Maximum rank of a Legendrian web

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Abstract

We propose the Legendrian web in a contact three manifold as a second order generalization of the planar web. An Abelian relation for a Legendrian web is analogously defined as an additive equation among the first integrals of its foliations. For a class of Legendrian $d$-webs defined by simple second order ODE’s, we give an algebraic construction of

$$\rho_d = \frac{(d-1)(d-2)(2d+3)}{6}$$

linearly independent Abelian relations. We then employ the method of local differential analysis and the theory of linear differential systems to show that $\rho_d$ is the maximum rank of a Legendrian $d$-web. In the complex analytic category, we give a possible projective geometric interpretation of $\rho_d$ as an analogue of Castelnuovo bound for degree $2d$ surfaces in the 3-quadric $Q^3 \subset P^4$ via the duality between $P^3$ and $Q^3$ associated with the rank two complex simple Lie group $Sp(2, \mathbb{C})$.

The Legendrian 3-webs of maximum rank three are analytically characterized, and their explicit local normal forms are found. For an application, we give an alternative characterization of a Darboux super-integrable metric as a two dimensional Riemannian metric $g_+$ which admits a mate metric $g_-$ such that a Legendrian 3-web naturally associated with the geodesic foliations of the pair $g_{\pm}$ has maximum rank.

Keywords: contact three manifold, Legendrian web, Abelian relation, maximum rank

1. Introduction

A $d$-web of codimension $\nu$ in a manifold is a set of $d$ foliations by submanifolds of codimension $\nu$ whose $d$ tangent spaces are in general position at each point. [13]. Owing to the basic nature of its definition, webs can be found in diverse areas of differential geometry. [2] [12] [14] [13] [32] [1] [18].

As pointed out by Chern in the survey [12], Lie’s theorem on the surfaces of double translation can be restated in web geometry terms as a result that a planar 4-web of curves of maximum rank three (see below) is algebraic, which is also related to a particular case of the converse of Abel’s theorem. For an example in projective geometry, the classical Chasles’ theorem on the intersecting cubics is linked with the hexagonality of algebraic planar 3-webs. According to [1], a three dimensional semi-simple Frobenius manifold carries a family of characteristic hexagonal planar 3-webs. Planar 3-webs also naturally arise as the asymptotic webs of Legendrian surfaces in the projective space $P^5$, [18]. Although in a possibly different context, the author’s own interest in web geometry originated from Gelfand & Zakharevich’s work on the Veronese webs attached to bi-Hamiltonian systems, [18]. Much of the developments of web geometry up until 1930’s are recorded in the book of Blaschke & Bol “Geometrie der Gewebe”, [2]. For the more recent literature on the subject, we refer to [22] and the references therein.

Let $W$ be a $d$-web of codimension $\nu$. A single regular foliation (of fixed codimension) in a finite dimensional manifold has no local invariants, and the local geometry of $W$ lies entirely in the relative position among the set of $d$ foliations. A pertinent notion in this aspect is the Abelian $k$-relation (equation), $1 \leq k \leq \nu$, an additive relation among the closed basic $k$-forms with respect to each foliation. The set of Abelian $k$-relations of $W$ naturally form a vector space, and the $k$-rank of $W$ denoted by $r^k(W)$ is defined as the dimension of the vector space of Abelian $k$-relations. The sequence of integers $r^k(W)$, $k = 1, 2, ... \nu$, are the basic local invariants of a web.

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1 An Abelian 1-relation represents an additive relation among the first integrals of foliations.

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The $k$-rank of a $d$-web of codimension $\nu$ is not arbitrary. There generally exists an optimal bound determined in terms of $k$, $d$, $\nu$, and the dimension of the ambient space. It is a classical result of Bol that the maximum rank (1-rank) for a planar $d$-web is $\frac{(d-1)(d-2)}{2}$, which is the well known genus bound for a degree $d$ plane algebraic curve. Chern in his thesis generalized this and showed that the maximum rank for a $d$-web of dimension $1$ is equal to the Castelnuovo bound for a degree $d$ algebraic curve.

One of the main geometrical ideas behind these results is the projective duality, Fig. 1.1 and the corresponding Abel’s theorem. A degree $d$ curve $\Sigma \subset (\mathbb{P}^n)^*$ induces by the standard dual construction an associated $d$-web of hyperplanes $\mathcal{W}_\Sigma$ in an open subset of $\mathbb{P}^n$. By Abel’s theorem, the trace of a holomorphic 1-form on $\Sigma$ gives rise to an Abelian relation for $\mathcal{W}_\Sigma$. The dimension of the subspace of such algebraic Abelian relations for $\mathcal{W}_\Sigma$ is then bounded by the Castelnuovo bound for the given algebraic curve.

With this background geometry, the results of Bol, and Chern admit the following interpretation. The vector space of Abelian relations (for a web of codimension 1) can be analytically defined as the space of solutions to a linear differential system for the sections of a direct sum bundle canonically associated with a web. The differential system is generally over-determined, see Section 2. It is not difficult to imagine that such a uniform rank bound, if exists, is determined entirely by the symbol of the linear differential system. The results of Bol, and Chern indicate that even if one considers the general webs of codimension 1, and not just the kind of algebraic webs $\mathcal{W}_\Sigma$ described above, the relevant symbol of the linear differential system for Abelian relations is isomorphic in an appropriate sense to that of the algebraic webs. The canonical linear differential system for Abelian relations does not get any more over-determined at the level of symbol when one extends the category from algebraic webs to general webs.

We shall consider in this paper a generalization of web geometry to the geometric situation where the foliations are subject to a first order constraint. Let $X$ be a three dimensional manifold. A contact structure on $X$ is a 2-plane field $\mathcal{D} \subset TX$ locally defined by $\mathcal{D} = \langle \theta \rangle^\perp$ for a 1-form $\theta$ that satisfies the non-degeneracy condition

$$d\theta \wedge \theta \neq 0.$$ 

Given a contact structure, a Legendrian curve is an immersed $D$-horizontal curve. A Legendrian $d$-web is a set of $d$ pairwise transversal foliations by Legendrian curves. The Pfaff-Darboux local normal form theorem for contact structures shows that a Legendrian $d$-web can be locally described by a set of $d$ second order ODE’s.

Recall that a planar $d$-web is a set of $d$ transversal foliations by curves on a two dimensional manifold, which is locally described by a set of $d$ first order ODE’s. The purpose of this paper is to propose the Legendrian web in a contact three manifold for a second order generalization of the planar web. An Abelian relation, which represents an additive functional relation among the first integrals of foliations, is analogously defined for the Legendrian web. The rank of a Legendrian web is the dimension of the vector space of its Abelian relations.

In analogy with the discussion above, in the complex analytic category the projective geometric model for our investigation is the duality associated with the rank two symplectic group $\text{Sp}(2, \mathbb{C})$, Fig. 1.2, see [6] for the details. Here the 3-quadratic $\mathbb{Q}^3 \subset \mathbb{P}^4$ is the space of Legendrian lines in $\mathbb{P}^3$, and dually $\mathbb{P}^3$ is the space of null lines in $\mathbb{Q}^3$ with respect to the $\text{Sp}(2, \mathbb{C})$ invariant contact, and conformal structures respectively. The double fiber bundle $Z$ is the incidence space.

Consider a degree $2d$ analytic surface $\Sigma \subset \mathbb{Q}^3 \subset \mathbb{P}^4$. Then generically $\Sigma$ intersects a null line at $d$ distinct points, and it induces by duality an associated $d$-web $\mathcal{W}_\Sigma$ of Legendrian lines on an open subset of $\mathbb{P}^3$. This suggests to consider a Legendrian $d$-web as a generalization of null degree $d$ surface in $\mathbb{Q}^3 \subset \mathbb{P}^4$.

Let $\Omega$ be a closed meromorphic 1-form on a null degree $d$ surface $\Sigma$. In consideration of Abel’s theorem, one may...
declare that $\Omega$ is holomorphic when the trace $(\pi_1)_* \circ (\pi_2)^* \Omega$ vanishes. By definition of trace, such a closed holomorphic 1-form gives rise to an Abelian relation for $\mathcal{V}_{\mathbb{C}}$. An Abelian relation for a Legendrian $d$-web can therefore be considered as a generalization of closed holomorphic 1-form on a null degree $d$ surface in $\mathbb{Q}^3$.

One of the initial motivations for the present work came from the observation that the bound for the rank of a Legendrian $d$-web would imply the Castelnuovo type bound for a null degree $d$ surface in $\mathbb{Q}^3$; determining the rank bound by a direct local analysis of the linear differential system for Abelian relations, one may derive conversely a global Castelnuovo bound for the analytic surfaces in $\mathbb{Q}^3$.

Main results.

1. We give a local analytic characterization of the Legendrian 3-webs of maximum rank three and determine their local normal forms, Proposition 3.10, Theorem 3.16. As an application we give an alternative characterization of a Darboux superintegrable metric in terms of a Legendrian 3-web associated with the geodesic flows of a pair of two dimensional Riemannian metrics, Theorem 3.23.

2. We give an algebraic construction of \frac{(d-1)(d-2)(2d+3)}{6} linearly independent Abelian relations for a class of simple Legendrian $d$-webs, Section 4.

3. The maximum rank of a Legendrian $d$-web is

$$\rho_d := \frac{(d-1)(d-2)(2d+3)}{6}. \quad (1.1)$$

Let us mention some of the closely related works on web geometry. In addition to the works of Bol, and Chern cited above, Chern & Griffiths determined the bound for the $\nu$-rank of the webs of codimension $\nu$ in a manifold of dimension $\nu m$, \cite{14}. Hénaut further generalized Chern & Griffiths’ bound to the bound for the $k$-rank for $1 \leq k \leq \nu$. We refer to \cite[28, 33, 36]{} for the analysis of the webs of maximum rank. Our main results can be considered as a partial generalization of these works to the Legendrian web. See also \cite{19} for a recent survey on the rank problems and linearizability for planar web.

Recently Shurygin gave an analysis of Legendrian 2 and 3 webs applying Kruglikov’s description of differential invariants for the scalar second order ODE up to point transformation, \cite[26]{34, 35}.

In complex projective geometry the rank bound (1.1) in our main results calls for the study of the extremal null degree $d$ surfaces in $\mathbb{Q}^3$ with the maximum number $\rho_d$ of (generalized) closed holomorphic 1-forms. The extremal null degree 3 surfaces are classified in \cite{39}.

We shall work within the real smooth category. Most of the results are valid in the complex analytic category with minor modification. All of the arguments in this paper are local and they are concerned with the appropriate germs of geometric structures in a neighborhood of a reference point. The over-determined PDE machinery are used throughout the paper without any specific references. For the standard references, we refer the reader to \cite{8, 25}.

We assume the following well known property of a Vandermonde matrix: given a set of $d$ distinct numbers $q^a$, $a = 1, 2, ... d$, the Vandermonde matrix \( V_d^a = ((q^a)^{b-1})_{a,b=1} \) has full rank.

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\footnote{The relevant technical details are being ignored here. Since we are attempting to define holomorphic objects on generally singular surfaces, there are a few different choices. \cite{24}. Note also that a holomorphic 1-form on a smooth compact complex surface is necessarily closed.}

\footnote{The maximum 1-rank of a 3-web of curves in a three dimensional manifold without any constraints is five, \cite[Section 36, 37]{8}.}
1.1. Contact three manifold

This section recalls some basic definitions and properties of contact three manifolds. The Pfaff-Darboux local normal form theorem (in dimension three) recorded in the below is a basic structure theorem which gives a local uniformization of contact structures. For our purpose, it provides an analytic representation of a Legendrian web in terms of a finite set of second order ODE’s for one scalar function of one variable.

Let $X$ be a three dimensional manifold. Let $TX \to X$ be the tangent bundle, and let $T^*X \to X$ be the cotangent bundle of $X$.

Definition 1.2. A contact structure on a three dimensional manifold $X$ is a 2-plane field $D \subset TX$ such that it is locally defined by $D = \langle \theta \rangle ^\perp$ for a 1-form $\theta$ which satisfies the non-degeneracy condition $d\theta \wedge \theta \neq 0$.

The contact line bundle $C \subset T^*X$ is the dual line bundle $C = D ^\perp$. A contact three manifold is a three dimensional manifold with a given contact structure. A (local) contact transformation of a contact three manifold is a (local) diffeomorphism which preserves the contact structure.

The contact three manifold in the following example will be used throughout the paper.

Example 1.4. The standard model of contact three manifold is the first jet space $J^1(\mathbb{R}, \mathbb{R})$ of scalar functions on the real line $\mathbb{R}$. Let $x$ be the standard coordinate of $\mathbb{R}$, and let $y, p$ denote the scalar function $y$ of $x$, and the first derivative $p = \frac{dy}{dx}$ respectively. The set of three functions $(x, y, p)$ form the standard coordinate of $J^1(\mathbb{R}, \mathbb{R})$. The contact line bundle is generated by the canonical contact 1-form

$$\theta = dy - p\,dx.$$  

(1.5)

Note the contact transformation $(x, y, p) \to (p, y - px, -x)$. The full group of contact transformations is thus strictly larger than the subgroup of contact transformations induced by the point transformations of the form $(x, y, p) \to (\hat{x}(x, y), \hat{y}(x, y), \hat{p}(x, y) = \frac{\hat{y} + py}{\hat{x} + px})$.

There exists a distinguished class of curves in a contact three manifold.

Definition 1.6. Let $X$ be a contact three manifold with the contact structure $D \subset TX$. A Legendrian curve in $X$ is an immersed $D$-horizontal curve.

From the non-degeneracy condition (1.3), it is clear that there do not exist immersed $D$-horizontal surfaces.

Example 1.7 (continued from Example 1.4). Let $f(x)$ be an arbitrary scalar function on $\mathbb{R}$. By definition of the canonical contact 1-form (1.5), the first jet graph

$$\left(x, f(x), \frac{df}{dx}(x) = f'(x) \right) \subset J^1(\mathbb{R}, \mathbb{R})$$

is a Legendrian curve.

The following Pfaff-Darboux local normal form theorem states that every Legendrian curve in a contact three manifold is locally isomorphic to Example 1.7.

