^t r(t - \tau, \lambda_n^\gamma)^{\frac{1}{2}}[r(t - \tau, \lambda_n^\gamma)^{\frac{1}{2}} f_n(\tau)]d\tau \right)^2
\leq \sum_{n=1}^\infty \lambda_n^{\gamma} \left( \int_0^t r(t - \tau, \lambda_n^\gamma)d\tau \right) \left( \int_0^t r(t - \tau, \lambda_n^\gamma)|f_n(\tau)|^2d\tau \right)
\[ \leq \sum_{n=1}^{\infty} \int_0^t r(t-\tau, \lambda_1^n) |f_n(\tau)|^2 d\tau \]
\[ = \int_0^t r(t-\tau, \lambda_1^\gamma) \|f(\tau)\|^2 d\tau, \]

here we used the fact that
\[ \mu \int_0^t r(t-\tau, \mu) d\tau = \mu \int_0^t r(\tau, \mu) d\tau = 1 - s(t, \mu) \leq 1, \forall \mu > 0. \]

Hence we get (13).

**Remark 1.** Since the embedding \( V_\gamma \subset L^2(\Omega) \) is compact, it follows from (11) that, for any \( t > 0 \), \( S(t) : L^2(\Omega) \to L^2(\Omega) \) is compact. Furthermore, let \( Q_t(f) = (R\ast f)(t) \), then (13) ensures that \( Q_t \) is also compact as an operator from \( C([0, T]; L^2(\Omega)) \) into \( L^2(\Omega) \), by reasoning that the embedding \( V_2^\gamma \subset L^2(\Omega) \) is compact.

In what follows, we use the notation \( u(t) \) for \( u(t, \cdot) \) and consider \( u \) as a function defined on \([0, T] \), taking values in space \( V_\beta \) for some \( \beta \in \mathbb{R} \). Denote \( A = (-\Delta)^\gamma \) with domain \( D(A) = V_\gamma \). Given \( f \in C([0, T]; L^2(\Omega)) \) and \( g \in L^2(\Omega) \), consider the linear problem
\[ k \ast u'(t) + Au(t) = f(t), t \in (0, T], \quad u(T) = g. \] \hspace{1cm} (15)
\[ \] \hspace{1cm} (16)

Let
\[ u(t) = \sum_{n=1}^{\infty} u_n(t)e_n. \]

Then (15) is formally reduced to the relaxation equation
\[ k \ast u'_n(t) + \lambda_n^\gamma u_n(t) = f_n(t), \]
where \( f_n(t) = (f(t), e_n) \). The last equation can be rewritten as
\[ \frac{d}{dt} [k \ast (u_n - u_n(0))](t) + \lambda_n^\gamma u_n(t) = f_n(t), \]
which implies that
\[ u_n(t) = s(t, \lambda_n^\gamma)u_n(0) + \int_0^t r(t-\tau, \lambda_n^\gamma) f_n(\tau) d\tau, \]
thanks to Proposition 1(e). Hence
\[ u(t) = \sum_{n=1}^{\infty} s(t, \lambda_n^\gamma)u_n(0)e_n + \sum_{n=1}^{\infty} \int_0^t r(t-\tau, \lambda_n^\gamma) f_n(\tau) d\tau e_n. \]

The final condition \( u(T) = g \) leads to
\[ g_n = (g, e_n) = s(T, \lambda_n^\gamma)u_n(0) + \int_0^T r(T-\tau, \lambda_n^\gamma) f_n(\tau) d\tau, \]
which yields
\[ u_n(0) = \frac{1}{s(T, \lambda_n^\gamma)} \left( g_n - \int_0^T r(T-\tau, \lambda_n^\gamma) f_n(\tau) d\tau \right). \]
So we have the following representation for the solution of (15)-(16).

\[ u(t) = \sum_{n=1}^{\infty} \frac{s(t, \lambda_n^\gamma)}{s(T, \lambda_n^\gamma)} \left( g_n - \int_0^T r(T - \tau, \lambda_n^\gamma) f_n(\tau) d\tau \right) e_n + \sum_{n=1}^{\infty} \int_0^t r(t - \tau, \lambda_n^\gamma) f_n(\tau) d\tau e_n \]

\[ = S(T)^{-1} S(t) g - S(T)^{-1} S(t) \int_0^T R(T - \tau) f(\tau) d\tau + \int_0^t R(t - \tau) f(\tau) d\tau. \]  

(17)

Let \( P(t) = S(T)^{-1} S(t) \), then

\[ P(t)v = \sum_{n=1}^{\infty} \frac{s(t, \lambda_n^\gamma)}{s(T, \lambda_n^\gamma)} (v, e_n) e_n, \ v \in L^2(\Omega). \]  

(18)

**Proposition 2.** We have the following estimates:

(a) For \( v \in L^2(\Omega) \) and \( t \in (0, T] \),

\[ \| P(t)v \| \leq \frac{k(T)^{-1}}{(1 * l)(t)} \| v \|. \]  

(19)

(b) If \( v \in V_\gamma \) and \( t \in [0, T] \), then

\[ \| P(t)v \| \leq (\lambda_1^{-\gamma} + k(T)^{-1}) \| v \|_{V_\gamma}. \]  

(20)

**Proof.** (a) In view of Proposition 1(a), one gets

\[ \frac{s(t, \lambda_n^\gamma)}{s(T, \lambda_n^\gamma)} \leq \frac{1 + \lambda_n^\gamma k(T)^{-1}}{1 + \lambda_n^\gamma (1 * l)(t)} \leq \max \left\{ 1, \frac{k(T)^{-1}}{(1 * l)(t)} \right\}, \ \forall t > 0. \]

For \( 0 < t \leq T \), it follows from the relation \( k * l = 1 \) that

\[ 1 = \int_0^t k(s)l(t - s) ds \geq k(t) \int_0^t l(t - s) ds = k(t)(1 * l)(t) \geq k(T)(1 * l)(t), \]

due to the fact that \( k \) is nonincreasing. Then

\[ \frac{k(T)^{-1}}{(1 * l)(t)} \geq 1 \text{ and } \frac{s(t, \lambda_n^\gamma)}{s(T, \lambda_n^\gamma)} \leq \frac{k(T)^{-1}}{(1 * l)(t)}, \ \forall t \in (0, T], \]

hence (19) holds.

(b) Observe that

\[ \frac{s(t, \lambda_n^\gamma)}{\lambda_n^\gamma s(T, \lambda_n^\gamma)} \leq \frac{1 + \lambda_n^\gamma k(T)^{-1}}{\lambda_n^\gamma [1 + \lambda_n^\gamma (1 * l)(t)]} \leq \lambda_1^{-\gamma} + k(T)^{-1}. \]  

(21)

Let \( v \in V_\gamma \). Then

\[ \| P(t)v \|^2 = \sum_{n=1}^{\infty} \left( \frac{s(t, \lambda_n^\gamma)}{s(T, \lambda_n^\gamma)} \right)^2 (v, e_n)^2 = \sum_{n=1}^{\infty} \left( \frac{s(t, \lambda_n^\gamma)}{\lambda_n^\gamma s(T, \lambda_n^\gamma)} \right)^2 \lambda_n^{2\gamma} (v, e_n)^2 \]

\[ \leq [\lambda_1^{-\gamma} + k(T)^{-1}]^2 \sum_{n=1}^{\infty} \lambda_n^{2\gamma} (v, e_n)^2, \]

which ensures (20).

The proof is complete. \( \square \)

The following result shows the condition for problem (15)-(16) to have a unique solution.
Proposition 3. Let \( f \in C([0,T];L^2(\Omega)) \) and \( g \in L^2(\Omega) \). Then
(a) The problem (15)-(16) has a unique solution if and only if
\[
\sum_{n=1}^{\infty} \frac{1}{s(T,\lambda_n^2)} \left( g_n - \int_0^T r(T-\tau,\lambda_n^2) f_n(\tau) d\tau \right)^2 < \infty, \tag{22}
\]
where \( f_n(t) = (f(t),e_n) \) and \( g_n = (g,e_n) \).
(b) If \( f \in C([0,T];V_r) \) and \( g \in V_r \), then condition (22) holds for \( d < 4\gamma \).

Proof. The proof is similar to that in [11] and we omit it. \( \square \)

Based on (17)-(18), we introduce the following definition.

Definition 2.2. Let \( f : [0,T] \times L^2(\Omega) \to L^2(\Omega) \) and \( g : C([0,T];L^2(\Omega)) \to L^2(\Omega) \). A function \( u \in C([0,T];L^2(\Omega)) \) is said to be a mild solution of the problem (1)-(3) if it satisfies
\[
u(t) = P(t)g(u) - P(t) \int_0^T R(T-\tau) f(\tau,u(\tau)) d\tau + \int_0^t R(t-\tau) f(\tau,u(\tau)) d\tau.
\]

In the next sections, we will analyze sufficient conditions for the solvability of (1)-(3).

