Sharp and plain estimates for Schrödinger perturbation of Gaussian kernel

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Abstract

We investigate whether a fundamental solution of the Schrödinger equation $\partial_t u = (\Delta + V) u$ has local in time sharp Gaussian estimates. We compare that class with the class of $V$ for which local in time plain Gaussian estimates hold. We concentrate on $V$ that have fixed sign and we present certain conclusions for $V$ in the Kato class.

1 Introduction and main results

Let $d = 1, 2, \ldots$. We consider the Gauss-Weierstrass kernel,

$$g(t, x, y) = (4\pi t)^{-d/2} e^{-|x - y|^2 / 4t}, \quad t > 0, \ x, y \in \mathbb{R}^d.$$ 

It is well known that $g$ is the fundamental solution of the equation $\partial_t u = \Delta u$, and time-homogeneous probability transition density – the heat kernel of $\Delta$. Throughout the paper we let $V: \mathbb{R}^d \to \mathbb{R}$ to be a Borel measurable function. We call $G: (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$ the heat kernel of $\Delta + V$ or the Schrödinger perturbation of $g$ by $V$, if the following Duhamel or perturbation formula holds for $t > 0$, $x, y \in \mathbb{R}^d$,

$$G(t, x, y) = g(t, x, y) + \int_0^t \int_{\mathbb{R}^d} G(s, x, z)V(z)g(t - s, z, y)dzds.$$ 

One of the directions in the study of $G(t, x, y)$ is to find its estimates or bounds. It is natural to ask if there are positive numbers, i.e., constants $0 < c_1 \leq c_2 < \infty$ such that the following two-sided bound holds,

$$c_1 \leq G(t, x, y) \leq c_2, \quad t > 0, \ x, y \in \mathbb{R}^d. \quad (1)$$

We call (1) sharp Gaussian estimates (or bounds) global (or uniform) in time. One can also ponder a weaker property – if for a given $T \in (0, \infty)$,

$$c_1 \leq \frac{G(t, x, y)}{g(t, x, y)} \leq c_2, \quad 0 < t \leq T, \ x, y \in \mathbb{R}^d. \quad (2)$$

We call (2) sharp Gaussian estimates local in time. We observe that the inequality in (1) is stronger than the plain Gaussian estimates global in time

$$c_1 (4\pi t)^{-d/2} e^{-|x - y|^2 / 4t} \leq G(t, x, y) \leq c_2 (4\pi t)^{-d/2} e^{-|x - y|^2 / 4t}, \quad t > 0, \ x, y \in \mathbb{R}^d. \quad (3)$$
where $0 < \varepsilon_1 \leq 1 \leq \varepsilon_2 < \infty$. Similarly, (2) is stronger than the plain Gaussian estimates local in time
\[ c_1 (4\pi t)^{-d/2} e^{-|y-x|^2 / 4\varepsilon_1} \leq G(t, x, y) \leq c_2 (4\pi t)^{-d/2} e^{-|y-x|^2 / 4\varepsilon_2}, \quad 0 < t \leq T, \ x, y \in \mathbb{R}^d. \] (4)

We refer the reader to [3] and [6] for a brief survey on the literature concerning (1), (2), (3) and (4), in particular, on the results of [25], [20] and [9]. In the present paper our main focus is on the distinction between local sharp Gaussian estimates (2) and local plain Gaussian estimates (4).

Theorem 1. Let $V \leq 0$. Then, (2) holds if and only if (4) holds according to the ‘local in time’ column of Table 1. Similarly, (1) holds if and only if (3) holds according to the ‘global in time’ column.

| dimension | local in time | global in time |
|-----------|--------------|---------------|
| $d \geq 4$ | No           | No            |
| $d = 3$   | No$^1$       | Yes$^2$       |
| $d = 2$   | Yes$^3$      | Yes$^3$       |
| $d = 1$   | Yes$^4$      | Yes$^5$       |

Table 1: Equivalence of sharp and plain Gaussian bounds for $V \leq 0$.

At this point we enclose some comments and references that complete Table 1 and which can also be tracked in other places in the paper.

Remark 1. Let $V \leq 0$. We list the superscripts of Table 1.

1) we refer the reader to [20, Theorem 1B];

2) (1) and (3) are equivalent to the potential boundedness of $V$ if $d = 3$, see [3];

3) (2) and (4) are equivalent to the enlarged Kato class condition on $V$ if $d = 2$, see (8) and Corollary 6;

4) (2) and (4) are equivalent to Kato class condition on $V$ (uniform local integrability of $V$) if $d = 1$, see (7) and Corollary 7;

5) (1) as well as (3) are impossible for non-trivial $V$ if $d \leq 2$, see [3, page 3].

In the literature there exist several intrinsic quantities that are used to characterize $V \leq 0$ for which (2) holds, and to formulate necessary and (separately) sufficient conditions for (2) if $V \geq 0$. Let us start with one that derives from Zhang [25, Lemma 3.1 and Lemma 3.2] and from Bogdan, Jakubowski and Hansen [4, (1)]. For $t > 0$ and $x, y \in \mathbb{R}^d$ we define
\[ S(V, t, x, y) = \int_0^t \int \frac{g(s, x, z)g(t - s, z, y)}{g(t, x, y)} |V(z)| \, dz \, ds. \] (5)

Further, we let
\[ \|S(V, t)\|_\infty = \sup_{x, y \in \mathbb{R}^d} S(V, t, x, y), \quad \|S(V)\|_{T, \infty} = \sup_{0 < t \leq T} \|S(V, t)\|_\infty. \]

Other quantities are surveyed in Section 2.1. The following lemma is an excerpt from [3] that exposes the relation between $\|S(V)\|_{T, \infty}$ and (2), and will suffice for our discussion and purposes. We write as usually $f^+ = \max\{0, f\}, f^- = \max\{0, -f\}$.
Lemma 2. We have

1) If \( V \leq 0 \), then for each \( T \in (0, \infty) \), \([2]\) is equivalent to \( \|S(V)\|_{T, \infty} < \infty \).

2) If \( V \geq 0 \), then \([2]\) implies \( \|S(V)\|_{T, \infty} < \infty \) for each \( T \in (0, \infty) \).

3) If for some \( h > 0 \) and \( 0 \leq \eta < 1 \) we have \( \|S(V +)\|_{h, \infty} \leq \eta \) and if \( S(V^-, t, x, y) \) is bounded on bounded subsets of \((0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d\), then
   \[
   e^{-S(V^-, t, x, y)} \leq \frac{G(t, x, y)}{g(t, x, y)} \left( \frac{1}{1 - \eta} \right)^{1+t/h}, \quad t > 0, \ x, y \in \mathbb{R}^d.
   \] (6)

The relation between the bound of \((5)\) and the upper bound in \((6)\), in the framework of integral kernels, can be found in \([5]\). For some other variants see \([13]\). Recall that the celebrated sufficient condition for the local plain Gaussian estimates \((4)\) is that \( V \) belongs to the Kato class \((\[2\], \[22\], \[15\], \[14\])\), which we abbreviate to \( V \in \mathcal{K}_d \). More precisely, \( V \in \mathcal{K}_d \) if
   \[
   \lim_{t \to 0^+} \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} g(s, x, z) |V(z)| \, dz \, ds = 0.
   \] (7)

We say that \( V \) belongs to the enlarged Kato class, which we denote by \( V \in \hat{\mathcal{K}}_d \), if
   \[
   \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} g(s, x, z) |V(z)| \, dz \, ds < \infty,
   \] (8)

holds for some (every) \( t > 0 \) (see \([23, Proposition 5.1]\)). The class \( \hat{\mathcal{K}}_d \) is also known as the Dynkin class in a measure theory context. We refer the reader to \([27], [16]\) and \([11]\) for a wider perspective on the Kato class; and to \([24], [21], [18], [20], [19], [12], [3]\) for a corresponding class and results for time-dependent \( V \). We will also use the following notation
   \[
   \Delta^{-1} V(x) = - \int_0^\infty \int_{\mathbb{R}^d} g(s, x, z) V(z) \, dz \, ds, \quad \|\Delta^{-1} V\|_{\infty} = \sup_{x \in \mathbb{R}^d} |\Delta^{-1} V(x)|.
   \]

We give main results concerning the difference between sharp and plain Gaussian estimates. We distinguish four cases: \( d \geq 4, d = 3, d = 2, d = 1 \).

**Theorem 2.** Let \( d \geq 4 \). There exists \( V \leq 0 \) with the following properties

(a) \( \text{supp}(V) \subseteq B(0, 1) \),

(b) \( V \in \mathcal{K}_d \),

(c) \( \|\Delta^{-1} V\|_{\infty} < \infty \),

(d) \( \|S(V, t)\|_{\infty} = \infty \) for every \( t > 0 \).

