Noncommutative Geometry of Super-Jordanian $OSp_h(2/1)$ Covariant Quantum Space

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Abstract

Extending a recently proposed procedure of construction of various elements of differential geometry on noncommutative algebras, we obtain these structures on noncommutative superalgebras. As an example, a quantum superspace covariant under the action of super-Jordanian $OSp_h(2/1)$ is studied. It is shown that there exist a two-parameter family of torsionless connections, and the curvature computed from this family of connections is bilinear. It is also shown that the connections are not compatible with the metric.

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I Introduction

Noncommutative geometry is one of the active fields in recent theoretical physics and mathematics. The physical interests seem to be focused on noncommutative differential geometry, since it plays crucial roles in the context of string theory and quantum gravity [1, 2]. There exist various approaches to noncommutative differential calculus. For instance, Connes' approach involves the Dirac operator [1], Dubois-Violette's method is based on derivations [3], and studies based on quantum groups also exist [4, 5, 6]. Analogues of Riemannian connection, curvature and metric on noncommutative algebra $\mathcal{A}$ were introduced in Ref. [7] where the authors used only the left $\mathcal{A}$-module structure of the differential forms. On the other hand, the $\mathcal{A}$-bimodule structure of an algebra of differential forms was used to define a linear connection for a particular differential calculus based on derivations [8]. Mourad also made essential use of $\mathcal{A}$-bimodule structure to define linear connection, torsion and curvature on a noncommutative algebra $\mathcal{A}$. In the procedure used in [9], a noncommutative generalization of the permutation operator on two copies of one-forms was introduced. The generalized permutation operator plays a role in defining the noncommutative differential geometry. This methodology was found to be useful in other approaches to the noncommutative differential calculi. Furthermore, it was extended to other studies concerning differential calculi on noncommutative algebras such as $SL_q(2)$ covariant quantum plane [10], two-parameter quantum plane [11], Jordanian $h$-deformed quantum plane [12, 13], matrix geometries [14], and so on. The curvatures corresponding to the connections [8, 9] were also studied [15] in this context.

In the present work, we follow the line of investigation developed in Refs. [8, 9, 15] to study the noncommutative differential geometry associated with the quantized supergroups. It will be seen that the ideas used there are also appropriate for studying noncommutative differential geometry on quantum superspaces. Following Refs. [8, 9, 15] we can naturally define linear connections, torsions, curvatures and metrics on noncommutative quantum superspaces. As an example, we will study the quantum superspace covariant under the action of super-Jordanian deformation of $OSp(2/1)$. The super-Jordanian $OSp_h(2/1)$ is introduced as a Hopf algebra dually related to the recently obtained triangular deformation of Lie superalgebra $osp(2/1)$ [16]; and it coincides with the deformed $OSp(2/1)$ supergroup studied by Juszczak and Sobczyk [17].

We obtain the most general form of linear connections on the quantum superspace covariant under the action of $OSp_h(2/1)$. The curvatures and the metric will also be studied in the sequel.

We first briefly review the construction of a linear connection in a commutative geometry [9]. Let $\mathcal{M}$ be a manifold, and $C(\mathcal{M})$ be an algebra of functions on $\mathcal{M}$. The set of $k$-forms is denoted by $\Omega^k$. The covariant derivative $D$ is a linear map from $\Omega^1$ to $\Omega^1 \otimes C(\mathcal{M}) \Omega^1$ obeying the Leibnitz rule

$$D(f\xi) = df \otimes \xi + fD\xi, \quad f \in C(\mathcal{M}), \quad \xi \in \Omega^1.$$  (I.1)
Since functions commute with forms, the Leibnitz rule may also be recast as

\[ D(\xi f) = \sigma(\xi \otimes df) + (D\xi)f, \quad (I.2) \]

where \( \sigma \) is a permutation acting on \( \Omega^1 \otimes_{C(M)} \Omega^1 \)

\[ \sigma(\xi \otimes \eta) = \eta \otimes \xi, \quad (\xi, \eta) \in \Omega^1 \quad (I.3) \]

and the exterior derivative \( d \) is nilpotent: \( d^2 = 0 \). In the context of commutative geometry the two Leibnitz rules (I.1) and (I.2) are equivalent. However, in the noncommutative setting this is not the case, as functions and forms do not commute. The noncommutative covariant derivative is constructed in such a way that it is required to satisfy [9] the two Leibnitz rules. Reflecting the noncommutative nature of functions and forms, the operator \( \sigma \) can no longer be represented by a simple permutation element. It needs to be modified for noncommutative quantum spaces.

Suppose the manifold \( M \) is parallelizable and let \( \omega^i \) be an arbitrary basis element of \( \Omega^1 \). The covariant derivative of a one-form is then uniquely determined by \( D\omega^i \). The linear connection is defined by \( \Gamma^i = -D\omega^i \). Namely, the covariant derivative defines the linear connection. Throughout this article, we use the terms ‘linear connection’ and ‘covariant derivative’ synonymously. Let \( \pi \) be a projection of \( \Omega^1 \otimes_{C(M)} \Omega^1 \) onto \( \Omega^2 \) such that \( \pi(\xi \otimes \eta) = \xi \wedge \eta \). Then the map \( \Theta : \Omega^1 \rightarrow \Omega^2 \) defined by \( \Theta = d - \pi \circ D \) is a bimodule homomorphism, that is, it maintains \( \Theta(f\xi) = f\Theta(\xi) \) and \( \Theta(\xi f) = \Theta(\xi)f \), where \( f \in C(M) \). The torsion is defined by \( \Theta(\omega^i) \). This construction of linear connections will be extended to noncommutative superspaces associated with quantum supergroups.

This paper is organized as follows: In the next section, we extend the differential geometry on noncommutative algebras to noncommutative superalgebras. The super-Jordanian deformation of \( OSp(2/1) \) is introduced in Section III. The quantum superspace which is covariant under the action of super-Jordanian \( OSp_{\hbar}(2/1) \) is introduced, and the differential calculus on it in the sense of Wess-Zumino is constructed in Section IV. The linear connection on the quantum superspace is studied in Section V, and it is observed that the most general torsionless connection is a member of a two-parameter family. In Section VI the curvature obtained from the linear connection is calculated and it is shown that the curvature is bilinear. The metric of the quantum superspace is also studied. We show that the covariant derivative is not compatible with the metric. Section VII contain the concluding remarks.

