EMBEDDINGS OF DECOMPOSITION SPACES INTO SOBOLEV AND BV SPACES

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ABSTRACT. In the present paper, we investigate whether an embedding of a decomposition space $D(Q, L^p, Y)$ into a given Sobolev space $W^{k,q}(\mathbb{R}^d)$ exists. As special cases, this includes embeddings into Sobolev spaces of (homogeneous and inhomogeneous) Besov spaces, $(\alpha)$-modulation spaces, shearlet smoothness spaces and also of a large class of wavelet coorbit spaces, in particular of shearlet-type coorbit spaces.

Precisely, we will show that under extremely mild assumptions on the covering $Q = (Q_i)_{i \in I}$, the decomposition space $D(Q, L^p, Y) \hookrightarrow W^{k,q}(\mathbb{R}^d)$ as soon as $p \leq q$ and $Y \hookrightarrow \ell^p_u(k_p, q_p)$ hold. Here, $q^q = \min \{q, q'\}$ and the weight $u(k_p, q_p)$ can be easily computed, only based on the covering $Q$ and on the parameters $k, p, q$.

Conversely, a necessary condition for existence of the embedding is that $p \leq q$ and $Y \cap I_0(I) \hookrightarrow \ell^p_u(k_p, q_p)$ hold, where $I_0(I)$ denotes the space of finitely supported sequences on $I$. Thus, even though our approach applies to (almost) arbitrary decomposition spaces, we can completely characterize existence of the embedding $D(Q, L^p, Y) \hookrightarrow W^{k,q}(\mathbb{R}^d)$ for the range $q \in (0, 2)$, since this implies $q = q^q$. Furthermore, this complete characterization also remains valid for the important special case $q = \infty$. In this case, we will even see that the embedding $D(Q, L^p, Y) \hookrightarrow W^{k,\infty}(\mathbb{R}^d)$ is equivalent to an embedding into the space $C^I_1(\mathbb{R}^d)$.

As a further result, we show that a decomposition space embeds into BV if and only if it embeds into $W^{1,1}(\mathbb{R}^d)$. Hence, our results also yield a complete characterization of embeddings of decomposition spaces into BV.

Finally, for the most important special case of a weighted Lebesgue sequence space $Y = \ell^q_u(I)$, we simplify the conditions from above even further: In this case, $Y \hookrightarrow \ell^q_u(I)$ is equivalent to $\|f\|_{\ell^q_u(I)} \leq \|f\|_{L^p}$ for all $f \in L^p(I)$. As a consequence, verification of the previously mentioned necessary or sufficient conditions reduces to an exercise in calculus which can be solved without requiring any knowledge of Fourier analysis.

To indicate the power and simplicity of our criteria, we apply them to all of the classes of decomposition spaces mentioned in the first paragraph above.

1. Introduction

Decomposition spaces were first introduced in full generality by Feichtinger and Gröbner in the 80s. They were then used by Gröbner in his PhD thesis to define the so-called $\alpha$-modulation spaces.

Recently, decomposition spaces received increased attention. This began with the work of Borup and Nielsen, who constructed Banach frames for certain decomposition spaces. Note, however, that Borup and Nielsen consider a more restricted class of decomposition spaces than Feichtinger and Gröbner. It is this class of decomposition spaces that we will consider in the present paper. Roughly, these decomposition spaces are defined as follows: One fixes a covering $Q = (Q_i)_{i \in I}$ of a subset $O$ of the frequency space $\mathbb{R}^d$. Then, given a suitable partition of unity $\Phi = (\varphi_i)_{i \in I}$ subordinate to $Q$, the decomposition space (quasi)-norm is defined as

$$\|f\|_{D(Q, L^p, Y)} := \left\|\left\|s^{-1}(\varphi_i f)\right\|_{L^p}\right\|_{Y},$$

where $Y \subseteq C^I$ is a suitable sequence space. Thus, these spaces are defined analogously to Besov spaces, in contrast Triebel Lizorkin spaces, where the “order” of the $L^p$ and $Y$ norms would be reversed.

Recent papers which study concrete examples of decomposition spaces are, in which the respective authors consider embeddings between decomposition spaces for different values of $\alpha$. Furthermore, we would like to mention, in which Labate et al. use the construction of decomposition spaces to introduce a new class of function spaces, the so-called shearlet smoothness spaces. In the same paper, the authors establish certain embeddings between Besov spaces and shearlet smoothness spaces.

As a more abstract application of decomposition spaces, we mention the paper, in which Hartmut Führ and the present author showed that a large class of wavelet coorbit spaces are naturally isomorphic to certain decomposition spaces, i.e., we have

$$\text{Co} \left( L^{p,q}_u(\mathbb{R}^d \times H) \right) \cong D(Q_H, L^p, \ell^q_u)$$

for a certain weight $u = u_{Q,v,q}$.

Key words and phrases. Decomposition spaces, Sobolev spaces, Coorbit spaces, Smoothness spaces, Embeddings, Shearlets, $\alpha$-modulation spaces, Besov spaces, BV spaces.
Here, the covering $Q_H$ used to define the decomposition spaces (which is called an induced covering) depends crucially on the dilation group $H \leq \text{GL}(\mathbb{R}^d)$ through which the wavelet coorbit space under consideration is defined. This decomposition space view on coorbit spaces is often superior to the original description as coorbit spaces; for example, the decomposition space view makes it possible to consider embeddings between wavelet coorbit spaces “living” on different groups.

Indeed, in my PhD thesis [21] and in the upcoming paper [22], I developed a general theory of embeddings between different decomposition spaces, i.e. embeddings of the form

$$D(Q, L^{p_1}, \ell^{q_1}_v) \hookrightarrow D(P, L^{p_2}, \ell^{q_2}_v)$$

for (possibly) different coverings $Q, P$. Here, we write $Y \hookrightarrow X$ if $Y \subset X$ and $\|x\|_X \leq C \cdot \|y\|_Y$ for all $y \in Y$.

Using this embedding theory and the interpretation of wavelet coorbit spaces as decomposition spaces, the existence of embeddings between coorbit spaces with respect to the shearlet-type dilation groups

$$H(c) := \left\{ \begin{pmatrix} a & b \\ 0 & a^c \end{pmatrix} \bigg| a \in (0, \infty), c > 0, b \in \mathbb{R} \right\}$$

for different values of $c \in \mathbb{R}$ could be characterized, cf. [21 Theorem 6.38]. Furthermore, I considered embeddings between shearlet-type coorbit spaces and (inhomogeneous) Besov spaces, achieving a complete characterization in many cases [21, Theorem 6.3.11].

As further applications, the embedding results for $\alpha$-modulation spaces given in [11] can be obtained (and even generalized) using the theory developed in [21, cf. [21 Theorem 6.1.7]. Likewise, the results in [13] for embeddings between shearlet smoothness spaces and Besov spaces can be improved to obtain a complete characterization of the existence of such embeddings, cf. [21 Theorem 6.4.3).

Despite these results, the thesis [21] only considers embeddings between different decomposition spaces. Thus, embeddings of decomposition spaces into Sobolev spaces are not covered. In the present paper, we will initiate the development of such an embedding theory. Indeed, we will completely characterize the existence of an embedding $D(Q, L^p, Y) \hookrightarrow W^{k,q}(\mathbb{R}^d)$ for $q \in (0,2] \cup \{\infty\}$. For the remaining range $q \in (2,\infty)$, we also obtain sufficient criteria and necessary criteria, but these two types of conditions do not coincide. Investigating this gap is a valuable topic for future research.

1.1. Our results. The precise formulation of the main result of this paper needs the notion of a regular covering $Q = (Q_i)_{i \in I} = (T_iQ_i' + b_i)_{i \in I}$, which will be fully explained in Section 2. At the moment, we only remark that this means that the sets $Q_i = T_iQ_i' + b_i$ are obtained from the normalized sets $Q'_i$ by means of the (invertible) affine maps $x \mapsto T_i x + b_i$. Furthermore, we require the normalized sets $Q'_i$ to be uniformly bounded, since without an assumption of this kind there would not be any meaningful relationship between the set $Q_i$ and the affine map $x \mapsto T_i x + b_i$.

In addition to the properties from the previous paragraph, a regular covering has to satisfy certain additional technical assumptions, which are explained in detail in Section 2. These assumptions, however, are fulfilled for any reasonable covering occurring in practice, in particular for the coverings used to define (homogeneous and inhomogeneous) Besov spaces, ($\alpha$)-modulation spaces, Shearlet smoothness spaces and all induced coverings $Q_H$ which are used when interpreting wavelet coorbit spaces as decomposition spaces (as in equation (1.1)).

Finally, the following theorem uses the notion of a $Q$-regular sequence space $Y$, which is also introduced in Section 2. For simplicity, the reader might simply think of the case of a weighted Lebesgue sequence space $Y = \ell^q(I)$, as defined in Subsection 1.4. Using these notions, our main result reads (slightly simplified) as follows: \[1\]

**Theorem 1.1.** Let $Q = (Q_i)_{i \in I} = (T_iQ_i' + b_i)_{i \in I}$ be a regular covering of the open set $\emptyset \neq \mathcal{O} \subset \mathbb{R}^d$. Let $k \in \mathbb{N}_0$ and $q \in (0, \infty]$ and let $Y \subset \ell^q$ be a $Q$-regular sequence space on $I$. Define the weight $u^{(k,p,q)}(i)$ by

$$u^{(k,p,q)}_i := |\det T_i|^{-\frac{1}{p} - \frac{1}{q} - \frac{1}{2}} \cdot \left(1 + |b_i|^k + \|T_i\|^k\right)^{p-1}$$

for $i \in I$ and let $q^\# := \min\{q, q'\}$. Then the following hold:

1. Actually, in the present generality, the inclusion $Y \subset X$ is sometimes not satisfied. In this case, we still write $Y \hookrightarrow X$ if there is a bounded linear map $\iota : Y \rightarrow X$ which (in some sense) deserves to be called an embedding. This is made more precise in Definition 3.3. As a concrete example, note that homogeneous Besov spaces are usually defined as subspaces $B^s_{p,q} \subset S^\prime$ where $S^\prime$ is the space of polynomials. Thus, the inclusion $B^s_{p,q} \subset B^s_{p,q} \subset S^\prime$ for an inhomogeneous Besov space $B^s_{p,q}$ can never be satisfied. Nevertheless, a (bounded linear) map $\iota : B^s_{p,q} \rightarrow B^s_{p,q}$ “deserves” to be called an embedding if $\iota f + P = f$ for all $f \in B^s_{p,q}$.
• If $p \leq q$ and if
\[ Y \hookrightarrow \ell^q_{u(k,p,q)} (I), \]
then
\[ \mathcal{D} (Q, L^p, Y) \hookrightarrow W^{k,q} (\mathbb{R}^d). \]

• Conversely, if
\[ \mathcal{D} (Q, L^p, Y) \hookrightarrow W^{k,q} (\mathbb{R}^d) \]
holds, then we have $p \leq q$ and
\[ Y \cap \ell_0 (I) \hookrightarrow \ell^q_{u(k,p,q)} (I), \]
where $\ell_0 (I)$ denotes the space of finitely supported sequences on $I$. In case of $q = \infty$, we also get
\[ Y \cap \ell_0 (I) \hookrightarrow \ell^q_{u(k,p,q)} (I) = \ell^q_{u(k,p,q)} (I). \]

As noted above, for $q \in (0,2] \cup \{ \infty \}$, we thus obtain a complete characterization of the existence of an embedding of the decomposition space $\mathcal{D} (Q, L^p, Y)$ into the Sobolev space $W^{k,q} (\mathbb{R}^d)$, at least if we ignore the slight difference between the spaces $Y$ and $Y \cap \ell_0 (I)$. Note that the restriction $q \in (0,2] \cup \{ \infty \}$ (which is crucial for sharpness) is a restriction on the ‘target space’ $W^{k,q} (\mathbb{R}^d)$, not on the ‘source space’ $\mathcal{D} (Q, L^p, Y)$.

As a further simplification, for the case of a weighted Lebesgue space $Y = \ell^p_v (I)$, we establish the equivalence
\[ \ell^p_v (I) \cap \ell_0 (I) \hookrightarrow \ell^p_u (I) \]
\[ \iff \ell^p_u (I) \hookrightarrow \ell^p_v (I) \]
\[ \iff (u_i/v_i)_{i \in I} \in E^{s(r/s)} (I). \]

Thus, verification of the embedding $Y = \ell^p_v (I) \hookrightarrow \ell^p_u (I)$ reduces to verifying finiteness of a certain $\ell^s (I)$-norm of a single sequence.

Finally, we remark that to prove the above result, we actually establish a stronger statement which might be of independent interest: Given $n \in \mathbb{N}_0$, we give sufficient criteria and also necessary criteria for boundedness of the family of all (suitably defined) partial derivative operators $\partial^\alpha : \mathcal{D} (Q, L^p, Y) \rightarrow L^q (\mathbb{R}^d)$ with $|\alpha| = n$.

We will see that a sufficient criterion is $p \leq q$ and $Y \hookrightarrow \ell^q_{u(k,p,q)} (I)$ with
\[ v_i^{(n,p,q)} = |\det T_i|^{\frac{1}{p} - \frac{1}{q}} \cdot (|b_i|^n + ||T_i||^n). \]

Similarly, a necessary criterion is $p \leq q$ and $Y \cap \ell_0 (I) \hookrightarrow \ell^q_{u(k,p,q)} (I)$. The theorem above is then a corollary by applying these criteria simultaneously for $n = 0,1,\ldots,k$.

1.2. Comparison with earlier results. There are only two types of published results of which I am aware in which embeddings into Sobolev spaces of some kind of decomposition space are considered.

The first result is the complete characterization of the embeddings $M_{s,0}^{p,q} (\mathbb{R}^d) \hookrightarrow W^{k,p} (\mathbb{R}^d)$ of modulation spaces into Sobolev spaces (or more precisely, the Bessel potential spaces $L^q_{s} (\mathbb{R}^d)$ which coincide with $W^{k,p} (\mathbb{R}^d)$ for $p \in (1,\infty)$ and $s = k \in \mathbb{N}_0$) given in [12] by Kobayashi and Sugimoto. Compared to the present results, we note the following differences:

• Kobayashi and Sugimoto obtain a complete characterization, even for $p \in (2,\infty)$. Using our results (cf. also example 2.3 below), we obtain some sufficient and some necessary conditions, but these only yield a complete characterization for $p \in (0,2] \cup \{ \infty \}$. Furthermore, the approach in [12] also characterizes embeddings into the spaces $L^q_{s} (\mathbb{R}^d)$ for non-integer values $s \in \mathbb{R} \setminus \mathbb{N}_0$.

• Our results apply to arbitrary embeddings $M_{s,0}^{p,q} (\mathbb{R}^d) \hookrightarrow W^{k,r} (\mathbb{R}^d)$, while in [12], only the case $r = p$ is considered. Furthermore, the present results can be applied to general $\alpha$-modulation spaces $M_{s,\alpha}^{p,q} (\mathbb{R}^d)$, not only to modulation spaces (i.e. not only for $\alpha = 0$).

The second known result of which I am aware are the embeddings of (inhomogeneous) Besov spaces $B^{s,q}_{p} (\mathbb{R}^d)$ into the Triebel-Lizorkin spaces $F^{p,q}_{s} (\mathbb{R}^d)$. This yields embeddings into Sobolev spaces as follows: As shown in [19] Section 2.3.5, equation (2), we have
\[ H^{s,p} (\mathbb{R}^d) = F^{p,2}_{s} (\mathbb{R}^d) \text{ for } p \in (1,\infty) \text{ and } s \in \mathbb{R}, \]
where $H^{s,p} (\mathbb{R}^d)$ is a Bessel potential space. Furthermore, because of $H^{s,p} (\mathbb{R}^d) = W^{s,p} (\mathbb{R}^d)$ for $s \in \mathbb{N}_0$ (cf. [20] Section 2.3.1, Definition 1(7) and Remark 2)), we finally get $W^{s,p} (\mathbb{R}^d) = F^{p,2}_{s} (\mathbb{R}^d)$ for $p \in (1,\infty)$ and $s \in \mathbb{N}_0$. 

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Now, in [19] Section 2.3.2, Proposition 2(iii), the embedding
\[ B^{p,\min(p,q)}_s (\mathbb{R}^d) \hookrightarrow F^{p,q}_s (\mathbb{R}^d) \hookrightarrow B^{p,\max(p,q)}_s (\mathbb{R}^d) \]
is established for \( p \in (0, \infty) \), \( q \in [0, \infty] \) and \( s \in \mathbb{R} \). Hence,
\[ B^{p,\min(p-2)}_s (\mathbb{R}^d) \hookrightarrow W^{s,p}_\ast (\mathbb{R}^d) = F^{p,2}_s (\mathbb{R}^d) \hookrightarrow B^{p,\max(p-2)}_s (\mathbb{R}^d) \]  \hspace{1cm} (1.4)
for \( p \in (1, \infty) \) and \( s \in \mathbb{N}_0 \). Finally, [19] Section 2.3, Proposition 2(ii) yields
\[ B^{p,q_0}_{s+\varepsilon} (\mathbb{R}^d) \hookrightarrow B^{p,q_1}_{s} (\mathbb{R}^d) \]
for all \( s \in \mathbb{R} \), \( \varepsilon > 0 \) and \( p, q_0, q_1 \in (0, \infty) \),
so that we also get
\[ B^{p,q}_{s+\varepsilon} (\mathbb{R}^d) \hookrightarrow B^{p,\min(p-2)}_s (\mathbb{R}^d) \hookrightarrow W^{s,p}_\ast (\mathbb{R}^d) \hookrightarrow B^{p,\max(p-2)}_s (\mathbb{R}^d) \hookrightarrow B^{p,q}_{s-\varepsilon} (\mathbb{R}^d) \]  \hspace{1cm} (1.5)
for arbitrary \( p \in (1, \infty) \), \( s \in \mathbb{N}_0 \), \( \varepsilon > 0 \) and \( q \in (0, \infty] \).

Apart from establishing inclusion relations between Besov spaces and Sobolev spaces, the above embeddings can also be used to derive sufficient and necessary criteria for embeddings of decomposition spaces into Sobolev spaces. Indeed, if \( D (Q, L^p, Y) \hookrightarrow W^{s,q}_\ast (\mathbb{R}^d) \) for some \( q \in (1, \infty) \), equation (1.5) yields
\[ D (Q, L^p, Y) \hookrightarrow B^{q,r}_{s-\varepsilon} (\mathbb{R}^d) \]
for all \( r \in (0, \infty) \) and \( \varepsilon > 0 \).

Conversely, if \( D (Q, L^p, Y) \hookrightarrow B^{q,r}_{s+\varepsilon} (\mathbb{R}^d) \) holds for some \( s \in \mathbb{N}_0 \), \( q \in (1, \infty) \), \( r \in (0, \infty) \) and \( \varepsilon > 0 \), then also \( D (Q, L^p, Y) \hookrightarrow W^{s,q}_\ast (\mathbb{R}^d) \).

In particular, since the Besov spaces are decomposition spaces, i.e. \( B^{p,q}_s (\mathbb{R}^d) = D (P, L^p, \ell^d_{u(\cdot)}) \) for a certain dyadic covering \( P = (P_j)_{j \in \mathbb{N}_0} \) of \( \mathbb{R}^d \) (cf. example 7.2), many of the results for embeddings of the form
\[ D (Q, L^p, \ell^d_{u(\cdot)}) \hookrightarrow D (P, \ell^d_{u(\cdot)}) = B^{p,q}_{s} (\mathbb{R}^d) \]
from [21] can be used to establish embeddings between decomposition spaces and Sobolev spaces. In the following, we will summarize arguments of this type under the term “Besov detour”, since to establish necessary/sufficient conditions for embedding into Sobolev spaces, we are making a “detour” through Besov spaces.

In comparison to the results in this paper, the “Besov detour” differs as follows:
- The identity \( F^{k-2}_s (\mathbb{R}^d) = W^{k,q}_\ast (\mathbb{R}^d) \) can fail for \( q \notin (1, \infty) \), so that the “Besov detour” is not applicable.
- As seen above, the “detour” via Besov spaces can yield sufficient criteria and necessary criteria for existence of the embedding \( D (Q, L^p, Y) \hookrightarrow W^{k,q}_\ast (\mathbb{R}^d) \), but these criteria will not be sharp in general. For example, usage of equation (1.5) causes a loss of an (arbitrary) \( \varepsilon > 0 \) in the smoothness parameter \( s \). Thus, embedding results obtained in this way will – at least for \( q \in (0,2] \cup \{ \infty \} \) – be inferior to those obtained using the results in the present paper. For \( q \in (2, \infty) \), however, we will see some examples where the “Besov detour” yields better results than those from the present paper.
- For the results from [21] to be applicable, the covering \( Q \) has to fulfill certain geometric properties. Roughly speaking, \( Q \) has to be finer than the dyadic covering \( P \) (or vice versa). More precisely, \( Q \) has to be almost subordinate to \( P \) or vice versa, see [21] Definition 3.3.1. But this is not fulfilled in all cases: For example, the covering \( Q_{H(c)} \) – which is induced by the shearlet-type group \( H(c) \) from equation (1.2) – is not almost subordinate to the dyadic covering \( P \) for \( c \in \mathbb{R} \setminus [0,1] \). Since \( P \) is not almost subordinate to \( Q_{H(c)} \) for this range of \( c \), the embedding results from [21] are not applicable in this case.
- In contrast to the results from [21], in the present paper, no “compatibility” between the covering \( Q \) and the dyadic covering \( P \) is required.
- Even if \( Q = (Q_i)_{i \in I} \) is almost subordinate to the dyadic covering \( P \), for the verification of the criteria given in [21], one first has to compute the so-called intersection sets
\[ I_j = \{ i \in I \mid Q_i \cap P_j \neq \emptyset \} \]
for each \( j \in \mathbb{N}_0 \). Then, one has to check finiteness of an expression of the form
\[ \left\| \left\| (u_i)_{i \in I_j} \right\|_{\ell^r (I_j)} \right\|_{\ell^t (\mathbb{N}_0)} \]
for certain \( r, t \in (0, \infty] \) and a certain weight \( (w_i)_{i \in I} \).

In comparison, the criteria developed in the present paper will be much more convenient to verify.
1.3. Structure of the paper. We begin our exposition in Section 2 by clarifying the assumptions on the covering $\mathcal{Q}$ and reviewing the definitions of decomposition spaces and Sobolev spaces. In particular, we formally introduce the class of regular coverings which was already used in Theorem 1.1. We remark that for the Quasi-Banach regime $q \in (0, 1)$, it seems that there is no general consensus on how the Sobolev spaces $W^{k,q}(\mathbb{R}^d)$ (for $k \geq 1$) should be defined. Thus, readers interested in this case should pay close attention to the definition of these spaces which we adopt.

The development of criteria for embeddings of decomposition spaces into Sobolev spaces begins in Section 3, where we show that the embedding $Y \hookrightarrow \ell^q_k(\mathbb{R}^d)$ indeed suffices for the existence of such an embedding. Necessity of the (slightly different) embedding $Y \cap \ell^q_0(\mathbb{R}^d) \hookrightarrow \ell^q_k(\mathbb{R}^d)$ is established in Section 4.

In view of these criteria, it is desirable to have a painless way of deciding whether an embedding of the form $Y \hookrightarrow \ell^q_k(\mathbb{R}^d)$ is actually true. For the case of a weighted Lebesgue sequence space $Y = \ell^q_k(\mathbb{R}^d)$, this problem is solved completely in Section 5, where we show (cf. also equation (1.3) above) that all one has to check is finiteness of a certain $\ell^d_k$-norm of a single sequence.

We complete our abstract results in Section 6, where we show that a decomposition space embeds into a $BV$ space if and only if it embeds into $W^{1,1}(\mathbb{R}^d)$. Thus, our previous criteria yield a complete characterization of when this is true.

Finally, we illustrate our results by considering embeddings into Sobolev spaces of several classes of decomposition spaces, namely of (homogeneous and inhomogeneous) Besov spaces, $(\alpha)$-modulation spaces, Shearlet-type coorbit spaces and coorbit spaces of the diagonal group.

1.4. Notation. In this paper, we use the convention

$$ Ff(\xi) := \hat{f}(\xi) := \int_{\mathbb{R}^d} f(x) \cdot e^{-2\pi i (x, \xi)} \, dx $$

for the Fourier transform of a function $f \in L^1(\mathbb{R}^d)$. As is well known (see [8, Theorem 8.29]), with this normalization, the Fourier transform extends to a unitary automorphism of $L^2(\mathbb{R}^d)$, where the inverse is the unique extension to $L^2(\mathbb{R}^d)$ of the inverse Fourier transform given by

$$ F^{-1}f(x) := f^\vee(x) = \hat{f}(-x) $$

for $f \in L^1(\mathbb{R}^d)$.

For $n \in \mathbb{N}_0$, we write $\mathfrak{n} := \{ k \in \mathbb{N} \mid k \leq n \}$. In particular, $\mathfrak{n} = \emptyset$. We denote the usual standard basis of $\mathbb{R}^d$ by $e_1, \ldots, e_d$. For a matrix $A \in \mathbb{R}^{d \times d}$, we write

$$ \|A\| := \max_{|x|=1} \|Ax\|, $$

where (as in the remainder of the paper), we write $|x|$ for the usual euclidean norm of a vector $x \in \mathbb{R}^d$.

For an integrability exponent $p \in (0, \infty]$, we define its conjugate exponent $p' \in [1, \infty]$ by

$$ p' := \begin{cases} \frac{p}{p' - 1}, & \text{if } p \in [1, \infty), \vspace{1mm} \\
\infty, & \text{if } p \in (0, 1), \end{cases} $$

where for $p \in [1, \infty]$, $p'$ satisfies $\frac{1}{p} + \frac{1}{p'} = 1$. Furthermore, we define the lower conjugate exponent of $p$ by

$$ p^\# := \min \{ p, p' \}. $$

For a function $f : \mathbb{R}^d \to \mathbb{C}$ and $x, \omega \in \mathbb{R}^d$, we define

$$ \|f\|_{\sup} = \sup_{y \in \mathbb{R}^d} |f(y)| $$

and

$$ L_x f : \mathbb{R}^d \to \mathbb{C}, y \mapsto f(y - x), $$

$$ M_x f : \mathbb{R}^d \to \mathbb{C}, y \mapsto e^{2\pi i \langle \omega, y \rangle} \cdot f(y). $$

We denote by $\mathcal{S}(\mathbb{R}^d)$ the space of Schwartz functions on $\mathbb{R}^d$. Its topological dual space $\mathcal{S}'(\mathbb{R}^d)$ is the space of tempered distributions on $\mathbb{R}^d$. Furthermore, we denote by $\mathcal{D}(\mathcal{O}) := C_c^\infty(\mathcal{O})$ the space of $C^\infty$ functions $f : \mathcal{O} \to \mathbb{C}$ with compact support in the open set $\emptyset \neq \mathcal{O} \subset \mathbb{R}^d$. With a suitable topology (cf. [10, Definition 6.3]), this space becomes a locally convex topological vector space such that its topological dual coincides with the space of distributions $\mathcal{D}'(\mathcal{O})$ on $\mathcal{O}$ as defined (without introducing the topology just
mentioned) for example in [8, Section 9.1]. We generally equip \( D'(\mathcal{O}) \) with the weak-\(^*\)-topology, i.e. with the topology of pointwise convergence on \( D(\mathcal{O}) \).

Finally, if \( I \neq \emptyset \) is an index set, \( q \in (0, \infty) \) and if \( u = (u_i)_{i \in I} \) with \( u_i > 0 \) for all \( i \in I \) is a weight on \( I \), we define the \textit{weighted Lebesgue sequence space} \( \ell^q_u(I) \) as
\[
\ell^q_u(I) := \{(x_i)_{i \in I} \in \mathbb{C}^I \mid (u_i x_i)_{i \in I} \in \ell^q(I)\},
\]
with \( \|(x_i)_{i \in I}\|_{\ell^q_u} = \|(u_i x_i)_{i \in I}\|_{\ell^q} \).

2. Decomposition Spaces and Sobolev Spaces

In this section, we introduce the notion of decomposition spaces and Sobolev spaces that we will use. In Subsection 2.1, we begin by explaining our convention regarding the Sobolev spaces \( W^{k,q}(\mathbb{R}^d) \). For \( q \in [1, \infty] \), this definition is entirely standard, but for \( q \in (0, 1) \), the situation changes dramatically. Readers only interested in the case \( q \in [1, \infty) \) can safely skip this first subsection if they are familiar with the usual definition of Sobolev spaces.

As we will see in Subsection 2.2, in order to obtain well-defined decomposition spaces \( D(Q, L^p, Y) \), we have to impose certain assumptions on the covering \( Q \). These different assumptions are discussed in detail in Subsection 2.2. In particular, we introduce the new notion of \textit{regular coverings} with which we will mainly work in the remainder of the paper.

As mentioned above, in Subsection 2.2, we recall the definition of the decomposition space \( D(Q, L^p, Y) \). Our definition – which is based on [21] and [22] – is slightly different from the usual one, e.g. as in [3]. The main difference is that we use a reservoir different from the space \( \mathcal{S}'(\mathbb{R}^d) \) to define our decomposition spaces. The reason for this is twofold: First, we want to allow coverings \( Q \) which cover a proper subset \( \mathcal{O} \subset \mathbb{R}^d \) and second, with the usual definition, it can happen that the resulting decomposition space is not complete.

In the final subsection, we recall from [21] [22] and [19] some results concerning convolution in the Quasi-Banach regime \( q \in (0, 1) \) which we will need. In a nutshell, the problem is that Young’s convolution relation \( L^1 \ast L^q \hookrightarrow L^q \) fails completely for \( q \in (0, 1) \). Instead, we get an estimate of the form
\[
\|f \ast g\|_{L^q} \leq C \cdot \|f\|_{L^p} \cdot \|g\|_{L^q},
\]
but only under the additional assumption that the Fourier supports \( \hat{f} \subset Q_1 \) and \( \hat{g} \subset Q_2 \) are compact. Furthermore, the constant \( C \) will depend in a nontrivial way on the sets \( Q_1, Q_2 \). Again, readers who are only interested in the case \( q \in [1, \infty] \) may safely skip this subsection, apart from Corollary 2.14. Note that for \( p \in [1, \infty] \), the proof of this Corollary is independent of the rest of Subsection 2.4.

2.1. The Sobolev spaces \( W^{k,q}(\mathbb{R}^d) \). The definition of the Sobolev spaces \( W^{k,q}(\mathbb{R}^d) \) for \( q \in [1, \infty] \) is entirely standard, i.e. we define
\[
W^{k,q}(\mathbb{R}^d) := \{ f \in L^q(\mathbb{R}^d) \mid \partial^\alpha f \in L^q(\mathbb{R}^d) \text{ for all } \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \leq k \}.
\]
Here, the partial derivative \( \partial^\alpha f \) denotes the distributional derivative of \( f \), which is well-defined, since every \( f \in L^q(\mathbb{R}^d) \subset D'(\mathbb{R}^d) \) defines a distribution – in fact even a tempered distribution. Note that we crucially use \( q \in [1, \infty] \) for the inclusion \( L^q(\mathbb{R}^d) \subset D'(\mathbb{R}^d) \) to be true. To be sure, \( \partial^\alpha f \in L^q(\mathbb{R}^d) \) means that there is a (uniquely determined) function \( f_\alpha \in L^q(\mathbb{R}^d) \) such that
\[
(-1)^{|\alpha|} \int_{\mathbb{R}^d} f(x) \cdot \partial^\alpha \varphi(x) \, dx = (-1)^{|\alpha|} \langle f, \partial^\alpha \varphi \rangle = \langle \partial^\alpha f, \varphi \rangle = \int_{\mathbb{R}^d} f_\alpha(x) \cdot \varphi(x) \, dx
\]
holds for all \( \varphi \in C_0^\infty(\mathbb{R}^d) \). In this case, we simply write \( \partial^\alpha f \) instead of \( f_\alpha \).

Finally, we equip \( W^{k,q}(\mathbb{R}^d) \) with the norm
\[
\|f\|_{W^{k,q}} := \sum_{\alpha \in \mathbb{N}_0^d \mid |\alpha| \leq k} \|\partial^\alpha f\|_{L^q},
\]
which makes it a Banach space. Note that since the weak partial derivatives \( \partial^\alpha f \) are uniquely determined by \( f \), the inclusion map
\[
\iota : W^{k,q}(\mathbb{R}^d) \rightarrow L^q(\mathbb{R}^d), \, f \mapsto f
\]
is injective. In case of \( q \in (0,1) \), we can not proceed as above, since in this case \( L^q (\mathbb{R}^d) \not\subset L^1_{\text{loc}} (\mathbb{R}^d) \), so that a function \( f \in L^q (\mathbb{R}^d) \) does not define a distribution in general. Hence, we cannot define the (weak) derivative using distribution theory. Instead, we proceed as follows: For \( q \in (0,1) \), we define \( W^{k,q} (\mathbb{R}^d) \) as the closure of

\[
W^{k,q}_* (\mathbb{R}^d) := \left\{ (\partial^\alpha f)_{\alpha \in \mathbb{N}^d_0, |\alpha| \leq k} \mid f \in C^\infty (\mathbb{R}^d) \text{ with } \partial^\alpha f \in L^q (\mathbb{R}^d) \text{ for all } \alpha \in \mathbb{N}^d_0 \text{ with } |\alpha| \leq k \right\}
\]

in the product \( \prod_{\alpha \in \mathbb{N}^d_0, |\alpha| \leq k} L^q (\mathbb{R}^d) \). Here, \( \partial^\alpha f \) denotes the classical derivative of \( f \in C^\infty (\mathbb{R}^d) \). Note that if we were to adopt the same definition of \( W^{k,q} (\mathbb{R}^d) \) also for \( q \in [1, \infty) \), we would obtain the same spaces as defined above (up to obvious identifications).

We finally remark that the space \( W^{k,q} (\mathbb{R}^d) \) for \( q \in (0,1) \) – as defined above – behaves quite pathologically in some respects. Indeed, in [14] and in the related paper [23], it is shown that the “inclusion” map

\[
W^{k,q}_* (\mathbb{R}^d) \to L^q (\mathbb{R}^d), (f_\alpha)_{\alpha \in \mathbb{N}^d_0, |\alpha| \leq k} \mapsto f_0
\]

is not injective for \( k \geq 1 \). Furthermore, the dual space of \( W^{k,q} (\mathbb{R}^d) \) is trivial. There are other (nonequivalent) conventions for defining \( W^{k,q} (\mathbb{R}^d) \) for \( q \in (0,1) \), but the present formulation will turn out to be most convenient for the results in this paper.

In any case, for \( k = 0 \), we have \( W^{k,q} (\mathbb{R}^d) = L^q (\mathbb{R}^d) \) for all \( q \in (0, \infty) \). For \( q \in [1, \infty) \), this is clear and for \( q \in (0,1) \), we use that the set of simple functions of the form \( \sum_{i=1}^n \alpha_i \chi_{A_i} \) with measurable, bounded sets \( A_i \subset \mathbb{R}^d \) is dense in \( L^q (\mathbb{R}^d) \). Now, for any indicator function \( \chi_A \) with measurable bounded \( A \subset \mathbb{R}^d \), we can find (e.g. by density of \( C^\infty_c (\mathbb{R}^d) \) in \( L^1 (\mathbb{R}^d) \)) a sequence \( (f_n)_{n \in \mathbb{N}} \in C^\infty_c (\mathbb{R}^d) \) with \( f_n (x) \to \chi_A (x) \) almost everywhere and such that \(-10^{-n} \chi_B \leq f_n \leq 10^{-n} \cdot \chi_B \) for all \( n \in \mathbb{N} \) and some fixed bounded set \( B \). Using the dominated convergence theorem, this implies \( f_n \to \chi_A \) in \( L^q (\mathbb{R}^d) \).

We had to use this somewhat involved argument, since Young’s convolution relation \( L^1 \ast L^q \to L^2 \) fails for \( q \in (0,1) \). Hence, approximating \( L^q \) functions by convolution with an approximate identity fails for \( q \in (0,1) \). A more detailed discussion of the failure of Young’s inequality for \( q \in (0,1) \) will be given in Subsection 2.4.

2.2. Structured, semi-structured and regular coverings. In this subsection, we introduce several different classes of coverings. All our coverings will always be of the form

\[
Q = (Q_i)_{i \in I} = (T_i Q_i' + b_i)_{i \in I}
\]

for suitable subsets \( Q_i' \subset \mathbb{R}^d \), invertible matrices \( T_i \in \text{GL} (\mathbb{R}^d) \) and shifts \( b_i \in \mathbb{R}^d \). Furthermore, we always assume that the set \( Q = \bigcup_{i \in I} Q_i \subset \mathbb{R}^d \) is fixed, i.e. \( Q \) will always be a covering of the set \( Q \). For the sake of brevity, we will not repeat these assumptions every time.

It is most important to keep in mind that the covering \( Q \) is a covering of (the subset \( Q \) of) the frequency domain \( \mathbb{R}^d \), and not of the space domain \( \mathbb{R}^d \).

The type of covering which is easiest to understand is that of a structured admissible covering, essentially as introduced by Borup and Nielsen in [3]. The only difference between their definition and ours is that we allow coverings of proper subsets \( Q \subsetneq \mathbb{R}^d \), whereas Borup and Nielsen only consider coverings of the whole frequency space \( \mathbb{R}^d \).

**Definition 2.1.** The covering \( Q \) is called an admissible covering of \( Q \) if \( Q_i \neq \emptyset \) for all \( i \in I \) and if the constant

\[
N_Q := \sup_{i \in I} |i^*|
\]

is finite, where

\[
i^* := \{ j \in I \mid Q_i \cap Q_j \neq \emptyset \}
\]

denotes the set of \( Q \)-neighbors of the index \( i \in I \).

An admissible covering \( Q \) is called a structured admissible covering of \( Q \), if the following hold:

1. We have \( Q_i = Q \) for all \( i \in I \), where \( Q \subset \mathbb{R}^d \) is a fixed open, bounded set.
2. There is a an open set \( P \subset Q \) with \( P \subset Q \) and with

\[
\bigcup_{i \in I} (T_i P + b_i) = Q.
\]
3. We have

\[
C_Q := \sup_{i \in I} \sup_{j \in i^*} \| T_i^{-1} T_j \| < \infty.
\]

\[\text{(2.1)}\]
Remark. The following notation related to the set of \( Q \)-neighbors will be frequently convenient: For \( M \subset I \), we define
\[
M^* := \bigcup_{\ell \in M} \ell^* \subset I.
\]
Now, we inductively define \( M^{0*} := M \) and \( M^{(n+1)*} := (M^n)^* \) for \( n \in \mathbb{N}_0 \). Finally, we set \( i^{n*} := \{i\}^{n*} \) and
\[
Q_i^{n*} := \bigcup_{\ell \in i^{n*}} Q_\ell
\]
for \( i \in I \) and \( n \in \mathbb{N}_0 \).

Now, in words, admissibility of a covering means that the number of neighbors of a set \( Q_i \) of the covering \( Q \) is uniformly bounded. For a structured admissible covering, we additionally assume all sets \( Q_i \) to be “of a similar shape/form”, in the sense that every set \( Q_i \) is of the form \( Q_i = T_i Q + b_i \) for a fixed set \( Q \subset \mathbb{R}^d \). Furthermore, we assume that we can shrink the set \( Q \) slightly, while still covering all of \( O \). This assumption— together with the technical condition \( C_Q < \infty \)—ensures existence of suitable partitions of unity subordinate to \( Q \), see Theorem \ref{thm:existence} below.

In some cases, the notion of a structured admissible covering turns out to be too restrictive. Thus, in Definition \ref{def:semi-structured}, I introduced the notion of a semi-structured admissible covering for which the assumption \( Q_i' = Q \) for all \( i \in I \) is dropped:

Definition 2.2. The covering \( Q \) is called a semi-structured covering of \( O \) if the following conditions hold:

1. \( Q \) is admissible,
2. The set \( \bigcup_{i \in I} Q_i' \subset \mathbb{R}^d \) is bounded,
3. The constant \( C_Q \) as defined in equation (2.1) is finite.

Finally, we say that the semi-structured covering \( Q \) is tight if we additionally have the following:

4. There is some \( \varepsilon > 0 \) such that for each \( i \in I \), there is some \( c_i \in \mathbb{R}^d \) with \( B_\varepsilon (c_i) \subset Q_i' \).

Remark. Note that every structured admissible covering is a tight semi-structured admissible covering.

As noted above, the definition of a structured admissible covering ensures existence of certain partitions of unity subordinate to the covering. The special type of partitions of unity which we introduce now will turn out to be suitable for defining decomposition spaces, cf. Subsection 2.3.

Definition 2.3. (cf. \cite{3} Definition 2)
Let \( Q \) be a semi-structured admissible covering of \( O \). We say that \( \Phi = (\varphi_i)_{i \in I} \) is a partition of unity subordinate to \( Q \) if the following hold:

1. \( \varphi_i \in C_c^\infty (O) \) for all \( i \in I \),
2. \( \varphi_i \equiv 0 \) on \( \mathbb{R}^d \setminus Q_i \) for all \( i \in I \),
3. \( \sum_{i \in I} \varphi_i \equiv 1 \) on \( O \).

Furthermore, for \( p \in [1, \infty] \), we say that \( \Phi \) is an \( L^p \)-BAPU (bounded admissible partition of unity) for \( Q \), if \( \Phi \) is a partition of unity subordinate to \( O \) for which the constant
\[
C_{\Phi,p} := \sup_{i \in I} \| F^{-1} \varphi_i \|_{L^1}
\]
is finite. For \( p \in (0, 1) \), we instead require finiteness of
\[
C_{\Phi,p} := \sup_{i \in I} \| \text{det} T_i \|_{\frac{1}{2}-1} \cdot \| F^{-1} \varphi_i \|_{L^p}.
\]

Finally, we say that \( Q \) is an \( L^p \)-decomposition covering if there is an \( L^p \)-BAPU for \( Q \).