Such a strong result is not possible if \( d = 3 \). Indeed, in this dimension the condition \( \|\Delta^{-1} V\|_{\infty} < \infty \) implies (is equivalent to) \( \sup_{t \geq 0} \|S(V, t)\|_{\infty} < \infty \), see \([3, (7)\) and (8)]\). In particular, if \( V \in \mathcal{K}_d \) has compact support, then \( \|\Delta^{-1} V\|_{\infty} < \infty \).

**Theorem 3.** Let \( d = 3 \). There exists \( V \leq 0 \) with the following properties

(a) \( V \in \mathcal{K}_3 \),

...
Theorems 2 and 3 yield that for \( d \geq 3 \) there is a function \( V \leq 0 \) such that (1) holds with \( \varepsilon_1 < 1 < \varepsilon_2 \) arbitrarily close to 1 and (2) does not hold. Additionally, for \( d \geq 4 \) the function \( V \) may be chosen in such a way that \( \text{supp} V \) is compact and (3) holds, see Corollaries 4 and 5. We note that the latter cannot be done in the dimension 3. In fact, if \( d = 3 \) and \( V \leq 0 \), the global plain Gaussian estimates (3) hold if and only if global sharp Gaussian estimates (1) hold, see [3, Page 6]. From Theorem 3 we deduce that such phenomenon does not occur for local in time bounds.

The situation is radically different if \( d \leq 2 \). In this case the condition \( V \in \mathcal{K}_d \) yields \( \|S(V,t)\|_{\infty} < \infty \). It is a consequence of the following theorem.

**Theorem 4.** Let \( d = 2 \) or \( d = 1 \). There exists an absolute constant \( c > 0 \) such that for all \( T > 0 \) and \( V \) we have

\[
c^{-1} \sup_{x \in \mathbb{R}^d} \int_0^T \int_{\mathbb{R}^d} g(s,x,z)|V(z)|dzds \leq \|S(V)\|_{T,\infty} \leq c \sup_{x \in \mathbb{R}^d} \int_0^T \int_{\mathbb{R}^d} g(s,x,z)|V(z)|dzds.
\]

(9)

As a corollary of Theorem 4 we characterize classes \( \mathcal{K}_d \) and \( \mathcal{\hat{K}}_d \) for \( d \leq 2 \), by using the quantity \( \|S(V)\|_{T,\infty} \), see Corollaries 6 and 7. Additionally, we obtain that for \( d \leq 2 \) and \( V \leq 0 \), (2) holds if and only if \( V \in \mathcal{\hat{K}}_d \). For \( d = 1 \) the same property holds for \( V \geq 0 \). See Corollaries 6 and 8.

The rest of the paper is organized as follows. In Section 2 we collect other quantities used in the literature to analyse (2), and we show that they are comparable. We also discuss analogies with various descriptions of the Kato class. In Section 3 we introduce an explicit kernel \( K(t,x,y) \) and use it to propose another test for (2) to hold. In that section we also formulate and prove Theorem 5. In Section 4 we prove Theorems 2 – 4. In Section 5 we give corollaries of the main results of the paper and the proof of Theorem 1.

Throughout the paper \( B(x,r) \) denotes a ball of radius \( r > 0 \) in \( \mathbb{R}^d \) centred at \( x \in \mathbb{R}^d \). In short we write \( B_r = B(0,r) \).

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2 Preliminaries

2.1 An overview of tests for sharp bounds

We have already seen in Lemma 2 how to use a test based on \( S(V,t,x,y) \) to analyse (2). In [25] Zhang introduced yet another object, for \( t > 0 \) and \( x, y \in \mathbb{R}^d \),

\[
N(V, t, x, y) = \int_{\mathbb{R}^d} \int_0^{t/2} e^{-|z-y+(\tau/t)(y-x)|^2/(4\tau)} \frac{V(z)}{\tau^{d/2}} dz d\tau + \int_{t/2}^t \int_{\mathbb{R}^d} e^{-|z-y+(\tau/t)(y-x)|^2/(4(t-\tau))} \frac{V(z)}{(t-\tau)^{d/2}} dz d\tau.
\]

It is actually comparable with \( S \) in the following sense,

\[
S(V,t,x,y) \geq m_1 N(V,t/2,x,y), \quad (L)
\]
\[
S(V,t,x,y) \leq m_2 N(V,t,x,y), \quad (U)
\]

(10)
where constants $m_1, m_2$ depend only on $d$, see [3] (L) and (U) on page 5. The quantity $N$ gives rise to

$$
\|N(V,t)\|_\infty = \sup_{x_0 \in \mathbb{R}^d} N(V,t,0) \quad \text{and} \quad \|N(V)\|_{T,\infty} = \sup_{0 < t \leq T} \|N(V,t)\|_\infty.
$$

On the other hand, in [20] Milman and Semenov (for $d \geq 3$) proposed to use for $\lambda > 0$,

$$
e_*(V,\lambda) = \sup_{\alpha \in \mathbb{R}^d} \|((\lambda - \Delta + 2\alpha \cdot \nabla)^{-1}V\|_\infty.
$$

The operator $(\lambda - \Delta + 2\alpha \cdot \nabla)^{-1}$ is an integral operator with a kernel equal to $\int_0^\infty e^{-\lambda s} p_\alpha(s, x, y) ds$, where for $\alpha \in \mathbb{R}^d$ and $t > 0, x, y \in \mathbb{R}^d$, the function $p_\alpha(t, x, y)$ is the fundamental solution of the equation $\partial_t = \Delta - 2\alpha \cdot \nabla$, i.e.,

$$p_\alpha(t, x, y) = g(t, x - 2\alpha t, y).$$

We will show that $e_*$ is also comparable with $S$ and $N$. To this end we will use

$$r_*(V, t) = \sup_{\alpha, x \in \mathbb{R}^d} \int_0^t \int p_\alpha(s, x, z) |V(z)| d\tau dz.$$

**Lemma 3.** For all $t > 0$ and $V$ we have

$$r_*(V, t/2) \leq (4\pi)^{-d/2} \|N(V, t)\|_\infty \leq 2 r_*(V, t/2).$$

**Proof.** Note that

$$
\sup_{x, y \in \mathbb{R}^d} \int_0^{t/2} \int_{\mathbb{R}^d} e^{-|z - y + (\tau/t)(y - x)|^2/(4\tau)} |V(z)| d\tau dz = \sup_{x, y \in \mathbb{R}^d} \int_0^{t/2} \int_{\mathbb{R}^d} e^{-|z - x + 2\alpha \tau|^2/(4\tau)} |V(z)| d\tau dz
$$

$$= (4\pi)^{d/2} \sup_{x, y \in \mathbb{R}^d} \int_0^{t/2} \int_{\mathbb{R}^d} p_\alpha(\tau, x, z) |V(z)| d\tau dz.$$

The assertion of the lemma follows from [3] Lemma 3.1].

**Lemma 4.** For all $\lambda > 0$, $\alpha \in \mathbb{R}^d$ and $V$ we have

$$(1 - e^{-1}) \|((\lambda - \Delta + 2\alpha \cdot \nabla)^{-1} V\|_\infty \leq \sup_{x \in \mathbb{R}^d} \int_0^{1/\lambda} \int_{\mathbb{R}^d} p_\alpha(s, x, z) |V(z)| d\tau dz,$$

$$e \|((\lambda - \Delta + 2\alpha \cdot \nabla)^{-1} V\|_\infty \geq \sup_{x \in \mathbb{R}^d} \int_0^{1/\lambda} \int_{\mathbb{R}^d} p_\alpha(s, x, z) |V(z)| d\tau dz.$$

**Proof.** For $t > 0$, $x \in \mathbb{R}^d$ we let $P_t f(x) = \int_{\mathbb{R}^d} p_\alpha(t, x, z) f(z) dz$. Note that

$$\|((\lambda - \Delta + 2\alpha \cdot \nabla)^{-1} V\|_\infty = \sup_{x \in \mathbb{R}^d} \int_0^\infty e^{-\lambda t} |V(x)| dt,$$

and

$$\sup_{x \in \mathbb{R}^d} \int_0^{1/\lambda} \int_{\mathbb{R}^d} p_\alpha(s, x, z) |V(z)| d\tau dz = \sup_{x \in \mathbb{R}^d} \int_0^{1/\lambda} P_t |V(x)| dz.$$

Therefore, the desired inequalities follow from [11] Lemma 3.3].
Recall from [3, Corollary 2.3] that for all \( T > 0 \) and \( V \) we have

\[
\|S(V)\|_{2T,\infty} \leq 2\|S(V)\|_{T,\infty}.
\]

(10)

Now, (L), (U), (10), Lemma 3 and Lemma 4 provide the following comparability.