Theorem 1.8 (Pfaff-Darboux local normal form). Let $X$ be a contact three manifold. Let $x_0 \in X$ be a reference point. There exists an adapted local coordinate $(x, y, p)$ of $X$ in a neighborhood of $x_0$ such that the contact line bundle is generated by

$$C = \langle \, dy - p\, dx \rangle.$$  

(1.9)

Let $\gamma \hookrightarrow X$ be a Legendrian curve through $x_0$. Then the adapted local coordinate $(x, y, p)$ can be chosen so that

$$(x, y, p) = (0, 0, 0) \quad \text{at} \quad x_0,$$

$$(y, p) = (0, 0) \quad \text{along} \quad \gamma.$$
Proof. The existence of an adapted local coordinate \((x, y, p)\) for \textsuperscript{1.9} is well known, \textsuperscript{8} p38. Given the reference point \(x_0 \in X\), let \(\gamma(\mathbb{R}) \ni x \mapsto (x, y, p(x))\) be such a coordinate system adapted at \(x_0\). From

\[
\frac{dy}{dx} - p(x) dx = \frac{dy}{p_0 x - y_0 + p_0 x_0} - (p - p_0) dx.
\]

one may take \((\hat{x}, \hat{y}, \hat{p}) = (x - x_0, y - p_0 x - y_0 + p_0 x_0, p - p_0)\) as the desired coordinate adapted at \(x_0\).

Let \((x, y, p)\) be such a coordinate system adapted at \(x_0\). Given a Legendrian curve \(\gamma\) through \(x_0\), there are three cases (they are not mutually exclusive):

- case \(\text{d}x|_{\gamma} \neq 0\) at \(x_0\).

Let \(\gamma^* y = f(x)\), \(\gamma^* p = f'(x)\) locally in a neighborhood of \(x_0\). From \(\frac{dy}{dx} - p(x) dx = \frac{dy}{x - f(x)} - (p - f'(x)) dx\), one may take \((\hat{x}, \hat{y}, \hat{p}) = (x, y - f(x), p - f'(x))\) as the desired coordinate adapted along \(\gamma\).

- case \(\text{d}y|_{\gamma} \neq 0\) at \(x_0\).

One may first translate \((x, y, p) \to (x, y + x, p + 1)\) so that \((x, y, p)|_{x_0} = (0, 0, 1)\). The contact 1-form \(\text{d}y - p(x) dx\) can then be written as \(\text{d}y - p(x) dx = -p(x) dx - \frac{1}{p} dy\). By interchanging \(x\) and \(y\), and up to a translation similar as above, this is reduced to the first case.

- case \(\text{d}p|_{\gamma} \neq 0\) at \(x_0\).

Under the contact transformation \((x, y, p) \to (p, y - px, -x)\) from Example \textsuperscript{1.4} this is also reduced to the first case.

\[
\square
\]

Pfaff-Darboux local normal form theorem reflects the ampleness of local group of contact transformations. We will see in the next section (Lie’s theorem) that the local group of contact transformations is in fact large enough to normalize not just a single Legendrian curve but a foliation by Legendrian curves.

2. Legendrian web

This section introduces the main subjects of the paper: Legendrian webs and their Abelian relations. In Section 2.1 a Legendrian web is defined as a finite set of foliations by Legendrian curves in a contact three manifold. In Section 2.2 an Abelian relation for a Legendrian web is defined as an additive equation among the first integrals of the foliations.

2.1. Legendrian web

Let \(d \geq 1\) be an integer.

Definition 2.1. A Legendrian \textit{d-web} on a contact three manifold is a set of \(d\) pairwise transversal foliations by Legendrian curves\textsuperscript{6}.

Analytically a Legendrian web is described as follows. Let \(X\) be a contact three manifold with the contact line bundle \(C = \mathcal{T}^* X\). Let \(\mathcal{W}\) be a Legendrian \(d\)-web on \(X\). By duality, \(\mathcal{W}\) is defined by a set of \(d\) rank two sub-bundles \(\mathcal{I}^a \subset \mathcal{T}^* X\), \(a = 1, 2, \ldots, d\), such that \(C \subset \mathcal{I}^a\) and from the transversality condition that \(\mathcal{I}^a \cap \mathcal{I}^b = C\) for \(a \neq b\). This dual representation of a Legendrian web is convenient for the differential analysis for Abelian relations later on.

The Pfaff-Darboux local normal form theorem gives a concrete analytic formulation of a Legendrian web. For a generic choice of adapted local coordinate \((x, y, p)\) for the given contact structure on \(X\), there exist a set of \(d\) functions \(q^a(x, y, p), a = 1, 2, \ldots, d\), \(q^a \neq q^b\) for \(a \neq b\), such that the defining sub-bundle \(\mathcal{I}^a\)'s for \(\mathcal{W}\) are locally generated by

\[
\mathcal{I}^a = \langle dx, p dx, q^a dx \rangle, \quad a = 1, 2, \ldots, d
\]

(and consequently on each Legendrian leaf we have \(dx \neq 0\)). The geometry of a Legendrian \(d\)-web is thus locally equivalent to the geometry of a set of \(d\) distinct second order ODE’s

\[
y'' = q^a(x, y, y'), \quad a = 1, 2, \ldots, d\]

\[
q^a \neq q^b \quad \text{for} \quad a \neq b,
\]

up to contact transformation.

Let us first examine the Legendrian \(d\)-webs for \(d = 1, 2\).

The following theorem of Lie states that there is no local invariant for a single Legendrian foliation and a Legendrian 1-web admits a unique local normal form up to contact transformation.

\[
\text{As remarked earlier, our analysis is essentially local and the issues such as global well-definedness of foliations shall be ignored.}
\]
**Example 2.3 (Lie’s theorem).** It is a classical theorem of Lie that a second order ODE for one scalar function of one variable is locally equivalent to the equation \( y'' = 0 \) up to contact transformation, \[31\]. The corresponding dual rank two sub-bundle \( \mathcal{J} \) on \( J^1(\mathbb{R}, \mathbb{R}) \) is
\[
\mathcal{J} = \langle dy - p\, dx, \, dp \rangle.
\]
Under the contact transformation \((x, y, p) \rightarrow (p, y - px, -x)\), one has
\[
\mathcal{I} = \langle dy, \, dx \rangle.
\]
Lie’s theorem thus shows that a Legendrian 1-web is locally equivalent to the foliation on \( J^1(\mathbb{R}, \mathbb{R}) \) given by the fibers of canonical projection \( J^1(\mathbb{R}, \mathbb{R}) \rightarrow J^0(\mathbb{R}, \mathbb{R}) \simeq \mathbb{R}^2 \).

As a consequence this implies that the geometry of a Legendrian web essentially lies in the relative position among the set of Legendrian foliations.

A contact transformation of \( J^1(\mathbb{R}, \mathbb{R}) \) which preserves the fibers of projection \( J^1(\mathbb{R}, \mathbb{R}) \rightarrow J^0(\mathbb{R}, \mathbb{R}) \simeq \mathbb{R}^2 \) is locally induced from (the prolongation of) a point transformation of \( \mathbb{R}^2 \). Lie’s theorem above thus also implies that the geometry of a Legendrian \( d \)-web is locally equivalent to the geometry of a set of \( d-1 \) scalar second order ODE’s up to point transformation. In this regard, Cartan’s work \[10\] on the second order ODE’s can be considered as a study of Legendrian 2-webs.

A class of Legendrian 3-webs are examined in detail in Section 3.

Let us indicate in passing a geometrical construction for Legendrian webs. They are generalization to our setting of the well known construction in web geometry without constraints, \[32\].

**Example 2.4 (continued from Example 1.4).** Let \( J^1(\mathbb{R}, \mathbb{R}) \rightarrow J^0(\mathbb{R}, \mathbb{R}) \simeq \mathbb{R}^2 \) be the canonical projection. In terms of the standard adapted coordinate \((x, y, p)\) of \( J^1(\mathbb{R}, \mathbb{R}) \), the projection map is given by
\[
(x, y, p) \rightarrow (x, y).
\]
Consider a two parameter family of curves in \( \mathbb{R}^2 \), each of which is locally a graph of a function \( y = y(x) \). Assuming this family of curves are sufficiently generic to second order, their Legendrian lifts \((x, y(x), y'(x)) \rightarrow J^1(\mathbb{R}, \mathbb{R})\) define a Legendrian foliation on an open subset of \( J^1(\mathbb{R}, \mathbb{R}) \).

**Example 2.5 (continued).** Let \( \{ p^1, p^2, \ldots p^d \} \subset \mathbb{R}^2 \) be a set of \( d \) distinct points. Consider for each \( p^a \) the two parameter pencil of \( y \) as a quadratic polynomial of \( x \) based at \( p^a \). A set of \( d \) distinct points in \( \mathbb{R}^2 \) in this way gives rise to a Legendrian \( d \)-web on an open subset of \( J^1(\mathbb{R}, \mathbb{R}) \).

**Example 2.6 (continued).** Let \( \{ p^1, p^2, p^3 \} \subset \mathbb{R}^2 \) be a set of three distinct points. In addition to the quadratic pencil \( 3 \)-web based at these points from Example 2.5 consider the Legendrian foliation induced by the two parameter family of conics through the three points. A set of three distinct points in \( \mathbb{R}^2 \) in this way gives rise to a Legendrian \( 4 \)-web on an open subset of \( J^1(\mathbb{R}, \mathbb{R}) \). This is a Legendrian analogue of Bol’s exceptional \( 5 \)-web, \[15\].

### 2.2. Abelian relation

In planar web geometry, the projective duality suggests to consider a planar \( d \)-web as a generalized degree \( d \) plane curve. An Abelian relation for a planar web formally corresponds to a holomorphic 1-form through Abel’s theorem, \[13\], \[23\].

We shall follow this line of idea and give a definition of Abelian relation for a Legendrian web.

**Definition 2.7.** Let \( X \) be a contact three manifold. Let \( \mathcal{W} \) be a Legendrian \( d \)-web on \( X \). Let \( \mathcal{I}^a \subset T^*X, \, a = 1, \, 2, \ldots d \), be the dual rank two sub-bundles which define \( \mathcal{W} \). An Abelian relation of \( \mathcal{W} \) is a \( d \)-tuple of 1-forms \( \Omega^a \in H^0(\mathcal{I}^a) \), such that
\[
\sum_b \Omega^b = 0. \tag{2.9}
\]
The \( \mathbb{R} \)-vector space of Abelian relations of \( \mathcal{W} \) is denoted by \( \mathcal{A}(\mathcal{W}) \). The rank of a Legendrian web \( \mathcal{W} \) is the dimension of \( \mathcal{A}(\mathcal{W}) \).

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\(^{3}\)See the argument below Definition 2.7 A Legendrian 2-web has no nontrivial Abelian relations.

\(^{8}\)In order to remove the ambiguity of adding by constants, we adopt here the differentiated version of first integrals for Legendrian foliation. See the remark below Proposition 4.3 for an equivalent alternative definition.
For a dual interpretation of Abelian relation for a Legendrian web as a generalized holomorphic 1-form, we refer to Figure 1.2 and the related remarks in Section 1. The idea to extend the notion of Abelian relation to Legendrian webs came from the exposition of the dualities associated with the rank two simple Lie groups, see also Section 6.

Rank provides a basic local numerical invariant of a Legendrian web under contact transformation.

Similarly as before, let us first examine the Abelian relations of Legendrian $d$-webs for $d = 1, 2$.

A Legendrian 1-web has rank zero by definition.

Let $\mathcal{W}$ be a Legendrian 2-web defined by a pair of rank two sub-bundles $\mathcal{I}^a \subset T^* X$, $a = 1, 2$. Let $(\Omega^1, \Omega^2)$ be an Abelian relation of $\mathcal{W}$. By definition we have $\Omega^1 + \Omega^2 = 0$ and, since $\mathcal{I}^1 \cap \mathcal{I}^2 = \mathcal{C}$, $\Omega^1$ must be a multiple of contact 1-form. The defining equation $d\Omega^1 = 0$ then forces $\Omega^1 = \Omega^2 = 0$. Hence a Legendrian 2-web also has rank zero. Abelian relation is nontrivial for the Legendrian $d$-webs for $d \geq 3$. In view of the main theme of this paper, the Abelian relations of a Legendrian web, we shall restrict our attention to the Legendrian $d$-webs for the case $d \geq 3$ from now on.

The following example shows that the Legendrian $d$-webs for $d \geq 3$ with at least one Abelian relation exist in abundance.

**Example 2.10.** Let $X$ be a contact three manifold with the contact line bundle $\mathcal{C} \subset T^* X$. Let $\mathcal{W}_0$ be a Legendrian $d$-web ($d \geq 2$) defined by the $d$ rank 2 sub-bundles $\mathcal{I}^a \subset T^* X$, $a = 1, 2, \ldots, d$. Let $\Omega^a \in H^0(\mathcal{I}^a)$ be a closed 1-form for each $a = 1, 2, \ldots, d$. Assuming that $\mathcal{I}^a$’s are sufficiently generic, define another distinct rank 2 sub-bundle $\mathcal{I}^0 \subset T^* X$ by $\mathcal{I}^0 = \mathcal{C} \oplus (\sum_a \Omega^a)$. The Legendrian $(d + 1)$-web defined by $\mathcal{I}^a$’s, $a = 0, 1, \ldots, d$, possesses by construction at least one nontrivial Abelian relation.

One of the general problems in Legendrian web theory would be to give a characterization of Legendrian webs of maximal rank; the class of Legendrian webs with as many Abelian relations as possible. Specifically, the defining equation for Abelian relations (2.8), (2.9) is a system of linear differential equations for the sections of the direct sum bundle $\oplus_{a=1}^d \mathcal{I}^a$. A computation for the cases $d = 3, 4$ shows that it is over-determined, and one suspects that a generic Legendrian web has rank zero. The Legendrian webs we shall be interested in are those with maximal number of independent Abelian relations, and this imposes a stringent set of compatibility equations that such a Legendrian web must satisfy. Moreover in this case it turns out that the vector space $\mathcal{A}(\mathcal{W})$ inherits rich geometric structures from the layered feature of the integrability conditions for (2.8), (2.9), see Section 5. The geometry of $\mathcal{A}(\mathcal{W})$ would be important in understanding the Legendrian webs of maximal rank.

On the other hand, it is not obvious if the rank of a Legendrian $d$-web is bounded in terms of $d$, or even finite (for example, the characteristic variety argument in [8, p285] does not directly apply to Legendrian web). One of the basic problems would be to determine an effective bound on the rank, if such a bound exists. We shall show in Section 5 that the rank of a Legendrian $d$-web admits the optimal bound

$$\rho_d := \frac{(d - 1)(d - 2)(2d + 3)}{6}.$$

With regard to the duality associated with the rank two simple Lie group $\text{Sp}(2, \mathbb{C})$ mentioned earlier, this number should have a meaning in complex projective geometry as an analogue of Castelnuovo bound for surfaces in the 3-quadratic $\mathbb{Q}^3 \subset \mathbb{P}^4$.

Before we proceed to the differential analysis for the proof of rank bound, let us examine in some detail the first nontrivial case of Legendrian 3-webs. A direct computation gives an independent proof that the maximum rank of a Legendrian 3-web is $\rho_3 = 3$. It also gives a taste of higher order differential analysis involved in analyzing the linear differential system for Abelian relations.

