A Critical Look at Rescattering Effects on $\gamma$ from $B^+ \to K\pi$

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ABSTRACT

Three ways are compared of dealing with rescattering effects in $B^\pm \to K^0\pi^\pm$, in order to determine the weak phase $\gamma$ from these processes and $B^\pm \to K^\pm\pi^0$. We find that neglecting these contributions altogether may involve sizable errors in $\gamma$, depending on the rescattering amplitude and on the value of a certain measurable strong phase. We show that an attempt to eliminate these effects by using the charge-averaged rate of $B^\pm \to K^\pm K^0$ suffers from a large theoretical error due to SU(3) breaking, which may be resolved when using also the processes $B^\pm \to \pi^\pm\eta_8$.

PACS codes: 12.15.Hh, 12.15.Ji, 13.25.Hw, 14.40.Nd
1 Introduction

The weak phase $\gamma = \text{Arg}(V_{ub}^*)$ is presently the least well known quantity among the four parameters (three angles and a phase) of the Cabibbo-Kobayashi-Maskawa (CKM) matrix. Its determination, which is regarded to be more difficult than that of the other two angles of the CKM unitarity triangle [1, 2], can provide a crucial test of the CKM mechanism for CP violation in the Standard Model. Several methods have been proposed to determine $\gamma$ from hadronic two-body $B$ decays. The methods which seem to be experimentally most feasible in the near future are based on applications of SU(3) flavor symmetry in $B$ decays into two light charmless pseudoscalars [3]. These methods involve certain theoretical uncertainties, which are expected to be reduced when more data become available and when better theoretical understanding of hadronic $B$ decays is achieved.

In a first paper in a series, Gronau, London and Rosner (GLR) [4] proposed to extract $\gamma$ by combining decay rate measurements of $B^+ \to K\pi$, $B^+ \to \pi\pi$ with their charge-conjugates. SU(3) breaking, occurring in a relation between $B \to \pi\pi \ I = 2$ and $B \to K\pi \ I = 3/2$ amplitudes, was introduced through a factor $f_K/f_{\pi}$ when assuming that these amplitudes factorize. In its original version, suggested before the observation of the heavy top quark, the method of Ref. [4] neglected electroweak penguin (EWP) contributions and certain rescattering effects. Subsequently, model-calculation showed that due to the heavy top quark the neglected EWP terms were significant [5]; and recently these terms were related by SU(3) to the $B \to K\pi \ I = 3/2$ current-current amplitudes [6, 7]. This led to a modification [8] of the GLR method, to be referred to as the GLRN method, which in the limit of flavor SU(3) symmetry includes EWP effects in a model-independent way. Corrections from SU(3) breaking, affecting the relation between EWP terms and current-current terms, were argued to be small [6, 9].

Assuming that the above SU(3) breaking effects are indeed under control, there is still an uncertainty due to rescattering effects. To determine $\gamma$ from the above rates, one takes the $B^+ \to K^0\pi^+$ amplitude to be pure penguin, involving no term with weak phase $\gamma$. This assumption, which neglects quark annihilation and rescattering contributions from charmless intermediate states, was challenged by a large number of authors [10]. Several authors proposed ways of controlling rescattering effects in $B^\pm \to K^0\pi^\pm$ by relating them through SU(3) to the much enhanced effects in $B^\pm \to K^\pm\bar{K}^0$ [11, 12, 13] (see also [14, 15, 16]). The charge-averaged rate of the latter processes can be used to set an upper limit on the rescattering amplitude in $B^\pm \to K^0\pi^\pm$. While present limits are at the level of 20 – 30% of the dominant penguin amplitude [7, 9] (depending somewhat on the value of $\gamma$), they are expected to be improved in the future. The smaller the rescattering amplitude is, the more precisely can $\gamma$ be determined from the GLRN method. A recent demonstration [1].
based on a few possible rate measurements, seems to show that if the rescattering amplitude is an order of magnitude smaller than the dominant penguin amplitude in $B^+ \to K^0\pi^+$, the uncertainty in $\gamma$ is only about 5 degrees.