3. Solvability in regular setting. We first consider the simple case, where \( f \) and \( g \) are Lipschitzian and take values in \( V_r \).

Theorem 3.1. Assume that
\[(F1) \quad \text{The function } f : [0,T] \times L^2(\Omega) \to V_r \text{ is continuous and there exists } L_f > 0 \text{ such that } \|f(t,v_1) - f(t,v_2)\|_{V_r} \leq L_f \|v_1 - v_2\|, \forall t \in [0,T], v_1, v_2 \in L^2(\Omega).\]
\[(G1) \quad \text{The function } g : C([0,T];L^2(\Omega)) \to V_r \text{ is Lipschitzian with constant } L_g, \text{ i.e., } \|g(u_1) - g(u_2)\|_{V_r} \leq L_g \|u_1 - u_2\|_\infty, \forall u_1, u_2 \in C([0,T]; L^2(\Omega)), \text{ where } \|u\|_\infty = \sup_{t \in [0,T]} \|u(t)\| \text{ for } u \in C([0,T]; L^2(\Omega)).\]

Then the problem (1)-(3) has a unique mild solution \( u \in C([0,T]; L^2(\Omega)) \), provided that
\[
\ell := (\lambda_1^{-\gamma} + k(T)^{-1})(L_g + \lambda_1^{-\gamma} L_f) + \lambda_1^{-2\gamma} L_f < 1. \tag{23}
\]

Proof. The proof is done by using the contraction mapping principle. Let
\[
\Phi(u)(t) = P(t)g(u) - P(t) \int_0^T R(T-\tau) f(\tau,u(\tau)) d\tau + \int_0^t R(t-\tau) f(\tau,u(\tau)) d\tau. \tag{24}
\]
We show that \( \Phi : C([0,T];L^2(\Omega)) \to C([0,T];L^2(\Omega)) \) has a unique fixed point. For \( u_1, u_2 \in C([0,T];L^2(\Omega)) \), we have the following estimates
\[
\|P(t)g(u_1) - P(t)g(u_2)\| \leq (\lambda_1^{-\gamma} + k(T)^{-1}) \|g(u_1) - g(u_2)\|_{V_r} \leq (\lambda_1^{-\gamma} + k(T)^{-1}) L_g \|u_1 - u_2\|_\infty, \tag{25}
\]
thanks to Proposition 2(b) and the assumption (G1). In addition,
\[
\|P(t) \int_0^T R(T-\tau)[f(\tau,u_1(\tau)) - f(\tau,u_2(\tau))]|d\tau\|.
\]
\begin{align*}
\leq (\lambda_1^{-\gamma} + k(T)^{-1}) \int_0^T r(T - \tau, \lambda_1^\gamma) \|f(\tau, u_1(\tau)) - f(\tau, u_2(\tau))\|_V \, d\tau \\
\leq (\lambda_1^{-\gamma} + k(T)^{-1}) \int_0^T r(T - \tau, \lambda_1^\gamma) L_f \|u_1(\tau) - u_2(\tau)\| \, d\tau \\
\leq (\lambda_1^{-\gamma} + k(T)^{-1}) \lambda_1^{\gamma} (1 - s(T, \lambda_1^\gamma)) L_f \|u_1 - u_2\|_\infty, \tag{26}
\end{align*}

here we have employed Proposition 2(b), assumption (F1), Proposition 1(b) and Lemma 2.1(b). Similarly, one has

\begin{align*}
\int_0^t R(t - \tau) &\|f(\tau, u_1(\tau)) - f(\tau, u_2(\tau))\| \, d\tau \\
\leq &\int_0^t r(t - \tau, \lambda_1^\gamma) \|f(\tau, u_1(\tau)) - f(\tau, u_2(\tau))\| \, d\tau \\
\leq &\lambda_1^{-\gamma} \int_0^t r(t - \tau, \lambda_1^\gamma) \|f(\tau, u_1(\tau)) - f(\tau, u_2(\tau))\|_V \, d\tau \\
\leq &\lambda_1^{-\gamma} \int_0^t r(t - \tau, \lambda_1^\gamma) L_f \|u_1(\tau) - u_2(\tau)\| \, d\tau \\
\leq &\lambda_1^{-\gamma} (1 - s(t, \lambda_1^\gamma)) L_f \|u_1 - u_2\|_\infty, \tag{27}
\end{align*}

where we have used the fact that \(\|v\| \leq \lambda_1^{-\gamma} \|v\|_V\) for any \(v \in V_r\).

Combining (25), (26) and (27), we arrive at

\[ \|\Phi(u_1)(t) - \Phi(u_2)(t)\| \leq \ell \|u_1 - u_2\|_\infty, \quad \forall t \in [0, T], \]

which implies that \(\Phi\) is a contraction mapping. The proof is complete. \(\square\)

In the next theorem, we show that if the kernel function \(l\) is nonincreasing, then \(f\) can take values in \(L^2(\Omega)\).

**Theorem 3.2.** Let \(g\) satisfy (G1). Assume that the kernel function \(l\) is nonincreasing and \(f : [0, T] \times L^2(\Omega) \rightarrow L^2(\Omega)\) is continuous and obeys the estimate

\[ \|f(t, v_1) - f(t, v_2)\| \leq L_f(t) \|v_1 - v_2\|, \quad \forall t \in [0, T], v_1, v_2 \in L^2(\Omega), \tag{28} \]

where \(L_f \in C([0, T])\) is a nonnegative function such that

\[ \int_0^T \left[ \frac{l(T - \tau)L_f(\tau)}{(1 \ast l)(T - \tau)} \right]^2 \, d\tau < \infty. \tag{29} \]

Then the problem (1)-(3) has a unique mild solution \(u \in C([0, T]; L^2(\Omega))\), provided that \(g(f, g) < 1\), where

\[ g(f, g) = (\lambda_1^{-\gamma} + k(T)^{-1}) L_g + \sup_{t \in [0, T]} \int_0^t r(t - \tau, \lambda_1^\gamma) L_f(\tau) \, d\tau \]

\[ + T^{\frac{1}{2}} k(T)^{-1} \left( \int_0^T \left[ \frac{l(T - \tau)L_f(\tau)}{(1 \ast l)(T - \tau)} \right]^2 \, d\tau \right)^{\frac{1}{2}}. \]

**Proof.** Since \(l\) is nonincreasing, we have

\[ l(t) = r(t, \mu) + \mu \int_0^t l(t - \tau) r(\tau, \mu) \, d\tau \]

\[ \geq r(t, \mu) + l(t) \mu \int_0^t r(\tau, \mu) \, d\tau \]
\[ = r(t, \mu) + l(t)(1 - s(t, \mu)) \]
\[ \geq r(t, \mu) + l(t) \frac{\mu(1 + l)(t)}{1 + \mu(1 + l)(t)}, \forall t > 0, \]

thanks to (6). Then
\[ r(t, \mu) \leq \frac{l(t)}{1 + \mu(1 + l)(t)}, \forall t > 0. \] (30)

Consider the solution operator defined by (24). Let \( u_1, u_2 \in C([0, T]; L^2(\Omega)) \). We get
\[ \|P(t)g(u_1) - P(t)g(u_2)\| \leq (\lambda_1^{-\gamma} + k(T)^{-1})L_g\|u_1 - u_2\|_\infty, \] (31)
as in (25). In addition, one has
\[
\begin{align*}
\| & \int_0^t R(t - \tau)[f(\tau, u_1(\tau)) - f(\tau, u_2(\tau))]d\tau \| \\
\leq & \int_0^t r(t - \tau, \lambda_1^\gamma)\|f(\tau, u_1(\tau)) - f(\tau, u_2(\tau))\|d\tau \\
\leq & \left( \int_0^t r(t - \tau, \lambda_1^\gamma)L_f(\tau)d\tau \right)\|u_1 - u_2\|_\infty.
\end{align*}
\] (32)

