Abstract

From the analyticity properties of the equation governing infinitesimal perturbations, it is shown that all stability properties of spatially extended 1D systems can be derived from a single function that we call entropy potential since it gives directly the Kolmogorov-Sinai entropy density. Such a function allows determining also Lyapunov spectra in reference frames where time-like and space-like axes point in general directions in the space-time plane. The existence of an entropy potential implies that the integrated density of positive exponents is independent of the reference frame.

KEY WORDS: Spatiotemporal chaos, coupled map lattices, entropy potential, spatiotemporal and comoving Lyapunov exponents.

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I. INTRODUCTION

The first part of this work [1] (hereafter referred to as LPT) was devoted to the definition and discussion of the properties of temporal (TLS) and spatial (SLS) Lyapunov spectra of 1D extended dynamical systems. In this second part we first show how the two approaches are mutually related by proving, in some simple cases, and conjecturing, in general, that all stability properties can be derived from a single observable: the entropy potential, which is a function of two independent variables, the spatial and the temporal growth rates \( \mu, \lambda \), respectively. Legendre transforms represent the right tool to achieve a complete description of linear stability properties in the space-time plane. In fact, we find that equivalent descriptions can be obtained by choosing any pair of independent variables in the set \( \{ n_\mu, n_\lambda, \mu, \lambda \} \), where \( n_\mu \) and \( n_\lambda \) are the integrated densities of spatial, resp. temporal, Lyapunov exponents. The corresponding potentials are connected via suitable Legendre transformations involving pairs of conjugated variables.

A further type of connection between spatial and temporal Lyapunov exponents discussed in this paper is found in connection with the evolution of perturbations along generic “world-lines” in the space-time, i.e. along directions other than the space and time axes considered in LPT. The extension of the usual definition of Lyapunov exponents to this more general class of frames, already discussed in [2], is rather appropriate for characterizing patterns with some anisotropy. Here, we show that this seemingly more general class of spatiotemporal exponents can be derived from the knowledge of spatial and temporal Lyapunov spectra, which thus confirm to contain all the relevant information.

For the sake of completeness, finally, we recall the last class of exponents introduced to describe convectively unstable states, comoving Lyapunov exponents [3], and their relationship again with SLS and TLS. In particular, we discuss the structure of the spectra in a simple case of a stationary random state.

Let us now briefly introduce the notations with reference to some specific models. Spatiotemporal chaos and instabilities in extended systems have been widely studied with the aid of models of reaction-diffusion processes, whose general 1D form is of the type [4,5]

\[
\partial_t y = F(y) + D \partial^2_x y, \quad (1)
\]

with the state variable \( y(x,t) \) defined on the domain \([0, L]\) (periodic boundary conditions \( y(0, t) = y(L, t) \) are generally assumed). The nonlinear function \( F \) accounts for the local reaction dynamics, while the diffusion matrix \( D \) represents the strength of the spatial coupling. The introduction of coupled map lattices (CML) has been of great help for understanding the statistical properties of spatio-temporal chaos, especially by means of numerical simulations.

In its standard form [3] a CML dynamics reads as

\[
y_{n+1}^i = f \left( (1 - \varepsilon)y_n^i + \frac{\varepsilon}{2} [y_{n-1}^i + y_{n+1}^i] \right), \quad (2)
\]

where \( i, n \) being the space, resp. time, indices labelling each variable \( y_n^i \) of a lattice of length \( L \) (with periodic boundary conditions \( y_n^{i+L} = y_n^i \)), and \( \varepsilon \) gauges the diffusion strength. The function \( f \), mapping a given interval \( I \) of the real axis onto itself, simulates a local nonlinear reaction process.
A generalization of model (2) has been proposed [3] to mimic 1D open-flow systems, namely
\[ y_{n+1}^i = f \left( (1 - \varepsilon)y_n^i + \varepsilon \left[ (1 - \alpha)y_{n-1}^i + \alpha y_n^{i+1} \right] \right). \] (3)

The parameter \( \alpha \) (bounded between 0 and 1) accounts for the possibility of an asymmetric coupling, corresponding to first order derivatives in the continuum limit.

The present paper is organized as follows. In Section II we introduce the entropy potential and derive its explicit expression in some simple cases. Sec. III is devoted to Lyapunov analysis in tilted reference frames, while Sec. IV deals with the relationships between temporal, spatial and spatiotemporal exponents. Comoving exponents are reviewed in Sec. V within the framework introduced in this paper. Some conclusive remarks are finally reported in Sec. VI.

**II. ENTROPY POTENTIAL**

The simplest context, where a discussion on the entropy potential can be set in, is provided by the linear diffusion equation for the field \( u(x, t) \)
\[ \partial_t u = \gamma u + D \partial_x^2 u, \]
(4)
which can be interpreted as the linearization of (the scalar version of) Eq. (1) around a uniform stationary solution \( y(x, t) = \text{const} \). The linear stability analysis amounts to assuming a perturbation of the form
\[ u(x, t) \sim \exp \left( \bar{\mu} x + \bar{\lambda} t \right), \]
(5)
where \( \bar{\lambda} = \lambda + i\omega \) and \( \bar{\mu} = \mu + i k \) are complex numbers the real parts of which denote temporal and spatial Lyapunov exponents, respectively, while the imaginary parts represent the integrated densities of spatial (\( \omega \)) and temporal (\( k \)) exponents. Substituting Eq. (5) in Eq. (4) we obtain
\[ \bar{\lambda} = \gamma + D \bar{\mu}^2. \]
(6)

By separating real and imaginary parts, we get
\[ \lambda(\omega, k) = \gamma + \left( \omega^2 - 4D^2 k^4 \right) / (4Dk^2) \]
(7)
\[ \mu(\omega, k) = \omega / (2Dk) \].

