Learning Quantum Finite Automata with Queries

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Abstract

Learning finite automata (termed as model learning) has become an important field in machine learning and has been useful realistic applications. Quantum finite automata (QFA) are simple models of quantum computers with finite memory. Due to their simplicity, QFA have well physical realizability, but one-way QFA still have essential advantages over classical finite automata with regard to state complexity (two-way QFA are more powerful than classical finite automata in computation ability as well). As a different problem in quantum learning theory and quantum machine learning, in this paper, our purpose is to initiate the study of learning QFA with queries (naturally it may be termed as quantum model learning), and the main results are regarding learning two basic one-way QFA: (1) We propose a learning algorithm for measure-once one-way QFA (MO-1QFA) with query complexity of polynomial time; (2) We propose a learning algorithm for measure-many one-way QFA (MM-1QFA) with query complexity of polynomial-time, as well.

Keywords: Quantum computing, Quantum finite automata, Learning from queries, Quantum model learning, SD oracles

1. Introduction

Learning finite automata has become an important field in machine learning and has been applied to wide-ranging realistic problems, for example, smartcards, network protocols, legacy software, robotics and control systems, pattern recognition, computational linguistics, computational biology, data compression, data mining, etc. In learning finite automata is termed as model learning. In fact, model learning and model checking as well as and model-based testing have intrinsic connections (see the pioneering contribution of Peled et al. and Steffen et al.).

Learning finite automata was first considered by Moore and an exponential-time query algorithm was proposed. In particular, Angluin in 1987 proposed the so-called membership and equivalence queries, a ground-breaking method for learning the models of finite automata. For learning deterministic finite automata (DFA), according to Angluin’s algorithm, the learner initially only knows the inputs (i.e. alphabet)
of the model to be learned (say $M$), and the aim of the learner is to learn the model through two types of queries, that is, membership queries (MQ) and equivalence queries (EQ). MQ means that the learner asks what the result (accepting or rejecting) of output is in response to an input sequence, and the oracle answers with accepting or rejecting, while EQ signifies the learner whether a hypothesized machine model (say $H$) is the same as the learned machine, and the oracle answers yes if this is the case. Otherwise ‘no’ is replied and an input string is provided as a counterexample to distinguish $H$ and $M$.

The complexity of queries of Angluin’s algorithm [1] is polynomial for learning DFA and Mealy machines. Angluin [2] proved that DFA cannot be learned in polynomial time by membership queries (or equivalence queries) only. Since Angluin’s algorithm was proposed [1], learning other models of finite automata has been investigated. Tzeng [39] studied learning probabilistic finite automata (PFA) and Markov chains via SD oracle, where SD oracle can answer state distribution, i.e., probability distribution of states for each input string, so it is more powerful than membership queries (MQ). For learning DFA via SD oracle, a state is replied for each input string, and the query complexity of learning DFA via SD oracle is polynomial [39].

Then Bergadano and Varricchio [14] used membership queries (MQ) and equivalence queries (EQ) to learn appropriately probabilistic finite automata, and a probably approximately correctly learning algorithm (i.e. PAC algorithm) was presented. Learning nondeterministic finite automata (NFA) was studied by Bollig et al. [8]. In recent years, Angluin et al. [4] initiated the research of learning alternating automata, and Berndt et al. [11] further solved the learning problem of residual alternating automata.

A natural inquiry is that SD oracle seems too strong. However, it was showed by Tzeng [39] that SD oracle is actually not too strong for learning DFA and PFA if the query complexity is required to be polynomial, because learning a consistency problem related to DFA and PFA via SD oracle is still NP-complete [39]. In this paper, we use an AD oracle for learning quantum finite automata (QFA) in polynomial time, that is, AD oracle can answer a state of superposition for each input string, i.e., amplitude distribution of states. If we relax AD oracle to weaker one that can only answer amplitudes of accepting states, instead of all amplitudes of a superposition state, then using this oracle to learn a consistency problem related to reversible finite automata (RFA) and quantum finite automata is NP-complete.

Quantum machine learning was early considered by Bshouty and Jackson [9] with learning from quantum examples, and then quantum learning theory [6] as an important theoretical subject of quantum machine learning has been deeply developed. Quantum learning theory [6] includes models of quantum exact learning, quantum PAC learning, and quantum agnostic learning; there models are combinations of corresponding classical learning models with quantum computing (in a way, quantum query algorithms).

However, learning quantum finite automata is still a pending problem to be studied, and this is the main goal of this paper. QFA can be thought of as a theoretical model of quantum computers in which the memory is finite and described by a finite-dimensional state space [10, 13, 34]. This kind of theoretical models was firstly proposed and studied by Moore and Crutchfield [26], Kondacs and Watrous [22], and then
Ambainis and Freivalds [5], Brodsky and Pippenger [13], and other authors (e.g., the references in [10, 34]). The decision problems regarding equivalence of QFA and minimization of states of QFA have been studied in [23, 24, 28, 34, 36].

According to the measurement times in a computation, one-way QFA have two types: measure-once one-way QFA (MO-1QFA) initiated by Moore and Crutchfield [26] and measure-many one-way QFA (MM-1QFA) studied first by Kondacs and Watrous [22]. In MO-1QFA, there is only a measurement for computing each input string, performing after reading the last symbol; in contrast, in MM-1QFA, measurement is performed after reading each symbol, instead of only the last symbol. Other one-way QFA include Latvian QFA [3], QFA with control language [12], Ancilla QFA [30], and one-way quantum finite automata together with classical states (1QFAC) [35], etc [10, 34].

MO-1QFA have advantages over crisp finite automata in state complexity for recognizing some languages [10, 34]. Mereghetti and Palano et al. [27] realized an MO-1QFA with optic implementation and the state complexity of this MO-1QFA has exponential advantages over DFA and NFA as well as PFA [31]. MM-1QFA have stronger computing power than MO-1QFA [13], but both MO-1QFA and MM-1QFA accept with bounded error only proper subsets of regular languages. Indeed, Brodsky and Pippenger [13] proved that the languages accepted by MO-1QFA with bounded error are exactly reversible languages that are accepted by reversible finite automata (RFA). RFA have three different definitions and were named as group automata, BM-reversible automata, and AF-reversible automata (see [33]), respectively. In particular, these three definitions are proved to be equivalent in [33].

