Quantum Stochastic Models
with Hydrodynamical Behaviour

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Abstract

We construct a class of quantum stochastic models of reservoir driven many-particle systems that are the natural counterparts of certain extensively studied classical ones, which have been shown to exhibit good hydrodynamical behaviour. Our treatment of these models achieves two main aims. The first is to show that they enjoy the hydrodynamical properties of their classical counterparts. The second is to show that they satisfy the key assumptions of the general quantum macrostatistical scheme, presented in earlier works by the author, which served to expose certain generic large scale features of nonequilibrium steady states, e.g. the long range hydrodynamical correlations that they carry. In this way we establish the viability of that scheme.

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I. Introduction

The purpose of this note is to bring together two developments in the theory of the relationship between the hydrodynamical and the microscopic pictures of reservoir driven macroscopic systems. The first of these developments, which we shall refer to as (I), is a body of work concerned with the derivation of hydrodynamics from the microscopic dynamics of a class of classical stochastic models [1-5]. Most interestingly, the hydrodynamical fluctuations of these models about their nonequilibrium steady states have been shown to carry long range spatial correlations [3-5] and to conform to a generalised version [5] of the Onsager-Machlup process [6]. The second development, which we shall refer to as (II), is a general, model-independent, quantum macrostatistical treatment [7,8] of hydrodynamical fluctuations about nonequilibrium steady states, that is based on certain hypotheses of chaoticity and local equilibrium, together with a generalised version of Onsager’s regression hypothesis [9]. On this very general basis, we have shown that, as in the special classical models of [3-5], the hydrodynamical fluctuations about nonequilibrium steady states carry long range spatial correlations and execute a generalised Onsager-Machlup process.

Our objectives here are to extend the constructive developments of (I) to the quantum regime and to show that the resultant models satisfy the assumptions of the general quantum macrostatistical theory of (II). The latter objective is thus designed to show that the ‘axiomatic’ scheme of (II) is viable.

Our approach to these objectives is based on a construction whereby we extend the generic classical stochastic model, $\Sigma_{cl}$, of (I) to a quantum system, $\Sigma$, in such a way that

(i) the abelian algebra of observables, $\mathcal{B}$, of $\Sigma_{cl}$ is a subalgebra of the nonabelian one, $\mathcal{A}$, of $\Sigma$;

(ii) $\mathcal{B}$ is stable under the dynamics of $\Sigma$;

(iii) the nonequilibrium steady state of $\Sigma_{cl}$ is just the restriction to $\mathcal{B}$ of that of $\Sigma$; and

(iv) the hydrodynamical observables of $\Sigma$ are precisely those of $\Sigma_{cl}$.

Thus, the construction of the quantum model $\Sigma$ in this way permits us to exploit some of the powerful results obtained for the hydrodynamical properties of its classical counterpart, $\Sigma_{cl}$. In particular, it enables us to verify that $\Sigma$ enjoys the hydrodynamical properties of $\Sigma_{cl}$ and, moreover, that it satisfies the basic assumptions of (II). This latter result therefore establishes that the scheme (II) is viable.

We present the formulation of the models $\Sigma_{cl}$ and $\Sigma$ in Section 2, and establish there the above properties (i)-(iv). Thus the hydrodynamical picture of $\Sigma$ reduces to that of $\Sigma_{cl}$. We formulate this picture explicitly in Section 3 and extend it to fluctuations about a nonequilibrium steady state in Section 4, where we specify the regression, chaoticity and local equilibrium hypotheses on which the theory of (II) was based. In Section 5 we prove the validity of these hypotheses for the present stochastic model and thereby establish the viability of the scheme of [II]. We conclude in Section 6 with a brief comment about an outstanding problem in the theory of quantum stochastic processes.
2. The Classical and Quantum Stochastic Models

The generic model with which we shall be concerned, whether classical or quantum, is a system of \( N \) identical particles that live in a bounded region \( \Omega_N \) of the \( d \)-dimensional lattice \( \mathbb{Z}^d \) and are coupled to reservoirs at its boundary. More specifically, \( \Omega_N \) is assumed to be the subset of \( \mathbb{Z}^d \) contained within the dilation by a certain factor, \( L_N \), of a fixed, \( N \)-independent bounded open connected region \( \Omega \) of the Euclidean space \( \mathbb{R}^d \). Thus, \( \Omega_N = \mathbb{Z}^d \cap (L_N \Omega) \). We define its boundary, \( \partial \Omega_N \), to comprise the sites in \( \Omega_N \) with at least one nearest neighbour that lies outside that region, and we define \( \text{Int}(\Omega_N) \), the interior of \( \Omega_N \), to be \( \Omega_N \setminus \partial \Omega_N \). This latter region thus consists of the sites in \( \Omega_N \) whose nearest neighbours also lie in \( \Omega_N \).

