General Massive Spin-2 on de Sitter Background

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Abstract: We study linearised massive gravity on the de Sitter background. With a small-enough graviton mass this may have had relevance to inflation, or the present-day cosmic acceleration. Higuchi has shown that this theory has a ghost as long as the dS curvature exceeds the graviton mass, $2H^2 > m^2$; this would entail rapid instabilities. In this work we extend the model and observe that the helicity-0 mode which is a ghost, can be turned into a positive energy state via kinetic mixing with the conformal mode. The latter gets restricted here by the requirement that the Bianchi identities be satisfied. This eliminates the ghost from the linearized theory. However, the spectrum still contains scalar tachyonic instabilities with the time scale $\sim 1/H$. This would have been problematic for the early universe, however, may be acceptable for the present-day accelerated expansion as the tachyon instability would take the age of the universe to develop.

Keywords: dark energy theory, gravity
1. Introduction, discussions and summary

There are at least two compelling reasons to study theories that modify gravity at large distances: the old cosmological constant problem [1] may be solved in this approach, and the observed cosmic acceleration [2] may be explained, see respectively [3, 4, 5, 6, 7] and [8, 9]. Presently, there is no satisfactory theory of modified gravity that could complete the above tasks (see [5, 10, 11] and [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22], for discussions of various issues and controversies); the search for a consistent theory continues [23, 20, 24, 25, 26].

One example of the large distance modified theory is four-dimensional massive gravity with cosmologically large graviton Compton wavelength. In the linearized approximation around Minkowski space this model is consistent theoretically, but fails to describe the observable world because of the vDVZ discontinuity [27]. Although strong non-linear interactions could have cured the discontinuity problem [28, 29], the non-linear theory of massive gravitons by itself is inconsistent [30, 31, 32, 33].

Certain Lorentz-violating massive theories [34, 35, 36] may have better non-linear behavior [37], as well as interesting phenomenology, however, they exhibit an unusual property of long-range instantaneous interactions [37].

Furthermore, the Lorentz-violating ghost condensation model that exhibits rich physics [38], as well as phenomenologically motivated $f(R)$-type models (see, e.g., [39] and references therein) have been actively discussed in the literature.

The main topic of the present work is massive gravity on the de Sitter background [40]. The interest in this is two-fold:
(A) One could be curious to know what happens to inflation if the graviton has a mass \( m \) such that \( H \gg m \), where \( H \) is the Hubble parameter. Very naively, one may expect that inflation proceeds as in the conventional setup as long as the physical size of the inflationary region is smaller than the graviton Compton wavelength, while for larger size the evolution would be altered, perhaps along the lines of [3].

(B) Some of the known examples of the self-accelerated universe [8, 9] contain in the linearized approximation a massive graviton on a dS background with \( H \sim m \). This may well be a generic property of a class of self-accelerated solutions, in which case understanding of massive gravity on the dS background may have observationally interesting consequences for the late time cosmological evolution.

However, before starting to address physically meaningful questions on inflation or late time accelerated expansion, one needs to deal with the theoretical consistency problems of massive gravity on dS. It has been shown by Higuchi [40] that for \( 2H^2 > m^2 \) the helicity-0 mode of the massive graviton on the dS background becomes a ghost. The ghost would entail very rapid instabilities of the background. This eliminates any hope to study inflation with massive gravity, as well as models of the late time cosmic acceleration that lie in the parameter range \( 2H^2 > m^2 \).

Our aim here is to address precisely these theoretical issues. Namely, we will try to modify the linearized theory of massive gravity on dS background such that the ghost in the case \( 2H^2 > m^2 \) is avoided. We will achieve our goal of eliminating the ghost, however, for \( 6H^2 > m^2 \) we are still left with two spin-0 tachyons of negative mass squared \( m^2 - 6H^2 \), one of which is decoupled from matter sources and the other has the coupling strength measured by the ratio \( m^2/(m^2 + 2H^2) \). Is this an improvement over the ghost?

The answer would depend on a concrete physical circumstance at hand. For instance, for the issue (A) above, if we deal with an inflationary scenario and allow for a small graviton mass of the order of the present days Hubble scale \( m \sim 10^{-33}\text{eV} \), then \( H \gg m \) and the tachyon instability could be very severe; unless further changes are made, it would destabilize the inflationary background in the time scale \( \sim 1/H \), spoiling the inflationary scenario. Hence, if graviton is massive, it better acquired its mass only during the late-time evolution of the universe.

On the other hand, if we have in mind applications to the present-day accelerated universe as in (B) above, then for \( 2H^2 > m^2 \) but \( H \sim m \) the tachyon instability time is of the order of the age of the Universe. In this context, replacing the ghost by the tachyon is a huge improvement as the ghost would have led to a catastrophic instability.

It is worth mentioning that in our model we achieve a continuous transition to the massless limit \( m \to 0 \), as the vDVZ discontinuity will be absent (in a way similar to what happens in the AdS case [11]). Finally, for \( m^2 > 6H^2 \) our model is also ghost and tachyon free.

In the remainder of this section we will try to summarize, in a less cumbersome (but still somewhat technical) way, the approach and results of the present work.

To start with, ghosts present a formidable problem in field theories. In the classical limit
they could lead to unbounded negative energy solutions. In the full theory ghosts can be quantized as positive-norm negative energy states or, alternatively, as negative-norm positive energy states. In the former case they lead to a rapid vacuum instability via a particle-ghost creation process, while in the latter case the negative norm states violate unitarity. Typically, in a theory with a given field content, if ghosts are present, there are no tools to avoid the above problems without violating analyticity and causality, or locality [42].

