DEFORMED QUANTUM COHOMOLOGY AND (0,2) MIRROR SYMMETRY

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Abstract. We compute instanton corrections to correlators in the genus-zero topological subsector of a (0, 2) supersymmetric gauged linear sigma model with target space $\mathbb{P}^1 \times \mathbb{P}^1$, whose left-moving fermions couple to a deformation of the tangent bundle. We then deduce the theory’s chiral ring from these correlators, which reduces in the limit of zero deformation to the (2, 2) ring. Finally, we compare our results with the computations carried out by Adams et al. [ABS04] and Katz and Sharpe [KS06]. We find immediate agreement with the latter and an interesting puzzle in completely matching the chiral ring of the former.

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## Contents

1. Introduction 3

2. Background 4
   2.1. Classical Operators 4
   2.2. Operators in Instanton Backgrounds 6

3. Elements of the Algorithm 8
   3.1. Classical Lifts 9
   3.2. Instanton Lifts 10
   3.3. The Minor Map 11
   3.4. Bundle Degenerations 13
   3.5. Čech Cohomology of the Canonical Bundle 14
   3.6. The Trace Map 16
   3.7. Applicability 17

4. Chiral Ring Relations 18
   4.1. Calculation 18
   4.2. Further Evidence 20

5. Comparison with Previous Results 21
   5.1. Cohomological Computations 21
   5.2. Computation via Mirror Symmetry 21

6. Conclusions 25

Acknowledgments 26

Appendix A. Lifts 27

Appendix B. Correlators 28

References 29
1. Introduction

Twisted non-linear sigma models (NLSM) with $\mathcal{N} = (2, 2)$ supersymmetry have a rich and fascinating structure. One particularly interesting facet is that a subset of their chiral operators form a ring, which is a quantum-corrected version of the classical cohomology ring of the target space [Wit88, LVW89, Wit90]. Generically, NLSMs with $(2, 2)$ supersymmetry may be deformed to $(0, 2)$ theories, with a ring of ground-state operators generalizing the $(2, 2)$ chiral ring [ABS04, ADE06]. These operators and their correlation functions have been recently discussed in [Sha05, Wit05, Tan06, Sha06].

Mathematically, the $(2, 2)$ chiral ring is described by quantum cohomology. Although most $(0, 2)$ chiral rings do not comprise deformations of $(2, 2)$ rings, we study a specific example fulfilling this condition, and comment on the general class to which it belongs. In spite of the reduced supersymmetry, the subsector of operators under consideration remains topological [ADE06]. In particular, this ring is a Frobenius algebra, a fact which we will exploit in our computations in §4.1.

A particularly interesting kind of $(0, 2)$ model describes a theory whose left-moving fermions couple to a holomorphic vector bundle on the target space. Such $(0, 2)$ heterotic theories, once believed to have no interpretation as string vacua due to instabilities in instanton sectors [DSWW86, DSWW87, Dis87, DG88], have been shown to in fact be stable and flow to a conformal field theory [SW95, B+95, BS03, BW03]. These theories may be twisted by a non-anomalous $U(1)$ current, with anomaly cancellation as in [DGM96, BW03].

In the following sections, we compute the chiral ring of a $(0, 2)$ NLSM coupled to a deformation $\mathcal{E}$ of the tangent bundle of $\mathbb{P}^1 \times \mathbb{P}^1$ by utilizing a gauged linear sigma model (GLSM) description [MRP95, DK94]. In §2 we review the field content and operators for trivial and non-trivial instanton sectors. The bundle to which the left-moving fermions couple is derived and shown to arise as the cokernel in a short exact sequence of sheaves on the target space. Finally, the instanton moduli space and the sheaf induced upon it by fermionic couplings are described. As this sheaf is unobstructed, the GLSM correlators will have a simple interpretation in the geometric phase as an integral over the GLSM moduli space of zero modes, as in [AM93].

Each step in the algorithm for computation of individual correlators is described in §3: finding Čech representatives of operators, computing their wedge product, and application of the isomorphism from the top Čech cohomology to $\mathbb{C}$. This isomorphism is simply the chiral ring’s Frobenius form. Additionally, we discuss the applicability of the algorithm to
more general toric varieties.

Once we have calculated correlators for instanton sectors of overall degree \( \leq 2 \), we compute the chiral ring by deducing a quadratic relation between the m. We perform this analysis in \( \S 4 \) and offer further evidence of our derived relations from the Coulomb branch of the GLSM. Finally in \( \S 5 \) we offer a comparison between our results and those of [KS06] and [ABS04].

2. Background

2.1. Classical Operators. For convenience, we review a modicum of necessary information about the particular half-twisted \((0, 2)\) GLSM considered here. GLSMs with \((2, 2)\) supersymmetry are described in [MRP95, Wit93], those with \((0, 2)\) in [DK94], and half-twisted heterotic GLSMs in [SW95].

Our model includes four bosonic fields \( \phi_1, \phi_2, \phi_3, \phi_4 \), charged under \( U(1) \times U(1) \) with respective charges \((1, 0), (1, 0), (0, 1), (0, 1)\). The space of vacua is given by the vanishing of the D-terms

\[
|\phi_1|^2 + |\phi_2|^2 - r_1 = 0, \quad |\phi_3|^2 + |\phi_4|^2 - r_2 = 0
\]

with FI terms \( r_1, r_2 \), leading in the usual way to \( M = \mathbb{P}^1 \times \mathbb{P}^1 \) in the geometric phase where \( r_1 \) and \( r_2 \) are positive.

Before twisting, we have right-moving fermions \( \psi_i \), superpartners of the \( \phi_i \), as well as four left-moving fermions \( \rho_1, \rho_2, \rho_3, \rho_4 \). The \( \rho_i \) have the same charges as the corresponding \( \phi_i \), but are not superpartners of these fields. The superfields \( \Gamma_i \), for which \( \rho_i \) are the respective lowest components, are not chiral. Their deviation from chirality is measured by functions \( E_i \) of the superfields \( \Phi_i \), whose lowest components are the \( \phi_i \), and other chiral superfields \( \Sigma, \tilde{\Sigma} \): [Wit93, ABS04]:

\[
\bar{\nabla}_+ \Gamma_i = \sqrt{2} E_i(\Phi_j, \Sigma, \tilde{\Sigma}).
\]

If there is \((2, 2)\) supersymmetry, then the \( \rho_i \) are the superpartners of the \( \phi_i \) under the additional supersymmetries.

In the geometric phase, the \( \psi_i \) fill out the tangent bundle of \( M \) after twisting. It will be useful to be explicit about this: consider the tangent bundle of \( M \), described as a cokernel
in the exact sequence

\[
\begin{pmatrix}
 x_0 & 0 \\
 x_1 & 0 \\
 0 & y_0 \\
 0 & y_1
\end{pmatrix} : 0 \to O \oplus O \to O(1,0)^2 \oplus O(0,1)^2 \to T_M \to 0.
\]

We will write \( \{x_0, x_1, y_0, y_1\} \) for the local coordinates on the moduli space induced by the fields \( \{\phi_1, \phi_2, \phi_3, \phi_4\} \): more details about this relationship will be described in §2.2. Here, \( O(m,n) \) is the sheaf on \( M \) defined as \( \pi_1^*O(m) \otimes \pi_2^*O(n) \), with \( \pi_1 \) and \( \pi_2 \) the natural projection maps to each \( \mathbb{P}^1 \), and \( O(m,n)^2 \equiv O(m,n) \oplus O(m,n) \).

Before imposing gauge equivalence, the \( \psi_i \) fill out the bundle \( O(1,0)^2 \oplus O(0,1)^2 \) on \( M \), and (3) says that after gauge equivalence is taken into account, the \( \psi_i \) actually fill out the quotient bundle \( T_M \).

The situation for the \( \rho_i \) is similar. Prior to imposing fermionic gauge symmetries, the \( \rho_i \) fill out the bundle \( O(1,0)^2 \oplus O(0,1)^2 \) on \( M \) as before. Subsequent to imposition, however, the fermionic and ordinary gauge symmetries do not necessarily coincide; in fact \( \{\rho_1, \rho_2, \rho_3, \rho_4\} \) fill out the quotient bundle \( E \) of \( O(1,0)^2 \oplus O(0,1)^2 \). In the (2, 2) situation we have that \( E = T_M \), but in general \( E \) is a deformation of \( T_M \).

The first cohomology group valued in the sheaf of endomorphisms of \( T_M \) describes the space of all first-order deformations. Since \( T_M \simeq O(-2,0) \oplus O(0,-2) \), we compute that

\[
\text{End}(T_M) \simeq O(-2,2) \oplus O(0,0)^2 \oplus O(2,-2),
\]

leading to

\[
H^1(M, \text{End}(T_M)) \simeq \left[ H^1(\mathbb{P}^1, O(-2)) \otimes H^0(\mathbb{P}^1, O(2)) \right] \\
\quad \oplus \left[ H^0(\mathbb{P}^1, O(2)) \otimes H^1(\mathbb{P}^1, O(-2)) \right],
\]

a six-dimensional complex vector space. We therefore introduce six complex parameters \( \{\epsilon_1, \epsilon_2, \epsilon_3, \gamma_1, \gamma_2, \gamma_3\} \) as a basis for the space of deformations and a matrix

\[
F \equiv \begin{pmatrix}
 x_0 & x_1 & \gamma_1 y_0 + \gamma_2 y_1 & \gamma_3 y_0 \\
 \epsilon_1 x_0 + \epsilon_2 x_1 & \epsilon_3 x_0 & y_0 & y_1
\end{pmatrix},
\]

which encodes the most general deformation \( E \) of the tangent bundle as the cokernel of \( \text{coker}(F) \):

\[
0 \to O \oplus O \xrightarrow{F} O(1,0)^2 \oplus O(0,1)^2 \to E \to 0.
\]

\(^1\)All commutative diagrams and some exact sequences in this paper were typeset using Paul Taylor's diagrams package, available at [http://www.cs.man.ac.uk/~pt/diagrams/](http://www.cs.man.ac.uk/~pt/diagrams/)
In other words, the data of $F$ describes the fermionic gauge symmetries, and the fermions $\rho_i$ fill out the bundle $\mathcal{E}$ after imposition of the symmetries. Physically, the functions $E_i$ in (2) encode the matrix [3], and thus the deformation as well. Careful readers may note that $\mathcal{E}$ is not necessarily a bundle; we will explore this detail further in §3.4, but for this section and the next we will assume that it is.

The topological sector of the twisted theory, described mathematically by the cohomology groups $H^p(\Lambda^q\mathcal{E}^*)$ [DG88], is comprised of operators corresponding to massless Ramond-Ramond states in the untwisted theory. In the twisted theory, massless states are expressed in terms of the fields as

$$f_{\Omega}(\phi)\psi^J\rho^\Omega,$$

where $J$ and $\Omega$ are multi-indices for $T_M$ and $\mathcal{E}^*$, respectively. Correlation functions of products of these fields correspond naturally to cup/wedge products of the cohomology representatives. For our particular theory, the relevant operators are all elements of $H^1(\Lambda^1\mathcal{E}^*)$, and non-vanishing classical correlation functions correspond to elements of $H^2(\Lambda^2\mathcal{E}^*)$.

