PROFILE OF SOLUTIONS FOR NONLOCAL EQUATIONS WITH CRITICAL AND SUPERCRITICAL NONLINEARITIES

MOUSOMI BHAKTA, DEBANGANA MUKHERJEE AND SANJIBAN SANTRA

Abstract. We study the fractional Laplacian problem

\[ \begin{align*}
(-\Delta)^s u &= u^p - \varepsilon u^q \quad \text{in } \Omega, \\
u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \\
u &\in H^s(\Omega) \cap L^{q+1}(\Omega);
\end{align*} \]

where \( s \in (0,1) \), \( q > p \geq \frac{N+2}{N-2s} \) and \( \varepsilon > 0 \) is a parameter. \( \Omega \subset \mathbb{R}^N \) is a bounded star-shaped domain with smooth boundary and \( N > 2s \). We establish existence of a variational positive solution \( u_\varepsilon \) and characterise the asymptotic behaviour of \( u_\varepsilon \) as \( \varepsilon \to 0 \). When \( p = \frac{\frac{N+2}{N-2s}}{N-2s} \), we describe how the solution \( u_\varepsilon \) blows up at an interior point of \( \Omega \). Furthermore, we prove the local uniqueness of solution of the above problem when \( \Omega \) is a convex symmetric domain of \( \mathbb{R}^N \) with \( N > 4s \) and \( p = \frac{\frac{N+2}{N-2s}}{N-2s} \).

1. Introduction

There has been considerable interest in understanding the asymptotic behavior of positive solutions of the elliptic problem

\[ \begin{align*}
\varepsilon^{2s}(-\Delta)^s u &= f(u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{in } \Omega, \\
u &\in H^s(\Omega) \cap L^{q+1}(\Omega);
\end{align*} \]

where \( \varepsilon > 0 \) is a parameter, \( s \in (0,1) \) and \( f \) is having superlinear nonlinearity with \( f(0) = 0 \). \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^N \). The existence and asymptotic behavior of solutions to (1.1) depend crucially on the behavior of \( f \) near 0. It is easy to check that problem (1.1) may not have any nontrivial solutions for small \( \varepsilon > 0 \) if \( f'(0) > 0 \). The case of \( f'(0) < 0 \) has been studied by many authors. To mention a few of them in the local case, we refer the papers [16], [25] and the references therein. In the nonlocal case, not much is known. Multi-peak solutions of a fractional Schrödinger equation in the whole of \( \mathbb{R}^N \) was considered in [13]. In [14], Dávila, et al constructed a family of solutions which have the properties that, when \( \varepsilon \to 0 \), those solutions concentrate at an interior point of the domain in the form of a scaling ground state in entire space. Bubble solutions for the fractional problems involving the almost critical or almost supercritical powers were considered in Dávila et al et al [12].

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In this paper, we consider the problem in the zero mass case i.e., when $f(0) = 0$ and $f'(0) = 0$. The problem (1.1) can be viewed as borderline problems. When $s = 1$, Berestycki and Lions in [4] proved the existence of ground state solutions if $f(u)$ behaves like $|u|^p$ for large $u$ and $|u|^q$ for small $u$ where $p$ and $q$ are respectively supercritical and subcritical.

In this paper, we consider the following family of problems:

\begin{equation}
\begin{aligned}
(-\Delta)^s u &= u^p - \varepsilon u^q \quad \text{in } \Omega, \\
u &> 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \\
u &\in H^s(\Omega) \cap L^{q+1}(\Omega),
\end{aligned}
\end{equation}

where $s \in (0, 1)$ is fixed, $(-\Delta)^s$ denotes the fractional Laplace operator defined, up to a normalisation factor, as

\begin{equation}
(-\Delta)^s u(x) = \frac{1}{2} \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} dy, \quad x \in \mathbb{R}^N.
\end{equation}

In (1.2), $q > p > 2^* - 1 = \frac{N+2s}{N-2s}$, $\varepsilon > 0$ is a parameter, $\Omega \subseteq \mathbb{R}^N$ is a bounded star-shaped domain with smooth boundary and $N > 2s$. Note under a suitable change of variable (1.2) can be transformed in the form of (1.1).

We denote by $H^s(\Omega)$, the usual fractional Sobolev space endowed with the so-called Gagliardo norm

\begin{equation}
\|g\|_{H^s(\Omega)} = \|g\|_{L^2(\Omega)} + \left( \int_{\Omega \times \Omega} \frac{|g(x) - g(y)|^2}{|x-y|^{N+2s}} dxdy \right)^{1/2}.
\end{equation}

For further details on the fractional Sobolev spaces we refer to [20] and the references therein. Note that, in problem (1.2) the Dirichlet datum is given in $\mathbb{R}^N \setminus \Omega$ and not simply on $\partial \Omega$ and therefore we need to introduce a new functional space $X_0$, which, in our opinion, is the suitable space to work with.

\begin{equation}
X_0(\Omega) := \{ v \in H^s(\mathbb{R}^N) : v = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega \}.
\end{equation}

By [31] Lemma 6 and 7, it follows that

\begin{equation}
\|v\|_{X_0} = \left( \int_Q \frac{|v(x) - v(y)|^2}{|x-y|^{N+2s}} dxdy \right)^{1/2},
\end{equation}

where $Q = \mathbb{R}^{2N} \setminus (\Omega^c \times \Omega^c)$ is a norm on $X_0$ and $(X_0, \|\cdot\|_{X_0})$ is a Hilbert space, with the inner product

\begin{equation}
<u, v>_{X_0} = \int_Q \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} dxdy.
\end{equation}

We observe that, norms in (1.4) and (1.6) are not same in general, since $\Omega \times \Omega$ is strictly contained in $Q$ (see [20] 31) but (1.4) and (1.6) are equivalent in some cases, such as $s > 1/2$. Clearly, the integral in (1.6) can be extended to whole of $\mathbb{R}^{2N}$ as $v = 0$ in $\mathbb{R}^N \setminus \Omega$. It follows from [31] Lemma 8 that the embedding $X_0 \hookrightarrow L^r(\mathbb{R}^N)$ is compact, for any $r \in [1, 2^*)$ and from [30] Lemma 9] that $X_0 \hookrightarrow L^2(\mathbb{R}^N)$ is continuous.
**Definition 1.1.** We say that \( u \in X_0 \cap L^{q+1}(\Omega) \) is a weak solution of Eq. (1.2), if 
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy = \int_{\Omega} u^p \varphi \, dx - \varepsilon \int_{\Omega} u^q \varphi \, dx.
\]

In recent years, a great deal of attention has been devoted to equations of elliptic/parabolic type with fractional and non-local operators because these kind of equations play important role in the real world and many perfect techniques which have been developed by well-known mathematicians during the past decades can not be directly applied to the fractional case. These equations arise from models in physics, engineering (see [24]), optimisation and finance (see [11]), obstacle problem (see [32]), conformal geometry and minimal surface (see [7]) and many more, see for instance, [2, 3, 35] and the references therein.

Nonlinear nonlocal problems of the form \((-\Delta)^s u = f(u)\) were studied by many authors where \( f : \mathbb{R}^N \to \mathbb{R} \) is a certain function. Since it is almost impossible to describe all the works involving them, we explain only few of them, which are related to our problem. In [30], Servadei and Valdinoci studied the Brezis-Nirenberg problem in the nonlocal case. More precisely, they considered the nonlinearity of the form \( \lambda u + u^{2^* - 1} \), with \( \lambda > 0 \). On the other hand, in [31] the same authors studied mountain-pass solutions for the equation with general integro-differential operator and with the nonlinearities of subcritical growth. In [5], first and second authors of this paper studied the equation in whole of \( \mathbb{R}^N \) with nonlinearities involving critical and supercritical growth. They established decay estimate of solution and the gradient of the solution at infinity and using that they prove nonexistence result via Pohozaev identity.

In the local case, \( s = 1 \), Merle and Pelletier [23] considered the equation (1.2). They proved that for \( N \geq 3 \), problem (1.2) possesses a family of solutions concentrating at a point \( \xi_0 \), which is a critical point of the Robin function \( R \). In this paper we extend the result to the fractional Laplacian case.

For the supercritical case \((p > 2^* - 1)\), define,
\[
F(u, \Omega) = \frac{1}{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy + \frac{1}{q + 1} \int_{\Omega} |u|^{q+1} \, dx,
\]
where \( l = \frac{2s(q+1)-N(p-1)}{2s(p+1)-N(p-1)} \), \( u \in X_0(\Omega) \cap L^{q+1}(\Omega) \) and
\[
K := \inf \left\{ F(u, \mathbb{R}^N) : u \in D^{s,2}(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N), \int_{\mathbb{R}^N} |u|^{p+1} = 1 \right\},
\]
where \( D^{s,2}(\mathbb{R}^N) \) is the closure of \( C_0^\infty(\mathbb{R}^N) \) w.r.t. to the norm \( \left( \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \right)^{1/2} \).

For the critical case \((p = 2^* - 1)\), we consider the usual functional
\[
S(u) = \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \left( \int_{\Omega} |u|^{p+1} \, dx \right)^{\frac{q+1}{2}},
\]
where \( u \in X_0(\Omega) \).

Define, the Sobolev constant

\[
S := \inf_{u \in D^{s,2}(\mathbb{R}^N) \setminus \{0\}} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy \left( \int_{\mathbb{R}^N} |u|^2^* \, dx \right)^{\frac{1}{2}}
\]

or, equivalently,

\[
S = \inf \left\{ S(v) : v \in D^{s,2}(\mathbb{R}^N), \int_{\mathbb{R}^N} |v|^{2^*} \, dx = 1 \right\}.
\]

It is well known by \([22]\) that \( S \) is achieved by

\[
U(x) = c_{N,s} \left( 1 + |x|^2 \right)^{-\left( \frac{N+2s}{N-2s} \right)},
\]

where

\[
c_{N,s} = 2^{\frac{N-2s}{2}} \left( \frac{\Gamma \left( \frac{N+2s}{2} \right)}{\Gamma \left( \frac{N-2s}{2} \right)} \right)^{\frac{N-2s}{4}}.
\]

By \([8]\) and \([21]\), a direct computation implies that for all \( \varepsilon > 0 \) and for any \( a \in \mathbb{R}^N \), \( U \) is the unique solution satisfying

\[
U_{\varepsilon,a}(x) = \varepsilon^{-\frac{N-2s}{2}} U \left( \frac{x - a}{\varepsilon} \right)
\]

and verifies the following equation

\[
\begin{cases}
(-\Delta)_x^s U = U^{2^*-1} & \text{in } \mathbb{R}^N, \\
U > 0 & \text{in } \mathbb{R}^N, \\
U \in D^{s,2}(\mathbb{R}^N).
\end{cases}
\]

Define the Green’s function \( G = G(x,y) \) of the operator \((-\Delta)_x^s\) in \( \Omega \) for \( x, y \in \Omega \) as

\[
\begin{cases}
(-\Delta)_x^s G(x,y) = \delta_y & \text{in } \Omega, \\
G(x,y) = 0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}
\]

It is convenient to introduce the regular part of \( G \), which is often denoted by \( H \), defined by

\[
G(x,y) := F(x,y) - H(x,y),
\]

where the function \( H \) satisfies

\[
\begin{cases}
(-\Delta)_x^s H(x,y) = 0 & \text{in } \Omega, \\
H(x,y) = F(x,y) & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

for any fixed \( y \in \Omega \) and

\[
F(x,y) = a_{N,s} \frac{|x - y|^{N-2s}}{|x-y|^{N-2s}},
\]

is the fundamental solution of the elliptic operator \((-\Delta)_x^s\). In \([17]\), \( a_{N,s} \) is

\[
a_{N,s} := \frac{\Gamma \left( \frac{N}{2} + s \right)}{2^{2s} \pi^{\frac{N}{2}} \Gamma(s)}.
\]
Define the Robin function as
\[ R(x) = H(x, x). \]
For the continuity of \( R \), see Abatangelo \cite{[1]}.\(^{1}\)

**Definition 1.2.** We say \( \Omega \) is strictly star-shaped with respect to the point \( y \), if
\[ \langle x - y, n(x) \rangle > 0 \quad \forall \ x \in \partial \Omega, \]
where \( n(x) \) is the unit outward normal to \( \partial \Omega \) at \( x \).

We recall here the general Pohozaev identity in the nonlocal case due to Ros-Oton and Serra \cite{[27]}: Let \( u \) be a bounded solution of
\[ \begin{cases} (-\Delta)^s u = f(u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \]
where \( \Omega \subset \mathbb{R}^N \) is a bounded \( C^{1,1} \) domain, \( f \) is locally Lipschitz and \( d(x) = \text{dist}(x, \partial \Omega) \). Then \( u \) satisfies the following identity:
\[ (2s - N) \int_{\Omega} uf(u) \, dx + 2N \int_{\Omega} F(u) \, dx = \Gamma(1+s)^2 \int_{\partial \Omega} \left( \frac{u(x)}{d^s(x)} \right)^2 \langle x, \nu(x) \rangle dS(x), \]
where \( F(t) = \int_0^t f(s) \, ds \), \( \nu(x) \) is the unit outward normal to \( \partial \Omega \) at \( x \) and \( \Gamma \) is the Gamma function.

Translating the function \( u \), it is easy to see that, when \( \Omega \) is a \( C^{1,1} \) bounded domain, the following general identity holds:
\[ (2s - N) \int_{\Omega} uf(u) \, dx + 2N \int_{\Omega} F(u) \, dx = \Gamma(1+s)^2 \int_{\partial \Omega} \left( \frac{u(x)}{d^s(x)} \right)^2 \langle x - y, \nu(x) \rangle dS(x), \]
for every \( y \in \mathbb{R}^N \).

Note that, by the above Pohozaev identity \cite{[12]} does not have any solution in a star-shaped domain when \( \varepsilon = 0 \).

We turn now to a brief description of the results presented below.

