Witnessing criticality in non-Hermitian systems via entropic uncertainty relation

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Non-Hermitian systems with exceptional points lead to many intriguing phenomena due to the coalescence of both eigenvalues and corresponding eigenvectors, in comparison to Hermitian systems where only eigenvalues degenerate. In this paper, we have investigated entropic uncertainty relation (EUR) in a non-Hermitian system and revealed a general connection between the EUR and the exceptional points of non-Hermitian system. Compared to the unitary dynamics determined by a Hermitian Hamiltonian, the behaviors of EUR can be well defined in two different ways depending on whether the system is located in unbroken or broken phase regimes. In unbroken phase regime, EUR undergoes an oscillatory behavior while in broken phase regime the oscillation of EUR breaks down. The exceptional points mark the oscillatory and non-oscillatory behaviors of the EUR. In the dynamical limit, we have identified the witness of critical behavior of non-Hermitian systems in terms of the EUR. Our results reveal that the EUR witness can exactly detect the critical points of non-Hermitian systems beyond (anti-) PT-symmetric systems. Our results may have potential applications to witness and detect phase transition in non-Hermitian systems.

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I. INTRODUCTION

Historically, the uncertainty principle $\Delta x\Delta p \geq \hbar/2$, originally proposed by Heisenberg [1] in 1927, captures that one can’t simultaneously predict the measurement outcomes of position and momentum with certainty. Subsequently, it has been further formulated by Robertson [2] and generalized to arbitrary pairs of incompatible observables $R$ and $Q$: $\Delta R \cdot \Delta Q \geq \frac{1}{2}|\langle \psi | [R, Q] | \psi \rangle |$, here $\Delta R \Delta Q$ denotes the standard deviations and $[R, Q] = RQ - QR$ is the commutator. However, this uncertainty relation in terms of the commutator is not always optimal to quantify the measuring uncertainty, because the bound is strongly related to the state $| \psi \rangle$ of system. In particular, there is a trivial result when $\langle \psi | [R, Q] | \psi \rangle$ is equal to zero for some specific states even though $R$ and $Q$ do not share any common eigenvectors. To overcome this drawback, Deutsch [3] in 1983 originally developed the so-called entropy uncertainty relation (EUR) in an information theoretical framework instead of the standard deviation, which later was proven by Maassen and Uffink [4] in 1988

$$H(R) + H(Q) \geq -2 \log_2 c$$

(1)

where $H(R)$/$H(Q)$ denotes the Shannon entropy of the probability distribution of the outcomes when $R$/$Q$ is performed on the pure state $| \psi \rangle$ of system, respectively. $c = \max_{i,j} \langle \phi_i | \phi_j \rangle$ quantifies the complementarity of $R$ and $Q$ with their corresponding eigenvectors $| \phi_i \rangle$ and $| \psi_j \rangle$. However, for any mixed state $\rho$ of system, the EUR can be formulated in general form as

$$H(R) + H(Q) \geq -2 \log_2 c + S(\rho),$$

(2)

where $S(\rho) = -Tr(\rho \log_2 \rho)$ is the von Neumann entropy. A distinct advantage of the EUR superior to the standard deviations is that the lower bound of EUR does not depend on specific states to be measured. In fact, the EUR can be further improved with the assistance of a quantum memory, so that the outcomes of two incompatible measurements can be predicted precisely by an observer with access to the quantum memory if the initial states are maximally entangled [5, 6]. At present, the EUR has received a great deal of attention [7–18] due to potential applications in quantum information processing tasks such as quantum entanglement witnessing [19–22], and quantum key distribution [23–24].

