Supersymmetric and Deformed Harry Dym Hierarchies

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Abstract

In this talk, we describe our recent results [1, 2] on the supersymmetric Harry Dym hierarchy as well as a newly constructed deformed Harry Dym hierarchy which is integrable with two arbitrary parameters. In various limits of these parameters, the deformed hierarchy reduces to various known integrable systems.
1 Introduction:

Supersymmetric extensions of a number of well known bosonic integrable models have been studied extensively in the past. The supersymmetric Korteweg-de Vries (sKdV) equation \[3\], the supersymmetric nonlinear Schrödinger (sNLS) equation \[4\] and the supersymmetric Two-Boson (sTB) equation \[5\] represent just a few in this category. A simple supersymmetric covariantization of bosonic integrable models, conventionally known as the B supersymmetrization (susy-B), has also attracted a lot of interest because of the appearance of such models in string theories. We have, for instance, the B extensions of the KdV (sKdV-B) equation \[6\], the supersymmetric TB (sTB-B) equation \[7\] and so on. Supersymmetric extensions of integrable models using a number \(N\) of Grassmann variables greater than one \[8\] and supersymmetric construction of dispersionless integrable models \[9\] have also been studied extensively in the past few years. The extended supersymmetric models are particularly interesting because, in the bosonic limit, they yield new classical integrable systems.

A classic bosonic integrable equation, the so called Harry Dym (HD) equation \[10\], has attracted much interest recently. The proprieties of this equation are discussed in detail in Ref. \[11\], and we simply emphasize that this equation shares the properties typical of solitonic equations, namely, it can be solved by the inverse scattering transform, it has a bi-Hamiltonian structure and infinitely many symmetries. In fact, the HD equation is one of the most exotic solitonic equations and the hierarchy to which it belongs, has a very rich structure \[12\]. In this hierarchy we also have nonlocal integrable equations such as the Hunter-Zheng (HZ) equation \[13\], which arises in the study of massive nematic liquid crystals as well as in the study of shallow water waves. The HD equation, on the other hand, is relevant in the study of the Saffman-Taylor problem which describes the motion of a two-dimensional interface between a viscous and a non-viscous fluid \[14\].

In this talk we will describe systematically the question of supersymmetrization of the HD hierarchy as well as a deformation of this hierarchy by two parameters that still is integrable. The talk is organized as follows. We will first review some of the essential features of the HD equation and its hierarchy. Then, we will briefly describe the new deformed model that is integrable. The simpler susy-B extension (sHD-B) of the HD hierarchy as well its bi-Hamiltonian formulation and Lax pairs will be discussed next. We will then derive the \(N=1\) supersymmetric extensions of the HD (sHD) equation. We will show that, in this case, there exist two nontrivial \(N=1\) supersymmetrizations. In the case of one of them, we have a bi-Hamiltonian description (we have not found a Lax representation yet) while in the second case, we have a Lax description (we have not found a Hamiltonian structure yet that satisfies the Jacobi identity). We also describe the supersymmetric extension for the HZ equation. Continuing on, we will describe the \(N=2\) supersymmetrization of the HD hierarchy which yields four possibilities and we discuss their properties.
2 The Harry Dym Hierarchy:

The Harry Dym equation

\[ w_t = \left( w^{-1/2} \right)_{xxx} , \tag{1} \]

appears in many disguised forms, namely,

\begin{align*}
  v_t &= \frac{1}{4} v^3 v_{xxx} , \\
  u_t &= \frac{1}{4} u^{3/2} u_{xxx} - \frac{3}{8} u^{1/2} u_x u_{xx} + \frac{3}{16} u^{-1/2} u_x^3 , \\
  r_t &= \left( r_{xx}^{-1/2} \right)_x ,
\end{align*} \tag{2}

where \( v = -2^{1/3} w^{-1/2} \), \( u = v^2 \) and \( r_{xx} = w \), respectively. In this paper, as in [12], we will confine ourselves, as much as is possible, to the form of the HD equation as given in (1).

