STABILITY AND BIFURCATION ON PREDATOR-PREY SYSTEMS WITH NONLOCAL PREY COMPETITION

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Abstract. In this paper, we investigate diffusive predator-prey systems with nonlocal intraspecific competition of prey for resources. We prove the existence and uniqueness of positive steady states when the conversion rate is large. To show the existence of complex spatiotemporal patterns, we consider the Hopf bifurcation for a spatially homogeneous kernel function, by using the conversion rate as the bifurcation parameter. Our results suggest that Hopf bifurcation is more likely to occur with nonlocal competition of prey. Moreover, we find that the steady state can lose the stability when conversion rate passes through some Hopf bifurcation value, and the bifurcating periodic solutions near such bifurcation value can be spatially nonhomogeneous. This phenomenon is different from that for the model without nonlocal competition of prey, where the bifurcating periodic solutions are spatially homogeneous near such bifurcation value.

1. Introduction. Since diffusive predator-prey and competing systems have much significant roles in population dynamics, they have been investigated extensively in the literature. We refer to [5, 6, 7, 8, 10, 13, 14, 30, 31, 33, 34, 39, 41, 42, 43] on the aspect of existence and nonexistence of nonconstant steady state solutions, periodic solutions and traveling wave solutions. These results could be used to explain the complex pattern formation in ecology. For example, Yi et al. [42] considered the following diffusive Rosenzweig-MacArthur predator-prey model,

\[
\begin{align*}
\frac{\partial u}{\partial t} - d_1 \Delta u &= u \left( a - \frac{u}{k} \right) - \frac{buv}{1 + mu}, \quad x \in \Omega, \ t > 0, \\
\frac{\partial v}{\partial t} - d_2 \Delta v &= -dv + \frac{cuv}{1 + mu}, \quad x \in \Omega, \ t > 0, \\
\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} &= 0, \quad x \in \partial \Omega, \ t > 0,
\end{align*}
\]

(1.1)

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and investigated Hopf and steady state bifurcations to explain the complex spatiotemporal dynamics. Here \( u(x,t) \) and \( v(x,t) \) are the densities of the prey and predator at time \( t > 0 \) and a spatial position \( x \in \Omega \) respectively; \( a > 0 \) and \( k > 0 \) represent the intrinsic growth rate and carrying capacity of the prey respectively; \( d > 0 \) represents the death rate of the predator; \( d_1, d_2 > 0 \) are the diffusion coefficients of the predator and prey respectively; \( b, c > 0 \) measure the interaction strength between the predator and prey; and \( m > 0 \) represents the prey’s ability to evade attack [14]. Through a \textit{priori} estimates of positive steady states and the implicit function theorem, Peng and Shi [33] also proved the nonexistence of nonconstant positive steady states when the conversion rate \( c \) is sufficiently large, which implies that the global bifurcating branches of steady states obtained in [42] are bounded loops. We refer to [44] for the dynamics of system (1.1) with the homogeneous Dirichlet boundary conditions and [25] for a modification of (1.1) with a prey refuge.

When the self-growth of the predator is logistic type, Du and Lou [14] analyzed the following model

\[
\begin{aligned}
\frac{\partial u}{\partial t} - d_1 \Delta u &= u(a-u) - \frac{buv}{1+mu}, \quad x \in \Omega, \ t > 0, \\
\frac{\partial v}{\partial t} - d_2 \Delta v &= v(d-v) + \frac{cuv}{1+mu}, \quad x \in \Omega, \ t > 0, \\
\nu u &= \nu v = 0, \quad x \in \partial \Omega, \ t > 0,
\end{aligned}
\]  

(1.2)

and investigated the positive steady states for large \( m \). Here all the parameters are positive constants except \( d \), which may change sign. They also analyzed the positive steady states with the homogeneous Dirichlet boundary conditions [12]. We remark that, for \( m = 0 \), system (1.2) was investigated in [26] and it was shown that every positive solution of system (1.2) converges to a constant steady state solution when time goes to infinity, which implies that system (1.2) has no nonconstant positive steady states. Moreover, Peng and Shi [33] proved the nonexistence of nonconstant positive steady states for \( m \neq 0 \).

During the past thirty years, many researchers have focused on the nonlocal interspecific competition of a species for resources [4, 18, 20, 21, 23]. In [4], Britton firstly proposed the following single population model with nonlocal competition effect

\[
\frac{\partial u(x,t)}{\partial t} = d\Delta u + \lambda u \left( 1 + \alpha u - \beta \int_{\Omega} K(x-y)u(y,t)dy \right),
\]  

(1.3)

and the periodic traveling waves were investigated when \( \Omega = (-\infty, \infty) \) and \( K(x) = \frac{1}{2} ae^{-a|x|} \). Then traveling wave solutions of Eq. (1.3) were studied extensively for more general kernel functions [2, 3, 15, 16, 18, 22]. When domain \( \Omega \) is bounded, Furter and Grinfeld [17] replaced \( K(x-y) \) in Eq. (1.3) with \( K(x,y) \), and used

\[
\pi = \int_{\Omega} K(x,y)u(y,t)dy
\]

to model the nonlocal completion of a single population. For the simplest case, where

\[
K(x,y) = \frac{1}{|\Omega|},
\]  

(1.4)
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they studied the the steady state bifurcation of Eq. (1.3) with the homogeneous Neumann boundary condition. Sun, Shi and Wang [38] also chose this spatial homogeneous kernel and investigated the steady states of Eq. (1.3) under the homogeneous Dirichlet boundary conditions. Then, the existence and bifurcations of steady states of diffusive logistic population models were investigated in [1, 9, 40] for more general kernel functions. We remark that for one-dimension domain \( \Omega = (0, l) \), the following kernel function

\[
K(x, y) = \frac{1}{l} + \frac{2}{l} \sum_{n=1}^{\infty} e^{-an^2\pi^2/l^2} \cos \frac{n\pi x}{l} \cos \frac{n\pi y}{l}
\]  

was used to model the growth of Nicholson’s blowflies with age structure [37], which was also used in [20] for a food-limited population model. Moreover, researchers have also concentrated on the nonlocal intraspecific competition of prey for predator-prey systems [19, 32]. For example, Merchant and Nagata [32] introduced the nonlocal competition of prey into the diffusive Rosenzweig-MacArthur predator-prey model and proposed the following model

\[
\begin{align*}
\frac{\partial u}{\partial t} - d_1 \Delta u &= u \left( a - \int_{\Omega} K(x, y) u(y, t) dy \right) - \frac{buv}{1+mu}, \quad x \in \Omega, \ t > 0, \\
\frac{\partial v}{\partial t} - d_2 \Delta v &= -dv + \frac{cuv}{1+mu}, \quad x \in \Omega, \ t > 0, \\
\partial_{\nu} u = \partial_{\nu} v &= 0, \quad x \in \partial \Omega, \ t > 0.
\end{align*}
\]  

(1.6)

When \( \Omega = (-\infty, \infty) \), they found that the nonlocal competition can destabilize the spatially homogeneous steady state and induce complex spatiotemporal patterns.

In this paper, we mainly consider model (1.6) when domain \( \Omega \) is bounded. Firstly, we consider the case that \( m = 0 \), which means that the prey’s ability to evade attack is weak, and then the interaction between the prey and predator is classical Lotka-Volterra type. For \( m = 0 \), model (1.6) has the following form

\[
\begin{align*}
\frac{\partial u}{\partial t} - d_1 \Delta u &= u \left( a - \int_{\Omega} K(x, y) u(y, t) dy \right) - buv, \quad x \in \Omega, \ t > 0, \\
\frac{\partial v}{\partial t} - d_2 \Delta v &= -dv + \frac{cuv}{1+mu}, \quad x \in \Omega, \ t > 0, \\
\partial_{\nu} u = \partial_{\nu} v &= 0, \quad x \in \partial \Omega, \ t > 0.
\end{align*}
\]  

(1.7)

Here \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) \((N \leq 3)\) with a smooth boundary \( \partial \Omega \), parameters \( d_1, d_2, a, b, c, \) and \( d \) are all positive constants, which have the same meanings as in Eq. (1.1), and the kernel function satisfies the following assumption:

(K) \( K(x, y) \) is nonnegative and belongs to \( C^1(\overline{\Omega} \times \overline{\Omega}) \).

Then, we also consider the following model

\[
\begin{align*}
\frac{\partial u}{\partial t} - d_1 \Delta u &= u \left( a - \int_{\Omega} K(x, y) u(y, t) dy \right) - buv, \quad x \in \Omega, \ t > 0, \\
\frac{\partial v}{\partial t} - d_2 \Delta v &= v(d - v) + cuv, \quad x \in \Omega, \ t > 0, \\
\partial_{\nu} u = \partial_{\nu} v &= 0, \quad x \in \partial \Omega, \ t > 0.
\end{align*}
\]  

(1.8)

Here the predator can also survive without the specific prey, and all the parameters are positive constants except \( d \), which may change sign. Actually, this model can also be obtained by introducing the nonlocal intraspecific competition of prey into model (1.2) for \( m = 0 \). Our main results for models (1.7) and (1.8) are as follows.

