Equimultiplicity of families of map germs from $\mathbb{C}^2$ to $\mathbb{C}^3$

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Abstract

In 1971, Zariski proposed some questions in Theory of Singularities. One of such problems is the so-called, nowadays, “Zariski’s multiplicity conjecture”. In this work, we consider the version of this conjecture for families.

We answer positively Zariski’s multiplicity conjecture for a special class of non-isolated singularities.

1 Introduction

In 1971, Zariski proposed some questions in Theory of Singularities [32]. One of such problems is the so-called, nowadays, “Zariski’s multiplicity conjecture”. In this work, we consider the version of this conjecture for families. More precisely, let $g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a reduced germ of holomorphic function, $V(g) = g^{-1}(0)$ the corresponding germ of hypersurface in $\mathbb{C}^n$. Let

$$G : (\mathbb{C}^n \times \mathbb{C}, \{0\} \times \mathbb{C}) \to (\mathbb{C}, 0) \quad G(z, t) = G_t(z)$$

be a deformation of $g$, that is, $G$ is a germ of holomorphic function such that $G_0 = g$ and, for all $t$ near 0, the germ $G_t$ is reduced. The multiplicity of $V(G_t) = G_t^{-1}(0)$ at 0 is the number of points of the intersection of $V(G_t)$ with a generic line passing close to the origin but not through the origin. Denote by $m(V(G_t))$ the multiplicity of $V(G_t)$ at 0. One says that the family $V(G_t)$ is equimultiple if, for all $t$ near 0, $m(V(G_t)) = m(V(G_0))$. One says that the family $V(G_t)$ is topologically trivial if, for all $t$ near 0, there is a germ of homeomorphism $h_t : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $h(V(G_t)) = V(G_0)$. Now, we can state:

**Conjecture 1.1** (Zariski’s multiplicity conjecture for families) If $G_t$ is topologically trivial, then is it equimultiple?

Almost fifty years later, Conjecture 1.1 is, in general, still unsettled (even for isolated hypersurfaces singularities). The answer is, nevertheless, known to be yes in several special cases. We cite for example, Zariski in [31], Greuel in [13], O’Shea in [21], Trotman in [28] and [29]. More recently, we can cite [1], [7], [5], [6], [18], [22], [27] and [30]. In order to know more about this conjecture and a more complete list, see the survey [8].

In this work, we provide a positive answer to Zariski’s multiplicity conjecture for some families of singular surfaces in $\mathbb{C}^3$ with non-isolated singularities under the condition that the family has a smooth normalization. For a family of surfaces $V(G_t)$ in $\mathbb{C}^3$ whose normalization is smooth, we can associate a family of parametrizations $f_t : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ whose images are $V(G_t)$. In this situation, it follows from [9] that the $A$-topological equisingularity of $f_t$ implies that the Milnor numbers of the sets $f_t^{-1}(\Sigma_t)$ remain constant, where $\Sigma_t$ is the singular set of $V(G_t)$. The converse also holds with some additional hypothesis on $G_t$.

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A particular class of parametrized singular surfaces consists on surfaces which are the image of an $\mathcal{A}$-finitely determined map germ $f : (\mathbb{C}^2,0) \rightarrow (\mathbb{C}^3,0)$. Let $F : (\mathbb{C}^2 \times \mathbb{C},0) \rightarrow (\mathbb{C}^3 \times \mathbb{C},0)$, $F(x,y,t) = (f_t(x,y),t)$ be a 1-parameter unfolding of a finitely determined map germ $f : (\mathbb{C}^2,0) \rightarrow (\mathbb{C}^3,0)$. In this case, the family of surfaces $F(\mathbb{C}^2 \times \mathbb{C})$ is topologically trivial in a neighbourhood of 0 if and only if $\mu(D(f_t))$ is constant, where $D(f_t)$ is the double point locus of $f_t$, for all sufficiently small $t \in \mathbb{C}$, and $\mu$ is the Milnor number of these sets (see [9] and also [2]). In this context, we can ask the following question:

**Question:** What kind of conditions do we need on $f$ that imply equimultiplicity of a topologically trivial family $F(\mathbb{C}^2 \times \mathbb{C})$?

