Monodromies of rational elliptic surfaces and extremal elliptic $K3$ surfaces

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Abstract

We give a systematic method to calculate some homological data from the global monodromy of a topological elliptic surface. We apply this method to the cases 1) the transcendental lattice of an extremal elliptic $K3$ surface, 2) the torsion part of Mordell-Weil group of a general elliptic surface, and 3) the Mordell-Weil lattice of a rational elliptic surface.

We present possible list of global monodromies of rational elliptic surface and extremal elliptic $K3$ surface which reproduce to the classification obtained by Oguiso-Shioda and Shimada-Zhang respectively.
1 Introduction

In our previous paper [5] we analyze the trivial lattices (which is essentially the combinations of the singular fibres), the Mordell-Weil lattices and the torsion parts of Mordell-Weil group of rational elliptic surfaces using the method of string junctions in string theory, and reproduce the classification of the Mordell-Weil lattices due to Oguiso-Shioda by constructing the corresponding global monodromy. In this paper we apply the idea of [5] to elliptic K3 surfaces.

Let \( f : X \rightarrow \mathbb{P}^1 \) be a general topological elliptic surface which has a smooth cross section and no multiple fibre. Let \( T \) be the sublattice of \( H_2(X, \mathbb{Z}) \) which is generated by irreducible components of each singular fibre, a smooth cross section and a generic fibre. We denote the orthogonal complement of \( T \) in \( H_2(X, \mathbb{Z}) \) and the primitive closure of \( T \) in \( H_2(X, \mathbb{Z}) \) by \( T^\perp \), \( \hat{T} \) respectively.

The results of this paper is as follows:
1. We give a systematic method to calculate the homological data $T^\perp$, $\hat{T}/T$ from the global monodromies.

2. We reproduce the classification of rational elliptic surfaces and extremal elliptic $K3$ surfaces due to Oguiso-Shioda and Shimada-Zhang by constructing the corresponding global monodromy.

About 1. We give these algorithms in section 4. These algorithms are essentially reduced to the calculation of the elementary divisor of integral matrix.

About 2. Let $f : X \to \mathbb{P}^1$ be a complex elliptic $K3$ surface with a section $O$ which satisfies the following conditions:
(i) The Picard number is 20.
(ii) The Mordell-Weil group is finite.
Then $f : X \to \mathbb{P}^1$ is said to be an extremal elliptic $K3$ surface.

The classification of the pairs (trivial lattice, Mordell-Weil group) of semi-stable extremal elliptic $K3$ surface has been done by Miranda and Persson [11] and complemented by Artal-Bartolo, Tokunaga and Zhang [13]. The classification of the pairs (trivial lattice, Mordell-Weil group) of unsemi-stable extremal elliptic $K3$ surface has been done by Ye [14]. The complete classification of the triplets (trivial lattice, transcendental lattice, Mordell-Weil group) of extremal elliptic $K3$ surfaces has been done by Shimada and Zhang [4].

One of the purpose of this paper is to give the list of global monodromies which corresponds to the list given by Shimada and Zhang (called Shimada-Zhang’s table). The main theorem of this paper is as follows:

**Theorem.** If there exist an extremal elliptic $K3$ surface with the global monodromy given in Table 4, then the corresponding triplet (trivial lattice, transcendental lattice, Mordell-Weil group) of the surface is given by the data of the same Table 4.

### 2 Preliminaries
2.1 Monodromy of elliptic surfaces

In this section we will recall some basic facts on the monodromy of elliptic surfaces following the reference [1].

Let \( f : X \to \mathbf{P}^1 \) be an elliptic surface that has \( n \) singular fibres \( F_{u_1}, F_{u_2}, \ldots, F_{u_n} \) on \( S = \{u_1, u_2, \ldots, u_n\} \). We take an base point \( u_0 \in \mathbf{P}^1 \setminus S \), and let \( \rho_1, \rho_2, \ldots, \rho_n \) be disjoint smooth paths which connect \( u_0 \) with \( u_1, u_2, \ldots, u_n \) respectively; \( \rho_1', \rho_2', \ldots, \rho_n' \) be generators of \( \pi_1 (\mathbf{P}^1 \setminus S, u_0) \) which correspond to \( \rho_1, \rho_2, \ldots, \rho_n, \rho_1, \rho_2, \ldots, \rho_n \) be automorphisms of \( H_1 (f^{-1}(u_0), \mathbf{Z}) \) corresponding to \( \rho'_1, \rho'_2, \ldots, \rho'_n \) and \( \langle \alpha, \beta \rangle \) be the basis of \( H_1 (f^{-1}(u_0), \mathbf{Z}) \).

We put \( K_i (1 \leq i \leq n) \) the circuit matrices along the path \( \rho'_i \), and call \( (K_1, K_2, \ldots, K_n) \) the global monodromy on \( (\rho'_1, \rho'_2, \ldots, \rho'_n) \) of \( X \). Namely if \( \rho'_i (\alpha) = a_i \alpha + c_i \beta \) and \( \rho'_i (\beta) = b_i \alpha + d_i \beta \) then \( K_i = \left( \begin{array}{cc} a_i & b_i \\ c_i & d_i \end{array} \right) \). It is clear also that \( K_n K_{n-1} \cdots K_1 = I \). For \( i = 1, 2, \ldots, n - 1 \) let \( u_i : \pi_1 (\mathbf{P}^1 \setminus S, u_0) \to \pi_1 (\mathbf{P}^1 \setminus S, u_0) \) be an automorphism of \( \pi_1 (\mathbf{P}^1 \setminus S, u_0) \) defined by \( u_i (\rho'_1) = \rho'_1, \ldots, u_i (\rho'_{i-1}) = \rho'_{i-1}, u_i (\rho'_i) = \rho'_{i+1}, u_i (\rho'_{i+1}) = \rho'^{-1}_{i+1} i \rho'_i, \rho'_{i+1}, u_i (\rho'_{i+2}) = \rho'_{i+2}, \ldots, u_i (\rho'_n) = \rho'_n \) and \( R_i (R_i^{-1}) \) be an transformation of global monodromy which is induced by \( u_i (u_i^{-1}) \). Then we have

\[
R_i ((K_1, \ldots, K_n)) = (K_1, \ldots, K_{i-1}, K_{i+1}, K_{i+1}K_{i+1}^{-1}, K_{i+2}, \ldots, K_n)
\]

\[
R_i^{-1} ((K_1, \ldots, K_n)) = (K_1, \ldots, K_{i-1}, K_i^{-1}K_{i+1}, K_i, K_{i+1}, K_{i+2}, \ldots, K_n).
\]

We call the transformations \( R_i, R_i^{-1} (1 \leq i \leq n - 1) \) the elementary transformations of global monodromy. If there exists a finite sequence of elementary transformations and \( SL(2, \mathbf{Z}) \) transformations corresponding to the base change of \( H_1 (f^{-1}(u_0), \mathbf{Z}) \) which transfer \( (K_1, K_2, \ldots, K_n) \) to \( (K'_1, K'_2, \ldots, K'_n) \), then we identify \( (K_1, K_2, \ldots, K_n) \) with \( (K'_1, K'_2, \ldots, K'_n) \) and denote \( (K_1, K_2, \ldots, K_n) \sim (K'_1, K'_2, \ldots, K'_n) \).

2.2 Oguiso-Shioda’s table and Shimada-Zhang’s table

In this section we recall the results of the references [3] and [4].

Let \( f : X_k \to \mathbf{P}^1 \) be a topological elliptic surface that has \( n \) singular fibres \( F_{u_1}, F_{u_2}, \ldots, F_{u_n} \) on \( S = \{u_1, u_2, \ldots, u_n\} \) and the Euler number is \( 12k \). Through out of this paper we assume \( f \) has a smooth cross section \( O \). Let \( O, F \) be a section and a generic fibre. We

\footnote{We remark that this definition of the circuit matrix is inverse of the ordinary one.}
Table 1: the correspondence of the type of $F_v$ and $T_v$

| the type of $F_v$ | $T_v$          |
|------------------|---------------|
| $I_n (1 \leq n)$ | $-A_{n-1}$ ($A_0 = 0$) |
| $II$             | $0$           |
| $III$            | $-A_1$        |
| $IV$             | $-A_2$        |
| $I^*_n (0 \leq n)$ | $-D_{n+4}$   |
| $II^*$           | $-E_8$        |
| $III^*$          | $-E_7$        |
| $IV^*$           | $-E_6$        |

put $U' = \langle O, F \rangle$. Then it is known that

$$H_2(X_k, \mathbb{Z}) \cong (-E_8)^{\oplus k} \oplus U^{\oplus 2k-2} \oplus U',$$

where $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $U' = \begin{pmatrix} -k & 1 \\ 1 & 0 \end{pmatrix}$. We set the sublattice $L_k$ of $H_2(X_k, \mathbb{Z})$;

$$L_k := (-E_8)^{\oplus k} \oplus U^{\oplus 2k-2}.$$

For each $v \in S = \{u_1, u_2, \ldots, u_n\}$, let

$$F_v = f^{-1}(v) = \Theta_{v,0} + \sum_{i=1}^{m_v-1} \mu_{v,i} \Theta_{v,i},$$

where $\Theta_{v,i} (0 \leq i \leq m_v - 1)$ are the irreducible components of $F_v$, $m_v$ being their multiplicity. $\Theta_{v,0}$ is the unique component of $F_v$ meeting the zero section. And we set the sublattice of $L_k$

$$V := \bigoplus_{v \in S} T_v,$$

where $T_v = \langle \Theta_{v,i} \mid 1 \leq i \leq m_v - 1 \rangle$. Then it is known that the opposite lattice $-T_v$ is a root lattice of rank $m_v - 1$ which is determined by the type of the singular fibre $F_v$. Then it is known that the type of the singular fibre $F_v$

We can summarize the correspondence of the type of $F_v$ and $T_v$ in Table 1.

Consider the lattice embedding

$$i_V : V \hookrightarrow L_k,$$ (2)
and let $V^\perp$ be the orthogonal complement of $V$ in $L_k$ and $\hat{V}$ the primitive closure of $V$ in $L_k$. Now we describe the geometric meaning of $V^\perp$, and $\hat{V}/V$. First, with respect to the geometric meaning of $\hat{V}/V$ the following is known.

**Theorem 1.** ([3]) The torsion part of Mordell-Weil group of $X_k$ is isomorphic to $\hat{V}/V$.

For $k = 1$, $X_k$ is a rational elliptic surface, and the following is known.

**Theorem 2.** ([3]) The Mordell-Weil lattice of a rational elliptic surface is isomorphic to the opposite lattice of $(V^\perp)^*$.

One of the main results of the reference [3] is the classification of all the possible triplets

$$(V, \text{Mordell-Weil lattice}, \text{torsion part of Mordell-Weil group})$$

of rational elliptic surfaces. (We call this table *Oguiso-Shioda’s table*.)

For $k = 2$, $X_k$ is an elliptic $K3$ surface. In particular an elliptic $K3$ surface is said to be *extremal* if the Picard number is 20 and Mordell-Weil group is finite. For an extremal elliptic $K3$ surface it is known that

**Theorem 3.** ([4]) The transcendental lattice of an extremal elliptic $K3$ surface is isomorphic to $V^\perp$. And the Mordell-Weil group is isomorphic to $\hat{V}/V$.

One of the main results of the reference [4] is to classify all the possible triplets

$$(V, \text{transcendental lattice}, \text{Mordell-Weil group})$$

of extremal elliptic $K3$ surfaces. (We call this table *Shimada-Chang’s table*.)

### 3 Junction
3.1 Tadpole junction lattice and rational tadpole junction lattice

In this section we define tadpole junction lattice and rational tadpole junction lattice by using the notations of the previous section and the descriptions of the reference [2].

Let \( D_i \) (\( 1 \leq i \leq n \)) be small closed disjoint 2-disks in \( P^1 \) with the center \( u_i \) (\( 1 \leq i \leq n \)). We can assume that the sets \( \rho_i \cap D_j \) (\( 1 \leq i \leq n, 1 \leq j \leq n, i \neq j \)) are empty and that each of the sets \( \rho_i \cap \partial D_i \) (\( 1 \leq i \leq n \)) has only one point. Let \( u'_i = \rho_i \cap \partial D_i \) and \( \bar{\rho}_i \) be the part of \( \rho_i \) from \( u_0 \) to \( u'_i \). We can assume that \( \rho'_i = \bar{\rho}_i \cdot \partial D_i \cdot \bar{\rho}_{i-1} \) (\( \partial D_i \) is oriented as the boundary of \( D_i \)).

Now we construct 2-cycles on \( X \). For a curve \( \rho \) starting from \( u_0 \) and an 1-cycle \( \gamma \) on \( f^{-1}(u_0) \), by the locally flatness of the fibering \( f : X \to P^1 \), there is a continuous family of 1-cycle \( \gamma_u \) of \( f^{-1}(u) \) (\( u \in \rho \)) such that \( \gamma_{u_0} = \gamma \); the union \( \bigcup_{u \in \rho} \gamma_u \), which is a real 2-dimensional surface, is denoted by \( \rho(\gamma) \). It is clear that

\[
\rho \mu(\gamma) = \rho(\gamma) + \mu(\rho^* \gamma) \quad \text{(as the element of } H_2(X, \mathbb{R}) \text{)} \quad (3)
\]

Consider \( n \) open surfaces

\[
\rho'_1(\gamma_1), \rho'_2(\gamma_2), \ldots, \rho'_n(\gamma_n) \quad (\gamma_i \in H_1(f^{-1}(u_0), \mathbb{R}))
\]

which have common boundary in \( f^{-1}(u_0) \); glue them together to get a closed surface \( J \) which is denoted by

\[
J = \rho'_1(\gamma_1) + \rho'_2(\gamma_2) + \cdots + \rho'_n(\gamma_n).
\]

We also use notations such as

\[
J = [x_1, y_1; x_2, y_2; \ldots; x_n, y_n]
\]

\[
= (x'_1, y'_1; x'_2, y'_2; \ldots; x'_n, y'_n),
\]

where \( \gamma_i = x_i \alpha + y_i \beta \) (\( 1 \leq i \leq n \)) and \( \gamma_i - \rho'_i(\gamma_i) = x'_i \alpha + y'_i \beta \). Then we have

\[
\begin{pmatrix} x'_i \\ y'_i \end{pmatrix} = (I - K_i) \begin{pmatrix} x_i \\ y_i \end{pmatrix}
\]

(4)

For \( J = [x_1, y_1; x_2, y_2; \ldots; x_n, y_n] = (x'_1, y'_1; x'_2, y'_2; \ldots; x'_n, y'_n) \), \( J \) is said to be a \textit{tadpole junction} if all \( x_i, y_i \) are integral and a \textit{rational tadpole junction} if all \( x'_i, y'_i \) are integral (See Figure [3]).
Now we define the *tadpole junction lattice* of $X$. We put $T = \{ u_i \in S \mid f^{-1}(u_i) \text{ is of type I} \}$ and $l = 2n - |T|$. If $u_i \in T$, we can assume that $f^{-1}(u_i)$ is of type $I_{n_i}$ and

$$K_i = \begin{pmatrix} 1 + n_i p_i q_i & -n_i p_i^2 \\ n_i q_i^2 & 1 + n_i p_i q_i \end{pmatrix}.$$

Then $x_i$ and $y_i$ are redundant as variables and we can use variable $t_i$ defined by

$$t_i = \left( \frac{p_i}{q_i} \right) \times \left( \begin{array}{c} x_i \\ y_i \end{array} \right),$$

where $\left( \begin{array}{c} a \\ b \end{array} \right) \times \left( \begin{array}{c} c \\ d \end{array} \right)$ means $ad - bc$. It is clear that

$$\left( \begin{array}{c} x'_i \\ y'_i \end{array} \right) = n_i t_i \left( \begin{array}{c} p_i \\ q_i \end{array} \right).$$

For $u_i \in T$ we will take $t_i$ as the independent variable instead of $x_i, y_i$ and we use the notation of $J$ such as $J = [s_1; s_2; \ldots; s_n]$ where $s_i = (x_i, y_i)$ ($u_i \notin T$) and $s_i = t_i$ ($u_i \in T$).

By the boundary condition, we have

$$\sum_{u_i \notin T} (I - K_i) \left( \begin{array}{c} x_i \\ y_i \end{array} \right) + \sum_{u_i \in T} n_i t_i \left( \begin{array}{c} p_i \\ q_i \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right). \quad (5)$$

Let

$$W_1 = \{ J = [x_1, y_1; \ldots; t_i; \ldots; x_n, y_n] \in \mathbb{Z}^l \mid x_i, y_i, t_i \text{ satisfies the condition (5)} \}$$
and
\[ \langle J_1, J_2 \rangle : W_1 \times W_1 \to \mathbb{Z}, \quad (J_1, J_2) \mapsto \langle J_1, J_2 \rangle \]

be the symmetric bilinear pairing defined by the intersection number of the tadpole junctions $J_1$ and $J_2$. Then, for $J_1 = [x_1, y_1; \ldots; t_i; \ldots; x_n, y_n]$ and $J_2 = [u_1, v_1; \ldots; w_i; \ldots; u_n, v_n]$, we have
\[
\langle J_1, J_2 \rangle = \sum_{u_i \notin T} \left( \frac{x_i}{y_i} \right) \times (I - K_i) \begin{pmatrix} u_i \\ v_i \end{pmatrix} - \sum_{u_i \in T} n_i t_i w_i \\
+ \sum_{1 \leq i < j \leq n} (I - K_i) \begin{pmatrix} x_i \\ y_i \end{pmatrix} \times (I - K_j) \begin{pmatrix} u_j \\ v_j \end{pmatrix} .
\]

(6)

Hence we can define the treadmill junction lattice of $X$ by $(W_1, \langle \cdot, \cdot \rangle)$. That is,

**Proposition 3.1** Let $(W_1, \langle \cdot, \cdot \rangle_K)$ (resp. $(\tilde{W}_1, \langle \cdot, \cdot \rangle_{\tilde{K}})$) be a tadpole junction lattice for a global monodromy $K = (K_1, K_2, \ldots, K_n)$ (resp. $\tilde{K} = (\tilde{K}_1, \tilde{K}_2, \ldots, \tilde{K}_n)$). If $K \sim \tilde{K}$, then there exists an isomorphism between $(W_1, \langle \cdot, \cdot \rangle_K)$ and $(\tilde{W}_1, \langle \cdot, \cdot \rangle_{\tilde{K}})$.

**Proof.** We can assume that $K$ transfers $\tilde{K}$ by the elementary transformation $R_i$. If $u_i \notin T$ and $u_{i+1} \notin T$, then we let the $4 \times 4$ matrix $P$
\[
P = \begin{pmatrix} I_2 - K_i & I_2 \\ K_{i+1} & 0 \end{pmatrix} .
\]

If $u_i \in T$ and $u_{i+1} \notin T$, then we let the $3 \times 3$ matrix $P$
\[
P = \begin{pmatrix} n_i p_i & 1 & 0 \\ n_i q_i & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} .
\]

If $u_i \notin T$ and $u_{i+1} \in T$, then we let the $3 \times 3$ matrix $P$
\[
P = \begin{pmatrix} (-q_{i+1}, p_{i+1}) (I_2 - K_i) & 1 \\ K_{i+1} & 0 \end{pmatrix} .
\]

If $u_i \in T$ and $u_{i+1} \in T$, then we let the $2 \times 2$ matrix $P$
\[
P = \begin{pmatrix} n_i (p_{i+1} q_i - p_i q_{i+1}) & 1 \\ 1 & 0 \end{pmatrix} .
\]
Let the $l \times l$ matrix $P_i$

$$P_i = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}.$$ 

It is clear that

$$\det(P_i) = \pm 1.$$ 

For $J = [x_1, y_1; \ldots; x_n, y_n] \in W_1$, we let $\tilde{J} = t_i P_i J$. Then we have $\tilde{J} \in W_2$ and

$$\langle J, J \rangle_k = \langle \tilde{J}, \tilde{J} \rangle_{\tilde{k}}.$$ 

Q.E.D.