As already discussed in LPT, \( \omega \) and \( k \) play the same role as the integrated densities \( n_\mu \) and \( n_\lambda \) and can be explicitly obtained by inverting Eqs. (7),
\[ n_\lambda \equiv k = \sqrt{\mu^2 - \frac{\lambda - \gamma}{D}} \]
(8)
\[ n_\mu \equiv -\omega = -2D\mu \sqrt{\mu^2 - \frac{\lambda - \gamma}{D}}. \]
The minus sign in the definition of \( n_\mu \) is just a matter of convention: we adopt this choice for consistency reasons with LPT.

The above sets of equations (7,8) stress, in a particular instance, the general observation reported in LPT that either the pair \( (n_\mu, n_\lambda) \) or \( (\mu, \lambda) \) suffices to identify a given perturbation, the remaining two variables being determined from the Lyapunov spectra. However, any two items in the set \( \{ \mu, \lambda, n_\mu, n_\lambda \} \) can be chosen to be the independent variables. The above two choices are preferable for symmetry reasons; however, the pairs \( (\mu, n_\lambda) \) and \( (\lambda, n_\mu) \) turn out to be the best ones for the identification of a single function, the entropy potential, which determines all stability properties.

In fact, as it is clear from Eq. (6), we can condense the two real functions needed for a complete characterization of the stability properties into a single analytic complex expression. Now, the mere circumstance that \( \lambda(\mu, n_\lambda) \) and \( n_\mu(\mu, n_\lambda) \) are the real and imaginary parts of the analytic function \( \tilde{\lambda}^*(\tilde{\mu}) \) has an immediate and important consequence: Cauchy-Riemann conditions are satisfied and it is possible to write \( \lambda \) and \( n_\mu \) as partial derivatives of the same real function,

\[
\frac{\partial \Psi}{\partial n_\lambda} = \lambda \quad (9) \\
\frac{\partial \Psi}{\partial \mu} = -n_\mu, 
\]

where \( \Psi \) is the imaginary part of the formal integral \( \tilde{\Psi} \) of \( \tilde{\lambda} \) with respect to \( \tilde{\mu} \). Equivalently, one might call into play the real part of \( \tilde{\Psi} \), as it is known that the latter contains the same amount of information.

In the case under investigation, we find

\[
\Psi(\mu, n_\lambda) = n_\lambda (\gamma + D\mu^2) - \frac{D}{3} n_\lambda^3, \quad (10)
\]

which, together with Eq. (9), provides a complete characterization of the system.

Another, less trivial, example where the linearized problem leads to an analytic function for the eigenvalues is the the 1D complex Ginzburg-Landau equation

\[
\partial_t A = (1 + ic_1)\partial_x^2 A + A - (1 - ic_3)A|A|^2, \quad (11)
\]

where \( A(x, t) \) is a complex field and \( c_1 \) and \( c_3 \) are real positive parameters. The stability of the “phase winding” solutions \( A(x, t) = A_0 \exp(i(\nu x - \omega_0 t)) \), with \( A_0 = \sqrt{1 - \nu^2} \) and \( \omega_0 = -c_3 + (c_1 + c_3)\nu^2 \), are ruled by the following equation for the (complex) perturbation \( u(x, t) \)

\[
\partial_t u = (1 + ic_1)(\partial_x^2 u + 2i\partial_x u) - (1 - ic_3)(1 - \nu^2)(u + u^*) \quad , \quad (12)
\]

together with its complex conjugate for \( u^* \) considered as an independent variable. The eigenvalue problem is solved assuming again

\footnote{The reference to the complex conjugate variable again follows from the convention adopted for \( n_\mu \).}
\begin{equation}
    u(x,t) = u_0 \exp(\mu x + \lambda t) \quad ; \quad u^*(x,t) = u_0^* \exp(\bar{\mu} x + \bar{\lambda} t),
\end{equation}
and equating to zero the determinant of the resulting linear system, to get the analytic (implicit) relation between $\bar{\lambda}$ and $\bar{\mu}$
\begin{equation}
    \left(\lambda + 1 - \nu^2 - \bar{\mu}^2 + 2c_1 \nu \bar{\mu}\right)^2 + \left(c_1 \bar{\mu}^2 + 2\nu \bar{\mu} + (1 - \nu^2) c_3\right)^2 = (1 + c_3^2)(1 - \nu^2)^2,
\end{equation}
which is analogous to Eq. (6)

On the basis of the examples discussed here and in the Appendix, one can convince himself that the analyticity property of the eigenvalue equation appears to be very general. Periodicity in time simply leads to multiply several r.h.s.'s all depending on $\bar{\mu}$, while periodicity in space requires distinguishing between different sites on the lattice. In the latter case, the equivalent of Eq. (A7) is obtained by equating to zero a suitable determinant, where the only variable is $\bar{\mu}$. There is no reason to expect that different conditions should hold in aperiodic regimes.

The above approach is, in some sense, a generalization of dispersion relations which are normally introduced for the characterization of elliptic equations. In that case, the only acceptable linear solutions are propagating plane waves, that is $\lambda = \mu = 0$ for (almost) all wavenumbers. From our point of view, this implies a strong simplification since the mutual relationships among $\{n_\lambda, n_\mu, \lambda, \mu\}$ reduce to the link between spatial and temporal wavenumbers. Moreover, it is obvious that, because of the degeneracy, not all representations are equivalent (in particular, the $(\mu, \lambda)$ plane is totally useless).

We must stress that the methodology that we are trying to develop in these two papers applies to general systems where propagation coexists with amplification (or damping). This is by no means a limitation, as all models introduced for the characterization of space-time chaos are in this class.

However, the most serious obstacle to a rigorous proof of the general validity of Eq. (3) is represented by the identification of the integrated densities $n_\mu$, $n_\lambda$ with the wavenumbers $\omega$ and $\kappa$, respectively. In the presence of spatial disorder, the Lyapunov vectors are no longer Fourier modes: one can at most determine an average wavenumber by counting the number of nodes in the eigenfunctions. This is not a problem in the absence of temporal disorder, when the node theorem applies [8]. However, in more general cases, it is no longer possible to speak of eigenfunctions and we are not aware of any generalization to overcome the difficulty. For such a reason, we have performed some direct numerical check to verify the correctness of our conjectures.