The remainder of the paper is organized as follows. In Section 2, in the interest of readability, we first introduce basics in quantum computing, then one-way QFA are recalled and we focus on reviewing MO-1QFA and MM-1QFA. The main contributions are in Sections 3 and 4. In Section 3, we first show the appropriate oracle to be used, that is, \textit{AA} is not strong enough for learning RFA and MO-1QFA with polynomial time, and a more powerful oracle (named as \textit{AD} oracle) is thus employed. With \textit{AD} oracle we design an algorithm for learning MO-1QFA with polynomial time, and the correctness and complexity of algorithm are proved and analyzed in detail. Afterwards, in Section 4 we continue to design an algorithm for learning MM-1QFA with polynomial time. Finally, the main results are summarized in Section 5, and further problems are mentioned for studying.

2. Preliminaries on quantum computing and QFA

For the sake of readability, in this section we outline basic notations and principles in quantum computing and review the definitions of MO-1QFA, MM-1QFA, and RFA. For more details, we can refer to [26] and [7, 10, 34, 38].
2.1. Basics in quantum computing

Let $\mathbb{C}$ denote the set of all complex numbers, $\mathbb{R}$ the set of all real numbers, and $\mathbb{C}^{n \times m}$ the set of $n \times m$ matrices having entries in $\mathbb{C}$. Given two matrices $A \in \mathbb{C}^{n \times m}$ and $B \in \mathbb{C}^{p \times q}$, their tensor product is the $np \times mq$ matrix, defined as

$$A \otimes B = \begin{bmatrix}
A_{11}B & \cdots & A_{1m}B \\
\vdots & \ddots & \vdots \\
A_{n1}B & \cdots & A_{nm}B
\end{bmatrix}. $$

$(A \otimes B)(C \otimes D) = AC \otimes BD$ holds if the multiplication of matrices is satisfied.

If $MM^\dagger = M^\dagger M = I$, then matrix $M \in \mathbb{C}^{n \times n}$ is unitary, where $\dagger$ denotes conjugate-transpose operation. $M$ is said to be Hermitian if $M = M^\dagger$. For $n$-dimensional row vector $x = (x_1, \ldots, x_n)$, its norm $\|x\|$ is defined as $\|x\| = (\sum_{i=1}^n x_i^*)^2$, where symbol $^*$ denotes conjugate operation. Unitary operations preserve the norm, i.e., $\|xM\| = \|x\|$ for each $x \in \mathbb{C}^{1 \times n}$ and any unitary matrix $M \in \mathbb{C}^{n \times n}$.

According to the basic principles of quantum mechanics [29], a state of quantum system can be described by a unit vector in a Hilbert space. More specifically, let $B = \{q_1, \ldots, q_n\}$ associated with a quantum system denote a basic state set, where every basic state $q_i$ can be represented by an $n$-dimensional row vector $\langle q_i | = (0 \ldots 1 \ldots 0)$ having only 1 at the $i$th entry (where $\langle \cdot |$ is Dirac notation, i.e., bra-ket notation).

At any time, the state of this system is a superposition of these basic states and can be represented by a row vector $\langle \phi | = \sum_{i=1}^n c_i \langle q_i |$ with $c_i \in \mathbb{C}$ and $\sum_{i=1}^n |c_i|^2 = 1$; $\langle \phi |$ represents the conjugate-transpose of $\langle \phi |$. So, the quantum system is described by Hilbert space $\mathcal{H}_Q$ spanned by the base $\{|q_i| : i = 1, 2, \ldots, n\}$, i.e. $\mathcal{H}_Q = \text{span}\{|q_i| : i = 1, 2, \ldots, n\}$.

The state evolution of quantum system complies with unitarity. Suppose the current state of system is $|\phi \rangle$. If it is acted on by some unitary matrix (or unitary operator) $M_1$, then $|\phi \rangle$ is changed to the new current state $M_1|\phi \rangle$; if the second unitary matrix, say $M_2$, is acted on $M_1|\phi \rangle$, then $M_1|\phi \rangle$ is changed to $M_2M_1|\phi \rangle$. So, after a series of unitary matrices $M_1, M_2, \ldots, M_k$ are performed in sequence, the system’s state becomes $M_kM_{k-1} \cdots M_1|\phi \rangle$.

To get some information from the quantum system, we need to make a measurement on its current state. Here we consider projective measurement (i.e. von Neumann measurement). A projective measurement is described by an observable that is a Hermitian matrix $O = c_1 P_1 + \cdots + c_s P_s$, where $c_i$ is its eigenvalue and, $P_i$ is the projector onto the eigenspace corresponding to $c_i$. In this case, the projective measurement of $O$ has result set $\{c_i\}$ and projector set $\{P_i\}$. For example, given state $|\psi \rangle$ is made by the measurement $O$, then the probability of obtaining result $c_i$ is $\|P_i|\psi \rangle\|^2$ and the state $|\psi \rangle$ collapses to $P_i|\psi \rangle \|P_i|\psi \rangle\|$. 

2.2. Review of one-way QFA and RFA

For non-empty set $\Sigma$, by $\Sigma^*$ we mean the set of all finite length strings over $\Sigma$, and $\Sigma^n$ denotes the set of all strings over $\Sigma$ with length $n$. For $u \in \Sigma^*$, $|u|$ is the length of $u$; for example, if $u = x_1x_2 \ldots x_m \in \Sigma^*$
where \( x_i \in \Sigma \), then \(|u| = m\). For set \( S \), \(|S|\) denotes the cardinality of \( S \).

