VARIATIONAL BIHAMILTONIAN COHOMOLOGIES AND INTEGRABLE HIERARCHIES I: FOUNDATIONS

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ABSTRACT. This series of papers is devoted to the study of deformations of Virasoro symmetries of the principal hierarchies associated to semi simple Frobenius manifolds. The main tool we use is a generalization of the bihamiltonian cohomology called the variational bihamiltonian cohomology. In the present paper, we give its definitions and compute the associated cohomology groups that will be used in our study of deformations of Virasoro symmetries. To illustrate its application, we classify the conformal bihamiltonian structures with semisimple hydrodynamic limits.

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1. Introduction

Since Dubrovin introduced the notion of Frobenius manifold in the beginning of 90’s of the last century inspired by the development of 2d topological field theory...
its relationship with hierarchies of integrable PDEs has become an
important research subject in the theory of integrable systems and its applications in the
study of Gromov-Witten invariants, singularity theory and quantum field theory, see some
of the related works \[1\] \[3\] \[4\] \[15\] \[16\] \[17\] \[18\] \[19\] \[21\] \[22\] \[23\] \[24\] \[28\] \[35\] \[38\] and references
therein. In \[15\], Dubrovin and Zhang, the third author of the present paper, proposed a
project to classify hierarchies of integrable PDEs which possess hydrodynamic limits and
satisfy the following properties: each integrable hierarchy has a bihamiltonian structure, a
tau function (also called a tau structure), and an infinite set of Virasoro symmetries that
act linearly on the tau function. The bihamiltonian structure of the integrable hierarchy
is assumed to be represented as a formal power series w.r.t. the dispersion parameter $\varepsilon$,
its coefficients are given by homogeneous differential polynomials of the dependent vari-
ables of the integrable hierarchy, and its dispersionless limit is a bihamiltonian structure
of hydrodynamic type. The existence of a bihamiltonian structure and of a tau function
implies that the dispersionless limit of the integrable hierarchy can be described by a
Frobenius manifold structure or a degenerate one. On the other hand, starting from
any Frobenius manifold, one can construct an integrable hierarchy of hydrodynamic type
which is called the Principal Hierarchy of the Frobenius manifold. The flat metric and
the intersection form of the Frobenius manifold yield a bihamiltonian structure of the
Principal Hierarchy which also possesses a tau function and an infinite set of Virasoro
symmetries. The Virasoro symmetries, apart from the first two, of the Principal Hierarchy
do not act linearly on the tau function. Under the semisimplicity condition of the
Frobenius manifold, the requirement of linear actions of the Virasoro symmetries on the
tau function leads to a dispersionful deformation, called the topological deformation, of
the Principal hierarchy. As it is shown in \[15\], the topological deformation of the Prin-
cipal Hierarchy is constructed via a quasi-Miura transformation determined by the loop
equation of the semisimple Frobenius manifold.

To classify dispersionful deformations of the Principal Hierarchy, Dubrovin and Zhang
introduced in \[15\] the notion of bihamiltonian cohomology $BH^k(M; P_0, P_1)$ for a bihamil-
tonian structure $(P_0, P_1)$ of hydrodynamic type defined on the jet space of a smooth
manifold $M$. The cohomology groups $BH^2(M; P_0, P_1)$ and $BH^3(M; P_0, P_1)$ character-
ize the equivalence classes of infinitesimal deformations of $(P_0, P_1)$ under Miura type
transformations and the obstruction of extending an infinitesimal deformations to a full
deformation respectively. Here we emphasize that the deformed bihamiltonian structures
are required to have the property that the coefficients of the powers of the deformation
parameter are given by homogeneous differential polynomials of the dependent variables,
this property is called the \textit{polynomiality} of the deformed Hamiltonian structures.

In \[34\] Lorenzoni studied the deformations of the bihamiltonian structure of the dis-

cersionless KdV hierarchy and showed, up to the fourth order approximation of the de-

formation parameter, that they are parametrized by a function of one variable. The very
first computation of the bihamiltonian cohomology groups for a semisimple bihamiltonian
structure of hydrodynamic type is given in \[10\] \[31\], from which the cohomology group
$BH^2_2(M; P_0, P_1)$ can be obtained, and it shows that the equivalence classes of the in-

finitesimal deformations of $(P_0, P_1)$ with $n$ dependent variables are parametrized by a set
of smooth functions $c_1(u_1), \ldots, c_n(u_n)$, where $u_1, \ldots, u_n$ are the canonical coordinates of
the semisimple bihamiltonian structure of hydrodynamic type. These functions are called the central invariants of the deformed bihamiltonian structure. It is also conjectured that the cohomology group $BH^3(M; P_0, P_1)$ is trivial, and that any infinitesimal deformation of $(P_0, P_1)$ can be extended to a full deformation.

An important observation toward computing the cohomology group $BH^3(M; P_1, P_2)$ was made by the first and third author of the present paper in [33], where they showed that one can work in the space of differential polynomials instead of the space of the local functionals to compute the bihamiltonian cohomology. By using this idea they proved the aforementioned conjecture for the bihamiltonian structure of hydrodynamic type associated to the one dimensional Frobenius manifold (or the dispersionless KdV hierarchy). Base on the method of [33] and on a clever utilizing of the technique of spectral sequences and some other tools from homological algebra, Carlet, Posthuma and Shadrin proved this conjecture for a general semisimple bihamiltonian structure of hydrodynamic type in [5].

A deformation of the bihamiltonian structure of the Principal hierarchy of a semisimple Frobenius manifold yields a deformation of the integrable hierarchy. It is shown in [11] that the deformed integrable hierarchy possesses a tau structure when the central invariants of the deformed bihamiltonian structure are constant functions. However, it remains to be unclear whether this deformed integrable hierarchy also possesses an infinite set of Virasoro symmetries. In particular, it is conjectured that the deformation of the Principal Hierarchy with all the central invariants being equal to $1/24$ possesses an infinite set of Virasoro symmetries which act linearly on the tau function of the deformed integrable hierarchy, or in other words, it coincides with the topological deformation of the Principal Hierarchy. An approach to prove the validity of this conjecture is as follows: one needs to show that the quasi-Miura transformation, which transforms the Principal Hierarchy to its topological deformation, also transforms the bihamiltonian structure of hydrodynamic type to a deformed bihamiltonian structure which satisfies the polynomiality property. In [2, 3] Buryak, Posthuma and Shadrin proved the polynomiality property of the first deformed Hamiltonian structure of the Principal Hierarchy, and in the recent preprint [25] Iglesias and Shadrin provide a certain evidence of the validity of the polynomiality property of the second Hamiltonian structure.

