BIG FREE GROUPS ACTING ON Λ-TREES

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Abstract. The set of homotopy classes of based paths in the Hawaiian earring has a natural \( \mathbb{R} \)-tree structure, but under that metric the action by the fundamental group is not by isometries. Following a suggestion by Cannon and Conner, this paper defines an \( \mathbb{R}^\omega \)-metric that does admit for an isometric action by the fundamental group. The space does not become an \( \mathbb{R}^\omega \)-tree but is 0-hyperbolic and embeds in an \( \mathbb{R}^\omega \)-tree.

Cannon and Conner define big free groups \( BF(c) \) for cardinal number \( c \) which are a generalization of the fundamental group of the Hawaiian earring. They define a big Cayley graph which coincides with the set of homotopy classes of paths in the case of the Hawaiian earring. Instead of inserting real intervals to obtain the Cayley graph, we can insert \( \mathbb{R}^\omega \)-intervals and obtain a new \( \mathbb{R}^\omega \)-tree which admits an isometric action. In fact we do not need all of \( \mathbb{R}^\omega \); we can insert \( \mathbb{Z}^\omega \)-intervals and obtain a \( \mathbb{Z}^\omega \)-tree. In the case of the Hawaiian earring we give a combinatorial description of the \( \mathbb{Z}^\omega \)-tree and the corresponding action.

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1. Introduction

Free groups have the property that their Cayley graphs are trees. Equivalently, the word metric on free groups is 0-hyperbolic. We can view free groups as the fundamental group of a wedge of circles. Let \( c \) be an arbitrary cardinal number and \( J \) be an indexing set of cardinality \( c \). Set \( W(c) = \bigvee_j S^1 \). Then \( \pi_1(W(c)) \) is isomorphic to \( F(c) \), the free group on \( c \) generators. The generators can be realized as the equivalence classes of loops. For each \( j \in J \) let \( a_j \) denote the equivalence class of a loop in \( W(c) \) that goes once around the \( j \)th circle. Then \( A = \{a_j\} \) is a generating set for \( \pi_1(W(c)) \).

The Hawaiian earring \( E \) is a space that stands opposed to the wedge of countably infinitely many circles. We can similarly define a “generating” set \( A \) but this set will not generate \( \pi_1(E) \) since a loop in \( E \) may traverse infinitely many of the circles.

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Motivated by this situation, Cannon and Conner define the big free group BF(c) as the set of possibly infinite products of elements of a generating set of cardinality c \[2\]. They show that if c is countably infinite then BF(c) is isomorphic to \(\pi_1(E)\).

Since words in BF(c) may have infinitely many letters the word metric on BF(c) is not ideal—it would take on infinite values. For the same reason the Cayley graph of BF(c) is not connected and therefore is not a tree. This paper defines a new word metric on BF(c) that distinguishes between different generators and takes on values in \(\mathbb{Z}^c\). It turns out that BF(c) is 0-hyperbolic under this metric and therefore acts on a \(\mathbb{Z}^c\)-tree by isometries.

2. Big free groups and \(\Lambda\)-trees

We start by recalling the definition of big free group.

2.1. Big free groups. The notion of a group defined as the set of possibly infinite products of generators has been developed by several authors. We follow the theory of Cannon and Conner \[2\]. See that paper for a review of other treatments.

Let \(A\) be an alphabet of arbitrary cardinality c and let \(A^{-1}\) denote a formal inverse set for \(A\). A transfinite word is any function \(w : S \rightarrow A \cup A^{-1}\) where \(S\) is totally ordered and \(w^{-1}(a)\) is finite for all \(a \in A \cup A^{-1}\). The condition that each \(a \in A \cup A^{-1}\) appears finitely many times is imposed to avoid the following calculation: \(a^\infty = aa^\infty \Rightarrow a = 1\). Also the Hawaiian earring has the property that no circle can be traversed by a path infinitely many times and its fundamental group is the motivation for the theory. Note that if \(A\) is countable then \(S\) is always countable as the countable union of finite sets.

Two transfinite words \(w_1 : S_1 \rightarrow A \cup A^{-1}\) and \(w_2 : S_2 \rightarrow A \cup A^{-1}\) are identified if there is an order preserving bijection \(\phi : S_1 \rightarrow S_2\) such that \(w_2 \circ \phi = w_1\).

A theory of infinite cancellation is required. Given a totally ordered set \(S\) and \(s, t \in S\) let \([s, t]_S\) denote the interval \(\{r \in S : s \leq r \leq t\}\). A transfinite word \(w : S \rightarrow A \cup A^{-1}\) admits a cancellation * if there is a subset \(T\) of \(S\) and an involution \(* : T \rightarrow T\) such that for each \(t \in T\), \([t, t*]_S = [t, t*]_T\) (\(*\) is complete), \(([t, t*]_T)* = [t, t*]_T\) (\(*\) is noncrossing) and \(w(t*) = w(t)^{-1}\) (\(*\) is an inverse pairing). The restriction of \(w\) to \(S - T\) is a transfinite word that arises from \(w\) via the cancellation *. The symmetric transitive closure of this relation gives an equivalence relation on transfinite words. We say a transfinite word is reduced if it admits no nonempty cancellations. Every word admits a maximal cancellation by Zorn’s Lemma and the resulting word is reduced. There may be more than one maximal cancellation but the resulting word is always the same.