(I) There exists \( c_0 > 0 \) such that system (1.7) has a unique positive steady state for \( c > c_0 \).
(II) If $a > bd$, then there exists $c_0 > 0$ such that system (1.8) has a unique positive steady state for $c > c_0$, and if $a < bd$, then there exists $c_0 > 0$ such that system (1.8) has no positive steady states for $c > c_0$.

We remark that the method used here is motivated by [33, 36]. However it cannot be used to model (1.6) for $m \neq 0$. The main reason is the lack of “order preserving property” of the nonlocal equation [11], and hence we can not estimate the upper bound for prey $u$. Finally, for model (1.6) with $m \neq 0$, we give the bifurcation analysis to show the existence of complex spatiotemporal patterns for a special kernel function, and find that the nonlocal competition of prey can induce some new dynamical behaviors.

The rest of the paper is organized as follows. In Section 2, we prove the existence and uniqueness of positive steady states for model (1.7) and (1.8), respectively. In section 3, we investigate the Hopf bifurcation for (1.6) with $m \neq 0$ and the spatially homogeneous kernel function.

2. Uniqueness of steady states. In this section, we will show the uniqueness of the steady states of models (1.7) and (1.8) for large conversion rate $c$. Therefore, complex pattern formation is impossible for large conversion rate.

2.1. Preliminaries. We first cite several results for later application. The first is the following result [27, 33].

**Lemma 2.1.** Suppose that $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^N$, and $d$ is a non-negative constant. If $z \in W^{1,2}(\Omega)$ is a non-negative weak solution of the following inequalities

$$\begin{cases}
-\Delta z + d z \geq 0, & x \in \Omega, \\
\partial_{\nu} z \leq 0, & x \in \partial \Omega,
\end{cases}$$

then, for any $q \in [1, \frac{N}{N-2})$, there is a positive constant $C$, which is determined only by $q$, $d$, and $\Omega$, such that

$$\|z\|_q \leq C \inf_{x \in \Omega} z.$$ 

The second is Harnack inequality [27, 28, 33, 35].

**Lemma 2.2.** Suppose that $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^N$, and $c(x) \in L^q(\Omega)$ for some $q > N/2$. If $z \in W^{1,2}(\Omega)$ is a non-negative weak solution of the following problem

$$\begin{cases}
\Delta z + c(x) z = 0, & x \in \Omega, \\
\partial_{\nu} z = 0, & x \in \partial \Omega,
\end{cases}$$

then there is a positive constant $C$, which is determined only by $\|c(x)\|_q$, $q$, and $\Omega$, such that

$$\sup_{x \in \Omega} z \leq C \inf_{x \in \Omega} z.$$ 

Finally, we cite the following maximum principles [27, 29, 35].

**Lemma 2.3.** Suppose that $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$, and $g \in C(\overline{\Omega} \times \mathbb{R})$. If $z \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies the inequalities

$$\begin{cases}
\Delta z + g(x, z) \geq 0, & x \in \Omega, \\
\partial_{\nu} z \leq 0, & x \in \partial \Omega,
\end{cases}$$

and there is a constant $K$ satisfying $g(x, z) < 0$ for $z > K$, then $\max_{x \in \overline{\Omega}} z \leq K$. 

Lemma 2.4. Suppose that $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$, and $g \in C(\overline{\Omega} \times \mathbb{R})$. If $z \in C^2(\overline{\Omega}) \cap C^1(\Omega)$ satisfies the inequalities
\[
\begin{aligned}
\Delta z + g(x, z) &\leq 0, \quad x \in \Omega, \\
\partial_{\nu} z &\geq 0, \quad x \in \partial \Omega,
\end{aligned}
\]
and there is a constant $K$ satisfying $g(x, z) > 0$ for $z < K$, then $\min_{x \in \Omega} z \geq K$.

2.2. Analysis of model (1.7). In this subsection, we will analyze the positive steady states of system (1.7), which satisfy
\[
\begin{aligned}
-d_1 \Delta u &= u \left( a - \int_{\Omega} K(x, y)u(y)dy \right) - buv, \quad x \in \Omega, \\
-d_2 \Delta v &= -dv + cuv, \quad x \in \Omega, \\
\partial_{\nu} u &= \partial_{\nu} v = 0, \quad x \in \partial \Omega.
\end{aligned}
\] (2.1)

Let $w = cu$, $z = bv$ and $\rho = \frac{1}{c}$. Then $w$ and $z$ satisfy
\[
\begin{aligned}
-d_1 \Delta w &= w \left( a - \rho \int_{\Omega} K(x, y)w(y)dy \right) - wz, \quad x \in \Omega, \\
-d_2 \Delta z &= -dz + wz, \quad x \in \Omega, \\
\partial_{\nu} w &= \partial_{\nu} z = 0, \quad x \in \partial \Omega.
\end{aligned}
\] (2.2)

Based on Lemmas 2.2-2.4, we first obtain a priori estimates for positive solutions of system (2.2).

Lemma 2.5. Assume that $d_1$, $d_2$, $a$, $d$, and $\rho$ are all positive constants, $\Omega$ is a bounded domain in $\mathbb{R}^N$ ($N \leq 3$) with a smooth boundary $\partial \Omega$, and $K(x, y)$ satisfies assumption (K). Let $(w_\rho, z_\rho)$ be a positive solution of system (2.2). Then the following two statements are true.

(I) For any $M_1 > 0$, there exists $C > 0$, depending on $M_1$, such that
\[
\sup_{0 \leq \rho \leq M_1} \sup_{x \in \Omega} w_\rho, \quad \sup_{0 \leq \rho \leq M_1} \sup_{x \in \Omega} z_\rho \leq C.
\]

(II) There exists $M_2 > 0$ such that
\[
\inf_{0 \leq \rho \leq M_2} \inf_{x \in \Omega} w_\rho > 0 \quad \text{and} \quad \inf_{0 \leq \rho \leq M_2} \inf_{x \in \Omega} z_\rho > 0.
\]

Proof. We first derive the existence of the upper bound. It follows from Eq. (2.2) that
\[
\begin{aligned}
d \int_{\Omega} z_\rho dx &= \int_{\Omega} w_\rho \left( a - \rho \int_{\Omega} K(x, y)w_\rho dy \right) dx, \\
\int_{\Omega} (-d + w_\rho) dx &= -d_2 \int_{\Omega} \frac{|\nabla z_\rho|^2}{z_\rho^2} dx \leq 0,
\end{aligned}
\] (2.3)

which imply that
\[
\int_{\Omega} z_\rho dx \leq a|\Omega|, \quad \int_{\Omega} w_\rho dx \leq d|\Omega|.
\] (2.4)

It follows from Lemma 2.1 that, for $q > N/2$, there exists $C_1 > 0$ such that
\[
\|z_\rho\|_q \leq C_1 \inf_{x \in \Omega} z \quad \text{for} \quad \rho \geq 0,
\]
which implies that \( \|z_\rho\|_q \leq aC_1 \) for \( \rho \geq 0 \). Here \( q \in [1, \infty) \) for \( N = 1, 2 \), and \( q \in (2, 3) \) for \( N = 3 \). Then

\[
\left\| a - \rho \int_\Omega K(x, y)w_\rho(y)dy - z_\rho \right\|_q \\
\leq \left( a + d|\Omega|M_1 \max_{(x,y) \in \overline{\Omega} \times \overline{\Omega}} |K(x, y)| \right) |\Omega|^{1/q} + aC_1
\]

(2.5)

for \( \rho \in [0, M_1] \). It follows from Lemma 2.2 that there exists a constant \( C_2 > 0 \), depending on \( M_1 \), such that

\[
\sup_{x \in \Omega} w_\rho \leq C_2 \inf_{x \in \Omega} w_\rho \quad \text{for} \quad 0 \leq \rho \leq M_1.
\]

(2.6)

Since \( \inf_{x \in \Omega} w_\rho|\Omega| \leq \int_\Omega w_\rho dx \leq d|\Omega| \) for \( \rho \geq 0 \), there is a constant \( C_3 > 0 \), depending on \( M_1 \), such that

\[
\sup_{x \in \Omega} w_\rho \leq C_3 \quad \text{for} \quad 0 \leq \rho \leq M_1,
\]

(2.7)

and consequently,

\[
\| - d + w_\rho \|_\infty \leq d + C_3 \quad \text{for} \quad 0 \leq \rho \leq M_1.
\]

Then, due to Lemma 2.2, there exists a constant \( C_4 > 0 \), depending on \( M_1 \), such that

\[
\sup_{x \in \Omega} z_\rho \leq C_4 \inf_{x \in \Omega} z_\rho \quad \text{for} \quad 0 \leq \rho \leq M_1.
\]