Finally, we observe that, for all \( t \in [0, T] \),
\[
\begin{align*}
& \|P(t)\int_0^T R(T - \tau)[f(\tau, u_1(\tau)) - f(\tau, u_2(\tau))]d\tau\|^2 \\
= & \sum_{n=1}^\infty \left[ \int_0^T \frac{s(t, \lambda_n^\gamma)}{s(T, \lambda_n^\gamma)} r(T - \tau, \lambda_n^\gamma)(f(\tau, u_1(\tau)) - f(\tau, u_2(\tau)), e_n)d\tau \right]^2 \\
\leq & \sum_{n=1}^\infty \left[ \int_0^T \frac{(1 + \lambda_n^\gamma k(T)^{-1})l(T - \tau)}{1 + \lambda_n^\gamma (1 + l)(T - \tau)}\|f(\tau, u_1(\tau)) - f(\tau, u_2(\tau)), e_n]\|d\tau \right]^2 \\
\leq & \sum_{n=1}^\infty \int_0^T \frac{k(T)^{-1}l(T - \tau)}{(1 + l)(T - \tau)}\left[ \|f(\tau, u_1(\tau)) - f(\tau, u_2(\tau)), e_n\| \right]^2 d\tau \\
\leq & Tk(T)^{-2}\sum_{n=1}^\infty \int_0^T \left[ \frac{l(T - \tau)}{(1 + l)(T - \tau)}\|f(\tau, u_1(\tau)) - f(\tau, u_2(\tau)), e_n\| \right]^2 d\tau,
\end{align*}
\]

here we utilized (6), (30), and the estimates
\[ s(t, \lambda_n^\gamma) \leq 1, \quad \frac{1 + \lambda_n^\gamma k(T)^{-1}}{1 + \lambda_n^\gamma (1 + l)(T - \tau)} \leq \frac{k(T)^{-1}}{(1 + l)(T - \tau)}, \forall n. \]

Therefore,
\[
\begin{align*}
& \|P(t)\int_0^T R(T - \tau)[f(\tau, u_1(\tau)) - f(\tau, u_2(\tau))]d\tau\|^2 \\
\leq & Tk(T)^{-2}\int_0^T \left[ \frac{l(T - \tau)}{(1 + l)(T - \tau)}\|f(\tau, u_1(\tau)) - f(\tau, u_2(\tau))\| \right]^2 d\tau \\
\leq & Tk(T)^{-2}\int_0^T \left[ \frac{l(T - \tau)}{(1 + l)(T - \tau)}L_f(\tau)\|u_1(\tau) - u_2(\tau)\| \right]^2 d\tau
\end{align*}
\]
It follows from (31)-(33) that

$$||\Phi(u_1) - \Phi(u_2)||_\infty \leq g(f,g)||u_1 - u_2||_\infty,$$

which shows that $\Phi$ is a contraction mapping. The proof is complete.

Remark 2. 1. It was shown in [1] that, if $\lambda$ is nonincreasing, then condition (29) is testified. Indeed, we see that

$$|l(T - \tau)L_f(\tau) - l(T - \tau)L_f(\tau)| \leq Nl(T - \tau)|T - \tau|\delta.$$

Then

$$\frac{l(T - \tau)L_f(\tau)}{(1 * l)(T - \tau)} \leq \frac{N}{(T - \tau)^{1-\delta}},$$

which guarantees (29).

2. Let us discuss about condition (29). If $L_f$ is a nonnegative function satisfying that $L_f(T) = 0$ and that

$$|L_f(t_1) - L_f(t_2)| \leq N|t_1 - t_2|\delta, \forall t_1, t_2 \in [0,T],$$

then condition (29) is testified. Indeed, we see that

$$\int_0^{T-\tau} l(z)dz \leq (T - \tau)l(T - \tau)$$

and

$$l(T - \tau)L_f(\tau) = l(T - \tau)|L_f(T) - L_f(\tau)| \leq Nl(T - \tau)|T - \tau|\delta.$$

3. If the function $g$ takes “less regular” values, e.g. $g(u) \in V_\beta$ with $\beta < \gamma$, the problem (1)-(3) may fail to admit solution in $C([0,T]; L^2(\Omega))$. Indeed, we consider a simple case, when $g$ is independent of $u$. Assume the opposite, that the problem has a solution $u \in C([0,T]; L^2(\Omega))$, i.e. $u(0)$ is well-defined. Then

$$u(t) = P(t)g - P(t)\int_0^T R(T - \tau)f(\tau, u(\tau))d\tau + \int_0^t R(t - \tau)f(\tau, u(\tau))d\tau.$$

Let the assumptions (28)-(29) hold and $f(\tau, 0) = 0$. Then the last two terms are defined at $t = 0$. Regarding the first term, let

$$g = \sum_{n=1}^{\infty} \frac{e_n}{n^\delta \gamma_n^\gamma}, \text{ with } \delta = \frac{1}{2} + 2d(\gamma - \beta).$$

Then $g \in V_\beta$ and

$$||P(0)g||^2 = \sum_{n=1}^{\infty} \frac{(g, e_n)^2}{s(T, \lambda_n^\gamma)^2} \geq \sum_{n=1}^{\infty} \frac{(1 + \lambda_n^\gamma(1 * l)(T))^2}{n^{2\delta} \lambda_n^{2\beta}}$$

$$\geq [(1 * l)(T)]^2 \sum_{n=1}^{\infty} \frac{\lambda_n^{2(\gamma - \beta)}}{n^{2\delta}}.$$

Note that $\lambda_n \sim Cn^{\frac{\gamma}{2}}$ as $n \to \infty$, for some $C > 0$, then

$$\frac{\lambda_n^{2(\gamma - \beta)}}{n^{2\delta}} \sim C^{2(\gamma - \beta)} n^{\frac{\gamma}{2}(\gamma - \beta) - 2\delta} = C^{2(\gamma - \beta)} n^{-1} \text{ as } n \to \infty,$$

which implies that $||P(0)g|| = \infty$. This is a contradiction.
We are now concerned with the case when $f$ and $g$ are non-Lipschitzian. In order to deal with the problem in this case, we need a regularity of the operators $S(\cdot)$ and $R(\cdot)$. Recall that $S(t)$, defined by (8), is the resolvent operator of the problem
\begin{align}
k \ast \partial_t u + (-\Delta)\gamma u &= 0, \quad t > 0, \\
u(0) &= u_0,
\end{align}
that is, $u(t) = S(t)u_0$. This problem is equivalent to the Volterra equation
\begin{equation}
u(t) + \int_0^t l(t-s)Au(s)ds = u_0,
\end{equation}
with $A = (-\Delta)\gamma$. This can be seen by convoluting (34) with $l$.

In what follows, for $l \in L^1_{loc}(\mathbb{R}^+)$, we denote by $\hat{l}$ the Laplace transform of $l$. We recall some notions and facts stated in [9].

**Definition 3.3.** Let $l \in L^1_{loc}(\mathbb{R}^+)$ be a function of subexponential growth, i.e.
\begin{equation}
\int_0^\infty |l(t)|e^{-\epsilon t}dt < \infty \quad \text{for every } \epsilon > 0.
\end{equation}

• Suppose that $\hat{l}(\lambda) \neq 0$ for all $\text{Re} \lambda > 0$. For $\theta > 0$, $l$ is said to be $\theta$-sectorial if $|\text{arg } \hat{l}(\lambda)| \leq \theta$ for all $\text{Re} \lambda > 0$.

• For given $m \in \mathbb{N}$, $l$ is called $m$-regular if there exists a constant $c > 0$ such that
\begin{equation}
|\lambda^{n}\hat{l}(\lambda)| \leq c|\hat{l}(\lambda)| \quad \text{for all } \text{Re} \lambda > 0, 1 \leq n \leq m.
\end{equation}

**Definition 3.4.** Equation (36) is called parabolic if the following conditions hold:
1. $\hat{l}(\lambda) \neq 0, 1/\hat{l}(\lambda) \in \rho(-A)$ for all $\text{Re} \lambda \geq 0$.
2. There is a constant $M \geq 1$ such that $U(\lambda) = \lambda^{-1}(I + \hat{l}(\lambda)A)^{-1}$ satisfies
\begin{equation}
\|U(\lambda)\| \leq \frac{M}{|\lambda|} \quad \text{for all } \text{Re} \lambda > 0.
\end{equation}

Denote by $\Sigma(\omega, \theta)$ the open sector with vertex $\omega \in \mathbb{R}$ and angle $2\theta$ in the complex plane, i.e.
\begin{equation}
\Sigma(\omega, \theta) = \{ \lambda \in \mathbb{C} : |\text{arg } (\lambda - \omega)| < \theta \}.
\end{equation}

We have the following sufficient condition for equation (36) to be parabolic.

**Proposition 4.** [9, Proposition 3.1] Assume that $l \in L^1_{loc}(\mathbb{R}^+)$ is of subexponential growth and $\theta$-sectorial for some $\theta < \pi$. If $A$ is closed linear densely defined, such that $\rho(-A) \supset \Sigma(0, \theta)$, and
\begin{equation}
\|(\lambda I + A)^{-1}\| \leq \frac{M}{|\lambda|} \quad \text{for all } \lambda \in \Sigma(0, \theta),
\end{equation}
then equation (36) is parabolic.