**Proposition 5.** For all \( T > 0 \) and \( V \) we have

\[
\frac{m_1}{2}\|N(V)\|_{T,\infty} \leq \|S(V)\|_{T,\infty} \leq m_2\|N(V)\|_{T,\infty},
\]

as well as

\[
r_*(V,T/2) \leq (4\pi)^{-d/2}\|N(V)\|_{T,\infty} \leq 2r_*(V,T/2),
\]

(11)

and

\[
(1 - e^{-1})e_*(V,1/T) \leq r_*(V,T) \leq ee_*(V,1/T).
\]

(12)

Thus, from Proposition 5 and Lemma 2 we conclude that the four tests on \( V \), for the local sharp Gaussian estimates (2) to hold, based on \( S, N, r_* \) and \( e_* \) are equivalent if \( V \leq 0 \); and comparable if \( V \geq 0 \) (in that case the exact magnitudes of quantities used in those tests matter, see part 3) of Lemma 2). In this context we highly recommend the reader to get familiar with [20, Theorem 1B and Theorem 1C]), where \( e_* \) is brought into play.

We end this subsection by one more observation on \( S \) and \( N \). Due to Lemma 3, (10), (11) and (L) the supremum over \( 0 < t \leq T \) in \( \|S(V)\|_{T,\infty} \) and \( \|N(V)\|_{T,\infty} \) is, in a sense, dispensable.

**Corollary 1.** For all \( T > 0 \) and \( V \) we have

\[
\|N(V,T)\|_{\infty} \leq \|N(V)\|_{T,\infty} \leq 2\|N(V,T)\|_{\infty},
\]

and

\[
\|S(V,T)\|_{\infty} \leq \|S(V)\|_{T,\infty} \leq 4(m_2/m_1)\|S(V,T)\|_{\infty}.
\]

2.2 Kato class analogies

It is well known that \( V \in K_d \) if and only if

\[
\lim_{\lambda \to \infty} \|\lambda - \Delta\|^{-1/2}V\|_{\infty} = 0.
\]

Actually, taking \( \alpha = 0 \) in Lemma 4 for all \( \lambda > 0 \) and \( V \) we get

\[
(1 - e^{-1})^{1/2}\|\lambda - \Delta\|^{-1/2}V\|_{\infty} \leq \sup_{x \in \mathbb{R}^d} \int_0^{1/\lambda} \int_{\mathbb{R}^d} g(s,x,z)\|V(z)\|\,dz\,ds \leq e\|\lambda - \Delta\|^{-1/2}V\|_{\infty},
\]

which is rather a general relation between a semigroup and its resolvent, see [11, Lemma 3.3]. In particular, \( V \) belongs to the enlarged Kato class if and only if \( \|\lambda - \Delta\|^{-1/2}V\|_{\infty} < \infty \) for some (every) \( \lambda > 0 \). In view of our main discourse on sharp Gaussian estimates a counterpart of those inequalities is given in (13), also as a consequence of Lemma 4.

The following result leads to an alternative description of the Kato class (see [8, Theorem 1.27]).
Lemma 6. There are constants $C_1$ and $C_2$ that depend only on dimension $d$ and such that for all $t > 0$ and $V$ we have

$$C_1 A(t) \leq \sup_{x \in \mathbb{R}^d} \int_{|z-x|< \sqrt{4t}} \frac{|V(z)|}{|z-x|^d} \, dz \leq C_2 A(t), \quad d \geq 3; \quad (14)$$

$$C_1 A(t) \leq \sup_{x \in \mathbb{R}^2} \int_{|z-x|< \sqrt{4t}} |V(z)| \log \frac{4t}{|z-x|^2} \, dz \leq C_2 A(t), \quad d = 2; \quad (15)$$

$$C_1 A(t) \leq \sup_{x \in \mathbb{R}} \sqrt{t} \int_{|z-x|< \sqrt{4t}} |V(z)| \, dz \leq C_2 A(t), \quad d = 1; \quad (16)$$

where

$$A(t) = \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} g(s,x,z)|V(z)| \, dz \, ds.$$ 

Proof. First note that the heat kernel $p_{0,d}$ defined in [8, page 47] has a different time scaling than $g$, i.e., $g(t,x,y) = p_{0,d}(2t,x,y)$ and $\int \int_{\mathbb{R}^d} g(s,x,z)|V(z)| \, dz \, ds = \frac{1}{2} \int_0^{2t} \int_{\mathbb{R}^d} p_{0,d}(s,x,z)|V(z)| \, dz \, ds$. The inequalities (14) are now deduced from [8, Theorem 1.28(a)]. The upper bound in (15) follows from the lower bound in [8, Theorem 1.28(b)]. To prove the lower bound in (15) we note that

$$\int_{|z-x|< \sqrt{4t}} |V(z)| \, dz \leq 5 \sup_{x \in \mathbb{R}^2} \int_{|z-x|< \sqrt{t}} |V(z)| \, dz \leq \frac{5}{2 \log(4/3)} \sup_{x \in \mathbb{R}^2} \int_{|z-x|< \sqrt{4t}} |V(z)| \log \frac{4t}{|z-x|^2} \, dz,$$

and apply the upper bound in [8, Theorem 1.28(b)]. Finally we look at (16), and due to [8, Theorem 1.28(b)] it suffices to show the upper bound in (16). To this end we observe that

$$\int_{|z-x|< \sqrt{4t}} \left( \sqrt{4t} - |z-x| \right) |V(z)| \, dz \geq \sqrt{t} \int_{|z-x|< \sqrt{4t}} |V(z)| \, dz \geq \sqrt{t} \int_{\sqrt{4t}-|z-x|< \sqrt{4t}} |V(z)| \, dz,$$

and use the lower bound in [8, Theorem 1.28(c)].

Therefore, $V$ belongs to the Kato class if the expressions in the square brackets of Lemma 6 converge to 0, see also [2, Theorem 4.5], [22, Proposition A.2.6], [7, Theorem 3.6], [3, Proposition 4.3]. In Section 3 we establish a counterpart of Lemma 6 describing sharp Gaussian estimates (2).

At least in high dimensions the latter description of the Kato class may be viewed through the prism of the following property: for every $d \geq 3$ there exists a constant $c > 0$ that depends only on $d$ and such that for all $t > 0$ and $x, y \in \mathbb{R}^d$ satisfying $|z - x| \leq \sqrt{4t}$ we have

$$c^{-1} \int_0^\infty g(s,x,z) \, ds \leq \int_0^t g(s,x,z) \, ds \leq \int_0^\infty g(s,x,z) \, ds.$$

In the context of sharp Gaussian estimates an analogue of that observation is proven in Proposition 8, more precisely in [19].
3 A new test for sharp bounds

Each of the tests based on $S$, $N$, $r_\ast$ or $e_\ast$ may have various advantages and disadvantages when applying to particular functions $V$. The utility of the condition based on $S$ has already been exposed in [3, Section 1.2] for functions $V$ that factorize. We use this paper as an opportunity to propose another equivalent test based on a function $K(t, x, y)$, which originates in $r_\ast(V, T)$. More precisely, we will estimate $r_\ast(V, T)$ by investigating the kernel $\int_0^T p_\alpha(s, x, z) ds$ on a certain crucial region. In what follows the notation is chosen to be consistent with [3]. For $t > 0$, $x, y \in \mathbb{R}^d$ we let:

\[
K(t, x, y) = e^{-\frac{|x||y|-(x,y)}{2}} \frac{1}{|x|^{d-2}} (1 + |x||y|)^{d/2-3/2} 1_{|x|\leq t|y|}, \quad \text{if } d \geq 3;
\]

\[
K(t, x, y) = e^{-\frac{|x||y|-(x,y)}{2}} \log \left(1 + \frac{1}{\sqrt{|x||y|}}\right) 1_{|x| \leq t|y|}, \quad \text{if } d = 2;
\]

\[
K(t, x, y) = e^{-\frac{|x||y|-(x,y)}{2}} \sqrt{t} (1 + t|y|^2)^{-1/2} 1_{|x| \leq t|y|}, \quad \text{if } d = 1.
\]

We further define

\[
K(V, t, x, y) = \int_{\mathbb{R}^d} K(t, z - x, y)V(z) dz, \quad \|K(V, t)\|_{\infty} = \sup_{x, y \in \mathbb{R}^d} K(V, t, x, y).
\]

**Theorem 5.** There are constants $0 < C_1 < C_2 < \infty$ that depend only on $d$ and such that for all $T > 0$ and $V$ we have

\[
C_1 \|K(V, T)\|_{\infty} \leq \|S(V)\|_{T, \infty} \leq C_2 \|K(V, T)\|_{\infty}.
\]

Before giving the proof of Theorem 5 we provide consequences, comments and auxiliary results.

**Corollary 2.** Let $V \leq 0$. Then [2] holds if and only if $\|K(V, T)\|_{\infty} < \infty$ for some (every) $T > 0$.

**Remark 7.** If $d \geq 3$, using Proposition 3, Theorem 5 and letting $T \to \infty$ we recover the result of [3, Theorem 1.4] that concerns global sharp Gaussian estimates [1].

