ASYMPTOTICS OF EIGENVALUES AND EIGENFUNCTIONS OF ENERGY-DEPENDENT STURM–LIOUVILLE EQUATIONS

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Abstract. We study asymptotics of eigenvalues, eigenfunctions and norming constants of singular energy-dependent Sturm–Liouville equations with complex-valued potentials. The analysis essentially exploits the integral representation of solutions, which we derive using the connection of the problem under study and a Dirac system of a special form.

1. Introduction

The main objective of the present paper is to study asymptotics of eigenvalues and eigenfunctions of Sturm–Liouville equations on (0,1) with energy-dependent potentials, viz.

\[ -y'' + qy + 2\lambda py = \lambda^2 y. \]

Here \( \lambda \in \mathbb{C} \) is the spectral parameter, \( p \) is a complex-valued function in \( L_2(0,1) \) and \( q \) is a complex-valued distribution in the Sobolev space \( W_{2}^{-1}(0,1) \), i.e. \( q = r' \) with a complex-valued \( r \in L_2(0,1) \). We consider equation (1.1) mostly under the Dirichlet boundary conditions

\[ y(0) = y(1) = 0. \]

We restrict our attention to such boundary conditions just to concentrate on the ideas and to avoid unnecessary technicalities. Other separated boundary conditions can be treated analogously; in particular, in the last section we formulate some results for the case of the mixed boundary conditions.

Energy-dependent Sturm–Liouville equations are of importance in classical and quantum mechanics. For instance, they are used for modelling mechanical systems vibrations in viscous media. The Klein–Gordon equations, which describe the motion of massless particles such
as photons, can also be reduced to the form (1.1). The corresponding evolution equations are used to model the interactions between colliding relativistic spinless particles. In such mechanical models the spectral parameter $\lambda$ is related to the energy of the system, which explains the terminology “energy-dependent” used for the spectral equation (1.1).

Asymptotic behaviour of eigenvalues, eigenfunctions, and other spectral characteristics for usual Sturm–Liouville operators are studied sufficiently well (see, e.g. [10, 12, 16]). The spectral problem for energy-dependent Sturm–Liouville equation (1.1) with $p \in W^1_2[0, \pi]$ and $q \in L_2[0, \pi]$ was considered by M. Gasymov and G. Guseinov in their short paper [3] of 1981. Some results on asymptotics are formulated there without proofs. Analogous problems under more general boundary conditions were also considered by I. Nabiev in [15]. The asymptotics of eigenvalues and eigenfunctions for Sturm–Liouville equations with singular potentials $q \in W^{-1}_2(0, 1)$ were studied by A. Savchuk and A. Shkalikov in [19].

Our aim in this paper is to investigate the corresponding asymptotics for energy-dependent Sturm–Liouville equations (1.1) under minimal smoothness assumptions on the potentials $p$ and $q$, including, e.g. the case when $q$ contains Dirac delta-functions and/or Coulomb-like singularities. Our approach consists in establishing a strong connection between the spectral problem (1.1) and the spectral problem for a Dirac system of a special form. We then study the latter and, in particular, derive the integral representation for the solution of (1.1) which, in turn, is a basis for the subsequent asymptotic analysis.

The paper is organised as follows. In the next section, we rigorously set the spectral problem under study and give main definitions. Connection between the spectral problem (1.1) and that for a special Dirac system is discussed in Section 3. In Section 4, we construct the transformation operator relating the solution of the obtained system with that of the Dirac system with zero potential. Next in Section 5 we derive the asymptotics of eigenvalues, eigenfunctions and the corresponding norming constants for the problem (1.1), (1.2) and justify the factorization formula for its characteristic function. In the last section we formulate analogous results for the spectral problem (1.1) under more general boundary conditions. Appendix A contains main definitions from the spectral theory for operator pencils. We also prove there that the algebraic multiplicity of $\lambda$ as an eigenvalue of the problem (1.1), (1.2) coincides with that of $\lambda$ as an eigenvalue of the corresponding operator pencil.

Notations. Throughout the paper, we denote by $\mathcal{M}_2 = \mathcal{M}_2(\mathbb{C})$ the linear space of $2 \times 2$ matrices with complex entries endowed with
the Euclidean operator norm. Next, \( p_0 \) will stand for \( \int_0^1 p(s) ds \). The superscript \( t \) will signify the transposition of vectors and matrices, e.g. \( (c_1, c_2)^t \) is the column vector \( \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \).

2. Preliminaries

In this section, we recall main definitions and formulate the spectral problem under study more rigorously. To start with, introduce the differential expression

\[ \ell(y) := -y'' + qy. \]

Since the potential \( q \) is a complex-valued distribution, we need to define the action of \( \ell \) in detail. To do this, we use the regularization by quasi-derivative method due to Savchuk and Shkalikov \[19,20\]. Take a function \( r \) from \( L_2(0,1) \) such that \( q = r' \) and for every absolutely continuous \( y \) introduce its quasi-derivative \( y^{[1]} := y' - ry \). Then define \( \ell(y) \) as

\[ \ell(y) = -\left(y^{[1]}\right)' - ry^{[1]} - r^2 y \]

on the domain

\[ \text{dom } \ell = \{ y \in AC[0,1] \mid y^{[1]} \in AC[0,1], \ell(y) \in L_2(0,1) \}. \]

A straightforward verification shows that so defined \( \ell(y) \) coincides with \( -y'' + qy \) in the distributional sense. Therefore we can recast the equation (1.1) as

(2.1)

\[ \ell(y) + 2\lambda py = \lambda^2 y. \]

A number \( \lambda \in \mathbb{C} \) is called an eigenvalue of the problem (1.1), (1.2) if equation (2.1) possesses a nontrivial solution satisfying the boundary conditions (1.2). This solution is then called an eigenfunction of the problem (1.1), (1.2) corresponding to \( \lambda \).

Let \( y(x,z) \) be the solution of (2.1) with \( z \) instead of \( \lambda \) subject to the initial conditions \( y(0) = 0, y^{[1]}(0) = 1 \). This solution exists and is unique \[19\], so that \( \lambda \) is an eigenvalue of the problem (1.1), (1.2) if and only if it is a zero of the characteristic function \( \varphi(z) := y(1,z) \). The corresponding eigenfunction then coincides up to a constant factor with \( y(\cdot, \lambda) \). The multiplicity of \( \lambda \) as a zero of \( \varphi(z) \) is called an algebraic multiplicity of the eigenvalue \( \lambda \) of (1.1), (1.2).

As we shall see further \( \varphi \) is an analytic nonconstant function, and so the set of its zeros is a discrete subset of \( \mathbb{C} \). This shows that the set of eigenvalues of (1.1), (1.2) is discrete. Without loss of generality, we shall make the following standing assumption:

(A) \( 0 \) is not a zero of the characteristic function \( \varphi(z) \), i.e. it is not an eigenvalue of the problem (1.1), (1.2).
In fact, (A) is achieved by shifting the spectral parameter $\lambda$ if necessary; then for $\lambda_0$ such that $\varphi(\lambda_0) \neq 0$ the problem (1.1), (1.2) with new spectral parameter $\mu := \lambda - \lambda_0$ and with $p$ and $q$ replaced by $p + \lambda_0$ and $q - 2\lambda_0 p - \lambda_0^2$ respectively the assumption (A) holds.