**Remark 3.** Let us mention that, $-A = (-\Delta)\gamma$ generates a contraction $C_0$-semigroup in $L^2(\Omega)$, which is given by
\begin{equation}
e^{-tA}v = \sum_{n=1}^{\infty} e^{-t\lambda_n^2} (v, e_n)e_n, \quad t \geq 0, v \in L^2(\Omega).
\end{equation}

Then (37) holds for $M = 1$ and for any $\theta < \pi$, due to the Hille-Yosida theorem (see, e.g. [16]).

The following result on the regularity of resolvent operator for equation (36) will be used in the sequel.
Proposition 5. [9, Theorem 3.1] Assume that \((36)\) is parabolic and the kernel function \(l\) is \(m\)-regular for some \(m \geq 1\). Then there is a resolvent family \(S(\cdot) \in C^{(m-1)}((0, \infty); \mathcal{L}(L^2(\Omega)))\) for \((36)\), and a constant \(M \geq 1\) such that

\[
\|t^n S^{(n)}(t)\| \leq M, \quad \text{for all } t > 0, n \leq m - 1,
\]

here \(\mathcal{L}(L^2(\Omega))\) denotes the space of bounded linear operators on \(L^2(\Omega)\).

By this proposition, we are able to obtain a regularity of the operator \(R(\cdot)\).

Lemma 3.5. Assume that \(l \in L^1_{loc}(\mathbb{R}^+)\) is of subexponential growth, \(\theta\)-sectorial for some \(\theta < \pi\) and \(3\)-regular. Then the operator \(R(\cdot)\) defined by \((9)\) is differentiable on \((0, \infty)\), that is, \(R(\cdot) \in C^1((0, \infty); \mathcal{L}(L^2(\Omega)))\). Moreover, the following estimate holds

\[
\|R'(t)\| \leq M\lambda^{-\gamma}t^{-2}, \quad \forall t > 0.
\]

Proof. According to Proposition 4 and Remark 3, equation \((36)\) is parabolic. Since \(l\) is \(3\)-regular, the resolvent operator \(S(\cdot)\) is twice continuously differentiable, that is, \(S(\cdot) \in C^2((0, \infty); \mathcal{L}(L^2(\Omega)))\). By Proposition 1(b), one has

\[
\int_0^t r(\tau, \lambda_n^2) d\tau = \frac{1}{\lambda_n^2} (1 - s(t, \lambda_n^2)),
\]

then \(r(t, \lambda_n^2) = -\frac{1}{\lambda_n^2} \partial_s s(t, \lambda_n^2)\) for all \(t > 0\). This ensures \(R(t) = -A^{-1}S'(t)\) for all \(t > 0\), and then \(R(\cdot)\) is differentiable on \((0, \infty)\), thanks to the fact that \(A^{-1}\) is a bounded operator. In addition, we get

\[
\|R'(t)\| = \|A^{-1}\|\|S''(t)\| \leq M\|A^{-1}\|t^{-2}, \quad \forall t > 0,
\]

thanks to Proposition 5. Noting that

\[
\|A^{-1}v\|^2 = \sum_{n=1}^{\infty} \lambda_n^{-2\gamma} (v, e_n)^2,
\]

we get \(\|A^{-1}\| \leq \lambda_1^{-\gamma}\), which completes the proof. \(\square\)

Put

\[
\Phi_1(u)(t) = P(t)g(u) - P(t) \int_0^t R(T - \tau)f(\tau, u(\tau)) d\tau,
\]

\[
\Phi_2(u)(t) = \int_0^t R(t - \tau)f(\tau, u(\tau)) d\tau.
\]

Lemma 3.6. Let the function \(l\) satisfy the assumptions of Lemma 3.5. Assume, in addition, that

(F2) The function \(f : [0, T] \times L^2(\Omega) \to V_{2\gamma}\) is continuous and there exists a non-decreasing function \(\Psi_f \in C(\mathbb{R}^+)\) such that

\[
\|f(t, v)\|_{V_{2\gamma}} \leq \Psi_f(\|v\|), \quad \forall t \in [0, T], v \in L^2(\Omega);
\]

(G2) The function \(g : C([0, T]; L^2(\Omega)) \to V_{2\gamma}\) is continuous and there is a non-decreasing function \(\Psi_g \in C(\mathbb{R}^+)\) such that

\[
\|g(u)\|_{V_{2\gamma}} \leq \Psi_g(\|u\|_{\infty}), \quad \forall u \in C([0, T]; L^2(\Omega)).
\]

Then the operators \(\Phi_1\) and \(\Phi_2\) defined by \((39)\) and \((40)\), respectively, are compact as mappings on \(C([0, T]; L^2(\Omega))\).
We first show that $\Phi_1$ is a compact operator. Observe that, for $v \in V_{2\gamma}$, we have

$$
\|P(t)v\|_{V_{2\gamma}}^2 = \sum_{n=1}^{\infty} \left( \frac{s(t, \lambda_n^2)}{s(T, \lambda_n^2)} \right)^2 \lambda_n^{2\gamma}(v, e_n)^2 = \sum_{n=1}^{\infty} \left( \frac{s(t, \lambda_n^2)}{\lambda_n^2 s(T, \lambda_n^2)} \right)^2 \lambda_n^{4\gamma}(v, e_n)^2
$$

$$
\leq (\lambda_1^{-\gamma} + k(T)^{-1})^2 \|v\|_{V_{2\gamma}}^2, \quad \forall t \geq 0,
$$

(41)

here we used estimate (21). Let $D \subset C([0, T]; L^2(\Omega))$ be a bounded set, i.e. there is $\rho > 0$ such that $\|u\|_\infty \leq \rho$, $\forall u \in D$. Then by (G2), $g(D)$ is bounded in $V_{2\gamma}$. According to (41), $P(t)g(D)$ is bounded in $V_{\gamma}$, which implies that the set $P(t)g(D)$ is compact in $L^2(\Omega)$ for any $t \in [0, T]$. Similarly, let

$$
F(D) = \left\{ -\int_0^T R(T - \tau)f(\tau, u(\tau))d\tau : u \in D \right\},
$$

then $F(D)$ is bounded in $V_{2\gamma}$, thanks to (F2). Thus the set $P(t)F(D)$ is compact in $L^2(\Omega)$ as well. We have proved that $\Phi_1(D)$ is compact in $L^2(\Omega)$ for any $t \in [0, T]$. We now testify the equicontinuity of $\Phi_1(\Omega)$ as well. We have proved that $\Phi_2(\Omega)$ is a bounded set in $V_2(\Omega)$ for any $t \in [0, T)$.

In order to show the relative compactness of $\Phi_2(D)$, we have

$$
\|P(t_2) - P(t_1)v\|^2 = \sum_{n=1}^{\infty} \left( \frac{s(t_2, \lambda_n^2) - s(t_1, \lambda_n^2)}{s(T, \lambda_n^2)} \right)^2 (v, e_n)^2
$$

$$
= \sum_{n=1}^{\infty} \left( \frac{\lambda_n^2 \int_{t_1}^{t_2} r(\tau, \lambda_n^2)d\tau}{s(T, \lambda_n^2)} \right)^2 (v, e_n)^2
$$

$$
\leq \left( \int_{t_1}^{t_2} r(\tau, \lambda_n^2)d\tau \right)^2 \sum_{n=1}^{\infty} \left( \frac{\lambda_n^{2\gamma}(v, e_n)}{\lambda_n^2 s(T, \lambda_n^2)} \right)^2
$$

$$
\leq \left( \int_{t_1}^{t_2} r(\tau, \lambda_n^2)d\tau \right)^2 (\lambda_1^{-\gamma} + k(T)^{-1})^2 \|v\|_{V_{2\gamma}}^2,
$$

here we used Proposition 1(b) and the estimate

$$
\frac{1}{\lambda_n^2 s(T, \lambda_n^2)} \leq \frac{1 + \lambda_n^2 k(T)^{-1}}{\lambda_n^2} \leq \lambda_1^{-\gamma} + k(T)^{-1}.
$$

Hence

$$
\|P(t_2) - P(t_1)v\| \leq \left( \int_{t_1}^{t_2} r(\tau, \lambda_n^2)d\tau \right) (\lambda_1^{-\gamma} + k(T)^{-1})R \to 0 \text{ as } t_2 - t_1 \to 0,
$$

uniformly in $v \in g(D) + F(D)$. Thus, $\Phi_1(D)$ is an equicontinuous set and then it is relatively compact in $C([0, T]; L^2(\Omega))$, according to the Arzelà-Ascoli theorem.

