Locally unidentifiable subset of quantum states and its resourcefulness in secret password distribution

Pratik Ghosal, Arkarabha Ghosal, Subhendu B. Ghosh, and Amit Mukherjee

1Department of Physical Sciences, Bose Institute, EN 80, Sector V, Bidhan Nagar, Kolkata 700 091, India
2Optics and Quantum Information Group, The Institute of Mathematical Sciences, CIT Campus, Taramani, Chennai 600113, India
3Physics and Applied Mathematics Unit, Indian Statistical Institute, Kolkata, 203 B. T. Road, Kolkata 700108, India
4Indian Institute of Technology, Jodhpur-342030, India

We introduce a hitherto unexplored form of quantum nonlocality, termed local subset unidentifiability, that arises from the limitation of spatially separated parties to perfectly identify a subset of mutually orthogonal multipartite quantum states, randomly chosen from a larger known set, using Local Operations and Classical Communication (LOCC). We show that this nonlocality is stronger than other existing forms of quantum nonlocality, such as local indistinguishability and local unmarkability. If more than one multipartite states from a locally indistinguishable set are distributed between spatially separated parties in a sequentially ordered fashion, they may or may not mark which state is which using LOCC. However, we show that even when the parties cannot mark the states, they may still locally identify the particular states given to them, though not their order—i.e., they can identify the elements of the given subset of states. Then we prove the existence of such subsets that are not even locally identifiable, thereby manifesting a stronger nonlocality. We also present the genuine version of this nonlocality—genuine subset unidentifiability—where the provided subset remains unidentifiable unless all the parties come together in a common location and perform global measurements. We anticipate potential applications of this nonlocality for future quantum technologies. We discuss one such application in a certain secret password distribution protocol, where this nonlocality outperforms its predecessors as a resource.

Introduction. In any operational theory, if two or more spatially separated communicating agents are able to perform a distributed task then it is trivially executable when they collaborate or come together in the same lab. The converse statement is also true in classical world. However, in the quantum regime, the scenario gets more complex. There exist distributed quantum tasks that can not be carried out locally, even though they are perfectly accomplishable when the agents collaborate. This phenomenon exhibits a stark deviation of quantum theory from the classical worldview. Usually, this anomaly is termed as quantum nonlocality [1, 2]. This is different from the celebrated Bell nonlocality that deals with the incapability of explaining correlations by local-realism [3]. Rather, this quantum nonlocality arises solely due to the intricate structure and topology of the quantum state space. It has huge significance in both application [4–8] and theory. The indistinguishability of certain mutually orthogonal multipartite quantum states by local operations and classical communication (LOCC) is one of the basic quantum nonlocal phenomena. In the past decades, a plethora of intriguing results were developed along this line [9–46]. Very recently, some other distributed quantum tasks have also been developed that manifest stronger forms of quantum nonlocality than the local indistinguishability viz., local state irreducibility [47], local unmarkability [48] etc. As mentioned earlier, impossibility of locally realising some distributed tasks gives rise to these nonlocalities. These tasks can broadly be divided into two paradigms—(i) local state discrimination and, (ii) local state elimination. Local indistinguishability and unmarkability are examples of the former, whereas local state irreducibility is for the later. In this work, we focus on the first paradigm and present an even stronger form of quantum nonlocality, exhibited by certain sets of mutually orthogonal entangled states. To demonstrate it we introduce a distributed task—Local Subset Identification (LSI). From a known set of mutually orthogonal states, a subset of more than one state is chosen and shared among multiple spatially separated agents (see Fig. 1). The objective of the task is to perfectly identify the subset, i.e., the identity of the states within it, via LOCC. For example, take a set of bipartite orthogonal states. Consider that any two states from the set are shared between spatially separated agents. Here, LSI asks the agents to perfectly locally identify which two states are given to them using LOCC. We show that the inability to perfectly accomplish this task for a set demonstrates a stronger form of quantum nonlocality than those discussed in the existing literature, termed Local Subset Unidentifiability. At the core of this proposed nonlocality lies the inability to perfectly distinguish certain sets of subspaces of rank more than one using LOCC. Since the task of addressing (in)distinguishability of sets of subspaces is naturally more complex than that of sets of vectors, our proposed nonlocality stands out compared to the previous nonlocalities. Moreover, we also deliver an information processing application of the proposed nonlocality.

Dispersal of information to maintain its secrecy is cru-
cial in information processing. Our proposed nonlocality also provides an interesting application in this line. Local subset unidentifiability furnishes heightened security in some secret password sharing scheme. We demonstrate that other nonlocal resources viz. local unmarkability if used in this task keeps the possibility of significant information leakage, whereas local unidentifiability gives significant advantage over local unmarkability. This makes local unidentifiability potentially useful in distributed protocols to support the emerging quantum internet technology [49, 50].

Furthermore, any nonlocal feature of quantum systems gets more intricate when multiparty scenario comes into the picture. In case of LSI, we also explore scenarios involving more than two spatially separated agents. Interestingly, we come up with a further stronger version of the nonlocality we are introducing here. In particular, we present sets of multipartite states that shows local subset unidentifiability (or locally unidentifiable) when all the agents are spatially separated. We show that these sets retain local unidentifiability in all possible bi-partitions. Therefore, to perfectly accomplish the LSI task, all the agents need to come together or must resort to additional quantum resources. We term it as Genuine Unidentifiability. We also illustrate that any set that is genuinely locally unidentifiable must also show genuine unmarkability which is hitherto an uncharted notion.

\textbf{Local Subset Identification.} We are now in a position to articulate the formal definition of our task (see Fig. 1).

\textbf{Definition 1 (}((n,|S'|))-Local Subset Identification). Consider a set \(S = \{ |\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle \} \subset \bigotimes_{k=1}^{n} \mathbb{C}^{d_k}\) of \(n\) \(N\)-party orthogonal quantum states. A subset \(S' \subset S\) containing \(1 \leq |S'| < n\) quantum states is randomly chosen and distributed among spatially separated classically communicating agents keeping its identity hidden. The task of \((n,|S'|)-local identification is to perfectly determine the elements of the set \(S'\).

We call a set \(S\) to be locally \((n,|S'|)-identifiable or simply \((n,|S'|)-\)identifiable if all possible subsets of \(S\) with the same cardinality as \(S'\) are locally identifiable, otherwise we call it locally \((n,|S'|)-unidentifiable or simply \((n,|S'|)-unidentifiable. It is evident that \(|S'| = n\) is a trivial question in the LSI framework, as the elements of the entire set \(S\) is already known to the agents. On the other extreme, when \(|S'| = 1\), this boils down to a well known task:

\textbf{Observation 1.} For a given set \(S\) the task of \((n,1)-LSI\) corresponds to the well-known task of Local State Discrimination (LSD) \([51]\).

Another related distributed task is Local State Marking (LSM) \([48]\). For the sake of completeness and comparison, we briefly describe it here. Consider the set \(S = \{ |\psi_{iA}\rangle\}_{i=1}^{n}\) of \(n\) bipartite orthogonal states from which a subset \(S'\) is shared between two spatially separated agents. Whereas the task of LSI is to locally identify just the elements of the subset, the task of \((n,|S'|)-LSM\) is to perfectly determine the order as well in which the elements are given to the agents. For example, consider \(S' = \{ |\psi_j\rangle, |\psi_k\rangle \}\) for any \(j < k \in \{1, \ldots, n\}\) \([52]\). These two states can be shared between the agents in two distinct orders: \(|\psi_j\rangle_{A_1B_1} \otimes |\psi_k\rangle_{A_2B_2}\) and \(|\psi_k\rangle_{A_1B_1} \otimes |\psi_j\rangle_{A_2B_2}\) both representing the same subset \(S'\). LSM asks the agents just the identity of the two states \(|\psi_j\rangle\) and \(|\psi_k\rangle\). LSM additionally asks whether they share \(|\psi_j\rangle_{A_1B_1} \otimes |\psi_k\rangle_{A_2B_2}\) or \(|\psi_k\rangle_{A_1B_1} \otimes |\psi_j\rangle_{A_2B_2}\). Thus, the LSM task can be formulated as a higher-dimensional LSD problem of the set \(\{ |\psi_{iA}\rangle \otimes |\psi_{i'A}\rangle \}_{i,i'=1}^{n}\) containing \(n^2\) states. This formalism can be extended to accommodate any number of parties and subsets with larger cardinalities. For a specific set of states this distributed task can not be accomplished perfectly locally, the set is said to exhibit a stronger quantum nonlocality than the local indistinguishability known as local unmarkability. If the \((n,|S'|)-LSM\) task can be perfectly accomplished, we call the set \(S\) as \((n,|S'|)-\)markable, otherwise it is \((n,|S'|)-\)unmarkable. Clearly, LSM is an easier task than LSM. Hence, local unidentifiability is a stronger nonlocal phenomenon than local unmarkability as well as local indistinguishability \([53]\).