In this sense, we prove the following result:

**Theorem 1.2** Let $f : (\mathbb{C}^2,0) \rightarrow (\mathbb{C}^3,0)$ be a finitely determined map germ, homogeneous with corank 1. Let $F(x,y,t) = (f_t(x,y),t)$ a topologically trivial 1-parameter unfolding of $f$. Then, the family $F(\mathbb{C}^2 \times \mathbb{C})$ is equimultiple.

We note that the class of topologically trivial unfoldings of a homogeneous finitely determined map germ $f$ was studied recently by Ruas and the author in [24], and by the author in his PhD. thesis [25], where the first known counter-example to a conjecture by Ruas [23] on the equivalence between Whitney equisingularity and topological triviality was obtained.

We remark that in Conjecture 1.1, if in addition $G_0 = g$ has an isolated singularity and is quasi-homogeneous, then a topologically trivial family $V(G_t)$ is equimultiple (see Greuel [13] and O’Shea [21], see also Trotman [28] and [29]).

In our case, when $f = (f_1, f_2, f_3)$ is homogeneous, that is, each $f_i$ is a homogeneous polynomial, the image $f(\mathbb{C}^2)$ is a surface in $\mathbb{C}^3$ defined by a quasi-homogeneous map germ $g : (\mathbb{C}^3,0) \rightarrow (\mathbb{C},0)$ (see [20]). Furthermore, as illustrated in Example 2.1, the surface $f(\mathbb{C}^2)$ always has an analytic curve (not necessarily smooth or irreducible) as its singular set. Finally, we note that some important contributions for the non-isolated case of Conjecture 1.1 was were recently given by Eyral and Ruas in [5] and [6]. In particular, Eyral showed in [7] that if $V(G_t)$ is a topologically trivial family of line singularities hypersurfaces in $\mathbb{C}^n$ and $G_0$ is homogeneous, then $V(G_t)$ is equimultiple.

## 2 Preliminaries

Consider a finite and holomorphic map $f : U \rightarrow \mathbb{C}^3$, where $U$ is a small enough neighbourhood of 0 in $\mathbb{C}^2$. Throughout the paper, $(x,y)$, $(x,y,t)$, $(X,Y,Z)$ and $(X,Y,Z,t)$ are used to denote systems of coordinates in $\mathbb{C}^2$, $\mathbb{C}^2 \times \mathbb{C}$, $\mathbb{C}^3$ and $\mathbb{C}^3 \times \mathbb{C}$, respectively. Also, $\mathbb{C}\{x_1, \ldots, x_n\} \simeq \mathcal{O}_n$ denotes the local ring of convergent power series in $n$ variables.

### 2.1 Finite Determinacy

Consider a finite map germ $f : (\mathbb{C}^2,0) \rightarrow (\mathbb{C}^3,0)$. By Mather-Gaffney criterion ([33]), $f$ is finitely determined if and only if there is a finite representative $f : U \rightarrow V$, where $U \subset \mathbb{C}^2$, $W \subset \mathbb{C}^3$ are open neighbourhoods of the origin, such that $f^{-1}(0) = \{0\}$ and the restriction $f : U \setminus \{0\} \rightarrow W \setminus \{0\}$ is stable. This means that the only singularities of $f$ on $U \setminus \{0\}$ are cross-caps (or Whitney umbrellas), transverse double and triple points. By shrinking $U$ if necessary, we can assume that there are no cross-caps nor triple points in $U$. In this sense, the set of transverse double points of $f$ plays a fundamental role in the study of the finite determinacy. The double point curve of $f$, denoted by $D(f)$, is defined as the following set:

$$D(f) := \{ (x,y) \in U : f^{-1}(f(x,y)) \neq \{(x,y)\} \} \cup \Sigma(f),$$
where $\Sigma(f)$ is the singular set of $f$. If $f$ is finite and generically $1 - 1$, then $D(f)$ is a closed analytic set of dimension 1 and it is possible to provide a convenient analytic structure on it (non-necessarily reduced) with the following property:

$f$ is finitely determined if and only if $D(f)$ is reduced (see [15] and [16]). The Milnor number $\mu(D(f))$ is called the Mond Number of $f$.