Unfortunately, previous efforts have attempted to discuss the EUR in Hermitian systems. Few detailed investigations concerning the EUR in non-Hermitian systems are available. In the present work, we extend the EUR dynamics of Hermitian systems to non-Hermitian ones. The primary motivation for such an extension is at least two-fold. First, there has been a growing interest in non-Hermitian systems due to the fact that many intriguing phenomena inaccessible in Hermitian systems can be observed in non-Hermitian systems, e.g., the violation of no-signaling principle [25, 26], loss-induced lasing [27, 51], and the optimal brachistochrone problem [28] and so on. Therefore, it is natural to ask whether the EUR can display some new phenomena in the non-Hermitian system. Second, in contract to Hermitian systems, there exists so-called exceptional points in non-Hermitian systems where not only eigenvalues, but also their corresponding eigenvectors coalesce. It has been proved that such points are relevant to describe dynamical phase transitions from an unbroken phase to a spontaneous broken phase [30, 31]. Therefore, it is urgently to wander whether this critical phenomena of non-Hermitian system can be confirmed by the behaviors of EUR.
Motivated by the above issues, we first focus on the behaviors of EUR in non-Hermitian systems and reveal a general connection between the EUR and the critical points of non-Hermitian system. Compared with the dynamics of EUR in Hermitian system case, we find that the dynamics of EUR can be well defined in the unbroken phase to the broken phase regimes. In the unbroken phase regime, the EUR exhibits an oscillatory behavior, while for the broken phase regime, the oscillation of the EUR breaks down. In this respect, this provides us a powerful method to witness criticality in non-Hermitian systems. Particularly, we identify unique criticality based on the EUR in the time limit infinity around the critical point, above which the EUR witness increases asymptotically but below which the EUR witness decays asymptotically. Therefore, there exists a sudden change of the EUR witness at critical points which are referred to the exceptional points of non-Hermitian system.

II. DYNAMICS OF A GENERAL TWO-LEVEL NON-HERMITIAN SYSTEM

In order to demonstrate the behaviors of EUR in a non-Hermitian system, for clarity and without loss of generality, we take a general two-level non-Hermitian system whose Hamiltonian reads as

$$\hat{H}_{NH} = \begin{pmatrix} re^{i\phi} & \sigma^{\dagger} \\ \sigma & re^{-i\phi} \end{pmatrix}. \quad (3)$$

The energy eigenvalues of $\hat{H}_{NH}$ are $E_{\pm} = r \cos \phi \pm \sqrt{s^2 - r^2 \sin^2 \phi}$ and the corresponding eigenvectors are $|E_{\pm}\rangle = \frac{1}{\sqrt{2 \cos \Theta}} \begin{pmatrix} e^{\pm i\Theta/2} \\ \pm e^{\mp i\Theta/2} \end{pmatrix}$ respectively. Here, defining $\sin \Theta = \sin \phi / \sqrt{s^2 - r^2 \sin^2 \phi}$. Obviously, two regions that are separated by the point $s = r^2 \sin^2 \phi$ where the phase transition from an unbroken phase to a broken phase happens. For $s > r^2 \sin^2 \phi$, the system is denoted in unbroken phase regime and exhibits real spectra, while for $s < r^2 \sin^2 \phi$, the system is termed in broken phase regime where complex conjugate eigenvalues emerge.

On the other hand, for the case of $s = \sigma$, Eq. (3) reduces to PT symmetric Hamiltonian which is invariant under the PT transformation $(PT)H(PT)^{-1} = H$ with operator $P$ denotes Pauli matrix $\sigma^x$, and $T$ represents the complex conjugation [33]. The corresponding eigenvalues are purely real in the unbroken phase regime ($s > |\sin \phi|$) and become complex in the broken phase regime ($s < |\sin \phi|$) [34].

Using the non-Hermitian Hamiltonian of Eq. (3), the dynamics of the non-Hermitian system can be easily obtained by solving the time dependent Schrödinger equation

$$i \frac{\partial |\Psi(t)\rangle}{\partial t} = \hat{H}_{NH} |\Psi(t)\rangle, \quad (4)$$

with the non-unitary time evolution operator $U(t) = \exp[-i\hat{H}_{NH}t]$ which is obtained as,

$$U(t) = e^{-itr \cos \phi} \begin{pmatrix} \cos \omega t + \frac{r \sin \phi \sin \omega t}{\omega} & -i \frac{\sigma \sin \omega t}{\omega} \\ i \frac{\sigma \sin \omega t}{\omega} & \cos \omega t - \frac{r \sin \phi \sin \omega t}{\omega} \end{pmatrix}, \quad (5)$$

where $\omega = \sqrt{s^2 - r^2 \sin^2 \phi}$. Particularly, at the exceptional point, namely, $s = r^2 \sin^2 \phi$, the time evolution operator becomes

$$U(t) \approx e^{-itr \cos \phi} \begin{pmatrix} 1 + tr \sin \phi & -itr \sin \phi \\ -itr \sin \phi & 1 - tr \sin \phi \end{pmatrix}. \quad (6)$$