Remark. The term “\( L^p \)-BAPU” does not refer to the fact that the \( L^p \) norm of \( \varphi_i \) or \( F^{-1} \varphi_i \) is uniformly bounded. Instead, the point is that the \( (\varphi_i)_{i \in I} \) define a uniformly bounded family of \( L^p \) Fourier multipliers. For \( p \in [1, \infty] \), this is a direct consequence of Young’s inequality, whereas for \( p \in (0, 1) \), this statement needs to be taken with a grain of salt, as explained in Subsection 2.4.

While existence of an \( L^p \)-BAPU is sufficient for obtaining well-defined decomposition spaces (cf. Subsection 2.3), we will need to impose more restrictive conditions on the partition of unity \( \Phi \) in order to establish embeddings into Sobolev spaces. Our next definition explains exactly which properties \( \Phi \) needs to have.
Definition 2.4. Let $Q$ be a semi-structured covering of $\mathcal{O}$ and let $\Phi = (\varphi_i)_{i \in I}$ be a partition of unity subordinate to $Q$. For $i \in I$, define the normalized version of $\varphi_i$ by
\[
\varphi_i^# : \mathbb{R}^d \to \mathbb{C}, \xi \mapsto \varphi_i(T_i\xi + b_i).
\]

We say that $\Phi$ is a regular partition of unity subordinate to $Q$ if $\varphi_i \in C_c^\infty(Q_i)$ for all $i \in I$ and if additionally $\gamma_i^# : \mathbb{R}^d \to \mathbb{C}, \xi \mapsto \gamma_i(T_i\xi + b_i)$ satisfies
\[
C_{\Phi, \alpha} := \sup_{i \in I} \left\| \partial^\alpha \varphi_i^# \right\|_{\text{sup}} < \infty
\]
is finite for all $\alpha \in \mathbb{N}_0^d$.

The covering $Q$ is called a regular covering of $\mathcal{O}$ if there exists a regular partition of unity $\Phi$ subordinate to $Q$. ▶

We will now show that every regular partition of unity is also an $L^p$-BAPU for all $p \in (0, \infty]$. In fact, we establish a slightly stronger claim.

Lemma 2.5. Let $Q$ be a semi-structured covering of $\mathcal{O} \subset \mathbb{R}^d$ and let $(\gamma_i)_{i \in I}$ be a family in $C_c^\infty(\mathcal{O})$ with $\gamma_i \equiv 0$ on $\mathcal{O} \setminus Q_i$ for all $i \in I$ and so that the normalized family $(\gamma_i^#)_{i \in I}$ given by
\[
\gamma_i^# : \mathbb{R}^d \to \mathbb{C}, \xi \mapsto \gamma_i(T_i\xi + b_i)
\]
satisfies
\[
C_{\alpha} := \sup_{i \in I} \left\| \partial^\alpha \gamma_i^# \right\|_{\text{sup}} < \infty \tag{2.2}
\]
for all $\alpha \in \mathbb{N}_0^d$.

Then we have
\[
\left\| \partial^\alpha \left[ F^{-1} \gamma_i \right] \right\|_{L^p} \leq K_{\alpha} \cdot \left| \det T_i \right|^{1 - \frac{d}{p}} \cdot \left( \left\| T_i \right\|^{\left| \alpha \right|} + \left| b_i \right|^{\left| \alpha \right|} \right)
\]
for all $\alpha \in \mathbb{N}_0^d$, $p \in (0, \infty]$ and $i \in I$, where the constant $K_{\alpha} = K_{\alpha}(d, p, Q, (\gamma_i)_{i \in I})$ is independent of $i \in I$. ▶

Remark. Actually, the proof establishes the stronger estimate
\[
\left| (\partial^\alpha \left[ F^{-1} \gamma_i \right] (x)) \right| \leq C_{\alpha, N} \cdot \left| \det T_i \right| \left( 1 + \left| T_i^T x \right| \right)^{-N} \cdot \left( \left| b_i \right|^{\left| \alpha \right|} + \left| T_i \right|^{\left| \alpha \right|} \right)
\]
for all $x \in \mathbb{R}^d$ for all $i \in I$ and arbitrary $N \in \mathbb{N}$ for some constant $C_{\alpha, N} = C_{\alpha, N}(Q, (\gamma_i)_{i \in I})$ which is independent of $i \in I$.

Proof. For brevity, we define $g_i := F^{-1} \gamma_i^#$ and $h_i := F^{-1} \gamma_i$ for $i \in I$. We begin with showing that for every $N \in \mathbb{N}$ and $\gamma \in \mathbb{N}_0^d$, there is a constant $K_{\gamma, N} = K_{\gamma, N}(Q, (\gamma_i)_{i \in I}) > 0$ with
\[
\left| \partial^\gamma g_i (x) \right| \leq K_{\gamma, N} \cdot (1 + |x|)^{-N}
\]
for all $x \in \mathbb{R}^d$ and $i \in I$. \hfill \bullet

Here it is crucial that the constant $K_{\gamma, N}$ is independent of $i \in I$ and $x \in \mathbb{R}^d$. A high-level proof of this estimate is as follows: Since $\sup \gamma_i \subset Q_i = T_iQ'_i + b_i$, we get the uniform inclusion $\sup \gamma_i^# \subset Q'_i \subset B_R(0)$ for some fixed $R = R(Q) > 0$ and all $i \in I$. In conjunction with the prerequisite from equation (2.2), we conclude that the family $(\gamma_i^#)_{i \in I}$ is bounded with respect to each of the norms
\[
\vartheta_N (f) := \max_{\alpha \in \mathbb{N}_0^d} \sup_{x \in \mathbb{R}^d} \left[ \left( 1 + |x| \right)^N \cdot \left| \partial^\alpha f (x) \right| \right].
\]
Thus, $\{ \gamma_i^# | i \in I \} \subset \mathcal{S} (\mathbb{R}^d)$ is a bounded subset of the topological vector space $\mathcal{S} (\mathbb{R}^d)$ (see [15] Section 1.6] for the relevant definition and also [15] Theorem 1.37 for the equivalent characterization which we use here). Since $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is a homeomorphism, we conclude that
\[
\{ g_i | i \in I \} = \{ F^{-1} \gamma_i^# | i \in I \} \subset \mathcal{S} (\mathbb{R}^d) \leftrightarrow L^p (\mathbb{R}^d)
\]
is also bounded for each $p \in (0, \infty]$. In the next two paragraphs (about 1 + $\frac{1}{2}$ pages), we provide a more direct proof of estimate (2.3). Readers who are willing to take this estimate on faith or who are satisfied with the preceding abstract argument should thus skip these two paragraphs.
For the explicit proof of equation (2.3), recall that standard properties of the Fourier transform (cf. e.g. [8] Theorem 8.22) yield for each multiindex $\kappa \in \mathbb{N}_0^d$ that

$$
x^\kappa \cdot \partial^\gamma g_i (x) = x^\kappa \cdot \partial^\gamma \left[ \mathcal{F}^{-1} \gamma_i^\# \right] (x)
= x^\kappa \cdot \mathcal{F}^{-1} \left[ \xi \mapsto (2\pi j \xi)^\gamma \cdot \gamma_i^\# (\xi) \right] (x)
= \left( \frac{j}{2\pi} \right)^\kappa \cdot \mathcal{F}^{-1} \left[ \xi \mapsto \partial^\gamma_\xi \left( (2\pi j \xi)^\gamma \cdot \gamma_i^\# (\xi) \right) \right] (x)
.$$ 

Here, we used the notation $j$ instead of $i$ for the imaginary unit to avoid possible confusion with the index $i \in I$. Since $\mathcal{F}^{-1} : L^1 (\mathbb{R}^d) \to C_0 (\mathbb{R}^d)$ is bounded, we conclude

$$
\| x^\kappa \cdot \partial^\gamma g_i \|_{L^1 (\mathbb{R}^d)} \leq \left\| \xi \mapsto \partial^\gamma_\xi \left[ (2\pi j \xi)^\gamma \cdot \gamma_i^\# (\xi) \right] \right\|_{L^1 (\mathbb{R}^d)}
(by \ Leibniz’\ formula) = \left\| \xi \mapsto \sum_{\lambda \in \mathbb{N}_0^d, \lambda \leq \kappa} \binom{\kappa}{\lambda} \cdot \left[ \partial^\gamma_\xi \left( (2\pi j \xi)^\gamma \cdot \left( \partial^{\kappa-\lambda} \gamma_i^\# \right) (\xi) \right) \right] \right\|_{L^1 (\mathbb{R}^d)}.
$$

Now, note that we have $\gamma_i (\xi) = 0$ for all $\xi \in \mathbb{R}^d \setminus Q_i = \mathbb{R}^d \setminus [T_i Q_i' + b_i]$ and hence

$$
\gamma_i^\# (\xi) = \gamma_i (T_i \xi + b_i) = 0 \text{ for all } \xi \in \mathbb{R}^d \setminus Q_i.
$$

Here, we implicitly used that $T_i \in \text{GL} (\mathbb{R}^d)$ is invertible. Since $Q$ is a semi-structured covering, there is some $R \geq 1$ with $Q_i' \subset B_R (0)$ for all $i \in I$ and we conclude $\text{supp} \gamma_i^\# \subset \overline{B_R (0)}$ and thus also $\text{supp} \partial^\gamma \gamma_i^\# \subset \overline{B_R (0)}$ for all $i \in I$. Now, observe

$$
| \partial^\gamma_\xi (2\pi j \xi)^\gamma | = |(2\pi j)^\gamma \cdot \partial^\gamma_\xi \gamma | = \begin{cases} K_{\gamma, \lambda} \cdot | \xi^{\gamma-\lambda} | \leq K_{\gamma, \lambda} R^{\gamma-\lambda} \leq K_{\gamma} \cdot R^{\gamma} , & \text{if } \lambda \leq \gamma, \\
0, & \text{else} \end{cases}
$$

for $|\xi| \leq R$. Here, $K_{\gamma} > 0$ is an absolute constant only depending on $\gamma \in \mathbb{N}_0^d$. Altogether, we conclude

$$
\| x^\kappa \cdot \partial^\gamma g_i \|_{L^1 (\mathbb{R}^d)} \leq \left\| \xi \mapsto \sum_{\lambda \in \mathbb{N}_0^d, \lambda \leq \kappa} \binom{\kappa}{\lambda} \cdot \left[ \partial^\gamma_\xi \left( (2\pi j \xi)^\gamma \cdot \left( \partial^{\kappa-\lambda} \gamma_i^\# \right) (\xi) \right) \right] \right\|_{L^1 (\mathbb{R}^d)}
\leq K_{\gamma} \cdot R^{\gamma} \cdot \sum_{\lambda \in \mathbb{N}_0^d, \lambda \leq \kappa} \binom{\kappa}{\lambda} \cdot \left\| \partial^{\kappa-\lambda} \gamma_i^\# \right\|_{L^1 (\overline{B_R (0)})}
\leq K_{\gamma} \cdot R^{\gamma} \cdot \sum_{\lambda \in \mathbb{N}_0^d, \lambda \leq \kappa \text{ and } \lambda \leq \gamma} \binom{\kappa}{\lambda} \cdot \left( B_R (0) \right) \cdot \left\| \partial^{\kappa-\lambda} \gamma_i^\# \right\|_{L^1 (\overline{B_R (0)})}
\leq K_{\gamma} \cdot R^{\gamma} \cdot \sum_{\lambda \in \mathbb{N}_0^d, \lambda \leq \kappa \text{ and } \lambda \leq \gamma} \binom{\kappa}{\lambda} \cdot C_{\kappa-\lambda}
\leq K_{\gamma, \kappa, R},
$$

10
where the constant $K_{\gamma, \kappa, R} = K (\gamma, \kappa, Q, (\gamma_i)_{i \in I}) > 0$ is independent of $i \in I$. But because of

$$
(1 + |x|)^N \leq (1 + d \|x\|_{\infty})^N
$$

(with suitable $\ell = \ell (x) \in D$) $d N \cdot (1 + |x\ell|)^N$

$$
= d N \cdot \sum_{m=0}^{N} \binom{N}{m} |x\ell|^m
$$

$$
= d N \cdot \sum_{m=0}^{N} \binom{N}{m} |x^{\ell_m}| |
$$

$$
\leq d N \cdot \sum_{m=0}^{N} \binom{N}{m} \sum_{\ell=1}^{d} |x^{\ell_m}| ,
$$

this finally implies

$$
\left\| (1 + |x|)^N \cdot \partial^n \varphi \right\|_{L^\infty} \leq d N \cdot \sum_{m=0}^{N} \binom{N}{m} \sum_{\ell=1}^{d} \left\| x^{\ell_m} \cdot \partial^n \varphi \right\|_{L^\infty}
$$

$$
\leq d N \cdot \sum_{m=0}^{N} \binom{N}{m} \sum_{\ell=1}^{d} K_{\gamma, \ell_m, R}
$$

$$
=: K_{\gamma, N}
$$

with $K_{\gamma, N} = K (\gamma, N, Q, (\gamma_i)_{i \in I})$. This completes the explicit proof of equation (2.3).

Now, note

$$
\gamma_i (\xi) = \gamma_i^# (T_i^{-1} (\xi - b_i)) = \left[ L_{b_i} (\gamma_i^# \circ T_i^{-1}) \right] (\xi)
$$

for all $\xi \in \mathbb{R}^d$, where $L_{b} f (y) = f (y - x)$ denotes the left-translation of $f$ by $x$.

By standard properties of the Fourier transform (cf. e.g. [8, Theorem 8.22]), this implies

$$
\theta_i = \mathcal{F}^{-1} \gamma_i = M_{b_i} \mathcal{F}^{-1} \left( \gamma_i^# \circ T_i^{-1} \right)
$$

$$
= |\det T_i| \cdot M_{b_i} \left[ \mathcal{F}^{-1} \gamma_i^# \circ T_i^{-1} \right]
$$

$$
= |\det T_i| \cdot M_{b_i} \gamma_i (T_i^T)
$$

(2.4)

where we recall $\theta_i = \mathcal{F}^{-1} \gamma_i^#$ and where $M_{b} f (y) = e^{2\pi j \langle b, y \rangle}$, $f (y)$ denotes the modulation of $f$ by $b$. Here, we used as above the notation $j$ for the imaginary unit. Using Leibniz’s formula, we derive

$$
\partial^\alpha \theta_i = |\det T_i| \cdot \partial^\alpha \left( e^{2\pi j \langle b_i, \cdot \rangle} \cdot [\theta_i \circ T_i^T] \right)
$$

$$
= |\det T_i| \cdot \sum_{\beta \leq \alpha} \left( \frac{\alpha}{\beta} \right) (2\pi j \cdot b_i)^{\alpha - \beta} e^{2\pi j \langle b_i, \cdot \rangle} \cdot \partial^\beta [\theta_i \circ T_i^T]
$$

(2.5)

We will now estimate each of the summands individually.

To this end, fix $\beta \leq \alpha$. We first consider the case $k := |\beta| > 0$. In this case, there are $i_1, \ldots, i_k \in D$ with $|\beta| = \sum_{\ell=1}^{k} \epsilon_{i_\ell}$, so that formula (2.7) from Lemma 2.9 below yields

$$
|\partial^\beta [\theta_i \circ T_i^T]| = \left| \sum_{\ell_1, \ldots, \ell_k \in \mathbb{Z}} (T_i^T)_{\ell_1, i_1} \cdots (T_i^T)_{\ell_k, i_k} \cdot (\partial_{\ell_1} \cdots \partial_{\ell_k} \theta_i) (T_i^T) \right|
$$

$$
\leq \sum_{\ell_1, \ldots, \ell_k \in \mathbb{Z}} \left| (T_i)_{i_1, \ell_1} \cdots (T_i)_{i_k, \ell_k} \right| \cdot |(\partial_{\ell_1} \cdots \partial_{\ell_k} \theta_i) (T_i^T)|
$$

(by eq. (2.7)) $\leq \|T_i\|^k \cdot C_{\ell, N} \cdot (1 + |T_i^T|)^{-N}$

$$
\leq C_{|\beta|, N} \cdot \|T_i\|^{|\beta|} \cdot (1 + |T_i^T|)^{-N}
$$

(2.6)
for some constant $C'_{|\beta|, N} = C'_{|\beta|, N} \left( \mathcal{Q}, (\gamma_i)_{i \in I} \right)$ which is independent of $i \in I$. In case of $k = |\beta| = 0$, i.e. for $\beta = 0$, an analogous argument yields

$$|\partial^\beta \left[ q_i \circ T_i^T \right]| = |q_i \circ T_i^T|$$

(by eq. (2.53)) \leq C_{0,N} \cdot (1 + \left| T_i^T \right|)^{-N} = C_{0,N} \cdot \left| T_i \right|^{|\beta|} \cdot (1 + \left| T_i^T \right|)^{-N},$$

so that estimate (2.50) also holds for $|\beta| = 0$.

In conjunction with equation (2.5), we conclude

$$|\partial^\alpha \theta_i| \leq |\det T_i| \cdot \sum_{\beta \leq \alpha} \left[ \left( \alpha \right)_\beta \cdot \left| (2\pi j \cdot b_i)^{\alpha - \beta} \right| \cdot |\partial^\beta \left[ q_i \circ T_i^T \right]| \right].$$

$$\leq (2\pi)^{|\alpha|} \cdot |\det T_i| \cdot (1 + \left| T_i^T \right|)^{-N} \cdot \sum_{\beta \leq \alpha} \left[ \left( \alpha \right)_\beta \cdot \alpha_{|\beta|,N} \cdot \left| b_i \right|^{\alpha - |\beta|} \left| T_i \right|^{|\beta|} \right].$$

Now, Young’s inequality $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ for $r,s > 0$ with $\frac{1}{r} + \frac{1}{s} = 1$ implies for $0 \neq \beta < \alpha$ and $r = r_{\alpha,\beta} = \frac{|\alpha|}{|\alpha| - |\beta|}$, as well as $s = s_{\alpha,\beta} = \frac{|\beta|}{|\alpha| - |\beta|}$ that

$$|b_i|^{\alpha - |\beta|} \left| T_i \right|^{|\beta|} \leq \frac{|b_i|^{\alpha}}{r} + \frac{||T_i||^{\alpha}}{s} \leq C_{\alpha} \cdot \left( |b_i|^{\alpha} + ||T_i||^{\alpha} \right),$$

where the last step used that for each fixed $\alpha \in \mathbb{N}_0$, there are only finitely many possible values of $r_{\alpha,\beta}$ and $s_{\alpha,\beta}$ where $\beta$ runs through all $0 \neq \beta < \alpha$.

In case of $\beta = 0$, we trivially have $|b_i|^{\alpha - |\beta|} \left| T_i \right|^{|\beta|} = |b_i|^{\alpha} \leq |b_i|^{\alpha} + ||T_i||^{\alpha}$ and in case of $\beta = \alpha$, we also have $|b_i|^{\alpha - |\beta|} \left| T_i \right|^{|\beta|} = ||T_i||^{\alpha} \leq |b_i|^{\alpha} + ||T_i||^{\alpha}$, so that all in all, we arrive at

$$|\partial^\alpha \theta_i| \leq C_{\alpha,N} \cdot |\det T_i| \cdot (1 + \left| T_i^T \right|)^{-N} \cdot \left( |b_i|^{\alpha} + ||T_i||^{\alpha} \right)$$

for all $i \in I$, where $C_{\alpha,N} = C_{\alpha,N} \left( \mathcal{Q}, (\gamma_i)_{i \in I} \right)$ is independent of $i \in I$.

We have thus established the estimate which was claimed in the remark. It remains to show that this implies the desired estimate for the $L^p$ norm claimed in the statement of the lemma. For $p = \infty$, this is clear. For $p \in (0, \infty)$, choose $N = N \left( p, d \right) \geq \frac{d+1}{p}$. Then, the change of variables formula yields

$$\left\| \left(1 + \left| T_i^T \right| \right)^{-N} \right\|_{L^p}^p = |\det T_i|^{-N} \int_{\mathbb{R}^d} \left(1 + \left| T_i^T x \right| \right)^{-Np} \cdot |\det T_i| \ dx$$

$$= |\det T_i|^{-N} \int_{\mathbb{R}^d} \left(1 + |y| \right)^{-Np} \ dy$$

$$\leq |\det T_i|^{-N} \int_{\mathbb{R}^d} \left(1 + |y| \right)^{-(d+1)} \ dy$$

$$= C_d \cdot |\det T_i|^{-N}$$

and hence $\left\| \left(1 + \left| T_i^T \right| \right)^{-N} \right\|_{L^p} \leq C_{p,d} \cdot |\det T_i|^{-1/p \cdot N}$. This yields

$$|\partial^\alpha \theta_i|_{L^p} \leq C_{\alpha,N(p),d,p} \cdot |\det T_i|^{-1/p} \cdot \left( |b_i|^{\alpha} + ||T_i||^{\alpha} \right),$$

where $C_{\alpha,N(p),d,p} > 0$ is independent of $i \in I$, as desired. 

In the preceding proof, we used a special form of the chain rule to compute higher derivatives of a function composed with a linear transformation. The following lemma formally establishes this elementary result.

**Lemma 2.6.** Let $A \in \mathbb{R}^{d \times d}$ be arbitrary and $f \in C^k \left( \mathbb{R}^d \right)$ for some $k \in \mathbb{N}$. Let $i_1, \ldots, i_k \in \mathbb{d}$ be arbitrary and let $\alpha = \sum_{m=1}^k e_{i_m} \in \mathbb{N}_0^d$, where $(e_1, \ldots, e_d)$ is the standard basis of $\mathbb{R}^d$.

Then $|\alpha| = k$ and

$$(\partial^\alpha [f \circ A]) (x) = \sum_{\ell_1, \ldots, \ell_k \in \mathbb{d}} [A_{\ell_1, i_1} \cdots A_{\ell_k, i_k} \cdot (\partial_{\ell_1} \cdots \partial_{\ell_k} f) (Ax)]$$

for every $x \in \mathbb{R}^d$. 

\[\blacksquare\]
Proof. We show the claim by induction on $k \in \mathbb{N}$. For $k = 1$, we have $\alpha = e_{i_1}$. Now, the chain rule implies

$$(\partial^\alpha [f \circ A]) (x) = (D (f \circ A))_{1,i_1} (x) = ((Df) (Ax) \cdot A)_{1,i_1}$$

$$= \sum_{\ell = 1}^d (Df (Ax))_{1,\ell} \cdot A_{\ell,i_1}$$

$$= \sum_{i_1 = 1}^d [A_{i_1,i_1} \cdot (\partial_{\ell_1} f) (Ax)],$$

so that the claim holds for $k = 1$.

Now, assume that the claim holds for some $k \in \mathbb{N}$ and let $f \in C^{k+1} (\mathbb{R}^d)$ and $i_1, \ldots, i_{k+1} \in I$. By using the case $k = 1$ and setting $\beta := \sum_{m=1}^k e_{i_m}$, we get

$$(\partial^\alpha [f \circ A]) (x) = (\partial^\beta (\partial_{i_{k+1}} [f \circ A])) (x)$$

(case $k = 1$) $= \partial^\beta \left( \sum_{\ell_{k+1} = 1}^d (A_{\ell_{k+1}, i_{k+1}} \cdot (\partial_{\ell_{k+1}} f) (Ax)) \right)$

(with $f_\ell := \partial_{\ell} f \in C^k (\mathbb{R}^d)$) $= \sum_{\ell_{k+1} = 1}^d (A_{\ell_{k+1}, i_{k+1}} \cdot (\partial^\beta [f_{\ell_{k+1}} \circ A]) (x))$

(by induction) $= \sum_{\ell_{k+1} \in I} \sum_{\ell_1, \ldots, \ell_k \in I} \sum_{\ell_{k+1} \in I} [A_{\ell_1,i_1} \cdots A_{\ell_k,i_k} \cdot (\partial_{\ell_1} \cdots \partial_{\ell_k} f_{\ell_{k+1}}) (Ax)]$

$$= \sum_{\ell_1, \ldots, \ell_{k+1} \in I} [A_{\ell_1,i_1} \cdots A_{\ell_{k+1},i_{k+1}} \cdot (\partial_{\ell_1} \cdots \partial_{\ell_{k+1}} f) (Ax)],$$

so that the claim also holds for $k+1$ instead of $k$. \hfill \square

As a consequence of Lemma 2.5, it is now straightforward to show that indeed every regular partition of unity is an $L^p$-partition of unity, for every $p \in (0, \infty]$.

**Corollary 2.7.** Let $\Phi = (\varphi_i)_{i \in I}$ be a regular partition of unity subordinate to $Q$. Then $\Phi$ is an $L^p$-BAPU for $Q$ for every $p \in (0, \infty]$.

In particular, every regular covering is also an $L^p$-decomposition covering for all $p \in (0, \infty]$.

**Proof.** Lemma 2.5 (with $\alpha = 0$) yields

$$\left\| F^{-1} \varphi_i \right\|_{L_p} \leq C_{d,q,Q,\Phi} \cdot |\det T_i|^{1-\frac{1}{p}}$$

for all $i \in I$ and $q \in (0, \infty]$. For $p \in [1, \infty]$, taking $q = 1$ shows that $\Phi$ is an $L^p$-decomposition covering. For $p \in (0, 1)$, we get the same conclusion by choosing $q = p$. \hfill \square

It is an important fact established by Borup and Nielsen in [3] Proposition 1 that every structured admissible covering is an $L^p$-decomposition covering. In fact, their proof even shows that every structured admissible covering is a regular covering. Our next result slightly generalizes this.

**Theorem 2.8.** Every structured admissible covering is a regular covering.

More generally, the following holds: If $Q = (Q_i)_{i \in I} = (T_i Q_i' + b_i)_{i \in I}$ is a semi-structured open covering of the open set $\emptyset \neq \mathcal{O} \subset \mathbb{R}^d$, and if for each $i \in I$, there is an open set $P_i'$ with $\overline{P_i'} \subset Q_i'$ and with $\mathcal{O} = \bigcup_{i \in I} (T_i P_i' + b_i)$ and such that the sets $\{P_i' \mid i \in I\}$ and $\{Q_i' \mid i \in I\}$ are finite, then $Q$ is a tight regular covering of $\mathcal{O}$. Furthermore, $I$ is countably infinite. \hfill \blacktriangleleft

**Remark.** All in all, we have the following inclusions/implications between the different types of coverings:

structured admissible $\subset$ regular $\subset$ semi-structured.

**Proof.** Essentially, a proof of this result is contained in the proof (but not in the statement) of [3] Proposition 1. Here, we give a different proof which is (at least in spirit) close to that of [21] Theorem 3.2.17.
For brevity, set \( P_i := T_i P'_i + b_i \). Note that \( P'_i \subset Q'_i \) is bounded, so that \( \overline{P_i} \subset Q_i \subset \mathbb{O} \) is compact. Thus, \( I \) has to be finite; indeed, if \( I \) was finite, then \( \mathcal{O} \) would be compact, because of

\[
\mathcal{O} = \bigcup_{i \in I} P_i \subset \bigcup_{i \in I} \overline{P_i} \subset \bigcup_{i \in I} Q_i = \mathcal{O},
\]

where the set \( \bigcup_{i \in I} \overline{P_i} \) is compact if \( I \) is finite. But this would imply that \( \mathcal{O} \) is open and closed, so that \( \mathcal{O} \in \{ \emptyset, \mathbb{R}^d \} \), which contradicts compactness of \( \mathcal{O} \neq \emptyset \).

Furthermore, \( I \) is countable. To see this, note that since \( \mathcal{O} \subset \mathbb{R}^d \) is second countable and \((P_i)_{i \in I} \) is an open cover of \( \mathcal{O} \), there is a countable subcover \((P_{i_n})_{n \in \mathbb{N}} \). But this implies that \( I = \bigcup_{n \in \mathbb{N}} i_n \) is countable as a countable union of finite sets. To see the equality, note that for \( i \in I \) and arbitrary \( x \in Q_i \subset \mathcal{O} \), there is \( n \in \mathbb{N} \) with \( x \in P_{i_n} \subset Q_{i_n} \), so that we get \( Q_i \cap Q_{i_n} \neq \emptyset \) and hence \( i \in i_n \).

Tightness of \( \mathcal{Q} \) is immediate: Since all \( Q'_i \subset \mathbb{R}^d \) are nonempty and open, and since \( \{Q'_i \mid i \in I\} \) is finite, there is some \( \varepsilon > 0 \) and for each \( i \in I \) some \( c_i \in \mathbb{R}^d \) with \( B_{\varepsilon}(c_i) \subset Q'_i \).

It remains to show that \( \mathcal{Q} \) is regular. To this end, we can assume \( I = \mathbb{N} \), since \( I \) is countably infinite. By assumption\(^2\), there are certain open sets \( U_1, \ldots, U_N, V_1, \ldots, V_N \subset \mathbb{R}^d \) with \( \bigcup_{m \in \mathbb{N}} U_m \subset U_m \) for all \( m \in \mathbb{N} \) and such that for each \( i \in I \), there is some \( m_i \in \mathbb{N} \) with \( P'_i \subset V_{m_i} \subset \overline{V_{m_i}} \subset U_{m_i} = Q'_i \). Now, for every \( m \in \mathbb{N} \) choose some \( \psi_m \in C^\infty_c(U_m) \) with \( \psi_m \equiv 1 \) on \( V_m \).

Finally, for \( i \in I \), define \( \gamma_i := L_{b_i} \left( \psi_{m_i} \circ T_i^{-1} \right) \), i.e.

\[
\gamma_i : \mathbb{R}^d \to \mathbb{C}, \xi \mapsto \psi_{m_i} \left( T_i^{-1}(\xi - b_i) \right)
\]

and note \( \gamma_i \in C^\infty_c(Q_i) \) with \( \gamma_i \equiv 1 \) on \( P_i \). In view of our assumption \( I = \mathbb{N} \) from above, we can now define

\[
\varphi_n := \gamma_n \cdot \prod_{j=1}^{n-1} \left( 1 - \gamma_j \right) \quad \text{for } n \in \mathbb{N}.
\]

With this definition, a straightforward induction yields \( \sum_{j=1}^{n} \varphi_j = 1 - \prod_{j=1}^{n} (1 - \gamma_j) \) for all \( n \in \mathbb{N} \). Now, for arbitrary \( x \in \mathcal{O} \), there is some \( n \in \mathbb{N} = I \) with \( x \in P_n \). Hence, \( 1 - \gamma_n(x) = 0 \), so that we get

\[
\sum_{j=1}^{\infty} \varphi_j(x) = 1 - \prod_{j=1}^{\infty} (1 - \gamma_j(x)) = 1 - 0 = 1.
\]

All in all, we have shown that \( \{\varphi_j\}_{j \in \mathbb{N}} \) is a smooth partition of unity on \( \mathcal{O} \), subordinate to \( \mathcal{Q} \).

It remains to show that for each \( \alpha \in \mathbb{N}_0^d \), the constant

\[
C_\alpha := \sup_{n \in \mathbb{N}} \| \partial^\alpha \varphi_n^\# \|_{\text{sup}}
\]

is finite, where \( \varphi_n^\#(\xi) = \varphi_n(T_n \xi + b_n) \). To see this, note that for arbitrary \( n \in \mathbb{N} \) and \( m \in \mathbb{N} \setminus n^* \), we have

\[
Q_n \cap \text{supp } \gamma_m \subset Q_n \cap Q_m = \emptyset
\]

and hence \( 1 - \gamma_m \equiv 1 \) on \( Q_n \). Because of \( \text{supp } \gamma_m \subset Q_n \), this implies

\[
\varphi_n = \gamma_n \cdot \prod_{j=1}^{n-1} \left( 1 - \gamma_j \right) = \gamma_n \cdot \prod_{j \in n \cap m^*} \left( 1 - \gamma_j \right).
\]

Note that (by admissibility of \( \mathcal{Q} \)) the number of factors in the product is uniformly bounded, independent of \( n \in \mathbb{N} \). By the Leibniz rule, it thus suffices to show that the constant

\[
C'_\alpha := \sup_{n \in \mathbb{N} \setminus j \in n^*} \| \partial^\alpha (\xi \mapsto \gamma_j(T_n \xi + b_n)) \|_{\text{sup}}
\]

is finite for all \( \alpha \in \mathbb{N}_0^d \). But this easily follows from the assumptions on a semi-structured admissible covering: Indeed, for \( n \in \mathbb{N} \) and \( j \in n^* \), we have

\[
\gamma_j(T_n \xi + b_n) = \psi_{m_j} \left( T_j^{-1}(T_n \xi + b_n) - b_j \right)
\]

= \psi_{m_j} \left( T_j^{-1}T_n \xi + T_j^{-1}(b_n - b_j) \right)

\(\text{The claim made here is not completely obvious. To see it, note that if we set } P'_i := \bigcup \{ P'_j \mid \overline{P'_j} \subset Q'_i \}, \text{ then the set } \{P'_i \mid i \in I\} \text{ is finite, since } \{P'_j \mid i \in I\} \text{ is. Furthermore, } P'_i \subset P''_i \subset \overline{P''_i} \subset Q'_i. \text{ By suitably numbering the } (Q'_i)_{i \in I} \text{ and } (P''_i)_{i \in I} \text{ as } U_1, \ldots, U_N \text{ and } V_1, \ldots, V_N \text{ (possibly with repetitions), we obtain the claim.} \)
and thus
\[ \| \partial^\alpha \left( \xi \mapsto \gamma_j \left( T_n \xi + b_n \right) \right) \|_{\sup} = \| \partial^\alpha \left( \xi \mapsto \psi_m \left( T_j^{-1} T_n \xi \right) \right) \|_{\sup} . \]
But using Lemma 2.6, we easily get
\[ \left| \left[ \partial^\alpha \left( \psi_m \circ T_j^{-1} T_n \right) \right] \left( \xi \right) \right| \leq \| T_j^{-1} T_n \|_{\sup} \max_{\beta \in \mathbb{N}_0^d} \sum_{|\beta| = |\alpha|} \| \partial^\beta \psi_m \|_{\sup} \]
for all \( \xi \in \mathbb{R}^d \). Here, we used \( \| T_j^{-1} T_n \| \leq C_n \), because of \( j \in \mathbb{N}^* \).

2.3. Decomposition spaces. Using the notion of \( L^p \)-decomposition coverings from the previous subsection, we are almost ready to define decomposition spaces. We only need one more definition which pertains to the sequence space \( Y \) used to define \( D \left( Q, L^p, Y \right) \).

**Definition 2.9.** (cf. [7] Definition 2.5) Let \( Q = (Q_i)_{i \in I} \) be an admissible covering. We say that a quasi-normed vector space \( (Y, \| \cdot \|_Y) \) is a quasi-regular sequence space if the following hold:

1. \( Y \) is a sequence space over \( I \), i.e. \( Y \) is a subspace of the space \( \mathbb{C}^I \) of all (complex) sequences over \( I \).
2. \( Y \) is a Quasi-Banach space, i.e. \( (Y, \| \cdot \|_Y) \) is complete.
3. \( Y \) is solid, i.e. if \( x = (x_i)_{i \in I} \in \mathbb{C}^I \) is a sequence with \( |x_i| \leq |y_i| \) for all \( i \in I \) and some sequence \( y = (y_i)_{i \in I} \in Y \), then \( x \in Y \) with \( \|x\|_Y \leq \|y\|_Y \).
4. \( Y \) is quasi-regular, i.e., the clustering map
   \[ \Psi : Y \to Y, x = (x_i)_{i \in I} \mapsto x^* = \left( \sum_{i \in I^*} x_i \right)_{i \in I} \]
   is well-defined and bounded.

**Remark.** That \( (Y, \| \cdot \|_Y) \) is a quasi-normed vector space means that \( \| \cdot \|_Y \) is a norm on \( Y \), with the exception that the usual triangle inequality is replaced by
\[ \|x + y\| \leq C \left( \|x\| + \|y\| \right) \]
for all \( x, y \in Y \) and some fixed constant \( C \geq 1 \).

The most important class of \( Q \)-regular sequence spaces is given by weighted Lebesgue spaces \( Y = \ell_u^\prime \left( I \right) \), if the weight \( u \) is \( Q \)-moderate, i.e. if
\[ C_{u, Q} := \sup_{i \in I} \sup_{j \in I^*} \frac{u_i}{u_j} \]
is finite. That this indeed yields a \( Q \)-regular sequence is shown e.g. in [21] Lemma 3.4.2, see also [7] Lemma 3.2.

Finally, using completeness of \( Y \) and a variant of the closed graph theorem (see e.g. [15] Theorem 2.1.5), one can show that the clustering map \( \Psi \) is bounded if and only if it is well-defined. This claim uses admissibility of \( Q \), since this implies that the set \( i^* \subset I \) is finite for all \( i \in I \), so that \( \Psi \) has a closed graph.

It turns out to be easiest to first define decomposition spaces on the Fourier side and to introduce the space-side versions of these spaces only afterwards.

**Definition 2.10.** Let \( p \in (0, \infty] \) and assume that \( Q = (Q_i)_{i \in I} = (T_i Q_i' + b_i)_{i \in I} \) is an \( L^p \)-decomposition covering of the open set \( \emptyset \neq O \subset \mathbb{R}^d \). Finally, let \( Y \leq \mathbb{C}^I \) be a \( Q \)-regular sequence space and let \( \Phi = (\varphi_i)_{i \in I} \) be an \( L^p \)-BAPU for \( Q \).

For a distribution \( f \in \mathcal{D}' (O) \), define the Fourier-side decomposition space (quasi)-norm of \( f \) (with respect to \( Q, L^p, Y \)) as
\[ \| f \|_{D_F(Q, L^p, Y)} := \left\| \left( \left\| F^{-1} \left( \varphi_i f \right) \right\|_{L^p} \right)_{i \in I} \right\|_Y , \tag{2.8} \]
with the convention that \( \| (c_i)_{i \in I} \|_Y = \infty \) if \( c_i = \infty \) for some \( i \in I \) and furthermore \( \| F^{-1} \left( \varphi_i f \right) \|_{L^p} = \infty \) if \( F^{-1} \left( \varphi_i f \right) \notin L^p \left( \mathbb{R}^d \right) \).

Finally, define the Fourier-side decomposition space (with respect to \( Q, L^p, Y \)) as
\[ D_F(Q, L^p, Y) := \left\{ f \in \mathcal{D}' (O) \mid \| f \|_{D_F(Q, L^p, Y)} < \infty \right\} . \]
Wiener theorem (cf. [8, Proposition 9.11]), the inverse Fourier transform $F^{-1}(\varphi_if) \in S'(\mathbb{R}^d)$ is given by (integration) a smooth function with polynomially bounded derivatives of all orders. Thus, it makes sense to write $\|F^{-1}(\varphi_if)\|_{L^p}$, with the caveat that we might have $\|F^{-1}(\varphi_if)\|_{L^p} = \infty$. In the remainder of the paper, we will always identify the (tempered) distribution $F^{-1}(\varphi_if)$ with the smooth function obtained from the Paley Wiener theorem. Hence,

$$[F^{-1}(\varphi_if)](x) = \left\langle \varphi_if, e^{2\pi i(x,\cdot)} \right\rangle = \left\langle f, \varphi_i, e^{2\pi i(x,\cdot)} \right\rangle$$

for all $x \in \mathbb{R}^d$.

Finally, note that the right-hand side of equation (2.8) uses the $L^p$-BAPU $\Phi$, whereas $\Phi$ is not mentioned on the left-hand side. This is justified by [21, Corollary 3.4.11], where it is shown that different choices of $\Phi$ yield the same (Fourier side) decomposition space with equivalent quasi-norms. Furthermore, [21, Theorem 3.4.13] shows that the resulting space $D_F(Q,L^p,Y)$ is complete and satisfies $D_F(Q,L^p,Y) \hookrightarrow D'(O)$. ♦

Now that we have introduced the Fourier-side version of decomposition spaces, we also want to define their space-side counterpart. To this end, we first have to define a suitable reservoir which takes the role of the space of distributions $D'(O)$ from above. We remark that our notation is heavily influenced by Triebel, in particular by [18].

**Definition 2.11.** For an open set $\varnothing \neq O \subset \mathbb{R}^d$, we set

$$Z(O) := \mathcal{F}(C_c^\infty(O)) = \left\{ \hat{f} \mid f \in C_c^\infty(O) \right\} \subseteq S(\mathbb{R}^d)$$

and endow this space with the unique topology which makes the Fourier transform

$$\mathcal{F} : C_c^\infty(O) = \mathcal{D}(O) \rightarrow Z(O)$$

a homeomorphism.

We equip the topological dual space $Z'(O) := [Z(O)]'$ of $Z(O)$ with the weak*-topology, i.e. with the topology of pointwise convergence on $Z(O)$. Finally, as on the Schwartz space, we extend the Fourier transform by duality to $Z'(O)$, i.e. we define

$$\mathcal{F} : Z'(O) \rightarrow \mathcal{D}'(O), f \mapsto f \circ \mathcal{F}.$$  

For $f \in Z'(O)$, we also write $\hat{f} := \mathcal{F} f \in \mathcal{D}'(O)$.

Since the Fourier transform $\mathcal{F} : C_c^\infty(O) \rightarrow Z(O)$ is bijective (even a homeomorphism), it is easily seen that $\mathcal{F} : Z'(O) \rightarrow \mathcal{D}'(O)$ is also a homeomorphism. ♦

Using the reservoir $Z'(O)$ that we just introduced, we can finally define the space-side decomposition spaces.

**Definition 2.12.** Under the general assumptions of Definition 2.10, we define the (space-side) Decomposition space (with respect to $Q, L^p, Y$) as

$$D(Q,L^p,Y) := \left\{ f \in Z'(O) \mid \mathcal{F} f \in D_F(Q,L^p,Y) \right\},$$

with (quasi)-norm

$$\|f\|_{D(Q,L^p,Y)} := \|\hat{f}\|_{D_F(Q,L^p,Y)} = \left\| \left( \|F^{-1}(\varphi_if)\|_{L^p} \right)_{i \in I} \right\|_Y,$$

where $\Phi = (\varphi_i)_{i \in I}$ is an $L^p$-BAPU for $Q$.

