A NEW PROOF OF THE THEOREMS OF LIN-ZAIDENBERG AND ABHYANKAR-MOH-SUZUKI

KAROL PALKA

Abstract. Using the theory of minimal models of quasi-projective surfaces we give a new proof of the theorem of Lin-Zaidenberg which says that every topologically contractible algebraic curve in the complex affine plane has equation $X^n = Y^m$ in some algebraic coordinates on the plane. This gives also a proof of the theorem of Abhyankar-Moh-Suzuki concerning embeddings of the complex line into the plane. Independently, we show how to deduce the latter theorem from basic properties of $\mathbb{Q}$-acyclic surfaces.

The following result is a homogeneous formulation of the theorems proved by Lin-Zaidenberg [ZL83] and Abhyankar-Moh [AM75] and Suzuki [Suz74]. Curves are assumed to be irreducible.

**Theorem A.** If a complex algebraic curve in $\mathbb{C}^2$ is topologically contractible then in some algebraic coordinates $\{x, y\}$ on $\mathbb{C}^2$ its equation is $x^n = y^m$ for some coprime $n > m > 0$.

The part proved by Lin-Zaidenberg concerns singular curves ($m \geq 2$). The part proved by Abhyankar-Moh and Suzuki concerns smooth curves ($m = 1$) and is usually stated in the following form.

**Theorem B.** If a complex algebraic curve in $\mathbb{C}^2$ is isomorphic to $\mathbb{C}^1$ then in some algebraic coordinates $\{x, y\}$ on $\mathbb{C}^2$ its equation is $x = 0$.

Theorem B has now several published proofs using variety of methods, from algebraic to topological. The easiest we know is by Gurjar [Gur02]. As for the singular case of Theorem A, the original proof relies on Teichmüller theory. A topological proof based on properties of knots was given in [NR87, NR88]. Proofs of both theorems using algebraic geometry can be found in [GM96] and [Kor07]. The latter two use the tools of the theory of open algebraic surfaces including the logarithmic Bogomolov-Miyaoka-Yau inequality established for surfaces of log general type by Kobayashi [Kob90] and Kobayashi-Nakamura-Sakai [KNS89]. Our proof of Theorem A also uses the theory of minimal models for log surfaces. We believe it is quite short and geometric. Both theorems are deduced from the following result.

**Theorem 1.** If $A \subseteq \mathbb{C}^2$ is a topologically contractible curve then there exists a minimal smooth completion $(X, D)$ of $\mathbb{C}^2 \setminus A$, such that the proper transform of $A$ is a fiber of a $\mathbb{P}^1$-fibration of $X$, whose restriction to $\mathbb{C}^2 \setminus A$ has irreducible fibers.

The basic new ingredient in the proof is to shift the focus from the surface $\mathbb{C}^2 \setminus A$, where $A$ is the contractible curve, to the surface $X = (\mathbb{C}^2 \setminus A) \cup C$, where $C$ is (some naturally defined open subset of) the last $(-1)$-curve created by the minimal log resolution of the singularity at infinity (see section 1). While the boundary of $X$ is not any more connected,
the important property is that in general the Euler characteristic of $X$ is negative. A similar idea was used in [PK13] and will be used in forthcoming papers (coauthored with M. Koras and P. Russell) finishing the classification of closed $\mathbb{C}^*$-embeddings into $\mathbb{C}^2$. Another new ingredient is that we rely on a more general version of the log BMY inequality which works for surfaces of non-negative logarithmic Kodaira dimension. We tried to make the article self-contained. In section 3 we give an independent, direct proof of Theorem B using some basic properties of $\mathbb{Q}$-acyclic surfaces.

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1. Preliminaries and notation

We work in the category of complex algebraic varieties. The results of this section are well known, we give short proofs for completeness. Let $D = \sum_{i=1}^{n} D_i$ be a reduced effective divisor on a smooth projective surface, which has smooth components $D_i$ and only normal crossings (i.e. $D$ is an snc-divisor). The number of (irreducible) components of $D$ is denoted by $\#D$. A component $C$ of $D$ is branching if it meets more than two components of $D - C$. A $(k)$-curve is a curve isomorphic to $\mathbb{P}^1$ which has self-intersection $k$. We say that $D$ is snc-minimal if after a contraction of any $(-1)$-curve contained in $D$ the image of $D$ is not an snc-divisor, or equivalently, if every $(-1)$-curve of $D$ is branching. We define the discriminant of $D$ by $d(D) = \det([-D_i \cdot D_j]_{i,j \leq n})$. We put $d(0) = 1$.

A $\mathbb{P}^1$- (or $\mathbb{C}^1$-) fibration is a surjective morphism whose general fibers are isomorphic to $\mathbb{P}^1$ (respectively to $\mathbb{C}^1$). If $(X, D)$ is a smooth completion of $X$ and $p$ is some fixed $\mathbb{P}^1$-fibration of $\overline{X}$ we put $\Sigma_X = \sum_{F \not\subseteq D} (\sigma(F) - 1)$, where the sum is taken over all fibers of $p$ not contained in $D$ and $\sigma(F)$ is the number of components of $F$ not contained in $D$. Clearly, $\Sigma_X \geq 0$ and the equality holds if and only if the restriction $p|_X$ has irreducible fibers. Let $\nu$ and $h$ be respectively the number of fibers contained in $D$ and the number
of horizontal components of $D$ (i.e. those whose push-forward by $p$ does not vanish). The following lemma is due to Fujita \cite[4.16]{Fuj82}.

**Lemma 2.** Let $(X, D)$ be a smooth completion of a smooth surface $X$. With the above notation for every $\mathbb{P}^1$-fibration of $X$ we have

\begin{equation}
(1.2) \quad h + \nu + \rho(X) = \Sigma_X + \# D + 2.
\end{equation}

**Proof.** Having a $\mathbb{P}^1$-fibration, $X$ dominates some $\mathbb{P}^1$-bundle over a projective curve. The latter has $\rho = 2$, so we have $\rho(X) = \Sigma_X + \# D + 2 = 2(\# F - 1) = \Sigma F \# F \cap D + \Sigma F (\sigma(F) - 1) = \Sigma F - h + \Sigma_X - \nu. \quad \square$

It is well known that every singular fiber of a $\mathbb{P}^1$-fibration of a smooth projective surface can be inductively reconstructed from a 0-curve by blowing up. In particular, we deduce by induction the following lemma.