Note that the bundle $\mathcal{E}$, as a deformation of the tangent bundle, satisfies

$$\Lambda^2\mathcal{E}^* \cong K_M,$$

so that $\Lambda^2\mathcal{E}^* \cong \mathcal{O}(-2, -2)$ on $M$ and $H^2(\Lambda^2\mathcal{E}^*) \cong H^2(K_M) \cong \mathbb{C}$. Examination of the exact sequence (6) further reveals that the bundle satisfies the heterotic anomaly cancellation conditions $c_1(\mathcal{E}) = c_1(T_M)$ and $\text{ch}_2(\mathcal{E}) = \text{ch}_2(T_M)$.

As we will be working with $\mathcal{E}^*$, the relevant sequences will be the dual of (6),

$$0 \longrightarrow \mathcal{E}^* \longrightarrow \mathcal{O}(-1, 0)^2 \oplus \mathcal{O}(0, -1)^2 \xrightarrow{F} \mathcal{O} \oplus \mathcal{O} \longrightarrow 0,$$

and its induced long exact sequence in cohomology,

$$0 \longrightarrow H^0(\mathcal{E}^*) \longrightarrow H^0(\mathcal{O}(-1, 0)^2 \oplus \mathcal{O}(0, -1)^2) \longrightarrow H^0(\mathcal{O} \oplus \mathcal{O})$$

$$\xrightarrow{H^1(\mathcal{E}^*)} H^1(\mathcal{O}(-1, 0)^2 \oplus \mathcal{O}(0, -1)^2) \longrightarrow H^1(\mathcal{O} \oplus \mathcal{O}) \longrightarrow \cdots.$$

Since $\mathcal{O}(-1, 0)^2 \oplus \mathcal{O}(0, -1)^2$ has neither global sections nor degree 1 cohomology, we see that

$$H^1(\mathcal{E}^*) \cong H^0(\mathcal{O} \oplus \mathcal{O}) \cong \mathbb{C}^2.$$

We explicitly construct this first isomorphism in §3.4 while finding Čech representatives of operators.

2.2. Operators in Instanton Backgrounds. The preceding analysis discussed the zero-instanton (classical) case, when the image of $\Sigma$ under the map $\varphi$ is homologous to a point
in $M$. We would also like to compute correlation functions in the presence of non-trivial instanton backgrounds.

A non-trivial instanton is a non-homologically trivial map. The space of algebraic maps $\mathbb{P}^1 \to \mathbb{P}^1$ is $\mathbb{Z}$ graded, with negative grading corresponding to the empty set, and zero grading corresponding to the set of trivial maps (that is, constant maps to a point). For positive $n$, the set of degree $n$ maps to $\mathbb{P}^1$ consists of pairs $\{(\phi_1(z), \phi_2(z))\}$ of homogeneous degree $n$ polynomials in the worldsheet variables collectively denoted $z$. Thus, for maps $\mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree $(m, n)$, $\phi_1$ and $\phi_2$ become sections of $\mathcal{O}_{\mathbb{P}^1}(m)$, while $\phi_3$ and $\phi_4$ become sections of $\mathcal{O}_{\mathbb{P}^1}(n)$.

We write the moduli space of such maps as $\mathcal{M}$, and we can use the $\phi_i$ to define local coordinates. In terms of the worldsheet homogeneous coordinates $z_0, z_1$, maps of instanton degree $m$ are written as

$$\phi_i(z_0, z_1) = \sum_{j=0}^{m} a_{ij} z_0^j z_1^{m-j}.$$ 

Here, the $a_{ij}$ are complex numbers. Imposing gauge equivalence, we see that in the geometric phase, the pairs of polynomials $(\phi_1, \phi_2)$ and $(\phi_3, \phi_4)$ are to be considered up to independent scalar multiplications. Thus, we combine the collections $a_{1,i}$ and $a_{2,j}$ as the set $\{x_0, \ldots, x_{2m+1}\}$ and the collections $a_{3,i}$ and $a_{4,j}$ as the set $\{y_0, \ldots, y_{2n+1}\}$. By gauge equivalence, each collection is defined up to independent scalar multiplications, so that the $x_i$ and $y_j$ behave like homogeneous coordinates on the product of two projective spaces.

We thus conclude that the degree $(m, n)$ moduli space is

$$\mathcal{M} = \mathbb{P}^{2m+1} \times \mathbb{P}^{2n+1}.$$ 

Following the customary notational abuse, we use non-linear sigma model language and think of $((\phi_1, \phi_2), (\phi_3, \phi_4))$ as a map from $\mathbb{P}^1$ to $\mathbb{P}^1 \times \mathbb{P}^1$ of degree $(m, n)$, although the map does not exist at points of the worldsheet where either $\phi_1$ and $\phi_2$ or $\phi_3$ and $\phi_4$ have simultaneous zeros.

In an instanton background, the operators of the theory become sections of sheaves on the instanton moduli space $\mathcal{M}$: expanding out $\rho_1$ and $\rho_2$ in terms of the instanton moduli space coordinates, we see that they in fact become sections of a sheaf $\mathcal{F}$ on $\mathcal{M}$. There is a natural description of $\mathcal{F}^*$ in terms of a short exact sequence on $\mathcal{M}$,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F}^* & \longrightarrow & \mathcal{O}(-1,0)^{2(m+1)} & \oplus & \mathcal{O}(0,-1)^{2(n+1)} & \longrightarrow & \mathcal{F}_{mn} & \longrightarrow & \mathcal{O}^2 & \longrightarrow & 0,
\end{array}$$

(12)
where $F_{mn}$ is the $(2m + 2n + 4) \times 2$ matrix

$$(13) \quad F_{mn} = \begin{pmatrix} x_0 & x_1 & \cdots & x_m & x_{m+1} & \cdots & x_{2m+1} \\ \epsilon_1 x_0 + \epsilon_2 x_{m+1} & \epsilon_1 x_1 + \epsilon_2 x_{m+2} & \cdots & \epsilon_1 x_{m+1} + \epsilon_2 x_{2m+1} & \epsilon_3 x_0 & \cdots & \epsilon_3 x_m \\ \gamma_1 y_0 + \gamma_2 y_{n+1} & \gamma_1 y_1 + \gamma_2 y_{n+2} & \cdots & \gamma_1 y_{n+1} + \gamma_2 y_{2n+1} & \gamma_3 y_0 & \cdots & \gamma_3 y_n \end{pmatrix},$$

where $x_i$ (resp. $y_j$) are coordinates on $\mathbb{P}^m$ (resp. $\mathbb{P}^n$).

Morally, in non-linear sigma model language, the sheaf $\mathcal{F}$ is defined on the moduli space $\mathcal{M} = \mathbb{P}^{2m+1} \times \mathbb{P}^{2n+1}$ in terms of the evaluation map $ev : \Sigma \times \mathcal{M} \rightarrow \mathcal{M}$ and the projection $\pi : \Sigma \times \mathcal{M} \rightarrow \mathcal{M}$ to be

$$\mathcal{F} \equiv \pi_* ev^* \mathcal{E}.$$ 

Note that the matrices (5) and (13) are of equal rank. As alluded to earlier, we will see that for some values of the parameters, $\mathcal{E}$ ceases to be locally free. For such values, $\mathcal{F}$ also ceases to be a bundle.

The stalk of this sheaf at a map $\varphi$ is $H^0(\Sigma, \varphi^* \mathcal{E})$. As explained in §5.3 of [KS06], this is the correct sheaf to describe operators as sections on the moduli space. One should take this statement with a grain of salt, since as noted earlier, the evaluation map is not defined everywhere.

When dealing with the unobstructed case, $R^1 \pi_* ev^* \mathcal{E} = R^1 \pi_* ev^* T_\mathcal{M} = 0$, we will have that

$$(14) \quad \det \mathcal{F}^* = \Lambda^{\text{top}} \mathcal{F}^* \cong K_\mathcal{M},$$

so that non-trivial instantons also satisfy the anomaly cancellation condition. Note that in the linear sigma model description, (14) can be verified directly: taking determinants in (12) yields

$$\Lambda^{\text{top}} \mathcal{F}^* \cong \mathcal{O}(-2m - 2, -2n - 2) \cong K_\mathcal{M}.$$ 

As with classical operators, correlation functions in instanton backgrounds correspond to cup/wedge products of the cohomology representatives. However, due to the larger $U(1)$ anomaly on the instanton moduli space, more operator insertions are required to yield a non-vanishing correlator.

### 3. Elements of the Algorithm

We now give an account of the various steps in the computation of correlation functions. As mentioned before, elements of $H^1(\mathcal{E}^*)$ comprise the space of topological operators of interest. We construct Čech representatives of these operators: in particular, we follow the
construction of the coboundary homomorphism (from the long exact sequence in cohomology induced by (11) to the penultimate step in order to express operators as cochains in $C^1(\mathcal{O}(-1,0)^2 \oplus \mathcal{O}(0,-1)^2)$. By omitting the final step, finding an element of $H^1(\mathcal{E}^*)$, we will obtain significant computational simplification when computing correlation functions.

We also work out the injection from $C^\text{top}(\det \mathcal{E}^*)$ to $C^\text{top}(\det(\mathcal{O}(-1,0)^2 \oplus \mathcal{O}(0,-1)^2))$, in the form of factors that must be divided from the correlation functions. Finally, we construct the isomorphism $H^\text{top}(\det \mathcal{E}^*) \cong H^\text{top}(K) \cong \mathbb{C}$, which evaluates the correlator.

For the purposes of Čech cohomology, we will take the usual algebraic cover of $\mathbb{P}^n$ by open sets $U_i = \{(x_0, x_0, \cdots, x_n) \in \mathbb{P}^n | x_i \neq 0\}$, and cover $\mathbb{P}^m \times \mathbb{P}^n$ by open sets

\begin{equation}
U_{i,j} \equiv \{(x_0, x_1, \cdots, x_m), [y_0, y_1, \cdots, y_n] \in \mathbb{P}^m \times \mathbb{P}^n | x_i \neq 0 \text{ and } y_j \neq 0\}.
\end{equation}

3.1. Classical Lifts. In this section, we carry out the steps in the construction of the coboundary map (11). Consider the complex of short exact sequences

\begin{equation}
\begin{array}{cccccc}
0 & \rightarrow & C^0(\mathcal{E}^*) & \rightarrow & C^0(\mathcal{O}(-1,0)^2 \oplus \mathcal{O}(0,-1)^2) & \rightarrow & C^0(\mathcal{O} \oplus \mathcal{O}) & \rightarrow & 0 \\
\delta_0 & & \delta_0 & & \delta_0 & & \\
0 & \rightarrow & C^1(\mathcal{E}^*) & \rightarrow & C^1(\mathcal{O}(-1,0)^2 \oplus \mathcal{O}(0,-1)^2) & \rightarrow & C^1(\mathcal{O} \oplus \mathcal{O}) & \rightarrow & 0 \\
\downarrow \delta_1 & & \downarrow \delta_1 & & \downarrow \delta_1 & & \\
\vdots & & \vdots & & \vdots & &
\end{array}
\end{equation}

Beginning with each of the elements $(0, 1), (1, 0) \in \ker \delta_0 \subset C^0(\mathcal{O} \oplus \mathcal{O})$, we use the fact that the matrix $F$ defines a surjective map to find an element of $C^0(\mathcal{O}(-1,0)^2 \oplus \mathcal{O}(0,-1)^2)$ which lifts it. That is, for each $U_i \subset M$, we must find a column vector $V$ whose first (respectively, last) two rows are rational functions of overall degree $-1$ in the first (respectively, second) $\mathbb{P}^1$'s variables;

\begin{equation}
\begin{pmatrix}
x_0 & x_1 & \gamma_1 y_0 + \gamma_2 y_1 & \gamma_3 y_0 \\
\epsilon_1 x_0 + \epsilon_2 x_1 & \epsilon_3 x_0 & y_0 & y_1
\end{pmatrix}
V = \begin{pmatrix} 1 \\ 0 \end{pmatrix},
\end{equation}

and similarly for $(0, 1)$. See the end of this section for a further discussion of this computation, and Appendix A for the exact form of the lifts.