Another important space to study the topology of $f(C^2)$ is the double point curve in the target, that is, the image of $D(f)$ by $f$, denoted by $f(D(f))$. Following Mond and Pellikaan [19], given a finite morphism of complex spaces $f : X \to Y$ the push-forward $f_*\mathcal{O}_X$ is a coherent sheaf of $\mathcal{O}_Y$-modules and $\mathcal{F}_k(f_*\mathcal{O}_X)$ denotes the $k$th Fitting ideal sheaf. Notice that the support of $\mathcal{F}_0(f_*\mathcal{O}_X)$ is just the image $f(X)$. Analogously, if $f : (X,x) \to (Y,y)$ is a finite map germ then we denote also by $\mathcal{F}_k(f_*\mathcal{O}_X)$ the $k$th Fitting ideal of $\mathcal{O}_{X,x}$ as $\mathcal{O}_{Y,y}$-module. In this way, given a finite map germ $f : (C^2,0) \to (C^3,0)$, the image of $f$ is an analytic surface in $C^3$ given by $f(C^2) = V(\mathcal{F}_1(f_*\mathcal{O}_2))$ and the double point space in the target is the analytic complex curve $f(D(f)) = V(\mathcal{F}_1(f_*\mathcal{O}_2))$. Notice that the underlying set germ of $f(D(f))$ is the image of $D(f)$ by $f$.

**Example 2.1** Consider the map germ $f(x,y) = (x, y^2, x^7y - 14x^5y^3 + 49x^3y^5 - 36xy^7)$. We have that

$$f(C^2) = V(Z^2 - 1294X^4Y^2 + 3528X^7Y^6 - 3049X^8Y^6 + 1444X^8Y^4 - 294X^{10}Y^3 + 28X^{12}Y^2 - X^{14}Y).$$

Also, $D(f) = V(x^2 - 14x^5y^2 + 49x^3y^4 - 36xy^6)$ is a reduced curve, hence $f$ is finitely determined. Note that $D(f)$ has seven irreducible components (see Figure 1), namely,

$$D(f)^1 = V(x+y), D(f)^2 = V(x-y), D(f)^3 = V(x+2y), D(f)^4 = V(x-2y),$$

$$D(f)^5 = V(x+3y), D(f)^6 = V(x-3y) \text{ and } D(f)^7 = V(x).$$

Furthermore, $f(D(f)^1) = f(D(f)^2), f(D(f)^3) = f(D(f)^4), f(D(f)^5) = f(D(f)^6)$ and

$$f(D(f)) = V(Z, 36XY^3 - 49X^3Y^2 + 14X^5Y - X^7) \subset C^3.$$
such that $I = \Psi^{-1} \circ F \circ \Phi$, where $I(x,t) = (f(x),t)$ is the trivial unfolding of $f$.

The following definition was given by Gaffney in [12] for finitely determined map germs in dimensions $(n,p)$. We restrict ourselves to the case $(n,p) = (2,3)$.

**Definition 2.3** We say that $F$ is a good unfolding of $f$ if there exist neighbourhoods $U$ of 0 in $\mathbb{C}^2 \times \mathbb{C}$, $W$ of 0 in $\mathbb{C}^3 \times \mathbb{C}$ and $T$ of 0 in $\mathbb{C}$ (the parameter space), such that the following hold:

(a) $F^{-1}(\{0\} \times T) = \{0\} \times T$, that is, $F$ maps $U \setminus (\{0\} \times T)$ into $W \setminus (\{0\} \times T)$.

(b) For all $(z_0,t_0) \in W \setminus (\{0\} \times T)$ the map $f_{t_0} : (\mathbb{C}^2,S) \to (\mathbb{C}^3,0)$ is stable, where $S = F^{-1}(z_0,t_0) \cap \Sigma(F) \cap U$ (which is a finite set) and $\Sigma(F)$ denotes the singular set of the unfolding $F$.

Let $F$ be a 1-parameter unfolding of a finitely determined map germ $f : (\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$. As in [24], we denote by $D(F)$ the double point locus of $F$ and by $D(f_i)$ the double point curve of each $f_i$. Also, we denote by $F(D(F))$ the image of $D(F)$ by $F$, and by $f(D(f_i))$ the double point in the target.