In order to show the relative compactness of $\Phi_2(D)$, one observes that

$$
\Phi_2(u)(t) = Q_t(N_f(u)), t \geq 0,
$$

where $N_f(u)(t) = f(t, u(t))$, the Nemtyski operator corresponding to $f$. In view of (F2), $N_f(D)$ is a bounded set in $C([0, T]; L^2(\Omega))$. So $Q_t(N_f(D))$ is relatively compact in $L^2(\Omega)$, due to Remark 1. This means that $\Phi_2(D)(t)$ is compact for any $t \in [0, T]$. It remains to check the equicontinuity of $\Phi_2(D)$. For $h \in (0, T), t \in$
\[ [0, T - h], \text{ one has} \]

\[
\|\Phi_2(u)(t + h) - \Phi_2(u)(t)\| \leq \| \int_t^{t+h} R(t + h - \tau)f(\tau, u(\tau))d\tau \|
\]

\[
+ \| \int_0^t [R(t + h - \tau) - R(t - \tau)]f(\tau, u(\tau))d\tau \|
\]

\[= E_1 + E_2.\]

Regarding \( E_1 \), we get

\[
E_1 \leq \int_t^{t+h} r(t + h - \tau, \lambda_1^\gamma)\|f(\tau, u(\tau))\|d\tau
\]

\[
\leq \lambda_1^{-2\gamma} \int_t^{t+h} r(t + h - \tau, \lambda_1^\gamma)\|f(\tau, u(\tau))\|_{V_2}d\tau
\]

\[
\leq \lambda_1^{-2\gamma} \int_t^{t+h} r(t + h - \tau, \lambda_1^\gamma)\Psi_f(\|u(\tau)\|)d\tau
\]

\[
\leq \lambda_1^{-2\gamma} \Psi_f(\rho) \int_0^h r(z, \lambda_1^\gamma)dz \to 0 \text{ as } h \to 0,
\]

uniformly in \( u \in D \), here we used the estimate \( \|v\| \leq \lambda_1^{-2\gamma} \|v\|_{V_2} \), for \( v \in V_2 \).

Dealing with \( E_2 \), one can assume that \( t > 0 \) and \( 0 < \sqrt{h} < t \). Then

\[
E_2 \leq \int_t^{t+h} \| [R(t + h - \tau) - R(t - \tau)]f(\tau, u(\tau)) \|d\tau
\]

\[
+ \int_0^t \| [R(t + h - \tau) - R(t - \tau)]f(\tau, u(\tau)) \|d\tau
\]

\[
\leq \int_t^{t+h} \| R(t + h - \tau)f(\tau, u(\tau)) \|d\tau + \int_{t-h}^t \| R(t - \tau)f(\tau, u(\tau)) \|d\tau
\]

\[
+ \int_0^t \| [R(t + h - \tau) - R(t - \tau)]f(\tau, u(\tau)) \|d\tau
\]

\[= E_{2a} + E_{2b} + E_{2c}.\]

Using the same estimates as those for \( E_1 \), we have

\[
E_{2a} \leq \lambda_1^{-2\gamma} \Psi_f(\rho) \int_{h}^{h+\sqrt{h}} r(z, \lambda_1^\gamma)dz \to 0 \text{ as } h \to 0,
\]

\[
E_{2b} \leq \lambda_1^{-2\gamma} \Psi_f(\rho) \int_{0}^{\sqrt{h}} r(z, \lambda_1^\gamma)dz \to 0 \text{ as } h \to 0,
\]

uniformly in \( u \in D \). For the last integral \( E_{2c} \), we see that

\[
E_{2c} \leq \int_0^t \| \int_{t-\tau}^{t+h-\tau} R'(z)f(\tau, u(\tau))dz \|d\tau
\]

\[
\leq M\lambda_1^{-\gamma} \int_0^t \| \int_{t-\tau}^{t+h-\tau} z^{-2}\|f(\tau, u(\tau))\|dzd\tau
\]

\[
\leq M\lambda_1^{-3\gamma} \Psi_f(\rho) \int_0^{t-h} \int_{t-h}^{t-h-\tau} z^{-2}dzd\tau
\]
Proof. Let the hypotheses of Lemma 3.6 hold. Then the problem

\[ \begin{aligned}
& (\lambda_1^{-\gamma} + k(T)^{-1})\lambda_1^{-\gamma}\liminf_{p \to \infty} \frac{\Psi_g(p)}{p} + (2\lambda_1^{-\gamma} + k(T)^{-1})\lambda_1^{-2\gamma}\liminf_{p \to \infty} \frac{\Psi_f(p)}{p} < 1. \quad (42)
\end{aligned} \]

The proof is complete. \(\square\)

**Theorem 3.7.** Let the hypotheses of Lemma 3.6 hold. Then the problem (1)–(3) has at least one mild solution in \(C([0, T]; L^2(\Omega))\), provided that

\[ (\lambda_1^{-\gamma} + k(T)^{-1})\lambda_1^{-\gamma}\liminf_{p \to \infty} \frac{\Psi_g(p)}{p} + (2\lambda_1^{-\gamma} + k(T)^{-1})\lambda_1^{-2\gamma}\liminf_{p \to \infty} \frac{\Psi_f(p)}{p} < 1. \]

Proof. We show that the solution operator

\[ \Phi(u)(t) = P(t)g(u) - P(t) \int_0^t R(T - \tau)f(\tau, u(\tau))d\tau + \int_0^t R(t - \tau)f(\tau, u(\tau))d\tau, \]

admits a fixed point in \(C([0, T]; L^2(\Omega))\) by using the Schauder fixed point theorem. By Lemma 3.6, we see that \(\Phi\) is a compact mapping. It suffices to prove that, there exists \(\rho > 0\) such that \(\Phi(B_\rho) \subset B_\rho\), where \(B_\rho\) is the closed ball in \(C([0, T]; L^2(\Omega))\) with radius \(\rho\) and center at origin.

Assume to the contrary that, for each \(n \in \mathbb{N}\), there is \(u_n \in C([0, T]; L^2(\Omega))\) such that \(\|u_n\|_\infty \leq n\) and \(\|\Phi(u_n)\|_\infty > n\). Then we get

\[ \|\Phi(u_n)(t)\| \leq \|P(t)g(u_n)\| + \|P(t) \int_0^t R(T - \tau)f(\tau, u_n(\tau))d\tau\|
\]

\[ + \int_0^t \|R(t - \tau)f(\tau, u_n(\tau))\|d\tau
\]

\[ \leq (\lambda_1^{-\gamma} + k(T)^{-1})\|g(u_n)\|_{V_1}
\]

\[ + (\lambda_1^{-\gamma} + k(T)^{-1})\int_0^t \|R(T - \tau)f(\tau, u_n(\tau))\|d\tau
\]

\[ + \int_0^t \|R(t - \tau)f(\tau, u_n(\tau))\|d\tau,
\]

thanks to Proposition 2(b) and Lemma 2.1(b). Using the estimates \(\|v\| \leq \lambda_1^{-2\gamma}\|v\|_{V_2}, \|v\|_{V_1} \leq \lambda_1^{-\gamma}\|v\|_{V_2}\), for \(v \in V_2\), and the assumptions (F2)–(G2), we obtain

\[ \|\Phi(u_n)(t)\| \leq (\lambda_1^{-\gamma} + k(T)^{-1})\lambda_1^{-\gamma}\Psi_g(\|u_n\|_\infty)
\]

\[ + (\lambda_1^{-\gamma} + k(T)^{-1})\lambda_1^{-\gamma}\Psi_f(\|u_n\|_\infty) \int_0^T r(T - \tau, \lambda_1^\gamma)d\tau
\]

\[ + \lambda_1^{-2\gamma}\Psi_f(\|u_n\|_\infty) \int_0^T r(t - \tau, \lambda_1^\gamma)d\tau
\]

\[ \leq (\lambda_1^{-\gamma} + k(T)^{-1})\lambda_1^{-\gamma}\Psi_g(\|u_n\|) + [(\lambda_1^{-\gamma} + k(T)^{-1})\lambda_1^{-2\gamma} + \lambda_1^{-3\gamma}]\Psi_f(\|u_n\|).
\]

It follows that

\[ \frac{\|\Phi(u_n)\|}{n} \leq (\lambda_1^{-\gamma} + k(T)^{-1})\lambda_1^{-\gamma}\frac{\Psi_g(\|u_n\|)}{n} + (2\lambda_1^{-\gamma} + k(T)^{-1})\lambda_1^{-2\gamma}\frac{\Psi_f(\|u_n\|)}{n}.
\]
Passing to the limit as \( n \to \infty \) in the last relation, we get a contradiction, due to condition (42). The proof completes.

4. Solvability in singular setting. Define

\[
C_l((0,T]; L^2(\Omega)) = \{ u \in C((0,T]; L^2(\Omega)) : \sup_{t \in (0,T]} (1 \ast l)(t) \| u(t) \| < \infty \}.
\]

Then \( C_l((0,T]; L^2(\Omega)) \) is a Banach space with the norm

\[
\| u \|_{C_l} = \sup_{t \in (0,T]} (1 \ast l)(t) \| u(t) \|.
\]

In this section, we consider the case that \( f \) and \( g \) take less regular values than those assumed in Section 3, and find a solution of the problem (1)-(3) in \( C_l((0,T]; L^2(\Omega)) \).

