Generic conformally flat hypersurfaces in $\mathbb{R}^4$

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Abstract
In this paper, we study generic conformally flat hypersurfaces in the Euclidean 4-space $\mathbb{R}^4$ using the framework of Möbius geometry. First, we classify locally the generic conformally flat hypersurfaces with closed Möbius form under the Möbius transformation group of $\mathbb{R}^4$. Such examples come from cones, cylinders, or rotational hypersurfaces over the surfaces with constant Gaussian curvature in 3-spheres, Euclidean 3-spaces, or hyperbolic 3-spaces, respectively. Second, we investigate the global behavior of the generic conformally flat hypersurface and give some integral formulas about these hypersurfaces.

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1 Introduction
A Riemannian manifold $(M^n, g)$ is conformally flat, if every point has a neighborhood which is conformal to an open set in the Euclidean space $\mathbb{R}^n$. A hypersurface of the Euclidean space $\mathbb{R}^{n+1}$ is said to be conformally flat if so it is with respect to the induced metric. The dimension of the hypersurface seems to play an important role

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in the study of conformally flat hypersurfaces. For $n \geq 4$, the immersed hypersurface $f : M^n \to \mathbb{R}^{n+1}$ is conformally flat if and only if at least $n - 1$ of the principal curvatures coincide at each point by the result of Cartan-Schouten [1,10]. Cartan-Schouten’s result is no longer true for three dimensional hypersurfaces. Lancaster [6] gave some examples of conformally flat hypersurfaces in $\mathbb{R}^4$ having three different principal curvatures. For $n = 2$, the existence of isothermal coordinates means that any Riemannian surface is conformally flat.

A conformally flat hypersurface $f : M^3 \to \mathbb{R}^4$ in $\mathbb{R}^4$ is said to be generic, if the second fundamental form has three distinct eigenvalues everywhere on $M^3$. Standard example of generic conformally flat hypersurface comes from cone, cylinder, or rotational hypersurface over a surface with constant Gaussian curvature in 3-sphere $S^3$, Euclidean 3-space $\mathbb{R}^3$, or hyperbolic 3-space $\mathbb{H}^3$, respectively. The (local) classification of these hypersurfaces is far from complete. However, several partial classification results of generic conformally flat hypersurfaces were given in [2], [3], [4], [7], [11], [12] and [13].

It is known that the conformal transformation group of $\mathbb{R}^n$ is isomorphic to its Möbius transformation group if $n \geq 3$. As conformal invariant objects, generic conformally flat hypersurfaces are investigated in this paper using the framework of Möbius geometry. If an immersed hypersurface in $\mathbb{R}^{n+1}$ has not any umbilical point, then we can define the so-called Möbius metric on the hypersurface, which is invariant under Möbius transformations [14]. Together with another quadratic form (called the Möbius second fundamental form) they form a complete system of invariants for hypersurfaces ($dim \geq 3$) in Möbius geometry [14]. Other important Möbius invariants of the hypersurface are the Möbius form and the Blaschke tensor. First, we find that the standard examples of generic conformally flat hypersurface has closed Möbius form, and vice versa.

**Theorem 1.1.** Let $f : M^3 \to \mathbb{R}^4$ be a generic conformally flat hypersurface. The Möbius form is closed if and only if the hypersurface $f$ is locally Möbius equivalent to one of the following hypersurfaces in $\mathbb{R}^4$:

1. a cylinder over a surface in $\mathbb{R}^3$ with constant Gaussian curvature,
2. a cone over a surface in $S^3$ with constant Gaussian curvature,
3. a rotational hypersurface over a surface in $\mathbb{H}^3$ with constant Gaussian curvature.
Second, we investigate the global behavior of compact generic conformally flat hypersurfaces by the M"obius invariants. Let $(M^n, g)$ be a Riemannian manifold. $K(P)$ denotes the sectional curvature of sectional plane $P(\in \wedge^2 TM^n)$. We call the sectional curvature $K(P)$ have sign if $K(P) \geq 0$ for all $P \in \wedge^2 TM^n$, or $K(P) \leq 0$ for all $P \in \wedge^2 TM^n$.

**Theorem 1.2.** Let $f : M^3 \to \mathbb{R}^4$ be a generic conformally flat hypersurface. If the hypersurface $M^3$ is compact, then the sectional curvature of the M"obius metric can not have sign.

**Theorem 1.3.** Let $f : M^3 \to \mathbb{R}^4$ be a generic conformally flat hypersurface. If the hypersurface $M^3$ is compact, then

$$\int_{M^3} \{ |\tilde{A}|^2 + \frac{1}{3} R^2 - |Ric|^2 - \frac{2}{27} \} dv_g = 0,$$

where $\tilde{A} := A - \frac{1}{3} tr(A) g$ denotes the trace-free Blaschke tensor, $|Ric|$ denotes the norm of the Ricci curvature of $g$, and $R$ denotes the scalar curvature of $g$.

**Corollary 1.1.** Let $f : M^3 \to \mathbb{R}^4$ be a generic conformally flat hypersurface. If the hypersurface $M^3$ is compact, then

$$\int_{M^3} \{ |\tilde{A}|^2 - \frac{2}{27} \} dv_g > 0,$$

where $\tilde{A} := A - \frac{1}{3} tr(A) g$ denotes the trace-free Blaschke tensor.

The paper is organized as follows. In section 2, we review the elementary facts about M"obius geometry of hypersurfaces in $\mathbb{R}^{n+1}$. In section 3, we investigate local behavior of generic conformally flat hypersurfaces in $\mathbb{R}^4$ and prove Theorem 1.1. In section 4, we investigate global behavior of generic conformally flat hypersurfaces in $\mathbb{R}^4$ and prove Theorem 1.2 and Theorem 1.3.

