THE MOONSHINE ANOMALY

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Abstract. By definition, the anomaly for the Monster group \(M\) acting on its natural (aka moonshine) representation \(V^\natural\) is a particular cohomology class \(\omega^\natural \in H^3(M, U(1))\) that arises as a conformal field theoretic generalization of the second Chern class of a representation. We show in this paper that \(\omega^\natural\) has order exactly 24 and is not a Chern class. In order to perform this computation, we introduce a finite-group version of T-duality, which we use to relate \(\omega^\natural\) to the anomaly for the Leech lattice CFT.

1. Introduction

Whenever a finite group \(G\) acts on a holomorphic conformal field theory \(V\), there is a corresponding anomaly \(\omega_V \in H^3(G, U(1))\) that measures the obstruction to gauging (also called orbifolding) the \(G\)-action. We will review the construction of \(\omega_V\) in Section 2: Section 2.1 describes the physical picture; Section 2.2 discusses mathematical foundations; Section 2.3 provides a method to calculate anomalies; and Section 2.4 addresses the fermionic version. Anomalies can be thought of as a type of “characteristic class” for the action of a group not on a vector space but on a conformal field theory, and have played an increasingly important role in recent work on moonshine-type phenomena [Gan09, Tho10, GPRV13, CilLW16]. The most famous example of a finite group acting on a conformal field theory is certainly the Monster group \(M\) acting on its natural “moonshine” representation \(V^\natural\). The main result of this paper is a calculation of the corresponding moonshine anomaly \(\omega^\natural \in H^3(M, U(1))\), showing in particular that it does not vanish:

**Theorem 1.** The order of \(\omega^\natural \in H^3(M, U(1))\) is exactly 24. Although Chern and fractional Pontryagin classes can arise as anomalies, \(\omega^\natural\) is neither a Chern nor a fractional Pontryagin class of any representation of \(M\).

The proof of Theorem 1 occupies most of Section 3. Specifically, in Section 3.1 we check that \(H^3(M, U(1))\) has no elements with order divisible by the primes \(p = 11\) or \(p \geq 17\); Section 3.2 shows that the order of \(\omega^\natural\) is not divisible by the primes \(p = 5, 7, 13\), and is divisible by 3 but not 9; and we show that 8 but not 16 divides the order of \(\omega^\natural\) in Section 3.3. Section 3.4 proves that \(\omega^\natural\) is not a Chern class. Section 3.5 contains some further remarks about the value of \(\omega^\natural\).

A main step in the proof is the following result due jointly to the author and D. Treumann. We let \(\text{Co}_0 = 2, \text{Co}_1\) denote Conway’s largest group, and \(\Lambda\) the Leech lattice.

**Theorem 2 ([JFT17]).** \(H^3(\text{Co}_0, U(1))\) is cyclic of order 24, generated by \(\frac{1}{24} (\Lambda \otimes R)\), the first fractional Pontryagin class of the 24-dimensional defining representation of \(\text{Co}_0\). \(\square\)

As explained in Example 2.4.1, the class \(\frac{1}{24} (\Lambda \otimes R) \in H^3(\text{Co}_0, U(1))\) is also the anomaly for the action of \(\text{Co}_0\) on the “super moonshine” module Fer(24)//\(\mathbb{Z}_2\) of [Dun07, DMC15], and Pontryagin and Chern classes of finite-dimensional representations of a group \(G\) can always be realized as the anomalies of actions of \(G\) on free fermion models; moreover, by Example 2.1.1, the anomalies of permutation orbifolds are always Pontryagin classes. There is an a priori upper bound on the orders of Pontryagin and Chern classes of representations that are (like \(\Lambda\)) defined over \(\mathbb{Z}\), coming from the fact that the fourth integral group cohomology of \(\text{GL}(N, \mathbb{Z})\) for \(N \gg 0\) is \(\mathbb{Z}_{24} \oplus \mathbb{Z}_2^3\) [Arl84]. Generalizing from characteristic classes to CFT anomalies suggests:

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Conjecture 1. Let $V$ be a holomorphic conformal field theory and $G \subset \text{Aut}(V)$ a finite group of automorphisms such that both $V$ and the action of $G$ are defined over $\mathbb{Z}$. Then the corresponding anomaly $\omega_V \in H^3(G, U(1))$ has order dividing 24.

We will not try to make precise in this paper what it should mean to say that a conformal field theory is “defined over $\mathbb{Z}$.” Almost surely the correct notion is that of a vertex operator algebra equipped with an integral form as in [DG12]. However, for technical reasons we do not model conformal field theories in this paper as vertex operator algebras, using instead conformal nets of von Neumann algebras (see Section 2.2), and we do not claim to know what a “conformal net over $\mathbb{Z}$” should be.

As discussed in [HC11, Gan16], one can predict the order of the moonshine anomaly $\omega^5$ using vertex operator algebraic methods familiar to moonshine theorists. We do not follow that route: the calculations in this paper depend instead on methods native to the cohomology of finite groups (e.g. maximal subgroups, spectral sequences) and to topological field theory (e.g. fusion categories). The inputs we use from conformal field theory are the existence of $V^\natural$, and hence of $\omega^5$, and its relationship to the Leech lattice CFT $V_\Lambda$. Specifically, there is a well-known [CN79] agreement between centralizers of certain elements of $\mathcal{M}$ and of certain elements of $\text{Aut}(V_{\Lambda}) = \text{hom}(\Lambda, U(1)).\text{Co}_0$, which we will understand as an example of finite group T-duality introduced in Section 2.3. T-duality allows information about anomalies to be moved between orbifold-equivalent conformal field theories. We use it extensively in the proof of Theorem 1.

Our proof of Theorem 1 avoids computing much about the group $H^3(\mathcal{M}, U(1))$. Nevertheless, it is very tempting to speculate the following analog of Theorem 2:

Conjecture 2. $H^3(\mathcal{M}, U(1)) \cong \mathbb{Z}_{24}$.

We address this conjecture in Section 3.5 without providing much evidence to support it. In Section 3.4, we show that Conjecture 2 implies:

Conjecture 3. For every complex representation $V$ of $\mathcal{M}$, $c_2(V) = 0 \in H^3(\mathcal{M}, U(1))$.

1.1. Notation and conventions. We mostly follow the ATLAS [CCN+85] for notation for finite groups. For example, when referring to a group, “$n$” denotes the cyclic group of order $n$. We also call this group $\mathbb{Z}_n$, which is its standard name in physics texts, and $\mathbb{F}_n$ when $n$ is prime and we are thinking of it as a field. (Mathematicians sometimes use $\mathbb{Z}_n$ to denote instead the ring of $n$-adic integers.) Elementary abelian groups are denoted $n^k$ and extraspecial groups $n^{1+k}$. An extension with normal subgroup $N$ and cokernel $G$ is denoted $N.G$ or occasionally $NG$; an extension which is known to split is written $N : G$. The conjugacy classes of elements of order $n$ in a group $G$ are named $nA$, $nB$, . . . , ordered by increasing size of the class (decreasing size of the centralizer).

The exponent of a finite group $G$ is the smallest $n$ such that $g^n = 1$ for all $g \in G$. If $G$ is a finite group and $A$ a module thereof, we write $H^\bullet(G, A)$ for the group cohomology of $G$ with coefficients in $A$; this should cause no confusion, since the cohomology of $G$-as-a-space is trivial. When $G$ is a Lie group, we write $H^\bullet(BG)$ to emphasize that the cohomology we consider depends just on the classifying space $BG$ of $G$. We primarily use $U(1)$-coefficients instead of the more mathematically-common $\mathbb{Z}$-coefficients because the former is more physically meaningful. Of course for a finite group $H^\bullet(G, U(1)) \cong H^{\bullet+1}(G, \mathbb{Z})$ is finite of exponent dividing the exponent of $G$.

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2. Orbifolds of conformal field theories

In this section, we first review the basic theory of orbifolds of conformal field theories and the corresponding anomalies, beginning with the physical description in Section 2.1 and then mentioning the appropriate mathematical details in Section 2.2. Although this story is well-known to experts (see for example [DW90, FFRS10]), it bears repeating since the intuition involved is often left out of mathematics papers, leading for example to confusion about what construction deserves the name “the orbifold.” We then introduce a “T-duality” for finite groups in Section 2.3. In Section 2.4 we discuss anomalies and T-duality when fermions are present.

2.1. Physical picture. Before focusing on a precise mathematical model, some physics. Any quantum field theory has an “algebra” $A$ of local observables. We put “algebra” in quotes because different axiomatizations involve different types of algebraic operations. In many situations — when the quantum field theory does not have too much gauge symmetry — the algebra $A$ encodes all data of the field theory. In the special case of (two-dimensional) conformal field theory, $A$ has two subalgebras $V$ and $W$ consisting respectively of the chiral and antichiral observables. The conformal field theory is heterotic when $V$ and $W$ are different.

Just as every harmonic function is a sum of holomorphic and antiholomorphic pieces, so too is $A$ generated by its subalgebras $V$ and $W$, but just as there’s a constant ambiguity in the choice of splitting of a function, so too are there relations between $V$ and $W$ in $A$. To describe these relations involves moving briefly into three dimensions. The algebras $V$ and $W$ are each algebras of local observables, but they do not typically define full quantum field theories the way $A$ does. Instead, each of $V$ and $W$ describes a (chiral or antichiral) boundary condition for a three-dimensional quantum field theory, which we will denote $Z(V)$ or $Z(W)$. Often this three-dimensional theory is topological; this happens in particular when the conformal field theory is rational, in which case the 3D theory is the Reshetikhin–Turaev theory corresponding to the modular category of vertex modules for the (anti)chiral algebra. When $V$ and $W$ are the chiral and antichiral halves of a full conformal field theory $A$, their 3D bulk theories agree $Z(V) \cong Z(W)$, and $A$ can be recovered as a sandwich of its two chiral halves separated by their common bulk.

$$Z(V) \cong Z(W)$$

A special case occurs when $W$ is trivial and $A = V$; this can occur only when the 3D bulk theory for $V$ is trivial. Such a conformal field theory is called holomorphic for obvious reasons.

