VISCOUS SINGULAR SHOCK PROFILES FOR THE KEYFITZ-KRANZER SYSTEM

TING-HAO HSU

December 4, 2015

Abstract. It was shown by Schecter [Sch04], using the methods of Geometric Singular Perturbation Theory, that the Dafermos regularization

\( u_t + f(u)_x = \epsilon u_{xx} \)

for the Keyfitz-Kranzer system admits an unbounded family of solutions. Inspired by that work, in this paper we provide a more intuitive approach which leads to a stronger result. In addition to the existence of viscous profiles, we also prove the weak convergence and show that the maximum of the solution is of order \( \epsilon^{-2} \). This asymptotic behavior is distinct from that obtained in the author’s recent work [Hsu15] on a system modeling two-phase fluid flow, for which the maximum of the viscous solution is of order \( \exp(\epsilon^{-1}) \).

1. Introduction

The Keyfitz-Kranzer system

\[
\begin{align*}
  u_1, t + (u_1^2 - u_2)_x &= 0, \\
  u_2, t + (\frac{1}{3}u_1^3 - u_1)_x &= 0,
\end{align*}
\]

was first introduced in [KK89, KK90]. It is a strictly hyperbolic, genuinely nonlinear system of conservation laws. A significant feature is that this model provides an example for singular shocks. A singular shock, roughly speaking, is a measure which contains delta functions and is the weak limit of some approximate solutions. For details of the definition, we refer to [Sev07, Key11].

The existence of singular shocks for (1.1) was proved by Keyfitz and Kranzer [KK95]. In that work, for certain Riemann data

\[
(u_1, u_2)(x, 0) = \begin{cases} (u_{1L}, u_{2L}), & x < 0, \\
(u_{1R}, u_{2R}), & x > 0,
\end{cases}
\]

they construct approximate solutions of the regularized system via Dafermos regularization

\[
\begin{align*}
  u_{1, t} + (u_1^2 - u_2)_x &= \epsilon u_{1,xx}, \\
  u_{2, t} + (\frac{1}{3}u_1^3 - u_1)_x &= \epsilon u_{2,xx}.
\end{align*}
\]

In particular, they proved that there are approximate solution of (1.3e) that converges to a step function away from the discontinuity as \( \epsilon \to 0 \), and approaches a combination of delta functions near the discontinuity.

A family of exact solutions of (1.3e), rather than approximate solutions, is called a viscous profile of (1.1). The existence of viscous profiles of (1.1) was proved in [Sch04] using Geometric Singular Perturbation Theory (GSPT). In that work, existence of solutions of (1.3e) and (1.2) were proved, and the solutions approach infinity near the discontinuity as \( \epsilon \to 0 \), but convergence of solutions was not considered. We enhance that pioneering work in the following respects: First, we simplify the process of blowing-up in [Sch04], and construct solutions in a more intuitive way. Second, we prove the weak convergence of the solutions, which confirms the conjecture in [KK90].

The system (1.1) can be derived from a single space dimensional model for isentropic gas dynamics equations

\[
\begin{align*}
  \rho_t + (\rho u)_x &= 0, \\
  (\rho u)_t + (\rho u^2 + \rho \gamma)_x &= 0,
\end{align*}
\]

2010 Mathematics Subject Classification. 35L65, 35L67, 34E15, 34C37.

Key words and phrases. Conservation laws; Singular shocks; Viscous profiles; Dafermos regularization; Geometric Singular Perturbation Theory.
with $\gamma = 1$, which corresponds to isothermal gas dynamics. By subtracting $u$ times the first equation in (1.4) from the second equation, one obtains (1.1) with $u_1 = u$ and $u_2 = \frac{1}{2}u^2 - \log \rho$ (see [Key11]). This means that (1.1) is equivalent to the isothermal gas dynamics (1.4) for smooth solutions, but conservation of mass and momentum has been replaced by conservation of velocity and a quantity that is an entropy for the original system.

The system (1.4) with any $\gamma$ between 1 and $5/3$ was considered in [KT12], and the existence of viscous profiles for singular shock was also proved. Some other generalizations of (1.1) were systematically analyzed in [Sev07].

In Section 2, we state our main result. In Sections 3 we sketch the construction of the solutions. In Section 4, we recall some tools in geometric singular perturbation theory, including Fenichel’s Theorems and the Exchange Lemma. In Sections 5 we verify that the conditions of GSPT for our construction. The proof for the Main Theorem is given in Section 6.

2. MAIN RESULT

In standard notation for conservation laws, we write (1.1) as

\[ u_t + f(u)_x = 0, \]

where $u = (\beta, v)$, and write Riemann data for Riemann problems in the form

\[ u(x, 0) = u_L + (u_R - u_L)H(x), \]

where $H(x)$ is the step function taking value 0 if $x < 0; 1$ if $x > 0$.

We study the systems that approximate (2.5) via the Dafermos regularization:

\[ u_t + f(u)_x = \epsilon u_{xx} \]

for small $\epsilon > 0$. Using the self-similar variable $\xi = x/t$, the system is converted to

\[ -\xi \frac{d}{d\xi} u + \frac{d}{d\xi} (f(u)) = \epsilon \frac{d^2}{d\xi^2} u, \]

and the initial condition (2.6) becomes

\[ u(-\infty) = u_L, \quad u(+\infty) = u_R. \]

The system (2.8) is equivalent to

\[ -\epsilon u_\xi = f(u) - \xi u - w \]

\[ w_\xi = -u \]

or, up to a rescaling of time,

\[ \dot{u} = f(u) - \xi u - w \]

\[ \dot{w} = -\epsilon u \]

\[ \dot{\xi} = \epsilon. \]

The time variable in (2.11) is implicitly defined by the equation of $\dot{\xi}$. When $\epsilon = 0$, (2.11) is reduced to

\[ \dot{u} = f(u) - \xi u - w \]

\[ \dot{w} = 0, \quad \dot{\xi} = 0. \]

Returning to the $(u_1, u_2)$ notation, the system (2.11) is written as

\[ \dot{u}_1 = u_1^2 - u_2 - \xi u_1 - w_1 \]

\[ \dot{u}_2 = \frac{1}{3}u_1^3 - u_1 - \xi u_2 - w_2 \]

\[ \dot{w}_1 = -\epsilon u_1, \quad \dot{w}_2 = -\epsilon u_2, \quad \dot{\xi} = \epsilon. \]

and (2.12) becomes

\[ \dot{u}_1 = u_1^2 - u_2 - \xi u_1 - w_1 \]

\[ \dot{u}_2 = \frac{1}{3}u_1^3 - u_1 - \xi u_2 - w_2 \]

\[ \dot{w}_1 = 0, \quad \dot{w}_2 = 0, \quad \dot{\xi} = 0. \]
At any equilibrium \( u_0 = (u_{10}, u_{20}) \) of (2.14), the eigenvalues for the linearized system are
\[
\lambda_-(u_0) = u_{10} - 1, \quad \lambda_+(u_0) = u_{10} + 1.
\]

**Main Theorem.** Consider the Riemann problem (1.1) and (1.2). Let
\[
\begin{align*}
& (2.16a) \quad s = f_1(u_L) - f_1(u_R) \\
& (2.16b) \quad w_L = f(u_L) - su_L, \quad w_R = f(u_R) - su_R \\
& (2.16c) \quad e_0 = w_{2L} - w_{2R}.
\end{align*}
\]
Assume
\begin{itemize}
    \item[(H1)] \( \text{Re}(\pm(u_R)) < s < \text{Re}(\pm(u_L)) \).
    \item[(H2)] \( e_0 > 0 \).
\end{itemize}

Then there exists a Dafermos profile for a singular shock from \( u_L \) to \( u_R \). That is, for each small \( \epsilon > 0 \), there is a solution \( \bar{u}_\epsilon(\xi) \) of (2.8) and (2.9), and this solution becomes unbounded as \( \epsilon \to 0 \). Indeed,
\[
\begin{align*}
& (2.17a) \quad \max_\xi \pm \bar{u}_{1\epsilon}(\xi) = \left( \omega_0 + o(1) \right) \epsilon^{-1} \\
& (2.17b) \quad \max_\xi \bar{u}_{2\epsilon}(\xi) = \left( \kappa_0^2 + o(1) \right) \epsilon^{-2},
\end{align*}
\]
as \( \epsilon \to 0 \), where \( \kappa_0 \) and \( \omega_0 \) are positive constant defined later in (3.36) and (6.102). Moreover, if we set \( u_\epsilon(x, t) = \bar{u}_\epsilon(x/t) \), then \( u_\epsilon(x, t) \) is a solution of (1.3c) and (2.18a)
\[
\begin{align*}
& (2.18a) \quad u_{1\epsilon} \to u_{1L} + (u_{1R} - u_{1L})H(x - st) \\
& (2.18b) \quad u_{2\epsilon} \to u_{2L} + (u_{2R} - u_{2L})H(x - st) + \frac{e_0}{\sqrt{1 + s^2}} t \delta_{\{x = st\}}
\end{align*}
\]
in the sense of distributions.

The notation \( t \delta_{\{x = st\}} \) in (2.18b) denotes the linear functional defined by
\[
(2.19) \quad (t \delta_{\{x = st\}}, \varphi) = \int_0^\infty t \varphi(st, t) \sqrt{1 + s^2} \, dt.
\]
The weight \( \sqrt{1 + s^2} \) is the arc length of the parametrized line \( \{ x = st \} \), so that the definition of the functional is independent of parametrizations.

A set of sample data for which (H1) and (H2) hold is
\[
(2.20) \quad u_L = (2, 6), \quad u_R = (-1.6, 4.56),
\]
for which (2.16) gives \( s = 0 \) and \( e_0 = 0.423 \). A numerical solution for this Riemann data using a finite difference scheme is shown in Fig. 1. Observe that both \( u_1 \) and \( u_2 \) appear to grow unboundedly near the shock. This is consistent with the theorem.
3. Compactification and Desingularization

To find solutions of (2.13c) connecting \(u_L\) and \(u_R\), we first consider the limiting system (2.14) with \((w_1, w_2, \xi) = (w_{1L}, w_{2L}, s)\) and \((w_1R, w_{2R}, s)\), where \(s, w_L \) and \(w_R\) are as defined in (2.16).

