Time periodic solutions of Cahn–Hilliard system with dynamic boundary conditions

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Abstract

The existence problem for Cahn–Hilliard system with dynamic boundary conditions and time periodic conditions is discussed. We apply the abstract theory of evolution equations by using viscosity approach and the Schauder fixed point theorem in the level of approximate problem. One of the key point is the assumption for maximal monotone graphs with respect to their domains. Thanks to this, we obtain the existence result of the weak solution by using the passage to the limit.

Key words: Cahn–Hilliard system, dynamic boundary condition, time periodic solutions.

AMS (MOS) subject classification: 35K25, 35A01, 35B10, 35D30.

1 Introduction

In this paper, we consider the following Cahn–Hilliard system with dynamic boundary condition and time periodic condition, say (P), which consists of the following:

\[
\frac{\partial u}{\partial t} - \Delta \mu = 0 \quad \text{in } Q := \Omega \times (0, T), \quad (1.1)
\]

\[
\mu = -\kappa_1 \Delta u + \xi + \pi(u) - f, \quad \xi \in \beta(u) \quad \text{in } Q, \quad (1.2)
\]

\[
u \Gamma = u|_\Gamma, \quad \mu \Gamma = \mu|_\Gamma \quad \text{on } \Sigma := \Gamma \times (0, T), \quad (1.3)
\]

\[
\frac{\partial u \Gamma}{\partial t} + \partial_\nu \mu - \Delta_\Gamma \mu \Gamma = 0 \quad \text{on } \Sigma, \quad (1.4)
\]

\[
\mu \Gamma = \kappa_1 \partial_\nu u - \kappa_2 \Delta_\Gamma u \Gamma + \xi \Gamma + \pi \Gamma (u \Gamma) - f \Gamma, \quad \xi \Gamma \in \beta \Gamma (u \Gamma) \quad \text{on } \Sigma, \quad (1.5)
\]

\[
u \Gamma (0) = u(T) \quad \text{in } \Omega, \quad u \Gamma (0) = u \Gamma (T) \quad \text{on } \Gamma \quad (1.6)
\]

where \(0 < T < +\infty\), \(\Omega\) is a bounded domain of \(\mathbb{R}^d \ (d = 2, 3)\) with smooth boundary \(\Gamma := \partial \Omega\), \(\kappa_1, \kappa_2\) are positive constants, \(\partial_\nu\) is the outward normal derivative on \(\Gamma\), \(u|_\Gamma, \mu|_\Gamma\) stand for the trace of \(u\) and \(\mu\) to \(\Gamma\), respectively, \(\Delta\) is the Laplacian, \(\Delta_\Gamma\) is the Laplace–Beltrami operator (see, e.g., [20]), and \(f : Q \to \mathbb{R}, \ f \Gamma : \Sigma \to \mathbb{R}\) are given data. Moreover,
in the nonlinear diffusion terms, \( \beta, \beta_\Gamma : \mathbb{R} \to 2^\mathbb{R} \) are maximal monotone graphs and \( \pi, \pi_\Gamma : \mathbb{R} \to \mathbb{R} \) are Lipschitz perturbations.

The Cahn–Hilliard equation [8] is a description of mathematical model for phase separation, e.g., the phenomenon of separating into two phases from homogeneous composition, the so-called spinodal decomposition. In (1.1)–(1.2), \( u \) is the order parameter and \( \mu \) is the chemical potential. Moreover, it is well known that the Cahn–Hilliard equation is characterized by the nonlinear term \( \beta + \pi \). It play an important role as derivative of double-well potential \( W \). As pioneering results, Kenmochi, Niezgódka and Pawlow study the Cahn–Hilliard equation with constraint by subdifferential operator approach [13]. In addition, Kubo investigates the Cahn–Hilliard equation [25]. Recently, in [21], it is discussed the strong solution of the Cahn–Hilliard equation in bounded domains with permeable and non-permeable walls in maximal \( L_p \) regularity spaces.

In terms of (1.3)–(1.5), we consider the dynamic boundary condition as being \( u_\Gamma, \mu_\Gamma \) unknown functions such that satisfied trace condition (1.3). The dynamic boundary condition was treated in recent years, for example, for the Stefan problem [1–3, 14], wider the degenerate parabolic equation [15, 16] and the Cahn–Hilliard equation [10, 11, 17, 19, 29].

To the best our knowledge, the type of dynamic boundary condition on the Cahn–Hilliard equation like (P) is formulated in [18]. As you can see, we consider the same type of equations (1.1)–(1.2) on the boundary. In other words, (1.1)–(1.5) is a transmission problem connecting \( \Omega \) and \( \Gamma \).

The nonlinear term \( \beta_\Gamma + \pi_\Gamma \) is also the derivative of double-well potential \( W_\Gamma \). As a well-known example, \( W = W_\Gamma = (1/4)(r^2 - 1)^2 \), namely, \( W' = W'_\Gamma = r^3 - r \) for \( r \in \mathbb{R} \). This is called the prototype double well potential. Note that we do not have to take different nonlinear terms \( W' \) and \( W'_\Gamma \) in \( \Omega \) and on \( \Gamma \), respectively. However, in problem (P), we treat it differently to generalize. In this case, it is necessary to assume the compatibility condition (see e.g., [9, 10]), stated (A4). The other example of \( W, W_\Gamma \) is stated later.

About the problem (P), Colli and Fukao study the Cahn–Hilliard system with dynamic boundary condition and initial value condition [10]. They set a function space that the total mass is zero. This idea is arised from the property of dynamic boundary condition. The property is called the total mass conservation.

Moreover, focusing on (1.6), the study respect to existence of time periodic solutions of the Cahn–Hilliard equation is not much. For example, [26, 28, 31]. In particular, Wang and Zheng discuss the existence of time periodic solution of the Cahn–Hilliard equation with Neumann boundary condition [31]. They employ the method of [4]. Note that they impose two assumptions for a maximal monotone graph, specifically, restricted domains and the following growth condition for maximal monotone graph \( \beta : \)

\[
\tilde{\beta}(r) \geq cr^2 \quad \text{for all } r \in \mathbb{R},
\]

for some positive constant \( c \). However, the above assumption is too restrictive for some physical applications.

In this paper, following the method of [31], we apply the abstract theory of evolution equations by using the viscosity approach and the Schauder fixed point theorem in the level of approximate problem. Moreover, by virtue of the viscosity approach, we can apply the abstract result [4]. Noting that, in [10], they do not impose the assumption of growth condition for maximal monotone graphs \( \beta \) and \( \beta_\Gamma \). Therefore, by setting the
appropriate convex functional and using the Poincaré–Wirtinger inequality, we can also avoid imposing the growth condition even though we consider the time periodic problem. Thanks to this result, we can choose wider kinds of nonlinear diffusion terms $\beta + \pi$ and $\beta_{\Gamma} + \pi_{\Gamma}$. However, respect to the assumption of restricted domains (see (A5)), this is a essential point to slove the problem (P). Nevertheless, we can infer that the assumption (A5) is not necessary when we choose the prototype double well potential. We sketch it Remark 4.1.

The present paper proceeds as follows.

In Section 2, main theorem and definition of solution are stated. At first, we prepare the notation used in this paper and set appropriate function spaces. Next, we introduce the definition of weak solution of (P) and the main theorems are given there. Also, we give the example of double-well potentials.

In Section 3, in order to prove convergence theorem, at first, we set convex functionals and consider approximate problems. Next, we obtain the solution of (P)$_{\varepsilon}$ by using the Schauder fixed point theorem. Finally, we deduce uniform estimates of the solution of (P)$_{\varepsilon}$.

In Section 4, we prove the existence of weak solutions by passing to the limit $\varepsilon \to 0$. Finally, we sketch the case that we choose the derivative of prototype double well potential as nonlinear diffusion terms.

A detailed index of sections and subsections follows.

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## 2 Main results

### 2.1 Notation

We introduce the spaces $H := L^2(\Omega)$, $H_{\Gamma} := L^2(\Gamma)$, $V := H^1(\Omega)$, $V_{\Gamma} := H^1(\Gamma)$ with standard norms $|\cdot|_H$, $|\cdot|_{H_{\Gamma}}$, $|\cdot|_V$, $|\cdot|_{V_{\Gamma}}$ and inner products $(\cdot, \cdot)_H$, $(\cdot, \cdot)_{H_{\Gamma}}$, $(\cdot, \cdot)_V$, $(\cdot, \cdot)_{V_{\Gamma}}$, respectively. Moreover, we set $H := H \times H_{\Gamma}$ and

$$V := \{ z := (z, z_\Gamma) \in V \times V_{\Gamma} : z|_\Gamma = z_{\Gamma} \text{ a.e. on } \Gamma \}.$$
\(H\) and \(V\) are then Hilbert spaces with inner products

\[
(u, z)_H := (u, z)_H + (u_\Gamma, z_\Gamma)_{H_\Gamma}, \quad \text{for all } u := (u, u_\Gamma), z := (z, z_\Gamma) \in H,
\]

\[
(u, z)_V := (u, z)_V + (u_\Gamma, z_\Gamma)_{V_\Gamma}, \quad \text{for all } u := (u, u_\Gamma), z := (z, z_\Gamma) \in V.
\]

Note that \(z \in V\) implies that the second component \(z_\Gamma\) of \(z\) is equal to the trace of the first component \(z\) on \(\Gamma\), and \(z \in H\) implies that \(z \in H\) and \(z_\Gamma \in H_\Gamma\) are independent. Throughout this paper, we use the bold letter \(u\) to represent the pair corresponding to the letter; i.e., \(u := (u, u_\Gamma)\).

Let \(m : H \to \mathbb{R}\) be the mean function defined by

\[
m(z) := \frac{1}{|\Omega| + |\Gamma|} \left\{ \int_{\Omega} z dx + \int_{\Gamma} z_\Gamma d\Gamma \right\} \quad \text{for all } z \in H,
\]

where \(|\Omega| := \int_{\Omega} 1 dx, |\Gamma| := \int_{\Gamma} 1 d\Gamma\). Then, we define \(H_0 := \{ z \in H ; m(z) = 0 \}\), \(V_0 := V \cap H_0\). Moreover, \(V^*, V_0^*\) denote the dual spaces of \(V, V_0\), respectively; the duality pairing between \(V_0^*\) and \(V_0\) is denoted \(\langle \cdot, \cdot \rangle_{V_0^*, V_0}\). We define the norm of \(H_0\) by \(|z|_{H_0} := |z|_H\) for all \(z \in H_0\). Now, we define the bilinear form \(a(\cdot, \cdot) : V \times V \to \mathbb{R}\) by

\[
a(u, z) := \kappa_1 \int_{\Omega} \nabla u \cdot \nabla z dx + \kappa_2 \int_{\Gamma} \nabla_\Gamma u_\Gamma \cdot \nabla_\Gamma z_\Gamma d\Gamma \quad \text{for all } u, z \in V.
\]

Then, for all \(z \in V_0\), \(|z|_{V_0} := \sqrt{a(z, z)}\) is the norm of \(V_0\). Also, for all \(z \in V_0\), we let \(F : V_0 \to V_0^*\) be the duality mapping defined by

\[
\langle Fz, \tilde{z} \rangle_{V_0^*, V_0} := a(z, \tilde{z}) \quad \text{for all } \tilde{z} \in V_0.
\]

Then, the following the Poincaré–Wirtinger inequality holds: there exists a positive constant \(c_P\) such that

\[
|z|^2_{V} \leq c_P |z|^2_{V_0} \quad \text{for all } z \in V
\]

(see [10, Lemma C]). Moreover, we define the inner product of \(V_0^*\) by

\[
(z^*, \tilde{z}^*)_{V_0^*} := \langle z^*, F^{-1} \tilde{z}^* \rangle_{V_0^*, V_0} \quad \text{for all } z^*, \tilde{z}^* \in V_0^*.
\]

Also, we define the projection \(P : H \to H_0\) by

\[
Pz := z - m(z)1 \quad \text{for all } z \in H,
\]

where \(1 := (1, 1)\). Then, since \(P\) is a linear bounded operator, note that the following property holds: let \(\{z_n\}_{n \in \mathbb{N}}\) be a sequence in \(H\) such that \(z_n \to z\) weakly in \(H\) for some \(z\), then we infer that

\[
Pz_n \to Pz \quad \text{weakly in } H_0,
\]

(2.2) because \(P\) is a single-valued operator. Then, we have \(V_0 \hookrightarrow H_0 \hookrightarrow V_0^*\), where \(\hookrightarrow\) stands for compact embedding (see [10, Lemmas A and B]).
2.2 Definition of the solution and main theorem

In this subsection we define our solution for (P) and then we state the main theorem. Firstly, from (1.1), (1.4) and the property of dynamic boundary condition, it holds the following total mass conservation:

\[ \int_{\Omega} u(t) dx + \int_{\Gamma} u_{\Gamma}(t) d\Gamma = \int_{\Omega} u(0) dx + \int_{\Gamma} u_{\Gamma}(0) d\Gamma := m_0 \quad \text{for all } t \in [0, T]. \]

To define solutions, we use the following notation: the variable \( \boldsymbol{v} := \boldsymbol{u} - m_0 \mathbf{1} \); the datum \( \boldsymbol{f} := (f, f_{\Gamma}) \); the function \( \pi(z) := (\pi(z), \pi_{\Gamma}(z_{\Gamma})) \) for \( z \in H \). Moreover, we set the space \( W := H^2(\Omega) \times H^2(\Gamma) \).