Suppose now that the full quantum field theory $A$ is acted on by a group $G \subset \text{Aut}(A)$, which we assume finite for convenience. This is equivalent to an extension of the theory $A$ defined in the presence of a background $G$ principal bundle. One may try to orbifold (also called gauge) the symmetry, producing a new full quantum field theory $A//G$. The path integral description of gauging consists of choosing an integration measure for which one can integrate out the choice
of gauge field. But one might fail to be able to construct a consistent integration measure. The obstruction to doing so is called the anomaly of the action. Indeed, the action of $G$ on $A$ produces an action of $G$ on the 3D bulk theory $Z(A)$ determined by $A$. Although $Z(A)$ is trivial (since $A$ is a full theory; when calling a field theory “trivial,” we will allow gravitational anomalies), the action of $G$ on it might not be. Only when the action of $G$ on $Z(A)$ is trivial can the orbifold $A//G$ be defined. The actions of a finite group $G$ on the trivial 3D bosonic oriented field theory are indexed by $H^3(G, U(1))$, and so the anomaly of $G$ action on $A$ is the corresponding class $\omega \in H^3(G, U(1))$. In arbitrary dimension, one finds the relation between gauge anomalies and symmetry protected topological phases from [Wen13].

One can describe the orbifold $A//G$ by a sandwich construction similar to the description of $A$ in terms of its chiral halves. Consider the $G$-fixed subalgebra $A^G \subset A$; this is most common when $A = V$ is holomorphic, in which case this subalgebra is called its chiral orbifold. (Continuing as above, we will conflate the quantum field theory $A$ with its algebra of local observables, and assume that the latter determines the former.) Like the chiral halves $V$ and $W$, the algebra $A^G$ does not determine a full conformal field theory. Rather, it lives at the boundary of a nontrivial 3D theory $Z(A^G)$. This, in turn, is automatically the result of gauging the trivial theory $Z(A)$ by the $G$ symmetry — this produces the Dijkgraaf–Witten theory corresponding to the class $\alpha$, with Wilson lines forming the modular tensor category $Z(Vect^\omega[G])$. Suppose now that one chooses a trivialization of $\omega$. Such a choice determines a special topological boundary condition for $Z(A^G) = Z(Vect^\omega[G])$, called the Neumann boundary condition, whose Wilson lines form the fusion category $Rep(G)$; the Neumann boundary exists because of the equivalence $Z(Vect[G]) \cong Z(Rep(G))$. The full orbifold is by definition the sandwich with $Z(A^G) = Z(Vect[G])$ in the bulk, the conformal boundary condition $A^G$ on one side, and the topological Neumann boundary condition on the other.

For any value of the anomaly $\omega$, the bulk theory $Z(A^G) = Z(Vect^\omega[G])$ admits a Dirichlet boundary condition with Wilson lines $Vect^\omega[G]$, and a similar sandwich of $A^G$ with Dirichlet boundary produces the unorbifolded field theory $A$. When $\omega = 0$, there is a Morita equivalence of fusion categories $Vect[G] \simeq Rep(G)$, which is to say an invertible topological defect separating the Dirichlet and Neumann boundary conditions for $Z(Vect[G]) \cong Z(Rep(G))$. By using this defect in a sandwich with $A^G$, one produces an invertible topological defect between $A$ and $A//G$:

\[
Z(A^G) \cong Z(Vect[G]) \cong Z(Rep(G))
\]

The description of $A^G$ as a conformal boundary condition for $Z(Vect^\omega[G])$ explains the relationship between anomalies and the multipliers for twisted-twining genera [GPRV13, Section 3]. Indeed, following [Gan09], let $M = M_1$ denote the moduli stack of elliptic curves and let $M_G$ denote the moduli stack of elliptic curves equipped with a principal $G$-bundle, and let $L_\omega$ the line-bundle thereon constructed from $\omega$. The Hilbert space that the Dijkgraaf–Witten theory $Z(Vect^\omega[G])$ assigns to an elliptic curve $E$ is the fiber over $E \in M$ of the pushforward of $L_\omega$ along $M_G \to M$. The “twisted-twining genera” are the (genus-one) conformal blocks of $A^G$; abstract nonsense of boundary field theories says that they are precisely sections of $L_\omega$. The moduli space $M_G$ breaks up into components: for a generic $G$-bundle on $E$, the monodromy map $\mathbb{Z}^2 \cong \pi_1 E \to G$ has image

\[
\xymatrix{ A \ar@{..}[r] & A//G \\
A^G \ar@{..}[r] \ar@{..}[ur] & \quad Z(A^G) \cong Z(Vect[G]) \cong Z(Rep(G)) \ar@{..}[u] \\
& \text{Dirichlet} = Vect[G] \ar@{..}[ur] & \quad \text{Neumann} = Rep[G] \ar@{..}[ur] }\]
a rank-2 abelian subgroup, but for certain bundles it is rank-1 (or trivial). Consider the point \( x \in \mathcal{M}_G \) corresponding to the elliptic curve \( E_\tau = C/(\mathbb{Z} \oplus \tau \mathbb{Z}) \) equipped with the \( G \)-bundle that is trivial in the \( \mathbb{Z} \) direction and has monodromy \( g \in G \) in the \( \tau \mathbb{Z} \)-direction. The corresponding twisted-twining genus is \( g \)-twined but untwisted; as a function of \( g = \exp(2\pi i \tau) \), it is the graded character of the action of \( g \) on \( A \). Suppose \( g \) has order \( n \). Then within the based fundamental group \( \pi_1(\mathcal{M}_G, x) \) for this point \( x \in \mathcal{M}_G \) is a subgroup generated by \( ST^nS^{-1} \), where \( S, T \in SL(2, \mathbb{Z}) \) are the standard generators. As explained in [GPR13, §3.3], the monodromy along this loop in the line bundle \( L_\omega \) — the “multiplier” for the action of \( ST^nS^{-1} \) on the character of \( g \) — is readily calculated to be the value of \( \omega \) on the 3-cycle represented by \( \sum_{k=0}^{n-1} g \otimes g^k \otimes g \).

**Example 2.1.1.** Suppose that \( V \) is a holomorphic conformal field theory and consider the permutation action of the symmetric group \( G = S_n \) on \( A = V^\otimes n \). It is commonly believed (see e.g. the conjecture of Müger [Tur10, Appendix 5, Conjecture 6.3] or the remark on p. 2 of [Dav13]) that such permutation actions are nonanomalous. This belief is false. The correct statement is that the anomaly is \( cp_1 \), where \( c \) is the central charge of \( V \) and \( p_1 \in H^2(S_n, U(1)) \) is the Pontryagin class of the permutation representation of \( S_n \). When \( n \geq 6 \), the class \( p_1 \) has order 12 in \( H^2(S_n, U(1)) \cong \mathbb{Z}_{12} \oplus \mathbb{Z}^2_2 \) [Tho86], and, for bosonic conformal field theories, \( c \) is divisible by 8, so the anomaly has order dividing 3. It is nontrivial, for example, for \( V = E_{8,1} = V_{E_8} \), the lattice conformal field theory corresponding to the \( E_8 \) lattice.

Consider, for example, the action of an \( n \)-cycle \((1 \ldots n) \in S_m \) of order \( n \). Let \( \chi_V(q) = \text{tr}(q^{L_0 - c/24}; V) \) denote the graded dimension of \( V \). Standard combinatorics implies that

\[
\text{tr}(q^{L_0 - nc/24}; V^\otimes n) = \chi_V(q^n).
\]

Note that the central charge of \( V^\otimes n \) is \( nc \) if \( V \) has central charge \( c \). If \( V \) is bosonic and holomorphic, then \( S \in SL(2, \mathbb{Z}) \) fixes \( \chi_V \) but \( T \) acts on \( \chi_V \) with eigenvalue \( \exp(2\pi i c/24) \). Recalling that \( q = \exp(2\pi i \tau) \) and that \( S : \tau \mapsto -1/\tau \) and \( T : \tau \mapsto \tau + 1 \), it is not hard to compute that \( ST^nS^{-1} \) acts on \( \text{tr}(q^{L_0 - nc/24}; V^\otimes n) \) with eigenvalue \( \exp(2\pi i c/24) \).

In summary, for permutation orbifolds, the “gauge anomaly” \( \omega \) is precisely the “gravitational anomaly” \( \exp(2\pi i c/24) \).

2.2. **Mathematical foundations.** There are two popular mathematical formalisms for the algebras of observables in chiral conformal field theories. A *vertex operator algebra* is the “Taylor expansion” at a point of the chiral algebra of all observables. Living in the purely algebraic world of power series, vertex operator algebras are very good for explicit computation. A *chiral conformal net* encodes instead those observables that are “smeared” along intervals in the unit circle surrounding the expansion point. (Since the theory is conformal, it doesn’t matter which circle one takes.) Chiral conformal nets come from the world of von Neumann algebras and algebraic quantum field theory. The axioms of both formalisms are widely available and will not be reviewed.

