Interferometer Response to Scalar Gravitational Waves

Ch. Corda
INFN - Sezione di Pisa and Università di Pisa, Via F. Buonarroti 2, I - 56127 PISA

S. A. Ali and C. Cafaro
Department of Physics, University at Albany - SUNY, 1400 Washington Avenue, Albany, NY, 12222, USA

It was recently suggested that the magnetic component of Gravitational Waves (GWs) is relevant in the evaluation of frequency response functions of gravitational interferometers. In this paper we extend the analysis to the magnetic component of the scalar mode of GWs which arise from scalar-tensor gravity theory. In the low-frequency approximation, the response function of ground-based interferometers is calculated. The angular dependence of the electric and magnetic contributions to the response function is discussed. Finally, for an arbitrary frequency range, the proper distance between two test masses is calculated and its usefulness in the high-frequency limit for space-based interferometers is briefly considered.

PACS numbers: 04.80.Nn, 04.80.-y, 04.25.Nx

Keywords: scalar gravitational waves; interferometers; magnetic components

I. INTRODUCTION

The study of the so-called magnetic component of Gravitational Waves (GWs) is an active field of research. In a series of recent works, the significance of these magnetic components to the total frequency response function of omnidirectional gravitational wave interferometers was emphasized [1, 2, 3, 4]. There are currently a number of gravitational wave detectors at various stages of development worldwide. While several such projects are in their design phase, others like the VIRGO detector (Cascina, Italy [5, 6]) are already under construction. Yet other projects, such as the GEO 600 (Hannover, Germany [7, 8]) the two LIGO detectors (Washington and Louisiana, USA [9, 10]) and the TAMA 300 detector (Tokyo, Japan [11, 12]) are functional and already taking scientific data. In view of such enormous international effort, it is prudent to carry out a thorough analysis of the expected frequency response associated with gravitational waves predicted by General Relativity and its various competing theoretical models, of which scalar-tensor gravity is one. Indeed, GW detectors will prove invaluable in the confirmation (or contradiction) of the physical consistency of the various theories of gravity [13, 14, 15, 16, 17, 18, 19, 20, 21].

Baskaran and Grishchuk have recently discussed the existence and relevance of the so-called magnetic components of GWs, which must be taken into account in the context of the total response functions (angular patterns) of interferometers for GWs propagating from arbitrary directions [3]. In [4], more detailed angular and frequency dependence of the magnetic contribution to interferometer response functions was given with specific application to the parameters of the LIGO and VIRGO interferometers.

In this paper the analysis is extended to the magnetic component of the scalar mode of gravitational waves which arise from scalar-tensor theories of gravity [14, 15, 16, 19, 21]. The angular dependence of the response function of interferometers for this magnetic component is given in the approximation of wavelength much larger than the linear dimensions of the interferometer. The results of this paper generalize the works of [3, 4] where it was shown that the electric and magnetic contributions are unambiguous in the long-wavelength approximation for ordinary GWs arising from General Relativity. In the high frequency regime, the division into electric and magnetic components becomes ambiguous. For this reason, in order to calculate the response of a GW interferometer in this regime, we shall use the exact (i.e., without low-frequency approximation) expressions for distance measurements.

The paper is organized as follows. In Section 2, a review of GWs from scalar-tensor theories of gravity is presented. An analysis of GWs arising from scalar-tensor gravity theory in the frame of a local observer is implemented in Section 3. In Section 4, we obtain the distance function between the

*Electronic address: corda@ego-gw.it
†Electronic address: alis@alum.rpi.edu
‡Electronic address: carlocafar2000@yahoo.it
two test particles of the interferometer by integrating the geodesic deviation equation. The variation of distances between test masses and the response of interferometers are considered in Section 5. In Section 6, we obtain the interferometer test mass distance function in absence of low-frequency approximation. It is then shown that the electric and magnetic contributions of the distance function arise from the series expansion of this exact result. Our conclusions are presented in Section 7.

II. SCALAR GRAVITATIONAL WAVES FROM SCALAR-TENSOR THEORIES OF GRAVITY

If the gravitational Lagrangian is nonlinear in the curvature invariants, the corresponding Einstein field equations have an order higher than second \cite{14, 15, 16, 19, 21}. For this reason such theories are often called higher-order gravitational theories. In the most general case, such higher order theories arise from an action of form

\[ S = \int d^4x \sqrt{-g} \left[ F(R, \Box R, \Box^2 R, \phi) - \epsilon \frac{\phi}{2} g^{\mu \nu} \phi_{,\mu} \phi_{,\nu} + \mathcal{L}_m \right], \]  

where \( F \) is an unspecified function of curvature invariants and a scalar field \( \phi, D_\mu = \partial_\mu + \Gamma_\mu \) is the covariant derivative with respect to the Christoffel connection coefficients \( \Gamma_\mu \) and \( \Box := g^{\mu \nu} D_\mu D_\nu \) is the d'Alembertian operator. The semicolon in \( \phi_{,\mu} \) represents \( D_\mu \)-differentiation. The term \( \mathcal{L}_m \) is the minimally coupled ordinary matter contribution.