**Theorem 2.1 (2 3.9).** Each equivalence class of transfinite words in contains exactly one reduced word.

The product of two transfinite words \(w_1 : S_1 \rightarrow A \cup A^{-1}\) and \(w_2 : S_2 \rightarrow A \cup A^{-1}\) is defined as the transfinite word \(w_1w_2 : S_1S_2 \rightarrow A \cup A^{-1}\) where \(S_1S_2\) is the disjoint union of \(S_1\) and \(S_2\) (given the obvious ordering) and \(w_1w_2|S_1 \equiv w_1\) and \(w_1w_2|S_2 \equiv w_2\). The inverse of a transfinite word \(w : S \rightarrow A \cup A^{-1}\) is the word \(w^{-1} : S \rightarrow A \cup A^{-1}\) where \(S\) reverses the ordering on \(S\) and \(w^{-1}(s) = w(s)\) for all \(s \in S\). Thus we have a group BF(c), the big free group on an alphabet of cardinality c. We typically represent BF(c) as the set of all reduced transfinite words.

Cannon and Conner define the big Cayley graph \(\Gamma(BF(c))\) \(\text{(2 Section 6)}\) as follows. Given a reduced transfinite word \(w : S \rightarrow A \cup A^{-1}\), form the Dedekind
cut space $\text{Cut}(w) = \text{Cut}(S)$ and then insert the real open interval $(0, 1)$ between adjacent points to form the “big interval” $I_w$. The adjacent points are of the form $(-\infty, s)$ and $(-\infty, s]$ so they correspond to the element $w(s) \in A \cup A^{-1}$. Each inserted interval is labeled by this element. Then $\Gamma(BF(c))$ is formed by taking the disjoint union of the $I_w$ and, for each pair, identifying the largest initial segment on which all of the labels agree. There is an action of $BF(c)$ on $\Gamma(BF(c))$ (see Section 3 of this paper).

In the case of the countably infinite cardinal $\aleph_0$, $\Gamma(BF(\aleph_0))$ is in one-to-one correspondence with the space of homotopy classes of paths in the Hawaiian earring and this correspondence suggests a metric for $\Gamma(BF(\aleph_0))$ where the action is by isometries (see [4, Lemma 2.11]). However, this metric loses the large scale structure of $\Gamma(BF(\aleph_0))$ and does not make it an $\mathbb{R}$-tree. In fact there is no $\mathbb{R}$-tree metric on $\Gamma(BF(\aleph_0))$ for which the action is by isometries (see Proposition 3.1).

Motivated by these issues, Cannon and Conner suggest a “big metric” for $\Gamma(BF(c))$. Their definition uses the tree structure of $\Gamma(BF(c))$ to find the shortest big interval between points $x, y \in \Gamma(BF(c))$. The big metric $d : \Gamma(BF(c)) \times \Gamma(BF(c)) \to \mathbb{R}_{\geq 0}$ counts, for each $a \in A$, the number of occurrences of $a$ and $a^{-1}$ (with fractions occurring at the ends) in that interval. This definition is reminiscent of the notion of a $\Lambda$-metric space with $\Lambda = \mathbb{R}$.

2.2. $\Lambda$-metric spaces. The theory of $\Lambda$-metric spaces is developed in [3] and that text is the reference for the facts stated in this section. Given an Abelian group $\Lambda$ and a total order $\leq$ on $\Lambda$, $\Lambda$ is an ordered Abelian group if for all $a, b, c \in \Lambda$, $a \leq b$ implies $a + c \leq b + c$. Given a set $X$, a $\Lambda$-metric on $X$ is a function $d : X \times X \to \Lambda$ such that the usual conditions are satisfied. For all $x, y, z \in X$,

\begin{enumerate}
  \item $d(x, y) \geq 0$
  \item $d(x, y) = 0$ if and only if $x = y$
  \item $d(x, y) = d(y, x)$
  \item $d(x, y) \leq d(x, z) + d(z, y)$.
\end{enumerate}

The topology induced by the metric is defined by the basic elements $B(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$ where $\epsilon \in \Lambda$ and $\epsilon > 0$.

Let $F(c)$ be the free group on $c$ generators where $c$ is a cardinal number. Given words $w, v \in F(c)$, the word metric counts the number of letters in the reduced form of $w^{-1}v$. Then $F(c)$ under the word metric is a $\mathbb{Z}$-metric space. Notice any $\mathbb{Z}$-metric space is discrete since $\mathbb{Z}$ has a smallest positive element.

As mentioned above the word metric does not work well for $BF(c)$ because words may contain infinitely many letters. Given $w, v \in BF(c)$ we count, for each $a \in A$, the occurrences of $a$ and $a^{-1}$ in $w^{-1}v$ and therefore wish to define a $\mathbb{Z}^c$-metric. Thus we need an order on $\mathbb{Z}^c$. To define the order we use an order on $A$. In the case of the fundamental group of the Hawaiian earring it is natural to consider the order on $A$ when defining a metric since these elements represent circles of decreasing size. In fact we will require $A$ to be well ordered so we define the big free group $BF(o)$ for the ordinal number $o$ of the well ordered set $A$. We start by giving a general definition for lexicographic orders.