2.2.1. MO-1QFA

We recall the definition of MO-1QFA. An MO-1QFA with \( n \) states and input alphabet \( \Sigma \) is a five-tuple

\[
\mathcal{M} = (Q, |\psi_0\rangle, \{U(\sigma)\}_{\sigma \in \Sigma}, Q_a, Q_r)
\]

(1)

where

- \( Q = \{|q_1\rangle,\ldots,|q_n\rangle\} \) consist of an orthonormal base that spans a Hilbert space \( \mathcal{H}_Q \) (\( |q_i\rangle \) is identified with a column vector with the \( i \)th entry 1 and the others 0); at any time, the state of \( \mathcal{M} \) is a superposition of these basic states;
- \( |\psi_0\rangle \in \mathcal{H} \) is the initial state;
- for any \( \sigma \in \Sigma \), \( U(\sigma) \in \mathbb{C}^{n \times n} \) is a unitary matrix;
- \( Q_a, Q_r \subseteq Q \) with \( Q_a \cup Q_r = Q \) and \( Q_a \cap Q_r = \emptyset \) are the accepting and rejecting states, respectively, and it describes an observable by using the projectors \( P(a) = \sum_{|q_i\rangle \in Q_a} |q_i\rangle \langle q_i| \) and \( P(r) = \sum_{|q_i\rangle \in Q_r} |q_i\rangle \langle q_i| \), with the result set \( \{a, r\} \) of which ‘a’ and ‘r’ denote “accepting” and “rejecting”, respectively. Here \( Q \) consists of accepting and rejecting sets.

Given an MO-1QFA \( \mathcal{M} \) and an input word \( s = x_1 \ldots x_n \in \Sigma^* \), then starting from \( |\psi_0\rangle \), \( U(x_1), \ldots, U(x_n) \) are applied in succession, and at the end of the word, a measurement \( \{P(a), P(r)\} \) is performed with the result that \( \mathcal{M} \) collapses into accepting states or rejecting states with corresponding probability. Hence, the probability \( L_M(x_1 \ldots x_n) \) of \( \mathcal{M} \) accepting \( w \) is defined as:

\[
L_M(x_1 \ldots x_n) = \|P(a)U_s|\psi_0\rangle\|^2
\]

(2)

where we denote \( U_s = U_{x_n}U_{x_{n-1}} \ldots U_{x_1} \).

2.2.2. RFA

Now we recollect reversible finite automata (RFA). As mentioned above, there are three equivalent definitions for RFA \[33\], that is group automata, BM-reversible automata, and AF-reversible automata. Here we describe group automata. First we review DFA. A DFA \( G = (S, s_0, \Sigma, \delta, S_a) \), where \( S \) is a finite state set, \( s_0 \in S \) is its initial state, \( S_a \subseteq S \) is its accepting state set, \( \Sigma \) is and input alphabet, and \( \delta \) is a transformation function, i.e., a mapping \( \delta : S \times \Sigma \to S \).

An RFA \( G = (S, s_0, \Sigma, \delta, S_a) \) is DFA and satisfies that for any \( q \in S \) and any \( \sigma \in \Sigma \), there is unique \( p \in S \) such that \( \delta(p, \sigma) = q \).

The languages accepted by MO-1QFA with bounded error is exactly the languages accepted by RFA \[13\]. In fact, RFA are the special cases of MO-1QFA, and this is showed by the following proposition.
**Proposition 1.** For any MO-1QFA \( M = (Q, |\psi_0\rangle, \{U(\sigma)\}_{\sigma \in \Sigma}, Q_a, Q_r) \) with \( |\psi_0\rangle \in Q \), if all entries in \( U(\sigma) \) for each \( \sigma \in \Sigma \) are either 0 or 1, then \( M \) is actually a group automaton.

**Proof.** Suppose the base states \( Q = \{|q_i\rangle : i = 1, 2, \ldots, n\} \), where \( |q_i\rangle \) is an \( n \)-dimensional column vector with the \( i \)th entry 1 and the others 0. Let \( |\psi_0\rangle = |q_{i_0}\rangle \) for some \( i_0 \in \{1, 2, \ldots, n\} \). It is clear that \( U(\sigma) \) (for each \( \sigma \in \Sigma \)) is a permutation matrix and therefore \( U(\sigma) \) is also a bijective mapping from \( Q \) to \( Q \). So, \( M \) is a group automaton.

\( \square \)

### 2.2.3. MM-1QFA

We review the definition of MM-1QFA. Formally, given an input alphabet \( \Sigma \) and an end-maker \( \$ \notin \Sigma \), an MM-1QFA with \( n \) states over the working alphabet \( \Gamma = \Sigma \cup \{\$\} \) is a six-tuple \( M = (Q, |\psi_0\rangle, \{U(\sigma)\}_{\sigma \in \Gamma}, Q_a, Q_r, Q_g) \), where

- \( Q, |\psi_0\rangle \), and \( U(\sigma) \ (\sigma \in \Gamma) \) are defined as in the case of MO-1QFA, \( Q_a, Q_r, Q_g \) are disjoint to each other and represent the accepting, rejecting, and going states, respectively.
- The measurement is described by the projectors \( P(a), P(r) \) and \( P(g) \), with the results in \( \{a, r, g\} \) of which ‘\( a \)’, ‘\( r \)’ and ‘\( g \)’ denote “accepting”, “rejecting” and “going on”, respectively.

Any input word \( w \) to MM-1QFA is in the form: \( w \in \Sigma^* \$, with symbol \$ \) denoting the end of a word. Given an input word \( x_1 \ldots x_n \$ \) where \( x_1 \ldots x_n \in \Sigma^n \), MM-1QFA \( M \) performs the following computation:

1. Starting from \( |\psi_0\rangle \), \( U(x_1) \) is applied, and then we get a new state \( |\phi_1\rangle = U(x_1)|\psi_0\rangle \). In succession, a measurement of \( O \) is performed on \( |\phi_1\rangle \), and then the measurement result \( i \ (i \in \{a, g, r\}) \) is yielded as well as a new state \( |\phi_i^1\rangle = \frac{P(i)|\phi_1\rangle}{\sqrt{p_i}} \) is obtained, with corresponding probability \( p_i^1 = ||P(i)|\phi_1\rangle||^2 \).