Given these elements of $C^0(\mathcal{O}(-1,0)^2 \oplus \mathcal{O}(0,-1)^2)$, we apply $\delta_0$ to obtain elements $Y$ and $\tilde{Y}$ of $C^1(\mathcal{O}(-1,0)^2 \oplus \mathcal{O}(0,-1)^2)$. By commutativity of the diagram, and the fact that $\delta_0(1, 0) = \delta_0(0, 1) = 0$, we see that these elements vanish upon application of $F$. The chains
Y and \( \tilde{Y} \) are therefore in the image of \( a \), so that there are elements \( \psi, \tilde{\psi} \in C^1(\mathcal{E}^*) \) satisfying \( a(\psi) = Y, a(\tilde{\psi}) = \tilde{Y} \). We will not obtain the explicit form of \( \psi \) and \( \tilde{\psi} \), opting rather to pursue computations using the \( C^1(\mathcal{O}(-1, 0)^2 \oplus \mathcal{O}(0, -1)^2) \) representatives: in §3.3 we will use the map \( a \) to find the proper element of \( H^2(\Lambda^2\mathcal{E}^*) \).

We now show that upon lifting to elements in \( C^1(\mathcal{E}^*) \), we will in fact obtain an element of \( H^1(\mathcal{E}^*) \). This is a standard diagram chase argument, which we reproduce for convenience of the reader. For the proof of this statement, we add numerical indices to the maps \( a \) and \( F \), explicitly indicating to which row in diagram (16) we are referring. Let \( \psi \) and \( \tilde{\psi} \) be elements of \( C^1(\mathcal{E}^*) \) satisfying

\[
a_1(\psi) = Y \quad a_1(\tilde{\psi}) = \tilde{Y}.
\]

Exactness of the diagram implies that \( \delta_1 Y = \delta_1 \tilde{Y} = 0 \), while commutativity tells us that \( a_2 \circ \delta_1 (\psi) = a_2 \circ \delta_1 (\tilde{\psi}) = 0 \), so that by injectivity of \( a \), \( \delta_1 \psi = \delta_1 \tilde{\psi} = 0 \). This tells us that \( \psi, \tilde{\psi} \) represent cohomology classes in \( H^1(\mathcal{E}^*) \). We reiterate, our computational intent lies with the elements \( Y \) and \( \tilde{Y} \), not \( \psi \) and \( \tilde{\psi} \).

Thus far, we have treated the lifts abstractly. Let us mention a few details of the analysis. On each open set, we must construct a solution to the system of simultaneous equations (17). Such a solution will be a well-defined rational function in terms of local variables. This implies that only monomials in the non-vanishing variables on that open set may appear in the denominator.

For the simple deformation that appeared in [KS06], solutions were constructable by hand. In the fully deformed theory, computations are considerably more complicated: a computer algebra system is called for. We attempted to use Mathematica [Mat05], specifically its “Solve” function. Unfortunately, Solve could not be instructed to restrict to rational functions with monomials of certain variables in the denominator. A custom solver was written to account for this condition.

3.2. Instanton Lifts. In instanton backgrounds, the operators in the topological subsector (7) become elements of \( H^1(\mathcal{M}, \mathcal{F}^*) \) by virtue of the reinterpretation of the \( \rho \) fermions as sections of \( \mathcal{F}^* \). Naïvely, one might think that we must reapply the algorithm for classical lifts to the new sheaf. However, a trick allows us to use the degree \((0, 0)\) solutions as lifts in instanton sectors of arbitrary degree.

We begin by constructing, as in (16), an exact sequence of complexes from the data of the short exact sequence of sheaves (12). These complexes allow us find elements of

\[
C^0 \left( \mathcal{O}(-1, 0)^{2(m+1)} \oplus \mathcal{O}(0, -1)^{2(n+1)} \right)
\]
that lift basis elements of $C^0(\mathcal{O})$ on each open set of $\mathbb{P}^{2m+1} \times \mathbb{P}^{2n+1}$. The deformation matrix $F_{mn}$ describing the sheaf $\mathcal{F}$ on $\mathbb{P}^{2m+1} \times \mathbb{P}^{2n+1}$ appears in [13].

On the open set where $x_i \neq 0$ and $y_j \neq 0$, we can use the solution from the degree $(0,0)$ case by noting that only columns $(i+1)$ and $(m+2+i)$ columns of $F_{mn}$ involve $x_i$, and only columns $[2m+2+(j+1)]$ and $[2m+2+(n+2+j)]$ involve $y_j$. The other variables appearing those columns do not appear elsewhere, so we obtain the desired lift by letting $\{x_0, x_1, y_0, y_1\}$ in the degree $(0,0)$ lift go to $\{x_{i+1}, x_{m+2+i}, y_{j+1}, y_{n+2+j}\}$ and writing the new solution as a column vector

$$
\begin{align*}
(m + 2 + i)\text{th} & \quad (2m + 3 + j)\text{th} \\
(\cdots, V_1, \cdots, V_2, \cdots, V_3, \cdots, V_4, \cdots), \\
(i + 1)\text{th} & \quad (2m + n + 4 + j)\text{th}
\end{align*}
$$

where the elements of $V$ occupy the indicated positions, and all other entries are zero. With this realization, lifts are quickly and simply computed on all open sets in the moduli space.

3.3. The Minor Map. As noted before, the only non-vanishing correlators in the zero-instanton sector are two-point functions. The operator product inside the correlation function must correspond to $\psi_i \cup \psi_j \in H^2(\Lambda^2\mathcal{E}^*)$, for some $\psi_i, \psi_j \in H^1(\mathcal{E}^*)$. In higher instanton sectors, the condition for a correlator to be non-vanishing will change. Since we are examining a theory with fields associated to an unobstructed sheaf, the appropriate number of operators will simply be the complex dimension of the moduli space.

We first discuss the zero instanton case, before giving a general formula. With our bundle restriction [8], we see that the second exterior power $\Lambda^2\mathcal{E}^* \cong \mathcal{O}(-2, -2)$. Since we will be dealing with representatives $Y$ and $\tilde{Y}$ of $\psi$ and $\tilde{\psi}$, respectively, their cup/wedge product will be an element of

$$
C^2(\Lambda^2 [\mathcal{O}(-1,0)^2 \oplus \mathcal{O}(0,-1)^2]) \cong C^2(\mathcal{O}(-2,0) \oplus \mathcal{O}(-1,-1)^4 \oplus \mathcal{O}(0,-2)),
$$

and we will obtain our final representative in $H^2(\mathcal{O}(-2,-2))$ by specifying the inclusion

$$
i: \mathcal{O}(-2,-2) \hookrightarrow \mathcal{O}(-2,0) \oplus \mathcal{O}(-1,-1)^4 \oplus \mathcal{O}(0,-2).
$$

Any such map can be specified by giving an element of $\mathcal{O}(0,2) \oplus \mathcal{O}(1,1)^4 \oplus \mathcal{O}(2,0)$. How can we find the proper element? Since the diagram [9] is exact, the kernel of the matrix defining the map between $\mathcal{O}(-1,0)^2 \oplus \mathcal{O}(0,-1)^2$ and $\mathcal{O}^2$ must contain the image of such an element. This implies that the corresponding map in the exact sequence of determinant bundles also contains the image, so that on each open set the lifts will have a common factor – an element of $\mathcal{O}(0,2) \oplus \mathcal{O}(1,1)^4 \oplus \mathcal{O}(2,0)$.  

We compute this element, and thus the map, by finding the maximal minor determinants of the kernel of (13). A local basis \( \{ e_1, e_2, f_1, f_2 \} \) for \( \mathcal{O}(0,1)^2 \oplus \mathcal{O}(1,0)^2 \) induces a basis \( \{ e_1 \wedge e_2, \ldots, f_1 \wedge f_2 \} \) for \( \mathcal{O}(0,2) \oplus \mathcal{O}(1,1)^4 \oplus \mathcal{O}(2,0) \) on the same open set, so that the explicit form of the map is given by the determinant of the appropriate maximal minor multiplied by a basis element. A basis for the kernel of (13) is furnished by the column space of

\[
\begin{pmatrix}
  x_1 y_1 - x_0 y_0 \gamma_3 \epsilon_3 & x_1 y_0 - x_0 (y_0 \gamma_1 + y_1 \gamma_2) \epsilon_3 \\
  -x_0 y_1 + x_0 y_0 \gamma_3 \epsilon_1 + x_1 y_0 \gamma_3 \epsilon_2 & x_0 (y_1 \gamma_2 \epsilon_1 + y_0 (\gamma_1 \epsilon_1 - 1)) + x_1 (y_0 \gamma_1 + y_1 \gamma_2) \epsilon_2 \\
  0 & x_3 x_0^2 - x_1 \epsilon_1 x_0 - x_1^2 \epsilon_2 \\
  \epsilon_3 x_0^2 - x_1 \epsilon_1 x_0 - x_1^2 \epsilon_2 & 0
\end{pmatrix},
\]

and the maximal minor determinants of this matrix are

\[
\begin{pmatrix}
  (\gamma_3 y_0^2 - y_1 \gamma_1 y_0 - y_1^2 \gamma_2) (\epsilon_3 x_0^2 - x_1 \epsilon_1 x_0 - x_1^2 \epsilon_2), \\
  - (\epsilon_3 x_0^2 - x_1 \epsilon_1 x_0 - x_1^2 \epsilon_2) (x_0 y_0 \gamma_3 \epsilon_3 - x_1 y_1), \\
  (\epsilon_3 x_0^2 - x_1 \epsilon_1 x_0 - x_1^2 \epsilon_2) (x_0 (y_0 \gamma_1 + y_1 \gamma_2) \epsilon_3 - x_1 y_0), \\
  - (x_0 (y_1 \gamma_2 \epsilon_1 + y_0 \gamma_1 \epsilon_1 - 1) + x_1 (y_0 \gamma_1 + y_1 \gamma_2) \epsilon_2) (\epsilon_3 x_0^2 - x_1 \epsilon_1 x_0 - x_1^2 \epsilon_2), \\
  - (\epsilon_3 x_0^2 - x_1 \epsilon_1 x_0 - x_1^2 \epsilon_2)^2
\end{pmatrix}.
\]

Note that each term contains a common factor \(-\epsilon_3 x_0^2 + x_1 \epsilon_1 x_0 + x_1^2 \epsilon_2\). Upon division by this polynomial, we find that the element of \( \mathcal{O}(0,2) \oplus \mathcal{O}(1,1)^4 \oplus \mathcal{O}(2,0) \) encoding the desired map takes the form

\[
(\gamma_3 y_0^2 + y_1 \gamma_1 y_0 + y_1^2 \gamma_2) e_1 \wedge e_2 + (x_0 y_0 \gamma_3 \epsilon_3 - x_1 y_1) e_1 \wedge f_1 \\
+ (x_1 y_0 - x_0 (y_0 \gamma_1 + y_1 \gamma_2) \epsilon_3) e_1 \wedge f_2 + [x_0 (y_1 - y_0 \gamma_3 \epsilon_1) - x_1 y_0 \gamma_3 \epsilon_2] e_2 \wedge f_1 \\
+ [x_0 (y_1 \gamma_2 \epsilon_1 + y_0 (\gamma_1 \epsilon_1 - 1)) + x_1 \epsilon_2 (y_0 \gamma_1 + y_1 \gamma_2)] e_2 \wedge f_2 \\
+ (\epsilon_3 x_0^2 - x_1 \epsilon_1 x_0 - x_1^2 \epsilon_2) f_1 \wedge f_2.
\]

Each of the correlators we compute contains this term as a multiplicative factor.