Both formalisms provide precise definitions for adjectives like “rational” and “holomorphic”, and it is widely believed but not yet known that the rational conformal field theories in the two formalisms agree. Many special cases of their agreement are known. First, as proved in [Bis12], the two formalisms are equivalent in the case when one has a chiral conformal field theory \( V \) of central charge \( c \in \mathbb{1}/2 \mathbb{Z} \) together with an injection \( \text{Vir}_{1/2} \hookrightarrow V \), where \( \text{Vir}_{1/2} \) denotes the \( c = 1/2 \) Virasoro algebra. Such algebras are called framed [DGH98], which is both well-deserved and also dangerous, since they are defined on oriented, not framed, complex curves. The conformal field theories we will use in this paper — the Leech lattice CFT \( \mathcal{V}_\Lambda \) and the moonshine CFT \( V^3 \) — are framed [KL06, Bis12], and so results proved about them in one formalism can be moved to the other. Second, as proved in [CKLW15], the two formalisms are equivalent when the CFT is “strongly local.” The moonshine CFT is strongly local [CKLW15], and there are no rational CFTs which are known not to be strongly local.
So far, the formalism of conformal nets has provided more powerful structural and category-theoretic results about conformal field theories. In particular, the orbifold/anomaly story of the previous section can be made fully rigorous in that formalism; on the vertex algebra side, the final missing pieces have been announced in conferences but have not yet appeared in print. On the conformal net side, the existence of an anomaly in $H^3(G, U(1))$, together with how it controls the orbifold problem for holomorphic conformal field theories, was announced in [Miūg10], and the detailed proofs are contained in [DMNO13, BKL15]. The connection between the anomaly and three-dimensional topological field theory is clarified by the series [BDH09, BDH13, BDH15, BDH16, BDH17]. The anomaly $\omega$ can be defined directly in terms of the fusion category of $G$-twisted $V$-modules. By definition, this is the monoidal category of *solitons* for the holomorphic conformal net $V$ — i.e. Doplicher–Haag–Roberts modules for the pullback of the net $V$ on $S^1$ along $\frac{1}{n}\arctan : \mathbb{R} \to S^1 = \mathbb{R}/\mathbb{Z}$ — that, when restricted to $V^G$, become isomorphic to vertex modules (i.e. modules for $V^G$ thought of as a net on $S^1$). This monoidal category is automatically fusion of shape $\text{Vect}^\omega[G]$, and $\omega$ is its associator [Miūg10].

Example 2.1.1, asserting the equivalence of gauge and gravity anomalies, can be mathematically proven in the conformal net setting by using the main results of [GL96, LX04].

### 2.3. T-duality for finite groups

Suppose now that the group $G$ acting on a holomorphic conformal field theory $V$ has a cyclic normal subgroup of order $n$, so that $G$ is an extension of shape $G = n.J$ for some finite group $J = G/n$. (We continue to follow the ATLAS [CCN⁺85] and let “$n$” denote a cyclic group of order $n$.) In practice, such groups arise as the normalizers of cyclic subgroups $n \leftrightarrow \text{Aut}(V)$. Suppose further that the anomaly $\omega \in H^3(G, U(1))$ of the action of $G$ on $V$ trivializes when restricted to the subgroup $n$. Then one can produce the orbifold conformal field theory $V//n$. An interesting question to ask is: what group acts on $V//n$, and with what anomaly? The special case $\omega = 0$ of this question is answered in [BT17]. In general, we have:

**Proposition 2.3.1.** Fix a finite group $J$ and a number $n \in \mathbb{N}$. There is an involution, called finite group T-duality, on the set of pairs $(G, \omega)$ where $G = n.J$ is an extension of $J$ by a cyclic group of order $n$, and $\omega \in H^3(G, U(1))$ is such that $\omega|_n = 0 \in H^3(n, U(1))$. Given a pair $(G, \omega)$, let $(G^\vee, \omega^\vee)$ denote the T-dual pair. Then $G^\vee$ is an extension of $J$ by the Pontryagin dual cyclic group $n^\vee = \text{hom}(n, U(1))$. If $G$ acts on a holomorphic conformal field theory $V$ with anomaly $\omega$, then $G^\vee$ acts on $V//n$ with anomaly $\omega^\vee$.

Explicitly, let $\kappa \in H^2(J, n)$ classify the extension $G$. Since $\omega|_n = 0$ and $H^2(n, U(1)) = 0$, $\omega$ determines a class $\alpha \in H^3(J, H^1(n, U(1))) = H^3(J, n^\vee)$. Under T-duality, the classes $\kappa$ and $\alpha$ are exchanged. The remaining datum of $\omega$ is a 3-cochain $\beta$ on $J$ satisfying $d\beta = \langle \alpha \cup \kappa \rangle$, where $\langle , \rangle$ denotes the pairing $n^\vee \otimes n \to U(1)$. This datum $\beta$ is preserved by T-duality.

The name “finite group T-duality” comes from thinking of the cyclic group $n$ as a “finite circle,” the extension $G = n.J$ as a “finite circle bundle,” the Pontryagin dual group $n^\vee$ as the “dual circle,” and the anomaly $\omega \in H^3(G, U(1))$ as a “Kalb–Ramond field.”

Since T-duality is involutive, one immediately finds the following examples. Suppose $g \in \text{Aut}(V)$ is an element of order $n$, generating the subgroup $\langle g \rangle \cong n$, and suppose that this subgroup acts nonanomalously on $V$. The *T-dual element* $g^\vee$ of $g$ is the generator of the dual $n^\vee$ acting on $V//\langle g \rangle$ (meaning that $g^\vee$ is normalized to that $\langle g^\vee, g \rangle = \exp(2\pi i/n) \in U(1)$). Then the centralizer $C(g) \subset \text{Aut}(V)$ of $g$ is T-dual to the centralizer $C(g^\vee) \subset \text{Aut}(V//\langle g \rangle)$, and the normalizer $N(\langle g \rangle) \subset \text{Aut}(V)$ of the subgroup $\langle g \rangle$ is T-dual to $N(\langle g^\vee \rangle) \subset \text{Aut}(V//\langle g \rangle)$.

**Proof.** By working with twisted modules, statements about actions on holomorphic conformal field theories may be translated directly into statements about fusion categories. For example, the data of $(G, \omega)$ is the data of the fusion category $\text{Vect}^\omega[G]$, and saying that $G$ acts on $V$ with anomaly $\omega$ is the same as saying that the category of $G$-twisted $V$-modules is $\text{Vect}^\omega[G]$. The requirements
$G = n.J$ and $\omega|_n = 0$ are equivalent to saying that $\text{Vect}^\omega[G]$ is a $J$-graded extension of $\text{Vect}[n]$ in the language of [ENO10]. The group $n^\omega$ automatically acts on $V/n$, and this automatically extends to an action of some $J$-graded extension of $\text{Vect}[n^\omega] = \text{Rep}(n)$.

The main theorem of [ENO10] classifies $J$-graded extensions of $\text{Vect}[n]$ in a way that is manifestly symmetric under Pontryagin duality $n \leftrightarrow n^\vee$; see also [JL99]. (Indeed, the classification depends only on the Morita-equivalence class of the fusion category $\text{Vect}[n]$.) The first datum needed is a map $J \to O(n \oplus n^\vee)$, where the latter denotes the group of transformations that are orthogonal for the canonical “hyperbolic” pairing. The actions corresponding to group extensions $G = n.J$ are precisely those where the map $J \to O(n \oplus n^\vee)$ factors through the map $J \to GL(n) \cong GL(n^\vee)$ giving the outer action of $J$ on $n$. The next datum is the class $\kappa \oplus \alpha \in H^2(J, n \oplus n^\vee)$. This datum cannot be arbitrary: the obstruction $\langle \alpha \cup \kappa \rangle \in H^4(J, U(1))$ must vanish. The third and final datum needed is a choice of primitive $\beta \in C^3(J, U(1))$ (coboundaries) for this obstruction.

The appendix to [ENO10] identifies these data with the components of the expansion of $\omega$ via the Lyndon–Hochschild–Serre (LHS) spectral sequence for $H^*(n.J, U(1))$. Recall that the LHS spectral sequence for an extension $G = N.J$ has $E_2$ page $H^*(J, H^*(N, U(1)))$ and converges to $H^*(G, U(1))$. In outline, its construction is as follows. Pick a cocycle $\kappa$ for the canonical “hyperbolic” pairing. The actions corresponding to group extensions $G = n.J$ are precisely those where the map $J \to O(n \oplus n^\vee)$ factors through the map $J \to GL(n) \cong GL(n^\vee)$ giving the outer action of $J$ on $n$. The next datum is the class $\kappa \oplus \alpha \in H^2(J, n \oplus n^\vee)$. This datum cannot be arbitrary: the obstruction $\langle \alpha \cup \kappa \rangle \in H^4(J, U(1))$ must vanish. The third and final datum needed is a choice of primitive $\beta \in C^3(J, U(1))$ (coboundaries) for this obstruction.

