Learning Equational Theorem Proving

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Abstract

We develop Stratified Shortest Solution Imitation Learning (3SIL) to learn equational theorem proving in a deep reinforcement learning (RL) setting. The self-trained models achieve state-of-the-art performance in proving problems generated by one of the top open conjectures in quasigroup theory, the Abelian Inner Mapping (AIM) Conjecture. To develop the methods, we first use two simpler arithmetic rewriting tasks that share tree-structured proof states and sparse rewards with the AIM problems. On these tasks, 3SIL is shown to significantly outperform several established RL and imitation learning methods. The final system is then evaluated in a standalone and cooperative mode on the AIM problems. The standalone 3SIL-trained system proves in 60 seconds more theorems (70.2%) than the complex, hand-engineered Waldmeister system (65.5%). In the cooperative mode, the final system is combined with the Prover9 system, proving in 2 seconds what standalone Prover9 proves in 60 seconds.

1 Introduction

Machine learning (ML) has recently proven its worth in many separate fields. From computer vision [1], to speech recognition [2] to game playing [3, 4, 5] using reinforcement learning (RL), machine learning has made immense progress. In automated mathematics and automated theorem proving, machine learning has also recently achieved some encouraging results.

Learned models have been used in saturation-based theorem provers to improve given clause selection [6], to synthesize functions in higher-order logic [7] and to guide a connection-tableau prover using reinforcement learning [8]. There have also been efforts to use reinforcement learning in interactive theorem proving [9, 10].

In general, the knowledge base of mathematics and the complexity of mathematical proofs will grow in the future. The need for proof checking and higher computer mathematics performance will therefore increase. Simultaneously, the complexity of software and the need for formal verification to prevent failure modes have also been growing [11]. Automated theorem proving and mathematics will benefit from advanced ML integration. One of the mathematical subfields where automated theorem provers are heavily used is the field of quasigroup and loop theory [12]. A quasigroup is similar to a group, but it does not guarantee associativity. A loop is a quasigroup with an identity element.

In this paper, we present Stratified Shortest Solution Imitation Learning (3SIL), a new machine learning approach that can learn equational theorem proving. We compare our 3SIL algorithm on two simpler arithmetic tasks with established reinforcement learning baselines and then apply it on theorems generated in the context of the Abelian Inner Mapping (AIM) Conjecture for loops [13] using the AIMLEAP proof environment [14].

The structure of the paper is as follows. Sections 2 and 3 describe the datasets used for the arithmetic and AIM conjecture tasks respectively. The RL methods used as baselines and the proposed 3SIL method are discussed in Section 4. Afterwards, Section 5 describes the tree neural network models used to process the mathematical expressions. Section 6 focuses on the experiments performed on the arithmetic tasks and comparison with RL methods. Our results on the AIM conjecture dataset and comparison with state-of-the-art automated theorem provers are in Section 7. Discussion and conclusions are in Sections 8 and 9. Our contributions are the following.

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1. We release a Python reinforcement learning environment for arithmetic rewriting tasks.
2. We propose 3SIL, stratified shortest solution imitation learning, a method that outperforms established reinforcement learning methods on the arithmetic rewriting tasks in our environments.
3. We use 3SIL to learn equational theorem proving in the AIMLEAP proof environment. The system performs competitively with established equational theorem provers, that contain many hours of work and lines of code. This is shown on a significant mathematical domain: the theory surrounding the open AIM conjecture, one of the top open problems in quasigroup theory.
4. We combine the learned system with an existing state-of-the-art automated theorem prover. The combined system outperforms the original prover, even with a smaller time limit.

2 Arithmetic Tasks

To develop a method that can solve equational theorem proving problems, we considered two simpler arithmetic tasks, which also have a tree-structured input and a sparse reward structure: Robinson arithmetic and polynomial arithmetic. In both cases, the task is to normalize a mathematical expression to one specific form. These two tasks are implemented as Python RL environments, which we make available.

The learning environments incorporate two existing datasets. For the Robinson arithmetic normalization task, we use a dataset that was constructed for reinforcement learning experiments in the interactive theorem prover HOL4 [15]. For the polynomial normalization task, we employ a dataset introduced for experiments in symbolic rewriting using recurrent language models [16].

2.1 Normalization in Robinson Arithmetic

The first formalism that we use as an RL environment is Robinson arithmetic (RA). RA is a simple arithmetic theory. Its language contains the successor function $S$, addition $+$ and multiplication $*$ and one constant, the 0. The theory considers only non-negative numbers. Numbers are represented by the constant 0 with the appropriate number of successor functions applied to it. The task for the agent is to rewrite an expression until there are only nodes of the successor or 0 types. Effectively, we are asking the agent to calculate the value of the expression. As an example, $S(S(0)) + S(0)$, representing $2 + 1$, needs to be rewritten to $S(S(S(0)))$. The setup for this RA normalization task is modeled after [15].

The expressions are represented as a tree data structure. Within the environment, there are seven different rewrite actions available to the agent. The four equations defining these actions are $x + 0 = x$, $x + S(y) = S(x + y)$, $x * 0 = 0$ and $x * S(y) = (x * y) + x$, where the agent can apply the equations in either direction. There is one exception: the multiplication by 0 cannot be applied from right to left, as this would require the agent to supply a term that would function as the x. The place where the rewrite is applied is denoted by the location of a cursor in the expression tree.

In addition to the rewrite actions, the agent can move the cursor to one of the children of the current cursor node. This gives a total number of nine actions. Moving to a child of a node with only one child counts as moving to the left child. After a rewriting action, the cursor is reset to the root of the expression. More details on the actions are in Appendix 1.

