TWO DEFINITE INTEGRALS
INVOLVING PRODUCTS OF FOUR LEGENDRE FUNCTIONS

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ABSTRACT. The definite integrals \( \int_{-1}^{1} xP_\nu(x)^4 \, dx \) and \( \int_{0}^{1} x(P_\nu(x))^2[(P_\nu(x))^2 - (P_\nu(-x))^2] \, dx \) are evaluated in closed form, where \( P_\nu \) stands for the Legendre function of degree \( \nu \in \mathbb{C} \). Special cases of these integral formulae have appeared in arithmetic studies of automorphic Green’s functions and Epstein zeta functions.

Keywords: Legendre functions, Bessel functions, asymptotic expansions, automorphic Green’s functions

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1. INTRODUCTION

In [15, §6, Eq. 77] and [19, Remark 3.3.4.1, Eq. 3.3.38], we stated the following definite integral formulae

\[
\int_{-1}^{1} xP_\nu(x)^4 \, dx = \frac{2 \sin 4(\nu \pi) [\psi(2)(\nu + 1) + \psi(2)(\nu - 2) + 28 \zeta(3)]}{(2\nu + 1)^2 \pi^4}
\] (1.1)

and

\[
\int_{0}^{1} x(P_\nu(x))^2 - (P_\nu(-x))^2 \, dx = \frac{4 \sin^2(\nu \pi) [\psi(0)(\nu + 1) + \psi(0)(\nu - 2) + \gamma_0 + 2 \log 2]}{(2\nu + 1)^2 \pi^2}
\] (1.2)

without elaborating on their proof. Here, the Legendre function of the first kind is defined by the Mehler–Dirichlet integral [2, p. 22, Eq. 1.6.28]

\[
P_\nu(\cos \theta) = \frac{2}{\pi} \int_{0}^{\theta} \frac{\cos ((2\nu + 1)\beta)}{\sqrt{2(\cos \beta - \cos \theta)}} \, d\beta, \quad \theta \in (0, \pi), \nu \in \mathbb{C};
\] (1.3)

the polygamma functions \( \psi^{(m)}(z) = d^{m+1} \log \Gamma(z)/dz^{m+1}, m \in \mathbb{Z}_{\geq 0} \) are logarithmic derivatives of the Euler gamma function; the Euler–Mascheroni constant \( \gamma_0 := \lim_{n \to \infty} \left( -\log n + \sum_{k=1}^{n} \frac{1}{k} \right) \) is given by \(-\psi^{(0)}(1/2) - 2\log 2 \) and Apéry’s constant \( \zeta(3) = \sum_{n=1}^{\infty} n^{-3} \) is \(-\psi^{(2)}(1/2)/14 \). On the right-hand sides of (1.1) and (1.2), the expressions are well-defined for \( \nu \in \mathbb{C} \setminus \{ \mathbb{Z} \cup \{-1/2\} \} \), and extend to all \( \nu \in \mathbb{C} \), by continuity.

Some special cases of (1.1) and (1.2) have shown up in arithmetic studies of automorphic Green’s functions (see [17, Eqs. 2.3.18 and 2.3.40] as well as [19, Remark 3.3.4.1, Eqs. 3.3.34–3.3.37]) and Epstein zeta functions [18, Eqs. 4.18 and 4.28], namely,

\[
\zeta(3) = -\frac{2\pi^4}{189} \int_{-1}^{1} x[P_{-1/6}(x)]^4 \, dx = -\frac{\pi^4}{168} \int_{-1}^{1} x[P_{-1/4}(x)]^4 \, dx = -\frac{\pi^4}{243} \int_{-1}^{1} x[P_{-1/3}(x)]^4 \, dx,
\] (1.4)

\[
\zeta(5) = -\frac{\pi^4}{372} \int_{-1}^{1} x[P_{-1/2}(x)]^4 \, dx,
\] (1.5)

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and

\[
\frac{8\pi^2}{9} \int_0^1 x[P_{-1/6}(x)]^2 \left( [P_{-1/6}(x)]^2 - [P_{-1/6}(-x)]^2 \right) \, dx = -3 \log 3, 
\] (1.6)

\[
\frac{\pi^2}{8} \int_0^1 x[P_{-1/4}(x)]^2 \left( [P_{-1/4}(x)]^2 - [P_{-1/4}(-x)]^2 \right) \, dx = -\log 2, 
\] (1.7)

\[
\frac{\pi^2}{27} \int_0^1 x[P_{-1/3}(x)]^2 \left( [P_{-1/3}(x)]^2 - [P_{-1/3}(-x)]^2 \right) \, dx = 2 \log 2 - \frac{3}{2} \log 3, 
\] (1.8)

\[
\frac{\pi^2}{7} \int_0^1 x[P_{-1/2}(x)]^2 \left( [P_{-1/2}(x)]^2 - [P_{-1/2}(-x)]^2 \right) \, dx = -\zeta(3). 
\] (1.9)

Here, \(\zeta(5) = \sum_{n=1}^{\infty} n^{-5}\). These special cases have already been verified by independent methods in the aforementioned references. For example, (1.6)–(1.8) evaluate, respectively, the following weight-6 automorphic Green’s functions [cf. 17, Eqs. 2.3.18–2.3.20]:

\[
G_3^{5/7}(1) \left( \frac{1 + i \sqrt{3}}{2}, i \right), \quad G_3^{5/7}(2) \left( i - \frac{1}{2}, i \frac{1}{\sqrt{2}} \right) \quad \text{and} \quad G_3^{5/7}(3) \left( 3 + i \sqrt{3}, \frac{i}{\sqrt{3}} \right), 
\] (1.10)

where

\[
G_3^{5/7}(n)(z_1, z_2) := - \sum_{a, b, c, d \in \mathbb{Z}} Q_2 \left( 1 + \frac{|z_1 - \frac{ab + c + d}{Nc + d}|^2}{2 \Im z_1 \Im z_2} \right), 
\] (1.11)

with \(Q_2(t) = -\frac{3t}{2} + \frac{3t^2 - 4}{4} \log t + \frac{1}{t} \) for \(t > 1\). One can compute these special values of automorphic Green’s functions by other means (see [4, Chap. IV, Proposition 2] and [10, Remark 3.3.4.1]). In addition to descending from the symmetry of Epstein zeta functions [18, Eq. 4.18], the evaluation in (1.5) has also appeared in the studies of lattice sums by Wan and Zucker [13, Eq. 42].