Consider for example a scalar ghost. We note that a kinetic mixing of two ghosts may eliminate one of them. To see this we look at a Lagrangian which on top of the conventional fields contains two additional fields $\sigma$ and $\tau$, with kinetic terms

$$ (\partial_\mu \sigma)^2 + 2\alpha (\partial_\mu \sigma)(\partial_\mu \tau) + (\partial_\mu \tau)^2, \quad (1.1) $$

where the parameter $\alpha$ sets the mixing strength. In the $\alpha \to 0$ limit both $\sigma$ and $\tau$ have wrong-sign (ghost-like) kinetic terms (in our conventions of the signature $\eta_{\mu\nu} = \text{diag}(-1,1,1,1)$). However, due to a large enough mixing, one of the ghosts can be turned into a particle by the field redefinition: $\sigma \to \sigma - \alpha \tau$. The resulting Lagrangian takes the form:

$$ (\partial_\mu \sigma)^2 - [\alpha^2 - 1](\partial_\mu \tau)^2. \quad (1.2) $$

For $|\alpha| > 1$ the kinetic term of $\tau$ flips the sign to the “right one”\(^1\).

Note that, if the kinetic term of $\sigma$ had an opposite sign in (1.1) (i.e., if it were not ghost-like) one would not be able to flip the ghost-like sign of the $\tau$ kinetic term via the diagonalization: It takes a ghost to kill a ghost!

Is the above exercise meaningful? After all we are still left with one ghost $\sigma$ that is bad-enough to give rise to all the known ghost-related problems.

The answer to the above question would be positive if the $\sigma$ field is constrained further, so that in the end, this field is left non-dynamical. We will argue below, and show in the text, that such a mechanism can be at work in a theory of massive gravity on the dS background with an additional scalar (the latter can be set to decouple from the matter stress-tensor).

In this work we will be discussing the Pauli-Fierz (PF) mass term, which has a virtue of being ghost-free in the flat space limit:

$$ \mathcal{L}_{\text{PF}} = -\frac{m^2}{4} (h_{\mu\nu}^2 - h^2). \quad (1.3) $$

Here the indices are contracted via the background dS metric $\gamma_{\mu\nu}$ and its inverse. Let us look at the decomposition of the metric perturbation on dS space, in terms of the transverse-traceless tensor $h^{TT}_{\mu\nu}$, transverse vector $V^T_{\mu}$, conformal scalar $\sigma$ and longitudinal scalar $\tau$:

$$ h_{\mu\nu} = h^{TT}_{\mu\nu} + \nabla_{\mu}V^T_{\nu} + \nabla_{\nu}V^T_{\mu} + \gamma_{\mu\nu}\sigma + \nabla_{\mu}\nabla_{\nu}\tau. \quad (1.4) $$

\(^1\)In the space of the mixing parameter, $\alpha = 1$ is a singular point where interactions with other fields would in general become infinitely strong. The two theories with different signs of the kinetic term of $\tau$ are separated by this singular point.
Unlike in Einstein’s gravity where the $\tau$ field is gauge removable, in the massive theory it acquires physical meaning of the helicity-0 state of the massive spin-2.

The Bianchi identities combined together with the equations of motion following from the massive theory with the term (1.3) necessarily imply that

$$\nabla^\mu \nabla^\nu h_{\mu\nu} = \Box h.$$ \hspace{1cm} (1.5)

Using (1.4) and (1.5) we will get

$$\Box \sigma = H^2 \Box \tau.$$ \hspace{1cm} (1.6)

Although there will be kinetic and mixing terms for $\sigma$ in the Lagrangian, the $\sigma$ field in the end should be supplemented by the above constraint which expresses it via $\tau$ and excludes it from the counting of the physical degrees of freedom.

Let us step back for a moment and return to the Lagrangian in which the constraint (1.6) has not been enforced yet. The relevant properties for us are encoded in the conformal mode $\sigma$ and helicity-0 state described by $\tau$, so we focus on these two fields. The $\tau$ field does not enter the EH Lagrangian, but enters the PF terms (1.3) in two different ways. First it acquires a kinetic mixing term with $\sigma$, and second it gets its own kinetic term due to the fact that the background is non-trivial (the covariant derivative does not commute with the dS space d’Alambertian, while it does so in Minkowski space). The kinetic term for $\tau$ arising from the PF terms (1.3) is (dropping the overall factor of $3/4$ here and below in this section):

$$m^2 H^2 (\nabla_\mu \tau)^2.$$ \hspace{1cm} (1.7)

This is a wrong-sign (ghost-like) kinetic term. Hence, if $m^2$ had no mixing with other fields, it would be a ghost. In the present case, $\tau$ does mix with $\sigma$, and the latter by itself has a ghost-like kinetic term that arises from the EH Lagrangian. We can diagonalize the $\sigma - \tau$ kinetic terms by the shift $\sigma = \overline{\sigma} + (m^2/2) \tau$ (leaving the mass terms mixed since this is not important here). The result reads:

$$2(\nabla_\mu \overline{\sigma})^2 - \frac{m^2}{2} (m^2 - 2H^2)(\nabla_\mu \tau)^2.$$ \hspace{1cm} (1.8)

As we can see, if $m^2 > 2H^2$, the helicity-0 mode $\tau$ acquires a positive-sign kinetic term. When $m^2 = 2H^2$ its kinetic term disappears (see discussions and references in [13, 14]). But when $2H^2 > m^2$, the helicity-0 mode becomes a ghost. The kinetic term of $\overline{\sigma}$ remains ghost-like all the time. However, as before $\overline{\sigma}$ is not an independent dynamical field as it gets related to $\tau$ via the constraint (1.4). Hence, for $2H^2 > m^2$, we are left with two fields with the ghost-like kinetic terms in the Lagrangian, and a constraint that relates them, which cannot help to circumvent any of the problems caused by the ghost. This is a convenient way of capturing some of the key results of Higuchi [10].

