Non-Abelian Aharonov-Bohm Caging in Photonic Lattices

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Aharonov-Bohm (AB) caging is the localization effect in translational-invariant lattices due to destructive interference induced by penetrated magnetic fields. While current research focuses mainly on the case of Abelian AB caging, here we go beyond and develop the non-Abelian AB caging concept by considering the particle localization in a 1D multi-component rhombic lattice with non-Abelian background gauge field. In contrast to its Abelian counterpart, the non-Abelian AB cage depends on both the form of the nilpotent interference matrix and the initial state of the lattice. This phenomena is the consequence of the non-Abelian nature of the gauge potential and thus has no Abelian analog. We further propose a circuit quantum electrodynamics realization of the proposed physics, in which the required non-Abelian gauge field can be synthesized by the parametric conversion method, and the non-Abelian AB caging can be unambiguously demonstrated through the pumping and the steady-state measurements of only a few sites on the lattice. Requiring only currently available technique, our proposal can be readily tested in experiment and may pave a new route towards the investigation of exotic photonic quantum fluids.

I. INTRODUCTION

Synthetic gauge fields in artificial atomic and photonic systems has been studied extensively in the past decades [1,2]. The motivation is to investigate exotic topological physics [3] in a well-controlled quantum simulator [4,5]. Inspired by the rapid progress in this field, research attention has been recently devoted to the realization of Aharonov-Bohm (AB) caging, where the interplay of the external magnetic field and the lattice geometry leads to completely flat bands (FB) in the 1D rhombic lattice [6] and 2D dice lattice [7] and consequently exotic localization on the perfectly periodic lattices. Being distinct from the localization due to disorder, this effect is interpreted by the destructive interference induced by the Peierls phase of the penetrated magnetic flux. The interaction mechanism then becomes dominant on the dispersionless band, making the AB caging lattices ideal platforms exploring strongly correlated physics. Following the experimental realization of AB caging in photonic systems [8,9], a variety of related theoretical works has been presented, including the effects of nonlinearity [10,11], topological pumping and edge states [14,15], FB laser [17], and influence of non-Hermicity [18,19].

Meanwhile, subtlety still exists in the sense that, discussions up to now are mainly based on the assumption that an Abelian background gauge potential is imposed. On the other hand, non-Abelian gauge field [20] has already manifested its essence in condensed matter physics and quantum optics, partially by the role of spin-orbital coupling (SOC) [21] in the physics of topological quantum matters [8]. In addition, synthetic SOC has been experimentally implemented in various artificial system during the past few years, ranging from ultra-cold atoms [22] and exciton-polariton microcavities [23] to coupled pendula chains [24]. These exciting advances thus raise the curious questions that whether the AB caging concept can be extended to the realm of non-Abelian gauge, and whether the non-Abelian nature of the gauge field would bring any new physics that has no Abelian correspondence.

In this manuscript, we propose the concept non-Abelian AB caging (i.e. AB caging in the presence of non-Abelian background gauge field) in a 1D rhombic lattice where the lattice sites contain multiple (pseudo)spin components and the background gauge potential becomes matrix-valued. In particular, the non-Abelian AB caging is defined by the emergence of nilpotent interference matrix, which is the matrix generalization of the destructive interference condition in the Abelian AB caging case. Detailed analysis further implies that particle localization in this situation is drastically different from its Abelian counterpart: The non-Abelian feature of the background gauge field results in exotic spatial configuration of non-Abelian AB cage, which is sensitive to both the nilpotent power of the interference matrix and the initial state of the lattice.

In addition, we consider the circuit quantum electrodynamics (QED) system [25] as a promising candidate of implementing the proposed physics. While the Abelian AB caging has been realized in optical waveguides [8,9], our proposal takes the advantages of flexibility and tunability of superconducting quantum circuit, which allow the future incorporation of photon-photon interaction and disorder in a time- and site- resolved manner [26,27]. The required link variables can be synthesized by the parametric frequency conversion (PFC) approach [28,30], which is feasible with current technology and can lead to in situ tunability of the synthesized non-Abelian gauge potential. Our numerical simulations pin-
point that localization-in-continuum dynamics can be observed through the steady-state photon number (SSPN) detection of only few sites on the lattice, which can serve as unambiguous evidence of the non-Abelian AB caging.