**Theorem 1.1.** There exists \( \varepsilon_n > 0 \) and \( \lambda_n > 0 \) with \( \varepsilon_n \to 0 \) as \( n \to \infty \) and \( \lambda_n \) uniformly bounded above and away from zero, such that
(i) there exists a solution \( u_n \) to Eq. \((1.2)\) corresponding to \( \varepsilon = \varepsilon_n \);
(ii) if \( p > 2^* - 1 \), then \( F(\lambda_n u_n) \to K \) and \( \int_{\Omega} u_n^{p+1} \, dx \to 0 \) as \( n \to \infty \);
(iii) if \( p = 2^* - 1 \), then \( S(u_n) \to S \) as \( n \to \infty \) and there exist constants \( A, B > 0 \) such that for all \( n \geq 1 \), it holds \( A < \int_{\Omega} u_n^{p+1} \, dx < B \),
where \( F(\cdot), S(\cdot), K \) and \( S \) are defined as in \cite{[17], [19], [18]} and \cite{10} respectively.

**Theorem 1.2.** Let \( \Omega \) be a smooth bounded star-shaped domain with respect to 0, \( 2^* - 1 = p < q \). Suppose \( u_\varepsilon \in X_0(\Omega) \) is a solution of Eq. \((1.2)\) such that
\[ S(u_\varepsilon) \to S \quad \text{and} \quad A < \int_{\Omega} u_\varepsilon^{p+1} \, dx < B, \]
where $S(\cdot)$, $\mathcal{S}$ are as in (1.9) and (1.10) respectively. Let $x_\varepsilon$ be a point such that $\|u_\varepsilon\|_{L^\infty} = u_\varepsilon(x_\varepsilon)$ Assume that, up to a subsequence $x_\varepsilon \to x_0$ as $\varepsilon \to 0$. Then $x_0$ is an interior point of $\Omega$ and along a subsequence

$$\lim_{\varepsilon \to 0} \varepsilon \|u_\varepsilon\|_{\infty}^{q-p+2} = \frac{\omega_N c_{N,s}^2}{2} \frac{(q + 1) R_{N,s,x_0}}{q(N - 2s) - (N + 2s)} s^2 \Gamma(s)^2 B \left( \frac{N}{2}, s \right)^2 \times$$

$$B \left( \frac{N}{2}, \left( \frac{N - 2s}{2} \right) q - s \right)^{-1},$$

where $c_{N,s}$ is defined in (1.12) and $B(a,b)$ is the Beta function defined by

$$B(a,b) = \int_0^\infty t^{a-1}(1+t)^{-a-b}. \tag{1.23}$$

Here

$$R_{N,s,x_0} = \int_{\partial \Omega} \left( \frac{G(x,x_0)}{d^s(x)} \right)^2 (x - x_0, \nu) dS.$$

Furthermore,

$$\lim_{\varepsilon \to 0} \frac{u_\varepsilon(x)\|u_\varepsilon\|_\infty}{d^s(x)} = \frac{\omega_N c_{N,s}^2}{2} \frac{\Gamma(\frac{N}{2}) \Gamma(s)}{\Gamma(\frac{N+2s}{2})} \frac{G(x,x_0)}{d^s(x)} \ in \ C_{\text{loc}}(\Omega \setminus \{x_0\}), \tag{1.24}$$

where $G(x,x_0)$ is the Green function as defined in (1.14) and $d(x) = \text{dist}(x, \partial \Omega)$.

Remark 1.1. Under a suitable modification to the Theorem 1.2 a similar blow-up type result for the equation with $(-\Delta)^s$ operator in a smooth bounded domain $\Omega$ with outside zero Dirichlet boundary condition can be obtained for the nonlinearity $f_1(u) = u^{2^*-1} - \varepsilon$ under the assumption

$$\tilde{F}(u_\varepsilon) := \int_{R^N \times R^N} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x - y|^{N+2s}} dxdy \to S \text{ whenever } N > 2s$$

and for the nonlinearity $f_2(u) = u^{2^*-1} + \varepsilon u$ under the assumption

$$S(u_\varepsilon) \to S \text{ whenever } N > 4s.$$

Concerning the uniqueness problem, the shape of the domain plays an important role and hence some assumptions on $\Omega$ is needed, see [18]. To prove uniqueness theorem, our assumption on the domain are the following:

(A1) $\Omega$ is symmetric with respect to the hyperplanes $\{x_i = 0\}, i = 1, 2, \cdots, N$.
(A2) $\Omega$ is convex in the $x_i$ directions, $i = 1, 2, \cdots, N$.

Remark 1.2. By (A1), (A2) and in virtue [17] Theorem 3.1 (also see [20] Corollary 1.2), every solution $u_\varepsilon$ of (1.2) is symmetric with respect to the hyperplanes $\{x_i = 0\}, i = 1, \cdots, N$ and strictly decreasing in the $x_i$ direction, $i = 1, \cdots, N$. Therefore

$$\max_{x \in \Omega} u_\varepsilon(x) = u_\varepsilon(0).$$
**Theorem 1.3.** Let $2^* - 1 = p < q$ and $\Omega$ be smooth bounded star-shaped domain in $\mathbb{R}^N$ with respect to 0, $N > 4s$, satisfying (A1) and (A2). Suppose $u_{\varepsilon}$ and $v_{\varepsilon}$ are two solutions of (1.2) with $\max_{x \in \Omega} u_{\varepsilon} = \max_{x \in \Omega} v_{\varepsilon}$ and satisfy (1.22). Then, there exists $\varepsilon_0 > 0$ such that $\forall \varepsilon \in (0, \varepsilon_0)$,

$$u_{\varepsilon} \equiv v_{\varepsilon} \ \text{in} \ \Omega.$$ 

The rest of the paper is organised as follows. In Section 2, we prove Theorem 1.1. Section 3 deals with the proof of Theorem 1.2. Section 4 is devoted to the study of uniqueness result. The last section is the Appendix. Laplace

**Notations:** Throughout this paper $C$ denotes the generic constants which may vary from line to line. Below are few notations which we use throughout the paper:

- $\omega_N$ = surface measure of unit ball in $\mathbb{R}^N$,
- $G(x, y)$ denotes the Green function of $(-\Delta)^s$ in $\Omega$,
- $B(.,.)$ and $\Gamma(.)$ denote the Beta function and the Gamma function respectively,
- $D^{s,2}(\mathbb{R}^N)$ denotes the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to $\left(\int_{\mathbb{R}^N} \frac{|w(x) - w(y)|^2}{|x-y|^{N+2s}} dxdy\right)^{1/2}$.

2. Asymptotic behavior

**Proposition 2.1.** Let $2^* - 1 \leq p < q$. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, the problem

$$\begin{cases}
(-\Delta)^s v = \lambda_\varepsilon v^p - \varepsilon v^q & \text{in} \ \Omega, \\
v > 0 & \text{in} \ \Omega, \\
v(x) = 0 & \text{in} \ \mathbb{R}^N \setminus \Omega,
\end{cases}$$

(2.1)

admits a solution $v_{\varepsilon}$, with the property that

$$A < \lambda_{\varepsilon} < B,$$

for some constants $A, B > 0$, independent of $n$. In addition

(i) if $p > 2^* - 1$, then $F(v_{\varepsilon}) \to K$ and $\int_\Omega v_{\varepsilon}^{p+1} dx \to 0$ as $\varepsilon \to 0$;

(ii) if $p = 2^* - 1$, then $S(v_{\varepsilon}) \to S$ as $\varepsilon \to 0$ and $\int_\Omega v_{\varepsilon}^{p+1} dx = 1$,

where $K$ and $S$ are defined as in (1.8) and (1.10) respectively.

**Proof.** Let $\Omega_\varepsilon = \frac{1}{\varepsilon} \setminus \Omega$ and $X_0(\Omega_\varepsilon) = \{ w \in H^s(\mathbb{R}^N) : w = 0 \ \text{in} \ \mathbb{R}^N \setminus \Omega_\varepsilon \}$. Clearly $\Omega_\varepsilon \to \mathbb{R}^N$ as $\varepsilon \to 0$. Let us consider the manifold $N_\varepsilon$ defined by:

$$N_\varepsilon = \{ w \in X_0(\Omega_\varepsilon) \cap L^{q+1}(\Omega_\varepsilon) : \int_{\Omega_\varepsilon} w^{p+1} dx = 1 \}.$$ 

On $N_\varepsilon$, the functional $F$ can be written as:

$$F(w) = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w(x) - w(y)|^2}{|x-y|^{N+2s}} dxdy + \frac{1}{q+1} \int_{\Omega_\varepsilon} w^{q+1} dx$$

(2.2)

$$=: \hat{F}(w).$$

For $p \geq 2^* - 1$, define

$$S_\varepsilon := \inf_{w \in N_\varepsilon} \hat{F}(w) = \inf_{w \in N_\varepsilon} F(w).$$

(2.3)
Let \( \{w_{n, \varepsilon}\} \subset N_\varepsilon \) be a minimizing sequence for \( \text{(2.3)} \). Therefore, we have,

\[
\hat{F}(w_{n, \varepsilon}) \to S_\varepsilon \text{ as } n \to \infty, \int_{\Omega_\varepsilon} w_{n, \varepsilon}^{p+1} dx = 1.
\]

Proceeding as in [5, Theorem 1.5], we can show that there exists \( w_{\varepsilon} \in X_0(\Omega_\varepsilon) \cap L^{q+1}(\Omega_\varepsilon) \) such that \( w_{n, \varepsilon} \rightharpoonup w_{\varepsilon} \) in \( X_0(\Omega_\varepsilon) \) and \( w_{\varepsilon} \) satisfies,

\[
(-\Delta)^s w_{\varepsilon} = \lambda_{\varepsilon} w_{\varepsilon}^p - w_{\varepsilon}^q \text{ in } \Omega_\varepsilon \quad \text{and} \quad \hat{F}(w_{\varepsilon}) = S_\varepsilon.
\]

This yields,

\[
\lambda_{\varepsilon} = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w_{\varepsilon}(x) - w_{\varepsilon}(y)|^2}{|x - y|^{N+2s}} dxdy + \int_{\Omega_\varepsilon} w_{\varepsilon}^{q+1} dx.
\]

Since, \( \hat{F}(w_{\varepsilon}) = S_\varepsilon \) we have, \( 2S_\varepsilon < \lambda_{\varepsilon} < (q+1)S_\varepsilon \). In Theorem A.1 (see Appendix), let \( \rho = \varepsilon - \frac{(p-1)}{2(q+1-p)} \), then \( N_\rho \) and \( S_\rho \) are exactly same as \( N_\varepsilon \) and \( S_\varepsilon \) defined here. Letting \( \varepsilon \to 0 \) we have,

\[
(2.4) \quad S_\varepsilon \to K \text{ if } p > 2^*_s - 1, \quad S_\varepsilon \to \frac{S}{2} \text{ if } p = 2^*_s - 1.
\]

Hence, there exists \( \varepsilon_0 > 0 \) and \( A, B > 0 \) such that \( A < \lambda_{\varepsilon} < B \) for all \( \varepsilon \in (0, \varepsilon_0) \).

Using the transformation

\[
v_{\varepsilon}(x) = \varepsilon^{-\frac{1}{p-1}} w_{\varepsilon}(\varepsilon^{-\frac{s-1}{2(s+p)}} x),
\]

we observe that \( v_{\varepsilon} \) is a solution of \( \text{(2.1)} \). Moreover, \( \int_{\Omega_\varepsilon} w_{\varepsilon}^{p+1} dx = 1 \) implies

\[
\int_{\Omega_\varepsilon} v_{\varepsilon}^{p+1} dx = \varepsilon^{-\frac{2(N-2s)(N+2s)}{2(N-p)}} \int_{\Omega_\varepsilon} w_{\varepsilon}^{q+1} dx.
\]

Hence,

\[
\int_{\Omega} v_{\varepsilon}^{p+1} dx = 1 \text{ if } p = 2^*_s - 1
\]

and

\[
\int_{\Omega} v_{\varepsilon}^{p+1} dx \to 0 \text{ as } \varepsilon \to 0, \text{ } p > 2^*_s - 1.
\]

A simple calculation yields

\[
F(w_{\varepsilon}) = \hat{F}(w_{\varepsilon}) = F(v_{\varepsilon}) \quad \text{when } p > 2^*_s - 1,
\]

where \( F \) and \( \hat{F} \) are defined as in \( \text{(1.7)} \) and \( \text{(2.2)} \). This along with \( \text{(2.4)} \) and the fact that \( F(w_{\varepsilon}) = S_\varepsilon \) implies

\[
F(v_{\varepsilon}) \to K \text{ if } p > 2^*_s - 1.
\]

Moreover when \( p = 2^*_s - 1 \),

\[
S \leq S(v_{\varepsilon}) \leq 2\hat{F}(v_{\varepsilon}, \Omega) = 2\hat{F}(w_{\varepsilon}, \Omega_\varepsilon) = 2S_\varepsilon \to S.
\]

Hence

\[
S(v_{\varepsilon}) \to S \text{ if } p = 2^*_s - 1.
\]

This completes the proof. \( \square \)
Proof of Theorem 1.1. Let \( v_\varepsilon \) and \( \lambda_\varepsilon \) be as in Proposition 2.1. Define, \( u_\varepsilon = \lambda_\varepsilon^{-1} v_\varepsilon \). Then it is easy to see that \( u_\varepsilon \) satisfies
\[
(-\Delta)^s u_\varepsilon = u_\varepsilon^p - \varepsilon \lambda_\varepsilon^{-1} u_\varepsilon \text{ in } \Omega.
\]
Using the bounds on \( \lambda_\varepsilon \) from Proposition 2.1, we can conclude that there exist solutions \( u_n \) of problem (1.2) along a sequence \( \{\varepsilon_n\}_{n \geq 1} \) of values \( \varepsilon \) which tends to \( 0 \) as \( n \to \infty \). Set \( \lambda_n := \frac{\lambda_{\varepsilon_n}}{\varepsilon_n} \). Thus, from Proposition 2.1 it follows
\[
F(\lambda_n u_n) \to K \quad \text{and} \quad \int_{\Omega} u_n^{p+1} \to 0 \quad \text{when} \quad p > 2^* - 1
\]
and
\[
S(\lambda_n u_n) \to S \quad \text{and} \quad A < \int_{\Omega} u_n^{p+1} < B \quad \text{when} \quad p = 2^* - 1,
\]
for some \( A, B > 0 \). Since \( S(\lambda_n u_n) = S(u_n) \), theorem follows. \( \square \)

3. The case \( p = 2^* - 1 \) and the proof of Theorem 1.2

Lemma 3.1. Let \( u_\varepsilon \) be as in Theorem 1.2. Then \( \|u_\varepsilon\|_\infty \to \infty \) as \( \varepsilon \to 0 \).

