The coexistence of $p$-wave spin triplet superconductivity and itinerant ferromagnetism

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Abstract

A model for coexistence of $p$-wave spin-triplet superconductivity (SC) and itinerant ferromagnetism (FM) is presented. The Hamiltonian can be diagonalized by using the $so(5)$ algebraic coherent state. We obtain the coupling equations of the magnetic exchange energy and superconducting gaps through the double-time Green function. It is found that the ferromagnetisation gives rise to the phase transitions of $p$-wave superconducting states or superfluid of $^3He$. 

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I. INTRODUCTION

Since Ginzburg\[1\] pointed out a possibility of the coexistence between ferromagnetism (FM) and superconductivity (SC) for the magnetization less than the thermodynamic critical field, many experimental investigations were made, for example, for impurity ferromagnetism in a superconductor[2]. As well-known, the coexistence between antiferromagnetism (AFM) and SC looks easy to be realized and observed in several compounds\[3\] because AFM moments spatially averaged over the SC coherent length vanish, but difficult for FM case. The rare exceptions are rare-earth ternary compounds $HoMo_6S_8$ and $ErRh_4B_4$ where in narrow region just below the Curie temperature $T_{FM}$ the coexistence of FM and SC was attained[4]. When the rare-earth 4$f$ moments completely align at lower temperature, SC is wiped out by a strong internal field. It seems that so far not known SC can fully sustain such a large molecular field. As for the theoretical developments, starting by Anderson and Suhl\[5\] there had been discussing the possible coexistence\[6, 7, 8\]. The possibility of a finite momentum paring state coexisting with the long range FM order was presented and SC in metals with a spin-exchange field produced by FM aligned impurities was considered. In such a SC ferromagnet there are two kinds of electrons respectively, responsible for FM and SC. One is localized electrons forming a FM background in metal indirect exchange coupling through itinerant electrons, the other forms Cooper pairs due to the effective attractive interaction. Recently, the theoretical works show the coexistence of weak itinerant FM with s-wave SC\[9, 10\].

On the other hand, the recent discovery of the coexistence FM and SC in $VGe_2$\[11, 12\], and subsequently in $ZrZn_2$\[13\] and $URhGe$\[14\], have shown clearly that the long awaited
spin-triplet SC state is realized in the nature. In these experiments it looks to have a weak and itinerant nature of electrons involved in both FM and SC. This naturally renews our interest in the relationship between FM and SC for p-wave SC (or superfluid) because a spin-triplet may form the Anderson-Morel state with combination of $|↑↑\rangle$ and $|↓↓\rangle$ [15, 16].

In this paper we will consider a general model for the coexistence between p-SC and itinerant FM. As the known result there is so(5) structure in p-SC [17, 18] that is formed by two $su(2)'s$ not commuting with each other where one $su(2)$ describes the attractive BCS interaction and the other for the usual spin operators as well as other 4 generators relating to the transitions. Motivated by Ref. [10], we write the Hamiltonian in two parts:

$$H = H_{SC} + H_{FM}$$

where $H_{SC}$ is the BW type of p-SC Hamiltonian i.e. $H_{SC} = \sum_{k,\alpha} \epsilon_k a_{k,\alpha}^\dagger a_{k,\alpha} + \frac{1}{2} \sum_{k,k',\alpha,\beta} V_{kk'} a_{-k,\alpha}^\dagger a_{k',\beta}^\dagger a_{k,\beta} a_{-k,\alpha}$. After taking the mean-field approximation we have the reduced Hamiltonian of the coexistence of p-SC and itinerant FM:

$$H = \sum_k \epsilon_k (a_{k,\uparrow}^\dagger a_{k,\uparrow} + a_{k,\downarrow}^\dagger a_{k,\downarrow}) - \sum_k (\Delta_{\alpha\beta}(k)a_{k,\beta}^\dagger a_{-k,\alpha}^\dagger) + H.c.$$  

$$+ \sum_k \Delta_{\alpha\beta}(k) <a_{k,\alpha}^\dagger a_{-k,\beta}> - \frac{J M}{2} \sum_k (a_{k,\alpha}^\dagger a_{k,\beta} - a_{k,\beta}^\dagger a_{k,\alpha}) + \frac{1}{2} J M^2 \tag{1}$$

