Achievable rates for pattern recognition

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Abstract

Biological and machine pattern recognition systems face a common challenge: Given sensory data about an unknown object, classify the object by comparing the sensory data with a library of internal representations stored in memory. In many cases of interest, the number of patterns to be discriminated and the richness of the raw data force recognition systems to internally represent memory and sensory information in a compressed format. However, these representations must preserve enough information to accommodate the variability and complexity of the environment, or else recognition will be unreliable. Thus, there is an intrinsic tradeoff between the amount of resources devoted to data representation and the complexity of the environment in which a recognition system may reliably operate.

In this paper we describe a general mathematical model for pattern recognition systems subject to resource constraints, and show how the aforementioned resource-complexity tradeoff can be characterized in terms of three rates related to number of bits available for representing memory and sensory data, and the number of patterns populating a given statistical environment. We prove single-letter information theoretic bounds governing the achievable rates, and illustrate the theory by analyzing the elementary cases where the pattern data is either binary or Gaussian.

I. INTRODUCTION

PATTERN recognition is the problem of inferring the nature of unknown objects from incoming and previously stored data. In real-world operating environments, the volume of raw data available often exceeds a recognition system’s resources for data storage and representation. Consequently, data stored in memory only partially summarizes the properties of physical objects, and internal representations of incoming sensory data are likewise imperfect approximations. In other words, pattern recognition with physical systems is frequently a problem of inference from compressed data. However, excessive data compression precludes reliable pattern recognition. In this paper we attempt to answer the following question: In a given environment, what are the least amounts of memory data and sensory data consistent with reliable pattern recognition?

The paper is organized as follows. In section II we introduce the general problem qualitatively. Relationships between the present work and other pattern recognition research is briefly described in section III. In section IV we formalize our problem as that of determining which combinations of three key rates are achievable, that is, which rate combinations are consistent with the possibility of reliable pattern recognition. These rates are directly related to number of bits available for representing memory and sensory data, and the number of distinct patterns which the recognition system must be able to discriminate. The main results of the paper are single letter formulas providing inner and outer bounds on the set of achievable rates, given in section VI and discussed in section VII. The theory is illustrated by applying it to the Binary case in section VIII and the Gaussian case in IX.

II. INFORMAL PROBLEM DESCRIPTION

In general, statistical pattern recognition problems may be specified in terms of a probabilistic model of the environment (‘nature’) 1; a pattern recognition system; and the interactions of the system with the environment during two distinct modes of operation, a training (‘offline’) phase and a testing (‘online’) phase. Informal descriptions for the environment and system models we study are given below, and formalized in section V. Our model and viewpoint are similar to others in the statistical pattern recognition literature (see, e.g. [6], [8], [10], [14], [21]), but fits most closely within the framework of Pattern Theory (see e.g. [12], [19], [20], [23], [24]). Please refer to the block diagram in figure I while reading the following description.

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1Non-probabilistic models have also been considered. Arguments for preferring the probabilistic formulation are discussed in [25].
Fig. 1. Block diagram for a generic pattern recognition system.

A. Environment

Training patterns and the training phase. The environment for a pattern recognition system is defined as the set of distinct entities that the system must learn to reliably distinguish. These entities are hereafter referred to simply as patterns, and may include, for example, distinct physical objects, properties of objects, or arrangements of multiple objects. We assume each pattern can be represented by an \( n \)-vector \( \mathbf{x} = (x_1, x_2, \ldots, x_n) \) whose elements take values in some alphabet \( \mathcal{X} \). Of the \( |\mathcal{X}|^n \) possible patterns, the environment contains only a small subset \( \{ \mathbf{X}(1), \mathbf{X}(2), \ldots, \mathbf{X}(M_c) \} \), \( M_c \ll |\mathcal{X}|^n \). However, before entering the environment, the system does not know which specific patterns will be present, but rather knows only their number \( M_c \) and that they are generated according to some probability distribution \( p(\mathbf{x}) \).

After being introduced into the environment, the system initially enters the training phase. During training the system attempts to form and store an internal representation (memory) of each pattern along with a semantic label, \( w \in M_c = \{1, 2, \ldots, M_c\} \). In concrete terms, the labels might correspond to a set of actions the system should undertake when it encounters each pattern, ‘pointers’ to additional stored information, or ‘names’ for the patterns. For simplicity, we take the labels to be integers, and denote the training set by \( C_x = \{ (\mathbf{X}(1), 1), (\mathbf{X}(2), 2), \ldots, (\mathbf{X}(M_c), M_c) \} \).

Observations and the testing phase. After the training phase, the system enters an ‘online’ testing phase. During testing the observed data is generated as follows. Nature randomly selects a pattern \( W \) according to some distribution \( p(w), w \in M_c \), retrieves the corresponding pattern \( \mathbf{x}(W) \in C_x \), and subjects it to a random transformation \( p(y|x) \) to produce a signal \( y = (y_1, y_2, \ldots, y_n) \) with elements in some alphabet \( \mathcal{Y} \). The patterns \( \mathbf{x} \) in \( C_x \) thus represent ‘pure signals’ or prototypes, and the observations \( y \in \mathcal{Y} \) represent distorted and noise-corrupted variations or signatures of the underlying patterns. The random map \( p(y|x) \) models two major intrinsic sources of difficulty in real-world pattern recognition problems: signature variation, differences between the sensory signals generated on different occasions by the same underlying object; and signature ambiguities, the fact that distinct objects often produce similar or identical signatures\(^2\).

\(^2\)Grenander [12] and Mumford [24] have argued that four ‘universal transformations’ (noise and blur, superposition, domain warping, and interruptions) account for most of the ambiguity and variability in naturally occurring signals.
B. Recognition system

A recognition system consists of three components (functions): A memory encoder \( f \); a sensory encoder \( \phi \); and a classifier. Since we assume the system must be designed prior to insertion into its environment, the functions \( (f, \phi, g) \) must be defined independent of the specific realizations of the training data \( C_x \) and sensory data encountered during online operation. On the other hand, the system design can take account of statistical information about the environment, i.e. knowledge of the distributions \( p(x) \) and \( p(y|x) \).

**Encoders.** The memory and sensory encoders \( f \) and \( \phi \) are mappings from the domains of the raw training and sensory data, respectively, into some form of approximate *internal representations*. Encoding may comprise several distinct operations, such as smoothing and noise reduction, segmentation, normalization, dimensionality reduction, etc., often collectively referred to as ‘feature extraction’ procedures [14]. In principle, the role of the resulting internal data representations may be played by any distinct set of physical configurations or ‘states’ of the system, provided that mechanisms exist for associating the training data with these memory states; inducing appropriate internal states from the sensory data; and retrieving memorized data, comparing it with compressed sensory data, and reporting a recognition decision.

Conceptually, we can alternatively regard the internal states of the system as ‘codewords,’ denoted \( C_u = \{u(1), u(2), \ldots, u(M_u)\} \) for the memory encoder; and \( C_v = \{v(1), v(2), \ldots, v(M_v)\} \) for the sensory encoder, where the codeword alphabets \( U \) and \( V \) are dictated by the physical nature of the system’s memory and sensory systems.

The sensory encoder is then defined as a mapping from the entire observation space onto the indices \( M_y = \{1, 2, \ldots, M_y\} \) of the sensory codebook \( \phi : y^n \rightarrow M_y, \phi(y) = \mu \), or equivalently, onto the codewords \( C_v \). The memory encoder \( f \) is similar, except that it receives labeled inputs and produces labeled outputs: Given a labeled training pattern \( (x(W), W) \), \( f \) associates to it both a memory index \( m \in M_x = \{1, 2, \ldots, M_x\} \) and reproduces the class label \( w \in M_c \), representing its storage in memory. Thus, \( f \) is a mapping from the product of the entire training data space and the set of training labels onto the product of the memory indices and class labels \( f : \mathcal{X}^n \times M_c \rightarrow M_x \times M_c \), \( f(x, w) = (m, w) \).

**Classifier.** The classifier, \( g \), attempts to infer the class label of an encountered pattern on the basis of the compressed sensory information and data stored in memory. Abstractly, the inference process may take be thought of as a search through the codebook \( C_u \) for the memory codeword best matching the current sensory codeword \( v \in C_v \). Physical implementations of the matching process may take the form of computational algorithms; the dynamics of some physical medium (e.g. a biological neural network); or an abstract decision rule. Mathematically, a classifier is a mapping \( g \) from the encoded sensory data \( \mu = \phi(y) \in M_y \) and the memory data \( C_u \) to a class label \( \hat{w} \in M_c \), i.e. \( g : M_y \times C_u \rightarrow M_c \), \( g(\mu, C_u) = \hat{w} \).

C. Figures of merit

For given distributions \( p(x) \) and \( p(y|x) \) and data dimension \( n \), there is clearly an intrinsic tradeoff between the number of internal memory and sensory states, \( M_x \) and \( M_y \), and the number of patterns \( M_c \) that can be reliably recognized. For our purposes it is preferable to characterize this tradeoff in a dimensionless manner, that is, in terms of *rates*. The rates of the memory and sensory encoders \( f \) and \( \phi \) are given respectively by \( R_x = \log_2 M_x/n \), \( R_y = \log_2 M_y/n \), where standard interpretations apply (see, e.g. [?], [?], [?]): Viewing the indices of the memory codebook \( M_x = \{1, 2, \ldots, M_x\} \) as binary strings of length \( N_x = \log_2 M_x \), the rate \( R_x \) is simply the cost, in bits/symbol, of representing each \( n \)-length training pattern \( x \in \mathcal{X}^n \) by a length-\( N_x \) binary string, \( R_x = N_x/n \). The analogous interpretation applies to the sensory codebook. We also quantify the amount of data in the training set by defining a rate \( R_c = \log_2 M_c/n \), interpreted as the number of training patterns discriminated *per-symbol* of encoded memory and sensory data.

D. The meaning of large \( n \)

Some of the results below (specifically, the ‘achievability’ proofs) rely on asymptotic arguments, requiring the parameter \( n \) to grow large. Physically, ‘large-\( n \)’ may correspond to representing the sensory and memory data at high resolution; collecting more of it; or making repeated measurements [28]. On the other hand,
though our proofs employ asymptotic arguments, the theorems themselves are stated in terms of single letter formulas, and in this sense they are independent of $n$. Hence, the ‘large-$n$’ assumption in the achievability proofs is not necessarily a fundamental limitation of the theory.

III. RELATED ISSUES

Before formalizing our problem, we briefly comment on some relationships between the present work and other issues in pattern recognition.

**Probabilistic modeling.** Our analysis supposes the existence of probabilistic models for the recognition environment, and that these distributions are available for use in designing the recognition system. For some types of random patterns, such as the pattern of grains on a wooden surface or of magnetic particles on magnetic tape, estimating the probability distributions is relatively straightforward [28]. Substantial progress has also been made in modeling more challenging objects, such as textures in natural imagery [7], [13], [26], [29], [31], and speech signals [15]. Nevertheless, in many cases of interest the development of accurate probabilistic models remains a challenge, and is an active research focus in pattern recognition research.

**Data compression.** The importance of data compression in pattern recognition systems appears most clearly articulated in the neuroscience literature, due largely to the pioneering work of Horace Barlow. Barlow has written extensively about experimental evidence and theoretical reasons for believing that principles of efficient data compression underly the capacity of animal brains for learning and intelligent behavior (see, e.g. [1]–[4], [11]). Additionally, in the past few decades much additional work in neurobiology has provided experimental evidence for efficient coding mechanisms in the sensory systems of diverse animals, including monkeys, cats, frogs, crickets, and flies [27]. More recently, data compression has come to be viewed as essential for managing metabolic energy costs in animal brains [22].

In the engineering pattern recognition literature, data compression usually arises in the context of **feature extraction.** Feature extractors are typically designed with the objectives of transforming the raw data available to the system into a format which facilitates easy matching or storage, and is robust (“invariant”) with respect to characteristic signature variations in sensory data [14], [21]. With respect to these goals, the volume of data used for internal data representations is present as an implicit constraint, since efficient data manipulation is often best achieved by compact representations. For complex environments, the cost of data representation becomes critical as an explicit design constraint. Whatever the motivations are, the crucial common aspect of all data encoding operations for our present purposes is that they reduce the amount of data available to the system as compared with the original data (usually in a lossy manner).

**Performance prediction vs. normalization.** Performance prediction is the problem of characterizing the performance for specific classes of recognition systems, often with the goal of discovering the optimal member (e.g. best parameter settings) of a given class [8]. By contrast, our objective is to characterize the requirements for the existence of reliable pattern recognition systems, and to describe absolute performance limits governing all such systems. In this sense, we aim to provide normalized performance bounds, with respect to which the performance of any actual or proposed recognition system may be evaluated.

