AN EVEN EXTREMAL LATTICE OF RANK 64

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Abstract. We construct an even extremal lattice of rank 64 by means of a generalized quadratic residue code.

1. Introduction

A lattice is a free Z-module L of finite rank with a symmetric bilinear form
\[ \langle , \rangle : L \times L \to \mathbb{Z} \]
that makes \( L \otimes \mathbb{R} \) a positive-definite real quadratic space. Let L be a lattice. The group of automorphisms of L is denoted by \( O(L) \). For simplicity, we write \( x^2 \) instead of \( \langle x, x \rangle \) for \( x \in L \). We say that L is even (or of type II) if \( x^2 \in 2\mathbb{Z} \) holds for all \( x \in L \). In this paper, we treat only even lattices. Since \( \langle , \rangle \) is non-degenerate, the mapping \( x \mapsto \langle x, -\rangle \) embeds \( L \) into the dual lattice
\[ L^\vee := \{ x \in L \otimes \mathbb{Q} \mid \langle x, y \rangle \in \mathbb{Z} \text{ for all } y \in L \} \].
We say that L is unimodular if this embedding is an isomorphism. We put
\[ \min(L) := \min \{ x^2 \mid x \in L \setminus \{0\} \} \].
It is well-known that, if \( L \) is an even unimodular lattice, then its rank \( n \) is divisible by 8 and \( \min(L) \) satisfies
\[ \min(L) \leq 2 + 2 \left\lfloor \frac{n}{24} \right\rfloor \].

Definition 1.1. We say that an even unimodular lattice L of rank \( n \) is extremal if the equality holds in (1.1).

Extremal lattices are important and interesting, because they give rise to dense sphere-packings. Extremal lattices of rank \( \leq 24 \) are completely classified. The famous Leech lattice is characterized as the unique (up to isomorphism) extremal lattice of rank 24. On the other hand, the classification of extremal lattices of rank \( \geq 32 \) seems to be very difficult. The known examples of extremal lattices are listed in the website [12] administrated by Nebe and Sloane, in Conway and Sloane [4, Chapter 1], or in Gaborit [5, Table 3].

As is extensively described in Conway and Sloane [4], there exist various methods of constructing a lattice from a code. The binary extended quadratic residue codes play an important role in these constructions. The most classical examples are that the extended Hamming code yields the extremal lattice \( E_8 \) of rank 8, and that the extended Golay code yields the Niemeier lattice of type 24A1. Various generalizations of quadratic residue codes are investigated. In particular, in Bonnecaze, Solé and Calderbank [1], the Leech lattice is constructed by a generalized

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quadratic residue code of length 24 with components in \( \mathbb{Z}/4\mathbb{Z} \). See also Chapman and Solé \([2]\) and Harada and Kitazume \([7]\).

In this paper, we consider a quadratic residue code with components in the discriminant group \( D_R := R'/R \) of an even lattice \( R \) of small rank, and construct a lattice \( L \) of large rank as an even overlattice of the orthogonal direct-sum of copies of \( R \) by using the code as the gluing data. As an application, we obtain the following:

**Theorem 1.2.** There exists an extremal lattice \( L_Q \) of rank 64 whose automorphism group \( O(L_Q) \) is of order 119040. This group \( O(L_Q) \) contains a subgroup \( \Gamma_Q \) of index 2 that fits in the exact sequence

\[
0 \rightarrow (\mathbb{Z}/2\mathbb{Z})^2 \rightarrow \Gamma_Q \rightarrow \text{PSL}_2(31) \rightarrow 1.
\]

The code \( Q \) that is used in the construction of \( L_Q \) is a generalized quadratic residue code of length 32 with components in the discriminant group \( D_R \approx \mathbb{Z}/35\mathbb{Z} \) of the lattice

\[
R = \begin{bmatrix} 6 & 1 \\ 2 & 6 \end{bmatrix}.
\]

In \([14]\), Quebbemann constructed (possibly several) extremal lattices of rank 64 as overlattices of the orthogonal direct-sum \( E_8(3)^8 \) of 8 copies of \( E_8(3) \). (See also \([4, \text{Chapter 8.3}]\).) Here \( E_8(3) \) denotes the lattice obtained from the lattice \( E_8 \) by multiplying the intersection form by 3. We have the following:

**Proposition 1.3.** The lattice \( L_Q \) does not contain \( E_8(3) \) as a sublattice.

**Corollary 1.4.** The lattice \( L_Q \) cannot be obtained by Quebbemann’s construction.

In \([11]\), Nebe discovered an extremal lattice

\( N_{64} := L_{8,2} \otimes L_{32,2} \)

of rank 64, and showed that \( O(N_{64}) \) contains a subgroup of order 587520 generated by 6 elements. (See the website \([12]\).) Since \( |O(L_Q)| < 587520 \), we obtain the following:

**Corollary 1.5.** The lattices \( L_Q \) and \( N_{64} \) are not isomorphic.

In Harada, Kitazume and Ozeki \([8]\) and Harada and Miezaki \([9]\), they also constructed several extremal lattices of rank 64. The relation of these lattices with our lattice has not yet been clarified.

We found the lattice \( L_Q \) by an experimental search. We hope that several more extremal lattices can be obtained by the same method.