2.4. Super case. Although it will not play a role in this paper, we mention for completeness the description of orbifolds of possibly-fermionic conformal field theories. If $V$ is allowed to have fermions, then the anomaly of an action of $G$ on $V$ is not a class in $H^3(G, U(1))$. It is instead a class in (extended) “supercohomology” [WG17], which, as explained in [GJF17], is the generalized cohomology theory corresponding to the spectrum $S = \text{Superalg}_C^\infty$ with homotopy groups $\pi_2S = \pi_1S = Z_2$ and $\pi_0S = U(1)$ and k-invariants $\text{Sq}^2 : Z_2 \to Z_2$ and $\exp(\pi_1 \text{Sq}^2) : Z_2 \to U(1)$. (The two possible j-invariants in this case give equivalent spectra.) A cocycle $\omega \in Z^3(S)$ consists of cochains $\omega_1 \in C^1(S, Z_2)$, $\omega_2 \in C^2(S, Z_2)$, and $\omega_3 \in C^3(S, U(1))$. The k-invariants say that these cochains should solve $d\omega_1 = 0$, $d\omega_2 = \text{Sq}^2 \omega_1$, and $d\omega_3 = \exp(\pi_1 \text{Sq}^2 \omega_2 + \text{(term involving } \omega_1))$. Let $S'$ denote the spectrum with homotopy groups $\pi_1S' = \pi_0S' = Z_2$ and k-invariant $\text{Sq}^2$. Then $H^3(S)$ fits into a long exact sequence like

$$\cdots \to H^3(G, U(1)) \to H^3(G, S) \to H^2(G, S') \to \cdots$$

The map $H^3(G, U(1)) \to H^3(G, S)$ corresponds to considering a bosonic conformal field theory as a possibly-fermionic one.

One can understand the anomaly in various ways. In terms of the induced action of $G$ on the trivial 3D theory $Z(V)$, the spectrum $S$ appears as a connective cover of the Brown–Comenetz dual to Spin Cobordism. The anomaly $\omega \in H^3(G, S)$ also has meaning in terms of the super fusion category of solitons on $V$ that restrict to honest modules on $V^G$. As in the even case, there is a
“unique” irreducible $g$-graded soliton (aka twisted sector) $V_g$ for each $g \in G$, where “unique” means up to possibly-odd isomorphism. In terms of these solitons, $\omega_1(g)$ measures whether whether the endomorphism algebra of $V_g$ is $\mathbb{C}$ or $\text{Cliff}_C(1)$ (objects of the latter type are called Majorana), and $\omega_2(g_1, g_2)$ measures whether the isomorphism $V_{g_1} \otimes V_{g_2} \cong V_{g_1 g_2}$ is even or odd. Finally, $\omega_3$ is the associator familiar from the non-super case.

As in the bosonic case, if $G$ acts on $V$ with anomaly $\omega$, then orbifolds $V//G$ depend on a choice of trivialization of $\omega$. Unlike in the bosonic case, if $G$ is cyclic of even order, there is a choice in the trivialization. To trivialize $\omega$, one first must trivialize $\omega_1 \in H^1(G, \mathbb{Z}_2)$; this either trivializes or it doesn’t. Such a choice makes $\omega_2$, which originally lives in a torsor for $H^2(G, \mathbb{Z}_2)$, into an honest cohomology class; one must then trivialize $\omega_2$, and there are $H^1(G, \mathbb{Z}_2)$-many ways to do so — this group is a copy of $\mathbb{Z}_2$ when $G = \mathbb{Z}_{2n}$, whereas in the bosonic case there was no $\omega_2$ needing trivialization. Finally, any such choice makes $\omega_3$ into an honest cohomology class, and one must trivialize it — there are $H^2(G, U(1))$ many choices for the trivialization, and this group vanishes for $G$ cyclic.

Super T-duality is clearest when $\omega_1 = 0$. Then $\omega = (\omega_2, \omega_3)$ is a class in the restricted supercohomology of [GW14]. More pedantically, a cocycle representative of $\omega$ consists of a cocycle $\omega_2 \in Z^2(G, \mathbb{Z}_2)$ and a cochain $\omega_3 \in Z^3(G, U(1))$ solving $d\omega_3 = \exp(\pi i \text{Sq}^2 \omega_2)$. Noting that $\text{Sq}^2 \omega_2 = \omega_2 \cup \omega_2$, since $\omega_2$ has degree 2, in the notation of Section 2.3 we can write the required equation as $d\omega_3 = (\omega_2 \cup \omega_2)$.

The super fusion category $\text{SuperVect}^\omega[G]$ has an underlying non-super fusion category. It is again group-like of shape $\text{Vect}^\omega[G]$, where $\tilde{G} = 2G$ is the extension classified by $\omega_2 \in H^2(G, \mathbb{Z}_2)$. The associator is $\tilde{\omega} = \omega_2 + \omega_3 \in H^3(\tilde{G}, U(1))$; it exists because of the equation $d\omega_3 = (\omega_2 \cup \omega_2)$, just as in Section 2.3. The pair $(\tilde{G}, \tilde{\omega})$ is called the bosonic shadow of $(G, \omega)$ in [BGK16].

Suppose now that $G = n.J$ and that $\omega_1 \equiv 0$ (meaning of course that a trivialization has been chosen). Construct its bosonic shadow $\tilde{G} = 2.n.J$. Since $\omega_1 \equiv 0$, the subgroup $2.n = n \subset \tilde{G}$ is a direct product: $\tilde{G} = (2 \times n).J = n.2.J = n.J$. Applying the bosonic T-duality from Section 2.3 produces a new group $\tilde{G}^\vee = n^\vee.J = n.J$, one has $n^\vee.2 = n^\vee \times 2$, and so we can write $\tilde{G}^\vee = 2.n^\vee.J$. Define $G^\vee = \tilde{G}^\vee/2 = n^\vee.J$. To complete the construction, one checks that the T-dual bosonic anomaly $\tilde{\omega}^\vee \in H^3(2.G^\vee, U(1))$ splits as $\omega_2^\vee + \omega_3^\vee$ with $d\omega_3^\vee = (\omega_2^\vee \cup \omega_2^\vee)$; then $\omega^\vee = (\omega_2^\vee, \omega_3^\vee)$ is the dual super anomaly for $G^\vee$.

**Example 2.4.1.** The most important fermionic conformal field theories are the free Majorana fermion models $\text{Fer}(n)$ of central charge $c = \frac{3}{2}$. The automorphism group of $\text{Fer}(n)$ is $O(n)$. According to [DH11], the anomaly of an action $G \to O(n)$ is precisely the string obstruction — the obstruction to lifting the action to a map $G \to \text{String}(n)$, which by definition is the 3-connected cover of $O(n)$. $\text{String}(n)$ fits into a tower $\cdots \to \text{String}(n) \to \text{Spin}(n) \to SO(n) \to O(n)$. The obstruction $\omega$ can be understood in pieces by trying to lift into the intermediate groups, and these pieces match the ones above: $\omega_1$ is the first Stiefel–Whitney class $w_1$, obstructing the lift from $O(n)$ to $SO(n)$; $\omega_2$ is the second Stiefel–Whitney class $w_2$, obstructing the lift from $SO(n)$ to $\text{Spin}(n)$; and $\omega_3 = \frac{p_1}{2}$ is the first fractional Pontryagin class, obstructing the lift from $\text{Spin}(n)$ to $\text{String}(n)$. In general, for an action of $G$ on a fermionic conformal field theory $V$, trivializations of $\omega_1$ and $\omega_2$ can be thought of as choices of “orientation” and “spin structure.”

Consider the central subgroup $\mathbb{Z}_2 \subset O(n) = \text{Aut}(\text{Fer}(n))$ switching the signs of all fermions simultaneously. Its string obstruction — its anomaly — vanishes exactly when $n$ is divisible by 8. Set $n = 8k$ and $G = 2.J \subset SO(n)$, so that we can ignore $\omega_1$. What is the T-dual group $G^\vee$ acting on $\text{Fer}(n)/\mathbb{Z}_2$? To make the question well-posed we need to choose a trivialization of $\omega_2$. The choices are a torsor for $H^2(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$. The two choices correspond to the two nontrivial double covers other than $SO(n)$ of $\text{PSO}(n)$. These double covers, called $SO^+(n)$ and $SO^-(n)$, are the images of $\text{Spin}(n)$ under the two spin-module actions. Choose either $SO^+(n)$. Then $G^\vee$ is nothing but the preimage in $SO^\pm(n) = 2.\text{PSO}(n)$ of $J \subset \text{PSO}(n)$.
Although the groups $SO^\pm(n)$ are isomorphic — they are related by a reflection in $O(n)$ — the preimages $G^\vee = 2.J$ may not be. A main example is $G = Co_0 = 2. Co_1 \subset SO(24)$. Then one of the preimages is again isomorphic to $Co_0$ and the other is isomorphic to $2 \times Co_1$. These choices show up when studying the “super moonshine” modules of [Dun07, DMC15]. Both papers study a fermionic CFT isomorphic to $\text{Fer}(24)//\mathbb{Z}_2$, with automorphism group $SO^\pm(24)$. According to [Dun07], this CFT admits $\mathcal{N} = 1$ supersymmetry, and as a supersymmetric conformal field theory its automorphism group is $Co_1$. According to [DMC15], the genus-zero phenomena central to Monstrous Moonshine appear when studying instead a different, faithful $Co_0$ action on $\text{Fer}(24)//\mathbb{Z}_2$. Fix a copy of $Co_0 \subset SO(24)$. One can think of the two “super moonshine” CFTs as two different choices of the orbifold $\text{Fer}(24)//\mathbb{Z}_2$. If, when defining $\text{Fer}(24)//\mathbb{Z}_2$, we use the spin structure for the $\mathbb{Z}_2$-action that extends to all of $Co_0$, the T-dual group is $Co_0$, and so we get the version from [DMC15]; if we use the other spin structure, the T-dual group is $2 \times Co_1$, of which only $Co_1$ preserves the $\mathcal{N} = 1$ supersymmetry, and we get the version from [Dun07].

If follows from Theorem 2 that the $Co_0$-action on the CFT from [DMC15] has anomaly of order 24 in $H^3(Co_0, U(1)) \cong \mathbb{Z}_{24}$. The anomaly for the $Co_1$-action on the CFT from [Dun07] is honestly super: it is a class in the (restricted) supercohomology of $Co_1$. This is a copy of $\mathbb{Z}_{24} = \mathbb{Z}_{12} \cdot \mathbb{Z}_2 = H^2(Co_1, U(1)).H^2(Co_1, \mathbb{Z}_2)$, and the anomaly again has order 24.

