A Fixed-Point Theorem For Mapping Satisfying a General Contractive Condition Of Integral Type Depended an Another Function *

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Abstract

In this paper, we study the existence of fixed points for mappings defined on complete metric space \((X, d)\) satisfying a general contractive inequality of integral type depended on another function. This conditions is analogous of Banach conditions and Branciari Theorem.

Keywords: Fixed point, contraction mapping, contractive mapping, subsequently convergent, subsequently convergent, integral type.

1 Introduction

The first important result on fixed points for contractive-type mapping was the well-known Banach’s Contraction Principle appeared in explicit form in Banach’s thesis in 1922, where it was used to establish the existence of a solution for an integral equation [1]. In the general setting of complete metric space this theorem runs as follows (see [5, Theorem 2.1] or [10, Theorem 1.2.2]).

**Theorem 1.1. (Banach’s Contraction Principle)** Let \((X, d)\) be a complete metric space and \(f : X \rightarrow X\) be a contraction (there exists \(k \in (0, 1)\) such that for each \(x, y \in X\); \(d(fx, fy) \leq kd(x, y)\)). Then \(f\) has a unique fixed point in \(X\), and for each \(x_0 \in X\) the sequence of iterates \(\{f^n x_0\}\) converges to this fixed point.

After this classical result Kannan in [4] analyzed a substantially new type of contractive condition. Since then there have been many theorems dealing with mappings satisfying various types of contractive inequalities. Such conditions involve linear and nonlinear expressions (rational, irrational, and of general type). The interested reader who wants to know more about this matter is recommended to go deep into the survey articles by Rhoades [7,8,9] and Meszaros [6], and into the references therein. Another result on fixed points for contractive-type mapping is generally attributed to Edelstein (1962) who actually obtained slightly more general versions. In the general setting of compact metric spaces this result runs as follows (see [5, Theorem 2.2]).

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Theorem 1.2. Let \((X, d)\) be a compact metric space and \(f : X \longrightarrow X\) be a contractive (for every \(x, y \in X\) such that \(x \neq y\); \(d(fx, fy) < d(x, y)\)). Then \(f\) has a unique fixed point in \(X\), and for any \(x_0 \in X\) the sequence of iterates \(\{f^n x_0\}\) converges to this fixed point.

Also in 2002 in [3] A. Branciari analyzed the existence of fixed point for mapping \(f\) defined on a complete metric space \((X, d)\) satisfying a contractive condition of integral type. (see the following theorem).

Theorem 1.3. Let \((X, d)\) be a complete metric space, \(\alpha \in (0, 1)\) and \(f : X \longrightarrow X\) be a mapping such that for each \(x, y \in X\), \(\int_0^d(fx, fy) \phi(t) dt \leq \alpha \int_0^d(x, y) \phi(t) dt\), where \(\phi : [0, +\infty) \longrightarrow [0, +\infty)\) is a Lebesgue-integrable mapping which is summable (i.e., with finite integral) on each compact subset of \([0, +\infty)\), nonnegative, and such that for each \(\epsilon > 0\), \(\int_0^\epsilon \phi(t) dt > 0\); then \(f\) has a unique fixed point \(a \in X\) such that for each \(x \in X\), \(\lim\limits_{n \to \infty} f^n x = a\).

The aim of this paper is to study the existence of fixed point for mapping \(f\) defined on a compact metric space \((X, d)\) such that is \(T_{f\phi}\)-contraction. In particular, we extend the main theorem due to A. Branciari [3] (Theorem 1.3) and the main theorem in [2] (2008). First we introduce the \(T_{f\phi}\)-contraction function and then extended the A. Branciari Theorem and the main theorem in [2] and Banach-contraction principle, by the same method for proof of the A. Branciari Theorem. At the end of paper some examples and applications concerning this kind of contractions. In [3] A. Branciari gave an example (Example 3.6) such that we can conclude this example by theorem 1.2 (because \(X = \{1/n : n \in \mathbb{N}\} \cup \{0\}\), with metric induced by \(\mathbb{R}\), \(d(x, y) = |x - y|\), is a compact metric space and \(f\) is a contractive mapping).