\textbf{Observation 2.} If a set \(S\) is \((n, m)-\)markable, then it readily
follows that it is also \((n,m)\)-identifiable.

If the \(N\) spatially separated agents can locally deter-
mine the order of the states of the given subset \(S\), then
the identity of the states becomes trivially known to
them. However, the converse of this statement is not nec-
essarily true.

**Theorem 1.** If a set \(S\) is \((n,m)\)-identifiable, then it is not
necessarily \((n,m)\)-markable.

**Proof.** The proof is constructive. Consider a set of
three Bell states: \(S = \{|B_1\rangle \otimes |B_2\rangle, |B_2\rangle \otimes |B_1\rangle, |B_3\rangle \otimes |B_3\rangle\}\),
where \(|B_1\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle), |B_2\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle), |B_3\rangle = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle)\). This set is \((3,2)\)-identifiable, but
\((3,2)\)-unmarkable.

\((3,2)\)-identifiability: A subset containing two distinct
Bell states from the set \(S\) can be chosen in \(\binom{3}{2} = 3\) ways,
namely, \(S_1' = \{|B_1\rangle, |B_2\rangle\}, S_2' = \{|B_2\rangle, |B_3\rangle\}, S_3' = \{|B_3\rangle, |B_1\rangle\}\). One such subset is randomly selected and the
states within it are distributed among two spatially
separated agents, Alice and Bob (see Fig. 1). They can
identify the elements of the given subset by performing
the following measurement \(M_a := \{|P^{I}\rangle \sum_{i=1}^{6} P_i = 1\}, \) where \(P_i := |B_i\rangle \langle B_i|, \alpha \in \{A1, A2, B1, B2\}\) and \(|B_1\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle)\), on their respective share of two qubits,
and comparing their measurement outcomes via classi-
cal communication. See Appendix A for details. Similar
result holds for any of three sets of Bell states.

\((3,2)\)-unmarkability: However, Alice and Bob can never locally mark (i.e., distinguish between the orders
of) two states from any set of three Bell states. This is
because the task involves locally distinguishing a state from a set of \(\binom{3}{2} = 6\) maximally entangled states (MESs)
\(\in \mathbb{C}^4 \otimes \mathbb{C}^4\), a scenario deemed impossible by Ref. [1].
The reference asserts that perfect LSD of MESs becomes
unattainable when the number of states exceeds the local
dimension of the shared state.

In Appendix A, we present additional examples of lo-
cally identifiable yet unmarkable sets. It’s worth not-
ing that if a set \(S\) is \((n,m)\)-identifiable, it is inherently
\((n,m)\)-unmarkable, highlighting local subset identifi-
bility as a stronger form of nonlocality. Here are some
examples of locally unidentifiable sets.

**Theorem 2.** The set of four two-qubit Bell states is \((4,2)\-
unidentifiable.

**Proof.** If a subset \(S'\) containing two states from a set \(S\) of
four two-qubit Bell states is shared among two spatially
separated agents, Alice and Bob, they will never be able
to locally distinguish between the \(S'\)s, i.e., know the
identity of the two given Bell states, let alone their ordering.
Following is the detailed proof.

Consider that \(S := \{|B_1\rangle, |B_2\rangle, |B_3\rangle, |B_4\rangle\}\). A subset
containing any two Bell states can be chosen in
\(\binom{4}{2} = 6\) ways, namely, \(S_1' := \{|B_1\rangle, |B_2\rangle\}, S_2' := \{|B_1\rangle, |B_3\rangle\}, S_3' := \{|B_1\rangle, |B_4\rangle\}, S_4' := \{|B_2\rangle, |B_3\rangle\}, S_5' := \{|B_2\rangle, |B_4\rangle\}, S_6' := \{|B_3\rangle, |B_4\rangle\}\). Now, if \(S_1'\) is
distributed among the agents, the shared states can ex-
ist in two distinct orders, either \(|B_1\rangle_{k_1B_1} \otimes |B_2\rangle_{k_2B_2}\) or
\(|B_1\rangle_{k_1B_1} \otimes |B_2\rangle_{k_2B_2}\) and \(|B_2\rangle_{k_1B_1} \otimes |B_1\rangle_{k_2B_2}\), both with equal probability, represent-
ing the same subset \(S_1'\). Consequently, the description of
the subset according to the agents corresponds to a mixed
state:

\[
\rho_1 := \frac{1}{2} |B_1\rangle \langle B_1|_{k_1B_1} \otimes |B_2\rangle \langle B_2|_{k_2B_2} + \frac{1}{2} |B_2\rangle \langle B_2|_{k_1B_1} \otimes |B_1\rangle \langle B_1|_{k_2B_2}.
\]

More generally, Alice and Bob’s description of subset
\(S_i' := \{|B_i\rangle, \langle B_i|\}\) is given by

\[
\rho_i := \frac{1}{2} |B_i\rangle \langle B_i|_{k_1B_1} \otimes |B_\beta\rangle \langle B_\beta|_{k_2B_2} + \frac{1}{2} |B_\alpha\rangle \langle B_\alpha|_{k_1B_1} \otimes |B_i\rangle \langle B_i|_{k_2B_2},
\]

for \(i \in \{1, \cdots, 6\}\).

Clearly, the task of locally distinguishing the subsets \(S_i'\) of \(S\) is equivalent to distinguishing between the mixed
states \(\{\rho_i\}_{i=1}^{6}\) by LOCC. We now prove our claim by reductio ad absurdum.

We assume the set \(\{\rho_i\}_{i=1}^{6}\) are locally distinguishable.
Consequently, their supports are also distinguishable
via LOCC. Hence, any set of pure states \(\{|\xi_i\rangle \langle \xi_i| \in \text{Support}(\rho_i)\}_{i=1}^{6}\) must also be locally
distinguishable. However, the set \(\{|B_1\rangle_{k_1B_1} \otimes |B_2\rangle_{k_2B_2}\}^{4}_{k=1}\),
each state of which \(\in \text{Support}(\rho_i)\), is locally indistin-
guishable as the number of MESs exceeds their local di-
dimension [1]. This readily contradicts our assumption that
\(\{\rho_i\}_{i=1}^{6}\) are locally distinguishable, thus concluding
our proof.

We have also identified several sets containing orthog-
inal general MESs that fail to achieve this task, thus
demonstrating local subset unidentifiability.

**Theorem 3.** Consider a set \(S\) of \(2 < d \leq d^2\) MESs in \(\mathbb{C}^d \otimes \mathbb{C}^d\).
There are \(\binom{d}{2}\) possible subsets, each containing \(1 < k < D\) distinct states from \(S\). The set \(S\) is \((D,k)\)-unidentifiable, if \(\binom{d}{2} > d^k\).

The proof of this is provided in Appendix B.

Now, as a corollary of the above theorem, we present
the following result, which is in spirit of the theorem stat-
ing that any complete basis set of MESs is locally indistin-
guishable [1].

**Corollary 3.1.** Any two distinct states selected from a com-
plete basis set of MESs of arbitrary dimensions cannot be lo-
cally identified.

**Proof.** Consider a complete basis set of \(d^2\) mutually or-
thogonal MESs in \(\mathbb{C}^d \otimes \mathbb{C}^d\). From this set, any two states
can be chosen in \(\binom{d^2}{2} = \frac{d^4(d^2-1)}{2}\) ways. This quantity
is always greater than \(d^2\) for any \(d \in \mathbb{Z}^+ \setminus \{1\}\). There-
fore, the subset of two states is always locally unidentifi-

\(\square\)
Now, we extend our analysis to the multipartite framework. Similar extensions exist for other known forms of nonlocality, such as local indistinguishability and local irreducibility [47, 56, 57].