**Proposition 3.1.** Assume (H1). Then there exists a unique solution of (2.14) of the form \(\gamma_1(\sigma) = (u^{(1)}(\sigma), w_L, s)\) satisfying

\[
\lim_{\sigma \to -\infty} u^{(1)}(\sigma) = u_L, \quad \lim_{\sigma \to 0^-} \left( \frac{u^{(1)}_2}{\sqrt{u^{(1)}_2}} \right)(\sigma) = \left( +\infty, \sqrt{3} - \sqrt{3} \right)
\]

and a unique solution of the form \(\gamma_2(\sigma) = (u^{(2)}(\sigma), w_R, s)\) satisfying

\[
\lim_{\sigma \to +\infty} u^{(1)}(\sigma) = u_R, \quad \lim_{\sigma \to 0^+} \left( \frac{u^{(2)}_2}{\sqrt{u^{(2)}_2}} \right)(\sigma) = \left( +\infty, -\sqrt{3} - \sqrt{3} \right).
\]

**Proof.** See [SSS93, Theorem 3.1]. \(\square\)

Motivated by Proposition 3.1 we compactify the state space by defining

\[
(3.21) \quad \beta = \frac{u_1}{\sqrt{u_2}}, \quad r = \frac{1}{\sqrt{u_2}}.
\]

In this definition we have assumed \(u_2\) to be positive. This is just for convenience and has no loss of generality. In general cases, since the value of \(u_2\) is bounded from below along \(\gamma_1\) and \(\gamma_2\), say \(u_2 > -M\), we may replace \(u_2\) by \(u_2 + M\).

In \((\beta, r, w_1, w_2, \xi, \epsilon)\)-coordinates, (2.13c) becomes, after multiplying by \(r\),

\[
\begin{align*}
\dot{\beta} &= \frac{-1}{6}\left( \beta^4 - 6\beta^2 + 6 \right) + r \left( \frac{-\beta^2}{2} + r \left( \frac{\beta^2}{2} - w_1 \right) + \frac{\beta^2}{2} \beta w_2 \right) \\
\dot{r} &= -\frac{\beta^2}{6} r + \frac{\beta}{2} \left( \xi + r \beta + r^2 w_2 \right) \\
\dot{w}_1 &= -\beta \epsilon \\
\dot{w}_2 &= \frac{\epsilon}{r} \\
\dot{\xi} &= r \epsilon \\
\dot{\epsilon} &= 0.
\end{align*}
\]

(3.22)

Note that the time scale in (3.22) is different from that of (2.13c), but we use the same notation \(\cdot\) to denote derivatives in time. This should cause no ambiguity since the time scales can be distinguished by the equations for \(\xi\).

In (3.22), the equation for \(\dot{w}_2\) is not defined when \(r = 0\). To make sense of it, one naive way is to multiply the system by \(r\), but this will make the set \(\{ r = 0 \}\) non-normally hyperbolic. To avoid this degeneracy, our remedy is to replace \(\epsilon\) by \(\kappa = \epsilon/r\). Then the system (3.22) becomes

\[
\begin{align*}
\dot{\beta} &= \frac{-1}{6}\left( \beta^4 - 6\beta^2 + 6 \right) + r \left( \frac{-\beta^2}{2} + r \left( \frac{\beta^2}{2} - w_1 \right) + \frac{\beta^2}{2} \beta w_2 \right) \\
\dot{r} &= -\frac{\beta^2}{6} r + \frac{\beta}{2} \left( \xi + r \beta + r^2 w_2 \right) \\
\dot{w}_1 &= -\kappa \beta r \\
\dot{w}_2 &= -\kappa \epsilon \\
\dot{\xi} &= \kappa r^2 \\
\dot{\kappa} &= \frac{\beta^2}{6} \kappa + \frac{r}{2} \left( -\kappa \xi - r \beta \kappa - r^2 \kappa w_2 \right)
\end{align*}
\]

(3.23)

Note that the first two equations in (3.22) and (3.23) are identical.

The sets \(\{ u_2 = +\infty \}\) and \(\{ \epsilon = 0 \}\) correspond to \(\{ r = 0 \}\) and \(\{ \kappa = 0 \}\). Taking \(r = 0\) and \(\kappa = 0\), the system (3.23) reduces to a single equation for \(\beta\), namely

\[
(3.24) \quad \dot{\beta} = \frac{-1}{6}\left( \beta^4 - 6\beta^2 + 6 \right).
\]
For this equation, the equilibria are $\beta = \rho_j$, $j = 1, \ldots, 4$, where

\begin{equation}
(3.25) \quad \rho_1 = -\sqrt{3 + \sqrt{3}}, \quad \rho_2 = -\sqrt{3 - \sqrt{3}}, \quad \rho_3 = \sqrt{3 - \sqrt{3}}, \quad \rho_4 = \sqrt{3 + \sqrt{3}}.
\end{equation}

Let

\begin{equation}
(3.26) \quad \mathcal{P}_L = \{ (\beta, r, \kappa, w_1, w_2, \xi) : \beta = \rho_3, r = 0, \kappa = 0 \}
\end{equation}

\begin{equation}
(3.27) \quad \mathcal{P}_R = \{ (\beta, r, \kappa, w_1, w_2, \xi) : \beta = \rho_2, r = 0, \kappa = 0 \}.
\end{equation}

The trajectory $\gamma_1$ given in Proposition 3.1 connects $u_L$ and $\mathcal{P}_L$, and $\gamma_2$ connects $u_R$ and $\mathcal{P}_R$. Next we shall find connections between $\mathcal{P}_L$ and $\mathcal{P}_R$.

We will find a trajectory on $\{ r = 0 \}$ connecting $(\beta, \kappa, w_1, w_2, \xi) = (\rho_3, 0, w_{1L}, w_{2L}, s)$ and $(\rho_2, 0, w_{1R}, w_{2R}, s)$. When $r = 0$, the system reduces to

\begin{equation}
(3.28a) \quad \dot{\beta} = \frac{1}{6} (\beta^4 - 6\beta^2 + 6)
\end{equation}

\begin{equation}
(3.28b) \quad \dot{\kappa} = \frac{\beta^3}{6} \kappa
\end{equation}

\begin{equation}
(3.28c) \quad \dot{w}_2 = -\kappa
\end{equation}

\begin{equation}
(3.28d) \quad \dot{w}_1 = 0, \quad \dot{\xi} = 0.
\end{equation}

Observe that the system (3.28) is only weakly coupled, so we can solve it by integration:

**Proposition 3.2.** There exist positive smooth functions $u_1$, $u_2$ and $u_3$ which satisfy the following: For any parameters $(\bar{\kappa}, \bar{w}_1, \bar{w}_2, \xi)$, the system (3.28) with boundary conditions

\begin{equation}
(3.29) \quad (\beta, \kappa)(0) = (0, \bar{\kappa}), \quad (w_1, w_2, \xi)(-\infty) = (\bar{w}_1, \bar{w}_2, \bar{\xi}),
\end{equation}

has a unique solution

\begin{equation}
(3.30) \quad (\beta^-, \kappa^-, w_1^-, w_2^-, \xi^-)(\sigma) = (u_1(\sigma), \bar{\kappa}u_2(\sigma), \bar{w}_1, \bar{w}_2 + \bar{\kappa}u_3(\sigma), \bar{\xi}).
\end{equation}

For any parameters $(\bar{\kappa}, \bar{w}_1, \bar{w}_2, \xi)$, the system (3.28) with boundary conditions

\begin{equation}
(3.31) \quad (\beta, \kappa)(0) = (0, \bar{\kappa}), \quad (w_1, w_2, \xi)(+\infty) = (\bar{w}_1, \bar{w}_2, \bar{\xi}),
\end{equation}

has a unique solution

\begin{equation}
(3.32) \quad (\beta^+, \kappa^+, w_1^+, w_2^+, \xi^+)(\sigma) = (u_1(-\sigma), \bar{\kappa}u_2(-\sigma), \bar{w}_1, \bar{w}_2 - \bar{\kappa}u_3(\sigma), \bar{\xi}).
\end{equation}
Figure 3. $\gamma_1$, $\gamma_2$ and $\gamma_0$ displayed in $(\beta, r, w_2)$-space.

Figure 4. Near the singular configuration $\gamma_1 \cup \gamma_0 \cup \gamma_2$, we will find trajectories for (3.23) lying in the hyper-surface $\{rk = \epsilon\}$ for each small $\epsilon > 0$.

Proof. First we solve (3.28a) by setting

$$t_1(\sigma)$$

to be the solution of (3.28a) satisfying $t_1(0) = 0$.

Let

$$t_2(\sigma) = \exp \left( \int_0^\sigma \frac{t_1(\tau)^3}{6} d\tau \right) , \quad t_3(\sigma) = \int_\infty^\sigma \frac{t_2(\tau)}{\epsilon} d\tau .$$

Then a direct calculation shows that (3.30) and (3.32) are solutions of (3.28) satisfying (3.29) and (3.31). □
See Fig 2 for the trajectories given in Proposition 3.2. Note that \( \iota_1(\sigma) \) defined in (3.33) satisfies \( \iota_1(-\infty) = \rho_3 \) and \( \iota_2(+\infty) = \rho_2 \).

**Proposition 3.3.** If we set

\[
(\tilde{\kappa}, \tilde{w}_1, \tilde{w}_2, \tilde{\xi}) = (\kappa_0, w_{1L}, w_{2L}, s), \quad (\kappa, \bar{w}_1, \bar{w}_2, \bar{\xi}) = (\kappa_0, w_{1R}, w_{2R}, s),
\]

with

\[
\kappa_0 = \frac{w_{2R} - w_{2L}}{2\iota_3(0)},
\]

where \( \iota_3(\sigma) \) is as defined in (3.34), then (3.30) and (3.32) coincide, and this gives a solution of (3.28), denoted by \( \gamma_0(\sigma) \), satisfying

\[
\gamma_0(-\infty) = (\rho_3, 0, w_{1L}, w_{2L}, s), \quad \gamma_0(+\infty) = (\rho_2, 0, w_{1R}, w_{2R}, s).
\]

**Proof.** First we set

\[
(\tilde{w}_1, \tilde{w}_2, \tilde{\xi}) = (w_{1L}, w_{2L}, s), \quad (\bar{w}_1, \bar{w}_2, \bar{\xi}) = (w_{1R}, w_{2R}, s).
\]

From the definitions in (2.16a) and (2.16b) we have \( w_{1L} = w_{1R} \). Solving

\[
(\beta^-, \kappa^-, w_{1L}, w_{2L}, \xi^-)(0) = (\beta^+, \kappa^+, w_{1R}, w_{2R}, \xi^+)(0)
\]

in (3.30) and (3.32) for \( \kappa \) and \( \bar{\kappa} \), we obtain the solution \( \tilde{\kappa} = \bar{\kappa} = \kappa_0 \) as defined in (3.36). This gives a trajectory \( \gamma_0(\sigma) \) satisfying (3.37). From the uniqueness of solutions of boundary value problems, this trajectory is unique. \( \square \)

We will show that the for the system (3.23) there are trajectories close to \( \gamma_1 \cup \gamma_0 \cup \gamma_2 \) lying in hyper-surfaces \( \{rk = \epsilon\}, \epsilon > 0 \). See Fig 3 and 4.