In the end of this paper we give an example (Example 3.5) such that we can not conclude this example by Theorem 1.1, Theorem 1.2. Branciari Theorem and the main theorem in [2], but we can conclude this example by the main theorem (Theorem 2.5 ) in this paper. In the sequel, \(\mathbb{N}\) will represent the set of natural numbers, \(\mathbb{R}\) the set of real number and \(\mathbb{R}^+\) the set of nonnegative real number.

## 2 Definitions and Main Result

The following theorem (Theorem 2.5) is the main result of this paper. In the first, we define some new definitions.

**Definition 2.1.** Let \((X, d)\) be a metric space and \(f, T : X \longrightarrow X\) be two functions and \(\phi : [0, +\infty) \longrightarrow [0, +\infty)\) be a Lebesgue-integrable mapping. A mapping \(f\) is said to be a \(T_{f\phi}\)-contraction if there exists \(\alpha \in (0, 1)\) such that for all \(x, y \in X\)

\[
\int_0^{d(Tf, Tf)} \phi(t) dt \leq \alpha \int_0^{d(x, y)} \phi(t) dt
\]

**Remark 2.2.** By taking \(Tx = x\) and \(\phi = 1\), \(T_{f\phi}\)-contraction and contraction are equivalent. Also by taking \(Tx = x\) we can define \(f\) is contraction.

**Example 2.3.** Let \(X = [1, +\infty)\) with metric induced by \(\mathbb{R}\): \(d(x, y) = |x - y|\). We consider two mappings \(T, f : X \longrightarrow X\) by \(Tx = \frac{1}{x} + 1\) and \(fx = 2x\). Obviously \(f\) is not contraction but \(f\) is \(T_{f1}\)-contraction.

**Definition 2.4.** [2] Let \((X, d)\) be a metric space. A mapping \(T : X \longrightarrow X\) is said sequentially convergent if we have, for every sequence \(\{y_n\}\), if \(\{Ty_n\}\) is convergence then \(\{y_n\}\) also is convergence. \(T\) is said subsequentially convergent if we have, for every sequence \(\{y_n\}\), if \(\{Ty_n\}\) is convergence then \(\{y_n\}\) has a convergent subsequence.
Theorem 2.5. [Main theorem] Let $(X,d)$ be a complete metric space, $\alpha \in (0,1)$, $T,f : X \rightarrow X$ be mapping such that $T$ is continuous, one-to-one and subsequentially convergent and $f$ is $T\phi$-contraction where $\phi : [0,+,\infty) \rightarrow [0,+,\infty)$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0,+,\infty)$, nonnegative and such that for each $\epsilon > 0$, $\int_0^\epsilon \phi(t)dt > 0$; then $f$ has a unique fixed point $a \in X$. Also if $T$ is sequentially convergent, then for each $x_0 \in X$, the sequence of iterates $\{f^nx_0\}$ converges to this fixed point.

Proof. **STEP 1.** Let $\alpha \in (0,1)$ such that for all $x,y \in X$

$$\int_0^d(Tfx, Tf'y) \phi(t)dt \leq \alpha \int_0^d(Tx,Ty) \phi(t)dt.$$  

(2.1)

So if for $a,b > 0$, $\int_0^a \phi(t)dt \leq \alpha \int_0^b \phi(t)dt$ then $a < b$.

**STEP 2.** We show that $f$ is a continuous mapping. If $\lim_{n \to \infty} x_n = x$ then by $\int_0^d(Tfx_n, Tf'x) \phi(t)dt \leq \alpha \int_0^d(Tx_n,Tx) \phi(t)dt$ and $\lim_{n \to \infty} d(Tx_n,Tx) = 0$, we conclude that:

$$\lim_{n \to \infty} d(Tfx_n, Tf'x) = 0.$$ 

Since $T$ is subsequentially convergent, $\{fx_n\}$ has a subsequence such $\{fx_{n_k}\}_{k=1}^\infty$ converge to a $y \in X$. So $d(Ty, Tf'x) = 0$. Since $T$ is one-to-one, $y = fx$. Hence, $\{fx_n\}$ has a subsequence converge to $fx$.