Proof. Note that as \( u_\varepsilon \in C(\overline{\Omega}) \) (see [5, Theorem 1.2]), for each fixed \( \varepsilon > 0 \), we have \( \|u_\varepsilon\|_\infty < \infty \). Furthermore, since \( u_\varepsilon \) is as in Theorem 1.2 we have
\[
\int_{\Omega} u_\varepsilon^{2^*} \, dx = c,
\]
where \( c \in (A, B) \). Suppose, \( \|u_\varepsilon\|_\infty \) is uniformly bounded. Therefore, by the Schauder estimate (see [29, 28]), \( u_\varepsilon \to u \) in \( C^{2s-\delta}_\text{loc}(\Omega) \cap C^s(\mathbb{R}^N) \), for any \( \delta > 0 \). By the definition of weak solution, we have
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_\varepsilon(x) - u_\varepsilon(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy = \int_{\Omega} (u_\varepsilon^{2^*-1} - \varepsilon u_\varepsilon^p) \varphi \, dx \quad \forall \varphi \in C^\infty_0(\Omega).
\]
Moreover, as \( \|u_\varepsilon\|_{C^s(\mathbb{R}^N)} \) is uniformly bounded (see [28, Proposition 1.1]), we get
\[
\frac{(u_\varepsilon(x) - u_\varepsilon(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \leq C \frac{|x - y|^s \|\nabla \varphi\|_{L^\infty} |x - y|}{|x - y|^{N+2s}} \leq C \frac{1}{|x - y|^{N-1+s}}.
\]
Therefore using the dominated convergence theorem, we can pass to the limit in (3.2) and get,
\[
\begin{cases}
(-\Delta)^s u = u^{2^*-1} & \text{in } \Omega, \\
u \geq 0 & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]
where
\[
A < \int_{\Omega} u^{2^*} \, dx < B.
\]
As \( A > 0 \), the above expression implies \( u \) is a nontrivial solution in a bounded star-shaped domain. Since, \( u \in C(\mathbb{R}^N) \) and \( u = 0 \) in \( \mathbb{R}^N \setminus \Omega \), clearly \( u \) is a bounded solution. By the maximum principle ([22, Proposition 2.17]), we also have \( u > 0 \) in \( \Omega \). This gives a contradiction due to the Pohozaev identity ([27, Corollary 1.3]). Hence the lemma follows. \( \square \)
Let \( x_\varepsilon \) be a local maximum point of \( u_\varepsilon \) and \( \gamma_\varepsilon \in \mathbb{R}^+ \) such that

\[
(3.5) \quad u_\varepsilon(x_\varepsilon) = \|u_\varepsilon\|_\infty = \gamma_\varepsilon \frac{N-2s}{2}. 
\]

Then \( \gamma_\varepsilon \to 0 \) as \( \varepsilon \to 0 \).

**Lemma 3.2. (Blow-up at an interior point)** Let \( x_0 := \lim_{\varepsilon \to 0} x_\varepsilon \), then \( x_0 \) is an interior point of \( \Omega \).

*Proof.* Let \( \lambda_1 \) be the first eigenvalue of \((-\Delta)^s \) in \( \Omega \) and \( \varphi_1 \) be a corresponding eigenfunction (see [30]), that is, \( \varphi_1 \) satisfies

\[
(-\Delta)^s \varphi_1 = \lambda_1 \varphi_1 \text{ in } \Omega, \\
\varphi_1 = 0 \text{ in } \mathbb{R}^N \setminus \Omega.
\]

Moreover, as \( u_\varepsilon \) is a classical solution (see [5, Proposition 3.1])

\[
\lambda_1 \int_{\Omega} \varphi_1 u_\varepsilon dx = \int_{\Omega} (-\Delta)^s \varphi_1 u_\varepsilon dx = \int_{\mathbb{R}^N} (-\Delta)^s \varphi_1 u_\varepsilon dx = \int_{\mathbb{R}^N} \varphi_1 (-\Delta)^s u_\varepsilon dx \\
= \int_{\Omega} \varphi_1 u_\varepsilon^2 dx - \int_{\Omega} \varphi_1 u_\varepsilon^2 dx \\
\leq \int_{\Omega} \varphi_1 u_\varepsilon^2 dx \\
\leq \left( \int_{\Omega} u_\varepsilon^{2^*} dx \right)^{\frac{2^*}{2^* - 1}} \left( \int_{\Omega} \varphi_1^{2^*} dx \right)^{\frac{2^*-1}{2^*}} \\
\leq B^{\frac{2^* - 1}{2^*}} \left( \int_{\Omega} \varphi_1^{2^*} dx \right)^{\frac{1}{2^*}} \leq C',
\]

for some constant \( C' \). Hence \( \int_{\Omega} \varphi_1 u_\varepsilon dx \leq \frac{C'}{\lambda_1} \). Since, \( \varphi_1 \geq C \) on \( \Omega' \subset \subset \Omega \), we obtain

\[
(3.6) \quad \int_{\Omega'} u_\varepsilon \leq C(\Omega'),
\]

for any \( \Omega' \subset \subset \Omega \).

Define

\[
O(\delta) := \{ z \in \Omega : \text{dist}(z, \partial \Omega) < \delta \}
\]

and

\[
I(\delta) := \{ z \in \Omega : \text{dist}(z, \partial \Omega) > \delta \}.
\]

**Claim:** There exists \( C > 0 \) such that

\[
\sup_{O(\delta)} u_\varepsilon(x) \leq C \quad \forall \, \varepsilon > 0.
\]

If \( \Omega \) is strictly convex, the moving plane argument (which is given in the proof of [17] Theorem 3.1) (also see [20] Corollary 1.2) yields the fact that each solution \( u_\varepsilon \) increases along an arbitrary straight line toward inside of \( \Omega \) emanating from a point on \( \partial \Omega \). (see for instance [10] Lemma 3.1]). Hence following an argument as in [19], we can find \( \gamma, \delta > 0 \) such that for any \( x \in O(\delta) \), there exists a measurable set \( \Gamma_x \) with (i) \( \text{meas}(\Gamma_x) \geq \gamma \), (ii) \( \Gamma_x \subset I(\delta) \), and (iii) \( u_\varepsilon(y) \geq u_\varepsilon(x) \) for any \( y \in \Gamma_x \).
In particular, \( \Gamma_x \) can be taken as a cone with vertex at \( x \). Let \( \Omega' = I(\frac{\delta}{2}) \). Then for any \( x \in O(\delta) \), we have

\[
 u(x) \leq \frac{1}{\text{meas}(\Gamma_x)} \int_{\Gamma_x} u(y)dy \leq \gamma^{-1} \int_{\Omega'} u \leq C(\Omega').
\]

This proves the claim when \( \Omega \) is strictly convex. The general case can be proved using Kelvin transform in the extended domain (see, for instance, [19], [10], [?]).

From Lemma 3.1 we have \( u_\varepsilon(x_\varepsilon) \to \infty \) as \( \varepsilon \to 0 \). On the other hand, the above claim implies \( u_\varepsilon \) is uniformly bounded near the boundary for all small \( \varepsilon > 0 \). Hence passing to a subsequence, the point \( x_\varepsilon \) converges to an interior point \( x_0 \in \Omega \).

\[ \square \]

Define

\[
 z_\varepsilon(x) = \gamma_{\varepsilon}^{\frac{N-2s}{2}} u_\varepsilon(\gamma_{\varepsilon} x + x_\varepsilon).
\]

Then \( \|z_\varepsilon\|_\infty = 1 \) and satisfies

\[
 (-\Delta)^s z_\varepsilon = z_\varepsilon^{2^*_s - 1} - \varepsilon \gamma_{\varepsilon}^{\frac{N+2s-\delta(N-2s)}{2}} z_\varepsilon^\delta \quad \text{in } \Omega_\varepsilon,
\]

\[
 z_\varepsilon > 0 \quad \text{in } \Omega_\varepsilon,
\]

\[
 z_\varepsilon = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega_\varepsilon,
\]

where \( \Omega_\varepsilon = \frac{\Omega - x_\varepsilon}{\gamma_{\varepsilon}} \).

**Lemma 3.3.** Suppose \( z_\varepsilon \) is as in (3.8). Then

(i) \( \lim_{\varepsilon \to 0} \varepsilon z_\varepsilon \in C_N \) and \( z(\varepsilon) = \left[ 1 + \frac{|x|^2}{\mu_{N,s}} \right]^{-\frac{N-2s}{2}} \), where \( \mu_{N,s} = c_{N,s}^{-\frac{1}{2}} \).

(ii) There exists \( Z \in D^{s,2}(\mathbb{R}^N) \) such that \( z_\varepsilon \to Z \) in \( C^{2s-\delta}_{\text{loc}}(\mathbb{R}^N) \) as \( \varepsilon \to 0 \), for any \( \delta > 0 \).

(iii) \( Z \) satisfies Eq. (1.13) and \( Z(x) = \left[ 1 + \frac{|x|^2}{\mu_{N,s}} \right]^{-\frac{N-2s}{2}} \), where \( \mu_{N,s} = c_{N,s}^{-\frac{1}{2}} \).

**Proof.** Using Lemma 3.2 we obtain \( \Omega_\varepsilon \to \mathbb{R}^N \) as \( \varepsilon \to 0 \). We know \( z_\varepsilon \) satisfies Eq. (3.9). Note that, \( \max_{\Omega} u_\varepsilon(x) = u_\varepsilon(x_\varepsilon) \) implies \( z_\varepsilon \) attains maximum at 0 and \( z_\varepsilon(0) = 1 \). Therefore, applying the definition of fractional Laplace operator, it is easy to see that \( (-\Delta)^s z_\varepsilon(0) \geq 0 \). Thus from (3.9), we have \( 1 - \varepsilon \gamma_{\varepsilon}^{\frac{N+2s-\delta(N-2s)}{2}} \geq 0 \).

This in turn implies \( \lim_{\varepsilon \to 0} z_\varepsilon \in C_{N}^0(\mathbb{R}^N) \in [0,1] \). Consequently, using Schauder estimate [28], \( z_\varepsilon \to Z \) in \( C^{2s-\delta}_{\text{loc}}(\mathbb{R}^N) \), for some \( \delta > 0 \). Let \( \phi \in C_0^\infty(\mathbb{R}^N) \). Thus, \( \phi \in C_0^\infty(\Omega_\varepsilon) \) for \( \varepsilon \) small. Taking \( \phi \) as the test function, from Eq. (3.9) we have

\[
 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(z_\varepsilon(x) - z_\varepsilon(y))(\phi(x) - \phi(y))}{|x - y|^{N+2s}}dxdy = \int_{\Omega_\varepsilon} z_\varepsilon^{2_s^* - 1}\phi dx - \varepsilon \gamma_{\varepsilon}^{\frac{(N+2s)-\delta(N-2s)}{2}} \int_{\Omega_\varepsilon} z_\varepsilon^\delta \phi dx.
\]

As \( \|z_\varepsilon\|_{L^\infty} = 1 \) and \( \phi \) has compact support, using dominated convergence theorem as in the proof of Lemma 3.1 we can pass to the limit \( \varepsilon \to 0 \) in the above integral
where $c = \lim_{\varepsilon \to 0} \varepsilon \gamma \varepsilon$. Since $z_{\varepsilon} \in H^{s}(\Omega_{\varepsilon})$ and $z_{\varepsilon} = 0$ in $\mathbb{R}^{N} \setminus \Omega_{\varepsilon}$, multiplying $(3.9)$ by $z_{\varepsilon}$ and integrating over $\mathbb{R}^{N}$, we have
\[
\|z_{\varepsilon}\|_{L^{2}(\mathbb{R}^{N})}^{2} = \int_{\mathbb{R}^{N}} z_{\varepsilon}^{2} \, dx - \varepsilon \gamma \varepsilon \int_{\mathbb{R}^{N}} z_{\varepsilon}^{q(x)-1} \, dx \leq \int_{\Omega_{\varepsilon}} z_{\varepsilon}^{2} \, dx < B.
\]
Therefore, up to a subsequence $z_{\varepsilon} \rightarrow \tilde{Z}$ in $D^{s,2}(\mathbb{R}^{N})$. By the uniqueness of limit, $Z = \tilde{Z}$. Thus $Z \in D^{s,2}(\mathbb{R}^{N})$. Consequently, multiplying $(3.11)$ by $z_{\varepsilon}$ and integrating over $\mathbb{R}^{N}$, we get $Z = L^{q+1}(\mathbb{R}^{N})$. Hence, if $c \neq 0$, we get a contradiction by Pohozaev identity (see [5, Theorem 1.4]). This implies $c = 0$ and $Z$ satisfies $(1.13)$. As a consequence, $Z$ must be of the form $\tilde{Z} = U(\xi)$, for some $\xi > 0$, where $U$ is as in $(1.11)$. As, $\max_{\Omega} z_{\varepsilon} = z_{\varepsilon}(0) = 1$, we get $Z(0) = 1$ and $0 \leq Z \leq 1$. Using this fact, it is easy to see that $\xi = c_{N,s}^{-1}$, where $c_{N,s}$ is as defined in $(1.12)$. From this, a computation yields $Z(x) = \left[1 + \frac{|x|^{2}}{\mu_{N,s}^{2}}\right]^{-\frac{N-2s}{2}}$, where $\mu_{N,s} = c_{N,s}^{\frac{1}{2}}$.

Now we show that there exists $C > 0$ independent of $\varepsilon$ such that $z_{\varepsilon}(x) \leq CZ(x)$ for all $x \in \Omega_{\varepsilon}$.

The local behavior of $z_{\varepsilon}$ is known. Next, we need to check the behavior of $z_{\varepsilon}$ near $\infty$. For this, define the Kelvin transform of $z_{\varepsilon}$ as
\[
\hat{z}_{\varepsilon}(x) = \left|\frac{x}{|x|^{2}}\right| \in \Omega_{\varepsilon} \setminus \{0\}.
\]
From $(3.9)$, it follows that $\hat{z}_{\varepsilon}$ satisfies
\[
\begin{cases}
(\Delta)^{s} \hat{z}_{\varepsilon} = \hat{z}_{\varepsilon}^{2^{s}-1} - \varepsilon \gamma \varepsilon \frac{(N+2s)-q(N-2s)}{2} |x|^{q(N-2s)-(N+2s)} \hat{z}_{\varepsilon}^{q} \quad \text{in} \quad \Omega_{\varepsilon}^* \\
\hat{z}_{\varepsilon} = 0 \quad \text{in} \quad \mathbb{R}^{N} \setminus \Omega_{\varepsilon}^*,
\end{cases}
\]
where $\Omega_{\varepsilon}^*$ is the image $\Omega_{\varepsilon}$ under the Kelvin transform. Hence the behavior of $z_{\varepsilon}$ near $\infty$ amounts to study the behavior of $\hat{z}_{\varepsilon}$ near $0$.

