Evolution of disturbances that propagate in viscoelastic metamaterial

D A Kolesov\textsuperscript{1,2}, V I Erofeev\textsuperscript{1}, V L Krupenin\textsuperscript{2} and A O Malkhanov\textsuperscript{1}

\textsuperscript{1}Mechanical engineering Research Institute of RAS, Russia, 603024, Nizhny Novgorod, Belinsky, 85
\textsuperscript{2}Institute of Mechanical Engineering named after A.A. Blagonravov of RAS, Russia, 119334, Moscow, Bardina Ulitsa, 4

alandess@yandex.ru

Abstract. We study the features of the propagation of a longitudinal wave in an acoustic (mechanical) metamaterial modeled as a one-dimensional chain containing the same masses connected by elastic elements (springs) having the same stiffness, each mass containing a series connection of another mass and a viscous element (damper). It is shown that the model under consideration allows us to describe the dispersion and frequency-dependent attenuation of a longitudinal wave, the nature of which substantially depends on the ratio of the external and internal masses of the metamaterial.

1. Introduction

Acoustic metamaterials, being, in fact, not materials, but cellular periodic structures, in the long-wave range behave like continuous materials. The study of the characteristics of dispersion, dissipation, and the demonstration of nonlinearity of acoustic waves in metamaterials is of interest. This interest is due to both the cognitive side of the issue and the prospects for practical applications of metamaterials\cite{1, 2}, among which the possibility of creating sound absorbers on their basis\cite{3–10} is increasingly mentioned.

Guided by a mathematical analogy between acoustic and electromagnetic waves, the researchers modeled acoustic (mechanical) materials. However, unlike natural materials with negative dielectric constant, these are artificial materials with a negative mass or negative elastic modulus. Obviously, the appearance of a negative effective mass is not a new physical effect, but just the result of inaccurate modeling of acoustic metamaterials.

2. A mathematical model of the metamaterial

In this study, we consider one-dimensional chains consisting of masses, each of which contains an internal mass. This simple system is used to illustrate how a negative effective mass can be obtained and what effect it has on the propagation of waves in a chain. We also note that the usual continuum theory is not ideal in describing micromotion in this system, and the Cosserat continuum model\cite{11–13} is more suitable for representing the chain.

The structural modeling method used in our work does not give absurd results (negative mass, etc.). To describe the dynamic properties of metamaterials, a model was proposed in\cite{14}, which is a one-dimensional chain containing the same masses $m_1$ connected by elastic elements (springs) having the
same stiffness $k_1$, while each mass inside itself contained another mass $m_2$ and another elastic element - spring with stiffness $k_2$. Such a model was called the mass-in-mass chain.

In [15, 16], this model was generalized by taking into account the quadratic nonlinearity of the external and internal elastic elements; it was shown that spatially localized nonlinear deformation waves (solitons) can form in the metamaterial when dynamically applied to it. However, within the framework of a purely elastic statement of the problem, it is not possible to investigate the dissipative properties of the metamaterial. To solve this problem, we replace an elastic element with rigidity $k_1$ with a viscous element (see Figure 1). The equations of dynamics of the modified mass-in-mass chain in the long wavelength range will have the form:

$$\frac{m_1}{L} \frac{\partial^2 u_1}{\partial t^2} - k_1 \frac{\partial^2 u_1}{\partial x^2} - \alpha \frac{\partial}{\partial t} (u_2 - u_1) = 0$$

(1)

$$\frac{m_2}{L} \frac{\partial^2 u_2}{\partial t^2} + \alpha \frac{\partial}{\partial t} (u_2 - u_1) = 0$$

(2)

Note that system (2), (3) can be reduced to a single equation with respect to displacement:

$$\frac{\partial^2 u_1}{\partial t^2} - c_0^2 \frac{\partial^2 u_1}{\partial x^2} + \frac{m_1 m_2}{L \alpha (m_1 + m_2)} \frac{\partial^3 u_1}{\partial t^3} - \frac{c_0^2 m_2}{L \alpha} \frac{\partial^3 u_1}{\partial x^2 \partial t} = 0$$

