A note on products involving $\zeta(3)$ and Catalan’s constant

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Abstract

In a recent paper Kachi and Tzermias give elementary proofs of four product formulas involving $\zeta(3)$, $\pi$, and Catalan’s constant. They indicate that they were not able to deduce these products directly from the values of a function introduced in 1993 by Borwein and Dykshoorn. We provide here such a proof for two of these formulas. We also give a direct proof for the other two formulas, by using a generalization of the Borwein-Dykshoorn function due to Adamchik. Finally we give an expression of the Borwein-Dykshoorn function in terms of the “parameterized-Euler-constant function” introduced by Xia in 2013, which happens to be a particular case of the “generalized Euler constant function” introduced by K. and T. Hessami Pilehrood in 2010.

Keywords: zeta function, Catalan constant, Glaisher-Kinkelin constant, generalized Euler constants, Borwein-Dykshoorn function

MSC Classes: 11Y60, 33B99, 11M06, 11M99

1 Introduction

In a recent paper Kachi and Tzermias prove in an elementary way four nice formulas involving $\zeta(3)$, $\pi$, and Catalan’s constant (see [6] Propositions 1 and 2]), namely

$$\lim_{k \to \infty} \prod_{k=1}^{2n+1} e^{-1/4} \left(1 - \frac{1}{k+1}\right)^{\frac{k(k+1)}{2}(-1)^k} = \exp \left(\frac{7\zeta(3)}{4\pi^2} + \frac{1}{4}\right)$$

(1)

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\[
\lim_{k \to \infty} \prod_{k=1}^{2n} e^{1/4} \left(1 - \frac{1}{k + 1}\right)^{k(-1)^k} = \exp \left(\frac{7\zeta(3)}{4\pi^2} - \frac{1}{4}\right)
\]  \hspace{1cm} (2)

\[
\lim_{k \to \infty} \prod_{k=1}^{2n} \left(1 - \frac{2}{2k + 1}\right)^{k(-1)^k} = \exp \left(\frac{2G}{\pi} - \frac{1}{2}\right)
\]  \hspace{1cm} (3)

\[
\lim_{k \to \infty} \prod_{k=1}^{2n+1} \left(1 - \frac{2}{2k + 1}\right)^{k(-1)^k} = \exp \left(\frac{2G}{\pi} + \frac{1}{2}\right)
\]  \hspace{1cm} (4)

where \( G = \sum_{n \geq 0} \frac{(-1)^n}{(2n+1)^2} \) is the Catalan constant.

At the end of their article [6] the authors recall the Borwein-Dykshoorn function

\[ D(x) = \lim_{n \to \infty} \prod_{k=1}^{2n+1} \left(1 + \frac{x}{k}\right)^{k(-1)^{k+1}} \]

which was introduced in [2] as a generalization of a result of Melzak [7] proving that

\[ \lim_{n \to \infty} \prod_{k=1}^{2n+1} \left(1 + \frac{2}{k}\right)^{k(-1)^{k+1}} = \frac{\pi e}{2}. \]

Kachi and Tzermas indicate that they were not able to prove any of the relations (1), (2), (3), and (4) directly from the expression of the values of \( D(x) \) for \( x \) any rational number with denominator 1, 2, or 3 proven in [2] (though the constant \( e^{G\pi} \) for example occurs both in [2] and in [6]).

In this paper we give a direct proof of relations (3) and (4) using the function \( D(x) \) (actually we only need the values \( D(1) \) and \( D(\frac{1}{2}) \)). We then use a function similar to \( D(x) \) introduced by Adamchik in [1, p. 284], namely

\[ E(x) = \lim_{n \to \infty} \prod_{k=1}^{2n} \left(1 - \frac{4x^2}{k^2}\right)^{-k(-1)^k} \]

(actually we only use the value \( E(\frac{1}{2}) \)), to prove directly relations (1) and (2).

2 Formulas (3) and (4)

**Proposition 1** Formulas (3) and (4) can be deduced directly from the values of the Borwein-Dykshoorn function \( D(1) \) and \( D(1/2) \), and from classical results for the function \( \Gamma \).
Proof.

We first note that
\[
\prod_{k=1}^{2n+1} \left(1 - \frac{2}{2k + 1}\right)^{k(-1)^k} = \prod_{k=1}^{2n} \left(1 - \frac{2}{2k + 1}\right)^{k(-1)^k} \left(1 - \frac{2}{4n + 3}\right)^{-(2n+1)}
\]

Since \(\lim_{n \to \infty} \left(1 - \frac{2}{4n+3}\right)^{-(2n+1)} = e\) (take the logarithm) it is clear that (4) is readily deduced from (3). It thus suffices to prove (3).