### 3. Legendrian 3-webs of maximum rank

It is well known in planar web geometry that there exists essentially a unique local model for the 3-webs of maximum rank one up to diffeomorphism; a set of three families of parallel lines in the plane. As the analysis in this section will show,
this uniqueness result is not true for the case of Legendrian 3-webs and the local moduli space of Legendrian 3-webs of maximum rank three is generally two dimensional\footnote{A heuristic explanation for the dimension count of local moduli is as follows. Consider the 3-quadric $Q^3 \subset \mathbb{P}^4$. The configuration space of three hyperplane sections in $Q^3$ up to Sp(2, $\mathbb{C}$) motion has general dimension $3 \dim (\mathbb{P}^4)^* - \dim \text{Sp}(2, \mathbb{C}) = 3 \cdot 4 - 10 = 2$. By projective duality, a set of three hyperplane sections in $Q^3$ corresponds to a linear Legendrian 3-web on an open subset of $\mathbb{P}^4$, and this correspondence is generically injective. A direct computation shows that this class of Legendrian 3-webs have the maximum rank three. See\cite{39} for the details.}

In this section, we employ the equivalence method of E. Cartan and give a local analytic characterization of the Legendrian 3-webs of maximum rank. Theorem\footnote{5.16} For an application, we consider the Legendrian 3-web associated with a pair of two dimensional Riemannian metrics and give a new characterization of Darboux super-integrable metrics, Theorem\footnote{5.23}

3.1. Local normal forms
Let us give a summary of the main results in this section.

A Legendrian 3-web $\mathcal{W}$ on a contact three manifold $X$ can be formulated as a $G_0$-structure on $X$ for a subgroup $G_0 \subset \text{GL}_3(\mathbb{R})$; the collection of frames suitably adapted to the given 3-web $\mathcal{W}$ at each point of $X$ form a principal $G_0$-sub-bundle in the principal frame bundle of $X$. (3.6). Under the normalization process of equivalence method this is further reduced to a $G \subset G_0$ structure with one dimensional structure group $G$. The torsion coefficients of the resulting structure equation and their successive derivatives are the basic local invariants of a Legendrian 3-web, Proposition\footnote{3.10}

The Abelian relations of $\mathcal{W}$ are by definition the solutions to a canonically attached linear differential equation. When expressed in terms of the adapted coframe of the reduced $G_0$-structure above, the compatibility conditions for $\mathcal{W}$ to have maximal rank are written as a set of linear equations in the basic local invariants of $\mathcal{W}$, (3.14). A differential analysis of these equations shows that the torsion coefficients are necessarily covariant constant and that a Legendrian 3-web of maximum rank three is homogeneous for certain local Lie group structure, Theorem\footnote{5.16}. For readable references on the equivalence method we refer to\cite{17}\cite{25}.

3.1.1. Equivalence problem for Legendrian 3-webs
Let $\mathcal{W}$ be a Legendrian 3-web on a contact three manifold $X$. Arguing locally, let $\theta$ be a contact 1-form, and let $\theta^a$, $a = 1, 2, 3$, be a set of 1-forms such that $\mathcal{W}$ is defined by the rank two sub-bundles $\mathcal{I}^a \subset T^* X$ locally generated by

$$\mathcal{I}^a = \langle \theta, \theta^a \rangle, \quad a = 1, 2, 3. \quad (3.1)$$

Up to scaling and adding multiples of $\theta$, one may arrange so that

$$\theta^1 + \theta^2 + \theta^3 = 0. \quad (3.2)$$

The most general transformation of $\{\theta, \theta^a\}$ preserving the relation (3.2) is of the following form;

$$(\theta, \theta^a) \to (\epsilon \theta, \epsilon \theta^+(a) + t^a \theta), \quad (3.3)$$

for nonzero scaling functions $\epsilon, \epsilon$, a permutation $\tau \in S_3$ (the symmetric group on three letters), and translating factors $t^a$'s such that $t^1 + t^2 + t^3 = 0$. This can be utilized to obtain a refined set of generators.

Lemma 3.4. There exists a transformation of 1-forms $\{\theta, \theta^a\}$ as described above such that

$$d\theta \equiv \theta^1 \wedge \theta^2 \mod \theta, \quad d\theta^a \equiv 0 \mod \theta, \quad a = 1, 2, 3. \quad (3.5)$$

Proof. For a given $\{\theta, \theta^a\}$ let

$$d\theta \equiv s\theta^1 \wedge \theta^2 \mod \theta, \quad d\theta^a \equiv s^a \theta^1 \wedge \theta^2 \mod \theta, \quad a = 1, 2, 3,$$

for coefficients $s, s^a$. Since $\theta$ is a contact 1-form, $s$ is a nonzero function. Consider the transformation

$$\theta \to s \theta, \quad \theta^a \to \theta^a - \frac{s^a}{s} \theta. \quad \square$$
Note that for the refined set of 1-forms \{\theta, \theta^a\} in Lemma 3.4 the most general transformation preserving the relation \(3.5\) is
\[
\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} \to \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ 2\alpha_3 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} + \begin{pmatrix} R \theta_1 + S \theta_2 \\ T \theta_1 - R \theta_2 \\ \theta_1 \theta_2 \end{pmatrix},
\]
for a nonzero scaling function \(\epsilon\), and a permutation \(\tau \in S_3\).

The analysis above can be recast in the language of Cartan’s equivalence method as follows. Let \(\mathcal{W}\) be a Legendrian 3-web on a contact three manifold \(X\). Let \(F \to X\) be the principal \(\text{GL}_3\mathbb{R}\)-frame bundle. With an abuse of notation, let \(\{\theta, \theta^1, \theta^2\}\) denote the tautological 1-forms on \(F\) and set \(\theta^3 = -(\theta^1 + \theta^2)\). Then, there exists a principal sub-bundle \(B \subset F\) with one dimensional structure group \(G\) acting on \(\{\theta, \theta^1, \theta^2\}\) by \(3.6\) such that, for any local section \(\sigma : X \to B\)
\[a)\ \sigma^* \theta\text{ is a contact 1-form, and }\mathcal{W}\text{ is defined by the ideals generated by }\mathcal{I}^a = \{\sigma^* \theta, \sigma^* \theta^a\}, a = 1, 2, 3, \]
b) the 1-forms \(\{\sigma^* \theta, \sigma^* \theta^a\}\) satisfy the relations \(3.2, 3.5\).

From the general theory, it is not difficult to show that the sub-bundle \(B\) can be chosen so that the tautological 1-forms satisfy the following structure equation on \(B\) for a choice of pseudo-connection 1-form \(\alpha\). We leave the details to the reader.
\[
d(\theta^1, \theta^2) = -\begin{pmatrix} \alpha \\ \alpha \\ 2\alpha \end{pmatrix} \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta \end{pmatrix} + \begin{pmatrix} R \theta^1 + S \theta^2 \\ T \theta^1 - R \theta^2 \\ \theta_1 \theta_2 \end{pmatrix}.
\]
(3.7)
Here \(\cdot \cdot \cdot\) denotes 0, and \(R, S, T\) are torsion coefficients.\(^{11}\)

Note that \(\alpha\) is uniquely determined by this structure equation. The exterior derivative identity \(d(d(\theta)) = 0\) furthermore implies
\[
d\alpha = \theta \wedge (N \theta^1 + L \theta^2),
\]
(3.9)
for coefficients \(N, L\).

Let us summarize the analysis so far.

**Proposition 3.10.** Let \(\mathcal{W}\) be a Legendrian 3-web on a contact three manifold \(X\). There exists a sub-bundle \(B\) of the \(\text{GL}_3\mathbb{R}\) principal frame bundle of \(X\) on which the tautological 1-forms \(\theta, \theta^a\), \(a = 1, 2, 3\), satisfy the equations \(3.1, 3.2, 3.7, 3.9\). The functional relations among the torsion coefficients \(R, S, T, N, L\) and their successive derivatives are the basic local invariants of \(\mathcal{W}\).

The algebra of local symmetry vector fields of \(\mathcal{W}\) is at most four dimensional, and it is four dimensional whenever the torsion coefficients \(R, S, T, N, L\) all vanish.

**Proof.** The algebra of local symmetry vector fields of \(\mathcal{W}\) is four dimensional when the torsion coefficients \(R, S, T, N, L\) are all constants. Since they scale along the fibers of the bundle \(B\), e.g., \(dR \equiv 2R\alpha\mod \theta, \theta^1, \theta^2\), this occurs only when \(R, S, T, N, L\) all vanish. The rest follows from the general theory of equivalence method. \(\square\)

The differential analysis for Abelian relations in the next sub-section will be carried out on the bundle \(B\).

We record in passing the following corollary of Proposition 3.10 on the intersection of local algebra of symmetry vector fields of two distinct path geometries on a surface.

Let \(M\) be a two dimensional manifold. Let \(P(TM) \to M\) be the projective tangent bundle equipped with the canonical contact structure. Recall that a path geometry on \(M\) is a (local) Legendrian foliation on \(P(TM)\) transversal to the fibers of projection \(P(TM) \to M\). \cite{3.5]}

**Corollary 3.11.** Let \(\mathcal{F}_\pm\) be a pair of distinct path geometries on a surface \(M\). Let \(p_+ \subset \mathfrak{X}(M)\) be the algebra of symmetry vector fields for \(\mathcal{F}_\pm\) respectively (here \(\mathfrak{X}(M)\) is the algebra of vector fields on \(M\)). Then
\[
\dim p_+ \cap p_- \leq 4.
\]

---

\(^{11}\)The structure group of the principal bundle \(B\) acts on \((\theta, \theta^a)\) by \((\theta, \theta^a) \to (\theta, \theta^a) + (\text{sgn}(\tau)\epsilon^2 \theta, \epsilon \theta^a)\) for a nonzero scaling parameter \(\epsilon\), and a permutation by \(\tau \in S_3\). For example, under the permutation \((\theta^1, \theta^2, \theta^3) \to (\theta^2, \theta^3, \theta^1)\), the torsion coefficients transform to
\[
\begin{pmatrix} R \\ S \\ -R \end{pmatrix} \to \begin{pmatrix} R \\ -S \\ -R \end{pmatrix},
\]
(3.8)
The equality holds whenever the pair $F_\pm$ is locally equivalent to the pair of flat path geometries defined by the second order ODE's

$$y'' = q_\pm, \quad q_+, q_- \text{ are distinct constants.}$$

**Proof.** The prolongation of the intersection $p_+ \cap p_-$ to $\mathbb{P}(TM)$ is the symmetry algebra of vector fields for the Legendrian 3-web defined by $F_\pm$, and the fibers of projection $\mathbb{P}(TM) \to M$. □

### 3.1.2. Abelian relations

We continue the analysis from Proposition 3.10.

For three functions $f, g^1, g^2$ on $B$, set

$$\Omega^1 = f \theta^1 + g^1 \theta, \quad \Omega^2 = f \theta^2 + g^2 \theta, \quad \Omega^3 = -\Omega^1 - \Omega^2,$$

denote a section of $T^a$ for $a = 1, 2, 3$, which satisfy the relation $\sum_b \Omega^b = 0$. The covariant derivatives of $f, g^1, g^2$ are written by

$$df = f \alpha + f_1 \theta^1 + f_2 \theta^2 + f_0 \theta, \quad dg^1 = 2g^1 \alpha + g^1_1 \theta^1 + g^1_2 \theta^2 + g^1_0 \theta, \quad dg^2 = 2g^2 \alpha + g^2_1 \theta^1 + g^2_2 \theta^2 + g^2_0 \theta.$$

We shall adopt the similar notational convention for the covariant derivatives of the structure coefficients $R, S, T, N, L$, e.g., $dR = 2R_\alpha + R_1 \theta^1 + R_2 \theta^2 + R_0 \theta$, etc.

By direct computation one finds that the defining equation for Abelian relation

$$d\Omega^a = 0, \quad a = 1, 2,$$

is equivalent to that

$$f_2 = g^1, \quad f_1 = -g^2, \quad g^1_2 = Sf, \quad g^1_1 = Tf, \quad g^1_0 = -Lf + Sg^1, \quad g^2_0 = Nf - Rg^2 + Tg^1.$$  \quad (3.12)

With these relations imposed, the equation $d(d(f)) \alpha \theta = 0$ implies $f_0 = 0$. Differentiating again, the exterior derivative identity $d(d(f)) = 0$ gives

$$g^1_1 = f_0 + Rf, \quad g^2_0 = f_0 - Rf.$$  \quad (3.13)

The linear differential system for the coefficients $\{f, g^1, g^2\}$ closes up at this step. It follows from the theory of linear differential equations that a Legendrian 3-web admits at most three linearly independent Abelian relations.

We proceed to examine the consequences of (3.12), (3.13). A direct computation shows that the identities $d(d(g^1)) = 0, d(d(g^2)) = 0$ give the following three compatibility equations.

$$(S_0 + L_2)f + (-S_1 + 4L)g^1 - S_2g^2 = 0, \quad (3.14)$$

$$(R_0 + L_1)f + (2N - R_1)g^1 + (-S_1 - L)g^2 = 0,$$

$$(-T_0 + N_1)f + T_1g^1 + (-3N - R_1)g^2 = 0.$$

In order for a Legendrian 3-web to have the maximal rank three, these equations should hold identically and each of the nine coefficients of $f, g^1, g^2$ in (3.13) must vanish, otherwise the rank is strictly less than three. This implies, after a short computation, that the structure coefficients of a Legendrian 3-web must satisfy

$$dR \equiv dS \equiv dT \equiv 0, \mod \alpha, \quad N = L = 0.$$  \quad (3.15)
Conversely, it is clear that when these equations hold a Legendrian 3-web admits three dimensional space of Abelian relations.

The question remains regarding the existence of Legendrian 3-webs satisfying (3.15). The structure equation (3.7), (3.9) with (3.15) imposed is easily seen to be compatible, i.e., \( d^2 = 0 \) is a formal identity of the structure equation, and it follows from the general theory of \( G \)-structures, [7, Appendix], that there exist two parameter family of Legendrian 3-webs with the structure coefficients satisfying (3.15). In particular, the maximum rank of a Legendrian 3-web is \( \rho_3 = 3 \).

We wish to integrate the structure equation (3.7), (3.9) with (3.15) and determine a local normal form for the Legendrian 3-webs of maximum rank in terms of a pair of second order ODE’s. Since \( d\alpha = 0 \), for simplicity let us take a section of the frame bundle \( B \to X \) for which \( \alpha = 0 \). Then \( R, S, T \) become constants (up to nonzero scale depending on the choice of section), and (3.7) is the structure equation of a Lie group. Hence a Legendrian 3-web of maximum rank three is necessarily locally homogeneous.

