Efficient coherent control by sequences of pulses of finite duration

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Abstract. Reliable long-time storage of arbitrary quantum states is a key element of quantum information processing. In order to dynamically decouple a spin or quantum bit from a dephasing environment by non-instantaneous pulses, we introduce an optimized sequence of $N$ control $\pi$ pulses that are realistic in the sense that they have a finite duration and a finite amplitude. We show that optimized dynamical decoupling is still applicable and that higher-order decoupling can be reached if shaped pulses are implemented. The sequence suppresses decoherence up to the order $O(T^{-M}) + O(\tau_{mx}^M)$, with $T$ being the total duration of the sequence and $\tau_{mx}$ the maximum length of the pulses. The exponent $M \in \mathbb{N}$ depends on the shape of the pulse. Based on existing experiments, a concrete setup for the verification of the properties of the advocated sequence is proposed.

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1. Introduction

In quantum information processing (QIP) and in nuclear magnetic resonance (NMR), it is essential to be able to decouple the quantum bit or the spin, respectively, from its environment. Both fields of research are of widespread interest and relevance. In the former, the ultimate goal is to realize reliable long-time storage of quantum information with as low as possible error rates. This is a prerequisite for QIP [1, 2]. In the latter, the high-precision measurement of nuclear spin dynamics is a long-standing goal [3, 4].

Besides choosing well-isolated systems the application of appropriately tailored sequences of control pulses [3, 5, 6], i.e. dynamic decoupling (DD), is one of the promising paths to achieve this goal. The basic idea goes back to Hahn’s spin echo pulse, which averages a static perturbation to zero [7]. For a dynamic environment, or bath, sequences of pulses are required [5, 6], [8]–[10]. The early suggestions are essentially periodic in time.

Recently, additional advantages of sequences with non-equidistant pulses were discovered. Concatenation (CDD) can suppress unwanted couplings in a high power $T^\ell$ of the length of the sequence [11]. But for the method used in [11], the required number $N$ of pulses grows exponentially with $4^\ell$. For pure dephasing, it was shown that this growth can be reduced decisively to a linear one $l \propto N$ if the instants of $\pi$ pulses were chosen according to

$$t_j = T \sin^2(j\pi/(2N + 2)), \quad (1)$$

which is called UDD (Uhrig DD). Relation (1) was derived for a spin–boson model [12] where it was observed that no details of the model entered. On the basis of numerical evidence and finite order recursion it was conjectured that (1) is applicable to any dephasing model [13, 14]. This claim was finally proven [15] for any order in $T$. For various simulated classical noise spectra the experimental verification of the theoretical results was achieved [16]–[18] by microwave control of the transition in Be ions. The suppression of the decoherence of the electron spin of hydrogen radicals was investigated by using electron spin resonance in crystals of irradiated malonic acid [19]. The decoherence was due to the quantum noise induced by nuclear spins. Again, the UDD proved superior to standard sequences.

For general decoherence, concatenation of the UDD sequence (CUDD) can be used [20]. For a suppression of the decoherence up to $T^\ell$, the number of pulses grows as $2^\ell$, which is an...
improvement by a square root with respect to the CDD of Khodjasteh and Lidar [11]. A more efficient scheme, called quadratic DD (QDD), which requires only a quadratically growing number of pulses, has been proposed very recently based on numerical [21] and analytical evidence [22].

All these sequences (periodic DD [10], CDD [11], UDD [12], CUDD [20] and QDD [21, 22]) rely on instantaneous, and thus idealized, pulses. This problem was realized early on and ongoing research investigates pulses of finite duration $\tau_p$ [23]–[27] and sequences of such pulses [28]–[32]. Eulerian DD [28, 33] is designed to annihilate the first order of a Magnus expansion over the whole sequence. Thus corrections of the order of $T \tau_p$ are not excluded. Similar caveats apply to many other sequences [29]–[32]. Also the experimental realizations in [16]–[18] have to take into account that real pulses cannot be instantaneous because the control amplitudes are necessarily bounded.

Our aim here is to derive an optimized sequence with UDD properties which relies on realistic pulses of finite duration and which is adapted to these real pulses. We do not provide a general scheme to use pulses of bounded control for arbitrary DD sequences. If the shape is appropriately designed, the pulse can be approximated as an instantaneous one up to $O(\tau_p^M)$. For $M = 3$, explicit results were derived in [34], while a recursive scheme for arbitrary $M$ has been proposed recently [35]. As far as the correction $O(\tau_p^M)$ is negligible, the proposed sequence displays the same exact analytic properties as the UDD sequence of ideal, instantaneous pulses.

