SPONTANEOUS SYMMETRY BREAKING OF $q$-GAUGE FIELD THEORY

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Abstract. In non-Abelian field theories with $q$-symmetry groups the massive particles have a non-local interpretation with a stringlike spectrum. It is shown that a massless vector similarly acquires a tower of masses by spontaneous symmetry breaking.
1. Introduction.

It is formally possible to generalize any non-Abelian field theory by replacing its Lie group by the corresponding quantum group. Since the Lie algebras are completely rigid with respect to small deformations of the structure constants, it is clear that any new theory obtained in this way must be very different from the corresponding Lie theory and in particular it will contain additional degrees of freedom. While conventional Yang-Mills theories may describe internal properties of point particles such as isotopic spin they do not describe extension, i.e. solitonic or stringy properties of an elementary particle. The quantum gauge theories may be able to do this and at the same time are approximated by a Yang-Mills theory in a correspondence limit. Here we pursue the correspondence further by considering spontaneous symmetry breaking.

2. The $q$-Yang-Mills Theory.

The basic Lagrangian may be expressed as follows:

$$S = \int d^4x \left\{ -\frac{1}{4} Q_\ell F_{\mu\nu} F^{\mu\nu} Q_r + i \psi^t C \gamma^\mu \nabla_\mu \psi + \frac{1}{2} [(\nabla_\mu \varphi)^t \epsilon (\nabla^\mu \varphi) + \varphi^t \epsilon \varphi] \right\} \quad (2.1)$$

where kinetic terms in $Q_\ell$ and $Q_r$ have been omitted. Here

$$Q'_\ell = Q_\ell T^{-1} \quad \psi' = T\psi \quad (2.2)$$

and

$$Q'_r = T Q_r \quad \varphi' = T\varphi \quad (2.3)$$

$\psi$ is also a Dirac field while $\varphi$ is a scalar with respect to the Lorentz group. The covariant derivative transforms as

$$\nabla'_\mu = T \nabla_\mu T^{-1} \quad (2.4)$$

The gauge field $A_\mu$ is defined by

$$A_\mu = \nabla_\mu - \partial_\mu \quad (2.5)$$

and transforms as

$$A'_\mu = T A_\mu T^{-1} + T \partial_\mu T^{-1} \quad (2.6)$$

The field strength is defined by

$$F_{\mu\nu} = [\nabla_\mu, \nabla_\nu] \quad (2.7)$$

and transforms as

$$F'_{\mu\nu} = T F_{\mu\nu} T^{-1} \quad (2.8)$$

The fundamental new assumption is that the gauge group is $SL_q(2)$. Then if $T$ belongs to $SL_q(2)$

$$T^t \epsilon T = T \epsilon T^t = \epsilon \quad (2.9)$$

where $t$ means transpose and

$$\epsilon = \begin{pmatrix} 0 & q^{-1/2} \\ -q^{1/2} & 0 \end{pmatrix} \quad (2.10)$$
In (2.1)
\[ Q = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix} \]  
(2.11)

\( \epsilon \) is analogous to the charge conjugation matrix \( C \) intertwining the Lorentz transformations \( L^t \) and \( L^{-1} \)

\[ L^t C L = C^t = C \]  
(2.12)

For closure one requires

\[ (T_1T_2)^t \epsilon(T_1T_2) = T_2^t T_1^t \epsilon T_1 T_2 = \epsilon \]  
(2.13)

which depends on

\[ (T_1T_2)^t = T_2^t T_1^t \]  
(2.14)

but (2.14) will be satisfied in general only if the matrix elements of \( T_1 \) commute with those of \( T_2 \). To ensure this property one may take \( T_1 \) and \( T_2 \) at spatially (causally) separated points with respect to the light cone. The resulting groupoid is therefore non-local.

We shall discuss the action (2.1) as the basis for a non-Abelian gauge theory. The treatment will differ from the conventional Lie theories since the interacting fields in (2.1) are operator valued even before quantization and it is necessary to introduce not only the usual Fock space but also a second state space. To describe the theory more completely one must assign quantum group representations to the constituent fields. Finally in order to compare with conventional theories we shall add a symmetry breaking term that becomes important only near the minimum in the field energy.