IV. NOTATION

We adopt the following notational conventions. Random variables are denoted by capital letters (e.g. $U$), and their values by lowercase letters (e.g. $u$). The alphabet in which a random variable takes values is denoted by a script capital letter (e.g. $\mathcal{U}$). Sequences of symbols are denoted either by boldface letters or with a superscript, interchangeably (e.g. $u = u^n = (u_1, u_2, \ldots, u_n)$ denotes a vector which takes values in the product alphabet $\mathcal{U}^n$). The probability mass function (p.m.f) for a random variable $U \in \mathcal{U}$ is denoted by $p_U(u)$, $u \in \mathcal{U}$. When the appropriate subscript is clear from context, we omit it to simplify notation; e.g. we usually write $p_U(u)$ simply as $p(u)$. Given random variables $U, V, W$, we denote the entropy of $U$ by $H(U)$, the mutual information between $U$ and $V$ by $I(U; V)$, and the conditional mutual information between $U$ and $V$ given $W$ by $I(U; V|W)$. The standard acronym ‘i.i.d.’ will stand for the phrase ‘independent and identically distributed.’ To express statements like ‘$U$ and $V$ are strongly jointly
delta typical‘ write \((U, V) \in T_{UV}\). The definition of strong (delta) joint typicality will be reviewed in the section where it first appears. Finally, to express statements like: \(X\) and \(Z\) are conditionally independent given \(Y\), i.e. \(p(x, y, z) = p(y)p(x|y)p(z|y)\), we write ‘\(X - Y - Z\) form a Markov chain,’ or simply \(X - Y - Z\).

V. Formal problem statement

**Definition 5.1:** The environment for a pattern recognition system, denoted by

\[
\mathcal{E} = (\mathcal{M}_c, p(w), \mathcal{X}, p(x), p(y|x), \mathcal{Y}),
\]

consists of three finite alphabets \(\mathcal{M}_c, \mathcal{X}, \mathcal{Y}\), probability distributions \(p(w)\) and \(p(x)\) over \(\mathcal{M}_c\) and \(\mathcal{X}\), and a collection of probability distributions \(p(y|x)\) on \(\mathcal{Y}\), one for each \(x \in \mathcal{X}\).

The interpretations are those given in the preceding section: \(\mathcal{M}_c = \{1, 2, \ldots, M_c\}\) is the set of class labels; patterns vectors are written in the symbols of \(\mathcal{X}\); and sensory data vectors in the symbols of \(\mathcal{Y}\). For our analysis we assume:

- the distribution over class labels is uniform, \(p(w) = 1/|\mathcal{M}_c|\) for all \(w \in \mathcal{M}_c\);
- the pattern components are i.i.d., \(p(x) = \prod_{i=1}^{n} p(x_i)\);
- the observation channel is memoryless, \(p(y|x) = \prod_{i=1}^{n} p(y_i|x_i)\).

**Definition 5.2:** An \((\mathcal{M}_c, \mathcal{M}_x, \mathcal{M}_y, n)\) pattern recognition code for an environment \(\mathcal{E}\) consists of three sets of integers

\[
\begin{align*}
\mathcal{M}_c &= \{1, 2, \ldots, M_c\} \\
\mathcal{M}_x &= \{1, 2, \ldots, M_x\} \\
\mathcal{M}_y &= \{1, 2, \ldots, M_y\}
\end{align*}
\]

a set of length-\(n\) sequences \(X(i) \in \mathcal{X}^n, i = 1, 2, \ldots, M_c\), where all components are drawn independently from \(p(x)\) and each sequence is paired with a distinct index from \(\mathcal{M}_c\)

\[
\mathcal{C}_x = \{(X(1), 1), (X(2), 2), \ldots, (X(M_c), M_c)\};
\]

a memory encoder

\[
f : \mathcal{X}^n \times \mathcal{M}_c \rightarrow \mathcal{M}_x \times \mathcal{M}_c; f(x, w) = (m, w);
\]

a sensory data encoder

\[
\phi : \mathcal{Y}^n \rightarrow \mathcal{M}_y; \phi(y) = \mu;
\]

and a classifier

\[
g : \mathcal{M}_y \times \mathcal{C}_u \rightarrow \mathcal{M}_c; g(\mu, C_u) = \hat{w}
\]

composed of two submappings \(g = g_2 \circ g_1\)

\[
\begin{align*}
g_1 &: \mathcal{M}_y \rightarrow \mathcal{M}_x; g_1(\mu) = \hat{m} \\
g_2 &: \mathcal{M}_x \times \mathcal{C}_u \rightarrow \mathcal{M}_c; g_2(\hat{m}, C_u) = \hat{w},
\end{align*}
\]

where \(C_u\) denotes the encoded training data

\[
\mathcal{C}_u = f(\mathcal{C}_x) = \{(m(1), 1), \ldots, (m(M_c), M_c)\}.
\]

For convenience hereafter, we refer to an \((\mathcal{M}_c, \mathcal{M}_x, \mathcal{M}_y, n)\) pattern recognition code by its three constituent mappings \((f, \phi, g)_n\), or simply as \((f, \phi, g)\) when the integer \(n\) is clear from context.

The rate \(R = (R_c, R_x, R_y)\) of an \((\mathcal{M}_c, \mathcal{M}_x, \mathcal{M}_y, n)\) code is

\[
R_c = \frac{1}{n} \log_2 M_c
\]
\[ R_c = \frac{1}{n} \log_2 M_x \]
\[ R_e = \frac{1}{n} \log_2 M_y, \]

where the units are bits per symbol.

For each pattern-label pair \((x(w), w) \in \mathcal{C}_x\), let \(\hat{m}(w)\) be the memory index assigned to \(x(w)\) by the memory encoder \(f\), and let the corresponding sensory data be \(y\). Define two error events

\[ \varepsilon_1(w) = \{ \hat{m} \neq m(w) \} \]
\[ \varepsilon_2(w) = \{ \hat{w} \neq w \}, \]

where \(\hat{m} = g_1(\mu) = g_1(\phi(y))\) and \(\hat{w} = g(\mu, C_u) = g_2(\hat{m}, C_u) = g_2(g_1(\phi(y)), C_u)\); and denote the union by

\[ \varepsilon(w) = \varepsilon_1(w) \cup \varepsilon_2(w). \]

During the testing phase of operation, if the pattern index \(w \in \mathcal{M}_c\) is selected, let

\[ P_{n}^n(w) = \Pr(\varepsilon(w)) \]

denote the probability of error. Note that these probabilities depend only on the random vectors \(X(w)\) and \(Y\) and hence are determined by the joint distribution \(p(x, y) = p(x)p(y|x)\). We define the average probability of error of the code as

\[ P_{n}^n = \frac{1}{M_c} \sum_{w \in \mathcal{M}_c} P_{n}^n(w). \]

Note that this probability is calculated under a uniform distribution on the pattern indices, \(p(w) = 1/M_c\). That is, we assume that every pattern index \(w \in \mathcal{M}_c\), and hence every pattern \(X(w)\), is selected with equal probability during the testing phase.

**Comment 5.3:** Expanding the probability of error in two ways

\[ P_{n}^n = \Pr(\varepsilon_1 \cup \varepsilon_2) \]
\[ = \Pr(\varepsilon_1) + \Pr(\varepsilon_1^c) \Pr(\varepsilon_2|\varepsilon_1^c) \]
\[ = \Pr(\varepsilon_2) + \Pr(\varepsilon_2^c) \Pr(\varepsilon_1|\varepsilon_2^c). \]

we see that \(P_{n}^n = 0\) if and only if

\[ \Pr(\varepsilon_1) = \Pr(\varepsilon_2) = \Pr(\varepsilon_1|\varepsilon_2^c) = \Pr(\varepsilon_2|\varepsilon_1^c) = 0. \]

The interpretation is that in a reliable pattern recognition system both components \(g_1\) and \(g_2\) of the classifier \(g\) must function reliably.

**Definition 5.4:** A rate \(\mathbf{R} = (R_x, R_y, R_c)\) is achievable in a recognition environment \(\mathcal{E}\) if for any \(\epsilon > 0\) and for all \(n\) sufficiently large, there exists an \((M_c, M_x, M_y, n)\) code \((f, \phi, g)_n\) with

\[ M_c \geq 2^{n R_c} \]
\[ M_x \leq 2^{n R_x} \]
\[ M_y \leq 2^{n R_y} \]

such that \(P_{n}^n < \epsilon\).

**Definition 5.5:** The achievable rate region \(\mathcal{R}\) for a recognition environment \(\mathcal{E}\) is the set of all achievable rate triples.

The primary goal of this paper is to characterize the achievable rate region \(\mathcal{R}\) in a way that does not involve the unbounded parameter \(n\), that is, to exhibit a single letter characterization of \(\mathcal{R}\).
VI. MAIN RESULTS

In this section we present inner and outer bounds on the achievable rate region \( \mathcal{R} \). The bounds are expressed in terms of sets of ‘auxiliary’ random variable pairs \( UV \), defined below. In these definitions we assume that \( U \) and \( V \) take values in finite alphabets \( \mathcal{U} \) and \( \mathcal{V} \) and have a well defined joint distribution with the ‘given’ random variables \( XY \). To each such pair of auxiliary random variables \( UV \) we associate a set of rates \( \mathcal{R}_{UV} \) defined by

\[
\mathcal{R}_{UV} = \{ R : \begin{align*}
    R_x &\geq I(U;X) \\
    R_y &\geq I(V;Y) \\
    R_c &\leq I(U;V) - I(U;V|X,Y). 
\end{align*}\}
\]

Next, we define two sets of random variable pairs,

\[
\mathcal{P}_{in} = \{ UV : U - X - Y, X - Y - V, U - (X,Y) - V \}.
\]

and

\[
\mathcal{P}_{out} = \{ UV : U - X - Y, X - Y - V \}.
\]

When convenient hereafter, we express the three independence constraints in \( \mathcal{P}_{in} \) as a single ‘long’ Markov chain, \( U - X - Y - V \).

Finally, we define two additional sets of rates

\[
\mathcal{R}_{in} = \{ R : R \in \mathcal{R}_{UV} \text{ for some } UV \in \mathcal{P}_{in} \}
\]

\[
\mathcal{R}_{out} = \{ R : R \in \mathcal{R}_{UV} \text{ for some } UV \in \mathcal{P}_{out} \}.
\]

Comment 6.1: Note that for rates in \( \mathcal{R}_{in} \), the long Markov constraint \( U - X - Y - V \) implies that the second term in the third inequality of \( \mathcal{R}_{UV} \) vanishes, i.e. \( I(U;V|XY) = 0 \).

Our main results are the following.

**Theorem 6.2 (Positive theorem: Inner bound):**

\( \mathcal{R}_{in} \subseteq \mathcal{R} \)

That is, every rate \( R \in \mathcal{R}_{in} \) is achievable.

**Theorem 6.3 (Negative theorem: Outer bound):**

\( \mathcal{R}_{out} \supseteq \mathcal{R} \)

That is, no rate \( R \notin \mathcal{R}_{out} \) is achievable.

The proofs appear in Appendices II and III.

Remark 6.4: If either \( X = U \) or \( Y = V \), or both, then the inner and outer bounds are identical, since in this case the extra Markov condition \( U - (X,Y) - V \) in the definition of \( \mathcal{P}_{in} \) is automatically satisfied. For example, if \( U = X \), then the condition is equivalent to \( I(U;V|XY) = I(X;V|XY) = 0 \), which is obviously true. Similar comments apply if \( U \) and \( V \) are any deterministic functions of \( X \) and \( Y \).

VII. DISCUSSION OF THE MAIN RESULTS

A. The gap between bounds

The true achievable rate region is sandwiched between the sets \( \mathcal{R}_{in} \) and \( \mathcal{R}_{out} \), i.e. \( \mathcal{R}_{in} \subseteq \mathcal{R} \subseteq \mathcal{R}_{out} \). The gap between \( \mathcal{R}_{in} \) and \( \mathcal{R}_{out} \) is due to the different independence constraints in the definitions of \( \mathcal{P}_{in} \).
and $P_{\text{out}}$: Whereas distributions in $P_{\text{in}}$ satisfy three Markov-chain constraints $U - X - Y$, $X - Y - V$, and $U - (X,Y) - V$ or, equivalently, the single ‘long chain’ constraint $U - X - Y - V$, distributions in $P_{\text{out}}$ need only satisfy the first two ‘short chain’ constraints. Hence, $R_{\text{out}}$ is the larger rate region and, in general, we expect a gap between the two regions.

B. Convexity

One manifestation of the difference between $R_{\text{out}}$ and $R_{\text{in}}$ is that $R_{\text{out}}$ is convex, while $R_{\text{in}}$ generally is not. We state this here as a lemma:

Lemma 7.1: $R_{\text{out}}$ is convex set, in the sense that all rates along the line connecting any two rates $R_1$ and $R_2$ contained in $R_{\text{out}}$ are also contained in $R_{\text{out}}$.

The convexity of $R_{\text{out}}$ is proved in Appendix III. The nonconvexity of $R_{\text{in}}$ is apparent from the examples studied in sections VIII and IX.

C. Berger’s observation and implications

At least in part, the reason for the gap can be appreciated more concretely using the following observation made by Berger when discussing the distributed source coding problem, for which the currently known inner and outer bounds on the achievable rates are separated by a similar gap [2]. Observe that the long-chain Markov constraint on $P_{\text{in}}$ implies that each corresponding joint distribution over $UV$ given $XY$ must factorize into a product of marginal distributions, $p(u,v|x,y) = p(u|x)p(v|y)$. By contrast, the less restrictive constraints on $P_{\text{out}}$ admit pairs whose joint distributions are convex mixtures of product marginals; that is, distributions of the form

$$p(uv|xy) = \sum_{q \in Q} p(q)p(u|x|q)p(v|y|q).$$

More explicitly, we can represent the set of all such auxiliary random variable pairs as follows.