2.2 Normalization in Polynomial Arithmetic

The polynomial normalization task can be seen as a more challenging extension of the RA normalization task. The polynomial arithmetic implemented follows the axioms known as Tarski’s high school axioms [17]. These introduce exponentiation, distributivity and associativity and free variables (for example $x, y, z$). We still represent numbers as successor functions applied to the constant 0. An example of a problem is the normalization of $((z * y)^{4}) * (((2 * 0) + y) * ((1 + x) + z))$ to $xy^{4}z^{3} + y^{4}z^{4}$. Due to the larger number of axioms in this arithmetic, there are more rewriting actions. In total, the polynomial arithmetic environment allows the agent to perform 28 different actions, including two actions that involve cursor movement (see Appendix 2 for details).

Both RL environments incorporate all the necessary logic to rewrite the tree correctly. The agents only supply an action to change the tree or to move the cursor to another node in the tree. Both normalization tasks only give a reward at the very end, when the expression is normalized. This reward structure is generally difficult to deal with, as there is no indication for the agent to know that it is on the right track, or to come up with partial solutions. It is also the reward structure for many theorem proving settings. For these normalization problems, there is an efficient algorithm that is guaranteed to correctly rewrite every expression, but this algorithm is not necessarily trivial to learn.
3 AIM Conjecture Task

Automated theorem proving has been applied in the theory surrounding the Abelian Inner Mapping Conjecture, known as the AIM Conjecture. This is one of the top open conjectures in quasigroup theory. Work on the conjecture has been going on for more than a decade. Automated theorem provers use hundreds of thousands of inference steps when run on problems from this theory. Theorem proving is undecidable, meaning that there cannot possible be a computer program that always gives the correct answer.

There is a dataset of theorems generated by this conjecture, which we use as a testbed for our machine learning methods [14]. The dataset comes with a simple prover called AIMLEAP that can take machine learning advice. We use this system as a reinforcement learning environment. AIMLEAP keeps the state and carries out the cursor movements and rewrites that a model chooses.

The AIM conjecture concerns specific structures in loop theory [13]. A loop is a quasigroup with an identity element. A quasigroup is a generalization of a group that does not preserve associativity. This mostly manifests in the presence of two different ‘division’ operators, one left-division and one right-division. We will briefly explain the conjecture because it indicates the nature of the data.

For loops, three inner mapping functions (left-translation L, right-translation R, and the mapping T) are:

\[
L(u, x, y) := (y * x) \backslash ((y * (x * u)) \\
R(u, x, y) := ((u * x) * y) / (x * y) \\
T(u, x) := x \backslash (u * x).
\]

These mappings can be seen as measures of the deviation from commutativity and associativity. The conjecture concerns the consequences of these three inner mapping functions forming an Abelian (commutative) group. There are two additional notions, that of the associator function a and the commutator function K:

\[
a(x, y, z) := (x * (y * z)) \backslash ((x * y) * z) \\
K(x, y) := (y * x) / (x * y).
\]

From these definitions, the conjecture can be stated. There are two parts to the conjecture. For both parts, the following equalities need to hold for all \( u, v, x, y, \) and \( z \):

\[
a(a(x, y, z), u, v) = 1 \\
a(x, a(y, z, u), v) = 1 \\
a(x, y, a(z, u, v)) = 1
\]

where 1 is the identity element. These are necessary, but not sufficient for the two main parts of the conjecture. The first part of the conjecture asks whether a loop modulo its center is a group. In this context, the center is the set of all elements that commute with all other elements. This is the case if:

\[K(a(x, y, z), u) = 1.\]

The second part of the conjecture asks whether a loop modulo its nucleus is an Abelian group. The nucleus is the set of elements that associate with all other elements. This is the case if:

\[a(K(x, y), z, u) = 1 \]
\[a(x, K(y, z), u) = 1 \]
\[a(x, y, K(z, u)) = 1.\]

Currently, work in this area is done using automated theorem provers such as Prover9 [18][13]. This has led to some promising results, but the search space is enormous. The main strategy for proving the AIM conjecture thus far has been to prove weaker versions of the conjecture (using additional assumptions) and then import crucial proof steps into the stronger version of the proof. The Prover9 theorem prover is especially suited to this approach because of its well-established hints mechanism [19]. The dataset is derived from this Prover9 approach and contains around 3500 theorems that can be proven with the supplied definitions and lemmas [14].

There are 177 possible actions in the environment. Three actions are cursor movements, where the cursor can be moved to an argument of the current position. The other actions all rewrite the current term at the cursor position with various axioms, definitions and lemmas that hold in the AIM context (see Appendix 3 for a list of these actions).

[https://github.com/ai4reason/aimleap](https://github.com/ai4reason/aimleap)
We handle the proof state as a tree, with the root node being an equality node. This is one of the theorems in the dataset:

\[
T(T(T(x, T(x, y)\setminus 1), T(x, y)\setminus 1), y) = \\
T((T(x, y)\setminus 1), y) \
\]

The task of the machine learning model is to process the proof state and recognize which actions are most likely to lead to a proof, meaning that the two sides of the starting equation are equal according to the AIMLEAP system.

4 Methods

This section describes the model training methods, and the next section explains the structure of the neural network models. Here, we briefly explain the approaches that we used as RL baselines, after which we go into detail about the method that we are proposing.

4.1 RL Baselines

For comparison, we used implementations of 4 different reinforcement learning baseline methods from the literature. The first method, Advantage Actor-Critic, or A2C \cite{19}, contains ideas on which the other three baseline methods build, so we will go into more detail for this method, while keeping the explanation for the other methods brief. For details, the corresponding papers could be consulted. Reinforcement learning attempts the maximization of the reward obtained when operating in an environment \cite{5}. In our setting, one proof or normalization attempt is an episode. Such an episode consists of lists of states \( s \) encountered, actions \( a \) taken and rewards \( r \) received after taking those actions. Our setting is such that there is only a reward of 1 at the end of a successful episode. The lists \( a, s \) and \( r \) are indexed by the timestep \( t \).