**Remark 2.4** (a) We have that the projections of $D(F)$ and $F(D(F))$ into the parameter space $\mathbb{C}$ are flat deformations of $D(f)$ and $f(D(f))$ (see [16, Lemma 4.2]), that is, $D(F)$ and $F(D(F))$ are flat families of reduced curves in the sense of [4].

(b) Suppose that $F$ is topologically trivial. Since the homeomorphisms $\Phi$ and $\Psi$ must preserve the double point curves, then $D(F)$ and $F(D(F))$ are topologically trivial families of curves (in the sense of [4]).

(c) Furthermore, Callejas Bedregal, Houston and Ruas in [2] and Fernández de Bobadilla and Pe Pereira in [9] (independently) showed that $F$ is topologically trivial if and only if $\mu(D(f_i))$ is constant.

(d) If $\mu(D(f_i))$ is constant, then $F$ is good (see for instance [2]).

(e) When $f$ is finitely determined and $F$ is a 1-parameter unfolding of $f$, then $D(F)$ is a reduced surface in $\mathbb{C}^2 \times \mathbb{C}$ given by a polynomial map $\lambda : \mathbb{C}^2 \times \mathbb{C} \to \mathbb{C}$, that is, $D(F) = V(\lambda)$. In the other hand, we have that the following equality between ideals in $\mathbb{C}\{x,y,t\}$

$$\langle F^*(F_1(F_2(O_2 \times O_1))) \rangle = \langle \lambda \rangle,$$

that is, in this case $D(F)$ can be also defined by the ideal $\langle F^*(F_1(F_2(O_2 \times O_1))) \rangle$.

### 3 Equimultiplicity of families of map germs

The following result shows us how we can relate the equimultiplicity of a family $f_i$ in terms of the Cohen-Macaulay property of a certain local ring. In the sequel, as in [17], $l(R)$, $e(q,A)$ and $A_p$ denotes the length of the 0-dimensional local ring $R$, the Hilbert-Samuel multiplicity of the ideal $q$ in the local ring $A$ and the localization of $A$ in the prime ideal $p$, respectively. As before, $m(f_i(\mathbb{C}^2))$ denotes the multiplicity of $f_i(\mathbb{C}^2)$ at 0.

**Lemma 3.1** Let $f : (\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$ be a finitely determined map germ with corank 1. Let $F(x,y,t) = (f(x,y),t)$ be a 1-parameter unfolding of $f$. Then

(a) the ideal $\langle t \rangle$ in $\mathbb{C}\{x,y,t\}/\langle F^*(X,Y,Z) \rangle$ is a parameter ideal and
\[ m(f(C^2)) = l \left( \frac{\mathbb{C}[x, y, t]}{(F^*(X, Y, Z), t)} \right) \quad \text{and} \quad m(f_t(C^2)) = e \left( \left( \frac{\mathbb{C}[x, y, t]}{(F^*(X, Y, Z))} \right) \right) \text{ for } t \neq 0. \]

(b) \( m(f_t(C^2)) \) is constant if and only if \( \mathbb{C}[x, y, t]/(F^*(X, Y, Z)) \) is a Cohen-Macaulay ring.

**Proof.** Take a small representative \( F : U \to W \) of \( F \), where \( U, W \) are neighbourhoods of 0 in \( \mathbb{C}^2 \times \mathbb{C}, \mathbb{C}^3 \times \mathbb{C} \), respectively, and let \( T \subset U, W \) be a neighborhood of 0 in \( \mathbb{C} \).

