DISTRIBUTED ALGORITHMS FOR FRACTIONAL COLORING

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Abstract. In this paper we study fractional coloring from the angle of distributed computing. Fractional coloring is the linear relaxation of the classical notion of coloring, and has many applications, in particular in scheduling. It is known that for every real \( \alpha > 1 \) and integer \( \Delta \), a fractional coloring of total weight at most \( \alpha (\Delta + 1) \) can be obtained deterministically in a single round in graphs of maximum degree \( \Delta \), in the LOCAL model of computation. However, a major issue of this result is that the output of each vertex has unbounded size. Here we prove that even if we impose the more realistic assumption that the output of each vertex has constant size, we can find fractional colourings with a weight arbitrarily close to known tight bounds for the fractional chromatic number in several cases of interest. Moreover, we improve on classical bounds on the chromatic number by considering the fractional chromatic number instead, without significantly increasing the output size and the round complexity of the existing algorithms.

1. Introduction

A (proper) \( k \)-coloring of a graph \( G \) is an assignment of colors to the vertices of \( G \), such that adjacent vertices receive different colors. This is the same as a partition or covering of the vertices of \( G \) into \( k \) independent sets. This has many applications in physical networks; for instance most scheduling problems can be expressed into a coloring problem in the underlying graph. When the resources at play inside the network are fractionable, it is more relevant to consider the linear relaxation of this problem, where one wants to assign weights \( x_S \in [0,1] \) to the independent sets \( S \) of \( G \), so that for each vertex \( v \) of \( G \), the sum of the weights \( x_S \) of the independent sets \( S \) containing \( v \) is at least 1, and the objective is to minimize the sum of the weights \( x_S \). The solution of this linear program is the fractional chromatic number of \( G \), denoted by \( \chi_f(G) \). The definition shows that \( \chi_f \) is rational and that \( \omega(G) \leq \chi_f(G) \leq \chi(G) \) for any graph \( G \) where \( \omega(G) \) denotes the clique number of \( G \) (maximum size of a set of pairwise adjacent vertices), and \( \chi(G) \) denotes the usual chromatic number (minimum \( k \) such that \( G \) admits a proper \( k \)-coloring). A more polyhedral definition of \( \chi_f \), which is not difficult to derive from the definition above, is that \( \chi_f(G) \) is the minimum \( x \) such that there is a probability distribution on the independent sets of \( G \), such that each vertex appears in a random independent set (drawn from this probability distribution) with probability at least \( \frac{1}{x} \).

In this paper, we study fractional coloring from the angle of distributed algorithms. In this context, each vertex outputs its “part” of the solution, and in a Locally Checkable Labelling (LCL) this part should be of constant size. For instance, in a distributed algorithm for proper \( k \)-coloring of \( G \), each vertex can output its color (an integer in \( [k] = \{1, \ldots, k\} \)), and in a distributed algorithm for (maximal or maximum) independent set, each vertex can output a bit saying whether it belongs to the independent set. In both cases the fact that the solution is correct can then be checked locally, in the sense that adjacent vertices only need to compare...
their outputs, and if there is no local conflict then the global solution is correct. For more details about the distributed aspects of graph coloring, the reader is referred to the book [3].

Looking back at the polyhedral definition of fractional coloring introduced above, a first possibility would be to design a randomized distributed algorithm producing a (random) independent set, in which each vertex has a large probability to be selected (in this case, the output of each vertex is a single bit, telling whether it belongs to the chosen independent set). A classical algorithm in this vein is the following: Each vertex is assigned a random identifier, and joins the independent set if its identifier is smaller than that of all its neighbors. This clearly produces an independent set, and it is not difficult to prove that each vertex is selected with probability at least \( \frac{1}{d(v) + 1} \) [4], so in particular this 1-round randomized algorithm witnesses the fact that the fractional chromatic number of graphs of maximum degree \( \Delta \) is at most \( \Delta + 1 \). Note that factor-of-IID algorithms for independent sets introduced in the past years are of this form (see for instance [9, 10]). This leaves the question of how to produce a deterministic distributed algorithm for fractional coloring. A solution explored in [14] is to assign to each independent set \( S \) of total weight \( x_S \) (or a finite union of intervals of total length \( x_S \)), where \( x_S \) is the weight defined above in the linear programming definition of fractional coloring, such that all \( I_S \) are pairwise disjoint. The output of each vertex \( v \) is then the union of all subsets \( I_S \) of \( S \) such that \( x_S \) is at least \( 1 \). Each vertex can check that its output (which is a finite union of intervals) has total length at least 1, and pairs of adjacent vertices can check that their outputs are disjoint, so the fact that this is a fractional coloring can be checked locally. If the set of identifiers of the vertices of \( G \) is known to all the vertices in advance (for instance if there are \( n \) vertices and the identifiers are \( \{1, \ldots, n\} \)), it is not difficult to transform the 1-round randomized algorithm described above into a 1-round deterministic algorithm producing such an output and with total weight at most \( \Delta + 1 \). The main result of [14] is a 1-round deterministic algorithm producing such an output and with total weight at most \( \alpha \Delta + 1 \) (for any \( \alpha > 1 \)), when the set of identifiers are not known in advance. As observed in [14], the unbounded size of the output implies that this algorithm is unusable in practice. In this paper, we explore a different way to design deterministic distributed algorithms producing fractional colorings of small total weight.

