HOMOGENEOUS POLYNOMIALS: HARMONIC MEANS AND COMPLETELY PARTITIONED WEIGHTED GEOMETRIC MEANS

C. SCHWANKE

Abstract. We provide two new characterizations of bounded orthogonally additive polynomials from a uniformly complete vector lattice into a convex bornological space using harmonic means and completely partitioned weighted geometric means. Our result involving completely partitioned weighted geometric means generalizes a recent theorem on bounded orthogonally additive polynomials by Z.A. Kusraeva as well as parts of related theorems by G. Buskes and the author.

1. Introduction

For \( r, s \in \mathbb{N} \), the \( s \)th root mean power \( S_s \) and the \( s \)th geometric mean \( G_s \) are given by

\[
S_s(x_1, \ldots, x_r) = \sqrt[s]{\sum_{k=1}^{r} x_k^s} \quad (x_1, \ldots, x_r \in \mathbb{R})
\]

and

\[
G_s(x_1, \ldots, x_s) = \sqrt[s]{\prod_{k=1}^{s} |x_k|} \quad (x_1, \ldots, x_s \in \mathbb{R}).
\]

These means have enjoyed extensive study in the setting of Archimedean vector lattices recently, see e.g. [2, 3, 6, 8, 10, 13]. Indeed, these means can be defined in uniformly complete vector lattices using the Archimedean vector lattice functional calculus developed in [5]. Of particular interest to this paper, it is proven in [10] that if (i) \( E \) is a uniformly complete vector lattice, (ii) \( Y \) is a convex bornological space, and (iii) \( P : E \to Y \) is a bounded orthogonally additive \( s \)-homogeneous polynomial with unique corresponding symmetric \( s \)-linear map \( \tilde{P} \), then the following hold:

\[
P(S_s(f_1, \ldots, f_r)) = \sum_{k=1}^{r} P(f_k) \quad (f_1, \ldots, f_r \in E_+, r \in \mathbb{N} \setminus \{1\})
\]
and

\[ P(\mathfrak{G}_s(f_1, \ldots, f_s)) = \tilde{P}(f_1, \ldots, f_s) \quad (f_1, \ldots, f_s \in E_+). \]  

(2)

The intimate relationship between means and orthogonally additive polynomials was further explored in \[8, \text{Theorems 2.3\&2.4}\] and \[13, \text{Theorem 2.3}\], where it was shown that the identities (1) and (2) in fact characterize bounded orthogonally additive polynomials \( P: E \to Y \).

The results mentioned above illustrate how the root mean power and geometric mean play an intriguing role in vector lattice theory. It is perhaps puzzling then why another well-known mean, the harmonic mean, has yet to be explored in this setting. This paper aims to further illustrate the interrelation between means and orthogonally additive polynomials by investigating the harmonic mean as well as certain weighted versions of the geometric mean.

Let \( s \in \mathbb{N} \). Given \( x_1, \ldots, x_s \in \mathbb{R} \), the \( s \)-th harmonic mean \( \eta_s \) is defined as

\[ \eta_s(x_1, \ldots, x_s) = \begin{cases} \frac{s \prod_{i=1}^{s} |x_i|}{\sum_{j=1}^{s} \left( \frac{1}{|x_j|} \prod_{i=1}^{s} |x_i| \right)} & x_1, \ldots, x_s \neq 0 \\ 0 & \text{else} \end{cases} \]

Given \( p \in \mathbb{N} \), \( x_1, \ldots, x_p \in \mathbb{R} \), and \( t_1, \ldots, t_p \in (0, 1) \) such that \( \sum_{k=1}^{p} t_k = 1 \), the weighted geometric mean \( \gamma_{t_1, \ldots, t_p} \) is given by

\[ \gamma_{t_1, \ldots, t_p}(x_1, \ldots, x_p) = \prod_{k=1}^{p} |x_k|^{t_k}. \]

Next let \( p, s \in \mathbb{N} \) with \( p \leq s \). By a \textit{completely partitioned weighted geometric mean}, we mean a weighted geometric mean of the form \( \gamma_{r_1/s, \ldots, r_p/s} \), where \( (r_1, r_2, \ldots, r_p) \) is a complete partition of \( s \). This means that

(i) \( r_1, \ldots, r_p \in \mathbb{N} \),
(ii) \( \sum_{k=1}^{p} r_k = s \), and
(iii) for every \( q \in \{1, \ldots, s\} \), there exists \( \alpha_k \in \{0, 1\} \) \((k = 1, \ldots, p)\) such that

\[ q = \sum_{k=1}^{p} \alpha_k r_k, \]

see \[12, \text{Definition 2.2}\].
Example 1.1. Given \( s \in \mathbb{N} \), the geometric mean \( G_s \) is a completely partitioned weighted geometric mean. For \( s \geq 2 \) and \( p = s - 1 \), the weighted geometric mean \( \gamma_{r_1/s,\ldots,r_p/s} \) such that \( r_1 = 2 \) and \( r_k = 1 \) for all \( k \in \{2,\ldots,p\} \) is also a completely partitioned weighted geometric mean.