In order to actually compute a correlation function, we must take the cup/wedge product of two cohomology representatives. Since any such element will be proportional to (20), we need only compute the \( e_1 \wedge e_2 \) term. This may be accomplished by first multiplying one of the cohomological representatives by \( f_1 \wedge f_2 \), and then dividing by \(-\gamma_3 y_0^2 + y_1 \gamma_1 y_0 + y_1^2 \gamma_2\). Truncating the product in this way relieves us of the computational burden of the other five wedge coefficients, a savings that vastly increases in higher instanton sectors. To check, a few correlators were completely computed, and as expected were exactly proportional to the map.

For higher-degree instanton sectors, the process applies *mutatis mutandis* to the kernel and minors of the matrix (13).
3.4. **Bundle Degenerations.** Let us emphasize that the parameter space spanned by the \( \epsilon \)'s and \( \gamma \)'s describes deformations of the tangent sheaf: there may be configurations where \( \mathcal{E}^* \) ceases to be a bundle. Such a configuration occurs precisely at the points on the instanton moduli space where each of the minors vanishes identically at some point \( p \in M \). At such a point, the deformation matrix \([13]\) would no longer be of rank two: the dimension of the stalk of \( \mathcal{E}^* \) changes, so it cannot be a bundle.

A polynomial in the deformation parameters detects the possibility of such points. We will restrict our attention to the open set in \( H^1(\text{End}(T\mathcal{M})) \) where this polynomial does not vanish, so that we have an honest bundle.

In order to detect the of points where the six deformation matrix minors

\[
\begin{align*}
\epsilon_3 x_0^2 - \epsilon_1 x_0 x_1 - \epsilon_2 x_1^2 & \quad x_0 y_0 - (y_0 \gamma_1 + y_1 \gamma_2) (x_0 \epsilon_1 + x_1 \epsilon_2) \\
x_0 y_1 - y_0 \gamma_3 (x_0 \epsilon_1 + x_1 \epsilon_2) & \quad x_1 y_0 - x_0 (y_0 \gamma_1 + y_1 \gamma_2) \epsilon_3 \\
x_1 y_1 - x_0 y_0 \gamma_3 \epsilon_3 & \quad y_1 (y_0 \gamma_1 + y_1 \gamma_2) - y_0^2 \gamma_3
\end{align*}
\]

vanish, we construct a Gröbner basis for the ideal \( I \) generated by the polynomials \([21] \), using an elimination order to eliminate the variables \( \{x_0, x_1, y_0, y_1\} \). In other words, we choose an ordering of the monomials so that the part of the Gröbner basis for \( I \) which does not include any of \( \{x_0, x_1, y_0, y_1\} \) forms a basis for the intersection of \( I \) with the subring of polynomials in the \( \epsilon \)'s and the \( \gamma \)'s. The resulting equations in the parameter variables define by construction the locus of parameter values for which \( \mathcal{E} \) fails to be a bundle at some point of \( \mathbb{P}^1 \times \mathbb{P}^1 \), i.e. values for which \( \mathcal{E} \) is not a bundle.

The computation is actually slightly more complicated. The homogeneity of the ideal \( I \) requires extraneous factors of the variables to be eliminated in order to get the degree high enough to achieve the desired elimination. To explain the result of the computation, we define a polynomial in the deformation parameters, \( \phi \):

\[
\phi \equiv (\gamma_1 + \gamma_2 \gamma_3 \epsilon_1)(\epsilon_1 + \gamma_1 \epsilon_2 \epsilon_3) - (\gamma_2 \gamma_3 \epsilon_2 \epsilon_3 - 1)^2
\]

Then we find that the only polynomial in \( I \) consisting of powers of \( x_1 \) and \( y_1 \) multiplied by a polynomial in the \( \epsilon \)'s and \( \gamma \)'s is \( x_1 y_1 \phi \). Furthermore, we can compute directly that any common zeros of the six minors satisfying either \( x_1 = 0 \) or \( y_1 = 0 \) necessarily satisfy \( \phi = 0 \). The conclusion is that the locus in the moduli space parametrizing sheaves \( \mathcal{E} \) which are not bundles is precisely given by \( \phi = 0 \).

We will find powers of \( \phi \) in the denominators of some of our expressions later in this paper. This is not a concern, since we have just shown that \( \phi \) is nonzero whenever the theory is well-defined, so that \( \mathcal{E} \) is actually a bundle.
3.5. Čech Cohomology of the Canonical Bundle. Once we have interpreted our correlators as elements of $H_{\text{top}}^n(M, \Lambda_{\text{top}} F^*) = H_{\text{top}}^n(M, K_M)$, the last step is to evaluate the result as a number via a trace isomorphism

$$\text{(23)} \quad \text{Tr} : H_{\text{top}}^n(M, K_M) \to \mathbb{C}.$$ 

We will describe how to find a trace map, passing to a broader context in which $M$ is replaced by an arbitrary smooth projective toric variety $X$.

We begin by recalling the description of homogeneous coordinates on $X$ [Cox95]. Thinking of $X$ as being described by a fan $\Sigma$, we associate variables $x_0, \ldots, x_N$ to the $N + 1$ edges of $\Sigma$. The homogeneous coordinate ring is the polynomial ring generated by the $x_i$, and is graded by the Chow group $A_{n-1}(X)$ of divisor classes on $X$, where $n = \dim(X)$. In this grading, the degree of $x_i$ is the class of the divisor $D_i$ defined by $x_i = 0$.

Illustrating with projective space $\mathbb{P}^n$, there are $n + 1$ edges in the fan and hence $n + 1$ coordinates $x_0, \ldots, x_n$. Each $D_i$ is a hyperplane, of degree 1; so we assign each $x_i$ the degree 1 and we recover the usual homogeneous coordinates on projective space.

The canonical class of $X$ is given by $K_X = -\sum_{i=0}^N D_i$, which we denote as $-D$. This description makes clear that $1/(x_0 \cdots x_N)$ is a (meromorphic) section of $K_X$. The complement of $D$ is the torus $T = (\mathbb{C}^*)^n$, defined by $x_i \neq 0$ for all $i$.

We know that $H^n(X, \mathcal{O}(-D)) = H^n(X, K_X)$ is one dimensional, and we try to find a generator. We take the standard open cover of $X$ described by the top dimensional cones in the fan. Explicitly, if $\sigma$ is a top dimensional cone, then the open set $U_\sigma$ is given as the locus where $x_i \neq 0$, for all $x_i$ corresponding to edges of the fan not contained in $\sigma$. For example, in the case $\mathbb{P}^{2n+1} \times \mathbb{P}^{2n+1}$ considered earlier, the $U_\sigma$ are precisely the $U_{ij}$ defined in equation (15).

Computing $H^n(X, \mathcal{O}(-D))$ by Čech cohomology using the above cover, we see that each of the intersections $U_{\sigma_0} \cap \ldots \cap U_{\sigma_k}$ needed in the Čech description necessarily contain the torus $T$. Since the only possible denominators of rational functions on $X$ which are holomorphic on all of $T$ are monomials, we arrive at an important observation:

In the Čech description of $H^n(X, \mathcal{O}(-D))$, we only need to consider expressions which are sums of Laurent monomials.

---

2If $X$ is the space of vacua of a GLSM, the set $\Sigma(1)$ of one-dimensional edges of $\Sigma$ is in one to one correspondence with the set of chiral fields in the model.
A *Laurent monomial* is a monomial where negative exponents are allowed.

A bit more pedagogically, we fix a multidegree \( r = (r_0, \ldots, r_N) \) and let \( C^i(X, \mathcal{O}(-D))_r \) denote those Čech \( i \)-cochains consisting entirely of components which are scalar multiples of \( x_0^{r_0} \cdots x_N^{r_N} \). Then we have

\[
C^i(X, \mathcal{O}(-D)) = \bigoplus_r C^i(X, \mathcal{O}(-D))_r.
\]

Since the coboundary maps

\[
\delta_i : C^i(X, \mathcal{O}(-D)) \to C^{i+1}(X, \mathcal{O}(-D))
\]

preserve the degree of monomials, we get an induced decomposition

\[
H^i(X, \mathcal{O}(-D)) = \bigoplus_r H^i(X, \mathcal{O}(-D))_r.
\]

Note that since the monomials occurring in representatives of \( H^n(X, \mathcal{O}(-D)) \) must be sections of \( \mathcal{O}(-D) \), we see by considering the grading of the homogeneous coordinate ring of \( X \) that \( H^n(X, \mathcal{O}(-D))_r \) can be nonzero only if the divisor \( \sum r_i D_i \) is homologous to \(-D\). An obvious way to satisfy this necessary condition is if \( r_i = -1 \) for all \( i \). We put \(-1 = (-1, \ldots, -1)\). Since \( H^n(X, \mathcal{O}(-D)) \) is one dimensional, our conclusion is that

\[
\text{If } H^n(X, \mathcal{O}(-D))_{-1} \neq 0, \text{ then dim } H^n(X, \mathcal{O}(-D))_{-1} = 1 \text{ and } H^n(X, \mathcal{O}(-D))_{-1} \simeq H^n(X, \mathcal{O}(-D)) \simeq \mathbb{C}.
\]

This means that if we can verify that the group \( H^{\text{top}}(\mathcal{M}, \mathcal{O}(-D))_{-1} \) is nonzero for the instanton moduli space \( \mathcal{M} \), then it is one dimensional, and we only have to consider Čech cocycles whose components are multiples of \( 1/(x_0 \cdots x_N) \) in the computation of Čech cohomology and the determination of a trace map. We conjecture that this condition always holds, a conjecture implicitly made in developing the algorithm in [KS06].

For the purposes of computing the cohomology, not every intersection \( U = U_{\sigma_0} \cap \ldots \cap U_{\sigma_k} \) is relevant, since \( 1/(x_0 \cdots x_N) \) is required to be holomorphic there. In terms of the fan, the condition is that for each edge \( \rho \), there must be at least one top-dimensional cone \( \sigma_i \) used in indexing \( U \) which does not contain \( \rho \). We call these open sets \( U \) *good* open sets.