The main purpose of the present paper and of its subsequents is to show that the deformation of the Principal Hierarchy of a semisimple Frobenius manifold which is associated to a deformed bihamiltonian structure with constant central invariants possesses an infinite set of Virasoro symmetries, and these Virasoro symmetries can be lifted to the actions on the tau function of the deformed integrable hierarchy; moreover, for the case when all the central invariants equal to $1/24$, the Virasoro symmetries act linearly on the tau function, and so this deformed integrable hierarchy is equivalent to the topological deformation of the Principal Hierarchy. In particular, this implies that the topological deformation of the Principal Hierarchy of a semisimple Frobenius manifold possesses a bihamiltonian structure which satisfies the polynomiality property. To this end, we first generalize in the present paper the notion of bihamiltonian cohomology by considering a certain cohomology, called the variational bihamiltonian cohomology, on the space of variational 1-forms of the infinite jet space of the dependent variables, and apply it to
the study of Virasoro symmetries of the deformed integrable hierarchy in the subsequent papers.

The paper is organized as follows. In Sect. 2, we recall the basic notions of bihamiltonian structures in terms of local functionals on the infinite jet bundle of a super manifold, and we present the main results of this paper. In Sect. 3, we first define the variational Hamiltonian cohomology and prove the triviality of the cohomology groups, and then we define the variational bihamiltonian cohomology. Sect. 4 and Sect. 5 are both devoted to the computation of the variational bihamiltonian cohomology groups. As an illustration of the application of this theory, we classify the conformal bihamiltonian structures and their deformations in Sect. 6. In Sect. 7 we give some concluding remarks.

2. Basic notions and Main results

In this section, we first recall some basic definitions and constructions on bihamiltonian structures in terms of the language of super variables. The readers may refer to [32, 33] and the lecture note [27] for a detailed introduction to this topic.

Let $M$ be an $n$-dimensional smooth manifold, and $\hat{M} = \Pi(T^* M)$ be the super manifold of dimension $(n | n)$ obtained by reversing the parity of the fiber of the cotangent bundle of $M$. Locally, one may choose a coordinate system $(u^1, \ldots, u^n)$ for an open neighborhood $U \subset M$ and we will use $(\theta_1, \ldots, \theta_n)$ to denote the coordinate system of the fibers. The variables $\theta_i$ are fermion, meaning that they satisfy the anti-commutation relations

$$\theta_i \theta_j + \theta_j \theta_i = 0.$$ 

Let $J_\infty(\hat{M})$ be the infinite jet bundle of $\hat{M}$ and let $\hat{A}$ be the ring of differential polynomials. In local coordinates, we can write it as

$$\hat{A} = C^\infty(U[[u^{i,s+1}, \theta_i^s | i = 1, \ldots, n; s \geq 0]].$$

Here and henceforth we denote $u^{i,0} = u^i$ and $\theta_i^0 = \theta_i$. The jet variables $u^{i,s}, \theta_i^s$ for $s \geq 1$ may be regarded as $s$-th order derivatives with respect to an independent variable, which we usually denote as $x$, when regarding $u^i$ and $\theta_i$ as dependent variables. This identification is possible due to the existence of the global vector field

$$\partial_x = \sum_{s \geq 0} u^{i,s+1} \frac{\partial}{\partial u^{i,s}} + \theta_i^{s+1} \frac{\partial}{\partial \theta_i^s}. \quad (2.1)$$

Sometimes we will use a prime to denote $\partial_x$, i.e. $f' := \partial_x f$. The elements in the quotient space $\mathcal{F} = \hat{A}/\partial_x$ are called local functionals and for an element $f \in \hat{A}$, we will use $\int f$ to denote its class in $\mathcal{F}$. Let us define two degrees $\deg_x$ and $\deg_\theta$ on the space $\hat{A}$, which are called respectively the differential gradation and the super gradation. On the generators of $\hat{A}$, these degrees are defined by

$$\deg_x u^{i,s} = \deg_x \theta_i^s = s; \quad \deg_\theta u^{i,s} = 0, \quad \deg_\theta \theta_i^s = 1.$$ 

We denote the space of homogeneous elements by

$$\hat{A}_d = \{ f \in \hat{A} | \deg_x f = d \}, \quad \hat{A}^p = \{ f \in \hat{A} | \deg_\theta f = p \}, \quad \hat{A}_d^p = \hat{A}_d \cap \hat{A}_d^p.$$

It is clear that the vector field $\partial_x$ is homogeneous with respect to both degrees, hence the gradations on $\hat{A}$ induce a gradation on $\mathcal{F}$ and we will denote the space of homogeneous elements by $\mathcal{F}_d, \mathcal{F}^p$ and $\mathcal{F}_d^p$ accordingly.
A nontrivial construction called the Schouten-Nijenhuis bracket equips the space $\hat{F}$ with a graded Lie algebra structure. Define the variational derivative of an element in $\hat{A}$ by

$$
\frac{\delta f}{\delta u^i} = \sum_{s \geq 0} (-\partial_x)^s \frac{\partial f}{\partial u^{i,s}}, \quad \frac{\delta f}{\delta \theta^i} = \sum_{s \geq 0} (-\partial_x)^s \frac{\partial f}{\partial \theta^{i,s}}.
$$

It can be shown that the variational derivative annihilates the image of $\partial_x$ so we can define the variational derivative of a local functional as follows:

$$
\frac{\delta F}{\delta u^i} = \int \frac{\delta f}{\delta u^i}, \quad \frac{\delta F}{\delta \theta^i} = \int \frac{\delta f}{\delta \theta^i}, \quad F = \int f \in \hat{F}.
$$

Then the Schouten-Nijenhuis bracket is defined to be the following bilinear map

$$
[\cdot, \cdot] : \hat{F}^p \times \hat{F}^q \to \hat{F}^{p+q-1}, \quad [P, Q] = \int \sum_i \frac{\delta P}{\delta \theta^i} \frac{\delta Q}{\delta u^i} + (-1)^p \frac{\delta P}{\delta u^i} \frac{\delta Q}{\delta \theta^i}.
$$

This bracket defines a graded Lie algebra structure on $\hat{F}$, whose sign convention is different from the usual definition, for more details one may refer to [32].

