Long heterochromatic paths in heterochromatic triangle free graphs *

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Abstract

In this paper, graphs under consideration are always edge-colored. We consider long heterochromatic paths in heterochromatic triangle free graphs. Two kinds of such graphs are considered, one is complete graphs with Gallai colorings, i.e., heterochromatic triangle free complete graphs; the other is heterochromatic triangle free graphs with k-good colorings, i.e., minimum color degree at least k. For the heterochromatic triangle free graphs $K_n$, we obtain that for every vertex $v \in V(K_n)$, $K_n$ has a heterochromatic $v$-path of length at least $d^c(v)$; whereas for the heterochromatic triangle free graphs $G$ we show that if, for any vertex $v \in V(G)$, $d^c(v) \geq k \geq 6$, then $G$ a heterochromatic path of length at least $\frac{3k}{4}$.

Keywords: Gallai coloring, k-Good coloring, Long heterochromatic path, Heterochromatic triangle free

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1. Introduction

We use Bondy and Murty [3] for terminology and notations not defined here and consider simple graphs only.

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Let $G = (V, E)$ be a graph. By an edge coloring of $G$ we will mean a function $C : E \to \mathbb{N}$, the set of natural numbers. If $G$ is assigned such a coloring, then we say that $G$ is an edge-colored graph. Denote the edge-colored graph by $(G, C)$, and call $C(e)$ the color of the edge $e \in E$. We say that $C(uv) = \emptyset$ if $uv \notin E(G)$ for $u, v \in V(G)$. For a subgraph $H$ of $G$, we denote $C(H) = \{C(e) \mid e \in E(H)\}$ and $c(H) = |C(H)|$. For a vertex $v$ of $G$, we say that color $i$ is presented at vertex $v$ if some edge incident with $v$ has color $i$. The color degree $d^c(v)$ is the number of different colors that are presented at $v$, and the color neighborhood $CN(v)$ is the set of different colors that are presented at $v$. All graphs considered in this paper are edge-colored. For a positive integer $k$, a coloring of a graph is called $k$-good if the minimum color degree of the graph is at least $k$. A path, or a cycle, or any subgraph is called heterochromatic (rainbow, or multicolored) if any two edges of it have different colors. A graph is called heterochromatic triangle free if it does not contain any (induced) heterochromatic triangles. If $u$ and $v$ are two vertices on a path $P$, $uPv$ will denote the segment of $P$ from $u$ to $v$, whereas $vP^{-1}u$ will denote the same segment but from $v$ to $u$. A path is called a $v$-path if it starts from the vertex $v$.

There are a lot of existing literature dealing with the existence of paths and cycles with special properties in edge-colored graphs. The heterochromatic Hamiltonian cycle or path problem was studied by Hahn and Thomassen [13], Rödl and Winkler (see [11]), Frieze and Reed [11], and Albert, Frieze and Reed [1]. In [2], Axenovich, Jiang and Tuza gave the range of the maximum $k$ such that there exists a $k$-good coloring of $E(K_n)$ that contains no properly colored copy of a path with fixed number of edges, no heterochromatic copy of a path with fixed number of edges, no properly colored copy of a cycle with fixed number of edges and no heterochromatic copy of a cycle with fixed number of edges, respectively. In [9], Erdős and Tuza studied the heterochromatic paths in infinite complete graph $K_\omega$. In [10], Erdős and Tuza studied the values of $k$, such that every $k$-good coloring of $K_n$ contains a heterochromatic copy of $F$ where $F$ is a given graph with $m$ edges ($m < n/k$). In [14], Manoussakis, Spyros and Tuza studied $(s, t)$-cycles in 2-edge-colored graphs, where an $(s, t)$-cycle is a cycle of length $s + t$ and $s$ consecutive edges are in one color and the remaining $t$ edges are in the other color. In [15], Manoussakis, Spyros, Tuza and Voigt studied conditions on the minimum number $k$ of colors, sufficient for the existence of given types (such as families of internally pairwise vertex-disjoint paths with common endpoints, Hamiltonian
paths and Hamiltonian cycles, cycles with a given lower bound of their length, spanning trees, stars, and cliques of properly edge-colored subgraphs in a k-edge-colored complete graph. In [8], Chou, Manoussakis, Megalaki, Spyroatos and Tuza showed that for a 2-edge-colored graph G and three specified vertices x, y and z, to decide whether there exists a color-alternating path from x to y passing through z is NP-complete. Many results in these mentioned papers were proved by using probabilistic methods.