In this article, we establish (1.1) and (1.2) in full generality, drawing on an analytic technique explored extensively in [15, 16]: the Hansen–Heine scaling limits that relate Legendre functions \(P_\nu\) of large degrees \(|\nu| \to \infty\) to Bessel functions (see [14, §5.7 and §5.71] as well as (2.4) and (2.5) in this article). The article is organized as follows. In §2 we present a collection of integrals, old and new, over products of four Bessel functions. In §3 we investigate the asymptotic behavior of the definite integrals in (1.1) and (1.2), based on the analysis in §2 and verify the integral formulae for all \(\nu \in \mathbb{C}\) by contour integrations.

\section*{2. Some definite integrals involving products of four Bessel functions}

We use \(P_\nu\) to define the Legendre functions of the second kind, as follows:

\[
Q_\nu(x) := \frac{\pi}{2 \sin(\nu \pi)} \{ \cos(\nu \pi)P_\nu(x) - P_\nu(-x) \}, \quad \nu \in \mathbb{C} \setminus \mathbb{Z}; \quad Q_n(x) := \lim_{\nu \to n} Q_\nu(x), \quad n \in \mathbb{Z}_{\geq 0} 
\] (2.1)

for \(-1 < x < 1\). For \(\nu \in \mathbb{C}, \, -\pi < \arg z \leq \pi\), the Bessel functions \(J_\nu\) and \(Y_\nu\) are defined by

\[
J_\nu(z) := \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k + \nu + 1)} \left( \frac{z}{2} \right)^{2k+\nu}, \quad Y_\nu(z) := \lim_{\nu \to -\nu} \frac{J_{\mu}(z) \cos(\mu \pi) - J_{-\mu}(z)}{\sin(\mu \pi)},
\] (2.2)

in parallel to the modified Bessel functions \(I_\nu\) and \(K_\nu\):

\[
I_\nu(z) := \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(k + \nu + 1)} \left( \frac{z}{2} \right)^{2k+\nu}, \quad K_\nu(z) := \frac{\pi}{2} \lim_{\nu \to -\nu} \frac{I_{-\mu}(z) - I_{\mu}(z)}{\sin(\mu \pi)}. 
\] (2.3)

As in [12, §3], our arguments in this paper draw heavily on the following asymptotic formulæ that connect Legendre functions \(P_\nu\) and \(Q_\nu\) to Bessel functions \(J_0\) and \(Y_0\) (see [12, Eqs. 43 and 46]...
The formula above follows directly from the asymptotic behavior [14, §7.2, Eq. 1]

\[ P_\nu(\cos \theta) = \sqrt{\frac{\theta}{\sin \theta}} J_0 \left( \frac{(2\nu + 1)\theta}{2} \right) + O \left( \frac{1}{2\nu + 1} \right), \quad (2.4) \]

\[ Q_\nu(\cos \theta) = -\frac{\pi}{2} \sqrt{\frac{\theta}{\sin \theta}} Y_0 \left( \frac{(2\nu + 1)\theta}{2} \right) + O \left( \frac{1}{2\nu + 1} \right), \quad (2.5) \]

where the error bounds are uniform for \( \theta \in (0, \pi/2) \), so long as \(|\nu| \to \infty, -\pi < \arg \nu < \pi \) [5, 7].

**Lemma 2.1.** We have the following evaluations:

\[ \int_0^\infty x J_0(x) Y_0(x)^3 \, dx = \int_0^\infty x [J_0(x)]^3 Y_0(x) \, dx = -\frac{1}{4\pi}, \quad (2.6) \]

and a vanishing identity:

\[ \int_0^\infty x [J_0(x)]^2 \left\{ [J_0(x)]^2 - 3[Y_0(x)]^2 \right\} \, dx = 0, \quad (2.7) \]

where it is understood that \( \int_0^\infty f(x) \, dx := \lim_{M \to +\infty} \int_{-M}^M f(x) \, dx. \)

**Proof.** We first point out that the Hankel function \( H_0^{(1)}(z) = J_0(z) + i Y_0(z) \) satisfies

\[ \mathcal{P} \int_{-\infty+i0^+}^{+\infty} z [H_0^{(1)}(z)]^4 \, dz = 0, \quad (2.8) \]

where the Cauchy principal value is taken: \( \mathcal{P} \int_{-\infty+i0^+}^{+\infty} f(z) \, dz := \lim_{M \to +\infty} \left( \int_{-M-i0^+}^{-M+i0^+} + \int_{M+i0^+}^{M-i0^+} \right) f(z) \, dz. \)

The formula above follows directly from the asymptotic behavior [14, §7.2, Eq. 1]

\[ H_0^{(1)}(z) = \sqrt{\frac{2}{\pi z}} e^{i(z-\frac{\pi}{4})} \left[ 1 + O \left( \frac{1}{|z|} \right) \right], \quad |z| \to +\infty, -\pi < \arg z < 2\pi, \quad (2.9) \]

and an application of Jordan’s lemma to a semi-circle in the upper half-plane. Noting that \( H_0^{(1)}(-x+i0^+) = -J_0(x) + i Y_0(x) \) for \( x > 0 \), we can convert (2.9) into

\[ 8i \left\{ \int_0^\infty x [J_0(x)]^3 Y_0(x) \, dx - \int_0^\infty x J_0(x) [Y_0(x)]^3 \, dx \right\} = 0, \quad (2.10) \]

which implies the first equality in (2.6).

We now use asymptotic analysis and residue calculus to establish a formula

\[ \mathcal{P} \int_{-\infty+i0^+}^{+\infty} \left\{ z [J_0(z)]^2 [H_0^{(1)}(z)]^2 - \frac{1-e^{4iz}}{\pi^2 z} \right\} \, dz = 0, \quad (2.11) \]

and read off its imaginary part as

\[ 4 \int_0^\infty x [J_0(x)]^3 Y_0(x) \, dx + \frac{1}{\pi^2} \int_{-\infty}^{+\infty} \frac{\sin(4x) \, dx}{x} = 0. \quad (2.12) \]

Thus we arrive at the evaluation

\[ \int_0^\infty x [J_0(x)]^3 Y_0(x) \, dx = -\frac{1}{4\pi}, \quad (2.13) \]

upon invoking the familiar Dirichlet integral \( \int_{-\infty}^{\infty} \frac{\sin x \, dx}{x} = \pi. \)