Definition 7.1: Let

$$P_{\text{mix}} = \{UV : U = (UQ, Q), V = (VQ, Q)\},$$

where $Q$ is any discrete random variable with a finite alphabet $Q$ which is independent of $X$ and $Y$, and for each $q \in Q$ the pair $U_qV_q \in P_{\text{in}}$.

Clearly, there is potentially a much larger set of distributions for $UV$ pairs in $P_{\text{mix}}$ than in $P_{\text{in}}$.

However, while $P_{\text{mix}}$ is clearly contained in $P_{\text{out}}$, it is unknown whether or under what conditions $P_{\text{mix}} = P_{\text{out}}$. Further, if we define the additional rate region

$$R_{\text{mix}} = \{R : R \in R_{\text{UV}} \text{ for some } UV \in P_{\text{mix}}\},$$

and let $\mathcal{C}(R_{\text{in}})$ denote the convex hull of $R_{\text{in}}$

$$\mathcal{C}(R_{\text{in}}) = \{R : R = \theta R_1 + \tilde{\theta} R_2, \ R_1, R_2 \in R_{\text{in}}, \ 0 \leq \theta \leq 1\}$$

where $\tilde{\theta} = 1 - \theta$, then it is easy to verify that the following logical statement holds:

If $P_{\text{out}} = P_{\text{mix}}$

then $R_{\text{out}} = R_{\text{mix}} = \mathcal{C}(R_{\text{in}})$.

Thus, it is unknown whether the presence of mixture distributions in $P_{\text{out}}$ is enough to account for all of the gap between $R_{\text{in}}$ and $R_{\text{out}}$. As discussed below in subsection VII-F (1) has interesting implications for closing the gap.

D. Relationship with distributed source coding

Some interesting connections hold between the results of Tung and Berger [5], [30] for the distributed source coding (DSC) problem and our results in theorems 6.2 and 6.3. Briefly, the situation treated in the
DSC problem, diagrammed in figure 2, is as follows. Two correlated sequences, X and Y, are encoded separately as \( m = f(X) \), \( \mu = \phi(Y) \), and the decoder \( g \) must reproduce the original sequences subject to a fidelity constraint, \( (Ed_x(X, X), Ed_y(\hat{Y}, Y)) \leq D \), where \( D = (D_x, D_y) \). The problem is to characterize, for any given distortion \( D \), the set of achievable rates \( \mathcal{R}(D) \).

![Fig. 2. The distributed source coding problem.](image)

The known inner and outer bounds for the DSC problem are as follows. Let \( \mathcal{P}_{\text{in}} \) and \( \mathcal{P}_{\text{out}} \), be defined as above, and define two new sets incorporating the distortion constraint

\[
\mathcal{P}_{\text{in}}(D) = \mathcal{P}_{\text{in}} \cap \mathcal{P}_{UU}(D) \\
\mathcal{P}_{\text{out}}(D) = \mathcal{P}_{\text{out}} \cap \mathcal{P}_{UU}(D),
\]

where

\[
\mathcal{P}_{UU}(D) = \{UV : \exists \hat{X}(U, V), \hat{Y}(U, V) \text{ s.t. } (Ed_x(\hat{X}, X), Ed_y(\hat{Y}, Y)) \leq D \}.\]

Paralleling equation 1 also define the sets of rates

\[
\tilde{\mathcal{R}}_{UU} = \{R : R_x \geq I(U; X|V) \\
R_y \geq I(V; Y|U) \\
R_x + R_y \geq I(UV; XY)\}
\]

and

\[
\mathcal{R}_{\text{in}}(D) = \{R : R \in \tilde{\mathcal{R}}_{UU} \text{ for some } UV \in \mathcal{P}_{\text{in}}(D)\} \\
\mathcal{R}_{\text{out}}(D) = \{R : R \in \tilde{\mathcal{R}}_{UU} \text{ for some } UV \in \mathcal{P}_{\text{out}}(D)\}.
\]

Then the Berger-Tung bounds for the DSC problem can be expressed as \( \mathcal{R}_{\text{in}}(D) \subseteq \mathcal{R}(D) \), and \( \mathcal{R}_{\text{out}}(D) \supseteq \mathcal{R}(D) \).

With the results presented in this way, the formal similarities between our pattern recognition problem and the DSC problem are obvious. Additionally, ignoring the distortion constraints for the moment, the pattern recognition problem can be thought of as a kind of generalization of the DSC problem, with the added complication that the ‘decoder’ receives not one sequence \( X \) but \( M_x = 2^{nR_x} \) such sequences, and must first determine which is the appropriate one with which to jointly decode the second received sequence \( Y \). This extra discrimination evidently requires extra information to be included at the encoders. This ‘rate excess’ is the difference between the minimum encoding rates required for the DSC and pattern recognition problems. Using the the short-chain Markov constraints \( U \rightarrow X \rightarrow Y \) and \( X \rightarrow Y \rightarrow V \), the rate excess for the \( X \) encoder is

\[
I(X; U) - I(X; U|V) = I(X; U) - I(XY; U|V) \\
eq I(X; U) - [I(XY; UV) - I(XY; V)] \\
eq I(X; U) + I(Y; V) - I(XY; UV) \\
eq I(U; V) - I(U; V|XY)
\]

and, by symmetry, at the \( Y \) encoder the excess required rate is

\[
I(Y; V) - I(Y; V|U) = I(U; V) - I(U; V|XY).
\]

Thus, the excess rate required at either terminal is directly related to the maximum number of patterns.
that must be discriminated, \( M_c = 2^{nR_c} \), \( R_c = I(U;V) - I(U;V|XY) \).

**E. Extension of the inner bound**

The following results provide a way to reduce the gap between \( R_{in} \) and \( R_{out} \) ‘from below,’ by improving on the inner bound.

**Theorem 7.2:** If the point \( \mathbf{R} = (R_c, R_x, R_y) \) is achievable, then for any \( 0 < \theta \leq 1 \), the point \( \mathbf{R}' = \theta \mathbf{R} \) is achievable.

**Corollary 7.2:** Let

\[
\mathcal{R}' = \{ \mathbf{R} : \mathbf{R} = \theta \mathbf{R}', \, \mathbf{R}' \in \mathcal{R}_{in}, \, 0 \leq \theta \leq 1 \}.
\]

Then \( \mathcal{R}' \subseteq \mathcal{R} \).

The theorem and corollary are proved in Appendix IV. As discussed in the next subsection, this extension of the inner bound may in some cases allow us to close the gap, i.e. in cases where the expression for the convex hull of \( \mathcal{R}_{in} \) simplifies such that \( \text{Co}(\mathcal{R}_{in}) = \mathcal{R}' \). Specific examples where this appears to be the case include the binary and Gaussian examples discussed in sections VIII and IX.

**F. On closing the gap**

What additional results would be needed to determine the true achievable rate region \( \mathcal{R} \)? To explore this question, consider the following hypothetical statements and their implications.

(a) \( P_{out} = P_{mix} \)
(b) \( \text{Co}(\mathcal{R}_{in}) = \mathcal{R}' \)
(c) \( \mathcal{R} \) is convex
(d) \( \mathcal{R} = \mathcal{R}_{out} \)

We emphasize that none of these statements have been proven. Nevertheless, the following Lemmas, stated in ‘if-then’ form, are true.

**Lemma 7.3:** (a),(b) \( \Rightarrow \) (d)

**Lemma 7.4:** (a),(c) \( \Rightarrow \) (d)

The proof of Lemma 7.3 is as follows. Assuming \( \text{Co}(\mathcal{R}_{in}) = \mathcal{R}' \), then by corollary 7.2 the convex hull is achievable, \( \text{Co}(\mathcal{R}_{in}) \subseteq \mathcal{R} \). But by (1) our assumption (a) implies \( \mathcal{R}_{out} = \mathcal{R}_{mix} = \text{Co}(\mathcal{R}_{in}) \), hence \( \mathcal{R}_{out} \subseteq \mathcal{R} \). Combining this with theorem 6.3 we have \( \mathcal{R}_{out} \subseteq \mathcal{R} \) and \( \mathcal{R}_{out} \supseteq \mathcal{R} \), or \( \mathcal{R} = \mathcal{R}_{out} \).

Lemmas 7.4 follows from straightforward timesharing arguments, as shown in Appendix V.

Both Lemmas 7.3 and 7.4 suggest potential routes for establishing the true achievable rate region \( \mathcal{R} \) by expanding the inner bound \( \mathcal{R}_{in} \). While we expect that premises (a) and (b) hold in certain cases, we suspect that they are not true in general; we have no current guess about (c). On the other hand, if \( \mathcal{R}_{out} \) is larger than \( \mathcal{R}_{mix} \), then it may still be possible to establish the true achievable rate region \( \mathcal{R} \) by tightening the outer bound, possibly down to \( \mathcal{R}_{mix} \). Thus \( \mathcal{R}_{out} \) and \( \mathcal{R}_{mix} \) are presently the most promising candidates for \( \mathcal{R} \).

**G. Degenerate cases**

We now briefly examine the degenerate cases where either \( X = U \), or \( Y = V \), or both. These simple cases have clear interpretations and are thus useful for building intuition about the general results of theorems 6.2 and 6.3. Note that in these cases \( I(U;V|XY) = 0 \), hence the third inequality in the definition of \( \mathcal{R}_{UV} \) simplifies to \( R_c \leq I(U;V) \). Additionally, in these cases there is no gap, i.e. the inner and outer bounds are equal; see Remark 6.4.

**Unlimited senses and memory.** First, consider a system in which the budgets for memory and sensory representations are unrestricted, i.e. no compression is required. In this case, we can effectively treat the
memories and sensory representations as if they were veridical; i.e. we can set \( U = X \) and \( V = Y \). The theorem constraints then become \( R_c \geq I(X;X) = H(X), \) \( R_c \geq I(Y;Y) = H(Y) \), and
\[
R_c \leq I(U;V) = I(X;Y).
\] (2)

This result indicates that, in the absence of compression, the recognition problem is formally equivalent to the following classical communication problem: Transmit one of \( M_c = 2^nR_c \) possible messages (patterns) to a receiver (the recognition module) \([28]\). In this case, the objects can be thought of as codewords which are stored without compression for direct comparison with the sensory data. This is the setup of the random coding proof of Shannon’s channel coding theorem, which gives the rates at which reliable communication is possible as those below the mutual information between the source (analogous to the memory here) and the received signals, \( I(X;Y) \) \([?], [?] \). This is exactly the condition expressed by \( \Phi \). The condition specifies an upper bound on the number of objects the system may be trained to recognize through the relation \( M_c = 2^nR_c \).

**Unlimited memory, limited senses.** Next, suppose that memory is effectively unlimited, so that we can put \( U = X \), but sensory data may be compressed. In this case, we can readily rewrite the condition on \( R_c \) as
\[
R_c \leq I(X;Y) - I(X;Y|V).
\] (3)

We check the extreme cases: If \( Y \) is fully informative about \( V \), \( Y = \phi^{-1}(V) \), then \( I(X;Y|V) = H(Y|V) - H(Y|X,V) = 0 \), and we recover the case discussed above. For intermediate cases where \( V \) is partially informative, then the effect of \( V \) is to degrade the achievable performance of the system below that possible with ‘perfect senses,’ and the reduction incurred is \( I(X;Y|V) \). In the extreme case that \( V \) is utterly uninformative (i.e. independent of \( Y \)), then \( I(X;Y|V) = I(X;Y) \), and we get \( R_c = 0 \), or \( M_c \leq 2^nR_c = 1 \), hence the system is useless.

**Limited memory, unlimited senses.** In the case of limited memory but unrestricted resources for sensory data representation, we get an expression symmetric with the previous case:
\[
R_c \leq I(X;Y) - I(X;Y|U).
\] (4)

As before, if the memory is perfect \((U = X)\), we get \( I(X;Y|U) = I(X;Y|X) = 0 \), recovering the channel coding constraint \( R_c \leq I(X;Y) \); assuming useless memories yields \( R_c \leq I(X;Y) - I(X;Y) = 0 \); and intermediate cases place the system between these extremes.