(2.8)

This, combined with Eq. (2.4), implies that there exists a constant \( C_5 > 0 \), depending on \( M_1 \), such that

\[
\sup_{x \in \Omega} z_\rho < C_5 \quad \text{for} \quad 0 \leq \rho \leq M_1.
\]

(2.9)

Letting \( \overline{C} = \max\{C_3, C_5\} \), we have

\[
\sup_{x \in \Omega} w_\rho, \sup_{x \in \Omega} z_\rho \leq \overline{C} \quad \text{for} \quad 0 \leq \rho \leq M_1.
\]

Next, we prove part (II). We first claim that there exists \( M_2 > 0 \) such that

\[
\inf_{0 \leq \rho \leq M_2} \inf_{x \in \Omega} w_\rho > 0.
\]

(2.10)

By way of contradiction, there exists a subsequence \( \{\rho_j\}_{j=1}^\infty \) such that \( \lim_{j \to \infty} \rho_j = 0 \) and

\[
\lim_{j \to \infty} \inf_{x \in \Omega} w_{\rho_j} = 0.
\]

Then, it follows from Eq. (2.6) that \( w_{\rho_j} \to 0 \) uniformly on \( \overline{\Omega} \) as \( j \to \infty \). Hence, for sufficiently large \( j \),

\[
\int_\Omega z_{\rho_j} (d - w_{\rho_j}) \ dx > 0,
\]

which is a contradiction. Therefore, the claim is proved and Eq. (2.10) holds. Finally, we claim that there exists \( M_2 > 0 \) such that

\[
\inf_{0 \leq \rho \leq M_2} \inf_{x \in \Omega} z_\rho > 0.
\]

(2.11)

Similarly, we argue indirectly and assume that there exists a subsequence \( \{\rho_k\}_{k=1}^\infty \) such that \( \lim_{k \to \infty} \rho_k = 0 \) and

\[
\lim_{k \to \infty} \inf_{x \in \Omega} z_{\rho_k} = 0.
\]
It follows from Eq. (2.8) that \( \lim_{k \to \infty} \sup_{x \in \Omega} z_{\rho_k} = 0 \). Since \( \lim_{k \to \infty} \rho_k = 0 \) and \( \int_{\Omega} w_{\rho_k} \, dx \leq d|\Omega| \), we see that, for sufficiently large \( k \),
\[
\int_{\Omega} w_{\rho_k} \left( a - \rho_k \int_{\Omega} K(x, y)w_{\rho_k}(y) \, dy - z_{\rho_k} \right) \, dx > 0,
\]
which is a contradiction. Therefore, Eq. (2.11) holds. Letting \( M_2 = \min\{\overline{M}_2, \tilde{M}_2\} \), we obtain
\[
\inf_{0 \leq \rho \leq M_2} \inf_{x \in \Omega} w_{\rho} > 0 \quad \text{and} \quad \inf_{0 \leq \rho \leq M_2} \inf_{x \in \Omega} z_{\rho} > 0.
\]
This completes the proof. \( \square \)

Then, we can obtain the following results on positive solutions of system (2.2) through the implicit function theorem.

**Theorem 2.6.** Assume that \( d_1, d_2, a, d, \) and \( \rho \) are all positive constants, \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) \((N \leq 3)\) with a smooth boundary \( \partial \Omega \), and \( K(x, y) \) satisfies the assumption \((K)\). Then there exists \( \rho_0 > 0 \) such that system (2.2) has a unique positive solution for \( \rho < \rho_0 \).

**Proof.** As in [36], we denote
\[
W^{2,p}_\rho(\Omega) = \{ u(x) \in W^{2,p}(\Omega) : \partial_\nu u = 0 \text{ on } \partial \Omega \}, \quad \text{where } p > N,
\]
and define an operator
\[
A(\rho, w, z) = (g_1(\rho, w, z), g_2(\rho, w, z))^T,
\]
where
\[
g_1(\rho, w, z) = d_1 \Delta w + w \left( a - \rho \int_{\Omega} K(x, y)w(y) \, dy \right) - wz,
\]
\[
g_2(\rho, w, z) = d_2 \Delta z - dz + wz.
\]
It follows from the embedding theorems that \( A \) is a continuously differentiable from \([0, M_2] \times W^{2,p}_\rho(\Omega) \times W^{2,p}_\rho(\Omega) \) to \( L^p(\Omega) \times L^p(\Omega) \), where \( M_2 \) is defined as in Lemma 2.5. Then \((w_\rho, z_\rho)\) is a solution of system (2.2) if and only if \( A(\rho, w_\rho, z_\rho) = (0, 0)^T \).

According to Lemma 2.4 [33], we see that \( A(0, w_*, z_*) = 0 \), where \((w_*, z_*) = (d, a)\), and \((w_*, z_*)\) is non-degenerate in the sense that zero is not the eigenvalue of the linearized problem with respect to \((w_*, z_*)\). The Fréchet derivative of \( A \) with respect to \((w, z)\) at \((0, w_*, z_*)\) is given by
\[
D_{(w,z)}A(0, w_*, z_*)[\phi, \psi]^T = \begin{pmatrix} d_1 \Delta \phi - w_* \psi \\ d_2 \Delta \psi + z_* \phi \end{pmatrix}.
\]
Since \((w_*, z_*)\) is non-degenerate, it follows that \( D_{(w,z)}A(0, w_*, z_*)\) is injective. As a consequence, noticing that \( D_{(w,z)}A(0, w_*, z_*)\) is a Fredholm operator, we see that \( D_{(w,z)}A(0, w_*, z_*)\) is an isomorphism. Then, it follows from the implicit function theorem that there exists \( \rho_0 > 0 \) such that, for \( 0 < \rho < \rho_0 \), system (2.2) has a positive solution \((w_\rho, z_\rho)\) \( \in W^{2,p}_\rho(\Omega) \times W^{2,p}_\rho(\Omega) \), which belongs to \( C^2(\overline{\Omega}) \times C^2(\overline{\Omega}) \) from the regularity theory and embedding theorems. Then it remains to prove the uniqueness. From the implicit function theorem, we only need to verify that if \((w^\rho, z^\rho)\) is a positive solution of system (2.2), then
\[
(w^\rho, z^\rho) \to (w_*, z_*) \text{ in } C^2(\overline{\Omega}) \times C^2(\overline{\Omega}) \text{ as } \rho \to 0.
\]
From Lemma 2.5, we find that
\[
\underline{C} \leq w^\rho, z^\rho \leq \overline{C} \text{ for } x \in \overline{\Omega} \text{ and } \rho \in [0, M_2],
\]
(2.12)
where $M_2$ is defined as in Lemma 2.5 and
\[
\mathcal{C} = \min \{ \inf_{0 < \rho \leq M_2} \inf_{x \in \Omega} w_\rho, \inf_{0 < \rho \leq M_2} \inf_{z \in \mathbb{R}} \inf \sup_{\rho \leq M_2} w_\rho \} > 0, \\
\overline{\mathcal{C}} = \max \{ \sup_{0 < \rho \leq M_2} \sup_{x \in \Omega} w_\rho, \sup_{0 < \rho \leq M_2} \sup_{z \in \mathbb{R}} \sup \inf_{\rho \leq M_2} z_\rho \}.
\]

It follows from Eq. (2.12) and the $L^p$ theory that \( \{ w_\rho : 0 \leq \rho \leq M_2 \} \) and \( \{ z_\rho : 0 \leq \rho \leq M_2 \} \) are bounded in $W^{2,p}(\Omega)$. Due to the embedding theorems, we find that \( \{ w_\rho : 0 \leq \rho \leq M_2 \} \) and \( \{ z_\rho : 0 \leq \rho \leq M_2 \} \) are bounded and precompact in $C^1(\Omega)$. Then, for any sequence \( \{ \rho_i \}_{i=1}^\infty \) satisfying \( \lim_{i \to \infty} \rho_i = 0 \), there exist a subsequence \( \{ \rho_{i_k} \}_{k=1}^\infty \) and \( (w^*, z^*) \in C^1(\Omega) \times C^1(\Omega) \) such that
\[
(w_{\rho_{i_k}}, z_{\rho_{i_k}}) \to (w^*, z^*) \quad \text{in} \quad C^1(\Omega) \times C^1(\Omega) \quad \text{as} \quad k \to \infty.
\]