Genuine Unidentifiability.- We begin by presenting an example of set of tripartite orthogonal states demonstrating local unidentifiability in every bipartition. We show that even when all but one agent collaborates in a single laboratory, with the freedom to perform the strongest possible joint operations on their composite subsystems, they still fail to achieve perfect LSI. We term this multipartite notion of unidentifiability as genuine. Furthermore, our examples also represent genuine multipartite instances of the existing concept of local unmarkability [48]. Consider a set of eight 3-qubit GHZ states, \(S \equiv \{\frac{1}{\sqrt{2}}(|000\rangle \pm |111\rangle), \frac{1}{\sqrt{2}}(|001\rangle \pm |110\rangle), \frac{1}{\sqrt{2}}(|010\rangle \pm |101\rangle), \frac{1}{\sqrt{2}}(|100\rangle \pm |011\rangle}\).

**Theorem 4.** The set \(S\) is locally \((D,k)\)-unidentifiable if \(\binom{D}{k} > 2^{2k}\), where \(2 < D \leq 8\) and \(1 < k < D\), even when two out of three parties collaborate in a same lab.

Proof of this is given in Appendix C.

The scenario gets further involved with increasing number of parties. For instance, when we add another spatially separated agent to the tripartite scenario, we encounter two distinct types of bi-partitions: 1 party vs. 3 parties \((1:3)\), and 2 parties vs. 2 parties \((2:2)\). Within each type, there exist bi-partitions that are related by party permutations. Sets of orthogonal four-party states that exhibit local unidentifiability in all such possible bi-partitions within both types bring out the true genuineness of the nonlocality discussed here.

Consider the set \(S' \equiv \{\Omega_{\alpha}\}_{\alpha=1}^{16}\),

\[
\begin{align*}
|\Omega_{1,2}\rangle &= \frac{1}{2}(|000\rangle \pm |111\rangle \pm |101\rangle \pm |110\rangle),
|\Omega_{3,4}\rangle &= \frac{1}{2}(|000\rangle \pm |111\rangle \mp |101\rangle \mp |110\rangle),
|\Omega_{5,6}\rangle &= \frac{1}{2}(|001\rangle \pm |010\rangle \mp |110\rangle \mp |101\rangle),
|\Omega_{7,8}\rangle &= \frac{1}{2}(|001\rangle \mp |010\rangle \mp |110\rangle \mp |101\rangle),
|\Omega_{9,10}\rangle &= \frac{1}{2}(|010\rangle \pm |001\rangle \pm |100\rangle \mp |111\rangle),
|\Omega_{11,12}\rangle &= \frac{1}{2}(|010\rangle \pm |001\rangle \mp |100\rangle \mp |111\rangle),
|\Omega_{13,14}\rangle &= \frac{1}{2}(|100\rangle \pm |001\rangle \mp |101\rangle \mp |110\rangle),
|\Omega_{15,16}\rangle &= \frac{1}{2}(|100\rangle \pm |001\rangle \pm |101\rangle \mp |110\rangle).
\end{align*}
\]

**Theorem 5.** The set \(S'\) is locally \((D,k)\)-unidentifiable if \(\binom{D}{k} > 2^{2k}\), where \(2 < D \leq 16\) and \(1 < k < D\), in all \(1 : 3\) as well 2 : 2 bi-partitions.

The proof of this theorem is provided in Appendix D. Additionally, Theorems 4 and 5 also present examples of genuine unmarkability. The concerned sets are locally unmarkable in all possible bipartitions.

Application in secret password distribution.- Our notion of stronger nonlocality, Local Subset Unidentifiability, has potential significance in various information processing technologies. In this context, we focus on a specific application (see Fig. 2). In this scenario, a sender aims to distribute hidden information, such as a locked password, among several spatially separated receivers who can communicate classically among themselves. The sender’s objective is to ensure that the receivers remain unaware of the password’s identity as long as they are separated, even if they have unrestricted classical communication. The password can only be unlocked when all the receivers physically gather in a common location.

Precisely, consider that the sender wish to share a password – a string \(X := x_1x_2 \cdots x_m\) of \(m\) letters, \(x_i\) being the \(i\)th letter in the string – among the receivers. Each letter in the string is to be chosen without repetition from an alphabet \(A = \{a_k\}_{k=1}^n\) of \(n > m\) letters which is known to the receivers as well. Now, the sender and the receivers agree upon an encoding scheme: the letters of the alphabet \(A\) are encoded in a set \(S := \{|\psi_k\rangle\}_{k=1}^n\) of pairwise orthogonal pure multipartite quantum states. Accordingly, the sender encodes their password \(\lambda\) into a string of quantum states: \(X \mapsto |\xi_1\rangle \otimes |\xi_2\rangle \otimes \cdots \otimes |\xi_m\rangle\), where the state \(|\xi_i\rangle\) can be any state from \(S\) with the only restriction that \(|\xi_i\rangle \neq |\xi_j\rangle\), \(\forall i, j\). Subsequently, the
sender shares this composite state among the receivers (see Fig. 2). If $S$ is locally $(n, m)$-unmarkable, then the spatially separated parties will not be able to perfectly discriminate the received string from the $n P_m = \frac{n!}{(n-m)!}$ possible strings of quantum states by LOCC. However, it may so happen that the receivers can perfectly predict the identity of the $m$ individual states $\{\xi_i\}$ if the encoding is done in a locally $(n, m)$-unmarkable, but locally $(n, m)$-identifiable set of quantum states. Then, they will be able to guess the correct permutation of the letters with success probability $\frac{1}{m!}$. Another alternative which they may opt for is to imperfectly discriminate (i.e., with a nonzero probability $P_{\text{imp}} < 1$, bounded by an upper limit discussed in [58]) the received string. The encoding set $S$ determines which of the success probabilities is higher.

However, if the sender encodes the password in a locally $(n, m)$-unidentifiable set of states, then the receivers will not be able to even identify the individual letters perfectly by LOCC, and hence they will not follow the former strategy. Furthermore, if, for such sets, $P_{\text{imp}} < \frac{1}{m!}$, then the security of the password is enhanced significantly. We have found examples of $(n, m)$-unidentifiable sets, $S_d \subset \mathbb{C}^d \otimes \mathbb{C}^d$ containing MESs for which $P_{\text{imp}} < \frac{1}{d!}$ (follows from Theorem 3) as long as $|S_d| \geq d + 1$ [58, 59].

**Discussion.** In summary, we come up with a new distributed task – LSI. We show that impossibility of accomplishing this task gives birth to a unique version of quantum nonlocality – local subset unidentifiability. We show that this is by far the strongest quantum nonlocality in the state discrimination paradigm, that arises from the impossibility of discriminating certain mutually orthogonal subspaces of rank more than one. In the multipartite framework, we introduce the notion of genuine unidentifiability which says that a set of quantum states may remain locally unidentifiable even in all possible bipartitions. Along this line, we also introduce the notion of genuine unmarkability in multipartite scenarios. Interestingly, we also propose a cryptographic application of this proposed nonlocality. In secret password distribution scheme, we demonstrate that local unidentifiability provides a strictly better encoding for protecting password secrecy than its predecessors. While we explore local unidentifiability only in entangled states, we believe it is not necessarily an exclusive characteristic of entanglement. Exploring the same feature in orthogonal product states would be quite intriguing.

We thank Guruprasad Kar, Somshubhro Bandyopadhyay, Sibasish Ghosh, Manik Banik, and Arup Roy for discussions.

