Simplicial Hochschild cochains as an Amitsur complex

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Abstract

It is demonstrated that the cochain complex of relative Hochschild $A$-valued cochains of a depth two extension $A | B$ under cup product is isomorphic as a differential graded algebra with the Amitsur complex of the coring $S = \text{End}_B A_B$ over the centralizer $R = A^B$ with grouplike element $1_S$, which itself is isomorphic to the Cartier complex of $S$ with coefficients in the $(S, S)$-bicomodule $R^e$. This specializes to finite dimensional algebras, H-separable extensions and Hopf-Galois extensions.

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1 Introduction

Relative Hochschild cohomology of a subring $B \subseteq A$ or ring homomorphism $B \to A$ is set forth in [4]. The coefficients of the general form of the cohomology theory are taken in a bimodule $M$ over $A$. If $M = A^*$ is the $k$-dual of the $k$-algebra $A$, this gives rise to a cyclic symmetry exploited in cyclic cohomology. If $M = A$, this has been shown to be related to the simplicial cohomology of a finitely triangulated space via barycentric subdivision, the poset algebra of incidence relations and the separable subalgebra of simplices by Gerstenhaber and Schack in a series of papers beginning with [3]. The $A$-valued relative cohomology groups of $(A, B)$ are also of interest in deformation theory. We refer to the relative Hochchild cochains with cohomology groups $H^n(A, B; A)$ as simplicial Hochschild cochains with cohomology.

In this note we will extend the following algebraic result in [7]: given a depth two ring extension $A | B$ with centralizer $R = A^B$ and endomorphism ring $S = \text{End}_B A_B$, the simplicial Hochschild cochains under cup product are isomorphic as a graded algebra to the tensor algebra of the $(R, R)$-bimodule $S$. Since $S$ is a left bialgebroid over $R$, it is in particular an $R$-coring with grouplike element $1_S = \text{id}_A$. The Amitsur complex of such a coring is a differential graded algebra explained in [2, 29.2]. We note below that the algebra isomorphism in [7] extends to an isomorphism of differential graded algebras. We also note that the Amitsur complex of the underlying coring of a bialgebroid is a Cartier complex with coefficients in a bicomodule formed from source and target homomorphisms. We remark on the consequences for relative Hochschild cohomology of various types of Galois extensions with bialgebroid action or coaction.

2 Preliminaries on depth two extensions

All rings and algebras are unital associative; homomorphisms and modules are unital as well. Let $R$ be a ring, and $M_R$, $N_R$ be right $R$-modules. The notation $M/N$ denotes that $M$ is
Recall that $M$ and $N$ are similar \cite[p. 208]{ref} if $M/N$ and $N/M$. A ring homomorphism $B \to A$ is sometimes called a ring extension $A|B$ (proper ring extension if $B \hookrightarrow A$).

**Definition 2.1.** A ring homomorphism $B \to A$ is said to be a right depth two (rD2) extension if the natural $(A, B)$-bimodules $A \otimes_B A$ and $A$ are similar.

Left D2 extension is defined similarly using the natural $(B, A)$-bimodule structures: a D2 extension is both rD2 and lD2. Note that in either case any ring extension satisfies $A/A \otimes_B A$.

Note some obvious cases of depth two: 1) $A$ a finite dimensional algebra, $B$ the ground field. 2) $A|B$ an H-separable extension. 3) $A|B$ a finite Hopf-Galois extension, since the Galois isomorphism $A \otimes_B A \xrightarrow{\sim} A \otimes H$ is an $(A, B)$-bimodule arrow (and its twist by the antipode shows $A|B$ to be lD2 as well).

Fix the notation $S := \text{End}_B A_B$ and $R = A^B$. Equip $S$ with $(R, R)$-bimodule structure

$$r \cdot \alpha \cdot s = r \alpha(-) s = \lambda_r \circ ho_s \circ \alpha$$

where $\lambda, \rho : R \to S$ denote left and right multiplication of $r, s \in R$ on $A$.

**Lemma 2.2.** \cite[3.11]{ref} If $A|B$ is rD2, then the module $S_R$ is a projective generator and

$$f_2 : S \otimes_R S \xrightarrow{\sim} \text{Hom}(B A \otimes_B A_B, B A_B)$$

via $f_2(\alpha \otimes_R \beta)(x \otimes_B y) = \alpha(x) \beta(y)$ for $x, y \in A$.

For example, if $A$ is a finite dimensional algebra over ground field $B$, then $S = \text{End} A$, the linear endomorphism algebra. If $A|B$ is H-separable, then $S \cong R \otimes_Z R^{op}$, where $Z$ is the center of $A$ \cite[4.8]{ref}. If $A|B$ is an $H^*$-Hopf-Galois extension, then $S \cong R\#H$, the smash product where $H$ has dual action on $A$ restricted to $R$ \cite[4.9]{ref}.