Before discussing numerical simulations, let us come back to the problem of the representation. In the above part, we have seen that the choice of the pair of independent variables $(\mu, n_\lambda)$ was very fruitful for the identification of a potential. However, the asymmetry of such a choice calls for transferring the above result in either representation proposed in LPT. This step can be easily done with the help of Legendre transforms. We discuss the transformation to the plane $(\mu, \lambda)$, any other transformation being a straightforward generalization of the same procedure.

From the first of Eq. (1), we see that $\lambda$ and $n_\lambda$ can, indeed, be considered as conjugate variables in a Legendre transform involving $\Psi$. The conjugate potential is naturally
\begin{equation}
    \Phi \equiv \lambda n_\lambda - \Psi.
\end{equation}
It is easily seen that in the new representation, the following relations hold

\[ \partial_\lambda \Phi = n_\lambda \]
\[ \partial_\mu \Phi = n_\mu . \]  

Accordingly, the potential \( \Phi \) is the appropriate function which allows determining the two integrated densities in the symmetric representation \((\lambda, \mu)\). We call \( \Phi \) the entropy potential since it coincides with the Kolmogorov-Sinai entropy density along a suitable line (see Sec. III).

In Fig. 1, we present a numerical reconstruction of \( \Phi(\mu, \lambda) \) in terms of its contour levels for the homogenoeus CML, namely model (2) with \( f(x) = rx \mod 1 \). The entropy potential is obviously known up to an additive arbitrary constant that we have fixed by imposing that the value attained on the upper border is equal to zero. The potential increases monotonously from top to bottom. It is clear that outside the allowed region delimited by the solid curves, \( \Phi \) is a linear function of \( \mu \) and \( \lambda \).

The structure of the potential does not substantially change for more general CMLs. We have tested Eq. (16) for a lattice of logistic maps \((f(x) = 4x(1-x)) \) with \( \varepsilon = 1/3 \) by integrating along two different paths in the \((\mu, \lambda)\) plane (see Table 1). The difference is so small that we can confirm that the relations are valid, within the numerical error.

### III. SPATIOTEMPORAL EXPONENTS

In the perspective of a complete characterization of space-time chaos, one should consider the possibility of viewing a generic pattern as being generated along directions other than time and space axes. In fact, once a pattern is given, any direction can, a priori, be considered as an appropriate “time” axis. Accordingly, questions can be addressed about the statistical properties of the pattern when viewed in that way.

#### A. Definitions

For the moment, we assume that the pattern is continuous both along space and time directions; we shall discuss later how the definitions can be extended to CML models. Let us consider a given spatiotemporal configuration of the field \( y(x, t) \), generated, say, by integrating Eq. (1). When arbitrary directions are considered in the \((x, t)\) plane, the coordinates must be properly scaled in order to force them to have the same dimension. We choose to multiply the time variable by \( c \), where \( c \) is a suitable constant with the dimension of a velocity. Moreover, let \( \vartheta \) denote the rotation angle of the tilted frame \((x', ct')\) with respect to the initial one \((x, ct)\), adopting the convention that positive angles correspond to clockwise rotations. Sometimes, it will be more convenient to identify the new frame by referring to the velocity \( v = c \tan \vartheta \). The limit cases \( v = 0 \) (\( \vartheta = 0 \)) and \( v = +\infty \) (\( \vartheta = \pi/2 \)) correspond to purely temporal and purely spatial propagations, respectively. The coordinate transformation reads as

\[ ct' = \beta \left( ct + \frac{v}{c} x \right) \]
\[ x' = \beta (-vt + x) , \]

(17)
where $\beta \equiv 1/\sqrt{1 + v^2/c^2}$. The physical meaning of $v$ is transparent: it can be interpreted as the velocity in the old frame of a point stationary in the tilted frame (constant $x'$).

The new field $y(x', ct')$ can be thought of as being the result of the integration of the model derived from the original one after the change of variables. Although it is not obvious whether the invariant measure in the initial frame is still attracting in the new frame (see Ref. [2] for a discussion of this point), one can anyhow study the stability properties by linearizing and defining the Lyapunov exponents in the usual way.

In CML models, the discreteness of both the space and the time lattice leads to some difficulties in the practical construction of tilted frames. In fact, only rational values of the velocity $v$ can be realized in finite lattices (in this case, it is natural to assume that the lattice spacing is the “same” along the spatial and the temporal directions and, accordingly, to set $c = 1$). Moreover, writing the explicit expression of the model requires introducing different site types. For this reason, we discuss in the following the simplest nontrivial case $v = 1/2$, the generalization to other rational velocities being conceptually straightforward.

A generic spatial configuration in the tilted frame is defined by sites of the spatiotemporal lattice $(i, n)$ connected by alternating horizontal (as in the usual case) and diagonal bonds (see Fig. 2). By suitably adjusting the relative fraction of the two types of links all rotations between 0 and $\pi/4$ can be reproduced. The explicit expression of the updating rule requires a proper numbering of the consecutive sites. Moreover, as seen in Fig. 2, it involves the “memory” of two previous states.

Finally, an exact implementation of the mapping rule requires acausal boundary conditions, since the knowledge of future (in the original frame) states is required (this is a general problem occurring also in the continuous case). As we are interested in the thermodynamic limit, we bypass the problem by choosing periodic boundary conditions. Such a choice has been shown not to affect the bulk properties of the dynamical evolution.