**Definition 2.1.** The triplet \( (\boldsymbol{v}, \mu, \xi) \) is called the weak solution of (P) if

\[ \boldsymbol{v} \in H^1(0, T; V_0^*) \cap L^\infty(0, T; V_0) \cap L^2(0, T; W), \]
\[ \mu \in L^2(0, T; V), \]
\[ \xi = (\xi, \xi_{\Gamma}) \in L^2(0, T; H), \]

and they satisfy

\[ \langle \boldsymbol{v}'(t), z \rangle_{V_0^*, V_0} + a(\mu(t), z) = 0 \quad \text{for all } z \in V_0 \]
\[ (\mu(t), z)_H = a(\boldsymbol{v}(t), z) + (\xi(t) - m(\xi(t)) + \pi(\boldsymbol{v}(t) + m_0 \mathbf{1}) - f, z)_H \quad \text{for all } z \in V \]

(2.4)

for a.a. \( t \in (0, T) \), and

\[ \xi \in \beta(v + m_0) \quad \text{a.e. in } Q, \quad \xi_{\Gamma} \in \beta_{\Gamma}(v_{\Gamma} + m_0) \quad \text{a.e. on } \Sigma \]

with

\[ \boldsymbol{v}(0) = \boldsymbol{v}(T) \quad \text{in } H_0. \]

(2.5)

**Remark 2.1.** We can see that \( \mu := (\mu, \mu_{\Gamma}) \) satisfies

\[ \mu = -\kappa_1 \Delta u + \xi - m(\xi) + \pi(\mu) - f \quad \text{a.e. in } Q, \]
\[ \mu_{\Gamma} = \kappa_1 \partial_{\nu} u - \kappa_2 \Delta_{\Gamma} u_{\Gamma} + \xi_{\Gamma} - m(\xi) + \pi_{\Gamma}(u_{\Gamma}) - f_{\Gamma} \quad \text{a.e. on } \Sigma, \]

where \( u = v + m_0 \) and \( u_{\Gamma} = v_{\Gamma} + m_0 \), because of the regularity \( \boldsymbol{v} \in L^2(0, T; W) \).

**Remark 2.2.** In (2.4), this is different from the following definition of [10] Definition 2.1:

\[ (\mu(t), z)_H = a(\boldsymbol{v}(t), z) + (\xi(t) + \pi(\boldsymbol{v}(t) + m_0 \mathbf{1}) - f, z)_H \quad \text{for all } z \in V \]

(2.6)

for a.a. \( t \in (0, T) \). However, by setting \( \tilde{\mu} := \mu + m(\xi) \mathbf{1} \), \( \tilde{\mu} \) satisfies \( \tilde{\mu} \in L^2(0, T; V) \) and (2.6). Hence, in other words, we can employ (2.6) as definition of (P) replaced by (2.4).

We assume that
(A1) $f \in L^2(0,T;V)$ and $f(t) = f(t + T)$ for a.a. $t \in (0,T)$.

(A2) $\pi, \pi_\Gamma : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz continuous functions with Lipschitz constants $L, L_\Gamma$, respectively.

(A3) $\beta, \beta_\Gamma : \mathbb{R} \to 2^\mathbb{R}$ are maximal monotone graphs, which is the subdifferential

$$\beta = \partial_\mathbb{R} \beta, \quad \beta_\Gamma = \partial_\mathbb{R} \beta_\Gamma$$

of some proper lower semicontinuous convex functions $\beta, \beta_\Gamma : \mathbb{R} \to [0, +\infty]$ satisfying $\beta(0) = \beta_\Gamma(0) = 0$ with domains $D(\beta)$ and $D(\beta_\Gamma)$, respectively.

(A4) $D(\beta_\Gamma) \subseteq D(\beta)$ and there exist positive constant $\rho, c_0 > 0$ such that

$$|\beta^\ast(r)| \leq \rho|\beta_\Gamma^\ast(r)| + c_0 \quad \text{for all } r \in D(\beta_\Gamma):$$

(A5) $D(\beta), D(\beta_\Gamma)$ are bounded domains with non-empty interior and, i.e., $\overline{D(\beta)} = [\sigma_\ast, \sigma^\ast]$ and $\overline{D(\beta_\Gamma)} = [\sigma_{\Gamma \ast}, \sigma_{\Gamma}^\ast]$ for some constants $\sigma_\ast, \sigma^\ast, \sigma_{\Gamma \ast}$ and $\sigma_{\Gamma}^\ast$ with $-\infty < \sigma_\ast \leq \sigma_{\Gamma \ast} < \sigma_{\Gamma}^\ast \leq \sigma^\ast < \infty$.

In particular, (A3) yields $0 \in \beta(0)$. The assumption (A5) is not imposed \cite{10}. However, it is essential to obtain uniform estimates in Section 3. This is the greatest difficulties of time periodic problem. Also, the assumption of compatibility of $\beta$ and $\beta_\Gamma$ (A4) is the same as in \cite{9,10}. Now, we give the example respect to nonlinear diffusion terms under the above assumptions:

- $\beta(r) = \beta_\Gamma(r) = (\alpha_1/2)\ln((1 + r)/(1 - r))$, $\pi(r) = \pi_\Gamma(r) = -\alpha_2r$ for all $r \in D(\beta) = D(\beta_\Gamma) = (-1,1)$ and $0 < \alpha_1 < \alpha_2$. for the logarithmic double well potential $W(r) = W_\Gamma(r) = (\alpha_1/2)\{((1 - r)\ln((1 - r)/2) + (1 + r)\ln((1 + s)/2)) + (\alpha_2/2)(1 - r^2)\}$. The condition $\alpha_1 < \alpha_2$ ensures that $W, W_\Gamma$ have double-well forms (see e.g., \cite{12}).
- $\beta(r) = \beta_\Gamma(r) = \partial I_{[-1,1]}(r)$, $\pi(r) = \pi_\Gamma(r) = -r$ for all $r \in D(\beta) = D(\beta_\Gamma) = [-1,1]$ for the singular potential $W(r) = W_\Gamma(r) = I_{[-1,1]}(r) - r^2/2$.
- $\beta(r) = \beta_\Gamma(r) = \partial I_{[-1,1]}(r) + r^3$, $\pi(r) = \pi_\Gamma(r) = -r$ for all $r \in D(\beta) = D(\beta_\Gamma) = [-1,1]$ for the modified prototype double well potential $W(r) = W_\Gamma(r) = I_{[-1,1]}(r) + (1/4)(r^2 - 1)^2 - r^2/2$.

The third example is modified due to the assumption (A5). However, we can choose the original prototype double well potential (see Remark 4.1).

Our main theorem is given now.

**Theorem 2.1.** Under the assumptions (A1)–(A4), for any given $m_0 \in \text{int}D(\beta_\Gamma)$, there exists a weak solution of (P) such that

$$\frac{1}{|\Omega| + |\Gamma|} \left\{ \int_\Omega u(0)dx + \int_\Gamma u_\Gamma(0) \right\} = m_0.$$
3 Approximate problem and uniform estimates

In section, we consider the approximate problem and obtain the uniform estimates to show the existence of weak solutions of (P).

3.1 Abstract formulation

In order to prove the main theorem, we apply the abstract theory of evolution equation. To do so, we define a proper lower semicontinuous convex functional $\varphi : H_0 \to [0, +\infty]$ by

$$
\varphi(z) := \begin{cases}
\frac{\kappa_1}{2} \int_\Omega |\nabla z|^2 dx + \frac{\kappa_2}{2} \int_\Gamma |\nabla z_\Gamma|^2 d\Gamma \\
+ \int_\Omega \hat{\beta}(z + m_0) dx + \int_\Gamma \hat{\beta}_\Gamma(z_\Gamma + m_0) d\Gamma \\
+ \infty & \text{if } z \in V_0 \text{ with } \hat{\beta}(z + m_0) \in L^1(\Omega), \hat{\beta}_\Gamma(z_\Gamma + m_0) \in L^1(\Gamma),
\end{cases}
$$

Next, for each $\varepsilon \in (0, 1]$, we define a proper lower semicontinuous convex functional $\varphi_\varepsilon : H_0 \to [0, +\infty]$ by

$$
\varphi_\varepsilon(z) := \begin{cases}
\frac{\kappa_1}{2} \int_\Omega |\nabla z|^2 dx + \frac{\kappa_2}{2} \int_\Gamma |\nabla z_\Gamma|^2 d\Gamma \\
+ \int_\Omega \hat{\beta}_\varepsilon(z + m_0) dx + \int_\Gamma \hat{\beta}_{\Gamma, \varepsilon}(z_\Gamma + m_0) d\Gamma \\
+ \infty & \text{if } z \in V_0 \text{ with } \hat{\beta}_\varepsilon(z + m_0) \in L^1(\Omega), \hat{\beta}_{\Gamma, \varepsilon}(z_\Gamma + m_0) \in L^1(\Gamma),
\end{cases}
$$

where $\hat{\beta}_\varepsilon, \hat{\beta}_{\Gamma, \varepsilon}$ defined as follows are Moreau–Yosida regularizations of $\hat{\beta}, \hat{\beta}_\Gamma$, respectively:

$$
\hat{\beta}_\varepsilon(r) := \inf_{s \in \mathbb{R}} \left\{ \frac{1}{2\varepsilon} |r - s|^2 + \hat{\beta}(s) \right\} = \frac{1}{2\varepsilon} |r - J_\varepsilon(r)|^2 + \hat{\beta}(J_\varepsilon(r)),
$$

$$
\hat{\beta}_{\Gamma, \varepsilon}(r) := \inf_{s \in \mathbb{R}} \left\{ \frac{1}{2\varepsilon\rho} |r - s|^2 + \hat{\beta}_\Gamma(s) \right\} = \frac{1}{2\varepsilon\rho} |r - J_{\Gamma, \varepsilon}(r)|^2 + \hat{\beta}_\Gamma(J_{\Gamma, \varepsilon}(r)),
$$

where $\rho$ is a constant as in (2.7) and $J_\varepsilon, J_{\Gamma, \varepsilon} : \mathbb{R} \to \mathbb{R}$ is resolvent operator given by

$$
J_\varepsilon(r) := (I + \varepsilon\beta)^{-1}(r), \quad J_{\Gamma, \varepsilon}(r) := (I + \varepsilon\rho\beta_\Gamma)^{-1}(r)
$$

