REMARKS ON TRIGONOMETRIC FUNCTIONS AFTER EISENSTEIN

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Abstract. We modify the Whittaker-Watson account of the Eisenstein approach to the trigonometric functions, basing these functions independently on the Eisenstein function $\varepsilon_2$.

0. Introduction

Eisenstein [E] initiated a novel approach to the theory of the trigonometric functions, based on the meromorphic functions defined by

$$\varepsilon_k(z) = \sum_{n \in \mathbb{Z}} \frac{1}{(z+n)^k}$$

for $k$ a positive integer and $z \in \mathbb{C} \setminus \mathbb{Z}$. These functions were named in honour of Eisenstein by Weil, who elaborated details of the somewhat mystical calculations and further developed the theory in [W]. Of course, this novel approach to the trigonometric functions was but an offshoot or a shadow of the larger theory of elliptic functions. In their account of the Weierstrassian elliptic function theory, Whittaker and Watson [WW] include a very brief introduction to this trigonometric theory by way of illustration.

A little more explicitly, the approach of [E] as explicated in [W] develops the theory of trigonometric functions from the fundamental formula

$$\varepsilon_1(z) = \pi \cot \pi z.$$ 

This formula is intended as a definition of the cotangent function in terms of the positive constant $\pi$ defined by

$$\pi^2 = 6 \sum_{n=1}^{\infty} \frac{1}{n^2}.$$ 

For the identification of $\varepsilon_1(z)$ with $\pi \cot \pi z$ as it is ordinarily understood, we refer to Remmert [R]; this reference also contains an outline of the Eisenstein approach and places it in historical context.

Our purpose here is to modify the approach adopted in [WW] so as to develop the trigonometric functions from the Eisenstein series $\varepsilon_2$. The approach in [WW] does not lend itself directly to a wholly independent construction of the trigonometric functions, as it incorporates $\pi$ with its ordinary meaning and makes use of the classical formulae

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$ 

When the approach in [WW] is reformulated so as not to assume $\pi$ with its ordinary meaning, the proof given there requires independent knowledge of the identity

$$2 \left( \sum_{n=1}^{\infty} \frac{1}{n^2} \right)^2 = 5 \sum_{n=1}^{\infty} \frac{1}{n^4}.$$ 

Our modification circumvents the need for this independent knowledge and indeed has this identity as a consequence. The approach in [WW] essentially identifies $\varepsilon_2(z)$ as $\pi^2 \csc^2 \pi z$ by virtue of its satisfying certain nonlinear differential equations of first and second order. Our
modification goes beyond this: the reciprocal of $\varepsilon^2$ satisfies the second-order linear differential equation

$$g'' + \left(24 \sum_{n=1}^{\infty} \frac{1}{n^2}\right) g = 2$$

from which the elementary trigonometric functions are immediately in evidence. Our approach has other benefits: for example, it eliminates the need for such tools as the Herglotz trick and the maximum modulus principle, which feature in some accounts of the theory.

1. A MODIFIED APPROACH

Our starting point is the second Eisenstein series, which we rename $f$ for simplicity:

$$f(z) = \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2}$$

for $z \in \mathbb{C} \setminus \mathbb{Z}$. The indicated series is normally convergent: let $K \subseteq \mathbb{C} \setminus \mathbb{Z}$ be compact and choose $R > 0$ so that $K$ lies in the disc $D_R(0)$; if $z \in K$ and $|n| > R$ then $|z-n| > |n| - R$ so that

$$\sum_{|n|>R} \frac{1}{|z-n|^2} \leq \sum_{|n|>R} \frac{1}{(|n|-R)^2}$$

and the uniformly majorizing series on the right converges by the limit comparison test. As a consequence, $f : \mathbb{C} \setminus \mathbb{Z} \to \mathbb{C}$ is holomorphic; moreover, $f$ is plainly even and of period one. At each integer, $f$ has a double pole: around zero,

$$f(z) = z^{-2} + \sum_{0 \neq n \in \mathbb{Z}} (z-n)^{-2}$$

where the second summand on the right is holomorphic in the open unit disc, there having Taylor expansion

$$\sum_{0 \neq n \in \mathbb{Z}} (z-n)^{-2} = \sum_{d=0}^{\infty} a_d z^{2d}$$

with

$$a_d = 2(2d+1) \sum_{n=1}^{\infty} n^{-(2d+2)}$$

as follows from the derived geometric series.