The HD equation is a member of the bi-Hamiltonian hierarchy of equations given by

\[ w_t^{(n+1)} = D_1 \frac{\delta H_{n+1}}{\delta w} = D_2 \frac{\delta H_n}{\delta w} , \tag{3} \]

for \( n = -2 \), where the bi-Hamiltonian structures are

\[ D_1 = \partial^3, \quad D_2 = w \partial + \partial w , \tag{4} \]

and the Hamiltonians for the HD equation are

\[ H_{-1} = \int dx \left( 2 w^{1/2} \right) , \quad H_{-2} = \int dx \left( \frac{1}{8} w^{-5/2} w_x^2 \right) . \tag{5} \]

We note here that the second structure in (4) corresponds to the centerless Virasoro algebra while

\[ D = D_2 + c D_1 \tag{6} \]

represents the Virasoro algebra with a central charge \( c \). We note also that the recursion operator following from (4), \( R = D_2 D_1^{-1} \), can be explicitly inverted to yield

\[ R^{-1} = \frac{1}{2} \partial^2 w^{-1/2} \partial^{-1} w^{-1/2} . \tag{7} \]

Furthermore, the conserved charges

\begin{align*}
  H_0 &= - \int dx \ w , \\
  H_0^{(1)} &= \int dx \ \left( \partial^{-1} w \right) , \\
  H_0^{(2)} &= \int dx \ \left( \partial^{-2} w \right) ,
\end{align*} \tag{8}
are Casimirs (or distinguished functionals) of the Hamiltonian operator $\mathcal{D}_1$ (namely, they are annihilated by the Hamiltonian structure $\mathcal{D}_1$). As a consequence of this, it is possible to obtain, in an explicit form, equations from (3) for integers $n$ both positive and negative, i.e., $n \in \mathbb{Z}$. As shown in [12], for $n > 0$, we have three classes of nonlocal equations. However, in this paper we will only study the hierarchy associated with the local Casimir $H_0$ in (5). In this way, for $n = 1$, we obtain from (3), with the conserved charges

$$H_1 = \int dx \dfrac{1}{2}(\partial^{-1}w)^2, \quad H_2 = \int dx \dfrac{1}{2}(\partial^{-2}w)(\partial^{-1}w)^2, \quad (9)$$

the Hunter-Zheng (HZ) equation

$$w_t = - (\partial^{-2}w)w_x - 2(\partial^{-1}w)w, \quad (10)$$

which is also an important equation that belongs to the Harry Dym hierarchy.

The integrability of the HD equation (1) also follows from its nonstandard Lax representation

$$L = \dfrac{1}{w}\partial^2, \quad \dfrac{\partial L}{\partial t} = -2[B, L], \quad (11)$$

where

$$B = (L^{3/2})_\geq 2 = w^{-3/2}\partial^3 - \dfrac{3}{4}w^{-5/2}w_x\partial^2. \quad (12)$$

Conserved charges, for $n = 1, 2, 3, \ldots$, are obtained from

$$H_{-(n+1)} = \text{Tr} L^{2n+1}. \quad (13)$$

A Lax representation for the HZ equation (10) is also known and is given by (11) with

$$B = \dfrac{1}{4}(\partial^{-2}w)\partial + \dfrac{1}{4}(\partial^{-1}w)\partial^2. \quad (14)$$

However, in this case, the operator $B$ is not directly related to $L$, and, consequently, the Lax equation is not of much direct use (in the construction of conserved charges etc).

3 Deformed Harry Dym Hierarchy:

Before going on to describe the supersymmetrization of this model, we discuss briefly a new integrable model that can be thought of as a deformation of the Harry Dym hierarchy. Identifying $w = u_{xx}$, we note that the deformed Harry Dym and the deformed Hunter-Zheng equations take the forms

$$u_t^{\text{dHD}} = \sqrt{2}\left(\dfrac{1 - \lambda u_{xx}}{\sqrt{2u_{xx} - \alpha - \lambda u_{xx}^2}}\right)_x,$$
Here $\alpha, \lambda$ are two constant parameters that cannot be simultaneously scaled away. When $\alpha = \lambda = 0$, these equations reduce respectively to the Harry Dym and the Hunter-Zheng equations. For $\lambda = 0, \alpha \neq 0$, the deformed Hunter-Zheng equation corresponds to the system studied by Alber et al [15]. For $\alpha = 1, \lambda \neq 0$, this system was considered by Manna et al [16] and argued to be integrable. We have shown [2] that the system of equations in (15) are bi-Hamiltonian and are integrable for arbitrary values of $\alpha, \lambda$ and belong to the same hierarchy.