This paper is organized as follows. In Section 2.1, we fix notions and notation about codes with components in a finite abelian group. In Section 2.2, we explain how to construct an even unimodular lattice from a code with components in the discriminant group \( D_R \) of an even lattice \( R \). In Section 3, we give the definition of a generalized quadratic residue code, and investigate its automorphisms. In Section 4, we construct the lattice \( L_Q \), and prove that \( L_Q \) is extremal and that \( O(L_Q) \) contains a subgroup \( \Gamma_Q \) of order 59520 that fits in the exact sequence \((1.2)\).

In particular, a brute-force method of the proof of \( \min(L_Q) = 6 \) is explained in detail. In Section 5, we calculate the set \( S \) of vectors of square-norm 6 in \( L_Q \). Using this set, we prove Proposition 1.3, and calculate the order of \( O(L_Q) \). In the last section, we give another construction of \( L_Q \).
The computational data obtained in this article is available from the author’s website [17]. In particular, the Gram matrix of $L_Q$ is found in [17]. A generating set of $O(L_Q)$ is available from in [12], though it is not minimal. For the computation, we used GAP [6].

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Conventions. The action of a group on a set is from the right, unless otherwise stated.

2. Preliminaries

2.1. Codes over a finite abelian group.

Definition 2.1. Let $A$ be a finite abelian group. A code of length $m$ over $A$ is a subgroup of $A^m$.

Let $G$ be a group. Then the symmetric group $S_m$ acts on $G^m$ by permutations of components. We denote by $G \wr S_m$ the wreath product $G^m \rtimes S_m$. Then we have a splitting exact sequence

\[ 1 \to G^m \to G \wr S_m \to S_m \to 1. \]

Suppose that $G$ acts on a set $X$. Since $S_m$ acts on $X^m$ by permutations of components and $G^m$ acts on $X^m$ by

\[(x_1, \ldots, x_m)^{(g_1, \ldots, g_m)} = (x_1^{g_1}, \ldots, x_m^{g_m}), \quad \text{where} \quad x_i \in X, \quad g_i \in G,\]

the group $G \wr S_m$ acts on $X^m$ in a natural way.

Let $H$ be a subgroup of the automorphism group $\text{Aut}(A)$ of a finite abelian group $A$. Then $H \wr S_m$ acts on $A^m$. For a code $C$ of length $m$ over $A$, we put

\[ \text{Aut}_H(C) := \{ g \in H \wr S_m \mid C^g = C \}. \]

2.2. Discriminant forms and overlattices. Let $R$ be an even lattice. We define the dual lattice of $R$ by

\[ R^\vee := \{ x \in R \otimes \mathbb{Q} \mid \langle x, v \rangle \in \mathbb{Z} \text{ for all } v \in R \}, \]

and the discriminant group $D_R$ of $R$ by

\[ D_R := R^\vee / R. \]

Note that $R^\vee$ has a natural $\mathbb{Q}$-valued symmetric bilinear form that extends the $\mathbb{Z}$-valued symmetric bilinear form of $R$. Hence $D_R$ is naturally equipped with a quadratic form

\[ q_R: D_R \to \mathbb{Q} / 2\mathbb{Z} \]

defined by $q_R(x \mod R) := x^2 \mod 2\mathbb{Z}$. We call $q_R$ the discriminant form of $R$. We denote by $O(q_R)$ the automorphism group of the finite quadratic form $(D_R, q_R)$. Then we have a natural homomorphism

\[ \eta_R: O(R) \to O(q_R). \]

Remark 2.2. The notion of discriminant forms was introduced by Nikulin [13] for the study of $K3$ surfaces, and it has been widely used in the investigation of $K3$ surfaces and Enriques surfaces. (See, for example, [16].)
The discriminant form of the orthogonal direct-sum $R^m$ of $m$ copies of $R$ is the orthogonal direct-sum $(D_R^m, q_R^m)$ of $m$ copies of $(D_R, q_R)$. Let $C$ be a code of length $m$ over $D_R$ that is totally isotropic with respect to the quadratic form

$$q_R^m: D_R^m \to \mathbb{Q}/2\mathbb{Z}.$$  

Then the pull-back

$$L_C := \text{pr}^{-1}(C)$$

of $C$ by the natural projection $\text{pr}: R^\vee m \to D_R^m$ with the restriction of the natural $\mathbb{Q}$-valued symmetric bilinear form of $R^\vee m$ is an even lattice that contains $R^m$ as a sublattice of finite index; that is, $L_C$ is an even overlattice of $R^m$. Moreover, since the index of $R^m$ in $L_C$ is equal to $|C|$, if $C$ satisfies

$$|C|^2 = |D_R|^m,$$

then $L_C$ is unimodular.

Let $C$ be a code of length $m$ over $D_R$ totally isotropic with respect to $q_R^m$. We put

$$H(R) := \text{Im}(\eta_R: \text{O}(R) \to \text{O}(q_R)) \subset \text{Aut}(D_R),$$

and consider the group $\text{Aut}_{H(R)}(C)$. Each element $g$ of $\text{Aut}_{H(R)}(C)$ is uniquely written as

$$g = \sigma \cdot (h_1, \ldots, h_m) \quad (\sigma \in \mathfrak{S}_m, \ h_i \in H(R)).$$

By the definition of $H(R)$, there exist elements $\tilde{h}_i \in \text{O}(R)$ such that $\eta_R(\tilde{h}_i) = h_i$ for $i = 1, \ldots, m$. Since $g$ preserves the code $C$, the action of

$$\tilde{g} := \sigma \cdot (\tilde{h}_1, \ldots, \tilde{h}_m) \in \text{O}(R) \wr \mathfrak{S}_m$$

on $(R^\vee)^m$ preserves the submodule $L_C \subset (R^\vee)^m$, and hence we obtain a lift $\tilde{g} \in \text{O}(L_C)$ of $g$. If $\eta_R$ is injective, then the lift $\tilde{g}$ of $g$ is unique. Therefore we have the following:

**Lemma 2.3.** Let $C$ and $L_C$ be as above. If the natural homomorphism $\eta_R$ is injective, then we have an injective homomorphism $\text{Aut}_{H(R)}(C) \hookrightarrow \text{O}(L_C)$.