**Lemma 3.** Let $F$ be a reduction of a singular complete fiber of a $\mathbb{P}^1$-fibration of some smooth projective surface. Then $F$ is a rational snc-tree and its $(-1)$-curves are non-branching. Assume $F$ contains a unique $(-1)$-curve $L_F$. Then $F - L_F$ has at most two connected components and if it has two then one of them is a chain of rational curves. Moreover, $F$ contains exactly two components of multiplicity 1, they are tips of $F$ and in case $F$ is not a chain they belong to the same connected component of $F - L_F$.

Note that sections meet only vertical components of multiplicity 1. We need a description of snc-minimal boundary divisors of $\mathbb{C}^2$. We follow the proof by Daigle and Russell \cite[5.12]{Dai91}, \cite[§1]{Rus02} (which works for any surface completable by a chain). The lemma was originally proved by Ramanujam \cite{Ram71} using only the fact that $\mathbb{C}^2$ is a smooth contractible surface which is simply connected at infinity.

**Lemma 4.** If $(\overline{X}, D)$ is a smooth snc-minimal completion of $\mathbb{C}^2$ then $D$ is a chain.

**Proof.** First of all, consider a reduced divisor $B$ (on some smooth projective surface) which can be transformed into a 0-divisor by a sequence of blowups and blowdowns by taking reduced total transforms and push forwards. Assume also that $D_0$ is either a zero divisor or a smooth curve not in $B$, such that the transformation can be done modulo $D_0$, i.e. that under all steps of the process the proper transform of $D_0$ is not contracted and stays smooth. We claim that $B$ contains a $(-1)$-curve in $B$ which is non-branching in $B_0 + B$ (in particular, $D_0 + B$ is not snc-minimal). To see this let $B_0$ be the first component of $B$ which is contracted by the transformation, i.e. the transformation starts with a sequence of blowups and then it contracts the proper transform $B'_0$ of $B_0$. If follows that $B'_0$ is a $(-1)$-curve which is non-branching in the total reduced transform of $D_0 + B$ and hence that $B_0$ is a curve with $B'_0 \geq -1$, non-branching in $D_0 + B$. But the intersection matrix of $B$, and hence of all its transforms, is negative definite, so $B'_0 = -1$ and we are done.

Suppose $D$ has a branching component $D_0$. By the factorization theorem for birational morphisms between smooth projective surfaces we know that $D$, being a boundary of $\mathbb{C}^2$, can be transformed into a chain of rational curves. Therefore, there is a connected component $B$ of $D - D_0$ which can be transformed to 0 modulo $D_0$. By the above argument there is a $(-1)$-curve $B_0$ in $B$ which is non-branching in $D_0 + B$. But then $D$ is not snc-minimal; a contradiction. \hfill $\square$

The type of an ordered chain of rational curves $T = T_1 + \ldots + T_k$ is the sequence $[-T_1^2, \ldots, -T_k^2]$. We say that the chain is in a standard form if it is of type $[0]$, $[1]$ or $[0, 0, a_1, \ldots, a_k]$ for some $a_i \geq 2$. It is an elementary exercise to show that by blowing
up and down on T we can bring it into a standard form (the number of zeros is at most two by the Hodge index theorem). The formula [1.1] implies that if T is a boundary of $\mathbb{C}^2$ in a standard form then it is of type $[0, 0]$.

An snc-divisor is of \textit{quotient type} if it can be contracted algebraically to a quotient singularity, i.e. to a smooth or an isolated singular point which is locally analytically of type $\mathbb{C}^2/G$, where $G$ is a finite subgroup of $GL(2, \mathbb{C})$. As a consequence, the intersection matrix of such a divisor is negative definite. Snc-minimal divisors of quotient type are well known, they are either negative definite chains of rational curves (corresponding to cyclic singularities $\mathbb{C}^2/\mathbb{Z}_k$) or special rational trees with unique branching components (forks). It is known that they do not contain $(-1)$-curves (see [Bri68]). For a general snc-divisor $D$ we denote the set of its connected components of quotient type by $\text{qt}(D)$.

Let $(X, D)$ be a smooth pair. For it we can run a minimal model program to obtain a birational morphism onto a log terminal surface $(V, \Delta)$ such that there is no curve $L$ on $V$ for which $L^2 < 0$ and $L \cdot (K_V + \Delta) < 0$. The pair $(V, \Delta)$ is called a \textit{minimal model} of $(X, D)$. If $(X, D)$ is snc-minimal (i.e. $D$ is an snc-minimal divisor) and the resulting morphism contracts only curves with supports in $D$ and its push-forwards then we say that $(X, D)$ is \textit{almost minimal} (for another, more direct definition see [Miy01, §2.3.11]). Recall that a \textit{log resolution} of a pair $(V, \Delta)$ with reduced $\Delta$ is a proper birational morphism from a smooth pair $r: (\overline{X}, \overline{D}) \to (V, \Delta)$ such that $D$ is the total reduced transform of $\Delta$.

Let $c(D)$ denote the number of connected components of $D$. The following proposition follows from [Fuj82, 6.20]. We denote the logarithmic Kodaira-Iitaka dimension by $\kappa$.

**Proposition 5.** Let $(X, D)$ be a smooth snc-minimal pair which is not almost minimal. Then there exists a $(-1)$-curve $\ell$ on $X$ which meets at most two connected components of $D$, each at most once and transversally and for which $\kappa(X \setminus D) = \kappa(X \setminus (D \cup \ell))$. In particular, if $p: (X, D) \to (X', D')$, with $D' = p_* D$, is the contraction of $\ell$ then

$$\chi(X' \setminus D') + c(D') = \chi(X \setminus D) + c(D) - 1.$$ 

Note that $(X', D')$ is a smooth pair and that if $(X', D')$ is not snc-minimal then the sum $\chi(X' \setminus D') + c(D')$ does not change when we snc-minimalize $D'$.

