Boundary Asymptotic Analysis for an Incompressible Viscous Flow: Navier Wall Laws
Mustapha El Jarroudi, Alain Brillard

To cite this version:
Mustapha El Jarroudi, Alain Brillard. Boundary Asymptotic Analysis for an Incompressible Viscous Flow: Navier Wall Laws. Applied Mathematics Optimization, Springer, 2008, 57 (3), pp.371-400. <10.1007/s00245-007-9026-5>. <hal-00539904>

HAL Id: hal-00539904
https://hal.archives-ouvertes.fr/hal-00539904
Submitted on 25 Nov 2010

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Boundary asymptotic analysis for an incompressible viscous flow: Navier wall laws

M. El Jarroudi, A. Brillard
Université Abdelmalek Essaâdi, FST Tanger
Département de Mathématiques, B.P. 416, Tanger, Morocco
Université de Haute-Alsace
Laboratoire de Gestion des Risques et Environnement
25 rue de Chemnitz, F-68200 Mulhouse, France

2008

Abstract

We consider a new way of establishing Navier wall laws. Considering a bounded domain \( \Omega \) of \( \mathbb{R}^N \), \( N = 2, 3 \), surrounded by a thin layer \( \Sigma_\varepsilon \), along a part \( \Gamma_2 \) of its boundary \( \partial \Omega \), we consider a Navier-Stokes flow in \( \Omega \cup \partial \Omega \cup \Sigma_\varepsilon \) with Reynolds’ number of order \( 1/\varepsilon \) in \( \Sigma_\varepsilon \). Using \( \Gamma \)-convergence arguments, we describe the asymptotic behaviour of the solution of this problem and get a general Navier law involving a matrix of Borel measures having the same support contained in the interface \( \Gamma_2 \). We then consider two special cases where we characterize this matrix of measures. As a further application, we consider an optimal control problem within this context.

Navier law, Navier-Stokes flow, \( \Gamma \)-convergence, asymptotic behaviour, optimal control problem.

AMS Classification. 76D05, 76D10, 76M45, 35Q30.

1 Introduction

A common hypothesis used in fluid mechanics is that, at the interface between a solid and a fluid, the velocity \( u \) of the fluid is equal to that of the solid. If the solid is at rest, the velocity of the fluid must thus vanish: \( u = 0 \), on the boundary of the solid. These are the so-called rigid boundary conditions. When writing this condition, one assumes that the fluid perfectly adheres to the solid.

This hypothesis has not always been accepted for a viscous fluid, although some verifications have been made through experiments. G. Taylor indeed verified in 1923 the correctness of this hypothesis, when studying the stability of the motion of a fluid flowing between two cylinders in rotation (Taylor-Couette’s problem).

Another approach has then been suggested. A thin layer adhering to the solid exists with a tangential velocity different from 0 on the surface of the solid. Navier suggested that this tangential velocity is proportional to the shearing strains and thus is given through

\[
(\text{Id} - n \otimes n) \nu \frac{\partial u}{\partial n} = \kappa u, \quad u \cdot n = 0,
\]

where \( \text{Id} \) is the identity matrix, \( n \) is the unit outer normal vector to the surface of the solid, \( \nu \) is the viscosity of the fluid and \( \kappa \) is a proportionality coefficient.

Many works have already been devoted to the derivation of Navier boundary conditions, see for example [2], [3], [13] and [14]. In [2] and [3], the authors considered a viscous and incompressible fluid, whose Reynolds number is of order \( 1/\varepsilon \), flowing in a domain with rugosities of thinness \( \varepsilon \) and \( \varepsilon \)-periodically
distributed on its boundary surface, and assuming an homogeneous Dirichlet boundary condition on the boundary of these rugosities. Using the asymptotic expansion method, they deduced, at the first-order level, a kind of Navier wall law

\[ \varepsilon (I - n \otimes n) \nu \frac{\partial u}{\partial n} = \kappa u, \]
\[ u \cdot n = 0. \]

In [13], the authors considered the laminar flow in a pipe with rough pieces \( \varepsilon \)-periodically distributed on the surface of the pipe, and imposing an homogeneous Dirichlet boundary condition on the boundary of these rough pieces. They used an homogenization process and obtained a Navier wall law, computing a corrector term. In [14], the author considered an \( \varepsilon \)-periodic geometry built with rough pieces of thinness \( \varepsilon_m \) and imposed there a boundary condition of the type

\[ \begin{cases} (I - n \otimes n) \nu \frac{\partial u}{\partial n} = \varepsilon k (g - \kappa u), \\ u \cdot n = 0. \end{cases} \]

The following limit law was obtained, depending on \( k \) and \( m \)

\[ \begin{cases} (I - n \otimes n) \nu \frac{\partial u}{\partial n} = \lambda (g - \kappa u), \\ u \cdot n = 0. \end{cases} \]

Throughout the present work, we consider a bounded domain \( \Omega \subset \mathbb{R}^N \), \( N = 2, 3 \), whose boundary \( \partial \Omega \) is Lipschitz continuous. We suppose that \( \partial \Omega = \Gamma_1 \cup \Gamma_2 \), with \( |\Gamma_1|, |\Gamma_2| > 0 \), where \( |\Gamma_i| \) denotes the Lebesgue measure of \( \Gamma_i \). We suppose that near \( \Gamma_2 \) there exists a thin layer \( \Sigma \varepsilon \) of thinness \( \varepsilon > 0 \), which extends \( \Omega \) into \( \Omega_{\varepsilon} = \Omega \cup \Gamma_2 \cup \Sigma \varepsilon \).

![Figure 1](image-url)  
Figure 1: The domain under consideration.

We consider the steady-state, viscous and incompressible Navier-Stokes flow in \( \Omega_{\varepsilon} \)

\[ \begin{cases} -\nu \Delta u_{\varepsilon} + (u_{\varepsilon} \cdot \nabla) u_{\varepsilon} + \nabla p_{\varepsilon} = f & \text{in } \Omega, \\ -\nu_{\varepsilon} \Delta u_{\varepsilon} + (u_{\varepsilon} \cdot \nabla) u_{\varepsilon} + \nabla p_{\varepsilon} = f & \text{in } \Sigma_{\varepsilon}, \\ \text{div} (u_{\varepsilon}) = 0 & \text{in } \Omega_{\varepsilon}, \\ (u_{\varepsilon})^+ = (u_{\varepsilon})^- & \text{on } \Gamma_2, \\ \nu \frac{\partial u_{\varepsilon}}{\partial n}^+ = \nu_{\varepsilon} (\frac{\partial u_{\varepsilon}}{\partial n})^- & \text{on } \Gamma_2, \\ u_{\varepsilon} = 0 & \text{on } \partial \Omega_{\varepsilon}, \end{cases} \tag{1} \]

where the superscript + (resp. –) denotes the trace seen from \( \Omega \) (resp. from \( \Sigma_{\varepsilon} \)) on \( \Gamma_2 \). The thin layer \( \Sigma_{\varepsilon} \) is here considered as an unstable thin boundary layer whose Reynolds’ number \( R_{\varepsilon} \) is of order \( 1/\varepsilon \) (see [12] pages 239-240), where Reynolds’ number is allowed to depend on the thinness of the layer). In the problem (1), we suppose that the density \( f \) of volumic forces belongs to \( L^\infty (\mathbb{R}^N, \mathbb{R}^N) \).

Our purpose is to describe the asymptotic behavior of the solution \( u_{\varepsilon} \) of (1) when \( \varepsilon \) goes to 0, in order to derive the Navier wall law. We use \( \Gamma \)-convergence arguments (see [3] for the definition and the
properties of the $\Gamma$-convergence) in order to characterize the limit problem. Our approach is based on the tools developed in [1], [4], [7], [8] and [9]. On $\Gamma_2$, we will get a general Navier law of the kind

$$\begin{cases}
(Id - n \otimes n) \nu \frac{\partial u}{\partial n} + \mu^* u & = 0, \\
u \cdot n & = 0,
\end{cases}$$

where $\mu^*$ is a symmetric matrix $(\mu_{ij})_{i,j=1,...,N}$ of Borel measures having their support contained in $\Gamma_2$, which do not charge the polar subsets of $\mathbb{R}^N$ and which satisfy $\mu_{ij} (B) \zeta_i \zeta_j \geq 0$, $\forall \zeta \in \mathbb{R}^N$, $\forall B \in \mathcal{B} (\mathbb{R}^N)$, where $\mathcal{B} (\mathbb{R}^N)$ denotes the set of all Borel subsets of $\mathbb{R}^N$ and where we have used the summation convention with respect to repeated indices.

As a first special case, we prove that when $\Omega \subset \{ x_3 > 0 \}$, $\Gamma_2 = \partial \Omega \cap \{ x_3 = 0 \}$ and $h$ is a periodic function, we get on $\Gamma_2$ the Robin type boundary conditions

$$\begin{cases}
\frac{\partial u_1}{\partial x_3} (x', 0) & = -c_1 u_1 (x', 0), \\
\frac{\partial u_2}{\partial x_3} (x', 0) & = -c_2 u_2 (x', 0), \\
u_3 (x', 0) & = 0,
\end{cases}$$

where $c_m$, $m = 1, 2$, are constants which will be computed in terms of the solution of appropriate local thin layer problems [21]. This situation can be generalized to the case of a general open and bounded set $\Omega$, surrounded on a part of its boundary by such a rough thin layer.

As a second example, we will consider the case where

$$\Sigma_\varepsilon = \{ s + t n (s) \mid s \in \Gamma_2, \ - \varepsilon h (s) < x_3 < 0 \},$$

where $h$ is a Lipschitz continuous and positive function on $\Gamma_2$. We here prove that Navier’s law takes the following expression on $\Gamma_2$

$$\begin{cases}
(Id - n \otimes n) \frac{\partial u}{\partial n} + \frac{1}{h} u & = 0, \\
u \cdot n & = 0.
\end{cases}$$

In the last part of this work, we consider an optimal control problem. Choosing $m > 0$, we consider the set $\Xi_m$ of all the matrices $h = \text{Diag} (h_i)_{i=1,...,N}$ of functions $h_i : \Gamma_2 \rightarrow [0, +\infty]$, which are $d\Gamma_2$-measurable and satisfy $\int_{\Gamma_2} h_i d\Gamma_2 = m$, $\forall i = 1, \ldots, N$. We suppose that $\Omega$ is smooth enough and consider the following problem with Navier conditions on $\Gamma_2$

$$\begin{cases}
-\nu \Delta u^h + (u^h \cdot \nabla) u^h + \nabla p^h & = f \text{ in } \Omega, \\
\text{div} (u^h) & = 0 \text{ in } \Omega, \\
h (Id - n \otimes n) \frac{\partial u^h}{\partial n} + u^h & = 0 \text{ on } \Gamma_2, \\
u^h \cdot n & = 0 \text{ on } \Gamma_2.
\end{cases}$$

(2)

Let $(u^h, p^h)$ be the solution of (2) and define the functional $F$ through

$$F (h, u) = \left\{ \begin{array}{ll}
\frac{\nu}{2} \int_{\Omega} |\nabla u|^2 \, dx & + \frac{1}{2} \sum_{i=1}^N \int_{\Gamma_2} (u_i)^2 \, d\Gamma_2 \\
& + \int_{\Omega} (u^h \cdot \nabla) u^h \cdot v \, dx - \int_{\Omega} f \cdot v \, dx & \text{ if } u \in V_{0, \Gamma_2} (\Omega), \\
& + \infty & \text{ otherwise,}
\end{array} \right.$$
where $V_{0,1}(\Omega)$ is the functional space defined in (6). We consider the optimal control problem

$$\min_{h \in \Xi} \min_{u \in V_{0,1}(\Omega)} F(h, u). \quad (3)$$

In the last section of this work, we describe the asymptotic behavior of the solution of (3), when $m$ goes to 0, and characterize the zones where some thin boundary layer appears. A problem of this kind has been considered in [11], but for a linear diffusion problem.

2 Functional framework

We define the $(H^1(\mathbb{R}^N))$ capacity of any compact subset $K$ of $\mathbb{R}^N$ as

$$\text{Cap}(K) = \inf \left\{ \int_{\mathbb{R}^N} |\nabla \varphi|^2 \, dx + \int_{\mathbb{R}^N} |\varphi|^2 \, dx \mid \varphi \in C^\infty_c(\mathbb{R}^N), \varphi \geq 1 \text{ on } K \right\}.$$  

If $U$ is an open subset of $\mathbb{R}^N$, then we define

$$\text{Cap}(U) = \sup \{ \text{Cap}(K) \mid K \subset U, K \text{ compact} \}.$$  

If $B \subset \mathbb{R}^N$ is a Borel subset of $\mathbb{R}^N$, then we define

$$\text{Cap}(B) = \inf \{ \text{Cap}(U) \mid B \subset U, U \text{ open} \}.$$  

**Definition 1** Let $B(\mathbb{R}^N)$ be the $\sigma$-algebra of all Borel subsets of $\mathbb{R}^N$.

1. A property is said to be true quasi-everywhere (q.e.) on $B \in B(\mathbb{R}^N)$ if it is true except on a subset of $B$ of capacity Cap equal to 0.
2. A function $u : B \to \overline{\mathbb{R}}$, with $B \in B(\mathbb{R}^N)$, is quasi-continuous on $B$ if, for every $\varepsilon > 0$, there exists an open subset $U \subset B$ with $\text{Cap}(U) < \varepsilon$ and such that the restriction of $u$ on $B \setminus U$ is continuous.
3. Every function $u \in H^1(\mathbb{R}^N)$ has a quasi-continuous representative $\tilde{u}$, which is unique for the equality quasi-everywhere in $\mathbb{R}^N$, (see [7], for example). $\tilde{u}$ is given through

$$\tilde{u}(x) = \lim_{r \to 0^+} \frac{1}{|B(x, r)|} \int_{B(x, r)} u(y) \, dy,$$

for q.e. $x \in \mathbb{R}^N$, where $|B(x, r)|$ is the Lebesgue measure of the ball $B(x, r)$ of $\mathbb{R}^N$ of radius $r > 0$ and centered at $x$.