We assume that the volume of \( \Omega \) is unity, that its boundary, \( \partial \Omega \), is smooth and that the mean particle number density, \( \nu \), of the system is \( N \)-independent. Thus

\[
L_N = (N/\nu)^{1/d}.
\]

We assume that the dynamics of the model corresponds to a stochastic process whereby the particles jump between nearest neighbouring lattice sites according to probabilistic laws that will be prescribed below.

2.1. The Classical Model.

For the classical model, \( \Sigma_{cl} \), we denote by \( n_x \) the number of particles at the site \( x \). In the case where an exclusion principle is operative, \( n_x \) is restricted to the values 0 and 1; otherwise it may take any non-negative integral value. Thus a particle configuration is a map \( n : x \mapsto n_x \) of \( \Omega_N \) into a set \( K \), which is either \( \{0,1\} \) or \( \mathbb{N} \) according to whether or not an exclusion principle is operative. We take the algebra, \( \mathcal{B} \), of bounded observables of the system to comprise the bounded, complex valued functions on the configuration space \( \Gamma = K^{\Omega_N} \), with supremum norm. Thus, equipping \( \Gamma \) with the discrete topology, \( \mathcal{B} = \mathcal{C}(\Gamma) \), the \( C^* \)-algebra of bounded continuous functions on \( \Gamma \). For \( x, y \in \Omega_N \), we define \( n \mapsto n_x^y \) to be the transformation of \( \Gamma \) corresponding to the transfer of a particle from \( x \) to \( y \), provided that that transfer is kinematically admissible, i.e. that \( (n_x - 1) \) and \( (n_y + 1) \) lie in \( K \): otherwise we define \( n_x^y \) to be simply \( n \). Likewise we define \( n_x^{\pm} \) to be the modifications of \( n \) corresponding to increments \( \pm 1 \) in \( n_x \), provided that \( (n_x \pm 1) \in K \); otherwise we define \( n_x^{\pm} = n \). We assume that the dynamics of the system is given by a continuous one-parameter semigroup, \( \phi_{cl}(\mathbb{R}_+) = \{ \phi_{cl}(t) | t \in \mathbb{R}_+ \} \) of linear, positivity preserving transformations of \( \mathcal{B} \). We denote its generator by \( \mathcal{G}_{cl} \) and we shall presently specify its explicit form for two models, namely those conventionally termed [1-5] the simple exclusion model and the zero range model.

The Simple Exclusion Model. For this model, \( K = \{0,1\} \) and \( \mathcal{G}_{cl} \) takes the following form

\[
\mathcal{G}_{cl}f(n) = \sum_{x,y \in \Omega_N} n_x(1 - n_y)[f(n_x^y) - f(n)] + \sum_{b \in \partial \Omega_N} r_b n_b (f(n_b^-) - f(n)) + \sum_{b \in \partial \Omega_N} h(b/L_N)(1 - n_b)(f(n_b^+) - f(n)),
\]

where the prime over the first sum signifies that summation is confined to nearest neighbours, \( h \) is a smooth positive-valued function on \( \partial \Omega \) and \( r_b \) is the number of nearest
neighbours of \( b(\in \partial \Omega_N) \) on the lattice \( \mathbb{Z}^d \) that lie outside \( \Omega_N \). Thus, the first sum represents the jumps between nearest neighbouring sites of the particles in the interior of \( \Omega_N \), the second the escape of particles across its boundary and the third the supply of particles by external sources at the boundary.

The Zero Range Model. For this model, \( K = \mathbb{N} \) and, in the same notation as in Eq. (2.2), \( G_d \) takes the following form.

\[
G_d f(n) = \sum_{x,y \in \Omega_N} g(n_x)(f(n^{x,y}) - f(n)) + \\
\sum_{b \in \partial \Omega_N} r_b g(n_b)(f(n^{b}) - f(n)) + \sum_{b \in \partial \Omega_N} h(b/L_N)(f(n^{b}) - f(n_b)), \tag{2.3}
\]

where \( g \) is a positive valued, non-increasing function on \( \mathbb{N} \) for which \( g(0) = 0 \) and \( \sup_k (g(k+1) - g(k)) \) is finite.

Note. It follows from Eqs. (2.2) and (2.3) that in the cases of the simple exclusion and the zero range models \( G n_x \) takes the forms \( (\Delta n)_x \) for the and \( (\Delta(g \circ n))_x \), respectively, for \( x \in \text{Int}(\Omega_N) \), where \( \Delta \) is the discrete Laplacian defined by the formula

\[
(\Delta f)_x = \sum_{y \in \Omega_N} (f_y - f_x),
\]

and the prime over \( \Sigma \) again indicates that the sum is taken over sites \( y \) that are the nearest neighbours of \( x \). Hence, for both models, the dynamics of the field \( n \) is diffusive.

2.2. The Quantum Model

We take the quantum model \( \Sigma \) to be a system of fermions or bosons according to whether or not the exclusion principle is operative. In either case we formulate the model in a standard way in terms of the Fock space \( \mathcal{H} \) and the creation and destruction operators \( \{a^*_x, a_x | x \in \Omega_L\} \) that act therein according to the following defining conditions.