### 3. Generalized quadratic residue codes

**3.1. Definition.** Let $A$ be a finite abelian group, and $p$ an odd prime. We consider the set of rational points

$$\mathbb{P}^1(\mathbb{F}_p) = \mathbb{F}_p \cup \{\infty\} = \{0, 1, \ldots, p-1, \infty\}$$

of the projective line over $\mathbb{F}_p$, and let $A^{p+1}$ denote the abelian group of all mappings

$$\mathbf{v}: \mathbb{P}^1(\mathbb{F}_p) \to A$$

from $\mathbb{P}^1(\mathbb{F}_p)$ of $A$. Let $\chi_p: \mathbb{F}_p^\times \to \{\pm 1\}$ denote the Legendre character of the multiplicative group $\mathbb{F}_p^\times := \mathbb{F}_p \setminus \{0\}$.

**Definition 3.1.** Let $a, b, d, s, t, e$ be elements of $A$. A generalized quadratic residue code of length $p+1$ over $A$ with parameter $(a, b, d, s, t, e)$ is the subgroup $Q$ of $A^{p+1}$ generated by the elements $\mathbf{v}_\infty, \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{p-1} \in A^{p+1}$ defined as follows:

$$\mathbf{v}_\infty(\nu) = \begin{cases} a & \text{if } \nu \in \mathbb{F}_p, \\ b & \text{if } \nu = \infty, \end{cases}$$

$$\mathbf{v}_0 = \begin{cases} a & \text{if } \nu = 0, \\ b & \text{if } \nu \neq 0, \end{cases}$$

$$\mathbf{v}_i = \mathbf{v}(\nu)$$

for $i = 1, \ldots, p-1$. The group $Q$ contains $\mathbf{v}_\infty$ and $\mathbf{v}_0$, and is invariant under the action of $\mathbf{v}_\infty(\nu)$ on $Q$. We say that $Q$ is $r$-homogeneous if $\mathbf{v}_\infty(\nu)^r = \mathbf{v}_\infty(\nu)$ for all $\nu \in \mathbb{F}_p$. The group $Q$ is the generalized quadratic residue code of length $p+1$ over $A$ with parameter $(a, b, d, s, t, e)$ if it is $r$-homogeneous for some $r$.
and, for \( \mu \in \mathbb{F}_p \),
\[
\nu_\mu(\nu) = \begin{cases} 
  d & \text{if } \nu = \mu, \\
  s & \text{if } \nu \in \mathbb{F}_p \setminus \{\mu\} \text{ and } \chi_p(\mu - \nu) = 1, \\
  t & \text{if } \nu \in \mathbb{F}_p \setminus \{\mu\} \text{ and } \chi_p(\mu - \nu) = -1, \\
  e & \text{if } \nu = \infty.
\end{cases}
\]

3.2. Automorphisms of a generalized quadratic residue code. Let \( A \) and \( p \) be as above. For simplicity, we put
\[
\mathcal{S} := \mathcal{S}(\mathbb{P}^1(\mathbb{F}_p)) \cong \mathcal{S}_{p+1}.
\]
The linear fractional transformation embeds \( \text{PSL}_2(p) \) into \( \mathcal{S} \). Let \( \alpha \) be a generator of \( \mathbb{F}_p^\times \). Then \( \text{PSL}_2(p) \) is generated by the three elements
\[
\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix},
\]
which correspond respectively to the permutations of \( \mathbb{P}^1(\mathbb{F}_p) \) defined as follows:
\[
\xi: \nu \mapsto -1/\nu, \quad \eta: \nu \mapsto \nu + 1, \quad \zeta: \nu \mapsto \alpha^2 \nu,
\]
with the understanding that \(-1/0 = \infty, -1/\infty = 0, \infty + 1 = \infty, \) and \( \alpha^2 \infty = \infty \).

Let \( Q \subset A^p+1 \) be a generalized quadratic residue code of length \( p+1 \) over \( A \), and \( H \) a subgroup of \( \text{Aut}(A) \). Let \( f_Q \) be the composite homomorphism of the natural inclusion \( \text{Aut}_H(Q) \hookrightarrow H \wr \mathcal{S} \) and the surjection \( H \wr \mathcal{S} \twoheadrightarrow \mathcal{S} \) in \( (2.1) \):
\[
f_Q: \text{Aut}_H(Q) \hookrightarrow H \wr \mathcal{S} \twoheadrightarrow \mathcal{S}.
\]

**Lemma 3.2.** The image of \( f_Q \) contains \( \eta \) and \( \zeta \).

**Proof.** The permutation of components given by \( \eta \) (resp. by \( \zeta \)) preserves the generating set \( \{v_\infty, v_0, \ldots, v_{p-1}\} \) of \( Q \). \( \square \)