$$M = \frac{1}{2} \sum_k (<a_{k,\uparrow}^\dagger a_{k,\downarrow}> - <a_{k,\downarrow}^\dagger a_{k,\uparrow}>) \tag{2}$$

$$\Delta_{\alpha\beta}(k) = -\frac{1}{2} \sum_{k'} V_{kk'} <a_{k',\alpha}^\dagger a_{-k',\beta}> \tag{3}$$

where $\epsilon_k = \frac{p^2}{2m} - \mu$ is the band energy measured from the chemical potential, and for p-wave attraction pair interaction potential $V_{kk'} = -3V_1(k,k') \mathbf{n} \cdot \mathbf{n'}$ ($\mathbf{n} = \frac{k}{|k|}$). $<\cdots>$ represents the thermodynamic average, $M$ defines the magnetization of the system, and $\Delta_{\alpha\beta}$ is the superconducting gap. We note that the eq.(1) is made up of p-wave SC terms and FM term. The two constant terms in eq.(1) result from the mean-field approximation, the first constant term comes from the BCS interaction and the second one from the exchange coupling. Here
the magnetization defined in eq. 2 arises from a spontaneously breaking of spin rotation
symmetry of the itinerant electrons, which is different from a paramagnetic response to a
magnetic field caused by localized spins. Therefore, both the gap and the magnetic exchange
energy are determined by eq. 3 and eq. 2 self-consistently, unlike in the conventional metal
with magnetic impurities where the exchange energy is considered as an external parameter.

Next we diagonalize the reduced Hamiltonian using Lie algebra so(5) coherent state
approach. The generators of Lie algebra so(5) is expressed as

$$I_{ab}(k) = \begin{pmatrix}
0 & -\frac{1}{2}(T_1^+(k) + T_1(k)) & -F_3(k) & 0 \\
-\frac{1}{2}(T_2^+(k) + T_2(k)) & F_2(k) & -F_1(k) & 0 \\
-\frac{1}{2}(T_3^+(k) + T_3(k)) & Q(k) & \frac{1}{2}i(T_1(k) - T_1^+(k)) & \frac{1}{2}i(T_2(k) - T_2^+(k)) \\
\end{pmatrix}$$

where $F(k) = \frac{1}{2}[S(k) + S(-k)]$, $Q(k) = \frac{1}{2}[S_0(k) + S_0(-k) - 2]$ and $S_i(k) = a_{k\sigma}^\dagger (\sigma_1)_{\alpha\beta} a_k\beta$ and
$T_i(k) = a_{-k\sigma}(\sigma_2\sigma_i)_{\alpha\beta} a_k\beta$ as well as their conjugates ($i = 0, 1, 2, 3, \sigma_0 = 1$ and summation
over the repeated $\alpha$ and $\beta$) is given in . It can be proved that $I_{ab}$ obey the following
commutation relation:

$$[I_{ab}(k), I_{cd}(k')] = -i\delta(k - k') \left( \delta_{ac} I_{bd}(k) + \delta_{bd} I_{ac}(k) - \delta_{ad} I_{bc}(k) - \delta_{bc} I_{ad}(k) \right)$$

and $I_{ab}(k) = -I_{ba}(k)$ ($a, b = 1, 2, 3, 4, 5$) is antisymmetric matrix element.

Therefore, we can rewrite eq. in terms of the generators of the Lie algebra so(5) as
follows:

$$H = \sum_k H(k) - E_0$$

$$H(k) = \epsilon_k Q(k) + \Delta(k) \cdot T^+(k) + \Delta^+(k) \cdot T(k) + JMS_3(k)$$
\[
E_0 = \sum_k [\epsilon_k - \Delta(k) \cdot <T^\dagger(k)>] + \frac{1}{2}JM^2
\]  
(7)

\[
\Delta(k) = \frac{1}{4} \sum_{k'} V_{kk'} <T(k')>
\]  
(8)

\[
M = \frac{1}{2} \sum_k <F_3(k)>
\]  
(9)