We now employ a familiar device, combining suitable derivatives and powers of $f$ so as to eliminate the poles. The Laurent expansion of $f(z)$ about the origin reads

$$f(z) = z^{-2} + a_0 + a_1 z^2 + \ldots$$

so that

$$f'(z) = -2z^{-3} + 2a_1 z + \ldots$$

and

$$f''(z) = 6z^{-4} + 2a_1 + \ldots$$

while

$$f(z)^2 = z^{-4} + 2a_0 z^{-2} + (a_0^2 + 2a_1) + \ldots$$

The combination $f'' - 6f^2 + 12a_0 f$ is of course holomorphic in $\mathbb{C} \setminus \mathbb{Z}$ and has period one; its singularities at the integers are removable, in view of the expansion

$$f''(z) - 6f(z)^2 + 12a_0 f(z) = (6a_0^2 - 10a_1) + \ldots$$

about the origin, where the ellipsis indicates a power series involving terms of degree two or greater. Removing these singularities, $f'' - 6f^2 + 12a_0 f$ becomes an entire function.
To proceed further, we examine the behaviour of \( f \) in the vertical strip
\[ S = \{ z \in \mathbb{C} \mid \text{Re } z \leq 1 \}. \]

**Theorem 1.** \( f(z) \to 0 \) as \( z \to \infty \) in the strip \( S \).

**Proof.** Let \( z = x + iy \in S \) so that \( |x| \leq 1 \) and if \( n \in \mathbb{Z} \) then \( |z - n|^2 = (n - x)^2 + y^2 \). If \( |n| \leq 1 \) then \( |z - n|^2 \geq y^2 \) while if \( |n| > 1 \) then \( |z - n|^2 \geq \left(|n| - 1\right)^2 + y^2 \). Accordingly, it follows that
\[
|f(z)| \leq \frac{3}{y^2} + 2 \sum_{n=1}^{\infty} \frac{1}{n^2 + y^2}.
\]
As \( z \to \infty \) in \( S \) we need only inspect the second summand on the right. For any \( N \) we have
\[
\sum_{n=1}^{\infty} (n^2 + y^2)^{-1} = \sum_{1 \leq n < N} (n^2 + y^2)^{-1} + \sum_{n > N} (n^2 + y^2)^{-1}.
\]
Let \( \varepsilon > 0 \): choose \( N \) so that \( \sum_{n > N} n^{-2} < \varepsilon \); it follows that if \( |y| > \sqrt{N/\varepsilon} \) then
\[
\sum_{n=1}^{\infty} (n^2 + y^2)^{-1} \leq N y^{-2} + \sum_{n > N} n^{-2} < 2\varepsilon.
\]
\[ \square \]

We now see that the entire function \( f'' - 6f^2 + 12a_0f \) is as trivial as can be.

**Theorem 2.** The meromorphic function \( f \) satisfies
\[
f'' - 6f^2 + 12a_0f = 0.
\]

**Proof.** The argument of Theorem 1 adapts easily to show that the second derivative \( f''(z) = 6 \sum_{n \in \mathbb{Z}} (z - n)^{-4} \) also tends to 0 as \( z \to \infty \) in \( S \). The entire function \( f'' - 6f^2 + 12a_0f \) is thus bounded in \( S \) and so bounded on \( \mathbb{C} \) by periodicity. According to the Liouville theorem, \( f'' - 6f^2 + 12a_0f \) is constant; the value of this constant is 0 because \( f'' - 6f^2 + 12a_0f \) vanishes at infinity. \( \square \)

Thus the constant term \( 6a_0^2 - 10a_1 \) in the expansion of \( f'' - 6f^2 + 12a_0f \) about the origin is zero. When we substitute the expressions for \( a_0 \) and \( a_1 \) and then simplify, we obtain the identity
\[
2 \left[ \sum_{n=1}^{\infty} \frac{1}{n^2} \right]^2 = 5 \sum_{n=1}^{\infty} \frac{1}{n^4}.
\]

**Theorem 3.** The meromorphic function \( f \) satisfies
\[
(f')^2 - 4f^3 + 12a_0f^2 = 0.
\]

**Proof.** Multiply the equation of Theorem 2 by \( 2f' \) to obtain
\[
2f'f'' - 12ff' + 24a_0f'f = 0
\]
and then integrate to obtain
\[
(f')^2 - 4f^3 + 12a_0f^2 = c
\]
for some \( c \in \mathbb{C} \). As \( f \) and (similarly) \( f' \) vanish at infinity, \( c = 0 \). \( \square \)