To prove (2.7), simply rewrite

\[ \mathcal{P} \int_{-\infty+i0^+}^{+\infty} z J_0(z) [H_0^{(1)}(z)]^3 \, dz = 0 \quad (2.14) \]

with the knowledge that \([H_0^{(1)}(x)]^3 - [H_0^{(1)}(-x+i0^+)]^3 = [J_0(x) + i Y_0(x)]^3 - [-J_0(x) + i Y_0(x)]^3 = 2 [J_0(x)]^3 - 6J_0(x) [Y_0(x)]^2. \)
Remark We note that the following integrals for \( \Re \nu > -1/2 \)
\[
\int_0^\infty x[J_\nu(bx)]^2 J_\nu(cx)Y_\nu(cx) \, dx = 0, \quad 0 < b < c \tag{2.15a}
\]
and
\[
\int_0^\infty x[J_\nu(bx)]^2 J_\nu(cx)Y_\nu(cx) \, dx = -\frac{1}{2\pi bc}, \quad 0 < c < b \tag{2.15b}
\]
have been tabulated in [10, item 2.13.24.2]. Formally, our (2.13) is an average of the two integral formulae displayed above.

Lemma 2.2. We have the following evaluation:
\[
\int_0^\infty x \left\{ [J_0(x)]^4 - 6[J_0(x)Y_0(x)]^2 + [Y_0(x)]^4 \right\} \, dx = -\frac{14\zeta(3)}{\pi^4}. \tag{2.16}
\]

Proof: We recast the stated integral into
\[
\Re \int_0^\infty z[H_0^{(1)}(z)]^4 \, dz = -\frac{14\zeta(3)}{\pi^4}. \tag{2.17}
\]
To show the identity above, deform the contour to the positive Im \( z \)-axis, use the relation \( H_0^{(1)}(iy) = 2iK_0(y)/\pi, \forall y > 0 \), along with the following formula:
\[
\int_0^\infty t[K_0(t)]^4 \, dt = \frac{7\zeta(3)}{8}, \tag{2.18}
\]
which is found in [8, p. 6], [3, Eq. 202] and [1, p. 19, Table 1]. (Thanks to the extensive and systematic research on “Bessel moments” in the references just mentioned, it is now possible to evaluate \( \int_0^\infty t^{\ell+mn}[K_m(t)]^n \, dt \) for \( \ell, m \in \mathbb{Z}_{>0}, n \in \{1, 2, 3, 4\} \) in closed form, as implemented in symbolic computation software like Mathematica. In contrast, only sporadic results are known when five or more Bessel factors are involved [1, §5 and §6].)

Lemma 2.3. We have the following integral identity:
\[
\int_0^\infty \left( x[J_0(x)]^2 ([Y_0(x)]^2 - [J_0(x)]^2) + \frac{1 - \cos(4x)}{\pi^2 x} \right) \, dx = 0. \tag{2.19}
\]

Proof. The claimed formula is equivalent to the statement that
\[
\Re \int_0^\infty \left\{ \frac{1 - e^{4iz}}{\pi^2 z} - z[J_0(z)]^2[H_0^{(1)}(z)]^2 \right\} \, dz = 0. \tag{2.20}
\]
Deforming the contour of integration to the positive Im \( z \)-axis, we see that it suffices to verify another vanishing identity:
\[
\int_0^\infty \left\{ \frac{1 - e^{-4y}}{y} - 4y[I_0(y)]^2[K_0(y)]^2 \right\} \, dy = 0. \tag{2.21}
\]
To put the formula above in broader context, we will demonstrate that
\[
\int_0^\infty y \left\{ I_{\nu-1/2}(y)I_{\nu+1/2}(y)K_{\nu-1/2}(y)K_{\nu+1/2}(y) - [I_\nu(y)]^2[K_\nu(y)]^2 \right\} \, dy = 0 \tag{2.22}
\]
holds as long as \( \Re \nu > -1/2 \). We start from the following identity [cf. 14, §13.6, Eq. 3]:
\[
I_\nu(y)K_\nu(y) = \int_0^{\pi/2} J_{2\nu}(2y \tan \phi) \, d\phi \cos \phi, \quad \Re \nu > -\frac{1}{2}, \tag{2.23}
\]
and transform it with the aid of a standard recurrence formula \( 2 \partial J_v(z) / \partial z = J_{v-1}(z) - J_{v+1}(z) \) for Bessel functions \([14, \S 3.2, \text{Eq. 2}]\). In particular, we have

\[
I_{v-1}(y)K_{v-1}(y) - I_v(y)K_v(y) = \int_0^{\pi/2} \frac{J_{2v-1}(2y \tan \theta) \sin \theta \, d\theta}{y}, \quad \Re v > 0, \tag{2.24}
\]

\[
I_{v-1/2}(y)K_{v-1/2}(y) - I_{v+1/2}(y)K_{v+1/2}(y) = \int_0^{\pi/2} \frac{J_{2v}(2y \tan \theta) \sin \theta \, d\theta}{y}, \quad \Re v > - \frac{1}{2}, \tag{2.25}
\]

after integration by parts. If we now denote the left-hand side of (2.22) by \( f(v) \), then

\[
f(v) - f(v - 1/2) = \int_0^\infty dy \int_{0<\theta<\pi/2, 0<\phi<\pi/2} d\theta \, d\phi \frac{\sin \theta}{\cos \phi} \times [J_{2v-1}(2y \tan \theta)J_{2v}(2y \tan \phi) - J_{2v}(2y \tan \theta)J_{2v-1}(2y \tan \phi)], \tag{2.26}
\]

where the integration over \( y \) has a closed form \([14, \S 13.42, \text{Eq. 8}]\):

\[
\int_0^\infty J_{\mu}(at)J_{\mu-1}(bt) \, dt = \begin{cases} 
  b^{\mu-1/a^\mu}, & a > b > 0, \\
  0, & 0 < a < b,
\end{cases} \tag{2.27}
\]

and one may indeed interchange the order of integrations, upon introducing a convergent factor \( e^{-ey}, e \to 0^+ \). This brings us

\[
f(v) - f(v - 1/2) = \int_0^\infty d\theta \int_{0<\phi<\pi/2} \left( \frac{\tan \phi}{\tan \theta} \right)^{2v} \frac{\cos \theta}{2 \cos \phi} \, d\theta \, d\phi - \int_0^\infty d\phi \int_0^{\pi/2} \left( \frac{\tan \phi}{\tan \theta} \right)^{2v} \frac{\sin \theta}{2 \sin \phi} \, d\theta \, d\phi. \tag{2.28}
\]

