We show that two apparently contradictory theories on the existence of Griffiths-McCoy singularities in magnetic metallic systems are in fact mathematically equivalent. We discuss the generic phase diagram of the problem and show that there is a non-universal crossover temperature range \( T^* < T < \omega_0 \) where power law behavior (Griffiths-McCoy behavior) is expected. For \( T < T^* \) power law behavior ceases to exist due to the destruction of quantum effects generated by the dissipation in the metallic environment. We show that \( T^* \) is an analogue of the Kondo temperature and is controlled by non-universal couplings.

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**Figure 1:** Schematic phase diagram \( T \times x \) where \( x \) controls the phase transition. Also shown is the behavior of the magnetic susceptibility, \( \chi(T) \), in the proximity of the quantum critical point.

Using the Hertz theory of quantum critical phenomena and treating the problem of an Ising field theory in the presence of a single impurity, Millis, Morr, and Schmalian (MMS) have cast doubts on the possibility of Griffiths singularities in these materials based on the over-damping of the cluster (droplet) dynamics due to the electronic degrees of freedom. This effect is the result of the slow decay of the droplet profile in the single impurity case. MMS have extended their calculation to a finite density of impurities and conclude that over-damping destroys quantum Griffiths singularities leading to \( \chi(T) \propto 1/T \) times \( \ln(T) \) corrections at all \( T < \omega_0 \) where \( \omega_0 \) is the high temperature cut-off of the theory above which the droplets are thermally activated. This result is in agreement with predictions in ref. [1] that \( \chi(T) \propto 1/(T \ln(1/T)) \) at low temperatures. MMS, however, do not observe any crossover with \( T \) (as it was found...
in ref. [1]) and conclude that power law behavior should not be observed.

In what follows we demonstrate the mathematical equivalence of the theory proposed in ref. [1] and by MMS [2]. Both theories have a non-universal energy scale, $T^* = \omega_0 e^{-C^2} < \omega_0$, so that for $T^* < T < \omega_0$ one has $\chi(T) \propto T^{-1+\lambda}$ and therefore quantum Griffiths singularities. $T^*$ plays the role of an effective Kondo temperature of the droplet that is a non-universal quantity dependent on the size of the Griffiths region (power law behavior) as a function of $T$. In this paper we revisit the theory proposed by MMS and study the size of the Griffiths region (power law behavior) as a function of $C_2$. In this paper we revisit the theory proposed by MMS and study the size of the Griffiths region (power law behavior) as a function of $C_2$. We will show that this region shrinks very fast with the decrease of $C_2$.

The starting point is the Hertz action for a field theory with Ising symmetry [3], $S = S_0 + S_I$, where:

$$S_0 = \frac{E_0 c^2 T}{8\pi} \sum_{\mathbf{k}, \mathbf{n}} \left( \xi^{-2} + k^2 + \frac{\omega^2}{c^2} + \frac{8\pi|\omega_n|}{TE_0} \right) |\phi(\mathbf{k}, \omega_n)|^2$$

$$S_I = \frac{E_0}{16\pi} \int_0^\beta d\tau \int d\mathbf{r} \phi^4(\mathbf{r}, \tau),$$

here $E_0$ is a characteristic energy scale (assumed to be of the order of the Kondo temperature of the system), $\xi_0$ is a characteristic length scale (of the order of the lattice spacing), $\xi$ is the correlation length, $c$ is a characteristic velocity, $\Gamma$ is the damping coefficient, and $\phi(\mathbf{k}, \omega_n)$ is its Fourier transform in momentum and Matsubara frequency. In principle the quantities that appear in the action have to be obtained from a microscopic theory or to be used as fitting parameters to the experiments.

The main difference between the two approaches discussed in ref. [1] and ref. [2] is that ref. [2] describes a hard spin model in the continuum while in ref. [1] a hard spin model in a lattice is used. In the first case the amplitude of the droplets, $\phi_0$, can fluctuate statistically while in the second case the amplitude of a cluster of $N$ spins is fixed by the spin $S$ of the spins in the cluster. Using the single impurity solution discussed in ref. [2] as a variational ansatz, MMS derived the probability distribution, $P(R, \phi_0)$ for a droplet of size $R$ and amplitude $\phi_0$, and the renormalized droplet tunneling splitting, $\omega_{\text{tunn}}$.