(3)

If we introduce the dimensionless displacement in (3) $U = u_1/u_0$, the coordinate $X = x/a$ and time $T = t/b$, where $b = m_2/L \alpha$, $a = c_0 b$, the equation takes the following form:

$$\frac{\partial^2 U}{\partial T^2} - \frac{\partial^2 U}{\partial X^2} + \delta \frac{\partial^3 U}{\partial X^2 \partial T} = 0.$$  

(4)

Here $\delta = m_1/(m_1 + m_2)$. This parameter belongs to the interval $\delta = [0, 1]$, involving two limiting cases: $\delta \to 1$, if $m_1 > m_2$ and $\delta \to 0$, if $m_2 > m_1$.

Figure 1. The mechanical model of a viscoelastic metamaterial.

We consider equation (4) with the following initial conditions:

$$U(X, 0) = Asech(yX) = \frac{2A}{e^{yX} + e^{-yX}},$$

(5)

$$\frac{\partial U(X, 0)}{\partial T} = 0,$$  

(6)

$$U^{(j)}(X, 0) = Asech(yX) = \frac{2A}{e^{yX} + e^{-yX}},$$

(7)

$$U^{(j+1)}(X, 0) = Asech(yX) = \frac{2A}{e^{yX} + e^{-yX}},$$

(8)

$$\frac{\partial U^{(j+1)}(X, 0)}{\partial T} = 0,$$  

(9)
where $A$ – is an amplitude, $\gamma$ – is a spatial parameter.

The development of the initial ($T_0 = 0$) disturbance (5), (6) can be traced over the next three time instants (Figure 2).

The solution to the problem is symmetric with respect to $X = 0$, because the initial value (5) is an even function. Solutions obtained at $\delta = 0.05$, are plotted to the left of the axis of symmetry (dashed line), and the solutions obtained at $\delta = 0.5$, are plotted to the right of the axis.

3. Research of disturbances in viscoelastic metamaterial

Comparison of these cases shows the difference in dispersion. The character of the attenuation of disturbances can vary and depends on the value of $\delta$. In the case of a small value of parameter $\delta$ the attenuation is much faster than when $\delta$ is greater. The initial sections (at $T = T_0$) in both cases are qualitatively similar. This is explained by the fact that in both cases anomalous dispersion takes place at large values of the wavenumber $k$. The main difference between the presented cases arises when considering the tail of the curves. At small values of parameter $\delta$ the solution behaves more like a solution of the diffusion equation, but for large values of parameter $\delta$ the solution behaves similarly to the solution of the wave equation. The presence of a more bulky terms at low values of parameter $\delta$ due to the superposition of the effects of normal dispersion and negative group velocity.

![Figure 2. Instant wave profiles at $A = 1$ and $\gamma = 3$, at the moments $T_0 = 0, T_1 = 7/3, T_2 = 14/3, T_3 = 21/3$, which calculated for two values of parameter $\delta$.](image)

In the Figure 3 it is shown that the wave profiles for asymmetric development for four consecutive time instants. The initial value consists of the sum of two perturbations that have different fundamental frequencies. The initial perturbation has the form:

$$U(x, 0) = A\sech(\gamma x) + B\sech[g(x + 1)],$$

$$\frac{\partial U(x, 0)}{\partial t} = 0.$$  

Here, the main perturbation, marked in Figure 3 as $a$, has an amplitude $A = 1$ and the spatial parameter $\gamma = 1.6$ (corresponds to a disturbance with a low fundamental frequency). The secondary
disturbance, which is marked in Figure 3 as $b$, shifts to the left with a certain step with respect to the main disturbance. The secondary disturbance has an amplitude of $B = 0.55$ and a spatial parameter of $g = 10$, which corresponds to a high fundamental frequency.

In the Figure 3 it is shown that the high-frequency perturbation $b$ propagates faster than the main perturbation, which has a lower frequency. For example, in position $b_2$ and at the corresponding moment in time $T_2$, the maximum of perturbation $b_0$ reaches the maximum of the main perturbation, and in position $b_3$ the maximum of perturbation $b_0$ is ahead of the maximum of the main perturbation. This phenomenon is explained by anomalous dispersion, which is expressed in the fact that the group velocity exceeds the phase velocity.