**Theorem 3.16.** Let \( X \) be a contact three manifold. Let \( \mathcal{W} \) be a Legendrian 3-web on \( X \).

a) The maximum rank of \( \mathcal{W} \) is three.

b) Suppose \( \mathcal{W} \) has the maximum rank. Then the structure invariants of \( \mathcal{W} \) satisfy the relation (3.15), and \( \mathcal{W} \) is locally equivalent to a left invariant homogeneous Legendrian 3-web on a three dimensional Lie group.

There exists a local contact isomorphism of \( X \) with the 1-jet space \( J^1(\mathbb{R}, \mathbb{R}) \) equipped with the canonical contact structure such that \( \mathcal{W} \) is defined by the integral curves of the following set of second order ODE’s. Note that \( R, S, T \)’s in the expressions below are the (constant) structure coefficients of (3.7) with \( \alpha = 0 \).

- **Case** \( R^2 + ST = 0 \). When \( R = S = 0 \),

\[
\begin{align*}
y'' + Ty &= 0, \\
y'' + Ty + 1 &= 0,
\end{align*}
\]

and the Legendrian foliation by the fibers of projection \( J^1(\mathbb{R}, \mathbb{R}) \to J^0(\mathbb{R}, \mathbb{R}) \). The general cases are obtained by applying the linear action (3.17) to (3.18).

- **Case** \( R^2 + ST > 0 \). When \( R > 0, S = T = 0 \),

\[
\begin{align*}
y'' + R(y')^2 &= 0, \\
y'' - R(y')^2 &= 0, \\
y'' + R \tanh(Ry)(y')^2 &= 0.
\end{align*}
\]

The general cases are obtained by applying the linear action (3.17) to (3.19).

- **Case** \( R^2 + ST < 0 \). When \( R = 0, -S = T \neq 0 \),

\[
\begin{align*}
y'' + \tan(Ty)(1 - T(y')^2) &= 0, \\
y'' - \cot(Ty)(1 - T(y')^2) &= 0, \\
y'' - \frac{\cos(Ty) - \sin(Ty)}{\cos(Ty) + \sin(Ty)}(1 - T(y')^2) &= 0.
\end{align*}
\]

The general cases are obtained by applying the linear action (3.17) to (3.20).

**Proof.** The proof is by direct computation. Set the contact 1-form \( \theta = dy - pdx \). Then \( \theta \) and \( \theta^1, \theta^2 \) below satisfy the structure equation (3.7), (3.15) with \( \alpha = 0 \). Note that under the linear action by an element \( g \in \text{SL}_2\mathbb{R} \),

\[
\begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix} \rightarrow g \begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix}, \quad \text{then} \quad \begin{pmatrix} R & S \\ T & -R \end{pmatrix} \rightarrow g \begin{pmatrix} R & S \\ T & -R \end{pmatrix} g^{-1}. \tag{3.17}
\]

- **Case** \( R^2 + ST = 0 \). Up to the conjugation (3.17), one may assume \( R = S = 0 \). Set

\[
\begin{align*}
\theta^1 &= dx, \\
\theta^2 &= dp + Tydx,
\end{align*}
\]

\[
\theta^1 + \theta^2 = dp + (Ty + 1)dx.
\]

\[\text{See [4, p29]}\] for a classification of three dimensional Lie groups.
• **Case** $R^2 + ST > 0$. Up to (3.17), one may assume $R > 0$, $S = T = 0$. Set
\[ \theta^1 = \frac{\exp (Ry)}{\sqrt{2R}} (dp + Rp^2 dx), \]
\[ \theta^2 = \frac{\exp (-Ry)}{\sqrt{2R}} (dp - Rp^2 dx), \]
\[ \theta^1 + \theta^2 = \frac{2 \cosh (Ry)}{\sqrt{2R}} (dp + R \tanh (Ry)p^2 dx). \]

• **Case** $R^2 + ST < 0$. Up to (3.17), one may assume $R = 0$. Set
\[ \theta^1 = \frac{\cos(Ty)}{\sqrt{1 - Tp^2}} (dp + \tan(Ty)(1 - Tp^2) dx), \]
\[ \theta^2 = \frac{\sin(Ty)}{\sqrt{1 - Tp^2}} (dp - \cot(Ty)(1 - Tp^2) dx), \]
\[ \theta^1 + \theta^2 = \frac{\cos(Ty) + \sin(Ty)}{\sqrt{1 - Tp^2}} (dp - \frac{\cos(Ty) - \sin(Ty)}{\cos(Ty) + \sin(Ty)} (1 - Tp^2) dx). \]

3.2. Geodesic Legendrian webs of maximum rank

There is a geometric situation where Legendrian webs naturally occur. Let $M$ be a two dimensional surface. Let $\mathbb{P}(TM) \to M$ be the projective tangent bundle equipped with the canonical contact structure. Consider on $M$ a finite set of Riemannian metrics $\{ g^a \}_{a=1}^d$. Each metric $g^a$ defines the geodesic Legendrian foliation on $\mathbb{P}(TM)$. Combined with the foliation by fibers of projection $\mathbb{P}(TM) \to M$, which are Legendrian, a set of $d$ Riemannian metrics on a surface $M$ gives rise to a Legendrian $(d + 1)$-web $\mathcal{W}_{\{g^a\}}$ on $\mathbb{P}(TM)$.

In this section, we give an application of the preceding analysis to the case a Legendrian 3-web is defined by a pair of two dimensional Riemannian metrics.

**Definition 3.21.** Let $g_{\pm}$ be a pair of Riemannian metrics on a two dimensional surface $M$. Under an appropriate transversality condition\(^{13}\) let $\mathcal{W}_{g_{\pm}}$ be the Legendrian 3-web on $\mathbb{P}(TM)$ defined by the geodesic foliations of $g_{\pm}$ and the fibers of projection $\mathbb{P}(TM) \to M$. The pair of Riemannian metrics are maximally geodesically compatible when the associated Legendrian 3-web $\mathcal{W}_{g_{\pm}}$ has the maximum rank three. In this case, $g_{\pm}$ are the maximally geodesically compatible mates to each other.

Theorem 3.16 implies that a maximally geodesically compatible pair of metrics share at least three dimensional algebra of projective vector fields. The classification result of [9] then immediately gives the local normal forms for such pair of metrics, Theorem 3.23.

Before we state the main result, let us recall the relevant known facts on the two dimensional Riemannian metrics. We refer to [9] [30] [29] for the details.

Let $M$ be a two dimensional oriented Riemannian manifold equipped with the metric $g$. Let $\pi : FM \to M$ be the $SO_2$-bundle of oriented orthonormal frames. Let $\{ \omega^1, \omega^2 \}$ be the tautological 1-forms on $FM$ such that $\pi^* g = (\omega^1)^2 + (\omega^2)^2$. The Levi-Civita connection 1-form $\rho$, and the Gauß curvature $K$ are uniquely defined on $FM$ by the equations
\[ d\omega^1 = -\rho \wedge \omega^2, \]
\[ d\omega^2 = \rho \wedge \omega^1, \]
\[ dp = K \omega^1 \wedge \omega^2. \]

Differentiating the last equation, the higher order derivatives of $K$ are inductively defined by
\[ dK = K_1 \omega^1 + K_2 \omega^2, \]
\[ dK_1 = -K_2 \rho + K_{11} \omega^1 + K_{12} \omega^2, \]
\[ dK_2 = K_1 \rho + K_{21} \omega^1 + K_{22} \omega^2, \] where $K_{12} = K_{21}$, etc.

---

\(^{13}\)See Step 1 in Section 3.2.1
The set of unparameterized, oriented geodesics of \( g \) defines the geodesic flow (foliation) \( \mathcal{F}_g \) on \( \mathbb{P}(TM) \), which is naturally extended to the tangent bundle \( TM \to M \).

Let \( \otimes^k(T^*M) \to M \) be the bundle of symmetric \( k \)-forms on \( M \), which we consider as the functions on \( TM \) homogeneous of degree \( k \) on the fibers. Set

\[
I_k(g) = \{ \sigma \in H^0(\otimes^k(T^*M)) \mid \sigma \text{ is a first integral for } \mathcal{F}_g \},
\]

\[
\varrho_k(g) = \text{dimension}(I_k(g)).
\]

The following results on \( I_k(g) \) for \( k = 1, 2, 3 \), are known. The class of Riemannian metrics we shall be interested in are the Darboux super-integrable metrics for which \( \varrho_2(g) = 4 \).

\( I_1(g) \):

An element of \( I_1(g) \) corresponds to a Killing vector field of \( g \). For any metric \( g \), one necessarily has \( \varrho_1(g) = 3 \), 1, or 0 (generic case);

- \( \varrho_1(g) = 3 \) when \( g \) is a metric of constant curvature.
- \( \varrho_1(g) = 1 \) when \( g \) admits a Killing vector field, which is unique up to scale. For an analytic characterization of this class of metrics, see [16]. For instance, the curvature must satisfy the equation \( K_1K_2(K_{11} - K_{22}) - (K_1^2 - K_2^2)K_{21} = 0 \).

\( I_2(g) \):

An element of \( I_2(g) \) corresponds to a Riemannian metric projectively equivalent to \( g \), [29]. For any metric \( g \), \( \varrho_2(g) = 6, 4, \ldots, \) or 1 (generic case);

- \( \varrho_2(g) = 6 \) when \( g \) is a metric of constant curvature.
- \( \varrho_2(g) = 4 \) when \( g \) admits a Killing field (\( \varrho_1(g) = 1 \)), and the curvature of \( g \) satisfies an additional set of second, and third order differential equations. This class of metrics are called Darboux super-integrable, [9].

Darboux super-integrable metrics are equivalently characterized by having three dimensional algebra of projective vector fields. Combining this with a classical result of Lie on the classification of the algebra of projective vector fields on the plane, [9] determined the explicit local normal forms for this class of metrics.

\( I_3(g) \):

The cubic integrals are studied in [30]. For any metric \( g \), \( \varrho_3(g) \leq 10 \), and \( \varrho_3(g) = 10 \) when \( g \) is a metric of constant curvature. It is possible that the next admissible value of \( \varrho_3(g) \) is 4, which is attained by the Darboux super-integrable metrics.

Let us now state the main results of this section.

**Theorem 3.23.** Let \( g_+ \) be a two dimensional Riemannian metric. \( g_+ \) admits a maximally geodesically compatible mate \( g_- \) whenever \( g_+ \) is either of constant curvature, or Darboux super-integrable.

Let \( \mathcal{R}_{g_+} \) denote the moduli space of Riemannian metrics maximally geodesically compatible with the given metric \( g_+ \).

a) when \( g_+ \) is of constant curvature, \( \mathcal{R}_{g_+} \) consists of 10 parameter family of metrics of constant curvature, and 8 parameter family of Darboux super-integrable metrics.

b) when \( g_+ \) is Darboux super-integrable, \( \mathcal{R}_{g_+} \) consists of 5 parameter family of metrics of constant curvature, and 4 parameter family of Darboux super-integrable metrics.

Many of the metrics in this theorem are geodesically equivalent. For example, for the class of Darboux super-integrable metrics \( g_+ \) to be considered in Section 3.2.2, all of the 5 parameter family of mates of constant curvature are geodesically equivalent, and the 4 parameter family of Darboux super-integrable mates are foliated by 3 parameter families of geodesically equivalent metrics.

### 3.2.1. Differential analysis

The differential analysis for the proof of Theorem 3.23 is a straightforward application of the over-determined PDE machinery. Due to the size of algebraic expressions involved, the computation was performed on the computer algebra system Maple. Let us record the relevant steps of the analysis, only for the case of Darboux super-integrable metrics. The analysis for the constant curvature metrics is similar, and shall be omitted.

Given a Darboux super-integrable metric \( g_+ \) on a two dimensional surface \( M \), let \( FM \to M \) denote the associated oriented orthonormal frame bundle. Let \( \{ \omega^1, \omega^2, \rho \} \) be the canonical 1-forms on \( FM \) which satisfy the structure equation
From the general theory, the geodesic foliation of $g_+$ on $FM$ is defined by the rank 2 sub-bundle

$$\mathcal{I}^+ = (\omega^2, \rho) \subset T^*FM.$$  

Here $\omega^2$ defines the canonical contact structure on $FM$.

**Proof of Theorem 3.23**

**Step 0.** Let $g_-$ be another metric on $M$. When pulled back to $FM$, one may write

$$g_- = (\eta^1)^2 + (\eta^2)^2,$$

for a $g_-$-orthonormal coframe $\{ \eta^1, \eta^2 \}$, where

$$\eta^1 = a\omega^1 + b\omega^2,$$

$$\eta^2 = c\omega^2,$$

for coefficients $a, b, c$. Let $\psi$ denote the connection 1-form for $\{ \eta^1, \eta^2 \}$ so that

$$d\eta^1 = -\psi \wedge \eta^2,$$

$$d\eta^2 = \psi \wedge \eta^1,$$

$$d\psi = Q\eta^1 \wedge \eta^2,$$

where $Q$ is the curvature of the metric $g_-$. Differentiating (3.24), the structure equation (3.25) implies that

$$d\psi = 2c_1\rho - a_1\psi + t\omega^1 + t_2\omega^2.$$  

Let us remark here that we shall adopt the similar notational convention for the covariant derivatives, i.e., $da_1 = a_{11} \omega^1 + a_{12} \omega^2$, mod $\rho, \psi$, etc. The curvature $Q$ for the metric $g_-$ is for example given by

$$Q = \frac{-c_{11} - t_2 + Kc}{a^2 c}.  \quad (3.26)$$

**Step 1.** Set the rank 2 sub-bundles

$$\mathcal{I}^0 = (\omega^2, \omega^1),$$

$$\mathcal{I}^- = (\eta^2, \psi),$$

$$= (\omega^2, c\rho + t\omega^1).$$

Then the set of three foliations defined by $\{ \mathcal{I}^\pm, \mathcal{I}^0 \}$ is the Legendrian 3-web $\mathcal{W}_{g_{\pm}}$ on $FM$, under the non-degeneracy condition that $a, c, t \neq 0$, which we assume from now on.

**Step 2.** Following the analysis of Section 3.1, it is straightforward to determine the associated coframe $\theta, \theta^a, a = 1, 2, 3,$ and $\alpha$, for $\mathcal{W}_{g_{\pm}}$, which fit into the structure equation (3.7), (3.9). The torsion coefficients $R, S, T, N, L$ are then expressed as the rational functions in the successive derivatives of $a, b, c, t$. 

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Step 3. The idea is that the integrability condition (3.15) for the Legendrian 3-web \( W_{g^\pm} \) to have maximum rank three allows one to close up the structure equations for \( a, b, c, t \).