For solutions \( (u_{1\epsilon}, u_{2\epsilon})(\xi) \) of (2.8a) and (2.9), from the equation for \( \dot{\xi} \) in (2.13a), we know the \( \xi \)-interval corresponding to any compact segment of \( \gamma_1 \) or \( \gamma_2 \) has length of order \( O(\epsilon) \). We will see at the end of Section 6.1 that the length of the \( \xi \)-interval corresponding to any compact segment of \( \gamma_0 \) is of order \( O(\epsilon^2) \).

4. Geometric Singular Perturbation Theory

Our main goal is to solve the boundary value problem (2.8a) and (2.9). Note that (2.8a) is a singularly perturbed equation since the perturbation \( \epsilon \frac{d^2}{dx^2} u \) has a higher order derivative than the other terms in the equation. We will apply GSPT to deal with singularly perturbed equations. The idea of GSPT is to first study a set of subsystems which forms a decomposition of a system, and then to use the information for the subsystems to conclude results for the original system.

In Section 4.1 and 4.2 we recall some fundamental theorems in GSPT. We only briefly state necessary theorems because it is similar to Hsu 15, Section 4. In Section 4.3 we state and give new proofs for a version of the Corner Lemma.

4.1. Fenichel’s Theory for Fast-Slow Systems. Note that (2.11c) is a fast-slow system, which means that the system is of the form

\[
\begin{align*}
\dot{x} &= f(x, y, \epsilon) \\
\dot{y} &= \epsilon g(x, y, \epsilon).
\end{align*}
\]

where \( (x, y) \in \mathbb{R}^n \times \mathbb{R}^l \), and \( \epsilon \) is a parameter. In order to deal with fast-slow systems, Fenichel’s Theory was developed in Fen74, Fen77, Fen79. Some expositions for that theory can be found in Wig94, Jon95.

An important feature of a fast-slow system is that the system can be decomposed into two subsystems: the limiting fast system and the limiting slow system. The limiting fast system is obtained by taking \( \epsilon = 0 \) in (3.38a): that is,

\[
\begin{align*}
\dot{x} &= f(x, y, 0) \\
\dot{y} &= 0.
\end{align*}
\]

On the other hand, note that the system (3.38c) can be converted to, after a rescaling of time,

\[
\begin{align*}
x' &= f(x, y, \epsilon) \\
y' &= g(x, y, \epsilon).
\end{align*}
\]
Taking $\epsilon = 0$ in (4.40), we obtain the limiting slow system

\[(4.41)\]

\[
\begin{align*}
0 &= f(x, y, 0) \\
\frac{dy}{dt} &= g(x, y, 0).
\end{align*}
\]

Note that the limiting slow system (4.41) describes dynamics on the set of critical points of the limiting fast system (4.39), so we will need to piece together the information of the limiting fast system and the limiting slow system in the vicinity of the set of critical points. To piece this information together, normal hyperbolicity defined below will be a crucial condition.

**Definition 1.** A critical manifold $S_0$ for (4.39) is an $t$-dimensional manifold consisting of critical points of (4.39). A critical manifold is normally hyperbolic if $D_x f(x, y, 0)|_{S_0}$ is hyperbolic. That is, at any point $(x_0, y_0) \in S_0$, all eigenvalues of $D_x f(x, y, 0)|(x_0, y_0)$ have nonzero real part.

Fenichel’s Theory is a center manifold theory for fast-slow systems. For a normally hyperbolic critical manifold $S_0$ for (4.39), the stable and unstable manifolds $W^s(S_0)$ and $W^u(S_0)$ can be defined in the natural way. We denote them by $W^s_0(S_0)$ and $W^u_0(S_0)$ to indicate their invariance under (4.38) with $\epsilon = 0$. Fenichel’s Theory assures that the hyperbolic structure of $S_0$ persists under perturbation (4.38). Below we state three fundamental theorems of Fenichel’s Theory following [Jon95].

**Theorem 4.1** (Fenichel’s Theorem 1). Consider the system (4.38e), where $(x, y) \in \mathbb{R}^n \times \mathbb{R}^l$, and $f$, $g$ are $C^r$ for some $r \geq 2$. Let $S_0$ be a compact normally hyperbolic manifold for (4.39). Then for any small $\epsilon \geq 0$ there exist locally invariant $C^r$ manifolds, denoted by $S_\epsilon$, $W^u_\epsilon(S_\epsilon)$ and $W^s_\epsilon(S_\epsilon)$, which are $C^1$ O($\epsilon$)-close to $S_0$, $W^u_0(S_0)$ and $W^s_0(S_0)$, respectively. Moreover, for any continuous families of compact sets $I_\epsilon \subset W^u_\epsilon(S_\epsilon)$, $J_\epsilon \subset W^s_\epsilon(S_\epsilon)$, there exist positive constants $C$ and $\nu$ such that

\[
\begin{align*}
\text{(4.42a)} \quad &\text{dist}(z \cdot t, S_\epsilon) \leq Ce^{\nu t} \quad \forall z \in I_\epsilon, \ t \leq 0 \\
\text{(4.42b)} \quad &\text{dist}(z \cdot t, S_\epsilon) \leq Ce^{-\nu t} \quad \forall z \in J_\epsilon, \ t \geq 0,
\end{align*}
\]

where $\cdot$ denotes the flow for (4.38e).

**Proof.** See [Jon95] Theorem 3].

**Remark 1.** If $S_0$ is locally invariant under (4.38e) for each $\epsilon$, then the $S_\epsilon$ can be chosen to be $S_0$ because of the construction in the proof of [Jon95] Theorem 3].

Note that $W^u(S_\epsilon)$ and $W^s(S_\epsilon)$ can be interpreted as a decomposition in a neighborhood of $S_0$ in $(x, y)$-space. The following theorem asserts that this induces a change of coordinates $(a, b, c)$ such that $W^u(S_\epsilon)$ and $W^s(S_\epsilon)$ correspond to $(a, c)$-space and $(b, c)$-space, respectively.

**Theorem 4.2** (Fenichel’s Theorem 2). Suppose the assumptions in Theorem 4.1 hold. Then under a $C^r$ $\epsilon$-dependent coordinate change $(x, y) \mapsto (a, b, c)$, the system (4.38e) can be brought to the form

\[
\begin{align*}
\dot{a} &= A^u(a, b, c, \epsilon)a \\
\dot{b} &= A^s(a, b, c, \epsilon)b \\
\dot{c} &= \epsilon \left(b(c) + E(a, b, c, \epsilon)\right)
\end{align*}
\]

in a neighborhood of $S_\epsilon$, where the coefficients are $C^{r-2}$ functions satisfying

\[
\text{(4.44)} \quad \inf_{\lambda \in \text{Spec}A^u(a, b, c, 0)} \text{Re} \lambda > 2\nu, \quad \sup_{\lambda \in \text{Spec}A^s(a, b, c, 0)} \text{Re} \lambda < -2\nu
\]

for some $\nu > 0$ and

\[
\text{(4.45)} \quad E = 0 \quad \text{on} \quad \{a = 0\} \cup \{b = 0\}.
\]

**Proof.** See [Jon95] Section 3.5] or [JT09] Proposition 1].

The family of trajectories for (4.41) forms a foliation of $S_0$. The following theorem says that this induces a foliation of $W^u_\epsilon(S_\epsilon)$ and $W^s_\epsilon(S_\epsilon)$.
Theorem 4.3 (Fenichel’s Theorem 3). Suppose the assumptions in Theorem 4.1 hold. Let $\Lambda_0$ be a submanifold in $S_0$ which is locally invariant under (4.41). Then there exist locally invariant manifolds $\Lambda_\epsilon$, $W^s_\epsilon(\Lambda_\epsilon)$, and $W^u_\epsilon(\Lambda_\epsilon)$ for $\epsilon > 0$ which are $C^{r-2}$ O(\epsilon)$-close to $\Lambda_0$, $W^s_\epsilon(\Lambda_\epsilon)$, and $W^u_\epsilon(\Lambda_0)$, respectively. Moreover, for any continuous families of compact sets $\mathcal{S}_\epsilon \subset W^s_\epsilon(\Lambda_\epsilon)$, $\mathcal{J}_\epsilon \subset W^u_\epsilon(\Lambda_\epsilon)$, $\epsilon \in [0, \epsilon_0]$, there exist positive constants $C$ and $\nu$ such that (4.42) holds with $S_\epsilon$ replaced by $\Lambda_\epsilon$. Suppose in addition that $S_0$ is invariant under (4.38) for each $\epsilon$. Then $\Lambda_\epsilon$ can be chosen to be $\Lambda_0$. 

Proof. Using Fenichel’s coordinates $(a, b, c)$ in Theorem 4.2 for the splitting of $S_0$, we can take $W^s_\epsilon(\Lambda_\epsilon)$ and $W^u_\epsilon(\Lambda_\epsilon)$ to be the pre-images of the sets $\{(a, b, c) : a = 0, c \in \Lambda_0\}$ and $\{(a, b, c) : b = 0, c \in \Lambda_0\}$, respectively, in $(x, y)$-space. From (4.44) we obtain (4.42) with $S_\epsilon$ replaced by $\Lambda_\epsilon$. Suppose $S_0$ is invariant under (4.38) for each $\epsilon$, then from the remark after Theorem 4.1 we can take $S_\epsilon = S_0$ and hence $\Lambda_\epsilon = \Lambda_0$

The system (4.43) is called a Fenichel normal form for (4.38), and the variables $(a, b, c)$ are called Fenichel coordinates.

4.2. Silnikov Boundary Value Problem. We have seen in Section 4.1 that fast-slow systems $\epsilon > 0$ can locally be converted into normal forms $\epsilon = 0$, where $A^p$ and $A^s$ satisfy the gap condition (4.44), and $E$ is a small term satisfying (4.45). If we append the system with the equation $\dot{c} = 0$ and then replace $c$ by $\check{c} = (c, \epsilon)$, we obtain a system of the form

$$\begin{align*}
\dot{a} &= A^p(a, b, \check{c})a \\
\dot{b} &= A^s(a, b, \check{c})b \\
\dot{\check{c}} &= h(\check{c}) + E(a, b, \check{c}),
\end{align*}$$

(4.46)

for which (4.44) and (4.45) are satisfied with $E$ replaced by $\tilde{E}$. For convenience, we will drop the tilde notation in (4.46) in the remaining discussion.

A Silnikov problem is the system (4.46) along with boundary data of the form

$$\begin{align*}
(b, c)(0) &= (b_0, c_0), & a(T) &= a_1,
\end{align*}$$

(4.47)

where $T \geq 0$.

The critical manifold for (4.46) is $\{a = 0, b = 0\}$, on which the system is governed by the limiting slow system

$$\dot{\check{c}} = h(c).$$

(4.48)

For a solution $(a(t), b(t), c(t))$ to the Silnikov boundary value problem (4.46) and (4.47), from conditions (4.44) and (4.45), it is natural to expect that $a(t)$ and $b(t)$ decay to 0 in backward time and forward time, respectively, and that $c(t)$ is approximately the solution of (4.48). A theorem from [Sch08b] asserts that this is the case:

Theorem 4.4 (Generalized Deng’s Lemma [Sch08b]). Consider the system (4.46) satisfying (4.44) and (4.45) with $C^r$ coefficients, $r \geq 1$, defined on the closure of a bounded open set $B_{k, \Delta} \times B_{m, \Delta} \times V \subset \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^l$, where $B_{k, \Delta} = \{a \in \mathbb{R}^k : |a| < \Delta\}$, $\Delta > 0$, and $V$ is a bounded open set in $\mathbb{R}^l$.