In scalar-tensor theories of gravity, both the metric tensor \( g_{\mu \nu} \) and a fundamental scalar field \( \phi \) are involved \cite{14, 19}. The action of scalar-tensor gravity theory can be recovered from (1) with the choice

\[ F(R, \phi) = f(\phi)R - V(\phi) \]  

Note that in this article we work with geometrized units \( G = c = \hbar = 1 \). Considering the choice (2), the most general action of scalar-tensor gravity theory in four dimensions is given by

\[ S = \int d^4x \sqrt{-g} \left[ f(\phi)R + \frac{1}{2} g^{\mu \nu} \phi_{,\mu} \phi_{,\nu} - V(\phi) + \mathcal{L}_m \right]. \]  

By choosing

\[ \phi = f(\phi), \quad \omega(\phi) = \frac{f'(\phi)}{2f(\phi)}, \quad W(\phi) = V(\phi(\phi)), \]  

equation (3) reduces to

\[ S = \int d^4x \sqrt{-g} \left[ \omega(\phi) R + \frac{\omega(\phi)}{\phi} g^{\mu \nu} \phi_{,\mu} \phi_{,\nu} - W(\phi) + \mathcal{L}_m \right]. \]  

Action (5) represents a generalization of Brans-Dicke theory \cite{22}. Variation of action (5) with respect to \( g_{\mu \nu} \) and \( \phi \) results in the Einstein-like equation

\[ G_{\mu \nu} = -\frac{4\pi \tilde{G} T^{(m)}_{\mu \nu}}{\phi} + \frac{\omega(\phi)}{\phi^2} \left( \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} g_{\mu \nu} g^{\alpha \beta} \phi_{,\alpha} \phi_{,\beta} \right) + \frac{1}{\phi} \left( \phi_{,\mu} \phi_{,\nu} - g_{\mu \nu} \Box \phi \right) + \frac{1}{2\phi} g_{\mu \nu} W(\phi) \]  

and Klein-Gordon equation

\[ \Box \phi = \frac{1}{2\omega(\phi) + 3} \left( -4\pi \tilde{G} T^{(m)} + 2W(\phi) + \sqrt{\phi} W'(\phi) + \frac{d\omega(\phi)}{d\phi} g^{\mu \nu} \phi_{,\mu} \phi_{,\nu} \right), \]

respectively. The quantity \( T^{(m)}_{\mu \nu} \) appearing in (6) is the ordinary stress-energy tensor of matter while \( \tilde{G} \) is a dimensional, positive definite constant \cite{14, 17, 19}. The Newton constant is replaced by the effective coupling

\[ G_{\text{eff}} = -\frac{1}{2\phi} G, \text{ where } G = 1 \]  

which is, in general, different from \( G \). General Relativity is obtained when the scalar field coupling is

\[ \phi = \text{constant} = -\frac{1}{2}. \]
When $\omega = \omega(\varphi) = \text{const.}$ in equations (6) and (7) the field equations describe the so-called string-dilaton gravity [14, 17, 19]. Since we study gravitational waves in this work, the linearized theory in vacuum ($T_{\mu\nu}^{(m)} = 0$) with a small perturbation $h_{\mu\nu}$ of the background will be analyzed. It is assumed that the tensorial sector of the background is Minkowskian $\eta_{\mu\nu}$ while the scalar sector is $\varphi = \varphi_0$ where $\varphi_0$ is assumed to be a minimum of $W$,

$$W \approx \frac{1}{2} \alpha \delta \varphi^2 \Rightarrow W' \approx \alpha \delta \varphi.$$

(10)

For the perturbed background specified by

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \varphi = \varphi_0 + \delta \varphi,$$

(11)

the linearized field equations are given by [14, 19, 23]

$$\ddot{R}_{\mu\nu} - \frac{\ddot{R}}{2} \eta_{\mu\nu} = -\partial_{\mu} \partial_{\nu} \Phi + \eta_{\mu\nu} \Box \Phi, \quad \Box \Phi = m^2 \Phi,$$

(12)

with

$$\Phi \equiv -\frac{\delta \varphi}{\varphi_0}, \quad m^2 \equiv \frac{\alpha \varphi_0}{2\omega + 3},$$

(13)

where $\ddot{R}_{\mu\nu\rho\sigma}, \dddot{R}_{\mu\nu}$ and $\dddot{R}$ are the linearized versions of $R_{\mu\nu\rho\sigma}, R_{\mu\nu}$ and $R$ respectively, and are moreover, computed to first order in $h_{\mu\nu}$ and $\delta \varphi$.