We note that the kernels of $S$ and $N$ are given explicitly, but they are of rather complex structure that involve three parameters $0 < t \leq T$, $x, y \in \mathbb{R}^d$ that the supremum is taken of. Corollary 1 makes it possible to remove one parameter from $S$ and $N$. Certain reduction is also made in $e_\ast$ and $r_\ast$, where only two parameters $\alpha, x \in \mathbb{R}^d$ appear. It is also known and results from a simple substitution (see [10, 8.432, formula 6.]) that for $\lambda > 0$ and $x, z, \alpha \in \mathbb{R}^d$,

\[
\int_0^\infty e^{-\lambda s} p_\alpha(s, x, z) ds = (2\pi)^{-d/2} e^{-(z-x,\alpha)} \left(\frac{\sqrt{\lambda + |\alpha|^2}}{|z - x|}\right)^{d/2-1} K_{d/2-1} \left(|z - x|\sqrt{\lambda + |\alpha|^2}\right), \quad (17)
\]

where $K_{\nu}$ is the modified Bessel function of the second kind. Thus,

\[
e_\ast(V, \lambda) = (2\pi)^{-d/2} \sup_{\alpha, x \in \mathbb{R}^d} \int_{\mathbb{R}^d} e^{-(z-x,\alpha)} \left(\frac{\sqrt{\lambda + |\alpha|^2}}{|z - x|}\right)^{d/2-1} K_{d/2-1} \left(|z - x|\sqrt{\lambda + |\alpha|^2}\right) |V(z)| dz.
\]
It is well known that $K_{d/2-1}$ admits the following estimates $K_{d/2-1} \approx z^{1-d/2}e^{-z}(1+z)^{d/2-3/2}$, $d \geq 3$, $K_0 \approx \ln(1+z^{-1/2})e^{-z}$ (see [1] formulas 9.6.6, 9.6.8, 9.6.9, 9.7.2, [2] page 11) and additionally $K_{1/2}(z) = \sqrt{2/\pi}e^{-z}z^{-1/2}$ (see [1] formula 10.2.16, 10.2.17). Hence,

$$e_{\ast}(V,\lambda) \approx \sup_{\alpha, x \in \mathbb{R}^d} \int_{\mathbb{R}^d} e^{-(z,\alpha) - |z|\sqrt{\lambda + |\alpha|^2}} |V(z + x)| dz, \quad \text{if } d \geq 3;$$

$$e_{\ast}(V,\lambda) \approx \sup_{\alpha, x \in \mathbb{R}^d} \int_{\mathbb{R}^d} e^{-(z,\alpha) - |z|\sqrt{\lambda + |\alpha|^2}} \log \left(1 + |z|\sqrt{\lambda + |\alpha|^2}^{-1/2}\right) |V(z + x)| dz, \quad \text{if } d = 2;$$

$$e_{\ast}(V,\lambda) = \sup_{\alpha, x \in \mathbb{R}} \frac{1}{2} \int_{\mathbb{R}} e^{-(z,\alpha) - |z|\sqrt{\lambda + |\alpha|^2}} \left(\sqrt{\lambda + |\alpha|^2}\right)^{-1} |V(z + x)| dz, \quad \text{if } d = 1. \quad (18)$$

Here $\approx$ means that the ratio of both sides is bounded above and below by positive constants independent of $\lambda$ and $V$. Actually, the comparability constants in the above depend only on $d$.

The relation between the exponents of the kernel $K(t, x, y)$ and in the explicit estimates of $e_{\ast}$ becomes more visible when putting $\alpha = -y/2$ and after noticing that

$$\langle z, \alpha \rangle + |z|\sqrt{\lambda + |\alpha|^2} = \langle z, \alpha \rangle + |z|\alpha + |z|\frac{\lambda}{\sqrt{\lambda + |\alpha|^2} + |\alpha|}. \quad (18)$$

What is more, on its support $K(t, x, y)$ coincides with the above explicit estimates of $e_{\ast}$ with $\lambda = 0$ if $d \geq 2$, and a similar comparability holds with $\lambda = 1/t$ if $d = 1$. This is not a coincidence and it becomes clear by the next proposition, which plays a key role in the proof of Theorem [3] and which reveals the origin of the function $K(t, x, y)$.

**Proposition 8.** For all $t > 0$, $\alpha, x, z \in \mathbb{R}^d$ satisfying $|z - x| \leq 2|\alpha|t$ we have

$$\frac{1}{2} \int_0^t p_\alpha(s, x, z) ds \leq \int_0^t p_\alpha(s, x, z) ds \leq \int_0^\infty p_\alpha(s, x, z) ds, \quad d \geq 2; \quad (19)$$

$$\frac{e}{e + 1} \int_0^\infty e^{-s/t}p_\alpha(s, x, z) ds \leq \int_0^t p_\alpha(s, x, z) ds \leq e \int_0^\infty e^{-s/t}p_\alpha(s, x, z) ds, \quad d = 1. \quad (20)$$

There are constants $0 < n_1 \leq n_2 < \infty$ that depend only on $d$ and such that for all $t > 0$, $\alpha, x, z \in \mathbb{R}^d$ satisfying $|z - x| \leq 2|\alpha|t$ we have

$$n_1 K(t, z - x, -2\alpha) \leq \int_0^t p_\alpha(s, x, z) ds \leq n_2 K(t, z - x, -2\alpha). \quad (21)$$

**Proof.** For simplicity we let $\tilde{x} = z - x$ and $y = -2\alpha$. Then we have

$$\int_0^t p_\alpha(s, x, z) ds = (4\pi t)^{-d/2} \int_0^1 s^{-d/2}e^{-\frac{|\tilde{x}-s|y|^2}{4s}} ds.$$  

Since for $|\tilde{x}| \leq |y|t$ and $s \in (0, 1)$, we have

$$|\tilde{x}|^2 + s|y|^2 \leq \frac{|ty|^2}{s} + s|\tilde{x}|^2.$$
For $d \geq 2$ we get
\[
\int_0^1 s^{-d/2} e^{-\frac{|x-ty|^2}{4ts}} \, ds = e^{\frac{(\hat{x},y)}{2}} \int_0^1 s^{-d/2+1} e^{-\left(\frac{|x|^2}{s} + s|y|^2\right)/(4t)} \, ds \\
\geq e^{\frac{(\hat{x},y)}{2}} \int_0^d s^{d/2-1} e^{-\left(\frac{|x|^2}{s} + s|y|^2\right)/(4t)} \, ds = \int_1^\infty u^{-d/2+1} e^{-\frac{|x-ty|^2}{4u}} \, du .
\]

Therefore, for $|z-x| \leq 2|\alpha|t$,
\[
\int_0^\infty p_\alpha(s,x,z) \, ds \leq 2 \int_0^t p_\alpha(s,x,z) \, ds .
\]

This proves (19). For $d = 1$ we have
\[
\int_0^1 s^{-1/2} e^{-\frac{|x-ty|^2}{4ts}} \, ds = e^{\frac{(\hat{x},y)}{2}} \int_0^1 s^{1/2} e^{-\left(\frac{|x|^2}{s} + s|y|^2\right)/(4t)} \, ds \\
\geq e^{\frac{(\hat{x},y)}{2}} \int_0^1 s^{1/2} e^{-\left(\frac{|y|^2}{s} + s|\hat{y}|^2\right)/(4t)} \, ds = \int_1^\infty u^{-1/2} e^{-\frac{|x-ty|^2}{4u}} \, du \\
\geq e \int_1^\infty u^{-1/2} e^{-\frac{|x-ty|^2}{4u}} \, du .
\]

Therefore, for $|z-x| \leq 2|\alpha|t$,
\[
\int_0^\infty e^{-s/t} p_\alpha(s,x,z) \, ds \leq (1 + 1/e) \int_0^t p_\alpha(s,x,z) \, ds .
\]

This ends the proof of (20). Now, note that we can take $\lambda = 0$ in (17) by passing with $\lambda > 0$ to zero. Then (21) follows from (17) and the estimates of $K_\nu$ mentioned above; and from (18) for $d = 1$. □

**Lemma 9.** For all $T > 0$ and $V$ we have
\[
r_* (V,T) \geq \frac{n_1}{2} ||K(V,T)||_\infty + \frac{1}{2} \sup_{x \in \mathbb{R}^d} \int_0^T \int_{\mathbb{R}^d} g(s,x,z) |V(z)| \, dz \, ds ,
\]
\[
r_* (V,T) \leq n_2 ||K(V,T)||_\infty + 2^{d-2} \sup_{x \in \mathbb{R}^d} \int_0^T \int_{\mathbb{R}^d} g(s,x,z) |V(z)| \, dz \, ds .
\]

The constants $0 < n_1 \leq n_2 < \infty$ are taken from (21).