4. An extremal lattice \( L_Q \) of rank 64

We construct an extremal lattice \( L_Q \) of rank 64. Let \( R \) be the lattice of rank 2 with a basis \( e_1, e_2 \) such that the Gram matrix of \( R \) with respect to \( e_1, e_2 \) is
\[
\begin{pmatrix} e_1 \cdot e_1 & e_1 \cdot e_2 \\ e_2 \cdot e_1 & e_2 \cdot e_2 \end{pmatrix} = \begin{bmatrix} 6 & 1 \\ 1 & 6 \end{bmatrix}.
\]

Let \( e_1^\vee, e_2^\vee \) be the basis of \( R^\vee \) dual to \( e_1, e_2 \). Then \( D_R \) is a cyclic group of order 35 generated by
\[
u := 6e_1^\vee + 2e_2^\vee = \frac{1}{35}(34e_1 + 6e_2).
\]
For simplicity, we denote by \( n \in \mathbb{Z}/35\mathbb{Z} \) the element
\[
n(6e_1^\vee + 2e_2^\vee) \in D_R.
\]
Then the discriminant form \( q_R: D_R \to \mathbb{Q}/2\mathbb{Z} \) is given by
\[
q_R(n) = 6n^2/35 \equiv 6 \mod 2\mathbb{Z}.
\]
We have
\[
O(q_R) = \{ k \in (\mathbb{Z}/35\mathbb{Z})^\times \mid 6k^2 \equiv 6 \mod 70 \} = \{\pm 1, \pm 6\}.\]
Table 4.1. The matrix $B$  

\[
\begin{bmatrix}
32 & 30 & 15 & 11 & 7 & 29 & 19 & 10 & 26 & 11 & 31 & 33 & 28 & 22 & 22 & 12 \\
16 & 13 & 23 & 21 & 19 & 30 & 25 & 3 & 11 & 21 & 31 & 32 & 12 & 9 & 9 & 4 \\
34 & 6 & 30 & 22 & 22 & 19 & 20 & 32 & 17 & 30 & 30 & 24 & 10 & 33 & 0 & 20 \\
34 & 26 & 1 & 17 & 9 & 6 & 4 & 17 & 14 & 12 & 25 & 19 & 34 & 8 & 33 & 20 \\
34 & 26 & 21 & 23 & 4 & 28 & 26 & 1 & 34 & 9 & 7 & 14 & 29 & 32 & 8 & 18 \\
34 & 24 & 21 & 8 & 10 & 23 & 13 & 23 & 18 & 29 & 4 & 31 & 24 & 27 & 32 & 28 \\
34 & 34 & 19 & 8 & 30 & 29 & 8 & 10 & 5 & 13 & 24 & 28 & 6 & 22 & 27 & 17 \\
34 & 23 & 29 & 6 & 30 & 14 & 14 & 5 & 27 & 0 & 8 & 13 & 3 & 4 & 22 & 12 \\
34 & 18 & 18 & 16 & 28 & 14 & 34 & 11 & 22 & 22 & 30 & 32 & 23 & 1 & 4 & 7 \\
34 & 13 & 13 & 5 & 3 & 12 & 34 & 31 & 28 & 17 & 17 & 19 & 7 & 21 & 1 & 24 \\
34 & 30 & 8 & 0 & 27 & 22 & 32 & 31 & 13 & 23 & 12 & 6 & 29 & 5 & 21 & 21 \\
34 & 27 & 25 & 30 & 22 & 11 & 7 & 29 & 13 & 8 & 18 & 1 & 16 & 27 & 5 & 6 \\
34 & 12 & 22 & 12 & 17 & 6 & 31 & 4 & 11 & 8 & 3 & 7 & 11 & 14 & 27 & 25 \\
34 & 31 & 7 & 9 & 34 & 1 & 26 & 28 & 21 & 6 & 3 & 27 & 17 & 9 & 14 & 12 \\
34 & 18 & 26 & 29 & 31 & 18 & 21 & 23 & 10 & 16 & 1 & 27 & 2 & 15 & 9 & 34 \\
34 & 5 & 13 & 13 & 16 & 15 & 3 & 18 & 5 & 5 & 11 & 25 & 2 & 0 & 15 & 29 \\
\end{bmatrix}
\]

On the other hand, the group $O(R)$ is of order 4 and is generated by 

$g_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad g_2 := \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$

The natural homomorphism $\eta_R: O(R) \to O(q_R)$ maps $g_1$ to $-6$ and $g_2$ to $-1$. Hence $\eta_R$ is an isomorphism. In particular, the image $H(R)$ of $\eta_R$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Hence we obtain an even unimodular overlattice $L_Q = \text{pr}^{-1}(Q)$ of $R^{32}$ by (2.2).

We investigate the generalized quadratic residue code $Q$ of length 32 over $D_R$ with parameter 

$(a, b, d, s, t, e) = (0, 0, 1, 7, 3, 2).$

Note that $F_{31}^\times$ is generated by 3. We arrange the elements of $\mathbb{P}^1(F_{31})$ as 

$\begin{bmatrix} \infty, 0, 1, 3^2, 3^4, \ldots, 3^{28}, 3, 3^3, 3^5, \ldots, 3^{29} \end{bmatrix},$

and write elements of $D_R^{32}$, $H(R)^{32}$, and $(R \otimes \mathbb{Q})^{32}$ as row vectors according this arrangement.