Given two integers \( p \geq q \geq 1 \), a \((p:q)\)-coloring of a graph \( G \) is an assignment of \( q \)-element subsets of \( [p] \) to the vertices of \( G \), such that the sets assigned to any two adjacent vertices are disjoint. An alternative view is that a \((p:q)\)-coloring of \( G \) is precisely a homomorphism from \( G \) to the Kneser graph \( KG(p, q) \), which is the graph whose vertices are the \( q \)-element subsets of \( [p] \), and in which two vertices are adjacent if the corresponding subsets of \( [p] \) are disjoint. The weight of a \((p:q)\)-coloring \( c \) is \( w(c) = p/q \).

The fractional chromatic number \( \chi_f(G) \) can be equivalently defined as the infimum of \( \{ \frac{q}{q} | G \text{ has a } (p:q)\text{-coloring} \} \) (as before, it can be proved that this infimum is indeed a minimum). Observe that a \((p:1)\)-coloring is a (proper) \( p \)-coloring. It is well known that the Kneser graph \( KG(p, q) \) has fractional chromatic number \( \frac{p}{q} \), while Lovász famously proved [16] that its chromatic number is \( p - 2q + 2 \) using topological methods. This shows in particular that \( \chi_f(G) \) and \( \chi(G) \) can be arbitrarily far apart.

This definition of the fractional chromatic number gives a natural way to produce distributed fractional colorings, while keeping the output of each vertex bounded. It suffices to fix the integer \( q \geq 1 \), and ask for the smallest integer \( p \) such that \( G \) has a \((p:q)\)-coloring; then the output of each vertex is the sequence of its \( q \) colors from \( [p] \), which can be encoded in a bit-string with at most \( q \log p = q(\log q + \log w(c)) \) bits (in the remainder of the paper, the output size always refers to the number of bits in this string). The requirement that the output
of each vertex has bounded size is critical in the case of fractional colourings, since in general the smallest integers $p$ and $q$ such that $\chi_f(G) = \frac{p}{q}$ can be exponential in $|V(G)|$ \cite{11}.

In addition to the classical applications of fractional coloring in scheduling (see \cite{14}), there is another more theoretical motivation for studying this problem (or the relaxation above where $q$ is fixed). In $n$-vertex graphs of maximum degree $\Delta$ (which is assumed to be a constant), a coloring with $\Delta + 1$ colors can be found in $O(\log^* n)$ rounds \cite{13, 15} in the LOCAL model of computation (which will be introduced formally below). On the other hand, Brooks’ Theorem says that if $\Delta \geq 3$, any graph of maximum degree $\Delta$ with no clique $K_{\Delta+1}$ is $\Delta$-colorable, and finding such a coloring has proved to be an interesting problem of intermediate complexity in distributed computing. It was proved that the round complexity for computing such a coloring is $\Omega(\log \log n)$ for randomized algorithms \cite{7} and $\Omega(\log n)$ for deterministic algorithms \cite{5}. Thus a large complexity gap appears between $\Delta$ and $\Delta + 1$ colors, and since the values are integral it is all that can be said about this problem. However, if the number of colors is real, or rational, we can try to have a more precise idea of the precise location of the complexity threshold in the interval $[\Delta, \Delta + 1]$. In Section 3 we will show that for fractional coloring, the complexity threshold is arbitrarily close from $\Delta$: namely finding a fractional $\Delta$-coloring is as difficult as finding a $\Delta$-coloring, but for any fixed $\varepsilon > 0$ and $\Delta$, a fractional $(\Delta + \varepsilon)$-coloring with output size of $O(\frac{1}{\varepsilon} \log \frac{\Delta}{\varepsilon})$ bits per vertex can be found in $O_{\varepsilon}(\log^* n)$ rounds in graphs of maximum degree $\Delta$ with no $K_{\Delta+1}$.

**Theorem 1.** For any integer $q \geq 1$, and any $n$-vertex graph $G$ of maximum degree $\Delta \geq 3$, without $K_{\Delta+1}$, a $(q\Delta + 1:q)$-coloring of $G$ can be computed in $O(q\Delta^2 + q \log^* n)$ rounds deterministically in the LOCAL model.

There are other similar complexity thresholds in distributed graph coloring. For instance, it was proved that $d$-dimensional grids can be colored with 4 colors in $O(\log^* n)$ rounds, while computing a 3-coloring in a 2-dimensional $n \times n$-grid takes $\Omega(n)$ rounds \cite{6}. For (almost) vertex-transitive graphs like grids finding minimum fractional colorings is essentially equivalent to finding maximum independent sets, and simple local randomized algorithms approaching the optimal independent set in grids can be used to produce $(2 + \varepsilon)$-fractional colorings with small output (see for instance \cite{10}). In Section 4 we will show that for any fixed $\varepsilon > 0$ and $d \geq 1$, a fractional $(2 + \varepsilon)$-coloring of the $d$-dimensional grid of dimension $n \times \cdots \times n$ with output size of $O(\frac{1}{\varepsilon} \log(\frac{d}{\varepsilon}))$ bits per vertex can be computed deterministically in $O_{\varepsilon,d}(\log^* n)$ rounds, while it can be easily observed that finding a $(2q:q)$-coloring takes $\Omega(n)$ rounds (even if $d = 1$, i.e. when the graph is a path).

**Theorem 2.** For every integers $d \geq 1$ and $q \geq 1$, a $(2q+4 \cdot 6^d : q)$-coloring of the $d$-dimensional grid $G(n,d)$ can be found in $O(d\ell(2\ell)^d + d\ell \log^* n)$ rounds deterministically in the LOCAL model, where $\ell = q + 2 \cdot 6^d$.