In the present Letter we reexamine in detail the uncertainty in $\gamma$ due to rescattering effects. Using a geometrical interpretation for the extraction of $\gamma$, we perform in Section 2 numerical simulations which cover the entire parameter space of the two relevant strong phases, the rescattering phase $\phi_A$ and the relative phase $\phi$ between $I = 3/2$ current-current and penguin amplitudes. We find that, contrary to the demonstration made in [9], a 10% rescattering amplitude leads to an uncertainty in $\gamma$ as large as about 14$^\circ$ around $\phi \sim 90^\circ$. For certain singular cases no solution can be found for $\gamma$. We show that $\phi$ can be determined rather precisely from the $B^{\pm} \to K\pi$ rate measurements [9], which could reduce substantially the error in $\gamma$ if values far apart from $\phi = 90^\circ$ were found.

It has been suggested [13] to go one step beyond setting limits on rescattering contributions in $A(B^{\pm} \to K^0\pi^{\pm})$ and to completely eliminate them by using the charge-averaged rate measurement of $B^{\pm} \to K^{\pm}K^0$. Applying our geometrical formulation, we will show in Section 3 that the resulting determination of $\gamma$ is unstable under SU(3) breaking which can introduce very large uncertainties in $\gamma$.

Finally, in order to overcome these uncertainties, we have recently proposed to use in addition to $B^\pm \to K^{\pm}K^0$ also the processes $B^\pm \to \pi^\pm\eta_8$ [17]. Although this may be considered an academic exercise, mainly due to complicating $\eta - \eta'$ mixing effects, we will examine in Section 4 the precision of this method. We will show that, when neglecting $\eta - \eta'$ mixing, the theoretical error in $\gamma$ is reduced to a few degrees.

We conclude in Section 5. An algebraic condition, used in Section 3 to eliminate rescattering effects by $B^\pm \to K^{\pm}K^0$ decays, is derived in an Appendix.

2 Rescattering uncertainty in $\gamma$ from $B^\pm \to K\pi$

The amplitudes for charged $B$ decays can be parameterized in terms of graphical contributions representing SU(3) amplitudes (we use the notations of [6]):

\[
A(B^+ \to K^0\pi^+) = |\lambda_u^{(s)}|e^{i\gamma}(A + P_{uc}) + \lambda_t^{(s)}(P_{ct} + P_3^{EW}) , \\
\sqrt{2}A(B^+ \to K^+\pi^0) = |\lambda_u^{(s)}|e^{i\gamma}(-T - C - A - P_{uc}) + \lambda_t^{(s)}(-P_{ct} + \sqrt{2}P_4^{EW}) , \\
\sqrt{2}A(B^+ \to \pi^+\pi^0) = |\lambda_u^{(s)}|e^{i\gamma}(-T - C) ,
\]

where $\lambda_q^{(g)} = V_{qs}^*V_{qs}'$ are the corresponding CKM factors. These amplitudes satisfy a triangle relation [4, 8]

\[
\sqrt{2}A(B^+ \to K^+\pi^0) + A(B^+ \to K^0\pi^+) = \sqrt{2}f_u|A(B^+ \to \pi^+\pi^0)|e^{i(\gamma + \xi)}(1 - \delta_{EW}e^{-i\gamma}) .
\]
Here we denote $\tilde{r}_u = (f_K/f_\pi)\lambda/(1 - \lambda^2/2) \simeq 0.28$, $\delta_{EW} = -(3/2)|\lambda_i^{(s)}/\lambda_u^{(s)}|\kappa \simeq 0.66$ ($\kappa \equiv (c_0 + c_{10})/(c_1 + c_2) = -8.8 \times 10^{-3}$), while $\xi$ is an unknown strong phase. The second term in the brackets represents the sum of EWP contributions to the amplitudes on the left-hand side [3, 4]. The factor $f_K/f_\pi$ accounts for factorizable SU(3) breaking effects.

The relation (4), together with its charge-conjugate counterpart, written for $\tilde{A}(\tilde{B} \to \tilde{f}) \equiv e^{2i\gamma}A(\tilde{B} \to \tilde{f})$, are represented graphically by the two triangles $OAA'$ and $OBB'$ in Fig. 1. Here all amplitudes are divided by a common factor $\mathcal{A} \equiv \sqrt{2}\tilde{r}_u|A(B^+ \to \pi^0)|e^{i(\gamma + \xi)}$, such that the horizontal line $OI$ is of unit length and the radius of the circle is $\delta_{EW}$. Four of the sides of the two triangles are given by

$$x_{0+} = \frac{1}{\sqrt{2}\tilde{r}_u} \frac{|A(B^+ \to K^0\pi^+)|}{|A(B^+ \to \pi^+\pi^0)|}, \quad \bar{x}_{0-} = \frac{1}{\sqrt{2}\tilde{r}_u} \frac{|A(B^+ \to K^0\pi^-)|}{|A(B^+ \to \pi^+\pi^0)|}.$$  

