Quantum many-body tunneling of attractive Bose-Einstein condensate through double asymmetric barrier

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We study the stability of attractive atomic Bose-Einstein condensate in the anharmonic trap using a correlated many-body method. The anharmonic parameter ($\lambda$) is slowly tuned from harmonic to weak and then to strong anharmonicity. For each choice of $\lambda$ the many-body equation is solved adiabatically. The use of the van der Waals interaction gives realistic picture which substantially differs from the mean-field results. For weak anharmonicity, we observe that the attractive condensate gains stability with larger number of bosons compared to that in the pure harmonic trap. The transition from resonances to bound states with weak anharmonicity also differs significantly from the earlier study of Moiseyev et.al.[J. Phys. B: At. Mol. Opt. Phys. 37, L193 (2004)]. We also study the tunneling of the metastable condensate very close to the critical number $N_{cr}$ of collapse. For intermediate anharmonicity, we observe dual tunneling through the two adjacent barriers. We also calculate the critical value of $\lambda$ where the left-sided transmission coefficient $T_{left}$ and the right-sided transmission coefficient $T_{right}$ become equal. This is very special feature which has not been observed in earlier calculation. For strong anharmonicity we see sharp decrease in the stability of the condensate. We observe two separate branches in the stability diagram where we plot $N_{cr}$ with tuned $\lambda$.

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I. INTRODUCTION

The decay and tunneling of the metastable states is an old quantum mechanical problem. However in the context of Bose-Einstein condensate (BEC) \(^1\), \(^2\), it re-merges new interest in the study of decay and tunneling of interacting trapped bosons through a potential barrier. Optical traps of finite width can support both the bound and resonance states \(^3\), \(^8\) and the macroscopic quantum tunneling of a BEC through such finite size barrier is directly experimentally observed \(^8\), \(^10\). Theoretically the treatment of transport within the mean-field has been attempted by many groups \(^3\), \(^8\), \(^11\). Many interesting phenomena are observed due to the nonlinear interaction term in the Gross-Pitaeveskii (GP) equation. Moiseyev et.al. \(^3\) have also studied the transition from resonance to bound states of trapped attractive BEC when the interatomic attraction is increased. Carr et.al. \(^8\) have studied the macroscopic quantum tunneling in a finite potential well in one, two and three dimensions. Time dependent GP equation has been used to study the decay process of the BEC \(^1\), \(^11\), \(^12\). In another attempt by the group of Heidelberg \(^12\), the decay and tunneling dynamics of few interacting bosons through one-dimensional barrier is studied from the first principle and compared with the mean-field results. But the quantum many-body calculation with realistic interaction for 3D shallow trap and the study of stability of attractive BEC with large number of bosons is still an open problem.

In the present work we consider quantum many-body tunneling of the attractive BEC near the criticality and trapped in a quadratic with a quartic confinement. The quartic term takes care of the shallow Gaussian potential of finite width. The external potential is modeled as $V(r) = \frac{1}{2} m \omega^2 r^2 - \lambda r^4$. This represents the optical trap used in many experimental BEC \(^13\), \(^14\). This type of study is important for various reasons. First, near the critical point, as the condensate becomes highly correlated the mean-field GP equation does not produce good results. This is pointed earlier that the choice of contact interaction in 3D attractive condensate does not represent the true interatomic interaction \(^15\), \(^16\). So near the critical point where the BEC has high probability of tunneling through the adjacent barrier one needs to apply an approximate many-body technique which takes care the interatomic correlation. Second, the choice of van der Waals interaction in the many-body theory with a short range hard sphere and a $-\frac{C}{r^6}$ tail gives the realistic picture. In the many-body effective potential the metastable BEC is now bounded by double Gaussian-like barriers of different height in two sides. Thus the BEC suffers tunneling through both the barriers simultaneously. This is distinctly different from the tunneling in the mean-field approach where the BEC suffers tunneling through the right side barrier which is the effect of finite trap size only. Although a preliminary attempt has been made in our earlier published work in this direction \(^17\) but it needs further study for the following reasons. In our earlier study we have considered tunneling for a fixed $\lambda$ parameter. In the present work we continuously tune $\lambda$ and study the tunneling process adiabatically. This needs to control the height of the confining potential well in a controlled fashion. As the trapping potential is imposed optically by laser, in the routine experiments of BEC, the

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height of the external potential is controlled by reducing the laser intensity. Thus our present study is quite important as it could be obtained in the laboratory with present day set-up.