Since
\[
w_{\rho_{i_k}} = [-d_1 \Delta + I]^{-1} \left[ w_{\rho_{i_k}} + w_{\rho_{i_k}} \left( a - \rho_{i_k} \int_{\Omega} K(x,y)w_{\rho_{i_k}}(y)dy \right) - w_{\rho_{i_k}} z_{\rho_{i_k}} \right],
\]
\[
z_{\rho_{i_k}} = [-d_2 \Delta + I]^{-1} \left[ z_{\rho_{i_k}} - d_2 z_{\rho_{i_k}} + w_{\rho_{i_k}} z_{\rho_{i_k}} \right],
\]
and \( \lim_{k \to \infty} \rho_{i_k} \int_{\Omega} K(x,y)w_{\rho_{i_k}}(y)dy = 0 \) in $C^1(\Omega)$, we see that, for some $\alpha \in (0, 1)$,
\[
(w_{\rho_{i_k}}, z_{\rho_{i_k}}) \to (w^*, z^*) \quad \text{in} \quad C^{2,\alpha}(\Omega) \times C^{2,\alpha}(\Omega) \quad \text{as} \quad k \to \infty,
\]
and \( (w^*, z^*) \) is a positive solution of Eq. (2.2) for $\rho = 0$. It follows from Lemma 2.4 [33] that \( (w^*, z^*) = (w_*, z_*) \). This completes the proof. \( \square \)

2.3. Analysis of model (1.8). In this subsection, we will analyze the positive steady states of system (1.8), which satisfy
\[
\begin{aligned}
-d_1 \Delta u &= u (a - \int_{\Omega} K(x,y)u(y)dy) - bwv, \quad x \in \Omega, \\
-d_2 \Delta v &= v (d - v) + cvw, \quad x \in \Omega, \\
\partial_{\nu} u &= \partial_{\nu} v = 0, \quad x \in \partial \Omega.
\end{aligned}
\]
Let $w = cu$, $z = bv$ and $\rho = \frac{1}{c}$. Then $w$ and $z$ satisfy
\[
\begin{aligned}
-d_1 \Delta w &= w (a - \rho \int_{\Omega} K(x,y)w(y)dy) - wz, \quad x \in \Omega, \\
-d_2 \Delta z &= z \left( d - \frac{z}{b} \right) + wz, \quad x \in \Omega, \\
\partial_{\nu} w &= \partial_{\nu} z = 0, \quad x \in \partial \Omega.
\end{aligned}
\]
Similarly, the existence/nonexistence of positive solutions of system (2.13) for large $c$ is also equivalent to that of system (2.14) for small $\rho$. As in Lemma 2.4 of [33], we first consider the positive solutions of system (2.14) when $\rho = 0$.

**Lemma 2.7.** Assume that $\rho = 0$, constants $d_1$, $d_2$, $a$, and $b$ are all positive, and $\Omega$ is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial \Omega$. If $a > bd$, then system (2.14) has a unique positive solution $\left( \left( \frac{a}{b} - d \right), a \right)$.

**Proof.** We construct the following functional:
\[
V(w, z) = \int_{\Omega} \left\{ \left( w - w_* \right) [d_1 \Delta w + w(a - z)] + \frac{z - z_*}{z} \left[ d_2 \Delta z + z \left( \frac{a}{b} - d \right) \right] \right\} dx,
\]
where
\[
(w_*, z_*) = \left( \left( \frac{a}{b} - d \right), a \right).
\]
An easy calculation implies that
\[
V(w, z) = -\int_\Omega \left[ d_1 \frac{w_z |\nabla w|^2}{w^2} + d_2 \frac{z_z |\nabla z|^2}{z^2} + \frac{(z - z_s)^2}{b} \right] dx.
\]

Therefore, if \((w(x), z(x))\) is a positive solution of system (2.14), then \(V(w, z) = 0\) which leads to \((w(x), z(x)) = (w_*, z_*)\).

\[\square\]

Then, from Lemmas 2.2-2.4, we obtain a priori estimates for positive solutions of system (2.14).

**Lemma 2.8.** Assume that \(d_1, d_2, a, b, \) and \(\rho\) are all positive constants, \(\Omega\) is a bounded domain in \(\mathbb{R}^N (N \leq 3)\) with a smooth boundary \(\partial \Omega\), and \(K(x, y)\) satisfies assumption (K). Let \((w_{\rho}, z_{\rho})\) be a positive solution of system (2.14). Then the following two statements are true.

(I) For any \(M_1 > 0\), there exists \(\overline{C} > 0\), depending on \(M_1\), such that
\[
\sup_{0 \leq \rho \leq M_1} \sup_{x \in \Omega} w_{\rho}, \sup_{0 \leq \rho \leq M_1} \sup_{x \in \Omega} z_{\rho} \leq \overline{C}.
\]

(II) If \(a > bd\), then there exists \(M_2 > 0\) such that
\[
\inf_{0 \leq \rho \leq M_2} \inf_{x \in \Omega} w_{\rho} > 0 \text{ and } \inf_{0 \leq \rho \leq M_2} \inf_{x \in \Omega} z_{\rho} > 0.
\]

**Proof.** We first prove part (I). From Eq. (2.14), we see that
\[
\int_\Omega z_{\rho} \left( \frac{z_{\rho}}{b} - d \right) dx = \int_\Omega w_{\rho} \left( a - \rho \int_\Omega K(x, y) w_{\rho} dy \right) dx,
\]
\[
\int_\Omega \left( d - \frac{z_{\rho}}{b} + w_{\rho} \right) dx = -d_2 \int_\Omega \frac{|\nabla z_{\rho}|^2}{z_{\rho}^2} dx \leq 0,
\]
which imply that
\[
\| z_{\rho} \|^2_{L^2} \leq (a + bd) \| z_{\rho} \|_{L^2(\Omega)}^{1/2} - abd |\Omega|.
\]

Therefore, there exists a constant \(C_1 > 0\) such that
\[
\int_\Omega w_{\rho} dx, \quad \| z_{\rho} \|_{L^2} \leq C_1 \text{ for all } \rho \geq 0,
\]
which leads to
\[
\left\| a - \rho \int_\Omega K(x, y) w_{\rho} (y) dy - z_{\rho} \right\|_{L^2} < \left( a + C_1 M_1 \max_{(x, y) \in \Omega} |K(x, y)| \right) |\Omega|^{1/2} + C_1
\]
for \(\rho \in [0, M_1]\). Then, it follows from Lemma 2.2 that there exists a constant \(C_2 > 0\), depending on \(M_1\), such that
\[
\sup_{x \in \Omega} w_{\rho} < C_2 \inf_{x \in \Omega} w_{\rho} \text{ for } 0 \leq \rho \leq M_1.
\]

Since \(\inf_{x \in \Omega} w_{\rho} |\Omega| \leq \int_\Omega w_{\rho} dx \leq C_1 \) for \(\rho \geq 0\), there is a constant \(C_3 > 0\), depending on \(M_1\), such that
\[
\sup_{x \in \Omega} w_{\rho} \leq C_3 \text{ for } 0 \leq \rho \leq M_1
\]
and consequently,
\[
-d_2 \Delta z_{\rho} = z_{\rho} \left( d - \frac{z_{\rho}}{b} \right) + w_{\rho} z_{\rho} \leq z_{\rho} \left( d + C_3 - \frac{z_{\rho}}{b} \right)
\]
for $0 \leq \rho \leq M_1$. Then, due to Lemma 2.3, there exists a constant $C_4 > 0$, depending on $M_1$, such that
\[ \sup_{x \in \Omega} z_\rho < C_4 \quad \text{for} \quad 0 \leq \rho \leq M_1. \] (2.19)

Let $\overline{C} = \max\{C_3, C_4\}$. Then we have
\[ \sup_{x \in \Omega} w_\rho, \sup_{x \in \Omega} z_\rho \leq \overline{C} \quad \text{for} \quad 0 \leq \rho \leq M_1. \]

Now, we prove part (II). We first claim that there exists $M_2 > 0$ such that
\[ \inf_{0 \leq \rho \leq M_2} \inf_{x \in \Omega} w_\rho > 0. \] (2.20)

We argue indirectly and assume that there exists a subsequence $\{\rho_j\}_{j=1}^{\infty}$ such that
\[ \lim_{j \to \infty} \rho_j = 0 \quad \text{and} \quad \lim_{j \to \infty} \inf_{x \in \Omega} z_{\rho_j} = 0. \]

It follows from Eq. (2.17) that $w_{\rho_j} \to 0$ uniformly on $\overline{\Omega}$ as $j \to \infty$. If $d < 0$, then, for sufficiently large $j$,
\[ \int_{\Omega} w_{\rho_j} \left( \frac{z_{\rho_j}}{b} - d - w_{\rho_j} \right) dx > \int_{\Omega} z_{\rho_j} (-d - w_{\rho_j}) dx > 0, \]
which is a contradiction. If $d \geq 0$, then, for sufficiently small $\epsilon \in \left(0, \frac{a - bd}{b}\right)$, there exists $j_0 > 0$ such that $\max_{x \in \Omega} |w_{\rho_j}| < \epsilon$ and
\[ -d_2 \Delta z_{\rho_j} \leq z_{\rho_j} \left( d - \frac{z_{\rho_j}}{b} + \epsilon \right) \]
for $j > j_0$. As a consequence, we have $z_{\rho_j} \leq b(d + \epsilon)$ and
\[ -d_1 \Delta w_{\rho_j} \geq w_{\rho_j} \left[ a - b(d + \epsilon) - \rho_j \int_{\Omega} K(x, y) w_{\rho_j}(y) dy \right] \]
for any $j > j_0$, which implies that
\[ \lim_{j \to \infty} \int_{\Omega} \left[ a - b(d + \epsilon) - \rho_j \int_{\Omega} K(x, y) w_{\rho_j}(y) dy \right] dx \leq 0. \]

