Cesàro operator on Hardy spaces associated with the Dunkl setting \((\frac{2\lambda}{2\lambda + 1} < p < \infty)\) \(\dagger\)

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Abstract

For \(p > \frac{2\lambda}{2\lambda + 1}\) with \(\lambda > 0\), the Hardy spaces \(H^p_\lambda(\mathbb{R}^d_+\!\!\!)\) associated with the Dunkl transform \(\mathcal{F}_\lambda\) and the Dunkl operator \(D_\lambda\) on the line, where \(D_\lambda f(x) = f'(x) + \frac{1}{\lambda} f(x) - f(-x)\), is the set of function \(F = u + iv\) on the upper half plane \(\mathbb{R}^d_+ = \{(x,y) : y > 0\}\), satisfying the \(\lambda\)-Cauchy-Riemann equations: \(D_\lambda u - \partial_y v = 0, \partial_y u + D_\lambda v = 0\), and \(\sup_{y>0} \|f(x,y)\|\|x^\lambda\|dx < \infty\). In this paper, we will study the boundedness of Cesàro operator on \(H^p_\lambda(\mathbb{R}^d_+)\). We will prove the following inequality

\[\|C_\alpha f\|_{H^p_\lambda(\mathbb{R}^d_+)} \leq C\|f\|_{H^p_\lambda(\mathbb{R}^d_+)},\]

for \(\frac{2\lambda}{2\lambda + 1} < p < \infty\), where \(C\) is dependent on \(\alpha, p, \lambda\), and the average function for the Cesàro operator \(C_\alpha\) is \(\phi_\alpha(t) = \alpha(1-t)^{\alpha-1}\) with \(\alpha > 0\).

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1 Introduction

The Hausdorff operators on the real Hardy spaces \(H^p(\mathbb{R})\) was initially studied by Kanjin in [6]. Miyachi studied the Boundedness of the Cesàro Operator on the Real Hardy Spaces in [5], and the Cesàro operator on the \(H^p\) on the Unit Disk was studied in [10]. A brief history of the study of the Cesàro operator and the Hausdorff operators can be found in [5], [6],[7],[8] and [10]. The Cesàro operator in this paper is defined as in [5].

The purpose of this paper is to consider the boundedness of Cesàro operator on the Hardy spaces associated with the Dunkl setting on the upper half plane \(H^p_\lambda(\mathbb{R}^d_+)\), \(\frac{2\lambda}{2\lambda + 1} < p \leq \infty\). More details associated with the Dunkl setting and \(H^p_\lambda(\mathbb{R}^d_+)\) will be introduced in Section 3. One difference about \(H^p_\lambda(\mathbb{R}^d_+)\) and \(H^p(\mathbb{R}^d_+)\) is that, \(H^p(\mathbb{R})\) related to \(H^p(\mathbb{R}^d_+)\) is a Homogeneous Hardy spaces, that we have an atomic decomposition for any function in \(H^p(\mathbb{R})\), but similar properties is not known on the \(H^p_\lambda(\mathbb{R}^d_+)\) including the related real \(H^p_\lambda(\mathbb{R})\). Thus we could not use the same way in [5] to prove the boundedness of Cesàro operator on \(H^p_\lambda(\mathbb{R}^d_+)\).

Our main result is Theorem 3.12. we will prove the following inequality

\[\|C_\alpha f\|_{H^p_\lambda(\mathbb{R}^d_+)} \leq C\|f\|_{H^p_\lambda(\mathbb{R}^d_+)},\]

for \(\frac{2\lambda}{2\lambda + 1} < p < \infty\), where \(C\) is a constant dependent on \(\alpha, p\) and \(\lambda\).

Similar result like Theorem 3.12 could not be extended to the case \(\mathbb{R}^d\) where \(d \geq 3\), because many properties of \(H^p_\lambda(\mathbb{R}^d_+)\) in [1] and [2] remain unknown.

Note: We use \(A \lesssim B\) to denote the estimate \(|A| \leq CB\) for some absolute universal constant \(C > 0\), which may vary from line to line. \(A \gtrsim B\) to denote the estimate \(|A| \geq CB\) for some absolute universal constant \(C > 0\). \(A \approx B\) or \(A \sim B\) to denote the estimate \(|A| \leq C_1 B, |A| \geq C_2 B\) for some absolute universal constant \(C_1, C_2\).

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2 Cesàro operator on the spaces $L^p(\mathbb{R}_+^{N+1}, d\mu(x))$

In this Section, we will discuss the Cesàro operator on $L^p(\mathbb{R}_+^{N+1}, d\mu(x))$, which is the set of vector function $\vec{f}(\bar{x}) = (u_0(\bar{x}), u_1(\bar{x}), u_2(\bar{x}), \ldots, u_N(\bar{x}))$ defined on $\mathbb{R}_+^{N+1}$ with $\bar{x} = (x, y) = (x_1, \ldots, x_N, y) \in \mathbb{R}_+^{N+1}$, where $y > 0$ and $x \in \mathbb{R}^N$, $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$. Let the measure $d\mu(x)$ to denote as $d\mu(x) = |x_1|^{2\lambda_1}|x_2|^{2\lambda_2} \cdots |x_N|^{2\lambda_N} dx_1 dx_2 \cdots dx_N$, where $0 < \lambda_i < \infty$ for $1 \leq i \leq N$.

Then $L^p(\mathbb{R}_+^{N+1}, d\mu(x))$ can be defined as the set of vector function $\vec{f}(\bar{x})$ satisfying

$$\|\vec{f}\|_{L^p(\mathbb{R}_+^{N+1}, d\mu(x))} = \sup_{y > 0} \left( \int_{\mathbb{R}^N} |\vec{f}(\bar{x})|^p d\mu(x) \right)^{1/p} < \infty,$$

where $0 < p < \infty$.

$L^p(\mathbb{R}^N)$ denote as the set of measurable function $f(x)$ on $\mathbb{R}^N$ satisfying $\|f\|_{L^p(\mathbb{R}^N)} = \left( c_\lambda \int_{\mathbb{R}^N} |f(x)|^p d\mu(x) \right)^{1/p} < \infty$, with $c_\lambda^{-1} = 2^{\lambda+1/2} \Gamma(\lambda + 1/2)$. To avoid confusion, we will use the symbol $\vec{f}$ to denote as a vector function, and $f$ to denote as the ordinary measurable function.

