NEW DESCRIPTION OF SELF-DUAL METRICS

J. Tafel
Institute of Theoretical Physics, University of Warsaw, Hoża 69, 00-681 Warsaw, Poland, email: tafel@fuw.edu.pl

Abstract. We show that Plebański’s equation for self-dual metrics is equivalent to a pair of equations describing canonical transformations in 2-dimensional phase spaces. Examples of linearizations of these equations are given.

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1 Introduction

Plebański [1] showed that complex metrics with the self-dual Riemann tensor [2], to be referred to as self-dual metrics, can be described in terms of one function Ω satisfying so-called Plebański’s first equation

\[ \Omega_{,q} \Omega_{,p} = \Omega_{,p} \Omega_{,q} = 1, \]  

(1)

Given Ω the corresponding metric reads

\[ g = \Omega_{,q} dq dq + \Omega_{,p} dp dp + \Omega_{,\tilde{q}} d\tilde{q} d\tilde{q} + \Omega_{,\tilde{p}} d\tilde{p} d\tilde{p}. \]  

(2)

He also proposed an equivalent equation known as Plebański’s second equation. At present there are several other descriptions of self-dual metrics [3, 4, 5, 6] (note that an erroneous sign in Husain’s equation was corrected and an equivalence of the latter equation to (1) was proved [7]). Much attention was payed to possible complete integrability of the self-duality conditions (see [8] and references therein).

In this paper we represent self-dual metrics in the form

\[ g = \{u, v\}_{q \tilde{q}} dq dq + \{u, v\}_{p \tilde{p}} dp dp + \{u, v\}_{\tilde{q} \tilde{p}} d\tilde{q} d\tilde{p} + \{u, v\}_{q \tilde{p}} dq d\tilde{p}, \]  

(3)

where the Poisson brackets are taken with respect to indicated coordinates, e.g. \( \{u, v\}_{q \tilde{q}} = u_{,q} v_{,\tilde{q}} - u_{,\tilde{q}} v_{,q} \). Functions u and v are subject to the equations

\[ \{u, v\}_{qp} = 1 \]  

(4)

\[ \{u, v\}_{\tilde{q} \tilde{p}} = 1 . \]  

(5)

Thus, u and v are related to q, p and to \( \tilde{q}, \tilde{p} \) by canonical transformations. Solving (4) and (5) requires correlating these transformations. In section 3 we show that these equations can be linearized under some assumptions on functions generating the canonical transformations.
It is convenient to denote coordinates $q, p$ by $z^A, A = 1, 2,$ and coordinates $\tilde{q}, \tilde{p}$ by $\tilde{z}^A$. Partial derivatives of any function $f$ with respect to these coordinates will be denoted by $f_{,A}$ and $f_{,\tilde{A}}$, respectively. Instead of $\{u, v\}_{z^A\tilde{z}^B}$ we will write $\{u, v\}_{AB}$. In this notation metric (2) reads
\[ g = \Omega_{,A\tilde{B}} dz^A d\tilde{z}^B \] (6) and metric (3) reads
\[ g = \{u, v\}_{A\tilde{B}} dz^A d\tilde{z}^B. \] (7)

2 Equivalence of formulations

Let $u$ and $v$ satisfy equations (4) and (5). Then
\[ \{u, v\}_{A[B, C]} = \{u, v\}_{B[A, C]} = 0, \] (8) hence there are functions $\Omega_A$ such that
\[ \{u, v\}_{AB} = \Omega_{A\tilde{B}} \] (9)
and $\Omega_{[A,C]}$ do not depend on coordinates $\tilde{z}^A$. One can shift $\Omega_A$ by functions of $z^A$ to achieve $\Omega_{[A,C]} = 0$. From this property it follows that there is a function $\Omega$ such that $\Omega_A = \Omega_{,A}$. Thus,
\[ \{u, v\}_{A\tilde{B}} = \Omega_{,A\tilde{B}} \] (10) and metric (7) takes the form (6). Substituting (4), (5) and (10) into the identity
\[ \{u, v\}_{q\tilde{q}}\{u, v\}_{p\tilde{p}} - \{u, v\}_{q\tilde{p}}\{u, v\}_{p\tilde{q}} = \{u, v\}_{qp}\{u, v\}_{\tilde{q}\tilde{p}} \] (11) shows that $\Omega$ satisfies Plebański’s equation (1).