Here, both double integrals are convergent when \( \Re v > 0 \), and they exactly cancel each other (as is evident from variable substitutions \( \theta \to \frac{\pi}{2} - \theta \) and \( \phi \to \frac{\pi}{2} - \phi \)). Thus, we have \( f(v) = f(v - 1/2) \) as long as \( \Re v > 0 \). Therefore, the relation \( f(v) = \lim_{n \to -\infty} f(v + n/2) \) holds for \( \Re v > -1/2 \). When \( v > -1/2 \), we can compute the limit \( \lim_{n \to -\infty} f(v + n/2) = 0 \) through the dominated convergence theorem. Finally, we claim that \( f(0, \Re v > -1/2) \) follows from analytic continuation. \( \square \)

**Remark** It is worth noting that the “Bessel moments” in the form of \( \int_0^\infty t^k[I_0(t)]^2[K_0(t)]^{n-2} \, dt \) (where \( n \geq 5, k \geq 0 \)) appear in the investigations of two-dimensional quantum field theory \([1, \text{Eq. 13}]\). More generally, in \( D \)-dimensional space, the free propagator of a massive particle is expressible in terms of \( K_{(D-2)/2} \) \([2, \text{Eq. 6}]\), and the product of Bessel functions \( I_{(D-2)/2} K_{(D-2)/2} \) arises from angular average of the aforementioned propagator \([3, \text{Eq. 227}]\). Thus, it might be appropriate to ask for some physical interpretations or implications of the cancellation formula in (2.22).

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### 3. Proof of the Main Identities

In this section, we shall refer to the left- and right-hand sides of (1.1) (resp. (1.2)) as \( A_L(v) \) and \( A_R(v) \) (resp. \( B_L(v) \) and \( B_R(v) \)). It is clear that all these four functions are invariant under the variable substitution \( v \to -v - 1 \). Furthermore, one can show that the Taylor expansions

\[(2v + 1)^2 A_R(v) = 4(v - n) - \frac{8\pi^2}{3}(v - n)^3 + O((v - n)^4), \tag{3.1}\]

\[(2v + 1)^2 B_R(v) = 2(v - n) + O((v - n)^2) \tag{3.2}\]

hold around every non-negative integer \( n \). Before achieving our final goal of proving \( A_L(v) = A_R(v) \) and \( B_L(v) = B_R(v) \), we need to check that the Taylor expansions of \( (2v + 1)^2 A_L(v) \) and \( (2v + 1)^2 B_L(v) \) agree with their respective counterparts in (3.1) and (3.2).

**Lemma 3.1.** (a) We have an integral formula for \( v \in \mathbb{C} \):

\[(2v + 1)^2 \int_{-1}^{1} x [P_v(x)]^3 P_v(-x) \, dx = \frac{\sin(2v\pi) \cos(v\pi)}{\pi} \tag{3.3}\]
and a vanishing identity for \( n \in \mathbb{Z}_{\geq 0} \):

\[
\int_{-1}^{1} xP_n(x)Q_n(x) \left\{ \frac{4}{\pi^2} [Q_n(x)]^2 - [P_n(x)]^2 \right\} \, dx = 0.
\]  

(3.4)

(b) We have the following Taylor expansions

\[
(2v + 1)^2 A_L(v) = 4(v - n) - \frac{8\pi^2}{3} (v - n)^3 + O((v - n)^4),
\]

(3.5)

\[
(2v + 1)^2 B_L(v) = 2(v - n) + O((v - n)^2)
\]

(3.6)
as \( v \) approaches \( n \in \mathbb{Z}_{\geq 0} \).

**Proof.** (a) The proof of (3.3) is found in [15, Corollary 5.3]. It was based on the Tricomi pairing (see [12, §4.3, Eq. 2] or [6, Eq. 11.237]) of the following formulae [15, Proposition 5.2 and Corollary 5.3]

\[
\begin{align*}
2 \sin(v\pi) & \quad \mathcal{P} \int_{-1}^{1} \frac{P_v(\xi)P_v(-\xi) \, d\xi}{x - \xi} = [P_v(x)]^2 - [P_v(-x)]^2, \\
2 \sin(v\pi) & \quad \mathcal{P} \int_{-1}^{1} \frac{P_v(\xi)P_v(-\xi) \, d\xi}{x - \xi} = x([P_v(x)]^2 - [P_v(-x)]^2) - \frac{2\sin(2v\pi)}{(2v + 1)\pi},
\end{align*}
\]

(3.8)

(3.9)

which are valid for \( x \in (-1, 1), v \in \mathbb{C} \), along with an integral evaluation [15, Eq. 19(0,v)]

\[
\int_{-1}^{1} P_v(\xi)P_v(-\xi) \, d\xi = \frac{2\cos(v\pi)}{2v + 1}, \quad v \in \mathbb{C} \setminus \{-1/2\}.
\]

(3.10)

To justify (3.4), we need the Tricomi pairing of the following formulae for \( n \in \mathbb{Z}_{\geq 0} \):

\[
\begin{align*}
\frac{4}{\pi^2} & \quad \mathcal{P} \int_{-1}^{1} \frac{P_n(\xi)Q_n(\xi) \, d\xi}{x - \xi} = \frac{4}{\pi^2} [Q_n(x)]^2 - [P_n(x)]^2, \\
\frac{4}{\pi^2} & \quad \mathcal{P} \int_{-1}^{1} \frac{\xi P_n(\xi)Q_n(\xi) \, d\xi}{x - \xi} = \frac{4}{\pi^2} [Q_n(x)]^2 - [P_n(x)]^2.
\end{align*}
\]

(3.11)

(3.12)

To prove (3.11), apply the Hardy–Poincaré–Bertrand formula (see [12, §4.3, Eq. 4] or [6, Eq. 11.52]) to the Neumann integral representation of \( Q_n \) [6, Eq. 11.269]:

\[
Q_n(x) = \mathcal{P} \int_{-1}^{1} \frac{P_n(\xi) \, d\xi}{2(x - \xi)}, \quad \forall x \in (-1, 1), \forall n \in \mathbb{Z}_{\geq 0}.
\]

(3.13)

To deduce (3.12) from (3.11), use a vanishing identity \( \int_{-1}^{1} P_n(x)Q_n(x) \, dx = 0, \forall n \in \mathbb{Z}_{\geq 0} \) [11, item 2.18.13.3].