For a local functional $P \in \hat{F}^p$, we can associate a derivation on $\hat{A}$ defined by

$$
D_P = \sum_i \sum_{s \geq 0} \partial_x^s \left( \frac{\delta P}{\delta \theta^i} \right) \frac{\partial}{\partial u^{i,s}} + (-1)^p \partial_x^p \left( \frac{\delta P}{\delta u^i} \right) \frac{\partial}{\partial \theta^i}.
$$

We see that $D_P$ is a derivation of super degree $p - 1$ and that $[D_P, \partial_x] = 0$. The following property is important for the construction of the cohomologies that will be given later in this and the next section:

$$
(-1)^{p-1} D_{[P, Q]} = D_P \circ D_Q - (-1)^{(p-1)(q-1)} D_Q \circ D_P; \quad P \in \hat{F}^p, \quad Q \in \hat{F}^q.
$$

This relation shows that the map $D$ induces a (graded) Lie algebra homomorphism

$$
\hat{F} \to \text{Der}(\hat{A}) : \quad P \mapsto D_P,
$$

where the natural graded Lie algebra structure on the space of derivations on $\hat{A}$, denoted by $\text{Der}(\hat{A})$, is given by the graded commutators. In what follows, we will also call an element of $\text{Der}(\hat{A})$ a vector field on $J^\infty(M)$.

Using the above notions, a Hamiltonian structure is defined to be a local functional $P \in \hat{F}^2$ satisfying $[P, P] = 0$, and it is called of hydrodynamic type if $P \in \hat{F}_1^2$. A bihamiltonian structure is a pair of Hamiltonian structure $(P_0, P_1)$ satisfying an additional compatibility condition $[P_0, P_1] = 0$. Now assume that we have a bihamiltonian structure $(P_0, P_1)$ of hydrodynamic type [27], locally we represent them as follows:

$$
P_a = \frac{1}{2} \int \sum g_{ij}^a(u) \theta_i \theta_j + \Gamma_{a,k}^{ij}(u) u^{k,1} \theta_i \theta_j, \quad a = 1, 2.
$$

This bihamiltonian structure is called semisimple if the roots $\lambda^1(u), \ldots, \lambda^n(u)$ of the characteristic equation $\det(g_{ij}^a - \lambda q_{ij}^a) = 0$ are distinct and not constant.

It is proved in [20] that if $(P_0, P_1)$ is semisimple, then the roots $\lambda^1(u), \ldots, \lambda^n(u)$ can serve as local coordinates of the manifold $M$ and in such a coordinate system $(P_0, P_1)$
can be represented in the following forms:

\[ P_0 = \frac{1}{2} \int \sum f^i(\lambda) \theta_i \theta^1_i + A^{ij} \theta_i \theta_j, \quad P_1 = \frac{1}{2} \int \sum g^i(\lambda) \theta_i \theta^1_i + B^{ij} \theta_i \theta_j, \]

where \( f^i \) are non-vanishing functions, \( g^i = \lambda^i f^i \) and the functions \( A^{ij} \) and \( B^{ij} \) are given by

\[ A^{ij} = \frac{1}{2} \left( \frac{f^i \partial f^j}{f^j} \lambda^{j,1} - \frac{f^j \partial f^i}{f^j} \lambda^{i,1} \right), \quad B^{ij} = \frac{1}{2} \left( \frac{g^i \partial f^j}{f^i} \lambda^{j,1} - \frac{g^j \partial f^i}{f^i} \lambda^{i,1} \right). \]

The coordinates \( \lambda^1, \cdots, \lambda^n \) are called the canonical coordinates of \((P_0, P_1)\).

In what follows, when we consider semisimple bihamiltonian structures of hydrodynamic type, it is always assumed that we choose the canonical coordinates as local coordinates. We will use \( u^1, \cdots, u^n \) instead of \( \lambda^1, \cdots, \lambda^n \) to denote them. In these coordinates, we represent a bihamiltonian structure \((P_0, P_1)\) of hydrodynamic type as

\[ (2.4) \quad P_0 = \frac{1}{2} \int \sum f^i(u) \theta_i \theta^1_i + A^{ij} \theta_i \theta_j, \quad P_1 = \frac{1}{2} \int \sum u^i f^i(u) \theta_i \theta^1_i + B^{ij} \theta_i \theta_j. \]

Given a hydrodynamic bihamiltonian structure \((P_0, P_1)\), we construct the variational bihamiltonian cohomology as follows. Consider the space of 1-forms \( \Omega \) of the infinite jet space \( J^\infty(\hat{M}) \), locally it is an \( \hat{A} \)-module generated by \( \delta u_{i,s} \) and \( \delta \theta_{s}^i \) for \( i = 1, \cdots, n \) and \( s \geq 0 \):

\[ \Omega = \left\{ \sum_{i,s \geq 0} g_{i,s} \delta u_{i,s} + h_{s}^i \delta \theta_{s}^i \mid g_{i,s}, h_{s}^i \in \hat{A} \right\}. \]

Each derivation of \( \hat{A} \) induces an action on \( \Omega \) by the Lie derivative (see Sect. 3 for details). In particular we consider the action of Lie derivative of \( \partial_x \) on \( \Omega \), which we still denote by \( \partial_x \), and we denote by \( \Omega \) the quotient space \( \Omega/\partial_x \Omega \). We can verify that the actions on \( \Omega \) given by the Lie derivatives of \( D_{P_0} \) and \( D_{P_1} \), which we denote by \( \tilde{D}_0 \) and \( \tilde{D}_1 \) respectively, equip \( \Omega \) a structure of double complex. We grade the space \( \Omega \) similarly as we do for the space \( \hat{F} \) by setting

\[ \deg_\infty \delta u_{i,s} = \deg_\infty \delta \theta_{s}^i = s; \quad \deg_0 \delta u_{i,s} = 0, \quad \deg_0 \delta \theta_{s}^i = 1. \]

Then we define the variational bihamiltonian cohomology groups as follows:

\[ \text{VBH}^p_d(\Omega, \tilde{D}_0, \tilde{D}_1) = \frac{\Omega^p_d \cap \ker \tilde{D}_0 \cap \ker \tilde{D}_1}{\Omega^p_d \cap \text{Im} \tilde{D}_0 \tilde{D}_1}. \]

We prove the following theorem in Sect. 4 and Sect. 5.