Let $\alpha > 0$, we write $\hat{\phi}_\alpha(t) = \alpha(1-t)^{\alpha-1}$ for $0 < t < 1$. For $\vec{f}(\bar{x}, y) = (u_0(x, y), u_1(x, y), \ldots, u_N(x, y)) \in L^p(\mathbb{R}_+^{N+1}, d\mu(x))$, we define the Cesàro operator $C_\alpha$ by the form as

$$(C_\alpha f)(x, y) = \int_0^1 t^{-1} u_j (t^{-1} x, t^{-1} y) \phi_\alpha(t) dt, \quad \text{for}, \quad 0 \leq j \leq N$$

where $x \in \mathbb{R}_+^{N+1}, t^{-1} x = (t^{-1} x_1, t^{-1} x_2, \ldots, t^{-1} x_N)$. Then $C_\alpha \vec{f}$ and $|C_\alpha \vec{f}|$ can be given by

$$C_\alpha \vec{f} = (C_\alpha u_0, C_\alpha u_1, \ldots, C_\alpha u_N),$$

$$|C_\alpha \vec{f}| = \left( (C_\alpha u_0)^2 + (C_\alpha u_1)^2 + \cdots + (C_\alpha u_N)^2 \right)^{1/2}.$$

It is easy to verify the following Proposition 2.1 and Proposition 2.2:

**Proposition 2.1.** For $\vec{f}(\bar{x}, y) = (u_0(x, y), u_1(x, y), \ldots, u_N(x, y)) \in L^p(\mathbb{R}_+^{N+1}, d\mu(x))$, where $0 < p < \infty$,

$$|C_\alpha \vec{f}| \leq C_\alpha |\vec{f}|.$$

**Proposition 2.2.** [Minkowski’s inequality] For $1 \leq p < \infty$, $x = (x_1, \ldots, x_N)$, if

$$\int_0^1 \left( \int_{\mathbb{R}^N} |f(x, t)|^p d\mu(x) \right)^{1/p} dt = M < \infty,$$

then

$$\left( \int_{\mathbb{R}^N} \left( \int_0^1 f(x, t)^p d\mu(x) \right)^{1/p} dt \right)^{1/p} \leq \int_0^1 \left( \int_{\mathbb{R}^N} |f(x, t)|^p d\mu(x) \right)^{1/p} dt = M < \infty.$$

By Proposition 2.2, we could prove the following Proposition 2.3

**Proposition 2.3.** Cesàro operator $C_\alpha$ is a bounded linear operator on $L^p(\mathbb{R}_+^{N+1}, d\mu(x))$ for $1 \leq p < \infty$:

$$\|C_\alpha \vec{f}\|_{L^p(\mathbb{R}_+^{N+1}, d\mu(x))} \leq C \|\vec{f}\|_{L^p(\mathbb{R}_+^{N+1}, d\mu(x))},$$

$C$ is dependent on $\alpha$, $p$ and $d\mu$. 

Proof. From Proposition 2.1 and the Minkowski’s inequality in Proposition 2.2, we could conclude:

\[ \|C_\alpha \overrightarrow{f}\|_{L^p(\mathbb{R}^{N+1}_+, d\mu(x))} = \sup_{y > 0} \left[ \int_{\mathbb{R}^n} \left| C_\alpha \overrightarrow{f}(x, y) \right|^p d\mu(x) \right]^{1/p} \]

\[ = \sup_{y > 0} \left[ \int_{\mathbb{R}^n} \int_0^1 t^{-1} \overrightarrow{f}(t^{-1}x, t^{-1}y) \phi_\alpha(t) dt \right]^p d\mu(x) \]

\[ \leq \sup_{y > 0} \left[ \int_{\mathbb{R}^n} \int_0^1 \left| t^{-1} \overrightarrow{f}(t^{-1}x, t^{-1}y) \right| t^{-1} \phi_\alpha(t) dt \right]^p d\mu(x) \]

\[ \leq \int_0^1 \sup_{y > 0} \left[ \int_{\mathbb{R}^n} \left| \overrightarrow{f}(t^{-1}x, t^{-1}y) \right|^p d\mu(x) \right]^{1/p} t^{-1} \phi_\alpha(t) dt \]

\[ \leq \int_0^1 \| \overrightarrow{f}\|_{L^p(\mathbb{R}^{N+1}_+, d\mu(x))} t^{-1} \phi_\alpha(t) dt \]

\[ \leq C \| \overrightarrow{f}\|_{L^p(\mathbb{R}^{N+1}_+, d\mu(x))} \] where \( \beta = N + \sum_{i=1}^N 2\lambda_i. \]

\[ \square \]

Lemma 2.4. For \( 0 < p \leq 1, \ i \in \mathbb{N}, \) if \( \sum |a_i|^p < \infty, \) then

\[ \left( \sum |a_i| \right)^p \leq \sum |a_i|^p. \]

Proof. Without loss of generality, we may assume that

\[ \sum |a_i|^p = 1. \]

In this case, we have \( |a_i| \leq 1 \) for any \( i \in \mathbb{N}, \) so

\[ \left( \sum |a_i| \right) \leq \sum |a_i|^p |a_i|^{1-p} \leq \sum |a_i|^p = 1. \]

Then we could obtain:

\[ \left( \sum |a_i| \right)^p \leq \sum |a_i|^p. \]

This proves the Lemma. \( \square \)

Lemma 2.5. For \( 0 < p \leq 1, \ k \in \mathbb{Z}, \ k \leq -1, \ x \in \mathbb{R}^N \) there exists \( \xi_k \in [2^{k-1}, 2^k] \) and \( \xi_k' \in [1 - 2^k, 1 - 2^{k-1}] \) such that the following holds:

\[ \left| \int_0^1 t^{-1} \overrightarrow{f}(t^{-1}x, t^{-1}y) \phi_\alpha(t) dt \right|^p \leq C_\alpha \sum_{k=-\infty}^{-1} \left| \overrightarrow{f}(\xi_k^{-1}x, \xi_k^{-1}y) \right|^p + C_\alpha \sum_{k=-\infty}^{-1} 2^{k \rho_0} \left| \overrightarrow{f}((\xi_k')^{-1}x, (\xi_k')^{-1}y) \right|^p, \]

where \( C_\alpha \) is a constant independent on \( x, y, k \) and \( \overrightarrow{f}. \)

Proof. By Proposition 2.1 and Lemma 2.4,

\[ \left| \int_0^1 t^{-1} \overrightarrow{f}(t^{-1}x, t^{-1}y) \phi_\alpha(t) dt \right|^p \leq \left| \int_0^{1/2} t^{-1} \overrightarrow{f}(t^{-1}x, t^{-1}y) \phi_\alpha(t) dt \right|^p + \left| \int_{1/2}^1 t^{-1} \overrightarrow{f}(t^{-1}x, t^{-1}y) \phi_\alpha(t) dt \right|^p. \] (1)