Let $K_0$ and $K_1$ be compact subsets of $V$ such that $K_0 \subset \text{Int}(K_1)$. For each $c_0 \in K_0$ let $J_{c_0}$ be the maximal interval such that $\phi(t, c_0) \in \text{Int}(K_1)$ for all $t \in J_{c_0}$, where $\phi(t, c_0)$ is the solution of (4.48) with initial value $g_0$. Let $\nu > 0$ be the number in (4.44). Suppose there exists $\beta > 0$ such that $\dot{v} := v - r\beta > 0$ and

$$|\phi(t, c_0)| \leq Me^{r(t-1)} \quad \forall t \in J_{c_0}.$$

Then there is a number $\delta_0 > 0$ such that if $|a_l| < \delta_0$, $|b_0| < \delta_0$, $c_0 \in V_0$, and $T > 0$ is in $J_{c_0}$, then the Silnikov boundary value problem (4.46) and (4.47) has a solution $(a(t), b(t), c(t), c_0)$ on the interval $0 \leq t \leq T$. Moreover, there is a number $K > 0$ such that for all $(t, T, a_l, b_0, c_0)$ as above and for all multi-indices $i$ with $|i| \leq r$,

$$\begin{align*}
|D_ia(t, T, a_l, b_0, c_0)| &\leq Ke^{-\nu(T-t)} \\
|D_ib(t, T, a_l, b_0, c_0)| &\leq Ke^{-\nu t} \\
|D_ic(t, T, a_l, b_0, c_0) - D_0c(t, c_0)| &\leq Ke^{-\nu T}.
\end{align*}$$

(4.49)
4.3. The Corner Lemma. The Corner Lemma was first asserted in [Sch04], but its author later pointed out [Sch08a, Remark 2.4] that the proof was flawed and needed to be reworked. In Theorem 4.6 we modify both the statement and the proof of the original lemma. In our modified version, the required assumptions are more restricted, but they are already enough for our purpose.

First we state the special case of Theorem 4.4 with \( h \equiv 0 \) in (4.46) as follows.

**Theorem 4.5.** Consider a system of the form

\[
\begin{align*}
\dot{a} &= A^u(a, b, c)a \\
\dot{b} &= A^s(a, b, c)b \\
\dot{c} &= E(a, b, c),
\end{align*}
\]

satisfying (4.44) and (4.45) with \( C^r \) coefficients, \( r \geq 1 \), defined on the closure of a bounded open set \( B = B_{k,\Delta} \times B_{m,\Delta} \times V \subset \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^l \). Then for any \( (a^1, b^0, c^0) \in B \) and \( T \geq 0 \), the Silnikov boundary value problem (4.50) and (4.47) has a unique solution, denoted by \( (a, b, c)(t; T, a^1, b^0, c^0) \), \( t \in [0, T] \). Moreover, if we set

\[
\begin{align*}
p_T &= (a, b, c)(0; T, a^1, b^0, c^0), & q_T &= (a, b, c)(T; T, a^1, b^0, c^0)
\end{align*}
\]

and write \( p_T = (a^p_T, b^0, c^0) \) and \( q_T = (a^1, \hat{b}_T, \hat{c}_T) \), then

\[
\| (a^p_T, b_T, \hat{c}_T - c^0) \|_{C^r(B)} \leq \tilde{C} e^{-\mu T}
\]

for some positive constants \( \tilde{C} \) and \( \mu \).

We will consider special cases of the system (4.50) for which there is an invariant manifold of codimension 1 which is transverse to an unstable direction. For definiteness, we assume \( \{a_k = 0\} \) to be invariant under (4.50), and the matrix-valued function \( A^u(a, b, c) \) is of the form

\[
A^u = \begin{pmatrix} A^u_0 & * \\ 0 & \lambda_k \end{pmatrix}
\]

where \( A^u_0 \) is a \((k-1) \times (k-1)\) matrix function and \( \lambda_k \) is a positive scalar function.
Using Theorem 4.5 we will prove the following:

**Theorem 4.6 (Corner Lemma).** Consider \( (4.50) \) defined on the closure of a bounded open set \( B_{k,\Delta} \times B_{m,\Delta} \times V \subset \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^l \), where the coefficients \( A^u, A^s \) and \( E \) are \( C^r \) for some \( r \geq 3 \), and \( A^u \) is of the form \( (4.52) \). Assume \( (4.44) \) and

\[
E(a, b, c) = 0 \quad \text{on} \quad \{a = 0\} \cap \{b = 0\}.
\]

Let \( \Lambda \subset V \) be a \( \sigma \)-dimensional \( C^r \) manifold, \( 0 \leq \sigma \leq 1 \), and let \( \mathcal{I} \) be a \( C^r \) manifold of the form

\[
\mathcal{I} = \{(a, b, c) : |a| < \Delta_1, b = b^0, c = c^0 + \theta(a, c^0)a, c^0 \in \Lambda\},
\]

where \( 0 < \{\Delta_1, |b^0|\} < \Delta \), and \( \theta \) is a \((l \times k)\)-matrix function. Let \( \mathcal{I}_e = \mathcal{I} \cap \{a_k = \epsilon\} \). Denote \( \mathcal{I}_e^* = \mathcal{I}_e \cdot [0, \infty) \).

Then the following holds: Fix any \( q_0 \in W^u(\Lambda) \) with positive \( a_k \)-coordinate. Then there exists a neighborhood \( V_0 \) of \( q_0 \) satisfying that

\[
\mathcal{I}_e^* \cap V_0 \text{ is } C^{r-3} \text{ close to } W^u(\Lambda) \cap V_0
\]
as \( \epsilon \to 0 \). See Fig. [3].

Furthermore, given any sequence of points \( q_\epsilon \in \mathcal{I}_e^* \cap V_0, \epsilon \in [0, \epsilon_0] \), which converges to a point \( q_0 \in W^u(\Lambda) \) as \( \epsilon \to 0 \), let \( p_\epsilon \in \mathcal{I}_e^* \) and \( T_\epsilon > 0 \) be such that \( q_\epsilon = p_\epsilon \cdot T_\epsilon \), and let \( p_0 \) be the unique point in \( \mathcal{I}_0 \) satisfying \( \pi^s(p_0) = \pi^s(q_0) \), where \( \pi^s, \pi^u \) are the projections along stable/unstable fibers. Then \( p_\epsilon \to p_0 \) as \( \epsilon \to 0 \), and

\[
\tilde{C}^{-1} \log \frac{1}{\epsilon} \leq T_\epsilon \leq \tilde{C} \log \frac{1}{\epsilon}
\]

for some \( \tilde{C} > 0 \).

**Proof.** Under the assumption \( (4.53) \), from [Den50] Lemma 2.2, there exists a \( C^{r-2} \) change of variables of the form \((a, b, c) \mapsto (a, b, \hat{c})\) so that the new system converted from \( (4.60) \), still denoted by \( (4.50) \), satisfies \( (4.45) \). The change of coordinate is a modification only on \( c \), so \( \mathcal{I} \) is still parametrized as \( (4.54) \) in the new coordinates. Therefore, by dropping the hat in \( \hat{c} \), we assume \( (4.45) \) holds for the system \( (4.50) \), and the coefficients are \( C^{r-2} \) functions.

The stable/unstable manifolds for \( (4.50) \) are

\[
W^s(\Lambda) = \{(a, b, c) : b = 0\}, \quad W^u(\Lambda) = \{(a, b, c) : a = 0\}.
\]

From \( (4.45) \), the slow variable \( c \) is constant on \( \{a = 0\} \cup \{b = 0\} \), which implies

\[
\pi^u(a, 0, c) = (0, 0, c), \quad \pi^s(0, b, c) = (0, 0, c).
\]

Let

\[
\mathcal{A} = \{a \in \mathbb{R}^k : |a - a(q_0)| < \Delta_2\}
\]

for some positive number \( \Delta_2 < \frac{1}{2} \min \{\Delta, |a(q_0)|, a_k(q_0)\} \), so that \( \mathcal{A} \subset B_{k,\Delta} \), where \( a(q_0) \) and \( a_k(q_0) \) denote the \( a \)- and \( a_k \)-coordinates of \( q_0 \). Choose a smooth real-valued function \( \chi(b) \) so that \( \chi(b^0) = 1 \) and \( \chi(0) = 0 \). Let

\[
\hat{c} = c - \chi(b)\theta(a, c^0)a.
\]

Then from \( \chi(b^0) = 0 \) we have

\[
\hat{c} = c - \theta(a, c^0)a \quad \text{on} \quad \{b = b^0\}
\]

and from \( \chi(0) = 0 \) we have

\[
\hat{c} = c \quad \text{on} \quad \{a = 0\} \cup \{b = 0\}.
\]

From \( (4.61) \), the image of \( \mathcal{I} \) in \((a, b, \hat{c})\)-space is

\[
\tilde{\mathcal{I}} = \{(a, b, \hat{c}) : |a| < \Delta_1, b = b^0, \hat{c} = c^0, c^0 \in \Lambda\}.
\]

From \( (4.44) \) we know

\[
\tilde{C}^{-1} < \lambda_k < \tilde{C}
\]
for some positive constant \( \tilde{C} \). In \((a, b, \tilde{c})\)-coordinates, the system (4.50) is converted to, after dividing the equation by \( \lambda_k \),

\[
(4.65) \quad a' = \begin{pmatrix} \dot{A}^u & \ast \\ 0 & 1 \end{pmatrix} a, \quad b' = \dot{A}^s b, \quad \tilde{c}' = \tilde{E},
\]

for some \( C^r \)-coefficients \( \dot{A}^u, \dot{A}^s \) and \( \tilde{E} \), where \( t \) denotes the derivative with respect to the time variable \( \zeta \) defined by

\[
(4.66) \quad d\zeta/d\sigma = \lambda_k,
\]

where \( \sigma \) is the time variable for (4.50). Clearly (4.44) holds with \( \lambda \) replaced by \( \dot{A}^u, \dot{A}^s \) and \( \tilde{\nu} := \nu/\tilde{C} \). Note that the condition (4.45) means \( \tilde{c} \) is constant on \( \{a = 0\} \cup \{b = 0\} \). From (4.62) we see that \( \tilde{c} \) is also constant on that set. Hence (4.45) holds with \( E \) replaced by \( \tilde{E} \). Thus Theorem 4.5 can be applied to (4.65).