**Lemma 3.4.** There exist $R > 0$ and $C > 0$ independent of $\varepsilon > 0$ such that any solution of $(3.14)$ satisfy
\[
\|\hat{z}_{\varepsilon}\|_{L^{\infty}(B_{R})} \leq C\left(\int_{B_{R}} \hat{z}_{\varepsilon}^{2} \, dx\right)^{\frac{1}{2}}.
\]

**Proof.** The proof follows along the same line of arguments as in [5, Theorem 1.1] (see also [34]) with a suitable modification and we skip the proof. \qed

For $(3.12)$, note that $\|z_{\varepsilon}\|_{\infty} = 1$ and this implies that $z_{\varepsilon} \leq CZ(x)$ locally. From $(1.22)$ and $(3.8)$, it follows
\[
A < \int_{\Omega_{\varepsilon}} z_{\varepsilon}^{2^{s}} \, dx < B.
\]
But this implies that
\[ \int_{B_\epsilon \cap \Omega} \frac{\partial^2}{\partial x^2} w \, \text{d}x \leq \int_{\Omega} \frac{\partial^2}{\partial x^2} w \, \text{d}x = \int_{\Omega} \frac{\partial^2}{\partial x^2} w \, \text{d}x < B. \]

Consequently from Lemma 3.4 we obtain \( z_\epsilon(x) \leq \frac{C}{|x|^{N-2s}} \) as \(|x| \to \infty\). Moreover, since at infinity \( Z \) decays as \(|x|^{-(N-2s)}\), we conclude \( z_\epsilon \leq CZ(x) \) near infinity. Hence, we have \( z_\epsilon \leq CZ(x) \) for all \( x \in \Omega_\epsilon \). As a conclusion, from (3.8) we obtain that there exists \( C > 0 \) independent of \( \epsilon \) such that

\[ (3.16) \quad u_\epsilon(x) \leq C\gamma \left( \frac{N}{2s} \right) \left( \frac{N-2s}{\epsilon} \right) Z \left( \frac{x-x_\epsilon}{\gamma \epsilon} \right). \]

Define \( w_\epsilon(x) = \|u_\epsilon\|_{\infty}u_\epsilon(x) = \gamma \left( \frac{N}{2s} \right) u_\epsilon(x) \). Then \( w_\epsilon \) satisfies

\[ \begin{cases} (-\Delta)^s w_\epsilon = \gamma \left( \frac{N}{2s} \right) u_\epsilon^{2s-1} - \epsilon \gamma \left( \frac{N}{2s} \right) u_\epsilon^q & \text{in } \Omega, \\ w_\epsilon = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \]

Lemma 3.5. The Green function associated to the fractional Laplacian \((-\Delta)^s\) satisfy the following inequalities.

(i) \( G(x, y) \leq \frac{C}{|x-y|^{N-2s}} \) and

(ii) \( G(x, y) \leq \frac{C \gamma \left( \frac{N}{2s} \right)}{|x-y|^{N-2s}} \),

where \( C > 0 \) is a constant depending on \( \Omega \) and \( s \) and \( N > 2s \).

Proof. This follows from Chen and Song [3], Theorem 1.1. \( \square \)

Lemma 3.6. Let \( w_\epsilon \) be as in (3.17). Then for every \( r > 0 \), there exists a constant \( C = C(r) > 0 \) such that

\[ \|w_\epsilon\|_{L^\infty(\Omega \setminus B_{r}(x_\epsilon))} \leq C. \]

Proof. From the Green function representation and Lemma 3.5 we have

\[ |w_\epsilon(x)| \leq \gamma \left( \frac{N}{2s} \right) \int_{\Omega} G(x, y)u_\epsilon^{2s-1} \, dy + \epsilon \gamma \left( \frac{N}{2s} \right) \int_{\Omega} G(x, y)u_\epsilon^q \, dy \]

\[ \leq C\gamma \left( \frac{N}{2s} \right) \int_{\Omega} |x-y|^{2s-N} u_\epsilon^{2s-1} \, dy + C \epsilon \gamma \left( \frac{N}{2s} \right) \int_{\Omega} |x-y|^{2s-N} u_\epsilon^q \, dy. \]

Moreover,

\[ \gamma \left( \frac{N-2s}{2} \right) \int_{\Omega} |x-y|^{2s-N} u_\epsilon^{2s-1} \, dy = \gamma \left( \frac{N-2s}{2} \right) \int_{\Omega \setminus B_{\epsilon} \cap \{x_\epsilon\}} |x-y|^{2s-N} u_\epsilon^{2s-1} \, dy \]

\[ + \gamma \left( \frac{N-2s}{2} \right) \int_{\Omega \setminus (\Omega \setminus B_{\epsilon} \cap \{x_\epsilon\})} |x-y|^{2s-N} u_\epsilon^{2s-1} \, dy. \]

Using (3.10) along with that fact that \( Z(x) = |x|^{-(N-2s)} \) at infinity, we have

\[ \gamma \left( \frac{N-2s}{2} \right) |x-y|^{2s-N} u_\epsilon^{2s-1}(y) \leq \frac{C \gamma \left( \frac{N}{2s} \right)}{|x-y|^{N-2s} |y-x_\epsilon|^{N+2s}} \quad \text{if } y \in \Omega \setminus B_{|x-x_\epsilon|}(x_\epsilon) \]

and

\[ \epsilon \gamma \left( \frac{N-2s}{2} \right) |x-y|^{2s-N} u_\epsilon^q(y) \, dy \leq \frac{C \epsilon \gamma \left( \frac{N}{2s} \right)}{|x-y|^{N-2s} |y-x_\epsilon|^{d+2s}} \quad \text{if } y \in \Omega \setminus B_{|x-x_\epsilon|}(x_\epsilon). \]
Hence,
\[
\gamma \varepsilon^{\frac{N-2s}{2}}\int_{\Omega \setminus B_{\frac{|x-x_{\varepsilon}|}{2}}(x_{\varepsilon})} |x-y|^{2s-N} u_{\varepsilon}^{2s-1}(y)dy \\
\leq \frac{C}{|x-x_{\varepsilon}|^{N+2s}} \int_{\Omega \setminus B_{\frac{|x-x_{\varepsilon}|}{2}}(x_{\varepsilon})} 1 \frac{1}{|x-y|^{N-2s}}dy \\
\leq \frac{C}{|x-x_{\varepsilon}|^{N+2s}}
\]
and
\[
\varepsilon \gamma \varepsilon^{\frac{N-2s}{2}}\int_{\Omega \setminus B_{\frac{|x-x_{\varepsilon}|}{2}}(x_{\varepsilon})} |x-y|^{2s-N} u_{\varepsilon}^{q}(y)dy \\
\leq \frac{C\varepsilon \gamma \varepsilon^{\frac{(N-2s)(q-1)}{2}}}{|x-x_{\varepsilon}|^{N-2s}} \int_{\Omega \setminus B_{\frac{|x-x_{\varepsilon}|}{2}}(x_{\varepsilon})} 1 \frac{1}{|x-y|^{N-2s}}dy \\
\leq \frac{C}{|x-x_{\varepsilon}|^{N-2s}}
\]

When \( y \in \Omega \cap B_{\frac{|x-x_{\varepsilon}|}{2}}(x_{\varepsilon}) \), we have \(|x-y| \geq |x-x_{\varepsilon}| - |y-x_{\varepsilon}| \geq \frac{1}{2}|x-x_{\varepsilon}|\).

Therefore applying (3.16) we obtain
\[
\gamma \varepsilon^{\frac{N-2s}{2}}\int_{\Omega \setminus B_{\frac{|x-x_{\varepsilon}|}{2}}(x_{\varepsilon})} |x-y|^{2s-N} u_{\varepsilon}^{2s-1}(y)dy \\
\leq \frac{C\gamma \varepsilon^{\frac{N-2s}{2}}}{|x-x_{\varepsilon}|^{N-2s}} \int_{\Omega \setminus B_{\frac{|x-x_{\varepsilon}|}{2}}(x_{\varepsilon})} u_{\varepsilon}^{2s-1}(y)dy \\
\leq \frac{C\gamma \varepsilon^{N}}{|x-x_{\varepsilon}|^{N-2s}} \int_{\mathbb{R}^{N}} Z^{2s-1}(y-x_{\varepsilon})dy \\
\leq \frac{C}{|x-x_{\varepsilon}|^{N-2s}} \int_{\mathbb{R}^{N}} Z^{2s-1}(x)dx \\
\leq \frac{C}{|x-x_{\varepsilon}|^{N-2s}}.
\]

Similarly applying Lemma 3.3 we obtain
\[
\varepsilon \gamma \varepsilon^{\frac{N-2s}{2}}\int_{\Omega \setminus B_{\frac{|x-x_{\varepsilon}|}{2}}(x_{\varepsilon})} |x-y|^{2s-N} u_{\varepsilon}^{q}(y)dy \\
\leq \frac{C\varepsilon \gamma \varepsilon^{\frac{(N-2s)(q-1)}{2}}}{|x-x_{\varepsilon}|^{N-2s}} \int_{\Omega \setminus B_{\frac{|x-x_{\varepsilon}|}{2}}(x_{\varepsilon})} u_{\varepsilon}^{q}(y)dy \\
\leq \frac{C\varepsilon \gamma \varepsilon^{N-q+1}}{|x-x_{\varepsilon}|^{N-2s}} \int_{\mathbb{R}^{N}} Z^{q}(y)dy \\
\leq \frac{C}{|x-x_{\varepsilon}|^{N-2s}}.
\]

where \( C > 0 \) is a uniform constant. Hence for any small \( r > 0 \) fixed, \( \Omega \setminus B_{r}(x_{0}) \subseteq \Omega \setminus \{x_{\varepsilon}\} \), for \( \varepsilon > 0 \) small enough and therefore, we have \( \|w_{\varepsilon}\|_{L^{\infty}(\Omega \setminus B_{r}(x_{0}))} \leq C \). \( \square \)

Note that (3.17) can be rewritten as
\[
\begin{align*}
(-\Delta)^{s}w_{\varepsilon} &= \gamma \varepsilon^{2s}w_{\varepsilon}^{2s-1} - \varepsilon \gamma \varepsilon^{\frac{N-2s}{2}}(q-1)u_{\varepsilon}^{q} \quad \text{in} \quad \Omega \\
&w_{\varepsilon} = 0 \quad \text{in} \quad \mathbb{R}^{N} \setminus \Omega.
\end{align*}
\]
Lemma 3.7.

\begin{equation}
\lim_{\varepsilon \to 0} \frac{w_\varepsilon(x)}{d(x)^s} = \gamma_0 \frac{G(x,x_0)}{d(x)^s} \text{ in } C(\Omega \setminus B_r(x_0)),
\end{equation}

for any $r > 0$. Here, $\gamma_0$ is same as in Lemma 3.8.

**Proof:** Choose $r > 0$ such that $\Omega' = \Omega \setminus B_r(x_0)$ is connected. Thus by Lemma 3.6 $|w_\varepsilon| \leq C$ for all $x \in \Omega'$.

Then for any $r > 0$ small and the fact that $\gamma_\varepsilon \to 0$ we have

\begin{equation}
\frac{w_\varepsilon(x)}{d(x)^s} = \frac{\gamma_\varepsilon^{- \frac{N-2s}{2}}}{d(x)^s} \int_\Omega G(x,y)u_\varepsilon^{2s-1}dy - \varepsilon \frac{\gamma_\varepsilon^{- \frac{N-2s}{2}}}{d(x)^s} \int_\Omega G(x,y)u_\varepsilon^2dy
\end{equation}

\begin{align*}
&= \frac{\gamma_\varepsilon^{- \frac{N-2s}{2}}}{d(x)^s} \int_{B_r(x_0)} G(x,y)u_\varepsilon^{2s-1}(y)dy + \frac{\gamma_\varepsilon^{- \frac{N-2s}{2}}}{d(x)^s} \int_{\Omega \setminus B_r(x_0)} G(x,y)u_\varepsilon^{2s-1}(y)dy \\
&- \varepsilon \frac{\gamma_\varepsilon^{- \frac{N-2s}{2}}}{d(x)^s} \int_{B_r(x_0)} G(x,y)u_\varepsilon^2dy - \varepsilon \frac{\gamma_\varepsilon^{- \frac{N-2s}{2}}}{d(x)^s} \int_{\Omega \setminus B_r(x_0)} G(x,y)u_\varepsilon^2dy.
\end{align*}

Using the second estimate in Lemma 3.6, 3.10 and the fact that $Z$ decays at infinity of the order $|y|^{-(N-2s)}$, we estimate the 2nd term on RHS as follows

\begin{align*}
\frac{\gamma_\varepsilon^{- \frac{N-2s}{2}}}{d(x)^s} \int_{\Omega \setminus B_r(x_0)} G(x,y)u_\varepsilon^{2s-1}(y)dy &\leq \frac{\gamma_\varepsilon^{- \frac{N-2s}{2}}}{d(x)^s} \int_{B_r(x_0)} u_\varepsilon^{2s-1}(y)dy \\
&= \frac{\gamma_\varepsilon^{- N}}{d(x)^s} \int_{B_r(x_0)} Z^{2s-1} \left( \frac{y-x_\varepsilon}{\gamma_\varepsilon} \right) |x-y|^{-N}dy \\
&\leq C \gamma_\varepsilon^{- N} \int_{B_r(x_0)} \left( \frac{y-x_\varepsilon}{\gamma_\varepsilon} \right) |x-y|^{-(N+2s)} |x-y|^{-N}dy \\
&= C \gamma_\varepsilon^{- \frac{2s}{N}} \int_{B_r(x_0)} |y-x_\varepsilon|^{-(N+2s)} |x-y|^{-N}dy \\
&= o_{r,\varepsilon}(1),
\end{align*}

where $o_{r,\varepsilon}(1)$ denote the term going to 0 as $r \to 0$ or $\varepsilon \to 0$. Note that we have used the fact that $|x-y|^{s-N}$ is integrable in $\Omega$. Similarly, it can be shown that,

\begin{align*}
\frac{\gamma_\varepsilon^{- \frac{N-2s}{2}}}{d(x)^s} \int_{\Omega \setminus B_r(x_0)} G(x,y)u_\varepsilon^2(y)dy &\leq \frac{\gamma_\varepsilon^{- \frac{N-2s}{2}}}{d(x)^s} \int_{B_r(x_0)} u_\varepsilon^2(y) |x-y|^{N-s}dy \\
&\leq C \gamma_\varepsilon^{- \frac{(N-2s)(q-1)}{2}} \int_{\Omega \setminus B_r(x_0)} \frac{1}{|y-x_\varepsilon|^{(N-2s)q} |x-y|^{-N}}dy \\
&= o_{r,\varepsilon}(1).
\end{align*}