2. Proof of Theorem 1

Assume that $A \subseteq \mathbb{C}^2$ is a topologically contractible curve. Let $k \geq 0$ be the number of singular points of $A$. We write $\mathbb{C}^2$ as $\mathbb{P}^2 \setminus L_\infty$, where $L_\infty$ is the line at infinity. Let $\overline{A} \subseteq \mathbb{P}^2$ be the closure $A$ and let

$$\pi: (\overline{X}', D') \to (\mathbb{P}^2, L_\infty + \overline{A})$$

be the minimal log resolution of singularities. We denote the proper transforms of $\overline{A}$ and $L_\infty$ on $\overline{X}'$ by $E$ and $L_\infty'$ respectively. Since the germ of $\overline{A}$ at infinity is analytically irreducible, the reduced total transform of $L_\infty$ contains a unique component $C'$ meeting $E$. Moreover, their difference can be written as $(\pi^* L_\infty)_{\text{red}} - C' = D'_1 + D_2$, where $D'_1$ and $D_2$ are connected and $D'_1$ contains $L'_\infty$. We may, and shall, assume that $\overline{A}$ does not meet $L_\infty$ transversally, otherwise $\overline{A}$ is a line in which case the above theorems obviously hold. By the minimality of the resolution it follows that $D_2$ is a rational chain with negative definite intersection matrix and with no $(-1)$-curves. In particular, $d(D_2) \geq 2$. Let $U$ be the reduced exceptional divisor over the singular points of $A$. Put $D_3 = E + U$. We have

$$D' = D'_1 + C' + D_2 + D_3.$$
Figure 1. The divisor $D'$ on $\overline{X}'$. Lines denote chains of rational curves.

It may happen that $L'_\infty$ is a $(-1)$-curve (necessarily non-branching in $D'_1$). Moreover, its contraction may introduce new non-branching $(-1)$-curves in the boundary. Let $\psi: (\overline{X}', D') \to (\overline{X}, D)$ be the composition of successive contractions of non-branching $(-1)$-curves contained in $D'_1$ and its images. Since the curves contracted by $\psi$ are disjoint from $D_2 + D_3$, we denote $D_2$, $E$, $U$, $D_3$ and their images on $\overline{X}$ by the same letters. We have $D = D_1 + C + D_2 + D_3$. Put $B = D_1 + D_2 + D_3 = D - C$ and $X = \overline{X} \setminus B$.

Clearly, $(\overline{X}, D_1 + D_2 + D_3)$ is a smooth completion of $X$ and the $D_i$'s are the connected components of the boundary. It may happen that $D_1 = 0$. Also, $X \setminus C = C^2 \setminus A$, with the smooth completion $(\overline{X}, D)$. It follows that $\chi(X) = \chi(C^2 \setminus A) + \chi(C \setminus (D_1 \cup D_2 \cup E)) = 2 - \#C \cap (D_1 \cup D_2 \cup E) = -\#C \cap D_1$. Thus, $\chi(X) = -1$, unless $D_1 = 0$.

Because the log resolution $\pi: (\overline{X}', D') \to (\mathbb{P}^2, L_\infty + \tilde{A})$ is minimal, each connected component of $U$ contains a unique component $U_i$, $i = 0, \ldots, k$, meeting $E$. Moreover, each $U_i$ is a $(-1)$-curve and $U_i \cdot E = 1$. The divisor $B$ is snc-minimal except the case when $E^2 = -1$ and $U$ has at most two connected components. Note however, that the minimality of the resolution implies that the only components of $B$ which meet $E$ are the $(-1)$-curves of $U$. So, even if $E$ is a non-branching $(-1)$-curve in $B$, its contraction does not introduce new non-branching $(-1)$-curves.

Proof of Theorem 1.

Claim 1. If $D_1 = 0$ then Theorem 1 holds.

Proof. We have $C^2 \geq (C')^2 + 1 \geq 0$. Let $D_C$ be the component of $D_2$ meeting $C$. Now [1.1] gives $-C^2d(D_2) - d(D_2 - D_C) = -1$, so $C^2d(D_2) + d(D_2 - D_C) = 1$. Because $d(D_2) \geq 2$, we obtain $C^2 = 0$ and $d(D_2 - D_C) = 1$, hence $D_2$ is irreducible. If we blow up once on $E \cap C$ and contract the proper transform of $C$ the new boundary of $X$ has the same dual graph but the self-intersection of $D_2$ increases. Repeating this elementary transformation we may assume $D_2^2 = 0$. Then the contraction of $U$ maps $\overline{X}$ to a smooth
surface $\mathbb{F}$ with $\rho = 2$ and the linear system of (the proper transform of) $D_2$ induces a projection $p: \mathbb{F} \to \mathbb{P}^1$. Moreover, (the proper transform of) $E$ is disjoint from $D_2$, so it is a 0-curve. Then $U = 0$ and Theorem [1] holds.

Thus from now on we assume $D_1 \neq 0$ (and $D_2 \neq 0$).

**Claim 2.** There is no curve $\ell \not\subseteq D_1 + D_2 + U$ for which the intersection matrix of $\ell + D_1 + D_2 + U$ is negative definite.