We define some notions concerning families of subsets of $\mathbb{R}^N$.

**Definition 2**

1. A subset $\mathcal{D} \subset B(\mathbb{R}^N)$ is a dense family in $B(\mathbb{R}^N)$ if, for every $A, B \in B(\mathbb{R}^N)$ with $\overline{A} \subset B$, there exists $D \in \mathcal{D}$ such that: $\overline{D} \subset \overline{\mathcal{D}} \subset \overline{B}$, where $\overline{A}$ (resp. $\overline{\mathcal{A}}$) denotes the interior (resp. the closure) of $A$.
2. A subset $\mathcal{R} \subset B(\mathbb{R}^N)$ is a rich family in $B(\mathbb{R}^N)$ if, for every family $(A_t)_{t \in [0, 1]} \subset B(\mathbb{R}^N)$ such that $\overline{A_t} \subset \overline{\mathcal{A}}$, for every $s < t$, the set $\{ t \in [0, 1] \mid A_t \notin \mathcal{R} \}$ is at most countable.

Let $O(\mathbb{R}^N)$ be the set of all open subsets of $\mathbb{R}^N$. We consider the class $F$ of functionals $F$ from $H^1(\mathbb{R}^N, \mathbb{R}^N) \times O(\mathbb{R}^N)$ to $[0, +\infty]$ satisfying:

4
i) (Lower semi-continuity): for every open subset $\omega \in \mathcal{O}(\mathbb{R}^N)$, the functional $u \mapsto F(u, \omega)$ is lower semi-continuous with respect to the strong topology of $H^1(\mathbb{R}^N, \mathbb{R}^N)$;

ii) (Measure property): for every $u \in H^1(\mathbb{R}^N, \mathbb{R}^N)$, $\omega \mapsto F(u, \omega)$ is the restriction to $\mathcal{O}(\mathbb{R}^N)$ of some Borel measure still denoted $F(u, \omega)$;

iii) (Localization): for every $\omega \in \mathcal{O}(\mathbb{R}^N)$ and every $u, v \in H^1(\mathbb{R}^N, \mathbb{R}^N)$:

$$u|_{\omega} = v|_{\omega} \Rightarrow F(u, \omega) = F(v, \omega);$$

iv) ($C^1$-convexity): for every $\omega \in \mathcal{O}(\mathbb{R}^N)$, the functional $u \mapsto F(u, \omega)$ is convex on $H^1(\mathbb{R}^N, \mathbb{R}^N)$ and moreover

$$\forall \varphi \in C^1(\mathbb{R}^N), 0 \leq \varphi \leq 1 : F(\varphi u + (1-\varphi) v, \omega) \leq F(u, \omega) + F(v, \omega).$$

Example 3 Let us define $\Gamma_{2,\varepsilon} = \partial \Omega_{\varepsilon} \cap \Sigma_{\varepsilon}$, for some thin layer $\Sigma_{\varepsilon}$, as defined above. We consider the functional $F^\varepsilon$ defined on the space $H^1(\mathbb{R}^N, \mathbb{R}^N) \times \mathcal{O}(\mathbb{R}^N)$ through

$$F^\varepsilon(u, \omega) = \begin{cases} 0 & \text{if } \tilde{u} = 0, \text{ q.e. on } \Gamma_{2,\varepsilon} \cap \omega, \\ +\infty & \text{otherwise.} \end{cases} \quad (4)$$

One can prove that $F^\varepsilon$ belongs to $\mathcal{F}$, for every $\varepsilon > 0$.

Let us set the following definitions.

Definition 4 Let $\text{Cap}$ be the above-defined capacity.

1. A Borel measure $\lambda$ is absolutely continuous with respect to the capacity $\text{Cap}$ if

$$\forall B \in \mathcal{B}(\mathbb{R}^N) : \text{Cap}(B) = 0 \Rightarrow \lambda(B) = 0.$$

2. $\mathcal{M}_0$ is the set of nonnegative Borel measures $\mathbb{R}^N$ which are absolutely continuous with respect to the capacity $\text{Cap}$.

We have the following example of measure in $\mathcal{M}_0$.

Example 5 For every $E \subset \mathbb{R}^N$ such that $\text{Cap}(E) > 0$, we define the measure $\infty_E$ through

$$\infty_E(B) = \begin{cases} 0 & \text{if } \text{Cap}(B \cap E) = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Then $\infty_E \in \mathcal{M}_0$.

Notice that, for every $u \in H^1(\mathbb{R}^N, \mathbb{R}^N)$ and every $\omega \in \mathcal{O}(\mathbb{R}^N)$, the functional $F^\varepsilon$ defined in (4) can be written as

$$F^\varepsilon(u, \omega) = \int_\omega |\tilde{u}|^2 d\infty_{\Gamma_{2,\varepsilon}} = \int_\omega |u|^2 d\infty_{\Gamma_{2,\varepsilon}}.$$

One has the following representation theorem for the functionals of $\mathcal{F}$.

Theorem 6 (see [4]) For every $F \in \mathcal{F}$, there exist a finite measure $\lambda \in \mathcal{M}_0$, a nonnegative Borel measure $\nu$ and a Borel function $g : \mathbb{R}^N \times \mathbb{R}^N \rightarrow [0, +\infty]$, with $\zeta \mapsto g(x, \zeta)$ convex and lower semi-continuous on $\mathbb{R}^N$, such that

$$\forall u \in H^1(\mathbb{R}^N, \mathbb{R}^N), \forall \omega \in \mathcal{O}(\mathbb{R}^N) : F(u, \omega) = \int_\omega g(x, \tilde{u}(x)) d\lambda + \nu(\omega).$$
Throughout the paper, we will need the following Corollary (see [2, Corollary 8.4]).

**Corollary 7** Let $F \in \mathcal{F}$. If $F(\cdot, \omega)$ is quadratic for every $\omega \in \mathcal{O}(\mathbb{R}^N)$, there exist $\lambda \in \mathcal{M}_0$ finite, a symmetric matrix $(a_{ij})_{i,j=1,\ldots,N}$, of Borel functions from $\mathbb{R}^N$ to $\mathbb{R}$ satisfying $a_{ij}(x)\zeta_1\zeta_j \geq 0$, $\forall \zeta \in \mathbb{R}^N$ and for q.e. $x \in \mathbb{R}^N$, for every $x \in \mathbb{R}^N$ a subspace $V(x)$ of $\mathbb{R}^N$, such that, for every $u \in H^1(\mathbb{R}^N, \mathbb{R}^N)$ and every $\omega \in \mathcal{O}(\mathbb{R}^N)$:

a) if $F(u, \omega) < +\infty$, then $u(x) \in V(x)$, for q.e. $x \in \omega$,

b) if $u(x) \in V(x)$, for q.e. $x \in \omega$

\[ F(u, \omega) = \int_\omega a_{ij}u_iu_j d\lambda. \]  

(5)

**Remark 8** Let $F \in \mathcal{F}$, $\lambda \in \mathcal{M}_0$ be the associated measure and $\Lambda$ be the set defined as $\Lambda = \cup_{\omega \in \Lambda(F)} \omega$, where

\[ A(F) = \left\{ \omega \in \mathcal{O}(\mathbb{R}^N) \mid F(\cdot, \omega) < +\infty, \text{ for q.e. } x \in \omega \right\}. \]

We define the matrix $\mu^* = (\mu_{ij}) = (a_{ij}\lambda)_{i,j=1,\ldots,N} + \infty\mathbb{R}^N \setminus \Lambda \text{Id}$ of measures, and, for every $x \in \mathbb{R}^N$, the subspace $V(x)$ through

\[ V(x) = \left\{ \begin{array}{ll} \mathbb{R}^N & \text{if } x \in \Lambda, \\ \{0\} & \text{if } x \in \mathbb{R}^N \setminus \Lambda. \end{array} \right. \]  

(6)

For every $u \in H^1(\mathbb{R}^N, \mathbb{R}^N)$ and every $\omega \in \mathcal{O}(\mathbb{R}^N)$, one has, using the preceding definition of $\mu^*$

\[ \int_\omega u_iu_j d\mu_{ij} = \left\{ \begin{array}{ll} \int_{\omega \setminus \Lambda} a_{ij}u_iu_j d\lambda & \text{if } \omega \subset \Lambda, \\ \int_{\omega \cap \Lambda} a_{ij}u_iu_j d\lambda & \text{if } \left\{ \begin{array}{l} u(x) = 0, \forall x \in \omega \cap \mathbb{R}^N \setminus \Lambda \\ \text{and } \text{Cap}(\omega \cap \mathbb{R}^N \setminus \Lambda) > 0, \end{array} \right. \text{otherwise.} \end{array} \right. \]

Thanks to [4], this expression can be written as

\[ \int_\omega u_iu_j d\mu_{ij} = \left\{ \begin{array}{ll} \int_{\omega \setminus \Lambda} a_{ij}u_iu_j d\lambda & \text{if } u(x) \in V(x), \text{ for q.e. } x \in \omega, \\ +\infty & \text{otherwise.} \end{array} \right. \]

We can thus write the functional $F$ defined in (3) as

\[ F(u, \omega) = \int_\omega u_iu_j d\mu_{ij}. \]

### 3 Study of the problem (II)

We here suppose that the "outer" boundary $\Gamma_{2, \varepsilon}$ of $\Sigma_\varepsilon$ can be defined as

\[ \Gamma_{2, \varepsilon} = \{(s, t) \mid s \in \Gamma_2, t = -\varepsilon h_\varepsilon(s)\}, \]

where $h_\varepsilon$ is a locally Lipschitz continuous function satisfying

\[ \|h_\varepsilon\|_{L^\infty(\Gamma_2)} \leq C, \forall \varepsilon > 0, \]

for some constant $C$ independent of $\varepsilon$. The Lipschitz continuity of $h_\varepsilon$ ensures the almost everywhere existence of a unit outer normal vector to $\Gamma_{2, \varepsilon}$, thanks to Rademacher’s Theorem, and ensures the
existence of an extension of every function of $H^1(\Omega, \mathbb{R}^N)$ in a function of $H^1(\mathbb{R}^N, \mathbb{R}^N)$. Let us define the functional spaces

$$
L^2(\mathbb{R}^N, \text{div}) = \{u \in L^2(\mathbb{R}^N, \mathbb{R}^N) \mid \text{div}(u) = 0 \text{ in } \mathbb{R}^N\},
$$

$$
H^1_{\Gamma_1}(\mathbb{R}^N, \text{div}) = \{u \in H^1(\mathbb{R}^N, \mathbb{R}^N) \mid \text{div}(u) = 0 \text{ in } \mathbb{R}^N, \ u = 0 \text{ on } \Gamma_1\},
$$

$$
H^1_{\Gamma_1}(\Omega, \text{div}) = \{u \in H^1(\Omega, \mathbb{R}^N) \mid \text{div}(u) = 0 \text{ in } \Omega, \ u = 0 \text{ on } \Gamma_1\},
$$

$$
V_{\Gamma_1}(\Omega) = L^2(\mathbb{R}^N, \text{div}) \cap H^1_{\Gamma_1}(\Omega, \text{div}),
$$

$$
V_{0,\Gamma_1}(\Omega) = H^1_{\Gamma_1}(\Omega, \text{div}) \cap \{u \in H^1(\Omega, \mathbb{R}^N) \mid u \cdot n = 0 \text{ on } \Gamma_2\}.
$$

In (7), let us replace throughout this section the homogeneous Dirichlet boundary condition $u^\varepsilon = 0$, on $\partial \Omega_\varepsilon$ by a combination between the homogeneous Dirichlet boundary condition $u^\varepsilon = 0$, on $\Gamma_{2,\varepsilon} \cap \omega$, for a given $\omega \in \mathcal{C}(\mathbb{R}^N)$, and homogeneous Neumann boundary conditions on $\Gamma_{2,\varepsilon} \setminus (\Gamma_{2,\varepsilon} \cap \omega)$. We introduce the functional space adapted to (7), with these modified boundary conditions

$$
V_{0,\omega}(\Omega_\varepsilon) = \{v \in H^1(\Omega_\varepsilon, \mathbb{R}^N) \mid \text{div}(v) = 0 \text{ in } \Omega_\varepsilon, \ v = 0 \text{ on } \Gamma_1 \cup (\Gamma_{2,\varepsilon} \cap \omega)\}.
$$

The variational formulation of (7) can be written as

$$
\forall \varphi \in V_{0,\omega}(\Omega_\varepsilon) : \nu \int_{\Omega} \nabla u^\varepsilon \cdot \nabla \varphi dx + \nu \varepsilon \int_{\Sigma_\varepsilon} \nabla u^\varepsilon \cdot \nabla \varphi dx
+ \int_{\Omega_\varepsilon} (u^\varepsilon \cdot \nabla) u^\varepsilon \cdot \varphi dx = \int_{\Omega_\varepsilon} f \cdot \varphi dx.
$$

(8)

Thanks to [[12]], for example, we deduce that (7) has a unique solution $(u^\varepsilon, p^\varepsilon)$ belonging to the space $V_{0,\omega}(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon)/\mathbb{R}$.