The case in which $\omega = \text{const.}$ and $W = 0$ in (6) and (7) has been analyzed in [19] in a manner that generalizes the canonical linearization of General Relativity [23]. In particular, the transverse-traceless (TT) gauge has been generalized to scalar-tensor gravity obtaining the total perturbation of a gravitational wave propagating in the $z$-direction (in the TT gauge) as,

$$h_{\mu\nu}(t + z) = A^+(t + z)e_{\mu\nu}^{(+)} + A^x(t + z)e_{\mu\nu}^{(x)} + \Phi(t + z)e_{\mu\nu}^{(s)}.$$  

(14)

The term $A^+(t + z)e_{\mu\nu}^{(+)} + A^x(t + z)e_{\mu\nu}^{(x)}$ describes the two standard (i.e. tensorial) polarizations of gravitational waves which arises from General Relativity in the TT gauge [23], while the term $\Phi(t + z)e_{\mu\nu}^{(s)}$ is the extension of the TT gauge to the scalar case. For a purely scalar GW, the metric perturbation (13) reduces to

$$h_{\mu\nu} = \Phi e_{\mu\nu}^{(s)},$$

(15)

and the corresponding line element is [19, 21]

$$ds^2 = dt^2 - dz^2 - (1 + \Phi) dx^2 - (1 + \Phi) dy^2,$$

(16)

with $\Phi = \Phi_0 e^{i\omega(t + z)}$.

The worldline $x, y, z = \text{const.}$ is a timelike geodesic representing the history of a free test mass (see the analogy with tensorial waves in [3, 4, 23]). In other words, in scalar-tensor theories of gravity the scalar field generates a third tensorial polarization for gravitational waves. This is because in equations (32) of [19] three different degrees of freedom are present, in contrast to standard General Relativity where gravitational waves exhibit two degrees of freedom. For details concerning the generation of this third tensorial polarization, see Section 3 of [19].

### III. ANALYSIS IN THE FRAME OF THE LOCAL OBSERVER

In a laboratory environment on Earth, a coordinate system (frame of local observer) in which space-time is locally flat is typically used [2, 4, 19, 20, 21, 23]. In this frame, the distance between any two points is given simply by the difference in their coordinates (in the sense of Newtonian physics). Moreover, in this frame, SGWs manifest themselves by exerting tidal forces on masses (the mirror and the beam-splitter in the case of an interferometer, see Figure 1).
Photons launched from the beam-splitter reflected back by the mirror.

A detailed analysis of the frame of the local observer is given in ref. [23], Section 13.6. Here only the essential features of this frame are emphasized. In particular, the coordinate $x_0$ is the proper time of the observer $O$; spatial axes are centered in $O$; in the special case of zero acceleration and zero rotation, the spatial coordinates $x_j$ are the proper distances along the axes and the frame of the local observer reduces to a local Lorentz frame. In this case the line element reads

$$ds^2 = -(dx^0)^2 + \delta_{ij}dx^i dx^j + \mathcal{O}(x^2),$$

where

$$\mathcal{O}(x^2) = \frac{1}{2} \left( \frac{\partial^2 g_{\mu \nu}}{\partial x^\sigma \partial x^\rho} \right)_{\rho_0} (x^\sigma - x^\sigma_{\rho_0}) (x^\rho - x^\rho_{\rho_0}) dx^\mu dx^\nu$$

with $P_0$ being the point where the Lorentz frame is placed. The effect of SGWs on test masses is described by the equation for geodesic deviation in this frame

$$\ddot{x}^i = -\ddot{R}^i_{0k0} x^k,$$

where $\ddot{R}^i_{0k0}$ are components of the linearized Riemann tensor [4, 21, 22]. Labelling the coordinates of the TT gauge as $t_{tt}, x_{tt}, y_{tt}, z_{tt}$, the coordinate transformation $x^\alpha = x^\alpha(x^0_{tt})$ from TT coordinates to the frame of the local observer is

$$t = t_{tt} + \frac{1}{2}(x^2_{tt} - y^2_{tt}) \dot{\Phi}(t),$$

$$x = x_{tt} + \frac{1}{2}x_{tt} \dot{\Phi}(t) + \frac{1}{2}x_{tt}z_{tt} \dot{\Phi}(t),$$

$$y = y_{tt} + \frac{1}{2}y_{tt} \dot{\Phi}(t) + \frac{1}{2}y_{tt}z_{tt} \dot{\Phi}(t),$$

$$z = z_{tt} - \frac{1}{2}(x^2_{tt} - y^2_{tt}) \dot{\Phi}(t),$$

where $\dot{\Phi}(t) \equiv \frac{\partial \Phi(t)}{\partial t}$ (see the analogy with tensorial waves in [3, 4, 24, 25]). The coefficients of this transformation (components of the metric and its first time derivative) are taken along the central worldline of the local observer $t = 0, x = 0, y = 0, z = 0$. [3, 24, 25]. The linear and quadratic terms, as powers of $x^0_{tt}$, are unambiguously determined by the conditions of the frame of the local observer, while the cubic and higher-order corrections are not determined by these conditions [3, 4, 24, 25]. **Cubic and higher-order corrections are not determined by these conditions.**