We approach the problem hierarchically. This means that we use the pulses and the periods of free evolution as building blocks for the sequence. Firstly, the properties of the pulses are derived and discussed. Secondly, these properties are used in the sequence. To this end, we exploit scaling in two independent variables, namely the durations $\tau_p$ of the pulses, whose maximum is $\tau_{mx}$, and the total duration of the sequence denoted by $T$. Note that these two timescales are largely independent, both in theory and in experiment, because the pulses are not applied back to back. The only constraint is that $T$ must be larger than the sum of the pulse durations $\tau_{p,j}$ (belonging to pulse $j$)

$$T \geq \sum_{j=1}^{N} \tau_{p,j}. \quad (2)$$

Relying on pulses, which cancel all orders $m < M$, the whole sequence avoids all mixed terms $T^n \tau_{mx}^m$, where $n \leq N + 1$ and $m < M$. Hence, significant progress over existing proposals [28]–[32] is achieved.

2. Model

We start from the Hamiltonian

$$H = \hat{A}_0 + \sigma_z \hat{A}_1 \tilde{F}(t), \quad (3)$$

where $\sigma_z$ is the $z$-component of the Pauli matrices. It is acting on the $S = 1/2$ spin or, generally, on the two-level system that represents a qubit. The operators $\hat{A}_i$ only act on the bath; they may also be $c$-numbers. We consider any kind of bath with bounded operators for the sake of the mathematical argument $||H|| \leq \gamma < \infty$, where $|| \cdot ||$ is any appropriate operator norm that remains invariant under unitary transforms. We expect that the order of suppression of the decoherence holds for any bath that can be approximated by bounded baths, i.e. the bath should have a hard high-energy cutoff.

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No spin-flip terms are included in (3) implying an infinite spin–lattice relaxation time $T_1$. It is an excellent approximation if $T_1 \gg T_2$, where $T_2$ is the dephasing time. Such a situation is achieved in the rotating reference frame of a system where the two levels with eigenvalues $\pm 1$ of $\sigma_z$ lie energetically far apart. Longitudinal relaxation and general decoherence will be addressed below.

Moreover, equation (3) is the effective Hamiltonian in the interaction picture of the short control pulses [12]–[15]. Thus the switching function $\tilde{F}(t) \in \mathbb{R}$ appears. The simplest example is an instantaneous $\pi$ pulse at $t = t_j$ which realizes a rotation about an axis perpendicular to $\sigma_z$. Then $\tilde{F}(t)$ changes its sign at $t = t_j$ abruptly, while it is constant elsewhere. A sequence of such pulses at the instants $\{t_j\}$ with $j \in \{1, 2, \ldots, N\}$ implies $\tilde{F}(t) = (-1)^j$ for $t \in \{t_j, t_j+1\}$, where we define $t_0 = 0$ and $t_{N+1} = T$.

Our derivation is based on the bounded quantum model (3). Thereby, classical Gaussian noise is treated to the extent that it can be approximated by the quantum model [14]. Certainly, non-Gaussian classical noise, see for instance [36], should be considered separately which is beyond the scope of the present article.

3. Derivation

Optimization of the sequence means to pose the question which switching instants $t_j$ make the sequence $\{t_j\}$ most efficient. For ideal instantaneous pulses, it was shown that the UDD instants (1) are optimum in the sense that the time evolution depends on the spin weakly [12]–[15]. The time evolution operator

$$\hat{U}_0 = \prod_{j=0}^{N} e^{-i[\hat{A}_0 \pm \hat{A}_1 (t_j)-(t_{j+1}-t_j)]} \quad (4)$$

for the eigenstates of $\sigma_z$ with eigenvalues $\pm 1$ depends on the spin only in a high power of $T$:

$$\hat{U}_+ - \hat{U}_- = \mathcal{O}((\gamma T)^{N+1}). \quad (5)$$

The analytical derivation of (5) is achieved by direct time-dependent perturbation theory (TDPT) [15] in powers of $t H$. Thus, if the $(N+1)$st power does not vanish it is of order $(\gamma T)^{N+1}$. The iterated time integrations of TDPT are conveniently expressed by the substitution $t := T \sin^2(\vartheta/2)$ as integrations over the variable $\vartheta$. The instants (1) are equidistant if expressed in $\vartheta$ because for $F(\vartheta) := \tilde{F}(T \sin^2(\vartheta/2))$ we have

$$F_{\text{UDD}}(\vartheta) = (-1)^j \quad \text{for} \quad \vartheta \in \left(\frac{j \pi}{N+1}, \frac{(j+1) \pi}{N+1}\right) \quad (6)$$

with $j \in \{0, N\}$. Allowing $j$ to take all integer values $j \in \mathbb{Z}$ the function $F_{\text{UDD}}$ becomes an odd function with $F_{\text{UDD}}(\vartheta + \pi/(N+1)) = - F_{\text{UDD}}(\vartheta)$. Hence, the Fourier series of $F_{\text{UDD}}(\vartheta)$ comprises only odd sin harmonics $\sin(l/(N+1)\vartheta)$ with $l \in \{1, 3, \ldots\}$. The coefficients are $4/((\pi l)^2)$. From this property, equation (5) is derived by exploiting trigonometric addition theorems recursively [15].

