\textbf{n-ANGULATED CATEGORIES FROM SELF-INJECTIVE ALGEBRAS}

ZENGQIANG LIN

Abstract. Let $C$ be a $k$-linear category with split idempotents, and $\Sigma : C \to C$ an automorphism. We show that there is an $n$-angulated structure on $(C, \Sigma)$ under certain conditions. As an application, we obtain a class of examples of $n$-angulated categories from self-injective algebras.

1. Introduction

Let $n$ be an integer greater than or equal to three. Geiss, Keller and Oppermann introduced the notion of $n$-angulated categories, which is a “higher dimensional” analogue of triangulated categories, and gave the standard construction of $n$-angulated categories from $(n-2)$-cluster tilting subcategories of a triangulated category which are closed under the $(n-2)$-nd power of the suspension functor \cite{8}. For $n = 3$, an $n$-angulated category is nothing but a classical triangulated category. Another examples of $n$-angulated categories from local algebras were given in \cite{4}. Let $R$ be a commutative local ring with maximal principal ideal $m = (p)$ satisfying $m^2 = 0$. Then the category of finitely generated free $R$-modules has a structure of $n$-angulation whenever $n$ is even, or when $n$ is odd and $2p = 0$ in $R$. The theory of $n$-angulated categories has been developed further, we can see \cite{3, 5, 12, 13, 14, 15} for reference. In this note, we devote to provide new class of examples of $n$-angulated categories.

Throughout this paper let $k$ be an algebraically closed field, and let $C$ be a $k$-linear category with split idempotents and $\Sigma : C \to C$ an automorphism. It is natural to ask under which conditions does the category $(C, \Sigma)$ has an $n$-angulation.

We first note that if $C$ is an $n$-angulated category, then the category $\text{mod} C$ of contravariant finitely presented and exact functors from $C$ to $\text{mod} k$ is a Frobenius category. Now we assume that $\text{mod} C$ is a Frobenius category. Then the stable category $\text{mod} C$ is a triangulated category and the suspension is the cosyzygy functor $\Omega^{-1}$. The automorphism $\Sigma$ can be extended to an exact functor from $\text{mod} C$ to $\text{mod} C$ and thus to a triangle functor of $\text{mod} C$. In this case, $(\Sigma, \sigma)$ and $(-\Omega^{-n}, (-1)^n)$ are two triangle endofunctors of $\text{mod} C$, where $\sigma : \Sigma \Omega^{-1} \to \Omega^{-1} \Sigma$ is a natural isomorphism. Heller showed in \cite{11} that there is a bijection between the class of pre-triangulations of $(C, \Sigma)$ and the class of isomorphisms of triangle functors from $(\Sigma, \sigma)$ to $(-\Omega^{-3}, -1)$. Since Heller did not succeed in proving the octahedral axiom, Amiot gave a necessary condition on the functor $\Sigma$ such that $(C, \Sigma)$ has

\textit{2010 Mathematics Subject Classification.} 16G20, 18E30, 18E10.

\textit{Key words and phrases.} $n$-angulated category; Frobenius category; self-injective algebra; periodic algebra.

This work was supported by the Natural Science Foundation of Fujian Province (Grants No. 2013J05009) and the Science Foundation of Huaqiao University (Grants No. 2014KJTD14).
a triangulated structure, which is applied to deformed preprojective algebras \[1\]. In \[8\] it is showed that Heller’s parametrization of pre-triangulations extends to pre-\(n\)-angulations. Our first main result is as follows.

**Theorem 1.1.** Let \(C\) be a \(k\)-linear category with split idempotents and \(\Sigma : C \to C\) an automorphism. If \(\text{mod} C\) is a Frobenius category and there exists an exact sequence of exact endofunctors of \(\text{mod} C\)

\[
0 \to \text{Id} \to X^1 \to X^2 \to \cdots \to X^n \to \Sigma \to 0,
\]

where all the \(X^i\) take values in \(\text{proj} C\). Then \((C, \Sigma)\) has an \(n\)-angulation structure.

Theorem 1.1 is a higher version of \[1, \text{Theorem 8.1}\]. Since \(n\)-angulated categories are more complex than triangulated categories, we should make some technological modifications in the proof.

Let \(A\) be a finite-dimensional \(k\)-algebra. Given an automorphism \(\sigma\) of \(A\), we denote by \(1_A\) the bimodule structure on \(A\) where the action on right is twisted by \(\sigma\). It is easy to check that \(1_A\) is quasi-periodic if \(A\) has a quasi-periodic projective resolution over the enveloping algebra \(A^e = A^{\text{op}} \otimes_k A\), i.e., \(\Omega^n_{A^e}(A) \equiv 1_A\) as \(A\)-bimodules for some natural number \(n\) and some automorphism \(\sigma\) of \(A\). In particular, \(A\) is periodic if \(\Omega^n_{A^e}(A) \equiv A\) as bimodules. In this case, if \(n\) is minimal, we say \(A\) is a periodic algebra of periodicity \(n\). It is well known that quasi-periodic algebras are self-injective algebras \[10\].

Our second main result is as follows.

**Theorem 1.2.** (=Theorem 4.3.) Let \(A\) be a finite-dimensional indecomposable quasi-periodic \(k\)-algebra. Assume that \(\Omega^n_{A^e}(A) \equiv 1_A\) as \(A\)-bimodules for an automorphism \(\sigma\) of \(A\). Then for each positive integer \(m\), the category \((\text{proj} A, \Sigma)\) has an \(mn\)-angulation structure, where \(\Sigma\) is the functor \(- \otimes A_{\sigma} (\cdot) : \text{proj} A \to \text{proj} A\). In particular, if \(\sigma\) is of finite order \(l\), then \((\text{proj} A, \text{Id}_{\text{proj} A})\) has an \(ln\)-angulation structure.