for all $r \in \mathbb{R}$. Moreover, $\beta_\varepsilon, \beta_{\Gamma, \varepsilon} : \mathbb{R} \to \mathbb{R}$ defined as follows are Yosida approximatioon for maximal monotone operators $\beta, \beta_\Gamma$, respectively:

$$
\beta_\varepsilon(r) := \frac{1}{\varepsilon} (r - J_\varepsilon(r)), \quad \beta_{\Gamma, \varepsilon}(r) := \frac{1}{\varepsilon\rho} (r - J_{\Gamma, \varepsilon}(r))
$$

for all $r \in \mathbb{R}$, where $J_\varepsilon, J_{\Gamma, \varepsilon} : \mathbb{R} \to \mathbb{R}$ are resolvent operators. Then, it holds $\beta_\varepsilon(0) = \beta_{\Gamma, \varepsilon}(0) = 0$. It is well known that $\beta_\varepsilon, \beta_{\Gamma, \varepsilon}$ are Lipschitz continuous with Lipschitz constant $1/\varepsilon, 1/(\varepsilon\rho)$, respectively. Here, we have following properties:

$$
0 \leq \hat{\beta}_\varepsilon(r) \leq \hat{\beta}(r), \quad 0 \leq \hat{\beta}_{\Gamma, \varepsilon}(r) \leq \hat{\beta}_\Gamma(r) \quad \text{for all } r \in \mathbb{R}.
$$
Hence, it holds $0 \leq \varphi_{\varepsilon}(z) \leq \varphi(z)$ for all $z \in H_0$. These properties of Yosida approximation and Moreau–Yosida regularizations are as in [5,6,23]. Moreover, thanks to [9, Lemma 4.4], it holds
\begin{equation}
|\beta_{\varepsilon}(r)| \leq \rho|\beta_{\Gamma,\varepsilon}(r)| + c_0 \quad \text{for all } r \in \mathbb{R}
\end{equation}
with the same constants $\rho$ and $c_0$ as in (2.7).

Now, for each $\varepsilon \in (0,1]$, we also define two proper lower semicontinuous convex functionals $\tilde{\varphi}, \psi_{\varepsilon} : H_0 \to [0, +\infty]$ by
\begin{align*}
\tilde{\varphi}(z) & := \begin{cases} 
\frac{\kappa_1}{2} \int_{\Omega} |\nabla z|^2 dx + \frac{\kappa_2}{2} \int_{\Gamma} |\nabla_{\Gamma} z_{\Gamma}|^2 d\Gamma & \text{if } z \in V_0, \\
+\infty & \text{otherwise}
\end{cases} \\
\psi_{\varepsilon}(z) & := \begin{cases} 
\int_{\Omega} \tilde{\beta}_{\varepsilon}(z + m_0) dx + \int_{\Gamma} \tilde{\beta}_{\Gamma,\varepsilon}(z_{\Gamma} + m_0) d\Gamma & \text{if } z \in V_0, \\
+\infty & \text{otherwise}
\end{cases}
\end{align*}
and
Then, from [10, Lemma C], the subdifferential $A := \partial_{H_0} \tilde{\varphi}$ on $H_0$ is characterized by
\begin{equation}
A(z) = (-\Delta z, \partial_{\nu} z - \Delta_{\Gamma} z_{\Gamma}) \quad \text{with } z = (z, z_{\Gamma}) \in D(A) = W \cap V_0.
\end{equation}

Moreover, the representation of the subdifferential $\partial_{H_0} \psi_{\varepsilon}$ is given by
\begin{equation}
\partial_{H_0} \psi_{\varepsilon}(z) = P \beta_{\varepsilon}(z + m_0 1) \quad \text{for all } z \in H_0.
\end{equation}
This is proved by the same way as [16, Lemma 3.2]. Noting that it holds $D(\partial_{H_0} \psi_{\varepsilon}) = H_0$ and $A$ is a maximal monotone operator, it follows from the abstract monotonicity methods (see e.g., [5, Sect. 2.1]) that $A + \partial_{H_0} \psi_{\varepsilon}$ is also a maximal monotone operator. Moreover, by a simple calculation, we deduce that $(A + \partial_{H_0} \psi_{\varepsilon}) \subset \partial_{H_0} \varphi_{\varepsilon}$. Hence, it follows that, for any $z \in H_0$,
\begin{equation}
\partial_{H_0} \varphi_{\varepsilon}(z) = (A + \partial_{H_0} \psi_{\varepsilon})(z)
\end{equation}
(see e.g., [13]).

### 3.2 Approximate problem for (P)

Now, we consider the following approximate problem, say $(P)_{\varepsilon}$: for each $\varepsilon \in (0,1]$ find $v_{\varepsilon} := (v_{\varepsilon}, v_{\Gamma,\varepsilon})$ satisfying
\begin{align}
\varepsilon v'_{\varepsilon}(t) + F^{-1} v'_{\varepsilon}(t) + \partial_{H_0} \varphi_{\varepsilon}(v_{\varepsilon}(t)) \\
+ P(\pi(v_{\varepsilon}(t) + m_0 1)) = P f(t) & \quad \text{in } H_0 \quad \text{for a.a. } t \in (0,T) 
\end{align}
\begin{equation}
v_{\varepsilon}(0) = v_{\varepsilon}(T) \quad \text{in } H_0.
\end{equation}
where, for all $z \in H$, $\bar{\pi}(z) := (\bar{\pi}(z), \bar{\pi}_{\Gamma}(z_{\Gamma}))$ is cut-off function of $\pi, \pi_{\Gamma}$ given by
\begin{equation}
\bar{\pi}(r) := \begin{cases} 
0 & \text{if } r \leq \sigma_\ast - 1, \\
\pi(\sigma_\ast)(r - \sigma_\ast + 1) & \text{if } \sigma_\ast - 1 \leq r \leq \sigma_\ast, \\
\pi(r) & \text{if } \sigma_\ast \leq r \leq \sigma^\ast, \\
-\pi(\sigma^\ast)(r - \sigma^\ast - 1) & \text{if } \sigma^\ast \leq r \leq \sigma^\ast + 1, \\
0 & \text{if } r \geq \sigma^\ast + 1
\end{cases}
\end{equation}
and

$$
\tilde{\pi}_\Gamma(r) := \begin{cases} 
0 & \text{if } r \leq \sigma_{\Gamma^*} - 1, \\
\pi_\Gamma(\sigma_{\Gamma^*} - 1) & \text{if } \sigma_{\Gamma^*} - 1 \leq r \leq \sigma_{\Gamma^*}, \\
\pi_\Gamma(r) & \text{if } \sigma_{\Gamma^*} \leq r \leq \sigma_{\Gamma^*}^*, \\
-\pi_\Gamma(\sigma_{\Gamma^*}^*) & \text{if } \sigma_{\Gamma^*}^* \leq r \leq \sigma_{\Gamma^*}^* + 1, \\
0 & \text{if } r \geq \sigma_{\Gamma^*}^* + 1 
\end{cases}
$$

(3.6)

for all $r \in \mathbb{R}$, respectively. We establish the above cut-off function by referring to [31].

From now, we show the next proposition.

**Proposition 3.1.** Under the assumptions (A1)–(A5), for each $\varepsilon \in (0, 1]$, there exists a unique function

$$
v_\varepsilon \in H^1(0, T; H_0) \cap L^\infty(0, T; V_0) \cap L^2(0, T; W)
$$

such that $v_\varepsilon$ satisfies (3.3) and (3.4).

In order to show the Proposition 3.1, we use the method in [31], that is, we employ the viscosity approach. Conforming to the method, we consider the following problem: for each $\varepsilon \in (0, 1]$ and $\tilde{f} \in L^2(0, T; V_0)$,

$$
(F^{-1} + \varepsilon I)v_\varepsilon'(t) + \partial \varphi_\varepsilon(v_\varepsilon(t)) = \tilde{f}(t) \quad \text{in } H_0 \quad \text{for a.a. } t \in (0, T),
$$

(3.7)

$$
v_\varepsilon(0) = v_\varepsilon(T) \quad \text{in } H_0.
$$

(3.8)

Now, we can apply the abstract theory of doubly nonlinear evolution equation respect to time periodic problem [4] for (3.7), (3.8) because the operator $\varepsilon I + F^{-1}$ and $\partial \varphi_\varepsilon$ are coercive in $H_0$. This is an important assumption to use Theorem 2.2 in [4]. It is an advantage of the viscosity approach. Hence, we obtain the next proposition.

**Proposition 3.2.** For each $\varepsilon \in (0, 1]$ and $\tilde{f} \in L^2(0, T; V_0)$, there exists a unique function $v_\varepsilon$ such that (3.7) and (3.8).

Hereafter, we apply the Schauder fixed point theorem to prove the existence of the solution of the problem $(P_\varepsilon)$. To this aim, we set

$$
Y_1 := \{ \tilde{v}_\varepsilon \in H^1(0, T; H_0) \cap L^\infty(0, T; V_0); \tilde{v}_\varepsilon(0) = \tilde{v}_\varepsilon(T) \}.
$$

Firstly, for each $\tilde{v}_\varepsilon \in Y_1$, we consider the following problem, say $(P_\varepsilon; \tilde{v}_\varepsilon)$:

$$
\varepsilon v_\varepsilon'(s) + F^{-1}v_\varepsilon'(s) + \partial \varphi_\varepsilon(v_\varepsilon(s)) + P(\tilde{\pi}(v_\varepsilon(s) + m_01)) = Pf(s) \quad \text{in } H_0
$$

(3.9)

for a.a. $s \in (0, T)$, with

$$
v_\varepsilon(0) = v_\varepsilon(T) \quad \text{in } H_0.
$$

Next, we obtain the estimates of the solution of $(P_\varepsilon; \tilde{v}_\varepsilon)$ to apply the Schauder fixed point theorem. Note that we can allow the dependent of $\varepsilon \in (0, 1]$ for estimates of Lemma 3.1 because we use the Schauder fixed point theorem in the level of approximation.
Lemma 3.1. Let \( v_\varepsilon \) be the solution of problem (P_\varepsilon; \bar{v}_\varepsilon), it holds the following estimates. There exist positive constants \( C_{1\varepsilon}, C_{2\varepsilon}, C_{3\varepsilon} \) such that

\[
\varepsilon \int_0^T |v_\varepsilon'(s)|^2_{H_0} ds + \int_0^T |v_\varepsilon'(s)|^2_{V_0} ds \leq C_{1\varepsilon}, \tag{3.10}
\]

\[
\int_0^T |v_\varepsilon(s)|^2_{V_0} ds + \int_0^T \int_\Omega \beta_\varepsilon (v_\varepsilon(s) + m_0) dx ds + \int_0^T \int_\Gamma \beta_{\Gamma,\varepsilon} (v_{\Gamma,\varepsilon}(s) + m_0) ds \leq C_2 \tag{3.11}
\]

and

\[
\frac{1}{2} |v_\varepsilon(t)|^2_{V_0} + \int_0^T \beta_\varepsilon (v_\varepsilon(t) + m_0) dt + \int_\Gamma \beta_{\Gamma,\varepsilon} (v_{\Gamma,\varepsilon}(t) + m_0) d\Gamma \leq C_{3\varepsilon} \tag{3.12}
\]

for a.a. \( t \in (0, T) \).