**Theorem 4.1.** Assume that

(F3) The function \( f : (0,T] \times L^2(\Omega) \to L^2(\Omega) \) obeys the estimate

\[
\| f(t, u_1(t)) - f(t, u_2(t)) \| \leq L_f(t) \| u_1 - u_2 \|_{C_l},
\]

for all \( t \in (0,T], u_1, u_2 \in C_l((0,T]; L^2(\Omega)), \) where \( L_f \in C((0,T]) \) is a nonnegative function;

(G3) The function \( g : C_l((0,T]; L^2(\Omega)) \to L^2(\Omega) \) satisfies the Lipschitz condition

\[
\| g(u_1) - g(u_2) \| \leq L_g \| u_1 - u_2 \|_{C_l}, \quad \forall u_1, u_2 \in C_l((0,T]; L^2(\Omega)),
\]

where \( L_g \) is a nonnegative number.

Then the problem (1)-(3) has a unique mild solution in \( C_l((0,T]; L^2(\Omega)) \), provided that

\[
k(T)^{-1} [L_0 + \Lambda(T)] + \sup_{t \in (0,T]} (1 \ast l)(t) \Lambda(t) < 1,
\]

where

\[
\Lambda(t) = \int_0^t r(t - \tau, \lambda_1^*) L_f(\tau) d\tau.
\]

**Proof.** Recall that the solution operator \( \Phi \) is given by \( \Phi(u) = \Phi_1(u) + \Phi_2(u) \), where \( \Phi_1, \Phi_2 \) are defined by (39)-(40). Let \( u, v \in C_l((0,T]; L^2(\Omega)) \), then

\[
\| \Phi_1(u)(t) - \Phi_1(v)(t) \| \leq \| P(t)[g(u) - g(v)] \|
\]

\[
+ \| P(t) \int_0^T R(T - \tau)[f(\tau, u(\tau)) - f(\tau, v(\tau))] d\tau \|
\]

\[
\leq \frac{k(T)^{-1}}{(1 \ast l)(t)} \| g(u) - g(v) \|
\]

\[
+ \frac{k(T)^{-1}}{(1 \ast l)(t)} \int_0^T r(T - \tau, \lambda_1^*) \| f(\tau, u(\tau)) - f(\tau, v(\tau)) \| d\tau
\]

\[
\leq \frac{k(T)^{-1}}{(1 \ast l)(t)} \left( L_g + \int_0^T r(T - \tau, \lambda_1^*) L_f(\tau) d\tau \right) \| u - v \|_{C_l},
\]

thanks to Proposition 2(a) and the assumptions (F3)-(G3). This implies

\[
\| \Phi_1(u) - \Phi_1(v) \|_{C_l} \leq k(T)^{-1} [L_g + \Lambda(T)] \| u - v \|_{C_l}.
\]

(45)

Regarding \( \Phi_2 \), we see that

\[
\| \Phi_2(u)(t) - \Phi_2(v)(t) \| \leq \int_0^t r(t - \tau, \lambda_1^*) \| f(\tau, u(\tau)) - f(\tau, v(\tau)) \| d\tau
\]
first analyze a compactness condition in $C$.

(b) If $B_{18}$ DINH-KE TRAN AND TRAN-PHUONG-THUY LAM

Lemma 4.2.

(a) following assertions holds.

(b) We first prove that $u \in C((0, T]; L^2(\Omega))$, where $u(t) = (1 * l)(t)u(t)$. Obviously, $u \in C((0, T]; L^2(\Omega))$, so we have to check that the limit $\lim_{t \to 0} u(t)$ exists. Let $u(t) = P(t)v, v \in B$, then

$$
\|u(t)(t)\| = [(1 * l)(t)]^2 \sum_{n=1}^{\infty} \left( \frac{s(t, \lambda_k^2)}{s(T, \lambda_k^2)} \right)^2 (v, e_n)^2
\leq [(1 * l)(t)]^2 \sum_{n=1}^{\infty} \left( \frac{s(t, \lambda_k^2)}{s(T, \lambda_k^2)} \right)^2 \lambda_k^{2\gamma} (v, e_n)^2
\leq [(1 * l)(t)]^2 (\lambda_1^{2\gamma} + k(T)^{-1})^2 \|v\|_{V_\gamma}^2.
$$

Hence $\lim_{t \to 0} u(t) = 0$. Now observe that $\{u_n\} \subseteq B$ converges to $u$ in $C((0, T]; L^2(\Omega))$ if $u_n(t) \to u(t)$ in $C((0, T]; L^2(\Omega))$. So our goal now is to show that $B[l] = \{u[l] : u \in B\}$ is relatively compact in $C((0, T]; L^2(\Omega))$. For $v \in B$ and $u[l] = (1 * l)P(\cdot)v \in B[l]$, we have

$$
\|u[t](t)\|_{\gamma} = (1 * l)(t)\|P(t)(-\Delta)\gamma v\|
\leq (1 * l)(t) \frac{k(T)^{-1}}{(1 * l)(t)} \|(-\Delta)\gamma v\| = k(T)^{-1} \|v\|_{V_\gamma}, \forall t > 0,
$$

thanks to Proposition 2. Therefore, $B[l](t)$ is bounded in $V_\gamma$ for all $t \in (0, T)$, which ensures that $B[l](t), t > 0$, is relatively compact in $L^2(\Omega)$. Clearly, $B[l](0) = \{0\}$ is also a compact set and hence, $B[l](t)$ is relatively compact for all $t \in [0, T]$.

It remains to check that $B[l]$ is equicontinuous in $C((0, T]; L^2(\Omega))$. For $t > 0, h > 0$, we see that

$$
\|u[t](t + h) - u[t](t)\| \leq [(1 * l)(t + h) - (1 * l)(t)]\|P(t)v\|
+ (1 * l)(t + h)\|P(t + h)v - P(t)v\|
= J_1 + J_2.
$$
For $J_1$, we have
\[ J_1 \leq [(1 \ast l)(t + h) - (1 \ast l)(t)](\lambda_1^{-\gamma} + k(T)^{-1})\|v\|_{\nu}, \rightarrow 0 \text{ as } h \rightarrow 0, \tag{47} \]
uniformly in $v \in B$. Regarding $J_2$, observe that $P(t) = S(t)S(T)^{-1}$ and
\[
\|S(T)^{-1}v\|^2 = \sum_{n=1}^{\infty} \frac{\lambda_n^{2\gamma}(v, c_n)^2}{\lambda_n} \leq \sum_{n=1}^{\infty} \left( \frac{1 + \lambda_n^\gamma k(T)^{-1}}{\lambda_n} \right)^2 \lambda_n^{2\gamma}(v, c_n)^2 \leq (\lambda_1^{-\gamma} + k(T)^{-1})\|v\|_{\nu}^2.
\]
So
\[
J_2 = (1 \ast l)(t + h)\|S(t + h) - S(t)\|S(T)^{-1}v\|
\leq (1 \ast l)(t + h)(\lambda_1^{-\gamma} + k(T)^{-1})\|v\|_{\nu} \|S(t + h) - S(t)\| \int_t^{t+h} S'(\tau) d\tau
\leq M(1 \ast l)(t + h)(\lambda_1^{-\gamma} + k(T)^{-1})\|v\|_{\nu} \int_t^{t+h} \frac{d\tau}{\tau}
= M(1 \ast l)(t + h)(\lambda_1^{-\gamma} + k(T)^{-1})\|v\|_{\nu} \ln \left( 1 + \frac{h}{t} \right)
\rightarrow 0 \text{ as } h \rightarrow 0,
\tag{48}
\]
uniformly in $v \in B$, here we employed Proposition 5. Finally, if $t = 0$ and $h > 0$, then
\[
\|u[t](h) - u[t](0)\| = \|u[t](h)\| = (1 \ast l)(h)\|P(h)v\| \leq (1 \ast l)(h)(\lambda_1^{-\gamma} + k(T)^{-1})\|v\|_{\nu}
\rightarrow 0 \text{ as } h \rightarrow 0,
\tag{49}
\]
uniformly in $v \in B$. Combining (47)-(49), we obtain the equicontinuity of $B[l]$. The proof is complete. \hfill \Box

Theorem 4.3. Let the kernel function $l$ satisfy the assumptions of Lemma 3.5. Assume that

(F4) The function $f : (0, T] \times L^2(\Omega) \rightarrow V_\gamma$ is continuous and satisfies
\[
\|f(t, u(t))\|_{\nu} \leq L_f(t)\Psi_f\|u\|_{C(\lambda)}, \forall t \in (0, T], u \in C_1((0, T]; L^2(\Omega)),
\]
where $L_f \in C((0, T])$ is nonnegative and $\Psi_f \in C(\mathbb{R}^+)$ is nondecreasing.