Here we emphasized that the set \( \Lambda = \{ \sqrt{2}T_3(k), -\sqrt{2}T_3^\dagger(k), Q(k) \} \) i.e. \( \{ -i\sqrt{2}\pi z, i\sqrt{2}\pi^\dagger z, -Q \} \) in [17] forms the quasi-spin \( \Lambda \). \( \Lambda \) does not commute with spin operators \( S(k) \) that give rise to \( T^\dagger_\pm(k) \) and \( T^\pm_\pm(k) \) which are beyond two \( su(2) \) and the total set forms \( so(5) \). In order to perform the diagonalization of eq.(6) we introduce the unitary transformation \( U(\xi_k) \) such that \( U(\xi_k)H(k)U(\xi_k) = E_{k\downarrow}n_{k\downarrow} + E_{k\uparrow}n_{k\uparrow} \) becomes diagonal for each given momentum \( k \). Following the general strategy [19] we introduce the \( so(5) \)-coherent operators:

\[
U(\xi_k) = \exp\{\xi_k [d(n) \cdot T^\dagger(k)] - H.c.\}
\]  
(10)

where \( \xi_k \) is called the coherent parameter, \( d(n) = (\sin \psi_k \cos \phi_k, \sin \psi_k \sin \phi_k, \cos \psi_k) \) corresponding to the direction of zero spin projection, \( \psi_k \) and \( \phi_k \) are angles in spin space for a given momentum \( k \). Taking the commutation relations for \( so(5) \) into account after lengthy but elementary calculations we derive the following two different solutions for \( d(n) \).

1. When \( \cos \psi_k = 0 \), the direction of the pair orbital angular momentum \( L \) is perpendicular to the direction of zero spin projection \( d \), the energy is split into:

\[
E_{k\uparrow} = (1 + \frac{JM}{\epsilon_k})E_k
\]  
(11)

\[
E_{k\downarrow} = (1 - \frac{JM}{\epsilon_k})E_k
\]  
(12)

\[
E_k = \sqrt{\epsilon_k^2 + 4\left[ \frac{\Delta_{\uparrow\uparrow}(k)^2}{(1 + \frac{JM}{2\epsilon_k})^2} + \frac{\Delta_{\downarrow\downarrow}(k)^2}{(1 - \frac{JM}{2\epsilon_k})^2} \right]}
\]  
(13)
This result exhibits that the SC energy is split by the magnetization $M$. For finite temperature $T$, making use of the double-time Green function we find $M$ and the non-vanishing components of $\Delta_{\alpha\beta}(k)$:

$$M = \frac{1}{4} \sum_k \frac{\epsilon_k}{E_k} \left[ \tanh \frac{\beta}{2} \left( 1 + \frac{JM}{2\epsilon_k} \right) E_k \uparrow - \tanh \frac{\beta}{2} \left( 1 - \frac{JM}{2\epsilon_k} \right) E_k \downarrow \right]$$

$$\Delta_{\uparrow\downarrow}(k) = -\frac{1}{2} \sum_{k'} V_{kk'} \frac{\Delta_{\uparrow\downarrow}(k')}{(\frac{JM}{2\epsilon_{k'}} + 1) E_{k'}} \tanh \frac{\beta E_{k'} \uparrow}{2}$$

$$\Delta_{\downarrow\downarrow}(k) = 0$$

From the gap equations (15) and (16) we read that under finite temperature, FM and $p$-wave SC may coexist in the $p$-wave equal spin pairing state (ABM state), i.e. $\Psi_{AM} \sim e^{\phi} \sin 2|\xi|(|\uparrow\uparrow\rangle + e^{\chi}|\downarrow\downarrow\rangle)$. But if choosing $p_{F}^{\pm} = \sqrt{m^*(2\mu \pm JM)}$, then the phase $A$ of the equal spin pairing state will turn into phase $A_1$ (with only spin up pairing $|\uparrow\uparrow\rangle$) and phase $A_2$ (with only spin down pairing $|\downarrow\downarrow\rangle$) respectively. At temperature $T = 0$, let us distinguish two cases for eqs. (14)-(16):

(a) If $JM = 2\epsilon_k$ or $p_{F}^{\pm} = \sqrt{m^*(2\mu + JM)}$, then the above consistent equations reduce to

$$M = \frac{1}{4} \sum_k \frac{\epsilon_k}{\sqrt{\epsilon_k^2 + |\Delta_{\uparrow\downarrow}(k)|^2}}$$

$$\Delta_{\uparrow\downarrow}(k) = -\frac{1}{2} \sum_{k'} V_{kk'} \frac{\Delta_{\uparrow\downarrow}(k')}{\sqrt{\epsilon_{k'}^2 + |\Delta_{\uparrow\downarrow}(k')|^2}}$$