**Remark.** From the properties of the Fourier-side decomposition spaces, it is immediate that $D(Q,L^p,Y)$ is a Quasi-Banach space with $D(Q,L^p,Y) \hookrightarrow Z'(O)$ which is independent of the choice of $\Phi$, with equivalent quasi-norms for different choices.

Readers familiar with the work of Borup and Nielsen (e.g. [3]) might object to the seemingly overcomplicated choice of the reservoir $Z'(O)$, when one could simply use $S'(\mathbb{R}^d)$. This choice, however, has two serious limitations:

1. We want to allow for the case that $Q$ only covers a proper subset $O \subset \mathbb{R}^d$ of the frequency space $\mathbb{R}^d$. In this case, the expression $\|f\|_{D(Q,L^p,Y)}$ does not define a (quasi)-norm on $S'(\mathbb{R}^d)$, since it is not positive definite. For example, the inverse Fourier transform of any Dirac delta distribution, $f = F^{-1}\delta_{x_0}$ with $x_0 \in \mathbb{R}^d \setminus O$, would satisfy $\|f\|_{D(Q,L^p,Y)} = 0$, although $f \neq 0$. 

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Thus, to obtain a proper norm, one would have to factor out a certain subspace of $\mathcal{S}'(\mathbb{R}^d)$. For example, the homogeneous Besov space $\mathcal{B}^{p,q}_r(\mathbb{R}^d)$ is usually defined as a subspace of $\mathcal{S}'(\mathbb{R}^d)/\mathcal{P}$, where $\mathcal{P}$ denotes the space of polynomials. Note that $\mathcal{P}$ is exactly the set of inverse Fourier transforms of distributions supported at the origin. Instead of factoring out a subspace of $\mathcal{S}'(\mathbb{R}^d)$ depending on $\mathcal{O}$, we prefer to work with the space $\mathcal{Z}'(\mathcal{O})$.

(2) Even in case of $\mathcal{O} = \mathbb{R}^d$, the space
\[
\mathcal{D}_{\mathcal{S}_R}(\mathcal{Q},L^p,Y) := \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \mid \|f\|_{\mathcal{D}(\mathcal{Q},L^p,Y)} < \infty \right\}
\]
is in general not complete, as observed in [21, Example 3.4.14] and [22, Example after Definition 21]. In contrast, the space $\mathcal{D}(\mathcal{Q},L^p,Y)$ defined above is complete. Thus, the choice of the reservoir $\mathcal{Z}'(\mathcal{O})$ turns out to be superior to $\mathcal{S}'(\mathbb{R}^d)$ even for $\mathcal{O} = \mathbb{R}^d$. ♦

Having properly introduced the classes of Sobolev spaces and decomposition spaces, the next section is dedicated to deriving sufficient conditions for an embedding of the form $\mathcal{D}(\mathcal{Q},L^p,Y) \hookrightarrow \mathcal{W}^{k,q}(\mathbb{R}^d)$. But beforehand, we briefly gather some facts concerning convolution relations for the Lebesgue spaces $L^q(\mathbb{R}^d)$ in the Quasi-Banach regime $q \in (0,1)$.

2.4. Convolution in the Quasi-Banach regime. As observed above, the convolution relation $L^1* L^q \subset L^q$, which is valid for $q \in [1,\infty]$, fails for $q \in (0,1)$. In this subsection, we recall from [19] and [21] some alternative convolution relations which are valid in the Quasi-Banach regime. All of these theorems, however, will require the “factors” of the convolution product to be bandlimited.

We begin with the following result, which shows that bandlimited functions in $L^q$ are automatically contained in $L^r$ for all $r \geq q$. Intuitively, this reflects the fact that bandlimited functions are locally well behaved, so that the only obstruction to membership in $L^q$ is insufficient decay at infinity. We first state the result for $p \in (0,2]$. Afterwards, we present a version for general $p \in (0,\infty]$ for the special case where the frequency support $\text{supp} \hat{f}$ is contained in $Q_i$ for a semi-structured covering $Q$.

Lemma 2.13. Let $\emptyset \neq \Omega \subset \mathbb{R}^d$ be compact and assume that $f \in \mathcal{S}'(\mathbb{R}^d)$ is a tempered distribution with compact Fourier support $\hat{f} \subset \Omega$.

If $f \in L^p(\mathbb{R}^d)$ for some $p \in (0,2]$, then
\[
\|f\|_{L^q} \leq [\lambda(\Omega)]^{\frac{1}{p} - \frac{1}{q}} \cdot \|f\|_{L^p}
\]
holds for all $q \in [p,\infty]$.

Proof. For a proof, see [21, Corollary 3.1.3]. The proof given there is strongly based on that of [19, 1.4.1(3)], where the same statement is shown, but with an unspecified constant $C_\Omega$ instead of $[\lambda(\Omega)]^{\frac{1}{p} - \frac{1}{q}}$. This constant – which can be extracted from the proof – will be important for us below. □

Now, we specialize the above result to functions which are bandlimited to sets $Q_i$ of a semi-structured covering $Q$. Note that the proof for $p \in [1,\infty]$ is self-contained, independent of Lemma 2.13.

Corollary 2.14. Let $Q = (Q_i)_{i \in I} = (T_i Q'_i + b_i)_{i \in I}$ be a semi-structured covering of the open set $\emptyset \neq \mathcal{O} \subset \mathbb{R}^d$.

Let $p,q \in (0,\infty]$ with $p \leq q$. Then there is a constant $C = C(Q,d,p,q)$ such that for each $i \in I$, we have
\[
\|\mathcal{F}^{-1} f\|_{L^q} \leq C \cdot |\det T_i|^{\frac{1}{p} - \frac{1}{q}} \cdot \|\mathcal{F}^{-1} f\|_{L^p}
\]
for all $f \in \mathcal{D}'(\mathcal{O})$ with compact support $\text{supp} \hat{f} \subset Q_i$.

Remark. Since $\text{supp} \hat{f} \subset Q_i \subset \mathcal{O}$ is compact, $f$ is actually a (tempered) distribution on all of $\mathbb{R}^d$, so that the tempered distribution $\mathcal{F}^{-1} f$ is given by (integration against) a smooth function, by the Paley Wiener theorem. In particular, Lemma 2.13 is applicable to $\mathcal{F}^{-1} f$. ♦

Proof. The proof given here is essentially that of [21, Lemma 5.1.3].

By definition of a semi-structured covering, there is some $R > 0$ with $\overline{Q_i} \subset B_R(0)$ for all $i \in I$. Now, we distinguish the two cases $p \in (0,1)$ and $p \in [1,\infty]$. 17
For $p \in (0, 1)$, we have
\[
\lambda(Q_i) \leq \lambda\left(T_i \overline{Q_i} + b_i\right) \\
\leq \lambda\left(T_i B_R(0) + b_i\right) \\
= \lambda(B_R(0)) \cdot |\det T_i|
\]
and hence
\[
\left[\lambda(Q_i)\right]^\frac{1}{p} - \frac{1}{q} \leq C_{R,p,q} \cdot |\det T_i|^{\frac{1}{p} - \frac{1}{q}},
\]
since $p \leq q$ implies $\frac{1}{p} - \frac{1}{q} \geq 0$. Thus, Lemma \ref{lem:convolution} yields $F^{-1}f \in L^q(\mathbb{R}^d)$ with
\[
\|F^{-1}f\|_{L^q} \leq \left[\lambda(Q_i)\right]^\frac{1}{p} - \frac{1}{q} \cdot \|F^{-1}f\|_{L^p} \leq C_{R,p,q} \cdot |\det T_i|^{\frac{1}{p} - \frac{1}{q}} \cdot \|F^{-1}f\|_{L^p},
\]
as desired.

For $p \in [1, \infty]$, we give a self-contained proof. Choose $\psi \in C_c^\infty(\mathbb{R}^d)$ with $\psi \equiv 1$ on $B_R(0)$ and let $\psi_i : \mathbb{R}^d \to C, \xi \mapsto \psi\left(T_i^{-1}(\xi - b_i)\right)$, i.e. $\psi_i = L_{b_i}(\psi \circ T_i^{-1})$. Note $\psi_i \equiv 1$ on a neighborhood of $Q_i$, so that $f = \psi_i f$, because of $\text{supp} f \subset Q_i$.

Using the general form of Young's inequality for convolutions (cf. \cite{Stein} Proposition 8.9), we get
\[
\|F^{-1}f\|_{L^q} = \|F^{-1}(\psi_i f)\|_{L^q} \\
= \|\left(\mathcal{F}^{-1} \psi_i\right) \ast (F^{-1}f)\|_{L^q} \\
\leq \|\mathcal{F}^{-1} \psi_i\|_{L^r} \cdot \|F^{-1}f\|_{L^p},
\]
where $r \in [1, \infty]$ has to be chosen such that $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{p}$, i.e. $\frac{1}{r} = 1 + \frac{1}{q} - \frac{1}{p}$. This is indeed possible, since $1 \leq p \leq q$, which yields $0 \leq 1 - \frac{1}{p} \leq 1 + \frac{1}{q} - \frac{1}{p} \leq 1$. Finally, standard properties of the Fourier transform (cf. \cite{Stein} Theorem 8.22) imply
\[
\mathcal{F}^{-1} \psi_i = |\det T_i| \cdot M_{b_i} \left[\mathcal{F}^{-1} \psi \circ T_i^T\right]
\]
and hence (using the change-of-variables formula)
\[
\|\mathcal{F}^{-1} \psi_i\|_{L^r} = |\det T_i| \cdot |\det T_i^T|^{-\frac{1}{r}} \cdot \|\mathcal{F}^{-1} \psi\|_{L^r} \\
= |\det T_i|^{1 - \left(1 + \frac{1}{r} - \frac{1}{q}\right)} \cdot \|\mathcal{F}^{-1} \psi\|_{L^r} \\
= C \cdot |\det T_i|^{\frac{1}{p} - \frac{1}{q}}.
\]
Altogether, this yields the desired estimate for the case $p \in [1, \infty]$. \hfill \Box

Now that we have established embeddings of bandlimited $L^p$-functions into $L^q$ for $p \leq q$, we give an overview over the alternatives to the convolution relation $L^1 \ast L^q \hookrightarrow L^q$ in case of $q \in (0, 1)$. As above, we first state a general result and then specialize to the case in which the factors of the convolution product are supported in sets $Q_i$ of a semi-structured covering $\mathcal{Q}$.

**Theorem 2.15.** (cf. \cite{Hansen} Proposition 1.5.1)

Let $Q_1, Q_2 \subset \mathbb{R}^d$ be compact and let $p \in (0, 1]$. Furthermore, assume $\psi \in L^1(\mathbb{R}^d)$ with $\text{supp} \psi \subset Q_1$ and such that $F^{-1} \psi \in L^p(\mathbb{R}^d)$. Then, for $f \in L^p(\mathbb{R}^d) \cap S' (\mathbb{R}^d)$ with $\text{supp} \hat{f} \subset Q_2$, we have
\[
\mathcal{F}^{-1}\left((\psi \cdot \hat{f})\right) = \left(\mathcal{F}^{-1} \psi\right) \ast f \in L^p(\mathbb{R}^d)
\]
with
\[
\|\mathcal{F}^{-1} (\psi \cdot \hat{f})\|_{L^p} \leq \lambda(Q_1 - Q_2)^{\frac{1}{p} - 1} \cdot \|\mathcal{F}^{-1} \psi\|_{L^p} \cdot \|f\|_{L^p},
\]
where $\lambda$ is the usual Lebesgue measure on $\mathbb{R}^d$ and where
\[
Q_1 - Q_2 := \{q_1 - q_2 \mid q_1 \in Q_1, q_2 \in Q_2\}
\]
is the **difference** set of $Q_1, Q_2$, which is compact and hence measurable, since $Q_1, Q_2$ are compact. \hfill \blacktriangleleft

**Remark.** Observe that $\psi \cdot \hat{f} \in S' (\mathbb{R}^d)$ is well-defined, even though $\psi$ might not be smooth. Indeed, since $p \leq 1$, Lemma \ref{lem:convolution} shows $f \in L^1(\mathbb{R}^d)$ and hence $\hat{f} \in C_0(\mathbb{R}^d) \subset L^\infty (\mathbb{R}^d)$ by the Riemann-Lebesgue lemma, so that $\psi \cdot \hat{f} \in L^1(\mathbb{R}^d) \subset S' (\mathbb{R}^d)$, because of $\psi \in L^1(\mathbb{R}^d)$. \hfill \blacktriangleleft
\textbf{Proof.} A proof of this statement can be found in the proof of [19 Proposition 1.5.1]. Note, however, that the constant $[\lambda (Q_1 - Q_2)]^{\frac{1}{p-1}}$ becomes apparent from the proof, but is not stated explicitly in the statement of [19 Proposition 1.5.1].

Another proof (stating the constant explicitly) can be found in [21 Theorem 3.1.4]. \hfill \Box

We close this section by specializing the above result to a more convenient version which applies if the sets $Q_1, Q_2$ from above are in fact members of a semi-structured covering $Q$. We remark that a version of the following convolution relation is implicitly used repeatedly in [3] and [11], without stating it explicitly.

\textbf{Corollary 2.16.} Let $p \in (0,1]$ and let $Q = (Q_i)_{i \in I} = (T_i Q_i' + b_i)_{i \in I}$ be an $L^p$-decomposition covering of the open set $\emptyset \neq O \subset \mathbb{R}^d$.

For arbitrary $n \in \mathbb{N}$, there is a constant $C = C(Q,n,d,p) > 0$ with the following property: If $i \in I$ and

- if $\psi \in L^1(R^d)$ with $\text{supp } \psi \subset Q_i^{n^*}$ and $F^{-1} \psi \in L^p(R^d)$ and
- if $f \in \mathcal{D}(O)$ with $\text{supp } f \subset Q_i^{n^*}$ and $F^{-1} f \in L^p(R^d)$,

then $F^{-1}(\psi f) \in L^p(R^d)$ with

$$\|F^{-1}(\psi f)\|_{L^p} \leq C \cdot |\det T_i|^{\frac{1}{p-1}} \cdot \|F^{-1} \psi\|_{L^p} \cdot \|F^{-1} f\|_{L^p}. \hfill \Box$$

\textbf{Proof.} A complete proof of this result is given in [21 Corollary 3.2.15]. Here, we use without proof the result of [21 Lemma 3.2.13] which implies that there is a constant $L = L(n,Q) > 0$ such that

$$Q_i^{n^*} \subset T_i (B_L(0)) + b_i$$

holds for all $i \in I$. Hence, we can apply Theorem 2.15 with $Q_1 = Q_2 = \overline{Q}_i$, since we have

$$Q_1 - Q_2 \subset T_i \left(B_{2L}(0)\right)$$

and hence $\lambda (Q_1 - Q_2) \leq |\det T_i| \cdot \lambda (B_{2L}(0))$. \hfill \Box

Now, we are equipped with a solid definition of the decomposition space $\mathcal{D}(Q,L,p,Y)$ and certain convolution relations for $L^p$ in case of $p \in (0,1)$. These will be put to use in the next section, where we derive sufficient conditions for existence of an embedding of a decomposition space into a Sobolev space. Actually, we will derive sufficient conditions for boundedness of (suitable defined) derivative operators $\partial^\alpha : \mathcal{D}(Q,L,p,Y) \to L^q(R^d)$.

### 3. Sufficient Conditions

In this section, we will show that the two conditions $p \leq q$ and $Y \hookrightarrow \ell^q_{u^{(k,p,q)}}(I)$ for a suitable weight $u^{(k,p,q)}$ are sufficient for the existence of the embedding

$$\mathcal{D}(Q,L,p,Y) \hookrightarrow W^{k,q}(R^d).$$

The main ingredient for the proof of this result is the following lemma which allows to sum a sequence of functions $f_i$, each bandlimited to the set $Q_i$. The important fact is that the $L^p$-norm of the sum $\sum_{i \in I} f_i$ can be controlled by the $L^p$-norm of the individual norms $\|f_i\|_{L^p}$. In most cases, this is a huge improvement over the obvious estimate obtained by the triangle inequality, which would yield an estimate in terms of the $\ell^1$-norm of the individual functions. In a second step, we will then estimate the $L^q$-norms of the derivatives of the pieces $f_i$ in terms of the norms $\|f_i\|_{L^p}$. This is possible, since each of the “pieces” $f_i$ is bandlimited to the set $Q_i$, cf. Lemma 3.2.

We remark that the proof heavily relies on Plancherel’s theorem (for the case $p = 2$) and interpolation. A different (more complicated) proof of a very similar result was given in [21 Lemma 5.1.2].

\textbf{Lemma 3.1.} Let $Q = (Q_i)_{i \in I} = (T_i Q_i' + b_i)_{i \in I}$ be an $L^1$-decomposition covering of the open set $\emptyset \neq O \subset \mathbb{R}^d$. Furthermore, let $p \in (0,\infty]$ and $k \in \mathbb{N}_0$ and assume that for each $i \in I$, we are given $f_i \in \mathcal{S}'(R^d) \cap L^p(R^d)$ with Fourier support $\text{supp } \hat{f}_i \subset Q_i^{k^*}$ and such that

$$\|\langle f_i \rangle_{L^p}\|_{\ell^p} < \infty,$$

where $p^* := \min\{p,p'\}$.
Then \( \sum_{i \in I} f_i \in L^p(\mathbb{R}^d) \) with unconditional convergence of the series in \( L^p(\mathbb{R}^d) \). Here, we identify \( f_i \) with its continuous (even smooth) version which exists by the Paley-Wiener theorem (cf. [8, Proposition 9.11]). Finally, we have
\[
\left\| \sum_{i \in I} f_i \right\|_{L^p(\mathbb{R}^d)} \leq C \cdot \left\| \left( \| f_i \|_{L^p} \right)_{i \in I} \right\|_{\ell^p}\quad (3.1)
\]
for some constant \( C = C(\mathcal{Q}, k) > 0 \).

**Proof.** Note that we always have \( p^\vee \leq 2 < \infty \), as can be seen by distinguishing the cases \( p \leq 2 \) and \( p \geq 2 \). In particular, since \( \left\| \left( \| f_i \|_{L^p} \right)_{i \in I} \right\|_{\ell^p} < \infty \), we easily get \( \| f_i \|_{L^p} = 0 \) for all \( i \in I \setminus I_0 \) for an at most countable set \( I_0 \subset I \). Now, since \( f_i \) is continuous for each \( i \in I \) by assumption, \( \| f_i \|_{L^p} = 0 \) actually implies \( f_i \equiv 0 \) everywhere (and not only almost everywhere). Thus, we can assume in the following that \( I = I_0 \) is countable.

Let us first handle the (easier) case \( p \in (0, 1] \). In this case, we have \( p^\vee = p \). Furthermore, it is well-known that \( \| \cdot \|_{L^p} \) is a \( p \)-norm, i.e. we have \( \| f + g \|_{L^p}^p \leq \| f \|_{L^p}^p + \| g \|_{L^p}^p \) for all measurable \( f, g \). Indeed, this is an immediate consequence of the estimate
\[
|a + b|^p \leq (|a| + |b|)^p \leq |a|^p + |b|^p
\]
which is valid for \( a, b \in \mathbb{C} \) since \( p \in (0, 1] \).

In view of the above and since \( I = I_0 \) is countable, the monotone convergence theorem yields
\[
\left\| \sum_{i \in I} |f_i|^p \right\|_{L^p} \leq \sum_{i \in I} \| f_i \|_{L^p}^p = \left\| \left( \| f_i \|_{L^p} \right)_{i \in I} \right\|_{\ell^p}^p = \left\| \left( \| f_i \|_{L^p} \right)_{i \in I} \right\|_{\ell^p} < \infty.
\]
By solidity of \( L^p(\mathbb{R}^d) \), this implies the claim. In addition to unconditional convergence in \( L^p \), we even get absolute convergence almost everywhere of the series \( \sum_{i \in I} f_i \).

Now, we handle the case \( p \in [1, 2] \). To this end, let \( \Phi = (\varphi_i)_{i \in I} \) be an \( L^1 \)-BAPU for \( \mathcal{Q} \). Such a family exists by assumption. For \( i \in I \), let \( \varphi_i^{(k+1)\ast} := \sum_{j \in (k+1)^\ast} \varphi_j \). Now, for \( p \in [1, 2] \), define the map \( \Phi_p : L^p(I_0; L^p(\mathbb{R}^d)) \to L^p(\mathbb{R}^d) \), \((g_i)_{i \in I_0} \mapsto \sum_{i \in I_0} F^{-1} \left( \varphi_i^{(k+1)\ast} g_i \right) \).

We will show that this map is well-defined and bounded (with unconditional convergence of the series in \( L^p(\mathbb{R}^d) \)) for \( p = 1 \) and \( p = 2 \). By complex interpolation (for vector-valued \( L^p \)-spaces, cf. [11 Theorems 5.1.1 and 5.1.2]), it then follows\(^3\) that this indeed holds for all \( p \in [1, 2] \). Note that each summand of the series defining \( \Phi_p \left( (g_i)_{i \in I_0} \right) \) is a well-defined (even smooth) function in \( L^p(\mathbb{R}^d) \), since \( \varphi_i^{(k+1)\ast} \in \mathcal{F}L^1(\mathbb{R}^d) \) is an \( L^p \)-Fourier multiplier by Young’s inequality (cf. [8, Theorem 8.7]). Here, we used that \( F^{-1} \varphi_i \in L^1(\mathbb{R}^d) \) for all \( i \in I \), by definition of an \( L^p \)-BAPU for \( p \in [1, \infty] \).

For \( p = 1 \), boundedness of \( \Phi_p \) is easy. By definition of an \( L^1 \)-BAPU, the constant \( K := \sup_{i \in I} \left\| F^{-1} \varphi_i \right\|_{L^1} \) is finite, as is \( N := \sup_{i \in I} |(k+2)\ast| \). Hence, \( \left\| F^{-1} \varphi_i^{(k+1)\ast} \right\|_{L^1} \leq \sum_{\ell \in (k+1)^\ast} \left\| F^{-1} \varphi_\ell \right\|_{L^1} \leq NK \) for all \( i \in I \), so that we get
\[
\left\| \sum_{i \in I_0} F^{-1} \left( \varphi_i^{(k+1)\ast} g_i \right) \right\|_{L^1} \leq \sum_{i \in I_0} \left\| F^{-1} \left( \varphi_i^{(k+1)\ast} g_i \right) \right\|_{L^1} \leq \sum_{i \in I_0} \left\| F^{-1} \varphi_i^{(k+1)\ast} \right\|_{L^1} \left\| g_i \right\|_{L^1} \leq NK \cdot \sum_{i \in I_0} \left\| g_i \right\|_{L^1} = NK \cdot \left( \left\| (g_i)_{i \in I_0} \right\|_{\ell^1(I_0; L^1(\mathbb{R}^d))} \right).
\]
This even establishes “absolute” – and hence unconditional – convergence of the series.

\(^3\)Actually, if each \( Q_i \) is open, then \( I \) is necessary countable, as seen in the proof of Theorem 2.5. But using the argument from the present proof, we can avoid assuming openness of the \( Q_i \).

\(^4\)Complex interpolation shows at least that the series \( \Phi_p(g) = \sum_{i \in I_0} F^{-1} \left( \varphi_i^{(k+1)\ast} \hat{g}_i \right) \) is a well-defined element of \( L^1 + L^2 \) for \( g = (g_i)_{i \in I_0} \in \ell^p(I_0; L^p(\mathbb{R}^d)) \). But we also get \( \left\| \Phi_p(g) \right\|_{L^p} \leq \left\| \left( \left\| g_i \right\|_{L^p} \right)_{i \in I_0} \right\|_{\ell^p} \). Because of \( p < \infty \), this easily yields unconditional convergence of the series, since for \( \varepsilon > 0 \), there is a finite set \( J_\varepsilon \subset I_0 \) with \( \left\| \left( \left\| g_i \right\|_{L^p} \right)_{i \in I_0 \setminus J_\varepsilon} \right\|_{\ell^p} < \varepsilon \).
For \( p = 2 \), we employ Plancherel’s theorem to get for arbitrary finite subsets \( J \subset I_0 \)

\[
\left\| \sum_{i \in J} \mathcal{F}^{-1} \left( \varphi_i^{(k+1)*} \hat{g}_i \right) \right\|_{L^2}^2 = \left\| \sum_{i \in J} \varphi_i^{(k+1)*} \hat{g}_i \right\|_{L^2}^2 = \int_{\mathbb{R}^d} \left| \sum_{i \in J} \varphi_i^{(k+1)*} (\xi) \cdot \hat{g}_i (\xi) \right|^2 d\xi.
\]

But for arbitrary \( \xi \in \mathcal{O} \), there is some \( i_\xi \in I \) with \( \xi \in Q_{i_\xi} \). For arbitrary \( i \in J \) with \( \varphi_i^{(k+1)*} (\xi) \neq 0 \), this implies \( \xi \in Q_{i_\xi}^{(k+1)*} \cap Q_{i_\xi} \neq \emptyset \) and hence \( i \in J \cap Q_{i_\xi}^{(k+1)*} \). By Cauchy-Schwarz, we conclude

\[
\left| \sum_{i \in J} \varphi_i^{(k+1)*} (\xi) \cdot \hat{g}_i (\xi) \right|^2 \leq \sum_{i \in J \cap Q_{i_\xi}^{(k+1)*}} \left| \varphi_i^{(k+1)*} (\xi) \right|^2 \cdot \left| \hat{g}_i (\xi) \right|^2 \leq |J \cap Q_{i_\xi}^{(k+1)*}| \cdot \left\| \varphi_i^{(k+1)*} \right\|_{L^2}^2 \cdot \sum_{i \in J} \left| \hat{g}_i (\xi) \right|^2 \leq N^3 K^2 \cdot \sum_{i \in J} \left| \hat{g}_i (\xi) \right|^2.
\]

Here, the last step used the easily verifiable estimates \( |i_{i_\xi}^{(k+2)*}| \leq \sup_{i \in I} |i^{(k+2)*}| = N \) and

\[
\left\| \varphi_i^{(k+1)*} \right\|_{L^2}^2 \leq \left\| \mathcal{F} \mathcal{F}^{-1} \varphi_i^{(k+1)*} \right\|_{L^2}^2 \leq \left\| \mathcal{F}^{-1} \varphi_i^{(k+1)*} \right\|_{L^1} \leq N K.
\]

If \( \xi \in \mathbb{R}^d \setminus \mathcal{O} \), then \( \varphi_i^{(k+1)*} (\xi) = 0 \), so that the above estimate trivially holds in this case. Altogether, another application of Plancherel’s theorem yields

\[
\left\| \sum_{i \in J} \mathcal{F}^{-1} \left( \varphi_i^{(k+1)*} \hat{g}_i \right) \right\|_{L^2}^2 \leq N^3 K^2 \cdot \sum_{i \in J} \int_{\mathbb{R}^d} \left| \hat{g}_i (\xi) \right|^2 d\xi \leq (N^2 K)^2 \cdot \sum_{i \in J} \left\| \hat{g}_i \right\|_{L^2}^2 \leq (N^2 K)^2 \cdot \left\| \sum_{i \in J} \left( g_i \cdot \chi_{J(i)} \right) \right\|_{L^2(I_0;L^2(\mathbb{R}^d))}^2 \leq (N^2 K)^2 \cdot \left\| \left( g_i \right)_{i \in I_0} \right\|_{L^2(I_0;L^2(\mathbb{R}^d))}^2.
\]

Now, since \( \left( g_i \right)_{i \in I_0} \in L^2 \left( I_0; L^2 \left( \mathbb{R}^d \right) \right) \), we can choose for arbitrary \( \varepsilon > 0 \) a finite subset \( J_0 \subset I_0 \) with

\[
\left\| \left( g_i \cdot \chi_{J(i)} \right)_{i \in I_0} \right\|_{L^2(I_0;L^2(\mathbb{R}^d))} < \varepsilon.
\]

Together with the above estimate and since \( I_0 \) is countable, this easily entails that the series \( \Phi_2 \left( \left( g_i \right)_{i \in I_0} \right) = \sum_{i \in I_0} \mathcal{F}^{-1} \left( \varphi_i^{(k+1)*} \hat{g}_i \right) \) converges unconditionally in \( L^2 \left( \mathbb{R}^d \right) \), with

\[
\left\| \Phi_2 \left( \left( g_i \right)_{i \in I_0} \right) \right\|_{L^2(\mathbb{R}^d)} \leq N^2 K \cdot \left\| \left( g_i \right)_{i \in I_0} \right\|_{L^2(I_0;L^2(\mathbb{R}^d))}.
\]

Note that the constant \( N^2 K \) only depends on \( N = N (\mathcal{Q}, k) \) and on \( K = K (\varphi_i)_{i \in I} = K (\mathcal{Q}) \).

Because of \( \left\| \Phi_1 \right\| \leq N K \leq N^2 K \) and \( \left\| \Phi_2 \right\| \leq N^2 K \), complex interpolation implies that each map \( \Phi_p \) is well-defined and bounded with \( \left\| \Phi_p \right\| \leq N^2 K = C' (\mathcal{Q}, k) \) for \( p \in [1, 2] \). To complete the proof for \( p \in [1, 2] \), it remains to show that boundedness of \( \Phi_p \) implies validity of equation 3.1 (with unconditional convergence of the series). But since \( \text{supp} \hat{f}_i \subset Q_i^{k*} \) holds for all \( i \in I \) by assumption and because of \( \varphi_i^{(k+1)*} = 1 \) on \( Q_i^{k*} \), we get

\[
f_i = \mathcal{F}^{-1} \hat{f}_i = \mathcal{F}^{-1} \left( \varphi_i^{(k+1)*} \hat{f}_i \right)
\]

for all \( i \in I \), so that

\[
\sum_{i \in I} f_i = \sum_{i \in I_0} f_i = \sum_{i \in I_0} \mathcal{F}^{-1} \left( \varphi_i^{(k+1)*} \hat{f}_i \right) = \Phi_p \left( \left( f_i \right)_{i \in I_0} \right).
\]
Hence,
\[ \left\| \sum_{i \in I} f_i \right\|_{L^p(\mathbb{R}^d)} \leq \left\| \Phi_p \cdot \left( (f_i)_{i \in I} \right) \right\|_{C^*(I;L^p(\mathbb{R}^d))} \leq C' \cdot \left\| (f_i)_{i \in I} \right\|_{L^p(\mathbb{R}^d)}, \]

since \( p \in [1, 2] \) implies \( p^\vee = p \).

Thus, it remains to consider the case \( p \in [2, \infty] \). Here, instead of the map \( \Phi_p \) from above, we consider
\[ \Psi_p : L^{p'}(I_0; L^p(\mathbb{R}^d)) \to L^p(\mathbb{R}^d), \quad (g_i)_{i \in I_0} \mapsto \sum_{i \in I_0} \mathcal{F}^{-1} \left( \varphi_i^{(k+1)*} g_i \right). \]
Note that for \( p = 2 \), we have \( p' = p \), so that \( \Psi_2 = \Phi_2 \) is bounded with unconditional convergence of the defining series. Thus, another complex interpolation argument shows that it suffices to prove boundedness of \( \Psi_{\infty} \) (with unconditional convergence of the series). Once this is done, validity of (3.1) for \( p \in [2, \infty] \) follows exactly as for \( p \in [1, 2] \), since we have \( p^\vee = p' < \infty \) for \( p \in [2, \infty] \). Note that the complex interpolation argument uses that the conjugate exponent “commutes” with interpolation.

To show boundedness of \( \Psi_{\infty} \), note that \( \alpha' = 1 \). Hence, we can argue as for \( p = 1 \). Indeed,
\[
\left\| \sum_{i \in I_0} \mathcal{F}^{-1} \left( \varphi_i^{(k+1)*} g_i \right) \right\|_{L^\infty} \leq \sum_{i \in I_0} \left\| \mathcal{F}^{-1} \left( \varphi_i^{(k+1)*} g_i \right) \right\|_{L^\infty} \\
\leq \sum_{i \in I_0} \left\| \mathcal{F}^{-1} \varphi_i^{(k+1)*} \right\|_{L^1} \| g_i \|_{L^\infty} \\
\leq NK \cdot \sum_{i \in I_0} \| g_i \|_{L^\infty} \\
= NK \cdot \left\| (g_i)_{i \in I_0} \right\|_{L^{\infty'}(I_0; L^\infty(\mathbb{R}^d))}.
\]

Here, we get “absolute” and hence unconditional convergence of the series. This completes the proof. □

Now, to derive a sufficient condition for boundedness of the derivative operator
\[ \partial_{\alpha}^* : \mathcal{D}(Q, L^p, Y) \to L^q(\mathbb{R}^d), \]

we need a way to estimate \( \left\| \partial_{\alpha} \left( \mathcal{F}^{-1}(\varphi_i \tilde{g}) \right) \right\|_{L^q} \) in terms of \( \left\| \mathcal{F}^{-1}(\varphi_i \tilde{g}) \right\|_{L^p} \). Such an estimate is established in our next lemma.

**Lemma 3.2.** Let \( Q = (Q_i)_{i \in I} = (T_i Q'_i + b_i)_{i \in I} \) be a regular covering of the open set \( \emptyset \neq \mathcal{O} \subset \mathbb{R}^d \). Let \( p \in (0, \infty] \) and assume that \( \Phi = (\varphi_i)_{i \in I} \) is a regular partition of unity subordinate to \( Q \).

Then
\[ \left\| \partial_{\alpha} \left[ \mathcal{F}^{-1}(\varphi_i f) \right] \right\|_{L^p} \leq C_{\alpha, p, Q, \Phi} \cdot \left( |b_i|^{\alpha} + \| T_i \|^{\alpha} \right) \cdot \left\| \mathcal{F}^{-1}(\varphi_i f) \right\|_{L^p} \tag{3.2} \]

holds for each \( f \in \mathcal{D}(\mathcal{O}) \) and each \( i \in I \).

**Proof.** Let us first handle the case \( p \in [1, \infty] \). Clearly, we may suppose that the right-hand side of equation (3.2) is finite. But \( \varphi_i^* \equiv 1 \) on \( Q_i \), whereas \( \varphi_i \) vanishes outside of \( Q_i \). Hence, \( \varphi_i = \varphi_i^* \varphi_i \) and thus
\[
\left\| \partial_{\alpha} \left[ \mathcal{F}^{-1}(\varphi_i f) \right] \right\|_{L^p} = \left\| \partial_{\alpha} \left[ \mathcal{F}^{-1}(\varphi_i^* \varphi_i f) \right] \right\|_{L^p} \\
= \left\| \left[ \partial_{\alpha} \left( \mathcal{F}^{-1}(\varphi_i) \right) \right] \ast \mathcal{F}^{-1}(\varphi_i f) \right\|_{L^p} \\
(\text{by Young’s inequality}) \leq \left\| \partial_{\alpha} \left( \mathcal{F}^{-1}(\varphi_i) \right) \right\|_{L^1} \cdot \left\| \mathcal{F}^{-1}(\varphi_i f) \right\|_{L^p} \\
(\text{by Lemma 2.46 with } p = 1) \leq C_{\alpha} \cdot \left( \| T_i \|^{\alpha} + |b_i|^{\alpha} \right) \left\| \mathcal{F}^{-1}(\varphi_i f) \right\|_{L^p}
\]

for a constant \( C_{\alpha} = C_{\alpha} (Q, \Phi, d) > 0 \) which is independent of \( i \in I \). Here, it is worth noting that \( \varphi_i^* \in C_c^\infty (\mathcal{O}) \), so that \( \varphi_i f \) is a well-defined (tempered) distribution on \( \mathbb{R}^d \) with compact support.

It remains to consider \( p \in (0, 1) \). The argument for proving equation (3.2) is analogous to the one before, with the important exception that Young’s inequality \( L^1 \ast L^p \to L^p \) fails for \( p \in (0, 1) \). But Corollary 2.14 shows that for \( f \in C_c^\infty (\mathbb{R}^d) \) with \( \supp f \subset Q_i^* \) and a tempered distribution \( g \in S' (\mathbb{R}^d) \) with \( \supp g \subset Q_i^* \) and with \( \mathcal{F}^{-1} g \in L^p (\mathbb{R}^d) \), we have
\[ \left\| \mathcal{F}^{-1}(fg) \right\|_{L^p} \lesssim_{k, p, Q} \left| \det T_i \right|^{\frac{1}{p} - 1} \cdot \left\| \mathcal{F}^{-1} f \right\|_{L^p} \cdot \left\| \mathcal{F}^{-1} g \right\|_{L^p}. \tag{3.3} \]
Note that this means that the Fourier multiplier $f$ acts boundedly on those $L^p$ functions with Fourier support in $Q^*_i$ for each fixed $k \in \mathbb{N}_0$, but not in general on arbitrary $L^p$ functions. Furthermore, the operator norm of the resulting Fourier multiplier depends in a nontrivial way on $|\det T_i| \approx \lambda(Q_i)$.

In the present case, this convolution relation implies (similar to the argument above)

$$\|\partial^{\alpha} [F^{-1} (\varphi_i f)]\|_{L^p} \leq C_{Q,p} \cdot |\det T_i|^{\frac{1}{p} - 1} \cdot \|\partial^{\alpha} (F^{-1} \varphi_i)\|_{L^p} \cdot \|F^{-1} (\varphi_i f)\|_{L^p}$$

(by Lemma 2.25) \leq C_{\alpha, p, Q, \varphi} \cdot \left(|\det T_i|^{\alpha_1} + |b_i|^{\alpha_1}\right) \cdot \|F^{-1} (\varphi_i f)\|_{L^p},$$

as desired. Here, we used $\text{supp} (\varphi_i f) \subset Q^*_i$ and $\varphi_i \subset Q_i \subset Q^*_i$ to justify application of the convolution relation from equation (3.3). □

Before we continue the development of our sufficient condition for boundedness of certain partial derivative operators, we first introduce a convenient notation.

**Definition 3.3.** Let $Q = (Q_i)_{i \in I}$ be an $L^p$-decomposition covering of an open set $\emptyset \neq O \subset \mathbb{R}^d$ and let $Y \leq C^l$ be a $Q$-regular sequence space. We define

$$\mathcal{S}^{p,Y}_O (\mathbb{R}^d) := \mathcal{S}_O (\mathbb{R}^d) \cap \mathcal{D}(Q, L^p, Y),$$

where we use the notation

$$\mathcal{S}_O (\mathbb{R}^d) := \left\{ f \in \mathcal{S} (\mathbb{R}^d) \bigg| \hat{\bar{f}} \in C^\infty_c (O) \right\}.$$

We say that the decomposition space $\mathcal{D}(Q, L^p, Y)$ embeds into a function space $Z$ on $\mathbb{R}^d$, written

$$\mathcal{D}(Q, L^p, Y) \hookrightarrow Z,$$

if there is a bounded linear map $\iota : \mathcal{D}(Q, L^p, Y) \to Z$ which satisfies $\iota f = f$ for all $f \in \mathcal{S}^{p,Y}_O (\mathbb{R}^d)$. In case of $Z = W^{k,q} (\mathbb{R}^d)$ with $q \in (0, 1)$, we instead require $\iota f = (\partial^{\alpha} f)|_{\alpha \leq k}$ for all $f \in \mathcal{S}^{p,Y}_O (\mathbb{R}^d)$.

We say that $\mathcal{D}(Q, L^p, Y)$ embeds injectively into $Z$, written $\mathcal{D}(Q, L^p, Y) \hookrightarrow \text{inj} Z$, if the map $\iota$ can be chosen to be injective. ▲

Now, we finally state and prove our sufficient conditions for boundedness of the partial derivative maps $\partial^{\alpha} : \mathcal{D}(Q, L^p, Y) \to L^q (\mathbb{R}^d)$. We use the notation $\partial^*_\alpha$ instead of $\partial^{\alpha}$ to distinguish the map which we define in the following theorem from the usual (weak) partial derivatives $\partial^{\alpha}$. ▲

**Theorem 3.4.** Let $Q = (Q_i)_{i \in I} = (T_i Q'_i + b_i)_{i \in I}$ be a regular covering of the open set $\emptyset \neq O \subset \mathbb{R}^d$ and let $p, q \in (0, \infty]$ with $p \leq q$ and $k \in \mathbb{N}_0$. Define the weight

$$u^{(k,p,q)} := \left(|\det T_i|^{\frac{1}{p} - \frac{1}{q}} \cdot (|b_i|^k + \|T_i\|^k)\right)_{i \in I}.$$

Assume that $Y \leq C^l$ is a $Q$-regular sequence space on $I$ satisfying $Y \hookrightarrow \ell^q_{(k,p,q)} (I)$ with $q = \min \{ q, q' \}$. Let $\Phi = (\varphi_i)_{i \in I}$ be a regular partition of unity for $Q$. Then for each $\alpha \in \mathbb{N}_0^d$ with $|\alpha| = k$, the map

$$\partial^*_\alpha : \mathcal{D}(Q, L^p, Y) \to L^q (\mathbb{R}^d),$$

is well-defined and bounded, with unconditional convergence of the series in $L^q (\mathbb{R}^d)$.

Furthermore, we have $\partial^*_\alpha f = \partial^\alpha f$ for all $f \in \mathcal{S}^{p,Y}_O (\mathbb{R}^d)$. ▲

**Remark.** We will see in Lemma 1.6 that

$$|b_i| + \|T_i\| \asymp \sup_{x \in Q_i} |x|$$

is (for a given family $Q = (Q_i)_{i \in I}$) asymptotically independent of the specific choice of $T_i, b_i, Q'_i$, as long as the resulting semi-structured covering $Q = (T_i Q'_i + b_i)_{i \in I}$ is tight. ▲
Proof. Since \( Q \) is admissible, the constant \( N := \sup_{i \in I} |i^*| \) is finite. By Corollary 2.7 \( \Phi \) is an \( L^p \)-BAPU for \( Q \). Fix \( \alpha \in \mathbb{N}_0^d \) with \( |\alpha| = k \).

Let \( f \in \mathcal{D}(Q, L^p, Y) \) and note \( g := \hat{f} \in \mathcal{D}_F(Q, L^p, Y) \subset \mathcal{D}'(O) \). Set \( g_i := \partial^\alpha [F^{-1}(\varphi_i g)] \) for \( i \in I \).