By Theorem 4.5, for any sufficiently large number \( T \) and any \((a^1, c^0) \in A \times \Lambda \), we can set \((a, b, \tilde{c})(t; T, a^1, b^0, c^0), t \in [0, T] \), to be the solution of (4.65) satisfying

\[
(4.67) \quad (b, \tilde{c})(0) = (b^0, c^0), \quad \tilde{c}(T) = a^1.
\]

Since the equation for \( a_k \) in (4.66) is \( a_k = a_k \), by choosing \( T = \zeta_\epsilon := \log(a_k/\epsilon) \), where \( a_k \) is the \( a_k \)-coordinate of \( a^1 \), the solution corresponding to (4.67) satisfies \( a_k(0) = \epsilon \). We set

\[
\tilde{p}_\epsilon = (a, b, \tilde{c})(0; \zeta_\epsilon, a^1, b^0, c^0), \quad \tilde{q}_\epsilon = (a, b, \tilde{c})(\zeta_\epsilon; \zeta, a^1, b^0, c^0),
\]

and let \( p_\epsilon \) and \( q_\epsilon \) be the images of \( \tilde{p}_\epsilon \) and \( \tilde{q}_\epsilon \), respectively, in \((a, b, c)\)-space. From (4.63) we see that \( \tilde{p}_\epsilon \in \tilde{I} \), and hence \( p_\epsilon \in I \). Since the \( a_k \)-coordinate of \( p_\epsilon \) is \( a_k(0) = \epsilon \), we conclude that \( p_\epsilon \in I_\epsilon \).

Regarding \( \tilde{p}_\epsilon \) and \( \tilde{q}_\epsilon \) as functions of \((a^1, c^0) \in A \times \Lambda \), using (4.51) with \( T \) and \( \nu \) replaced by \( \zeta_\epsilon \) and \( \tilde{\nu} \), we have

\[
(4.68) \quad \|\tilde{p}_\epsilon - (0, b^0, c^0)\|_{C^{r-3}(A \times \Lambda)} + \|\tilde{q}_\epsilon - (a^1, 0, c^0)\|_{C^{r-3}(A \times \Lambda)} \leq C \epsilon^\delta.
\]

From (4.62) it follows that the \( \tilde{c} \)-coordinates of \( \tilde{p}_\epsilon \) and \( \tilde{q}_\epsilon \) are \( O(\epsilon^\delta) \)-close to \( c^0 \) in \( C^{r-2} \)-norm. Hence (4.68) holds with \( \tilde{p}_\epsilon \) and \( \tilde{q}_\epsilon \) replaced by \( p_\epsilon \) and \( q_\epsilon \). Since \( p_\epsilon \) and \( q_\epsilon \) parametrize \( I_\epsilon \) and \( I_\epsilon^* \) in neighborhoods of \( p_0 \) and \( q_0 \), by (4.57) this proves (4.55).

Next we consider the sequences \( q_\epsilon \) and \( p_\epsilon \) described in the statement. Write

\[
p_\epsilon = (a^{\text{in}}_\epsilon, b^{\text{in}}_\epsilon, c^{\text{in}}_\epsilon), \quad q_\epsilon = (a_\epsilon, b_\epsilon, c_\epsilon),
\]

and \( q_0 = (a^1, 0, c^0) \) in \((a, b, c)\)-coordinates. By the definition of \( I \), we have \( b^{\text{in}} = b^0 \). The assumption \( q_\epsilon \to q_0 \) gives \( c_\epsilon \to c^0 \), and then by (4.58) the assumption \( \pi^u(q_0) = \pi^s(p_0) \) implies \( p_0 = (0, b^0, c^0) \). From (4.68) we have \( a^{\text{in}}_\epsilon = o(1) \) and \( c^{\text{in}}_\epsilon = c_\epsilon + o(1) \). It follows that \( c^{\text{in}}_\epsilon \to c^0 \), and hence \( p_\epsilon \to p_0 \).

Let \( T_\epsilon > 0 \) be the number such that \( q_\epsilon = p_\epsilon \cdot T_\epsilon \). Since \( p_\epsilon \in I_\epsilon \), the \( a_k \)-coordinate of \( p_\epsilon \) equals \( \epsilon \), so from (4.66) we have

\[
(4.69) \quad T_\epsilon = \int_0^{\zeta_\epsilon} \frac{1}{\lambda_k} d\zeta, \quad \text{where} \quad \zeta_\epsilon = \log \frac{a_k(q_\epsilon)}{\epsilon} = \log \frac{a_k(q_0) + o(1)}{\epsilon}.
\]

Inserting (4.64) in (4.69), we then obtain (4.56). \( \square \)

5. Singular Configuration

We will find trajectories of limiting subsystems of the fast-slow system (2.13c) such that the union of those trajectories forms a singular configuration joining the end states \( u_L \) and \( u_R \).

5.1. End States \( U_L \) and \( U_R \). Observe that the system (2.14) has a normally hyperbolic critical manifold (5.70)

\[
S_0 = \{(u, w, \xi) : f(u) - \xi u - w = 0, \xi \neq \Re(\lambda_\pm(u))\},
\]

where \( \lambda_\pm(u) \) are the eigenvalues of \( Df(u) \), as defined in (2.15). The limiting slow system for (2.13c) is

\[
\begin{align*}
0 &= f(u) - \xi u - w \\
w' &= -u \\
\xi' &= 1.
\end{align*}
\]
From (H1) we have $s < \text{Re}(\lambda_{\pm}(u_L))$, so $(u_L, w_L, s) \in S_0$. Choose $\delta > 0$ so that $s + 2\delta < \text{Re}(\lambda_{\pm}(u_L))$, and set
\begin{equation}
\mathcal{U}_L = (u_L, w_L, s) \bullet (\infty, -\delta]
\end{equation}
(5.72)
\begin{equation}
= \{(u, w, \xi) : u = u_L, w = w_L - \alpha_1 u_L, \xi = s + \alpha_1, \alpha_1 \in (\infty, \delta]\},
\end{equation}
where $\bullet$ denotes the flow for (5.71). It is clear that $\mathcal{U}_L \subset S_0$ is normally hyperbolic with respect to (2.14), and is locally invariant with respect to (2.11). Note that each point in $\mathcal{U}_L$ is a hyperbolic equilibrium for the 2-dimensional system (2.12), and the unstable manifold $W^u_{\alpha}(\mathcal{U}_L)$ is naturally defined.

**Proposition 5.1.** Assume (H1). Let $\mathcal{U}_L$ be defined in (5.72). Fix any $r \geq 1$. There exists a family of invariant manifolds $W^u(\mathcal{U}_L)$ which are $C^k O(\epsilon)$-close to $W^u_{\alpha}(\mathcal{U}_L)$ such that for any continuous family \{I_0\} of compact sets $I_0 \subset W^u_{\alpha}(\mathcal{U}_L)$,
\begin{equation}
\text{dist}(\mathcal{U}_L(\epsilon_t), I_0) \leq Ce^{\mu t} \quad \forall \epsilon \in I_0, t \leq 0, \epsilon \in [0, \epsilon_0],
\end{equation}
for some positive constants $C$ and $\mu$.

**Proof.** This follows from Theorem 4.3 by taking $\mathcal{U}_L$ to be $\mathcal{U}_0$. Although $\mathcal{U}_L$ is not compact, it is uniformly normally hyperbolic since $\xi - \text{Re}(\lambda_{\pm}(u_L)) < -\delta$ on $\mathcal{U}_L$, and the proof of Theorem 4.3 in [Jon95, Theorem 4] is still valid. \hfill \Box

**Remark 2.** Proposition 5.1 was also asserted in [Sch04, Liu04, KTT12].

From (H1) we also have, by decreasing $\delta$ if necessary, $s - 2\delta > \text{Re}(\lambda_{\pm}(u_L))$, and hence a similar result holds for for the set $\mathcal{U}_R$ defined by
\begin{equation}
\mathcal{U}_R = (u_R, w_R, s) \bullet [-\delta, \infty]
\end{equation}
(5.74)
\begin{equation}
= \{(u, w, \xi) : u = u_L, w = w_L - \alpha_2 u_R, \xi = s + \alpha_2, \alpha_2 \in [-\delta, \infty]\}.
\end{equation}

**Proposition 5.2.** Assume (H1). Let $\mathcal{U}_R$ be defined by (5.74). Fix any $r \geq 1$. There exists a family of invariant manifolds $W^s(\mathcal{U}_R)$ which are $C^k O(\epsilon)$-close to $W^s_{\alpha}(\mathcal{U}_R)$ such that for any continuous family \{J_0\} of compact sets $J_0 \subset W^s_{\alpha}(\mathcal{U}_R)$,
\begin{equation}
\text{dist}(\mathcal{U}_R(\epsilon_t), J_0) \leq Ce^{-\mu t} \quad \forall \epsilon \in J_0, t \geq 0, \epsilon \in [0, \epsilon_0],
\end{equation}
for some positive constants $C$ and $\mu$.

### 5.2 Intermediate States $\mathcal{P}_L$ and $\mathcal{P}_R$.

It is easy to see that $\mathcal{P}_L$ defined in (3.26) is a normally hyperbolic critical manifold for (3.23), so $C^k$ unstable and stable manifolds $W^u(\mathcal{P}_L)$ and $W^s(\mathcal{P}_L)$ of $\mathcal{P}_L$ exist for any fixed $k \geq 1$. Note that $\{r = 0\}$ and $\{\kappa = 0\}$ are invariant under (3.23) while $\{\beta = \rho_3\}$ is not. We can straighten $W^u(\mathcal{P}_L)$ and $W^s(\mathcal{P}_L)$ by modifying $\beta$:

**Proposition 5.3.** Let $W^{u,s}(\mathcal{P}_L)$ be $C^k$ unstable/stable manifolds of $\mathcal{P}_L$ for (3.23), $k \geq 1$. There exists a $C^k$ function $\hat{\beta} = \hat{\beta}(\beta, r, w_1, w_2, \xi)$ such that
\begin{equation}
\hat{\beta} = \beta \quad \text{when} \quad r = 0
\end{equation}
and $(\hat{\beta}, r, \kappa, w_1, w_2, \xi)$ is a change of coordinates near $\mathcal{P}_L$ satisfying
\begin{equation}
W^s(\mathcal{P}_L) = \{((\hat{\beta}, r, \kappa, w_1, w_2, \xi) : \hat{\beta} = \rho_3, \kappa = 0\}
\end{equation}
(5.77)
\begin{equation}
W^u(\mathcal{P}_L) = \{((\hat{\beta}, r, \kappa, w_1, w_2, \xi) : r = 0\}.
\end{equation}
Proof. At each point of $\mathcal{P}_L$ defined in (3.26), the linearized system corresponds to the matrix represented in $(\beta, r, \kappa)$-coordinates as

$$
\begin{pmatrix}
\frac{-2}{3} \rho_3 (\rho_3^2 - 3) & -\frac{1}{6} \rho_3 \xi & 0 \\
0 & \frac{1}{\rho_3^2} & 0 \\
0 & 0 & \frac{1}{\rho_3^3}
\end{pmatrix},
$$

which has eigenvalues

$$
\frac{-2}{3} \rho_3 (\rho_3^2 - 3) > 0, \quad \frac{-1}{6} \rho_3 < 0, \quad \frac{1}{\rho_3^3} > 0,
$$

and eigenvectors

$$(1, 0, 0)^\top, \quad \left(\frac{1}{\rho_3} \rho_3 \xi, \frac{-2}{3} \rho_3 (\rho_3^2 - 3) + \frac{1}{3} \rho_3, 0\right)^\top, \quad (0, 0, 1)^\top.
$$