**Proposition 9** The solution $(u^\varepsilon, p^\varepsilon)$ of (7) satisfies the following estimates

$$
\sup_{\varepsilon} \left( \int_{\Omega} |\nabla u^\varepsilon|^2 dx + \varepsilon \int_{\Sigma_\varepsilon} |\nabla u^\varepsilon|^2 dx \right) < +\infty,
$$

$$
\sup_{\varepsilon} \int_{\mathbb{R}^N} |u^\varepsilon|^2 dx < +\infty,
$$

$$
\sup_{\varepsilon} \|p^\varepsilon\|_{L^2(\Omega_\varepsilon)/\mathbb{R}} < +\infty.
$$

**Proof.** 1. Taking $u^\varepsilon$ as test-function in (8), we obtain

$$
\nu \int_{\Omega} |\nabla u^\varepsilon|^2 dx + \nu \varepsilon \int_{\Sigma_\varepsilon} |\nabla u^\varepsilon|^2 dx
= \int_{\Omega} f \cdot u^\varepsilon dx + \int_{\Sigma_\varepsilon} f \cdot u^\varepsilon dx
\leq \|f\|_{L^\infty(\mathbb{R}^N, \mathbb{R}^N)} \|u^\varepsilon\|_{L^1(\Omega_\varepsilon, \mathbb{R}^N)}
\leq \|f\|_{L^\infty(\mathbb{R}^N, \mathbb{R}^N)} C(\Omega) \|\nabla u^\varepsilon\|_{L^1(\Omega_\varepsilon, \mathbb{R}^N)},
$$

using Poincaré’s inequality. Cauchy-Schwarz’ inequality implies

$$
\int_{\Omega} |\nabla u^\varepsilon|^2 dx + \varepsilon \int_{\Sigma_\varepsilon} |\nabla u^\varepsilon|^2 dx
\leq C(f, \Omega) \left( \left( \int_{\Omega} |\nabla u^\varepsilon|^2 dx \right)^{1/2} + \left( \varepsilon \int_{\Sigma_\varepsilon} |\nabla u^\varepsilon|^2 dx \right)^{1/2} \right),
$$

whence, using the trivial inequality $(a + b)^2 \leq 2(a^2 + b^2)$

$$
\int_{\Omega} |\nabla u^\varepsilon|^2 dx + \varepsilon \int_{\Sigma_\varepsilon} |\nabla u^\varepsilon|^2 dx \leq C \Rightarrow \|\nabla u^\varepsilon\|_{L^1(\Omega_\varepsilon, \mathbb{R}^N)} \leq C.
$$
The continuous embedding from $W^{1,1}_{\Gamma_1} (\Omega, \mathbb{R}^N)$ to $L^2 (\Omega, \mathbb{R}^N)$ implies the existence of a constant $C$ independent of $\varepsilon$ such that
\[
\int_{\Omega_\varepsilon} |u^\varepsilon|^2 \, dx \leq C.
\]
2. Let us define the zero mean value pressure $\bar{p}^\varepsilon = p^\varepsilon - \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} p^\varepsilon \, dx$, and let $\psi_\varepsilon$ be the solution of the following problem (see [16]),
\[
\begin{aligned}
\text{div} (\psi_\varepsilon) &= \bar{p}^\varepsilon & \text{in } \Omega_\varepsilon, \\
\psi_\varepsilon &= 0 & \text{on } \Gamma_1 \cup (\Gamma_2 \cap \omega), \\
\|\nabla \psi_\varepsilon\|_{L^2 (\Omega, \mathbb{R}^{N^2})} &\leq C (\Omega) \|\bar{p}^\varepsilon\|_{L^2 (\Omega_\varepsilon)},
\end{aligned}
\]
for some constant $C (\Omega)$ independent of $\varepsilon$. Multiplying (1) by $\psi_\varepsilon$ and using Green’s formula, one obtains
\[
\nu \int_{\Omega} \nabla u^\varepsilon \cdot \nabla \psi_\varepsilon \, dx + \nu \varepsilon \int_{\Sigma_\varepsilon} \nabla u^\varepsilon \cdot \nabla \psi_\varepsilon \, d\nu + \int_{\Omega_\varepsilon} (u^\varepsilon \cdot \nabla) u^\varepsilon \cdot \psi_\varepsilon \, dx \\
= \int_{\Omega_\varepsilon} f \cdot \psi_\varepsilon \, dx + \int_{\Omega_\varepsilon} (\bar{p}^\varepsilon \cdot \nabla) \psi_\varepsilon \, dx.
\]
Because
\[
\begin{aligned}
\left| \int_{\Omega_\varepsilon} f \cdot \psi_\varepsilon \, dx \right| &\leq \|f\|_{L^2 (\mathbb{R}^N, \mathbb{R}^N)} \|\psi_\varepsilon\|_{L^2 (\Omega, \mathbb{R}^N)} \\
\left| \int_{\Omega_\varepsilon} (u^\varepsilon \cdot \nabla) u^\varepsilon \cdot \psi_\varepsilon \, dx \right| &\leq C \|\psi_\varepsilon\|_{L^2 (\Omega, \mathbb{R}^N)} \|\nabla u^\varepsilon\|_{L^2 (\Omega, \mathbb{R}^N)} \\
\left| \int_{\Omega} \nabla u^\varepsilon \cdot \nabla \psi_\varepsilon \, dx \right| &\leq C \|\bar{p}^\varepsilon\|_{L^2 (\Omega_\varepsilon)} \|\nabla u^\varepsilon\|_{L^2 (\Omega, \mathbb{R}^N)}.
\end{aligned}
\]
thanks to [3] and using Poincaré’s inequality, we obtain
\[
\|\bar{p}^\varepsilon\|_{L^2 (\Omega_\varepsilon)}^2 \leq C \left( \|\nabla u^\varepsilon\|_{L^2 (\Omega, \mathbb{R}^N)}^2 + 1 \right) \|\bar{p}^\varepsilon\|_{L^2 (\Omega_\varepsilon)},
\]
which proves the third estimate. ■

Remark 10 We can observe that, when we impose an homogeneous Dirichlet boundary condition on the whole $\Gamma_2 \varepsilon$, for example when $\omega = \mathbb{R}^N$, the above estimates can be obtained in a simpler way, assuming only that $f \in L^2 (\mathbb{R}^N, \mathbb{R}^N)$.

4 Convergence

Every function $u \in H^{1,1}_{\Gamma_1} (\Omega, \text{div})$ can be extended in a function of the space $H^{1,1}_{\Gamma_1} (\mathbb{R}^N, \text{div})$, still denoted $u$ (see [4], Theorem 4.3.3, for example). We define the functional $\Phi^\varepsilon$ on $L^2 (\mathbb{R}^N, \mathbb{R}^N)$ associated to $\bar{p}$, with the above-described modified boundary conditions on $\Gamma_2 \varepsilon$ through
\[
\Phi^\varepsilon (u) = \begin{cases} 
\nu \int_{\Omega} |\nabla u|^2 \, dx + \nu \varepsilon \int_{\mathbb{R}^N \setminus \Omega} |\nabla u|^2 \, dx & \text{if } u \in H^{1,1}_{\Gamma_1} (\mathbb{R}^N, \text{div}), \\
+\infty & \text{otherwise},
\end{cases}
\]
and the functional $\Phi^0$ defined on $L^2 (\mathbb{R}^N, \mathbb{R}^N)$ through
\[
\Phi^0 (u) = \begin{cases} 
\nu \int_{\Omega} |\nabla u|^2 \, dx & \text{if } u \in V^{1,1}_{\Gamma_1} (\Omega), \\
+\infty & \text{otherwise}.
\end{cases}
\]
From the estimates given in Proposition 3, we can deduce that the asymptotic behaviour of the problem (3) is obtained when studying the \( \Gamma \)-limit of the associated energy functional for the following topology.

**Definition 11** A sequence \( (u_\varepsilon)_\varepsilon \) \(-\varepsilon\)-converges to \( u \), if it converges to \( u \) in the strong topology of \( L^2(\mathbb{R}^N, \mathbb{R}^N) \) and if \( \sup_\varepsilon \Phi^\varepsilon (u_\varepsilon) < +\infty \).

We first present the \( \Gamma \)-convergence result for \( (\Phi^\varepsilon)_\varepsilon \).

**Proposition 12** When \( \varepsilon \) goes to 0, the sequence \( (\Phi^\varepsilon)_\varepsilon \) \( \Gamma \)-converges to \( \Phi^0 \), in the topology \( \tau \).

**Proof.** Step 1: verification of the \( \Gamma \)-lim sup. Take any \( u \in V_{\Gamma_1}(\Omega) \) and consider the set \( \Omega^{0,\varepsilon} = \Omega \cup \partial \Omega \cup \Sigma^{0,\varepsilon} \), with

\[
\Sigma^{0,\varepsilon} = \{ x \in \mathbb{R}^N \mid 0 < d(x, \partial \Omega) < \varepsilon \},
\]

where \( d(x, \partial \Omega) \) denotes the euclidean distance between \( x \) and the boundary \( \partial \Omega \). Let \( u^{1,\varepsilon} \) be such that \( \text{div} (u^{1,\varepsilon}) = 0 \) in \( \mathbb{R}^N \) and

\[
\| u - u^{1,\varepsilon} \|_{L^2(\mathbb{R}^N \setminus \Omega^{0,\varepsilon}, \mathbb{R}^N)} < \varepsilon.
\]

We define \( \overline{u}^{1,\varepsilon} \) through

\[
\overline{u}^{1,\varepsilon} = \begin{cases}
  u^{1,\varepsilon} & \text{in } \mathbb{R}^N \setminus \Omega^{0,\varepsilon}, \\
  0 & \text{on } \partial \Omega^{0,\varepsilon}.
\end{cases}
\]

We then take a nonnegative and smooth function \( \rho_\varepsilon \in C_\infty^\varepsilon (\mathbb{R}^N) \) with support in \( B(0, \varepsilon) \) and satisfying \( \int_{\mathbb{R}^N} \rho_\varepsilon(x) \, dx = 1 \). We define the function \( \overline{u}^{1,\varepsilon} \) through \( \overline{u}^{0,\varepsilon} = (\rho_\varepsilon \ast \overline{u}^{1,\varepsilon})_{|_{R \setminus \Omega^{0,\varepsilon}}} \). There exists \( \widehat{u} \in L^2 (\mathbb{R}^N, \mathbb{R}^N) \) such that \( \text{curl}(\widehat{u}) = u \) in \( \mathbb{R}^N \) (see [3], for example). We finally define the function \( u^{0,\varepsilon} \) through

\[
u^{0,\varepsilon} = \begin{cases}
  \overline{u}^{0,\varepsilon} & \text{in } \mathbb{R}^N \setminus \Omega^{0,\varepsilon}, \\
  u & \text{in } \Omega.
\end{cases}
\]

We immediately satisfy that \( u^{0,\varepsilon} \in H^1_{\Gamma_1}(\mathbb{R}^N, \text{div}) \), that the sequence \( (u^{0,\varepsilon})_\varepsilon \) converges to \( u \) in the topology \( \tau \) and that

\[
\limsup_{\varepsilon \to 0} \Phi^\varepsilon (u^{0,\varepsilon}) \leq \nu \int_\Omega |\nabla u|^2 \, dx = \Phi^0 (u).
\]

Step 2: verification of the \( \Gamma \)-lim inf. We take any sequence \( (u_\varepsilon)_\varepsilon \) contained in \( H^1_{\Gamma_1}(\mathbb{R}^N, \text{div}) \) which converges to \( u \) in the topology \( \tau \). We trivially have

\[
\Phi^0 (u) \leq \liminf_{\varepsilon \to 0} \Phi^\varepsilon (u_\varepsilon) \leq \liminf_{\varepsilon \to 0} \Phi^\varepsilon (u_\varepsilon),
\]

thanks to the lower semi-continuity property of \( \Phi^0 \) for the weak topology of \( H^1 (\mathbb{R}^N, \mathbb{R}^N) \).

We define the functional \( G^\varepsilon \) on \( L^2 (\mathbb{R}^N, \mathbb{R}^N) \times \mathcal{O} (\mathbb{R}^N) \) through

\[
G^\varepsilon (u, \omega) = \begin{cases}
  \Phi^\varepsilon (u) + F^\varepsilon (u, \omega) & \text{if } u \in H^1_{\Gamma_1}(\mathbb{R}^N, \text{div}), \\
  +\infty & \text{otherwise},
\end{cases}
\]

where \( F^\varepsilon \) is defined in [4]. Our main result is the following.

**Theorem 13** There exist a rich family \( \mathcal{R} \subset \mathcal{B} (\mathbb{R}^N) \) and a symmetric matrix \( \mu^* = (\mu^*_{ij})_{i,j=1,...,N} \) of Borel measures having their support contained in \( \Gamma_2 \), which are absolutely continuous with respect to the above-defined capacity \( \text{Cap} \), and satisfying \( \mu^*_{ij} (B) \zeta_i \zeta_j \geq 0, \forall \zeta_i \in \mathbb{R}^N, \forall B \in \mathcal{B} (\mathbb{R}^N) \), such that, for every \( u \in V_{\Gamma_1}(\Omega) \) and every \( \omega \in \mathcal{R} \cap \mathcal{O} (\mathbb{R}^N) \)

\[
\left( \Gamma_{\varepsilon \to 0} \text{lim} G^\varepsilon \right) (u, \omega) = \nu \int_\Omega |\nabla u|^2 + \int_{\Gamma_{\varepsilon \cap \omega}} u_i u_j d\mu_{ij} =: G^0 (u, \omega),
\]

where the \( \Gamma \)-limit is taken with respect to the topology \( \tau \).
Proof. The upper and lower $\Gamma$-limits of the sequence $(G^\varepsilon)_\varepsilon$, with respect to the topology $\tau$, exist, which are respectively defined through

$$
\forall u \in V_{\Gamma_1}(\Omega), \forall B \in \mathcal{B}(\mathbb{R}^N): \begin{cases} G^\varepsilon(u, B) = \inf_{\omega \in \mathcal{O}} \limsup_{\varepsilon \to 0} G^\varepsilon(u, B), \\
G^\varepsilon(u, B) = \inf_{\omega \in \mathcal{O}} \liminf_{\varepsilon \to 0} G^\varepsilon(u, B).
\end{cases}
$$

(11)

Because $F^\varepsilon$ takes nonnegative values and thanks to Proposition [12], we observe that, for every $B \in \mathcal{B}(\mathbb{R}^N)$, one has

$$
G^\varepsilon(\cdot, B) \geq \Phi^0(\cdot); G^\varepsilon(\cdot, B) \geq \Phi^0(\cdot).
$$

Let us define the functionals $F^s$ and $F^i$ on $L^2(\mathbb{R}^N, \mathbb{R}^N) \times \mathcal{B}(\mathbb{R}^N)$ through

$$
(F^0)^{\alpha}(u, B) = \begin{cases} G^\varepsilon(u, B) - \Phi^0(u) & \text{if } u \in V_{\Gamma_1}(\Omega), \\
+\infty & \text{otherwise},
\end{cases}
$$

with $\alpha = s, i$. Let $u \in V_{\Gamma_1}(\Omega)$ and $(u_\varepsilon)_{\varepsilon} \subset H^1_1(\mathbb{R}^N, \text{div})$ be such that $(u_\varepsilon)_{\varepsilon}$ converges to $u$ in the topology $\tau$. We define $z_\varepsilon = u_\varepsilon - u$. Thus $(z_\varepsilon)_{\varepsilon} \subset H^1(\mathbb{R}^N, \mathbb{R}^N)$ and $(z_\varepsilon)_{\varepsilon}$ converges to 0 in the topology $\tau$. Replacing $u_\varepsilon$ by $z_\varepsilon + u$ in $[11]$, one obtains, using the quadratic property of $\Phi^\varepsilon$

$$
(F^0)^{s}(u, B) = \inf_{\varepsilon \to 0} \limsup_{\varepsilon \to 0} (\Phi^\varepsilon(z_\varepsilon) + F^\varepsilon(u + z_\varepsilon, B)),
$$

$$
(F^0)^{i}(u, B) = \inf_{\varepsilon \to 0} \liminf_{\varepsilon \to 0} (\Phi^\varepsilon(z_\varepsilon) + F^\varepsilon(u + z_\varepsilon, B)).
$$

The functionals $(F^0)^{s}$ and $(F^0)^{i}$ satisfy the following properties.

