TOWARDS A PROOF OF THE CLASSICAL SCHOTTKY UNIFORMIZATION CONJECTURE

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Abstract. As a consequence of Koebe’s retrosection theorem, every closed Riemann surface of genus \( g \geq 2 \) is uniformized by a Schottky group. Marden observed that there are Schottky groups which are not classical ones, that is, they cannot be defined by a suitable collection of circles. This opened the question to if every closed Riemann surface can be uniformized by a classical Schottky group. Recently, Hou has observed this question has an affirmative answer by first noting that every closed Riemann surface can be uniformized by a Schottky group with limit set of Hausdorff dimension < 1 and then by proving that such a Schottky group is necessarily a classical one. In this paper, we provide another argument, based on the density of Belyi curves on the the moduli space \( \mathcal{M}_g \) and the fact that the locus \( \mathcal{M}^c_g \subset \mathcal{M}_g \) of those Riemann surfaces uniformized by classical Schottky groups is a non-empty open set. We show that every Belyi curve can be uniformized by a classical Schottky group, so \( \mathcal{M}^c_g = \mathcal{M}_g \).

1. Introduction

Let \( S \) be a closed Riemann surface of genus \( g \geq 2 \). An uniformization of \( S \) is a tuple \((G, \Delta, P : \Delta \to S)\), where \( G \) is a Kleinain group, \( \Delta \) is a \( G \)-invariant connected component of its region of discontinuity (that is, \((G, \Delta)\) is a function group) and \( P \) is a regular holomorphic cover with \( G \) as its deck group; in particular, \( G \) acts freely on \( \Delta \). The collection of uniformizations of \( S \) has a natural partial order defined as follows. An uniformization \((G_1, \Delta_1, P_1 : \Delta_1 \to S)\) is higher than \((G_2, \Delta_2, P_2 : \Delta_2 \to S)\) if there is a holomorphic map \( Q : \Delta_1 \to \Delta_2 \) so that \( P_1 = P_2 \circ Q \) (in particular, \( Q \) is a regular covering map, that is, is defined by a subgroup \( N \) of \( G_1 \) and \( G_2 = G_1/N \)). The highest uniformizations correspond to \( \Delta \) being simply connected (so isomorphic to the unit disc). The lowest uniformizations corresponds to the case that \( G \) is a Schottky group.

Geometrically, a Schottky group of rank \( g \geq 2 \) is defined as (equivalent definitions can be found in [10]) a group \( G \) generated by \( g \)loxodromic elements \( A_1, \ldots, A_g \), where there exists a collection of \( 2g \) pairwise disjoint simple loops \( C_1, \ldots, C_g, C'_1, \ldots, C'_g \) on the the Riemann sphere \( \hat{\mathbb{C}} \) bounding a common region \( \mathcal{D} \) of connectivity \( 2g \) so that \( A_j(C_j) = C'_j \) and \( A_j(\mathcal{D}) \cap \mathcal{D} = \emptyset \), for all \( j = 1, \ldots, g \). The domain \( \mathcal{D} \) is called a standard fundamental domain for \( G \), the collection of loops \( C_1, \ldots, C_g, C'_1, \ldots, C'_g \) a fundamental set of loops and the Möbius transformations \( A_1, \ldots, A_g \) a Schottky set of generators. It is known that \( G \) is a free group of rank \( g \) and Chuckrow [3] proved that every set of \( g \) generators of it is a Schottky set of generators. Its region of discontinuity \( \Omega \) is connected and the quotient space \( \Omega/G \) is a closed Riemann surface of genus \( g \) [9]. Koebe’s retrosection theorem states that every closed Riemann surface can be obtained, up to isomorphisms, as above by a suitable Schottky group. A simple proof of this fact was also given by L. Bers in [2] using quasiconformal

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mappings. A Schottky uniformization of a closed Riemann surface \( S \) is a uniformization \((G, \Omega, P : \Omega \to S)\), where \( G \) is a Schottky group with region of discontinuity \( \Omega \).