By Lemma 2.4, we could conclude:

\[ \left| \int_0^{1/2} t^{-1} \overrightarrow{f}(t^{-1}x, t^{-1}y) \phi_\alpha(t) dt \right|^p = \sum_{k=-\infty}^{-1} \int_{2^{k-1}}^{2^k} t^{-1} \overrightarrow{f}(t^{-1}x, t^{-1}y) \phi_\alpha(t) dt \]

\[ \leq \sum_{k=-\infty}^{-1} \int_{2^{k-1}}^{2^k} t^{-1} \overrightarrow{f}(t^{-1}x, t^{-1}y) \phi_\alpha(t) dt \leq \sum_{k=-\infty}^{-1} 2^{k \rho_0} \left| \overrightarrow{f}(\xi_k^{-1}x, \xi_k^{-1}y) \right|^p. \]
We could deduce that there exists $\xi_k \in [2^{k-1}, 2^k]$ and a constant $C$ ($C = 200$ for example) independent on $k$, $\mathcal{F}$, $x$, and $y$, such that the following holds:

$$\int_{2^{k-1}}^{2^k} t^{-1} |\mathcal{F} (t^{-1} x, t^{-1} y) | \phi_\alpha(t) dt \leq C \xi_k^{-1} |\mathcal{F} (\xi_k^{-1} x, \xi_k^{-1} y) | \int_{2^{k-1}}^{2^k} \phi_\alpha(t) dt$$

$$\leq (1/2)^{\rho_0} C \xi_k^{-1} |\mathcal{F} (\xi_k^{-1} x, \xi_k^{-1} y) | \leq \mathcal{C}_\alpha |\mathcal{F} (\xi_k^{-1} x, \xi_k^{-1} y) |.$$

Then

$$\left| \int_{0}^{1/2} t^{-1} |\mathcal{F} (t^{-1} x, t^{-1} y) | \phi_\alpha(t) dt \right|^p \leq C \sum_{k=-\infty}^{-1} \left\| \int_{2^{k-1}}^{2^k} t^{-1} |\mathcal{F} (t^{-1} x, t^{-1} y) | \phi_\alpha(t) dt \right\|^p \leq C \sum_{k=-\infty}^{-1} \left| \mathcal{F} (\xi_k^{-1} x, \xi_k^{-1} y) \right|^p. \quad (2)$$

In the same way we could conclude that:

$$\left| \int_{1/2}^{1} t^{-1} |\mathcal{F} (t^{-1} x, t^{-1} y) | \phi_\alpha(t) dt \right|^p = \left| \mathcal{C} \sum_{k=-\infty}^{-1} \int_{2^{k-1}}^{2^k} t^{-1} |\mathcal{F} (t^{-1} x, t^{-1} y) | \phi_\alpha(t) dt \right|^p \leq \left| \mathcal{C} \sum_{k=-\infty}^{-1} \left| \mathcal{F} (\xi_k^{-1} x, \xi_k^{-1} y) \right|^p. \right.$$
From Proposition 2.3 and Proposition 2.6, we could obtain the following theorem:

**Theorem 2.7.** For $0 < p < \infty$, the Cesàro operator $C_{\alpha}$ is bounded on $L^p(\mathbb{R}^N_+, d\mu(x))$ spaces:

$$\|C_{\alpha}f\|_{L^p(\mathbb{R}^N_+, d\mu(x))} \leq C\|f\|_{L^p(\mathbb{R}^N_+, d\mu(x))}.$$ 

$C$ is a constant dependent on $\alpha$, $p$ and $d\mu$. Thus the Cesàro operator $C_{\alpha}$ is well defined on $L^p(\mathbb{R}^N_+, d\mu(x))$ spaces.

## 3 Cesàro operator on the spaces $H^p_{\lambda}(\mathbb{R}^2_+)$

In this section, we will discuss the Cesàro operator on the spaces $H^p_{\lambda}(\mathbb{R}^2_+)$. First, we will introduce some basic concept associated with the Dunkl setting and $H^p_{\lambda}(\mathbb{R}^2_+)$, then, with the Theorem 2.7 we obtained in previous section, we will prove the boundedness of Cesàro operator on the spaces $H^p_{\lambda}(\mathbb{R}^2_+)$.

### 3.1 The $\lambda$-translation and the $\lambda$-convolution

In [9], a theory about the Hardy spaces on the half-plane $\mathbb{R}^2_+ = \{(x, y) : x \in \mathbb{R}, y > 0\}$ associated with the Dunkl transform on the line $\mathbb{R}$ is developed. For $0 < p < \infty$, $L^p_{\lambda}(\mathbb{R})$ is the set of measurable functions satisfying $\|f\|_{L^p_{\lambda}} = \left(\int_{\mathbb{R}} |f(x)|^p |x|^{2\lambda} dx\right)^{1/p} < \infty$ with $c_{\lambda}^{-1} = 2^{\lambda+1/2}\Gamma(\lambda + 1/2)$, and $p = \infty$ is the usual $L^\infty(\mathbb{R})$ space. For $\lambda \geq 0$, The Dunkl operator on the line is (cf. [3],[4]):

$$D_{x} f(x) = f'(x) + \frac{\lambda}{x} [f(x) - f(-x)]$$

involving a reflection part. We assume $\lambda > 0$ in what follows. The Dunkl transfor for $f \in L^1_{\lambda}(\mathbb{R})$ is given by:

$$(\mathcal{F}_{\lambda} f)(\xi) = c_{\lambda} \int_{\mathbb{R}} f(x) E_{\lambda}(-ix\xi)|x|^{2\lambda} dx, \quad \xi \in \mathbb{R},$$

where $E_{\lambda}(-ix\xi)$ is the Dunkl kernel

$$E_{\lambda}(iz) = j_{\lambda - 1/2}(z) + \frac{iz}{2\lambda + 1} j_{\lambda + 1/2}(z), \quad z \in \mathbb{C}$$

and $j_{\alpha}(z)$ is the normalized Bessel function

$$j_{\alpha}(z) = 2^\alpha \Gamma(\alpha + 1) \frac{J_{\alpha}(z)}{z^\alpha} = \Gamma(\alpha + 1) \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n! \Gamma(n + \alpha + 1)}.$$ 

Since $j_{\lambda - 1/2}(z) = \cos z$, $j_{\lambda + 1/2}(z) = z^{-1} \sin z$. It follows that $E_0(iz) = e^{iz}$, and $\mathcal{F}_0$ agrees with the usual Fourier transform. $E_{\lambda}(iz)$ can also be represented by the following [12]

$$E_{\lambda}(iz) = c_{\lambda}' \int_{-1}^{1} e^{izt} (1 + t)(1 - t^2)^{\lambda - 1} dt, \quad c_{\lambda}' = \frac{\Gamma(\lambda + 1/2)}{\Gamma(\lambda)\Gamma(1/2)},$$

where $E_{\lambda}(ix\xi)$ satisfies

$$D_{x}[E_{\lambda}(ix\xi)] = i\xi E_{\lambda}(ix\xi), \quad \text{and} \quad E_{\lambda}(ix\xi)|_{x=0} = 1.$$ 

