KAM Theorem for a Hamiltonian system with Sublinear Growth Frequencies at Infinity

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Abstract

We prove an infinite-dimensional KAM theorem for a Hamiltonian system with sublinear growth frequencies at infinity. As an application, we prove the reducibility of the linear fractional Schrödinger equation with quasi-periodic time-dependent forcing.

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1 Introduction

We study a Hamiltonian system with frequencies that grow sublinearly at infinity. That is, we consider

\[ H = N + P = \sum_{1 \leq j \leq d} \omega_j(\xi) I_j + \sum_{n \in \mathbb{Z}} \Omega_n(\xi)|z_n|^2 + P(\xi, I, z, \bar{z}), \]

where \( \Omega_n = |n|^\alpha + \lambda + \tilde{\Omega}_n \) with \( 0 < \alpha < 1, \lambda > 0 \). For such a Hamiltonian, the gaps between the frequencies are decreasing. For example, let \( \alpha = \frac{1}{2} \) and \( \tilde{\Omega}_n = 0 \); one then has \( \Omega_n = |n|^\frac{1}{2} + \lambda, n \in \mathbb{Z} \). For any fixed \( a \in \mathbb{Z} \), one has \( \lim_{n \to \infty} \Omega_{n+a} - \Omega_n = 0 \). This is considerably different from the suplinear growth cases; that is,

\[ \Omega_n = |n|^\alpha + o(|n|^{\alpha-1}), \quad \begin{cases} \alpha \geq 2, & d > 1 \\ \alpha \geq 1, & d = 1 \end{cases} \]

\( \text{(1.2)} \)

We refer the reader to \([3, 4, 5, 6, 9, 11, 12, 15, 16, 17, 19, 21, 22, 23, 25, 26, 28] \) for more information. Nevertheless, there are few results on the Kolmogorov-Arnold-Moser (KAM) theorem for the Hamiltonian (1.1).
The study of the above Hamiltonian (1.1) is motivated by Zakharov [30] and Craig-Sulem [7], who introduced the Hamiltonian structure of a water wave in a channel of infinite depth. We note that there have been many important studies on water waves. The time-periodic and time-quasi-periodic solutions and the standing gravity water waves were derived. For details, see [2].

In regard to Hamiltonian system with sublinear growth of frequencies, Craig and Worfolk [10] presented a Birkhoff normal form. Craig and Sulem [8] studied the function-space mapping properties of Birkhoff-normal-form transformations of the Hamilton for the equations for water waves. Wu-Xu [27] gave an infinite-dimensional KAM theorem for the Hamiltonian (1.1). In their work, the perturbation maintains conservation of momentum and a strong regularity condition, $X_P : \mathcal{P}_C^{\rho,p} \to \mathcal{P}_C^{\rho,\bar{p}}$ with $\bar{p} > p$ (see Section 2 for the definition of space). Following this work, Xu [29] relaxed the regularity of the perturbation, that is $X_P : \mathcal{P}_C^{\rho,p} \to \mathcal{P}_C^{\rho,p}$. Using the property of Töplitz–Lipschity by [12], they presented a new KAM theorem. However, the condition concerning momentum conservation is necessary in both results. There are significant differences if the perturbation does not satisfy momentum conservation. Recently, Baldi et al. [2] obtained a result for (1.1) without such restriction. They developed a regularization procedure performance on the linearized PDE at each approximate quasi-periodic solution. Moreover, they used their theory to study the time-quasi-periodic solutions for finite-depth gravity water waves. We also mention the interesting work by Duclos et al. [24], in which the energy growth of similar Hamiltonians is given.

Motivated by [2], we aim to prove an infinite-dimensional KAM Theorem for the Hamiltonian (1.1). Our method is different from that in [2]. We emphasize that the perturbation does not satisfy momentum conservation. Nevertheless, following the idea by [15], the regularity condition $X_P : \mathcal{P}_C^{\rho,p} \to \mathcal{P}_C^{\rho,\bar{p}}$ ($\bar{p} > p$) is replaced by Assumption B2 (see Section 2 for definition). As a simple application, this theorem is applied to the reducibility of the fractional nonlinear Schrödinger equation (2.10). We believe our method helps in understanding the dynamics of such Hamiltonian systems. The general strategy in proofing the KAM Theorem 1 is explained below. Generally, the existence of multiple normal frequencies leads to a complex normal form. To show the main idea, we only consider (1.1) for simplicity.

As usual, let $R$ (see (3.6)) be the truncation of $P$. The smallness of $P - R$ is obvious if we reduce the weight $\rho$ (see (2.2)) a little. We then need to solve the so called homological equation. Concerning the solution of the homological equation, the estimations of $F_{k,n,m}^{11}$ ($k \neq 0$) are standard if we
have diophantine condition
\[
|\langle k, \omega \rangle + \Omega \cdot \ell| \geq \frac{\gamma}{K^{4\tau}}, \quad 0 < |k| \leq K,
\]
where \( \Omega = (\cdots, \Omega_n, \cdots)_{n \in \mathbb{Z}}, \ell \in \mathbb{Z}^N, \) and \( |\ell| \leq 2. \) However, there is a great difference if \( k = 0. \) For the suplinear growth \((\alpha \geq 1)\) cases, as an example, we set \( \Omega_n = |n|^2 + \tilde{\Omega}_n (n \in \mathbb{Z}), \) then \( |\Omega_n - \Omega_m| \geq 1/2 \) if \( |n| \neq |m|. \) Following this computation, the regularity of the vector field \( X_F \) is obvious. However, this fact is not appropriate for the sublinear growth \((0 < \alpha < 1)\) cases we consider. If the vector field \( X_P \) only satisfies condition \( X_P : \mathcal{P}^{\rho, p} \rightarrow \mathcal{P}^{\rho, p}, \) we have
\[
|F_{0,n,m}^{11}| = \left| \frac{P_{0,n,m}^{11}}{\Omega_n - \Omega_m} \right| \approx \varepsilon e^{-|n-m|^\rho |n|^{1-\alpha}}.
\]
We obtain an unbounded vector field \( X_F. \) That is, there is a strong loss of regularity in the KAM scheme. This is very similar to the claim in [2], the presence of a sublinear \((\alpha < 1)\) growth of the linear frequencies produce strong losses of derivatives in the iterative KAM scheme. To overcome this problem and motivated by [15], we assume additionally that \( P \) satisfies Assumption \( B2 \) (see Section 2 for a definition). We then have
\[
|F_{0,n,m}^{11}| \leq \frac{\varepsilon e^{-|n-m|^\rho}}{\langle n \rangle^{\alpha} \langle m \rangle^\beta}.
\]
With the restriction \( |n - m| \leq K \) and condition \( \alpha + \beta \geq 1, \) we have
\[
|F_{0,n,m}^{11}| \leq \frac{\varepsilon e^{-|n-m|^\rho}}{\langle n \rangle^{\beta} \langle m \rangle^{\beta} |n|^{\alpha} - |m|^{\alpha}} \leq \frac{\varepsilon e^{-|n-m|^\rho}}{|n|^\beta |m|^\beta} \leq \frac{\varepsilon e^{-|n-m|^\rho}}{|n|^\beta}. \quad (1.3)
\]
Thus \( X_F \) is a regular vector field from \( \mathcal{P}^{\rho, p} \) into itself.

At the same time, using the estimate (1.3) on \( F_{0,n,m}^{11}, \) one can observe that the homological solution \( F \) does not satisfy Assumption \( B2. \) However, we prove that \( \{P, F\} \) still satisfies Assumption \( B2 \) and that the new perturbation \( P_+ \) also satisfies Assumption \( B2. \)

Finally, we introduce a method for estimating the measure of the excluded parameters. We first identify a parameter set \( \mathcal{O} \) with positive measure, such that for any \( \xi \in \mathcal{O} \) and \( k \in \mathbb{Z}^d \) with \( 0 < |k| \leq K, \) one has
\[
|\langle k, \omega \rangle| \geq \frac{\gamma}{K^{4\tau}}.
\]
We next focus mainly on the sets of resonances like
\[
\bigcup_{0 < |k| \leq K \atop |n - m| \leq K} \{ \xi \in \mathcal{O} : |\langle k, \omega \rangle \pm (\Omega_n - \Omega_m)| \leq \frac{\gamma}{K^{4\tau}} \}.
\]
Following an easy computation, one has $|n|^\alpha - |m|^\alpha \leq \frac{\gamma}{4K^r}$ if $|n| \geq \frac{K^r}{\gamma}$ and $|n - m| \leq K$. Recalling the drift of frequencies, one has $|\tilde{\Omega}_n| \leq \frac{\gamma}{|m|^{2r}} (n \in \mathbb{Z})$. Therefore, for any $\xi \in \mathcal{O}$, if $|n| \geq \frac{K^r}{\gamma}$ and $|n| \leq \frac{K^r}{\gamma}$ and $0 < |k| \leq K$, one obtains

$|\langle k, \omega \rangle \pm (\Omega_n - \tilde{\Omega}_m)| \geq |\langle k, \omega \rangle| - ||n|\alpha - |m|\alpha| - |\tilde{\Omega}_n| - |\tilde{\Omega}_m| \geq \frac{\gamma}{2K^r}$.

Therefore, we only need to consider the resonance sets restricted by $0 < |k| \leq K$, $|n| \leq \frac{K^r}{\gamma}$ and $|n - m| \leq K$. We then prove that the measure of the excluded parameters is bounded by $\gamma$ in the standard way.

# 2 An Infinite-Dimensional KAM Theorem

Let $\mathcal{O}$ be a positive-measure parameter set in $\mathbb{R}^d$. We consider small perturbations of an infinite-dimensional Hamiltonian in the parameter-dependent normal form

$$N = \langle \omega(\xi), I \rangle + \sum_{n \in \mathbb{Z}} \Omega_j z_n \bar{z}_n$$

(2.1)

on phase space $\mathcal{P}^{\rho,p} = T^d \times \mathbb{R}^d \times \ell^{\rho,p} \times \ell^{\rho,p}$

with coordinate $(\theta, I, z, \bar{z})$, where $\xi \in \mathcal{O}$, $T^d = \mathbb{R}^d / 2\pi \mathbb{Z}^d$ and $\ell^{\rho,p}$ is the Hilbert space of all real (later complex) sequences $w = (\cdots, w_n, \cdots)_{n \in \mathbb{Z}}$ with norm

$$\|w\|_{\rho,p}^2 = \sum_{n \in \mathbb{Z}} |w_n|^{2(2\rho)|n|} |n|^{2p}, \quad p > 0, \rho > 0.$$

(2.2)

The complexification of $\mathcal{P}^{\rho,p}$ is denoted by $\mathcal{P}_{\mathbb{C}}^{\rho,p}$. The symplectic structure is $dI \wedge d\theta + i \sum_{n \in \mathbb{Z}} dz_n \wedge d\bar{z}_n$.

The perturbation term $P = P(I, \theta, z, \bar{z}; \xi)$ is real analytic in $I, \theta, z, \bar{z}$ and Lipschitz in the parameters $\xi$. For each $\xi \in \mathcal{O}$, its Hamiltonian vector field $X_P = (-P_\theta, P_I, iP_z, -iP_{\bar{z}})$ defines a real analytic map from $\mathcal{P}^{\alpha,\rho}$ into itself near $\mathcal{T}_0^d = T^d \times \{0, 0, 0\}$. To make this quantitative, we introduce the complex $\mathcal{T}_0^d$-neighborhoods

$$D(s, r) = \{(\theta, I, z, \bar{z}) : |\text{Im}\theta| < s, |I| < r^2, |z|_{\rho,p} < r, |\bar{z}|_{\rho,p} < r\},$$

(2.3)

where $|\cdot|$ denotes the sup-norm of complex vectors, and the weighted phase space norms are defined as

$$|W|_{r,p} =: |W|_{r,\rho,p} = |X| + \frac{1}{r^2} |Y| + \frac{1}{r} |U|_{\rho,p} + \frac{1}{r} |V|_{\rho,p}$$

(2.4)
for \( W = (X, Y, U, V) \).

Let \( P \) be real analytic in \( D_{\rho}(s, r) \) for some \( s, r > 0 \) and Lipschitz in \( O \). We then define the norms

\[
\|P\|_{D_{\rho}(s, r)} = \sup_{D_{\rho}(s, r) \times O} |P| < \infty
\]

and

\[
\|P\|_{D_{\rho}(s, r)}^{\xi} = \sup_{\xi \in O \setminus \eta} \sum_{\xi \neq \eta} \frac{|\triangle_{\xi\eta} P|}{|\xi - \eta|} < \infty
\]

where \( \triangle_{\xi\eta} P = P(\cdot, \xi) - P(\cdot, \eta) \). We also define the semi-norms

\[
\|X P\|_{r, D_{\rho}(s, r)} = \sup_{D_{\rho}(s, r) \times O} \|X P\|_{r, \rho}
\]

and

\[
\|X P\|_{r, D_{\rho}(s, r)}^{\xi} = \sup_{D_{\rho}(s, r) \times O, \xi \neq \eta} \frac{|\triangle_{\xi\eta} X P|}{|\xi - \eta|},
\]

where \( \triangle_{\xi\eta} X P = X P(\cdot, \xi) - X P(\cdot, \eta) \). For simplicity, we usually write

\[
\|P\|_{D_{\rho}(s, r)}^{*} = \|P\|_{D_{\rho}(s, r)} + \|P\|_{D_{\rho}(s, r)}^{\xi},
\]

\[
\|X P\|_{r, D_{\rho}(s, r)}^{*} = \|X P\|_{r, D_{\rho}(s, r)} + \|X P\|_{r, D_{\rho}(s, r)}^{\xi}.
\]

In the sequel, the semi-norm of any function \( f(\xi) \) on \( \xi \in O \) is defined as

\[
|f|_{r, O} = |f|_{O} + |f|_{r, O}^{E},
\]

where the Lipschitz semi-norm is defined analogously to \( \|X P\|_{r, D_{\rho}(s, r)}^{E} \).