A Schottky group is called classical if it has a Schottky set of generators with a fundamental set of loops consisting of circles. Marden [8] showed that, for every \( g \geq 2 \), there are non-classical Schottky groups of rank \( g \); see also [7]. An explicit family of examples of non-classical Schottky groups of rank two was constructed by Yamamoto [11] and a theoretical construction of an infinite collection of non-classical Schottky groups is provided in [4]. This opened the question to if every closed Riemann surface has a classical Schottky uniformization (I think it was Lipman Bers who first asked this question?).

**Theorem 1 (Classical Schottky uniformization conjecture).** *Every closed Riemann surface can be uniformized by a classical Schottky group.*

An affirmative answer to the above conjecture were known in either one of the following two situations.

1. \( S \) admits an anticonformal automorphism of order two with fixed points (Koebe).
2. \( S \) has \( g \) pairwise disjoint homologically independent short loops (McMullen).

In [5] Hou proved that every closed Riemann surface can be uniformized by a Schottky group whose limit set has Hausdorff dimension < 1 and, in [6], he also proved that such a Schottky group is necessarily a classical one; so this provides a proof to Theorem 1.

In this paper we describe a different argument. The main idea is the following. Let \( \mathcal{M}_g \) be the moduli space of closed Riemann surfaces of genus \( g \geq 2 \), \( \mathcal{M}_g^{cs} \) be its locus consisting of classes of Riemann surfaces which can be uniformized by classical Schottky groups and \( \mathcal{M}_g^{b} \) be the locus of classes of Belyi curves. It is well known that \( \mathcal{M}_g^{cs} \) is a non-empty open set and that \( \mathcal{M}_g^{b} \) is a dense subset (as a consequence of Belyi’s theorem [1]). In this way, in order to prove Theorem 1, it is enough to check that every Belyi curve can be uniformized by a classical Schottky group (Theorem 2).

**Remark 1.** It is important to remark that we are dealing with all Belyi curves and not just the regular Belyi curves (also called quasiplatonic curves); as these last ones only provides a finite collection of points in \( \mathcal{M}_g \).

Moduli space \( \mathcal{M}_g \) is non-compact. A compactification, the Deligne-Mumford compactification, is obtained by considering stable Riemann surfaces of genus \( g \). Every stable Riemann surface of genus \( g \geq 2 \) can be uniformized by a noded Schottky group of rank \( g \) (a geometrically finite Kleinian group isomorphic to the free group of rank \( g \)). These noded Schottky groups are geometric limits of Schottky groups. Similarly to Schottky groups, noded Schottky groups can be defined geometrically using a system of \( 2g \) simple loops, but one now permits tangencies at parabolic fixed points. A neoclassical Schottky group is one for which these system of loops can be chosen to be circles. In [4] it was observed that there are noded Riemann surfaces which cannot be uniformized by a neoclassical Schottky group.
2. Preliminaries

2.1. Annuli and their modules. An annulus (or ring domain) is a doubly connected domain $A$ in $\hat{\mathbb{C}}$. There exists a unique $r > 1$ so that $A$ is biholomorphically equivalent to a circular annulus $A_r := \{ z \in \mathbb{C} : r^{-1} < |z| < r \}$. The modulus of $A$ is defined as $\text{mod}(A) = \frac{1}{\pi} \log r$. Next, we list some known results on the modulus of annuli.

**Lemma 1** (Grötzch inequality). If $A$ is an annulus and $B \subset A$ an essential annulus (i.e., the inclusion map induces an injective map on the fundamental group), then $\text{mod}(B) \leq \text{mod}(A)$.

**Lemma 2.** If $A$ and $B$ are annuli and $Q : A \to B$ is a degree $d$ covering map, then $\text{mod}(A) = d \text{mod}(B)$.

**Lemma 3.** Every annulus $A$ satisfying that $\text{mod}(A) > 1/2$ contains an euclidean circle separating its borders.

**Lemma 4.** Let $A$ and $B$ annuli and $Q : A \to B$ a finite degree surjective holomorphic map (with a finite set of critical points on $A$). We also assume that $Q$ sends a central loop of $A$ onto a central loop of $B$ and this restriction is a covering map of loops. Then there exists a positive constant $C(Q)$, only depending on $Q$, so that $\text{mod}(A) > C(Q) \text{mod}(B)$.