There are numerous examples of periodic algebras. The most notable examples are preprojective algebras of Dynkin graphs, whose periodicity at most 6. These results have been generalized to deformed preprojective algebras \[2\]. Dugas showed that each self-injective algebra of finite representation type is periodic \[6\]. We also can obtain periodic algebras as endomorphism algebras of periodic \(d\)-cluster-tilting objects in a triangulated category \[7\]. Therefore, by Theorem 1.2 we can construct a large class of examples of \(n\)-angulated categories from self-injective algebras.

This paper is organized as follows. In Section 2, we recall the definition of \(n\)-angulated category and make some preliminaries to prove our first main result. In Section 3, we prove Theorem 1.1. In Section 4, we prove Theorem 1.2 and give some examples.

## 2. Definitions and Preliminaries

Let \(C\) be an additive category equipped with an automorphism \(\Sigma : C \to C\). An \(n\)-\(\Sigma\)-sequence in \(C\) is a sequence of morphisms

\[X_\bullet = (X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1).\]
Its left rotation is the $\Sigma$-sequence

$$X_2 \to X_1 \to X_0 \to \cdots \to X_{-1}, X_0 \to \Sigma X_1 \to \cdots \to \Sigma X_2.$$  

We can define right rotation of an $\Sigma$-sequence similarly. An $\Sigma$-sequence $X_\bullet$ is exact if the induced sequence

$$
\cdots \to C(-,X_1) \to C(-,X_2) \to \cdots \to C(-,X_n) \to C(-,\Sigma X_1) \to \cdots
$$

is exact. A morphism of $\Sigma$-sequences is a sequence of morphisms $\varphi_\bullet = (\varphi_1, \varphi_2, \cdots, \varphi_n)$ such that the following diagram commutes

$$
\begin{array}{c}
\begin{array}{cccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\
| & \varphi_1 & | & \varphi_2 & | & \varphi_3 & | & \varphi_n & | & \Sigma \varphi_1 \\
Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma Y_1 \\
\end{array}
\end{array}
$$

where each row is an $\Sigma$-sequence. It is an isomorphism if $\varphi_1, \varphi_2, \cdots, \varphi_n$ are all isomorphisms in $C$.

**Definition 2.1.** ([8]) An $\Sigma$-angulated category is a triple $(C, \Sigma, \Theta)$, where $C$ is an additive category, $\Sigma$ is an automorphism of $C$, and $\Theta$ is a class of $\Sigma$-sequences satisfying the following axioms:

(N1) (a) The class $\Theta$ is closed under direct sums and direct summands.

(b) For each object $X \in C$ the trivial sequence

$$X \xrightarrow{1_X} X \to 0 \to \cdots \to 0 \to \Sigma X$$

belongs to $\Theta$.

(c) For each morphism $f_1 : X_1 \to X_2$ in $C$, there exists an $\Sigma$-sequence in $\Theta$ whose first morphism is $f_1$.

(N2) An $\Sigma$-sequence belongs to $\Theta$ if and only if its left rotation belongs to $\Theta$.

(N3) Each commutative diagram

$$
\begin{array}{c}
\begin{array}{cccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\
| & \varphi_1 & | & \varphi_2 & | & \varphi_3 & | & \varphi_n & | & \Sigma \varphi_1 \\
Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma Y_1 \\
\end{array}
\end{array}
$$

with rows in $\Theta$ can be completed to a morphism of $\Sigma$-sequences.

(N4) In the situation of (N3), the morphisms $\varphi_3, \varphi_4, \cdots, \varphi_n$ can be chosen such that the mapping cone

$$X_2 \oplus Y_1 \xrightarrow{(-f_2,0) \ \ \ \ varphi_2 \ \ \ \ g_1} X_3 \oplus Y_2 \xrightarrow{(-f_3,0) \ \ \ \ varphi_3 \ \ \ \ g_2} \cdots \xrightarrow{(-f_n,0) \ \ \ \ varphi_n \ \ \ \ g_{n-1}} \Sigma X_1 \oplus Y_n \xrightarrow{-\Sigma f_1 \ \ \ \ \ \ \ \ g_n} \Sigma X_2 \oplus \Sigma Y_1$$

belongs to $\Theta$.

**Remark 2.2.** (a) If $(C, \Sigma, \Theta)$ is an $\Sigma$-angulated category, then $\Sigma$ is called a suspension functor and $\Theta$ is called an $\Sigma$-angulation of $(C, \Sigma)$ whose elements are called $\Sigma$-angles. If $\Theta$ only satisfies the three axioms (N1),(N2) and (N3), then $\Theta$ is called a pre-$\Sigma$-angulation of $(C, \Sigma)$ and the triple $(C, \Sigma, \Theta)$ is called a pre-$\Sigma$-angulated category. In this case, an element of $\Theta$ is also called an $\Sigma$-angle.

(b) An $\Sigma$-complex is a complex $X_\bullet = (X_i, f_i)_{i \in \mathbb{Z}}$ over $C$ such that $X_{k+n} = \Sigma X_k$ and $f_{k+n} = \Sigma f_k$ for all $k \in \mathbb{Z}$. Let $X_\bullet = (X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n}$
Throughout this section, we make the following assumption.

**Assumption 2.3.** Let $\mathcal{C}$ be a $k$-linear category with split idempotents and $\Sigma : \mathcal{C} \to \mathcal{C}$ an automorphism, which satisfying the following two conditions:

1. The category $\text{mod}\mathcal{C}$ is a Frobenius category.
2. There exists an exact sequence of exact endofunctors of $\text{mod}\mathcal{C}$

$$0 \to \text{Id} \to X^1 \to X^2 \to \cdots \to X^n \to \Sigma \to 0$$

(2.1)

where all the $X^i$ take values in $\text{proj}\mathcal{C}$.