**Proposition 4.1.** The code $Q$ is totally isotropic with respect to $q_R^{32}$, and satisfies $|Q| = 35^{16}$.

**Proof.** The code $Q$ is generated by the row vectors of the matrix $[I_{16}|B]$, where $I_{16}$ is the identity matrix of size 16, and $B$ is the $16 \times 16$ matrix in Table 4.1 (The components of $B$ are in $D_R = \mathbb{Z}/35\mathbb{Z}$.) It is easy to confirm that $Q$ is totally isotropic with respect to $q_R^{32}$, and that $|Q| = 35^{16}$ holds. \hfill \Box

Hence we obtain an even unimodular overlattice $L_Q = \text{pr}^{-1}(Q)$ of $R^{32}$ by (2.2).

We will show that $L_Q$ is extremal, and that $O(L_Q)$ contains a subgroup $\Gamma_Q$ with the properties stated in Theorem 1.2.

**Proposition 4.2.** The kernel of the homomorphism $f_Q: \text{Aut}(H(R)^{32}) \to \mathcal{S}$ is equal to the image of the diagonal homomorphism $\delta: H(R) \to H(R)^{32}$. The image of $f_Q$ contains the permutation $\xi \in \mathcal{S}$.
Proof. Let \( \sigma \) be an element of \( G \). Let \( Z \) be the \( 32 \times 32 \) matrix
\[
\begin{pmatrix}
I_{16} & B \\
0 & 35 I_{16}
\end{pmatrix},
\]
where \( B \) is regarded as a matrix with components, not in \( \mathbb{Z}/35\mathbb{Z} \), but in \( \mathbb{Z} \). Let \( Z^\sigma \) be the matrix obtained by applying the permutation \( \sigma \) of components to the row vectors of \( Z \). For \( x = (x_1, \ldots, x_{32}) \in H(R)^{32} \subset H(R) \rtimes S \), let \( \Delta(x) \) denote the diagonal matrix with components being the representatives in \( \mathbb{Z} \) of \( x_1, \ldots, x_{32} \). Then we have \( Q^\sigma z = Q \) only when
\[
Z^\sigma \cdot \Delta(x) \cdot Z^* \equiv O \mod 35.
\] We can calculate the set
\[
\Lambda(\sigma) := \{ \gamma \in H(R)^{32} \mid Q^{\sigma\gamma} = Q \}
\]
by solving the congruence linear equation (4.3) with unknowns \( x_1, \ldots, x_{32} \). By this method, we obtain \( \Lambda(\text{id}) = \delta(H(R)) \). On the other hand, we have \( \Lambda(\xi) \neq \emptyset \). Indeed, we see that \( \Lambda(\xi) \) contains the element
\[
(1, -1, -6, \ldots, -6, 6, \ldots, 6) \in H(R)^{32} \quad (15 \text{ times of } -6 \text{ and } 15 \text{ times of } 6).
\]
Hence \( \text{Im } f_Q \) contains \( \xi \). □

Combining Proposition 4.2 with Lemma 3.2, we see that the image of \( f_Q \) includes the subgroup \( \text{PSL}_2(31) \subset G \). We put
\[
\Gamma_Q := f_Q^{-1}(\text{PSL}_2(31)).
\]
By Lemma 2.3 we have a natural embedding \( \Gamma_Q \hookrightarrow O(L_Q) \) of the subgroup \( \Gamma_Q \) of \( \text{Aut}_{H(R)}(Q) \) into \( O(L_Q) \). Let \( \Gamma_Q \) be the image of this embedding. Then \( \Gamma_Q \) satisfies the exact sequence (1.2) in Theorem 1.2. In particular, \( \Gamma_Q \) is of order 59520.

Proposition 4.3. We have \( \min(L_Q) = 6 \).

Proof. It is easy to calculate a basis of \( L_Q \) and the associated Gram matrix. Therefore the minimal norm \( \min(L_Q) \) can be calculated by, for example, the function \texttt{ShortestVectors} of \textsc{GAP} [6]. However, this method did not give an answer in reasonable time. Hence we adopt the following method.

For \( n \in D_R = \mathbb{Z}/35\mathbb{Z} \), we put
\[
\lambda(n) := \min \{ \ x^2 \mid x \in R^\vee, \ x \mod R = n \}.
\]
Then the values of \( \lambda(n) \) are calculated as in Table 4.2. For a codeword
with the following properties:

\[ w = [ n_{\infty}, n_0, n_1, n_3^2, \ldots, n_{328}, n_3, n_3^3, n_3^5, \ldots, n_{329} ] \in (\mathbb{Z}/35\mathbb{Z})^{32}, \]

we put

\[ \mu(w) := \lambda(n_\infty) + \lambda(n_0) + \sum_{k=0}^{14} \lambda(n_{32k}) + \sum_{k=0}^{14} \lambda(n_{32k+1}). \]

In order to prove Proposition 4.3, it is enough to show that there exists no non-zero codeword \( w \) in \( Q \) with \( \mu(w) \leq 4 \).