(a) Since \( F \) is topologically trivial, by Remark 2.4 (c) and (d) we have that \( V((F^*(X, Y, Z))) \cap U = (0, 0) \times T \simeq T \) in the source. Thus, we have the following equality between ideals

\[ \langle \sqrt{F^*(X, Y, Z)} \rangle \mathbb{C}[x, y] = (x, y)\mathbb{C}[x, y], \]

and therefore \( x^n, y^m \in (F^*(X, Y, Z)) \) for some \( n, m \in \mathbb{N} \). Therefore,

\[ \sqrt{(t)} \left( \frac{\mathbb{C}[x, y, t]}{(F^*(X, Y, Z))} \right) = (x, y, t) \left( \frac{\mathbb{C}[x, y, t]}{(F^*(X, Y, Z))} \right) \]

and therefore \( t \) is a parameter in \( \mathbb{C}[x, y, t]/(F^*(X, Y, Z)) \) (in the sense of [17, Ch. 14]). Since \( f \) is finitely determined, \( f \) is the normalization of its image \( f(C^2) \), thus (see for instance [12, p. 210]) we have that:

\[ m(f(C^2)) = e((f^*(X, Y, Z)), \mathbb{C}[x, y]). \]  

(1)

Since \( f \) has corank 1, thus \( (f^*(X, Y, Z)) \mathbb{C}[x, y] = (x, y^k)\mathbb{C}[x, y] \) for some \( k \), therefore

\[ e((f^*(X, Y, Z)), \mathbb{C}[x, y]) = e((x, y^k), \mathbb{C}[x, y]) = l \left( \frac{\mathbb{C}[x, y]}{(x, y^k)} \right) = l \left( \frac{\mathbb{C}[x, y, t]}{(F^*(X, Y, Z), t)} \right), \]

(2)

it follows by (1) and (2) the first equality of (a). Since \( f \) has corank 1, we can write \( f(x, y) = (x, p(x, y), q(x, y)) \). Now, write

\[ f_s(x, y) = (x + h_1(x, y, t), p(x, y) + h_2(x, y, t), q(x, y) + h_3(x, y, t)) \]

(3)

Since \( F \) is also corank 1, after a change of coordinates we can assume that \( h_1 = 0 \) in (3). Since \( f_t \) is birational over its image, we have that

\[ m(f_t(C^2)) = e((f^*_t(X, Y, Z)), \mathbb{C}[x, y]) = e((x, y^s), \mathbb{C}[x, y]) =
\]

\[ = l \left( \frac{\mathbb{C}[x, y]}{(x, y^s)} \right) = l \left( \frac{\mathbb{C}[x, y, t]}{(F^*(X, Y, Z))} \right) = e \left( \left( \frac{\mathbb{C}[x, y, t]}{(F^*(X, Y, Z))} \right) \right) \]

for some \( s \geq 1 \), where the last equality follows by [17, Th. 14.7].

(b) Since the ideal \( (t) \) is a parameter ideal, by (a) and [17, Th. 17.11] we have that \( \mathbb{C}[x, y, t]/(F^*(X, Y, Z)) \) is Cohen-Macaulay if and only if

\[ l \left( \frac{\mathbb{C}[x, y, t]}{(F^*(X, Y, Z), t)} \right) = e \left( \left( \frac{\mathbb{C}[x, y, t]}{(F^*(X, Y, Z))} \right) \right), \]

if and only if \( m(f(C^2)) = m(f_t(C^2)) \), for all \( t \).

For the next lemma, we will need the following result:
Proposition 3.2 (3) Let $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^p, 0)$ be a analytic map germ and $J$ an ideal of $\mathcal{O}_p$. Let $f^* : \mathcal{O}_p \rightarrow \mathcal{O}_n$ be the ring homomorphism induced by $f$ such that $I = f^*(J)$. If $\mathcal{O}_p/J$ is Cohen-Macaulay and the $\text{codim}(V(I)) = \text{codim}(V(J))$, then $\mathcal{O}_n/I$ is also Cohen-Macaulay.

Lemma 3.3 Let $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^4, 0)$ be a finitely determined map germ and $F(x, y, t) = (f_1(x, y), t)$ a topologically trivial 1-parameter unfolding of $f$. Suppose that $F(D(F))^j$ is equimultiple along the parameter space for some irreducible component $F(D(F))^j$ of $F(D(F))$. Then,

$$\frac{\mathbb{C}[x, y, t]}{F^*(X, Y, Z)}$$

is a Cohen-Macaulay ring.