The relative orientation of the two triangles depends on $\gamma$ and is not determined from measurements of the sides alone. Assuming that the rescattering amplitude with weak phase $\gamma$ in $B^+ \to K^0\pi^+$ can be neglected, one takes the amplitude (5) to be given approximately by the second (penguin) term [4, 5], which implies $OB = e^{2i\gamma}OA$ in Fig. 1. In this approximation, the weak phase $\gamma$ is determined by requiring that the angle ($2\gamma$) between $OA$ and $OB$ is equal to the angle ($2\gamma$) at the center of the circle [3].

In order to study the precision of determining in this way the phase $\gamma$ as function of the rescattering contribution which is being neglected, let us rewrite (6) in the form

$$A(B^+ \to K^0\pi^+) = -V_{cb} \left(1 - \frac{\lambda^2}{2}\right) p(1 + \epsilon_A e^{i\phi_A} e^{i\gamma}) \quad p \equiv P_{ct} + P_{3EW}$$

where $\epsilon_A$ measures the magnitude of rescattering effects. In Fig. 1 the magnitude of these effects has a simple geometrical interpretation in terms of the distance of the point $Y$ from the origin $O$, $\epsilon_A = |YO|/|YA|$, where $YO$ and $YA$ are the two components in the $B^+ \to K^0\pi^+$ amplitude carrying weak phases $\gamma$ and zero, respectively

$$YO = |\lambda_u^{(s)}|e^{i\gamma} [(A + P_{uc}) - p]/\mathcal{A}, \quad YA = V_{cb} \left(1 - \frac{\lambda^2}{2}\right) p/\mathcal{A}.$$  

The rescattering phase $\phi_A$ is given by $\phi_A = \text{Arg}(YO/YZ)$, where $Z$ is any point on the line bisecting the angle $AYB$. A second strong phase which affects the determination of $\gamma$ is $\phi$, the relative strong phase between the penguin amplitude $p$ and the $I = 3/2$ current-current amplitude $T + C$. In Fig. 1 this phase is given by $\phi = \text{Arg}(YZ/OI)$. 

4
Let us now investigate the dependence of the error in \( \gamma \) when neglecting rescattering on the relevant hadronic parameters. Our procedure will be as follows. First we generate a set of amplitudes based on the geometry of Fig. 1 and on given values of the parameters \( \gamma, \epsilon, \epsilon_A, \phi, \phi_A \); then we solve the equation \( \cos 2\gamma = \cos(BOA) \) and compare the output value of \( \gamma \) with its input value. Here \( \epsilon \) is given in terms of the ratio of charge-averaged branching ratios \[ \epsilon \equiv \lambda_1 - \lambda_2 (f_K f_\pi \sqrt{2B(B^\pm \rightarrow \pi^\pm \pi^0)} / f_\pi) , \]

The geometrical construction in Fig. 1 is described by \[ Y_A = \frac{\epsilon e^{i(\phi - \gamma)}}{\epsilon \sqrt{1 + 2\epsilon_A \cos \phi_A \cos \gamma + \epsilon_A^2}} OI , \quad OY = \epsilon_A e^{i(\phi_A + \gamma)} Y_A , \]

implying a rate asymmetry between \( B^+ \rightarrow K^0\pi^+ \) and \( B^- \rightarrow \bar{K}^0\pi^- \).

For illustration, we take \( \gamma = 76^\circ, \epsilon = 0.24, \epsilon_A = 0.1 \) (which is a reasonable guess \[ \epsilon_A \approx 0.1 \]), and we vary \( \phi \) and \( \phi_A \) in the range \( 0^\circ \leq \phi \leq 180^\circ, -90^\circ \leq \phi_A \leq 270^\circ \). The results of a search for solutions in the interval \( 65^\circ \leq \gamma \leq 90^\circ \) are presented in Fig. 2 which displays a twofold ambiguity. Fig. 2(a) shows the solution as function of \( \phi_A \) for two values of \( \phi \), \( \phi = 60^\circ \) and \( \phi = 90^\circ \). Whereas for \( \phi_A = 90^\circ \) the solution is very close to the input value, the deviation becomes maximal for \( \phi_A = 0^\circ \) and \( 180^\circ \). This agrees with the geometry of Fig. 1, in which the largest rescattering effects are expected when \( YO \) is parallel or anti-parallel to the line bisecting the angle \( BYA \).