Let \(D_n(x) := \prod_{k=1}^{2n+1} \left(1 + \frac{x}{k}\right)^{k(-1)^k}\) and \(A_n := \prod_{k=1}^{2n} \left(1 - \frac{2}{2k + 1}\right)^{k(-1)^k}\). Then

\[
A_n = \prod_{k=1}^{2n} \left(\frac{2k - 1}{2k + 1}\right)^{k(-1)^k} = \prod_{k=1}^{2n} \left(\frac{2k - 1}{2k}\right)^{k(-1)^k} \prod_{k=1}^{2n} \left(\frac{2k + 1}{2k}\right)^{k(-1)^k+1}
\]

But
\[
\prod_{k=1}^{2n} \left(\frac{2k - 1}{2k}\right)^{k(-1)^k} = \prod_{\ell=0}^{2n-1} \left(\frac{\ell + 1}{\ell + 2}\right)^{\ell(-1)^{\ell+1}}
\]

\[
= \prod_{\ell=0}^{2n-1} \left(\frac{\ell + 1}{\ell + 2}\right)^{\ell(-1)^{\ell+1}} \prod_{\ell=0}^{2n-1} \left(\frac{\ell + 1}{\ell + 2}\right)^{(-1)^{\ell+1}}
\]

\[
= \prod_{\ell=0}^{2n-1} \left(\frac{\ell + 1}{\ell + 2}\right)^{\ell(-1)^{\ell+1}} \prod_{\ell=0}^{2n-1} \left(\frac{\ell + 1}{\ell + 2}\right)^{\ell(-1)^{\ell+1}} \prod_{\ell=0}^{2n-1} \left(\frac{\ell + 1}{\ell + 2}\right)^{\ell(-1)^{\ell+1}}
\]

\[
= \prod_{\ell=0}^{2n-1} \left(\frac{\ell + 1}{2\ell}\right)^{\ell(-1)^{\ell+1}} \prod_{\ell=0}^{2n-1} \left(\frac{\ell + 1}{\ell + 2}\right)^{\ell(-1)^{\ell+1}} \prod_{\ell=0}^{2n-1} \left(\frac{\ell + 1}{\ell + 2}\right)^{\ell(-1)^{\ell+1}}
\]

\[
= \prod_{\ell=0}^{2n-1} \left(1 + \frac{1}{2\ell}\right)^{\ell(-1)^{\ell+1}} \prod_{\ell=0}^{2n-1} \left(\frac{4\ell + 2 4\ell + 3}{4\ell + 1 4\ell + 4}\right)
\]

Hence

\[
A_n = \frac{D_{n-1} \left(\frac{1}{2}\right)}{D_{n-1}(1)} \prod_{\ell=0}^{2n-1} \left(\frac{4\ell + 2 4\ell + 3}{4\ell + 1 4\ell + 4}\right)
\]

We note that \(\lim_{n \to \infty} \left(\frac{4n}{4n + 1}\right)^{2n} = e^{-1/2}\) (take the logarithm), and that

\[
\lim_{n \to \infty} \prod_{\ell=0}^{2n-1} \left(\frac{4\ell + 2 4\ell + 3}{4\ell + 1 4\ell + 4}\right) = \lim_{n \to \infty} \prod_{\ell=0}^{2n-1} \left(\frac{\ell + 1/2 \ell + 3/4}{\ell + 1/4 \ell + 1}\right) = \frac{\Gamma \left(\frac{1}{2}\right)}{\Gamma \left(\frac{3}{4}\right)}
\]
(see, e.g., [9, Section 12-13]). Furthermore, from [2]

\[ \lim_{n \to \infty} D_{n-1} \left( \frac{1}{2} \right) = D \left( \frac{1}{2} \right) = \frac{2^{1/6} \sqrt{\pi} A^3 e^{G/\pi}}{\Gamma \left( \frac{1}{4} \right)} \]

and

\[ \lim_{n \to \infty} D_{n-1} (1) = D (1) = \frac{A^6}{2^{1/6} \sqrt{\pi}} \]

where \( A \) is the Glaisher-Kinkelin constant \( (A = \exp \left( \frac{1}{12} - \zeta'(-1) \right) \), where \( \zeta \) is the Riemann zeta function). Putting these relations together yields

\[ \lim_{n \to \infty} A_n = \frac{\sqrt{2} \pi^{3/2}}{\Gamma \left( \frac{1}{4} \right) \Gamma \left( \frac{3}{4} \right) e^{\frac{2G}{\pi}} \frac{1}{2}}. \]