2. In the above step, if \( |\phi_g^2\rangle \) is obtained, then starting from \( |\phi_g^2\rangle \), \( U(x_2) \) is applied and a measurement \( \{P(a), P(r), P(g)\} \) is performed. The evolution rule is the same as the above step.

3. The process continues as far as the measurement result ‘\( g \)’ is yielded. As soon as the measurement result is ‘\( a \)’ (‘\( r \)’), the computation halts and the input word is accepted (rejected).

Thus, the probability \( L_M(x_1 \ldots x_n) \) of \( M \) accepting \( w \) is defined as:

\[
L_M(x_1 \ldots x_n) = \sum_{k=1}^{n+1} ||P(a)U(x_k)\prod_{i=1}^{k-1} (P(g)U(x_i))|\psi_0\rangle||^2,
\]

\[
(3)
\]
or equivalently,

\[ L_{\lambda_k} (x_1 \ldots x_n) \]

\[ = \sum_{k=0}^{n} ||P(a)U(x_{k+1}) \prod_{i=1}^{k} (P(g)U(x_i))|\psi_0>|^2, \]

where, for simplicity, we can denote $ by $ x_{n+1} $ if no confusion results.

3. Learning MO-1QFA

First we recall a definition concerning model learning with an oracle in polynomial time.

**Definition 1.** Let $ R $ be a class to be learned and $ O_R $ be an oracle for $ R $. Then $ R $ is said to be polynomially learnable using the oracle $ O_R $ if there is a learning algorithm $ L $ and a two-variable polynomial $ p $ such that for every target $ r \in R $ of size $ n $ to be learned, $ L $ runs in time $ p(n, m) $ at any point and outputs a hypothesis that is equivalent to $ r $, where $ m $ is the maximum length of data returned by $ O_R $ so far in the run.

In order to learn a model with polynomial time via an oracle, we hope this oracle is as weaker as possible. For learning MO-1QFA, suppose an oracle can only answer the amplitudes of accepting states for each input string, then can we learn MO-1QFA successfully with polynomial time via such an oracle? We name such an oracle as $ AA $ oracle. For clarifying this point, we try to use $ AA $ oracle to learning DFA. In this case, $ AA $ oracle can answer if it is either an accepting state or a rejecting state for each input string. Equivalently, $ AA $ oracle is exactly membership query for learning DFA as the target model. Therefore, learning DFA via $ AA $ oracle is not polynomial by virtue of the following Angluin’s result [2].

**Theorem 1.** DFA are not polynomially learnable using the membership oracle only.

Since RFA are special DFA and MO-1QFA, we have the following proposition.

**Proposition 2.** DFA and RFA as well as MO-1QFA are not polynomially learnable using $ AA $ oracle only.

So, we consider a stronger oracle, named as $ AD $ oracle that can answer all amplitudes (instead of the amplitudes of accepting states only) of the superposition state for each input string. For example, for quantum state $ |\psi\rangle = \sum_{i=1}^{n} \alpha_i |q_i\rangle $ where $ \sum_{i=1}^{n} |\alpha_i|^2 = 1 $, $ AA $ oracle can only answer the amplitudes of accepting states in $ \{|q_1\rangle, |q_2\rangle, \ldots, |q_n\rangle\} $, but $ AD $ oracle can answer the amplitudes for all states in $ \{|q_1\rangle, |q_2\rangle, \ldots, |q_n\rangle\} $. Using $ AD $ oracle, we can prove that MO-1QFA and MM-1QFA are polynomially learnable. Therefore, for learning DFA or RFA, $ AD $ oracle can answer a concrete state for each input string, where the concrete state is the output state of the target automaton to be learned.

First we can easily prove that RFA are linearly learnable via $ AD $ oracle, and this is the following proposition.
Proposition 3. Let $RFA \ G = (S, s_0, \Sigma, \delta, S_a)$ be the target to be learned. Then $G$ is linearly learnable via using $AD$ oracle with query complexity at most $|S||\Sigma|$.

**Proof.** First, $AD$ oracle can answer the initial state $s_0$ via inputting empty string. Then by taking $s_0$ as a vertex, we use pruning algorithm of decision tree to obtain all states in $G$ while accepting states are marked as well. It is easy to know that the query complexity is $O(|S||\Sigma|)$.

Our main concern is whether $AD$ oracle is too strong, that is to say, whether $AD$ oracle possesses too much information for our learning tasks. To clarify this point, we employ a consistency problem that, in a way, demonstrates $AD$ oracle is really not too strong for our model learning if the time complexity is polynomial. So, by virtue of similar ideas and methods of proof in [39] where the deterministic problems are for DFA and PFA, respectively, we have the following two propositions.

Proposition 4. For any alphabet $\Sigma$ and finite set $S = \{q_1, q_2, \ldots, q_n\}$, the following problem is NP-complete: Given a set $D$ with $D \subseteq \Sigma^* \times S$, determine whether there is a $RFA \ G = (S_1, s_0, \Sigma, \delta, S_a)$ such that for any $(x, q) \in D$, $\delta(s_0, x) = q$, where $p \in S_1$ if and only if $(x, p) \in D$ for some $x \in \Sigma^*$.

Proposition 5. For any alphabet $\Sigma$ and finite set $S = \{q_1, q_2, \ldots, q_n\}$, the following problem is NP-complete: Let $Q(S) = \{\sum_{i=1}^n c_i q_i : \sum_{i=1}^n |c_i|^2 = 1, c_i \in \mathbb{C}, i = 1, 2, \ldots, n\}$. Given a set $D$ with $D \subseteq \Sigma^* \times Q(S)$, determine whether there is an $MO-1QFA \ M = (Q, |\psi_0\rangle, \{U(\sigma)\}_{\sigma \in \Sigma}, Q_a, Q_r)$ such that for any $(x, \sum_{i=1}^n c_i q_i) \in D$, $\langle \psi_0 | U(x) = (c_1, c_2, \ldots, c_n)$.