3. The Quantum Groups \( SL_q(2) \), \( SU_q(2) \), and \( SO_q(2) \).

Set

\[ T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \quad T \in SL_q(2) . \]  
(3.1)

Then Eqs. (2.9) require that the matrix elements of \( T \) satisfy the following relations

(a) \( ab = qba \)  
(b) \( ac = qca \)  
(c) \( bc = cb \)  
(d) \( cd = qdc \)  
(e) \( bd = qdb \)  
(f) \( ad - qbc = 1 \)  
(g) \( ad - da = (q - q_1)bc \)  
\[ q_1 = q^{-1} \]  

where \( q \) may be complex.

A quantum subgroup is obtained by setting

\[ d = \bar{a} \]  
(3.3a)

\[ c = -q_1 \bar{b} \]  
(3.3b)

Then

(a) \( ab = qba \)  
(b) \( \bar{a} = q\bar{a} \) \( \bar{b} = q\bar{b} \)  
(c) \( \bar{b} = \bar{b} \)  
(d) \( \bar{b} = q\bar{b} \)  
(e) \( \bar{b} = q\bar{b} \)  
(f) \( a\bar{a} + \bar{b} = 1 \)  
(g) \( \bar{a}a + q_1\bar{b}b = 1 \)  

(3.4)
Note that (3.4) is invariant under involution only if
\[ \bar{q} = q . \] (3.5)

In the hermitian case, \( \bar{q} = q^* \) (= complex conjugate) and by (3.5) \( q \) must be real. The corresponding quantum group is \( SU_q(2) \) or \( T^{-1} = (T^t)^* \).

If, however, \( \bar{a} = a^t \), \( \bar{b} = b^t \), and \( \bar{q} = q^t \), where \( t \) means simple transposition, and if \( q \) is a scalar matrix, then \( \bar{q} = q^t = q \) where \( q \) may be complex. Then (3.4) and (3.5) are satisfied and the corresponding quantum group is \( SO_q(2) \) or \( T^{-1} = T^t \).

4. Representations of the \( SU_q(2) \) and the \( SO_q(2) \) Algebras.

A matrix representation of the algebra should be understood in the following discussion.

(a) Hermitian Involution \( (SU_q(2)) \).

Let
\[ T = \begin{pmatrix} \alpha & \beta \\ -q_1 \bar{\beta} & \bar{\alpha} \end{pmatrix} \] (4.1)

Then permitting the operation of hermitian conjugation with real \( q \) we may abbreviate (3.4) as follows:
\[
\begin{align*}
\alpha \beta &= q \beta \alpha \\
\alpha \bar{\beta} &= q \bar{\beta} \alpha \\
\alpha \bar{\alpha} + \beta \bar{\beta} &= 1 \\
\bar{\alpha} + q_1^2 \beta \bar{\beta} &= 1 \\
\beta \bar{\beta} &= \bar{\beta} \\
q_1 &= q^{-1}
\end{align*}
\] (4.2)

If \( q = 1 \), the equations (4.2) may be satisfied by complex numbers \( \alpha \) and \( \beta \) subject to
\[ |\alpha|^2 + |\beta|^2 = 1 . \] (4.3)

If \( q = -1 \), (4.2) may be satisfied by
\[
\begin{align*}
\bar{\alpha} &= \alpha = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1 \\
\bar{\beta} &= \beta = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_2
\end{align*}
\] (4.4)

In both cases, \( q = \pm 1 \), \( T \in SU(2) \). In addition to these finite representations there is an infinite dimensional representation. To derive this representation let us introduce the infinite dimensional state space associated with the algebra (4.2).

Denote the basis states by \( |n \rangle \), \( n = 0, 1, 2 \ldots \). Since \( \beta \) and \( \bar{\beta} \) commute we may require
\[
\begin{align*}
\alpha |0 \rangle &= 0 \\
\beta |0 \rangle &= b |0 \rangle \\
\bar{\beta} |0 \rangle &= b^* |0 \rangle \\
\bar{\beta} |0 \rangle &= b^* |0 \rangle
\end{align*}
\] (4.5-4.7)
where $b$ and $b^*$ are complex conjugates. Then the remaining states are obtained with the aid of the raising operator $\bar{\alpha}$ by

$$\bar{\alpha}|n\rangle = \lambda_n|n + 1\rangle .$$  \hfill (4.8)

It is also consistent to set

$$\alpha|n\rangle = \mu_n|n - 1\rangle .$$  \hfill (4.9)

Then the quadratic relations determine $\lambda_n$ and $\mu_n$ as follows:

$$|\lambda_n| = (1 - |b|^2q^{2n})^{\frac{1}{2}}$$  \hfill (4.10)

$$\mu_n = \lambda_{n-1}$$  \hfill (4.11)

One also finds

$$\beta|n\rangle = q^n b|n\rangle$$  \hfill (4.12)

$$\bar{\beta}|n\rangle = q^n b^*|n\rangle$$  \hfill (4.13)

$$|b|^2 = q^2 > 0 \rightarrow q \text{ is real} .$$  \hfill (4.14)

Eq. (4.10) implies negative norms unless $q^2 \leq 1$. The $|n\rangle$ are eigenstates of the self-adjoint operator

$$H = \frac{1}{2}(\alpha\bar{\alpha} + \bar{\alpha}\alpha)$$  \hfill (4.15)

and are hence orthogonal.