1. For every $u \in V_{\Gamma_1}(\Omega)$, $(F^0)^{s}(u, \cdot)$ and $(F^0)^{i}(u, \cdot)$ are nonnegative measures, because $F^\varepsilon(u + z_\varepsilon, \cdot)$ is a measure for every $\varepsilon > 0$ and for every sequence $(z_\varepsilon)_{\varepsilon} \subset V_{\Gamma_1}(\Omega)$ which converges to 0 in the topology $\tau$.

2. $(F^0)^{s}(\cdot, B)$ and $(F^0)^{i}(\cdot, B)$ are lower semi-continuous on $H^1(\mathbb{R}^N, \mathbb{R}^N)$, when equipped with its strong topology, because $G^\varepsilon(\cdot, B), G^i(\cdot, B)$ and $\Phi^0$ are lower semi-continuous as upper, lower, or $\Gamma$-limits of functionals which are lower semi-continuous for this strong topology.

3. Let $\omega \in O(\mathbb{R}^N)$ and $u, v \in V_{\Gamma_1}(\Omega)$ be such that $u|\omega = v|\omega$. Then $(F^0)^{s}(\cdot, \omega) = (F^0)^{s}(\cdot, \omega)$ and $(F^0)^{i}(\cdot, \omega) = (F^0)^{i}(\cdot, \omega)$, because $F^\varepsilon(u + z_\varepsilon, \omega) = F^\varepsilon(v + z_\varepsilon, \omega)$, for every sequence $(z_\varepsilon)_{\varepsilon}$ such that $u + z_\varepsilon \in H^1_1(\mathbb{R}^N, \text{div})$, for every $\varepsilon > 0$.

4. Take any $\varphi \in C^1(\mathbb{R}^N)$ such that $0 \leq \varphi \leq 1$, $u, v \in V_{\Gamma_1}(\Omega)$ and $B \in \mathcal{B}(\mathbb{R}^N)$. One has, for every sequence $(z_\varepsilon)_{\varepsilon} \subset V_{\Gamma_1}(\Omega)$ converging to 0 in the topology

$$
F^\varepsilon(z_\varepsilon + \varphi u + (1 - \varphi) v, B) = F^\varepsilon((z_\varepsilon + u) \varphi + (1 - \varphi)(z_\varepsilon + v), B) \leq F^\varepsilon(z_\varepsilon + u, B) + F^\varepsilon(z_\varepsilon + v, B),
$$

because $F^\varepsilon$ is $C^1$-convex. Because $\Phi^\varepsilon$ takes nonnegative values, for every $\varepsilon > 0$, one has

$$
\limsup_{\varepsilon \to 0} (\Phi^\varepsilon(z_\varepsilon) + F^\varepsilon(z_\varepsilon + \varphi u + (1 - \varphi) v, B)) \leq \limsup_{\varepsilon \to 0} (\Phi^\varepsilon(z_\varepsilon) + F^\varepsilon(z_\varepsilon + u, B) + \Phi^\varepsilon(z_\varepsilon) + F^\varepsilon(z_\varepsilon + v, B)) \leq \limsup_{\varepsilon \to 0} (\Phi^\varepsilon(z_\varepsilon) + F^\varepsilon(z_\varepsilon + u, B)) + \limsup_{\varepsilon \to 0} (\Phi^\varepsilon(z_\varepsilon) + F^\varepsilon(z_\varepsilon + v, B)).
$$

Taking the infimum over all sequences $(z_\varepsilon)_{\varepsilon} \subset H^1(\mathbb{R}^N, \mathbb{R}^N)$ which converge to 0 in the topology $\tau$, one obtains

$$
(F^0)^{s}(\varphi u + (1 - \varphi) v, B) \leq (F^0)^{s}(u, B) + (F^0)^{s}(v, B).
$$

We prove in a similar way that $(F^0)^{s}$ is convex. Thus $(F^0)^{s}$ is $C^1$-convex.
Thanks to the compactness theorem of [10], there exist a subsequence \((\varepsilon_k)_k\) and a dense and countable family \(\mathcal{D} \subset \mathcal{B}(\mathbb{R}^N)\) such that, for every \(u \in \mathbf{V}_{\Gamma_1}(\Omega)\) and every \(B \in \mathcal{D}\)

\[
\left(\Gamma-\lim_{k \to +\infty}G^{\varepsilon_k}\right)(u, B) = G^0(u, B),
\]

where the \(\Gamma\)-limit is taken with respect to the topology \(\tau\). We then define the functional \(F^0\) on \(L^2(\mathbb{R}^N, \mathbb{R}^N) \times \mathcal{D}\) as

\[
F^0(u, B) = \begin{cases} G^0(u, B) - \Phi^0(u) & \text{if } u \in \mathbf{V}_{\Gamma_1}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}
\]

We have \(F^0 = (F^0)^* = (F^0)^i\) on \(L^2(\mathbb{R}^N, \mathbb{R}^N) \times \mathcal{D}\). We then extend \(F^0\) on \(L^2(\mathbb{R}^N, \mathbb{R}^N) \times \mathcal{B}(\mathbb{R}^N)\) defining

\[
F^0(u, B) = \sup_{D \in \mathcal{D}, \mathcal{D} \subset B} (F^0)^s(u, D) = \sup_{D \in \mathcal{D}, \mathcal{D} \subset B} (F^0)^i(u, D). \tag{13}
\]

We define the family \(\mathcal{R}(F)\) of Borel subsets of \(\mathbb{R}^N\) through

\[
\mathcal{R}(F) = \left\{ B \in \mathcal{B}(\mathbb{R}^N) \mid \forall u \in L^2(\mathbb{R}^N, \mathbb{R}^N) : (F^0)^s_u(u, B) = \sup_{D \in \mathcal{D}, \mathcal{D} \subset B} (F^0)^s(u, D) \right\}.
\]

Then we prove (see [3, Proposition 14.14]) that \(\mathcal{R}(F^0)\) is a rich family in \(\mathcal{B}(\mathbb{R}^N)\) and \(F^0 = (F^0)^* = (F^0)^s = (F^0)^i = (F^0)^i = (F^0)^*\) on \(\mathcal{R}(F^0)\). One obtains, for every \(u \in \mathbf{V}_{\Gamma_1}(\Omega)\) and every \(B \in \mathcal{R}(F^0)\)

\[
F^0(u, B) = \inf \limsup_{z_k \to 0} (\Phi^*_{\varepsilon_k}(z_k) + F^*_{\varepsilon_k}(u + z_k, B))
\]

\[
= \inf \liminf_{z_k \to 0} (\Phi^*_{\varepsilon_k}(z_k) + F^*_{\varepsilon_k}(u + z_k, B)).
\]

Let now \(\varepsilon'\) denote any subsequence of \(\varepsilon\). Thanks to the above method, there exist a subsequence \((\varepsilon'_k)_k\), a functional \(F^0\) and a rich family \(\mathcal{R}(F^0)\) such that, for every \(u \in \mathbf{V}_{\Gamma_1}(\Omega)\) and every \(B \in \mathcal{R}(F^0)\)

\[
F^0(u, B) = \inf \limsup_{z_{k'} \to 0} (\Phi^*_{\varepsilon'}(z_{k'}) + F^*_{\varepsilon'}(u + z_{k'}, B))
\]

\[
= \inf \liminf_{z_{k'} \to 0} (\Phi^*_{\varepsilon'}(z_{k'}) + F^*_{\varepsilon'}(u + z_{k'}, B)).
\]

Because \(\mathcal{R}(F^0) \cap \mathcal{R}(F^0)\) is still a rich family, one has

\[\forall u \in \mathbf{V}_{\Gamma_1}(\Omega), \forall B \in \mathcal{R} : F^0(u, \cdot) = F^0(u, \cdot), \text{ on } \mathcal{R}(F^0) \cap \mathcal{R}(F^0).\]

Because the countable intersection of rich families is a rich family too, one can repeat the above reasoning and deduce the existence of a rich family \(\mathcal{R}\) in \(\mathcal{B}(\mathbb{R}^N)\) on which the above limits coincide. One thus obtains, for every \(u \in \mathbf{V}_{\Gamma_1}(\Omega)\) and every \(B \in \mathcal{R}\)

\[
\left(\Gamma-\lim_{\varepsilon \to 0}G^\varepsilon\right)(u, \omega) = \Phi^0(u) + F^0(u, B), \tag{14}
\]

where the \(\Gamma\)-limit is taken with respect to the topology \(\tau\).

Thanks to the above properties 1., 2., 3. and 4. and to the relations (12) and (13), \(F^0\) belongs to \(\mathbb{F}\). Because \(\Phi^\varepsilon\) and \(F^\varepsilon\) are quadratic, thanks to Corollary [3] and to Remark [3] there exist \(\lambda \in \mathcal{M}_0\) finite, a
symmetric matrix $(a_{ij})_{i,j=1,...,N}$ of Borel functions from $\mathbb{R}^N$ to $\mathbb{R}$ with $a_{ij}(x)\zeta_i\zeta_j \geq 0$, $\forall \zeta \in \mathbb{R}^N$ and for q.e. $x \in \mathbb{R}^N$, such that, for every $u \in V_{\Gamma_2}(\Omega)$ and every $\omega \in \mathcal{R} \cap \mathcal{O}(\mathbb{R}^N)$

$$F^0(u,\omega) = \int_\omega u_i u_j d\mu_{ij},$$

with $\mu^* = (\mu_{ij})_{i,j=1,...,N} = (a_{ij}\lambda)_{i,j=1,...,N} + \infty_{\mathbb{R}^N \setminus \lambda} \text{Id}$, where $\lambda$ is defined as in Remark 3.

Let us now precise the support of $\mu^*$. For every $u,v \in H^1_1(\mathbb{R}^N,\text{div})$, such that $v|\Omega = u|\Omega$, one has

$$F^0(u,R^N) = \int_{\mathbb{R}^N} v_i u_j d\mu_{ij},$$

because $F^0$ is local ($\mathbb{R}^N$ belongs to $\mathcal{R}$ because every rich family is dense, and every dense family contains $\mathbb{R}^N$). One deduces that $\text{supp}(\mu^*) \subset \Omega \cup \Gamma_2$. Thanks to (13), one has

$$0 \leq \int_{\mathbb{R}^N} u_i u_j d\mu_{ij} + F^0(u) \leq \liminf_{\varepsilon \to 0^+} \left(\Phi^\varepsilon(u) + F^\varepsilon(u,R^N)\right).$$

Taking $u \in H^1_0(\Omega,\text{div}) = \{u \in H^1_0(\Omega,R^N) \mid \text{div}(u) = 0\}$, then, for every $\varepsilon > 0$, $F^\varepsilon(u,R^N) = 0$, and $\liminf_{\varepsilon \to 0^+} \Phi^\varepsilon(u) = \Phi^0(u)$. One deduces, using (13), that $\int_\Omega u_i u_j d\mu_{ij} = 0$, and thus that $\text{supp}(\mu^*) \subset \Gamma_2$, which ends the proof.

**Remark 14**

1. We thus get Navier’s wall law at the zeroth-order limit of the problem (1).

2. Theorem 13 can be extended to every kind of obstacle functional in $\mathcal{F}$, using Theorem 2 for the integral representation. One can define, for example, sequences of obstacle functionals on $H^1(\mathbb{R}^N,\mathbb{R}^N) \times \mathcal{O}(\mathbb{R}^N)$ of the kind

$$(F^\varepsilon)^+(u,\omega) = \begin{cases} 0 & \text{if } \tilde{u} \geq 0 \text{ q.e. on } \Gamma_2 \cap \omega, \\ +\infty & \text{otherwise,} \end{cases}$$

the limit $(F^0)^+$ of which is defined on $V_{\Gamma_1}(\Omega) \times (\mathcal{R}^+ \cap \mathcal{O}(\mathbb{R}^N))$ (for some rich family $\mathcal{R}^+$) as

$$(F^0)^+(u,\omega) = \int_{\omega \cap \Gamma_2} u^+_i u^+_j d\mu_{ij},$$

where $u^+_i = \max\{0,u_i\}$, $i = 1,...,N$.

3. One proves that $\mu_{ij} \in H^{-1/2}(\Gamma_2)$, $\forall i,j = 1,...,N$, where $\mu_{ij}$ is the measure defined in Theorem 2. One first observes that the measure $\lambda$ defined in Theorem 2 belongs to $H^{-1/2}(\Gamma_2)^+$. $\lambda$ is indeed finite. Because for every compact subset $K \subset \Gamma_2$, one has $\lambda(K) < +\infty$, hence $\lambda$ is a Radon nonnegative measure. Moreover, because $\lambda$ is absolutely continuous with respect to the capacity $\text{Cap}$, we deduce from [2, Theorem 2.2], the existence of a Radon measure $\varpi \in H^{-1/2}(\Gamma_2)$ and of a Borel function $f : \Gamma_2 \to [0, +\infty[$ such that $f = \frac{d\varpi}{d\text{Cap}}$.