Recall that a left $R$-bialgebroid $H$ is a type of bialgebra over a possibly noncommutative base ring $R$. More specifically, $H$ and $R$ are rings with “target” and “source” ring anti-homomorphism and homomorphism $R \to H$, commuting at all values in $H$, which induce an $(R, R)$-bimodule structure on $H$ from the left. W.r.t. this structure, there is an $R$-coring structure $(H, R, \Delta, \varepsilon)$ such that $1_R$ is a grouplike element (see the next section) and the left $H$-modules form a tensor category with fiber functor to the category of $(R, R)$-bimodules. One of the main theorems in depth two theory is

**Theorem 2.3.** \cite[3.10, 4.1]{ref} Suppose $A|B$ is a left or right D2 ring extension. Then the endomorphism ring $S := \text{End}_B A_B$ is a left bialgebroid over the centralizer $A^B := R$ via the source map $\lambda : R \hookrightarrow S$, target map $\rho : R^{op} \hookrightarrow S$, coproduct

$$f_2(\Delta(\alpha))(x \otimes_B y) = \sum_{(\alpha)} f_2(\alpha_{(1)} \otimes_R \alpha_{(2)})(x \otimes_B y) = \alpha(xy).$$

Also $A$ under the natural action of $S$ is a left $S$-module algebra with invariant subring $A^S \cong \text{End}_E A$ where $E \cong A_B$, $A \otimes_R A \xrightarrow{\sim} A \# S$ via $\alpha \otimes \lambda_a \circ \alpha$.

We note in passing the measuring axiom of module algebra action from eq. (2): in Sweedler notation, $\sum_{(\alpha)} \alpha_{(1)}(x) \alpha_{(2)}(y) = \alpha(xy)$. Note too that $\Delta(\lambda_r) = \lambda_r \otimes 1_S$ and $\Delta(\rho_s) = 1_S \otimes_R \rho_s$ for $r, s \in R$. 

2
3 Amitsur complex of a coring with grouplike

An $R$-coring $\mathcal{C}$ has coassociative coproduct $\Delta : \mathcal{C} \to \mathcal{C} \otimes_R \mathcal{C}$ and counit $\varepsilon : \mathcal{C} \to R$, both mappings being $(R, R)$-bimodule homomorphisms. We assume that $\mathcal{C}$ also has a grouplike element $g \in \mathcal{C}$, which means that $\Delta(g) = g \otimes_R g$ and $\varepsilon(g) = 1$. The Amitsur complex $\Omega(\mathcal{C})$ of $(\mathcal{C}, g)$ has $n$-cochain modules

$$\Omega^n(\mathcal{C}) = \mathcal{C} \otimes_R \cdots \otimes_R \mathcal{C}$$

($n$ times $\mathcal{C}$), the zero’th given by $\Omega^0(\mathcal{C}) = R$. The Amitsur complex is the tensor algebra

$$\Omega(\mathcal{C}) = \bigoplus_{n=0}^{\infty} \Omega^n(\mathcal{C})$$

with a compatible differential $d = \{d^n\}$ where $d^n : \Omega^n(\mathcal{C}) \to \Omega^{n+1}(\mathcal{C})$. These are defined by $d^0 : R \to \mathcal{C}$, $d^0(r) = rg - gr$, and

$$d^n(c_1 \otimes \cdots \otimes c^n) = g \otimes c_1 \otimes \cdots \otimes c^n + (-1)^{n+1}c_1 \otimes \cdots \otimes c^n \otimes g + \sum_{i=1}^{n}(-1)^i c_1 \otimes \cdots \otimes c^{i-1} \otimes \Delta(c_i) \otimes c^{i+1} \otimes \cdots \otimes c^n$$

Some computations show that $\Omega(\mathcal{C})$ is a differential graded algebra $[2]$, with defining equations, $d \circ d = 0$ as well as the graded Leibniz equation on homogeneous elements,

$$d(\omega \omega') = (d\omega)\omega' + (-1)^{|\omega|}\omega d\omega'.$$

The name Amitsur complex comes from the case of a ring homomorphism $B \to A$ and $A$-coring $\mathcal{C} := A \otimes_B A$ with coproduct $\Delta(x \otimes_B y) = x \otimes_B 1_A \otimes_B y$ and counit $\varepsilon(x \otimes_B y) = xy$. The element $g = 1 \otimes_B 1$ is a grouplike element. We clearly obtain the classical Amitsur complex, which is acyclic if $A$ is faithfully flat over $B$. In general, the Amitsur complex of a Galois $A$-coring $(\mathcal{C}, g)$ is acyclic if $A$ is faithfully flat over the $g$-coinvariants $B = \{b \in A \mid bg = gb\} [2, 29.5]$.