A2C attempts to find suitable parameters for an agent by minimizing a loss function consisting of two parts:

\[
\mathcal{L} = \mathcal{L}^\text{A2C}_{\text{policy}} + \mathcal{L}^\text{A2C}_{\text{value}}
\]

This loss is calculated by using the states and actions encountered in the environment. The first part of the loss is the policy loss, which for one sample has the form

\[
\mathcal{L}^\text{A2C}_{\text{policy}} = - \log \pi_\theta(a|s) A(s, a),
\]

where \( A(s, a) \) is the advantage function and \( \pi_\theta \) denotes the policy model \( \pi \) with parameters \( \theta \). \( A(s, a) \) denotes the advantage in terms of future rewards of taking action \( a \) in state \( s \) compared to the value of the current state. A2C uses the approximation \( V^n_i - V(s_i) \) for the advantage, where \( V^n_i = \sum_{i=0}^{n-1} \gamma^i r_{t+i} + \gamma^n V(s_{t+n}) \), the \( n \)-step value estimate. Here \( \gamma \) is the discount factor and \( n \) is the number of states used for the estimate, indexed by timestep \( t \). The value estimates \( V(s) \) for computing the advantage function are supplied by the value predictor \( V_\mu \) with parameters \( \mu \), which is trained using the loss:

\[
\mathcal{L}^\text{A2C}_{\text{value}} = \frac{1}{2} (V^n_i - V_\mu(s_i))^2,
\]

which minimizes the squared error between the bootstrapped value estimate which takes into account environment rewards and the predicted value. Note that the sets of parameters \( \mu \) and \( \theta \) may share parameters.

The second RL baseline method we tested in our experiments is ACER, Actor-Critic with Experience Replay \cite{21}. This approach can make use of data from older episodes to train the current model. ACER applies corrections to the value estimates so that data from old episodes may be used to train the current policy. It also uses trust region policy optimization to limit the size of the policy updates. This method is included as a baseline to check whether making use of a larger replay buffer to update the parameters would be advantageous.

Our third RL baseline is the widely used proximal policy optimization (PPO) algorithm \cite{22}. It restricts the size of the parameter update to avoid causing a large difference between the original model and the updated model. In this way, PPO addresses the training instability of many reinforcement learning approaches. It has been used in various settings, for example complex video games \cite{23}. With the potential instability of neural network outputs on tree-structured data, the PPO algorithm is well-positioned. We use the PPO algorithm with clipped objective, as described in \cite{22}.

Our final RL baseline uses only the transitions with positive advantage to train on for a portion of the training procedure, to learn more from good episodes. This was proposed as self-imitation learning (SIL) \cite{24}, but to avoid confusion with the method that we are proposing, we expand the acronym to SIL-PAAC, for positive advantage actor-critic. This algorithm outperformed A2C on the sparse-reward task Montezuma’s Revenge. As theorem proving has a sparse reward structure, we included SIL-PAAC as a baseline. More information about the implementations for the baselines can be found in Appendix 4.
4.2 3SIL

We introduce stratified shortest solution imitation learning (3SIL) to perform well in the equational theorem proving domain. It learns to explicitly imitate the actions taken during the shortest solutions found for each problem in the dataset. We do this by minimizing the cross-entropy between the model output and the actions taken in the shortest solution. During the procedure no value function is trained. Instead of this, we build upon the assumption for data selection that shorter proofs are better in the context of theorem proving and expression normalization. We keep a history for each problem, where we store the current shortest solution (states seen and actions taken) found for that problem in the training dataset. We can also store multiple shortest solutions for each problem if there are multiple strategies for a proof (the number of solutions kept is governed by the parameter $k$).

During training, we sample state-action pairs from each problem’s current shortest solution at an equal probability (if a solution was found). This directly counters one of the phenomena that we had observed: the training examples for the baseline methods tend to be dominated by very long episodes (as they contribute more states and actions). This stratified sampling method ensures that problems with short proofs get represented equally in the training process.

The 3SIL algorithm is described in more detail in Algorithm 2. Sampling from a noisy version of policy $\pi_\theta$ means that actions are sampled from the model-defined distribution and in 5% of cases a random valid action is selected. Our approach is similar to the imitation learning algorithm DAGGER (Dataset Aggregation), which was used for several games [25] and modified for branch-and-bound algorithms in [26]. The behavioral cloning (BC) technique used in robotics [27] also shares elements. 3SIL significantly differs from DAGGER and BC because it does not use an outside expert to obtain useful data, because of the stratified sampling procedure and because of the keeping of the shortest solutions for each problem in the training dataset. We include as an additional baseline an implementation of behavioral cloning, where we regard proofs already encountered as coming from an expert. We minimize cross-entropy between the actions in proofs we have found and model predictions.

For the polynomial and AIM tasks, we introduce two other techniques, biased sampling and episode pruning. In biased sampling, problems without a solution in the history are sampled 5 times more during episode collection than solved problems to accelerate progress. This was determined by testing 1, 2, 5 and 10 as sampling proportions. For episode pruning, when the agent encountered the same state twice, we prune the episode to exclude the looping before storing the episode. This helps the model learn to avoid these loops.

5 Models

The tree-structured states representing expressions occurring during the tasks will be processed by a neural network. The neural network takes the tree-structured state and predicts an action to take that will bring the expression closer to being normalized or the theorem closer to being proven.
There are two main components to the neural network we use: an *embedding* in the form of a tree neural network that outputs a numerical vector representing the tree-structured proof state and a second *predictor* network that takes this vector representation of the state and outputs a distribution of the actions possible in the environment (in the reinforcement learning baselines that we use, this predictor network has the additional task of predicting the value of a state).