Consider now the perturbed Hamiltonian

\[
H = \sum_{1 \leq j \leq d} \omega_{j}(\xi) I_{j} + \sum_{n \in \mathbb{Z}} \Omega_{n} z_{n} \bar{z}_{n} + P(I, \theta, z, \bar{z}; \xi). \tag{2.5}
\]

The assumptions imposed on the frequency and the perturbation are given.

**Assumption A (Frequency)**

(A1) **Nondegeneracy**: The map \( \xi \to \omega(\xi) \) is Lipschitz between \( O \) and its image with \( |\omega|_{O}^{*} \leq M \).

(A2) **Sublinear growth of normal frequencies**:

\[
\Omega_{n} = |n|^\alpha + \lambda + \tilde{\Omega}_{n}(\xi), \quad n \in \mathbb{Z}, \tag{2.6}
\]

where \( 0 < \alpha < 1, \beta > 0, \lambda > 0 \) and \( \sup_{n \in \mathbb{Z}} |n|^{2\beta} \tilde{\Omega}_{n}|_{O}^{*} \leq L \) with \( L \ll 1 \) and \( LM < 1 \).
Remark 2.1 The positive number $\lambda$ is given to avoid some technical problems.

Assumption $\mathcal{B}$ (Perturbation) (B1) $P$ is real analytic in $I, \theta, z, \bar{z}$ and Lipschitz in $\xi$; in addition, there exist $r, s > 0$ so that $\|X_P\|^*_{r,D_\rho(s,r)} < \infty$.

As in [15], we define the space $\Gamma^\beta_{r,D_\rho(s,r)}$. We say that $P \in \Gamma^\beta_{r,D_\rho(s,r)}$ if $\tilde{\|}[P]^\beta_{r,D_\rho(s,r)} = \max \|P\|^\beta_{r,D_\rho(s,r)} < \infty$. The norm $\tilde{\|}[\cdot]_{r,D_\rho(s,r)}$ is defined by the conditions

$$
\|P\|_{D_\rho(s,r)} \leq r^2 \tilde{\|}[P]_{r,D_\rho(s,r)}^\beta,
$$

$$
\max_{1 \leq j \leq d} \left\| \frac{\partial P}{\partial I_j} \right\|_{D_\rho(s,r)} \leq \tilde{\|}[P]_{r,D_\rho(s,r)}^\beta
$$

$$
\left\| \frac{\partial P}{\partial w_n} \right\|_{D_\rho(s,r)} \leq r \tilde{\|}[P]_{r,D_\rho(s,r)}^\beta e^{-|n| r \langle n \rangle} \langle n \rangle^{-\beta},
$$

$$
\left\| \frac{\partial P}{\partial w_n^2} \right\|_{D_\rho(s,r)} \leq \tilde{\|}[P]_{r,D_\rho(s,r)}^\beta e^{-|n| r \langle n \rangle} \langle n \rangle^{-\beta},
$$

where $n, m \in \mathbb{Z}$, $\langle n \rangle = \max \{\frac{1}{2}, |n|\}$, $\ell = \pm 1$ and $w_1^1 = \bar{z}_n$, $w_n^{-1} = z_n$. Hence, the semi-norm $\tilde{\|}[\cdot]_{r,D_\rho(s,r)}^\beta$.

$$
\|P\|_{D_\rho(s,r)}^\ell \leq r^2 \tilde{\|}[P]_{r,D_\rho(s,r)}^\beta^\ell,
$$

$$
\max_{1 \leq j \leq d} \left\| \frac{\partial P}{\partial I_j} \right\|_{D_\rho(s,r)}^\ell \leq \tilde{\|}[P]_{r,D_\rho(s,r)}^\beta^\ell
$$

$$
\left\| \frac{\partial P}{\partial w_n} \right\|_{D_\rho(s,r)}^\ell \leq r \tilde{\|}[P]_{r,D_\rho(s,r)}^\beta^\ell e^{-|n| r \langle n \rangle} \langle n \rangle^{-\beta},
$$

$$
\left\| \frac{\partial P}{\partial w_n^2} \right\|_{D_\rho(s,r)}^\ell \leq \tilde{\|}[P]_{r,D_\rho(s,r)}^\beta^\ell e^{-|n| r \langle n \rangle} \langle n \rangle^{-\beta},
$$

where $n, m \in \mathbb{Z}$, $\langle n \rangle = \max \{\frac{1}{2}, |n|\}$, $\ell = \pm 1$ and $w_1^1 = \bar{z}_n$, $w_n^{-1} = z_n$.

(B2) $P \in \Gamma^\beta_{r,D_\rho(s,r)}$ for $\beta > 0$.

Now we are ready to state the infinite-dimensional KAM theorem.

Theorem 1 Let $0 < \alpha < 1$ and $\beta > 0$ such that $\alpha + \beta \geq 1$. The Hamiltonian $H = N + P$ is defined on $\mathcal{P}_x^\beta$ for any $\xi \in \mathcal{O}$. Suppose that the normal
form $N$ satisfies Assumption A and the perturbation $P$ satisfies Assumption B given $s,r,\rho,\gamma > 0$. Then there is a positive constant $\varepsilon_0 \leq ce^{-\frac{4\rho}{\gamma}}$, such that if
\[
\|X_P\|_{D_\rho(s,r)}^* + \|P\|_{D_\rho(s,r)}^{\beta,*} \leq \varepsilon_0,
\]
the following holds:

1) a Cantor like set $O_\gamma$ of $O$ with $\text{meas}(O \setminus O_\gamma) = O(\gamma^{\frac{1}{4}})$;

2) a family of real analytic symplectic maps $\Phi : D_{\rho/2}(s/2,r/2) \times O_\gamma \to \mathcal{P}_{\mathbb{C}}^{a,p}$ with
\[
\|\Phi - id\|_{D_{\rho/2}(s/2,r/2)}^* \leq \varepsilon_0^{\frac{3}{4}}; \tag{2.7}
\]

3) a family of normal forms
\[
N^* + A^* = \langle \omega^*, I \rangle + \sum_{j \in \mathbb{Z}} \Omega_j^*(\xi)z_n \bar{z}_n + \sum_{n \in \mathbb{Z}} a_n^*(\xi)z_n \bar{z}_{-n}
\]
defined on $D_{\rho/2}(s/2,r/2) \times O_\gamma$, such that
\[
H \circ \Phi = N^* + A^* + P^*, \tag{2.8}
\]
where the Taylor series expansion of $P^*$ only contains monomials of the form $I^m z^q \bar{z}^\bar{q}$ with $2|m| + |q + \bar{q}| \geq 3$, and
\[
|\omega^*|_{O_\gamma}, \sup_{n \in \mathbb{Z}} |n|^{2\beta}(\Omega_n^* - \Omega_n)|_{O_\gamma}, \sup_{n \in \mathbb{Z}} |n|^{2\beta}e^{|n|\rho}|a_{n,-n}|_{O_\gamma} \leq c\varepsilon_0. \tag{2.9}
\]

2.1 Application to the fractional NLS equation

Imposing periodic boundary conditions, we apply Theorem 1 to the fractional NLS equation
\[
iu_t - |\partial_x|^\frac{\beta}{2}u + \lambda u = e\Psi(V(t\omega,x;\xi)\Psi u), \quad x \in \mathbb{T}, t \in \mathbb{R}, \tag{2.10}
\]
where the convolution operator $\Psi : u \to \Psi * u$ is given with function $\psi(x)$, which is smooth and of order $\beta > 0$. More precisely, $\|\Psi u\|_{\rho,\rho+\beta} \leq c\|u\|_{\rho,\rho}$. The parameter $\lambda$ is positive, $\lambda > 0$. The function $V : \mathbb{T}^d \times \mathbb{T} \times O \ni (\theta,x;\xi) \mapsto \mathbb{R}$ is real analytic in $\theta$ and $x$, and Lipschitz in $\xi$. For $\rho > 0$, function $V(\theta,x;\xi)$ extends analytically to the domain $\mathbb{T}_\rho^d \times \mathbb{T}_\rho$, with $\mathbb{T}_\rho^d = \{a + ib \in \mathbb{C}^d / 2\pi \mathbb{Z}^d : |b| \leq \rho\}$. It is noted that, in the physics literature, the fractional Schrödinger equation was introduced by Laskin [20] in deriving a
fractional version of the classical quantum mechanics. Subsequently, many works have been done on such equations; see [14, 18] for details.

Let \( \{ \phi_n(x) = \sqrt{\frac{1}{2\pi}} e^{i\langle n,x \rangle} \}_{n \in \mathbb{Z}} \) denote the standard Fourier basis of operator \(|\partial_x|^{\frac{1}{2}} + \lambda\) and \( \{ \Omega_n = |n|^{\frac{1}{2}} + \lambda \}_{n \in \mathbb{Z}} \) be its eigenvalues. Expanding \( u \) and \( \bar{u} \) in this basis, specifically, \( u = \sum_{n \in \mathbb{Z}} z_n \phi_n(x) \) and \( \bar{u} = \sum_{n \in \mathbb{Z}} \bar{z}_n \phi_n(x) \), and the equation (2.10) can be written as a non-autonomous Hamiltonian system

\[
\begin{aligned}
\dot{z}_n &= -i \Omega_n z_n - i \partial_{\bar{z}_n} P(\omega t, \varphi, z, \bar{z}; \xi), \quad n \in \mathbb{Z}, \\
\dot{\bar{z}}_n &= i \Omega_n \bar{z}_n + i \partial_{z_n} P(\omega t, \varphi, z, \bar{z}; \xi), \quad n \in \mathbb{Z}.
\end{aligned}
\] (2.11)

We then re-interpret (2.11) as an autonomous Hamiltonian system in the extended phase space \( \mathcal{R}^{a, \rho} \),

\[
\begin{aligned}
\dot{I} &= -\partial_\theta P(\theta, z, \bar{z}; \xi), \\
\dot{\theta} &= \omega, \\
\dot{z}_n &= -i \Omega_n z_n - i \epsilon \partial_{\bar{z}_n} P(\theta, z, \bar{z}; \xi), \quad n \in \mathbb{Z}, \\
\dot{\bar{z}}_n &= i \Omega_n \bar{z}_n + i \epsilon \partial_{z_n} P(\theta, z, \bar{z}; \xi), \quad n \in \mathbb{Z},
\end{aligned}
\] (2.12)

with perturbation

\[
P(\theta, z, \bar{z}; \xi) = \epsilon \int_T V(\theta, x; \xi)(\sum_{n \in \mathbb{Z}} z_n \phi_n(x))(\sum_{n \in \mathbb{Z}} \bar{z}_n \bar{\phi}_n(x)) dx.
\]

The last three equations of (2.12) are independent of \( I \) and are equivalent to (2.10). Furthermore, (2.12) determines a Hamiltonian system associated with Hamiltonian

\[
H = N + P = \langle \omega, I \rangle + \sum_{n \in \mathbb{Z}} \Omega_n |z_n|^2 + P(\theta, z, \bar{z}; \xi)
\] (2.13)

with symplectic structure \( dI \wedge d\theta + i \sum_{n \in \mathbb{Z}} dz_n \wedge d\bar{z}_n \). The external parameters are explicitly the frequencies \( \omega \in [0, 2\pi]^d \).

We now verify that (2.13) satisfies all the assumptions of Theorem 1.

Verification of Assumption A: It is obvious.

Verification of Assumption B: Since \( V(\theta, x; \xi) \) is analytic in \( x \) and \( \theta \), for any \( n \in \mathbb{Z} \), one has uniformly

\[
|\frac{\partial P}{\partial z_n}| = |\epsilon \int_T \Psi(\theta, x; \xi) \Psi \phi_n(x) dx| \leq c e^{-|n|^\rho \langle n \rangle^{-\beta}}, \quad \forall \theta \in \mathbb{T}^d, \forall \xi \in \mathcal{O}.
\]
Similarly, \(|\frac{\partial P}{\partial z_n}| \leq c e^{-|n|\rho} < |n|^{-\beta}, \ \forall \theta \in \mathbb{T}^d, \ \forall \xi \in \mathcal{O}, \ \forall n \in \mathbb{Z}\). Clearly,

\[
\frac{\partial^2 P}{\partial z_m \partial z_n} = \frac{\partial^2 P}{\partial \bar{z}_m \partial \bar{z}_n} = 0, \ \forall n, m \in \mathbb{Z}.
\]

If we write \(\hat{V}(\theta, x; \xi) = \sum_{k \in \mathbb{Z}} \hat{V}_k(\theta; \xi) e^{i(k,x)} dx\), then

\[
\left| \frac{\partial^2 P}{\partial z_m \partial z_n} \right| = \left| \frac{\partial \hat{V}_{m-n}}{(n)\beta(n)\beta} \right| \leq c e^{-|n-m|\rho} \frac{(n)\beta(n)\beta}{(n)\beta(n)\beta}, \ \forall \theta \in \mathbb{T}_\rho^d, \ \forall \xi \in \mathcal{O}, \ \forall n, m \in \mathbb{Z}.
\]

Thus, the Assumption \(B\) obtains if we set \(\epsilon\) sufficiently small.