\footnote{Note that the same conclusions will be reached if $\overline{\sigma}$ is expressed from (1.6) in terms of $\tau$ and substituted into (1.8).}
The main idea of this paper is to turn the $\tau$ field into a field with a right-sign kinetic term via additional mixing between $\sigma$ and $\tau$, in analogy with the scalar model (1.1). For this we introduce, as in [44], a scalar field $\phi$ which facilitates a new kinetic mixing between $\sigma$ and $\tau$ in (1.8). This mixing is designed such that after the diagonalization of the kinetic terms the $\tau$ field acquires an additional right-sign contribution and turns into a field with a right-sign kinetic term even for $2H^2 > m^2$. The $\sigma$ retains its ghost-like nature, which again is harmless, since this field is constrained.

One could integrate out the additional scalar $\phi$, in which case, one would be left with a Lagrangian in which both the EH and PF mass terms are modified. For convenience we will retain the scalar $\phi$, since the Lagrangian is manifestly local in this case.

In the rest of the paper we will work with general expressions without separating conformal and helicity-0 modes, although our results may be more conveniently understood as described above.

What is left out of the present work is the discussion of a nonlinear completion of the models that we are discussing here. One such possibility could be the DGP model [3] endowed with an additional scalar dynamics, that at the linearized level would reduce to the theory presented in this article. Related to the previous comment, we will also not discuss any implications of potential quantum-loop corrections – there are challenging issues to be understood already at the classical level in the linearized theory.

2. Action and Equations of Motion

We start by considering a quadratic action for a graviton of mass $m$ and a scalar field with kinetic mixing on de Sitter space, coupled to a conserved matter stress-tensor:

$$L_{\text{eff}} = L^{(2)}_{\text{EH}}(h_{\mu\nu}) - \frac{1}{4} m^2 (h_{\mu\nu}^2 - h^2) - \phi O_{\mu\nu} h_{\mu\nu} + \phi K + h_{\mu\nu} T^{\mu\nu} + q \phi T ,$$

where $L^{(2)}_{\text{EH}}$ is the second order expansion of the Einstein-Hilbert action around de Sitter space with the cosmological constant $\Lambda = 3H^2$, and the operator

$$O_{\mu\nu} = \nabla_\mu \nabla_\nu - \gamma_{\mu\nu} \Box - 3H^2 \gamma_{\mu\nu} .$$

We have introduced a coefficient $q$ which is assumed to be a constant whose value is chosen later. This form of the operator $O_{\mu\nu}$ is motivated by its transversality on the de Sitter background. For future convenience, we define

$$Q \equiv - \Box - 4H^2 .$$

The operator $K$ that appears in (2.1) also remains undetermined at this stage. We shall find that a particular choice of this operator gives rise to special simplifications of the theory. At this point we just assume that $K$ is a scalar operator and contains at most second derivatives. Hence, it takes the following form

$$K = A \Box + B .$$
Since $A$ and $B$ are assumed to be constants, then $K$ commutes with $Q$.

There are both kinetic and mass mixings between graviton and scalar in (2.1), and for non-vanishing value of the parameter $q$ both fields are sourced by matter. The equations of motion resulting from this action are

$$G_{\mu\nu}^{\text{cl}} - \frac{m^2}{2}(h_{\mu\nu} - \gamma_{\mu\nu} h) - O_{\mu\nu}\phi = -T_{\mu\nu},$$

(2.5)

$$\mathcal{O}^\mu_{\nu\rho\sigma}h_{\mu\rho} - 2K\phi = qT.$$  

(2.6)

As in the case of the pure PF gravity, the Bianchi identities give rise to the following relations

$$\nabla^\mu h_{\mu\nu} = \nabla^\nu h,$$  

(2.7)

which can be used to reduce the equations of motion to the system:

$$\frac{1}{2}\Box h_{\mu\nu} - (2H^2 + m^2)h_{\mu\nu} - \gamma_{\mu\nu}(H^2 - m^2)h - \nabla_\mu \nabla_\nu h = -T_{\mu\nu} + O_{\mu\nu}\phi,$$  

(2.8)

with the trace equation being

$$(3H^2 - \frac{3}{2}m^2)h + 3Q\phi = T.$$  

(2.9)

The field $h_{\mu\nu}$ is not traceless, therefore, to derive its propagator, one must use the Lichnerowicz operator $\Delta_L$ defined in the Appendix by equation (A.6), with which one can write the equations of motion (2.5) as

$$\frac{1}{2}(\Delta_L - 6H^2 + m^2)h_{\mu\nu} = T_{\mu\nu} - O_{\mu\nu}\phi - \frac{1}{2}[(3H^2 - m^2)\gamma_{\mu\nu} + \nabla_\mu \nabla_\nu] h,$$  

(2.10)

or, by defining

$$M_{\mu\nu} \equiv (3H^2 - m^2)\gamma_{\mu\nu} + \nabla_\mu \nabla_\nu,$$  

(2.11)

as

$$\frac{1}{2}(\Delta_L - 6H^2 + m^2)h_{\mu\nu} = T_{\mu\nu} - O_{\mu\nu}\phi - \frac{1}{2}M_{\mu\nu} h.$$  

(2.12)

Finally, the equation of motion for $\phi$, (2.6), implies that

$$-3H^2 h - 2K\phi = qT,$$  

(2.13)

and that, together with equation (2.9), yields the following solutions for $\phi$ and $h$:

$$\phi = \frac{(1 + q)H^2 - \frac{1}{2}gm^2}{3H^2 Q + (m^2 - 2H^2)K} T,$$

(2.14)

$$h = \frac{qQ + \frac{2}{3}K}{\frac{2}{3}gm^2 - (1 + q)H^2} \phi.$$  

(2.15)

It can be easily checked that if we set $m^2 = 2H^2$, we recover the results for both $\phi$ and $h$ studied in [14].
3. Solutions

The goal here is to obtain a theory free of ghosts. Our approach will be similar to the one adopted in [44], in which the parameters of the original action are chosen so that there are no ghost-like poles in the propagators of the physical fields.