We introduce an ordering \( < \) on \( \mathbb{Z}/35\mathbb{Z} \) by

\[ m < m' \iff \tilde{m} < \tilde{m}', \]

where \( \tilde{m} \in \mathbb{Z} \) is the representative of \( m \in \mathbb{Z}/35\mathbb{Z} \) satisfying \( 0 \leq \tilde{m} < 35 \). For \( n \in \mathbb{Z}/35\mathbb{Z} \), we denote by \( \text{Stab}(n) \) the stabilizer subgroup of \( n \) in \( H(R) = \{ \pm 1, \pm 6 \} \subset (\mathbb{Z}/35\mathbb{Z})^{\times} \). Then, for each codeword \( w \) of \( Q \), the orbit

\[ w^\Gamma_Q := \{ w^n \mid \gamma \in \Gamma_Q \} \]

of \( w \) under the action of \( \Gamma_Q \) contains at least one element \( [ n_{\infty}, n_0, n_1, n_9, \ldots, n_3, n_{27}, \ldots ] \)

with the following properties:

(i) \( \lambda(n_{\infty}) \geq \lambda(n_\nu) \) for any \( \nu \in \mathbb{F}_p \),

(ii) \( n_{\infty} \geq kn_{\infty} \) for any \( k \in H(R) = \{ \pm 1, \pm 6 \} \),

(iii) \( \lambda(n_0) \geq \lambda(n_\nu) \) for any \( \nu \in \mathbb{F}_p^{\times} \),

(iv) \( n_0 \geq kn_0 \) for any \( k \in \text{Stab}(n_\infty) \),

(v) \( \lambda(n_1) \geq \lambda(n_{32k}) \) for \( k = 1, \ldots, 14 \), and if \( \lambda(n_1) = \lambda(n_{32k}) \), then \( n_1 \geq n_{32k} \),

(vi) \( n_1 \geq kn_1 \) for any \( k \in \text{Stab}(n_{\infty}) \cap \text{Stab}(n_0) \).

By backtrack searching, we look for a non-zero codeword satisfying \( \mu(w) \leq 4 \) and the properties (i)-(vi), and confirm that there exist no such codewords in \( Q \). (The arrangement \( L_2 \) of the points of \( \mathbb{F}^3(\mathbb{F}_p) \) is convenient for this backtrack searching.)

This task was carried out by distributed computation on eight CPUs of 3 GHz. It took us about 75 days.

Thus Theorem 1.2 is proved, except for the fact that \( \Gamma_Q \) is of index 2 in \( O(L_Q) \).

5. Short vectors of \( L_Q \)

In this section, we prove Proposition 1.3 and complete the proof of Theorem 1.2 by showing \( |O(L_Q)| = 119040 \).

By the theory of modular forms (see, for example, [15 Chapter 7]), we see that the theta function of \( L_Q \) is equal to

\[ \sum_{v \in L_Q} q^{v^2/2} = 1 + 2611200q^3 + 19525860480q^4 + 19715393260800q^5 + \cdots. \]

In particular, the size of the set \( S \) of vectors \( v \in L_Q \) of square-norm \( v^2 = 6 \) is 2611200. We calculate the set \( S \) and its orbit decomposition by \( \Gamma_Q \) by the following random search method. The result is given in Table 5.1 and presented more explicitly in [17].

**Random search method.** Let \( G \) be the Gram matrix of \( L_Q \). We set

\[ S = \{ \}, \quad O = \{ \}. \]
While $|S| \leq 2611200$, we do the following calculation. Let $U \in \text{GL}_{64}(\mathbb{Z})$ be a random unimodular matrix of size 64 with integer components. We apply the LLL algorithm by Lenstra, Lenstra and Lovász [10] (see also [3, Chapter 2]) to
\[
U G := U \cdot G \cdot T U
\]
with the sensitivity parameter 1. Suppose that we find a vector $v' \in \mathbb{Z}^{64}$ such that $v' \cdot U G \cdot T v' = 6$. Then $v := v' \cdot U$ is a vector of square-norm 6 in $L_\mathcal{Q}$. If $v$ is not yet in $S$, then we append its orbit $o := \{v^\gamma \mid \gamma \in \Gamma_\mathcal{Q}\}$ to $S$, and add the set $o$ to $O$. When $|S|$ reaches 2611200, the set $S$ is equal to the set of vectors in $L_\mathcal{Q}$ of square-norm 6 and $O$ gives the orbit decomposition of $S$ by $\Gamma_\mathcal{Q}$.

The set $S$ is decomposed into 56 orbits by $\Gamma_\mathcal{Q}$. We choose an element $v^{(i)}$ from each orbit $o_i$ for $i = 1, \ldots, 56$. Let $\varepsilon_1, \ldots, \varepsilon_8$ be the standard basis of $E_8(3)$. We have $\varepsilon_2^{\otimes i} = 6$ for $i = 1, \ldots, 8$. If $L_\mathcal{Q}$ contained a sublattice isomorphic to $E_8(3)$, then there would exist an embedding $\iota: \{\varepsilon_1, \ldots, \varepsilon_8\} \hookrightarrow S$ that preserves the intersection form. By the action of $\Gamma_\mathcal{Q}$, we can assume that $\iota(\varepsilon_1)$ is equal to the representative element $v^{(i)}$ of some orbit $o_i$. By backtrack searching, we confirm that there exists no such embedding $\iota$. Thus Proposition 1.3 is proved.

