Topological Invariance at Finite Temperature

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Abstract

We examine the thermal behavior of a theory with charged massive vector matter coupled to Chern-Simons gauge field. We obtain a critical temperature $T_c$, at which the effective mass of vector field vanishes, and the system transfers from a symmetry broken phase to topological phase. The phase transition is suggested to be of the zeroth order, as the free energy of the system is discontinuous at $T_c$. Application to the $(2 + 1)$ dimensional quantum gravity is briefly discussed.

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1 Introduction

Symmetries and symmetry breaking serve as a basic principle in modern physics. For instance, it is generally appreciated now that all the four basic interactions in Nature can be formulated as theories with gauge invariance. However, there are still more to be understood, such as how gravity may be quantized, as quantum gravity is believed, for instance, to have dominated for a period (thought short) after big bang and determined the evolution of Universe. There seems a need for larger symmetries to set up a framework for quantum gravity. There has been argument that a fundamental theory of quantum gravity will not involve a space-time metric [1]. A theory depending not on a choice of space-time metric is invariant under diffeomorphism transformation, and is known as topological theory. This topological symmetry rules out any local dynamical degree of freedom, and the number of states of such a system could be as few as one - the vacuum - only.

Topological quantum field theories are studied in the context of low dimensional topology [2]. In particular, the topological quantum Yang-Mill theory is used to interpret the Donaldson theory which is regarded as a key to understanding the four dimensional topology, and thus as a key to understanding the geometry structure of the space-time [3]. In the three dimensions, the topological quantum Chern-Simons gauge theory is shown to provide a natural framework of studying the Jones polynomial of knot theory [4], and to have important implications for two dimensional rational conformal field theories [4][5].

Progress is also made in understanding quantum gravity. It is shown that $(2 + 1)$ dimensional general relativity is equivalent to a topological field theory - the Chern-Simons gauge theory for the Poincare group, and thus, as a quantum field theory, is exactly soluble [6]. Various topological models relating to gravity in different space-time dimensions have been suggested in the literature [6][7][8]. On the other hand, since any theory one uses to describe physics at low energy and/or low temperature involves a spacetime metric and local dynamical degrees of freedom, there must be some means that connects a topological theory and its low energy/temperature limit. In particular, if a topological quantum field theory describes a phase of unbroken diffeomorphism of quantum gravity, a metric and local dynamics might arise as some form of symmetry breaking. Such a possible topological phase transition might happen in the early stage of Universe when it was extremely hot and small in size.
In this Paper, following a recent work [10], we examine a system that exhibits a topological phase transition. The system we work with is a \((2 + 1)\) dimensional one with a charged massive vector matter coupled to a \(U(1)\) gauge field. Both the vector and gauge fields are governed by a Chern-Simons term. While the Chern-Simons gauge field does not carry local dynamical degree of freedom (and is known as a topological field), the vector matter field carries them due to its mass term. Moreover, as the mass term involves a space-time metric, the massive vector field theory and thus the interacting theory are obviously topologically variant. Now, however, if somehow the vector field becomes massless, the local dynamical degrees of freedom, the space-time metric, and the difference between the gauge and matter fields disappear. And then the system, now with non-Abelian gauge symmetry, turns out to be topologically invariant. Namely, a topological phase transition would happen. The mass of vector field naturally serves as the order parameter characterizing the possible transition. To realize it, we heat up the system and see there indeed exists a critical temperature at which the effective vector mass vanishes.

Breaking and restoration of various symmetries at high energy and/or finite temperature have been known for a long time in quantum field theories. Notable theoretical observations include the finite temperature restorations of \(SU(2) \times U(1)\) gauge symmetry for the weak electric interactions [11][12][13] and the deconfinement transition in the quark-gluon plasma (for a review, see for instance [14]). Here we provide an example of phase transition concerning topological and gauge symmetries. The transition, recognized as of the zeroth order, is thus far as we know the most discontinuous - even the free energy density discontinues at the critical temperature. Our conclusions are drawn based on perturbation expansion over the Chern-Simons coupling to the second order. We also observe that, with the thermal fluctuation, the effective Chern-Simons coupling in the symmetry broken phase is just a slowly increasing function of temperature \(T\) with 
\[
g(T = 0) = g_r \quad \text{and} \quad g(T = T_c) = 2g_r
\]
at the second order, as we shall see below. Therefore the perturbation expansion and our conclusions should be reliable in the whole range of temperature \(T\), as long as the renormalized zero temperature Chern-Simons coupling \(g_r\) is small.

In the next section, we review the model in some details with emphasis on symmetries. Section 3 is for deriving the partition function in path integral formalism. In section 4, we calculate one loop corrections to two- and three-point correlation functions. From these we obtain the critical temperature for the transition, and verify
the reliability of perturbation expansion. We then calculate in section 5 free energy density to two loops. From it, the order of topological phase transition is determined. Section 6 is devoted to summarizing our results.