Thus, when taking cup/wedge products to obtain Čech \( n \)-cochain representatives of correlation functions, we do not account for their value on every \( n + 1 \)-fold intersection of open sets. For example, on \( \mathbb{P}^1 \times \mathbb{P}^1 \), there are four open sets: \( U_{0,0}, U_{1,0}, U_{0,1} \), and \( U_{1,1} \), with \( U_{i,j} \) defined as in [15]. The “good” two-fold intersections would be \( U_{0,0} \cap U_{1,1}, \) and \( U_{1,0} \cap U_{0,1} \). The intersection \( U_{0,0} \cap U_{1,0} \) would be excluded, for example, since \( y_1 \) is not invertible on this set.
3.6. The Trace Map. For any given instanton sector, we have $H^{\text{top}}(F^*) \cong \mathbb{C}$, as follows from (14). In order to complete our computations, we will need the explicit form of this isomorphism, i.e. a trace map. For the reasons discussed in the previous section, we can and will replace $H^{\text{top}}(F^*)$ with $H^{\text{top}}(F^*)^{-1}$. Since we will be dealing with cup/wedge products of elements of $C^1\left(\mathcal{O}(-1,0)^{2m+2} \oplus \mathcal{O}(0,-1)^{2n+2}\right)$ rather than explicit elements of $H^{\text{top}}$, we will actually construct a map from $H^{\text{top}}(K_M)^{-1}$ to $\mathbb{C}$ and use the minor map to remove the extraneous terms introduced by using cochains rather than cocycles to compute.

The desired map is determined up to a non-zero complex multiple by the following algorithm. The $\delta$-map between cochain groups can be described by specifying the weight with which the components of an element of $C^{i-1}$ on each $i$-fold intersection contribute to the components on each $(i+1)$-fold intersection in $C^i$. For any element $A$ in $C^{i-1}$, and on each $(i+1)$-fold intersection in $C^i$, we obtain the representative of $\delta A$ on that intersection by summing elements in $C^{i-1}$:

$$(\delta A)_{j_1, \cdots, j_{i+1}} = \sum_{k=1}^{i+1} (-1)^{k-1} A_{j_1, \cdots, \hat{j}_k, \cdots, j_{i+1}}.$$ 

In this expression, $\hat{j}_k$ denotes the exclusion of the $j_k$th index from the collection. Writing elements of $C^i$ as a column vector, with each slot in the vector denoting the element’s value on an $(i+1)$-fold intersection, the $\delta$ map is simply a matrix with elements 0, 1, or $-1$. Then it is straightforward to obtain $\delta$: if the $i$-fold intersection denoted by $j_1, \cdots, \hat{j}_k, \cdots, j_{i+1}$ is in the $n$th row of the vector, and the $(i+1)$-fold intersection $j_1, \cdots, j_{i+1}$ is in the $m$th row, then the $(m,n)$th element of $\delta_{i-1}$ is $(-1)^k$. Let us reiterate: since we are interested in cohomology, and invalid intersections will not contribute, we can ignore them when computing $\delta$.

Next, we construct a matrix $Z^i$, whose rows span the nullspace of $\delta_i$. We consider the matrix $Z^i$ as a projection map from $C^i$ to the basis of ker($\delta_i$) given by the rows of $Z^i$. In addition, if we multiply this matrix by its transpose, we obtain a projection map from $C^i$ to ker($\delta_i$) $\subset C^i$, expressed in the basis for $C^i$: by composing $\delta_{i-1}$ with $^tZ^i \cdot Z^i$, we obtain a map

$$(^tZ^i \cdot Z^i) \circ \delta_{i-1} : C^{i-1} \rightarrow \ker(\delta_i) \subset C^i.$$ 

The transpose of $Z^i \circ \delta_{i-1}$ is a map from ker($\delta_i$) to $C^{i-1}$, whose kernel $H_i$ spans the subspace of elements of ker($\delta_i$) which are not in the image of $\delta_{i-1}$: the row space of this matrix is exactly the $i^{th}$ cohomology group we seek,

$${^tZ^i \circ \delta_{i-1}} \equiv \ker \left[({^tZ^i \circ \delta_{i-1}})\right]$$ 

$${H^i = \text{rowspace}(H_i)}.$$
In the same manner as $Z^i$, we think of $H_i$ as a map from $\ker(\delta_i)$ to $H^i$. Thus, the composition $H_i \cdot Z^i$ is a projection from $C^i$ to $H^i$, the desired trace map.

Mathematica\cite{Mat05} was sufficient for computing $\delta$ kernels for most instanton sectors. However, for the degree $(1,1)$ sector, the cocycle space’s large dimension required the use of specialized software to find null spaces. We used routines from the “Integer Matrix Library”\cite{CS05, Che05} to compute bases for various kernels over $\mathbb{Z}_p$ for $p = 5279, 4409, 3571$, before applying the Chinese Remainder Theorem to obtain bases over the integers. For a map $\delta_i$, the result is the matrix form of $Z^i$; its rows form a basis for $\ker(\delta_i)$.

3.7. Applicability. The algorithm described in this section may be utilized for any compact toric target space with at worst orbifold singularities. See \cite{MRP95} for generalities on toric varieties and their relationship to the GLSM.

Given a general GLSM containing bosonic fields $\phi_i, i = 0 \ldots N$ with respective charges $Q_{ik}, k = 1 \ldots r$ under a $U(1)^r$ gauge group, we denote by $M$ the vacuum moduli space in a geometric phase. Geometrically, $M$ can be described as a toric variety with edges $v_i$ in one to one correspondence with the $\phi_i$. Each $v_i$ is associated to a divisor $D_i$ on $M$, along with a line bundle $\mathcal{O}(D_i)$. There are sections $x_i$ of the bundle $\mathcal{O}(D_i)$ that serve as homogeneous coordinates on $M$, from which $M$ can be recovered from $\mathbb{C}^{N+1}$ by symplectic reduction in the usual way.

There is an exact sequence, proven in \cite{BC94}, which generalizes \cite{BC94} for any quasi-smooth compact toric variety $M$:

\begin{equation}
0 \longrightarrow \mathcal{O}^r \longrightarrow \oplus_i \mathcal{O}(D_i) \longrightarrow \mathcal{T}_M \longrightarrow 0.
\end{equation}

The map $\oplus \mathcal{O}(D_i) \rightarrow \mathcal{T}_M$ takes a collection of sections $s_i$ of $\mathcal{O}(D_i)$ to the tangent vector $\sum_i s_i(\partial/\partial x_i)$, which is a well-defined vector field on $M$ since it is neutral under $U(1)^r$ by construction. The $(i,k)^{th}$ entry of the map $\oplus \mathcal{O}^r \rightarrow \oplus_i \mathcal{O}(D_i)$ is given by $Q_{ik}x_i$. Note that $\sum_i Q_{ik}x_i(\partial/\partial x_i)$ is the Euler vector field in $\mathbb{C}^{N+1}$ corresponding to the action of the $k^{th}$ $U(1)$ on $\mathbb{C}^{N+1}$ that gives rise to the trivial vector field on $M$, which is one of the requirements of the exactness of \cite{BC94}.

The tangent bundle $\mathcal{T}_M$ can be deformed as in the case of $M = \mathbb{P}^1 \times \mathbb{P}^1$ by deforming the entries of the first non-trivial map in \cite{BC94}, giving an exact sequence

\begin{equation}
0 \longrightarrow \mathcal{O}^r \longrightarrow \oplus_i \mathcal{O}(D_i) \longrightarrow \mathcal{E} \longrightarrow 0.
\end{equation}

Dualizing, we have an exact sequence

\begin{equation}
0 \longrightarrow \mathcal{E}^* \longrightarrow \oplus_i \mathcal{O}(-D_i) \longrightarrow \mathcal{O}^r \longrightarrow 0.
\end{equation}
The coboundary map in (30) is $H^0(O^r) \to H^1(\mathcal{E}^*)$, which can be used to describe elements of $H^1(\mathcal{E}^*)$ as in the $\mathbb{P}^1 \times \mathbb{P}^1$ case, by taking lifts. While the lifts cannot be written in closed form as simply as in the $\mathbb{P}^1 \times \mathbb{P}^1$ case, they can be found algorithmically.

In an instanton sector, we get a sheaf $\mathcal{F}$ on the instanton moduli space $\mathcal{M}$. As described in [MRP95, Section 3.7], $\mathcal{M}$ is itself a toric variety. To each edge $v_i$ in the fan for $\mathcal{M}$ is associated $b_i$ edges in the fan for $\mathcal{M}$, with the number $b_i$ depending on the particular instanton sector under consideration. Since $\mathcal{M}$ is a toric variety, we have analogous to (30) a short exact sequence

$$0 \longrightarrow \mathcal{F}^* \longrightarrow \bigoplus_i O(-D_i)^{b_i} \longrightarrow O^r \longrightarrow 0,$$

generalizing (12). Here we are abusing notation by denoting with $D_i$ the divisor class on $\mathcal{M}$ associated with the divisor class $D_i$ on $\mathcal{M}$ as described in [MRP95]. Taking determinants in (31) gives

$$\det(\Lambda^\text{top} F^*) \cong O(-\sum_i b_i D_i) \cong K_M$$

as before.

The coboundary map of (31) is $H^0(O^r) \to H^1(\mathcal{F}^*)$. Since the domain of this map is naturally $\mathbb{C}^r$, as is the domain of the other coboundary map $H^0(O^r) \to H^1(\mathcal{E}^*)$, we have a natural way to map our elements of $H^1(\mathcal{E}^*)$ to elements of $H^1(\mathcal{F}^*)$.

To compute correlation functions of our elements of $H^1(\mathcal{E}^*)$, we map them to $H^1(\mathcal{F}^*)$ as just explained and take cup products to get an element of $H^\text{top}(K_M) \cong \mathbb{C}$. The methods of [KS06] together with the methods developed here for computing the trace map allow for the completion of the desired computation in principle, constrained only by the limitations of machine computation.

4. Chiral Ring Relations

4.1. Calculation. The quantum cohomology ring is the polynomial ring $\mathbb{C}[\psi, \tilde{\psi}]$, modulo two relations, which in the $(2,2)$ limit reduce to

$$\psi \ast \psi = q$$
$$\tilde{\psi} \ast \tilde{\psi} = \tilde{q}.$$

We can partially deduce the form of the relations by making the ansätze

$$a (\psi \ast \psi) + b (\psi \ast \tilde{\psi}) + c (\tilde{\psi} \ast \tilde{\psi}) = d$$
$$\tilde{a} (\psi \ast \psi) + \tilde{b} (\psi \ast \tilde{\psi}) + \tilde{c} (\tilde{\psi} \ast \tilde{\psi}) = \tilde{d},$$

where for example $\tilde{a} \to 0$ in the $(2,2)$ limit. The problem possesses an inherit symmetry, wherein

$$\epsilon_i \leftrightarrow \gamma_i \quad \psi \leftrightarrow \tilde{\psi} \quad x^i \leftrightarrow y^i \quad q \leftrightarrow \tilde{q},$$
which implies that the first and second ring relations in (33) are exchanged under the
symmetry with the identifications
\[(35) \quad a \leftrightarrow \tilde{c} \quad b \leftrightarrow \tilde{b} \quad c \leftrightarrow \tilde{a} \quad d \leftrightarrow \tilde{d}.\]

Thus, only one ring relation need be computed. The relations may be deduced by identifying
cohomological inner product and correlation functions in the physical theory:
\[(\psi \star \cdots \star \tilde{\psi}, \psi) = \left\langle \psi \cdots \tilde{\psi} \psi \right\rangle.\]