Since $\mathcal{C}$ has split idempotents, the Yoneda functor gives a natural equivalence between $\mathcal{C}$ and $\text{proj}\mathcal{C}$, which is the subcategory of $\text{mod}\mathcal{C}$ consisting of projectives. For convenience we identify $\mathcal{C}$ with $\text{proj}\mathcal{C}$. Since mod$\mathcal{C}$ is a Frobenius category, we get $\text{proj}\mathcal{C} = \text{inj}\mathcal{C}$ and the quotient category mod$\mathcal{C}$ is a triangulated category with the suspension functor $\Omega^{-1}$. In this case, the automorphism $\Sigma$ of $\mathcal{C}$ in fact induces a triangle functor of mod$\mathcal{C}$ which is also denoted by $\Sigma$. For each $M \in \text{mod}\mathcal{C}$, we fix a short exact sequence $0 \to M \to I_M \to \Omega^{-1}M \to 0$ with $I_M \in \mathcal{C}$. Thus we obtain a standard injective resolution

$$I_M \rightarrow I_{\Omega^{-1}M} \rightarrow I_{\Omega^{-2}M} \rightarrow \cdots$$

(2.2)

of $M$.

**Lemma 2.4.** There exists a functorial isomorphism $\alpha : \Sigma \to \Omega^{-n}$.

**Proof.** For each $M \in \text{mod}\mathcal{C}$, by (2.1) and (2.2) we obtain the following commutative diagram

$$
\begin{array}{cccccccc}
0 & \rightarrow & M & \rightarrow & X^1M & \rightarrow & X^2M & \rightarrow & \cdots & \rightarrow & X^nM & \rightarrow & \Sigma M & \rightarrow & 0 \\
0 & \rightarrow & M & \rightarrow & I_M & \rightarrow & I_{\Omega^{-1}M} & \rightarrow & \cdots & \rightarrow & I_{\Omega^{-n}M} & \rightarrow & \Omega^{-n}M & \rightarrow & 0 \\
\end{array}
$$

with exact rows, which implies that $\alpha_M : \Sigma M \xrightarrow{\sim} \Omega^{-n}M$ in mod$\mathcal{C}$. For each morphism $f : M \rightarrow M'$ in mod$\mathcal{C}$, we can easily deduce that $\Omega^{-n}f \cdot \alpha_M = \alpha_{M'} \cdot \Sigma f$ by comparison theorem. □

Let

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1$$

(2.3)

be an exact $n$-$\Sigma$-sequence in $\mathcal{C}$, and $M = \ker f_1$. We note that (2.3) can be seen as the beginning of an injective resolution of $M$. Since $f_n$ has a factorization $X_n \rightarrow \Sigma M \rightarrow \Sigma X_1$, there exists an isomorphism $\beta_M : \Sigma M \xrightarrow{\sim} \Omega^{-n}M$ in mod$\mathcal{C}$.

**Definition 2.5.** Denote by $\Phi$ the class of exact $n$-$\Sigma$-sequences

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1$$

in $\mathcal{C}$ such that $\beta_{\ker f_1} = \alpha_{\ker f_1}$. 
We will show in next section that \((\mathcal{C}, \Sigma, \Phi)\) is an \(n\)-angulated category. In the rest of this section, we will give an effective description on the elements in \(\Phi\), see Proposition 2.10 for detail.

For each \(M \in \text{mod}\mathcal{C}\), we denote by \(T_M\) the \(n\)-\(\Sigma\)-sequence
\[
X^1M \rightarrow X^2M \rightarrow \cdots \rightarrow X^nM \rightarrow \Sigma X^1M
\]
induced by the exact sequence (2.1). It is easy to see that \(T_M \in \Phi\). We call \(T_M\) a standard \(n\)-angle.

We denote by \(C_{n,\Sigma}^{\text{ex}}(\text{proj}\mathcal{C})\) the category of acyclic complexes over \(\text{proj}\mathcal{C}\). Denote by \(C_{n,\Sigma}^{\text{ex}}(\text{proj}\mathcal{C})\) the non-full subcategory of \(C_{n,\Sigma}^{\text{ex}}(\text{proj}\mathcal{C})\) whose objects are acyclic \(n\)-\(\Sigma\)-complexes of the following form
\[
(X_\bullet, f_\bullet) = \cdots \rightarrow X_1 f_1 X_2 f_2 X_3 f_3 \cdots f_{n-1} X_n f_n \Sigma X_1 \cdots \Sigma X_2 \rightarrow \cdots ,
\]
and whose morphisms are \(n\)-\(\Sigma\)-periodic. We note that the category \(C_{n,\Sigma}^{\text{ex}}(\text{proj}\mathcal{C})\) is a Frobenius category and the projective-injectives are the \(n\)-\(\Sigma\)-contractible complexes, i.e., the complexes homotopic to zero with an \(n\)-\(\Sigma\)-periodic homotopy. The functor \(Z_1 : C_{n,\Sigma}^{\text{ex}}(\text{proj}\mathcal{C}) \rightarrow \text{mod} \mathcal{C}\) which sends a complex \((X_\bullet, f_\bullet)\) to \(\ker f_1\) and the functor \(T : \text{mod} \mathcal{C} \rightarrow C_{n,\Sigma}^{\text{ex}}(\text{proj}\mathcal{C})\) which sends an object \(M\) to \(T_M\) are exact functors. Both of the two functors preserve the projective-injectives, thus we get the following lemma.