**Proof.** We have $\#(\ell + D_1 + D_2 + U) = \#D - 1 = \rho(\mathbb{X})$, so the claim follows from the Hodge index theorem. □

**Claim 3.** If $D_1$ is not negative definite then it is not a chain and $C$ is a $(-1)$-curve.

**Proof.** Suppose $D_1$ is a chain. We change it into a standard form $\tilde{D}_1$, so that the zero-curve is a tip of $D$. Denote the proper transform of $C$ by $\tilde{C}$. If $\tilde{D}_1$ is irreducible then it is a 0-curve, so by (1.1) $-1 = d(\tilde{D}_1 + \tilde{C} + D_2) = -d(D_2) \leq -2$, which is impossible. Thus $\tilde{D}_1$ is not irreducible. Then $E + \tilde{C} + D_2$ is vertical for the $\mathbb{P}^1$-fibration induced by the 0-tip, so either $\tilde{C}$ is a branching component of a fiber or it meets two components of a fiber and the section contained in $\tilde{D}_1$. By Lemma [3] $\tilde{C}$ cannot be a $(-1)$-curve, hence $(\tilde{C})^2 \leq -2$. Then $\tilde{D}_1 + \tilde{C} + D_2$ is a boundary of $\mathbb{C}^2$ in a standard form, so it is of type $[0, 0]$. But then $D_2 = 0$; a contradiction. Thus $D_1$ is not a chain. By Lemma [4] $D_1 + C + D_2$ is not snc-minimal, so $C^2 = -1$. □

**Claim 4.** $D_3$ is not a $(-1)$-curve.

**Proof.** Suppose $D_3$ is a $(-1)$-curve. Then $U = 0$ and $D_3 = E$. Taking $\ell = E$ in Claim 2 we see that $D_1$ is not negative definite. By Claim 3 $D_1$ is not a chain and $C$ is a $(-1)$-curve. Consider the $\mathbb{P}^1$-fibration given by the linear system of $E + C$. We have $h = 2$, so $\Sigma_X = \nu$. The divisor $B$ contains no fibers. Indeed, otherwise $D$ would contain more than one fiber $(E + C$ is one of them), hence $D$ would contain a loop, which is false. We obtain $\Sigma_X = \nu = 0$. Let $F$ be a singular fiber and $L_F$ its unique component not contained in $B$. The two sections contained in $B$ belong to different connected components of $B$, so the two components of $F$ of multiplicity one meeting them belong to different connected components of $F - L_F$. By Lemma [8] it follows that $F$ is a chain and meets the sections in tips. Since the vertical part of $D_2$ is connected, there is at most one singular fiber other than $E + C$. It follows that $D_1$ is a chain; a contradiction. □

**Claim 5.** If $U \neq 0$ then $D_3$ is not contained in a divisor of quotient type.

**Proof.** Suppose $Q$ is a divisor of quotient type containing $D_3$. The $(-1)$-curve $U_1$ is branching in $D_3$, hence in $Q$. Because the self-intersection of $U_1$ is $(-1)$, the snc-minimalization of $Q$ does not touch $U_1$, hence leads to an snc-minimal divisor of quotient type which contains a branching $(-1)$-curve. But there are no such divisors. A contradiction. □

We now analyze the creation of the almost minimal model of $(\mathbb{X}, B)$. Let $\epsilon$ be the contraction of $E$ in case it is a non-branching $(-1)$-curve, otherwise put $\epsilon = \text{id}_{\mathbb{X}}$. Let

$$(\mathbb{X}, B) \xrightarrow{\epsilon} (\mathbb{X}_0, B_0) \xrightarrow{p_1} (\mathbb{X}_1, B_1) \xrightarrow{p_2} \ldots \xrightarrow{p_n} (\mathbb{X}_n, B_n)$$

be a sequence of birational morphisms leading to the almost minimal model $(\mathbb{X}_n, B_n)$ of $(\mathbb{X}, B)$ grouped so that $p_{i+1}$ is a composition of a contraction of some $(-1)$-curve $\ell_i \not\subseteq B_i$, witnessing the non-almost minimalty of $(\mathbb{X}_i, B_i)$ followed by the snc-minimalization of the image of $B_i$. Put $p = p_n \circ \ldots \circ p_1 \circ \epsilon$ and $X_i = \mathbb{X}_i \setminus B_i$. We denote $\ell_i$’s and their proper transforms on $\mathbb{X}$ by the same letters.
Claim 6. \( \chi(X) = \chi(X_0) \) and \( \chi(X_i) \geq \chi(X_{i+1}) \).

Proof. Since \( D_3 \) is not a \((-1)\)-curve, we have \( \chi(X_0) = \chi(X) \). Suppose \( \chi(X_{i+1}) > \chi(X_i) \). Since \( B_i \) is snc-minimal, \( \ell_i \) meets two connected components of \( B_i \), each transversally in a unique point, and together with these components contracts to a smooth point on \( X_{i+1} \). Suppose \( U \neq 0 \). By Claim 5 these two connected components do not contain the image of \( D_3 \), so they contain the images of \( D_1 \) and \( D_2 \). Thus \( D_1 + \ell + D_2 \) is contained in a divisor of quotient type disjoint from \( D_3 \), which contradicts Claim 1. Thus \( U = 0 \) and Claim 1 implies that the connected components met by \( \ell \) do not contain the image of \( D_1 \), hence contain images of \( D_2 \) and \( E \). Let \( X \to \tilde{X} \) be the contraction of \( D_2 \) and \( E \). We have \( -1 = d(D_1) \cdot d(C + D_2) - d(D_2) \), so since \( d(D_2) \geq 2 \), we see that \( d(D_1) \) and \( d(D_2) \) are coprime. In particular, \( d(D_1) \neq 0 \). Then the components of \( D_1 \) generate \( H_2(\tilde{X}, \mathbb{Q}) \). Using Nakai’s criterion we show easily that \( D \) supports an ample divisor (see [Fuj82, 2.4]), so \( \tilde{X} \setminus D_1 \) is affine. But it contains the image of \( \ell \), which is projective; a contradiction.

Claim 7. \( \kappa(X) = -\infty \).