In the updating procedure, two different cases are recognized: the variable $y$ is either determined from the past values in the neighbouring sites, or it requires the newly updated $y$-value on the right neighbour (see Fig. 3). For $v = 1/2$, this can be done by simply distinguishing between even and odd sites,

$$X_{n+1}^{2i} = f \left( (1 - \varepsilon)X_n^{2i} + \frac{\varepsilon}{2} \left[ Y_n^{2i-1} + X_n^{2i+1} \right] \right)$$

$$X_{n+1}^{2i+1} = f \left( (1 - \varepsilon)X_n^{2i+1} + \frac{\varepsilon}{2} \left[ X_n^{2i} + X_n^{2i+2} \right] \right)$$

where $i = 1, \ldots, L/2$ ($L$ is assumed to be even for simplicity), while

$$Y_n^{j+1} \equiv X_n^j$$

are additional variables introduced to account for the dependence at time $n - 1$. Taking into account that $X_{n+1}^{2i+2}$ can be determined from the $X$ and $Y$ variables at time $n$, the mapping can be finally expressed in the usual synchronous form $(X_n^i, Y_n^i) \rightarrow (X_{n+1}^i, Y_{n+1}^i)$, but with an asymmetric spatial coupling with next and next-to-next nearest neighbours. The Lyapunov exponents $\eta_j$ can now be computed with the usual technique.

In analogy with the original model, we expect again that, in the limit of infinitely extended systems, the set of exponents $\eta_j(v)$ will converge to an asymptotic form,
\[ \eta_j(v) \rightarrow \eta(v, n_\eta) \quad , \]  

where \( n_\eta \) is the corresponding integrated density. We will refer to this function as the spatiotemporal Lyapunov spectrum (STLS). In the limit cases \( v = 0, +\infty \) (\( \vartheta = 0, \pi/2 \)), the STLS reduces to the standard temporal and spatial spectrum, respectively.

The recursive scheme (18) implies an increase of the phase-space dimension by a factor \((1+1/2)\) (in general \(1 + v\)). Actually, as we will argue, these new degrees of freedom are not physically relevant. However, for consistency reasons with the original rescaling of the spatial variable, we choose to normalize the spatiotemporal density between 0 and \(1 + v\) (the time units are, instead, left unchanged by the above construction).

The generalization to asymmetric maps (3) is straightforward: it removes the degeneracy \(v \rightarrow -v\). Numerical results for logistic maps, indicate that the dependence of the positive exponents on the velocity is quite weak in the fully symmetric case \(\alpha = 1/2\) (for instance, the maximum exponent exhibits a 20\% variation in the whole \(v\) range), while it is remarkable for asymmetric couplings. In every case, the negative part of the spectrum sharply changes with the velocity. This is consistent with the results obtained for delayed maps in Ref. [2].

**B. Representation in the \((\mu, \lambda)\) plane**

Spatiotemporal exponents can be put in relation with \(\mu\) and \(\lambda\) by rewriting the general expression for a perturbation in a frame rotated by an angle \(\vartheta\),

\[ \exp(\mu x + \lambda t) = \exp(\mu' x' + \lambda t') \quad . \]

Such an equation induces a rotation of the same angle in the \((c\mu, \lambda)\) variables,

\[ \lambda' = \beta (\lambda + v\mu) \]
\[ \mu' = \beta \left( -(v/c^2)\lambda + \mu \right) \quad . \]

The above equations allow studying the stability with respect to generic perturbations with an exponential profile along \(x'\). For simplicity, we shall consider only uniform perturbations,

\[ \exp(\mu x + \lambda t) = \exp(\eta t') \quad , \]

where the growth rate \(\eta\) denotes the spatiotemporal exponent. Notice that we have changed notations from \(\lambda'\) to \(\eta\), to understand that the condition \(\mu' = 0\) is fulfilled. From the second of Eq. (22), uniform perturbations in the rotated frame correspond to points along the line \(L\)

\[ \lambda = c^2 \mu/v \quad , \]

in the \((\mu, \lambda)\) plane.

Whenever the evolution of an exponentially localized perturbation of type (24) is considered, it is natural to introduce the quantity \(\hat{V} = \lambda/\mu\), which can be interpreted as the velocity of the front [10]. Eq. (24) connects this velocity with that of the rotated frame,

\[ \hat{V} = c^2/v \quad . \]
Therefore, on the basis of definition (23), \((x', ct')\) can be interpreted as the reference frame in which the front associated with the perturbation propagates with an “infinite” velocity.

The explicit expression for \(\eta\) is

\[
\eta = \sqrt{\lambda^2 + (c\mu)^2} = \lambda/\beta .
\]  

(26)

Such a relation can be turned into a self-consistent equation for the maximum Lyapunov exponent \(\eta_{\text{max}}\) by imposing the constraint that the pair \((\mu, \lambda)\) lies on the line \(\lambda = \lambda_{\text{max}}(\mu)\), namely

\[
\eta_{\text{max}} = \frac{1}{\beta} \lambda_{\text{max}} \left( \frac{v}{c^2} \beta \eta_{\text{max}} \right) .
\]  

(27)

Some ambiguities arise when velocities \(v > c\) are considered, since the line \(L\) intersects \(\lambda_{\text{max}}(\mu)\) in two points as seen in Fig. 3. This phenomenon was already noticed in Ref. [10], while discussing the propagation of exponentially localized disturbances in the original reference frame. Moreover, it has been shown that only the front corresponding to the smaller value of \(\mu\) is stable, except for some cases where a nonlinear mechanism intervenes dominating the propagation process [10].

At \(v = c^2/V_\ast\) the two intersections degenerate into a single tangency point. This condition defines \(V_\ast\), which can be interpreted as the slowest propagation velocity of initially localized disturbances [10].

The extension of Eq. (27) to the rest of the spectrum requires to connect \(n_\lambda\) and \(n_\mu\) with \(n_\eta\). In the next section, we will show how to perform such a step with the help of the entropy potential. Here, we limit ourselves to discuss the structure of the STLS for different values of the tilting angle \(\vartheta\). In Fig. 3, we report the borders of the bands, which can be determined from the intersections of \(L\) with the border \(\partial D\) of the domain of allowed perturbations (see Fig. 1 of LPT and Fig. 3). For \(\vartheta = 0\) (temporal case) a single band is present but, as soon as \(\vartheta > 0\), a second negative band arises from the intersections with the branch diverging to \(-\infty\) at \(\mu = -\mu_c\). For \(\vartheta > \pi/4\), the negative band disappears and a positive band arises from the intersections with the branch diverging to \(+\infty\) with slope \(v = c\). A single band spectrum is again recovered for \(\vartheta \geq \vartheta_\ast = \tan(c/V_\ast)\).