**Lemma 2.6.** The functors \(Z_1\) and \(T\) induce triangle functors \(Z_1 : K_{n,\Sigma}^{\text{ex}}(\text{proj}\mathcal{C}) \rightarrow \text{mod} \mathcal{C}\) and \(T : \text{mod} \mathcal{C} \rightarrow K_{n,\Sigma}^{\text{ex}}(\text{proj}\mathcal{C})\). Moreover, \(Z_1 T = \text{Id}_{\text{mod} \mathcal{C}}\).

An object in \(C_{n,\Sigma}^{\text{ex}}(\text{proj}\mathcal{C})\) is simply called an \(n\)-\(\Sigma\)-complex in \(\mathcal{C}\). Since an exact \(n\)-\(\Sigma\)-sequences in \(\mathcal{C}\) can naturally extend to an \(n\)-\(\Sigma\)-complex, we can view \(\Phi\) as a full subcategory of \(C_{n,\Sigma}^{\text{ex}}(\text{proj}\mathcal{C})\). In this sense an object in \(\Phi\) is called a \(\Phi\)-\(n\)-\(\Sigma\)-complex.

**Lemma 2.7.** Let \((X_\bullet, f_\bullet)\) be an \(n\)-\(\Sigma\)-complex and \((Y_\bullet, g_\bullet)\) a \(\Phi\)-\(n\)-\(\Sigma\)-complex. If \(\varphi_\bullet : X_\bullet \rightarrow Y_\bullet\) is homotopy-equivalent, then \(X_\bullet\) is also a \(\Phi\)-\(n\)-\(\Sigma\)-complex.

**Proof.** Let \(M = \ker f_1\) and \(N = \ker g_1\), then \(\varphi_\bullet\) induces a morphism \(h = Z_1(\varphi_\bullet) : M \rightarrow N\). The morphism \(h\) is an isomorphism in \(\text{mod} \mathcal{C}\) since \(\varphi_\bullet\) is an isomorphism in \(K_{n,\Sigma}^{\text{ex}}(\text{proj}\mathcal{C})\). By comparison theorem we have \(\Omega^{-n} h \cdot \beta_M = \beta_N \cdot \Sigma h\). We also have \(\Omega^{-1} h \cdot \alpha_M = \alpha_N \cdot \Sigma h\) by the naturality of \(\alpha\). Since \(\beta_N = \alpha_N\), we obtain that \(\beta_M = (\Omega^{-n} h)^{-1} \cdot \alpha_M \cdot \Sigma h = \alpha_M\), so \(X_\bullet\) is a \(\Phi\)-\(n\)-\(\Sigma\)-complex. \(\square\)

**Lemma 2.8.** Each commutative diagram
\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \cdots & f_{n-1} & \xrightarrow{f_n} & X_n \\
\downarrow{\varphi_1} & & \downarrow{\varphi_2} & & \ downarrow{\varphi_3} & & \ & \downarrow{\varphi_{n-1}} & \downarrow{\varphi_n} \\
Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \cdots & g_{n-1} & \xrightarrow{g_n} & Y_n
\end{array}
\]
whose rows are \(\Phi\)-\(n\)-\(\Sigma\)-complexes can be extended to an \(n\)-\(\Sigma\)-periodic morphism.

**Proof.** Since the \(Y_i\)'s are projective-injectives, by the factorization property of cokernel and the definition of injective we can find morphisms \(\varphi_i : X_i \rightarrow Y_i\) such that 
\[
\varphi_i f_{i-1} = g_{i-1} \varphi_{i-1}, \quad \text{where} \quad i = 3, 4, \ldots, n.
\]
Let \(M = \ker f_1, N = \ker g_1,\) and \(h : M \rightarrow N\) be the morphism induced by the left commutative square. Assume that \(f_n\) has a factorization \(l_n \pi_n : X_n \rightarrow \Sigma X_1\) and \(g_n\) has a factorization \(l_n' \pi_n' : Y_n \rightarrow \Sigma Y_1\). The morphism \(\varphi_n\) induces a morphism \(p : \Sigma M \rightarrow \Sigma N\)
such that $p\pi_n = \pi'_n\phi_n$. It should be noted that we do not have $\Sigma\phi_1 \cdot l_n = l'_n p$, but we have $\Sigma\phi_1 \cdot l_n = l'_n \cdot \Sigma h$.

Note that $\beta_N \cdot p = \Omega^{-n}h \cdot \beta_M$ by comparison theorem. On the other hand, we have $\alpha_N \cdot \Sigma h = \Omega^{-n}h \cdot \alpha_M$ by the naturality of $\alpha$. Since $\beta_M = \alpha_M$ and $\beta_N = \alpha_N$, we obtain that $p = \alpha^{-1}_N \cdot \Omega^{-n}h \cdot \alpha_M = \Sigma h$ in $\mod C$. Thus there exists a projective-injective $I$ in $\mod C$ and morphisms $a : \Sigma M \to I$ and $b : I \to \Sigma N$ such that $p - \Sigma h = ba$. As $I$ is projective, there exists a morphism $c : I \to Y_n$ such that $b = \pi'_n c$. We put $\phi'_n = \phi_n - ca\pi_n$, then $\phi'_n f_{n-1} = \phi_n f_{n-1} = g_{n-1}\phi_{n-1}$ and $g_n\phi'_n = l'_n\phi'_{n}(\phi_n - ca\pi_n) = l'_n(p - ba)\pi_n = l'_n \cdot \Sigma h \cdot \pi_n = \Sigma\phi_1 \cdot l_n \pi_n = \Sigma\phi_1 \cdot f_n$. Thus $(\phi_1, \phi_2, \phi_3, \cdots, \phi_{n-1}, \phi'_n)$ is a morphism in $C^\infty_{\Sigma \text{proj} C}$.