Next we define the rational tadpole junction lattice of $X$. Consider a tadpole junction $J' = (x_1', y_1'; \ldots; x_n', y_n') \in \mathbb{Z}^l$. If $u_i \in T$, we let

$$t_i' = n_i t_i.$$ 

By substituting (4) and (7) to (5), we have

$$\sum_{u_i \notin T} \left( \begin{array}{c} x_i' \\ y_i' \end{array} \right) + \sum_{u_i \in T} t_i' \left( \begin{array}{c} p_i \\ q_i \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right).$$ 

Let

$$W_2 = \{ J' = (x_1', y_1'; \ldots; t_i'; \ldots; x_n', y_n') \in \mathbb{Z}^l \mid x_i', y_i', t_i' \text{ satisfies the condition (8)} \}$$

and

$$\langle , \rangle : W_2 \times W_2 \to \mathbb{Q}, \quad (J'_1, J'_2) \mapsto \langle J'_1, J'_2 \rangle$$
be the symmetric bilinear pairing defined by the intersection number of the rational tadpole junctions $J'_1$ and $J'_2$. Then we have

$$\langle J'_1, J'_2 \rangle = \sum_{u_i \notin T} (I - K_i)^{-1} \begin{pmatrix} x'_i \\ y'_i \end{pmatrix} \times \begin{pmatrix} u'_i \\ v'_i \end{pmatrix} - \sum_{u_i \in T} \frac{t'_iw'_i}{n_i}
$$

\[(9)\]

for $J'_1 = (x'_1, y'_1; \ldots; t'_i; \ldots; x'_n, y'_n)$ and $J'_2 = (u'_1, v'_1; \ldots; w'_i; \ldots; u'_n, v'_n)$. We can define the rational tadpole junction lattice of $X$ by $(W_2, \langle , \rangle)$. That is,

**Proposition 3.2** Let $(W_2, \langle , \rangle_K)$ (resp. $(\tilde{W}_2, \langle , \rangle_{\tilde{K}})$) be a tadpole junction lattice for a global monodromy $K = (K_1, K_2, \ldots, K_n)$ (resp. $\tilde{K} = (\tilde{K}_1, \tilde{K}_2, \ldots, \tilde{K}_n)$). If $K \sim \tilde{K}$, then there exists an isomorphism between $(W_2, \langle , \rangle_K)$ and $(W_2, \langle , \rangle_{\tilde{K}})$.

**Proof.** We can assume that $K$ transfer $\tilde{K}$ by the elementary transformation $R_i$. If $u_i \notin T$ and $u_{i+1} \notin T$, then we let the $4 \times 4$ matrix $Q$

$$Q = \begin{pmatrix} I_2 - K_{i+1} & I_2 \\ K_{i+1} & O \end{pmatrix}.$$ 

If $u_i \in T$ and $u_{i+1} \notin T$, then we let the $3 \times 3$ matrix $Q$

$$Q = \begin{pmatrix} (I_2 - K_{i+1}) \begin{pmatrix} p_i \\ q_i \end{pmatrix} & I_2 \\ 1 & O_{1,2} \end{pmatrix}.$$ 

If $u_i \notin T$ and $u_{i+1} \in T$, then we let the $3 \times 3$ matrix $Q$

$$Q = \begin{pmatrix} -n_{i+1}q_{i+1}, n_{i+1}p_{i+1} & 1 \\ K_{i+1} & 0 \\ 0 & 0 \end{pmatrix}.$$ 

If $u_i \in T$ and $u_{i+1} \in T$, then we let the $2 \times 2$ matrix $Q$

$$Q = \begin{pmatrix} n_{i+1}(p_{i+1}\bar{y}_i - p_iq_i) & 1 \\ 1 & 0 \end{pmatrix}.$$
Let the $l \times l$ matrix $Q_i$

$$Q_i = \begin{pmatrix} 1 & \cdots & 1 \\ \cdots & Q & \cdots \\ 1 & \cdots & 1 \end{pmatrix}.$$ 

It is clear that

$$\det(Q_i) = \pm 1.$$ 

For $J' = (x'_1, y'_1; \ldots; x'_n, y'_n) \in W_2$, we let $\tilde{J}' = 'Q_iJ'$. Then we have $\tilde{J}' \in W_2$ and

$$\langle J', J' \rangle_K = \langle \tilde{J}', \tilde{J}' \rangle_{\tilde{K}}.$$ 

Q.E.D.

### 3.2 Null junction and rational null junction

Let $\rho_0$ be an path which is homotopic to constant mapping $id_{a_0}$. We define a null junction to be a tadpole junction which is homologous to $\rho_0(\gamma) (\gamma' \in H_1(f^{-1}(u_0), Z))$ and let

$$U_1 = \{ J = [x_1, y_1; \ldots; x_n, y_n] = (x'_1, y'_1; \ldots; x'_n, y'_n) \in Z^l \mid J \text{ is a null junction} \},$$

which is called the null junction $Z$-module.

**Lemma 3.1** The null junction $Z$-module $U_1$ is generated by $J_a = [x_1, y_1; \ldots; x_n, y_n] \in Z^{2n}$ and $J_b = [u_1, v_1; \ldots; u_n, v_n] \in Z^{2n}$, where

$$\begin{pmatrix} x_i \\ y_i \end{pmatrix} = K_{i-1}K_{i-2} \cdots K_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} u_i \\ v_i \end{pmatrix} = K_{i-1}K_{i-2} \cdots K_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1 \leq i \leq n).$$

**Proof.** Because $\rho_0 = \rho_1\rho_2 \cdots \rho_n$, by using (3) repeatedly, we obtain

$$\rho_0(\gamma) = \rho_1(\gamma) + \cdots + \rho_i(\rho_{i-1}^* \cdots \rho_1^*(\gamma)) + \cdots + \rho_n(\rho_{n-1}^* \cdots \rho_1^*(\gamma)).$$
We put \( \gamma = p\alpha + q\beta \) \((p, q \in \mathbb{Z})\). We have \( \rho_0(\gamma) = \rho_0(\alpha) + \rho_0(\beta) \). And, by the definition of the tadpole junction, we have

\[
J_a = \rho_1(\alpha) + \cdots + \rho_i(\rho^*_{i-1} \cdots \rho^*_1(\alpha)) + \cdots + \rho_n(\rho^*_{n-1} \cdots \rho^*_1(\alpha)) \\
J_b = \rho_1(\beta) + \cdots + \rho_i(\rho^*_{i-1} \cdots \rho^*_1(\beta)) + \cdots + \rho_n(\rho^*_{n-1} \cdots \rho^*_1(\beta)).
\]

We have thus proved the lemma. \(Q.E.D.\)

Similarly we define a rational null junction to be a rational tadpole junction which is homologous to \( \rho_0(\gamma') \) \((\gamma' \in H_1(f^{-1}(u_0), \mathbb{Q}))\) and let

\[
U_2 = \{J' = (x'_1, y'_1; \ldots; x'_n, y'_n) \in \mathbb{Z}^l \mid J' \text{ is a rational null junction}\},
\]

which is called the rational null junction \(\mathbb{Z}\)-module.

For \(i \leq i \leq n\), if \(u_i \in T\), then let \(N_i = n_i(-q_i, p_i)\) else \(N_i = I - K_i\). Let the \(l \times 2\) matrix \(N\)

\[
N = \begin{pmatrix}
N_1 \\
\vdots \\
N_n
\end{pmatrix},
\]

and the elementary divisor type of \(N\) be \((n_1, n_2)\). That is, there are a \(l \times l\) unimodular matrix \(A\) and a \(2 \times 2\) unimodular matrix \(B = (b_1, b_2)\) such that

\[
ANB = \begin{pmatrix}
n_1 & 0 \\
0 & n_2
\end{pmatrix}.
\]  

(10)

**Lemma 3.2** The rational null junction \(\mathbb{Z}\)-module \(U_2\) is generated by \(J'_a = (x'_1, y'_1; \ldots; x'_n, y'_n)\) and \(J'_b = (u'_1, v'_1; \ldots; u'_n, v'_n)\), where

\[
\begin{pmatrix} x'_i \\ y'_i \end{pmatrix} = \frac{1}{n_1} (I - K_i) K_{i-1} \cdots K_1 b_1, \quad \begin{pmatrix} u'_i \\ v'_i \end{pmatrix} = \frac{1}{n_2} (I - K_i) K_{i-1} \cdots K_1 b_2 \quad (1 \leq i \leq n).
\]  

(11)
Proof. Let \( rJ_a + sJ_b (r, s \in \mathbb{Q}) \) be a rational null junction, and put \( t (r', s') = B^{-1} t (r, s) \). For \( i (1 \leq i \leq n) \), if \( u_i \in T \), then let
\[
M_i = n_i \left( \left( \frac{p_i}{q_i} \right) \times K_{i-1} \cdots K_1 \left( \begin{array}{c} 1 \\ 0 \end{array} \right) , \left( \frac{p_i}{q_i} \right) \times K_{i-1} \cdots K_1 \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \right)
\]
else
\[
M_i = (I - K_i) K_{i-1} \cdots K_1.
\]
Let the \( l \times 2 \) matrix \( M \)
\[
M = \begin{pmatrix} M_1 \\ \vdots \\ M_n \end{pmatrix}.
\]
By the definition of the rational tadpole junction, we get \( M^t (r, s) \in \mathbb{Z}^l \). This is equivalent to the condition \( ANB^t (r', s') \in \mathbb{Z}^l \) because \( A \) is unimodular. By (10) it means \( n_1 x', n_2 y' \in \mathbb{Z} \), and which means \( n_1 x, n_2 y \in \mathbb{Z} \) because \( B \) is unimodular. That is, \( x = a/n_1, y = b/n_2 (a, b \in \mathbb{Z}) \). Therefore \( J'_a = (1/n_1) J_a, J'_b = (1/n_2) J_b \). By Lemma 3.1 we have (11). Q.E.D.

From the definition, we can identify the tadpole junction with the element of \( H_2 (X, \mathbb{Z}) \) by the class mapping
\[
h : W_1 \to V^\perp \subset L_k, \ J \mapsto [J].
\]

Lemma 3.3

\[\text{Ker } h = U_1\]

Proof. It is enough to show \( \text{Ker } h \subset U_1 \). Let \( J = [x_1, y_1; \ldots; x_n, y_n] \in \mathbb{Z}^{2n} \) be the element of \( \text{Ker } h \). For \( i (1 \leq i \leq n - 1) \), define the tadpole junction \( \tilde{J}_i, \hat{J}_i \) by
\[
\tilde{J}_i = \begin{pmatrix} 1 \\ 2 \\ i \\ i+1 \\ n-1 \\ n \end{pmatrix} \begin{pmatrix} 0, 0; 0, 0; \ldots; 1, 0; -1, 0; \ldots; 0, 0; 0, 0 \end{pmatrix}
\]
\[
\hat{J}_i = \begin{pmatrix} 1 \\ 2 \\ i \\ i+1 \\ n-1 \\ n \end{pmatrix} \begin{pmatrix} 0, 0; 0, 0; \ldots; 0, 1; 0, -1; \ldots; 0, 0; 0, 0 \end{pmatrix}.
\]
We have \( \langle J, \hat{J}_i \rangle = 0 \) and \( \langle J, \hat{J}_i \rangle = 0 \) because \( J \) is the element of \( \text{Ker} \ h \). On the other hand, by an easy calculation, we obtain
\[
\langle J, \hat{J}_i \rangle = \left( \begin{pmatrix} x_i \\ y_i \end{pmatrix} - K_{i-1} \begin{pmatrix} x_{i-1} \\ y_{i-1} \end{pmatrix} \right) \times \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad \langle J, \hat{J}_i \rangle = \left( \begin{pmatrix} x_i \\ y_i \end{pmatrix} - K_{i-1} \begin{pmatrix} x_{i-1} \\ y_{i-1} \end{pmatrix} \right) \times \begin{pmatrix} 0 \\ -1 \end{pmatrix}.
\]

Therefore we have \( \begin{pmatrix} x_i \\ y_i \end{pmatrix} = K_{i-1} \begin{pmatrix} x_{i-1} \\ y_{i-1} \end{pmatrix} \). And then we have \( \begin{pmatrix} x_i \\ y_i \end{pmatrix} = K_{i-1} \cdots K_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \).

By Lemma [3,3] we have \( J = x_1 J_a + y_1 J_b \in U_1 \). Q.E.D.

### 3.3 Brane configuration and junction

In this section we define the brane configuration and junction lattice and prove the necessary proposition and the lemmas.

It is well known that the local monodromy matrix \( K \) of the singular fibre \( F \) of type \( I_1 \)
\[
K = \begin{pmatrix} 1 + pq & -p^2 \\ q^2 & 1 - pq \end{pmatrix}
\]
if the vanishing cycle of \( F \) is \( p\alpha + q\beta \). Then we denote the singular fibre \( F \) by \( X_{[p,q]} \) and the matrix \( K \) by \( K_{[p,q]} \). In particular we denote \( X_{[1,0]}, X_{[1,-1]}, X_{[1,1]} \) by \( A, B, C \) and \( K_{[1,0]}, K_{[1,-1]}, K_{[1,1]} \) by \( K_A, K_B, K_C \) respectively. We call the finite sequence of \( X_{[p,q]} \) the **brane configuration**. Let \( X = X_{[p_1,q_1]} \cdots X_{[p_n,q_n]} \). By (F), we can define the elementary transformations \( T_i, T_i^{-1} \) of brane configurations
\[
T_i \left( X_{[p_1,q_1]} \cdots X_{[p_i,q_i]} X_{[p_{i+1},q_i+1]} \cdots X_{[p_n,q_n]} \right) = X_{[p_1,q_1]} \cdots X_{[p_{i+1},q_i+1]} X_{[r,s]} \cdots X_{[p_n,q_n]} \quad (12)
\]
\[
T_i^{-1} \left( X_{[p_1,q_1]} \cdots X_{[p_i,q_i]} X_{[p_{i+1},q_i+1]} \cdots X_{[p_n,q_n]} \right) = X_{[p_1,q_1]} \cdots X_{[t,u]} X_{[p_{i+1},q_i+1]} \cdots X_{[p_n,q_n]},
\]
where
\[
[r, s] = [p_i, q_i] + (p_i q_{i+1} - p_{i+1} q_i) [p_{i+1}, q_{i+1}]
\]
\[
[t, u] = [p_{i+1}, q_{i+1}] + (p_i q_{i+1} - p_{i+1} q_i) [p_i, q_i].
\]

If there exists a finite sequence of elementary transformation which transfer \( X_{[p_1,q_1]} \cdots X_{[p_n,q_n]} \) to \( X_{[p_1,q_1]} \cdots X_{[r_n,s_n]} \), then we identify \( X_{[p_1,q_1]} \cdots X_{[p_n,q_n]} \) with \( X_{[r_1,s_1]} \cdots X_{[r_n,s_n]} \) and denote \( X_{[p_1,q_1]} \cdots X_{[p_n,q_n]} \sim X_{[r_1,s_1]} \cdots X_{[r_n,s_n]} \).
Now we define the *junction lattice* of the brane configuration $X$. We define the $\mathbb{Z}$-module $Z_X$ of rank $n - 2$ by

$$Z_X = \left\{ Q = (Q_1, \ldots, Q_n) \in \mathbb{Z}^n \mid Q_1 \left( \frac{p_1}{q_1} \right) + \cdots + Q_n \left( \frac{p_n}{q_n} \right) = \left( \frac{0}{0} \right) \right\},$$

and the symmetric bilinear $(Q, Q)$ pairing by

$$(Q, Q) = -\sum_{i=1}^{n} Q_i^2 + \sum_{i<j} (p_i q_j - p_j q_i) Q_i Q_j.$$ 

We call the lattice $(Z_X, (\ , \ ))$ *junction lattice* of $X$ and call the element of $Z_X$ *junction*. This definition is well-defined \footnote{[3]}. We remark that the junction $Q = (Q_1, \ldots, Q_n)$ can be geometrically interpreted as the tadpole junction $J = (Q_1(p_1, q_1); \ldots; Q_n(p_n, q_n))$. By an easy calculation we obtain

$$\langle Q, Q \rangle = \langle J, J \rangle.$$ \hfill (13)

**Theorem 4.**\footnote{[1]}  
Let $X_{\left[p_1, q_1\right]} \cdots X_{\left[p_n, q_n\right]}$ be the brane configuration which satisfies

$$K_{\left[p_1, q_1\right]} \cdots K_{\left[p_n, q_n\right]} = I.$$ \hfill (14)

Then there exists a natural number $k$ such that $l = 12k$ and

$$X_{\left[p_1, q_1\right]} \cdots X_{\left[p_{12k}, q_{12k}\right]} \sim \underbrace{A^8_{\text{BCBC}}}_{12} \cdots \underbrace{A^8_{\text{BCBC}}}_{12} = \{A^8_{\text{BCBC}}\}^k.$$ \hfill (15)

**Theorem 5.**\footnote{[10]}  
Let $X$ be the brane configuration $X_{\left[p_1, q_1\right]} \cdots X_{\left[p_n, q_n\right]}$ and $K_X$ be the matrix $K_{\left[p_1, q_1\right]} \cdots K_{\left[p_n, q_n\right]}$. Let $M_X$ be the Gram matrix of the junction lattice $Z_X$ of $X$ and $d$ is the G.C.D of $x_i y_j - x_j y_i$ ($1 \leq i < j \leq n$). Except for the case $X = X_{\left[p, q\right]}$, we have

$$\det(M_X) = \frac{\det(I - K_X)}{d^2}.$$
Proposition 3.3 Let \( X = X_{[p_1, q_1]} \cdots X_{[p_{12k}, q_{12k}]} \) be the brane configuration which satisfies the condition (15) and \( Z_X \) the junction lattice of \( X \). Let \( U \) be a \( \mathbb{Z} \)-module generated by
\[
a_i = \left( \frac{p_i}{q_i} \right) \times K_{[p_{i-1}, q_{i-1}]} \cdots K_{[p_1, q_1]} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad b_i = \left( \frac{p_i}{q_i} \right) \times K_{[p_{i-1}, q_{i-1}]} \cdots K_{[p_1, q_1]} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]
Then the quotient lattice \( Z_X / U \) is isomorphic to \( L_k \).

Proof.

Claim 1. \((Q, Q)\) is even for arbitrary \( Q = (Q_1, \ldots, Q_{12k}) \in Z_X \).

proof. By induction, we can prove the following lemma.

Lemma 3.4 Let \( Q = (Q_1, \ldots, Q_{12k}) \) is an element of \( \mathbb{Z}^{12k} \). If
\[
Q_1 \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} + \cdots + Q_{12k} \begin{pmatrix} p_{12k} \\ q_{12k} \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (\text{mod} \ 2),
\]
then \((Q, Q)\) is even, else \((Q, Q)\) is odd.

Claim 1. follows from this lemma and the definition of \( Z_X \). q.e.d.

For a brane configuration \( Y \), we denote the positive signature, the negative signature and the nullity of the Gram matrix of \( Z_Y \) by \( \sigma_+ (Y) \), \( \sigma_- (Y) \) and \( \sigma_0 (Y) \). For a matrix
\[
A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix}, \quad \text{we denote} \quad \begin{pmatrix} a_{1,1} & \cdots & a_{1,i} \\ \vdots & \ddots & \vdots \\ a_{i,1} & \cdots & a_{i,i} \end{pmatrix} \quad \text{by} \ A_i \quad (1 \leq i \leq n).
\]

Claim 2. \( \sigma_+ (X) = 2k - 2, \ \sigma_- (X) = 10k - 2, \ \sigma_0 (X) = 2. \)

proof. By (12), we have
\[
X \sim X_1 X_2
\]
where \( X_1 = A^{8k} B^{2k} \) and \( X_2 = X_{[4k-3,1-4k]} X_{[4k-5,3-4k]} \cdots X_{[1,-3]} C \).

First we show that \( \sigma_- (X) \geq 10k - 2. \)
In fact we have \( Z_{X_1} \cong (-A_{8k-1}) \oplus (-A_{2k-1}) \), and then \( \sigma_- (X) \geq \sigma_- (X_1) = 10k - 2. \)
Next we show that \( \sigma_+ (X) \geq 2k - 2. \)
It is enough to show that $Z_{X_2}$ is positive definite. Let $Y_n = X_{[2n-3,1-2n]}X_{[2n-5,3-2n]}\cdots X_{[1,-3]}C$ and $K_{Y_n} := \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} = K_{[-1,-1]} \cdots K_{[2n-5,3-2n]}K_{[2n-3,1-2n]}$ ($n \geq 3$). Then by induction we have

$$a_n = (-1)^{n+1}(4n^2 - n - 1), \quad b_n = (-1)^{n+1}(4n^2 - 5n),$$

$$c_n = (-1)^{n+1}n, \quad d_n = (-1)^{n+1}(n - 1).$$

Let $b_{n,i} = (0, \ldots, 0, i, \ldots, 0, 1 - i, 2 - i)$ ($3 \leq i \leq n$). It is easy to see that $B_i = \langle b_{i,3}, \ldots, b_{i,i} \rangle$ is the basis of $Z_{Y_i}$. Let $M_{Y_i}$ be the Gram matrix of $Z_{Y_i}$ about the basis $B_i$ ($3 \leq i \leq n$). Then by Theorem 5. and (16) we have

$$\det ((M_{Y_n})_i) = \det (M_{Y_i})$$

$$= \left(4i^2 - 2 + (-1)^i2\right)/16 > 0 \quad (3 \leq i \leq n).$$

Therefore it follows that $Z_{Y_n}$ is positive definite. We see that $Z_{X_2}$ is positive definite because $X_2 = Y_{2k}$.

Lastly we have $\sigma_0(X) \geq 2$ by Lemma 3.1 and Lemma 3.3. Therefore we have proved Claim 2. q.e.d.