We define the physical metric perturbation

\[ h_{\mu\nu}^{\text{phy}} = h_{\mu\nu} - q\gamma_{\mu\nu}\phi, \]  

that captures the entire response of the system to \( T_{\mu\nu} \) (after performing this shift, the scalar \( \phi \) is no longer sourced by \( T \)). The value of the field is given by:

\[
\frac{1}{2}h_{\mu\nu}^{\text{phy}} = \frac{1}{\Delta L - 6H^2 + m^2} \left\{ T_{\mu\nu} - \mathcal{O}_{\mu\nu}\phi - \frac{1}{2}M_{\mu\nu}h \right\} + \frac{1}{2}\gamma_{\mu\nu}q\phi 
\]

\[
= \frac{1}{\Delta L - 6H^2 + m^2} \left\{ T_{\mu\nu} - \mathcal{O}_{\mu\nu}\phi - \frac{1}{2}M_{\mu\nu}h + \frac{q(Q - 2H^2 + m^2)}{2}\gamma_{\mu\nu}\phi \right\},
\]

where \( q \) and \( \mathcal{K} \) are still to be determined. Notice that generically \( \phi \) and \( h \) contain single poles.

Following the strategy of [44] to cancel all the unwanted poles, we first focus on the terms inside the curly brackets on the r.h.s. of equation (3.2). There, the single poles are carried by the two terms including \( \phi \) and \( h \) respectively. Their pole parts must cancel each other out. The finite remnant of this cancellation, once taken out of the curly brackets, turns into a single pole term that eventually can be canceled by \( \gamma_{\mu\nu}q\phi \). The terms proportional to \( \nabla_\mu \nabla_\nu T \) are harmless, at least at the tree-level, since they lead to vanishing contributions when contracted with a conserved \( T_{\mu\nu} \).

Notice that there is nothing we can do about the operator \( M_{\mu\nu} \) in front of \( h \). Therefore for any cancellation to become possible, the term \( \mathcal{O}_{\mu\nu}\phi \) must somehow contain a term of the form of \( M_{\mu\nu}h \) (or \( M_{\mu\nu}\phi \) since \( \phi \) and \( h \) are related by equation (2.14)). The solution is then to demand that

\[ \mathcal{O}_{\mu\nu}\phi = \frac{1}{3}\gamma_{\mu\nu}T + M_{\mu\nu}\phi. \]

The coefficient in front of \( M_{\mu\nu} \) is fixed to be one. This leads to

\[ (Q - 2H^2 + m^2)\frac{(1 + q)H^2 - \frac{1}{2}qm^2}{3H^2Q - (2H^2 - m^2)\mathcal{K}} = \frac{1}{3}, \]

and we find

\[ (2H^2 - m^2)\mathcal{K} = 3q\left(\frac{1}{2}m^2 - H^2\right)Q + 6(H^2 - \frac{1}{2}m^2)\left(1 + q\right)H^2 - \frac{1}{2}qm^2. \]

Notice that for \( m^2 = 2H^2 \), the above equation is automatically satisfied and \( \mathcal{K} \) remains completely arbitrary. That is why for this special case one finds additional freedom as discussed in [44].

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\(^3\)One may attempt to change the coefficient of the \( \nabla_\mu \nabla_\nu \) term in the definition of \( \mathcal{O} \), but this only amounts to a rescaling of \( \phi \), \( q \) and \( \mathcal{K} \), which are at this stage not fixed.
Figure 1: a. single graviton (wiggly line) exchange with no scalar mixing. b. Case with scalar (dashed line) mixing. The general case would contain a chain alternating scalar and graviton.

Things are very different when $2H^2 > m^2$. In this case, the form of $K$ is fixed to be

$$K = -\frac{3q}{2}Q + 3\left((1 + q)H^2 - \frac{1}{2}qm^2\right).$$

Such a choice of $K$ leads to an immediate consequence that $h$ is directly proportional to $\phi$. Indeed, we have

$$h = -2\phi.$$  

Therefore the single poles inside the curly brackets, besides the $q(Q - 2H^2 + m^2)\phi$ term, cancel exactly. Furthermore, we find

$$\phi = -\frac{1}{3(\Box + 6H^2 - m^2)}T,$$

which in its turn implies that

$$\frac{1}{2}h^{\text{phy}}_{\mu\nu} = \frac{1}{\Delta L - 6H^2 + m^2}\left(T_{\mu\nu}^{(1/2)} + \frac{1 + q}{6}\gamma_{\mu\nu}T\right).$$

Recall now that $h_{\mu\nu} = h^{\text{phy}}_{\mu\nu} - q\gamma_{\mu\nu}\phi$, and if we rewrite the original action in terms of the “physical” metric perturbation, we get

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{EH+PF}}^{(2)}(h^{\text{phy}}_{\mu\nu}) - (1 + q)\phi\mathcal{O}^{\mu\nu}h^{\text{phy}}_{\mu\nu} - \frac{3}{2}q(q + 1)\phi\Box\phi - \frac{3}{2}qm^2h^{\text{phy}}_{\mu\nu}\phi +$$

$$+ \frac{3}{2}(2q - 1)(qm^2 - 2(q + 1)H^2)\phi^2 + h^{\text{phy}}_{\mu\nu}T^{\mu\nu}.$$  