Considering a free mass with a timelike geodesic ($x = l_1, y = l_2, z = l_3$) [3, 4], equations (20) define the motion of this mass with respect to the introduced frame of the local observer, namely

$$x(t) = l_1 + \frac{1}{2}l_1 \dot{\Phi}(t) + \frac{1}{2}l_1l_3 \dot{\Phi}(t),$$

$$y(t) = l_2 + \frac{1}{2}l_2 \dot{\Phi}(t) + \frac{1}{2}l_2l_3 \dot{\Phi}(t),$$

$$z(t) = l_3 - \frac{1}{4}(l_1^2 - l_2^2) \dot{\Phi}(t).$$
In absence of GWs the position of the mass is \( \tilde{x}_{\text{mass}} = (l_1, l_2, l_3) \). The effect of the SGW is to drive the mass to have oscillations. Thus, in general all three components of motion are present in (21). Neglecting the terms with \( \dot{\Phi} \) in (21), the “traditional” equations for the motion of the mass are obtained \([19, 21]\),

\[
\begin{align*}
x(t) &= l_1 + \frac{1}{2} l_1 \Phi(t), \\
y(t) &= l_2 + \frac{1}{2} l_2 \Phi(t), \\
z(t) &= l_3.
\end{align*}
\]

Clearly, this is analogous to the electric component of motion in electrodynamics \([3, 4]\), while equations

\[
\begin{align*}
x(t) &= l_1 + \frac{1}{2} l_1 l_3 \Phi(t), \\
y(t) &= l_2 + \frac{1}{2} l_2 l_3 \Phi(t), \\
z(t) &= l_3 - \frac{1}{4} (l_1^2 - l_2^2) \Phi(t),
\end{align*}
\]

are analogues of the magnetic component of motion. One might suspect the presence of this magnetic component is a “frame artifact” due to transformation (20). This issue will be addressed in the next Section where it will be shown that (24) is obtained directly from the geodesic deviation equation, verifying that the magnetic component becomes significant when the frequency of the wave increases, but only in the low-frequency regime. This can be understood directly from equations (21). In fact, recalling that \( \Phi = \Phi_0 e^{i \omega (t + z)} \), (21) becomes

\[
\begin{align*}
x(t) &= l_1 + \frac{1}{2} l_1 \Phi(t) + \frac{1}{2} l_1 l_3 \omega \Phi(\omega t - \frac{\pi}{2}), \\
y(t) &= l_2 + \frac{1}{2} l_2 \Phi(t) + \frac{1}{2} l_2 l_3 \omega \Phi(\omega t - \frac{\pi}{2}), \\
z(t) &= l_3 - \frac{1}{4} (l_1^2 - l_2^2) \omega \Phi(\omega t - \frac{\pi}{2}).
\end{align*}
\]

Thus, terms with \( \dot{\Phi} \) in (21) can be neglected only when the wavelength goes to infinity, while at high-frequencies expansion terms proportional to \( \omega l_i l_j \ (i, j = 1, 2, 3) \) corrections in (24) break down.

### IV. EQUATIONS OF MOTION FROM GEODESIC DEVIATION

In this Section we consider the geodesic deviation extended to second order approximation from which it is shown that the magnetic component of a SGW is not a frame artifact. Consider a two-parameters congruence of geodesics. Let \( \gamma_1 (\tau, r) \) and \( \gamma_2 (\tau, r) \) be two neighboring geodesics with initially-parallel tangent vectors \( u^\alpha \) and connecting vectors \( n^\beta \). The selector parameter \( r \) tells “which” geodesic is being considered, while the affine parameter \( \tau \) tells “where” on a given geodesic someone is. Then, the relative acceleration of these two neighboring geodesics is given, at the lowest order of approximation \([23, 24]\), by the Jacobi-Levi-Civita (JLC equation) of geodesic spread,

\[
\frac{D^2 n^\delta}{d\tau^2} = R_{\alpha\beta\gamma}^\delta u^\alpha n^\beta u^\gamma
\]

(25)

where

\[
u^\alpha(\tau, r) = \frac{\partial x^\alpha(\tau, r)}{\partial \tau} \bigg|_{r=\text{const}} \]

(26)
is the unit vector tangent to the geodesic \( \gamma (\tau, r) \) and

\[
n^\alpha(\tau, r) = \frac{\partial x^\alpha(\tau, r)}{\partial r} \bigg|_{\tau=\text{const}} \delta r
\]