(a) \( \mathcal{H} \) contains a vector \( \Phi \) that is annihilated by the action of each of the \( a_x \)‘s and is cyclic with respect to the polynomials in the \( a^*_x \)‘s; and

(b) the operators \( a_x \) and \( a^*_x \) satisfy the canonical commutation or anticommutation relations, namely

\[
[a_x, a^*_y]_z = \delta_{x,y} I; \ [a_x, a_y] = 0 \ \forall \ x, y \in \Omega_N, \tag{2.4}
\]

according to whether the system consists of bosons or fermions. For either case, we define the number operator

\[
\hat{n}_x = a^*_x a_x \ \forall \ x \in \Omega_N. \tag{2.5}
\]

It follows immediately from Eqs. (2.4) and (2.5) that the \( \hat{n}_x \)‘s intercommute and thus constitute a classical field \( \hat{n} := \{\hat{n}_x | x \in \Omega_N\} \). We denote by \( \psi(n) \) the simultaneous eigenvector of these operators \( \hat{n}_x \) with corresponding eigenvalues \( n_x \), i.e.

\[
\hat{n}_x \psi(n) = n_x \psi(n) \ \forall \ x \in \Omega_N. \tag{2.6}
\]
It follows from this formula and our specifications of \( \mathcal{H} \) that the vectors \( \psi(n) \) form a complete orthogonal basis for this space as \( n \) runs through the classical configuration space \( \Gamma = K^{\Omega_N} \), with \( K = \mathbb{N} \) or \{0, 1\} according to whether the particles are bosons or fermions. Further, by Eqs. (2.4)-(2.6),

\[
a_x \psi(n) = n_x^{1/2} \psi(n^x) \quad \text{and} \quad a_x^* \psi(n) = (1 \pm n_x)^{1/2} \psi(n^{x, \pm}) \tag{2.7}
\]

and hence

\[
a_x^* a_x \psi(n) = (n_x(1 \pm n_y))^{1/2} \psi(n^{x,y}), \tag{2.8}
\]

where \( n^{x,y} \) and \( n^{x,\pm} \) are defined as in Section 2.1 and \( \pm \) signifies the boson-fermion alternatives.

We denote by \( \mathcal{F} \) the additive group of bounded continuous real-valued functions \( \theta : x \to \theta_x \) on \( \Omega_N \) and we define the unitary representation \( U \) of \( \mathcal{F} \) by the formula

\[
U(\theta) = \exp \left( i \sum_{x \in \Omega_N} \theta_x \hat{n}_x \right), \tag{2.9}
\]

We take the algebra, \( \mathcal{A} \), of bounded observables of \( \Sigma \) to be that of the bounded operators in \( \mathcal{H} \) and we define \( \gamma \) to be the representation of \( \mathcal{F} \) implemented by \( U \) in \( \text{Aut}(\mathcal{A}) \), i.e.

\[
\gamma(\theta)A = U(\theta)AU(\theta)^{-1} \quad \forall A \in \mathcal{A}. \tag{2.10}
\]

Thus, by Eqs. (2.4)-(2.6), (2.9) and (2.10), \( \gamma(\theta) \) is the local gauge automorphism given by the formula

\[
\gamma(\theta)a_x = a_x \exp(-i\theta_x) \tag{2.11}
\]

We define \( \mathcal{B} \) to be the locally gauge invariant subalgebra of \( \mathcal{A} \) i.e., by Eq. (2.10), the set of elements of \( \mathcal{A} \) that commute with all the \( U(\theta)'s \). It follows from this definition and Eq. (2.9) that \( \mathcal{B} \) comprises the elements \( B \) of \( \mathcal{A} \) for which

\[
U(\theta)B\psi(n) = BU(\theta)\psi(n) = \exp \left( i \sum_{x \in \Omega_N} n_x \theta_x \right)B\psi(n) \quad \forall n \in \Gamma, \ \theta \in \mathcal{F},
\]

i.e. for which \( B\psi(n) \) is a simultaneous eigenvector of the \( \hat{n}_x \)'s with corresponding eigenvalues \( n_x \). This signifies that \( B\psi(n) = F(n)\phi(n) \), where \( F \) is some bounded complex-valued function on \( \Gamma \), i.e. that \( B \) is the operator \( F(\hat{n}) \), defined by the formula

\[
F(\hat{n})\psi(n) = F(n)\psi(n) \quad \forall n \in \Gamma. \tag{2.12}
\]

Thus we have established the following proposition.