\footnote{In [2] Theorem 6.1] the authors claim there is metric on $\Gamma(BF(\aleph_0))$ for which the action is by isometries. They meant no $\mathbb{R}$-tree metric.}

\footnote{Fischer and Zastrow show that while there is an $\mathbb{R}$-tree metric on $\Gamma(BF(\aleph_0))$, there is no such metric that induces the standard topology where the action is by isometries ([4, Example 4.14]).}
Definition 2.2. Let $A$ be a totally ordered indexing set and for each $a \in A$ let $S_a$ be a partially ordered set. Given $(s_a), (t_a) \in \prod_A S_a$, define $(s_a) \leq (t_a)$ if there exists $a \in A$ with $s_a \leq t_a$ and $s_b = t_b$ for all $b < a$.

The set $\prod_A S_a$ is partially ordered under $\leq$, the lexicographic order ($A$ is required to be totally ordered so that $\leq$ will be transitive). In the case that the $S_a$ are partially ordered Abelian groups (partially ordered sets that satisfy $a \leq b \implies a + c \leq b + c$) then $\prod_A S_a$ is a partially ordered Abelian group. In the case that the $S_a$ is totally ordered, the product may not be totally ordered. It is totally ordered provided $A$ is well ordered.

Lemma 2.3. Let $A$ be an indexing set and for each $a \in A$ let $S_a$ be a totally ordered Abelian group. If $A$ is well ordered then the lexicographic order on $\prod_A S_a$ is a total ordering.

Proof. Suppose $A$ is well ordered. Suppose $(s_a), (t_a) \in \prod_A S_a$. Let $B = \{ a \in A : s_a \neq t_a \}$. If $B = \emptyset$ then $(s_a) = (t_a)$. Otherwise $B$ has a least element $a$ and either $s_a < t_a$ or $s_a > t_a$. \hfill \Box

Now that we have an order on $\mathbb{Z}^a$ we are in a position to define a $\mathbb{Z}^a$-metric on $\text{BF}(a)$. Given $w, v \in \text{BF}(a)$, define $d(w, v) = (n_a)$ where $n_a$ counts the number of occurrences of $a$ and $a^{-1}$ in the reduced form of $w^{-1}v$. It is obviously symmetric and positive definite. To see the triangle inequality holds, suppose $w, v, u \in \text{BF}(a)$. Consider the reduced form of $w^{-1}v$ and the reduced form of $v^{-1}u$ concatenated. Since for each $a \in A$ the reduced form of a word has the same or fewer occurrences of $a$ and $a^{-1}$ as the nonreduced form, we must have $d(w, u) \leq d(w, v) + d(v, u)$.

2.3. Geodesic $\Lambda$-metric spaces. An important example of a $\Lambda$-metric space is $\Lambda$ itself where $d(a, b) = |a - b|$ for $a, b \in \Lambda$ ($|a|$ is defined in the usual way). Then we can define a $\Lambda$-geodesic in a $\Lambda$-metric space $X$ as an isometry $\alpha : [a, b]_{\Lambda} \rightarrow X$ (the interval $[a, b]_{\Lambda}$ is also defined in the usual way). We can assume $a = 0$ (\textit{\cite{3}}, p.8).

We call the image a segment. We will sometimes refer to a $\Lambda$-geodesic as a geodesic if the context is clear. A space is called $\Lambda$-geodesic if every pair of points can be joined by a geodesic.

The free group $F(c)$ under the word metric is $\mathbb{Z}$-geodesic. Given reduced words $w, v \in F(c)$, let $l$ be the number of initial letters that $w$ and $v$ have in common (we could have $l = 0$). Suppose $w$ has $n$ letters and $v$ has $m$ letters. For each $1 \leq l \leq n$ let $w_{l}$ be the word obtained by removing the last $n - l$ letters from $w$ and for each $1 \leq i \leq m$ let $v_{i}$ be the word obtained by removing the last $m - i$ letters from $v$. Then $\{w_{n}, w_{n-1}, \ldots, w_{l}, v_{l+1}, v_{l+2}, \ldots, v_{m}\}$ is a segment with endpoints $w$ and $v$.

On the other hand, the big free group $\text{BF}(o)$ is not $\mathbb{Z}^{o}$-geodesic provided $o > 1$. Let $a$ be the first generator and suppose $\alpha : [(0, 0, \ldots), (1, 0, 0, 0, \ldots)]_{\mathbb{Z}^{o}} \rightarrow \text{BF}(o)$ is a geodesic from $i$ to $a$ where $i$ is the empty word. But $(1, -1, 0, 0, \ldots) \in [(0, 0, \ldots), (1, 0, 0, \ldots)]_{\mathbb{Z}^{o}}$ and there is no word $w \in \text{BF}(o)$ with $d(i, w) = (1, -1, 0, 0, \ldots)$. We will see that $\text{BF}(o)$ embeds isometrically in a geodesic $\mathbb{Z}^{o}$-metric space and that it acts on that space by isometries.

2.4. $\Lambda$-trees. A $\Lambda$-tree is defined to be a $\Lambda$-geodesic $\Lambda$-metric space such that the following conditions are satisfied.

1. If two segments in $X$ intersect in a single point, which is an endpoint of both, then their union is a segment.
If two segments in $X$ have a common endpoint, then their intersection is also a segment.