Following [13], we have

**Theorem 2** For any \(0 < \epsilon \leq \epsilon_0\), where \(\epsilon_0\) is sufficiently small, there exists a Cantor-like set \(\mathcal{O}_\epsilon\) of positive measure and \(\text{meas}(\mathcal{O}_\epsilon) \to (2\pi)^d\) as \(\epsilon \to 0\), such that for \(\omega \in \mathcal{O}_\epsilon\) and \(\varphi \in \mathbb{T}^d\), there exists a complex-linear isomorphism \(\Psi = \Psi(\varphi; \omega)\) in the space \(L^2(\mathbb{T}^d)\), which depends analytically on \(\varphi \in \mathbb{T}_{\rho/2}^d\) and a bounded Hermitian matrix \(A_{\mathbb{Z} \times \mathbb{Z}}\) with

\[
A_{n,m} = 0, \quad n \neq -m. \quad (2.14)
\]

The following holds: a curve \(v(t) = v(t, \cdot) \in L^2(\mathbb{T}^d)\) satisfies the autonomous equation

\[
\dot{v} = i\Delta v + i\epsilon Av \quad (2.15)
\]

if and only if \(v(t, \cdot) = \Psi(\varphi_0 + t\omega)v(t, \cdot)\) is a solution of \((2.10)\).

As \(A\) is Hermitian and satisfies \((2.14)\), then the spectrum of the linear operator on the r.h.s. of \((2.15)\) is a pure point and is imaginary. Hence, all the solutions \(v(t) \in L^2(\mathbb{T}^d)\) of \((2.15)\) are almost-periodic functions of \(t\).

### 3 KAM STEP

Theorem 1 is proved by a KAM iteration, which involves an infinite sequence of changes in variables. Each step of the KAM iteration makes the perturbation smaller than before in a narrower parameter set and analytic domain. The main task to show is that the new perturbation still satisfies the Assumption \(B2\).

At the \(\nu\)-step of the KAM iteration, we consider a Hamiltonian

\[
H_{\nu} = H_0 + A_{\nu} + P_{\nu}
\]
defined on $D\rho(\nu, s, r) \times O\nu$, where the Assumption $A$ and the $B$ are satisfied. We construct a symplectic change of variables

$$ \Phi_\nu : D\rho_{\nu+1}(r_{\nu+1}, s_{\nu+1}) \times O\nu+1 \to D\rho_\nu(r, s), $$

such that the vector field $X_{H\nu} \circ \Phi_\nu$ defined on $D\rho_{\nu+1}(r_{\nu+1}, s_{\nu+1}) \times O\nu+1$ satisfies

$$ \|X_{P_{\nu+1}}\|_{r_{\nu+1}, D\rho_{\nu+1}} \leq \varepsilon_{\nu+1}, $$

with new normal form $N_{\nu+1} + A_{\nu+1}$. Moreover, the new perturbation $P_{\nu+1}$ still satisfies the Assumption $B$.

For simplicity of notation in the following, the quantities without subscripts refer to the quantities at the $\nu$th step, whereas the quantities with subscripts $+$ denote the corresponding quantities at the $(\nu+1)$th step.

Let us then consider Hamiltonian $H = N + A + P$ with

$$ N = \langle \omega(\xi), I \rangle + \sum_{n \in \mathbb{Z}} \Omega_n(\xi) z_n \bar{z}_n, \quad A = \sum_{1 \leq |n| \leq K} a_{n,-n}(\xi) z_n \bar{z}_n, \quad (3.1) $$

on $D\rho(s, r) \times O$, where $|a_{n,-n}|^* \leq \varepsilon_0 e^{-2|n|^2} (n)^{-2\beta}$ and $K$ is the truncation parameter. The corresponding symplectic structure is $dI \wedge d\theta + i \sum_{n \in \mathbb{Z}} dz_n \wedge d\bar{z}_n$. The normal frequencies are assumed to satisfy

$$ ||n||_{H\nu}^{2\beta} (\Omega_n - |n|^\alpha - \lambda) |\Omega_n| \leq \varepsilon, \forall n \in \mathbb{Z}. \quad (3.2) $$

For ease of notation, we set

$$ a_{n,-n} = 0 \text{ if } |n| > K \quad (3.3) $$

and define

$$ A_0 = \Omega_0, \quad A_n = \begin{pmatrix} \Omega_n & a_n \\ a_{-n,n} & \Omega_{-n} \end{pmatrix}, \quad |n| \geq 1. \quad (3.4) $$

Let

$$ \tau = 12\tau_1 + 16\zeta, \quad \tau_1 \geq d + 3 + \frac{4}{\alpha^2}; \zeta = \frac{\tau_1 + 1}{1 - \alpha}. \quad (3.5) $$

The parameter $\tau_1$ is only used in the section on the measure estimate. We
now assume that, for \( \xi \in \mathcal{O} \) and \( |k| \leq K \), there is
\[
\| (k, \omega)^{-1} \| < \frac{K^7}{\gamma}, \quad k \neq 0
\]
\[
\| (k, \omega) I_n + A_n \|^{-1} < \frac{K^2}{\gamma},
\]
\[
\| (k, \omega) I_{nm} \pm (A_n \otimes I_n + I_m \otimes A_m) \|^{-1} < \frac{K^4}{\gamma},
\]
\[
\| (k, \omega) I_{nm} \pm (A_n \otimes I_n - I_m \otimes A_m) \|^{-1} < \frac{K^4}{\gamma}, \quad k \neq 0 \& |n - m| < K.
\]

where \( I_n \) and \( I_{nm} \) are identity matrices, \( \text{diam} I_n = \text{diam} A_n \) and \( \text{diam} I_{nm} = \text{diam} A_n \times \text{diam} A_m \).

Let \( R \) be the truncation of \( P(\theta, I, z, \bar{z}; \xi) \) with \( K \),
\[
R(\theta, I, z, \bar{z}; \xi) = \sum_{|k| \leq K, |k| \pm |q| \leq 2, \sum_{i,j} |i_j| \leq K} R_{klqq} e^{i(k, \theta)} I^l z^q \bar{z}^q,
\]
(3.6)

where \( R_{klqq} = P_{klqq} \). For ease of notation, we rewrite it as
\[
R = R^0 + R^1 + R^{10} + R^{01} + R^{20} + R^{11} + R^{02}
\]
\[
= \sum_{|k| \leq K} R_{k}^0 e^{i(k, \theta)} + \sum_{|k| \leq K} (R_{k}^1, I) e^{i(k, \theta)} + \sum_{|k| \leq K, n \in \mathbb{Z}} R_{k,n}^{10} z_n e^{i(k, \theta)}
\]
\[
+ \sum_{|k| \leq K, n \in \mathbb{Z}} R_{k,n}^{01} z_n e^{i(k, \theta)} + \sum_{|k| \leq K, n \in \mathbb{Z}} R_{k,n}^{20} z_n z_m e^{i(k, \theta)}
\]
\[
+ \sum_{|k| \leq K, n \in \mathbb{Z}} R_{k,n}^{11} z_n z_m e^{i(k, \theta)} + \sum_{|k| \leq K, n \in \mathbb{Z}} R_{k,n}^{02} z_n z_m e^{i(k, \theta)},
\]

where \( R_{k,n}^{10} = R_{k0q_00}, \quad q_n = (\cdots, 0, \cdots, 1, 0, \cdots, 0, \cdots) \) and 1 is at the \( n^{th} \) position; \( R_{k,n}^{01} = R_{k00q_n} \); \( R_{k,n}^{20} = R_{k0q_0n} \) with \( q_{nm} = q_n + q_m \); \( R_{k,n}^{11} = \)

\(^{1}\)The tensor product (or direct product) of two \( m \times n \) matrices \( A = (a_{ij}) \), \( B \) is \( (mk) \times (nl) \) matrix defined by
\[
A \otimes B = (a_{ij} B) = \begin{pmatrix}
a_{11} B & \cdots & a_{1n} B \\
\vdots & \ddots & \vdots \\
a_{m1} B & \cdots & a_{mn} B
\end{pmatrix}.
\]

Let \( a_{n-m} = 0, \quad |n| \neq |m| \), and \( n, m \neq 0 \), then \( \text{diam} I_n = 2 \) and \( \text{diam} I_{nm} = 2 \)
\[
(k, \omega) I_{nm} \pm (A_n \otimes I_n - I_m \otimes A_m) = \text{diag}(k, \omega) + \Omega_n - \Omega_{nm}.
\]

One may refer to [6] for more information on this symbol.
\( R_{k0q_n} ; R_{k,nm}^{02} = R_{k00q_m} \) with \( q_{nm} = q_n + q_m \). The generalized mean part of \( R \) is defined as

\[
\langle R \rangle := \langle R_0^1, I \rangle + \sum_{n \in \mathbb{Z}} R_{0,nn}^{11} |z_n|^2 + \sum_{n \in \mathbb{Z}} R_{0,n,-n}^{11} z_n \tilde{z}_{-n}.
\]  

(3.7)

Let \( F(\theta, I, z, \tilde{z}; \xi) \) be the solution of the so-called homological equation

\[
\{ N + A, F \} + R - \langle R \rangle = 0.
\]  

(3.8)

As usual, the function \( F \) is assumed to have the same form as \( R \); that is,

\[
F = F_0 + F^1 + F^{01} + F^{10} + F^{11} + F^{02}.
\]  

(3.9)

Once we can solve equation (3.8) in a proper space, let \( X_F^1 \) be the flow of \( X_F \) at time \( t \) associated with the vector field of \( F \). We have a new Hamiltonian,

\[
H \circ X_F^1 = (N + A + R) \circ X_F^1 + (P - R) \circ X_F^1
\]  

(3.10)

\[
= N + \{ N + A, F \} + R + \int_0^1 (1 - t) \{ \{ N + A, F \}, F \} \circ X_F^t dt
\]  

\[
+ \int_0^1 \{ R, F \} \circ X_F^t dt + (P - R) \circ X_F^1
\]  

\[
= N_+ + A_+ + P_+,
\]

where the new perturbation,

\[
P_+ =: \int_0^1 \{ (1 - t) \langle R \rangle + tR, F \} \circ X_F^t dt + (P - R) \circ X_F^1,
\]  

(3.11)

and the new normal forms \( N_+ \) and \( A_+ \) have the same form as (3.1) with

\[
\omega_+(\xi) = \omega + R_0^3, \quad \Omega_+ = \Omega_n + R_{0,nn}^{11}, \quad a_{n,-n}^+ = a_{n,-n} + R_{0,nn}^{11}.
\]  

(3.12)

It is easy to check that the function \( F \) is not in the space \( \Gamma^{\beta,\alpha}_{r,D(s,r)} \) because the growth of frequencies is sublinear (see 3.26). We shall prove that the homological solution \( F \) is in class \( \Gamma^{\beta,\alpha}_{r,D(s,r)} \). We say that \( F \in \Gamma^{\beta,\alpha}_{r,D(s,r)} \) if \( \| F \|_{r,D(s,r)}^{\beta,\alpha} < \infty \). Like \( \| . \|_{r,D(s,r)}^{\beta,\alpha} \), the semi-norm \( \| . \|_{r,D(s,r)}^{\beta,\alpha} \) is defined by the
conditions

\[ \| F \|_{D_{\rho}(s,r)} \leq r^2 \sqrt{\| F \|_{r,D_{\rho}(s,r)}^{\beta,\alpha}}, \]
\[ \max_{1 \leq j \leq d} \| \frac{\partial F}{\partial I_j} \|_{D_{\rho}(s,r)} \leq r \sqrt{\| F \|_{r,D_{\rho}(s,r)}^{\beta,\alpha}} e^{-\frac{|n|\rho}{\gamma}}, \]
\[ \| \frac{\partial F}{\partial z_n} \|_{D_{\rho}(s,r)} \| \frac{\partial F}{\partial z_m} \|_{D_{\rho}(s,r)} \leq \frac{\| F \|_{r,D_{\rho}(s,r)}^{\beta,\alpha}}{\gamma}, \]
\[ \| \frac{\partial^2 F}{\partial z_n \partial z_m} \|_{D_{\rho}(s,r)} \leq \frac{\| F \|_{r,D_{\rho}(s,r)}^{\beta,\alpha}}{\gamma^2}, \]
\[ \| \frac{\partial^2 (F - [F])}{\partial z_n \partial z_m} \|_{D_{\rho}(s,r)} \leq \frac{\| F \|_{r,D_{\rho}(s,r)}^{\beta,\alpha}}{\gamma}, \]
\[ \| \frac{\partial^2 [F]}{\partial z_n \partial z_m} \|_{D_{\rho}(s,r)} \leq \frac{\| F \|_{r,D_{\rho}(s,r)}^{\beta,\alpha}}{\gamma^2}, \]

where \([F(\theta, I, z, \bar{z}; \xi)] = \int_{\mathbb{T}} F(\theta, I, z, \bar{z}; \xi) d\theta\) and \(n, m \in \mathbb{Z}\).