**Theorem 2.1.** The variational bihamiltonian cohomology groups for a semisimple bihamiltonian structure \((P_0, P_1)\) of hydrodynamic type have the following properties:

i) \( \text{VBH}^2_d(\Omega, \tilde{D}_0, \tilde{D}_1) \cong \oplus_{i=1}^n C^\infty(\mathbb{R}) \).
ii) If \( d \geq 2 \) and if both \((p,d)\) and \((p+1,d)\) are NOT in the index set \( I \), then \( \text{VBH}^p_d(\bar{\Omega}, \bar{D}_0, \bar{D}_1) = 0 \), where the index set is defined by \( I = I_1 \cup I_2 \cup I_3 \) with:

\[
\begin{align*}
I_1 &= \{(i,j) \mid j = 0, 1; \ i = j + 1, \cdots, j + n + 1 \}; \\
I_2 &= \{(i,j) \mid j = 2, \cdots, n; \ i = j, \cdots, j + n + 1 \}; \\
I_3 &= \{(i,j) \mid j = n + 1, n + 2, n + 3; \ i = j, \cdots, j + n \}.
\end{align*}
\]

iii) \( \text{VBH}^1_d(\bar{\Omega}, \bar{D}_0, \bar{D}_1) = 0 \).

To illustrate the application of the variational bihamiltonian cohomology, in Sect. [4] we study the conformal property of bihamiltonian structures which is shared by the bihamiltonian structures associated with Frobenius manifolds. Let us introduce the notations:

\[
\text{Der}(\bar{\mathcal{A}})^{p} := \{ X \in \text{Der}(\bar{\mathcal{A}}) \mid X(\bar{\mathcal{A}}^k) \subset \bar{\mathcal{A}}^{k+p} \},
\]

\[
\text{Der}(\bar{\mathcal{A}})^{p}_{d} := \{ X \in \text{Der}(\bar{\mathcal{A}}) \mid X(\bar{\mathcal{A}}^k) \subset \bar{\mathcal{A}}^{k+d} \}.
\]

Then the conformal property of a bihamiltonian structure can be described as the follows.

**Definition 2.1.** A bihamiltonian structure \((P_0, P_1)\) is called conformal if there exists a nonzero derivation \( E \in \text{Der}(\bar{\mathcal{A}})^0 \) and real numbers \( \mu, \lambda_0, \lambda_1 \) such that:

\[
[E, \partial_\lambda] = \mu \partial_\lambda, \quad [E, D_{P_a}] = \lambda_a D_{P_a}, \quad a = 0, 1.
\]

**Example 2.1.** The bihamiltonian structure of the dispersionless KdV hierarchy is given by

\[
P_0 = \frac{1}{2} \int \theta^1; \quad P_1 = \frac{1}{2} \int u \theta^1.
\]

One can check that it is a conformal bihamiltonian structure with \( E \) given by

\[
E = \sum_{s \geq 0} (\lambda_1 - \lambda_0 + s \mu) u^{(s)} \frac{\partial}{\partial u^{(s)}} + (\lambda_1 + (s-1) \mu) \theta^s \frac{\partial}{\partial \theta^s}.
\]

**Theorem 2.2.** A semisimple bihamiltonian structure \((P_0, P_1)\) of hydrodynamic type is conformal if and only if there exist real numbers \( d^1, \cdots, d^n \) such that the functions \( f^1, \cdots, f^n \) given in [2.4] satisfy the following identities:

\[
\sum_{i} w^i \frac{\partial f^i}{\partial u^j} = d^i f^j, \quad \forall i;
\]

\[
(d^i - d^j) \frac{\partial f^i}{\partial u^j} = 0, \quad \forall i, j.
\]

In such a case, if we require that the derivation \( E \) has differential degree 0, then \( E \) is an Euler type vector field given by

\[
E = \sum_{i,s \geq 0} (\lambda_1 - \lambda_0 + s \mu) u^{i,s} \frac{\partial}{\partial u^{i,s}} + (\lambda_1 - (s-1) \mu) \theta^s \frac{\partial}{\partial \theta^s}.
\]

Given a semisimple and conformal bihamiltonian structure of hydrodynamic type, we consider whether its deformations are still conformal. It turns out that there is only a certain family of deformations that preserve the conformal property, and the equivalence
classes of these deformations under Miura type transformations are parametrized by $n$ constants, as it is shown by the following theorem.

**Theorem 2.3.** Let $(P_0, P_1)$ be a semisimple and conformal bihamiltonian structure of hydrodynamic type, then its deformation $(\tilde{P}_0, \tilde{P}_1)$ is conformal if and only if the central invariants are given by:

$$c_i(u^i) = C_i(u^i)^{m_i}, \quad m_i = \frac{\lambda_1 - \lambda_0 - 2\mu - (\lambda_1 - \lambda_0)\delta^i}{\lambda_1 - \lambda_0},$$

where $C_i$ are arbitrary constants.

3. **Variational Bihamiltonian Cohomology**

In this section, we start with the definition of variational Hamiltonian cohomology for a single Hamiltonian structure of hydrodynamic type and prove its triviality. Then we define the variational bihamiltonian cohomology and make some necessary algebraic preparations for computing the cohomology groups.

3.1. **Variational forms on the infinite jet bundle.** We recall the constructions of the variational forms on the infinite jet bundle $J^\infty(\hat{M})$ (cf. [15] and reference therein). Let $\mathcal{E}$ be the space of differential forms on $J^\infty(\hat{M})$, locally is just the space

$$\mathcal{E} = \bigwedge^* (\text{Span}_\hat{A}\{\delta\theta^i, \delta u^{i,s} \mid i = 1, \cdots, n, s \geq 0\}).$$

Here we adopt Deligne’s sign rule for these variational forms, for example

$$\theta_i^s \delta\theta_j^t = -\delta\theta_i^s \theta_j^t, \quad \theta_i^s \delta\theta_j^t = \delta\theta_i^s \theta_j^t, \quad \delta u^{i,s} \delta u^{j,t} = -\delta u^{i,s} \delta u^{j,t} = -\delta u^{i,s} \delta u^{j,t}.$$

Any vector field $X \in \text{Der}(\hat{A})$ induces an action on $\mathcal{E}$ by the Lie derivative, which is defined by the Cartan formula

$$L_X = \delta \circ \iota_X + \iota_X \circ \delta,$$

where the de Rham differential is given by

$$\delta = \sum_{i,s \geq 0} \delta u^{i,s} \frac{\partial}{\partial u^{i,s}} + \delta \theta_i^s \frac{\partial}{\partial \theta_i^s}.$$ 

In particular, the vector field $\partial_x$ defined in (2.1) induces an action on $\mathcal{E}$, which we still denote by $\partial_x$. We define the space of local functionals of forms by

$$\bar{\mathcal{E}} = \mathcal{E} / \partial_x \mathcal{E},$$

and represent its elements in the form $\int f$ with $f \in \mathcal{E}$. In addition, if a vector field $X \in \text{Der}(\hat{A})$ commutes with $\partial_x$, then its Lie derivative $L_X$ also commutes with $\partial_x$ acting on $\mathcal{E}$, hence it induces an action on $\bar{\mathcal{E}}$ which we still denote by $L_X$.