In the case that $\Lambda = \mathbb{Z}$ or $\Lambda = \mathbb{R}$, condition (2) is automatically satisfied ([3, Lemma 1.2.3]). A $\Lambda$-tree is uniquely geodesic ([3, Lemma 1.3.6]) and if $X$ is a $\Lambda$-tree, for $x, y \in X$ we write $[x, y]$ to denote the unique segment between $x$ and $y$.

The definition above is formulated in terms of basic facts about classical trees. There is another characterization that relies on the concept of a metric space being $\delta$-hyperbolic. We recount the definition and then comment on the case of $\delta = 0$ in relation to trees.

Let $X$ be a $\Lambda$-metric space and let $v \in X$ be a basepoint. Given $x, y \in X$, the Gromov product of $x$ and $y$ with respect to $v$ is $(x \cdot y)_v = \frac{1}{2}(d(v, x) + d(v, y) - d(x, y))$. We usually suppress the notation of the basepoint and write $x \cdot y$. Notice in general we may have $x \cdot y \notin \Lambda$ (it is in $\frac{1}{2}\Lambda$). However, in the case of a $\Lambda$-tree, $x \cdot y \in \Lambda$ and it measures how long the segments $[v, x]$ and $[v, y]$ coincide. For there is a $u \in X$ with $[v, x] \cap [v, y] = [v, u]$ and $[x, u] \cup [u, y] = [x, y]$ (see Figure 1). Then

$$x \cdot y = \frac{1}{2}(d(v, x) + d(v, y) - d(x, y))$$

$$= \frac{1}{2}(d(v, u) + d(u, x) + d(v, u) + d(u, y) - d(u, x) - d(u, y))$$

$$= d(v, u).$$

Denote the point $u$ in the above argument as $Y(v, x, y)$. It does not depend on the order of $v, x, y$.

In the case of the the free group $F(c)$, if $w, v \in F(c)$ and $v$ is the basepoint then $w \cdot v$ counts the number of initial letters that $w$ and $v$ have in common. Similarly, for $w, v \in BF(a)$, $w \cdot v$ counts, for each $a \in A$, the number of occurrences of $a$ and $a^{-1}$ in the initial letters that $w$ and $v$ have in common.

Given $\delta \in \Lambda$ and $v \in X$, the $\Lambda$-metric space $X$ is $\delta$-hyperbolic with respect to $v$ if for all $x, y, z \in X$, $x \cdot y \geq \min\{x \cdot z, y \cdot z\} - \delta$. We say $X$ is $\delta$-hyperbolic if it is $\delta$-hyperbolic for all $v \in X$. If $X$ is $\delta$-hyperbolic with respect to one basepoint then it is $2\delta$-hyperbolic with respect to any other basepoint. Thus if a space is $0$-hyperbolic with respect to one basepoint then it is $0$-hyperbolic.

In the case of $0$-hyperbolicity the requirement becomes $x \cdot y \geq \min\{x \cdot z, y \cdot z\}$. But we also have $x \cdot z \geq \min\{x \cdot y, y \cdot z\}$ and $y \cdot z \geq \min\{x \cdot y, x \cdot z\}$. By choosing the smallest of $x \cdot y, x \cdot z$, and $y \cdot z$ we see that it must be equal to one of the other

![Figure 1. The Gromov product](image-url)
two. In other words, two of \( x \cdot y, x \cdot z \), and \( y \cdot z \) are equal and they are less than or equal to the third.

Both \( F(c) \) and \( BF(o) \) are 0-hyperbolic. Suppose \( w, v, u \in F(c) \). Suppose without loss of generality that \( w \cdot v \leq v \cdot u \). Then \( v \) and \( u \) have at least as many initial letters in common as \( w \) and \( v \) so \( w \) and \( u \) must have the same number of initial letters in common, that is, \( w \cdot u = w \cdot v \). A similar argument shows that \( BF(o) \) is 0-hyperbolic.

The following theorem combines [3 Lemma 2.1.6] and [3 Lemma 2.4.3].

**Theorem 2.4.** Suppose \( X \) is a geodesic \( \Lambda \)-metric space. The following are equivalent.

1. \( X \) is 0-hyperbolic and there is a \( v \in X \) such that \( (x \cdot y)_v \in \Lambda \) for all \( x, y \in X \).
2. \( X \) is a \( \Lambda \)-tree.

Thus \( F(c) \) is a \( \mathbb{Z} \)-tree. Given any \( \mathbb{Z} \)-tree \( X \) there is a classical tree \( \Gamma \) with \( X \) as the set of vertices and the \( \mathbb{Z} \)-metric of \( X \) is the path metric on \( \Gamma \). In the case of \( F(c) \), \( \Gamma \) is the Cayley graph.

2.5. **Groups acting on \( \Lambda \)-trees.** We know that \( BF(o) \) is not a \( \mathbb{Z}^o \)-tree since it is not \( \mathbb{Z}^o \)-geodesic. However there is a standard construction of a \( \Lambda \)-tree from a \( \Lambda \)-metric space that satisfies (1) in the above theorem. In the context of the space being a group we obtain an isometric action of the group on the \( \Lambda \)-tree. It is convenient to use the notation of a length function.

Given a group \( G \) and an ordered Abelian group \( \Lambda \), a length function is a function \( \text{L} : G \rightarrow \Lambda \) such that the following conditions are satisfied.