4 The Susy-B Harry Dym (sHD-B) Equations:

The most natural generalization of an equation to a supersymmetric one is achieved simply by working in a superspace. We note, from the HD equation (1), that by a simple dimensional analysis, we can assign the following canonical dimensions to various quantities

\[ [x] = -1, \quad [t] = 3, \quad \text{and} \quad [w] = 4. \]  

(16)

The $N=1$ supersymmetric equations are best described in the superspace parameterized by the coordinates $z = (x, \theta)$, where $\theta$ represents the Grassmann coordinate ($\theta^2 = 0$). In this space, we can define

\[ D = \partial_x + \theta \partial_x, \quad D^2 = \partial, \quad [\theta] = -\frac{1}{2}, \]  

(17)

representing the supercovariant derivative. Let us introduce the fermionic superfield

\[ W = \psi + \theta w, \quad [W] = [\psi] = \frac{7}{2}. \]  

(18)

A simple supersymmetrization of a bosonic system, conventionally known as the B supersymmetric (susy-B) extension [6], is obtained by simply replacing the bosonic variable $w$, in the original equation, by

\[ (DW) = w + \theta \psi', \]  

(19)

where $W$ represents a fermionic superfield. This leads to a manifestly supersymmetric equation and following this for the case of the equation (1), we obtain the susy-B HD (sHD-B) equation

\[ W_t = \partial^2 D \left( (DW)^{-1/2} \right), \]  

(20)

where $W$ is the fermionic superfield (18).
This system is bi-Hamiltonian with the even Hamiltonian operators

\[ \mathcal{D}_1 = \partial^2, \quad \mathcal{D}_2 = D(DW)D^{-1} + D^{-1}(DW)D, \]  

(21)

and the odd Hamiltonians (which follow from under the substitution \( w \to (DW) \))

\[ H_{-1} = \int dz \,(DW)^{1/2}, \quad H_{-2} = \int dz \,\frac{1}{8}(DW)^{-5/2}(DW_x)^2. \]  

(22)

The Casimirs of \( \mathcal{D}_1 \) can be easily identified with the ones following from \([5]\).

The sHD-B equation \((20)\) has two possible nonstandard Lax representations. Let

\[ L = (DW)^{-1}D^4 + cW_x(DW)^{-2}D^3. \]  

(23)

Then, it can be easily checked that the nonstandard Lax equation

\[ \frac{\partial L}{\partial t} = \left[(L^{3/2})_{\geq 3}, L\right], \]  

(24)

leads to the sHD-B equation \((20)\) for \( c = 0, -1 \). Here the projection \( (\cdot)_{\geq 3} \) is defined with respect to the powers of the supercovariant derivative \( D \). In a similar manner, we can obtain the SHD-BB equations \([\Pi]\).

5 The Supersymmetric \( N=1 \) Harry Dym (sHD) and Hunter-Zheng (sHZ) Equations:

In this section, we will discuss \( N=1 \) supersymmetrization of the system and correspondingly, it is appropriate to work in the superspace defined in \([17]-[18]\).

There are two basic ways one can study the \( N = 1 \) supersymmetrization of the Harry Dym equation. First, we note that it is possible to supersymmetrize the two Hamiltonian structures of the Harry Dym equation in \([3]\), which is easily seen from the fact that the second Hamiltonian structure is the centerless Virasoro algebra. Thus, the supersymmetrized Hamiltonian structures follow to be

\[ \mathcal{D}_1 = D\partial^2, \quad \mathcal{D}_2 = \frac{1}{2}[W\partial + 2\partial W + (DW)D], \]  

(25)

and they are compatible. Next, we write the most general local equation in superspace that is consistent with the dimensional counting which leads to a one parameter family of equations. Requiring this to be bi-Hamiltonian with respect to \((25)\), namely, requiring

\[ W_t = \mathcal{D}_1 \frac{\delta H_{-1}}{\delta W} = \mathcal{D}_2 \frac{\delta H_{-2}}{\delta W}, \]  

(26)

determines the parameter to be \( a = 6 \). The Hamiltonians in \((26)\), in this case have the
forms \((dz = dx d\theta \text{ with } \int d\theta = 0 \text{ and } \int d\theta \theta = 1)\)

\[
H_{-1} = \int dz\, 2W(DW)^{-1/2},
\]
\[
H_{-2} = \int dz\, \frac{1}{8} [W_x(DW_x)(DW)^{-5/2} - 15WW_xW_{xx}(DW)^{-7/2}] ,
\]
and the \(N=1\) sHD equation assumes the simple form

\[
W_t = D\partial^2 \left(2(DW)^{-1/2} - 3WW_x(DW)^{-5/2}\right) .
\]