The spectral problem (1.1), (1.2) can be regarded as the spectral problem for the quadratic operator pencil $T$ (see [14, 17]) defined by

$$(2.2) \quad T(\lambda)y := \lambda^2 y - 2\lambda py + y'' - qy$$
on the $\lambda$-independent domain

$$\text{dom } T := \{ y \in \text{dom } \ell \mid y(0) = y(1) = 0 \}.$$  

For the pencil $T$ one can introduce the notions of the spectrum, the eigenvalues and corresponding eigenvectors, their geometric and algebraic multiplicities (see, e.g. [14] and Appendix A). The spectral properties of $T$ were discussed in [17]. In particular, it was proved therein that the spectrum of $T$ consists only of eigenvalues, which can easily be shown to coincide with the eigenvalues of the problem (1.1), (1.2) defined above. In Appendix A, we show that the algebraic multiplicity of $\lambda$ as an eigenvalue of (1.1), (1.2) coincides with that of $\lambda$ as an eigenvalue of $T$.

3. Reduction to the Dirac system

In this section we reduce equation (1.1) to a $\lambda$-linear Dirac-type system of the first order. We shall further use the connection between (1.1) and this system to derive the asymptotics of interest.

The following observation plays an important role in the reduction procedure.

**Lemma 3.1.** The equation $\ell(y) = 0$ possesses a complex-valued solution which does not vanish on $[0, 1]$.

**Proof.** Note firstly that for every complex $a, b$ and every $x_0$ from $[0, 1]$ the equation $\ell(y) = 0$ possesses a unique solution satisfying the conditions $y(x_0) = a$ and $y^{[1]}(x_0) = b$ (see, e.g. [19]).

Assume that $y$ is a solution of $\ell(y) = 0$. We introduce the polar coordinates $\rho$ and $\theta$ via $y(x) = \rho(x) \sin \theta(x)$ and $y^{[1]}(x) = \rho(x) \cos \theta(x)$. Clearly, the solution $y$ vanishes at some point $x_0$ from $[0, 1]$ if and only if $\theta(x_0) = \pi k$ for some $k \in \mathbb{Z}$. The function $\theta$ is called the Prüfer angle (see [4, 20]) and can be defined to be continuous; it then satisfies the equation

$$\theta'(x) = (\cos \theta(x) + r(x) \sin \theta(x))^2.$$
Equation (3.1) has the form $\theta' = f(x, \theta)$ with the right-hand side $f$ that is not continuous in $x$. However, (3.1) is a so-called Caratheodory equation and under the initial condition $\theta(\xi) = 0$ with $\xi \in [0, 1]$ it possesses a unique solution $\theta(x, \xi)$ (see e.g. [2], Theorem 1.1.2); this solution depends continuously on $\xi$ (see [2], Theorem 2.8.2). Therefore the mapping $\xi \mapsto \theta(0, \xi)$ is continuous and its image $I_0$ is a compact subset of $\mathbb{C}$ containing 0 as a continuous image of a compactum $[0, 1]$. Note further that for any $k \in \mathbb{Z}$ the solutions $\theta_k(x, \xi)$ of the problem (3.1) with $\theta(\xi) = \pi k$ are equal to $\theta_k(x, \xi) = \theta(x, \xi) + \pi k$. The images $I_k$ of the mappings $\xi \mapsto \theta_k(x, \xi)$ are compact and are the shifts of $I_0$ by $\pi k$ along the real axis.

Let us now take any complex number, say $\theta_0$, outside the union of all the compacta $I_k, k \in \mathbb{Z}$, and consider the problem (3.1) with $\theta(0) = \theta_0$. In view of the above arguments and uniqueness of the solution of the corresponding initial-value problem, the solution of (3.1) with $\theta(0) = \theta_0$ can equal to $\pi k$, $k \in \mathbb{Z}$, at no point of the interval $[0, 1]$. Consider the solution $y_0$ of $\ell(y) = 0$ subject to the initial conditions $y(0) = 1$, $y^{[1]}(0) = \cot \theta_0$. Then $y_0$ does not vanish on $[0, 1]$, which is the assertion of the lemma. □

Denote by $y_0$ any solution of $\ell(y) = 0$ not vanishing on $[0, 1]$ and set $v = y_0'y/y_0$. Observe that $v \in L^2(0, 1)$ and $q = v' + v^2$, i.e. $q$ is a Miura potential (see [7]). Then the differential expression $-y'' + qy$ can be written in the factorized form, viz.

$$ (3.2) \quad -y'' + qy = -\left(\frac{d}{dx} + v\right)\left(\frac{d}{dx} - v\right)y. $$

Remark 3.2. Observe that the function $v$ satisfies the equality $v - r = y_0^{[1]}/y_0$, so that $v - r$ is a continuous function on $[0, 1]$ and $(v - r)(x) = \cot \theta(x)$ for every $x \in [0, 1]$, where $\theta$ is the Prüfer angle corresponding to $y_0$.

For $\lambda \neq 0$ consider the functions $u_2 := y$ and $u_1 := (y' - vy)/\lambda$ and recast the spectral equation (1.1) as the following first order system for $u_1$ and $u_2$:

$$ (3.3) \quad u'_2 - vu_2 = \lambda u_1, $$

$$ (3.4) \quad -u'_1 - vu_1 + 2pu_2 = \lambda u_2. $$

Setting

$$ (3.5) \quad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad P := \begin{pmatrix} 0 & -v \\ -v & 2p \end{pmatrix}, \quad u(x) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, $$

we see that the above system is the spectral problem \( \ell(P)u = \lambda u \) for a Dirac differential expression \( \ell(P) \) acting in \( L_2(0, 1) \times L_2(0, 1) \) via

\[
(3.6) \quad \ell(P)u := J \frac{du}{dx} + Pu
\]
on the domain

\[
\text{dom } \ell(P) = \{ u = (u_1, u_2)^t \mid u \in W^1_2(0, 1) \times W^1_2(0, 1) \}.
\]

It was shown in [18] that the spectral problem (1.1), (1.2) is closely related to the spectral problem for the Dirac operator \( D(P) \) defined by the differential expression \( \ell(P) \) on the domain \( \text{dom } D(P) = \{ u = (u_1, u_2)^t \mid u \in \text{dom } \ell(P), u_2(0) = u_2(1) = 0 \} \).

In particular, the nonzero spectra for both problems coincide counting with multiplicity. Moreover, \( u = (u_1, u_2)^t \) is an eigenfunction of the operator \( D(P) \) corresponding to the eigenvalue \( \lambda \neq 0 \) if and only if \( y = u_2 \) is an eigenfunction of (1.1), (1.2) corresponding to \( \lambda \) and \( u_1 = (u_2 - vu_2)/\lambda \).

By assumption (A), \( \lambda = 0 \) is not an eigenvalue of (1.1), (1.2); however, it is in the spectrum of \( D(P) \):

**Lemma 3.3.** Under assumption (A), \( \lambda = 0 \) is an eigenvalue of \( D(P) \) of algebraic multiplicity one.

**Proof.** A straightforward verification shows that \( \lambda = 0 \) is an eigenvalue of \( D(P) \), and that every corresponding eigenfunction must be collinear to \( u_0 = (u_1, u_2)^t \), where \( u_1 = \exp\{-\int v\}, u_2 \equiv 0 \). Therefore the eigenvalue \( \lambda = 0 \) is geometrically simple.