For $x, t, z \in \mathbb{R}$, we set $W_{\lambda}(x, t, z) = W_0(x, t, z)(1 - \sigma_{z,t,z} + \sigma_{z,x,t} + \sigma_{z,t,z})$, where

$$W_0(x, t, z) = \frac{c_{\lambda}''|xtz|^{|z| \leq 2\lambda} \chi_{\{(|x| - |t|) \leq |z| \leq (|x| - |t|)^2\}}(1)}{[(|x| - |t|)^2 - z^2]^2 (z^2 - (|x| - |t|)^2)^{\lambda - 1}},$$

$$c_{\lambda}'' = 2^{3/2-\lambda}(\Gamma(\lambda + 1/2))^2/|\sqrt{\Gamma(\lambda)}|, \quad \text{and} \quad \sigma_{z,t,z} = \frac{x^2 + t^2 - z^2}{2zt} \text{ for } x, t \in \mathbb{R} \setminus \{0\}, \text{ and } 0 \text{ otherwise.}$$

From [12], we have
Proposition 3.1. The Dunkl kernel $E_{\lambda}$ satisfies the following product formula:

$$E_{\lambda}(x\xi)E_{\lambda}(t\xi) = \int_{\mathbb{R}} E_{\lambda}(z\xi)dv_{x,t}(z), \quad x, t \in \mathbb{R} \text{ and } \xi \in \mathbb{C},$$

where $\nu_{x,t}$ is a signed measure given by $dv_{x,t}(z) = c_{\lambda} W_{\lambda}(x, t, z) |z|^{2\lambda}dz$ for $x, t \in \mathbb{R} \setminus \{0\}$, $dv_{x,t}(z) = d\delta_{x}(z)$ for $t = 0$, and $dv_{x,t}(z) = d\delta_{t}(z)$ for $x = 0$.

If $t \neq 0$, for an appropriate function $f$ on $\mathbb{R}$, the $\lambda$-translation is given by

$$(r_{\lambda}f)(x) = c_{\lambda} \int_{\mathbb{R}} f(z)W_{\lambda}(x, t, z)|z|^{2\lambda}dz;$$

and if $t = 0$, $(\tau_{0}f)(x) = f(x)$. If $(\tau_{\lambda}f)(x)$ is taken as a function of $t$ for a given $x$, we may set $(\tau_{\lambda}f)(0) = f(t)$ as a complement. An unusual fact is that $\tau_{\lambda}$ is not a positive operator in general. For $(x, t) \neq (0, 0)$, an equivalent form of $(\tau_{\lambda}f)(x)$ is given by [12],

$$(\tau_{\lambda}f)(x) = c_{\lambda} \int_{0}^{\infty} \left( f_{c}(x, t, \theta) + f_{o}(x, t, \theta) \frac{x + t}{(x, t, \theta)}(1 + \cos \theta) \sin^{2\lambda-1} \theta d\theta \right)$$

where $x, t \in \mathbb{R}$, $f_{c}(x) = (f(x) + f(-x))/2$, $f_{o}(x) = (f(x) - f(-x))/2$, $(x, t, \theta) = \sqrt{x^{2} + t^{2} + 2xt \cos \theta}$. For two appropriated function $f$ and $g$, their $\lambda$-convolution $f \ast_{\lambda} g$ is defined by

$$(f \ast_{\lambda} g)(x) = c_{\lambda} \int_{\mathbb{R}} (\tau_{\lambda}f)(-t)g(t)|t|^{2\lambda}dt.$$
3.2 Some facts about $H^p_\lambda(\mathbb{R}^2_+)$

When $u$ and $v$ satisfy the $\lambda$-Cauchy-Riemann equations:

\[
\begin{cases}
D_x u - \partial_y v = 0, \\
\partial_y u + D_x v = 0
\end{cases}
\]

(13)

the function $F(z) = F(x,y) = u(x,y) + iv(x,y)$ ($z = x + iy$) is said to be a $\lambda$-analytic function on the upper half plane $\mathbb{R}^2_+$. It is easy to see that $F = u + iv$ is $\lambda$-analytic on $\mathbb{R}^2_+$ if and only if

\[T_2 F \equiv 0, \quad \text{with} \quad T_z = \frac{1}{2}(D_x + i\partial_y).
\]

If $u$ and $v$ are $C^2$ functions and satisfy (13), then

\[(\triangle_\lambda u)(x, y) = 0, \quad \text{with} \quad \triangle_\lambda = D^2_x + \partial^2_y.
\]

(14)

A $C^2$ function $u(x, y)$ satisfying Formula (14) is said to be $\lambda$-harmonic. In [9], the Hardy space $H^p_\lambda(\mathbb{R}^2_+)$ is defined to be the set of $\lambda$-analytic functions on $\mathbb{R}^2_+$ satisfying

\[\|F\|_{H^p_\lambda(\mathbb{R}^2_+)} = \sup_{y > 0} \left\{ c_\lambda \int_\mathbb{R} |F(x + iy)|^p |x|^{2\lambda} dx \right\}^{1/p} < +\infty.
\]

When $p > \frac{2\lambda}{2\lambda + 1}$, some basic conclusions on $H^p_\lambda(\mathbb{R}^2_+)$ are obtained in [9], together with the associated real Hardy space $H^p(\mathbb{R})$ on the line $\mathbb{R}$, the collection of the real parts of boundary functions of $F \in H^p_\lambda(\mathbb{R}^2_+)$.

The following results about $\lambda$-Poisson integral and Conjugate $\lambda$-Poisson integral are obtained in [9]. For $1 \leq p \leq \infty$, the $\lambda$-Poisson integral of $f \in L^p_\lambda(\mathbb{R})$ in the Dunkl setting is given by:

\[(Pf)(x, y) = c_\lambda \int_\mathbb{R} f(t)(\tau_x P_y)(-t)|t|^{2\lambda} dt = c_\lambda (f * \lambda P_y)(x), \quad \text{for} \quad x \in \mathbb{R}, \ y \in (0, \infty),
\]

where $(\tau_x P_y)(-t)$ is the $\lambda$-Poisson kernel with $P_y(x) = m_\lambda y(y^2 + x^2)^{-\lambda-1}$, $m_\lambda = 2^{\lambda+1/2}\Gamma(\lambda + 1)/\sqrt{\pi}$. Similarly, the $\lambda$-Poisson integral for $du \in B_\lambda(\mathbb{R})$ can be given by $(P(mu))(x, y) = c_\lambda \int_\mathbb{R} (\tau_x P_y)(-t)|t|^{2\lambda} dt(t).

Proposition 3.4. [9] (i) $\lambda$-Poisson kernel $(\tau_x P_y)(-t)$ can be represented by

\[
(\tau_x P_y)(-t) = \frac{\Gamma(\lambda + 1/2)}{2\lambda^{1/2}\pi} \int_0^\pi \frac{y(1 + \text{sgn}(xt) \cos \theta)}{(y^2 + x^2 + t^2 - 2|xt| \cos \theta)^{\lambda+1}} \sin^{2\lambda-1} \theta d\theta.
\]

(15)

(ii) The Dunkl transform of the function $P_y$ is $(\mathcal{F}_\lambda P_y)(\xi) = e^{-\xi|\xi|}$, and $(\tau_x P_y)(-t) = c_\lambda \int_\mathbb{R} e^{-\xi|\xi|} E_\lambda(i\xi x) E_\lambda(-it\xi)|\xi|^{2\lambda} d\xi$.