[1] Asher Peres and William K. Wootters, “Optimal detection of quantum information,” Phys. Rev. Lett. 66, 1119–1122 (1991).
[2] Charles H Bennett, David P DiVincenzo, Christopher A Fuchs, Tal Mor, Eric Rains, Peter W Shor, John A Smolin, and William K Wootters, “Quantum nonlocality without entanglement,” Physical Review A 59, 1070 (1999).
[3] J. S. Bell, “On the einstein podolsky rosen paradox,” Physics Physique Fizika 1, 195–200 (1964).
[4] Barbara M. Terhal, David P. DiVincenzo, and Debbie W. Leung, “Hiding bits in bell states,” Phys. Rev. Lett. 86, 5807–5810 (2001).
[5] D.P. DiVincenzo, D.W. Leung, and B.M. Terhal, “Quantum data hiding,” IEEE Transactions on Information Theory 48, 580–598 (2002).
[6] T. Eggeling and R. F. Werner, “Hiding classical data in multipartite quantum states,” Phys. Rev. Lett. 89, 097905 (2002).
[7] William Matthews, Stephanie Wehner, and Andreas Winter, “Distinguishability of quantum states under restricted families of measurements with an application to quantum data hiding,” Communications in Mathematical Physics 291, 813–843 (2009).
[8] Damian Markham and Barry C. Sanders, “Graph states for quantum secret sharing,” Phys. Rev. A 78, 042309 (2008).
[9] S Virmani, M FSacchi, M BPlenio, and D Markham, “Optimal local discrimination of two multipartite pure states,” Phys. Lett. A 288, 62–68 (2001).
[10] Berry Groisman and Lev Vaidman, “Nonlocal variables with product-state eigenstates,” J. Phys. A Math. Gen. 34, 6881–6889 (2001).
[11] Jonathan Walgate, Anthony J. Short, Lucien Hardy, and Vlatko Vedral, “Local distinguishability of multipartite orthogonal quantum states,” Phys. Rev. Lett. 85, 4972–4975 (2000).
[12] Michal Horodecki, Aditi Sen(De), Ujjwal Sen, and Karol Horodecki, “Local indistinguishability: More nonlocality without less entanglement,” Phys. Rev. Lett. 90, 047902 (2003).
[13] Heng Fan, “Distinguishability and indistinguishability by local operations and classical communication,” Phys. Rev. Lett. 92, 177905 (2004).
[14] Sibasish Ghosh, Guruprasad Kar, Anirban Roy, and Debasis Sarkar, “Distinguishability of maximally entangled states,” Phys. Rev. A 70, 022304 (2004).
[15] S. De Rinaldis, “Distinguishability of complete and extendible product bases,” Phys. Rev. A 70, 022309 (2004).
[16] John Watrous, “Bipartite subspaces having no bases distinguishable by local operations and classical communication,” Phys. Rev. Lett. 95, 080505 (2005).
[17] J. Niset and N. J. Cerf, “Multipartite nonlocality without entanglement in many dimensions,” Phys. Rev. A 74, 052103 (2006).
[18] Ming-Yong Ye, Wei Jiang, Ping-Xing Chen, Yong-Sheng Zhang, Zheng-Wei Zhou, and Guang-Can Guo, “Local distinguishability of orthogonal quantum states and generators of $su(N)$,” Phys. Rev. A 76, 032329 (2007).
[19] Heng Fan, “Distinguishing bipartite states by local operations and classical communication,” Phys. Rev. A 75, 042305 (2007).
[20] Runyao Duan, Yuan Feng, Zhengfeng Ji, and Mingsheng Ying, “Distinguishing arbitrary multipartite basis unam-
biguously using local operations and classical communication,” Phys. Rev. Lett. 98, 230502 (2007).
[21] Somshubhro Bandyopadhyay and Jonathan Walgate, “Local distinguishability of any three quantum states,” J. Phys. A. Math. Theor. 42, 072002 (2009).
[22] Yuan Feng and Yaoyun Shi, “Characterizing locally indistinguishable orthogonal product states,” IEEE Trans. Inf. Theory 55, 2799–2806 (2009).
[23] Runyao Duan, Yu Xin, and Mingsheng Ying, “Locally indistinguishable subspaces spanned by three-qubit unextendible product bases,” Phys. Rev. A 81, 032329 (2010).
[24] Nengkun Yu, Runyao Duan, and Mingsheng Ying, “Four locally indistinguishable ququad-ququad orthogonal maximally entangled states,” Phys. Rev. Lett. 109, 020506 (2012).
[25] Ying-Hui Yang, Fei Gao, Guo-Jing Tian, Tian-Qing Cao, and Qiao-Yan Wen, “Local distinguishability of orthogonal quantum states in a $2 \otimes 2 \otimes 2$ system,” Phys. Rev. A 88, 024301 (2013).
[26] Andrew M Childs, Debbie Leung, Laura Mančinska, and Maris Ozols, “A framework for bounding nonlocality of state discrimination,” Commun. Math. Phys. 323, 1121–1153 (2013).
[27] Zhi-Chao Zhang, Fei Gao, Guo-Jing Tian, Tian-Qing Cao, and Qiao-Yan Wen, “Nonlocality of orthogonal product basis quantum states,” Phys. Rev. A 90, 022313 (2014).
[28] Suxia Yu and C. H. Oh, “Detecting the local indistinguishability of maximally entangled states,” (2015), arXiv:1502.01274 [quant-ph].
[29] Zhi-Chao Zhang, Fei Gao, Su-Juan Qin, Ying-Hui Yang, and Qiao-Yan Wen, “Nonlocality of orthogonal product states,” Phys. Rev. A 92, 012332 (2015).
[30] Yan-Ling Wang, Mao-Sheng Li, Zhu-Jun Zheng, and Shao-Ming Fei, “Nonlocality of orthogonal product basis quantum states,” Phys. Rev. A 92, 032313 (2015).
[31] Jianxin Chen and Nathaniel Johnston, “The minimum size of unextendible product bases in the bipartite case (and some multipartite cases),” Commun. Math. Phys. 333, 351–365 (2015).
[32] Ying-Hui Yang, Fei Gao, Guang-Bao Xu, Hui-Juan Zuo, Zhi-Chao Zhang, and Qiao-Yan Wen, “Characterizing unextendible product bases in qutrit-ququad system,” Sci. Rep. 5, 11963 (2015).
[33] Zhi-Chao Zhang, Fei Gao, Ya Cao, Su-Juan Qin, and Qiao-Yan Wen, “Local indistinguishability of orthogonal product states,” Phys. Rev. A 93, 012314 (2016).
[34] Guang-Bao Xu, Qiao-Yan Wen, Su-Juan Qin, Ying-Hui Yang, and Fei Gao, “Quantum nonlocality of multipartite orthogonal product states,” Phys. Rev. A 93, 032341 (2016).
[35] Xiaoqian Zhang, Xiaoping Tan, Jian Weng, and Yongjun Li, “LOCC indistinguishable orthogonal product quantum states,” Sci. Rep. 6 (2016).
[36] Guang-Bao Xu, Ying-Hui Yang, Qiao-Yan Wen, Su-Juan Qin, and Fei Gao, “Locally indistinguishable orthogonal product bases in arbitrary bipartite quantum system,” Sci. Rep. 6 (2016).
[37] Yan-Ling Wang, Mao-Sheng Li, Shao-Ming Fei, and Zhu-Jun Zheng, “Constructing unextendible product bases from the old ones,” (2017), arXiv:1703.06542 [quant-ph].
[38] Zhi-Chao Zhang, Ke-Jia Zhang, Fei Gao, Qiao-Yan Wen, and C. H. Oh, “Construction of nonlocal multipartite quantum states,” Phys. Rev. A 95, 052344 (2017).
[39] Guang-Bao Xu, Qiao-Yan Wen, Fei Gao, Su-Juan Qin, and Hui-Juan Zuo, “Local indistinguishability of multipartite orthogonal product bases,” Quantum Inf. Process. 16 (2017).
[40] Yan-Ling Wang, Mao-Sheng Li, Zhu-Jun Zheng, and Shao-Ming Fei, “The local indistinguishability of multipartite product states,” Quantum Inf. Process. 16 (2017).
[41] Xiaoqian Zhang, Jian Weng, Xiaoqing Tan, and Weiqi Luo, “Indistinguishability of pure orthogonal product states by LOCC,” Quantum Inf. Process. 16 (2017).
[42] Xiaoqian Zhang, Cheng Guo, Weiqi Luo, and Xiaoqing Tan, “Local distinguishability of quantum states in bipartite systems,” (2017), arXiv:1712.08830 [quant-ph].
[43] Xiaoqian Zhang, “Local distinguishability of quantum states in multipartite system,” (2017), arXiv:1712.09870 [quant-ph].
[44] Sarah Croke and Stephen M. Barnett, “Difficulty of distinguishing product states locally,” Phys. Rev. A 95, 012337 (2017).
[45] Subhendu B. Ghosh, Tathagata Gupta, A. V. Ardra, Anandamay Das Bhowmik, Sutapa Saha, Tamal Guha, and Amit Mukherjee, “Activating strong nonlocality from local sets: An elimination paradigm,” Phys. Rev. A 106, L010202 (2022).
[46] Tathagata Gupta, Subhendu B. Ghosh, A. V. Ardra, Anandamay Das Bhowmik, Sutapa Saha, Tamal Guha, Ramij Rahaman, and Amit Mukherjee, “Genuine activation of quantum nonlocality: Stronger than local indistinguishability,” (2022).
[47] Saronath Halder, Manik Banik, Sristy Agrawal, and Somshubhro Bandyopadhyay, “Strong quantum nonlocality without entanglement,” Physical review letters 122, 040403 (2019).
[48] Samrat Sen, Edwin Peter Lobo, Sahl Gopalkrishna Naik, Ram Krishna Patra, Tathagata Gupta, Subhendu B Ghosh, Sutapa Saha, Mir Alamuddin, Tamal Guha, Some Sankar Bhattacharya, et al., “Local quantum state marking,” Physical Review A 105, 032407 (2022).
[49] H. J. Kimble, “The quantum internet,” Nature 453, 1023–1030 (2008).
[50] Stephanie Wehner, David Elkouss, and Ronald Hanson, “Quantum internet: A vision for the road ahead,” Science 362, eaam9288 (2018), https://www.science.org/doi/pdf/10.1126/science.aam9288.
[51] Local state discrimination (LSD) is a well known distributed task where one state randomly chosen from a known set $S$ of orthogonal multipartite states is shared among spatially separated agents and their objective is to determine the identity of the given state by LOCC.
[52] Since both $\{ |\psi_j \rangle, |\psi_k \rangle \}$ and $\{ |\tilde{\psi}_k \rangle, |\tilde{\psi}_j \rangle \}$ represent the same set, we take $j < k$ to avoid double counting.
[53] Note that if a set $S$ is locally distinguishable then obviously the set is locally markable as well as locally identifiable. Also, if the set $S$ is $(n,m)$-unidentifiable, for $n > m > 1$, it must be $(n,m)$-unmarkable. However, $(n,m)$-LSI or $(n,m)$-LSM task is not comparable in straightforward sense to the LSD task for $m > 1$ as the last one is defined only for $m = 1$. Nevertheless, it is obvious that if a set is $(n,m)$-unidentifiable or $(n,m)$-unmarkable it must be locally indistinguishable.
[54] 1] M. Hayashi, D. Markham, M. Murao, M. Owari, and S. Virmani, “Bounds on multipartite entangled orthogonal state discrimination using local operations and classical communication,” Phys. Rev. Lett. 96, 040501 (2006).
[55] Somshubhro Bandyopadhyay, “Entanglement, mixedness, and perfect local discrimination of orthogonal quan-
tum states,” *Phys. Rev. A* **85**, 042319 (2012).