Suppose that the algebraic multiplicity of \( \lambda = 0 \) is greater than one. Then there exists a vector \( w = (w_1, w_2)^t \) from the domain of \( D(P) \) associated with \( u_0 \), i.e. satisfying the equality \( D(P)w = u_0 \). Then \( \left( \frac{d}{dx} - v \right) w_2 = u_1 \) and thus \( \ell w_2 = -\left( \frac{d}{dx} + v \right) \left( \frac{d}{dx} - v \right) w_2 = -\left( \frac{d}{dx} + v \right) u_1 = 0 \). Moreover, \( w \in \text{dom } D(P) \) requires that \( w_2(0) = w_2(1) = 0 \) and thus either \( w_2 \) is an eigenfunction of (1.1), (1.2) for the eigenvalue \( \lambda = 0 \) or \( w \equiv 0 \). The first possibility is ruled out by assumption (A), while the second one is impossible in view of the relation \( u_2' - vu_2 = u_1 \). The contradiction derived shows that no such \( w \) exists and finishes the proof. \( \square \)

4. Transformation operator

In this section we construct the so-called transformation operator relating the solution of the system \( \ell(P)u = \lambda u \) and that of \( \ell(P_0)u = \lambda u \) with zero matrix potential \( P_0 \) (i.e. with matrix potential having all components zero).
Denote by $U(x, \lambda)$ a $2 \times 2$ matrix-valued function satisfying the equation
\begin{equation}
J \frac{dU}{dx} + PU = \lambda U
\end{equation}
and the initial condition $U(0) = I$.

**Theorem 4.1.** Let $P$ in (4.1) be of the form (3.5) with $p$ and $v$ from $L_2(0,1)$. Then
\begin{equation}
U(x, \lambda) = e^{a(x)J} + \int_0^x e^{-\lambda(x-s)J} K(x,s)ds,
\end{equation}
where $a(x) = a(x, \lambda) = \int_0^x p(s)ds - \lambda x$ and $K$ is a matrix-valued function such that for every $x \in [0,1]$ the function $K(x, \cdot)$ is from $L_2((0,1), \mathcal{M}_2)$. Moreover, the mapping
\begin{equation}
x \mapsto K(x, \cdot) \in L_2((0,1), \mathcal{M}_2)
\end{equation}
is continuous on $[0,1]$.

This theorem is very similar to the corresponding theorem of [1] and its proof requires only minor modifications.

**Proof.** Observe that the system (4.1) can be rewritten as
\[J \frac{dU}{dx} + QU = (\lambda - p)U\]
with
\[Q = \begin{pmatrix} -p & -v \\ -v & p \end{pmatrix} = pJ_1 - vJ_2, \quad J_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]
The variation of constants method shows that $U$ satisfies the following integral equation:
\[U(x) = e^{a(x)J} + \int_0^x e^{(a(x)-a(s))J} JQ(s)U(s)ds.
\]
This equation can be solved by the method of successive approximation. Setting
\begin{equation}
U_0(x) = e^{a(x)J}, \quad U_n(x) = \int_0^x e^{(a(x)-a(s))J} JQ(s)U_{n-1}(s)ds,
\end{equation}
we see that the solution of the above equation can formally be given by the sum $\sum_{n=0}^\infty U_n$. We shall prove below that
\begin{equation}
\sum_{n=0}^\infty \|U_n\|_\infty < \infty
\end{equation}
(here $\|U_n\|_\infty := \sup_{x \in [0,1]} |U_n(x)|$, and $|U_n(x)|$ is the Euclidean norm of the matrix $U_n(x)$) whence the series $\sum_{n=0}^{\infty} U_n$ converges in the space $L_\infty([0,1],\mathcal{M}_2)$ to the solution of the equation (4.1) which satisfies the initial condition $U(0) = I$.

Let us now prove (4.5). Set $\tilde{Q}(y) := \exp\{-2\int_0^t p J\}JQ(t)$ and $Q_n(t_1,\ldots,t_n) := \tilde{Q}(t_n)\tilde{Q}(t_{n-1})\ldots\tilde{Q}(t_1)$.

Observe firstly that the matrix $Q$ anti-commutes with $J$ and therefore

$$e^{-tJ}\tilde{Q}(s) = \tilde{Q}(s)e^{tJ}.$$ 

Using this in the relations (4.4), we obtain by induction that

$$U_n(x) = e^{\int_0^x p(s)dsJ} \int_{\Pi_n(x)} e^{-\lambda(x-2\xi_n(t))J} Q_n(t_1,\ldots,t_n)dt_1\ldots dt_n,$$

where

$$\Pi_n(x) = \{t := (t_1,\ldots,t_n) \in \mathbb{R}^n \mid 0 \leq t_1 \leq \cdots \leq t_n \leq x\};$$

$$\xi_n(t) = \sum_{j=1}^n (-1)^{n-j}t_j.$$ 

Setting $s = \xi_n(t)$, we can rewrite the equality for $U_n$ as

$$U_n(x) = \int_0^x e^{-\lambda(x-2s)J} K_n(x,s)ds,$$

where $K_1(x,s) \equiv e^{\int_0^s p J}\tilde{Q}(s)$ and for $n \geq 2$

$$K_n(x,s) = e^{\int_0^s p J} \int_{\Pi_n(x,s)} Q_n(t_1,\ldots,t_{n-1},s+\xi_{n-1}(t))dt_1\ldots dt_{n-1}$$

with $0 \leq s < x \leq 1$ and

$$\Pi_n(x,s) = \{t := (t_1,\ldots,t_{n-1}) \in \mathbb{R}^{n-1} \mid 0 \leq t_1 \leq \cdots \leq t_{n-1} \leq \xi_{n-1}(t)+s \leq x\}.$$
Let us estimate the $L_2$-norm of $K_n(x, \cdot)$:

$$\|K_n(x, \cdot)\|_2^2 = \int_0^1 |K_n(x, s)|^2 ds$$

$$\leq \frac{1}{(n-1)!} \int_0^1 ds \int_{\Pi_n^{-1}(x,s)} e^{\frac{x}{2} \int_0^1 |\mathcal{Q}_n(t_1, \ldots, t_{n-1}, s + \xi_{n-1}(t))|^2 dt_1 \ldots dt_{n-1}}$$

$$\leq \frac{1}{(n-1)!} \int_{\Pi_n(x)} e^{\frac{x}{2} \int_0^1 |\mathcal{Q}_n(t_1, \ldots, t_n)|^2 dt_1 \ldots dt_n}$$

$$= \frac{1}{((n-1)!)^2 n!} e^{\frac{x}{2} \int_0^1 |\tilde{Q}(t)|^2 dt} \left( \int_0^x |\tilde{Q}|^2 \right)^n \leq \frac{e^{2\|p\|_1 \|\tilde{Q}\|_2^n}}{(n-1)!}$$