The power of the UDD sequence has been demonstrated experimentally [16, 17]. The noise, i.e. the coupling to the bath $\hat{A}_1$, is simulated, so that it can be switched off during the pulse. Thereby, a partial solution of the finiteness of the pulse amplitudes is achieved. But generally, decoherence processes cannot be switched off. Using pulses of duration $\tau_p$ with
constant amplitude instead of instantaneous pulses introduces an unwanted term of the order \(\gamma \tau_p\) at each rotation, i.e. linear in the pulse length. For a sequence of length \(N\) these corrections can accumulate to \(N\gamma \tau_p\) unless the contributions of subsequent pulses cancel each other.

An improvement by one order in \(\gamma \tau_p\) is achieved by the ersatz \(\pi\) pulse, which makes the linear correction vanish. Then the time evolution operator \(\hat{U}_p\) for a pulse reads

\[
\hat{U}_p(t + \tau_p, t) = \hat{U}^{\text{ideal}}_p(t + \tau_p, t) + \mathcal{O}((\gamma \tau_p)^M), \tag{7a}
\]

\[
\hat{U}^{\text{ideal}}_p(t + \tau_p, t) = e^{-i(t - \tau_p)H} \hat{P}_\theta e^{-it\tau_p H}, \tag{7b}
\]

where \(M = 2\). It is understood that \(t\) marks the beginning of the pulse and \(t + \tau_p\) its end. It is important that \(H\) is the Hamiltonian of the total system, i.e. spin, bath and their mutual coupling, \(\hat{P}_\theta\) the ideal pulse with \(\theta = \pi\) and \(\tau_s\) is the instant when the approximated ideal pulse occurs. In a sequence \(\{\tau_j\}\), the instant \(\tau_s\) is to be identified with the switching instants \(\tau_j\). No adjustment of the sequence takes place. The relation (7) can be achieved by shaping the pulse appropriately [25]–[27]. Hence, we can set up a UDD sequence with more realistic pulses of the kind (7) for which the deviations read

\[
\hat{U}^{\text{UDD}}_+ - \hat{U}^{\text{UDD}}_- = \mathcal{O}(\gamma T)^{N+1} + \mathcal{O}(N\gamma \tau_{mx})^M \tag{8}
\]

with \(M = 2\). The additivity of the corrections is a straightforward property of the unitary evolution operators. If we denote the UDD sequence made from the ideal pulses \(\hat{U}^{\text{ideal}}_p(t + \tau_p, t)\) in (7b) by \(\hat{U}^{\text{UDD,ideal}}_{\pm}\), we know from (7a)

\[
\hat{U}^{\text{UDD}}_{\pm} = \hat{U}^{\text{UDD,ideal}}_{\pm} + \mathcal{O}(N\gamma \tau_{mx})^M \tag{9}
\]

for \(N\) pulses. The unitary invariance of the norm \(\gamma\) is used for each pulse. Next, we know from the properties of the UDD sequence [13]–[15]

\[
\hat{U}^{\text{UDD,ideal}}_+ - \hat{U}^{\text{UDD,ideal}}_- = \mathcal{O}(\gamma T)^{N+1}. \tag{10}
\]

Combined with (9), this equation implies (8).

The bound \((\gamma T)^{N+1}\) resulting from the sequence can be improved systematically by enlarging \(N\). The bound \(N(\gamma \tau_{mx})^2\) resulting from the pulses can be improved by making it shorter. But if this is not possible, one is stuck because the exponent of 2 cannot be incremented for \(\theta = \pi\) pulses as implied by mathematical no-go theorems [25, 26]. Hence, we are facing here a serious conceptual obstacle.

Recently, a variant of (7)

\[
\hat{U}_p(t + \tau_p, t) = \hat{U}^{\text{zero}}_p(t + \tau_p, t) + \mathcal{O}((\gamma \tau_p)^M), \tag{11a}
\]

\[
\hat{U}^{\text{zero}}_p(t + \tau_p, t) = e^{-i\tau_p \hat{A}_0} \hat{P}_\theta \tag{11b}
\]

with \(M = 3\) was shown [34] to reduce the correction to \((\gamma \tau_p)^3\). Note that in (11) only the Hamiltonian \(\hat{A}_0\) of the bath occurs without coupling to the spin. Hence \([\hat{A}_0, \hat{P}_\theta] = 0\) holds and no \(\tau_s\) needs to be introduced.

Explicit solutions are obtained for pure dephasing [34]. The correlation time of the dephasing bath should not be much smaller than \(\tau_p\). Moreover, no no-go theorem was found that prevents us from achieving higher orders as well. Indeed, a recursive scheme based on
concatenation is proposed that achieves arbitrary order $M$ at exponential cost [35], i.e. each composite $\pi$ pulse consists of $> 17^{M-1}$ elementary pulses. This demonstrates that in principle arbitrary $M$ can be achieved although the exponential cost may spoil its practical usefulness. But due to the shortness of the pulses compared to the whole sequence ($\tau_p \ll T$), we do not expect that particularly large values of $M$ are required.