**H. Rate region surfaces**

An equivalent way to characterize the sets \( \mathcal{R}, \mathcal{R}_{in} \) and \( \mathcal{R}_{out} \) that will be useful in sections \[VIII] and \[IX] is to specify the boundary or *surface* of each region. For \( \mathcal{R} \), the surface is
\[
\sigma(r_x,r_y) = \max_{R \in \mathcal{C}(r_x,r_y)} R_c, \quad \text{where}
\]
\[
\mathcal{C}(r_x,r_y) = \{ \mathbf{R} : \mathbf{R} \in \mathcal{R}, \ R_x = r_x, \ R_y = r_y \}.
\]

Similarly, by direct extension of theorems \[62] and \[63] the surfaces of \( \mathcal{R}_{in} \) and \( \mathcal{R}_{out} \) are specified by
\[
\sigma_{in}(r_x,r_y) = \max_{UV \in \mathcal{C}_{in}(r_x,r_y)} I(U;V) - I(U;V|XY) \] (5)
\[
\sigma_{out}(r_x,r_y) = \max_{UV \in \mathcal{C}_{out}(r_x,r_y)} I(U;V) - I(U;V|XY),
\]
where
\[
\mathcal{C}_{in}(r_x,r_y) = \{ UV \in \mathcal{P}_{in} : r_x \geq I(U;X), \ r_y \geq I(V;Y) \}
\]
\[
\mathcal{C}_{out}(r_x,r_y) = \{ UV \in \mathcal{P}_{out} : r_x \geq I(U;X), \ r_y \geq I(V;Y) \}.\]
A useful alternative form comes from rewriting the right hand side of \(5\) as
\[
I(U;V) - I(U;V|XY) = I(U;V) - H(U|XY) - H(V|XY) + H(UV|XY)
\]
\[
= I(U;V) - H(U|X) - H(V|Y) + H(UV|XY)
\]
\[
= I(X;U) + I(Y;V) - I(XY;UV).
\]
(6)

The second line follows from the Markov constraints \(U - X - Y\) and \(X - Y - V\). Hence,
\[
r^*_s(r_x, r_y) = \max I(X;U) + I(Y;V) - I(XY;UV)
\]
(7)

where the subscript * stands for in or out} and the maximization is over \(C_{in}(r_x, r_y)\) or \(C_{out}(r_x, r_y)\), respectively.

In what follows we seek explicit formulas for \(r_{in}(r_x, r_y)\) and \(r_{out}(r_x, r_y)\), which do not involve the optimization over the sets \(C_{in}(r_x, r_y)\) and \(C_{out}(r_x, r_y)\).

VIII. Binary case

In this section we study a simple case in which the alphabets for the training patterns and sensory data are binary, \(X = Y = \{0, 1\}\). The training patterns \(X = (X_1, \ldots, X_n)\) are generated by \(n\)-independent drawings from a uniform Bernoulli distribution, \(X \sim B(1/2)\). Observations \(Y = (Y_1, \ldots, Y_n)\) are outputs of a binary symmetric channel with crossover probability \(q\)
\[
p(y|x) = \left( \begin{array}{cc} \bar{q} & q \\ q & \bar{q} \end{array} \right),
\]
where \(\bar{q} = 1 - q\). Equivalently, we can represent \(Y\) as \(Y = X \oplus W\), where \(W \sim B(q)\) and is independent of \(X\).

We now propose explicit formulas for \(r_{in}(r_x, r_y)\) and \(r_{out}(r_x, r_y)\) in this binary case. Our formulas involve the following two functions. First, define
\[
g(r_x, r_y) = 1 - h(q * q_x * q_y),
\]
where \(q_x\) and \(q_y\) are specified implicitly by
\[
r_x = 1 - h(q_x) \\
r_y = 1 - h(q_y);
\]
\(h(\cdot)\) is the binary entropy function \(h(x) = -x \log(x) - (1-x) \log(1-x)\); and \(q_x, q_y \in [0, 1/2]\) to ensure that \(h(\cdot)\) is invertible. Next, let \(g^*(r_x, r_y)\) denote the upper concave envelope of \(g(r_x, r_y)\),
\[
g^*(r_x, r_y) = \sup \theta g(r_{x1}, r_{y1}) + \bar{\theta} g(r_{x2}, r_{y2}),
\]
where \(\bar{\theta} = 1 - \theta\). The supremum is over all combinations \((\theta, r_{x1}, r_{y1}, r_{x2}, r_{y2})\) such that
\[
(r_x, r_y) = \theta(r_{x1}, r_{y1}) + \bar{\theta}(r_{x2}, r_{y2}),
\]
and each variable in the optimization is restricted to the unit interval \([0, 1]\). As explained in Appendix VII in both the binary case and the corresponding Gaussian case considered in the next section, the expression for the convex hull of the inner bound simplifies to
\[
g^*(r_x, r_y) = \sup \theta g(r'_x, r'_y),
\]
and the supremum is over all combinations \((\theta, r'_x, r'_y)\) such that
\[
(r_x, r_y) = \theta(r'_x, r'_y).
\]

Conjecture 8.1: In the binary case the surfaces of \(R_{in}\) and \(R_{out}\) are
\[
r_{in}(r_x, r_y) = g(r_x, r_y) \\
r_{out}(r_x, r_y) = g^*(r_x, r_y).
\]
From Theorem 7.2, \( g^*(r_x, r_y) \) is in fact achievable. Thus, if the conjecture on the outer bound is true, then there is no gap between the inner and outer bounds, and \( g^*(r_x, r_y) \) defines the achievable rate region. Figure 3 shows the inner and outer bounds and their difference.

To establish these conjectures we must prove both the ‘forward’ inequalities \( r_{in} \geq g, r_{out} \geq g^* \), and the ‘backward’ inequalities \( r_{in} \leq g, r_{out} \leq g^* \). The backward inequalities remain to be proven, whereas the forward inequalities can be proven by relatively straightforward constructions, as we now show.

**Proof:** \( (r_{in}(r_x, r_y) \geq g(r_x, r_y)) \) Let \( W_x \sim B(q_x) \), \( W_y \sim B(q_y) \) be binary random variables independent of \( X \) and \( Y \), and define

\[
U = X \oplus W_x \\
V = Y \oplus W_y.
\]

The pair \( UV \) is obviously in \( P_{in} \). Furthermore,

\[
I(X; U) = H(X) - H(X|U) \\
= 1 - H(U \oplus W_x|U) \\
= 1 - h(q_x),
\]

\[
I(V; Y) = H(Y) - H(Y|V) \\
= 1 - H(V \oplus W_y|V) \\
= 1 - h(q_y),
\]

\[
I(U; V) = H(V) - H(V|U) \\
= 1 - H(U \oplus W_x \oplus W \oplus W_y|U) \\
= 1 - h(q_x \ast q \ast q_y).
\]
Setting \( r_x = I(U; X) = 1 - h(q_x), \) and \( r_y = I(Y; V) = 1 - h(q_y) \), we have \( UV \in \mathcal{P}_{in} \) and \( UV \in \mathcal{C}(r_x, r_y) \). Hence,

\[
r_{in}(r_x, r_y) = \max_{UV \in \mathcal{C}(r_x, r_y)} I(U; V) \geq 1 - h(q * q_x * q_y) = g(r_x, r_y). \]

\[ \blacksquare \]

**Proof:** \( (r_{out}(r_x, r_y) \geq g^*(r_x, r_y)) \) Using the same construction as in the forward proof for the inner bound formula, define two pairs of random variables \((U_1V_1), (U_2V_2) \in \mathcal{P}_{in} \subset \mathcal{P}_{out}\) such that

\[
\begin{align*}
  r_{x_1} &= I(U_1; X) = 1 - h(q_{x_1}), \\
  r_{y_1} &= I(V_1; Y) = 1 - h(q_{y_1}), \\
  r_{x_2} &= I(U_2; X) = 1 - h(q_{x_2}), \\
  r_{y_2} &= I(V_2; Y) = 1 - h(q_{y_2}).
\end{align*}
\]

Let \( (r_x, r_y) = \theta(r_{x_1}, r_{y_1}) + \bar{\theta}(r_{x_2}, r_{y_2}), \theta \in [0, 1] \). Since \( r_{out}(r_x, r_y) \) is convex, we have

\[
\begin{align*}
  r_{out}(r_x, r_y) &\geq \theta r_{out}(r_{x_1}, r_{y_1}) + \bar{\theta} r_{out}(r_{x_2}, r_{y_2}) \\
  &\geq \theta g(r_{x_1}, r_{y_1}) + \bar{\theta} g(r_{x_2}, r_{y_2}).
\end{align*}
\]

The inequalities above hold for all valid choices of \( \theta, r_{x_1}, r_{x_2}, r_{y_1}, r_{y_2} \), hence \( r_{out}(r_x, r_y) \geq g^*(r_x, r_y) \), as desired. \[ \blacksquare \]

**IX. Gaussian Case**

We now consider a Gaussian version of our problem. Let \( X \) and \( Y \) be zero-mean Gaussian random variables with correlation coefficient \( \rho_{xy} \). In parallel with our discussion of the binary case, we propose explicit formulas for the surfaces of \( R_{in} \) and \( R_{out} \) for the Gaussian case, this time in terms of the following two functions. In both formulas, let

\[
\begin{align*}
  r_x &= -\frac{1}{2} \log(1 - \rho_{xu}^2) \\
  r_y &= -\frac{1}{2} \log(1 - \rho_{yu}^2).
\end{align*}
\]

Note that these expressions determine the correlation coefficients \( \rho_{xu} \) and \( \rho_{yu} \). Define

\[
G(r_x, r_y) = -\frac{1}{2} \log(1 - \rho_{xy}^2 \rho_{xu}^2 \rho_{yu}^2). \tag{8}
\]

and

\[
G^*(r_x, r_y) = r_x + r_y + \frac{1}{2} \log(1 + \frac{2 \rho_{xy}^2 \gamma - \beta}{1 - \rho^2}), \tag{9}
\]

where

\[
\begin{align*}
  \gamma &= \rho_{xy} \rho_{xu} \rho_{yu}, \\
  \beta &= \rho_{xu}^2 + \rho_{yu}^2 - (1 - \rho_{xy}^2) \rho_{xu}^2 \rho_{yu}^2, \\
  \rho &= \frac{\beta}{2 \gamma} - \sqrt{\left(\frac{\beta}{2 \gamma}\right)^2 - 1}.
\end{align*} \tag{10}
\]

**Conjecture 9.1:** In the Gaussian case the surfaces of \( R_{in} \) and \( R_{out} \) are

\[
\begin{align*}
  r_{in}(r_x, r_y) &= G(r_x, r_y) \\
  r_{out}(r_x, r_y) &= G^*(r_x, r_y).
\end{align*}
\]

Figure 5 shows plots of the inner and outer bounds and their difference, as well as the difference between the outer bound and the convex hull of the inner bound. Interestingly, unlike the binary case, for the Gaussian case the outer bound is not equal to the convex hull of the inner bound.
The following proof relies on some basic properties of the mutual information between Gaussian random variables, given as Lemmas in Appendix [VIII]

![Diagram](image)

**Fig. 4.** Surfaces of the Gaussian inner bound $z = r_{in}$ (a) and outer bound $z = r_{out}$ (b) regions; and differences between the outer bound and inner bounds $z = r_{in} - r_{out}$ (c) and between the outer bound and the convex hull of the inner bound $z = r_{out} - H(r_{in})$ (d). In these plots $\rho_{xy} = 0.8$.

In the analysis that follows, we assume that the true distributions are Gaussian. Under this assumption, we solve the inner and outer bounds. If the true distributions are Gaussian, then our conjecture is true.

**Proof:** ($r_{in}(r_x, r_y) = G(r_x, r_y)$) As noted in Appendix [VIII] mutual informations between jointly Gaussian random variables are completely determined by their correlation coefficients. For a length-4 Markov chain $U - X - Y - V$ of jointly Gaussian random variables $I(U; V|XY) = 0$ and, applying Lemma 8.2 from Appendix [VIII] we have $\rho_{uv} = \rho_{xu}\rho_{xy}\rho_{yv}$, hence

$$I(U; V) - I(U; V|XY) = -\frac{1}{2} \log(1 - \rho_{xu}^2 \rho_{xy}^2 \rho_{yv}^2).$$

This mutual information is maximized when the constraints $I(X; U) \leq r_x$, $I(Y; V) \leq r_y$ are satisfied with equality, hence when $\rho_{xu}$ and $\rho_{yv}$ satisfy $r_x = -\frac{1}{2} \log(1 - \rho_{xu}^2)$ and $r_y = -\frac{1}{2} \log(1 - \rho_{yv}^2)$. This proves the theorem.

The following proof for the surface of the outer bound region uses the form of $r_{out}(r_x, r_y)$ in (7). We assume that the constraints on $r_x$ and $r_y$ are satisfied with equality. In this case, the optimization problem reduces to that of minimizing the $I(XY; UV)$ subject to the length-3 Markov constraints $U - X - Y$, $X - Y - V$. **Proof:** ($r_{out}(r_x, r_y) = G^*(r_x, r_y)$) Using Lemma 8.3 from appendix [VIII] we have

$$C_{xy,uv} = \begin{bmatrix} \rho_{xu} & \rho_{xy} \\ \rho_{yu} & \rho_{yv} \end{bmatrix} = \begin{bmatrix} 1 & \rho_{xy} \\ \rho_{xy} & 1 \end{bmatrix} \begin{bmatrix} \rho_{xu} & 0 \\ 0 & \rho_{xu} \end{bmatrix}.$$
The left hand matrix in this decomposition is $C_{xy,xy}$, denoted hereafter simply as $C$, and we denote the right hand matrix by $D$. Then applying Lemma 8.1 from appendix VIII yields

$$I(XY;UV) = \frac{1}{2} \log |C| - \frac{1}{2} \log |C - C_{xy,uv}C^{-1}_{uv,uv}C_{uv,yx}|$$

$$= \frac{1}{2} \log |C| - \frac{1}{2} \log |C - CDC^{-1}_{uv,uv}DC|$$

$$= -\frac{1}{2} \log |C| - \frac{1}{2} \log |C - DC^{-1} D|.$$  

Substituting for the $2 \times 2$ matrices in this last expression and rearranging terms yields

$$I(XY;UV) = -\frac{1}{2} \log[1 + 2\rho_{uv}^2 \gamma - \beta],$$

where $\gamma$ and $\beta$ are defined in (10).