(27)
is the separation vector between two nearby geodesics. The quantity \( R_{\alpha\beta\gamma}^\delta \) in (25) is the Riemann curvature tensor defined in the standard way as \( R_{\alpha\beta\gamma}^\delta = \partial_\beta \Gamma^\delta_{\alpha\gamma} - \partial_\gamma \Gamma^\delta_{\alpha\beta} + \Gamma^\delta_{\sigma\beta} \Gamma^\sigma_{\alpha\gamma} - \Gamma^\delta_{\sigma\gamma} \Gamma^\sigma_{\alpha\beta} \) and is calculated along the central geodesic.
The Christoffel connection coefficients $\Gamma_{\beta\gamma}^\alpha$ are defined by $\Gamma_{\beta\gamma}^\alpha = \frac{1}{2}g^{\alpha\sigma}(\partial_\beta g_{\sigma\nu} + \partial_\nu g_{\sigma\beta} - \partial_\sigma g_{\beta\nu})$. The operator $\frac{D}{dt}$ is the covariant derivative calculated along that line. It is assumed that $r = 0$ corresponds to the central geodesic line while a second nearby geodesic corresponds to $r = r_0$. To discuss the magnetic component of motion in the field of a SGW we require the geodesic deviation equations be extended to the next order approximation. These equations were obtained in [27], while a modified derivation can be found in [28].

Consider the vector $w^\alpha$ defined as

$$w^\alpha = \frac{Dn^\alpha}{dr} = n^\alpha ;_\beta n^\beta = \frac{\partial^2 x^\alpha}{\partial r^2} + \Gamma_{\beta\gamma}^\alpha u^\beta u^\gamma.$$  

(28)

This vector obeys the equations [3, 27, 28]

$$\frac{D^2 w^\delta}{dr^2} = R_{\alpha\beta\gamma}^{\delta} u^\alpha u^\gamma N^\beta + (R_{\alpha\beta\gamma}^{\delta} ;_\xi - R_{\gamma\alpha\beta}^{\delta} ;_\beta)u^\alpha u^\beta u^\gamma u^\epsilon + 4R_{\alpha\beta\gamma}^{\delta} u^\beta \frac{Dn^\alpha}{dr} n^\gamma.$$  

(29)

Defining the vector

$$N^\alpha \equiv r_0 n^\alpha + \frac{1}{2}r_0^2 w^\alpha,$$  

(30)

equations [24] and [28] can be combined to give [3, 27, 28]

$$\frac{D^2 N^\delta}{dr^2} = R_{\alpha\beta\gamma}^{\delta} u^\alpha u^\gamma N^\beta + (R_{\alpha\beta\gamma}^{\delta} ;_\xi - R_{\gamma\alpha\beta}^{\delta} ;_\beta)u^\alpha u^\beta u^\gamma N^\epsilon + 2R_{\alpha\beta\gamma}^{\delta} u^\beta \frac{Dn^\alpha}{dr} N^\gamma + O(r_0^3).$$  

(31)

It is then possible to write the expansion of $x^\alpha(\tau, r_0)$ in terms of $N^\alpha$ as

$$x^\alpha(\tau, r_0) = x^\alpha(\tau, 0) + N^\alpha - \Gamma_{\beta\gamma}^\alpha N^\beta N^\gamma + O(r_0^3).$$  

(32)

This formula shows that in the frame of the local observer (in which the Christoffel connection coefficients $\Gamma_{\beta\gamma}^\alpha = 0$ along the central geodesic line [24]) the spatial components of $N^\alpha$ will directly give the time-dependent position of the nearby test mass. According to [31], these positions include the next-order corrections as compared with solutions to [24].

We now specialize to the SGW metric [16], and take into account only linear perturbations in the SGW amplitude $\Phi$. The first test mass is described by the central timelike geodesic $x^i(t) = 0$, its tangent vector being $u^\alpha \equiv (1, 0, 0, 0)$. The second test mass is situated at the unperturbed position $x^i(0) = l^i$, having zero unperturbed velocity. Assuming that the frame of the local observer is located along the central geodesic, the task is to find the trajectory of the second test mass using the generalized geodesic deviation equation [31]. The deviation vector can be written as

$$N^i(t) = l^i + \delta l^i(t)$$  

(33)

where the variation in distance $\delta l^i(t)$ is caused by the SGW. In the frame of the local observer one can replace all covariant derivative in [31] by ordinary derivatives [22]. In the lowest approximation [31] reduces to [22] and specializes to

$$\frac{d^2 \delta l^i(t)}{dt^2} = -\frac{1}{2}l^j \frac{\partial^2}{\delta l^j} \Phi \delta_j^i = \frac{1}{2} \omega^2 \Phi \epsilon^{(s)}_{i}.$$  

(34)

in the field of [16]. The relevant solution to [31] coincides exactly with the usual electric part of the motion given by [22]. Since we want to identify the magnetic part of the gravitational force arising from a SGW, all terms in [31] must be considered. Since $\frac{d\delta \omega^2}{dt^2}$ is of order $\Phi$, the third term of [31] is of order $\Phi^2$ and can be neglected. Computing the derivatives of the curvature tensor and substituting them into [31] specialized in the field of [16], the correct equations of motions read,

$$\frac{d^2 \delta l^i(t)}{dt^2} = \frac{1}{2} \omega^2 i^j \Phi \epsilon^{(s)}_{j} - \frac{1}{2} \omega^2 i^k \epsilon^{(s)}_{ij} (k_j \delta^i_j + \frac{1}{2} k^i_j) \delta^s_j \Phi \epsilon^{(s)}. $$  