Differentiating \( R \) and evaluating modulo \( \alpha, \omega^2 \), one may solve for \( \{ c_{21}, t_{21} \} \) in terms of the rest of the variables. Differentiating \( S \) and evaluating modulo \( \alpha, \omega^2, \omega^1 \) with these relations, one may solve for \( \{ b_2 \} \). Successively differentiating this equation for \( b_2 \), one may solve for \( \{ b_{21}, b_{22}, c_2 \} \). Successively differentiating again the equation for \( c_2 \), one may solve for \( \{ b_{13}, c_{22} \} \).

Differentiating \( R \) and evaluating modulo \( \alpha, \omega^1, \rho \), one may solve for \( \{ t_{22} \} \). This implies that \( dR, dS \equiv 0 \), mod \( \alpha \).

Step 4. At this step, the remaining undetermined second derivatives of \( a, b, c, t \) are \( a_{11}, c_{11}, t_{11} \). Introduce the third order derivatives \( da_{11} \equiv a_{111} \omega^3 + a_{112} \omega^2 \), mod \( \rho \), etc, for these variables.

From the exterior derivative identities \( d(d(a_1)), d(d(c_1)), d(d(t_1)) = 0 \), one may solve for \( \{ a_{112}, c_{112}, t_{112} \} \). From the exterior derivative identity \( d(d(b_1)) = 0 \), one may solve for \( \{ c_{111} \} \). Differentiating \( T \) and evaluating modulo \( \alpha, \omega^2 \) with these relations, one may solve for \( \{ a_{111}, t_{111} \} \). The structure equations for \( a, b, c, t \) close up at this step. We remark that one already has \( d\alpha = 0 \).

Step 5. A key integrability equation is obtained from the identity \( d(d(\frac{1}{2}a_{11} - 2c_{11})) = 0 \), which gives

\[ 3atK_1 + (tb + ca_1 - 2ac_1)K_2 = 0. \]

Assuming the metric \( g_+ \) is not of constant curvature, one may solve for \( \{ c_1 \} \). Differentiating this equation for \( c_1 \), one may solve for \( \{ c_{11} \} \).

Step 6. Differentiating \( T \) and evaluating modulo \( \alpha \), one may solve for \( \{ t_{11} \} \). Differentiating this equation for \( t_{11} \), one may solve for \( \{ t_1, t_2 \} \). Note that the set of remaining independent variables at this step are \( \{ a, b, c, t; a_1, b_1, a_{11} \} \).

Taking the exterior derivative \( d(d(a_{11})) = 0 \), the resulting single integrability equation factors into two parts. The vanishing of the one part is equivalent to that the curvature \( Q \) of the metric \( g_- \) is constant. Assuming this is not the case, the vanishing of the other integrability equation allows one to solve for \( \{ b_1 \} \). Differentiating this equation for \( b_1 \), one may solve for \( \{ a_{11} \} \).

At this stage, the remaining independent variables are \( \{ a, b, c, t; a_1 \} \), and the structure equations for these variables are compatible. Since the metric \( g_- \) is well defined on the surface \( M \), and is invariant under the SO(3) action along the fibers of \( FM \rightarrow M \), from the general theory of differential equations it follows that the moduli space of these maximally geodesically compatible mates is generically \( 5 - 1 = 4 \) dimensional.

The analysis for the case when the metric \( g_- \) has constant curvature follows from the similar analysis.

3.2.2. Example

Lie classified the possible local symmetry Lie algebras, and the representations thereof, of projective vector fields on the plane. Bryant, Manno, & Matveev used this to give explicit local normal forms for the two dimensional (pseudo) Riemannian metrics admitting a transitive algebra of projective vector fields. [9]. In this sub-section, we apply this result to give examples of the maximally geodesically compatible pairs of Darboux super-integrable metrics. We shall closely follow [9].

Let \( (x, y) \) be a local coordinate of \( \mathbb{R}^2 \). Let \( g_+ \) be the Darboux super-integrable metric

\[ g_+ = e^{3x}dx^2 - 2D_+ e^x dy^2. \]  

The projective connection associated with \( g_+ \), or equivalently the equation of un-parameterized geodesics of \( g_+ \), is given by the second order ODE

\[ y'' = \frac{1}{2}y' + D_+ e^{-2x}(y')^3. \]  

The local symmetry algebra of \( \mathfrak{g}_+ \) is generated by

\[ \left\langle \frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, 2y \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} \right\rangle. \]  

One may verify by direct computation that the projective connections invariant under \( \mathfrak{g}_+ \) are defined by the second order ODE’s of the following form

\[ y'' = \frac{1}{2}y' + De^{-2x}(y')^3, \quad D \text{ is constant.} \]  

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This projective connection is flat whenever $D = 0$. Since we are considering the pair of Darboux super-integrable metrics, assume $D \neq 0$.

The maximally geodesically compatible mates of $g_+$ share the local symmetry algebra of projective vector fields. It thus suffices to find the metrics which has \[(3.30)\] as the equation of un-parameterized geodesics. Consider for example the metrics of the form

$$g_- = E(x, y)dx^2 + G(x, y)dy^2.$$ 

Then $g_-$ is a mate of $g_+$ with the equation of geodesics \[(3.30)\] whenever

$$E(x, y) = E(x) = \frac{e^{2x+c_1}}{-2De^{-x+c_1} + c_2},$$

$$G(x, y) = G(x) = \frac{1}{-2De^{-x+c_1} + c_2}, \quad D \neq D_+; \quad c_1, c_2 \text{ are constants.}$$

The three (un-differentiated) Abelian relations for the Legendrian 3-web $W_{g_\pm}$ are given by;

\[
\left(\frac{-2D_+e^{-x}y^2p^2 + 4p^2e^x + e^x y^2 - 4pe^xy}{2p^2(D - D_+)}\right), \quad \left(\frac{2De^{-x}y^2p^2 - 4p^2e^x - e^x y^2 + 4pe^xy}{2p^2(D - D_+)}\right), \quad \left(\frac{-e^{-x}y^2}{2p^2(D - D_+)}\right);
\]

\[
\left(\frac{-2D_+e^{-x}yp^2 - 2pe^x + e^x y}{2p^2(D - D_+)}\right), \quad \left(\frac{2De^{-x}yp^2 + 2pe^x - e^x y}{2p^2(D - D_+)}\right), \quad \left(\frac{-e^{-x}y}{2p^2(D - D_+)}\right);
\]

\[
\left(\frac{-2D_+e^{-x}p^2 + e^x}{2p^2(D - D_+)}\right), \quad \left(\frac{2De^{-x}p^2 - e^x}{2p^2(D - D_+)}\right), \quad \left(\frac{-e^{-x}}{2p^2(D - D_+)}\right).
\]

4. Legendrian $d$-webs of rank $\geq \rho_d$

We now turn our attention to the Legendrian $d$-webs for general $d \geq 3$.

As the analysis in Section 3 will show, the essential step in establishing the rank bound for a Legendrian $d$-web is to understand the layers of compatibility equations for the linear differential system for Abelian relations. In this section we consider the class of simple Legendrian webs defined by the second order ODE’s (4.1) below.

A generating set of first integrals for this class of Legendrian webs are written explicitly as polynomials in the adapted local coordinates. In hindsight the aforementioned compatibility equations are reflected in the higher degree algebraic relations among them, and this leads one to conclude that there exists at least $\rho_d$ linearly independent Abelian relations, Proposition 4.3

The analysis in Section 5 will show that these account for all the Abelian relations and the class of Legendrian $d$-webs defined by (4.1) have rank $\rho_d$.

Let $X = J^1(\mathbb{R}, \mathbb{R})$ equipped with the canonical contact structure, Example 1.4 Let $(x, y, p)$ be the adapted coordinate so that the contact line bundle is generated by $C = \langle dy - pdx \rangle$. Consider a Legendrian $d$-web $W$ on $X$ defined by the following set of second order ODE’s;

\[
y'' = q^a, \quad a = 1, 2, \ldots d,
\]

\[
q^a \text{'s are distinct constants.}
\]

The corresponding dual rank 2 sub-bundles of $T^*X$ are

\[
T^a = \langle dy - pdx, dp - q^a dx \rangle, \quad a = 1, 2, \ldots d.
\]

We claim that $W$ has at least $\rho_d$ linearly independent Abelian relations.\[15\]
Proposition 4.3. Let \( X = J^1(\mathbb{R}, \mathbb{R}) \) be the 1-jet space of scalar functions on \( \mathbb{R} \) equipped with the canonical contact structure. Let \( \mathcal{W} \) be the Legendrian \( d \)-web on \( X \) defined by the second order ODE’s \((4.1)\), \( d \geq 3 \). Then \( \mathcal{W} \) has at least \( \rho_d \) linearly independent Abelian relations, and the rank of \( \mathcal{W} \) is bounded below by \( \rho_d \).

We present a proof of the proposition in the following five steps. In the first four steps, we construct \( \rho_d \) Abelian relations. In the last step, we show that they are linearly independent.

Let us indicate here a minor technical point for the analysis in this section. An Abelian relation of a Legendrian web defined by a set of rank 2 sub-bundles \( \{ I^a \}_{a=1}^d \) is by definition a \( d \)-tuple of closed 1-forms \( (\Omega^1, \Omega^2, ... \Omega^d) \), \( \Omega^a \in H^0(I^a) \), such that \( \sum_a \Omega^b = 0 \). Fix a reference point \( x_0 \in X \), and let \( h^a \) be the unique local anti-derivative \( dh^a = \Omega^a \) in a neighborhood of \( x_0 \) such that \( h^a(x_0) = 0 \). This defines an isomorphism from \( \mathcal{A}(\mathcal{W}) \) to the space of local un-differentiated Abelian relations, which consist of \( d \)-tuple of respective first integrals \( (h^1, h^2, ... h^d) \) such that they vanish at \( x_0 \), and that \( \sum_b h^b = 0 \).

We shall freely use either of these equivalent forms as convenient from now on.

Proof of Proposition \([4.3]\)
Recall \( X = J^1(\mathbb{R}, \mathbb{R}) \) with the adapted coordinate \((x, y, p)\). Choose \( x_0 = (0, 0, 0) \in X \) for a reference point. The objects of analysis below are all algebraic (analytic) and it suffices to prove the claim locally in a neighborhood of \( x_0 \).

**Step 1.** Universal first integrals:

Consider the second order ODE

\[
y'' = q, \tag{4.4}
\]

where \( q \) is an indeterminate constant. We wish to construct the first integrals for \((4.4)\) which are polynomials in \( q \) of the form

\[
h(q) = h_0 + h_1 q + h_2 q^2 + \ldots + h_m q^{m-1}, \tag{4.5}
\]

where \( h_j|_{x_0} = 0 \).

Set the following three first integrals for \((4.4)\) which are linear in \( q \):

\[
\begin{align*}
u_0^3 &= (y - px) + q \frac{x^2}{2}, \\
u_1^3 &= px, \\
u_2^3 &= \frac{p^2}{2} - qy.
\end{align*}
\]

Note that \((u_j^3)^2 = 2q u_0^3 + 2u_2^3\).

Inductively define \( u_j^{m+1} \) for \( m \geq 3 \), \( 0 \leq j \leq 2m - 2 \), by

\[
\begin{align*}
u_j^{m+1} &= u_0^m u_j^m, & j &= 0, 1, \ldots, 2m - 4, \\
u_j^{m+1} &= u_2 u_{j-2}^{m-1} & j &= 2m - 3, \ldots, m, \\
u_j^{m+1} &= u_0^m u_2 u_{j-2}^{m-4}. & j &= 2m - 2, \ldots, m - 2.
\end{align*}
\]

One has the formula

\[
u_j^{m+1} = (u_0^j)^{j_0} (u_1^j)^{j_1} (u_2^j)^{j_2}, \quad 0 \leq j \leq 2m - 2, \tag{4.7}
\]

where the indices \((j_0, j_1, j_2)\) are uniquely determined by

\[
j_0 + j_1 + j_2 = m - 1, \tag{4.8}
\]

\[
j_0, j_2 \geq 0, \quad j_1 = 0, \text{ or } 1, \\
&j = 0 \cdot j_0 + 1 \cdot j_1 + 2 \cdot j_2.
\]

Since these are polynomials in \( \{ u_0^3, u_1^3, u_2^3 \} \), they are all first integrals for \((4.4)\).

**Step 2.** Basic properties:

Assign the weights

\[
\text{weight}(x, y, p, q) = (-1, 0, 1, 2)
\]

respectively. We extend the weight to the polynomials in an obvious way. Let \( \mathbb{R}[q] \) be the polynomial ring in the indeterminate \( q \).

The following lemma follows immediately from the construction of \( u_j^{m+1} \). We shall omit the proof.
Lemma 4.9.

a) Each \( u_j^{m+1} \) is homogenous of weight \( j \). For a fixed \( m \geq 2 \), the set of elements \( \{ u_j^{m+1} \} \) is linearly independent over \( \mathbb{R} \) as \( \mathbb{R}[q] \)-valued functions.

b) Each \( v_j^{m+1} \) has degree \( m - 1 \) in \( q \), and the set of elements \( \cup_{m=2}^{\infty} \{ v_j^{m+1} \} \) are linearly independent over \( \mathbb{R} \) as \( \mathbb{R}[q] \)-valued functions.

The following refined linear independence property will be used in Step 4.

Lemma 4.10. Let \( u = u(q) \) be a finite linear combination of the elements in \( \cup_{m=2}^{\infty} \{ u_j^{m+1} \} \). If \( u(q_0) = 0 \) for a value \( q_0 \), then \( u = 0 \).