Furthermore $\frac{G(x,\gamma_\varepsilon)}{s(x)^r}$ is continuous in $\overline{\Omega} \setminus \{x\}$, (see 3.6 Lemma 6.5). Therefore, from 3.20 we obtain

\begin{equation}
\frac{w_\varepsilon(x)}{d(x)^s} = \gamma_\varepsilon^{- \frac{N-2s}{2}} G(x,x_0) \int_{B_r(x_0)} u_\varepsilon^{2s-1}dy + L + o_{r,\varepsilon}(1),
\end{equation}

where

\begin{align*}
L = \varepsilon \gamma_\varepsilon^{- \frac{N-2s}{2}} G(x,x_0) \int_{B_r(x_0)} u_\varepsilon^2dy.
\end{align*}
Doing a straightforward computation using (3.16), we have
\[ L \leq \varepsilon \gamma_0 \frac{(N+2) - q(N+2)}{2} \int_{B_\varepsilon(x_0)} G(x, x_0) \frac{d(x)^q}{d(x)^s} d(x)^s \int_{B_\varepsilon(x_0)} Z^q(y) dy \]
\[ \leq \varepsilon \gamma_0 \frac{(N+2) - q(N+2)}{2} \int_{\mathbb{R}^N} Z^q(y) dy \]

Thus, using Lemma 3.8, it is not difficult to check that \( L = o_{\varepsilon, r}(1) \). Define
\[ \gamma_0 = \lim_{r \to 0} \lim_{\varepsilon \to 0} \frac{N - 2s}{2} \int_{B_\varepsilon(0)} u_\varepsilon^2 - 1 dy. \]

This argument actually goes through for uniform convergence, i.e., we get
\[ \sup_{x \in \Omega \setminus B_r(x_0)} \left| \frac{w_\varepsilon(x)}{d(x)^s} - \gamma_0 \frac{G(x, x_0)}{d(x)^s} \right| \to 0. \]

Furthermore, note that for each fixed \( \varepsilon > 0 \), \( \sup_{x \in \Omega} |u_\varepsilon(x)| < C_\varepsilon \). Thus, from the definition of \( w_\varepsilon \), we obtain that for each fixed \( \varepsilon > 0 \), RHS of (3.13) is in \( L^\infty(\Omega) \).

Hence for each fixed \( \varepsilon > 0 \), applying [28, Theorem 1.2] we have \( \frac{d}{dx} \in C^\alpha(\Omega) \), for some \( \alpha \in (0, 1) \). On the other hand, from [9, Lemma 6.5], it follows that \( \frac{G(x, x_0)}{d(x)^s} \) is continuous up to \( \partial \Omega \). Hence, a straightforward elementary analysis yields
\[ \sup_{x \in \Omega \setminus B_r(x_0)} \left| \frac{w_\varepsilon(x)}{d(x)^s} - \gamma_0 \frac{G(x, x_0)}{d(x)^s} \right| = \sup_{x \in \Omega \setminus B_r(x_0)} \left| \frac{w_\varepsilon(x)}{d(x)^s} - \gamma_0 \frac{G(x, x_0)}{d(x)^s} \right| \to 0. \]

Clearly, \( \gamma_0 \) is positive as
\[ \gamma_0 \geq \lim_{r \to 0} \lim_{\varepsilon \to 0} \frac{N - 2s}{2} \int_{B_\varepsilon(0)} u_\varepsilon^2 - 1 dy \]
\[ \geq \lim_{r \to 0} \lim_{\varepsilon \to 0} \int_{B_\varepsilon(0)} u_\varepsilon^2 dy \]
\[ \geq A. \]

This completes the proof. \( \square \)

Lemma 3.8. Let \( u_\varepsilon \) be as in Theorem 1.2 and \( \gamma_\varepsilon \) be as defined in (3.5). Define
\[ \gamma_0 := \lim_{r \to 0} \lim_{\varepsilon \to 0} \frac{N - 2s}{2} \int_{B_\varepsilon(x_0)} u_\varepsilon^2 - 1 dy. \]

Then
\[ \gamma_0 = \frac{\omega_N c_{N,s} \Gamma \left( \frac{N}{2} \right) \Gamma \left( s \right)}{2 \Gamma \left( \frac{N - 2s}{2} \right)}, \]

where \( c_{N,s} \) is as defined in (1.12).

Proof. We define \( I_{\varepsilon, r} := \gamma_\varepsilon \frac{N - 2s}{2} \int_{B_\varepsilon(x_0)} u_\varepsilon^2 - 1 dy \). Using (3.8), we obtain \( u_\varepsilon(x) = \gamma_\varepsilon \frac{N - 2s}{2} \varepsilon \left( \frac{x - x_0}{\varepsilon} \right) \). Thus
\[ I_{\varepsilon, r} = \gamma_\varepsilon \frac{N - 2s}{2} \frac{N - 2s + N}{2} \int_{B_\varepsilon(x_0)} \frac{z_\varepsilon^2 - 1}{x_\varepsilon^2} dx = \int_{B_\varepsilon(x_0)} \frac{z_\varepsilon^2 - 1}{x_\varepsilon^2} dx. \]
Note that, $\varepsilon \to 0$ implies $\gamma_\varepsilon \to 0$. Therefore,

$$
\gamma_0 = \lim_{r \to 0} \lim_{\varepsilon \to 0} I_{\varepsilon,r} = \int_{\mathbb{R}^N} Z^{2^*-1} \, dx,
$$

where $Z$ is as in Lemma 3.3. Hence, by doing a straightforward computation, we obtain

$$
\gamma_0 = \frac{\omega_N c_{N,s}^2}{2} B \left( \frac{N}{2}, s \right),
$$

where $B(a, b) = \int_0^\infty t^{a-1} (1 + t)^{-a-b} \, dt$ is the Beta function, $c_{N,s}$ is as defined in (3.12) and $\omega_N$ is the surface measure of unit ball. Recall that $B(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)}$.

Thus $B \left( \frac{N}{2}, s \right) = \frac{\Gamma \left( \frac{N}{2} \right) \Gamma(s)}{\Gamma \left( \frac{N}{2} + s \right)}$ and the lemma follows.

**Proof of Theorem 1.2.** Applying (1.21) to $u_\varepsilon$ yields

$$
\Gamma(1+s)^2 \int_{\partial \Omega} \left( \frac{w_\varepsilon(x)}{d^a(x)} \right)^2 \delta(x-x_0, \nu) \, dS = 2\varepsilon \left( \frac{N-2}{q} - \frac{N}{q+1} \right) \int_{\Omega} \| u_\varepsilon \|_\infty^2 \, u_\varepsilon^{q+1} \, dx.
$$

Using $w_\varepsilon = \| u_\varepsilon \|_\infty u_\varepsilon$ in the above expression, we have

$$
\Gamma(1+s)^2 \int_{\partial \Omega} \left( \frac{w_\varepsilon(x)}{d^a(x)} \right)^2 \delta(x-x_0, \nu) \, dS = 2\varepsilon \left( \frac{N-2}{q} - \frac{N}{q+1} \right) \| u_\varepsilon \|_\infty^2 \int_{\Omega} \| u_\varepsilon \|^2 \, u_\varepsilon^{q+1} \, dx.
$$

Thanks to Lemma 3.7, applying dominated convergence theorem, we have

$$
\lim_{\varepsilon \to 0} \Gamma(1+s)^2 \int_{\partial \Omega} \left( \frac{w_\varepsilon(x)}{d^a(x)} \right)^2 \delta(x-x_0, \nu) \, dS = \gamma_0^2 \Gamma(1+s)^2 \int_{\partial \Omega} \left( \frac{G(x, x_0)}{d^a(x)} \right)^2 \delta(x-x_0, \nu) \, dS.
$$

Moreover, using the relations (3.8) and (3.5), the RHS of (3.26) reduces to

$$
\text{RHS of (3.26)} = 2\varepsilon \left( \frac{N-2}{q} - \frac{N}{q+1} \right) \| u_\varepsilon \|_\infty^2 \gamma_0^2 \Gamma(1+s)^2 \int_{\Omega} \| u_\varepsilon \|^2 \, u_\varepsilon^{q+1} \, dx
$$

\begin{equation}
= 2\varepsilon \left( \frac{N-2}{q} - \frac{N}{q+1} \right) \| u_\varepsilon \|_\infty^2 \gamma_0^2 \int_{\Omega} \| u_\varepsilon \|^2 \, u_\varepsilon^{q+1} \, dx.
\end{equation}

Since $z_\varepsilon \to Z$ a.e and $z_\varepsilon \leq CZ$, by the dominated convergence theorem it follows

$$
\int_{\Omega} z_\varepsilon^{q+1} \, dx \to \int_{\mathbb{R}^N} Z^{q+1} \, dx.
$$

We substitute back (3.28) into (3.26) and take the limit $\varepsilon \to 0$. Therefore, using (3.27) we obtain

\begin{equation}
\lim_{\varepsilon \to 0} \varepsilon \| u_\varepsilon \|_\infty^2 \gamma_0^2 \Gamma(1+s)^2 \int_{\partial \Omega} \left( \frac{G(x, x_0)}{d^a(x)} \right)^2 \delta(x-x_0, \nu) \, dS
\end{equation}

\begin{equation}
= \frac{\gamma_0^2 \Gamma(1+s)^2}{2} \left( \frac{N-2}{q} - \frac{N}{q+1} \right) \int_{\mathbb{R}^N} Z^{q+1} \, dx.
\end{equation}

From Lemma 3.3 we know $Z(x) = \left( 1 + \frac{|x|^2}{\mu_{N,s}} \right)^{-\frac{N-2}{2}}$, where $\mu_{N,s} = \frac{4}{N-2}$. Thus, a straightforward calculation yields

$$
\int_{\mathbb{R}^N} Z^{q+1} \, dx = \frac{c_{N,s}^2 \omega_N}{2} B \left( \frac{N}{2}, \left( \frac{N-2}{2} q - s \right) \right).
$$
From Lemma 3.8, it is known that 
\[ \gamma_0 = \frac{\omega N c^2 N,s}{2} \left( \frac{\Gamma(N)}{\Gamma(N-2s)} \right). \]
Substituting the value of \( \gamma_0 \) and \( \int_{\mathbb{R}^N} Z^{s+1} dx \) in (3.29) we have,
\[
\lim_{\varepsilon \to 0} \varepsilon \| u_\varepsilon \|_{\infty}^{\frac{(N-2s)+q-N(q-s)}{2}} = \frac{\omega N c^2 N,s}{2} \frac{(q+1) R_{N,s,x_0}}{q(N-2s)-(N+2s)} \varepsilon^2 \Gamma(s)^2 B \left( \frac{N}{2}, \frac{N-2s}{q} \right) \]
\[
\times B \left( \frac{N}{2}, \frac{(N-2s)}{q} \right)^{-1}.
\]
\[\square\]

4. Uniqueness result for \( p = 2^* - 1 \)

**Proof of Theorem 1.3** We break the proof into few steps.

**Step 1:** Let \( u_\varepsilon \) and \( v_\varepsilon \) be two solutions of (1.2) with
\[
\max_{\Omega} u_\varepsilon = \max_{\Omega} v_\varepsilon.
\]
Let \( \gamma_\varepsilon \) be as in (3.3). Then by the assumptions of the theorem, we have
\[
\gamma_\varepsilon = \| u_\varepsilon \|_{L^{\infty}^{N-2s}} = u_\varepsilon(0)^{-\frac{2}{N-2s}} = \| v_\varepsilon \|_{L^{\infty}^{N-2s}} = v_\varepsilon(0)^{-\frac{2}{N-2s}}.
\]
Note that, by Lemma 3.1 we have \( \gamma_\varepsilon \to 0 \) as \( \varepsilon \to 0 \). Define,
\[
\theta_\varepsilon(x) = u_\varepsilon(\gamma_\varepsilon x) - v_\varepsilon(\gamma_\varepsilon x), \quad x \in \Omega_{\varepsilon} = \frac{\Omega}{\gamma_\varepsilon},
\]
and
\[
\psi_\varepsilon(x) = \frac{\theta_\varepsilon(x)}{\| \theta_\varepsilon \|_{L^{\infty}(\Omega_{\varepsilon})}} = \frac{\theta_\varepsilon(x)}{\| u_\varepsilon - v_\varepsilon \|_{L^{\infty}(\Omega)}}.
\]
Therefore,
\[
(-\Delta)^s \psi_\varepsilon = \frac{\gamma_\varepsilon^{2s}}{\| u_\varepsilon - v_\varepsilon \|_{L^{\infty}(\Omega)}} \left[ (u_\varepsilon^p(\gamma_\varepsilon x) - v_\varepsilon^p(\gamma_\varepsilon x)) - \varepsilon (u_\varepsilon^q(\gamma_\varepsilon x) - v_\varepsilon^q(\gamma_\varepsilon x)) \right].
\]
It is easy to see that,
\[
u_\varepsilon^p(\gamma_\varepsilon x) - v_\varepsilon^p(\gamma_\varepsilon x) = p \int_0^1 \left( t u_\varepsilon(\gamma_\varepsilon x) + (1-t) v_\varepsilon(\gamma_\varepsilon x) \right)^{p-1} \theta_\varepsilon(x) dt.
\]
Using the fact that \( p = 2^* - 1 = \frac{N+2s}{2} \) and \( \gamma_\varepsilon^{2s} = \| u_\varepsilon \|_{L^{\infty}(\Omega)}^{-(p-1)} = \| v_\varepsilon \|_{L^{\infty}(\Omega)}^{-(p-1)} \), a straight forward computation yields
\[
\begin{cases}
(-\Delta)^s \psi_\varepsilon = \left( c_\varepsilon^1(x) - \varepsilon c_\varepsilon^2(x) \right) \psi_\varepsilon & \text{in } \Omega_{\varepsilon}, \\
\psi_\varepsilon = 0 & \text{in } \mathbb{R}^N \setminus \Omega_{\varepsilon},
\end{cases}
\]
where
\[
c_\varepsilon^1(x) = p \int_0^1 \left[ \frac{u_\varepsilon(\gamma_\varepsilon x)}{\| u_\varepsilon \|_{L^{\infty}(\Omega)}} + (1-t) \frac{v_\varepsilon(\gamma_\varepsilon x)}{\| v_\varepsilon \|_{L^{\infty}(\Omega)}} \right]^{p-1} dt,
\]
\[
c_\varepsilon^2(x) = q \int_0^1 \left[ \frac{u_\varepsilon(\gamma_\varepsilon x)}{\| u_\varepsilon \|_{L^{\infty}(\Omega)}} + (1-t) \frac{v_\varepsilon(\gamma_\varepsilon x)}{\| v_\varepsilon \|_{L^{\infty}(\Omega)}} \right]^{q-1} dt.
\]
Here we observe that, \( \frac{u_\varepsilon(x)}{|u_\varepsilon|_{L^\infty(\Omega)}} = z_\varepsilon(x) \), where \( z_\varepsilon \) is as defined in (3.8) (since here \( x_\varepsilon = 0 \)). Consequently, using Lemma 3.3 and (3.12), we obtain