(35)

The second term on the right hand side of [35] is responsible for the magnetic component of motion and can be interpreted as the gravitational analogue of the magnetic part of the Lorentz force (see also the analogy for ordinary tensorial waves in [2] and [4]).
V. VARIATION OF DISTANCES BETWEEN TEST MASSES AND RESPONSE OF INTERFEROMETERS

In this Section we are interested in the distance between the central particle, located at the coordinate origin and a particle located on average at some position \((l_1, l_2, l_3)\). This model represents the situation of the beam-splitter and an interferometer [3, 4]. In the frame of the local observer the line element is given by equation (17). In this frame, the Galilean distance - accurate to terms of order \(\Phi l\) and \(\Phi l^2\omega\) while neglecting terms quadratic in \(\Phi\) - is given by,

\[
d(t) = \sqrt{x^2 + y^2 + z^2 + O((\Phi l^2\omega)^2)}. \tag{36}
\]

Letting

\[
x = l_1 + \delta x, \quad y = l_2 + \delta y, \quad z = l_3 + \delta z \tag{37}
\]

in (36) we get

\[
d(t) = l + \frac{1}{l}(l_1\delta x + l_2\delta y + l_3\delta z). \tag{38}
\]

By using the time dependent positions (24), the distance \(d(t)\) takes the form (with the required approximation \(\omega l \ll 1\))

\[
d(t) = l + \frac{1}{2l}(l_1^2 - l_2^2)\Phi(t) - \frac{1}{4l}\omega l_3(l_1^2 - l_2^2)\Phi(\omega t - \frac{\pi}{2}). \tag{39}
\]

The first correction to \(l\) is due to the electric contribution, while the second correction to \(l\) is due to the magnetic contribution. According to (24), the magnetic component of motion is present even if the mean position of the second test mass is such that \(l_3 = 0\), in perfect analogy with the pure General Relativity case shown in [3, 4].

Now we consider the response of a laser interferometer. To compute the response function of the interferometer to a SGW from arbitrary propagating directions we recall that the arms of the interferometer are in the \(\vec{u}\) and \(\vec{v}\) directions, while the \(x, y, z\) frame is adapted to the propagating SGW. Therefore, performing a spatial rotation of the coordinate system, we obtain

\[
u = -x \cos \theta \cos \phi + y \sin \phi + z \sin \theta \cos \phi, \\
v = -x \cos \theta \sin \phi - y \cos \phi + z \sin \theta \sin \phi, \\
w = x \sin \theta + z \cos \theta, \tag{40}
\]

or, in terms of the \(x, y, z\) frame,

\[
x = -u \cos \theta \cos \phi - v \cos \theta \sin \phi + w \sin \theta, \\
y = u \sin \phi - v \cos \phi, \\
z = u \sin \theta \cos \phi + v \sin \theta \sin \phi + w \cos \theta. \tag{41}
\]

In this way the SGW is propagating from an arbitrary direction \(\vec{r}\) to the interferometer (see Figure 2).
A SGW propagating from an arbitrary direction.

The response function $\delta d(t)$ is defined by

$$
\delta d(t) \equiv d_u(t) - d_v(t), \quad (42)
$$

where $d_u(t)$ and $d_v(t)$ are the distances in the $u$ and $v$ direction. Using equations (39), (40), (41) and (42), we obtain

$$
\delta d(t) = -\Phi(t) \omega l \sin^2 \theta \cos 2\phi + \frac{1}{4} \Phi(t) \omega l^2 \cos \theta \left\{ \left( \frac{1 + \sin^2 \theta}{2} \right) + \sin^2 \theta \sin 2\phi \right\} (\cos \phi - \sin \phi). \quad (43)
$$

The first term in (43) is due to the electric contribution, while the second term is due to the magnetic contribution. Our response function (43) is more complete than the previously derived expression in [19, 21, 29] since it includes an intrinsic magnetic contribution. The function $\frac{1}{4} \omega l \cos \theta \left\{ \left( \frac{1 + \sin^2 \theta}{2} \right) + \sin^2 \theta \sin 2\phi \right\} (\cos \phi - \sin \phi)$ represents the so-called “angular pattern” [19] of interferometers for the magnetic contribution of SGWs. A plot of this function is shown for the Virgo and LIGO interferometers in Figure 3 and 4 respectively, for a frequency of $f = 100 \text{Hz}$ in each case.
The angular dependence of the total response function of the Virgo interferometer to the magnetic component of a SGW for $f = 100\,\text{Hz}$. 
The angular dependence of the total response function of the LIGO interferometer to the magnetic component of a SGW for $f = 100\,Hz$.