As continuous and positively homogeneous functions, these harmonic and completely partitioned weighted geometric means are defined in any uniformly complete Archimedean vector lattice \( E \) using the Archimedean vector lattice functional calculus, see [5]. Moreover, we have the following explicit formula for completely partitioned weighted geometric means of elements in \( E_+ \) (which coincides with the functional calculus definition):

\[
\gamma_{r_1/s,\ldots,r_p/s}(f_1,\ldots,f_p) = \frac{1}{s} \inf \left\{ \sum_{k=1}^{p} r_k \theta_k f_k : \theta_k \in (0, \infty), \prod_{k=1}^{p} \theta_k^{r_k/s} = 1 \right\} (f_1,\ldots,f_p \in E_+).
\]

This formula is obtained from [6, Theorem 3.7], which follows from the fact that the classical weighted geometric means are concave on \( \mathbb{R}_+^p \) and are thus the infimum of their tangents. A more general version of this formula can also be found in [7, Section 4].

Noting that the classical harmonic mean is concave on \( \mathbb{R}_+^s \), it also follows from [6, Theorem 3.7] that

\[
\eta_s(f_1,\ldots,f_s) = s \inf \left\{ \sum_{k=1}^{s} a_k f_k : 0 \leq a_1,\ldots,a_s \leq 1, \sum_{k=1}^{s} \sqrt{a_k} = 1 \right\} (f_1,\ldots,f_s \in E_+).
\]

Turning to orthogonally additive polynomials on vector lattices, we note that all vector spaces in this manuscript are real, and all vector lattices are Archimedean. For any unexplained terminology, notation, or basic theory regarding vector lattices, we refer the reader to the standard texts, e.g. [1, 11, 14].

Let \( E \) be a uniformly complete vector lattice, let \( V \) be a vector space, and let \( s \in \mathbb{N} \). Recall that a map \( P : E \to V \) is called an \( s \)-homogeneous polynomial if there exists a (unique) symmetric \( s \)-linear map \( \hat{P} : E^s \to V \) such that \( P(f) = \hat{P}(f,\ldots,f) \) (\( f \in E \)). (We denote the symmetric \( s \)-linear map associated with an \( s \)-homogeneous polynomial \( P \) by \( \hat{P} \) throughout.) Recall that an \( s \)-homogeneous polynomial \( P : E \to V \) is said to be orthogonally additive if

\[
P(f + g) = P(f) + P(g)
\]

holds whenever \( f, g \in E \) are disjoint. We will also say that \( P \) is positively orthogonally additive if \( P(f + g) = P(f) + P(g) \) holds whenever \( f, g \in E_+ \) are disjoint.
2. Main Results

We begin this section with the following proposition regarding the harmonic mean, which will aid our proof of Theorem 2.4.

**Proposition 2.1.** Let \( E \) be a uniformly complete vector lattice, and put \( s \in \mathbb{N} \). Then
\[
\bigwedge_{k=1}^{s} f_k \leq \eta_s(f_1, \ldots, f_s) \leq s \bigwedge_{k=1}^{s} f_k
\]
holds for all \( f_1, \ldots, f_s \in E_+ \).