Let us come back to the study of problem (1). The solution $u^\varepsilon$ of (1), with the homogeneous Dirichlet boundary conditions on $\partial \Omega_\varepsilon$ is also the solution of the minimization problem

$$\inf_{v \in L^1(\mathbb{R}^N,\mathbb{R}^N)} \left(G^\varepsilon(v,R^N) + 2\int_{\Omega_\varepsilon} (u^\varepsilon \cdot \nabla) u^\varepsilon \cdot v dx - 2\int_{\Omega_\varepsilon} f \cdot v dx\right).$$

From Theorem 13, one deduces the following asymptotic behaviour of the solution of (1).
Corollary 15  The solution \((u^\varepsilon, p^\varepsilon)\) of (14), is such that \((u^\varepsilon)_\varepsilon\) converges to \(u^0\) in the topology \(\tau\) and \(((p^\varepsilon))_{\varepsilon}\) converges to \(p^0\) in the strong topology of \(L^2(\Omega)/\mathbb{R}\), where \((u^0, p^0)\) belongs to \(V_{0, \Gamma_1}(\Omega) \times L^2(\Omega)/\mathbb{R}\), and is the solution of the limit minimization problem

\[
\inf_{v \in L^2(\mathbb{R}^N, \mathbb{R}^N)} \left( G^0(v, \mathbb{R}^N) + 2 \int_{\Omega} \langle u^0 \cdot \nabla \rangle v^0 \cdot vdx - 2 \int_{\Omega} f \cdot vdx \right),
\]

or of the limit problem with Navier law

\[
\begin{align*}
-\nu \Delta u_0^0 + (u^0_0 \cdot \nabla) u^0_0 + \nabla p^0_0 &= f & \text{in } \Omega, \\
\operatorname{div}(u^0_0) &= 0 & \text{in } \Omega, \\
u_0 &= 0 & \text{on } \Gamma_1, \\
0 & \text{on } \Gamma_2, \\
(I - n \otimes n) \frac{\partial u_0^0}{\partial n} + \mu \cdot u^0_0 &= 0 & \text{on } \Gamma_2.
\end{align*}
\]

**Proof.** We first observe that, for every sequence \((v_\varepsilon)_\varepsilon\) converging to \(v\) in the topology \(\tau\)

\[
\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} f \cdot v_\varepsilon dx = \int_{\Omega} f \cdot vdx,
\]

Thanks to the properties of the \(\Gamma\)-convergence, \((u^\varepsilon)_\varepsilon\) converges to \(u^0\) in the topology \(\tau\), with \(u^0 \in V_{\Gamma_1}(\Omega)\), and

\[
\lim_{\varepsilon \to 0} G^\varepsilon(u^\varepsilon, \mathbb{R}^N) = G^0(u^0, \mathbb{R}^N) = \nu \int_{\Omega} |\nabla u_0^0|^2 dx + \int_{\Gamma_2} (u_0^0)^2 \mu d\mu_{ij}.
\]

Then

\[
\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} (u^\varepsilon \cdot \nabla) u^\varepsilon \cdot v_\varepsilon dx = \int_{\Omega} (u^0 \cdot \nabla) u^0 \cdot vdx,
\]

for every sequence \((v_\varepsilon)_\varepsilon\) converging to \(v\) in the topology \(\tau\). For every \(\varphi \in C^1(\mathbb{R}^N)\), one has

\[
\left| \int_{\Sigma_\varepsilon} u^\varepsilon \cdot \nabla \varphi dx \right| \leq \left( \int_{\Sigma_\varepsilon} |\nabla \varphi|^2 dx \right)^{1/2} \left( \int_{\mathbb{R}^N} |u^\varepsilon|^2 dx \right)^{1/2},
\]

and thus \(\lim_{\varepsilon \to 0} \int_{\Sigma_\varepsilon} u^\varepsilon \cdot \nabla \varphi dx = 0\). Because \(\operatorname{div}(u^\varepsilon) = \operatorname{div}(u^0) = 0\), and \(u^\varepsilon = 0\) q.e. on \(\Gamma_2\), one has

\[
0 = \int_{\Omega_\varepsilon} u^\varepsilon \cdot \nabla \varphi dx = \int_{\Omega} u^\varepsilon \cdot \nabla \varphi dx + \int_{\Sigma_\varepsilon} u^\varepsilon \cdot \nabla \varphi dx.
\]

Taking the limit of this equality, we obtain

\[
0 = \int_{\Omega} u^0 \cdot \nabla \varphi dx = \int_{\Gamma_2} u^0 \cdot n \varphi d\Gamma_2,
\]

which proves that \(u^0 \cdot n = 0\) on \(\Gamma_2\). Thus \(u^0 \in V_{0, \Gamma_1}(\Omega)\) is the solution of the problem (17). The variational formulation of (17) can be written as

\[
\forall \varphi \in V_{0, \Gamma_1}(\Omega) : \int_{\Omega} \left(-\nu \Delta u^0 + (u^0 \cdot \nabla) u^0\right) \cdot \varphi dx + \int_{\Gamma_2} \nu \frac{\partial u^0}{\partial n} \varphi d\Gamma_2 + \int_{\Gamma_2} (u^0)^2 \mu d\mu_{ij} = \int_{\Omega} f \cdot \varphi dx.
\]

There exists \(p_0 \in L^2(\Omega)/\mathbb{R}\) such that \(-\nu \Delta u^0 + (u^0 \cdot \nabla) u^0 - f = -\nabla p_0\). Thanks to Proposition 3 the sequence \(((p^\varepsilon))_{\varepsilon}\) converges to \(p^0\) in the strong topology of \(L^2(\Omega)/\mathbb{R}\). Because \(\varphi \cdot n = 0\) on \(\Gamma_2\), with \(n = (0, 0, 1)\), one has: \(\nu \frac{\partial u^0}{\partial n} \varphi = (Id - n \otimes n) \nu \frac{\partial u^0}{\partial n} \varphi\), which ends the proof. \qed
5 Special cases

We intend to specialize the general result obtained in Theorem 13 in two cases where the boundary \( \Gamma_{2, \varepsilon} \) can be defined through some Lipschitz continuous function.

5.1 Periodic case

In this section, we suppose that \( \Omega \subset \{ x_3 > 0 \} \) with \( \partial \Omega \cap \{ x_3 = 0 \} = \Gamma_2 \), \( \Gamma_2 \) containing 0. We define \( Y = (-1/2, 1/2)^3 \) and consider a \( Y \)-periodic function \( h \in C^2_r (Y, \mathbb{R}_+) \). For every \( k \in \mathbb{Z}^2 \), we define \( Y^k_\varepsilon = (-\varepsilon/2, \varepsilon/2)^2 + (k_1 \varepsilon, k_2 \varepsilon) \), and let \( I_\varepsilon = \{ k \in \mathbb{Z}^2 \mid Y^k_\varepsilon \subset \Gamma_2 \} \). We define \( h_\varepsilon \) on \( \Gamma_2 \) through

\[
  h_\varepsilon (x') = \begin{cases} 
    \frac{h \left( \frac{x'}{\varepsilon} \right)}{m^2} & \text{if there exists } k \in I_\varepsilon \text{ such that } x' = (x_1, x_2) \in Y^k_\varepsilon, \\
    0 & \text{otherwise}
  \end{cases}
\]

and \( \Sigma_\varepsilon \) through

\[
  \Sigma_\varepsilon = \{ x \in \mathbb{R}^3 \mid x' = (x_1, x_2) \in \Gamma_2, \ -\varepsilon h_\varepsilon (x') < x_3 < 0 \}.
\]

Thanks to Theorem 13, there exist a rich family \( \mathcal{R} \subset \mathcal{B} (\mathbb{R}^3) \), a symmetric matrix \( (\mu_{ij})_{i,j=1,\ldots,N} \) of Borel measures having the same support contained in \( \Gamma_2 \), absolutely continuous with respect to the capacity \( \text{Cap} \), and satisfying \( \mu_{ij} (B) \zeta_i \zeta_j \geq 0 \), \( \forall \zeta_i \in \mathbb{R}^3 \), \( \forall B \in \mathcal{B} (\mathbb{R}^3) \), such that, for every \( u \in \mathcal{V}_{1, \varepsilon} (\Omega) \) and every \( \omega \in \mathcal{R} \cap \mathcal{O} (\mathbb{R}^3) \)

\[
  \inf \left\{ \liminf_{\varepsilon \to 0} \Phi^\varepsilon (z_\varepsilon) \mid u + z_\varepsilon = 0 \text{ on } \{ x_3 = -\varepsilon h_\varepsilon (x') \} \cap \omega \text{ and } z_\varepsilon \xrightarrow{\varepsilon \to 0} 0 \right\} = \int_{\omega \cap \Gamma_2} u_i u_j d\mu_{ij},
\]

where \( \Phi^\varepsilon \) is the energy functional defined in (13).

Because the lower boundary \( \Gamma_{2, \varepsilon} \) of \( \Sigma_\varepsilon \), defined through the equality \( \Gamma_{2, \varepsilon} = \{ (x', x_3) \mid x_3 = -\varepsilon h_\varepsilon (x') \} \), has a periodic structure, the measures \( \mu_{ij} \), \( i, j = 1, \ldots, N \), are invariant under translations on \( \Gamma_2 \). This implies \( \mu_{ij} = K_{ij} dx' \), where \( K_{ij}, i, j = 1, 2, 3 \), are constants in \( \mathbb{R} \) satisfying \( K_{ij} \zeta_i \zeta_j \geq 0 \), \( \forall \zeta_i \in \mathbb{R}^3 \).

The purpose of this section is to identify these constants \( K_{ij} \), \( i, j = 1, 2, 3 \). We observe that we do not have to determine \( K_{33} \), \( i = 1, 2, 3 \), because, in the limit problem, one has \( u \cdot n = u \cdot e_3 = u_3 = 0 \).

**Theorem 16** The limit Navier wall law of the limit problem (20) is in this case

\[
  \frac{\partial (u^0)^m}{\partial x_3} = c_m (u^0)^m, \quad \text{on } \Gamma_2, \ m = 1, 2,
\]

where the constants \( c_m \) are defined in (24).

**Proof.** We define the set \( Z_h = \{ x \mid x' \in Y, \ -h (x') < x_3 < 0 \} \) and consider in \( Z_h \) the local Stokes problems for \( m = 1, 2 \)

\[
  \begin{cases} 
    -\Delta w^m + \nabla q^m = e^m & \text{in } Z_h, \\
    \text{div} (w^m) = 0 & \text{in } Z_h, \\
    w^m = e^m & \text{on } \{ x_3 = -h (x') \}, \\
    w^m = 0 & \text{on } \{ x_3 = 0 \}, \\
    w^m, q^m & \text{\( Y \)-periodic,}
  \end{cases}
\]

where \( e^m \) is the \( m \)-th vector of the canonical basis of \( \mathbb{R}^3 \). Lax-Milgram' Theorem implies that (20) has a unique solution \( (w^m, q^m) \) with

\[
  w^m \in \mathcal{V} (Z_h) = \left\{ u \in \mathcal{H}^1 (Z_h, \mathbb{R}^3) \mid \text{div} (u) = 0 \text{ in } Z_h, \right. \\
  u = 0 \text{ on } \{ x_3 = 0 \}, u \text{ \( Y \)-periodic}
\]

and

\[
  q^m \in \mathcal{L}^2 (Z_h) / \mathbb{R}, q^m \text{ \( Y \)-periodic.}
\]
Let \( z_h = \max_{x' \in Y} h(x') \) and choose \( H > z_h \). We define

\[ \tilde{Z}_h = \{ x \mid x' \in Y, -H < x_3 < -h(x') \} \]

and consider in \( \tilde{Z}_h \) problems similar to (20) except that we impose \( \tilde{w}^m = e^m \) on \( \{ x_3 = -h(x') \} \) and \( \tilde{w}^m = 0 \) on \( \{ x_3 = -H \} \). Let us define

\[
\begin{align*}
\tilde{S}_\varepsilon &= \{ x \in \mathbb{R}^3 \mid x' = (x_1, x_2) \in \Gamma_2, -\varepsilon H < x_3 < -\varepsilon h_\varepsilon (x') \}, \\
B_\varepsilon &= \{ x \in \mathbb{R}^3 \mid x' = (x_1, x_2) \in \Gamma_2, -\varepsilon H < x_3 < 0 \}
\end{align*}
\]

and the functions \( (w^m, q^m) \) and \( (\tilde{w}^m, \tilde{q}^m) \) through

\[
\begin{align*}
\begin{cases}
w^m (x) &= w^m \left( \frac{x}{\varepsilon} \right), & q^m (x) &= q^m \left( \frac{x}{\varepsilon} \right), \\
\tilde{w}^m (x) &= \tilde{w}^m \left( \frac{x}{\varepsilon} \right), & \tilde{q}^m (x) &= \tilde{q}^m \left( \frac{x}{\varepsilon} \right).
\end{cases}
\end{align*}
\]

We finally build the function \( z^0_\varepsilon \), on \( B_\varepsilon \), through

\[
\begin{cases}
w^m (x) & \text{if } x \in \Sigma_\varepsilon, \\
\tilde{w}^m (x) & \text{on } \{ x_3 = -\varepsilon h_\varepsilon (x') \},
\end{cases}
\]