Since the sets $\{r = 0\}$ and $\{\kappa = 0\}$ are invariant under (3.23), it follows that

$$W^s(\mathcal{P}_L) = \{ (\beta, r, \kappa, w_1, w_2) : r = 0 \}$$

and $W^s(\mathcal{P}_L)$ can be parameterized as

$$W^s(\mathcal{P}_L) = \{ (\beta, r, \kappa, w_1, w_2) : \kappa = 0 \text{ and } \beta = \rho_3 + \phi(r, w_1, w_2, \xi) r \}$$

where $\phi$ is a $C^k$ function satisfying

$$\phi(r, w_1, w_2, \xi) = \frac{1}{2} \rho_3 \xi = \frac{-2}{3} \rho_3 (\rho_3^2 - 3) + \frac{1}{3} \rho_3 + O(r).
$$

Set

$$\hat{\beta} = \beta - \phi(r, w_1, w_2, \xi)r.
$$

Then (5.86) implies (5.76). Now (5.77) follows from (5.85) and (5.87), and (5.78) follows from (5.84). $\square$

A similar result holds for $\mathcal{P}_R$. We omit it here.

5.3. Transversal Intersections. To prove the Main Theorem, we need to find trajectories for (3.23) connecting $U_L$ and $U_R$ in $(\beta, r, \kappa, w_1, w_2, \xi)$-space satisfying $r \kappa = \epsilon$ for each small $\epsilon > 0$. Note that the trajectories $\gamma_1$ and $\gamma_2$ given in Proposition 3.1 satisfy $\gamma_1 \subset W_0^u(U_L) \cap W^s(\mathcal{P}_L)$ and $\gamma_2 \subset W_0^u(U_R) \cap W^s(\mathcal{P}_R)$. Our strategy is to first define 2-dimensional manifolds $\mathcal{I}_\epsilon$ and $\mathcal{J}_\epsilon$, $\epsilon \in [0, \epsilon_0]$, contained in $W^u_\epsilon(U_L)$ and $W^u_\epsilon(U_R)$, respectively, such that $\cup \mathcal{I}_\epsilon$ and $\cup \mathcal{J}_\epsilon$ are transverse to $\gamma_1$ and $\gamma_2$, and then track forward/backward trajectories from $\mathcal{I}_\epsilon$ and $\mathcal{J}_\epsilon$. An illustration with $\epsilon = 0$ is shown in Fig 6.

To track trajectories evolving from $\mathcal{I}_\epsilon$ and $\mathcal{J}_\epsilon$, we will apply the Corner Lemma stated in Section 4.3. The key idea is to show that the manifolds that evolve from $\mathcal{I}_\epsilon$ and $\mathcal{J}_\epsilon$, denoted by $\mathcal{I}_\epsilon^*$ and $\mathcal{J}_\epsilon^*$, respectively, are $C^1$ close to $W^u(\Lambda_L)$ and $W^u(\Lambda_R)$, where $\Lambda_L \subset \mathcal{P}_L$ and $\Lambda_R \subset \mathcal{P}_R$ are projections of $\mathcal{I}_0$ and $\mathcal{J}_0$. Hence transversal intersection of $W^u(\Lambda_L)$ and $W^u(\Lambda_R)$ will imply that of $\mathcal{I}_\epsilon^*$ and $\mathcal{J}_\epsilon^*$.

Fix a small $r^0 > 0$ so that $\gamma_1$ intersects $\{r = r^0\}$ at a unique point. Denote this point by $p^0_{\text{in}}$. That is,

$$p^0_{\text{in}} = \gamma_1 \cap \{r = r^0\}.$$
Figure 6. The 1D intervals $\Lambda_L$ and $\Lambda_R$ are projections of $I_0$ and $J_0$, respectively, on the critical manifolds $P_L$ and $P_R$. In the 5D space $\{r = 0\}$, the 3D manifolds $W^u(\Lambda_L)$ and $W^s(\Lambda_R)$ intersect transversally at $q_0$, and their intersection is the curve $\gamma_0$, which is transversal to $\Gamma$.

We set
\[
I_\epsilon = W^u(U_L) \cap \{r = r^0\} \cap V_1,
\]
where $V_1$ is an open neighborhood of $p_0^{in}$ to be specified below: From the expression (5.72), $U_L$ is 1-dimensional, so from (H1) we see that $W^u(U_L)$ is 3-dimensional. Hence we can choose $V_1$ so that $I_\epsilon$ is parametrized, in $(\beta, r, \kappa, w_1, w_2, \xi)$-coordinates given in Proposition 5.3 by
\[
I_\epsilon = \{(\beta, r, \kappa, w_1, w_2, \xi) : r = r^0, \kappa = \epsilon/r^0,
\]
\[
(w_1, w_2, \xi) = (w_{1L}, w_{2L}, s) + \alpha_1(-u_{1L}, -u_{2L}, 1) + \epsilon\theta(\beta, \alpha_1, \epsilon),
\]
\[
|\alpha_1| < \Delta_1\}
\]
for some $\Delta_1 > 0$ and some $C^4$ function $\theta$. (The order of differentiability of $\theta$ is chosen so that the Corner Lemma applies.) Note that $I_0$ is a affine surface, and $I_\epsilon$ can be viewed as a perturbation of $I_0$.

Let
\[
I = \bigcup_{\epsilon \in [0, \epsilon_0]} I_\epsilon.
\]
Since $p_0^{in} \in \gamma_1 \subset W^u(U_L) \cap W^s(P_L)$, from (5.85) and (5.90) we see that $I$ and $W^s(P_L)$ intersect transversally at $p_0^{in}$, and the projection into $P_L$ of their intersection along stable fibers is, by (5.80),
\[
\Lambda_L = \{(\beta, r, \kappa, w_1, w_2, \xi) : \beta = \rho_3, r = 0, \kappa = 0,
\]
\[
(w_1, w_2, \xi) = (w_{1L}, w_{2L}, s) + \alpha_1(-u_{1L}, -u_{2L}, 1),
\]
\[
|\alpha_1| < \Delta_1\}.
\]
Also we let $p_0^{out}$, $J_\epsilon$, $J$ and $\Lambda_R$ be analogously defined.

Since $\Lambda_L$ is a subset of the normally hyperbolic critical manifold $P_L$ for (5.79), the unstable manifold $W^u(\Lambda_L)$ can be defined in the natural way. From (5.78) we see that $W^s(\Lambda_L) \subset \{r = 0\}$. Similarly, $\Lambda_R$ and
$W^s(\Lambda_R)$ are defined, and $W^s(\Lambda_R) \subset \{r = 0\}$. Note that the trajectory $\gamma_0$ given in Proposition 3.3 satisfies $\gamma_0 \subset W^u(\Lambda_L) \cap W^s(\Lambda_R)$.

To track the intersection of $W^u(\Lambda_L)$ and $W^s(\Lambda_R)$ along $\gamma_0$, we fix a hyperplane $\Gamma = \{(\beta, r, \kappa, w_1, w_2, \xi) : \beta = 0\}$
and set $q_0 = \gamma_0 \cap \Gamma$.

**Proposition 5.4.** $W^u(\Lambda_L)$ and $W^s(\Lambda_R)$ intersect transversally at $q_0$ in the space $\{r = 0\}$, and their intersection near $q_0$ is a portion of the curve $\gamma_0$ given in Proposition 3.3 and hence is transverse to $\Gamma$ at $q_0$.

**Proof.** From Proposition 3.2 and 3.3 we have

$$T_{q_0}W^u(\Lambda_L) = \text{Span} \begin{pmatrix} 1 \\ 0 \\ 0 \\ * \\ 2\iota_3(0) \\ 0 \end{pmatrix}$$

and

$$T_{q_0}W^s(\Lambda_R) = \text{Span} \begin{pmatrix} 1 \\ 0 \\ 0 \\ * \\ -2\iota_3(0) \\ 0 \end{pmatrix}$$
in $(\beta, r, \kappa, w_1, w_2, \xi)$ coordinates, where $\iota_3(\sigma)$ is the positive function defined in (3.34). Since $\iota_3(0) \neq 0$ and $u_{1L} \neq u_{1R}$, from (5.95) and (5.96) we see that $T_{q_0}W^u(\Lambda_L)$ and $T_{q_0}W^s(\Lambda_R)$ span $(\beta, \kappa, w_1, w_2, \xi)$-space and they have a 1-dimensional intersection which is transversal to $\Gamma$. Since $q_0 \in \gamma_0 \subset W^u(\Lambda_L) \cap W^s(\Lambda_R)$, the desired result follows.

Now we have obtained the singular configuration $\gamma_1 \cup \gamma_0 \cup \gamma_2$, which joins the end states $u_L$ and $u_R$. In the next section we will show that there are solutions of (2.11ε) which are close to the singular configuration.

**6. Completing the Proof of the Main Theorem**

We split the proof of the main theorem into two parts. In Section 6.1 we prove the existence of solutions of the boundary value problem (2.8ε) and (2.9). In Section 6.2 we derive the weak convergence (2.18).