II Noncommutative extension of superspace geometry

Let \( \mathcal{A} \) be a noncommutative algebra with \( \mathbb{Z}_2 \) grading. The grading is specified by parity of elements of \( \mathcal{A} \). An even (odd) element \( f \in \mathcal{A} \) has a parity \( \hat{f} = 0 \) (1). It is assumed that a differential calculus over \( \mathcal{A} \), describing, in particular, the one-forms and their commutation
relations with the elements of $A$, has been constructed. Let $\Omega^k(A)$ and $d$ denote the space of $k$-forms over $A$ and the exterior derivative, respectively. The covariant derivative $D$ is defined as a map $D : \Omega^1 \to \Omega^1 \otimes_A \Omega^1$ subject to the following Leibnitz rules

$$D(f\xi) = df \otimes \xi + (-1)^jfD\xi,$$  \hspace{2cm} (II.1)

$$D(\xi f) = (-1)^j\sigma(\xi \otimes df) + (D\xi)f, \hspace{2cm} f \in \Omega^0, \hspace{0.5cm} \xi \in \Omega^1,$$  \hspace{2cm} (II.2)

where $\sigma : \Omega^1 \otimes_A \Omega^1 \to \Omega^1 \otimes_A \Omega^1$ refers to a noncommutative generalization of the permutation map. The covariant derivative changes the parity of a $k$-form $\xi$ by unity:

$$\hat{D}\xi = \hat{\xi} + 1 \pmod{2}.$$  \hspace{2cm} (II.3)

Using the definition of the covariant derivative, one can show that the map $\sigma$ is $A$-bilinear

$$\sigma(f\xi \otimes \eta) = f\sigma(\xi \otimes \eta), \hspace{1cm} \sigma(\xi \otimes \eta f) = \sigma(\xi \otimes \eta)f, \hspace{1cm} f \in \Omega^0, \hspace{0.5cm} (\xi, \eta) \in \Omega^1.$$  \hspace{2cm} (II.4)

We now demonstrate the first relation in (II.4). The second relation in (II.4) follows similarly. For arbitrary elements $(f, g) \in \Omega^0$ and $\xi \in \Omega^1$, we compute $D(f\xi g)$ in two different ways. Regarding it as $D(f \cdot \xi g)$, we apply (II.1) and obtain

$$D(f\xi g) = df \otimes \xi g + (-1)^j fD(\xi g) = df \otimes \xi g + (-1)^j f(\xi \otimes dg) + (-1)^j f(D\xi)g.$$  

Alternately, for the choice $D(f\xi g) = D(f\xi \cdot g)$ the Leibnitz rule (II.2) yields

$$D(f\xi g) = (-1)^j f\sigma(\xi \otimes dg) + df \otimes \xi g + (-1)^j f(D\xi)g.$$  

As the above two computations must give identical results, it follows $\sigma(f\xi \otimes dg) = f\sigma(\xi \otimes dg)$.

The covariant derivative may be extended as a linear map from the $n$-fold tensored space $\otimes^n\Omega^1$ to the $(n+1)$-fold tensored space $\otimes^{(n+1)}\Omega^1$. This is done recurrently while maintaining the following extension of the Leibnitz rule

$$D(\omega \otimes \omega') = D\omega \otimes \omega' + (-1)^{\hat{\omega}}\sigma_{12}(\omega \otimes D\omega'),$$  \hspace{2cm} (II.5)

where $\omega \in \Omega^1$, $\omega' \in \otimes^{n-1}\Omega^1$ and $\sigma_{12}$ has a nontrivial structure in the first two sectors:

$$\sigma_{12} = \sigma \otimes 1 \otimes 1 \otimes \cdots \otimes 1.$$  \hspace{2cm} (II.6)

Let $\pi$ be a projection of $\Omega^1 \otimes_A \Omega^1$ onto $\Omega^2$ defined by the wedge product on the forms

$$\pi(\xi \otimes \eta) = \xi \wedge \eta.$$  \hspace{2cm} (II.7)
The noncommutativity of $A$, in general, demands $\xi \wedge \eta \neq -\eta \wedge \xi$. Employing the projection operator $\pi$, we define the torsion $\Theta$ of the covariant derivative $D$ as a map $\Theta : \Omega^1 \to \Omega^2$

$$\Theta : \Omega^1 \to \Omega^2, \quad \Theta = d - \pi \circ D. \quad (\text{II.8})$$

The torsion is always left $A$-linear, whereas the condition

$$\pi \circ (\sigma - 1) = 0 \quad (\text{II.9})$$

is necessary for it to be right $A$-linear. More explicitly, the torsion satisfies the relations

$$\Theta(f\xi) = (-1)^\hat{f} f\Theta(\xi), \quad \Theta(\xi f) = \Theta(\xi)f, \quad f \in \Omega^0, \ \xi \in \Omega^1. \quad (\text{II.10})$$

The condition (II.9) is necessary for the validity of the second relation in (II.10). Note that the relation (II.9) has a sign difference from the nongraded case [10, 14]. Since the proof is straightforward, we show only the second relation. The exterior derivative acts on the one-form $\xi f$ as follows

$$d(\xi f) = (d\xi)f + (-1)^\hat{\xi}\xi \wedge df = (d\xi)f + (-1)^\hat{\xi}\pi(\xi \otimes df),$$

while the action of $\pi \circ D$ on $\xi f$ reads

$$\pi \circ D(\xi f) = \pi((-1)^\hat{\xi}\sigma(\xi \otimes df) + (D\xi)f).$$

Consequently, it follows

$$\Theta(\xi f) = \Theta(\xi)f - (-1)^\hat{\xi}\pi \circ (\sigma - 1)(\xi \otimes df).$$

It is thus evident that the condition (II.9) need to be satisfied for the torsion $\Theta$ to be right $A$-linear.

The curvature is defined by the following map [15]

$$\pi_{12}D^2 : \Omega^1 \to \Omega^2 \otimes_A \Omega^1, \quad (\text{II.11})$$

where $\pi_{12} = \pi \otimes 1$. The torsionless condition $\Theta = 0$ and the validity of the constraint (II.9) requires the curvature to be left $A$-linear:

$$\pi_{12}D^2(f\xi) = f\pi_{12}D^2(\xi), \quad f \in \Omega^0, \ \xi \in \Omega^1. \quad (\text{II.12})$$

We demonstrate this below. Employing the Leibnitz rule (II.5), we compute

$$D^2(f\xi) = Ddf \otimes \xi + (-1)^\hat{d}\sigma_{12}(df \otimes D\xi) + (-1)^\hat{\xi}df \otimes D\xi + fD^2\xi.$$
The first and the second terms in the above expression vanish because of the torsionless condition and the constraint (II.9), respectively. Thus the curvature is left $\mathcal{A}$-linear. In general, the curvature is not right $\mathcal{A}$-linear. It is, however, known that there exist some cases for nongraded $\mathcal{A}$ where the curvature is right $\mathcal{A}$-linear [10]. We will find such an example for graded $\mathcal{A}$ in the following sections.

Now let us define a metric. A metric $g$ is a non-degenerate $\mathcal{A}$-bilinear map
\[
g : \Omega^1 \otimes_\mathcal{A} \Omega^1 \rightarrow \mathcal{A}.
\] (II.13)

The metric is said to be non-degenerate if the following conditions hold: $g(\xi \otimes \eta) = 0$ for all $\eta \in \Omega^1$ implies $\xi = 0$, and, simultaneously, $g(\xi \otimes \eta) = 0$ for all $\xi \in \Omega^1$ implies $\eta = 0$. Symmetry of the metric is defined by using the extended permutation $\sigma$. A metric satisfying $g \circ \sigma = g (g \circ \sigma = -g)$ is known to be symmetric (skew-symmetric) in nature. If the following diagram is commutative, the covariant derivative $D$ is said to be compatible with the metric $g$, or, in short, $D$ is said to be metric:
\[
\begin{array}{ccc}
\Omega^1 \otimes_\mathcal{A} \Omega^1 & \xrightarrow{D} & \Omega^1 \otimes_\mathcal{A} \Omega^1 \otimes_\mathcal{A} \Omega^1 \\
\downarrow g & & \downarrow 1 \otimes g \\
\mathcal{A} & \xrightarrow{d} & \Omega^1 \\
\end{array}
\]

More explicitly, the above compatibility condition reads
\[
d \circ g = (1 \otimes g) \circ D.
\] (II.14)

In the following sections an example of the differential geometry described here will be presented. In this example, the algebra $\mathcal{A}$ is taken to be a quantum superspace covariant under the action of a quantum supergroup $OSp_h(2/1)$. The example will be constructed so as to keep the covariance of all relations.