2 The Model

The model of interests is a massive complex vector field $B_\mu$ minimally coupled to a gauge field $a_\mu$ who’s dynamics is governed by a Chern-Simons term,

$$I = \int_\Omega \frac{1}{2} \left( \epsilon_{\mu\nu\lambda} B_\mu^* (\partial_\nu - ig a_\nu) B_\lambda + M B_\mu^* B_\mu + \epsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda \right),$$

(2.1)

where the three space-time manifold $\Omega$ has Lorentzian signature. This theory was studied in a different context (anyon) [15][16]. This is an Abelian gauge field theory, violating parity and time reversal symmetries. Though the $B_\mu$ field involves a Chern-Simons term as well, its nature is rather different from that of $a_\mu$ field, because of its non-vanishing mass term. Under a $U(1)$ transformation parametrized by $\alpha(x, t)$, the Chern-Simons field $a_\mu$ varies as a gauge field

$$a_\mu(x, t) \rightarrow a_\mu(x, t) + \partial_\mu \alpha(x, t),$$

(2.2)

and the massive $B_\mu$ field transforms like a charged matter field

$$B_\mu(x, t) \rightarrow e^{i\alpha(x, t)} B_\mu(x, t).$$

(2.3)

As a consequence of the $U(1)$ symmetry, the Chern-Simons gauge field $a_\mu$ does not carry dynamical degree of freedom, but the massive vector field $B_\mu$ does. To see this, we consider the equations of motion. For the Chern-Simons field $a_\mu$, it is

$$\epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda = j_\mu,$$

(2.4)

where the current $j_\mu = -i \epsilon_{\mu\nu\lambda} B_\mu^* B_\lambda$. From (2.4), one obtains the current conservation

$$\partial_\mu j_\mu = 0.$$  

(2.5)

Moreover, from (2.4), one can solve $a_\mu$ by expressing it in integrals over the current $j_\mu$, with a use of gauge fixing to $a_\mu$. This implies the Chern-Simons field is completely
determined by the matter field. On the other hand, the equation of motion for the massive vector field $B_{\mu}$ is

$$\epsilon_{\mu\nu\lambda}(\partial_\nu - ig_\lambda)B_\lambda + iMB_\mu = 0. \tag{2.6}$$

Acting $(\partial_\mu - ig_\mu)$ on (2.6), for $M \neq 0$, one obtains

$$(\partial_\mu - ig_\mu)B_\mu = 0. \tag{2.7}$$

As $a_\mu$ is not an independent field, (2.7) ensures that one of the three components of $B_\mu$ is eliminated in a covariant way. Another way to see the difference between $a_\mu$ and $B_\mu$ fields is via the canonical approach, in which an independent degree of freedom must be accompanied by a non-vanishing canonical momentum conjugate, and vice versa. Let’s write down the canonical momentum conjugate of the field variable $a_\mu$

$$\pi_\mu = \frac{\delta L}{\delta a_\mu} = \frac{1}{2}\epsilon_0\epsilon_{\mu\lambda}a_\lambda. \tag{2.8}$$

Now one sees $a_0$ is not an independent degree of freedom as $\pi_0 = 0$. Moreover, the gauge transformation (2.2) implies one gauge degree of freedom of $a_\mu$ should be fixed. A simple choice is the axil gauge $a_2 = 0$. Then, as $a_1$’s conjugate $\pi_1 = \frac{1}{2}a_2 = 0$ [see (2.8)], $a_1$ is not an independent degree of freedom either. Therefore, the Chern-Simons field $a_\mu$ has no local dynamical degree of freedom at all. On the other hand, the canonical momentum conjugate of the massive $B_\mu$ field takes the same form as given in (2.8). Therefore $B_0$ is not an independent field variable. However, there is no gauge freedom associated to the massive $B_\mu$ field, and $B_1$ is the independent field variable and $B_2$ the canonical momentum conjugate, or vice versa.

On the other hand, when $M = 0$, the difference between $a_\mu$ and $B_\mu$ disappears, except the latter is complex. Then, for convenience, we set $gB_\mu = A^1_\mu + iA^2_\mu$ and $ga_\mu = A^0_\mu$. Substituting these into (2.4), we arrive at an action

$$I_{CS} = \frac{1}{8\pi\alpha} \int_{\Omega} \epsilon_{\mu\nu\lambda} \left( A^a_\mu \partial_\nu A^a_\lambda + \frac{1}{3} \epsilon^{abc} A^a_\mu A^b_\nu A^c_\lambda \right), \tag{2.9}$$

where $g^2 = 4\pi\alpha$. Carrying no local dynamical degree of freedom, this is known as a topological field theory. Now, the gauge invariance of the theory is not simply $U(1)$, but a non-Abelian one. The gauge group can be $SU(2)$ or $ISO(2,1)$, with structure constant $\epsilon^{abc}$. The variation of $A^a_\mu$ under a gauge transformation is

$$\delta A^a_\mu = D_\mu e^a, \tag{2.10}$$
where \( D_\mu \tau^a = \partial_\mu \tau^a + \epsilon^{abc} A^b_\mu \tau^c \).

Moreover, independent of a choice of spacetime metric, (2.9) is invariant under diffeomorphism transformations. Such a transformation can be generated by a vector on the three manifold \( \Omega \), \( V^\mu \), via Lie derivative \( \mathcal{L}_V \). Under it, the Chern-Simons field \( A_\mu \) transforms as

\[
\mathcal{L}_V A^a_\mu = D_\mu (A^a_\nu V^\nu) + V^\nu F^a_{\nu\mu},
\]

so that

\[
\mathcal{L}_V I_{CS}[A] = 0,
\]

provided \( V^\mu \) vanishes on the boundaries of \( \Omega \) or the space-time manifold has no boundary [9][17]. However, subject to the flat connection condition \( F_{\mu\nu} = [D_\mu, D_\nu] = 0 \), diffeomorphism transformations completely fall into gauge transformations. In fact, the first term of (2.11) can be identified as the gauge transformation (2.10) with gauge parameter \( \tau^a = A^a_\nu V^\nu \), and the second term is proportional to the flat connection condition. Therefore, on the constraint surface, a generator generates simultaneously both diffeomorphism and gauge transformation. Consequently, a gauge choice fixes both the gauge and diffeomorphism. As a topological quantum field theory, (2.9) is exactly soluble [4].