(b) The symmetry of Legendre polynomials \( P_n(-x) = (-1)^n P_n(x), n \in \mathbb{Z}_{\geq 0} \) gives \( (2n + 1)^2 A_L(n) = (2n + 1)^2 B_L(n) = 0 \). So it remains to check the derivatives of \( A_L(v) \) and \( B_L(v) \) at \( v = n \in \mathbb{Z}_{\geq 0} \). In what follows, we define

\[
P_n^{(m)}(x) := \frac{\partial^n}{\partial v^m} \bigg|_{v=n} P_v(x), \quad m \in \mathbb{Z}_{\geq 0},
\]

(3.14)
to simplify notations.

For the linear order Taylor expansions of \( (2v + 1)^2 A'_L(v) \) and \( (2v + 1)^2 B'_L(v) \), we compute

\[
(2n + 1)^2 A'_L(n) = 4(2n + 1)^2 \int_{-1}^{1} x[P_n(x)]^3 P_n^{(1)}(x) \, dx
\]

(3.15)
and
\[(2n + 1)^2 B'_L(n) = 2(2n + 1)^2 \int_0^1 x[P_n(x)]^2 (P_n(x)P_n^{[1]}(x) - P_n(-x)P_n^{[1]}(-x)) \, dx \]
\[= 2(2n + 1)^2 \int_{-1}^1 x[P_n(x)]^3 P_n^{[1]}(x) \, dx. \tag{3.16} \]

Differentiating (3.3) at \(v = n \in \mathbb{Z}_{\geq 0}\), we obtain
\[3(2n + 1)^2 \int_{-1}^1 x[P_n(x)]^2 P_n(-x)P_n^{[1]}(x) \, dx + (2n + 1)^2 \int_{-1}^1 x[P_n(x)]^3 P_n^{[1]}(-x) \, dx \tag{3.17} \]
\[= 2(2n + 1)^2 (-1)^n \int_{-1}^1 x[P_n(x)]^3 P_n^{[1]}(x) \, dx = \frac{\partial}{\partial v} \bigg|_{v=n} \frac{\sin(2v\pi) \cos(v\pi)}{\pi} = 2(-1)^n, \tag{3.18} \]
which yields \((2n + 1)^2 A'_L(n) = 2(2n + 1)^2 B'_L(n) = 4\) for all \(n \in \mathbb{Z}_{\geq 0}\).

For the quadratic order expansion of \((2n + 1)^2 A_L(v)\), we compute
\[(2n + 1)^2 A''_L(n) + 8(2n + 1)A'_L(n) \]
\[= 4(2n + 1)^2 \int_{-1}^1 x[P_n(x)]^3 P_n^{[2]}(x) \, dx + 12(2n + 1)^2 \int_{-1}^1 x[P_n(x)]^2 [P_n^{[1]}(x)]^2 \, dx + \frac{32}{2n + 1}, \tag{3.19} \]
and compare it to the second-order derivative of (3.3):
\[6(2n + 1)^2 \left\{ \int_{-1}^1 x[P_n(x)]P_n(-x)[P_n^{[1]}(x)]^2 \, dx + \int_{-1}^1 x[P_n(x)]^2 P_n^{[1]}(x)P_n^{[1]}(-x) \, dx \right\} \]
\[+ (2n + 1)^2 \left\{ 3 \int_{-1}^1 x[P_n(x)]^2 P_n(-x)P_n^{[2]}(x) \, dx + \int_{-1}^1 x[P_n(x)]^3 P_n^{[2]}(-x) \, dx \right\} + \frac{16(-1)^n}{2n + 1} \]
\[= (-1)^n \left\{ 2(2n + 1)^2 \int_{-1}^1 x[P_n(x)]^3 P_n^{[2]}(x) \, dx + 6(2n + 1)^2 \int_{-1}^1 x[P_n(x)]^2 [P_n^{[1]}(x)]^2 \, dx + \frac{16}{2n + 1} \right\} \]
\[= \frac{\partial^2}{\partial v^2} \bigg|_{v=n} \frac{\sin(2v\pi) \cos(v\pi)}{\pi} = 0. \tag{3.20} \]

Here, the integral
\[\int_{-1}^1 x[P_n(x)]^2 P_n^{[1]}(x)P_n^{[1]}(-x) \, dx \tag{3.21} \]
vanishes because the integrand is an odd function. This shows that \((2n + 1)^2 A_L(v) = 4(v - n) + O((v - n)^3)\).

To prepare for the computation of
\[A''_L(n) \]
\[= 24 \int_{-1}^1 xP_n(x)[P_n^{[1]}(x)]^3 \, dx + 36 \int_{-1}^1 x[P_n(x)]^2 P_n^{[1]}(x)P_n^{[2]}(x) \, dx + 4 \int_{-1}^1 x[P_n(x)]^3 P_n^{[3]}(x) \, dx \tag{3.22} \]
occuring in the cubic order expansion, we rewrite (3.4) with \(2Q_n(x) = P_n^{[1]}(x) - (-1)^nP_n^{[1]}(-x)\) (a consequence of (2.1)), which leads us to
\[\int_{-1}^1 xP_n(x)[P_n^{[1]}(x) - (-1)^nP_n^{[1]}(-x)]^3 \, dx = \pi^2 \int_{-1}^1 x[P_n(x)]^3 [P_n^{[1]}(x) - (-1)^nP_n^{[1]}(-x)] \, dx \]
\[= 2\pi^2 \int_{-1}^1 x[P_n(x)]^3 P_n^{[1]}(x) \, dx = \frac{2\pi^2}{2n + 1} \tag{3.23} \]
It is then clear that
\[\frac{\partial^3}{\partial v^3} \bigg|_{v=n} \left\{ \int_{-1}^1 x[P_v(x)]^3 P_v(-x) \, dx - \int_{-1}^1 x[P_v(-x)]^3 P_v(x) \, dx \right\} + \frac{12(-1)^n\pi^2}{(2n + 1)^2} \]
that the predominant contribution to formulae in (2.4) and (2.5), as well as the integral evaluations in (2.6), (2.7) and (2.16), we see \( N \) the uniform bounds do not depend on \( N \).

\[
\int_{-\infty}^{\infty} \frac{2\sin(\nu\pi)}{\nu^2} \frac{1}{\sin(\nu\pi)} \frac{1}{\nu^2} = 0.
\]

This gives the formula

\[
A''(n) = \frac{288}{(2n+1)^4} - \frac{16\pi^2}{(2n+1)^2}.
\]

The Taylor expansion in (3.3) can now be verified.