The preceding operator representation may be displayed as an infinite matrix representation in the usual way:

\[
\begin{align*}
\langle n'|\alpha|n\rangle &= \mu_n \delta(n', n - 1) \\
\langle n'|\bar{\alpha}|n\rangle &= \lambda_n \delta(n', n + 1) \\
\langle n'|\beta|n\rangle &= q^n b \delta(n', n) \\
\langle n'|\bar{\beta}|n\rangle &= q^n b^* \delta(n', n)
\end{align*}
\]  \hfill (4.16-4.19)

where $\lambda_n$ and $\mu_n$ are given by (4.10) and (4.11). By restricting the action to a finite subspace of the state space one may search for finite dimensional representations. The 3-dimensional representation illustrates the general case.

\[
\bar{\alpha} = \begin{pmatrix}
0 & 0 & 0 \\
(1 - q^2)^{1/2} & 0 & 0 \\
0 & (1 - q^4)^{1/2} & 0
\end{pmatrix} \quad \alpha = \begin{pmatrix}
0 & (1 - q^2)^{1/2} & 0 \\
0 & 0 & (1 - q^4)^{1/2} \\
0 & 0 & 0
\end{pmatrix}
\]  \hfill (4.20)

\[
\beta = \begin{pmatrix}
q & 0 & 0 \\
0 & q^2 & 0 \\
0 & 0 & q^3
\end{pmatrix} \quad \bar{\beta} = \beta^* .
\]  \hfill (4.21)
Then $\alpha \bar{\alpha} + \beta \bar{\beta} = 1$ implies
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & q^6
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
\text{(4.22)}

or $q^6 = 1$ and $q = \exp \left( \frac{2\pi i}{6} n \right)$

But $q$ must be real, therefore
\[
q = \pm 1 .
\]
\text{(4.23)}

There are no additional finite-dimensional representations to be obtained in this way.

(b) Transposition Involution ($SO_q(2)$).

The matrix equations (3.3) and (3.4) and (4.2) may be read slightly differently by substituting simple transposition for hermitian conjugation. Then the equations (4.2) retain their same form but now
\[
\bar{\alpha} = \alpha^t \\
\bar{\beta} = \beta^t = \beta \\
\bar{q} = q^t = q
\]
\text{(4.24)}

and $q$ may be complex.

The equations describing the state space are also the same except for changes required by (4.24). For example
\[
\beta \bar{\beta} |0\rangle = b^2 |0\rangle \quad \text{instead of} \quad \beta \bar{\beta} |0\rangle = |b|^2 |0\rangle \\
b^2 = q^2 \quad \text{instead of} \quad |b|^2 = q^2
\]
\text{(4.25)}

The principal equations now read as follows:
\[
\begin{align*}
\alpha |0\rangle &= 0 \\
\beta |0\rangle &= b |0\rangle \\
\bar{\beta} |0\rangle &= b |0\rangle \\
\bar{\alpha} |n\rangle &= \lambda_n |n+1\rangle \\
\alpha |n\rangle &= \mu_n |n-1\rangle
\end{align*}
\]
\text{(4.26)}

where
\[
\begin{align*}
\lambda_n &= (1 - q^{2n+2})^\frac{1}{2} , \\
\mu_n &= \lambda_{n-1} .
\end{align*}
\]
\text{(4.27)}
Eqs. (4.20) and (4.21) for the 3-dimensional representation are also unchanged except that \( \bar{\beta} = \beta^t \) and \( q \) may now be complex.