Tree neural networks have been used in natural language processing [28] and also in Robinson arithmetic expression embedding [29]. These networks consist of smaller neural networks, each representing one of the possible functions that occur in the expressions. For example, there will be separate networks representing addition and multiplication. The cursor is a special unary operation node with its own network that we insert into the tree at the current location. For each unique leaf, such as the constant 0, the variables \(x, y, z\), or identity element 1, we generate a random vector (from a standard normal distribution) that will represent this leaf. Aside from the 0 in the RA experiments, these vectors are trainable.

The numerical representation of a tree is constructed by starting at the leaves of the tree, for which we can look up the generated vectors. These vectors act as input to the neural networks that represent the parent node’s operation, yielding a new vector, which now represents the subtree of the parent node. The process repeats until there is a single vector for the entire tree after the root node is processed.

The neural networks representing each operation consist of a linear transformation, a non-linearity in the form of a rectified linear unit (ReLU) and another linear transformation. In the case of binary operations, the first linear transformation will have an input dimension of \(2n\) and an output dimension of \(n\), where \(n\) is the dimension of the vectors representing the leaves of the tree (the *internal representation size*).

When we have obtained a single vector embedding representing the entire tree data structure, this vector serves as the input to the predictor neural network, which consists of three linear layers, with non-linearities (Sigmoid/ReLU) in between these layers. The last layer has an output dimension equal to the number of possible actions in the environment. We obtain a probability distribution over the actions by applying the softmax function to the output of this last layer. In the cases where we also need a value prediction, there is a parallel last layer that predicts the state’s value (usually referred to as a two-headed network [40]).

The internal representation size \(n\) for the Robinson arithmetic experiments is set to 16, for the polynomial and AIM tasks this is 32. The number of neurons in each layer (except for the last one) of the predictor networks is 64.

In the AIM dataset task, an arbitrary number of variables can be introduced during the proof. These are represented by untrainable random vectors. We add a special neural network (with the same architecture as the networks representing unary operations, so from size \(n\) to \(n\)) that processes these vectors before they are processed by the rest of the tree neural network embedding. The idea is that this neural network learns to project these new variable vectors into a subspace and that an arbitrary number of variables can be handled. The vectors are resampled at the start of each episode, so the agent cannot learn to recognize variables. This approach was inspired by the *prime* mechanism in [29], but we use separate vectors for all variables instead of building vectors sequentially.

## 6 Arithmetic Experiments

### 6.1 Robinson Normalization

The Robinson arithmetic dataset [15] is split into three distinct sets, based on the number of steps that it takes a fixed rewriting strategy to normalize the expression. This fixed strategy, LOPL, which stands for *left outermost proof length*, always rewrites the leftmost possible element. If it takes this strategy less than 90 steps to solve the problem, it is in the *low* difficulty category. Problems with a difficulty between 90 and 130 are in the *medium* category and a greater difficulty than 130 leads to the *high* category. The *high* dataset also contains problems the LOPL strategy could not solve within the timelimit. The *low* dataset is split into a training and testing set. We train on the *low* difficulty problems, but after training we also test on problems with a higher difficulty. Because we have a difficulty measure for this dataset, we use a curriculum setup. We start by learning to normalize the expressions that a fixed strategy can normalize in the smallest number of steps. This setup is similar to [15].

The 400 problems with the lowest difficulty are the starting point. Every time an agent reaches 95 percent success rate when evaluated on a sample of size 400 from these problems, we add 400 more difficult problems to set of training problems \(P\). Agents are evaluated after every epoch. The blocks of size 400 are called *levels*. The number of episodes \(m\) and \(f\) are set to 1000. For 3SIL and BC, the batch size BS is 32 and the number of batches NB is 250. The baselines are configured so that the number of episodes and training transitions is at least as many as the 3SIL/BC approaches. For more details on this, see Appendix 4. If an episode takes more than 100 steps, it is stopped. ADAM is used as an
Figure 1: The level in the curriculum reached by each method. Each method was run three times. The bold line shows the mean performance and the shaded region shows the minimum and maximum performance.

Table 1: Generalization performance with greedy evaluation for test set problems for the Robinson arithmetic normalization tasks. Generalization is high on the low and medium difficulty datasets (the agents are trained on problems of the same difficulty as the low difficulty dataset). For the high difficulty dataset, performance drops.

| Method        | Low       | Medium    | High      |
|---------------|-----------|-----------|-----------|
| 3SIL (k=1)    | 1.00 ± 0.01 | 0.98 ± 0.03 | 0.77 ± 0.10 |
| 3SIL (k=2)    | 0.99 ± 0.00 | 0.96 ± 0.01 | 0.66 ± 0.08 |
| BC            | 0.98 ± 0.01 | 0.98 ± 0.01 | 0.56 ± 0.05 |

optimizer. For 3SIL and BC, the evaluation is done greedily (always take the highest probability actions). For the other methods, we performed experiments with both greedy and non-greedy (sample from the model distribution and add 5% noise) evaluation and show the results of the configuration which performed best (which in most cases was the non-greedy evaluation, except for PPO).

In Figure 1, we show the progression through the training curriculum for behavioral cloning, the RL methods and two configurations of 3SIL. Behavioral cloning simply imitates actions from successful episodes. Of the RL baselines, PPO reaches the second level in one run, while ACER steadily solves the first level and in the best run solves around 80% of the second level. Both methods do not learn enough solutions for the second level to advance to the third. A2C and SIL-PAAC do not reach the second level, so these are left out of the plot. However, they do learn to solve about 70-80% of the first 400 problems. From these results we can conclude that the RL methods do not perform well on this task in our experiment. We attribute this to the difficulty of learning a stable value function with the tree neural network as an embedding and the sparse rewards. Our hypothesis is that because this value estimate influences the policy updates, the RL methods do not learn well on this task with the current setup. Note that the two methods with a trust region update mechanism, ACER and PPO, perform better than the methods without this mechanism. From these results, it is clear that 3SIL with $k = 1$ is the best-performing configuration. It reaches the end of the training curriculum of about 5000 problems in 40 epochs.