**Proposition 3.2.** The integral identity in (1.1) holds for all \( v \in \mathbb{C} \), where the evaluations of the right-hand side at \( v \in \{-1/2\} \cup \mathbb{Z} \) are interpreted as limits of a continuous function in \( v \).

**Proof.** Similar to [15, Proposition 3.1], we need to bound the expression

\[
\mathcal{A}(v) := \frac{(2v+1)^2}{\sin^4(v\pi)} [A_L(v) - A_R(v)]
\]

on a square contour \( C_N \) in the complex \( v \)-plane, with vertices \( \frac{1-4N}{4} - iN, \frac{1+4N}{4} - iN, \frac{1+4N}{4} + iN \) and \( \frac{1-4N}{4} + iN \), where \( N \in \mathbb{Z}_{>0} \). Observe that both \( |\cot(v\pi)| \) and \( 1/|\sin(v\pi)| \) are bounded on \( C_N \), and the uniform bounds do not depend on \( N \).

First, we consider the case where \( v = \frac{1+4N}{4} \) for a large positive integer \( N \). By the asymptotic formulae in (2.4) and (2.5), as well as the integral evaluations in (2.6), (2.7) and (2.16), we see that the predominant contribution to

\[
A_L(v) = \int_0^1 x \left[ (P_n(x))^4 - (P_n(-x))^4 \right] dx
\]

is

\[
\int_0^\infty \theta \left[ J_0 \left( \frac{(2v+1)\theta}{2} \right) \right]^4 \left[ \cos(v\pi)J_0 \left( \frac{(2v+1)\theta}{2} \right) + \sin(v\pi)Y_0 \left( \frac{(2v+1)\theta}{2} \right) \right]^4 \sin\theta \cos\theta d\theta
\]

\[
= -\sin^4(v\pi) \int_0^\infty \left[ J_0 \left( \frac{(2v+1)\theta}{2} \right) \right]^4 \left[ \cos(v\pi)J_0 \left( \frac{(2v+1)\theta}{2} \right) + \sin(v\pi)Y_0 \left( \frac{(2v+1)\theta}{2} \right) \right]^4 \sin\theta \cos\theta d\theta
\]

\[
+ 2\sin^2(v\pi) \int_0^\infty \theta \left[ J_0 \left( \frac{(2v+1)\theta}{2} \right) \right]^2 \left[ \cos(v\pi)J_0 \left( \frac{(2v+1)\theta}{2} \right) + \sin(v\pi)Y_0 \left( \frac{(2v+1)\theta}{2} \right) \right]^2 \sin\theta \cos\theta d\theta
\]

\[
- 2\sin(v\pi) \cos(v\pi) \int_0^\infty \theta J_0 \left( \frac{(2v+1)\theta}{2} \right) Y_0 \left( \frac{(2v+1)\theta}{2} \right) \left[ J_0 \left( \frac{(2v+1)\theta}{2} \right) \right]^2 \sin\theta \cos\theta d\theta
\]

\[
= \left[ \frac{14\zeta(3)}{\pi^4} \sin^4(v\pi) + \frac{\cos(v\pi)}{\pi^4} \frac{1}{\sin^3(v\pi)} + O\left( \frac{1}{(2v+1)^2} \right) \right] \frac{4}{(2v+1)^2},
\]

which is compatible with the expansion

\[
\frac{A_R(v)}{\sin^4(v\pi)} = \left[ \frac{14\zeta(3)}{\pi^4} + \frac{\cos(v\pi)}{\pi^4} \frac{1}{\sin^3(v\pi)} + O\left( \frac{1}{(2v+1)^2} \right) \right] \frac{4}{(2v+1)^2}.
\]

We break down the error bound for \( \mathcal{A}(v) \) into three parts.
(i) By the uniform approximations in the Hansen–Heine scaling limits (2.4) and (2.5), we have

\[
\int_0^{\pi/2} \left\{ J_0 \left( \frac{(2v + 1)\theta}{2} \right) \right\}^4 - \left[ \cos(v\pi)J_0 \left( \frac{(2v + 1)\theta}{2} \right) + \sin(v\pi)Y_0 \left( \frac{(2v + 1)\theta}{2} \right) \right]^4 \frac{\theta^2 \cot \theta \, d\theta}{\sin^4(v\pi)}
\]

\[
= \int_0^{\pi/2} \left\{ [P_v(\cos \theta)]^4 - \left[ \cos(v\pi)P_v(\cos \theta) - \frac{2\sin(v\pi)}{\pi}Q_v(\cos \theta) \right]^4 \right\} \frac{\sin \theta \cos \theta \, d\theta}{\sin^4(v\pi)} + O \left( \frac{1}{(2v + 1)^{5/2}} \right).
\]

(3.30)

Concretely speaking, the error term is bounded by

\[
O \left( \frac{1}{2v + 1} \int_0^{\pi/2} p \left( \frac{(2v + 1)\theta}{2} \right) \theta^2 \cot \theta \, d\theta \right)
\]

\[
= O \left( \frac{1}{(2v + 1)^{11/4}} \right).
\]

(3.32)

The rationale behind this is similar to (i) one may dissect the error term

\[
O \left( \frac{1}{2v + 1} \int_0^{\sqrt{2v+1}} p \left( \frac{(2v + 1)\theta}{2} \right) \theta \, d\theta \right) = O \left( \frac{1}{(2v + 1)^3} \int_0^{\sqrt{2v+1}/2} p(x) \, dx \right)
\]

(3.33)

into contributions from the ranges \( x \in (0,1) \) and \( x \in [1, \sqrt{2v + 1}/2] \).