### III Super-Jordanian deformation of $OSp(2/1)$

In this Section we introduce a quantum deformation of the supergroup $OSp(2/1)$. The conventions adopted here regarding the graded Yang-Baxter equation are same as in the Refs. [18, 19]. The quantum supergroup discussed here is the dual Hopf algebra to the super-Jordanian deformed $U_h(osp(2/1))$ algebra introduced recently. The study of super-Jordanian $osp(2/1)$ algebra was initiated by Kulish [20]. It was further developed by the works of the present authors [16] and Borowiec et. al. [21]. In Ref. [16], the universal $\mathcal{R}$ matrix of the $U_h(osp(2/1))$ algebra was obtained up to $O(h^3)$ where $h$ is the deformation parameter. Its limiting classical value is described by $h \rightarrow 0$. The fundamental representation of the generators of the $U_h(osp(2/1))$ algebra is obtained by mapping the deformed algebra on its classical counterpart. Although
the two deformation maps given in Ref. [16] provide two distinct sets of matrices for the fundamental representation, the pertinent $R$ matrices computed for these two cases are identical. All the terms in the universal $\cal R$ matrix $O(h^3)$ and above vanish in the fundamental representation, and, therefore, the $R$ matrix in the said representation is determined by the terms up to $O(h^2)$. The $R$ matrix, thus obtained, is given by

$$R = \begin{pmatrix} 1 & -h & h & h^2/2 \\ 1 & -h & h & h \\ 1 & -h & h & h \\ 1 & -h & h & h \end{pmatrix}$$ \hspace{1cm} (III.1)

where the dot ($\cdot$) is used instead of 0 for better readability. The $R$ matrix (III.1) solves the graded Yang-Baxter equation. The inverse of this $R$ matrix is given by $R^{-1} = R(-h)$; and identifying $h = -p$ in (III.1) the $R$-matrix given in Ref. [17] is reproduced. In Ref. [20], a contraction technique is applied to the $R$ matrix in the fundamental representation of the $U_q(osp(2/1))$ algebra to obtain a triangular $\hat{R}$ matrix, which maintains the relation $\hat{R}_{ij}^{st} = R_{ji}^{st}$ with the $R$ matrix given in (III.1).

Now we explicitly write down the nonstandard deformed supergroup $OSp_h(2/1)$. Since the $R$ matrix (III.1) is the inverse of the one used in Ref. [17], the quantum supergroup $OSp_h(2/1)$ is identical to the one given in Ref. [17] where the deformed supergroup $OSp_h(2/1)$ is constructed by the FRT [22] method. Let the inverse scattering matrix $T$ in the fundamental representation of the super-Jordanian deformed $OSp_h(2/1)$ is given by

$$T = (t_{ij}^t) = \begin{pmatrix} a & \alpha & b \\ \gamma & e & \beta \\ c & \delta & d \end{pmatrix}$$ \hspace{1cm} (III.2)

where $\hat{i} = 0(1)$ for $i = \{1, 3\}(\{2\})$, and $\hat{i} \hat{j} = \hat{i} + \hat{j}$. Thus the entries $a, b, c, d, e$ are even elements and $\alpha, \beta, \gamma, \delta$ are odd ones. The RTT relation and deformed orthosymplectic conditions

$$T^{st}JT = J, \hspace{1cm} TJ^{-1}T^{st} = J^{-1}, \hspace{1cm} J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & -h/2 \end{pmatrix}$$ \hspace{1cm} (III.3)

determine the relations among the entries of $T$. The supertransposition of $T$ is defined by

$$(T^{st})^i_j = (-1)^{(\hat{j}+\hat{i})}t^{ji}.$$ It follows from this definition that $(AB)^{st} = B^{st}A^{st}, ((A^{st})^{st})^i_j = (-1)^{\hat{i}+\hat{j}}A^{ji}$. Note that the matrix $J$ has the property

$$(-1)^{\hat{i}+\hat{j}}J_{ab} = J_{ab}.$$ \hspace{1cm} (III.4)
This simplifies many relations in the later computations. Following Ref. [17] we express the elements $e, \beta$ and $\gamma$ in terms of the remaining elements $a, b, c, d, \alpha$. The commutation relations satisfied by the elements $a, b, c, d, \alpha$ and $\delta$ are summarized as

\begin{align*}
[a, b] &= h(1 - a^2), & [a, c] &= hc^2, & [a, d] &= h(cd - ca), \\
[a, \alpha] &= 0, & [a, \delta] &= hc\delta, & [b, c] &= h(ca + dc), \\
[b, d] &= h(d^2 - 1), & [b, \alpha] &= haa, & [b, \delta] &= h(d\delta + c\alpha), \\
[c, d] &= -hc^2, & [c, \alpha] &= -hc\delta, & [c, \delta] &= 0, \\
[d, \alpha] &= h(\delta a - \delta d), & [d, \delta] &= h\delta c, & \{\alpha, \delta\} &= h(ac - \delta^2),
\end{align*}

(III.5)

The other entries of $T$ may be algebraically solved as follows:

\begin{align*}
e &= 1 + \alpha\delta - \frac{h}{2}ac, & \beta &= \alpha d - \delta b - h\delta d - \frac{h}{2}\gamma, & \gamma &= \alpha c - \delta a - h\delta c.
\end{align*}

(III.6)

Relations analogous to the classical supergroup $OSp(2/1)$ exist for the nonstandard deformation:

\begin{align*}
ad - bc + \alpha\delta + \frac{h}{2}ac &= 1, & e^{-1} &= (1 - \alpha\delta + \frac{h}{2}ac)(1 - \frac{h^2}{4}c^2)^{-1}, & \alpha\delta + \beta\gamma &= \frac{h}{2}(ac - dc).
\end{align*}

(III.7)

For completeness, we also give the commutation relations involving the elements $e, \beta$ and $\gamma$:

\begin{align*}
[a, e] &= h\gamma\delta, & [b, e] &= h(\beta\delta + \gamma\alpha), & [c, e] &= 0, \\
[d, e] &= h\gamma\delta, & [e, \alpha] &= h(e\delta + \gamma a), & [e, \beta] &= h(d\delta + \gamma e), \\
[e, \gamma] &= h\delta\gamma, & [e, \delta] &= h\gamma\delta, & [a, \beta] &= h(\gamma d - \gamma a), \\
[b, \beta] &= h\beta d, & [c, \delta] &= -h\gamma\delta, & [d, \beta] &= 0, \\
\{\alpha, \beta\} &= h(ea - ed), & \{\beta, \gamma\} &= -h(dc + \gamma^2), & \{\beta, \delta\} &= hce, \\
[a, \gamma] &= h\gamma c, & [b, \gamma] &= h(\beta c + \gamma a), & [c, \gamma] &= 0, \\
[d, \gamma] &= h\gamma c, & \{\alpha, \gamma\} &= -hce, & \{\gamma, \delta\} &= 0,
\end{align*}