Note that \( g_i \in C^\infty(\mathbb{R}^d) \cap \mathcal{S}'(\mathbb{R}^d) \) is a well-defined polynomially bounded function by the Paley Wiener theorem (cf. 3 Proposition 9.11), since \( \varphi_i g \) is compactly supported. Furthermore, Lemma 3.2 (with \( q \) instead of \( p \)) implies \(-\) with \( u_i^{(k)} := |b_i|^k + \|T_i\|^k \) for \( i \in I \) \(-\) that

\[
\|g_i\|_{L^p} = \|\partial^\alpha [F^{-1}(\varphi_i g)]\|_{L^p}
\]

(by Lemma 3.2) \( \leq C_{\alpha,p,q} \cdot \left( |b_i|^\alpha + \|T_i\|^\alpha \right) \cdot \|F^{-1}[\varphi_i g]\|_{L^p}
\]

(by Corollary 2.14 since \( p \leq q \)) \( \leq C_{k,p,q} \cdot \prod_i \left( \|T_i\|^{-\frac{1}{4}} \cdot u_i^{(k)} \cdot \|F^{-1}[\varphi_i g]\|_{L^p} \right)
\]

\[
= C_{k,p,q} \cdot \prod_i u_i^{(k)} \cdot \|F^{-1}[\varphi_i g]\|_{L^p}
\]

(3.4) for all \( i \in I \).

Note that taking partial derivatives can not increase the Fourier support, so that

\[
supp g_i = \text{supp} \left( F \left( \partial^\alpha [F^{-1}(\varphi_i g)] \right) \right) \subset \text{supp} \left( F \left[ F^{-1}(\varphi_i g) \right] \right) \subset Q_i.
\]

Hence, Lemma 3.1 yields unconditional convergence in \( L^q(\mathbb{R}^d) \) of the series

\[
\sum_{i \in I} g_i = \sum_{i \in I} \partial^\alpha \left[ F^{-1}(\varphi_i \cdot \hat{f}) \right] = \partial^\alpha f,
\]

with

\[
\|\partial^\alpha f\|_{L^q} \leq \left\| \sum_{i \in I} g_i \right\|_{L^q} \leq \left\| \left( \sum_{i \in I} \|g_i\|_{L^q} \right) \right\|_{L^q} \leq C_{k,p,q} \cdot \left( \sum_{i \in I} \|F^{-1}[\varphi_i g]\|_{L^p} \right) \leq C_{k,p,q} \cdot \prod_i \left( \|F^{-1}[\varphi_i g]\|_{L^p} \right)
\]

(since \( |i^*| \leq N \) and \( L^p \) is quasi-normed) \( \leq \left\| \left( \sum_{i \in I} \|F^{-1}[\varphi_i g]\|_{L^p} \right) \right\|_{L^q} \)

\[
= \left\| \left( \sum_{i \in I} \|F^{-1}[\varphi_i g]\|_{L^p} \right) \right\|_{L^q} \leq C_{k,p,q} \cdot \prod_i \left( \|F^{-1}[\varphi_i g]\|_{L^p} \right)
\]

(since \( Y \hookrightarrow L^p(\mathbb{R}^d) \)) \( \leq \left\| \left( \sum_{i \in I} \|F^{-1}[\varphi_i g]\|_{L^p} \right) \right\|_{L^p} \)

\[
= \left\| \left( \sum_{i \in I} \|F^{-1}[\varphi_i g]\|_{L^p} \right) \right\|_{L^q} \leq \left\| \left( \sum_{i \in I} \|F^{-1}[\varphi_i g]\|_{L^p} \right) \right\|_{L^p} \)
\]

(since \( Y \) is \( Q \)-regular) \( \leq \left\| \left( \sum_{i \in I} \|F^{-1}[\varphi_i g]\|_{L^p} \right) \right\|_{L^p} \)

\[
= \|g\|_{\mathcal{D}'(Q, L^p, Y)} = \|f\|_{\mathcal{D}(Q, L^p, Y)} < \infty.
\]

It remains to show \( \partial^\alpha f = \partial^\alpha f \) for \( f \in \mathcal{S}^Y_0(\mathbb{R}^d) \). Note that \( \sum_{i \in I} \varphi_i \equiv 1 \) on \( \mathcal{O} \) implies \( \mathcal{O} \subset \bigcup_{i \in I} U_i \) with \( U_i := \varphi_i^{-1}(\mathbb{C} \setminus \{0\}) \). Since \( K := \text{supp} \hat{f} \subset \mathcal{O} \) is compact, there is thus a finite subset \( I_0 \subset I \) such that \( U := \bigcup_{i \in I_0} U_i \supset K \). Now, \( \varphi_i \equiv 0 \) on \( U \setminus K \) if \( i \notin I_0 \). Hence, \( \varphi_i \equiv 1 \) on \( U \cap K = \text{supp} \hat{f} \) and hence \( \hat{f} = \varphi_{I_0} \cdot \hat{f} \). Furthermore, \( \varphi_i \hat{f} \equiv 0 \) for all \( i \in I \setminus I_0 \). All in all, this yields

\[
\partial^\alpha f = \sum_{i \in I} \partial^\alpha \left[ F^{-1}(\varphi_i \cdot \hat{f}) \right] = \sum_{i \in I_0} \partial^\alpha \left[ F^{-1}(\varphi_i \cdot \hat{f}) \right] = \partial^\alpha \left[ F^{-1}(\varphi_{I_0} \cdot \hat{f}) \right] = \partial^\alpha \left[ F^{-1} \hat{f} \right] = \partial^\alpha f,
\]

as claimed. Note that this calculation is justified, since \( I_0^* \subset I \) is finite. \( \square \)
As a corollary of the preceding theorem, we derive sufficient conditions for embeddings of decomposition spaces into Sobolev spaces.

**Corollary 3.5.** Let $Q = (Q_i)_{i \in I} = (T_i Q_i^1 + b_i)_{i \in I}$ be a regular covering of the open set $\emptyset \neq O \subset \mathbb{R}^d$ and let $p, q \in (0, \infty]$ and $k \in \mathbb{N}_0$.

For $i \in I$, define

$$v_i := |\det T_i|^{\frac{1}{p} - \frac{1}{q}},$$

$$w_i := |\det T_i|^{\frac{1}{p} - \frac{1}{q}} \left( |b_i|^k + \|T_i\|^k \right).$$

If we have $p \leq q$ and if the $Q$-regular sequence space $Y \leq C^I$ satisfies

$$Y \hookrightarrow l^q_v(I) \quad \text{and} \quad Y \hookrightarrow l^q_w(I),$$

then $D(Q, L^p, Y) \hookrightarrow W^{k,q}(\mathbb{R}^d)$.

More precisely, we have the following:

1. For $q \geq 1$, the map $\iota_q^{(k)}$ from Theorem 3.4 is injective and bounded as a map

$$\iota_q^{(k)} : D(Q, L^p, Y) \to W^{k,q}(\mathbb{R}^d).$$

Furthermore, we have the following:

(a) $\iota_q^{(k)} f = f$ for all $f \in S^0_Y(\mathbb{R}^d)$ and

(b) $\partial^\alpha \left( \iota_q^{(k)} f \right) = \partial^\alpha f$ (with $\partial^\alpha$ as in Theorem 3.4) for all $f \in D(Q, L^p, Y)$ and $|\alpha| \leq k$.

(c) If $q = \infty$, then $\iota_\infty^{(k)}$ is even well-defined and bounded as a map into

$$C^k_b(\mathbb{R}^d) := \left\{ f \in C^k(\mathbb{R}^d) \mid \|f\|_{C^k_B} := \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{\text{sup} < \infty} \right\}.$$

2. For $q < 1$, the map

$$\iota_q^{(k)} : D(Q, L^p, Y) \to W^{k,q}(\mathbb{R}^d), f \mapsto (\partial^\alpha f)_{|\alpha| \leq k}$$

is well-defined and bounded, with the $\partial^\alpha$ as in Theorem 3.4. Furthermore, $\iota_q^{(k)} f = (\partial^\alpha f)_{|\alpha| \leq k}$ for all $f \in S^0_Y(\mathbb{R}^d)$.

3. If $q < 1$ and if there are $r \geq 1$ and $\ell \in \mathbb{N}_0$ such that $\iota_r^{(\ell)}$ is bounded (with unconditional convergence of the series), then $\iota_q^{(k)}$ is injective with

$$\iota_q^{(k)} \bigg|_{|\alpha| \leq \min(k, \ell)} = \left( \partial^\alpha \left( \iota_q^{(k)} f \right) \right)_{|\alpha| \leq \min(k, \ell)}$$

for all $f \in D(Q, L^p, Y)$. 

\[\Box\]

**Proof.** Let $\Phi = (\varphi_i)_{i \in I}$ be the regular partition of unity subordinate to $Q$ which is used in Theorem 3.4 to define the maps $\partial^\alpha_q$. Using the weights $u^{(n,p,q)}$ for $n \in \mathbb{N}_0$ which were defined in Theorem 3.4, we have $v = u^{(0,p,q)}$ so that the map

$$\iota_q^{(0)} = \partial^\alpha_q : D(Q, L^p, Y) \to L^q(\mathbb{R}^d), f \mapsto \sum_{i \in I} F^{-1} \left( \varphi_i \cdot \hat{f} \right)$$

is well-defined with unconditional convergence of the series in $L^q(\mathbb{R}^d)$ and with $\iota_q^{(0)} f = f$ for $f \in S^0_Y(\mathbb{R}^d)$.

We first show that $\iota_q^{(0)}$ is injective for $q \geq 1$. To this end, let $f \in D(Q, L^p, Y)$ with $\iota_q^{(0)} f = 0$. Since convolution with the Schwartz function $F^{-1} \varphi_j$ for arbitrary $j \in I$ is a bounded linear operator on $L^q(\mathbb{R}^d)$
Thus, an application of Theorem 3.4 shows that the map for every $f$.

Furthermore, for $f \in S_{\mathcal{C}}^{0,Y}(\mathbb{R}^d)$, Theorem 3.4 shows $\iota_q^{(k)} f = (\partial^\alpha f)|_{\alpha| \leq k} = (\partial^\alpha f)|_{\alpha| \leq k}$, as claimed.

Now, assume $q \geq 1$ and let $\phi \in C^\infty_c(\mathbb{R}^d)$ be arbitrary. We have $\partial^\beta \phi \in L^q(\mathbb{R}^d) \subset [L^q(\mathbb{R}^d)]'$ for all $\beta \in \mathbb{N}_0^d$, since $q \geq 1$. Thus, using the unconditional convergence in $L^q(\mathbb{R}^d)$ of the series defining $\iota_q^{(0)} f$ and
where the closure is taken in $L^q$ so that the weak derivative $\partial^\alpha f$ is given by $\partial^\alpha f \in L^q (\mathbb{R}^d)$. Since this holds for all $\alpha \in \mathbb{N}_0^d \setminus \{0\}$ with $|\alpha| \leq k$, we finally get $\iota_q^{(k)} f = \iota_q^{(0)} f \in W^{k,q} (\mathbb{R}^d)$ with $\|\iota_q^{(k)} f\|_{W^{k,q}} \lesssim \|\iota_q^{(0)} f\|_{L^q} + \sum_{\alpha \in \mathbb{N}_0^d \setminus \{0\}} \|\partial^\alpha f\|_{L^q} \lesssim \|f\|_{D(\mathcal{Q},L^p,Y)} < \infty$.

Now, we show that for $q = \infty$, the map $\iota_q^{(k)}$ is actually well-defined and bounded as a map into $C_b^k (\mathbb{R}^d)$. To see this, simply note that by the Paley-Wiener theorem, each of the functions $\mathcal{F}^{-1} (\varphi_i \hat{f})$ is smooth, so that the (finite) partial sums of the series $\sum_{\alpha \in \mathbb{N}_0^d} \mathcal{F}^{-1} (\varphi_i \hat{f})$ lie in $W^{k,\infty} (\mathbb{R}^d) \cap C^\infty (\mathbb{R}^d) \subset C_b^k (\mathbb{R}^d)$. Since the series converges unconditionally in $W^{k,\infty} (\mathbb{R}^d)$ and since $C_b^k (\mathbb{R}^d) \subset W^{k,\infty} (\mathbb{R}^d)$ is closed (with $\|\cdot\|_{C_b^k} \simeq \|\cdot\|_{W^{k,\infty}}$ on $C_b^k (\mathbb{R}^d)$), we easily see that $\iota_q^{(k)} : D(\mathcal{Q},L^p,Y) \rightarrow C_b^k (\mathbb{R}^d)$ is indeed well-defined and bounded.

It remains to prove the third statement of the corollary. By assumption, the series $\sum_{\alpha \in \mathbb{N}_0^d} \mathcal{F}^{-1} (\varphi_i \hat{f})$ converges to $\left(\iota_q^{(k)} f\right)_0$ in $L^q (\mathbb{R}^d)$ and to $\iota_q^{(0)} f$ in $L^r (\mathbb{R}^d)$. Since convergence in $L^q (\mathbb{R}^d)$ for arbitrary $s \in (0,\infty]$ implies convergence in measure, we get $\left(\iota_q^{(k)} f\right)_0 = \iota_q^{(0)} f$ (almost everywhere). In particular, we see that $\iota_q^{(k)} f = 0$ yields $\iota_q^{(0)} f = \left(\iota_q^{(k)} f\right)_0 = 0$ and hence $f = 0$ (by injectivity of $\iota_q^{(0)}$), so that $\iota_q^{(k)}$ is injective. A completely analogous argument using unconditional convergence of the series $\sum_{\alpha \in \mathbb{N}_0^d} \mathcal{F}^{-1} (\varphi_i \hat{f})$ shows $\iota_q^{(k)} f |_{|\alpha| \leq \min(k,t)} = \left(\mathcal{F}^{-1} (\varphi_i \hat{f})\right) |_{|\alpha| \leq \min(k,t)}$ for all $f \in D(\mathcal{Q},L^p,Y)$, as claimed.

Now that we have obtained sufficient criteria for the embedding $D(\mathcal{Q},L^p,Y) \hookrightarrow W^{k,q} (\mathbb{R}^d)$, it is natural to ask whether these criteria are sharp. This is the goal of the next section.

### 4. Necessary Conditions

In this section, we will assume that we have the following, for some $k \in \mathbb{N}_0$: For every multiindex $\alpha \in \mathbb{N}_0^d$ with $|\alpha| = k$, the map

$$\iota_\alpha : S^p_\mathcal{O}^Y (\mathbb{R}^d) \rightarrow L^q (\mathbb{R}^d), \ f \mapsto \partial^\alpha f$$

is bounded, where the space $S^p_\mathcal{O}^Y (\mathbb{R}^d)$ is as in Definition 3.3. Our general goal is to show that this implies conditions very similar to the sufficient conditions given in Theorem 3.4. In case of $q \in (0,2] \cup \{\infty\}$, we will even see that the necessary and sufficient criteria will coincide.

Our first aim is to show that the assumption $p \leq q$ from Theorem 3.4 is necessary.

The proofs of all our necessary conditions (not only of $p \leq q$) will be based on the following – relatively elementary – *arbitrage* result [10]. Since it is so central to our approach, we provide a proof, even though the result is probably well known.

**Lemma 4.1.** Let $n \in \mathbb{N}$, $p \in (0,\infty]$ and let $f_1, \ldots, f_n \in L^p (\mathbb{R}^d)$. For $p = \infty$, assume additionally that

$$f_i \in \{f \in L^\infty (\mathbb{R}^d) \mid \sup f \text{ compact}\} \quad \text{for all } i \in \{1,\ldots,n\},$$

where the closure is taken in $L^\infty (\mathbb{R}^d)$.

Then we have

$$\left\| \sum_{i=1}^n L_{x_i} f_i \right\|_{L^p} \overset{\text{min}_{x_i,|x_i| \rightarrow \infty} \rightarrow} \left\| \left(\|f_i\|_{L^p}\right)_{i \in \mathbb{N}} \right\|_{\ell^p}.$$
In particular, there exists $R = R(p, (f_i)_{i \in \mathbb{N}}) > 0$ with
\[
\frac{1}{2} \cdot \left\| (\|f_i\|_{L^p})_{i \in \mathbb{N}} \right\|_{\ell^p} \leq \left\| \sum_{i=1}^{n} L_x f_i \right\|_{L^p} \leq 2 \cdot \left\| (\|f_i\|_{L^p})_{i \in \mathbb{N}} \right\|_{\ell^p}
\]
for all $x_1, \ldots, x_n \in \mathbb{R}^d$ with $|x_i - x_j| \geq R$ for all $i, j \in \mathbb{N}$ with $i \neq j$.

**Proof.** If $\left\| (\|f_i\|_{L^p})_{i \in \mathbb{N}} \right\|_{\ell^p} > 0$, then the second part of the lemma is a trivial consequence of the first part. If $\left\| (\|f_i\|_{L^p})_{i \in \mathbb{N}} \right\|_{\ell^p} = 0$, then all quantities in the last part of the lemma vanish, so that the claim is trivial. Thus, it remains to prove the first part.

To this end, let us first assume that $f_1, \ldots, f_n$ are compactly supported. Since there are only finitely many $f_i$, there is some $R > 0$ with $\text{supp} f_i \subset B_{R/2}(0)$ for all $i \in I$. For $x_1, \ldots, x_n \in \mathbb{R}^d$ with $|x_i - x_j| \geq R$ for $i \neq j$, we then have
\[
\text{supp} (L_x f_i) \cap \text{supp} (L_x f_j) \subset B_{R/2}(x_i) \cap B_{R/2}(x_j) = \emptyset \quad \text{if } i \neq j.
\]
For $p \in (0, \infty)$, this implies
\[
\left\| \sum_{i=1}^{n} L_x f_i \right\|_{L^p}^p = \int_{\mathbb{R}^d} \left( \sum_{i=1}^{n} |(L_x f_i)(x)| \right)^p \, dx \\
\leq \int_{\mathbb{R}^d} \sum_{i=1}^{n} |(L_x f_i)(x)|^p \, dx \\
= \left\| (\|f_i\|_{L^p})_{i \in \mathbb{N}} \right\|_{\ell^p}^p,
\]
where the step marked with $(\ast)$ used that at most one summand of the sum does not vanish for each $x \in \mathbb{R}^d$. For $p = \infty$, we similarly have
\[
\left\| \sum_{i=1}^{n} L_x f_i \right\|_{L^\infty} = \text{ess sup}_{x \in \mathbb{R}^d} \left\| \sum_{i=1}^{n} (L_x f_i)(x) \right\| = \max \text{ ess sup}_{x \in \mathbb{R}^d} \| (L_x f_i)(x) \| = \left\| (\|f_i\|_{L^\infty})_{i \in \mathbb{N}} \right\|_{\ell^\infty}
\]
for $x_1, \ldots, x_n \in \mathbb{R}^d$ with $|x_i - x_j| \geq R$ for $i \neq j$, since for each $x \in \mathbb{R}^d$, at most one summand of the sum does not vanish.

For the general case, we use an approximation argument. Let us first consider the case $p \geq 1$. For $\varepsilon > 0$, there are compactly supported $g_1, \ldots, g_n \in L^p(\mathbb{R}^d)$ with $\|f_i - g_i\|_{L^p} < \varepsilon$ for all $i \in \mathbb{N}$. For $p < \infty$, this follows by density of $C_c(\mathbb{R}^d)$ in $L^p(\mathbb{R}^d)$; for $p = \infty$, it is a consequence of our additional assumptions on the $f_i$. Hence, for $R > 0$ large enough and $|x_i - x_j| \geq R$ for $i \neq j$, the considerations from above yield
\[
\left\| \sum_{i=1}^{n} L_x g_i \right\|_{L^p} = \left\| (\|g_i\|_{L^p})_{i \in \mathbb{N}} \right\|_{\ell^p}.
\]
Using the (second) triangle inequality for $L^p$ and $\ell^p$, this yields
\[
\left\| \sum_{i=1}^{n} L_x f_i \right\|_{L^p} - \left\| (\|f_i\|_{L^p})_{i \in \mathbb{N}} \right\|_{\ell^p} \\
\leq \left\| \sum_{i=1}^{n} L_x f_i \right\|_{L^p} - \sum_{i=1}^{n} \left\| L_x f_i \right\|_{L^p} + \left\| (\|g_i\|_{L^p})_{i \in \mathbb{N}} \right\|_{\ell^p} - \left\| (\|f_i\|_{L^p})_{i \in \mathbb{N}} \right\|_{\ell^p} \\
\leq \sum_{i=1}^{n} \left\| L_x (f_i - g_i) \right\|_{L^p} + \left\| (\|g_i\|_{L^p} - \|f_i\|_{L^p})_{i \in \mathbb{N}} \right\|_{\ell^p} \\
\leq \sum_{i=1}^{n} \left\| L_x (f_i - g_i) \right\|_{L^p} + \left\| (\varepsilon)_{i \in \mathbb{N}} \right\|_{\ell^p} \\
\leq 2n\varepsilon.
\]
Here, the last step used translation invariance of $\|\cdot\|_{L^p}$ and the embedding $\ell^1 \hookrightarrow \ell^p$.

For the case $p \in (0, 1)$, the above argument is invalid, since $\|\cdot\|_{L^p}$ and $\|\cdot\|_{\ell^p}$ do not satisfy the triangle inequality anymore. Instead, we note that we have the $p$-triangle inequality $\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$ which
implies that \( d(f, g) := \|f - g\|_{L^p}^p \) defines a metric on \( L^p(\mathbb{R}^d) \). The same also holds for \( \ell^p \). In particular, \( \|\cdot\|_{L^p} \) is continuous. By density of \( C_c(\mathbb{R}^d) \subset L^p(\mathbb{R}^d) \), this allows us to choose \( g_1, \ldots, g_n \in C_c(\mathbb{R}^d) \) with \( \|f_i\|_{L^p} - \|g_i\|_{L^p} \leq \varepsilon \) and \( \|f_i - g_i\|_{L^p} < \varepsilon \) for all \( i \in \mathbb{N} \). For \( R > 0 \) large enough and \( |x_i - x_j| \geq R \) for \( i \neq j \), this yields

\[
\left\| \sum_{i=1}^n L_{x_i} f_i \right\|_{L^p}^p - \left\| \sum_{i=1}^n L_{x_i} g_i \right\|_{L^p}^p \leq \left\| \sum_{i=1}^n L_{x_i} (f_i - g_i) \right\|_{L^p}^p + \left\| \left( \|g_i\|_{L^p} - \|f_i\|_{L^p} \right) \right\|_{\ell^p}^p \leq 2n \cdot \varepsilon^p.
\]

Since \( \varepsilon > 0 \) was arbitrary and because the map \( [0, \infty) \to [0, \infty) \), \( x \mapsto x^{1/p} \) is continuous, this implies the claim also for \( p \in (0, 1) \).

As a consequence, we obtain the following corollary which provides a generalization of the fact that bounded, translation invariant operators from \( L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d) \) can only be nontrivial if \( p \leq q \). Note that this is decidedly false if \( \mathbb{R}^d \) is replaced by any compact topological group. No originality regarding this result is claimed. A slightly less general form of the present result can be found in [10] around equation (11).

**Corollary 4.2.** Let \( p, q \in (0, \infty] \) and \( x_0 \in \mathbb{R}^d \setminus \{0\} \). Assume that the (not necessarily closed) subspace \( V \leq L^p(\mathbb{R}^d) \) is invariant under the translation \( L_{x_0} \) and that

\[
T : \left( V, \|\cdot\|_{L^p} \right) \to L^q(\mathbb{R}^d)
\]

is linear and bounded with \( T(L_{x_0} f) = L_{x_0} (T f) \) for all \( f \in V \).

Finally, assume that

\[
\{ f \in V \mid T f \neq 0 \} \cap \{ f \in L^p(\mathbb{R}^d) \mid \text{supp } f \text{ compact} \} \neq \emptyset,
\]

where the closure is taken in \( L^p(\mathbb{R}^d) \). Then we have \( p \leq q \).

**Remark.** Note that assumption 4.1 is always satisfied for \( p < \infty \) if \( T \neq 0 \), since in this case, we have

\[
\{ f \in L^p(\mathbb{R}^d) \mid \text{supp } f \text{ compact} \} = L^p(\mathbb{R}^d).
\]

**Proof.** In case of \( q = \infty \), the claim is trivial, so that we can assume \( q < \infty \) in the following.

By assumption, there is some

\[
f \in \left\{ g \in L^p(\mathbb{R}^d) \mid \text{supp } f \text{ compact} \right\} \cap V
\]

with \( T f \neq 0 \). In particular, \( f \neq 0 \).

Let \( n \in \mathbb{N} \) be arbitrary. Applying Lemma 4.1 to the family \((f_1, \ldots, f_n) := (f, \ldots, f)\), we obtain some \( R_1 > 0 \) such that

\[
\left\| \sum_{i=1}^n L_{x_i} f \right\|_{L^p} \leq 2 \cdot \left\| (\|f\|_{L^p})_{i \in \mathbb{N}} \right\|_{\ell^p} = 2 \|f\|_{L^p} \cdot n^{1/p}
\]

for all \( x_1, \ldots, x_n \in \mathbb{R}^d \) satisfying \( |x_i - x_j| \geq R_1 \) for all \( i, j \in \mathbb{N} \) with \( i \neq j \).

Note that because of \( q < \infty \), we have

\[
g := T f \in \left\{ h \in L^q(\mathbb{R}^d) \mid \text{supp } h \text{ compact} \right\},
\]

so that we can apply Lemma 4.1 to the family \((g_1, \ldots, g_n) := (g, \ldots, g)\). This yields \( R_2 > 0 \) such that

\[
\left\| \sum_{i=1}^n L_{x_i} g \right\|_{L^q} \geq \frac{1}{2} \cdot \left\| (\|g\|_{L^q})_{i \in \mathbb{N}} \right\|_{\ell^q} = \frac{\|g\|_{L^q}}{2} \cdot n^{1/q}
\]

holds for all \( x_1, \ldots, x_n \in \mathbb{R}^d \) satisfying \( |x_i - x_j| \geq R_2 \) for all \( i, j \in \mathbb{N} \) with \( i \neq j \).
Let $R := \max \{ R_1, R_2 \} > 0$. Because of $x_0 \neq 0$, there is $N \in \mathbb{N}$ with $|N x_0| > R$. If we set $x_{\ell} := \ell \cdot N x_0$ for $\ell \in \mathbb{N}$, this implies

$$|x_i - x_j| = |(i - j) \cdot N x_0| = |i - j| \cdot |N x_0| \geq |N x_0| \geq R \geq R_k$$

for $k \in \{ 1, 2 \}$ and $i, j \in \mathbb{N}$ with $i \neq j$. Hence,

$$\frac{\| g \|_{L^q} \cdot n^{1/q}}{2} \leq \left\| \sum_{i=1}^{n} L_{x_i} g \right\|_{L^q} \leq \left\| \sum_{i=1}^{n} L_{\ell N \cdot x_0} (T f) \right\|_{L^q} = \left\| T \left( \sum_{i=1}^{n} L_{\ell \cdot N \cdot x_0} f \right) \right\|_{L^q} \leq \left\| T \right\| \cdot \left\| \sum_{i=1}^{n} L_{x_i} f \right\|_{L^p} \leq 2 \left\| T \right\| \| f \|_{L^p} \cdot n^{1/p},$$

where we used that $T$ commutes with $L_{n x_0}$ for all $n \in \mathbb{N}$, since it commutes with $L_{x_0}$. All in all, we get

$$n^{\frac{1}{q} - \frac{1}{p}} \leq 4 \left\| T \right\| \cdot \frac{\| f \|_{L^p}}{\| g \|_{L^q}}$$

for all $n \in \mathbb{N}$, where the right-hand side is independent(!) of $n \in \mathbb{N}$. Thus, $\frac{1}{q} - \frac{1}{p} \leq 0$ which easily implies the claim $p \leq q$. \qed

For later use, we will also need the following fact about “richness” of $C_c^\infty(U)$ for arbitrary open $U \subset \mathbb{R}^d$.

**Lemma 4.3.** Let $\emptyset \neq U \subset \mathbb{R}^d$ be open and let $k \in \mathbb{N}$ be arbitrary. Then, for each $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq k$, there is a function $f_{\alpha, k} \in C_c^\infty(U)$ with

$$\partial^\beta \left( \mathcal{F}^{-1} f_{\alpha, k} \right)(0) = \delta_{\alpha, \beta} \quad \text{for all } \beta \in \mathbb{N}_0^d \text{ with } |\beta| \leq k.$$

**Proof.** We first show the following intermediate result:

**Claim.** Let $\emptyset \neq I \subset \mathbb{R}$ be an open, bounded interval and let $N \in \mathbb{N}$ be arbitrary. Then the map

$$\Phi : C_c^\infty(I) \to \mathbb{C}^N, f \mapsto \left( \left[ \partial^{\ell - 1} \left( \mathcal{F}^{-1} f \right) \right](0) \right)_{\ell \in \mathbb{N}}$$

is surjective.

**Proof.** Assume that the claim fails, so that $V := \Phi(C_c^\infty(I)) \subset \mathbb{C}^N$ is a strict subspace. This yields some $a \in \mathbb{C}^N \setminus \{ 0 \}$ with $\langle \Phi(f), a \rangle = 0$ for all $f \in C_c^\infty(I)$.

But by standard properties of the Fourier transform (see [3, Theorem 8.22]), we have for $f \in C_c^\infty(I)$ that

$$\left[ \partial^{\ell - 1} \left( \mathcal{F}^{-1} f \right) \right](0) = \int_{\mathbb{R}} f(\xi) \cdot (2\pi i \xi)^{\ell - 1} \, d\xi,$$

so that we get

$$0 = \int_{\mathbb{R}} f(\xi) \cdot \sum_{\ell=1}^{N} \frac{1}{a_{\ell}} (2\pi i \xi)^{\ell - 1} \, d\xi = \int_{I} f(\xi) \cdot \sum_{\ell=1}^{N} \frac{1}{a_{\ell}} (2\pi i \xi)^{\ell - 1} \, d\xi$$

for all $f \in C_c^\infty(I)$. Since $I$ is a bounded interval, we have $g \in L^2(I)$ for

$$g : I \to \mathbb{C}, \xi \mapsto \sum_{\ell=1}^{N} (2\pi i)^{\ell - 1} \frac{1}{a_{\ell}} \cdot \xi^{\ell - 1}.$$
But since $C_c^\infty (I) \subset L^2 (I)$ is dense, we conclude $g = 0$ as an element of $L^2 (I)$. By continuity of $g$, we get $g \equiv 0$ and hence (by uniqueness of the coefficients of a polynomial) $(2\pi)^{\ell -1} \frac{\pi}{\sin \frac{\pi \ell}{2}} = 0$ for all $\ell \in \mathbb{Z}$, in contradiction to $a \neq 0$. □

Now, we return to the actual proof of the lemma. Since $U \neq \emptyset$ is open, there is for each $\ell \in I$ an open, bounded interval $I_\ell \subset \mathbb{R}$ with $\prod_{\ell=1}^p I_\ell \subset U$. Now, for each $\ell \in I$, we have $\alpha_\ell \in \{0\} \cup \mathbb{K}$, so that the claim yields a function $f_\ell \in C_c^\infty (I_\ell)$ with

$$\left[ \partial^m (F^{-1} f_\ell) \right](0) = \delta_{m, \alpha_\ell} \quad \text{for all } m \in \{0\} \cup \mathbb{K}.$$ 

We have $f := f_1 \otimes \cdots \otimes f_d \in C_c^\infty (U)$ for

$$(f_1 \otimes \cdots \otimes f_d) (x) = f_1 (x_1) \cdots f_d (x_d) \quad \text{where } x = (x_1, \ldots, x_d).$$

But it is easy to see that the Fourier transform commutes with tensor products, i.e.

$$F^{-1} f = (F^{-1} f_1) \otimes \cdots \otimes (F^{-1} f_d).$$

From this, we easily get for $\beta \in \mathbb{N}_0^d$ with $|\beta| \leq k$ (which implies $\beta_\ell \in \{0\} \cup \mathbb{K}$ for all $\ell \in I$) that

$$\left[ \partial^\beta (F^{-1} f) \right](0) = \left[ \partial^{\beta_1} (F^{-1} f_1) \right](0) \cdots \left[ \partial^{\beta_d} (F^{-1} f_d) \right](0)$$

$$= \delta_{\beta_1, \alpha_1} \cdots \delta_{\beta_d, \alpha_d} = \delta_\beta, \alpha.$$

This completes the proof. □

We can now finally show that $p \leq q$ is a necessary condition for boundedness of $\partial^\alpha : S^0_{O,Y} (\mathbb{R}^d) \to L^q (\mathbb{R}^d)$.

**Theorem 4.4.** Let $p, q \in (0, \infty]$ and let $\mathcal{Q} = \{ Q_i \}_{i \in I}$ be an $L^p$-decomposition of the open set $\mathcal{O} \subset \mathbb{R}^d$. Assume that $\{0\} \neq Y \subset \mathcal{C}^I$ is a $\mathcal{Q}$-regular sequence space on $I$.

Finally, assume that $\iota_\alpha : S^0_{O,Y} (\mathbb{R}^d) \to L^q (\mathbb{R}^d), f \mapsto \partial^\alpha f$ is bounded for some $\alpha \in \mathbb{N}_0^d$. Then $p \leq q$.

**Proof.** By assumption, $Y \neq \{0\}$ is nontrivial. Since $Y \subset \mathcal{C}^I$ is solid, there is thus some $i_0 \in I$ with $\delta_{i_0} \in Y$. Since $Y$ is $\mathcal{Q}$-regular, this even implies $\chi_{i_0^*} = \delta_{i_0} \in Y$ and thus also $\sum_{\ell \in i_0^*} \chi_{\ell^\ast} \in Y$, which − by solidity − shows $\chi_{i_0^*} \in Y$.

Let $\Phi = (\varphi_i)_{i \in I}$ be an $L^p$-BAPU for $\mathcal{Q}$. By admissibility of $\mathcal{Q}$, we have $Q_{i_0} \neq \emptyset$. Since $\varphi_{i_0} \equiv 1$ on $Q_{i_0}$, this implies that the open(!) set $U := \{ x \in \mathbb{R}^d \mid \varphi_{i_0} (x) \neq 0 \} \subset Q_{i_0} \subset \mathcal{O}$ is nonempty.

Now, define

$$V := \{ f \in \mathcal{S} (\mathbb{R}^d) \mid \text{supp } \widehat{f} \subset U \} \subset \mathcal{S} (\mathbb{R}^d).$$

We claim that the operator

$$T : (V, || \cdot ||_{L^p}) \to L^q (\mathbb{R}^d), f \mapsto \partial^\alpha f$$

is bounded. Indeed, let $f \in V \subset \mathcal{S} (\mathbb{R}^d)$ be arbitrary. Because of $\text{supp } \widehat{f} \subset U \subset Q_{i_0}^*$, we have $\varphi_{i_0} \widehat{f} \equiv 0$ for all $i \in I \setminus i_0^*$. But for $i \in i_0^*$, we have

$$|| F^{-1} (\varphi_i \cdot \widehat{f}) ||_{L^p} \leq || F^{-1} \widehat{f} ||_{L^p} = || f ||_{L^p}.$$ 

For $p \in [1, \infty]$, this follows from Young’s inequality, since $|| F^{-1} \varphi_i \cdot \widehat{f} ||_{L^1} \leq C$ by definition of an $L^p$-BAPU. In case of $p \in (0, 1)$, we use again the definition of an $L^p$-BAPU and the support restrictions $\text{supp } \widehat{f} \subset Q_{i_0}^* \subset Q_{i_0}^{1*}$, as well as $\text{supp } \varphi_i \subset Q_i \subset Q_{i_0}^{1*}$, together with Corollary 2.10. Note that for $p \in (0, 1)$, the above estimate would fail in general without an a priori support estimate for $\widehat{f}$.

Altogether, we have shown

$$|| F^{-1} (\varphi_i \cdot \widehat{f}) ||_{L^p} \leq C \cdot || f ||_{L^p} \cdot \chi_{i_0^*} (i) \quad \text{for all } i \in I,$$

where $C = C (\Phi, p)$ is an absolute constant. Thus, solidity of $Y$ yields

$$|| f ||_{D (\mathcal{Q}, L^p, Y)} = \left( || F^{-1} (\varphi_i \cdot \widehat{f}) ||_{L^p} \right)_{i \in I} \in Y \leq C \left( || \chi_{i_0^*} \cdot ||_{Y} \cdot || f ||_{L^p} < \infty$$

and hence $f \in \mathcal{S}^p_{O,Y} (\mathbb{R}^d)$. Since $\iota_\alpha$ is bounded by assumption, we conclude

$$|| T f ||_{L^q} = || \partial^\alpha f ||_{L^q} = || \iota_\alpha f ||_{L^q} \leq || \iota_\alpha || \cdot || f ||_{D (\mathcal{Q}, L^p, Y)} \leq C || \iota_\alpha || \left( || \chi_{i_0^*} \cdot ||_{Y} \cdot || f ||_{L^p},$$

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so that $T$ is indeed bounded.

Since translation corresponds to modulation on the Fourier side, i.e.
\[ \hat{L_x f} = M_x \hat{f} \] for $f \in S(\mathbb{R}^d)$ and since modulation does not change the support, we see that $V$ is invariant under arbitrary translations. Furthermore, it is clear that $T$ commutes with arbitrary translations. Now, since $U \neq \emptyset$ is open, Lemma 4.3 shows that there is some $g \in C^\infty_0(U)$ such that $f := F^{-1}g \in V$ satisfies $\partial^\alpha f(0) = \partial^\alpha [F^{-1}g](0) \neq 0$. Since $\partial^\alpha f$ is continuous, this implies $Tf = \partial^\alpha f \neq 0$ in $L^p(\mathbb{R}^d)$. Finally, we have

\[ f \in S(\mathbb{R}^d) \subset \{ h \in L^p(\mathbb{R}^d) \ | \ \text{supp} \ h \ \text{compact} \} , \]

so that all prerequisites of Corollary 4.2 are satisfied. But this corollary implies $p \leq q$ as desired. \hfill $\square$

Our next major goal is to show that boundedness of all $\ell_\alpha$ for $|\alpha| = k$ already implies boundedness of the embedding $Y \cap \ell_0(I) \hookrightarrow \ell^q_{u(k,p,q)}(I)$, where $u(k,p,q)$ is defined as in Theorem 3.3. Note that (ignoring the intersection with $\ell_0(I)$) this condition coincides with the one given in Theorem 3.3 if $q = q^0 = \min \{q, q', p\}$, i.e. if $q \leq 2$. In general, for $q \in (2, \infty)$, the embedding $Y \cap \ell_0(I) \hookrightarrow \ell^q_{u(k,p,q)}(I)$ is not a necessary condition for boundedness of all $\ell_\alpha$ with $|\alpha| = k$, as we will see in Section 7 using the concrete examples of modulation spaces and Besov spaces. This raises the question whether one can find less restrictive sufficient conditions for boundedness of the $\ell_\alpha$. For modulation spaces\cite{12} and Besov spaces\cite{19}, this is indeed the case. Nevertheless, I do not know of any results which apply in the full generality considered here. This is a valuable topic for further research.

To prove necessity of the embedding $Y \cap \ell_0(I) \hookrightarrow \ell^q_{u(k,p,q)}(I)$, we will make the technical assumption that $Q = (Q_i)_{i \in I} = (T_i Q'_i + b_i)_{i \in I}$ is tight, i.e. that there is some $\varepsilon > 0$ such that for each $i \in I$, some ball $B_\varepsilon(c_i)$ is contained in the “normalized” set $Q'_i$. In principle, the points $(c_i)_{i \in I}$ are allowed to vary wildly with $i \in I$. But the next lemma shows that we can actually choose the $c_i$ only from a fixed finite set.

This technical fact will become important for constructing suitable functions to “test” boundedness of the maps $\ell_\alpha$ in order to derive boundedness of the embedding $Y \cap \ell_0(I) \hookrightarrow \ell^q_{u(k,p,q)}(I)$.

**Lemma 4.5.** Let $Q = (T_i Q'_i + b_i)_{i \in I}$ be a tight semi-structured covering of the open set $\emptyset \neq O \subset \mathbb{R}^d$.

Then there is some $\varepsilon > 0$ and finitely many points $a_1, \ldots, a_N \in \mathbb{R}^d$, such that for each $i \in I$, there is some $\ell_i \in N$ with $B_{\varepsilon}(a_i) \subset Q'_i$. \hfill $\blacksquare$

**Proof.** Fix some $\varepsilon > 0$ and for each $i \in I$ some $c_i \in \mathbb{R}^d$ with $B_\varepsilon(c_i) \subset Q'_i$. Existence is ensured by tightness of $Q$. Furthermore, since $Q$ is semi-structured, there is some $R > 0$ with $Q'_i \subset B_R(0)$ for all $i \in I$.

Now, using Zorn’s Lemma (or in fact using an induction, since $I$ is countable), we can choose a maximal subset $J \subset I$ with the property $B_{\varepsilon/3}(c_i) \cap B_{\varepsilon/3}(c_j) = \emptyset$ for all $i, j \in J$ with $i \neq j$. For an arbitrary finite subset $L \subset J$, we now have – because the sets $B_{\varepsilon/3}(c_i) \subset B_\varepsilon(c_i) \subset Q'_i \subset B_R(0)$ for $i \in L$ are pairwise disjoint – that

\[ |L| \cdot \lambda \left( B_{\varepsilon/3}(0) \right) = \sum_{i \in L} \lambda \left( B_{\varepsilon/3}(c_i) \right) = \lambda \left( \bigcup_{i \in L} B_{\varepsilon/3}(c_i) \right) \leq \lambda \left( B_R(0) \right) \]

and hence $|L| \leq \lambda \left( B_R(0) \right) / \lambda \left( B_{\varepsilon/3}(0) \right)$. Since this holds for every finite subset $L \subset J$, $J$ must be finite, say $J = \{i_1, \ldots, i_N\}$. Define $a_\ell := c_{i_\ell}$ for $\ell \in N$.