(ii) As \( 1 - \theta \cot \theta = O((2v + 1)^{-1}) \) for \( \theta \in (0, 1/\sqrt{2v + 1}) \), we have an estimate

\[
\int_0^{\sqrt{2v+1}} \left\{ J_0 \left( \frac{(2v + 1)\theta}{2} \right) \right\}^4 - \left[ \cos(v\pi)J_0 \left( \frac{(2v + 1)\theta}{2} \right) + \sin(v\pi)Y_0 \left( \frac{(2v + 1)\theta}{2} \right) \right]^4 \frac{\theta^2 \cot \theta \, d\theta}{\sin^4(v\pi)}
\]

\[
= \int_0^{\sqrt{2v+1}/2} \theta \left\{ J_0 \left( \frac{(2v + 1)\theta}{2} \right) \right\}^4 - \left[ \cos(v\pi)J_0 \left( \frac{(2v + 1)\theta}{2} \right) + \sin(v\pi)Y_0 \left( \frac{(2v + 1)\theta}{2} \right) \right]^4 \frac{\theta \, d\theta}{\sin^4(v\pi)}
\]

+ \( O \left( \frac{1}{(2v + 1)^{11/4}} \right). \)

\]

(3.34)

along with integration by parts, we are able to verify that

\[
\int_0^{\pi/2} \left\{ J_0 \left( \frac{(2v + 1)\theta}{2} \right) \right\}^4 - \left[ \cos(v\pi)J_0 \left( \frac{(2v + 1)\theta}{2} \right) + \sin(v\pi)Y_0 \left( \frac{(2v + 1)\theta}{2} \right) \right]^4 \frac{\theta^2 \cot \theta \, d\theta}{\sin^4(v\pi)}
\]

\[
= O \left( \frac{1}{(2v + 1)^{5/2}} \right)
\]

and

\[
\int_0^{\pi/2} \left\{ J_0 \left( \frac{(2v + 1)\theta}{2} \right) \right\}^4 - \left[ \cos(v\pi)J_0 \left( \frac{(2v + 1)\theta}{2} \right) + \sin(v\pi)Y_0 \left( \frac{(2v + 1)\theta}{2} \right) \right]^4 \frac{\theta \, d\theta}{\sin^4(v\pi)}
\]

\[
= O \left( \frac{1}{(2v + 1)^{5/2}} \right).
\]

(3.35)

(3.36)
In sum, we have the following bound estimate

\[ \mathcal{A}(\nu) = O\left( \frac{1}{\sqrt{2\nu + 1}} \right) \]  \hspace{1cm} (3.37)

when \( \nu = \frac{1+4N}{4} \) for a large positive integer \( N \).

Then, for generic \( \nu \in C_N \) satisfying \( \text{Re} \nu \geq -1/2 \), we argue that (3.37) remains valid, upon modifying steps (i)–(iii) by contour deformations. For example, a variation on the derivations in step (i) brings us

\[ O\left( \frac{1}{(2\nu+1)^3} \int_0^{2\nu+1/\pi/4} p(x) \frac{2x/(2\nu+1)}{\tan(2x/(2\nu+1))} x \, dx \right) = O\left( \frac{1}{(2\nu+1)^{5/2}} \right), \]  \hspace{1cm} (3.38)

while we can bound the following integral on a circular arc in the complex \( z \)-plane

\[ \int_{(2\nu+1)/\pi/4}^{(2\nu+1)/\pi/4} p(z) \frac{2z/(2\nu+1)}{\tan(2z/(2\nu+1))} z \, dz \]  \hspace{1cm} (3.39)

in the spirit of Jordan’s lemma.

By virtue of the reflection symmetry \( \mathcal{A}(\nu) = \mathcal{A}(-\nu - 1) \), we have thus confirmed the error bound in (3.37) for all \( \nu \in C_N \). Furthermore, in view of (3.41) and (3.43), we are sure that \( \mathcal{A}(\nu) \) is analytic in the region bounded by the contour \( C_N \).

Finally, by Cauchy’s integral formula, we see that

\[ \mathcal{A}(\nu) = \frac{1}{2\pi i} \lim_{N \to \infty} \oint_{C_N} \frac{\mathcal{A}(z) \, dz}{z - \nu} \]  \hspace{1cm} (3.40)

vanishes identically, hence \( A_L(\nu) = A_R(\nu), \forall \nu \in \mathbb{C} \).

**Proposition 3.3.** The integral identity in (1.2) holds for all \( \nu \in \mathbb{C} \), where the evaluations of the right-hand side at \( \nu \in (-1/2) \cup \mathbb{Z} \) are interpreted as limits of a continuous function in \( \nu \).

**Proof:** We will only show that

\[ \mathcal{B}(\nu) := \frac{(2\nu+1)^2}{\sin^2(\nu\pi)} [B_L(\nu) - B_R(\nu)] = O\left( \frac{1}{\sqrt{2\nu + 1}} \right) \]  \hspace{1cm} (3.41)

for \( \nu = \frac{1+4N}{4} \), where \( N \) is a large positive integer. The rest of the procedures are essentially similar to those in Proposition 3.2.

Through direct expansions of the digamma functions, we have

\[ \frac{(2\nu+1)^2}{\sin^2(\nu\pi)} B_R(\nu) = \frac{2\cot(\nu\pi)}{\pi} + \frac{4(\gamma_0 + 2\log 2 + 2\log \nu)}{\pi^2} + O\left( \frac{1}{\nu} \right). \]  \hspace{1cm} (3.42)

In the meantime, we approximate \( B_L(\nu) \) by

\[ \tilde{B}(\nu) = \int_0^{(2\nu+1)/\pi/4} \left[ J_0(x) \right]^2 \left\{ [J_0(x)]^2 - [\cos(\nu\pi)J_0(x) + \sin(\nu\pi)Y_0(x)]^2 \right\} \frac{2x/(2\nu+1)}{\tan(2x/(2\nu+1))(2\nu+1)} \]  \hspace{1cm} (3.43)

and estimate the error bounds. By analogy to the proof of Proposition 3.2, we assert that

\[ \frac{(2\nu+1)^2}{\sin^2(\nu\pi)} [B_L(\nu) - \tilde{B}(\nu)] = O\left( \frac{1}{\sqrt{2\nu + 1}} \right). \]  \hspace{1cm} (3.44)

As in the proof of Proposition 3.2(iii–iii), we have

\[ -8 \int_0^{(2\nu+1)/\pi/4} [J_0(x)]^3 Y_0(x) \frac{2x/(2\nu+1)}{\tan(2x/(2\nu+1))} x \, dx = \frac{2\cot(\nu\pi)}{\pi} + O\left( \frac{1}{\sqrt{2\nu + 1}} \right), \]  \hspace{1cm} (3.45)
This eventually demonstrates that follows from (2.19) and integration by parts on \( x = -\frac{1}{\sqrt{2v + 1}} \) according to (2.13), while

\[
2 \cos x = 2(2k + 1)^2 + O\left(\frac{1}{(2v + 1)^{3/4}}\right)
\]

analog for the researches on “Bessel moments” [8, 3, 1] for (up to) four Bessel factors, which were framework for evaluating “Legendre moments” for (up to) four Legendre factors. This forms an four Legendre factors. In a sense, our previous work [15, 16] and this article have provided a

\[
\int_0^{(2v+1)^{3/4}} \left( x \left[ J_0(x) \right]^2 \left[ Y_0(x) \right]^2 - \left[ J_0(x) \right]^2 \right) + \frac{1 - \cos(4x)}{\pi^2 x} \frac{2x/(2v + 1)}{\tan(2x/(2v + 1))} \, dx = O\left(\frac{1}{\sqrt{2v + 1}}\right)
\]