By assumption, $\rho_{xu}$ and $\rho_{yu}$ are being held fixed, so we are optimizing $I(XY;UV)$ only with respect to $\rho_{uv}$. Setting $\partial I(XY;UV)/\partial \rho_{uv} = 0$ and solving, we obtain that, if $\beta > 2\gamma > 0$, then the maximum is achieved at $\rho_{uv}^* = \rho$, where $\rho$ is defined in (10).

To complete the proof we must show that $\beta > 2\gamma > 0$. Noting that $\beta, \gamma > 0$ and substituting, the desired inequality becomes

$$\rho_{xu}^2 + \rho_{yu}^2 - \rho_{xz}^2 \rho_{yz}^2 > 2\rho_{xy}\rho_{xu}\rho_{yu} - \rho_{xy}^2 \rho_{xz}^2 \rho_{yz}^2.$$  

Subtracting 1 from each side and factoring yields the equivalent inequality

$$-(1 - \rho_{xu}^2)(1 - \rho_{yu}^2) > -(1 - \rho_{xy}\rho_{xu}\rho_{yu})^2.$$  

To show that this holds for all $\rho_{xy}$, note that the maximum of the right hand side is achieved by $\rho_{xy} = 1$, so that the inequality becomes

$$(1 - \rho_{xu}^2)(1 - \rho_{yu}^2) - (1 - \rho_{xy}\rho_{xu}\rho_{yu})^2 < 0.$$  

This inequality holds, since

$$\begin{align*}
(1 - \rho_{xu}^2)(1 - \rho_{yu}^2) - (1 - \rho_{xy}\rho_{xu}\rho_{yu})^2 &= 1 - \rho_{yu}^2 - \rho_{xu}^2 + \rho_{xz}^2 \rho_{yz}^2 - (1 - 2\rho_{xy}\rho_{xu}\rho_{yu} + \rho_{xz}^2 \rho_{yz}^2) \\
&= -\rho_{xz}^2 \rho_{yz}^2 + 2\rho_{xy}\rho_{xu}\rho_{yu} \\
&= (\rho_{xu} - \rho_{yu})(\rho_{yu} - \rho_{xz}) \\
&= -(\rho_{xu} - \rho_{yu})^2 < 0.
\end{align*}$$

\[\blacksquare\]

**APPENDIX I**

**PROOF OF THE INNER BOUND**

In this section we prove the inner bound $R_{in} \subseteq R$, theorem 6.2. The proof relies on standard random coding arguments and properties of strongly jointly typical sets [?]. Given a joint distribution $p(xyuuv)$, the strongly jointly $\delta$-typical set is defined by

$$T_{\delta} = \left\{xyuv : \frac{N(xyuv|xuuv)}{n} - p(xyuv) \leq \delta \forall xuuv \in XVU \right\},$$

where $N(xyuv|xuuv)$ is the number of times the symbol combination $xyuv$ occurs in $xyuv$. Likewise, we write e.g. $T_{\delta}^X, T_{\delta}^{XY}, T_{\delta}^{XUV}$ for singles, pairs, and triples. We will also use conditionally strongly jointly $\delta$-typical sets, for example

$$\mathcal{T}_{\delta} = \{u : (xu) \in T_{\delta}^X\}.$$
The subscripts are omitted when context allows. We will also need the fact that for any positive numbers \( \delta, \epsilon > 0 \), fixed vector \( x \), and large enough \( n \),
\[
2^{-n[I(X;Y)+\epsilon]} \leq Pr(xY \in T_{XY}^\delta) \leq 2^{-n[I(X;Y)-\epsilon]}.
\]  

**Proof:** To begin, let \( R = (R_c, R_x, R_y) \) be any rate triple in \( R_{in} \), and let \( \epsilon > 0 \) be any positive constant. Then there exists a pair of random variables \( UV \in P_{in} \) such that \( R \in R_{UV} \). We wish to prove \( R \in R \). To this end, we will use \( UV \) to construct an \( (M_x, M_y, M_c, n) \) pattern recognition code \((f, \phi, g)\), with \( M_c \geq 2^{nR_c}, M_x \leq 2^{nR_x}, \) and \( M_y \leq 2^{nR_y} \), such that \( P_e^n \leq \epsilon \) for a sufficiently large integer \( n \).

For concreteness, we will suppose that the mappings \( f, \phi \) and \( g \) are implemented in distinct memory, sensory, and recognition ‘modules,’ respectively, each of which ‘knows’ the joint distribution \( p(x,yuv) \).

**Random codebook generation.** To serve as codewords, select \( M_x \) length-\( n \) vectors by sampling with replacement from a uniform distribution over the set \( T_{XY}^\delta \). Assign each codeword a unique index \( i \in M_x \), where \( M_x = \{1,2,\ldots,M_x\} \). Denote the resulting codebook
\[
B_u = \{u(1), u(2), \ldots, u(M_x)\},
\]
where the \( u(i) \) are the indexed codewords.

Similarly, for the sensory module generate \( M_y \) length-\( n \) codewords by sampling with replacement from a uniform distribution on \( T_{Y}^\delta \). Assign each codeword a unique index \( j \in M_y \), where \( M_y = \{1,2,\ldots,M_y\} \). Denote the resulting codebook
\[
B_v = \{v(1), v(2), \ldots, v(M_y)\},
\]
where the \( v(j) \) are the indexed codewords.

Provide copies of both codebooks \( B_u \) and \( B_v \) to the recognition module.

**Memory encoding rule** \( f \). Let \( C_x = \{(X(1),1), (X(2),2), \ldots, (X(M_c),M_c)\} \) be the set of labeled random patterns to be encoded into memory during the training phase. We define the memory encoder \( f \) in terms of the following procedure. Given a labeled pattern \((x(w), w)\), the memory module searches through the memory codebook \( B_u \) for a codeword \( u \) such that \((x(w), u) \in T_{XY}^\delta \). If such a codeword is found we denote it by \( u(w) \), and denote its index in the codebook \( B_u \) by \( m(w) \). If \( B_u \) has no codeword that is strongly jointly \( \delta \)-typical with \( x(w) \), an error is declared and the label \( w \) is associated with the first codeword of \( B_u \). Denoting the event that the above procedure fails by \( E_1 \) and its complement by \( E_1^c \), let
\[
f(x(w), w) = \begin{cases} (1, u) & \text{if } E_1 \text{ occurs;} \\ (m(w), w) & \text{if } E_1^c \text{ occurs.} \end{cases}
\]

An error is also declared if the above procedure results in assigning more than one pattern label to the same memory codeword; denote this second error event \( E_2 \). The training phase corresponds formally to applying \( f \) to all \( M_c \) patterns in \( C_x \), inducing the set
\[
C_u = f(C_x) = \{(m(1),1), \ldots, (m(M_c),M_c)\}.
\]

Note that not all of the codewords in \( B_u \) have been used in the encoding procedure. Likewise, in the decoding algorithm described below, we need only consider the subset of codewords \( u \in B_u \) whose indices in \( B_u \) also appear in \( C_u \). We denote the set of indices for these ‘active’ codewords \( L = L(C_u) = \{m(1), m(2), \ldots, m(M_c)\} \).

After training, reveal the memory codebook \( B_u \), the compressed data \( C_u \), and the mapping \( f \) to the recognition module.

**Sensory encoding rule** \( \phi \). The sensory encoding rule \( \phi \) is defined as follows. Let \( y \) be an input to the sensory module during the testing phase. The sensory module searches sequentially through the sensory codebook \( B_v \) for a codeword \( v \) such that \( yv \in T_{Y}^\delta \). If the search succeeds, denote the found codeword by \( v(y) \) and denote its index by \( \mu(y) \). If the search fails, declare an error, and let the sensory encoder
output be $\mu = 1$. Letting $E_3$ be the error event and $E_3^c$ its complement, let

$$\phi(y) = \begin{cases} 1, & \text{if } E_3 \text{ occurs;} \\ \mu(y), & \text{if } E_3^c \text{ occurs.} \end{cases}$$

Reveal the sensory codebook $B_n$ and the mapping $\phi$ to the recognition module.

**Classifier: $g_1$.** We next specify $g_1$, the first part of the classifier $g = g_1 \circ g_2$. Upon receiving the index $\mu = \mu(y)$ from the sensory module, the recognition module retrieves the $\mu$-th codeword $v(y)$ from the sensory codebook $B_n$, then searches the ‘active’ portion of the memory codebook $B_n(L) \subset B_n$ for a codeword $u$ such that $uv(y) \in T_{UV}^\delta$. If such a $u$ exists and is unique, denote it by $\hat{u} = \hat{u}(\mu)$ and its index in the codebook $B_n$ by $\hat{m} = \hat{m}(\hat{u})$. If no such $u$ exists, declare an error, $E_4$; if more than one such $u$ exists, declare an error $E_5$; and in case of either $E_4$ or $E_5$ let $\hat{m} = 1$. Thus, set

$$g_1(\mu) = \begin{cases} 1, & \text{if either } E_4 \text{ or } E_5 \text{ or both occur;} \\ \hat{m}, & \text{if both } E_4^c \text{ and } E_5^c \text{ occur.} \end{cases}$$

**Classifier: $g_2$.** After determining the codeword index $\hat{m} = g_1(\mu)$, the recognition module searches the set of stored data $C_n$ for a pair $(m, w)$ whose first entry is $m = \hat{m}$ and retrieves the associated class label. Note that if none of the errors $E_i, i = 1, \ldots, 5$ occurs, there pair $(\hat{m}, \hat{w})$ is in fact unique. If there is more than one such pair, then to ensure uniqueness choose the first. Denoting the retrieved label by $\hat{w}$, let

$$g_2(\hat{m}, C_n) = \hat{w}.$$ 

**Analysis of the probability of error**

We now show that the probability of recognition error using the code $(f, \phi, g)_n$ developed above vanishes as $n \to \infty$. The following list qualitatively describes all possible sources of error using the code $(f, \phi, g)_n$:

- $E_0$: The sensory data is too ambiguous—i.e. it is not strongly jointly typical with the training pattern;
- $E_1$: The training pattern is unencodable;
- $E_2$: Two or more training patterns are associated with same memory codeword;
- $E_3$: The sensory data is unencodable;
- $E_4$: The encoded sensory data matches no codeword in memory;
- $E_5$: The encoded sensory data matches one or more incorrect memory codewords.

More formally, the possible errors are

$$E_0 = \{ (x(w), y) \notin T_{XY}^\delta \}$$

$$E_1 = E_0^c \cap \left\{ \bigcap_{i=1}^{M_\alpha} \{ (x, u(i)) \notin T_{XU}^\delta \} \right\}$$

$$E_2 = \left( \bigcap_{n=0}^{1} E_n^c \right) \cap \left\{ \bigcup_{x(w')} \{ (x(w'), u(w)) \in T_{XY}^\delta \} \right\}$$

$$E_3 = E_0^c \cap \left\{ \bigcap_{i=1}^{M_\alpha} \{ (y, v(i)) \notin T_{YV}^\delta \} \right\}$$

$$E_4 = \left( \bigcap_{n=0}^{3} E_n^c \right) \cap \{ (x(w), u(w), y, v(y)) \notin T_{XYZY}^\delta \}$$

$$E_5 = \left( \bigcap_{n=0}^{4} E_n^c \right) \cap \left\{ \bigcup_{u(m') \in B_n} \{ (u(m'), v(y)) \in T_{UY}^\delta \} \right\} ,$$

where in the last line the set $L^*$ includes all indices in $L$ except $m(w)$, i.e. $L^* = L \setminus m(w)$. The average
total probability of error is upper-bounded as
\[
P^n_e \leq P \left\{ \bigcup_{\ell=1}^{5} \right\} \leq \sum_{\ell=0}^{5} P(E_{\ell}).
\]

Hence to show \( P^n_e \leq \epsilon \) it suffices to show that each term in the sum vanishes as \( n \to \infty \).