In a second plot, Fig. 2(b), we fix \( \phi_A = 0^\circ \) and vary \( \phi \) over its entire range, which illustrates the maximal rescattering effect. We find two branches of the solution for \( \gamma \), both of which deviate strongly from the input value \( \gamma = 76^\circ \) for values of \( \phi \) around \( 90^\circ \). At \( \phi = 90^\circ \) there is no solution for \( \epsilon_A = 0.1 \) in the considered interval. We checked that the solution is restored and approaches the input value as the magnitude of \( \epsilon_A \) decreases to zero, as it should. Thus, the uncertainty in \( \gamma \), seen both in Fig. 2(a) and Fig. 2(b) at \( \phi_A = 0^\circ \) and around \( \phi = 90^\circ \), is about \( 14^\circ \). It can even be worse in the singular cases where no solution for \( \gamma \) can be found.

A variant of this method for determining \( \gamma \), proposed recently in \[ \text{(9)} \], was formulated in terms of two quantities \( R_* \) and \( A \) defined by \[ R_* \equiv \frac{B(B^\pm \rightarrow K^0\pi^\pm)}{2B(B^\pm \rightarrow K^\pm\pi^0)} , \quad \bar{A} \equiv \frac{B(B^+ \rightarrow K^+\pi^0) - B(B^- \rightarrow K^-\pi^0)}{B(B^\pm \rightarrow K^0\pi^\pm)} - \frac{B(B^+ \rightarrow K^0\pi^+ - B(B^- \rightarrow K^0\pi^-)}{2B(B^\pm \rightarrow K^0\pi^\pm)} . \]

These quantities do not contain \( O(\epsilon_A) \) terms; their dependence on the rescattering parameter \( \epsilon_A \) appears only at order \( O(\epsilon\epsilon_A) \). Therefore, it was argued in \[ \text{(10)} \], the
The determination of $\gamma$, by setting $\epsilon_A = 0$ in the expressions for $R_*$ and $\tilde{A}$, is insensitive to rescattering effects. This procedure gives two equations for $\gamma$ and $\phi$ which can be solved simultaneously from $R_*$ and $\tilde{A}$. Using two pairs of input values for $(R_*, \tilde{A})$ (corresponding to a restricted range for $\phi_A$ and $\phi$) seemed to indicate that the error in $\gamma$ for $\epsilon_A = 0.08$ is only about $5^\circ$. (The relations between the parameters used in [9] and ours are $\phi = -\phi, \eta = \phi_A + \pi, \tilde{\epsilon}_{3/2} = \epsilon$ and $\epsilon_a = \epsilon_A$).

In Fig. 3 we show the results of such an analysis carried out for the entire parameter space of $\phi_A$ and $\phi$. Whereas the angle $\phi$ can be recovered with small errors, the results for $\gamma$ show the same large rescattering effects for values of $\phi$ around $90^\circ$ as in Fig. 2. (A slight improvement is the absence of a discrete ambiguity in the value of $\gamma$.) These results show that the large deviation of $\gamma$ from its physical value for $\phi = 90^\circ$ is a general phenomenon, common to all variants of this methods. Some information about the size of the expected error can be obtained by first determining $\phi$. Values not too close to $90^\circ$ would be an indication for a small error.

### 3 Eliminating rescattering by $B^\pm \to K^\pm K^0$

The amplitude for $B^+ \to K^+ K^0$ is obtained from $A(B^+ \to K^0 \pi^+)$ in (1) by a $U$-spin rotation

$$A(B^+ \to K^+ K^0) = |\lambda^{(d)}| e^{i\gamma}(A + P_{uc}) + |\lambda^{(d)}| e^{-i\beta}(P_{ct} + P_{3EW}).$$

In the limit of SU(3) symmetry the amplitudes in (11) are exactly the same as those appearing in (1). In Fig. 4 $A(B^+ \to K^+ K^0)$, scaled by the factor $\lambda/(1 - \lambda^2/2)$ (and divided by $A$ as in Fig. 1), is given by the line $OC$ and its charge-conjugate is given by $OD$. We have shown in [17] that knowledge of these two amplitudes allows one to completely eliminate the rescattering contribution $A + P_{uc}$ from the determination of $\gamma$. This is achieved by effectively replacing in the GLRN method the origin $O$ by the intersection $Y$ of the lines $AC$ and $BD$. $\gamma$ is determined by requiring that the angle $(2\gamma)$ between $YA$ and $YB$ is equal to the angle $(2\gamma)$ at the center of the circle.