Therefore for every sequence $\{x_n\}$ converge to $x$, the sequence $\{fx_n\}$ has a subsequence converge to $fx$. This shows that $f$ is continuous at $x$.

**STEP 3.** Since (2.1) is holds, for all $n \in \mathbb{N}$:

$$\int_0^d(Tfx_n, Tf^n'x) \phi(t)dt \leq \alpha^n \int_0^d(Tfx, Tf'x) \phi(t)dt \quad \forall x \in X.$$ 

As a consequence, since $\alpha \in (0,1)$, we further have

$$\int_0^d(Tfx_n, Tf^n'x) \phi(t)dt \to 0^+ \quad \text{as} \quad n \to \infty$$  

(2.2)

Since

$$\int_0^\epsilon \phi(t)dt > 0, \quad \forall \epsilon > 0$$  

(2.3)

is holds we conclude that

$$\lim_{n \to \infty} d(Tfx_{n+1}, Tf^n'x) = 0$$  

(2.4)

**Step 4.** $\{Tf^n'x\}$ is a bounded sequence.

If $\{Tf^n'x\}_{n=1}^\infty$ is not a bounded sequence then, we choose the sequence $\{n_k\}_{k=1}^\infty$ such that $n_1 = 1$ and for each $k \in \mathbb{N}$, $n_{k+1}$ is ”minimal” in the sense that

$$d(Tfx_{n+1}, Tf^n'x) > 1.$$
So,
\[
1 < d(Tf^{n_k+1}x, Tf^{n_k}x) \\
\leq d(Tf^{n_k+1}x, Tf^{n_k+1-1}x) + d(Tf^{n_k+1-1}x, Tf^{n_k}x) \\
\leq d(Tf^{n_k+1}x, Tf^{n_k+1-1}x) + 1. \tag{2.5}
\]

Hence, by (2.4) and (2.5) we conclude that
\[
d(Tf^{n_k+1}x, Tf^{n_k}x) \to 1 \text{ as } k \to \infty \tag{2.6}
\]

Also by step 1,
\[
d(Tf^{n_k+1}x, Tf^{n_k+1}x) \leq d(Tf^{n_k+1-1}x, Tf^{n_k}x).
\]

Therefore,
\[
1 - d(Tf^{n_k+1}x, Tf^{n_k}x) < d(Tf^{n_k+1}x, Tf^{n_k}x) - d(Tf^{n_k+1}x, Tf^{n_k}x) \\
\leq d(Tf^{n_k+1}x, Tf^{n_k+1}x) \\
\leq d(Tf^{n_k+1-1}x, Tf^{n_k}x) \\
\leq 1.
\]

Hence, by (2.4),
\[
d(Tf^{n_k+1}x, Tf^{n_k+1}x) \to 1 \text{ as } k \to \infty. \tag{2.7}
\]

Therefore,
\[
\int_0^1 d(Tf^{n_k+1}x, Tf^{n_k+1}x) \phi(t) dt \leq \alpha \int_0^1 d(Tf^{n_k+1}x, Tf^{n_k+1}x) \phi(t) dt \\
\leq \alpha \int_0^1 \phi(t) dt. \tag{2.8}
\]

By (2.7) and (2.8) we conclude that
\[
\int_0^1 \phi(t) dt = \lim_{k \to \infty} \int_0^1 d(Tf^{n_k+1}x, Tf^{n_k+1}x) \phi(t) dt \\
\leq \alpha \int_0^1 \phi(t) dt.
\]

So \( \int_0^1 \phi(t) dt = 0 \) and this is contradiction.