[56] Sumit Rout, Ananda G Maiti, Amit Mukherjee, Sarathon Halder, and Manik Banik, “Genuinely nonlocal product bases: Classification and entanglement-assisted discrimination,” *Physical Review A* **100**, 032321 (2019).

[57] Sumit Rout, Ananda G Maiti, Amit Mukherjee, Sarathon Halder, and Manik Banik, “Multiparty orthogonal product states with minimal genuine nonlocality,” *Physical Review A* **104**, 052433 (2021).

[58] (4) A set $S = \{\ket{\psi_i}\}_{i=1}^n$ is complete if it is unmarkable that means there exists a set $\tilde{S}$ of locally indistinguishable states $\{\otimes_{i=1}^n \ket{\xi_i}\}$ with cardinality $\tilde{P}_m$, where every $\ket{\xi_i} \in S$. The probability $P_{\text{imp}}$ for marking the above set by LOCC has an upper bound, $P_{\text{imp}} \leq \frac{d_n}{P_{\text{imp}}}$ (see Lemma 8 of [60]). Here $d$ represents local dimension of the multipartite state $S$.

[59] (1) If a set $S = \{\ket{\psi_i}\}_{i=1}^n$ is complete set of $d \times d$ maximally entangled bases then such set is $(n, m)$-unidentifiable if $d^m > d^n$ (see Theorem 3) which automatically implies that $P_{\text{imp}} < \frac{1}{m^2}$ holds for such set.

[60] Michael Nathanson, “Distinguishing bipartite orthogonal states using LOCC: Best and worst cases,” *Journal of Mathematical Physics* **46**, 062103 (2005).

**Appendix A: Locally identifiable subsets of Two-qubit Bell states**

(3, 2) — **Bell identifiability**: Alice and Bob can successfully identify any subset of two Bell states $\{\ket{B_i} , \ket{B_j}\}_{i \neq j}$ given from a known set of three Bell states, $\{\ket{B_i} , \ket{B_j} , \ket{B_k}\}_{i \neq j \neq k}$ by LOCC only, where

$$
\ket{B_{1,2}} = \frac{1}{\sqrt{2}}(\ket{00} \pm \ket{11}), \quad \ket{B_{3,4}} = \frac{1}{\sqrt{2}}(\ket{01} \pm \ket{10})
$$

The protocol is the following. Each agent (Alice or Bob) performs a Bell basis measurement $M_\alpha := \{P_\alpha^i : \sum_{i=1}^4 P_\alpha^i = \mathbb{I}_4\}$, where $P_\alpha^i := \proj{B_i}{\alpha}$, $\alpha \in \{A_1A_2, B_1B_2\}$, on the two qubits in their respective labs. Afterwards, the agents can know the identity of the shared subset by tallying their measurement outcomes via classical communication as the following table represents,

| $S_1'$ | $S_2'$ | $S_3'$ |
|--------|--------|--------|
| (1, 2) | (1, 4) | (1, 3) |
| (1, 2) | (2, 3) | (2, 4) |
| (3, 4) | (3, 2) | (3, 1) |
| (4, 3) | (4, 1) | (4, 2) |

TABLE I. Here, $(m, n)$ represents that $m$th local projector $P_{m|A_2}$ clicked in Alice’s lab and $n$th local projector $P_{n|B_1}$ clicked in Bob’s lab, when both of them performs Bell basis measurement $M_\alpha$, where $P_\alpha^i := \proj{B_i}{\alpha}$, $\alpha \in \{A_1A_2, B_1B_2\}$. The states for a specific subset, say, $S_1'$, the joint outcome $(m, n)$ can be anything from the set $\{(1, 2), (2, 1), (3, 4), (4, 3)\}$.

(4, 3) — **Bell identifiability**: Alice and Bob can successfully identify any subset of three Bell states given from the set $\{\ket{B_i}\}_{i=1}^4$ using only LOCC. The protocol is the following. Each agent (Alice or Bob) performs a projective measurement $M_\alpha := \{P_\alpha^i : \sum_{i=1}^4 P_\alpha^i = \mathbb{I}_4\}$, where $P_\alpha^i := \proj{G_i}{\alpha}$, $\alpha \in \{A_1A_2A_3, B_1B_2B_3\}$, on the three qubits in their respective labs. Here, $\ket{G_i}$ is any of the 8 orthogonal three-qubit GHZ states: $\frac{1}{\sqrt{2}}(\ket{000} \pm \ket{111}), \frac{1}{\sqrt{2}}(\ket{010} \pm \ket{101}), \frac{1}{\sqrt{2}}(\ket{100} \pm \ket{011})$. Afterwards, the agents can know the identity of the shared subset by tallying their measurement outcomes via classical communication.