**Proposition 2.1.** (1) \( \mathcal{B} \) comprises the functions of \( \hat{n} \); and

(2) the mapping \( F(\hat{n}) \to F(n) \) of \( \mathcal{B} \) onto \( C(\Gamma) \), the algebra of observables of \( \Sigma_{cl} \), is a \( C^* \)-isomorphism. Hence \( \mathcal{B} \) may be identified with the latter algebra.
We assume that the dynamics of the model $\Sigma$ is given by a strongly continuous one-parameter semigroup $\phi(t) = \{\phi(t)\}_{t \in \mathbb{R}_+}$ of completely positive identity preserving contractions of $\mathcal{A}$. Its generator $\mathcal{G}$ therefore takes the standard form for that of a quantum dynamical semigroup, namely $[10, 11]$

$$\mathcal{G}A = i[H, A] - \sum_j \left( V_j^* A V_j - \frac{1}{2} [V_j^* V_j, A]_+ \right) \forall A \in \mathcal{A}, \quad (2.13)$$

where $H$ is a self-adjoint element of $\mathcal{A}$ and $V_j$ and $\sum_j V_j^* V_j$ also belong to this algebra. Thus, the dynamics of the model is determined by $H$ and the $V$'s. We shall now specify these operators for quantum versions of the simple exclusion and zero range models.

The Quantum Simple Exclusion Model. In view of the requirement of an exclusion principle, we take the particles of this model to be fermions. We construct its dynamical semigroup $\phi(t)$ in such a way as to obtain a natural correspondence between its generator and that of $\phi(t)$, as given by Eq. (2.2). Specifically we choose the operators $H$ and the $V$'s of Eq. (2.13) in the following way.

(a) We take $H$ to be zero, since $\mathcal{G}_{cl}$ contains no Hamiltonian part.

(b) Since the indices involved in the structure of $\mathcal{G}_{cl}$ comprise the nearest neighbouring pairs $(x, y)$ of sites of $\Omega_N$ together with the boundary sites $b$ of $\partial \Omega_N$, we assume that the index $j$ of Eq. (2.13) also runs through just these sets.

(c) For $j = (x, y)$, we choose $V_j$ to be $a_x^* a_y$, since the first summand of Eq. (2.13) then represents the transfer of a particle from $x$ to $y$, with probability rate that corresponds to that of Eq. (2.2).

(d) For each $b \in \partial \Omega$, we introduce two separate $V$'s, namely $h(b/L_N)^{1/2} a_b^*$ and $r_b^{1/2} a_b$, which lead to the creation and annihilation, respectively, of the particle at $b$, with weights corresponding to those of Eq. (2.2).

Thus, under these specifications, the formula (2.13) takes the following form.

$$\mathcal{G}A = \sum_{x,y \in \Omega_N} (a_x^* a_y A a_y^* a_x - \frac{1}{2} [a_x^* a_y a_y^* a_x, A]_+ + \sum_{b \in \partial \Omega_N} r_b (a_b^* A a_b - \frac{1}{2} [a_b^* a_b, A]) + \sum_{b \in \partial \Omega_N} h(b/L_N) (a_b A a_b^* - \frac{1}{2} [a_b a_b^*, A]_+). \quad (2.14)$$

The Quantum Zero Range Model. Since no exclusion principle is operative for this model, we take its particles to be bosons. Thus, the operators $a_x$ and $a_x^*$ are unbounded here. In order to keep the formulation of the model in terms, exclusively, of bounded ones, we introduce the operators

$$a_x = (I + \hat{n}_x)^{-1/2} a_x, \quad a_x^* = (I + \hat{n}_x)^{-1/2} \quad (2.15)$$

and note that, by Eqs. (2.6), (2.7) and (2.15), their actions on $\psi(n)$ are given by the formula

$$a_x \psi(n) = (1 - \delta_{n_x, 0}) \psi(n^{x^-}); \quad a_x^* \psi(n) = \psi(n^{x^+}). \quad (2.16)$$

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Hence, $\alpha_x^*$ and $\alpha_x$ serve as bounded creation and annihilation operators.

In order to formulate the quantum version of the generator of the dynamical semigroup of the model, we proceed along the same lines as for the simple exclusion model, simply replacing $a_x$ by $\alpha_x$. Thus we obtain the following formula for the quantum version of Eq. (2.3), as applied to bosons.

$$G_A = \sum_{x,y \in \Omega} (g(\hat{n}_x)^{1/2}\alpha_x^*\alpha_y A\alpha_y^*\alpha_x g(\hat{n}_x)^{1/2}, A|_+ \rangle + \sum_{b \in \partial \Omega \cap N} r_b(g(\hat{n}_b)^{1/2}\alpha_b^*\alpha_b g(\hat{n}_b)^{1/2}, A|_+ \rangle + \sum_{b \in \partial \Omega \cap eN} h(b/L)(\alpha_b^*\alpha_b - \frac{1}{2}[\alpha_b\alpha_b^*, A|_+ \rangle). \tag{2.17}$$

2.3. The Quantum System $\Sigma$ as an Extension of the Classical one $\Sigma_{cl}$

Since the subalgebra $B$ of $\mathcal{A}$ is identified with that of the observables of $\Sigma_{cl}$, the following proposition establishes that, for the simple exclusion and zero range models, the dynamics of $\Sigma$ induces an autonomous subdynamics on the classical observables $B$ that is precisely that of the system $\Sigma_{cl}$. In other words the quantum system $\Sigma$ is an extension of the classical one, $\Sigma_{cl}$.