**Proof.** Let \( f_1, \ldots, f_s \in E_+ \), and let \( i \in \{1, \ldots, s\} \) be arbitrary. Then we have
\[
\bigwedge_{k=1}^{s} f_k = \eta_s(1, \ldots, 1) \left( \bigwedge_{k=1}^{s} f_k \right)
\]
\[
= s \inf \left\{ \sum_{k=1}^{s} a_k \cdot 1 : 0 \leq a_1, \ldots, a_s \leq 1, \sum_{k=1}^{s} \sqrt{a_k} = 1 \right\} \left( \bigwedge_{k=1}^{s} f_k \right)
\]
\[
= s \inf \left\{ \sum_{k=1}^{s} a_k \left( \bigwedge_{k=1}^{s} f_k \right) : 0 \leq a_1, \ldots, a_s \leq 1, \sum_{k=1}^{s} \sqrt{a_k} = 1 \right\}
\]
\[
\leq s \inf \left\{ \sum_{k=1}^{s} a_k f_k : 0 \leq a_1, \ldots, a_s \leq 1, \sum_{k=1}^{s} \sqrt{a_k} = 1 \right\}
\]
\[
= \eta_s(f_1, \ldots, f_s)
\]
\[
\leq s f_i,
\]
where the last inequality follows from taking \( a_i = 1 \) and \( a_j = 0 \) for all \( j \in \{1, \ldots, s\} \setminus \{i\} \) in the last infimum above. We conclude that
\[
\bigwedge_{k=1}^{s} f_k \leq \eta_s(f_1, \ldots, f_s) \leq s \bigwedge_{k=1}^{s} f_k.
\]
\( \square \)

We next present a corollary which immediately follows from Proposition 2.1.

**Corollary 2.2.** Let \( E \) be a uniformly complete vector lattice, and put \( s \in \mathbb{N} \). Then
\[
\eta_s(f_1, \ldots, f_s) = 0
\]
holds whenever \( f_1, \ldots, f_s \in E_+ \) and \( f_i \perp f_j \) for some \( i, j \in \{1, \ldots, s\} \).
Notation 2.3. In the theorem below as well as its proof, we at times for brevity will write
\[ \eta(f_i)_{i=1}^s := \eta_s(f_1, \ldots, f_s). \]
We also denote the vector lattice \( s \)-power of \( E \) by \( (E^s, \circ_s) \), see \[4\].

We next present the following theorem which provides a relationship between bounded orthogonally additive polynomials and the harmonic mean.

Theorem 2.4. Let \( s \in \mathbb{N} \setminus \{1\} \). Suppose \( E \) is a uniformly complete vector lattice, \( Y \) is a convex bornological space, and \( P : E \to Y \) is a bounded \( s \)-homogeneous polynomial. Then \( P \) is orthogonally additive if and only if
\[
\tilde{P}(f_1, \ldots, f_s) = \frac{1}{s} \left( \tilde{P}(\eta_s(f_1, \ldots, f_s), f_1, \ldots, f_s) + \tilde{P}(f_1, \eta_s(f_1, \ldots, f_s), f_2, \ldots, f_s) + \cdots + \tilde{P}(f_1, \ldots, f_{s-1}, \eta_s(f_1, \ldots, f_s)) \right)
\]
holds for every \( f_1, \ldots, f_s \in E_+ \).

Proof. First suppose that \( P \) is orthogonally additive. Let \( E^u \) denote the universal completion of \( E \), and denote the \( f \)-algebra multiplication on \( E^u \) by juxtaposition. Suppose that \( f_1, \ldots, f_s \in E_+ \).

Step 1. Given \( j \in \{1, \ldots, s\} \), we denote the product
\[ f_1 f_2 \cdots f_{j-1} f_{j+1} \cdots f_{s-1} f_s \]
by
\[ f_1 \cdots \tilde{f}_j \cdots f_s \]
for short. We claim that in \( E^u \) we have
\[
\eta_s(f_1, \ldots, f_s) \sum_{j=1}^s (f_1 \cdots \tilde{f}_j \cdots f_s) = s(f_1 \cdots f_s).
\]
To verify this claim, let \( C \) be the Archimedean \( f \)-subalgebra of \( E^u \) generated by
\[
\{f_1, \ldots, f_s, \eta_s(f_1, \ldots, f_s)\}.
\]
Suppose that $\omega : C \to \mathbb{R}$ is a nonzero multiplicative vector lattice homomorphism. Using [6, Theorem 3.7] in the second and fourth equalities below, we obtain

$$\omega \left( \eta_s(f_1, \ldots, f_s) \sum_{j=1}^s \bar{f}_j \right) = \omega(\eta(f_1, \ldots, f_s)) \omega \left( \sum_{j=1}^s \bar{f}_j \right)$$

$$= \eta_s(\omega(f_1), \ldots, \omega(f_s)) \sum_{j=1}^s (\omega(f_1) \cdots \omega(f_j) \cdots \omega(f_s))$$

$$= s(\omega(f_1) \cdots \omega(f_s))$$

$$= \omega(s(f_1 \cdots f_s)).$$

Since the set of all nonzero multiplicative vector lattice homomorphisms $\omega : C \to \mathbb{R}$ separates the points of $C$ (see [5, Corollary 2.7]), we have

$$\eta_s(f_1, \ldots, f_s) \sum_{j=1}^s \bar{f}_j \cdot \bar{f}_j \cdots \bar{f}_s = s(f_1 \cdots f_s),$$

as claimed.