In [2], Axenovich, Jiang and Tuza considered the local variation of anti-Ramsey problem. Namely, they studied the maximum integer k, denoted by $g(n, H)$, such that there exists a k-good edge coloring of $K_n$ that does not contain any heterochromatic copy of a given graph H. They showed that for a fixed integer $k \geq 2, k - 1 \leq g(n, P_{k+1}) \leq 2k - 3$, i.e., if $K_n$ is edge-colored by a $(2k - 2)$-good coloring, then there must exist a heterochromatic path $P_{k+1}$, there exists a $(k - 1)$-good coloring of $K_n$ such that no heterochromatic path $P_{k+1}$ exists.

In [4], the authors considered the long heterochromatic paths in general graphs with a k-good coloring and showed that if G is an edge-colored graph with $d^c(v) \geq k$ (color degree condition) for every vertex v of G, then G has a heterochromatic v-path of length at least $\lceil \frac{k+1}{2} \rceil$. In [5, 6], we got some better bound of the length of longest heterochromatic paths in general graphs with a k-good coloring.

**Theorem 1.1** [5] Let G be an edge-colored graph and $3 \leq k \leq 7$ an integer. Suppose that $d^c(v) \geq k$ for every vertex v of G. Then G has a heterochromatic path of length at least $k - 1$.

**Theorem 1.2** [6] Let G be an edge-colored graph. If $d^c(v) \geq k \geq 7$ for any vertex $v \in V(G)$, then G has a heterochromatic path of length at least $\lceil \frac{2k}{3} \rceil + 1$.

In [7], we showed that if $|CN(u) \cup CN(v)| \geq s$ (color neighborhood union condition) for every pair of vertices u and v of G, then G has a heterochromatic path of length at least $\lceil \frac{s+1}{2} \rceil$, and gave examples to show that the lower bound is best possible in some sense.

Some special edge colorings have also been studied, such as Gallai colorings, which is defined to be the edge colorings of complete graphs in which no heterochromatic triangles exist. In [12], Gyárfás and Simonyi studied the existence
of special monochromatic spanning trees in such colorings, they also determined the size of largest monochromatic stars guaranteed to occur.

In this paper, we consider long heterochromatic paths in complete graphs $K_n$ with Gallai colorings, i.e., heterochromatic triangle free complete graphs, and in heterochromatic triangle free graphs with $k$-good colorings. We obtain that if $K_n$ is heterochromatic triangle free, then for every vertex $v \in V(K_n)$, $K_n$ has a heterochromatic $v$-path of length at least $d_c(v)$. For the heterochromatic triangle free general graphs $G$, we show that if $d_c(v) \geq k \geq 6$ for any vertex $v \in V(G)$, then $G$ has a heterochromatic path of length at least $\frac{3k}{4}$.

2. Heterochromatic triangle free complete graphs

In this section, we consider a heterochromatic triangle free complete graph $G$, and try to find a long heterochromatic path from it.

**Theorem 2.1** Suppose $G$ is a heterochromatic triangle free complete graph. Then for every vertex $u$ in $G$, $G$ has a heterochromatic $u$-path of length at least $d_c(u)$.

**Proof.** Let $u$ be any vertex of $G$ and let $d_c(u) = k$. Suppose $v_1, v_2, \ldots, v_k$ are $k$ different neighbors of $u$ such that the $k$ edges $uv_1, uv_2, \ldots, uv_k$ all have distinct colors (see Figure 2.1).

![Figure 2.1](image)

At first, we will construct a path $P$ by the following inductive algorithm.

**Algorithm**

**Step 1.** If the two edges $v_1v_2$ and $uv_1$ have the same color, we let $w_1 = v_1$, $w_2 = v_2$; otherwise, we let $w_1 = v_2$, $w_2 = v_1$. Set $P_1 = w_1w_2$. 

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If **Step** \( i - 1 \) is finished and we have obtained the path \( P_{i-1} = w_1w_2\ldots w_i \). Then

**Step** \( i \). If the two edges \( w_iv_{i+1} \) and \( uw_i \) have the same color, we let \( w_{i+1} = v_{i+1} \).

Otherwise, if the color of the edge \( w_{i-1}v_{i+1} \) is the same as the color of the edge \( uw_{i-1} \), we let \( w_{i+1} = w_i \) and \( w_i = v_{i+1} \).