Proof. Suppose \( \kappa(X) \geq 0 \). Then \( \kappa(X_n) = \kappa(X) \geq 0 \), so since \( (X_n, B_n) \) is almost minimal, the log BMY inequality (see [Lan03, 3.4, §9] and [Pal11, 2.5]) gives

\[
0 \leq \frac{1}{3}((K_{X_n} + B_n)^{+})^2 \leq \chi(X_n) + \frac{1}{2} \# \text{qt}(B_n).
\]

The divisor \( B_n \) is snc-minimal, so each \( \Gamma(T) \) is nontrivial, hence \( 0 \leq \chi(X_n) + \frac{1}{2} \# \text{qt}(B_n) \leq \chi(X) + \frac{1}{2} \# \text{qt}(B_n) \). If \( B_n \) has more than two connected components of quotient type then \( D_1 \), \( D_2 \) and \( D_3 \) are contained in disjoint divisors of quotient type, which is impossible by Claim 1 (take \( \ell = E \)). It follows that all the above inequalities become equalities, so \( B_n \) has exactly three connected components, two of them are of quotient type with \( |G_i| = 2 \) and \( \chi(X_n) = \chi(X) = -1 \). It follows that two connected components of \( B_n \) are \((-2)\)-curves and \( \chi(X_i) = \chi(X_{i+1}) \) for every \( i \). By Proposition \( 5 \) \( n = 0 \), i.e. \( (X_0, B_0) \) is almost minimal.

If \( U \neq 0 \) then by Claim 5, \( D_1 \) and \( D_2 \) are of type \([2]\). But as we have seen in the proof of Claim 6, \( d(D_1) \) and \( d(D_2) \) are coprime. Thus \( U = 0 \). By Claim 1 \( E + D_1 \) is not negative definite, so the only possibility is that \( D_2 \) and \( E \) are \((-2)\)-curves and \( D_1 \) is not negative definite. By Claim 3 \( C^2 = -1 \). Consider the \( \mathbb{P}^1 \)-fibration of \( X \) induced by the linear system of \( D_2 + 2C + E \). We have \( h = 1 \) hence \( 0 \leq \Sigma_X = \nu - 1 \). Since \( D \) contains no loop, the 2-section contained in \( D_1 \) meets \( F_\infty \) in one point. Since \( D \) is snc-minimal we infer that \( F_\infty \) is of type \([2,1,2]\). Denote the middle \((-1)\)-curve by \( L \). When we snc-minimize \( D_1 + C + D_2 \) starting from the contraction of \( C \) and \( D_2 \) we do not touch \( F_\infty - L \). By Lemma \([4]\) the result of this minimalization is of type \([2,a,2]\) for some \( a \leq 0 \). However, the discriminant of \([2,a,2]\) is even, hence a chain of this type cannot be a boundary of \( \mathbb{C}^2 \); a contradiction.

Claim 8. \( X \) has a \( \mathbb{C}^1 \)-fibration.

Proof. Suppose \( X \) has a \( \mathbb{P}^1 \)-fibration. It extends to a \( \mathbb{P}^1 \)-fibration of \( \overline{X} \). Then \( D_3 \) is vertical, so it cannot contain a branching \((-1)\)-curve. It follows that \( U = 0 \). We have now \( \Sigma_X = \nu - 2 \), so there are at least 2 fibers contained in \( B \). It follows that \( D_1 \) is a fiber, hence \( d(D_1) = 0 \) and \( d(D_1) \) and \( d(D_2) \) are not coprime; a contradiction. Thus \( X \) has no \( \mathbb{P}^1 \)-fibration. Suppose it also has no \( \mathbb{C}^1 \)-fibration. Because \( \kappa(X) = -\infty \), the structure theorems for smooth surfaces of negative logarithmic Kodaira dimension imply
that \((\overline{X}_n, B_n)\) is a minimal log-resolution of a log del Pezzo surface \([\text{Miy01}, 2.3.15]\).
Moreover, since not all connected components of \(B\) are of quotient type, this log del Pezzo is open, hence has a structure of a Platonic \(\mathbb{C}^*\)-fibration by \([\text{MTS1}]\). In particular, \(\chi(X_n) = 0\). Then \(\chi(X_n) > \chi(X) = -1\), which contradicts Claim 6.

**Claim 9.** \(X\) has a \(\mathbb{C}^1\)-fibration onto \(\mathbb{C}^1\) with irreducible fibers.

**Proof.** Let \(\tilde{\pi}: (\tilde{X}, \tilde{B}) \rightarrow (\overline{X}_0, B_0)\) be a minimal modification of \((\overline{X}_0, B_0)\) such that the above \(\mathbb{C}^1\)-fibration can be written as \(r|_X\), where \(r: \tilde{X} \rightarrow \mathbb{P}^1\) is a \(\mathbb{P}^1\)-fibration. Because the base point of \(r: \overline{X}_0 \rightarrow \mathbb{P}^1\) (if exists) belongs to \(B_0\), we have \(\rho(\tilde{X}) = \rho(\overline{X}_0) - \#B_0 = 0\), hence \(h + \nu = 2 + \Sigma_X\) by \([1,2]\).
But because \(r|_X\) is a \(\mathbb{C}^1\)-fibration, \(h = 1\), so \(\nu \geq 1\), i.e. \(\tilde{B}\) contains a fiber \(F_\infty\) of \(r\). Suppose there is more than one fiber contained in \(\tilde{B}\). Since the reduced total transform of \(D\), \(\tilde{D} = \tilde{B} + C\), contains no loop, \(C\) is vertical. In particular, \(C^2 \leq 0\). But \(C\) is a branching component of \(D\), since \(h = 1\), it cannot be a fiber. Thus \(C\) is a \((-1)\)-curve, and hence it is a non-branching component of a fiber containing it. But \(C\) is branching in \(\tilde{D}\), so it meets exactly two other vertical components of \(\tilde{D}\) and the section contained in \(\tilde{D}\). However, the former implies that its multiplicity in the fiber is at least two and the latter implies that its multiplicity is one; a contradiction. Thus \(\nu = 1\) and hence \(\Sigma_X = 0\), so \(r(X) \cong \mathbb{C}^1\) and \(r|_X\) has irreducible fibers. \(\square\)