**Proof.** Recall that $p_\alpha(t,x,z) = g(t,x-2\alpha t,z)$. If we put $\alpha = 0$, we get that $r_* (V,T)$ is bounded below by $\sup_{x \in \mathbb{R}^d} \int_0^T \int_{\mathbb{R}^d} g(s,x,z) |V(z)| \, dz \, ds$, while by reducing the domain of integration in space variable $z$ to $|z-x| \leq 2|\alpha|t$ and by (21) we have $r_* (V,T) \geq n_1 ||K(V,T)||_\infty$. That proves the lower bound (22). Now, let $y = -2\alpha$. For the upper bound we consider two regions of integration,
\[
A_1 = \{z \in \mathbb{R}^d: |z-x| > t|y|\} , \\
A_2 = \{z \in \mathbb{R}^d: |z-x| \leq t|y|\} .
\]

Note that if $z \in A_1$ and $s \in (0,t)$, then
\[
|z-x-ty| \leq |z-x-sy| + (t-s)|y| \\
< |z-x-sy| + |z-x| - |sy| \leq 2|z-x-sy| .
\]
On the set $A_2$ we apply (21). This ends the proof of (23). 

We are now ready to justify Theorem 5.

**Proof of Theorem 5.** We will actually prove that

$$c_1 \| K(V,T) \|_\infty \leq r_\ast(V,T) \leq c_2 \| K(V,T) \|_\infty,$$

for all $T > 0$ with constants $0 < c_1 < c_2 < \infty$ that depend only on $d$. The result will then follow from Proposition 5 and (10). The lower bound holds by (22). We focus on the upper bound and due to (23) it suffices to show that

$$\sup_{x \in \mathbb{R}^d} \int_{0}^{4T} \int_{\mathbb{R}^d} g(s, x, z) |V(z)| \, dz \, ds \leq c \| K(V,T) \|_\infty.$$

For the whole proof we let $y = (4r^{-1/2}, 0, \ldots, 0) \in \mathbb{R}^d$. Then for $d \geq 3$, since $- (x,y) \leq |x| |y|$, we have

$$K(t,x,y) \geq e^{-4|x|^{-1/2}} |x|^{-d-2} 1_{|x| \leq \sqrt{16T}} \geq e^{-16} |x|^{-d-2} 1_{|x| \leq \sqrt{16T}}.$$

Therefore, by (14) (cf. [6, (4.3)]) there is a constant $c > 0$ that depends only on $d$ such that

$$\| K(V,T) \|_\infty \geq e^{-16} \sup_{x \in \mathbb{R}^d} \int_{|z-x| \leq \sqrt{16T}} \frac{|V(z)|}{|z-x|^{d-2}} \, dz \geq c \sup_{x \in \mathbb{R}^d} \int_{0}^{4T} g(s, x, z) |V(z)| \, dz \, ds.$$

For $d = 2$ we first note that $\log(1 + r/2) \geq (1/3) \log(r)$ if $r \geq 1$. Therefore,

$$K(t,x,y) \geq e^{-16} \log \left(1 + \frac{1}{2} \left(\frac{16T}{|x|^2}\right)^{1/4}\right) 1_{|x| \leq \sqrt{16T}} \geq (e^{-16}/3) \log \left(\frac{16T}{|x|^2}\right) 1_{|x| \leq \sqrt{16T}}.$$

Finally, by (15) there is an absolute constant $c > 0$ such that

$$\| K(V,T) \|_\infty \geq (e^{-16}/3) \sup_{x \in \mathbb{R}^2} \int_{|z-x| \leq \sqrt{16T}} \log \left(\frac{16T}{|z-x|^2}\right) |V(z)| \, dz \geq c \sup_{x \in \mathbb{R}^2} \int_{0}^{4T} g(s, x, z) |V(z)| \, dz \, ds.$$

For $d = 1$ we have

$$K(t,x,y) \geq e^{-16} \sqrt{17} 1_{|x| \leq \sqrt{16T}},$$

and by (16) there is an absolute constant $c > 0$ such that

$$\| K(V,T) \|_\infty \geq \frac{e^{-16}}{\sqrt{17}} \sup_{x \in \mathbb{R}} \sqrt{T} \int_{|z-x| < \sqrt{16T}} |V(z)| \, dz \geq c \sup_{x \in \mathbb{R}} \int_{0}^{4T} g(s, x, z) |V(z)| \, dz \, ds.$$

□
4 Proofs of Theorems 2 – 4

4.1 Proof of Theorem 2

In the proof we construct a function $V$ with the desired properties. The construction is based on another function defined in [3, Proposition 1.6], and uses truncations and dilatations.

Proof. For $s > 0$ we let $\tau_s f(x) = sf(\sqrt{s}x)$. Note that such dilatation does not change the norm

$$\|\Delta^{-1}(\tau_s f)\|_\infty = \|\Delta^{-1} f\|_\infty.$$

Moreover, $\text{supp}(\tau_s f) \subseteq B(0, r/\sqrt{s})$ if $\text{supp}(f) \subseteq B(0, r)$, $r > 0$, and for $t > 0$,

$$\|S(\tau_s f, t)\|_\infty = \|S(f, st)\|_\infty.$$

Now, let $U: \mathbb{R}^d \to \mathbb{R}$ be non-positive and such that

$$\|U\|_\infty \leq 1, \quad \|\Delta^{-1} U\|_\infty = C < \infty, \quad \sup_{t > 0} \|S(U, t)\|_\infty = \infty.$$

Such $U$ exists by [3, Proposition 1.6 and Theorem 1.4]. By the definition of the supremum norm and the monotone convergence theorem, for $n \in \mathbb{N}$ there are $t_n, r_n > 0$ such that

$$\|S(U_{1_{B_{r_n}}}, t_n)\|_\infty > (4m_2/m_1)4^n.$$

For simplicity we define

$$U_n = U_{1_{B_{r_n}}}.$$

Let $s_n = \max\{r_n^2, n t_n\}$ and define

$$V_n = \tau_{s_n}(U_n).$$

Then $\text{supp}(V_n) \subseteq B(0, 1)$, $V_n \in L^\infty(\mathbb{R}^d)$, $\|\Delta^{-1} V_n\|_\infty \leq C$ and by Corollary 1

$$\|S(V_n, 1/n)\|_\infty = \|S(U_n, s_n/n)\|_\infty$$

$$\geq \left(\frac{m_1}{4m_2}\right) \|S(U_n)\|_{s_n/n, \infty}$$

$$\geq \left(\frac{m_1}{4m_2}\right) \|S(U_n)\|_{t_n, \infty} \geq \left(\frac{m_1}{4m_2}\right) \|S(U_n, t_n)\|_\infty > 4^n.$$

Finally, let

$$V := \sum_{n=1}^{\infty} \frac{V_n}{2^n}.$$

Obviously, part (a) holds. Further, again by Corollary 1 for $t > 0$ we get

$$\|S(V, t)\|_\infty \geq \left(\frac{m_1}{4m_2}\right) \lim_{n \to \infty} \|S(V, 1/n)\|_\infty \geq \left(\frac{m_1}{4m_2}\right) \lim_{n \to \infty} 2^{-n}\|S(V_n, 1/n)\|_\infty = \infty.$$

This proves part (d). The statement (c) holds by

$$\|\Delta^{-1} V\|_\infty \leq \sum_{n=1}^{\infty} \frac{\|\Delta^{-1} V_n\|_\infty}{2^n} \leq C < \infty.$$
Next,

\[
\sup_{x \in \mathbb{R}^d} \int_0^t \int \frac{g(s, x, z)}{2^n} \, dz \, ds
\]

\[
\leq \sum_{n=1}^N \sup_{x \in \mathbb{R}^d} \int_0^t \int \frac{|V_n(z)|}{2^n} \, dz \, ds + \sum_{n=N+1}^\infty \sup_{x \in \mathbb{R}^d} \int_0^\infty \int \frac{|V_n(z)|}{2^n} \, dz \, ds
\]

\[
\leq N \sum_{n=1}^N t \|V_n\|_\infty + \sum_{n=N+1}^\infty \|\Delta^{-1}V_n\|_\infty/2^n
\]

\[
\leq N \sum_{n=1}^N \|V_n\|_\infty + \frac{C}{2^N},
\]

which can be made arbitrary small by the choice of \(N\) and \(t\), and proves part (b).

\[\square\]

### 4.2 Proof of Theorem 3

Similarly to the proof of Theorem 2 we construct a function \(V\) with the desired features. We will choose a decreasing function \(f \geq 0\) satisfying \(\int_0^{1/25} f(r) \, dr = \infty\). The function \(V\) will be given by a series based on certain functions \(V_n\). Each \(V_n\) will be supported on a union of properly chosen cylinders \(C_{k,r}\) and will have values according to the function \(f\). In particular, the choice will be such that on the support of \(V_n\), the function \(K(t, x, y)\) with \(|y| = 25n\) will be comparable to \(1/|x|\) and such that for a sequence \(n_i \in \mathbb{N}\) diverging to infinity we will have

\[
\|K(V_{n_i}, 1)\|_\infty \geq c \int_{1/(25n_i)}^{1/25} f(r) \, dr \geq 4^i.
\]

In the first lemma we investigate a function \(U_r\) that is supported on a cylinder \(C_r \subset \mathbb{R}^3\) and takes values related to the size of the cylinder. To simplify the notation, for \(z = (z_1, z_2, z_3) \in \mathbb{R}^3\) we write \(z = (z_1, z_2)\), where \(z_2 = (z_2, z_3) \in \mathbb{R}^2\).