Then, using the commutativity of the ring, correlators are related by (33). For example,
since \(d\) is a constant, we will have that \(\langle d \rangle = 0\) by anomaly considerations, and the two-point
functions may be related as
\[(36) \quad a \langle \psi \psi \rangle + b \langle \psi \tilde{\psi} \rangle + c \langle \tilde{\psi} \psi \rangle = 0.\]

In general, we will have
\[(37) \quad a \left\langle \cdots \psi \psi \psi \tilde{\psi} \psi \cdots \right\rangle + b \left\langle \cdots \psi \psi \tilde{\psi} \tilde{\psi} \psi \cdots \right\rangle + c \left\langle \cdots \tilde{\psi} \psi \psi \tilde{\psi} \psi \cdots \right\rangle = d \left\langle \cdots \psi \tilde{\psi} \psi \cdots \right\rangle\]
where “…) on the left side of the correlator is an equal, but arbitrary, number of \(\psi\)'s, and
similarly for \(\tilde{\psi}\)'s on the right. Each computed correlation function (up to instanton degree
2) was consistent, but only 2 were independent. Thus, we deduce the constants in the ring
relation (33) by substitution into some of these four-point functions.

\[a \langle \psi \psi \psi \psi \rangle = -b \langle \psi \psi \psi \tilde{\psi} \rangle - c \langle \psi \psi \tilde{\psi} \psi \rangle + d \langle \psi \psi \rangle\]
\[a \langle \psi \psi \psi \tilde{\psi} \rangle = -b \langle \psi \psi \psi \psi \rangle - \tilde{c} \langle \psi \tilde{\psi} \psi \psi \rangle + \tilde{d} \langle \psi \tilde{\psi} \rangle\]
\[a \langle \psi \psi \tilde{\psi} \psi \tilde{\psi} \rangle = -b \langle \psi \psi \tilde{\psi} \tilde{\psi} \rangle - \tilde{c} \langle \tilde{\psi} \psi \psi \tilde{\psi} \rangle + \tilde{d} \langle \tilde{\psi} \tilde{\psi} \rangle\]

Plugging in the values for the correlation functions appearing in Appendix 13 and solving
these relations, we determine that the coefficients are related by the equations
\[b (1 - \gamma_2 \gamma_3 \epsilon_2 \epsilon_3) = (a + c \gamma_2 \gamma_3) \epsilon_1 + \gamma_1 (c + a \epsilon_2 \epsilon_3)\]
\[d (1 - \gamma_2 \gamma_3 \epsilon_2 \epsilon_3) = c (\tilde{q} + q \gamma_2 \gamma_3) + a (q + \tilde{q} \epsilon_2 \epsilon_3)\]

In the (2,2) limit, the ring relation constants must reduce to
\[a = 1 \quad b = 0 \quad c = 0 \quad d = q,\]
so we make the substitutions \(a \rightarrow 1 + a\) and \(d \rightarrow q + d\) and the equations become
\[b (1 - \gamma_2 \gamma_3 \epsilon_2 \epsilon_3) = [(1 + a) + c \gamma_2 \gamma_3] \epsilon_1 + \gamma_1 [c + (1 + a) \epsilon_2 \epsilon_3]\]
\[(q + d) (1 - \gamma_2 \gamma_3 \epsilon_2 \epsilon_3) = c (\tilde{q} + q \gamma_2 \gamma_3) + (1 + a) (q + \tilde{q} \epsilon_2 \epsilon_3),\]
where now \(a, b, c, \) and \(d\) all vanish in the (2,2) limit. We will assume that these constants
are independent of \(q\) and \(\tilde{q},\) and will provide arguments partially justifying this assumption.
at the end of §5.2. We thus collect the coefficients of \( q \) and \( \tilde{q} \), set them equal to zero, and solve for \( \{a, b, c, d\} \). This procedure produces an astonishingly simple relation: coupled with the symmetry described in (35), we have that

\[
\psi \ast \psi + \epsilon_1 (\psi \ast \tilde{\psi}) - \epsilon_2 \epsilon_3 (\tilde{\psi} \ast \psi) = q
\]

(38)

\[
\tilde{\psi} \ast \tilde{\psi} + \gamma_1 (\tilde{\psi} \ast \psi) - \gamma_2 \gamma_3 (\psi \ast \psi) = \tilde{q}.
\]

(39)

4.2. Further Evidence. Here, we offer some further evidence for our ring relations. Starting with the GLSM description of the (0,2) theory as outlined in §2.1, one would suspect that the classical ring relations (those with \( q = 0 \)) might be encoded into the classical action somehow. We investigate the Coulomb branch of the GLSM in search of these relations.

The action for the left-moving fermions contains the term \( \sum_i |E_i|^2 \). Here, \( E_i \) is the function in (2), which upon performing the superspace integrals depends only on the lowest components of the superfields. After integrating out the \( D \) field, the bosonic interactions that remain arise from this expression. Writing these terms in matrix form, we have

\[
\begin{pmatrix}
\tilde{\phi}_1 \\
\tilde{\phi}_2
\end{pmatrix} =
\begin{pmatrix}
|\Sigma|^2 + \epsilon_1 \bar{\Sigma} \Sigma + \epsilon_1 \Sigma \bar{\Sigma} + |\epsilon_3 \Sigma|^2 & \epsilon_1 \epsilon_2 |\Sigma|^2 + \epsilon_2 \Sigma \bar{\Sigma} \\
\epsilon_2 \Sigma \bar{\Sigma} + \epsilon_1 \epsilon_2 |\Sigma|^2 + \epsilon_3 \Sigma \bar{\Sigma}
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix}
\]

(40)

and

\[
\begin{pmatrix}
\tilde{\phi}_1 \\
\tilde{\phi}_2
\end{pmatrix} =
\begin{pmatrix}
|\Sigma|^2 + \gamma_1 \bar{\Sigma} \Sigma + \gamma_1 \Sigma \bar{\Sigma} + |\gamma_3 \Sigma|^2 & \gamma_3 \bar{\Sigma} \Sigma + \gamma_1 \gamma_2 |\Sigma|^2 + \gamma_2 \Sigma \bar{\Sigma} \\
\gamma_2 \Sigma \bar{\Sigma} + \gamma_1 \gamma_2 |\Sigma|^2 + \gamma_3 \Sigma \bar{\Sigma}
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix}
\]

(41)

Let us concentrate in particular on the contribution of the bosonic zero modes to the \( \Sigma \) effective action. From the matrix forms (40) and (41), one can read off that the zero modes’ contribution appears in the combination

\[
|\Sigma|^2 + \gamma_1 \bar{\Sigma} \Sigma + \gamma_1 \Sigma \bar{\Sigma} + |\gamma_3 \Sigma|^2 \left| \Sigma^2 + \epsilon_1 \Sigma \bar{\Sigma} - \epsilon_2 \epsilon_3 \Sigma^2 \right|^2.
\]

(42)

These bosonic modes precisely cancel the fermionic zero mode contribution, which arise from Yukawa interactions of the form

\[
\sigma \rho_a \frac{\partial E^a}{\partial \phi_i} \psi_i + \text{c.c.,}
\]

(43)

where \( \sigma \) is the lowest component of the superfield \( \Sigma \). Note that the multiplicands are exactly the classical limit of the relations in (38) and (39). These results provide strong indications that our ring relations are correct.
Two questions naturally arise: do the chiral ring relations [38] and [39] match the results in [ABS04], and do the correlation functions outlined in (56)–(58) match those in [KS06]? Let us first address the latter question.

5.1. Cohomological Computations. The deformation described in [KS06] relied on two parameters, $\epsilon_1$ and $\epsilon_2$, yet the ring relations produced therein depended only on the difference of these two parameters. This dependence arises from the fact that deformations described by $\epsilon_2$ are not independent from those described by $\epsilon_1$: consider their matrix,

$$
\begin{pmatrix}
 x_0 & x_1 & 0 & 0 \\
 \epsilon_1 x_0 & \epsilon_2 x_1 & y_1 & y_2 
\end{pmatrix}.
$$

By adding arbitrary multiples of the first row to the second, one generates the same translation of both parameters. Since both describe the same deformation, we set the $\epsilon_2$ in [KS06] to zero so that the deformation is fully described by $\epsilon_1$.

In order to match our results, the $\epsilon$ and $\gamma$ parameters must be adjusted so that the deformation described by (5) matches that of [KS06], under the condition on their parameters imposed above. In particular, the deformations match when

$$
\epsilon_2 = \epsilon_3 = \gamma_1 = \gamma_2 = \gamma_3 = 0.
$$

In this limit, one can check that the four-point functions appearing in §7.1 of [KS06] match those in Appendix B. The results of our six-point computations do not appear due to their size, but they do match the results of [KS06], as well as being consistent with the relations [38] and [39].

5.2. Computation via Mirror Symmetry. We now begin our comparison with the results of [ABS04]. Their work centered on the construction of dual theories, and the exploitation of this symmetry to deduce the form of the chiral ring relations, among other properties. In §6.2, the authors construct a GLSM coupled to a deformation of the tangent bundle of $\mathbb{P}^1 \times \mathbb{P}^1$.

The fermi superfields in the sigma model are not quite chiral, with their deviation from chirality measured by combinations $E_i$ and $\tilde{E}_i$ of chiral fields $\Phi_i, \tilde{\Phi}_i, \Sigma$ and $\tilde{\Sigma}$. These functions encode the deformation matrix in a straightforward way; the $E$’s may be interpreted as the image in $\Gamma(O(1,0)^2 \oplus O(0,1)^2)$ of the basis elements $\Sigma$ and $\tilde{\Sigma}$ by the injection in the short exact sequence (11).
Compare the functions of chiral fields in §6.2 with our deformation mapping \((5)\). The parameters \(\alpha'_2\) and \(\beta'_2\) are dependent on the other \(\alpha\)'s and \(\beta\)'s, since the space of deformations is only six-dimensional: the first of these variables corresponds to \(\epsilon_2\) of [KS06]. We have argued previously that \(\epsilon_2\) does not measure an independent deformation, and under the symmetry described in \((34)\), \(\alpha'_2\) and \(\beta'_2\) are exchanged. Upon setting these extraneous parameters to zero, we find agreement to our deformation when
\[
\alpha_i = \epsilon_i \quad \alpha'_1 = \epsilon_3 \quad \beta_i = \gamma_i \quad \beta'_1 = \gamma_3.
\]
Chiral ring relations were computed by comparing effective potentials for those fields unaffected by duality in both the GLSM and its mirror dual, resulting in
\[
X + p(\epsilon, \gamma)\frac{e^{it_1}}{X} + q(\epsilon, \gamma)\tilde{X} + s(\epsilon, \gamma)\frac{e^{it_2}}{X} = 0
\]
\[
\tilde{X} + \tilde{p}(\epsilon, \gamma)\frac{e^{it_2}}{X} + \tilde{q}(\epsilon, \gamma)X + \tilde{s}(\epsilon, \gamma)\frac{e^{it_1}}{X} = 0.
\]
Here, \(e^{it_1}\) corresponds to our \(q\) and \(e^{it_2}\) to our \(\tilde{q}\). The chiral fields \(X\) and \(\tilde{X}\), as elements of the quantum cohomology ring, must be related by some simple linear combination to our operators \(\psi\) and \(\tilde{\psi}\),
\[
\begin{pmatrix} X \\ \tilde{X} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \psi \\ \tilde{\psi} \end{pmatrix},
\]
where each element of the matrix is some polynomial in the \(\epsilon\)'s and \(\gamma\)'s. Assuming that the ring relation \((38)\) will take the form \(\alpha X^2 + \beta X\tilde{X} + \gamma\tilde{X}^2 = q\), we plug in the change of variables above and find that
\[
\alpha, \beta, \gamma \propto \frac{1}{(BC - AD)^2}.
\]
At first glance this is discouraging, since the ring relation expected from the physics seems to be ill-defined on certain configurations of parameters. All is not lost, however: we have encountered this type of situation before. To alleviate our conceptual difficulties, we make the ansatz that the only combination of the deformation parameters that could possibly appear in the denominator of such a change of variables is the polynomial \(\phi\) defined in \((22)\):
\[
BC - AD = \phi.
\]
Let us now impose our knowledge of the behavior of \((47)\) in the limiting case examined in [KS06]. There, it was found that \(X = \psi + \epsilon_1\tilde{\psi}, \tilde{X} = \psi\). Notice that due to symmetry, had we chosen \(\gamma_1\) to be the non-vanishing parameter, the correct change of variables would have been \(X = \psi, \tilde{X} = \gamma_1\psi + \tilde{\psi}\). With this observation, we make the ansatz that the correct limiting transformation when \(\epsilon_1\) and \(\gamma_1\) are non zero is
\[
X = \psi + \epsilon_1\tilde{\psi} \quad \tilde{X} = \gamma_1\psi + \tilde{\psi}.
\]
We make a change of variables to account for this limiting behavior:

\[
\begin{align*}
A &\rightarrow 1 + A' \\
B &\rightarrow \epsilon_1 + B' \\
C &\rightarrow 1 + C' \\
D &\rightarrow \gamma_1 + D'.
\end{align*}
\]