The last observation implies in particular that $(2q:q)$-coloring trees takes $\Omega(n)$ rounds. On the other hand, trees can be colored with 3 colors in $O(\log n)$ rounds and this is best possible (even with 3 replaced by an arbitrary number of colors) \cite{13, 15}. In Section 5 we prove that, for every $\varepsilon > 0$, graphs of maximum average degree at most $2 + \varepsilon/40$ and large girth can be $(2 + \varepsilon)$-colored in $O_{\varepsilon}(\log n)$ rounds with output size of $O(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon})$ bits per vertex.

**Theorem 3.** Let $G$ be an $n$-vertex graph with girth at least $2q + 2$, and mad($G$) $\leq 2 + \frac{1}{40q}$, for some fixed $q \geq 1$. Then a $(2q + 1:q)$-coloring of $G$ can be computed deterministically in $O(q \log n + q^2)$ rounds in the LOCAL model.
This implies that trees, and more generally graphs of sufficiently large girth from any minor-closed class can be \((2 + \varepsilon)-colored in \(O(\log n)\) rounds, for any \(\varepsilon > 0\). Note that the assumption that the girth is large cannot be avoided, as a cycle of length \(2q - 1\) has fractional chromatic number equal to \(2 + \frac{1}{2q - 1} > 2 + \frac{1}{q}\).

We conjecture that more generally, graphs of girth \(\Omega(q)\) and maximum average degree \(k + O(1/q)\) (where \(k \geq 2\) and \(q\) are integers) have a \((kq + 1:q)\)-coloring that can be computed efficiently by a deterministic algorithm in the \(\text{LOCAL}\) model (the fact that such a coloring exists is a simple consequence of [17] but the proof there uses flows and does not seem to be efficiently implementable in the \(\text{LOCAL}\) model).

2. Preliminaries

2.1. The \(\text{LOCAL}\) model of computation. All our results are proved in the \(\text{LOCAL}\) model, introduced by Linial [15]. We consider a network, in the form of an \(n\)-vertex graph \(G\) whose vertices have unbounded computational power, and whose edges are communication links between the corresponding vertices. We are given a combinatorial problem that we need to solve in the graph \(G\). In the case of deterministic algorithms, each vertex of \(G\) starts with an arbitrary identifier (an integer between 1 and \(n^c\), for some constant \(c \geq 1\)). For randomized algorithms, each vertex starts instead with a collection of (private) random bits. The vertices then exchange messages (of unbounded size) with their neighbors in synchronous rounds, and after a fixed number of rounds (the \(\text{round complexity of the algorithm}\)), each vertex outputs its local “part” of the global solution of the problem. This could for instance be the color of the vertex in a proper \(k\)-coloring. In \(\text{Locally Checkable Labelling (LCL)}\) problems, this output has to be of constant size, and should be checkable locally, in the sense that the solution is correct globally if and only if it is correct in all neighborhoods of some (constant) radius. \(\text{LCL}\) problems include problems like \(k\)-coloring (with constant \(k\)), or maximal independent set, but not maximum independent set (for instance), and are central in the field of distributed algorithms.

It turns out that with the assumption that messages have unbounded size, vertices can just send to their neighbors at each round all the information that they have received so far, and in \(t\) rounds each vertex \(v\) “knows” its neighborhood \(B_t(v)\) at distance \(t\) (the set of all vertices at distance at most \(t\) from \(v\)). More specifically \(v\) knows the labelled subgraph of \(G\) induced by \(B_t(v)\) (where the labels are the identifiers of the vertices), and nothing more, and the output of \(v\) is based solely on this information.

The goal is to minimize the round complexity. Since in \(t\) rounds each vertex sees its neighborhood at distance \(t\), after a number rounds equal to the diameter of \(G\), each vertex sees the entire graph. Since each vertex has unbounded computational power, a distinguished vertex (the vertex with the smallest identifier, say) can compute an optimal solution of the problem and communicate this solution to all the vertices of the graph. This shows that any problem can be solved in a number of rounds equal to the diameter of the graph, which is at most \(n\) when \(G\) is connected. The goal is to obtain algorithms that are significantly more efficient, i.e. of round complexity \(O(\log n)\), or even \(O(\log^* n)\), where \(\log^* n\) is the number of times we have to iterate the logarithm, starting with \(n\), to reach a value in \((0, 1]\).

2.2. A useful list-coloring result. We will need the following consequence of a result of Aubry, Godin and Togni [2, Corollary 8] (see also [8]).

**Theorem 4** ([2]). Let \(q \geq 1\) be an integer and let \(P = v_1, v_2, \ldots, v_{2q+1}\) be a path. Assume that for \(i \in \{1, 2q + 1\}\) the vertex \(v_i\) has a list \(L(v_i)\) of at least \(q + 1\) colors, and for any \(2 \leq i \leq 2q\),
$v_i$ has a list $L(v_i)$ of at least $2q + 1$ colors. Then each vertex $v_i$ of $P$ can be assigned a subset $S_i \subseteq L(v_i)$ of $q$ colors, so that adjacent vertices are assigned disjoint sets.

3. Maximum degree

In a graph $G$, we say that a path $P$ is an induced path if the subgraph of $G$ induced by $V(P)$, the vertex set of $P$, is a path. Note that shortest paths are induced paths, and in particular every connected graph that has no induced path on $k$ vertices has diameter at most $k - 2$.