The theory described by (3.11) contains six propagating degrees of freedom: five polarizations of a massive spin two, plus the extra scalar that we introduced in (2.1). After the shift that defines $h^{\text{phy}}_{\mu\nu}$, $T_{\mu\nu}$ only sources five degrees of freedom – the polarizations of $h^{\text{phy}}_{\mu\nu}$. Therefore, if we neglect for a moment the mixing between $h^{\text{phy}}_{\mu\nu}$ and $\phi$ we get that the former propagates five degrees of freedom, two of which do not couple to a conserved source, giving rise to the usual propagator of massive gravity on dS. A single graviton exchange between two sources leads in this case to a diagram like the one of figure 1a.

Including the mixing terms perturbatively, would give diagrams of the type depicted in figure 1b, in which a combination of the helicity-0 component of $h^{\text{phy}}_{\mu\nu}$ oscillates into the scalar $\phi$ and back again. But at every instant it is only one scalar degree of freedom that is propagating between the sources, on top of the ordinary helicity-2 components.
The resummation of all such tree-level diagrams gives the propagator (3.10). In the latter the contributions of the two helicity-2 states are put together in the term proportional to $T_{\mu\nu}^{(1/2)}$, while the contribution of the single scalar mode (which is a superposition of the helicity-0 mode of $h_{\mu\nu}^{\text{phy}}$ and $\phi$) is in the term proportional to $T$. Notice that the latter has positive residue as long as $q > -1$.

In order to determine whether the remaining sixth degree of freedom – the other combination of the helicity-0 mode and $\phi$, which is not sourced by $T_{\mu\nu}$ – is a ghost or not we use the following trick: we temporarily set to zero $T_{\mu\nu}$ and include a putative source $J$ via the term $+J\phi$ in the action. $J$ excites different combinations of the helicity-0 mode and $\phi$; then, if there is a ghost-like excitation in the sixth mode that’s not sourced by $T$, it would be sourced by $J$. Performing this analysis, we find that for

$$q > \frac{2H^2}{m^2 - 2H^2},$$

(3.12)

no ghosts are excited by $J$ either. Hence, this linearized theory is free of ghosts. Note also that for $2H^2 > m^2$ the condition (3.12) is less restrictive than the one for the other scalar combination ($q > -1$).

Another point to be mentioned is that both the field $\phi$ (for $q$ satisfying (3.12)) and the scalar component of $h_{\mu\nu}^{\text{phy}}$ have positive mass squared for $m^2 > 6H^2$, while they become tachyonic when $m^2 < 6H^2$ (see, also eq. (4.2) below. One of these tachyons is decoupled from the matter stress-tensor and another one couples to it with the strength that is suppressed as $m^2/(m^2 + 2H^2)$. The cases where this may not lead to a problem were discussed in Section 1. In particular, such a tachyon is a big improvement over the ghost, as for $2H^2 > m^2$ but $H \sim m$ the tachyon instability time is of the order of the age of the Universe, while in the case of a ghost the instability would have been catastrophically fast.

In our model there is a continuous transition to the massless limit $m \to 0$ and the vDVZ discontinuity is absent for some choices of $q$, one of which we discuss in the next section.

4. An example with no vDVZ discontinuity

At this point we make a choice for the value of the parameter $q$. We take

$$q = \frac{m^2 - 2H^2}{m^2 + 2H^2},$$

(4.1)

and that renders the theory ghost free for all positive values of $m^2$ and $H^2$.

Indeed, with the choice (4.1) we find that the physical graviton field has the following structure:

$$\frac{1}{2} h_{\mu\nu}^{\text{phy}} = \frac{1}{\Delta_L - 6H^2 + m^2 T_{\mu\nu}^{(1/2)}} + \frac{1}{m^2 + 2H^2} - \frac{m^2}{3} \frac{1}{m^2 + 2H^2} - \square - 6H^2 + m^2 T,$$

(4.2)

the first term describes the propagation of two tensor polarizations of squared mass $m^2$, while the second one shows the propagation of a scalar component with a positive residue $m^2/3(m^2 + 2H^2)$. 

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A scan of the possible values of $H^2$ in (4.1) is also interesting: For $2H^2 = m^2$ we have $q = 0$ which coincides with the special massive spin-2 theory described in [44].

For $H^2 \gg m^2$, making an expansion in powers of $\epsilon = \sqrt{m^2/2H^2}$ in (3.11), we get in the first order:

$$L_{\text{eff}} = L_{EH+PF}^{(2)}(h_{\mu\nu}^{\text{phy}}) + \frac{1}{2} \varepsilon(\Box + 9H^2)\varphi + h_{\mu\nu}^{\text{phy}}T^{\mu\nu} - \sqrt{2} \epsilon \varphi(\mathcal{O}^{\mu\nu} - \frac{3}{2} H^2)h_{\mu\nu}^{\text{phy}} , \quad (4.3)$$

where we introduced the canonically normalized field $\varphi = \sqrt{3m^2/4H^2} \varphi$. If we keep $H^2$ finite in the $\epsilon \to 0$ limit (i.e., a massless limit) the mixing disappears between $\varphi$ and $h_{\mu\nu}^{\text{phy}}$, so does the PF term (proportional to $H^2 \epsilon^2$). Thus, the theory becomes GR plus a decoupled scalar $\varphi$. Furthermore, the limit is smooth, since from (4.2) we can see the second term becoming negligible leaving only the two polarizations of a massless graviton.