(G4) The function $g : C_1((0, T]; L^2(\Omega)) \rightarrow V_\gamma$ satisfies
\[
\|g(u)\|_{\nu} \leq \Psi_g\|u\|_{C(\lambda)}, \forall u \in C_1((0, T]; L^2(\Omega)),
\]
where $\Psi_g \in C(\mathbb{R}^+)$ is a nondecreasing function.

Then the problem (1)-(3) has at least one mild solution in $C_1((0, T]; L^2(\Omega))$, provided that
\[
\liminf_{p \to \infty} \frac{\Psi_g(p)}{p} + \left( \Lambda(T) + k(T) \sup_{t \in (0, T]} (1 \ast l)(t)\Lambda(t) \right) \liminf_{p \to \infty} \frac{\Psi_f(p)}{p} < k(T)\lambda_1, \tag{50}
\]
where $\Lambda$ is defined by (44).

Proof. We make use of the Schauder fixed point theorem to show that, the solution operator $\Phi(u) = \Phi_1(u) + \Phi_2(u)$ has a fixed point in $C_t((0, T]; L^2(\Omega))$, where $\Phi_1$ and $\Phi_2$ are defined in (39)-(40). We first show that $\Phi_1$ and $\Phi_2$ are compact operators. Let $D \subset C_t((0, T]; L^2(\Omega))$ be a bounded set. Put $B = g(D) + F(D)$ with $F$ defined by

$$F(u) = -\int_0^T R(T - \tau) f(\tau, u(\tau)) d\tau.$$  

Then $B$ is a bounded set in $V_\gamma$ and $\Phi_1(D) = P(\cdot) B$. By Lemma 4.2, $\Phi_1(D)$ is relatively compact. Reasoning as in the proof of Lemma 3.6, we see that $\Phi_2$ is also relatively compact in $C([0, T]; L^2(\Omega))$, which ensures that $\Phi_2(D)$ is relatively compact in $C_t((0, T]; L^2(\Omega))$. Indeed, in the opposite case, for each $n \in \mathbb{N}$, there exists $u_n \in C_t((0, T]; L^2(\Omega))$ such that $\|u_n\|_{C_t} \leq n$ and $\|\Phi(u_n)\|_{C_t} > n$. For all $t \in (0, T]$, we have

$$(1 * l)(t) \|\Phi(u_n)(t)\| \leq (1 * l)(t) \|P(t)g(u_n)\|$$

$$+ (1 * l)(t) \|P(t) \int_0^T R(T - \tau) f(\tau, u(\tau)) d\tau\|$$

$$+ (1 * l)(t) \|\int_0^t R(t - \tau) f(\tau, u_n(\tau)) d\tau\|$$

$$\leq k(T)^{-1} \|g(u_n)\| + k(T)^{-1} \|\int_0^T r(T - \tau, \lambda_1^\gamma) f(\tau, u(\tau)) d\tau\|$$

$$+ (1 * l)(t) \int_0^t r(t - \tau, \lambda_1^\gamma) \|f(\tau, u_n(\tau))\| d\tau$$

$$\leq k(T)^{-1} \lambda_1^\gamma \|g(u_n)\|_{V_\gamma}$$

$$+ k(T)^{-1} \lambda_1^\gamma \|\int_0^T r(T - \tau, \lambda_1^\gamma) \|f(\tau, u(\tau))\|_{V_\gamma} d\tau\|$$

$$+ \lambda_1^\gamma (1 * l)(t) \int_0^t r(t - \tau, \lambda_1^\gamma) \|f(\tau, u_n(\tau))\|_{V_\gamma} d\tau,$$

thanks to Proposition 2. Using (F4)-(G4), we get

$$(1 * l)(t) \|\Phi(u_n)(t)\| \leq k(T)^{-1} \lambda_1^\gamma \Psi_p(n) + k(T)^{-1} \lambda_1^\gamma \Psi_f(n) \Lambda(T)$$

$$+ \lambda_1^\gamma \Psi_f(n) (1 * l)(t) \Lambda(t).$$

It follows that

$$1 < \frac{1}{n} \|\Phi(u_n)\|_{C_t} \leq k(T)^{-1} \lambda_1^\gamma \frac{\Psi_p(n)}{n}$$

$$+ \lambda_1^\gamma \left( k(T)^{-1} \Lambda(T) + \sup_{t \in (0, T]} (1 * l)(t) \Lambda(t) \right) \frac{\Psi_f(n)}{n},$$

Passing to the limit as $n \to \infty$, one gets a contradiction with (50). The proof is complete. □
5. Examples.

5.1. Example 1. (Ultra-slow diffusion equation) Let \( k(t) = \int_0^1 t^{\beta-1} \frac{d\beta}{\Gamma(\beta)} \). Consider the problem

\[
k \ast \partial_t u - \partial_x^2 u = H(t) \ln(1 + u^2), \ t \in (0, T], \ x \in (0, 1),
\]

\[
u(t, 0) = u(t, 1) = 0,
\]

\[
u(T, x) = \int_0^T \int_0^1 K(t, x, y) u(t, y) dy,
\]

where \( H \) and \( K \) are given functions. In this case, \( V_\gamma = V = H^2(0, 1) \cap H^1_0(0, 1) \).

Note that the associate kernel function \( l(t) = \int_0^\infty e^{-pt} \frac{dp}{1+p} \) is nonincreasing on \((0, \infty)\). Let \( f(t, v)(x) = H(t) \ln(1 + v(x)^2) \) for \( v \in L^2(0, 1) \), then

\[
||f(t, v_1) - f(t, v_2)||^2 \leq |H(t)|^2 \int_0^1 |v_1(x) - v_2(x)|^2 dx = |H(t)|^2 \|v_1 - v_2\|^2,
\]

thanks to the fact that, the function \( s \mapsto \ln(1 + s^2) \) is Lipschitzian with constant 1.

Let \( g(u)(x) = \int_0^T dt \int_0^1 K(t, x, y) u(t, y) dy \). We assume that

1. \( K(t, 0, y) = K(t, 1, y) = 0, \) for all \( t \in [0, T], \ y \in (0, 1) \).
2. The function \( x \mapsto K(t, x, y) \) is twice differentiable.

Then using the H"older inequality, we get

\[
||g(u_1) - g(u_2)||_V^2 = \int_0^1 \left( \int_0^T dt \int_0^1 \partial_x K(t, x, y) |u_1(t, y) - u_2(t, y)| dy \right)^2 dx
\]

\[
+ \int_0^1 \left( \int_0^T dt \int_0^1 \partial_x^2 K(t, x, y) |u_1(t, y) - u_2(t, y)| dy \right)^2 dx
\]

\[
\leq T^2 \int_0^1 \sup_{t \in [0, T]} \|\partial_x K(t, x, \cdot)\|^2 \|u_1(t, \cdot) - u_2(t, \cdot)\|^2 dx
\]

\[
+ T^2 \int_0^1 \sup_{t \in [0, T]} \|\partial_x^2 K(t, x, \cdot)\|^2 \|u_1(t, \cdot) - u_2(t, \cdot)\|^2 dx
\]

\[
\leq T^2 \|u_1 - u_2\|^2 \int_0^1 \left( \sup_{t \in [0, T]} \|\partial_x K(t, x, \cdot)\|^2 + \sup_{t \in [0, T]} \|\partial_x^2 K(t, x, \cdot)\|^2 \right) dx.
\]