where $\|K(\cdot)\|_2 := \left( \int_0^1 |K(s)|^2 ds \right)^{1/2}$ and $|K(x)|$ is the Euclidean norm of the matrix $K(x)$. Put $C := \max_{x \in [1,1]} |e^{-\lambda x}|$. Then

$$|U_n(x)| \leq C \int_0^x |K_n(x, s)| ds \leq C \|K_n(x, \cdot)\|_2 \leq C \left( \frac{e^{2\|p\|_1 \|\tilde{Q}\|_2^n}}{(n-1)!} \right),$$

which yields (4.5). The estimate of the norm $\|K_n(x, \cdot)\|_2$ implies also the convergence of the series $K(x, \cdot) := \sum_{n=1}^\infty K_n(x, \cdot)$ in $L_2([0,1], \mathcal{M}_2)$ with

$$\|K(x, \cdot)\|_2 \leq \sum_{n=1}^\infty \frac{e^{\|p\|_1 \|\tilde{Q}\|_2^n}}{(n-1)!} \leq \|\tilde{Q}\|_2 \exp\{\|\tilde{Q}\|_2 + \|p\|_1\}.$$

Therefore the first statement of the theorem is proved.

To prove continuity of (4.3) it is enough to verify continuity of the mapping $x \mapsto \exp\left\{-\int_0^x pJ\right\}K(x, \cdot) =: \tilde{K}(x, \cdot)$. Take $x_1, x_2 \in [0,1]$ such that $x_1 < x_2$. Set $\tilde{K}_n(x, s) := \exp\left\{-\int_0^s pJ\right\}K_n(x, \cdot)$; then

$$\tilde{K}_n(x_2, s) - \tilde{K}_n(x_1, s) = \int_{\Pi^{\bullet}_{n-1}(x_1, x_2, s)} \mathcal{Q}_n(t_1, \ldots, t_{n-1}, s + \xi_{n-1}(t)) dt_1 \ldots dt_{n-1},$$

where

$$\Pi^{\bullet}_{n-1}(x_1, x_2, s) := \{ t := (t_1, \ldots, t_{n-1}) \in \mathbb{R}^{n-1} \mid 0 \leq t_1 \leq \cdots \leq t_{n-1} \leq \xi_{n-1}(t) + s, x_1 \leq \xi_{n-1}(t) + s \leq x_2 \}.$$
Therefore,
\[
\int_0^1 |\bar{K}_n(x_2, s) - \bar{K}_n(x_1, s)|^2 ds \leq \frac{1}{(n-1)!} \int_0^1 \int_0^{\Pi_{n-1}^*(x_1, x_2, s)} |Q_n(t_1, \ldots, t_{n-1}, s + \xi_{n-1}(t))|^2 dt_1 \ldots dt_{n-1} ds
\]
\[
\leq \frac{1}{(n-1)!} \int_{x_1}^{x_2} dt_n \int_{\Pi_{n-1}(x_2)}^{\Pi_{n-1}^*(x_1, x_2, s)} |Q_n(t_1, \ldots, t_n)|^2 dt_1 \ldots dt_{n-1}
\]
\[
\leq \frac{1}{((n-1)!)^2} \|\bar{Q}\|_2^{2(n-1)} \cdot \int_{x_1}^{x_2} |\bar{Q}(t)|^2 dt.
\]
This yields the estimate of the norms
\[
\|\bar{K}_n(x_2, \cdot) - \bar{K}_n(x_1, \cdot)\|_2 \leq \frac{1}{(n-1)!} \|\bar{Q}\|_2^{n-1} \left[ \int_{x_1}^{x_2} |\bar{Q}(t)|^2 dt \right]^{1/2}
\]
and so
\[
\|\bar{K}(x_1, \cdot) - \bar{K}(x_2, \cdot)\| \leq C \left[ \int_{x_1}^{x_2} |\bar{Q}(t)|^2 dt \right]^{1/2} \exp\{\|\bar{Q}\|\}
\]
with some constant $C$ depending on $p$. This shows that the mapping $x \mapsto \bar{K}(x, \cdot)$ is continuous from $[0, 1]$ to $L_2((0, 1), \mathcal{M}_2)$. The proof is complete. \hfill \Box

Observe that the $2 \times 2$ matrix $U_0 = e^{-\lambda x J}$ is a solution of the system $\ell(P_0)U = \lambda U$ with zero potential $P_0$. Therefore, in view of the last theorem, the solution $U$ of the problem (4.4) can be obtained from $U_0$ by means of the transformation operator $\mathcal{T} = \mathcal{R} + \mathcal{K}$, where $\mathcal{R}$ is an operator of multiplication by $\exp\{J \int_0^x p\}$ and $\mathcal{K}$ is an integral operator acting as follows
\[
\mathcal{K} f(x) = \int_0^x f(x - 2s)K(x, s)ds.
\]
This transformation operator also performs similarity of the corresponding differential expressions, namely,
\[
\ell(P)\mathcal{T} = \mathcal{T}\ell(P_0).
\]
5. Asymptotics

In this section, we derive the asymptotics of eigenvalues and eigenfunctions and the corresponding norming constants of the problem (1.1) under the Dirichlet boundary conditions (1.2). We also obtain the factorization of the characteristic function for the problem under study.

5.1. Asymptotics of the eigenvalues. Consider the vector \( u(x, \lambda) = U(x, \lambda)(1, 0)^t = (u_1(x, \lambda), u_2(x, \lambda))^t \). In view of (4.2), the second component \( u_2(x, \lambda) \) of the vector \( u(x, \lambda) \) is given by

\[
u_2(x, \lambda) = -\sin a(x) + \int_0^x k_{11}(x, s) \sin(\lambda(x-2s))ds + \int_0^x k_{21}(x, s) \cos(\lambda(x-2s))ds.
\]

Observe that the function \( u_2(x, \lambda) \) solves equation (1.1) and satisfies the initial condition \( u_2(0, \lambda) = 0 \). However,

\[
u_2^{[1]}(0, \lambda) = \lambda(u_1(0, \lambda) + cu_2(0, \lambda)) = \lambda,
\]

where \( c = (v-r)(0) \) (see Remark 3.2). Therefore \( \varphi(\lambda) = u_2(1, \lambda)/\lambda \) is the characteristic function for the spectral problem (1.1), (1.2).

Further observe that \( u_2(1, \lambda) \) can be written as

\[
u_2(1, \lambda) = \sin(\lambda - p_0) + \int_0^1 f(s)e^{i\lambda(1-2s)}ds
\]

with \( f(s) = \frac{1}{2}[k_{21}(1, s) + k_{21}(1, 1-s) - ik_{11}(1, s) + ik_{11}(1, 1-s)] \). Recall also that \( p_0 := \int_0^1 p(s)ds \). Let us make the change of variables \( z := \lambda - p_0 \) and consider the function

\[
\delta(z) = \sin z + \int_0^1 \tilde{f}(s)e^{iz(1-2s)}ds,
\]

where \( \tilde{f}(s) = f(s)e^{ip_0(1-2s)} \). Clearly, \( \delta(z) = u_2(1, z + p_0) \). By Theorem 4 of [11], the zeros of \( \delta(z) \) can be labelled according to their multiplicities as \( z_n, n \in \mathbb{Z} \), so that \( z_n = \pi n + \lambda_n \), where the sequence \( (\lambda_n)_n \in \mathbb{Z} \) belongs to \( l_2(\mathbb{Z}) \). Hence the zeros of the function \( u_2(1, \lambda) \) can be labelled according to their multiplicities as \( \lambda_n, n \in \mathbb{Z} \), so that

\[
\lambda_n = \pi n + p_0 + \lambda_n.
\]

In view of this asymptotics, all but finitely many zeros of \( u_2(1, \lambda) \) are simple.