To prove that the reverse transformation exists is less straightforward. Let us use the antisymmetric tensors $\epsilon^{AB}$ and $\epsilon^{A\tilde{B}}$ to raise indices in a standard way, e.g. $p^A = \epsilon^{AB}p_B$. This notation allows to write equations (4) and (5) in the form
\[ v^{A}u_{,A} = 1 \] (12)
\[ v^{A}\tilde{u}_{,A} = 1. \] (13)
Assume that a function $\Omega$ satisfies the equation
\[ \Omega_{,AB}\Omega_{,\tilde{C}}\delta^{\tilde{B}C} = \epsilon_{AC}, \] (14) which is equivalent to (1). Our aim is to prove the existence of solutions $u, v$ of equations (10), (12) and (13).

Taking contractions of (10) with $v^{A}$ or $u^{A}$ and using (12) we can replace equations (10) by
\[ v_{,\tilde{B}} = v^{A}\Omega_{,AB} \] (15)
and

\[ u, \dot{B} = u^A \Omega_{,A\dot{B}}. \tag{16} \]

It follows from (14)-(16) that

\[ v^B u, \dot{B} = v^A u^C (\Omega_{,A\dot{B}} \Omega_{,C\dot{B}}) = v^A u_A. \tag{17} \]

Thus, equation (13) is satisfied if equations (12) and (14)-(16) are satisfied.

Let us write equations (15) and (16) as

\[ D_{\dot{B}} v = 0 \tag{18} \]

\[ D_{\dot{B}} u = 0, \tag{19} \]

where operators \( D_{\dot{B}} \) are given by

\[ D_{\dot{B}} = \partial_{\dot{B}} + \Omega^A_{,\dot{B}} \partial_A. \tag{20} \]

Vectors \( D_{\dot{B}} \) are part of a null basis for metric (2). In view of twistor constructions related to self-dual metrics [9] it is not surprising that \( D_{\dot{B}} \) commute provided equation (14) is satisfied,

\[ [\partial_{\dot{B}} + \Omega^A_{,\dot{B}} \partial_A, \partial_{\dot{C}} + \Omega^D_{,\dot{C}} \partial_D] = 2 \Omega^A_{,[\dot{B}\dot{C}],A} \partial_D = 2(\Omega^A_{,[\dot{B}\dot{C}],A} \partial_D = 0. \tag{21} \]

Let \( v \) be a solution of (18). It remains to solve equations (12) and (19), which form a linear system for a function \( u \). The system is integrable since all the operators \( D_{\dot{B}} \), \( v^A \partial_A \) commute due to (21) and the equality

\[ [\partial_{\dot{B}} + \Omega^C_{,\dot{B}} \partial_C, v^A \partial_A] = (v^A \partial_A) = 0 \tag{22} \]

forced by (18). Thus, equations (10), (12), (13) have solutions and we obtain the following theorem.

**Theorem 1.** All complex self-dual metrics can be locally represented in the form (3), where \( u \) and \( v \) are subject to equations (4) and (5).

Note that if \( u, v \) and all the coordinates are real, then metric (3) is real with the signature \(+ + --\). The same signature is obtained when \( u \) and \( iv \) are real and

\[ \tilde{q} = \epsilon \tilde{q}, \quad \tilde{p} = -\epsilon \tilde{p}, \quad \epsilon = \pm 1, \tag{23} \]

where the bar denotes the complex conjugate. Simple conditions of this kind are not available for the Euclidean signature of \( g \). This difficulty appears also in other approaches to self-duality [5, 6].

Equations (11) and (15) possess point symmetries which are easy to obtain if we denote \( u, v \) by \( f^A \) and write equations (11), (15) as

\[ \det(\frac{\partial f^A}{\partial z^B}) = \det(\frac{\partial f^A}{\partial z^B}) = 1. \tag{24} \]
Let new variables $f'^A$ be functions of $f^B$, new coordinates $z'^A$ be functions of $z^B$ and $\tilde{z}'^A$ be functions of $\tilde{z}^B$. This transformation preserves equations (24) provided

$$\text{det} \left( \frac{\partial f'^A}{\partial f^B} \right) = \text{det} \left( \frac{\partial z'^A}{\partial z^B} \right) = \text{det} \left( \frac{\partial \tilde{z}'^A}{\partial \tilde{z}^B} \right) = c, \quad (25)$$

where $c$ is a constant. In terms of Poisson brackets equations (25) read

$$\{u', v'\}_{uv} = \{q', p'\}_{qp} = \{\tilde{q}', \tilde{p}'\}_{\tilde{q}\tilde{p}} = c. \quad (26)$$

There are also natural discrete symmetries given by an appropriate interchange of variables. All above transformations preserve the metric modulo a constant factor.