Relating to $P_1(x)$, the conjugate $\lambda$-Poisson integral for $f \in L^p_\lambda(\mathbb{R})$ is given by:

\[(Qf)(x, y) = c_\lambda \int_\mathbb{R} f(t)(\tau_x Q_y)(-t)|t|^{2\lambda} dt = c_\lambda (f * \lambda Q_y)(x), \quad \text{for} \quad x \in \mathbb{R}, \ y \in (0, \infty),
\]

where $(\tau_x Q_y)(-t)$ is the conjugate $\lambda$-Poisson kernel with $Q_y(x) = m_\lambda x(y^2 + x^2)^{-\lambda-1}$.

Proposition 3.5. [9] (i) The conjugate $\lambda$-Poisson kernel $(\tau_x Q_y)(-t)$ can be represented by

\[
(\tau_x Q_y)(-t) = \frac{\Gamma(\lambda + 1/2)}{2\lambda^{1/2}\pi} \int_0^\pi \frac{(x - t)(1 + \text{sgn}(xt) \cos \theta)}{(y^2 + x^2 + t^2 - 2|xt| \cos \theta)^{\lambda+1}} \sin^{2\lambda-1} \theta d\theta.
\]

(ii) The Dunkl transform of $Q_y(x)$ is given by $(\mathcal{F}_\lambda Q_y)(\xi) = -i(\text{sgn} \xi) e^{-\xi|\xi|}$, for $\xi \neq 0$, and $(\tau_x Q_y)(-t) = -ic_\lambda \int_\mathbb{R} (\text{sgn} \xi)e^{-\xi|\xi|} E_\lambda(i\xi x) E_\lambda(-it\xi)|\xi|^{2\lambda} d\xi$.

Then we can define the associated maximal functions as

\[
(P^+_\lambda f)(x) = \sup_{|s-x|<y} \|(Pf)(s, y)\|, \quad (P^{-}_\lambda f)(x) = \sup_{y>0} \|(Pf)(x, y)\|
\]

\[
(Q^+_\lambda f)(x) = \sup_{|s-x|<y} \|(Qf)(s, y)\|, \quad (Q^{-}_\lambda f)(x) = \sup_{y>0} \|(Qf)(x, y)\|
\]
Cesàro operator on Hardy spaces associated with the Dunkl setting \((\frac{2\lambda}{2\lambda+1} < p < \infty)\)

**Proposition 3.6.** [9] (i) For \(f \in L^p_{u}(\mathbb{R}), 1 \leq p < \infty\), \(u(x, y) = (Pf)(x, y)\) and \(v(x, y) = (Qf)(x, y)\) on \(\mathbb{R}^3_+\) satisfy the \(\lambda\)-Cauchy-Riemann equations (13), and are both \(\lambda\)-harmonic on \(\mathbb{R}^3_+\).

(ii) (semi-group property) If \(f \in L^p_{u}(\mathbb{R}), 1 \leq p \leq \infty\), and \(y_0 > 0\), then \((Pf)(x, y) = P[(Pf)(\cdot, y_0)](x, y)\), for \(y > 0\).

(iii) If \(1 < p < \infty\), then there exists some constant \(A'_p\) for any \(f \in L^p_{u}(\mathbb{R}), \| (Q_{\gamma}^*) f \|_{L^p_{\lambda}} \leq A'_p \| f \|_{L^p_{\lambda}}\).

(iv) \(P_{\lambda}^*\) and \(P^*\) are both \((p, p)\) type for \(1 < p \leq \infty\) and weak-(1, 1) type.

(v) If \(dp \in \mathcal{B}_{\lambda}(\mathbb{R}), \| (P(dp)) \|_{L^p_{\lambda}} \leq \| dp \|_{\mathcal{B}_{\lambda}}\) as \(y \to 0^+\), \([P(dp)](\cdot, y)\) converges \(\ast\)-weakly to \(dp\).

(vi) If \(f \in L^p_{u}(\mathbb{R}), 1 \leq p \leq 2\), and \([\mathcal{F}_{\lambda}(P(f(\cdot, y)))](\xi) = e^{-\frac{\text{v}}{2}|\xi|}(\mathcal{F}_{\lambda}f)(\xi)\), and

\[
(Pf)(x, y) = c_{\lambda} \int_{\mathbb{R}} e^{-\frac{y}{2}|\xi|}(\mathcal{F}_{\lambda}f)(\xi)E_{\lambda}(ix\xi)|\xi|^{2\lambda} d\xi, \quad (x, y) \in \mathbb{R}^3_+,
\]

therefore, Formula (16) is true when we replace \(f \in L^p_{u}(\mathbb{R})\) with \(dp \in \mathcal{B}_{\lambda}(\mathbb{R})\).

(vii) If \(1 \leq p < \infty\) and \(F = u + iv \in H^p_{\lambda}(\mathbb{R}^3_+)\), then \(F\) is the \(\lambda\)-Poisson integral of its boundary values \(F(x), x \in L^p_{u}(\mathbb{R})\).

**Lemma 3.7.** [9] Let \(F \in H^p_{\lambda}(\mathbb{R}^3_+), \frac{2\lambda}{2\lambda+1} \leq p \leq 1\). Then there exists a function \(\phi\) on \(\mathbb{R}\), satisfying

(i) For all \((x, y) \in \mathbb{R}^3_+\),

\[
F(x, y) = c_{\lambda} \int_{\mathbb{R}} e^{-\frac{y}{2}|\xi|}(\mathcal{F}_{\lambda}f)(\xi)E_{\lambda}(ix\xi)|\xi|^{2\lambda} d\xi;
\]

(ii) \(\phi\) is continuous on \(\mathbb{R}\), and \(\phi(\xi) = 0\) for \(\xi \in (-\infty, 0]\);

(iii) For \(y > 0\), the function \(\xi \mapsto e^{-\frac{y}{2}|\xi|}\phi(\xi)\) is bounded on \(\mathbb{R}\) satisfying \(\xi \mapsto e^{-\frac{y}{2}|\xi|}\phi(\xi) \in L^p_{\lambda}(\mathbb{R})\);

(iv) Obviously, the function \(\phi\) satisfying the following inequality:

\[
\sup_{y > 0} c_{\lambda} \int_{\mathbb{R}} e^{-\frac{y}{2}|\xi|}(\mathcal{F}_{\lambda}f)(\xi)E_{\lambda}(ix\xi)|\xi|^{2\lambda} d\xi \leq \frac{1}{|x|^{2\lambda}} dx < \infty.
\]

Thus we could deduce that for \(\frac{2\lambda}{2\lambda+1} \leq p \leq 1, F \in H^p_{\lambda}(\mathbb{R}^3_+)\) if and only if there exits a \(\phi\) satisfying (i), (ii), (iii) , (iv).