**Example 3.1.** The space of 1-forms is given by

$$\mathcal{E}^1 = \left\{ \sum_{i,s \geq 0} g_{i,s} \delta u^{i,s} + h_i^s \delta\theta^s_i \mid g_{i,s}, h_i^s \in \hat{A} \right\}.$$
Let us grade $E^1$ as we do for $\hat{A}$ as follows:

$$(E^1)_d = \left\{ \sum_{i,s \geq 0} g_{i,s} \delta u^i_s + h^i_s \delta \theta^s_i \mid g_{i,s}, h^i_s \in \hat{A}_{d-s} \right\},$$

$$(E^1)^p = \left\{ \sum_{i,s \geq 0} g_{i,s} \delta u^i_s + h^i_s \delta \theta^s_i \mid g_{i,s} \in \hat{A}^p, h^i_s \in \hat{A}^{p-1} \right\}.$$

For any element $\omega \in \bar{E}^1$, we can uniquely represent it in the form

$$\omega = \int \sum_{i,s \geq 0} g_{i,s} \delta u^i_s + h^i_s \delta \theta^s_i, \quad g_{i,s}, h^i_s \in \hat{A}.$$

So we can identify the space $\bar{E}^1$ with the space of $\hat{A}$-valued differential 1-forms on $\hat{M}$.

**Example 3.2.** Let $X \in \text{Der}(\hat{A})^q$ and $\omega \in (E^1)^p$ be given by

$$\omega = \sum_{i,s \geq 0} g_{i,s} \delta u^i_s + h^i_s \delta \theta^s_i, \quad g_{i,s} \in \hat{A}^p, \quad h^i_s \in \hat{A}^{p-1},$$

then the action of $L_X$ on $\omega$ has the expression

$$L_X \omega = \sum_{i,s \geq 0} X(g_{i,s}) \delta u^i_s + (-1)^p g_{i,s} \delta \left( X(u^i_s) \right) + X(h^i_s) \delta \theta^s_i + (-1)^{(p-1)q} h^i_s \delta \left( X(\theta^s_i) \right).$$

Let us consider the space $\text{Der}(\hat{A})^q$ of derivations on $\hat{A}$ which commute with $\partial_x$. Take an element $X \in \text{Der}(\hat{A})^q$, since $[X, \partial_x] = 0$ we can regard $X$ as an $\hat{A}$-valued vector field on $\hat{M}$ instead of on $J^\infty(\hat{M})$. Note that $\hat{M}$ admits a canonical symplectic structure

$$\varpi = \sum u^i \wedge \delta \theta_i,$$

hence the space of the $\hat{A}$-valued vector fields on $\hat{M}$ is canonically identified with the space of $\hat{A}$-valued differential 1-forms on $\hat{M}$, which is the same as the space $\bar{E}^1$ as we explained in the Example 3.1. Let us write down this identification explicitly. An element $X$ of $\text{Der}(\hat{A})^q$ can be represented as

$$X = \sum_{i,s \geq 0} \partial^s_i \left( X(u^i_s) \right) \frac{\partial}{\partial u^i_s} + \partial^s_i \left( X(\theta_i) \right) \frac{\partial}{\partial \theta_i}.$$ 

Restricting it on $\hat{M}$, we get the following $\hat{A}$-valued vector field on $\hat{M}$ which we still denote by $X$:

$$X = \sum_i X(u^i) \frac{\partial}{\partial u^i} + X(\theta_i) \frac{\partial}{\partial \theta_i}.$$ 

Using the canonical symplectic structure $\varpi$, we identify $X$ with the 1-form

$$W = \iota_X \varpi = \sum_i X(u^i) \delta \theta_i - X(\theta_i) \delta u^i,$$

which corresponds to a unique element $\omega \in \bar{E}^1$ given by

$$\omega = \int \sum_i X(u^i) \delta \theta_i - X(\theta_i) \delta u^i.$$
We note that for an element \( X \in (\text{Der}(\mathcal{A})^\partial)^d \), the corresponding element \( \omega \) belongs to \((\mathcal{E}^1)^{d+1}_d\). Let us denote this correspondence by

\[
\Phi : \text{Der}(\mathcal{A})^\partial \rightarrow \mathcal{E}^1; \quad \Phi : X \mapsto \int_{\Omega} (X|_{\Omega}).
\]

Now let \( P \) be a Hamiltonian structure of hydrodynamic type. By using the identity (3.2), we see that the space \( \text{Der}(\mathcal{A})^\partial \) becomes a cochain complex with the differential given by adjoint action of \( D_P \). The question is that when we identify elements in \( \text{Der}(\mathcal{A})^\partial \) as elements in \( \mathcal{E}^1 \), then how the action of \( D_P \) induces a differential on \( \mathcal{E}^1 \)?

**Lemma 3.1.** Let \( P \) be a Hamiltonian structure of hydrodynamic type, then the following identity holds true for any \( X \in \text{Der}(\mathcal{A})^\partial \):

\[
\Phi ([D_P, X]) = \mathcal{L}_{D_P} \Phi(X).
\]

**Proof.** Since both the definition (3.1) and the identity (3.2) are coordinate free, we can choose a system of local coordinates \( (v^1, \ldots, v^n; \phi_1, \ldots, \phi_n) \) on \( \hat{M} \) such that the Hamiltonian structure \( P \) has the expression

\[
P = \frac{1}{2} \int \eta^{\alpha\beta} \phi_\alpha \phi^1_\beta,
\]

here \( \eta^{\alpha\beta} \) is a constant non-degenerate matrix, and summation over repeated lower and upper Greek indices is assumed. We call such coordinates the flat coordinates of \( P \).