## 2 M"obius invariants of hypersurfaces in $\mathbb{R}^{n+1}$

In [14], Wang has defined M"obius invariants of submanifolds in $\mathbb{S}^{n+1}$ and given a congruent theorem of hypersurfaces in $\mathbb{S}^{n+1}$. In this section, we define M"obius invariants
and give a congruent theorem of hypersurfaces in $\mathbb{R}^{n+1}$ in the same way in [14]. For details we refer to [9], [14].

Let $\mathbb{R}^{n+3}_1$ be the Lorentz space, i.e., $\mathbb{R}^{n+3}$ with inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle x, y \rangle = -x_0 y_0 + x_1 y_1 + \cdots + x_{n+2} y_{n+2},$$

for $x = (x_0, x_1, \cdots, x_{n+2}), y = (y_0, y_1, \cdots, y_{n+2}) \in \mathbb{R}^{n+3}$. Let $f : M^n \to \mathbb{R}^{n+1}$ be a hypersurface without umbilical points and assume that $\{e_i\}$ is an orthonormal basis with respect to the induced metric $I = df \cdot df$ with $\{\theta_i\}$ the dual basis. Let $II = \sum_{ij} h_{ij} \theta_i \theta_j$ and $H = \sum_i \frac{h_{ii}}{n}$ be the second fundamental form and the mean curvature of $f$, respectively. We define the Möbius position vector $Y : M^n \to \mathbb{R}^{n+3}_1$ of $f$ by

$$Y = \rho \left( \frac{1 + |f|^2}{2}, \frac{1 - |f|^2}{2}, f \right), \quad \rho^2 = \frac{n}{n - 1}(|II|^2 - nH^2).$$

**Theorem 2.1.** [14] Two hypersurfaces $f, \tilde{f} : M^n \to \mathbb{R}^{n+1}$ are Möbius equivalent if and only if there exists $T$ in the Lorentz group $O(n + 2, 1)$ such that $\tilde{Y} = YT$.

It follows immediately from Theorem 2.1 that

$$g = \langle dY, dY \rangle = \rho^2 df \cdot df$$

is a Möbius invariant, called the Möbius metric of $f$.

Let $\Delta$ be the Laplacian with respect to $g$. Define

$$N = -\frac{1}{n} \Delta Y - \frac{1}{2n^2} \langle \Delta Y, \Delta Y \rangle Y,$$

which satisfies $\langle Y, Y \rangle = 0 = \langle N, N \rangle, \quad \langle N, Y \rangle = 1$.

Let $\{E_1, \cdots, E_n\}$ be a local orthonormal basis for $(M^n, g)$ with dual basis $\{\omega_1, \cdots, \omega_n\}$. Write $Y_i = E_i(Y)$. Then we have

$$\langle Y_i, Y \rangle = \langle Y_i, N \rangle = 0, \quad \langle Y_i, Y_j \rangle = \delta_{ij}, \quad 1 \leq i, j \leq n.$$

Let $\xi$ be the mean curvature sphere of $f$ written as

$$\xi = \left( \frac{1 + |f|^2}{2} H + f \cdot e_{n+1}, \frac{1 - |f|^2}{2} H - f \cdot e_{n+1}, H f + e_{n+1} \right),$$

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where $e_{n+1}$ is the unit normal vector field of $f$ in $\mathbb{R}^{n+1}$. Thus \( \{Y, N, Y_1, \cdots, Y_n, \xi\} \) forms a moving frame in $\mathbb{R}^{n+3}$ along $M^n$. We will use the following range of indices in this section: $1 \leq i, j, k \leq n$. We can write the structure equations as following:

\[
\begin{align*}
    dY &= \sum_i Y_i \omega_i, \\
    dN &= \sum_{ij} A_{ij} \omega_i Y_j + \sum_i C_i \omega_i \xi, \\
    dY_i &= -\sum_j A_{ij} \omega_j Y_i + \omega_i N + \sum_j B_{ij} \omega_j \xi, \\
    d\xi &= -\sum_i C_i \omega_i Y - \sum_{ij} \omega_i B_{ij} Y_j,
\end{align*}
\]

where $\omega_{ij}$ is the connection form of the Möbius metric $g$ and $\omega_{ij} + \omega_{ji} = 0$. The tensors

\[
\begin{align*}
    A &= \sum_{ij} A_{ij} \omega_i \otimes \omega_j, \\
    B &= \sum_{ij} B_{ij} \omega_i \otimes \omega_j, \\
    C &= \sum_i C_i \omega_i
\end{align*}
\]

are called the Blaschke tensor, the Möbius second fundamental form and the Möbius form of $f$, respectively. The covariant derivative of $C_i, A_{ij}, B_{ij}$ are defined by

\[
\begin{align*}
    \sum_j C_{i,j} \omega_j &= dC_i + \sum_j C_j \omega_{ji}, \\
    \sum_k A_{ij,k} \omega_k &= dA_{ij} + \sum_k A_{ik} \omega_{kj} + \sum_k A_{kj} \omega_{ki}, \\
    \sum_k B_{ij,k} \omega_k &= dB_{ij} + \sum_k B_{ik} \omega_{kj} + \sum_k B_{kj} \omega_{ki}.
\end{align*}
\]