Therefore, $[a - b(d + \epsilon)] |\Omega| \leq 0$, which is a contradiction. So the claim is proved and Eq. (2.20) holds. Finally, we claim that there exists $M_2 > 0$ such that
\[ \inf_{0 \leq \rho \leq M_2} \inf_{x \in \Omega} z_\rho > 0. \] (2.21)

Similarly, by way of contradiction, there exists a subsequence $\{\rho_k\}_{k=1}^{\infty}$ such that
\[ \lim_{k \to \infty} \rho_k = 0 \quad \text{and} \quad \lim_{k \to \infty} \inf_{x \in \Omega} z_{\rho_k} = 0. \]

It follows from Lemma 2.2 that $\lim_{k \to \infty} \sup_{x \in \Omega} z_{\rho_k} = 0$. Since $\lim_{k \to \infty} \rho_k = 0$ and $\int_{\Omega} w_{\rho_k} dx \leq C_1$, we see that, for sufficiently large $k$,
\[ \int_{\Omega} w_{\rho_k} \left( a - \rho_k \int_{\Omega} K(x, y) w_{\rho_k}(y) dy - z_{\rho_k} \right) dx > 0, \]
which is a contradiction. Therefore, Eq. (2.21) holds. This completes the proof. \( \square \)

In what follows, we give our main results on solutions of system (2.14) for small $\rho$. 

Theorem 2.9. Assume that $d_1$, $d_2$, $a$, $b$, and $\rho$ are all positive constants, $\Omega$ is a bounded domain in $\mathbb{R}^N$ ($N \leq 3$) with a smooth boundary $\partial \Omega$, and $K(x,y)$ satisfies assumption (K). Then the following two statements are true.

(I) If $a > bd$, then there exists $\rho_0 > 0$ such that system (2.14) has a unique positive solution, which is locally asymptotically stable for $\rho < \rho_0$.

(II) If $a < bd$, then there exists $\rho_1 > 0$ such that system (2.14) has no positive solutions for $\rho < \rho_1$.

Proof. We first consider the case that $a > bd$. As in Theorem 2.6, we define an operator

$$\mathcal{A}(\rho, w, z) = (\mathcal{G}_1(\rho, w, z), \mathcal{G}_2(\rho, w, z))^T,$$

where

$$\mathcal{G}_1(\rho, w, z) = d_1 \Delta w + w \left( a - \rho \int_{\Omega} K(x, y) w(y) dy \right) - wz,$$

$$\mathcal{G}_2(\rho, w, z) = d_2 \Delta z + z \left( d - \frac{z}{b} \right) + wz.$$

Then $\mathcal{A}$ is a continuously differentiable from $[0, \alpha_2] \times W^{2,p}_0(\Omega) \times W^{2,p}_0(\Omega)$ to $L^p(\Omega) \times L^p(\Omega)$, where $\alpha_2$ is defined as in Lemma 2.8, and $(w_\rho, z_\rho)$ is a positive solution of system (2.14) if and only if $\mathcal{A}(\rho, w_\rho, z_\rho) = (0, 0)^T$. According to Lemma 2.7, we see that $\mathcal{A}(0, w_\rho, z_\rho) = 0$, where $(w_\rho, z_\rho)$ is defined as in Eq. (2.15). An easy calculation implies that the Fréchet derivative of $\mathcal{A}$ with respect to $(w, z)$ at $(0, w_\rho, z_\rho)$ is given by

$$D_{(w,z)} \mathcal{A}(0, w_\rho, z_\rho)[\phi, \psi]^T = \begin{pmatrix} d_1 \Delta \phi - w_\rho \psi \\ d_2 \Delta \psi - \frac{z_\rho}{b} \psi + z_\rho \phi \end{pmatrix},$$

and the eigenvalue problem with respect to $(w_\rho, z_\rho)$ for the corresponding parabolic equations is

$$\begin{pmatrix} d_1 \Delta \phi - w_\rho \psi \\ d_2 \Delta \psi - \frac{z_\rho}{b} \psi + z_\rho \phi \end{pmatrix} = \lambda \begin{pmatrix} \phi \\ \psi \end{pmatrix}.$$}

Therefore, the sequence of the characteristic equations is

$$\lambda^2 + \left[ (d_1 + d_2) \mu_i + \frac{z_\rho}{b} \right] \lambda + d_1 \mu_i \left( d_2 \mu_i + \frac{z_\rho}{b} \right) = 0, \quad i = 0, 1, 2, \cdots ,$$

where $0 = \mu_0 < \mu_1 < \mu_2 < \cdots < \mu_i < \cdots$ are the eigenvalues of operator $-\Delta$ on $\Omega$ with the homogeneous Neumann boundary condition. It follows that $(w_\rho, z_\rho)$ is linearly stable, and $D_{(w,z)} \mathcal{A}(0, w_\rho, z_\rho)$ is an isomorphism. Then, from the implicit function theorem, we see that there exists $\rho_0 > 0$ such that, for $0 < \rho < \rho_0$, system (2.14) has a solution $(w_\rho, z_\rho)$, which is locally asymptotically stable. Hence the existence is proved, and it remains to prove the uniqueness. From the implicit function theorem, we need to verify that if $(w^{\rho}, z^{\rho})$ is a solution of system (2.14), then

$$(w^{\rho}, z^{\rho}) \to (w_\rho, z_\rho) \quad \text{in} \quad C^2(\Omega) \times C^2(\Omega) \quad \text{as} \quad \rho \to 0.$$}

As in Theorem 2.6, we can prove that, for any sequence $\{\rho_i\}_{i=1}^\infty$ satisfying $\lim_{i \to \infty} \rho_i = 0$, there exists a subsequence $\{\rho_{i_k}\}_{k=1}^\infty$ such that

$$(w^{\rho_{i_k}}, z^{\rho_{i_k}}) \to (w^*, z^*) \quad \text{in} \quad C^{2,\alpha}(\Omega) \times C^{2,\alpha}(\Omega) \quad \text{as} \quad k \to \infty,$$

where $(w^*, z^*)$ is a positive solution of Eq. (2.14) for $\rho = 0$. It follows from Lemma 2.7 that $(w^*, z^*) = (w_\rho, z_\rho)$. This completes the proof of part (I).
In the following, we consider the case that \( a < bd \). We argue indirectly and assume that there exists a sequence \( \{ \rho_i \}_{i=1}^{\infty} \) such that \( \lim_{i \to \infty} \rho_i = 0 \) and system (2.14) has a positive solution \((w_{\rho_i}, z_{\rho_i})\) for \( \rho = \rho_i \). It follows from Lemma 2.4 and the second equation of system (2.14) that \( z_{\rho_i} \geq bd \), and consequently,

\[
-d_1 \Delta w_{\rho_i} \leq w_{\rho_i} \left( a - \rho_i \int_{\Omega} K(x, y) w_{\rho_i}(y) dy - bd \right).
\]

Since \( \lim_{i \to \infty} \rho_i = 0 \), we see that, for sufficiently large \( i \),

\[
a - \rho_i \int_{\Omega} K(x, y) w_{\rho_i}(y) dy - bd < 0.
\]

As a consequence, \( w_{\rho_i} \leq 0 \) for sufficiently large \( i \). This is a contradiction. \( \square \)

3. Hopf bifurcation for spatially homogeneous kernel. To show the existence of complex spatiotemporal patterns, in this section, we consider the Hopf bifurcation for model (1.6) with the homogeneous Neumann boundary conditions. Here we do not assume that \( m = 0 \), and kernel function \( K(x, y) \) is chosen as in Eq. (1.4) for simplicity. Moreover, to compare with the work of [42] and show the effect of nonlocal competition, we also choose \( \Omega = (0, \ell \pi) \) as in [42]. Then Eq. (1.6) has the following form

\[
\begin{aligned}
\frac{\partial u}{\partial t} - d_1 u_{xx} &= u \left( 1 - \frac{u}{K} \right) - \frac{\gamma uv}{1 + u}, \\
\frac{\partial v}{\partial t} - d_2 v_{xx} &= -dv + \frac{cuv}{1 + mu}, \\
u_x(0, t) &= v_x(0, t) = 0, \\
u_x(\ell \pi, t) &= v_x(\ell \pi, t) = 0, \\
u(0, t) &= v(0, t) = 0.
\end{aligned}
\]  
(3.1)

Obviously, model (3.1) has a constant positive steady state \((u_*, v_*)\) if and only if \( c - dm > \frac{d}{a} \), where

\[
u_* = \frac{d}{c - dm}, \quad v_* = \frac{1}{b}(a - u_*)(1 + mu_*).
\]  
(3.2)

An easy calculation implies that \((u_*, v_*)\) is locally asymptotically stable for \( m = 0 \). This, combined with Theorem 2.6, also implies that model (3.1) has a unique positive steady state, which is constant for large conversion rate \( c \).