**Encoding Errors**

**Error event \( E_0 \):** By the Asymptotic Equipartition Property, \( Pr(E_0) \to 0 \) [?].

**Error event \( E_1 \):** For \( E_1 \), we use the well known fact that if \( R_x \geq I(X;U) \), then the \( M_x = 2^{nR_x} \) codewords in \( B_x \) are sufficient to cover the pattern source \( X \). Explicitly, let \( R_x = I(X;U) + \alpha \), for any \( \alpha > 0 \). Then for any \( \epsilon > 0 \) and sufficiently large \( n \),
\[
Pr(E_1) = \sum_{xy \in T^\delta} Pr(E_1|x)Pr(x)Pr(y|x) \\
\leq \sum_{x \in T^\delta} Pr(E_1|x)Pr(x) \\
= \sum_{x \in T^\delta} \{1 - Pr(xU \in T^\delta|x)\}M_x Pr(x) \\
\leq \{1 - 2^{-n[I(X;U) + \alpha/2]}\}M_x \\
\leq 2^{-M_x2^{-n[I(X;U) + \alpha/2]}} \\
\leq 2^{-2^{\alpha n/2}} \\
\leq \epsilon,
\]

where (a) is due to the property of strongly jointly typical sets in equation 11 and in (b) we have used \((1 - \alpha)^3 \leq 2^{-\alpha^3}\). Hence, \( Pr(E_1) \to 0 \).

**Error event \( E_2 \):** Conditioned on \( E_0^c \cap E_1^c \), we have \( u(w) \in T^\delta_0 \). The sequences \( X(u') \in C_x \), \( u' \neq w \) are generated independently of \( u(w) \). Thus
\[
P(E_2) = \sum_{X(u') \in C_x, u' \neq w} Pr \left( X(u') \in T^\delta_0 | u(w) \in T^\delta_0 \right) \\
\leq |C_x|2^{-nI(X;U) + ne} \\
\leq 2^{nR_x2^{-nI(X;U) + ne}} \\
\leq \epsilon
\]
for large enough \( n \), since \( R_x \leq I(U;V) \leq I(X;U) \) under the Markov assumption \( U - X - Y - V \). Hence, \( P(E_2) \to 0 \).

**Error event \( E_3 \):** By a covering argument analogous to the one used in the analysis of event \( E_1 \), having \( M_y \geq 2^{nI(Y;V)} \) codewords in \( C_u \) is sufficient to ensure \( P(E_3) \to 0 \).

**Decoding errors**

**Error event \( E_4 \):** To analyze the probability of event \( E_4 \), we invoke the following uniform version of the well-known Markov Lemma [5], [17], [18], [30].

**Lemma 1.1:** Let \( A - B - C \) be a Markov chain; let \( ab \in T^\delta_{AB} \); let \( C \) be chosen from a uniform distribution over \( T^\delta_{BC} \); and let \( \epsilon > 0 \) be any positive constant. Then \( Pr(abC \notin T^\delta_{ABC}) \leq \epsilon \) for \( n \) sufficiently large.

To bound the probability of event \( E_4 \), we condition on \( \cap_{i=0}^{n-1} E_i^c \) and apply the Markov lemma twice in succession to establish the following two claims:

i) \[
Pr(xyV(y) \notin T^\delta_{XYV} | xy \in T^\delta, V(y) \in T^\delta_y) \leq \epsilon
\]

ii) \[
Pr(U(w)xyV(y) \notin T^\delta_{XYUV} | xyV(y) \in T^\delta, U(w) \in T^\delta_x) \leq \epsilon
\]
To prove (i), note that the conditions of the Markov Lemma can be satisfied making the following substitutions in the Lemma: \((a, b, C) \rightarrow (x, y, V(y))\). Similarly, to prove (ii), put \((a, b, C) \rightarrow (yv, x, U(w))\). Combining (i) and (ii), we conclude that \(\Pr(E_4) \rightarrow 0\).

**Error event \(E_5\):**

Given \(\bigcap_{n=0}^{3} E_n^c\), we have \(v(y) \in T^c_V\). The sequences \(U(n') \in B_u, m' \in L^* = L \setminus m(w)\) were generated independently of \(v(y)\). Thus

\[
P(E_5) = \sum_{U(n') \in B_u, m' \in L^*} \Pr(U(n')v(y) \in T^c_V | v(y) \in T_V)
\]

\[
\leq |L^*| 2^n |I(U;V) - \epsilon|
\]

\[
\leq 2^n R_c 2^{-n |I(U;V) - \epsilon|}
\]

\[
\leq \epsilon
\]

for large enough \(n\), since \(R_c \leq I(U;V)\). Hence, \(P(E_5) \rightarrow 0\).

We have constructed a rate \(R\) code for which \(P_e^n \leq \sum_{n=0}^{3} \Pr(E_n) \rightarrow 0\). Consequently, \(R \in R\), completing the proof.

\[\blacksquare\]

**APPENDIX II**

**PROOF OF THE OUTER BOUND**

In this section we prove theorem which states the outer bound \(R \subseteq R_{out}\). In the proof let \(W\) be the test index, selected from a uniform distribution \(p(w)\) over the pattern indices \(\mathcal{M}_c\); let \(X = X(W)\) be the selected test pattern from the set of training patterns \(\mathcal{C}_x\); let \(m = m(W)\) be the compressed, memorized form of \(X\) computed from \(f\) as \((m, W) = f(X, W)\); let \(C_u = f(C_x)\) be the memorized data; let \(Y\) be the sensory data; and let \(\mu = \mu(W) = \phi(Y)\) be the encoded form of sensory data. Note that \(m\) and \(\mu\) are random variables through their dependence on \(X\) and \(Y\). The mutual informations in the proof are calculated with respect to the joint distribution over \((W, C_x, C_u, X, Y, m, \mu, w)\). We can verify that this distribution is well-defined by writing it out explicitly:

\[
p(w, C_x, C_u, x, y, m, \mu) = p(w) p(C_x) p(C_u | C_x) p(x | w, C_x) p(y | x) p(m | x, w) p(\mu | y),
\]

where

\[
p(w) = \begin{cases} \frac{1}{M_c}, & w \in \mathcal{M}_c, \\ 0, & \text{otherwise}; \end{cases}
\]

\[
p(C_x) = \prod_{i=1}^{M_c} \prod_{n=1}^{N} p(x_i)
\]

\[
p(C_u | C_x) = \begin{cases} 1, & C_u = f(C_x) \\ 0, & \text{otherwise}; \end{cases}
\]

\[
p(x | w, C_x) = \begin{cases} 1, & x = x(w), (x, w) \in C_x \\ 0, & \text{otherwise}; \end{cases}
\]

\[
p(y | x) = \prod_{i=1}^{N} p(y_i | x_i)
\]

\[
p(m | x, w) = \begin{cases} 1, & f(x, w) = (m, w) \\ 0, & \text{otherwise}; \end{cases}
\]

\[
p(\mu | y) = \begin{cases} 1, & \mu = \phi(y) \\ 0, & \text{otherwise}. \end{cases}
\]
\[ p(\hat{w}|\mu, C_u) = \begin{cases} 
1 & \hat{w} = g(\mu, C_u) \\
0 & \text{otherwise.} 
\end{cases} \]

The independence relationships underlying the structure of this distribution are clear from the block diagram of figure 1. They are also usefully displayed using a directed graphical model (‘Bayes’ net’) [9], [16].

\[ \begin{array}{c}
\begin{array}{c}
W \\
X^n \\
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
C_x \\
C_u \\
m \\
Y^n \\
\mu \\
\hat{w} \\
\end{array}
\end{array} \]

Fig. 5. Independence relationships for \((W, C_x, C_u, X, Y, m, \mu, \hat{w})\)

**Proof:** (Theorem 6.3)

Assume \(R = (R_x, R_y, R_c) \in \mathcal{R}\). Then there exists a sequence of \((M_x, M_y, M_c, n)\) codes \((f, \phi, g)_n\), such that for any \(\epsilon > 0\),

\[
\begin{align*}
M_c &\geq 2^n R_c \\
M_x &\leq 2^n R_x \\
M_y &\leq 2^n R_y
\end{align*}
\]

and \(P^n_c = Pr(\hat{W} \neq W) \leq \epsilon\). To show that \(R \in \mathcal{R}_{out}\), we must construct a pair of auxiliary random variables \(U V\) such that \(UV \in \mathcal{P}_{out}\) and \(R \in \mathcal{R}_{UV}\).

We construct the desired pair \(U V\) in three steps: (1) We introduce a set of intermediate random variable pairs \(U_iV_i, i = 1, 2, \ldots, n\), individually contained in \(\mathcal{P}_{out}\); (2) we derive mutual information inequalities for \(R_x, R_y,\) and \(R_c\) involving sums of the intermediate variables; (3) we convert the sum inequalities into inequalities in the final pair \(U V\) by applying Lemma 2.1.

**Step 1:**

Let the intermediate auxiliary random variables be

\[
\begin{align*}
U_i &= (m, W, X_i^{i-1}) \\
V_i &= (\mu, Y_i^{i-1})
\end{align*}
\]

for \(i = 1, 2, \ldots, n\). Each pair is in \(\mathcal{P}_{out}\). This is verified for the \(U_i\) by calculating

\[
I(U_i; Y_i|X_i) = H(Y_i|X_i) - H(Y_i|m, W, X_i^{i-1}, X_i)
= H(Y_i|X_i) - H(Y_i|m, W, X_i^{i-1})
\leq a H(Y_i|X_i) - H(Y_i|m, W, X^n)
\leq b H(Y_i|X_i) - H(Y_i|X^n)
\leq c H(Y_i|X_i) - H(Y_i|X_i)
= 0,
\]

where the reasons for the lettered steps are (a) conditioning reduces entropy, (b) the \(Y_i\) are independent of all other variables given \(X^n\), and (c) the pairs \(X_iY_i\) are i.i.d. Hence, \(U_i - X_i - Y_i\) is a Markov chain. By a similar argument, \(X_i - Y_i - V_i\) is also a Markov chain. Hence, \(U_iV_i \in \mathcal{P}_{out}\) for each \(i = 1, 2, \ldots, n\).

**Step 2:**
First,

\[ M_c (n R_x) \geq M_c \log M_x \]
\[ = H(C_u) \]
\[ = H(C_u) - H(C_u | C_x) \]
\[ = I(C_u ; C_x) \]
\[ = H(C_x) - H(C_x | C_u) \]
\[ = \sum_{w=1}^{M_c} [H(X^n(w), w) - H(X^n(w)|m(w), w)] \]
\[ = \sum_{w=1}^{M_c} [H(X^n(W)|W = w) - H(X^n(W)|m(w), W = w)] \]
\[ = \sum_{w=1}^{M_c} [H(X^n(w)) - H(X^n(W)|m(w), W = w)] \]
\[ = \sum_{w=1}^{M_c} [H(X^n) - H(X^n|W = w)] \]
\[ = \sum_{w=1}^{M_c} \sum_{i=1}^{n} [H(X_i) - H(X_i|m, W = w, X^{i-1})] \]
\[ = M_c \sum_{i=1}^{n} \sum_{w=1}^{M_c} [H(X_i) - H(X_i|m, W = w, X^{i-1})]p(w) \]
\[ = M_c \sum_{i=1}^{n} [H(X_i) - \sum_{w=1}^{M_c} p(w)H(X_i|m, W = w, X^{i-1})] \]
\[ = M_c \sum_{i=1}^{n} [H(X_i) - H(X_i|m, W, X^{i-1})] \]
\[ = M_c \sum_{i=1}^{n} [H(X_i) - H(X_i|U_i)] \]
\[ = M_c \sum_{i=1}^{n} I(X_i; U_i), \]

or

\[ n R_x \geq \sum_{i=1}^{n} I(X_i; U_i), \]

where the justifications are (a) \( C_u = f(C_x) \); (b) the pairs \((X^n(w), w)\) are independent; (c) in this expression \( w \) is a deterministic variable (i.e. \( H(w) = H(W|W = w) = 0 \)); (d) the \( X^n(w) \) are i.i.d. and independent of their index \( w \); (e) to simplify notation, we have written \( m = m(w), X^n = X^n(w) \); (f) the \( X_i \) are i.i.d.; and (g) \( W \) is distributed according to \( p(w) = 1/M_c, w = 1, 2, \ldots, M_c \).

Next,

\[ n R_y \geq H(\mu) \]
\[ = H(\mu) - H(\mu|Y^n) \]
\[ = \sum_{i=1}^{n} H(Y_i) - H(Y_i|Y^{i-1}\mu) \]
\[
\sum_{i=1}^{n} H(Y_i) - H(Y_i|V_i) = \sum_{i=1}^{n} I(Y_i; V_i).
\]

Step (a) follows from \( \mu = \phi(Y^n) \).

Finally,

\[
nR_c \leq \log M_c \\
= H(W) \\
= I(W; C_u, \mu) + H(W|C_u, \mu) \\
\stackrel{a}{\leq} I(W; C_u, \mu) + n\epsilon_n \\
= I(W; C_u) + I(W; \mu | C_u) + n\epsilon_n \\
\stackrel{b}{=} 0 + I(W; \mu | C_u) + n\epsilon_n \\
= I(W; C_u; \mu) - I(\mu; C_u) + n\epsilon_n \\
\leq I(W; C_u; \mu) + n\epsilon_n \\
\leq I(W; m; \mu) + n\epsilon_n \\
\stackrel{d}{=} \sum_{i=1}^{n} I(X_i; U_i) + I(Y_i; V_i) - I(X_i Y_i; U_i V_i) + 2n\epsilon_n \\
\stackrel{e}{=} \sum_{i=1}^{n} I(U_i; V_i) - I(U_i; V_i|X_i V_i) + 2n\epsilon_n,
\]

The lettered steps are justified as follows.