It is worthwhile to mention that, the non-Hermitian Hamiltonian given by Eq. (3) with real spectra can be mapped into a Hermitian one via the Hermitian transformation $\eta H_{NH} \eta^{-1}$, e.g., $\eta H_{NH} \eta^{-1} = H$, where the Hermitian matrix is $\eta = \frac{1}{\sqrt{\cos \Theta}} \begin{pmatrix} \cos(\Theta/2) & -i \sin(\Theta/2) \\ i \sin(\Theta/2) & \cos(\Theta/2) \end{pmatrix}$. Therefore, the non-unitary time evolution operator given by Eq. (5) reduces to an unitary evolution one as a consequence of Hermiticity of $H$

$$U(t) = e^{-itr \cos \phi} \begin{pmatrix} \cos \omega t & -i \sin \omega t \\ -i \sin \omega t & \cos \omega t \end{pmatrix}. \quad (7)$$

Given an arbitrary initial state $|\Psi(0)\rangle$, one can express it as a superposition of $|E_{\pm}\rangle$, namely, $|\Psi(0)\rangle = \sin \phi \frac{1}{2} |E_{+}\rangle + \cos \phi \frac{1}{2} |E_{-}\rangle$ which evolves in time according to $|\Psi(t)\rangle = U(t)|\Psi(0)\rangle = \sin \phi \frac{1}{2} e^{-iE_{-}t}|E_{+}\rangle + \cos \phi \frac{1}{2} e^{-iE_{-}t}|E_{-}\rangle$. However, due to the nonorthogonality of eigenvectors $|E_{\pm}\rangle$, we have to transform this eigenvector representation into the computational orthonormal vectors $|0\rangle, |1\rangle$ via the similarity transformation

$$|0\rangle = \frac{1}{\sqrt{2 \cos \Theta}} \begin{pmatrix} e^{i\Theta/2} |E_{+}\rangle + e^{-i\Theta/2} |E_{-}\rangle \\ e^{-i\Theta/2} |E_{+}\rangle - e^{i\Theta/2} |E_{-}\rangle \end{pmatrix} \quad (8a)$$

$$|1\rangle = \frac{1}{\sqrt{2 \cos \Theta}} \begin{pmatrix} e^{-i\Theta/2} |E_{+}\rangle - e^{i\Theta/2} |E_{-}\rangle \\ e^{i\Theta/2} |E_{+}\rangle + e^{-i\Theta/2} |E_{-}\rangle \end{pmatrix}. \quad (8b)$$

In this respect, the time evolution state $|\Psi(t)\rangle$ can be rewritten as

$$|\Psi(t)\rangle = (\alpha e^{-iE_{-}t} - e^{-i\Theta} \beta e^{-iE_{-}t}) |0\rangle \quad (9)$$

$$+ (e^{-i\Theta} \alpha e^{-iE_{-}t} + \beta e^{-iE_{-}t}) |1\rangle,$$

where $\alpha = \frac{1}{2} \sec \Theta (\cos \phi + e^{i\Theta} \sin \phi)$ and $\beta = \frac{1}{2} \sec \Theta (e^{i\Theta} \cos \phi - \sin \phi)$. In general, the evolution of a non-Hermitian system is not trace-preserving. Therefore, a normalized form should be given

$$\rho(t) = \frac{|\Psi(t)\rangle \langle \Psi(t)|}{\text{tr}(|\Psi(t)\rangle \langle \Psi(t)|)}. \quad (10)$$
III. ENTROPIC UNCERTAINTY RELATION IN NON-HERMITIAN SYSTEMS

To make calculations of the EUR = $-\Sigma_{X}p_{X} \log_{2} p_{X}$ which is only depending on the outcomes of probability, some measuring process is necessary to carry out. For simplify, we examine a pair of projective operators as observables $X \in \{R,Q\}$ which are represented by $P_{X} = 1/2(I + \vec{n} \cdot \vec{\sigma})$, where $\vec{n} = (n_{1}, n_{2}, n_{3})$ is a unit vector and $\vec{\sigma} = (\sigma_{x}, \sigma_{y}, \sigma_{z})$. Therefore, the probability distribution of measurement outcome becomes