For example, if a subset $S_1'$ is distributed among the agents then their shared three Bell states can be either of the following six orders— $|B_{1A_1B_1} \otimes |B_{3A_2B_2} \otimes |B_{3A_3B_3} \otimes |B_{2A_2B_3} \otimes |B_{2A_3B_2} \otimes |B_{1A_1B_2} \otimes |B_{2A_1B_3} \otimes |B_{1A_2B_3} \otimes |B_{1A_3B_2} \otimes |B_{2A_2B_3} \otimes |B_{2A_3B_2} \otimes |B_{3A_1B_3} \otimes |B_{3A_2B_2} \otimes |B_{3A_3B_1} \otimes |B_{1A_1B_3} \otimes |B_{2A_1B_2} \otimes |B_{3A_1B_2} \otimes |B_{1A_2B_2} \otimes |B_{1A_3B_1} \otimes |B_{2A_3B_1} \otimes |B_{3A_2B_1} \otimes |B_{3A_3B_1}$, if Alice and Bob ignores the order of the distributed state then such shared state can be written as a convex mixture of each of the six possible orders. Hence, Alice and Bob has to discriminate a set of $\binom{4}{3}$ density matrices $\{\rho_i\}$, where

$$
\rho_i = \frac{1}{6} \text{Perm}_{\mu} |\alpha_\mu\rangle \langle \beta_\mu| \otimes |\beta_\mu\rangle \langle \beta_\gamma|, \quad \mu = 1, 2, \ldots, 6, \quad \alpha_i \neq \beta_i \neq \gamma_i \in \{1, 2, 3, 4\}
$$

and further we have $Tr(\rho_i \rho_j) = \delta_{ij}$ for all $(i, j)$. The set of states $\{\rho_i\}$ is said to be LOCC indistinguishable if and only if the support vectors of individual $\rho_i$ are LOCC indistinguishable. Now we must show whether the set of vectors


\[ |B_{\alpha_i}\rangle_{A_1B_1} |B_{\beta_j}\rangle_{A_2B_2} |B_{\gamma_k}\rangle_{A_3B_3} = \frac{1}{\sqrt{2}} \sum_{m_i=1}^{8} |G_{m_i}\rangle_{A_1A_2A_3} \otimes U_{m_i} |G_{m_i}\rangle_{B_1B_2B_3}, \quad \forall i \in \{1, 2, 3, 4\} \]

where \( U_{m_i} \) is local unitary acting on \( |G_{m_i}\rangle \) such that \( U_{m_i} |G_{m_i}\rangle = \mu_{m_i} |G_{m_i}\rangle \) where \( \mu_{m_i} \in \{-1, +1\} \). Now from Eq. (2) it is clear that if Alice measures her three-qubit in \( \{ |G_{m_i}\rangle \} \) basis then for a given outcome \( m_i = m_j = m \in \{1, 2, ..., 8\}, \) where \( i \neq j \), the pair of collapsed states at Bob’s side, i.e. \( \{ |G_{m_i}\rangle, |G_{m_j}\rangle \} \) must be pairwise orthogonal for all \( (i, j) \in \{1, 2, 3, 4\} \). Hence, the set of vectors \( \{ |B_{\alpha_i}\rangle_{A_1B_1} |B_{\beta_j}\rangle_{A_2B_2} |B_{\gamma_k}\rangle_{A_3B_3}\}_{i=1}^{4} \) are locally distinguishable which implies that \( \{ \rho_i \}_{i=1}^{8} \) is LOCC distinguishable.



\[
\begin{array}{cccc}
S'_1 & S'_2 & S'_3 & S'_4 \\
(1, 4) & (1, 3) & (1, 7) & (1, 8) \\
(2, 3) & (2, 4) & (2, 8) & (2, 7) \\
(3, 2) & (3, 1) & (3, 5) & (3, 6) \\
(4, 1) & (4, 2) & (4, 6) & (4, 5) \\
(5, 8) & (5, 7) & (5, 3) & (5, 4) \\
(6, 7) & (6, 8) & (6, 4) & (6, 3) \\
(7, 6) & (7, 5) & (7, 1) & (7, 2) \\
(8, 5) & (8, 6) & (8, 2) & (8, 1) \\
\end{array}
\]

TABLE II: Here, \((m, n)\) represents the joint outcome for the \(m^{th}\) projective measurement \(P_{m_1k_2k_3}\) performed by Alice in \(\{ |G_{m_i}\rangle \}_{m=1}^{8}\) basis and \(i^{th}\) projective measurement \(P_{n_1k_2k_3}\) of Bob in \(\{ |G_{n_i}\rangle \}_{n=1}^{8}\) basis. Here each subset \(S'_1\) is chosen in a way, where \(S'_1 = \{ |B_1\rangle, |B_2\rangle, |B_3\rangle\}, S'_2 = \{ |B_1\rangle, |B_2\rangle, |B_4\rangle\}, S'_3 = \{ |B_1\rangle, |B_3\rangle, |B_4\rangle\} \) and \(S'_4 = \{ |B_2\rangle, |B_3\rangle, |B_4\rangle\}\). The table states that for a specific subset, say, \(S'_1\), the joint outcome \((m, n)\) can be anything from the set \((1, 4), (2, 3), (3, 2), (4, 1), (5, 8), (6, 7), (7, 6), (8, 5)\).

(4, 3) – Bell unmarkability: Although Alice and Bob can identity any given subset of three Bell states out of four but cannot mark the three states \(\{ |B_i\rangle \}\) with LOCC only. The proof is very simple and directly follows from [1]. The local dimension of three given Bell states is \(d = 8\) whereas, the cardinality for \((4, 3)\) Bell marking is \(4 \times 3! = 24\) which is greater than the local dimension. Hence, such set is locally unmarkable.

Appendix B: Proof of Theorem 3

Theorem 3. Consider a complete basis set \(S\) of maximally entangled states (MES) in \(\mathbb{C}^d \otimes \mathbb{C}^d\). There are \(d^2\) possible subsets, each containing \(k\) distinct states from \(S\). The set \(S\) is \((d^2, k)\)-unidentifiable, if \(\binom{d^2}{k} > d^k\). Moreover, the set of \(D\) maximally entangled states \((D < d^2)\) will also be \((D, k)\)-unidentifiable, provided \(\binom{D}{k} > d^k\).

Proof: Consider a maximally entangled state in \(\mathbb{C}^d \otimes \mathbb{C}^d\) as \(|\Gamma\rangle := \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |jj\rangle\). A complete basis set of pairwise orthogonal maximally entangled states, \(S = \{ |\Gamma_i\rangle \}_{i=1}^{d^2}\) can be generated from the unique state \(|\Gamma\rangle\) as \(|\Gamma_i\rangle = (U_i \otimes 1) |\Gamma\rangle\), where \(\{U_i\}_{i=1}^{d^2}\) represents a complete set of Hilbert-Schmidt orthogonal unitary operators from SU\((d)\). A subset of \(k\) distinct maximally entangled states can be chosen from \(S\) in \(\kappa := \binom{d^2}{k}\) ways. We denote the subsets as \(S_i^{(d^2, k)}\), where \(i\) runs from 1 to \(\kappa\).

Now, the objective is to distinguish the subsets via LOCC. This actually amounts to locally distinguishing corresponding mixed states \(\{\rho_i^{(\kappa)}\}_{i=1}^{\kappa}\), where each state \(\rho_i^{(\kappa)}\) is an equal classical mixture of all possible permutations of the composition of the \(k\) pure states from \(S_i^{(d^2, k)}\). These mixed states are clearly orthogonal to each other because, by construction, each subset contains at least one different element from the other. If the \(\rho_i^{(\kappa)}\)s are distinguishable, then any set of \(\kappa\) vectors, each chosen from the support of different \(\rho_i^{(\kappa)}\)s in \(\mathbb{C}^{d^k} \otimes \mathbb{C}^{d^k}\), are also distinguishable. However, any \(\kappa\) pure maximally entangled states in \(\mathbb{C}^{d^k} \otimes \mathbb{C}^{d^k}\) are locally indistinguishable if \(\kappa > d^k\) [1]. Clearly, the
set \( \{ \rho_i \} \) is locally indistinguishable, otherwise it leads to a contradiction. Evidently, if we consider a set of \( D < d^2 \) maximally entangled states, it can be easily shown following a similar approach as above, that the set will be \((D, k)\)-unidentifiable for \( (D)_k > d^k \). This completes our proof.

\[ \square \]

### Appendix C: Proof of Theorem 4

**Theorem 4.** Consider a complete basis set \( S \) of GHZ states in \( \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \). The set \( S \) is locally \((D, k)\)-unidentifiable if \( (D)_k > 2^{2k} \) (where \( D \in \{2, 8\} \) and \( k \in \{1, D\} \)) even when two out of three parties come together in a same lab (collaborate).

We start by providing a sketchy version of the proof involving \( D = 8, k = 2 \) for the sake of readability. The general version of the proof is given below.