For $v \in S$, we define its type $\tau(v)$ by
\[
\tau(v) := [t_0(v), t_1(v), t_2(v), t_3(v), t_6(v)],
\]
where $t_m(v)$ is the size of the set
\[
\{ x \in S \mid \langle x, v \rangle = m \}.
\]
Then we have $t_0(v) = 1$ and
\[
t_0(v) + 2(t_1(v) + t_2(v) + t_3(v) + t_6(v)) = 2611200
\]
for any $v \in S$. The set $S$ is decomposed into the disjoint union
\[
S = \bigsqcup \mathcal{S}_\tau, \quad \text{where } \mathcal{S}_\tau := \{ v \in S \mid \tau(v) = \tau \},
\]
according to the types, and each $\mathcal{S}_\tau$ is a disjoint union of orbits $o_i$ of the action of $\Gamma_\mathcal{Q}$. In Table 5.2 we give the list of all possible types $\tau$ and the size of each set $\mathcal{S}_\tau$. Note that the action of $O(L_\mathcal{Q})$ preserves each $\mathcal{S}_\tau$. Let $\mathcal{S}_0$ be the set of vectors of type
\[
[1377392, 578256, 38343, 304, 1].
\]
The size 23808 of $\mathcal{S}_0$ is minimal among all $\mathcal{S}_\tau$. (See the last line of Table 5.2.) This subset $\mathcal{S}_0$ is a union of two orbits $o_{k_1}$ and $o_{k_2}$ of size 11904. By direct calculation, we confirm the following fact:
\[
\text{(5.1) For each } v \in \mathcal{S}_0, \text{ there exist exactly seven vectors } v' \in \mathcal{S}_0 \text{ such that } \langle v, v' \rangle = -3.\]

We find a sequence $V_0 = [v_1, \ldots, v_{64}]$ of vectors $v_i$ of $\mathcal{S}_0$ satisfying the following:
Table 5.2. Decomposition of $S$ by types

| $t_0$  | $t_1$  | $t_2$  | $t_3$  | $t_6$  | the size of $S_r$ |
|--------|--------|--------|--------|--------|--------------------|
| 1368552 | 583866 | 37323  | 134    | 1      | 39680              |
| 1370112 | 582876 | 37503  | 164    | 1      | 39680              |
| 1371152 | 582216 | 37623  | 184    | 1      | 119040             |
| 1371880 | 581754 | 37707  | 198    | 1      | 59520              |
| 1372088 | 581556 | 37743  | 204    | 1      | 119040             |
| 1372192 | 581292 | 37791  | 212    | 1      | 119040             |
| 1372504 | 581358 | 37779  | 210    | 1      | 119040             |
| 1372608 | 581292 | 37791  | 212    | 1      | 119040             |
| 1372816 | 581160 | 37815  | 216    | 1      | 59520              |
| 1372920 | 581094 | 37827  | 218    | 1      | 158720             |
| 1373128 | 580962 | 37851  | 222    | 1      | 59520              |
| 1373232 | 580896 | 37863  | 224    | 1      | 59520              |
| 1373440 | 580764 | 37887  | 228    | 1      | 59520              |
| 1373648 | 580632 | 37911  | 232    | 1      | 119040             |
| 1373752 | 580566 | 37923  | 234    | 1      | 119040             |
| 1373960 | 580434 | 37947  | 238    | 1      | 59520              |
| 1374168 | 580302 | 37971  | 242    | 1      | 119040             |
| 1374272 | 580236 | 37983  | 244    | 1      | 59520              |
| 1374480 | 580104 | 38007  | 248    | 1      | 75648              |
| 1374584 | 580038 | 38019  | 250    | 1      | 59520              |
| 1374688 | 579972 | 38031  | 252    | 1      | 59520              |
| 1374896 | 579840 | 38055  | 256    | 1      | 178560             |
| 1375000 | 579774 | 38067  | 258    | 1      | 59520              |
| 1375104 | 579708 | 38079  | 260    | 1      | 119040             |
| 1376872 | 578586 | 38283  | 294    | 1      | 71424              |
| 1377392 | 578256 | 38343  | 304    | 1      | 23808              |

(i) $\langle v_i, v_j \rangle = -3$ if and only if $|i - j| = 1$, and
(ii) $v_1, \ldots, v_{64}$ form a basis of $L_Q$.

See [17] for the explicit vector representations of these vectors $v_1, \ldots, v_{64}$. We then enumerate all the sequences $V' = [v'_1, v'_2, \ldots, v'_{64}]$ of vectors of $S_0$ such that

(a) $v'_1$ is either $v^{(k_1)}$ or $v^{(k_2)}$, where $v^{(k_\nu)}$ is the fixed representative of the orbit $o_{k_\nu}$ contained in $S_0$, and

(b) $\langle v_i, v_j \rangle = \langle v'_i, v'_j \rangle$ for $i, j = 1, \ldots, 64$.

Then we obtain exactly 10 sequences $V_1, \ldots, V_{10}$ with these properties. Since the action of $O(L_Q)$ preserves $S_0 = o_{k_1} \cup o_{k_2}$ and the action of $\Gamma_Q$ is transitive on each of $o_{k_1}$ and $o_{k_2}$, we see that, for each $g \in O(L_Q)$, there exists an element $h \in \Gamma_Q$ such that

$$V_0^{gh} = [v_1^{gh}, v_{64}^{gh}] \in \{V_1, \ldots, V_{10}\}.$$ 

For each $i = 1, \ldots, 10$, we calculate the matrix $g_i \in O(L_Q \otimes \mathbb{Q})$ such that $V_0^{g_i} = V_i$. It turns out that these $g_i$ preserve $L_Q \subset L_Q \otimes \mathbb{Q}$, and hence we have $g_i \in O(L_Q)$. By construction, the group $O(L_Q)$ is generated by $\Gamma_Q$ together with $g_1, \ldots, g_{10}$. 
We calculate the order of $|O(L_Q)|$. It turns out that
\[ |O(L_Q)| = 119040 = 2|\Gamma_Q|. \]
Thus the proof of Theorem 5.2 is completed.