$$p_{X} = \frac{1}{N} \left(2\alpha\beta^{2}[n_{3}\cos \Theta - i(n_{2} - \sin \Theta)] + 2\alpha^{2}[n_{3}\cos \Theta + i(n_{2} - \sin \Theta)]e^{2i\Theta E_{t}} + [(n_{2} \sin \Theta - 1)\sec^{2} \Theta - n_{1} \sin \theta]e^{i\Theta E_{t}}\right),$$

where $N = 4i\sin \Theta(\alpha^{2} - \alpha^{4}e^{2i\Theta E_{t}}) - 2\sec^{2} \Theta e^{i\Theta E_{t}}$. From above expression, the outcome probability is associated with the difference of energy eigenvalues $\Delta E \equiv E_{c} - E_{c} = 2\omega$ and leads to two completely different dynamical behaviors. For $s\sigma > r^{2}\sin^{2}\phi$, the probability distribution exhibits an oscillatory behavior with the period $T = \pi/\omega$. In contrary, for $s\sigma < r^{2}\sin^{2}\phi$, the probability distribution with oscillation breaks down. Particularly, for $s\sigma = r^{2}\sin^{2}\phi$ which is referring to the exceptional point where the eigenvalues as well as their corresponding eigenvectors coalesce, and the probability distribution becomes

$$p_{X} = \frac{1}{4} \left[\frac{2 - 2n_{2}}{1 + r^{2}t^{2} + rt(2\cos \Theta \sin \phi - rt \cos 2\phi)}\right].$$

To reveal a relationship between the EUR and the exceptional point of non-Hermitian system, let us consider the initial state to prepare a maximal coherence pure state, $|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ at time $t = 0$. According to the Eq. (10), a normalized density at a later time $t$ is

$$\rho(t) = \left(\begin{array}{cc}
\frac{s^{2}\sin^{2}\omega t + (\omega \cos \omega t + r \sin \phi \sin \omega t)^{2}}{27\omega^{2}} & [\omega \cos \omega t + (is - r \sin \phi \sin \omega t)]
\frac{\omega \cos \omega t - (is + r \sin \phi \sin \omega t)}{27\omega^{2}}
\end{array}\right)^{2} \left(\begin{array}{cc}
\frac{s^{2}\sin^{2}\omega t + (\omega \cos \omega t - r \sin \phi \sin \omega t)^{2}}{27\omega^{2}} & [\omega \cos \omega t - (is + r \sin \phi \sin \omega t)]
\frac{\omega \cos \omega t + (is - r \sin \phi \sin \omega t)}{27\omega^{2}}
\end{array}\right),$$

where $T = \cos^{2}\omega t + \frac{(s^{2} + \pi^{2} + 2r^{2}\sin^{2}\phi) \sin^{2}\omega t}{2s^{2}}$. Considering the two complementary observables $P = \sigma_{x}$ and $P = \sigma_{z}$ as measured observables (i.e., choosing $\vec{n} = (0,0,0)$ and $\vec{n} = (0,0,\pm 1)$ for $\sigma_{x}$ and $\sigma_{z}$ respectively), the lower bound of the EUR in Eq.(1) is always 1, while its EUR takes the form as

$$EUR = \sum_{i=1}^{2} p_{i} \log_{2} p_{i} = -\sum_{i=1}^{2} p_{i} \log_{2} p_{i}$$

with

$$p_{1} = \frac{4\omega^{2}\cos^{2}\omega t + (s + r)\sin^{2}\omega t}{47\omega^{2}},$$

$$p_{2} = \frac{4\omega^{2}\sin^{2}\omega t + (s - r)\sin^{2}\omega t}{47\omega^{2}},$$

$$p_{1} = \frac{\sigma^{2}\sin^{2}\omega t + (\omega \cos \omega t - r \sin \phi \sin \omega t)^{2}}{27\omega^{2}},$$

$$p_{2} = \frac{\sigma^{2}\sin^{2}\omega t + (\omega \cos \omega t + r \sin \phi \sin \omega t)^{2}}{27\omega^{2}}.$$
A. Criticality witness in PT-symmetric non-Hermitian systems

