SCHUBERT PRODUCTS FOR PERMUTATIONS WITH SEPARATED DESCENTS

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Abstract. We say that two permutations \( \pi \) and \( \rho \) have separated descents at position \( k \) if \( \pi \) has no descents before position \( k \) and \( \rho \) has no descents after position \( k \). We give a counting formula, in terms of reduced word tableaux, for computing the structure constants of products of Schubert polynomials indexed by permutations with separated descents, and recognize that these structure constants are certain Edelman-Greene coefficients. Our approach uses generalizations of Schützenberger’s jeu de taquin algorithm and the Edelman-Greene correspondence via bumpless pipe dreams.

1. Introduction

Schubert polynomials are representatives of Schubert classes of the full flag variety. Products of Schubert polynomials expand into a positive \( \mathbb{Z} \)-linear combination of Schubert polynomials, and the coefficients in this expansion are known as the Schubert structure constants. It is a famous open problem in algebraic combinatorics to model the structure constants nonnegatively using combinatorial objects that bypass geometry in full generality; or in other words, to find a \#P-algorithm that computes the structure constants. For a more extensive introduction of this problem, see e.g. [Len10].

The goal of this paper is to make some progress on this problem. We say that a pair of permutations \( \pi \) and \( \rho \) have separated descents at position \( k \) if \( \pi \) has no descents before position \( k \) and \( \rho \) has no descents after position \( k \). This notion was first brought forth by Knutson and Zinn-Justin [KZJ], where they derived a puzzle rule for this problem (that generalizes to equivariant \( K \)-theory) from studying quiver varieties. This perspective of studying Schubert and related problems is surveyed in [Knu]. The rule presented in our work is a tableaux rule and uses only elementary combinatorial methods. In particular, our result generalizes results of Kogan [Kog00], Knutson–Yong [KY04], and Lenart [Len10] where different rules were given for solving the separated descents problem when one of the permutations is Grassmannian (i.e., has only a single descent). One advantage of our approach is that it naturally recognizes the separated-descent structure constants as Edelman-Greene coefficients, which are the coefficients of the expansion of Stanley symmetric functions in the Schur basis. Our rule is also natural for computing products of multiple Schubert polynomials indexed by permutations with separated descents. This gives a generalization of Purbhoo–Sottile [PS09], where products of multiple Grassmannian permutations were considered. We note that even under the condition that every permutation in the indexing set is Grassmannian, our rule is still more general, since it is able to compute the coefficients of all terms in the product, whereas in [PS09] only those coefficients of Grassmannian permutations were considered.
Our main theorem (Theorem 4.2) is as follows:

**Theorem.** Suppose \( \pi, \rho \in S_n \) such that \( \pi \) has no descents before position \( k \) and \( \rho \) has no descents after position \( k \). Define

\[
\pi \star \rho(i) = \begin{cases} 
\pi(i + k) - k & \text{if } i \in [1 - k, 0] \\
\rho(i) + n - k & \text{if } i \in [1, k] \\
\pi(i) - k & \text{if } i \in [k + 1, n] \\
\rho(i - (n - k)) + n - k & \text{if } i \in [n + 1, 2n - k]
\end{cases}
\]

where \( \pi \star \rho \in S_{[1-k,2n-k]} \). Let \( \sigma \in S_{2n-k} \) such that \( \ell(\pi \star \rho) - \ell(\sigma) = \ell((\pi \star \rho)\sigma^{-1}) = k(n-k) \). Let \( \lambda_{k \times (n-k)} \) be the partition of the \( k \times (n-k) \) rectangular shape. The Schubert structure constant \( c_{\pi,\rho}^\sigma \) is equal to the Edelman-Greene coefficient \( j_{\lambda_{k \times (n-k)}}^{\pi \star \rho} \), which is the number of reduced word tableaux \( T \) of shape \( \lambda_{k \times (n-k)} \) such that the permutation given by the reading word of \( T \) (see Section 2.2 for convention) is \( (\pi \star \rho)\sigma^{-1} \). Furthermore, \( c_{\pi,\rho}^\sigma = 0 \) for all other \( \sigma \) (even though the number of tableaux may be nonzero).

We show an example that illustrates the statement of the theorem. Let \( n = 6, k = 3, \pi = 135264, \rho = 513246 \). Then \( \pi \star \rho = [-2, 0, 2, 8, 4, 6, -1, 3, 1, 5, 7, 9] \). We have the following Schubert product expansion:

\[
\mathcal{S}_\pi \mathcal{S}_\rho = \mathcal{S}_{615243} + \mathcal{S}_{534162} + \mathcal{S}_{625134} + \mathcal{S}_{526143} + 2\mathcal{S}_{624153} + \mathcal{S}_{7152346} + \mathcal{S}_{7142536} + \mathcal{S}_{7231546}.
\]

For \( \sigma = 624153 \), there are two Coxeter-Knuth classes of reduced words for \( (\pi \star \rho)\sigma^{-1} \) whose reduced word tableaux are of shape \( 3 \times 3 \). These are:

\[
\begin{array}{ccc}
-1 & 1 & 3 \\
0 & 4 & 5 \\
2 & 6 & 7
\end{array}
\]

\[
\begin{array}{ccc}
-1 & 1 & 3 \\
0 & 2 & 5 \\
1 & 4 & 7
\end{array}
\]

For \( \sigma = 7142536 \), there is one Coxeter-Knuth class of reduced words for \( (\pi \star \rho)\sigma^{-1} \) whose reduced word tableau is of shape \( 3 \times 3 \):

\[
\begin{array}{ccc}
-1 & 1 & 3 \\
0 & 2 & 5 \\
1 & 4 & 7
\end{array}
\]

When \( \pi \) and \( \rho \) both have a single descent at position \( k \), our theorem gives a combinatorial interpretation for the classical Littlewood-Richardson coefficients in terms of certain reduced word tableaux of a rectangular shape, which does not seem to directly match an existing rule in the literature. However, a version of the theorem in terms of the generalization of jeu de taquin on semi-standard Young tableaux can also be stated, and this is given in Theorem 4.4.

We take a moment to elaborate on the journey that led to the discovery. The first idea was to generalize Schützenberger’s jeu de taquin algorithm using bumpless pipe dreams. The key challenge here was to come up with a bumpless pipe dream analogue of skew tableaux. To achieve this, we introduce a simulation of jeu de taquin using semi-standard tableaux, and use the direct bijection between bumpless pipe dreams for \( k \)-Grassmannian permutations and semi-standard Young tableaux of the corresponding shape to obtain a
version of jeu de taquin with bumpless pipe dreams. The second challenge was to figure out to what extent the algorithm can be generalized beyond Grassmannian permutations. It turns out that the separated descents condition fits the bill, and the rectification process by jeu de taquin with bumpless pipe dreams for $k$-Grassmannian permutations uses moves very much like those described in [LLS21], where a bumpless pipe dream version of Edelman-Greene correspondence is given. The full result is hence achieved by considering a generalization of the Edelman-Greene correspondence using bumpless pipe dreams. The obstruction to extending our techniques beyond the separated descent case seems to be the difficulty of finding a bumpless pipe dream analogue of skew semi-standard Young tableaux in direct bijection of the product of the bumpless pipe dreams for the two input permutation. The details of this construction in the separated descent case is introduced in Section 4.1.

The organization of the paper is as follows. Section 2 provides background and sets conventions for the paper. A combinatorial interpretation of the divided difference operators is also given in this section. Section 3 develops the main technical tools, the rectification and insertion algorithms on bumpless pipe dreams, and establishes some necessary properties of these algorithms. Section 4 states and proves our main result, Theorem 4.2, as well as the extension of our result to Schubert products for for multiple permutations with separated descents. Section 5 spells out the connection of our construction to the original jeu de taquin, and Section 6 ends with some concluding remarks.

2. Preliminaries

We introduce some notations. For $m, n \in \mathbb{Z}$, $m \leq n$ we write $[m, n] := \{m, m+1, \cdots, n\}$. Let $S_{[m, n]}$ denote the set of permutations on the alphabet $[m, n]$. As usual, $S_n$ denotes the set of permutations on $\{1, \cdots, n\}$.

2.1. Bumpless pipe dreams and Schubert polynomials. Bumpless pipe dreams were introduced in [LLS21]. A bumpless pipe dream for a permutation $\pi \in S_{[m, n]}$ is a tiling of an $(n-m+1) \times (n-m+1)$ grid with rows and columns indexed by numbers in $[m, n]$ by the following six kinds of tiles, such that $n-m+1$ pipes that travel from the south border and exit from the east border are formed. The tile where two pipes “bump” is forbidden, and hence the name “bumpless pipe dream.”

We also impose the “reducedness” condition, that no two pipes are allowed to cross twice. The pipes are labelled from $m$ to $n$ on the south border, and reading off the labels from top to bottom on the east border gives a permutation in $S_{[m, n]}$. Given a permutation $\pi \in S_{[m, n]}$, we denote the set of bumpless pipe dreams of this permutation by BPD($\pi$). The tiles in a bumpless pipe dreams are indexed with matrix coordinates. The Rothe bumpless pipe dream Rothe($\pi$) is the unique bumpless pipe dream of $\pi$ that does not contain any $\varepsilon$-tiles$^{1}$, which directly corresponds to the Rothe diagram of the permutation $\pi$. It was shown in [LLS21, Section 5.2] any $D \in$ BPD($\pi$) can be obtained from Rothe($\pi$)

$^{1}$These are pronounced “jay” as in “j”, whereas the $\varepsilon$-tiles are pronounced “are".
by performing a sequence of droop moves, which we briefly recall. A droop is a local move that swaps an \( \sqcup \)-tile of pipe \( p \) at position \((a, b)\) with a strictly-southeast blank tile at position \((a + i, b + j)\). Let \( R \) be the rectangle with NW corner \((a, b)\) and SE corner \((a + i, b + j)\). After the droop, \( p \) travels along row \( a + i \) and column \( b + j \), \((a, b)\) becomes blank, and \((a + i, b + j)\) becomes a \( \sqcup \)-tile. The droop is only allowed if every tile in row \( a \) and column \( b \) of \( R \) contains \( p \), \( R \) contains only one \( \sqcup \)-tile at \((a, b)\), and after the droop we obtain a valid bumpless pipe dream. The reverse operation of a droop is called an undroop. We note here that a more general version of droops and undroops will sometimes be used and defined later.