Because \( h = 0 \) on \( \partial Y \), one can suppose that \( z^0_\varepsilon = 0 \) on \( \partial \Gamma_2 \times (-\varepsilon H, 0) \). This implies that \( z^0_\varepsilon \in H^1_{\Gamma_1} (\mathbb{R}^3, \text{div}) \) and \( z^0_\varepsilon = 0 \) on \( \partial B_\varepsilon \). Moreover

\[
\begin{align*}
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} |z^0_\varepsilon|^2 \, dx &= \lim_{\varepsilon \to 0} \int_{B_\varepsilon} |z^0_\varepsilon|^2 \, dx \\
&= \lim_{\varepsilon \to 0} \sum_{k \in I_\varepsilon} \int_{Y^3_k} \int_{-\varepsilon H}^{\varepsilon H} |z^0_\varepsilon|^2 \, dx \\
&= \lim_{\varepsilon \to 0} \left( \sum_{k \in I_\varepsilon} \int_{Y^3_k} \int_{-H}^{0} |\tilde{w}^m (x)|^2 \, dx \\
& \quad + \sum_{k \in I_\varepsilon} \int_{Y^3_k} \int_{-h(x')}^{0} |w^m (x)|^2 \, dx \right) \\
&= 0
\end{align*}
\]

and

\[
\begin{align*}
\lim_{\varepsilon \to 0} \Phi^\varepsilon (z^0_\varepsilon) &= \lim_{\varepsilon \to 0} \nu \int_{\Sigma_\varepsilon} |\nabla z^0_\varepsilon|^2 \, dx \\
&= \nu \sum_{k \in I_\varepsilon} \int_{Y^3_k} \int_{-h(x')}^{0} |\nabla w^m (x)|^2 \, dx \\
&= \nu |\Gamma_2| c_m,
\end{align*}
\]

with

\[
c_m = \int_{Z_h} |\nabla w^m|^2 \, dx.
\]

Taking \( u = -e^m \) on \( \Sigma_\varepsilon \), in (19), one obtains

\[
K_{mm} |\Gamma_2| = \inf \left\{ \lim_{\varepsilon \to 0} \inf_{z_\varepsilon \in \tilde{S}_\varepsilon} \Phi^\varepsilon (z_\varepsilon) \mid z_\varepsilon = e^m \text{ on } \{ x_3 = -\varepsilon h_\varepsilon (x') \}, \right\}
\]

\[
\leq \lim_{\varepsilon \to 0} \Phi^\varepsilon (z^0_\varepsilon) = \nu c_m |\Gamma_2|.
\]

This implies

\[
K_{mm} |\Gamma_2| \leq \nu c_m |\Gamma_2|.
\]
Take any sequence \((z_\varepsilon)_\varepsilon \subset H^1_\Omega, (\mathbb{R}^3, \text{div})\) such that \(z_\varepsilon = e^m\) on the surface \(\{x_3 = -\varepsilon h_\varepsilon (x')\}\) and \((z_\varepsilon)_\varepsilon\) converges to 0 in the topology \(\tau\). We write the subdifferential inequality
\[
\Phi^\varepsilon (z_\varepsilon) \geq \Phi^\varepsilon (z_\varepsilon^0) + 2\nu \varepsilon \int_{\Sigma_\varepsilon} \nabla z_\varepsilon^0 \cdot \nabla (z_\varepsilon - z_\varepsilon^0) \, dx.
\] (23)

We observe that
\[
\varepsilon \int_{\Sigma_\varepsilon} \nabla z_\varepsilon^0 \cdot \nabla (z_\varepsilon - z_\varepsilon^0) \, dx = -\varepsilon \int_{\Sigma_\varepsilon} \Delta z_\varepsilon^0 \cdot (z_\varepsilon - z_\varepsilon^0) \, dx - \varepsilon \int_{\Gamma_2} \frac{\partial z_\varepsilon^0}{\partial n} \cdot (z_\varepsilon - z_\varepsilon^0) \, d\Gamma_2.
\]
Using the regularity (at least \(H^2\)) of \(w^m\), we obtain
\[
\varepsilon \Delta z_\varepsilon^0 \rightharpoonup 0_{\varepsilon \to 0} \int_{Z_\varepsilon} \Delta w^m (x) \, dx,
\]
where the convergence takes place in the weak topology of \(L^2 (\mathbb{R}^3, \mathbb{R}^3)\) and \(1_{\Gamma_2}\) is the characteristic function of \(\Gamma_2\). Then
\[
\left| \varepsilon \int_{\Gamma_2} \frac{\partial z_\varepsilon^0}{\partial n} \cdot (z_\varepsilon - z_\varepsilon^0) \, d\Gamma_2 \right| \leq \left( \int_{\Gamma_2} |\partial w^m|^2 \, d\Gamma_2 \right)^{1/2} \left( \int_{\mathbb{R}^3} |z_\varepsilon - z_\varepsilon^0|^2 \, dx \right)^{1/2}.
\]
Because \((z_\varepsilon - z_\varepsilon^0)_\varepsilon\) converges to 0 in the strong topology \(L^2 (\mathbb{R}^3, \mathbb{R}^3)\), we have
\[
\lim_{\varepsilon \to 0} \varepsilon \int_{\Sigma_\varepsilon} \nabla z_\varepsilon^0 \cdot \nabla (z_\varepsilon - z_\varepsilon^0) \, dx = 0.
\]

Taking the \(\lim \inf\) in (23), one obtains
\[
\liminf_{\varepsilon \to 0} \Phi^\varepsilon (z_\varepsilon) \geq \liminf_{\varepsilon \to 0} \Phi^\varepsilon (z_\varepsilon^0) = \nu c_m |\Gamma_2|.
\]

In this last inequality, taking the infimum with respect to all sequences \((z_\varepsilon)_\varepsilon\) satisfying the imposed conditions, one obtains: \(K_{mn} \Gamma_2 \geq \nu c_m |\Gamma_2|\). This inequality and (22) imply: \(K_{mn} = \nu c_m\). Taking now \(u = e^1 + e^2\) on \(\Sigma_\varepsilon\) in (19), one obtains
\[
(K_{11} + 2K_{12} + K_{22}) |\Gamma_2| = \inf \left\{ \liminf_{\varepsilon \to 0} \Phi^\varepsilon (z_\varepsilon) \mid z_\varepsilon = e^1 + e^2 \text{ on } \{x_3 = -\varepsilon h_\varepsilon (x')\}, z_\varepsilon \rightharpoonup 0_{\varepsilon \to 0} \right\}
\leq \liminf_{\varepsilon \to 0} \Phi^\varepsilon (z_\varepsilon^0 + z_\varepsilon^2).
\]

Because \(\int_{Z_\varepsilon} \nabla w^1 \cdot \nabla w^2 \, dz = 0\), we have
\[
\lim_{\varepsilon \to 0} \Phi^\varepsilon (z_\varepsilon^0 + z_\varepsilon^2) = \nu |\Gamma_2| (c_1 + c_2).
\]

This implies: \(K_{12} \leq 0\), through the above expression of \(K_{mn}\). Writing a subdifferential inequality as in (23), one obtains: \(K_{12} \geq 0\), which implies: \(K_{12} = 0\). □

16
5.2 Case where $h_ε$ is independent of $ε$

As in the previous section, we still suppose that $Ω \subset \{x_3 > 0\}$ and $\partial Ω \cap \{x_3 = 0\} = Γ_2$. But, we here suppose that the boundary $Γ_{2, ε}$ is given as

$$Γ_{2, ε} = \{(x', x_3) | x_3 = -εh(x')\}$$

where $h$ is a Lipschitz continuous function satisfying $h(x') > 0, \forall x' \in Γ_2$. We have the following result.

**Theorem 17** Under the preceding hypothesis, the Navier wall law is in this case

$$(Id - n \otimes n) \frac{∂ u_0}{∂ n} + \frac{u_0}{h} = 0, \text{ on } Γ_2.$$

**Proof.** Thanks to Theorem 13, there exist a rich family $R_{Γ_2} \subset B(Σ)$, a symmetric matrix $(μ_{ij})_{i,j=1,\ldots,N}$ of Borel measures having their support contained in $Γ_2$, which are absolutely continuous with respect to the capacity $Cap$, and satisfying $μ_{ij}(B) \zeta_i \zeta_j ≥ 0, \forall ζ_i ∈ R^3, ∀ B ∈ B(Σ)$, such that, for every $u ∈ V_{Γ_1}(Ω)$ and every $ω ∈ R_{Γ_2} \cap C(Γ_2)$

$$\int_ω u_i u_j \mu_{ij} = \inf \left\{ \lim_{ε→0} \Phi(ε) | u + z_ε = 0 \text{ on } \{x_3 = -εh(x')\} \cap \omega, z_ε \to 0 \right\}.$$  

(24)

Take $u = -ε^1$ on $\{x_3 = -εh(x')\}$. Then choose $ω ∈ R_{Γ_2} \cap C(Γ_2)$, an open subset $ω^ε$ of $R^2$ such that $ω^ε \setminus \Sigma = \{x' ∈ R^2 | 0 < d(x', ∂ω) < ε\}$ and $φ^ε ∈ C^1(R^2)$ with $0 ≤ φ^ε ≤ 1$ such that

$$\begin{cases} φ^ε = 1 \text{ in } ω, \\ φ^ε = 0 \text{ on } ∂ω^ε.\end{cases}$$

We define the function $w^{1ε}$ through

$$\begin{cases} (w^{1ε})_1(x) = \frac{x_3}{εh(x')} φ^ε(x'), \\ (w^{1ε})_2(x) = 0, \\ (w^{1ε})_3(x) = \frac{ε}{2} \left( \frac{∂ h}{∂ x_1}(x') φ^ε(x') - \frac{∂ φ^ε}{∂ x_1}(x') h(x') \right) + \frac{(x_3)^2}{2} \left( \frac{1}{εh(x')} \frac{∂ φ^ε}{∂ x_1}(x') - \frac{φ^ε(x')}{ε h^ε(x') ∂ x_1(x')} \right).\end{cases}$$

One has $\text{div}(w^{1ε}) = 0, ∀ ε > 0$, and $w^{1ε} = ε^1$ on $\{x_3 = -εh(x')\} \cap (ω × (-∞, 0))$. We now consider the problem

$$\begin{cases} -Δζ^{1ε} + ξω^{1ε} = ε^1 \quad \text{in } Ω, \\ \text{div}(ζ^{1ε}) = 0 \quad \text{in } Ω, \\ ζ^{1ε} = 0 \quad \text{on } Γ_1, \\ ζ^{1ε} = \left(0, 0, \frac{ε}{2} \left( \frac{∂ h}{∂ x_1}(x') φ^ε(x') - \frac{∂ φ^ε}{∂ x_1}(x') h(x') \right) \right) \quad \text{on } Γ_2. \end{cases}$$

(25)

The problem (25) has a unique solution $(ζ^{1ε}, ω^{1ε}) ∈ H^1_{Γ_1}(Ω, \text{div}) × L^2(Ω)/R$, satisfying

$$\int_Ω |∇ζ^{1ε}|^2 dx ≤ C; \int_Ω |ζ^{1ε}|^2 dx ≤ C,$$

17
where $C$ is a constant independent of $\varepsilon$. Let $H > z_h$, with $z_h = \max_{\Omega} h$. We define the function $\tilde{w}^{1\varepsilon}$ in $D_\varepsilon = \{x \mid -H < x_3 < -\varepsilon h(x')\}$ through

\[
\begin{cases}
(\tilde{w}^{1\varepsilon})_1(x) = \frac{x_3 + H}{\varepsilon (H - h(x'))} \varphi^\varepsilon(x'), \\
(\tilde{w}^{1\varepsilon})_2(x) = 0, \\
(\tilde{w}^{1\varepsilon})_3(x) = \varepsilon \left( 2 \frac{\partial h}{\partial x_1} (x') \varphi^\varepsilon(x') - \frac{\partial \varphi^\varepsilon}{\partial x_1} (x') (H - h(x')) \right) - \frac{(x_3 + H)^2}{2} \left( \frac{1}{\varepsilon (H - h(x'))} \frac{\partial \varphi^\varepsilon}{\partial x_1} (x') + \frac{\varepsilon (H - h(x'))}{\varepsilon (H - h(x'))^2} \frac{\partial h}{\partial x_1} (x') \right).
\end{cases}
\]

We consider the bounded, smooth and open subset $\Omega_H = \{x \mid x_3 > -H\}$ and $\partial \Omega_H \cap \{x \mid x_3 = -H\} = \Gamma_2$, and the solution $(\zeta^{1\varepsilon}_H, \omega^{1\varepsilon}_H)$ of the problem

\[
\begin{cases}
-\Delta \zeta^{1\varepsilon}_H + \nabla \omega^{1\varepsilon}_H = e^1 & \text{in } \Omega_H, \\
\text{div} (\zeta^{1\varepsilon}_H) = 0 & \text{in } \Omega_H, \\
\zeta^{1\varepsilon}_H = 0 & \text{on } \Omega_H \setminus \Gamma_2, \\
\zeta^{1\varepsilon}_H = \left( 0, 0, \frac{1}{2} \left( \frac{\partial h}{\partial x_1} (x') \varphi^\varepsilon(x') - \frac{\partial \varphi^\varepsilon}{\partial x_1} (x') (H - h(x')) \right) \right) & \text{on } \Gamma_2.
\end{cases}
\]

Let us define the function $z^{1\varepsilon}_0$ through

\[
z^{0,1}_\varepsilon = \begin{cases}
\varepsilon \zeta^{1\varepsilon}_H & \text{in } \Omega, \\
w^{1\varepsilon}_H & \text{in } \Sigma_\varepsilon, \\
\tilde{w}^{1\varepsilon} & \text{in } D_\varepsilon, \\
\varepsilon \zeta^{1\varepsilon}_H & \text{in } \Omega_H.
\end{cases}
\]

One immediately verifies that $z^{0,1}_\varepsilon \in H^1_{\Gamma_1}(\mathbb{R}^3, \text{div})$, $z^{0,1}_\varepsilon = e^1$ on the surface $\{x_3 = -\varepsilon h(x')\} \cap (\omega \setminus (-\infty, 0))$, $(z^{0,1}_\varepsilon)_\varepsilon$ converges to 0 in the strong topology of $L^2(\mathbb{R}^3, \mathbb{R}^3)$ and

\[
\lim_{\varepsilon \to 0} \Phi^\varepsilon (z^{0,1}_\varepsilon) = \lim_{\varepsilon \to 0} \nu \varepsilon \iint_{\omega \times (-\varepsilon h(x'), 0)} |\nabla z^{0,1}_\varepsilon|^2 \, dx = \nu \int_{\omega} \frac{dx'}{h(x')}.
\]