Proof. Take a small representative $F : U \rightarrow W$ of $F$, where $U, W$ are neighbourhoods of 0 in $\mathbb{C}^2 \times \mathbb{C}$, $\mathbb{C}^3 \times \mathbb{C}$, respectively, and $T \subset U, W$ be a neighborhood of 0 in $\mathbb{C}$, such that the representatives $p_1 : D(F) \cap U \rightarrow T$ and $p_2 : F(D(F)) \cap W \rightarrow T$ are good in the sense of [4], where $p_1$ and $p_2$ are projections in the last factor. Take an irreducible component $D(F)^j$ of $D(F)$ and its image $F(D(F))^j$ which also is an irreducible component of $F(D(F))$. Since $F$ is topologically trivial, by Remark 2.4 (b), the families of curves $p_1 : D(F) \cap U \rightarrow T$ and $p_2 : F(D(F)) \cap W \rightarrow T$ are also topologically trivial. Hence, the restrictions

$$p_1 : D(F)^j \cap U \rightarrow T \quad \text{and} \quad p_2 : F(D(F))^j \cap W \rightarrow T$$

are also topologically trivial. Note that, $F(D(F))^j$ is reduced. However in general the special fiber $p_2^{-1}(0)$ of the restriction in (4) can have an embedded zero dimensional component, hence $p_2^{-1}(0)$ can be not reduced. In this case the family $F(D(F))^j \cap W$ in (4) is a family of generically reduced curves (in the sense of [14] or [26]). Since $D(F)^j$ and $F(D(F))^j$ are topologically trivial, by [10, Th. 4.4] their normalizations are smooth. Let

$$n_1 : U_1 \rightarrow D(F)^j \cap U \quad \text{and} \quad n_2 : U_2 \rightarrow F(D(F))^j \cap W,$$

be the normalization maps of $D(F)^j \cap U$ and $F(D(F))^j \cap W$, respectively, where $U_1$ and $U_2$ are open neighborhoods of 0 in $\mathbb{C}^2$. Consider the restriction of $F$ to $D(F)^j \cap U$ which we also will denote by $F$. Since $F : D(F)^j \cap U \rightarrow F(D(F))^j \cap W$ is surjective and finite, by the Universal Property of the normalization (see for instance [11]) there is a map $\tilde{F} : U_1 \rightarrow U_2$ such that the following diagram is commutative:

$$\begin{array}{ccc}
U_1 & \xrightarrow{\tilde{F}} & U_2 \\
\downarrow n_1 & & \downarrow n_2 \\
D(F)^j \cap U & \xrightarrow{F} & F(D(F))^j \cap W
\end{array}$$

Let $I_{D(F)^j} \subset \mathbb{C}[x, y, t]$ and $I_{F(D(F))^j} \subset \mathbb{C}[X, Y, Z, t]$ be the defining ideals of $D(F)^j$ and $F(D(F))^j$ (locally), respectively. Thus, we have the following induced commutative diagram of local rings:

$$\frac{\mathbb{C}[x, y, t]}{I_{D(F)^j}} \simeq \mathcal{O}_{D(F)^j} \xleftarrow{f^*} \frac{\mathbb{C}[x, y, t]}{I_{F(D(F))^j}} \simeq \mathcal{O}_{F(D(F))^j} \xrightarrow{f^*} \frac{\mathbb{C}[X, Y, Z, t]}{I_{F(D(F))^j}}$$
Since \( F(D(F)) \) is equimultiple along \( T \), by [26, Th. 3.3] (see also [25]) \( F(D(F)) \) is Whitney equisingular. Thus, by [26, Lemma 4.3] the local ring

\[
\frac{\mathbb{C}\{s, t\}}{\langle n_2^*(X, Y, Z) \rangle}
\]

is Cohen-Macaulay. Since \( codim(V(n_2^*(X, Y, Z))) = codim(V(\tilde{F}(n_2^*(X, Y, Z)))) \), by Proposition 3.2, we have that the local ring

\[
\frac{\mathbb{C}\{u, t\}}{\langle F^*(n_2^*(X, Y, Z)) \rangle}
\]

is also Cohen-Macaulay. By the commutativity of the diagram, it follows that

\[
\frac{\mathbb{C}\{u, t\}}{\langle n_1^*(F^*(X, Y, Z)) \rangle}
\]