(III.8)

\begin{align*}
\beta^2 &= \frac{h}{2}(1 - d^2), & \gamma^2 &= -\frac{h}{2}c^2.
\end{align*}

As a consequence of the grading the RTT relation involves extra sign factors in the tensor products of $T$ and the identity matrix:

\begin{align*}
\sum_{x,y} (-1)^{\hat{y}(x+i)} R^{k\ell}_{xy} t^x_i t^y_j = \sum_{x,y} (-1)^{\hat{y}(k+i)} t^x_i R^{xy}_{ij}.
\end{align*}

(III.9)

The coalgebra mappings of the quantum supergroup $OSp_h(2/1)$ are, as usual, given by

\begin{align*}
\Delta(T) &= T \otimes T, & \epsilon(T) &= \text{diag}(1, 1, 1).
\end{align*}

(III.10)

The antipode is obtained from the coproduct:

\begin{align*}
S(T) &= \begin{pmatrix}
d + \frac{h}{2}c & -\beta - \frac{h}{2}\gamma & -b - \frac{h}{2}(a - d) + \frac{h^2}{4}c \\
d & -\alpha + \frac{h}{2}\delta & a - \frac{h}{2}c \\
-\alpha & \frac{h}{2}c & \gamma
\end{pmatrix} = J^{-1}T^{st} J.
\end{align*}

(III.11)

It is easy to see that $TS(T) = S(T)T = \text{diag}(1, 1, 1)$. 

Differential calculus on quantum superspace

In this Section, a quantum superspace covariant under the action of $OSp_h(2/1)$ is introduced and a differential calculus in the sense of Wess and Zumino [6] is constructed. The quantum superspace is a graded algebra, denoted by $\mathcal{A}$, generated by two odd ($\theta_1, \theta_2$) and one even ($x$) elements. The defining relations of the algebra $\mathcal{A}$ read

$$\begin{align*}
[\theta_1, x] &= -hx\theta_2, \quad \{\theta_1, \theta_2\} = 0, \quad [\theta_2, x] = 0, \\
\theta_1^2 &= \frac{-h}{2}(x^2 - 2\theta_1\theta_2), \quad \theta_2^2 = 0.
\end{align*}$$

(IV.1)

It is straightforward to verify that the relations (IV.1) are preserved under the action of $OSp_h(2/1)$ from the left

$$\begin{pmatrix}
\theta_1' \\
x'
\theta_2'
\end{pmatrix} = \begin{pmatrix}
a & \alpha & b \\
\gamma & e & \beta \\
c & \delta & d
\end{pmatrix} \begin{pmatrix}
\theta_1 \\
x \\
\theta_2
\end{pmatrix}.$$

(IV.2)

The quantum superspace (IV.1) has an important difference from that associated to the Jordanian quantum supergroup $GL_h(1/1)$ discussed in Refs. [23, 24]. In the quantum superspace covariant under the action of $GL_h(1/1)$, the deformation parameter $h$ is a Grassmann variable, whereas in (IV.1) the quantity $h$ commute with all elements of the quantum superspace.

A scalar element $\varphi$ exists in the quantum superspace $\mathcal{A}$:

$$\varphi \equiv X^{st} J X = x^2 - 2\theta_1\theta_2,$$

(IV.3)

where

$$X = \begin{pmatrix}
\theta_1 \\
x \\
\theta_2
\end{pmatrix}, \quad X^{st} = (-\theta_1, x, -\theta_2).$$

(IV.4)

Then it is easy to show that $\varphi$ is preserved by the left action of $OSp_h(2/1)$ Note that the parity of components of $X$ is $\tilde{X}^i = 1 + \hat{i}$ (mod 2). The fourth relation in (IV.1) implies that $\theta_2^2$ is also a scalar in the algebra $\mathcal{A}$. Employing a solution of the nongraded Yang-Baxter equation the defining relations (IV.1) of the algebra $\mathcal{A}$ may be written in a compact form:

$$X^i X^j = \sum_{k,\ell} B_{k\ell}^{ij} X^\ell X^k, \quad B_{12} B_{13} B_{23} = B_{23} B_{13} B_{12},$$

(IV.5)

where the matrix $B$ reads

$$B(h) = \begin{pmatrix}
-1 & \cdot & \cdot & -h & \cdot & \cdot & h & -h^2/2 \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & -1 & \cdot & \cdot & \cdot & h & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & -1 & h \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1
\end{pmatrix}.$$

(IV.6)
The matrix $B$ is related to the $R$ matrix of the deformed supergroup $OSp_{h}(2/1)$ as

$$R(h)_{xy}^{kl} = (-1)^{1+k+(1+x)\bar{y}} (B(h)^{-1})_{xy}^{kl} = (-1)^{1+k+\bar{y}+kl} B(-h)_{xy}^{kl}, \quad (IV.7)$$

where the last equality follows from the relation $(B(h)^{-1})_{xy}^{kl} = (-1)^{k\bar{e}+\bar{x}y} B(-h)_{xy}^{kl}$.

The differential calculus on quantum space is an algebra generated by coordinates $X^i$, differentials $\Xi^i \equiv dX^i$ and derivatives $\partial_i = \frac{\partial}{\partial X^i}$. The parity of differentials and derivatives are, in general, $\hat{\Xi}^i = 1 + \hat{X}^i$, $\hat{\partial}_i = \hat{X}^i$. The differential calculus on quantum superspace using solutions of the nongraded Yang-Baxter equation is developed in Ref. [25]. These authors require the exterior derivative $d$, which maps a $k$-form to a $(k+1)$-form, to maintain three properties: (i) nilpotency, (ii) graded Leibnitz rule

$$d(f \wedge g) = (df) \wedge g + (-1)^{\hat{f}} f \wedge dg, \quad f \in \Omega^p, \ g \in \Omega^q \quad (IV.8)$$

and (iii) its action on a function $f(X^i)$ is given by $df = \sum_i \Xi^i \partial_i f$. Employing these properties, the following commutation relations among $X^i$, $\Xi^i$ and $\partial_i$ may be determined:

$$\Xi^i \wedge \Xi^j = \sum_{k,\ell} (-1)^{\hat{X}^i + \hat{X}^\ell} B_{\ell k}^{ij} \Xi^\ell \wedge \Xi^k, \quad X^i \Xi^j = \sum_{k,\ell} (-1)^{\hat{X}^i} B_{k \ell}^{ij} \Xi^\ell X^k,$$

$$\partial_j X^i = \delta_{ij} + \sum_{k,\ell} B_{k j}^{\ell i} X^k \partial_\ell, \quad \partial_j \Xi^i = \sum_{k,\ell} (-1)^{\hat{X}^i} (B^{-1})_{\ell j}^{ki} \Xi^\ell \partial_k,$$

$$\partial_\ell \partial_j = \sum_{k,\ell} B_{\ell j}^{k \ell} \partial_k \partial_\ell. \quad (IV.9)$$