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**A. Conventions and definitions**

On the de Sitter background the Riemann tensor is given by

$$R_{\mu\nu\rho\sigma} = H^2(\gamma_{\mu\rho}\gamma_{\nu\sigma} - \gamma_{\mu\sigma}\gamma_{\nu\rho}), \quad (A.1)$$

therefore the Ricci tensor takes the form

$$R_{\mu\nu} = 3H^2\gamma_{\mu\nu}, \quad (A.2)$$

and the Ricci curvature scalar equals to $R = 12H^2$. Furthermore,

$$\Box \nabla_\mu \nabla_\nu \varphi = (\nabla^\rho \nabla_\mu \nabla_\rho \nabla_\nu + \nabla^\rho [\nabla_\rho, \nabla_\mu] \nabla_\nu] \varphi$$

$$= (\nabla_\mu \Box \nabla_\nu + \nabla^\rho [\nabla_\rho, \nabla_\mu] \nabla_\nu + [\nabla^\rho, \nabla_\mu] \nabla_\rho \nabla_\nu] \varphi$$

$$= (\nabla_\mu \nabla_\nu \Box - \nabla^\rho R^{\lambda}_{\mu\nu\rho\lambda} \nabla_\lambda - R^{\lambda}_{\mu\rho\lambda} \nabla_\lambda \nabla_\nu - R^{\lambda}_{\nu\rho\lambda} \nabla_\lambda \nabla_\rho - \nabla_\mu R^{\lambda}_{\rho\nu\lambda} \nabla_\lambda) \varphi . \quad (A.3)$$

Using (A.4) we get

$$(\Box \nabla_\mu \nabla_\nu - \nabla_\mu \nabla_\nu \Box) \varphi = 8H^2 \left( \nabla_\mu \nabla_\nu - \frac{1}{4} \gamma_{\mu\nu} \Box \right) \varphi . \quad (A.4)$$

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The Lichnerowicz operator acting on a general rank-2 tension is given by

\[ \Delta_L h_{\mu \nu} = -\Box h_{\mu \nu} + 2 R^\rho_{\mu \nu \sigma} h^\sigma_{\rho} + R_{\rho \mu} h^\rho_{\nu} + R_{\rho \nu} h^\rho_{\mu} . \]  

(A.5)

Using the Riemann tensor given in (A.1), this becomes

\[ \Delta_L h_{\mu \nu} = -\Box h_{\mu \nu} + 2 H^2 (h_{\mu \nu} - \gamma_{\mu \nu} h) + 6 H^2 h_{\mu \nu} \]

\[ = -\Box h_{\mu \nu} + 8 H^2 h_{\mu \nu} - 2 H^2 \gamma_{\mu \nu} h . \]  

(A.6)

Define

\[ P_{\mu \nu} = \nabla_\mu \nabla_\nu - \frac{1}{4} \gamma_{\mu \nu} \Box , \quad Q = -\Box - 4 H^2 , \quad S = -\Box + 4 H^2 . \]  

(A.7)

It is easy to verify that

\[ \Delta_L P_{\mu \nu} \phi = (-\Box + 8 H^2) P_{\mu \nu} \phi = -P_{\mu \nu} \Box \phi , \]  

(A.8)

and therefore

\[ (\Delta_L - 4 H^2) P_{\mu \nu} \phi = P_{\mu \nu} Q \phi . \]  

(A.9)

Similarly

\[ \Delta_L \gamma_{\mu \nu} \phi = -\Box \gamma_{\mu \nu} \phi . \]  

(A.10)

Some other useful identities are:

\[ [\nabla_\mu , \nabla_\nu] \phi = 0 , \]  

(A.11)

\[ [\Box , \nabla_\mu] \phi = (\nabla^\mu \nabla_\nu \nabla_\mu - \nabla^\mu \nabla_\nu \nabla_\nu) \phi = [\nabla^\nu , \nabla_\mu] \nabla_\nu \phi = 3 H^2 \nabla_\mu \phi , \]  

(A.12)

and following the second identity given above, we also find

\[ \nabla^\mu \Box \nabla_\mu \phi = \Box^2 \phi + 3 H^2 \Box \phi . \]  

(A.13)

As a consequence of this, the operator \( O^{\mu \nu} \) annihilates \( \nabla_\mu \nabla_\nu \phi \) for any scalar \( \phi \), since

\[ O^{\mu \nu} \nabla_\mu \nabla_\nu \phi = \nabla^\mu \nabla^\nu \nabla_\mu \nabla_\nu \phi - \Box^2 \phi - 3 H^2 \Box \phi = \nabla^\mu \Box \nabla_\mu \phi - \Box^2 \phi - 3 H^2 \Box \phi = \Box^2 \phi + 3 H^2 \Box \phi - \Box^2 \phi - 3 H^2 \Box \phi = 0 . \]  

(A.14)

This is consistent with the linearized perturbations of the Ricci scalar on the dS background being proportional to \( O^{\mu \nu} h_{\mu \nu} \).

**B. Some more discussions**

Let us perform the following transformation

\[ h^{\text{phy}}_{\mu \nu} = \tilde{h}_{\mu \nu} + c \nabla_\mu \nabla_\nu \phi , \]  

(B.1)

where \( c \) is a constant to be determined.
In what follows we discuss how the effective Lagrangian (3.11) varies under such a transformation. Clearly \( \mathcal{L}_{\text{EH}}^{(2)}(\tilde{h}_{\mu\nu}) = \mathcal{L}_{\text{EH}}^{(2)}(h_{\text{phy}}^{\mu\nu}) \), because what we have done is just a gauge transformation for \( h_{\text{phy}}^{\mu\nu} \) in the case of pure gravity.