The property (11) is promising, but it cannot be used in standard DD, or in UDD in particular, as ersatz for an instantaneous pulse. This is so because any standard DD sequence presupposes that between the pulses $\hat{P}_\pi$ the full Hamiltonian $H$, not only $\hat{A}_0$, is active. This conceptual obstacle cannot be solved by pulse shaping because the no-go theorems block further progress [25, 26]. To overcome this obstacle is the main goal of the present paper. We find that an adjustment of the sequence to the pulses of finite duration is required.

Our present fundamental observation is that relation (11b) translates to $\tilde{F}(t)$ for a single realistic pulse between $t^-$ and $t^+=t^-+\tau_p$ in the form

$$\tilde{F}(t) = \begin{cases} 1 & \text{for } t < t^-, \\ 0 & \text{for } t^- < t < t^+, \\ -1 & \text{for } t > t^+. \end{cases}$$

(12)

The $\pi$ pulse implies the inversion of the sign. But during the pulse itself the relation (11) implies that the coupling between the spin (qubit) and the bath is effectively averaged to zero up to $O((\gamma \tau_p)^M)$. This is so since $\hat{A}_0$ in (11b) does not comprise the spin–bath coupling; it only comprises the bath dynamics. This implies that the switching function $\tilde{F}(t)$ takes the value zero during the pulse. Note that there are jumps in the switching function even though the pulse is generated by bounded control. The reason for this behavior is that unitary time evolution is considered over finite time intervals, not over infinitesimal intervals. This means that from the hierarchical level of the sequence we do not look into the pulses. The description with $\tilde{F}(t)$ is only valid on the level of the sequence, not within the pulse interval. Furthermore, the correction term in equation (11a) may not be forgotten.

Next we look for a sequence with $\tilde{F}(t) \in \{-1, 0, 1\}$ that corresponds to an odd function $F(\vartheta) \in \{-1, 0, 1\}$ with the antiperiodic behavior $F(\vartheta + \pi/(N+1)) = -F(\vartheta)$. Such a sequence, RUDD (realistic UDD), allows for the same mathematical argument as UDD (6), ensuring that the effective time evolution of the spin is the identity up to corrections of order $(\gamma T)^{N+1}$. The reason is the antiperiodicity of the switching function, which is the fundamental reason for the annihilation of the preceding orders [15, 22]. Note that this argument holds only for UDD and similar optimized sequences. Hence, we do not provide a general scheme for the incorporation of pulses of finite duration into arbitrary sequences.

The sequence fulfilling the requirement of antiperiodicity reads

$$F_{\text{RUDD}}(\vartheta) = \begin{cases} (-1)^j & \text{for } \vartheta \in \left(\frac{j\pi}{N+1} + \vartheta_p, \frac{(j+1)\pi}{N+1} - \vartheta_p\right), \\ 0, & \text{otherwise} \end{cases}$$

(13)

for $j \in \mathbb{Z}$. The Fourier series comprises only the odd sin harmonics $\sin(l(N+1)\vartheta)$ with coefficients $4\cos(l(N+1)\vartheta_p)/(l\pi l)$. The parameter $0 \leq \vartheta_p \leq \pi/(2N+2)$ determines the duration of the pulses. Except for the given inequality it is independent of $N$. Note that pulses of equal duration in $\vartheta$ do not correspond to pulses of equal duration in time $t$.

$$\tilde{F}_{\text{RUDD}}(t) = \begin{cases} (-1)^j & \text{for } t \in (t^+_j, t^+_{j+1}), \\ 0, & \text{otherwise}, \end{cases}$$

(14)
Figure 1. Amplitudes $v(t)$ (black solid and blue short-dashed lines) for pulses rotating about axes in the $xy$ plane and the resulting switching functions $\tilde{F}(t)$ (red dashed lines) for $N = 1, 2, 3$ and $4\pi$ pulses. For clarity, fairly large values $\theta_p = 0.7\pi/(2N + 2)$ are chosen. For rendering purposes, $\theta_p$ is chosen here to depend on $N$. Generally, it only has to fulfill $\theta_p \leq \pi/(2N + 2)$.