From the above two propositions we use $AD$ oracle for learning $MO-1QFA$ and $MM-1QFA$. Let $M = (Q, |\psi_0\rangle, \{U(\sigma)\}_{\sigma \in \Sigma}, Q_a, Q_r)$ be the target $MO-1QFA$ to be learned, where, as the case of learning PFA [39], the learner is supposed to have the information of $Q, Q_a, Q_r$, but the other parameters are to be learned by mean of querying the oracle for achieving an equivalent $MO-1QFA$ (more concretely, for each $\sigma \in \Sigma$, unitary matrix $V(\sigma)$ corresponding to $U(\sigma)$ needs to be determined, but it is possible that $V(\sigma) \neq U(\sigma)$). For any $x \in \Sigma^*$, $AD(x)$ can answer an amplitude distribution that is exactly equivalent to a state of superposition corresponding to the input string $x$, more exactly, $AD(x)$ can answer the same state as $U(\sigma_k)U(\sigma_{k-1})\cdots U(\sigma_1)|\psi_0\rangle$ where $x = \sigma_1\sigma_2\cdots\sigma_k$. From now on, we denote $U(x) = U(\sigma_k)U(\sigma_{k-1})\cdots U(\sigma_1)$ for $x = \sigma_1\sigma_2\cdots\sigma_k$.

We outline the basic idea and method for designing the learning algorithm of $MO-1QFA \ M$. First, the initial state can be learned from $AD$ oracle by querying empty string $\varepsilon$. Then by using $AD$ oracle we continue to search for a base of the Hilbert space spanned by $\{v^* = U(x)|\psi_0\rangle : x \in \Sigma^*\}$. This procedure will be terminated since the dimension of the space is at most $|Q|$. In fact, we can prove this can be finished in polynomial time. Finally, by virtue of the learned base and solving groups of linear equations we can
conclude $V(\sigma)$ for each $\sigma \in \Sigma$. We prove these results in detail following the algorithm, and now present Algorithm 1 for learning MO-1QFA as follows.

**Algorithm 1** Algorithm for learning MO-1QFA $\mathcal{M} = (Q, |\psi_0\rangle, \{U(\sigma)\}_{\sigma \in \Sigma}; Q_a, Q_r)$

1: $|\psi_0^*\rangle \leftarrow AD(\varepsilon)$;
2: Set $B$ to be the empty set;
3: Set $Nod \leftarrow \{\text{node}(\varepsilon)\}$;
4: while $Nod$ is not empty do
5: begin Take an element $\text{node}(x)$ from $Nod$;
6: \quad $v^*(x) \leftarrow AD(x)$;
7: \quad if $v^*(x) \notin \text{span}(B)$ then
8: \quad \quad begin Add $\text{node}(x^\sigma)$ to $Nod$ for all $\sigma \in \Sigma$;
9: \quad \quad $B \leftarrow B \cup \{v^*(x)\}$ end;
10: \quad end if
11: end while
12: end;
13: Let $V(\sigma) = [x_{ij}(\sigma)]$ for any $\sigma \in \Sigma$, $1 \leq i, j \leq n$;
14: Define a linear system:
15: \quad for any $v^*(x) \in B$ and any $\sigma \in \Sigma$, $V(\sigma)v^*(x) = v^*(x^\sigma) = AD(x^\sigma)$,
16: \quad for $1 \leq i_1, i_2 \leq n$, if $i_1 \neq i_2$, then $\sum_{j=1}^{n} x_{i_1,j}(\sigma)x_{i_2,j}(\sigma) = 0$; otherwise, it is 1,
17: Find a suitable solution for $x_{ij}(\sigma)$’s,
18: if there is a solution then
19: \quad return $\mathcal{M}^* = (Q, |\psi_0^*\rangle, \{V(\sigma)\}_{\sigma \in \Sigma}; Q_a, Q_r)$
20: else
21: \quad return (not exist);
22: end if

Next we prove the correctness of Algorithm 1 and then analyze its complexity. First we prove that Step 1 to Step 12 in Algorithm 1 can produce a set of vectors $B$ consisting of a base of space spanned by \{v^*(x)|x \in \Sigma^*\}, where $v^*(x) = AD(x)$ is actually the vector replied by oracle $AD$ for input string $x$, that is $v^*(x) = AD(x) = U(\sigma_1)U(\sigma_{k-1})\ldots U(\sigma_1)|\psi_0\rangle$, for $x = \sigma_1\sigma_2\ldots \sigma_k$.

**Proposition 6.** In Algorithm 1 for learning MO-1QFA, the final set of vectors $B$ consists of a base of Hilbert space span\{v^*(x)|x \in \Sigma^*\} that is spanned by \{v^*(x)|x \in \Sigma^*\}.

**Proof.** From the algorithm procedure we can assume that $B = \{v^*(x_1), v^*(x_2),\ldots, v^*(x_m)\}$ for some $m$, where it is clear that some $x_i$ equals to $\varepsilon$, and for any $x \in \Sigma^*$, there are $x_j$ and $y \in \Sigma^*$ such that $x = x_jy$. 
The rest is to show that $v^*(x)$ can be linearly represented by the vectors in $B$ for any $x \in \Sigma^*$. Let $x = x_j y$ for some $x_j$ and $y \in \Sigma^*$. By induction on the length $|y|$ of $y$. If $|y| = 0$, i.e., $y = \varepsilon$, then it is clear for $x = x_j$. If $|y| = 1$, then due to the procedure of algorithm, $v^*(x_j y)$ is linearly dependent on $B$. Suppose that it holds for $|y| = k \geq 0$. Then we need to verify it holds for $|y| = k + 1$. Denote $y = z \sigma$ with $|z| = k$. Then with induction hypothesis we have $v^*(x_j z) = \sum_k c_k v^*(x_k)$. Therefore we have

$$
v^*(x) = v^*(x_j z \sigma) = U(\sigma)v^*(x_j z) = U(\sigma)\sum_k c_k v^*(x_k) = \sum_k c_k v^*(x_k \sigma). \hspace{1cm} (7)
$$

Since $v^*(x_k \sigma)$ is linearly dependent on $B$ for $k = 1, 2, \ldots, m$, the proof is completed.