(a) By assumption, \( \Pr(\hat{w} \neq W) = P^n_c \rightarrow 0 \), where \( \hat{w} = g_n(\mu, C_u) \). Thus, applying Fano’s inequality yields

\[ H(W|C_u, \mu) \leq H(P^n_c) + P^n_c \log(M_c - 1) \leq n\epsilon_n, \]

where \( \epsilon_n \rightarrow 0 \).

(b) The test index \( W \) and patterns \( C_u \) are drawn independently, hence \( W \) and \( C_u = f(C_x) \) are independent and \( I(W; C_u) = 0 \).

(c) Writing \( C_u = C_u^* \cup \{(m, W)\} \) , \( C_u^* = C_u \setminus \{(m, W)\} \), we have

\[
I(W; C_u; \mu) = I(W, (m, W); C_u^*; \mu) \\
= I(W, m; \mu) + I(W, C_u^*; \mu | W, m) \\
= I(W, m; \mu) + I(C_u^*; \mu | W, m) \\
= I(W, m; \mu) + 0,
\]

since the \((m(i), i)\) are independent of \( \mu \) for \( i \neq W \).

(d) To justify this step we invoke the following two results, proved in Appendix VI. Let \( A, \alpha, B, \beta, \) and \( \gamma \) be arbitrary discrete random variables. Then:

**Theorem 2.1:**

\[ I(\alpha; \beta) \geq I(A; \alpha) + I(B; \beta) - I(AB; \alpha \beta), \]

with equality if and only if \( I(A\alpha; B\beta) = I(A; B) \).
Theorem 2.2: Let $Z_i = (\gamma; A^{i-1})$, $i = 1, 2, \ldots, n$, where the $A_i$ are i.i.d. Then

$$\sum_{i=1}^{n} I(A_i; Z_i) = I(A^n; \gamma).$$

To apply Theorem 2.1, make the substitution $(\alpha, \beta, A, B) \rightarrow (mW, \mu, X^n, Y^n)$. Then the condition for equality is satisfied:

$$I(X^n, m, W; Y^n, \mu) = I(X^n, W; Y^n) + I(m, W; X^n, W) \overset{a}{=} I(X^n, W; Y^n, \mu) + 0 = I(X^n, W; Y^n) + I(X^n, W; \mu|Y^n) \overset{b}{=} I(X^n, W; Y^n) + 0 = I(X^n, Y^n) + I(W; Y^n|X^n) \overset{c}{=} I(X^n, Y^n) + 0,$$

since (a) $(m, W) = f(X^n, W)$, (b) $\mu = \phi(Y^n)$, and (c) $Y^n$ only depends on $W$ through $X^n = X^n(W)$, so that $H(Y^n|X^n, W) = H(Y^n|X^n)$. Thus Theorem 2.1 yields

$$I(m, W; \mu) = I(X^n; m, W) + I(Y^n; \mu) - I(X^n, Y^n; m, W, \mu). \quad (12)$$

Next, apply Theorem 2.2 three times with the substitutions:

$$(Z_i, \gamma, A^{i-1}) \rightarrow (U_i, mW, X^{i-1})$$

$$(V_i, \mu, Y^{i-1})$$

$$(U_iV_i, mW\mu, X^{i-1}Y^{i-1})$$

to obtain

$$\sum_{i=1}^{n} I(X_i; U_i) = I(X^n; m, W)$$

$$\sum_{i=1}^{n} I(Y_i; V_i) = I(Y^n; \mu)$$

$$\sum_{i=1}^{n} I(X_iY_i; U_iV_i) = I(X^nY^n; m, W, \mu)$$

Adding the first two expressions and subtracting the third yields

$$\sum_{i=1}^{n} [I(X_i; U_i) + I(Y_i; V_i) - I(X_i, Y_i; U_i, V_i)] = [I(X^n; m, W) + I(Y^n; \mu) - I(X^n, Y^n; m, W, \mu)]. \quad (13)$$

Combining (12) and (13) yields

$$I(m, W; \mu) = \sum_{i=1}^{n} I(X_i; U_i) + I(Y_i; V_i) - I(X_i, Y_i; U_i, V_i),$$

as claimed.
In the following, let

In this Appendix we prove a slightly more general version of Lemma 2.1 from section II, and demonstrate

Applying Lemma 2.1 to the results of steps 1 and 2, we obtain

for each \( i = 1, 2, \ldots, n \), where in the second-to-last step we have used the fact that \( U_i - X_i - Y_i \) and \( X_i - Y_i - V_i \) are Markov chains for \( i = 1, 2, \ldots, n \), as shown above in Step 1.

Step 3:

For this step we use the following Lemma, proved in Appendix III.

Lemma 2.1: Suppose \( U_iV_i \in \mathcal{P}_{out}, i = 1, 2, \ldots, n \). Then there exists \( UV \in \mathcal{P}_{out} \) such that

Applying Lemma 2.1 to the results of steps 1 and 2, we obtain

where \( UV \in \mathcal{P}_{out} \). With respect to this \( UV \), by definition we have \( \mathbf{R} \in \mathcal{R}_{UV} \). Hence, \( \mathbf{R} \in \mathcal{R}_{out} \), and the proof is complete.

\[ \blacksquare \]

APPENDIX III

CONVEXITY OF THE OUTER BOUND

In this Appendix we prove a slightly more general version of Lemma 2.1 from section II and demonstrate that the outer bound rate region \( \mathcal{R}_{out} \) is convex.

In the following, let \( Q \) be any finite alphabet, and assume that we have pairs \( X_qY_q \) for all \( q \in Q \) which are i.i.d. \( \sim p(xy) \).

Lemma 3.1: Suppose \( U_qV_q \in \mathcal{P}_{out} \) for all \( q \in Q \), and let let \( Q \sim p(q), q \in Q \) be any discrete random variable independent of the pairs \( \{X_qY_q\} \). Then there exists a pair of discrete random variables \( UV \in \mathcal{P}_{out} \) such that

\[ \sum_{q \in Q} p(q)I(X_q; U_q) = I(X; U) \]

\[ \sum_{q \in Q} p(q)I(Y_q; V_q) = I(Y; V) \]
Lemma 3.1: Let $R$ be any set of rates such that $R_q \in \mathcal{R}_{out}$ for all $q \in \mathcal{Q}$, where $\mathcal{Q} = \{1, 2, \ldots, n\}$ and $p(q) = 1/n$ for all $q \in \mathcal{Q}$. Then

$$\sum_{q \in \mathcal{Q}} p(q) [I(U_q; V_q) - I(U_q; V_q|X_qY_q)] = I(U; V) - I(U; V|XY).$$

Remark 3.1: Lemma 2.1 in section III follows immediately from the above Lemma, by choosing $\mathcal{Q} = \{1, 2, \ldots, n\}$ and $p(q) = 1/n$ for all $q \in \mathcal{Q}$.

Proof: As a candidate for the pair $UV$ in the Lemma, consider $U = (U_Q, Q)$ and $V = (V_Q, Q)$, i.e.

$$U = \{U_q \text{ if } Q = q\}$$

$$V = \{V_q \text{ if } Q = q\}.$$  

To verify that $UV \in \mathcal{P}_{out}$, we proceed to check that $U - X - Y$ and $X - Y - V$ are Markov chains. By the assumption $U_qV_q \in \mathcal{P}_{out}$ for each $q \in \mathcal{Q}$, we have $I(U_q; Y_q|X_q) = 0$ and $I(V_q; X_q|Y_q) = 0$. Hence

$$0 = \sum_{q \in \mathcal{Q}} p(q) I(U_q; Y_q|X_q)$$

$$= \sum_{q \in \mathcal{Q}} p(q) I(U_q; Y_q|X_q, Q = q)$$

$$= I(U_Q; Y_Q|X_QQ)$$

$$= I(U_Q; Y_Q|X, Q)$$

$$= I(U_QQ; Y|X) - I(Q; Y|X)$$

$$= I(U_QQ; Y|X)$$

$$= I(U; Y|X),$$

where in (a) we are able to drop the subscript $Q$ on $X_Q$ and $Y_Q$ because the $X_q$ and $Y_q$ are i.i.d. and independent of $Q$; and similarly (b) is because $I(Q; Y|X) = 0$, due to the independence of $Q$ and $Y$. By an analogous calculation, we also find $I(V; X|Y) = 0$. Hence, $U - X - Y$ and $X - Y - V$, and $UV \in \mathcal{P}_{out}$ as desired.

It remains to demonstrate the three equalities in the Lemma. For the first equality, we write

$$I(X; U) = I(X; U_QQ)$$

$$= I(X; U_Q|Q) + I(X; Q)$$

$$= I(X; U_Q|Q)$$

$$= I(X_Q; U_Q|Q)$$

$$= \sum_{q \in \mathcal{Q}} p(q) I(X_q; U_q),$$

where, as above, (a) and (b) follow from the facts that the $X_q$ are i.i.d. and independent of $Q$. Similar calculations yield

$$I(Y; V) = \sum_{q \in \mathcal{Q}} p(q) I(Y_q; V_q),$$

which is the second required equality, and

$$I(XY; UV) = \sum_{q \in \mathcal{Q}} p(q) I(X_qY_q; U_qV_q).$$

This last equality can be combined with the first two to yield the third required equality using

$$I(X; U) + I(Y; V) - I(XY; UV) = I(U; V) - I(U; V|XY),$$

which follows from the two short Markov chains $U - X - Y$ and $X - Y - V$, as shown in subsection [VII-H] of [3]. The proof is complete.

The convexity of $\mathcal{R}_{out}$ follows readily from the preceding Lemma.

Lemma 3.2: $\mathcal{R}_{out}$ is convex. That is, let $R_q$ be any set of rates such that $R_q \in \mathcal{R}_{out}$ for all $q \in \mathcal{Q}$, where $\mathcal{Q}$
is a finite alphabet, and let \( p(q) \) be any probability distribution over \( Q \). Then \( R = \sum_{q \in Q} p(q)R_q \in \mathcal{R}_{\text{out}} \).

**Proof:** Fix an arbitrary distribution \( p(q) \) and rates \( R_q \in \mathcal{R}_{\text{out}} \) for all \( q \in Q \). By the definition of \( \mathcal{R}_{\text{out}} \), for each rate \( R_q \), there exists a pair \( U_qV_q \in \mathcal{P}_{\text{out}} \) such that \( R_q \in \mathcal{R}_{U_qV_q} \). Consequently,

\[
\begin{align*}
R_x &= \sum_{q \in Q} p(q)R_{x,q} \geq \sum_{q \in Q} p(q)I(X_q;U_q) \\
R_y &= \sum_{q \in Q} p(q)R_{y,q} \geq \sum_{q \in Q} p(q)I(Y_q;V_q) \\
R_c &= \sum_{q \in Q} p(q)R_{c,q} \leq \sum_{q \in Q} p(q)I(U_q;V_q) - I(U_q;V_q|U_qV_q).
\end{align*}
\]

As in the proof of Lemma 5.1, use these pairs to construct a new pair \( UV \), by defining \( U = (U_Q,Q) \), \( V = (V_Q,Q) \). From the proof of Lemma 5.1 we know (1) that \( UV \in \mathcal{P}_{\text{out}} \), and (2) the sums on the right hand sides of the inequalities above can be replaced with expressions in \( U \) and \( V \), yielding

\[
\begin{align*}
R_x &\geq I(X;U) \\
R_y &\geq I(Y;V) \\
R_c &\leq I(U;V) - I(U;V|UV),
\end{align*}
\]

which means that \( R \in \mathcal{R}_{UV} \) for the given \( UV \). Hence, \( R = \sum_{q \in Q} p(q)R_q \in \mathcal{R}_{\text{out}} \). Since \( p(q) \) and \( R_q \in \mathcal{R}_{\text{out}} \) were arbitrary, we conclude that \( \mathcal{R}_{\text{out}} \) is convex. \( \blacksquare \)

**APPENDIX IV**

**PROOF OF THEOREM 7.2**

In this section we prove theorem 7.2. The argument is based on time sharing. Consider a sequence of codes of lengths \( n_i \) that achieve \((R_c,R_x,R_y)\). Corresponding to this sequence is a sequence of codes of lengths \( m_i \) that satisfy \( \theta m_i = n_i \), constructed as follows. For each \( m_i \), select any \( \theta m_i \) components; reveal the indices of the selected components to the memory encoder and the sensory encoder. Use the corresponding code from the first sequence on these components, ignoring all other components. For \( m_i \), there are \( 2^{m_i \theta R_c} \) patterns, \( 2^{m_i \theta R_y} \) memory states, and \( 2^{m_i \theta R_x} \) sensory states.

The corollary 7.2 follows immediately from the inner bound, theorem 6.2.

**APPENDIX V**

**PROOF OF LEMMA 7.4**

In this Appendix we prove the ‘if-then’ statement asserted in Lemma 7.4.

The assumptions of the statement are that (a) \( \mathcal{P}_{\text{mix}} = \mathcal{P}_{\text{out}} \); and (b) that the achievable rate region \( \mathcal{R} \) is convex. We wish to show that these imply \( \mathcal{R} = \mathcal{R}_{\text{out}} \).