As previously mentioned that \( S \) is a complete set of orthonormal tripartite GHZ basis defined as \( S := \{ \frac{1}{\sqrt{2}} (|000\rangle \pm |111\rangle), \frac{1}{\sqrt{2}} (|011\rangle \pm |101\rangle), \frac{1}{\sqrt{2}} (|100\rangle \pm |011\rangle) \} \). First, note that the set \( S \) has the following property. If any two of the three spatially separated agents sharing an arbitrary state, say \(|G_\alpha\rangle \in S\) collaborate then they can transform \(|G_\alpha\rangle \) to any other state, say \(|G_\beta\rangle \in S\) via implementing two-qubit unitary operations on their subsystems. This property makes \( S \) locally indistinguishable, even if two agents collaborate \([2]\). Now, if the three agents share a pair of states from \( S \), say \(|G_\alpha\rangle \otimes |G_\beta\rangle\) with \( \alpha \neq \beta \) then they can transform the pair of shared states to any of the possible \( 56 \) choices from the set \( S = \{ |G_\alpha\rangle \otimes |G_\beta\rangle \mid \alpha \neq \beta, \forall \{\alpha, \beta\} \in \{1, \ldots, 8\} \} \), with any two agents collaborating and implementing four-qubit unitaries on their subsystems of the pair of three-qubit states. Following the arguments in \([2]\), the set \( \{ |G_\alpha\rangle \otimes |G_\beta\rangle \}_{\alpha < \beta = 1}^8 \) is locally indistinguishable even with two collaborating agents, as one can write \( 26 > 2^4 \). On the other hand, perfect local identification of the subset of any two states from \( S \) necessarily implies perfect local discrimination of the set of 26 pairwise orthogonal mixed states, \( \{ \rho_{\alpha, \beta} := \frac{1}{2} (|G_\alpha\rangle \langle G_\alpha| \otimes |G_\beta\rangle \langle G_\beta| + |G_\beta\rangle \langle G_\beta| \otimes |G_\alpha\rangle \langle G_\alpha|) \}_{\alpha < \beta = 1}^8 \) with certainty. But, perfect distinguishability of the set \( \{ \rho_{\alpha, \beta} \}_{\alpha < \beta = 1}^8 \) ensures perfect discrimination of its support vectors, i.e., discrimination of the set of vectors, \( \{ |G_\alpha\rangle \otimes |G_\beta\rangle \}_{\alpha < \beta = 1}^8 \). Hence, the set \( \{ \rho_{\alpha, \beta} \}_{\alpha < \beta = 1}^8 \) is locally indistinguishable and consequently, \( S \) is locally \((8, 2)\)-unidentifiable even if any two of the three agents perform any two-qubit operation. This readily rules out the scope of local identification when all three parties are separated. Now, we discuss the general proof.

**Proof.** Firstly, note that elements of the set \( S \) follow the relation below,

\[ |G_\alpha\rangle = (\mathbb{I} \otimes U_{\alpha, \beta}) |G_\beta\rangle, \quad \alpha, \beta \in \{1, \ldots, 8\}, \]

where, \( U_{\alpha, \beta} \) is a unitary acting on two qubits and \( \mathbb{I} \) is the identity operator acting on \( \mathbb{C}^2 \). Naturally, any subset of \( S \) inherits the same property.

Let \( S := \{ |G_\alpha\rangle \}_{\alpha = 1}^D, \quad 2 < D \leq 8 \). There are \( k := \binom{D}{k} \) different ways in which \( k \) states can be chosen from \( S \) to form subsets that we label as \( S'_i, \quad i = 1, \ldots, k \). Locally identifying \( S'_i \) tantamounts to locally distinguishing the set of mixed states \( \{ \rho_i \}_{i = 1}^k \) where

\[ \rho_i := \frac{1}{k!} \sum_{\mu=1}^{k!} |\Phi^i_{\mu}\rangle \langle \Phi^j_{\mu}|, \]

\[ |\Phi^i_{\mu}\rangle = \text{Perm}_\mu |G_{\alpha_1}\rangle |G_{\alpha_2}\rangle \ldots |G_{\alpha_k}\rangle, \quad \mu = 1, \ldots, k!, \]

\[ \alpha_1 \neq \alpha_2 \ldots \neq \alpha_k \in \{1, \ldots, D\}, \]

such that, \( |G_{\alpha_1}\rangle, |G_{\alpha_2}\rangle, \ldots, |G_{\alpha_k}\rangle \in S'_i \).

Here, \( \text{Perm}_\mu |G_{\alpha_1}\rangle |G_{\alpha_2}\rangle \ldots |G_{\alpha_k}\rangle \) means different possible permutations of the state \(|G_{\alpha_1}\rangle |G_{\alpha_2}\rangle \ldots |G_{\alpha_k}\rangle \). For example, \( \text{Perm}_1 |G_{\alpha_1}\rangle |G_{\alpha_2}\rangle |G_{\alpha_3}\rangle = |G_{\alpha_1}\rangle |G_{\alpha_2}\rangle |G_{\alpha_3}\rangle \), \( \text{Perm}_2 |G_{\alpha_1}\rangle |G_{\alpha_2}\rangle |G_{\alpha_3}\rangle = |G_{\alpha_1}\rangle |G_{\alpha_2}\rangle |G_{\alpha_3}\rangle \) and so on. Moreover, another property must hold for the set \( \{ \rho_i \}_{i = 1}^k \) given by \( \text{Tr}(\rho_i \rho_j) = \delta_{ij} \) since the support of \( \rho_i \) i.e., \( \text{supp}(\rho_i) \) is orthogonal to \( \text{supp}(\rho_j) \) for all \( (i, j) \). We examine the distinguishability of \( S'_i \)'s (equivalently, distinguishability of the \( \rho_i \)'s) when two out of the three qubits of each of the \( k \) \( |G_{\alpha}\rangle \)'s are in the same lab. Under such condition, it is possible to transform one permutation to any other by implementing
local unitaries on the two qubits of each of the $k$ $|G_\alpha\rangle$s. Therefore,

$$|G_{\alpha_1}'\rangle|G_{\alpha_2}'\rangle \ldots |G_{\alpha_k}'\rangle = (I \otimes U_{\alpha_1'\alpha_1})|G_{\alpha_1}\rangle (I \otimes U_{\alpha_2'\alpha_2})|G_{\alpha_2}\rangle \ldots (I \otimes U_{\alpha_k'\alpha_k})|G_{\alpha_k}\rangle.$$ 

Clearly, the set $\{|\Phi_i\rangle\}_{i=1}^\kappa \subset \mathbb{C}^{2^k} \otimes \mathbb{C}^{2^k}$ has the property: $|\Phi_i\rangle = (I \otimes U_{ij,\mu\nu})|\Phi_j\rangle$ for $i, j = 1, \ldots, \kappa$ and $\mu, \nu = 1, \ldots, k!$ where $U_{ij,\mu\nu}$s are 2$q$-qubit unitaries. So, if $\kappa > 2^{2k}$, $\{|\Phi_i\rangle\}_{i=1}^\kappa$ is not perfectly distinguishable [2].

On the other hand, if we assume that $\rho_i$s are distinguishable, then so are their supports. This implies that the set $\{|\Phi_i\rangle\}_{i=1}^\kappa$ is also distinguishable, as for each $i$, $|\Phi_i\rangle \in \text{Support}(\rho_i)$. But this leads to contradiction for $\kappa > 2^{2k}$. So, the $S'_i$s are locally indistinguishable in each bi-partition, provided $\binom{D}{k} > 2^{2k}$, that is, the sets $S'_i$s are genuinely unidentifiable.

**Appendix D: Proof of Theorem 5**

![Diagram](image)

**FIG. 3.** This diagram represents the possible transformation from the state $|\Omega_i\rangle \in S$ to the remaining states $\{|\Omega_{i\neq 1}\rangle\} \in S$ through a set of unitary operations $\{|V_{\alpha\gamma}\rangle = \sigma_\mu \otimes \sigma_\nu\}$, where $\mu, \nu = 0, 1, 2, 3$ that satisfies the condition $|\Omega_i\rangle = (I_4 \otimes V_{\alpha\gamma})|\Omega_i\rangle$, where $I_4$ is an identity operation on $\mathbb{C}^2 \otimes \mathbb{C}^2$. One can trivially find other possible set of unitaries $\{|W_{\alpha\beta}\rangle\}$, for which the transformation $|\Omega_i\rangle = (I_2 \otimes W_{\alpha\beta})|\Omega_\beta\rangle$ is satisfied.

**Theorem 5.** Consider a complete basis set $S$ (set of states given in equation 5) of genuinely entangled states in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. The set $S$ is locally $(D, k)$-unidentifiable if $\binom{D}{k} > 2^{2k}$ (where $D \in \{2, 16\}$ and $k \in \{1, D\}$) in all $1 : 3$ as well $2 : 2$ bi-partitions.