**Proposition 2.2.** For the models under consideration,

(1) The algebra $B$ is stable under the semigroup $\phi(R_+)$; and

(2) the restriction of $\phi(R_+)$ to $B$ is just the dynamical semigroup $\phi_{cl}(R_+)$ of $\Sigma_{cl}$.

**Proof.** Since $G$ and $G_{cl}$ are the generators of $\phi(R_+)$ and $\phi_{cl}(R_+)$, respectively, it suffices to show that, for the models concerned, $G_{cl}$ is just the restriction of $G$ to $B$. By Prop. 2.1, this condition is just that

$$[G F(\hat{n})] \psi(n) = [G_{cl} F(n)] \psi(n) \quad \forall F \in \mathcal{C}(\Gamma), \quad n \in \Gamma. \tag{2.18}$$

It is now a straightforward matter to check that, by Eqs. (2.5)-(2.8), (2.12), (2.14), (2.16) and (2.17), this condition is satisfied by both the simple exclusion and zero range models.

**Proposition 2.3.** The dynamical transformations $\phi(R_+)$ of the models under consideration commute with the gauge automorphisms $\gamma(\theta)$. Hence the dynamics of these models are locally gauge covariant.

**Proof.** Since $G$ is the generator of $\phi(R_+)$, it suffices to show that $G$ commutes with $\gamma(\theta)$; and it follows from Eqs. (2.4)-(2.6), (2.10), (2.11), (2.15) and (2.17) that it does so.

2.4. Steady States of $\Sigma_{cl}$ and $\Sigma$

Assume now that $\Sigma_{cl}$ has a unique steady state $\omega_{cl}$, as has been established for the simple exclusion and zero range models [1-5]. We shall now show that $\omega_{cl}$ extends to
a locally gauge invariant stationary state $\omega$ of $\Sigma$. To this end we introduce the conditional expectation, $P$, of $A$ onto $B$ that defines $PA$ as the mean over all the local gauge transformations $\gamma(\theta)$, i.e.

$$PA = \left[ \Pi_{x \in \Omega_N} (2\pi)^{-1} \int_0^{2\pi} d\theta_x \right] \gamma(\theta) A \quad \forall \ A \in \mathcal{A}. \quad (2.19)$$

We then define $\omega$ to be the state of $\Sigma$ given by the formula

$$\omega(A) = \omega_{cl}(PA) \quad \forall \ A \in \mathcal{A}. \quad (2.20)$$

In view of Props. 2.2 and 2.3, it follows immediately from the last two equations that $\omega$ is indeed a locally gauge invariant stationary state of $\Sigma$, and that it is the only one that reduces to $\omega_{cl}$ on $B$. Moreover, in the case of the models under consideration, it is the only stationary state* of $\Sigma$, for the following reasons. Frigerio [12, Theorem 3.2] has shown that a quantum dynamical semigroup $\phi$ cannot admit more than one stationary state if the commutant of the operators $H$ and $\{V_j\}$ appearing in the formula (2.13) for its generator consists of the scalar multiples of the identity; and it follows easily from Eqs. (2.14) and (2.17) that, in view of the assumed strict positivity of the functions $h$ and $g$, this condition is satisfied by the models under consideration. We remark that the steady states $\omega$ and $\omega_{cl}$ are ones of equilibrium or nonequilibrium according to whether or not the function $h$ is constant over $\partial \Omega_N$.

In all cases, it follows from the above considerations that the quantum dynamical system $\Sigma$, as represented now by $(\mathcal{A}, \phi, \omega)$, is an extension of the classical one $(\mathcal{B}, \phi_{cl}, \omega_{cl})$.

Since we shall be concerned with properties of the model in certain limits where $N$ tends to infinity, we shall henceforth indicate the $N$-dependence of $\Sigma$, $\omega$, $\omega_{cl}$, $\phi$, $\phi_{cl}$, $\mathcal{G}$ and $\mathcal{G}_{cl}$ by attaching the superscript $(N)$ to these symbols.

3. The Hydrodynamic Picture.

We shall now investigate the large scale dynamical properties of the field $\hat{n}$, with the aim of showing that it exhibits good hydrodynamical behaviour. To this end, we make the following two observations.

(a) Since the field $\hat{n}$ of $\Sigma^{(N)}$ is built from the observables $\hat{n}_x$, which are affiliated to the algebra $\mathcal{B}$, it follows from Props. 2.1 and 2.2 that the dynamics of this field reduces to that of its classical counterpart, $n$ (of Section 2.1), as governed by the dynamical semigroup $\phi_{cl}^{(N)}(\mathbb{R}_+)$ of $\Sigma^{(N)}$.