II. NON-ABELIAN AB CAGING: CONCEPT AND EXAMPLES

A. Definition of non-Abelian AB

In this section, the physics of non-Abelian AB caging is illustrated in the context of a periodic 1-D rhombic lattice sketched in Fig. 1(a), with each site consisting of $N$ (pseudo)spin modes. The Hamiltonian of the lattice in the presence of an $U(N)$ background gauge field $A$ takes the form

$$H = -J \sum_{(n,\alpha,\beta)} \alpha_n^\dagger U_{n\alpha,\beta} \beta_n,$$

where $J$ is the uniform positive hopping strength between the linked sites shown in Fig. 1(a), $\alpha_n = [\alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,N}]^T$ with $\alpha = A, B, C$ is the multi-component annihilation operator vector of the $n$th site in the $n$th unit-cell, and $U_{n\alpha,\beta} = \exp\left[i \int [n,\alpha] \ dx \cdot \mathbf{A}(x)\right]$ is the translational-invariant link variable describing the unitary transformation experienced by a particle when it hops from site $[m,\beta]$ to site $[n,\alpha]$.

We then define the non-Abelian AB caging by the condition that the interference matrix

$$I = \frac{1}{2} (U_2 U_1 + U_4 U_3),$$

is nilpotent, i.e.

$$\exists \ m \in N, \ \text{s.t.} \ I^{m-1} \neq 0, \ I^m = 0.$$  \hfill (3)

Here $U_1$, $U_2$, $U_3$, and $U_4$ are the rightward link variables labeled in Fig. 1(a). This definition can be explained intuitively as follow: Imagine that there is a particle initially populated in the $[n,A]$ site highlighted in Fig. 1(a). Due to the geometry of the lattice, this particle can move rightward to $[n+1,A]$ via only two paths. Along these two paths, it will gain the unitary transformations $U_{\text{up}} = U_2 U_1$ and $U_{\text{down}} = U_4 U_3$ respectively, and consequently an interference described by the interference matrix $I$ in Eq. (2).

For the Abelian situation $N = 1$, AB caging happens exactly when $\pi$ magnetic flux is penetrated in each loop of the lattice. The two up and down paths shown in Fig. 1(a) then interfere with each other destructively and result in an vanishing $I = 0$. Therefore, the particle initially in the site $[n,A]$ becomes localized as it cannot spread outward to sites further than $[n \pm 1,A]$. For the non-Abelian situation $N > 1$, the localization of the particle can still happen when $I = 0$, just the same as the Abelian case. However, the matrix feature of $A$ now offers possibilities of much more rich physics: A matrix-formed interference matrix $I$ takes the advantage that it can be nilpotent, with c-number $I = 0$ being its $N = 1$ special case. This point lies in the heart of our generalization of AB caging to the non-Abelian gauge situation.

B. Two examples of nilpotent interference matrices

To validate the proposed non-Abelian AB caging concept, we consider a special example

$$I = \sum_{n=1}^{N-1} |n+1\rangle \langle n| = \frac{1}{2} \left[ \sum_{n=1}^{N-1} |n\rangle \langle n+1| + |1\rangle \langle N| \right]$$

$$+ \sum_{n=1}^{N-1} |n\rangle \langle n+1| - |1\rangle \langle N| \right].$$

with nilpotent power $m = N$. Here $|n\rangle$ is the $N$-column with only its $n$th item being unity and others being zero, and a candidate decomposition of $I$ into two $U(N)$ matrices (i.e. $U_{\text{up}}$ and $U_{\text{down}}$) is implied. The evolution of particles on the lattice still exhibits localization feature in this case. However, the non-Abelian AB cage now has an enlarged size and a spatial location depending on the initial state of the system: Let us assume that a particle is initially prepared in the $[n,A,l]$ mode with $l \in [1,N]$. Due to the form of $I$ shown in Eq. (4), the
particle will hop into the \((l-1)\)th mode when it arrives at the \([n+1,A]\) site. After it reaches the \([n+1,A]\) site, it can move rightward further. The rightmost \(A\) site that the particle can approach is \([n+l-1,A]\), and its further moving towards \([n+l,A]\) is suppressed. The leftward moving of the particle can be investigated in the similar way, with \(I^\dagger = (U_{\text{up}}^{-1} + U_{\text{down}}^{-1})/2\) being used as the interference matrix due to the reversed moving direction. The particle can thus reach as far as the \([n+l-N,A,N]\) mode. The above analysis therefore indicates that the particle will experience a spin dependent asymmetric directional breath dynamics. This is different from the Abelian case, in which a particle initially prepared in \([n,A]\) site will breath in a symmetric manner between \([n,A]\) and its four neighboring \(B\) and \(C\) sites. This symmetry is however manifestly broken in the considered non-Abelian situation.