Proof. Note that \( u_j^{m+1} \equiv y^{j_0} p_0 \left( \frac{p_2}{x} - qy \right)^{j_2} \mod x \), where \( (j_0, j_1, j_2) \) is determined by (4.8). Evaluating at \( q = q_0 \), the highest weight term of \( u_j^{m+1} \mod x \) (now forgetting the weight of \( q \)) is \( y^{j_0} p_0^{j_1 + 2 j_2} \). From (4.8), since \( j_1 = 0 \), or 1, the associated index map \( (m, \tilde{j}) \to (j_0, \tilde{j} = j_1 + 2 j_2) \) is injective. Lemma follows from this by applying induction on decreasing weights. \( \square \)

Step 3. Abelian relations:

Let \( q^l \equiv (\{ q_1 \}^l, \{ q_2 \}^l, \ldots, \{ q_d \}^l) \) denote a vector in \( \mathbb{R}^d \) for \( l = 0, 1, \ldots, d - 2 \). Since the constants \( q^a \)'s are distinct, by Vandermonde identity these vectors are linearly independent and there exists a linearly independent set of vectors \( \mu^l = (v_{1\mu}^l, v_{2\mu}^l, \ldots, v_{d\mu}^l), \mu = 1, 2, \ldots, d - 1 \), which are complimentary to \( \{ q^l \}_{l=0}^{d-2} \) such that

\[
(\mu^l, q^l) = \sum_a \mu_a^l (q^a)^l = 0, \quad \text{for } l = 0, 1, \ldots, d - \mu - 1. \tag{4.11}
\]

Take an element \( h = v_j^{m+1}, m \leq d - 1 \), and consider the first integral \( h(q^a) \) for \( \mathbb{I}^n, \tag{4.2} \). By definition of \( v_j^{m+1} \), it is a polynomial of the form

\[
h(q^a) = h_0 + h_1 q^a + h_2 (q^a)^2 + \ldots, h_{m-1} (q^a)^{m-1},
\]

which is of degree \( m - 1 \) in \( q^a \). It follows that for each \( \mu^l, 1 \leq \mu \leq d - m \), the \( d \)-tuple of first integrals

\[
\left( v_1^\mu h(q^1), v_2^\mu h(q^2), \ldots, v_d^\mu h(q^d) \right) \tag{4.12}
\]

gives an Abelian relation for \( W \) (the sum of components vanish by (4.11)).

In summary, each \( u_j^{m+1}, 2 \leq m \leq d - 1 \) and \( 0 \leq j \leq 2m - 2 \), gives rise to \( d - m \) Abelian relations.

Step 4. Decomposition of \( \rho_d \):

The preceding analysis suggests the following decomposition of \( \rho_d \).

\[
\begin{align*}
\rho_3 & : \quad 1 \cdot 3 \\
\rho_4 & : \quad 2 \cdot 3 + 1 \cdot 5 \\
\rho_5 & : \quad 3 \cdot 3 + 2 \cdot 5 + 1 \cdot 7 \\
\rho_6 & : \quad 4 \cdot 3 + 3 \cdot 5 + 2 \cdot 7 + 1 \cdot 9 \\
\vdots & \\
\rho_d & : \quad (d - 2) \cdot 3 + (d - 3) \cdot 5 + (d - 4) \cdot 7 + \ldots + 1 \cdot (2d - 3).
\end{align*}
\tag{4.13}
\]

The set of universal first integrals \( \{ u_j^{m+1} \}_{j=0}^{2m-2} \) for \( m = 2, 3, \ldots \) are identified with the diagonal entries \( 1 \cdot 3, 1 \cdot 5, 1 \cdot 7, 1 \cdot 9, \ldots \), in the above decomposition. The entries on the column below each diagonal element are identified with those generated by the appropriate vectors \( v^l, (4.11), (4.12) \). One may check that this yields the correct formula for \( \rho_d \).

Step 5. Linear independence:

From Step 3, for fixed \( d \geq 3 \) the given set of \( \rho_d \) Abelian relations are explicitly written by

\[
\Gamma_j^{m;\mu} = \left( v_1^{\mu} u_j^{m+1}(q^1), v_2^{\mu} u_j^{m+1}(q^2), \ldots, v_d^{\mu} u_j^{m+1}(q^d) \right), \tag{4.14}
\]

for \( 2 \leq m \leq d - 1 \), \( 0 \leq j \leq 2m - 2 \), \( 1 \leq \mu \leq d - m \).
Suppose for a set of coefficients $c_{m;\mu}^j$, a linear combination
\[ \sum_{m;\mu,j} c_{m;\mu}^j \Gamma_j^{m;\mu} = 0. \]
In component-wise this is equivalent to that for each $a = 1, 2, \ldots, d$,
\[ \sum_{m;\mu} (\sum_{\mu} c_{m;\mu}^j v_{a}^{m+1}) u_j^{m+1}(q^a) = 0. \]
By Lemma $4.10$ this implies $\sum_{m;\mu} (\sum_{\mu} c_{m;\mu}^j v_{a}^{m+1}) u_j^{m+1} = 0$. By Lemma $4.9$ this then implies for each $a, m, j$ that
\[ \sum_{\mu} c_{m;\mu}^j v_{a}^{m+1} = 0. \]
The linear independence of the vectors $v^\mu = (v_1^\mu, v_2^\mu, \ldots, v_d^\mu)$, (4.11), then forces $c_{m;\mu}^j = 0$. □

Proposition $4.3$ shows that the maximal rank of a Legendrian $d$-web is at least $\rho_d$.

The class of Legendrian webs (4.1) will also play a role in the analysis for the rank bound in the next section. The linear differential equation for Abelian relations will be examined on this class of Legendrian webs to show that the associated symbol is nondegenerate.

5. Maximum rank of a Legendrian web

5.1. Overview

In sub-section $5.1.1$, we record a theorem on the moduli of solutions to a closed linear differential system with constraint, Theorem $5.8$. The theorem is well known and follows from an elementary application of Frobenius theorem. We record and emphasize it here because it is the main conceptual ingredient to our proof of the rank bound for Legendrian webs.

In sub-section $5.1.2$ we explain a rough idea of how to apply Theorem $5.8$ to the linear differential system for Abelian relations and obtain a upper-bound on the rank.

5.1.1. Linear differential system with constraints

Let $X$ be a finite dimensional, simply connected manifold. On $X$ suppose there be an $n$-by-$n$ matrix valued ($n \geq 1$) 1-form $\phi$. For a $\mathbb{R}^n$-valued function $f = (f^1, f^2, \ldots, f^n)$ on $X$, consider the following closed linear differential system associated with $\phi$;
\[ df = f \phi. \]

Let $S$ be the $\mathbb{R}$-vector space of solutions to (5.1).

Set
\[ \Phi := d\phi + \phi \wedge \phi. \]

Differentiating (5.1), one gets the compatibility equation that a solution $f$ necessarily satisfies,
\[ f \Phi = 0. \]

Suppose the differential equation (5.1) is compatible in the sense that the $n$-by-$n$ matrix valued 2-form $\Phi$ vanishes
\[ \Phi = 0, \]
and the equation (5.3) holds identically. Then the following existence and uniqueness theorem is well known.

\[ ^{16}\text{Here 'closed' means that the derivatives of the unknown function } f \text{ are determined as the functions of } f \text{ itself and do not involve any new variables.} \]
Theorem 5.5 (Moduli of solutions to a compatible, closed linear differential system). Let $X$ be a finite dimensional, simply connected manifold. Consider a compatible, closed linear differential system (5.1) as described above, for which (5.4) holds. Let $x_0 \in X$ be a reference point. Then for any finite value $f_0 \in \mathbb{R}^n$ there exists a unique solution to (5.1) satisfying the initial value condition $f(x_0) = f_0$. One consequently has that $\dim(S) = n$.

For our intended proof of the rank bound for Legendrian webs, we shall make use of two variants of the above theorem of the following kinds.

Consider now the case when the two form $\Phi$ in (5.2) does not vanish identically, and it imposes a set of $m$ linear constraint equations (not necessarily independent)

$$ E^i := \sum_a C^i_a f^a = 0, \quad i = 1, 2, \ldots, m. \quad (5.6) $$

Here $(C^i_a)$ is an $m$-by-$n$ matrix valued function on $X$. It is possible that the successive derivatives of (5.6) impose a sequence of new linear compatibility equations. A closed linear differential system (5.1) with constraint (5.6) is called compatible when this does not occur, and symbolically one has that

$$ dE^i \equiv 0 \mod \{E^j\}_{j=1}^m, \quad i = 1, 2, \ldots, m. \quad (5.7) $$

The following theorem is an immediate application of Theorem 5.5.

Theorem 5.8 (Moduli of solutions to a closed linear differential system with constraints). Let $X$ be a finite dimensional, simply connected manifold. Consider a closed linear differential system (5.1) as described above, but with constraint (5.6).

a) Suppose $\text{rank}(C^i_a)$ is constant. Then

$$ \dim(S) \leq n - \text{rank}(C^i_a). $$

b) Suppose the linear differential system with constraints is compatible and (5.7) holds. Let $x_0 \in X$ be a reference point. Then for any finite value $f_0 \in \mathbb{R}^n$ which satisfies the constraint equation (5.6) at $x_0$, i.e.,

$$ \sum_a C^i_a(x_0) f^a_0 = 0, \quad i = 1, 2, \ldots, m, $$

there exists a unique solution to (5.1) with constraint (5.6) satisfying the initial value condition $f(x_0) = f_0$. Suppose furthermore that $\text{rank}(C^i_a)$ is constant. Then one has that

$$ \dim(S) = n - \text{rank}(C^i_a). $$

The proof is by an application of Frobenius theorem. We shall omit the details and refer the reader to [20].

5.1.2. Sketch of idea

The argument for the proof of rank bound consists of the following four steps: Initial problem, Prolongation, Closing up, and Non-degeneracy of symbol and rank bound.

Initial problem. Recall $X$ is a contact three manifold with the contact line bundle $\mathcal{C} \subset T^*X$. Let $\mathcal{W}$ be a Legendrian $d$-web on $X$ defined by a set of $d$ rank two sub-bundles $T^a \supset \mathcal{C}, a = 1, 2, \ldots, d$. An Abelian relation is by definition a section of the direct sum bundle $\bigoplus_{b=1}^d T^b$ which satisfies the equations

$$ E_0 : \sum_b \Omega^b = 0, \quad (5.9) $$

$$ P_1 : \quad d\Omega^b = 0. \quad (5.10) $$

Here $E_0$ is a 0-th order compatibility equation, and $P_1$ is a 1-st order linear differential equation.

Let us use $\mathcal{F}_w$ to denote symbolically the $w$-th order derivatives of the section of $\bigoplus_{b=1}^d T^b$. Then $E_0$ imposes a set of linear relations on $\mathcal{F}_0$, and $P_1$ imposes a set of linear relations on $\mathcal{F}_1 \mod \mathcal{F}_0$.

\[\text{In the actual analysis we shall not use the filtration by order but by depth for the higher order derivatives. See Section 5.2 for the details.}\]
**Prolongation.** Differentiating $E_0$ using $P_1$ we get a new compatibility equation denoted by $E_1$, which imposes a set of linear relations on $\mathcal{F}_1 \mod \{F_0\}$. The equation $P_1$ is a first order linear partial differential equation for the sections of $\omega_{\mathbb{R}^1} \oplus \mathbb{I}^b$. Let $P_2$ denote the second order equation obtained by differentiating $P_1$ once by the standard method of prolongation. It imposes a set of linear relations on $\mathcal{F}_2 \mod \{F_1,F_0\}$.

Differentiating $E_1$ using $P_2$ we get a new compatibility equation denoted by $E_2$, which imposes a set of linear relations on $\mathcal{F}_2 \mod \{F_1,F_0\}$.

Continuing in this manner, one obtains a sequence of pairs $(E_w, P_{w+1})$ of higher order compatibility equations $E_w$ and differential equations $P_{w+1}$. The pair $(E_w, P_{w+1})$ is the $w$-th prolongation of $(E_0, P_1)$.

**Closing up.** Counting the number of variables in $\mathcal{F}_w$ and the number of equations in $E_w$, one finds that there exists $w = w(d)$ such that

\[
\text{(number of variables in } \mathcal{F}_w(d)) = \text{(number of equations in } E_w(d)).
\]

It is at this point that we apply (a) of Theorem [5.8]. If the set of equations $E_w(d) \mod \{\mathcal{F}_w\}_{w=0}^{w(d)-1}$ has full rank on $\mathcal{F}_w(d)$, then one may solve for $\mathcal{F}_w(d)$ in terms of $\{\mathcal{F}_w\}_{w=0}^{w(d)-1}$. The linear differential system for Abelian relations closes up at this order $w(d)$. By (a) of Theorem [5.8] the rank bound is obtained by computing the rank of the set of equations $E_w(d)$,

\[
\dim(A(W)) \leq \sum_{w=0}^{w(d)} \text{(number of variables in } \mathcal{F}_w) - \text{rank}(\bigcup_{w=0}^{w(d)} E_w).
\]

The proof of rank bound for the Legendrian webs is now reduced to checking the rank of a set of linear compatibility equations $\bigcup_{w=d}^{w(d)} E_w$.

**Non-degeneracy of symbol and rank bound.** Let us consider for a moment the analogous problem for the case of planar webs for comparison. The similar strategy as described above works without much change. It is easily checked that the relevant symbol for the set of compatibility equations in this case is a standard Vandermonde matrix. Non-degeneracy of the symbol follows and one gets the desired rank bound.

For Legendrian webs on the other hand, it turns out that the relevant symbol consists of the layers of matrices that contain blocks of Vandermonde-like sub-matrices. Although these are explicit integer matrices, we are currently not able to show directly that they are non-degenerate.

It is at this point that we apply (b) of Theorem [5.8] and prove the non-degeneracy of symbol indirectly. We consider another related system of linear partial differential equation with the following properties;

a) the vector space of solutions are exactly the space of polynomial Abelian relations for the Legendrian webs described in Section [4]

b) the relevant symbol of compatibility equations is isomorphic to that of $\bigcup_{w=0}^{w(d)} E_w$.

The non-degeneracy of the entire set of compatibility equations $\bigcup_{w=0}^{w(d)} E_w$ follows from (b) of Theorem [5.8] by counting the number of dependent variables for the new linear differential system, and by observing that the space of solutions has the expected dimension $\rho_d$, which also gives the rank bound for our problem.

The remaining analysis is divided into two parts. **Initial problem** and **Prolongation** are examined in Section [5.2]. **Closing up** and **Non-degeneracy of symbol and rank bound** are examined in Section [5.3].

5.2. **Structure equation**

We continue the analysis from (2.2), Section [2.1]

5.2.1. **Initial problem**

Let $\mathcal{W}$ be a Legendrian $d$-web on a contact three-manifold $X$. Let $(x, y, p)$ be an adapted local coordinate of $X$ so that $\mathcal{W}$ is defined by the $d$ second order ODE’s (2.2). The corresponding rank 2 sub-bundles of $T^*X$ are locally generated by

\[
I^a = \{dy - pdz, dp - q^a(x, y, p)dx\}, \quad a = 1, 2, \ldots, d,
\]  

(5.11)

where $q^a \neq q^b$ for $a \neq b$. 

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Set

\[ \theta = dy - p \, dx, \quad \theta^a = dp - q^a \, dx, \quad a = 1, 2, \ldots d, \]  

and let

\[ \Omega^a = f_{10}^a \theta^a + f_{01}^a \theta, \quad a = 1, 2, \ldots d, \]  

denote a section of \( \mathcal{I}^a \) for a set of variables \( \{ f_{10}^a, f_{01}^a \}_{a=1}^d \).