Now we assume that \( X \) is an element of \( \text{Der}(\mathcal{A})^\partial \) with super degree \( p \), which means that \( X(\mathcal{A}^q) \subset \mathcal{A}^{p+q} \), then it is straightforward to show that

\[
[D_P, X]v^\alpha = D_P (X(v^\alpha)) + (-1)^{p+1} \eta^{\alpha\beta} (X(\phi^1_\beta)), \quad [D_P, X]\phi_\alpha = D_P (X(\phi_\alpha)).
\]

Then we arrive at

\[
\Phi ([D_P, X]) = \int (D_P (X(v^\alpha)) + (-1)^{p+1} \eta^{\alpha\beta} (X(\phi^1_\beta))) \, \delta \phi_\alpha - D_P (X(\phi_\alpha)) \, \delta v^\alpha.
\]

On the other hand we have

\[
\mathcal{L}_{D_P} \Phi(X) = \mathcal{L}_{D_P} \int X(v^\alpha) \, \delta \phi_\alpha - X(\phi_\alpha) \, \delta v^\alpha
\]

\[
= \int D_P (X(v^\alpha)) \, \delta \phi_\alpha - D_P (X(\phi_\alpha)) \, \delta v^\alpha - (-1)^{p+1} X(\phi_\alpha) \delta (\eta^{\alpha\beta} \phi^1_\beta)
\]

\[
= \int (D_P (X(v^\alpha)) + (-1)^{p+1} \eta^{\alpha\beta} (X(\phi^1_\beta))) \, \delta \phi_\alpha - D_P (X(\phi_\alpha)) \, \delta v^\alpha.
\]

Therefore we prove the lemma. \( \square \)

3.2. Variational Hamiltonian cohomology and its triviality. From now on we will use \( \Omega \) to denote the space of 1-forms \( \mathcal{E}^1 \) and \( \hat{\Omega} \) to denote its quotient space \( \mathcal{E}^1 \). The space of homogeneous elements with differential degree \( d \) and super degree \( p \) will be denoted by \( \Omega^p_d \) and \( \hat{\Omega}^p_d \) respectively.

Let \( P \) be a Hamiltonian structure of hydrodynamic type, we will use \( \hat{D}_P \) to denote the action \( \mathcal{L}_{D_P} \) on the space \( \Omega \) and \( \hat{\Omega} \). By using the identity (3.2) we conclude that \( \hat{D}_P \circ \hat{D}_P = 0 \), so \( \hat{D}_P \) is a differential on the spaces \( \Omega \) and \( \hat{\Omega} \).
Definition 3.1. The variational Hamiltonian cohomology of $\Omega$ (and of $\bar{\Omega}$, respectively) is defined to be the cohomology of the complex $(\Omega^\bullet, \bar{\Omega}^\bullet, \tilde{D}_P)$ (and of $(\Omega^\bullet, \bar{\Omega}^\bullet, \tilde{D}_P)$, respectively) given by

$$H^p_d(\Omega, \tilde{D}_P) = \frac{\Omega^p_d \cap \ker \tilde{D}_P}{\Omega^p_d \cap \text{Im} \tilde{D}_P}; \quad H^p_d(\bar{\Omega}, \tilde{D}_P) = \frac{\bar{\Omega}^p_d \cap \ker \tilde{D}_P}{\bar{\Omega}^p_d \cap \text{Im} \tilde{D}_P}.$$

By using the fundamental facts of the homological algebra we have the following lemma.

Lemma 3.2. The short exact sequence

$$0 \to \Omega \xrightarrow{\partial_x} \Omega \xrightarrow{\pi} \bar{\Omega} \to 0$$

induces the following long exact sequence of the cohomology groups for $d \geq 1$:

$$\cdots \to H^p_d(\Omega, \tilde{D}_P) \to H^p_d(\bar{\Omega}, \tilde{D}_P) \to H^{p+1}_d(\Omega, \tilde{D}_P) \to \cdots.$$

Remark 3.1. On the space $\hat{A}$, the map $\partial_x$ has the kernel $\mathbb{R}$, however on the space $\Omega$ the map $\partial_x$ is injective.

Theorem 3.3 (Triviality of the variational Hamiltonian cohomology). We have

$$H^p_d(\Omega, \tilde{D}_P) = 0$$

for $p > 0$, $d > 0$.

Proof. We choose locally a system of flat coordinates $(v^1, \cdots, v^n; \phi_1, \cdots, \phi_n)$ on $\hat{M}$ such that $P = \frac{1}{2} \int \eta^{\alpha\beta} \phi_\alpha \phi^\beta$. Then for a 1-form

$$\omega = \sum_{s \geq 0} f_{\alpha,s} \delta v^{\alpha,s} + g_{\alpha}^s \delta \phi_\alpha^s \in \Omega^1_d$$

we have

$$\tilde{D}_P \omega = \sum_{s \geq 0} D_P(f_{\alpha,s}) \delta v^{\alpha,s} + (-)^p \eta^{\alpha\beta} f_{\alpha,s} \delta \phi_\beta^{s+1} + D_P(g_{\alpha}) \delta \phi_\alpha^s.$$

If $\omega \in \ker \tilde{D}_P$, then we see that

$$D_P(g_{\alpha}) = 0; \quad D_P(g_{\alpha}^s) = (-)^p \eta^{\alpha\beta} f_{\beta,s} = 0, \quad s \geq 0,$$

so we can write $\omega$ in the form

$$\omega = \tilde{D}_P \left( \sum_{s \geq 0} (-1)^p \eta^{\alpha\beta} g_{\alpha}^{s+1} \delta v^{\beta,s} \right) + g_{\alpha}^0 \delta \phi_\alpha.$$

For $g_{\alpha}^0 \in \hat{A}^{p-1}_d$, by using the triviality of the Hamiltonian cohomology (see, for example [27, 32]) we can represent it as $g_{\alpha}^0 = D_P(h_{\alpha}^0)$. Thus the cocycle $\omega$ must also be a coboundary. The theorem is proved. \qed

From Lemma 3.2 we have the following corollary.

Corollary 3.4. We have $H^p_d(\bar{\Omega}, \tilde{D}_P) = 0$ for $p > 0$, $d > 0$. 