We further study a more general situation in which \(I\) has nilpotent power smaller than \(N\). Without loss of generality, we assume

\[
I = \sum_{n=1}^{m-1}|n\rangle\langle n + N - m + 1| + \sum_{n=1}^{N-m+1}|n\rangle\langle N + 1 - n| + \sum_{n=1}^{m-1}|n\rangle\langle n + N - m + 1| - \sum_{n=1}^{N-m+1}|n\rangle\langle N + 1 - n|,
\]

with nilpotent power \(m \in (1,N)\). Re-performing the previous analysis, we find that now the spatial configuration of the non-Abelian AB cage is not only determined by the initial mode number \(l\) of the particle, but also the nilpotent power \(m\) of \(I\). Explicitly speaking, there are six situations, with main results summarized in Tab. I. For each situation, the AB cage is characterized by its size defined by the number of \(A\) sites that can be populated by the particle during its evolution, and its right and left edges, defined by the rightmost and leftmost \(A\) sites that can be populated by the particle during its evolution. We should emphasize that Eqs. (4) and (5) obviously do not exhaust the possible forms of the nilpotent interference matrix \(I\). Meanwhile, these two illuminating examples have already revealed several properties of the non-Abelian AB caging which are significantly different from its Abelian counterpart: The AB cage in the non-Abelian situation becomes larger, with its size and edge location depending on both the nilpotent power of the interference matrix and the initial mode populated by the particle. From the previous derivation, we can see that these exotic new features stem from the matrix nature of the gauge field \(A\) and thus have not Abelian analog.

C. A minimal \(U(2)\) model

We then turn to a \(U(2)\) design which can be regarded as the minimal realization of the proposed non-Abelian AB caging. Also, this is partially due to recent advances of realizing two-component SOC in artificial systems. The link variables can be carefully set as

\[
U_1 = U_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad U_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},
\]

such that a nilpotent \(I\) with \(m = 2\) can be achieved:

\[
I = \frac{1}{2}(U_2U_1 + U_4U_3) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.
\]

If the particle is initially prepared in the \([n,A,1]\) mode, it can move rightward and arrive the \([n+1,A,2]\) mode, but it can reach neither the \([n+2,A]\) nor the \([n-1,A]\) sites. Meanwhile, If the particle is initially prepared in the \([n,A,2]\) mode, it can move leftward and arrive the \([n+1,A,1]\) mode, but it cannot arrive the \([n+2,A]\) or the \([n-1,A]\) sites.

An alternative perspective is to calculate the band structure of the lattice. As shown in Fig. 2(a), the six bands of the lattice become all completely flat with the link variables in Eq. (3) imposed, implying that a particle on the lattice should move with velocity \(v = \partial E/\partial k = 0\). In addition, the compact localized eigenstate (CLES) for each eigenenergy is calculated and shown in Figs. 2(b)-(g). For the two-fold degenerate middle bands with \(E_3 = E_4 = 0\), the CLESs do not have \(A\) site component (Figs. 2(b) and (c)). Meanwhile, for the other four bands with eigenenergies \(E_1 = -E_6 = \sqrt{6J}\) and \(E_2 = -E_5 = \sqrt{2J}\), the corresponding CLESs containing the \([n,A,1]\) and \([n+1,A,2]\) mode components are explicitly shown in Figs. 2(d)-(g). The particle initially populated in the \([n,A,1]\) mode can then be viewed as the superposition of these four CLESs, and the non-Abelian AB caging dynamics mentioned in previous paragraphs can thus be simply interpreted as the time evolution of the superposition of these four CLESs.

III. A CANDIDATE CIRCUIT QED IMPLEMENTATION

A. The circuit QED lattice proposal

In this section, we consider the circuit QED lattice shown in Fig. 1(b) as the candidate realization of the proposed non-Abelian AB caging described in Eqs. (5) and (7). This circuit QED lattice consists of three types of superconducting transmissionline resonators (TLRs) differed by their lengths and placed in an interlaced form. These TLRs play the corresponding roles of the A, B, and C sites of the rhombic lattice depicted in Fig. 1(a). At their ends, the TLRs are commonly grounded by inductors with inductances much smaller than those of the
TABLE I: Spatial configuration of the non-Abelian AB cage versus the nilpotent power

| Initial mode | Size of the cage | Right edge | Left edge |
|--------------|-----------------|------------|-----------|
| \( l \in (1, m) \) | \( l \in (m, N-m) \) | \([n + l - 1, A]\) | \([n, A]\) |
| \( l \in (N-m, N) \) | \([n + l - 1, A]\) | \([n, A]\) | \([n + l - N, A]\) |
| \( l \in (1, N-m) \) | \( N-l+1 \) | \([n, A]\) | \([n, A]\) |
| \( l \in (N-m, N) \) | \( N-l+1 \) | \([n, A]\) | \([n + l - N, A]\) |