Then, Euler’s reflection formula \( \Gamma(z) \Gamma(1 - z) = \pi / \sin(\pi z) \) (see, e.g., [9, Section 12-14]) yields the classical relations \( \Gamma(1/2) = \sqrt{\pi} \) and \( \Gamma(1/4) \Gamma(3/4) = \pi \sqrt{2} \). Thus

\[ \lim_{n \to \infty} A_n = e^{\frac{2G}{\pi} \frac{1}{2}} \]

which is Formula (3). \( \square \)

### 3 Formulas (1) and (2)

**Proposition 2** Formulas (1) and (2) can be deduced directly from the value of the Adamchik function \( E(1/2) \), and from classical results for the function \( \Gamma \).

**Proof.** We first note that

\[ \prod_{k=1}^{2n+1} e^{-1/4} \left( 1 - \frac{1}{k+1} \right)^{\frac{k(k+1)(-1)^k}{2}} = \left( \prod_{k=1}^{2n} e^{1/4} \left( 1 - \frac{1}{k+1} \right)^{\frac{k(k+1)(-1)^k}{2}} \right) \alpha_n \]

with \( \alpha_n = e^{-n \frac{1}{4} (2n+1) - n(1/2)(2n+1)} \). Since \( \alpha_n \) tends to \( e^{1/2} \) (take the logarithm), it is clear that Formula (2) implies Formula (1). It thus suffices to prove Formula (2).

Let \( E_n = \prod_{k=1}^{2n} e^{1/4} \left( 1 - \frac{1}{k+1} \right)^{\frac{k(k+1)(-1)^k}{2}} \). We write \( E_n^2 \) in two different ways. On one hand

\[ E_n^2 = \prod_{k=1}^{2n} e^{1/2} \left( 1 - \frac{1}{k+1} \right)^{(k^2+k)(-1)^k} = e^n \prod_{k=1}^{2n} \left( 1 - \frac{1}{k+1} \right)^{k^2(-1)^k} \prod_{k=1}^{2n} \left( 1 - \frac{1}{k+1} \right)^{k(-1)^k} \]

\[ = e^n \prod_{k=1}^{2n} \left( 1 - \frac{1}{k+1} \right)^{k^2(-1)^k} \prod_{\ell=2}^{2n+1} \left( 1 - \frac{1}{\ell} \right)^{(\ell-1)(-1)^{\ell+1}} \].

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On the other hand

\[
E_n^2 = \prod_{k=1}^{2n} e^{1/2} \left( 1 - \frac{1}{k + 1} \right)^{(k^2 + k)(-1)^k} = \prod_{\ell=2}^{2n+1} e^{1/2} \left( 1 - \frac{1}{\ell} \right)^{(\ell^2 - \ell)\ell}.
\]

Multiplying out the two expressions obtained for \(E_n^2\) yields

\[
E_n^4 = 2e^{2n} \left( \frac{2n}{2n+1} \right)^{(2n+1)^2} \prod_{k=2}^{2n} \left( 1 - \frac{1}{k^2} \right)^{-k^2(-1)^k} = 2e^{2n} \left( \frac{2n}{2n+1} \right)^{(2n+1)^2} \prod_{k=2}^{2n} \left( 1 - \frac{1}{k^2} \right)^{-k^2(-1)^k}.
\]

But \(2e^{2n} \left( \frac{2n}{2n+1} \right)^{(2n+1)^2}\) tends to \(2e^{-3/2}\) when \(n\) tends to infinity (take the logarithm). We also have (see [1] p. 287)

\[
\lim_{n \to \infty} \prod_{k=2}^{2n} \left( \frac{k^2 - 1}{k^2} \right)^{-k^2(-1)^k} = \lim_{n \to \infty} \prod_{k=2}^{2n} \left( 1 - \frac{1}{k^2} \right)^{-k^2(-1)^k} = \frac{\pi}{4} \exp \left( \frac{1}{2} + \frac{7\zeta(3)}{\pi^2} \right)
\]

and

\[
\lim_{n \to \infty} \prod_{\ell=2}^{2n+1} \left( 1 - \frac{1}{\ell} \right)^{-\ell} = \lim_{n \to \infty} \left( \prod_{k=1}^{n} \left( 1 - \frac{1}{2k} \right) \prod_{k=1}^{n} \left( 1 - \frac{1}{2k+1} \right)^{-1} \right) = \lim_{n \to \infty} \prod_{k=1}^{n-1} \left( \frac{(2k+1)(2k+3)}{(2k)^2} \right) \frac{\Gamma(1)^2}{\Gamma \left( \frac{3}{2} \right) \Gamma \left( \frac{5}{2} \right)} = \frac{2}{\frac{2}{\pi}} = \frac{2}{\pi}.
\]