Now, we see that the mass parameter \( M \) plays the role of order parameter. When \( M > 0 \), the system described by (2.1) has \( U(1) \) symmetry only; however when \( M = 0 \), the system is invariant under non-Abelian gauge and topological transformations. In the following sections, we shall consider the quantization of the massive theory, i.e. the system in a symmetry broken phase, in a heat bath. We shall see then that the mass of the vector field \( B_\mu \) becomes temperature dependent, \( M = M(T) \), and there exists a critical temperature \( T_c \) at which \( M(T) = 0 \), and a topological phase transition of the zeroth order happens.

Another parameter of the theory is the Chern-Simons coupling \( g \). It is dimensionless as the Chern-Simons interaction operator is marginal. This implies that the Chern-Simons coupling may need potentially non-trivial renormalization. However, due to the topological nature of the Chern-Simons term, \( g \) is not sensitive to changes of energy scale. In other words, the beta function of \( g \) vanishes identically, and \( g \) is not a running coupling constant. This fact makes a ‘small’ Chern-Simons coupling a perfect controlling parameter in perturbation expansion. On the other hand, however, when the thermal effect is taken into account, the concern is now that the Chern-Simons coupling may be temperature dependent, and if it went up rapidly with
temperature, perturbation would break down. We shall see below that fortunately it is not the case. The effective (temperature dependent) Chern-Simons coupling is just a slowly increasing function of temperature, and at the critical temperature it is only twice as large as the renormalized zero temperature one.

3 Partition Function

Now we attach the system to a heat bath with temperature $T$. As is known, finite temperature behavior of any theory is specified by the partition function

$$Z = \text{Tr} e^{-\beta H} ;$$

(3.1)

and the thermal expectations of physical observables

$$< O > = \frac{1}{Z} \text{Tr}[O e^{-\beta H}] ,$$

(3.2)

where $\beta = 1/T$ is the inverse temperature ($k_B = 1$) [18].

As our system is relativistic, we like to work out its covariant formalism. It is clear from the discussion in the previous section, the Chern-Simons theory involves both the primary (first class) and secondary (second class) constraints. Therefore, to derive the partition function in the path integral formalism, special cares are needed. We shall outline the derivation in the zero temperature field theory [19], and then switch to the finite temperature case. We shall focus on the Chern-Simons field $a_\mu$, i.e. consider the theory

$$L_{cs} = -\frac{1}{2} \epsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + a_\mu j_\mu .$$

(3.3)

Generalization to the massive field $B_\mu$ will be then straightforward. With the momentum conjugates $\pi_\mu$ given in (2.8), it is easy to check that the theory (3.3) is subject to the following constraints:

$$\chi_0 = \pi_0 ,$$

(3.4)

$$\chi_i = \pi_i - \frac{1}{2} \epsilon_{ij} a_j ,$$

(3.5)

$$\chi = \epsilon_{ij} \partial_i a_j + j_0 .$$

(3.6)

The last one is from $\delta L_{cs}/\delta a_0 = 0$. Then the total Hamiltonian density is

$$\mathcal{H} = \mathcal{H}_c + \lambda_0 \chi_0 + \lambda_i \chi_i + \lambda \chi ,$$

(3.7)
where \( \lambda \)'s are the lagrange multipliers, and the canonical Hamiltonian density \( H_c \) is

\[
H_c = \dot{a}_\mu \pi^\mu - L_{cs}
\]

(3.8)

\[
= \dot{a}_0 \pi_0 + \dot{a}_i (\pi_i - \frac{1}{2} \epsilon_{ij} a_j) + a_0 (\epsilon_{ij} \partial_j a_j + j_0) - a_i j_i.
\]

(3.9)

From (3.7) and (3.9), one sees that \( \dot{a}_0, \dot{a}_i \) and \( a_0 \) in \( H \) can be absorbed by the lagrange multiplier \( \lambda_0, \lambda_i \) and \( \lambda \), respectively. Doing so, \( a_0 \) disappears, and so one can discard \( \pi_0 \) by setting \( \lambda_0 = 0 \). The Hamiltonian density is now

\[
H = \lambda_i \chi_i + \lambda \chi - a_i j_i.
\]

(3.10)

Now two constraints remain. The first, \( \chi_i \), is of second class. And the second, \( \chi \), seems to be of second class too. However, upon a linear combination with \( \chi_i \), a redefined constraint, \( \partial_i \pi_i + \epsilon_{ij} \partial_j a_j + j_0 \), turns out to be of first class. As a matter of fact, this first class constraint is due to the gauge freedom of the Chern-Simons field. A nice approach to deal with a quantum system with first and second class constraints is to replace the Poison brackets with the Dirac brackets [20] so that the second class constraints can be absorbed into the measure of the path integrals [21] and the first class ones are taken care by using gauge fixing terms. For this purpose, let us choose a fixed time \( t \). The constraint matrix element \( C_{ij} \) at the time \( t \) is

\[
C_{ij}(x,y) = \{\chi_i(x), \chi_j(y)\} = -2 \epsilon_{ij} \delta(x - y);
\]

(3.11)

and the inverse

\[
C^{-1}_{ij}(x,y) = \frac{1}{2} \epsilon_{ij} \delta(x - y).
\]

(3.12)

Then the basic (equal time) Dirac brackets (also called star brackets) are

\[
\{a_i(x), a_j(y)\}_D = \frac{1}{2} \epsilon_{ij} \delta(x - y),
\]

(3.13)

\[
\{\pi_i(x), \pi_j(y)\}_D = 2 \epsilon_{ij} \delta(x - y),
\]