Otherwise, let \( j_0 \) be the maximum integer \( j \) such that the two edges \( w_jv_{i+1} \) and \( uv_{i+1} \) have the same color and the colors of the two edges \( w_{j-1}v_{i+1} \) and \( uv_{i+1} \) are distinct. If all the \( i \) edges \( w_1v_{i+1}, w_2v_{i+1}, \ldots, w_{i-1}v_{i+1}, uv_{i+1} \) have the same color, we set \( j_0 = 1 \).

If \( j_0 = 1 \), let \( w_{i+1} = w_i \), \( w_i = w_{i-1} \), \( w_{i-1} = w_{i-2} \), \ldots, \( w_3 = w_2 \), \( w_2 = w_1 \), \( w_1 = v_{i+1} \). Otherwise, \( 2 \leq j_0 \leq i - 1 \), let \( w_{i+1} = w_i \), \( w_i = w_{i-1} \), \ldots, \( w_{j_0+1} = w_{j_0} \), \( w_{j_0} = v_{i+1} \). Set \( P_i = w_1w_2\ldots w_{i+1} \).

Continue the process till \( i = k \) and we obtain the path \( P = P_k \).

Then, we will prove the following claim about the path \( P \) obtained from the algorithm.

**Claim.** The path \( P = w_1w_2\ldots w_k \) obtained from the algorithm is heterochromatic, the vertex set of path \( P \) is actually the set \( \{v_1, v_2, \ldots, v_k\} \), and for each \( 1 \leq l \leq k - 1 \), the two edges \( w_lw_{l+1} \) and \( uw_l \) have the same color.

**Proof.** To prove the claim, we will show that for each \( i \) \((1 \leq i \leq k - 1)\), the path \( P_i = w_1w_2\ldots w_{i+1} \) we obtained after Step \( i \) satisfies that the vertex set of \( P_i \) is actually the set \( \{v_1, v_2, \ldots, v_{i+1}\} \), and for each \( 1 \leq l \leq i \), the two edges \( w_lw_{l+1} \) and \( uw_l \) have the same color.

When \( i = 1 \), since there is no heterochromatic triangle in \( K_n \), the edge \( v_1v_2 \) has the same color as the color of the edge \( uv_1 \) or \( uv_2 \). From Step 1, we can easily see that no matter which color the edge \( v_1v_2 \) has, the path \( P = w_1w_2 \) that is obtained after Step 1 contains actually two vertices \( v_1 \) and \( v_2 \), and the two edges \( w_1w_2 \) and \( uw_1 \) have the same color.

Suppose Step \( i - 1 \) has been finished, and the path \( P = w_1w_2\ldots w_i \) we have obtained now contains actually \( i \) vertices \( v_1, v_2, \ldots, v_i \), and for each \( 1 \leq l \leq i - 1 \), the two edges \( w_lw_{l+1} \) and \( uw_l \) have the same color, which is named to be color \( c_l \).
See Figure 2.2. Now we consider the path $P_i = w_1w_2...w_{i+1}$ we obtained after Step $i$.

For convenience, if the path $P_i = w_1w_2...w_{i+1}$ satisfies that the vertex set of $P_i$ is actually the set \{v_1, v_2, ..., v_{i+1}\}, and for each $1 \leq l \leq i$, the two edges $w_lw_{l+1}$ and $uw_l$ have the same color, we say that $P_i$ satisfies **Condition A**.

If the two edges $w_{i}v_{i+1}$ and $uw_i$ have the same color, then $w_{i+1} = v_{i+1}$. So, $P_i$ obviously satisfies Condition A.

Otherwise, since the triangle $uw_iw_{i+1}$ is not heterochromatic, the two edges $w_iw_{i+1}$ and $uw_{i+1}$ have the same color. In this case, if the color of the edge $w_{i−1}v_{i+1}$ is the same as the color of the edge $uw_{i−1}$, then $w_{i+1} = w_i$ and $w_i = v_{i+1}$ by Step $i$. Thus, $P_i$ satisfies Condition A.

Now we consider the case when the two edges $w_iv_{i+1}$ and $uw_{i+1}$ have the same color, and the two edges $w_{i−1}v_{i+1}$ and $uw_{i−1}$ have two distinct colors. Noticing that the triangle $uw_{i−1}v_{i+1}$ is not heterochromatic, we can conclude that the two edges $w_{i−1}v_{i+1}$ and $uw_{i+1}$ have the same color.