In our present calculation we choose $^{85}\text{Rb}$ condensate in the anharmonic trap. We study the various tunneling phenomena for different choices of anharmonic parameter $\lambda$: weak, intermediate and strong. We observe new physics which substantially differs from mean-field picture. For very weak anharmonicity the stability of the attractive Bose condensate is substantially improved whereas for very strong anharmonicity the condensate collapses quickly. For intermediate anharmonicity we observe dual tunneling of the condensate through the adjacent barriers.

The paper is organized as follows. In Section II, we introduce the many-body calculation with the correlated harmonic potential basis. Section III discusses the numerical results and Section IV concludes the summary of our work.

II. METHODOLOGY

A. Many-body calculation with correlated potential harmonic basis

The earlier theoretical studies on attractive BEC in the harmonic trap used the mean-field approximation which results the Gross-Pitaevskii (GP) equation for contact $\delta$-interaction \cite{18}. As the total condensate wave function is taken as the product of single particle wave functions, the effect of interatomic correlation is completely ignored. However specially for the attractive condensate, as the atoms come closer and closer, the central density becomes high and the condensate becomes highly correlated near the critical point. Naturally the interatomic correlation can no longer be ignored and one needs a full quantum many-body calculation which takes care of the effect of interatomic correlation.

In our present study we solve the many-body Schrödinger equation by potential harmonic expansion method (PHEM), which basically uses a truncated two-body basis set which keeps all possible two-body correlation \cite{14} and we go beyond the mean-field approximation. The potential harmonic expansion method with an additional short range correlation function, called CPHEM, has already been established as a very useful technique for the study of attractive BEC \cite{14, 20, 21}. Here we describe the methodology briefly for the interested readers. Details are found in our earlier work \cite{22, 23}. The Hamiltonian for a system of $A = (N + 1)$ identical bosons (each of mass $m$) interacting via two-body potential $V(\vec{r}_{ij}) = V(\vec{r}_i - \vec{r}_j)$ and confined in an external trap (which is modeled as a harmonic potential with a quartic term) has the form

$$H = -\frac{\hbar^2}{2m} \sum_{i=1}^{A} \nabla^2_i + \sum_{i=1}^{A} V_{\text{trap}}(\vec{r}_i) + \sum_{i,j>i}^{A} V(\vec{r}_i - \vec{r}_j). \quad (1)$$

After elimination of the center of mass motion and using standard Jacobi coordinates \cite{14, 24, 25}, the Hamiltonian describing the relative motion of the atoms is given by

$$H = -\frac{\hbar^2}{m} \sum_{i=1}^{N} \nabla^2_{\zeta_i} + V_{\text{trap}} + V_{\text{int}}(\zeta_1, \ldots, \zeta_N), \quad (2)$$

$V_{\text{int}}$ is the sum of all pair-wise interactions expressed in terms of the Jacobi vectors. It is to be noted that Hyper-spherical harmonic expansion method (HHEM) is an \textit{ab-initio} tool to solve the many-body Schrödinger equation where the total wave function is expanded in the complete set of hyperspherical basis \cite{24}. Although HHEM is a complete many-body approach which includes all correlations, due to large degeneracy of the HH basis, HHEM can not be applied to a typical BEC which contains few thousands to few millions of atoms. However in the context of experimentally achieved BEC, as the interparticle separation is very large compared to the range of interatomic interaction, we can safely ignore the effect of three-body and higher-body correlation and can keep only the two-body correlation. This is perfectly justified for dilute BEC where the probabilities of three and higher body collision is negligible. It permits us to decompose the total wave function $\Psi$ into two-body Faddeev component for the interacting $(ij)$ pair as

$$\Psi = \sum_{i,j>i}^{A} \phi_{ij}(\vec{r}_{ij}, r). \quad (3)$$

It is worth to note that $\phi_{ij}$ is a function of two-body separation $(\vec{r}_{ij})$ only and also includes the global hyperradius $r$, which is given by $r = \sqrt{\sum_{i=1}^{N} \zeta_i^2}$. Thus the effect of two-body correlation comes through the two-body interaction in the expansion basis. $\phi_{ij}$ is symmetric under $P_{ij}$ for bosonic atoms and satisfy the Faddeev equation