By (12), we have

$$X \sim X_3X_{[3,1]}A,$$

where $X_3 = (A^8BCBC)^{n-1}A^7BC^2$. Let $M_{X_3}$ be the Gram matrix of $Z_{X_3}$ and $C = \langle c_1, \ldots, c_{12k-4} \rangle$ the basis of $Z_{X_3}$. By Theorem 5. we have $\det (M_{X_3}) = 1$. Hence it follows from Lemma 3.1 and Lemma 3.3 that $\langle c_1, \ldots, c_{12k-4}, Q_a, Q_b \rangle$ is the basis of $Z_X$. Therefore $Z_X/U$ is isomorphic to $Z_{X_3}$. By Claim 1. and Claim 2. we see that $Z_{X_3}$ is even unimodular lattice and the signature of $Z_{X_3}$ is $(2k - 2, 10k - 2)$. We have $Z_{X_3} \cong L_k$ by lattice theory [15]. We have thus proved the proposition. Q.E.D.

In the following we describe the confluence of singular fibres by the brane configuration. Consider the case that the singular fibre $F$ is of type $I^*_0$ for instance. In this case it is known that the local monodromy matrix of $F$ is $-I$ [4]. The brane configuration $X_F = A^4BC$ corresponds to $F$ because $-I = K_{[1,1]}K_{[1,-1]}(K_{[1,0]})^4$. We remark that this kind of expression of the monodromy matrix as a confluence is not unique. However,
Table 2: the brane configuration $X_F$ and the junction lattice $Z_{X_F}$ of the singular fibre $F$

| $I_n$ $(1 \leq n)$ | $II$ | $III$ | $IV$ | $I_n^*$ $(0 \leq n)$ | $II^*$ | $III^*$ | $IV^*$ |
|---------------------|------|-------|------|----------------------|-------|--------|-------|
| $A^n$               | $AC$ | $A^2C$| $A^3C$| $A^{n+1}BC$          | $A^7BC^2$| $A^6BC^2$| $A^6BC^2$|
| $-A_{n-1}$ ($A_0 = 0$) | 0    | $-A_1$| $-A_2$| $-D_{n+4}$           | $-E_8$ | $-E_7$ | $-E_6$ |

all such expressions are equivalent (see for instance \[9\].) And then it turns out that the junction lattice $Z_{X_F}$ is $Z_{X_F} = -D_4$ \[8\], which corresponds to the fact that $-T_v$ of the singular fibre of type $I_0^*$ is $D_4$. In general, the brane configuration $X_F$ and the junction lattice $Z_{X_F}$ of the singular fibre $F$ can be summarized in Table 2 \[8\].

Consider the elliptic surface $X_k$ such that the global monodromy is $K = (K_1, \ldots, K_n)$. Let the brane configuration $X_{F_{u_i}}$ which correspond to the singular fibre $F_{u_i}$

$$X_{F_{u_i}} = X_{[p_{i,1},q_{i,1}]} \cdots X_{[p_{i,m_i},q_{i,m_i}]} \quad (1 \leq i \leq n),$$

and then we have

$$K_i = K_{[p_{i,m_i},q_{i,m_i}]} \cdots K_{[p_{i,1},q_{i,1}]}.$$

We associate the brane configuration $X = (X_{F_{u_1}}) \cdots (X_{F_{u_n}})$ with the global monodromy $K$, and then we call $X$ the \textit{brane configuration} of $X_k$. We put $\tilde{X} = X_{F_{u_1}} \cdots X_{F_{u_n}} = X_{[p_{1,1},q_{1,1}]} \cdots X_{[p_{1,m_1},q_{1,m_1}]} \cdots \cdots \cdots X_{[p_{n,1},q_{n,1}]} \cdots X_{[p_{n,m_n},q_{n,m_n}]}$. The junction lattice $Z_{\tilde{X}}$ of $\tilde{X}$ is called \textit{junction lattice} of $X_k$ and denoted by $\tilde{L}_k$. Note that the parenthesis have the essential meaning in the representation of brane configurations, namely $X = (X_{F_{u_1}}) \cdots (X_{F_{u_n}})$ is a confluence of $\tilde{X} = X_{F_{u_1}} \cdots X_{F_{u_n}}$.

We define a mapping $v : W_1 \rightarrow \tilde{L}_k$, $J = [x_1, y_1; \ldots; t_k; \ldots; x_n, y_n] \mapsto v(J)$ by

$$v(J) = (Q_{1,1}, \ldots, Q_{1,m_1}; \ldots; Q_{n,1}, \ldots, Q_{n,m_n}),$$
where
\[ Q_{i,j} = \begin{cases} 
  t_i & \text{if } u_i \in T, \\
  (p_{i,j}^{q_{i,j}}) \times K_{p_{i,j-1,q_{i,j-1}}} \cdots K_{p_{i,1,q_{i,1}}} \left( x_i \right) & (1 \leq j \leq m_i) \text{ if } u_i \notin T. 
\end{cases} \] (17)

for \( 1 \leq i \leq n \). By a calculation we can show the following fact.

\[ \langle J_1, J_2 \rangle = (v(J_1), v(J_2)). \]

We put \( U_0 = v(U_1) \). Let

\[ \tilde{W}_i = \left\{ Q = (Q_{1,1}, \ldots, Q_{1,m_1}; \ldots; Q_{n,1}, \ldots, Q_{n,m_n}) \in \mathbb{Z}^{12k} \mid \sum_{j=1}^{m_i} Q_{i,j} \left( p_{i,j}^{q_{i,j}} \right) = \left( 0 \right) \right\} (1 \leq i \leq n), \]

\[ \tilde{W} = \bigoplus_{i=1}^n \tilde{W}_i, \text{ and } \tilde{V} = \{ J + U_0 \mid J \in \tilde{W} \}. \]

**Lemma 3.5** The quotient lattice \( \tilde{L}_k/U_0 \) is isomorphic to \( L_k \).

**Proof.** In Proposition 3.3 we take \( X_{[p_1, q_1]} \cdots X_{[p_{12k-1}, q_{12k}]} = X_{F_{u_1}} \cdots X_{F_{u_n}} \) instead of \( X = X_{[p_{1}, q_{1}]} \cdots X_{[p_{12k-1}, q_{12k}]} \). It is enough to show \( U = U_0 \) by Proposition 3.3. Let \( \tilde{K}_i = K_{[p_{i}, q_{i}]} (1 \leq i \leq n) \), and put \( l = \sum_{k=1}^{i-1} m_k + j \). Then we obtain

\[ a_l = \left( p_l^{q_l} \right) \times \tilde{K}_l \cdots K_{\sum_{k=1}^{i-1} m_k + 1} \tilde{K}_{\sum_{k=1}^{i-1} m_k} \cdots K_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

\[ = \left( p_{i,j}^{q_{i,j}} \right) \times K_{[p_{i,j-1}, q_{i,j-1}]} \cdots K_{[p_{i,1}, q_{i,1}]} K_{i-1} \cdots K_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

By Lemma 3.3 and (17) we obtain \( Q_a = v(J_a) \). Similarly we can show \( Q_b = v(J_b) \). By \( U = \langle Q_a, Q_b \rangle \) and \( U_0 = \langle v(J_a), v(J_b) \rangle \), we have \( U = U_0 \). Q.E.D.

**Lemma 3.6** The quotient lattice \( \tilde{V}/U_0 \) is isomorphic to \( V \).

**Proof.** By the definition of \( \tilde{V} \) and of \( \tilde{W} \) we have

\[ \tilde{V}/U_0 \simeq \tilde{W} = \bigoplus_{i=1}^n \tilde{W}_i. \]
On the other side we have \( \tilde{W}_i \cong T_{u_i} \) \( (1 \leq i \leq n) \) by Table 1 and Table 2. Therefore we have \( \tilde{V}/U_0 \cong V \). Q.E.D.

Consider the lattice embedding

\[
i_K : \tilde{V}/U_0 \hookrightarrow \tilde{L}_k/U_0.
\]

**Theorem 3.1** The lattice embedding \( i_K : \tilde{V}/U_0 \hookrightarrow \tilde{L}_k/U_0 \) is equivalent to the lattice embedding \( i_V : V \hookrightarrow L_k \), that is, the following diagram commutes.

\[
\begin{array}{ccc}
i_K : \tilde{V}/U_0 & \hookrightarrow & \tilde{L}_k/U_0 \\
\downarrow & & \downarrow \\
i_V : V & \hookrightarrow & L_k
\end{array}
\]

**Proof.** There exists a topological elliptic surface \( \tilde{X}_k \) such that the brane configuration is \( \tilde{X} = X_{F_{u_1}} \cdots X_{F_{u_n}} \). We denote \( L_k \) of \( \tilde{X}_k \) by \( L_k' \). By definition \( \tilde{L}_k \) is the junction lattice of \( \tilde{X}_k \) and \( U_0 \) is the null junction \( \mathbb{Z} \)-module of \( \tilde{X}_k \). Let \( \tilde{h} : \tilde{L}_k \rightarrow L_k' \) be the class mapping and put \( \tilde{h}(\tilde{V}) = V' \). By Lemma 3.3, Lemma 3.5 and Lemma 3.6, \( i_K \) is equivalent to the lattice embedding \( i : V \hookrightarrow L_k' \). On the other hand, because there exists a topological deformation from \( X_k \) to \( \tilde{X}_k \), \( i \) is equivalent to \( i_V \). Q.E.D.

Let \( \tilde{V}^\perp \) be the orthogonal complement of \( \tilde{V} \) in \( \tilde{L}_k \).

**Lemma 3.7** The mapping \( v : W_1 \rightarrow \tilde{V}^\perp \) is isomorphic.

**Proof.** By (13) we have \( v(W_1) \subset \tilde{V}^\perp \). Let \( Q = (Q_{1,1}, \ldots, Q_{1,m_1}; \ldots; Q_{n,1}, \ldots, Q_{n,m_n}) \) be an element of \( \tilde{V}^\perp \). In the same way as the proof of Lemma 3.3 we can show that \( Q \) satisfies the condition (17). Hence we have \( v(W_1) = \tilde{V}^\perp \).

We define the mapping \( v' : \tilde{V}^\perp \rightarrow W_1, Q = (Q_{1,1}, \ldots, Q_{1,m_1}; \ldots; Q_{n,1}, \ldots, Q_{n,m_n}) \mapsto v'(Q) \) by

\[
v'(Q) = \left( \sum_{j=1}^{m_1} Q_{1,j} (p_{1,j}, q_{1,j}); \ldots; \sum_{j=1}^{m_n} Q_{n,j} (p_{n,j}, q_{n,j}) \right).
\]

We have \( v' \circ v = id_{W_1} \) and \( v \circ v' = id_{\tilde{V}^\perp} \) by (4) and (4). We have thus proved the lemma. Q.E.D.
4 Torsion part of Mordell-Weil group, Mordell-Weil lattice, and transcendental lattice

4.1 Junctions and homologies

In this section we explain the relation between tadpole junction, rational tadpole junction and homology.

First we describe the relation between a tadpole junction and homology.

**Theorem 4.1** Let $W_1$ be the tadpole junction lattice of $X_k$ and $U_1$ the null tadpole junction $\mathbb{Z}$-module of $X_k$. Then the quotient lattice $W_1/U_1$ is isomorphic to $V^\perp$.

**Proof.** By Lemma 3.3, it is enough to show that the mapping $h$ is surjective. We can define the mapping $u = h \circ v^{-1} : \tilde{V}^\perp \to V^\perp$ by Lemma 3.7. It is clear that $u$ conserve the intersection number. Because $\text{Ker} u = U_0$, we have $\tilde{V}^\perp/U_0 \cong u(\tilde{V}^\perp)$. On the other side, by the meaning of $U_0$ and Theorem 3.1, we have $\tilde{V}^\perp/U_0 \cong \left(\tilde{V}/U_0\right)^\perp \cong V^\perp$. Therefore we see that $u$ is surjective. It follows from this that $h$ is surjective immediately. Q.E.D.

The following corollary follows from the theorem proved above and Theorem 3.

**Corollary 4.1** The transcendental lattice of extremal elliptic $K3$ surfaces is isomorphic to the quotient lattice $W_1/U_1$.

In the following we describe the relation between a rational tadpole junction and homology.

**Lemma 4.1** The quotient lattice $W_2/U_1$ is isomorphic to $L_k/V$:

$$W_2/U_1 \cong L_k/V.$$  \hfill (19)

**Proof.** Let the mapping $w : \tilde{L}_k \to W_2$, $Q = (Q_{1,1}, \ldots, Q_{1,m_1}; \ldots; Q_{n,1}, \ldots, Q_{n,m_n}) \mapsto w(Q)$ defined by

$$w(Q) = \left( \sum_{j=1}^{m_1} Q_{1,j} (p_{1,j}, q_{1,j}) ; \ldots ; \sum_{j=1}^{m_n} Q_{n,j} (p_{n,j}, q_{n,j}) \right)$$
and the mapping \( \bar{w} : \tilde{L}_k/U_0 \to W_2/U_1 \) defined by \( J + U_0 \mapsto w(J) + U_1 \). Let the mapping 
\( \bar{v} : W_2/U_1 \to (\tilde{L}_k \otimes \mathbb{Q})/U_0 = (\tilde{L}_k/U_0) \otimes \mathbb{Q} \) defined by \( J + U_1 \mapsto \bar{v}(J) + U_1 \), where \( \bar{v} \) is defined by the same formula (17) for \( v \). Let \( p = \bar{v} \circ \bar{w} : \tilde{L}_k/U_0 \to (\tilde{L}_k/U_0) \otimes \mathbb{Q} \). Then 
we see that \( p \) is the orthogonal projection to the direction of \( \tilde{V}/U_0 \). By Theorem 2.1, 
we have \( p(\tilde{L}_k/U_0) \cong L_k/V \). On the other hand we have \( W_2/U_1 \cong p(\tilde{L}_k/U_0) \) because 
\( \bar{w}(\tilde{L}_k/U_0) = (W_2/U_1) \) and \( \bar{v} : W_2/U_1 \to \bar{v}(W_2/U_1) \) is isomorphic. Therefore we obtain (19). Q.E.D.

**Theorem 4.2** Let \( W_2 \) be the rational tadpole junction lattice of \( X_k \) and \( U_2 \) the rational null junction \( \mathbb{Z} \)-module of \( X_k \). Then the quotient lattice \( W_2/U_2 \) is isomorphic to \( (V^\perp)^* \).

**Proof.** We know the following fact [3]:

\[ L_k/V \cong (V^\perp)^* \oplus \hat{V}/V. \]

The theorem follows from taking the free part of (19). Q.E.D.

It follows from the theorem proved above and Theorem 2

**Corollary 4.2** The Mordell-Weil lattice of rational elliptic surfaces \( X_k \) is isomorphic to the opposite lattice of the quotient lattice \( W_2/U_2 \).

The following theorem follows from taking the torsion part of (19).

**Theorem 4.3** Let \( U_1 \) be the null junction \( \mathbb{Z} \)-module of \( X_k \) and \( U_2 \) the rational null junction \( \mathbb{Z} \)-module of \( X_k \). Then \( U_2/U_1 \) is isomorphic to \( \hat{V}/V \).

It follows from the theorem proved above and Theorem 1

**Corollary 4.3** The torsion part of Mordell-Weil group of elliptic surfaces is isomorphic to \( U_2/U_1 \).
4.2 Torsion part of Mordell-Weil group, Mordell-Weil lattice, transcendental lattice

In this section we explain how to calculate the torsion part of Mordell-Weil group of elliptic surfaces, the Mordell-Weil lattice of rational elliptic surfaces, and the transcendental lattice and the Mordell-Weil group of extremal elliptic $K3$ surfaces from the given data of global monodromy.

First we describe the method to calculate the torsion part of Mordell-Weil group of elliptic surfaces from the given data of global monodromy.

**Theorem 4.4** The torsion part of Mordell-Weil group of elliptic surfaces is isomorphic to $\mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_2\mathbb{Z}$.  

The theorem follows from Lemma 3.1, Lemma 3.2 and Corollary 4.3, and the explicit algorithm to calculate the numbers $n_1, n_2$ is already given in Section 3.2.

Next we describe the method to calculate the transcendental lattice of extremal elliptic $K3$ surfaces from the given data of global monodromy.

It is sufficient to get the basis of $W_1/U_1$ by Corollary 4.1 and (6).

1. Construction of the basis $\langle J_1, J_2, \ldots, J_{l-2} \rangle$ of $W_1$.

   We define the $2 \times l$ matrix $R$ such that the following equation is equivalent to the equation (6).

   $$R^t (x_1, y_1; \ldots; t_i; \ldots; x_n, y_n) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$  

   By using the elementary divisor theory we can have a $2 \times 2$ unimodular matrix $H$ and a $l \times l$ unimodular matrix $G = (g_1, \ldots, g_l)$ such that  

   $$HRG = \begin{pmatrix} u_1 & 0 & 0 & \cdots & 0 \\ 0 & u_2 & 0 & \cdots & 0 \end{pmatrix}.$$  

   Then the basis are obtained as  

   $$J_1 = ^t g_3, \ J_2 = ^t g_4, \ldots, \ J_{l-2} = ^t g_l.$$  

2. Construction of the basis $\langle J_a, J_b \rangle$ of $U_1$.

   This is given in Lemma 3.1.
3. Construction of the basis of $W_1/U_1$.

Let

$$D = \begin{pmatrix} J_1 \\ \vdots \\ J_{l-2} \end{pmatrix}, \quad O = \begin{pmatrix} J_a \\ J_b \end{pmatrix}.\]

Then there exists a unique $2 \times (l-2)$ matrix $L$ such that

$$LD = O.$$

Because the embedding $U_1 \subset W_1$ is primitive, by the elementary divisor theory, there exists a $2 \times 2$ unimodular matrix $P$ and a $(l-2) \times (l-2)$ unimodular matrix $Q$ such that

$$PLQ = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Let the $(l-4) \times l$ matrix $V$

$$V := \begin{pmatrix} v_1 \\ \vdots \\ v_{l-4} \end{pmatrix} := \begin{pmatrix} 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} Q^{-1} D.$$

Then $\langle v_1, \ldots, v_{l-4} \rangle$ is the basis of $W_1/U_1$.

Finally we describe the method to calculate Mordell-Weil lattice of rational elliptic surfaces from the given data of global monodromy.

It is sufficient to get the basis of $W_2/U_2$ by Corollary 4.2 and (9).

1. Construction of the basis $\langle J'_1, J'_2, \ldots, J'_{l-2} \rangle$ of $W_2$.

We define the $2 \times l$ matrix $R'$ such that the following equation is equivalent to the equation (8).

$$R'^t (x'_1, y'_1; \ldots; t'_i; \ldots; x'_n, y'_n) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

By using the elementary divisor theory we can have a $2 \times 2$ unimodular matrix $H'$ and a $l \times l$ unimodular matrix $G' = (g'_1, \ldots, g'_l)$ such that

$$H'R'G' = \begin{pmatrix} u'_1 & 0 & 0 & \cdots & 0 \\ 0 & u'_2 & 0 & \cdots & 0 \end{pmatrix}.$$
Then we can take
\[ J'_1 = t g'_3, \ J'_2 = t g'_4, \ldots, \ J'_{l-2} = t g'_l. \]

2. Construction of the basis \( \langle J'_a, J'_b \rangle \) of \( U_2 \)
   This is given in Lemma 3.2.

3. Construction of the basis of \( W_2/U_2 \).
   Let
   \[
   D' = \begin{pmatrix}
   J'_1 \\
   \vdots \\
   J'_{l-2}
   \end{pmatrix}, \quad O' = \begin{pmatrix}
   J''_a \\
   J'_b
   \end{pmatrix}.
   \]
   Then there exists a unique \( 2 \times (l-2) \) matrix \( L' \) such that
   \[ L'D' = O'. \]
   Because the embedding \( U_2 \subset W_2 \) is primitive, by the elementary divisor theory,
   there exists a \( 2 \times 2 \) unimodular matrix \( P' \) and a \( (l-2) \times (l-2) \) unimodular matrix \( Q' \) such that
   \[
   P'L'Q' = \begin{pmatrix}
   1 & 0 & 0 & \cdots & 0 \\
   0 & 1 & 0 & \cdots & 0
   \end{pmatrix}.
   \]
   Let the \( (l-4) \times l \) matrix \( V' \)
   \[
   V' := \begin{pmatrix}
   v'_1 \\
   \vdots \\
   v'_{l-4}
   \end{pmatrix} := \begin{pmatrix}
   0 & 0 & 1 & 0 & \cdots & 0 \\
   0 & 0 & 0 & 1 & \cdots & 0 \\
   \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
   0 & 0 & 0 & 0 & \cdots & 1
   \end{pmatrix} Q'^{-1} D'.
   \]
   Then \( \langle v'_1, \ldots, v'_{l-4} \rangle \) is the basis of \( W_2/U_2 \).

**Example 4.1** (Table \( \mathbf{3} \) : No.34)

\( X = (A^4BC)(A^2)(A^2)BC \).

\( K = \left( \begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}, \begin{pmatrix}
1 & -2 \\
0 & 1
\end{pmatrix}, \begin{pmatrix}
1 & -2 \\
0 & 1
\end{pmatrix}, \begin{pmatrix}
0 & -1 \\
1 & 2
\end{pmatrix}, \begin{pmatrix}
2 & -1 \\
1 & 0
\end{pmatrix} \right) \).