Proof. At first, for each \( \bar{v}_\varepsilon \in Y_1 \), note that there exists a positive constant \( M \), depending only on \( \sigma_*, \sigma_\Gamma, \sigma^* \) and \( \sigma^*_\Gamma \), such that

\[
|\tilde{\pi} (\bar{v}_\varepsilon(t) + m_0 1)|^2_{H_0} \leq M \quad \text{for all} \ t \in [0, T]. \tag{3.13}
\]

Now, testing (3.9) at time \( s \in (0, T) \) by \( v_\varepsilon'(s) \) and using the Young inequality, we infer that

\[
v_\varepsilon'(s) = (P f(s) - P(\tilde{\pi} (\bar{v}_\varepsilon(s) + m_0 1)), v_\varepsilon'(s))_{H_0}
\]

\[
\leq \frac{1}{2} |f(s)|^2_{V} + \frac{1}{2} |v_\varepsilon'(s)|^2_{V_0} + \frac{M}{2\varepsilon} + \frac{\varepsilon}{2} |v_\varepsilon'(s)|^2_{H_0}
\]

for a.a. \( s \in (0, T) \). Therefore, we have that

\[
|v_\varepsilon'(s)|^2_{H_0} + |v_\varepsilon'(s)|^2_{V_0} + \frac{d}{ds} \varphi_\varepsilon (v_\varepsilon(s)) \leq |f(s)|^2_{V} + \frac{M}{\varepsilon}. \tag{3.14}
\]

Then, integrating it over \( (0, T) \) with respect to \( s \) and using the periodic property, we see that

\[
\varepsilon \int_0^T |v_\varepsilon'(s)|^2_{H_0} ds + \int_0^T |v_\varepsilon'(s)|^2_{V_0} ds \leq \int_0^T |f(s)|^2_{V_0} ds + \frac{MT}{\varepsilon},
\]

which implies that it follows the first estimate (3.10).

On the other hand, testing (3.9) at time \( s \in (0, T) \) by \( v_\varepsilon(s) \) and from (2.1), we deduce that

\[
\frac{1}{2} \frac{d}{ds} |v_\varepsilon(s)|^2_{V_0} + \frac{\varepsilon}{2} \frac{d}{ds} |v_\varepsilon(s)|^2_{H_0} + \varphi_\varepsilon (v_\varepsilon(s))
\]

\[
\leq (P f(s) - P(\tilde{\pi} (\bar{v}_\varepsilon(s) + m_0 1)), v_\varepsilon(s))_{H_0} + \varphi_\varepsilon (0)
\]

\[
\leq 2c_p |f(s)|^2_{H_0} + \frac{1}{4c_p} |v_\varepsilon(s)|^2_{H_0} + 2c_p M + \varphi(0)
\]

\[
\leq 2c_p |f(s)|^2_{H_0} + \frac{1}{4} |v_\varepsilon(s)|^2_{V_0} + 2c_p M + \varphi(0)
\]

\[
\leq 2c_p |f(s)|^2_{H_0} + \frac{1}{2} \varphi_\varepsilon (v_\varepsilon(s)) + 2c_p M + \varphi(0)
\]
for a.a. $s \in (0, T)$, thanks to the definition of the subdifferential. From the definition of $\varphi_\varepsilon$, it follows that

$$
\frac{1}{2} \frac{d}{ds} |v_\varepsilon(s)|_V^2 + \frac{\varepsilon}{2} \frac{d}{ds} |v_\varepsilon(s)|_{H_0}^2 + \frac{1}{4} |v_\varepsilon(s)|_V^2 \\
+ \frac{1}{2} \int_\Omega \tilde{\beta}_\varepsilon(v_\varepsilon(s) + m_0) \, dx + \frac{1}{2} \int_\Gamma \tilde{\beta}_{T,\varepsilon}(v_{T,\varepsilon}(s) + m_0) \, d\Gamma \\
\leq 2c_p |f(s)|_{H_0}^2 + 2c_p M + \int_\Omega \tilde{\beta}_\varepsilon(m_0) \, dx + \int_\Gamma \tilde{\beta}_{T,\varepsilon}(m_0) \, d\Gamma
$$

for a.a. $s \in (0, T)$. Integrating it over $(0, T)$ and using the periodic property, we see that

$$
\frac{1}{2} \int_0^T |v_\varepsilon(s)|_V^2 \, ds + \int_0^T \int_\Omega \tilde{\beta}_\varepsilon(v_\varepsilon(s) + m_0) \, dx \, ds + \int_0^T \int_\Gamma \tilde{\beta}_{T,\varepsilon}(v_{T,\varepsilon}(s) + m_0) \, d\Gamma \, ds \\
\leq 4c_p |f|_{L^2(0,T;H_0)}^2 + 4c_p TM + T|\Omega| |\tilde{\beta}(m_0)| + T|\Gamma| |\tilde{\beta}_{T}(m_0)|.
$$

Hence, there exist a positive constant $C_2$ such that the second estimate (3.11) holds.

Next, for each $s, t \in [0, T]$ such that $s \leq t$, we integrate (3.14) over $[s, t]$ with respect to $s$. Then, by neglecting the first two positive terms, we have

$$
\varphi_\varepsilon(v_\varepsilon(t)) \leq \varphi_\varepsilon(v_\varepsilon(s)) + \frac{1}{2} \int_s^t |f(s)|_V^2 \, ds + \frac{MT}{2\varepsilon}
$$

for a.a. $s, t \in [0, T]$, namely,

$$
\frac{1}{2} |v_\varepsilon(t)|_V^2 + \int_\Omega \tilde{\beta}_\varepsilon(v_\varepsilon(t) + m_0) \, dx + \int_\Gamma \tilde{\beta}_{T,\varepsilon}(v_{T,\varepsilon}(t) + m_0) \, d\Gamma \\
\leq \frac{1}{2} |v_\varepsilon(s)|_V^2 + \int_\Omega \tilde{\beta}_\varepsilon(v_\varepsilon(s) + m_0) \, dx + \int_\Gamma \tilde{\beta}_{T,\varepsilon}(v_{T,\varepsilon}(s) + m_0) \, d\Gamma \\
+ \frac{1}{2} \int_0^T |f(s)|_V^2 \, ds + \frac{MT}{2\varepsilon}
$$

(3.15)

for a.a. $s, t \in [0, T]$. Now, integrating it over $(0, t)$ with respect to $s$, we deduce that

$$
\frac{t}{2} |v_\varepsilon(t)|_V^2 + t \int_\Omega \tilde{\beta}_\varepsilon(v_\varepsilon(t) + m_0) \, dx + t \int_\Gamma \tilde{\beta}_{T,\varepsilon}(v_{T,\varepsilon}(t) + m_0) \, d\Gamma \\
\leq \frac{1}{2} |v_\varepsilon(s)|_V^2 + \int_\Omega \tilde{\beta}_\varepsilon(v_\varepsilon(s) + m_0) \, dx ds + \int_\Gamma \tilde{\beta}_{T,\varepsilon}(v_{T,\varepsilon}(s) + m_0) \, d\Gamma ds \\
+ \frac{T}{2} \int_0^T |f(s)|_V^2 \, ds + \frac{MT^2}{2\varepsilon}
$$

(3.16)

for a.a. $t \in [0, T]$. In particular, putting $t := T$ and dividing (3.16) by $T$, it follows that

$$
\frac{1}{2} |v_\varepsilon(T)|_V^2 + \int_\Omega \tilde{\beta}_\varepsilon(v_\varepsilon(T) + m_0) \, dx + \int_\Gamma \tilde{\beta}_{T,\varepsilon}(v_{T,\varepsilon}(T) + m_0) \, d\Gamma \\
\leq \frac{1}{2T} \int_0^T |v_\varepsilon(s)|_V^2 \, ds + \frac{1}{T} \int_\Omega \tilde{\beta}_\varepsilon(v_\varepsilon(s) + m_0) \, dx ds \\
+ \frac{1}{T} \int_\Gamma \tilde{\beta}_{T,\varepsilon}(v_{T,\varepsilon}(s) + m_0) \, d\Gamma ds + \frac{1}{2} \int_0^T |f(s)|_V^2 \, ds + \frac{MT}{2\varepsilon}.
$$

(3.17)
Hence, combining the second estimate \((3.11)\) and \((3.17)\), we see that
\[
\frac{1}{2} \| \mathbf{v}_\varepsilon(T) \|^2_{\mathbf{V}_0} + \int_{\Omega} \mathcal{\beta}_\varepsilon(v_\varepsilon(T) + m_0) \, dx + \int_{\Gamma} \mathcal{\beta}_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(T) + m_0) \, d\Gamma \\
\leq \frac{C_2}{T} + \frac{1}{2} \int_0^T |f(s)|^2_{\mathbf{V}} \, ds + \frac{MT}{2\varepsilon}.
\]

Moreover, from the periodic property, we infer that
\[
\frac{1}{2} \| \mathbf{v}_\varepsilon(0) \|^2_{\mathbf{V}_0} + \int_{\Omega} \mathcal{\beta}_\varepsilon(v_\varepsilon(0) + m_0) \, dx + \int_{\Gamma} \mathcal{\beta}_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(0) + m_0) \, d\Gamma \\
\leq \frac{C_2}{T} + \frac{1}{2} \int_0^T |f(s)|^2_{\mathbf{V}} \, ds + \frac{MT}{2\varepsilon}.
\] (3.18)

Now, let \(s = 0\) in \((3.15)\). Then, owing to \((3.18)\), we deduce that
\[
\frac{1}{2} \| \mathbf{v}_\varepsilon(t) \|^2_{\mathbf{V}_0} + \int_{\Omega} \mathcal{\beta}_\varepsilon(v_\varepsilon(t) + m_0) \, dx + \int_{\Gamma} \mathcal{\beta}_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(t) + m_0) \, d\Gamma \\
\leq \frac{C_2}{T} + |f|^2_{L^2(0,T;\mathbf{V})} + \frac{MT}{\varepsilon}
\]
for a.a. \(t \in [0,T]\). Therefore, there exisits a positive constant \(C_{3\varepsilon}\) such that the final estimate \((3.12)\) holds.

In terms of \((3.10)\), the key point to prove the estimate is exploiting \((3.13)\). \((3.13)\) is arised from the form of cut-off functions \((3.5), (3.6)\). The form of cut-off functions depend on the assumption \((A5)\) essentially. However, considered the same estimate in \([10, Lemma 4.1]\), They do not impose the assumption. They use the Gronwall inequality to obtain the estimate because the initial value is given data. On the other hand, we can not obtain it even though we use the Gronwall inequality, because the initial value is not given. For this reason, it is nacessary to impose \((A5)\). This is a difficult point to solve the time periodic problem.

Now, we show the existence of solutions of approximate problem \((P)_\varepsilon\).