$$[T + V_{\text{trap}} - E_R] \phi_{ij} = -V(\vec{r}_{ij}) \sum_{k,l>k}^{A} \phi_{kl} \quad (4)$$

where $T$ is the total kinetic energy. Operating $\sum_{i,j>i}$ on both sides of equation (4), we get back the original Schrödinger equation. In this approach, we assume that when $(ij)$ pair interacts, the rest of the bosons are inert spectators. Thus the total hyperangular momentum quantum number as also the orbital angular momentum of the whole system is contributed by the interacting pair only. Next the $(ij)$th Faddeev component is expanded in the set of potential harmonics (PH) (which is a subset of HH basis and sufficient for the expansion of $V(\vec{r}_{ij})$) appropriate for the $(ij)$ partition as

$$\phi_{ij}(\vec{r}_{ij}, r) = r^{-\frac{(2N-1)}{2}} \sum_{K} P_{2K+l}^{lm} \Omega_{N}^{ij}(\zeta_{K}) u_{K}^{l}(r). \quad (5)$$
$\Omega_{ij}^N$ denotes the full set of hyperangles in the 3N-dimensional space corresponding to the (ij)th interacting pair and $P_{2K+l}^m(\Omega_{ij}^N)$ is called the PH basis. It has an analytic expression:

$$P_{2K+l}^m(\Omega_{ij}^N) = Y_{lm}(\omega_{ij}) \left( \begin{array}{c} \Omega_{ij}^N \\ \omega_{ij} \end{array} \right) \left( \begin{array}{c} P_0^l(\phi) Y_0(D-3) \\ D = 3N. \end{array} \right)$$

$Y_0(D-3)$ is the HH of order zero in the $(3N-3)$ dimensional space spanned by $\{\zeta_1, ..., \zeta_{N-1}\}$ Jacobi vectors; $\phi$ is the hyperangle given by $r_{ij} = r \cos \phi$. For the remaining $(N-1)$ noninteracting bosons we define hyperradius as

$$\rho_{ij} = \sum_{K=1}^{N-1} \Omega_{ij}^N,$$

such that $r^2 = r_{ij}^2 + \rho_{ij}^2$ and $r$ represents the global hyperradius of the condensate. The set of $(3N-1)$ quantum numbers of HH is now reduced to only 3 as for the $(N-1)$ non-interacting pair

$$l_1 = l_2 = ... = l_{N-1} = 0,$$
$$m_1 = m_2 = ... = m_{N-1} = 0,$$
$$n_2 = n_3 = ... = n_{N-1} = 0,$$

and for the interacting pair $l_N = l$, $m_N = m$ and $n_N = K$. Thus the 3N dimensional Schrödinger equation reduces effectively to a four dimensional equation with the relevant set of quantum numbers: hyperradius $r$, orbital angular momentum quantum number $l$, azimuthal quantum number $m$ and grand orbital quantum number $2K + l$ for any $N$. Substituting in Eq(4) and projecting on a particular PH, a set of coupled differential equations (CDE) for the partial wave $u_{ij}^l(r)$ is obtained

$$\left[ - \frac{n^2}{m} \frac{d^2}{dr^2} + V_{trap}(r) + \frac{n^2}{m^2} \{ \mathcal{L}(\mathcal{L} + 1) + 4(K + \alpha + \beta + 1) - E_{R} \} U_{ki}(r) + \sum_{K'} f_{Kl} V_{KK'}(r) f_{K'l} U_{K'l}(r) \right] = 0,$$

where $\mathcal{L} = l + \frac{3A-s}{2}$, $U_{ki} = f_{Kl} u_{ij}^l(r)$, $\alpha = \frac{3A-s}{2}$ and $\beta = l + 1/2$.