**Proof.** Notice that for the operator \(T_{\tau} = \frac{1}{2}(D_x + i\partial_y)\), with \(\phi(\xi)\) satisfying (i), (ii), (iii) , (iv), we have

\[
T_{\tau} c_{\lambda} \int_{\mathbb{R}} e^{-\frac{y}{2}|\xi|}\phi(\xi)E_{\lambda}(ix\xi)|\xi|^{2\lambda} d\xi = c_{\lambda} \int_{\mathbb{R}} \frac{1}{2}(-i\xi + i\xi^2)e^{-\frac{y}{2}|\xi|}(\mathcal{F}_{\lambda}f)(\xi)E_{\lambda}(ix\xi)|\xi|^{2\lambda} d\xi = 0.
\]

Thus the function \(c_{\lambda} \int_{\mathbb{R}} e^{-\frac{y}{2}|\xi|}\phi(\xi)E_{\lambda}(ix\xi)|\xi|^{2\lambda} d\xi\) is \(\lambda\)-analytic on the upper half plane \(\mathbb{R}^3_+\).

**Lemma 3.8.** For \(y > 0, x, z \in \mathbb{R}\), we have the following estimations for the \(\lambda\)-Poisson kernel \((\tau_{x} P_y)(-z)\) and the conjugate \(\lambda\)-Poisson kernel \((\tau_{x} Q_y)(-z)\)

\[
|\partial_{x}(\tau_{x} P_y)(-z)| \lesssim \left(\frac{1}{y^2 + (|x| - |z|)^2}\right)^{\lambda+1}, \quad |\partial_{y}(\tau_{x} P_y)(-z)| \lesssim \left(\frac{1}{y^2 + (|x| - |z|)^2}\right)^{\lambda+1},
\]

\[
|\partial_{\bar{y}}(\tau_{x} P_y)(-z)| \lesssim \left(\frac{1}{y^2 + (|x| - |z|)^2}\right)^{\lambda+\frac{1}{2}}, \quad |\partial_{\bar{x}}(\tau_{x} P_y)(-z)| \lesssim \left(\frac{1}{y^2 + (|x| - |z|)^2}\right)^{\lambda+\frac{1}{2}},
\]

\[
|\partial_{x}(\tau_{x} Q_y)(-z)| \lesssim \left(\frac{1}{y^2 + (|x| - |z|)^2}\right)^{\lambda+1}, \quad |\partial_{y}(\tau_{x} Q_y)(-z)| \lesssim \left(\frac{1}{y^2 + (|x| - |z|)^2}\right)^{\lambda+1},
\]

\[
|\partial_{\bar{y}}(\tau_{x} Q_y)(-z)| \lesssim \left(\frac{1}{y^2 + (|x| - |z|)^2}\right)^{\lambda+\frac{1}{2}}, \quad |\partial_{\bar{x}}(\tau_{x} Q_y)(-z)| \lesssim \left(\frac{1}{y^2 + (|x| - |z|)^2}\right)^{\lambda+\frac{1}{2}}.
\]
Proof. For the case when $z = 0$ (or $x = 0$), we have $(\tau_x P^s_0)(0) = P^s_0(x) = m_\lambda y(y^2 + x^2)^{-\lambda - 1}$ and $(\tau_x Q_0)(0) = Q_0(x) = m_\lambda x(y^2 + x^2)^{-\lambda - 1}$, where $m_\lambda = 2^{\lambda + 1/2}\Gamma(\lambda + 1)/\sqrt{\pi}$. We could obtain the Formulas (18, 19, 20, 21) directly.

Then, we will only consider the case when $xz > 0$ (The case $xz < 0$ can be discussed in the same way). By Proposition 3.4, we could deduce that

$$
\int_0^\pi \left| \frac{\partial}{\partial x} \frac{y(1 + \cos \theta) \sin^{2\lambda - 1} \theta}{(y^2 + x^2 + z^2 - 2xz \cos \theta)^{\lambda + 1}} \right| d\theta = \int_0^\pi \left| \frac{y(\lambda + 1)(2x - 2z \cos \theta)(1 + \cos \theta) \sin^{2\lambda - 1} \theta}{(y^2 + x^2 + z^2 - 2xz \cos \theta)^{\lambda + 2}} \right| d\theta 
\leq C \left( \frac{1}{y^2 + (|x| - |z|)^2} \right)^{\lambda + 1}.
$$

Thus

$$
|\partial_x (\tau_x P^s_0)(-z)| \lesssim \left( \frac{1}{y^2 + (|x| - |z|)^2} \right)^{\lambda + 1}.
$$

By Proposition 3.4, we also deduce that

$$
\int_0^\pi \left| \frac{\partial}{\partial y} \frac{y(1 + \cos \theta) \sin^{2\lambda - 1} \theta}{(y^2 + x^2 + z^2 - 2xz \cos \theta)^{\lambda + 1}} \right| d\theta 
\leq \int_0^\pi \left| \left( \frac{y^2(1 + \cos \theta) \sin^{2\lambda - 1} \theta}{(y^2 + x^2 + z^2 - 2xz \cos \theta)^{\lambda + 2}} \right) \right| d\theta + \int_0^\pi \left| \frac{(1 + \cos \theta) \sin^{2\lambda - 1} \theta}{(y^2 + x^2 + z^2 - 2xz \cos \theta)^{\lambda + 1}} \right| d\theta 
\leq C \left( \frac{1}{y^2 + (|x| - |z|)^2} \right)^{\lambda + 1}.
$$

Thus

$$
|\partial_y (\tau_x P^s_0)(-z)| \lesssim \left( \frac{1}{y^2 + (|x| - |z|)^2} \right)^{\lambda + 1}.
$$

In the same way as Formulas (22, 23), by Proposition 3.4 and Proposition 3.5, we could obtain Formulas (18, 19, 20, 21) for the case $xz > 0$. This proves the Lemma.

By the Lemma 3.8, we could obtain the following Lemma:

**Lemma 3.9.** For $y > 0$, $t > 0$, $x, u \in \mathbb{R}$, we have the following estimations for the $\lambda$-Poisson kernel $((\tau_{t-1} P_{1-1})y)(-u)$ and the conjugate $\lambda$-Poisson kernel $((\tau_{t-1} Q_{1-1})y)(-u)$

$$
|\partial_x (\tau_{t-1} P_{1-1})y)(-u)| + |\partial_y (\tau_{t-1} P_{1-1})y)(-u)| \lesssim t^{-1} \left( \frac{1}{(t^{-1}y)^2 + (|t^{-1}x| - |u|)^2} \right)^{\lambda + 1},
$$

$$
|\partial_y (\tau_{t-1} P_{1-1})y)(-u)| + |\partial_x (\tau_{t-1} P_{1-1})y)(-u)| \lesssim t^{-2} \left( \frac{1}{(t^{-1}y)^2 + (|t^{-1}x| - |u|)^2} \right)^{\lambda + \frac{1}{2}},
$$

$$
|\partial_x (\tau_{t-1} Q_{1-1})y)(-u)| + |\partial_y (\tau_{t-1} Q_{1-1})y)(-u)| \lesssim t^{-1} \left( \frac{1}{(t^{-1}y)^2 + (|t^{-1}x| - |u|)^2} \right)^{\lambda + 1},
$$

$$
|\partial_y (\tau_{t-1} Q_{1-1})y)(-u)| + |\partial_x (\tau_{t-1} Q_{1-1})y)(-u)| \lesssim t^{-2} \left( \frac{1}{(t^{-1}y)^2 + (|t^{-1}x| - |u|)^2} \right)^{\lambda + \frac{1}{2}}.
$$