**Proof of Proposition 3.1** To this aim, we apply the Schauder fixed point theorem. To do so, we set
\[
\mathbf{Y}_2 := \left\{ \bar{\mathbf{v}}_\varepsilon \in \mathbf{Y}_1; \sup_{t \in [0,T]} |\bar{\mathbf{v}}_\varepsilon|_{\mathbf{V}_0}^2 + \varepsilon |\bar{\mathbf{v}}_\varepsilon|_{H^1(0,T;\mathbf{H}_0)}^2 \leq M_\varepsilon \right\},
\]
where \(M_\varepsilon\) is a positive constant and be determined by Lemma 3.1. Then, the set \(\mathbf{Y}_2\) is non-empty compact convex on \(C(0,T;\mathbf{H}_0)\). Now, from Proposition 3.2, for each \(\bar{\mathbf{v}}_\varepsilon \in \mathbf{Y}_2\), there exists a unique solution \(\mathbf{v}_\varepsilon\) of \((P_\varepsilon; \bar{\mathbf{v}}_\varepsilon)\). Moreover, from Lemma 3.1, it holds \(\mathbf{v}_\varepsilon \in \mathbf{Y}_2\). Here, we define the mapping \(S : \mathbf{Y}_2 \to \mathbf{Y}_2\) such that, for each \(\bar{\mathbf{v}}_\varepsilon \in \mathbf{Y}_2\), corresponding \(\bar{\mathbf{v}}_\varepsilon\) to the solution \(\mathbf{v}_\varepsilon\) of \((P_\varepsilon; \bar{\mathbf{v}}_\varepsilon)\). Then, the mapping \(S\) is continuous on \(\mathbf{Y}_2\) with respect to topology of \(C(0,T;\mathbf{H}_0)\). Indeed, let \(\{\mathbf{v}_{\varepsilon,n}\}_{n \in \mathbb{N}} \subset \mathbf{Y}_2\) be \(\mathbf{v}_{\varepsilon,n} \to \mathbf{v}_\varepsilon\) in \(C(0,T;\mathbf{H}_0)\) and
\{v_{\varepsilon, n}\}_{n \in \mathbb{N}} be the sequence of the solution of \((P_\varepsilon; \bar{v}_{\varepsilon, n})\). From Lemma 3.1, there exist a subsequence \(\{n_k\}_{k \in \mathbb{N}}, \) with \(n_k \to \infty\) as \(k \to \infty\), and \(v_{\varepsilon} \in H^1(0, T; H_0) \cap L^\infty(0, T; V_0)\) such that
\[
v_{\varepsilon, n_k} \to v_{\varepsilon}\ \text{ weakly star in } H^1(0, T; H_0) \cap L^\infty(0, T; V_0). \tag{3.19}\]
Hence, from (3.19) and the Ascoli–Alzela theorem (see e.g., [30]), there exists a subsequence (not relabeled) such that
\[
v_{\varepsilon, n_k} \to v_{\varepsilon} \quad \text{in } C([0, T]; H_0) \tag{3.20}\]
as \(k \to \infty\). Also, we have
\[
v'_{\varepsilon, n_k} \to v'_\varepsilon \quad \text{weakly in } L^2(0, T; H_0) \tag{3.21}\]
as \(k \to \infty\). Because we have \(v_{\varepsilon, n_k}(0) = v_{\varepsilon, n_k}(T)\), it holds \(v_{\varepsilon}(0) = v_{\varepsilon}(T)\) in \(H_0\). Hereafter, we show that this \(v_{\varepsilon}\) is the solution of \((P_\varepsilon; \bar{v}_\varepsilon)\). Since \(v_{\varepsilon, n_k}\) is the solution of \((P_\varepsilon; \bar{v}_{\varepsilon, n_k})\), we see that
\[
\int_0^T (P f(s) - P(\bar{\pi}(v_{\varepsilon, n_k}(s) + m_0)) - \varepsilon v'_{\varepsilon, n_k}(s) - F^{-1}v'_{\varepsilon, n_k}(s), \eta(s) - v_{\varepsilon, n_k}(s))_{H_0} ds \\
\leq \int_0^T \varphi_\varepsilon(\eta(s)) ds - \int_0^T \varphi_\varepsilon(v_{\varepsilon, n_k}(s)) ds \tag{3.22}\]
for all \(\eta \in L^2(0, T; H_0)\), thanks to the definition of subdifferential \(\partial \varphi_\varepsilon\). Moreover, it follows from \(\bar{v}_{\varepsilon, n_k} \to \bar{v}_\varepsilon\) in \(C(0, T; H_0)\) that
\[
P(\bar{\pi}(v_{\varepsilon, n_k} + m_0)) \to P(\bar{\pi}(\bar{v}_\varepsilon + m_0)) \quad \text{in } C([0, T]; H_0). \tag{3.23}\]
Thus, on account of (3.19)–(3.23), by taking the upper limit as \(k \to \infty\) in (3.22) and using
\[
\liminf_{k \to \infty} \int_0^T \varphi_\varepsilon(v_{\varepsilon, n_k}(s)) ds \geq \int_0^T \varphi_\varepsilon(v_{\varepsilon}(s)) ds,
\]
we infer that
\[
\int_0^T (P f(s) - P(\bar{\pi}(\bar{v}_\varepsilon(s) + m_0)) - \varepsilon v'_{\varepsilon}(s) - F^{-1}v'_{\varepsilon}(s), \eta(s) - v_{\varepsilon}(s))_{H_0} ds \\
\leq \int_0^T \varphi_\varepsilon(\eta(s)) ds - \int_0^T \varphi_\varepsilon(v_{\varepsilon}(s)) ds
\]
for all \(\eta \in L^2(0, T; H_0)\). Hence, we see that the function \(v_{\varepsilon}\) is the solution of \((P_\varepsilon; \bar{v}_\varepsilon)\). As a result, it follows from the uniqueness of the solution of \((P_\varepsilon; \bar{v}_\varepsilon)\) that
\[
S(\bar{v}_{\varepsilon, n_k}) = v_{\varepsilon, n_k} \to v_{\varepsilon} = S(\bar{v}_\varepsilon) \quad \text{in } C([0, T]; H_0)
\]
as \(k \to \infty\). Therefore, the mapping \(S\) is continuous with respect to \(C([0, T]; H_0)\). Thus, from the Schauder fixed point theorem, there exists a fixed point on \(Y_2\), namely, the problem \((P)_\varepsilon\) admits a solution \(v_{\varepsilon}\). Finally, from the fact that \(\partial \varphi_\varepsilon(v_{\varepsilon}) \in L^2(0, T; H_0)\),
which implies $\mathbf{v}_\varepsilon \in L^2(0, T; W)$. \hfill \Box

Now, we consider the chemical potential $\mathbf{\mu} := (\mu, \mu_\Gamma)$ by approximating. For each $\varepsilon \in (0, 1]$, we set the approximate sequence

$$\mathbf{\mu}_\varepsilon(s) := \varepsilon \mathbf{v}_\varepsilon'(s) + \partial \varphi_\varepsilon(\mathbf{v}_\varepsilon(s)) + \tilde{\pi}(\mathbf{v}_\varepsilon(s) + m_0 \mathbf{1}) - \mathbf{f}(s)$$

(3.24)

for a.a. $s \in (0, T)$. From (3.2), we can rewrite (3.24) as

$$\mathbf{\mu}_\varepsilon(s) = \varepsilon \mathbf{v}_\varepsilon'(s) + \mathbf{A}\mathbf{v}_\varepsilon(s) + P\beta_\varepsilon(\mathbf{v}_\varepsilon(s) + m_0 \mathbf{1}) + \tilde{\pi}(\mathbf{v}_\varepsilon(s) + m_0 \mathbf{1}) - \mathbf{f}(s)$$

(3.25)

for a.a. $s \in (0, T)$. Then, we rewrite (3.3) as

$$F^{-1}\mathbf{v}_\varepsilon'(s) + \mathbf{\mu}_\varepsilon(s) - \omega_\varepsilon(s)\mathbf{1} = 0 \text{ in } V$$

for a.a. $s \in (0, T)$, where

$$\omega_\varepsilon(s) := m(\tilde{\pi}(\mathbf{v}_\varepsilon(s) + m_0 \mathbf{1}) - \mathbf{f}(s))$$

for a.a. $s \in (0, T)$. Therefore, we have $P\mathbf{\mu}_\varepsilon = \mathbf{\mu}_\varepsilon - \omega_\varepsilon \mathbf{1} \in L^2(0, T; V_0)$ and $\omega_\varepsilon \in L^2(0, T)$. Then, it holds $\mathbf{\mu}_\varepsilon \in L^2(0, T; V)$ and

$$\mathbf{v}_\varepsilon'(s) + FP\mathbf{\mu}_\varepsilon(s) = 0 \text{ in } V_0^*$$

(3.26)

for a.a. $s \in (0, T)$.

### 3.3 Uniform estimates

In this subsection, we obtain uniform estimates independent of $\varepsilon \in (0, 1]$. We refer to [31] to obtain uniform estimates.

**Lemma 3.2.** There exists a positive constant $M_1$, independent of $\varepsilon \in (0, 1]$, such that

$$\frac{1}{2} \int_0^T |\mathbf{v}_\varepsilon(s)|^2_{V_0} ds + \int_0^T \int_\Omega \tilde{\beta}_\varepsilon(\mathbf{v}_\varepsilon(s) + m_0) dx ds + \int_0^T \int_\Gamma \tilde{\beta}_{\Gamma, \varepsilon}(v_{\Gamma, \varepsilon}(s) + m_0) d\Gamma ds \leq M_1.$$  

(3.27)

**Proof** From (3.5), (3.6) and the assumption (A3), note that $\tilde{\pi}, \tilde{\pi}_\Gamma$ is globally Lipschitz continuous on $\mathbb{R}$. We denote the Lipschitz constant of $\tilde{\pi}, \tilde{\pi}_\Gamma$ by $\tilde{L}, \tilde{L}_\Gamma$, respectively. Moreover, we can take primitive functions $\tilde{\tilde{\pi}}$ and $\tilde{\tilde{\pi}}_\Gamma$ of $\tilde{\pi}$ and $\tilde{\pi}_\Gamma$ satisfying

$$\int_\Omega \tilde{\tilde{\pi}}(\mathbf{v}_\varepsilon(s)) dx \geq 0, \quad \int_\Gamma \tilde{\tilde{\pi}}_\Gamma(v_{\Gamma, \varepsilon}(s)) d\Gamma \geq 0$$

\hfill \Box
for a.a. \( s \in (0, T) \), respectively. Now, we test (3.3) at time \( s \in (0, T) \) by \( v_\varepsilon(s) \) and use the Young inequality. Then, we deduce that

\[
\begin{align*}
\frac{1}{2} \frac{d}{ds} |v_\varepsilon(s)|^2_{V_0} + \varepsilon \frac{d}{ds} |v_\varepsilon(s)|^2_{H_0} + \varphi_\varepsilon(v_\varepsilon(s))
\leq (P f(s) - P(\pi'(v_\varepsilon(s) + m_0 1)) , v_\varepsilon(s))_{H_0} + \varphi_\varepsilon(0)
\leq c_p |f(s)|^2_H + \frac{1}{4c_p} |v_\varepsilon(s)|^2_{H_0} + c_p M + \varphi(0)
\leq \frac{1}{4} |v_\varepsilon(s)|^2_{V_0} + c_p |f(s)|^2_H + c_p M + \varphi(0)
\leq \frac{1}{2} \varphi_\varepsilon(v_\varepsilon(s)) + c_p |f(s)|^2_H + c_p M + \varphi(0)
\end{align*}
\]

for a.a. \( s \in (0, T) \). Namely, we have

\[
\begin{align*}
\frac{1}{2} \int_0^T |v_\varepsilon(s)|^2_{V_0} + \varepsilon \frac{1}{2} \int_0^T |v_\varepsilon(s)|^2_{H_0} + \frac{1}{4} \int_0^T |v_\varepsilon(s)|^2_{V_0}
+ \frac{1}{2} \int_0^T \beta_\varepsilon(v_\varepsilon(s) + m_0) dx + \frac{1}{2} \int_0^T \hat{\beta}_T_\varepsilon(v_\varepsilon(s) + m_0) dT
\leq c_p |f(s)|^2_H + c_p M + \varphi(0)
\end{align*}
\]

for a.a. \( s \in (0, T) \). Integrating it over \( (0, T) \) and using the periodic property, we see that

\[
\begin{align*}
\frac{1}{2} \int_0^T |v_\varepsilon(s)|^2_{V_0} + \int_0^T \int_0^T \beta_\varepsilon(v_\varepsilon(s) + m_0) dx ds + \int_0^T \int_0^T \hat{\beta}_T_\varepsilon(v_\varepsilon(s) + m_0) dT ds
\leq 2c_p |f|^2_{L^2(0, T; H)} + 2c_p TM + 2T \varphi(0).
\end{align*}
\]

This yields that the estimate (3.27) holds. \( \square \)

**Lemma 3.3.** There exists a positive constant \( M_2 \), independent of \( \varepsilon \in (0, 1] \), such that

\[
\varepsilon \int_0^T |v_\varepsilon'(s)|^2_{H_0} ds + \frac{1}{2} \int_0^T |v_\varepsilon'(s)|^2_{V_0} ds \leq M_2.
\]

**Proof** We test (3.3) at time \( s \in (0, T) \) by \( v_\varepsilon'(s) \). Then, by using the Young inequality, we see that

\[
\begin{align*}
\varepsilon |v_\varepsilon'(s)|^2_{H_0} + |v_\varepsilon'(s)|^2_{V_0} + \frac{d}{ds} \varphi_\varepsilon(v_\varepsilon(s))
+ \frac{d}{ds} \int_0^T \pi'(v_\varepsilon(s) + m_0) dx + \frac{d}{ds} \int_0^T \hat{\pi}_T (v_\varepsilon(s) + m_0) dT
= (P f(s), v_\varepsilon'(s))_{H_0}
\leq \frac{1}{2} |f(s)|^2_H + \frac{1}{2} |v_\varepsilon'(s)|^2_{V_0}
\end{align*}
\]
for a.a. $s \in (0, T)$. This implies that
\[
\varepsilon |v'_\varepsilon(s)|^2_{H_0} + \frac{1}{2} |v'_\varepsilon(s)|^2_{V_0^*} + \frac{d}{ds} \phi (v_\varepsilon(s)) + \frac{d}{ds} \int_{\Omega} \widehat{\pi}(v_\varepsilon(s) + m_0) dx + \frac{d}{ds} \int_{\Gamma} \widehat{\pi}_T (v_{T,\varepsilon}(s) + m_0) d\Gamma \\
\leq \frac{1}{2} |f(s)|^2_{V} \tag{3.28}
\]
for a.a. $s \in (0, T)$. Therefore, by integrating it over $(0, T)$ with respect to $s$ and using the periodic property, we can conclude. \hfill \Box

**Lemma 3.4.** There exists a positive constant $M_3$, independent of $\varepsilon \in (0, 1]$, such that
\[
\frac{1}{2} |v_\varepsilon(t)|^2_{V_0} + \int_{\Omega} \widehat{\pi}(v_\varepsilon(t) + m_0) dx + \int_{\Gamma} \widehat{\pi}_T (v_{T,\varepsilon}(t) + m_0) d\Gamma \leq M_3 \tag{3.29}
\]
for a.a. $t \in [0, T]$.