While our approach works well on the training set, we must check if the models generalize to unseen examples. Only the methods that reached the end of the curriculum are tested. In Table 1 we show the results of evaluating the performance of our models on the three different test sets. Because we expect to require longer solutions, the episode limits are expanded from 100 steps to 200 and 250 for the medium and high datasets respectively. For the low and medium datasets, the second of which contains problems with more difficult solutions than the training data, the models solve almost all test problems. For the high difficulty dataset, the performance drops by at least 20 percentage points. Our method outperforms the Monte Carlo Tree Search approach used in [15] on the same datasets.
Table 2: Test set performance of 3SIL (k=1) as fraction of problems solved for the polynomial rewriting task with the biased sampling and loop pruning extensions. The modifications lead to better test set performance.

| Test Set Performance                  |               |
|---------------------------------------|---------------|
| NO BIAS, NO PRUNING                   | 0.56 ± 0.04   |
| BIAS, NO PRUNING                      | 0.58 ± 0.03   |
| BIAS, PRUNING                         | 0.63 ± 0.02   |

6.2 Polynomial Normalization

We also tested our method on the more challenging polynomial normalization task. We introduce two extra techniques, biased problem sampling and episode pruning, that improve the results (see Section 4.2). These techniques are used on top of the best performing configuration on the Robinson arithmetic dataset (3SIL with \( k = 1 \)). We now train for 750 epochs and the next level is reached when 90% of the evaluation problems are solved.

We use an even mixture of the six datasets provided in [16], with 1000 samples from each dataset. Again, we measure the difficulty of each problem as the number of steps that a fixed strategy takes. For the polynomial normalization, we counted the number of steps needed by the \texttt{expand} function of the mathematical Python library \textit{Sympy} [31]. For testing, we take 1000 problems from each of the six test sets.

In Figure 2, we show the training progress for three different configurations. The results show that both bias and pruning leads to a faster progression on the curriculum. The generalization performance of the method is shown in Table 2. The success rate on the polynomial dataset is worse than on the Robinson dataset, but this is expected as the task is more difficult. We are solving a much more challenging task than [16] with the same dataset, as we need to supply a proof instead of doing a one-shot transformation, but still can solve 63% of test problems. Especially the successor representations of large numbers make the task hard for our model. The exponentiation operation can cause large numbers to appear, which are represented as deep trees.

7 AIM Dataset Experiments

We will now show our results on the AIM Conjecture dataset. The setup is similar to the one for the polynomial expression normalization tasks. We apply 3SIL (\( k = 1 \)) to train models in the AIMLEAP environment. Ten percent of the AIM dataset is used as a hold-out test set, not seen during training. As there is no estimate for the difficulty of the problems in terms of the actions available to the model, we do not use a curriculum ordering for these experiments. The number of episodes collected before training \( m \) is 2,000,000. These random proof attempts result in about 300 proofs. The model learns from these proofs and afterwards the search for new proofs is also guided by the model’s predictions. For the AIM experiments, episodes are stopped after 30 steps in the AIMLEAP environment. The models are trained
for 100 epochs. The number of collected episodes per epoch \( f \) is 10,000. The successful proofs are stored, and the shortest proof for each theorem is kept. NB is 500 and BS is set to 32. The number of problems with a solution in the history after each epoch of the training run is shown in Appendix 6.

After 100 epochs, about 2500 of around 3100 problems in the training dataset have a solution in their history. To test the generalization capability of the models, we inspect their performance on the holdout test set problems. In Table 3, we compare the success rate of the trained models on a separate test set with three different automated theorem provers: E, Waldmeister and Prover9. E is currently one of the best overall automated theorem provers [32]. Waldmeister is a prover specialized in memory-efficient equational theorem proving [33] and Prover9 is the theorem prover that is used for AIM conjecture research and the prover that the dataset was generated by. Waldmeister and E are the best performing solvers in competitions for the relevant unit equality (UEQ) category [32].

The results show that a single greedy evaluation of the model is not as strong as the theorem proving software. However, the theorem provers got 60 seconds of execution time, and the execution of the model, including interaction with AIMLEAP, takes on average less than 1 second. We allowed the model to use 60 seconds, by running attempts until the time was up, sampling actions from the model distribution with 5% noise, instead of using greedy execution. With this approach, the model outperforms Waldmeister.

Figure 3 shows the overlap between the problems solved by each prover. The diagram shows that each theorem prover found a few solutions that no other prover could find within the time limit. Almost half of all problems from the test set that are solved are solved by all four systems.

Table 3: Theorem proving performance on the hold-out test set in fraction of problems solved. Means and standard deviations are the results of evaluations of 3 different models from 3 different training runs.
Table 4: Prover 9 theorem proving performance on the hold-out test set when injecting lemmas suggested by the learned model. Prover 9’s performance increases when using the model’s suggested lemmas.

| Method                                      | Success Rate |
|---------------------------------------------|--------------|
| Prover 9 (1S)                               | 0.715        |
| Prover 9 (2S)                               | 0.746        |
| Prover 9 (1S) + Model Lemmas (1S)           | 0.841 ± 0.019|

We also combine the model with Prover 9. The model modifies the starting form of the goal, for a maximum of 1 second in the AIMLEAP environment. This produces new expression on one or both sides of the equality. We then add, as lemmas, equalities between the left-hand side of the goal before the model’s rewriting and after the rewriting. The same is done for the right-hand side. For each problem, this procedure yields new lemmas that are added to the problem specification file that is given to Prover 9.