1. \( \text{L}(g) = 0 \) if and only if \( g = 1 \).
2. \( \text{L}(g) = \text{L}(g^{-1}) \) for all \( g \in G \).
3. For all \( g, h, k \in G \), \( c(g, h) \geq \min\{c(g, k), c(h, k)\} \) where \( c(g, h) = \frac{1}{2}(\text{L}(g) + \text{L}(h) - \text{L}(g^{-1}h)) \).

This definition is that of a Lyndon length function in [3] except that only the reverse direction in condition (1) is assumed there. A length function \( \text{L} \) induces a metric \( d \) on \( G \) where \( d(g, h) = \text{L}(g^{-1}h) \) for \( g, h \in G \) (condition (3) implies that the triangle inequality holds). Notice \( c(g, h) = (g \cdot h)_1 \). Because of condition (3), \( G \) is 0-hyperbolic under \( d \).

Given \( w \in F(c) \), let \( \text{L}(w) \) be the number of letters in the reduced word \( w \). Then the induced metric is the word metric. Similarly, for \( w \in BF(c) \), set \( \text{L}(w) = (n_a) \in \mathbb{Z}^o \) where for each \( a \in A \), \( n_a \) is the number of occurrences of \( a \) and \( a^{-1} \) in the reduced word \( w \).

If a group \( G \) has a length function such that \( c(g, h) \in \Lambda \) for all \( g, h \in G \), then there is a canonical \( \Lambda \)-tree on which it acts by isometries. The \( \Lambda \)-tree \( T(G) \) is constructed in [3 Theorem 2.4.6] by taking the disjoint union of \( \Lambda \)-intervals \([0, \text{L}(g)]\) for each \( g \in G \) and then identifying \( n \in [0, \text{L}(g)] \) and \( n \in [0, \text{L}(h)] \) if \( n \leq c(g, h) \). Denote the equivalence class of \( n \in [0, \text{L}(g)] \) by \( \langle n, g \rangle \).

To see the appropriate metric to put on \( T(G) \) let us examine the metric of \( \Lambda \)-trees more closely. The following is [3 Lemma 2.1.2(2)]. Suppose \( X \) is a \( \Lambda \)-tree with basepoint \( v \). Let \( x, y \in X \) and set \( u = Y(v, x, y) \) as in Figure[1] Let \( x_n, y_n \in [v, x] \) be the point that is distance \( n \) from \( v \) and \( y_m \in [v, y] \) be the point that is distance \( m \) from \( v \). Then \( d(x_n, y_m) = n + m - 2 \min\{n, m, x \cdot y\} \). For if \( n \leq x \cdot y \) then \( x_n, y_m \in [v, y] \) so \( d(x_n, y_m) = |n - m| \). A symmetric statement holds if \( m \leq x \cdot y \).
We use the same formula to define the metric on \( T(G) \) where \( d(\cdot, \cdot) \) is by isometries.

The action of \( h \) of as changing the basepoint from 1 to \( \{ g, h \} \) is instructive.

The theorem follows Theorems 2.4.4, 2.4.5, and 2.4.6 in [3] but a direct proof is instructive.

Let \( h, g, k \in G \) and notice \( c(g, k) = (hg \cdot hk)h \). Thus the action can be thought of as changing the basepoint from 1 to \( h \) (see Figure 2).

We first show the action of \( h \) on \( T(G) \) is well defined. Suppose \( \langle n, g \rangle, \langle m, h \rangle \in T(G) \), so \( n \leq c(g, k) = (hg \cdot hk)h \). Thus \( h\langle n, g \rangle, h\langle m, k \rangle \in [h, u] \) so we must have \( h\langle n, g \rangle = h\langle m, k \rangle \).

To see that the action is an isometry, suppose \( \langle n, g \rangle, \langle m, k \rangle \in T(G) \). If \( n \leq c(g, k) \), then \( h\langle n, g \rangle, h\langle m, k \rangle \in [h, k] \) so \( d(h\langle n, g \rangle, h\langle m, k \rangle) = |n - m| \). A symmetric statement holds if \( m \leq c(g, k) \). If \( m > c(g, k) \) and \( n > c(g, k) \) then \( h\langle n, g \rangle \in [u, hg] \) and \( h\langle m, k \rangle \in [u, hk] \) so \( d(h\langle n, g \rangle, h\langle m, k \rangle) = n + m - 2c(g, h) \).

Finally for surjectivity, suppose \( \langle m, k \rangle \in T(G) \). Set \( w = Y(1, h, k) \). If \( m \leq c(h, k) \) then \( \langle m, k \rangle \in [1, w] \) and \( h(L(h) - m, h^{-1}) = \langle m, k \rangle = \langle m, k \rangle \). If \( m \geq c(h, k) \), then \( \langle m, k \rangle \in [w, k] \) and \( h(L(h) + m - 2c(h, k), h^{-1}k) = \langle m, k \rangle \).

We can read a formula for the action from Figure 2. If \( n \leq c(g, h^{-1}) \) then \( h \cdot \langle n, g \rangle = \langle L(h) - n, h \rangle \). If \( n \geq c(g, h^{-1}) \) then \( h \cdot \langle n, g \rangle = \langle L(h) + n - 2c(g, h^{-1}), hg \rangle \).

Of course there are other ways of measuring the first coordinate of \( h \cdot \langle n, g \rangle \) in the second case.