**STEP 5.** By (2.1) for every \( m, n \in \mathbb{N}(m > n) \),
\[
\int_0^1 d(Tf^m x, Tf^n x) \phi(t) dt \leq \alpha^n \int_0^1 d(Tf^{m-n} x, Tx) \phi(t) dt. \tag{2.9}
\]
By step 4, (2.9) and \( \alpha \in (0,1) \),
\[
\lim_{m,n \to \infty} \int_0^d(T f^m x, T f^n x) = 0 \quad (2.10)
\]
Since (2.3) is hold \( \lim_{m,n \to \infty} d(T f^m x, T f^n x) = 0 \), and this shows that \( \{T f^n x\}_{n=1}^\infty \) is a Cauchy sequence. Hence there exists \( a \in X \) such that
\[
\lim_{n \to \infty} T f^n x = a \quad (2.11)
\]
**STEP 6.** Since \( T \) is a subsequentially convergent, \( \{f^x\} \) has a convergent subsequence. So there exists \( b \in X \) and \( \{n_k\}_{k=1}^\infty \) such that \( \lim_{k \to \infty} f^{n_k} x = b \). Since \( T \) is continuous
\[
\lim_{k \to \infty} T f^{n_k} x = T b,
\]
and by (2.11) we conclude that
\[
T b = a. \quad (2.12)
\]
Since \( f \) is continuous (step 2) and \( \lim_{k \to \infty} f^{n_k} x = b \), \( \lim_{k \to \infty} f^{n_k+1} x = f b \) and so \( \lim_{k \to \infty} T f^{n_k+1} x = T f b \).

Again by (2.11) we have
\[
\lim_{k \to \infty} T f^{n_k+1} x = a
\]
and therefore, \( T f b = a \). So by (2.12), \( T f b = T b \). Since \( T \) is one-to-one, \( f b = b \). Therefore \( f \) has a fixed point.

**STEP 7.** Since \( T \) is one-to-one and \( f \) is \( T \) continuous – contraction, \( f \) has a unique fixed point. \( \square \)

### 3 Examples and Applications

In this section, we give some applications and some examples concerning these contractive mapping of integral type, which clarify the connection between our result and the classical ones.

**Remark 3.1.** Theorem 2.5 is a generalization of the Banach’s contraction principle (Theorem 1.1), letting \( \phi(t) = 1 \) for each \( t \geq 0 \) and \( T x = x \) for each \( x \in X \) in Theorem 2.5, we have
\[
\int_0^d(T f x, T f y) \phi(t) dt = d(f x, f y) \\
\leq \alpha d(x, y) \\
= \alpha \int_0^d(T x, T y) \phi(t) dt
\]

**Remark 3.2.** Theorem 2.5 is a generalization of the A. Branciari theorem (Theorem 1.3), letting \( T x = x \) for each \( x \in X \) in Theorem 2.5, so
\[
\int_0^d(T f x, T f y) \phi(t) dt = \int_0^d(f x, f y) \phi(t) dt \\
\leq \alpha \int_0^d(f x, f y) \phi(t) dt \\
= \alpha \int_0^d(T x, T y) \phi(t) dt.
\]
We can conclude the following theorem (the main Theorem in [2]) by Theorem 2.5.

**Theorem 3.3.** Let \((X, d)\) be a complete metric space and \(T : X \rightarrow X\) be a one-to-one, continuous and subsequentially convergent mapping. Then for every \(T -\) contraction function \(f : X \rightarrow X\), \(f\) has a unique fixed point. Also if \(T\) is sequentially convergent, then for each \(x_0 \in X\), the sequence of iterates \(\{f^n x\}\) converges to this fixed point. \((f : X \rightarrow X\) is \(T -\) contraction if there exist \(\alpha \in (0, 1)\) such that for all \(x, y \in X\)

\[ d(Tfx, Tfy) \leq \alpha d(Tx, Ty). \]

**Proof.** By taking \(\phi(t) = 1\) for each \(t \in [0, +\infty)\) in Theorem 2.5 we can conclude this theorem.

**Example 3.4.** Let \(X = [1, +\infty)\) with metric induced by \(R : d(x, y) = |x - y|\), thus, since \(X\) is a closed subset of \(R\), it is a complete metric space. We define \(T, f : X \rightarrow X\) by \(T x = \ln x + 1\) and \(f x = k\sqrt{x}\) such that \(k \geq 1\) be a fixed element of \(R\). Obviously \(f\) is not contraction, but \(f\) is \(T -\) contraction and \(T\) is one-to-one, continuous and sequentially convergent. So \(f\) has a unique fixed point by Theorem 2.5.