is also a Cohen-Macaulay ring. Since the dimension of the ring in (5) is 1 and \( V(n_1^*(F^*(X, Y, Z))) = 0 \times T \subset \mathbb{C}^2 \), it follows that \( t \) is not a zero divisor in (5). Since \( F \) is topologically trivial, we have that \( V(F^*(X, Y, Z)) = T \cong (0, 0) \times T \subset \mathbb{C}^3 \). Now, we have the following inclusion between ideals in \( \mathbb{C}\{X, Y, Z, t\} \)

\[
(F_1(F_*(\mathcal{O}_2 \times \mathcal{O}_1))) \subset (X, Y, Z).
\]

hence, we have the following inclusion between ideals in \( \mathbb{C}\{x, y, t\} \)

\[
(F^*(F_1(F_*(\mathcal{O}_2 \times \mathcal{O}_1)))) \subset (F^*(X, Y, Z)).
\]

Recall by Remark 2.4 (e) that \( \langle F^*(F_1(F_*(\mathcal{O}_2 \times \mathcal{O}_1))) \rangle = I_{D(F)} \). Thus, we can define the quotient morphisms

\[
q_1 : \frac{\mathbb{C}\{x, y, t\}}{I_{D(F)}(F)} \to \frac{\mathbb{C}\{x, y, t\}}{\langle F^*(X, Y, Z) \rangle} \quad \text{and} \quad q_2 : \frac{\mathbb{C}\{u, t\}}{\langle n_1^*(F^*(X, Y, Z)) \rangle} \to \frac{\mathbb{C}\{u, t\}}{\langle n_1^*(F^*(X, Y, Z)) \rangle}
\]

Consider the composition \( q_2 \circ n_1^* \). Since \( q_2(n_1^*(F^*(X, Y, Z))) = 0 \) in the local ring in (5), by the Universal property of quotient, there is a unique morphism \( \varphi \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
\mathbb{C}\{x, y, t\} & \xrightarrow{n_1^*} & \mathbb{C}\{u, t\} \\
I_{D(F)}(F) & \xrightarrow{q_1} & \frac{\mathbb{C}\{u, t\}}{\langle n_1^*(F^*(X, Y, Z)) \rangle} \\
\end{array}
\]

\[
q_2 \circ n_1^* \quad \circ \quad \varphi
\]

\[
\begin{array}{ccc}
\mathbb{C}\{x, y, t\} & \xrightarrow{\varphi} & \mathbb{C}\{x, y, t\} \\
\langle F^*(X, Y, Z) \rangle & \xrightarrow{q_1} & \frac{\mathbb{C}\{x, y, t\}}{\langle F^*(X, Y, Z) \rangle} \\
\end{array}
\]

and \( \varphi(h) = q_2(n_1^*(h)) \), thus \( \varphi(t) = t \). Suppose that \( \mathbb{C}\{x, y, t\}/\langle F^*(X, Y, Z) \rangle \) is not Cohen-Macaulay, then \( t \) is a zero divisor in \( \mathbb{C}\{x, y, t\}/\langle F^*(X, Y, Z) \rangle \), thus there is \( g \neq 0 \) such that \( g.t = 0 \) in \( \mathbb{C}\{x, y, t\}/\langle F^*(X, Y, Z) \rangle \). We have that \( 0 = \varphi(0) = \varphi(gt) = \varphi(g)t \). Since \( g \notin \langle F^*(X, Y, Z) \rangle \), thus \( \varphi(g) \neq 0 \), hence \( t \) is a zero divisor in \( \mathbb{C}\{u, t\}/\langle n_1^*(F^*(X, Y, Z)) \rangle \), a contradiction. \( \blacksquare \)

We say that a polynomial map germ \( f(x, y) = (f_1(x, y), f_2(x, y), f_3(x, y)) \) is homogeneous if each \( f_i \) is a homogeneous function. We are now in conditions to state our main result.
Theorem 3.4 Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be a finitely determined map germ, homogeneous with corank 1. Let $F(x,y,t) = (f_1(x,y), t)$ a topologically trivial 1-parameter unfolding of $f$. Then, $m(f_1(\mathbb{C}^2))$ is constant.