The vector space of Abelian relations \( \mathcal{A}(\mathcal{W}) \) is by definition the space of solutions to the linear differential system \((2.8), (2.9)\). In terms of \( \{ f_{10}^a, f_{01}^a \}_{a=1}^d \), these equations are written as a system of first order linear differential equations with constraint as follows.

\[ \sum f_{10}^a = 0, \quad \sum q^b f_{10}^b = 0, \quad \sum f_{01}^a = 0. \]  

\[ (df_{10}^a + (f_{01}^a + q_{10}^a f_{10}^a) \, dx) \wedge \theta^a + (df_{01}^a + q_{01}^a f_{10}^a \, dx) \wedge \theta = 0, \quad a = 1, 2, \ldots d. \]  

Here \( q_{10}^a, q_{01}^a \) denote the partial derivatives.

### 5.2.2. Prolongation

Following the standard theory of exterior differential systems, [8, Chapter VI], we shall determine the infinite prolongation of the equations \((5.14), (5.15)\).

By Cartan’s lemma, there exist the prolongation variables \( \{ f_{20}^a, f_{11}^a, f_{02}^a \}_{a=1}^d \) such that for any solution \( \{ f_{10}^a, f_{01}^a \}_{a=1}^d \) one has that

\[ df_{10}^a = f_{20}^a \theta^a + f_{11}^a \theta - (f_{01}^a + \tau_{10}^a) \, dx, \]

\[ df_{01}^a = f_{11}^a \theta^a + f_{02}^a \theta - (\tau_{01}^a) \, dx, \quad a = 1, 2, \ldots d, \]  

where \( \tau_{10}^a = q_{10}^a f_{10}^a, \quad \tau_{01}^a = q_{01}^a f_{10}^a \). Note the identity

\[ \tau_{10}^a, \tau_{01}^a \equiv 0 \mod f_{10}^a. \]

By successive application of Cartan’s lemma (or equivalently since the mixed partials commute), one may inductively define the higher order prolongation variables \( \{ f_{ij}^a \}_{a=1}^d \) for \( i, j \geq 0, \ (i, j) \neq (0,0) \), by

\[ df_{ij}^a \equiv f_{i+1,j}^a \theta^a + f_{i,j+1}^a \theta, \quad \text{mod} \ dx, \]  

or equivalently

\[ f_{(i+1)j}^a = \frac{\partial}{\partial p} f_{ij}^a, \quad f_{i(j+1)}^a = \frac{\partial}{\partial y} f_{ij}^a. \]  

Set \( f_{ij}^a = 0 \) for \( i < 0, \ j < 0, \) or \( (i, j) = (0,0) \).

In order to facilitate the computation, let us introduce the following three filtrations on the variables \( f_{ij}^a \)'s.

**Definition 5.19.** Let \( \{ f_{ij}^a \}_{a=1}^d, \ i, j \geq 0, \ (i, j) \neq (0,0) \), be the sequence of prolongation variables (or derivatives) defined by \((5.16), (5.17)\). For a single element \( f_{ij}^a \), assign **height, weight, and depth** by

\[ \text{height}(f_{ij}^a) = i, \]  

\[ \text{weight}(f_{ij}^a) = i + j, \]  

\[ \text{depth}(f_{ij}^a) = i + 2j. \]

For a finite linear combination of \( f_{ij}^a \)'s, weight (depth) is defined as the maximum weight (depth) of nonzero terms. The weight (depth) of an Abelian relation \((5.13)\), or more generally a first integral, is the maximum weight (depth) of the associated nonzero derivatives \( f_{ij}^a \).
Figure 5.1: Υ_{ij} for the case \((i,j) = (3,1)\): upper-left chamber.

A node with \((ij)\) on top represents the set \(\{ f_a^{ij} \}_{a=1}^d \). The weight increases as one moves to the right, the height increases as one moves upward, and the depth increases as one moves to the lower-right. The chamber \(\Upsilon_{ij}\) consists of the \(f_{i'j'}\)'s represented by the node \((i'j')\) which lie in the upper-left chamber bounded by the vertical weight line, and the slant depth line.

The partial filtration associated with the following subsets turns out to be convenient for our analysis.

**Definition 5.21.** The fundamental chamber \(\Upsilon_{ij}\) at \((ij)\) is the subset

\[
\Upsilon_{ij} = \{ f_a^{ij} \mid \text{weight}(f_a^{i'j'}) \leq i + j, \text{ depth}(f_a^{i'j'}) \leq i + 2j \}. \tag{5.22}
\]

The subset of elements \(\{ f_a^{ij} \}_{a=1}^d \subset \Upsilon_{ij}\) are the dominant coefficients in that they are the elements for which the equalities hold in the definition \(5.22\).

Note the following differential relations (with an abuse of notation)

\[
\frac{\partial}{\partial p} \Upsilon_{ij} \subset \Upsilon_{(i+1)j}, \tag{5.23}
\]

\[
\frac{\partial}{\partial y} \Upsilon_{ij} \subset \Upsilon_{i(j+1)}, \tag{5.24}
\]

\[
\left( \frac{\partial}{\partial x} + p \frac{\partial}{\partial y} \right) \Upsilon_{ij} \subset \Upsilon_{(i+1)j}, \tag{5.25}
\]

(These are obvious except the third one, see \(5.37\)). These inclusion relations will be used implicitly for the rest of this section.

The following refinement of \(5.17\) suffices for the purpose of our analysis.

**Lemma 5.24.** The rough structure equation \(5.17\) can be refined to

\[
df_a^{ij} = f_a^{i+1,j} \theta^p + f_a^{i,j+1} \theta - (if_a^{i-1,j+1} + \tau_a^{ij}) dx, \tag{5.25}
\]

where

\[
\tau_a^{ij} \equiv 0, \mod \Upsilon_{ij}. \tag{5.26}
\]

**Proof.** We apply induction argument on increasing depth, and weight.
Differentiating \( (5.24) \) by the Lie derivative \( L_{\frac{\partial}{\partial x}} \), one gets
\[
d f_{l(i+1)}^a = f_{l(i+2)}^a \theta^a + f_{l(i+1)(j+1)}^a \vartheta - (i f_{l(i+1)}^a + L_{\frac{\partial}{\partial x}} \tau_j^a) \, dx
- (f_{l(i+1)}^a) \partial_{\theta}^a + f_{l(i+1)}^a \, dx.
\]
By the induction hypothesis and \( (5.23) \), \( \{ f_{l(i+1)}^a, L_{\frac{\partial}{\partial x}} \tau_j^a \} \subset \Upsilon_{(i+1)j} \). Hence \( \tau_j^a \equiv 0 \mod \Upsilon_{(i+1)j} \).

Differentiating \( (5.24) \) by the Lie derivative \( L_{\frac{\partial}{\partial x}} \), one gets
\[
d f_{l(i+1)(j+1)}^a = f_{l(i+1)(j+1)}^a \theta^a + f_{l(i+1)(j+2)}^a \vartheta - (i f_{l(i+1)(j+2)}^a + L_{\frac{\partial}{\partial x}} \tau_j^a) \, dx
- (f_{l(i+1)}^a) \partial_{\theta}^a + f_{l(i+1)}^a \, dx.
\]
By the induction hypothesis, one gets \( \tau_j^a \equiv 0 \mod \Upsilon_{(j+1)} \).

The claim \( (5.26) \) now follows from the initial condition \( (5.16) \), \( \tau_{10}^a, \tau_{01}^a \equiv 0 \mod \Upsilon_{10} \). □

5.2.3. Compatibility equations

In this subsection, we apply Lemma \( (5.24) \) repeatedly to compute the sequence of higher order compatibility equations from \( (5.14) \).

There are two preliminaries. Firstly, let us introduce a simplified notation
\[
\langle q^i, f_{jk} \rangle = \sum_b (q^b)^i f_{jk}^b.
\]
For example, the initial equation \( (5.14) \) is written in this notation as
\[
\langle q^0, f_{10} \rangle = 0, \quad (\langle q^1, f_{10} \rangle = 0,
\langle q^0, f_{01} \rangle = 0.
\]

Secondly, set an \( \infty \)-by-\( \infty \) integer matrix \( (c_I^J) \) for \( I \geq 0 \) as follows.
\[
c_0^I = 1, \quad (5.28)
c_I^J = 0, \quad \text{for } J < 0, \text{ or } J > I/2,
\]
\[
c_I^J = (I - 2J + 1) c_{I-1}^{J-1} + c_{I}^{J-1},
= \frac{I!}{2^J (I - 2J)! J!}.
\]
The first few terms are (for \( J = 0, 1, 2, 3 \)),
\[
\begin{array}{c|cccc}
I & 0 & 0 & 0 & 0 \\
J & 1 & 0 & 0 & 0 \\
\hline
0 & 1 & 1 & 0 & 0 \\
1 & 1 & 3 & 0 & 0 \\
\end{array}
\]

The sequence of higher order compatibility equations obtained by differentiating the initial equations \( (5.14) \) is recorded in the below. This is a result of straightforward computation.

The symbol structure of these compatibility equations are key to establishing the rank bound. Note that the equation \( (5.30) \) is written modulo lower depth terms.

**Lemma 5.29.** The successive derivatives of \( (5.14) \) by using \( (5.25) \) yield the following sequence of higher order compatibility equations.
\[
E_{ij}^I : \quad c_0^I \langle q^I, f_{ij} \rangle + c_1^I \langle q^{I-1}, f_{(i-2)(j+1)} \rangle \ldots + c_k^I \langle q^{I-k}, f_{(i-2k)(j+k)} \rangle \ldots \equiv 0,
\mod \Upsilon_{(i+1)(j-1)}, \quad \text{for } 0 \leq I \leq i.
\]
Proof. We apply induction argument on increasing depth, and weight.

Differentiating $(5.30)$ by the Lie derivative $L_{\frac{\partial}{\partial y}}$, and noting from $(5.23)$ that $L_{\frac{\partial}{\partial y}} \mathcal{Y}_{(i+1)(j-1)} \subset \mathcal{Y}_{(i+1)j}$, one gets

$$E^I_{(j+1):} \quad c^I_0 (q^I, f_{(j+1)}) + c^I_1 (q^{I-1}, f_{(i-2)(j+2)}) \ldots + c^I_k (q^{I-k}, f_{(i-2k)(j+k+1)}) \ldots \equiv 0,$$

mod $\mathcal{Y}_{(i+1)j}$, for $0 \leq I \leq i$.

It thus suffices to verify the case $j = 0$.

Differentiating $(5.30)$ by the Lie derivative $L_{\frac{\partial}{\partial y}}$, and noting from $(5.23)$ that $L_{\frac{\partial}{\partial y}} \mathcal{Y}_{(i+1)(j-1)} \subset \mathcal{Y}_{(i+2)(j-1)}$, one gets

$$E^I_{(i+1)j}: \quad c^I_0 (q^I, f_{(i+1)j}) + c^I_1 (q^{I-1}, f_{(i-1)(j+1)}) \ldots + c^I_k (q^{I-k}, f_{(i-2k)(j+k)}) \ldots \equiv 0,$$

mod $\mathcal{Y}_{(i+2)(j-1)}$, for $0 \leq I \leq i$.

It thus suffices to verify the case $I = i$.

Differentiate $(5.30)$ now for the case $I = i$ by the Lie derivative $L_{\frac{\partial}{\partial y} + p \frac{\partial}{\partial y}}$. One notes from $(5.23)$ the relation $L_{\frac{\partial}{\partial y} + p \frac{\partial}{\partial y}} \mathcal{Y}_{(i+1)(j-1)} \subset \mathcal{Y}_{(i+2)(j-1)}$, and this shows from $(5.37)$ that

$$E^{i+1}_{(i+1)j} \quad c^{i+1}_0 (q^{i+1}, f_{(i+1)j}) + c^i (q^{i-1}, f_{(i-1)(j+1)}) + c^i (q^i, f_{(i-1)(j+1)}) + (i - 2) (q^{i-1}, f_{(i-3)(j+2)}) \ldots c^i_k (q^{i-k}, f_{(i-k)(j+k)}) \ldots \equiv 0,$$

mod $\mathcal{Y}_{(i+2)(j-1)}$.

This also implies the recursive relation for the structural constants $c_I^i$,

$$c^{i+1}_0 = c^i_0,$$

$$c^{i+1}_1 = i c^i_0 + c^i_1,$$

$$\ldots$$

$$c^{i+1}_k = (i - 2k + 2) c^i_{k-1} + c^i_k,$$

$$\ldots$$

The claim $(5.30)$ now follows from the initial condition $(5.27)$. □

Let us summarize the analysis so far.

- the sequence of higher order compatibility equations for Abelian relations $\{E^I_{ij}\}$ propagates in the direction of increasing depth,
- for each fundamental chamber $\mathcal{Y}_{ij}$, there are $(i + 1)$ linear compatibility equations for the dominant coefficients $\{ f^I_{ij} \}_{I=1}^d$.

In the next section, we shall give a geometric interpretation of these results and give a proof of rank bound.

### 5.3. Proof of rank bound

In this section we give a proof that the linear differential system for Abelian relations closes up at depth $2d - 4$, and the rank of a Legendrian $d$-web is bounded by the expected value $\rho_d$.

#### 5.3.1. Model case

In this sub-section we shall consider the class of Legendrian webs discussed in Section 4.1, which are defined by the second order ODE’s (4.1). We shall use them as a model for our indirect analysis of the symbol of linear constraint equations (5.30) via (b) of Theorem (5.8).

Let us recall some notations.

$$\begin{align*}
q^a's \text{ are distinct constants,} \\
\theta = dy - pdx, \quad \theta^n = dp - q^n dx, \\
q^I = (q^1)^I, (q^2)^I, \ldots (q^n)^I, \\
f_{ij} = (f_{ij}^1, f_{ij}^2, \ldots f_{ij}^d).
\end{align*}$$

Consider the following linear differential system for Abelian relations which is truncated at depth $(2d - 4) + 1$. 

Let $S_d$ denote the vector space of solutions to this equation.

For the class of Legendrian webs under consideration, one finds that

$$\tau_{ij}^a = 0, \quad \text{in (5.25)},$$

$$E_{ij}^I \text{ holds in (5.30) without mod } \Upsilon_{(i+1)(j-1)}.$$

It easily follows from this that $(*)$ is a closed linear differential system with constraint, which is moreover compatible. By (b) of Theorem 5.8, the rank of the set of linear compatibility equations is determined by the number of dependent variables $f_{ij}^a$’s and the dimension of the moduli space of solutions.