VI. VARIATION OF DISTANCE FROM THE THEORY OF SCALAR GRAVITATIONAL WAVES

It is important to verify that the approximate expression (39) follows directly from the exact definition of distance which arises from the theory of SGWs. For this purpose we make use of the time of flight measurement of a photon traveling from one test mass to another and back. Indeed, such considerations of photon transit time is an integral part of the more general problem of finding the null geodesics of light in the presence of weak gravitational waves $[3, 4, 19, 20, 23, 24, 25]$. This definition of distance $d(t)$ is valid independent of the relationship between $l$ and $\omega$ and does not require the introduction of the frame of the local observer. Consider a photon emitted from the beam-splitter of an interferometer at instant $t_0$. Furthermore, assume the same photon is reflected by a mirror and returns to the beam-splitter at instant $t_1$. The proper distance $d(t)$ between the two test masses at time $t$ is defined as

$$d(t) = \frac{t_1 - t_0}{2},$$ (44)

where $t = \frac{t_1 + t_0}{2}$ is the mean time between the departure and arrival of the photon. In the field of a SGW, the trajectories of photons are described by the null geodesic $ds^2 = 0$, which from (16) leads to

$$dt^2 = (1 + \Phi)dx^2 + (1 + \Phi)dy^2 + dz^2.$$ (45)

The unperturbed outgoing photon can be parametrized as $[3]$

$$t = t_0 + l\zeta, \quad x = l_1\zeta, \quad y = l_2\zeta, \quad z = l_3\zeta,$$ (46)

with $0 \leq \zeta \leq 1$. Thus, according to equation (45) we obtain

$$dt = l \left[ 1 + \Phi(\zeta) \frac{l_1^2 + l_2^2}{l^2} \right] d\zeta.$$ (47)
Integrating both the sides of this equation we find the time of arrival of the photon at the mirror. In an analogous manner, the unperturbed reflected photon can be parametrized as

\[ t = t_1 - t_0 + l \zeta, \quad x = l_1 - l_3 \zeta, \quad y = l_2 - l_3 \zeta, \quad z = l_3 - l_4 \zeta, \]  

where it is \( 0 \leq \zeta \leq 1 \) again. An analogous integration of (45) gives the time of arrival of the photon back at the beam splitter (from the mirror). Combining the two pieces of the transit of the photon from the beam splitter to mirror and back enables one to obtain an exact expression for the distance \( d(t) \), namely

\[ d(t) = t + \frac{l_1^2 - l_2^2}{4l} \left( \frac{\Phi(\omega t + \omega l_3) - \Phi(\omega t + \omega l)}{\omega(l - l_3)} - \frac{\Phi(\omega t + \omega l_3) - \Phi(\omega t - \omega l)}{\omega(l + l_3)} \right). \]  

In the low frequency approximation and retaining only the first two terms in the expansion of \( \Phi \), (49) reduces to (9). Equation (39) is sufficient only for ground based interferometers, for which the condition \( \omega \ll 1/l \) is in general satisfied. In the case of space-based interferometers for which the above condition is not satisfied in the high-frequency region of the sensitivity band \([3, 4]\), the theory of scalar gravitational waves (49) must be employed. Having shown that formula (39) follows from the exact distance (49) suggests the magnetic contribution to the distance is an universal physical phenomenon which must to be taken into account in the data analysis of GW interferometers.

VII. CONCLUSIONS

In this paper the motion of a free test particle in the field of a scalar gravitational wave arising from scalar-tensor theories of gravity is considered in detail by solving higher order geodesic deviation equations. The presence and significance of the so-called electric and magnetic contributions to the distance function between interferometer test masses have been shown. The angular dependence of the response function of interferometers due the magnetic correction to the distance function in the low-frequency approximation was obtained. These results generalize the previous works in the literature, where it has been shown that the electric and magnetic contributions are unambiguous in the long-wavelength approximation for ordinary GWs arising from General Relativity. Finally, for arbitrary frequency range, the proper distance between the two interferometer test masses is calculated and its usefulness in the high-frequency limit for space based interferometers is briefly considered. The response function (i.e., variation of distance) which has been obtained is more complete than the previous derived expressions in the literature (see [16, 19, 21, 30] for example), because such a response function includes the magnetic contribution.

In this high-frequency regime the division between electric and magnetic components becomes ambiguous, thus requiring use of the theory of scalar gravitational waves without (low-frequency) approximation. The exact expression for the response function was derived from which it was verified that the electric and magnetic corrections follow from a series expansion of the result. We also emphasize that the presence of the higher order terms in the frequency dependent response function become crucial for space-based interferometers, since the (low-frequency) condition \( \omega \ll 1/l \) is not satisfied in the high-frequency portion of the sensitivity band of the interferometer. The response function contains contributions from two terms, namely the electric and magnetic, both of which have been explicitly derived and explained. In this case, if one neglects the magnetic component of SGWs, approximately 10% of currently observable signal could, in principle, be lost. Our findings suggest the magnetic component of SGWs is an universal physical phenomenon that must be taken into account so as not to discard potentially relevant gravitational wave interferometer data.