Notice that in symplectic maps, the STLS is symmetric for any value of \(\vartheta\) (see LPT) so that positive and negative bands appear and disappear simultaneously.

It is worthwhile to illustrate some of the above considerations in the simple case of the linear diffusion equation (4). The expression for \(\lambda_{\text{max}}(\mu)\) can be obtained from Eq. (6), by setting \(k\) and \(\omega\) equal to 0. Accordingly, Eq. (27) reads as

\[
\beta \eta_{\text{max}} = \gamma + D \left( \frac{v}{c^2} \beta \eta_{\text{max}} \right)^2 .
\]  

(28)

On the other hand, the model equation in the rotated frame can be obtained from the substitutions

\[
\partial_t \rightarrow \beta \left( \partial_{t'} - v \partial_{x'} \right) ,
\]

\[
\partial_x \rightarrow \beta \left( \frac{v}{c^2} \partial_{t'} + \partial_{x'} \right) .
\]  

(29)
By introducing the usual Ansatz for the shape of the perturbation,
\[ u(x', t') \sim \exp [i\kappa x' + (\eta + i\Omega)t'] \] , \hspace{1cm} (30)
separating the real from the imaginary part, and eliminating \( \Omega \), we obtain the integrated density of spatiotemporal exponents
\[ \kappa(\eta, v) = \beta \left( 1 - 2D \frac{v^2}{c^4} \beta \eta \right) \sqrt{\frac{v^2}{c^4} (\beta \eta)^2 - \frac{\beta \eta}{D}} + \frac{\gamma}{D} \] (31)
and the corresponding STLS \( \eta(\kappa, v) \). Dimensional analysis shows that \( \kappa \) is an inverse length, as expected for a density of exponents. Notice that Eq. (28) is recovered, by setting \( \kappa = \Omega = 0 \) in Eq. (31).

In this and more general continuous models, we should remark that the line \( \mathcal{L} \) intersects \( \lambda_{\text{max}}(\mu) \) twice for any arbitrarily small \( v \). This is because the Laplacian operator sets no upper limit to the propagation velocity of disturbances.

IV. FROM THE ENTROPY POTENTIAL TO DYNAMICAL INVARIANTS

The present section is devoted to establish the consequences of the existence of the entropy potential on the Lyapunov spectra and other dynamical indicators such as the Kolmogorov-Sinai entropy and the Kaplan-Yorke dimension of the attractor. In order to keep the notations as simple as possible, we assume that time and space coordinates are scaled in such a way that \( c = 1 \).

A. Spatiotemporal exponents

The very existence of the entropy potential \( \Phi \) implies that the Lyapunov spectrum in a frame tilted at an angle \( \vartheta \) (recall that \( \vartheta \) is the angle from the \( \lambda \)-axis) can be obtained by computing the derivative of \( \Phi \) along the direction \( \vec{u} = (\sin \vartheta, \cos \vartheta) \) in the \((\mu, \lambda)\) plane. In fact, this is a straightforward generalization of the previous findings that \( n_\mu \) and \( n_\lambda \) are the derivatives of \( \Phi \) along the \( \mu \) and \( \lambda \) direction, respectively. Accordingly, the STLS is linked to the TLS and SLS by the following general equation
\[ n_\eta(v, \eta) = \vec{u} \cdot \nabla \Phi = \beta [vn_\mu + n_\lambda] \] , \hspace{1cm} (32)
where \( \nabla = (\partial_\mu, \partial_\lambda) \) is the gradient in the \((\mu, \lambda)\) plane, and the r.h.s of the above formula is evaluated for
\[ \mu = v\beta \eta \]  \hspace{1cm} (33)
\[ \lambda = \beta \eta \]  \hspace{1cm} (34)
Such a relation can be directly verified for the diffusion equation from Eqs. (8),(31). Further, more significative tests have been performed by checking numerically the validity of Eq. (32) in some lattice models involving, e.g., logistic and homogeneous chains, the spectra of which are reported in see Fig. 5 (notice that, since the time axis has not been renormalized in the tilted frame, the factor \( \beta \) need not be introduced).
B. Entropy

Kolmogorov-Sinai entropy $H_{KS}$ is a measure of the information-production rate during a chaotic evolution. An estimate of $H_{KS}$ is given by the Pesin formula \[12\] as the sum $H_{\lambda}$ of the positive Lyapunov exponents. While it is rigorously proven that $H_{KS} \leq H_{\lambda}$, numerical simulations indicate that, in general, an equality holds. In spatially extended systems, $H_{KS}$ is believed to be proportional to the system size \[13\]. For this reason, it is convenient to introduce the entropy density $h_{\lambda}$ which, in the thermodynamic limit, is computed as the integral of the positive part of the Lyapunov spectrum.

Therefore, it is natural to extend the definition of entropy to tilted frames as an integral along the line $L$,

$$h_{\eta} = \int_{n_{\min}}^{n_{\max}} \eta(n_{\mu}, n_{\lambda})dn_{\eta} ,$$  \hspace{1cm} (35)

where $n_{\min}$ is the integrated density in the point where the line $L$ intersects $\partial D$, i.e. where $\eta = \eta_{\max}$, while $n_{\max}$ is measured in the origin, i.e. where $\eta = 0$. In the limit $v \to 0$, the above equation reduces to the definition of the density $h_{\lambda}$, which refers to the original reference frame. For $v \to \infty$, instead, we obtain the “spatial” entropy density $h_{\mu}$.

Numerical simulations performed with different CML models indicate that $h_{\lambda} < h_{\mu}$. This can be explained by the following argument. The patterns obtained asymptotically by iterating the model in the original reference frame are, in general, unstable if generated along the spatial direction \[14\]. In other words, the spatiotemporal attractor is a (strange) repellor of the spatial dynamics. Accordingly, part of the local instability accounted for by the sum of positive spatial Lyapunov exponents is turned into a contribution to the escape rate from the repellor \[15\] and $h_{\mu}$ must be larger than the entropy $h_{\lambda}$ of the original pattern.