Given \( \pi \in S_n \), define the bumpless pipe dream polynomial \( P_\pi \) by

\[
P_\pi(x) := \sum_{D \in \text{BPD}(\pi)} \text{wt}(D),
\]

where \( \text{wt}(D) := \prod_{(i,j) \in \text{blank}(D)} x_{i} \) and \( \text{blank}(D) \) is the set of blank tiles in \( D \).

Meanwhile, Schubert polynomials were defined by Lascoux and Schützenberger via divided difference operators \( \partial_i \), defined as

\[
\partial_i f := \frac{f - s_i f}{x_i - x_{i+1}},
\]

where \( f \) is a polynomial, and \( s_i \) acts by swapping the variables \( x_i \) with \( x_{i+1} \). The Schubert polynomial \( \mathcal{S}_\pi \) for \( \pi \in S_n \) is defined as follows:

\[
\mathcal{S}_\pi := \begin{cases} 
  x_1^{n-1}x_2^{n-2}\cdots x_{n-1} & \text{if } \pi = n, (n - 1), \cdots, 2, 1 \\
  \partial_i \mathcal{S}_{\pi s_i} & \text{if } \pi(i) < \pi(i + 1).
\end{cases}
\]

It was proved in [LLS21] and in [Wei21] that \( P_\pi = \mathcal{S}_\pi \). We give another proof by directly computing how divided difference operators act on bumpless pipe dream polynomials. The operations defined on bumpless pipe dreams in the proof are analogous to the “mitosis” operations for pipe dreams. See [KM05] and [Mil03].

**Proposition 2.1.** Let \( \pi \in S_n \). If \( \pi(i) < \pi(i + 1) \), then \( \partial_i P_\pi = 0 \). Otherwise, \( \partial_i P_\pi = P_{\pi s_i} \).

**Proof.** Suppose \( D \in \text{BPD}(\pi) \). We define a row move within row \( i \) and \( i + 1 \), which is a slight generalization of the droop move of [LLS21] and recalled above, but for simplicity we also call it a droop. Let \((i, c)\) be the coordinate of an \( \sqcup \)-tile in row \( i \), and \( p \) the pipe that contains this \( \sqcup \). If there is a blank tile \((i + 1, y)\) in row \( i + 1 \) directly below \( p \), the row move can change \( p \) so that \((i, c)\) becomes blank, \((i + 1, y)\) becomes \( \sqcup \), and \( p \) travels in row \( i + 1 \) instead of \( i \) strictly between columns \( c \) and \( y \), and any “kinks” of other pipes between columns \( c \) and \( y \) and blank tiles get shifted up.

We also define a reverse row move, which for simplicity we also call an undroop. Let \((i + 1, y)\) be the coordinate of a \( \sqcup \)-tile in row \( i + 1 \), and let \( p \) be the pipe that contains this \( \sqcup \). If there is a blank tile \((i, c)\) in row \( i \) directly above \( p \), the reverse move can change \( p \) so that \((i + 1, y)\) becomes blank, \((i, c)\) becomes \( \sqcup \), and \( p \) travels in row \( i \) instead of \( i + 1 \) strictly between columns \( c \) and \( y \), and any “kinks” of other pipes between columns \( c \) and \( y \) get shifted down. We also call this an undroop. In Figure 1, we show some examples of these row moves and reverse row moves.
We then define an equivalence relation on BPD(\(\pi\)), where \(D_1 \sim D_2\) if \(D_1\) and \(D_2\) are the same outside of row \(i\) and \(i+1\), and are related by a sequence of droops/undroops as defined above within row \(i\) and \(i+1\). Let \(\mathcal{D}\) be an equivalence class. Consider a bumpless pipe dream \(D\) in an equivalence class \(\mathcal{D}\). If \(e\) is a blank tile in row \(i\) or \(i+1\), there are three cases:

(a) \(e\) is above or below another blank tile,

(b) \(e\) is above or below a pipe \(p\) that exits from row \(j\) with \(j \leq i\) (i.e., \(p = \pi(j)\)), in which case we say “\(e\) is assigned to \(p\)”.

(c) \(e\) is above the pipe \(p = \pi(i+1)\), the pipe that exits from row \(i+1\).

Let \(Q\) denote the set of pipes to which a droop or undroop is available within row \(i\) and \(i+1\) of \(D\). By definition of these moves, we notice that by restricting to row \(i\) and \(i+1\), if a blank tile is assigned to \(q\), it can only be moved by \(q\) and not other pipes. If we fix other pipes and only perform droops/undroops on \(q\), the set of blank tiles that can be moved by \(q\) generate symmetric polynomial \(x_i^\alpha + x_i^{\alpha-1}x_{i+1} + \cdots + x_i^{\alpha+1}\), where \(\alpha\) is the number of such blank tiles. Blank tiles of types (a) and (c) are not affected any droop or undroops.

Let \(m\) be the number of blank tiles of type (c). We define \(P_{\mathcal{D}} := \sum_{D \in \mathcal{D}} \prod_{(i,j) \in \text{blank}(D)} x_i\), and notice that \(P_{\mathcal{D}} = p(x_1, \ldots, x_{i-1}, x_{i+2}, \ldots, x_{n-1})(f(x_i, x_{i+1}) + x_i^m)\), where \(p\) is a monomial in variables not involving \(x_i\) and \(x_{i+1}\), \(f\) is symmetric in \(x_i\) and \(x_{i+1}\). (In fact, \(f\) can be written as a product of symmetric polynomials in \(x_i\) and \(x_{i+1}\) indexed by elements in \(Q\), times \((x_i x_{i+1})^b\), where \(b\) is the number of vertical stacks of two blank tiles in row \(i\) and \(i+1\).) We will show that

\[
\widehat{c}_i P_{\mathcal{D}} := \begin{cases} 
0 & \text{if } m = 0 \\
\frac{P_{\mathcal{D}^*}}{P_{\mathcal{D}}} & \text{otherwise},
\end{cases}
\]

where \(\mathcal{D}^*\) is defined in the argument to follow.

If \(\pi(i) < \pi(i+1)\), the pipes \(\pi(i)\) and \(\pi(i+1)\) do not cross in any element of BPD(\(\pi\)). This implies \(m = 0\) and \(P_{\mathcal{D}}\) is symmetric in \(x_i\) and \(x_{i+1}\), so \(\widehat{c}_i P_{\mathcal{D}} = 0\). Since \(\mathcal{D}\) is arbitrary, \(\widehat{c}_i P_{\pi} = 0\).

Otherwise \(\pi(i) > \pi(i+1)\). Pick \(D \in \mathcal{D}\), let \(y_i\) be the column index of the last \(\uparrow\)-tile on row \(i\) in \(D\), and \(y_{i+1}\) be the column index of the last \(\downarrow\)-tile on row \(i+1\) in \(D\). By assumption, \(y_{i+1} < y_i\). Again, if \(m = 0\) then \(\widehat{c}_i P_{\mathcal{D}} = 0\). If \(m > 0\), we construct a (multi-valued) map \(\phi\) that sends a \(D \in \text{BPD}(\pi)\) to a \(\mathcal{D}' \in \text{BPD}(\pi s_i)\) as follows. Replace the crossing of pipes \(\pi(i)\) and \(\pi(i+1)\) with a \(\uparrow\)-tile, and undroop the NW-elbow into one of these \(m\) boxes in row \(i\), as illustrated in Figure 2. We argue the set of diagrams \(\mathcal{D}'\) that can be obtained from \(D \in \mathcal{D}\), where \(\mathcal{D}\) is an equivalence class in BPD(\(\pi\)) via \(\phi\) form an equivalence class \(\mathcal{D}'\). First notice that if \(D'_1, D'_2 \in \phi(D)\), then \(D'_1 \sim D'_2\) by droops/undroops of the pipe.
Corollary 2.2 \((\text{LLS}21)\). For every \(\pi \in S_n\), \(P_\pi = \mathcal{S}_\pi\).

\begin{proof}
When \(\pi = n, n-1, \ldots, 1\), \(\text{BPD}(\pi)\) contains a singleton Rothe bumpless pipe dream, and \(P_\pi = x_1^{n-1}x_2^{n-2}\ldots x_{n-1}\). The inductive case is given by Proposition 2.1. \qedhere
\end{proof}

2.2. Edelman-Greene correspondence. We recall some results about the Edelman-Greene correspondence, and set some conventions for this paper.

Definition 2.3 (Coxeter-Knuth insertion). Suppose \(P\) is a row-and-column strict increasing tableau. Let \(x\) be a number to be inserted into \(P\). Initialize the row index \(i = 1\), and follow the steps below.