**Proof** For each $s, t \in [0, T]$ such that $s \leq t$, we integrate (3.28) over $[s, t]$. Then, by neglecting the first two positive terms, we see that
\[
\phi (v_\varepsilon(t)) + \int_{\Omega} \widehat{\pi}(v_\varepsilon(t) + m_0) dx + \int_{\Gamma} \widehat{\pi}_T (v_{T,\varepsilon}(t) + m_0) d\Gamma \\
\leq \phi (v_\varepsilon(s)) + \int_{\Omega} \widehat{\pi}(v_\varepsilon(s) + m_0) dx + \int_{\Gamma} \widehat{\pi}_T (v_{T,\varepsilon}(s) + m_0) d\Gamma + \frac{1}{2} \int_0^T |f(s)|^2_{V_0} ds
\]
for a.a. $s, t \in [0, T]$. Now, integrating it over $(0, t)$ with respect to $s$, it follows that
\[
\frac{t}{2} |v_\varepsilon(t)|^2_{V_0} + t \int_{\Omega} \widehat{\beta}_\varepsilon(v_\varepsilon(t) + m_0) dx + t \int_{\Gamma} \widehat{\beta}_{T,\varepsilon}(v_{T,\varepsilon}(t) + m_0) d\Gamma \\
\leq \frac{1}{2} \int_0^T |v_\varepsilon(s)|^2_{V_0} ds + \int_0^T \int_{\Omega} \widehat{\beta}_\varepsilon(v_\varepsilon(s) + m_0) dx ds + \int_0^T \int_{\Gamma} \widehat{\beta}_{T,\varepsilon}(v_{T,\varepsilon}(s) + m_0) d\Gamma ds \\
+ \int_0^T \int_{\Omega} \widehat{\pi}(v_\varepsilon(s) + m_0) dx ds + \int_0^T \int_{\Gamma} \widehat{\pi}_T (v_{T,\varepsilon}(s) + m_0) d\Gamma ds \\
+ \frac{T}{2} \int_0^T |f(s)|^2_{V_0} ds \tag{3.30}
\]
for a.a. $t \in [0, T]$. Here, Note that we have
\[
|\widehat{\pi}(r)| \leq \int_0^r |\widehat{\pi}(\tau)| d\tau \\
\leq \tilde{L} \int_0^r |\tau| d\tau + \int_0^r |\widehat{\pi}(0)| d\tau \\
= \frac{\tilde{L}}{2} r^2 + |\widehat{\pi}(0)|r
\]
for all \( r \in \mathbb{R} \). and similarly,

\[
|\tilde{\pi}_\Gamma(r)| \leq \frac{\overline{L}_\Gamma}{2} r^2 + |\tilde{\pi}_\Gamma(0)| r \quad \text{for all } r \in \mathbb{R}.
\]

Then, by using the Young inequality, we infer that

\[
\int_\Omega \tilde{\pi}(v_\varepsilon(s) + m_0) dx \leq \int_\Omega \left( \frac{\overline{L}}{2} |v_\varepsilon(s) + m_0|^2 + |\tilde{\pi}(0)||v_\varepsilon(s) + m_0| \right) dx
\]

\[
\leq \overline{L} \int_\Omega |v_\varepsilon(s) + m_0|^2 dx + \frac{1}{2\overline{L}} |\tilde{\pi}(0)|^2 |\Omega|
\]

\[
\leq 2\overline{L} \int_\Omega |v_\varepsilon(s)|^2 dx + 2m_0^2 |\Omega| + \frac{1}{2\overline{L}} |\tilde{\pi}(0)|^2 |\Omega| \tag{3.31}
\]

for a.a. \( s \in [0, T] \). Similarly, we have

\[
\int_\Gamma \tilde{\pi}_\Gamma(v_{T,\varepsilon}(s) + m_0) d\Gamma \leq 2\overline{L}_\Gamma \int_\Gamma |v_{T,\varepsilon}(s)|^2 d\Gamma + 2m_0^2 |\Gamma| + \frac{1}{2\overline{L}_\Gamma} |\tilde{\pi}_\Gamma(0)|^2 |\Gamma| \tag{3.32}
\]

for a.a. \( s \in [0, T] \). Thus, on account of (3.30) - (3.32), we deduce that

\[
\frac{t}{2} \left| \frac{v_\varepsilon(t)}{V_0} \right|^2 + t \int_\Omega \widehat{\beta}_\varepsilon(v_\varepsilon(t) + m_0) dx + t \int_\Gamma \widehat{\beta}_{T,\varepsilon}(v_{T,\varepsilon}(t) + m_0) d\Gamma
\]

\[
\leq \frac{1}{2} \int_0^T \left| v_\varepsilon(s) \right|^2 |V_0| ds + \int_0^T \int_\Omega \widehat{\beta}_\varepsilon(v_\varepsilon(s) + m_0) dx ds + \int_0^T \int_\Gamma \widehat{\beta}_{T,\varepsilon}(v_{T,\varepsilon}(s) + m_0) d\Gamma ds
\]

\[
+ 2\overline{L} \int_\Omega |v_\varepsilon(s)|^2 dx + 2\overline{L}_\Gamma \int_\Gamma |v_{T,\varepsilon}(s)|^2 d\Gamma + \frac{T}{2} \int_0^T |f(s)|^2 |V_0| ds + M_4
\]

\[
\leq \left( \frac{1}{2} + \overline{L}_{CP} \right) \int_0^T \left| v_\varepsilon(s) \right|^2 |V_0| ds + \int_0^T \int_\Omega \widehat{\beta}_\varepsilon(v_\varepsilon(s) + m_0) dx ds
\]

\[
+ \int_0^T \int_\Gamma \widehat{\beta}_{T,\varepsilon}(v_{T,\varepsilon}(s) + m_0) d\Gamma ds + \frac{T}{2} \int_0^T |f(s)|^2 |V_0| ds + M_4
\]

for a.a. \( t \in [0, T] \), where \( \tilde{L} := \max\{2\overline{L}, 2\overline{L}_\Gamma \} \) and

\[
M_4 := 2m_0^2 |\Omega| + \frac{1}{2\overline{L}} |\tilde{\pi}(0)|^2 |\Omega| + 2m_0^2 |\Gamma| + \frac{1}{2\overline{L}_\Gamma} |\tilde{\pi}_\Gamma(0)|^2 |\Gamma|.
\]

In particular, putting \( t := T \) and dividing it by \( T \), it follows that

\[
\frac{1}{2} \left| \frac{v_\varepsilon(T)}{V_0} \right|^2 + \int_\Omega \widehat{\beta}_\varepsilon(v_\varepsilon(T) + m_0) dx + \int_\Gamma \widehat{\beta}_{T,\varepsilon}(v_{T,\varepsilon}(T) + m_0) d\Gamma
\]

\[
\leq \frac{1}{T} \left( \frac{1}{2} + \overline{L}_{CP} \right) \int_0^T \left| v_\varepsilon(s) \right|^2 |V_0| ds + \frac{1}{T} \int_0^T \int_\Omega \widehat{\beta}_\varepsilon(v_\varepsilon(s) + m_0) dx ds
\]

\[
+ \frac{1}{2} \int_0^T \int_\Gamma \widehat{\beta}_{T,\varepsilon}(v_{T,\varepsilon}(s) + m_0) d\Gamma ds + \frac{T}{2} \int_0^T |f(s)|^2 |V_0| ds + \frac{M_4}{T}. \tag{3.33}
\]
Combining (3.27) and (3.33), there exists a positive constant $\tilde{M}_3$ such that

$$\frac{1}{2} \left| v_\varepsilon(T) \right|^2_{V_0} + \int_\Omega \hat{\beta}_\varepsilon(v_\varepsilon(T) + m_0) dx + \int_\Gamma \hat{\beta}_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(T) + m_0) d\Gamma \leq \tilde{M}_3.$$ 

From the periodic property, we have

$$\varphi_\varepsilon(v_\varepsilon(0)) = \frac{1}{2} \left| v_\varepsilon(0) \right|^2_{V_0} + \int_\Omega \hat{\beta}_\varepsilon(v_\varepsilon(0) + m_0) dx + \int_\Gamma \hat{\beta}_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(0) + m_0) d\Gamma \leq \tilde{M}_3.$$ 

(3.34)

Now, integrating (3.28) by $(0, t)$ with respect to $s$, it follows from (3.31)–(3.32) that

$$\varphi_\varepsilon(v_\varepsilon(t)) + \int_\Omega \tilde{\pi}(v_\varepsilon(t) + m_0) dx + \int_\Gamma \tilde{\pi}_\Gamma(v_{\Gamma,\varepsilon}(t) + m_0) d\Gamma$$

$$\leq \varphi_\varepsilon(v_\varepsilon(0)) + \int_\Omega \tilde{\pi}(v_\varepsilon(0) + m_0) dx + \int_\Gamma \tilde{\pi}_\Gamma(v_{\Gamma,\varepsilon}(0) + m_0) d\Gamma + \frac{1}{2} \int_0^T |f(s)|^2_{V_0} ds$$

$$\leq (1 + 2\tilde{L}_{CP}) \varphi_\varepsilon(v_\varepsilon(0)) + \frac{1}{2} \int_0^T |f(s)|^2_{V_0} ds + M_4$$

(3.35)

for a.a. $t \in [0, T]$. Therefore, by virtue of (3.34)–(3.35), there exists a positive constant $M_3$ such that the estimate (3.29) holds.

\[ \square \]

**Lemma 3.5.** There exists a positive constant $M_4$, independent of $\varepsilon \in (0, 1]$, such that

$$\delta_0 \int_0^T \left| \beta_\varepsilon(v_\varepsilon(s) + m_0) \right|_{L^1(\Omega)}^2 ds + \delta_0 \int_0^T \left| \beta_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(s) + m_0) \right|_{L^1(\Gamma)}^2 ds \leq M_4$$

(3.36)

for some positive constants $\delta_0$.