$f_{Kl}$ is a constant and represents the overlap of the PH for interacting partition with the sum of PHs corresponding to all partitions $[23]$. The potential matrix element $V_{KK'}(r)$ is given by

$$V_{KK'}(r) = \int P_{2K+l}^m(\Omega_{ij}^N) V(r_{ij}) P_{2K'+l}^m(\Omega_{ij}^N) d\Omega_{ij}^N.$$

B. Introduction of additional short range correlation

As pointed earlier in the mean-field GP equation the two-body interaction is represented by a single parameter, the s-wave scattering length $a_{sc}$ only. It disregards the detailed structure. The presence of essential singularity as $r \to 0$ for the attractive contact $\delta$-interaction makes the Hamiltonian unbound from below. So in our present many-body calculation, we use a realistic interatomic potential like the van der Waals potential with an attractive $\frac{-1}{r^6}$ tail at large separation and a strong short range repulsion. The inclusion of detailed structure in the two-body potential with the short range repulsive core needs to include an additional short range correlation in the PH basis. This short range behavior is represented by a hard core of radius $r_c$ and we calculate the two-body wave function $\eta(r_{ij})$ by solving the zero-energy two-body Schrödinger equation

$$- \frac{\hbar^2}{m} \frac{1}{r_{ij}^2} \frac{d}{dr_{ij}} \left( r_{ij}^2 \frac{d\eta(r_{ij})}{dr_{ij}} \right) + V(r_{ij}) \eta(r_{ij}) = 0.$$

This zero-energy two-body wave function $\eta(r_{ij})$ is a good representation of the short range behavior of $\phi_{ij}$ as in the experimental BEC, the energy of the interacting pair is negligible compared with the depth of the interatomic potential. It is taken as the two-body correlation function in the PH expansion basis. The value of $r_c$ is obtained from the requirement that the calculated $a_{sc}$ has the expected value [20]. We introduce this as a short-range correlation function in the expansion basis. This also improves largely the rate of convergence of the PH basis and we call it as correlated Potential Harmonic expansion method (CPHEM). We replace Eq(5) by

$$\phi_{ij}(\vec{r}_{ij}, r) = r^{-(3N-s-1)} \sum_{K} P_{2K+l}^m(\Omega_{ij}^N) u_{ij}^l(r) \eta(r_{ij}) \cdot (14)$$

and the correlated PH (CPH) basis is given by

$$[P_{2K+l}^m(\Omega_{ij}^N)]_{\text{correlated}} = P_{2K+l}^m(\Omega_{ij}^N) \eta(r_{ij}), (15)$$

The correlated potential matrix $V_{KK'}(r)$ is now given by

$$V_{KK'}(r) = (h_{KK'}^{\alpha} h_{KK'}^{\beta} - 1) \times \int f^{-1}(P_{K}^{\alpha}(z) V \left( r \sqrt{1 + r^2} \right) P_{K}^{\beta}(z) \eta \left( r \sqrt{1 + r^2} \right) W_{i}(z)) dz. (16)$$

Here $P_{K}^{\beta}(z)$ is the Jacobi polynomial, and its norm and weight function are $h_{KK'}^{\beta}$ and $W_{i}(z)$ respectively [27].

One may note that the inclusion of $\eta(r_{ij})$ makes the PH basis non-orthogonal. One may surely use the standard procedure for handling non-orthogonal basis. However in the present calculation we have checked that $\eta(r_{ij})$ differs from a constant value only by small amount and the overlap $\langle P_{2K+l}^m(\Omega_{ij}^N) | P_{2K+l}^m(\Omega_{ij}^N) \eta(r_{kl}) \rangle$ is quite small. Thus we get back the Eq(11) approximately when the correlated potential matrix is calculated by Eq(16).
III. RESULTS

A. Choice of two-body potential and calculation of many-body effective potential

The interatomic potential has been chosen as the van der Waals potential with a hard core of radius \( r_c \), viz., \( V(r_{ij}) = \infty \) for \( r_{ij} \leq r_c \) and \( = -\frac{C_6}{r_{ij}^6} \) for \( r_{ij} > r_c \). \( C_6 \) is known for a specific atom and in the limit of \( C_6 \to 0 \), the potential becomes a hard sphere and the cutoff radius exactly coincides with the \( s \)-wave scattering length \( a_{sc} \). For our present study we consider \( ^{85}\text{Rb} \) atoms in the JILA trap where the controlled collapse experiment has been observed \([28],[29]\). By utilizing the Feshbach resonance one can effectively tune the negative scattering length and can study the stability in controlled fashion. In our choice of two-body potential we tune \( r_c \) to reproduce the experimental scattering length. As we decrease \( r_c \), \( a_{sc} \) decreases and at a particular critical value of \( r_c \) it passes through \(-\infty \to 0\). For our present calculation we choose \( ^{85}\text{Rb} \) atoms with \( C_6 = 6.4898 \times 10^{-4} \) o.u. \([1]\) and \( a_{sc} = -1.832 \times 10^{-4} \) o.u., which is one of the choices of scattering length \( a_{sc} \) in the controlled collapse experiment of Roberts et al. \([28],[29]\). With this \( V(r_{ij}) \) we solve the zero-energy two-body Schrödinger equation and tune \( r_c \) to obtain correctly \( a_{sc} = -1.832 \times 10^{-4} \) o.u. We choose \( r_c \) such that it corresponds to the zero node in the two-body wave function. The chosen parameter for our calculation is \( r_c = 1.3955 \times 10^{-3} \) o.u. With these set of parameters we solve the coupled differential equation by hyperspherical adiabatic approximation (HAA) \([30]\).