**Proof of Theorem 7.** The first step is to construct the graph $H_1$, whose nodes are all the induced paths on $2q + 1$ vertices in $G$, with an edge between two nodes of $H_1$ if the corresponding paths in $G$ share at least one vertex. So $H_1$ can be seen as the intersection graph of the induced paths of length $2q + 1$ of $G$. Note that any communication in $H_1$ can be emulated in $G$, by incurring a multiplicative factor of $O(q)$ on the round complexity. Note that $H_1$ has at most $n \cdot \Delta^{2q}$ vertices and maximum degree $O(q^2 \Delta^{2q})$ (given a path $P$, one has $O(q^2)$ possible choices for the position of the intersection vertex $x$ on $P$ and on an intersecting path, then at most $\Delta^{2q}$ possible ways of extending $x$ into such an intersecting path). It follows that a maximal independent set $S_1$ in $H_1$ can be computed in $O(\Delta(H_1) + \log^*(|V(H_1)|)) = O(q^2 \Delta^{2q} + \log^*(\Delta^{2q} n)) = O(q^2 \Delta^{2q} + \log^* n)$ rounds in $G$ [15], and thus in $O(q^3 \Delta^{2q} + q \log^* n)$ rounds in $G$.

Observe that the set $S_1$ corresponds to a set of vertex-disjoint induced paths of length $2q + 1$ in $G$. Let $\mathcal{P} = \bigcup_{P \in S_1} V(P)$. By maximality of $S_1$, the graph $G - \mathcal{P}$ has no induced path of length $2q + 1$, and in particular each connected component $C$ of $G - \mathcal{P}$ has diameter at most $2q$. Each such component $C$ has a $(q\Delta + 1: q)$-coloring $c$ (indeed, $C$ even has a $(q\Delta: q)$-coloring, by Brooks’ theorem), which can be computed in $O(q)$ rounds. Our next step is to extend this coloring $c$ of $G - \mathcal{P}$ to $\mathcal{P}$. To this end, define a graph $H_2$ whose nodes are the elements of $S_1$, in which two nodes are adjacent if the corresponding paths are adjacent in $G$ (i.e. some edge of $G$ has a vertex in each of these paths). Observe that $H_2$ has at most $n$ nodes and maximum degree $O(q\Delta)$. So a proper coloring $c_2$ of $H_2$ with $t = O(q\Delta)$ colors $1, 2, \ldots, t$ can be found in $O(q\Delta(H_2) + \log^* n)$ rounds in $H_2$ (and thus in $O(q^2 \Delta + q \log^* n)$ rounds in $G$).

For each color $i$, consider the paths of $\mathcal{P}$ of color $i$ in $c_2$. We will extend the current partial coloring of $G$ to these paths (which are pairwise non-adjacent by definition) using Theorem 4. For each of these paths, each vertex starts with $q\Delta + 1$ available colors and the coloring of the neighborhood of this path forbids at most $q(\Delta - 2)$ colors for each internal vertex of the path, and at most $q(\Delta - 1)$ colors for each endpoint of the path. Thus each internal vertex of a path of $\mathcal{P}$ has a list of $2q + 1$ colors and each endpoint has a list of $q + 1$ colors, as required for the application of Theorem 4. The extension thus takes $t = O(q\Delta)$ steps, each taking $O(q)$ rounds, and thus the final round complexity is $O(q^3 \Delta^{2q} + q \log^* n)$.

We now prove that finding a $(q\Delta: q)$-coloring is significantly harder. We will use a reduction to the sinkless orientation problem: given a bipartite $n$-vertex 3-regular graph $G$, we have to find an orientation of the edges of $G$ so that each vertex has at least one outgoing edge. It was proved that in the LOCAL model this takes $\Omega(\log \log n)$ rounds for a randomized algorithm [7] and $\Omega(\log n)$ for a deterministic algorithm [5].

**Theorem 5.** For any integer $q \geq 1$, obtaining a $(3q: q)$-coloring of an $n$-vertex 3-regular graph with no $K_4$ takes $\Omega(\log \log n)$ rounds for a randomized algorithm and $\Omega(\log n)$ rounds for a deterministic algorithm in the LOCAL model.

**Proof.** Consider a bipartite 3-regular $n$-vertex graph $G$, in which we want to compute a sinkless orientation. Let $H$ be the $3n$-vertex 3-regular graph obtained from $G$ by replacing each vertex $v$ by a triangle $x_v, y_v, z_v$ (see Figure 1). Assume that we can compute a $(3q: q)$-coloring $c$ of
Figure 1. The construction of $H$ from $G$ in the proof of Theorem 5.

$H$. Note that each triangle of $H$ uses all $3q$ colors, and thus the set $S$ of vertices of $H$ whose set of colors in $c$ contains the color 1 intersects each triangle $x_v, y_v, z_v$ of $H$ in a single vertex (say $x_v$, up to renaming of the vertices of $H$). Let $e_v$ be the edge of $H$ that contains $x_v$, but is disjoint from $y_v$ and $z_v$. Then for every vertex $v$ of $G$ we orient $e_v$ from $v$ to the other endpoint of $e_v$ in $H$. Note that this gives a partial orientation of $H$ (an edge cannot be oriented in both directions, since otherwise the two endpoints contain color 1, which is a contradiction), which can be transferred to a partial orientation of $G$ by contracting each triangle $x_v, y_v, z_v$ back to the vertex $v$. Since $S$ intersects each triangle of $H$, the resulting partial orientation of $G$ is sinkless, as desired.

4. Coloring the grid

Let $G(n, d)$ be the $d$-dimensional grid with vertex set $[n]^d$, i.e. two distinct vertices $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ in $[n]^d$ are adjacent in $G(n, d)$ if and only if $d_1(x, y) = \sum_{i=1}^{d} |x_i - y_i| = 1$ (where $d_1$ denotes the usual taxicab distance). We assume that all the vertices have their identifier (but do not have access to their own coordinates in the grid). We note that our results do not assume any knowledge of directions in the grid (i.e. a consistent orientation of edges, such as South→North and West→East in 2 dimensions), in contrast with the results of [6].