1. If \(x\) is larger than or equal to all numbers in row \(i\) or if row \(i\) is empty, append \(x\) at the end of the row, and stop.
2. Otherwise, let \(z\) be the leftmost number in row \(i\) larger than \(x\).
   a. If \(z = x + 1\) and the value of \(x\) is already present in row \(i\), leave row \(i\) unchanged, increment \(x\) by 1, increment \(i\) by 1, and go to Step 1. Otherwise go to Step 2(b).
   b. Put \(x\) in the position of \(z\) in \(P\) and let \(z\) be the new value of \(x\). Increment \(i\) by 1, then go to Step 1.

Definition 2.4 (Edelman-Greene map). Suppose \(\pi\) is a permutation and \(i = i_1i_2\cdots i_\ell\) is a reduced word of \(\pi\). Initialize \(P^0\) and \(Q^0\) to both be empty tableaux. For each \(j = 1, 2, \ldots, \ell\), construct \(P^j\) by inserting the number \(i_{\ell+1-j}\) into \(P^{j-1}\) using the Coxeter-Knuth insertion, and \(Q_j\) by adding a new box to \(Q^{j-1}\) so that \(P^j\) and \(Q^j\) have the same shape, and fill this box with \(i\). Let the insertion tableau \(P(i)\) be \(P^\ell\) and the recording tableau \(Q(i)\) be \(Q^\ell\). We also say that \(P(i)\) is the reduced word tableau of \(i\).
Remark. Note that we insert a reduced word from right to left into a tableau. This choice is made for the convenience of stating our main theorem, and is consistent with the convention used in [LLS21]. For this reason, we will define our reading order for the insertion tableau to be row-by-row from top to bottom, and right-to-left within each row.

Definition 2.5. The Coxeter-Knuth equivalence on the set of reduced words of \( \pi \) is generated by the following elementary relations:

(a) \( \cdots jik \cdots \sim jki \cdots \) if \( i < j < k \),
(b) \( \cdots ikj \cdots \sim \cdots kij \cdots \) if \( i < j < k \),
(c) \( \cdots i(i+1)i \cdots \sim (i+1)i(i+1) \).

Recall that the descent sets of a standard tableau of shape \( \lambda \) is defined as \( \text{Des}(S) := \{ j : j + 1 \text{ appears in a lower row than } j \} \), and the descent set of a reduced word \( i = i_1 \cdots i_l \) is \( \text{Des}(i) = \{ j : i_j > i_{j+1} \} \). We write the reverse of a reduced word \( i \) as \( i^{-1} \).

Theorem 2.6 (Edelman-Greene correspondence [EG87]). The map

\[
\text{EG} : i \mapsto (P(i), Q(i))
\]

is an injective map from the set of reduced words for \( \pi \), \( \text{red}(\pi) \), to the set of pairs of tableaux \((P, Q)\) where \( P \) is a row-and-column strict increasing tableau and \( Q \) a standard tableau. This map has the following properties:

(a) For each \( P \), every standard tableau \( Q \) with the same shape as \( P \) appears in the image.

(b) Two reduced words of \( \pi \) are Coxeter-Knuth equivalent if and only if they have the same insertion tableau.

(c) If \( \text{EG}(i) = (P, Q) \) and \( Q' \) is another standard tableau, then \( \text{EG}^{-1}(P, Q') \) is Coxeter-Knuth equivalent to \( i \).

(d) For each \( i \in \text{red}(\pi) \), \( \text{Des}(i^{-1}) = \text{Des}(Q(i)) \).

We now recall some basic facts about Stanley symmetric functions. For \( \pi \) a permutation, the set of reduced compatible sequences \( \text{RCS}(\pi) \) is defined as \( \text{RCS}(\pi) := \{(i, r) : i = i_1i_2\cdots i_{\ell(\pi)} \in \text{red}(\pi), r = r_1\cdots r_{\ell(\pi)}, 1 \leq i_1 \leq \cdots \leq i_{\ell(\pi)}, i_j < i_{j+1} \implies r_j < r_{j+1}\} \). The Stanley symmetric functions \( F_\pi \) is defined as

\[
F_\pi = \sum_{(i, r) \in \text{RCS}(\pi)} x_r,
\]

where \( x_r = x_{r_1} \cdots x_{r_{\ell(\pi)}} \) for \( r = r_1 \cdots r_{\ell(\pi)} \). It is well-known that the expansion \( F_\pi = \sum_\lambda j_\lambda^\pi s_\lambda \) of \( F_\pi \) into the Schur basis has positive coefficients known as the Edelman-Greene coefficients. The coefficient \( j_\lambda^\pi \) has the combinatorial interpretation of counting the number of reduced word tableaux of shape \( \lambda \) for \( \pi \). This can be proved by a variant of Theorem 2.6 that bijects RCS(\( \pi \)) with the set of pairs of \((P, Q)\) where \( P \) is a row-and-column strict increasing tableau whose reading word is a reduced word of \( \pi \), \( Q \) a decreasing semi-standard Young tableau of the same shape as \( P \). (\( Q \) being decreasing is due to our convention that inserts a reduced compatible sequence from right to left.)

We also recall some facts about the recording tableaux for the Edelman-Greene correspondence.
**Definition 2.7.** Let $Q \in \text{SYT}(\lambda)$. For $1 \leq i \leq |\lambda| - 2$, define the **elementary dual equivalence** $h_i$ as an action on $Q$ such that $h_i$ fixes all $j$ for $j \notin \{i, i+1, i+2\}$, and if the reading word of $Q_{\lfloor i,i+1,i+2 \rfloor}$ is of the form $xiy$ or $x(i+2)y$, swaps the entries $x$ and $y$ in $Q$, and fixes it otherwise.

It is known that the set of standard Young tableaux are connected by the elementary dual equivalences, see e.g. [Hai92, Proposition 2.14]. Furthermore, dual equivalence characterizes the Coxeter-Knuth moves on reduced words, see e.g. [EG87, Theorem 6.24], [Ham14, Proposition 2.2(a)].

2.3. Jeu de taquin. We recall Schützenberger’s **jeu de taquin** algorithm on semi-standard skew tableaux. Our reference is [Ful97, Chapter 1].

Let $T$ be a semi-standard skew tableau of shape $\nu/\mu$. The jeu de taquin algorithm on $T$ rectifies $T$ into a semi-standard tableau by iterating the following steps.

1. Pick a SE-most empty box in the NW empty region to be “active”.
2. Repeat this step until the current active empty box is on the SE border: pick the smaller number from the box to the right and the box directly below the active empty box, and slide that box into the active empty box. If there is a tie, pick the box below. The newly created empty box after sliding becomes the new active empty box.

It is well-known that the final output is independent of the choice made in in Step (1). Given partitions $\lambda, \mu, \nu$, the Littlewood-Richardson coefficient $c_{\lambda,\mu}^\nu$ is the number of skew semi-standard Young tableaux of shape $\lambda \ast \mu$ that are jeu de taquin equivalent to some fixed semi-standard Young tableau of shape $\nu$, where $\lambda \ast \nu$ is obtained by placing the partition diagram of $\mu$ immediately NE of the diagram of $\lambda$.

3. Rectification and Insertion

In this section, we introduce a reverse insertion algorithm, which we call rectification, on bumpless pipe dreams with a partition of marked blank tiles at the NW corner. The terminology “rectification” comes from the rectification algorithm jeu de taquin on skew semi-standard tableaux, and this connection is explained in Section 5. We also introduce an insertion algorithm of a reduced word into a bumpless pipe dream and when successful, produces a bumpless pipe dream with a partition of marked blank tiles at the NW corner. Rectification and insertion are inverses of each other. Our rectification and insertion generalize those introduced in [LLS21] that realize the Edelman-Greene correspondence with bumpless pipe dreams, and will be the main technical tool to give a combinatorial interpretation of the separated descent Schubert structure constants, as well as relating those to the Edelman-Greene coefficients.

In the rest of the paper, when we say $\sigma \in S_{[a,b]}$ and $F \in \text{BPD}(\sigma)$, we specifically consider $F$ as a bumpless pipe dream with rows and columns indexed by $[a,b]$. 
3.1. Rectification. Let $\sigma \in S_{[a,b]}$. Define $\text{BPD}^x(\sigma)$ to be the set of bumpless pipe dreams of $\sigma$ such that each tile in a (possibly empty) partition contained inside the connected region of blank tiles at the NW corner of $F$ is marked with “\$”. We require the NW corner of this partition to be anchored at the NW corner of the bumpless pipe dream. Given $F \in \text{BPD}^x(\sigma)$, let $\lambda(F)$ denote the partition formed by tiles marked with “\$”, and let $Q \in \text{SYT}(\lambda(F))$. We will now describe a rectification process on $F$ that iteratively removes these marked tiles according to the order specified by $Q$ and produces a reduced word of length $|\lambda|$. During the entire rectification process, we also will make sure to maintain the invariance of the number of non-marked blank tiles in each row.

To start, let $i$ be the empty reduced word. Let $Q'$ be a working copy of $Q$. In each iteration of the rectification, find the coordinate $(x,y)$ of $Q'$ with the largest value, and delete this box from $Q'$. We then pick the marked blank tile at $(x,y)$ to be the active one and perform the following operations.