In HAA, we assume that the hyperradial motion is slow compared to the hyperangular motion and the potential matrix together with the hypercentrifugal repulsion is diagonalized for a fixed value of \( r \). Thus the effective potential for the hyperradial motion is obtained as a parametric function of \( r \). We choose the lowest eigen potential \( \omega_0(r) \) as the effective potential in which the condensate moves collectively. The energy and wave function of the condensate are finally obtained by solving the adiabatically separated hyperradial equation in the extreme adiabatic approximation (EAA)

\[
-\frac{\hbar^2}{m} \frac{d^2}{dr^2} + \omega_0(r) - E_R \psi_0(r) = 0 , \tag{17}
\]

subject to approximate boundary condition on \( \psi_0(r) \). For our numerical calculation we fix \( l = 0 \) and truncate the CPH basis to a maximum value \( K = K_{\text{max}} \) requiring proper convergence. In Fig.1 we plot the many-body effective potential \( \omega_0(r) \) as a function of hyperradius \( r \) for \( N = 1000 \) atoms in the pure harmonic trap \( (\Lambda = 0) \). For \( N < N_{cr} (\sim 2385) \), the condensate is metastable and is associated with a deep and narrow attractive well (NAW) on the left side. For \( r \to 0 \), there is a strong repulsive wall which is the reflection of the hard core van der Waals interaction. For \( N \) less than the critical number \( N_{cr} \) a metastable region (MSR) appears for larger \( r \). An intermediate barrier (IB) separates the NAW from the MSR. The panel (a) of Fig. 1, the NAW together with the repulsive core is shown. The IB and MSR have been shown in panel (b) of Fig. 1. With increase in \( N \), we observe that the height of IB decreases, together with a decrease in the difference \( (\Delta \omega) \) between the maximum of IB \( (\omega_{\text{max}}) \) and minimum of MSR \( (\omega_{\text{min}}) \) and the NAW starts to be more negative and narrower. As \( N \to N_{cr} \), \( \omega_{\text{max}} \) and \( \omega_{\text{min}} \) coincides, the MSR disappears which corresponds to the collapse of metastable BEC. For \( N > N_{cr} \), there will be only the NAW and the condensate will settle in the deep attractive well. In the attractive \( \delta \)-function potential, as \( r \to 0 \), the effective potential rapidly goes to \(-\infty \) which is called a pathological singularity. Here the many-body picture is in sharp contrast with the GP mean-field picture with contact interaction.

The advantages of the CPHEM are as follows.

i) As for the attractive BEC, the condensate becomes highly correlated, the choice of CPH basis (which keeps all possible two-body correlations) is per-
fectly justified for the description of correlated dilute BEC.

ii) By using the HAA we basically reduced the 3N dimensional problem into an effective one-dimensional problem in hyperradial space which provides both qualitative and quantitative picture of the system. As the effective quantum numbers are always four (for any N) we can run our many-body code for very high values of N.

iii) The many-body effective potential strongly differs from the mean-field potential. In GP, the choice of contact $\delta$-interaction in the two-body potential gives rise to the pathological singularity in the effective potential. Thus the study of post-collapse scenario of the attractive condensate is beyond the scope of mean-field theory. Whereas the presence of short-range hard core in the van der Waals interaction does not only remove the singularity, it gives the realistic scenario and can describe the formation of atomic cluster after collapse.