Since the complex roots of unity are now allowed, there is now an infinite class of finite-dimensional representations. In the \( N \)-dimensional representation

\[
\beta = \begin{pmatrix} q & q^2 & \cdots & q^N \end{pmatrix}, \quad q = \exp \left[ \frac{\pi i}{N} \right].
\]

For example, if \( N = 2 \),

\[
q = \exp \left[ \frac{\pi i}{2} \right] = i
\]

\[
\alpha = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix} \quad \bar{\alpha} = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} i & 0 \\ 0 & i^2 \end{pmatrix}.
\]

There is the following correspondence between the \( q \)-algebra and the conventional notation for the Cartan subalgebra in both cases (a) and (b):

\[
\bar{\alpha} \sim E_+, \quad \alpha \sim E_- \quad \beta, \bar{\beta} \sim H.
\]

5. A Different Reduction of the State Space of \( SL_q(2) \).

By not making the second assumption (3.3b) one obtains a less restrictive state space. Eqs. (3.2) may be rewritten to bring out the following algebraic automorphism:

\[
\alpha \leftrightarrow \delta, \quad q \leftrightarrow q_1.
\]

\[
\alpha \beta = q \beta \alpha \quad \alpha \gamma = q \gamma \alpha \quad \alpha \delta - q \beta \gamma = 1 \quad \beta \gamma = \gamma \beta
\]

\[
\delta \beta = q_1 \beta \delta \quad \delta \gamma = q_1 \gamma \delta \quad \delta \alpha - q_1 \beta \gamma = 1
\]

Since \( \beta \) and \( \gamma \) commute they have a common set of eigenstates. Let the ground state, \( |0\rangle \), be defined by

\[
\alpha |0\rangle = 0
\]

\[
\beta |0\rangle = b_o |0\rangle
\]

\[
\gamma |0\rangle = c_o |0\rangle.
\]

Define the state \( |n\rangle \) by the recursive relations:

\[
\delta |n\rangle = \lambda_n |n + 1\rangle
\]

\[
\alpha |n\rangle = \mu_n |n - 1\rangle.
\]
Then by (5.1)
\[
\begin{align*}
\beta|n\rangle &= q^n b_o|n\rangle \\
\gamma|n\rangle &= q^n c_o|n\rangle
\end{align*}
\] (5.4)

The algebraic symmetry may be used to define the dual states according to
\[
\begin{align*}
\langle 0|\delta &= 0 \\
\langle n|\alpha &= \langle n + 1|\lambda_n \\
\langle n|\delta &= \langle n - 1|\mu_n
\end{align*}
\] (5.5)

We also assume
\[
\begin{align*}
\langle 0|\beta &= \langle 0|b_o \\
\langle 0|\gamma &= \langle 0|c_o
\end{align*}
\] (5.6)

Then by (4.1)
\[
\begin{align*}
\langle n|\beta &= \langle n|q^n b_o \\
\langle n|\gamma &= \langle n|q^n c_o
\end{align*}
\] (5.7)

Set
\[
\begin{align*}
\delta &= \bar{\alpha} \\
\beta &= \bar{\beta} \\
\gamma &= \bar{\gamma}
\end{align*}
\] (5.8)

where the bar means either (a) hermitian or (b) transposition conjugation.

We discuss (a) first. Then the dual equations (5.5)-(5.7) are also hermitian conjugates
of (5.2)-(5.4) if \( q \) and the other numerical factors are all real as we shall assume.

We also have by (5.1)
\[
\begin{align*}
\bar{\alpha}\alpha - q_1 \beta\gamma &= 1 \\
\alpha\bar{\alpha} - q\beta\gamma &= 1
\end{align*}
\] (5.9a, 5.9b)

Under the algebraic automorphism (5.9a) \( \leftrightarrow \) (5.9b) and under hermitian conjugation, these
equations go into themselves.

We may compute
\[
\langle n|\alpha\bar{\alpha}|n\rangle = \langle n|\alpha\lambda_n|n + 1\rangle = \lambda_n\mu_{n+1}\langle n|n\rangle = \lambda_n\mu_{n+1}
\] (5.10)

and also by (5.3), (5.5) and (5.8)
\[
\langle n|\alpha\bar{\alpha}|n\rangle = \langle n|\alpha\cdot\bar{\alpha}|n\rangle = |\lambda_n|^2\langle n + 1|n + 1\rangle = |\lambda_n|^2.
\] (5.11)

where we have set \( \langle n|n\rangle = 1 \) for all \( n \). Then
\[
\lambda_n^* = \mu_{n+1}.
\] (5.12)

By (5.9a)
\[
b_o c_o = q.
\] (5.13)
By (5.9b)\[\langle n|\alpha\bar{\alpha} - q\beta\gamma|n\rangle = 1\]
and by (5.11) and (5.13)\[|\lambda_n|^2 - q q^{2n}b_\sigma c_\sigma = 1 .\]
Then
\[|\lambda_n|^2 = 1 + q^{2n+2} \quad \text{(5.14)}\]
\[|\mu_n|^2 = 1 + q^{2n} \quad \text{(5.15)}\]
by (5.12).