(see, e.g., [3] Section 12-13]). Hence, finally,

\[
\lim_{n \to \infty} E_n^4 = \exp \left( -1 + \frac{7\zeta(3)}{\pi^2} \right), \quad \text{thus} \quad \lim_{n \to \infty} E_n = \exp \left( -\frac{1}{4} + \frac{7\zeta(3)}{4\pi^2} \right)
\]

which is Formula (2). □

**Remark 1** In [6] the authors note that multiplying Formulas (1) and (2) together and squaring imply the following relation

\[
\lim_{k \to \infty} \left( \frac{2^{2k} \cdot 4^{4k} \cdot 6^{6k} \cdots (2k)^{(2k)^2}}{1^{1^2} \cdot 3^{3^2} \cdot 5^{5^2} \cdots (2k-1)^{(2k-1)^2}} \right)^4 \left( \frac{(2k+2)^{4k+5}}{(2k+1)^{12k+9}} \right)^k = \exp \left( \frac{7\zeta(3)}{\pi^2} \right)
\]
which they show equivalent to the formula given by Guillera and Sondow in [3, Example 5.3]
\[
\left( \frac{2^1}{1^1} \right) \frac{1}{2^3} \left( \frac{2^2}{1^1 \cdot 3^1} \right) \frac{2^4}{2^3} \left( \frac{2^3 \cdot 4^1}{1^1 \cdot 3^3} \right) \frac{4^4}{2^3} \left( \frac{2^4 \cdot 4^1}{1^1 \cdot 3^6 \cdot 5^1} \right) \frac{4^4}{2^3} \ldots = \exp \left( \frac{7\zeta(3)}{4\pi^2} \right).
\]
The authors of [6] also note that Formula (3) is a rearrangement of a formula given by
Guillera and Sondow in [3, Example 5.5]
\[
\left( \frac{3^1}{1^1} \right) \frac{1}{2^3} \left( \frac{3^2}{1^1 \cdot 5^1} \right) \frac{2^4}{2^3} \left( \frac{3^3 \cdot 7^1}{1^1 \cdot 5^3} \right) \frac{7^7}{2^3} \left( \frac{3^4 \cdot 7^1}{1^1 \cdot 5^6 \cdot 9^1} \right) \frac{7^7}{2^3} \ldots = \exp \left( \frac{G}{\pi} \right).
\]
which is in turn equivalent to
\[
\lim_{k \to \infty} \left( \frac{3^3 \cdot 7^7 \cdot 11^{11} \ldots (4k - 1)^{4k-1}}{1^1 \cdot 5^5 \cdot 9^9 \ldots (4k - 3)^{4k-3}} \right)^2 \frac{(4k + 3)^{2k+1}}{(4k + 1)^{6k+1}} = \exp \left( \frac{4G}{\pi} \right).
\]
We thus see that both formulas in [3, Example 5.3] and [3, Example 5.5] can be deduced from known values of the functions \( D \) and \( E \) in [2] and [1].

4 Conclusion

In [8] the authors note that Borwein-Dykshoorn formulas
\[
\lim_{n \to \infty} \prod_{n=1}^{2N+1} \left( 1 + \frac{1}{n} \right)^{n(-1)^{n+1}} = e \lim_{n \to \infty} \prod_{n=1}^{2N} \left( 1 + \frac{1}{n} \right)^{n(-1)^{n+1}} = \frac{A^6}{2^{1/6} e^{\sqrt{\pi}}}
\]
can be written
\[
\prod_{n=1}^{\infty} \left( \frac{\frac{1}{n}}{1 + \frac{\frac{1}{n}}{n}} \right)^{(-1)^{n-1}} = \frac{2^{1/6} e^{\sqrt{\pi}}}{A^6}.
\]
A similar reasoning proves that
\[
D(x) = \lim_{k \to \infty} \prod_{k=1}^{2n+1} \left( 1 + \frac{x}{k} \right)^{k(-1)^{k+1}} = e^x \prod_{k=1}^{\infty} \left( e^{-x} \left( 1 + \frac{x}{k} \right) \right)^{(-1)^{k+1}}.
\]
This in turn implies that
\[
\log D(x) = x + \sum_{k=1}^{\infty} (-1)^{k+1} \left( -x + k \log \left( 1 + \frac{x}{k} \right) \right)
\]
Now recall the definition of the “parameterized-Euler-constant function” \( \gamma_\alpha(z) \) defined in [10, Definition 3.1] for \(|z| \leq 1\) and \( \alpha > -1 \) by
\[
\gamma_\alpha(z) = \sum_{n=1}^{\infty} z^{n-1} \left( \frac{\alpha}{n} - \log \left( 1 + \frac{\alpha}{n} \right) \right).
\]
For $|z| < 1$ we have