**B. Weak anharmonicity ($\lambda < 9.36 \times 10^{-6}$ o.u)**

In our earlier work we have successfully applied CPHEM to describe controlled collapse experiment and calculated the stability factor more accurately than the mean-field results [10]. Thus the main aim of our present work is to study the stability of $^{85}$Rb atoms with fixed interaction when the shallowness of external trapping potential ($\lambda$) is continuously tuned. Experimentally the tuning is done by controlling the intensity of laser beam, which provides the optical trap. The many-body effective potential for weak anharmonicity is plotted in Fig. 2(a) for $N = 2395$ which is very close to criticality $N_{cr} = 2400$. We slowly change $\lambda$, from very close to harmonic to $\lambda = 9.35 \times 10^{-6}$ o.u. and numerically solve the many-body equation adiabatically for each choice of $\lambda$. For such weak anharmonicity, the metastable condensate in the MSR region is tightly bounded by the high intermediate barrier on the right side (RIB) and does not suffer any tunneling through RIB. Whereas the intermediate barrier on the left side (LIB) is of finite height and the condensate suffers tunneling through the LIB only. The LIB is again associated with a deep negative well (NAW) on the left side as shown in the Fig. 1 and the condensate after tunneling settles down in the NAW and forms cluster. By increasing $\lambda$ gradually, we observe the greater stability of the attractive condensate as the metastable region (MSR) becomes deeper. The stability of the condensate with gradually increasing values of $\lambda$ is also clear from Fig. 2(b) where we plot the wavefunction for various choices of $\lambda$. The wave function expands with increasing $\lambda$. From the condensate wave function we calculate the condensate radius $r_{av}$ as

$$r_{av} = \left( \frac{1}{A} \sum_{i=1}^{A} (\vec{x}_i - \vec{R})^2 \right)^{1/2} = \frac{\sqrt{\rho^2}}{\sqrt{2A}},$$

where $\vec{R}$ is the center of mass coordinate. And we plot it in Fig. 2(c). We observe that $r_{av}$ increases gradually with increase in $\lambda$. We also calculate the WKB tunneling probability $T$ as

$$T = \exp\left(-\frac{2}{r_{av}} \int_{r_1}^{r_2} \sqrt{2|\omega_0(r) - E|} \, dr \right).$$

where the limits of integration $r_1$ and $r_2$ are the inner and outer turning points of the intermediate barrier on the left (LIB) of $\omega_0(r)$, $E$ is the energy of the metastable condensate. We plot it as a function of $\lambda$ in the Fig. 3. It nicely presents that for larger $\lambda$ the condensate gets better stability and less tunneling. Thus tuning $\lambda$ gradually from very close to harmonic to weak anharmonicity we observe increasing stability of the condensate. Thus simply changing shape of the external trap one can produce highly stable condensate even for the attractive bosons. Whereas with decreasing $\lambda$ gradually as the transmission coefficient $T$ increases, the system will collapse for a certain number of atoms. This is similar to the collapse of attractive BEC in pure harmonic trap. In our present calculation with the choice of parameters we observe greater stability till $\lambda < 9.36 \times 10^{-6}$ o.u.

**C. Intermediate anharmonicity ($9.36 \times 10^{-6} \leq \lambda \leq 9.53 \times 10^{-6}$)**

This is very interesting situation when the condensate in the MSR is bounded by two barriers (LIB and RIB) of finite width and may suffer tunneling from the MSR through both the barriers simultaneously. For better understanding we plot the many-body effective potential in Fig. 4 for some specific choices of $\lambda$ within the above range and for fixed $N=2665$. For $\lambda = 9.36 \times 10^{-6}$, the height of LIB is smaller than the height of RIB. Thus although the condensate will suffer dual tunneling, however the transmission probability through the LIB is higher compared to that through RIB. Increasing $\lambda$ to $9.40 \times 10^{-6}$, we observe that the height of LIB and RIB are comparable and the tunneling probability of the condensate through LIB and RIB are of equal weightage. Increasing $\lambda$ further, we observe that although the LIB remains of significant height, RIB drastically falls. Naturally the condensate will suffer larger tunneling through the RIB. As stated before there is a large deep negative well on the left side of the LIB, which is not shown in the figure for better presentation. Thus when the condensate leaks through the LIB, it is not lost. The highly correlated bosons will form cluster within the deep well. Whereas after tunneling through the RIB the interacting bosons will be trapless and will behave as uniform
Bose gas. The calculated WKB tunneling probability $T$ is plotted as a function of $\lambda$ in this intermediate region in Fig. 5. The double branches definitely signify the dual possibility of tunneling through the double barrier. However unlike the double Gaussian barrier, we have two asymmetric barrier of different height. In Fig. 5, we observe that the tunneling through LIB ($T_{left}$) gradually decreases with $\lambda$ which is similar to the case of weak anharmonicity as discussed previously. Whereas the transmission probability through RIB ($T_{right}$) sharply increases with $\lambda$ which is similar to the case of strong anharmonicity (described in the next section). We see that $T_{left}$ and $T_{right}$ cross at some value of $\lambda$. The crossing point is shown in the inset of Fig. 5. The corresponding value of $\lambda$ is $9.398 \times 10^{-6}$ o.u. At this critical $\lambda$ ($\lambda_{critical}$), $T_{left}$ becomes equal to $T_{right}$, which signifies that the two asymmetric barriers become two symmetric Gaussian barriers and the condensate will suffer equal tunneling. Thus our observation nicely demonstrate that by simply controlling the external laser beam, one can control the nature of anharmonicity and stability of the condensate. The value of $\lambda_{critical}$ depends on the scattering length $a_{sc}$. Thus taking different choices of $a_{sc}$ from controlled collapse experiments of Roberts [28], one can calculate different critical values of $\lambda$.