**Example 2.4.2.** Another class of examples of fermionic orbifolds comes from lattice conformal field theories. If $L$ is an even unimodular lattice and $L' \subset L$ has index 2, then there is a unique even lattice other than $L$ that contains $L'$; this uniqueness reflects the uniqueness in trivializing the corresponding $\mathbb{Z}_2$-action on the bosonic conformal field theory $V_L$. But there is also an odd lattice containing $L'$ in index 2, which comes from the other choice of spin structure — the conformal field theory built from an odd lattice has fermions.

### 3. Proof of Theorem 1

We now prove Theorem 1, which asserts that the action of the Monster group $M$ on its natural representation $V^2$ has an anomaly $\omega^2$ of order exactly 24. The conformal field theoretic inputs we use are:

1. the existence of $V^2$ and $M = \text{Aut}(V^2)$, and hence of the anomaly $\omega^2 \in H^3(M, U(1));$
2. isomorphisms $V^2//\mathbb{Z}_p \cong V_\Lambda$ for $p = 2, 3, 5, 7, 13$, and hence $T$-dualities between appropriate subgroups of $M$ and $\text{Aut}(V_\Lambda) = U(1)^{24}.Co_0$, as explained in Section 3.2, where $\Lambda$ denotes the Leech lattice and $V_\Lambda$ the corresponding lattice conformal field theory.

We use one other CFT calculation — namely, a phase discrepancy in a certain twisted sector of $V^2$ — in the proof of Lemma 3.4.4. This Lemma shows that Conjecture 2 implies Conjecture 3, and is not used in the proof of Theorem 1.

When calculating cohomology classes of finite groups, one may proceed prime-by-prime, because of the following standard result.

**Lemma 3.0.1.** Let $G$ be a finite group. Then $H^k(G, U(1))$ is finite abelian for $k \geq 1$, and so splits as $H^k(G, U(1)) = \bigoplus_p H^k(G, U(1))_{(p)}$ where the sum ranges over primes $p$ and $H^k(G, U(1))_{(p)}$ has order a power of $p$. Fix a prime $p$ and suppose that $S \subset G$ is a subgroup such that $p$ does not divide the index $|G|/|S|$, i.e. such that $S$ contains the $p$-Sylow of $G$. Then the restriction map $\alpha \mapsto \alpha|_S : H^k(G, U(1)) \rightarrow H^k(S, U(1))$ is an injection onto a direct summand.

The Sylow subgroups of $M$ are known (and listed for example in the ATLAS [CCN+85]). For reference sake, the order of the Monster is

$$|M| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71.$$  

We consider the primes 17–71, and also the prime 11, as “large,” since they can be dispensed with quickly in Section 3.1. The primes 3, 5, 7, and 13 are “small,” and we handle them using $T$-duality.
in Section 3.2. We also handle the prime 2 using T-duality, but it is sufficiently complicated that we give it its own Section 3.3.

3.1. The “large” primes $p = 11, 17, 19, 23, 29, 31, 41, 47, 59, 71$.

**Lemma 3.1.1.** $H^3(M, U(1))_{(p)} = 0$ for $p \geq 17$ or $p = 11$.

**Proof.** Let $p \geq 17$ be a prime dividing $|M|$. Then the $p$-Sylow in $M$ is cyclic and contained in a subgroup of shape $p : n$ for $n = \frac{p-1}{2} > 2$. It is easy to see that $H^3(p : n, U(1))_{(p)} = 0$. Indeed, it is a direct summand of $H^3(p, U(1)) \cong \mathbb{Z}_p$, and it is fixed by the $n$-action. The automorphism $g \mapsto g^n$ of $\mathbb{Z}_p$ acts on $H^3(\mathbb{Z}_p, U(1))$ by multiplication by $a^2$. But $n \subset \text{Aut}(p) = \mathbb{Z}_p^\times$ contains elements other than $\pm 1$, and so acts nontrivially on $H^3(p, U(1))$, and so leaves only the 0 subgroup fixed.

For $p = 11$, the 11-Sylow in $M$ is contained in a subgroup of shape $(11 : 5)^{\times 2}$, and $H^3((11 : 5)^{\times 2}, U(1))_{(11)}$ vanishes by Küneth’s formula. □

3.2. The “small” primes $p = 3, 5, 7, 13$. Let $\Lambda$ denote the Leech lattice, $\Lambda^\vee = \text{hom}(\Lambda, U(1))$ its dual torus, and $V_\Lambda$ its lattice conformal field theory. By construction, $\text{Aut}(V_\Lambda)$ contains, and is in fact equal to, a group of shape $\Lambda^\vee.\text{Co}_0$. In the sense of vertex algebras, the following result is definitional for $p = 2$, conjectured in [FLM88, Tui92] for all $p$, announced for $p = 3$ in [DM94], and proved in full in [ALY17]:

**Proposition 3.2.1 ([ALY17]).** Let $g^\vee \in \Lambda^\vee.\text{Co}_0$ be an automorphism of $V_\Lambda$ of prime order $p$ that projects to a fixed-point-free automorphism of $\Lambda$. Then $p$ is one of 2, 3, 5, 7, or 13, and for fixed $p$ all such automorphisms are conjugate; moreover, every fixed-point-free automorphism of $\Lambda$ of order $p$ lifts to an order-$p$ automorphism of $V_\Lambda$. The action of $(g^\vee) \cong \mathbb{Z}_p$ on $V_\Lambda$ is nonanomalous and the orbifold $V_\Lambda//\mathbb{Z}_p$ is isomorphic to $V^\vee$. Let $g \in M = \text{Aut}(V^\vee)$ denote the T-dual to $g^\vee \in \Lambda^\vee.\text{Co}_0$. Then $g$ is of $M$-conjugacy class $pB$.

**Proof.** Since the statement is verified in the sense of vertex algebras in [ALY17], it suffices to transfer the claim to the conformal net setting. The work on “framed CFTs” from [Bis12] verifies analytically the isomorphism $(V_\Lambda)//\mathbb{Z}_2 \cong V^\vee$. With this, the claim is equivalent to the assertion that $(V^\vee)//\mathbb{Z}_{2p} \cong V^\vee$ for $p = 3, 5, 7, 13$. But $V^\vee$, hence all of its subalgebras, is strongly local in the sense of [CKLW15], and so the algebraic isomorphism $(V^\vee)//\mathbb{Z}_{2p} \cong V^\vee$ from [ALY17] transfers to the analytic setting. □

The primes 2, 3, 5, 7, and 13 are precisely those primes $p$ for which $p - 1$ divides 24. For $p \neq 2$, image of $g^\vee$ in $\text{Co}_1$ is in $\text{Co}_1$-conjugacy class $pA$; for $p = 2$, the image of $g^\vee$ in $\text{Co}_0$ is the central element.

**Corollary 3.2.2.** As in the notation of Proposition 3.2.1, let $g \in M$ denote an element of $M$-conjugacy class $pB$ and $g^\vee \in \Lambda^\vee.\text{Co}_0$ the T-dual element, covering (when $p$ is odd) an element of $\text{Co}_0$-conjugacy class $pA$. The centralizers $C(g) \subset M$ and $C(g^\vee) \subset \Lambda^\vee.\text{Co}_0$ are $T$-dual.

These centralizers are

| $p$ | $C(g) \subset M$ | $C(g^\vee) \subset \Lambda^\vee.\text{Co}_0$ |
|-----|-------------------|------------------------------------------|
| 2   | $2^{1+24}.\text{Co}_1$ | $2^{24}.\text{Co}_0$                   |
| 3   | $3^{1+12}.2\text{Suz}$ | $3^{12} : 6\text{Suz}$                  |
| 5   | $5^{1+6} : 2J_2$ | $5^6 : (5 \times 2J_2)$                |
| 7   | $7^{1+4} : 2A_7$ | $7^4 : (7 \times 2A_7)$                |
| 13  | $13^{1+2} : 2A_4$ | $13^2 : (13 \times 2A_4)$               |

As in the ATLAS [CCN+85], Suz denotes the Suzuki group and $J_2 = HJ$ denotes the Hall–Janko group. □

The conjugacy classes $pB$ in $M$ are denoted $p$— in [CN79], where the coincidence between their centralizers and the centralizers of the classes $pA$ in $\text{Co}_0$ is already observed.
Lemma 3.2.3. The order of \( \omega^5 \) is not divisible by 5, 7, or 13.

Proof. Throughout this proof, we let \( p = 5, 7, \) or \( 13 \) and \( d = 24/(p-1) \). For \( p = 5 \) and 13, we work with the T-dual pairs \( G = C(g) \subset M \) and \( G^\vee = C(g^\vee) \subset \Lambda^\vee \Co_0 \) from Corollary 3.2.2. For \( p = 7 \), we need a pair of slightly larger groups. The normalizer \( N(7B) \) of an element of M-conjugacy class \( 7B \) has shape \( 7^{1+4} : (3 \times 2S_7) \); we will use the subgroup \( G = 7^{1+4} : (3 \times 2A_7) \) and its dual \( G^\vee = 7^3 : (7 : 3 \times 2A_7) \). The factor of 3 acts on \( 7^4 \) by scalars and rescales the symplectic pairing.