**Theorem 4.** The function \( f \) is nowhere zero.

**Proof.** Theorem 2 and Theorem 3 tell us that if we write \( 2p(w) = 4w^3 - 12a_0w^2 \) then \( f'' = p' \circ f \) and \( (f')^2 = 2p \circ f \). An elementary induction shows that each even-order derivative of \( f \) is a polynomial in \( f \) with vanishing constant term: for the inductive step, if \( f^{(2d)} = q \circ f \) then \( f^{(2d+1)} = (q' \circ f)f' \) and \( f^{(2d+2)} = (2q'p + q'p') \circ f \); the square of each odd-order derivative of \( f \) is then also a polynomial in \( f \) with vanishing constant term. Finally, if \( f \) were to vanish at \( a \in C \setminus \mathbb{Z} \) then all its derivatives would vanish at \( a \); the Identity Theorem would then force \( f \) itself to vanish, which is absurd. \( \square \)
We may now introduce the reciprocal function $g = 1/f$: as $f$ is a nowhere-zero meromorphic function with a double pole at each integer, $g$ is an entire function with a double zero at each integer; as $f$ is even and of period one, $g$ is even and of period one.

**Theorem 5.** The entire function $g$ satisfies
\[ g'' + 12a_0 g = 2. \]

Proof. Simply differentiate and then substitute from Theorem 2 and Theorem 3 $g' = -f^{-2} f'$ so that $g'' = 2f^{-3}(f')^2 - f^{-2} f'' = 2f^{-3}(4f^3 - 12a_0 f^2) - f^{-2}(6f^2 - 12a_0 f) = 2 - 12a_0 g$ as required. □

Recall that $g$ has a double zero at the origin; accordingly, the second-order differential equation displayed in Theorem 5 is supplemented by the initial data $g(0) = 0$ and $g'(0) = 0$.

At this point, it is quite clear that our approach has made contact with the elementary trigonometric functions. Define the positive number $\pi$ by
\[ \pi^2 := 3a_0 = 6 \sum_{n=1}^{\infty} n^{-2}. \]
Define the function $c : \mathbb{C} \to \mathbb{C}$ by the rule that if $z \in \mathbb{C}$ then
\[ c(z) := 1 - 2\pi^2 g(z/2\pi). \]
The entire function $c$ has period $2\pi$; this inbuilt periodicity is a special feature of the Eisenstein approach. Further, a direct calculation reveals that it satisfies the initial value problem
\[ c'' + c = 0; \quad c(0) = 1, \quad c'(0) = 0. \]

As an entire function, its Taylor series about the origin is consequently
\[ c(z) = \sum_{n=0}^{\infty} (-)^n \frac{z^{2n}}{(2n)!}. \]
Thus $c$ is precisely the cosine function, from which flows the whole theory of trigonometric functions. Incidentally, a duplication formula for the cosine function shows that $f(z) = \pi^2 \csc^2 \pi z$.

We close by remarking on ways in which our approach varies from the approach in [WW]. First of all, [WW] incorporates $\pi$ in the theory from the very start; its removal from the function there analyzed yields $f$. Our Theorem 5 improves the [WW] observation that $f(z)$ is bounded as $z \to \infty$ in the strip \{ $z \in \mathbb{C} : \text{Re} \ z \leq 1/2$ \}; the weaker result means that [WW] must assume the identity $2[\sum_{n=1}^{\infty} n^{-2}]^2 = 5 \sum_{n=1}^{\infty} n^{-4}$ in order to conclude that the bounded function $f'' - 6f^2 + 12a_0 f$ is identically zero. Our Theorem 5 to the effect that $f$ never vanishes permits us to pass directly to its reciprocal $g$ and thence to the elementary second-order linear differential equation in Theorem 5 by contrast, [WW] essentially stops short at the nonlinear differential equations that we display in Theorem 2 and Theorem 3.

**REFERENCES**

[E] F. G. M. Eisenstein, *Genaue Untersuchung der unendlichen Doppelprodukte ...* , Jour. für Reine und Angew. Math. 35 (1847) 153-274.

[R] R. Remmert, *Theory of Complex Functions*, Graduate Texts in Mathematics 122, Springer-Verlag (1991).

[W] A. Weil, *Elliptic Functions according to Eisenstein and Kronecker*, Ergebnisse der Mathematik 88, Springer-Verlag (1976).

[WW] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Second Edition, Cambridge University Press (1915).