The Mordell-Weil group \( MW \cong A_1^{\oplus 2} \oplus \mathbb{Z}/2\mathbb{Z} \) is computed as follows:
Therefore we have

\[ M^W = 3. \]

\[ n_1 = 1, n_2 = 2. \]

\[ J'_a = (2; 0; 0; -1; -1), J'_b = (-1; 1; -1; -1; 2; 1) \]

\[ \begin{align*}
L' &= \begin{pmatrix} 0 & 0 & -1 & -1 \\ -1 & -1 & 2 & 1 \end{pmatrix},
P' &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
Q' &= \begin{pmatrix} -1 & 0 & -1 & -1 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 \end{pmatrix}. 
\end{align*} \]

\[ v'_1 = (-1, 1, 0, 0, 1, 0), v'_2 = (0, -1, -1, 0, 0, 1). \]

For \( J' = (x'_1, y'_1; t'_2; t'_3; t'_4; t'_5) \), we have

\[ \langle J', J' \rangle = -t'_2^2/2 - t'_3^2/2 - t'_4^2 - t'_5^2 - x'_1t'_4 + x'_1t'_5 - y'_1t'_2 - y'_1t'_3 - y'_1t'_4 - y'_1t'_5 - t'_2t'_4 + t'_2t'_5 - t'_3t'_4 + t'_3t'_5 + 2t'_4t'_5. \]

By using reduction theory, we have

\[ \langle v'_1, v'_2 \rangle = \begin{pmatrix} -1/2 & 3/2 \\ 3/2 & -5/2 \end{pmatrix} \cong \begin{pmatrix} -1/2 & 0 \\ 0 & -1/2 \end{pmatrix} = \langle v'_a, v'_b \rangle, \]

where \( v'_a = v'_1 + v'_2 = (-1, 0; -1; 0; 1, 1), v'_b = -2v'_1 - v'_2 = (2, -1; 1; 0; -2; -1). \)

Therefore we have \( M^W \cong A_1^{\oplus 2} \oplus \mathbb{Z}/2\mathbb{Z}. \)
Example 4.2 (Table 4: No.61 (a))

\[ X = (A^{10}) (X_{[0,1]}^6) (C^4) (X_{[3,1]}^2) \] \(X_{[41,11]}X_{[5,1]}\).

For \(J = [t_1; t_2; t_3; t_4; t_5; t_6]\), we have

\[
\langle J, J \rangle = -10t_1^2 - 6t_2^2 - 4t_3^2 - 2t_4^2 - t_5^2 - t_6^2 + 60t_1t_2 + 40t_1t_3 + 20t_1t_4 + 110t_1t_5 + 10t_1t_6 \\
- 24t_2t_3 - 36t_2t_4 - 246t_2t_5 - 30t_2t_6 - 16t_3t_4 - 120t_3t_5 - 16t_3t_6 - 16t_4t_5 \\
- 4t_4t_6 - 14t_5t_6.
\]

\[
\langle v_a, v_b \rangle = \begin{pmatrix} 10 & 0 \\ 0 & 12 \end{pmatrix} \text{ (transcendental lattice),}
\]

where \(v_a = [0; -7; 37; -176; 23; -7]\), \(v_b = [0; 0; -1; 4; 0; -4]\).

\(U_1 = \langle J_a, J_b \rangle\), where \(J_a = [0; -1; 5; -23; 3; -1]\), \(J_b = [1; 10; -49; 225; -29; 5]\).

\(U_2 = \langle J'_a, J'_b \rangle\), where \(J'_a = (0; -6; 20; -46; 3; -1)\), \(J'_b = (5; 15; -48; 110; -7; 0)\).

\(J'_a = J_a \in U_1, \ J'_b \notin U_1, \ 2J'_b = 5J_a + J_b \notin U_1, \ MW \cong \mathbb{Z}/2\mathbb{Z}\).

Example 4.3 (Table 4: No.61 (b))

\[ X = (A^{10}) (B^6)(X_{[4,-3]}^4)(X_{[3,-4]}^2)X_{[1,-3]}C. \]

For \(J = [t_1; t_2; t_3; t_4; t_5; t_6]\), we have

\[
\langle J, J \rangle = -10t_1^2 - 6t_2^2 - 4t_3^2 - 2t_4^2 - t_5^2 - t_6^2 - 60t_1t_2 - 200t_1t_3 - 80t_1t_4 - 30t_1t_5 + 10t_1t_6 \\
- 24t_2t_3 - 12t_2t_4 - 12t_2t_5 + 12t_2t_6 - 8t_3t_4 - 28t_3t_5 + 36t_3t_6 - 10t_4t_5 \\
+ 14t_4t_6 + 4t_5t_6.
\]

\[
\langle v_a, v_b \rangle = \begin{pmatrix} 10 & 0 \\ 0 & 12 \end{pmatrix} \text{ (transcendental lattice),}
\]

where \(v_a = [0; 3; -1; 0; 0; -2]\), \(v_b = [0; 2; -3; 6; 0; 0]\).

\(U_1 = \langle J_a, J_b \rangle\), where \(J_a = [0; 1; -1; 2; -1; -1]\), \(J_b = [1; -9; 8; -15; 5; 1]\).

\(U_2 = \langle J'_a, J'_b \rangle\), where \(J'_a = (10; -54; 32; -30; 5; 1)\), \(J'_b = (-5; 24; -14; 13; -2; 0)\).

\(J'_a = J_b \in U_1, \ J'_b \notin U_1, \ 2J'_b = -J_a - J_b \in U_1, \ MW \cong \mathbb{Z}/2\mathbb{Z}\).
Remark 4.1 Let $X_1$ be the extremal semi-stable elliptic $K3$ surface of Example 4.2 and $X_2$ be the one of Example 4.3. Then the triplet $\left( V, V^\perp, \hat{V}/V \right)$ of $X_1$ coincides with the one of $X_2$, however the embeddings $i_V$ of these surface are not equivalent. The difference of these embeddings can be seen by looking at the structure of sections.

Let $X$ be an extremal semi-stable elliptic $K3$ surface which have $I_{n_1}, \ldots, I_{n_6}$ type fibres, and $\psi$ be the homomorphism from $MW(X)$ to $\mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_6\mathbb{Z}$ given in [11], i.e., $\psi(s) = (a_1, \ldots, a_6)$, where $a_i$ specify the irreducible component that the section $s$ intersects the corresponding singular fibre. Let $s_1$ be the generator of $MW(X_1)$, and then we have $\psi(s_1) = (5, 3, 0, 0, 0, 0)$ [11][13]. We can consider that $J'_b = (5; 15; -48; 110; -7; 0)$ corresponds to $s_1$ at this case. In fact we have $J'_b \equiv \psi(s_1) \mod (10, 6, 4, 2, 1, 1)$. Similarly for the surface of $X_2$, the generator of $MW(X_2)$ $s_2$ satisfy $\psi(s_2) = (5, 0, 2, 1, 0, 0)$ and $J'_b = (-5; 24; -14; 13; -2; 0)$. Then we also have $J'_b \equiv \psi(s_1) \mod (10, 6, 4, 2, 1, 1)$.

5 The global monodromies of rational elliptic surfaces and of extremal elliptic $K3$ surfaces

5.1 Some working hypotheses for classifications of global monodromies

In this section we discuss the global monodromies $K = (K_1, \ldots, K_n)$ of elliptic surfaces.

Let

$$M_1 = \{ K \mid K \text{ satifies the condition (a)} \}, \quad M_2 = \{ K \mid K \text{ satifies the condition (b)} \},$$

where the conditions (a), (b) are as follows :

(a) There exists a complex elliptic surface such that the global monodromy is $K$ and the Euler number is $12k$.

(b) There exists a topological elliptic surface such that the global monodromy is $K$ and the Euler number is $12k$. 

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Let

\[ E_1 = \{ i_V : V \leftrightarrow L_k \mid i_V \text{ satifies the condition (i)} \} , \]
\[ E_2 = \{ i_V : V \leftrightarrow L_k \mid i_V \text{ satifies the condition (ii)} \} , \]

where the conditions (i), (ii) is as follows :

(i) There exists a complex elliptic surface which realize the embedding \( i_V \).

(ii) There exists a topological elliptic surface which realize the embedding \( i_V \).

Let the mapping \( F : M_2 \rightarrow E_2 \) defined by \( K \mapsto i_K : V \leftrightarrow L_k \).

We define the following three kinds of operations I, II, III acting on monodromy data \( K = (K_1, \ldots, K_n) \in M \).

I. The elementary transformation :
\[ R_i : K \mapsto R_i (K) \quad (1 \leq i \leq n - 1). \]

II. The action of \( P \in SL(2, \mathbb{Z}) \):
\[ K \mapsto K' = (K'_1, \ldots, K'_{n}), \text{ where } K'_i = PK_iP^{-1} (1 \leq i \leq n). \]

III. \( K \mapsto K' = (K'_1, \ldots, K'_{n+1}) \).
\[ (K_1, \ldots, K_{i-1}, K_C K_A^h, K_{i+1}, \ldots, K_n) \mapsto (K_1, \ldots, K_{i-1}, K_A^h, K_C, K_{i+1}, \ldots, K_n), \]
where \( h = 2, 3, 4. \)

If \( K \) transfer to \( K' \) by a finite sequence of the operations I, II, III and of the inverse operations of them, then we denote \( K \approx K' \). It is easy to see that \( F(K) = F(K') \) if \( K \approx K' \). We anticipate that the converse is true.

**Hypothesis 1.**

(H\_1) Let \( K, K' \) be the elements of \( M_2 \). Then we have \( K \approx K' \) if \( F(K) = F(K') \).

Let
\[ \tilde{M}_2 = \{ K \mid K \text{ satifies the condition (d)} \} \subseteq M_2, \]

where the condition (c) is as follows :
there exists a topological elliptic surface such that the global monodromy is $K$ and the Euler number is $12k$, and which has no singular fibre of type $II$, $III$, $IV$.

It follows from Hypothesis 1. immediately.

**Hypothesis 2.** (H2) The mapping $F|_{\tilde{M}_2} : \tilde{M}_2 \to E_2$ is bijective.

**Hypothesis 3.** (H3) The mapping $F|_{M_1 \cap \tilde{M}_2} : M_1 \cap \tilde{M}_2 \to E_1$ is surjective.

Let

$$M = M_1 \cap \tilde{M}_2, \quad M' = F^{-1}(E_1) \cap \tilde{M}_2.$$

We assume Hypothesis 1. and Hypothesis 2.

We discuss the classification of $K \in M_1$. It follows from (H2) and (H3) that the mapping $F|_M : M \to E_1$ is a bijection. On the other hand we see that the mapping $F|_{M'} : M' \to E_1$ is a bijection by (H2). Therefore, under (H2) and (H3), we conclude

$$M = M'. \quad (20)$$

Let $V$ be the direct sum of some root lattices of type $A,D,E$.

Let

$$D = \{ V \mid V \text{ satisfies the condition (iii)} \},$$

where

(iii) there exists a complex elliptic surface such that the trivial lattice is $V \oplus U'$.

For $V \in D$, we define the equation $e_V$ as follows:

$$e_V : K_n \cdots K_1 = I,$$

where each $K_i$ is conjugate to the local monodromy matrix of the singular fibre of type $I_i, I_i^*, III^*, IV^*$ corresponding to the lattice $V$. By the definition of $M'$, we have

$$M' = \{ K = (K_1, \ldots, K_n) \mid K \text{ is a solution of } e_V, V \in D \}.$$
Let

\[ R = \left\{ (V, V^\perp, \hat{V}/V) \mid (V, V^\perp, \hat{V}/V) \text{ satisfies the condition (iv)} \right\}, \]

where

(iv) there exists a complex elliptic surface which realizes \((V, V^\perp, \hat{V}/V)\).

In the case of rational elliptic surfaces \((k = 1)\) and of extremal elliptic K3 surfaces (special case of \(k = 2)\) \(R\) is Oguiso-Shioda’s table and Shimada-Zhang’s table respectively. Our problem is to obtain the list of global monodromies which reproduce Oguiso-Shioda’s table and Shimada-Zhang’s table.

### 5.2 A classification table

For the problem raised at the end of previous subsection, we obtained a (tentative) result, which is summarized in Table 3 and Table 4. The problem is to find the configuration which reproduce the Oguiso-Shioda’s table and Shimada-Zhang’s table. Let us explain how we solved this problem.

Some of the configurations are obtained simply by the confluence process of known solutions \(A^8BCBC\) and \(A^8BCBCA^8BCBC\) (Table 3 No.1 - 8 etc). To obtain the other configurations we use the elementary transformations. For example, the configuration of No.34 in Table 3, the transformation is given as follows:

\[
A^8BCBC \sim A^4A^3ABCBC \sim A^4A^3BX_{[0,1]}CBC \sim A^4A^3BCABC \sim A^4A^2BCA^2BC \sim A^4BCA^2A^2BC
\]

We obtained almost all the configurations in the tables by this method. For each configuration, we have calculated the pair \((V^\perp, \hat{V}/V)\) by the method of section 4.2. For the cases such as the case No.18 in Table 4, we need to find the configuration corresponding for each data of \((V^\perp, \hat{V}/V)\). In these cases, one of the configurations is sometimes hard to obtain and we need a lot of try and error. The systematic method of the calculation of the data \((V^\perp, \hat{V}/V)\) was useful to complete this procedure.
In the following we consider the meanings of Table 3 and of Table 4 under the above hypotheses.

Let

\[ S_1 = \{ K \mid K \text{ appears in Table 3} \}, \quad S_2 = \{ K \mid K \text{ appears in Table 4} \}. \]

First we consider the case of rational elliptic surfaces. We assume \((H_2)\). One can show that \((H_3)\) is true in this case \([12]\). On the other hand it is known that the classification of \(i : V \hookrightarrow L_1 \in E_1\) coincide with the classification of \((V, V^\perp, \hat{V}/V) \in R_1\) (= Oguiso-shioda’ table) \([3]\). Hence we have \(M' = S_1\). Therefore, by \((20)\), under \((H_2)\) we conclude

\[ M = S_1. \]

In the following we consider the case of extremal elliptic \(K3\) surfaces. We assume \((H_2)\). In this case there are no possibility of the difference between \(M\) and \(M_1\) except for No.297 \([14]\). Hence we see that \((H_3)\) is true except for No.297. Therefore we have \(M_1 = M'\) except for No.297. On the other hand it turns out that the classification of \((V, V^\perp, \hat{V}/V) \in R_1\) is slightly diffrent from the classification of \(i : V \hookrightarrow L_2 \in E_1\). See Remark \([11]\).

**Assumption.** (A) There are no difference between the classification of \((V, V^\perp, \hat{V}/V) \in R_1\) and the one of \(i : V \hookrightarrow L_2 \in E_1\) except for No.61.

Under \((H_2)\) and \((A)\) we conclude

\[ M_1 = S_2 \quad \text{except for No.297 and No.61}. \]

In the Table 4 the data \((a, b, c)\) in the last column represent the transcendental lattice, namely

\[ \text{matrix...} \]

The trivial lattice \(V\) is recovered from the brane configurations by the following rule:

\[ (X^n) \rightarrow A_{n-1} \]

The numbering is the same as that of Simada-Zhang.
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Table 3: Global monodromies of rational elliptic surfaces

| No. | $r$ | $V$ | global monodromy | $MW$ |
|-----|-----|-----|-------------------|------|
| 1   | 8   | 0   | $A^8B$ | $E_8$ |
| 2   | 7   | $A_1$ | $(A^2)^3A^6B$ | $E_7^*$ |
| 3   | 6   | $A_2$ | $(A^3)^2A^5B$ | $E_6^*$ |
| 4   | $A_1^{g2}$ | $(A^2)^2A^4$ | $D_6^*$ |
| 5   | 5   | $A_3$ | $(A^4)^2A^4B$ | $D_5^*$ |
| 6   | $A_2 \oplus A_1$ | $(A^3)^3A^3B$ | $A_5^*$ |
| 7   | $A_1^{g3}$ | $(A^3)^3A^2B$ | $D_4^* \oplus A_1^*$ |
| 8   | 4   | $A_4$ | $(A^3)^2A^3B$ | $A_4^*$ |
| 9   | $D_4$ | $(A^4BC)A^4B$ | $A_3^* \oplus A_1^*$ |
| 10  | $A_3 \oplus A_1$ | $(A^5)(A^2)^2A^2B$ | $A_3^* \oplus A_1^*$ |
| 11  | $A_2^{g2}$ | $(A^2)^2A^2B$ | $A_2^{g2}$ |
| 12  | $A_2 \oplus A_1^{g2}$ | $(A^3)^2A^2B$ | $\Lambda_{12}$ |
| 13  | $A_1^{g4}$ | $(A^3)^2A^2B$ | $D_4^* \oplus Z/2Z$ |
| 14  | $A_1^{g4}$ | $(A^3)^2A^2BC$ | $A_1^{g4}$ |
| 15  | 3   | $A_5$ | $(A^5)^2A^2B$ | $A_2^* \oplus A_1^*$ |
| 16  | $D_5$ | $(A^5)^2A^2B$ | $A_3^*$ |
| 17  | $A_4 \oplus A_1$ | $(A^6)(A^2)^2A^2B$ | $A_3^{g2}$ |
| 18  | $D_4 \oplus A_1$ | $(A^4)^2A^2B$ | $A_1^{g3}$ |
| 19  | $A_3 \oplus A_2$ | $(A^4)^2A^2B$ | $\Lambda_{19}$ |
| 20  | $A_2^{g2} \oplus A_1$ | $(A^3)^2A^2B$ | $A_2^* \oplus \{1/6\}$ |
| 21  | $A_3 \oplus A_1^{g2}$ | $(A^4)^2A^2B$ | $A_3^* \oplus Z/2Z$ |
| 22  | $A_3 \oplus A_1^{g2}$ | $(A^4)^2A^2B$ | $A_3^* \oplus \{1/4\}$ |
| 23  | $A_2 \oplus A_1^{g3}$ | $(A^3)^2A^2B$ | $A_1^* \oplus \Lambda_{12}$ |
| 24  | $A_1^{g5}$ | $(A^3)^2A^2B$ | $A_1^{g5} \oplus Z/2Z$ |
| 25  | 2   | $A_6$ | $(A^7)^2A^2B$ | $\Lambda_{25}$ |
| 26  | $D_6$ | $(A^6)^2A^2B$ | $A_1^{g2}$ |
| 27  | $E_6$ | $(A^5)^2A^2B$ | $A_2^*$ |
| 28  | $A_5 \oplus A_1$ | $(A^6)^2A^2B$ | $A_2^* \oplus Z/2Z$ |
| 29  | $A_4 \oplus A_1$ | $(A^5)^2A^2B$ | $A_1^* \oplus \{1/6\}$ |
| 30  | $D_5 \oplus A_1$ | $(A^5)^2A^2B$ | $A_1^* \oplus \{1/4\}$ |
| No. | $r$ | $V$ | global monodromy | $MW$ |
|-----|-----|-----|------------------|-----|
| 31  | 2   | $A_4 \oplus A_2$ | $(A^5)(A^3)BCBC$ | $A_{(31)}$ |
| 32  |     | $D_4 \oplus A_2$ | $(A^4BC)(A^3)ABC$ | $A_{(23)}$ |
| 33  |     | $A_4 \oplus A_1^{\oplus 2}$ | $(A^5)(A^2)^2 X_{[2,-1]} X_{[1,-2]} C$ | $\Lambda_{(33)}$ |
| 34  |     | $D_4 \oplus A_1^{\oplus 2}$ | $(A^4BC)(A^2)^2 BC$ | $A_{1}^{\oplus 2} \oplus \mathbb{Z}/2\mathbb{Z}$ |
| 35  |     | $A_3 \oplus A_3$ | $(A^4)^2 BCBC$ | $A_{1}^{\oplus 2} \oplus \mathbb{Z}/2\mathbb{Z}$ |
| 36  |     | $A_3 \oplus A_3$ | $(A^4)^2 AX_{[2,-1]} X_{[1,-2]} C$ | $\langle 1/4 \rangle ^{\oplus 2}$ |
| 37  |     | $A_3 \oplus A_2 \oplus A_1$ | $(A^4)(A^3)(B^2) X_{[1,-3]} C A$ | $A_{1}^* \oplus (1/12)$ |
| 38  |     | $A_3 \oplus A_1^{\oplus 3}$ | $(A^4)(A^2)(B^2)(X_{[0,1]}^2) BC$ | $A_{1}^* \oplus (1/4) \oplus \mathbb{Z}/2\mathbb{Z}$ |
| 39  |     | $A_2^{\oplus 3}$ | $(A^3)^2 X_{[2,-1]} X_{[1,-2]} C$ | $A_{2}^* \oplus \mathbb{Z}/3\mathbb{Z}$ |
| 40  |     | $A_2^{\oplus 2} \oplus A_1^{\oplus 2}$ | $(A^3)^2 (B^2)(X_{[0,1]}^2) BC$ | $\langle 1/6 \rangle ^{\oplus 2}$ |
| 41  |     | $A_2 \oplus A_1^{\oplus 4}$ | $(A^3)(B^2)(X_{[0,1]}^2)(B^2)(X_{[0,1]}^2) A$ | $A_{(23)} \oplus \mathbb{Z}/2\mathbb{Z}$ |
| 42  |     | $A_1^{\oplus 6}$ | $(A^2)^2 (B^2)^2 (X_{[0,1]}^2)(X_{[2,1]}^2)$ | $A_1^{\oplus 2} \oplus (\mathbb{Z}/2\mathbb{Z})^2$ |
| 43  | 1   | $E_7$ | $(A^6BC)^2 X_{[3,1]} A^2$ | $A_1^*$ |
| 44  |     | $A_7$ | $(A^8)BCBC$ | $A_1^* \oplus \mathbb{Z}/2\mathbb{Z}$ |
| 45  |     | $A_7$ | $(A^8)AX_{[2,-1]} X_{[1,-2]} C$ | $\langle 1/8 \rangle$ |
| 46  |     | $D_7$ | $(A^7)BC ABC$ | $\langle 1/4 \rangle$ |
| 47  |     | $A_6 \oplus A_1$ | $(A^7)(B^2) X_{[1,-3]} C A$ | $\langle 1/14 \rangle$ |
| 48  |     | $D_6 \oplus A_1$ | $(A^6BC)(A^2) BC$ | $A_{1}^* \oplus \mathbb{Z}/2\mathbb{Z}$ |
| 49  |     | $E_6 \oplus A_1$ | $(A^2)(A^5BC)^2 X_{[3,1]} A$ | $\langle 1/6 \rangle$ |
| 50  |     | $D_5 \oplus A_2$ | $(A^6BC)(A^3) BC$ | $\langle 1/12 \rangle$ |
| 51  |     | $A_5 \oplus A_2$ | $(A^6)(A^3)X_{[2,-1]} X_{[1,-2]} C$ | $A_{1}^* \oplus \mathbb{Z}/3\mathbb{Z}$ |
| 52  |     | $D_5 \oplus A_1^{\oplus 2}$ | $(A^5BC)(B^2)(X_{[0,1]}^2) A$ | $\langle 1/4 \rangle \oplus \mathbb{Z}/2\mathbb{Z}$ |
| 53  |     | $A_5 \oplus A_1^{\oplus 2}$ | $(A^6)(B^2)(X_{[0,1]}^2) BC$ | $\langle 1/6 \rangle \oplus \mathbb{Z}/2\mathbb{Z}$ |
| 54  |     | $D_4 \oplus A_3$ | $(A^4BC)(A^4) BC$ | $\langle 1/4 \rangle \oplus \mathbb{Z}/2\mathbb{Z}$ |
| 55  |     | $A_4 \oplus A_3$ | $(A^5)(A^4)X_{[2,-1]} X_{[1,-2]} C$ | $\langle 1/20 \rangle$ |
| 56  |     | $A_4 \oplus A_2 \oplus A_1$ | $(A^5)(A^3)(B^2) X_{[1,-3]} C$ | $\langle 1/30 \rangle$ |
| 57  |     | $D_4 \oplus A_1^{\oplus 3}$ | $(A^4BC)(A^2)(B^2)(X_{[0,1]}^2)$ | $A_{1}^* \oplus (\mathbb{Z}/2\mathbb{Z})^2$ |
| 58  |     | $A_3^{\oplus 2} \oplus A_1$ | $(A^4)^2 (B^2) X_{[1,-3]} C$ | $A_{1}^* \oplus \mathbb{Z}/4\mathbb{Z}$ |
| 59  |     | $A_3 \oplus A_2 \oplus A_1^{\oplus 2}$ | $(A^4)(B^3)(X_{[0,1]}^2)(X_{[2,1]}^2) X_{[3,1]}$ | $\langle 1/12 \rangle \oplus \mathbb{Z}/2\mathbb{Z}$ |
| 60  |     | $A_3 \oplus A_1^{\oplus 4}$ | $(A^4)(B^2)(X_{[0,1]}^2)(B^2)(X_{[0,1]}^2)$ | $\langle 1/4 \rangle \oplus (\mathbb{Z}/2\mathbb{Z})^2$ |
Table 3: Global monodromies of rational elliptic surfaces