The big free group \( BF(o) \) acts by isometries on the \( \mathbb{Z}^o \)-tree \( T(BF(o)) \). The action is free and without inversions.
3. Big free groups acting on the big Cayley graph

In addition to BF(\(o\)) acting on the \(Z^o\)-tree \(T(BF(o))\), we also have BF(\(o\)) acting on the big Cayley graph \(\Gamma(BF(o))\). The action can be described in a way similar to the former action, but can also be described using a combinatorial description of \(\Gamma(BF(o))\). A description of the space of homotopy classes of paths in the Hawaiian earring is given in [4 Example 4.15] and can be extended to any \(\Gamma(BF(o))\). An element of \(\Gamma(BF(o))\) is an equivalence class of some \(t\) in a real interval labeled by some \(a^p \in A \cup A^{-1}\) and can be represented by a triple \((w, a^p, t)\) where \(w\) is the word that is read before the interval containing \(t\). In order to obtain a unique representation, given a triple \((w, a^p, t)\), we assume that \(w\) does not end in the letter \(a^{-p}\). Also, if \(t = 0\) then no second coordinate is used and we may just write \(w\). Then the action is defined as follows. Given \(u \in BF(o)\) and \((w, a^p, t) \in \Gamma(BF(o))\), 
\[ u \cdot (w, a^p, t) = (uw, a^p, t) \] provided \(uw\) does not have \(a^{-p}\) as a last letter. In the case that it does, 
\[ u \cdot (w, a^p, t) = (uwa^p, a^{-p}, 1-t). \]

**Proposition 3.1.** There is no \(R\)-tree metric on \(\Gamma(BF(o))\) for infinite \(o\) for which the action by BF(\(o\)) is by isometries.

**Proof.** Suppose there is an \(R\)-tree metric. Let \(\{a_i\}_{i \in N} \subset A\) and set \(d_i = d(a_i, t)\). Then there are \(p_i \in N\) so that \(\sum p_i d_i = \infty\). Set \(w = a_1^{p_1} a_2^{p_2} a_3^{p_3} \cdots\). Then \([i, w] = [i, a_i^{p_i}] \cup [a_i^{p_i}, a_i^{p_i+1}] \cup [a_i^{p_i+1}, a_i^{p_i+2}] \cdots\) and these intervals are disjoint except for at endpoints by construction. Since the action is by isometries, \(L(w) = \sum p_i d_i\), a contradiction.

The \(Z^o\)-metric that Cannon and Conner describe for \(\Gamma(BF(o))\) is an extension of the \(Z^o\)-metric on BF(\(o\)). Given \((w, a^p, t), (v, b^q, s) \in \Gamma(BF(o))\), define 
\[ d((w, a^p, t), (v, b^q, s)) = L(w) + t L(a) + L(v) + s L(b) - 2 \min\{L(w) + t L(a), L(v) + s L(b), c(wa, vb)\} \]
In the case that \(c(wa, vb) \leq L(w)\) and \(c(wa, vb) \leq L(v)\), the formula becomes \(L(a^{-1} w^{-1} vb) - (1-t)L(a) - (1-s)L(b)\). In the case that \(L(w) + t L(a) \leq c(wa, wv)\) (which implies \(wa\) is in the big interval \(I_{vb}\), the formula becomes \(L(v) + s L(b) - L(w) - t L(a)\). A similar formula holds if \(L(v) + s L(b) \leq c(wa, vb)\). Under this \(Z^o\)-metric, \(\Gamma(BF(o))\) is 0-hyperbolic and the action of BF(\(o\)) on \(\Gamma(BF(o))\) is by isometries.

**Remark 3.2.** Since \(\Gamma(BF(o))\) is 0-hyperbolic and \(x \cdot y \in Z^o\) for all \(x, y \in \Gamma(BF(o))\), it embeds in an \(R^o\)-tree and the action of BF(\(o\)) on \(\Gamma(BF(o))\) extends to an isometric action on that \(R^o\)-tree. Also, using the inclusion \(Z^o \hookrightarrow R^o\), we obtain an isometric embedding of \(T(BF(o))\) into an \(R^o\)-tree. While this \(R^o\)-tree differs from the one mentioned just before, in both a word \(w \in BF(o)\) is associated with an \(R^o\)-interval \([0, 0, \ldots), L(w)\) \(R^o\). Given a letter \(a \in A\), the embedding of the real interval between \(w\) and \(wa\) in \(\Gamma(BF(o))\) into the \(R^o\)-interval \([0, 0, \ldots), L(wa)\) is given by \(L(w) + t L(a)\) for \(0 \leq t \leq 1\). The embedding of the \(Z^o\)-interval between \(w\) and \(wa\) in \(T(BF(o))\) into the same \(R^o\)-interval is given by \(L(w) + t\) where \(t \in [0, 0, \ldots), L(a)\) \(R^o\). Thus the two embeddings intersect only at the endpoints.