**D. Strong anharmonicity ($\lambda > 9.53 \times 10^{-6}$)**

This is the other end of anharmonicity where the condensate is tightly bound in the MSR by a high barrier on the left side (LIB) and suffers no tunneling through it. But the RIB is of finite height and the condensate suffers tunneling through RIB only. With increase in $\lambda$ value, the barrier height of RIB quickly decreases which enhances the tunneling probability rapidly. The effective potential is plotted in the Fig. 6(a). Unlike the previous case of weak anharmonicity, here the MSR has no deep well on the right side. So after tunneling the condensate will behave as trapless, uniform Bose gas. The feature is also clear from Fig. 6(b) where we observe that the condensate wave function is associated with oscillation on the right side. For better clarity the oscillatory part of the wave function is presented in the inset of Fig. 6(b). It clearly shows that the condensate is basically confined in the MSR but a partlikages through the adjacent RIB. With increasing $\lambda$ the oscillation also increases. The corresponding WKB tunneling probability is presented in Fig. 7 which shows sharp increase in $T$ even with small change in $\lambda$ which indicates the quicker collapse of the condensate in a strong anharmonic trap.

**E. Collapse of atomic BEC in anharmonic trap**

Here we study the collapse of attractive Bose-Einstein condensate in the shallow trap which is produced with an additional quartic anharmonic potential with a harmonic potential. In the BEC experiments of attractive condensate [28, 29], it is observed that the condensate is metastable for a finite number of atoms. This is called critical number ($N_{cr}$) and the condensate collapses when $N > N_{cr}$. In the controlled collapse experiment of $^{85}$Rb condensate, the $N_{cr}$ is calculated both by the GP mean-field theory [13] and ab-initio quantum many-body calculation [10]. The critical stability factor is more accurately calculated in the correlated many-body approach [26] and is closer to the experimental value compared to the mean-field result. Thus in the present study we are interested to calculate $N_{cr}$ when the external trap is slowly tuned from harmonic to weak anharmonic and then strong anharmonic. In Fig. 8 we plot the many-body effective potential for both harmonic and with weak anharmonicity. For $\lambda = 0$, the trap is purely isotropic, harmonic and the condensate collapses when $N > N_{cr}$. In Fig. 8 we observe that for $\lambda = 0$, the intermediate barrier on the left side (IB) just vanishes for $N = 2385$ and the metastable condensate confined in the MSR settles down in the negative deep well on the left side of the intermediate barrier (IB) (not shown in the figure). It corresponds to collapse and the corresponding critical number for collapse is $N_{cr} = 2385$. Now with the harmonic potential if we add anharmonic potential with small anharmonicity ($\lambda = 1.0 \times 10^{-6}$), the MSR starts to develop and the intermediate barrier reappears. For this $\lambda$, the condensate collapses for $N_{cr} = 2400$. Thus by increasing $\lambda$ by very small amount, the critical number increases by 15 atoms and gives better stability of the condensate. It corresponds to the metastable condensate which is confined in MSR. Thus the stability is enhanced by controlling $\lambda$. The large $r$ behavior of the eigen potential almost remains unchanged. Thus by increasing $\lambda$ very slowly we observe the condensate becomes more stable and $N_{cr}$ gradually increases with $\lambda$. It is shown in Fig. 9, which nicely describes that $N_{cr}$ smoothly increases with $\lambda$, when $\lambda$ is small. In the earlier study of Moiseyev et.al. [8] the transition from resonance to bound state has been studied when the attractive interaction gradually increases. However to study such transition, an additional negative offset ($V_{0}$) was required to prevent collapse. For $V_{0} = 0$, the authors found in the ref. [8] that resonance never turns into bound state as the nonlinearity becomes more negative, the condensate collapses. However due to the use of hard core van der Waals potential we do not require any such external offset to create the metastable condensate. Thus the increasing stability and the transition from resonance to quasi-bound states for very weak anharmonicity strongly differs from the earlier observation of Moiseyev et.al. However with further increase in $\lambda$, we enter in the zone of intermediate anharmonicity and strong anharmonicity which shows that $N_{cr}$ now sharply decreases with $\lambda$. Thus we observe two branches in Fig. 9. We very carefully studied and observed that these two branches never meet. The magnified graph near the point of bifurcation is also presented in the inset of Fig. 9.
IV. CONCLUSION