The purpose of Algorithm 1 is to learn the target MO-1QFA $M = (Q, |\psi_0\rangle, \{U(\sigma)\}_{\sigma \in \Sigma}, Q_a, Q_r)$, so we need to verify $M^* = (Q, |\psi_0^*\rangle, \{V(\sigma)\}_{\sigma \in \Sigma}, Q_a, Q_r)$ obtained is equivalent to $M$. For this it suffices to check $V(x)|\psi_0^*\rangle = U(x)|\psi_0\rangle$ for any $x \in \Sigma^*$, where $V(x) = V(\sigma_s)V(\sigma_{s-1})\ldots V(\sigma_1)$ and $U(x) = U(\sigma_s)U(\sigma_{s-1})\ldots U(\sigma_1)$ for $x = \sigma_1 \sigma_2 \ldots \sigma_s$.

**Theorem 2.** In Algorithm 1 for learning MO-1QFA, for any $x \in \Sigma^*$,

$$
V(x)|\psi_0^*\rangle = U(x)|\psi_0\rangle. \hspace{1cm} (8)
$$

**Proof.** For $x = \varepsilon$, $V(\varepsilon) = U(\varepsilon) = I$, and $|\psi_0^*\rangle = AD(\varepsilon) = |\psi_0\rangle$, so it holds.

For any $\sigma \in \Sigma$ and for any $v^*(x) \in B$, according to Algorithm 1, we have $V(\sigma)v^*(x) = v^*(x \sigma) = AD(x \sigma)$. In particular, taking $x = \varepsilon$, then we have $V(\sigma)|\psi_0^*\rangle = v^*(\sigma) = AD(\sigma) = U(x)|\psi_0\rangle$.

Suppose it holds for $|x| = k$. The rest is to prove that it holds for $|x| = k + 1$. Denote $y = x \sigma$ where $|x| = k$ and $\sigma \in \Sigma$. Due to Proposition $\Box$, $v^*(x)$ can be linearly represented by $B = \{v^*(x_1), v^*(x_2), \ldots, v^*(x_m)\}$, i.e., $v^*(x) = \sum_k c_k v^*(x_k)$ for some $c_k \in \mathbb{C}$. With the induction hypothesis, $V(x)|\psi_0^*\rangle = U(x)|\psi_0\rangle$ holds.
Then by means of Algorithm 1 we have

\[ V(y|\psi_0^*) = V(x|\psi_0^*) \]
\[ = V(\sigma)V(x|\psi_0^*) \]
\[ = V(\sigma)U(x)|\psi_0 \rangle \]
\[ = V(\sigma)v^*(x) \]
\[ = V(\sigma)\sum_k c_kv^*(x_k) \]
\[ = \sum_k c_kV(\sigma)v^*(x_k) \]
\[ = \sum_k c_kv^*(x_k\sigma). \]  

(9)

On the other hand, since \( v^*(z) = AD(z) = U(z)|\psi_0 \rangle \) for any \( z \in \Sigma^* \), we have

\[ \sum_k c_kv^*(x_k\sigma) = \sum_k c_kU(x_k\sigma)|\psi_0 \rangle \]
\[ = \sum_k c_kU(\sigma)U(x_k)|\psi_0 \rangle \]
\[ = \sum_k c_kU(\sigma)v^*(x_k) \]
\[ = U(\sigma)\sum_k c_kv^*(x_k) \]
\[ = U(\sigma)v^*(x) \]
\[ = U(x\sigma)|\psi_0 \rangle \]
\[ = U(y)|\psi_0 \rangle. \]  

(10)

So, the proof is completed.

\[ \square \]

From Theorem 2 it follows that Algorithm 1 returns an equivalent MO-1QFA to the target MO-1QFA \( M = (Q, |\psi_0 \rangle, \{U(\sigma)\}_{\sigma \in \Sigma}, Q_a, Q_r) \) to be learned. Next we analyze the computational complexity of Algorithm 1.

**Proposition 7.** Let the target MO-1QFA to be learned have \( n \)'s bases states. The the computational complexity of Algorithm 1 is \( O(n^5|\Sigma|) \).

**Proof.** We consider it from two parts.
(I) The first part of Algorithm 1 to get $B$: The complexity to determine the linear independence of some $n$-dimensional vectors is $O(n^3)$ \[16\], and there are at most $n$ time to check this, so the first part of Algorithm 1 to get $B$ needs time $O(n^4)$.

(II) The second part of finding the feasible solutions for $V(\sigma)$ for each $\sigma \in \Sigma$: For any $\sigma \in \Sigma$, Step 15 defines $|B|$’s matrix equations and these equations are clearly equivalent to a group of linear equations, but are subject to the restriction conditions in Step 16. So, this part is actually a problem of linear programming and we can refer to \[15, \text{ and } 20\] to get the time complexity is $O(n^5|\Sigma|)$.

Therefore, by combining (I) and (II) we have the complexity of Algorithm 1 is $O(n^5|\Sigma|)$.

4. Learning MM-1QFA

In this section, we study learning MM-1QFA via $\text{AD}$ oracle. Let $\mathcal{M} = (Q, |\psi_0\rangle, \{U(\sigma)\}_{\sigma \in \Gamma}, Q_a, Q_r, Q_g)$ be the target QFA to be learned, where $\Gamma = \Sigma \cup \{\$\}$, and $\$ \notin \Sigma$ is an end-maker. As usual, $Q, Q_a, Q_r, Q_g$ are supposed to be known, and the goal is to achieve unitary matrices $V(\sigma)$ for each $\sigma \in \Gamma$ in order to get an equivalent MM-1QFA $\mathcal{M}^* = (Q, |\psi_0\rangle, \{V(\sigma)\}_{\sigma \in \Gamma}, Q_a, Q_r, Q_g)$. $\text{AD}$ oracle can answer an amplitude distribution $\text{AD}(x)$ for any $x \in \Gamma^*$. MM-1QFA performs measuring after reading each input symbol, and then only the non-halting (i.e. going on) states continues to implement computing for next step, and the amplitude distribution for the superposition state after performing each unitary matrix needs to be learned from oracle.