**Proof.** The basic approach for this proof will be similar as of Theorem 4 but with a more general picture. So let us write a complete set of pairwise orthogonal four party states as $S := \{|\Omega_\alpha\rangle\}_{\alpha=1}^{16}$, where

\begin{align*}
|\Omega_{1,2}\rangle &= \frac{1}{2}(|0000\rangle \pm |0111\rangle + |1010\rangle \pm |1101\rangle), \\
|\Omega_{3,4}\rangle &= \frac{1}{2}(|0000\rangle \pm |0111\rangle - |1010\rangle \mp |1101\rangle), \\
|\Omega_{5,6}\rangle &= \frac{1}{2}(|0001\rangle \pm |0110\rangle + |1011\rangle \pm |1100\rangle), \\
|\Omega_{7,8}\rangle &= \frac{1}{2}(|0001\rangle \pm |0110\rangle - |1011\rangle \mp |1100\rangle), \\
|\Omega_{9,10}\rangle &= \frac{1}{2}(|0010\rangle \pm |0101\rangle + |1000\rangle \pm |1111\rangle), \\
|\Omega_{11,12}\rangle &= \frac{1}{2}(|0010\rangle \pm |0101\rangle - |1000\rangle \mp |1111\rangle).
\end{align*}
\[ |\Omega_{11,12}\rangle = \frac{1}{2}(|0010\rangle \pm |0101\rangle - |1000\rangle \pm |1111\rangle), \]
\[ |\Omega_{13,14}\rangle = \frac{1}{2}(|0011\rangle \pm |0100\rangle + |1001\rangle \pm |1110\rangle), \]
\[ |\Omega_{15,16}\rangle = \frac{1}{2}(|0011\rangle \pm |0100\rangle - |1001\rangle \pm |1110\rangle). \]

At first we need to prove that \( S := \{|\Omega_\alpha\rangle\}_{\alpha=1}^{16} \) is locally indistinguishable. Thus we need to show that this set has the following properties:
\[ |\Omega_\alpha\rangle = (I \otimes W_{\alpha\beta}) |\Omega_\beta\rangle = (I \otimes I \otimes V_{\alpha\gamma}) |\Omega_\gamma\rangle, \quad \forall \alpha \neq \beta \neq \gamma \in \{1, 2, 3, \ldots, 16\} \]

where \( I \) implies identity operation on a single party, \( W_{\alpha\beta} \) implies a joint unitary acting on any three party or equivalently on any \( (1:3) \) bi-partition and \( V_{\alpha\gamma} \) implies a joint unitary action on any two party. Now we will approach in a similar way as the proof of Theorem 4.

Let \( S := \{|\Omega_\alpha\rangle\}_{\alpha=1}^D, 2 < D \leq 16 \). There are \( \kappa := \binom{D}{k} \) different ways in which \( k \) states can be chosen from \( S \) to form subsets that we label as \( S'_i, i = 1, \ldots, \kappa \). Thus we have to see whether we can locally distinguish the set of mixed states \( \{\rho_i\}_{i=1}^\kappa \) where
\[ \rho_i := \frac{1}{k!} \sum_{\mu=1}^{k!} |\Phi^i_\mu\rangle \langle \Phi^i_\mu|, \]
\[ |\Phi^i_\mu\rangle = \text{Perm}_\mu |\Omega_{\alpha_1}\rangle |\Omega_{\alpha_2}\rangle \ldots |\Omega_{\alpha_k}\rangle, \mu = 1, \ldots, k!, \]
\[ \alpha_1 \neq \alpha_2 \ldots \neq \alpha_k \in \{1, \ldots, D\}, \]
such that, \( |\Omega_{\alpha_1}\rangle, |\Omega_{\alpha_2}\rangle, \ldots, |\Omega_{\alpha_k}\rangle \in S'_i \).

Here, \( \text{Perm}_\mu |\Omega_{\alpha_1}\rangle |\Omega_{\alpha_2}\rangle \ldots |\Omega_{\alpha_k}\rangle \) means different possible permutations of the state \( |\Omega_{\alpha_1}\rangle |\Omega_{\alpha_2}\rangle \ldots |\Omega_{\alpha_k}\rangle \). We examine the distinguishability of \( S'_i \)’s (equivalently, distinguishability of the \( \rho_i \)'s) when either two \((2:2) \) bipartition or three \((1:3) \) bipartition out of the four qubits of each of the \( k \) elements from \( \{|\Omega_\alpha\rangle\}_{\alpha=1}^D \) are in the same lab. Under such condition, it is possible to transform one permutation, say \( \mu \) to any other, say \( \mu' \) by implementing local unitaries on the two qubits of each of the shared state \( |\Omega_\alpha\rangle \) as
\[ |\Omega'_{\alpha_1}\rangle |\Omega'_{\alpha_2}\rangle \ldots |\Omega'_{\alpha_k}\rangle \]
\[ = (I \otimes W'_{\alpha_1\alpha_1}) |\Omega_{\alpha_1}\rangle (I \otimes W'_{\alpha_2\alpha_2}) |\Omega_{\alpha_2}\rangle \ldots (I \otimes W'_{\alpha_k\alpha_k}) |\Omega_{\alpha_k}\rangle \]
\[ = (I \otimes I \otimes V_{\alpha_1\gamma_1}) |\Omega_{\gamma_1}\rangle (I \otimes I \otimes V_{\alpha_2\gamma_2}) |\Omega_{\gamma_2}\rangle \ldots (I \otimes I \otimes V_{\alpha_k\gamma_k}) |\Omega_{\gamma_k}\rangle \]

Clearly, the support vectors of \( \rho_i \) or the set \( \{\{\Phi^i_\mu\}\}_{i=1}^\kappa \) is a strict subset of \( \mathbb{C}^{2^k} \otimes \mathbb{C}^{2^k} \) as well as of \( \mathbb{C}^{2^k} \otimes \mathbb{C}^{2^k} \) and has the property:
\[ |\Phi^i_\mu\rangle = (I \otimes \bigcup_{ij,\mu\nu} U_{ij,\mu\nu}) |\Phi^i_{\delta}\rangle \quad \forall i, j = 1, \ldots, \kappa, \mu, \nu = 1, \ldots, k! \]
\[ = (I \otimes I \otimes \bigcup_{\delta_{i\delta}} W_{\delta_{i\delta}}) |\Phi^i_{\delta}\rangle \quad \forall i, l = 1, \ldots, \kappa, \mu, \delta = 1, \ldots, k! \]

where \( U_{ij,\mu\nu} \) are 3k-qubit unitaries and \( W_{\delta_{i\delta}} \) are 2k qubit unitaries. Thus the local dimension required for implementing the unitary \( U_{ij,\mu\nu} \) is \( 2^{3k} \) whereas for \( W_{\delta_{i\delta}} \), it is \( 2^{2k} \) and we know that \( 2^{3k} > 2^{2k} \) for any \( k > 0 \). Hence, if cardinality \( \kappa \) of the set \( \{\rho_i\}_{i=1}^\kappa \) satisfies the strict condition, \( \kappa > 2^{2k} \), then \( \{\{\Phi^i_\mu\}\}_{i=1}^\kappa \) is not perfectly distinguishable [2].

On the other hand, if we assume that \( \rho_i \)'s are distinguishable, then so are their supports. This implies that the set \( \{\Phi^i_\mu|\}_{i=1}^\kappa \) is also distinguishable, as for each \( i, |\Phi^i_\mu\rangle \in \text{Support}(\rho_i) \). But this leads to contradiction for \( \kappa > 2^{2k} \). So, the \( S'_i \)'s are locally unidentifiable in each bipartition, provided \( \binom{D}{k} > 2^{2k} \), that is, the sets \( S'_i \)'s are genuinely unidentifiable.

[1] M. Hayashi, D. Markham, M. Murao, M. Owari, and S. Virmani, Bounds on multipartite entangled orthogonal state discrimination using local operations and classical communication, Phys. Rev. Lett. 96, 040501 (2006).

[2] M. Nathanson, Distinguishing bipartite orthogonal states using locc: Best and worst cases, Journal of Mathematical Physics 46, 062103 (2005), https://doi.org/10.1063/1.1914731.