In all cases, we set \( G = p.J \) and \( G^\vee = p^\vee.J \), and we have \( J = p^d : 2X \) with:

| \( p \) | \( X \) | \( J \) |
|------|-------|----|
| 5    | \( J_2 \) |   |
| 7    | \( 3 \times A_7 \) |   |
| 13   | \( A_4 \) |   |

In all cases \( G = p^{1+d} : X \) contains the \( p \)-Sylow of \( M \) and so the \( p \)-part of the order of \( \omega^5 \) is detected by the order of \( \omega^5|G \).

Following the notation from Proposition 2.3.1, let \( \kappa \in H^2(J, \mathbb{Z}_p) \) classify the extension \( G = p.J \) and expand \( \omega^5|G = \alpha + \beta \) with \( \alpha \in H^2(J, \mathbb{Z}_p) \) classifying the extension \( G^\vee = p^\vee.J \). In all cases, the extension \( G^\vee = p^\vee : J \) splits (as, indeed, \( N(pA) = p^\vee : (N(pA)/p^\vee) \) splits), and so \( \alpha = 0 \). It follows that \( \omega^5|G = \beta \) is pulled back from a class \( \beta \in H^3(J, U(1)) \) and \( \omega^5 = \kappa \cup \beta \in H^2(J, p) \cup H^3(J, U(1)) \subset H^6(G^\vee, U(1)) \). By construction, \( \omega^5 \) is the anomaly of the action of \( G^\vee \) on \( V_\Lambda \), and \( \beta = \omega^5|J \). We will compute \( H^*(J, U(1)) \) by using the LHS spectral sequence \( H^*(2X, H^*(p^d, U(1))) \Rightarrow H^*(J, U(1)) \).

The K"unneth formula gives \( H^1(p^d, U(1)) \cong p^d, H^2(p^d, U(1)) \cong \text{Alt}^2(p^d), \) and \( H^3(p^d, U(1)) \cong \text{Sym}^2(p^d). \text{Alt}^3(p^d) \). The central \( 2 < 2X \) acts nontrivially on \( H^1(p^d, U(1)) \) and on \( \text{Alt}^3(p^d) \), and so \( 2X \) can have no cohomology with these coefficients. The module \( p^d \) carries a symplectic form but is not symmetrically self-dual over \( 2X \), and so \( \text{Sym}^2(p^d) \) also has no fixed points; thus \( H^0(2X, H^3(p^d, U(1))) = 0 \).

We claim furthermore that \( H^1(2X, \text{Alt}^2(p^d)) = 0 \). The case \( p = 13 \) is automatic since 13 does not divide \( |2A_4| \). For \( p = 7 \), note that the central \( 3 < 2X \) acts nontrivially on \( \text{Alt}^2(7^4) \), and so \( H^1(3 \times 2A_7, 7^5) = 0 \) for all \( i \). For \( p = 5 \), one can check \( H^1(2J_2, \text{Alt}^2(5^6)) = 0 \) using Holt’s GAP program Cohomolo.

It follows that the restriction \( H^3(J, U(1)) \to H^3(2X, U(1)) \) is an isomorphism, and so it suffices to compute the order of \( \omega^5|_{2X} \), where again \( \omega^5 \) is the anomaly for the Leech lattice conformal field theory \( V_\Lambda \). But the action of \( 2X \) on \( V_\Lambda \) factors through a group of shape \( 2^{24}.\Co_0 \) — indeed, the centralizer in \( \Lambda^\vee . \Co_0 = \text{Aut}(V_\Lambda) \) has this shape — and so \( \omega^5|_{2X} \) is the restriction of a class in \( H^3(2^{24}.\Co_0, U(1)) \). By Theorem 2, \( H^3(2^{24}.\Co_0, U(1))|_{(p)} = 0 \) for \( p \geq 5 \).

\( \square \)

Lemma 3.2.4. \( \omega^5 \) has order \( 3 \times 2^k \) for some \( k \).

Proof. Set \( p = 3, d = 12, J = 3^{1+12} : 2Suz, G = 3^{1+12} : 2Suz, G^\vee = 3^{1+12} : 6Suz, \) and \( X = \text{Suz} \). It suffices to show that the component of \( \omega^5|G \) in \( H^3(G, U(1))_{(3)} \) has order exactly 3. Unlike in the \( p = 5 \) case, the extension \( G^\vee = 3.J \) does not split, and so \( \alpha \neq 0 \in H^2(J, \mathbb{Z}_3) \). It follows that \( \omega^5|G \) is non-zero and has order divisible by 3.

We study the cohomology of the group \( G = 3^{1+12} : 2Suz \) using a LHS spectral sequence. Set \( E = p^d = 3^{1+12} \) and \( 3E = 3^{1+12} \). Then \( E \) is symplectically but not symmetrically self-dual over \( 2Suz \). We have \( H^1(3E, U(1)) \cong E \) and \( H^2(3E, U(1)) \cong \text{Alt}^2(E)/3 = 3^{64}.3 \). It follows that \( H^*(2Suz, H^1(3E, U(1))) = 0 \), and the spectral sequence has \( E_2 \) page:

\[
\begin{array}{cccc}
\text{H}^0(2\text{Suz}, \text{H}^3(3E, U(1))) & * & \text{H}^1(2\text{Suz}, 3^{64}.3) & \text{H}^3(2\text{Suz}, U(1)) \\
0 & 0 & 0 & \\
* & * & * & \\
\end{array}
\]

\( \square \)
Furthermore, $3E$ is T-dual to a group of shape $E \times 3$. It follows that $\omega^2|_{3E}$ is pulled back from $H^3(E, U(1)) \cong \text{Sym}^2(E) \oplus \text{Alt}^3(E)$. As in Lemma 3.2.3, $H^0(2\text{Suz}, \text{Sym}^2(E) \oplus \text{Alt}^3(E)) = 0$. Therefore $\omega^2|_G$ lives in an extension of subquotients of $H^1(2\text{Suz}, 3^{64}.3)$ and $H^3(2\text{Suz}, U(1))$.

Since $H^1(2\text{Suz}, 3^{64}.3)$ has exponent 3, to complete the proof it suffices to show that $H^3(2\text{Suz}, U(1))_{(3)} = H^3(\text{Suz}, U(1))_{(3)} = 0$. According to the ATLAS [CCN+85], the 3-Sylow in Suz is contained in a maximal subgroup of shape $3^5 : M_{11}$. The group $M_{11}$ has two 5-dimensional modules over $\mathbb{F}_3$, dual to each other. Matching the ATLAS’s letters, we will call the second modules $3^{5a}$ and $3^{5b}$. Holt’s GAP program Cohomolo quickly computes

$$H^1(M_{11}, 3^{5a}) = 0,$$
$$H^1(M_{11}, 3^{5b}) = 3.$$

But $H^2(\text{Suz}, U(1))_{(3)} = 3$, since the Schur multiplier of Suz is $\mathbb{Z}_6$. Since $H^2(M_{11}, U(1))_{(3)} = 0$, and $H^2(\text{Suz}, U(1))_{(3)} \subset H^2(3^5 : M_{11}, U(1))$, we must have $H^1(M_{11}, H^1(3^5, U(1))) = 3$, which is possibly only when $H^1(3^5, U(1)) = 3^{5b}$, which forces the maximal subgroup of Suz to be $3^{5a} : M_{11}$.

Cohomolo also confirms that

$$H^2(M_{11}, H^1(3^{5a}, U(1))) = H^2(3^{5a}, 3^{5b}) = 0.$$

Furthermore, since $3^{5a}$ is not self-dual and does not admit an invariant 3-form, $H^0(M_{11}, H^3(3^{5a}, U(1))) = 0$. One can quickly compute the 10-dimensional $M_{11}$-module $H^2(3^{5a}, U(1))$ and check that

$$H^1(M_{11}, H^2(3^{5a}, U(1))) = 0.$$

Finally, that $H^3(M_{11}, U(1))_{(3)} = 0$ is classically known, and quick to compute with Ellis’s GAP program HAP. This confirms that that $H^3(2\text{Suz2}, U(1))_{(3)} = 0$. 

3.3. The prime $p = 2$. To complete the proof of Theorem 1, we must study the 2-part of $\omega^2$: we wish to show that the order of $\omega^2$ is $8 \times (\text{odd})$. The 2-Sylow in $\mathbb{M}$ is contained in a maximal subgroup of shape $G = 2^{1+24}.\text{Co}_1$, the centralizer of an element of $\mathbb{M}$ of conjugacy class 2B. As mentioned already in Proposition 3.2.1, $G$ is T-dual to the centralizer in $\text{Aut}(V_\Lambda) = \Lambda^\vee.\text{Co}_0$ of any order-2 lift of the central element of $\text{Co}_0$. This dual group has shape $G^\vee = 2^{24}.\text{Co}_0$. It is known that the extension $G^\vee$ does not split [Iva09].

Lemma 3.3.1. $8\omega^2|_G$ is pulled back from $\text{Co}_1$.

Proof. The LHS spectral sequence for $G = 2^{1+24}.\text{Co}_1$ asserts that $H^3(G, U(1))$ is an extension of subquotients of:

1. $H^0(\text{Co}_1, H^3(2^{1+24}, U(1)))$, and $H^3(2^{1+24}, U(1))$ has exponent 4;
2. $H^1(\text{Co}_1, H^2(2^{1+24}, U(1)))$, and $H^2(2^{1+24}, U(1)) = 2^{274}.2$ has exponent 2;
3. $H^2(\text{Co}_1, H^1(2^{1+24}, U(1)))$, and $H^1(2^{1+24}, U(1)) = 2^{24}$ has exponent 2;
4. $H^3(\text{Co}_1, U(1))$, which by [JFT17] is isomorphic to $\mathbb{Z}_{12}$.