\[
\begin{aligned}
(4.4) \quad \frac{u_\varepsilon(\gamma_\varepsilon x)}{|u_\varepsilon|_{L^\infty(\Omega)}} & \rightarrow Z \quad \text{in } C^s_{\text{loc}}(\mathbb{R}^N) \quad \text{and} \quad \frac{u_\varepsilon(\gamma_\varepsilon x)}{|u_\varepsilon|_{L^\infty(\Omega)}} \leq \frac{C}{(1 + |x|^2)^{\frac{N-2s}{2}}},
\end{aligned}
\]

where \( Z \) is the solution of (1.13) with \( Z(0) = 1 \) and \( 0 < Z \leq 1 \). Hence \( Z(x) = (1 + \frac{|x|^2}{\mu_{N,s}})^{-\frac{N-2s}{2}} \), where \( \mu_{N,s} = c_{N,s}^\ast \), (see Lemma 3.3). As a consequence, thanks to Lemma 3.3(i), from (4.2) and (4.3) we have

\[
(4.5) \quad c_\varepsilon^1(x) \rightarrow \left( \frac{N + 2s}{N - 2s} \right) \frac{1}{(1 + \frac{|x|^2}{\mu_{N,s}})^{2s}} \quad \text{and} \quad \varepsilon c_\varepsilon^2(x) \rightarrow 0,
\]

uniformly on compact subsets of \( \mathbb{R}^N \). Applying Schauder estimates 28 to the equation (1.1), it follows there exists \( \psi \in C^s(\mathbb{R}^N) \) such that \( \psi_\varepsilon \rightarrow \psi \) in \( C^s_{\text{loc}}(\mathbb{R}^N) \).

Since, from Remark 1.2, we have \( \psi_\varepsilon \) is radially symmetric, we obtain \( \psi \) is radially symmetric too. Passing to the limit in (4.10) (as in Lemma 3.3) yields

\[
(4.6) \quad \begin{cases}
(-\Delta)^s \psi = \left( \frac{N + 2s}{N - 2s} \right) \frac{\psi}{(1 + \frac{|x|^2}{\mu_{N,s}})^{2s}} \quad \text{in } \mathbb{R}^N, \\
||\psi||_{L^\infty(\mathbb{R}^N)} \leq 1.
\end{cases}
\]

**Step 2:** In this step, we will prove that \( \psi \in D^{s,2}(\mathbb{R}^N) \).

Since \( \psi_\varepsilon \in H^s(\Omega_\varepsilon) \), \( \psi_\varepsilon = 0 \) in \( \mathbb{R}^N \setminus \Omega_\varepsilon \) and \( u_\varepsilon, v_\varepsilon = 0 \) in \( \mathbb{R}^N \setminus \Omega \), taking \( \psi_\varepsilon \) as a test function in (4.1), we have

\[
(4.7) \quad ||\psi_\varepsilon||_{D^{s,2}(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} c_\varepsilon^1(x)\psi_\varepsilon^2 dx - \varepsilon \int_{\mathbb{R}^N} c_\varepsilon^2(x)\psi_\varepsilon^2 dx \leq \int_{\Omega_\varepsilon} c_\varepsilon^1(x)\psi_\varepsilon^2 dx.
\]

Thus applying the Sobolev inequality, we have

\[
(4.8) \quad S \left( \int_{\Omega_\varepsilon} ||\psi_\varepsilon||^2 dx \right)^{\frac{2}{N}} \leq \int_{\Omega_\varepsilon} c_\varepsilon^1(x)\psi_\varepsilon^2 dx.
\]

Let us fix \( \delta > 0 \), will be chosen later. Since \( ||\psi_\varepsilon||_{L^\infty(\Omega_\varepsilon)} = 1 \), Hölder inequality yields

\[
(4.9) \quad \int_{\Omega_\varepsilon} c_\varepsilon^1(x)\psi_\varepsilon^2 dx \leq \int_{\Omega_\varepsilon} c_\varepsilon^1(x)\psi_\varepsilon^2 dx \leq \left( \int_{\Omega_\varepsilon} ||\psi_\varepsilon||^2 dx \right)^{\frac{2-\delta}{2}} \left( \int_{\Omega_\varepsilon} |c_\varepsilon^1|^{\frac{2(2-\delta)}{2-\delta}} dx \right)^{\frac{2-\delta}{2}}.
\]

Combining (4.8) and (4.9) we have

\[
\int_{\Omega_\varepsilon} ||\psi_\varepsilon||^2 dx \leq \left( \int_{\Omega_\varepsilon} |c_\varepsilon^1|^{\frac{2}{2-\delta}} dx \right)^{\frac{2-\delta}{2}} \leq C \left( \int_{\Omega_\varepsilon} \left[ \frac{1}{(1 + \frac{|x|^2}{\mu_{N,s}})^{2s}} \right]^{\frac{2}{2-\delta}} dx \right)^{\frac{2-\delta}{2}}
\]

\[
(4.10) \quad \leq C,
\]

for some constant \( C > 0 \), if we choose \( \delta < \frac{4}{N-2s} \). For this choice of \( \delta \), substituting back (4.10) into (4.1) yields

\[
\int_{\Omega_\varepsilon} c_\varepsilon^1(x)\psi_\varepsilon^2 dx \leq C. \quad \text{As a result, from (4.7) we have}
\]
Claim: For $\psi$ which implies

$$\liminf_{\varepsilon \to 0} ||\psi\varepsilon||_{D^{s,2}(\mathbb{R}^N)} \leq C,$$

which implies $\psi \in D^{s,2}(\mathbb{R}^N)$.

**Step 3:** In this step we will establish that

$$\tag{4.11} |\psi\varepsilon(x)| \leq \frac{C}{|x|^{N-2s}}, \quad x \in \Omega_{\varepsilon} \setminus B_r(0),$$

for $\varepsilon > 0$ small enough and for some constant $C > 0$ and $r > 0$ independent of $\varepsilon$.

To prove this step, define $\hat{\psi}\varepsilon$ as the Kelvin transform of $\psi\varepsilon$, that is,

$$\hat{\psi}\varepsilon(x) = \frac{1}{|x|^{N-2s}} \psi\varepsilon\left(\frac{x}{|x|^2}\right), \quad x \in \Omega_{\varepsilon} \setminus \{0\}.$$

Let $\Omega^*_\varepsilon$ be the image $\Omega_{\varepsilon}$ under the Kelvin transform. Since

$$(-\Delta)^s \hat{\psi}\varepsilon(x) = \frac{1}{|x|^{N+2s}} (-\Delta)^s \psi\varepsilon\left(\frac{x}{|x|^2}\right),$$

doing a straightforward computation we obtain,

$$\tag{4.12} \begin{cases} (-\Delta)^s \hat{\psi}\varepsilon = \frac{1}{|x|^{4s}} \left( c_1^s \left( \frac{x}{|x|^2} \right) - \varepsilon c_2^s \left( \frac{x}{|x|^2} \right) \right) \hat{\psi}\varepsilon \quad \text{in} \quad \Omega^*_\varepsilon, \\ \hat{\psi}\varepsilon = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega^*_\varepsilon. \end{cases}$$

We set,

$$a_\varepsilon(x) := \frac{1}{|x|^{4s}} \left( c_1^s \left( \frac{x}{|x|^2} \right) - \varepsilon c_2^s \left( \frac{x}{|x|^2} \right) \right).$$

Thus, (4.12) reduces to

$$(-\Delta)^s \hat{\psi}\varepsilon = a_\varepsilon(x) \hat{\psi}\varepsilon \quad \text{in} \quad \Omega^*_\varepsilon.$$

Claim: For $N > 4s$, the function $a_\varepsilon \in L^t(\Omega^*_\varepsilon)$, for some $t > \frac{N}{2s}$.

Assuming the claim, let us first complete the proof of step 3. Thanks to the above claim, using Moser iteration technique in the spirit of the proof of [3] Theorem 1.1 (see also [31] and [33] Lemma B.3)], it can be shown that

$$\sup_{\Omega^*_\varepsilon \cap B_t(0)} |\hat{\psi}\varepsilon| \leq C \left( \int_{\Omega^*_\varepsilon \cap B_t(0)} |\hat{\psi}\varepsilon|^2 \right)^{\frac{1}{2}}.$$

Moreover,

$$\int_{\Omega^*_\varepsilon \cap B_t(0)} |\hat{\psi}\varepsilon|^2 \leq \int_{\Omega^*_\varepsilon} |\hat{\psi}\varepsilon|^2 = \int_{\Omega_{\varepsilon}} |\psi\varepsilon|^2 \leq C.$$

The last inequality is due to (4.10). Hence $\sup_{\Omega^*_\varepsilon \cap B_t(0)} |\hat{\psi}\varepsilon| \leq C$. This in turn implies,

$$|\psi\varepsilon(x)| \leq \frac{C}{|x|^{N-2s}}, \quad x \in \Omega_{\varepsilon} \setminus B_r(0),$$

for $\varepsilon > 0$ small enough and for some constant $C > 0$ and $r > 0$.

Now, let us prove the claim.

Using (4.4), it is easy to see that $\frac{1}{|x|^{4s}} c_1^s \left( \frac{x}{|x|^2} \right) \leq \frac{C}{(\mu_{N, s}^a + |x|^2)}$. Hence for $t > \frac{N}{2s}$,

$$\tag{4.13} \int_{\Omega^*_\varepsilon} \frac{1}{|x|^{4st}} c_1^s \left( \frac{x}{|x|^2} \right) dx \leq C \int_{\mathbb{R}^N} \frac{dx}{(\mu_{N, s}^a + |x|^2)^{2st}} < \infty.$$
On the other hand, \( \|u_\varepsilon \|_{L^\infty(\Omega)} \) ≤ 1 implies \( |\varepsilon c_\varepsilon^2| \leq \varepsilon c_\varepsilon^{(N+2s)+(N-2s)} \) . Note that, boundedness of \( \Omega \) implies there exists \( R > 0 \) such that \( \Omega \subseteq B_R(0) \). Hence \( \Omega_\varepsilon \subseteq B_{\frac{R}{\varepsilon}}(0) \) and \( \Omega_\varepsilon^* \subseteq \mathbb{R}^N \setminus B_{\frac{R}{\varepsilon}}(0) \). Therefore,

\[
\int_{\Omega_\varepsilon^*} \frac{1}{|x|^{4st}} \varepsilon c_\varepsilon^2 (\frac{x}{|x|^2})^t \, dx \leq C \left( \varepsilon c_\varepsilon \right)^{(N+2s)+(N-2s)} \int_{\Omega_\varepsilon^*} \frac{dx}{|x|^{4st}} \leq C \left( \varepsilon c_\varepsilon \right)^{(N+2s)+(N-2s)} \left( \frac{\gamma_\varepsilon}{R} \right)^{N-4st} \tag{4.14} \]

Since \( p = 2^* - 1 \), from Theorem 1.2 it follows that \( \varepsilon \| u_\varepsilon \|_{L^\infty(\Omega)} \) \( \frac{q(N-2s)+N-6s}{N-2s} \) = \( C' \), that is, \( \varepsilon c_\varepsilon^{(N-6s)+N} \) = \( C' \). As a result,

\[
\text{RH}\, \text{S of (4.14)} \leq C \varepsilon c_\varepsilon^{t(N-6s)+N}. \tag{4.15} \]

Clearly, \( N \geq 6s \) implies \( \varepsilon c_\varepsilon^{t(N-6s)+N} \leq C \) for some constant \( C > 0 \). If \( 4s < N < 6s \), then choose \( t \in \left( \frac{N}{N-6s}, \frac{N}{N-2s} \right) \) to get \( t(N-6s) + N \geq 0 \).

Hence, combining (4.13) and (4.15) the claim follows.

**Step 4:** Thanks to [15, Theorem 1.1], the linear space of solutions to equation (4.6) can be spanned by the following \((N+1)\) functions:

\[
\psi_i(x) = \frac{2x_i}{(1 + \frac{|x|^2}{\mu_{N,s}})^{\frac{N-2s+2}{2}}} \quad i = 1, \ldots, N
\]

and

\[
\psi_{N+1}(x) = \frac{1 - |x|^2}{(1 + \frac{|x|^2}{\mu_{N,s}})^{\frac{N-2s+2}{2}}}. \]

That is, general solution of (4.6) can be written as

\[
\psi(x) = \alpha \frac{1 - |x|^2}{(1 + \frac{|x|^2}{\mu_{N,s}})^{\frac{N-2s+2}{2}}} + \sum_{i=1}^{N} \beta_i \frac{2x_i}{(1 + \frac{|x|^2}{\mu_{N,s}})^{\frac{N-2s+2}{2}}},
\]

where \( \alpha, \beta_i \in \mathbb{R} \). Since \( \psi \) is a symmetric function, each \( \beta_i = 0 \).