Black solid lines stand for $\theta = \pi$ pulses with $v_{\pi}(t)$ as in (15) with $a_\pi = 10.804433[1/\tau_p]$, $b_\pi = 6.831344[1/\tau_p]$ and $c_\pi = 2.174538[1/\tau_p]$. The first and the last pulse rotate by $2\pi$ (blue short-dashed lines) with $a_{2\pi} = 10.236155[1/\tau_p]$, $b_{2\pi} = 2.9661717[1/\tau_p]$ and $c_{2\pi} = 0.889052[1/\tau_p]$. For clarity, $v_{2\pi}(t)/5$ is plotted.

with $t_j^\pm := T \sin^2 \left[ \frac{j\pi}{2N+2} \pm \theta_p/2 \right]$. This is illustrated for $N = \{1, 2, 3, 4\}$ in figure 1, where the necessary time-dependent amplitudes $v(t)$ defining the control Hamiltonian $H_C(t) = v(t)\sigma_y$ are also shown; $\sigma_y$ is the $y$ component of the Pauli matrices. For instance, the amplitudes can be parameterized by

$$v_\theta(t) = \theta/2 + (a_\theta - \theta/2) \cos(2\pi t/\tau_p) + (b_\theta - a_\theta) \cos(4\pi t/\tau_p) + (c_\theta - b_\theta) \cos(6\pi t/\tau_p) - c_\theta \cos(8\pi t/\tau_p).$$

(15)

There is one subtlety about the beginning and the end of the sequence. In order to generate the switching function (14), there must be a first and a last pulse that averages the coupling between the spin and the bath to zero while inducing no net rotation. For this purpose $\theta$ can take any multiple of $2\pi$. Solving the equations derived in [34], which imply that the pulse fulfills the relation (11), leads to the parameters given in the caption of figure 1.

The duration of the pulses is shortest towards the ends of the interval $T$ for which the quantum state of the spin is to be stored. Concomitantly, the amplitudes are largest for the first and the last pulse. In practice, the initial and the final pulse can be combined with the pulses...
by which the quantum state of the spin (the qubit) is generated, for instance a $\pi/2$ pulse (for solutions see [34]).

What has been achieved by the sequence (14) depicted in figure 1? This sequence is an optimized DD scheme made from pulses of finite duration and finite amplitudes with analytically founded properties. Bounded control is a crucial aspect of realistic sequences, so that the proposed sequence is an important step closer to a realistic scenario. Nevertheless, the sequence is still optimized in the sense that it shares the same power-law property as the UDD built from instantaneous pulses.

There are two sources of corrections in the unitary time evolution $\hat{U}_{\pm}^{\text{RUDD}}$ of the RUDD sequence. The first kind of corrections stems from the pulses that are only close to $\hat{U}_p^{\text{zero}}(t + \tau_p, t)$ but not identical to it, see equation (11). Denoting the time evolution of the RUDD sequence made from the pulses $\hat{U}_p^{\text{zero}}(t + \tau_p, t)$ in (11b) by $\hat{U}_{\pm}^{\text{RUDD,zero}}$, we know, from (11a),

$$\hat{U}_{\pm}^{\text{RUDD}} = \hat{U}_{\pm}^{\text{RUDD,zero}} + O(N(\gamma \tau_{\text{mix}})^M)$$

for $N$ pulses. The second kind of corrections stems from the sequence itself. The time evolution $\hat{U}_{\pm}^{\text{RUDD,zero}}$ is rigorously governed by $F_{\text{RUDD}}(\vartheta)$ defined in (13). Then we know from the paper by Yang and Liu [15] that

$$\hat{U}_+^{\text{RUDD,zero}} - \hat{U}_-^{\text{RUDD,zero}} = O((\gamma T)^{N+1}).$$

The total correction is given by the sum of both kinds of corrections because their norm is invariant under unitary transformations. So, in analogy to (8), we obtain

$$\hat{U}_+^{\text{RUDD}} - \hat{U}_-^{\text{RUDD}} = O((\gamma T)^{N+1}) + O(N(\gamma \tau_{\text{mix}})^M).$$

We stress that this relation excludes mixed terms $(\gamma T)^n(\gamma \tau_{\text{mix}})^m$ with $n \leq N + 1$ and $m < M$ because each pulse complies with (11a) separately. We point out that $M$ does not need to be as large as $N$ because the pulses are much shorter anyway. So the relatively short and simple pulse found in [34] realizing $M = 3$ may often be completely sufficient.

For later reference, we point out that the above derivation also holds if we allow for an explicit analytic time dependence of the operators $\hat{A}_0$ and $\hat{A}_1$ in (3). This was recently shown by us in the context of optimized DD for time-dependent Hamiltonians [22] relying only on the mathematical properties of the switching function $F(\vartheta)$. Hence, the same argument also applies to the RUDD sequence if the pulses are shaped to realize zero coupling during their duration, see equation (12). This is definitely the case if there is no time dependence during the pulses because the pulses suggested in [34] and [35] can be used. This is indeed a relevant case as we will discuss below.