The amplitude (11) can be decomposed into two terms carrying definite weak phases in form very similar to (9),

$$\frac{\lambda}{1 - \lambda^2/2}A(B^+ \to K^+ K^0) = V_{cb} \left(1 - \frac{\lambda^2}{2}\right) p \left(-\frac{\lambda^2}{(1 - \lambda^2/2)^2} + \epsilon_A e^{i\phi_A} e^{i\gamma}\right),$$

The ratio $|CY|/|AY| = \lambda^2/(1 - \lambda^2/2)^2$ implies that the triangle $AYB$ is about 25 times larger than the triangle $CYD$. This will result in a large uncertainty in $\gamma$ also when the equality between the corresponding terms in $B^+ \to K^0 \pi^+$ and $B^+ \to K^+ K^0$ amplitudes involves relatively small SU(3) violation.
The geometrical construction by which rescattering amplitudes can be completely eliminated in the SU(3) limit consists of three steps. (See Fig. 4. For an alternative suggestion, see [13].)

a) Determine the position of the point $Y$ as a function of the variable angle $2\gamma$ and the decay rates of $B^\pm \to K\pi$ and $B^+ \to \pi^+\pi^0$. The point $Y$ is chosen on the mid-perpendicular of $AB$ such that the equality of the angles marked $2\gamma$ is preserved for any value of $\gamma$.

b) Draw two circles of radii $\lambda/(1 - \lambda^2/2)|A(B^\pm \to K^0K^\pm)|$ centered at the origin $O$ (dashed-dotted circles in Fig. 4). The intersections of the lines $AY$ and $BY$ with these circles determine $C$ and $D$ respectively (up to a two-fold ambiguity), again as functions of $\gamma$.

c) The physical value of $\gamma$ is determined by the requirement $|AC| = |BD|$ [17]. This condition on $\gamma$ can be formulated in an algebraic form, showing that only the charge-averaged rate of $B^\pm \to K^\pm K^0$ is needed. The condition is given by Eq. (15) in the Appendix.

Let us examine the precision of this method for $\epsilon_A = 0.1$ at $\phi \approx 90^\circ$, for which the simpler method of Sec. 2 receives large rescattering corrections. In Fig. 5(a) we show the left-hand side of Eq. (15) as a function of variable $\gamma$ at $\phi = 90^\circ$ for several values of $\phi_A$. The value of $\gamma$ is obtained from the condition that the left-hand side of this equation vanishes. In the absence of SU(3) breaking this method reproduces precisely the physical value of $\gamma$ ($\gamma = 76^\circ$) for all values of $\phi_A$. However SU(3) breaking effects can become important, to the point of completely spoiling this method. We simulate these effects by taking the amplitudes $p$ and $a \equiv A + P_{uc} - p$ in $B^\pm \to K^\pm K^0$ (Eq. (11)) to differ by at most 30% from those in $B^\pm \to K^0\pi^\pm$ (Eq. (4)). This expands the lines of Fig. 5(a) into bands of finite width, which give a range for the output value of $\gamma$.

In Fig. 5(b) we show the effects of SU(3) breaking on the determination of $\gamma$ as function of $\phi_A$ for $\phi = 90^\circ$. We see that for values of $|\phi_A|$ larger than about 25$^\circ$ the error on $\gamma$ is quite large. Thus, we conclude that for certain values of the strong phases the determination of $\gamma$ using this method is unstable under SU(3) breaking in the relation between $B^+ \to K^0\pi^+$ and $B^+ \to K^+\bar{K}^0$.