**Claim 10.** The \(\mathbb{C}^1\)-fibration of \(X\) has no base points on \(\overline{X}_0\).

**Proof.** Denote the unique fiber and the unique section of \(r\) contained in \(\tilde{B}\) respectively by \(F_\infty\) and \(H\). The divisor \(B_0\) is snc-minimal and, since \(D_3 \neq [1]\) and \(D_1 \neq 0\), it has three connected components. Let \(T_1, T_2, T_3\) be the connected components of \(B\), say \(T_3\) contains \(H\). Then \(T_3\) contains \(F_\infty\) and the divisors \(T_1\) and \(T_2\) are vertical and snc-minimal. After snc-minimalizing \(T_3 - H\) if necessary we may assume \(\tilde{B} - H\) contains only branching \((-1)\)-curves. But then arguing as in the proof of the previous claim we see that in fact \(T_3 - H\) contains no \((-1)\)-curves at all. Let \(F\) be a singular fiber other than \(F_\infty\). Since \(\Sigma_X = 0\), we infer that \(F\) contains a unique \((-1)\)-curve \(L_F\). By Lemma \([3]\) \(F - L_F\) has at most two connected components and one of them meets \(H\). Since \(T_1, T_2\) and \(T_3\) are disjoint, there are at least two singular fibers other than \(F_\infty\). It follows that \(H\) is a branching component of \(\tilde{B}\). But then \(\tilde{\pi} = \text{id}\), i.e. the \(\mathbb{C}^1\)-fibration of \(X\) is a restriction of a \(\mathbb{P}^1\)-fibration of \(\overline{X}_0\). \(\square\)

Since \(B_0\) is snc-minimal, \(B_0 - H\) contains no \((-1)\)-curves, so \(F_\infty\) is a 0-curve. It remains to prove that \(F_\infty = E\). Since \(C\) is a branching component of \(\epsilon_*D\) with \(C^2 \geq -1\), it follows that it is horizontal. Then \(F_\infty\) meets \(C\). In particular, \(F_\infty\) is a tip of \(B_0\) and \(C\) is a section. If \(F_\infty \subseteq D_1\) then the snc-minimalization of \(D_1 + C + D_2\) does not contract \(F_\infty\), hence by Claim 3 leads to an snc-minimal boundary of \(\mathbb{C}^2\) which is not a chain. But the latter is impossible by Lemma \([4]\). Since \(D_2\) is negative definite, we get \(F_\infty \subseteq \epsilon_*D_3\). Since \(U_i\) is branching in \(D_3\), \(\epsilon_*U_i\) is branching in \(\epsilon_*U_i\), which implies that \(F_\infty\) is not one of the \(\epsilon_*U_i\)’s. Therefore, \(\epsilon = \text{id}\) and hence \(F_\infty = E\). \(\square\)

We now show how Theorem A follows from Theorem \([1]\)

**Proof of Theorem A.** Let \((\overline{X}, D)\) and \(r: \overline{X} \rightarrow D\) be respectively a minimal smooth completion of \(\mathbb{C}^2 \setminus A\) and a \(\mathbb{P}^1\)-fibration as in Theorem \([1]\). Denote the proper transform of \(A\) on \(\overline{X}\) by \(E\). Since \((\overline{X}, D - E)\) is a smooth completion of \(\mathbb{C}^2 \setminus \text{Sing } A\), \(D - E\) has a unique connected component \(D_\infty\) which is a rational tree with non-negative definite intersection
matrix, and such that $U = D - E - D_\infty$ consists of $\# \text{Sing} A$ connected components contractible to smooth points (of $\mathbb{C}^2$). In particular, connected components of $U$ are negative definite rational trees. By the minimality of $(\mathcal{X}, D)$ and by the analytical irreducibility of the singularities of $A$, each such tree contains a unique $(-1)$-curve $U_i$ and $E$ meets $U$ exactly in $U_i$’s, each once and transversally. Since the analytic branch of $A$ at infinity (considered, say, in $\mathbb{P}^2$) is irreducible, there is a unique component $C$ of $D_\infty$ meeting $E$. We obtain that $D$ is a rational tree with $\rho(\mathcal{X}) + 1$ irreducible components. Note $D$ has $h = \# \text{Sing} A + 1$ horizontal components. We have $\Sigma_{\mathcal{X}/D} = 0$, so by (1.2) $h = 3 - \nu \leq 2$, so $A$ has at most one singular point. Clearly, $C$ is a horizontal component of $D$ and $D_\infty - C$ has at most two connected components, call them $D_1$ and $D_2$. If $A$ is singular ($U \neq 0$) then $U_1$, the $(-1)$-curve of $U$ meeting $E$, is the second horizontal component of $D$. They are both sections of $r$. Because $(\mathcal{X}, D)$ is minimal, $D - C - U_1$ contains no $(-1)$-curves. Indeed, such a curve would be a non-branching component of a fiber and, since the horizontal components of $D$ are sections, also a non-branching $(-1)$-curve in $D$.

Up to this point we just reproved for $(\mathcal{X}, D)$ what could be obtained by taking the special minimal completion of $\mathbb{C}^2 \setminus A$ as defined in section 1. Suppose $U = 0$. Then $h = 1$, so $\nu = 2$, i.e. there is a unique fiber of $r$ contained in $D_1 + D_2$. Since $D-C$ contains no $(-1)$-curves, the fiber is a 0-curve. It follows that, say, $D_1$ is a 0-curve. Making an elementary transformation on $D_1$ we may assume $C^2 = -1$. The snc-minimization of $D_\infty$ does not contract $D_1$, which by Lemma 4 implies that $D_2$ is a chain (negative definite or empty). Since $D_\infty$ is a boundary of $\mathbb{C}^2$, (1.1) gives $-1 = d(D_1) \cdot (d(C + D_2) - d(D_2)) = -d(D_2)$, so $d(D_2) = 1$. Thus $D_2 = 0$ and hence $\mathcal{X}$ is a Hirzebruch surface. The contraction of $C$ maps it to $\mathbb{P}^2$ and $C + D_1$ into a pair of lines, so we are done.