**6.1. Existence of Viscous Profile.**

**Proposition 6.1.** Let $p_0^\text{in}, p_0^\text{out}, q_0, \mathcal{I}_\epsilon, \mathcal{J}_\epsilon$ and $\Gamma$ be defined in Section 5.3. For each small $\epsilon > 0$, there exist $p_\epsilon^\text{in} \in \mathcal{I}_\epsilon, p_\epsilon^\text{out} \in \mathcal{J}_\epsilon, q_\epsilon \in \Gamma$ and $T_{1\epsilon}, T_{2\epsilon} > 0$ such that

$$q_\epsilon = p_\epsilon^\text{in} \cdot T_{1\epsilon}, \quad q_\epsilon = p_\epsilon^\text{out} \cdot (-T_{2\epsilon}),$$

where $\cdot$ denotes the flow for (3.23), satisfying

$$\begin{align} p_\epsilon^\text{in} \cdot q_\epsilon^\text{out} \cdot q_\epsilon & = (p_0^\text{in} \cdot p_0^\text{out} \cdot q_0) + o(1) \end{align}$$
as $\epsilon \rightarrow 0$, and

$$C^{-1} \log \frac{1}{\epsilon} \leq T_{i\epsilon} \leq C \log \frac{1}{\epsilon}, \quad i = 1, 2,$$

for some $C > 0$. Moreover, if we set $\beta_\epsilon(\sigma)$ and $\kappa_\epsilon(\sigma)$ to be the $\beta$- and $\kappa$-coordinates of $q_\epsilon \cdot \sigma, \sigma \in [-T_{1\epsilon}, T_{2\epsilon}]$, then

$$\max_{\sigma \in [-T_{1\epsilon}, T_{2\epsilon}]} \kappa_\epsilon(\sigma) = \kappa_0 + o(1)$$

as $\epsilon \rightarrow 0$. Note that the trajectory $\gamma_0$ given in Proposition 3.3, and hence is transverse to $\Gamma$ at $q_0$. □
and
\begin{equation}
\max_{\sigma \in [-T_1, T_2]} \pm \beta_\epsilon(\sigma)\kappa_\epsilon(\sigma) = \omega_0 + o(1),
\end{equation}
as \epsilon \to 0, where \kappa_\epsilon is defined in \((3.36)\), and
\begin{equation}
\omega_0 = \kappa_0 \iota_2(\sigma_0),
\end{equation}
where \sigma_0 is the unique number such that \iota_1(\sigma_0) = 1, and \iota_1(\sigma), \iota_2(\sigma) are positive functions defined in \((3.33)\) and \((3.34)\).

**Proof.** Let \(I = \bigcup I_\epsilon\). Since \(I_\epsilon \subset \{ r = r^0 \}\), from the relation \(\kappa = \epsilon/r\) we have
\[ I_\epsilon = I \cap \{ \kappa = \epsilon/r^0 \}. \]
From the construction of \(p_{0_0}, p_{0_1}\) and \(q_0\), we have
\[ \pi_{p_0}^w(q_0) = \pi_{p_0}^s(p_{0_0}^w) = (0, 0, 0), \]
in \((\beta, \epsilon, w_1, \epsilon, \xi)\)-coordinates, where \(\pi_{p_0}^s\) is the projection onto \(\mathcal{P}_{L,R}\) along stable/unstable fibers. For the system \((5.79)\), the conditions for the Corner Lemma are satisfied. Hence there exists a neighborhood \(V_0\) of \(q_0\) such that
\begin{equation}
\mathcal{J}_r^* \cap V_0 = C^1 \text{ close to } T_{q_0} W^u(\Lambda_L) \cap V_0,
\end{equation}
where \(\mathcal{J}_r^* = \mathcal{I}_r \cdot [0, \infty)\). Similarly, setting \(\mathcal{J}_r^* = \mathcal{J}_r \cdot (-\infty, 0]\), we have
\begin{equation}
\mathcal{J}_r^* \cap V_0 = C^1 \text{ close to } T_{q_0} W^s(\Lambda_L) \cap V_0.
\end{equation}
From \((6.103)\), \((6.104)\) and Proposition 5.4 it follows that the projections of \(\mathcal{J}_r^*\) and \(\mathcal{J}_r^*\) in the 5-dimensional space \(\{ r = 0 \}\) intersect transversally at a unique point in \(\Gamma\) near \(q_0\). From the relation \(r = \epsilon/\kappa\), we then recover a unique intersection point
\begin{equation}
q_\epsilon \in \mathcal{I}_r^* \cap \mathcal{J}_r^* \cap \Gamma
\end{equation}
in \((\beta, \epsilon, \kappa, w_1, w_2, \xi)\)-space. By the construction we have \((6.97)\) and \((6.98)\). The estimate \((6.99)\) follows from \((4.56)\).

The unstable fiber containing \(q_0\) in \(W^u(\mathcal{P}_L)\) is the trajectory \(\gamma_0\) defined in Proposition 3.3. The \(\beta\)- and \(\kappa\)-coordinates on \(\gamma_0\) are \(\iota_1(\sigma)\) and \(\kappa_0 \iota_2(\sigma)\), respectively. From \((3.34)\) we know \(\iota_2(\sigma) \leq \iota_2(0) = 1\). Hence \((6.100)\) follows. To prove \((6.101)\), by symmetry of \(\gamma_0\), it suffices to show that
\begin{equation}
\max_{\sigma \in [-\infty, 0]} \iota_1(\sigma) \iota_2(\sigma) = \iota_2(\sigma_0),
\end{equation}
where \(\sigma_0\) is defined by \(\iota_1(\sigma_0) = 1\). Note that the values of \(\iota_1(\sigma)\) and \(\iota_2(\sigma)\) are positive on \((-\infty, 0)\), and \(\iota_1(0) \iota_2(0) = 0, \quad \iota_1(-\infty) \iota_2(-\infty) = \rho_3 \cdot 0 = 0\).

By taking the derivative of \(\iota_1(\sigma) \iota_2(\sigma)\) it can be readily seen that the maximum of this function occurs at a unique number \(\sigma_0\) satisfying \(\iota_1(\sigma_0) = 1\). Indeed, from the definition \((3.33)\) and \((3.34)\), we have
\[ d \overline{d} \iota_1(\sigma) \iota_2(\sigma) \bigg| \sigma = \frac{\dot{\gamma}_1(\sigma) + \frac{1}{4} \iota_1(\sigma)^2}{\iota_2(\sigma)} \left[ -\left( (\beta^2 - 6 \beta^2 + 6) + \beta^3 \right) \right], \]
where we write \(\beta = \iota_1(\sigma)\). Since \(0 < \iota_1(\sigma) < \rho_3\) for \(\sigma \in (-\infty, 0)\), this derivative has a unique zero, which occurs when \(\beta = 1\). This proves \((6.106)\) and hence \((6.101)\). \qed

**Proposition 6.2.** Let \(q_\epsilon = (\beta^0, r^0, \kappa^0, w_1^0, w_2^0, \xi^0) \in \Gamma\) be defined in Proposition 6.1. Let \((u^0_1, u^0_2) = (\beta^0/r^0, 1/(v^0_\epsilon)^2)\) and
\begin{equation}
(\tilde{u}_1, \tilde{u}_2, \tilde{w}_1, \tilde{w}_2)(\xi) = (u^0_1, u^0_2, w^0_1, w^0_2, \xi^0) \bullet (\xi - \xi^0),
\end{equation}
or equivalently,
\begin{equation}
(\tilde{u}_1, \tilde{u}_2, \tilde{w}_1, \tilde{w}_2, \xi) = (u^0_1, u^0_2, w^0_1, w^0_2, \xi^0) \bullet \left( \frac{\xi - \xi^0}{\epsilon} \right).
\end{equation}
Then \((\tilde{u}_1, \tilde{u}_2)\) is a solution of \((2.8a)\) and \((2.9)\), and it satisfies \((2.17)\).
Proof. Since (2.8c) and (2.10c) are equivalent and \((u_{1\epsilon}, u_{2\epsilon}, \tilde{w}_{1\epsilon}, \tilde{w}_{2\epsilon})(\xi)\) is a solution of (2.10c), we know \((u_{1\epsilon}, u_{2\epsilon})\) is a solution of (2.8c).

Let \(T_{1\epsilon}\) and \(T_{2\epsilon}\) be as defined in Proposition 6.1. Then

\[ q_{\epsilon} \cdot (-T_{1\epsilon}) \in \mathcal{I}_{\epsilon}, \quad q_{\epsilon} \cdot (T_{2\epsilon}) \in \mathcal{I}_{\epsilon}, \]

where \(\cdot\) denotes the flow for (3.22). Since \(\mathcal{I}_{\epsilon} \subset W_{\epsilon}(U_{\epsilon})\) and \(\mathcal{J}_{\epsilon} \subset W_{\epsilon}(U_{\epsilon}),\) from (5.73) and (5.75) it follows that

\[ \lim_\epsilon \to \infty \text{dist}(p_{\epsilon}^0 \cdot t, U_{\epsilon}) = 0, \quad \lim_\epsilon \to \infty \text{dist}(p_{\epsilon}^0 \cdot t, U_{\epsilon}) = 0, \]

which implies (2.9). Since \(\tilde{u}_{2\epsilon} = (1/\tilde{r}_{\epsilon})^2 = (\tilde{r}_{\epsilon}/\epsilon)^2\) and \(\tilde{u}_{1\epsilon} = \tilde{\beta}_{\epsilon}/\tilde{r}_{\epsilon} = \tilde{\beta}_{\epsilon}/\epsilon,\) from (6.100) and (6.101) we obtain (2.17).

Here we justify the assertion made at the end of Section 5. From the equation for \(\dot{\xi}\) in (3.23), we have

\[ \dot{\xi}_{\epsilon} = \tilde{r}_{\epsilon} \tilde{r}_{\epsilon}^2 = \epsilon^2 / \tilde{r}_{\epsilon}. \]

Since the integral of \(1/\kappa\) along any compact segment of \(\gamma_0\) is finite, the change in \(\xi\) near such a segment is of order \(O(\epsilon^2)\).

6.2. Convergence of Viscous Profile.

Proposition 6.3. Let \(\tilde{u}_{\epsilon} = (\tilde{u}_{1\epsilon}, \tilde{u}_{2\epsilon})\) be the solution of (2.8c) and (2.9) given in Proposition 6.2. Let \(p_{\epsilon}^{0\text{in}}\) and \(p_{\epsilon}^{0\text{out}}\) be defined in Proposition 6.1, and \(s\) defined in (2.16a). Then

\[
\begin{align*}
|\xi_{\epsilon}^{\text{in}} - s| + |\xi_{\epsilon}^{\text{out}} - s| &= o(1) \\
\int_{-\infty}^{\xi_{\epsilon}^{\text{in}}} |\tilde{u}_{\epsilon}(\xi) - u_{\epsilon}| \, d\xi + \int_{\xi_{\epsilon}^{\text{out}}}^{\infty} |\tilde{u}(\xi) - u_{\epsilon}| \, d\xi &= o(1) \\
\int_{\xi_{\epsilon}^{\text{in}}}^{\xi_{\epsilon}^{\text{out}}} \tilde{u}_{\epsilon}(\xi) \, d\xi &= (0, e_0) + o(1)
\end{align*}
\]

as \(\epsilon \to 0\), where \(\xi_{\epsilon}^{\text{in, out}}\) and are the \(\xi\)-coordinates of \(p_{\epsilon}^{\text{in, out}}\).

Proof. Note that \(s\) is the \(\xi\)-coordinate of \(p_{\epsilon}^{0\text{in}}\). From the triangular inequality we have

\[ |\xi_{\epsilon}^{\text{in}} - s| \leq |p_{\epsilon}^{0\text{in}} - p_{\epsilon}^{0\text{in}}| = o(1), \]

where the second inequality follows from (6.98). A similar inequality holds for \(\xi_{\epsilon}^{\text{out}}\), so we obtain (6.109).