In order to check the efficiency of the proposed criticality witness $W$, we first consider the non-Hermitian systems with Hamiltonian given by Eq.(3). A quantum phase transition occurs at the exceptional point $r_0 = \sqrt{s\sigma/\sin^2\phi}$. In particular, for the case of $s = \sigma$, the system with Eq.(3) reduces to a standard PT-symmetric case, and a quantum phase transition occurs at $r_0 = s/\sin\phi$. To test the validity of the above mentioned criticality witness $W$, we resort to numerical simulation. Fig. 2(a) displays the behaviors of criticality witness $W$ for non-Hermitian system with PT symmetry for $s = \sigma = 2$ and $\phi = \pi/2$. As expected, the behaviors of criticality witness $W$ increases asymptotically in the unbroken phase $r < r_0 = 2$, while in the broken phase $r > r_0 = 2$, the $W$ displays asymptotically decay. A abrupt change of criticality witness $W$ happens at the transition point $r = r_0 = 2$. This result implies the signature of a quantum critical point is confirmed by $W$. On the other hand, the similar behavior occurs for a general non-Hermitian system without PT symmetry, e.g., $\sigma = \sqrt{2}$, $s = \sqrt{2}/2$ and $\phi = \pi/2$ as shown in Fig. 2(b). A sudden change of $W$ occurs at the critical point $r_0 = 1$. This fact proves the critical behavior of the non-Hermitian systems is faithfully detected and witnessed by the EUR witness.

To find a deeper insight into the nature of EUR with the critical point in the non-Hermitian system, we have also investigated the rate of change of EUR in dynamics limit which can be realized experimentally and which can detect the exceptional point of non-Hermitian systems defined as

$$\beta = \lim_{t \to \infty} \left\{ \frac{d}{dt} [H(R) + H(Q)] \right\}. \quad (16)$$

The non-vanishing parameter $\beta$ is corresponding to the unbroken phase regime, while the vanishing of $\beta$ is corresponding to the broken phase regime. Hence $\beta$ is similar to the local order parameter characterizing quantum phase transitions. Results are presented in Fig. 3 where a sudden change of $\beta(r)$ occurs at critical point, above which the non-vanishing value of $\beta(r)$ is observed in the unbroken phase regime and below which the value of $\beta(r)$ vanishes in the broken phase regime. Therefore, $\beta(r)$ can be used as an efficient tool to detect and characterize the phase transition which can be realized experimentally.

B. Criticality witness in anti-PT-symmetric non-Hermitian systems

The generalized form of a single-qubit anti-PT symmetric Hamiltonian can be expressed as

$$H_{NH} = \begin{pmatrix} \lambda e^{i\phi} & i s \\ i s & -\lambda e^{-i\phi} \end{pmatrix}, \quad (17)$$

![FIG. 1. (Color online) The EUR for initial state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ with observables $Q = \sigma_x$ and $R = \sigma_z$ with respect to $t$ in both Hermitian and non-Hermitian systems. Blue dashed curve depicts the behavior of EUR under the Hermitian Hamiltonian. Blue solid, green solid and red solid curves correspond to $s\sigma > r^2 \sin^2\phi$ (e.g., $s = \sigma = 2$ and $r = 1$), $s\sigma < r^2 \sin^2\phi$ (e.g., $s = \sigma = 1$ and $r = 2$) and exceptional points $s\sigma = r^2 \sin^2\phi$, respectively. Other parameters are fixed $\phi = \pi/2$.

IV. ENTRepH uncertainty relation as a signature of the exceptional points of non-Hermitian systems

In traditional quantum information approaches, quantum phase transitions can be identified by the critical behavior of quantities such as entanglement [42, 43], quantum Fisher information [45, 46], fidelity [47–49] and so on. These quantities are sensitive to the quantum criticality, and often serve as a signature of transition points. However, these quantities are more than a mere theoretical construct, especially they have difficult in experimental accessibility. Therefore, it is interesting to investigate the quantum criticality beyond Hermitian systems, and with a criticality witness that is feasible and experimentally accessible.

Recall that quantum phase transitions have their infrastructure to the Heisenberg uncertainty relation, we identify a criticality witness based on the long-time average of the EUR

$$W = \lim_{T \to \infty} \frac{1}{T} \int_0^T EUR(t) dt. \quad (15)$$

In analogy to what has been performed for quantum entanglement as a signature of the quantum criticality [44], a sudden change of $W$ with respect to varying system parameters occurs at the critical point implying that there is a quantum phase transition occurs, and this critical point is referred to as an exceptional point in non-Hermitian systems.
FIG. 2. (Color online) (a) The criticality witness $\mathcal{W}$ in a $PT$ symmetric non-Hermitian system with $\sigma = s = 2$ and $\phi = \pi/2$. Therefore, a quantum critical point occurs at $r_0 = 2$. (b) The criticality witness $\mathcal{W}$ in a general non-Hermitian system with $\sigma = \sqrt{2}$, $s = \sqrt{2}/2$ and $\phi = \pi/2$. Therefore, there is a quantum critical point at $r_0 = 1$.