The following example is the main example of this paper. In the following we show that, we can not conclude this example by Theorem 1.1, Theorem 1.2, Theorem 1.3 (Branciari Theorem) and Theorem 3.3.

**Example 3.5.** Let \(X := \{\frac{1}{n} \mid n \in N\} \cup \{0\}\) with metric induced by \(R : d(x, y) := |x - y|\), thus, since \(X\) is a closed subset of \(R\), it is a complete metric space. We consider a mapping \(f : X \rightarrow X\) defined by

\[
fx = \begin{cases} 
\frac{1}{n+3} & : x = \frac{1}{n}, \text{n is odd} \\
0 & : x = 0 \\
\frac{1}{n-1} & : x = \frac{1}{n}, \text{n is even}
\end{cases}
\]

and defined \(\phi : [0, +\infty) \rightarrow [0, +\infty)\) by

\[
\phi(t) = \begin{cases} 
t^{\frac{1}{2}}[1 - \log t] & : t > 0 \\
0 & : t = 0
\end{cases}
\]

we have \(\int_0^\tau \phi(t)dt = \tau^{\frac{1}{2}}\).

By taking \(n = 2\) and \(m = 4\), \(|f(1/m) - f(1/n)| > |1/m - 1/n|\), so \(f\) is not contraction and contractive. Hence, we can not conclude that, \(f\) has a fixed point by Theorem 1.1 and Theorem 1.2.

Now we show that we can not use Branciari Theorem for this example. For \(x = 1/m, y = 1/n\) where \(m\) and \(n\) are even if

\[
\int_0^{|fx-fy|} \phi(t)dt \leq \alpha \int_0^{|x-y|} \phi(t)dt
\]

then

\[
|\frac{1}{m} - \frac{1}{n}|^{\frac{1}{2}}|\log |\frac{1}{m} - \frac{1}{n}|^{\frac{1}{2}}| \leq \alpha |\frac{1}{m} - \frac{1}{n}|^{\frac{1}{2}}\]
\[
\Rightarrow \quad \left| \frac{m-n}{(m-1)(n-1)} \right|^{(m-1)(n-1)} \leq \alpha \left| \frac{m-n}{mn} \right|^{m-n} 
\]

For \( m = 4 \) and \( n = 2 \) we conclude that \( 1 < \alpha \). So we can not use Branciari Theorem.

Now we defined \( T : X \to X \) by

\[
T x = \begin{cases} 
\frac{1}{n-1} &; x = \frac{1}{n}, \ n \text{ is even} \\
0 &; x = 0 \\
\frac{1}{n+1} &; x = \frac{1}{n}, \ n \text{ is odd} 
\end{cases}
\]

Obviously \( T \) is one-to-one and sequentially convergent and continuous.

we have

\[
T f x = \begin{cases} 
\frac{1}{n+2} &; x = \frac{1}{n}, \ n \text{ is odd} \\
0 &; x = 0 \\
\frac{1}{n} &; x = \frac{1}{n}, \ n \text{ is even} 
\end{cases}
\]

Since \( \sup \frac{|T f x - T f y|}{|T x - T y|} = 1 \), \( f \) is not \( T \)-contraction, and so we can not use Theorem 3.3 for this example. Now we show that the condition of Theorem 2.5 are holds. We show that \( f \) is \( T \)-contraction and

\[
\int_0^\alpha \phi(t) dt \leq \frac{1}{2} \int_0^\alpha \phi(t) dt \quad \text{for all } x, y \in X. \quad (2.13)
\]