4. A combinatorial description

We wish to give a combinatorial description of \(T(BF(o))\) similar to the one given above for \(\Gamma(BF(o))\). That is, given \((n, w) \in T(BF(o))\), we wish to find \(v \in BF(o)\) and \(a \in A\) so that \((n, w) \in [v, va^{\pm 1}]\). We will see that such a description may not exist for \(o > \omega\), the first infinite ordinal, and show that one does exist for \(o = \omega\).
We will need the concept of a subword of a word in $\text{BF}(o)$. First note that for any $T(G)$, $h \in [1, g]$ if and only if $L(h) + L(h^{-1}g) = L(g)$. The reverse direction is a direct calculation, and if $h \in [1, g]$ then $L(h) \leq c(g, h) = \frac{1}{2}(L(g) + L(h) - L(g^{-1}h))$ so $L(h) + L(h^{-1}g) \leq L(g)$. But we always have $L(h) + L(h^{-1}g) \geq L(g)$ by the triangle inequality. Now consider $v, w \in \text{BF}(o)$. The equation $L(v) + L(v^{-1}w) = L(w)$ means that there is no reduction in the product of $v$ and $v^{-1}w$. In that case if we write $v : S_v \to A \cup A^{-1}$ and $w : S_w \to A \cup A^{-1}$ then we may assume $S_v \subset S_w$ and it follows that $S_w$ is a Dedekind cut of $S_v$. We call such a $v$ a subword of $w$. Cannon and Conner note that there is a one-to-one correspondence between the Dedekind cuts of $S_w$ and the subwords of $w$. This correspondence induces a linear order on the set of subwords of $w$.

If $\langle n, w \rangle \in [v, va^p]$, then $v$ is a subword of $w$. For then $L(v) \leq n \leq c(w, va^p)$ and if $v^{-1}w$ does not begin with $a^p$, $c(w, va^p) = c(v, w)$ so $L(v) = c(v, w)$. If $v^{-1}w$ does begin with $a^p$ then $c(w, va^p) = c(v, w) + L(a)$ and since two of $c(w, va^p)$, $c(v, w)$, and $c(v, va^p)$ are equal and not greater than the third, we must have $c(v, w) = c(v, va^p) = L(v)$.

We now show that if $o > \omega$ then there may be $\langle n, w \rangle \in T(\text{BF}(o))$ such that there are no $v \in \text{BF}(o)$ and $a \in A$ with $\langle n, w \rangle \in [v, va^\pm 1]$. We assume $o = \omega + 1$ for convenience of notation but the argument extends to any $o > \omega$.

**Example 4.1.** Set $o = \omega + 1$, say $A = \{a_1, a_2, \ldots, b\}$. Define the word $w = \cdots a_3 a_2 a_1$. Suppose there is a subword $v$ of $w$ and $a \in A$ so that $(L(b), w) \in [v, va^\pm 1]$. Then $L(v) \leq L(b)$ and the only such $v$ is $\iota$. Thus $(L(b), w) \in [\iota, a^\pm 1] \cap [\iota, w]$ so $L(b) \leq c(a^\pm 1, w) = (0, 0, \ldots, 0)$, a contradiction.

The situation in the above example is that there is not a first subword of $w$ that has length at least $L(b)$. We now show that in $\text{BF}(\omega)$ there are no problem elements like $(L(b), w)$; given $(n, w)$ we can always find the first subword of $w$ that has length at least $n$.

**Proposition 4.2.** Given $\langle n, w \rangle \in T(\text{BF}(\omega))$, there are $v \in \text{BF}(o)$ and $a \in A$ so that $\langle n, w \rangle \in [v, va^\pm 1]$.

**Proof.** We may assume $n > 0$ since $\langle 0, w \rangle = \iota$. Let $a \in A$ be the first nonzero coordinate of $n$. Since $w$ has length at least $n$ that length must have a positive value in the $b$ coordinate for some $b \leq a$. Now there are only finitely many occurrences of $b$ and $b^{-1}$ in $w$ for $b \leq a$ so we may list them. If any of these occurrences causes the corresponding subword to have length at least $n$, then we pick the first one and we are done. Suppose all of them still yield subwords that have length less than $n$. Let $u$ be the last such subword. Then $u_b = w_b$ for all $b \leq a$; we do not have any more occurrences of $b$ or $b^{-1}$ in $w$ for $b \leq a$. Also, $u_b = n_b$ for all $b \leq a$; otherwise we would never obtain $L(w) \geq n$. Thus $u_b = n_b$ for all $b \leq a$.

We have $L(u) < n$; let $a_2 \in A$ be the first coordinate with $L(u)_{a_2} < n_{a_2}$. Note by the claim above that $n_{a} > a$. Again, since $w$ has length at least $n$ it must have occurrences of $b$ or $b^{-1}$ for $a < b \leq a_2$. Again there are only finitely many such occurrences and if any one of them yields a subword with length at least $n$ then we are done. Otherwise we obtain $u_b = n_b$ for all $b \leq a_2$. If the process never terminates we obtain $w_b = n_b$ for all $b \in A$, thus $\langle n, w \rangle = w$.

Thus we can represent an element of $T(\text{BF}(\omega))$ as a triple $(w, ap^p, t)$ where $w \in \text{BF}(o)$, $a \in A$, $p = \pm 1$, and $t \in [0, L(a))$ (we set $t = n - L(v)$ in the statement.
of the previous proposition). We have restrictions analogous to those for $\Gamma(BF(o))$ and can give a description of the action and distance function in the same fashion as well. Given $u \in BF(\omega)$ and $(w, a^p, t) \in T(BF(\omega))$, $u \cdot (w, a^p, t) = (uw, a^p, t)$ unless $uw$ ends in $a^{-p}$ in which case $u \cdot (w, a^p, t) = (uwa^p, a^{-p}, L(a) - t)$. Given $(w, a^p, t), (v, b^q, s) \in T(BF(\omega))$, $d((w, a^p, t), (v, b^q, s)) = L(w^{-1}v) + t + s$ unless $w = v$ and $a^p = b^q$ in which case we simply have $|t - s|$.