The integrability conditions for the structure equations are given by

\[
\begin{align*}
    (2.1) \quad & A_{ij,k} - A_{ik,j} = B_{ik} C_j - B_{ij} C_k, \\
    (2.2) \quad & C_{i,j} - C_{j,i} = \sum_k (B_{ik} A_{kj} - B_{jk} A_{ki}), \\
    (2.3) \quad & B_{ij,k} - B_{ik,j} = \delta_{ij} C_k - \delta_{ik} C_j, \\
    (2.4) \quad & R_{ijkl} = B_{ik} B_{jl} - B_{il} B_{jk} + \delta_{ik} A_{jl} + \delta_{il} A_{jk} - \delta_{ij} A_{kl} - \delta_{jk} A_{il}, \\
    (2.5) \quad & R_{ij} := \sum_k R_{ikjk} = -\sum_k B_{ik} B_{kj} + (\text{tr} A) \delta_{ij} + (n - 2) A_{ij}, \\
    (2.6) \quad & \sum_i B_{ii} = 0, \quad \sum_{ij} (B_{ij})^2 = \frac{n-1}{n}, \quad \text{tr} A = \sum_i A_{ii} = \frac{1}{2n} (1 + \frac{n}{n-1} R),
\end{align*}
\]
Here $R_{ijkl}$ denotes the curvature tensor of $g$, and $R = \sum_{ij} R_{ijij}$ is the Möbius scalar curvature. We know that all coefficients in the structure equations are determined by $\{g, B\}$ when $n \geq 3$. Thus we have

**Theorem 2.2.** \cite{[4]} Two hypersurfaces $f : M^n \to \mathbb{R}^{n+1}$ and $\tilde{f} : M^n \to \mathbb{R}^{n+1} (n \geq 3)$ are Möbius equivalent if and only if there exists a diffeomorphism $\varphi : M^n \to M^n$ which preserves the Möbius metric and the Möbius second fundamental form.

By equation (2.2), we have

\[(2.7) \quad dC = 0 \iff \sum_k (B_{ik}A_{kj} - B_{jk}A_{ki}) = 0.\]

For the second covariant derivative of $B_{ij}$ defined by

\[dB_{ij,k} + \sum_m B_{mj,k}\omega_{mi} + \sum_m B_{im,k}\omega_{mj} + \sum_m B_{ij,m}\omega_{mk} = \sum_m B_{ij,km}\omega_m,\]

we have the following Ricci identities

\[B_{ij,kl} - B_{ij,lk} = \sum_m B_{mj,R_{mikl}} + \sum_m B_{im,R_{mjkl}}.\]

We call eigenvalues of $(B_{ij})$ as Möbius principal curvatures of $f$. Clearly the number of distinct Möbius principal curvatures is the same as that of its distinct Euclidean principal curvatures.

Let $k_1, \ldots, k_n$ be the principal curvatures of $f$, and $\{\lambda_1, \ldots, \lambda_n\}$ the corresponding Möbius principal curvatures, then the curvature sphere of principal curvature $k_i$ is

\[\xi_i = \lambda_i Y + \xi = \left(\frac{1 + |f|^2}{2}k_i + f \cdot e_{n+1}, \frac{1 - |f|^2}{2}k_i - f \cdot e_{n+1}, k_i f + e_{n+1}\right).\]

Note that $k_i = 0$ if and only if,

\[< \xi_i, (1, -1, 0, \cdots, 0) > = 0.\]

This means that the curvature sphere of principal curvature $k_i$ is a hyperplane in $\mathbb{R}^{n+1}$.

### 3 Generic conformally flat hypersurfaces in $\mathbb{R}^4$

In this section, we give some local properties of the Möbius invariants of generic conformally flat hypersurfaces in $\mathbb{R}^4$. 

Let \((M^n, g)\) be an \(n\)-dimensional Riemannian manifold, and \(\{e_1, \cdots e_n\}\) be a local orthonormal frame field on \((M^n, g)\), and \(\{\omega_1, \cdots, \omega_n\}\) its dual coframe field. The Weyl conformal tensor \(W = \sum_{ijkl} W_{ijkl} \omega_i \otimes \omega_j \otimes \omega_k \otimes \omega_l\) and the Schouten tensor \(S = \sum_{ij} S_{ij} \omega_i \otimes \omega_j\) of \((M^n, g)\) are defined by, respectively,

\[
W_{ijkl} = R_{ijkl} - \frac{1}{n-2} \{R_{ik} \delta_{jl} - R_{jk} \delta_{il} - \delta_{ik} R_{jl} - \delta_{jk} R_{il} - \frac{R}{(n-1)}(\delta_{ik} \delta_{jl} - \delta_{jk} \delta_{il})\},
\]

\[
S_{ij} = R_{ij} - \frac{R}{2(n-1)} \delta_{ij},
\]

where \(R_{ij}\) denotes the Ricci curvature and \(R\) the scalar curvature of \((M^n, g)\).

A result of Weyl states that a Riemannian manifold \((M^n, g)\) of dimension \(n(\geq 4)\) is conformally flat if and only if the Weyl conformal tensor vanishes, and a Riemannian manifold \((M^n, g)\) of dimension 3 is conformally flat if and only if the Schouten tensor is a Codazzi tensor (i.e., \(S_{ij,k} = S_{ik,j}\)). Using the Weyl’s result, we can prove the following lemma (or see [15]),

**Lemma 3.1.** [15] A Riemannian product \((M_1, g_1) \times (M_2, g_2) = (M_1 \times M_2, g_1 + g_2)\) is conformally flat if and only if either

1. \((M_i, g_i)\) is one dimensional curve, and \((M_j, g_j), (i \neq j)\) is a space form, or
2. \((M_1, g_1)\) and \((M_2, g_2)\) are space forms of dimension at least two, with non-zero opposite curvatures.

For hypersurfaces in \(\mathbb{R}^{n+1}\), when \(n \geq 4\), it is well-known from the Cartan-Schouten that a hypersurface \(f : M^n \to \mathbb{R}^{n+1}\) is conformally flat if and only if at least \(n-1\) of the principal curvatures coincide at each point. But Cartan-Schouten’s result is no longer true in dimension 3, since there exist generic conformally flat hypersurfaces.