In case (b) the bar in (5.8) and following equations means transposition conjugation. The dual state equations are also transposition conjugate. Then \(q\) and other numerical factors are not constrained to be real. The scalar products, measured by \(\lambda_n^2\) rather than \(|\lambda_n|^2\), may still be chosen positive for a proper choice of \(q\):
\[\lambda_n^2 = 1 + q^{2n+1} . \quad \text{(5.16)}\]

In reductions of \(SL_q(2)\) to \(SU_q(2)\) and \(SO_q(2)\) we have assumed (3.3b). In that case one must also assume \(q^2 \leq 1\) according to (4.10) and (4.27), but we are now assuming that \(\gamma\) and \(\beta\) are separately hermitian (symmetric) rather than hermitian (symmetric) conjugates and \(q\) may therefore be greater than unity by (5.14) and (5.16).

In terms of \(T\) the equation (5.8) means\[T \sigma_1 T^{-1} = \sigma_1 \quad \text{(5.17)}\]
where
\[\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} . \quad \text{(5.18)}\]

6. The Choice of Gauge.

Since the theory is \(T\)-invariant, all fields may lie in the algebra (3.2). Every \(T\)-transformation will introduce new powers of the generators \((\alpha, \bar{\alpha}, \beta, \bar{\beta})\). The theory may be completed by introducing a privileged gauge that breaks the \(T\)-invariance. In the Higgs model the symmetry breaking is accomplished by associating the privileged gauge with the minimum in the field energy. One may imagine a similar mechanism here. In any case we would like the \(q\)-theory to lie close enough to the Lie theory to permit a plausible correspondence limit. In the Lie case \((SU(2))\) the vector and neutral scalar fields may be written
\[W_\mu = W_\mu^+ \tau_- + W_\mu^- \tau_+ + W_\mu^3 \tau_3 \quad \text{(6.1)}\]
\[\Phi = \Phi^3 \tau_3 + \Phi^0 \quad \text{(6.2)}\]
To go over from the Lie algebra to the $q$-algebra we adopt the correspondence (4.31) and write

$$W_{\mu} = W_{\mu}^+ \alpha + W_{\mu}^- \bar{\alpha} + W_{\mu}^\beta \beta + W_{\mu}^{\bar{\beta}} \bar{\beta}$$

$$\Phi = \Phi^\beta \beta + \Phi^{\bar{\beta}} \bar{\beta}$$

(6.3)

(6.4)
since the diagonal elements correspond to $(\beta, \bar{\beta})$.

In the $SU_q(2)$ theory $q$ is real $q^2 \leq 1$, and $(\alpha, \bar{\alpha}, \beta, \bar{\beta})$ are regarded as operators on the infinite dimensional internal state space. Then there are 4 kinds of $W$ particles: $W^+, W^-, W^\beta, W^{\bar{\beta}}$, and there are 2 kinds of $\Phi$ particles: $\Phi^\beta$ and $\Phi^{\bar{\beta}}$; and each of these carries an infinite set of excited states labelled by the eigenvalue of $\beta$.

In the $SO_q(2)$ theory the transition from $SL_q(2)$ to a subgroup is accomplished with the aid of the transposition involution (which is the more economical choice since $SL_q(2)$ is itself defined in terms of a transposition). In this case, $q$ is an $N^{th}$ root of unity. Then $\beta = \bar{\beta}$ and each of the 4 particles $(W^+, W^-, W^\beta, \Phi^\beta)$ carries a finite set of excited states (of height $N$ since the eigenvalues of $\beta$ will run from $q$ to $q^N$).

One next has the option of regarding $q$ (a) as a basic parameter of the second theory or (b) as a new quantum number that will be different for the different particle multiplets. With option (b) one would be able to include the $q = \pm 1$ particles, the familiar $SU(2)$ particles. In this case we may regard $q$ as an operator commuting with the rest of the algebra and equal to the product of a hermitian and a unitary matrix so that its eigenvalues may be either real or roots of unity, depending on the state on which it operates. In this way one would arrive at a more general formalism that would subsume the separate cases already mentioned. The enlarged state space is then divided into disjoint sectors labelled by the eigenvalues of $q$. Since $q$ commutes with the complete algebra, transitions between the separate sectors are not possible within the algebra.

