THE AVERAGE SIZE OF A CONNECTED VERTEX SET OF A $k$-CONNECTED GRAPH

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Abstract. The topic is the average order $A(G)$ of a connected induced subgraph of a graph $G$. This generalizes, to graphs in general, the average order of a subtree of a tree. In 1984, Jamison proved that the average order, over all trees of order $n$, is minimized by the path $P_n$, the average being $A(P_n) = \frac{(n+2)}{3}$. In 2018, Kroeker, Mol, and Oellermann conjectured that $P_n$ minimizes the average order over all connected graphs $G$ - a conjecture that was recently proved. In this short note we show that this lower bound can be improved if the connectivity of $G$ is known. If $G$ is $k$-connected, then $A(G) \geq \frac{n}{2} \left(1-\frac{1}{2^k+1}\right)$.

1. Introduction

Although connectivity is a basic concept in graph theory, problems involving the enumeration of the connected induced subgraphs of a given graph have only recently received attention. The topic of this paper is the average order of a connected induced subgraph of a graph. Let $G$ be a connected finite simple graph with vertex set $V = \{1, 2, ..., n\}$, and let $U \subseteq V$. The set $U$ is said to be a connected set if the subgraph of $G$ induced by $U$ is connected. Denote the collection of all connected sets, excluding the emptyset, by $\mathcal{C} = \mathcal{C}(G)$. The number of connected sets in $G$ will be denoted by $N(G)$. Let $S(G) = \sum_{U \in \mathcal{C}} |U|$ be the sum of the sizes of the connected sets. Further, let $A(G) = \frac{S(G)}{N(G)}$ and $D(G) = \frac{A(G)}{n}$ denote, respectively, the average size of a connected set of $G$ and the proportion of vertices in an average size connected set. The parameter $D(G)$ is referred to as the density of connected sets of vertices. The density allows us to compare the average size of connected sets of graphs of different orders. If, for example, $G$ is the complete graph $K_n$, then $A(K_n)$ is the average size of a subset of an $n$-element set, which is $n/2$ (counting the empty set for simplicity), and the density is $1/2$.

Papers [2, 4, 5, 7, 8, 9, 12, 13] on the average size and density of connected sets of trees appeared beginning with Jamison’s 1984 paper [4]. The invariant $A(G)$, in this case, is the average order of a subtree of a tree. Concerning lower bounds, Jamison proved that the average size of a subtree of a tree of order $n$ is minimized by the path $P_n$. In particular $A(T) \geq \frac{(n+2)}{3}$ for all trees $T$ with equality only for $P_n$. Therefore $D(T) > 1/3$ for all trees $T$. Vince and Wang [9] proved that if $T$ is a tree all of whose non-leaf vertices have degree at least three, then $\frac{1}{2} \leq D(T) < \frac{3}{4}$, both bounds being best possible.

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Although results are known for trees, little was known until recently for graphs in general. Krocker, Mol, and Oellermann conjectured in their 2018 paper [6] that the lower bound of Jamison for trees extends to graphs in general.

**Conjecture 1.** The $P_n$ minimizes the average size of a connected set over all connected graphs.

In [1] Balodis, Mol, and Oellermann verified the conjecture for block graphs of order $n$, i.e., for graphs each maximal 2-connected component of which is a complete graph. The conjecture was proved recently for connected graphs in general in [11] and independently shortly thereafter in [3].

**Theorem 1.** If $G$ is a connected graph of order $n$, then
\[ A(G) \geq \frac{n+2}{3}, \]
with equality if and only if $G$ is a path. In particular, $D(G) > 1/3$ for all connected graphs $G$.

In [10] it was conjectured that the lower bound of Vince and Wang for trees extends to graphs in general.