In order to make the connection between the different theories we notice that the number of spins in a droplet is approximately given by: $\int d\mathbf{r} \phi(\mathbf{r})/(4\pi\xi_0^3/3) \propto \phi_0(R/\xi_0)^3$, where it is assumed that the droplet exists in a region of size $R$ only. In what follows, instead of using MMS’s reduced variables: $y = R/\xi_0$, $f = \phi_0(\xi/\xi_0)$, and $u = V_0^2(\xi/\xi_0)$ ($V_0$ is the strength of the disorder) we define a new variable:

$$N(y, f) = f^2 \frac{\xi}{\xi_0} y^3,$$

and new parameters:

$$\nu = c_\gamma a C_2,$$

$$N_c = (c_\gamma a)^{-1},$$

where

$$a(y) = 1 + 3/y^2,$$

$$c_\gamma = E_0/(6\Gamma),$$

$$C_2 = 1 + \ln \left[ E_0^2 c^2/(6e^2) \right].$$

Notice that the parameters defined in [4] are independent of $R$ and $\phi_0$. Using these variables we can rewrite the main results of ref. [2], namely, the droplet tunneling splitting (eq. (36) of ref. [2]),

$$\omega_{\text{tunn}}(N) = \omega_0 \exp \left\{ -\frac{\nu N}{1-N/N_c} \right\},$$

and the probability distribution $(N[y^3, f^2]$ in eq. (15) of ref. [2]),

$$P(N, f) \propto N^{1-\theta} e^{-N/N_\xi(f)}$$

where $\theta = 5/2$ and

$$N_\xi(f) = \frac{f^2 u}{(f^2 + a)^2} \frac{\xi}{\xi_0}.$$

We can now make a direct comparison between the two different theories. Notice that the tunneling splitting is only dependent on the number of spins in the droplet, $N$, and it is identical to eq. (5.6) of ref. [1] if we assume a single cut-off energy scale (that is, set $W = \omega_0$ and $\varphi = 1$, see eq. (4.84) in ref. [1]). This result allows us to identify $N_c$ as the maximum number of spins in the droplet such that, above this number, tunneling ceases to occur, and $\nu$ is the effective damping coefficient (the equivalent of $\gamma$ in eq. (5.6) of ref. [1]). The probability distribution [9] is the equivalent of eq. (5.7) of ref. [1] except that in ref. [2] this probability distribution depends not only on the number of spins in the droplet but also on their amplitude. Although in a soft spin model there are fluctuations in the size of the droplet, those fluctuations are limited by the exponential term in [6]. In fact, MMS find that the droplets that contribute most to the magnetic susceptibility are such that

$$f_{\text{MMS}} \propto \left( \frac{\xi_0}{\xi} \right)^{1/2}.$$
\[ N_{\xi,MMS} \propto V_0^2 \frac{\xi}{\xi_0} \] (9)

In ref. [1] it is shown that (see eq. (4.6) of ref. [1]):

\[ N_{\xi,\text{CNJ}} \propto \left( \frac{\xi}{\xi_0} \right)^D, \] (10)

where \( D = 2.54 \) is the fractal dimension of the cluster in three dimensions. Therefore, [9] and [10] are very similar (the difference in the exponents is due to the different nature of the percolation in the two problems) and show that \( N_\xi \) diverges as \( \xi \to \infty \).

It should be clear at this point that the two main elements of the theory, namely the tunneling splitting and its distribution, although calculated in very different way, are essentially the same in both theories. The main question is why MMS reach the conclusion that power law behavior, that is, quantum Griffiths singularities cannot be observed while in ref. [1] it was found that there is a wide range in \( T \) where power law behavior should be observed. To answer this question let us consider the distribution of tunneling splittings, that is, the probability of finding a droplet with tunneling splitting between \( \Delta \) and \( \Delta + d\Delta \), (instead of the distribution of cluster sizes).

It is easy to convert from one to the other using (4) and (5) (we define \( \Delta = \omega_{\text{run}} \)):

\[ P(\Delta) = \frac{dN}{d\Delta} \propto \left( \frac{\ln(\omega_0/\Delta)}{\Delta (N,\nu + \ln(\omega_0/\Delta))} \right)^{1-\theta} \] (11)