We emphasize that the RUDD with (18) provides an efficient scheme for DD based on bounded control. It is the main result of our paper. The previously obstructive no-go theorems [25, 26] can be circumvented by the RUDD approach. The qualitatively novel finding in the present work is that the sequence has to be adjusted in a precise way in order to allow for realistic pulses while preserving the properties of the sequence of ideal pulses. Above, we constructed a precise prescription that achieves the necessary adjustment. We expect that this observation extends beyond the case of UDD and RUDD. This expectation is illustrated in section 4.
We emphasize that the number of pulses \(N\) cannot be made infinite without using shorter and shorter pulses with larger and larger amplitudes. Hence, a given bound to the available power of the control pulses limits the maximum possible number of pulses for a given interval \(T\). But such limits exist in any experimental setup anyway [16, 17, 19] and we expect that the RUDD approach will prove its usefulness for a moderate number of pulses. The limit \(T \to 0\) is studied here to characterize the mathematical properties of the idealized situation. An achievement of the RUDD over the UDD sequence is that for any finite duration \(T\) and finite number of pulses \(N\), only pulses of finite amplitude are needed.

4. Iterated sequences

In view of the above, we expect that the famous CPMG sequence [8, 9] can be improved for realistic pulses as follows. The CPMG is given by the \(n\)-fold iteration of the two-pulse cycle \(t - \pi - 2t - \pi - t\), where \(\pi\) stands for a \(\pi\) pulse and \(t\) for free evolution of time \(t\). This two-pulse cycle is the UDD sequence for \(N = 2\) pulses [12]. Hence, for pulses of finite duration, the iteration of the \(N = 2\) panel in figure 1 suggests itself. A slight modification is possible by replacing two \(2\pi\) pulses, where two cycles meet, by one \(2\pi\) pulse of double the length. Hence, it is promising to use the sequence

\[
(2\pi)_1 \left[ -t_2 - \pi_p - 2t_2 - \pi_p - t_2 - (2\pi)_{2n} \right]^{n-1} - t_2 - \pi_p - 2t_2 - \pi_p - t_2 - (2\pi)_1, \tag{19}
\]

with \(t_1 = 2t (1 - \cos(\theta_p))\), \(t_2 = 2t \sin((\pi/6) - \theta_p)\) and \(\tau_p = 4t \cos(\pi/6) \sin(\theta_p)\). The subscripts indicate the pulse durations. We iterate that the advocated recipe to account for bounded control only applies to UDD-type sequences.

5. Simulation of a RUDD sequence

The advocated RUDD sequence relies on its mathematical properties, which have a certain beauty in themselves. But the ultimate check will be its experimental usefulness. A crucial step achieving this goal is an experiment with simulated noise such as the one performed for UDD [16]–[18]. There, the simulated noise was switched off during the pulse. The theoretical calculations took this dead time of the noise into account. But variable pulse lengths such as in RUDD were not considered.

We propose to implement the RUDD according to (14) with pulses of finite, constant amplitudes during the intervals where \(\tilde{F}(t) = 0\). No pulse shaping is required if the noise is switched off during the pulse so that \(\tilde{F}(t) = 0\) is fulfilled by construction. Hence, we have

\[
\hat{U}_{+}^{\text{RUDD}} - \hat{U}_{-}^{\text{RUDD}} = \mathcal{O}((\gamma T)^{N+1}) \tag{20}
\]

for this particular experiment instead of (18).

The pulse intervals have to be chosen as in (14). Concomitantly, the amplitudes have to vary to ensure that the pulses are \(\pi\) pulses. In this way, any deviation resulting from the pulses is eliminated. It is highly interesting to investigate if such a RUDD sequence is more powerful than existing realizations.
6. Longitudinal relaxation

A UDD sequence can also suppress longitudinal relaxation [15]. Pulses of angle $\pi$ about the $z$-axis can suppress terms proportional to $x$- and $y$-components, i.e. $\sigma_x$ and $\sigma_y$, of the Pauli matrices up to order $(\gamma T)^{N+1}$ for $\{t_j\}$ as in (1). Concatenation of such UDD sequences (CUDD) can be used to suppress any kind of relaxation [20]. The QDD appears to be the most efficient scheme to fulfill this purpose [21, 22].

The pulses depicted in figure 1 and computed in [34] also work to order $(\gamma T)^3$ if used for rotations $\hat{P}_z^\pi$ around the $z$-axis for arbitrary couplings to $\sigma_x$, $\sigma_y$ and $\sigma_z$. The pulse $\hat{P}_z^\pi$ induces an inversion of the sign of the couplings along $\sigma_x$ and $\sigma_y$.

