Twins and Vertex- Identification on Graphs

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Abstract

Recently, several vertex identifying notions were introduced (identifying coloring, lid-coloring, ...), these notions were inspired by identifying codes. All of them, as well as original identifying code, are based on separating two vertices according to some conditions on their closed neighborhood. Therefore, twins can not be identified. So most of known results focus on twin-free graph. Here, we show how twins can modify optimal value of vertex-identifying parameters for identifying coloring and locally identifying coloring.

Keyword : Identifying coloring, locally identifying coloring, twins, separating.

1 Introduction

In this paper, we are interested on vertex-colorings allowing to distinguish the vertices of a graph. Given a graph $G$ and a coloring $c$, a pair of vertices $u$ and $v$ of $G$ is identified if and only if $c(N[u]) \neq c(N[v])$ (where $N[u]$ denotes the closed neighborhood of $u$). Two vertices $u$ and $v$ with $N[u] = N[v]$ are called twins. Observe that two twins can not be identified.
During the discrete week of Institute Fourier at Grenoble in 2009, Eric Duchêne and Jullien Moncel presented the concept of identifying coloring of a graph as a vertex coloring such that any pair of vertices is identified. Clearly, an identifying coloring of $G$ exists if and only if $G$ has no twins.

Since, several authors [1, 2] introduced two notions of locally identifying where identified pair of vertices is required only for adjacent vertices. Moreover, in order to incorporate graph with twins, the identifying condition concerns only pair of non-twin vertices.

Since only few results are known on identifying coloring (see [6]), and in order to be uniform, we propose here to modify the definition of identifying coloring of a graph as a vertex coloring such that any pair of non-twin vertices is identified.

In [1], a relaxed locally identifying coloring, rlid-coloring for short, of a graph $G = (V, E)$ is defined as a mapping $c : V \rightarrow \mathbb{N}$ such that any pair of adjacent non-twin vertices is identified. Aline Parreau et al. [2] introduced the notion of locally identifying coloring, lid-coloring for short, as a rlid-coloring $c$ which is proper that is $c(u) \neq c(v)$ for all pair of adjacent vertices $u, v$.

Given a graph $G$, $\chi_{ld}(G)$ (respectively $\chi_{lid}(G), \chi_{rlid}(G)$) denotes the smallest number of colors needed to have an identifying coloring (resp. lid-coloring, rlid-coloring) of $G$.

The $\chi_{lid}$ is the most studied of these parameters, see for instance [2, 3, 4]. Nevertheless, except in [1], most of the results concern twin-free graphs. The aim of this paper is to show that twins may have significant influence on these parameters.

In order to state our results, we will need additional definitions. Let $\mathcal{R}$ be the equivalence relation defined as follows: for all vertices $u, v \in V(G)$, we have $u \mathcal{R} v$ iff $N[u] = N[v]$. Denote by $G\setminus\mathcal{R}$, the maximal twin-free subgraph of $G$ (that is the quotient of $G$ by relation $\mathcal{R}$). The number of equivalence-classes having at least two vertices in $G$ is denoted by $t(G)$. Let $T(G)$ be the cardinality of a largest equivalence-class.

In [1], the authors proved the following theorem:

**Theorem 1.1.** Let $G$ be a graph. Then we have

$$\chi_{rlid}(G \setminus \mathcal{R}) - t(G) \leq \chi_{rlid}(G) \leq \chi_{rlid}(G \setminus \mathcal{R}).$$

Moreover, the authors in [1] exhibit graphs for which the bounds are tight. In this paper, we give analogous results for identifying colorings and lid-colorings.
Theorem 1.2. Let $G$ be a graph. Then we have
\[\chi_{id}(G \setminus \mathcal{R}) - t(G) \leq \chi_{id}(G) \leq \chi_{id}(G \setminus \mathcal{R}).\]

Theorem 1.3. Let $G$ be a graph. Then we have
\[\chi_{lid}(G \setminus \mathcal{R}) - t(G) \leq \chi_{lid}(G) \leq \chi_{lid}(G \setminus \mathcal{R}) + (T(G) - 1)t(G).\]

Proofs of Theorems 1.2 and 1.3 are given in Section 2. In Section 3 we exhibit graphs for which the bounds in Theorems 1.2 and 1.3 are tight.