In the $SL_q(2)$ case $\beta$ and $\gamma$ are separately hermitian rather than hermitian conjugates. There are again 4 kinds of vector particles but there are no finite representations. The important distinction is now that $q^2 \leq 1$ no longer holds.

7. Vector Masses for $SU_q(2)$ and $SO_q(2)$.

Following the usual ideas about spontaneous symmetry breaking, we may assume a neutral scalar field, a functional potential leading to a non-invariant lowest state, the existence of Goldstone bosons if the symmetry is global, and the replacement of Goldstone bosons by massive vectors if the symmetry is position-dependent. We shall accordingly add to the original Lagrangian a symmetry breaking term that becomes important near the minimum in the field energy and couples a neutral scalar field to the vector field.

The interaction term is assumed to be invariant under $SU(2)$ only, while the interacting fields continue to lie in the $SU_q(2)$ or the $SO_q(2)$ algebra. Then the vector mass term is buried in the following kinetic energy term:

$$\left(\nabla_\mu \Phi \right)^\dagger \left(\nabla^\mu \Phi \right)$$

(7.1)

where the dagger indicates hermitian conjugation. Here $\nabla_\mu$ is the covariant derivative:

$$\nabla_\mu = \partial_\mu + W_\mu^+ \alpha + W_\mu^- \bar{\alpha} + W_\mu^\beta \beta + W_\mu^{\bar{\beta}} \bar{\beta}$$

(7.2)
and Φ is a neutral scalar field:

$$\Phi = (\bar{\varphi} + \frac{1}{\varphi} + \ldots)\beta$$  \hspace{1cm} (7.3)

where $\bar{\varphi}$ minimizes the energy and determines the vacuum expectation value. $\bar{\varphi}$ will be taken real.

We are interested only in the lowest order contribution to (7.1) and only in terms determining the vector masses. Then the terms contributing to the mass are

$$\nabla_\mu \Phi \rightarrow (W_\mu(+)\alpha\beta + W_\mu(-)\bar{\alpha}\beta + W_\mu^\beta\beta^2) \bar{\varphi}$$  \hspace{1cm} (7.4)

where for simplicity the last term in (7.2) has also been dropped and

$$(\nabla_\mu \Phi)^\dagger \rightarrow (W_\mu(-)\beta^+\alpha^+ + W_\mu(+)\beta^+\bar{\alpha}^+ + (W_\mu^\beta)^*(\beta^2)^+) \bar{\varphi}.$$  \hspace{1cm} (7.5)

To keep a correspondence with the standard case we have set

$$W_\mu(\pm) = \frac{1}{2}(W_1\mu \pm iW_2\mu).$$  \hspace{1cm} (7.6)

The expectation value of (7.1) in the $n^{th}$ state contributes

$$\langle n|W_\mu(-)W_\mu^\mu(+)\beta^+\alpha^+\alpha\beta + W_\mu(+)W_\mu^\mu(-)\beta^+\bar{\alpha}\beta^\dagger + (W_\mu^\beta)^*(W_\mu^\beta)(\beta^2)^+\beta^2|n\rangle \varphi_o^2.$$  \hspace{1cm} (7.7)

There are two cases:

(a) hermitian conjugation, $q$ is real and the group is $SU_q(2)$

(b) transposition conjugation, $q$ is root of unity and the group is $SO_q(2)$.

In case (a) expression (7.7) becomes

$$\langle n|W_\mu(-)W_\mu^\mu(+)\beta\bar{\alpha}\alpha\beta + W_\mu(+)W_\mu^\mu(-)\bar{\beta}\alpha\bar{\alpha}\beta + (W_\mu^\beta)^*(W_\mu^\beta)(\beta^2)^+\beta^2|n\rangle \varphi_o^2.$$  \hspace{1cm} (7.8)

But

$$W_\mu(-)W_\mu^\mu(+) = \frac{1}{4}(W_1^2 + W_2^2).$$  \hspace{1cm} (7.9)

Write

$$W_\mu^\beta = Z_\mu.$$  \hspace{1cm} (7.10)