Let \(f : M^3 \to \mathbb{R}^4\) be a generic hypersurface. We choose an orthonormal basis \(\{E_1, E_2, E_3\}\) with respect to the Möbius metric \(g\) such that

\[
(B_{ij}) = \text{diag}\{b_1, b_2, b_3\}, \quad b_1 < b_2 < b_3.
\]

Let \(\{\omega_1, \omega_2, \omega_3\}\) be the dual of \(\{E_1, E_2, E_3\}\). The conformal fundamental forms of \(f\) are defined by

\[
\Theta_1 = \sqrt{(b_3 - b_1)(b_2 - b_1)} \omega_1, \quad \Theta_2 = \sqrt{(b_3 - b_2)(b_2 - b_1)} \omega_2, \quad \Theta_3 = \sqrt{(b_3 - b_1)(b_3 - b_2)} \omega_3.
\]
Using the equation (2.5) and (2.6), the Schouten tensor of $f$ is

$$S = \sum_{ij} \left( -\sum_l B_{il}B_{lj} + A_{ij} + \frac{1}{6}\delta_{ij} \right) \omega_i \wedge \omega_j.$$ 

Thus

$$S_{ij,k} = -\sum_l \left( B_{il,k}B_{lj} + B_{il}B_{lj,k} \right) + A_{ij,k}.$$ 

If the hypersurface $f$ is conformally flat, then $S_{ij,k} = S_{ik,j}$. Combining the equations (2.1) and (3.8), we obtain the following equation

$$b_k B_{ik,j} - b_j B_{ij,k} = 2(B_{ij}C_k - B_{ik}C_j).$$ 

Using the equation (2.3), we have the following equations,

$$B_{12,3} = B_{13,2} = 0,$$

$$B_{ij,i} = \frac{3b_i}{b_j - b_i} C_j, \quad B_{ii,j} = \frac{b_i - b_k}{b_j - b_i} C_j, \quad i \neq j, j \neq k, i \neq k.$$ 

Using $dB_{ij} + \sum_k B_{kj}\omega_{ki} + \sum_k B_{ik}\omega_{kj} = \sum_k B_{ij,k}\omega_k$ and (3.8), we get

$$\omega_{ij} = \sum_k \frac{B_{ij,k}}{b_i - b_j} \omega_k = \frac{B_{ij,i}}{b_i - b_j} \omega_i + \frac{B_{ij,j}}{b_i - b_j} \omega_j.$$ 

The following lemma is trivial by the equation (3.11) and (3.12), (or see [3],[13]).

**Lemma 3.2.** Let $M^3 \to \mathbb{R}^4$ be a generic hypersurface. The following are equivalent:

(1), the hypersurface is conformally flat;

(2), the schouten tensor is a Codazzi tensor;

(3), the conformal fundamental forms $\Theta_1, \Theta_2, \Theta_3$ are closed.

Next, we give the standard examples of generic conformally flat hypersurfaces in $\mathbb{R}^4$.

**Example 3.1.** Let $u : M^2 \to \mathbb{R}^3$ be an immersed surface. We define the cylinder over $u$ in $\mathbb{R}^4$ as

$$f = (id, u) : \mathbb{R}^1 \times M^2 \to \mathbb{R}^1 \times \mathbb{R}^3 = \mathbb{R}^4, \quad f(t, y) = (t, u(x)),$$

where $id : \mathbb{R}^1 \to \mathbb{R}^1$ is the identity map.
The first fundamental form $I$ and the second fundamental form $II$ of the cylinder $f$ are, respectively,

$$I = I_{\mathbb{R}^1} + I_u, \quad II = II_u,$$

where $I_u, II_u$ are the first and second fundamental forms of $u$, respectively, and $I_{\mathbb{R}^1}$ denotes the standard metric of $\mathbb{R}^1$. Let $\{k_1, k_2\}$ be principal curvatures of surface $u$. Obviously the principal curvatures of hypersurface $f$ are $\{0, k_1, k_2\}$. The Möbius metric $g$ of hypersurface $f$ is

$$(3.13) \quad g = \rho^2 I = \frac{n}{n-1} (||II||^2 - nH^2) I = (4H_u^2 - 3K_u) (I_{\mathbb{R}^1} + I_u),$$

where $H_u, K_u$ are the mean curvature and Gaussian curvature of $u$, respectively. Therefore combining Lemma 3.1 we have the following result.

**Proposition 3.1.** Let $f : M^3 \to \mathbb{R}^4$ be a cylinder over a surface $u : M^2 \to \mathbb{R}^3$, then the cylinder $f$ is conformally flat if and only if the surface $u$ is of constant Gaussian curvature.

**Example 3.2.** Let $u : M^2 \to S^3 \subset \mathbb{R}^4$ be an immersed surface. We define the cone over $u$ in $\mathbb{R}^4$ as

$$f : \mathbb{R}^+ \times M^2 \to \mathbb{R}^4, \quad f(t, x) = tu(x).$$

The first and second fundamental forms of the cone $f$ are, respectively,

$$I = I_{\mathbb{R}^1} + t^2 I_u, \quad II = t II_u,$$

where $I_u, II_u, I_{\mathbb{R}^1}$ are understood as before. Let $\{k_1, k_2\}$ be principal curvatures of surface $u$. The principal curvatures of hypersurface $f$ are $\{0, \frac{1}{t}k_1, \frac{1}{t}k_2\}$. The Möbius metric $g$ of hypersurface $f$ is

$$(3.14) \quad g = \frac{1}{t^2} \left[4H_u^2 - 3(K_u - 1)\right] (I_{\mathbb{R}^1} + t^2 I_u) = \left[4H_u^2 - 3(K_u - 1)\right] (I_{\mathbb{R}^1} + I_u),$$

where $H_u, K_u$ are the mean curvature and Gaussian curvature of $u$, respectively, $I_{\mathbb{R}^1} = \frac{dt^2}{t^2}$. Therefore combining Lemma 3.1 we have the following result.