Next note that \( u_2(1, \lambda) \) is a characteristic function of the operator \( D(P) \) defined in Section 3, whence \( \lambda = 0 \) is a zero of \( u_2(1, \lambda) \) of order 1 (see Lemma 3.3). However, under assumption (A) it is not a zero of the characteristic function \( \varphi(\lambda) \). Therefore the set of all the eigenvalues of the problem (1.1), (1.2) coincides with the set of zeros...
of the function \( u_2(1, \lambda) \) different from 0. Thus the following theorem holds true.

**Theorem 5.1.** The eigenvalues of the problem \( (1.1), (1.2) \) can be labelled according to their multiplicities as \( \lambda_n \) with \( n \in \mathbb{Z}^* := \mathbb{Z} \setminus \{0\} \) so that

\[
\lambda_n = \pi n + p_0 + \tilde{\lambda}_n
\]

with an \( \ell_2 \)-sequence \( (\tilde{\lambda}_n) \).

Next we construct the factorization of the characteristic function \( \varphi(\lambda) \), which allows to determine \( \varphi(\lambda) \) via the eigenvalues of \( (1.1), (1.2) \). We use this factorization to derive the formula determining the norming constants of \( (1.1), (1.2) \) via two spectra of the equation \( (1.1) \) under two types of boundary conditions (see [17]).

**Theorem 5.2.** Suppose that \( \lambda_n, n \in \mathbb{Z}^* \), are the eigenvalues of the spectral problem \( (1.1), (1.2) \). Then the characteristic function \( \varphi(\lambda) \) can be factorized in the following way:

\[
\varphi(\lambda) = \begin{cases} 
\text{V.p. } \prod_{n=-\infty \atop n \neq 0}^{\infty} \frac{\lambda_n - \lambda}{\pi n}, & \text{if } p_0 \neq \pi l, \ l \in \mathbb{Z}, \\
(-1)^l \text{V.p. } \prod_{n=-\infty \atop n \neq 0}^{\infty} \frac{\lambda_n - \lambda}{\pi n}, & \text{if } p_0 = \pi l, \ l \in \mathbb{Z}.
\end{cases}
\]

**Proof.** Suppose firstly that \( p_0 \neq \pi l, \ l \in \mathbb{Z} \). Observe that the function \( u_2(1, \lambda) \) is given by (5.1) and so it is of exponential type 1. Recall also that, by Lemma 3.3, \( \lambda = 0 \) is a zero of \( u_2(1, \lambda) \) of order 1. Therefore by Hadamard factorization theorem (see e.g. [21]) we have

\[
u_2(1, \lambda) = \lambda e^{A\lambda + B} \prod_{n=-\infty \atop n \neq 0}^{\infty} \left( 1 - \frac{\lambda}{\lambda_n} \right) e^{\frac{\lambda}{\lambda_n}},
\]

where \( A \) and \( B \) are some constants and \( \lambda_n \) are the zeros of the function \( u_2(1, \lambda) \). In view of the asymptotic distribution (5.2), the series \( \sum_{n=1}^{\infty} \frac{\lambda_n}{\lambda_n} \) converges. Indeed,

\[
\sum_{n=-\infty}^{\infty} \frac{1}{\lambda_n} = \sum_{n=1}^{\infty} \left( \frac{1}{\lambda_n} + \frac{1}{\lambda_{-n}} \right) = \sum_{n=1}^{\infty} \frac{\pi^2 n^2}{\lambda_n \lambda_{-n}} \cdot \frac{\lambda_n + \lambda_{-n}}{\pi^2 n^2},
\]

\footnote{Here and hereafter all infinite products and sums are understood in the principal value sense and the symbol V.p. will be omitted.}
and we note that the series \( \sum_{n=1}^{\infty} \frac{\lambda_n + \lambda_{-n}}{\pi n^2} \) is absolutely convergent and the sequence \( \left( \frac{\pi^2 n^2}{\lambda_n \lambda_{-n}} \right) \) is uniformly bounded. Therefore we can write

\[
 u_2(1, \lambda) = \lambda e^{A' \lambda + B} \prod_{n=\infty}^{\infty} \left( 1 - \frac{\lambda}{\lambda_n} \right)
\]

with some constant \( A' \).

To find the values \( A' \) and \( B \), consider the ratio \( \frac{u_2(1, \lambda)}{\sin(\lambda - p_0)} \) and find its limits along the ray \( \lambda = re^{i\theta}, \theta \neq 0, \pi \). In view of (5.1) and a refined version of the Riemann–Lebesgue lemma \([13, \text{Lemma 1.3.1}]\), we have

\[
 (5.3) \quad \frac{u_2(1, re^{i\theta})}{\sin(re^{i\theta} - p_0)} = 1 + o(1), \quad r \to \infty.
\]

Recall (see e.g. \([21]\)) that the function \( \sin(\lambda - p_0) \) can be factorized as follows:

\[
 \sin(\lambda - p_0) = (\lambda - p_0) \prod_{n=\infty}^{\infty} \frac{\nu_n - \lambda}{\pi n},
\]

where \( \nu_n \) are the zeros of \( \sin(\lambda - p_0) \), i.e. \( \nu_n = \pi n + p_0 \). Therefore we have

\[
 (5.4) \quad \frac{u_2(1, \lambda)}{\sin(\lambda - p_0)} = \lambda e^{A' \lambda + B} \prod_{n=\infty}^{\infty} \frac{\pi n}{\lambda_n} \cdot \prod_{n=\infty}^{\infty} \frac{\lambda_n - \lambda}{\nu_n - \lambda}.
\]

Let us show that \( A' = 0 \). If \( A \) were not 0, then one could choose the direction \( \theta \) such that \( \text{Re} A' re^{i\theta} \) tends to infinity as \( r \to \infty \). Next note that by Lemma [5.3] given below the product \( \prod_{n=\infty}^{\infty} \frac{\pi n}{\lambda_n} \) is convergent and by Lemma [5.4] the product \( \prod_{n=\infty}^{\infty} \frac{\lambda_n - re^{i\theta}}{\nu_n - re^{i\theta}} \) converges to 1 as \( r \to \infty \) and \( \theta \neq 0, \pi \). These arguments together with (5.3) and (5.4) give a contradiction. Thus \( A' = 0 \) and

\[
 e^{B} \prod_{n=\infty}^{\infty} \frac{\pi n}{\lambda_n} = 1,
\]

yielding that

\[
 \varphi(\lambda) = \prod_{n=\infty}^{\infty} \frac{\lambda_n - \lambda}{\pi n}.
\]
If \( p_0 = \pi l \) for some \( l \in \mathbb{Z} \), then
\[
\sin(\lambda - p_0) = (-1)^l \lambda \prod_{n=-\infty \atop n \neq 0}^{\infty} \frac{\pi n - \lambda}{\pi n},
\]
and so
\[
\frac{u_2(1, \lambda)}{\sin(\lambda - p_0)} = (-1)^l e^{A' \lambda + B} \prod_{n=-\infty \atop n \neq 0}^{\infty} \frac{\lambda_n - \lambda}{\pi n}.
\]
Using this, (5.3) and the arguments analogous to above, we obtain that \( A' = 0 \) and \( e^B = (-1)^l \prod_{n=-\infty \atop n \neq 0}^{\infty} \frac{\lambda_n - \lambda}{\pi n} \). Thus for \( p_0 = \pi l \)
\[
\varphi(\lambda) = (-1)^l \prod_{n=-\infty \atop n \neq 0}^{\infty} \frac{\lambda_n - \lambda}{\pi n}.
\]
The proof is complete.