Note that the distance between vertices in the grid coincide with the taxicab distance $d_1$ in $\mathbb{R}^d$. In this section it will be convenient to work instead with the Chebyshev distance $d_\infty(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$, since balls with respect to $d_\infty$ are grids themselves, as well as their (possibly empty) pairwise intersections. We observe that any communication between two nodes at distance $d_\infty$ at most $\ell$ can be emulated within $d \cdot \ell$ rounds, hence working with $d_\infty$ rather than with $d_1$ does not have a sensible impact in terms of round complexity. In the remainder of this section, all distances refer to the $d_\infty$-distance.

Proof of Theorem 2. Let $G = G(n, d)$ and $\ell = q + 2 \cdot 6^d$. We assume that $n \geq 2\ell$, for otherwise $G$ has diameter at most $O(\ell)$ and a desired coloring can be found in $O(\ell)$ rounds. We start by finding an inclusionwise maximal independent set $I$ of $G[\ell]$, the graph obtained from $G$ by linking each pair of vertices at $(d_\infty)$-distance at most $\ell$ from each other, which are therefore linked by a path of length at most $d\ell$ in $G$. Note that the maximum degree of $G[\ell]$ corresponds to the maximum size of a ball of radius $\ell$ in a $d$-dimensional grid, that is at most $(2\ell + 1)^d$; therefore $I$ can be constructed in $O((2\ell + 1)^d + \log^* n)$ rounds in $G[\ell]$ [15], and this can be emulated in $O(d\ell(2\ell)^d + d\ell \log^* n)$ rounds in $G$. 

A given vertex $x$ of the grid can find the list $L_x$ of vertices of coordinates $x + e$ for every $e \in \{-1, 1\}^d$ within $d$ rounds (with no possibility of distinguishing any pair $y, z$ of elements in that list when $y$ is the image of $z$ by an automorphism of the rooted ball of radius $d$ centered in $x$). This can be done with the following procedure. Given a vertex $y \in V(G)$, we say that a vertex $x \in V(G)$ is a 1-neighbour of $y$ if $N(z)$ contains $y$, and that it is an $i$-neighbour of $y$ if $N(z)$ contains at least two $(i-1)$-neighbours of $y$, for every $2 \leq i \leq d$. Then $L_x$ is the list of vertices $y \in V(G)$ such that $x$ is a $d$-neighbour of $y$, and this list can be found within $d$ rounds. Given $x$ and $y = x + e \in L_x$, finding $z = x + 2e$ can be done in $2d$ rounds, since $z$ is the vertex in $L_y$ furthest away from $x$ with respect to the $d_1$-distance in $G$.

The next step of the coloring procedure is as follows. Every vertex $x \in I$ chooses any vertex $y = x + e \in L_x$ in such a way that $x[i] := x + i \cdot e_x \in [n]^d$ is well-defined for every $i \leq \ell$ (this is possible since $n \geq 2\ell$, and such a direction can be chosen in $\ell \cdot d$ rounds).

For any $x \in I$ and $1 \leq i \leq \ell$, we define a set $B(x, i)$ as follows. Each vertex $x$ considers the vertex $y \in I$ of smallest identifier such that $d_{\infty}(y, x[i]) \leq 2\ell$ and joins the set $B(x, i)$. Equivalently, $B(x, i)$ is the ball of center $x[i]$ and radius $2\ell$, in which we remove all vertices at distance at most $2\ell$ from $x'[i]$ for some $x' \in I$ of smaller identifier than that of $x$. We also let $\overline{B}(x, i)$ be obtained from $B(x, i)$ after removing all vertices at distance exactly $2\ell$ from $x[i]$. When some vertex $v$ is in $B(x, i) \setminus \overline{B}(x, i)$, we say that $v$ is a boundary vertex for $x[i]$.

Every vertex $v \in V(G)$ is at distance at most $\ell$ from at least one vertex $x \in I$ by maximality of $I$, and therefore at distance at most $2\ell$ from $x[i]$ for every $i \leq \ell$. It follows that for every fixed $1 \leq i \leq \ell$, the collection of the sets $B(x, i)$ over all $x \in I$ forms a partition of $V(G)$.

We now show that for each $i$, no two distinct sets $\overline{B}(x, i)$ and $\overline{B}(y, i)$ (with $x, y \in I$) are connected by an edge. Indeed, assume for the sake of contradiction that there is an edge $uv$ with $u \in \overline{B}(x, i)$ and $v \in \overline{B}(y, i)$, for two distinct vertices $x$ and $y$ in $I$ with $\text{ID}(x) < \text{ID}(y)$, where $\text{ID}(x)$ denotes the identifier of $x$. Then $d_{\infty}(x, u) \leq 2\ell - 1$ and thus $d_{\infty}(x, v) \leq 2\ell$ which contradicts the fact that $v \notin B(x, i)$ since $\text{ID}(x) < \text{ID}(y)$. This shows that no two distinct sets $\overline{B}(x, i)$ and $\overline{B}(y, i)$ are connected by an edge, as desired. This implies that the subgraph $G_i$ of $G$ induced by $\bigcup_{x \in I} \overline{B}(x, i)$ (which is bipartite) has all its components of diameter at $O(\ell)$, and in particular for every $1 \leq i \leq \ell$, we can find a proper 2-coloring $c_i$ of $G_i$ with colors in $\{2i-1, 2i\}$ within $O(\ell)$ rounds. We now show that the union of these colorings over all $1 \leq i \leq \ell$ yields a $(2q + 4 \cdot 6^d : q)$-coloring of $G$.