Now, let $i \in I$ be arbitrary. If $i \notin J$, i.e. $i = i_\ell$ for some $\ell \in N$, then

\[ B_{\varepsilon/3}(a_\ell) = B_{\varepsilon/3}(c_{i_\ell}) \subset B_\varepsilon(c_{i_\ell}) \subset Q'_i = Q'_i, \]

Otherwise, if $i \notin J$, then by maximality of $J$, we have $B_{\varepsilon/3}(c_i) \cap B_{\varepsilon/3}(c_j) \neq \emptyset$ for some $j \in J$, say $j = i_\ell$ for some $\ell \in N$. For $x \in B_{\varepsilon/3}(c_i)$, we now have

\[ |x - c_i| \leq |x - c_j| + |c_j - y| + |y - c_i| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \]

where $y \in B_{\varepsilon/3}(c_i) \cap B_{\varepsilon/3}(c_j) \neq \emptyset$ was chosen arbitrarily. This means

\[ B_{\varepsilon/3}(a_\ell) = B_{\varepsilon/3}(c_{i_\ell}) = B_{\varepsilon/3}(c_i) \subset B_\varepsilon(c_i) \subset Q'_i \]

We have thus shown that for every $i \in I$, there is some $\ell \in N$ with $B_{\varepsilon/3}(a_\ell) \subset Q'_i$, as desired. \hfill $\square$
As a further technical result, we need the following:

**Lemma 4.6.** Let \( Q = (T_iQ'_i + b_i)_{i \in I} \) be a tight semi-structured covering. Then the following hold:

1. We have \( |b_i| + \|T_i\| > \sup_{x \in Q_i} |x| \) uniformly in \( i \in I \).
2. If \( I^{(0)} \subset I \) and \( M > 0 \) satisfy \( \|T_i^{-1}\| < M \) for all \( i \in I^{(0)} \), then there is some \( \delta > 0 \) and for each \( i \in I^{(0)} \), some \( y_i \in \mathbb{R}^d \) with \( B_\delta (y_i) \subset Q_i \) and with

\[
|y_i| \asymp |b_i| + \|T_i\|.
\]

**Proof.** By definition of a semi-structured covering, there is some \( R > 0 \) with \( Q'_i \subset B_R (0) \) for all \( i \in I \). Furthermore, by tightness, there is some \( \varepsilon > 0 \) and for each \( i \in I \) some \( x_i \in \mathbb{R}^d \) with \( B_\varepsilon (x_i) \subset Q'_i \). We clearly have

\[
\sup_{x \in Q_i} |x| = \sup_{x \in Q'_i} \|T_i x + b_i\| \leq |b_i| + \|T_i\| \sup_{x \in Q'_i} |x| \leq |b_i| + R \cdot \|T_i\|,
\]

so that for the first part of the lemma, it remains to show "\( \preccurlyeq \)".

Now, let \( i \in I \) be arbitrary. We distinguish two cases:

**Case 1.** We have \( |b_i| > 2R \cdot \|T_i\| \). In this case, we choose \( y_i := b_i + T_i x_i \in Q_i \). Note \( |x_i| \leq R \) because of \( x_i \in B_\varepsilon (x_i) \subset Q'_i \subset B_R (0) \). Hence,

\[
|b_i| + \|T_i\| \leq \frac{|b_i|}{4} + \frac{R}{2} \|T_i\| \leq \frac{|b_i|}{2} \leq \frac{|b_i|}{2} - \|T_i\| \leq |y_i| \leq \|T_i\| \cdot |x_i| \leq |y_i| \leq |b_i| + \|T_i\| |x_i| \leq |b_i| + R \cdot \|T_i\| \leq |b_i| + \|T_i\|.
\]

Because of \( y_i \in Q_i \), this shows \( \sup_{x \in Q_i} |x| \geq |y_i| \geq |b_i| + \|T_i\| \), as desired.

Now, assume that \( i \in I^{(0)} \), so that \( \|T_i^{-1}\| \leq M \). For \( \delta < \frac{\varepsilon}{3T} \) and \( x \in B_\delta (y_i) \), we then have

\[
\|T_i^{-1} (x - T_i x_i - b_i)\| \leq \|T_i^{-1}\| \cdot \|x - T_i x_i - b_i\| \leq M \cdot \|x - y_i\| < M \cdot \delta < \varepsilon.
\]

Thus,

\[
x = T_i x_i + b_i + T_i \left[ T_i^{-1} (x - T_i x_i - b_i) \right]
\]

\[
\in T_i x_i + b_i + T_i \left( B_\varepsilon (0) \right)
\]

\[
= T_i \left( B_\varepsilon (x_i) \right) + b_i
\]

\[
\subset T_i Q'_i + b_i = Q_i,
\]

so that we get \( B_\delta (y_i) \subset Q_i \) as claimed.

**Case 2.** We have \( |b_i| \leq 2R \cdot \|T_i\| \). Let \( r := \frac{\varepsilon}{2} > 0 \) and choose \( z \in \mathbb{R}^d \) with \( |z| = r \) and \( \|T_i z\| = r \cdot \|T_i\| \). Then we have

\[
diam \left( T_i \left( B_r (x_i) \right) + b_i \right) = diam \left( T_i \left( B_r (0) \right) \right)
\]

\[
= diam \left( T_i \left( B_\varepsilon (0) \right) \right)
\]

\[
\geq \|T_i z - T_i (-z)\| = 2r \cdot \|T_i\|.
\]

Now, if \( diam (A) > \alpha > 0 \) for some \( A \subset \mathbb{R}^d \), then there is some \( x \in A \) with \( |x| \geq \frac{\alpha}{2} \), since otherwise we would have \( |x - y| \leq |x| + |y| \leq \alpha \) for all \( x, y \in A \) and hence \( diam (A) \leq \alpha \).
Thus, by the above calculation, there is some \( z_i \in B_r (x_i) \) with \( |T_i z_i + b_i| \geq \frac{r}{2} \cdot \|T_i\| \). Recall \( B_r (x_i) \subset B_{\varepsilon} (x_i) \subset Q_i' \subset B_{R_{1}} (0) \). For \( y_i := T_i z_i + b_i \in Q_i \), we thus have
\[
|b_i| + \|T_i\| \leq \frac{r}{4} \cdot \left( \|T_i\| + \frac{|b_i|}{2R} \right) \leq \frac{r}{2} \cdot \|T_i\| \leq |y_i| \leq \|T_i\| |z_i| + |b_i| \leq |b_i| + \|T_i\|.
\]
As above, this shows \( \sup_{x \in Q_i} |x| \geq |b_i| + \|T_i\| \).

Now, assume again that \( i \in I^{(0)} \), so that \( \|T^{-1}_i\| \leq M \). For \( \delta < \frac{r}{2M} \) and arbitrary \( x \in B_{\delta} (y_i) \), this yields
\[
\|T^{-1}_i (x - T_i x_i - b_i)\| \leq \|T^{-1}_i (x - y_i)\| + \|T^{-1}_i (y_i - T_i x_i - b_i)\| \\
\leq M \cdot |x - y_i| + \|T^{-1}_i (T_i (z_i - x_i))\| \\
\leq M \cdot |x - y_i| + |z_i - x_i| \\
< M \delta + r \\
< \varepsilon.
\]
As in the first case, this implies \( B_{\delta} (y_i) \subset T_i (B_{\varepsilon} (x_i)) + b_i \subset Q_i \).

We have thus shown that any \( \delta > 0 \) with \( \delta < \frac{r}{2M} \) makes the second part of the lemma true. \( \square \)

Now, we can finally prove necessity of the embedding \( Y \cap \ell_0 (I) \hookrightarrow \ell^{q}_{u(k,p,q)} (I) \). Note that we will see in the next section that the restriction to \( \ell_0 (I) \) is superfluous if \( Y \) is a weighted Lebesgue space.

**Theorem 4.7.** Let \( p,q \in (0, \infty) \) and let \( Q = (Q_i)_{i \in I} = (T_i Q_i' + b_i)_{i \in I} \) be a tight semi-structured \( L^p \)-decomposition covering of the open set \( \emptyset \neq O \subset \mathbb{R}^d \). Assume that \( Y \leq C^\gamma \) is a \( Q \)-regular sequence space.

Finally, let \( k \in \mathbb{N}_0 \) and assume that
\[
\iota_{\alpha} : S^{\partial, Y} \hookrightarrow L^q (\mathbb{R}^d) : f \mapsto \partial^\alpha f
\]
is bounded for all \( \alpha \in \mathbb{N}_0^d \) with \( |\alpha| = k \). Define
\[
u^{(k,p,q)} := \left( |\det T_i|^{\frac{1}{p' - \frac{1}{q}}} \cdot \left( |b_i|^k + \|T_i\|^k \right) \right)_{i \in I}
\]
as in Theorem 3.4.

Then the following hold:

1. The map
\[
Y \cap \ell_0 (I) \hookrightarrow \ell^{q}_{u(k,p,q)} (I) \hookrightarrow \ell^{q}_{u(k,p,q)} (I), \ (c_i)_{i \in I} \hookrightarrow (c_i)_{i \in I}
\]
is well-defined and bounded, where \( \ell_0 (I) \) denotes the space of finitely supported sequences on \( I \).

2. If \( q = \infty \), then the map
\[
Y \cap \ell_0 (I) \hookrightarrow \ell^{\infty}_{u(k,p,q)} (I) = \ell^{1}_{u(k,p,q)} (I), \ (c_i)_{i \in I} \hookrightarrow (c_i)_{i \in I}
\]
is well-defined and bounded.

3. For each subset \( I^{(0)} \subset I \) and each \( M > 0 \) with \( \|T^{-1}_i\| \leq M \) for all \( i \in I^{(0)} \), the following hold:

   a. If \( q \in [2, \infty) \), then
\[
Y \cap \ell_0 (I^{(0)}) \hookrightarrow \ell^{2}_{u(k,p,q)} (I).
\]

   b. If \( q \in (0, \infty) \), then
\[
Y \cap \ell_0 (I^{(0)}) \hookrightarrow \ell^{2}_{u(k,p,q)} (I).
\]

**Proof.** Our argument is based on the following claim, whose proof we postpone to the end of the current proof.

**Claim 4.8.** Let
\[
I_1 := \left\{ i \in I \middle| \|T_i\|^k > (2\pi |b_i|)^k \right\},
\]
\[
I_2 := \left\{ i \in I \middle| \|T_i\|^k \leq (2\pi |b_i|)^k \right\}.
\]

Then there are a finite family of functions \( \mathcal{F} \subset C^\infty (\mathbb{R}^d) \), some \( \delta > 0 \) and \( C_1, C_2 > 0 \) such that the following hold:

1. For each \( i \in I_1 \), there is some \( \alpha^{(i)} \in \mathbb{N}_0^d \) with \( |\alpha^{(i)}| = k \) and a function \( \gamma_i \in C^\infty (Q_i) \) with
\[
|\partial^{\alpha^{(i)}} (\mathcal{F}^{-1} \gamma_i) (x)| \geq C_1 \cdot |\det T_i| \cdot \|T_i\|^k \cdot \chi_{B_{\delta} (0)} (T_i x) \quad \text{for all } x \in \mathbb{R}^d.
\]
(2) For each \( i \in I_2 \), there is some \( \alpha(i) \in \mathbb{N}_{0}^{d} \) with \( |\alpha(i)| = k \) and a function \( \gamma_i \in C_{c}^{\infty}(Q_i) \) with
\[
\left| \left[ a^{\alpha(i)} \right] \left( \mathcal{F}^{-1} \gamma_i \right) \right| (x) \geq C_2 \cdot |\det T_i| \cdot |b_i|^k \cdot \chi_{B_i(0)}(T_i^T x) \quad \text{for all } x \in \mathbb{R}^d.
\] (4.3)
Furthermore, each \( \gamma_i \) (for \( i \in I_1 \) as well as for \( i \in I_2 \)) is of the form
\( \gamma_i = L_b (f_i \circ T_i^{-1}) \), i.e.
\[
\gamma_i(\xi) = f_i(T_i^{-1}(\xi - b_i))
\] (4.4)
for all \( \xi \in \mathbb{R}^d \) and some \( f_i \in \mathcal{F} \).

Note that equations (4.2) and (4.3) - together with the definitions of \( I_1, I_2 \) - imply
\[
\left| \left[ a^{\alpha(i)} \right] \left( \mathcal{F}^{-1} \gamma_i \right) \right| (x) \geq C \cdot |\det T_i| \cdot (|b_i|^k + ||T_i||^k) \cdot \chi_{B_i(0)}(T_i^T x)
\] (4.5)
for all \( x \in \mathbb{R}^d \) and \( i \in I \) with \( C := \frac{1}{2} \cdot \min \left\{ C_1, (2\pi)^{-k} \cdot C_2 \right\} > 0 \).

Now, let \( c = (c_i)_{i \in I} \in Y \cap \ell_0 (I) \) be arbitrary and let \( I_0 := \text{supp } c \subset I \). For \( \alpha \in \mathbb{N}_{0}^{d} \) with \( |\alpha| = k \), set
\[
I(\alpha) := \left\{ i \in I \mid \alpha(i) = \alpha \right\}.
\]
Finally, let \( S^1 := \{ z \in \mathbb{C} \mid |z| = 1 \} \) and for arbitrary sequences \( x = (x_i)_{i \in I} \in (\mathbb{R}^d)^I, \ v = (v_i)_{i \in I} \in (S^1)^I \) and an arbitrary multiindex \( \alpha \in \mathbb{N}_{0}^{d} \) with \( |\alpha| = k \), define
\[
g_{x,v,\alpha} := \sum_{i \in I(\alpha)} (\varepsilon_i c_i |\det T_i|^{\frac{1}{\gamma_i}} \cdot M_{-x_i} \gamma_i) \in C_{c}^{\infty}(\mathcal{O}),
\]
\[
f_{x,v,\alpha} := \mathcal{F}^{-1} g_{x,v,\alpha} = \sum_{i \in I(\alpha)} (\varepsilon_i c_i |\det T_i|^{\frac{1}{\gamma_i}} \cdot L_{x_i} \left[ \mathcal{F}^{-1} \gamma_i \right] ) \in \mathcal{F}^{-1} (C_{c}^{\infty}(\mathcal{O})) = \mathcal{S}_{\mathcal{O}}(\mathbb{R}^d).
\]
Let us first show \( g_{x,v,\alpha} \in \mathcal{D}_{F}(Q, L^p, Y) \) and provide an estimate for the corresponding norm. To this end, we note that equation (4.4) - together with elementary properties of the Fourier transform - implies
\[
\| \mathcal{F}^{-1} \gamma_i \|_{L^p} = |\det T_i| \cdot \| M_{b_i} \left[ \mathcal{F}^{-1} f_i \circ T_i^T \right] \|_{L^p}
\]
\[
= |\det T_i| \cdot \| \left( \mathcal{F}^{-1} f_i \circ T_i^T \right) \|_{L^p}
\]
\[
= |\det T_i|^{1 - \frac{1}{p}} \cdot \| \mathcal{F}^{-1} f_i \|_{L^p}
\]
\[
\leq K \cdot |\det T_i|^{1 - \frac{1}{p}},
\] (4.6)
for the finite(!) constant \( K := \sup_{f \in \mathcal{F}} \| \mathcal{F}^{-1} f \|_{L^p} \). For finiteness of \( K \), we used that \( \mathcal{F} \subset C_{c}^{\infty}(\mathbb{R}^d) \) is a finite set.

Now, let \( \Phi = (\varphi_i)_{i \in I} \) be an \( L^p \)-BAPU for \( Q \) and set \( N := \sup_{i \in I} |i^*| \). For arbitrary \( \ell \in I \), we have
\[
\varphi_{\ell} \cdot g_{x,v,\alpha} = \sum_{i \in I(\alpha) \cap \ell^*} \varepsilon_i c_i |\det T_i|^{\frac{1}{\gamma_i}} \cdot M_{-x_i} (\varphi_{\ell} \gamma_i),
\] (4.7)
since multiplication by \( \varphi_{\ell} \) and modulation by \(-x_i\) commute and because of \( \gamma_i \in C_{c}^{\infty}(Q_i) \). Thus, the (quasi)-triangle inequality for \( L^p(\mathbb{R}^d) \), together with the uniform estimate \( |I(\alpha) \cap \ell^*| \leq |\ell^*| \leq N \), yields
\[
\| \mathcal{F}^{-1} (\varphi_{\ell} \cdot g_{x,v,\alpha}) \|_{L^p} \leq \sum_{i \in I(\alpha) \cap \ell^*} |c_i| |\det T_i|^{\frac{1}{\gamma_i}} \cdot \left\| \mathcal{F}^{-1} (M_{-x_i} (\varphi_{\ell} \gamma_i)) \right\|_{L^p}
\]
\[
= \sum_{i \in I(\alpha) \cap \ell^*} |c_i| |\det T_i|^{\frac{1}{\gamma_i}} \cdot \left\| \mathcal{F}^{-1} (\varphi_{\ell} \gamma_i) \right\|_{L^p}
\]
\[
\lesssim \sum_{i \in I(\alpha) \cap \ell^*} |c_i| |\det T_i|^{\frac{1}{\gamma_i}} \cdot \left\| \mathcal{F}^{-1} \gamma_i \right\|_{L^p}
\] (by equation (4.4))
\[
\lesssim \sum_{i \in I(\alpha) \cap \ell^*} |c_i| \leq (|\ell^*|)_\ell
\] (4.8)
for all \( \ell \in I \). Here, the sequence \( \{c_i\} \in Y \) is defined by \( |c_i| = |c_i| \) and furthermore \( (d^*)_\ell = \sum_{\ell \in \ell^*} d_{\ell} \) as usual. Above, the step marked with (*) can be justified as follows: For \( p \in [1, \infty] \), it is a direct consequence of Young’s convolution inequality \( L^1 \ast L^p \rightarrow L^p \), since the definition of an \( L^p \)-BAPU implies finiteness of
sup_{t \in I} \parallel F^{-1} \varphi_t \parallel_{L^1}$. In case of $p \in (0,1)$, it is a consequence of Corollary 2.14 and the definition of an $L^p$-BAPU, since we have the support estimate supp $\gamma_i \subset Q_i \subset Q_i^*$ for $i \in \ell^*$ and also supp $\varphi_\ell \subset \overline{Q_\ell} \subset Q_\ell^*$.

All in all, we conclude $g_{x,\varepsilon,\alpha} \in D_F(Q, L^p, Y)$, since the previous estimate yields – by solidity and $Q$-regularity of $Y$ – that

$$
\|g_{x,\varepsilon,\alpha}\|_{D_F(Q, L^p, Y)} = \left\| \left( \|F^{-1} (\varphi_\ell \cdot g_{x,\varepsilon,\alpha})\|_{L^p}\right)_{\ell \in I} \right\|_{Y} \\
\lesssim \|c\|_Y \lesssim \|c\|_Y = \|c\|_Y < \infty.
$$

Hence, $f_{x,\varepsilon,\alpha} \in S^{p,Y}_Q$. Recall that we assume $\epsilon_0 : S^{p,Y}_Q \to L^q(\mathbb{R}^d)$, $f \mapsto \partial^\alpha f$ to be bounded. Thus,

$$
\|\partial^\alpha f_{x,\varepsilon,\alpha}\|_{L^q} \lesssim \epsilon_0 f_{x,\varepsilon,\alpha} \|D_{(Q, L^p, Y)} = \|g_{x,\varepsilon,\alpha}\|_{D_F(Q, L^p, Y)} \lesssim \|c\|_Y,
$$

where the implied constants are independent of $x, \varepsilon$ and of $c$.

Finally, we will obtain a lower bound on $\|\partial^\alpha f_{x,\varepsilon,\alpha}\|_{L^q}$ which will then imply the claim. Indeed, we have

$$
\|\partial^\alpha f_{x,\varepsilon,\alpha}\|_{L^q} = \left\| \sum_{i \in I^{(\alpha)}} \left( \epsilon_i c_i |\det T_i|^{\frac{1}{p}-1} \cdot L_{x,i} \left( \partial^\alpha \left( F^{-1} \gamma_i \right) \right) \right) \right\|_{L^q}
$$

(by Lemma 4.1 for suitable $x = (x_i)_{\ell \in I} \in (\mathbb{R}^d)^I$)

$$
\geq \frac{1}{2} \left\| \left( |c_i| |\det T_i|^{\frac{1}{p}-1} \cdot \partial^\alpha \left( F^{-1} \gamma_i \right) \right)_{i \in I^{(\alpha)} \cap I_0} \right\|_{L^q}
$$

($\alpha = \alpha^{(i)}$ for $i \in I^{(\alpha)}$ and $c_i = 0$ for $i \notin I_0$)

$$
\left( \alpha = \alpha^{(i)} \right) \leq \frac{1}{2} \left\| \left( |c_i| |\det T_i|^{\frac{1}{p}-1} \cdot \partial^\alpha \left( F^{-1} \gamma_i \right) \right)_{i \notin I^{(\alpha)}} \right\|_{L^q}.
$$

Note that Lemma 4.1 is indeed applicable (even for $q = \infty$), since $I^{(\alpha)} \cap I_0 \subset I_0$ is finite and because of $\partial^\alpha \left( F^{-1} \gamma_i \right) \in S(\mathbb{R}^d) \subset C_0(\mathbb{R}^d)$.

Now, we finally employ estimate (4.5) to conclude – for $q \in (0, \infty)$ – that

$$
\|\partial^\alpha (F^{-1} \gamma_i)\|_{L^q} = \left[ \int_{\mathbb{R}^d} \left( \left( \partial^\alpha (F^{-1} \gamma_i) \right) (x) \right)^q \, dx \right]^{1/q}
$$

$$
\geq C \cdot |\det T_i| \cdot \left( |b_i|^k + |T_i|^k \right) \cdot \left( \int_{\mathbb{R}^d} \left[ \chi_{B_{\mathbb{R}^d}} \left( T_i T x \right) \right]^q \, dx \right)^{1/q}
$$

($y = T_i T x$)

$$
\geq C \cdot |\det T_i|^{-\frac{1}{q}} \cdot \left( |b_i|^k + |T_i|^k \right) \cdot \left( \int_{\mathbb{R}^d} \left[ \chi_{B_{\mathbb{R}^d}} (y) \right]^q \, dy \right)^{1/q}
$$

$$
\geq C \cdot |\lambda (B_{\mathbb{R}^d})|^{1/q} \cdot |\det T_i|^{-\frac{1}{q}} \cdot \left( |b_i|^k + |T_i|^k \right),
$$

(4.10)

where $\lambda$ denotes the Lebesgue measure. For $q = \infty$, equation (4.5) easily shows that equation (4.10) still holds. By putting everything together (recall equation (4.9)), we arrive at

$$
\|c\|_Y \gtrsim \|\partial^\alpha f_{x,\varepsilon,\alpha}\|_{L^q}
$$

(choose suitable $x = (x_i)_{\ell \in I} \in (\mathbb{R}^d)^I$)

$$
\geq \left\| \left( |c_i| |\det T_i|^{\frac{1}{p}-1} \cdot \partial^\alpha \left( F^{-1} \gamma_i \right) \right)_{i \in I^{(\alpha)}} \right\|_{L^q}
$$

$$
\geq \left\| \left( |c_i| |\det T_i|^{\frac{1}{p}-1} \cdot \partial^\alpha \left( F^{-1} \gamma_i \right) \right)_{i \notin I^{(\alpha)}} \right\|_{L^q}
$$

$$
= \left\| \left( c_i \cdot \chi_{I^{(\alpha)}} \right)_{i \in I} \right\|_{L^q_{\alpha^{(k,p,q)}}}.
$$

where the implied constants are independent of $c$. Furthermore, because of

$$
I = \bigcup_{\alpha \in \mathbb{N}_0^d, |\alpha| = k} I^{(\alpha)},
$$

the (quasi)-triangle inequality for $L^q_{\alpha^{(k,p,q)}} (I)$ finally implies

$$
\|c\|_{L^q_{\alpha^{(k,p,q)}}} \lesssim \sum_{\alpha \in \mathbb{N}_0^d, |\alpha| = k} \left\| \left( c_i \cdot \chi_{I^{(\alpha)}} \right)_{i \in I} \right\|_{L^q_{\alpha^{(k,p,q)}}} \lesssim \|c\|_Y,
$$

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where the implied constants are independent of $c \in Y \cap \ell_0 (I)$. This establishes the first part of the theorem.

For the second part, we use $x_i = 0$ for all $i \in I$ and we choose each $\varepsilon_i \in S^1$ in such a way that

$$
\varepsilon_i c_i \cdot \left[ \partial^\alpha (F^{-1} \gamma_i) \right] (0) = \left| c_i \cdot \left[ \partial^\alpha (F^{-1} \gamma_i) \right] (0) \right|
$$

(by equation (4.10)) $\geq C \cdot |c_i| \cdot |\det T_i| \cdot \left( |b_i|^k + \|T_i\|^k \right)$.

Now, let $\alpha \in \mathbb{N}_0^d$ with $|\alpha| = k$ be arbitrary. By continuity of $\partial^\alpha f_{x, \varepsilon, \alpha} \in S (\mathbb{R}^d)$, the $L^\infty$ norm of the partial derivative $\partial^\alpha f_{x, \varepsilon, \alpha}$ coincides with the genuine supremum norm. Hence,

$$
\| \partial^\alpha f_{x, \varepsilon, \alpha} \|_{L^\infty} \geq \| \partial^\alpha f_{x, \varepsilon, \alpha} (0) \|
$$

$$
(x_i = 0 \text{ for all } i \in I) = \left[ \partial^\alpha \sum_{i \in I^{(\alpha)}} (\varepsilon_i c_i \cdot |\det T_i|^{\frac{1}{p} - \frac{1}{q}} \cdot F^{-1} \gamma_i) \right] (0)
$$

$$
(\alpha = \alpha^{(i)} \text{ for } i \in I^{(\alpha)}) = \sum_{i \in I^{(\alpha)}} \varepsilon_i c_i \cdot |\det T_i|^{\frac{1}{p} - \frac{1}{q}} \cdot \left[ \partial^\alpha (F^{-1} \gamma_i) \right] (0)
$$

(all summands nonnegative) $\geq C \cdot \sum_{i \in I^{(\alpha)}} |c_i| \cdot |\det T_i|^{\frac{1}{p}} \cdot \left( |b_i|^k + \|T_i\|^k \right)$.

But as seen in equation (4.13) above, we have $\| \partial^\alpha f_{x, \varepsilon, \alpha} \|_{L^q} \lesssim \|c\|_Y$, where the implied constant does not depend on $\varepsilon, x$ and $c$. Thus, we can sum over all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| = k$, to obtain (recalling that $q = \infty$)

$$
\|c\|_Y \gtrsim C^{-1} \cdot \sum_{|\alpha| = k} \| \partial^\alpha f_{x, \varepsilon, \alpha} \|_{L^\infty}
$$

$$
\geq \sum_{i \in I} |c_i| \cdot |\det T_i|^{\frac{1}{p}} \cdot \left( |b_i|^k + \|T_i\|^k \right)
$$

$$
\left( \text{since } \frac{1}{q} = 0 \right) = \| (c_i)_{i \in I} \|_{\ell^1 (k, p, q)}.
$$

Since $(c_i)_{i \in I} \in Y \cap \ell_0 (I)$ was arbitrary, this completes the proof of the second part of the theorem.

For the first statement in the last part of the theorem, we assume $2 \leq q < \infty$ and we let $M > 0$ with $\|T_i^{-1}\| \leq M$ for all $i \in I^{(0)}$. Furthermore, we use $x_i = 0$ for all $i \in I$ and we assume $c = (c_i)_{i \in I} \in Y \cap \ell_0 (I^{(0)})$, i.e. $c_i = 0$ for all $i \in I \setminus I^{(0)}$ and hence $I_0 = \operatorname{supp} c \subset I^{(0)}$. Let $\alpha \in \mathbb{N}_0^d$ with $|\alpha| = k$. We want to apply Khintchine’s inequality (cf. [24 Proposition 4.5]), so that we consider $\varepsilon = (\varepsilon_i)_{i \in I^{(\alpha)} \cap I_0}$ as a random vector, with $\varepsilon_i$ independent and identically distributed Rademacher random variables. As noted above in equation (4.9), we have

$$
\| \partial^\alpha f_{x, \varepsilon, \alpha} \|_{L^q} \lesssim \|c\|_Y,
$$

where the implied constant does not depend on $\varepsilon, x$ and $c$. By raising this to the $q$-th power and taking expectations (with respect to $\varepsilon$), we conclude (cf. the definition of $f_{x, \varepsilon, \alpha}$ from above)

$$
\|c\|_Y^q \gtrsim \mathbb{E}_x \| \partial^\alpha f_{x, \varepsilon, \alpha} \|_{L^q}^q
$$

$$
(x_i = 0 \text{ and } \alpha = \alpha^{(i)} \forall i \in I^{(\alpha)}) = \int_{\mathbb{R}^d} \mathbb{E}_x \left[ \sum_{i \in I^{(\alpha)} \cap I_0} \varepsilon_i c_i \cdot |\det T_i|^{\frac{1}{p} - \frac{1}{q}} \cdot \partial^\alpha (F^{-1} \gamma_i) (x) \right] q \, dx
$$

(by Khintchine’s inequality) $\gtrsim \int_{\mathbb{R}^d} \left[ \sum_{i \in I^{(\alpha)} \cap I_0} |c_i| \cdot |\det T_i|^{\frac{1}{p} - \frac{1}{q}} \cdot \partial^\alpha (F^{-1} \gamma_i) (x) \right] q/2 \, dx$

(by eq. 4.13) $\gtrsim \int_{\mathbb{R}^d} \left[ \sum_{i \in I^{(\alpha)} \cap I_0} |c_i| \cdot |\det T_i|^{\frac{1}{p} - \frac{1}{q}} \cdot |\det T_i| \left( |b_i|^k + \|T_i\|^k \right) \cdot \chi_{B_\delta (0)} \left( T_i^T x \right)^2 \right] q/2 \, dx
$$

$$
= \int_{\mathbb{R}^d} \left[ \sum_{i \in I^{(\alpha)} \cap I_0} |c_i| \cdot |\det T_i|^{\frac{1}{p}} \left( |b_i|^k + \|T_i\|^k \right) \cdot \chi_{B_\delta (0)} \left( T_i^T x \right)^2 \right] q/2 \, dx.
$$

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Now, note that if $\chi_{B_\delta(0)}(T^T_i x) \neq 0$ for some $i \in I_0 \subset I^{(0)}$, we get $T^T_i x \in B_\delta(0)$, i.e. $x \in T^T_i (B_\delta(0))$ and hence $|x| \leq \delta \|T^T_i\| = \delta \|T^{-1}_i\| \leq M\delta$. Thus, we get

$$\|c\|^q \geq \int_{B_{M\delta}(0)} \left( \sum_{i \in I^{(0)} \cap I_0} |c_i| |\det T_i|^{\frac{q}{2}} \left( |b_i|^k + \|T_i\|^k \right) \cdot \chi_{B_\delta(0)}(T_i^T x) \right)^{q/2} \, d x$$

(by Jensen’s ineq., since $\frac{q}{2} \geq 1$)

$$\geq \left[ \int_{B_{M\delta}(0)} \sum_{i \in I^{(0)} \cap I_0} |c_i| |\det T_i|^{\frac{q}{2}} \left( |b_i|^k + \|T_i\|^k \right) \cdot \chi_{B_\delta(0)}(T_i^T x) \right]^{q/2} \, d x$$

$$\geq \left[ \sum_{i \in I^{(0)} \cap I_0} |c_i| |\det T_i|^{\frac{q}{2}} \left( |b_i|^k + \|T_i\|^k \right) \cdot \chi_{B_\delta(0)}(T_i^T x) \right]^{q/2} \, d x$$

$$\geq \left[ \sum_{i \in I^{(0)} \cap I_0} |c_i| |\det T_i|^{\frac{q}{2}} \left( |b_i|^k + \|T_i\|^k \right) \cdot \chi_{B_\delta(0)}(T_i^T x) \right]^{q/2} \, d x$$

$$\geq \left( \sum_{i \in I^{(0)} \cap I_0} |c_i| |\det T_i|^{\frac{q}{2}} \left( |b_i|^k + \|T_i\|^k \right) \right)^{q/2}$$

Now, the same steps as in the proof of the first part of the theorem show $Y \cap I_0 \sim I^{(0)} \hookrightarrow \ell^2_{u,(p,2)}(I)$, as desired. The step marked with (†) in the estimate above used exactly the same argument as before the displayed equation to justify changing the domain of integration from $B_{M\delta}(0)$ to all of $\mathbb{R}^d$.

Now, let us prove the second embedding in the last part of the theorem. To this end, recall from Lemma 4.6 that there is some $\delta > 0$ and for each $i \in I^{(0)}$ some $y_i \in \mathbb{R}^d$ with $|y_i| \approx |b_i| + \|T_i\|$ and with $B_\delta(y_i) \subset Q_i$. Choose $j^{(i)} \in \mathbb{N}^d$ with

$$L_i := \left| (y_i)_{j^{(i)}} \right| = \max_{j \in \mathbb{N}^d} \left| (y_i)_j \right| \geq \frac{|y_i|}{d} \approx |b_i| + \|T_i\|$$

and define $\alpha^{(i)} := k \cdot e_{j^{(i)}} \in \mathbb{N}^d$ with $|\alpha^{(i)}| = k$. To make the above estimate precise, let $L > 0$ satisfy $L_i \geq L \cdot (|b_i| + \|T_i\|)$ for all $i \in I^{(0)}$. Note that we have $1 = \|T_i^{-1}T_i\| \leq M \cdot \|T_i\|$ and thus

$$L_i \geq L \cdot (|b_i| + \|T_i\|) \geq L \cdot \|T_i\| \geq \frac{L}{M} \quad \forall i \in I^{(0)}.$$  (4.11)

Now, for $\alpha \in \mathbb{N}^d$ with $|\alpha| = k$, let

$$I_\alpha := \left\{ i \in I^{(0)} \left| \alpha^{(i)} = \alpha \right. \right\}.$$  

Fix $\alpha \in \mathbb{N}^d$ with $|\alpha| = k$. By Lemma 4.3 there is a function $\varphi \in C^\infty_c(B_\delta(0))$ with

$$(\partial^\beta \varphi)(0) = \delta_{0,\beta} \quad \text{for all } \beta \in \mathbb{N}_0^d \text{ with } |\beta| \leq k, \text{ where } \psi := F^{-1}\varphi. \quad (4.12)$$

Note $L_y, \varphi \in C^\infty_c(B_\delta(y_i)) \subset C^\infty_c(Q_i) \subset C^\infty_c(O)$ for each $i \in I^{(0)}$. Similar to the construction above, let $c = (c_i)_{i \in I} \in Y \cap I_0 \sim I^{(0)}$ and $\varepsilon = (\varepsilon_i)_{i \in I} \in \{\pm 1\}^I$ be arbitrary and define

$$g_{\varepsilon,\alpha} := \sum_{i \in I_\alpha} (\varepsilon_i c_i \cdot L_y, \varphi) \in C^\infty_c(O),$$

$$f_{\varepsilon,\alpha} := F^{-1}g_{\varepsilon,\alpha} = \sum_{i \in I_\alpha} (\varepsilon_i c_i \cdot M_y, \psi) \in \mathcal{S}_O(\mathbb{R}^d).$$
Using the same arguments as in equations (4.7) and (4.8) above, we see
\[
\| F^{-1}(\varphi \cdot g_{\varepsilon, \alpha}) \|_{L^p} \lesssim \sum_{i \in I_\alpha} |c_i| \cdot \| F^{-1}(\varphi \cdot L_y \varphi) \|_{L^p}
\]
\[
\lesssim \sum_{i \in I_\alpha} |c_i| \cdot \| F^{-1}(L_y \varphi) \|_{L^p}
\]
\[
\lesssim \| F^{-1}\varphi \|_{L^p} \cdot \sum_{i \in I_\alpha} |c_i|
\]
\[
\lesssim (|c|^\gamma)_{\ell}
\]
for all \( \ell \in I \) and thus \( f_{\varepsilon, \alpha} \in S^{0,\gamma}_0(\mathbb{R}^d) \) with \( \|f_{\varepsilon, \alpha}\|_{D^1(Q,L^p,Y)} \lesssim \|c\|_Y \).

Now, by continuity and by equation (4.12), there is some \( \delta > 0 \) with
\[
|\psi(x)| \geq \frac{1}{2} \quad \text{and} \quad |\partial^\beta \psi(x)| \leq \frac{1}{4} \cdot \left( \frac{M}{L} + 2\pi \right)^{-k} \quad \forall \|x\| \leq \delta \forall \beta \in \mathbb{N}_0^d \quad \text{with} \quad 0 < |\beta| \leq k.
\]
Using the Leibniz rule, this implies for \( |x| \leq \delta \) that
\[
\left| \left[ \partial^{\alpha^{(i)}}(M_y, \psi) \right](x) \right| = \left| \sum_{\beta \leq \alpha^{(i)}} \left( \alpha^{(i)} \beta \right) \cdot (2\pi y_i)^\alpha^{(i)} - \beta \cdot e^{2\pi j(y_i,x)} \cdot (\partial^\beta \psi)(x) \right|
\]
\[
\geq \left( (2\pi y_i)^\alpha^{(i)} \psi(x) \right) - \sum_{\beta \leq \alpha^{(i)}} \left( \alpha^{(i)} \beta \right) \cdot (2\pi y_i)^\alpha^{(i)} - \beta \cdot (\partial^\beta \psi)(x)
\]
\[
(\alpha^{(i)} = k \cdot e_j^{(i)} \quad \text{and} \quad L_i \geq |(y_i)_j| \quad \text{for all} \quad j \geq \frac{2\pi L_i}{2} \cdot \frac{1}{4} \left( \frac{M}{L} + 2\pi \right)^{-k} \cdot \sum_{\beta \leq \alpha} \left( \alpha^{(i)} \beta \right) \cdot (2\pi y_i)^\alpha^{(i)} - \beta
\]
\[
(\text{multinomial theorem}) = \frac{2\pi L_i}{2} \cdot \frac{1}{4} \left( \frac{M}{L} + 2\pi \right)^{-k} \cdot (1, \ldots, 1, \ldots, 2\pi L_i) \cdot (2\pi L_i)^{\alpha^{(i)}}
\]
\[
= \frac{2\pi L_i}{2} \cdot \frac{1}{4} \left( \frac{M}{L} + 2\pi \right)^{-k} \cdot (1 + 2\pi L_i)^k
\]
\[
(\text{since} \quad L_i \geq L/M \quad \text{by eq. (4.11)} \quad \geq \frac{2\pi L_i}{2} \cdot \frac{1}{4} \cdot \left( \frac{M}{L} + 2\pi \right)^{-k} \cdot \left( \left( \frac{M}{L} + 2\pi \right) L_i \right)^k
\]
\[
\geq L^k/4.
\]
But because of \( \alpha^{(i)} = \alpha \) for \( i \in I_\alpha \), we get (with another application of Khintchine’s inequality)
\[
E_\varepsilon \| \partial^\alpha f_{\varepsilon, \alpha} \|_{L^q}^q = \int_{\mathbb{R}^d} E_\varepsilon \left[ \sum_{i \in I_\alpha} \varepsilon_i c_i \left[ \partial^{\alpha^{(i)}}(M_y, \psi) \right](x) \right]^q \ dx
\]
\[
\leq \int_{\mathbb{R}^d} \left( \sum_{i \in I_\alpha} |c_i| \left[ \partial^{\alpha^{(i)}}(M_y, \psi) \right](x) \right)^2 \ dx^{q/2}
\]
\[
\leq \int_{B_4(0)} \left( \sum_{i \in I_\alpha} |c_i| \left[ \partial^{\alpha^{(i)}}(M_y, \psi) \right](x) \right)^2 \ dx^{q/2}
\]
\[
\geq \int_{B_4(0)} \left( \sum_{i \in I_\alpha} |c_i| \frac{L_k}{4} \right)^2 \ dx^{q/2}
\]
\[
\geq \left( \sum_{i \in I_\alpha} |c_i| \left( |b_i| + \|T_i\| \right)^2 \right)^{q/2}.
\]
All in all, we finally get
\[
\left\| \left( c_i \cdot u_i^{(k,p,p)} \right)_{i \in I_\alpha} \right\|_{L^2}^q \lesssim E_\varepsilon \| \partial^\alpha f_{\varepsilon, \alpha} \|_{L^q}^q \leq \| t_\alpha \|_q \cdot E_\varepsilon \| f_{\varepsilon, \alpha} \|_{D^1(Q,L^p,Y)}^q \lesssim \| t_\alpha \|_q \cdot \| c \|_Y^q,
\]
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which yields the desired embedding $Y \cap I_0 (I^{(0)}) \hookrightarrow \ell^2_{\omega(k,p,p)} (I)$ by summing over $\alpha \in \mathbb{N}_0^d$ with $|\alpha| = k$, since $I^{(0)} = \bigcup_{|\alpha| = k} I_\alpha$.

This completes the proof of Theorem 4.7 with the caveat that we postponed the proof of Claim 4.8 which we now give.

**Proof of Claim 4.8.** Let $\varepsilon > 0$, $N \in \mathbb{N}$ and $a_1, \ldots, a_N \in \mathbb{R}^d$ as in Lemma 4.3, i.e. such that for each $i \in I$, there is some $m^{(i)} \in \mathbb{N}$ with $B_\varepsilon (a_m^{(i)}) \subset Q_i$. Lemma 4.3 shows that for each $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq k$ and each $m \in \mathbb{N}$, there is some function $f_{\alpha,m} \in C^\infty_c (B_\varepsilon (a_m))$ with

$$
[\partial^\beta (F^{-1} f_{\alpha,m})] (0) = \delta_{\alpha,\beta} \text{ for all } \beta \in \mathbb{N}_0^d \text{ with } |\beta| \leq k.
$$

Let

$$
\mathcal{F} := \{ f_{\alpha,m} \mid m \in \mathbb{N} \text{ and } \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \leq k \}.
$$

Note that $\mathcal{F} \subset C^\infty_c (\mathbb{R}^d)$ is indeed finite.

We let $\eta = \eta (d,k) > 0$ be some unspecified constant, the precise value of which we will choose below.