### 3 Reductions to linear equations

In terms of a generating function $S(v, q, \tilde{q}, \tilde{p})$, where $\tilde{q}, \tilde{p}$ appear as parameters, solutions $u, v$ of (4) are given implicitly by

$$p = S_v, \quad u = S_w. \quad (27)$$

Similarly, equation (5) can be formally solved by means of a generating function $\tilde{S}(v, \tilde{q}, q, p)$,

$$\tilde{p} = \tilde{S}_{\tilde{q}}, \quad u = \tilde{S}_w. \quad (28)$$

Functions $S$ and $\tilde{S}$ should be correlated in such a way that solutions $u, v$ obtained from (27) or (28) coincide.

Assume that $S$ and $\tilde{S}$ are linear in $\tilde{p}$ and $p$, respectively,

$$S = S'(v, q, \tilde{q}) + \tilde{p}a(v, q, \tilde{q}), \quad \tilde{S} = \tilde{S}'(v, q, \tilde{q}) + p\tilde{a}(v, q, \tilde{q}). \quad (29)$$

Substituting (29) into (27) and (28) leads to the following conditions guaranteeing the uniqueness of $u$ and $v$

$$\tilde{a}_{\tilde{q}} = a_q^{-1}, \quad \tilde{a}_v = a_va^{-1}_q \quad (30)$$

$$S'_{vq} + a_q\tilde{S}'_{\tilde{q}} = 0, \quad S'_{v} - \tilde{S}'_{w} + a_w\tilde{S}'_{\tilde{q}} = 0. \quad (31)$$

Equations (30) form a nonlinear system for the functions $a$ and $\tilde{a}$. Given $a$ and $\tilde{a}$ it remains to solve linear equations (31) for functions $S'$ and $\tilde{S}'$.

Equations (30) are integrable with respect to $\tilde{a}$ provided $a$ satisfies the following quasilinear equation

$$a_{vq} + a_qa_{v\tilde{q}} - a_va_{q\tilde{q}} = 0. \quad (32)$$

Simple solutions of (32) can be found by assuming that one of the second derivatives of $a$ vanishes. For instance, for $a_{vq} = 0$ one obtains $a = \tilde{q}(v + q)$. For this function $a$ equations (31) yield

$$S = F_q + \tilde{p}\tilde{q}(v + q), \quad (33)$$

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where function $F(v, q, \tilde{q})$ is subject to the linear equation

$$F_{vq} + F_{v\tilde{q}} - F_{q\tilde{q}} = 0. \quad (34)$$

In this case the corresponding metric (3) possesses the translational Killing vector $\partial_p - \partial_{\tilde{p}}$. For this reason it must belong to the class of generalized Hawking metrics [6].

If $a_{,v\tilde{q}} = 0$ then the generating function $S$ can be put into the form

$$S = F_{,v} - e^q F_{,\tilde{q}} + \tilde{p}(q\tilde{q} + e^q v + c(q)). \quad (35)$$

Here $c$ is an arbitrary function of $q$, with its derivative denoted by $\dot{c}$, and $F$ has to satisfy the linear equation

$$F_{vq} + (\tilde{q} + e^q v + \dot{c})F_{v\tilde{q}} - e^q F_{vq} - e^q F_{\tilde{q}} = 0. \quad (36)$$

If $a_{,q\tilde{q}} = 0$ one obtains

$$S = F_{,\tilde{q}} + \tilde{p}(v\tilde{q} + e^{-v} q + c(v)) \quad (37)$$

and the following linear equation for $F$

$$e^{-v} F_{,v\tilde{q}} + (e^{-v} q - \tilde{q} - \dot{c})F_{,q\tilde{q}} + F_{,vq} + F_{,q} = 0. \quad (38)$$

Given a solution $F$ of equation (36) or (38) one can find functions $u$, $v$ from relations (27) and then construct the corresponding self-dual metric (3). Up to the best knowledge of the author the reduction of the Plebański equation, for some ansatz, to the linear equations (36) or (38) is new.

4 Conclusions

We have shown that all self-dual metrics can be represented in the form (3), where functions $u$, $v$ have to satisfy equations (4) and (5) defining canonical transformations in a 2-dimensional phase space.

The new formulation creates new possibilities of construction of self-dual metrics. Equations (4), (5) are equivalent to linear ones (34), (36) or (38) under the assumption that functions $u$, $v$ are given by (27) and $S$ takes the form (33), (35) or (37). An open problem is whether these linear reductions admit interesting solutions other than gravitational instantons of Hawking [10].

In our opinion the new formulation can be also useful as a new tool in classification of completely integrable equations. It is well known that many of these equations can be considered as reductions of the self-dual Yang-Mills equations [11] or the self-dual Einstein equations [12] (see also [8]). Perhaps, due to a new form of self-duality conditions, one can obtain in this way further completely integrable equations.

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