(3.14)

\[
\{a_i(x), \pi_j(y)\}_D = \eta_{ij} \delta(x - y).
\]

(3.15)

And the first class constraint satisfies

\[
\{\chi(x), \chi(y)\}_D = 0.
\]

(3.16)

Now, to fix the first class constraint \( \chi \), one introduces a pair of ghosts, \( c \) and \( \bar{c} \), and their momentum conjugates, \( \bar{f} \) and \( f \), and the momentum conjugate of the lagrange
multiplier $\lambda, \sigma$. With these new fields, the system has the global nilpotent (off-shell) BRST symmetry \cite{22} \cite{17}. The BRST charge

$$Q = \int d^2 x (c \chi - i f \sigma)$$

(3.17)

generates the BRST transformations for all the fields via the Dirac brackets. Now, the partition function is

$$Z_{cs} = N \int [D\mu] e^{i I_{eff}},$$

(3.18)

with

$$I_{eff} = \int d^3 x \left( a_i \pi_i + \dot{\lambda} \sigma + \dot{c} \bar{f} + \dot{f} + a_i j_i + \{\Phi, Q\}_D \right),$$

(3.19)

where $\Phi$ is a gauge fixing function, and the measure

$$[D\mu] = (\det |\chi_i, \chi_j|)^{1/2} \prod_i \delta(\chi_i) D a_i D \pi_i D \lambda D \sigma D c D \bar{c} D f D \bar{f}.$$  

(3.20)

The determinant factor in the measure can be absorbed into the normalization factor $N$. And the integrations over $\pi_i$ eliminate the second class constraint $\chi_i$. Moreover, as a covariant gauge fixing for the later use, the gauge function is so chosen

$$\Phi = i \bar{c} \left( \frac{1}{2} \rho \sigma - \partial_i a_i \right) + \bar{f} \lambda$$

(3.21)

that

$$\{\Phi, Q\}_D = -\frac{\rho}{2} \sigma^2 - i \bar{c} \partial_i \partial_i c + \sigma \partial_i a_i - i \bar{f} f - \lambda (j_0 + 2 \epsilon_{ij} \partial_i a_j),$$

(3.22)

where $\rho$ with dimension in the unit of mass parametrizes the covariant gauge fixing for $U(1)$ symmetry. Now, integrating out all the momenta ($\sigma, \bar{f}, f$ and $\pi_i$) and denoting the lagrange multiplier $\lambda$ as $a_0$ in the partition function, we obtain finally

$$Z_{cs} = N \int D a_\mu D c D \bar{c} \exp \left( i \int d^3 x \left( \frac{1}{2} \epsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + a_\mu j_\mu + i \bar{c} \partial^2 c + \frac{1}{2 \rho} (\partial_\mu a_\mu)^2 \right) \right).$$

(3.23)

Above, the ghost fields $c$ and $\bar{c}$ do not interact with other fields but only serve to cancel the non-physical, in fact all, degrees of freedom of The Chern-Simons field $a_\mu$.

The derivation of the partition function for the vector $B_\mu$ field can be done similarly. The only difference is that there is no gauge freedom and so no first class constraint for the $B_\mu$ field, since it is massive.

With these preparation, it is readily to work out the functional integral representation of partition function at finite temperature $T$. The trick is rather simple:
to replace the time variable $t$ with the imaginary time $i\tau$ via a Wick rotation, and to explain the final imaginary time as the inverse temperature $\beta = 1/T$. Then the partition function of system described by (2.1) is

$$Z = N \int \prod_\mu \prod_\nu \prod_\lambda D a_\mu D B^*_\nu D B_\lambda D c D \bar{c} \exp \left( - \int_0^\beta d\tau \int d^2 x L \right),$$  

(3.24)

with the Euclidean Lagrangian

$$L = -\frac{i}{2} \epsilon_{\mu\nu\lambda} B^*_\mu (\partial_\nu - i g a_\nu) B_\lambda + \frac{M}{2} B^*_\mu B_\mu - \frac{i}{2} \epsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + (\partial_\mu \bar{c})(\partial_\mu c) + \frac{1}{2\rho} (\partial_\mu a_\mu)^2.$$

(3.25)

Being vector bosons or ghosts with ghost number $\pm 1$, all fields in (3.24) are subject to the periodic boundary condition such as

$$a_\mu(\beta, x) = a_\mu(0, x) \quad \text{and} \quad B_\mu(\beta, x) = B_\mu(0, x).$$

(3.26)

From the partition function (3.24), it is easy to work out the finite temperature Feynman rules. The Chern-Simons propagator in the Landau gauge and the vertex are

$$D^0_{\mu\nu}(p) = \frac{\epsilon_{\mu\nu\lambda} p_\lambda}{p^2}, \quad \text{and} \quad \Gamma^0_{\mu\nu\lambda} = g \epsilon_{\mu\nu\lambda}.$$

(3.27)

And the vector propagator is

$$G^0_{\mu\nu} = \frac{\epsilon_{\mu\nu\lambda} p_\lambda + \delta_{\mu\nu} M + p_\mu p_\nu/M}{p^2 + M^2}.$$

(3.28)

According to the periodic boundary condition (3.26), the third component of momentum, the frequency, takes discrete values, $p_3 = 2\pi n T$ for integer $n$’s. Besides, each loop in a Feynman diagram carries an integration-summation $T \sum_n \int d^2 p/(2\pi)^2$ over the internal momentum-frequency $(p, 2\pi T n)$; and at each vertex, the momentum-frequency conservation is satisfied.