From theorem 6.4 we have \( \mathcal{R}_{\text{out}} \supseteq \mathcal{R} \). To prove the Lemma, we must demonstrate the converse, \( \mathcal{R}_{\text{out}} \subseteq \mathcal{R} \).

It suffices to show the boundary points of \( \mathcal{R}_{\text{out}} \) are achievable. Let \( R = (R_c,R_x,R_y) \) be an arbitrary rate on the boundary of \( \mathcal{R}_{\text{out}} \). Then there exists \( UV \in \mathcal{P}_{\text{out}} \) such that \( R_c = I(X;U) + I(Y;V) - I(XY;UV) \), \( R_x = I(X;U) \) and \( R_y = I(Y;V) \). In turn, assumption (a) \( \mathcal{P}_{\text{mix}} = \mathcal{P}_{\text{out}} \) implies that there exists \( Q \sim p(q), q \in Q \) independent of \( XY \) and pairs \( U_qV_q \in \mathcal{P}_{\text{out}}, q \in Q \) such that \( R_c = I(X;U_q,Q) \), \( R_y = I(Y;V_q,Q) \), and \( R_x = I(X;U_q,Q) + I(Y;V_q,Q) - I(XY;U_qV_q,Q) \). Hence, using the independence of \( Q \) from \( X \) and \( Y \) we have

\[
\begin{align*}
R_x &= \sum_{q \in Q} I(U_q;X)p(q) \\
R_y &= \sum_{q \in Q} I(V_q;Y)p(q)
\end{align*}
\]
\[ R_c = \sum_{q \in \mathcal{Q}} [I(U_q; X) + I(V_q; Y) - I(U_q V_q; XY)]p(q). \]

Next, let \( R_{xq} = I(U_q; X) \), \( R_{yq} = I(V_q; Y) \), \( R_q = I(X; U_q) + I(Y; V_q) - I(XY; U_q V_q) \), for \( q = 1, 2, \ldots, |\mathcal{Q}| \). Then, by definition, each rate \( R_q = (R_{xq}, R_{yq}, R_q) \) is in \( \mathcal{R}_{in} \). Since \( \mathcal{R}_{in} \subseteq \mathcal{R} \) by theorem 6.2, \( R_q \in \mathcal{R} \) for each \( q \in \mathcal{Q} \).

According to the preceding argument, \( \mathcal{R} = (R_c, R_x, R_y) \) is a convex combination of achievable rates. Consequently, if \( \mathcal{R} \) is convex as assumed, then \( \mathcal{R} \in \mathcal{R} \). Since the rate \( \mathcal{R} \) was an arbitrary boundary point of \( \mathcal{R}_{out} \), we conclude \( \mathcal{R}_{out} \subseteq \mathcal{R} \), hence \( \mathcal{R} = \mathcal{R}_{out} \) as desired.

**APPENDIX VI**

**PROOFS OF THEOREMS 2.1 AND 2.2**

Consider the elementary Shannon inequalities, stated in the following two Lemmas. The variables \( A, B, \alpha, \beta, \gamma, \delta \) appearing in the Lemmas denote arbitrary discrete random variables.

**Lemma 6.1:**

\[ I(A; \alpha) = I(A; \alpha, \gamma) - I(A, \alpha; \gamma) + I(\alpha; \gamma). \]

**Proof:**

\[
I(A; \gamma|\alpha) = I(A; \alpha, \gamma) - I(A; \alpha) = I(A, \alpha; \gamma) - I(\gamma; \alpha).
\]

**Lemma 6.2:**

\[
I(A; \alpha) + I(B; \beta) = I(A; B) + I(\alpha; \beta) - I(A, \alpha; B, \beta) + I(A, B; \alpha, \beta)
\]

**Proof:**

\[
I(A, \alpha; B, \beta) - I(A, B; \alpha, \beta) = H(A, \alpha) + H(B, \beta) - H(A, B) - H(\alpha, \beta) = -I(A; \alpha) - I(B; \beta) + I(A; B) + I(\alpha; \beta)
\]

Theorems 2.1 and 2.2 follow directly from the Lemmas above.

**Theorem 6.1 (Theorem 2.1):**

\[ I(\alpha; \beta) \geq I(A; \alpha) + I(B; \beta) - I(A, B; \alpha, \beta) \]

with equality if and only if \( I(A, \alpha; B, \beta) = I(A; B) \).

**Proof:** Rearrange Lemma 6.2 to get

\[
I(\alpha; \beta) = I(A; \alpha) + I(B; \beta) - I(A, B; \alpha, \beta) + [I(A, \alpha; B, \beta) - I(A; B)]
\]

The Lemma now follows readily from the preceding expression: We obtain equality in the Lemma if (and only if) the term in brackets is zero. Otherwise, the bracketed term is nonnegative, since

\[
I(A, \alpha; B, \beta) - I(A; B) = H(\alpha|A) + H(\beta|B) - H(\alpha, \beta|A, B) = H(\alpha|A) - H(\alpha|A, B) + H(\beta|B) - H(\beta|A, B, \alpha) \geq 0,
\]

where the inequality is due to the fact that conditioning reduces entropy.
Theorem 6.2 (Theorem 2.2): If \( U_i = (\gamma, A_i^{-1}) \), then

\[
I(A^n; \gamma) = \sum_{i=1}^{n} I(A_i; U_i) - \sum_{i=2}^{n} I(A_i; A_i^{-1})
\]

Proof: In Lemma 5.1 put \( A = A_i, \alpha = A_i^{-1} \). Note that \( U_1 = \gamma \). Hence, substituting and summing from 2 to \( n \) yields

\[
\sum_{i=2}^{n} I(A_i; A_i^{-1}) = \sum_{i=2}^{n} I(A_i; U_i) - I(A^n; \gamma) + I(A_i; \gamma)
\]

\[
= \sum_{i=2}^{n} I(A_i; U_i) - I(A^n; \gamma) + I(A_i; U_i)
\]

\[
= \sum_{i=1}^{n} I(A_i; U_i) - I(A^n; \gamma).
\]

\[\blacksquare\]

APPENDIX VII
SIMPLIFICATION OF CONVEX HULLS

In this section we argue geometrically that the expressions for convex hulls of the inner bound regions simplify to just one term in both the binary and Gaussian cases. To discuss both cases simultaneously, let us represent the surface of either inner bound by a positive valued function \( f : D \to \mathbb{R}_+ \). Here, \( D \) is a square region

\[
D = \{ r = (x, y) \in \mathbb{R}^2 : 0 \leq x \leq M, 0 \leq y \leq M \},
\]

and \( M \) is a positive constant. In the binary case, \( f(r) = g(r) \), and \( D = [0, 1] \times [0, 1] \); in the Gaussian case, \( f(r) = G(r) \), and \( D = [0, \infty) \times [0, \infty) \). Some important properties shared by both cases are that for all \( r = (x, y) \in D \),

\[
f(x, y) \geq 0, \quad f(0, y) = f(x, 0) = 0,
\]

\[
f_x(r), f_y(r) > 0, \quad f_{xx}(r), f_{yy}(r) < 0,
\]

where the subscripts denote partial derivatives.

Denote the convex hull of \( f(r) \) by \( c(r) \). Generically, the boundary of the convex hull is

\[
c(r) = \max \theta f(r_1) + \bar{\theta} f(r_2),
\]

where the maximum is over all triples \( (\theta, r_1, r_2) \) such that \( r = \theta r_1 + \bar{\theta} r_2, \theta \in [0, 1] \), and \( r_1, r_2 \in D \). However, as argued next, for the cases under study this simplifies to

\[
c(r) = \max \theta f(r'),
\]

where \( r = \theta r' \).

The convex hull of a surface can be characterized in terms of its tangent planes. Given any point \( r' = (x, y) \in D \), if its tangent plane lies entirely above the surface, then \( (r', f(r')) \) is on the convex hull. If the tangent plane cuts through the surface at one or more other points, then \( (r, f(r)) \) is not on the convex hull. If the tangent plane intersects the surface at exactly two points, then both points are on the convex hull.

The tangent plane at an arbitrary point \( r' = (x', y') \in D \) is the set of points satisfying

\[
z(x, y) = f_x(x - x') + f_y(y - y') + z',
\]

where the partial derivatives are evaluated at \( r' \), i.e. \( f_x = f_x(r') \), \( f_y = f_y(r') \), and \( z' = f(r') \). The tangent plane intersects the \( z = 0 \) plane in a line. Setting \( z(r) = 0 \) and solving

\[
y = mx + b,
\]

where
\[ \begin{align*}
m &= -(f_x/f_y) \\
b &= 1/f_y[z'f_x + y'f_y - z'].\end{align*} \]

Since \( f_x, f_y > 0 \), the slope \( m = -(f_x/f_y) \) is negative. This line intersects the positive orthant whenever the intercept \( b \geq 0 \), in which case the tangent plane cuts through the surface, since \( f \geq 0 \). Thus, the only points on the original surface \( f(x, y) \) that can be on the convex hull are those for which \( b \leq 0 \).

Next consider any path through \( D \) along a line segment \( y = \alpha x, \alpha > 0 \), starting from one of the 'outer edges' of \( D \), where \( x = M \) or \( y = M \), and consider what happens to the tangent plane's line of intersection \( \ell \) with the \( z = 0 \) plane as we move in along the path toward the origin \((0, 0)\). Initially, the tangent planes lie entirely above the surface, and the intercept \( \ell \) is negative, \( b < 0 \). This intercept increases along the path until \( b = 0 \), at which point \( \ell \) intersects \((0, 0)\). Here, the tangent plane contains a line segment attached on one end to the point of tangency, and at the other end to the point \((r, f(r)) = (0, 0, 0)\); everywhere else, the tangent plane is above the surface. Continuing toward the origin, all other points along the path have tangent planes such that \( \ell \) has a positive intercept \( b > 0 \), hence these points are excluded from the convex hull.

These considerations imply that the convex hull \( c(r) \) is composed entirely of two kinds of points. First, points which coincide with the original surface, \( c(r) = \theta f(r) \), with \( \theta = 1 \). These points occur at values of \( r = (x, y) \) 'up and to the right' of \((0, 0)\). Second, points along line segments connecting surface points 'up and to the right' \((r', f(r'))\) with the point \((r, f(r)) = (0, 0, 0)\), that is \( c(r) = \theta f(r') + \bar{\theta} f(0, 0) = \bar{\theta} f(r') \), where \( r = \theta r' \) and \( \theta \in [0, 1] \). Hence, for all \( r \in D \), \( c(r) \) has the desired form.

An example of another function that behaves in the same way just described is \( f(x, y) = (1 - (1 - x)^2)(1 - (1 - y)^2) \), with \( D = [0, 1] \times [0, 1] \).

**APPENDIX VIII**

**PROPERTIES OF GAUSSIAN MUTUAL INFORMATION**

Our analysis of the Gaussian pattern recognition problem relies on well-known results, stated below without proof.

**Lemma 8.1:** The mutual information between two Gaussian random vectors \( X \) and \( Y \) depends only on the matrices of correlation coefficients. Specifically,

\[ I(X; Y) = \frac{1}{2} \log (\det C_{x,x}) - \frac{1}{2} \log (\det C_{x,x|y}), \]

where

\[ C_{x,x|y} = C_{x,x} - C_{x,y}C_{y,y}^{-1}C_{y,x}. \]

In the most well known special case of \( Y = X + W \), where \( X \) and \( W \) are independent Gaussian random variables with variances \( P \) and \( N \), respectively, yields

\[ I(X; Y) = \frac{1}{2} \log \left( 1 + \frac{P}{N} \right) = -\frac{1}{2} \log \left( 1 - \rho_{x,y}^2 \right), \]

where the correlation coefficient \( \rho_{x,y} = \sqrt{P/(P + N)} \).

**Lemma 8.2:** If \( X, Y \) and \( Z \) are zero mean Gaussian random vectors that form a Markov chain \( X \rightarrow Y \rightarrow Z \), then

\[ C_{x,z} = C_{x,y}C_{y,y}^{-1}C_{y,z}. \]

Note that for dimension one, \( X \rightarrow Y \rightarrow Z \) implies \( \rho_{x,z} = \rho_{x,y}\rho_{y,z} \).

**Lemma 8.3:** Let \( X, Y, U, \) and \( V \) be jointly Gaussian random variables such that \( U \rightarrow X \rightarrow Y \) and \( X \rightarrow Y \rightarrow V \) are Markov chains. Then the matrix of correlation coefficients \( C_{xy,uv} \) decomposes as

\[
C_{xy,uv} = \begin{bmatrix}
1 & \rho_{xy} \\
\rho_{xy} & 1
\end{bmatrix}
\begin{bmatrix}
\rho_{xu} & 0 \\
0 & \rho_{yv}
\end{bmatrix}.
\]
This lemma follows immediately by using Lemma 8.2 to obtain the substitutions $C_{x,v} = C_{x,y}C_{y,v}^{-1}C_{y,v} = \rho_{xy}\rho_{vy}$ and $C_{u,y} = C_{u,x}C_{x,y}^{-1}C_{x,y} = \rho_{ux}\rho_{xy}$.

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