The total number of colors is $2\ell = 2q + 4 \cdot 6^d$, so there remains to show that each vertex $v \in V(G)$ is assigned at least $q$ colors in $c$. This is equivalent to showing that every vertex $v \in V(G)$ is a boundary vertex for $x[i]$ for at most $2 \cdot 6^d$ different combinations of $x$ and $i$. If $v$ is a boundary vertex for $x[i]$, then $x[i]$ lies on the boundary of the ball $B_v$ of center $v$ and radius $2\ell$. Note that every line intersects the boundary of a convex polytope in at most two points or in a segment, and in the latter case the line is contained in the hyperplane defining a facet. Since we have chosen the directions $e_x \in \{-1, 1\}^d$ while balls in $d_{\infty}$ are grids (bounded by axis-parallel hyperplanes), this shows that for every vertex $x \in I$, the set of vertices $\{x[i] : 1 \leq i \leq \ell\}$ intersects the boundary of $B_v$ at most twice, and if the intersection is non-empty then $x$ is at distance at most $3\ell$ from $v$. For a fixed vertex $x$ there can be at most $6^d$ vertices in $N_{G[\infty]}(v) \cap I$. To see this, for every $(i_1, \ldots, i_d) \in \{-3, -2, \ldots, 2\}^d$, we let $S_{(i_1, \ldots, i_d)}$ be the set of vertices $y \in V(G)$ of coordinates $(y_1, \ldots, y_d)$ satisfying $y_j + i_j \cdot \ell \leq y_j \leq y_j + (i_j + 1) \cdot \ell$ for every $1 \leq j \leq d$. It is straightforward that the diameter of $S_{(i_1, \ldots, i_d)}$ is $\ell$, so it contains at most one element of $I$. Since moreover the collection $(S_{(i_1, \ldots, i_d)})$ covers $N_{G[\infty]}(v)$, the results follows, which ends the proof. □
An example of the partition of the 2-dimensional grid into the sets $B(x, i)$; here $2\ell = 4$, and the labels of the vertices are increasing according to the lexicographical ordering of the coordinates. The regions containing the boundary vertices are lighter; the vertices in $I$ are colored in red.

5. Sparse graphs

The average degree of a graph $G = (V, E)$, denoted by $\text{ad}(G)$, is defined as the average of the degrees of the vertices of $G$ (it is equal to 0 if $V$ is empty and to $2|E|/|V|$ otherwise). The maximum average degree of a graph $G$, denoted by $\text{mad}(G)$, is the maximum of the average degrees of the subgraphs of $G$.

The girth of a graph $G$ is the length of a shortest cycle in $G$ (if the graph is acyclic we set its girth to $+\infty$). In this section we are interested in graphs of maximum average degree at most $2 + \varepsilon$, for some small $\varepsilon > 0$, and large girth. We first prove that they contain a linear number of vertices that are either of degree at most 1 or belong to long chains of vertices of degree 2, and such a set can be found efficiently. Note that the condition that the girth is large is necessary (a disjoint union of triangles has average degree 2 but no vertex of degree 1 and no long chain of vertices of degree 2).

**Lemma 1.** Let $G$ be an $n$-vertex graph with girth at least $2q + 2$, and $\text{ad}(G) \leq 2 + \frac{1}{40q}$, for some $q \geq 1$. Let $S$ be the set of vertices of degree at most 1 in $G$, and let $P$ be the set of vertices belonging to a path consisting of at least $2q + 1$ vertices, all of degree 2 in $G$ (in particular each vertex of $P$ has degree 2 in $G$). Then $|S \cup P| \geq \frac{1}{40q} n$. 
Proof. Let \( \varepsilon = \frac{1}{409} \). For \( i = 0, 1, 2 \), let \( V_i \) be the set of vertices of degree \( i \), and let \( V_3^+ \) be the set of vertices of degree at least 3. We denote by \( n_0, n_1, n_2, \) and \( n_3^+ \) the cardinality of these four sets. Since \( G \) has average degree at most \( 2 + \varepsilon \), we have
\[
n_1 + 2n_2 + 3n_3^+ \leq n_1 + 2n_2 + \sum_{v \in V_3^+} d_G(v) \leq (2 + \varepsilon)(n_0 + n_1 + n_2 + n_3^+),
\]
and thus \( n_3^+ \leq \frac{2 + \varepsilon}{1 - \varepsilon} \cdot n_0 + \frac{2 + \varepsilon}{1 - \varepsilon} \cdot n_1 + \frac{\varepsilon}{1 - \varepsilon} \cdot n_2 \leq \frac{5}{2} n_0 + \frac{3}{2} n_1 + \frac{3}{2} \varepsilon n_2 \) (since \( \varepsilon \leq \frac{1}{8} \)).