Therefore, for any $x = \sigma_1\sigma_2\ldots\sigma_k \in \Gamma^*$, since MM-1QFA $\mathcal{M}$ outputs the following state (un-normalized form) as the current state:

$$U(\sigma_k)P_nU(\sigma_{k-1})P_n\ldots U(\sigma_1)P_n|\psi_0\rangle,$$

we require $\text{AD}$ oracle can answer $\text{AD}(x) = U(\sigma_k)P_nU(\sigma_{k-1})P_n\ldots U(\sigma_1)P_n|\psi_0\rangle$. In particular, $\text{AD}(\varepsilon) = |\psi_0\rangle$.

Before presenting the algorithm of learning MM-1QFA, we describe the main ideas and procedure.

First the initial state can be learned from $\text{AD}$ oracle via querying empty string $\varepsilon$.

Then by using $\text{AD}$ oracle we are going to search for a base $\mathcal{B}$ of the Hilbert space spanned by $\{v^*(x) : x \in \Sigma^*\}$ where for any $x = \sigma_1\sigma_2\ldots\sigma_k \in \Sigma^*$,

$$v^*(x) = \text{AD}(x) = U(\sigma_k)P_nU(\sigma_{k-1})P_n\ldots U(\sigma_1)P_n|\psi_0\rangle.$$

This procedure will be terminated due to the finite dimension of the space (at most $|Q|$), and this can be completed with polynomial time.

Finally, by combining the base $\mathcal{B}$ and with groups of linear equations we can obtain $V(\sigma)$ for each $\sigma \in \Sigma$. These results can be verified in detail after Algorithm 2, and we now present Algorithm 2 for learning MM-1QFA in the following.

\[12\]
Algorithm 2 Algorithm for learning MM-1QFA $M = (Q, |\psi_0\rangle, \{U(\sigma)\}_{\sigma \in \Gamma}, Q_a, Q_r, Q_g)$

1. $|\psi_0\rangle \leftarrow AD(\varepsilon); v^*(\$) \leftarrow AD(\$)$;
2. Set $B$ to be the empty sets;
3. Set $Nod \leftarrow \{node(\varepsilon)\};$
4. while $Nods$ not empty do
5. begin Take an element $node(x)$ from $Nod;$
6. $v^*(x) \leftarrow AD(x);$
7. if $v^*(x) \notin \text{span}(B)$ then
8. begin Add $node(x\sigma)$ to $Nod$ for all $\sigma \in \Sigma;$
9. $B \leftarrow B \cup \{v^*(x)\}$ end;
10. end if
11. end while
12. end;
13. Let $V(\sigma) = [x_{ij}(\sigma)]$ for any $\sigma \in \Sigma \cup \{\$\}, 1 \leq i, j \leq n;$
14. Define linear systems:
15. for any $v^*(x) \in B$ and any $\sigma \in \Sigma \cup \{\$\}, $V(\sigma)P_n v^*(x) = v^*(x\sigma) = AD(x\sigma);$  
16. for $1 \leq i_1, i_2 \leq n,$ if $i_1 \neq i_2,$ then $\sum_{j=1}^{n} x_{i_1,j}(\sigma)x_{i_2,j}(\sigma) = 0;$ otherwise, it is 1,
17. Find a suitable solution for $x_{ij}(\sigma)$’s, and denote $V = \{V(\sigma) : \sigma \in \Sigma \cup \{\$\}\};$
18. if there is a solution then
19. return $M^* = (Q, |\psi_0\rangle, \{V(\sigma)\}_{\sigma \in \Sigma \cup \{\$\}}, Q_a, Q_r, Q_g)$
20. else
21. return (not exist);
22. end if
Next we first demonstrate that the algorithm can find out a base $B$ for Hilbert space $\text{span}\{v^*(x)|x \in \Sigma^*\}$.

**Proposition 8.** In Algorithm 2 for learning MM-1QFA, the final set of vectors $B$ consists of a base of Hilbert space $\text{span}\{v^*(x)|x \in \Sigma^*\}$, where $v^*(x)$ is actually the vector replied by oracle $AD$ for input string $x$, that is $v^*(x) = AD(x) = U(\sigma_k)P_nU(\sigma_{k-1})P_n \ldots U(\sigma_1)P_n|\psi_0\rangle$, for $x = \sigma_1\sigma_2\ldots\sigma_k \in \Sigma^*$.

**Proof.** Suppose that $B = \{v^*(x_1), v^*(x_2), \ldots, v^*(x_m)\}$, where it is clear that some $x_i$ equals to $\varepsilon$. So, for any $x \in \Sigma^*$, there are $x_j$ and $y \in \Sigma^*$ such that $x = x_zy$. It suffices to show that $v^*(x)$ can be linearly represented by the vectors in $B$. By induction on the length $|y|$ of $y$. If $|y| = 0$, i.e., $y = \varepsilon$, then it is obvious for $x = x_j$. In addition, for $|y| = 1$, $v^*(x_zy)$ is linearly dependent on $B$ in terms of the algorithm’s operation. Suppose that it holds for $|y| = k \geq 0$. Then we need to verify it holds for $|y| = k + 1$. Denote $y = z\sigma$ with $|z| = k$. Then by induction hypothesis $v^*(x_zy) = \sum_k c_kv^*(x_k)$ for some $c_k \in \mathbb{C}$ with $k = 1, 2, \ldots, m$. Therefore we have

$$v^*(x) = v^*(x_jz\sigma) = U(\sigma)P_nv^*(x_zy) = U(\sigma)P_n\sum_k c_kv^*(x_k) = \sum_k c_kU(\sigma)P_nv^*(x_k) = \sum_k c_kv^*(x_k\sigma).$$