In continuous models this inequality is brought to the extreme case, as $h_{\mu}$ is infinite.

Another explanation of the above result can be found by referring directly to the plane $(\mu, \lambda)$. Integrating by parts, Eq. (35) can be rewritten as

$$h_{\eta} = n_{\min}\eta_{\max} - \int_{0}^{\eta_{\max}} n_{\eta}(\mu, \lambda)d\eta .$$  \hspace{1cm} (37)

As $n_{\min} = 0$ along the upper border (see LPT), we can compute $h_{\eta}(v)$ for $v < 1$ by integrating the gradient of $\Phi$ along the line $L$ from $(0, 0)$ to the upper border itself. Since the upper
border is an equipotential line (see also Fig. 1), $h_\eta$ is independent of $v$. The same argument can be repeated for $v > 1/V^*$ by suitably shifting $n_\mu$ and $n_\lambda$.

The independency of $h_\eta$ of $v$ has also a physical interpretation. The Kolmogorov-Sinai entropy density, in fact, is the amount of information needed to characterize a space-time pattern (apart from the information flow through the boundaries [13]) divided by its temporal duration and the spatial extension, i.e. divided by the area. Therefore, $h_{KS}$ is independent of the way the axes are oriented in the plane, i.e. of the velocity $v$. As a consequence, $h_\eta = h_{KS}$ for all $v < 1$.

The above conclusion still holds when the STLS exhibits a positive band as well (which is always the case in continuous models), provided that the content of such a band is discarded. Accordingly, we can conclude that the new degrees of freedom, associated to the positive band, which appear in the rotated frame are just physically irrelevant directions which turn the original attractor into a repellor. If $v > c^2/V_*$, the two bands merge together and it is not anymore possible to distinguish between unstable but irrelevant directions and the unstable manifold of the original attractor. Presumably, this means that the repellor is turned into a strange repellor with a singular measure along some (all) unstable directions.

C. Dimension

A second important indicator of the “complexity” of a spatiotemporal dynamics is the fractal dimension. An upper bound $D_{KY}$ to it is given by the Kaplan-Yorke formula [12]. The existence of a limit Lyapunov spectrum, implies that $D_{KY}$ is proportional to the system size [13], so that it is convenient to introduce the dimension density $d_{KY}$. In the framework of the present paper, it is natural to extend the definition to generic velocities. The dimension density satisfies the integral equation,

$$\int_0^{d_{KY}} \eta(v, n_\eta) dn_\eta = 0 \quad .$$

(38)

As for the entropy density, Eq. (38) can be more easily interpreted with reference to the $(n_\mu, n_\lambda)$ plane. In fact, the curve implicitly defined by the above constraint is the equipotential line $C$

$$\tilde{\Phi}(n_\mu, n_\lambda) = 0 \quad .$$

(39)

The dimension density $d_{KY}(v)$ can, in turn, be determined from Eq. (32) at the intersection point between $C$ and the image of $L$ in the plane $(n_\mu, n_\lambda)$.

At variance with the entropy density, $d_{KY}(v)$ changes with $v$ (see Fig. 3) even if we avoid considering the second positive band. In fact, while $h_\eta$ is an information divided by a space-time area, $d_{KY}(v)$ is a number of degrees of freedom divided by a length, measured orthogonally to the propagation axis. Thus, at least from a dimensional point of view, it is meaningless to compare $d_{KY}(v)$ for different velocities. However, one can reduce temporal to spatial lengths by introducing the scaling factor $c$ and, in turn, ask himself how the dimension changes with $c$. It is easily seen that the scaling dependence on $c$ is expressed by the following relation,
\[ d_{KY}(v, c_1) \sqrt{1 + \left( \frac{v}{c_1} \right)^2} = d_{KY}(v, c_2) \sqrt{1 + \left( \frac{v}{c_2} \right)^2}, \quad (40) \]

The (completely arbitrary) choice of \( c \) reflects in different dependences of \( d_{KY} \) on \( v \). A natural procedure to fix \( c \) is by minimizing the dependence of \( d_{KY} \) on the observation angle. This amounts to choosing the time units in such a way as to make the 2D pattern as isotropic as possible. In homogeneous CMLs, the procedure is so effective that a suitable choice of \( c \) allows removing almost completely the velocity dependence as seen in Fig. 3a, where the results for the natural value \( c = 1 \) are compared with those for \( c = 3 \).

More in general, however, it is not possible to achieve such a complete success. This is, for instance, the case of the logistic CML, where the dimension drop for \( c = 1 \) is too large to be compensated by any choice of \( c \) (see Fig. 3b, where the curve for \( c = 1 \) is compared with the best results obtained for \( c = +\infty \)).

A further indicator which is sometimes useful in characterizing the chaoticity of a given extended system is the dimension density \( d_u \) of the unstable manifold. This dimension is nothing but \( n_\eta \) in the point where \( \eta = 0 \), i.e. in the origin, and its expression simply reads as

\[ d_u = \beta n_\lambda(0, 0) \quad . \quad (41) \]

being \( n_\mu(0, 0) \equiv 0 \). The choice \( c = +\infty \) of the scaling factor removes exactly the dependence on the orientation of the reference frame. This choice is equivalent to measuring lengths in the untilted frame.