\[\text{Lemma 2.9.} \quad \text{The functor} \quad Z_1 : K^e_{\Sigma \text{proj} C} \to \mod C \quad \text{is full and its kernel is an ideal whose square vanishes. Thus} \quad Z_1 \quad \text{detects isomorphisms, that is, if} \quad Z_1(f) \quad \text{is an isomorphism in} \quad \mod C, \quad \text{then} \quad f \quad \text{is a homotopy-equivalence.} \]

\[\text{Proof.} \quad \text{The last assertion follows from the first assumption and Lemma 8.6. By Lemma 2.9 we have} \quad Z_1 T = \text{Id}_{\mod C}, \quad \text{which implies that} \quad Z_1 \quad \text{is full. We only need to show that ker} \quad Z_1 \quad \text{is an ideal whose square vanishes.} \]

Let $\phi_\bullet : (X_\bullet, f_\bullet) \to (Y_\bullet, g_\bullet)$ be a morphism of $n$-$\Sigma$-complexes with $Z_1(\phi_\bullet) = 0$. Let $(M, I : M \to X_1)$ be the kernel of $f_1$ and $\Sigma^{-1}f_n = l \pi$. Similarly let $(N, N' : N \to Y_1)$ be the kernel of $g_1$ and $\Sigma^{-1}g_n = l' \pi'$. Then $h = Z_1(\phi_\bullet)$ has a factorization $M \overset{a}{\to} I \overset{b}{\to} N$, where $I$ is projective-injective. Thus there exist two morphisms $c : X_1 \to I$ and $d : I \to \Sigma^{-1}Y_1$ such that $a = cd$ and $b = \pi d$. Let $h_1 = dc$. Note that $(\phi_1 - \Sigma^{-1}g_n \cdot h_1)\Sigma^{-1}f_n = 0$, there exists a morphism $m_1 : M_1 \to Y_1$ such that $\phi_1 - \Sigma^{-1}g_n \cdot h_1 = m_1 \pi_1$. Since $Y_1$ is projective-injective, there exists a morphism $h_2 : X_2 \to Y_1$ such that $m_1 = h_2l_1$. Thus $\phi_1 = \Sigma^{-1}g_n \cdot h_1 + h_2 f_1$. Similarly we can show that there exist morphisms $h_{i+1} : X_{i+1} \to Y_i$ such that $\phi_i = h_{i+1} f_i + g_{i-1} h_i$, $i = 2, 3, \cdots, n$. We take $\phi'_n = (h_{n+1} - \Sigma h_1)f_n$, then $\phi_n - \phi'_n = g_{n-1} h_n + \Sigma h_1 \cdot f_n$. Hence the morphism $\phi_\bullet$ is homotopy to the morphism $\phi'_\bullet = (0, 0, \cdots, 0, \phi'_n)$ with
an \(n\)-\(\Sigma\)-periodic homotopy \((h_1, h_2, \ldots, h_n)\).

Let \(\varphi \mapsto (X_\bullet, f_\bullet) \rightarrow (Y_\bullet, g_\bullet)\) and \(\psi \mapsto (Y_\bullet, g_\bullet) \rightarrow (Z_\bullet, h_\bullet)\) be morphisms in the kernel of \(Z_1\). Up to homotopy, we assume that \(\varphi \mapsto (0, 0, \cdots, 0, \varphi_n)\) and \(\psi \mapsto (0, 0, \cdots, 0, \psi_n)\). Thus we get the following diagram.

\[
\begin{array}{cccccc}
\Sigma^{-1} X_n & \xrightarrow{\Sigma^{-1} f_n} & X_1 & \xrightarrow{f_1} & X_2 & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\
\Sigma^{-1} Y_n & \xrightarrow{h_1} & \Sigma^{-1} Y_1 & \xrightarrow{g_1} & Y_2 & \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma Y_1 \\
\Sigma^{-1} Z_n & \xrightarrow{h_2} & \Sigma^{-1} Z_1 & \xrightarrow{h_n} & Z_2 & \cdots & \xrightarrow{h_{n-1}} & Z_n & \xrightarrow{h_n} & \Sigma Z_1 \\
\end{array}
\]

Since \(g_n \varphi_n = 0\) and \(\psi_n g_{n-1} = 0\), we have \(\varphi_n\) factors through \(g_{n-1}\) and \(\psi_n\) factors through \(g_n\). Thus \(\psi_n \varphi_n = b_{n+1} g_n g_{n-1} a_n = 0\). So \(\psi \varphi = 0\).

The following proposition is a higher version of [1 Proposition 8.7].

**Proposition 2.10.** The category of \(\Phi-n\)-\(\Sigma\)-complexes is equivalent to the category of \(n\)-\(\Sigma\)-complexes which are homotopy-equivalent to standard \(n\)-angles.

**Proof.** Since standard \(n\)-angles are \(\Phi-n\)-\(\Sigma\)-complexes, Lemma [2.7] implies that each \(n\)-\(\Sigma\)-complex which is homotopy-equivalent to a standard \(n\)-angle is a \(\Phi-n\)-\(\Sigma\)-complex. Let \((X_\bullet, f_\bullet)\) be a \(\Phi-n\)-\(\Sigma\)-complex. Let \(M\) be the kernel of \(f_1\). Since \(X^1 M\) and \(X^2 M\) are projective-injective, we can find morphisms \(\varphi_1 : X_1 \rightarrow X^1 M\) and \(\varphi_2 : X_2 \rightarrow X^2 M\) and...
$X^2M$ such that the following diagram is commutative.

\[
\begin{array}{cccccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} \Sigma X_1 \\
\downarrow{\varphi_1} & & \downarrow{\varphi_2} & & & & & \downarrow{\Sigma\varphi_1} \\
X^1M & \xrightarrow{f_1} & X^2M & \xrightarrow{f_2} & X^3M & \cdots & \xrightarrow{f_{n-1}} & X^nM & \xrightarrow{f_n} \Sigma X^1M \\
\end{array}
\]

We can complete $(\varphi_1,\varphi_2)$ to an $n$-$\Sigma$-periodic morphism $\varphi_\bullet = (\varphi_1,\varphi_2,\cdots,\varphi_n)$ from $X_\bullet$ to $T_M$ by Lemma 2.8. Since $Z_1(\varphi_\bullet) = \text{Id}_M$, we obtain that $\varphi_\bullet$ is an homotopy-equivalence by Lemma 2.9 i.e., $X_\bullet$ is homotopy-equivalent to $T_M$. □

3. PROOF OF THEOREM 1.1

Proof of Theorem 1.1. We are going to show that $(C,\Sigma,\Phi)$ is an $n$-angulated category.