Our convention of the matrix $B$ differs from that in Ref. [25]. We use the Yang-Baxter equation of the form (IV.5), whereas the Yang-Baxter equation in the braid group form $F_{12}F_{23}F_{12} = F_{23}F_{12}F_{23}$ is used in Ref. [25]. They are related as $F_{ij}^{kl} = B_{ij}^{kl}$. The relations (IV.9) are covariant under the action of the super-Jordanian $OSp_{h}(2/1)$:

$$X^h = \sum_j t^i_j X^j, \quad \Xi^h = \sum_j (-1)^{\hat{j}+j} t^i_j \Xi^j, \quad \partial^h_j = \sum_j (-1)^{\hat{j}+j} ((T^{st})^{-1})^j_i \partial_j. \quad (IV.10)$$

To show the covariance, we need RTT type relations for $T$ and $T^{st}$ with the matrix $B$. They are obtained via (III.9) and (IV.5):

$$\sum_{i,j} (-1)^{\hat{b}+\hat{j}+i b \bar{t} + \hat{a} t^a c^i} t^i_j B_{cd}^{ij} = \sum_{i,j} (-1)^{\hat{c}+\hat{i}+\hat{c} \bar{d} + \hat{a} t^a} B_{ij}^{ab} t^a_d t^i_c, \quad (IV.11)$$

$$\sum_{i,j} (-1)^{\hat{i}+\hat{a}+i \bar{b} + \hat{b} t^b} \tau^a_i \tau^a_j B_{ij}^{cd} = \sum_{i,j} (-1)^{\hat{d}+\hat{j}+\hat{c} \bar{d} + \hat{a} \tau^a} B_{ij}^{ab} \tau^a_d \tau^a_c, \quad (IV.12)$$

$$\sum_{i,j} (-1)^{\hat{i}+\hat{b}+\hat{c} \bar{d} + \hat{b} t^b} \tau^a_i t^b_j B_{ij}^{cd} = \sum_{i,j} (-1)^{\hat{c}+\hat{i}+\hat{c} \bar{d} + \hat{b} t^b} B_{ji}^{\hat{a} c} \tau^a_d \tau^a_c, \quad (IV.13)$$

$$\sum_{i,j} (-1)^{\hat{c}+\hat{i}+\hat{c} \bar{d} + \hat{b} t^b} (B^{-1})_{ai} t^a_j \tau^a_i t^a_j (B^{-1})_{ic}, \quad (IV.14)$$
where $\tau = (T^*)^{-1}$. Introducing the notations $\xi_1 = d\theta_1$, $\eta = dx$, $\xi_2 = d\theta_2$, the explicit form of the $OSp_h(2/1)$ covariant differential calculus on the quantum superspace $A$ is summarized as follows:

- **coordinates**

  \[
  [\theta_1, x] = -hx\theta_2, \quad \{\theta_1, \theta_2\} = 0, \quad [\theta_2, x] = 0, \quad [\theta_1, \theta_2] = 0, \quad \theta_1^2 = -\frac{h}{2}(x^2 - 2\theta_1\theta_2), \quad \theta_2^2 = 0. \tag{IV.15}
  \]

- **differentials**

  \[
  \xi_1 \wedge \eta - \eta \wedge \xi_1 = h\eta \wedge \xi_2, \quad \xi_1 \wedge \xi_2 - \xi_2 \wedge \xi_1 = h\xi_2 \wedge \xi_2, \\
  \eta \wedge \xi_2 - \xi_2 \wedge \eta = 0, \quad \eta \wedge \eta = -\frac{h}{2}\xi_2 \wedge \xi_2. \tag{IV.16}
  \]

- **coordinates and differentials**

  \[
  [\theta_1, \xi_1] = h(\theta_1\xi_2 + x\eta - \theta_2\xi_1 - \frac{h}{2}\theta_2\xi_2), \quad \{\theta_1, \eta\} = h\xi_2, \\
  [\theta_1, \xi_2] = h\theta_2\xi_2, \quad [x, \xi_1] = -h\theta_2\eta, \quad [x, \eta] = -h\theta_2\xi_2, \\
  [x, \xi_2] = 0, \quad \{\theta_2, \xi_1\} = -h\theta_2\xi_2, \quad \{\theta_2, \eta\} = 0, \quad \{\theta_2, \xi_2\} = 0. \tag{IV.17}
  \]

- **derivatives and coordinates**

  \[
  \partial_1\theta_1 = 1 - \theta_2\partial_1, \quad \partial_1x = x\partial_1, \quad \partial_1\theta_2 = -\theta_2\partial_1, \\
  \partial_x\theta_1 = \theta_2\partial_x - hx\partial_1, \quad \partial_x x = 1 + x\partial_x + h\theta_2\partial_1, \quad \partial_x\theta_2 = \theta_2\partial_x, \\
  \partial_2\theta_1 = -\theta_2\partial_2 - h(\theta_1\partial_1 + x\partial_x + \theta_2\partial_2 + \frac{h}{2}\theta_2\partial_1), \quad \partial_2 x = x\partial_2 - h\theta_2\partial_2, \\
  \partial_2\theta_2 = 1 - \theta_2\partial_2 + h\theta_2\partial_1, \quad \text{where } \partial_1 = \frac{\partial}{\partial \theta_1}, \partial_x = \frac{\partial}{\partial x}, \partial_2 = \frac{\partial}{\partial \theta_2}. \tag{IV.18}
  \]

- **derivatives and differentials**

  \[
  \partial_1\xi_1 = \xi_1\partial_1 - h\xi_2\partial_1, \quad \partial_1\eta = -\eta\partial_1, \quad \partial_1\xi_2 = \xi_2\partial_1, \\
  \partial_x\xi_1 = \xi_1\partial_x - h\eta\partial_1, \quad \partial_x\eta = \eta\partial_x + h\xi_2\partial_1, \quad \partial_x\xi_2 = \xi_2\partial_x, \\
  \partial_2\xi_1 = \xi_1\partial_2 + h(\xi_1\partial_1 + \eta\partial_2 + \xi_2\partial_2 + \frac{h}{2}\xi_2\partial_1), \quad \partial_2\xi_2 = \xi_2\partial_2 - h\xi_2\partial_1. \tag{IV.19}
  \]

- **derivatives**

  \[
  \partial_1^2 = 0, \quad \partial_1\partial_x = \partial_x\partial_1, \quad \partial_1\partial_2 = -\partial_2\partial_1, \quad \partial_x^2 = h(\partial_1\partial_2 - \frac{1}{2}\partial_1^2). \tag{IV.20}
  \]
**V OSp\(_h(2/1)\)** symmetric torsionless connections

We have seen that a scalar \( \varphi (~\Theta^2_1) \) exists in the quantum superspace \( \mathcal{A} \). This scalar is an \( OSp\(_h(2/1)\) \) invariant zero-form. Invariant one and two-forms under the action of the deformed supergroup \( OSp\(_h(2/1)\) \) also exist in the differential calculus \( \mathcal{A} \):

\[
\varrho = \sum_{a,b} J_{ab} X^a \Xi^b = \theta_1 \xi_2 + x\eta - \theta_2 \xi_1 - \frac{\hbar}{2} \theta_2 \xi_2, \quad (V.1)
\]

\[
\chi = \sum_{a,b} J_{ab} \Xi^a \wedge \Xi^b = 0. \quad (V.2)
\]