| No. | r | $V$ | global monodromy | $MW$ |
|-----|---|-----|-----------------|------|
| 61  | 61 | $A_2 \oplus A_1$ | $(A^3)^2 (B^3) (X_{1,-2}^2) C$ | $(1/6) \oplus \mathbb{Z}/3\mathbb{Z}$ |
| 62  | 0  | $E_8$ | $(A^7BC^2) X_{[3,1]} A$ | 0 |
| 63  | 63 | $A_8$ | $(A^9) X_{[2,-1]} X_{[1,-2]} C$ | $\mathbb{Z}/3\mathbb{Z}$ |
| 64  | 64 | $D_8$ | $(A^8BC) BC$ | $\mathbb{Z}/2\mathbb{Z}$ |
| 65  | 65 | $E_7 \oplus A_1$ | $(A^2) (A^6BC^2) X_{[3,1]}$ | $\mathbb{Z}/2\mathbb{Z}$ |
| 66  | 66 | $A_5 \oplus A_2 \oplus A_1$ | $(A^6) (B^3) (X_{[1,-2]}^2) C$ | $\mathbb{Z}/6\mathbb{Z}$ |
| 67  | 67 | $A_4 \oplus A_1$ | $(A^5) (B^5) X_{[2,-3]} C$ | $\mathbb{Z}/5\mathbb{Z}$ |
| 68  | 68 | $A_2 \oplus A_1$ | $(A^3) (B^3) (X_{[0,1]}^3) C^3$ | $(\mathbb{Z}/3\mathbb{Z})^2$ |
| 69  | 69 | $E_6 \oplus A_2$ | $(A^3) (A^5BC^2) X_{[3,1]}$ | $\mathbb{Z}/3\mathbb{Z}$ |
| 70  | 70 | $A_7 \oplus A_1$ | $(A^8) (B^2) X_{[1,-3]} C$ | $\mathbb{Z}/4\mathbb{Z}$ |
| 71  | 71 | $D_6 \oplus A_1 \oplus A_1$ | $(A^6BC) (B^2) (X_{[0,1]}^2)$ | $(\mathbb{Z}/2\mathbb{Z})^2$ |
| 72  | 72 | $D_5 \oplus A_3$ | $(A^6BC) (B^4) X_{[1,-2]}$ | $\mathbb{Z}/4\mathbb{Z}$ |
| 73  | 73 | $D_4 \oplus A_1$ | $(A^4BC) (A^4BC)$ | $(\mathbb{Z}/2\mathbb{Z})^2$ |
| 74  | 74 | $(A_3 \oplus A_1) \oplus A_1$ | $(A^4) (B^4) (X_{[0,1]}^2) (X_{[2,1]}^2)$ | $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ |

\[\Lambda_{(12)} = \frac{1}{6} \begin{pmatrix} 2 & 1 & 0 & -1 \\ 1 & 5 & 3 & 1 \\ 0 & 3 & 6 & 3 \\ -1 & 1 & 3 & 5 \end{pmatrix}, \quad \Lambda_{(17)} = \frac{1}{10} \begin{pmatrix} 3 & 1 & -1 \\ 1 & 7 & 3 \\ -1 & 3 & 7 \end{pmatrix}, \]

\[\Lambda_{(19)} = \frac{1}{12} \begin{pmatrix} 7 & 1 & 2 \\ 1 & 7 & 2 \\ 2 & 2 & 4 \end{pmatrix}, \quad \Lambda_{(23)} = \frac{1}{6} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \]

\[\Lambda_{(25)} = \frac{1}{7} \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}, \quad \Lambda_{(31)} = \frac{1}{15} \begin{pmatrix} 2 & 1 \\ 1 & 8 \end{pmatrix}, \]

\[\Lambda_{(33)} = \frac{1}{10} \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}. \]
Table 4: Global monodromies of extremal elliptic $K3$ surfaces

| No | global monodromy                                                                 | $MW$                  | $a$ | $b$ | $c$ |
|----|----------------------------------------------------------------------------------|-----------------------|-----|-----|-----|
| 1  | $(A^4)(X_4^{[3,-2]})(X_4^{[10,-7]})(X_4^{[7,-5]})(X_4^{[4,-3]})(B^4)$            | $Z/4Z \times Z/4Z$   | 4   | 0   | 4   |
| 2  | $(A^5)(B^5)(X_5^{[2,-3]})(X_5^{[3,-4]})(X_5^{[2,1]})(X_5^{[2,3]})(B^5)$         | $Z/5Z$               | 10  | 0   | 10  |
| 3  | $(A^5)(X_5^{[3,-2]})(X_5^{[13,-9]})(X_5^{[10,-7]})(X_5^{[7,-5]})(B^5)$          | $(0)$                 | 60  | 0   | 60  |
| 4  | $(A^6)(B^6)(X_6^{[4,-5]})(X_6^{[3,-4]})(X_6^{[2,1]})(X_6^{[2,3]})(X_6^{0,1})$   | $Z/2Z \times Z/6Z$   | 2   | 0   | 6   |
| 5  | $(A^6)(X_6^{[1,2]})(X_6^{[3,1]})(X_6^{[3,5]})(C^3)$                              | $Z/3Z \times Z/3Z$   | 6   | 0   | 6   |
| 6  | $(A^6)(B^6)(X_6^{[82,-99]})(X_6^{[4,-5]})(X_6^{[29,-37]})(X_6^{[5,-7]})X$        | $Z/6Z$               | 4   | 0   | 6   |
| 7  | $(A^6)(B^6)(X_6^{[4,-5]})(X_6^{[1,2]})(X_6^{[3,5]})(X_6^{C})$                    | $Z/2Z \times Z/2Z$   | 12  | 0   | 12  |
| 8  | $(A^6)(X_6^{[0,1]})(X_6^{[1,2]})(X_6^{[3,5]})(C^3)(X_6^{[2,1]})(X_6^{[5,1]})$   | $Z/6Z$               | 6   | 0   | 12  |
| 9  | $(A^6)(X_6^{[0,1]})(X_6^{[1,2]})(X_6^{[3,5]})(C^3)(X_6^{[2,1]})(X_6^{[7,2]})$   | $(0)$                 | 30  | 0   | 30  |
| 10 | $(A^6)(B^6)(X_6^{[2,1]})(X_6^{[3,5]})(X_6^{[3,1]})(X_6^{[19,4]})$               | $Z/3Z$               | 6   | 0   | 30  |
| 11 | $(A^6)(X_6^{[8,5]})(B^3)(X_6^{[3,4]})(X_6^{[2,3]})(X_6^{[2,1]})(X_6^{C})$       | $Z/2Z$               | 12  | 0   | 30  |
| 12 | $(A^6)(X_6^{[0,1]})(X_6^{[1,2]})(C^4)(X_6^{[3,1]})(X_6^{[2,1]})(X_6^{[5,1]})$   | $Z/2Z$               | 24  | 12  | 36  |
| 13 | $(A^7)(X_7^{[0,1]})(X_7^{[1,2]})(X_7^{[13,24]})(X_7^{[9,34]})(X_7^{[3,5]})$     | $Z/7Z$               | 2   | 1   | 4   |
| 14 | $(A^7)(B^7)(X_7^{[3,5]})(X_7^{[7,-9]})(X_7^{[2,1]})(X_7^{[2,1]})$               | $(0)$                 | 42  | 0   | 42  |
| 15 | $(A^7)(X_7^{[3,5]})(X_7^{[4,29,33]})(C^4)(X_7^{[4,4]})(X_7^{[0,1]})$             | $(0)$                 | 28  | 7   | 28  |
| 16 | $(A^7)(B^7)(X_7^{[5,7]})(X_7^{[5,1]})(X_7^{[5,7]})(X_7^{[0,1]})(X_7^{[3,4]})$   | $(0)$                 | 28  | 7   | 28  |
| 17 | $(A^7)(X_7^{[5,9]})(X_7^{[5,9]})(C^4)(X_7^{[2,3]})(X_7^{[2,1]})(X_7^{[6,1]})$   | $(0)$                 | 50  | 20  | 50  |
| 18 | $(1)$ $(A^7)(X_7^{[3,-2]})(X_7^{[5,-7]})(B^4)(X_7^{[3,-10]})(C)$                | $(0)$                 | 10  | 0   | 140 |
| 19 | $(2)$ $(A^7)(X_7^{[3,-2]})(B^7)(X_7^{[5,-7]})(B^4)(X_7^{[3,-10]})(C)$           | $(0)$                 | 20  | 0   | 70  |
| 20 | $(A^7)(X_7^{[5,9]})(C^4)(X_7^{[8,5]})(X_7^{[5,3]})(X_7^{[7,1]})$                 | $(0)$                 | 30  | 0   | 84  |
| 21 | $(A^7)(B^6)(X_6^{[0,1]})(X_6^{[1,3]})(C^4)(X_6^{[1,2]})$                         | $Z/2Z$               | 12  | 6   | 24  |
| 22 | $(A^7)(B^6)(X_6^{[0,1]})(C^4)(X_6^{[2,1]})(X_6^{[1,2]})$                         | $Z/2Z$               | 4   | 0   | 84  |
| 23 | $(1)$ $(A^7)(C^6)(X_6^{[4,3]})(X_6^{[3,1]})(X_6^{[2,3]})(X_6^{[3,5]})$          | $(0)$                 | 30  | 0   | 42  |
| 24 | $(2)$ $(A^7)(X_6^{[6,-1]})(B^5)(X_6^{[0,1]})(X_6^{[2,3]})(X_6^{[9,11]})$        | $(0)$                 | 18  | 6   | 72  |
Table 4: Global monodromies of extremal elliptic $K3$ surfaces

| No  | global monodromy                                                                 | $MW$                | $a$ | $b$ | $c$ |
|-----|----------------------------------------------------------------------------------|---------------------|-----|-----|-----|
| 24  | $(A^7)(X^6_{[0,1]})(X^5_{[5,7]})(C^4)X_{[7,2]}X_{[20,3]}$                       | (0)                 | 12  | 0   | 70  |
| 25  | $(A^8)(X^8_{[0,1]})(X^2_{[1,7]})(X^2_{[1,5]})(X^2_{[1,3]})(C^2)$                | $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$        | 4   | 0   | 4   |
| 26. (1) | $(A^8)(B^8)(X^3_{[0,1]})(C^3)X_{[7,3]}X_{[4,1]}$                             | (0)                 | 24  | 0   | 24  |
|      (2) | $(A^8)(X^8_{[0,1]})(C^3)(X^3_{[3,3]}X_{[19,11]}X_{[5,1]}$             | $\mathbb{Z}/2\mathbb{Z}$                | 12  | 0   | 12  |
| 27  | $(A^8)(X^8_{[2,-1]})(B^4)(X^2_{[5,1]}X_{[9,7]}X_{[29,5]}$                   | $\mathbb{Z}/8\mathbb{Z}$                | 2   | 0   | 4   |
| 28  | $(A^8)(X^4_{[2,-1]})(X^4_{[5,3]})(X^4_{[8,-3]})(B^2)(X^2_{[3,1]}$         | $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$        | 4   | 0   | 8   |
| 29  | $(A^8)(B^4)(X^4_{[1,-2]})(X^4_{[2,-5]})(X^4_{[1,-3]}C$                      | $\mathbb{Z}/4\mathbb{Z}$                | 4   | 0   | 24  |
| 30  | $(A^8)(C^9)(X^4_{[4,3]})(X^3_{[1,8]})(X^3_{[3,2]}X_{[3,1]}$                 | (0)                 | 12  | 0   | 120 |
| 31  | $(A^8)(B^4)(X^4_{[6,-4]})(C^3)(X^2_{[2,1]})(X^2_{[1,1]}$                   | $\mathbb{Z}/2\mathbb{Z}$                | 20  | 0   | 24  |
| 32  | $(A^8)(B^6)(X^6_{[2,-3]})(X^2_{[20,-31]})(X_{[3,-5]}C$                      | $\mathbb{Z}/2\mathbb{Z}$                | 6   | 0   | 24  |
| 33  | $(A^8)(X^6_{[0,1]})(C^4)(X^2_{[5,1,1]})(X^2_{[7,5]})(X^2_{[2,1]}$         | $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$        | 8   | 0   | 12  |
| 34  | $(A^8)(X^6_{[2,-1]})(B^4)(C^3)(X^3_{[10,7]}X_{[3,1]}$                      | $\mathbb{Z}/2\mathbb{Z}$                | 12  | 0   | 24  |
| 35  | $(A^8)(B^6)(X^8_{[4,-5]})(X^2_{[4,-3]})(X^3_{[5,1]}X_{[4,1]}$             | $\mathbb{Z}/2\mathbb{Z}$                | 2   | 0   | 120 |
| 36. (1) | $(A^8)(X^6_{[0,1]})(X^5_{[3,23]})(C^3)X_{[8,5]}X_{[7,2]}$              | (0)                 | 6   | 0   | 120 |
|      (2) | $(A^8)(X^6_{[9,-1]})(X^5_{[13,-2]})(X^3_{[58,-9]}X_{[5,1]}X_{[4,-3]}$     | (0)                 | 24  | 0   | 30  |
| 37  | $(A^8)(X^7_{[9,-2]})(X^3_{[9,-1]})(X^3_{[4,-1]})(X^2_{[7,-2]}X_{[5,-3]}$   | (0)                 | 24  | 0   | 42  |
| 38  | $(A^8)(B^7)(X^4_{[5,-6]})(X^2_{[4,-5]})(X^2_{[1,-2]}C$                    | $\mathbb{Z}/2\mathbb{Z}$                | 12  | 4   | 20  |
| 39  | $(A^8)(B^7)(X^4_{[5,-6]})(X^2_{[4,-5]})(X^2_{[1,-2]}X_{[4,1]}$             | (0)                 | 50  | 16  | 4   |
| 40. (1) | $(A^8)(B^7)(X^5_{[4,-5]})(X^2_{[11,-14]})(X_{[2,-3]}X_{[5,1]}$            | (0)                 | 2   | 0   | 280 |
|      (2) | $(A^8)(B^7)(C^5)(X^3_{[60,49]}X_{[27,22]}X_{[3,2]}$                      | (0)                 | 18  | 4   | 32  |
| 41  | $(A^8)(X^7_{[2,-1]})(B^6)(X^6_{[12,-53]}X_{[11,-14]}X_{[2,-2]}$           | (0)                 | 16  | 4   | 22  |
| 42. (1) | $(A^9)(B^9)(X^5_{[0,1]})(X^2_{[2,1]})(X^2_{[7,2]}X_{[6,1]}$              | (0)                 | 18  | 0   | 18  |
|      (2) | $(A^9)(X^9_{[3,1]})(X^2_{[13,4]})(X^2_{[4,1]}X_{[40,7]}X_{[7,1]}$         | $\mathbb{Z}/3\mathbb{Z}$                | 4   | 2   | 10  |
| 43  | $(A^9)(X^4_{[3,5]})(X^4_{[11,18]})(X^3_{[7,11]})(C^3)(X^2_{[6,1]}$       | $\mathbb{Z}/3\mathbb{Z}$                | 12  | 0   | 18  |
| 44  | $(A^9)(B^5)(X^5_{[3,-5]})(X^2_{[42,-71]})(X^2_{[0,1]}X_{[7,2]}$           | (0)                 | 20  | 10  | 50  |
Table 4: Global monodromies of extremal elliptic $K3$ surfaces