Now, we consider the case that \( m \neq 0 \). Recall from [42] that the nondimensionalized form of (1.1) is

\[
\begin{aligned}
\frac{\partial u}{\partial t} - d_1 u_{xx} &= u \left( 1 - \frac{u}{K} \right) - \frac{\gamma uv}{1 + u}, \\
\frac{\partial v}{\partial t} - d_2 v_{xx} &= -dv + \frac{\gamma uv}{1 + u}, \\
u_x(0, t) &= v_x(0, t) = 0, \\
u_x(\ell \pi, t) &= v_x(\ell \pi, t) = 0, \\
u(0, t) &= v(0, t) = 0.
\end{aligned}
\]  
(3.3)

Similarly, the nondimensionalized form of (3.1) is

\[
\begin{aligned}
\frac{\partial u}{\partial t} - d_1 u_{xx} &= u \left( 1 - \frac{1}{k \ell \pi} \int_0^{\ell \pi} u(y, t) dy \right) - \frac{\gamma uv}{1 + u}, \\
\frac{\partial v}{\partial t} - d_2 v_{xx} &= -dv + \frac{\gamma uv}{1 + u}, \\
u_x(0, t) &= v_x(0, t) = 0, \\
u_x(\ell \pi, t) &= v_x(\ell \pi, t) = 0, \\
u(0, t) &= v(0, t) = 0.
\end{aligned}
\]  
(3.4)
where $\gamma$ is equivalent to conversion rate $c$. Model (3.4) has a constant positive steady state $(\lambda, v_\lambda)$, where

$$
\lambda = \frac{\theta}{\gamma - \theta} \quad v_\lambda = \frac{(k - \lambda)(1 + \lambda)}{k\gamma},
$$

(3.5)

if and only if $\gamma > \frac{(1 + k)}{k} \left(\text{or equivalently, } 0 < \lambda < k\right)$. Therefore, throughout this section, we always assume $\lambda \in (0, k)$, and use $\lambda$ (equivalent to conversion rate $c$) as a bifurcation parameter to study the Hopf bifurcation. Moreover, we will compare the bifurcation points between models (3.3) and (3.4) and investigate whether nonlocal competition can bring new dynamical behaviors.

We denote $\mathbb{N} = \{1, 2, 3, \cdots\}$ and $\mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}$. Let

$$
X := \{(u, v)^T \in H^2(0, \ell\pi) \times H^2(0, \ell\pi) : (u_x, v_x)|_{x=0}, \varepsilon = 0\},
$$

(3.6)

and define the complexification of $X$ to be $X_C := X \oplus iX = \{x_1 + ix_2 \mid x_1, x_2 \in X\}$. Linearizing system (3.4) at $(\lambda, v_\lambda)$, we have

$$
\begin{align*}
\frac{\partial w}{\partial t} - d_1 w_{xx} &= -\frac{\lambda}{k\ell\pi} \int_0^{\ell\pi} w(y, t)dy + \frac{\lambda(k - \lambda)}{k(1 + \lambda)} w - \theta z, \quad x \in (0, \ell\pi), \ t > 0, \\
\frac{\partial z}{\partial t} - d_2 z_{xx} &= \frac{k - \lambda}{k(1 + \lambda)} w, \quad x \in (0, \ell\pi), \ t > 0, \\
w_x(0, t) &= z_x(0, t) = 0, \ w_x(\ell\pi, t) = z_x(\ell\pi, t) = 0, \ t > 0.
\end{align*}
$$

(3.7)

Then we obtain the sequence of the characteristic equations with respect to $(\lambda, v_\lambda)$ as follows:

$$
\beta^2 - T_n(\lambda)\beta + D_n(\lambda) = 0, \ n \in \mathbb{N}_0,
$$

(3.8)

where

$$
T_0(\lambda) = \frac{\lambda(k - 1 - 2\lambda)}{k(1 + \lambda)}, \quad D_0(\lambda) = \frac{\theta(k - \lambda)}{k(1 + \lambda)},
$$

(3.9)

and for $n \in \mathbb{N}$,

$$
\begin{align*}
T_n(\lambda) &= \frac{\lambda(k - \lambda)}{(k - \lambda) k(1 + \lambda)} - \frac{(d_1 + d_2)n^2}{\ell^2}, \\
D_n(\lambda) &= \frac{\theta(k - \lambda)}{k(1 + \lambda)} - \left[\frac{d_2\lambda(k - \lambda)}{k(1 + \lambda)}\right] \frac{n^2}{\ell^2} + \frac{d_1d_2n^4}{\ell^4}.
\end{align*}
$$

(3.10)

Therefore, $(\lambda, v_\lambda)$ is locally asymptotically stable, if $T_n(\lambda) < 0$ and $D_n(\lambda) > 0$ for each $n \in \mathbb{N}_0$. Moreover, it follows from [24, 42] that Hopf bifurcation value $\lambda_0$ satisfies the following condition:

$$(\text{H}_1): \text{There exists } n \in \mathbb{N}_0 \text{ such that}$$

$$
T_n(\lambda_0) = 0, \ D_n(\lambda_0) > 0, \quad \text{and} \quad T_j(\lambda_0) \neq 0, \ D_j(\lambda_0) \neq 0 \quad \text{for} \ j \neq n,
$$

(3.11)

and the unique pair of complex eigenvalues $\alpha(\lambda) \pm i\omega(\lambda)$ near $\lambda_0$ satisfy

$$
\alpha'(\lambda_0) \neq 0.
$$

(3.12)

Denote

$$
C_1(\lambda) = \frac{\lambda(k - \lambda)}{k(1 + \lambda)}, \quad C_2(\lambda) = \lambda C_1(\lambda) = \frac{\lambda^2(k - \lambda)}{k(1 + \lambda)},
$$

(3.13)

and we can easily derive the following results for later application.

**Lemma 3.1.** Assume that $k > 0$. Then the following two statements are true.
that follows from Lemma 3.1 that when

\( \lambda \) 1. Recall from [42] that when

\( \ell \) follows from Lemma 3.1 that when

Then we consider the occurrence of Hopf bifurcation for \( k \leq 1 \).

**Theorem 3.2.** Suppose that \( d_1, d_2, \gamma > 0 \) and \( k \leq 1 \) satisfy

\[
\frac{d_1}{d_2} = \frac{C_2(\lambda_2)}{4\theta},
\]

and define

\[
\ell_n = n\sqrt{\frac{d_1 + d_2}{C_1(\lambda_1)}},
\]

where \( C_i(\lambda), \lambda_i (i = 1, 2) \) are defined as in Eq. (3.13) and Lemma 3.1 respectively. Then the following two statements are true.

(I) If \( \ell \in (0, \ell_1) \), then \( (\lambda, v_\lambda) \) is locally asymptotically stable for \( \lambda \in (0, k) \).

(II) If \( \ell \in (\ell_1, \infty) \), then there exist two points \( \lambda_{1,+}^H(\ell) \) and \( \lambda_{1,-}^H(\ell) \), satisfying

\[
0 < \lambda_{1,+}^H(\ell) < \lambda_1 < \lambda_{1,-}^H(\ell) < k,
\]

such that \( (\lambda, v_\lambda) \) is locally asymptotically stable for \( \lambda \in (\lambda_{1,+}^H(\ell), k) \cup (0, \lambda_{1,-}^H(\ell)) \) and unstable for \( \lambda \in (\lambda_{1,-}^H(\ell), \lambda_{1,+}^H(\ell)) \). Moreover, system (3.4) undergoes Hopf bifurcation at \( \lambda = \lambda_{1,+}^H(\ell) \) and \( \lambda = \lambda_{1,-}^H(\ell) \), and the bifurcating periodic solutions near \( \lambda_{1,+}^H(\ell) \) or \( \lambda_{1,-}^H(\ell) \) are spatially nonhomogeneous.