The above combinatorial representation is unique. First let us note that there are no words in $[v, va^p]$ other than the endpoints $v$ and $va^p$. For if $u \in [v, va^p]$ is a word then $L(a) = L(v^{-1}u) + L(a^{-p}v^{-1}u)$ which implies $L(v^{-1}u) = 0$ or $L(a^{-p}v^{-1}u) = 0$. Now for uniqueness, suppose $(v, a^p, t) = (u, b^q, t) \in T(BF(\omega))$, that is, $(L(v) + t, va^p) = (L(u) + s, ub^q)$. By definition we have $L(v) + t = L(u) + s$. Since $L(v)$ and $L(u)$ have integer coordinate values and $s$ and $t$ have only one possible coordinate with a non integer value we have two cases. If $t = s = 0$ then $L(v) = L(u)$ and $v = u$ since $v, u \in [t, \{n, q\}]$. If $t > 0$ and $s > 0$ then $t = s$ and we must have $a = b$ since that is the non integer coordinate. We also have $v = u$ as in the first case. Finally, we have $p = q$ since otherwise $L(v) + t \leq c(va, va^{-1}) = L(v)$, a contradiction.

We now describe the quotient of the tree $T(BF(\omega))$ under the action of $BF(\omega)$ as a wedge of $Z^2$-circles $C_a$ for $a \in A$. Given $a \in A$, let $C_a$ be the $Z^2$-interval $[0, L(a)]$ with the endpoints identified. Define the distance between points $s$ and $t$ in the circle to be $\min\{|s - t|, L(a) - |s - t|\}$. The formula works with both 0 and $L(a)$ used for the identification point.

The quotient is the set $T(BF(\omega))$ under the identification of elements that are in the same orbit. We map an element $(w, a^p, t)$ of the quotient to the point $t \in C_a$ if $p = 1$ and the point $L(a) - t$ if $p = -1$. We show that this mapping is a bijection.

Let $(w, a^p, t) \in T(BF(\omega))$ and $u \in BF(\omega)$. Then $u \cdot (w, a^p, t) = (uw, a^p, t)$ unless $uw$ ends in $a^{-p}$ in which case $u \cdot (w, a^p, t) = (uwa^p, a^{-p}, L(a) - t)$. First suppose $p = 1$. In the first case both $(w, a^t)$ and $u \cdot (w, a^t)$ are sent to $t \in C(a)$. In the second case $u \cdot (w, a^t) = (uw, a^{-1}, L(a) - t)$ and is sent to $t \in C(a)$ as is $(w, a^t)$. Symmetric statements hold for $p = -1$ and we see that the mapping is well defined. Now suppose $(w, a^p, t)$ and $(v, a^q, s)$ are mapped to the same point $r \in C_a$. If $p = q = 1$ we obtain $t = s$ and $(v, a^t) = vw^{-1} \cdot (w, a^t)$ since $(vw^{-1})w = v$ does not end in $a^{-p}$ by assumption. If $p = 1$ and $q = -1$ we have $(w, a^t)$ being sent to $t \in C(a)$ as is $(v, a^{-1}, s)$. Then $s = L(a) - t$ and $(v, a^{-1}, L(a) - t) = va^{-1}w^{-1} \cdot (w, a^t)$ since $(va^{-1}w^{-1})w$ does end in $a^{-1}$. The other cases for $p$ and $q$ are handled in a similar fashion. Finally, given $t \in C_a$ we have $(t, a^t) \in C_a$ in the quotient space which is sent to $t$.

We define a $Z^2$-metric on the quotient space. Given $(w, a^p, t)$ and $(v, b^q, s)$, define the distance in the quotient space to be $|t + s|$ unless $w = v$ and $a^p = b^q$ in which case we set the distance to be $|t - s|$. If the wedge $\vee C_a$ is given the standard wedge metric then the mapping is an isometry.

5. The induced topology

The fact that $BF(\omega)$ is isomorphic to the fundamental group of the Hawaiian earring suggests a topology for it that is inherited from a standard topology on the space of fixed endpoint homotopy classes of paths. Given a path $a$ in $X$ and a neighborhood $U$ of the endpoint of $a$, $B([a], U) = \{[\beta] : \beta = a\gamma$ for some path $\gamma$ whose image lies in $U$. We define the following topology on $BF(\omega)$ following the above model. Let $w \in BF(\omega)$ and $a \in A$. Define $B(w, a) = \{v : v = wu\}$
where \( u \in \text{BF}(o) \) has each letter greater than \( a \). We show that the \( \mathbb{Z}^a \)-metric on \( \text{BF}(o) \) induces this topology. Let \( a \in A \). Set \( \epsilon = (0,0,\ldots,1,0,0,\ldots) \) where the 1 appears in the \( a \) coordinate. Then \( B(w, \epsilon) \subset B(w, a) \). Now let \( \epsilon > (0,0,\ldots) \). Let \( a \) be the first nonzero coordinate in \( \epsilon \) and let \( b \) be the coordinate after \( a \). Then \( B(w, b) \subset B(w, \epsilon) \).

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