In [ABS04], a charge was assigned to each parameter: in that scheme, \(\epsilon_1\) was given charge \(k - \tilde{k}\), while \(\gamma_1\) was given charge \(\tilde{k} - k\). In order to match the charges of the limiting behaviour, \(A'\) and \(D'\) must be neutral, while \(B'\) and \(C'\) must have charges \(k - \tilde{k}\) and \(\tilde{k} - k\), respectively. Analyticity of the basis change requires the coefficients to be polynomials in the parameters, and relevant combinations of parameters and charges appear in Table 1.

| \(k - \tilde{k}\)       | \(\tilde{k} - k\)   |
|-------------------------|----------------------|
| \(\epsilon_1\)         | \(\gamma_1\)        |
| \(\gamma_1\epsilon_2\epsilon_3\) | \(\epsilon_1\gamma_2\gamma_3\) |

Table 1. Charge assignments for relevant combinations of deformation parameters.

Since we have already accounted for the \(\epsilon_1\) dependence of \(B\), the only other possibility is \(\gamma_1\epsilon_2\epsilon_3\). We make the ansätze that \(B = \epsilon_1 + \gamma_1\epsilon_2\epsilon_3\), and \(C = \gamma_1 + \epsilon_1\gamma_2\gamma_3\), so that upon substitution \(\phi = BC - AD\) reduces to

\[
0 = -(\gamma_2\gamma_3\epsilon_2\epsilon_3)^2 + 2\gamma_2\gamma_3\epsilon_2\epsilon_3 + A' + A'D' + D'.
\]

One observes that this equation is solved by \(A' = D' = -\gamma_2\gamma_3\epsilon_2\epsilon_3\), a combination of charge zero, so that the proper change of variables is

\[
\begin{pmatrix}
\bar{X} \\
\bar{X}
\end{pmatrix} = \begin{pmatrix}
1 - \gamma_2\gamma_3\epsilon_2\epsilon_3 & \epsilon_1 + \gamma_1\epsilon_2\epsilon_3 \\
\gamma_1 + \epsilon_1\gamma_2\gamma_3 & 1 - \gamma_2\gamma_3\epsilon_2\epsilon_3
\end{pmatrix} \begin{pmatrix}
\psi \\
\tilde{\psi}
\end{pmatrix}.
\]

This change of variables respects the interchange symmetry described by (34); exchange of the \(\psi\)’s and the \(\epsilon\) and \(\gamma\) parameters results in an exchange of the \(X\)’s.

With this change, the coefficients in the ring relation for the new variables are easily computed to be

\[
\begin{align*}
\alpha &= -\frac{1}{\phi} \\
\beta &= \frac{1}{\phi}(\epsilon_1) \\
\gamma &= \frac{1}{\phi}(\epsilon_2\epsilon_3)
\end{align*}
\]

\[
\Rightarrow -X^2 + \epsilon_1X\bar{X} + \epsilon_2\epsilon_3\bar{X}^2 = \phi q \\
-\bar{X}^2 + \gamma_1X\bar{X} + \gamma_2\gamma_3X^2 = \phi \tilde{q}.
\]

Let us now turn to a comparison of these results with those (46) deduced in [ABS04]. We find an apparent contradiction between our ring relation in the new variables (51) and those
We begin by multiplying the equations in (46) by $X\tilde{X}$, obtaining
\begin{align*}
X^2 \tilde{X} + p(\epsilon, \gamma) \tilde{X} e^{it_1} + q(\epsilon, \gamma) X \tilde{X}^2 + s(\epsilon, \gamma) X e^{it_2} &= 0 \\
X \tilde{X}^2 + p(\epsilon, \gamma) X e^{it_2} + q(\epsilon, \gamma) X^2 \tilde{X} + s(\epsilon, \gamma) \tilde{X} e^{it_1} &= 0.
\end{align*}
(52)

One can quickly observe that there is no linear combination of these two relations which would give those in (51), since there would need to be terms like $X^3$ and $\tilde{X}^3$ in (52).

Let us consider this problem from another direction; substituting $\psi$ and $\tilde{\psi}$ into the relations (52). We impose that upon restriction to the $(2,2)$ locus, the first (second) ring relation matches the A-model relation $X^2 = q (\tilde{X}^2 = \tilde{q})$. Then, $s(\epsilon, \gamma) = \tilde{s}(\epsilon, \gamma) = 0$, and up to multiplication by some other function of the $\epsilon$’s and $\gamma$’s, the relations become
\begin{align*}
X^2 + q(\epsilon, \gamma) X \tilde{X} &= -p(\epsilon, \gamma) e^{it_1} \\
\tilde{X}^2 + \tilde{q}(\epsilon, \gamma) X \tilde{X} &= -\tilde{p}(\epsilon, \gamma) e^{it_2}.
\end{align*}
(53)

Then, we make the change of variables
\begin{equation}
\begin{pmatrix}
\psi \\
\tilde{\psi}
\end{pmatrix}
= \begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
X \\
\tilde{X}
\end{pmatrix},
\end{equation}
(54)
and substitute into the ring relations (38) and (39). Matching the $X$ and $\tilde{X}$ dependence in (53), we must have that
\begin{equation}
\begin{aligned}
\tilde{B} &= -\frac{D}{2} \left( \epsilon_1 \pm \sqrt{\epsilon_1^2 + 4\epsilon_2\epsilon_3} \right) \\
\tilde{C} &= -\frac{A}{2} \left( \gamma_1 \pm \sqrt{\gamma_1^2 + 4\gamma_2\gamma_3} \right).
\end{aligned}
\end{equation}
(55)

Thus, we find that the functions $q(\epsilon, \gamma)$ and $p(\epsilon, \gamma)$ must depend on square roots of combinations of the parameters, which appear in both the numerator and denominator of these functions. Unfortunately, it is not possible to completely solve for $\{A, B, C, D\}$, since there are four unknowns and only two equations imposed by $X$ and $\tilde{X}$ dependence after substitution into (53). It may be that this is the correct change of variables, although it is not obvious that it is well defined when the bundle is, and is aesthetically quite displeasing besides.

It may be possible, however, to resolve the apparent contradiction between the “nice” change of variables (51) and the results of [ABS04]. The ring relations therein were computed by simply minimizing the superpotential in the theory dual to the GLSM. It may be that this is only true classically (in the field theory sense). Since the ratio of left- and right-moving determinants need not cancel, one may find left-moving one loop corrections. Indeed, some preliminary work [Set07] indicates that the appearance of terms quadratic in the deformation parameters qualitatively matches expectations of one-loop contributions.
Additionally, note that the degree \((0, 0)\) correlators appearing in (56) have terms which are cubic and quartic in the deformation parameters. Given the form of the interactions in the GLSM – outlined in equations (40), (41), and (43) – such terms would most likely arise from loop contributions.

Let us say a few more words about the coefficients of (33). We first note that the chiral ring relations in (46) are independent of \(q\) and \(\tilde{q}\). The Yukawa couplings contributing to possible one-loop additions to the dual superpotential do not involve \(q\) or \(\tilde{q}\) either, so that from a physical standpoint, the coefficients cannot develop dependence on these quantities. Consider this from a the point of view of the charges of the parameters, as in Table 2. Any \(q\) or \(\tilde{q}\) dependence in the coefficients \(\{a, b, c, d\}\) must arise in (possibly inverse) powers of \(q/\tilde{q}\). As we expect the ring relations to be analytic in the size of the \(\mathbb{P}^1\)’s, such terms should not contribute. As a zeroth-order check of this statement, we can add \(q\)- and \(\tilde{q}\)-dependent terms to each coefficient \(\{a, b, c, d\}\) – in such a way that the charges are respected – and show that first-order additions necessarily vanish. Consulting Table 2 we see that the combinations

\[
\epsilon_1 \gamma_1, \quad \epsilon_2 \epsilon_3 \gamma_2 \gamma_3, \quad \gamma_1^2 \epsilon_2 \epsilon_3, \quad \text{and} \quad \epsilon_1^2 \gamma_2 \gamma_3
\]

are of zero charge, so arbitrary functions of them may multiply any part of the \(q, \tilde{q}\) terms. Furthermore, we see that the only combinations of \(q\) and \(\tilde{q}\) compatible with matching coefficient charges are powers of \(q/\tilde{q}\). We perform our check by examining only \(O(\frac{q}{\tilde{q}}, \frac{\tilde{q}}{q})\) additions to the coefficients, with at most degree three polynomials in the parameters multiplying \(q\) and \(\tilde{q}\): for example, the allowed terms for \(a\) would be

\[
a \rightarrow 1 + \frac{q}{\tilde{q}} (a_1 \gamma_1 \epsilon_1 + a_2 \gamma_2 \epsilon_3) + \frac{\tilde{q}}{q} (a_3 \epsilon_2 \epsilon_3)
\]

for complex constants \(a_i\). One can check that each possible term must vanish in order to satisfy the relations in (37).