It is easy to check that the Hamiltonian \(H_{-1}\) is a Casimir of \(D^2\) and the conserved charge

\[
H_0 = -\int d\theta W
\]
is a Casimir of \(D_1\). As a result, the hierarchy can be extended to positive values of \(n\) and this leads to the \(N = 1\) susy Hunter-Zheng equation

\[
W_t = \mathcal{D}_1 \frac{\delta H_2}{\delta W} = \mathcal{D}_2 \frac{\delta H_1}{\delta W} = -\frac{3}{2} W(D^{-1}W) - W_x(D^{-3}W) - \frac{1}{2} (DW)(D^{-2}W) ,
\]
where

\[
H_1 = \int dz\, \frac{1}{4}(D^{-1}W)(D^{-2}W) ,
\]
\[
H_2 = \int dz\, \frac{1}{2}(D^{-1}W)(D^{-2}W)(D^{-3}W) .
\]

Both the sHD and the sHZ equations are bi-Hamiltonian systems and the infinite set of commuting conserved charges can be constructed recursively. As a result, they describe supersymmetric integrable systems.

The second approach to finding a nontrivial \(N=1\) supersymmetrization of the HD equation is to start with the Lax operator in (11) and generalize it to superspace. Let us start with the most general Lax operator involving non-negative powers of \(D\),

\[
L = a_0^2 D^4 + \alpha_1 D^3 + a_1 D^2 + \alpha_2 D + a_2 ,
\]
where Roman coefficients are bosonic and Greek ones are fermionic. It is easy to verify that, in this case, there are only three projections, \((L^4)_{\geq 0,1,3}\) (with respect to powers of \(D\)), that can lead to a consistent Lax equation. Using this ansatz, we have not yet been able to obtain the sHD equation using fractional powers of the Lax operator. The Lax pair for this system, therefore, remains an open question.

On the other hand, when

\[
a_0 = (DW)^{-1}, \quad \alpha_1 = cW_x(DW)^{-2}, \quad a_1 = a_2 = 0 = \alpha_2 ,
\]
where $c$ is an arbitrary parameter, the nonstandard Lax equation
\[
\frac{\partial L}{\partial t} = \left[ (L^{3/2})_{\geq 3}, L \right],
\] (34)
yields consistent equations only for $c = 0, -1, -\frac{1}{2}$. As we have pointed out in the last section, for the values of the parameter, $c = 0, -1$, we have the sHD-B equation. The third choice of the parameter, therefore, leads to a new nontrivial $N=1$ supersymmetrization of the HD equation. Namely, with
\[
L = (DW)^{-1}D^4 - \frac{1}{2} W_x(DW)^{-2}D^3,
\] (35)
the Lax equation
\[
\frac{\partial L}{\partial t} = \left[ (L^{3/2})_{\geq 3}, L \right],
\] (36)
leads to a second $N = 1$ supersymmetrization of the HD equation of the form
\[
W_t = \frac{1}{16} \left[ 8D^5((DW)^{-1/2}) - 3D(W_{xx}W_x(DW)^{-5/2}) 
+ \frac{3}{4}(DW_x)^2W_x(DW)^{-7/2} - \frac{3}{4}D^{-1}((DW_x)^3(DW)^{-7/2}) \right].
\] (37)
This is manifestly a nonlocal susy generalization in the variable $W$ which, however, is a completely local equation in the variable $(DW)$.

Since this system of equations has a Lax description, it is integrable and the conserved charges can be calculated in a standard manner and the first few charges take the forms
\[
H_1 = \int dz W_x(DW_x)(DW)^{-5/2}, 
\] (38)
\[
H_2 = \int dz W_x \left[ 16(DW_{xxx})(DW)^{-7/2} - 84(DW_{xx})(DW_x)(DW)^{-9/2} 
+ 77(DW_x)^3(DW)^{-11/2} \right],
\]
and so on. However, we have not yet succeeded in finding a Hamiltonian structure which satisfies Jacobi identity (it is clear that the Hamiltonian structure is nonlocal, since the Hamiltonian is local).