$$\gamma_\alpha(z) + z\gamma'_\alpha(z) = \sum_{n=1}^{\infty} z^{n-1} \left( \alpha - n \log \left( 1 + \frac{\alpha}{n} \right) \right).$$

Thus (with the same justification as in the proof of [8, Theorem 16]) we have

$$D(x) = e^{1+\gamma'_\alpha(-1)-\gamma_\alpha(-1)}.$$

After we put a first version of this paper on ArXiv, K. Hessami Pilehrood indicated to us that Xia’s function is actually a particular case of the function $\gamma_{a,b}(z)$ introduced and studied in [4]

$$\gamma_{a,b}(z) = \sum_{n=0}^{\infty} \left( \frac{1}{an+b} - \log \left( \frac{an+b+1}{an+b} \right) \right) z^n.$$

This definition is [4, Relation (14)] ($a, b \in \mathbb{N}, |z| \leq 1$), while [4, Theorem 1] gives the analytic continuation of $\gamma_{a,b}(z)$ for $a, b$ positive reals and $z \in \mathbb{C} \setminus [1, +\infty)$. It is clear that (the function on the left side is the one in [4], the one on the right side is the one in [10])

$$\gamma_{1\alpha,1/\alpha}(z) = \gamma_{\alpha}(z).$$

Note that, in view of [4, Corollary 3] (see also [10, 3.6]), this gives an expression of $D(x)$ in terms of the Lerch transcendent (see [3, 8]) $\Phi(z, s, u) = \sum_{n \geq 0} \frac{1}{(an+b)^u}$ and its derivatives. It is then no real surprise that the quantities $7\zeta(3)/4\pi^2$ and $G/\pi$ also occur in Examples 2.2 and 2.3 of [3] in the relations

$$\frac{\partial \Phi}{\partial s}(-1, -2, 1) = \frac{7\zeta(3)}{4\pi^2} \quad \text{and} \quad \frac{\partial \Phi}{\partial s}(-1, -1, \frac{1}{2}) = \frac{G}{\pi}.$$

**Remark 2** A product similar to the products studied in [6] is given by Holcombe in [5]:

$$\pi = e^{3/2} \prod_{n=2}^{\infty} e \left( 1 - \frac{1}{n^2} \right)^{n^2}.$$

**References**

[1] V. S. Adamchik, The multiple gamma function and its application to computation of series, *Ramanujan J.* 9 (2005) 271–288.

[2] P. Borwein, W. Dykshoorn, An interesting infinite product, *J. Math. Anal. Appl.* 179 (1993) 203–207.

[3] J. Guillera, J. Sondow, Double integrals and infinite products for some classical constants via analytic continuation of Lerch’s transcendent, *Ramanujan J.* 16 (2008) 243–270.
[4] K. Hessami Pilehrood, T. Hessami Pilehrood, Vacca-type series for values of the generalized Euler constant function and its derivative, *J. Int. Seq.* **13** (2010) Article 10.7.3.

[5] S. R. Holcombe, A product representation for $\pi$, Preprint (2013), available at the URL http://arxiv.org/abs/1204.2451

[6] Y. Kachi, P. Tzermias, Infinite products involving $\zeta(3)$ and Catalan’s constant, *J. Int. Seq.* **15** (2012) Article 12.9.4.

[7] Z. A. Melzak, Infinite products for $\pi e$ and $\pi/e$, *Amer. Math. Monthly* **68** (1961) 39–41.

[8] J. Sondow, P. Hadjicostas, The generalized-Euler-constant function $\gamma(z)$ and a generalization of Somos’s quadratic recurrence constant, *J. Math. Anal. Appl.* **332** (2007) 292–314.

[9] E. T. Whittaker, G. N. Watson, *A Course of Modern Analysis*, Fourth Edition, reprinted, Cambridge University Press, Cambridge, 1996.

[10] L.-m. Xia, The parameterized-Euler-constant function $\gamma_\alpha(z)$, *J. Number Theory* **133** (2013) 1–11.