Let \( H \) be the multigraph obtained from \( G \) by removing all connected components isomorphic to a cycle, and then replacing each maximal path of vertices of degree 2 in \( G \) by a single edge. Note that \( H \) has no vertices of degree 2, and it contains precisely \( n_0 + n_1 + n_3^+ \) vertices. Observe also that the number \( m_H \) of edges of \( H \) is precisely \( \frac{1}{2} \sum_{v \in V_0 \cup V_1 \cup V_3^+} d_G(v) \). It thus follows from the inequalities above that
\[
m_H \leq \frac{1}{2} (2 + \varepsilon)(n_0 + n_1 + n_2 + n_3^+) - n_2
= (1 + \frac{\varepsilon}{2}) (n_0 + n_1 + n_3^+) + \frac{\varepsilon}{2} \cdot n_2
\leq \frac{5}{4} n_0 + \frac{5}{4} n_1 + \frac{5}{4} (\frac{5}{2} n_0 + \frac{3}{2} n_1 + \frac{3}{2} \varepsilon n_2) + \frac{\varepsilon}{2} \cdot n_2.
\leq \frac{5}{2} (n_0 + n_1) + 3 \varepsilon n_2.
\]

We now set \( S = V_0 \cup V_1 \). Thus, if \( |S| = n_0 + n_1 \geq \frac{1}{409} n = \varepsilon n \) we have sets \( S \) and \( P = \emptyset \) satisfying all required properties. Hence, we can assume in the remainder of the proof that
\[
n_0 + n_1 \leq \varepsilon n \leq \varepsilon (n_0 + n_1 + n_2 + n_3^+)
\leq (n_0 + n_1) \left( \frac{7}{2} \varepsilon + n_2 \varepsilon (1 + \frac{3}{2} \varepsilon) \right)
\leq (n_0 + n_1) \left( \frac{7}{2} \varepsilon + 2n_2 \varepsilon \right)
\leq \frac{1}{1 - \frac{\varepsilon}{2}} \cdot 2n_2 \varepsilon \leq 3 \varepsilon n_2,
\]

since \( \varepsilon \leq \frac{1}{12} \). This implies that \( n_3^+ \leq 9 \varepsilon n_2 \) since otherwise the average degree would be larger than \( (2 + \varepsilon) \). And then we finally have \( n \leq (1 + 12 \varepsilon) n_2 \), and \( m_H \leq 18 \varepsilon n_2 \).

In \( G \), remove all the vertices of degree at most 1 and at least 3. We are left with \( m_H \) (possibly empty) paths \( P_1, P_2, \ldots, P_{m_H} \) of vertices of degree 2 in \( G \), each corresponding to an edge of \( H \) (each edge of \( H \) is either a path in \( G \) of vertices of degree two, or a real edge of \( G \) in which case the corresponding path is empty), plus a certain number of cycles (consisting of vertices of degree 2 in \( G \)). Since \( G \) has girth at least \( 2q + 2 \), each vertex of such a cycle is included in a path consisting of at least \( 2q + 1 \) vertices of degree 2 in \( G \), so all these vertices can be added to the set \( P \). We also add to \( P \) all the paths \( P_i \) (\( 1 \leq i \leq m_H \)) containing at least \( 2q + 1 \) vertices. As a consequence, the set \( P \) contains all the vertices of degree 2 in \( G \), except those which only belong to paths \( P_i \) of at most \( 2q \) vertices. So we have \( |P| \geq n_2 - 2q m_H \). By the inequalities above, we have
\[
n_2 - 2q m_H \geq n_2 (1 - 18 \varepsilon \cdot 2q) \geq \frac{1 - 2 \cdot 18 \varepsilon q}{1 + 12 \varepsilon} \cdot n \geq \frac{n}{40},
\]
where the last inequality follows from \( \varepsilon = \frac{1}{409} \). Since \( \frac{n}{40} \geq \varepsilon n \), the set \( P \) contains at least \( \varepsilon n \) vertices, as desired. \( \square \)
We now explain how to apply Lemma 1 to design a distributed algorithm for \((2q + 1:q)\)-coloring. Note that the next result requires that the maximum average degree of \(G\) is close to 2, while the previous result only required that the average degree is close to 2.

Proof of Theorem 3. The algorithm proceeds similarly as in \cite{13}. We set \(G_0 := G\) and for \(i = 1\) to \(\ell = O(\log n)\) we define \(S_{i-1} \cup P_{i-1}\) as the set of vertices of degree at most 2 given by applying Lemma 1 to \(G_{i-1}\) (which has average degree at most \(2 + \frac{1}{\log q}\) since \(\text{mad}(G) \leq 2 + \frac{1}{\log q}\)), and set \(G_i := G_{i-1} - (S_{i-1} \cup P_{i-1})\). Note that each \(S_i \cup P_i\) consists of a set of vertices of \(V(G_i)\) of size at least \(\frac{1}{\log q} |V(G_i)|\), and in particular we can choose \(\ell = O(\log n)\) such that \(G_\ell\) is empty. Note that the induced subgraph \(G[S_i \cup P_i] = G_i[S_i \cup P_i]\) consists of isolated vertices and edges, paths consisting of at least \(2k + 1\) vertices, all of degree 2 in \(G_i\), and cycles consisting of at least \(2q + 2\) vertices, all of degree 2 in \(G_i\).

Note that each \(S_i \cup P_i\) can be computed in \(O(q)\) rounds (each vertex only needs to look at its neighborhood at distance at most \(2q + 1\)), and thus the decomposition of \(G\) into \(S_1, P_1, \ldots, S_\ell, P_\ell\) (and the sequence of graphs \(G_1, \ldots, G_\ell\)) can be computed in \(O(q \log n)\) rounds.