As mentioned above, \( \mathcal{O}^{\mu\nu} \) annihilates \( \nabla_\mu \nabla_\nu \phi \), therefore, \( \phi \mathcal{O}^{\mu\nu} h_{\text{phy}}^{\mu\nu} = \phi \mathcal{O}^{\mu\nu} \tilde{h}_{\mu\nu} \) is also invariant. The term \( \mathcal{L}_{\text{PF}} \) of course breaks the gauge symmetry and it varies:

\[
\mathcal{L}_{\text{PF}}(h_{\text{phy}}^{\mu\nu}) = -\frac{1}{4} m^2 (h_{\text{phy}}^{\mu\nu} - h^{\mu\nu})^2 \\
= \mathcal{L}_{\text{PF}}(\tilde{h}_{\mu\nu}) \\
- \frac{1}{2} m^2 \tilde{c} \tilde{h}_{\mu\nu} \nabla_\mu \nabla_\nu \phi - \frac{1}{12} m^2 c^2 \phi \nabla_\mu \nabla_\nu \phi + \frac{1}{2} m^2 c \phi \Box \phi + \frac{1}{12} m^2 c^2 \phi \Box^2 \phi \\
= \mathcal{L}_{\text{PF}}(\tilde{h}_{\mu\nu}) - \frac{1}{2} m^2 \tilde{c} \tilde{h}_{\mu\nu} \nabla_\mu \nabla_\nu \phi - \frac{3}{4} m^2 H^2 c^2 \phi \Box \phi + \frac{1}{2} m^2 c \Box \phi,
\]

where we have used some of the identities for swapping operators given in Appendix A. The only other term in (3.11) that is not invariant is

\[
-\frac{3}{2} q m^2 h_{\text{phy}}^{\mu\nu} \phi = -\frac{3}{2} q m^2 \tilde{h} \phi - \frac{3}{2} q c m^2 \phi \Box \phi.
\]

Therefore we find

\[
\mathcal{L}_{\text{eff}}^{(2)} = \mathcal{L}_{\text{EH+PF}}^{(2)}(\tilde{h}_{\mu\nu}) - \left[ \frac{1}{2} m^2 c + (1 + q) \right] (\phi \Box_\mu \nabla_\nu \tilde{h}_{\mu\nu} - \phi \Box \tilde{h}) \\
+ \left[ 3(1 + q) H^2 - \frac{3}{2} q m^2 \right] \tilde{h} \phi - \left[ \frac{3}{2} q (q + 1 + m^2) + \frac{3}{4} m^2 H^2 c^2 \right] \phi \Box \phi \\
+ \frac{3}{2} (2q - 1) \left[ q m^2 - 2(q + 1) H^2 \right] \phi^2 + \tilde{h}_{\mu\nu} T^{\mu\nu}.
\]

It is not completely trivial that one can now set the value of a single constant \( c \) to remove all the derivative mixings between \( \tilde{h} \) and \( \phi \). To do so we must choose

\[
c = -\frac{2(1 + q)}{m^2},
\]

in which case the theory reduces to

\[
\mathcal{L}_{\text{eff}}^{(2)} = \mathcal{L}_{\text{EH+PF}}^{(2)}(\tilde{h}_{\mu\nu}) - \frac{3}{2} \left[ q m^2 - 2(1 + q) H^2 \right] \tilde{h} \phi + \frac{3}{2m^2} \left[ q m^2 - 2(1 + q) H^2 \right] \phi \Box \phi \\
+ \frac{3}{2} (2q - 1) \left[ q m^2 - 2(q + 1) H^2 \right] \phi^2 + \tilde{h}_{\mu\nu} T^{\mu\nu}.
\]

Here we find a curious result. All the terms that involve the scalar \( \phi \) contain a common factor. To remove the mixing between \( \tilde{h} \) and \( \phi \), one must set this factor to zero by choosing

\[
q = \frac{2 H^2}{m^2 - 2 H^2}.
\]

In this case, all the terms that involve \( \phi \) disappear simultaneously, and the theory becomes just a conventional massive spin-2 on de Sitter background without any additional degrees of
freedom! It means that with this choice of $q$ and the special form of operator $\mathcal{O}^{\mu\nu}$ and $\mathcal{K}$, the initial Lagrangian (2.1) is nothing but a pure linearized gravity on de Sitter background, but expressed after a certain conformal and gauge transformations have been performed. This was shown by Higuchi [40] to have a ghost for $m^2 < 2H^2$. Notice that our consistency region for $q$ (3.12) is just right above the value (3.7).

References

[1] S. Weinberg, Rev. Mod. Phys. 61, 1 (1989).

[2] A. G. Riess et al. [Supernova Search Team Collaboration], Astron. J. 116, 1009 (1998) [arXiv:astro-ph/9805201];
   S. Perlmutter et al. [Supernova Cosmology Project Collaboration], Astrophys. J. 517, 565 (1999) [arXiv:astro-ph/9812133].

[3] G. Dvali, G. Gabadadze and M. Porrati, Phys. Lett. B485, 208 (2000) [hep-th/0005016].

[4] G. Dvali, G. Gabadadze and M. Shifman, Phys. Rev. D 67, 044020 (2003) [arXiv:hep-th/0202174]; arXiv:hep-th/0208096.

[5] N. Arkani-Hamed, S. Dimopoulos, G. Dvali and G. Gabadadze, arXiv:hep-th/0209227.

[6] G. Gabadadze and M. Shifman, Phys. Rev. D 69, 124032 (2004) [arXiv:hep-th/0312289].

[7] G. Gabadadze, arXiv:hep-th/0408118; In Ian Kogan Memorial Volume, Shifman, M. (ed.) et al. World Scientific, 2004; vol.2, pp 1061-1130.