| No | global monodromy                                                                 | $MW$   | $a$ | $b$ | $c$ |
|----|---------------------------------------------------------------------------------|--------|-----|-----|-----|
| 45 | $(A^9)(X^5_{[2,-1]})(X^3_{[7,-4]})(X^3_{[5,-3]})(B^3)C$                        | $\mathbb{Z}/3\mathbb{Z}$ | 12  | 3   | 12  |
| 46 | $(A^9)(X^5_{[0,1]})(C^2)(X^3_{[3,2]})(X^2_{[7,4]})(X)_{[3,1]}$                | $(0)$   | 6   | 0   | 180 |
| 47 | $(A^9)(B^6)(X^3_{[0,1]})(C^3)(X^2_{[7,4]})(X)_{[4,1]}$                        | $\mathbb{Z}/3\mathbb{Z}$ | 6   | 0   | 18  |
| 48 | $(A^9)(B^6)(X^3_{[0,1]})(C^3)(X)_{[12,7]}X_{[3,1]}$                            | $\mathbb{Z}/3\mathbb{Z}$ | 4   | 0   | 18  |
| 49 | $(A^9)(B^6)(X^5_{[0,1]})(X^2_{[2,1]})(X)_{[2,4]}X_{[15,2]}$                   | $(0)$   | 18  | 0   | 30  |
| 50 | $(A^9)(B^7)(X^3_{[3,-5]})(X^2_{[0,1]})(X^2_{[4,1]})(X)_{[15,2]}$              | $(0)$   | 18  | 0   | 42  |
| 51 | $(A^9)(B^7)(X^3_{[0,1]})(C^3)(X)_{[12,7]}X_{[6,1]}$                            | $(0)$   | 10  | 4   | 52  |
| 52 | $(A^9)(B^7)(X^5_{[0,1]})(X)_{[219,-277]}X_{[83,-105]}X_{[7,-9]}$              | $(0)$   | 18  | 9   | 22  |
| 53 | $(A^9)(X^5_{[2,-1]})(C^3)(X^3_{[7,4]})(X)_{[3,1]}X_{[13,2]}$                  | $(0)$   | 18  | 0   | 24  |
| 54. 1 | $(A^{10})(B^{10})X_{[7,-8]}X_{[3,-4]}X_{[1,,-3]}C$                              | $(0)$   | 10  | 0   | 10  |
|      | $(A^{10})(C^{10})X_{[9,8]}X_{[13,11]}X_{[4,3]}X_{[3,1]}$                      | $\mathbb{Z}/5\mathbb{Z}$ | 2   | 0   | 2   |
| 55 | $(A^{10})(X^4_{[2,-1]})(A^4)(B^3)(X^2_{[1,-4]})(C)$                           | $\mathbb{Z}/2\mathbb{Z}$ | 4   | 0   | 60  |
| 56 | $(A^{10})(X^4_{[0,1]})(C^3)(X^3_{[3,1]})(X^2_{[4,1]})(X^2_{[9,1]})$           | $\mathbb{Z}/2\mathbb{Z}$ | 6   | 0   | 60  |
| 57 | $(A^{10})(B^5)(X^5_{[19,24]})(C^3)(X)_{[12,7]}X_{[7,2]}$                      | $\mathbb{Z}/5\mathbb{Z}$ | 2   | 0   | 10  |
| 58 | $(A^{10})(B^5)(C^3)(X^3_{[13,1]})(X)_{[10,7]}X_{[2,1]}$                       | $\mathbb{Z}/2\mathbb{Z}$ | 20  | 10  | 20  |
| 59 | $(A^{10})(X^5_{[2,-1]})(B^4)(X^2_{[12,-19]})(X^2_{[1,,-2]})(X)_{[4,1]}$      | $\mathbb{Z}/2\mathbb{Z}$ | 10  | 0   | 20  |
| 60 | $(A^{10})(X^3_{[0,1]})(X^3_{[3,1]})(X^2_{[5,7]})(C^3)(X)_{[5,1]}$             | $\mathbb{Z}/2\mathbb{Z}$ | 12  | 6   | 18  |
| 61. 1 | $(A^{10})(X^6_{[0,1]})(C^4)(X^2_{[3,1]})(X)_{[4,11]}X_{[5,1]}$               | $\mathbb{Z}/2\mathbb{Z}$ | 10  | 0   | 12  |
|      | $(A^{10})(B^6)(X^4_{[4,5]})(C^3)(X^2_{[3,-4]})(X)_{[1,,-3]}C$               | $\mathbb{Z}/2\mathbb{Z}$ | 10  | 0   | 12  |
| 62. 1 | $(A^{10})(C^6)(X^5_{[13,11]})(X)_{[32,27]}X_{[4,3]}X_{[3,1]}$                | $(0)$   | 10  | 0   | 30  |
|      | $(A^{10})(X^5_{[0,1]})(C^5)(X^2_{[4,1]})(X)_{[9,7]}X_{[5,3]}X_{[3,1]}$      | $\mathbb{Z}/2\mathbb{Z}$ | 10  | 5   | 10  |
| 63 | $(A^{10})(B^7)(X^3_{[3,-4]})(X^2_{[7,-10]})(X)_{[1,,-2]}C$                   | $\mathbb{Z}/2\mathbb{Z}$ | 4   | 2   | 36  |
| 64 | $(A^{10})(X^7_{[0,1]})(X^3_{[1,6]})(X^2_{[5,27]})(X)_{[11,59]}X_{[1,4]}$     | $(0)$   | 10  | 0   | 42  |
| 65 | $(A^{10})(X^8_{[0,1]})(C^4)(X)_{[7,5]}X_{[7,4]}X_{[3,1]}$                     | $(0)$   | 2   | 0   | 140 |
| 66 | $(A^{10})(C^8)(X^3_{[8,7]})(X)_{[97,84]}X_{[37,32]}X_{[6,5]}$                 | $(0)$   | 10  | 0   | 24  |
Table 4: Global monodromies of extremal elliptic $K3$ surfaces

| No | global monodromy | $MW$ | $a$ | $b$ | $c$ |
|----|------------------|------|----|----|----|
| 67 | $(A^{10})(B^3)(C^2)X_{[32,19]}X_{[12,7]}X_{[3,1]}$ | $(0)$ | 10 | 0 | 18 |
| 68 | $(A^{11})(B^4)X_{[1,-3]}^4X_{[3,-10]}^3X_{[1,-5]}C$ | $(0)$ | 24 | 12 | 28 |
| 69 | $(A^{11})(B^4)X_{[5,-7]}^3X_{[2,-3]}^2(C^2)X_{[3,1]}$ | $(0)$ | 12 | 0 | 66 |
| 70 | $(A^{11})(B^5)X_{[15,-19]}^2X_{[2,-3]}X_{[2,1]}X_{[11,2]}$ | $(0)$ | 10 | 5 | 30 |
| 71. (1) | $(A^{11})(B^5)X_{[31,-39]}^2(C^3)X_{[17,12]}X_{[8,5]}$ | $(0)$ | 6 | 3 | 84 |
| (2) | $(A^{11})(B^5)X_{[2,-3]}^3X_{[3,-5]}X_{[1,-3]}C$ | $(0)$ | 24 | 9 | 24 |
| 72 | $(A^{11})(B^5)X_{[3,1]}^3X_{[2,5]}^2(C^2)X_{[19,2]}$ | $(0)$ | 2 | 0 | 330 |
| 73. (1) | $(A^{11})(C^5)X_{[4,3]}^4X_{[2,3]}X_{[9,2]}X_{[6,1]}$ | $(0)$ | 20 | 0 | 22 |
| (2) | $(A^{11})(B^5)X_{[3,1]}^3X_{[20,11]}X_{[3,1]}$ | $(0)$ | 12 | 4 | 38 |
| 74. (1) | $(A^{11})X_{[6,0,1]}^6(C^3)X_{[3,1]}^X_{[5,1]}X_{[21,2]}$ | $(0)$ | 6 | 0 | 66 |
| (2) | $(A^{11})(B^6)X_{[9,-11]}^2X_{[13,16]}X_{[2,-3]}C$ | $(0)$ | 18 | 6 | 24 |
| 75. (1) | $(A^{11})X_{[6,0,1]}^6(C^4)X_{[11,8]}X_{[5,3]}X_{[3,1]}$ | $(0)$ | 4 | 0 | 66 |
| (2) | $(A^{11})X_{[17,-2]}^4X_{[18,-1]}^4X_{[2,-1]}X_{[1,-2]}X_{[7,3]}$ | $(0)$ | 12 | 2 | 26 |
| 76 | $(A^{11})(B^7)X_{[2,-2]}^2(C^2)X_{[20,11]}X_{[3,1]}$ | $(0)$ | 12 | 2 | 26 |
| 77. (1) | $(A^{11})X_{[-1,2]}^7X_{[6,-13]}X_{[2,-5]}X_{[1,-5]}C$ | $(0)$ | 4 | 1 | 58 |
| (2) | $(A^{11})X_{[0,1]}^7X_{[3,1,6]}X_{[3,2]}X_{[3,1]}$ | $(0)$ | 16 | 5 | 16 |
| 78. (1) | $(A^{11})X_{[3,1]}^8X_{[7,4]}X_{[57,11]}X_{[16,3]}X_{[7,1]}$ | $(0)$ | 2 | 0 | 88 |
| (2) | $(A^{11})X_{[0,1]}^8X_{[6,1]}X_{[46,7]}X_{[27,4]}X_{[9,1]}$ | $(0)$ | 10 | 2 | 18 |
| 79 | $(A^{11})(C^9)X_{[8,7]}X_{[17,14]}X_{[16,13]}X_{[3,2]}$ | $(0)$ | 10 | 1 | 10 |
| 80 | $(A^{12})(X_{[3,-3]}^3)(X_{[3,5,-2]}^5(B^3)X_{[6,1]}^2)Z/3Z$ | $(6)$ | 12 |
| 81 | $(A^{12})(X_{[4,-1]}^3)(X_{[3,1]}^2)(X_{[2,1]}^2)(X_{[2,12,5]}^2)(X_{[3,1]}^2)Z/6Z$ | $(2)$ | 0 | 12 |
| 82 | $(A^{12})(X_{[0,1]}^4)(C^4)X_{[2,1]}^2X_{[3,1]}$ | $(0)$ | 4 | 0 | 6 |
| 83. (1) | $(A^{12})(X_{[1,-1]}^3)(X_{[3,1]}^3)(X_{[3,6]}^3)(X_{[3,1]}^2)Z/3Z$ | $(4)$ | 0 | 12 |
| (2) | $(A^{12})(X_{[2,-3]}^3(B^3)X_{[1,-5]}^2)Z/6Z$ | $(4)$ | 4 | 0 | 6 |
| 84. (1) | $(A^{12})(X_{[0,1]}^3X_{[3,3]}^3)(X_{[3,7]}^2)(C^2)X_{[7,1]}Z/4Z$ | $(6)$ | 6 | 0 | 6 |
| (2) | $(A^{12})(C^4)X_{[9,7]}^2X_{[13,10]}^2X_{[3,2]}X_{[3,1]}Z/2Z$ | $(12)$ | 12 | 0 | 12 |
Table 4: Global monodromies of extremal elliptic $K3$ surfaces

| No   | global monodromy | $MW$ | $a$ | $b$ | $c$ |
|------|------------------|------|-----|-----|-----|
| 85   | $(A^{12})(B^{5})(X_{[3,-4]}^2)(X_{[2,-5]}^2)(X_{[0,1]}^2)X_{[5,1]}$ | $\mathbb{Z}/2\mathbb{Z}$ | 6   | 0   | 20  |
| 86   | $(A^{12})(C^5)(X_{[5,4]}^2)(X_{[3,2]}^2)X_{[3,1]}X_{[10,1]}$ | (0)  | 12  | 0   | 30  |
| 87.  | $(A^{12})(B^{6})(X_{[3,-5]}^2)(X_{[2,-3]}^2)X_{[1,-3]}C$ | $\mathbb{Z}/2\mathbb{Z}$ | 6   | 0   | 12  |
|      | $(A^{12})(X_{[2,4]}^6)(X_{[9,4]}^2)(X_{[2,3]}^2)X_{[24,5]}X_{[6,1]}$ | $\mathbb{Z}/6\mathbb{Z}$ | 2   | 0   | 4   |
| 88   | $(A^{12})(X_{[3,1]}^6)(X_{[4,1]}^3)X_{[23,5]}X_{[26,5]}X_{[11,2]}$ | $\mathbb{Z}/3\mathbb{Z}$ | 4   | 0   | 6   |
| 89   | $(A^{12})(X_{[3,-1]}^7)(X_{[7,1]}^2)X_{[53,7]}X_{[61,8]}X_{[17,2]}$ | (0)  | 4   | 0   | 42  |
| 90   | $(A^{13})(X_{[5,-1]}^3)(X_{[3,1]}^3)(X_{[4,1]}^2)(X_{[37,6]}^2)X_{[13,2]}$ | (0)  | 12  | 6   | 42  |
| 91   | $(A^{13})(C^4)(X_{[4,3]}^3)(X_{[2,1]}^2)X_{[7,1]}X_{[10,1]}$ | (0)  | 6   | 0   | 52  |
| 92.  | $(A^{13})(B^5)(C^2)(X_{[3,1]}^2)X_{[5,1]}X_{[23,2]}^2$ | (0)  | 2   | 0   | 130 |
|      | $(A^{13})(C^5)(X_{[3,2]}^2)(X_{[7,4]}^2)X_{[5,2]}X_{[4,1]}$ | (0)  | 18  | 8   | 18  |
| 93   | $(A^{13})(B^{5})(C^3)X_{[17,12]}X_{[5,3]}X_{[3,1]}$ | (0)  | 6   | 3   | 34  |
| 94   | $(A^{13})(B^{6})(C^2)X_{[17,10]}X_{[9,5]}X_{[3,1]}$ | (0)  | 10  | 2   | 16  |
| 95   | $(A^{13})(C^7)X_{[6,5]}X_{[7,5]}X_{[5,3]}X_{[3,1]}$ | (0)  | 2   | 1   | 46  |
| 96.  | $(A^{14})(C^3)(X_{[2,1]}^3)(X_{[9,1]}^2)X_{[86,9]}X_{[11,1]}$ | (0)  | 6   | 0   | 42  |
|      | $(A^{14})(B^3)(C^3)X_{[10,7]}^2X_{[5,3]}X_{[3,1]}$ | $\mathbb{Z}/2\mathbb{Z}$ | 6   | 3   | 12  |
| 97   | $(A^{14})(X_{[1,5]}^3)(X_{[2,9]}^2)(X_{[1,3]}^2)(C^2)X_{[13,1]}$ | $\mathbb{Z}/2\mathbb{Z}$ | 2   | 0   | 42  |
| 98   | $(A^{14})(B^4)(X_{[2,-3]}^2)(C^2)X_{[9,5]}X_{[3,1]}$ | $\mathbb{Z}/2\mathbb{Z}$ | 6   | 2   | 10  |
| 99   | $(A^{14})(C^4)(X_{[2,1]}^3)(X_{[21,8]}^2)X_{[11,4]}X_{[4,1]}$ | (0)  | 4   | 0   | 42  |
| 100. | $(A^{14})(C^5)(X_{[17,14]}^2)X_{[67,55]}X_{[50,41]}X_{[4,3]}$ | (0)  | 2   | 0   | 70  |
|      | $(A^{14})(X_{[9,-2]}^5)(X_{[3,-1]}^2)BX_{[9,2]}X_{[8,1]}$ | (0)  | 8   | 2   | 18  |
|      | $(A^{14})(B^5)(X_{[3,-4]}^2)X_{[5,3]}X_{[1,-3]}C$ | $\mathbb{Z}/2\mathbb{Z}$ | 2   | 1   | 18  |
| 101  | $(A^{14})(C^6)X_{[5,4]}X_{[5,3]}X_{[5,2]}X_{[4,1]}$ | (0)  | 4   | 2   | 22  |
| 102  | $(A^{15})(X_{[0,1]}^3)(C^3)X_{[7,4]}X_{[5,2]}X_{[4,1]}$ | $\mathbb{Z}/3\mathbb{Z}$ | 4   | 1   | 4   |
| 103. | $(A^{15})(X_{[0,1]}^3)(C^2)(X_{[7,1]}^2)X_{[15,4]}X_{[5,1]}$ | (0)  | 12  | 6   | 18  |
|      | $(A^{15})(X_{[3,1]}^3)(X_{[-1,2]})(C^2)X_{[4,1]}X_{[7,1]}$ | (0)  | 12  | 6   | 18  |
| 104  | $(A^{15})(X_{[0,1]}^4)(C^2)X_{[13,5]}X_{[11,4]}X_{[4,1]}$ | (0)  | 10  | 0   | 12  |
Table 4: Global monodromies of extremal elliptic K3 surfaces

| No | global monodromy | $MW$ | $a$ | $b$ | $c$ |
|----|------------------|------|-----|-----|-----|
| 105 | $(A^{15})(C^5)X_{[23,19]}X_{[28,23]}X_{[16,13]}X_{[3,2]}$ | (0) | 10 | 5 | 10 |
| 106 | $(A^{16})(B^2)(C^2)(X^2_{[5,1]}X_{[3,1]}X_{[13,1]}$ | $\mathbb{Z}/4\mathbb{Z}$ | 2 | 0 | 4 |
| 107. (1) | $(A^{16})(X^3_{[0,1]})(C^2)X_{[7,2]}X_{[6,1]}X_{[9,1]}$ | (0) | 10 | 2 | 10 |
| 107. (2) | $(A^{16})(B^3)(C^2)X_{[9,1]}X_{[9,5]}X_{[3,1]}$ | $\mathbb{Z}/2\mathbb{Z}$ | 4 | 0 | 6 |
| 108 | $(A^{16})(B^4)X_{[5,1]}X_{[3,5]}X_{[1,3]}C$ | $\mathbb{Z}/4\mathbb{Z}$ | 2 | 0 | 2 |
| 109. (1) | $(A^{17})(C^2)(X^2_{[3,1]}X_{[19,5]}X_{[21,5]}X_{[9,2]}$ | (0) | 2 | 0 | 34 |
| 109. (2) | $(A^{17})(X^2_{[3,1]})(X^2_{[21,5]}X_{[9,2]}X_{[8,1]}X_{[18,1]}$ | (0) | 4 | 2 | 18 |
| 110 | $(A^{17})(C^3)X_{[11,8]}X_{[10,7]}X_{[5,3]}X_{[3,1]}$ | (0) | 6 | 3 | 10 |
| 111. (1) | $(A^{18})(X^2_{[4,1]}X_{[17,5]}X_{[2,1]}X_{[5,2]}X_{[4,1]}$ | (0) | 4 | 2 | 10 |
| 111. (2) | $(A^{18})(C^2)X_{[8,5]}X_{[7,4]}X_{[5,2]}X_{[4,1]}$ | $\mathbb{Z}/3\mathbb{Z}$ | 2 | 0 | 2 |
| 112 | $(A^{19})X_{[4,1]}X_{[5,2]}X_{[0,1]}X_{[5,2]}X_{[4,1]}$ | (0) | 2 | 1 | 10 |
| 113 | $(A^{6})(BC)X_{[3,5]}X_{[1,2]}X_{[5,8]}X_{[1,2]}X_{[3,1]}$ | (0) | 20 | 0 | 20 |
| 114 | $(A^{6})(BC)(X^5_{[8,5]})(B^5)X_{[3,4]}X_{[5,8]}$ | $\mathbb{Z}/2\mathbb{Z}$ | 12 | 0 | 12 |
| 115 | $(A^{6})(BC)X_{[3,9]}X_{[3,5]}X_{[7,5]}X_{[7,12]}$ | (0) | 20 | 0 | 30 |
| 116 | $(A^{6})(BC)(B^6)X_{[5,4]}X_{[4,1]}X_{[3,5]}X_{[3,5]}$ | $\mathbb{Z}/2\mathbb{Z}$ | 12 | 0 | 20 |
| 117 | $(A^{6})(BC)(B^7)X_{[4,5]}X_{[3,4]}X_{[3,5]}X_{[3,5]}$ | (0) | 14 | 0 | 28 |
| 118 | $(A^{6})(BC)(B^7)X_{[4,5]}X_{[3,4]}X_{[3,5]}X_{[3,5]}$ | (0) | 12 | 0 | 84 |
| 119 | $(A^{6})(BC)(X^7_{[1,2]}X_{[3,5]}X_{[3,5]}X_{[3,5]}$ | (0) | 20 | 0 | 42 |
| 120. (1) | $(A^{6})(BC)(B^7)X_{[9,11]}X_{[4,5]}X_{[5,7]}X_{[5,7]}$ | (0) | 6 | 0 | 84 |
| 120. (2) | $(A^{6})(BC)(X^7_{[9,11]})(B^6)X_{[14,19]}X_{[8,11]}$ | (0) | 12 | 0 | 42 |
| 121 | $(A^{6})(BC)(X^5_{[7,4]}X_{[2,1]}X_{[16,9]}X_{[16,9]}X_{[2,5]}$ | $\mathbb{Z}/4\mathbb{Z}$ | 2 | 0 | 8 |
| 122 | $(A^{6})(BC)(B^8)X_{[4,5]}X_{[11,14]}X_{[3,4]}X_{[3,4]}$ | $\mathbb{Z}/4\mathbb{Z}$ | 6 | 0 | 8 |
| 123 | $(A^{6})(BC)(B^8)X_{[5,6]}X_{[2,3]}X_{[2,3]}X_{[4,7]}$ | $\mathbb{Z}/2\mathbb{Z}$ | 8 | 0 | 20 |
| 124 | $(A^{6})(BC)X_{[5,6]}X_{[2,3]}X_{[2,3]}X_{[5,6]}X_{[5,6]}$ | (0) | 8 | 4 | 20 |
| 125. (1) | $(A^{6})(BC)X_{[3,9]}X_{[3,9]}X_{[3,9]}X_{[3,9]}X_{[11,9]}$ | (0) | 2 | 0 | 180 |
| 125. (2) | $(A^{6})(BC)X_{[3,9]}X_{[3,9]}X_{[3,9]}X_{[3,9]}X_{[5,9]}$ | (0) | 18 | 0 | 20 |
Table 4: Global monodromies of extremal elliptic $K3$ surfaces