**Proposition 3.2.** Let $f : M^3 \to \mathbb{R}^4$ be a cone over a surface $u : M^2 \to S^3$, then the cone $f$ is conformally flat if and only if the surface $u$ is of constant Gaussian curvature.
Example 3.3. Let \( \mathbb{R}^3_+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_3 > 0\} \) be the upper half-space endowed with the standard hyperbolic metric
\[
 ds^2 = \frac{1}{x_3^2} [dx_1^2 + dx_2^2 + dx_3^2].
\]

Let \( u = (x_1, x_2, x_3) : M^2 \rightarrow \mathbb{R}^3_+ \) be an immersed surface. We define rotational hypersurface over \( u \) in \( \mathbb{R}^4 \) as
\[
 f : S^1 \times M^2 \rightarrow \mathbb{R}^4, \quad f(\phi, x_1, x_2, x_3) = (x_1, x_2, x_3\phi),
\]
where \( \phi : S^1 \rightarrow S^1 \) is the unit circle.

The first fundamental form and the second fundamental form of \( u \) is, respectively,
\[
 I_u = \frac{1}{x_3^2} (dx_1 \cdot dx_1 + dx_2 \cdot dx_2 + dx_3 \cdot dx_3),
\]
\[
 II_u = \frac{1}{x_3^2} (dx_1 \cdot d\eta_1 + dx_2 \cdot d\eta_2 + dx_3 \cdot d\eta_3) - \frac{\eta_3}{x_3} I_u.
\]

The first and the second fundamental forms of \( f \) is, respectively,
\[
 I = dx \cdot dx = x_3^2 (I_{S^1} + I_u), \quad II = x_3 II_u - \eta_3 I_u - \eta_3 I_{S^1}.
\]

Let \( \{k_1, k_2\} \) be principal curvatures of \( u \). Then principal curvatures of hypersurface \( f \) are \( \{-\eta_3, \frac{k_1}{x_3} - \frac{\eta_3}{x_3}, \frac{k_2}{x_3} - \frac{\eta_3}{x_3}\} \). Thus the Möbius metric of the rotational hypersurface \( f \) is
\[
 (3.15) \quad g = \rho^2 I = \left[ \frac{4H_u^2}{x_3^2} - 3(K_u + 1) \right] (I_{S^1} + I_u),
\]
where \( H_u, K_u \) are the mean curvature and Gaussian curvature of \( u \), respectively. Therefore combining Lemma 3.3 we have the following result,

Proposition 3.3. Let \( f : M^3 \rightarrow \mathbb{R}^4 \) be a rotational hypersurface over a surface \( u : M^2 \rightarrow \mathbb{R}^3_+ \), then the hypersurface \( f \) is conformally flat if and only if the surface \( u \) is of constant Gaussian curvature.

Proposition 3.4. Let \( f : M^3 \rightarrow \mathbb{R}^4 \) be one of generic conformally flat hypersurfaces given by above three examples (3.1) (3.2) (3.3). Then the Möbius form is closed (i.e., \( dC = 0 \)).
Proof. The Möbius metric $g$ in above three examples (3.1) (3.2) (3.3) can be unified in a single formula:

$$g = [4H_u^2 - 3(K_u - \epsilon)](ds^2 + I_u) = \rho^2(ds^2 + I_u),$$

where $I_u$, $K_u$, and $H_u$ are the induced metric, Gaussian curvature, and mean curvature of $u : M^2 \to N^3(\epsilon)$, respectively.

Let $\{e_2, e_3\}$ be a local orthonormal basis on $TM^2$ with respect to $I_u$, consisting of unit principal vectors of $u$ and $e_1 = \frac{\partial}{\partial s}$. Then $\{e_1, e_2, e_3\}$ is an orthonormal basis for $T(I \times M^2)$ with respect to $ds^2 + I_u$. Let $\tilde{R}_{ijkl}$ denote the curvature tensor of the metric $ds^2 + I_u$, and $R_{ijkl}$ the curvature tensor for $g = \rho^2[ds^2 + I_u]$. From Yau’s paper [15], we have

$$R_{ijij} = \rho^2 \tilde{R}_{ijij} + \rho \rho_{ii} + \rho \rho_{jj} - |\nabla \rho|^2, \quad i \neq j$$

$$R_{ijkj} = \rho^2 \tilde{R}_{ijkj} + \rho \rho_{jk}, \quad \text{when } \{i, j, k\} \text{ are distinct},$$

which implies that $(B_{ij}) = diag(b_1, b_2, b_3)$ and $(A_{ij}) = diag(a_1, a_2, a_3)$ under the local orthonormal basis $\{\rho^{-1}e_1, \rho^{-1}e_2, \rho^{-1}e_3\}$ by the equation (2.4). Thus $dC = 0$ by the equation (2.7).

Next, we prove Theorem 1.1. From Proposition 3.4, we prove the other hand of Theorem 1.1 and we assume $dC = 0$. From the equation (2.7), under the orthonormal basis $\{E_1, E_2, E_3\}$ in (3.8) we find

$$(A_{ij}) = diag\{a_1, a_2, a_3\}.$$ 

The equations (3.8) and (3.17) imply that

$$R_{ijkj} = A_{jk} = 0, \quad j \neq k,$$

by the equation (2.4).