In Table 4, it is shown that adding lemmas suggested by the rewriting actions of the trained models improves the performance of Prover 9. Running Prover 9 for 2 seconds results in better performance than running it for 1 second, as expected. The combined system improved on Prover 9’s 2-second performance, indicating that the model suggests useful lemmas. The best configuration even outperforms Prover 9 with a 60 second time limit (Table 3). These results indicate that using a learned model in combination with a standard theorem prover is possible and a viable direction.

8 Discussion

The experiments on the Robinson arithmetic task indicate that 3SIL outperforms the standard RL algorithms we tested. This could be due to the difficulty of learning a stable value function in an environment where the rewards are sparse and the input data has a tree structure. In some cases, a destabilizing update decreased performance. Our method, without a value function, exhibited more stable behavior.

Generalization performance for the tasks is high. We conclude that the structure and relatively low number of parameters in the neural networks prevent overfitting. We experimented with the transformer architectures to process the tree-structured input, as in [34], but the training of this model was unsuccessful. This is probably due to the low amount of data available at the start of training being insufficient for the transformer.

9 Conclusion and Future Work

Our experiments show that it is possible to use 3SIL to learn equational theorem proving on theorems related to the Abelian Inner Mapping Conjecture. The model outperforms some established theorem provers, indicating that it is possible for learned provers to compete with standard theorem provers on specific domains or serve as an empowering tool for existing provers. The general case of theorem proving is undecidable, indicating that the models are solving hard problems.

In future work, we will apply our method to equational reasoning tasks for other group-theory related mathematical structures. In addition, we plan to investigate the extent to which Prover 9 can be assisted by our method. An especially interesting research direction concerns selecting which proofs to learn from: some proofs might use sub-proofs that are more general than other sub-proofs.

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Appendix

1 Robinson arithmetic actions

Agents in the Robinson arithmetic environment can use these actions to modify the expressions. After application of actions 1-7, the cursor is reset to the root of the expression.

| Action | Description |
|--------|-------------|
| 1.      | $x + 0 \rightarrow x$ |
| 2.      | $x \rightarrow x + 0$ |
| 3.      | $x + S(y) \rightarrow S(x + y)$ |
| 4.      | $S(x + y) \rightarrow x + S(y)$ |
| 5.      | $x \ast 0 \rightarrow 0$ |
| 6.      | $x \ast (y+z) \rightarrow (x+y)+z$ or $x \ast (y \ast z) \rightarrow (x \ast y) \ast z$ |
| 7.      | $x \ast y + x \rightarrow x \ast S(y)$ |
| 8.      | Move cursor to left child |
| 9.      | Move cursor to right child |

2 Polynomial arithmetic actions

Agents in the Polynomial arithmetic environment can use these actions to modify the expressions. After any application that is not action 8 or 9, the cursor is reset to the root of the expression.

| Action | Description |
|--------|-------------|
| 1.      | $x + 0 \rightarrow x$ |
| 2.      | $x \rightarrow x + 0$ |
| 3.      | $x + S(y) \rightarrow S(x + y)$ |
| 4.      | $S(x + y) \rightarrow x + S(y)$ |
| 5.      | $x \ast 0 \rightarrow 0$ |
| 6.      | $x \ast S(y) \rightarrow x \ast y + x$ |
| 7.      | $x \ast y + x \rightarrow x \ast S(y)$ |
| 8.      | Move cursor to left child |
| 9.      | Move cursor to right child |
| 10.     | $x + y \rightarrow y + x$ or $x \ast y \rightarrow y \ast x$ |
| 11.     | $x^0 \rightarrow 1$ |
| 12.     | $x^S(y) \rightarrow x^y \ast x$ |
| 13.     | $x^y \ast x \rightarrow x^S(y)$ |
| 14.     | $(x+y)+z \rightarrow x+(y+z)$ or $(x \ast y) \ast z \rightarrow x \ast (y \ast z)$ |
| 15.     | $x \ast (y+z) \rightarrow x \ast y + x$ |
| 16.     | $x \ast (y+z) \rightarrow x \ast y + x \ast z$ |
| 17.     | $x \ast y + x \ast z \rightarrow x \ast (y+z)$ |
| 18.     | $x \ast 1 \rightarrow x$ |
| 19.     | $x \rightarrow x \ast 1$ |
| 20.     | $1^x \rightarrow 1$ |
| 21.     | $x^1 \rightarrow x$ |
| 22.     | $x \rightarrow x^1$ |
| 23.     | $x^{(y+z)} \rightarrow x^y \ast x^z$ |
| 24.     | $x^y \ast x^z \rightarrow x^{(y+z)}$ |
| 25.     | $(x \ast y)^z \rightarrow x^z \ast y^z$ |
| 26.     | $x^z \ast y^z \rightarrow (x \ast y)^z$ |
| 27.     | $(x^y)^z \rightarrow x^{(y \ast z)}$ |
| 28.     | $x^{(y \ast z)} \rightarrow (x^y)^z$ |

3 AIM Dataset actions

In the AIMLEAP environment, the agents are able to use the following axioms, definitions, identities and known propositions to modify the current expression. Note that the agent may apply these in either direction if applicable. In addition, the agent may move the cursor to the first, second or third argument of the current node.

Axiom $\text{lid}$: \( e \ast x = x \).
Axiom $\text{s1}$: \( (x \ast y)/y = x \).

Axiom $\text{rid}$: \( x \ast e = x \).
Axiom $\text{s2}$: \( (x/y) \ast y = x \).

Axiom $\text{b1}$: \( x \setminus (x \ast y) = y \).
Definition $\alpha(x,y,z) := (x \ast (y \ast z)) \setminus ((x \ast y) \ast z)$.