(N1a) and (N1b) are trivial.

(N1c). Let $f_1 : X_1 \to X_2$ be a morphism in $C$, $A = \ker f_1$ and $B = \coker f_1$. By sequence (2.1) we easily obtain the following commutative diagram

\[
\begin{array}{cccccccc}
0 & \xrightarrow{} & A & \xrightarrow{i} & X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X^1B & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-2}} & X^{n-3}B & \xrightarrow{\pi_{n-1}} & C & \xrightarrow{g} & 0 \\
0 & \xrightarrow{} & 0 & \xrightarrow{} & I_A & \xrightarrow{} & I_{\Omega^{-1}A} & \xrightarrow{} & I_{\Omega^{-2}A} & \cdots & \xrightarrow{} & I_{\Omega^{n-1}A} & \xrightarrow{} & \Omega^{1-n}A & \xrightarrow{} & 0 \\
\end{array}
\]

with exact rows. Since $g$ is an isomorphism in $\text{mod}C$, we take $h = (\Omega^{-1}g)^{-1}\alpha_A$. Consider the following commutative diagram

\[
\begin{array}{cccccccc}
0 & \xrightarrow{} & C & \xrightarrow{i_{n-1}} & X_n & \xrightarrow{\pi_n} & \Sigma A & \xrightarrow{h} & 0 \\
0 & \xrightarrow{} & C & \xrightarrow{\alpha_C} & I_C & \xrightarrow{p_C} & \Omega^{-1}C & \xrightarrow{\Omega^{-1}g} & 0 \\
0 & \xrightarrow{} & 0 & \xrightarrow{} & 0 & \xrightarrow{} & 0 & \xrightarrow{} & 0 \\
\end{array}
\]

where $X_n$ is the pullback of $h$ and $p_C$. It is easy to see that

\[
\begin{array}{cccccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X^1B & \xrightarrow{f_3} & X^2B & \xrightarrow{f_4} & \cdots & \xrightarrow{f_{n-2}} & X^{n-3}B & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{\Sigma l_{\pi_n}} & \Sigma X_1 \\
\end{array}
\]

is a $\Phi$-$n$-$\Sigma$-complex.

(N2). Let $X_\bullet$ be a $\Phi$-$n$-$\Sigma$-complex. Since $X_\bullet [1]$ is isomorphic to the left rotation of $X_\bullet$ and $X_\bullet [-1]$ is isomorphic to the right rotation of $X_\bullet$, we only need to show that $X_\bullet [1]$ and $X_\bullet [-1]$ are $\Phi$-$n$-$\Sigma$-complexes. In fact, $X_\bullet$ is homotopy-equivalent to $T_M$ for some object $M \in \text{mod}C$ by Proposition 2.10. Thus $X_\bullet [1]$ is homotopy-equivalent to $T_M[1]$. Since $T : \text{mod}C \to K^\text{ex}_{n-\Sigma}(\text{proj}C)$ is a triangle functor, we obtain
that \( T_{\Omega^{-1}M} \) is isomorphic to \( T_M[1] \). Now \( X_\bullet[1] \) is homotopy-equivalent to \( T_{\Omega^{-1}M} \), which implies that \( X_\bullet[1] \) is a \( \Phi-n-Sigma \)-complex. Similarly we can show \( X_\bullet[-1] \) is a \( \Phi-n-Sigma \)-complex.

N3. It follows from Lemma 2.8, (N4). Suppose we have a commutative diagram

\[
X_\bullet : \quad X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1
\]

\[
\phi_1 \quad \phi_2 \quad \phi_3 \quad \phi_4 \quad \cdots \quad \phi_{n-1} \quad \phi_n
\]

\[
Y_\bullet : \quad Y_1 \xrightarrow{g_1} Y_2 \xrightarrow{g_2} Y_3 \xrightarrow{g_3} \cdots \xrightarrow{g_{n-1}} Y_n \xrightarrow{g_n} \Sigma Y_1
\]