\[\square\]

**Lemma 5.3.** The product \( \prod_{n=-\infty \atop n \neq 0}^{\infty} \frac{\lambda_n}{\pi n} \) is convergent.

**Proof.** We firstly prove the result analogous to Lemma 3.1. of [5] for sequences of complex numbers. For every \( \varepsilon > 0 \), set \( \delta = \delta(\varepsilon) := \sup \frac{|z - \ln(1 + z)|}{z^2} \). Note that \( \delta \) is finite because the function \( f(z) := \frac{z - \ln(1 + z)}{z^2} \) is analytic in the domain \( D := \{ z \in \mathbb{C} \mid |1 + z| \geq \varepsilon \} \) and tends to zero as \( |z| \to \infty \). Using arguments analogous to those in the proof of the mentioned lemma, we obtain that if \( (a_n) \) is a sequence of complex numbers from \( \ell_2 \) such that \( \sum a_n \) converges and \( |1 + a_n| \geq \varepsilon \) for all \( n \in \mathbb{Z} \), then
\[
|\ln \prod_{n=-\infty}^{\infty} (1 + a_n)| \leq \sum_{n=-\infty}^{\infty} |a_n| + \delta \sum_{n=-\infty}^{\infty} a_n^2.
\]
Using arguments analogous to those in the proof of one can show that for each \( \varepsilon > 0 \), there exists \( \delta = \delta(\varepsilon) \) such that whenever the sequence \( (a_n) \) of numbers from \( \ell_2(\mathbb{Z}) \) with \( \sum |a_n| < \infty \) satisfies the inequality \( |a_n + 1| > \varepsilon \) for all \( n \in \mathbb{Z} \), the following estimate is valid

Therefore, for \( \varepsilon \) such that \( |\lambda_n / \pi n| > \varepsilon \)
\[
|\ln \prod_{n=-\infty \atop n \neq 0}^{\infty} \frac{\lambda_n}{\pi n}| \leq \sum_{n=-\infty \atop n \neq 0}^{\infty} \left( \frac{p_0}{\pi n} + \frac{\lambda_n}{\pi n} \right) + \delta \sum_{n=-\infty \atop n \neq 0}^{\infty} \left( \frac{p_0 + \lambda_n}{\pi n} \right)^2,
\]
with \( \delta := \max_{|z|+1>\varepsilon} \frac{\ln(1+z)}{z^2} \).

Observe that \( V_p \sum \frac{p_0}{\pi n} = 0 \). The sequences \( (\tilde{\lambda}_n) \) and \( (1/n) \) are from \( \ell_2 \), so that the series \( \sum \frac{\lambda_n}{\pi n} \) is convergent; as a result, the first summand in the righthand side of the last estimate is finite. Since the sequence \( (1/\pi n) \) is from \( \ell_2 \) and \( \left(\frac{p_0 + \tilde{\lambda}_n}{\pi n}\right)^2 \leq \frac{C}{\pi^2} \) with \( C := \sup_n (p_0 + \tilde{\lambda}_n)^2 \), the second summand is finite as well. All these arguments imply that the product \( \prod_{n \neq 0} \frac{\lambda_n}{\pi n} \) is convergent and so complete the proof. \( \square \)

**Lemma 5.4.** The product \( \prod_{n=-\infty}^{\infty} \frac{\lambda_n - re^{i\theta}}{\nu_n - re^{i\theta}} \) converges to 1, as \( r \to \infty \) with \( \theta \neq 0, \pi \).

**Proof.** Consider the series

\[
\sum_{n=-\infty}^{\infty} \ln \frac{\lambda_n - re^{i\theta}}{\nu_n - re^{i\theta}} = \sum_{n=-\infty}^{\infty} \ln \left(1 + \frac{\tilde{\lambda}_n}{\nu_n - re^{i\theta}}\right).
\]

One can find \( N \) sufficiently large such that \( |\tilde{\lambda}_n| < 1/2 \) if \( |n| > N \). Also, for all \( r > R_\theta \) with \( R_\theta = (1 + |p_0|)/\sin \theta \) we have \( |\nu_n - re^{i\theta}| > 1 \). Since \( |\ln(1+z)| \leq |z| \) if \( |z| \leq 1/2 \), for such \( n \) and \( r \)

\[
\left| \ln \left(1 + \frac{\tilde{\lambda}_n}{\nu_n - re^{i\theta}}\right) \right| \leq \left| \frac{\tilde{\lambda}_n}{\nu_n - re^{i\theta}} \right|.
\]

Next observe that

\[
|\nu_n - re^{i\theta}| \geq |\pi n - re^{i\theta}| - |p_0| \geq |\pi n \sin \theta| - |p_0|.
\]

Since \( |p_0| < \frac{1}{2} \pi N \sin \theta \) for sufficiently large \( N \), for all \( n \) with \( |n| > N \) we have

\[
|\nu_n - re^{i\theta}| > \frac{\sin \theta}{2} |\pi n|.
\]

Therefore,

\[
\left| \frac{\tilde{\lambda}_n}{\nu_n - re^{i\theta}} \right| < \frac{2}{\pi |\sin \theta|} \frac{|\tilde{\lambda}_n|}{|n|}.
\]

Since the sequences \( (\tilde{\lambda}_n) \) and \( (1/n) \) belong to \( \ell_2 \), the series \( \sum_{n=-\infty}^{\infty} \frac{\tilde{\lambda}_n}{|n|} \) is convergent and so the series \( (5.5) \) is convergent uniformly in \( r > R_\theta \).
for a fixed $\theta$, $\theta \neq 0, \pi$. Therefore,

$$\lim_{r \to \infty} \sum_{n=-\infty}^{\infty} \ln \frac{\lambda_n - re^{i\theta}}{\nu_n - re^{i\theta}} = \sum_{n=-\infty}^{\infty} \ln \frac{\lambda_n - re^{i\theta}}{\nu_n - re^{i\theta}} = 0,$$

which means that the product $\prod_{n \neq 0} \frac{\lambda_n - re^{i\theta}}{\nu_n - re^{i\theta}}$ converges to 1 as $r \to \infty$.