Axiom $\text{b2}$: \( x \ast (x \setminus y) = y \).
Definition $K(x,y) := (y \ast x) \setminus (x \ast y)$.
Definition $T(u, x) := x \setminus (u \ast x)$.

Definition $L(u, x, y) := (y \ast x) \setminus (y \ast (x \ast u))$.

Definition $R(u, x, y) := ((u \ast x) \ast y)/(x \ast y)$.

Axiom TT: $T(T(u, x), y) = T(T(u, y), x)$.

Axiom TL: $T(L(u, x, y), z) = L(T(u, z), x, y)$.

Axiom TR: $T(R(u, x, y), z) = R(T(u, z), x, y)$.

Axiom LR: $L(R(u, x, y), z, w) = R(L(u, z, w), x, y)$.

Axiom LL: $L(L(u, x, y), z, w) = L(L(u, z, w), x, y)$.

Axiom RR: $R(R(u, x, y), z, w) = R(R(u, z, w), x, y)$.

Known prop_034cf5c5: $x \ast T(y, x) = y \ast x$.

Known prop_66f8dd43: $T(x, y, y) = y \setminus x$.

Known prop_81894ca4: $(x \ast T(y, x))/x = y$.

Known prop_da4738e2: $T(x, x \setminus y) = (x \setminus y) \setminus y$.

Known prop_4a90a23f: $x \ast T(T(y, x), z) = T(y, z) \ast x$.

Known prop_73fa4877: $T(T(x/y, z), y) = T(y \setminus x, z)$.

Known prop_d10a3b1a: $(x \ast y) \ast L(z, y, x) = x \ast (y \ast z)$.

Known prop_293cbc16: $L(x \setminus y, x, z) = (z \ast x) \setminus (z \ast y)$.

Known prop_55464ec9: $R(x, y, z) \ast (y \ast z) = (x \ast y) \ast z$.

Known prop_611b8127: $R(x/y, y, z) = (x \ast z)/(y \ast z)$.

Known prop_dd1c86f: $x \ast ((x \setminus e) \ast y) = L(y, x \setminus e, x)$.

Known prop_0d7e7151: $(x \setminus e) \ast y = x \setminus L(y, x \setminus e, x)$.

Known prop_1aae4a83: $x \ast L(T(y, x), z, w) = L(y, z, w) \ast x$.

Known prop_1db0183a: $x \ast R(T(y, x), z, w) = R(y, z, w) \ast x$.

Known prop_55842885: $T(x/y, z) \ast y = y \ast T(y \setminus x, z)$.

Known prop_1a725917: $(x \ast y) \setminus (x \ast (z \ast y)) = L(T(z, y), y, x)$.

Known prop_c626af2d: $T(x, x \setminus y) = L(T(x, y), y, x)$.

Known prop_526a359c: $T((x \ast y)/(z \ast y), z) = R(z \setminus x, z, y)$.

Known prop_deeac89a: $T(R(x, y, z), w) \ast (y \ast z) = (T(x, w) \ast y) \ast z$.

Known prop_575d1ed9: $R(x, x \setminus e, y) = y/(x \setminus e) \ast y)$.

Known prop_f338e359: $(e/x) \ast (x \ast y) = L(y, x, e/x)$.

Known prop_5a914e30: $R(x, y, y \setminus e) = (x \ast y) \ast (y \setminus e)$.

Known prop_cc8a9ae6: $R(x, e/y, y) = (x \ast (e/y)) \ast y$.

Known prop_c2aa8580: $L(x \setminus (y \setminus z), x, y) = (y \ast x) \setminus z$.

Known prop_b3890d2c: $R(x/y, y, y \setminus e) = x \ast (y \setminus e)$.

Known prop_f14899ed: $T(L(x/y, z, w), y) = L(y \setminus x, z, w)$.

Known prop_f2b7d0ab: $L(x \setminus T(y, z), x, z) = (z \ast x) \setminus (y \ast z)$.
Known prop_be6cad0a: $R((x/y)z/z, y) = x/(z * y)$.

Known prop_1e558562: $L((x/y)z/z, x) = (z * ((x * z) \ y)) / z$.

Known prop_e9ee6f09: $K(x \ e, x) = (x \ e) \ x$.

Known prop_9ee87fb5: $K(x, e/x) = x * (e/x)$.

Known prop_da958b3f: $L(x \ e, x, y) = (y \ x) \ \ y$.

Known prop_c3fa51e8: $R(e/x, x, y) = y/(x \ y)$.

Known prop_ba6418d1: $R(e/x, x, y) \ \ y = x \ y$.

Known prop_d7dd57dd: $(x \ y) \ ((x \ y) \ y) = R(T(z, x/y), x, y)$.

Known prop_e1aa92ed: $(x \ y) * K(y, x) = y \ x$.

Known prop_fed9b2: $R(x/y, z, w) \ y = y \ R(y \ x, z, w)$.

Known prop_3d75df700: $(x \ y) \ R((x \ y) \ y, z, w) = (x \ y R(x / e, z, w) \ y$.

Known prop_acafcc6f0: $L(R(x \ y) \ z, w, u), x, y) = R((y \ x) \ y, z, w, u)$.

Known prop_203fe9515: $x \ (y \ R(y \ x) \ y, z, w) = (x \ y R(x / e, z, w) \ y$.

Known prop_2e844a29: $(x \ y) \ R((x \ y) \ z, w, u) = x \ (y \ R(y \ (x \ y), z, w, u))$.

Known prop_d9457e09: $(x \ y) \ R((x \ y) \ y, z, w) = R(x, z, w) \ (x \ y)$.

Known prop_ce2987245: $x \ \ R(x, y, z) = R(x, y) \ y \ (x \ e)$.

Known prop_b7fe5fbb: $K(x \ e, x) = (e/x) \ (x \ e)$. 