**Proof.** Since \( k \leq 1 \), it follows that \( T_0(\lambda) < 0 \) and \( D_0(\lambda) > 0 \) for \( \lambda \in (0, k) \). Noticing that

\[
\frac{d_1}{d_2} = \frac{C_2(\lambda_2)}{4\theta},
\]

we have

\[
\frac{d_1}{d_2} > \frac{\lambda^2(k - \lambda)}{4\theta k(1 + \lambda)}
\]

for any \( \lambda \in (0, k) \), which implies that \( D_n(\lambda) > 0 \) for \( \lambda \in (0, k) \) and each \( n \in \mathbb{N} \). It follows from Lemma 3.1 that when \( \ell \in (0, \ell_1) \), \( T_n(\lambda) < 0 \) for \( \lambda \in (0, k) \) and each \( n \in \mathbb{N} \). Therefore, \( (\lambda, v_\lambda) \) is locally asymptotically stable for \( \lambda \in (0, k) \).

When \( \ell \in (\ell_1, \infty) \), there exist two points \( \lambda = \lambda_{1,+}^H(\ell) \) and \( \lambda = \lambda_{1,-}^H(\ell) \) such that \( T_1(\lambda_{1,+}^H(\ell)) = 0 \) and \( T_1(\lambda) > 0 \) for \( \lambda \in (\lambda_{1,-}^H(\ell), \lambda_{1,+}^H(\ell)) \), and \( T_1(\lambda) < 0 \) for \( \lambda \in (0, \lambda_{1,-}^H(\ell)) \cup (\lambda_{1,+}^H(\ell), k) \). Consequently, when \( \lambda \in (0, \lambda_{1,-}^H(\ell)) \cup (\lambda_{1,+}^H(\ell), k) \), \( T_n(\lambda) < 0 \) for each \( n \in \mathbb{N} \), which implies \( (\lambda, v_\lambda) \) is locally asymptotically stable for \( \lambda \in (\lambda_{1,+}^H(\ell), k) \cup (0, \lambda_{1,-}^H(\ell)) \) and unstable for \( \lambda \in (\lambda_{1,-}^H(\ell), \lambda_{1,+}^H(\ell)) \). When \( \lambda \) is near \( \lambda_{1,+}^H(\ell) \) (or \( \lambda_{1,-}^H(\ell) \)), the unique pair of eigenvalues \( \alpha(\lambda) \pm i\omega(\lambda) \) satisfy \( \alpha(\lambda) = T_1(\lambda)/2 \) and \( \alpha'(\lambda_{1,+}^H(\ell)) = C_1(\lambda_{1,+}^H(\ell))/2 \) (or \( \alpha'(\lambda_{1,-}^H(\ell)) = C_1(\lambda_{1,-}^H(\ell))/2 \)). It follows from Lemma 3.1 that \( \alpha'(\lambda_{1,+}^H(\ell)) < 0 \) and \( \alpha'(\lambda_{1,-}^H(\ell)) > 0 \). Therefore, \( \lambda_{1,+}^H(\ell) \) and \( \lambda_{1,-}^H(\ell) \) are both Hopf bifurcation points, and the bifurcating periodic solutions are spatially nonhomogeneous. This completes the proof. \( \square \)

**Remark 3.3.** 1. Recall from [42] that when \( k \leq 1 \), \( (\lambda, v_\lambda) \) is locally asymptotically stable for model (3.3). Therefore, Hopf bifurcation is more likely to occur with the nonlocal competition of prey.
2. For \( \ell \in (\ell_1, \infty) \), when \( \lambda \) passes through \( \lambda_{1,\pm}^H(\ell) \) (or \( \lambda_{1,\pm}^H(\ell) \)) from left to right, \((\lambda, v_\lambda)\) will change its stability from stability to instability (or instability to stability) through Hopf bifurcation. Moreover, the bifurcating periodic solutions are spatially nonhomogeneous. This is also a new dynamical behavior induced by the nonlocal effect. For model (3.3), \((\lambda, v_\lambda)\) can also change its stability from stability to instability when \( \lambda \) passes through some Hopf bifurcation point, but the bifurcating periodic solutions near such bifurcation point are always spatially homogeneous.

3. As in [42], we can also prove that for \( \ell \in (\ell_n, \ell_{n+1}) \) with \( n \in \mathbb{N} \), there exist 2\( n \) Hopf bifurcation points \( \lambda_{j,\pm}^H(\ell) \) \((1 \leq j \leq n)\) satisfying

\[
0 < \lambda_{1,-}^H(\ell) < \lambda_{2,-}^H(\ell) < \cdots < \lambda_{N,0}^H(\ell) < \lambda_1 < \lambda_{1,+}^H(\ell) < \cdots < \lambda_{2,+}^H(\ell) < \lambda_{1,+,n}^H(\ell) < k. \tag{3.17}
\]

However, \((\lambda, v_\lambda)\) will not change its stability or instability when \( \lambda \) passes through these points \( \lambda_{j,\pm}^H(\ell) \) for \( 2 \leq j \leq n \).

The case that \( k > 1 \) is more complex. In this case \( T_0((k-1)/2) = 0 \), and consequently, \((k-1)/2\) is also a possible Hopf bifurcation point. As in Theorem 3.2, we only concern with some special Hopf bifurcation points. When \( \lambda \) passes through such points, \((\lambda, v_\lambda)\) will change its stability or instability.

**Theorem 3.4.** Suppose that \( d_1, d_2, \gamma, \theta > 0 \) and \( 1 < k < 3 \) satisfy Eq. (3.14), and define

\[
\tilde{\ell}_n = n\sqrt{\frac{2(d_1 + d_2)k}{k-1}}, \tag{3.18}
\]

where \( C_i(\lambda), \lambda_i \) \((i = 1, 2)\) are defined as in Eq. (3.13) and Lemma 3.1 respectively. Then the following three statements are true.

(I) If \( \ell \in (0, \ell_1) \), where \( \ell_1 \) is defined as in Eq. (3.15), then \((\lambda, v_\lambda)\) is locally asymptotically stable for \( \lambda \in ((k-1)/2, k) \) and unstable for \( \lambda \in (0, (k-1)/2) \). Moreover, system (3.4) undergoes Hopf bifurcation at \((k-1)/2\), and the bifurcating periodic solutions near \((k-1)/2\) are spatially homogeneous.

(II) If \( \ell \in (\ell_1, \tilde{\ell}_1) \), then there exist \( \lambda_{1,+}^H(\ell) \) and \( \lambda_{1,-}^H(\ell) \) \((\lambda_{1,+,\ell}^H(\ell) > \lambda_{1,-}^H(\ell) > (k-1)/2)\) such that \((\lambda, v_\lambda)\) is locally asymptotically stable for \( \lambda \in (\lambda_{1,+,\ell}(\ell), k) \cup ((k-1)/2, \lambda_{1,-}^H(\ell)) \) and unstable for \( \lambda \in (0, (k-1)/2) \cup (\lambda_{1,-}^H(\ell), \lambda_{1,+,\ell}(\ell)) \). Moreover, system (3.4) undergoes Hopf bifurcation at \( \lambda_{1,+,\ell}(\ell) \) and \((k-1)/2\), and the bifurcating periodic solutions near \( \lambda_{1,+,\ell}(\ell) \) (respectively, \((k-1)/2\)) are spatially nonhomogeneous (respectively, homogeneous).

(III) If \( \ell \in (\tilde{\ell}_1, \infty) \), then there exists \( \lambda_{1,+}^H(\ell) > (k-1)/2 \) such that \((\lambda, v_\lambda)\) is locally asymptotically stable for \( \lambda \in (\lambda_{1,+}^H(\ell), k) \) and unstable for \( \lambda \in (0, \lambda_{1,+}^H(\ell)) \). Moreover, system (3.4) undergoes Hopf bifurcation at \( \lambda = \lambda_{1,+}^H(\ell) \), and the bifurcating periodic solutions are spatially nonhomogeneous.

**Proof.** Firstly, we prove part (I). Since \( 1 < k < 3 \), we see that \((k-1)/2 < \sqrt{k+1-1, D_0(\lambda)} > 0 \) for \( \lambda \in (0, k) \), \( T_0(\lambda) < 0 \) for \( \lambda \in ((k-1)/2, k) \), and \( T_0(\lambda) > 0 \) for \( \lambda \in (0, (k-1)/2) \). For \( \ell \in (0, \ell_1) \), an easy calculation yields \( D_n(\lambda) > 0 \) and \( T_n(\lambda) < 0 \) for each \( n \in \mathbb{N} \) and \( \lambda \in (0, k) \), which implies that \((\lambda, v_\lambda)\) is locally asymptotically stable for \( \lambda \in ((k-1)/2, k) \) and unstable for \( \lambda \in (0, (k-1)/2) \). When \( \lambda \) near \((k-1)/2\), the unique pair of eigenvalues \( \alpha(\lambda) \pm \iota \omega(\lambda) \) satisfy \( \alpha(\lambda) = T_0(\lambda)/2 \) and
\( \alpha' ((k - 1)/2) = T_0'' ((k - 1)/2)/2 < 0. \) Therefore, \((k - 1)/2\) is the Hopf bifurcation point, and the bifurcating periodic solutions are spatially homogeneous.