| No | global monodromy | $MW$ | $a$ | $b$ | $c$ |
|----|------------------|------|-----|-----|-----|
| 126 | $(A^{5}BC)(X_{[3,-2]}^{9})(B^{6})X_{[5,-7]}X_{[3,-5]}$ | $(0)$ | 12 | 0 | 18 |
| 127 | $(A^{5}BC)(B^{10})(X_{[3,-8]}^{7})X_{[6,-7]}X_{[4,-5]}$ | $(0)$ | 6 | 0 | 60 |
| 128 | $(A^{5}BC)(B^{10})(X_{[3,-4]}^{2})X_{[5,-7]}X_{[2,-1]}$ | $\mathbb{Z}/2\mathbb{Z}$ | 2 | 0 | 60 |
| 129 | $(A^{5}BC)(B^{10})(X_{[7,-8]}^{4})X_{[13,-15]}X_{[5,-6]}$ | $\mathbb{Z}/2\mathbb{Z}$ | 8 | 4 | 12 |
| 130 | $(A^{5}BC)(X_{[3,-2]}^{10})(B^{5})X_{[5,-8]}X_{[4,-7]}$ | $(0)$ | 10 | 0 | 20 |
| 131 | $(A^{5}BC)(X_{[0,1]}^{11})(X_{[1,8]}^{11})X_{[1,7]}X_{[1,3]}$ | $(0)$ | 14 | 4 | 20 |
| 132 | $(A^{5}BC)(B^{12})(X_{[9,-10]}^{2})X_{[7,-8]}X_{[5,-6]}$ | $\mathbb{Z}/4\mathbb{Z}$ | 2 | 0 | 6 |
| 133 | $(A^{5}BC)(B^{12})(X_{[9,-10]}^{3})X_{[7,-8]}X_{[3,-4]}$ | $\mathbb{Z}/2\mathbb{Z}$ | 6 | 0 | 6 |
| 134. (1) | $(A^{5}BC)(B^{13})(X_{[10,-11]}^{2})X_{[17,-19]}X_{[7,-8]}$ | $(0)$ | 2 | 0 | 52 |
| (2) | $(A^{5}BC)(B^{13})(X_{[6,-7]}^{2})X_{[27,-32]}X_{[4,-5]}$ | $(0)$ | 6 | 2 | 18 |
| 135 | $(A^{5}BC)(B^{14})X_{[10,-11]}X_{[6,-7]}X_{[3,-4]}$ | $(0)$ | 6 | 2 | 10 |
| 136 | $(A^{6}BC)(B^{6})X_{[0,1]}X_{[2,-1]}X_{[6,0]}X_{[1,3]}$ | $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ | 2 | 0 | 2 |
| 137 | $(A^{6}BC)(X_{[1,-2]}^{6})X_{[1,-3]}B(X_{[1,-3]}^{4})X_{[0,1]}$ | $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ | 4 | 0 | 4 |
| 138 | $(A^{6}BC)(B^{5})(X_{[3,-4]}^{5})X_{[3,-2]}X_{[1,-3]}X_{[0,1]}$ | $(0)$ | 30 | 0 | 30 |
| 139 | $(A^{6}BC)(B^{6})(X_{[4,-5]}^{6})X_{[2,-1]}X_{[2,-1]}X_{[1,-4]}$ | $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ | 6 | 0 | 6 |
| 140 | $(A^{6}BC)(X_{[2,-1]}^{6})X_{[3,-2]}X_{[4,0]}X_{[1,-3]}X_{[1,-3]}$ | $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ | 4 | 0 | 12 |
| 141 | $(A^{6}BC)(X_{[7,-3]}^{6})X_{[2,-1]}X_{[2,-1]}X_{[2,-1]}$ | $\mathbb{Z}/2\mathbb{Z}$ | 4 | 0 | 30 |
| 142 | $(A^{6}BC)(B^{7})X_{[5,-6]}X_{[3,-4]}X_{[1,-3]}$ | $(0)$ | 14 | 0 | 14 |
| 143 | $(A^{6}BC)(X_{[2,-1]}^{7})X_{[5,-3]}X_{[3,-3]}B^{3}X_{[2,-7]}$ | $(0)$ | 6 | 0 | 70 |
| 144 | $(A^{6}BC)(X_{[2,-1]}^{8})X_{[5,-3]}X_{[3,-3]}B^{3}X_{[2,-7]}$ | $\mathbb{Z}/2\mathbb{Z}$ | 6 | 0 | 24 |
| 145 | $(A^{6}BC)(X_{[2,-1]}^{9})X_{[9,-14]}X_{[5,-8]}$ | $\mathbb{Z}/2\mathbb{Z}$ | 4 | 0 | 24 |
| 146 | $(A^{6}BC)(X_{[2,-1]}^{9})B^{3}X_{[9,-14]}X_{[5,-8]}X_{[1,-3]}$ | $\mathbb{Z}/2\mathbb{Z}$ | 6 | 2 | 14 |
| 147 | $(A^{6}BC)(B^{9})(X_{[1,-2]}^{0})X_{[19,-43]}X_{[2,-5]}$ | $(0)$ | 4 | 2 | 46 |
| 148. (1) | $(A^{6}BC)(X_{[2,-1]}^{10})B^{3}X_{[1,-3]}X_{[1,-5]}X_{[1,-5]}$ | $\mathbb{Z}/2\mathbb{Z}$ | 6 | 0 | 10 |
| (2) | $(A^{6}BC)(B^{10})(X_{[1,-2]}^{4})X_{[1,-2]}X_{[1,-2]}X_{[1,-2]}$ | $\mathbb{Z}/2\mathbb{Z}$ | 4 | 2 | 16 |
Table 4: Global monodromies of extremal elliptic K3 surfaces

| No | global monodromy | MW       | a  | b  | c  |
|----|------------------|----------|----|----|----|
| 149| \((A^6BC)(X_{[2,-1]}^{10})(B^4)X_{[3,-5]}X_{[1,-3]}\) | \(\mathbb{Z}/2\mathbb{Z}\) | 4  | 0  | 10 |
| 150| \((A^6BC)(X_{[2,-1]}^{11})(B^3)X_{[1,-4]}X_{[1,-7]}\) | (0)      | 6  | 0  | 22 |
| 151| \((A^6BC)(B^{12})(X_{[1,-2]}^2)X_{[1,-4]}X_{[1,2]}\) | \(\mathbb{Z}/2\mathbb{Z}\) | 4  | 0  | 6  |
| 152| \((A^6BC)(B^{13})X_{[10,-11]}X_{[5,-6]}X_{[2,-3]}\) | (0)      | 4  | 2  | 14 |
| 153| \((A^6BC)(X_{[1,-3]}^5X_{[0,1]}X_{[2,-3]}X_{[6,0,1]})(C^3)\) | \(\mathbb{Z}/2\mathbb{Z}\) | 6  | 0  | 12 |
| 154| \((A^6BC)(X_{[1,-2]}^5X_{[0,1]}X_{[2,-3]}X_{[8,0,1]}X_{[1,2]}\) | \(\mathbb{Z}/2\mathbb{Z}\) | 4  | 0  | 8  |
| 155| \((A^7BC)(B^7X_{[1,-2]}A)(X_{[1,-2]}^3)(X_{[0,1]}^3)\) | (0)      | 12 | 0  | 12 |
| 156| \((A^7BC)(X_{[1,-2]}^{14})(X_{[2,3]}^4)(C^4)(X_{[5,2]}^3)\) | \(\mathbb{Z}/4\mathbb{Z}\) | 8  | 4  | 8  |
| 157| \((A^7BC)(X_{[0,1]}^5)(X_{[2,9]}^5)(X_{[3,4]}^3)(X_{[1,2]}^2)\) | (0)      | 10 | 0  | 60 |
| 158| \((A^7BC)(X_{[2,-3]}^7)(X_{[2,1]}^4)(X_{[3,5]}^3)\) | (0)      | 8  | 4  | 44 |
| 159| \((A^7BC)(B^7)(X_{[3,4]}^5)(X_{[0,1]}^2X_{[8,5]}\) | (0)      | 4  | 0  | 70 |
| 160| \((A^7BC)(B^7)(X_{[4,-5]}^6)(X_{[15,-19]}X_{[1,2]}\) | (0)      | 2  | 0  | 84 |
| 161| \((A^7BC)(B^8)(X_{[3,4]}^3)(X_{[5,-7]}^2)(X_{[0,1]}^3)\) | \(\mathbb{Z}/2\mathbb{Z}\) | 4  | 0  | 24 |
| 162| \((A^7BC)(X_{[3,1]}^{14})(B^4)(X_{[1,2]}^2X_{[0,1]}^2)\) | \(\mathbb{Z}/4\mathbb{Z}\) | 2  | 0  | 8  |
| 163| \((A^7BC)(B^{10})(X_{[3,4]}^{14})(X_{[0,1]}^2X_{[7,4]}\) | \(\mathbb{Z}/2\mathbb{Z}\) | 4  | 0  | 10 |
| 164| \((A^7BC)(B^{10})(X_{[6,-9]}^3)(X_{[8,7]}X_{[1,2]}\) | (0)      | 2  | 0  | 60 |
| 165| \((A^7BC)(B^{11})(X_{[4,3]}^2X_{[5,3]}\) | (0)      | 4  | 0  | 22 |
| 166| \((A^7BC)(B^{12})(X_{[1,2]}X_{[1,2]}X_{[3,2]}\) | \(\mathbb{Z}/4\mathbb{Z}\) | 2  | 1  | 2  |
| 167| \((A^7BC)(X_{[3,2]}^5)(B^6)(X_{[0,1]}^2\) | \(\mathbb{Z}/2\mathbb{Z}\) | 4  | 0  | 12 |
| 168| \((A^7BC)(X_{[3,2]}^5)(B^6)(X_{[0,1]}^2)X_{[1,2]}\) | \(\mathbb{Z}/2\mathbb{Z}\) | 2  | 0  | 12 |
| 169| \((A^8BC)(X_{[3,1]}^8)(CB)(C^2)(X_{[3,1]}^2)\) | \(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\) | 2  | 0  | 2  |
| 170| \((A^8BC)(X_{[0,1]}^4)(X_{[1,2]}^4)(C^3)(X_{[3,1]}^3)\) | \(\mathbb{Z}/2\mathbb{Z}\) | 12 | 0  | 12 |
| 171| \((A^8BC)(B^6)(X_{[4,-5]}^6)(X_{[8,-14]}X_{[1,2]}\) | \(\mathbb{Z}/2\mathbb{Z}\) | 6  | 0  | 6  |
| 172| \((A^8BC)(B^6)(X_{[4,-5]}^8)(X_{[3,4]}^3)(X_{[0,1]}^3)\) | \(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\) | 2  | 0  | 12 |
| 173| \((A^8BC)(X_{[5,2]}^6)(B^5)(X_{[0,1]}^2X_{[7,3]}\) | \(\mathbb{Z}/2\mathbb{Z}\) | 2  | 0  | 30 |
| No | global monodromy | MW         | a | b | c |
|----|------------------|------------|---|---|---|
| 174 | \((A^8BC)(B^7)(X^3_{[4, -5]})(X^2_{[7, -9]}X_{[1, -2]})\) | \((0)\) | 12 | 6 | 24 |
| 175 | \((A^8BC)(B^8)(X^2_{[2, -3]})(X^2_{[0, 1]}X_{[8, 3]})\) | \(\mathbb{Z}/2\mathbb{Z}\) | 2 | 0 | 24 |
| 176 | \((A^8BC)(B^{10})(X^2_{[0, 1]}X_{[7, 0]}X_{[3, 2]})\) | \(\mathbb{Z}/2\mathbb{Z}\) | 2 | 0 | 10 |
| 177 | \((A^8BC)(X^5_{[3, 1]}AX_{[5, 2]})(X^5_{[5, 1]}AX_{[9, 2]})\) | \(\mathbb{Z}/2\mathbb{Z}\) | 4 | 0 | 4 |
| 178 | \((A^8BC)(B^6X_{[0, 1]}X_{[2, -1]}X^4_{[0, 1]})(C^2)\) | \(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\) | 2 | 0 | 4 |
| 179 | \((A^8BC)(X^9_{[4, 1]}X_{[5, 1]}X_{[3, 1]}X_{[9, 2]}X_{[7, 1]}\) | \((0)\) | 4 | 0 | 4 |
| 180 | \((A^8BC)(B^5)(X^3_{[7, -9]})(X^3_{[3, 4]})(X^2_{[0, 1]}\) | \((0)\) | 12 | 0 | 30 |
| 181 | \((A^8BC)(B^9)(X^4_{[1, -5]})(X^2_{[7, -9]}X_{[1, -2]}\) | \(\mathbb{Z}/2\mathbb{Z}\) | 4 | 0 | 12 |
| 182 | \((A^9BC)(B^6)(X^5_{[2, -3]}X_{[5, -8]}C\) | \((0)\) | 4 | 0 | 30 |
| 183 | \((A^9BC)(B^7)(X^3_{[4, -5]})(X^2_{[3, -4]}C\) | \((0)\) | 4 | 0 | 42 |
| 184 | \((A^9BC)(B^8)(X^2_{[6, -7]})(X^2_{[4, -5]}X_{[1, -2]}\) | \(\mathbb{Z}/2\mathbb{Z}\) | 4 | 0 | 8 |
| 185 | \((A^9BC)(X^9_{[2, -1]})(X^2_{[3, -2]}X_{[5, -4]}\) | \((0)\) | 4 | 0 | 18 |
| 186 | \((A^9BC)(B^{10})X_{[2, -3]}X_{[9, -20]}X_{[1, -3]}\) | \((0)\) | 4 | 0 | 10 |
| 187 | \((A^9BC)(X^6_{[3, -2]}BX_{[5, -3]})(B^5)\) | \((0)\) | 4 | 0 | 20 |
| 188 | \((A^{10}BC)(B^4)(X^4_{[2, -3]})(X^2_{[3, -5]})(X^2_{[0, 1]}\) | \(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\) | 4 | 0 | 4 |
| 189 | \((A^{10}BC)(X^7_{[4, -1]})(B^5)X_{[2, -3]}X_{[3, 2]}\) | \((0)\) | 10 | 0 | 10 |
| 190 | \((A^{10}BC)(B^5)(X^4_{[3, -4]})(X^2_{[2, -3]}C\) | \(\mathbb{Z}/2\mathbb{Z}\) | 2 | 0 | 20 |
| 191 | \((A^{10}BC)(X^6_{[0, 1]})(X^2_{[1, 5]})(X^2_{[1, 3]})(C^2\) | \(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\) | 4 | 2 | 4 |
| 192 | \((A^{10}BC)(X^6_{[3, -2]})(B^4)X_{[3, -7]}C\) | \(\mathbb{Z}/2\mathbb{Z}\) | 2 | 0 | 12 |
| 193 | \((A^{10}BC)(B^7)(X^3_{[2, -1]}X_{[3, -8]}C\) | \((0)\) | 2 | 0 | 42 |
| 194 | \((A^{10}BC)(B^9)X_{[6, -7]}X_{[2, -3]}C\) | \((0)\) | 2 | 0 | 18 |
| 195 | \((A^{10}BC)(X^5_{[3, 1]}X_{[4, -1]}X_{[6, -1]})(B^3)(X^2_{[1, -2]}\) | \(\mathbb{Z}/2\mathbb{Z}\) | 4 | 0 | 6 |
| 196 | \((A^{10}BC)(X^6_{[0, 1]}CB)(C^3)X_{[3, 1]}\) | \(\mathbb{Z}/2\mathbb{Z}\) | 2 | 0 | 6 |
| 197 | \((A^{10}BC)(X^7_{[4, -1]}AX_{[9, -2]})(X^2_{7, -2})X_{[2, -1]}\) | \(\mathbb{Z}/2\mathbb{Z}\) | 2 | 0 | 4 |
| 198 | \((A^{11}BC)(C^4)(X^3_{[3, 3]})(X^3_{[3, 5]}X_{[3, 1]}\) | \((0)\) | 12 | 0 | 12 |
### Table 4: Global monodromies of extremal elliptic K3 surfaces

| No | Global Monodromy | MW | a | b | c |
|----|------------------|----|---|---|---|
| 199 | $(A^{11}BC)(X_{[3,-1]}^5)(B^3)(X_{[0,1]}^2)X_{[7,2]}$ | (0) | 6 | 0 | 20 |
| 200 | $(A^{11}BC)(X_{[2,-1]}^6)(B^3)X_{[2,-5]}C$ | (0) | 6 | 0 | 12 |
| 201 | $(A^{11}BC)(B^7)(X_{[1,-2]}^2)X_{[2,-7]}C$ | (0) | 6 | 2 | 10 |
| 202 | $(A^{12}BC)(X_{[3,-1]}^3)(C^3)(X_{[3,9]}^2)(X_{[4,1]}^2)$ | $\mathbb{Z}/2\mathbb{Z}$ | 6 | 0 | 6 |
| 203 | $(A^{12}BC)(X_{[2,-1]}^4)(B^3)(X_{[1,-3]}^2)C$ | $\mathbb{Z}/2\mathbb{Z}$ | 4 | 0 | 6 |
| 204 | $(A^{12}BC)(B^5)(X_{[3,-4]}^2)(X_{[1,-2]}^2)C$ | $\mathbb{Z}/2\mathbb{Z}$ | 4 | 2 | 6 |
| 205 | $(A^{12}BC)(B^5X_{[2,-3]}X_{[0,1]})(C^2)X_{[3,1]}$ | $\mathbb{Z}/2\mathbb{Z}$ | 2 | 0 | 4 |
| 206 | $(A^{12}BC)(B^6X_{[3,-4]}X_{[1,-2]}X_{[1,-3]}X_{[8,1]}C$ | $\mathbb{Z}/2\mathbb{Z}$ | 2 | 0 | 2 |
| 207 | $(A^{13}BC)(B^5)(C^2)X_{[12,7]}X_{[3,1]}$ | (0) | 2 | 0 | 20 |
| 208 | $(A^{13}BC)(B^6)X_{[3,-4]}X_{[1,-3]}C$ | (0) | 2 | 0 | 12 |
| 209 | $(A^{13}BC)(X_{[5,1]}^5AX_{[9,2]}X_{[13,2]}X_{[8,1]}$ | (0) | 4 | 0 | 4 |
| 210 | $(A^{14}BC)(X_{[10,-1]}^3)(X_{[3,-1]}^3)X_{[3,1]}X_{[2,-5]}$ | (0) | 6 | 0 | 6 |
| 211 | $(A^{14}BC)(B^3)(X_{[2,-1]}^2)(C^2)X_{[3,1]}$ | $\mathbb{Z}/2\mathbb{Z}$ | 2 | 0 | 6 |
| 212 | $(A^{14}BC)(B^4)(X_{[2,-3]}^2)X_{[1,-3]}C$ | $\mathbb{Z}/2\mathbb{Z}$ | 2 | 0 | 4 |
| 213 | $(A^{14}BC)(B^5)X_{[2,-3]}X_{[1,2]}X_{[2,1]}$ | (0) | 4 | 2 | 6 |
| 214 | $(A^{15}BC)(X_{[2,-1]}^3)(B^2)X_{[3,2]}X_{[3,1]}$ | (0) | 4 | 0 | 6 |
| 215 | $(A^{16}BC)(B^2)(C^2)X_{[5,9]}X_{[3,1]}$ | $\mathbb{Z}/2\mathbb{Z}$ | 2 | 0 | 2 |
| 216 | $(A^{16}BC)(B^2)X_{[5,-5]}X_{[1,-3]}C$ | $\mathbb{Z}/2\mathbb{Z}$ | 2 | 1 | 2 |
| 217 | $(A^{17}BC)(X_{[3,-1]}^5BX_{[4,1]}X_{[7,1]}$ | (0) | 2 | 0 | 4 |
| 218 | $(A^{18}BC)X_{[2,1]}X_{[7,2]}X_{[9,2]}X_{[6,1]}$ | (0) | 2 | 0 | 2 |
| 219 | $(A^5BC^2)(A^5BC^2)(A^5BC^2)$ | $\mathbb{Z}/3\mathbb{Z}$ | 2 | 1 | 2 |
| 220 | $(A^5BC^2)(X_{[5,3]}X_{[7,4]}X_{[3,2]})(X_{[1,2]}^4)(X_{[5,1]}^4)$ | (0) | 12 | 0 | 12 |
| 221 | $(A^5BC^2)(A^5)(X_{[2,-1]}^5)(B^4)(X_{[1,-2]}^2)$ | (0) | 20 | 0 | 30 |
| 222 | $(A^5BC^2)(A^5X_{[4,-1]}X_{[2,-1]})(B^6)(X_{[0,1]}^2)$ | $\mathbb{Z}/3\mathbb{Z}$ | 2 | 0 | 6 |
| 223 | $(A^5BC^2)(A^6)(B^6)(X_{[9,-1]}^3)X_{[3,-4]}$ | $\mathbb{Z}/3\mathbb{Z}$ | 6 | 0 | 6 |
### Table 4: Global monodromies of extremal elliptic $K3$ surfaces