It is straightforward to see that there is a characteristic tunneling splitting \( \Delta_c \),

\[ \Delta_c = \omega_0 e^{-\nu N_c}, \] (12)

such that for \( \Delta \gg \Delta_c \) one has:

\[ P(\Delta) \propto \Delta^{1/(\nu N_c)-1}, \] (13)

which leads to \( \chi(T) \propto T^{-1+\lambda} \) for \( T \gg \Delta_c \), in agreement with eq. (6.1) of ref. [1]. Using (8) and (9) we find:

\[ \lambda_{\text{MMS}} = \frac{1}{\nu N_\xi} \propto \frac{\xi_0}{V_0^2 \xi}, \] (14)

which is eq. (44) of ref. [2]. This last result can be compared with eq. (6.1) of ref. [1] where:

\[ \lambda_{\text{CNJ}} \propto \left( \frac{\xi_0}{\xi} \right)^D, \] (15)

and both theories predict that \( \lambda \to 0 \) as one approaches the QCP (see fig.11). This behavior is characteristic of quantum Griffiths singularities. Once again the difference in the dependence of \( \lambda \) with the dimensionality comes from the fact that ref. [1] deals with a percolation problem in a lattice while ref. [2] studies an impurity problem in the continuum. Nevertheless, the overall conclusion that \( \lambda \) vanishes at the critical point is common to both theories. On the other hand, if \( \Delta \ll \Delta_c \) we have from (11):

\[ P(\Delta) \propto \frac{1}{\Delta \ln(\omega_0/\Delta)} \] (16)

which leads to the result that \( \chi(T) \propto 1/(T \ln(1/T)) \) in agreement with eq. (5.18) in ref. [1] and with the conclusion expressed in ref. [2] that the susceptibility is \( 1/T \) times logarithms. Thus, we have unequivocally shown that the theory presented by MMS has a crossover temperature \( T^* = \Delta_c \) from power law to \( 1/(T \ln(1/T)) \) from high to low temperatures and also proved the equivalence of the results of ref. [1] and ref. [2].

The reason why the crossover is not discussed in ref. [2] can be easily understood if we rewrite \( \Delta_c \) in terms of the parameters in (3):

\[ \Delta_c = \omega_0 e^{-C_2} \] (17)

which means that the crossover depends on the value of \( C_2 \), that is, depends on a non-universal number. By assuming that \( c/\xi_0 \approx E_0 \) MMS find \( C_2 \approx 1 \). This choice implies that \( \Delta_c/\omega_0 \approx 0.4 \) which is not particularly small and leads to a limited dynamical range for the power law to be observed (in Fig.3 of ref. [2] the plots stop at frequencies of order of \( 0.1 \omega_0 \) while the crossover should be observed at \( 0.4 \omega_0 \)). Thus, it is of great interest to understand how the magnetic response of the system depends on \( C_2 \).

We see from [1] that \( C_2 \) depends on the details of the lattice through \( \xi_0 \) and \( c \) as well as the Kondo energy scale through \( E_0 \). Any physical crossover, by definition, depends on non-universal parameters. A famous example is the Kondo problem where the Kondo temperature, \( T_K = \omega_0 e^{-1/g} \) (where \( g \) is the strength of the spin exchange interaction between localized moments and conduction electrons), is also a non-universal function of the cut-off and, like the problem at hand, is an exponential function of effective coupling. As in the Kondo case the crossover temperature in this problem is exponentially sensitive on the choice of \( C_2 \). The crossover can be easily visualized if one considers, for instance, the function \( J_{\text{res}}(\omega) \) defined in ref. [2]. This function is directly related with the contribution to \( \chi(T) \) coming from the fluctuating clusters through eq. (38) of ref. [2]:

\[ \chi(T) = \xi^{-3} \int d\omega \frac{J_{\text{res}}(\omega)}{\omega(\omega + T)}, \] (18)

which is the analogue of eq. (5.15) of ref. [1]. In fig.2 we plot \( \alpha C_2 J_{\text{res}}(\omega)/J_{\text{res}}(0) \) versus \( \omega/\omega_0 \) for \( V_0 = 0.5 \),
for particular realizations of Heisenberg quantum clusters [10] and in studies of metallic Heisenberg magnets [11] have shown that tunneling is not suppressed and that Griffiths-McCoy singularities are possible down to \( T = 0 \). Nevertheless, essentially all systems of experimental interest where non-Fermi liquid is observed have magnetic anisotropies due to spin-orbit coupling and crystal field effects [8] and therefore are in the Ising universality class discussed here.

In summary, we have shown that the theory proposed by MMS in ref. [2] has generically the same crossover behavior that we predicted in ref. [1] where for \( T^* < T < \omega_0 \) quantum Griffiths singularities with non-universal exponents, \( \chi(T) \propto T^{-1+\lambda} \), should be observed and that for \( T < T^* \) cluster freezing leads to \( \chi(T) \propto 1/(T \ln(1/T)) \) (see fig. 1). The value of \( T^* \), however, varies from system to system and cannot be obtained within the theory.

We have argued in ref. [1] that \( T^* \ll \omega_0 \) and therefore power law behavior should be clearly visible in a crossover region. This conclusion seems to be in agreement with the experimental data in various materials [2, 4, 5].

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