3. Belyi curves can be uniformized by classical Schottky groups

A closed Riemann surface $S$, of genus $g \geq 2$, is called a Belyi curve if it admits a non-constant meromorphic map $\beta : S \to \hat{\mathbb{C}}$ whose branch values are contained in the set $\{1, \omega_3, \omega_2^3\}$, where $\omega_3 = e^{2\pi i/3}$ (in this case $\beta$ is a called a Belyi map for $S$). Usually, many authors assume the branch values to be contained in the set $\{\infty, 0, 1\}$, but this is equivalent as the group of Möbius transformations acts 3-transitively on the Riemann sphere.

The group of Möbius transformations keeping invariant the set $\{1, \omega_3, \omega_2^3\}$ is generated by the transformations $A(z) = \omega_3 z$ and $B(z) = 1/z$ and it is isomorphic to the symmetric group in three letters $\mathfrak{S}_3$. The rational map

$$R(z) = \frac{(1 + 2\omega_3)(z^3 + z^{-3}) - 6}{(1 + 2\omega_3)(z^3 + z^{-3}) + 6}$$

provides a regular branched cover with deck group $\langle A, B \rangle$ and whose set of branch values is $\{1, \omega_3, \omega_2^3\}$.

If $\beta : S \to \hat{\mathbb{C}}$ is a Belyi map for $S$, then the composition map $R \circ \beta : S \to \hat{\mathbb{C}}$ still a Belyi map for $S$. A Belyi map for $S$ obtained as a finite sequence of compositions $R \circ R \circ \cdots \circ R \circ \beta$ is called a refining of $\beta$. 
Theorem 2. Every Belyi curve can be uniformized by a classical Schottky group.

Proof. Let us denote by $S^1 = \{a \in \mathbb{C} : |z| = 1\}$ the unit circle in the complex plane. If $\beta : S \to \mathbb{C}$ is a Belyi map for $S$, then $\beta^{-1}(S^1)$ provides a triangulation of $S$. By taking a refining of $\beta$, if necessary, we may assume the following two properties:

1. there are $g$ pairwise disjoint homologically independent simple loops $\alpha_1, \ldots, \alpha_g$ inside $\beta^{-1}(S^1)$;
2. for every $j = 1, \ldots, g$, $\beta(\alpha_j) = S^1$ and $\beta : \alpha_j \to S^1$ is a covering map.

Let us consider a Schottky uniformization $(G, \Omega, P : \Omega \to S)$ defined by the loops $\alpha_1, \ldots, \alpha_g$, that is, there is a fundamental set of loops $C_1, \ldots, C_g, C'_1, \ldots, C'_g$ for $G$ so that $P(C_j) = P(C'_j) = \alpha_j$, for $j = 1, \ldots, g$. Let $B_1, \ldots, B_g$ a corresponding Schottky set of generators.

Let us consider the annuli $A_r = \{z \in \mathbb{C} : r < |z| < r\}$, where $r > 1$. Then, the preimage of $A_r$ by the meromorphic map $Q = \beta \circ P : \Omega \to \mathbb{C}$ provides a neighborhood $\tilde{A}_r$ of the graph $\beta^{-1}(S^1)$. Inside $\tilde{A}_r$ there is a collection of $g$ pairwise disjoint annuli, say $\tilde{A}_r^1, \ldots, \tilde{A}_r^g$, where $C_j \subset \tilde{A}_r^j$. We chose them so that $Q : \tilde{A}_r^j \to A_r$ is surjective.

In this way, $\text{mod}(\tilde{A}_r^j) = C(Q)\text{mod}(A_r) = C(Q) \log(r)/\pi$ (see Lemma 4).

Next, we make $r$ approach to $+\infty$ in order to assume the module of each annuli $\tilde{A}_r^j$ to be as big as we want. Now Lemma 3 ensures that inside $\tilde{A}_r^j$ there is a circle $D_j$ (homotopic to $C_j$ in the corresponding annuli). It can be seen that the new loops $D_1, \ldots, D_g, D'_1 = B_1(D_1), \ldots, D'_g = B_g(D_g)$ define a new set of fundamental loops for $G$ making it a classical Schottky group.

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