By counting for each depth, we have the following table, see Table 1. The total sum gives

$$\left( \text{number of variables } f_{ij}^a \right) - \left( \text{number of compatibility equations } E_{ij}^I \right) \quad (5.31)$$

$$= \left( d + \sum_{k=2}^{d-1} 2kd \right) - \left( 2 + \sum_{k=2}^{d-1} k^2 + k(k+1) \right) = \rho_d.$$

We wish to show that

**Lemma 5.32.** For each $d \geq 3$,

$$\dim S_d = \rho_d.$$

Assuming this is true, we have the following corollaries.

**Corollary 5.33.** The set of constraint equations

$$\{ E_{ij}^I \mid i, j \geq 0, (i, j) \neq (0, 0), i + 2j \leq (2d - 4) + 1, 0 \leq I \leq i \}$$

has full rank for each $d \geq 3$.

**Proof.** This follows from (5.31) and (b) of Theorem 5.8. □

---

**Table 1**

| depth | number of variables $f_{ij}^a$ | number of compatibility equations $E_{ij}^I$ of depth $i + 2j$ |
|-------|-------------------------------|---------------------------------------------------------------|
| 1     | $d$                           | 2                                                             |
| 2     | $2d$                          | $1 + 3$                                                       |
| 3     | $2d$                          | $2 + 4$                                                       |
| 4     | $3d$                          | $1 + 3 + 5$                                                   |
| 5     | $3d$                          | $2 + 4 + 6$                                                   |
| ...   | ...                           | ...                                                           |
| $kd$  | $k^2$                         | $k(k + 1)$                                                   |
| $2d - 4$ | $(d - 1)d$                  | $(d - 1)^2$                                                  |
| $2d - 3$ | $(d - 1)d$                  | $(d - 1)d$                                                  |
Corollary 5.34. The set of constraint equations
\[ \{ E_{ij} | i, j \geq 0, (i, j) \neq (0, 0), i + 2j = \delta, 0 \leq I \leq i \} \]
has full rank for each \( \delta \geq 1 \).

Corollary 5.35. The class of Legendrian webs defined by the second order ODE’s (4.1) have rank \( \rho_d \).

Proof. From the table above, the number of variables match the number of equations at depth \( 2d - 3 \). By Corollary 5.34 and (5.30), the prolongation variables \( f_{ij}^3 \) of depth \( \geq 2d - 3 \) all vanish. The linear differential system for Abelian relations reduces to (\ast) in this case. \( \square \)

Proof of Lemma 5.32. We claim that \( \mathcal{S}_d \) is exactly the space of polynomial Abelian relations constructed in Section 4.

**Step 1.** An Abelian relation of depth \( \leq (2d - 4) + 1 \) has weight \( \leq (2d - 4) + 1 \) by definition of depth and weight. Since the weight equals the degree as a polynomial in the variables \{p, y\}, an Abelian relation of depth \( \leq (2d - 4) + 1 \) is a polynomial in the variables \{p, y\}.

The first integrals \{ \( u^0_1(q^a), u^1_1(q^a) \) \} for the foliation defined by the second order ODE \( y'' = q^a \) are functionally independent (note \( u^0_1(q^a) = y, u^1_1(q^a) = p, \) mod \( x \)). One may in fact linearly solve for the variables \{p, y\} in terms of \{ \( u^0_1(q^a), u^1_1(q^a) \) \}. A first integral for the given ODE which is a polynomial in \{p, y\} thus must be a polynomial (with constant coefficients) of \{ \( u^0_1(q^a), u^1_1(q^a) \) \}.

**Step 2.** From the identity \( (u^1_1)^2 = 2u^2_1 + 2qu^3_1 \), one may choose as a basis for the vector space of such polynomial first integrals;
\[ v^{j_0+j_1+j_2}_j(q^a) = (u^0_1)^{j_0}(u^1_1)^{j_1}(u^2_1)^{j_2}(q^a), \]
where
\[
\begin{align*}
    &j_0 + j_1 + j_2 = m - 1 \geq 1, \\
    &j_0, j_2 \geq 0, \quad j_1 = 0, \text{ or } 1, \\
    &j = 0 \cdot j_0 + 1 \cdot j_1 + 2 \cdot j_2 \leq 2m - 2.
\end{align*}
\]

Note that each pair \((m, j), m \geq 2, 0 \leq j \leq 2m - 2, \) uniquely determines \( (j_0, j_1, j_2) \).

**Step 3.** We claim that
\[
\begin{align*}
    \text{weight}(u^{m+1}_j) &= j_0 + j_1 + 2j_2, \\
    \text{depth}(u^{m+1}_j) &= 2j_0 + j_1 + 2j_2. \tag{5.36}
\end{align*}
\]
One checks that \( (u^0_1, u^1_1, u^2_1) \) has weight \( (1, 1, 2) \), and depth \( (2, 1, 2) \) respectively, and the claim follows by induction.

Since \( j_1 = 0, \) or \( 1, \) the condition depth \( (u^{m+1}_j) \leq (2d - 4) + 1 \) translates to the condition \( m \leq d \).

**Step 4.** Note that weight and depth uniquely determine \((m, j)\). It follows from Lemma 4.9 and the analysis in Section 4 that it suffices to show, for each \((m, j), 2 \leq m \leq d, 0 \leq j \leq 2m - 2, \) that there are at most \( d - m \) independent Abelian relations of the form
\[ (v_1 u^{m+1}_j(q^1), v_2 u^{m+1}_j(q^2), \ldots, v_d u^{m+1}_j(q^d)), \]
for constant coefficients \( (v_1, v_2, \ldots, v_d) \) (this implies \( \dim \mathcal{S}_d = \rho_d \)).

From Lemma 4.9 again, \( u^{m+1}_j(q) \) is homogeneous of weight \( j \) ("weight" used in Section 4), and has degree \( m \) in the indeterminate variable \( q \). One may check that each of \( m \) coefficients of \( u^{m+1}_j(q) \) as a polynomial in \( q \) is nonzero (see 4.6), and hence they are linearly independent functions\footnote{Evaluate \( u^0_1, u^1_1, u^2_1 \) at \((x, y, p) = (-2, -1, 2)\), then each of them are of the form \( \epsilon_0 + q \epsilon_1 \) with \( \epsilon_0, \epsilon_1 > 0 \).} By Vandermonde identity (since \( q^a \neq q^b \) for \( a \neq b \)), the coefficient \( (v_1, v_2, \ldots, v_d) \) as a vector in \( \mathbb{R}^d \) must satisfy \( m \) independent linear equations.

**Step 5.** From Step 4, only those \( u^{m+1}_j, 2 \leq m \leq d - 1, \) give rise to a nontrivial Abelian relation. By (5.36),
\[ \text{depth}(u^{m+1}_j) = 2j_0 + j_1 + 2j_2 \leq 2j_0 + 2j_1 + 2j_2 = 2m - 2 \leq 2d - 4. \] \( \square \)
5.3.2. Proof of rank bound

Corollary 5.34 when combined with the structure equation (5.25) and (b) of Theorem 5.8 immediately implies the desired rank bound for the Legendrian webs. Before we present a proof, let us give a possible geometric interpretation of the depth filtration and the associated layers of compatibility equations.

Recall $C \subset T^* X$ is the contact line bundle. Given a general Legendrian web $W$ defined by a set of rank two sub-bundles $\mathcal{T}^a \subset T^* X, a = 1, 2, \ldots, d$, the vector space of Abelian relations $\mathcal{A}(W)$ consists of closed sections of the direct sum bundle $\bigoplus_{a=1}^d \mathcal{T}^a$. Consider the projection

$$\pi : \mathcal{A}(W) \subset H^0(\bigoplus_{a=1}^d \mathcal{T}^a) \to H^0(\bigoplus_{a=1}^d \mathcal{T}^a / C).$$

By the defining non-degeneracy property of a contact 1-form, $\pi$ is easily seen to be injective. From this it follows that in order to apply Theorem 5.8 to the analysis of Abelian relations, it suffices to analyze, roughly speaking, the higher order jets of sections of $\bigoplus_{a=1}^d \mathcal{T}^a$. Consider the filtration and the associated layers of compatibility equations.

Proof. Theorem 5.39. Suppose it satisfies the equation

$$\dot{Z} \equiv 0, \mod Z; C.$$  

Then in fact

$$\dot{Z} \equiv 0, \mod Z.$$  

We leave it to the reader to verify this elementary claim.

The preceding consideration can be formulated analytically as the proposed depth filtration on the higher order jets of sections of the direct sum bundle $\bigoplus_{a=1}^d \mathcal{T}^a$. Consider the projection $\pi : \mathcal{A}(W) \subset H^0(\bigoplus_{a=1}^d \mathcal{T}^a) \to H^0(\bigoplus_{a=1}^d \mathcal{T}^a / C)$.

In hindsight, this is due to the following observation: let $X$ be a contact three manifold with the given contact line bundle $C \subset T^* X$. Let $Z = (Z_1, Z_2, \ldots, Z_r)$ be a $\mathbb{R}^r$-valued function on $X$ (here $r$ stands for the rank of a Legendrian web). Suppose it satisfies the equation

$$\dot{Z} \equiv 0, \mod Z; C.$$  

Then in fact

$$\dot{Z} \equiv 0, \mod Z.$$  

We leave it to the reader to verify this elementary claim.

The operation of differentiating modulo $C$ is easily seen to be injective. From this it follows that in order to apply Theorem 5.8 to the analysis of Abelian relations, it suffices to analyze, roughly speaking, the higher order jets of sections of $\bigoplus_{a=1}^d \mathcal{T}^a$ modulo $C$.

Recall the definition of depth $\text{depth}(f_{ij}^a) = i + 2j$. Let $W$ be a Legendrian web. Then in fact

$$\dot{Z} \equiv 0, \mod Z; C.$$  

We leave it to the reader to verify this elementary claim.

Theorem 5.39. For $d \geq 3$, the maximum rank of a Legendrian $d$-web is $\rho_d$.

Proof. Theorem 5.39. Suppose it satisfies the equation

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$$\dot{Z} \equiv 0, \mod Z; C.$$  

We leave it to the reader to verify this elementary claim.
In order to apply a) of Theorem 5.8, it now suffices to show that the set of remaining constraint equations
\[ \{ E_{ij}^I \mid i + 2j \leq 2d - 4, 0 \leq I \leq i \} \]
has full rank.

We proceed inductively on decreasing depths. At each depth \( \delta \leq 2d - 4 \), Corollary 5.34 and (5.30) show that the set of constraint equations
\[ \{ E_{ij}^I \mid i + 2j = \delta, 0 \leq I \leq i \} \]
has full rank when restricted to the set of variables of depth \( \delta \)
\{f_{ij}^I \mid i + 2j = \delta\} modulo the lower depth variables \( F^{\delta-1} \).

\[ \square \]

6. Concluding remarks

1. The purpose of the present paper is to propose Legendrian web as a second order generalization of planar web. Within the complex analytic category, the list of dualities associated with the rank 2 simple Lie groups in Fig. 6.1 provides a projective geometric perspective on this generalization. [6] Here \( Z', Z, Z'' \) are the respective incidence spaces.

\[ \begin{array}{ccc}
Z' & \rightarrow & (\mathbb{P}^2)^* \\
\mathbb{P}^2 & \rightarrow & \mathbb{P}^3 \\
SL(3, \mathbb{C}) & \rightarrow & \mathbb{Q}^3 \\
Q^5 & \rightarrow & \mathbb{N}^5 \\
G_2^C \\
\end{array} \]

**Figure 6.1:** Dualities associated with the rank 2 simple Lie groups.

From this it is a natural extension to consider the analogous problem for the webs on \( \mathbb{Q}^5 \) defined by a set of horizontal foliations with respect to the \( G_2^C \) invariant rank 2 distribution (Cartan distribution).

2. (In the complex analytic category) Is there a projective algebro-geometric proof of the bound on the dimension of the space of closed holomorphic 1-forms on surfaces in \( \mathbb{Q}^3 \) of given degree? What is the appropriate definition of holomorphic 1-forms in this case?

The projective duality above suggests one way of defining a holomorphic 1-form on a generally singular analytic surface \( \Sigma \subset \mathbb{Q}^3 \); a meromorphic 1-form on a surface in \( \mathbb{Q}^3 \) is holomorphic when its trace on a generic small open subset of \( \mathbb{P}^3 \) vanishes. Does this agree with the notion of holomorphic forms given by Henkin & Passare in [24]? In particular, is it true that a holomorphic 1-form on a surface in this sense is necessarily closed? (likely not)

3. Given a set of \( n + 2 \) points \( \{ p_1, p_2, ..., p^{n+2} \} \) in \( \mathbb{R}P^n \) in general position, the associated exceptional \( n + 3 \) web \( \mathcal{E}_{n+3} \) consists of the \( n + 2 \) bundles of lines with vertices \( p^a \)'s, and the family of rational normal curves through \( p^a \)'s. Damiano in his thesis gave a characterization that, up to diffeomorphism, \( \mathcal{E}_{n+3} \) is the unique non-linearizable quadrilateral web of curves in dimension \( n, [15] \). He also showed that \( \mathcal{E}_{n+3} \) is of maximum rank based on the observation that \( \mathcal{E}_{n+3} \) naturally occurs on the (smooth part of) configuration space of ordered set of \( n + 3 \) points in \( \mathbb{R}P^1 \), which is essentially the quotient space of the Grassmannian \( GR(2, \mathbb{R}^{n+3}) \) by the Cartan subgroup of \( SL(n+3, \mathbb{R}) \). Modulo the codimension one subspace of combinatorial Abelian relations, the remaining single non-combinatorial Abelian relation is given by the trace of the appropriate power of the harmonic form representing the Euler class of the canonical bundle of \( GR(2, \mathbb{R}^{n+3}) \). It would be interesting if such a geometric construction exists for Legendrian webs.

4. We currently do not have any nontrivial examples of the Legendrian 4-webs of maximum rank eleven. Let us give a description of one candidate.

Let \( (x^1, x^2, x^3, x^4) \) be the coordinate of \( \mathbb{R}^4 \). Let \( \varpi = dx^1 \wedge dx^2 + dx^3 \wedge dx^4 \) be the standard symplectic 2-form. Let \( S^3 \subset \mathbb{R}^4 \) be the unit sphere equipped with the \( U(2) \)-invariant induced contact structure defined by the contact 1-form
\[ \theta = -x^2 dx^1 + x^1 dx^2 - x^4 dx^3 + x^3 dx^4. \] Consider the Legendrian 4-web \( \mathcal{W} \) on a generic small open subset of \( S^3 \) defined by the rank 2 sub-bundles

\[ \mathcal{I}^a = \{ \theta, dx^a \}, \quad a = 1, 2, 3, 4. \]

Each of the four sub 3-webs of \( \mathcal{W} \) has rank 0, and \( \mathcal{W} \) is distinct from the examples in Section 4. Is \( \mathcal{W} \) algebraic in any way?

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