5.2. Asymptotics of eigenfunctions and norming constants.

Let us now consider the vectors $u_n := u(x, \lambda_n)$. Put $u_{n,0} = (\cos(\lambda_n x), \sin(\lambda_n x))^t$. Then, in view of Theorem 4.1, we have

$$u_n = R u_{n,0} + L u_{n,0},$$

where the operator $R$ was defined at the end of Section 4 and

$$L u(x) = \int_0^x L(x,s) u(x-2s) ds$$

with

$$L(x,s) = \begin{pmatrix} k_{11}(x,s) & -k_{21}(x,s) \\ k_{21}(x,s) & k_{11}(x,s) \end{pmatrix}$$

and $k_{ij}$ being the corresponding entries of $K$. This yields the following

**Theorem 5.5.** The eigenfunctions $y_n$ of the problem (1.1), (1.2) corresponding to the eigenvalues $\lambda_n$ satisfy the asymptotics

$$y_n(x) = \sin \left( \lambda_n x - \int_0^x p \right) + \tilde{y}_n(x)$$

with $\tilde{y}_n(x) = (0,1)^t L u_{n,0}.$

Note that the vectors $v_n := (\cos(\pi n + p_0)x, \sin(\pi n + p_0)x)^t$ form an orthonormal basis in $L_2(0,1) \times L_2(0,1)$. Next observe that

$$\|u_{n,0} - v_n\| = \left\| (e^{\lambda_n x} - I)v_n \right\| \leq \left\| \int_0^x e^{\lambda_n x} \frac{d}{dt}e^{\lambda_n t} dt \right\| \leq C \|\tilde{\lambda}_n\|,$$

where $C := \max_{n \in \mathbb{Z}} e^{\lambda_n}$. Since the sequence $(\tilde{\lambda}_n)$ belongs to $\ell_2$, this means that the sequence $u_{n,0}$ is quadratically close to the orthonormal basis and therefore it is a Bari basis [21]. Then

$$\|u_n - R u_{n,0}\| = \|L u_{n,0}\| \leq \|L(u_{n,0} - v_n)\| + \|L v_n\|.$$

Since $u_{n,0}$ is quadratically close to $v_n$, the sequence $(\|L(u_{n,0} - v_n)\|)$ belongs to $\ell_2$. Next observe that the operator $L$ is of Hilbert-Schmidt class. Therefore $(\|L v_n\|)$ also belongs to $\ell_2$ (see e.g. [8, V.2.4.]). All
these arguments imply that the sequence \( \|u_n - R u_{n,0}\| \) is from \( \ell_2 \). Since
\[
\|u_n\| - 1 \leq \|u_n - R u_{n,0}\|,
\]
we obtain that
\[
\|u_n\| = 1 + \tilde{\alpha}_n,
\]
where \((\tilde{\alpha}_n)\) belongs to \( \ell_2 \).

Observe also that if \( p \) and \( q \) are real-valued, the norming constants of (1.1), (1.2) corresponding to real and simple eigenvalues coincide with the norms of eigenvectors of the operator \( D(P) \) (see [6]). Therefore the following holds true.

**Theorem 5.6.** If \( p \) and \( q \) are real-valued, the norming constants \( \alpha_n \) of the problem (1.1), (1.2) corresponding to the eigenvalues \( \lambda_n \) satisfy the asymptotics
\[
\alpha_n = 1 + \tilde{\alpha}_n,
\]
where \((\tilde{\alpha}_n) \in \ell_2 \).

### 6. Results for the mixed boundary conditions

In this section, we consider the equation (1.1) under the boundary conditions which we call the mixed ones; namely,
\[
y(0) = y^{[1]}(1) + h y(1) = 0
\]
with some complex \( h \). As the proofs are analogous to those applied in the case of the Dirichlet boundary conditions, we shall only reformulate the results.

Without loss of generality, we assume that \( \mu = 0 \) is not an eigenvalue of the problem (1.1), (6.1). Consider the function
\[
\psi(\mu) := u_1(1, \mu) + \frac{(h_1 + h)u_2(1, \mu)}{\mu},
\]
where \( u_1 \) and \( u_2 \) are the solutions of the system (3.3), (3.4) and \( h_1 := (v - r)(1) \). The function \( y(\cdot, \mu) := u_2(\cdot, \mu) \) solves the equation \( \ell(y) + 2\mu py = \mu^2 y \) and satisfies the relation
\[
y^{[1]} + hy = (y' - vy) + (v - r)y + hy = \mu u_1 + (v - r + h)u_2;
\]
in particular, \( y(0, \mu) = 0 \) and \( y^{[1]}(1, \mu) + hy(1, \mu) = \mu \psi(\mu) \). Therefore \( \psi \) is a characteristic function for the problem (1.1), (6.1), i.e. the zeros of \( \psi(\mu) \) are the eigenvalues of the mentioned problem.
Using the asymptotics form of \( u_1 \) and \( u_2 \), we have

\[
\psi(\mu) = \cos a(1, \mu) - \frac{(h + h)}{\mu} \sin a(1, \mu) \\
- \int_0^1 k_{21}(1, s) \sin(\mu(1 - 2s))ds + \int_0^1 k_{11}(1, s) \cos(\mu(1 - 2s))ds \\
+ \frac{h + h}{\mu} \left( \int_0^1 k_{11}(1, s) \sin(\mu(1 - 2s))ds + \int_0^1 k_{21}(1, s) \cos(\mu(1 - 2s))ds \right).
\]

Taking into account Theorem 4 from [11], we obtain

**Theorem 6.1.** The eigenvalues of (1.1), (6.1) can be labelled according to their multiplicities as \( \mu_n, n \in \mathbb{Z} \), so that they satisfy the following asymptotics

\[
\mu_n = \pi \left( n + \frac{1}{2} \right) + p_0 + \bar{\mu}_n,
\]

with \( \ell_2 \)-sequence \( (\bar{\mu}_n) \). In particular, all eigenvalues \( \mu_n \) with large enough \( |n| \) are simple.

Analogously to the case of the Dirichlet boundary conditions, we prove the following

**Theorem 6.2.** Let \( \mu_n, n \in \mathbb{Z} \), be the eigenvalues of (1.1), (6.1). Then the characteristic function \( \psi(\mu) \) can be factorized in the following way

\[
\psi(\mu) = \begin{cases} 
-V.p. \prod_{n=-\infty}^{\infty} \frac{\mu_n - \mu}{\pi (n + 1/2)}, & \text{if } p_0 \neq \frac{\pi}{2} + \pi l, l \in \mathbb{Z}, \\
(-1)^{n+1}(\mu_0 - \mu)V.p. \prod_{n=-\infty \atop n \neq 0}^{\infty} \frac{\mu_n - \mu}{\pi n}, & \text{if } p_0 = \frac{\pi}{2} + \pi l, l \in \mathbb{Z}.
\end{cases}
\]

**Theorem 6.3.** The eigenfunctions \( y_n \) of (1.1), (6.1) corresponding to the eigenvalues \( \mu_n \) satisfy the asymptotics

\[
y_n(x) = \cos \left( \lambda_n x - \int_0^x p \right) + \tilde{y}_n(x),
\]

where the sequence \( (\|\tilde{y}_n(x)\|) \) is from \( \ell_2 \).