It is evident that the invariant two-form \( \chi \) is trivial. It is straightforward to verify the invariance of \( \varrho \) and \( \chi \) under the transformation (IV.10). Note that the \( \varrho \) appears on the right hand side of the first relation in (IV.17). It is easy to find the commutation relations between the invariant forms and the basis elements \( (X^a, \Xi^a) \). For the zero-form \( \varphi \) these relations read

\[
X^a \varphi = \varphi X^a, \quad \Xi^a \varphi = \varphi \Xi^a. \quad (V.3)
\]

The commutation properties of the invariant one-form \( \varrho \) are succinctly given by

\[
X^a \varrho = (-1) \hat{X}^a \varrho X^a, \quad \Xi^a \wedge \varrho = (-1) \hat{\Xi}^a \varrho \wedge \Xi^a. \quad (V.4)
\]

In a more expanded version the above relations read

\[
[x, \varrho] = \{\theta_i, \varrho\} = 0, \quad i = 1, 2 \quad (V.5)
\]

\[
\eta \wedge \varrho + \varrho \wedge \eta = 0, \quad \xi_i \wedge \varrho - \varrho \wedge \xi_i = 0. \quad (V.6)
\]

It is also straightforward to verify the relation

\[
\varrho \wedge \varrho = 0. \quad (V.7)
\]

In order to determine the covariant derivative, it is necessary to find the action of the extended permutation \( \sigma \) on \( \Omega^1 \otimes \Omega^1 \). This can be done by applying the covariant derivative \( D \) on the second relation in (IV.9). Using the Leibnitz rules, we obtain

\[
\Xi^i \otimes \Xi^j + (-1) \hat{X}^i \bar{X}^j D\Xi^j = \sum_{k,l} (-1)^{\hat{X}^i} B_{kl}^{ij} \{-(-1)^{\hat{\Xi}^l} \sigma(\Xi^l \otimes \Xi^k) + (D\Xi^l) X^k\}. \quad (V.8)
\]

This relation implies that the action of \( \sigma \) on \( \Xi^l \otimes \Xi^k \), and the commutation relations between \( X^i \) and \( D\Xi^j \) may be consistently described as

\[
\Xi^i \otimes \Xi^j = \sum_{k,l} (-1)^{\hat{X}^i} B_{kl}^{ij} \sigma(\Xi^l \otimes \Xi^k), \quad (V.8)
\]

\[
X^i D\Xi^j = \sum_{k,l} B_{kl}^{ij} (D\Xi^l) X^k. \quad (V.9)
\]
Using the property (IV.7) the exchange relation (V.8) may be solved yielding the action of $\sigma$ on the tensored space of one-forms as follows:

$$\sigma(\Xi^k \otimes \Xi^\ell) = \sum_{i,j} (-1)^{i\ell} R^k_{ij} \Xi^i \otimes \Xi^j \equiv \sum_{i,j} \tilde{R}^k_{ij} \Xi^i \otimes \Xi^j,$$

(V.10)

The matrix $\tilde{R}$ has two important properties, namely, $\tilde{R}$ is idempotent and satisfies a non-graded Yang-Baxter equation

$$\tilde{R}^2 = 1, \quad \tilde{R}_{12}\tilde{R}_{23}\tilde{R}_{12} = \tilde{R}_{23}\tilde{R}_{12}\tilde{R}_{23}.$$

(V.11)

As a consequence of the exchange of the superscripts in the definition (V.10) of $\tilde{R}$, it satisfies a different form of Yang-Baxter equation from the one obeyed by $R$. The operator $\sigma$, therefore, follows identical properties:

$$\sigma^2 = 1, \quad \sigma_{12}\sigma_{23}\sigma_{12} = \sigma_{23}\sigma_{12}\sigma_{23}.$$

(V.12)

The map $\sigma$ may now be explicitly written as follows

$$\sigma(\xi_1 \otimes \xi_1) = \xi_1 \otimes \xi_1 - h(\xi_1 \otimes \xi_2 + \eta \otimes \eta - \xi_2 \otimes \xi_1 - \frac{h}{2} \xi_2 \otimes \xi_2),$$

$$\sigma(\xi_1 \otimes \eta) = \eta \otimes \xi_1 + h\xi_2 \otimes \eta,$$

$$\sigma(\xi_1 \otimes \xi_2) = \xi_2 \otimes \xi_1 + h\xi_2 \otimes \xi_2,$$

$$\sigma(\eta \otimes \xi_1) = \xi_1 \otimes \eta - h\eta \otimes \xi_2,$$

$$\sigma(\eta \otimes \eta) = -\eta \otimes \eta - h\xi_2 \otimes \xi_2;$$

$$\sigma(\eta \otimes \xi_2) = \xi_2 \otimes \eta,$$

$$\sigma(\xi_2 \otimes \xi_1) = \xi_1 \otimes \xi_2 - h\xi_2 \otimes \xi_2,$$

$$\sigma(\xi_2 \otimes \eta) = \eta \otimes \xi_2,$$

$$\sigma(\xi_2 \otimes \xi_2) = \xi_2 \otimes \xi_2.$$

(V.13)

The map $\sigma$ being $A$-bilinear, the following relations hold:

$$\sigma(\eta \otimes \varphi) = \varphi \otimes \xi_1, \quad \sigma(\eta \otimes \varphi) = -\varphi \otimes \eta, \quad \sigma(\xi_2 \otimes \varphi) = \varphi \otimes \xi_2,$$

$$\sigma(\eta \otimes \eta) = -\eta \otimes \eta, \quad \sigma(\xi_2 \otimes \xi_2) = \xi_2 \otimes \varphi;$$

$$\sigma(\varphi \otimes \varphi) = -\varphi \otimes \varphi.$$

(V.14)

The explicit form of the map $\sigma$ being known, the relations (V.12) and (II.9) may be verified by direct computation.

To derive the action of the covariant derivative $D$ on $\Xi^a$, we compare the relation (V.9) with (IV.5) and the second relation in (IV.9). The comparison suggests that $D\Xi^a$ contains $X^a$ and $\Xi^a$ as factors. Another important observation is that $D\Xi^a$ has the same transformation property as $X^a$ under the action of $OSp_h(2/1)$ namely,

$$D\Xi^a = \sum_i t_i^a D\Xi^i.$$

(V.15)
Thus the most general form of $D\Xi^a$ may be given by

$$D\Xi^a = c_0X^a\varpi + c_1(-1)^{\hat{a}}\Xi^a \otimes \varrho + c_2\varrho \otimes \Xi^a,$$  

(V.16)

where $c_i$ ($i = 0, 1, 2$) are real parameters and $\varpi \in \Omega^1 \otimes \Omega^1$ satisfies

$$\varpi' = \varpi, \quad X^a\varpi = \varpi X^a.$$  

(V.17)

It is not difficult to see that each term on the right hand side of (V.16) has the same transformation property as $X^a$ under the action of the deformed supergroup $OSp_h(2/1)$. Furthermore, each term of (V.16) satisfies the same commutation relation as (V.9). As we have seen in the beginning of this Section, the $OSp_h(2/1)$ invariant two-form $\chi$ is trivial so that the only possible choice for $\varpi$ is given by

$$\varpi = \varrho \otimes \varrho.$$  

(V.18)

In this way, we have seen that (V.16) and (V.18) describe the most general linear connection.