**Proof** To show this lemma, we can employ the method of [10] Lemma 4.1, 4.3, because of imposing same assumptions as [10] for $\beta, \beta_\Gamma$ and being $m_0 \in \text{int}(D(\beta_\Gamma))$. Therefore, we can also exploit the following inequalities stated in [17] Sect. 5: for each $\varepsilon \in (0, 1]$, there exist two positive constants $\delta_0$ and $c_1$ such that

$$\beta_\varepsilon(r)(r - m_0) \geq \delta_0 \left| \beta_\varepsilon(r) \right| - c_1, \quad \beta_{\Gamma,\varepsilon}(r)(r - m_0) \geq \delta_0 \left| \beta_{\Gamma,\varepsilon}(r) \right| - c_1$$

for all $r \in \mathbb{R}$. Hence, it follows that

$$\left( \beta_\varepsilon(u_\varepsilon(s)), v_\varepsilon(s) \right)_H \geq \delta_0 \int_\Omega \left| \beta_\varepsilon(u_\varepsilon(s)) \right| dx - c_1 |\Omega| + \delta_0 \int_\Gamma \left| \beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon}(s)) \right| d\Gamma - c_1 |\Gamma|$$

(3.37)

for a.a. $s \in (0, T)$. On the other hand, we test (3.3) at time $s \in (0, T)$ by $v_\varepsilon(s)$. Then, from (3.2), we see that

$$\left( \varepsilon v'_\varepsilon(s), v_\varepsilon(s) \right)_{H_0} + \left( v'_\varepsilon(s), v_\varepsilon(s) \right)_{V_0} + \left( A v_\varepsilon(s), v_\varepsilon(s) \right)_{H_0} + \left( P \beta_\varepsilon(u_\varepsilon(s)), v_\varepsilon(s) \right)_{H_0}$$

$$\leq \left( f(s) - \tilde{\pi}(u_\varepsilon(s)), v_\varepsilon(s) \right)_H.$$ 

(3.38)
Hence, from (3.37)–(3.38) and the maximal monotonicity of $A$, by squaring we have
\[
\left( \delta_0 \int |\beta_{\varepsilon}(u_{\varepsilon}(s))| dx + \delta_0 \int |\beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon}(s))| d\Gamma \right)^2 \leq 3c_1^2(\|\Omega\| + |\Gamma|)^2
\]
\[
+ 9(\|f(s)\|^2_H + |\pi(u_{\varepsilon}(s))|^2_H + |v_{\varepsilon}'(s)|_{H_0}^2) + 3|v_{\varepsilon}'(s)|_{V_0}^2 |v_{\varepsilon}(s)|_{V_0}^2
\]
for a.a. $s \in (0, T)$. Therefore, from the Lipschitz continuity of $\tilde{\pi}, \tilde{\pi}_{\Gamma}$ and Lemma 3.4, by integrating it over $(0, T)$ with respect to $s$, there exists a positive constant $M_4$ such that the estimate (3.36) holds.

**Lemma 3.6.** There exists a positive constants $M_5$, independent of $\varepsilon \in (0, 1]$, such that
\[
\int_0^T |\mu_{\varepsilon}(s)|_V^2 ds \leq M_5.
\] (3.39)

**Proof** Firstly, by using the Lipschitz continuity of $\tilde{\pi}, \tilde{\pi}_{\Gamma}$ and the Hölder inequality, it follows from (2.1) and Lemma 3.4 that there exists a positive constant $M_5^*$ such that
\[
|m(\pi(v_{\varepsilon}(s) + m_0 1))| \leq \frac{1}{|\Omega| + |\Gamma|} \left\{ \int_{\Omega} |\tilde{\pi}(v_{\varepsilon}(s) + m_0)| dx + \int_{\Gamma} |\tilde{\pi}_{\Gamma}(v_{\Gamma,\varepsilon}(s) + m_0)| d\Gamma \right\}
\]
\[
\leq \frac{1}{|\Omega| + |\Gamma|} \left\{ \tilde{L}|\Omega|^\frac{1}{2} |v_{\varepsilon}(s)|^2_H + \tilde{L}|\Omega| |m_0| + |\Omega| |\tilde{\pi}(0)|
\]
\[
+ \tilde{L}_{\Gamma}|\Gamma|^\frac{1}{2} |v_{\Gamma,\varepsilon}(s)|^2_{H_{\Gamma}} + \tilde{L}_{\Gamma} |\Gamma| |m_0| + |\Gamma| |\tilde{\pi}_{\Gamma}(0)| \right\}
\]
\[
\leq \frac{1}{|\Omega| + |\Gamma|} M_5^* \left\{ |v_{\varepsilon}(s)|^2_{V_0} + 1 \right\}
\]
\[
\leq \frac{1}{|\Omega| + |\Gamma|} M_5^* (M_3 + 1) =: \tilde{M}_5
\] (3.40)
for a.a. $s \in (0, T)$. Therefore, owing to (3.40) we deduce that
\[
|m(\mu_{\varepsilon}(s))|^2 = |m(\pi(v_{\varepsilon}(s) + m_0 1) - f(s))|^2
\]
\[
\leq 2\tilde{M}_5^2 + \frac{4}{(|\Omega| + |\Gamma|)^2} (|f(s)|_{L^1(\Omega)} + |f_{\Gamma}(s)|_{L^1(\Gamma)}) =: \tilde{M}_5
\]
for a.a. $s \in (0, T)$. Next, from (2.1), (3.26) and the fact $P\mu_{\varepsilon}(s) = \mu_{\varepsilon}(s) - m(\mu_{\varepsilon}(s))1$ for a.a. $s \in (0, T)$, we deduce that
\[
\int_0^T |\mu_{\varepsilon}(s)|_V^2 ds \leq 2 \int_0^T |P\mu_{\varepsilon}(s)|^2_V ds + 2 \int_0^T |m(\mu_{\varepsilon}(s))1|^2_V ds
\]
\[
\leq 2c_p \int_0^T |P\mu_{\varepsilon}(s)|^2_V ds + 2(|\Omega| + |\Gamma|) \int_0^T |m(\mu_{\varepsilon}(s))|^2 ds
\]
\[
\leq 2c_p \int_0^T |\varepsilon_{\varepsilon}'(s)|^2_{V_0} ds + 2T(|\Omega| + |\Gamma|) \tilde{M}_5^2.
\]
Thus, from Lemma 3.3, there exist a positive constant $M_5$ such that the estimate (3.39) holds.
Lemma 3.7. There exists a positive constant $M_6$, independent of $\varepsilon \in (0,1]$, such that
\[
\frac{1}{2} \int_0^T |\beta_\varepsilon (v_\varepsilon (s) + m_0)|^2_{H_r} ds + \frac{1}{4\rho} \int_0^T |\beta_\varepsilon (v_{T,\varepsilon} (s) + m_0)|^2_{H_r} ds \leq M_6 \tag{3.41}
\]

Proof. From the definition of $\mu_\varepsilon$, we can infer that
\[
\mu_\varepsilon = \varepsilon \partial_t v_\varepsilon - \kappa_1 \Delta v_\varepsilon + \beta_\varepsilon (v_\varepsilon + m_0) - m(\beta (v_\varepsilon + m_0 1)) + \bar{\pi} (v_\varepsilon + m_0) - f \quad \text{a.e. in } Q, \tag{3.42}
\]
\[
\mu_{T,\varepsilon} = \varepsilon \partial_t v_{T,\varepsilon} + \kappa_1 \partial_r v_\varepsilon - \kappa_2 \Delta r v_{T,\varepsilon} + \beta_{T,\varepsilon} (v_{T,\varepsilon} + m_0) - m(\beta (v_\varepsilon + m_0 1)) + \pi (v_{T,\varepsilon} + m_0) - f_T \quad \text{a.e. on } \Sigma. \tag{3.43}
\]

Now, it follows from (3.36) that there exists a positive constant $M_6$ such that
\[
\frac{1}{2} \int_0^T |\beta_\varepsilon (v_\varepsilon (s) + m_0 1)|^2 \leq \left( \frac{2}{(|\Omega| + |\Gamma|)^2} \right) \left( |\beta_\varepsilon (v_\varepsilon + m_0)|_{L^1(\Omega)} + |\beta_{T,\varepsilon} (v_{T,\varepsilon} (s) + m_0)|_{L^1(\Gamma)} \right) \leq M_6 \tag{3.44}
\]
for a.a. $s \in (0, T)$. Moreover, we test (3.42) at time $s \in (0, T)$ by $v_\varepsilon (v_\varepsilon + m_0)$ and exploit (3.43). Then, on account of the fact $(\beta_\varepsilon (v_\varepsilon + m_0))_{|r} = \beta_\varepsilon (v_{T,\varepsilon} + m_0)$, by integrating over $\Omega$ we deduce that
\[
\kappa_1 \int_\Omega \beta_\varepsilon (v_\varepsilon (s) + m_0) |\nabla v_\varepsilon (s)|^2 dx + \kappa_2 \int_\Gamma \beta_{T,\varepsilon} (v_{T,\varepsilon} (s) + m_0) |\nabla v_{T,\varepsilon} (s)|^2 d\Gamma = \frac{1}{2} \int_\Omega \beta_\varepsilon (v_\varepsilon (s) + m_0) |\nabla v_\varepsilon (s)|^2 dx + \frac{1}{4\rho} \int_\Gamma \beta_{T,\varepsilon} (v_{T,\varepsilon} (s) + m_0) |\nabla v_{T,\varepsilon} (s)|^2 d\Gamma \leq \left( f (s) + \mu_\varepsilon (s) - \varepsilon \nu_\varepsilon (s) - \pi (v_\varepsilon (s) + m_0), \beta_\varepsilon (v_\varepsilon (s) + m_0) \right)_{H_r} + \left( m(\beta_\varepsilon (v_\varepsilon (s) + m_0 1)), \beta_\varepsilon (v_\varepsilon (s) + m_0) \right)_{H_r} \tag{3.45}
\]
for a.a. $s \in (0, T)$. Now, from (3.1), since the both sign of $\beta_\varepsilon (r)$ and $\beta_{T,\varepsilon} (r)$ is same for all $r \in \mathbb{R}$, we infer that
\[
\int_\Gamma \beta_{T,\varepsilon} (v_{T,\varepsilon} (s) + m_0) \beta_\varepsilon (v_{T,\varepsilon} (s) + m_0) d\Gamma = \int_\Gamma |\beta_{T,\varepsilon} (v_{T,\varepsilon} (s) + m_0) | |\beta_\varepsilon (v_{T,\varepsilon} (s) + m_0) | d\Gamma \geq \frac{1}{2\rho} \int_\Gamma |\beta_\varepsilon (v_\varepsilon (s) + m_0) |^2 d\Gamma \geq \frac{\rho^2}{2\rho} |\Gamma|. \tag{3.46}
\]
Also, it holds
\[
\int_\Omega \beta_\varepsilon (v_\varepsilon (s) + m_0) |\nabla v_\varepsilon (s)|^2 dx \geq 0, \quad \int_\Gamma \beta_{T,\varepsilon} (v_{T,\varepsilon} (s) + m_0) |\nabla v_{T,\varepsilon} (s)|^2 d\Gamma \geq 0. \tag{3.47}
\]
Moreover, by using the Young inequality, the Lipschitz continuity of $\tilde{\pi}, \tilde{\pi}_T$ and (3.44), there exists a positive constant $\hat{M}_6$ such that

$$
\left( f(s) + \mu_\varepsilon(s) - \varepsilon v_\varepsilon'(s) - \tilde{\pi}(v_\varepsilon(s) + m_0), \beta_\varepsilon(v_\varepsilon(s) + m_0) \right)_H \\
+ \left( m(\beta_\varepsilon(v_\varepsilon(s) + m_0)), \beta_\varepsilon(v_\varepsilon(s) + m_0) \right)_H \\
\leq \frac{1}{2} \beta_\varepsilon(v_\varepsilon(s) + m_0)^2_H + 4|f(s)|^2_H + 4|\mu_\varepsilon(s)|^2_H + 4\varepsilon^2|v_\varepsilon'(s)|^2_H + 4|\tilde{\pi}(v_\varepsilon(s) + m_0)|^2_H \\
+ m(\beta_\varepsilon(v_\varepsilon(s) + m_0)) \\
\leq \frac{1}{2} \beta_\varepsilon(v_\varepsilon(s) + m_0)^2_H + \hat{M}_6 \left( |f(s)|^2_H + |\mu_\varepsilon(s)|^2_H + \varepsilon^2|v_\varepsilon'(s)|^2_H + |v_\varepsilon(s)|^2_H + 1 \right)
$$