whose rows are \( \Phi-n-Sigma \)-complexes. Let \( (M,l : M \to X_1) \) be the kernel of \( f_1 \), \( (N,l' : N \to Y_1) \) be the kernel of \( g_1 \) and \( h : M \to N \) the induced morphism. Then there exist two homotopy-equivalences \( a_\bullet : X_\bullet \to T_M \) and \( b_\bullet : T_N \to Y_\bullet \) by the proof of Proposition 2.10. Let \( \phi_\bullet = b_\bullet \cdot T(h) \cdot a_\bullet = (\phi_1, \phi_2, \cdots, \phi_n) \) be the morphism from \( X_\bullet \) to \( Y_\bullet \). It is easy to see that \( (\phi_1 - \phi_1)l = 0 \), so there exists a morphism \( h_2 : X_2 \to Y_1 \) such that \( \phi_1 - \phi_1 = h_2f_1 \). Note that \( (\phi_2 - \phi_2 - g_1h_2)f_1 = 0 \), there exists a morphism \( h_3 \) such that \( \phi_2 - \phi_2 = h_2h_1 = h_3h_2 \), i.e., \( \phi_2 - \phi_2 = g_1h_2 + h_3f_2 \). Let \( \phi_3 = \phi_3 + g_2h_3 \), then \( g_3\phi_3 = g_3\phi_3 = \phi_4f_3 \). If we take \( \phi_4 = \phi_4, \cdots, \phi_n = \phi_n \), then \( g_1\phi_i = \phi_i f_1, i = 4, \cdots, n - 1 \), and \( g_n\phi_n = g_n\phi_n = \sum \phi_1 \cdot f_n = (\phi_1 - h_2f_1) \cdot f_n = \sum \phi_1 \cdot f_n \). Thus \( \phi_\bullet = (\phi_1, \phi_2, \phi_3, \cdots, \phi_n) \) is an \( n-Sigma \)-periodic morphism and \( \phi_\bullet \) is \( n-Sigma \)-homotopic to \( \phi_\bullet \) with the homotopy \( (0, h_2, h_3, 0, \cdots, 0) \).

It remains to show that the cone \( C(\phi_\bullet) \) is a \( \Phi-n-Sigma \)-complex. In fact, since \( \phi_\bullet \) is \( n-Sigma \)-homotopic to \( \phi_\bullet = b_\bullet \cdot T(h) \cdot a_\bullet \), where \( a_\bullet \) and \( b_\bullet \) are homotopy-equivalent, we obtain that the cones \( C(\phi_\bullet), C(\phi_\bullet) \) and \( C(T(h)) \) are isomorphisms in \( K_n^{\omega Sigma}(\text{proj} C) \).

Let \( M \xrightarrow{h} N \xrightarrow{C(h)} \Omega^{-1}M \) be a triangle in \( \text{mod} C \). Since \( T : \text{mod} C \to K_n^{\omega Sigma}(\text{proj} C) \) is a triangle functor, \( T_M \xrightarrow{C(h)} T_N \xrightarrow{T_{\Omega^{-1}M}} T_M[1] \) is a triangle in \( K_n^{\omega Sigma}(\text{proj} C) \). Thus \( C(T(h)) \cong T_{C(h)} \) in \( K_n^{\omega Sigma}(\text{proj} C) \). By these isomorphisms and Proposition 2.10 we get \( C(\phi_\bullet) \) is a \( \Phi-n-Sigma \)-complex.

4. APPLICATION TO SELF-INJECTIVE ALGEBRAS

In this section, we will apply Theorem 1.1 to self-injective algebras and give some examples.

Lemma 4.1. (11 Lemma 1.5) Let \( A \) be a finite-dimensional indecomposable quasi-periodic \( k \)-algebra, then \( A \) is a self-injective algebra.

Lemma 4.2. Let \( A \) be a finite-dimensional self-injective \( k \)-algebra. If there exists an exact sequence of \( A-A \)-bimodules

\[
0 \to 1A_\sigma \to P_n \to P_{n-1} \to \cdots \to P_1 \to A \to 0
\]

where \( \sigma \) is an automorphism of \( A \) and the \( P_i \)'s are projective as bimodules, then \( \text{proj} A \) has an \( n \)-angulation structure where the suspension functor is \( -\otimes_A A_{\sigma^{-1}} \).

Proof. Since \( A \) is self-injective, \( \text{mod} A \) is a Frobenius category. If one tensors the sequence (4.1) with \( 1A_{\sigma^{-1}} \), one obtain the following exact sequence of \( A-A \)-bimodules

\[
0 \to A \to 1A_{\sigma^{-1}} \otimes_A P_n \to 1A_{\sigma^{-1}} \otimes_A P_{n-1} \to \cdots \to 1A_{\sigma^{-1}} \otimes_A P_1 \to 1A_{\sigma^{-1}} \to 0
\]
where all the $1A_{σ−1} ⊗_A P_i$ are projective as bimodules. Thus we have the following exact sequence of exact endofunctors of $\text{mod}A$

\[ 0 \to \text{Id} \to - ⊗_A A_{σ−1} ⊗_A P_n \to - ⊗_A A_{σ−1} ⊗_A P_{n−1} \to \cdots \]
\[ \to - ⊗_A A_{σ−1} ⊗_A P_1 \to - ⊗_A A_{σ−1} \to 0. \]

Moreover, the functors $- ⊗_A A_{σ−1} ⊗_A P_1$ take values in $\text{proj}A$. Note that $e_i A ⊗_A A_{σ−1} \cong e_i A_{σ−1} \cong σ^{-1}(e_i) A$ for each idempotent $e_i$ of $A$, the functor $- ⊗_A A_{σ−1} : \text{proj}A \to \text{proj}A$ is an automorphism. By Theorem 4.1, we deduce that $(\text{proj}A, - ⊗_A A_{σ−1})$ has an $n$-angulation. □

**Theorem 4.3.** Let $A$ be a finite-dimensional indecomposable quasi-periodic $k$-algebra. Assume that $Ω^m_A(σ) \cong 1A_σ$ as $A$-$A$-bimodules for an automorphism $σ$ of $A$. Then for each positive integer $m$, the category $(\text{proj}A, Σ)$ has an $mn$-angulation structure, where $Σ$ is the functor $- ⊗_A A_{σ−m} : \text{proj}A \to \text{proj}A$. In particular, if $σ$ is of finite order $l$, then $(\text{proj}A, \text{Id}_{\text{proj}A})$ has an $ln$-angulation structure.