(b) As remarked in the Note following Eq. (2.3), the evolution of the field $n$ is diffusive. Hence, for a hydrodynamical description of this field on a length scale whose unit is $L_N$, the natural unit of the corresponding time scale is $L_N^2$.

In view of these observations, we formulate the hydrodynamical picture of this field on macroscopic length and time scales whose units are $L_N$ and $L_N^2$, respectively, as in Refs.

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* In fact, the theory that follows does not depend on this uniqueness.
[1-5, 7, 8]. Thus, in this scaling, the dynamics of \( n \) is represented formally by the classical field

\[
q_t^{(N)}(x) = \sum_{y \in \Omega_N} \phi^{(N)}(L_N^2 t) n_y \delta(y - L_N x) \quad \forall \ x \in \Omega, \ y \in \mathbb{R}_+.
\]

\( q_t^{(N)} \) is therefore a \( \mathcal{D}'(\Omega) \)-class distribution, in the sense of L. Schwartz [13]. To be precise, it is a continuous linear functional on the Schwartz space \( \mathcal{D}(\Omega) \) of infinitely differentiable functions on \( \mathbb{R}^d \) with support in \( \Omega \); and its action on the latter space is given by the formula

\[
q_t^{(N)}(f) = L^{-d} \sum_{y \in \Omega_N} \phi_{cl}^{(N)}(L_N^2 t) n_y f(L_N^{-1} y) \quad \forall \ f \in \mathcal{D}(\Omega).
\]

(3.1)

In fact this field \( q_t^{(N)} \) does indeed exhibit good hydrodynamical properties since (cf. [2]), for appropriate initial states \( \mu^{(N)} \) of \( \Sigma^{(N)}_{cl} \), the expectation value of \( q_t^{(N)}(f) \) converges in probability, as \( N \to \infty \), to the smeared form \( \int_{\Omega} dx q_t(x) f(x) \) of a smooth field \( q_t \), which evolves according to a phenomenological equation of the form

\[
\frac{\partial q_t}{\partial t} = \Delta \Phi(q_t),
\]

(3.2)

where the function \( \Phi \) is smooth and non-negative: in the case of the simple exclusion model it is the identity function. The spatial boundary condition for this evolution is given by the formula

\[
\Phi(q_t(x)) = h(x) \quad \forall \ x \in \partial \Omega, \ t \in \mathbb{R}_+,
\]

(3.3)

where \( h \) is the function that governs the boundary term in Eqs. (2.2) and (2.3). We denote by \( \mathcal{T} \) the stationary solution of Eqs. (3.2) and (3.3). Evidently it is just the expectation value of \( q_t^{(N)} \) for the nonequilibrium steady state \( \omega_{cl} \) in the limit \( N \to \infty \).

We note that it follows from Eq. (3.2) that the linearised equation of motion for a small perturbations \( \delta q_t \) of \( \mathcal{T} \) takes the form

\[
\frac{d}{dt} \delta q_t = \mathcal{L} \delta q_t
\]

(3.4)

where

\[
\mathcal{L} = \Delta [\Phi'(\bar{q}(x))(\cdot)].
\]

(3.5)

Hence, assuming that \( \mathcal{L} \) is the generator of a one-parameter semigroup \( \{ T_t \mid t \in \mathbb{R}_+ \} \) of linear transformations of \( \mathcal{D}'(\Omega) \), the solution of Eqs. (3.4) is simply

\[
\delta q_t = T_{t-t_0} \delta q_{t_0} \quad \forall t \geq t_0
\]

(3.6)

4. The Hydrodynamic Fluctuation Process

We define the field \( \xi_t^{(N)} \), which represent the fluctuations of \( q_t^{(N)} \) about its mean for the steady state \( \omega_{cl}^{(N)} \), by the formula

\[
\xi_t^{(N)}(f) = N^{1/2} [q_t^{(N)}(f) - \omega_{cl}^{(N)}(q_t^{(N)}(f))] \quad \forall \ f \in \mathcal{D}(\Omega).
\]

(4.1)
\(\xi^{(N)}\) is thus a classical stochastic process for the state \(\omega_{cr}\), indexed by \(\mathbb{R} \times \mathcal{D}(\Omega)\). Our aim now is to verify that it satisfies the following conditions, which were the hypotheses on which the macrostatistical theory of [7,8] was based.

(0) The hydrodynamic limit hypothesis. This asserts that the process \(\xi^{(N)}\) converges in law to a stationary stochastic process \(\xi\) as \(N \to \infty\).

(1). The Regression Hypothesis. This asserts that the dynamical law governing deviations of the hydrodynamical variable \(q_t\) from its steady state value is the same whether they arise from spontaneous fluctuations or from weak external perturbations. Thus, in view of the formula (3.6) for the perturbed hydrodynamics, the regression hypothesis is that

\[
E(\xi_t|\xi_{t_0}) = T_{t-t_0}\xi_{t_0} \quad \forall t \geq t_0.
\]  (4.2)

where \(E(\cdot|\xi_{t_0})\) denotes the conditional expectation, given \(\xi_{t_0}\).