**Step 2.** Notice that in $E^u$ we have the elementary identity

$$\eta_s(f_1, \ldots, f_s) \sum_{j=1}^s \bar{f}_j \cdot \bar{f}_j \cdots \bar{f}_s =$$

$$(\eta(f_i)_{i=1}^s f_2 \cdots f_s) + (f_1 \eta(f_i)_{i=1}^s f_3 \cdots f_s) + \cdots + (f_1 \cdots f_{s-1} \eta(f_i)_{i=1}^s).$$

**Step 3.** Arguing as in [10], we assert that

$$s \circ \varnothing(f_1, \ldots, f_s) =$$

$$\varnothing(\eta(f_i)_{i=1}^s f_2 \cdots f_s) + \varnothing(f_1 \eta(f_i)_{i=1}^s f_3 \cdots f_s) + \cdots + \varnothing(f_1 \cdots f_{s-1} \eta(f_i)_{i=1}^s).$$

Indeed, by [4, Theorem 4.1], there exists a uniformly complete vector sublattice $F$ of $E^u$ and a vector lattice isomorphism $i : E^\varnothing \to F$ such that both

$$x_1 \cdots x_s \in F$$

and

$$i \circ \varnothing(x_1, \ldots, x_s) = x_1 \cdots x_s$$

hold for all $x_1, \ldots, x_s \in E$. 

It thus follows from Step 1 and Step 2 that

\[
s(\mathcal{S})(f_1, \ldots, f_s) = i^{-1}\left(s(f_1 \cdots f_s)\right)
\]

\[
= i^{-1}\left(\eta_s(f_1, \ldots, f_s) \sum_{j=1}^{s} (f_1 \cdots \tilde{f}_j \cdots f_s)\right)
\]

\[
= i^{-1}\left((\eta(f_i)_{i=1}^{s} f_2 \cdots f_s) + (f_1 \eta(f_i)_{i=1}^{s} f_3 \cdots f_s) + \cdots + (f_1 \cdots f_{s-1} \eta(f_i)_{i=1}^{s})\right)
\]

\[
= i^{-1}(\eta(f_i)_{i=1}^{s} f_2 \cdots f_s) + i^{-1}(f_1 \eta(f_i)_{i=1}^{s} f_3 \cdots f_s) + \cdots + i^{-1}(f_1 \cdots f_{s-1} \eta(f_i)_{i=1}^{s})
\]

\[
= \mathcal{S}(\eta(f_i)_{i=1}^{s}, f_2, \ldots, f_s) + \mathcal{S}(f_1, \eta(f_i)_{i=1}^{s}, f_3, \ldots, f_s) + \cdots + \mathcal{S}(f_1, \ldots, f_{s-1}, \eta(f_i)_{i=1}^{s}).
\]

**Step 4.** By the proof of [9, Theorem 4], there exists a bounded linear operator \(S: E^{\mathcal{S}} \to Y\) for which

\[
\tilde{P}(x_1, \ldots, x_s) = S(\mathcal{S})(x_1, \ldots, x_s)
\]

holds for every \(x_1, \ldots, x_s \in E\). It thus follows from Step 3 that

\[
s\tilde{P}(f_1, \ldots, f_s) = s S(\mathcal{S})(f_1, \ldots, f_s)
\]

\[
= S\left(s(\mathcal{S})(f_1, \ldots, f_s)\right)
\]

\[
= S\left(\mathcal{S}(\eta(f_i)_{i=1}^{s}, f_2, \ldots, f_s) + \mathcal{S}(f_1, \eta(f_i)_{i=1}^{s}, f_3, \ldots, f_s) + \cdots + \mathcal{S}(f_1, \ldots, f_{s-1}, \eta(f_i)_{i=1}^{s})\right)
\]

\[
= S(\mathcal{S})(\eta(f_i)_{i=1}^{s}, f_2, \ldots, f_s) + S(\mathcal{S})(f_1, \eta(f_i)_{i=1}^{s}, f_3, \ldots, f_s) + \cdots + S(\mathcal{S})(f_1, \ldots, f_{s-1}, \eta(f_i)_{i=1}^{s})
\]

\[
= \tilde{P}(f_1, \eta(f_i)_{i=1}^{s}, f_3, \ldots, f_s) + \tilde{P}(f_1, \ldots, f_{s-1}, \eta(f_i)_{i=1}^{s}).
\]

Thus the first implication is proved.