### 6. Conclusions

In this work, we computed correlators and deduced chiral ring relations in the topological subsector of a \((0, 2)\) supersymmetric gauged linear sigma model, whose bosons in the geometric phase map to \(\mathbb{P}^1 \times \mathbb{P}^1\) and whose left-moving fermions couple to a deformation of the tangent bundle. We find that the ring relations in terms of deformation parameters \(\epsilon_i, \gamma_i\)

| Quantity | Charge | Quantity | Charge | Quantity | Charge |
|----------|--------|----------|--------|----------|--------|
| \(a\)    | 0      | \(\epsilon_1\) | \(k - \tilde{k}\) | \(q\)    | 2\(k\) |
| \(b\)    | \(k - \tilde{k}\) | \(\gamma_1\) | \(\tilde{k} - k\) | \(\tilde{q}\) | 2\(k\) |
| \(c\)    | 2\((k - \tilde{k})\) | \(\epsilon_2 \epsilon_3\) | 2\((k - \tilde{k})\) | \(\gamma_2 \gamma_3\) | 2\((k - k)\) |
| \(d\)    | 2\(k\) | \(\gamma_2 \gamma_3\) | 2\((k-k)\) |

Table 2. Charge assignments for parameters and coefficients.
and a basis \( \{ \psi, \tilde{\psi} \} \) of topological operators are

\[
\psi \star \psi + \epsilon_1 (\psi \star \tilde{\psi}) - \epsilon_2 \epsilon_3 (\tilde{\psi} \star \tilde{\psi}) = q
\]

\[
\tilde{\psi} \star \tilde{\psi} + \gamma_1 (\psi \star \tilde{\psi}) - \gamma_2 \gamma_3 (\psi \star \psi) = \tilde{q}.
\]

Our correlation functions and ring relations match those computed in [KS06] in the limit where the deformations match. We were neither able to verify nor refute the ring relations appearing in [ABS04]. The difficulty arises in finding a change of basis between our operators (\( \psi \) and \( \tilde{\psi} \)) and theirs (\( X \) and \( \tilde{X} \)). Without loop corrections, it seems that in order to match results, the basis change must involve square roots of polynomials in the deformation parameters.

The ring relations derived in this work exhibit some curious features. Note that some deformation parameters appear quadratically: if we set all parameters but \( \epsilon_2 \) and \( \gamma_2 \) to zero, the ring relations are the same as those of the \((2,2)\) theory. Indeed, the ring is apparently insensitive to those deformations parametrized by non-vanishing pairs \( \{ \epsilon_i, \gamma_j \} \) among the parameters \( \{ \epsilon_2, \epsilon_3, \gamma_2, \gamma_3 \} \), with the remaining two vanishing (for instance, \( \epsilon_2 = \gamma_2 = 0, \epsilon_3 \neq 0, \gamma_3 \neq 0 \)). From the point of view of [ABS04], the reason no modification to the ring relations occurs is clear: for any given pair of non-vanishing parameters, there is no combination such that the \( U(1) \) charge of the term will not depend on the \( U(1) \) charges of the bosonic fields.

The fact that each of these pairs of parameters give apparently non-trivial theory deformations with the same chiral ring is quite interesting. That there are multiple pairs with this property shows that the invariance of the ring is intrinsic to the problem, and not an artifact of the presentation of the deformation.

Another curiosity is the origin of the polynomial \( \phi \) in the physics. Since it measures those deformations of the tangent bundle that are not bundles, it is most likely that the GLSM of this deformation leads to a bad conformal field theory. Should the discrepancy between our results and those of the mirror symmetry computation be satisfactorily explained, and the mirror map understood, it would be interesting to compare the mirror theories in these singular limits. Presumably the mirror at these points becomes a badly behaved CFT as well.

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**APPENDIX A. LIFTS**

On $\mathbb{P}^1 \times \mathbb{P}^1$, we compute the following elements of $\mathcal{O}^0(\mathcal{O}(-1, 0)^2 \oplus \mathcal{O}(0, -1)^2)$, which are mapped to $(1, 0)$ via the matrix

$$
\begin{pmatrix}
    x_0 & x_1 & \gamma_1 y_0 + \gamma_2 y_1 & \gamma_3 y_0 \\
    \epsilon_1 x_0 + \epsilon_2 x_1 & \epsilon_3 x_0 & y_0 & y_1
\end{pmatrix}.
$$

The lifts are, in terms of the quantity $\phi$ defined in (22),

$$
U_{0,0} \Rightarrow \frac{1}{\phi x_0 y_0^2} \begin{pmatrix}
    \gamma_2 y_0^2 (\gamma_2 y_3 e_2 e_3 - 1) - y_0 y_1 (1 + e_2 e_3) \\
    y_0 e_2 y_3 [y_1 (1 + y_1)] - y_0 (1 - y_0 y_1) \\
    x_0 [y_0 e_2 y_3 (1 + y_1) + y_1 (1 - y_0 y_1)] \\
    x_1 e_2 [y_0 (1 + y_1) + y_1 (1 - y_0 y_1)] + x_0 y_1 e_2 (1 - y_0 y_1)
\end{pmatrix}
$$

$$
U_{0,1} \Rightarrow \frac{1}{\phi x_0 y_1^2} \begin{pmatrix}
    y_1^2 (1 + e_2 e_3) - y_0 y_1 (1 + e_2 e_3) \\
    y_0 e_2 y_3 [y_1 (1 + y_1) + y_1 (1 - y_0 y_1)] \\
    x_0 [y_0 y_1 (1 + y_1) + y_1 (1 - y_0 y_1)] \\
    x_1 e_2 [y_0 (1 + y_1) + y_1 (1 - y_0 y_1)] - x_0 y_1 e_2 (1 - y_0 y_1)
\end{pmatrix}
$$

$$
U_{1,0} \Rightarrow \frac{1}{\phi x_1 y_0} \begin{pmatrix}
    -y_1 y_2 e_3 (1 - y_2 e_3) - y_0 e_3 (1 + y_2 e_3) \\
    -y_1 y_2 e_3 (1 + y_2 e_3) + y_0 e_3 (1 - y_2 e_3) \\
    x_1 e_2 (1 + y_2 e_3) + x_0 e_3 (1 - y_2 e_3) \\
    x_0 [y_0 y_1 (1 + e_2 e_3) + x_1 y_2 e_3 (1 - y_2 e_3)]
\end{pmatrix}
$$

$$
U_{1,1} \Rightarrow \frac{1}{\phi x_1 y_1} \begin{pmatrix}
    y_0 y_1 e_3 (1 + y_1 e_2 e_3) - y_1 y_2 e_3 (1 + y_1 e_2 e_3) \\
    y_1 [y_0 y_1 e_3 (1 + y_1 e_2 e_3) + y_1 e_2 e_3 (1 + y_1 e_2 e_3)] \\
    x_0 y_1 e_3 (1 + y_2 e_3) + x_1 y_2 e_3 (1 - y_2 e_3) \\
    x_1 y_2 e_3 (1 + y_1 e_2 e_3) + x_0 e_3 (1 - y_2 e_3)
\end{pmatrix}
$$

Similarly, we compute lifts of $(0, 1)$:
Effectuation of the algorithm detailed in [33] yields the following correlators:

**Degree (0,0):**

\[ \langle \psi \psi \rangle = \frac{1}{\phi} (\epsilon_1 + \gamma_1 \epsilon_2 \epsilon_3) \]

\[ (56) \]

\[ \langle \tilde{\psi} \tilde{\psi} \rangle = \frac{1}{\phi} (\gamma_2 \gamma_3 \epsilon_2 \epsilon_3 - 1) \]

\[ \langle \tilde{\psi} \psi \rangle = \frac{1}{\phi} (\gamma_1 + \epsilon_1 \gamma_2 \gamma_3) \]

**Degree (1,0):**

\[ \langle \psi \psi \psi \rangle_{1,0} = \frac{1}{\phi^2} (\epsilon_1 + \gamma_1 \epsilon_2 \epsilon_3) [\gamma_1 (\epsilon_1 + \gamma_1 \epsilon_2 \epsilon_3) + 2(\gamma_2 \gamma_3 \epsilon_2 \epsilon_3 - 1)] \]

\[ \langle \psi \psi \bar{\psi} \rangle_{1,0} = \frac{1}{\phi^2} \left[ (\gamma_2 \gamma_3 \epsilon_2 \epsilon_3 - 1)^2 + \gamma_2 \gamma_3 (\epsilon_1 + \gamma_1 \epsilon_2 \epsilon_3)^2 \right] \]

\[ (57) \]

\[ \langle \tilde{\psi} \tilde{\psi} \tilde{\psi} \rangle_{1,0} = \frac{1}{\phi^2} \left[ (\gamma_1 + \gamma_2 \gamma_3 \epsilon_3 - 1) [2(\gamma_1 + \gamma_2 \gamma_3 \epsilon_3) - \gamma_1 (1 - \gamma_2 \gamma_3 \epsilon_2 \epsilon_3)] \right] \]

\[ \langle \bar{\psi} \bar{\psi} \bar{\psi} \rangle_{1,0} = \frac{1}{\phi^2} \left[ (\gamma_1 + \gamma_2 \gamma_3 \epsilon_3)^2 + \gamma_2 \gamma_3 (\gamma_2 \gamma_3 \epsilon_2 \epsilon_3 - 1)^2 \right] \]

\[ \langle \tilde{\psi} \tilde{\psi} \psi \rangle_{1,0} = \frac{1}{\phi^2} [\gamma_1 (\gamma_1 + \gamma_2 \gamma_3 \epsilon_3) - 2\gamma_2 \gamma_3 (\gamma_2 \gamma_3 \epsilon_2 \epsilon_3 - 1)] \]
Degree \((0,1)\):

\[
\langle \psi \psi \psi \psi \rangle_{0,1} = \frac{-1}{\phi^2} \left[ \epsilon_1 (\epsilon_1 + \gamma \epsilon_2 \epsilon_3) + 2 \epsilon_2 \epsilon_3 (1 - \gamma_2 \gamma_3 \epsilon_2 \epsilon_3) \right]
\]

\[
\langle \psi \psi \tilde{\psi} \tilde{\psi} \rangle_{0,1} = \frac{1}{\phi^2} \left[ \left( \epsilon_1 + \gamma \epsilon_2 \epsilon_3 \right)^2 + \epsilon_2 \epsilon_3 (\gamma_2 \gamma_3 \epsilon_2 \epsilon_3 - 1)^2 \right]
\]

\[
\langle \tilde{\psi} \tilde{\psi} \tilde{\psi} \tilde{\psi} \rangle_{0,1} = 1 \phi^2 \left[ (\gamma_2 \gamma_3 \epsilon_2 \epsilon_3 - 1) \left[ \epsilon_1 (\gamma_2 \gamma_3 \epsilon_2 \epsilon_3 - 1) + 2 (\epsilon_1 + \gamma \epsilon_2 \epsilon_3) \right] \right]
\]

\[
\langle \tilde{\psi} \tilde{\psi} \tilde{\psi} \psi \psi \rangle_{0,1} = \frac{1}{\phi^2} \left[ \epsilon_2 \epsilon_3 (\gamma_1 + \gamma_2 \gamma_3 \epsilon_1)^2 + (1 - \gamma_2 \gamma_3 \epsilon_2 \epsilon_3)^2 \right]
\]

\[
\langle \tilde{\psi} \tilde{\psi} \tilde{\psi} \tilde{\psi} \rangle_{0,1} = \frac{1}{\phi^2} \left[ \epsilon_1 (\gamma_1 + \gamma_2 \gamma_3 \epsilon_1) + 2 (\gamma_2 \gamma_3 \epsilon_2 \epsilon_3 - 1) \right]
\]

The overall four-point correlation functions are then of the form

\[
\langle \psi \psi \psi \psi \rangle = q \cdot \langle \psi \psi \psi \psi \rangle_{1,0} + \tilde{q} \cdot \langle \psi \psi \psi \psi \rangle_{0,1}.
\]

Six-point functions from the \((0,2)\), \((2,0)\), and \((1,1)\) sectors involve too many terms to express here. A Mathematica notebook containing all computed correlators may be obtained from the authors.

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