To see this, one has to modify the specific calculation for a rotation about a fixed axis in the paper by Pasini et al [34] according to $\hat{A}_0 \rightarrow \hat{B}_0 = \hat{A}_0 + \hat{A}_z \sigma_z$ and $\hat{\sigma} \cdot \hat{A} \rightarrow \hat{\sigma}_\perp \cdot \hat{A} = \sigma_x \hat{A}_x + \sigma_y \hat{A}_y$ for a Hamiltonian $\hat{H} = \hat{A}_0 + \hat{\sigma} \cdot \hat{A}$. Equation (11) becomes

$$\hat{U}_p(t + \tau_p, t) = e^{-i\tau_p \hat{B}_0} \hat{P}_z^\pi \hat{U}_G(\tau_p, 0),$$

where $\hat{U}_G(\tau_p, 0)$ encodes the corrections. It is given by

$$\hat{U}_G(\tau_p, 0) = T \{ e^{-i B_0^T G(t) dt} \},$$

where the time-dependent Hamiltonian of the corrections $G(t)$ stands for

$$G(t) := e^{i B_0 \hat{P}_z^\pi} (\hat{\sigma}_\perp \cdot \hat{A}) \hat{P}_z^\pi e^{-i B_0 t}$$

with $\hat{P}_z^\pi = \exp(-i \sigma_z \int_0^t ds \; v(s))$ resulting from $H_c(t) = v(t) \sigma_z$ representing the pure control rotation at instant $t$. We show that $\hat{U}_G(\tau_p, 0) = 1 + O((\gamma T)^3)$ if the shaped rotations about $\sigma_z$ proposed in Pasini et al [34] are applied about $\sigma_z$.

The Magnus expansion [3, 37] allows us to write the time evolution in equation (22) in terms of cumulants $U_G(\tau_p, 0) = \exp(-i \tau_p \sum_{i=1}^\infty \eta^{(i)})$. Each cumulant $\eta^{(i)}$ scales as $(\gamma T)^i$. Following the approach of Pasini et al [34], it is straightforward to find

$$\eta^{(1)} = \eta_{11}(\sigma_x \hat{A}_y - \sigma_y \hat{A}_x) + \eta_{12}(\sigma_x \hat{A}_x + \sigma_y \hat{A}_y),$$

with $\eta_{11}$ and $\eta_{12}$ being the first-order corrections. For the second order, one finds $\eta^{(2)} = \eta^{(2a)} + \eta^{(2b)}$ with

$$\eta^{(2a)} = \eta_{21}[\hat{B}_0, \sigma_x \hat{A}_y - \sigma_y \hat{A}_x] + \eta_{22}[\hat{B}_0, \sigma_x \hat{A}_x + \sigma_y \hat{A}_y],$$

(25a)

$$\eta^{(2b)} = 2\eta_{23} \sigma_z \left[ \hat{A}_y, \hat{A}_x \right] + i \left( \hat{A}_x^2 + \hat{A}_y^2 \right).$$

(25b)

The expressions for $\eta_{21}$, $\eta_{22}$ and $\eta_{23}$ are

$$\eta_{11} := \int_0^{\tau_p} dt \; \sin \psi(t),$$

(26a)

$$\eta_{12} := \int_0^{\tau_p} dt \; \cos \psi(t),$$

(26b)
\[ \eta_{21} := \int_0^{\tau_p} dt \, t \sin \psi(t), \quad (26c) \]
\[ \eta_{22} := \int_0^{\tau_p} dt \, t \cos \psi(t), \quad (26d) \]
\[ \eta_{23} := \int_0^{\tau_p} dt_1 dt_2 \sin(\psi(t_1) - \psi(t_2)) \text{sgn}(t_1 - t_2), \quad (26e) \]

where \( \psi(t) := 2 \int_0^t v(t')dt' \). These conditions are exactly the same as those reported by Pasini et al [34] for pure dephasing. Hence they have the same solutions and the pulses depicted in figure 1 make the first- and the second-order corrections vanish also for longitudinal relaxation. No changes in the pulse shapes are required. Up to the third order, the transverse coupling is suppressed and only the \( z \)-coupling survives unaltered. Between two subsequent pulses the sign of the \( x \) and \( y \) coupling is inverted. For pulses corrected in higher order \( M > 3 \) corrections, we again refer to [35] for a proof-of-principle construction. Hence, the RUDD sequence is equally applicable for the suppression of longitudinal relaxation.

Equations (26) hold generally for the suppression of decoherence perpendicular to the fixed axis of rotation of the pulse. The decohering coupling along this axis is not suppressed. The case of pure dephasing can be seen as a special case of the more general case discussed here: there is no coupling along the axis of rotation and only one (out of two possible) perpendicular coupling.

7. Concatenation of RUDD sequences

To tackle general decoherence, a combination of at least two sequences of rotations about perpendicular spin axes is used. Available schemes rely on recursive concatenation as for CDD [11] or CUDD [20] or on a single-step concatenation as for QDD [21, 22]. Hence, it is natural to consider concatenation of RUDD sequences of rotations about two perpendicular spin axes.

For simplicity we consider the QDD scheme that comprises two levels. On the first level, two (e.g. \( \hat{A}_x \) and \( \hat{A}_y \)) of the three couplings \( \hat{A} \) to the components of \( \hat{\sigma} \) are eliminated up to a certain order. This is exactly what is achieved by a RUDD of \( N_z \) rotations about \( z \) for longitudinal relaxation as discussed in the previous section\(^2\). Up to the corrections \( \mathcal{O}(\gamma T_z N_0^{z+1}) + \mathcal{O}(N_0^{y\tau_z,m_0})^M \) of the primary level, the resulting time evolution is given by an effective Hamiltonian that implies dephasing only. Note that \( T_z \) is the duration of the primary RUDD sequence.