Since every point in \(U_{\epsilon}\) has \(u\)-coordinate equal to \(u_{\epsilon}\),

\[ |\tilde{u}(\xi) - u_{\epsilon}| \leq \text{dist}((\tilde{u}(\xi), \tilde{w}(\xi), (\xi), U_{\epsilon})) = \text{dist}((u_{\epsilon}^0, w_{\epsilon}^0, \xi_{\epsilon}^0) \bullet \kappa_{\epsilon} (\xi), U_{\epsilon}), \]

where the last equality follows from (6.107). Using (5.73), the last term is \(\leq C \exp(\mu \xi_{\epsilon}^0 / \epsilon)\). Since \(\xi_{\epsilon}^{\text{in}} < \xi_{\epsilon}^0\), it follows that

\[
\int_{-\infty}^{\xi_{\epsilon}^{\text{in}}} |\tilde{u}(\xi) - u_{\epsilon}| \, d\xi \leq \int_{-\infty}^{\xi_{\epsilon}^{\text{in}}} C \exp(\mu \xi_{\epsilon}^0 / \epsilon) \, d\xi \leq \int_{-\infty}^{\xi_{\epsilon}^{\text{in}}} C \exp(\mu \xi_{\epsilon}^{\text{out}} / \epsilon) \, d\xi = \frac{\epsilon}{\mu} C.
\]

A similar inequality holds for \(\int_{\xi_{\epsilon}^{\text{out}}}^{\infty} |\tilde{u}(\xi) - u_{\epsilon}| \, d\xi\), so we obtain (6.110).

From the equation for \(\xi\) in (3.23), denoting the time variable by \(\sigma\), we can write \(\xi = \xi(\sigma)\) by (6.112)

\[ \xi(0) = \xi_{\epsilon}^0, \quad \frac{d\xi}{d\sigma} = \kappa_{\epsilon}(\xi, \tilde{r}_{\epsilon}(\xi))^2, \]

From (6.97) we have (6.113)

\[ \xi(-T_{1\epsilon}) = \xi_{\epsilon}^{\text{in}}, \quad \xi(T_{2\epsilon}) = \xi_{\epsilon}^{\text{out}}. \]

From (6.112) and (6.113), using the equation for \(\tilde{w}_{2\epsilon}\) in (3.23), it follows that

\[ \int_{\xi_{\epsilon}^{\text{in}}}^{\xi_{\epsilon}^{\text{out}}} \tilde{w}_{2\epsilon}(\xi) \, d\xi = \int_{\xi_{\epsilon}^{\text{in}}}^{\xi_{\epsilon}^{\text{out}}} \frac{1}{(\tilde{r}_{\epsilon}(\xi))^2} \, d\xi \leq \int_{-T_{1\epsilon}}^{T_{2\epsilon}} \frac{\kappa_{\epsilon}(\sigma) \tilde{r}_{\epsilon}(\sigma)^2}{\tilde{r}_{\epsilon}(\sigma)^2} \, d\sigma \]

\[ = - \int_{-T_{1\epsilon}}^{T_{2\epsilon}} \tilde{w}_{2\epsilon}(\sigma) \, d\sigma = w_{2}(p_{\epsilon}^{\text{out}}) - w_{2}(p_{\epsilon}^{\text{in}}) = w_{2L} - w_{2R} + o(1) \]
where \( w_2(p) \) denotes the \( w_2 \)-coordinate of \( p \), and that
\[
\int_{\xi_{in}}^{\xi_{out}} |\tilde{u}_{1\epsilon}(\xi)| \, d\xi = \int_{\xi_{in}}^{\xi_{out}} |\tilde{\beta}_\epsilon(\xi)| \, d\xi = \int_{-T_{1\epsilon}}^{T_{2\epsilon}} |\tilde{\beta}_\epsilon(\sigma)|\tilde{\eta}_\epsilon(\sigma) \tilde{r}_\epsilon(\sigma) \, d\sigma
\]
(6.115)
where the last inequality follows from (6.99). Now (6.114) and (6.115) give (6.111).

\[\] 

Proof. Given any smooth function \( \psi \in C^\infty_c(\mathbb{R}) \) with compact support, using Proposition 6.3 it can be readily seen that
\[
\int_{-\infty}^{\infty} \psi(\xi)\tilde{u}_\epsilon(\xi) \, d\xi = u_L \int_{-\infty}^{s} \psi(\xi) \, d\xi + u_R \int_{s}^{\infty} \psi(\xi) \, d\xi + (0, e_0)\psi(s) + o(1).
\]
This holds for all \( \psi \), so (6.116) holds. \( \square \)

Proposition 6.4. Let \( \tilde{u}_\epsilon = (\tilde{u}_{1\epsilon}, \tilde{u}_{2\epsilon}) \) be the solution of (2.8c) and (2.9) given in Proposition 6.2. Then
\[
\tilde{u}_\epsilon \rightarrow u_L + (u_R - u_L)\delta(\xi - s) + (0, e_0)\delta(\xi - s)
\]
in the sense of distributions as \( \epsilon \to 0 \).

Proof. Given any smooth function \( \psi \in C^\infty_c(\mathbb{R}) \) with compact support, using Proposition 6.3 it can be readily seen that
\[
\int_{-\infty}^{\infty} \psi(\xi)\tilde{u}_\epsilon(\xi) \, d\xi = u_L \int_{-\infty}^{s} \psi(\xi) \, d\xi + u_R \int_{s}^{\infty} \psi(\xi) \, d\xi + (0, e_0)\psi(s) + o(1).
\]
This holds for all \( \psi \), so (6.116) holds. \( \square \)

Proposition 6.5. Let \( \tilde{u}_\epsilon = (\tilde{u}_{1\epsilon}, \tilde{u}_{2\epsilon}) \) be the solution of (2.8c) and (2.9) given in Proposition 6.2. Let \( u_\epsilon(x, t) = \tilde{u}_\epsilon(x/t) \). Then the weak convergence (2.13) holds.

Proof. Given any \( \varphi \in C^\infty_c(\mathbb{R} \times \mathbb{R}_+) \), using Proposition 6.4 we have
\[
\int_{0}^{\infty} \int_{-\infty}^{\infty} \varphi(x, t)u_\epsilon(x, t) \, dx \, dt = \int_{0}^{\infty} t \int_{-\infty}^{\infty} \varphi(t\xi, t)\tilde{u}_\epsilon(\xi) \, d\xi \, dt
\]
\[
= \int_{0}^{\infty} t \left\{ u_L \int_{-\infty}^{s} \varphi(t\xi, t) \, d\xi + (0, e_0)\varphi(st, t) + u_R \int_{s}^{\infty} \varphi(t\xi, t) \, d\xi \right\} \, dt + o(1)
\]
\[
= u_L \int_{0}^{\infty} \int_{-\infty}^{st} \varphi(x, t) \, dx \, dt + u_R \int_{0}^{\infty} \int_{st}^{\infty} \varphi(x, t) \, dx \, dt + (0, e_0) \int_{0}^{\infty} t\varphi(st, t) \, dt + o(1).
\]
By the definition (2.19), this implies (2.18). \( \square \)

Proposition 6.2 and 6.5 complete the proof of the Main Theorem.

ACKNOWLEDGEMENTS

The author would express his gratitude to his advisor, Professor Barbara L. Keyfitz, without whose help this work would not have been possible.

REFERENCES

[Den90] Bo Deng. Homoclinic bifurcations with nonhyperbolic equilibria. SIAM J. Math. Anal., 21(3):693–720, 1990.
[Den77] Neil Fenichel. Asymptotic stability with rate conditions. H. Indiana Univ. Math. J., 26(1):81–93, 1977.
[Den77] Neil Fenichel. Geometric singular perturbation theory for ordinary differential equations. J. Differential Equations, 31(1):53–98, 1979.
[Fen77] Neil Fenichel. Asymptotic stability with rate conditions. Indiana Univ. Math. J., 23:1109–1137, 1973/74.
[Hsu15] Ting-Hao Hsu. Viscous singular shock profiles for a system of conservation laws modeling two-phase flow. Under review. Available at [http://arxiv.org/abs/1512.00394/](http://arxiv.org/abs/1512.00394/).
[Jon95] Christopher K. R. T. Jones. Geometric singular perturbation theory. In Dynamical systems (Montecatini Terme, 1994), volume 1609 of Lecture Notes in Math., pages 44–118. Springer, Berlin, 1995.
[JT09] Christopher K. R. T. Jones and Siu-Kei Tin. Generalized exchange lemmas and orbits heteroclinic to invariant manifolds. Discrete Contin. Dyn. Syst. Ser. S, 2(4):967–1023, 2009.
[Key11] Barbara Lee Keyfitz. Singular shocks: retrospective and prospective. Confluentes Math., 3(3):445–470, 2011.
[KK89] Barbara Lee Keyfitz and Herbert C. Kranzer. A viscosity approximation to a system of conservation laws with no classical riemann solution. In Nonlinear hyperbolic problems (Bordeaux, 1988), volume 1402 of Lecture Notes in Math., pages 185–197. Springer, Berlin, 1989.
[KK90] Herbert C. Kranzer and Barbara Lee Keyfitz. A strictly hyperbolic system of conservation laws admitting singular shocks. In Nonlinear evolution equations that change type, volume 27 of IMA Vol. Math. Appl., pages 107–125. Springer, New York, 1990.
Barbara Lee Keyfitz and Herbert C. Kranzer. Spaces of weighted measures for conservation laws with singular shock solutions. *J. Differential Equations*, 118(2):420–451, 1995.

Barbara Lee Keyfitz and Charis Tsikkou. Conserving the wrong variables in gas dynamics: a Riemann solution with singular shocks. *Quart. Appl. Math.*, 70(3):407–436, 2012.

Weishi Liu. Multiple viscous wave fan profiles for Riemann solutions of hyperbolic systems of conservation laws. *Discrete Contin. Dyn. Syst.*, 10(4):871–884, 2004.

Stephen Schecter. Existence of Dafermos profiles for singular shocks. *J. Differential Equations*, 205(1):185–210, 2004.

Stephen Schecter. Exchange lemmas. I. Deng’s lemma. *J. Differential Equations*, 245(2):392–410, 2008.

Stephen Schecter. Exchange lemmas. II. General exchange lemma. *J. Differential Equations*, 245(2):411–441, 2008.

Michael Sever. Distribution solutions of nonlinear systems of conservation laws. *Mem. Amer. Math. Soc.*, 190(889):viii+163, 2007.

David G. Schaeffer, Stephen Schecter, and Michael Shearer. Non-strictly hyperbolic conservation laws with a parabolic line. *J. Differential Equations*, 103(1):94–126, 1993.

Stephen Wiggins. *Normally hyperbolic invariant manifolds in dynamical systems*, volume 105 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1994. With the assistance of György Haller and Igor Mezić.

Department of Mathematics, The Ohio State University, Columbus, OH 43210

E-mail address: hsu.296@osu.edu