2 Proofs of bounds

Proof of Theorem 1.2. We present here a proof similar to the proof of Theorem 1.1 given in [1].
Consider an identifying coloring $c$ of $G \setminus \mathcal{R}$. Let define a coloring $c'$ of $G$ as follows : for each vertex $x$ in $G \setminus \mathcal{R}$ and its twin (if there exists) $y$, set $c'(x) = c'(y) = c(x)$. Since in $G$, we are not interested to distinguish the twins then $c$ defines an identifying coloring of $G$.

Now, let $c$ be an identifying coloring of $G$ using colors $\{1, \ldots, \chi_{id}(G)\}$. Consider the coloring $c'$ defined as follows : $c'(u) = c(u)$ if the vertex $u$ has no twin in $G$ and color the other $t(G)$ vertices of $G \setminus \mathcal{R}$ with different colors $\chi_{id}(G) + 1$ until $\chi_{id}(G) + t(G)$. This coloring gives an identifying coloring of $G \setminus \mathcal{R}$. \hfill \qed

Proof of Theorem 1.3.

Now, consider a $lid$-coloring $c$ of $G \setminus \mathcal{R}$ using colors $\{1, \ldots, \chi_{lid}(G \setminus \mathcal{R})\}$. Let $c'$ be a coloring obtained from $c$ as follow: $c'(u) = c(u)$ for all $u$ in $G \setminus \mathcal{R}$. By definition, there are at most $(T(G) - 1)t(G)$ vertices in $G$ which are not in $G \setminus \mathcal{R}$. For each of them assign a distinct color from $\{\chi_{lid}(G \setminus \mathcal{R}) + 1, \ldots, \chi_{lid}(G \setminus \mathcal{R}) + (T(G) - 1)t(G)\}$. This coloring gives an lid-coloring of $G$. Now, similarly as proof of Theorem 1.2 let $c$ be a $lid$-coloring of $G$ using colors $\{1, \ldots, \chi_{lid}(G)\}$. Consider the coloring $c'$ defined as follows : $c'(u) = c(u)$ if the vertex $u$ has no twin in $G$ and color the other $t(G)$ vertices of $G \setminus \mathcal{R}$ with different colors $\chi_{lid}(G) + 1$ until $\chi_{lid}(G) + t(G)$. This coloring gives an lid-coloring of $G \setminus \mathcal{R}$. \hfill \qed
3 Extremal graphs

First define the split graph $H_p = (S_p \cup K_p, E)$ for a given integer $p$ where $S_p = \{s_1, \ldots, s_p\}$ (respectively $K_p = \{k_0, \ldots, k_p\}$) induces a stable (resp. clique). The others edges of $H_p$ are $s_ik_i$ for all $i = 1, \ldots, p$.

Property 3.1. Let $p \geq 1$ be an integer. We have that

$$\chi_{id}(H_p) = p + 2 \text{ and } \chi_{lid}(H_p) = 2p + 1.$$ 

Proof. The coloring $c$ defined by $c(s_i) = i, c(k_i) = p + 1$ for all $i = 1, \ldots, p$ and $c(k_0) = p + 2$, is an identifying coloring of $H_p$.

Let prove now that $\chi_{id}(H_p) \geq p + 2$. Let $c$ be an identifying coloring of $H_p$.

First observe that $c(s_i) \neq c(s_j)$ for all $i \neq j$. Indeed, otherwise $c(N[k_i]) = c(N[k_j])$ which leads a contradiction. Second, suppose that $c(s_i) = c(k_j)$ for some integers $i, j$ ($i$ can be equal to $j$). Then $c(N[k_0]) = c(N[k_i])$, a contradiction. To conclude, check that if $|c(K)| = 1$ then $c(N[s_i]) = c(N[k_i])$ for all $i$, a contradiction which completes the proof of $\chi_{id}(H_p) \geq p + 2$.

Any coloring using $2p + 1$ distinct colors is a lid-coloring of $H_p$.

Let prove now that $\chi_{lid}(H_p) \geq 2p + 1$. Let $c$ be an lid-coloring of $H_p$. As previously, we have $c(s_i) \neq c(s_j)$ for all $i \neq j$ else $c(N[k_i]) = c(N[k_j])$ and $k_i$ and $k_j$ are adjacent. Again, $c(s_i) \neq c(k_j)$ for all pair $i, j$, otherwise $c(N[k_0]) = c(N[k_i])$ for some $i \neq 0$, a contradiction. To conclude, since $K$ is a clique, then $|c(K)| = p + 1$.