### 3.1 Homological Equation

We next solve the homological equation and then prove that \(F \in \Gamma_{r,D_{\rho}(s-s,r)}^{\beta,\alpha}\).

The regularity of \(F\) is also given.

**Lemma 3.1** Let \(0 < \sigma < s, 0 < \mu < \rho, K > 0\), and \(R \in \Gamma_{r,D_{\rho}(s,r)}^{\beta,\alpha}\) of the form

\[ R = \sum_{|k| \leq K, \sum_{j \in \mathbb{Z}} |q_j| \leq 2, \sum_{j \in \mathbb{Z}} |q_j| \leq K} R_{kq} e^{i(k, \theta)} I^l z^q \bar{z}^q. \]

Assume that for any \(\xi \in \mathcal{O}, |k| \leq K\) and \(n, m \in \mathbb{Z}\), we have

\[ \| (k, \omega)^{-1} \| \leq \frac{\gamma}{K^\tau}, k \neq 0 \]
\[ \| (k, \omega)(I_n + A_n)^{-1} \| \leq \frac{K^{2\tau}}{\gamma}, \]
\[ \| (k, \omega)(I_{nm} \pm (A_n \otimes I_n + I_m \otimes A_m))^{-1} \| \leq \frac{K^{4\tau}}{\gamma}, \]
\[ \| (k, \omega)(I_{nm} \pm (A_n \otimes I_n - I_m \otimes A_m))^{-1} \| \leq \frac{K^{4\tau}}{\gamma}, k \neq 0 \& |n - m| < K. \]
Then the homological equation (3.8) has a solution $F(\theta, I, z, \bar{z}; \xi)$ with $F \in \Gamma^{\beta,\alpha}_{r,D}(s-\sigma,r)$, such that

$$[F]_{r,D}(s-\sigma,r)^{\beta,\alpha,*} \leq \frac{CK^{8\tau} [R]_{r,D}(s,r)^{\beta,*}}{\gamma^2 \sigma^{d+1}}.$$  (3.14)

Proof: From the structure of $N$ and $R$, the homological equation (3.8) is equivalent to

$$\{N, F^0 + F^1\} + R^0 + R^1 - \langle R_0^1, I \rangle = 0,$$  (3.15)

$$\{N + A, F^{10}\} + R^{10} = 0,$$  (3.16)

$$\{N + A, F^{01}\} + R^{01} = 0,$$  (3.17)

$$\{N + A, F^{11}\} + R^{11} - \sum_{|n| = |m|} R_{0,nm}^1 z_n \bar{z}_m = 0,$$  (3.18)

$$\{N + A, F^{20}\} + R^{20} = 0,$$  (3.19)

$$\{N + A, F^{02}\} + R^{02} = 0.$$  (3.20)

Solving the homological equation.

Solving (3.15): Let $j = 0$ or $1$, then $F^j(\theta) = \sum_{0 < |k| \leq K} F_k^j e^{i(k,\theta)}$ are constructed by setting

$$F_k^j = \frac{1}{i(k,\omega)} R_k^j, 0 < |k| \leq K, j = 0, 1.$$  (3.22)

Given the assumption (3.13), for $0 < |k| \leq K$ and $\xi \in \mathcal{O}$, there is

$$\|\langle k, \omega(\xi) \rangle^{-1}\| < \frac{K^7}{\gamma}.$$  (3.23)

Since $R \in \Gamma^{\beta}_{r,D}(s,r)$, we have

$$|F_k^j|_{\mathcal{O}} \leq r^{2-2j} \gamma^{-2} K^{2\tau} \|R\|^{\beta,*}_{r,D}(s,r), 0 < |k| \leq K, j = 0, 1.$$  (3.24)

Solving (3.16): For any $n \in \mathbb{Z}$, we have

$$\langle (k,\omega) + \Omega_n \rangle F_{k,-n}^{10} + a_{n,-n} F_{k,-n}^{10} = -i R_{k,n}^{10},$$  (3.25)

$$\langle (k,\omega) + \Omega_{-n} \rangle F_{k,n}^{10} + a_{-n,n} F_{k,n}^{10} = -i R_{k,n}^{10}.$$  (3.26)

The above equations can be written as
with
\[
Q_{k,[n]}^{10} = (F_{k,n}^{10}, F_{k,-n}^{10}), \quad R_{k,[n]}^{10} = (R_{k,n}^{10}, R_{k,-n}^{10}).
\]
As \( R \in \Gamma_{r,D(s,r)}^{\beta} \), one has
\[
|R_{k,n}^{10}|_\mathcal{O}, |R_{k,n}^{10}|_\mathcal{O} \leq r[R]_{r,D(s,r)}^{\beta,*} e^{-|k|s} e^{-|n|}\rho(n)^{-\beta}.
\]
By the small-divisor assumptions (3.13),
\[
||\langle (k, \omega) \rangle_{n} + A_{n} \rangle^{-1} | \leq \frac{K^{2r}}{\gamma}, |k| \leq K,
\]
we obtain
\[
|F_{k,n}^{10}|_\mathcal{O} \leq \gamma^{-2} K^{4r} r[R]_{r,D(s,r)}^{\beta,*} e^{-|k|s} e^{-|n|}\rho(n)^{-\beta}.
\] (3.23)
The equation (3.17) can be done in the same way.

**Solving (3.18):** First, we consider instances with \( k \neq 0 \). Comparing the Fourier coefficients, we have \( F_{k,n,m}^{11}, F_{k,n,-m}, F_{k,-n,m}^{11}, F_{k,-n,-m}^{11} \) satisfying
\[
\begin{align*}
\langle (k, \omega) + \Omega_{n} - \Omega_{m} \rangle F_{k,n,m}^{11} & + a_{n} F_{k,n,m}^{11} - a_{m} F_{k,n,-m}^{11} = -i R_{k,n,m}^{11}, \\
\langle (k, \omega) + \Omega_{n} - \Omega_{-m} \rangle F_{k,n,-m}^{11} & + a_{n} F_{k,n,-m}^{11} - a_{m} F_{k,-n,m}^{11} = -i R_{k,n,-m}^{11}, \\
\langle (k, \omega) + \Omega_{-n} - \Omega_{m} \rangle F_{k,-n,m}^{11} & + a_{n} F_{k,-n,m}^{11} - a_{m} F_{k,-n,-m}^{11} = -i R_{k,-n,m}^{11}, \\
\langle (k, \omega) + \Omega_{-n} - \Omega_{-m} \rangle F_{k,-n,-m}^{11} & + a_{n} F_{k,-n,-m}^{11} - a_{m} F_{k,-n,-m}^{11} = -i R_{k,-n,-m}^{11}. 
\end{align*}
\]
These equations can be written as
\[
\langle (k, \omega) \rangle_{n,m} + A_{n} \otimes I_{m} - I_{m} \otimes A_{m} \rangle Q_{k,[n],[m]}^{11} = -i R_{k,[n],[m]}^{11} \quad (3.24)
\]
with
\[
Q_{k,[n],[m]}^{11} = (F_{k,n,m}^{11}, F_{k,n,-m}, F_{k,-n,m}^{11}, F_{k,-n,-m}^{11}), \\
R_{k,[n],[m]}^{11} = (R_{k,n,m}^{11}, R_{k,n,-m}, R_{k,-n,m}^{11}, R_{k,-n,-m}^{11}).
\]
As \( R \in \Gamma_{r,D(s,r)}^{\beta} \), one has
\[
|R_{k,n,m}^{11}|_{\mathcal{O}} \leq [R]_{r,D(s,r)}^{\beta,*} e^{-|k|s} e^{-|n-m|}\rho(n)^{-\beta} \rho(m)^{-\beta}, \forall n, m \in \mathbb{Z}.
\]
Thus with the small divisor assumption (3.13),

\[ \|(k, \omega)\mathbb{I}_{nm} \pm (A_n \otimes \mathbb{I}_n - \mathbb{I}_m \otimes A_m)\|^{-1} < \frac{K^{4\tau}}{\gamma}, \]

we have

\[ |F_{k,n,m}^{11}| \leq \gamma^{-2} K^{8\tau} \|R\|_{r,k(m,s,r)}^\beta e^{-|k|e^{-|n-m|\rho(n)^{-\beta}}(m)^{-\beta}} k \neq 0. \] (3.25)

Second, we solve (3.18) setting \( k = 0 \). By (3.18), we only need to consider instances \( |n| \neq |m| \), for which the equation (3.24) takes the form

\[ (A_n \otimes \mathbb{I}_n - \mathbb{I}_m \otimes A_m)Q_{0,n,m}^{11} = -i R_{0,n,m}^{11}. \]

Recall (3.4), \( \alpha + \beta \geq 1 \) and Lemma A.1, the matrix \( A_n \otimes \mathbb{I}_n - \mathbb{I}_m \otimes A_m \) is diagonally dominant. One has

\[ \|(A_n \otimes \mathbb{I}_n - \mathbb{I}_m \otimes A_m)\|^{-1} \leq \frac{1}{2} |n|^\alpha - |m|^\alpha \]

and then

\[ |F_{0,n,m}^{11}| \leq \|[R]_{r,k(m,s,r)}^\beta e^{-|n-m|^\rho(n)^{-\beta}}(m)^{-\beta}|n|^\alpha - |m|^\alpha \|^{-1}. \] (3.26)

Solving (3.19): Comparing the Fourier coefficients, we have that \( F_{k,n,m}^{20}, F_{k,n,-m}^{20}, F_{k,-n,m}^{20}, \) and \( F_{k,-n,-m}^{20} \) satisfy

\[
\begin{align*}
&((k, \omega) + \Omega_n + \Omega_m)F_{k,n,m}^{20} + a_{n,-n}F_{k,n,-m}^{20} + a_{m,-m}F_{k,n,m}^{11} = -iR_{k,n,m}^{20}, \\
&((k, \omega) + \Omega_n + \Omega_m)F_{k,n,-m}^{20} + a_{n,-n}F_{k,n,m}^{20} + \bar{a}_{m,-m}F_{k,n,m}^{20} = -iR_{k,n,-m}^{20}, \\
&((k, \omega) + \Omega_n + \Omega_m)F_{k,-n,m}^{20} + a_{n,-n}F_{k,-n,-m}^{20} + a_{m,-m}F_{k,n,m}^{20} = -iR_{k,-n,m}^{20}, \\
&((k, \omega) + \Omega_n + \Omega_m)F_{k,-n,-m}^{20} + a_{n,-n}F_{k,-n,m}^{20} + \bar{a}_{m,-m}F_{k,-n,-m}^{20} = -iR_{k,-n,-m}^{20}.
\end{align*}
\]

The above equations can be rewritten as

\[ ((k, \omega)\mathbb{I}_{nm} + A_n \otimes \mathbb{I}_n + \mathbb{I}_m \otimes A_m)Q_{k,n,m}^{20} = -i R_{k,n,m}^{20} \]

with

\[ Q_{k,n,m}^{20} = (F_{k,n,m}^{20}, F_{k,n,-m}^{20}, F_{k,-n,m}^{20}, F_{k,-n,-m}^{20}), \]

\[ R_{k,n,m}^{20} = (R_{k,n,m}^{20}, R_{k,n,-m}^{20}, R_{k,-n,m}^{20}, R_{k,-n,-m}^{20}). \]
As $R \in \Gamma^k_{r,D,s(r)}$, one has
\[
|R_{k,n,m}^0|_{\mathcal{O}} \leq |R|_{r,D,s(r)}^\beta e^{-|k|\sigma - |n+m|\rho} \langle n \rangle^{-\beta} \langle m \rangle^{-\beta}, \forall n, m \in \mathbb{Z}.
\]

Recalling the small divisor assumption \((3.13)\), one has
\[
\|(\langle k, \omega \rangle I_{nm} \pm (A_n \otimes I_n + I_m \otimes A_m))^{-1}\| < \frac{K^{4\tau}}{\gamma},
\]
and then
\[
|F_{k,n,m}^0|_{\mathcal{O}} \leq \gamma^{-2} K^{8\tau} |R|_{r,D,s(r)}^\beta e^{-|k|\sigma - |n+m|\rho} \langle n \rangle^\beta \langle m \rangle^\beta.
\]

The equation \((3.27)\) can be treated in the same way.