If all the $i$ edges $w_1v_{i+1}, w_2v_{i+1}, ..., w_{i−1}v_{i+1}, uv_{i+1}$ have the same color, we have that the two edges $w_1v_{i+1}$ and $uv_{i+1}$ have the same color and for each $l$ ($1 \leq l \leq i − 1$), the two edges $uw_l$ and $w_tw_{i+1}$ have the same color. Thus, the path $v_{i+1}w_1w_2...w_i$ satisfies Condition A. This implies that when we set $w_{i+1} = w_i$, $w_i = w_{i−1}$, $w_{i−1} = w_{i−2}$, ..., $w_3 = w_2$, $w_2 = w_1$, $w_1 = v_{i+1}$, the path $P_i = w_1w_2...w_iw_{i+1}$ satisfies Condition A.

Otherwise, we can find a maximum integer $j$, say $j_0$, such that the two edges $w_jv_{i+1}$ and $uv_{i+1}$ have the same color, and the colors of the two edges $w_{j−1}v_{i+1}$ and $uv_{i+1}$ are distinct. It is clear from the discussion above that in this case, $2 \leq j_0 \leq i − 1$. Then the vertex $w_{j_0−1} \in \{v_1, v_2, ..., v_i\}$. By the assumption
that the colors that the $k$ edges $uv_1, uv_2, \ldots, uv_k$ all have distinct edges, we have that the two edges $uw_{j_0-1}$ and $uv_{i+1}$ have two distinct colors. On the other hand, the two edges $w_{j_0-1}v_{i+1}$ and $uw_{i+1}$ have two distinct colors. So we can conclude that the two edges $w_{j_0-1}v_{i+1}$ and $uw_{j_0-1}$ have the same color, because the triangle $uw_{j_0-1}v_{i+1}$ is not heterochromatic. Then we can conclude that the path $w_1w_2\ldots w_{j_0-1}v_{i+1}w_{j_0}w_{j_0+1}\ldots w_i$ satisfies Condition A. This implies that if we set $w_{i+1} = w_i, w_i = w_{i-1}, \ldots, w_{j_0+1} = w_{j_0}, w_{j_0} = v_{i+1}$, the path $P_1 = w_1w_2\ldots w_iw_{i+1}$ satisfies Condition A. The Claim is thus proved by induction. 

Now we turn back to the proof of the theorem.

Since the path $P$ we obtained from the algorithm satisfies all the conditions in the Claim, the color of the edge $w_ku$ does not appear on the path $P$, and so the path $w_1Pw_ku$ is a heterochromatic $u$-path of length $k$. The proof is thus complete. 

Using the theorem above, we can easily get the following result as a corollary.

**Theorem 2.2** Suppose $G$ is a heterochromatic triangle free complete graph. If the maximum color degree among all the vertices in $G$ is $k$, i.e., $\max_{v \in V(G)} d^c(v) = k$, then there is a heterochromatic path of length at least $k$ in $G$.

**3. Heterochromatic triangle free general graphs**

In this section, we consider long heterochromatic paths in a heterochromatic triangle free general graph. Before we give our main theorem, we would like to give some properties about this special kind of edge colored graphs.

**Lemma 3.1** Suppose $G$ is a heterochromatic triangle free graph, and $P = u_0u_1\ldots u_l$ is a heterochromatic path of length $l \geq 5$. If the two edges $u_0u_i$ and $u_0u_j$ $(2 \leq i < i+1 < j \leq l)$ exist and their colors are distinct and do not appear on the path $P$, then either there exists an integer $s$, $i < s < j$, such that the edge $u_0u_s$ does not exist, or there exist two integers $s$ and $t$ $(i < s < t < j)$, such that the two edges $u_0u_s$ and $u_0u_t$ have the same color.

**Proof.** We will prove it by contradiction.
Suppose that we cannot get the conclusion, which implies that all the edges \( u_0 u_{i+1}, u_0 u_{i+2}, \ldots, u_0 u_{j-1}, u_0 u_j \) exist in \( G \) and they all have distinct colors.

First, we consider the triangle \( u_0 u_{j-1} u_j \). Since it is not heterochromatic, the color of the edge \( u_0 u_j \) does not appear on the path \( P \), and the two edges \( u_0 u_{j-1} \) and \( u_0 u_j \) have two distinct colors, we have that the two edges \( u_0 u_{j-1} \) and \( u_{j-1} u_j \) have the same color.