It follows that for any class $\alpha \in H^3(G, U(1))$, $16\alpha$ is pulled back from $H^3(\text{Co}_1)$. Moreover, the only way for $8\alpha$ to fail to be pulled back from $H^3(\text{Co}_1)$ is if $\alpha|_{2^{1+24}}$ has order 4. But $\omega^2|_{2^{1+24}}$ implements a T-duality between $2^{1+24}$ and $2 \times 2^{24}$, from which it follows that $\omega^2|_{2^{1+24}}$ has order 2 in $H^0(\text{Co}_1, H^3(2^{1+24}, U(1)))$. 

To complete the proof of Theorem 1, we use the same binary dihedral group used in [JFT17]. By Proposition 3.2.1, $\text{Co}_0$ has a subgroup (centralizing an order-7 fixed-point-free automorphism of $\Lambda$) of shape $2A_7$. Inside $A_7$, choose a dihedral group $D_8$ of order 8 — there is a unique conjugacy class of such subgroups. Its lift to $2A_7$ is a binary dihedral group $2D_8$ of order 16. This is a McKay group — the corresponding Dynkin diagram is $D_6$ — and its third cohomology is $H^3(2D_8, U(1)) \cong \mathbb{Z}_{16}$. In [JFT17] it is shown that the restriction map $H^3(\text{Co}_0, U(1))_{(2)} \to H^3(2D_8, U(1))$ is an injection onto the even subgroup $\mathbb{Z}_8 \subset \mathbb{Z}_{16}$. 


This binary dihedral group is also naturally a subgroup of \( M \), since it lives inside the centralizer of an element of \( M \)-conjugacy class 7B. In all cases, the central 2 \( \subset 2D_8 \) is (a lift of) the central element of \( Co_0 \). It follows that:

**Lemma 3.3.2.** \( \omega^5|_{2D_8} \) has order 8.

*Proof.* The above discussion implies that the subgroups \( 2D_8 \subset M \) and \( 2D_8 \subset \text{Aut}(V_\Lambda) \) are T-dual by the usual orbifold construction \( V_\Lambda = V^2//\mathbb{Z}_2 \). The classes in \( H^3(2D_8, U(1)) \) implementing such a T-duality are those with order 8.

Combining Lemmas 3.3.1 and 3.3.2 completes the proof of Theorem 1:

**Lemma 3.3.3.** \( \omega^5|_G \) has order \( 8 \times \text{(odd)} \).

*Proof.* By Lemma 3.3.1, \( 8\omega^5|_G \) is pulled back from \( Co_1 \). By \([\text{JFT17}], H^3(Co_1, U(1))(2) \cong \mathbb{Z}_4 \), and classes are detected by restricting to \( D_8 \), and so they are also detected by pulling back to \( 2D_8 \). But \( 8\omega^5|_{2D_8} = 0 \) by Lemma 3.3.2.

3.4. \( \omega^5 \) is not a Chern class. The first sentence of Theorem 1 having been verified, we turn to proving the second sentence, which asserts that \( \omega^5 \neq c_2(V) \) for every complex representation \( V \) of \( M \). The argument in this section was suggested to me by D. Treumann. Denote by \( R(M) \) the complex representation ring of \( M \). Since \( H^1(M, U(1)) = 0, c_2 : R(M) \rightarrow H^3(M, U(1)) \) is linear. Let \( 2D_8 \) denote the binary dihedral group centralizing some chosen element of conjugacy class 7B used in Section 3.3. We know that \( \omega^5|_{2D_8} \) has order 8. Our strategy will be to compute the restriction \( c_2|_{2D_8} : R(M) \rightarrow H^3(2D_8, U(1)) \cong \mathbb{Z}_{16} \) and show that \( \omega^5|_{2D_8} \) is not in the image. In fact, we will show that \( c_2|_{2D_8} \) is the zero map.

**Lemma 3.4.1.** The seven conjugacy classes in \( 2D_8 \) — the identity, the central element, three of order 4, and two of order 8 — merge in \( M \) into just four, determined by their orders. Specifically, these are the \( M \)-conjugacy classes 1, 2B, 4D, and 8F.

*Proof.* Choose any element of order 4 in \( 2D_8 \), and multiply it with the fixed element of conjugacy class 7B centralized by \( 2D_8 \). One produces an element of order 28 in \( M \) whose fourth power is of class 7B and whose seventh power is the chosen element of order 4. The unique fourth root of class 7B is class 28A, and \((28A)^7 = 4D\), so all order-4 classes in \( 2D_8 \) merge in \( M \). But 4D has a unique square root — 8F — so the two classes of order 8 must also merge. Finally, \((4D)^2 = 2B\).

The group \( 2D_8 \) is the McKay group corresponding to the Dynkin diagram \( D_6 \). Let \( V_0 \) denote the trivial representation, \( V_1 \) the one-dimensional representation in which the kernel of the \( 2D_8 \) action is cyclic of order 8, \( V_2 \) and \( V_3 \) the other two one-dimensional representations, \( V_4 \) the two-dimensional real representation of \( D_8 \), and \( V_5 \) and \( V_6 \) the two faithful irreps. Taking \( V_6 \) to be the “defining” 2-dimensional representation, the McKay graph is:

```
   V0 ----- V2
  /     \   /     \
V6 - V4 - V5
  \     /   \     /
      V1   V3
```

Given a representation \( V \) of \( 2D_8 \), let \( n_i \) be defined by \( V \cong \bigoplus_{i=0}^{6} V_i^{n_i} \). The basic theorem of characters of a finite group says that the numbers \( n_i \) can be computed from knowing the traces over \( V \) of all conjugacy classes. Suppose that \( V \) is restricted from an irrep of \( M \). Then, by Lemma 3.4.1, \( \text{tr}_V(g) \) depends only on the order of \( g \in 2D_8 \). A quick computation with the character table for \( 2D_8 \) reveals:
\[ \left( n_0, n_1, n_2, n_3, n_4, n_5, n_6 \right) = \begin{pmatrix} 1/16 & 1/16 & 1/16 & 1/16 & 1/8 & 1/8 & 1/8 \\ 1/16 & 1/16 & 1/16 & 1/16 & 1/8 & -1/8 & -1/8 \\ 5/8 & -3/8 & 1/8 & 1/8 & -1/4 & 0 & 0 \\ 1/4 & 1/4 & -1/4 & -1/4 & 0 & 0 & 0 \end{pmatrix} \]

Note that \( n_2 = n_3 \) and \( n_5 = n_6 \). Using the character table for \( M \) from the Atlas, we can now compute the map \( R(M) \to R(2D_8) \). Somewhat surprisingly:

**Lemma 3.4.2.** With notation as above, if \( V \) is a \( M \)-representation then \( n_4 \) is divisible by \( 8 \) and \( n_5 \) and \( n_6 \) are divisible by \( 16 \). The numbers \( n_0, \ldots, n_3 \) can all be odd. \( \square \)

The cohomology of McKay groups is easy to understand. It is easiest to work with \( \mathbb{Z} \)-coefficients, since \( H^*(G, \mathbb{Z}) = H^{*+1}(G, U(1)) \) is a graded ring. If \( G \) is McKay, this ring is supported only in even degrees. We give ourselves a variable \( t \) of degree 2 to track degrees of mixed-degree expressions, so that the total Chern character of a representation \( V \) is \( c(V) = 1 + c_1(V)t + c_2(V)t^2 + \ldots \); recall the Whitney sum formula that \( c(V \oplus W) = c(V)c(W) \). For \( G = 2D_8 \), we have \( H^2(2D_8, \mathbb{Z}) = \mathbb{Z}_2^2 \). The three non-zero elements are \( c_1(V_1), c_1(V_2), \) and \( c_1(V_3) = c_1(V_1) = c_1(V_2) + c_1(V_3) \). Furthermore, \( H^4(2D_8, \mathbb{Z}) = \mathbb{Z}_{16} \) comes with a distinguished generator, namely \( c_2(V_6) \); we work with this generator throughout, so that for example \( \langle c_2(V) = 3 \rangle \) means \( c_2(V) = 3c_2(V_6) \).

We may now compute the remaining \( c_2(V_i) \)’s. First, \( \text{Sym}^2(V_6) = V_4 \oplus V_4 \), and by working in \( SU(2) \) we know that \( c_1(\text{Sym}^2(V_6)) = 0 \) and \( c_2(\text{Sym}^2(V_6)) = 4 \). Then

\[ 1 + 4t^2 = (1 + tc_1(V_1))(1 + tc_1(V_4) + t^2c_2(V_4)). \]

But \( c_1(V_1)^2 = 0 \). Indeed, \( c_1(V_1)^2 \) is certainly 2-torsion, but \( V_1 \) is pulled back from a one-dimensional representation of \( 2D_{16} \), and the restriction map \( \mathbb{Z}_{32} \cong H^2(2D_{16}, \mathbb{Z}) \to H^2(2D_8, \mathbb{Z}) \cong \mathbb{Z}_{16} \) necessarily vanishes on the 2-torsion subgroup of the domain. It follows that \( c_2(V_1) = 4 \).

Continuing on, we have \( \text{Sym}^2(V_6) = V_5 \oplus V_6 \), and from \( SU(2) \) we learn that \( c_2(\text{Sym}^2(V_5)) = 10 \), so \( c_2(V_5) = 9 \).