Now, by continuity, there is some $\delta > 0$ such that

$$
|\partial^\alpha (F^{-1} f_{\alpha,m}) (x)| > \frac{1}{2} \quad \text{and} \quad |\partial^\beta (F^{-1} f_{\alpha,m}) (x)| < \eta \text{ for all } \beta \in \mathbb{N}_0^d \setminus \{ \alpha \} \text{ with } |\beta| \leq k \quad (4.13)
$$

holds for all $x \in \mathbb{R}^d$ with $|x| < \delta$ and all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq k$ and $m \in \mathbb{N}$.

Now, for $i \in I$ and $|\alpha| \leq k$, define $g_{i,\alpha} := F^{-1} f_{\alpha,m(i)}$, as well as

$$
\gamma_{i,\alpha} : \mathbb{R}^d \to \mathbb{C}, \xi \mapsto f_{\alpha,m(i)} (T_i^{-1} (\xi - b_i))
$$

and $\theta_{i,\alpha} := F^{-1} \gamma_{i,\alpha}$. Precisely the same calculation as in equation (2.4) yields

$$
\theta_{i,\alpha} = F^{-1} \gamma_{i,\alpha} = |\det T_i| \cdot M_{b_i} (g_{i,\alpha} \circ T_i^T) \quad (4.14)
$$

We first indicate the construction in case of $i \in I_1$. Note that $i \in I_1$ implies $k > 0$ and $\|T_i\| > 2\pi |b_i|$. Now, for each $i \in I_1$, choose $k^{(i)}, \ell^{(i)} \in d$ satisfying

$$
K_i := \bigg| (T_i)_{k^{(i)},\ell^{(i)}} \bigg| = \max_{k,\ell \in d} \bigg| (T_i)_{k,\ell} \bigg| \geq \frac{\|T_i\|}{d^2}. \quad (4.15)
$$

Here, the final (suboptimal) estimate is an easy consequence of the definitions. Define $\alpha^{(i)} := k \cdot e_{k^{(i)}} \in \mathbb{N}_0^d$ and $\beta^{(i)} := k \cdot e_{\ell^{(i)}}$, where $(e_1, \ldots, e_d)$ denotes the standard basis of $\mathbb{R}^d$. Note $|\alpha^{(i)}| = k = |\beta^{(i)}|$. Finally, set $\gamma_i := \gamma_{i,\beta^{(i)}}$ and $g_i := g_{i,\beta^{(i)}}$, as well as $\theta_i := \theta_{i,\beta^{(i)}}$. Note by definition of $\gamma_{i,\alpha}$ that $\gamma_i$ indeed satisfies equation (4.1) with $f_i = f_{\beta^{(i)},m(i)} \in \mathcal{F}$. Furthermore, note that we have

$$
\text{supp } \gamma_i = \text{supp } \gamma_{i,\beta^{(i)}} = T_i (\text{supp } f_{\beta^{(i)},m(i)}) + b_i \subset T_i (B_\varepsilon (a_m(i))) + b_i \subset T_i Q_i + b_i = Q_i
$$

and thus $\gamma_i \in C^\infty_c (Q_i)$ as desired. It thus remains to establish estimate (4.2).

For $i \in I_2$, we proceed similarly. We choose some $j^{(i)} \in d$ with

$$
L_i := \bigg| (b_i)_{j^{(i)}} \bigg| = \max_{j \in d} \bigg| (b_i)_j \bigg| \geq \frac{|b_i|}{d}.
$$

Furthermore, we define $\alpha^{(i)} := k \cdot e_{j^{(i)}} \in \mathbb{N}_0^d$ with $|\alpha^{(i)}| = k$ and we choose $\gamma_i := \gamma_{i,0}$ and $g_i := g_{i,0}$, as well as $\theta_i := \theta_{i,0}$. Precisely as above, we see that $\gamma_i$ indeed satisfies equation (4.4) with $f_i = f_{0,m(i)} \in \mathcal{F}$ and that $\gamma_i \in C^\infty_c (Q_i)$. Thus, for $i \in I_2$, it remains to establish estimate (4.3).

We start by establishing equation (4.2) for $i \in I_1$. To this end, first note that this estimate is trivial in case of $T_i^T x \notin B_\delta (0)$. Hence, we can (and will) assume for the following considerations that $T_i^T x \in B_\delta (0)$. Next,
note as above that $i \in I_1$ implies $k > 0$ and hence $\alpha^{(i)} \neq 0$, so that Lemma 2.6 (with $i_1 = \cdots = i_k = k^{(i)}$) is applicable and yields

$$
\left| \left( \partial^{\alpha^{(i)}} \left[ \vartheta_i \circ T_i^T \right] \right)(x) \right| = \sum_{\ell_1, \ldots, \ell_k \in \mathbb{D}} \left| (T_i)_{k^{(i)}, \ell_1} \cdots (T_i)_{k^{(i)}, \ell_k} \cdot (\partial_{\ell_1} \cdots \partial_{\ell_k} \vartheta_i)(T_i^T x) \right|
\geq \left| (T_i)_{k^{(i)}, \ell(i)} \cdot (\partial_{\ell(i)} \vartheta_i)(T_i^T x) \right| - \sum_{\ell_1, \ldots, \ell_k \in \mathbb{D}} \left| (T_i)_{k^{(i)}, \ell_1} \cdots (T_i)_{k^{(i)}, \ell_k} \cdot (\partial_{\ell_1} \cdots \partial_{\ell_k} \vartheta_i)(T_i^T x) \right|
\geq K_k^i \left[ \left| \left( \partial^{\beta^{(i)}} \vartheta_i \right)(T_i^T x) \right| - \sum_{\ell_1, \ldots, \ell_k \in \mathbb{D}} \left| (\partial_{\ell_1} \cdots \partial_{\ell_k} \vartheta_i, \beta^{(i)})(T_i^T x) \right| \right]
\geq K_k^i \left( \frac{1}{2} - \eta \cdot d^k \right).
\tag{4.16}
$$

Here, the step marked with $(\ast)$ used the definition of $K_i$ (cf. equation (4.16)) and of $\beta^{(i)} = k \cdot e_{\ell(i)}$, as well as of $\vartheta_i = \vartheta_{i, \beta^{(i)}}$. Finally, the step marked with $(1)$ made use of $T_i^T x \in B_\delta (0)$, of estimate (4.15) and of $\partial_{\ell_1} \cdots \partial_{\ell_k} \neq \partial^{\beta^{(i)}}$ for those indices $(\ell_1, \ldots, \ell_k)$ over which the sum is taken. Furthermore, this step also used that the sum has less than $d^k$ terms. The estimate given in equation (4.16) is nontrivial as soon as $\eta = \eta (d, k)$ satisfies $\eta < \frac{1}{2} d^{-k}$. We will assume this for the rest of the proof.

Now, we note that equation 4.14 and an application of Leibniz’s rule yield – as in equation 2.5 – that

$$
\left| \left( \partial^{\alpha^{(j)}} (F^{-1} \gamma_i) \right)(x) \right| = \left( \partial^{\alpha^{(i)}} \vartheta_i \right)(x) = \left| \det T_i \right| \sum_{\beta \leq \alpha^{(i)}} \left| \left( \frac{\alpha^{(i)}}{\beta} \right) \cdot (2\pi j \cdot b_i)^{\alpha^{(i)} - \beta} \cdot e^{2\pi j (b_i \cdot x)} \cdot (\partial^\beta \left[ \vartheta_i \circ T_i^T \right]) (x) \right|,
\tag{4.17}
$$

where we have written $j$ for the imaginary unit to avoid confusion with the index $i \in I_1$. We will keep this convention for the remainder of the proof. In view of this identity and of equation (4.16), our next step is to estimate for $\beta \leq \alpha^{(i)}$ with $\alpha^{(i)} \neq \beta$ the term

$$
\left| (2\pi j \cdot b_i)^{\alpha^{(i)} - \beta} \cdot e^{2\pi j (b_i \cdot x)} \cdot (\partial^\beta \left[ \vartheta_i \circ T_i^T \right]) (x) \right|
\leq K_k^i \cdot d^{k - |\beta|} \cdot \left| \det T_i \right| \sum_{\beta \leq \alpha^{(i)}} \left| \left( \frac{\alpha^{(i)}}{\beta} \right) \cdot (2\pi j \cdot b_i)^{\alpha^{(i)} - \beta} \cdot e^{2\pi j (b_i \cdot x)} \cdot (\partial^\beta \left[ \vartheta_i \circ T_i^T \right]) (x) \right|
\leq K_k^i \cdot d^{k - |\beta|} \cdot \left| \det T_i \right| \sum_{\beta \leq \alpha^{(i)}} \left| \left( \frac{\alpha^{(i)}}{\beta} \right) \cdot (2\pi j \cdot b_i)^{\alpha^{(i)} - \beta} \cdot e^{2\pi j (b_i \cdot x)} \cdot (\partial^\beta \left[ \vartheta_i \circ T_i^T \right]) (x) \right|
\leq K_k^i \cdot d^{k - |\beta|} \cdot \left| \det T_i \right| \sum_{\beta \leq \alpha^{(i)}} \left| \left( \frac{\alpha^{(i)}}{\beta} \right) \cdot (2\pi j \cdot b_i)^{\alpha^{(i)} - \beta} \cdot e^{2\pi j (b_i \cdot x)} \cdot (\partial^\beta \left[ \vartheta_i \circ T_i^T \right]) (x) \right|
\left( \text{by definition of } K_i \right)
\leq d^{k - |\beta|} \cdot K_k^i \cdot \left| \det T_i \right| \sum_{\beta \leq \alpha^{(i)}} \left| \left( \frac{\alpha^{(i)}}{\beta} \right) \cdot (2\pi j \cdot b_i)^{\alpha^{(i)} - \beta} \cdot e^{2\pi j (b_i \cdot x)} \cdot (\partial^\beta \left[ \vartheta_i \circ T_i^T \right]) (x) \right|
\tag{4.18}
$$

Here, the step marked with $(\ast)$ used that $\beta \leq \alpha^{(i)}$ together with $\beta \neq \alpha^{(i)}$ implies $|\beta| < |\alpha^{(i)}| = |\beta^{(i)}|$ and hence $\partial_{\ell_1} \cdots \partial_{\ell_{|\beta|}} \neq \partial^{\beta^{(i)}}$ for all $\ell_1, \ldots, \ell_{|\beta|} \in \mathbb{D}$ so that equation 4.15 – together with $T_i^T x \in B_\delta (0)$ – justifies the marked step. Note though, that the step marked with $(1)$ is only justified for $|\beta| \in \mathbb{N}$, i.e. for
\( \beta \neq 0 \) (cf. Lemma [2,6]). But in case of \( \beta = 0 \), we simply have
\[
\left| (2\pi j \cdot b_i)^{\alpha(i) - \beta} e^{2\pi i (b_i,x)} \cdot (\partial^\beta [\theta_i \circ T_i^T]) (x) \right|
\]

(since \( \theta_i = \theta_i, \alpha(i) = F^{-1} f_{\beta(i),m(i)} \))
\[
= \left| (2\pi j \cdot b_i)^{\alpha(i)} \cdot (F^{-1} f_{\beta(i),m(i)}) (T_i^T x) \right|
\]
\[
\leq d^2 K_i \cdot |(F^{-1} f_{\beta(i),m(i)}) (T_i^T x)|
\]
(by eq. [4.13] since \( |\beta(i)| = k > 0 \))
\[
\leq d^2 k \cdot d^k \eta.
\]

Thus, equation [4.18] also holds for \( \beta \neq 0 \). At \((*)\), we used that \( k > 0 \), since \( i \in I_1 \).

Altogether, equation [4.17] and estimates [4.16] and [4.18] yield
\[
\left| \left[ \partial^{\alpha(i)} F^{-1} \gamma_i \right] (x) \right| \geq |\text{det} T_i| \cdot \left[ K_i^k \left( \frac{1}{2} - d^k \eta \right) - \sum_{\substack{\beta \leq \alpha(i) \\ \beta \neq \alpha(i)}} \left( \frac{\alpha(i)}{\beta} \right) K_i^k d^k \eta \right]
\]
\[
\geq |\text{det} T_i| K_i^k \left[ \frac{1}{2} - \eta \left( d^k + d^k \sum_{\beta \leq \alpha(i)} \left( \frac{\alpha(i)}{\beta} \right) \right) \right]
\]
(by the multi-binomial theorem)
\[
= |\text{det} T_i| K_i^k \left[ \frac{1}{2} - \eta \left( d^k + d^k (1, \ldots, 1) (1, \ldots, 1)^{\alpha(i)} \right) \right]
\]
\[
= |\text{det} T_i| K_i^k \left[ \frac{1}{2} - \eta \left( d^k + d^k 2^{\alpha(i)} \right) \right]
\]
(since \( |\alpha(i)| = k \))
\[
= |\text{det} T_i| K_i^k \left[ \frac{1}{2} - \eta \left( d^k + d^k 2^k \right) \right].
\]

All in all, we see that if we choose \( \eta = \eta (d, k) := \frac{1}{4(d^k + d^k 2^k)} \), then
\[
\left| \left[ \partial^{\alpha(i)} F^{-1} \gamma_i \right] (x) \right| \geq \frac{|\text{det} T_i| \cdot K_i^k}{4} \geq \frac{|\text{det} T_i| \cdot (\|T_i\| / d)^k}{4} = (4d^k)^{-1} \cdot |\text{det} T_i| \cdot \|T_i\|^k
\]
holds for all \( x \in \mathbb{R}^d \) with \( T_i^T x \in B_\delta (0) \). This finally establishes equation [4.2] with \( C_1 = (4d^k)^{-1} \).

To complete the proof, we establish equation [4.3] for \( i \in I_2 \). As above, we see that this estimate trivially holds if \( T_i^T x \notin B_\delta (0) \). Thus, in the following, we will assume \( T_i^T x \in B_\delta (0) \).

Now, let \( \beta \in \mathbb{N}_0^d \) with \( 0 \neq \beta \leq \alpha(i) \) be arbitrary. Note that existence of such a multiindex \( \beta \) implies \( 0 < |\beta| = \alpha(i)| = k \) and thus \( 2\pi |b_i| \geq \|T_i\| \), since \( i \in I_2 \). Choose \( i_1, \ldots, i_{|\beta|} \in \mathbb{R}^d \) with \( \beta = \sum_{m=1}^{|\beta|} e_{im} \) and apply Lemma [2,4] to conclude
\[
\left| (\partial^\beta [\theta_i \circ T_i^T]) (x) \right| = \sum_{\ell_1, \ldots, \ell_{|\beta|} \in \mathbb{Z}} \left[ (T_i)_{i_1, \ell_1} \cdots (T_i)_{i_{|\beta|}, \ell_{|\beta|}} \cdot (\partial_{\ell_1} \cdots \partial_{\ell_{|\beta|}} \theta_i) (T_i^T x) \right]
\]
\[
\leq (2\pi d L_i)^{|\beta|} \sum_{\ell_1, \ldots, \ell_{|\beta|} \in \mathbb{Z}} \eta
\]
\[
= (2\pi d^2 \cdot L_i)^{|\beta|} \cdot \eta.
\]

Here, the step marked with \((*)\) used equation [4.13], which is applicable since we have \( T_i^T x \in B_\delta (0) \) and \( \theta_i, 0 = F^{-1} f_{0,m(i)} \), as well as \( 0 < |\beta| \leq k \).

\( ^6 \)Note that this choice of \( \eta \) also makes the assumption \( \eta < \frac{1}{d^k} \) from above true.
But for $\beta = 0$, we have

$$|\left(\partial^\beta [\varphi_i \circ T_i^T] \right)(x)| = |\varphi_{i,0} (T_i^T x)|$$

(since $\varphi_{i,0} = \mathcal{F}^{-1} f_{0,m(i)} = |(\mathcal{F}^{-1} f_{0,m(i)}) (T_i^T x)|$

(by eq. (4.17) since $T_i^T x \in B_3(0)$) $\geq \frac{1}{2}$.

Altogether, since equation (4.17) also holds for $i \in I_2$, we conclude

$$\left| \left(\partial^\alpha (\mathcal{F}^{-1} \gamma_i) \right)(x) \right| = \left| \det T_i \cdot \sum_{\beta \leq \alpha} \left(\frac{\alpha(i)}{\beta} \cdot (2\pi j \cdot b_j)^{\alpha(i) - \beta} \cdot e^{2\pi ij(b_j \cdot x)} \cdot (\partial^\beta [\varphi_i \circ T_i^T ])(x) \right) \right|$$

$$\geq |\det T_i| \left| (2\pi j b_j)^{\alpha(i)} \partial^\beta [\varphi_i \circ T_i^T ](x) - \sum_{0 \neq \beta \leq \alpha} \left(\frac{\alpha(i)}{\beta} \right)^2 |(2\pi j b_j)^{\alpha(i) - \beta} \cdot \partial^\beta [\varphi_i \circ T_i^T ](x) \right|$$

$$\geq |\det T_i| \left [ 2\pi (b_j)_{j(i)} \right]^k \cdot \left[ \frac{1}{2} - \sum_{0 \neq \beta \leq \alpha} \left(\frac{\alpha(i)}{\beta} \right)^2 \cdot (2\pi L_i)^{|\alpha(i)| - |\beta|} \cdot (2\pi d^2 \cdot L_i)^{|\beta|} \cdot \eta \right]$$

(since $|\alpha(i)| = k \geq |\det T_i| (2\pi L_i)^k$).

$$\left[ \frac{1}{2} - \eta \cdot \sum_{\beta \leq \alpha} \left(\frac{\alpha(i)}{\beta} \right)^2 \cdot (2\pi L_i)^{|\alpha(i)| - |\beta|} \cdot (2\pi d^2 \cdot L_i)^{|\beta|} \cdot \eta \right]$$

$$(|\beta| \leq |\alpha(i)| = k \geq |\det T_i| (2\pi L_i)^k \cdot \left[ \frac{1}{2} - \eta \cdot d^{2k} \sum_{\beta \leq \alpha} \left(\frac{\alpha(i)}{\beta} \right)^2 \cdot (1, \ldots, 1)^{\alpha(i)} \right]$$

$$(|\alpha(i)| = k \geq |\det T_i| (2\pi L_i)^k \cdot \left[ \frac{1}{2} - \eta \cdot d^{2k} \cdot 2^k \right].$$

Here, the step marked with (†) is justified by equation (4.19) and because of $|(b_i)_j| \leq L_i$ for all $j \in d$.

Now, recall

$$\eta = \eta(d, k) = \frac{1}{4 (d^k + 2 k d^k)} < \frac{1}{4 \cdot 2^k d^k} \leq \frac{1}{4 \cdot 2^k d^k}$$

and $L_i \geq \frac{|b_i|}{d}$, so that we finally get

$$\left[ \left(\partial^\alpha (\mathcal{F}^{-1} \gamma_i) \right)(x) \right] \geq |\det T_i| (2\pi L_i)^k \cdot \left[ \frac{1}{2} - \eta \cdot d^{2k} \cdot 2^k \right]$$

$$\geq \frac{1}{4} |\det T_i| (2\pi L_i)^k \cdot \left[ \frac{1}{2} - \eta \cdot d^{2k} \cdot 2^k \right]$$

for all $x \in \mathbb{R}^d$ with $T_i^T x \in B_3(0)$. As observed above, this establishes equation (4.3) with $C_2 = \frac{1}{4d^k}$. □

As a corollary, we get the following somewhat surprising – result.

**Corollary 4.9.** Let $Q = (Q_{i})_{i \in I} = (T_i Q' + b_i)_{i \in I}$ be a tight regular covering of the open set $\varnothing \neq O \subset \mathbb{R}^d$ such that $\sup_{i \in I} \|T_i^{-1}\| < \infty$.

Let $\ell \in \mathbb{N}_0$ and $p \in (0, \infty]$ and let $Y \subset C^l$ be a $Q$-regular sequence space for which $Y \cap \ell_0(I) \subset Y$ is dense.

If $D(Q, L^p, Y) \rightarrow W^{\ell,q} (\mathbb{R}^d)$ for some $q \in (2, \infty)$ and if $p \leq 2$, then $D(Q, L^p, Y) \hookrightarrow W^{\ell,2} (\mathbb{R}^d)$. ▲

**Remark.** This result is slightly surprising, since there is no embedding $W^{\ell,q} (\mathbb{R}^d) \rightarrow W^{\ell,2} (\mathbb{R}^d)$ for $q \neq 2$. ♦

**Proof.** The assumptions easily imply that the prerequisites of the last part of Theorem 4.7 are satisfied for each $0 \leq k \leq \ell$ with $I^{(0)} = I$, so that we get

$$Y \cap \ell_0(I) \rightarrow \ell_{u(k,p,2)}^2(I)$$
for all $0 \leq k \leq \ell$. Since $Y \cap \ell_0^0(I) \subset Y$ is dense and since all relevant spaces are sequence spaces (i.e., they embed continuously into the Hausdorff space $C^I$, equipped with the product topology), this implies

$$Y \hookrightarrow \ell_{u(k,p,2)}^0(I) = \ell_{(2^*,p)}^0(I)$$

for $0 \leq k \leq \ell$. Since we also have $p \leq 2$, Corollary 3.5 yields $D(Q, L^p, Y) \hookrightarrow W^{t,2}(\mathbb{R}^d)$. \hfill \Box

5. Characterization of the Embedding $\ell_v^r(I) \hookrightarrow \ell_{w}^r(I)$

As we saw in Sections 3 and 4 to decide existence of the embedding

$$D(Q, L^p, Y) \hookrightarrow W^{k,q}(\mathbb{R}^d),$$

we have to decide whether an embedding of the form $Y \hookrightarrow \ell_v^r(I)$ for certain $r \in (0, \infty]$ and a certain weight $u = (u_i)_{i \in I}$ is valid. In the present section, we will simplify this problem for the case $Y = \ell_v^r(I)$ to the point where only finiteness of a single sequence space norm has to be decided.

Lemma 5.1. Let $u = (u_i)_{i \in I}$ and $v = (v_i)_{i \in I}$ be positive weights on an index set $I \neq \emptyset$ and let $p, q \in (0, \infty]$. Then, the map

$$\iota : (\ell_0(I), \| \cdot \|_{\ell^0}) \hookrightarrow \ell_{v}^0(I), \,(c_i)_{i \in I} \mapsto (c_i)_{i \in I}$$

is bounded if and only if

$$(u_i/v_i)_{i \in I} \in \ell^q((p/q)')'(I).$$

Here, the exponent $q \cdot (p/q)'$ has to be calculated according to the following convention\footnote{Apart from the convention noted here, we have $p/q = \infty$ if $p = \infty$ and $q < \infty$ and (as in the rest of the paper) $r' = \infty$ if $r \leq 1$. Finally, $\infty' = 1$.}:

$$q \cdot (p/q)' = \begin{cases} \infty, & \text{if } q = \infty, \\ q \cdot (p/q)' & \text{if } q < \infty. \end{cases}$$

If equation (5.2) is fulfilled, we even have $\ell_v^r(I) \hookrightarrow \ell_q^r(I)$.

Remark. In short, the last part of the lemma shows that it suffices to verify an embedding on the (not necessarily dense) subspace $\ell_v^r(I) \cap \ell_0(I)$. Thus, the restriction to finitely supported sequences in the necessary conditions given in Theorem 4.7 is no essential restriction, at least if $Y$ is a weighted sequence space.

Furthermore, note

$$q \cdot (p/q)' = \infty \iff q = \infty \text{ or } (q < \infty \text{ and } (p/q)’ = \infty)$$

$$\iff q = \infty \text{ or } \left(q < \infty \text{ and } \frac{p}{q} \leq 1\right)$$

$$\iff q = \infty \text{ or } \left(q < \infty \text{ and } p \leq q\right)$$

$$\iff p \leq q.$$ (5.2)

Finally, we have

$$\frac{1}{q \cdot (p/q)'} = \left(\frac{1}{q} - \frac{1}{p}\right)_+ ,$$

(5.3)

where $x_+ = \max\{x, 0\}$. Indeed, there are two cases:

Case 1. $\frac{1}{q} - \frac{1}{p} \leq 0$, i.e., $p \leq q$. As seen in equation (5.2), this yields $q \cdot (p/q)' = \infty$ and hence

$$\frac{1}{q \cdot (p/q)'} = 0 = \left(\frac{1}{q} - \frac{1}{p}\right)_+$$

as claimed.

Case 2. $\frac{1}{q} - \frac{1}{p} > 0$, i.e., $p > q$. By equation (5.2) again, this yields $q \cdot (p/q)' < \infty$ and in particular $\frac{p}{q} < \infty$. Hence,

$$\frac{1}{q \cdot (p/q)} = \frac{1}{q} \cdot \left(1 - \frac{1}{p/q}\right) = \frac{1}{q} \cdot \left(1 - \frac{q}{p}\right) = \frac{1}{q} - \frac{1}{p} = \left(\frac{1}{q} - \frac{1}{p}\right)_+ .$$

The two properties from equations (5.2) and (5.3) will be used repeatedly in Section 7 for concrete applications of our embedding results.
Proof of Lemma 5.1. We first establish the implication “⇐”. To this end, it suffices to show $\ell_v^p(I) \to \ell_u^q(I)$. For the proof, we first make the following general observation: Hölder’s inequality

$$
\|\sum_{i \in I} |x_i y_i| \|_{\ell_1} = \sum_{i \in I} |x_i y_i| \leq \|x_i\|_{\ell_{p}} \cdot \|y_i\|_{\ell_{p'}} \ .
$$

which is well known for $p \in [1, \infty]$ also holds for $p \in (0,1)$, since in this case, we have the norm-decreasing embedding $\ell^p \to \ell^1$ and hence

$$
\sum_{i \in I} |x_i y_i| \leq \|x_i\|_{\ell_1} \cdot \|y_i\|_{\ell_{p}} \cdot \|y_i\|_{\ell_{p'}} = \|x_i\|_{\ell_1} \cdot \|y_i\|_{\ell_{p}} \cdot \|y_i\|_{\ell_{p'}} \ .
$$

Thus, for $c = (c_i)_{i \in I} \in \ell_v^p(I)$ and $q < \infty$, we have

$$
\|c\|_{\ell_u^q} = \sum_{i \in I} \|c_i\|_{\ell_1} \cdot \|c_i\|_{\ell_{q'}}^q \leq \sum_{i \in I} \|c_i\|_{\ell_{p/q}} \cdot \|v_i\|_{\ell_{p/q'}} \cdot \|c_i\|_{\ell_{p/q'}}^q < \infty.
$$

If otherwise $q = \infty$, we have $q \cdot (p/q)' = \infty$ and hence for each $i \in I$

$$
u_i \cdot c_i = \frac{u_i}{v_i} \cdot v_i \cdot c_i \leq \sum_{i \in I} \|u_i\|_{\ell_{p/q'}} \cdot \|v_i\|_{\ell_{p/q'}} \cdot \|c_i\|_{\ell_{p/q'}} \ .
$$

Thus, we have shown $\ell_v^p(I) \to \ell_u^q(I)$ in all possible cases.

It remains to show “⇒”. To this end, let us first prove $u/v \in \ell_\infty(I)$ with $(u/v)_i = u_i/v_i$ for $i \in I$. Indeed, for arbitrary $i \in I$, we have

$$
u_i \cdot \delta_i \leq \sum_{i \in I} \|c_i\|_{\ell_{p/q'}} \cdot \|c_i\|_{\ell_{p/q'}} \ .
$$

so that we get $0 \leq u_i/v_i \leq \|\|/\|$ and hence $u/v \in \ell_\infty(I)$.

Thus, we can assume in the following that $\alpha := q \cdot (p/q)' < \infty$ and thus $\beta := (p/q)' < \infty$, which means $p/q > 1$, i.e. $\infty \geq p > q$. Now, let $\emptyset \neq I_0 \subseteq I$ be an arbitrary finite subset and define

$$c_i := \left(\frac{u_i}{v_i} \right)^\beta \quad \text{for } i \in I_0$$

and $c_i = 0$ for $i \in I \setminus I_0$. Note $c = (c_i)_{i \in I} \in \ell_0(I)$. Because of $q < \infty$ and $\alpha = q \cdot \beta$, we have

$$\left\| \frac{u}{v} \cdot \chi_{I_0} \right\|_{\ell_\infty}^\beta = \left[ \sum_{i \in I_0} (u_i/v_i)^\beta \right]^{1/q} = \left[ \sum_{i \in I_0} (u_i c_i)^\beta \right]^{1/q} = \|c\|_{\ell_u^q} \leq \|c\| \cdot \|c\|_{\ell_v^p} = \|c\|_{\ell_v^p} \ .
$$

Now, let us first assume $p < \infty$. In this case, we have $1 < p/q < \infty$, so that $\beta = (p/q)'$ satisfies $\beta \in (1, \infty)$ and

$$\beta - 1 = \frac{1}{1/\beta} - 1 = \frac{1}{1 - \frac{1}{p/q}} - 1 = \frac{1}{1 - \frac{q}{p}} - 1 = \frac{p}{p - q} - 1 = \frac{q}{p - q} .$$
We claim \( p(\beta - 1) = \alpha \). Indeed, this is equivalent to
\[
\frac{1}{p} \cdot \frac{1}{\beta - 1} = \frac{1}{\alpha} = \frac{1}{q} \cdot \frac{1}{(p/q)} = \frac{1}{q} \left( 1 - \frac{1}{p/q} \right) \iff \frac{1}{p} \cdot \frac{p - q}{q} = \frac{1}{q} \left( 1 - \frac{q}{p} \right)
\]
which is a tautology. Hence,
\[
\left\| \left( u_i/v_j \right)^{\beta} \cdot v_j/u_i \right\|_{l_p} = \left\| \left( u_i/v_j \right)^{\beta - 1} \right\|_{l_p} = \| u/v \|_{\ell^{p/(\alpha - 1)(l_0)}}^{\beta - 1} = \| u/v \|_{\ell^{p/q}(l_0)}^{\beta - 1}.
\]
Altogether, we have thus shown
\[
\left\| u/v \cdot \chi_{I_0} \right\|_{l_p} \leq \| \ell \| \cdot \left\| u/v \cdot \chi_{I_0} \right\|_{l_{p/q}}^{\beta - 1},
\]
so that rearranging yields \( \| u/v \cdot \chi_{I_0} \|_{l_\alpha} \leq \| \ell \| \). Note that this used finiteness of \( \| u/v \cdot \chi_{I_0} \|_{l_\alpha} \) which holds, since \( I_0 \subset I \) is finite. But since \( I_0 \subset I \) was an arbitrary finite subset and because of \( \alpha < \infty \), we get \( \| u/v \|_{l_\alpha} \leq \| \ell \| < \infty \).

It remains to consider the case \( \alpha < \infty \), but \( p = \infty \). In this case, we have \( p/q = \infty \) and hence \( \beta = (p/q)' = 1 \), which finally yields \( \alpha = q \cdot \beta = q \). In this case, let again \( I_0 \subset I \) be an arbitrary finite subset and define \( c_i := v_i^{-1} \) for \( i \in I_0 \) and \( c_i = 0 \) for \( i \in I \setminus I_0 \). Then
\[
\left\| \frac{u}{v} \cdot \chi_{I_0} \right\|_{l_\alpha} = \left\| (u_i \cdot c_i)_{i \in I} \right\|_{l_\alpha} = \| c \|_{l_p} \leq \| \ell \| \| c \|_{l_\alpha} = \| \ell \| \| \chi_{I_0} \|_{l_\alpha} \leq \| \ell \|.
\]
As above, we conclude \( \| u/v \|_{l_\alpha} \leq \| \ell \| < \infty \). \( \square \)

We can now state a simplified version of our embedding results.

**Corollary 5.2.** Let \( Q = (Q_i)_{i \in I} = (T_iQ_i' + b_i)_{i \in I} \) be a tight regular covering of the open set \( \emptyset \neq \mathcal{O} \subset \mathbb{R}^d \).

Let \( n \in \mathbb{N}_0 \) and \( p, q, r \in (0, \infty) \) and let \( u = (u_i)_{i \in I} \) be a \( \mathcal{Q} \)-moderate weight. For \( t \in (0, \infty] \), define the weight \( w^{(t)} = (w_i^{(t)})_{i \in I} \) by
\[
w_i^{(t)} := |\det T_i|^{\frac{1}{\alpha}} \cdot (1 + |b_i|^n + \| T_i \|_n) \quad \text{for} \ i \in I.
\]

Then the following hold:
1. If \( p \leq q \) and if
\[
\frac{w(q)}{u} \in \ell^{p/\alpha'}(r/\gamma') \quad (I),
\]
then all assumptions of Corollary 3.5 (with \( k = n \)) are satisfied. In particular,
\[
\mathcal{D}(Q, L^p, \ell_u^{n}) \hookrightarrow W^{n,q}(\mathbb{R}^d).
\]
If \( q = \infty \), then \( \mathcal{D}(Q, L^p, \ell_u^{n}) \hookrightarrow C_0^n(\mathbb{R}^d) \).

2. Conversely, if
\[
\mathcal{D}(Q, L^p, \ell_u^{n}) \hookrightarrow W^{n,q}(\mathbb{R}^d),
\]
then \( p \leq q \) and the following hold:
   a) We have
\[
\frac{w(q)}{u} \in \ell^{p/\alpha'}(r/\gamma') \quad (I).
\]
   b) If \( q = \infty \), then
\[
\frac{w(q)}{u} \in \ell^{r/\gamma'}(I).
\]
   c) If \( I_0 \subset I \) satisfies \( \| T_i^{-1} \|_{l_\alpha} < \infty \), then we have the following:
   i) If \( q \in (0, \infty) \), then
\[
\frac{w(q)}{u} \in \ell^{2(1/\gamma')}(I_0).
\]
   (ii) If \( q \in [2, \infty] \), then
\[
\frac{w(q)}{u} \in \ell^{2(1/\gamma')}(I_0).\]
Proof. (1) To avoid confusion with the weights from the present corollary, let us write \( v^*, w^* \) for the weights \( v, w \) as defined in Corollary 3.5 (with \( k = n \)). With this notation, we have \( w^{(q)}_i \geq v^{(q)}_i \) and \( u^{(q)}_i \geq w^{(q)}_i \) for all \( i \in I \). Now, the assumption \( u^{(q)}_i / u \in T^q_i \) (I) implies by Lemma 5.1 that

\[
\ell^q_{u} (I) \mapsto \ell^{q_i} w^{(q)}_i (I) \mapsto \ell^{q_i} \cap \ell^{q_i} w^* (I).
\]

Since we also have \( p \leq q \), all assumptions of Corollary 3.5 are satisfied.

(2) Assume \( D(Q, L^p, \ell^q_u) \hookrightarrow W^{n,q} (\mathbb{R}^d) \). By Definition 3.2 and Theorem 4.4 this implies \( p \leq q \). Furthermore, it is easy to see that the given embedding implies boundedness of the maps \( \ell_a \) from Theorem 4.7 for every \( \alpha \in \mathbb{N}_0^d \) with \( |\alpha| \leq n \). Thus, by applying Theorem 4.7 (with \( Y = \ell^q_u (I) \) and \( k = 0 \) or \( k = n \), respectively), we get

\[
\ell^q_{u} (I) \cap \ell^q_0 (I) \mapsto \ell^q_{(0,p,q)} (I) \quad \text{and} \quad \ell^q_{u} (I) \cap \ell^q_0 (I) \mapsto \ell^q_{(0,n,q)} (I)
\]

for the weights \( u^{(0,p,q)}, u^{(n,p,q)} \) from that theorem. Using Lemma 5.1 we derive

\[
\frac{u^{(0,p,q)}}{u} \in \ell^q (r/q') (I) \quad \text{and} \quad \frac{u^{(n,p,q)}}{u} \in \ell^q (r/q') (I).
\]

But clearly, \( w^{(q)}_i \leq u^{(0,p,q)}_i + u^{(n,p,q)}_i \) for all \( i \in I \), so that \( u^{(q)}_i / u \in \ell^q (r/q') (I) \) as claimed.

In case of \( q = \infty \), since all of the maps \( \ell_a \) (with \( |\alpha| \leq n \)) from Theorem 4.7 are bounded, we get (with \( Y = \ell^p_u (I) \) and with \( k = 0 \) or \( k = n \), respectively) that

\[
\ell^p_{u} (I) \cap \ell^p_0 (I) \mapsto \ell^p_{(0,p,q)} (I) \quad \text{and} \quad \ell^p_{u} (I) \cap \ell^p_0 (I) \mapsto \ell^p_{(0,n,p,q)} (I)
\]

for the weights \( u^{(0,p,q)}, u^{(n,p,q)} \) from Theorem 4.7. Using Lemma 5.1 as well as 1. \( (r/1)' = r \), we get \( w^{(q)}_i / u \in \ell^p (r/q) \) using essentially the same arguments as above.

Finally if \( I_0 \subset I \) satisfies \( M := \sup_{t \in I_0} \| T^{-1}_t \| < \infty \), then the third part of Theorem 4.7 (with \( Y = \ell^p_u (I) \) and \( k = 0 \) or \( k = n \), respectively) yields

\[
\ell^p_{u} (I) \cap \ell^p_0 (I_0) \mapsto \ell^p_{(0,p,2)} (I_0) \quad \text{and} \quad \ell^p_{u} (I) \cap \ell^p_0 (I_0) \mapsto \ell^p_{(n,p,2)} (I_0) \quad \text{if} \quad q \in [2, \infty),
\]

as well as

\[
\ell^p_{u} (I) \cap \ell^p_0 (I_0) \mapsto \ell^p_{(0,p,p)} (I_0) \quad \text{and} \quad \ell^p_{u} (I) \cap \ell^p_0 (I_0) \mapsto \ell^p_{(n,p,p)} (I_0) \quad \text{if} \quad q \in (0, \infty).
\]

Using Lemma 5.1 and the estimate \( w^{(i)}_i \leq u^{(0,p)}_i + u^{(n,p)}_i \) for all \( i \in I \), we get \( w^{(p)} / u \in \ell^p (r/2) \) if \( q \in (0, \infty) \) and \( w^{(2)} / u \in \ell^p (r/2) \) if \( q \in [2, \infty) \).

6. EMBEDDINGS INTO BV (\mathbb{R}^d)

In this short section, we make the (perhaps surprising) observation that a decomposition space embeds into

\[
BV^k (\mathbb{R}^d) := \{ f : \mathbb{R}^d \to \mathbb{C} \mid f \in L^1 (\mathbb{R}^d) \quad \text{and} \quad \forall \alpha \in \mathbb{N}_0^d \setminus \{0\} \quad \text{with} \quad |\alpha| \leq k : \partial^\alpha f \text{ is a finite measure} \}
\]

if and only if it embeds into \( W^{k,1} (\mathbb{R}^d) \). Here, \( BV^k (\mathbb{R}^d) \) is equipped with the norm

\[
\| f \|_{BV^k} := \| f \|_{L^1} + \sum_{\alpha \in \mathbb{N}_0^d \setminus \{0\}} \| \partial^\alpha f \|_{TV},
\]

where \( \| \mu \|_{TV} := |\mu| (\mathbb{R}^d) \) denotes the total variation norm of a (finite) Borel measure \( \mu \) on \( \mathbb{R}^d \).

Corollary 6.1. Let \( Q = (Q_i)_{i \in I} = (T_i Q_i + b_i)_{i \in I} \) be a tight regular covering of the open set \( \emptyset \neq O \subset \mathbb{R}^d \). Let \( n \in \mathbb{N} \) and \( p \in (0, \infty) \) and let \( Y \leq C^1 \) be a \( Q \)-regular sequence space.

If \( Y \cap \ell^q_0 (I) \subset Y \) is dense or if \( Y = \ell^q_u (I) \) for some \( r \in [0, \infty] \) and a \( Q \)-moderate weight \( u = (u_i)_{i \in I} \), then

\[
D(Q, L^p, Y) \hookrightarrow BV^n (\mathbb{R}^d)
\]

holds if and only if

\[
D(Q, L^p, Y) \hookrightarrow W^{n,1} (\mathbb{R}^d)
\]

is true. In this case, we even have \( p \leq 1 \) and \( Y \hookrightarrow \ell^q_u (I) \) with

\[
v_i := |\det T_i|^{k-1} \cdot (1 + |b_i|^n + \| T_i \|^n),
\]

as well as \( D(Q, L^p, Y) \hookrightarrow L^p \hookrightarrow W^{n,1} (\mathbb{R}^d) \). △
Remark. Note that the corollary remains true (with the same proof) even if the definition of \( \text{BV}^k (\mathbb{R}^d) \) is changed such that the elements of \( \text{BV}^k (\mathbb{R}^d) \) are only required to be finite measures instead of \( L^1 \) functions. ♦

**Proof.** Note that \( \theta : W^{n,1} (\mathbb{R}^d) \hookrightarrow \text{BV}^n (\mathbb{R}^d) \), \( f \mapsto f \) is an isometric embedding. This makes the implication "\( \Leftarrow \)" trivial.

For "\( \Rightarrow \)", note that by assumption, there is a bounded linear map \( \iota : \mathcal{D} (Q, \mathbb{R}^d, Y) \to \text{BV}^n (\mathbb{R}^d) \) which satisfies \( \iota f = f \) for all \( f \in \mathcal{S}_0^Y \). But \( \mathcal{S}_0^Y \subset \mathcal{S} (\mathbb{R}^d) \subset W^{n,1} (\mathbb{R}^d) \), so that \( \iota : \mathcal{S}_0^Y \to \text{BV}^n (\mathbb{R}^d) \), \( f \mapsto f \) is well-defined and bounded since

\[
\| \iota f \|_{\text{BV}^n} = \| \theta \circ \iota f \|_{\text{BV}^n} = \| f \|_{\text{BV}^n} = \| \iota f \|_{\text{BV}^n} \leq \| f \|_{\mathcal{D} (Q, \mathbb{R}^d, Y)}.
\]

By Theorem 4.4 this implies \( p \leq 1 \). Furthermore, boundedness of \( \iota \) easily implies that each of the maps \( \iota_n \) (with \( |n| \leq n \)) from Theorem 4.7 are bounded. Thus, two applications of this theorem (for \( k = 0 \) and \( k = n \), respectively), show

\[
Y \cap \ell_0 (I) \hookrightarrow \ell_{u_{(0,p,1)}}^Y (I) = \ell_{u_{(0,p,1)}}^Y (I) \quad \text{and} \quad Y \cap \ell_0 (I) \hookrightarrow \ell_{u_{(n,p,1)}}^Y (I) = \ell_{u_{(n,p,1)}}^Y (I),
\]

with \( u_{(0,p,1)} \) and \( u_{(n,p,1)} \) as in Theorem 4.7. Here, we also used \( 1^Y = 1 \).