4 The use of $B^\pm \to \pi^\pm\eta_8$

In Ref. [17] we proposed to use in addition to $B^+ \to K^+\bar{K}^0$ also $B^+ \to \pi^+\eta_8$ and their charge-conjugates. Writing

$$A(B^+ \to \pi^+\eta_8) = |\lambda^{(d)}_w|e^{i\gamma}(-T - C - 2A - 2P_{uc}) + |\lambda^{(d)}_t|e^{-i\beta}(-P_{ct} + P_{5\text{EW}}),$$  \hspace{1cm} (13)
we find the triangle relation

\[ A(B^+ \to K^+\bar{K}^0) + \sqrt{3} \frac{2}{\sqrt{2}} A(B^+ \to \pi^+\eta_8) = \frac{1}{\sqrt{2}} A(B^+ \to \pi^+\pi^0) \, . \]  (14)

This relation and its charge-conjugate provide another condition which determines the positions of the points C and D. As in Section 3, the phase \( \gamma \) is determined by the equation \( \cos(BY A) = \cos 2\gamma \), where the point Y is fixed by the intersection of the lines AC and BD. General considerations, based on the relative sizes of the amplitudes involved, suggest that this method is relatively insensitive to SU(3) breaking effects [L7].

We illustrate this in Fig. 6 where we show on the same plot the two sides of the equation \( \cos(BY A) = \cos 2\gamma \) as functions of the variable \( \gamma \). As in method of Section 3, SU(3) breaking is simulated by taking the penguin \( (p) \) and annihilation \( (a) \) amplitudes in \( B^\pm \to K^\pm K^0 \) to differ by at most 30% (separately for their real and imaginary parts) from those in \( B^\pm \to K^0\pi^\pm \). The latter are used to construct the positions of the points C and D. In the example of Fig. 6 we take \( \epsilon_A = 0.1, \phi = 90^\circ, \phi_A = 45^\circ \), for which the two methods described in Sections 2 and 3 were shown to lead to large errors in \( \gamma \). For an input value \( \gamma = 76^\circ \), the output is given by the range \( 74^\circ < \gamma < 78^\circ \), obtained by the intersection of the solid line with the band formed by the diamond points. We see that the error in \( \gamma \) due to SU(3) breaking is less than \( \pm 2^\circ \), which confirms the general arguments of [L7]. This scheme, or rather its analogous version using \( B^0 \) and \( B_s \) decay [L7], may prove useful for a determination of \( \gamma \) in case that the strong phases \( (\phi, \phi_A) \) turn out to have values which preclude the use of the two simpler methods.

5 Conclusion

We compared three ways of dealing with rescattering effects in \( B^\pm \to K^0\pi^\pm \), in order to achieve a precise determination for the weak phase \( \gamma \) from these processes and \( B^\pm \to K^\pm\pi^0 \). In the simplest GLRN method we find that large errors in \( \gamma \) are possible for a particular range of the strong phases, \( \phi \sim 90^\circ \), even when the rescattering term is only at a level of 10%. \( B^+ \) and \( B^- \) decay rate measurements are expected to provide rather precise information on \( \phi \). Small errors in \( \gamma \) would be implied if \( \phi \) turns out to be far away from 90°. The second method, in which rescattering effects can be completely eliminated in the SU(3) limit by using also the charge-averaged \( B^\pm \to K^\pm K^0 \) rate, suffers from a sizable uncertainty due to SU(3) breaking. These uncertainties would be resolved in an ideal world, where \( B^\pm \to \pi^\pm\eta_8 \) can be measured, or alternatively by using corresponding \( B^0 \) and \( B_s \) decays.
6 Appendix

The weak angle $\gamma$ is fixed in the method described in Sec. 3 by the condition $|AC| = |BD|$, or equivalently $|YC| = |YD|$. Explicitly, this can be written after some algebra as an equation in $\gamma$

$$2(1 - x_0)^2 \bar{y}^2 + 2x_0(1 - x_0) \bar{y} \cdot (\vec{A} + \vec{B}) + x_0^2(x_0^2 + \bar{x}_0^2) - (y_{+0}^2 + \bar{y}_{-0}^2) = 0,$$

where $x_0$ is defined as the ratio of two CP rate differences

$$x_0 \equiv \frac{(y_{+0}^2 - \bar{y}_{-0}^2)/(x_0^2 - \bar{x}_0^2)}{\sqrt{2}(1 - \lambda^2/2)},$$

Here

$$y_{+0} = \frac{1}{\sqrt{2}} \frac{f_\pi}{f_K} \frac{|A(B^+ \to K^+K^0)|}{|A(B^+ \to \pi^+\pi^0)|}, \quad \bar{y}_{-0} = \frac{1}{\sqrt{2}} \frac{f_\pi}{f_K} \frac{|A(B^- \to K^-K^0)|}{|A(B^+ \to \pi^+\pi^0)|}$$

obey an SU(3) relation with the amplitudes (I) of $B^\pm \to K\pi^\pm$

$$y_{+0}^2 - \bar{y}_{-0}^2 = -\frac{\lambda^2}{(1 - \lambda^2/2)^2}(x_0^2 - \bar{x}_0^2).$$