**Remark 5.1.** If $g \in O(L_Q)$ is not contained in $\Gamma_Q$, then $g$ does not preserve the sublattice $R^{32} \subset L_Q$, and hence does not induce an automorphism of the code $Q$.

### 6. Another construction of $L_Q$

Let $o_1$ and $o_2$ be the two orbits of size 128 in $S$ (see Table 5.1). Let $\langle o_1 \rangle$ and $\langle o_2 \rangle$ be the sublattices of $L_Q$ generated by $o_1$ and by $o_2$, respectively. It is easily confirmed that both $\langle o_1 \rangle$ and $\langle o_2 \rangle$ are of rank 64 and that
\[ (o_1) + (o_2) = L_Q. \]
For simplicity, we put
\[ E := \{ e_1^{(1)}, e_2^{(1)}, \ldots, e_1^{(32)}, e_2^{(32)} \}, \]
where $e_1^{(i)}$ and $e_2^{(i)}$ are the standard basis of the $i$th component of $R^{32}$ satisfying (1.1). One of the two orbits of size 128, say $o_1$, is equal to the union of $E$ and $-E$. For each $e_1^{(i)} \in E \subset o_1$, there exists a unique vector $f_1^{(i)} \in o_2$ (resp. $f_{-1}^{(i)} \in o_2$) such that $\langle e_1^{(i)}, f_1^{(i)} \rangle = 2$ (resp. $\langle e_1^{(i)}, f_{-1}^{(i)} \rangle = -2$). The mapping
\[ e_1^{(i)} \mapsto f_1^{(i)}, \quad e_2^{(i)} \mapsto f_{-1}^{(i)} \]
induces an isometry
\[ \rho : \langle o_1 \rangle \overset{\sim}{\rightarrow} \langle o_2 \rangle, \]
and hence gives rise to $\rho \otimes Q \in O(R^{32} \otimes Q)$.

**Remark 6.1.** The orthogonal transformation $\rho \otimes Q$ of $R^{32} \otimes Q$ does not preserve $L_Q \subset R^{32} \otimes Q$. Indeed, the order of $\rho \otimes Q \in O(R^{32} \otimes Q)$ is infinite.

The matrix representation $M_{\rho}$ of $\rho \otimes Q$ with respect to the basis $E$ of $R^{32} \otimes Q$ is related to generalized quadratic residue codes as follows. Let $T$ be the $32 \times 32$ matrix whose rows and columns are indexed by $F^1(F_{31})$ sorted as in (1.2), and whose $(\mu, \nu)$th component is the string
\[ \begin{cases} 
"a" & \text{if } \mu = \infty \text{ and } \nu \neq \infty, \\
"b" & \text{if } \mu = \infty \text{ and } \nu = \infty, \\
"d" & \text{if } \mu = \nu \neq \infty, \\
"s" & \text{if } \mu \neq \infty, \nu \neq \infty, \mu \neq \nu, \text{ and } \chi_{31}(\mu - \nu) = 1, \\
"t" & \text{if } \mu \neq \infty, \nu \neq \infty, \mu \neq \nu, \text{ and } \chi_{31}(\mu - \nu) = -1, \\
"e" & \text{if } \mu \neq \infty \text{ and } \nu = \infty; 
\end{cases} \]
that is, $T$ is the template matrix of quadratic residue codes of length 32. We put
\[ m_a := \frac{1}{35} \begin{bmatrix} 1 & -6 \\ 6 & -1 \end{bmatrix}, \quad m_b := \frac{1}{35} \begin{bmatrix} 12 & -2 \\ 2 & -12 \end{bmatrix}, \quad m_d := \frac{1}{35} \begin{bmatrix} 12 & -2 \\ 2 & -12 \end{bmatrix}, \]
\[ m_s := \frac{1}{35} \begin{bmatrix} -6 & 1 \\ -1 & 6 \end{bmatrix}, \quad m_t := \frac{1}{35} \begin{bmatrix} 6 & -1 \\ 1 & -6 \end{bmatrix}, \quad m_e := \frac{1}{35} \begin{bmatrix} -1 & 6 \\ -6 & 1 \end{bmatrix}. \]
Proposition 6.2. The matrix representation $M_\rho$ of $\rho \otimes \mathbb{Q}$ with respect to the basis $E$ of $\mathbb{R}^{32} \otimes \mathbb{Q}$ is obtained from the template matrix $T$ by substituting "a" with $m_a$, "b" with $m_b$, "d" with $m_d$, "s" with $m_s$, "t" with $m_t$, and "e" with $m_e$.

By (6.1), we obtain another method of construction of $L_\mathbb{Q}$ as follows.

Proposition 6.3. The lattice $L_\mathbb{Q}$ is generated by $E$ and $E_\rho$ in $\mathbb{R}^{32} \otimes \mathbb{Q}$.

Note added on 2018/05/04: Masaaki Harada confirmed $\min(L_\mathbb{Q}) = 6$ by a direct computation using Magma. It took about 27 days. We thank Professor Masaaki Harada for this heavy computation.

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