But obviously \( v_i \leq u_{(0,p,1)} + u_{(n,p,1)} \) for all \( i \in I \), so that we get \( Y \cap \ell_0 (I) \hookrightarrow \ell_{v_i}^Y (I) = \ell_{v_i}^Y (I) \). There are now two cases:

**Case 1.** If \( Y \cap \ell_0 (I) \leq Y \) is dense, we get \( Y \hookrightarrow \ell_{v_i}^Y (I) \), since \( Y \) and \( \ell_{v_i}^Y (I) \) both embed continuously into the Hausdorff space \( \mathbb{C}^I \), so that the unique continuous extension of the embedding \( Y \cap \ell_0 (I) \hookrightarrow \ell_{v_i}^Y (I) \) has to be given by the identity.

**Case 2.** If \( Y = \ell_{v_i}^Y (I) \) for some \( r \in (0, \infty) \), then Lemma 5.21 yields \( Y \hookrightarrow \ell_{v_i}^Y (I) \).

Since we have \( Y \hookrightarrow \ell_{v_i}^Y (I) \) in each case and since we also have \( p \leq 1 \) as seen above, Corollary 3.4 implies \( \mathcal{D} (Q, \mathbb{R}^d, Y) \hookrightarrow W^{n,1} (\mathbb{R}^d) \), as desired. \( \square \)

7. Applications

In this section, we apply our general, simplified embedding results from Section 5 to a large collection of examples, namely to

1. homogeneous and inhomogeneous Besov spaces,
2. \( \alpha \)-modulation spaces,
3. shearlet smoothness spaces,
4. shearlet-type coorbit spaces and
5. coorbit spaces of the diagonal group.

**Example 7.1.** (homogeneous Besov spaces)

Homogeneous Besov spaces can be obtained as decomposition spaces with respect to a certain dyadic covering of \( \mathcal{O} := \mathbb{R}^d \setminus \{0\} \). More precisely, the covering is given by \( Q = \{ Q_n \}_{n \in \mathbb{Z}} \) where \( Q_n = \{ n \} \setminus \mathcal{O} \) for all \( n \in \mathbb{Z} \) with \( b_n := 0 \) and \( T_n := 2^n \cdot \text{id} \). It is easy to see that \( P := \{ B_2 (0) \} \cap \mathcal{O} \) is compactly contained in \( Q \) and that \( \mathbb{R}^d \setminus \{0\} = \bigcup_{n \in \mathbb{Z}} T_n P \). Finally, \( x \in Q_n \cap Q_m \) implies \( 2^{-n} \leq |x| \leq 2^{m+2} \) and thus \( n \leq m + 4 \). By symmetry, we arrive at \( |n - m| \leq 4 \), so that on the one hand \( |n^*| \leq 9 \) and on the other hand

\[
\sup_{n \in \mathbb{Z}} \sup_{m \in \mathbb{N}^*} \| T_n^{-1} T_m \| = \sup_{n \in \mathbb{Z}} \sup_{m \in \mathbb{N}^*} 2^{4 - |n^*|} \leq 2^4,
\]

so that \( Q \) is indeed a structured admissible covering of \( \mathbb{R}^d \setminus \{0\} \). By Theorem 2.8 this implies that \( Q \) is a tight regular covering of \( \mathbb{R}^d \setminus \{0\} \). Thus, the assumptions regarding \( Q \) in Corollary 5.2 are satisfied.

Using the covering \( Q \), the usual homogeneous Besov spaces are (up to certain identifications) given by

\[
\mathcal{B}^{p,r}_{s} (\mathbb{R}^d) = \mathcal{D} (Q, \mathbb{R}^d, \ell_{v_i}^Y)
\]

for \( u = u(s) = \{ 2^{n} \}_{n \in \mathbb{Z}} \) and \( p, r \in (0, \infty) \), as well as \( s \in \mathbb{R} \).

Let \( k \in \mathbb{N}_0 \) and \( q \in (0, \infty) \). We are interested in whether an embedding of the form \( \mathcal{B}^{p,r} (\mathbb{R}^d) \hookrightarrow W^{k,q} (\mathbb{R}^d) \) holds. To this end, note that the weight \( v := u(q) \) from Corollary 5.2 (with \( n = k \)) is in this case given by

\[
v_n = |\det T_n|^{\frac{1}{q}} \cdot \left( 1 + |b_n|^k + \| T_n \|^k \right) \sim 2^{d(\frac{1}{q} - \frac{1}{k})} (1 + 2^k)
\]
for \( n \in \mathbb{Z} \). Hence,
\[
\frac{v_n}{u_n} = 2^n[d(\frac{1}{p} + \frac{1}{q})] \left(1 + 2^{nk}\right) = 2^n[d(\frac{1}{p} + \frac{1}{q})] + 2^n[d(\frac{1}{p} + \frac{1}{q}) + k - s].
\]

Now, note that a weight of the form \((2^{\alpha n})_{n \in \mathbb{Z}}\) is unbounded (and thus not contained in any space \( \ell^\theta(\mathbb{Z}) \) for \( \theta \in (0, \infty) \)) as soon as \( \alpha \neq 0 \). For \( \alpha = 0 \), the weight is constant and thus contained in \( \ell^\infty(\mathbb{Z}) \), but in no space \( \ell^\theta(\mathbb{Z}) \) with \( 0 < \theta < \infty \).

Altogether, we get for \( \theta \in (0, \infty) \) that
\[
\frac{v}{u} \in \ell^\theta(\mathbb{Z}) \iff \left(2^n[d(\frac{1}{p} + \frac{1}{q}) + k - s]\right)_{n \in \mathbb{Z}} \in \ell^\theta(\mathbb{Z}) \text{ and } \left(2^n[d(\frac{1}{p} + \frac{1}{q}) + k - s]\right)_{n \in \mathbb{Z}} \in \ell^\theta(\mathbb{Z})
\]
\[
\iff \theta = \infty \text{ and } s = d \left(\frac{1}{p} - \frac{1}{q}\right) \text{ and } s = d \left(\frac{1}{p} - \frac{1}{q}\right) + k,
\]
which can only hold for \( k = 0 \). Thus, Corollary 5.2 implies that \( \bar{B}^p,q(\mathbb{R}^d) \to W^{k,q}(\mathbb{R}^d) \) can only hold for \( k = 0 \).

For \( k = 0 \), note that Corollary 5.2 implies that \( \bar{B}^p,q(\mathbb{R}^d) \to L^q(\mathbb{R}^d) = W^{0,q}(\mathbb{R}^d) \) holds as soon as we have \( p \leq q \) and
\[
\frac{v}{u} \in \ell^\infty(\mathbb{R}^d) \iff q^\infty \cdot (r/q)^\infty = \infty \text{ and } s = d \left(\frac{1}{p} - \frac{1}{q}\right).
\]
(by eq. 5.2) \( \iff r \leq q^\infty \) and \( s = d \left(\frac{1}{p} - \frac{1}{q}\right) \).

By Corollary 5.2 for \( q \in (0, 2] \cup \{\infty\} \), this condition is also necessary for \( \bar{B}^p,q(\mathbb{R}^d) \to L^q(\mathbb{R}^d) \) to hold. For \( q \in (2, \infty) \), however, Corollary 5.2 (together with similar considerations) only implies that
\[
r \leq q \text{ and } s = d \left(\frac{1}{p} - \frac{1}{q}\right)
\]
is a necessary condition for \( \bar{B}^p,q(\mathbb{R}^d) \to L^q(\mathbb{R}^d) \) to hold.

This raises the question of what happens for \( q^\infty < r \leq q \), i.e. if the necessary condition is fulfilled, but the sufficient condition is not. In general, I do not know if the embedding \( \bar{B}^p,q(\mathbb{R}^d) \to L^q(\mathbb{R}^d) \) holds in this case. But at least for \( p = q \in (2, \infty) \), we can say slightly more: In this case, \( s = 0 \). Furthermore, for \( I_0 := N_0 \), we have \( \sup_{I \in I_0} \|T_{I}^{-1}\| = \sup_{n \in N_0} 2^{-n} = 1 < \infty \), so that part 2(c) of Corollary 5.2 is applicable. Hence,
\[
(3)_{n \in N_0} = \left(\frac{3}{2^m}\right)_{n \in N_0} = \frac{w(u)}{u} \in \ell^{2(r/2)^\infty}(I_0) = \ell^{2(r/2)^\infty}(N_0),
\]
which can only hold for \( 2 \cdot (r/2)^\infty = \infty \), i.e. \( r \leq 2 \). Hence, for \( p \in (2, \infty) \), we see that \( \bar{B}^p,q(\mathbb{R}^d) \to L^p(\mathbb{R}^d) \) can only hold for \( s = 0 \) and \( r \in (0, 2] \) and does hold for \( r \in (0, q^\infty) \). We have thus reduced the “uncertain” region from \((q^\infty, q]\) to \((q^\infty, 2] \). ✷

**Example 7.2.** (inhomogeneous Besov spaces)

Inhomogeneous Besov spaces are defined using a similar covering than the homogeneous Besov spaces from the previous example. The only difference is that the sets \( Q_n \) for \( n < 0 \), i.e. the “small” sets of the homogeneous dyadic covering are replaced by a single ball which covers the low frequencies (including zero).

Precisely, we will use the covering \( Q = (Q_n)_{n \in N_0} = (T_n Q_n + b_n)_{n \in N_0} \) with \( T_n = 2^n \cdot \text{id} \) and \( b_n := 0 \), as well as \( Q'_n := B_4(0) \setminus \overline{B_4}(0) \) for \( n \in N \). Finally, for \( n = 0 \), we set
\[
T_0 := \text{id} \quad \text{and} \quad b_0 := 0 \quad \text{as well as} \quad Q'_0 := B_2(0).
\]

Note that \( Q \) is indeed a semi-structured admissible covering, since \( x \in Q_n \cap Q_m \) for \( m, n \in \mathbb{N} \) implies \( 2^{n-2} \leq |x| \leq 2^{n+2} \) and likewise \( 2^{m-2} \leq |x| \leq 2^{m+2} \). Thus,
\[
2^{n-2} \leq 2^{m+2} \quad \text{and} \quad 2^{m-2} \leq 2^{n+2},
\]
which yields \( n - 4 \leq m \leq n + 4 \), i.e. \( n^* \subset \{n - 4, \ldots, n + 4\} \cap N_0 \) and hence \( |n^*| \leq 10 \) for all \( n \in N \). Furthermore, \( \|T_{n}^{-1}T_{m}\| = 2^{m-n} \leq 2^4 \) and \( \|T_{m}^{-1}T_{n}\| = 2^{n-m} \leq 2^4 \).

Likewise, for \( x \in Q'_0 \cap Q_m \) with \( m \in \mathbb{N} \), we get
\[
2^{m-2} \leq |x| \leq 2 = 2^1,
\]
i.e. \( m \leq 3 \) and thus \( 0^* \subset \{0, \ldots, 3\} \), which implies \( |0^*| \leq 4 \). Furthermore, \( \|T_{0}^{-1}T_{m}\| = 2^m \leq 2^4 \) and \( \|T_{m}^{-1}T_{0}\| = 2^{-m} \leq 1 \).
Finally, let $P'_n := B_2(0) \setminus \overline{B_{1/2}(0)}$ for $n \in \mathbb{N}$ and $P'_0 := B_{3/2}(0)$. Then $P'_n$ is open with $\overline{P'_n} \subset Q'_n$ for all $n \in \mathbb{N}_0$. Finally,
\[ \bigcup_{n \in \mathbb{N}} T_n P'_n = \bigcup_{n \in \mathbb{N}} \left( B_{2^{n+1}}(0) \setminus \overline{B_{2^{n-1}}(0)} \right) \supset \mathbb{R}^d \setminus \overline{B_1(0)} \]
and hence $\bigcup_{n \in \mathbb{N}_0} T_n P'_n = \mathbb{R}^d$. Thus, $Q$ satisfies all assumptions of Theorem 2.3 so that $Q$ is a tight regular covering of $\mathbb{R}^d$.

With the dyadic covering $Q$, the usual inhomogeneous Besov spaces are given (up to trivial identifications) for $p, r \in (0, \infty]$ and $s \in \mathbb{R}$ by
\[ \mathcal{B}^s_{p,r} (\mathbb{R}^d) = \mathcal{D} (Q, L^p, \ell^r_{\infty}) \]
for $u = u^{(s)} = (2^{sn})_{n \in \mathbb{N}_0}$. Now, let $q \in (0, \infty]$ and $k \in \mathbb{N}_0$ be arbitrary. The weight $v := w^{(q)}$ from Corollary 5.2 (with $n = k$) is given by
\[ v_n = |\det T_n|^{\frac{1}{p}-1} \cdot \left( 1 + |b_n|^k + ||T_n||^k \right) = 2^{dn\left( \frac{1}{p} - \frac{1}{q} \right)} \cdot \left( 1 + 2^{nk} \right) < 2^{dn\left( \frac{1}{p} - \frac{1}{q} \right)} \cdot 2^{nk} = 2^{n(k+d(\frac{1}{p} - \frac{1}{q}))} \]
for $n \in \mathbb{N}_0$. Due to the exponential nature of the weights $u, v$, we have
\[ \frac{v}{u} \in \ell^q (r/q') \quad \text{if} \quad r \leq q, \quad \text{if} \quad r > q. \]

Recall from equation (7.2) that $q \cdot (p/q)' = \infty$ if and only if $p \leq q$. Thus, Corollary 5.2 shows that we have $\mathcal{B}^s_{p,r} (\mathbb{R}^d) \hookrightarrow W^{k,q} (\mathbb{R}^d)$ as soon as $p \leq q$ and
\[ \frac{v}{u} \in \ell^q (r/q') \quad \text{if} \quad r \leq q, \quad \text{if} \quad r > q. \]

Conversely, Corollary 5.2 shows for $q \in (0, 2] \cup \{ \infty \}$ that the above conditions are also necessary for existence of the embedding $\mathcal{B}^s_{p,r} (\mathbb{R}^d) \hookrightarrow W^{k,q} (\mathbb{R}^d)$.

For the analysis in case of $q \in (2, \infty)$, instead of Corollary 5.2, we employ the “Besov detour” as described in Subsection 1.2. This will show that the sufficient criterion from the present paper lacks sharpness for $q \in (2, \infty)$. Indeed, by equation (1.4), we have
\[ \mathcal{B}^{q^2}_{k,2} (\mathbb{R}^d) = \mathcal{B}^{q,2}_{k,\min\{q,2\}} (\mathbb{R}^d) \hookrightarrow \mathcal{F}^{q,2}_{k} (\mathbb{R}^d) = W^{k,q} (\mathbb{R}^d), \]
so that $\mathcal{B}^s_{p,r} (\mathbb{R}^d) \hookrightarrow W^{k,q} (\mathbb{R}^d)$ holds as soon as $\mathcal{B}^s_{p,r} (\mathbb{R}^d) \hookrightarrow \mathcal{B}^{q,2}_{k} (\mathbb{R}^d)$. By (the proof of) [11] Proposition 2.4] (with $\alpha = 1$), this holds as soon as $p \leq q$ and
\[ \begin{cases} s \geq k + d \left( \frac{1}{p} - \frac{1}{q} \right), \quad \text{if} \quad r \leq 2, \\ s > k + d \left( \frac{1}{p} - \frac{1}{q} \right), \quad \text{if} \quad r > 2. \end{cases} \]

Note that this condition is strictly weaker than the sufficient condition in equation (7.3).

At least for $p = q \in (2, \infty)$, this relaxed sufficient condition is sharp: If $\mathcal{B}^s_{p,r} (\mathbb{R}^d) \hookrightarrow W^{k,p} (\mathbb{R}^d)$, then part (c) of Corollary 5.2 (with $I_0 = I = \mathbb{N}_0$ and $\sup_{i \in I_0} \| T_{i}^{-1} \| = \sup_{n \in \mathbb{N}_0} 2^{-n} = 1 < \infty$) yields
\[ \left( \frac{2^{kn}}{2^{sn}} \right)_{n \in \mathbb{N}_0} \preceq \frac{w(p)}{u} \in \ell^q (r/q') (I_0), \]
which is easily seen to be equivalent to condition (7.2), since $p = q$. I do not know if the condition (7.2) is also necessary for existence of the embedding if $p < q$.

\begin{example} \textbf{(\(\alpha\)-modulation spaces)} \end{example}

In [2] Theorem 2.6], it was shown that for $0 \leq \alpha < 1$ and $d \in \mathbb{N}$, there is some $r_1 = r_1(\alpha, d) > 0$ such that the family
\[ \mathcal{O}^{(\alpha)} := \mathcal{O}_r^{(\alpha)} := \left( O_k^{(\alpha)}(0) \right)_{k \in \mathbb{Z}^d \setminus \{0\}} := \left( B_{r|k|^{\alpha_0}} \left( |k|^{\alpha_0} k \right) \right)_{k \in \mathbb{Z}^d \setminus \{0\}} = \left( T_k Q + b_k \right)_{k \in \mathbb{Z}^d \setminus \{0\}} \]
with $\alpha_0 := \frac{\alpha}{1 - \alpha}$, $Q := B_r(0)$ and
\[ T_k := |k|^{\alpha_0} \cdot \text{id} \quad \text{as well as} \quad b_k := |k|^{\alpha_0} k \]
for $k \in \mathbb{Z}^d \setminus \{0\}$ is an admissible covering of $\mathbb{R}^d$ for each $r > r_1$. In [21] Theorem 6.1.3, it was furthermore shown that this covering is indeed a structured admissible covering and hence – by Theorem 2.8 – a tight regular covering of $\mathbb{R}^d$.

For $p, s \in (0, \infty]$ and $\gamma \in \mathbb{R}$, the $\alpha$-modulation space with integrability exponents $p, s$ and and weight exponent $\gamma$ is given (up to certain identifications) by

$$M^{p,s}_{\gamma,\alpha}(\mathbb{R}^d) = D \left( \mathcal{O}^{(\alpha)}, L^p, \ell^s_{\gamma,\alpha} \right),$$

where $u_k^{(\gamma,\alpha)} := |k|^\gamma/(1-\alpha)$. That the weight $u^{(\gamma,\alpha)}$ is indeed $\mathcal{O}^{(\alpha)}$-moderate is shown in [21] Lemma 6.1.2.

Now, for $n \in \mathbb{N}_0$ and $q \in (0, \infty]$, we are interested in existence of an embedding

$$M^{p,s}_{\gamma,\alpha}(\mathbb{R}^d) \hookrightarrow W^{n,q}(\mathbb{R}^d).$$

Note that the relevant weight $v := w^{(q)}$ from Corollary [6.2] is given by

$$v_k = |\det T_k|^{\frac{1}{q} - \frac{1}{p}} \cdot (1 + |k|^n + \|T_k\|^n)
= |k|^{d\alpha(\frac{1}{q} - \frac{1}{p})} \cdot \left(1 + \left(|k|^{\alpha_0+1}\right)^n + |k|^{n\alpha_0}\right)
\asymp |k|^{d\alpha(\frac{1}{q} - \frac{1}{p})} \cdot |k|^{\frac{\alpha}{1-\alpha}}
= |k|^{\frac{\alpha}{1-\alpha}(n+\alpha d(\frac{1}{p} - \frac{1}{q}))}.
$$

Here, we used $|k| \geq 1$ for $k \in \mathbb{Z}^d \setminus \{0\}$, as well as $0 \leq \alpha_0 < 1 + \alpha_0 = 1 + \frac{\alpha}{1-\alpha} = \frac{1}{1-\alpha}$. We conclude

$$\frac{v_k}{u_k^{(\gamma,\alpha)}} \asymp |k|^{\frac{\alpha}{1-\alpha}(n-\gamma + \alpha d(\frac{1}{p} - \frac{1}{q}))}$$

for $k \in \mathbb{Z}^d \setminus \{0\}$.

Thus, for $\theta \in (0, \infty)$, we have

$$\left\| \frac{v}{u^{(\gamma,\alpha)}} \right\|_{L^p}^d \asymp \sum_{k \in \mathbb{Z}^d \setminus \{0\}} |k|^{\frac{\alpha}{1-\alpha}(n-\gamma + \alpha d(\frac{1}{p} - \frac{1}{q}))},$$

which is finite if and only if

$$\theta - \frac{n - \gamma + \alpha d(\frac{1}{p} - \frac{1}{q})}{\alpha} < 0
\iff n - \gamma + \alpha d(\frac{1}{p} - \frac{1}{q}) < \frac{d(\alpha - 1)}{\theta}
\iff \gamma > n + \alpha d(\frac{1}{p} - \frac{1}{q}) - \frac{d(\alpha - 1)}{\theta}
\iff \gamma > n + \alpha d(\frac{1}{p} - \frac{1}{q}) - \frac{d(\alpha - 1)}{\theta} + \frac{d}{\theta}
\iff \gamma > n + \alpha d(\frac{1}{p} - \frac{1}{q}) + \frac{d}{\theta}. \tag{7.3}$$

For the remaining case $\theta = \infty$, it is easy to see $v/u^{(\gamma,\alpha)} \in \ell^\theta(\mathbb{Z}^d \setminus \{0\})$ if and only if

$$\frac{1}{1-\alpha} \left( n - \gamma + \alpha d(\frac{1}{p} - \frac{1}{q}) \right) \leq 0
\iff n - \gamma + \alpha d(\frac{1}{p} - \frac{1}{q}) \leq 0
\iff \gamma \geq n + \alpha d(\frac{1}{p} - \frac{1}{q}),$$

which is precisely the same condition as in equation (7.3), except that the strict inequality is replaced by a non-strict one.

Now, Corollary [5.2] shows that the embedding $M^{p,s}_{\gamma,\alpha}(\mathbb{R}^d) \hookrightarrow W^{n,q}(\mathbb{R}^d)$ is valid as soon as $p \leq q$ and

$$\frac{v}{u^{(\gamma,\alpha)}} \in \ell^\theta(\mathbb{Z}^d \setminus \{0\}).$$

There are now two cases:
Similarly, for \( M_{\gamma, \alpha}^{p, s} \) if \( p = q \), then equation (5.2) shows \( q^q \cdot (s/q^q) = \infty \), so that our considerations above imply that \( v/u(\gamma, \alpha) \in L^{q^q \cdot (s/q^q)'}(\mathbb{Z}^d \setminus \{0\}) \) is equivalent to
\[
\gamma \geq n + \alpha d \left( \frac{1}{p} - \frac{1}{q} \right) = n + \alpha d \left( \frac{1}{p} - \frac{1}{q} \right) + d (1 - \alpha) \left( \frac{1}{q^q} - \frac{1}{s} \right)_+.
\]

Case 2. If \( s > q^q \), then equation (5.2) shows \( q^q \cdot (s/q^q)' < \infty \) and equation (5.3) yields
\[
\frac{1}{q^q \cdot (s/q^q)'} = \left( \frac{1}{q^q} - \frac{1}{s} \right)_+.
\]

By equation (7.3), we thus see that \( v/u(\gamma, \alpha) \in L^{q^q \cdot (s/q^q)'}(\mathbb{Z}^d \setminus \{0\}) \) is equivalent to
\[
\gamma \geq n + \alpha d \left( \frac{1}{p} - \frac{1}{q} - \left( \frac{1}{q^q} - \frac{1}{s} \right)_+ \right) + d \left( \frac{1}{q^q} - \frac{1}{s} \right)_+.
\]

Altogether, we have shown that \( M_{\gamma, \alpha}^{p, s}(\mathbb{R}^d) \rightarrow W^{n, q}(\mathbb{R}^d) \) holds as soon as \( p \leq q \) and
\[
\begin{cases}
\gamma \geq n + d \left[ \alpha \left( \frac{1}{p} - \frac{1}{q} \right) + (1 - \alpha) \left( \frac{1}{q^q} - \frac{1}{s} \right)_+ \right], & \text{if } s \leq q^q,
\gamma \geq n + d \left[ \alpha \left( \frac{1}{p} - \frac{1}{q} \right) + (1 - \alpha) \left( \frac{1}{q^q} - \frac{1}{s} \right)_+ \right], & \text{if } s > q^q.
\end{cases}
\]

(7.4)

In case of \( q \in (0, 2] \cup \{ \infty \} \), Corollary 5.2 also shows that this condition is necessary for existence of the embedding.

For the analysis in case of \( q \in (2, \infty) \), we again use the “Besov detour” as described in Subsection 1.2. If we have \( M_{\gamma, \alpha}^{p, s}(\mathbb{R}^d) \rightarrow W^{n, q}(\mathbb{R}^d) \), then equation (1.4) yields
\[
M_{\gamma, \alpha}^{p, s}(\mathbb{R}^d) \rightarrow W^{n, q}(\mathbb{R}^d) \rightarrow F_{n, q}^{\infty}(\mathbb{R}^d) \rightarrow B_{n, \max(q, 2)}(\mathbb{R}^d) = B_{n, q}^{q}(\mathbb{R}^d).
\]

But by [21] Theorem 6.2.8], the embedding \( M_{\gamma, \alpha}^{p, s}(\mathbb{R}^d) \rightarrow B_{n, q}^{q}(\mathbb{R}^d) \) holds if and only if \( p \leq q \) and
\[
\begin{cases}
\gamma \geq n + d \left[ \alpha \left( \frac{1}{p} - \frac{1}{q} \right) + (1 - \alpha) \left( \frac{1}{q^q} - \frac{1}{s} \right)_+ \right], & \text{if } s \leq q,
\gamma \geq n + d \left[ \alpha \left( \frac{1}{p} - \frac{1}{q} \right) + (1 - \alpha) \left( \frac{1}{q^q} - \frac{1}{s} \right)_+ \right], & \text{if } s > q.
\end{cases}
\]

(7.5)

Similarly, for \( q \in (2, \infty) \), the same theorem and equation (1.4) also show that we have
\[
M_{\gamma, \alpha}^{p, s}(\mathbb{R}^d) \rightarrow B_{n, q}^{q}(\mathbb{R}^d) = B_{n, \min(q, 2)}^{q}(\mathbb{R}^d) \rightarrow F_{n, q}^{\infty}(\mathbb{R}^d) = W^{n, q}(\mathbb{R}^d)
\]

if \( p \leq q \) and
\[
\begin{cases}
\gamma \geq n + d \left[ \alpha \left( \frac{1}{p} - \frac{1}{q} \right) + (1 - \alpha) \left( \frac{1}{q^q} - \frac{1}{s} \right)_+ \right], & \text{if } s \leq 2,
\gamma \geq n + d \left[ \alpha \left( \frac{1}{p} - \frac{1}{q} \right) + (1 - \alpha) \left( \frac{1}{q^q} - \frac{1}{s} \right)_+ \right], & \text{if } s > 2.
\end{cases}
\]

Thus, we obtain a complete characterization for \( q \in (0, 2] \cup \{ \infty \} \) and for \( q \in (2, \infty) \) if \( s \notin (2, q] \). To obtain this last result, however, we had to resort to [21], instead of the results in this paper.

I do not know if the condition (7.5) is sharp in general. But for \( \alpha = 0 \) and \( p = q \in (2, \infty) \), the characterization given in [12] shows that \( M_{\gamma, 0}^{p, q}(\mathbb{R}^d) \rightarrow W^{n, q}(\mathbb{R}^d) \) holds if and only if condition (7.5) (with \( p = q \) and \( \alpha = 0 \)) is satisfied.

Example 7.4. (Shearlet smoothness spaces)

Shearlet smoothness spaces have first been introduced by Labate et al. in [13]. These spaces are defined using a suitable covering of \( \mathbb{R}^2 \). This covering is (slightly modified) given by (cf. [21] Definition 6.4.1)
\[
S = (S_i)_{i \in I} = (T_i Q + b_i)_{i \in I},
\]
where \( I = I_0 \cup \{ 0 \} \) with
\[
I_0 = \{(m, \epsilon, \delta) \in \mathbb{N}_0 \times \mathbb{Z} \times \{0, 1\} \mid |m| \leq 2^n\}
\]

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and $T_0 := 4 \cdot \text{id}$, as well as $b_0 := \left( -\frac{4}{0} \right)$. Furthermore,

$$Q := \mathcal{U}_{(-1,1)}^{(\frac{1}{3},3)} := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \left( \frac{1}{3}, 3 \right) \times \mathbb{R} \bigg| \frac{y}{x} \in (-1, 1) \right\}$$

and

$$T_{n,m,\varepsilon,\delta} := \varepsilon \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \cdot \left( \begin{array}{cc} 2^{2n} & 0 \\ 2^n m & 2^n \end{array} \right)$$

as well as $b_{n,m,\varepsilon,\delta} := 0$ for $(n, m, \varepsilon, \delta) \in I_0$.

This covering is a slight modification of the one given by Labate et al., since the covering constructed in [13] is not a structured admissible covering, contrary to the statement of [13 Proposition 4.1]. In fact, the covering given in [13] does not admit an associated partition of unity, since the interiors of the sets fail to cover all of $\mathbb{R}^2$. The covering given here is indeed a structured admissible covering, cf. [21 Lemma 6.4.2].

Now, given $p, r \in (0, \infty]$ and $\beta \in \mathbb{R}$, the shearlet smoothness space with parameters $p, r, \beta$ is given by

$$\mathcal{S}^p_r (\mathbb{R}^2) = \mathcal{D} (\mathcal{S}, \mathcal{L}^p, \ell^r_{u^{(\beta)}})$$

with $u^{(\beta)}_0 := 1$ and $u^{(\beta)}_{n,m,\varepsilon,\delta} = 2^{2n} \beta$ for $(n, m, \varepsilon, \delta) \in I_0$. We are interested in existence of the embedding $\mathcal{S}^p_r (\mathbb{R}^2) \hookrightarrow W \cdot q (\mathbb{R}^2)$ for $q \in (0, \infty]$ and $k \in \mathbb{N}_0$. The relevant weight from Corollary 5.2 (with $n = k$) is given by

$$v_i := w_i^{(q)} = |\det T_i|^{\frac{1}{2} - \frac{1}{q}} \cdot \left( 1 + |b_i| \right)$$

for $i = (n, m, \varepsilon, \delta) \in I_0$. Here, we used $|m| \leq 2^n$ and $n \geq 0$ for $i \in I_0$, so that we get

$$\| T_i \| = \left\| \begin{array}{cc} 2^{2n} & 0 \\ 2^n m & 2^n \end{array} \right\| \leq \max \{ 2^{2n}, 2^n |m|, 2^n \} = 2^{2n} \geq 1$$

and $|\det T_i| = 2^{3n}$, as well as $b_i = 0$ for $i = (n, m, \varepsilon, \delta) \in I_0$. Altogether, we get

$$\frac{v_i}{u_i^{(\beta)}} \leq 2^{n[2k-2\beta+3(\frac{1}{p} + \frac{1}{q})]}$$

for $i = (n, m, \varepsilon, \delta) \in I_0$.

Since the single term for $i = 0$ is irrelevant for membership of $v/u^{(\beta)}$ in $\ell^\theta (I)$ for $\theta \in (0, \infty)$, we get

$$\frac{v}{u^3} \in \ell^\theta (I) \iff \left\| \left( \frac{v_i}{u_i^{(\beta)}} \right)_{i \in I_0} \right\| < \infty$$

$$\iff \sum_{n=0}^{\infty} \sum_{m=-2^n}^{2^n} 2^{6n[2k-2\beta+3(\frac{1}{p} + \frac{1}{q})]} < \infty$$

$$\iff \sum_{n=0}^{\infty} 2^{n[1+\theta(2k-2\beta+3(\frac{1}{p} + \frac{1}{q}))]} < \infty,$$

which is equivalent to

$$1 + \theta \left( 2k - 2\beta + 3 \left( \frac{1}{p} - \frac{1}{q} \right) \right) < 0$$

$$\iff 2k - 2\beta + 3 \left( \frac{1}{p} - \frac{1}{q} \right) < -\frac{1}{\theta}$$

$$\iff k + \frac{3}{2} \left( \frac{1}{p} - \frac{1}{q} \right) + \frac{1}{2\theta} < \beta.$$
Now, we are in a position to apply Corollary 5.2, which shows that $S^p_{q,r}(\mathbb{R}^2) \hookrightarrow W^{k,q}(\mathbb{R}^2)$ holds if we have $p \leq q$ and \( \frac{q-1}{p-1} \in \ell^{q-1}(r/q)' \) \( \{I\} \). There are now two cases:

Case 1. If \( r \leq q \), then equation (5.2) shows $q^\gamma \cdot (r/q)' = \infty$, so that our considerations above imply that \( \nu/\mu(\beta) \in \ell^{q-1}(r/q)' \) \( \{I\} \) is equivalent to

$$
\beta \geq k + \frac{3}{2} \left( \frac{1}{p} - \frac{1}{q} \right) = k + \frac{3}{2} \left( \frac{1}{p} - \frac{1}{q} \right) + \frac{1}{2} \left( \frac{1}{q^\gamma} - \frac{1}{r} \right) + .
$$

Case 2. If \( r > q \), then equation (5.2) shows $q^\gamma \cdot (r/q)' < \infty$ and equation (5.3) yields

$$
\frac{1}{q^\gamma \cdot (r/q)'} = \left( \frac{1}{q^\gamma} - \frac{1}{r} \right) + .
$$

By our considerations above, \( \nu/\mu(\beta) \in \ell^{q-1}(r/q)' \) \( \{I\} \) is thus equivalent to

$$
\beta > k + \frac{3}{2} \left( \frac{1}{p} - \frac{1}{q} \right) + \frac{1}{2} \left( \frac{1}{q^\gamma} - \frac{1}{r} \right) + .
$$

Altogether, we have shown that $S^p_{q,r}(\mathbb{R}^2) \hookrightarrow W^{k,q}(\mathbb{R}^2)$ holds as soon as $p \leq q$ and

$$
\begin{align*}
\beta \geq k + \frac{3}{2} \left( \frac{1}{p} - \frac{1}{q} \right) + \frac{1}{2} \left( \frac{1}{q^\gamma} - \frac{1}{r} \right) +, & \quad \text{if } r \leq q, \\
\beta > k + \frac{3}{2} \left( \frac{1}{p} - \frac{1}{q} \right) + \frac{1}{2} \left( \frac{1}{q^\gamma} - \frac{1}{r} \right) +, & \quad \text{if } r > q.
\end{align*}
$$

By Corollary 5.2 for $q \in (0,2] \cup \{\infty\}$, these conditions are also necessary for existence of the embedding.

In fact, also for $q \in (2,\infty)$, our sufficient condition turns out to be reasonably sharp. To see this, we again use the “Besov detour”, in conjunction with a result from [21]: If the embedding $S^p_{q,r}(\mathbb{R}^2) \hookrightarrow W^{k,q}(\mathbb{R}^2)$ holds, then equation (1.2) shows (recall $q \in (2,\infty) \subset (1,\infty)$)

$$
S^p_{q,r}(\mathbb{R}^2) \hookrightarrow W^{k,q}(\mathbb{R}^2) = L^{q^\gamma_2}(\mathbb{R}^2) \hookrightarrow B^{q,q}_{k,\max\{q,2\}}(\mathbb{R}^2) = B^{q,q}_{k}(\mathbb{R}^2).
$$

Using [21] Theorem 6.4.3, this implies that $p \leq q$ and

$$
\begin{align*}
\beta \geq k + \frac{3}{2} \left( \frac{1}{p} - \frac{1}{q} \right) + \frac{1}{2} \left( \frac{1}{q^\gamma} - \frac{1}{r} \right) +, & \quad \text{if } r \leq q, \\
\beta > k + \frac{3}{2} \left( \frac{1}{p} - \frac{1}{q} \right) + \frac{1}{2} \left( \frac{1}{q^\gamma} - \frac{1}{r} \right) +, & \quad \text{if } r > q.
\end{align*}
$$

Conversely, as in the case of $\alpha$-modulation spaces, the “Besov detour”, in conjunction with [21] Theorem 6.4.3, also shows that we have $S^p_{q,r}(\mathbb{R}^2) \hookrightarrow W^{k,q}(\mathbb{R}^2)$ as soon as $p \leq q$ and

$$
\begin{align*}
\beta \geq k + \frac{3}{2} \left( \frac{1}{p} - \frac{1}{q} \right) + \frac{1}{2} \left( \frac{1}{q^\gamma} - \frac{1}{r} \right) +, & \quad \text{if } r \leq 2, \\
\beta > k + \frac{3}{2} \left( \frac{1}{p} - \frac{1}{q} \right) + \frac{1}{2} \left( \frac{1}{q^\gamma} - \frac{1}{r} \right) +, & \quad \text{if } r > 2.
\end{align*}
$$

Thus, even for $q \in (2,\infty)$, the only difference between this improved sufficient condition and the improved necessary condition (7.6) from above is that the sufficient condition requires a strict inequality, while the necessary criterion only yields a non-strict inequality. Furthermore, this difference only occurs for $r \in (2,q]$. Again, I do not know if the necessary condition (7.6) is sharp in general. But in view of the results from [12] for modulation spaces (cf. the previous example), I conjecture (at least for $p = q \in (2,\infty)$) that the necessary condition (7.6) is also sufficient for existence of the embedding $S^p_{q,r}(\mathbb{R}^2) \hookrightarrow W^{k,q}(\mathbb{R}^2)$.

Example 7.5. (Shearlet coorbit spaces)

For $c \in \mathbb{R}$, let

$$
H^{(c)} := \left\{ \varphi \left( \begin{array}{cc} a & b \\ 0 & a^c \end{array} \right) \left| a \in (0,\infty), b \in \mathbb{R}, \varphi \in \{\pm 1\} \right. \right\},
$$

denote the Shearlet type group with parameter $c$ (as in equation (1.2)). In [21] Corollary 6.3.5 and Theorem 4.6.4, it was shown that the coorbit space

$$
\text{Co} \left( L^p_{a,r} \left( \mathbb{R}^2 \times H^{(c)} \right) \right)
$$

for $p,r \in (0,\infty]$ and a moderate weight $\omega : H^{(c)} \rightarrow (0,\infty)$, is canonically isomorphic to a certain decomposition space $D \left( S^{(c)}, L^p, \ell_{n(r)} \right)$, where the weight $\omega(r)$ depends on $r$ and on the weight $\omega$. Here, the coorbit space is
formed with respect to the quasi-regular representation of $\mathbb{R}^2 \times H^{(c)}$, which acts by translations and $(L^2$ normalized) dilations on $L^2(\mathbb{R}^2)$. Furthermore, the space $L^p_{v,H} \mathbb{R}^2 \times H^{(c)}$ is a mixed, weighted Lebesgue space, with (quasi)-norm given by

$$
\|f\|_{L^p_{v,H}} := \left[ \int_{H^{(c)}} \left| v(h) \cdot \|f(\cdot,h)\|_{L^p(\mathbb{R}^2)} \right|^r \frac{dh}{|\det h|} \right]^{1/r},
$$

with the usual modifications for $r = \infty$. Finally, the weight $v : H^{(c)} \to (0, \infty)$ is called moderate if it satisfies $v(xyz) \leq v_0(x)v(y)v_0(z)$ for all $x,y,z \in H^{(c)}$ and some locally bounded, submultiplicative weight $v_0 : H^{(c)} \to (0, \infty)$.

More precisely, the structured admissible covering $\mathcal{S}^{(c)} = \left( T^{(c)}_i Q \right)_{i \in I}$ of $\mathcal{O} = \mathbb{R}^* \times \mathbb{R}$ is defined as follows (cf. [21 Corollary 6.3.5]): We have $I = \mathbb{Z}^2 \times \{ \pm 1 \}$ and

$$
T^{(c)}_{n,m,\varepsilon} = \varepsilon \cdot \left( \begin{array}{cc} 2^n & 0 \\ 0 & 2^m \end{array} \right) \cdot \left( \begin{array}{cc} 1 & 0 \\ m & 1 \end{array} \right) \quad \text{for } (n,m,\varepsilon) \in I,
$$

as well as

$$
Q = \left\{ \left( \begin{array}{c} x \\ y \end{array} \right) \in \left( \frac{1}{2}, 2 \right) \times \mathbb{R} \bigg| \frac{y}{x} \in (-1,1) \right\}.
$$

Finally, the weight $u^{(r)}$ is given by

$$
u^{(r)}_{n,m,\varepsilon} = 2^{-n(1+c)(\frac{1}{2} - \frac{1}{r})} \cdot v \left( T^{(c)}_{n,m,\varepsilon} \right)^{-1},
$$

for $(n,m,\varepsilon) \in I$.

For the specific choice $c = \frac{1}{2}$ and weights $v = v^{(s)}$ of the form

$$
v^{(s)}(\varepsilon \cdot \left( \begin{array}{cc} a & b \\ 0 & a^{1/2} \end{array} \right)) = a^s,
$$

the coorbit space $\text{Co} \left( L^p_{v^{(s)}}(\mathbb{R}^2 \times H^{(1/2)}) \right)$ was studied in [4, 5]. In the second of those papers, the authors developed embeddings of a subspace of $\text{Co} \left( L^p_{v^{(s)}}(\mathbb{R}^2 \times H^{(1/2)}) \right)$ into a sum of two (homogeneous) Besov spaces. This is a very different result than what we are interested in, namely establishing an embedding

$$
\text{Co} \left( L^p_{v,H} \mathbb{R}^2 \times H^{(c)} \right) \hookrightarrow W^{k,q}(\mathbb{R}^2)
$$

of the whole coorbit space into a single Sobolev space.

For our setting, it will turn out to be natural to consider weights of the form $v = v^{(\alpha,\beta)}$ with

$$
v^{(\alpha,\beta)} : H^{(c)} \to (0, \infty), \quad A = \varepsilon \left( \begin{array}{cc} a & b \\ 0 & a^{1/2} \end{array} \right) \mapsto a^{\alpha} \cdot \| A^{-T} \|^{\beta}.
$$

Up to a slight transform in the parameters $\alpha, \beta$, this weight coincides with the one considered in [21 equation (6.3.15) and Theorem 6.3.11] and is thus moderate.