**Step 5:** In this step we will prove that \( \alpha = 0 \).

Suppose \( \alpha \neq 0 \). We aim to get a contradiction. For simplicity of the calculation, we can take \( \alpha = 1 \) and \( \mu_{N,s} = 1 \), that is,

\[
\psi(x) = \frac{1 - |x|^2}{(1 + |x|^2)^{\frac{N-2s+2}{2}}}. \tag{4.16} \]

Let \( \Omega' \) be any neighbourhood of \( \partial \Omega \), not containing the origin.

**Claim:** \( \|u_\varepsilon\|_{L^\infty(\Omega)} \frac{||u_\varepsilon(x) - v_\varepsilon(x)||_{L^\infty(\Omega)}^2}{||u_\varepsilon - v_\varepsilon\|_{L^\infty(\Omega)}^2} \rightarrow -c_0 \frac{G(x,0)}{\delta(x)^r} \) uniformly in \( \Omega' \),

for some constant \( c_0 > 0 \).
Indeed, 

\[
(-\Delta)^t \left( \frac{|u_\varepsilon(x)-v_\varepsilon(x)|^2}{L^2(\Omega)} \right) = \frac{||u_\varepsilon||^2_{L^2(\Omega)}}{||u_\varepsilon-v_\varepsilon||_{L^2(\Omega)}} \left( (u_\varepsilon^p - v_\varepsilon^p) - \varepsilon(u_\varepsilon^q - v_\varepsilon^q) \right) \\
= \frac{||u_\varepsilon||^2_{L^2(\Omega)}}{||u_\varepsilon-v_\varepsilon||_{L^2(\Omega)}} (d_\varepsilon^1(x) - \varepsilon d_\varepsilon^2(x)) (u_\varepsilon - v_\varepsilon)
\]

(4.17)

where

\[
d_\varepsilon^1(x) = p \int_0^1 (tu_\varepsilon(x) + (1-t)v_\varepsilon(x))^{p-1} dt
\]

and

\[
d_\varepsilon^2(x) = q \int_0^1 (tu_\varepsilon(x) + (1-t)v_\varepsilon(x))^{q-1} dt.
\]

Note that

\[
d_\varepsilon^1(\gamma_\varepsilon x) = \gamma_\varepsilon^{-2s} c_\varepsilon^1(x) \quad \text{and} \quad d_\varepsilon^2(\gamma_\varepsilon x) = \gamma_\varepsilon^{-2s} c_\varepsilon^2(x).
\]

Therefore, using (4.17), we have

\[
d_\varepsilon^1(x) \leq C \gamma_\varepsilon^{-2s} \frac{1}{(\mu N,s + |\gamma_\varepsilon|^2)^{2s}} \leq C \frac{\gamma_\varepsilon^{-2s}}{|x|^{4s}}.
\]

(4.18)

\[
d_\varepsilon^2(x) \leq C \frac{\gamma_\varepsilon^{-2s}}{(\mu N,s + |\gamma_\varepsilon|^2)^{(N-2s)/2} + 1} \leq C \frac{\gamma_\varepsilon^{(N-2s) - N}}{|x|^{(N-2s)(q-1)}}
\]

(4.19)

**Subclaim 1:** \( \lim_{\varepsilon \to 0} f_{\varepsilon}(x) = 0 \quad \forall \ x \in \Omega'. \)

As \( \gamma_\varepsilon \to 0 \), using (4.17), (4.18) and (4.19), for \( x \in \Omega' \) we obtain

\[
f_{\varepsilon}(x) = \frac{||u_\varepsilon||^2_{L^2(\Omega)}}{|x|^{2s}} (d_\varepsilon^1(\gamma_\varepsilon x)) (d_\varepsilon^2(x) - \varepsilon d_\varepsilon^2(x))
\]

\[
\leq C ||u_\varepsilon||^2_{L^2(\Omega)} \frac{1}{|x|^{N-2s}} \left( d_\varepsilon^1(x) + \varepsilon d_\varepsilon^2(x) \right)
\]

\[
\leq \frac{C}{|x|^{N-2s}} \left( \gamma_\varepsilon^{2s} + \gamma_\varepsilon^{(N-2s) - N} \right)
\]

\[
\to 0,
\]

since \( q > \frac{N+2s}{N-2s} \).

**Subclaim 2:** \( \lim_{\varepsilon \to 0} \int_{\Omega} f_{\varepsilon}(x) dx = -c_0 \), for some constant \( c_0 > 0 \).

To see this,

\[
\int_{\Omega} f_{\varepsilon}(x) dx = \frac{||u_\varepsilon||^2_{L^2(\Omega)}}{|x|^{N-2s}} \int_{\Omega} d_\varepsilon^1(x) (u_\varepsilon - v_\varepsilon) dx
\]

\[
- \frac{||u_\varepsilon||^2_{L^2(\Omega)}}{|x|^{N-2s}} \int_{\Omega} \varepsilon d_\varepsilon^2(x) (u_\varepsilon - v_\varepsilon) dx
\]

\[
= \int_{\Omega} c_\varepsilon^1(y) \psi_\varepsilon(y) dy - \varepsilon \int_{\Omega} c_\varepsilon^2(y) \psi_\varepsilon(y) dy.
\]
In the last step, we have used the change of variable $x = \gamma \varepsilon y$. Using (4.5) and (4.16) via dominated convergence theorem, we obtain

\[
\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \frac{1}{(|x|^{2s} + 1)} \, dx.
\]

Using change of variable the RHS of the above equality can be computed as follows:

\[
\int_{\Omega_\varepsilon} \frac{1}{(|x|^{2s} + 1)} \, dx = \omega_N \int_0^1 \left(1 - r^2\right) \frac{(1 - r^2)(1 - r^{N-2s})}{(1 + r^2)^{\frac{N+2s}{2}}} \, dr.
\]

As $s > 0$, $\int_0^1 \frac{(1 - r^2)(1 - r^{N-2s})}{(1 + r^2)^{\frac{N+2s}{2}}} \, dr \leq \int_0^1 r^{2s-1} \, dr < \infty$. Hence from (4.20), we get

\[
\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \frac{1}{(|x|^{2s} + 1)} \, dx = -c_0,
\]

for some $c_0 > 0$. Similarly it can be shown that

\[
|\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \frac{1}{(|x|^{2s} + 1)} \, dx| < \infty.
\]

Therefore,

\[
\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \frac{1}{(|x|^{2s} + 1)} \, dx = 0.
\]

Combining (4.22) and (4.23), Subclaim 2 follows.

Now we get back to (4.17). Define,

\[
\phi_\varepsilon(x) := \|u_\varepsilon\|_{L^\infty(\Omega)} \frac{u_\varepsilon(x) - v_\varepsilon(x)}{\|u_\varepsilon - v_\varepsilon\|_{L^\infty(\Omega)}}.
\]

Then $\phi_\varepsilon$ satisfies

\[
\begin{cases}
(-\Delta)^s \phi_\varepsilon = f_\varepsilon & \text{in } \Omega, \\
\phi_\varepsilon = 0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}
\]

Then for any $r > 0$ small and $x \in \Omega'$, we have

\[
\frac{\phi_\varepsilon(x)}{d^s(x)} = \int_{\Omega} G(x, y) f_\varepsilon(y) \frac{d^s(x)}{d^s(y)} \, dy = \int_{\Omega \setminus B_r(0)} G(x, y) f_\varepsilon(y) \frac{d^s(x)}{d^s(y)} \, dy + \int_{B_r(0)} G(x, y) f_\varepsilon(y) \frac{d^s(x)}{d^s(y)} \, dy.
\]

Using Lemma (5.3) and Subclaim 1, we estimate the 2nd term on RHS as follows:

\[
\left| \int_{\Omega \setminus B_r(0)} G(x, y) f_\varepsilon(y) \frac{d^s(x)}{d^s(y)} \, dy \right| \leq C \int_{\Omega \setminus B_r(0)} \frac{|f_\varepsilon(y)|}{|x - y|^{N+s}} \, dy = o_\varepsilon(1),
\]

\[
\left| \int_{B_r(0)} G(x, y) f_\varepsilon(y) \frac{d^s(x)}{d^s(y)} \, dy \right| \leq C r^{s-N} \int_{B_r(0)} |f_\varepsilon(y)| \, dy = o_\varepsilon(1).
\]
Thus, it follows

This proves the claim.

Note that by Theorem 1.2, \( \lim_{\varepsilon \to 0} \varepsilon \). Applying the change of variable as in (4.21), it can be proved that

\[
\lim_{\varepsilon \to 0} \frac{\phi_{\varepsilon}(x)}{d^{s}(x)} = \frac{G(x, 0)}{d^{s}(x)} \lim_{r \to 0} \int_{B_{r}(0)} f_{\varepsilon}(y)dy.
\]

Moreover, by Subclaim 2,

\[
\lim_{r \to 0} \lim_{\varepsilon \to 0} \frac{G(x, 0)}{d^{s}(x)} \int_{B_{r}(0)} f_{\varepsilon}(y)dy = -c_{0} \frac{G(x, 0)}{d^{s}(x)}.
\]

Thus, it follows

\[
\lim_{\varepsilon \to 0} \frac{\phi_{\varepsilon}(x)}{d^{s}(x)} = -c_{0} \frac{G(x, 0)}{d^{s}(x)}.
\]

This proves the claim.

In order to complete the proof of Step 5, we apply the Pohozaev identity (1.20) to \( u_{\varepsilon} \) and \( v_{\varepsilon} \).

\[
\Gamma(1 + s)^{2} \int_{\partial \Omega} \left( \frac{u_{\varepsilon}(x)}{d^{s}(x)} \right)^{2} (x \cdot \nu) dS = \varepsilon \left( (N - 2s) - \frac{2N}{q + 1} \right) \int_{\Omega} u_{\varepsilon}^{q+1} dx,
\]

\[
\Gamma(1 + s)^{2} \int_{\partial \Omega} \left( \frac{v_{\varepsilon}(x)}{d^{s}(x)} \right)^{2} (x \cdot \nu) dS = \varepsilon \left( (N - 2s) - \frac{2N}{q + 1} \right) \int_{\Omega} v_{\varepsilon}^{q+1} dx.
\]

Subtracting one from the other and multiplying by \( \frac{|u_{\varepsilon}|^{2}_{L^{\infty}(\Omega)}}{|u_{\varepsilon} - v_{\varepsilon}|_{L^{\infty}(\Omega)}d^{s}(x)} \) in both sides yields,

\[
\Gamma(1 + s)^{2} \int_{\partial \Omega} \frac{|u_{\varepsilon}|^{2}_{L^{\infty}(\Omega)}(u_{\varepsilon} - v_{\varepsilon}) (u_{\varepsilon} + v_{\varepsilon}) |u_{\varepsilon}|_{L^{\infty}(\Omega)}d^{s}(x)}{(x \cdot \nu) dS}
\]

\[
= \varepsilon \left( (N - 2s) - \frac{2N}{q + 1} \right) (q + 1) \int_{\Omega} \frac{|u_{\varepsilon}^{q+1}(u_{\varepsilon} - v_{\varepsilon})}{|u_{\varepsilon} - v_{\varepsilon}|_{L^{\infty}(\Omega)}} (u_{\varepsilon} - v_{\varepsilon}) \int_{0}^{1} (t u_{\varepsilon} + (1 - t) v_{\varepsilon})^{q} dt dx.
\]

By doing the change of variable \( x = \gamma_{\varepsilon}y \), RHS of (4.27) reduces as

\[
\text{RHS of (4.27)} = \varepsilon |u_{\varepsilon}|^{q+2} \left[ q(N - 2s) - (N + 2s) \right]
\]

\[
\times \int_{\Omega} \psi_{\varepsilon}(y) \left[ \int_{0}^{1} \left( t \frac{u_{\varepsilon}(\gamma_{\varepsilon})(q)}{|u_{\varepsilon}||\nu_{\varepsilon}|_{L^{\infty}(\Omega)}} + (1 - t) \frac{v_{\varepsilon}(\gamma_{\varepsilon})}{|v_{\varepsilon}||\nu_{\varepsilon}|_{L^{\infty}(\Omega)}} \right)^{q} dt \right] dy.
\]

Note that by Theorem 1.2 \( \lim_{\varepsilon \to 0} \varepsilon |u_{\varepsilon}|^{q+2} \left[ q(N - 2s) - (N + 2s) \right] = C_{1} \), for some constant \( C_{1} > 0 \). Therefore, using dominated convergence theorem via (4.4) and (4.16), we obtain

\[
\lim_{\varepsilon \to 0} \text{RHS of (4.27)} = C_{1} \int_{\mathbb{R}^{N}} \frac{1 - |x|^{2}}{(1 + |x|^{2})^{N/2}} dx.
\]

Applying the change of variable as in (4.21), it can be proved that

\[
\int_{\mathbb{R}^{N}} \frac{1 - |x|^{2}}{(1 + |x|^{2})^{N/2}} dx = \omega_{N} \int_{0}^{1} \frac{r^{N-1}(1 - r^{2})(1 - r^{q(N-2s)-2(N+2s)})}{(1 + r^{2})^{(N-2s)/2 + 1}} dr = C_{2},
\]
where $C_2 > 0$ is a constant. Hence,

$$\lim_{\varepsilon \to 0} \left[ \text{RHS of } (4.27) \right] > 0.$$  

(4.29)

On the other hand, applying (1.24) and (4.26) to LHS via dominated convergence theorem, we get

$$\lim_{\varepsilon \to 0} \left[ \text{LHS of } (4.27) \right] = 2\Gamma(1 + s)2^\frac{\omega_N C_{N,s}^{2 \ast - 1}}{2} \frac{\Gamma(\frac{N}{2})\Gamma(s)\Gamma(1 + s)^2}{\Gamma(\frac{N + 2s}{2})} \int_{\partial\Omega} \left( \frac{G(x,0)}{d^p(x)} \right)^2 (x \cdot \nu)dS < 0.$$  

(4.30)

Combining (4.29) along with (4.30) gives the contradiction. Hence $\alpha = 0$ and step 5 follows.