From the results of dispersion analysis, it follows that the high-frequency wave components must also decay faster than the low-frequency components. Indeed, this statement is confirmed in Figure 3. The peak value of the main perturbation decreased from the initial amplitude $A = 1$ at time $T_0$ to $A \approx 0.2$ at time $T_3$. On the other hand, the perturbation amplitude $b_0$ decreases more significantly, from $B = 0.55$ at time $T_0$ to $B \approx 0$ at time $T_3$.

![Figure 3. Instant wave profiles at $\delta = 0.5, \gamma = 1.6, g = 10, A = 1, B = 0.55$, at the moments $T_0 = 0, T_1 = 15/4 , T_2 = 30/4, T_3 = 45/4$. Markers $b_i$ depict the place of the peak $b_0$ when the disturbance propagates the to right, $b$ – when the disturbance propagates the to left.]

As a result of the studies, it was shown that the longitudinal wave in the viscoelastic metamaterial, defined as the mass-in-mass chain, has dispersion and frequency-dependent attenuation. The evolution of the wave profile is analyzed, both in the low-frequency and in the high-frequency ranges.

4. Acknowledgments
This work was financially supported by RSF grant 19-19-00065.

References
[1] Norris A, Haberman M 2012 Special issue on acoustic metamaterials J Acoust Soc Am Vol 132 No 4 Pt 2 pp 2783-2945
[2] Craster R V, Guenneau S 2013 Acoustic metamaterials: negative refraction, imaging, lensing and
cloaking (Dortrecht: Springer) 323 p

[3] Bobrovnitskii Yu I 2014 Effective parameters and energy of acoustic metamaterials and media. Acoustical Physics Vol 60 No 2 pp 134-141

[4] Bobrovnitskii Yu I 2015 Models and general wave properties of two-dimensional acoustic metamaterials and media. Acoustical Physics Vol 61 No 3 pp 255-264

[5] Bobrovnitskii Yu I, Tomolina T V, Laktionov M M 2016 A discrete model of damped acoustic metamaterials. Acoustical Physics Vol 62 No 1 pp 1-7

[6] Li J, Chan C T 2004 Double-negative acoustic metamaterial Phys Rev E 70 055602

[7] Fang N, Xi D, Xu J, Ambati M, Srituravanich W, Sun C, Zhang X 2006 Ultrasonic metamaterials with negative modulus Nat Mater Vol 5 pp 452–456

[8] Ding Y, Liu Z, Qiu C, Shi J 2007 Metamaterial with simultaneously negative bulk modulus and mass density Phys Rev Lett Vol 99 093904

[9] Cheng Y, Xu J Y, Liu X J 2008 One-dimensional structured ultrasonic metamaterials with simultaneously negative dynamic density and modulus Phys Rev B 77 045134.

[10] Chan C T, Li J, Fung K H 2006 On extending the concept of double negativity to acoustic waves JZUS A 7 pp 24–28

[11] Erofeev V I 2003 Wave processes in solids with microstructure (Singapore: World Scientific) 256 p

[12] Bagdoev A G, Erofeyev V I, Shekoyan A V 2016 Wave dynamics of generalized continua (Berlin: Springer) 274 p

[13] Vardoulakis I 2019 Cosserat continuum mechanics (Cham: Springer) 180 p

[14] Huang H H, Sun C T, Huang G L 2009 On the negative effective mass density in acoustic metamaterials Int J Eng Sci Vol 47 pp 610-617

[15] Erofeev V I, Kolesov D A, Malkhanov A O 2019 Nonlinear localized waves of deformation in the class of metamaterials as set as the mass-in-mass chain Advanced Structured Materials Vol 108 pp 105-116

[16] Erofeev V, Kolesov D, Malkhanov A 2019 Nonlinear strain waves in metamaterial defined a mass-to-mass IOP Conf Series: Materials Science and Engineering 2019 (in press)