With this choice, the weight $u^{(r)}$ from above is given by

$$
u^{(r)}_{n,m,\varepsilon} = 2^{-n(1+c)(\frac{1}{2} - \frac{1}{r})} \cdot 2^{-n\alpha} \cdot \| T^{(c)}_{n,m,\varepsilon} \|^{\beta}.
$$

Furthermore, the weight $v := w^{(q)}$ from Corollary 3.2 satisfies

$$
v_i = \left| \det T^{(c)}_i \right|^\frac{1}{2} \left( 1 + \| b_i \| + \| T^{(c)}_i \| \right) \quad \text{for } i \in I = \mathbb{Z}^2 \times \{ \pm 1 \},
$$

so that the quotient $w = (w_i)_{i \in I}$ of the two weights satisfies

$$
w_{n,m,\varepsilon} := \frac{w^{(r)}_{n,m,\varepsilon}}{u^{(r)}_{n,m,\varepsilon}} = 2^{n(\alpha + (1+c)(\frac{1}{2} - \frac{1}{r} - \frac{1}{q}))} \cdot \left( 1 + \| T^{(c)}_{n,m,\varepsilon} \| \right) \cdot \| T^{(c)}_{n,m,\varepsilon} \|^{\beta - \frac{k}{q}} = 2^{n(\alpha + (1+c)\gamma)} \cdot \left( \| T^{(c)}_{n,m,\varepsilon} \|^{\beta} + \| T^{(c)}_{n,m,\varepsilon} \|^{-(\beta - k)} \right) \quad \text{(7.7)}
$$
with $\gamma := \frac{1}{p} - \frac{1}{q} + \frac{1}{p} - \frac{1}{q}$. As a preparation for the application of Corollary 5.2 we want to characterize the condition $w \in \ell^\theta(I)$ in terms of $\alpha, \beta \in \mathbb{R}$ and $\theta \in (0, \infty]$.

To this end, note

$$\| T_{n,m}^{(c)} \| = \left\| \begin{pmatrix} 2^n & 0 \\ 2^{nc} & 2^{nc} \end{pmatrix} \right\| = 2^n + 2^{nc} + 2^{nc} |m|.$$

In particular $\| T_{0,m}^{(c)} \| \approx 1 + |m|$ and hence $w_{0,m} \approx (1 + |m|)^{\beta} + (1 + |m|)^{-(\beta - k)}$. Thus, if we have $w \in \ell^\theta(I) \subset \ell^\infty(I)$, then $\beta - k \geq 0$, i.e. $\beta \geq k \geq 0$.

In summary, our problem reduces to characterizing the condition $w^{(a,b)} \in \ell^\theta(\mathbb{Z}^2)$ for

$$w^{(a,b)}_{n,m} := 2^{an} \cdot \| T_{n,m}^{(c)} \|^{-b},$$

where $a \in \mathbb{R}$ and $b \geq 0$ are arbitrary. For this, we distinguish two cases:

**Case 1.** $c \geq 1$. Define

$$M_1 := \{(n, m) \in \mathbb{Z}^2 \mid n \geq 0 \text{ and } m \neq 0\},$$

$$M_2 := \{(n, m) \in \mathbb{Z}^2 \mid n \geq 0 \text{ and } m = 0\},$$

$$M_3 := \{(n, m) \in \mathbb{Z}^2 \mid n < 0 \text{ and } |m| \geq \left\lfloor 2^{n(1-c)} \right\rfloor \},$$

$$M_4 := \{(n, m) \in \mathbb{Z}^2 \mid n < 0 \text{ and } |m| \leq \left\lfloor 2^{n(1-c)} \right\rfloor - 1\}.$$

We now distinguish the four subcases corresponding to $(n, m) \in M_i$ for $i \in \{1, 2, 3, 4\}$.

1. For $(n, m) \in M_1$, note $cn \geq n$ and $|m| \geq 1$, so that $2^{nc} |m| \geq 2^{nc} \geq 2^n$. All in all, this yields $\| T_{n,m}^{(c)} \| \approx 2^{nc} |m|$.

2. For $(n, m) \in M_2$, note again $cn \geq n$ and hence $2^{nc} \geq 2^n \geq 0 = 2^{nc} |m|$. All in all, this yields $\| T_{n,m}^{(c)} \| \approx 2^{nc}$.

3. For $(n, m) \in M_3$, we have $|m| \geq \left\lfloor 2^{n(1-c)} \right\rfloor \geq 2^{n(1-c)}$ and thus $2^{nc} |m| \geq 2^n \geq 2^{nc}$, where the last step used that $n < 0$ and $c \geq 1$ imply $cn \leq n$. All in all, we get $\| T_{n,m}^{(c)} \| \approx 2^{nc} |m|$.

4. For $(n, m) \in M_4$, we have $|m| \leq \left\lfloor 2^{n(1-c)} \right\rfloor - 1$. Hence, $|m| \leq 2^{n(1-c)}$, which implies $2^{nc} |m| \leq 2^n$. Since $n < 0$ and $c \geq 1$, we also have $cn \leq n$ and thus $2^{nc} \leq 2^n$. Altogether, this yields $\| T_{n,m}^{(c)} \| \approx 2^n$.

In summary, we have shown

$$w^{(a,b)}_{n,m} = 2^{an} \cdot \| T_{n,m}^{(c)} \|^{-b} \times \begin{cases} 2^{an} \cdot (2^{nc} |m|)^{-b} = 2^{n(a-bc)} |m|^{-b}, & \text{if } (n, m) \in M_1, \\ 2^{an} \cdot (2^{nc})^{-b} = 2^{n(a-bc)}, & \text{if } (n, m) \in M_2, \\ 2^{an} \cdot (2^{nc} |m|)^{-b} = 2^{n(a-bc)} \cdot |m|^{-b}, & \text{if } (n, m) \in M_3, \\ 2^{an} \cdot (2^{n})^{-b} = 2^{n(a-bc)}, & \text{if } (n, m) \in M_4. \end{cases}$$

Based on this asymptotic behaviour, we can now characterize finiteness of $\| w^{(a,b)} \|_{\ell^\infty(M_i)}$ for each $i \in \{1, 2, 3, 4\}$.

1. On $M_1$, we see that

$$\| w^{(a,b)} \|_{\ell^\infty(M_1)} \propto \left\| \begin{pmatrix} 2^{n(a-bc)} \cdot |m|^{-b} \\ n \geq 0, m \neq 0 \end{pmatrix} \right\|_{\ell^\infty}$$

is finite if and only if $a - bc \leq 0$.

2. The norm $\| w^{(a,b)} \|_{\ell^\infty(M_2)} \propto \left\| \begin{pmatrix} 2^{n(a-bc)} \\ n \geq 0 \end{pmatrix} \right\|_{\ell^\infty}$ is finite iff the condition $a - bc \leq 0$ from the preceding case is satisfied.
(3) On $M_3$, we see that
\[
\left\| w^{(a,b)} \right\|_{\ell^\infty(M_3)} = \left\| \left( 2^{-n(a-bc)} \cdot |m|^{-b} \right)_{n<0, |m| \geq 2^n(1-c)} \right\|_{\ell^\infty}
\]
\[
(b \geq 0) = \left\| \left( 2^{n(1-c)} \cdot 2^{-n(1-c)} \right)_{n<0} \right\|_{\ell^\infty}
\]
\[
= \left\| 2^{-n(b-a)} \right\|_{\ell^\infty}
\]
is finite if and only if $b - a \leq 0$. To justify the step marked with $(*)$, note that we have $n(1-c) \geq 0$ for $n < 0$ and thus $2^{n(1-c)} \geq 1$, which easily yields
\[
2^{n(1-c)} \leq \left[ 2^{n(1-c)} \right] \leq 2^{n(1-c)} + 1 \leq 2 \cdot 2^{n(1-c)}.
\]

(4) On $M_4$, we see that
\[
\left\| w^{(a,b)} \right\|_{\ell^\infty(M_4)} = \left\| \left( 2^{n(1-b)} \right)_{n<0, |m| \leq 2^n(1-c)} \right\|_{\ell^\infty} = \left\| 2^{-n(b-a)} \right\|_{\ell^\infty}
\]
is finite if and only if $b - a \leq 0$.

In summary, we see (for $b \geq 0$) that $\| w^{(a,b)} \|_{\ell^\infty(\mathbb{Z}^2)}$ is finite if and only if we have $b \leq a \leq bc$.

Now, we characterize the condition $\| w^{(a,b)} \|_{\ell^\infty(\mathbb{Z}^2)} < \infty$ for $b \geq 0$ in terms of $a$, $b$ and $\theta \in (0, \infty)$.

As above, we distinguish four cases:

(1) On $M_1$, we see that
\[
\left\| w^{(a,b)} \right\|_{\ell^\theta(M_1)}^\theta = \sum_{n=0}^{\infty} \left( \sum_{m \in \mathbb{Z} \setminus \{0\}} \left| w^{(a,b)}_{n,m} \right|^\theta \right) \leq \sum_{n=0}^{\infty} 2^{-n(a-bc)} \leq 2^{-n(a-bc)}
\]
is finite if and only if $a - bc < 0$ and $\theta b > 1$.

(2) On $M_2$, we see that
\[
\left\| w^{(a,b)} \right\|_{\ell^\theta(M_2)}^\theta = \sum_{n=0}^{\infty} 2^{-n(a-bc)}
\]
is finite if and only if $a - bc < 0$, which is already implied by finiteness of $\| w^{(a,b)} \|_{\ell^\theta(M_1)}$.

(3) On $M_3$, we see that (since we can assume that $b\theta > 1$ by finiteness of $\| w^{(a,b)} \|_{\ell^\theta(M_1)}$), finiteness of
\[
\left\| w^{(a,b)} \right\|_{\ell^\theta(M_3)}^\theta \leq \sum_{n=-\infty}^{-1} \left( \sum_{m \geq 2^n(1-c)} \left| w^{(a,b)}_{n,m} \right|^\theta \right)
\]
\[
\leq \sum_{n=1}^{\infty} \left( 2^{-n \theta(1-bc)} \cdot \sum_{m \geq 2^{-n \theta(1-bc)}} m^{-b\theta} \right)
\]
(since $b\theta > 1$)
\[
\leq \sum_{n=1}^{\infty} \left( 2^{-n \theta(1-bc)} \cdot 2^{-(n-1)(1-b\theta)} \right)
\]
(since $n(1-c) \geq 0$)
\[
= \sum_{n=1}^{\infty} \left( 2^{(n-c)(b-a)-1} \right)
\]
is equivalent (under the assumption \( \|w^{(a,b)}\|_{\ell^p(M_1)} < \infty \)) to

\[ c + \theta(b - a) - 1 < 0. \]

Here, at (*), we used (for \( m_0 = \left[2^{n(c-1)}\right] \in \mathbb{N} \)) the estimate

\[
\sum_{m=m_0}^{\infty} m^\theta = \sum_{m=m_0}^{\infty} \int_m^{m+1} m^\theta \, dx \\
\overset{(*)}{=} \sum_{m=m_0}^{\infty} \frac{(1+x)^{1+\theta}}{1+\theta} \, dx \\
= \int_{m_0}^\infty x^\theta \, dx
\]

\[
\overset{(*)}{=} \begin{cases} \frac{m_0^{1+\theta}}{\theta} \approx \theta m_0^{1+\theta}, & \text{if } \theta < -1, \\ \infty, & \text{if } \theta \geq -1, \end{cases}
\]

which is justified, since inside the integral at \((*)\), we have \( 1 \leq m_0 \leq m \leq x \leq m + 1 \) and thus \( m \leq x \leq m + 1 \leq 2m \), which finally yields \( m^\theta \approx x^\theta \) for arbitrary \( \theta \in \mathbb{R} \).

(4) On \( M_4 \), the norm

\[
\|w^{(a,b)}\|_{\ell^p(M_4)}^{\theta} \approx \sum_{n=-\infty}^{-1} \sum_{m=-\left[2^{(n-c)}\right]+1}^{\left[2^{(n-c)}\right]-1} 2^{n\theta(a-b)}
\]

\[
= \sum_{\ell=1}^{\infty} 2^{-\ell\theta(a-b)} \left( 2 \cdot \left[2^{-\ell(1-c)}\right] - 1 \right)
\]

\[
\overset{(*)}{=} \sum_{\ell=1}^{\infty} 2^{-\ell\theta(a-b)} 2^{-\ell(1-c)}
\]

\[
\approx \sum_{\ell=1}^{\infty} 2^{-\ell(1-c+\theta(a-b))}
\]

is finite if and only if \( 1 - c + \theta(a - b) > 0 \). Here, the step marked with \((*)\) used that we have for \( \ell \in \mathbb{N} \) that \(-\ell(1-c) \geq 0\) and hence \(2^{-\ell(1-c)} \geq 1\), which implies

\[
2^{-\ell(1-c)} \leq \left[2^{-\ell(1-c)}\right] \leq 2 \cdot \left[2^{-\ell(1-c)}\right] - 1 \leq 2 \cdot \left[2^{-\ell(1-c)}\right] \leq 2 \cdot \left(2^{-\ell(1-c)} + 1\right) \leq 4 \cdot 2^{-\ell(1-c)}.
\]

Altogether, we have shown for \( \theta \in (0, \infty) \) and \( b \geq 0 \) that \( \|w^{(a,b)}\|_{\ell^p(\mathbb{Z}^2)} \) is finite if and only if we have

\[
b\theta > 1 \text{ and } a - bc < 0 \text{ and } 1 - c + \theta(a - b) > 0
\]

\[
\iff b\theta > 1 \text{ and } a \in \left(b + \frac{c - 1}{\theta}, cb\right).
\]

Case 2. \( c \in (-\infty, 1) \). Here, we use the modified sets

\[
M_1 := \{(n, m) \in \mathbb{Z}^2 \mid n \geq 0 \text{ and } |m| \leq \left[2^{(1-c)n}\right]\},
\]

\[
M_2 := \{(n, m) \in \mathbb{Z}^2 \mid n \geq 0 \text{ and } |m| \geq \left[2^{(1-c)n}\right] + 1\},
\]

\[
M_3 := \{(n, m) \in \mathbb{Z}^2 \mid n < 0 \text{ and } m \neq 0\},
\]

\[
M_4 := \{(n, m) \in \mathbb{Z}^2 \mid n < 0 \text{ and } m = 0\}.
\]
Here, we used

Finally, we are in a position to apply Corollary 5.2. A sufficient condition for existence of the embedding

Using this simplified expression for $w^{(a,b)}$, one can characterize finiteness of $\|w^{(a,b)}\|_{\ell^\infty(M_i)}$ for $i \in \mathbb{N}$ as in the previous case. The results are as follows:

1. On $M_1$, $\|w^{(a,b)}\|_{\ell^\infty(M_1)}$ is finite if and only if $a - b \leq 0$.
2. On $M_2$, $\|w^{(a,b)}\|_{\ell^\infty(M_2)}$ is finite if and only if $a - b \leq 0$.
3. On $M_3$, $\|w^{(a,b)}\|_{\ell^\infty(M_3)}$ is finite if and only if $a - bc \geq 0$.
4. On $M_4$, $\|w^{(a,b)}\|_{\ell^\infty(M_4)}$ is finite if and only if $a - bc \geq 0$.

Altogether, we see that $\|w^{(a,b)}\|_{\ell^\infty(\mathbb{Z}^2)}$ is finite if and only if we have $bc \leq a \leq b$.

Finally, one can characterize the condition $\|w^{(a,b)}\|_{\ell^\infty(\mathbb{Z}^2)} < \infty$ for $b \geq 0$ in terms of $a, b$ and $\theta \in (0, \infty)$, as for $c \geq 1$. The results are as follows:

1. On $M_1$, $\|w^{(a,b)}\|_{\ell^\infty(M_1)}^\theta$ is finite if and only if $1 - c + \theta (a - b) < 0$.
2. On $M_2$, $\|w^{(a,b)}\|_{\ell^\infty(M_2)}^\theta$ is finite if and only if we have $b\theta > 1$ and $\theta (a - b) + 1 - c < 0$.
3. On $M_3$, $\|w^{(a,b)}\|_{\ell^\infty(M_3)}^\theta$ is finite if and only if we have $b\theta > 1$ and $\theta (bc - a) < 0$, which is equivalent to $bc - a < 0$.
4. On $M_4$, $\|w^{(a,b)}\|_{\ell^\infty(M_4)}^\theta$ is finite if and only if we have $\theta (bc - a) < 0$, i.e. if and only if $bc - a < 0$.

Altogether, we see for $\theta \in (0, \infty)$ and $b \geq 0$ that $\|w^{(a,b)}\|_{\ell^\infty(\mathbb{Z}^2)}$ is finite if and only if we have

$$b\theta > 1 \text{ and } bc - a < 0 \text{ and } \theta (a - b) < 0$$

$$\iff b\theta > 1 \text{ and } a \in \left( cb, b + \frac{c - 1}{\theta} \right).$$

Finally, recall that we wanted to characterize finiteness of $\|w\|_{\ell^\theta(I)}$ with $w$ as in equation (7.9), i.e. with

$$w_{n,m,\varepsilon} \approx w_{n,m}^{(\alpha + (1 + c)\gamma, \beta)} + w_{n,m}^{(\alpha + (1 + c)\gamma, \beta - k)}.$$

Using the characterization that we just obtained, we see that $\|w\|_{\ell^\theta(I)}$ is finite if and only if the following holds:

$$\begin{cases} 
\beta \geq k \text{ and } \beta \leq \alpha + (1 + c)\gamma \leq c(\beta - k), & \text{if } c \geq 1 \text{ and } \theta = \infty, \\
\beta \geq k \text{ and } \max \{c\beta, c(\beta - k)\} \leq \alpha + (1 + c)\gamma \leq \beta - k, & \text{if } c < 1 \text{ and } \theta = \infty, \\
\beta > k + \frac{1}{\theta} \text{ and } \beta + \frac{1}{\theta} \leq \alpha + (1 + c)\gamma \leq c(\beta - k), & \text{if } c \geq 1 \text{ and } \theta < \infty, \\
\beta > k + \frac{1}{\theta} \text{ and } \max \{c\beta, c(\beta - k)\} \leq \alpha + (1 + c)\gamma \leq \beta - k + \frac{1}{\theta}, & \text{if } c < 1 \text{ and } \theta < \infty.
\end{cases}$$

Here, we used $\gamma = \frac{1}{\theta} + \frac{1}{\theta} = \frac{1}{\theta}$, cf. equation (7.9). Note

$$\max \{c\beta, c(\beta - k)\} = \begin{cases} 
c\beta, & \text{if } c \geq 0, \\
c(\beta - k), & \text{if } c < 0.
\end{cases}$$

Finally, we are in a position to apply Corollary 5.2. A sufficient condition for existence of the embedding

$$\text{Co} \left( L_{\nu}^{p,\tau} (\mathbb{R}^2 \times H^{(c)}) \right) \cong D \left( S^{(c)}, L^{p,\ell^\tau(a)} \right) \to W^{k,q} (\mathbb{R}^2)$$

(7.10)
is that \( p \leq q \) and \( w \in \ell^{p'}(r/q')' (\mathbb{Z}^2 \times \{ \pm 1 \}) \). In view of equations (7.9), (5.2) and (5.3), the second part of this condition is equivalent to the following:

\[
\begin{align*}
\beta \geq k & \quad \text{and} \quad \beta \geq \alpha + (1 + c) \gamma \leq c (\beta - k), & \text{if} \ c \geq 1 \text{ and } r \leq q^\gamma, \\
\beta \geq k & \quad \text{and} \quad \max \{ c \beta, c (\beta - k) \} \leq \alpha + (1 + c) \gamma \leq \beta - k, & \text{if} \ c < 1 \text{ and } r \leq q^\gamma, \\
\beta > k + \frac{1}{q^\gamma} - \frac{1}{r} & \quad \text{and} \quad \beta + (c - 1) \left( \frac{1}{q^\gamma} - \frac{1}{r} \right) < \alpha + (1 + c) \gamma < c (\beta - k), & \text{if} \ c \geq 1 \text{ and } r > q^\gamma, \\
\beta > k + \frac{1}{q^\gamma} - \frac{1}{r} & \quad \text{and} \quad \max \{ c \beta, c (\beta - k) \} < \alpha + (1 + c) \gamma < \beta - k + (c - 1) \left( \frac{1}{q^\gamma} - \frac{1}{r} \right), & \text{if} \ c < 1 \text{ and } r > q^\gamma.
\end{align*}
\]  

(7.11)

Furthermore, Corollary 5.2 shows for \( q \in (0, 2] \cup \{ \infty \} \) that these conditions are also necessary for existence of the embedding (7.10). Finally, a necessary (but (probably) not sufficient) condition for existence of the embedding in case of \( q \in (2, \infty) \) is obtained by replacing \( q^\gamma \) by \( q \) everywhere in the sufficient condition (7.11) (and by additionally requiring \( p \leq q \)).

In the present setting, I refrain from invoking the “Besov detour” for \( q \in (2, \infty) \), since the necessary and sufficient conditions for existence of the embedding \( D (S^{(c)}, L^p, \ell^q_w) \hookrightarrow B_1 \) from [21, Theorem 6.3.12] lack sharpness for \( q \in (2, \infty) \), since the covering \( S^{(c)} \) is not moderate with respect to the dyadic “Besov” covering. Thus, the results in [21] do not yield an improvement in this case.

Before closing this example, I want to draw attention to a special case of the characterization from above: The starting point for the present paper was the following question posed by Holger Rauhut:

When is there an embedding of a shearlet coorbit space into a BV space? In view of Corollary 5.1 this is equivalent to asking when \( \text{Co} \left( L^{p,r}_{\psi_{(\alpha,\beta)}}, (\mathbb{R}^2 \times H^{(1/2)}) \right) \) \( \hookrightarrow \) \( \text{W}^{1,1} (\mathbb{R}^2) \) is true. This amounts to choosing \( k = q = 1 \) and \( c = 1/2 \). Because of \( q = 1 \in (0, 2] \), we thus obtain a complete solution to this question: The desired embedding holds if and only if we have \( p \leq 1 \) and

\[
\begin{align*}
\beta & \geq 1 \quad \text{and} \quad \frac{\beta}{2} \leq \alpha + \frac{3}{2} \left( \frac{1}{p} - \frac{1}{r} - \frac{1}{2} \right) \leq \beta - 1, & \text{if} \ r \leq 1, \\
\beta > 2 - \frac{1}{r} \quad \text{and} \quad \frac{\beta}{2} < \alpha + \frac{3}{2} \left( \frac{1}{p} - \frac{1}{r} - \frac{1}{2} \right) < \beta - 1 - \frac{1}{2} (1 - \frac{1}{r}), & \text{if} \ r > 1.
\end{align*}
\]

Slightly simplified, this last condition reads

\[
\begin{align*}
\beta & \geq 1 \quad \text{and} \quad \frac{\beta}{2} + \frac{3}{4} \leq \alpha + \frac{3}{2} \left( \frac{1}{p} - \frac{1}{r} - \frac{1}{4} \right) \leq \beta - \frac{1}{4}, & \text{if} \ r \leq 1, \\
\beta > 2 - \frac{1}{r} \quad \text{and} \quad \frac{\beta}{2} + \frac{3}{4} < \alpha + \frac{3}{2} \left( \frac{1}{p} - \frac{1}{r} - \frac{1}{4} \right) < \beta - \frac{3}{4} + \frac{1}{2r}, & \text{if} \ r > 1.
\end{align*}
\]

(7.12)

Note that we have \( \left( \beta - \frac{1}{4} \right) - \left( \frac{\beta}{2} + \frac{3}{4} \right) = \frac{\beta}{2} - 1 \geq 0 \) if and only if \( \beta \geq 2 \). For \( r \leq 1 \), this shows that the condition in equation (7.12) can be satisfied (with a suitable choice of \( \alpha \in \mathbb{R} \)) if and only if \( \beta \geq 2 \). Likewise, \( \left( \beta - \frac{3}{4} + \frac{1}{2r} \right) - \left( \frac{\beta}{2} + \frac{3}{4} \right) = \frac{\beta}{2} - \frac{3}{4} + \frac{1}{2r} > 0 \) if and only if \( \beta > 3 - \frac{1}{r} \). This shows that for \( r > 1 \), the condition in equation (7.12) can be satisfied (for a suitable choice of \( \alpha \in \mathbb{R} \)) if and only if \( \beta > 3 - \frac{1}{r} \).

As concluding remarks, we note the following:

(1) The condition \( \beta \geq k \) is always necessary for the embedding \( \text{Co} \left( L^{p,r}_{\psi_{(\alpha,\beta)}}, (\mathbb{R}^2 \times H^{(c)}) \right) \) \( \hookrightarrow \) \( \text{W}^{k,q} (\mathbb{R}^2) \). Thus, using the weight \( v^{(c)} \) as in [4, 5] (which amounts to choosing \( \beta = 0 \)), one can never obtain an embedding into a Sobolev space \( \text{W}^{k,q} (\mathbb{R}^2) \) of positive smoothness \( k \geq 1 \).

(2) Using the approach in the present paper, we were able to handle arbitrary \( c \in \mathbb{R} \), whereas in [21, Theorem 6.3.11], only the range \( c \in (0, 1] \) was considered, since in the remaining range, the dyadic “Besov covering” is incompatible with the covering \( S^{(c)} \), i.e. neither is subordinate to the other one. This shows that, in addition to being easier to apply, the results in the present paper can be employed in some settings in which the findings from [21] are not applicable.

\[\text{Example 7.6. (Coorbit spaces of the diagonal group)}\]

In this example, we consider coorbit spaces of the diagonal group \( D \leq \text{GL} (\mathbb{R}^d) \) for \( d \in \mathbb{N} \), given by

\[
D = \left\{ \text{diag} (a_1, \ldots, a_d) = \begin{pmatrix} a_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_d \end{pmatrix} \middle| a_1, \ldots, a_d \in \mathbb{R}^* = \mathbb{R} \setminus \{0\} \right\}.
\]

\[\text{See [21, Definition 3.3.1]}\] for the definition of moderateness of one covering with respect to another one.

\[\text{60}\]
The general setup for the definition of coorbit spaces with respect to $D$ as in the case of shearlet-type coorbit spaces (cf. Example 2.5).

It is easy to see that the dual action $D \times \mathbb{R}^d \to \mathbb{R}^d$, $(h, \xi) \mapsto h^{-T} \xi$ has the unique open(!) orbit $O = (\mathbb{R}^*)^d$, on which it acts with trivial (and hence compact) stabilizers. Thus, $D$ is an admissible dilation group in the sense of [9] and [21 Chapter 4].

Now, let $I := \mathbb{Z}^d \times \{ \pm 1 \}^d$ and define

$$A_{k, \varepsilon} := \text{diag}(\varepsilon_1 \cdot 2^{k_1}, \ldots, \varepsilon_d \cdot 2^{k_d}) \in D \quad \text{for } (k, \varepsilon) \in I.$$  

We claim that $(A_i)_{i \in I}$ is well-spread in $D$. To this end, note that

$$U_0 := D \cap \left\{ A \in \mathbb{R}^{d \times d} \mid \|A - \text{id}\| < \frac{1}{2} \right\}$$

is a unit neighborhood in $D$. Thus, there is a (smaller) unit neighborhood $U \subset D$ with $U = U^{-1}$ and $UU \subset U_0$.

We claim that $(A_i)_{i \in I}$ is $U$-separated. Indeed, it is easy to see $A^{-1}_{k, \varepsilon} A_{\ell, \delta} = A_{k, \varepsilon, \ell - k}$ and hence

$$\left\| A^{-1}_{k, \varepsilon} A_{\ell, \delta} - \text{id} \right\| = \left\| \text{diag}\left((\varepsilon \delta)_{i} \cdot 2^{\ell - k_i} - 1, \ldots, (\varepsilon \delta)_{d} \cdot 2^{\ell - k_d} - 1\right) \right\| = \max\left\{ 1 - (\varepsilon \delta)_{i} \cdot 2^{\ell - k_i} \mid i \in \mathbb{Z}^d \right\}.$$  

Now, if $\varepsilon_i \neq \delta_i$ for some $i \in \mathbb{Z}^d$, this implies $(\varepsilon \delta)_{i} \cdot 2^{\ell - k_i} < 0$ and thus $\left\| A^{-1}_{k, \varepsilon} A_{\ell, \delta} - \text{id} \right\| > 1 > \frac{1}{2}$. If otherwise $\varepsilon_i = \delta_i$ for all $i \in \mathbb{Z}^d$, then $\left\| A^{-1}_{k, \varepsilon} A_{\ell, \delta} - \text{id} \right\| < \frac{1}{2}$ implies $2^{\ell - k_i} \in B_{1/2}(1) = (\frac{1}{2}, \frac{3}{2}) \subset (\frac{1}{2}, 2)$ and hence $\ell_i = k_i$ for all $i \in \mathbb{Z}^d$. We have thus shown $\left\| A^{-1}_{k, \varepsilon} A_{\ell, \delta} - \text{id} \right\| \geq \frac{1}{2}$ if $(k, \varepsilon) \neq (\ell, \delta)$. But $A_{k, \varepsilon} U \cap A_{\ell, \delta} U \neq \emptyset$ implies $A_{k, \varepsilon} U \subset U^{-1} = U \subset U_0$ and hence $\left\| A^{-1}_{k, \varepsilon} A_{\ell, \delta} - \text{id} \right\| < \frac{1}{2}$, i.e. $(k, \varepsilon) = (\ell, \delta)$. Thus, $(A_i)_{i \in I}$ is indeed $U$-separated.

Now, since $D$ is homeomorphic to $(\mathbb{R}^*)^d$, it is easy to see that

$$V := \left\{ \text{diag}(a_1, \ldots, a_d) \mid a_1, \ldots, a_d \in \left[ \frac{1}{2}, 2 \right) \right\} \subset D$$

is a compact unit neighborhood. But $(A_i)_{i \in I}$ is $V$-dense and thus well-spread in $D$. Indeed, for arbitrary $A = \text{diag}(a_1, \ldots, a_d) \in D$, we can choose

$$\varepsilon_i := \text{sgn}(a_i) \in \{ \pm 1 \} \quad \text{and} \quad k_i := \lfloor \log_2 (|a_i|) \rfloor \in \mathbb{Z} \quad \text{for } i \in \mathbb{Z}^d.$$  

With this choice, we easily see $-\log_2 (|a_i|) \leq -k_i \leq 1 - \log_2 (|a_i|)$ and thus $1 \leq 2^{-k_i} |a_i| \leq 2$ for all $i \in \mathbb{Z}^d$, which implies

$$A^{-1}_{k, \varepsilon} A = \text{diag}(2^{-k_1} |a_1|, \ldots, 2^{-k_d} |a_d|) \in V$$

and hence $A \in A_{k, \varepsilon} V$.

Now, let $P := \left( \frac{2}{3}, \frac{3}{2} \right)^d$ and $Q := \left( \frac{1}{2}, 2 \right)^d$. Then $P \subset Q$ and it is easy to see $\bigcup_{i \in I} A_i^{-T} P = (\mathbb{R}^*)^d = O$. Thus, $P := (A_{r}^{-T})_{i \in I}$ and $Q := Q_D := (A_{i}^{-T} Q)_{i \in I}$ are both coverings of $O$ induced by $D$ (cf. [21 Definition 4.4.3]) and hence semi-structured admissible coverings of $O$, by [21 Lemma 4.4.4]. Consequently, $Q$ is a structured admissible covering of $O$ and in particular a tight regular covering and thus also an $L^p$-decomposition covering of $O$ for all $p \in (0, \infty)$, cf. Theorem 2.8 and Corollary 2.7.

Now, by [21 Theorem 4.6.4], we get (up to canonical identifications) that

$$\text{Co} \left( L^{r, p} \left( \mathbb{R}^d \right) \right) = \mathcal{D} \left( Q_D, L^p, \ell^r_{u(r)} \right)$$

for every moderate weight $v : D \to (0, \infty)$, where the weight $u^{(r)}$ is given by

$$u_i^{(r)} := |\det A_i|^{-r} \cdot v (A_i) \quad \text{for } i \in I.$$  

See Example 2.5 for the definition of a moderate weight and note that each submultiplicative, continuous weight is moderate. Furthermore, [21 Lemma 4.2.7] shows that the set of submultiplicative, continuous weights is closed under multiplication and addition and also under taking maximums and inversions, where the inversion of a weight $v : G \to (0, \infty)$ is given by $v^{-1}(x) := v \left( x^{-1} \right)$.
Now, we describe the class of weights \( v : D \to (0, \infty) \) that we will consider in this example: For \( \ell \in \mathbb{Z} \), the weight

\[
v_1^{(\ell)} : D \to (0, \infty), \quad \text{diag} (a_1, \ldots, a_d) \mapsto |a_\ell|
\]

is continuous and (sub)multiplicative. This yields submultiplicativity of the weight \( v_2^{(\ell)} := \max \{1, v_1^{(\ell)}\} \), which satisfies

\[
v_2^{(\ell)}(\text{diag} (a_1, \ldots, a_d)) = \begin{cases} |a_\ell|, & \text{if } |a_\ell| \geq 1, \\ 1, & \text{if } |a_\ell| \leq 1. \end{cases}
\]

But since \( v_2^{(\ell)} \) is a submultiplicative weight on \( D \), the same is true of \( v_3^{(\ell)} : D \to (0, \infty), A \mapsto v_2^{(\ell)}(A^{-1}) \) and thus also of

\[
v^{(\ell, \alpha_1, \alpha_2)} : D \to (0, \infty), \quad A = \text{diag} (a_1, \ldots, a_d) \mapsto \left( v_2^{(\ell)}(A) \right)^{\alpha_1} \cdot \left( v_3^{(\ell)}(A) \right)^{-\alpha_2} = \begin{cases} |a_\ell|^{\alpha_1}, & \text{if } |a_\ell| \geq 1, \\ |a_\ell|^{-\alpha_2}, & \text{if } |a_\ell| < 1, \end{cases}
\]

for arbitrary \( \alpha_1, \alpha_2 \in \mathbb{R} \). Finally, for \( \alpha, \beta \in \mathbb{R}^d \), the weight

\[
v^{(\alpha, \beta)} : D \to (0, \infty), \quad A \mapsto \prod_{\ell=1}^d v^{(\ell, \alpha_\ell, \beta_\ell)}(A)
\]

is submultiplicative as a product of submultiplicative weights.

Now, we are interested in existence of an embedding

\[
\text{Co} \left( L^p_{\psi, \alpha} (\mathbb{R}^d \times D) \right) = D \left( \mathcal{Q}_D, L^p, \ell^{\alpha(r)} \right) \hookrightarrow W^{k,q} (\mathbb{R}^d)
\]

for \( k \in \mathbb{N}_0 \) and \( q \in (0, \infty] \), where

\[
u^{(r)}_{k, \varepsilon} = |\det A_{k, \varepsilon}|^{\frac{1}{2} - \frac{1}{q}} \cdot v^{(\alpha, \beta)}(A_{k, \varepsilon}) = \prod_{\ell=1}^d \begin{cases} 2^{k_\ell (\alpha_\ell + \frac{1}{2} - \frac{1}{q})}, & \text{if } k_\ell \geq 0, \\ 2^{k_\ell (\beta_\ell + \frac{1}{2} - \frac{1}{q})}, & \text{if } k_\ell < 0. \end{cases}
\]

In the present setting, the weight \( v := w^{(q)} \) from Corollary 5.2 is given by

\[
v_{k, \varepsilon} = |\det T_{k, \varepsilon}|^{\frac{1}{2} - \frac{1}{q}} \cdot (1 + |b_{k, \varepsilon}|^n + \|T_{k, \varepsilon}\|^n) \geq |\det A_{k, \varepsilon}|^{\frac{1}{2} - \frac{1}{q}} \cdot \left( 1 + \left\| A_{k, \varepsilon}^{-1} \right\|^n \right) = (1 + \max \{ 2^{-nk_\ell} \mid \ell \in \mathbb{Z} \}) \cdot \prod_{\ell=1}^d 2^{k_\ell \left( \frac{1}{2} - \frac{1}{q} \right)},
\]

where it is important to note \( T_i = A_i^{-T} \) for \( i \in I \), since \( \mathcal{Q}_D = (A_i^{-T} Q)_i \in l^r \).

Thus, the relevant quotient \( \frac{v_{k, \varepsilon}}{v^{(r)}_{k, \varepsilon}} \) satisfies

\[
\frac{v_{k, \varepsilon}}{v^{(r)}_{k, \varepsilon}} \asymp \left( 1 + \max \{ 2^{-nk_\ell} \mid \ell \in \mathbb{Z} \} \right) \cdot \prod_{\ell=1}^d \begin{cases} 2^{k_\ell (\gamma + \alpha_\ell)}, & \text{if } k_\ell \geq 0, \\ 2^{k_\ell (\gamma - \beta_\ell)}, & \text{if } k_\ell < 0, \end{cases} \]

\[
\asymp \left( 1 + \sum_{\ell=1}^d 2^{-nk_\ell} \right) \cdot v^{(\gamma - \alpha, \gamma - \beta)} \left( \text{diag} \left( 2^{k_1}, \ldots, 2^{k_d} \right) \right) + \sum_{\ell=1}^d v^{(\gamma - \alpha - n_\ell, \gamma - \beta - n_\ell)} \left( \text{diag} \left( 2^{k_1}, \ldots, 2^{k_d} \right) \right),
\]

with \( \gamma := \frac{1}{q} - \frac{1}{p} + \frac{1}{2} - \frac{1}{q} \) and \( \gamma_\ell := \gamma \cdot (1, \ldots, 1) \in \mathbb{R}^d \).

Now, due to the exponential nature and due to the product structure of the weight

\[
w^{(\alpha, \beta)} (k_1, \ldots, k_d) := v^{(\alpha, \beta)} \left( \text{diag} \left( 2^{k_1}, \ldots, 2^{k_d} \right) \right) = \prod_{\ell=1}^d 2^{k_\ell \alpha_\ell}, \quad \text{if } k_\ell \geq 0,
\]

\[
2^{k_\ell \beta_\ell}, \quad \text{if } k_\ell < 0,
\]

we see

\[
w^{(\alpha, \beta)} \in \ell^q (\mathbb{Z}^d) \iff \begin{cases} \alpha_\ell \leq 0 \text{ and } \beta_\ell \geq 0 \text{ for all } \ell \in \mathbb{Z}, & \text{if } \theta = \infty, \\ \alpha_\ell < 0 \text{ and } \beta_\ell > 0 \text{ for all } \ell \in \mathbb{Z}, & \text{if } \theta < \infty. \end{cases}
\]
For brevity, let us write $a \leq b$ for $a, b \in \mathbb{R}^d$ if $a_\ell \leq b_\ell$ for all $\ell \in \mathbb{N}$. The notation $a < b$ is defined analogously. In view of equation (7.14), we see
\[
\frac{\nu}{u(\ell)} \in \ell^\theta \left( Z^d \times \{ \pm 1 \}^d \right)
\]
\[\Leftrightarrow \begin{cases} 
\gamma_e - \alpha \leq 0 \text{ and } \gamma_e - \beta \geq 0 \text{ and } \gamma_e - \alpha - ne_\ell \leq 0 \text{ and } \gamma_e - \beta - ne_\ell \geq 0 \text{ for all } \ell \in \mathbb{N}, \quad &\text{if } \theta = \infty, \\
\gamma_e - \alpha < 0 \text{ and } \gamma_e - \beta > 0 \text{ and } \gamma_e - \alpha - ne_\ell < 0 \text{ and } \gamma_e - \beta - ne_\ell > 0 \text{ for all } \ell \in \mathbb{N}, \quad &\text{if } \theta < \infty
\end{cases}
\]
\[\Leftrightarrow \begin{cases} 
\alpha_\ell \geq \gamma \text{ and } \beta_\ell \leq \gamma - n \text{ for all } \ell \in \mathbb{N}, \quad &\text{if } \theta = \infty, \\
\alpha_\ell > \gamma \text{ and } \beta_\ell < \gamma - n \text{ for all } \ell \in \mathbb{N}, \quad &\text{if } \theta < \infty.
\end{cases}
\]

Now, Corollary 5.2 (in conjunction with equation (5.2)) shows that the embedding (7.13) holds if we have $p \leq q$ and
\[
\begin{cases} 
\alpha_\ell \geq \frac{1}{q} - \frac{1}{p} + \frac{1}{r} - \frac{1}{2} \quad \text{and} \quad \beta_\ell \leq \frac{1}{q} - \frac{1}{p} + \frac{1}{r} - \frac{1}{2} - n \text{ for all } \ell \in \mathbb{N}, \quad &\text{if } r \leq q^\gamma, \\
\alpha_\ell > \frac{1}{q} - \frac{1}{p} + \frac{1}{r} - \frac{1}{2} \quad \text{and} \quad \beta_\ell < \frac{1}{q} - \frac{1}{p} + \frac{1}{r} - \frac{1}{2} - n \text{ for all } \ell \in \mathbb{N}, \quad &\text{if } r > q^\gamma.
\end{cases}
\]
\[\text{(7.16)}
\]
For $q \in (0, 2] \cup \{ \infty \}$, these conditions are also necessary for existence of the embedding (7.13). For $q \in (2, \infty)$, a necessary condition for (7.13) is that $p \leq q$ and that equation (7.16) holds, with $q^\gamma$ replaced by $q$ throughout.

In summary, due to the exponential nature of the weight $u(\alpha, \beta)$, our criteria are again reasonably sharp, even for $q \in (2, \infty)$. The only difference between sufficient and necessary criteria is that for $q \in (2, \infty)$ and $r \in (q^\gamma, q]$, the sufficient condition requires a strict inequality, while the necessary condition only yields a non-strict estimate. We again refrain from using the “Besov detour”, since the methods from [21] are also not sharp, since the covering $Q = Q_{\mathcal{D}}$ is not moderate with respect to the dyadic “Besov” covering.

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