**Lemma 10.** For \(r > 0\) we define \(C_r = \left[0, \frac{1}{4}\right] \times D_r\), where \(D_r\) is a 2-dimensional ball of radius \(r\) centred at 0. For \(r \in (0, e^{-1})\), \(z \in \mathbb{R}^3\) put

\[
g(r) = \frac{1}{r^2 \ln r \, \ln |\ln r|} \quad \text{and} \quad U_r(z) = g(r)1_{C_r}(z).
\]

Then

\[
\lim_{\varepsilon \to 0^+} \sup_{r \in (0, 1/5)} \int_{|z - x| < \varepsilon} \frac{1}{|z - x|} |U_r(z)| \, dz = 0.
\]

**Proof.** Note that \(g(r)\) is decreasing on \((0, 1/5)\). On the other hand, \(r^2 g(r)\) and \(r^2 |\ln r| g(r)\) are increasing on \((0, 1/5)\). Let \(0 < \varepsilon < 1/5\) and

\[
I_r(\varepsilon) := \sup_{x \in \mathbb{R}^3} \int_{|z - x| < \varepsilon} \frac{1}{|z - x|} |U_r(z)| \, dz = g(r) \sup_{x \in \mathbb{R}^3} \int_{|z| < \varepsilon} \frac{1}{|z|} 1_{C_r}(z + x) \, dz.
\]
If $\varepsilon \leq r$, then
\[ I_r(\varepsilon) \leq g(r) \int_{|z|<\varepsilon} \frac{1}{|z|} dz \leq 2\pi\varepsilon^2 g(\varepsilon). \]

If $r \leq \varepsilon$, we use the symmetric rearrangement inequality [17, Chapter 3] and that $\varepsilon < 1/5$ to get
\[ \int_{|z|<\varepsilon} \frac{1}{|z|^2} 1_{C_r}(z+x)dz = \int_{-x_1}^{1/4-x_1} dz_1 \int_{\mathbb{R}^2} \frac{1}{|z|} 1_{D_r}(z_2 + x_2)dz_2 \leq \int_{-x_1}^{1/4-x_1} dz_1 \int_{\mathbb{R}^2} \frac{1}{|z|} 1_{D_r}(z_2)dz_2 \leq \int_{-1/4}^{1/4} dz_1 \int_{\mathbb{R}^2} \frac{1}{|z|} 1_{D_r}(z_2)dz_2 = \int_{|z|<\varepsilon} \frac{1}{|z|} 1_{C_r(-C_r)}(z)dz. \]

Now note that
\[ B(0, \varepsilon) \cap (C_r \cup (-C_r)) \subseteq B(0, \sqrt{2}r) \cup ([r, \varepsilon] \times D_r) \cup ([-\varepsilon, r) \times D_r). \]

Then
\[ I_r(\varepsilon) \leq g(r) \left( \int_{|z| \leq \sqrt{2}r} \frac{1}{|z|} dz \right) + 2 \int_{r}^{\varepsilon} \frac{|D_r|}{z_1} dz_1 \leq g(r) \left( 4\pi r^2 + 2\pi r^2 |\ln r| \right) \leq g(\varepsilon) \left( 4\pi \varepsilon^2 + 2\pi \varepsilon^2 |\ln \varepsilon| \right). \]

Thus
\[ \lim_{\varepsilon \to 0^+} \sup_{x \in \mathbb{R}^3 \cap (0,1/5)} \int_{|z-x|<\varepsilon} \frac{1}{|z-x|} |U_r(z)|dz = \lim_{\varepsilon \to 0^+} \sup_{r \in (0,1/5)} I_r(\varepsilon) = 0. \]

**Corollary 3.** For $k \in \mathbb{N}$ and $r > 0$ we define
\[ C_{k,r} = [k, k + \frac{1}{4}] \times D_r, \]
where $D_r$ is a 2-dimensional ball of radius $r$ centred at 0. For $r \in (0, e^{-1})$, $n \in \mathbb{N}$ and $z \in \mathbb{R}^3$ put
\[ f(r) = \frac{1}{r \ln r \ln \ln r} \quad \text{and} \quad V_n(z) = f \left( \frac{z_1}{25n} \right) \sum_{k=1}^{n} 1_{C_{k,\sqrt{k/(25n)}}}(z). \]

Then
\[ \lim_{\varepsilon \to 0^+} \sup_{x \in \mathbb{R}^3 \cap (0,1/5)} \int_{|z-x|<\varepsilon} \frac{1}{|z-x|} |V_n(z)|dz = 0. \]

**Proof.** We use $g(r)$ and $U_r$ as defined in Lemma 10. Note that $f(r)$ is decreasing on $(0, e^{-3})$ and $f(r^2) \leq g(r)$ on $(0, e^{-1})$. We record that every two cylinders $C_{k,\sqrt{k/(25n)}}$ that correspond to different values of $k \in \mathbb{N}$ are disjoint. Therefore if $z \in C_{k,\sqrt{k/(25n)}}$ we have
\[ V_n(z) = f \left( \frac{z_1}{25n} \right) \leq f \left( \frac{k}{25n} \right) \leq g \left( \sqrt{\frac{k}{25n}} \right) = U_{\sqrt{k/(25n)}}(z-(k,0)). \]
What is more, the distance between every two cylinders $C_{k, \sqrt{k/(25n)}}$ that correspond to different $k$ is at least $3/4$. Thus, for any $x \in \mathbb{R}^d$ and $0 < \varepsilon < 3/8$, the intersection of $B(x, \varepsilon)$ and $\text{supp}(V_n)$ is a subset of at most one cylinder $C_{k, \sqrt{k/(25n)}}$, and so by Lemma 10

$$\lim_{\varepsilon \to 0^+} \sup_{x \in \mathbb{R}^3, n \in \mathbb{N}} \int_{|z-x|<\varepsilon} \frac{1}{|z-x|}|V_n(z)|dz \leq \lim_{\varepsilon \to 0^+} \sup_{k=1, \ldots, n} \int_{|z-x|<\varepsilon} \frac{1}{|z-x|} U_{\sqrt{k/(25n)}}(z-(k,0))dz$$

$$\leq \lim_{\varepsilon \to 0^+} \sup_{x \in \mathbb{R}^3, r \in (0,1/5)} \int_{|z-x|<\varepsilon} \frac{1}{|z-x|}|U_r(z)|dz = 0.$$ 

\[\square\]

**Lemma 11.** Let $V_n$ be defined as in Corollary 3. There are $n_i \in \mathbb{N}$, $i \in \mathbb{N}$, such that for every $i \in \mathbb{N}$,

$$\|K(V_{n_i}, 1)\|_{\infty} \geq 4^i.$$

**Proof.** Let $\theta > 0$. Then

$$\theta(|z| - z_1) < 1 \iff z_1 > \frac{\theta}{2} |z_2|^2 - \frac{1}{2\theta}.$$ 

For $n \in \mathbb{N}$ we put

$$E_n := \left\{ z \in \mathbb{R}^3 : z_1 > \frac{25n}{2} |z_2|^2 - \frac{1}{50n} \right\}.$$

Thus, for $z \in E_n$ we have $25n(|z| - z_1) < 1$. Then, by taking $x = 0$ and $y = (25n,0)$ in the first inequality below, and using $\text{supp}(V_n) \subset E_n \cap B(0,25n)$ in the second one,

$$\|K(V_n, 1)\|_{\infty} = \sup_{x,y \in \mathbb{R}^3} \int_{|z-x|\leq|y|} e^{-\frac{|z-x|}{25n}} \frac{1}{|z-x|}|V_n(z)|dz$$

$$\geq \int_{|z|\leq25n} e^{-\frac{1}{2}25n(|z| - z_1)} |V_n(z)|dz \geq e^{-\frac{1}{2}} \int_{\mathbb{R}^3} \frac{1}{|z|} |V_n(z)|dz.$$ 

Further, by the definition of $V_n$ and $C_{k, \sqrt{k/(25n)}}$,

$$\|K(V_n, 1)\|_{\infty} \geq e^{-1/2} \sum_{k=1}^{n} \int_{\mathbb{R}^3} \frac{1}{|z|} f\left(\frac{z_1}{25n}\right) 1_{C_{k, \sqrt{k/(25n)}}}(z)dz$$

$$\geq e^{-1/2} \sum_{k=1}^{n} \frac{k+1}{k} f\left(\frac{z_1}{25n}\right) D_{\sqrt{k/(25n)}} dz_1$$

$$\geq \frac{\pi e^{-1/2}}{2} \sum_{k=1}^{n} \frac{k+1}{k} f\left(\frac{z_1}{25n}\right) \frac{dz_1}{25n}$$

$$\geq \frac{\pi e^{-1/2}}{8} \int f\left(\frac{z_1}{25n}\right) \frac{dz_1}{25n} = \frac{\pi e^{-1/2}}{8} \int_{1/(25n)}^{1/25} f(r)dr.$$ 