Now we consider the triangle \( u_0 u_{j-2} u_{j-1} \). As the two edges \( u_0 u_{j-1} \) and \( u_{j-1} u_j \) have the same color, and the path \( P \) is heterochromatic, we have that the two edges \( u_0 u_{j-1} \) and \( u_{j-2} u_{j-1} \) have two distinct colors. On the other hand, by the assumption, the triangle \( u_0 u_{j-2} u_{j-1} \) is not heterochromatic, and the two edges \( u_0 u_{j-2} \) and \( u_0 u_{j-1} \) have two distinct colors. So the two edges \( u_0 u_{j-2} \) and \( u_{j-2} u_{j-1} \) have the same color.

In the same way, we can get, orderly, the edge \( u_0 u_{j-3} \) has the same color as the edge \( u_{j-3} u_{j-2} \) has, \ldots, the edge \( u_0 u_{i-1} \) has the same color as the edge \( u_{i+1} u_{i+2} \) has. Then the triangle \( u_0 u_i u_{i+1} \) is heterochromatic, a contradiction, which completes the proof.

In a similar way, we can get the following property.

**Lemma 3.2** Suppose \( G \) is a heterochromatic triangle free graph, and \( P = u_0 u_1 \ldots u_l \) is a heterochromatic path of length \( l \geq 5 \). If the edge \( u_0 u_i \) exists and the color of it does not appear on the path \( P \), then \( i \geq 3 \), and either there exists an integer \( s, 2 \leq s < i \), such that the edge \( u_0 u_s \) does not exist, or there exist two integers \( s \) and \( t \) (\( 2 \leq s < t \leq i \)), such that the two edges \( u_0 u_s \) and \( u_0 u_t \) have the same color.

Now we can state our main theorem.

**Theorem 3.3** Suppose \( G \) is a heterochromatic triangle free graph. If \( d^e(v) \geq k \geq 6 \) for any vertex \( v \in V(G) \), then \( G \) has a heterochromatic path of length at least \( \frac{3k}{4} \).

**Proof.** Suppose \( P = u_0 u_1 u_2 \ldots u_l \) is one of the longest heterochromatic paths in \( G \). Assume that \( CN(u_0) \) has \( s \) different colors not appearing on \( P \), and \( CN(u_l) \) has \( t \) different colors not appearing on \( P \). Then there exist \( s \) different vertices \( u_{x_1}, u_{x_2}, \ldots, u_{x_s} \) on the path \( P \), where \( 2 \leq x_1 < x_2 < \ldots < x_s \leq l \), such
that the colors of the $s$ edges $u_0 u_{x_1}, u_0 u_{x_2}, \ldots, u_0 u_{x_3}$ are all distinct and do not appear on $P$. There also exist $t$ different vertices $u_{y_1}, u_{y_2}, \ldots, u_{y_t}$ on the path $P$, where $0 \leq y_1 < y_2 < \ldots < y_t \leq l - 2$, such that the colors of the $t$ edges $u_{y_1} u_t, u_{y_2} u_t, \ldots, u_{y_t} u_t$ are all distinct and do not appear on $P$. Since there exists no heterochromatic triangle in $G$, we have $x_1 \geq 3, x_{i+1} > x_i + 1$ for $i = 1, 2, \ldots, s - 1, y_t \leq l - 3, y_{j+1} > y_j + 1$ for $j = 1, 2, \ldots, t - 1$.

Since $k \geq 6$, we can conclude from Theorems 1.1 and 1.2 that the path $P$ is of length $l \geq 5$. By Lemma 3.2, we have that

$$|\{C(u_0 u_2), C(u_0 u_3), \ldots, C(u_0 u_{x_1})\}| \leq x_1 - 2.$$  

We can also get from Lemma 3.1 that for any $1 \leq i \leq s - 1$,

$$|\{C(u_0 u_{x_i+1}), C(u_0 u_{x_{i+2}}), \ldots, C(u_0 u_{x_{i+1}-1}), C(u_0 u_{x_{i+1}})\}| \leq x_{i+1} - x_i - 1.$$  