[8] C. Deffayet, Phys. Lett. B 502, 199 (2001) [arXiv:hep-th/0010186].

[9] C. Deffayet, G. R. Dvali and G. Gabadadze, Phys. Rev. D 65, 044023 (2002) [astro-ph/0105068].

[10] N. Kaloper and D. Kiley, JHEP 0705 (2007) 045 [arXiv:hep-th/0703190]. N. Kaloper, arXiv:0711.3210 [hep-th].

[11] G. Dvali, S. Hofmann and J. Khoury, Phys. Rev. D 76, 084006 (2007) [arXiv:hep-th/0703027].

[12] M. A. Luty, M. Porrati and R. Rattazzi, JHEP 0309, 029 (2003) [arXiv:hep-th/0303116].

[13] G. Gabadadze and A. Iglesias, Phys. Rev. D 72, 084024 (2005) [arXiv:hep-th/0407049]; Phys. Lett. B 632, 617 (2006) [arXiv:hep-th/0508201].

[14] K. Koyama, Phys. Rev. D 72, 123511 (2005) [arXiv:hep-th/0503191].

[15] D. Gorbunov, K. Koyama and S. Sibiryakov, Phys. Rev. D 73, 044016 (2006) [arXiv:hep-th/0512097].

[16] C. Charmousis, R. Gregory, N. Kaloper and A. Padilla, JHEP 0610, 066 (2006) [arXiv:hep-th/0604086].

[17] K. Izumi, K. Koyama and T. Tanaka, JHEP 0704, 053 (2007) [arXiv:hep-th/0610282].

[18] C. Deffayet, G. Gabadadze and A. Iglesias, JCAP 0608, 012 (2006) [arXiv:hep-th/0607099].

[19] G. Dvali, G. Gabadadze, O. Pujolas and R. Rahman, Phys. Rev. D 75, 124013 (2007) [arXiv:hep-th/0612016].
[20] G. Gabadadze, Nucl. Phys. Proc. Suppl. 171, 88 (2007) [arXiv:0705.1929 [hep-th]].
[21] R. Gregory, N. Kaloper, R. C. Myers and A. Padilla, JHEP 0710, 069 (2007) [arXiv:0707.2666 [hep-th]].
[22] K. Izumi, K. Koyama, O. Pujolas and T. Tanaka, Phys. Rev. D 76, 104041 (2007) [arXiv:0706.1980 [hep-th]].
[23] C. de Rham and A. J. Tolley, JCAP 0607, 004 (2006) [arXiv:hep-th/0605122].
[24] C. de Rham, G. Dvali, S. Hofmann, J. Khoury, O. Pujolas, M. Redi and A. J. Tolley, arXiv:0711.2072 [hep-th].
[25] O. Corradini, K. Koyama and G. Tasinato, arXiv:0712.0385 [hep-th].
[26] Z. Kakushadze, Phys. Rev. D 77, 024001 (2008) [arXiv:0710.1061 [hep-th]]; Int. J. Geom. Meth. Mod. Phys. 05, 157 (2008) [arXiv:0711.0386 [hep-th]].
[27] H. van Dam and M. J. G. Veltman, Nucl. Phys. B 22, 397 (1970); V. I. Zakharov, JETP Lett. 12 (1970) 312 [Pisma Zh. Eksp. Teor. Fiz. 12 (1970) 447].
[28] A. I. Vainshtein, Phys. Lett. B 39 (1972) 393.
[29] C. Deffayet, G. R. Dvali, G. Gabadadze and A. I. Vainshtein, Phys. Rev. D 65, 044026 (2002) [arXiv:hep-th/0106001].
[30] D. G. Boulware and S. Deser, Phys. Rev. D 6, 3368 (1972).
[31] G. Gabadadze and A. Gruzinov, Phys. Rev. D 72, 124007 (2005) [arXiv:hep-th/0312074].
[32] C. Deffayet and J. W. Rombouts, Phys. Rev. D 72, 044003 (2005) [arXiv:hep-th/0505134].
[33] P. Creminelli, A. Nicolis, M. Papucci and E. Trincherini, JHEP 0509, 003 (2005) [arXiv:hep-th/0505147].
[34] V. A. Rubakov, arXiv:hep-th/0407104.
[35] S. L. Dubovsky, JHEP 0410, 076 (2004) [arXiv:hep-th/0409124].
[36] S. L. Dubovsky, P. G. Tinyakov and I. I. Tkachev, Phys. Rev. Lett. 94, 181102 (2005) [arXiv:hep-th/0411158].
[37] G. Gabadadze and L. Grisa, Phys. Lett. B 617, 124 (2005) [arXiv:hep-th/0412332].
[38] N. Arkani-Hamed, H. C. Cheng, M. A. Luty and S. Mukohyama, JHEP 0405, 074 (2004) [arXiv:hep-th/0312099].
[39] R. Bean, D. Bernat, L. Pogosian, A. Silvestri and M. Trodden, Phys. Rev. D 75, 064020 (2007) [arXiv:astro-ph/0611321].
[40] A. Higuchi, Nucl. Phys. B 282, 397 (1987).
[41] M. Porrati, Phys. Lett. B 498, 92 (2001) [arXiv:hep-th/0011152].
[42] D. G. Boulware and D. J. Gross, Nucl. Phys. B 233, 1 (1984).
[43] S. Deser and R. I. Nepomechie, Annals Phys. 154, 396 (1984); S. Deser and A. Waldron, Phys. Rev. Lett. 87, 031601 (2001) [arXiv:hep-th/0102166].
[44] G. Gabadadze and A. Iglesias, JCAP 0802, 014 (2008) [arXiv:0801.2165 [hep-th]].