The Amitsur complex of interest to this note is the following derivable from the left bialgebroid $S = \text{End}_B A_B$ of a depth two ring extension $A \mid B$ with centralizer $A_B^c = R$. The underlying $R$-coring $S$ has grouplike element $1_S = 1d_A$, with $(R, R)$-bimodule structure, coproduct and counit defined in the previous section. In Sweedler notation, we may summarize this as follows:

$$\Omega(S) = R \oplus \mathcal{S} \oplus \mathcal{S} \otimes_R S \oplus \mathcal{S} \otimes_R S \otimes_R S \otimes_R S \oplus \cdots$$

$$d^0(r) = \lambda_r - \rho_r, \quad d^1(\alpha) = 1_S \otimes_R \alpha - \alpha(1) \otimes_R \alpha(2) + \alpha \otimes_R 1_S, \ldots$$

It is interesting to remark that this particular Amitsur complex is naturally isomorphic to a Cartier complex of the $R$-coring $S$ with coefficients in the $(S, S)$-bicomodule $R^c [2, 30.3]$. The right coaction is given by $\rho^R(r \otimes s) = r \otimes \rho_s$, left coaction by $\rho^L(r \otimes s) = \lambda_r \otimes s$, and we note that $\text{Hom}_{R^c \otimes R}(R^c, \Omega^n(S)) \cong \Omega^n(S)$, the differentials being preserved by the isomorphism.

4 Cup product in simplicial Hochschild cohomology

Let $A \mid B$ be an extension of $K$-algebras. We briefly recall the $B$-relative Hochschild cohomology of $A$ with coefficients in $A$ (for coefficients in a bimodule, see the source [4]). The zero’th cochain group $C^0(A, B; A) = A_B = B$, while the $n$’th cochain group

$$C^n(A, B; A) = \text{Hom}_{B \otimes B}(A \otimes_B \cdots \otimes_B A, A)$$
(n times A in the domain). In particular, \( C^1(A, B; A) = \text{End}_B A_B = S \). The coboundary \( \delta^n : C^n(A, B; A) \to C^{n+1}(A, B; A) \) is given by
\[
(\delta^n f)(a_1 \otimes \cdots \otimes a_{n+1}) = a_1 f(a_2 \otimes \cdots \otimes a_{n+1}) + (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) a_{n+1} + \sum_{i=1}^n (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) \tag{4}
\]
and \( \delta^0 : R \to S \) is given by \( \delta^0(r) = \lambda_r - \rho_r \). The mappings satisfy \( \delta^{n+1} \circ \delta^n = 0 \) for each \( n \geq 0 \).

Its cohomology is denoted by \( H^n(A, B; A) = \ker \delta^n / \text{im} \delta^{n-1} \), and might be referred to as a simplicial Hochschild cohomology, since this cohomology is isomorphic to simplicial cohomology if \( A \) is the poset algebra of incidence relation in a finite simplicial complex and \( B \) is the separable subalgebra of simplices, where \( A \) is embeddable in an upper triangular matrix algebra with \( B \) the diagonal matrices [4].

The cup product \( \cup : C^m(A, B; A) \otimes_K C^n(A, B; A) \to C^{n+m}(A, B; A) \) makes use of the multiplicative structure on \( A \) and is given by
\[
(f \cup g)(a_1 \otimes \cdots \otimes a_{m+n}) = f(a_1 \otimes \cdots \otimes a_m) g(a_{m+1} \otimes \cdots \otimes a_{m+n}) \tag{5}
\]
which satisfies the equation \( \delta^{n+m}(f \cup g) = (\delta^m f) \cup g + (-1)^m f \cup \delta^n g \) [3]. Cup product therefore passes to a product on the cohomology. We note that \( (C^*(A, B; A), \cup, +, \delta) \) is a differential graded algebra we denote by \( C(A, B) \).

**Theorem 4.1.** Suppose \( A \mid B \) is a right or left \( D2 \) algebra extension. Then the relative Hochschild \( A \)-valued cochains \( C(A, B) \) is isomorphic to a differential graded algebra to the Amitsur complex \( \Omega(S) \) of the \( B \)-coring \( S \).