1. If the current marked tile is not the rightmost tile of a contiguous block of blank tiles on a row, slide the mark to the rightmost tile.
2. Suppose the current mark is at position $(i,j)$. Let $p$ be the pipe that passes through $(i,j + 1)$.
   1. If pipe $p$ contains a $\square$ in column $j + 1$, let $(i',j+1)$ be the coordinate of this $\square$-tile. In other words, $i' > i$ is the smallest such that $(i',j+1)$ is a $\square$-tile. First temporarily ignore the pipes that cross $p$ in columns $j$ and $j+1$ and between row $i$ and $i'$. (Note that since no two pipes can cross twice, $p$ must be the only pipe that passes through $(i',j)$. ) Undroop $p$ at $(i',j+1)$ into $(i,j)$ so that $(i,j)$ becomes an $\square$-tile and $(i',j+1)$ becomes a blank tile. Move the mark at $(i,j)$ to $(i',j+1)$. Now adjust pipes that cross $p$ between row $i$ and $i'+1$ so their “kinks shift right”. See Figure 3. Go back to step (1).
   2. Otherwise, $p$ originates from column $j+1$, i.e., $p = j+1$. In this case, pipes $j$ and $j+1$ must cross at some tile $(i',j+1)$ with $i' > i$. replace this cross with a $\square$-tile, and resolve this bump by undrooping the NW-elbow into $(i,j)$, and adjust other pipes if necessary in a similar fashion as in Step (2a), as shown in Figure 4. This move decrements the number of marked tile by 1 and is the end of this iteration of rectification. Append $j$ to the end of the reduced word $i$. If $F'$ is the marked bumpless pipe dream in the beginning of this iteration, define $\text{pop}(F',(x,y))$ to be $j$ and $\text{\nabla}(F',(x,y))$ the resulting (marked) bumpless pipe dream with one fewer mark than $F'$.

The process completes when $Q' = \emptyset$ and there are no marked tiles in the grid. Denote the output reduced word $i$ as $\Psi(F,Q)$, and the bumpless pipe dream at the end of the algorithm $\text{rect}(F,Q)$.

Example 3.1. Figure 5 shows an example of an iteration of the rectification algorithm. Suppose the partition of marked tiles is anchored at $(1,1)$. This iteration produces the simple reflection $s_7$. If $Q \in \text{SYT}((3,2))$ determines that the order for removal of the outer boxes of marked tiles to be $(2,2)$, $(1,3)$, $(2,1)$, $(1,2)$, $(1,1)$, the resulting reduced word $i = (7,9,6,8,3)$ (or $s_7s_9s_8s_6s_8s_3$ in conventional notation).
Remark. These column moves are slight generalizations of (the backward direction of) the column moves defined in [LLS21, Section 5.7] for a bumpless pipe dream description of the Edelman-Greene correspondence. The only difference is that here blank tiles strictly between rows $i$ and $i'$ are allowed, whereas in [LLS21] they are forbidden. In Step (1) we have the extra sliding move that sends the marked tile across a contiguous block of blank tiles from one end to the other, which was not needed in their setting. This simple generalization has powerful consequences. In Section 5, we relate these moves to jeu de taquin on semi-standard Young tableaux. Because both jeu de taquin and the Edelman-Greene insertion are realized by these column moves, we are able to connect the separated-descent
Schubert structure constants to Edelman-Greene coefficients. The same column moves are also used in constructing the canonical bijection between pipe dreams and bumpless pipe dreams [GH21].

3.2. Insertion. We now describe an insertion algorithm, which is the reverse of the rectification algorithm. Let $F \in \text{BPD}^\times(\sigma)$ for some $\sigma \in S_{[a,b]}$. We emphasize that we consider the NW corner of $F$ to be position $(a,a)$. Let $a \leq j < b$. If pipes $j$ and $j+1$ already cross, the insertion of $j$ into $F$, denoted $F \leftarrow j$, is not defined. Otherwise, let $(i,j)$ be the location of the $j$-tile with $i$ largest, and $(i',j+1)$ the location of the $(j+1)$-tile with $i'$ largest. Perform the reverse of the terminal column move as described in Step (2b) of the rectification algorithm (Figure 4), which crosses pipes $j$ and $j+1$ at $(i',j+1)$ and creates a blank tile at $(i,j)$. Place a mark at $(i,j)$ and let it be the active one.

1. If the active mark is not the leftmost tile of a contiguous block of unmarked blank tiles in a row, slide it to the leftmost one.
2. Suppose the active marked tile is at $(i,j)$, and the pipe that passes through $(i,j-1)$ is $p$, if it exists.
   (a) Let $(i',j-1)$ be the coordinate of the $i'$-tile with $i' < i$ largest. Perform the reverse of the column move described in Step (2a) of the rectification algorithm (Figure 3) so that $p$ droops into $(i,j)$, pipes intersecting $p$ between row $i$ and $i+1$ have their “kinks shift left” within columns $j$ and $j+1$, and $(i',j-1)$ becomes a blank tile. Move the mark from $(i,j)$ to $(i',j-1)$, and go back to Step (1).
   (b) If $p$ does not exist and the active mark is connected to the existing partition of marked tiles, forming a new partition with one more box, (or at $(a,a)$ if no tiles were initially marked,) terminate the algorithm with success, and the resulting bumpless pipe dream with marked blank tile is denoted $F \leftarrow j$. If the active mark is disconnected to the existing partition of marked tiles, the insertion algorithm fails and $F \leftarrow j$ is undefined.

Define the insertion footprints for $F \leftarrow j$ as the set of positions the mark appears at that later become $j$-tiles during the insertion process, plus its final position. We observe that an insertion footprints set consists of tiles that run SW to NE, with no two tiles on the same row or column.

Example 3.2. If we reverse the arrows in Figure 5, we get an example of insertion $j$ into the bottom-left bumpless pipe dream. The insertion footprints consists of coordinates $(5,7), (4,6), (3,3), (2,2)$, namely the circled positions except for $(3,5)$.

Having defined the insertion of a single simple reflection, we define the insertion of a reduced word into a $F \in \text{BPD}^\times(\sigma)$. Let $w \in \mathfrak{S}_{[a,b]}$ and $i = i_1 i_2 \cdots i_l$ a reduced word for $w$. Suppose $\ell(w\sigma) = \ell(w) + \ell(\sigma)$. We define $F \leftarrow i$ as $(((F \leftarrow i_l) \leftarrow i_{l-1}) \leftarrow \cdots) \leftarrow i_1$, if every iteration of the insertion is successful. Otherwise, $F \leftarrow i$ is undefined. If $\lambda(F) = \emptyset$, i.e., $F \in \text{BPD}(\sigma)$, the saturated chain of partitions $\lambda(F \leftarrow i_l) \subset \lambda((F \leftarrow i_l) \leftarrow i_{l-1}) \subset \cdots \subset \lambda(F \leftarrow i)$ gives rise to a standard Young tableau of shape $\lambda(F \leftarrow i)$, which we denote by $Q(F \leftarrow i)$. 

When $F$ is the identity bumpless pipe dream, the insertion of $i$ into $F$ recovers the Edelman-Greene correspondence with bumpless pipe dreams, as described in [LLS21, Section 5]. The insertion algorithm described above is a generalization that allows $F$ to be an arbitrary bumpless pipe dream of an arbitrary permutation $\sigma \in S_{[a,b]}$ with the caveat that the insertion might fail, even when $\sigma$ and the reduced word being inserted are length-additive. However, this failure can always be remedied by considering $\sigma$ as a permutation in $S_{[a',b]}$ for some $a' < a$ small enough, or in other words, enlarge $F$ by adding enough pipes that give the identity permutation outside the NW corner of $F$. This is always possible given $i$. We refer to this process as back-stabilizing $F$. This idea will be useful later, cf. Lemma 3.13. The idea of back-stabilizing has showed up in recent work of Pechenik and Weigandt [PW22] where they gave a positive rule for Schubert products for inverse Grassmannian permutations, even though completely different combinatorial objects are used.

**Example 3.3.** In Figure 6, the bumpless pipe dream $F$ is the result of rectification of the top-left bumpless pipe dream $D$ in Figure 5. Therefore, the insertion $F \leftarrow (7, 9, 6, 8, 3)$ results in $D$. Note that the insertion $F \leftarrow (6, 7, 9, 6, 8, 3) = D \leftarrow 6$ is not defined. However, if we back-stabilize $F$ by adding one more pipe numbered 0 and get $F', F' \leftarrow (6, 7, 9, 6, 8, 3)$ is defined.

![Figure 6. Back-stabilization remedies failures of insertion](image-url)
Figure 7. Insertion footprints for $F \leftarrow i$ (green) and $(F \leftarrow i) \leftarrow j$ (red) for $i < j$ in the bumpless pipe dream $F \leftarrow i$

adjacent to any blank tiles, because each column move always replaces the top-left $\square$ in the bounding rectangle with a mark.

$\square$

3.3. Properties of insertion and rectification. In this subsection, we discuss properties of insertion and rectification. Corollary 3.7, Propositions 3.10 and 3.12 are the main properties of rectification and insertion required to prove Theorem 4.2, the main theorem. Lemmas 3.5, 3.6, 3.8, 3.9 are technical analysis of the insertion algorithms required to prove Propositions 3.10 and 3.12. Finally, Corollary 3.11 and Lemma 3.13 give some necessary conditions under which insertions are defined, which are necessary for the main theorem.

Lemma 3.5. Let $F \in \text{BPD}^\Lambda(\sigma)$ and suppose $i < j$. When all insertions are defined,

1. The insertion footprints for $(F \leftarrow i) \leftarrow j$ are strictly to the N/E/NE of the insertion footprints for $F \leftarrow i$. Or in other words, no insertion footprint of $F \leftarrow i$ is N/E/NE of any insertion footprint of $(F \leftarrow i) \leftarrow j$.

2. The insertion footprints for $(F \leftarrow j) \leftarrow i$ are strictly to the S/W/SW of the insertion footprints for $F \leftarrow j$.

Proof. We show (1); (2) is similar.