The lattice can thus be described by the Hamiltonian

\[ H = \sum_{m, n, \sigma} \omega_{m,n} \alpha_{m,n,\sigma}^{\dagger} \alpha_{m,n,\sigma}, \]  

where the eigenfrequencies of the cavity modes are specified as

\[ \begin{align*}
\left[ \omega_{A1}, \omega_{B1}, \omega_{C1} \right] &= \left[ \omega_0 - \Delta, \omega_0, \omega_0 + \Delta \right], \\
\left[ \omega_{A2}, \omega_{B2}, \omega_{C2} \right] &= \left[ 2 \omega_{A1}, \omega_{B1}, \omega_{C1} \right],
\end{align*} \]  

with \( \omega_0 / 2\pi \in [5, 6] \) GHz and \( \Delta/2\pi \in [1, 2] \) GHz. Such configuration is for the following application of the PFC method and can be experimentally realized through the length selection of the TLRs in the millimeter range.  

We further consider the implementation of the required link variables on the lattice, taking the form in Eq. (6) in the rotating frame of \( H_0 \). Here we employ the dynamic modulation method \( [28, 30, 37, 38] \). For this aim, the TLRs are connected at their ends by connecting superconducting quantum interference devices (SQUIDs) (Fig. 1(c)) \( [29, 39] \), which correspond to the linking bonds shown in Fig. 1(a) one-to-one. The connecting SQUIDs can be regarded as tunable inductances which can be a.c. modulated by external magnetic flux oscillating at very high frequencies \( [40] \). It will be figured out in the following that each of the bonds on the lattice can be independently controlled by the modulating pulse threaded in the corresponding connecting SQUID, leading to the site-resolved arbitrary control of the synthesized non-Abelian gauge field. In addition, estimations in our previous works have shown that this method can result in effective hopping strength in the range \( J/2\pi \in [5, 15] \) MHz, which is robust against unavoidably imperfection factors in realistic experiments including the fabrication errors of the circuit and the background low-frequency noises.\([33, 35]\).

**B. The PFC scheme of realizing non-Abelian gauge**

The essential idea of the PFC method can be illustrated step-by-step by investigating the a.c. modulation of a particular \( [n, \alpha] \leftrightarrow [m, \beta] \) coupling SQUID which...
introduces a time-dependent current-current coupling
\[ H_{\text{a.c.}}^{n,\alpha,m\beta} = g_{n,\alpha,m\beta}(t) \sum_{\sigma,\sigma'} (\alpha_n + \alpha_n^\dagger)(\beta_{m\sigma'} + \beta_{m\sigma'}^\dagger). \] (10)

Here we assume that the parameters in Eq. (10) satisfy the condition
\[ \omega_{n\alpha}, \omega_{n\alpha'} \gg |\omega_{n\alpha} - \omega_{n\alpha'}| \gg |g_{n,\alpha,m\beta}(t)|. \] (11)

In the first step, we aim at constructing the concrete \([n, \alpha, \sigma_1] \leftrightarrow [m, \beta, \sigma_2]\) photon hopping. As the \([n, \alpha, \sigma_1]\) and \([m, \beta, \sigma_2]\) modes are far off-resonant, the desired hopping can hardly be achieved with static \(g_{n,\alpha,m\beta}(t)\). Meanwhile, we can complete this task by dynamically modulating \(g_{n,\alpha,m\beta}(t)\) as
\[ g_{n,\alpha,m\beta}(t) = 2J \cos[(w_{n\alpha_1} - w_{n\beta_2})t - \theta_{n\alpha_1,m\beta\sigma_2}]. \] (12)