4 Critical Temperature

We now consider perturbation expansion of the theory. In this section, the two and three point correlation functions are calculated to the second order. For our purpose, we shall set the external momenta-frequencies zero. To start, we cite a useful equation
that maps the discrete summation $T \sum_{n=-\infty}^{\infty} f(p_3 = 2\pi T n)$ into continuous integrals. By a contour integral on a complex plane, one obtains

$$T \sum_{n=-\infty}^{\infty} f(p_3 = 2\pi T n) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dz}{2\pi} \left[ f(z) + f(-z) \right] + \int_{-\infty}^{\infty+i\epsilon} \frac{dz}{2\pi} \left[ f(z) + f(-z) \right] \frac{1}{e^{-i\beta z} - 1}.$$

(4.1)

The expression is correct as long as the function $f(p_3)$ has no singularities along the real $p_3$ axis. One advantage of this equation is that it naturally separates the temperature independent piece from the temperature dependent one. As an application, let's consider a typical integration-summation

$$J(M, T) = T \sum_{n=-\infty}^{\infty} \int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2 + M^2},$$

(4.2)

which will appear in loop calculations. By using (4.1), the frequency sum is converted into the integrals

$$J(M, T) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{p^2 + M^2} + 2 \int_{-\infty+i\epsilon}^{\infty+i\epsilon} \frac{dz}{2\pi} \int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2 + M^2} \frac{1}{e^{i\beta z} - 1}.$$

(4.3)

The first term above is ultraviolet divergent and a regularization is necessary. If a naive cutoff $\Lambda$ is introduced, the result is

$$\int_{\Lambda} d^3p \frac{1}{(2\pi)^3} \frac{1}{p^2 + M^2} = \frac{\Lambda}{2\pi^2} - \frac{M}{4\pi}.$$

(4.4)

On the other hand, if a gauge invariance reserved regularization such as the dimensional regularization is used, one ends up with the second term in (4.4) only, though the integral is power-counting linear divergent in three dimensions. Namely, the two regularization schemes differ one another only by a $\Lambda$-dependent term. This sort of linearly cutoff dependent terms can be absorbed by re-definition of the zero temperature mass and coupling constant, as we shall see below, and therefore no physics should be affected by the regularization procedure(s) used. The second term in (4.3) involves no divergent, thanks to the Bose-Einstein distribution function. This is a well-known feature of the finite temperature theory that all the divergences appear and thus can be taken care in the zero temperature field theories, and thermal effects are finite at any finite temperature $T$. The integrals on the complex $z$ plane and on the real two-dimensional $p$ space in the second term of (4.3) are readily to perform, and we obtain

$$J(M, T) = \frac{\Lambda}{2\pi^2} - \frac{M}{4\pi} - \frac{T}{2\pi} \ln(1 - e^{-M/T}).$$

(4.5)
Now we consider one loop corrections to the correlation functions with vanishing external momenta. The inverse two-point function of $B_\mu$ field is defined as

$$G_{\mu\nu}^{-1}(p) = G^{0}_{\mu\nu}^{-1}(p) - \Sigma_{\mu\nu}(p),$$

(4.6)

where $\Sigma_{\mu\nu}(p)$ denoting the self-energy. Then the effective mass of the vector field $B_\mu$ can be defined as

$$M(g, T) \delta_{\mu\nu} = G_{\mu\nu}^{-1}(p = 0).$$

(4.7)

In general, with only $O(2)$ symmetry in the space, the effective mass $M(g, T)$ of a vector field in the longitudinal direction is not necessarily the same with that in the transverse direction. For the massive vector field we are considering, it is the case.

It is not difficult to check that the tadpole diagrams have not contributions to the self-energy. Then calculating

$$\Sigma^{(2)}_{\mu\nu}(p) = g^2 T \sum_n \int \frac{d^2 q}{(2\pi)^2} \epsilon_{\mu\sigma\eta} G^0_{\sigma\lambda}(p + q) \epsilon_{\lambda\tau\nu} D^0_{\tau\eta}(q),$$

(4.8)

we have

$$\Sigma^{(2)}_{\mu\nu}(p = 0) = \frac{2}{3} g^2 \delta_{\mu\nu} J(M, T),$$

(4.9)

and the effective mass by using (4.7)

$$M(g_r, T) = M_r + \frac{1}{6\pi} g_r^2 \left( M_r + 2T \ln(1 - e^{-M_r/T}) \right) + \mathcal{O}(g^4),$$

(4.10)

where $M_r$ denotes the renormalized zero temperature mass. For instance, $M_r = M - \frac{1}{3\pi^2} g^2 \Lambda$ at one-loop in the regularization by a naive ultraviolet cutoff $\Lambda$, or $M_r = M$ in the dimensional regularization. In the bracket in (4.10), the bare mass $M$ has been replaced by the renormalized $M_r$, and the bare zero temperature Chern-Simons coupling $g$ by the renormalized $g_r$ (see below), this replacement affects at most higher orders. The first term in the bracket of (4.10) is the radiative correction at zero temperature. Without a symmetry to restrict the renormalized vector mass to particular values, especially zero, the vector mass is a free parameter. Indeed, as we have seen here that, even one starts with a massless theory $M = 0$, the radiation correction at zero temperature generates a vector mass via the Coleman-Weinberg mechanism [25]. The second term in the bracket of (4.10) is obviously due to exchanging energy with the heat reservoir. The low temperature limit $T \to 0$ is trivial, as $M(g, T) \to (1 + \frac{1}{6\pi} g^2)M_r > 0$. On the other hand, since $M(g, T)$ is a monotonically
decreasing function of temperature, as $T$ goes up, the thermal fluctuation tends to drive the effective mass to zero. Namely, there must exist a critical temperature $T_c$ at which $M(g, T) = 0$, and a phase transition happens. $T_c$ is readily to solve from (4.10). We obtain at this order

$$e^{-a M_r/T_c} = 1 - e^{-M_r/T_c},$$

with $a = (1/2 + 3\pi/g^2)$. In a linearized form, as a good approximation when $T_c \gg M_r$,

$$T_c \simeq 3(\frac{1}{2} + \frac{\pi}{g^2})M_r.$$  

Now we see that the Chern-Simons interaction is responsible to the phase transition as it should be, and a stronger interaction causes a transition at a lower temperature, with the renormalized mass $M_r$ fixed.