Suppose the first tile in the insertion footprints of $F \leftarrow i$ is $(x_1, y_1)$, and consider the bumpless pipe dream $F \leftarrow i$. We know that $y_1 \leq i$. We consider the set of tiles that either lies in the “upper hook” inside the width 2 rectangle that bounds the column move that has happened for inserting $i$, or in a contiguous block of blank tiles on a row that bounds the “sliding” move of the marked tile. Note that the insertion footprints are the tiles at the SW corners of the green “zigzag” strip illustrated in Figure 7.

Notice that the southernmost possible coordinate of the lowest $\square$-tile in column $i + 1$ is $(x_1, i + 1)$. Therefore, the first tile in the insertion path of $(F \leftarrow i) \leftarrow j$ with column index no greater than $i + 1$ is at or above row $x_1$. If it is at row $x_1$, its coordinate must be $(x_1', y_1') = (x_1, y_1 + 1)$. In this case, the insertion footprints for $(F \leftarrow i) \leftarrow j$ are exactly the set of tiles immediately to the right of each tile in the insertion footprints of
(F ← i). Otherwise, it is on (x', y') with x' < x. We consider the set of tiles that lie in the “lower hook” inside the width 2 rectangles that bound column moves that are about to happen for inserting j, union the contiguous blocks of blank tiles on a row that bound the sliding moves of the marked tile. The insertion footprints of (F ← i) ← j are the tiles immediately to the right of the SW corners of this zigzag strip. Now imagine travelling from (x', y') along this region. Since going west horizontally allows only the use of blank tiles, we may never cross past the green strip. In the case when two red and green vertical segments coincide, we end up in the same situation as the first case. Thus our claim about the insertion footprints follows. □

Running the proof of Lemma 3.5 backwards, we have the analogous statements for rectification.

Lemma 3.6. Let F ∈ BPD^x(σ), and b_1 := (x_1, y_1), b_2 := (x_2, y_2) be two outerboxes of the partition formed by marked blank tiles of F, such that x_1 > x_2. Namely, b_1 lies SW of b_2. Let F_1 := ∇(F, b_1) and F_2 := ∇(F, b_2). Then

(1) pop(F, b_1) < pop(F_1, b_2)
(2) pop(F, b_2) > pop(F_2, b_1).

Recall that Ψ(F, Q) denotes the reduced word obtained by the rectification algorithm on F in the order specified by Q. The following Corollary follows from Lemma 3.6.

Corollary 3.7. Let F ∈ BPD^x(σ) and Q ∈ SYT(λ(F)), and let i := Ψ(F, Q). Then Des(i^{-1}) = Des(Q).

Lemma 3.8. Let F ∈ BPD^x(σ) and suppose i < j < k. When all the insertions are defined,

(1) ((F ← j) ← i) ← k = ((F ← j) ← k) ← i
(2) ((F ← i) ← k) ← j = ((F ← k) ← i) ← j.

Proof. The proof is similar to the proof of [LLS21, Lemma 5.25]. In particular, we supply the details omitted in part (2) of their proof.

To show (1), we notice that since the insertion footprints of (F ← j) ← i lie SW of the insertion footprints of (F ← j) by Lemma 3.5, the area NE of the insertion footprints is not affected by the insertion of i. Similarly, the area SW of the insertion footprints (F ← j) is not affected by the insertion (F ← j) ← k. Therefore, after the insertion (F ← j), the insertions of i and k commute.

To show (2), first we consider the case when there is no tile (x, y) in the insertion footprints of F ← i such that (x, y + 1) is in the insertion footprints of (F ← i) ← k. In this case, the insertion footprints of (F ← i) still are $\Box$-tiles in the bumpless pipe dream (F ← i) ← k. This means that we can undo the insertion of i in (F ← i) ← k and get (F ← k). In other words, in this case (F ← i) ← k = (F ← k) ← i.

Otherwise, let (x, y) be the first tile in the insertion footprints of F ← i such that (x, y + 1) is in the insertion footprints of (F ← i) ← k. This means that the insertion of i and k before either path reaches (x, y) commute, and therefore the footprints of ((F ← i) ← k) ← j are the same as ((F ← k) ← i) ← j in this initial segment and is NE of the path for i and SW of the path for j.
Now consider the situation where we insert $i$ until the marked tile reaches $(x_i, y + 1)$ for some $x_i > x$, and then insert $k$ until the marked tile reaches the furthest tile $(x_k, y_k)$ where $x < x_k < x_i$ and $y_k > y + 1$. (Note that $(x_k, y_k)$ might be the rightmost tile of some contiguous block of blank tiles, in which case it will not become a footprint. It’s easy to check given our conditions these tiles must exist.) Then, we insert $j$ until it reaches the furthest tile $(x_j, y_j)$ with $y_j > y + 1$ and $x_j > x_k$. One can check that it must be the case that $x_j \leq x_i$ and $y_j \leq y_k$.

We now perform three droops moves that advance the marked tiles for $i, j, k$ in that order. Notice that all three moves belong to the same active pipe, and the results are the same as if the moves were done in the order $k, i, j$, as illustrated in Figure 8. From the diagram on the left, we continue the insertion process by completing the insertion of $i$, and then $k$, and finally $j$. From the diagram on the right, we do so in the order $k, i, j$. By similar reasoning as in part (1), we see that these result in the same diagram when finished. □

**Lemma 3.9.** Let $F \in \text{BPD}^\times (\sigma)$. When all the insertions are defined, $((F \leftarrow i) \leftarrow i + 1) \leftarrow i = ((F \leftarrow i + 1) \leftarrow i) \leftarrow i + 1$.

**Proof.** The same reasoning in the proof of [LLS21, Lemma 5.26] applies here. □

**Proposition 3.10.** Let $P$ be a bumpless pipe dream with a region of marked blank tiles of shape $\lambda$ at its NW corner, and let $Q$ be a standard tableau of shape $\lambda$. Then $C := \{\Psi(P, S) : S \in \text{SYT}(\lambda)\}$ forms a single Coxeter-Knuth equivalence class. Furthermore, the result of rectification $\text{rect}(P, Q)$ is independent of $Q$. 

![Figure 8. Advancing marks for $i, k, j$ vs. advancing marks for $k, i, j$. Blue represents $i$, red represents $k$, and green represents $j$. The $\square$-tile in the top diagram has coordinate $(x, y)$.](image-url)
Proof. By running the proof of Lemmas 3.8 and 3.9 backwards, we may show that if \( Q, Q' \in \text{SYT}(\lambda) \) are dual equivalent, then \( \text{rect}(P, Q) \) and \( \text{rect}(P, Q') \) are the same, and \( \Psi(P, Q) \) and \( \Psi(P, Q') \) differ by a single Coxeter-Knuth move. We omit the details. Therefore by the fact that dual equivalences connect all \( \text{SYT}(\lambda) \), the words in \( C \) are all Coxeter-Knuth equivalent, and \( \text{rect}(P, Q) \) is independent of \( Q \). \( C \) is a single Coxeter-Knuth class because it is in bijection with \( \text{SYT}(\lambda) \). \( \square \)

Because rectification is independent of order, we write \( \text{rect}(P) := \text{rect}(P, Q) \) for any \( Q \in \text{SYT}(\lambda(P)) \).

The following Corollary is immediate from Proposition 3.10.

Corollary 3.11. Let \( F \) be a bumpless pipe dream and \( i \) a reduced word. If \( F \leftarrow i \) is defined, then for any \( i' \) Coxeter-Knuth equivalent to \( i \), \( F \leftarrow i' \) is also defined.

The following proposition generalizes [LLS21, Theorem 5.19].

Proposition 3.12. Let \( \sigma, w \in S_{[a, b]} \) such that \( \ell(w \sigma) = \ell(w) + \ell(\sigma) \). Suppose \( i \) is a reduced word of \( w \) and \( F \in \text{BPD}(\sigma) \). Denote the shape of the reduced word tableau of \( i \) by \( \lambda(i) \).

The following statements hold.

1. For a fixed \( P \in \text{BPD}(w \sigma) \) with \( \ell(w) \) marked tiles, the set \( C_P = \{ j \in \text{red}(w) : F \leftarrow j = P \} \), when non-empty, is a single Coxeter-Knuth equivalence class.
2. If \( F \leftarrow i \) is defined, \( \lambda(F \leftarrow i) = \lambda(i) \).

Proof. For (1), that \( C_P \) is a union of Coxeter-Knuth equivalence classes follows from Lemmas 3.8, 3.9, and Corollary 3.11. It forms a single Coxeter-Knuth class follows from Proposition 3.10.

For (2), suppose \( F' := F \leftarrow i \) is defined. Let \( C(i) \) be the Coxeter-Knuth equivalence class of \( i \). By Proposition 3.10, \( C(i) = \{ \Psi(F', S) : S \in \text{SYT}(\lambda(F')) \} \).

By Corollary 3.7,

\[
\{ \text{Des}(S) : S \in \text{SYT}(\lambda(F')) \} = \{ \text{Des}(i^{-1}) : i \in C(i) \}.
\]

If \( C \) is Coxeter-Knuth equivalence class and \( \lambda_C \) the shape of the reduced words tableaux of words in \( C \), The equality of multisets \( \{ \text{Des}(j^{-1}) : j \in C \} = \{ \text{Des}(S) : S \in \text{SYT}(\lambda_C) \} \) is a property of Edelman-Greene correspondence. Also recall from the theory of fundamental quasi-symmetric functions that the multiset \( \{ \text{Des}(S) : S \in \text{SYT}(\lambda) \} \) uniquely determines \( \lambda \). It follows that \( \lambda(F') = \lambda(i) \). \( \square \)

Lemma 3.13. Let \( \sigma \in S_b, i \) a reduced word of \( w \in S_{[a, b]} \) where \( a \leq 0 < b \). Assume \( \ell(w \sigma) = \ell(w) + \ell(\sigma) \). Let \( P(i) \) be the reduced word tableau for \( i \). If the number of rows of \( P(i) \) is no greater than \( 1 - a \), then for any \( F \in \text{BPD}(\text{id}_{[a, 0]} \oplus \sigma) \) where \( \text{id}_{[a, 0]} \oplus \sigma \in S_{[a, b]} \), \( F \leftarrow i \) is defined.