**Theorem 6.4.** If \( p \) and \( q \) are real-valued, the norming constants \( \beta_n \) of (1.1), (6.1) corresponding to the eigenvalues \( \mu_n \) satisfy the asymptotics

\[
\beta_n = 1 + \tilde{\beta}_n,
\]

with an \( \ell_2 \)-sequence \( (\tilde{\beta}_n) \).
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Appendix A. Algebraic multiplicities of the eigenvalues

In this appendix we recall the main notions of the spectral theory for the operator pencils. We also show that the algebraic multiplicity of \( \lambda \) as an eigenvalue of the operator pencil \( T \) of (2.2) coincides with the corresponding multiplicity of \( \lambda \) as an eigenvalue of the problem (1.1), (1.2).

An operator pencil \( T \) is an operator-valued function on \( \mathbb{C} \). The spectrum of an operator pencil \( T \) is the set \( \sigma(T) \) of all \( \lambda \in \mathbb{C} \) such that \( T(\lambda) \) is not boundedly invertible, i.e.

\[
\sigma(T) = \{ \lambda \in \mathbb{C} \mid 0 \in \sigma(T(\lambda)) \}.
\]

A number \( \lambda \in \mathbb{C} \) is called an eigenvalue of \( T \) if \( T(\lambda)y = 0 \) for some non-zero function \( y \in \text{dom} T \), which is then the corresponding eigenfunction.

Vectors \( y_1, \ldots, y_{m-1} \) from \( \text{dom} T \) are said to be associated with an eigenvector \( y_0 \) corresponding to an eigenvalue \( \lambda \) if

\[
(A.1) \quad \sum_{k=0}^{j} \frac{1}{k!} T^{(k)}(\lambda)y_{j-k} = 0, \quad j = 1, \ldots, m - 1.
\]

Here \( T^{(k)} \) denotes the \( k \)-th derivative of \( T \) with respect to \( \lambda \). The number \( m \) is called the length of the chain \( y_0, \ldots, y_{m-1} \) of an eigenvector and associated vectors. The maximal length of a chain starting with an eigenvector \( y_0 \) is called the algebraic multiplicity of an eigenvector \( y_0 \).

For an eigenvalue \( \lambda \) of \( T \) the dimension of the null-space of \( T(\lambda) \) is called the geometric multiplicity of \( \lambda \). The eigenvalue is said to be geometrically simple if its geometric multiplicity equals one.

All the eigenvalues of the pencil \( T \) of (2.2) are geometrically simple (see [17]), and then the algebraic multiplicity of an eigenvalue is the algebraic multiplicity of the corresponding eigenvector. (If the eigenvalue \( \lambda \) is not geometrically simple, its algebraic multiplicity is the number of vectors in the corresponding canonical system, see [9, 14]).

An eigenvalue is said to be algebraically simple (or just simple) if its algebraic multiplicity is one.

In the next proposition we show that the order of \( \lambda \) as a zero of the characteristic function \( \varphi \) coincides with the algebraic multiplicity of \( \lambda \) as an eigenvalue of the operator pencil \( T \) defined by (2.2).
Proposition A.1. Suppose $\lambda$ is an eigenvalue of the spectral problem (1.1), (1.2). Then $\lambda$ is a zero of the characteristic function $\varphi$ of order $m$ if and only if $\lambda$ is an eigenvalue of the operator pencil $T$ given by (2.2) of algebraic multiplicity $m$.

Proof. Suppose that $y(x, z)$ is the solution of (1.1) subject to the initial conditions $y(0, z) = 0$, $y^{[1]}(0, z) = 1$ and that $\lambda$ is a zero of $\varphi(z) = y(1, z)$ of order $m$. Then $y(x, \lambda)$ is an eigenfunction of (1.1), (1.2) corresponding to $\lambda$. Clearly, $y(x, \lambda)$ is also an eigenfunction of the operator pencil $T$ corresponding to the eigenvalue $\lambda$. Consider the chain of the vectors $y_j$, $j = 0, 1, \ldots$, such that $y_0 = y(x, \lambda)$ and

$$y_j(x, \lambda) := \frac{1}{j!} \frac{\partial^j y(x, z)}{\partial z^j} \bigg|_{z=\lambda}, \quad j \geq 1.$$ 

Set

$$\tau(\lambda)y := \lambda^2 y - 2\lambda py - \ell(y).$$

Straightforward verification shows that $y_j$ satisfy equalities (A.1) with $\tau$ instead of $T$. Moreover, since $\lambda$ is a zero of $y(1, z)$ of order $m$, we have $y_1(1) = \cdots = y_{m-1}(1) = 0$, and all the functions $y_j$, $j = 0, \ldots, m-1$, belong to the domain of $T$ and so form a chain of eigen- and associated vectors of $T$ corresponding to $\lambda$. Therefore $m$ does not exceed the algebraic multiplicity of $\lambda$ as an eigenvalue of $T$.

Assume that $v_0, \ldots, v_l$ is a chain of eigen- and associated vectors corresponding to an eigenvalue $\lambda$ of $T$. Then $v_0$ solves the equation

$$\tau(\lambda)y = 0$$

and satisfies the boundary conditions (1.2), and thus coincides with $y_0$ up to a scalar factor. Without loss of generality, we assume that $v_0 = y_0$ and then show by induction that there exists a sequence $(c_k)_{k=1}^l$ such that

(A.2) $$v_k - y_k = \sum_{j=1}^{k} c_j v_{k-j}.$$ 

To start with, observe that

$$\tau'(\lambda)(v_0 - y_0) + \tau(\lambda)(v_1 - y_1) = 0.$$ 

But $v_0 - y_0 = 0$ and $(v_1 - y_1)(0) = 0$. Therefore $v_1 - y_1 = c_1 v_0$ giving the base of induction. Next suppose that the statement holds for all $k < n$ and prove it for $k = n$. Observe that

$$\tau(\lambda)(v_n - y_n) = -\sum_{j=1}^{n} \frac{1}{j!} \tau^{(j)}(v_{n-j} - y_{n-j}).$$
By assumption we obtain

\[
\tau(\lambda)(v_n - y_n) = -\sum_{j=1}^{n} \frac{1}{j!} \tau^{(j)}(\sum_{i=1}^{n-j} c_i v_{n-j-i})
\]

\[
= -\sum_{i=1}^{n-1} c_i \left( \sum_{j=1}^{n-i} \frac{1}{j!} \tau^{(j)}(v_{n-j-i}) \right) = \sum_{j=1}^{n-1} c_j \tau(\lambda)v_{n-j}.
\]

Therefore,

\[
\tau(\lambda) \left( v_n - y_n - \sum_{j=1}^{n-1} c_j v_{n-j} \right) = 0
\]

and

\[
\left( v_n - y_n - \sum_{j=1}^{n-1} c_j v_{n-j} \right) (0) = 0,
\]

giving that

\[
v_n - y_n - \sum_{j=1}^{n-1} c_j v_{n-j} = c_n v_0,
\]

i.e. [A.2] holds.

Assuming that \( l \geq m \), we see that \( v_m - y_m = \sum_{j=1}^{m} c_j v_{m-j} \) and so \( y_m(1) = 0 \). This contradicts the fact that \( \lambda \) is a zero of \( \varphi(z) \) of order \( m \). The proof is complete. \( \square \)

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