### 3.3 The boundedness of Cesàro operator on $H_p^\alpha(\mathbb{R}_+^2)$

**Proposition 3.10.** Cesàro operator $C_\alpha$ defined on $H_p^\alpha(\mathbb{R}_+^2)$ is a bounded linear operator on $H_p^\alpha(\mathbb{R}_+^2)$ for $1 \leq p < \infty$:

$$
\|C_\alpha F\|_{H_p^\alpha(\mathbb{R}_+^2)} \leq C\|F\|_{H_p^\alpha(\mathbb{R}_+^2)},
$$

where $C$ is dependent on $\alpha$, $p$ and $\lambda$. 
Thus we could have
\[
C_\alpha F(x, y) = \int_0^1 f \ast \lambda P_{\alpha/2}(x/t) t^{-1} \phi_\alpha(t) dt + i \int_0^1 f \ast \lambda Q_{\alpha/2}(x/t) t^{-1} \phi_\alpha(t) dt
\]  
(24)

Notice that we could deduce the following Formula (25) by Proposition 3.2 (i), (ii) and Holder inequality:
\[
\int_0^1 \left( \int_{\mathbb{R}} |f(u)\partial_y \tau_{-1} P_{\alpha/2}(t^{-1}x)| \lambda^2|u\lambda| t^{-1} \phi_\alpha(t) dt \right)
\leq \int_0^1 \left( \int_{\mathbb{R}} |f(u)|^p |u\lambda|^2 \lambda^2|u\lambda| t^{-1} \phi_\alpha(t) dt \right)
\]  
(25)

By Lemma 3.9 and Formula (25), we could deduce the following inequality:
\[
\int_0^1 \left( \int_{\mathbb{R}} |f(u)\partial_y \tau_{-1} P_{\alpha/2}(t^{-1}x)| \lambda^2|u\lambda| t^{-1} \phi_\alpha(t) dt \right)
\leq \int_0^1 \left( \int_{\mathbb{R}} |f(u)|^p |u\lambda|^2 \lambda^2|u\lambda| t^{-1} \phi_\alpha(t) dt \right)
\]  
(26)

Let \( u = t^{-1}z \), then we could deduce the following inequality by Formula (26)
\[
\int_0^1 \left( \int_{\mathbb{R}} |f(u)\partial_y \tau_{-1} P_{\alpha/2}(t^{-1}x)| \lambda^2|u\lambda| t^{-1} \phi_\alpha(t) dt \right)
\leq \int_0^1 \left( \int_{\mathbb{R}} |f(u)|^p |u\lambda|^2 \lambda^2|u\lambda| t^{-1} \phi_\alpha(t) dt \right)
\]  
(27)

Thus similar to Formula (27), by Lemma 3.9, we could obtain
\[
\int_0^1 \left( \int_{\mathbb{R}} |f(u)\partial_x \tau_{-1} P_{\alpha/2}(t^{-1}x)| \lambda^2|u\lambda| t^{-1} \phi_\alpha(t) dt \right)
\leq \int_0^1 \left( \int_{\mathbb{R}} |f(u)|^p |u\lambda|^2 \lambda^2|u\lambda| t^{-1} \phi_\alpha(t) dt \right)
\]  
(28)

\[
\int_0^1 \left( \int_{\mathbb{R}} |f(u)(\partial_y \partial_y + \partial_x \partial_x) \tau_{-1} P_{\alpha/2}(t^{-1}x)| \lambda^2|u\lambda| t^{-1} \phi_\alpha(t) dt \right)
\leq \int_0^1 \left( \int_{\mathbb{R}} |f(u)|^p |u\lambda|^2 \lambda^2|u\lambda| t^{-1} \phi_\alpha(t) dt \right)
\]  
(29)
In the same way, we could also deduce the following inequalities (30, 31, 32):

\[
\int_0^1 \left( \int_{\mathbb{R}} \left| f(u)(\partial_x) \tau_{-u} Q_{t^{-1}y}(t^{-1}x) \right| |u|^{2^\gamma} du \right) t^{-1} \phi_\alpha(t) dt < \infty, \quad (30)
\]

\[
\int_0^1 \left( \int_{\mathbb{R}} \left| f(u)(\partial_y) \tau_{-u} Q_{t^{-1}y}(t^{-1}x) \right| |u|^{2^\gamma} du \right) t^{-1} \phi_\alpha(t) dt < \infty, \quad (31)
\]

and

\[
\int_0^1 \left( \int_{\mathbb{R}} \left| f(u)(\partial_y \partial_y + \partial_x \partial_x) \tau_{-u} Q_{t^{-1}y}(t^{-1}x) \right| |u|^{2^\gamma} du \right) t^{-1} \phi_\alpha(t) dt < \infty. \quad (32)
\]

From inequalities (27, 28, 29, 30, 31, 32), we could take \( \triangle \), \( \partial_y \) and \( D_x \) under integration signs of Formula (24). Notice that we have the following three formulas hold:

\[
(\tau_x P_y)(-u) = c_\lambda \int_{\mathbb{R}} e^{-y|\xi|} E_\lambda(ix \xi) E_\lambda(-iu \xi)|\xi|^{2^\gamma} d\xi,
\]

\[
(\tau_x Q_y)(-u) = -ic_\lambda \int_{\mathbb{R}} (\text{sgn} \, \xi) e^{-y|\xi|} E_\lambda(ix \xi) E_\lambda(-iu \xi)|\xi|^{2^\gamma} d\xi,
\]

\[
D_x[E_\lambda(ix \xi)] = i \xi E_\lambda(ix \xi), \quad \text{and} \quad E_\lambda(ix \xi)|_{x=0} = 1.
\]