In our present work, we have studied the decay and tunneling of attractive Bose Einstein condensate in the finite trap by approximate but \textit{ab-initio} many-body calculation. Main attention has been paid in the study of stability of attractive condensate when the effective trap height is tuned from very close to harmonic to weak anharmonicity and then to strong anharmonicity. The use of correlated PH basis and the realistic van der Waals interaction correctly describes the many-body tunneling process and gives the realistic picture. Our result strongly differs from the earlier study of mean-field results through one-dimensional barrier. The greater stability of attractive Bose gas in the anharmonic trap with very weak anharmonicity and the dual tunneling of the attractive condensate for intermediate anharmonicity are of special interest. The use of hard core potential in the two-body interaction can always describe the post collapse scenario of the attractive condensate in the many-body picture. No additional negative offset potential is required at \( r = 0 \) to observe collapse-to-metastable transition. Such attractive condensates in the finite trap are created in the laboratory when the height of the confining potential is controlled by controlling the laser intensity in the optical trap or by controlling the electric current in magnetic trap. Thus the attractive BEC can be created in a shallow trap of chosen height and once it is materialized several experiments can be done. Thus our present many-body calculation is quite relevant experimentally and calculated observables like the tunneling probability \( T \) and critical value of \( \lambda \) parameter, \( N_{cr} \) can be easily observed in the present day experiments. The choice of scattering length \( a_{sc} \) is also taken from the experiment of \( ^{85}\text{Rb} \) atoms in JILA.

This is the first such many-body calculation where we compute resonances and decay of many-boson systems in three-dimension beyond the mean-field. A natural extension of the present work is to study the dynamics and to calculate the non-escape probability and various coherence properties for many-boson systems in 3D, which requires the solution of full time dependent many-body equation.

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FIG. 2: (color online) a) Plot of effective potential $\omega_0(r)$ of the condensate with $N = 2395$ for different $\lambda$; b) Plot of corresponding condensate wave functions and c) Plot of the average condensate size $r_{av}$ for $N = 2395$ against anharmonicity parameter $\lambda$. 
FIG. 3: (color online) Plot of Transmission coefficient $T$ vs the anharmonicity parameter $\lambda$ for $N = 2395$.

FIG. 4: (color online) Plot of effective potential $\omega_0(r)$ for different anharmonicity parameter $\lambda$ for $N = 2665$.

FIG. 5: (color online) Plot of Transmission coefficient $T$ vs anharmonicity parameter $\lambda$. The inset shows the crossing point of $T_{left}$ and $T_{right}$. 
FIG. 6: (color online) a) Plot of effective potential $\omega_0(r)$ of the condensate with $N = 160$ for different $\lambda$; b) Plot of corresponding wave functions of the condensate. In the inset the oscillatory part of the condensate wave function is shown.

FIG. 7: (color online) Plot of Transmission coefficient $T$ vs the anharmonicity parameter $\lambda$ for a condensate with $N = 160$. 
FIG. 8: (color online) Plot of effective potential \( \omega \) for different weak anharmonicity \( \lambda \) for a condensate with \( N = 2385 \).

FIG. 9: (color online) Plot of critical number \( N_{cr} \) vs the anharmonicity parameter \( \lambda \).