where both of the parameters $\lambda$ and $s$ denote real numbers. It is easy to show that this Hamiltonian satisfies the anti PT invariant Hamiltonian $(PT)H(PT) = -H$ [32, 51]. The anti-$PT$ symmetric non-Hermitian Hamiltonian has complex eigenvalues spectra $\varepsilon_{\pm} = i\lambda \sin \phi \pm \sqrt{\lambda^2 \cos^2 \phi - s^2}$. Therefore, the system is termed in the regime of unbroken phase when $|s| > |\lambda \cos \phi|$ [50]. Following the same calculation procedure as above, the analytical expression of EUR is obtained

$$EUR = -\sum_{i=1}^{2} p_x^i \log_2 p_x^i - \sum_{i=1}^{2} p_z^i \log_2 p_z^i$$  \hspace{1cm} (18)

with

\begin{align*}
    p_x^1 &= \frac{1}{2} \left( 1 + \frac{s \omega \sinh 2\omega t - \lambda^2 \cos^2 \phi}{s \cos 2\omega t - \lambda^2 \cos^2 \phi} \right), \\
    p_x^2 &= \frac{1}{2} \left( 1 - \frac{s \omega \sinh 2\omega t - \lambda^2 \cos^2 \phi}{s \cos 2\omega t - \lambda^2 \cos^2 \phi} \right), \\
    p_z^1 &= \frac{1}{2} \left( 1 - \frac{\omega^2}{s^2 \cosh 2\omega t - \lambda^2 \cos^2 \phi} \right), \\
    p_z^2 &= \frac{1}{2} \left( 1 + \frac{\omega^2}{s^2 \cosh 2\omega t - \lambda^2 \cos^2 \phi} \right),
\end{align*}

with $\omega = \sqrt{s^2 - \lambda^2 \cos^2 \phi}$.

As expected, the efficiency of the proposed criticality witness also works for anti-$PT$-symmetric non-Hermitian system where we take $\lambda = 1$ and $\phi = \pi/2$. A abrupt change of $\mathcal{W}$ happens at the transition point $s_0 = \lambda \cos \phi = 1$, as reported in Fig. 4(a). This indicates the critical behavior of the anti-parity-time symmetric non-Hermitian systems is also identified by criticality witness $\mathcal{W}$. On the other hand, the revival of $\beta(s)$ for an anti-parity-time symmetric non-Hermitian system is also observed in Fig. 4(b). The non-vanishing parameter $\beta(s)$ is corresponding to the broken phase regime, while the vanishing of $\beta(s)$ is corresponding to the unbroken phase regime. The signature of a quantum critical point is characterized by $\beta(s)$ from non-zero value to zero value.

C. Experimental feasibility

Finally, we give a brief discussion of the above predictions which are easy to implement in experiment according to our procedure. To accomplish this, we here restrict our discussions to the single-ion system where
In conclusion, we have investigated the dynamics of EUR in the non-Hermitian systems. Compared with the dynamics governed by a Hermitian Hamiltonian, there are three different types of behavior depending on the dynamics of EUR in the unbroken regime, at the exceptional point, or in the spontaneously broken regime. In the unbroken regime, the EUR undergoes an oscillatory behavior, while in broken phase regime where the oscillation breaks down. At the exceptional point, the EUR increases asymptotically to a stable value. The exceptional point marks the oscillatory and non-oscillatory behavior of the EUR. In addition, we also identify the witness of critical behavior in terms of the EUR in the dynamical limit where two kinds of non-Hermitian models, including (anti-)PT-symmetric systems are taken into consideration. We find that criticality witness can be an effective index to identify exceptional point in these two models. Finally, the experimental feasibility of our schemes is also discussed.

Our approach establishes a general connection between the criticality of non-Hermitian system and the behaviors of EUR. Therefore, our results may have potential applications to witness and detect criticality in non-Hermitian systems. Besides, our investigation could be helpful for understanding the dynamics features of EUR and further manipulating the EUR in the non-Hermitian system.

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