**Proof.** By Lemma 4.1 we know that $A$ is a self-injective algebra. We claim that there exists an exact sequence of $A$-$A$-bimodules

\[ 0 \to 1A_{σm} \to P_{mn} \to P_{mn−1} \to \cdots \to P_{n+1} \to P_n \to \cdots \to P_1 \to A \to 0 \]

where the $P_i$’s are projective as bimodules. Thus the theorem immediately follows from Lemma 4.2. We prove this claim by induction on $m$. Since $Ω^m_A(σ) \cong 1A_σ$ in $\text{mod}A^e$, there exists an exact sequence (4.1), where the $P_i$’s are projective as $A$-$A$-bimodules. Assume now that $m > 1$ and our claim holds for $m−1$. Applying the functor $1A_{σm−1} ⊗_A A$ to the sequence (4.1), we obtain the following exact sequence of $A$-$A$-bimodules

\[ 0 \to 1A_{σm} \to 1A_{σm−1} ⊗_A P_n \to \cdots \to 1A_{σm−1} ⊗_A P_1 \to 1A_{σm−1} \to 0. \] (4.2)

Take $P_{(m−1)n+i} = 1A_{σm−1} ⊗_A P_i$, where $i = 1, 2, \cdots, n$. Then the $P_{(m−1)n+i}$’s are projective as bimodules. By induction and (4.2), we obtain that our claim holds for each positive integer $m$. □

In particular, we get the following easy corollary.

**Corollary 4.4.** Let $A$ be a finite-dimensional periodic $k$-algebra of periodicity $n$. Then the category $(\text{proj}A, \text{Id}_{\text{proj}A})$ has an $n$-angulation structure. Moreover, for each positive integer $m$, the category $(\text{proj}A, \text{Id}_{\text{proj}A})$ has an $mn$-angulation structure.

**Example 4.5.** We will revisit [1 Corollary 9.3]. Let $Δ$ be a graph of generalized Dynkin type and $A = P^f(Δ)$ be the corresponding deformed preprojective algebra introduced by Białkowski-Erdmann-Skowroński [2]. We note that if $f$ is zero, then $P^f(Δ)$ is just the usual preprojective algebra introduced by Gelfand-Ponomarev [9]. By [2 Proposition 3.4], we get $Ω^m_A(σ) \cong 1A_{σ−1}$ as $A$-$A$-bimodules for an automorphism $σ$ of $A$ of finite order. Moreover, for each idempotent $e_i$ of $A$, we have $σ(e_i) = εf(i)$, where $ν$ is the Nakayama permutation. By Theorem 4.1, $\text{proj}A$ is a triangulated category, and the suspension functor $- ⊗_A A_σ$ turns out to be the Nakayama functor. Let $m$ be the order of $σ$, then $(\text{proj}A, \text{Id}_{\text{proj}A})$ has a $3m$-angulation structure.
Example 4.6. Let $A = kQ_n/I_s$ be a self-injective Nakayama $k$-algebra, where $n \geq 1$, $s \geq 2$, $Q_n$ is the quiver

![Quiver Diagram]

and $I_s$ is the ideal generated by paths of length $s$. It is easy to see that $A$ is of finite representation type. In the notation of Asashiba this is of type $(A_n, \frac{2}{s}, 1)$. By Table 5.2 in [7], we know the periodicity of $A$ is

$$p = \begin{cases} s, & k = 2, n = 1 \text{ and } 2 \nmid s; \\ \frac{2s}{(s, n+1)}, & \text{otherwise}. \end{cases}$$

Thus $(\text{proj} A, \text{Id}_{\text{proj} A})$ has a structure of $p$-angulated category.

Acknowledgements  The author thanks professor Xiaowu Chen for drawing his attention to the reference [1] and for valuable conversation on this topic.

REFERENCES

[1] C. Amiot, On the structure of triangulated categories with finitely many indecomposables, Bull. Soc. math. Frances, 135(3)(2007), 435-474.
[2] J. Białkowski, K. Erdmann and A. Skowroński, Deformed preprojective algebras of generalized Dynkin type, Trans. Amer. Math. Soc. 359 (2007), 2625-2650.
[3] P. A. Bergh and M. Thaule, The axioms for $n$-angulated categories, Algebr. Geom. Topol. 13(4)(2013), 2405-2428.
[4] P. A. Bergh, G. Jasso and M. Thaule, Higher $n$-angulations from local algebras, arXiv:1311.2089v2.
[5] P. A. Bergh and M. Thaule, The Grothendieck group of an $n$-angulated category, J. Pure Appl. Algebra, 218(2)(2014), 354-366.
[6] A. Dugas, Periodic resolutions and self-injective algebras of finite type, J. Pure Appl. Algebra, 214 (2010), 990-1000.
[7] A. Dugas, Periodicity of $d$-cluster-tilted algebras, J. Algebra, 368(15)(2012), 49-52.
[8] C. Geiss, B. Keller and S. Oppermann, $n$-angulated categories, J. Reine Angew. Math. 675(2013), 101-120.
[9] I. M. Gel’fand and V. A. Ponomarev, Model algebras and representations of graphs, Funkc. Anal. Priloz. 13 (1979), 1-12.
[10] E. L. Green, N. Snashall and O. Solberg, The Hochschild cohomology ring of a selfinjective algebra of finite type, Proc. Amer. Math. Soc. 131 (2003), 3387-3393.
[11] A. Heller, Stable homotopy categories, Bull. Amer. Math. Soc. 74(1968), 28-63.
[12] G. Jasso, $n$-abelian and $n$-exact categories, arXiv:1405.7895v2.
[13] P. Jørgenson, Torsion classes and t-structures in higher homological algebra, Internat. Math. Res. Notices, in press.
[14] Z. Lin, $n$-angulated quotient categories induced by mutation pairs, Czech. Math. J., in press.
[15] Z. Lin, Right $n$-angulated categories arising from covariantly finite subcategories, arXiv:1409.2928v1.