The effective Hamiltonian is of the form given in equation (3), but with time-dependent operators \( \hat{A}_0(t) \) and \( \hat{A}_z(t) \). The time dependence of these operators is analytical since it results from the time evolution for the time interval \( T_z \) given by the Schrödinger equation on the primary level. Note that it is understood that all the switching instants are chosen relative to \( T_z \). Then}\(^2\) We use the term ‘longitudinal relaxation’ for the couplings perpendicular to the \( z \)-coupling and eliminate them first. But we stress that in our theoretical treatment no spin axis is special. Thus the first suppression can also be done by \( x \)- or \( y \)-rotations suppressing the corresponding perpendicular couplings, i.e. the \( y \) and \( z \) couplings or the \( x \)- and \( z \)-couplings, respectively.

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a UDD sequence of duration $T_\perp$ can be applied on the secondary level [22], which consists of $N_\perp$ rotations about the spin $x$- or $y$-axis to suppress dephasing up to corrections $O((\gamma T_\perp)^{N_\perp+1})$. This means that general decoherence can be suppressed by a RUDD on the primary and a UDD on the secondary level.

To obtain a quadratic scheme using pulses of finite duration, we use a RUDD also on the secondary level, calling the resulting scheme QRUDD. This is possible since in the derivation of the RUDD for pure dephasing in section 3 we mentioned that the initial Hamiltonian may display an analytic time dependence. This effective time dependence results here from the pulse sequence on the primary level. Thus it is by construction not present during the secondary pulses. These secondary pulses have to be constructed in the presence of general decoherence $\hat{\sigma} \cdot \hat{A}$ so that the explicit solution in the paper by Pasini et al [34] cannot be used. But the equations to be solved are given in sufficient generality in this reference. For a proof-of-existence, we refer to the work of Khodjasteh et al [35] where concatenated solutions for such pulses are constructed recursively.

Hence, the known mathematical properties of UDD sequences and of $\pi$ pulses suffice to conclude that even general decoherence can be efficiently suppressed by DD with bounded control by means of this QRUDD scheme. It is a quadratic scheme of UDD sequences of bounded, and thus essentially realistic, control pulses.

8. Summary

We derived in this paper that an optimized sequence of realistic pulses (RUDD), i.e. of finite duration and amplitude, can be set up that suppresses dephasing or longitudinal relaxation up to $T^{N+1}$ in the length of the sequence and up to $\tau_{mx}^M$ in the maximum duration of the pulses, avoiding all mixed terms in contrast to previous proposals. This statement is based on rigorous analytical calculations for bounded baths and it is expected to apply to systems with hard high-energy cutoff. Our argument is based on the fundamental mathematical property of the optimized sequences of UDD type, namely a certain antiperiodicity in the auxiliary variable $\vartheta$. Thus it only applies to such sequences and not to arbitrary sequences.

We introduced and exploited the concept of double scaling in the durations $\tau_p$ of the pulses and in the duration $T$ of the whole sequence. We emphasize that both scales can be varied independently except for a certain constraint, see equation (2).

The key achievement is to establish a precise prescription on how the sequence has to be adjusted to allow for the use of pulses with bounded amplitudes, which are thus decisively more realistic. Only the adjustment of the sequence to the use of tailored pulses of finite duration allowed us to circumvent no-go theorems [25, 26] concerning the properties of tailored pulses.

The proposed RUDD can be used for suppressing pure dephasing, i.e. suppressing coupling of the bath to one spin component, or for suppressing longitudinal relaxation, i.e. suppressing coupling of the bath to two spin components. General decoherence, i.e. suppressing coupling of the bath to all three spin components, cannot be suppressed by a single RUDD but by a quadratic concatenated scheme (QRUDD) of two RUDDs, made from rotations about two perpendicular spin axes.

Based on the known properties of UDD [12, 13, 15] and QDD [21, 22], we think that the design of the sequences is very close to its optimum. But we expect that the design of the pulses can still be improved. Whereas for $M = 3$ (the leading non-vanishing correction is cubic
in $\tau_p$) rather simple pulse shapes are known [34], for higher order pulses with $M > 3$, recursive concatenation provides a recipe for their construction at the expense of an exponential increase in the number of elementary pulses [35].

Certainly, further research is called for to determine the performance of RUDD and QRUDD for specific models. One important issue is to determine the size of the prefactors of the neglected terms. Another issue on the way to the experimental application of RUDD and QRUDD is to investigate the robustness of both the tailored pulses and the sequences to imperfections such as imprecise timing.

To stimulate further research on the experimental side, we proposed an experimental setup to verify the RUDD for simulated noise that can be switched off [16]–[18], so that a RUDD can be checked without pulse shaping.

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