So

$$|\{C(u_0 u_1), C(u_0 u_2), \ldots, C(u_0 u_{l-1}), C(u_0 u_l)\}|$$

$$\leq |\{C(u_0 u_1)\}| + |\{C(u_0 u_2), C(u_0 u_3), \ldots, C(u_0 u_{x_1})\}|$$

$$+ |\{C(u_0 u_{x_1+1}), C(u_0 u_{x_2+2}), \ldots, C(u_0 u_{x_{l-1}-1}), C(u_0 u_{x_{l-1}})\}|$$

$$+ |\{C(u_0 u_{x_{l+1}}), \ldots, C(u_0 u_{x_l})\}|$$

$$\leq 1 + (x_1 - 2) + (x_2 - x_1 - 1) + (x_3 - x_2 - 1) + \ldots + (x_s - x_{s-1} - 1) + (l - x_s)$$

$$= l - s. \quad (3.1)$$

On the other hand, for any vertex $v$ which is adjacent to $u_0$ but does not belong to the path $P$, the color of the edge $u_0 v$ is not same as the color of the edge $u_{y_j} u_{y_{j+1}}$ for any $1 \leq j \leq t$, for otherwise, $v u_0 P u_{y_j} u_t P^{-1} u_{y_{j+1}}$ is a heterochromatic path of length $l + 1$, a contradiction. So we have $CN(u_0) \{C(u_0 u_i) : 1 \leq i \leq l\} \subseteq C(P) \{C(u_{y_j} u_{y_{j+1}}) : 1 \leq j \leq t\}$, and then

$$|CN(u_0) \{C(u_0 u_i) : 1 \leq i \leq l\}| \leq l - t. \quad (3.2)$$

From Inequalities 3.1 and 3.2, we have

$$k \leq |CN(u_0)|$$

$$\leq |CN(u_0) \{C(u_0 u_i) : 1 \leq i \leq l\}| + |\{C(u_0 u_i) : 1 \leq i \leq l\}|$$

$$\leq (l - t) + (l - s) = 2l - s - t. \quad (3.3)$$

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On the other hand, since the color degrees of the two vertices \( u_0 \) and \( u_l \) are both at least \( k \), and because of the assumption that \( P \) is one of the longest heterochromatic paths, we have that \( l + s \geq k, l + t \geq k \). This implies that \( s \geq k - l \) and \( t \geq k - l \). Now we can get from Inequality 3.3 that

\[
    k \leq 2l - s - t \leq 2l - 2(k - l),
\]

So, \( 4l \geq 3k \), and \( l \geq \frac{3k}{4} \), and the proof is thus complete.

\[\blacksquare\]

4. Concluding remarks

Finally, we give examples to show that our lower bounds given in Theorems 2.1 and 2.2 are best possible.

**Remark 4.1** For any integer \( k \geq 1 \), there is a heterochromatic triangle free complete graph \( G_k \) with the color degree of every vertex \( v \) in \( G_k \) is \( k \), i.e., \( d^c(v) = k \), such that any longest heterochromatic \( v \)-path in \( G_k \) is of length \( k \).

Let \( G_k \) be an edge colored complete graph whose vertices are the ordered \( k \)-tuples of 0’s and 1’s. An edge is in color \( j \) (\( 1 \leq j \leq k \)) if and only if the first \( j - 1 \) coordinates of its two ends are exactly the same and the \( j \)-th coordinates of its two ends are different.

It is not hard to see that there exist no heterochromatic triangles in \( G_k \). Otherwise, suppose \( uvw \) is a heterochromatic triangle, where \( u = (u_1, u_2, \ldots, u_k) \), \( v = (v_1, v_2, \ldots, v_k) \) and \( w = (w_1, w_2, \ldots, w_k) \), the edge \( uv \) is in color \( x \), the edge \( vw \) is in color \( y \), and the edge \( uw \) is in color \( z \). Without loss of generality, we can assume that \( 1 \leq x < y < z \leq k \). Since the edge \( uv \) is in color \( x \), we can conclude that the first \( x - 1 \) coordinates of the two vertices \( u \) and \( v \) are exactly the same, and the \( x \)-th coordinates of \( u \) and \( v \) are different. Similarly, we have that the first \( z - 1 \) coordinates of the two vertices \( u \) and \( w \) are exactly the same, and the \( z \)-th coordinates of \( u \) and \( w \) are different. Then the first \( x - 1 \) coordinates of the two vertices \( v \) and \( w \) are exactly the same, and the \( x \)-th coordinates of \( v \) and \( w \) are different. So the edge \( vw \) is in color \( x \), a contradiction.

It is obvious that for every vertex \( v \) in \( G_k \), its color degree is \( k \). So we can easily conclude that the longest heterochromatic \( v \)-path in \( G_k \) is of length at least
k by Theorem 2.1. On the other hand, any longest heterochromatic path in $G_k$ is not longer than $k$, since there are only $k$ different colors used in this graph. Hence, the conclusion in the remark is true.

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