Let us recall that our main interest is in torsionless connections, as the torsionfree condition is necessary for making the curvature left $\mathcal{A}$-linear. We restrict the linear connection obtained above to be torsionfree: $\Theta\Xi^a = 0$. As the nilpotency of $d$ constrains $d\Xi^a = 0$, we obtain

$$\Theta\Xi^a = -\pi \circ D\Xi^a = -c_0\varrho \wedge \varrho - c_1(-1)^{\hat{a}}\Xi^a \wedge \varrho - c_2\varrho \wedge \Xi^a$$

$$= -(c_1 + c_2)\varrho \wedge \Xi^a = 0,$$  

(V.19)

where we have used the relations (V.4) and (V.7). The torsionfree condition thus requires $c_2 = -c_1$. Therefore, the general form of the $OSp_h(2/1)$ symmetric torsionless connections is given by the following two-parameter family:

$$D\Xi^a = c_0X^a\varrho \otimes \varrho + c_1((-1)^{\hat{a}}\Xi^a \otimes \varrho - \varrho \otimes \Xi^a).$$  

(V.20)

More explicitly these connections read

$$D\xi_1 = c_0\theta_1\varrho \otimes \varrho + c_1(\xi_1 \otimes \varrho - \varrho \otimes \xi_1),$$

$$D\eta = c_0x\varrho \otimes \varrho - c_1(\eta \otimes \varrho + \varrho \otimes \eta),$$

$$D\xi_2 = c_0\theta_2\varrho \otimes \varrho + c_1(\xi_2 \otimes \varrho - \varrho \otimes \xi_2).$$  

(V.21)

For the torsionless connections (V.20), it is easy to see

$$D\varrho = \sum_{a,b} J_{ab} \Xi^a \otimes \Xi^b + (c_0\varphi - 2c_1)\varrho \otimes \varrho.$$  

(V.22)

Applying (II.7, V.2, V.7) it immediately follows that

$$\pi(D\varrho) = 0.$$  

(V.23)
VI Curvature and metric

A two-parameter family of $OSp_h(2/1)$ symmetric torsionfree connections was obtained in the previous section. Since the generalized permutation operator $\sigma$ satisfies the relation (II.9), the curvature computed from the connections are left $A$-linear. Recall that curvatures are, in general, not right $A$-linear. In the present case, however, the curvature is also right $A$-linear. We exhibit this by explicit computation. We also discuss the metric on the quantum superspace $A$. It, however, turns out that the connections are not compatible with the metric.

To obtain the curvature, we apply $\pi_{12}D$ on (V.20). Each term is computed separately and listed below:

$$\pi_{12}D(X^a \varrho \otimes \varrho) = \Xi^a \wedge \varrho \otimes \varrho - \sum_{b,c} (-1)^{\hat{\chi}_a} J_{bc} X^a \varrho \wedge \Xi^b \otimes \Xi^c,$$

$$\pi_{12}D((-1)^{\hat{a}} \Xi^a \otimes \varrho) = \sum_{b,c} J_{bc} \Xi^a \wedge \Xi^b \otimes \Xi^c + (c_0 \varphi - 2c_1) \Xi^a \wedge \varrho \otimes \varrho,$$

$$\pi_{12}D(\varrho \otimes \Xi^a) = -c_1 \Xi^a \wedge \varrho \otimes \varrho.$$

Combining the above results, the curvature is obtained as follows:

$$\pi_{12}D^2 \Xi^a = (c_0 - c_1^2 + c_0 c_1 \varphi) \Xi^a \wedge \varrho \otimes \varrho + (c_0 (-1)^{\hat{a}} X^a \varrho + c_1 \Xi^a) \wedge \Lambda,$$

where

$$\Lambda = \sum_{a,b} J_{ab} \Xi^a \otimes \Xi^b = \xi_1 \otimes \xi_2 + \eta \otimes \eta - \xi_2 \otimes \xi_1 - \frac{h}{2} \xi_2 \otimes \xi_2.$$

Note that $\pi(\Lambda) = \chi = 0$. Expanding the first term in the right hand side of (VI.1) as

$$\Xi^a \wedge \varrho \otimes \varrho = \sum_{b,c} (-1)^{\hat{\chi}_a} J_{bc} \Xi^a X^b \wedge \varrho \otimes \Xi^c,$$

we express the curvature in terms of a two-form $\omega$

$$\pi_{12}D^2 \Xi^a = \sum_b \omega_b^a \otimes \Xi^b,$$

where

$$\omega_b^a = \sum_k J_{kb} \left\{ (-1)^{\hat{k}} \{ c_0 (-1)^{\hat{a}} X^a \Xi^k - (c_0 - c_1^2 + c_0 c_1 \varphi) \Xi^a X^k \} \wedge \varrho + c_1 \Xi^a \wedge \Xi^k \right\}.$$

We now prove that the curvature obtained above is right $A$-linear. To this end, we note that the following relation may be established by direct computation:

$$[X^a, \Lambda] = 0.$$

Employing the second relation in (IV.9), in conjunction with the left $A$-linearity of the curvature, we obtain

$$\pi_{12}D^2(\Xi^b X^a) = \sum_{i,j} (-1)^{\hat{X}_i} (B^{-1})_{ij}^a X^i \pi_{12}D^2 \Xi^j.$$
Substituting (VI.1) into (VI.6), and then transferring \( X^i \) to the right via equations (IV.9, V.3, V.4, VI.5), we demonstrate the intended result
\[
\pi_{12} D^2(\Xi^b X^a) = (\pi_{12} D^2 \Xi^b) X^a, \tag{VI.7}
\]
establishing the right \( \mathcal{A} \)-linearity of the curvature. In the above computation we have used the fact that the matrices \( B^{ab}_{ij} \) and \((B^{-1})^{ab}_{ij}\) maintain the following relationship regarding the parity of their indices: \( \hat{a} + \hat{b} = \hat{i} + \hat{j} \).