**Proof.** We define a mapping \( f \) by \( f_0 = \text{id}_R, f_1 = \text{id}_S \), and for \( n > 1 \),
\[
f_n : S \otimes_R \cdots \otimes_R S \xrightarrow{\cong} \text{Hom}_{B-B}(A \otimes_B \cdots \otimes_B A, A). \tag{6}
\]

by \( f_n(a_1 \otimes \cdots \otimes a_n) = a_1 \cup \cdots \cup a_n \). (Note that \( f_2 \) is consistent with our notation in section 2.) We proved by induction on \( n \) in [7, Theorem 5.1] that \( f \) is an isomorphism of graded algebras. We complete the proof by noting that \( f \) is a cochain morphism, i.e., commutes with differentials.

For \( n = 0 \), we note that \( \delta^0 \circ f_0 = f_1 \circ d^0 \), since \( d^0 = \delta^0 \). For \( n = 1 \),
\[
\delta^1(f_1(a))(a_1 \otimes_B a_2) = a_1 \alpha(a_2) - \alpha(a_1 a_2) + \alpha(a_1) a_2
\]
using eq. [2]. The induction step is carried out in a similar but tedious computation: this completes the proof that \( C(A, B) \cong \Omega(S) \). \( \square \)

### 5 Applications of the theorem

We immediately note that the cohomology rings of the two differential graded algebras are isomorphic.

**Corollary 5.1.** Relative \( A \)-valued Hochschild cohomology is isomorphic to the cohomology of the \( A^B \)-coring \( S = \text{End}_B A_B \):
\[
H^*(A, B; A) \cong H^*(\Omega(S), d) \tag{7}
\]

if \( A \mid B \) is a left or right depth two extension.
For example, a depth two f.g. projective extension is separable iff its \( R \)-coring \( S \) is coseparable [6 Theorem 3.1]. Cartier cohomology of a coseparable coring with any coefficients vanishes in positive dimensions [2 30.4] as does Hochschild cohomology of a separable extension [4]. But cohomology of the Amitsur complex above is a particular case of Cartier cohomology as noted at the end of section 3:

\[
H^\ast(\Omega(S), d) \cong H^\ast_{\mathrm{Ca}}(S, R^c).
\] (8)

**Corollary 5.2.** If the ring extension \( A \mid B \) is \( H \)-separable and one-sided faithfully flat, then the relative Hochschild cohomology, \( H^n(A, B; A) \) vanishes in positive dimensions.

**Proof.** The extension is necessarily proper by faithful flatness. Note that \( S \cong R \otimes_Z R \) is a Galois \( R \)-coring, since \( \{ r \in R \mid r \cdot 1_S = 1_S \cdot r \} = Z \), the center of \( A \) and the isomorphism \( r \otimes s \mapsto \lambda_r \circ \rho_s \) is clearly an \( R \)-coring homomorphism. Whence \( \Omega(S) \) is acyclic by [2 29.5]. \( \square \)

This also follows from proving that an \( H \)-separable extensions is separable.

The next corollary may be stated more generally for algebras over a base ring which is hereditary, if the universal coefficient theorem is taken into account. Let \( K \) be a Hopf algebra.

**Corollary 5.3.** Suppose \( A \mid B \) is a finite Hopf-\( K \)-Galois extension of algebras over a field \( k \). Then relative Hochschild \( A \)-valued cohomology is isomorphic to the Cartier cohomology of the underlying coalgebra \( K \) with trivial coefficients: for \( n \geq 2 \),

\[
H^n(A, B; A) \cong A^B \otimes_k \Omega^n_{\mathrm{Ca}}(K, k).
\] (9)

**Proof.** This follows from the determination of \( R \otimes_k K \cong S \) via \( r \otimes h \mapsto \lambda_r \circ (h \triangleright \cdot) \), and that \( \Delta_S = R \otimes \Delta_K \) in [5]. The relation of action of \( K \) on \( A \) to coaction \( A \to A \otimes K^\ast \) is given by \( h \triangleright a = a_{(0)}(a_{(1)}, h) \). The \( K \)-bicomodule structure on \( k \) is given by the unit \( k \to K \). Note that \( \Omega^n(S) \cong R \otimes K \otimes \cdots \otimes K (n \text{ times } K) \), where \( d^n = R \otimes d^n_c \) and \( d^n_c \) is the differential for coalgebra cohomology of \( K \) with coefficients in \( k \) [2 30.3]. \( \square \)

For example, a finite dimensional Hopf algebra \( K \) is Galois over \( k1_k \) via its coproduct as coaction, where \( K^\ast \) acts on \( K \) via \( h^\ast \to h = h_{(1)}(h^\ast, h_{(2)}) \). In this case, relative cohomology recovers absolute cohomology and the corollary states something well-known in a somewhat different perspective: for \( n \geq 2 \), \( H^n(K, K) \cong K \otimes \Omega^n_{\mathrm{Ca}}(K^\ast, k) \) (also, \( \cong \Omega^n_{\mathrm{Ca}}(K^\ast, K^\ast) \)).

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