Then we take \( \triangle \), \( \partial_y \) and \( D_x \) under integration signs of Formula (24), and we could deduce that the following holds:

\[
T_x C_\alpha F(x,y) = T_x \int_0^1 f * \lambda P_{y/x}(x/t) t^{-1} \phi_\alpha(t) dt + iT_x \int_0^1 f * \lambda Q_{y/x}(x/t) t^{-1} \phi_\alpha(t) dt
\]

\[
= T_x \int_0^1 \left( \int_{\mathbb{R}} f(u) \tau_{-u} P_{t^{-1}y}(t^{-1}x) \right| |u|^{2^\gamma} du \right) t^{-1} \phi_\alpha(t) dt + iT_x \int_0^1 \left( \int_{\mathbb{R}} f(u) \tau_{-u} Q_{t^{-1}y}(t^{-1}x) \right| |u|^{2^\gamma} du \right) t^{-1} \phi_\alpha(t) dt
\]

\[
= c_\lambda \int_0^1 \int_{\mathbb{R}} f(u) \left( T_x \int_{\mathbb{R}} e^{-y|\xi|} E_\lambda(it^{-1}x \xi) E_\lambda(-iu \xi)|\xi|^{2^\gamma} d\xi \right) \left| |u|^{2^\gamma} du \right| t^{-1} \phi_\alpha(t) dt
\]

\[
- c_\lambda \int_0^1 \int_{\mathbb{R}} f(u) \left( T_x \int_{\mathbb{R}} (\text{sgn} \, \xi) e^{-y|\xi|} E_\lambda(it^{-1}x \xi) E_\lambda(-iu \xi)|\xi|^{2^\gamma} d\xi \right) \left| |u|^{2^\gamma} du \right| t^{-1} \phi_\alpha(t) dt
\]

\[
= 2c_\lambda \int_0^1 \int_{\mathbb{R}} f(u) \left( T_x \int_{\mathbb{R}} e^{-y|\xi|} E_\lambda(it^{-1}x \xi) E_\lambda(-iu \xi)|\xi|^{2^\gamma} d\xi \right) \left| |u|^{2^\gamma} du \right| t^{-1} \phi_\alpha(t) dt
\]

\[
= 0.
\]

Thus \( (C_\alpha F)(x,y) \) is a \( \lambda \)-analytic function, together with Theorem 2.7, we could deduce that Cesàro operator \( C_\alpha \) defined on \( L^p_\lambda(\mathbb{R}^2_+) \) is a bounded linear operator on \( L^p_\lambda(\mathbb{R}^2_+) \) for \( 1 \leq p < \infty \):

\[
\|C_\alpha F\|_{L^p_\lambda(\mathbb{R}^2_+)} \leq C \|F\|_{L^p_\lambda(\mathbb{R}^2_+)},
\]

where \( C \) is a constant dependent on \( \alpha \), \( p \), and \( \lambda \).

**Proposition 3.11.** Cesàro operator \( C_\alpha \) defined on \( L^p_\lambda(\mathbb{R}^2_+) \) is a bounded linear operator on \( L^p_\lambda(\mathbb{R}^2_+) \) for \( \frac{2^\gamma}{2^\lambda+1} < p < 1 \):

\[
\|C_\alpha F\|_{L^p_\lambda(\mathbb{R}^2_+)} \leq C \|F\|_{L^p_\lambda(\mathbb{R}^2_+)},
\]

\( C \) is dependent on \( \alpha \), \( p \), and \( \lambda \).

**Proof.** From Lemma 3.7, we could deduce that for \( F(x,y) \in H^p_\lambda(\mathbb{R}^2_+) \left( \frac{2^\gamma}{2^\lambda+1} < p \leq 1 \right) \), if and only if there exists a function \( \phi(\xi) \) satisfying (i), (ii), (iii), (iv) in Lemma 3.7. We define the operator \( B_\alpha \) as following:

\[
B_\alpha \phi(\xi) = \int_0^1 \phi(t \xi) \phi_\alpha(t)|t|^{2^\gamma} dt.
\]
Let $B(x, y)$ to denote the function

$$B(x, y) = c_\lambda \int_{0}^{\infty} e^{-y|\xi|} \phi(x, y) E_\lambda(x\xi)|\xi|^{2\lambda} d\xi.$$ 

Notice that for $0 < t < 1$, $|e^{-y|\xi|}\phi(\xi)| = |e^{-y|\xi|/2}\phi(t\xi)| < |e^{-y|\xi|/2}(e^{-y|\xi|/2}(e^{-y|\xi|/2}(e^{-y|\xi|/2}(\phi(t\xi)))).$ By Lemma 3.7, the function $e^{-y|\xi|/2}(\phi(t\xi))$ is bounded on $\mathbb{R}$, thus $e^{-y|\xi|}\phi(t\xi) \in L^1_\lambda(\mathbb{R}).$ Then by Fubini theorem, we could write $B(x, y)$ as:

$$B(x, y) = c_\lambda \int_{0}^{\infty} e^{-y|\xi|} \phi(x, y) E_\lambda(x\xi)|\xi|^2 d\xi.$$ 

Notice that

$$\frac{1}{t^{2\lambda+1}} F(t^{-1}x, t^{-1}y) = c_\lambda \int_{0}^{\infty} e^{-y|\xi|} \phi(t\xi) E_\lambda(x\xi)|\xi|^{2\lambda} d\xi,$$ 

where $B(x, y) \in H^p_\lambda(\mathbb{R}^n_+)$ for $\frac{2\lambda}{2\lambda + 1} < p \leq 1$, and that $F(x, y)$ and $\phi(\xi)$ satisfy (i), (ii), (iii) , (iv) in Lemma 3.7: $F(x, y) = c_\lambda \int_{0}^{\infty} e^{-y|\xi|} \phi(x, y) E_\lambda(x\xi)|\xi|^{2\lambda} d\xi.$ Thus by formulas (33, 34), we could have

$$B(x, y) = \int_{0}^{1} F(t^{-1}x, t^{-1}y) t^{-1} \phi(x, y) dt = (C_\alpha F)(x, y).$$

By Theorem 2.7, we could deduce that

$$\|C_\alpha F\|_{L^p(\mathbb{R}^n_+|x|^{2\lambda} dx)} \leq C\|F\|_{L^p(\mathbb{R}^n_+|x|^{2\lambda} dx)}.$$

It is easy to check that the function $C_\alpha \phi(\xi)$ satisfies (i), (ii), (iii) , (iv) in Lemma 3.7, thus we could deduce that $B(x, y) = (C_\alpha F)(x, y) \in H^p_\lambda(\mathbb{R}^n_+).$ Together with Theorem 2.7, we could deduce the Proposition 3.11 that Cesàro operator $C_\alpha$ defined on $H^p_\lambda(\mathbb{R}^n_+)$ is a bounded linear operator on $H^p_\lambda(\mathbb{R}^n_+)$ for $\frac{2\lambda}{2\lambda + 1} < p \leq 1.$

By Proposition 3.10 and Proposition 3.11, we could obtain the following theorem:

**Theorem 3.12.** Cesàro operator $C_\alpha$ defined on $H^p_\lambda(\mathbb{R}^n_+)$ is a bounded linear operator on $H^p_\lambda(\mathbb{R}^n_+)$ for $\frac{2\lambda}{2\lambda + 1} < p < \infty: \|C_\alpha f\|_{H^p_\lambda(\mathbb{R}^n_+)} \leq C\|f\|_{H^p_\lambda(\mathbb{R}^n_+)}.$

where $C$ is a constant dependent on $\alpha, p$ and $\lambda.$

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