(13)

Since $v^*(x_k\sigma)$ is linearly dependent on $B$ for $k = 1, 2, \ldots, m$, $v^*(x)$ can be linearly represented by the vectors in $B$ and the proof is completed.

\[\square\]

Then we need to verify that the MM-1QFA $M^*$ obtained in Algorithm 2 is equivalent to the target MM-1QFA $M$. This can be achieved by checking $V(\$)P_n|\psi_0\rangle = U(\$)P_n|\psi_0\rangle$ and for any $x = \sigma_1\sigma_2\ldots\sigma_k \in \Sigma^*$,

$$V(\sigma_k)P_nV(\sigma_{k-1})P_n \ldots V(\sigma_1)P_n|\psi_0\rangle = U(\sigma_k)P_nU(\sigma_{k-1})P_n \ldots U(\sigma_1)P_n|\psi_0\rangle.$$  

(14)

So we are going to prove the following theorem.

**Theorem 3.** In Algorithm 2 for learning MM-1QFA, we have

$$V(\$)P_n|\psi_0\rangle = U(\$)P_n|\psi_0\rangle;$$

and for any $x = \sigma_1\sigma_2\ldots\sigma_k \in \Sigma^*$, Eq. (14) holds, where $|x| \geq 1$.

**Proof.** Note $v^*(\varepsilon) \in B$, by means of Step 15 in Algorithm 2 and taking $\sigma = \$\$, we have

$$V(\$)P_nv^*(\varepsilon) = v^*(\$) = AD(\$) = U(\$)P_n|\psi_0\rangle.$$  

(16)
Since \( v^*(\varepsilon) = AD(\varepsilon) = |\psi_0\rangle \), and from Algorithm 2 we know \( AD(\varepsilon) = |\psi_0\rangle \), Eq. (15) holds.

Next we prove that Eq. (14) holds for any \( x = \sigma_1 \sigma_2 \ldots \sigma_k \in \Sigma^* \). We do it by induction method on the length of \(|x|\).

If \(|x| = 1\), say \( x = \sigma \in \Sigma \), then with Step 15 in Algorithm 2 and taking \( v^*(\varepsilon) \), we have
\[
V(\sigma)P_nv^*(\varepsilon) = AD(\sigma) = U(\sigma)P_n|\psi_0\rangle,
\]
so, Eq. (14) holds for \(|x| = 1\) due to \( v^*(\varepsilon) = |\psi_0\rangle \).

Assume that Eq. (14) holds for any \(|x| = k \geq 1\). The rest is to prove that Eq. (14) holds for any \(|x| = k + 1\). Let \( x = y\sigma \) with \( y = \sigma_1 \sigma_2 \ldots \sigma_k \). Suppose \( v^*(y) = \sum_i c_i v^*(x_i) \) for some \( c_k \in \mathbb{C} \). For each \( i \), by means of Step 15 in Algorithm 2, we have
\[
V(\sigma)P_nv^*(x_i) = AD(x_i\sigma) = U(\sigma)P_nAD(x_i),
\]
and therefore
\[
V(\sigma)P_n \sum_i c_i v^*(x_i) = U(\sigma)P_n \sum_i c_i AD(x_i).
\]
Since \( v^*(x_i) = AD(x_i) \), we further have
\[
V(\sigma)P_n v^*(y) = U(\sigma)P_n v^*(y).
\]
By using \( v^*(y) = U(\sigma_k)P_n U(\sigma_{k-1})P_n \ldots U(\sigma_1)P_n|\psi_0\rangle \), and the above induction hypothesis (i.e., Eq. (14) holds), we have
\[
V(\sigma_{k+1})P_nV(\sigma_k)P_n V(\sigma_{k-1})P_n \ldots V(\sigma_1)P_n|\psi_0\rangle = U(\sigma_{k+1})P_n U(\sigma_{k-1})P_n \ldots U(\sigma_1)P_n|\psi_0\rangle.
\]
Consequently, the proof is completed.

To conclude the section, we give the computational complexity of Algorithm 2.

**Proposition 9.** Let the target MM-1QFA to be learned have \( n \)'s bases states. The computational complexity of Algorithm 2 is \( O(n^5|\Sigma|) \).

**Proof.** It is actually similar to the proof of Proposition 7.

5. Concluding remarks

Quantum finite automata (QFA) are simple models of quantum computing with finite memory, but QFA have significant advantages over classical finite automata concerning state complexity \([7, 10, 38]\), and QFA can be realized physically to a considerable extent \([27]\). As a new topic in quantum learning theory and quantum machine learning, learning QFA via queries has been studied in this paper. As classical model learning \([40]\), we can term it as quantum model learning.
The main results we have obtained are that we have proposed two polynomial-query learning algorithms for measure-once one-way QFA (MO-1QFA) and measure-many one-way QFA (MM-1QFA), respectively. The oracle to be used is an \( AD \) oracle that can answer an amplitude distribution, and we have analyzed that a weaker oracle being only able to answer accepting or rejecting for any inputting string is not enough for learning QFA with polynomial time.

However, we still do not know whether there are a weaker oracle \( Q \) than \( AD \) oracle but by using \( Q \) one can learn MO-1QFA or MM-1QFA with polynomial time. This is an interesting and challenging problem to be solved. Of course, for learning RFA, similar to learning DFA \([1]\), we can get an algorithm of polynomial time by using membership queries (MQ) together with equivalence queries (EQ).

Besides MO-1QFA and MM-1QFA, there are other one-way QFA, including Latvian QFA \([3]\), QFA with control language \([12]\), Ancilla QFA \([30]\), one-way quantum finite automata together with classical states (1QFAC) \([35]\), and others, so one of the further problems worthy of consideration is to investigate learning these QFA via queries.

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