Next, we compute a definite integral with the help of integration by parts and a Fourier series expansion [9, item 5.4.2.1]:

\[
\int_0^{(2v+1)^{3/4}} \frac{1 - \cos(4x)}{\pi^2 x} \frac{2x/(2v + 1)}{\tan(2x/(2v + 1))} \, dx = \frac{2}{\pi^2} \int_0^{\pi/2} \sin^2((2v + 1)\theta) \cot \theta \, d\theta
\]

\[
= -\frac{2(2v + 1)}{\pi^2} \int_0^{\pi/2} \sin(2v + 1)\theta) \log \sin \theta \, d\theta = \frac{2(2v + 1)}{\pi^2} \int_0^{\pi/2} \sin(2v + 1)\theta) \left[ \log 2 + \sum_{k=1}^{\infty} \frac{\cos(2k\theta)}{k} \right] \, d\theta
\]

\[
= \frac{2 \cos^2(v\pi)}{\pi^2} \log 2 + \frac{1}{4} \sum_{k=1}^{\infty} \left( \frac{2}{k} - \frac{1}{k + 2v + 1} - \frac{1}{k - 2v - 1} \right) [1 + (-1)^k \cos(2v\pi)]
\]

\[
= \frac{3v + 1}{2\pi^2 v(2v + 1)} + \frac{\psi(0)(2v) + \gamma_0 + \log 2}{\pi^2} + \frac{\cos(2v\pi)}{4\pi^2} \left[ \psi(0)\left( v + \frac{1}{2} \right) - 2\psi(0)(v + 1) + \psi(0)\left( v + \frac{3}{2} \right) \right]
\]

\[
= \frac{\gamma_0 + 2 \log 2 + \log v}{\pi^2} + O\left(\frac{1}{\sqrt{2v + 1}}\right).
\]

This eventually demonstrates that

\[
\frac{(2v + 1)^2}{\sin^2(v\pi)} B_L(v) = \frac{2 \cot(v\pi)}{\pi} + \frac{4(\gamma_0 + 2 \log 2 + \log v)}{\pi^2} + O\left(\frac{1}{\sqrt{2v + 1}}\right),
\]

so (3.41) holds.

While our foregoing treatments of (1.1) and (1.2) were motivated by their significance in certain arithmetic problems, the methods just outlined can be equally applied to other integrals with four Legendre factors. In a sense, our previous work [15, 16] and this article have provided a framework for evaluating “Legendre moments” for (up to) four Legendre factors. This forms an analog for the researches on “Bessel moments” [8, 3, 1] for (up to) four Bessel factors, which were motivated by Feynman diagrams in quantum field theory.

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References

[1] D. H. Bailey, J. M. Borwein, D. J. Broadhurst, and M. L. Glasser. Elliptic integral evaluations of Bessel moments and applications. J. Phys. A, 41:205203 (46pp), 2008.
[2] Harry Bateman. *Table of Integral Transforms*, volume I. McGraw-Hill, New York, NY, 1954. (compiled by staff of the Bateman Manuscript Project: Arthur Erdélyi, Wilhelm Magnus, Fritz Oberhettinger, Francesco G. Tricomi, David Bertin, W. B. Fults, A. R. Harvey, D. L. Thomsen, Jr., Maria A. Weber and E. L. Whitney).

[3] S. Groote, J. G. Körner, and A. A. Privovarov. On the evaluation of a certain class of Feynman diagrams in $x$-space: Sunrise-type topologies at any loop order. *Ann. Phys.*, 322:2374–2445, 2007.

[4] B. Gross, W. Kohnen, and D. Zagier. Heegner points and derivatives of $L$-series. II. *Math. Ann.*, 278:497–562, 1987.

[5] D. S. Jones. Asymptotics of the hypergeometric function. *Math. Meth. Appl. Sci.*, 24:369–389, 2001.

[6] Frederick W. King. *Hilbert Transforms (Volume 1)*, volume 124 of *Encyclopedia of Mathematics and Its Applications*. Cambridge University Press, Cambridge, UK, 2009.

[7] F. W. J. Olver. *Asymptotics and Special Functions*. Computer Science and Applied Mathematics. Academic Press, New York, NY, 1974.

[8] Stéphane Ouvry. Random Aharonov–Bohm vortices and some exactly solvable families of integrals. *J. Stat. Mech.: Theory Exp.*, 1:P09004 (9pp), 2005.

[9] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev. *Integrals and Series*, volume 1: Elementary Functions. Gordon and Breach Science Publishers, New York, NY, 1986. (translated from the Russian by N. M. Queen).

[10] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev. *Integrals and Series*, volume 2: Special Functions. Gordon and Breach Science Publishers, New York, NY, 1986. (translated from the Russian by N. M. Queen).

[11] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev. *Integrals and Series*, volume 3: More Special Functions. Gordon and Breach Science Publishers, New York, NY, 1990. (translated from the Russian by G. G. Gould).

[12] Francesco Giacomo Tricomi. *Integral Equations*. Dover Publications, New York, NY, 1985.

[13] J. G. Wan and I. J. Zucker. Integrals of $K$ and $E$ from lattice sums. *Ramanujan J.*, 40:257–278, 2016.

[14] G. N. Watson. *A Treatise on the Theory of Bessel Functions*. Cambridge University Press, Cambridge, UK, 2nd edition, 1944.

[15] Yajun Zhou. Legendre functions, spherical rotations, and multiple elliptic integrals. *Ramanujan J.*, 34:373–428, 2014.

[16] Yajun Zhou. On some integrals over the product of three Legendre functions. *Ramanujan J.*, 35:311–326, 2014.

[17] Yajun Zhou. Kontsevich–Zagier integrals for automorphic Green’s functions. I. *Ramanujan J.*, 38:227–329, 2015.

[18] Yajun Zhou. Ramanujan series for Epstein zeta functions. *Ramanujan J.*, 40:367–388, 2016.

[19] Yajun Zhou. Kontsevich–Zagier integrals for automorphic Green’s functions. II. *Ramanujan J.*, 42:623–688, 2017 (see arXiv:1505.0318v3 [math.NT] for erratum/addendum).

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