that indicates the gap equation described by the Anderson-Morel state with only $|\uparrow\uparrow\rangle$ spin pairs. As a physical consequence the existence of FM may turn the p-wave EPS (phase $A$ in $^3He$) into phase $A_1$ with the only state $|\uparrow\uparrow\rangle$. 
(b). if \( JM = -2\epsilon_k \) or \( \overline{p_F} = \sqrt{m^*(2\mu - JM)} \), then we have \( \Delta_{\uparrow\uparrow}(k) = 0 \) and all the \( \uparrow\uparrow \) in (16) and (17) are replaced by \( \downarrow\downarrow \). The gap equation is described by ABM state with only \( \downarrow\downarrow \) spin-down pairs, i.e. the phase \( A \) can be turned into phase \( A_2 \) with only \( \downarrow\downarrow \).

Therefore, the coexistence of FM and \( p \)-wave SC gives rise to the phase transitions from phase \( A \) to phase \( A_1 \) or \( A_2 \). Such a phase transition may be observed in the coexistence of FM and SC for \( p \)-SC and \( ^3\text{He} \) superfluid.

2. When \( \sin \psi_k = 0 \), the direction of the pair orbital angular momentum \( L \) is parallel to the direction of zero spin projection \( d \) and the energy is split into

\[
E_{k\uparrow} = E_k + \frac{JM}{2} \tag{20}
\]

\[
E_{k\downarrow} = E_k - \frac{JM}{2} \tag{21}
\]

\[
E_k = \sqrt{\epsilon_k^2 + 4|\Delta_{\uparrow\downarrow}(k)|^2} \tag{22}
\]

that is the same as the s-wave case \([10]\) which exhibits a two-fold Zeeman splitting effect in the itinerant FM. For finite temperature \( T \) by making use of the double-time Green function the \( M \) and the non-vanishing components of \( \Delta_{\alpha\beta}(k) \) (\( \Delta_{\uparrow\downarrow} = \Delta_{\downarrow\uparrow} \)) can be calculated:

\[
M = \frac{1}{4} \sum_k (\tanh \frac{\beta E_{k\uparrow}}{2} - \tanh \frac{\beta E_{k\downarrow}}{2}) \tag{23}
\]

\[
\Delta_{\uparrow\downarrow}(k) = -\frac{1}{4} \sum_{k'} V_{kk'} \frac{\Delta_{\uparrow\downarrow}(k')}{E_{k'}} (\tanh \frac{\beta E_{k'\uparrow}}{2} + \tanh \frac{\beta E_{k'\downarrow}}{2}) \tag{24}
\]

The above gap equation can be described by the opposite spin pairing state of \( p \)-wave SC, i.e. \( \Psi_{AM} \sim e^{\phi} \sin |\xi| \left[ (\frac{1}{\sqrt{2}}) (\uparrow\downarrow + \downarrow\uparrow) \right] \).

At \( T = 0 \), we obtain the same relations as given by \([10]\): \[p_F^\pm = \sqrt{2m^*\mu \pm m^*\sqrt{(JM)^2 - 16|\Delta_{\uparrow\downarrow}|^2}} \tag{25} \]
\[ M = \frac{1}{4} \int \frac{d^3p}{(2\pi)^3} = \frac{1}{12\pi^2}[(p_F^+)^3 - (p_F^+)^3] \] (26)

\[ \Delta_{\uparrow\downarrow}(k) = -\frac{1}{4} \sum_{k'} V_{kk'} \frac{\Delta_{\uparrow\downarrow}(k')}{\sqrt{\epsilon_{k'}^2 + 4|\Delta_{\uparrow\downarrow}(k')|^2}} \] (27)

The only solutions of eqs. (26) and (27) for the coexistence of SC and FM occur for \( JM > 4|\Delta_{\uparrow\downarrow}| \) that can be discussed as the same as s-wave case given in Ref.[10]. Therefore, the opposite spin pairing triplet state behaviors very similar to the s-wave singlet state in the present of exchange splitting. This leads to the conclusion that the effect of the different exchange splitting of SC state is determined by whether the state contains OSP .(The spin singlet and the opposite spin pairing state of the \( p \)-wave triplet are, by definition, OSP states.) or ESP, and the FM can make transition of the phase \( A \) of SC state to the phase \( A_1 \) or phase \( A_2 \) under ESP state.

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