For each \(1 \leq i \leq \ell\), in parallel, compute a maximal independent set \(I_i\) at distance \(2q + 1\) in \(G[S_i \cup P_i]\). Recall that the vertices of \(S_i\) have degree at most 1 in \(G_i\), so they induce isolated vertices or isolated edges in \(G_i\) (and \(G\)), while \(P_i\) induces a disjoint union of cycles of length at least \(2q + 2\) and paths of at least \(2q + 1\) vertices, each consisting only of vertices of degree 2 in \(G_i\). In particular, by maximality of \(I_i\), the set \(P_i - I_i\) induces a collection of disjoint (and pairwise non-adjacent) paths of at least \(2q + 1\) and at most \(4q + 2\) vertices (except the first and last segment of each path of \(P_i\), which might contain fewer vertices). For each path of \(P_i\), discard from \(I_i\) the first and last vertex of \(I_i\) in the path (these two vertices might coincide if a path of \(P_i\) contains a single vertex of \(I_i\)), and call \(I_i'\) the resulting subset of \(I_i\) (note that each vertex \(x \in I_i\) can check in \(O(q)\) rounds if it belongs to \(I_i'\) by inspecting the lengths of the two subpaths of vertices of degree 2 adjacent to \(x\), if any of them is smaller than \(2q + 1\) then \(x \in I_i'\)). By maximality of \(I_i\) and the definition of \(I_i'\), the set \(P_i - I_i'\) induces a collection of disjoint (and pairwise non-adjacent) paths of length at least \(2q + 1\) and at most \(8q + 2\). Note that each graph \(G[S_i \cup P_i]\) has maximum degree at most 2, so a maximal independent set \(I_i\) at distance \(2q + 1\) can be computed in \(O(q^2 + \log^* n)\) rounds. Since the computation of the sets \(I_i\) is made in parallel in each \(G[S_i \cup P_i]\), this step takes \(O(q^2 + \log^* n)\) rounds.

We now color each \(S_i \cup P_i\) in reverse order, i.e. from \(i = \ell - 1\) to 0. For the components induced by \(S_i\) this can be done greedily, since the vertices have degree at most 1 in \(G_i\), they have at most one colored neighbor and thus at most \(q\) forbidden colors (and at least \(q + 1\) available colors). For the components induced by \(P_i\), we start by coloring \(I_i'\) arbitrarily, and then extend the coloring greedily to \(P_i - I_i'\) until each path of uncolored vertices has size precisely \(2q + 1\) (this can be done in \(O(q)\) rounds). We then use Theorem 1 (each endpoint of an uncolored path has a list of at least \(2q + 1 - q \geq q + 1\) available colors). Each coloring extension takes \(O(q)\) rounds, so overall this part takes \(O(q \log n)\) rounds. It follows that the overall round complexity is \(O(q^2 + q \log n)\), as desired. 

This immediately implies the following.

Corollary 6. Let \(G\) be an \(n\)-vertex tree. Then for any fixed \(q \geq 1\), a \((2q + 1:q)\)-coloring of \(G\) can be computed in \(O(q \log n + q^2)\) rounds.

Note that given any \((2q + 1:q)\)-coloring, we can deduce a \((q + 2)\)-coloring in a single round (each vertex chooses the smallest color in its set of \(q\) colors given by the \((2q + 1:q)\)-coloring), while coloring trees with a constant number of colors takes \(\Omega(\log n)\) rounds \cite{15}, so the round complexity in Corollary 6 is best possible.
For $k \geq 1$, a graph $G$ is $k$-path-degenerate if any non-empty subgraph $H$ of $G$ contains a vertex of degree at most 1, or a path consisting of $k$ vertices of degree 2 in $H$.

**Lemma 2.** If $G$ is $k$-path-degenerate, then $\text{mad}(G) \leq 2 + \frac{2}{k}$.

**Proof.** Let $H$ be a subgraph of $G$. Let $n$ and $m$ be the number of vertices and edges of $H$. We prove that $\text{ad}(H) = 2m/n \leq 2 + \frac{2}{k}$ by induction on $n$. If $H$ is empty, then the result is trivial, so assume that $n \geq 1$. Since $G$ is $k$-path-degenerate, $H$ contains a vertex of degree at most 1 or a path of $k$ vertices of degree 2 in $H$. Assume first that $H$ contains a vertex $v$ of degree at most 1. Then $H - v$ contains $n - 1$ vertices and at least $m - 1$ edges, and thus by induction $2 + \frac{2}{k} \geq \text{ad}(H - v) \geq 2\frac{m-1}{n-1}$. It follows that $2m \leq (n - 1)(2 + \frac{2}{k}) + 2 \leq n(2 + \frac{2}{k})$, and thus $\text{ad}(H) \leq 2 + \frac{2}{k}$, as desired. Assume now that $H$ contains a path $P$ of $k$ vertices of degree 2 in $H$. Then $H - P$ contains $n - k$ vertices and $m - k - 1$ edges, and by induction $2 + \frac{2}{k} \geq \text{ad}(H - P) = 2\frac{m-1}{n-1}$. It follows that $2m \leq (n - k)(2 + \frac{2}{k}) + 2k + 2 \leq n(2 + \frac{2}{k})$, thus $\text{ad}(H) \leq 2 + \frac{2}{k}$, as desired. □

It was proved by Gallucio, Goddyn, and Hell [12] that if $C$ is a proper minor-closed class, or a class closed under taking topological minors, then for any $k \geq 1$ there is a girth $g(k)$ such that any graph $G \in C$ with girth at least $g(k)$ is $k$-path-degenerate. Using Lemma 2 and Theorem 3 this immediately implies the following.

**Corollary 7.** For any graph $H$ and integer $q \geq 1$ there is an integer $g$ such that any $n$-vertex graph $G$ of girth at least $g$, excluding $H$ as a minor or topological minor, can be $(2q + 1; q)$-colored in $O(\log n)$ rounds.

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