As is mentioned in section 2, the Chern-Simons coupling $g$ receives only trivial correction at zero temperature so that the beta function of $g$ identically vanishes. Alternatively said, it is insensitive to the change of energy scale. However, the thermal effect at finite temperature will affect the strength of the Chern-Simons coupling. Namely, in the finite temperature field theory, the coupling constant is a function of temperature, $g = g(T)$. Then it is crucial to the perturbation expansion over the Chern-Simons coupling that $g(T)$ must be reasonably small for $T$’s in the range $[0, T_c]$. The effective (finite temperature) coupling constant $g(T)$ can be defined as

$$\Gamma_{\mu \nu \lambda}(p = 0) = g(T)\epsilon_{\mu \nu \lambda},$$

where $\Gamma_{\mu \nu \lambda}(p)$ is the three-point function. Calculating the one loop diagram for the three vertex, we have

$$\Gamma_{\mu \nu \lambda}^{(2)}(p = 0) = \frac{2}{3}g^3\epsilon_{\mu \nu \lambda}J(M, T),$$

and then the effective coupling is

$$g(T) = g_r \left(1 - \frac{1}{6\pi}g_r^2[1 + \frac{T}{M_r}\ln(1 - e^{-M_r/T})]\right) + O(g^5),$$

where the renormalized zero-temperature coupling $g_r = g(1 + \frac{g^2\Lambda}{3\pi^2M})$ in the regularization by a naive cutoff, or $g_r = g$ in the dimensional regularization. Again, in (4.13), we have replaced the bare parameters $g$ and $M$ with the renormalized ones $g_r$ and $M_r$, and this affects only higher orders. It shows that $g(T)$ is a monotonically
slowly increasing function of the temperature $T$, with $g(T = 0) = g_r(1 - \frac{1}{6\pi}g_r^2)$ at zero temperature. On the other hand, at the critical temperature $T = T_c$, the effective coupling is

$$g(T_c) = 2g_r.$$  \hspace{1cm} (4.16)

This implies that in the region $T < T_c$, the perturbation expansion over the Chern-Simons coupling is reliable, only if the renormalized zero temperature coupling constant $g_r$ is small.

Though the gauge fields at zero temperature have no dynamical mass due to the gauge symmetry, the quantum thermal fluctuations may endow them thermal masses. The electric and magnetic masses can be defined via the polarization tensor $\Pi_{\mu\nu}(p, g, M, T)$ \[26\]:

$$\mathcal{M}_{el}(g_r, M_r, T)\delta_{\mu0}\delta_{\nu0} + \mathcal{M}_{mag}(g_r, M_r, T)\delta_{\mu j}\delta_{\nu j} = -\Pi_{\mu\nu}(p = 0, g_r, M_r, T).$$ \hspace{1cm} (4.17)

Calculating the one-loop diagram for the polarization tensor at $p = 0$, we obtain

$$\Pi(2)_{\mu\nu}(p = 0, g, M, T) = \frac{2}{3}g^2\delta_{\mu\nu}\left(J(M, T) + \frac{M}{4\pi}\right),$$ \hspace{1cm} (4.18)

and the thermal masses $\mathcal{M}_{el}(g_r, M_r, T) = \mathcal{M}_{mag}(g_r, M_r, T) = \mathcal{M}(g_r, M_r, T)$ with

$$\mathcal{M}(g_r, M_r, T) = -\frac{2}{3\pi}g_r^2T\ln(1 - e^{-M_r/T}) + \mathcal{O}(g^4).$$ \hspace{1cm} (4.19)

Above we have set the renormalized zero temperature masses of the gauge field to zero by using counter terms when it is necessary (for instance in the regularization by a large momentum cutoff), so that the gauge symmetry is respected. And we have replaced the bare zero temperature coupling $g$ and vector mass $M$ in \[4.19\] with the renormalized ones. The thermal masses of the gauge field are monotonically increasing function of $T$. At the critical temperature $T_c$

$$\mathcal{M}(T_c) = M_r(1 + \frac{1}{6\pi}g_r^2) + \mathcal{O}(g^4).$$ \hspace{1cm} (4.20)

The electric and magnetic masses $\mathcal{M}_{el}(T)$ and $\mathcal{M}_{mag}(T)$ are known as the inverse screening length in the plasma \[14\] \[18\]. Unlike those in the $(3 + 1)$ dimensional $QED$ and $QCD$ where the magnetic screening is absent, our results here suggest that both the static electric and magnetic fields are screened by the plasma thermal excitations in the $(2 + 1)$ dimensional massive system. And thus, at least to the one loop order, the plasma of thermal excitations acts like a superconductor.
5 Zeroth Order Transition

We turn to calculation of free energy density for the system. As is known, the free energy is the single most important function in the thermodynamics. From it all other thermodynamic properties can be determined. In the present case, the free energy may tell us the nature of the topological phase transition. With the conventional Fourier transformation, we come to the momentum space. At the lowest order, performing the Gaussian integrals in (3.24), we have the partition function

$$Z_0 = [\det(-\epsilon_{\mu\nu\lambda}p_\lambda + M\delta_{\mu\nu})]^{-1}[\det(-\epsilon_{\mu\nu\lambda}p_\lambda + \frac{1}{\rho}p_\mu p_\nu)]^{-\frac{1}{2}}\det(p^2) . \quad (5.1)$$