This implies that CP rate differences in $B^\pm \to K^0\pi^\pm$ and $B^\pm \to K^\pm K^0$ are equal and of opposite sign [13]. We see that in the SU(3) limit the condition (13), which eliminates rescattering effects, requires only a measurement of the charge-averaged rate of $B^\pm \to K^\pm K^0$ and not the CP asymmetry in these processes [13].

To prove (13), let us consider two lines $AY$ and $BY$ cutting two circles of radii $R_1$, $R_2$ (centered at the origin) at points $C$ and $D$ respectively. The intersection points can be written as $\vec{C} = \vec{Y} + x_1(\vec{A} - \vec{Y})$ and $\vec{D} = \vec{Y} + x_2(\vec{B} - \vec{Y})$, where $x_1, x_2$ are solutions of the equations

$$(\vec{A} - \vec{Y})^2 x_1^2 + 2x_1 \vec{Y} \cdot (\vec{A} - \vec{Y}) + (\vec{Y}^2 - R_1^2) = 0$$

$$(\vec{B} - \vec{Y})^2 x_2^2 + 2x_2 \vec{Y} \cdot (\vec{B} - \vec{Y}) + (\vec{Y}^2 - R_2^2) = 0.$$}

The condition $|YC| = |YD|$ is equivalent to requiring that these two equations have a common solution $x_1 = x_2$. Obviously, if such a solution exists, it is given by

$$x_0 = \frac{R_2^2 - R_1^2}{2\vec{Y} \cdot (\vec{A} - \vec{B})} = \frac{R_2^2 - R_1^2}{\vec{A}^2 - \vec{B}^2},$$

where we used the equality $(\vec{A} - \vec{Y})^2 = (\vec{B} - \vec{Y})^2$. Taking the sum of (19) and (20) with the value (21) for $x$ leads immediately to the condition (13).

Acknowledgements. We thank R. Fleischer, M. Neubert and T.M. Yan for useful discussions. This work is supported by the National Science Foundation and by the United States - Israel Binational Science Foundation under Research Grant Agreement 94-00253/3.
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Figure 1: Relative orientation of $B^+ \rightarrow K\pi$ amplitude triangles.
Figure 2: The weak phase $\gamma$ is obtained as the solution to the equation $\cos(2\gamma) = \cos(BOA)$. (a) - the dependence of the solution on $\phi_A$, for two values of $\phi = 60^\circ$ and $\phi = 90^\circ$; (b) - the dependence of the solution on $\phi$, for $\phi_A = 0^\circ$. (both graphs correspond to $\epsilon_A = 0.1, \gamma = 76^\circ$)

Figure 3: (a) - the weak phase $\gamma$ extracted from the method using the parameters $(R_*, \tilde{A})$, as a function of the strong phase $\phi$ for several values of $\phi_A$ ($\epsilon_A = 0.1$). The horizontal line shows the assumed physical value of $\gamma = 76^\circ$. (b) - the strong phase $\phi$ can be reconstructed using the $(R_*, \tilde{A})$ data.
Figure 4: Geometric construction for the method described in Sec. 3. C and D denote the intersection points of the lines AY and BY determined as explained in the text, with the two circles of radii given by $|A(B^\pm \to K^\pm K^0)|$.

Figure 5:  (a) - The left-hand of Eq. (15) as a function of variable $\gamma$ for $\phi = 90^\circ$ and for different values of $\phi_A$. All these curves intersect at $\gamma = 76^\circ$, which is the assumed physical value. (b) - SU(3) breaking effects introduce an error on the extracted value of $\gamma$, here shown as function of $\phi_A$ at $\phi = 90^\circ$. 
Figure 6: Numerical results for the method of Sec. 4. The two sides of the equation $\cos(BY\phi) = 2\gamma$ as function of variable $\gamma$, including 30% SU(3) breaking effects in the $p$ and $a$ amplitudes. The physical value of $\gamma$ is determined by the intersection of the solid line with the wide band. The strong phases are taken as $(\phi, \phi_A) = (90^\circ, 45^\circ)$. 