Finally, \( \text{Sym}^4(V_6) = V_1 \oplus V_4 \oplus (V_2 \oplus V_3) \) and \( c_2(\text{Sym}^4(V_6)) = 20 \), and so

\[ c_2(V_2 \oplus V_3) = 16 = 0 \pmod{16}. \]

Given Lemma 3.4.2 and the fact that \( c_1(V_1)^2 = 0 \), we conclude:

**Lemma 3.4.3.** For every \( M \)-representation \( V \), \( c_2(V)|_{2D_8} = 0 \in H^2(2D_8, U(1)) \). \( \square \)

Since \( \omega^2|_{2D_8} \) has order 8, it cannot be a Chern class. The relation \( c_2 = -2p_2 \) verifies that \( \omega^3 \) also is not a fractional Pontryagin class. This completes the proof of Theorem 1.

Lemma 3.4.3 is our main evidence for Conjecture 3, which predicts that \( c_2(V) = 0 \) already in \( H^3(M, U(1)) \). We end by observing that Conjecture 3 follows from Conjecture 2:

**Lemma 3.4.4.** Suppose that \( H^3(M, U(1)) \cong \mathbb{Z}_{24} \). Then \( c_2(V) = 0 \) for every representation \( V \) of \( M \).

**Proof.** If \( H^3(M, U(1)) \cong \mathbb{Z}_{24} \), then restricting to \( 2D_8 \) gives an injection \( H^3(M, U(1)) \to H^3(2D_8, U(1)) \). The 2-part of the Lemma then follows from Lemma 3.4.3. According to [HC11], \( \omega^3|_{\langle 3C \rangle} \neq 0 \), where \( \langle 3C \rangle \) is the cyclic subgroup generated by an element of conjugacy class \( 3C \). Thus, if \( H^3(M, U(1)) \cong \mathbb{Z}_{24} \), restriction to \( \langle 3C \rangle \) is an injection on 3-parts. But a character table computation confirms that \( c_2(V)|_{\langle 3C \rangle} = 0 \) for every \( V \). \( \square \)
3.5. Further remarks on the value of $\omega^3$. We calculated the order of the Moonshine anomaly $\omega^3$ without needing to say much about its ambient cohomology group $H^3(M, U(1))$. In particular, it is possible that $H^3(M, U(1))$ has elements of order much higher than 24, and it is an interesting question to find out how close $\omega^3$ comes to saturating $H^3(M, U(1))$. The calculations in this section came from trying (and failing) to answer that question, and in particular to prove Conjecture 2 that $\omega^3$ generates $H^3(M, U(1))$.

Since $\omega^3|_{2^{1+24}}$ implements a T-duality between $2^{1+24}$ and $2 \times 2^{24}$, it pulls back from a class $\omega_0 \in H^3(2^{24}, U(1))$: specifically, $\omega_0$ is the anomaly of the $2^{24}$-action on $V_\Lambda = V^7//\mathbb{Z}_2$. The lattice conformal field theory $V_\Lambda$ is constructed so that the anomaly for $U(1)^{24} \subset \text{Aut}(V_\Lambda)$ is precisely the Leech lattice pairing thought of as a class in $H^4(BU(1)^{24}, \mathbb{Z}) = \text{Sym}^3(\mathbb{Z}^{24})$. It follows that $\omega_0$ the mod-2 reduction of the Leech pairing.

Although T-duality implies that $\omega^3|_{2^{1+24}}$ is pulled back from $2^{24}$, T-duality also guarantees that $\omega^3|_{2^{1+24}, Co_1}$ is not pulled back from $2^{24}.Co_1$, since $2^{24}.Co_0 \not\cong 2 \times (2^{24}.Co_1)$. It is interesting to compare the LHS spectral sequences for $2^{1+24}.Co_1$ and $2^{24}.Co_1$. We have

$$H^1(2^{1+24}, U(1)) = H^1(2^{24}, U(1)) = 2^{24},$$
$$H^2(2^{1+24}, U(1)) = 2^{274}.2, \quad H^2(2^{24}, U(1)) = 2.2^{274}.2.$$

The equality in the first line emphasizes that the pullback map $H^1(2^{24}, U(1)) \to H^1(2^{1+24}, U(1))$ is an isomorphism; the pullback $H^2(2^{24}, U(1)) \to H^2(2^{1+24}, U(1))$ is the obvious surjection.

Unfortunately, the group $Co_1$ and the modules $2^{274}.2$ and $2.2^{274}.2$ are much too large for programs like Cohomolo and HAP to handle. Fortunately, given a $G$-module $M$, $H^1(G, M)$ is “just a matrix computation” once one has a presentation of $G$. The ATLAS [CCN+85] does not claim a presentation for $Co_1$, but two presentations were found in [Soi87], and explicit matrices for one of these presentations were found by R. Parker via the methods of [HO06]; this presentation, and the generators satisfying it, are listed in Appendix A. Using this presentation, it is long but not difficult to compute:

$$H^1(Co_1, 2^{274}.2) \cong H^1(Co_1, 2.2^{274}.2) \cong \mathbb{Z}_2.$$

Consider now the following long exact sequence produced by taking $Co_1$ cohomology of the extension $2 \to 2.2^{274}.2 \to 2^{274}.2$:

\[
\begin{array}{cccc}
2 & 2.2^{274}.2 & 2^{274}.2 \\
H^0(Co_1, -) & 1 & \sim & 1 \\
H^1(Co_1, -) & 0 & 1 & \sim \longrightarrow 1 \\
H^2(Co_1, -) & 1 & \leftarrow & ? \\
\end{array}
\]

Here numbers denote the dimensions over $\mathbb{F}_2$ of the cohomology groups, and we do not draw in arrows that vanish.

One finds in particular that the pullback map $H^1(Co_1, 2.2^{274}.2) \to H^1(Co_1, 2^{274}.2)$ is an isomorphism. In total degree 3, therefore, the $E_2$ pages of the LHS spectral sequences for $H^* (2^{1+24}, Co_1, U(1))$ and $H^* (2^{24}.Co_1, U(1))$ agree except for the $H^0(Co_1, H^3(\ldots))$ entry, where the class we want is $\omega_0$, in the image of the pullback map:

\[
\begin{array}{cccc}
\omega_0 & Z_2 & H^2(2^{274}.2) & Z_2 \\
0 & 0 & H^2(2^{24}) & H^3(2^{24}) \\
* & 0 & Z_2 & Z_{12} \\
\end{array}
\quad
\begin{array}{cccc}
\omega_0 & Z_2 & H^2(2.2^{274}.2) & Z_2 \\
0 & 0 & H^2(2^{24}) & H^3(2^{24}) \\
* & 0 & Z_2 & Z_{12} \\
\end{array}
\]

\[
H^* (Co_1, H^* (2^{1+24}, U(1))) \quad H^* (Co_1, H^* (2^{24}, U(1)))
\]
To save space we abbreviate $H^i(C_0, M)$ by $H^i(M)$.

How is it possible, then, for $\omega^2|_{2^{1+24}C_0}$ not to be pulled back from $2^{24}C_0$? The only option is if, in fact, $\omega_0 \in H^3(2^{24}, U(1))$ does not extend to a class in $H^3(2^{24}.C_0, U(1))$. It fails to extend exactly when the $d_2$ differential, connecting the $E_2$ and $E_3$ pages, is such that $d_2\omega_0 \in H^2(2^{274}.2)$ is non-zero and in the kernel of $H^2(2^{274}.2) \rightarrow H^2(2^{274}.2)$. By the above long exact sequence analysis, this kernel is the image of $H^2(C_0, 2) = \mathbb{Z}_2$ in $H^2(2^{274}.2)$, i.e. it "is" the extension $C_0 = 2.C_0$. This is where $\omega^2$ "stores" the information that the T-dual to $2^{1+24}C_0$ is $2^{24}.C_0$ and not $2^{24}.C_0 \times 2$.

One way Conjecture 2 might be proved is if in fact $H^3(2^{1+24}.C_0, U(1)) \cong \mathbb{Z}_{24}$. For this to succeed, the $\mathbb{Z}_2 = H^1(C_0, 2^{274}.2)$ in the LHS for $H^*(2^{1+24}.C_0, U(1))$ will have to emit a nontrivial differential to either $H^3(C_0, 2^{24})$ or to $H^4(C_0, U(1))$. These groups seem far beyond what current computer power can handle.

Furthermore, one will need to handle the group $H^2(C_0, 2^{24})$. Direct computation of $H^2(C_0, 2^{24})$ requires not just a presentation of $C_0$ but also a complete list of syzygies, which is currently inaccessible. Certainly $H^2(C_0, 2^{24})$ is nontrivial: it contains the class $\kappa$ classifying the extension $2^{24}.C_0$, which is known not to split [Iva09]. It is reasonable to speculate that this is the only possible extension, so that $H^2(C_0, 2^{24}) \cong \mathbb{Z}_2$. In the notation of Section 2.3, the differential $d_2 : H^2(C_0, 2^{24}) \rightarrow H^4(C_0, U(1))$ is $\langle - \cup \kappa \rangle$ where $\langle , \rangle$ is the Leech pairing. A possible step towards proving Conjecture 2 would be to show that $\langle \kappa \cup \kappa \rangle \neq 0$. Using the arguments of Section 3.3, such a computation would already show that $H^3(2^{1+24}.C_0, U(1))$, and hence $H^3(M, U(1))$, has exponent $8 \times$ odd.

**APPENDIX A. SOICHER’S PRESENTATION FOR C0**

The presentation of $C_0$ from [Soi87] is as a quotient of the Coxeter group with diagram

$$a \rightarrow b \rightarrow c \rightarrow d \rightarrow e \rightarrow f \rightarrow g \rightarrow h \rightarrow i$$

by the further relations

$$a = (cd)^4, \quad 1 = (bcde)^8, \quad ((b(cd)^2efgh)^{13}i)^3 = 1.$$
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THE MOONSHINE ANOMALY

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