From the definition of $B_{ij,kl}$ and (2.3), (3.11) and (3.12), we have

$$B_{23,31} = \frac{3b_2B_{33,1} - 3b_3B_{22,1}}{(b_3 - b_2)^2}C_2 + \frac{3b_3}{b_2 - b_3}[C_{2,1} - \frac{B_{12,1}}{b_1 - b_2}C_1] + B_{31,3}\frac{B_{12,1}}{b_1 - b_2},$$

$$B_{23,13} = (B_{22,1} - B_{33,1})\frac{B_{23,3}}{b_2 - b_3} + (B_{11,2} - B_{33,2})\frac{B_{13,3}}{b_1 - b_3}. $$

(3.18)
Using Ricci identity $B_{23,31} - B_{23,13} = (b_3 - b_2)R_{2313} = 0$ and (3.18), we have

$$b_3 C_{1,2} = \frac{2b_1^2 + b_2 b_3}{(b_2 - b_1)(b_1 - b_3)} C_1 C_2 = -C_1 C_2.$$  

Similarly we have

$$b_1 C_{2,3} = -C_2 C_3, \quad b_2 C_{1,3} = -C_1 C_3.$$  

Therefore

$$b_k C_{i,j} = -C_i C_j, \text{ for } i \neq j, i \neq k, k \neq j.$$  

(3.19)

Now we define $\{C_{i,j,k}\}$ given by

$$dC_{i,j} + \sum_m C_{m,j} \omega_{mi} + \sum_m C_{i,m} \omega_{mj} = \sum_m C_{i,j,m} \omega_m.$$  

Let $\{i, j, k\}$ be distinct. Taking derivative for (3.19) along $E_k$ and invoking (3.11) and (3.12), we get

$$B_{kk,k} C_{i,j} + b_k [C_{i,j,k} - C_{j,k,i} - b_k \frac{B_{ki,k}}{b_k - b_i} - C_{i,k} \frac{B_{kj,k}}{b_k - b_j}] = -C_i [C_{j,k} - C_k \frac{B_{j,k,k}}{b_k - b_j}] - C_j [C_{i,k} - C_k \frac{B_{ik,k}}{b_k - b_i}].$$  

(3.20)

If $b_1 b_2 b_3 = 0$, we can assume that $b_1 = 0$, which implies that $b_2 = -b_3 = \sqrt{\frac{1}{3}}$ by the equation (2.6). Using (3.11), we have $C_1 = C_2 = C_3 = 0$ and $B_{ij,k} = 0$.

Next we assume that $b_1 b_2 b_3 \neq 0$. Because $b_1^2 + b_2^2 + b_3^2 = \frac{2}{3}$, $B_{ij,j} = B_{jj,i} - C_i$, from (3.19) and (3.21) we have

$$b_k C_{i,j,k} = \frac{4 C_i C_j C_k}{3 b_i b_j b_k} = \frac{4 C_1 C_2 C_3}{3 b_1 b_2 b_3}.$$  

(3.21)

Since $C_{i,j,k} = C_{j,i,k} = C_{k,i,j}$ and $b_i \neq b_j, i \neq j$, from (3.21) we get

$$C_{i,j,k} = C_{j,i,k} = C_{k,i,j} = 0, \quad C_1 C_2 C_3 = 0.$$  

We can assume that $C_1 = 0$, then

$$\omega_{12} = \frac{B_{12,1}}{b_1 - b_2} \omega_1, \quad \omega_{13} = \frac{B_{13,1}}{b_1 - b_3} \omega_1, \quad \omega_{23} = \frac{B_{23,2}}{b_2 - b_3} \omega_2 + \frac{B_{23,3}}{b_2 - b_3} \omega_3.$$  

(3.22)
Thus curvature sphere \( \tilde{F} \). From structure equation of the hypersurface and (3.22), we get

\[
-\frac{1}{2} \sum_{kl} R_{12kl} \omega_k \wedge \omega_l = d(B_{12.1}) \wedge \omega_1 + [(B_{13.1}B_{23.2})/(b_1-b_2)] \omega_1 \wedge \omega_2 + [(B_{13.1}B_{12.1}B_{23.3})/(b_1-b_3)(b_2-b_3)] \omega_1 \wedge \omega_3,
\]

which implies that

\[
\begin{align*}
E_3(B_{12.1}) - B_{13.1} & = 0, \\
E_2(B_{12.1}) - [(B_{13.1}B_{12.1})/(b_1-b_2)] & = R_{1212} = b_1 b_2 + a_1 + a_2.
\end{align*}
\]

Similarly we have

\[
\begin{align*}
E_3(B_{13.1}) - B_{12.1} & = 0, \\
E_2(B_{13.1}) - [(B_{13.1}B_{23.3})/(b_1-b_3)] & = R_{1313} = b_1 b_3 + a_1 + a_3.
\end{align*}
\]

Under the local basis above, \( \{Y, N, Y_1, Y_2, Y_3, \xi\} \) forms a moving frame in \( \mathbb{R}^6 \) along \( M^3 \). We define

\[
F = b_1 Y + \xi, \quad X_2 = \frac{B_{12.1}}{b_1-b_2} Y + Y_2, \quad X_3 = \frac{B_{13.1}}{b_1-b_3} Y + Y_3,
\]

\[
T = a_1 Y + N - \frac{B_{12.1}}{b_1-b_2} Y_2 - \frac{B_{13.1}}{b_1-b_3} Y_3 - b_1 \xi,
\]

\[
Q = 2a_1 + b_1^2 + (\frac{B_{12.1}}{b_1-b_2})^2 + (\frac{B_{13.1}}{b_1-b_3})^2.
\]

Clearly \( F \) is the curvature sphere of principal curvature \( k_1 \). And

\[
\begin{align*}
<F,X_2> &= <F,Y_2> = 0, \\
<T,X_2> &= <T,Y_2> = 0, \\
<F,F> &= <X_2,X_2> = 1, \quad <T,T> = Q.
\end{align*}
\]