6 The $N=2$ Supersymmetric Harry Dym Hierarchy:

The most natural way to discuss the $N = 2$ supersymmetric extension of the HD equation is in the $N = 2$ superspace. Just as we defined a superspace in the case of $N = 1$ supersymmetry, let us define a superspace parameterized by $z = (x, \theta_1, \theta_2)$, where $\theta_1, \theta_2$ define two Grassmann coordinates (anti-comuting and nilpotent, namely, $\theta_1 \theta_2 = -\theta_2 \theta_1$, ...
\( \theta_1^2 = \theta_2^2 = 0 \). In this case, we can define two supercovariant derivatives

\[
D_1 = \frac{\partial}{\partial \theta_1} + \theta_1 \frac{\partial}{\partial x}, \quad D_2 = \frac{\partial}{\partial \theta_2} + \theta_2 \frac{\partial}{\partial x},
\]

which satisfy

\[
D_1^2 = D_2^2 = \partial, \quad D_1 D_2 + D_2 D_1 = 0.
\]

Such a superspace naturally defines a system with \( N = 2 \) supersymmetry. Let us consider a bosonic superfield, \( W \), in this space which will have the expansion (we denote it by the same symbol as in the case of \( N = 1 \))

\[
W = w_0 + \theta_1 \chi + \theta_2 \psi + \theta_2 \theta_1 w_1.
\]

Looking at the bosonic superfield \( W \) in (41), we note that it has two bosonic components as well as two fermionic components. In the bosonic limit, when we set the fermions to zero, the \( N = 2 \) equation would reduce to two bosonic equations. Since we have only the single HD equation (1) to start with, the construction of such a system is best carried out in the Lax formalism. This also brings out the interest in such extended supersymmetric systems, namely, they lead to new bosonic integrable systems in the bosonic limit.

As in (32), let us consider the most general \( N = 2 \) Lax operator which contains differential operators in this superspace of the following form (taking a more general Lax involving only differential operators does not lead to equations which reduce to the HD equation),

\[
L = W^{-1} \partial^2 + (D_1 W^{-1})(\kappa_1 D_1 + \kappa_2 D_2) \partial + (D_2 W^{-1})(\kappa_3 D_1 + \kappa_4 D_2) \partial \\
+ (\kappa_5 (D_1 D_2 W) W^{-2} + \kappa_6 (D_1 W)(D_2 W) W^{-3}) D_1 D_2,
\]

where \( \kappa_i, i = 1, 2, \ldots, 6 \) are arbitrary constant parameters. The \( N = 2 \) supersymmetry corresponds to an internal \( O(2) \) invariance that rotates \( \theta_1 \rightarrow \theta_2, \theta_2 \rightarrow -\theta_1 \) and correspondingly \( D_1 \rightarrow D_2, D_2 \rightarrow -D_1 \) (thereby rotating the fermion components of the superfield into each other). This invariance, imposed on the Lax operator, identifies

\[
\kappa_4 = \kappa_1, \quad \kappa_3 = -\kappa_2.
\]

Using the computer algebra program REDUCE [17] and the special package SUSY2 [18], we are able to study systematically the hierarchy of equations following from the Lax equation

\[
\frac{\partial L}{\partial t} = [L^{3/2} \geq_2, L].
\]

The consistency of the equation (44) leads to four possible solutions for the values of the arbitrary parameters

1. \( \kappa_1 = \kappa_2 = \kappa_5 = \kappa_6 = 0 \),
2. $\kappa_2 = 0$, $\kappa_1 = \kappa_5 = -\frac{\kappa_6}{2} = 1$,
3. $\kappa_2 = \kappa_5 = \kappa_6 = 0$, $\kappa_1 = \frac{1}{2}$,
4. $\kappa_2 = 0$, $\kappa_1 = \kappa_5 = \frac{1}{2}$, $\kappa_6 = \frac{3}{4}$.

The first and the second cases can be checked to lead to the same dynamical equation which is nothing other than the sHD-BB equations we alluded to earlier.

For the third choice of parameters, we note that it leads to a nontrivial $N = 2$ supersymmetrization. However, in the bosonic sector, where we set all the fermions to zero, the Harry Dym equation becomes decoupled from the other and, therefore, is not very interesting. We also note that, under the $N = 1$ reduction, it is straightforward to see that the system goes over to the one discussed earlier.

The fourth case is probably the most interesting of all. Here, the Lax operator takes the form

$$L^{(4)} = - (W^{-1/2} D_1 D_2)^2 .$$

We note here that a similar situation also arises in the study of the $N = 2$ sKdV hierarchy [19] (for the case of the parameter $a = 4$). In this case, we have interesting nontrivial flows and, in the bosonic limit, we have a new integrable system which has recently been investigated further in [20]. We further note that this supersymmetric system has a Hamiltonian structure of the form

$$D = -2W^{1/2} D_1 D_2 \partial W^{1/2} .$$

7 Conclusions:

In this talk, we have described our recent results [1, 2] on the supersymmetrization as well as deformation of the Harry Dym hierarchy.

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