(2) The Chaoticity Hypothesis. In order to specify this hypothesis, we introduce Nelson’s [14] forward time derivative of \(\xi_t\), namely

\[
D\xi_t = \lim_{\tau \downarrow 0} \tau^{-1} E(\xi_{t+\tau} - \xi_t|\xi_t);
\]  (4.3)

and we infer from Eqs. (4.2) and (4.3) that, since \(\mathcal{L}\) is the generator of \(T(\mathbb{R}_+), \)

\[
D\xi_t = \mathcal{L}\xi_t.
\]  (4.4)

By the definition (4.3), \(D\xi_t\) is the instantaneous expectation value of the rate of change of \(\xi_t\). Accordingly, we designate \(\int_s^t du D\xi_u\) to be the secular part of the increment \((\xi_t - \xi_s)\) in \(\xi\) over the time interval \([s,t]\). Correspondingly, we designate the stochastic part of \((\xi_t - \xi_s)\) to be the remaining part, \(w_{t,s}\), of this increment; and, in view of Eq. (4.4), this takes the form

\[
w_{t,s} = \xi_t - \xi_s - \int_s^t du \mathcal{L}\xi_u.
\]  (4.5)

Thus \(w\) is a process indexed by \(\mathbb{R}_+^2 \times \Omega\). Our chaoticity hypothesis, which is designed to represent the stochasticity of this process, is that it is Gaussian and that its space-time correlations are of zero range, corresponding to ones of finite range on the microscopic scale. Thus the hypothesis is that \(w\) is Gaussian and that

\[
E(w_{t,s}(f)w_{t',s'}(g)) = 0 \text{ if either } [s,t]\cap[s',t'] = \emptyset \text{ or } \text{supp}(f)\cap\text{supp}(g) = \emptyset.
\]  (4.6)

(3) The Local Equilibrium Hypothesis. We formulate the local properties of the process \(\xi\) in terms of the transformation \(f \to f_{x_0,\epsilon}\) of \(\mathcal{D}(\Omega)\) defined by the formula

\[
f_{x_0,\epsilon} = \epsilon^{-d/2} f(\epsilon^{-1}(x-x_0)).
\]  (4.7)

This transformation corresponds to the spatial rescaling by the factor \(\epsilon\) around the point \(x_0\). Further, in thermal equilibrium, the static two-point function for \(\xi\) enjoy the properties [2,3]

\[
E(\xi(f)\xi(g)) = \chi(\mathbb{Q}) \int_\Omega df(x)g(x),
\]

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where $\chi$ represents the compressibility of the system; and

$$E(\xi(L^*(f)\xi(g)) = E(\xi(f)\xi(L^* g)) = \chi(\overline{\eta})\Phi'(\overline{\eta}) \int_\Omega dx \nabla f(x) \cdot \nabla g(x),$$

where $L^*$ is the dual of $L$. By Eq. (4.6), these last two equations are equivalent to the following ones.

$$E(\xi(f_{x_0, \epsilon})\xi(g_{x_0, \epsilon})) = \chi(\overline{\eta}) \int_\Omega dx f(x) g(x),$$

and

$$\epsilon^2 E(\xi(L^* f_{x_0, \epsilon})\xi(g_{x_0, \epsilon})) = \epsilon^2 E(\xi(f_{x_0, \epsilon})\xi(L^* g_{x_0, \epsilon})) = \chi(\overline{\eta})\Phi'(\overline{\eta}) \int_\Omega dx \nabla f(x) \cdot \nabla g(x).$$

The local equilibrium conditions, for fluctuations about nonequilibrium steady states, are just the limiting forms of these equations, as $\epsilon$ decreases to zero, with $\overline{\eta}$ replaced by $\overline{\eta}(x_0)$. Thus they are given by the formulae

$$\lim_{\epsilon \to 0} E(\xi(f_{x_0, \epsilon})\xi(g_{x_0, \epsilon})) = \chi(\overline{\eta}(x_0)) \int_\Omega dx f(x) g(x) \quad (4.8)$$

and

$$\lim_{\epsilon \to 0} \epsilon^2 E(\xi(L^* f_{x_0, \epsilon})\xi(g_{x_0, \epsilon})) = \lim_{\epsilon \to 0} \epsilon^2 E(\xi(L^* g_{x_0, \epsilon})) = \chi(\overline{\eta}(x_0))\Phi'(\overline{\eta}(x_0)) \int_\Omega dx \nabla f(x) \cdot \nabla g(x). \quad (4.9)$$

Note. These conditions represent local equilibrium on the hydrodynamic scale and are thus different from those formulated on the microscopic scale in Refs. [1, 2].