Physically speaking, this is the process depicted in Fig. 1(c), where a photon initially in the \([n, \alpha, \sigma_1]\) mode changes its energy by absorbing/emitting an energy quanta \([w_{n\beta_2} - w_{n\alpha_1}]\) from/to the oscillating \(g_{n,\alpha,m\beta}(t)\) and then hops into the \([m, \beta, \sigma_2]\) mode. More rigorously, this process can be mathematically treated in the rotating frame of \(H_S\). By using the rotating wave approximation, we can obtain the effective Hamiltonian:
\[ H_{\text{eff}}^{n\alpha_1,m\beta_2} = J_0^{n\alpha_1,m\beta_2} \exp(-i\theta_{n\alpha_1,m\beta_2}) + \text{h.c.}, \] (13)

with the other fast oscillating terms being neglected. Here we can see that both the phase and the amplitude of the effective \([n, \alpha, \sigma_1] \leftrightarrow [m, \beta, \sigma_2]\) hopping can be independently controlled by the \([n, \alpha] \leftrightarrow [m, \beta]\) modulating pulse \(g_{n,\alpha,m\beta}(t)\).

This parametric formalism can then be generalized to establish the matrix-form non-Abelian link variables required in Eq. (6). In this situation, the coupling strength \(g_{n,\alpha,m\beta}(t)\) in Eq. (10) contains multiple frequencies, with each tone controlling one hopping branch in the linking variable. This is schematically sketched in Fig. 1(d), where the red and blue lines label the eigenfrequencies of the \([n, \alpha]\) and \([m, \beta]\) sites, and the dashed and the solid arrows represent the component-preserving \(U_1\) and \(U_4\) and the component-mixing \(U_2\) and \(U_3\), respectively. The major obstacle of the generalization is the cross talk effect, i.e., a particular tone in \(g_{n,\alpha,m\beta}(t)\) may induce other unwanted hopping process. Meanwhile, this effect can be effectively suppressed as follow. As shown in Eq. (9) the link variables have been selected to take relatively simple forms. Each link variable matrix contains at most 2 non-zero items, implying that any coupling strength \(g_{n,\alpha,m\beta}(t)\) consists of at most 2 tones. For instance, the a.c. modulation of the \([n, A] \leftrightarrow [n, B]\) connecting SQUID have two tones with frequencies \(2\Delta\) and \(\Delta\), which induce the \([n, A_1] \leftrightarrow [n, B_1]\) and the \([n, A_2] \leftrightarrow [n, B_2]\) PFC bonds by bridging their frequency gaps, respectively. Moreover, the eigenfrequencies of the cavity modes are set such that the two tones are significantly different. This is already indicated in Eqs. (9) and (11) and shown in Fig. 1(d). In this situation, the controlling tone of one hopping branch can hardly influence the other, resulting in arbitrary control of the linking variables. With these strategies exploited, the leading effect is the a.c. Stark shifts of the cavity mode frequencies induced by the off-resonant pumping. This effect is of second-order and can be further compensated by the adjustment of the frequencies of \(g_{n,\alpha,m\beta}(t)\) which corresponds to the renormalization of \(H_S\) in Eq. (8), as implied in the derivation of Eq. (13).

C. Measurement of the non-Abelian AB caging

The proposed non-Abelian AB caging physics can be observed through the coherent monochromatic pumping of a particular \([n, A, \sigma]\) mode on the lattice, which can be described by
\[ H_{\text{pump}} = P \alpha e^{-i\Omega_P t} + \text{h.c.}, \] (14)

where \(\alpha\) is the vector of the annihilation operators of the whole lattice, \(P\) is the corresponding pumping strength vector, and \(\Omega_P\) is the detuning of the pumping frequency with respect to \(\omega_{A\sigma}\). The pumping can inject photons into the lattice which will then experience the exotic localization dynamics analyzed in the previous section. Therefore, we expect that information about the proposed non-Abelian AB caging can be extracted from the driven-dissipation steady state of the lattice.

FIG. 3: SSPN distribution of the lattice with the \([0, A]\) site being pumped. The detunings and the pumped modes are selected as: (a). \(\Omega_P = \sqrt{6}J, [0, A, 1]\); (b). \(\Omega_P = \sqrt{2}J, [0, A, 1]\); (c). \(\Omega_P = \sqrt{6}J, [0, A, 1]\); (d). \(\Omega_P = \sqrt{2}J, [0, A, 2]\). For each subfigure, the up and lower panels denote the SSPN in the first and the second components, respectively.