VIII. ACKNOWLEDGEMENTS

We thank Salvatore Capozziello, Mauro Francaviglia, Maria Felicia De Laurentis and Giancarlo Cella for helpful advice during this work. We thank an anonymous Referee for useful comments which led to the improvement of this article. Thanks is also extended to the European Gravitational Observatory (EGO) consortium for use of their computing facilities.

[1] C. Corda, Proceedings of the XLIIIInd Rencontres de Moriond, Gravitational Waves and Experimental Gravity, p. 95, Ed. J. Dumarchez and J. T. Tran, Than Van, THE GIOI Publishers (2007)
[2] C. Corda, Int. J. Mod. Phys. D 16, 9, 1497-1517 (2007)
[3] D. Baskaran and L. P. Grishchuk, Class. Quant. Grav. 21, 4041-4061 (2004)
[4] C. Corda, Int. J. Mod. Phys. A 22, n. 13, 2361-2381 (2007); C. Corda - gr-qc/0702080, accepted by Int. J. Mod. Phys. D (2007)
[5] C. Corda, Astropart. Phys. 27, No 6, 539-549 (2007)
[6] F. Acernese et al. (the Virgo Collaboration), Class. Quant. Grav. 23, 8 S63-S69 (2006)
[7] S. Hild (for the LIGO Scientific Collaboration), Class. Quant. Grav. 23, 19 S643-S651 (2006)
[8] B. Willke et al., Class. Quant. Grav. 23, S207-S214 (2006)
[9] D. Sigg (the LIGO Scientific Collaboration); online at: www.ligo.org/pdf/public/P050036.pdf
[10] B. Abbott et al. (the LIGO Scientific Collaboration), Phys. Rev. D 72, 042002 (2005)
[11] M. Ando and the TAMA Collaboration, Class. Quant. Grav. 19, 7 1615-1621 (2002)
[12] D. Tatsunami, Y. Tsumesada and the TAMA Collaboration, Class. Quant. Grav. 21, 5 S451-S456 (2004)
[13] K. S. Thorne in 300 years of gravitation p. 330, Ed. S.W. Hawking and W. Israel, Cambridge: Cambridge University Press (1987)
[14] S. Capozziello, Newtonian Limit of Extended Theories of Gravity in Quantum Gravity Research Trends Ed. A. Reimer, pp. 227-276 Nova Science Publishers Inc., NY (2005); online at: arXiv: gr-qc/0412088 (2004)
[15] C. Brans and R.H. Dicke, Phys. Rev. 124, 925 (1961)
[16] M. E. Tobar, T. Suzuki and K. Kuroda, Phys. Rev. D 59, 102002 (1999)
[17] G. Alemanni, M. Francaviglia, M. L. Ruggiero and A. Tartaglia, Gen. Rel. Grav. 37, 11 (2005)
[18] G. Alemanni, M. Capone, S. Capozziello and M. Francaviglia, Gen. Rel. Grav. 38, 1 (2006)
[19] S. Capozziello and C. Corda, Int. J. Mod. Phys. D 15, 1119-1150 (2006); C. Corda, Response of laser interferometers to scalar gravitational waves- talk in the Gravitational Waves Data Analysis Workshop in the General Relativity Trimester of the Inst. H. Poincaré, Paris 13 - 17 November 2006, online at: www.luth2.obspm.fr/ IHP06/workshops/gwdata/corda.pdf
[20] C. Corda, J. Cosmol. Astropart. Phys. JCAP04009 (2007); C. Corda, Astropart. Phys. doi:10.1016/j.astropartphys.2007.05.009
[21] M. E. Tobar, T. Suzuki and K. Kuroda, Phys. Rev. D 59, 102002 (1999); M. Maggiore and A. Nicolis, Phys. Rev. D 62, 024004 (2000)
[22] C. Brans and R. H. Dicke, Phys. Rev. 124(3) 925-935 (1961)
[23] C. W. Misner, K. S. Thorne and J. A. Wheeler, “Gravitation”, W. H. Feeman and Company (1973)
[24] L. P. Grishchuk, Sov. Phys. Usp. 20, 319 (1977)
[25] L. P. Grishchuk, Sov. Phys. JETP 39, 402 (1974)
[26] L. Landau and E. Lifšits, “Teoria dei campi”, Editori riuniti edition III (1999)
[27] S. L. Bazansky Ann. Inst. H. Poincaré A 27, 115 (1977)
[28] R. Kerner, J. W. van Holten and R. Colistete Jr., Class. Quant. Grav. 18, 4725 (2001)
[29] N. Bonasia and M. Gasperini, arXiv: gr-qc/0504079 (2005)
[30] M. Shibata, K. Nakao and T. Nakamura, Phys. Rev. D 50, 7304 (1994)