Proof. Recall that we read a word from right to left when performing insertion, which is the same as inserting \( i^{-1} \) from left to right. By Corollary 3.11, we may assume without loss of generality that the recording tableau \( Q(i) \) reads \( 1, \cdots, \ell(i) \) when read row-by-row from top to bottom, and within each row left to right. Suppose \( Q(i) \) has \( m \) entries in the last row. By Lemma 3.4, since there are no unmarked blank tiles in the non-positive rows, the first \( \ell(i) - m \) iterations of insertion of \( i \) must be successful. Since \( \text{Des}(i^{-1}) = \text{Des}(Q(i)) \)
the last \( m \) letters of \( 1^{-1} \) must be increasing. By the proof of Lemma 3.5 (1) as well as the fact that there are no unmarked blank tiles in non-positive rows, we may see that the insertion of the remaining letters must also be successful, by similar reasoning as in Lemma 3.4.

\[ \square \]

4. Schubert products for permutations with separated descents

In this section, we first introduce a construction that generalizes the construction of placing a semi-standard Young tableau NE of another and creating a skew tableau which can then be rectified, using bumpless pipe dreams with marked blank tiles as introduced in the previous section. The explicit connection will be explained in Section 5. We then prove our main theorem in Section 4.2 using the properties we developed for insertion and rectification in Section 3.1.

4.1. The star operation for permutations with separated descents. We say that a permutation \( \pi \in S_n \) has a descent at position \( i \) if \( \pi(i) > \pi(i + 1) \). We denote the set of \( \pi \)'s descents \( \text{Des}(\pi) := \{i \in \{1, 2, \cdots, n-1\} : \pi(i) > \pi(i + 1)\} \).

Suppose \( \pi, \rho \in S_n \) have separated descents at position \( k \); namely for all \( i \in \text{Des}(\pi) \), \( i \geq k \) and for all \( i \in \text{Des}(\rho) \), \( i \leq k \). We define an operation \( \ast \) on permutations \( \pi \) and \( \rho \), where \( \pi \ast \rho \in S_{[1-k,2n-k]} \), as follows:

\[
\pi \ast \rho(i) = \begin{cases} 
\pi(i + k) - k & \text{if } i \in [1 - k, 0] \\
\rho(i) + n - k & \text{if } i \in [1, k] \\
\pi(i) - k & \text{if } i \in [k + 1, n] \\
\rho(i - (n - k)) + n - k & \text{if } i \in [n + 1, 2n - k].
\end{cases}
\]

We emphasize that the symmetric group \( S_{[1-k,2n-k]} \) is on \( 2n \) numbers that involve non-positive numbers.

Now let \( D \in \text{BPD}(\pi) \) and \( E \in \text{BPD}(\rho) \). We define a procedure to produce a bumpless pipe dream with marked blank tiles \( D \ast E \) in the \( 2n \times 2n \) grid with indices \( 1-k, \cdots, 2n-k \) by the following construction.

We separate \( D \) into two blocks, \( D_{\text{top}} \) and \( D_{\text{bot}} \), where \( D_{\text{top}} \) contains the first \( k \) rows of \( D \) and \( D_{\text{bot}} \) contains the last \( n-k \) rows of \( D \). Also separate \( E \) into \( E_{\text{top}} \) and \( E_{\text{bot}} \) in a similar fashion. Pictorially, \( D = \frac{D_{\text{top}}}{D_{\text{bot}}} \) and \( E = \frac{E_{\text{top}}}{E_{\text{bot}}} \). Since the first \( k \) values of \( \pi \) are in increasing order, properties of bumpless pipe dreams ensure that the NE corner of \( D \) looks like

\[
\begin{array}{ccc}
\end{array}
\]

with a total of \( k \) rows. Call this \( \nabla_k \). We now build \( D \ast E \in \text{BPD}^\pi(\pi \ast \rho) \) according to the schema shown in Figure 9.

Notice that there are \( k \) vertical pipes in the region between \( E_{\text{top}} \) and \( E_{\text{bot}} \), connecting the two parts and making the diagram a valid bumpless pipe dream. The region marked with \( X \) in the NW corner are blank tiles of the \( k \times (n-k) \) rectangular shape, each marked with an “\( x \)”. 
Let $\text{BPD}^{\boxtimes}(\pi \star \rho) \subset \text{BPD}^X(\pi \star \rho)$ denote the set of bumpless pipe dreams with marked blank tiles of $\pi \star \rho$ such that the first $k$ rows are of the form that the initial $k$ columns consist of marked blank tiles, followed by $k$ non-intersecting hook-shaped pipes that completely occupy the $k \times (n + k)$ remaining columns. This looks like the first $k$ rows in Figure 9.

By construction, it is easy to see that $D \ast E \in \text{BPD}^{\boxtimes}(\pi \star \rho)$. For a concrete example, see Figure 10.

**Lemma 4.1.** There is a direct bijection between $\text{BPD}^{\boxtimes}(\pi \star \rho)$ and $\text{BPD}(\pi) \times \text{BPD}(\rho)$. In particular, if $F \in \text{BPD}^{\boxtimes}(\pi \star \rho)$ corresponds to $(D, E) \in \text{BPD}(\pi) \times \text{BPD}(\rho)$ under the bijection, then for each $i$ the number of unmarked blank tiles in row $i$ of $F$ is equal to the total number of blank tiles in row $i$ of $D$ and $E$.

**Proof.** Suppose $F \in \text{BPD}^{\boxtimes}(\pi \star \rho)$. By the definition of $\pi \star \rho$, the last descent of $\pi \star \rho$ is less than $n$, and therefore the rows $[n + 1, 2n - k]$ of $F$ do not contain any blank tiles and are completely determined by the rows above. First consider the $(n - k) \times n$ region of $F$ with rows $[k + 1, n]$ and columns $[1 - k, n - k]$. The pipes $\pi \star \rho(i) = \pi(i) - k \leq n - k$ for $i \in [k + 1, n]$. This means that these $n - k$ pipes travel from the south edge of the region and exit from the east edge of this region. The pipes $\pi(i + k) - k \leq n - k$ for $i \in [1 - k, 0]$ travel from the south edge and exit from the north edge of this region. Their configuration agrees with the last $(n - k)$ rows of a bumpless pipe dream of $\pi$. 
Now consider the $k \times n$ region of $F$ with rows $[1,k]$ and columns $[1-k,n-k]$. By the definition of $\pi \ast \rho$, since $\rho(i) + n - k > n$, the only pipes in this region have indices $\pi(i) + k - n - k$ for $i \in [1-k,0]$, and all pipes with indices $\pi(i) + k - n - k$ for $i \in [1-k,0]$ enter this region from the south edge and exit from columns $[n-k+1,n]$ of the north edge without ever crossing each other. If we reroute these pipes by modifying the tiles in the triangular region (weakly) above the $k$th diagonal counting from the NE corner of this region, so that the pipes exit from the east edge instead, we can make this region agree with the first $k$ rows of a bumpless pipe dream of $\pi$, which can be combined with the bottom region to make up a bumpless pipe dream of $\pi$.

Consider the $k \times n$ region of $F$ with rows $[1,k]$ and columns $[n-k+1,2n-k]$. By the definition of $\pi \ast \rho$ and the discussion above, the only pipes that show up in this region are $\rho(i) + n - k > n - k$, and these pipes enter from the south edge of this region and exit from the east. Therefore, this region agrees with the first $k$ rows of a bumpless pipe dream of $\rho$. Furthermore, there are no blank tiles in the $(n-k) \times n$ region with rows $[k+1,n]$ and columns $[n-k+1,2n-k]$. Since $\rho$ has no descents after $k$, its bumpless pipe dream is completely determined by the first $k$ rows. This gives us a bumpless pipe dream of $\rho$.

It is easy to see that the inverse of this process is the procedure of constructing $D \ast E$ as described above. We have established the desired bijection. \hfill $\square$

**Remark.** Lemma 4.1 is the reason why we need the “separated descent” assumption. If this condition is not satisfied, it seems very difficult to construct a set like $\text{BPD}(\pi \ast \rho)$ in weight-preserving bijection with $\text{BPD}(\pi) \times \text{BPD}(\rho)$.

### 4.2. Main theorem.

**Theorem 4.2.** Suppose $\pi, \rho \in S_n$ such that $\pi$ has no descents before position $k$ and $\rho$ has no descents after position $k$. Let $\pi \in S_{2n-k}$ such that $\ell(\pi \ast \rho) - \ell(\pi) = \ell((\pi \ast \rho)\sigma^{-1}) = k(n-k)$. Let $\lambda_{k \times (n-k)}$ be the partition of the $k \times (n-k)$ rectangular shape. The Schubert structure constant $c_{\pi,\rho}^{\sigma}$ is equal to the Edelman-Greene coefficient $f_{\lambda_{k \times (n-k)}}^{(\pi \ast \rho)\sigma^{-1}}$, which is the number of reduced word tableaux $T$ of shape $\lambda_{k \times (n-k)}$ such that the permutation given by the reading word of $T$ is $(\pi \ast \rho)\sigma^{-1}$. Furthermore, $c_{\pi,\rho}^{\sigma} = 0$ for all other $\sigma$ (even though the number of tableaux may be nonzero).