This ends the proof since $\int_0^{1/25} f(r)dr = \infty$. 

\[\square\]
Proof of Theorem 4. For $n \in \mathbb{N}$ let $V_n$ be as in Corollary 3 and $(n_i)_{i \in \mathbb{N}}$ be a sequence of natural numbers taken from Lemma 11. We take

$$V := -\sum_{i=1}^{\infty} V_n/2^i.$$ 

By Lemma 11 we have

$$\|K(V,1)\|_\infty \geq \sup_{i \in \mathbb{N}} 2^{-i} \|K(V_n,1)\|_\infty = \infty.$$ 

Therefore, by Theorem 5 and Corollary 1 part b) follows. Next, we have

$$\sup_{x \in \mathbb{R}^3} \int_{|z-x|<\epsilon} \frac{1}{|z-x|} |V(z)|dz \leq \sum_{i=1}^{\infty} 2^{-i} \sup_{x \in \mathbb{R}^3} \int_{|z-x|<\epsilon} \frac{1}{|z-x|} |V_n(z)|dz$$

$$\leq \sup_{x \in \mathbb{R}^3, i \in \mathbb{N}} \int_{|z-x|<\epsilon} \frac{1}{|z-x|} |V_n(z)|dz,$$

which can be made arbitrary small by the choice of $\epsilon$ due to Corollary 3. This proves part a). \(\square\)

4.3 Proof of Theorem 4

Before we pass to the proof of Theorem 4 we show the following auxiliary result in $d = 2$. For $z \in \mathbb{R}^2$ we write as usual $z = (z_1, z_2)$, where $z_1, z_2 \in \mathbb{R}$.

Lemma 12. Let $d = 2$. For $r \geq 2$ we let $D_r = \{z \in \mathbb{R}^2: z_1 \geq 0$ and $2 \leq |z| \leq r\}$. There exists a constant $c > 0$ such that for all Borel measurable $U: \mathbb{R}^2 \to [0, \infty]$ and $r \geq 2$,

$$\int_{D_r} K(1, z, (r, 0)) U(z)dz \leq c \sup_{w \in \mathbb{R}^2} \int_{|z| \leq 2} U(z + w)dz.$$ 

Proof. Note that for $r > 0$ and $n \in \mathbb{N} \cup \{0\}$,

$$r(|z| - z_1) \leq n \iff |z_2| \leq \frac{\sqrt{2nz_1r + n^2}}{r}.$$ 

In the rest of proof we consider $r \geq 2$ and $0 \leq z_1 \leq r$. For $n \in \mathbb{N} \cup \{0\}$ we let

$$f_n(z_1) := \frac{\sqrt{2nz_1r + n^2}}{r} \quad \text{and} \quad F_n := \{z \in \mathbb{R}^2: f_n(z_1) \leq |z_2| \leq f_{n+1}(z_1), 0 \leq z_1 \leq r\}.$$ 

Obviously, $f_n$ and $F_n$ depend on $r$, which we do not indicate explicitly to lighten the notation. In particular, $n \leq r(|z| - z_1) \leq n + 1 \iff z \in F_n$. A direct analysis of the derivative shows that for each $a \geq 0$ and $b > 0$ a function

$$h(t) = \sqrt{2(a+b)(t+1) + (t+1)^2} - \sqrt{2at + t^2}, \quad t \geq 0,$$

is decreasing on $[0, a/b]$ and increasing on $[a/b, \infty)$. This guarantees for each $\delta \in (0, 1)$ that

$$f_{n+1}(z_1 + \delta) - f_n(z_1) \leq \max \left\{ f_1(z_1 + \delta), \lim_{n \to \infty} (f_{n+1}(z_1 + \delta) - f_n(z_1)) \right\}$$

$$= \max \left\{ \frac{\sqrt{2z_1r + 1}}{r}, \delta + \frac{1}{r} \right\} \leq \max \left\{ \frac{\sqrt{2 + \frac{1}{r^2} \delta + \frac{1}{r}}}{r}, \frac{3}{2} \right\} \leq \frac{3}{2}$$

We fix $\delta \in (0, 1)$ (any $\delta \leq \sqrt{7}/2$ has that property) so that for all $n \in \mathbb{N} \cup \{0\}$,

$$\sqrt{(f_{n+1}(z_1 + \delta) - f_n(z_1))^2 + \delta^2} \leq 2.$$ (24)
For $n, k \in \mathbb{N} \cup \{0\}$ we define rectangles

$$P_{n,k} := \left[ k\delta, (k+1)\delta \right] \times \left[ f_n(k\delta), f_{n+1}((k+1)\delta) \right] \subset \mathbb{R}^2.$$  

The bottom left vertex of $P_{n,k}$ equals $a_{n,k} = (k\delta, f_n(k\delta))$ and satisfies $|a_{n,k}| = k\delta + \frac{n}{r}$. Furthermore, if $k \leq \left\lfloor \frac{r}{\delta} \right\rfloor$, then $k\delta \leq r$ and by (24) the diagonal of $P_{n,k}$ does not exceed 2. Hence $P_{n,k} \subseteq B(a_{n,k}, 2)$, where the latter is a 2-dimensional ball of radius 2 centred at $a_{n,k}$. Next, observe that

$$D_r \subseteq \bigcup_{n=0}^{\left\lfloor \frac{r}{\delta} \right\rfloor} F_n \quad \text{and} \quad F_n \subseteq \bigcup_{k=0}^{\left\lfloor \frac{r}{\delta} \right\rfloor} P_{n,k}.$$  

Finally, on $D_r \cap F_n \cap P_{n,k}$ we have

$$K(1, z, (r, 0)) = e^{-\frac{1}{2}r(|z|-z_1)} \log \left( 1 + \frac{1}{\sqrt{r|z|}} \right) \mathbf{1}_{|z| \leq r}$$

$$\leq e^{-\frac{1}{2}r(|z|-z_1)} \leq \frac{e^{-n/2}}{\sqrt{r \max\{|a_{n,k}|, 2\}}} \leq \frac{e^{-n/2}}{\sqrt{r(k\delta/2 + 1)}} \mathbf{1}_{B(a_{n,k}, 2)}(z).$$

This implies

$$\int_{D_r \cap F_n \cap P_{n,k}} K(1, z, (r, 0)) U(z)dz \leq \frac{e^{-n/2}}{\sqrt{r(k\delta/2 + 1)}} \int_{|z| \leq 2} U(z + a_{n,k})dz$$

$$\leq \frac{e^{-n/2}}{\sqrt{r(k\delta/2 + 1)}} \sup_{w \in \mathbb{R}^2} \int_{|z| \leq 2} U(z + w)dz.$$  