**Conjecture 2.** If $G$ is a connected graph all of whose vertices have degree at least 3, then
\[ D(G) \geq \frac{1}{2}. \]

Krocker, Mol, and Oellermann [6] verified Conjecture 2 for connected cographs. A cograph can be defined recursively: the one vertex graph is a cograph and, if $G$ and $H$ are cographs, then so is their disjoint union and their join. (The join is obtained by joining by an edge each vertex of $G$ to each vertex of $H$.) Complete bipartite graphs are examples of cographs. Conjecture 2 remains open, but in this note we prove a lower bound on $D(G)$ close to $1/2$ if $G$ is highly connected. More precisely:

**Theorem 2.** If $G$ is $k$-connected, then
\[ D(G) \geq \frac{1}{2} \left( 1 - \frac{1}{2^k + 1} \right). \]

### 2. Proof of Theorem 2

If $i \in V$, let $N(G, i), S(G, i),$ and $A(G, i)$ denote the number of connected sets in $G$ containing $i$, the sum of the sizes of all connected sets containing $i$, and the average size of a connected set containing $i$, respectively. The following result appears in [11, Corollary 3.2].

**Theorem 3.** If $i \in V$ is any vertex of a connected graph $G$ of order $n$, then
\[ A(G, i) \geq \frac{n+1}{2}. \]

**Corollary 4.** If $G$ is a connected graph of order $n$, then
\[ \sum_{U \in C} |U|^2 \geq \left( \frac{n+1}{2} \right) S(G). \]

**Proof.** From Theorem 3 we have
\[ S(G, i) \geq \frac{n+1}{2} N(i) \]
for all $i \in V$. Now count the number of pairs in the set $\{ (i, U) : U \in C, i \in U \subseteq V \}$ in two ways to obtain
\[ S(G) = \sum_{U \in C} |U| = \sum_{i \in V} N(G, i). \]

Similarly
\[ \sum_{U \in C} |U|^2 = \sum_{U \in C} \sum_{i \in U} |U| = \sum_{i \in V} \sum_{U \in C, i \in U} |U| = \sum_{i \in V} S(G, i) \geq \frac{n+1}{2} \sum_{i \in V} N(G, i) = \left( \frac{n+1}{2} \right) S(G). \]
Proof of Theorem 2 The proof is by induction on \( k \). When \( k = 1 \), the statement is \( D(G) \geq \frac{1}{3} \), which follows from Theorem 1 in the introduction.

Let

\[
a_k := \frac{1}{2} \left( 1 - \frac{1}{2^k + 1} \right).
\]

A straightforward calculation shows that

\[
a_k = \frac{2a_{k-1}}{2a_{k-1} + 1}.
\]

Assume that the statement of Theorem 2 holds for all \((k-1)\)-connected graphs and assume that \( G \) is \( k \)-connected. Let \( \mathcal{C} = \{ U \in \mathcal{C} : U \neq V \} \), and denote by \( N'(G) \) and \( S'(G) \) the number of connected sets and the sum of the sizes of the connected sets, respectively, not including \( V \). For \( i \in V \), denote by \( G_i \) the graph induced by the vertices \( V \setminus \{i\} \). Note that \( G_i \) is \((k-1)\)-connected for all \( i \in V \) and therefore, by the induction hypothesis, we have \( S(G_i) \geq a_{k-1}(n-1)N(G_i) \) for all \( i \in V \).

Now

\[
nS'(G) - \sum_{U \in \mathcal{C}'} \vert U \vert^2 = \sum_{U \in \mathcal{C}'} \vert U \vert(n - \vert U \vert) = \sum_{i \in V} S(G_i) \geq a_{k-1}(n-1) \sum_{i \in V} N(G_i)
\]

\[
= a_{k-1}(n-1) \sum_{U \in \mathcal{C}'} (n - \vert U \vert) = a_{k-1}n(n-1)N'(G) - a_{k-1}(n-1)S'(G).
\]

This implies

\[
(n + a_{k-1}(n-1))(S(G) - n) \geq a_{k-1}n(n-1)(N(G) - 1) + \sum_{U \in \mathcal{C}} \vert U \vert^2 - n^2,
\]

equivalently

\[
(n + a_{k-1}(n-1))S(G) \geq a_{k-1}n(n-1)N(G) + \sum_{U \in \mathcal{C}} \vert U \vert^2.
\]

By Corollary 4 this implies

\[
\left( \frac{n-1}{2} + a_{k-1}(n-1) \right)S(G) \geq a_{k-1}n(n-1)N(G),
\]

or

\[
D(G) = \frac{S(G)}{nN(G)} = \frac{a_{k-1}}{\frac{n}{2} + a_{k-1}} = \frac{2a_{k-1}}{2a_{k-1} + 1} = a_k.
\]

\[\square\]

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