Consider the first extension $H_{2^a}^{ext} = (S_{2^a} \cup K_{2^a}, E)$ for some integer $a \geq 1$ where $K_{2^a}$ induces a clique. One may define the vertices of $K_{2^a} = \{k_E \mid E \subseteq \{1, \ldots, a\}\}$. Now define $S_{2^a-1} = \{s_E \mid E \subseteq \{1, \ldots, a\}$ and $i \in E\}$. Observe that $|K_{2^a}| = 2^a$ and $|S_{2^a-1}| = a.2^a-1$. The others edges of $H_{2^a}^{ext}$ are $s_E,k_E$ for all $i \in E$ and $s_E,s_E$ for all $i, j \in E$.

Remark that $H_{2^a}^{ext}\backslash \mathcal{R} = H_{2^a-1}$ with $t(H_{2^a}^{ext}) = 2^a - 1 - a$ and $T(H_{2^a}^{ext}) = a$.

Property 3.2. Let $a \geq 1$ be an integer. We have that

$$\chi_{id}(H_{2^a}^{ext}) = a + 2 \text{ and } \chi_{lid}(H_{2^a}^{ext}) = a + 2^a.$$ 

Proof. The coloring $c$ defined by $c(s_E) = i, c(k_E) = a+1$ for all $E \subseteq \{1, \ldots, a\}$ and for all $i \in E$ and $c(k_0) = a + 2$, is an identifying coloring of $H_{2^a}^{ext}$.
By Theorem 1.2, we have $\chi_{id}(H_{2a}^{ext}) \geq \chi_{id}(H_{2a}^{ext}) - t(H_{2a}^{ext}) = 2^a - 1 + 2 - (2^a - 1 - a) = a + 2$.

For the lid-coloring the proof is similar except that we need distinct colors for each vertex in the clique $K_{2^a}$.

The two previous propositions show that lower bound of Theorems 1.2 and 1.3 are tight for graph $H_{2a}^{ext}$. Upper bound of Theorem 1.2 is reached for any twin-free graph.

Given integers $p \geq t \geq 1$ and $T \geq 1$, consider the graph $H_p^{(T,t)}$ obtained from $H_p$ by adding $T - 1$ twins to all vertices $k_i$ for all $i = 1, \ldots, t$.

Remark that $H_p^{(T,t)} \setminus \mathcal{R} = H_p$ with $t(H_p^{(T,t)}) = t$ and $T(H_p^{(T,t)}) = T$.

**Property 3.3.** Let $a \geq 1$ be an integer. We have that

$$\chi_{lid}(H_p^{(T,t)}) = 2p + 1 + (T - 1).t.$$

**Proof.** Any coloring using $2p + 1 + (T - 1).t$ distinct colors is a lid-coloring of $H_p^{(T,t)}$.

The proof of $\chi_{lid}(H_p^{(T,t)}) \geq 2p + 1 + (T - 1).t$ follows the one of Proposition 3.1.

For all triple of integers $p \geq t \geq 1$ and $T \geq 1$, Proposition 3.3 show that upper bound of Theorem 1.3 is reached.

**Concluding Remarks**

The main motivation of the present paper is to point out that twins may play a crucial role in identifying coloring problems using closed neighborhood. Probably it should be to difficult to re-consider all known results on twin-free graphs. But there are some special classes of graphs (e.g. split graphs) for which this work remains attractive.

Instead of coloring, one may ask what happens for identifying codes? An identifying code \cite{5} is a subset of vertices $C$, such that for any pair of distinct vertices $u, v$, $N[u] \cap C \neq N[v] \cap C$. This is the classical definition, and clearly, a graph having twins does not admit an identifying code. Consider, now the
new definition where the condition $N[u] \cap C \neq N[v] \cap C$ has to be verified only for non-twin pair of vertices.

It is not too difficult to see that the size of a minimum identifying codes in a graph $G$ with new definition is equal to the size of a minimum identifying codes in $G\backslash \mathcal{R}$. Therefore, it is not restrictive for identifying codes to consider only twin-free graphs.

Now, for coloring versions of identifying problems one may consider a weighted version. Given a graph $G$ and a weight function $w : V \rightarrow \mathbb{N}$, a mapping $c : V \rightarrow 2^\mathbb{N}$ is an weighted-identifying coloring of $G$ if and only if $|c(u)| \leq w(u)$ for all vertices $u$ and all distinct pairs of non-twin vertices are identified.

Observe that an optimal value for a pair $(w, G)$ is the same than the optimal value for the pair $(w', G\backslash \mathcal{R})$ where $w'(u) = w(u) + T(u) - 1$ where $T(u)$ is the number of twins of $u$. So for this new definition one may focus only on twin-free graphs.

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