5. Verification of the Hypotheses 0-3

We shall now show that the above hypotheses are verified by the simple exclusion and zero range models.

The Simple Exclusion Model. The fluctuation process for this model was worked out in detail by Spohn [3] for the case where $\Omega$ is the slab*, $(0, 1) \times \mathbb{R}^{d-1}$. Here we shall confine our attention to the one-dimensional case, where $\Omega$ is the linear segment $(0, 1)$.

For this case the following results have been established [3].

(i) The function $\Phi$ appearing in the phenomenological law (3.2) is just the identity function and correspondingly, by Eq. (3.5), the generator $L$ is the Laplacian $\Delta$, with Dirichlet boundary conditions.

* Of course, for $d > 1$, the slab does not meet our condition that $\Omega$ be bounded. However, it is a straightforward matter to extend our treatment and results to that situation.
(ii) The process $\xi^{(N)}_t$ converges in law, as $N \to \infty$, to a limit $\xi_t$ that is governed by a Langevin equation

$$dx_t = L\xi_t dt + dw_t,$$

where $w_t$ is the Wiener process for which

$$E([w_t(f) - w_s(f)][w_{t'}(g) - w_{s'}(g)]) = 2\int_\Omega dx \chi(q(x))\Phi'(\overline{q}(x))\nabla f(x) \cdot \nabla g(x)[s, t] [s', t'] \quad \forall \, f, g \in \mathcal{D}(\Omega), \, t, s(t, t'), s'(t', t') \in \mathbb{R}^+,$$

and

$$E([w_t - w_s]|\xi_u) = 0 \text{ for } t \geq s \geq u.$$

(iii) The static two-point function takes the form

$$E(\xi(f)\xi(g)) = \int_0^1 dx \chi(q(x))f(x)g(x) + [h(1) - h(0)] \int_0^1 dx f(x)\Delta^{-1} g(x),$$

where

$$\chi(q) = q(1 - q)$$

and $h$ is the function appearing in the boundary term of Eq. (2.2).

The result (ii) immediately substantiates the hypotheses (0) and (2). Moreover, since $L$ is the generator of $\mathcal{L}(\mathbb{R}^+)$, it follows from Eq. (5.1) that

$$\xi_t = T_{t-t_0}\xi_{t_0} + \int_{t_0}^{t} T_{t-u}dw(u) \quad \forall \, t \geq t_0$$

and hence, by Eq. (5.3), that the hypothesis (1) is also fulfilled. Finally, the local equilibrium properties (4.8) and (4.9) are simple consequences of the formulae (4.7) and (5.4), which signifies that the model also satisfies hypothesis (3).

The Zero Range Model. A key property of this model is that its steady state takes the simple product form

$$\omega^{(N)}_{cd} = \otimes_{x \in \Omega} n_{x, \overline{q}(x)},$$

where $m_{x, \overline{q}(x)}$ is a probability measure on the functions of $n_x$ that depends on the value of the stationary field $\overline{q}$ at the site $x$.

It follows now from a straightforward adaptation of the argument* of [2, Ch. 11] that, for this model too, the process $\xi^{(N)}_t$ converges in law to a Gaussian process $\xi_t$, which is

* For that argument, as applied to the present situation, $\xi_t$ lies in the Sobolev space $\mathcal{H}_{-r}(\Omega) := \{ f : \Omega \to \mathbb{R} \mid \int_\Omega dx f(1 - \Delta)^{-r} f < \infty \}$ for sufficiently large $r \in \mathbb{N}$. This space is a subspace of $\mathcal{D}'(\Omega)$. The operators $\mathcal{L}$ and $\Phi(\overline{q}(x))^1/2 \nabla$ on $\mathcal{H}_{-r}$ play the roles of those denoted, in Gothic script, by $A$ and $B$, respectively, in [2].
also represented by Eqs. (5.1)-(5.3), though with $\mathcal{L}$ now given by Eq. (3.5). Hence, by the same argument as for the simple exclusion model, we see that this model satisfies the hypotheses (0), (1) and (2). Further, it follows from (0) that

$$E(\xi(f)\xi(g)) = \lim_{N \to \infty} \omega_{\xi}^{(N)}(\xi^{(N)}(f)\xi^{(N)}(g))$$

and hence, by Eqs. (4.7) and (5.6), that the conditions (4.8) and (4.9) are fulfilled, with $\chi(\overline{q}(x))$ the variance of the particle number at the site $L_N x$ in the single particle state $\mu_x, \overline{\xi}(x)$.

6. Concluding Remarks

In this article we have constructed a quantum stochastic model that fulfills the hypotheses of our general macrostatistical picture [7,8] of nonequilibrium steady states. A rather unphysical feature of this model is that the density field $q^{(N)}_t$ is classical not only at the hydrodynamical level but also at the microscopic one. Thus the problem of constructing a quantum model whose classical properties emerge only at the hydrodynamical and thermodynamical levels remains a challenging and interesting one.

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