We then numerically calculate the SSPN distribution of this lattice in the presence of driving and dissipation, with results shown in Fig. 3. This corresponds to the solution of the equation
\[ i\frac{d(\alpha)}{dt} = \left[ B - \left( \Omega_P + \frac{1}{2} i \kappa \right) I \right] (\alpha) + P = 0, \] (15)
where the matrix $\mathcal{B}$ is defined by $\alpha^\dagger \mathcal{B} \alpha = H$ and $\kappa$ is the assumed uniform decay rate of the cavity modes. In this simulation we consider a chain of 10 unit-cells with the pumped $A$ site at the central position. The parameters are chosen as $J/2\pi = |P|/2\pi = 10$ MHz and $\kappa = 0.01J$. To maximize the injection of photons into the lattice, we set $\Omega_P$ to be resonant with one of the the eigenenergies shown in Fig. 2(a). Explicitly, Figs. 3(a)/(c) correspond to $\Omega_P = \sqrt{6}J$ and pumping the $|0, A, 1\rangle/|0, A, 2\rangle$ modes, and Figs. 3(b)/(d) correspond to $\Omega_P = \sqrt{2}J$ and pumping the $|0, A, 1\rangle/|0, A, 2\rangle$ modes. For each subfigure, the up and lower panel denote the SSPN distribution in the first and second components of the sites, respectively. It can be clearly observed that the calculated SSPN distributions exhibit several features of the discussed non-Abelian AB caging. In particular, the injected photons become localized with asymmetric SSPN distribution with respect to the pumped $A$ site. The SSPN distribution extend more rightward(leftward) if the pumped mode is the first(second) component, as predicted in Sec. III.

From another perspective, the calculated SSPN distribution reflects to some extent the spatial configuration of the corresponding CLESs. This can be intuitively understood because pumping a cavity mode with a particular eigenfrequency corresponds to exciting the corresponding CLES containing the component of the pumped mode. For instance, the superposition of the up and lower panel of Fig. 3(a) coincides exactly with the CLES shown in Fig. 2(d). Fig. 3(c) also coincides with Fig. 2(d) in this sense, with the SSPN been leftward translated by one unit-cell as the second component of the site is excited. In addition, it should be noticed that the SSPN patterns shown in Figs. 3(a) and (c) are different from those in Figs. 3(b) and (d). This difference can be attributed to the excitation of different CLESs shown in Figs. 2(d) and (e).

The calculated SSPN distribution can be experimentally detected by the following simple measurement scheme. Each of the lattice sites sufficiently involved in the SSPN calculation is capacitively connected to an external coil with input/output ports for pumping/measurement. The steady state of the lattice can be prepared by injecting microwave pulses through the input port for a sufficiently long time. During the steady-state period, energy will leak out from the coupling capacitance, which is proportional $\omega_{n\sigma}\langle \alpha_{n\sigma}^\dagger \alpha_{n\sigma} \rangle$ with the proportional constant determined by the coupling capacitance. The target observable $\langle \alpha_{n\sigma}^\dagger \alpha_{n\sigma} \rangle$ can therefore be measured by simply integrating the energy flowing to the output port in a given time duration. This measurement method has already been used in experiment with both the amplitude and the phase of a coherent state of a superconducting 3D cavity were measured [27]. Here we emphasize that what we want to measure is the expectation value $\langle \alpha_{n\sigma}^\dagger \alpha_{n\sigma} \rangle$, while the detailed probability of the multi-mode coherent steady state projected to the Fock basis is nevertheless not needed. It is this weak requirement that greatly simplify our measurement.

IV. DISCUSSION AND CONCLUSION

In conclusion, we propose in this manuscript the non-Abelian AB caging concept which is the multi-component matrix generalization of the existing Abelian AB caging concept. Distinct from its Abelian counterpart, the AB caging in this situation become sensitive to both the explicit form of the gauge potential and the initial state of the lattice. These features are the consequence of the matrix nature of our theory and thus have no Abelian analog. Moreover, we suggest a superconducting quantum circuit implementation of the proposed physics, in which the lattice sites are built by superconducting transmission-line resonators and the required non-Abelian gauge is constructed by the PFC method. The unambiguous verification of the non-Abelian AB caging can further be achieved through the steady-state manipulation of only few sites on the lattice.

While the localized steady states of the CLES excitations considered in this manuscript can be thoroughly understood in the single particle picture, what is more important is that the dispersionless flat band is an ideal platform of investigating correlated many-body states. The introduction of interaction will lead us to the realm where rich but less explored physics locates. On the other hand, as the strong coupling has already been achieved in circuit QED [25], the Bose-Hubbard type [41] and Jaynes-Cummings-Hubbard type photon-photon interaction [26, 27] can be incorporated by coupling the TLRs with superconducting qubits. Therefore, our further direction should be the characterization of nonequilibrium strongly-correlated photonic quantum fluids in the proposed architecture.

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