| No | Global Monodromy                  | MW    | a | b | c |
|----|-----------------------------------|-------|---|---|---|
| 224| $(A^5BC^2)(A^6)(X^4_{[3,-1]})(B^3)(X^3_{[1,-2]})$ | $\mathbb{Z}/3\mathbb{Z}$ | 6 | 0 | 12 |
| 225| $(A^6BC^2)(X^6_{[8,3]})(A^5)(X^4_{[2,-1]})(X_{[2,-3]})$ | $\mathbb{Z}/3\mathbb{Z}$ | 0 | 12 | 30 |
| 226| $(A^6BC^2)(A^5X_{[3,-1]}B^2)(X^7_{[0,1]})(X_{[1,2]})$ | $\mathbb{Z}/3\mathbb{Z}$ | 0 | 6 | 12 |
| 227. (1) | $(A^6BC^2)(A^7)(X^4_{[0,-3]})(X^3_{[4,-1]})(X^2_{[2,-1]})$ | $\mathbb{Z}/3\mathbb{Z}$ | 0 | 6 | 84 |
| 228. (2) | $(A^6BC^2)(X^5_{[3,-1]})(A^4)(X^3_{[1,-2]})(X^2_{[1,-3]})$ | $\mathbb{Z}/3\mathbb{Z}$ | 0 | 12 | 42 |
| 229| $(A^6BC^2)(A^7)(X^5_{[4,-1]})(B^2)(X^2_{[1,-3]})$ | $\mathbb{Z}/3\mathbb{Z}$ | 0 | 20 | 10 | 26 |
| 230| $(A^6BC^2)(X^7_{[3,1]})(A^5)(X^3_{[9,-5]})(X_{[5,-3]})$ | $\mathbb{Z}/3\mathbb{Z}$ | 0 | 18 | 3 | 18 |
| 231| $(A^6BC^2)(X^7_{[3,1]})(A^6)(X^2_{[2,-1]})(X_{[1,2]})$ | $\mathbb{Z}/3\mathbb{Z}$ | 0 | 6 | 0 | 42 |
| 232| $(A^6BC^2)(A^8)(X^5_{[2,-1]})(X^2_{[0,1]})(X_{[5,7]})$ | $\mathbb{Z}/3\mathbb{Z}$ | 0 | 2 | 0 | 120 |
| 233| $(A^6BC^2)(A^8)(X^6_{[5,-1]})(X_{[3,-1]})(X_{[1,2]})$ | $\mathbb{Z}/3\mathbb{Z}$ | 0 | 6 | 0 | 24 |
| 234| $(A^6BC^2)(A^9)(X^8_{[6,-1]})(X^3_{[5,-1]})(X_{[3,-1]})$ | $\mathbb{Z}/3\mathbb{Z}$ | 0 | 6 | 3 | 6 |
| 235| $(A^6BC^2)(A^9)(X^8_{[5,-1]})(X^2_{[2,-1]})(X^2_{[0,1]})$ | $\mathbb{Z}/3\mathbb{Z}$ | 0 | 2 | 0 | 18 |
| 236| $(A^6BC^2)(X^9_{[3,1]})(A^5X_{[3,-2]})(X_{[1,2]})$ | $\mathbb{Z}/3\mathbb{Z}$ | 0 | 12 | 3 | 12 |
| 237| $(A^6BC^2)(A^{10})(B^3)(X^2_{[1,-2]})(X_{[1,2]})$ | $\mathbb{Z}/3\mathbb{Z}$ | 0 | 12 | 6 | 18 |
| 238| $(A^6BC^2)(X^{10}_{[3,1]})(A^4)(X_{[2,-3]})(X_{[1,-3]})$ | $\mathbb{Z}/3\mathbb{Z}$ | 0 | 10 | 0 | 12 |
| 239| $(A^6BC^2)(A^{11})(X^2_{[2,-1]})(X^2_{[0,1]})(X_{[4,5]})$ | $\mathbb{Z}/3\mathbb{Z}$ | 0 | 2 | 0 | 66 |
| 240| $(A^6BC^2)(X^{11}_{[2,1]})(X^3_{[4,1]})(X_{[11,3]})(X_{[5,1]})$ | $\mathbb{Z}/3\mathbb{Z}$ | 0 | 6 | 3 | 18 |
| 241. (1) | $(A^6BC^2)(A^{12})(B^2)(X_{[1,-3]})(X_{[1,2]})$ | $\mathbb{Z}/3\mathbb{Z}$ | 0 | 6 | 0 | 12 |
| 242. (2) | $(A^6BC^2)(A^{12})(X^2_{[6,-1]})(X_{[3,-1]})$ | $\mathbb{Z}/3\mathbb{Z}$ | 0 | 2 | 0 | 4 |
| 243| $(A^6BC^2)(A^{13})(X^9_{[3,1]})(X_{[2,-1]})(X_{[1,2]})$ | $\mathbb{Z}/3\mathbb{Z}$ | 0 | 4 | 1 | 10 |
| 244| $(A^6BC^2)(X^9_{[5,1]})(AX_{[9,2]})(A^7)(X^2_{[0,1]})$ | $\mathbb{Z}/3\mathbb{Z}$ | 0 | 2 | 0 | 84 |
| 245| $(A^6BC^2)(X^5_{[4,1]})(AX_{[7,2]})(A^8)(B)$ | $\mathbb{Z}/3\mathbb{Z}$ | 0 | 8 | 0 | 12 |
| 246| $(A^6BC^2)(A^5X_{[0,1]})(X^2_{[2,1]})(A^6BC)$ | $\mathbb{Z}/3\mathbb{Z}$ | 0 | 6 | 0 | 6 |
Table 4: Global monodromies of extremal elliptic $K3$ surfaces

| No  | global monodromy                                                                 | $MW$ | $a$ | $b$ | $c$ |
|-----|---------------------------------------------------------------------------------|------|-----|-----|-----|
| 247 | $(A^5BC^2)(X^6_{[4,1]}AX_{[7,2]})(A^5)(X^3_{[0,1]})$                           | (0)  | 6   | 0   | 30  |
| 248 | $(A^5BC^2)(X^6_{[3,1]}AX_{[5,2]})(A^7)X_{[2,-1]}$                            | (0)  | 4   | 2   | 22  |
| 249 | $(A^5BC^2)(X^7_{[3,1]}X_{[4,1]}X_{[2,1]})(A^5)(B^2)$                          | (0)  | 4   | 0   | 30  |
| 250 | $(A^5BC^2)(A^7BC)(X^5_{[0,1]}AX_{[1,-2]})$                                    | (0)  | 4   | 0   | 12  |
| 251 | $(A^5BC^2)(A^8X_{[3,1]}X_{[5,1]})(X^5_{[0,1]}X_{[1,3]})$                      | (0)  | 8   | 2   | 8   |
| 252 | $(A^5BC^2)(A^8X_{[2,1]}X_{[4,1]})(X^3_{[2,-1]})(B^2)$                          | (0)  | 6   | 0   | 12  |
| 253 | $(A^5BC^2)(X^8_{[3,1]}X_{[10,3]}X_{[4,1]})(A^4)X_{[1,-2]}$                    | (0)  | 4   | 0   | 12  |
| 254 | $(A^5BC^2)(A^{11}BC)(X^5_{[0,1]}X_{[2,3]})$                                   | (0)  | 2   | 0   | 12  |
| 255 | $(A^5BC^2)(A^{12}X_{[2,-1]}X_{[0,1]}BX_{[1,2]}$                                | (0)  | 4   | 2   | 4   |
| 256 | $(A^6BC^2)(C^6X_{[3,2]}A^2)(C^3)(X^3_{[2,1]})$                                | (0)  | 6   | 0   | 6   |
| 257 | $(A^6BC^2)(A^6CX_{[3,1]})(A^4)(C^2)$                                          | $\mathbb{Z}/2\mathbb{Z}$ | 2   | 0   | 4   |
| 258 | $(A^6BC^2)(A^6X_{[0,1]}X^2_{[2,1]})(A^5)X_{[0,1]}$                            | (0)  | 4   | 2   | 6   |
| 259 | $(A^6BC^2)(C^5)(X^4_{[5,4]})(X^4_{[2,1]})(X^2_{[12,5]})$                      | $\mathbb{Z}/2\mathbb{Z}$ | 4   | 0   | 20  |
| 260 | $(A^6BC^2)(A^5)(X^4_{[2,-1]})(B^3)(X^3_{[0,1]})$                              | (0)  | 12  | 0   | 30  |
| 261 | $(A^6BC^2)(X^6_{[8,3]})(X^4_{[3,1]})(A^4)X_{[1,3]}$                            | $\mathbb{Z}/2\mathbb{Z}$ | 4   | 0   | 12  |
| 262 | $(A^6BC^2)(X^6_{[2,1]})(X^4_{[5,2]})(A^3)(X^2_{[0,1]})$                       | $\mathbb{Z}/2\mathbb{Z}$ | 6   | 0   | 12  |
| 263 | $(A^6BC^2)(A^6)(X^5_{[3,1]}X^2_{[2,1]})(X^2_{[3,1]})(X^2_{[2,3]})$           | $\mathbb{Z}/2\mathbb{Z}$ | 8   | 2   | 8   |
| 264 | $(A^6BC^2)(X^6_{[5,3]})(X^5_{[5,1]})(X^3_{[3,1]}X_{[9,2]})$                  | (0)  | 6   | 0   | 30  |
| 265 | $(A^6BC^2)(X^7_{[3,1]})(X^7_{[7,2]})(A^3)(X^2_{[2,3]})$                       | (0)  | 6   | 0   | 42  |
| 266 | $(A^6BC^2)(X^7_{[3,1]})(A^4)(X^3_{[7,5]}X_{[2,3]})(X_{[2,-3]}))$              | (0)  | 4   | 0   | 42  |
| 267.1 | $(A^6BC^2)(X^7_{[2,1]})(X^5_{[7,3]}X_{[2,1]}X_{[11,2]})$                     | (0)  | 2   | 0   | 70  |
| 268.2 | $(A^6BC^2)(X^7_{[3,1]})(A^5)(B^2)(X_{[1,2]})$                                | (0)  | 8   | 2   | 18  |
| 268 | $(A^6BC^2)(X^7_{[2,1]})(X^6_{[3,1]}X_{[10,3]}X_{[5,17]})$                    | (0)  | 4   | 2   | 22  |
| 269 | $(A^6BC^2)(X^8_{[2,1]})(X^3_{[13,6]})(X^3_{[3,1]}X_{[18,5]})$                | (0)  | 6   | 0   | 24  |
| 270 | $(A^6BC^2)(X^8_{[2,1]})(X^3_{[7,3]})(X^2_{[12,5]})(X^2_{[3,1]})$             | $\mathbb{Z}/2\mathbb{Z}$ | 2   | 0   | 24  |
Table 4: Global monodromies of extremal elliptic K3 surfaces

| No | global monodromy | MW  | a | b | c |
|----|------------------|-----|---|---|---|
| 271 | $A^6BC^2(X^8_{[2,1]} X^4_{[3,1]} X^2_{[7,2]} X_{[5,1]}$) | $\mathbb{Z}/2\mathbb{Z}$ | 4 | 0 | 8 |
| 272 | $A^6BC^2(X^8_{[7,4]} X^6_{[9,5]} X_{[20,11]} X_{[4,1]}$) | (0) | 6 | 2 | 14 |
| 273 | $(A^6BC^2(X^9_{[3,1]})(A^3X^2_{[1,2]}X_{[3,4]}$) | (0) | 6 | 0 | 18 |
| 274 | $(A^6BC^2(X^9_{[3,1]})(A^4X_{[1,2]}X_{[1,2]}$) | (0) | 4 | 0 | 18 |
| 275 | $(A^6BC^2(A^{10}X^2_{[7,1]})(B^2X_{[1,3]}$) | $\mathbb{Z}/2\mathbb{Z}$ | 2 | 0 | 10 |
| 276 | $(A^6BC^2)(X^{10}_{[3,1]})(A^3X_{[1,3]}X_{[2,3]}$) | (0) | 6 | 0 | 10 |
| 277 | $(A^6BC^2)(X^{11}_{[2,1]})(X^2_{[3,1]} X_{[11,3]} X_{[5,1]}$) | $\mathbb{Z}/2\mathbb{Z}$ | 4 | 1 | 4 |
| 278 | $(A^6BC^2)(X^{11}_{[7,4]})(X^2_{[9,5]} X_{[13,7]} X_{[4,1]}$) | (0) | 6 | 2 | 8 |
| 279 | $(A^6BC^2)(A^6BC^2)(A^4BC)$ | $\mathbb{Z}/2\mathbb{Z}$ | 2 | 0 | 2 |
| 280 | $(A^6BC^2)(X^5_{[8,3]} X_{[11,4]} X_{[5,2]} X^5_{[3,1]})(A^3)$ | (0) | 6 | 0 | 20 |
| 281 | $(A^6BC^2)(X^5_{[5,3]} X_{[7,4]} X_{[3,2]} X^6_{[2,1]} X^2_{[3,1]}$) | $\mathbb{Z}/2\mathbb{Z}$ | 2 | 0 | 12 |
| 282 | $(A^6BC^2)(X^5_{[0,1]} A X_{[1,2]})(C^7X_{[3,2]}$) | (0) | 6 | 2 | 10 |
| 283 | $(A^6BC^2)(X^6_{[8,5]} X_{[2,1]} X^4_{[2,1]} X^3_{[3,1]}$) | $\mathbb{Z}/2\mathbb{Z}$ | 4 | 0 | 6 |
| 284 | $(A^6BC^2)(C^6X_{[2,1]} X_{[0,1]} X^6_{[3,2]} X_{[5,3]}$) | $\mathbb{Z}/2\mathbb{Z}$ | 4 | 2 | 4 |
| 285 | $(A^6BC^2)(A^6BC)(C^6AX_{[1,2]}$) | $\mathbb{Z}/2\mathbb{Z}$ | 2 | 0 | 4 |
| 286 | $(A^6BC^2)(C^6X_{[2,1]} X_{[0,1]} X^4_{[3,2]})(X^2_{[3,2]}$) | $\mathbb{Z}/2\mathbb{Z}$ | 4 | 0 | 4 |
| 287 | $(A^6BC^2)(X^5_{[2,1]} X_{[3,3]} C)(X^5_{[3,1]} X_{[7,2]}$) | (0) | 2 | 0 | 20 |
| 288 | $(A^6BC^2)(X^8_{[2,1]} X_{[4,1]} C)(X^3_{[3,1]} X^2_{[4,1]}$) | $\mathbb{Z}/2\mathbb{Z}$ | 2 | 0 | 6 |
| 289 | $(A^6BC^2)(X^9_{[5,2]} X_{[9,1]} X_{[7,3]})(A^3X_{[1,3]}$) | (0) | 4 | 0 | 6 |
| 290 | $(A^6BC^2)(A^{10}X_{[2,1]} X_{[4,1]})(C^2X_{[5,3]}$) | $\mathbb{Z}/2\mathbb{Z}$ | 2 | 0 | 2 |
| 291 | $(A^6BC^2)(A^{11}X_{[2,1]} X_{[4,1]})(X_{[0,1]} X_{[3,2]}$) | (0) | 2 | 0 | 4 |
| 292 | $(A^6BC^2)(X^5_{[3,2]} X_{[5,3]} C^2)(X^3_{[3,1]} X^3_{[10,3]}$) | (0) | 6 | 0 | 12 |
| 293 | $(A^6BC^2)(A^6X_{[3,1]} B^2)(X^5_{[0,1]})(C^2$) | (0) | 2 | 0 | 30 |
Table 4: Global monodromies of extremal elliptic $K3$ surfaces

| No  | global monodromy                                                                 | $MW$ | $a$ | $b$ | $c$ |
|-----|---------------------------------------------------------------------------------|------|-----|-----|-----|
| 294 | $(A^6 BC^2)(A^5 X_{[2,-1]} X_{[0,1]}^2)(C^6) X_{[3,2]}$                         | (0)  | 6   | 0   | 6   |
| 295 | $(A^6 BC^2)(A^5 X_{[0,1]} X_{[2,1]}^2)(A^5 BC)$                                | (0)  | 2   | 0   | 12  |
| 296 | $(A^7 BC^2)(C^7 X_{[5,4]} X_{[3,2]}^2)(X_{[3,1]}^2)(X_{[4,1]}^2)$              | (0)  | 2   | 0   | 2   |
| 297 | $(\alpha)$                                                                     | (0)  | 2   | 1   | 2   |
|     | $(A^7 BC^2)(A^3 BC^2)(A^3)^2 C$                                                 | (0)  | 2   | 1   | 2   |
| 298 | $(A^7 BC^2)(A^7)^2 (X_{[2,1]}^4)(X_{[5,2]}^3)(X_{[8,3]}^3)$                    | (0)  | 12  | 0   | 12  |
| 299 | $(A^7 BC^2)(A^5)(X_{[2,1]}^2)(X_{[5,2]}^2)(X_{[11,4]}^2)$                       | (0)  | 10  | 0   | 10  |
| 300 | $(A^7 BC^2)(A^5)(C^4)(X_{[2,1]}^2)(X_{[8,3]}^2)$                               | (0)  | 6   | 0   | 20  |
| 301 | $(A^7 BC^2)(A^6)(X_{[2,-1]}^6)(X_{[7,4]}^2 X_{[0,1]}^2)$                       | (0)  | 6   | 0   | 6   |
| 302 | $(A^7 BC^2)(X_{[5,1]}^6)(A^4)(X_{[2,1]}^3)(X_{[13,5]})$                        | (0)  | 6   | 0   | 12  |
| 303 | $(A^7 BC^2)(A^6)(X_{[0,1]}^5)(X_{[2,1]}^2)(X_{[19,7]})$                        | (0)  | 2   | 0   | 30  |
| 304 | $(A^7 BC^2)(A^7)^2 (X_{[3,-1]}^3)(X_{[2,1]}^2)(X_{[19,8]})$                    | (0)  | 6   | 3   | 12  |
| 305 | $(A^7 BC^2)(A^7)(X_{[0,1]}^3)(X_{[2,1]}^2)(X_{[1,2]}^2)(X_{[2,1]}^2)$          | (0)  | 2   | 0   | 42  |
| 306 | $(A^7 BC^2)(A^7)(X_{[2,1]}^2)(X_{[5,-3]}^2)(X_{[0,1]})$                        | (0)  | 6   | 2   | 10  |
| 307 | $(A^7 BC^2)(A^7)^2 (X_{[2,-1]}^5)(X_{[5,3]}^2)^2 C$                            | (0)  | 2   | 1   | 18  |
| 308 | $(A^7 BC^2)(A^8)(X_{[2,1]}^3)(X_{[3,-2]}^2)^2 C$                               | (0)  | 2   | 0   | 24  |
| 309 | $(A^7 BC^2)(X_{[2,1]}^9)(X_{[7,3]}^2)(X_{[3,1]}^2)(X_{[7,1]}^2)$               | (0)  | 2   | 0   | 18  |
| 310 | $(A^7 BC^2)(X_{[2,1]}^5)(X_{[3,1]}^3)(X_{[11,3]})(X_{[5,1]}^2)$               | (0)  | 6   | 3   | 6   |
| 311 | $(A^7 BC^2)(X_{[0,2]}^7)(X_{[3,1]}^2)(X_{[21,5]})(X_{[9,2]}^2)$               | (0)  | 2   | 0   | 10  |
| 312 | $(A^7 BC^2)(A^{11})(X_{[6,-1]})(X_{[3,-1]}^{11})$                              | (0)  | 2   | 1   | 6   |
| 313 | $(A^7 BC^2)(A^{4} X_{[2,1]}(X_{[1,4]}^{2})(X_{[2,1]}^{2})(X_{[5,2]}^2 A)$     | (0)  | 4   | 0   | 4   |
| 314 | $(A^7 BC^2)(X_{[7,2]}^5)(X_{[4,1]}^2)(X_{[10,3]}^2)(X_{[4,1]}^5)(A^2)$        | (0)  | 2   | 0   | 20  |
| 315 | $(A^7 BC^2)(X_{[4,1]}^5 AX_{[7,2]})(A^6)$                                      | (0)  | 2   | 0   | 12  |
| 316 | $(A^7 BC^2)(X_{[4,1]}^6 AX_{[7,2]})(A^3)(X_{[2,1]}^3)$                         | (0)  | 6   | 0   | 6   |
| 317 | $(A^7 BC^2)(X_{[6,1]}^4 AX_{[5,2]})(A^5)(X_{[0,1]}^2)$                         | (0)  | 4   | 2   | 6   |
| 318 | $(A^7 BC^2)(A^7 CX_{[3,1]}^2)(X_{[0,1]}^6)(C^2)$                               | (0)  | 4   | 0   | 6   |
Table 4: Global monodromies of extremal elliptic $K3$ surfaces

| No  | global monodromy                                      | MW  | $a$ | $b$ | $c$ |
|-----|--------------------------------------------------------|-----|-----|-----|-----|
| 319 | $(A^7BC^2)(A^9BC)(X_{[2,-1]}^2X_{[0,1]}^2)(0)$         |     | 2   | 0   | 4   |
| 320 | $(A^7BC^2)(A^{10}BC)(X_{[-2,-1]}^2X_{[2,-1]}^2)(0)$   |     | 2   | 0   | 2   |
| 321 | $(A^7BC^2)(A^5X_{[2,1]}X_{[4,1]}^2)(X^2_{[2,1]}^2)(A^4)(0)$ |     | 2   | 0   | 12  |
| 322 | $(A^7BC^2)(A^5CX_{[3,1]}^2)(X^5)(0)$                  |     | 2   | 1   | 8   |
| 323 | $(A^7BC^2)(A^5X_{[0,1]}X_{[2,1]}^2)(X^4_{[0,1]}X_{[2,1]}^2)(0)$ |     | 4   | 2   | 4   |
| 324 | $(A^7BC^2)(A^6CX_{[3,1]}^2)(X^3)(X_{[2,1]}^2)(0)$     |     | 2   | 0   | 6   |
| 325 | $(A^7BC^2)(A^6X_{[0,1]}X_{[2,1]}^2)(X^3)(A^4)(0)$     |     | 2   | 0   | 4   |

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