From structure equation of the hypersurface and (3.22), we get

\[
E_1(F) = 0, \quad E_2(F) = (b_1-b_2)X_2, \quad E_3(F) = (b_1-b_3)X_3.
\]

Thus curvature sphere \( F \) induces a surface \( \tilde{M} = M^3/L \) in the de-Sitter space \( \mathbb{S}^5 \)

\[
F : \tilde{M} = M^3/L \rightarrow \mathbb{S}^5.
\]
where fibers $L$ are integral submanifolds of distribution $D = \text{span}\{E_1\}$. We define $V = \text{span}\{T, Y_1\}$. Clearly we have

$$F \perp V.$$  

Using (3.22), (3.23) and (3.24), we can get that

$$E_1(Y_1) = -T, \quad E_2(Y_1) = 0, \quad E_3(Y_1) = 0,$$

$$E_1(T) = QY_1, \quad E_2(T) = \frac{B_{12,1}}{b_1 - b_2} T, \quad E_3(T) = \frac{B_{13,1}}{b_1 - b_3} T.$$  

This implies that subspace $V$ is parallel along $M^3$. Similarly we have

$$E_1(Q) = 0, \quad E_2(Q) = 2 \frac{B_{12,1}}{b_1 - b_2} Q, \quad E_3(Q) = 2 \frac{B_{13,1}}{b_1 - b_3} Q.$$  

Regarding (3.28) as a linear first-order differential equation for $Q$, we see that $Q \equiv 0$ or $Q \neq 0$ on a connected manifold $M^n$. Therefore there are three possibilities for the induced metric on the fixed subspace $V \subset \mathbb{R}^6$.

**Case 1**, $Q = 0$, then $<T, T> = Q = 0$, therefore $V$ is endowed with a degenerate inner product.

By (3.27), $T$ determines a fixed light-like direction in $\mathbb{R}^6_1$. Up to a Möbius transformation, we may take to be

$$T = \lambda (1, -1, 0, 0, 0, 0), \quad \lambda \in C^\infty(M^3).$$

Since $V$ is a fixed degenerate subspace in $\mathbb{R}^6$, we can find a space-like vector $v$ such that $V = \text{Span}\{e = (1, -1, 0, 0, 0, 0), v\}$ and $<e, F> = <v, F> = 0$. We interpret the geometry of the hypersurface $f : M^3 \to \mathbb{R}^4$ as below:

1) $v$ determines a fixed hyperplane $\Sigma$ in $\mathbb{R}^4$ because of $<T, v> = 0$.

2) $F$ is a two parameter family of hyperplanes orthogonal to the fixed hyperplane $\Sigma$ in $\mathbb{R}^4$.

Therefore $f(M^3)$, as the envelope of this family of hyperplanes $F$, is clearly a cylinder over a surface $\tilde{M} \subset \mathbb{R}^3$.

**Case 2**, $Q < 0$, then $<T, T>$ is negative, and $V$ is a Lorentz subspace in $\mathbb{R}^6_1$. Up to a Möbius transformation, we can assume that

$$V = \text{Span}\{T, Y_1\} = \text{Span}\{p_0 = (1, 1, 0, 0, 0, 0), p_1 = (1, -1, 0, 0, 0, 0)\}.$$
Using the stereographic projection, $p_0, p_1$ correspond to the origin $O$ and the point at infinity $\infty$ of $\mathbb{R}^4$, respectively. Since $F \perp V$, $F$ is a two parameter family of hyperplanes (passing $O$ and $\infty$). Therefore $f(M^3)$, as the envelope of this family of hyperplanes $F$, is clearly a cone (with vertex $O$) over a surface $\tilde{M} \subset S^3$.

**Case 3**, $Q > 0$, then $< T, T >$ is positive, and $V$ is a space-like subspace in $\mathbb{R}^6_1$. Up to a Möbius transformation, we can assume that

$$V = \text{Span}\{T, Y_1\} = \text{Span}\{(0, 0, 1, 0, 0), (0, 0, 1, 0, 0)\} = \mathbb{R}^2.$$ 

Thus $V$ is a fixed two dimensional plane $\mathbb{R}^2 \subset \mathbb{R}^4$, and $F$ is a two parameter family of hyper-sphere orthogonal to this fixed plane $\mathbb{R}^2$ with centers locating on it. Thus $F$ envelopes a rotational hypersurface $f(M^3)$ (over a surface $\tilde{M} \subset \mathbb{R}^3_1$).

From Case 1, Case 2, Case 3, we prove that if $dC = 0$, then the hypersurface is Möbius equivalent to one of the standard examples of generic conformally flat hypersurface. thus we complete the proof of Theorem 1.1.

## 4 Global behavior of the generic conformally flat hypersurface

Let $f : M^3 \to \mathbb{R}^4$ be a generic conformally flat hypersurface. We say that the pair $(U, \omega)$ is admissible if

1. $U$ is an open subset of $M^3$,
2. $\omega = (\omega_1, \omega_2, \omega_3)$ is a orthonormal co-frame field on $U$ with respect to the Möbius metric $g$,
3. $\omega_1 \wedge \omega_2 \wedge \omega_3 = dv_g$,
4. $B = \sum_i b_i \omega_i \otimes \omega_i$.

Denote by $F = (E_1, E_2, E_3)$ the dual frame field of $\omega$. Then it is easily-seen that, $(U, \omega)$ is admissible if and only if $E_i$ is an unit principal vector associated to $b_i$ for each $1 \leq i \leq 3$, and $\{E_1, E_2, E_3\}$ is an oriented basis associated to the orientation of $M^3$. Denote by $\{\omega_{ij}\}$ the connection form with respect to $(U, \omega_1, \omega_2, \omega_3)$. Thus under the admissible frame field $\{E_1, E_2, E_3\}$,

$$(B_{ij}) = \text{diag}\{b_1, b_2, b_3\}.$$
Therefore the 2-forms $\Phi$ and $\Psi$ are well-defined on $M^3$.