It remains to notice that $\frac{1}{\sqrt{r}} \sum_{k=0}^{\left\lfloor \frac{r}{\delta} \right\rfloor} (k\delta/2 + 1)^{-1/2} \leq 1 + 4/\delta$ and $\sum_{n=0}^{\infty} e^{-n/2} < \infty$. \hfill \Box
Proof of Theorem 4. The lower bound in \([9]\) follows immediately from \([10], [11]\) and \([12]\). We focus on the upper bound. Due to Theorem \([5]\) it suffices to estimate \(\|K(V,t)\|_\infty, t > 0\). First we consider \(d = 2\). For \(|y| \leq 2t^{-1/2}\) we have
\[
K(t, x, y) \leq \log \left(1 + \frac{\sqrt{t}}{|x|}\right) 1_{|x| \leq \sqrt{4t}} \leq \left(1 + \log \frac{4t}{|x|^2}\right) 1_{|x| \leq \sqrt{4t}} .
\] (25)
Therefore, by \([15]\) there is an absolute constant \(c > 0\) such that
\[
\sup_{|y| \leq 2t^{-1/2}} \sup_{x \in \mathbb{R}^2} K(V, t, x, y) \leq c \sup_{x \in \mathbb{R}^2} \int_0^t \int_{\mathbb{R}^2} g(s, x, z) |V(z)| dz ds.
\]
We focus on \(|y| \geq 2t^{-1/2}\). Let
\[
A_1 = \{ z \in \mathbb{R}^2 : \langle z - x, y \rangle \leq 0 \} ,
\]
\[
A_2 = \{ z \in \mathbb{R}^2 : \langle z - x, y \rangle \geq 0 \text{ and } |z - x| \leq \sqrt{4t} \} ,
\]
\[
A_3 = \{ z \in \mathbb{R}^2 : \langle z - x, y \rangle \geq 0 \text{ and } \sqrt{4t} \leq |z - x| \leq 2t \} .
\]
On the set \(A_1\) we have \(|z - x - sy| \geq |z - x|\), hence by \([21]\) we get
\[
n_1 K(t, z - x, y) \leq \int_0^t p(-y/2)(s, x, z) ds = \int_0^t g(s, x + sy, z) ds \leq \int_0^t g(s, x, z) ds .
\]
Thus
\[
\sup_{|y| \geq 2t^{-1/2}} \sup_{x \in \mathbb{R}^2} \int_{A_1} K(t, z - x, y) |V(z)| dz \leq \left(1/n_1\right) \sup_{x \in \mathbb{R}^2} \int_0^t \int_{\mathbb{R}^2} g(s, x, z) |V(z)| dz ds.
\]
On the set \(A_2\) we argue like in \([25]\), therefore
\[
\sup_{|y| \geq 2t^{-1/2}} \sup_{x \in \mathbb{R}^2} \int_{A_2} K(t, z - x, y) |V(z)| dz \leq c \sup_{x \in \mathbb{R}^2} \int_0^t \int_{\mathbb{R}^2} g(s, x, z) |V(z)| dz ds.
\]
It remains now to consider
\[
\sup_{|y| \geq 2t^{-1/2}} \sup_{x \in \mathbb{R}^2} \int_{A_3} K(t, z - x, y) |V(z)| dz .
\]
Given \(|y| \geq 2t^{-1/2}\) we let
\[
\mathcal{O}_y = \begin{bmatrix} y_1 |y|^{-1} \\ -y_2 |y|^{-1} \end{bmatrix} .
\]
Note that \(\mathcal{O}_y\) is a rotation matrix in \(\mathbb{R}^2\) such that \(\mathcal{O}_y y = (|y|, 0)\). Then, substituting \(z\) by \(t^{1/2} \mathcal{O}_y^{-1} z\), we obtain
\[
\int_{A_3} K(t, z - x, y) |V(z)| dz = \int_{D_r} K(1, z, (r, 0)) U(z) dz ,
\] (26)
where
\[
r = t^{1/2}|y| , \quad D_r = \{ z \in \mathbb{R}^2 : z_1 \geq 0 \text{ and } 2 \leq |z| \leq r \} , \quad U(z) = t|V(t^{1/2} \mathcal{O}_y^{-1} z + x)| .
\]
Combining (26) and Lemma 12 we get for $|y| \geq 2t^{-1/2}$,
\[
\int \mathcal{A}_3 K(t, z - x, y)|V(z)|dz \leq c \sup_{w \in \mathbb{R}^2} \int |t|V(t^{1/2} \mathcal{O}_y^{-1} z + t^{1/2} \mathcal{O}_y^{-1} w + x)|dz
\leq c \sup_{w \in \mathbb{R}^2} \int |V(z + \tilde{w})|dz.
\]
Thus by (15),
\[
\sup_{|y| \geq 2t^{-1/2}} \sup_{x \in \mathbb{R}^2} \int \mathcal{A}_3 K(t, z - x, y)|V(z)|dz \leq c \sup_{x \in \mathbb{R}^2} \int_0^t \int_{\mathbb{R}^2} g(s, x, z)|V(z)|dzds.
\]
This finally gives the desired estimate and ends the proof for $d = 2$.

Now, let $d = 1$. Using [6, Lemma 4.2] with $k(x) = \sqrt{t} \left(1 + t|y|^2\right)^{-1/2} 1_{|z| \leq t|y|}$ and $K(x) = \sqrt{t}$ we get for $r > 0$,
\[
\|K(V, t)\|_\infty \leq \sup_{x, y \in \mathbb{R}} \int_\mathbb{R} \sqrt{t} \left(1 + t|y|^2\right)^{-1/2} 1_{|z - x| \leq t|y|}|V(z)|dz
\leq \sup_{x, y \in \mathbb{R}} \left(1 + \frac{\sqrt{4t}}{r} \left(1 + t|y|^2\right)^{1/2}\right) \int \sqrt{t}|V(z)|dz \leq \left(1 + \frac{\sqrt{4t}}{r}\right) \sup_{x \in \mathbb{R}} \sqrt{t} \int |V(z)|dz.
\]
Eventually, we put $r = \sqrt{4t}$ and use (16), which ends the proof.

5 Corollaries and proof of Theorem 1

We will now give corollaries of Theorems 2 – 4. We will separately consider the cases $d \geq 4$, $d = 3$, $d = 2$ and $d = 1$. We begin with $d \geq 4$ and an aftermath of Theorem 2.

Corollary 4. Let $d \geq 4$. There is compactly supported $V \leq 0$ such that

(i) holds with $\varepsilon_1 < 1 < \varepsilon_2$ arbitrarily close to 1,
(ii) holds,
(iii) does not hold.

By considering $-V$ we can obtain a similar non-negative example.

Proof. We take $V \leq 0$ from Theorem 2. We justify all statements by using parts (a), (b), (c) and (d) of the theorem along with the references indicated below. Namely,

• $V$ is compactly supported by (a),
• (i) follows from (b) and [20, Theorem 1A],
• (ii) follows from (c) and [20] p. 556 and Corollary A,
• (iii) follows from (d), Corollary 1 and Lemma 2.

For a non-negative example we may need to multiply $-V$ by a small constant to obtain $(e') \|\Delta^{-1}V\|_\infty < \varepsilon$ (small) and use for instance [6] Theorem 1.4] to get (ii).
A similar argumentation based on Theorem 3, [20, Theorem 1A and 1B], Corollary 1 and Lemma 2 gives consequences for \(d = 3\). As pointed out after Theorem 3 we cannot expect an example of \(V \leq 0\) that satisfies (3), but not (2).

**Corollary 5.** Let \(d = 3\). There is \(V \leq 0\) (of unbounded support) such that

(i) (1) holds with \(\varepsilon_1 < 1 < \varepsilon_2\) arbitrarily close to 1,

(ii) (2) fails to hold.

Here is what results from Theorem 4 for \(d = 2\).

**Corollary 6.** Let \(d = 2\). We have

1) \(V \in \mathcal{K}_2\) if and only if \(\lim_{T \to 0^+} \|S(V)\|_{T,\infty} = 0\).

2) \(V \in \hat{\mathcal{K}}_2\) if and only if \(\|S(V)\|_{T,\infty} < \infty\) for some (every) \(T > 0\).

3) If \(V \leq 0\), then (2) holds if and only if \(V \in \hat{\mathcal{K}}_2\).

**Proof.** The first two statements follow from Theorem 4 and the definitions of \(\mathcal{K}_2\) and \(\hat{\mathcal{K}}_2\). The last one follows from Lemma 2 and (2).

Finally we focus on \(d = 1\) in view of Theorem 4.

**Corollary 7.** Let \(d = 1\). The following conditions are equivalent

a) \(V \in \mathcal{K}_1\),

b) \(V \in \hat{\mathcal{K}}_1\),

c) \(\sup_{x \in \mathbb{R}^d} \int_{|z-x| \leq 1} |V(z)| < \infty\),

d) \(\lim_{T \to 0^+} \|S(V)\|_{T,\infty} = 0\),

e) \(\|S(V)\|_{T,\infty} < \infty\) for some (every) \(T > 0\).

**Proof.** The equivalence of a), b) and c) is well known and follows for instance from (16). Part a) is equivalent to d), and part b) to e) by Theorem 4.

**Corollary 8.** Let \(d = 1\). If \(V\) is of fixed sign, then (2) holds if and only if \(V \in \mathcal{K}_1\).

**Proof.** The equivalence follows from Lemma 2 and Corollary 7.

**Proof of Theorem 4.** We justify statements in Table 1. We refer to ’local in time’ and ’global in time’ column as the ’first’ and the ’second’ column, respectively. If \(d \geq 4\), the lack of the equivalence in both columns is an aftermath of Corollary 4 (also since (1) implies (3)). If \(d = 3\), the negative answer in the ’first’ column results from Corollary 5. Before we move forward, we note that for \(V \leq 0\), by the Duhamel formula,

\[
\int_0^t \int_{\mathbb{R}^d} G(s,x,z)|V(z)|g(t-s,z,y)dzds \leq g(t,x,y).
\]

Thus, by integrating in \(x\) variable over \(\mathbb{R}^d\), we see that (3) implies

\[
\sup_{t > 0, y \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} |V(z)|g(t-s,z,y)dzds = \sup_{x \in \mathbb{R}^d} \int_0^\infty \int_{\mathbb{R}^d} g(s,x,z)|V(z)|dzds < \infty, \quad (27)
\]
while (4) necessitates
\[
\sup_{0 < t \leq T, y \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} |V(z)| g(t - s, z, y) dz ds = \sup_{x \in \mathbb{R}^d} \int_0^T \int_{\mathbb{R}^d} g(s, x, z) |V(z)| dz ds < \infty. \tag{28}
\]
Therefore, the positive answer in dimension \(d = 3\) in the ‘second’ column follows from (27) and [3, Corollary 1.5 and (8)] (or see [3, Page 6]). The remaining two positive answers in 'global in time' column (dimensions \(d = 2, d = 1\)) also follow from (27), this time complemented with Theorem 4 and [3, Lemma 1.1]. The two positive answers in 'local in time' column (dimensions \(d = 2, d = 1\)) follow from (28), Theorem 4 and the first statement of Lemma 2 (see also [3, Lemma 1.1]).

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