**Step 5.** To prove the converse, suppose that

\[
\tilde{P}(f_1, \ldots, f_s) = \frac{1}{s} \left(\tilde{P}(\eta(f_i)_{i=1}^{s}, f_2, \ldots, f_s) + \tilde{P}(f_1, \eta(f_i)_{i=1}^{s}, f_3, \ldots, f_s) + \cdots + \tilde{P}(f_1, \ldots, f_{s-1}, \eta(f_i)_{i=1}^{s}, f_s)\right)
\]

holds for every \(f_1, \ldots, f_s \in E_+\). Let \(f, g \in E_+\) with \(f \perp g\). It follows immediately from our assumption and Corollary 2.2 that for any \(k \in \{1, \ldots, s-1\}\) we have

\[
\tilde{P}(f_1, \ldots, f_{k \text{ copies}}, \underbrace{g, \ldots, g}_{s-k \text{ copies}}) = 0.
\]
Then the binomial theorem yields
\[
P(f + g) = P(f) + P(g) + \sum_{k=1}^{s-1} \binom{s}{k} \hat{P}(f_1, \ldots, f, \ldots, g, \ldots, g) = P(f) + P(g).
\]

Hence \( P \) is positively orthogonally additive. Finally, from [13, Theorem 2.3] we obtain that \( \hat{P} \) is orthogonally additive. \( \square \)

We proceed to prove a similar result involving completely partitioned weighted geometric means. The following lemma is required. Its proof relies on [6, Theorem 3.7] and [5, Corollary 2.7] and is similar to Step 1 of the proof of Theorem 2.4. It is therefore left to the reader.

**Lemma 2.5.** Let \( p, s \in \mathbb{N} \setminus \{1\} \) with \( p \leq s \). Suppose \( E \) is a uniformly complete vector lattice, and let \( \gamma_{r_1/s, \ldots, r_p/s} \) be a completely partitioned weighted geometric mean. Then
\[
\gamma_{r_1/s, \ldots, r_p/s}(f_1, \ldots, f_p) = \mathcal{G}_s(f_1, \ldots, f_1, f_2, \ldots, f_2, \ldots, f_p, \ldots, f_p)
\]
holds for all \( f_1, \ldots, f_p \in E_+ \).

We proceed to our main result involving completely partitioned weighted geometric means.

**Theorem 2.6.** Let \( p, s \in \mathbb{N} \setminus \{1\} \) with \( p \leq s \). Suppose \( E \) is a uniformly complete vector lattice, \( Y \) is a convex bornological space, and \( P: E \to Y \) is a bounded \( s \)-homogeneous polynomial. Let \( \gamma_{r_1/s, \ldots, r_p/s} \) be a completely partitioned weighted geometric mean. Then \( P \) is orthogonally additive if and only if
\[
P(\gamma_{r_1/s, \ldots, r_p/s}(f_1, \ldots, f_p)) = \hat{P}(f_1, \ldots, f_1, f_2, \ldots, f_2, \ldots, f_p, \ldots, f_p)
\]
holds for every \( f_1, \ldots, f_p \in E_+ \).

**Proof.** To prove the first implication, suppose that \( P \) is orthogonally additive. Assume that \( f_1, \ldots, f_s \in E_+ \). By Lemma 2.5 and the main result of [10] (see also [8, Theorems 2.3&2.4] and [13, Theorem 2.3]), we have
\[
P(\gamma_{r_1/s, \ldots, r_p/s}(f_1, \ldots, f_p)) = \mathcal{G}_s(f_1, \ldots, f_1, f_2, \ldots, f_2, \ldots, f_p, \ldots, f_p)
\]
\[
= \hat{P}(f_1, \ldots, f_1, f_2, \ldots, f_2, \ldots, f_p, \ldots, f_p).
\]
Thus the first implication of the theorem holds.