To complete the proof, it suffices to estimate $\frac{\partial F}{\partial \bar{z}_n}$ and $\frac{\partial^2 F}{\partial \bar{z}_n \partial \bar{z}_m}$. Using \((3.21), (3.23), (3.25), (3.26), (3.27)\), and Lemma A.1, we take the sum in $m$ and $k$,
\[
\frac{\partial F}{\partial \bar{z}_n}|_{D_{\rho}(s-\sigma,r)}
\]
\[
= \sum_{|k| \leq K} F_{k,n}^{10} + \sum_{|k| \leq K, |n+m| \leq K} F_{k,n,m}^{20} z_m + \sum_{|k| \leq K, |n-m| \leq K} F_{k,n,m}^{11} z_m |_{D_{\rho}(s-\sigma,r)}
\]
\[
\leq \frac{K^{8\tau}|R|_{r,D,s(r)}^\beta e^{-|n|\rho}}{\gamma^{2\sigma d+1} \langle n \rangle^\beta} \left(r e^{-|n|\rho} + \sum_{|n+m| \leq K} \frac{e^{-|n+m|\rho} r e^{-|m|\rho}}{\langle m \rangle^\beta \langle n \rangle^\beta} + \sum_{|n-m| \leq K} \frac{e^{-|n-m|\rho} r e^{-|m|\rho}}{\langle m \rangle^\beta |n|^{\alpha} - |m|^{\alpha} \langle m \rangle^\beta} \right)
\]
\[
\leq \frac{r K^{8\tau+1}|R|_{r,D,s(r)}^\beta e^{-|n|\rho}}{\gamma^{2\sigma d+1} \langle n \rangle^\beta} \left(1 + \sum_{|n+m| \leq K} \frac{1}{\langle m \rangle^\beta \langle n \rangle^\beta} + \sum_{|n-m| \leq K} \frac{1}{\langle m \rangle^\beta |n|^{\alpha} - |m|^{\alpha} \langle m \rangle^\beta} \right)
\]
\[
\leq \frac{r K^{8\tau+1}|R|_{r,D,s(r)}^\beta e^{-|n|\rho}}{\gamma^{2\sigma d+1} \langle n \rangle^\beta}.
\]

The last inequality follows as $\alpha + \beta \geq 1$.

If $|n| \neq |m|$, we have
\[
\frac{\partial^2 (F - \langle F \rangle)}{\partial \bar{z}_n \partial \bar{z}_m}|_{D_{\rho}(s-\sigma,r)} \leq \sum_{0 < |k| \leq K} F_{k,n,m}^{11} e^{i(k,\theta)} |_{D_{\rho}(s-\sigma,r)}
\]
\[
\leq \frac{K^{8\tau}|R|_{r,D,s(r)}^\beta e^{-|n-m|\rho}}{\gamma^{2\sigma d+1} \langle n \rangle^\beta \langle m \rangle^\beta}.
\]
and
\[ |\partial^2[F]_{\partial z_n \partial z_m}|_{D_\rho(s-\sigma,r)} \leq |F^{11}_{0,n,m}|_{D(s-\sigma,r)} \leq C[R]_{r,D_\rho(s,r)}^{\beta,*} e^{-|n-m|\rho} \frac{C(\rho)}{(n)\beta(m)\beta|n|^\alpha - |m|^\alpha}. \tag{3.30} \]

If \( n = m \), then
\[ |\partial^2 F_{\partial z_n \partial z_n}|_{D_\rho(s-\sigma,r)} \leq \sum_{0 < |k| \leq K} F^{11}_{k,n} e^{i(k,\theta)} |\partial z|^\beta F^{11}_{n} 0 \leq R_{r,D_\rho(s,r)}^{\beta,*} K^{2r} \gamma^2 \sigma d+1 (n)\beta. \]

If \( n = -m \) and \( |n| \geq K^2 \), one has \( \partial^2 F_{\partial z_n \partial z_{-n}} = 0 \) by the restriction on (3.6).

With these observations, we have
\[ [F]_{r,D_\rho(s-\sigma,r)}^{\beta,\alpha} \leq CK^{8r+1} \frac{\rho - |\alpha|}{n} F^{11}_{k,n} e^{i(k,\theta)} |\partial z|^\beta F^{11}_{n} 0 \leq R_{r,D_\rho(s,r)}^{\beta,*} K^{2r} \gamma^2 \sigma d+1 (n)\beta. \]

The estimation on the Lipschitz semi-norm of \( F \) is standard. Here we consider only \( F^{11}_{0,n,m} \) as an example. Recall (3.4) and let \( |n|, |m| \geq K^3 \), (3.18) can be written as
\[ (\Omega_n - \Omega_m) F^{11}_{0,n,m} = -i R^{11}_{0,n,m}. \]

One has
\[ \Delta \xi \eta F^{11}_{0,n,m} = -\frac{i \Delta \xi \eta R^{11}_{0,n,m} + F^{11}_{0,n,m} \Delta \xi \eta (\Omega_n - \Omega_m)}{\Omega_n - \Omega_m} = -\frac{i \Delta \xi \eta R^{11}_{0,n,m} + F^{11}_{0,n,m} \Delta \xi \eta (\tilde{\Omega}_n - \tilde{\Omega}_m)}{\Omega_n - \Omega_m} \]

By (3.26), we have
\[ |\Delta \xi \eta F^{11}_{0,n,m}| \leq \frac{|\Delta \xi \eta R^{11}_{0,n,m}|}{|n|\alpha - |m|\alpha} + \frac{\rho - |\alpha|}{n} F^{11}_{k,n} e^{i(k,\theta)} |\partial z|^\beta F^{11}_{n} (n)\beta(m)\beta|n|^\alpha - |m|^\alpha^2 |\Delta \xi \eta (\tilde{\Omega}_n - \tilde{\Omega}_m)|. \]

Hence
\[ |\Delta \xi \eta F^{11}_{0,n,m}| \leq \frac{|\Delta \xi \eta R^{11}_{0,n,m}|}{|\xi - \eta|} + \frac{\rho - |\alpha|}{n} F^{11}_{k,n} e^{i(k,\theta)} |\partial z|^\beta F^{11}_{n} (n)\beta(m)\beta|n|^\alpha - |m|^\alpha^2 |\Delta \xi \eta (\tilde{\Omega}_n - \tilde{\Omega}_m)|. \]

Note that \( |\tilde{\Omega}_n| \leq \frac{L}{|n|^{2\alpha}} \) for \( n \in \mathbb{Z} \), we have
\[ |F^{11}_{0,n,m}|_{\tilde{\Omega}} \leq \frac{R^{\beta,*}_{r,D_\rho(s,r)} e^{-|n-m|\rho}}{(n)\beta(m)\beta} \left( \frac{1}{|n|\alpha - |m|\alpha} + \frac{1}{(n)\beta |m|\alpha^2 |m|^{2\alpha}} \right) L. \tag{3.31} \]
Recall that by (3.6), one has $|n - m| \leq K$. Let $a = n - m \neq 0$; then by Lemma A.1 and condition $\alpha + \beta \geq 1$, we have

$$\frac{1}{(|n|^\alpha - |m|^\alpha)^2} \frac{1}{|n|^{2\beta}} \leq \frac{L}{|n|^{2\beta + 2\alpha - 2}} \leq L.$$ 

Finally, there is

$$|F^1_{0,n,m}|_{\Theta} \leq C [R]^{\beta,s}_{r,D\rho(s,r)} e^{-|n-m|/\rho} \langle n \rangle^{\beta/2} \langle m \rangle^{\beta} |n|^\alpha - |m|^\alpha. \leq [R]^{\beta,s}_{r,D\rho(s,r)} e^{-|n-m|/\rho} \langle n \rangle^{\beta/2} \langle m \rangle^{\beta} |n|^\alpha - |m|^\alpha.$$  

(3.32)

Estimates of the others can be obtained in the same way and thus we immediately have our conclusion.

The regularity of $X_F^1$ is given by the following lemma.

**Lemma 3.2** Let $\alpha + \beta \geq 1$; if $F$ is the homological solution given in Lemma 3.1, we then have

$$\|X_F\|_{\mathcal{D}^\rho - \mu(s-2\sigma,r)} \leq \frac{CK^{8\tau+1}}{\gamma^2 \mu^2 \sigma^d+1} \|X_R\|_{\mathcal{D}^\rho(s,r)}.$$  

*Proof:* Following [23, 12], by (3.21), (3.23), (3.25), (3.26) and (3.27), the proof of this Lemma is standard once we can have a proper bound on (3.26), that is $|F^1_{0,n,m}|_{\Theta}$. Recall the restriction on (3.6), one has $|n-m| \leq K$. A similar restriction applies to $F$ from (3.8). Setting $a = n - m \neq 0$, then by (3.26),(3.31) and Lemma A.1,

$$|F^1_{0,n,m}|_{\Theta} \leq [R]^{\beta,s}_{r,D\rho(s,r)} \cdot \frac{e^{-|n-m|/\rho}}{(n)^{\beta/2} (m)^{\beta}} \leq [R]^{\beta,s}_{r,D\rho(s,r)} \cdot \frac{e^{-|n-m|/\rho}}{(n)^{\beta}}.$$  

The last inequality is possible because $\alpha + \beta \geq 1$. We then have our conclusion.

**Lemma 3.3** Let $\eta = \varepsilon^{\frac{1}{\gamma^2}}, D_{3\eta} = D_{\rho - \mu}(s_+ + \frac{i}{4} \sigma, \frac{i}{4} \eta r), 0 < i \leq 4$. If $\varepsilon \ll (\frac{1}{4}\gamma^2 K^{-8\tau - 1})$, we then have

$$X_F^t : D_{2\eta} \to D_{3\eta}, \quad 1 \leq t \leq 1.$$  

Moreover,

$$\|DX_F^t - Id\|_{\eta r,\eta r, D_{3\eta}}^s \leq \frac{CK^{8\tau+1} \varepsilon}{\gamma^2 \mu^{\rho+1} \sigma^d+2}.$$  

(3.34)

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In the above, following [23], we use the operator norm $|L|_{r,s} = \sup_{W \neq 0} \frac{|LW|}{|W|}$ with $| \cdot |_r$ defined in (2.4)

Indeed, following [13] and [15], we also prove the following:

**Corollary 1** The symplectic transformation $X_F^1$ reads

$$
\begin{pmatrix}
I \\
\theta \\
Z
\end{pmatrix} \mapsto \begin{pmatrix}
I + M(\theta, Z) + L(\theta)Z \\
\theta \\
T(\theta) + U(\theta)Z
\end{pmatrix}
$$

(3.35)

where $M(\theta, Z)$ is quadratic in $Z$, $L(\theta)$ and $U(\theta)$ are bounded linear operators from $\ell^{a,\rho} \times \ell^{a,\rho}$ in $\mathbb{R}^d$ and $\ell^{a,\rho} \times \ell^{a,\rho}$, respectively.

### 3.2 Estimate of the Poisson Bracket

**Lemma 3.4** Let $\alpha, \beta$ be positive numbers such that $\alpha + \beta \geq 1$. If $R \in \Gamma^\beta_{r,D_\rho(s,r)}$ and $F$ is the homological solution of (3.8). Then, for any $0 < 4\sigma < s$, $0 < \mu < \rho$ and $n \in \mathbb{N}$, we have the following

\[
\{ [R,F] \}_{r,D_\rho-m(s-2\sigma,r/2)} \leq \frac{C[F]^{\beta,\alpha,*}_{r,D_\rho(s-\sigma,r)} [R]^{\beta,\alpha,*}_{r,D_\rho(s,r)}}{\sigma \mu^{p+1}(\rho - \mu)^2},
\]

(3.36)

\[
\{ \cdots \{ R,F \} \cdots , F \}_{r,D_\rho-m(s-2\sigma,r/2)} \leq \left( \frac{C[F]^{\beta,\alpha,*}_{r,D_\rho(s-\sigma,r)} [R]^{\beta,\alpha,*}_{r,D_\rho(s,r)}}{\sigma \mu^{p+1}(\rho - \mu)^2} \right)^n [R]^{\beta,\alpha,*}_{r,D_\rho(s,r)}.
\]

(3.37)

**Proof:** The estimates (3.36) and (3.37) are proved in the same way. We show the first in detail. The expansion of $\{ R,F \}$ reads,

$$
\{ R,F \} = \sum_{1 \leq j \leq d} \left( \frac{\partial R}{\partial \theta_j} \frac{\partial F}{\partial I_j} - \frac{\partial F}{\partial \theta_j} \frac{\partial R}{\partial I_j} \right) + i \sum_{n \in \mathbb{Z}} \left( \frac{\partial R}{\partial z_n} \frac{\partial F}{\partial \bar{z}_n} - \frac{\partial F}{\partial z_n} \frac{\partial R}{\partial \bar{z}_n} \right).
$$

It remains to estimate each term of this expansion and its derivatives.

Note that $F$ is of degree 2; we have

$$
\frac{\partial^2 F}{\partial z \partial I} = \frac{\partial^2 F}{\partial I^2} = \frac{\partial^3 F}{\partial w^3} = 0, w = z or \bar{z}.
$$

(3.38)

By (3.8) and (3.6),

$$
\frac{\partial^2 F}{\partial z_n \partial \bar{z}_m} = 0, |n - m| > K, \frac{\partial^2 F}{\partial z_n \partial z_m} = \frac{\partial^2 F}{\partial \bar{z}_n \partial \bar{z}_m} = 0, |n + m| > K.
$$

(3.39)
These restrictions are crucially used in this proof.