V. COMOVING EXPONENTS

Another class of indicators, introduced to describe convective instabilities in open-flow systems, consists of the so-called comoving or velocity-dependent Lyapunov exponents [3]. They quantify the growth rate of a localized disturbance in a reference frame moving with constant velocity \( V \). Given an initial perturbation \( u(x, 0) \) which is different from zero only within the spatial interval \([-L_0/2, L_0/2]\], numerical analyses indicate

\[ u(x, t) \sim \exp \left( \Lambda(x/t) t \right) , \quad (42) \]

for \( t \) sufficiently large. Eq. (42) defines the comoving Lyapunov exponent \( \Lambda \) as a function of \( V = x/t \). The initial width \( L_0 \) of the disturbance is not a relevant parameter, since a generic perturbation grows with the maximum rate.\(^2\)

The definition of \( \Lambda \) can be extended to a whole spectrum of comoving exponents by looking not just at the local amplitude of the perturbation but also at its shape [17]. Since the physical meaning of the rest of the spectrum is still questionable, in the following we limit ourselves to discuss the maximum.

\(^2\)In the particular case of a \( \delta \)-like initial profile, the definition of local Lyapunov exponent introduced in Ref. [16] is recovered.
As a matter of fact, the limit $t \to \infty$ (required by a meaningful definition of an asymptotic rate) implies the infinite-size limit. Therefore, one must carefully keep under control the system size, when longer times are considered. This is perhaps the most severe limitation against an accurate direct measurement of $\Lambda$.

It can be easily shown that $\Lambda(V)$ is connected with the maximal temporal Lyapunov exponent $\lambda_{\text{max}}(\mu)$ by a Legendre-type transformation [14]. Eq. (42) implies that the perturbation has a locally exponential profile with a rate

$$\mu = \frac{d\Lambda(V)}{dV}$$

in the point $x = Vt$. On the other hand, we know that such a profile evolves as

$$u(Vt, t) \sim \exp[(\lambda_{\text{max}}(\mu) + \mu V)t]$$

(44)

By combining Eqs. (42) and (44), we obtain

$$\Lambda(V) = \lambda_{\text{max}}(\mu) + \mu\frac{d\lambda_{\text{max}}(\mu)}{d\mu}$$

(45)

which, together with Eq. (43) can be interpreted as a Legendre transform from the pair $(\Lambda, V)$ to the pair $(\lambda_{\text{max}}, \mu)$. The inverse transform reveals the further constraint

$$V = \frac{d\lambda_{\text{max}}(\mu)}{d\mu}$$

(46)

Eq. (43) states that $\Lambda(V)$ is the growth rate of an exponentially localized perturbation with a given $\mu$ value as determined from the condition Eq. (43). However, the perturbation itself propagates with yet another velocity, $\tilde{V} = \lambda_{\text{max}}(\mu)/\mu$. As a matter of fact, $\tilde{V}$ and $V$ correspond to phase and group velocities for propagating waves in linear dispersive media. In particular, the “phase” velocity $\tilde{V}$ can be larger than the “light” velocity ($c = 1$ in CML with nearest neighbour coupling), while $V$ is bounded to be smaller.

A simple geometrical interpretation of the above Legendre transformations can be given with reference to the $(\mu, \lambda)$ plane. The comoving Lyapunov exponent $\Lambda(V)$ is the distance between the origin and the intersection of the $\lambda$ axis with the straight line of slope $V$, tangent to the upper temporal border. If the system is chaotic, such an intersection remains positive for $V \leq V_*$. Therefore, as already remarked, $V_*$ is the maximum velocity of disturbance propagation. Indeed, along the worldlines with $V > V_*$, the disturbance does not vanish exactly, but decreases exponentially in time.

Whenever a Legendre transform comes into play, some attention must be payed to the concavity of the functions involved in the transformation. In the present context, this is the case of frozen random patterns where the border of the allowed region exhibits a change of concavity at $\mu = \mu_1$ (see Fig. 6 of LPT). This implies that for $\mu < \mu_1$, the maximal temporal exponent is constant and equal to $\lambda_{\text{max}}(\mu_1)$. The corresponding “phase transition” reflects itself as a linear dependence of the comoving Lyapunov exponent on the velocity for $|V| < V_1$,

$$\Lambda(V) = \lambda_{\text{max}}(\mu_1) - \mu_1 V$$

(47)
This is evident in Fig. 7, where the whole set of Λ values is reported.

It is clear that comoving, spatiotemporal and temporal exponents are related to one another. However, the link is not so straightforward as one might think. Indeed, the velocity $v$ of the rotated frame where $\eta_{\text{max}}$ coincides (up to a normalization factor) with $\lambda_{\text{max}}$ is equal to $c^2/\tilde{V}$ and thus differs from both $\tilde{V}$ and $V$.

VI. CONCLUSIONS

In the present paper we have shown that all instability properties of 1D chaotic systems can be derived from a suitable entropy potential expressed as a function of any pair of variables in the set $\{\mu, \lambda, n_\mu, n_\lambda\}$. The most appropriate representation depends on the problem under investigation. For instance, the properties of Kolmogorov-Sinai entropy are more naturally described with reference to $(n_\mu, n_\lambda)$. This is analogous to standard thermodynamics, where several potentials (Gibbs, Helmholtz, etc.) are introduced to cope with different physical conditions.

The very notion of entropy potential implies general relations among the classes of Lyapunov exponents introduced and discussed here and in LPT, namely spatial, temporal, spatiotemporal and comoving exponents. Another remarkable consequence of the existence of an entropy potential is the independency of the Kolmogorov-Sinai entropy density $h_{KS}$ (as determined from the the spatiotemporal spectrum) of the propagation direction in the space-time plane. Accordingly, $h_{KS}$ can be considered as a super-invariant dynamical indicator. This is not the case of the fractal dimension, the dependence of which provides information about the anisotropy of the pattern.

We should, however, point out that our statements are not rigorously proved (except for some simple test models). However, since our numerical simulations suggest their general validity, we strongly believe that systematic analytical investigations should eventually succeed in proving their validity. A final remark concerns the space dimensionality. The existence of the entropy potential stems from the analyticity of the complex dispersion relations which, in turn, is peculiar of 1D systems.

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APPENDIX A:

The crucial point in justifying the existence of the entropy potential is the analytic structure of the eigenvalue equation stemming from the linearized dynamics. To support the generality of this statement we consider in this Appendix two more examples, namely
the linear stability analysis of homogeneous solutions both of the 1D wave equation and of CML.