One thus deduces from (2) within this context

\[
\mu_{11}(\omega) \leq \nu \int_{\omega} \frac{dx'}{h(x')}.
\]

Furthermore, taking $(z_\varepsilon)_\varepsilon \subset H^1_{\Gamma_1}(\mathbb{R}^3, \text{div})$, $z_\varepsilon = e^1$ on $\{x_3 = -\varepsilon h(x')\} \cap (\omega \setminus (-\infty, 0))$, $(z_\varepsilon)_\varepsilon$ converges to 0 in the topology $\tau$, and using the subdifferential inequality

\[
\Phi^\varepsilon (z_\varepsilon) \geq \Phi^\varepsilon (z^{0,1}_\varepsilon) + \nu \varepsilon \iint_{\Sigma_\varepsilon} \nabla z^{0,1}_\varepsilon \cdot \nabla (z_\varepsilon - z^{0,1}_\varepsilon) \, dx + \nu \int_{\Omega} \nabla z^{0,1}_\varepsilon \cdot \nabla (z_\varepsilon - z^{0,1}_\varepsilon) \, dx,
\]

we prove that $\mu_{11}(\omega) \geq \nu \int_{\omega} \frac{dx'}{h(x')}$. This implies the equality: $\mu_{11}(\omega) = \nu \int_{\omega} \frac{dx'}{h(x')}$ and, since this equality is true for every $\omega \in \mathcal{R}_{\Gamma_1} \cap \Omega \setminus \Gamma_2$, we obtain $\mu_{11} = \nu \int_{\omega} \frac{dx'}{h(x')}$. Choosing now $u = -e^2$ on $\Sigma_\varepsilon$, we can build a test-function $z^{0,2}_\varepsilon$ in a similar way and prove: $\mu_{22} = \nu \int_{\omega} \frac{dx'}{h(x')}$. 18
Finally, taking \( u = -(e^1 + e^2) \) on \( \Sigma_e \), we consider the sequence \( (z^0_\varepsilon)_\varepsilon \) defined through: \( z^0_\varepsilon = z^{0,1}_\varepsilon + z^{0,2}_\varepsilon \). One deduces from the above computations that
\[
\lim_{\varepsilon \to 0} \Phi^e (z^0_\varepsilon) = \lim_{\varepsilon \to 0} \Phi^e (z^{0,1}_\varepsilon + z^{0,2}_\varepsilon) = 2\nu \int_{\Omega} \frac{dx'}{h(x')}
\]
and, as in the periodic case, that \( \mu_{1,2} = 0 \). The boundary conditions on \( \Gamma_2 \) can thus be written as
\[
\begin{align*}
\frac{\partial (u^0)_m}{\partial x_3} &= 0, \\
\frac{\partial (u^0)_m}{\partial x_3} &= \frac{1}{h} (u^0)_m, \quad m = 1, 2,
\end{align*}
\]
which ends the proof. \( \blacksquare \)

**Remark 18** In a general way, if \( \Sigma_e = \{ \sigma + t \sigma \mid \sigma \in \Gamma_2, -\varepsilon h (\sigma) < t < 0 \} \), with \( h \) positive and Lipschitz continuous on \( \Gamma_2 \), we can prove that the limit law is
\[
\begin{cases}
(Id - n \otimes n) \frac{\partial u^0}{\partial n} + \frac{u^0}{h} = 0, \\
\frac{u^0}{\partial n} = 0.
\end{cases}
\]

6 Optimal control problem

For a given real \( m > 0 \), we consider the set \( \Xi_m \) of all matrices \( h = \text{Diag} (h_i)_{i=1,\ldots,N} \) of functions \( h_i : \Gamma_2 \to [0, +\infty] \), \( \Gamma_2 \)-measurable and such that
\[
\int_{\Gamma_2} h_i d\Gamma_2 = m, \forall i = 1, \ldots, N.
\]

We suppose that \( \partial \Omega \) is \( C^2 \) and consider the Navier-Stokes problem, with Navier wall law, according to Theorem [7]
\[
\begin{cases}
-\nu \Delta u^h + (u^h \cdot \nabla) u^h + \nabla p^h = f & \text{in } \Omega, \\
\text{div} (u^h) = 0 & \text{in } \Omega, \\
h (Id - n \otimes n) \frac{\partial u^h}{\partial n} + u^h = 0 & \text{on } \Gamma_2, \\
u h^* \cdot \nabla u^h \cdot \nabla u^h = 0 & \text{on } \Gamma_2, \\
u^* = 0 & \text{on } \Gamma_1,
\end{cases}
\]
which has a unique solution \( (u^h, p^h) \in V_0, \Gamma_1 (\Omega) \times L^2 (\Omega) / R \). We define the functional \( F \) on \( \Xi_m \times H_{\text{diff}} (\Omega, \text{div}) \) associated to (26) through
\[
F (h, u) = \begin{cases}
\frac{\nu}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{2} \sum_{i=1}^{N} \int_{\Gamma_2} \frac{(u_i)^2}{h_i} d\Gamma_2 \\
+ \int_{\Omega} (u^h \cdot \nabla) u^h \cdot u^h dx - \int_{\Omega} f \cdot u^h dx & \text{if } u \in V_{0, \Gamma_1} (\Omega), \\
& \text{otherwise}.
\end{cases}
\]

We consider the optimal control problem (3), which means that the cost functional is here taken as the global energy. We observe that
\[
F (h, u^h) = -\int_{\Omega} f \cdot u^h dx.
\]

This implies that the minimization of \( F \), with respect to \( u \) on the set \( V_{0, \Gamma_1} (\Omega) \), is equivalent to the maximization of the work of the external forces on this set. The problem (3) has a unique minimizer when Poincaré's inequality
\[
\left( \int_{\Gamma_2} |u|_d\Gamma_2 \right)^2 \leq \int_{\Gamma_2} h_2 d\Gamma_2 \int_{\Gamma_2} \frac{(u_i)^2}{h_i} d\Gamma_2,
\]
becomes an equality, for every \(i = 1, \ldots, N\), that is when

\[
\hat{h}_i^m = m \frac{|u_i^m|_{\Gamma_2}}{\int_{\Gamma_2} |u_i^m| d\Gamma_2},
\]

where \((u^m, p^m)\) is the solution of

\[
\begin{cases}
-\nu \Delta u^m + (u^m \cdot \nabla) u^m + \nabla p^m = f \quad \text{in } \Omega, \\
\operatorname{div} (u^m) = 0 \quad \text{in } \Omega, \\
u m \cdot n = 0 \quad \text{on } \Gamma_2, \\
u m = 0 \quad \text{on } \Gamma_1,
\end{cases}
\]

\[
\left\{ \begin{array}{l}
(Id - n \otimes n) \frac{\partial u^m}{\partial n} \\
\frac{1}{m} \left( \sum_{i=1}^N \left( \int_{\Gamma_2} |v_i|^2 d\Gamma_2 \right) \right) \\
\frac{1}{m} \left( \int_{\Gamma_2} |(u^m)_i| d\Gamma_2 \right) \\
\end{array} \right\} = 0 \quad \text{on } \Gamma_2.
\]

Trivially, the study of the \(\Gamma\)-convergence of the sequence of the energies associated to (3), when \(m\) goes to 0 and relatively to the weak topology of \(H^1(\Omega, \mathbb{R}^N)\), will lead to the following conclusions: \((u^m)_m\) converges to \(u^0\) in the weak topology of \(H^1(\Omega, \mathbb{R}^N)\), \((p^m)_m\) converges to \(p^0\) in the strong topology of \(L^2(\Omega) / \mathbb{R}\), where \((u^0, p^0)\) is the solution of the problem

\[
\begin{cases}
-\nu \Delta u^0 + (u^0 \cdot \nabla) u^0 + \nabla p^0 = f \quad \text{in } \Omega, \\
\operatorname{div} (u^0) = 0 \quad \text{in } \Omega, \\
u^0 = 0 \quad \text{on } \partial \Omega.
\end{cases}
\] (27)

In order to study the asymptotic behavior of \(\left(\frac{u^m}{m}\right)_{m'}\) on \(\Gamma_2\), we introduce the following linearized perturbation of the Navier-Stokes problem (27)

\[
\begin{cases}
-\nu \Delta u^{0,m} + (u^{0,m} \cdot \nabla) u^{0,m} + \nabla p^{0,m} = f - (u^m \cdot \nabla) u^m \quad \text{in } \Omega, \\
\operatorname{div} (u^{0,m}) = 0 \quad \text{in } \Omega, \\
u^{0,m} = 0 \quad \text{on } \partial \Omega.
\end{cases}
\] (28)

The problem (28) is a Stokes system, the source term of which is \(f - (u^m \cdot \nabla) u^m\). Consider now the functional \(I_m\) defined on \(V_{0,\Gamma_1}(\Omega)\) through

\[
I_m(v) = \frac{m \nu}{2} \int_{\Omega} |\nabla v|^2 \, dx + \frac{1}{2} \sum_{i=1}^N \left( \int_{\Gamma_2} |v_i| d\Gamma_2 \right)^2 + \int_{\Gamma_2} (Id - n \otimes n) \frac{\partial u^{0,m}}{\partial n} \cdot v d\Gamma_2.
\]
Proposition 19

One has the following properties.

Proof.

\begin{align*}
\phi
\end{align*}

We observe that the couple \((v^m, q^m)\) defined through

\begin{align*}
v^m &= \frac{u^m - u_0}{m} ; q^m = \frac{p^m - p_0}{m},
\end{align*}

is the minimizer of \(J_m\). For every \(\varphi \in \mathbf{H}^{1/2} (\Gamma_2, \mathbb{R}^N)\), there exists a unique extension \(v_\varphi \in \mathbf{V}_{0, \Gamma_1} (\Omega)\) of \(\varphi\) defined through

\begin{align*}
\int_{\Omega} |\nabla v_\varphi|^2 \, dx &= \inf_{\{w \in \mathbf{V}_{0, \Gamma_1} (\Omega) \mid w|_{\Gamma_2} = \varphi\}} \int_{\Omega} |\nabla w|^2 \, dx.
\end{align*}

Let us denote \(\mathcal{M} (\Gamma_2, \mathbb{R}^N)\) the space of finite Radon measures on \(\Gamma_2\) with values in \(\mathbb{R}^N\). We consider the functional \(J_m\) defined on \(\mathcal{M} (\Gamma_2, \mathbb{R}^N)\) through

\begin{align*}
J_m (\varphi) &= \left\{ \begin{array}{ll}
\frac{mv}{2} \int_{\Omega} |\nabla v_\varphi|^2 \, dx + \frac{1}{2} \sum_{i=1}^{N} \left( \int_{\Gamma_2} |\varphi_i| \, d\Gamma_2 \right)^2 \\
+ \int_{\Gamma_2} (Id - n \otimes n) \frac{\partial u^m}{\partial n} \cdot \varphi \, d\Gamma_2 & \text{if } \varphi \in \mathbf{H}^{1/2} (\Gamma_2, \mathbb{R}^N) \\
\text{and } \varphi \cdot n = 0 \text{ on } \Gamma_2, \\
+ \infty & \text{otherwise.}
\end{array} \right.
\end{align*}

Then \((v^m)|_{\Gamma_2}\) is the unique minimizer of \(J_m\).

Proposition 19 One has the following properties.

1. \(\sup_m \sum_{i=1}^{n} \left( \int_{\Gamma_2} |v^m_i| \, d\Gamma_2 \right) < + \infty\).

2. The sequence \((J_m)_m\) \(\Gamma\)-converges, when \(m\) tends to 0 and with respect to the weak* topology of \(\mathcal{M} (\Gamma_2, \mathbb{R}^N)\), to the functional \(J\) defined from \(\mathcal{M} (\Gamma_2, \mathbb{R}^N)\) to \(\mathbb{R}\) through

\begin{align*}
J (\lambda) &= \sum_{i=1}^{N} (|\lambda_i| (\Gamma_2))^2 + \int_{\Gamma_2} (Id - n \otimes n) \frac{\partial u^0}{\partial n} \, d\lambda, \\
\text{where } |\lambda_i| (\Gamma_2) & \text{ is the total variation of } \lambda_i \text{ on } \Gamma_2.
\end{align*}

Proof. 1. Remark that a regularity property of the boundary \(\partial \Omega\) implies that

\begin{align*}
\sup_m \left\| (Id - n \otimes n) \frac{\partial u^m}{\partial n} \right\|_{L^\infty (\Gamma_2, \mathbb{R}^N)} < + \infty.
\end{align*}
One thus obtains

\[ J_m \left( (v^m)_{\Gamma_2} \right) \geq \frac{1}{2} \sum_{i=1}^{N} \left( \int_{\Gamma_2} |v_i^m| d\Gamma_2 \right)^2 - C \sum_{i=1}^{N} \left( \int_{\Gamma_2} |v_i^m| d\Gamma_2 \right). \]

Moreover

\[ \sup_m J_m \left( (v^m)_{\Gamma_2} \right) \leq \sup_m J_m (0) = 0 \Rightarrow \sup_m \sum_{i=1}^{N} \left( \int_{\Gamma_2} |v_i^m| d\Gamma_2 \right) \leq C. \]

This implies the existence of a subsequence of \( (v^m)_{\Gamma_2} \), still denoted \( (v^m)_{\Gamma_2} \), which converges to some \( \lambda \) in the weak* topology of \( \mathcal{M}(\Gamma_2, \mathbb{R}^N) \).