Corollary 3.5. We have $H^p_d(\bar{\Omega}, \tilde{D}_P) = 0$ for $p > 0$, $d > 0$. 

3.3. Definition of the variational bihamiltonian cohomology. We proceed to define the variational bihamiltonian cohomology and discuss its relations with the bihamiltonian cohomology. Let \((P_0, P_1)\) be a semisimple bihamiltonian structure of hydrodynamic type and \(u^1, \ldots, u^n\) be its canonical coordinates. We will use \(\tilde{D}_0\) and \(\tilde{D}_1\) to denote \(\tilde{D}_P_0\) and \(\tilde{D}_P_1\) respectively. By using the identity (3.2) we have \(\tilde{D}_i\tilde{D}_j + \tilde{D}_j\tilde{D}_i = 0\) for \(i, j = 0, 1\).

**Definition 3.2.** The variational bihamiltonian cohomology for \((P_0, P_1)\) is defined to be the following groups:

\[
VBH^p_d(\Omega, \tilde{D}_0, \tilde{D}_1) = \frac{\tilde{\Omega}_d^p \cap \ker \tilde{D}_0 \cap \ker \tilde{D}_1}{\tilde{\Omega}_d^p \cap \text{Im} \tilde{D}_0 \tilde{D}_1};
\]

\[
VBH^p_d(\Omega, \tilde{D}_0, \tilde{D}_1) = \frac{\tilde{\Omega}_d^p \cap \ker \tilde{D}_0 \cap \ker \tilde{D}_1}{\tilde{\Omega}_d^p \cap \text{Im} \tilde{D}_0 \tilde{D}_1}.
\]

**Lemma 3.5.** The cohomology groups of the cochain complex \(\tilde{\Omega}[\lambda] = \Omega \otimes \mathbb{R}[\lambda]\) with differential \(\partial_\lambda = \tilde{D}_1 + \lambda \tilde{D}_0\) is isomorphic to the variational bihamiltonian cohomology, i.e.

\[
H^p_d(\tilde{\Omega}[\lambda], \partial_\lambda) \cong VBH^p_d(\Omega, \tilde{D}_0, \tilde{D}_1), \quad d \geq 2.
\]

Similarly, we have the following isomorphisms for the corresponding quotient spaces:

\[
H^p_d(\tilde{\Omega}[\lambda], \partial_\lambda) \cong VBH^p_d(\tilde{\Omega}, \tilde{D}_0, \tilde{D}_1), \quad d \geq 2.
\]

**Proof.** The lemma follows from the triviality of the variational Hamiltonian cohomology. For details, one may refer to the proof of Lemma 4.4 of [33]. \(\square\)

**Lemma 3.6** (Salamander lemma). We have the following isomorphism induced by \(\tilde{D}_0\) for \(p > 0, d > 0\):

\[
\tilde{D}_0 : \frac{\tilde{\Omega}_d^p \cap \ker \tilde{D}_1 \tilde{D}_0}{\tilde{\Omega}_d^p \cap (\text{Im} \tilde{D}_0 + \text{Im} \tilde{D}_1)} \cong VBH^{p+1}_{d+1}(\tilde{\Omega}, \tilde{D}_0, \tilde{D}_1).
\]

**Proof.** The lemma follows easily from the triviality of the variational Hamiltonian cohomology. \(\square\)

The definition of the variational bihamiltonian cohomology is comparable with the definition of bihamiltonian cohomology. On the space \(\mathcal{A}\) there are differentials \(D_0\) and \(D_1\) which are defined in (2.2) by the bihamiltonian structure \((P_0, P_1)\), and on the space \(\mathcal{F}\) there are the induced differential which are denoted by \(d_0\) and \(d_1\). The bihamiltonian cohomology is then given by

\[
BH^p_d(\mathcal{A}, D_0, D_1) = \frac{\tilde{\mathcal{A}}_d^p \cap \ker D_0 \cap \ker D_1}{\tilde{\mathcal{A}}_d^p \cap \text{Im} D_0 D_1} \cong H^p_d(\tilde{\mathcal{A}}[\lambda], D_1 - \lambda D_0);
\]

\[
BH^p_d(\mathcal{F}, d_0, d_1) = \frac{\tilde{\mathcal{F}}_d^p \cap \ker d_0 \cap \ker d_1}{\tilde{\mathcal{F}}_d^p \cap \text{Im} d_0 d_1} \cong H^p_d(\tilde{\mathcal{F}}[\lambda], d_1 - \lambda d_0).
\]

There is a natural cochain map

\[
\delta : \tilde{\mathcal{F}}[\lambda] \to \tilde{\mathcal{O}}[\lambda]
\]

\[
(3.7) \quad \delta : \tilde{\mathcal{F}}[\lambda] \to \tilde{\mathcal{O}}[\lambda]
\]
between the complexes \(A[\lambda]\) and \(\Omega[\lambda]\) given by the de Rham differential \(\delta\). This map can be further illustrated as follows: We first note that the space \(\hat{\mathcal{F}}\) can be identified with

\[
\text{Der}(\hat{A})^D := \{X \in \text{Der}(\hat{A}) \mid \exists Y \in \hat{F}, X = DY\}
\]

by using the map (2.2). By applying the identity (2.3) we see that the differentials \(d_0, d_1\) on \(\mathcal{F}\) induce differentials \(\text{ad}_{d_0}, \text{ad}_{d_1}\) on \(\text{Der}(\hat{A})^D\). Since it is obvious that \(\text{Der}(\hat{A})^D \subseteq \text{Der}(\hat{A})^\theta\), the identity (3.2) shows that the map (3.7) can be viewed as a natural embedding (with a change of signs, see the remark below):

\[
i : \text{Der}(\hat{A}[\lambda])^D \hookrightarrow \text{Der}(\hat{A}[\lambda])^\theta.
\]

Therefore we conclude that the bihamiltonian cohomology can be viewed as the cohomology on the space \(\text{Der}(\hat{A}[\lambda])^D\), while the variational bihamiltonian cohomology is the cohomology on the space \(\text{Der}(\hat{A}[\lambda])^\theta\). In this sense, the bihamiltonian cohomology is just a restriction of the variational bihamiltonian cohomology onto a subcomplex.