and

$$
\left( f_T(s) + \mu_{T,\varepsilon}(s) - \varepsilon v_{T,\varepsilon}'(s) - \tilde{\pi}_T(v_{T,\varepsilon}(s) + m_0), \beta_\varepsilon(v_{T,\varepsilon}(s) + m_0) \right)_{H_T} \\
+ \left( m(\beta_\varepsilon(v_{T,\varepsilon}(s) + m_0)), \beta_\varepsilon(v_{T,\varepsilon}(s) + m_0) \right)_{H_T} \\
\leq \frac{1}{4\rho} |\beta_\varepsilon(v_{T,\varepsilon}(s) + m_0)|^2_{H_T} + 2\rho|f_T(s)|^2_{H_T} + 2\rho|\mu_{T,\varepsilon}(s)|^2_{H_T} + 2\rho^2|v_{T,\varepsilon}'(s)|^2_{H_T} \\
+ 2\rho|\tilde{\pi}_T(v_{T,\varepsilon}(s) + m_0)|^2_{H_T} + 2\rho m(\beta_\varepsilon(v_\varepsilon(s) + m_0)) \\
\leq \frac{1}{4\rho} |\beta_\varepsilon(v_{T,\varepsilon}(s) + m_0)|^2_{H_T} + 2\rho|\tilde{\pi}_T(v_{T,\varepsilon}(s) + m_0)|^2_{H_T} + 2\rho m(\beta_\varepsilon(v_\varepsilon(s) + m_0)) \\
+ \rho \hat{M}_6 \left( |f_T(s)|^2_{H_T} + |\mu_{T,\varepsilon}(s)|^2_{H_T} + \varepsilon^2|v_{T,\varepsilon}'(s)|^2_{H_T} + |v_{T,\varepsilon}(s)|^2_{H_T} + 1 \right)
$$

for a.a. $s \in (0, T)$. Thus, from Lemma 3.3, 3.4 and (2.1), by combining from (3.45)–(3.49) and integrating them over $(0, T)$, we can conclude existence of the constant $\hat{M}_6$ satisfying (3.41).}

**Lemma 3.8.** There exists a positive constant $M_7$, independent of $\varepsilon \in (0, 1)$, such that

$$
\kappa_1 \int_0^T |\Delta v_\varepsilon(s)|^2_H ds + \int_0^T |v_\varepsilon(s)|^2_{H^2(\Omega)} ds + \int_0^T |\partial_\nu v_\varepsilon(s)|^2_{H^1} ds \leq M_7.
$$

This lemma is proved exactly the same as in [10] Lemmas 4.4] because the necessary uniform estimates to prove it is obtained by Lemmas 3.3, 3.4, 3.6 and 3.7. Sketching simply, comparing in (3.42) we deduce that $|\Delta v_\varepsilon|_{L^2(0,T;H)}$ is uniformly bounded. Moreover, by using the theory of the elliptic regularity and the trace theory (see e.g., [7, Theorem 3.2, p. 1.79 and Theorem 2.25, p. 1.62], respectively), we can conclude that (3.50) holds.

**Lemma 3.9.** There exists a positive constant $M_8$, independent of $\varepsilon \in (0, 1)$, such that

$$
\int_0^T |\beta_{T,\varepsilon}(v_{T,\varepsilon}(s) + m_0)|^2_{H_T} ds \leq M_8.
$$
Proof We test \((3.43)\) at time \(s \in (0, T)\) by \(\beta_{\Gamma, \varepsilon}(v_{\Gamma, \varepsilon}(s) + m_0)\) and integrating it over \(\Gamma\). Then, by using the Young inequality and the Lipschitz continuity of \(\tilde{\pi}_\Gamma\), there exists a positive constant \(M_8\) such that

\[
\kappa_2 \int_\Gamma \beta_{\Gamma, \varepsilon}'(v_{\Gamma, \varepsilon}(s) + m_0) |\nabla v_{\Gamma, \varepsilon}(s)|^2 d\Gamma + |\beta_{\Gamma, \varepsilon}(v_{\Gamma, \varepsilon}(s) + m_0)|^2_{H^r} \\
= (f_\Gamma(s) + \mu_\Gamma(s) - \varepsilon v_{\Gamma, \varepsilon}'(s) - \partial_\nu v_{\varepsilon}(s) - \tilde{\pi}_\Gamma(v_{\Gamma, \varepsilon}(s) + m_0), \beta_{\Gamma, \varepsilon}(v_{\Gamma, \varepsilon}(s) + m_0))_{H^r} \\
\leq \frac{1}{2} |\beta_{\Gamma, \varepsilon}(v_{\Gamma, \varepsilon}(s) + m_0)|^2_{H^r} + \frac{1}{2} |m(\beta_{\varepsilon}(v_{\varepsilon}(s) + m_0))|^2_{H^r} \\
+ \tilde{M}_8(|f_\Gamma(s)|^2_{H^r} + |\mu_\Gamma(s)|^2_{H^r} + \varepsilon^2 |v_{\Gamma, \varepsilon}'(s)|^2_{H^r} + |\partial_\nu v_{\varepsilon}(s)|^2_{H^r} + |v_{\Gamma, \varepsilon}(s)|^2_{H^r} + 1) \\
\leq \frac{1}{2} |\beta_{\Gamma, \varepsilon}(v_{\Gamma, \varepsilon}(s) + m_0)|^2_{H^r} + |\Gamma| \tilde{M}_6 \\
+ \tilde{M}_8(|f_\Gamma(s)|^2_{H^r} + |\mu_\Gamma(s)|^2_{H^r} + \varepsilon^2 |v_{\Gamma, \varepsilon}'(s)|^2_{H^r} + |\partial_\nu v_{\varepsilon}(s)|^2_{H^r} + |v_{\Gamma, \varepsilon}(s)|^2_{H^r} + 1) \\
(3.52)
\]

for a.a. \(s \in (0, T)\). Noting that it holds

\[
\kappa_2 \int_\Gamma \beta_{\Gamma, \varepsilon}'(v_{\Gamma, \varepsilon}(s) + m_0) |\nabla v_{\Gamma, \varepsilon}(s)|^2 d\Gamma \geq 0.
\]

Thus, on account of Lemma 3.3, 3.4, 3.6 and 3.8, by integrating \((3.52)\) over \((0, T)\), we can find a positive constant \(M_7\) such that the estimate \((3.51)\) holds. \(\square\)

Lemma 3.10. There exists a positive constant \(M_9\), independent of \(\varepsilon \in (0, 1]\), such that

\[
\int_0^T |v_{\varepsilon}(s)|^2_w ds \leq M_9.
\]

This lemma is also proved the same as in \([10]\) Lemmas 4.5]. The key point to prove it is that we can obtain the uniform estimates of \(|\Delta_{\Gamma} v_{\Gamma, \varepsilon}|_{L^2(0, T; H^r)}\) by comparing in \((3.43)\). We omit the proof.

4 Proof of convergence theorem

In this section, we obtain the existence of weak solution of \((P)\) by performing passage to the limit for the approximate problem \((P)_\varepsilon\). The convergence theorem is also nearly the same \([10]\) Sect. 4]. The different point from \([10]\) is that the component of the weak solution of \((P)\) satisfies \((2.4)\) and the periodic property \((2.5)\).

Thanks to the previous estimates Lemmas from 3.3 to 3.10, there exist a subsequence \(\{\varepsilon_k\}_{k \in \mathbb{N}}\) with \(\varepsilon_k \to 0\) as \(k \to \infty\) and some limits functions \(v \in H^1(0, T; V_0^*) \cap L^\infty(0, T; V_0) \cap L^2(0, T; W), \mu \in H^1(0, T; V), \xi \in L^2(0, T; H)\) and \(\xi_\Gamma \in L^2(0, T; H_\Gamma)\) such that

\[
v_{\varepsilon_k} \rightharpoonup v \text{ weakly star in } H^1(0, T; V_0^*) \cap L^\infty(0, T; V_0) \cap L^2(0, T; W),
\]

(4.53)
\[ \varepsilon_k v_{\varepsilon_k} \to 0 \quad \text{strongly in } H^1(0, T; H_0), \]
\[ \mu_{\varepsilon_k} \to \mu \quad \text{weakly in } L^2(0, T; V), \]
\[ \beta_{\varepsilon_k}(u_{\varepsilon_k}) \to \xi \quad \text{weakly in } L^2(0, T; H), \]
\[ \beta_{\Gamma, \varepsilon_k}(u_{\varepsilon_k}) \to \xi_{\Gamma} \quad \text{weakly in } L^2(0, T; H_{\Gamma}) \]

as \( k \to \infty \). Owing to (4.53) and a well-known compactness results (see e.g., [30]), we obtain
\[ v_{\varepsilon_k} \to v \quad \text{strongly in } C([0, T]; H_0) \cap L^2(0, T; V_0) \] (4.56)
as \( k \to \infty \). This yeilds that
\[ u_{\varepsilon_k} \to u := v + m_0 1 \quad \text{strongly in } C([0, T]; H_0) \cap L^2(0, T; V_0) \] (4.57)
as \( k \to \infty \). Therefore, from (4.57) and the Lipschitz continuity of \( \tilde{\pi}, \tilde{\pi}_{\Gamma} \), we deduce that
\[ \tilde{\pi}(u_{\varepsilon_k}) \to \tilde{\pi}(u) \quad \text{strongly in } C([0, T]; H). \]

Hence, by passing to the limit in (3.25) and (3.26), we obtain (2.3) and the following weak formulation:
\[ (\mu(t), z)_H = a(v(t), z) + (\xi(t) - m(\xi(t)) 1 + \tilde{\pi}(v(t)) - f, z)_H \quad \text{for all } z \in V \] (4.58)
for a.a. \( t \in (0, T) \) where \( \xi := (\xi, \xi_{\Gamma}) \), because of the property (2.2) of linear bounded operator \( P \). Now, we can infer \( v + m_0 \in D(\beta) \) and \( v_{\Gamma} + m_0 \in D(\beta_{\Gamma}) \). Hence, from the form (3.5) and (3.6), we deduce that \( \tilde{\pi}(v + m_0) = \pi(v + m_0) \) a.e. in \( Q \) and \( \tilde{\pi}_{\Gamma}(v + m_0) = \pi_{\Gamma}(v_{\Gamma} + m_0) \) a.e. on \( \Sigma \). This implies that we obtain (2.4) replaced by (4.58). Moreover, it follows from (4.56) that
\[ v(0) = v(T) \quad \text{in } H_0. \]

Also, due to (4.54), (4.55), (4.57) and the monotonicity of \( \beta \), from the fact [5] Prop. 2.2, p. 38 we obtain
\[ \xi \in \beta(v + m_0) \quad \text{a.e. in } Q, \quad \xi_{\Gamma} \in \beta_{\Gamma}(v + m_0) \quad \text{a.e. on } \Sigma. \]

Thus, we complete the proof of Theorem 2.1.

**Remark 4.1.** In the previous sections, we impose the restricted assumption (A5) respect to the domains \( D(\beta) \) and \( D(\beta_{\Gamma}) \). This is an essential assumption if we treat the multivalued \( \beta, \beta_{\Gamma} \). However, referrring to [28], we can avoid the assumption (A5) when focusing only on prototype double potential, namely when we choose \( \beta(u) = u^3, \beta_{\Gamma}(u_{\Gamma}) = u_{\Gamma}^3, \pi(u) = -u \) and \( \pi_{\Gamma}(u_{\Gamma}) = -u_{\Gamma} \) in (1.2) and (1.5). The reason why we can avoid it is that we need not to use (3.13) to obtain estimates. The method to do so is used in [28]. Concretely, it is using the Hölder inequality mainly. We can employ the same method even if we consider the problem with dynamic boundary condition. Therefore, we can prove Theorem 2.1 without the assumption (A5) in the case of the prototype double potential.
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