2. Choose any sequence \((\varphi^m)_m \subset H^{1/2}(\Gamma_2, \mathbb{R}^N)\), satisfying \( \varphi^m \cdot n = 0 \), on \( \Gamma_2 \) and converging to \( \lambda \) in the weak* topology of \( \mathcal{M}(\Gamma_2, \mathbb{R}^N) \). The functional \( \mu \mapsto |\mu| \), where \( |\mu| \) is the total variation of \( \mu \), being lower semi-continuous on \( \mathcal{M}(\Gamma_2) \), one has

\[ \liminf_{m \to 0} \int_{\Gamma_2} |\varphi^m| d\Gamma_2 \geq |\lambda| (\Gamma_2). \]

Thanks to the regularity of the boundary, \( \left( (Id - n \otimes n) \frac{\partial u_{0,m}}{\partial n} \right)_m \) uniformly converges to \( (Id - n \otimes n) \frac{\partial u_0}{\partial n} \), hence

\[ \liminf_{m \to 0} \int_{\Gamma_2} (Id - n \otimes n) \frac{\partial u_{0,m}}{\partial n} : \varphi^m d\Gamma_2 \geq \int_{\Gamma_2} \left( (Id - n \otimes n) \frac{\partial u_0}{\partial n} \right) : d\lambda. \]

This implies

\[ \liminf_{m \to 0} J_m (\varphi^m) \geq J (\lambda). \quad (29) \]

In order to prove the \( \Gamma \)-lim sup property, let us suppose that \( \Omega \subset \{x_N < 0\} \) and \( \partial \Omega \cap \{x_N = 0\} = \Gamma_2 \) (in fact using a system of local coordinates, one can then study the case of every smooth surface \( \Gamma_2 \)). We define \( x' = (x_1, \ldots, x_{N-1}) \) and the nonnegative and smooth function \( \rho_\varepsilon \) through

\[ \rho_\varepsilon (x') = \begin{cases} \frac{C}{\varepsilon^{N-1}} \exp \left( -\frac{\varepsilon^2}{\varepsilon^2 - |x'|^2} \right) & \text{if } |x'| < \varepsilon, \\ 0 & \text{if } |x'| \geq \varepsilon, \end{cases} \]

where

\[ C = \left( \int_{B_{N-1}(0,1)} \exp \left( -\frac{1}{1 - |\zeta|^2} \right) d\zeta \right)^{-1}. \]

Let \( (\omega_{1/\varepsilon})_\varepsilon \), where \( [1/\varepsilon] \) denotes the entire part of \( 1/\varepsilon \), be a sequence of open subsets of \( \Gamma_2 \) such that

\[ \omega_1 \subset \omega_2 \subset \cdots \subset \omega_{[1/\varepsilon]} \subset \cdots \subset \Gamma_2, \]

\[ [\omega_{1/\varepsilon}]_\varepsilon = \Gamma_2, \]

\[ d (\omega_{1/\varepsilon}, \partial \Gamma_2) = \varepsilon. \]

We associate the partition of unity \( (\eta_\varepsilon)_\varepsilon \) through

\[ \begin{cases} \eta_\varepsilon \in C_c^\infty (\omega_{1/\varepsilon}), \\ \eta_\varepsilon (x') = 1 \in \omega_{1/\varepsilon-1} \setminus [1/\varepsilon] - [1/\varepsilon'], \text{ with } \varepsilon' = \frac{\varepsilon}{1 - \varepsilon}, \\ 0 \leq \eta_\varepsilon (x') \leq 1, \forall x' \in \Gamma_2, \forall \varepsilon > 0. \end{cases} \]

22
For $\lambda = (\lambda_1, \ldots, \lambda_{N-1}, 0) \in \mathcal{M}(\Gamma_2, \mathbb{R}^N)$, we define the vectorial measure $\lambda^\varepsilon$ through $\lambda^\varepsilon = (\lambda \ast \rho_\varepsilon) \eta_\varepsilon$.

We observe that $\lambda^\varepsilon \in \mathcal{C}_c^\infty(\Gamma_2, \mathbb{R}^N)$ and

$$
\lambda^\varepsilon \rightharpoonup \varepsilon \rightarrow 0 \quad w^* \mathcal{M}(\Gamma_2, \mathbb{R}^N),
$$

$$
|\nabla \lambda^\varepsilon|(x') \leq \frac{C}{\varepsilon^N} \quad \forall x' \in \Gamma_2.
$$

We build the function $w^\varepsilon$

\[
\begin{align*}
(w^\varepsilon)_i(x) &= \frac{\varepsilon - x_N}{\varepsilon} (\lambda^\varepsilon)_i(x') \quad i = 1, \ldots, N - 1, \forall x \in \Omega, \\
(w^\varepsilon)_N(x) &= \frac{\text{div} (\lambda^\varepsilon(x'))}{2} \left( \frac{(\varepsilon - x_N)^2}{\varepsilon} - \varepsilon \right).
\end{align*}
\]

We immediately observe that $w^\varepsilon \in H^1(\Omega, \mathbb{R}^N)$ and

\[
\begin{align*}
\text{div} (w^\varepsilon) &= 0 \quad \text{in } \Omega, \\
(w^\varepsilon)_N &= 0 \quad \text{on } \Gamma_2, \\
w^\varepsilon &= 0 \quad \text{on } \Gamma_1,
\end{align*}
\]

that is $w^\varepsilon \in V_{0,\Gamma_1}(\Omega)$, for every $\varepsilon > 0$. We now define

\[
\begin{align*}
\varepsilon &= m^{1/4} \\
w^m &= w^{m/\varepsilon} \\
\lambda^m &= \lambda^{m/\varepsilon}.
\end{align*}
\]

One has

\[
\begin{align*}
m \int_{\Omega} |\nabla w^m|^2 \, dx &\leq C \sqrt{m}, \\
J_m(\lambda^m) &= I_m(v^m) \leq I_m(w^m),
\end{align*}
\]

hence

$$
\limsup_{m \to 0} J_m(\lambda^m) \leq \limsup_{m \to 0} I_m(\lambda^m) = J(\lambda).
$$

This inequality and (29) end the proof. ■

This inequality and (31) end the proof.

One has the following result.

**Theorem 20** Let

\[
M_i = \max_{\sigma \in \Gamma_2} \left| \left( (\text{Id} - n \otimes n) \frac{\partial u^0}{\partial n} \right)_i (\sigma) \right|,
\]

\[
K^\pm_i = \left\{ \sigma \in \Gamma_2 \mid \left( (\text{Id} - n \otimes n) \frac{\partial u^0}{\partial n} \right)_i (\sigma) = \pm M_i \right\}.
\]

We have the following properties.

1. When $m$ goes to 0, the sequence $\left( (u^m/m) |_{\Gamma_2} \right)_m$ converges in the weak* topology of the space $\mathcal{M}(\Gamma_2, \mathbb{R}^N)$ to a vectorial measure $\lambda = (\lambda_i)_{i=1,\ldots,N}$ such that $\text{supp} (\lambda_i) \subseteq K^+_i \cup K^-_i$, with $\lambda_i$ positive on $K^+_i$ and negative on $K^-_i$, $i = 1, \ldots, N$.

2. $\int_{\Gamma_2} \left( (\text{Id} - n \otimes n) \frac{\partial u^0}{\partial n} \right)_i \, d\lambda_i = -M_i, \ i = 1, \ldots, N.$

3. $\lim_{m \to 0} \int_{\Gamma_2} |u^m/m| \, d\Gamma_2 = |\lambda_i| (\Gamma_2) = M_i, \ i = 1, \ldots, N.$
4. When \( m \) goes to 0, the sequence \((u^m/m)_m\) converges in the weak* topology of \( M(\Gamma_2, \mathbb{R}^N) \) to a measure \( \overline{\lambda}_i \) such that \( \text{supp}(\overline{\lambda}_i) \subseteq K_i^+ \cup K_i^- \), \( \overline{\lambda}_i \) is positive on \( K_i^- \) and negative on \( K_i^+ \), and \( |\overline{\lambda}_i| (\Gamma_2) = 1, i = 1, \ldots, N \).

**Proof.** One deduces from Proposition[23] and from the properties of the \( \Gamma \)-convergence that \( (u^m/m)|_{\Gamma_2} \) converges in the weak* topology of \( M(\Gamma_2, \mathbb{R}^N) \), when \( m \) goes to 0, to a measure \( \lambda = (\lambda_i)_{i=1, \ldots, N} \) such that \( J(\lambda) = \min_{v \in M(\Gamma_2, \mathbb{R}^N)} J(v) \). Define

\[
    \mathcal{M}_1(\Gamma_2, \mathbb{R}^N) = \{ \mu \in M(\Gamma_2, \mathbb{R}^N) \mid |\mu| (\Gamma_2) = 1, i = 1, \ldots, N \}
\]

and consider the functional \( \tilde{J} \) defined from \([0, +\infty]^N \times \mathcal{M}_1(\Gamma_2, \mathbb{R}^N)\) to \( \mathbb{R} \) through

\[
\tilde{J}(\mu_1, \ldots, \mu_N) = J((\mu_1, \ldots, \mu_N)) = \frac{1}{2} \sum_{i=1}^N (t_i)^2 + \sum_{i=1}^N \int_{\Gamma_2} \left( (Id - n \otimes n) \frac{\partial u^0}{\partial n} \right)_i \, d\mu_i.
\]

One has

\[
\min_{v \in M(\Gamma_2, \mathbb{R}^N)} J(v) = \min_{\mu \in \mathcal{M}_1(\Gamma_2, \mathbb{R}^N)} \min_{t_i \geq 0} \tilde{J}(\mu_1, \ldots, \mu_N).
\]

The minimum of (30) with respect to \( t = (t_1, \ldots, t_N) \) exists if

\[
\int_{\Gamma_2} \left( (Id - n \otimes n) \frac{\partial u^0}{\partial n} \right)_i \, d\mu_i \leq 0, \forall i = 1, \ldots, N.
\]

Let us now find the minimum with respect to \( \mu \in \mathcal{M}_1(\Gamma_2, \mathbb{R}^N) \). One has

\[
-\int_{\Gamma_2} \left( (Id - n \otimes n) \frac{\partial u^0}{\partial n} \right)_i \, d\mu_i \geq -M_i,
\]

for every \( \mu \in \mathcal{M}_1(\Gamma_2, \mathbb{R}^N) \) such that

\[
\int_{\Gamma_2} \left( (Id - n \otimes n) \frac{\partial u^0}{\partial n} \right)_i \, d\mu_i \leq 0, \forall i = 1, \ldots, N,
\]

the minimum being reached in the case of equality, that is if and only if \( \text{supp}(\mu_i) \subseteq K_i^+ \cup K_i^- \). One has \( \lambda_i = M_i \mu_i, i = 1, \ldots, N \). Remarking that \( \overline{\lambda}_i = \mu_i \), one observes that \((h^m/m)_m\) converges in the weak* topology of \( M(\Gamma_2, \mathbb{R}^N) \), when \( m \) tends to 0, to \( \overline{\lambda}_i \), and the same result occurs for the sequence \( (|u^m_1|_{\Gamma_2} / \int_{\Gamma_2} |u^m_1| \, d\Gamma_2) \). The sequence \((h^m/m)_m\) converges in \( M(\Gamma_2, \mathbb{R}^N) \)-weak* to a probability measure \( \overline{\lambda}_i \) with support in the set of points of \( \Gamma_2 \) where the shear motions, given through \((Id - n \otimes n) \frac{\partial u}{\partial n}\), are large for the limit flow described through (23).

**Remark 21** We thus think that, inside this flow, a thin boundary layer of thickness \( mh_i \) occurs in the \( i \)-th direction with a probability \( \overline{\lambda}_i \) (for every \( i \)).

**Acknowledgements.** This work has been supported by the Comité Mixte Franco-Marocain under the Action Intégrée MA/04/93.

24
References

[1] E. Acerbi and G. Buttazzo, Reinforcement problem in the calculus of variations. Ann. Inst. Henri
Poincaré, Anal. Non Linéaire 3 (1986), 273-284.

[2] Y. Achdou and O. Pironneau, Domain decomposition and wall laws. C. R. Acad. Sci., Paris, Sér. I
320, No.5 (1995), 541-547.

[3] Y. Achdou, O. Pironneau, and F. Valentin, Effective boundary conditions for laminar flows over
periodic rough boundaries. J. Comput. Phys. 147, No.1 (1998), 187-218.

[4] G. Buttazzo, G. Dal Maso, and U. Mosco, Asymptotic behavior for Dirichlet problems in domains
bounded by thin layers. Partial Differential Equations and the Calculus of Variations, Essays in
Honor of Ennio De Giorgi, pp. 193–249, Birkhäuser, Boston, 1989.

[5] G. Dal Maso, An introduction to Γ-convergence. Progress in NonLinear Differential Equations and
Applications, vol. 8, Birkhäuser, Basel, 1993.

[6] G. Dal Maso, On the integral representation of certain local functionals. Ric. Mat. 32 (1983), 85-113.

[7] G. Dal Maso and U. Mosco, Wiener criteria and energy decay for relaxed Dirichlet problems. Arch.
Ration. Mech. Anal. 95 (1986), 345-387.

[8] G. Dal Maso and U. Mosco, Wiener’s criterion and Γ-convergence. Appl. Math. Optimization 15
(1987), 15-63.

[9] G. Dal Maso, A. Defranceschi, and E. Vitali, Integral representation for a class of $C^1$-convex func-
tionals. J. Math. Pures Appl., IX. Sér. 73, No.1 (1994), 1-46

[10] E. De Giorgi and T. Franzoni, Su un tipo di convergenza variationale. Atti Accad. Naz. Lincei, VIII.
Ser., Rend., Cl. Sci. Fis. Mat. Nat. 58 (1975), 842-850.

[11] P. Esposito and G. Riey, Asymptotic behaviour of thin insulation problem. J. Convex Anal. 10, No.
2 (2003), 379-388.

[12] L.D. Landau and E.M. Lifschitz, Physique théorique Tome 6 : Mécanique des fluides. Second edition.
Editions Mir, Moscou, 1989.

[13] W. Jäger and A. Mikelic, On the roughness-induced effective boundary conditions for an incompress-
ible viscous flow. J. Differ. Equations 170, No.1 (2001), 96-122.

[14] E. Marusic-Paloka, Average of the Navier’s law on the rapidly oscillating boundary. J. Math. Anal.
Appl. 259, No. 2 (2001), 685-701.

[15] R. Temam, Navier-Stokes equations. Theory and numerical analysis. North-Holland, Amsterdam,
1984.

[16] H. Triebel, Theory of function spaces. Birkhäuser, Basel, 1983.

[17] W.P. Ziemer, Weakly differentiable functions. Springer-Verlag, Berlin, 1989.