Then (7.8) becomes

$$\left[ \frac{1}{4}(W_1^2 + W_2^2)\langle n|\bar{\beta}(\bar{\alpha}\alpha + \alpha\bar{\alpha})\beta|n\rangle + Z^2\langle n|(\bar{\beta}\beta)^2|n\rangle \right] \varphi_o^2$$  \hspace{1cm} (7.11)

where

$$\langle n|\bar{\beta}(\bar{\alpha}\alpha + \alpha\bar{\alpha})\beta|n\rangle = \langle n|2\bar{\beta}\beta - (1 + q_1^2)(\bar{\beta}\beta)^2|n\rangle$$  \hspace{1cm} (7.12)
and
\[ \langle n | \beta \beta | n \rangle = q^{2n+2} . \] (7.13)

Therefore (7.1) is reduced to
\[ M_{W}^2 \frac{1}{4} (W_1^2 + W_2^2) + M_{Z}^2 \] (7.14)

where
\[ M_{W}^2 = \frac{1}{4} (2q^{2n+2} - q^{4n+4} - q^{4n+2}) \varphi_o^2 \] (7.15)

and
\[ M_{Z}^2 = q^{4n+4} \varphi_o^2 . \] (7.16)

Here \( M_W \) and \( M_Z \) are the masses of the three vectors in this model.

In case (b) expression (7.7) becomes
\[ \langle n | W^{\mu} (-) \beta^{\dagger} \alpha \beta + W^{\mu} (+) \beta^{\dagger} \alpha^* \bar{\alpha} \beta + (W^{\mu}_{\bar{\alpha}})^* (W^{\beta})^\mu (\beta^{\dagger} \beta)^2 | n \rangle \varphi_o^2 . \] (7.17)

To reduce (7.17) we need \( \langle n | \beta^{\dagger} \alpha^* \bar{\alpha} \beta | n \rangle \), \( \langle n | \beta^{\dagger} \alpha^* \bar{\alpha} \beta | n \rangle \), and \( \langle n | (\beta^{\dagger} \beta)^2 | n \rangle \). Since the eigenvalues of \( \beta \) are the roots of unity, one has
\[ \beta^{\dagger} \beta = 1 . \] (7.18)

Then
\[ \beta^{\dagger} \alpha \beta = \alpha^{\dagger} \beta \beta \alpha = \alpha^{\dagger} \alpha \] \[ \beta^{\dagger} \alpha^* \bar{\alpha} \beta = \alpha^* (\beta^{\dagger} \beta) \bar{\alpha} = \alpha^* \bar{\alpha} . \] (7.19)

By (4.27)
\[ \langle n | \alpha^{\dagger} \alpha | n \rangle = \sum_{m} \langle n | \bar{\alpha} | m \rangle \langle m | \alpha | n \rangle = | \lambda_{n-1} |^2 \] (7.21)

\[ \langle n | \alpha^* \alpha^t | n \rangle = | \lambda_n |^2 \]

and
\[ | \lambda_n |^2 = \left[ (1 - q^{2n+2}) (1 - q_1^{2n+2}) \right]^{\frac{1}{2}} \] (7.22)

\[ = 2 - (q^{2n+2} + q_1^{2n+2}) \] \[ = [2 - (q^{2n+2} + q_1^{2n+2})]^{\frac{1}{2}} . \]

By (4.28)
\[ | \lambda_n |^2 = 2 \left| \sin \frac{\pi}{N} (n + 1) \right| . \] (7.23)

The expression (7.17) now becomes
\[ \left[ \frac{1}{4} (W_1^2 + W_2^2) (| \lambda_n |^2 + | \lambda_{n-1} |^2) + Z^2 \right] \varphi_o^2 \] \[ = \left[ \left( \sin \frac{\pi}{N} (n + 1) \right) \right] + \left( \sin \frac{\pi}{N} n \right) \] (7.24)

so that the masses appearing in the \( N \)-dimensional multiplet are
\[ M_W(n)^2 = \frac{1}{2} \varphi_o^2 \left[ \left( \sin \frac{\pi}{N} (n + 1) \right) \right] + \left( \sin \frac{\pi}{N} n \right) \] \[ M_Z^2 = \varphi_o^2 . \] (7.25)
8. Vector Masses: $SL_q(2)$ with $\bar{T}\sigma_1 = \sigma_1 T$.

We discuss only case (a). Then $\beta$ and $\gamma$ are separately hermitian rather than hermitian conjugates and $q$ is real but otherwise unrestricted. We repeat the calculation of mass for this case.