\* The estimations of \( \{F, R\} \) and \( \frac{\partial}{\partial I_k} \{F, R\} \). Using the Cauchy estimate, we obtain

\[
\left| \{F, R\}_{D_{\rho-\mu}(s-2\sigma, r)} \right|^* \leq \frac{C r^2}{\sigma} (2d + \sum_{n \in \mathbb{Z}} e^{-\frac{2|n|(\rho-\mu)}{\langle n \rangle 2\beta}}) \left[ F \right]^{\beta, \alpha,*}_{r, D_{\rho}(s-\sigma, r)} \left[ R \right]^{\beta,*}_{r, D_{\rho}(s, r)}
\]

\[
\leq \frac{C r^2}{\sigma(\rho - \mu)^2} \left[ F \right]^{\beta, \alpha,*}_{r, D_{\rho}(s-\sigma, r)} \left[ R \right]^{\beta,*}_{r, D_{\rho}(s, r)}.
\]

(3.40)

Similarly,

\[
\left| \frac{\partial}{\partial I_k} \{F, R\}_{D_{\rho-\mu}(s-2\sigma, r)} \right|^* \leq \frac{C}{\sigma(\rho - \mu)^2} \left[ F \right]^{\beta, \alpha,*}_{r, D_{\rho}(s-\sigma, r)} \left[ R \right]^{\beta,*}_{r, D_{\rho}(s, r)}
\]

(3.41)

\* The estimations of \( \frac{\partial}{\partial z_n} \{F, R\} \) and \( \frac{\partial}{\partial z_n} \{F, R\} \). Clearly,

\[
\frac{\partial}{\partial z_n} \left( \frac{\partial R}{\partial I_k} \frac{\partial F}{\partial \theta_k} \right) = \frac{\partial R}{\partial I_k} \frac{\partial^2 F}{\partial z_n \partial \theta_k} + \frac{\partial^2 R}{\partial z_n \partial I_k} \frac{\partial F}{\partial \theta_k}.
\]

We shall estimate each term one by one.

\* Using the Cauchy estimate in \( \theta_k \),

\[
\left| \frac{\partial R}{\partial I_k} \frac{\partial^2 F}{\partial z_n \partial \theta_k} \right|_{D_{\rho-\mu}(s-2\sigma, r)}^* \leq C \left[ R \right]^{\beta,*}_{r, D_{\rho}(s, r)} \left| \frac{\partial F}{\partial z_n} \right|_{D_{\rho}(s-\sigma, r)}^* \sigma^{-1}
\]

(3.42)

\[
\leq \frac{C r \left[ R \right]^{\beta,*}_{r, D_{\rho}(s, r)} \left[ F \right]^{\beta, \alpha,*}_{r, D_{\rho}(s-\sigma, r)} e^{-\frac{|n|(\rho-\mu)}{\sigma \langle n \rangle^\beta}}}{\langle n \rangle^\beta},
\]

and using the Cauchy estimate in \( I_k \) and \( \theta_k \),

\[
\left| \frac{\partial^2 R}{\partial z_n \partial I_k} \frac{\partial F}{\partial \theta_k} \right|_{D_{\rho-\mu}(s-2\sigma, r)}^* \leq C \left| \frac{\partial R}{\partial z_n} \right|_{D_{\rho}(s, r)}^* \left| F \right|_{D_{\rho}(s-\sigma, r)}^* \frac{\sigma}{r^2}
\]

(3.43)

\[
\leq \frac{C r \left[ R \right]^{\beta,*}_{r, D_{\rho}(s, r)} \left[ F \right]^{\beta, \alpha,*}_{r, D_{\rho}(s-\sigma, r)} e^{-\frac{|n|(\rho-\mu)}{\sigma \langle n \rangle^\beta}}}{\langle n \rangle^\beta}.
\]

The above two estimates yield

\[
\left| \frac{\partial}{\partial z_n} \left( \frac{\partial R}{\partial I_k} \frac{\partial F}{\partial \theta_k} \right) \right|_{D_{\rho-\mu}(s-2\sigma, r)}^* \leq \frac{C r \left[ F \right]^{\beta, \alpha,*}_{r, D_{\rho}(s-\sigma, r)} \left[ R \right]^{\beta,*}_{r, D_{\rho}(s, r)} e^{-\frac{|n|(\rho-\mu)}{\sigma \langle n \rangle^\beta}}}{\langle n \rangle^\beta}.
\]

Similarly,

\[
\left| \frac{\partial}{\partial z_n} \left( \frac{\partial F}{\partial I_k} \frac{\partial R}{\partial \theta_k} \right) \right|_{D_{\rho-\mu}(s-2\sigma, r)}^* \leq \frac{C r \left[ F \right]^{\beta, \alpha,*}_{r, D_{\rho}(s-\sigma, r)} \left[ R \right]^{\beta,*}_{r, D_{\rho}(s, r)} e^{-\frac{|n|(\rho-\mu)}{\sigma \langle n \rangle^\beta}}}{\langle n \rangle^\beta}.
\]
By the Leibniz’s rule and (3.39), one has
\[
\frac{\partial}{\partial z_n} \left( \frac{\partial R}{\partial z_m} \frac{\partial F}{\partial z_m} \right) = \left\{ \frac{\partial^2 R}{\partial z_n \partial z_m} \frac{\partial F}{\partial z_m} + \frac{\partial R}{\partial z_m} \frac{\partial^2 F}{\partial z_n \partial z_m} \right\} |m - n| \leq K, \\
\frac{\partial}{\partial z_n} \left( \frac{\partial R}{\partial z_m} \frac{\partial F}{\partial z_m} \right) |m - n| > K. \tag{3.44}
\]

Since
\[
\left| \frac{\partial^2 R}{\partial z_n \partial z_m} \frac{\partial F}{\partial z_m} \right|_{D_{\rho-\mu}(s-2\sigma,r)} \leq \frac{Cr[F]_{r,D_{\rho}(s-\sigma,r)}^{\beta,\alpha,\ast} \sigma, r \left[ R_{r,D_{\rho}(s-\sigma,r)}^{\beta,\ast} \sigma, r \right] e^{-|m|(\rho-\mu)}}{(\langle n \rangle)^{\beta}} \cdot \frac{e^{-|n|(\rho-\mu)}}{(\langle n \rangle)^{\beta}} \cdot \frac{e^{-|m-n|\mu}}{(\langle m \rangle)^{2\beta}},
\]
if \( |m| \neq |n| \), by Lemma A.1, one has
\[
\left| \frac{\partial^2 F}{\partial z_n \partial z_m} \frac{\partial F}{\partial z_m} \right|_{D_{\rho-\mu}(s-2\sigma,r)} \leq \frac{Cr[F]_{r,D_{\rho}(s-\sigma,r)}^{\beta,\alpha,\ast} \sigma, r \left[ R_{r,D_{\rho}(s-\sigma,r)}^{\beta,\ast} \sigma, r \right] e^{-|m|(\rho-\mu)}}{(\langle n \rangle)^{\beta}} \cdot \frac{e^{-|n|(\rho-\mu)}}{(\langle n \rangle)^{\beta}} \cdot \frac{e^{-|m-n|\mu}}{(\langle m \rangle)^{2\beta}}.
\]
If \( m = n \), then obviously,
\[
\left| \frac{\partial R}{\partial z_n} \frac{\partial^2 F}{\partial z_n \partial z_m} \right|_{D_{\rho-\mu}(s-2\sigma,r)} \leq Cr[F]_{r,D_{\rho}(s-\sigma,r)}^{\beta,\alpha,\ast} \sigma, r \left[ R_{r,D_{\rho}(s-\sigma,r)}^{\beta,\ast} \sigma, r \right] \frac{1}{\langle n \rangle^{3\beta}}.
\]
Taking the sum in \( m \), one has
\[
\sum_{m \in N} \left| \frac{\partial}{\partial z_n} \left( \frac{\partial R}{\partial z_m} \frac{\partial F}{\partial z_m} \right) \right|_{D_{\rho-\mu}(s-2\sigma,r)}^{\ast} \leq \frac{Cr[F]_{r,D_{\rho}(s-\sigma,r)}^{\beta,\alpha,\ast} \sigma, r \left[ R_{r,D_{\rho}(s-\sigma,r)}^{\beta,\ast} \sigma, r \right] e^{-|n|(\rho-\mu)}}{(\langle n \rangle)^{\beta}} \cdot \left( \sum_{m \in N} \frac{e^{-|m-n|\mu}}{(\langle m \rangle)^{2\beta}} + \frac{e^{-|m|\mu}}{(\langle m \rangle)^{\beta}} \right) \mu \langle n \rangle^{\beta}.
\]
This implies that
\[
\left| \frac{\partial}{\partial z_n} \{ R, F \} \right|_{D_{\rho-\mu}(r-2\sigma,s)}^{\ast} \leq \frac{Cr K[F]_{r,D_{\rho}(s-\sigma,r)}^{\beta,\alpha,\ast} \sigma, r \left[ R_{r,D_{\rho}(s-\sigma,r)}^{\beta,\ast} \sigma, r \right] e^{-|n|(\rho-\mu)}}{\mu \sigma \langle n \rangle^{\beta}}.
\]
The estimation on $\frac{\partial^2}{\partial z_n \partial z_m} \{ F, R \}$.

With (3.38), one has

$$\frac{\partial^2}{\partial z_n \partial z_m} \left( \frac{\partial F}{\partial I_k} \right) = \frac{\partial F}{\partial I_k} \frac{\partial^2 R}{\partial z_n \partial z_m}$$

and therefore the estimate below is straightforward,

$$\left| \frac{\partial^2}{\partial z_n \partial z_m} \left( \frac{\partial F}{\partial I_k} \right) \right|_{D_{\rho-\mu}(s-2\sigma, r)}^* \leq \frac{C \left[ F \right]_{r, D_\rho(s-\sigma, r)}^\beta, \alpha^* \left[ R \right]_{r, D_\rho(s-\sigma, r)}^\beta, \alpha^* e^{-|n-m|(|\rho-\mu|)}}{\langle n \rangle^\beta \langle m \rangle^\beta}.$$  

We have the expression

$$\frac{\partial^2}{\partial z_n \partial z_m} \left( \frac{\partial F}{\partial I_k} \right) = \frac{\partial F}{\partial I_k} \frac{\partial^2 R}{\partial z_n \partial z_m} + \frac{\partial F}{\partial I_k} \frac{\partial^2 R}{\partial z_n \partial z_m} + \frac{\partial F}{\partial I_k} \frac{\partial^2 R}{\partial z_n \partial z_m} + \frac{\partial F}{\partial I_k} \frac{\partial^2 R}{\partial z_n \partial z_m}.$$  

Let $[F(\theta, z)] = \int_{T^d} F(\theta, z) d\theta$ and recalling (3.29), then by the Cauchy estimate in $\theta_k$,

$$\left| \frac{\partial R}{\partial I_k} \frac{\partial^2 F}{\partial \theta_k \partial z_n \partial z_m} \right|_{D_{\rho-\mu}(s-2\sigma, r)}^* = \left| \frac{\partial R}{\partial I_k} \frac{\partial^2 (F - [F])}{\partial \theta_k \partial z_n \partial z_m} \right|_{D_{\rho-\mu}(s-2\sigma, r)}^* \leq \left| \frac{\partial R}{\partial I_k} \right|_{D_{\rho}(s-\sigma, r)}^* \left| \frac{\partial^3 F}{\partial \theta_k \partial z_n \partial z_m} \right|_{D_{\rho-\mu}(s-2\sigma, r)}^* \leq C \left[ R \right]_{r, D_\rho(s-\sigma, r)}^\beta, \alpha^* \left[ F \right]_{r, D_\rho(s-\sigma, r)}^\beta, \alpha^* e^{-|n-m|(|\rho-\mu|)}}{\langle n \rangle^\beta \langle m \rangle^\beta}.$$  

Using the Cauchy estimate in $I_k$ and $\theta_k$,

$$\left| \frac{\partial^2 R}{\partial I_k \partial \theta_k \partial z_n \partial z_m} \right|_{D_{\rho-\mu}(s-2\sigma, r/2)}^* \leq \left| \frac{\partial^2 R}{\partial I_k \partial \theta_k \partial z_n \partial z_m} \right|_{D_{\rho-\mu}(s-\sigma, r)}^* \leq C \left[ R \right]_{r, D_\rho(s-\sigma, r)}^\beta, \alpha^* \left[ F \right]_{r, D_\rho(s-\sigma, r)}^\beta, \alpha^* e^{-|n-m|(|\rho-\mu|)}}{\langle n \rangle^\beta \langle m \rangle^\beta}.$$  

The estimates of $\frac{\partial^2 R}{\partial I_k \partial \theta_k \partial z_n \partial z_m}$ can be obtained in the same way. Finally, we consider the last function on the right-hand side of formula (3.46). By
the Cauchy estimate in \( I_k \),

\[
|\frac{\partial^3 R}{\partial I_k \partial z_n \partial z_m \partial \theta_k} |_{D_{\rho - \mu}(s-2\sigma_r/2)}^* D_{\rho - \mu}(s-\sigma_r) |F|_{D_{\rho}(s-\sigma_r)}^* \leq \frac{|\partial^2 R|_{D_{\rho}(s-\sigma_r)} |F|_{D_{\rho}(s-\sigma_r)}^*}{r^2} \leq \frac{C[R]^\beta_{r, D_{\rho}(s-\sigma_r)} e^{-|n-m|\rho}}{r^2 \langle n \rangle^\beta \langle m \rangle^\beta} \frac{C[F]^\beta_{r, D_{\rho}(s-\sigma_r)} e^{-|n-m|\rho}}{\sigma \langle n \rangle^\beta \langle m \rangle^\beta}.
\]