Proof. Since $f$ has corank 1, we can assume that $f$ has the form $f(x,y) = (x, p(x,y), q(x,y))$. Let $n, m$ be the degrees of $p$ and $q$, respectively. Since $f$ is finitely determined, we have that $D(f) = V(\lambda(x,y))$ where $\lambda(x,y)$ is a homogeneous polynomial function with degree $d = nm - n - m + 1$ defined as

$$\lambda(x,y) = \prod_{i=1}^{d} (x - \alpha_i y)$$

where $\alpha_i \in \mathbb{C}$ are complex numbers all distinct (see [20]). Thus, $D(f)$ has $d$ smooth irreducible components. For $j = 1, \ldots, d$, denote by $D(f)^j$ the irreducible component of $D(f)$ defined by $x - \alpha_j y$. If $n = 1$ or $m = 1$, then $f$ is an immersion. Thus, we can assume that $n, m \geq 2$.

If $n = m = 2$, then $d = 1$ and $D(f)$ is a smooth curve. Therefore, by [15, Th. 2.14] we have that $f$ is stable and $\mathcal{A}$-equivalent to $(x,y^2,xy)$, the statement follows now by the stability of $f$.

Suppose now that $n \leq m$ and $3 \leq m$. In this case, there is at least one $\alpha_{j_0} \neq 0$. One parametrization of $D(f)^{j_0}$ is given by the map $\phi_{j_0} : (\mathbb{C}, 0) \to D(f)^{j_0}$ defined by $\phi_{j_0}(u) = (\alpha_{j_0} u, u)$. Hence, a parametrization of the image $f(D(f)^{j_0})$ of $D(f)^{j_0}$ is given by the map $f \circ \phi_{j_0}(u) = (\alpha_{j_0} u, p(\alpha_{j_0} u, u), q(-\alpha_{j_0} u, u))$ (see [24]). Thus, $f(D(f)^{j_0})$ is a smooth curve, hence the multiplicity of $f(D(f)^{j_0})$ is 1.

By upper-semicontinuity of the multiplicity, we have that the family of curves $F(D(F)^{j_0})$ is equimultiple (all fibers have multiplicity 1). By Lemma 3.3 we have that

$$\frac{\mathbb{C}\{x,y,t\}}{F^*(X,Y,Z)}$$

is a Cohen-Macaulay ring. The statement follows now by Lemma 3.1 (b).

Remark 3.5 (a) If $\gcd(n,m) \neq 2$ in Theorem 3.4, then Ruas and the author showed in [24] that $F$ is actually Whitney equisingular.

(b) The difficulty to extend Theorem 3.4 without any hypothesis about the corank of $f$ is that if $f$ has corank 2, then Lemma 3.1 (a) is not more true. In fact, from [16] consider the corank 2 map germ $f(x,y) = (x^2, y^2, x^3 + y^3 + xy)$ and the unfolding $F = (f_1(x,y), t)$ of $f$ where $f_1$ is defined as

$$f_1(x,y) = (x^2, y^2, x^3 + y^3 + xy + tx^3y + txy^3).$$

In [16], Marar, Nuño-Ballesteros and Peñafort-Sanchis showed that the unfolding $F$ is Whitney equisingular, in particular, $F$ is topologically trivial. Now, we have that

$$4 = m(f(\mathbb{C}^2)) \neq l\left(\frac{\mathbb{C}\{x,y,t\}}{(x^2, y^2, x^3 + y^3 + xy + tx^3y + txy^3, t)}\right) = 3$$

and

$$4 = m(f_1(\mathbb{C}^2)) \neq e\left(\frac{\mathbb{C}\{x,y,t\}}{(x^2, y^2, x^3 + y^3 + xy + tx^3y + txy^3)}\right) = 3$$

for $t \neq 0$.

However, note that $\mathbb{C}\{x,y,t\}/(x^2, y^2, x^3 + y^3 + xy + tx^3y + txy^3)$ is a Cohen-Macaulay ring.

Example 3.6 Consider the finitely determined map germ $f(x,y) = (x^2, y^2, x^7y - 14x^5y^3 + 49x^3y^5 - 36xy^7)$ from Example 2.1. Since $f$ is homogeneous and has corank 1, by Theorem 3.4 each topologically trivial unfolding of $f$ is equimultiple.
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