**Case 1.** Let \( x = \frac{1}{m}, y = \frac{1}{n} \) and \( m \) and \( n \) are even. Then

\[
\int_0^\alpha \phi(t) dt \leq \frac{1}{2} \int_0^\alpha \phi(t) dt \Leftrightarrow \left| \frac{m-n}{mn} \right|^{(m-1)(n-1)} \leq \frac{1}{2} \left| \frac{(m-1)(n-1)}{m-n} \right|^{(m-1)(n-1)} \leq \frac{1}{2} \]

Obviously the last inequality is holds, because

\[
\left| \frac{(m-1)(n-1)}{mn} \right| \leq 1 \text{ and } \left| \frac{(m-1)(n-1)}{m-n} \right| \geq 1
\]

and so

\[
\left| \frac{(m-1)(n-1)}{mn} \right|^{(m-1)(n-1)} \leq 1,
\]

and

\[
\left| \frac{m-n}{mn} \right|^{m-n} \leq \frac{1}{2}
\]

Therefore for this case (2.13) is holds.

**Case 2.** Let \( x = \frac{1}{m}, y = \frac{1}{n} \) and \( m \) and \( n \) are odd.
**Case 3.** Let \( x = \frac{1}{m}, y = \frac{1}{n} \) such that \( m \) is even and \( n \) is odd. By the same argument in case 1 we conclude that (2.13) for case 2 and case 3 is holds.

**Case 4.** Let \( x = 0, y = \frac{1}{n} \) such that \( n \) is even. Then

\[
\int_0^{T|x-Ty|} \phi(t)dt \leq \frac{1}{2} \int_0^{T|x-Ty|} \phi(t)dt
\]

\[
\Leftrightarrow \left( \frac{1}{n} \right)^n \leq \frac{1}{2} \left( \frac{1}{n} \right)^{n-1}
\]

\[
\Leftrightarrow \left( \frac{1}{n} \right)^n (n-1)^{n-1} \leq \frac{1}{2}
\]

\[
\Leftrightarrow \left( \frac{n-1}{n} \right)^{n-1} \frac{1}{n} \leq \frac{1}{2}
\]

The last inequality is holds, because,

\[
\left( \frac{n-1}{n} \right)^{n-1} \leq 1 \quad \text{and} \quad \frac{1}{n} \leq \frac{1}{2}
\]

Therefore (2.13) is true for this case.

**Case 5.** Let \( x = 0, y = \frac{1}{n} \) such that \( n \) is odd. By the same argument in case 4 we conclude that (2.13) is holds for this case.

Hence, (2.13) is holds for all \( x, y \in X \). Therefore the condition of Theorem 2.5 are hold and so \( f \) has a unique fixed point.

**References**

[1] S. Banach, *Sur Les Operations Dans Les Ensembles Abstraits et Leur Application Aux E’quations Inte’grales*, Fund. Math. 3(1922), 133-181(French).

[2] A. Beiranvand, S. Moradi, M. Omid and H. Pazandeh, *Two Fixed-Point Theorem For Special Mapping*, to appear.

[3] A. Branciari, *A fixed point theorem for mapping satisfying a general contractive condition of integral type* Int. J. and M. since, 29:9 (2002), 531-536.

[4] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc. 60(1968),71-76.

[5] Kazimierz Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Combridge University Press, New York, 1990.

[6] J. Meszaros, *A Comparison of Various Definitions of Contractive Type Mappings*, Bull. Calcutta Math. Soc. 84(1992), no. 2, 167-194.

[7] B. E. Rhoades, *A Comparison of Various Definitions of Contractive Mappings*, Trans. Amer. Math. Soc. 226(1977), 257-290.

[8] B. E. Rhoades, *Contractive definitions revisited*, Topological Methods in Nonlinear Functional Analysis (Toronto, Ont.,1982), Contemp. Math., Vol. 21, American Mathematical Society, Rhode Island, 1983, pp. 189-203.
[9] B. E. Rhoades, *Contractive Definitions*, Nonlinear Analysis, World Science Publishing, Singapore, 1987, pp. 513-526.

[10] O. R. Smart, Fixed Point Theorems, Cambridge University Press, London, 1974.

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