We conclude that

\[
|\frac{\partial^2}{\partial z_n \partial z_m} \frac{\partial R}{\partial I_k \partial \theta_k} |_{D_{\rho - \mu}(s-2\sigma_r)}^* \leq \frac{C[F]^\beta_{r, D_{\rho}(s-\sigma_r)} [R]^\beta_{r, D_{\rho}(s-\sigma_r)} e^{-|n-m|\rho}}{\sigma \langle n \rangle^\beta \langle m \rangle^\beta}.
\]

\* By (3.39), one has

\[
\frac{\partial^2}{\partial z_n \partial z_m} \frac{\partial R}{\partial I_k \partial \theta_k} = \left\{ \frac{\partial^2 R}{\partial z_n \partial z_k \partial z_m \partial z_k} + \frac{\partial^2 F}{\partial z_n \partial z_m \partial z_k \partial z_k}, \begin{array}{ll}
|m - k| \leq Kor |n - k| < K, \\
other.
\end{array}
\]

Straightforwardly, we have the estimate

\[
|\frac{\partial^2 R}{\partial z_n \partial z_k} \frac{\partial^2 F}{\partial z_n \partial z_k} |_{D_{\rho - \mu}(s-2\sigma_r)}^* \leq \left\{ \frac{\partial^2 R}{\partial z_n \partial z_k} |_{D_{\rho}(r-s)} \frac{\partial^2 F}{\partial z_n \partial z_k} |_{D_{\rho}(r,s)} \right\} \leq \frac{[R]^\beta_{r, D_{\rho}(s-\sigma_r)} e^{-|m-k|\rho}}{\langle m \rangle^\beta \langle k \rangle^\beta} \cdot \frac{[F]^\beta_{r, D_{\rho}(s-\sigma_r)} e^{-|n-k|\rho}}{\langle n \rangle^\beta \langle m \rangle^\beta} \leq C[F]^\beta_{r, D_{\rho}(s-\sigma_r)} [R]^\beta_{r, D_{\rho}(s-\sigma_r)} e^{-|n-m|\rho} \langle n \rangle^\beta \langle m \rangle^\beta, \]

and

\[
|\frac{\partial^2 R}{\partial z_n \partial z_k} \frac{\partial^2 F}{\partial z_n \partial z_k} |_{D_{\rho - \mu}(s-2\sigma_r)}^* \leq C[F]^\beta_{r, D_{\rho}(s-\sigma_r)} [R]^\beta_{r, D_{\rho}(s-\sigma_r)} e^{-|n-m|\rho} \langle n \rangle^\beta \langle m \rangle^\beta.
\]
Using the Cauchy estimate in \( z_k \), we have

\[
\begin{align*}
\left| \frac{\partial^3 R}{\partial z_n \partial z_m \partial z_k} \right|_{D_{\rho - \mu} (s - 2\sigma, r/2)} &\leq \frac{|k|^p e^{k|\rho - \mu|}}{r} \left| \frac{\partial^2 R}{\partial z_n \partial z_m} \right|_{D_{\rho} (r,s)} \left| \frac{\partial F}{\partial z_k} \right|_{D_{\rho - \mu} (r-2\sigma, s)} \\
&\leq \frac{|k|^p e^{k|\rho - \mu|}}{r} \left| R^{\beta, \ast}_{\sigma, r} \right|_{D_{\rho} (s, r)} e^{-n|\rho - \mu|} \frac{e^{-|k|p}}{\langle m \rangle^\beta \langle n \rangle^\beta}.
\end{align*}
\]

Therefore, by (3.50), (3.51), and (3.52), and taking the sum in \( k \), then

\[
\left| \frac{\partial^2 \{ F, R \}}{\partial z_n \partial z_m} \right|_{D_{\rho - \mu} (s - 2\sigma, r)} \leq \frac{C \left[ F \right]^{\beta, \ast}_{r, D_{\rho} (s - \sigma, r/2)} \left[ R \right]^{\beta, \ast}_{r, D_{\rho} (s, r)} e^{-n|\rho - \mu|} \frac{e^{-|k|p}}{\langle m \rangle^\beta \langle n \rangle^\beta}}{\sigma \mu^{p+1}}. (3.53)
\]

Similarly, one has

\[
\left| \frac{\partial^2 \{ F, R \}}{\partial z_n \partial z_m} \right|_{D_{\rho - \mu} (s - 2\sigma, r)} \leq \frac{C \left[ F \right]^{\beta, \ast}_{r, D_{\rho} (s - \sigma, r/2)} \left[ R \right]^{\beta, \ast}_{r, D_{\rho} (s, r)} e^{-n|\rho - \mu|} \frac{e^{-|k|p}}{\langle m \rangle^\beta \langle n \rangle^\beta}}{\sigma \mu^{p+1}}. (3.54)
\]

The estimations of \( \frac{\partial^2 \{ F, R \}}{\partial z_n \partial z_m} \{ F, R \} \) and \( \frac{\partial^2 \{ F, R \}}{\partial z_n \partial z_m} \{ F, R \} \) can be done in the same way. We then have immediately our conclusion. \( \square \)

### 3.3 Estimate of the New Perturbation

Recalling that

\[
\left[ P \right]^{\beta, \ast}_{r, D_{\rho} (s, r)} + \| X P \|_{D_{\rho} (s, r)}^* \leq \varepsilon,
\]

then by Lemma 3.2 and 3.3, there is a symplectic change of variables

\[
\Phi_+ : D_{\rho_+} (s_+, r_+) \times O_+ \to D_{\rho} (s, r),
\]

with \( s_+ = s - 4\sigma > 0, r_+ = \eta r, \eta = \epsilon^3, \) and \( \rho_+ = \rho - \mu > 0 \), such that the vector field \( X_{H \circ \Phi} \) defined on \( D_{\rho_+} (s_+, r_+) \) satisfies

\[
\| X_{P_+} \|_{D_{\rho_+} (s_+, r_+)}^* \leq c(\eta + e^{-K\mu}) \varepsilon + c\gamma^{-2} \mu^{-2-p} \sigma^{-d-1} K^{8r+2} \eta^{-4} e^{-K\mu}.
\]

Therefore, the remaining task is that \( P_+ \) satisfies Assumption B2. First, we have
Lemma 3.5 Let $P \in \Gamma_{\rho(D, s, r)}$ and consider its Taylor series approximation $R$ (see (3.6)). Then

$$\|R\|_{r, D}^{\beta, *}(s, r) \leq \|P\|_{r, D}^{\beta, *}(s, r).$$

By the Taylor series expansion, the new perturbation $P_+$ can be written as

$$P_+ = P - R + \{P, F\} + \frac{1}{2!} \{\{N + A, F\}, F\} + \frac{1}{2!} \{\{F, F\}, F\} + \cdots$$

Since $\{N + A, F\} = -R + (R)$, by Lemma 3.5 and Lemma 3.4, the new perturbation $P_+$ satisfies Assumption $B2$ with a suitable parameter setting.

Lemma 3.6 The new perturbation $P_+ \in \Gamma_{\rho, D}^{\beta, r}(s + \sigma, r + \epsilon)$ satisfies

$$\|P_+\|_{r, D}^{\beta, *}(s, r) \leq c(\eta + e^{-K\mu})(\eta + e^{-K\mu}) \varepsilon + c\gamma - 2\mu - 3\alpha - d - 1 K^8 T + 2 \eta - 4 \varepsilon e^{-K\mu}.$$
Lemma 4.1 Let $\varepsilon_0$ be small enough and $\nu \geq 0$. Suppose that

1. $N_\nu + A_\nu = \langle \omega_\nu, I \rangle + \sum_{n \in \mathbb{Z}} \Omega_\nu^n \bar{z}_n + \sum_{|n| \leq K_{\nu-1}} a^\nu_{n, -n}(\xi) z_n \bar{z}_{-n}$ is a normal form with parameters $\xi$ on a closed set $O_\nu$ of $\mathbb{R}^d$. For any $\xi \in O$, $|k| \leq K_\nu$ and $n, m \in \mathbb{Z}$ with $|n \pm m| \leq K_\nu$, there are

$$|\langle k, \omega_\nu(\xi) \rangle| < \frac{\gamma}{K_\nu}, \quad k \neq 0$$

$$\|\langle k, \omega_\nu \rangle I_n + A_\nu^n \| < \frac{\gamma}{K_\nu},$$

$$\|\langle k, \omega_\nu \rangle I_{nm} \pm (A_\nu^n \otimes I_m + I_n \otimes A_\nu^m) \| < \frac{\gamma}{K_\nu}, \quad k \neq 0 \& |n - m| < K,$$

where

$$A_0^\nu = \Omega_0^\nu, \quad A_n^\nu = \left( \begin{array}{cc} \Omega_n^\nu & a_n^\nu \nu \n \end{array} \right), \quad |n| \geq 1.$$

2. $\omega_\nu(\xi), \Omega_\nu^m(\xi)$ are Lipschitz in $\xi$ and satisfy

$$|\omega_\nu - \omega_{\nu-1}|_{O_\nu} \leq \varepsilon_{\nu-1}, \quad \|n|^{2}\beta(\Omega_n^\nu - \Omega_n^{\nu-1})|_{O_\nu} \leq \varepsilon_{\nu-1};$$

3. $N_\nu + A_\nu + P_\nu$ satisfies Assumption $\mathcal{A}, \mathcal{B}$ with $r_\nu, s_\nu, \rho_\nu, \varepsilon_\nu$ and

$$\|P_\nu\|_{r_\nu, D_\nu}^* + \|X_{P_\nu}\|_{r_\nu, D_\nu}^* \leq \varepsilon_\nu.$$

Then, there exists a new closed set $O_{\nu+1} =: O_\nu \setminus R_\nu^{\nu+1}$ (see (6.1) for the construction of $R_\nu^{\nu+1}$), and a symplectic transformation of variables,

$$\Phi_\nu : D_{\nu+1} \times O_\nu \to D_{\nu+1},$$

such that on $D_{\nu+1} \times O_\nu$, $H_{\nu+1} = H_\nu \circ \Phi_\nu$ takes the form

$$H_{\nu+1} = \langle \omega_{\nu+1}, I \rangle + \sum_{n \in \mathbb{Z}} \Omega_{\nu+1}^n z_n \bar{z}_n + \sum_{|n| \leq K_{\nu+1}} a^\nu_{n, -n}(\xi) z_n \bar{z}_{-n} + P_{\nu+1}. \quad (4.3)$$

The Hamiltonian $H_{\nu+1}$ satisfies all the assumptions of $H_\nu$ with $\nu+1$ in place of $\nu$.

5 Convergence

We follow the proofs in [15] and [13]. First, we have estimates,
Lemma 5.1 For \( \nu \geq 0 \) and \( n \in \mathbb{Z} \),

\[
\frac{1}{\sigma_\nu} |\Phi_{\nu+1} - iD|_{D_{\nu+1}}^* \leq c \gamma^{-1} d^{-1} K^2 \varepsilon_\nu.
\]

\[
|\omega_{\nu+1} - \omega_\nu|_{\mathcal{O}_\nu} \leq \varepsilon_\nu, \quad \sup_{n \in \mathbb{Z}}|n|^{2\beta} (\Omega^{\nu+1}_n - \Omega^\nu_n)_{\mathcal{O}_\nu} \leq \varepsilon_\nu, \quad \sup_{n \in \mathbb{Z}} e^{n|\rho_\nu| + 1} |n|^{2\beta} |\alpha^{\nu+1}_n| \leq \varepsilon_\nu.
\]

To apply Lemma 4.1 when \( \nu = 0 \), we set \( \varepsilon_0 = \varepsilon, r_0 = r, s_0 = s, \rho_0 = \rho, L_0 = L, N_0 = N, A_0 = 0, P_0 = P \). The smallness conditions are satisfied if we set \( \varepsilon_0 \) sufficiently small. The small divisor conditions are satisfied by setting \( \mathcal{O}_1 = \mathcal{O}\mathcal{O}^0(\text{see (6.1)}) \). Then the iterative Lemma applies, we obtain a sequence of transformations \( \Psi^\nu \) defined on \( D_{\nu+1} \times \mathcal{O}_{\nu+1} \) with

\[
\Psi^\nu = \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_\nu : D_{\nu+1} \times \mathcal{O}_{\nu+1} \rightarrow D(r_0, s_0), \nu \geq 0,
\]
such that \( \mathcal{H} \circ \Psi^\nu = N^\nu + P_{\nu+1} \). For \( \nu \geq 0 \), by the chain rule, we have

\[
|D\Phi^\nu|_{r_0, r_{\nu+1}, D_{\nu+1}} \leq \prod_{m=1}^{\nu+1} \left| D\Phi_m \right|_{r_m, r_{m-1}, D_m} \leq \prod_{m=1}^{\nu+1} (1 + \varepsilon_\nu^m) \leq 2. \quad (5.1)
\]

Therefore, with the mean-value theorem, we obtain

\[
|\Psi^\nu - \Psi^\nu|_{r_0, D_{\nu+1}} \leq |D\Psi^\nu|_{r_0, r_{\nu}, D_{\nu}} |\Phi_{\nu+1} - iD|_{r_{\nu}, D_{\nu+1}} \leq 2\varepsilon^2_\nu,
\]

and \( \Psi^\nu \) converges uniformly to \( \Psi^\infty \) on \( D_{\nu}^{1/2} (\mathcal{O}) \times \mathcal{O}_{\gamma} \). We have estimate (2.7) on \( D_{\nu}^{1/2} (\mathcal{O}) \times \mathcal{O}_{\gamma} \) with \( \mathcal{O}_\gamma = \bigcap_{\nu \geq 1} \mathcal{O}_\nu \).