Dropping the vector index one has

$$\nabla = \partial + C + N \quad (8.1)$$

$$C = W(+\alpha) + W(-\bar{\alpha})$$

$$N = W_\beta \beta + W_\gamma \gamma \quad (8.2)$$

$$\Phi = \rho_\beta \beta + \rho_\gamma \gamma .$$

The mass terms are

$$\langle n | [(C + N)\Phi]^\dagger [(C + N)\Phi] | n \rangle = \langle n | \Phi^\dagger (C^\dagger C + N^\dagger N) \Phi | n \rangle \quad (8.3)$$

where

$$C^\dagger C = \frac{1}{4}(W_1^2 + W_2^2)[2 + (q + q_1)\beta\gamma] . \quad (8.4)$$

Since $C^\dagger C$ and $N^\dagger N$ depend only on $\beta$ and $\gamma$ they commute with $\Phi$ and $\Phi^\dagger +$. Then (8.3) becomes

$$\langle n | (C^\dagger C + N^\dagger N) \Phi^\dagger \Phi | n \rangle \quad (8.5)$$

and the contribution of the “charged” vectors is

$$\langle n | C^\dagger C | n \rangle \langle n | \Phi^\dagger \Phi | n \rangle = \frac{1}{4}(W_1^2 + W_2^2)[2 + (q + q_1)q^{2n+1}]\langle n | \Phi^\dagger \Phi | n \rangle . \quad (8.6)$$

Hence the mass$^2$ of the “charged” vectors is

$$M_n(\pm)^2 = \frac{1}{4}[2 + q^{2n+2} + q^{2n}]\langle n | \Phi^\dagger \Phi | n \rangle . \quad (8.7)$$

The contribution of the neutral fields is

$$\langle n | N^\dagger N | n \rangle \langle n | \Phi^\dagger \Phi | n \rangle = (W_\beta b_o + W_\gamma c_o)^2 q^{2n} \langle n | \Phi^\dagger \Phi | n \rangle$$

$$= Z^2 q^{2n} \langle n | \Phi^\dagger \Phi | n \rangle \quad (8.8)$$

where

$$Z = W_\beta b_o + W_\gamma c_o \quad (8.9)$$

and the corresponding mass$^2$ is

$$M_n^2(Z) = q^{2n} \langle n | \Phi^\dagger \Phi | n \rangle . \quad (8.10)$$
The ratio of the masses of the charged and neutral vectors is then

\[
\left( \frac{M_{\pm}}{M_Z} \right)_n = \frac{1}{2} (2q^{-2n} + q^2 + 1)
\]

\[
\left( \frac{M_{\pm}}{M_Z} \right)_o = \frac{1}{2} (3 + q^2)
\]

(8.11)

By (8.2) \( N \) contains a \( \beta \)-field and also a \( \gamma \)-field; but since the eigenvalues of \( \beta \) and \( \gamma \) are proportional by (5.4), one of these fields is redundant and permits the replacement of \( W_\beta \) and \( W_\gamma \) by \( Z \) in Eq. (8.9).

There is always enough information in the lowest lying multiplet to fix \( q \). If \( q \) should turn out to be real and exceed unity in the case of other higher vector multiplets, then the formalism would suggest a phenomenological theory predicting the manner in which the low-lying pattern would be replicated at higher energies.

9. Summary.

We have discussed a formal extension of non-Abelian field theory with the following properties: the symmetry group is non-local; the dynamical fields do not lie in a Lie algebra. As a consequence, when the fields are resolved in normal modes, it is necessary to expand state space in order to make a particle interpretation. It has been shown earlier\(^1\) for free scalar fields that the particles associated with the expanded state space are characterized by a string-like spectrum.

If \( q \) is real, the \( SL_q(2) \) group is collapsed to the \( SU_q(2) \) group; if \( q \) is a root of unity \( SL_q(2) \) is collapsed to \( SO_q(2) \). If \( q \) is a root of unity, the particle multiplets are finite-dimensional; but if \( q \) is real, only an infinite-dimensional multiplet is allowed. In order to accommodate both kinds of multiplets, it is necessary that total state space be divided into disjoint sectors corresponding to different values of \( q \). Transition between sectors can be mediated by additional interactions lying outside the \( q \)-algebra. In both the \( SU_q(2) \) and \( SO_q(2) \) examples \( q^2 \leq 1 \). To obtain the possibility of \( q > 1 \) and real one may utilize \( SL_q(2) \) with \( T\sigma_1 = \sigma_1 T \).

The principal novelty of this approach lies in the alternative that it provides for introducing into field theory non-locality and solitonic particles in an algebraic rather than a geometric framework such as proposed in stringlike or other higher dimensional theories.

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References.

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