The determinants are readily calculated. The one for the CS term, the second in the above, is

$$[\det(-\epsilon_{\mu\nu\lambda}p_\lambda + \frac{1}{\rho}p_\mu p_\nu)]^{-\frac{1}{2}} = \sqrt{\beta\rho} \prod_n \prod_p (\beta^2 p^2)^{-1} . \quad (5.2)$$

The products are canceled by the determinant for ghost term, the last in (5.1), and the gauge parameter term \(\sqrt{\beta\rho}\) contributes the zero-point energy. This verifies Chern-Simons gauge field carries no local dynamical degree of freedom. On the other hand, the determinant for the free massive \(B_\mu\) field gives

$$[\det(-\epsilon_{\mu\nu\lambda}p_\lambda + M\delta_{\mu\nu})]^{-1} = \frac{1}{\beta M} \prod_n \prod_p [\beta^2 (p^2 + M^2)]^{-1} . \quad (5.3)$$

Then the leading contribution to the free energy density, \(\mathcal{F}_0 = -\ln Z_0/(\beta V)\), is

$$\mathcal{F}_0 = \frac{1}{2\beta V} \sum_n \sum_p \ln \left(\frac{\beta M^2}{\rho}\right) + 2T \int \frac{d^2p}{(2\pi)^2} \left[\beta\omega + \ln(1 - e^{-\beta\omega})\right] , \quad (5.4)$$

where \(\omega = \sqrt{p^2 + M^2}\). The second term is just the free energy of a gas of noninteracting, massive bosons. Namely that the massive vector \(B_\mu\) obeys the Bose-Einstein distribution. The factor “2” in the second term in (5.4) indicates two degrees of freedom, carried by the complex field \(B_\mu\), in the thermal equilibrium. Besides, like the gauge fixing parameter \(\rho\), the vector mass \(M\) has an extra contribution to the zero point energy, as shown in the first term of (5.4).

Dropping the zero-point energy, the free energy density \(\mathcal{F}_0\) can be re-written as

$$\mathcal{F}_0 = -\frac{3}{\pi} T^3 h_4\left(\frac{M_r}{T}\right) + \mathcal{O}(g^2) , \quad (5.5)$$
where
\[ h_n(x) = \frac{1}{\Gamma(n)} \int_0^\infty dy \frac{y^{n-1}}{\sqrt{y^2 + x^2}} e^{\frac{1}{\sqrt{y^2 + x^2}}} - 1. \] (5.6)

The perturbation corrections to the free energy may be calculated by expanding the partition function (3.24) in the Chern-Simons coupling \( g \) and calculating the resulting Feynman diagrams. At the second order, it is

\[ \ln Z_2 = -\frac{1}{2} \]

with the real (dashed) line standing for the \( B_\mu (a_\mu) \) propagator. Calculating the two-loop diagram, we obtain

\[ \ln Z_2 = -g^2 M \beta V (J(M, T))^2 , \] (5.7)

and the correction to the free energy density

\[ F_2 = \frac{1}{(4\pi)^2} g_r^2 M_r T^2 \left( \frac{M_r}{T} + 2\ln(1 - e^{-M_r/T}) \right)^2 + \mathcal{O}(g^3) . \] (5.8)

Being positive-definite, \( F_2 \) increases monotonically with \( T \). Physically, this implies the quantum thermal fluctuation tends to decrease the pressure \( P(T) = -F(T) \), contrary to the thermal behavior of the system in the free theory limit, as shown in (5.3). It would be interesting to calculate the higher order corrections, for instance, the next order \( g^3 \) from the ring diagrams. To our main purpose of looking at the nature of the phase transition at large \( T \sim T_c \), the result up to \( g^2 \) is sufficient.

At the critical \( T_c \), as \( T_c \sim 3(\frac{1}{2} + \frac{\pi}{g}) M_r \gg M_r \) for weak CS interactions, we consider the leading term in both \( F_0 \) and \( F_2 \). It is readily to see

\[ F_0(T_c) = -81\pi^2 h_4(0) M_r^3 \frac{1}{g_r^6} + \mathcal{O}(\frac{1}{g_r^4}) , \] (5.9)

\[ F_2(T_c) = \frac{9}{4} M_r^3 \frac{1}{g_r^2} + \mathcal{O}(g_r^2) , \] (5.10)

where \( h_4(0) = 2\zeta(3) \sim 2.404 \). This shows that near the critical temperature \( T_c \), \( F_0 \) dominates. In other words, the pressure of the system to the second order is positive-definite as it approaches the critical temperature from the below,

\[ P(T \sim T_c) = -F(T \sim T_c) > 0 . \] (5.11)
Now let us look into the symmetry phase, where the mass of $B_\mu$ field vanishes. Due to the topological nature, the free energy must identically vanish. This can be readily verified within the covariant formalism used in this work. The symmetry phase of the system is described by the pure non-Abelian Chern-Simons theory (2.9). In a covariant gauge fixing, as usual one introduces ghost term
\[ \partial_\mu \bar{c}_a D_\mu c^a, \]  
along with a gauge fixing term such as $(\partial_\mu A^a_\mu)^2/(2\rho)$. Doing so, one fixes diffeomorphism too. This ghost term includes the kinetic of ghost and gauge interaction between the ghost and Chern-Simons field. The Gaussian integral of the kinetic term of ghost $c^a$ with $a = 1, 2, 3$ cancel that of the Chern-Simons term for $A^a_\mu$, and so $\mathcal{F}_0 \equiv 0$. And then, order by order, the contributions to quantities like the free energy from the ghost loops cancel out completely those from the Chern-Simons field loops, at zero temperature [27] as well as in the thermal ensemble. In particular,
\[ P(T) \equiv 0. \]  
This seems a natural consequence of lacking of dynamical degree of freedom in a topological theory (or in a topological phase of a system).