Then, we prove part (II). Note that \( C_1 ((k - 1)/2) = (k - 1)/2k. \) Then, we see that, when \( \ell \in (\ell_1, \ell_1) \), there exist \( \lambda^H_+(\ell), \lambda^H_-(\ell) > (k - 1)/2 \) such that \( T_1(\lambda^H_+(\ell)) = T_1(\lambda^H_-(\ell)) = 0 \), then \( T_1(\lambda) > 0 \) for \( \lambda \in (\lambda^H_-(\ell), \lambda^H_+(\ell)) \), and \( T_1(\lambda) < 0 \) for \( \lambda \in (0, \lambda^H_-(\ell)) \cup (\lambda^H_+(\ell), k) \). Consequently, \( T_n(\lambda) < 0 \) for each \( n \geq 2 \) and \( \lambda \in (0, \lambda^H_-(\ell)) \cup (\lambda^H_+(\ell), k) \). Note that \( T_0(\lambda) < 0 \) for \( \lambda \in ((k - 1)/2, k) \), and \( T_0(\lambda) > 0 \) for \( \lambda \in (0, (k - 1)/2) \). Then, \((\lambda, v_\lambda)\) is locally asymptotically stable for \( \lambda \in (\lambda^H_-(\ell), k) \cup ((k - 1)/2, \lambda^H_-(\ell)) \) and unstable \( \lambda \in (0, (k - 1)/2) \cup (\lambda^H_-(\ell), \lambda^H_+(\ell)) \). When \( \lambda \) near \( \lambda^H_+(\ell) \) (or \( \lambda^H_-(\ell) \)), the unique pair of eigenvalues \( \alpha(\lambda) \pm i\omega(\lambda) \) satisfy \( \alpha(\lambda) = T_1(\lambda)/2 \) and \( \alpha'(\lambda^H_+(\ell)) = C_1'(\lambda^H_+(\ell))/2 \) (or \( \alpha'(\lambda^H_-(\ell)) = C_1'(\lambda^H_-(\ell))/2 \)). If follows from Lemma 3.1 that \( \alpha'(\lambda^H_+(\ell)) < 0 \) and \( \alpha'(\lambda^H_-(\ell)) > 0 \). Therefore, \( \lambda^H_+(\ell) \) and \( \lambda^H_-(\ell) \) are both Hopf bifurcation points, and the bifurcating periodic solutions are spatially nonhomogeneous. Moreover, similarly as in part (I), we see that \((k - 1)/2\) is also the Hopf bifurcation point, and the bifurcating periodic solutions are spatially homogeneous.

Finally, when \( \ell \in (\ell_1, \infty) \), there exists \( \lambda^H_-(\ell) > (k - 1)/2 \) such that \( T_1(\lambda^H_+(\ell)) = 0, T_1(\lambda) > 0 \) for \( \lambda \in ([k - 1]/2, \lambda^H_+(\ell)) \), and \( T_1(\lambda) < 0 \) for \( \lambda \in (\lambda^H_+(\ell), k) \). Consequently, \( T_n(\lambda) < 0 \) for each \( n \in \mathbb{N} \) and \( \lambda \in (\lambda^H_+(\ell), k) \), which implies that \((\lambda, v_\lambda)\) is locally asymptotically stable for \( \lambda \in (\lambda^H_+(\ell), k) \) and unstable \( \lambda \in (0, \lambda^H_+(\ell)) \). Similarly as in part (II), we see that the unique pair of eigenvalues \( \alpha(\lambda) \pm i\omega(\lambda) \) near \( \lambda^H_+(\ell) \) satisfy \( \alpha'(\lambda^H_+(\ell)) < 0 \). Therefore, \( \lambda^H_+(\ell) \) is the Hopf bifurcation point and the bifurcating periodic solutions are spatially nonhomogeneous. This completes the proof.

Then, we have the following results for \( k \geq 3 \), and omit the proof here.

**Theorem 3.5.** Suppose that \( d_1, d_2, \gamma, \theta > 0 \) and \( k \geq 3 \) satisfy Eq. (3.14), and \( \hat{\ell}_n \) is defined as in (3.18). Then the following two statements are true.

(I) If \( \ell \in (0, \hat{\ell}_1) \), then \((\lambda, v_\lambda)\) is locally asymptotically stable for \( \lambda \in ((k - 1)/2, k) \) and unstable for \( \lambda \in (0, (k - 1)/2) \). Moreover, system (3.4) undergoes Hopf bifurcation at \((k - 1)/2\), and the bifurcating periodic solutions are spatially homogeneous.

(II) If \( \ell \in (\hat{\ell}_1, \infty) \), then there exists \( \lambda^H_+(\ell) > (k - 1)/2 \) such that \((\lambda, v_\lambda)\) is locally asymptotically stable for \( \lambda \in (\lambda^H_+(\ell), k) \) and unstable for \( \lambda \in (0, \lambda^H_+(\ell)) \). Moreover, system (3.4) undergoes Hopf bifurcation at \( \lambda = \lambda^H_+(\ell) \), and the bifurcating periodic solutions are spatially nonhomogeneous.

**Remark 3.6.** Recall from [42] that when \( k > 1 \) and \( \lambda \in ((k - 1)/2, k) \), \((\lambda, v_\lambda)\) is locally asymptotically stable for model (3.3). It follows from Theorems 3.4 and 3.5 that when \( k > 1 \), there may also exist Hopf bifurcation points in interval \((k - 1)/2, k) \) for model (3.4). Therefore, Hopf bifurcation is also more likely to occur with the nonlocal competition of prey in this case.

Finally, we give some numerical examples to support our theoretical results, and show the existence of spatially nonhomogeneous periodic solutions.

**Example 3.7.** To visualize the results in Theorem 3.2, we choose

\[
d_1 = 0.1, \ d_2 = 0.2, \ k = 0.5, \ \theta = 1, \ \ell = 2. \tag{3.19}
\]
These parameters satisfy Eq. (3.14) and \( \ell > \ell_1 \). Then system (3.4) has a unique constant positive steady state if and only if \( \gamma > 3 \). It follows from Theorem 3.2 that there exist two Hopf bifurcation points \( \lambda_{H,-} \approx 0.1049 \) (or equivalently, \( \gamma \approx 10.3521 \)) and \( \lambda_{H,+} \approx 0.3576 \) (or equivalently, \( \gamma \approx 3.7972 \)) such that the unique constant steady state is locally asymptotically stable for \( \gamma \in (3.3972, 10.3521) \) and unstable for \( \gamma \in (3.3972, 10.3521) \). Moreover, the bifurcating periodic solutions are spatially nonhomogeneous near these two Hopf bifurcation points \( \lambda_{1,-} \) and \( \lambda_{1,+} \), see Fig. 1 for numerical simulation.

**Example 3.8.** To visualize the results in Theorem 3.4, we choose

\[
d_1 = 0.2, \ d_2 = 0.3, \ k = 1.5, \ \theta = 1.
\] (3.20)

An easy calculation yields \( \ell_1 \approx 1.4904 \) and \( \tilde{\ell}_1 \approx 2.1213 \), and (3.4) has a unique constant positive steady state if and only if \( \gamma > 5/3 \). We choose \( \ell = 1.5 \), and then \( \ell \in (\ell_1, \tilde{\ell}_1) \). It follows from Theorem 3.4 that there exist three Hopf bifurcation points \( \lambda_{H,-}^{H} = 2/3 \) (or equivalently, \( \gamma = 2.5 \)), \( \lambda_{H,-} = 1/2 \) (or equivalently, \( \gamma = 3 \)), and \( (k-1)/2 = 1/4 \) (or equivalently, \( \gamma = 5 \)) such that the unique constant steady state is locally asymptotically stable for \( \gamma \in (5/3, 2.5) \cup (3, 5) \) and unstable for \( \gamma \in (2.5, 3) \cup (5, \infty) \). Moreover, the bifurcating periodic solutions are spatially nonhomogeneous near \( \gamma = 2.5 \) or \( \gamma = 3 \), and spatially homogeneous near \( \gamma = 5 \), see Fig. 2 for numerical simulation.
Figure 2. The constant steady state loses its stability through Hopf bifurcation. (Upper): $\gamma = 2.7$, and the solution converges to the bifurcated spatially nonhomogeneous periodic solution. Here initial values: $u(x, 0) = 0.3 + 0.1 \cos^2 \frac{x}{3}$, $v(x, 0) = 0.2 + 0.1 \cos^2 \frac{x}{3}$, $x \in [0, 1.5\pi]$. (Lower): $\gamma = 6$, and the solution converges to the bifurcated spatially homogeneous periodic solution. Here initial values: $u(x, 0) = 0.7 + 0.5 \cos^2 \frac{x}{3}$, $v(x, 0) = 0.7 + 0.5 \cos^2 \frac{x}{3}$, $x \in [0, 1.5\pi]$.

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