**Step 6:** Step 5 implies that $\psi \equiv 0$. Therefore, by Step 1, $\psi_\varepsilon \to 0$ in $K$ for every compact set $K$ in $\mathbb{R}^N$. Let $y_\varepsilon \in \mathbb{R}^N$ such that $||\psi_\varepsilon||_{L^\infty(\Omega_\varepsilon)} = \frac{1}{||\psi_\varepsilon||_{L^\infty(\Omega_\varepsilon)}}$.

Since by definition of $\psi_\varepsilon$ it follows $||\psi_\varepsilon||_{L^\infty(\Omega_\varepsilon)} = 1$, we get

$$\psi_\varepsilon(y_\varepsilon) = 1.$$  

(4.31)

This in turn implies $y_\varepsilon \to \infty$ as $\varepsilon \to 0$. On the other hand, (4.11) yields that $\psi_\varepsilon(y_\varepsilon) \to 0$. This contradicts (4.31). Hence the uniqueness result follows. □

**Appendix A.**

Define

$$\tilde{F}(w) = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w(x) - w(y)|^2}{|x - y|^{N + 2s}} dx dy + \frac{1}{q + 1} \int w^{q + 1} dx,$$

where $q > p \geq 2^\ast - 1$. For $\rho > 0$, set

$$X_0(\rho \Omega) := \{ w \in H^s(\mathbb{R}^N) : w = 0 \text{ in } \mathbb{R}^N \setminus \rho \Omega \},$$

$$N_\rho = \{ w \in X_0(\rho \Omega) \cap L^{q + 1}(\rho \Omega) : \int_{\rho \Omega} w^{q + 1} dx = 1 \}.$$  

Define

$$S_\rho := \inf_{w \in N_\rho} \tilde{F}(w).$$

**Theorem A.1.** (i) If $p = 2^\ast - 1$, then $S_\rho \to \frac{\tilde{S}}{2}$ as $\rho \to \infty$, where $\tilde{S}$ is as defined in (1.10).

(ii) If $p > 2^\ast - 1$, then $S_\rho \to \mathcal{K}$ as $\rho \to \infty$, where $\mathcal{K}$ is as defined in (1.8).

**Proof.** **Step 1:** First we prove that $\lim_{\rho \to \infty} S_\rho \leq \frac{\tilde{S}}{2}$. Let us consider the function $U(x)$ defined as in (1.11). We know that $\mathcal{S}$ is achieved by $U$ and $U$ is the unique ground state solution of (1.13) with $\int_{\mathbb{R}^N} U^{2^\ast}(x)dx = 1$.

Define

$$U_\rho(x) := \rho \left( \frac{N - 2s}{4} \right) U \left( \frac{x}{\sqrt{\rho}} \right) \quad \text{and} \quad \phi_\rho(x) = \phi \left( \frac{x}{\rho} \right)$$
where \( \phi \in C_0^\infty(\mathbb{R}^N) \), \( \text{supp}(\phi) \subset \Omega \), and \( \phi \equiv 1 \) in \( \frac{\Omega}{2} \), \( 0 \leq \phi \leq 1 \), \( |\nabla \phi| \leq \frac{2}{\rho} \), where \( d = \text{diam}(\Omega) \). It is easy to see that \( U_\rho \) is also a solution of (1.13).

Set \( v_\rho(x) := U_\rho(x) \phi_\rho(x) \) and \( \hat{v}_\rho(x) = \frac{v_\rho}{|v_\rho|_{L^2(\rho \Omega)}} \). Then \( \hat{v}_\rho \in N_\rho \) and thus,

\[
(A.2) \quad S_\rho \leq \hat{F}(\hat{v}_\rho)
\]

Note that,

\[
(A.4) \quad \lim_{\rho \to \infty} \int_{\rho \Omega} v_\rho^2 \, dx = \rho^{-\frac{N}{2}} \int_{\rho \Omega} U^2 \left( \frac{x}{\sqrt{\rho}} \right) \phi^2 \left( \frac{x}{\sqrt{\rho}} \right) \, dx = \int_{\Omega} U^2(x) \phi^2 \left( \frac{x}{\sqrt{\rho}} \right) \, dx.
\]

Therefore,

\[
(A.3) \quad \lim_{\rho \to \infty} \int_{\rho \Omega} v_\rho^2 \, dx = \int_{\mathbb{R}^N} U^2(x) \, dx = 1.
\]

Similarly,

\[
(A.5) \quad \lim_{\rho \to \infty} \int_{\rho \Omega} \hat{v}_\rho^{q+1} \, dx = \lim_{\rho \to \infty} \rho^{\frac{N+2s}{q(N-2s)}} \int_{\mathbb{R}^N} U^{q+1}(x) \phi^{q+1} \left( \frac{x}{\sqrt{\rho}} \right) \, dx = 0,
\]

as \( q > \frac{N+2s}{N-2s} \). Hence, from (A.2),

\[
(A.6) \quad \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_\rho(x) - v_\rho(y)|^2}{|x - y|^{N+2s}} \, dx \, dy = I_1^1 + I_2^2 + I_3^3,
\]

where

\[
I_1^1 := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|U_\rho(x) - U_\rho(y)|^2}{|x - y|^{N+2s}} \phi_\rho^2(x) \, dy \, dx,
\]

\[ I_2^2 := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\phi_\rho(x) - \phi_\rho(y)|^2}{|x - y|^{N+2s}} U_\rho^2(y) \, dy \, dx,
\]

\[ I_3^3 := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(U_\rho(x) - U_\rho(y))(\phi_\rho(x) - \phi_\rho(y))U_\rho(y)\phi_\rho(x)}{|x - y|^{N+2s}} \, dx \, dy.
\]

A simple calculation yields

\[
(A.7) \quad \lim_{\rho \to \infty} I_1^1 = \lim_{\rho \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|U(x) - U(y)|^2}{|x - y|^{N+2s}} \phi^2 \left( \frac{x}{\sqrt{\rho}} \right) \, dy \, dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|U(x) - U(y)|^2}{|x - y|^{N+2s}} \, dy \, dx = S.
\]

Using change of variable, it is not difficult to see that

\[
(A.8) \quad I_2^2 = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} F_\rho(x, y) \, dy \, dx, \quad \text{where} \quad F_\rho(x, y) = \frac{|\phi \left( \frac{x}{\sqrt{\rho}} \right) - \phi \left( \frac{y}{\sqrt{\rho}} \right)|^2 U^2(x)}{|x - y|^{N+2s}}.
\]

Clearly, \( F_\rho(x, y) \to 0 \) pointwise as \( \rho \to \infty \). Using dominated convergence theorem, we aim to show that \( \lim_{\rho \to \infty} I_2^2 = 0 \). Let

\[
D_1 := \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : |x - y| \leq 1\},
\]

\[
D_2 := \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : |x - y| > 1\}.
\]
Thus,
\[ I_\rho^2 = \int_{D_1} F_\rho(x, y) dxdy + \int_{D_2} F_\rho(x, y) dxdy =: I_\rho^{2,1} + I_\rho^{2,2} \]

In \( D_1 \), we estimate \( F_\rho(x, y) \) as follows:
\[
F_\rho(x, y) = \frac{|\phi\left(\frac{x}{\sqrt{\rho}}\right) - \phi\left(\frac{y}{\sqrt{\rho}}\right)|^2 U^2(x)}{|x - y|^{N+2s}} \leq \frac{1}{\rho^2} \left| \frac{x}{\sqrt{\rho}} - \frac{y}{\sqrt{\rho}} \right|^2 \| \nabla \phi \|_{L^\infty(\mathbb{R}^N)} U^2(x) \| x - y \|^{N+2s} \\
\leq \frac{1}{\rho} |x - y|^{-2(N+2s)} \| \nabla \phi \|_{L^\infty(\mathbb{R}^N)} U^2(x) \| x - y \|^{N+2s} \\
\leq |x - y|^{-2(N+2s)} \| \nabla \phi \|_{L^\infty(\mathbb{R}^N)} U^2(x),
\]

for \( \rho > 1 \). Moreover,
\[
\int_{D_1} |x - y|^{-2(N+2s)} \| \nabla \phi \|_{L^\infty(\mathbb{R}^N)} U^2(x) dxdy \\
\leq \| \nabla \phi \|_{L^\infty(\mathbb{R}^N)} \int_{x \in \mathbb{R}^N} U^2(x) \int_{y \in \mathbb{R}^N, |x - y| \leq 1} |x - y|^{-2(N+2s)} dxdy \\
= \| \nabla \phi \|_{L^\infty(\mathbb{R}^N)} \| U \|_{L^2(\mathbb{R}^N)}^2 N w_N \int_0^1 r^{-1-2s} dr < \infty.
\]

Hence, by the dominated convergence theorem we see that \( \lim_{\rho \to \infty} I_\rho^{2,1} = 0 \). On the other hand, in \( D_2 \) we estimate \( F_\rho(x, y) \) as follows:
\[
F_\rho(x, y) \leq \frac{4 \| \phi \|_{L^\infty(\mathbb{R}^N)} U^2(x)}{|x - y|^{N+2s}}.
\]

Proceeding same way as above, we can show that RHS of (A.9) is in \( L^\infty(D_2) \). Hence, by the dominated convergence theorem we see that \( \lim_{\rho \to \infty} I_\rho^{2,2} = 0 \). Consequently,
\[
\lim_{\rho \to \infty} I_\rho^2 = 0.
\]

Using change of variable, we see that
\[
I_\rho^3 = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} H_\rho(x, y) dxdy,
\]
where,
\[
H_\rho(x, y) = \frac{|U(x) - U(y)||\phi\left(\frac{x}{\sqrt{\rho}}\right) - \phi\left(\frac{y}{\sqrt{\rho}}\right)||U(x)||\phi\left(\frac{y}{\sqrt{\rho}}\right)|}{|x - y|^{N+2s}}.
\]
Clearly \( H_\rho(x, y) \to 0 \) pointwise as \( \rho \to \infty \). Moreover,
\[
|H_\rho(x, y)| \leq \frac{|U(x) - U(y)||\phi\left(\frac{x}{\sqrt{\rho}}\right) - \phi\left(\frac{y}{\sqrt{\rho}}\right)||U(x)||\phi\left(\frac{y}{\sqrt{\rho}}\right)|}{|x - y|^{N+2s}} \leq \frac{1}{2} \frac{|U(x) - U(y)|^2}{|x - y|^{N+2s}} + \frac{1}{2} \frac{|\phi\left(\frac{x}{\sqrt{\rho}}\right) - \phi\left(\frac{y}{\sqrt{\rho}}\right)|^2 U^2(x)}{|x - y|^{N+2s}}
\]

The 1st term on RHS is in \( L^1(\mathbb{R}^N \times \mathbb{R}^N) \) and 2nd term can be dominated by \( L^1 \) function as before. Hence by dominated convergence theorem, we have
\[
\lim_{\rho \to \infty} I_\rho^3(x, y) = \lim_{\rho \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} H_\rho(x, y) dxdy = 0.
\]
As a result, combining \( (A.7), (A.10), (A.13) \), along with \( (A.6) \) and \( (A.5) \) we obtain

\[
\lim_{\rho \to \infty} S_\rho \leq \frac{S_2}{2}.
\]

**Step 2:** In this step we aim to show \( \lim_{\rho \to \infty} S_\rho \geq \frac{S_2}{2} \). Let \( \delta > 0 \) be arbitrary. As \( S_\rho = \inf_{w \in N_\rho} \tilde{F}(w) \), there exists \( w_{\rho, \delta} \in N_\rho \) such that

\[
(A.14) \quad \tilde{F}(w_{\rho, \delta}) < S_\rho + \delta.
\]

Let \( \eta(.) \) be the standard mollifier function, i.e, \( \eta(x) = C \exp(-\frac{1}{|x|^2}) \) if \( |x| < 1 \) and 0 otherwise. Set \( \eta_\rho(x) = \sigma^{-N} \eta(\frac{x}{\sigma}) \).

Define \( w_{\rho, \delta}^\sigma := w_{\rho, \delta} \ast \eta_\sigma \) and \( v_{\rho, \delta}^\sigma = \frac{w_{\rho, \delta}^\sigma}{|w_{\rho, \delta}^\sigma|_{L^{2^*}(\mathbb{R}^N)}} \).

We note that \( v_{\rho, \delta}^\sigma \in C_0^\infty(\mathbb{R}^N) \cap N \)

\[
N := \{ w \in D^{s,2}(\mathbb{R}^N) : w \in L^{q+1}(\mathbb{R}^N), \int_{\mathbb{R}^N} w^{q+1}dx = 1 \}
\]

and \( D^{s,2}(\mathbb{R}^N) \) is completion of \( C_0^\infty(\mathbb{R}^N) \) with the norm \( \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}}dxdy \right)^{\frac{1}{2}} \).

Note that \( v_{\rho, \delta}^\sigma \rightarrow w_{\rho, \delta} \) in \( D^{s,2}(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N) \) as \( \sigma \to 0 \).

Hence, we have

\[
\frac{S_2}{2} \leq \tilde{F}(v_{\rho, \delta}^\sigma) \rightarrow \tilde{F}(w_{\rho, \delta}).
\]

Combining this with \( (A.14) \) we have, \( \frac{S_2}{2} < S_\rho + \delta \). As \( \delta > 0 \) is arbitrary we have, \( \frac{S_2}{2} \leq \lim_{\rho \to \infty} S_\rho \). This implies \( \frac{S_2}{2} \leq \lim_{\rho \to \infty} S_\rho \). This completes the proof.

**Second part:**

Let \( w \in D^{s,2}(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N) \) be a minimizer for \( K \) (existence is guaranteed by \[\text{Theorem 1.4]}\) with \( \int_{\mathbb{R}^N} w^{q+1}dx = 1 \).

Define \( \phi_\rho \) as in step 1. Set \( w_\rho := \rho \phi_\rho \) and \( \hat{w}_\rho := \frac{w_\rho}{|w_\rho|_{L^{q+1}(\mathbb{R}^N)}} \). Then \( \hat{w}_\rho \in N_\rho \). Consequently, \( S_\rho \leq \tilde{F}(\hat{w}_\rho) \). Proceeding before as in step 1, we can show that \( \tilde{F}(\hat{w}_\rho) \to K \) as \( \rho \to \infty \). Hence, \( \lim_{\rho \to \infty} S_\rho \leq K \). To get the other sided inequality, we use the same idea as first part. Hence, the result follows. \( \square \)

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