**Proof.** We will prove the theorem by constructing a bijection between $\text{BPD}(\pi) \times \text{BPD}(\rho)$ with

$$\bigsqcup_{\sigma \in \Sigma} \text{BPD}(\sigma) \times \{C(i) : i \in \text{red}((\pi \ast \rho)\sigma^{-1}), \text{shape}(P(i)) = \lambda_{k \times (n-k)}\},$$

where $\Sigma = \{\sigma \in S_{2n-k} : \ell(\pi \ast \rho) - \ell(\pi) = \ell((\pi \ast \rho)\sigma^{-1}) = k(n-k)\}$ (in other words, $\sigma$ must be below $\pi \ast \rho$ in left weak Bruhat order and their lengths differ by $k \times (n-k)$), and $P(i)$ the reduced word tableau of $i$.

Since $\pi$ and $\rho$ have separated descents, the construction of $D \ast E$ is for any $D \in \text{BPD}(\pi)$ and $E \in \text{BPD}(\rho)$ is possible. Suppose rect$(D \ast E) = F$. Since all marked blank tiles in $D \ast E$ are in rows $[1-k,0]$, there are no unmarked blank tiles in rows $[1-k,0]$, and rectification of $D \ast E$ removes all the marked blank tiles while preserving the number of unmarked blank tiles in each row, the pipes numbered $1-k$ to $0$ of $F$ must all be for the
identity permutation on $[1 - k, 0]$. Therefore $F \in \text{BPD}(\text{id}_{[1-k,0]} \otimes \sigma)$ for some $\sigma \in S_{2n-k}$. That $\sigma \in \Sigma$ follows from the definition of rectification. Then by Proposition 3.10,
\[ C := \{ \Psi(D \ast E, U) : U \in \text{SYT}(\lambda_{k \times (n-k)}) \} \]
is a single Coxeter-Knuth class in reduced words of $(\pi \ast \rho)\sigma^{-1}$, and the corresponding reduced word tableaux is of shape $\lambda_{k \times (n-k)}$. So $D \ast E$ maps to $(F, C)$ under the bijection, and $D \ast E = F \leftarrow i$ for any $i \in C$.

For the other direction, let $\sigma \in \Sigma$, and suppose $i$ is a reduced word of $(\pi \ast \rho)\sigma^{-1}$ such that the shape of $P(i)$ is $\lambda_{k \times (n-k)}$. Consider $\text{id}_{[1-k,0]} \otimes \sigma \in S_{[1-k,2n-k]}$, so the NW corner of any bumpless pipe dream of $\text{id}_{[1-k,0]} \otimes \sigma$ is at $(1 - k, 1 - k)$. Let $F \in \text{BPD}(\text{id}_{1-k,0} \otimes \sigma)$. By Lemma 3.13 as well as the length condition defining $\Sigma$, $F \leftarrow i$ is defined, and therefore the permutation of $(F \leftarrow i)$ is $\pi \ast \rho$, by definition of $i$ and insertion. By Proposition 3.12, since the shape of the reduced word tableau of $i$ is $\lambda_{k \times (n-k)}$, the first $k$ rows of $(F \leftarrow i)$ must be of the form that the first $n - k$ columns consist of marked blank tiles, and the rest of the columns contain no blank tiles. This means $(F \leftarrow i) \in \text{BPD}^{\otimes}(\pi \ast \rho)$. If $j$ is another reduced word of $(\pi \ast \rho)\sigma^{-1}$ but in a different Coxeter-Knuth class than $i$, then $(F \leftarrow i) \neq (F \leftarrow j)$ by Proposition 3.12. Therefore, each pair $(F, C(i))$ corresponds to a unique element in $\text{BPD}^{\otimes}(\pi \ast \rho)$. The insertion algorithm preserves the number of unmarked blank tiles on each row, so $F$ and $F \leftarrow i$ have the same monomial weight.

Then since $\text{BPD}^{\otimes}(\pi \ast \rho)$ bijects with $\text{BPD}(\pi) \times \text{BPD}(\rho)$ in a weight-preserving way by Lemma 4.1, the desired bijection is established. Furthermore, since the Edelman-Greene coefficient $j_{\lambda_{k \times (n-k)}}^{(\pi \ast \rho)\sigma^{-1}}$ counts the number of elements in the set $\{ C(i) : i \in \text{red}((\pi \ast \rho)\sigma^{-1}), \text{shape}(P(i)) = \lambda_{k \times (n-k)} \}$, the statement $c_{\pi, \rho} = j_{\lambda_{k \times (n-k)}}^{(\pi \ast \rho)\sigma^{-1}}$ follows. The bijection accounts for all positive structure constants, so $c_{\sigma, \rho} = 0$ for all $\sigma \notin \Sigma$.

**Example 4.3.** Let $Q = ((1,3), (2,5), (4,6)) \in \text{SYT}(\lambda_{3 \times 2})$. Then $\Psi(D \ast E, Q) = s_1 s_2 s_3 s_5 s_0 s_3$, where $D \ast E$ is as in Figure 10. The result $\text{rect}(D \ast E) \in \text{BPD}(\pi = -2 -103254167)$ is shown in Figure 11. $\mathcal{S}_{32541}$ appears with multiplicity 1 in the Schubert product expansion of $\mathcal{S}_{13542} \mathcal{S}_{2143}$.

**Figure 11. rect(D \ast E)**

**Remark.** Notice that we specifically imposed a length condition relating $\pi, \rho, \sigma$ when stating our rule. To show that this is necessary, consider the example when $\pi = 13542$, $\rho = 21435$, $\pi \ast \rho = (-2, 0, 2, 4, 3, 6, 1, -1, 5, 7)$, and $\sigma = 5243167$. Then $(\pi \ast \rho)\sigma^{-1} =
$(-2,0,2,-1,3,1,6,4,5,7)$, and $i = s_1s_2s_{-1}s_5s_0s_4$ is a reduced word of $(\pi \circ \rho)\sigma^{-1}$ whose reduced word tableau is of shape $3 \times 2$. Here $\ell(\sigma) = 8$, $\ell(\pi \circ \rho) = 12$, and the required length condition is not satisfied. Clearly, $\mathfrak{S}_\sigma$ does not appear in the expansion of $\mathfrak{S}_\pi \mathfrak{S}_\rho$ for degree reasons. Notice that the insertion of $i$ into any $F \in \text{BPD}(\sigma)$ is not defined.

As a direct consequence of the proof of Theorem 4.2, we may state a different rule for the separated-descent structure constants that is more closely analogous to one of the classical rules for Littlewood-Richardson coefficients using semi-standard Young tableaux, see e.g. Corollary 2(v) of [Ful97, Section 5.1], without making a connection to the Edelman-Greene story. In this statement, we do not need to specify a length condition as in Theorem 4.2, as the bumpless pipe dreams that arise as results of rectification are automatically for the correct permutations.

**Theorem 4.4.** Let $\pi, \rho \in S_n$ be permutations with separated descents at position $k$ and $\sigma$ a permutation. Let $F \in \text{BPD}(id_{[1-k,0]}[\mathfrak{S}_\sigma])$. Then $e^{\mathfrak{S}_\sigma}_{\pi,\rho}$ is the number of elements $P$ in $\text{BPD}\!(\mathfrak{S}(\pi \circ \rho))$ such that $\text{rect}(P) = F$.

4.3. **Multiple permutations with separated descents.** Equipped with the techniques developed in the previous sections, we can easily generalize the result to Schubert products for multiple permutations with separated descents. To be precise, suppose $\pi_0, \pi_1, \ldots, \pi_m \in S_n$ and $1 \leq k_1 < k_2 < \ldots < k_m \leq n - 1$, where $\text{Des}(\pi_0) \subseteq [1, k_1], \text{Des}(\pi_1) \subseteq [k_1, k_2], \ldots, \text{Des}(\pi_i) \subseteq [k_i, k_{i+1}], \ldots, \text{Des}(\pi_m) \subseteq [k_m, n - 1]$. The star operation can be generalized to the $(m+1)$-ary version, as follows:

$$\star(\pi_m, \ldots, \pi_1, \pi_0)(i) = \begin{cases} 
\pi_m(i + \sum_{\alpha=1}^{m} k_\alpha) - (\sum_{\alpha=1}^{m} k_\alpha) & \text{if } i \in [1 - \sum_{\alpha=1}^{m} k_\alpha, -\sum_{\alpha=1}^{m-1} k_\alpha] \\
\pi_{m-1}(i + \sum_{\alpha=1}^{m-1} k_\alpha) - (\sum_{\alpha=1}^{m-1} k_\alpha) + n & \text{if } i \in [1 - \sum_{\alpha=1}^{m-1} k_\alpha, -\sum_{\alpha=1}^{m-2} k_\alpha] \\
\vdots & \\
\pi_1(i + k_1) - (\sum_{\alpha=1}^{1} k_\alpha) + (m - 1)n & \text{if } i \in [1 - k_1, 0] \\
\pi_0(i) + mn - \sum_{\alpha=1}^{1} k_\alpha & \text{if } i \in [1, k_1] \\
\pi_1(i) + (m - 1)n - \sum_{\alpha=1}^{1} k_\alpha & \text{if } i \in [k_1 + 1, k_2] \\
\vdots & \\
\pi_m(i) - \sum_{\alpha=1}^{m-1} k_\alpha & \text{if } i \in [k_m + 1, n] \\
\phi(i) & \text{if } i \in [n + 1, (m+1)n - \sum_{\alpha=1}^{m} k_\alpha] 
\end{cases}$$

where $\phi(i)$ is the $(i-n)$th smallest number in $[1 - \sum_{\alpha=1}^{m} k_\alpha, (m+1)n - \sum_{\alpha=1}^{m} k_\alpha \backslash \pi \circ \rho [1 - \sum_{\alpha=1}^{m} k_\alpha, n])$. Here $\star(\pi_m, \ldots, \pi_0) \in S_{[1 - \sum_{\alpha=1}^{m} k_\alpha, (m+1)n - \sum_{\alpha=1}^{m} k_\alpha]}$.