Known prop_7c96e347d4: $R(T(e/x, z), x, y) = T(y/(x \ y), z)$.

Known prov_3e047dc57d: $R(x, x \ e, y) \ y = (x \ e) * y$.

Known prov_ee7819246: $R(x, x, y) * (y * x) = (y * x) * T(x, y)$.

Known prov_062c221162: $T(x, y) = R(T(x, y * x), y, x)$.

Known prov_d18167f7: $x * (T(x, y) \ e) = T(x, y) \ x$.

Known prov_638527978: $(x \ y) / (y / x) = (e / (y / x)) * (x \ y)$.

Known prov_b192646899: $x \ T(x \ y, e) = K(y \ x/y, y)$.

Known prov_2e3b566bd_alt1: $K(x \ (x/e), y * y = T(y, x)$.

Known prov_1f5e2572: $L(T(x \ e, z), x, y) = T((y \ x) \ y, z)$.

Known prov_7fed2c3e64: $(x \ y) * T((x \ y) \ x, z) = x \ (y * T(y \ e, z))$.

Known prov_47e1e09dd: $(x \ y) * T((x \ y) \ y, z) = (x * T(x \ e, z)) \ y$.

Known prov_a06014e62d_com: $x * (x \ T(x \ e, y)) = (x * T(x \ e, y)) \ x$.

Known prov_a06014ce62d: $T(x, x \ T(x \ e, y)) = x$.

Known prov_49726cdecf0: $T(x, (x \ e) * x) = x$.

Known prov_a69214e69: $R(x, x \ e, x) = T(x, x \ e)$.

Known prov_3a3e9a39ee: $T(x, x \ e) \ e = T(x, x \ e)$.

Known prov_1cecad55d: $T(x, e / x) = (x \ e) \ e$.

Known prov_13e5e8ed0a: $T(e / (e / x), x \ e) = x$.

Known prov_183b179b43: $K(x \ e) \ x = T(x, x \ e)$.
4 Implementation details

All models were implemented using PyTorch. All experiments for the Robinson and polynomial arithmetic were run on a 16 core Intel(R) Xeon(R) CPU E5-2670 0 @ 2.60GHz. The AIM experiments were run on a 72 core Intel(R) Xeon(R) Gold 6140 CPU @ 2.30GHz. All calculations were done on CPU.

The PPO implementation was adapted from [https://github.com/nikhilbarhate99/PPO-PyTorch](https://github.com/nikhilbarhate99/PPO-PyTorch). The model was updated every 2000 timesteps, the PPO clip coefficient was set to 0.2. The number of epochs each update was set to 4. The learning rate was 0.002 and the discount factor $\gamma$ was set to 0.99.

The ACER implementation was adapted from [https://github.com/dchetelat/acer](https://github.com/dchetelat/acer). The replay buffer size was 20,000. The truncation parameter was 10 and the model was updated every 100 steps. The replay ratio was set to 4. Trust region decay was set to 0.99 and the constraint was set to 1. The discount factor was set to 0.99 and the learning rate to 0.001. Off-policy minibatch size was set to 1.

The A2C and SIL implementations were based on Pytorch reinforcement learning actor-critic example code available at [https://github.com/pytorch/examples/tree/master/reinforcement_learning](https://github.com/pytorch/examples/tree/master/reinforcement_learning). The code was modified to use the bootstrapped value estimate when doing on-policy updates. For SIL, we implemented the SIL loss on top of the A2C implementation. There is also a prioritized replay buffer with an exponent of 0.6, as in the original paper. Each epoch, 8000 (250 batches of size 32) transitions were taken from the prioritized replay buffer in the SIL step of the algorithm. The size of the prioritized replay buffer was 40,000. The critic loss weight was set to 0.01 as in the original paper.

For the 3SIL and behavioral cloning implementations, we sample 8000 transitions (250 batches of size 32) from the replay buffer or history. For the behavioral cloning, we used a buffer of size 40,000. The 3SIL history or replay buffer contains at a maximum $k$ solutions for each problem in the training set. When sampling transitions in 3SIL, we first sample a problem and then we sample a transition from the stored solutions for that problem in the buffer.

For all approaches the environment was queried to find out which actions were available in the current state. The softmax outputs corresponding to invalid actions were set to 0 and the distribution normalized.

5 Data processing and proving environment details

For the polynomial arithmetic normalization dataset, there were a handful of problems that contained numbers larger than 100 in the normalized expression. These were filtered out as these caused large slowdowns due to the successor representation of numbers. 20 out of 6000 problems were filtered out by this procedure.

AIMLEAP expects a distance estimate for each applicable action. This represents the estimated distance to a proof. This behavior was converted to a reinforcement learning setup by always setting the chosen action of the model to the minimum distance and all other actions to a distance larger than the maximum proof length. Only the chosen action is then carried out.

Versions of the automated theorem provers used: Version 2.5 of EProver ([https://www1lehre.dhbw-stuttgart.de/~sschulz/E/E.html](https://www1lehre.dhbw-stuttgart.de/~sschulz/E/E.html)), the Nov 2017 version of Prover9 ([https://github.com/ai4reason/Prover9](https://github.com/ai4reason/Prover9)) and the Feb 2018 version of Waldmeister ([https://www.mpi-inf.mpg.de/departments/automation-of-logic/software/waldmeister/download](https://www.mpi-inf.mpg.de/departments/automation-of-logic/software/waldmeister/download)).
6 Training of AIM models

![Graph showing the number of training problems solved over epochs. The graph demonstrates a rapid collection of solutions at the start, followed by a slowdown and stabilization at around 2500 problems with known solutions after 100 epochs.]

Figure 4: The number of training problems for which a solution was encountered and stored. At the start of the training, the model rapidly collects more solutions, but after 100 epochs, the process has slowed down and settled at about 2500 problems with known solutions.