To prove the second implication, suppose that

$$P\left(\frac{\gamma_{r_1/s}}{r_1 \text{ copies}}, \ldots, \frac{\gamma_{r_p/s}}{r_p \text{ copies}}(f_1, \ldots, f_p)\right) = \tilde{P}(f_1, \ldots, f_1, f_2, \ldots, f_2, \ldots, f_p, \ldots, f_p)$$

holds for every $f_1, \ldots, f_p \in E_+$. Let $f, g \in E_+$ be disjoint. From the binomial theorem we have

$$P(f + g) = P(f) + P(g) + \sum_{q=1}^{s-1} \binom{s}{q} \tilde{P}(f, \ldots, f, g, \ldots, g).$$

Next put $j \in \{1, \ldots, p-1\}$. By [6, Theorem 3.7], we have

$$\frac{\gamma_{r_1/s}}{j \text{ copies}} \frac{\gamma_{r_p/s}}{p-j \text{ copies}}(f, \ldots, f, g, \ldots, g) = \frac{1}{s} \inf_{\theta_k \in (0, \infty)} \left\{ \sum_{k=1}^{j} r_k \theta_k f + \sum_{k=j+1}^{p} r_k \theta_k g : \prod_{k=1}^{p} \theta_k^{r_k/s} = 1 \right\}.$$

Note that $\frac{\gamma_{r_1/s}}{j \text{ copies}} \frac{\gamma_{r_p/s}}{p-j \text{ copies}}(f, \ldots, f, g, \ldots, g) \geq 0$ follows from the positivity of $f$ and $g$. Suppose next that

$$a \leq \sum_{k=1}^{j} r_k \theta_k f + \sum_{k=j+1}^{p} r_k \theta_k g$$

for all $\theta_1, \ldots, \theta_p \in (0, \infty)$ such that $\prod_{k=1}^{p} \theta_k^{r_k/s} = 1$. Then $a = a_1 + a_2 \in I_f \oplus I_g$, where $I_f$ and $I_g$ are the principal ideals generated by $f$ and $g$, respectively. Then

$$a_1 \leq \sum_{k=1}^{j} r_k \theta_k f \quad \text{and} \quad a_2 \leq \sum_{k=j+1}^{p} r_k \theta_k g$$

both hold for all $\theta_1, \ldots, \theta_p \in (0, \infty)$ with $\prod_{k=1}^{p} \theta_k^{r_k/s} = 1$. Thus $a \leq 0$ and we obtain that

$$\frac{\gamma_{r_1/s}}{j \text{ copies}} \frac{\gamma_{r_p/s}}{p-j \text{ copies}}(f, \ldots, f, g, \ldots, g) = 0 \text{ for all } j \in \{1, \ldots, p-1\}.$$

Next let $q \in \{1, \ldots, s-1\}$ be arbitrary. Since $\frac{\gamma_{r_1/s}}{j \text{ copies}} \frac{\gamma_{r_p/s}}{p-j \text{ copies}}$ is a completely partitioned weighted geometric mean, there exist $\alpha_k \in \{0, 1\}$ ($k = 1, \ldots, p$) such that

$$q = \sum_{k=1}^{p} \alpha_k r_k.$$

For $\alpha \in \{0, 1\}$ define $\bar{\alpha} = \alpha + 1 \mod 2$. Then we have

$$s - q = \sum_{k=1}^{p} \bar{\alpha}_k r_k.$$
Since $\hat{P}$ is symmetric, we can without loss of generality suppose that
\[ \sum_{k=1}^{j} r_k = q \]
and
\[ \sum_{k=j+1}^{p} r_k = s - q, \]
for some $j \in \{1, \ldots, p - 1\}$. We then by assumption obtain
\[
\hat{P}(f, \ldots, f, g, \ldots, g) = \hat{P}(f, \ldots, f, f, \ldots, f, g, \ldots, g, \ldots, g) = P(\gamma_{r_1/s, \ldots, r_p/s}(f, \ldots, f, g, \ldots, g)) = 0.
\]
Since $q \in \{1, \ldots, s - 1\}$ was arbitrary, we have
\[
P(f + g) = P(f) + P(g) + \sum_{q=1}^{s-1} \left( \sum_{k=1}^{q} r_k \sum_{k=q+1}^{p} r_k \right) \hat{P}(f, \ldots, f, g, \ldots, g).
\]
Hence $P$ is positively orthogonally additive. Finally, it follows from [13, Theorem 2.3] that $\hat{P}$ is orthogonally additive.

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Department of Mathematics and Applied Mathematics, University of Pretoria, Private Bag X20, Hatfield 0028, South Africa and Unit for BMI, North-West University, Private Bag X6001, Potchefstroom, 2520, South Africa

*Email address: cmschwanke26@gmail.com*