Define also $\text{BPD}\!(\star(\pi_m, \ldots, \pi_0))$ as the set of $D \in \text{BPD}(\star(\pi_m, \ldots, \pi_0))$ such that rows $[1 - \sum_{\alpha=1}^{m} k_\alpha, 0]$ are of the form as shown in Figure 12. We denote the shape of the partition at the NW corner as $\lambda(k_1, \ldots, k_m, n)$.

We can easily generalize the procedure described in Section 4.1 and define the $\star$-operation on $D_m \in \text{BPD}(\pi_m, \ldots, D_0 \in \text{BPD}(\pi_0)$. The resulting bumpless pipe dream is an element of $\text{BPD}\!(\star(\pi_m, \ldots, \pi_0))$. Furthermore, $\text{BPD}\!(\star(\pi_m, \ldots, \pi_0))$ is in direct bijection with $\text{BPD}(\pi_m) \times \cdots \times \text{BPD}(\pi_0)$. See Figure 13.
Our main theorem generalizes to Schubert products for multiple permutations with separated descents.

**Theorem 4.5.** Suppose \( \pi_0, \pi_1, \ldots, \pi_m \in S_n \) and \( 1 \leq k_1 < k_2 < \cdots < k_m \leq n - 1 \), where \( \text{Des}(\pi_0) \subseteq [1, k_1] \), \( \text{Des}(\pi_1) \subseteq [k_1, k_2] \), \( \ldots \), \( \text{Des}(\pi_i) \subseteq [k_i, k_{i+1}] \), \( \ldots \), \( \text{Des}(\pi_m) \subseteq [k_m, n - 1] \). Let \( \sigma \in S_{(n+1)n - \sum_{\alpha=1}^{m} k_{\alpha}} \) such that \( \ell(\pi_{m}, \ldots, \pi_{0}) - \ell(\sigma) = \ell(\pi_{m}, \ldots, \pi_{0})\sigma^{-1} = [\lambda(k_1, \ldots, k_m, n)] \). The Schubert structure constant \( c_{\pi_{0},\ldots,\pi_{m}}^{\sigma} \) is equal to the Edelman-Greene coefficient \( j_{\lambda(k_1,\ldots,k_m,n)}^{\pi_{m},\ldots,\pi_{0}} \sigma^{-1} \), which is the number of reduced word tableaux \( T \) of shape \( \lambda(k_1, \ldots, k_m, n) \) such that the permutation given by the reading word of \( T \) is \( \pi_{m}, \ldots, \pi_{0} \sigma^{-1} \). Furthermore, \( c_{\pi_{m},\ldots,\pi_{0}}^{0} = 0 \) for all other \( \pi \).

**Remark.** Given \( \pi_0, \pi_1, \pi_2 \) that satisfy the separated descent conditions as stated in the theorem, it is not difficult to check that for any \( D_0 \in \text{BPD}(\pi_0) \), \( D_1 \in \text{BPD}(\pi_1) \), \( D_2 \in \text{BPD}(\pi_2) \), \( \text{rect}(\pi_0, D_0, D_1, D_2) = \text{rect}(D_0 \star (\text{rect}(D_1 \star D_2))) = \text{rect}(\text{rect}(D_0 \star D_1) \star D_2) \).
5. Connection to jeu de taquin

The construction of the star operation and the rectification process were inspired by the celebrated jeu de taquin algorithm on skew tableaux. We discuss the connection in this section.

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0)$ be a partition. Let $w(\lambda, k)$ be the $k$-Grassmannian permutation associated to partition $\lambda$, namely, $w(\lambda, k)(i) = \lambda_{k-i+1} + i$ for $i \leq k$ and $w(\lambda, k)$ has no descents after $k$. For any $D \in \text{BPD}(w(\lambda, k))$, associate a semi-standard Young tableau $T_D$ to $D$ as follows. Label each blank tile in $D$ with its row index, and perform all possible droop moves in $D$. Each droop moves a blank tile towards the NW direction together with its label. When finished, the semi-standard Young tableau at the NW corner is $T_D$. Denote the shape of $T_D$ as $\lambda$. It is known from [LLS21] that this map $D \mapsto T_D$ gives a bijection between $\text{BPD}(w(\lambda, k))$ and $SSYT_k(\lambda)$, where $SSYT_k(\lambda)$ denotes the set of semi-standard Young tableaux of shape $\lambda$ with entries no greater than $k$.

The relationship between bumpless pipe dreams and tableaux in some sense dates back to [KMY09], before the bumpless pipe dreams formulation was explicitly stated.

Let $\nu = (\nu_1, \ldots, \nu_m)$ and $\mu = (\mu_1, \ldots, \mu_l)$ be partitions such that $l < m$ and $\mu_i \leq \nu_i$ for each $i$. Suppose $k \geq m$. Let $T \in SSYT_k(\nu/\mu)$. We complete $T$ into a semi-standard tableau of shape $\nu$ by filling in for each row $i = l, l - 1, \ldots, 1$, the number $i - l$ in each box on row $i$ that belongs to $\mu$. We also mark these entries with a different color. Call the resulting marked semi-standard tableau $T'$. We may now use the correspondence between semi-standard Young tableaux of straight shapes and bumpless pipe dreams for Grassmannian permutations to get a bumpless pipe dream $D_{T'}$ for $T'$. To distinguish the non-positive entries from the original entries, we mark each of the blank tiles in the non-positive rows of $D_{T'}$ with an “$x$”.

![Figure 14. Connection of $T_D \ast T_E$ with $D \ast E$](image-url)
We will now demonstrate by an example how our construction connects to the classical construction for computing products of Schur polynomials with semi-standard Young tableaux, see Figure 14. In this example, \( k = 3 \) and \( n = 5 \). We let \( T_D \) denote the first tableau, \( D \in \text{BPD}(12534) \) its corresponding bumpless pipe dream, \( T_E \) the second tableau, and \( E \in \text{BPD}(13524) \). \( T_D \ast T_E \) is the skew tableau of shape \((4,3,2)/(2,2)\) formed by placing \( T_E \) immediately NE of \( T_D \). We fill in the boxes in the NW partition of shape \((2,2)\) with non-positive numbers as shown, and find its corresponding bumpless pipe dream \( F \) with marked blank tiles. Finally, we construct \( D \ast E \) and find its corresponding skew tableau. Notice that for both \( D \ast E \) and \( F \), drooping and undrooping within positive rows will generate a set in direct bijection with \( \text{BPD}(12534) \times \text{BPD}(13524) \). Therefore, our \( \ast \)-construction is not the unique choice, but a convenient one.

It remains to explain how jeu de taquin relates to our column moves. Let \( T' \) be a marked semi-standard Young tableau, as described prior to the example. Define a rectification algorithm on \( T' \) by iterating the following procedure until there are no more marked entries.

1. Pick a SE most marked number in the NW marked region to be “active” in this iteration.
2. Increment the active marked number by 1. If this number is equal to \( k \) and at the SE border, delete it and end the iteration. Otherwise:
   a. If the active marked number is equal to the number below, increment the number below as well and move the mark there. Repeat this until we get a valid tableau again.
   b. If the active marked number is equal to the number to the right, move the mark to the rightmost number with this value on this row. Repeat this step.

**Proposition 5.1.** The rectification algorithm on \( T' \) described above simulates jeu de taquin on \( T \). Furthermore, the algorithm agrees with the rectification algorithm defined on marked bumpless pipe dreams in each step when each intermediate marked tableau is mapped to its corresponding marked bumpless pipe dream.

**Proof.** Step 2(a) simulates consecutive vertical slides in jeu de taquin, and Step 2(b) simulates consecutive horizontal slides. Furthermore, Step 2(a) corresponds to an undroop column move as defined in Section 3.1, and Step 2(b) corresponds to moving a mark to the rightmost tile in a consecutive block of blank tiles on a row in a bumpless pipe dream. Finally, in a bumpless pipe dream of a \( k \)-Grassmannian permutation, the column move that removes a cross is only available when the marked tile is in row \( k \). This determines the terminating condition in Step 2.

\[ \square \]

6. Final remarks

It was shown in [LLS21] that bumpless pipe dreams compute double Schubert polynomials, and further explained in [Wei21] that they also compute double Grothendieck polynomials. We note that our construction cannot be easily extended to an equivariant setting, because the \( \ast \)-operation and the insertion/rectification algorithm are row-biased
and do not maintain the necessary invariance for columns. A similar phenomenon is exhibited in [KY04]. However, it should be possible to generalize our result to $K$-theory. A $K$-theoretic version of Edelman-Greene insertion (perhaps Hecke insertion [BKS+08]) for bumpless pipe dreams, would be necessary. We leave this for future work. An interpretation of the $K$-theoretic version of jeu de taquin [TY09] in the Schubert setting might also be relevant. It would also be interesting to see what other Schubert structure constants, if any, are also Edelman-Greene coefficients.

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