The wave equation

\[ \partial_t^2 u = -m^2 u + \partial_x^2 u, \quad (A1) \]

is the conservative analogous of Eq. (4) \((m)\) is a real parameter) and can be treated in a similar way, obtaining

\[ \tilde{\lambda}^2 = \tilde{\mu}^2 - m^2. \quad (A2) \]

The above expression justifies per se the existence of the entropy potential. Incidentally, notice that the Hamiltonian nature of Eq. (A1) implies the degeneracy of the standard TLS in zero, since the uniform solution is an elliptic fixed point. The entropy potential is determined as the real or, equivalently, the imaginary part of the formal integral

\[ \tilde{\Psi}(\tilde{\mu}) = \int \tilde{\lambda} d\tilde{\mu} = \frac{1}{2} \left[ \tilde{\mu}^2 - m^2 \cosh^{-1}(\frac{\tilde{\mu}}{m}) \right]. \quad (A3) \]

This can be verified in the limit of a “weak” instability \(m \to 0\), when Eq. (A3) approximately reads as

\[ \tilde{\Psi}(\tilde{\mu}) \approx \frac{1}{2} \left[ \tilde{\mu}^2 - m^2 \log \left( \frac{\tilde{\mu}}{m} \right) \right]. \quad (A4) \]

By also expanding to the lowest order in \(m\) the expressions of \(\lambda\) and \(\omega\) determined by Eq. (A2), we obtain

\[ \lambda(\mu, k) \approx |\mu| \left(1 - \frac{1}{2} \frac{m^2}{\mu^2 + k^2} \right) \quad (A5) \]

\[ \omega(\mu, k) \approx k \left(1 + \frac{1}{2} \frac{m^2}{\mu^2 + k^2} \right). \]

It is straightforward to verify that

\[ \partial_{\tilde{\mu}} \Re \tilde{\Psi} = -\partial_{\mu} \Im \tilde{\Psi} = \omega \]

\[ \partial_{\mu} \Re \tilde{\Psi} = \partial_k \Im \tilde{\Psi} = \lambda. \quad (A6) \]

For homogeneous solutions of CML models, we obtain

\[ e^{\lambda} = r \left[(1 - \varepsilon) + \varepsilon \cosh \tilde{\mu} \right], \quad (A7) \]

where \(r\) is the multiplier. Unfortunately, in this case it is not possible to write down an explicit expression for the integral \(\tilde{\Psi}\) for generic parameter values. We limit ourselves to discuss the problem in the limit of a small coupling, i.e. \(\varepsilon \to 0\). Expansion of (A7) to the first order in \(\varepsilon\), yields

\[ \tilde{\Psi}(\tilde{\mu}) \approx (\log r - \varepsilon)\tilde{\mu} + \varepsilon \sinh \tilde{\mu}, \quad (A8) \]

and

\[ \lambda(\mu, n_\lambda) \approx \log r - \varepsilon (1 - \cos k \cosh \mu) \]

\[ n_\mu(\mu, n_\lambda) \approx \varepsilon \sin k \sinh \mu, \quad (A9) \]

which should be compared with the corresponding expressions obtained by expanding to first order in \(\varepsilon\) Eqs. (16) and (20) of LPT. Moreover, one can verify that the relations analogous to Eqs. (A6) hold also in the present example.
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FIGURES

FIG. 1. Contour plot of the entropy potential $\Phi$ for a homogeneous chain ($r = 2, \varepsilon = 1/3$).

FIG. 2. Lattice implementation of the definition of spatiotemporal Lyapunov exponents for $v = 1/2$.

FIG. 3. Plot of the boundary $\partial D$ and of the line $\lambda = v\mu$ in the three velocity regimes for the logistic CML $\varepsilon = 1/3$. The three lines refer to the different cases $v < 1$ (solid), $1 < v < 1/V_*$ (dot-dashed) and $v = 1/V_*$ (dashed).

FIG. 4. Boundaries of the STLS versus the tilting angle $\vartheta$ for the homogeneous chain ($r = 2, \varepsilon = 1/3$).

FIG. 5. Comparison between the STLS obtained by direct numerical computation and formula (32) for (a) homogeneous ($r = 2$) with $v = 4/5$ and (b) logistic CML with $v = 3/5$ (in both cases $\varepsilon = 1/3$).

FIG. 6. Kaplan-Yorke dimension density $d_{KY}$ obtained from the STLS versus the tilting angle $\vartheta$ for (a) homogeneous $(r = 1.2)$ and (b) logistic CML models: in both cases $\varepsilon = 1/3$. Circles refer to the scaling factor $c = 1$, while crosses correspond to $c = 3, +\infty$ in (a), (b), respectively.

FIG. 7. Maximum comoving Lyapunov exponent $\Lambda(V)$ for a frozen random pattern obtained as Legendre transform versus $V$ (for comparison see also Fig. 6 in LPT). The vertical line indicates the position of the critical velocity $V_1$ (see the text for definition).
TABLE I. Entropy potential $\Phi$ computed by integrating along two different paths in two different points of the $(\mu, \lambda)$ plane. The difference is definitely smaller than the statistical error ($\approx 10^{-3}$).

| Path      | Integral | Path      | Integral | $\Phi$  |
|-----------|----------|-----------|----------|---------|
| $(0, 0) \rightarrow (0, 3)$ | 0.1011   | $(0, 3) \rightarrow (2, 3)$ | -0.3883  | -0.2872 |
| $(0, 0) \rightarrow (2, 0)$ | -0.5274  | $(2, 0) \rightarrow (2, 3)$ | 0.2406   | -0.2868 |
| $(0, 0) \rightarrow (0, 0.8)$ | 0.1069   | $(0, 0.8) \rightarrow (3, 0.8)$ | -0.8036  | -0.6967 |
| $(0, 0) \rightarrow (3, 0)$ | -1.4972  | $(3, 0) \rightarrow (3, 0.8)$ | 0.8      | -0.6972 |