Let us now turn to the metric, which is considered as a bilinear map \( g : \Omega^1 \otimes \mathcal{A} \Omega^1 \to \mathcal{A} \). To completely determine the metric we need to know the action of the map \( g \) on the basis elements of \( \Omega^1 \otimes \Omega^1 \). Setting \( g^{ab} = g(\Xi^a \otimes \Xi^b) \), we require that \( g^{ab} \) to be invariant under the action of \( OSp(2/1) \):
\[
g^{\hat{a} \hat{b}} = g(\Xi^a \otimes \Xi^b) = \sum_{k, \ell} g((\hat{a} + \hat{k}) t_k^a \Xi^k \otimes (-1)^{\hat{b} + \hat{\ell}} t_\ell^b \Xi^\ell)
= \sum_{k, \ell} (-1)^{\hat{a} + \hat{b} + \hat{k} + \hat{\ell}} t_k^a g^{k \ell} (t^b)^{\ell}.
\]
The above result, in conjunction with the identity (III.3), immediately yields \( g^{ab} = g^{\hat{a} \hat{b}} \), provided we choose \( g^{k \ell} = (-1)^{\hat{k} + \hat{\ell}} (J^{-1})_{k \ell} = (J^{-1})_{k \ell} \). The \( OSp(2/1) \) invariant metric, therefore, is given by
\[
g^{ab} = g(\Xi^a \otimes \Xi^b) = (J^{-1})_{ab} = \begin{pmatrix} \frac{h}{2} & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{VI.8}
\]
Denoting the components of \( g^{-1} \) by \( g_{ab} \), we note that the invariant one-form \( \varrho \) may be written in terms of the metric
\[
\varrho = \sum_{a, b} g_{ab} X^a \Xi^b.
\]
The structure of the metric (VI.8) implies
\[
d \circ g(\Xi^a \otimes \Xi^b) = 0. \tag{VI.9}
\]
The compatibility condition (II.14) now reads
\[
(1 \otimes g) \circ D(\Xi^a \otimes \Xi^b) = 0. \tag{VI.10}
\]
To compute the left hand side in (VI.10), we start by ordering the one-forms in the expression of \( \varrho \) to the left:
\[
\varrho = \xi_2 \theta_1 + \eta x - \xi_1 \theta_2 + \frac{h}{2} \xi_2 \theta_2 = \sum_{a, b} (-1)^{\hat{a} \hat{b}} J_{ab} \Xi^a X^b. \tag{VI.11}
\]
We now readily obtain
\[
g(\varrho \otimes \Xi^a) = X^a, \quad g(\Xi^a \otimes \varrho) = (-1)^{\hat{a} \hat{a}} X^a. \tag{VI.12}
\]
Following (II.5) the action of the covariant derivative on $\Omega^1 \otimes \Omega^1$ is given as

$$D(\Xi^a \otimes \Xi^b) = D\Xi^a \otimes \Xi^b + (-1)^{\hat{a}\hat{b}}\sigma_{12}(\Xi^a \otimes D\Xi^b). \quad (VI.13)$$

Substituting (V.20) into (VI.13), we observe that, as a consequence of the bilinearity of $g$, we may treat the first (proportional to $c_0$) and the second (proportional to $c_1$) terms in the right hand side of (VI.13) separately. For the choice $c_1 = 0$, we then obtain

$$D(\Xi^a \otimes \Xi^b) = c_0(X^a \otimes \theta \otimes \Xi^b + \theta \otimes \Xi^a \otimes \theta X^b),$$

which, in turn, yields

$$(1 \otimes g) \circ D(\Xi^a \otimes \Xi^b) = (-1)^{\hat{a}\hat{b}}2c_0\theta X^a X^b. \quad (VI.14)$$

For the alternate choice $c_0 = 0$, it follows that

$$D(\Xi^a \otimes \Xi^b) = c_1\{(-1)^{\hat{a}\hat{b}}\theta \otimes \Xi^b - 2\theta \otimes \Xi^a \otimes \Xi^b + (-1)^{\hat{a}+\hat{b}}\sigma_{12}(\Xi^a \otimes \Xi^b \otimes \theta)\}.$$  

The right hand side in (VI.10) now reads

$$(1 \otimes g) \circ D(\Xi^a \otimes \Xi^b) = c_1\{(-1)^{\hat{a}\hat{b}}\Xi^a X^b - 2g^{ab}\theta + (-1)^{\hat{a}+\hat{b}}(1 \otimes g) \circ \sigma_{12}(\Xi^a \otimes \Xi^b \otimes \theta)\}. \quad (VI.15)$$

The last term is computed by using (V.13) and (VI.12). The result is listed below:

$$(1 \otimes g) \circ D(\xi_1 \otimes \xi_1) = 0,$$

$$(1 \otimes g) \circ D(\xi_1 \otimes \eta) = c_1(\xi_1 x + \eta \theta_1 - h\xi_2 x),$$

$$(1 \otimes g) \circ D(\xi_1 \otimes \xi_2) = c_1(\eta x - \frac{h}{2}\xi_2 \theta_2 + \theta),$$

$$(1 \otimes g) \circ D(\eta \otimes \xi_1) = -c_1(\xi_1 x + \eta \theta_1 + h\eta \theta_2),$$

$$(1 \otimes g) \circ D(\eta \otimes \eta) = -c_1(2\eta x - h\xi_2 \theta_2 + 2\theta),$$

$$(1 \otimes g) \circ D(\eta \otimes \xi_2) = -c_1(\eta \theta_2 + \xi_2 x),$$

$$(1 \otimes g) \circ D(\xi_2 \otimes \xi_1) = -c_1(\eta x - \frac{h}{2}\xi_2 \theta_2 + \theta),$$

$$(1 \otimes g) \circ D(\xi_2 \otimes \eta) = c_1(\eta \theta_2 + \xi_2 x),$$

$$(1 \otimes g) \circ D(\xi_2 \otimes \xi_2) = 0.$$

Together with (VI.14), it has been shown that $(1 \otimes g) \circ D \neq 0$, except for the trivial choice $c_0 = c_1 = 0$. Thus the covariant derivative $D$ is not compatible with the metric.

VII Concluding remarks

In the present work we have studied noncommutative spaces, linear connections, curvatures and metrics associated with the quantized supergroups. Our approach is a naive extension of
the differential geometry developed in [8, 9, 15]. We have demonstrated that the ideas of these authors may be appropriately adapted to study the geometric objects related to the quantum supergroups. Specifically, we applied the extended differential geometry to the quantum superspace covariant under the quantum supergroup $OSp_h(2/1)$. We have seen that our particular example has a two-parameter family of $OSp_h(2/1)$ symmetric torsionfree connections. It turned out that the curvature of the connection was bilinear. The connection was, however, not compatible with the metric. These properties are specific to our example. There could be other quantum superspace endowed with linear connections compatible with metric.

It may be of interest to recall the results related to the quantum spaces covariant under quantized $SL(2)$ groups; and compare them with the present results. It is well-known that $SL(2)$ admits two inequivalent deformations: the standard $q$-deformation and the Jordanian $h$-deformation. The quantum space for $q$-deformed $SL(2)$ has a one-parameter family of torsionless linear connections and it has been shown that there can be no compatible metric [10], whereas the quantum space of $h$-deformed $SL(2)$ is more classical. It has a two-parameter family of torsionfree linear connections. A one-parameter subfamily of these connections is known to be compatible with a metric [13]. On the other hand, the Lie superalgebra $osp(2/1)$ admits three inequivalent deformations [20]. We are thus able to consider three deformations of the supergroup $OSp(2/1)$: $q$-deformation [19], $h$-deformation [27] and super-Jordanian deformation. The $q$ and $h$-deformations have the $SL(2)$ counterparts, while super-Jordanian does not. The super-Jordanian deformation can be regarded as an algebra intermediate between $q$ and $h$-deformations. We have seen that the quantum space for super-Jordanian $OSp_h(2/1)$ is less classical since the connections are not metric. This leads us to the anticipate that the quantum space for $h$-deformed $OSp(2/1)$ has connections which are metric, while the connections on the quantum spaces related to the standard $q$-deformed supergroup $OSp_q(2/1)$ are not metric. This will be presented in a future work.

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