So the assumptions of Theorem 3.2 are satisfied, provided that the function \( H(t), t \in [0, T], \) is H"older continuous with order \( \delta \in (\frac{1}{2}, 1] \), \( H(T) = 0, \) \( H \) and \( K \) are small enough in the sense that

\[
L_f(t) = |H(t)|, \ t \in [0, T],
\]

\[
L_g = T \left( \int_0^1 \left( \sup_{t \in [0, T]} \|\partial_x K(t, x, \cdot)\|^2 + \sup_{t \in [0, T]} \|\partial_x^2 K(t, x, \cdot)\|^2 \right) \right)^{\frac{1}{2}},
\]

obey the condition \( \varrho(f, g) < 1 \) stated in Theorem 3.2.
5.2. **Example 2.** (Multi-term fractional diffusion equation) Consider the following problem

$$
\partial_t^\alpha u(t, x) + \mu \partial_t^\beta u(t, x) - \partial_x^2 u(t, x) = \tilde{f}(t, x) \int_0^\pi u(t, y) \arctan u(t, y) dy,
$$

for $t \in (0, T)$, $x \in (0, \pi)$, with the boundary condition

$$
u(t, 0) = u(t, \pi) = 0, \quad \forall t \in [0, T],
$$

and the final condition

$$u(T, x) = \tilde{g}(x) \int_{[0, \pi] \times [0, T]} K(t, y) |u(t, y)|^\delta dy dt, \quad 0 < \delta < 1,
$$

where $0 < \alpha < \beta < 1$, $\mu > 0$, $\tilde{f}$, $\tilde{g}$, and $K$ are given functions such that the related integrals converge. In this example, we use $V = V = H^2(0, \pi) \cap H_0^1(0, \pi)$. Let

$$
k(t) = g_{1-\alpha}(t) + \mu g_{1-\beta}(t),
$$

$$f(t, u)(x) = \tilde{f}(t, x) \int_0^\pi u(t, y) \arctan u(t, y) dy,
$$

$$g(u)(x) = \tilde{g}(x) \int_{[0, \pi] \times [0, T]} K(t, y) |u(t, y)|^\delta dy dt.
$$

Then $f$ and $g$ fail to be globally Lipschitzian. Assume that

1. $\tilde{f}(t, 0) = \tilde{f}(t, \pi) = 0$, $x \mapsto f(t, x)$ is differentiable up to second order, for all $t \in [0, T]$.
2. $\tilde{g}(0) = \tilde{g}(\pi) = 0$, $\tilde{g}$ is twice differentiable.

It should be mentioned that, since $k$ is completely monotone, the associate kernel function $l$ exists and its Laplace transform is given by

$$\hat{l}(\lambda) = \lambda^{-1} k(\lambda)^{-1} = (\lambda^\alpha + \mu \lambda^\beta)^{-1}.
$$

For $\lambda \in \mathbb{C}$, Re$\lambda > 0$, one has

$$|\arg \lambda^\alpha| < |\arg \lambda^\beta| < \frac{\pi}{2},$$

$$|\arg(\lambda^\alpha + \mu \lambda^\beta)| \in (|\arg \lambda^\alpha|, |\arg \lambda^\beta|).
$$

Hence

$$|\arg \hat{l}(\lambda)| = |\arg(\lambda^\alpha + \mu \lambda^\beta)|^{-1} = |\arg(\lambda^\alpha + \mu \lambda^\beta)| \leq |\arg \lambda^\beta| < \frac{\pi}{2}.
$$

That is, $l$ is $\frac{\pi}{2}$-sectorial. On the other hand, we have

$$\lambda \hat{l}^\prime(\lambda) = - (\alpha \lambda^\alpha + \beta \mu \lambda^\beta)(\lambda^\alpha + \mu \lambda^\beta)^{-2}.
$$

Noting that, $z_1 = \lambda^\alpha$ and $z_2 = \mu \lambda^\beta$ belong to the same quadrant, so

$$|\eta_1 \lambda^\alpha + \eta_2 \mu \lambda^\beta| \leq |\lambda^\alpha + \mu \lambda^\beta| = |\hat{l}(\lambda)|^{-1}, \quad \forall \eta_1, \eta_2 \in (0, 1).
$$

This implies

$$|\lambda \hat{l}^\prime(\lambda)| = |\alpha \lambda^\alpha + \beta \mu \lambda^\beta| |\hat{l}(\lambda)|^2 \leq |\hat{l}(\lambda)|.
$$

Since $\hat{l}(\lambda) = - (\alpha \lambda^\alpha - 1 + \beta \mu \lambda^\beta)^{-1} |\hat{l}(\lambda)|^2$, we get

$$\lambda^2 \hat{l}^\prime(\lambda) = -[\alpha (\alpha - 1) \lambda^\alpha + \beta (\beta - 1) \mu \lambda^\beta] |\hat{l}(\lambda)|^2 - 2\lambda (\alpha \lambda^\alpha + \beta \mu \lambda^\beta) |\hat{l}(\lambda)| \hat{l}(\lambda).$$
Thus using (54)-(55), we obtain
\[ |\lambda^2 \hat{p}(\lambda)| \leq |\alpha(\alpha - 1)\lambda^\alpha + \beta(\beta - 1)\mu\lambda^\beta|\hat{l}(\lambda)|^2 + 2|\alpha\lambda^\alpha + \beta\mu\lambda^\beta|\lambda^2 \hat{p}(\lambda)| |\lambda^2 \hat{p}(\lambda)| \leq 3|\hat{l}(\lambda)|. \] 

(56)

Now by direct computation, one gets
\[ \lambda^3 \hat{p}''(\lambda) = -[\alpha(\alpha - 1)(\alpha - 2)\lambda^\alpha + \beta(\beta - 1)(\beta - 2)\mu\lambda^\beta]\hat{l}(\lambda)^2 \]
\[ - 4[\alpha(\alpha - 1)\lambda^\alpha + \beta(\beta - 1)\mu\lambda^\beta]\hat{l}(\lambda)\lambda^2 \hat{p}(\lambda) \]
\[ - 2(\alpha\lambda^\alpha + \beta\mu\lambda^\beta)\lambda^2 \hat{p}(\lambda)^2 \]
\[ - 2(\alpha\lambda^\alpha + \beta\mu\lambda^\beta)\hat{l}(\lambda)\lambda^2 \hat{p}(\lambda). \]

We have proved that the kernel function \( l \) is 3-regular.

Employing (54), (55) and (56) yields
\[ |\lambda^3 \hat{p}''(\lambda)| \leq 13|\hat{l}(\lambda)|. \]

We have proved that, the kernel function \( l \) is 3-regular.

Concerning the function \( f \), we see that
\[ \|f(t, u(t))\|^2 \leq \frac{\pi^3}{4} \left( \int_0^\pi |\partial_x \hat{f}(t, x)|^2 dx + \int_0^\pi |\partial_x^2 \hat{f}(t, x)|^2 dx \right) \int_0^\pi |u(t, y)|^2 dy \]
\[ = \left( \frac{L(t)}{(1 + l)(t)} \right)^2 \|u(t, \cdot)\|^2 \]
\[ \leq L_f(t)^2 \|u\|_{\tilde{C}_1}^2, \]

where
\[ L_f(t) = \frac{L(t)}{(1 + l)(t)}, \quad L(t) = \frac{\pi \sqrt{\pi}}{2} \left( \int_0^\pi |\partial_x \hat{f}(t, x)|^2 dx + \int_0^\pi |\partial_x^2 \hat{f}(t, x)|^2 dx \right)^{\frac{1}{2}}. \]

Thus \( f \) satisfies (F4) with \( \Psi_f(r) = r. \)

Concerning the function \( g \), applying the Hölder inequality yields
\[ \left( \int_{[0, \tau] \times [0, T]} K(t, y)|u(t, y)|^\delta dy dt \right)^2 \leq T \int_0^T dt \left( \int_0^\pi K(t, y)|u(t, y)|^\delta dy \right)^2 \]
\[ \leq T \int_0^T dt \left( \int_0^\pi |K(t, y)|^{\frac{2\delta}{2 - \delta}} dy \right)^{2 - \delta} \left( \int_0^\pi |u(t, y)|^2 dy \right)^\delta \]
\[ = T \int_0^T M(t)\|u(t, \cdot)\|^{2\delta} dt \]
\[ \leq \left( T \int_0^T \frac{M(t)}{(1 + l)(t)^{2\delta}} dt \right) \|u\|_{\tilde{C}_1}^{2\delta} \quad = \quad L_g^2 \|u\|_{\tilde{C}_1}^{2\delta}, \]

where
\[ L_g = \left( T \int_0^T \frac{M(t)}{(1 + l)(t)^{2\delta}} dt \right)^{\frac{1}{2}}, \quad M(t) = \left( \int_0^\pi |K(t, y)|^{\frac{2\delta}{2 - \delta}} dy \right)^{2 - \delta}. \]

Taking estimate (57) into account, we get
\[ \|g(u)\|_{\tilde{C}_1}^2 \leq (\|\hat{g}'\|^2 + \|\hat{g}''\|^2)\lambda^2 \hat{p}(\lambda)^2 \|u\|_{\tilde{C}_1}^2. \]

So \( g \) satisfies (G4) with \( \Psi_g(r) = (\|\hat{g}'\|^2 + \|\hat{g}''\|^2)^{\frac{1}{2}} \lambda^2 \hat{p}(\lambda)^2 L_g r^\delta. \)
According to Theorem 4.3, the problem (51)-(53) admits at least one mild solution in $C_l((0,T]; L^2(0,\pi))$, provided that

$$\Lambda(T) + k(T) \sup_{t \in (0,T]} (1 + l)(t) \Lambda(t) < k(T),$$

where $\Lambda$ is defined by (44).

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