Combining $d\omega_{ij} - \sum_k \omega_{ik} \wedge \omega_{kj} = \sum_{kl} R_{ijkl} \omega_k \wedge \omega_l$, we get

$$d\Phi = [\frac{9 b_1 b_2 C_3^2}{(b_1 - b_3)^2(b_2 - b_3)^2} - R_{1212} - R_{1313} - R_{2323}] \omega_1 \wedge \omega_2 \wedge \omega_3$$

(4.29)

Similarly we can compute $d(\omega_{23} \wedge \omega_1)$ and $d(\omega_{31} \wedge \omega_2)$. Therefore we have

$$d\Psi = -[(b_1 - b_2)^2 R_{1212} + (b_1 - b_3)^2 R_{1313} + (b_2 - b_3)^2 R_{2323}] dv_g$$

(4.30)

$$+ [\frac{18 b_1 b_2 C_3^2}{(b_1 - b_3)^2(b_2 - b_3)^2} + \frac{18 b_2 b_3 C_1^2}{(b_1 - b_2)^2(b_1 - b_3)^2} + \frac{18 b_1 b_3 C_2^2}{(b_1 - b_2)^2(b_2 - b_3)^2}] dv_g,$$

where $dv_g = \omega_1 \wedge \omega_2 \wedge \omega_3$. Combining the equation (4.29) and the equation (4.30), we have

$$2d\Phi - d\Psi = \{(b_1 - b_2)^2 - 2\} R_{1212} + [(b_2 - b_3)^2 - 2] R_{2323} + [(b_1 - b_3)^2 - 2] R_{1313} \} dv_g.$$

(4.31)

If $M^3$ is compact, the equation (4.31) implies that

$$\int_{M^3} \{(b_1 - b_2)^2 - 2\} R_{1212} + [(b_2 - b_3)^2 - 2] R_{2323} + [(b_1 - b_3)^2 - 2] R_{1313} \} dv_g = 0.$$

(4.32)

From $b_1 + b_2 + b_3 = 0$ and $b_1^2 + b_2^2 + b_3^2 = \frac{2}{3}$, we can derive that

$$(b_1 - b_2)^2 + (b_2 - b_3)^2 + (b_1 - b_3)^2 = 2.$$
which implies that

\[(4.33) \quad (b_1 - b_2)^2 - 2 < 0, \quad (b_2 - b_3)^2 - 2 < 0, \quad (b_1 - b_3)^2 - 2 < 0.\]

Now we assume that the sectional curvature of $M^3$ with respect to the M"obius metric have sign, for example, the sectional curvature is nonnegative. The equation (4.32) and (4.33) imply that the sectional curvature vanishes, i.e.,

\[R_{1212} = R_{2323} = R_{1313} = 0.\]

In [8], authors classify the hypersurfaces $f : M^3 \to \mathbb{R}^4$ with constant M"obius sectional curvature, which are non-compact. This is a contradiction, thus we finish the proof of Theorem 1.2.

Using the equation (2.4) and the equation (2.6), we have

\[(4.34) \quad [(b_1 - b_2)^2 - 2] R_{1212} + [(b_2 - b_3)^2 - 2] R_{2323} + [(b_1 - b_3)^2 - 2] R_{1313} = 2 - \frac{10}{3} tr(A) + 3[b_1^2 a_1 + b_2^2 a_2 + b_3^2 a_3].\]

On the other hand, the equation (2.5) implies that

\[(4.35) \quad |\text{Ric}|^2 = \frac{2}{9} + 5 tr(A)^2 + |A|^2 - \frac{4}{3} tr(A) - 2[b_1^2 a_1 + b_2^2 a_2 + b_3^2 a_3],\]

where $|\text{Ric}|$ denote the norm of the Ricci curvature. Combining the equation (4.34) and the equation (4.35), we can derive that

\[(4.36) \quad [(b_1 - b_2)^2 - 2] R_{1212} + [(b_2 - b_3)^2 - 2] R_{2323} + [(b_1 - b_3)^2 - 2] R_{1313} = \frac{5}{9} - \frac{16}{3} tr(A) + \frac{15}{2} tr(A)^2 + \frac{3}{2} |A|^2 - \frac{3}{2} |\text{Ric}|^2.\]

Let $\tilde{A} := A - \frac{1}{3} tr(A) g$ denote the trace-free Blaschke tensor, then $|\tilde{A}|^2 = |A|^2 - \frac{1}{3} tr(A)^2$. Thus from the equation (4.36), we have

\[(4.37) \quad [(b_1 - b_2)^2 - 2] R_{1212} + [(b_2 - b_3)^2 - 2] R_{2323} + [(b_1 - b_3)^2 - 2] R_{1313} = \frac{3}{2} |\tilde{A}|^2 - \frac{3}{2} |\text{Ric}|^2 - \frac{1}{3} R^2 - \frac{1}{9}.\]

Now if the hypersurface $M^3$ is compact, then

\[(4.38) \quad \int_{M^3} (|\tilde{A}|^2 + \frac{1}{3} R^2 - |\text{Ric}|^2 - \frac{2}{27}) dv_g = 0.\]
Therefore we finish the proof of Theorem 1.3.

Since $R = \text{tr}(\text{Ric})$, we have $\frac{1}{3}R^2 - |\text{Ric}|^2 \leq 0$ on $M^3$. If $\frac{1}{3}R^2 - |\text{Ric}|^2 \equiv 0$, then the sectional curvature $K = 0$, and there is a contradiction by the results in [8]. Thus Corollary 1.1 is proved.

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