It remains to prove that \( \Psi^\infty \) is indeed defined on \( D_{\nu}^{1/2} (\mathcal{O}) \times \mathcal{O}_{\gamma} \) with the same estimates. A similar discussion in [15] indicates that the estimate (2.7) can be extended to the domain \( D_{\nu}^{1/2} (\mathcal{O}) \). The estimates (2.9) are simple and hence we omit the details.

Note that \( H \) is analytic on \( D_{\nu}^{1/2} (\mathcal{O}) \), we deduce that \( H \circ \Psi^\infty = N^* + A^* + P^* \) is analytic on \( D_{\nu}^{1/2} (\mathcal{O}) \). Finally, we need to prove that

\[
\partial_y P^* = \partial_z P^* = 0, \quad \partial_{zi,j}^2 P^* = \partial_{zi,j}^2 P^* = \partial_{zi,j}^2 P^* = 0
\]
on \( D_{\nu}^{1/2} (\mathcal{O}) \times \mathcal{O}_{\gamma} \). In the following, we only give the proof for \( \partial_{zi,j}^2 P^* = 0 \); the others can be treated in the same way. Note that \( \|\partial_{zi,j}^2 P_{\nu} \|_{D_{\nu}^{1/2}} \leq \varepsilon_{\nu} \) and \( \|\partial_{zi,j}^2 (P_{\nu} - P_{\nu+1}) \|_{D_{\nu}^{1/2}} \leq \varepsilon_{\nu} + \varepsilon_{\nu+1} \). It follows that

\[
\|\partial_{zi,j}^2 (P_{\nu} - P^*) \|_{D_{\nu}^{1/2}} \leq \sum_{k=\nu}^{\infty} \|\partial_{zi,j}^2 (P_{\nu} - P_{\nu+1}) \|_{D_{\nu}^{1/2}} \leq 2\varepsilon_{\nu}
\]
and then
\[ \|\partial_{z_i z_j}^2 P^*\|_{D(s/2)} \leq \|\partial_{z_i z_j}^2 P_\nu\|_{D(s/2)} + \|\partial_{z_i z_j}^2 (P_\nu - P^*)\|_{D(s/2)} \leq 4\epsilon_\nu \]
for all \( \nu \geq 0 \), this means \( \partial_{z_i z_j}^2 P^* = 0 \) on \( D_\nu^2(\frac{\xi}{2}, \frac{r}{2}) \times O_\gamma \).

6 Measure Estimates

By (3.5), we have \( \tau_1 > d + 3 + \frac{4}{\alpha}, \xi = \frac{\tau_1 + 1}{1 - \alpha} \). For any \( \nu \geq 0 \), we define \( O_{\nu+1} = O_\nu \setminus \mathcal{R}_\nu \), the resonance set \( \mathcal{R}_\nu \) is defined to be
\[
\mathcal{R}_\nu = \mathcal{R}_{\nu,0} \cup \mathcal{R}_{\nu,1} \cup \mathcal{R}_{\nu,2} \cup \mathcal{R}_{\nu,11},
\]
where
\[
\mathcal{R}_{\nu,0} = \bigcup_{0 < |k| \leq K_\nu} \mathcal{R}_{\nu,0}^k = \bigcup_{0 < |k| \leq K_\nu} \{ \xi \in O_{\nu-1} : |\langle k, \omega_\nu(\xi) \rangle|^{-1} \geq \frac{K_{\nu}^{\tau_1}}{\gamma} \},
\]
\[
\mathcal{R}_{\nu,1} = \bigcup_{|k| \leq K_\nu} \mathcal{R}_{\nu,1}^k = \bigcup_{|k| \leq K_\nu} \{ \xi \in O_{\nu-1} : \| (k, \omega_\nu) \|_{\nu} + A_\nu^{-1} \| \geq \frac{K_{\nu}^{2\tau_1}}{\gamma} \},
\]
\[
\mathcal{R}_{\nu,2} = \bigcup_{|k| \leq K_\nu, n \in \mathbb{Z}} \mathcal{R}_{\nu,2}^k = \bigcup_{|k| \leq K_\nu, n \in \mathbb{Z}} \{ \xi \in O_{\nu-1} : \| (k, \omega_\nu) \|_{\nu} + A_\nu^{-1} \| \geq \frac{K_{\nu}^{4\tau_1}}{\gamma} \},
\]
\[
\mathcal{R}_{\nu,11} = \bigcup_{0 < |k| \leq K_\nu, |n-m| \leq K} \mathcal{R}_{\nu,11}^k = \bigcup_{0 < |k| \leq K_\nu, |n-m| \leq K} \{ \xi \in O_{\nu-1} : \| (k, \omega_\nu) \|_{\nu} + A_\nu^{-1} \| \geq \frac{K_{\nu}^{12\tau_1 + 16\sigma}}{\gamma} \}.
\]

Lemma 6.1

\[
\text{meas}(\mathcal{R}_{\nu,0}) \leq \frac{\gamma}{K_{\nu}^{\tau_1-d}}, \quad \text{meas}(\mathcal{R}_{\nu,1}) \leq \frac{\gamma^{\frac{1}{2}}}{K_{\nu}^{\tau_1-d-\frac{1}{\alpha}}}, \quad \text{meas}(\mathcal{R}_{\nu,2}) \leq \frac{\gamma^{\frac{1}{2}}}{K_{\nu}^{\tau_1-d-\frac{1}{\alpha}}}.
\]

The proof of this Lemma is standard and is omitted.
Lemma 6.2  (Lemma 7.6 of [6]) Let $M$ be a $N \times N$ non-singular matrix with $\|M\| < B$; then,

$$\{ \omega : \|M^{-1}\| \geq h \} \subset \{ \omega : |\det M| < \frac{cB^{N-1}}{h} \}.$$ 

Lemma 6.3

$$\text{meas}(R_{\nu,11}) \leq \frac{\gamma_1}{K_\nu}.$$ 

Proof: Recalling the truncation $R_\nu$ in (3.6) and the homological equation (3.8), one has $0 < |k| \leq K_\nu$ and $|n-m| \leq K_\nu$. Because $\alpha < 1$, then $||n|^\alpha - |m|^\alpha| \leq K_\nu$ and hence

$$\|\langle k, \omega_\nu \rangle I_{nm} \pm (A_\nu^\nu \otimes I_n - I_m \otimes A_\nu^\nu)\| \leq CK_\nu.$$

Then, by Lemma 6.2,

$$R_{\nu,11}^{\nu,0} \subset Q_{\nu,0}^{\nu,11}$$

$$= \{ \xi \in \mathcal{O}_\nu : \|\det(\langle k, \omega_\nu \rangle I_{nm} \pm (A_\nu^\nu \otimes I_n - I_m \otimes A_\nu^\nu))\| \leq \frac{K_\nu^{12\tau_1 + 16\sigma - 3}}{\gamma} \}.$$ 

Let $a = m - n$, then

$$\bigcup_{0 < |k| \leq K_\nu, \ n, m \in \mathbb{Z}} Q_{\nu,0}^{\nu,11} = \bigcup_{0 < |k| \leq K_\nu, \ n, m \in \mathbb{Z}} Q_{k,n,n+a}^{\nu,11}.$$ 

By Lemma A.1, for any $\xi \in R_{\nu,0}^{\nu,0}$ and $0 < |k| \leq K_\nu$, one has

$$\|\langle k, \omega \rangle\| \geq \gamma K_\nu^{-\tau_1}.$$ 

Now we will prove $Q_{\nu,0}^{\nu,11} = \emptyset$ if $|k|, |n-m| \leq K_\nu$ and $\max\{|n|, |m|\} \geq K_\nu^{\tau_1 + 2\zeta}$. For the set with such restrictions, one has $|n|, |m| \geq K_\nu^{\tau_1 + 2\zeta - 1}$ by Lemma A.1. Let $a = m - n$, then $|a| \leq K_\nu$. Note that $\zeta = \frac{\tau_1 + \alpha}{1 - \alpha}$, $\alpha + \beta \geq 1$ and

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\( \varepsilon_0 < e^{-\frac{4\rho}{\tau}} \), there is

\[
|\langle k, \omega \rangle + \Omega^\nu_n - \Omega^\nu_{n+a}| = |\langle k, \omega \rangle + |n|^\alpha + \tilde{\Omega}^\nu_n - |n + a|^\alpha - \tilde{\Omega}^\nu_{n+a}|
\]

\[
\geq |\langle k, \omega \rangle| - ||n|^\alpha - |n + a|^\alpha| - |\tilde{\Omega}^\nu_n| - |\tilde{\Omega}^\nu_{n+a}|
\]

\[
\geq \gamma K^\nu - \frac{\alpha|a|}{|n|^{1-\alpha}} - \frac{\varepsilon_0}{|n|^{2\beta}} - \frac{|n + a|^{2\beta}}{|n + a|^{2\beta}}
\]

\[
\geq \gamma K^\nu - \frac{\gamma K^\nu - \frac{\gamma K^\nu}{4} - \frac{\gamma K^\nu}{4} K^\nu}{4}
\]

\[
\geq \frac{1}{2} \gamma K^\nu.
\]

By (3.3), one has

\[
|\det(\langle k, \omega \rangle I_{nm} \pm (A^\nu_n \otimes I_n - I_m \otimes A^\nu_m))| \geq \frac{1}{32} \gamma^4 K^{4\tau_1}. \tag{6.2}
\]

Thus, we have following

\[
R^\nu_{11} \subset \bigcup_{0 < |k| \leq K^\nu, |n-m| \leq K^\nu} Q^\nu_{knm} = \bigcup_{0 < |k| \leq K^\nu, |n-m| \leq K^\nu} Q^\nu_{knm}.
\]

Let

\[
M = \det((\langle k, \omega \rangle I_{nm} \pm (A^\nu_n \otimes I_n - I_m \otimes A^\nu_m)),
\]

and then with a simple computation, one has

\[
\inf_{\xi \in \mathbb{O}} \max_{0 < d \leq 4} |\partial^d \xi M| \geq \frac{1}{2} |k|^4.
\]

In view of Lemma A.2, we have

\[
\text{meas}(Q^\nu_{knm}) \leq \frac{\gamma^4}{K^{3\tau_1 + 4d - 1}},
\]

and then

\[
\text{meas}(R^\nu_{11}) \leq \frac{\gamma^4}{K^{3\tau_1 + 4d - 1}} \ast K^d \ast K^d \leq \frac{\gamma^4}{K^{3\tau_1 - d - 1}}.
\]

Lemma 6.4 Let \( \tau_1 > d + 3 + \frac{4}{\alpha^2} \); then the total measure needed to be excluded in the KAM iteration is

\[
\text{meas}\left(\bigcup_{\nu \geq 0} R^\nu\right) \leq \text{meas}[R^\nu_{0} \cup R^\nu_{1} \cup R^\nu_{2} \cup R^\nu_{11}] \leq \sum_{\nu \geq 0} \frac{\gamma^4}{K^{3\tau_1 - d - 1}} \leq \frac{\gamma^4}{31}.
\]
A Appendix

Lemma A.1 For $K > 1$ and any $n, m \in \mathbb{Z}\setminus\{0\}$ such that $n \neq m$ and $|n - m| \leq K$, one has

$$\frac{|m|}{K} \leq |n| \leq K|m|$$

and

$$|n|^{\alpha} - |m|^{\alpha} \geq \frac{\alpha}{2|m|^{1-\alpha}}.$$ 

Lemma A.2 (Lemma 8.4 of [1]). Let $g : I \to \mathbb{R}$ be $b + 3$-times differentiable, and assume that

1. $\forall \sigma \in I$, there exists $s \leq b + 2$ such that $g^{(s)}(\sigma) > B$.
2. There exists $A$ such that $|g^{(s)}(\sigma)| \leq A$ for $\forall \sigma \in I$ and $\forall s$ with $1 \leq s \leq b + 3$.

Define

$$I_h \equiv \{ \sigma \in I : |g(\sigma)| \leq h \},$$

then

$$\frac{\text{meas}(I_h)}{\text{meas}(I)} \leq \frac{A}{B^{2 + 3 + \cdots + (b + 3) + 2B^{-1}}}h^{\frac{1}{b+3}}.$$ 

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