Since the the pressure (or free energy) discontinues at the critical temperature $T_c$, we conclude that the topological phase transition is of the zeroth order.

6 Summary

In conclusion, we have looked into the thermal behavior of the theory with a charged massive vector matter coupled to Chern-Simons gauge field. To the second order in perturbation expansion, we have obtained a critical temperature $T_c$, at which the effective mass of vector field vanishes, and the system transfers from a symmetry broken phase to topological phase. The phase transition seems to be of the zeroth order, as the free energy of the system is discontinuous at $T_c$. We have seen that the effective Chern-Simons coupling $g(T)$ is a slowly increasing function of temperature $T$ in the symmetry broken phase, and therefore perturbation expansion over the Chern-Simons coupling should be reliable in the whole range of $T$, provided the renormalized zero temperature coupling is small.
The present investigation presents to our knowledge the first example that exhibits restoration of topological invariance at some critical temperature. At the phase transition point, the local dynamical degrees of freedom and spacetime metric disappear. Meanwhile, the thermodynamical properties of the system subject to sudden changes. For instance, the pressure of the system promptly withdraws. And it is not difficult to check that (all) other thermodynamical quantities derivable from the free energy are discontinuous at the critical $T_c$ as well. This is a phenomenon that has not been observed in the Nature nor in theories, and deserves further studies.

It is interesting to notice that the pure Chern-Simons theory with the non-Abelian gauge group $ISO(2,1)$ is equivalent to the $(2 + 1)$ dimensional general relativity with a vanishing cosmological constant [6]. This is realized by identifying the vierbein and spin connection as the gauge field $A_a^a$, and mapping the Einstein action to the Chern-Simons action (2.9). Then, in this $(2 + 1)$ dimensional theoretical laboratory, we have seen a possible realization of the topological invariance in quantum gravity. Namely that the thermal fluctuation plays a key role to drive the dynamical system under consideration to a phase of pure gravity without a spacetime metric and local dynamical degree of freedom but with topological symmetry.

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References

[1] See, for instance, R. Penrose, in *Magic without Magic*, J. Klauder (ed.), San Francisco Freeman, 1972.

[2] It was suggested by M.F. Atiyah, in *The Mathematical Heritage of Hermann Weyl*, Proc. Symp. Pure Math. 48, R. Wells (ed.), Providence RI: American Mathematical Society 1988, 285-299.

[3] E. Witten, Commun. Math. Phys. 117, 353 (1988).

[4] E. Witten, Commun. Math. Phys. 121, 351 (1989).

[5] G. Moore and M. Seiberg, in *Physics, Geometry and Topology*, H.C. Lee (ed.) Plenum, New York, 1990) p.263.
[6] E. Witten, Nucl. Phys. B, 46 (1988).

[7] A. Tseytlin, J. Math. Phys. 15 L105 (1982).

[8] S. Deser, R. Jackiw, and S. Templeton, Phys. Rev. Lett. 48, 975 (1982).

[9] G.T. Horowitz, Commun. Math. Phys. 125, 417 (1989).

[10] W. Chen, *Phase Transition in (2+1) Dimensional Quantum Gravity*, IFT-490-UNC (1994).

[11] D.A. Kirzhnits and A.D. Line, Phys. Lett. 42B, 417 (1972).

[12] L. Dolan and R. Jackiw, Phys. Rev. D 9, 3320 (1973).

[13] S. Weinberg, Phys. Rev. D 9, 3357 (1973).

[14] D. Gross, R. Pisarski, and D. Yaffe, Rev. Modn. Phys. 53, 43 (1981).

[15] W. Chen and C. Itoi, Phys. Rev. Lett. 72, 2527 (1994); IFT-488-UNC/NUP-A-94-3.

[16] R. Jackiw and V.P. Nair, Phys. Rev. D 43, 1933 (1991).

[17] W. Chen, Phys. Rev. D 41, 1172 (1990).

[18] J.I. Kapusta, *Finite-Temperature Field Theory*, Cambridge Univ. Press, New York 1989.

[19] A similar discussion has been conducted also by R. Amorim and J. Barcelos-Neto, preprint IF-UFRJ-05/94.

[20] P.A.M. Dirac, *Lectures on Quantum Mechanics*, Ueshiva Univ, New York, 1964.

[21] A.P. Senjanovic, Ann. Phys. (N.Y.) 100 227 (1976).

[22] E.S. Fradkin and G.A. Wilkovisky, Phys. Lett. B55 224 (1975); I.A. Batalin and G. Vilkovisky, Phys. Lett. B69 309 (1977).

[23] S. Weinber, Phys. Rev. D 10, 2445 (1974).

[24] P. Morley and M. Kisslinger, Phys. Rep. 51C (2) (1979).
[25] S. Coleman and E. Weinberg, Phys. Rev. D 7, 1888 (1973).

[26] This is also valid for dynamical and/or external electromagnetic fields, if they are minimally coupled to the vector field $B_\mu$.

[27] W. Chen, G.W. Semenoff, and Y.-S. Wu, Phys. Rev. D 44, R1625 (1991); ibid 46, 5521 (1992).