We may therefore assume that $U \neq 0$. Then $A$ has a unique singular point. Let $F$ be a singular fiber of $r$. Its unique component $L_F$ not contained in $D$ is also the unique $(-1)$-curve in $F$. Now $C$ and $U_1$ are sections of $r$, so they meet components of $F$ of multiplicity one. By Lemma 3 $F - L_F$ has at most two connected components. It follows that $F$ is a chain. Indeed, otherwise only one connected component of $F - L_F$ contains components of multiplicity 1, which would imply that $D$ contains a loop. Thus, every singular fiber of $r$ is a chain with a unique $(-1)$-curve. Also, $F - L_F$ has exactly two connected components, both contained in $D$. Each such chain contains exactly two components of multiplicity one, which are tips of the chain. Since $U$ can be contracted to a point by iterating contractions of $(-1)$-curves, it follows that $U_1$ meets exactly two components of $U - U_1$, and hence $U_1 \cdot (D - U_1) = 3$. Thus $r$ has exactly two singular fibers, $F_1$ and $F_2$. Let $L_i$ be the unique $(-1)$-curve of $F_i$. We have $D - E = D_\infty + U$ and we can write $U - U_1 = V_1 + V_2$ and $D_\infty - C = D_1 + D_2$, so that $D_i$ and $V_i$ are connected and $F_i = V_i + L_i + D_i$. Put $Y = \mathcal{X} \setminus D_\infty$. The morphism $\pi_{|Y}: (Y, U + E \setminus \{\infty\}) \to (\mathbb{C}^2, A)$ is a log resolution of singularities. The curves $\pi(L_1 \cap Y)$ and $\pi(L_2 \cap Y)$ are isomorphic to $\mathbb{C}^1$ and meet in one point, transversally.

We claim there exist coordinates $\{x_1, x_2\}$ on $\mathbb{C}^2$ such that $\pi(L_i \cap Y)$ is given by $x_i = 0$. To see this first contract $U$. The images of $L_1$ and $L_2$ are smooth, meet transversally and have non-negative self-intersections. Now blow up on $(L_1 + L_2) \cap D_\infty$, so that the proper transforms of $L_1$ and $L_2$ are again $(-1)$-curves and denote the resulting projective surface by $\tilde{X}$, the total reduced transform of $D_\infty$ by $\tilde{D}_\infty$ and the proper transforms of $L_i$ by $\tilde{L}_i$. By construction, $\tilde{D}_\infty$ is a chain met by $\tilde{L}_1$ in a tip $W_1$ which is a $(-1)$-curve. Also, $(\tilde{X}, \tilde{D}_\infty)$ is a smooth completion of $\mathbb{C}^2$. For the $\mathbb{P}^1$-fibration $\tilde{X} \to \mathbb{P}^1$ induced by the linear system of $\tilde{L}_1 + \tilde{L}_2$ we have $\Sigma_{\mathbb{C}^2} \geq 1$ and $h = 2$, so by (1.2) $\nu = \Sigma_{\mathbb{C}^2} \geq 1$, i.e. $\tilde{D}_\infty$ contains.
a fiber $F$. Because $D_\infty - W_1 - W_2$ is connected and does not meet $\hat{L}_1 + \hat{L}_2$, it follows that $F = D_\infty - W_1 - W_2$. Let $W_3$ and $W_4$ be the (different) components of $F$ meeting $W_1$ and $W_2$ respectively. Contracting successively $(-1)$-curves in $F - W_3 - W_4$ if necessary we may assume there are no $(-1)$-curves in $F - W_3 - W_4$. Because $W_3$ and $W_4$ meet sections, they have multiplicity 1, so then $F$ is necessarily of type $[1, 2, \ldots, 2, 1]$, where the subchain of $(-2)$-curves has length $s \geq 0$. Since $W_1 + F + W_2$ is a boundary of $\mathbb{C}^2$, its discriminant is $-1$, which gives $s = 0$. Blowing up once on $W_1 + W_2$ we may assume $F$ is of type $[1, 2, 1]$. Then the contraction of $W_1 + W_2 + W_3 + W_4$ maps the completion of $\mathbb{C}^2$ onto $\mathbb{P}^2$ and $L_1 + L_2$ onto a pair of lines meeting transversally. This gives the coordinates \( \{x_1, x_2\} \).

Put $n = d(V_1)$ and $m = d(V_2)$. The morphism $r_Y : Y \to \mathbb{P}^1$ is a $\mathbb{C}^1$-fibration and $x_1^n/x_2^m$ is a coordinate on $\mathbb{C}^1$. Since $E \setminus \{\infty\}$ is a fiber of $r_Y$, it has equation $x_1^n/x_2^m = \alpha$ for some $\alpha \in \mathbb{C}^*$ and we may assume $\alpha = 1$. Then $A$ has equation $x_1^n = x_2^m$. \hfill \Box

3. Another proof of the Abhyankar-Moh-Suzuki theorem

We now give another, independent of section 2 proof of Theorem B. We need the following lemma. In case of contractible surfaces it was obtained by similar methods by Gurjar and Miyanishi [GM98].

**Lemma 6.** [Pal13, 3.1(vii)] Let $X \to X'$ be a log resolution of a rational $\mathbb{Q}$-acyclic normal surface, let $\hat{E}$ be the reduced exceptional divisor and $(\overline{X}, D)$ a smooth completion of $X$. If $\hat{E} + D$ is a sum of rational trees then

$$|d(D)| = d(\hat{E}) \cdot |H_1(X', \mathbb{Z})|^2.$$ 

**Proof.** Let $M_D$ and $M$ be the boundaries of closures of tubular neighborhoods of $D$ and $\hat{E}$. We may assume that $M_D$ and $M$ are disjoint oriented 3-manifolds. Since $\hat{E}$ is a sum of rational trees, $H_1(\hat{E}, \mathbb{Q}) = 0$. By the $\mathbb{Q}$-acyclicity of $X'$ the components of $D + \hat{E}$ freely generate $H_2(\overline{X}, \mathbb{Q})$, so $d(D + \hat{E}) \neq 0$. By [Mum61] $H_1(M_D, \mathbb{Z})$ and $H_1(M, \mathbb{Z})$ are finite groups of orders respectively $|d(D)|$ and $d(\hat{E})$. By the Poincare duality $H_2(M_D, \mathbb{Z})$ and $H_2(M, \mathbb{Z})$ are trivial. Let $K = \overline{X} \setminus (\text{Tub}(D) \cup \text{Tub}(\hat{E}))$. By the Lefschetz duality $H_i(K, M_D) \cong H^{4-i}(K, M) = H^{4-i}(X', \text{Sing } X')$, which for $i > 1$ implies that $H_i(K, M_D) \cong H^{4-i}(X') \cong H_{3-i}(X')$ by the universal coefficient formula. Thus the reduced homology exact sequence of the pair $(K, M_D)$ with $\mathbb{Z}$-coefficients gives:

$$0 \to H_2(K) \to H_1(X') \to H_1(M_D) \to H_1(K) \to H_2(X') \to 0.$$ 

On the other hand, since $H_i(K, M) \cong H_i(X', \text{Sing } X')$ and $H_1(X', \text{Sing } X') = H_1(X') \oplus \hat{H}_0(\text{Sing } X')$, the reduced homology exact sequence of the pair $(K, M)$ gives:

$$0 \to H_2(K) \to H_2(X') \to H_1(M) \to H_1(K) \to H_1(X') \to 0.$$ 

From the two exact sequences we obtain $|H_2(K)| \cdot |d(D)| \cdot |H_2(X')| = |H_1(X')| \cdot |H_1(K)|$ and $|H_2(K)| \cdot d(\hat{E}) \cdot |H_1(X')| = |H_2(X')| \cdot |H_1(K)|$, hence

$$|d(D)| \cdot |H_2(X', \mathbb{Z})|^2 = d(\hat{E}) \cdot |H_1(X', \mathbb{Z})|^2.$$ 

Because a rational $\mathbb{Q}$-acyclic surface is necessarily affine by an argument of Fujita ([Fuj82, 2.4]), $X$ is an affine variety, and hence has a structure of a CW-complex of real dimension 2. It follows that $H_2(X', \mathbb{Z})$ is torsionless, hence $H_2(X', \mathbb{Z}) = 0$. \hfill \Box
Assume $A \subseteq \mathbb{C}^2$ is a smooth contractible planar curve. Let $(\overline{X}, D)$ be a completion of $\mathbb{C}^2 \setminus A$ defined in section 2. We have $U = 0$, so $D_3 = E$. Assume that $D_1 \neq 0$. Let $D_C$ be the component of $D_1$ meeting $C$. Because $D_1 + C + D_2$ is a boundary of $\mathbb{C}^2$, its discriminant is $-1$. By (1.1)

$$-1 = d(D_1 + C + D_2) = d(D_1) \cdot d(C + D_2) - d(D_1 - D_C) \cdot d(D_2),$$

which gives $\text{gcd}(d(D_1), d(D_2)) = 1$. Since $d(D_2) \geq 2$, we infer that $d(D_1) \neq 0$.

Suppose $E^2 > 0$. By blowing up on $E \setminus C$ we may replace it with a chain $\hat{E} + H + D_4$, where $\hat{E}$, the proper transform of $E$, is a 0-curve, $H$ is a $(-1)$-curve and $D_4$ is a chain of $(-2)$-curves of length $E^2 - 1$. Now $\hat{E}$ induces a $\mathbb{P}^1$-fibration of the constructed projective surface, such that $H$ is the unique horizontal component of $\hat{B} = D_1 + D_2 + \hat{E} + H + D_4$. Since $H$ and $D_1 + D_2$ are disjoint, $\hat{E}$ is the unique fiber contained completely in $\hat{B}$. Then (1.2) gives $\Sigma_X = 0$. It follows that every singular fiber $F$ contains a unique component $L_F$ not contained in $\hat{B}$ and this component is a unique $(-1)$-curve of $F$. Since $H$ is a section of the fibration, by Lemma 3 $L_F \cdot H = 0$ and $F - L_F$ has at most two connected components. One of these components (the one meeting $H$, necessarily non-empty) is contained in $D_4$. But $D_4$ is connected, so we see that there is at most one singular fiber. Then $D_1 + D_2$ is contained in this fiber, hence it is connected, so $D_1 = 0$, in contradiction to the assumption.

Suppose $E^2 < 0$. Then $d(B) = d(D_1)\cdot d(D_2)\cdot d(E) \neq 0$, so the components of $B$ are independent in $H_2(\overline{X}, \mathbb{Q})$, hence they generate freely the latter space. Let $X \to X'$ be the contraction of $E$ and $D_2$. We check using Lefschetz duality and standard exact sequences that $X'$ is $\mathbb{Q}$-acyclic. Applying Lemma 6 to $(\overline{X}, D_1)$ and $\hat{E} = D_2 + E$ we get that $d(D_2 + E)$ divides $d(D_1)$. But $d(D_2 + E) = d(D_2) \cdot d(E)$, so $d(D_2)$ divides $d(D_1)$: a contradiction.

Thus $E$ is a 0-curve. Then $D_1$ and $D_2$ are vertical for the $\mathbb{P}^1$-fibration of $\overline{X}$ induced by the linear system of $E$. Since $D_2$ is negative definite, we have $\nu \leq 2$. By (1.2), $\nu = 2 + \Sigma_X$, so $\nu = 2$. This means that $D_1$, being snc-minimal, is a 0-curve, so $d(D_1) = 0$; a contradiction.

Therefore, we proved that $D_1 = 0$. As in the Claim 1 in the previous section we argue that $C^2 = 0$ and $D_2$ is irreducible. Then $\rho(\overline{X}) = 2$, so $\overline{X}$ is a Hirzebruch surface. Again, after making some elementary transformation we may assume that $D_2^2 = -1$. Then the contraction of $D_2$ maps $\overline{X}$ onto $\mathbb{P}^2$ and $E$ onto a line. Theorem B follows.

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Karol Palka: INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES, UL. ŚNIADECKICH 8, 00-656 WARSAW, POLAND

E-mail address: palka@impan.pl