**Remark 3.2.** The map \(\delta : \hat{F} \to \hat{\Omega}\) can be viewed as a generalization of the correspondence between the Hamiltonian functions and the Hamiltonian vector field on a finite dimensional symplectic manifold. In our case, this correspondence is twisted by a sign:

\[
\delta X = (-1)^{p-1} \Phi(D_X), \quad X \in \hat{F}^p.
\]

### 4. The Cohomology Group VBH_{3}^2

In this section, we compute the bihamiltonian cohomology group \(\text{VBH}_{3}^2(\hat{\Omega}, \hat{D}_0, \hat{D}_1)\). The computation of other cohomology groups will be covered in the next section. We fix a semisimple bihamiltonian \((P_0, P_1)\) of hydrodynamic type and work in the canonical coordinates such that the bihamiltonian structure is given by (2.4). Note that if we define a derivation

\[
D(g^1, \ldots, g^n) = \sum_{s \geq 0; i,j} \partial_x^s (g^i \eta^j) \frac{\partial}{\partial u^{i,s}}
\]

\[
+ \frac{1}{2} \sum_{s \geq 0; i,j} \partial_x^s \left( \partial_j g^i u^{i,1} \theta_i + g^i \partial_j g^i \frac{\partial}{\partial u^{i,s}} \right) \frac{\partial}{\partial u^{i,s}}
\]

\[
+ \frac{1}{2} \sum_{s \geq 0; i,j} \partial_x^s \left( \partial_j g^i \theta_j \theta_i + g^i \partial_j g^i \frac{\partial}{\partial \theta_i} \right) \frac{\partial}{\partial \theta_i}
\]

\[
+ \frac{1}{2} \sum_{s \geq 0; i,j,k} \partial_x^s \left( \partial_k g^i \partial_k g^i \frac{\partial}{\partial u^{i,s}} \right) \frac{\partial}{\partial u^{i,s}}
\]

for a set of functions \(g^1, \ldots, g^n\), then we know that

\[
D_{P_0} = D(f^1, \ldots, f^n), \quad D_{P_1} = D(u^1 f^1, \ldots, u^n f^n).
\]

Here and henceforth we will use \(\partial_t\) to denote \(\frac{\partial}{\partial u^t}\).

From Lemma 3.6 it follows that in order to compute the cohomology group \(\text{VBH}_{3}^2(\hat{\Omega}, \hat{D}_0, \hat{D}_1)\) we only need to consider the spaces

\[
\mathcal{Z} := \hat{\Omega}_2^1 \cap \ker(\hat{D}_0 \hat{D}_1), \quad \mathcal{B} := \hat{\Omega}_2^1 \cap (\text{Im} \hat{D}_0 + \text{Im} \hat{D}_1).
\]
For an element $\omega \in \bar{\Omega}^1_2$, we can represent it uniquely in the form

$$\omega = \int \sum_i X^i \delta u^i + Y^i \delta \theta_i, \quad X^i \in \hat{A}_2^1, \quad Y^i \in \hat{A}_2^0,$$

where the differential polynomials can be written as

$$X^i = \sum_j X_j^{(i)} \theta_j^2 + \sum_{j,k} \left( X_{kj}^{(i)} u^{j,1} \theta_k^1 + Z_{jk}^{(i)} u^{k,1} \theta_j^1 \right) + \sum_{j,k,l} Z_{jkl}^{(i)} u^{k,1} u^{l,1} \theta_j^1,$$

$$Y^i = \sum_j Y_j^{(i)} u^j + \sum_{j,k} Y_{jk}^{(i)} u^{j,1} u^{k,1}.$$

**Definition 4.1.** For an element $\omega \in \bar{\Omega}^1_2$ which is represented in the form (4.1) – (4.3), the indices $\text{ind}_i(\omega)$ for $i = 1, \cdots, n$ with respect to a semisimple bihamiltonian structure $(P_0, P_1)$ of hydrodynamic type are defined to be the functions

$$\text{ind}_i(\omega) := \frac{1}{f}(X_i^{(i)} + Y_i^{(i)}).$$

We will see later that the indices defined above are generalizations of the central invariants of deformations of bihamiltonian structures of hydrodynamic type. To compute the cohomology group $\text{VBH}^2_3(\bar{\Omega}) \cong \mathbb{Z}/\mathcal{B}$, we need the following two lemmas.

**Lemma 4.1.** For any given cocycle $\omega \in \mathcal{Z}$, the index $\text{ind}_i(\omega)$ for $i = 1, \cdots, n$ with respect to a semisimple bihamiltonian structure $(P_0, P_1)$ of hydrodynamic type is a function of single variable $u^i$ for any $i = 1, \cdots, n$.

**Lemma 4.2.** A cocycle $\omega \in \mathcal{Z}$ is a coboundary, i.e. $\omega \in \mathcal{B}$, if and only if

$$\text{ind}_i(\omega) = 0, \quad i = 1, \cdots, n.$$

**Theorem 4.3.** The quotient space $\mathcal{Z}/\mathcal{B} \cong \oplus_{i=1}^n C^\infty(\mathbb{R})$.

**Proof.** Given $n$ functions of single variable $c_1(u^1), c_2(u^2), \cdots, c_n(u^n)$, we construct an element $\tau \in \mathcal{Z}$ as follows:

$$\tau = \int \delta \left( D_1 \sum_i c_i(u^i) u^{i,1} \log u^{i,1} - D_0 \sum_i u^i c_i(u^i) u^{i,1} \log u^{i,1} \right).$$

One can check directly, or refer to [31], to confirm the fact that actually $\tau$ is a 1-form with differential polynomial coefficients and that $\tau \in \mathcal{Z}$.

By a straightforward computation we can show that

$$\text{ind}_i(\tau) = -3c_i(u^i), \quad i = 1, \cdots, n.$$

Therefore, it follows from Lemma 4.1 that we can choose suitable $c_i(u^i)$ such that

$$\text{ind}_i(\omega - \tau) = 0$$

for any given cocycle $\omega \in \mathcal{Z}$. Then we apply Lemma 4.2 to conclude that the class $[\omega]$ coincides with the class $[\tau]$ in the space $\mathcal{Z}/\mathcal{B}$. Lemma 4.2 also implies that $\tau$ is a coboundary if and only if $c_i(u^i) = 0$, hence different choices of functions $c_1(u^1), \cdots, c_n(u^n)$ give different classes $[\tau]$ in $\mathcal{Z}/\mathcal{B}$. The theorem is proved. □
The remaining part of this section is devoted to the proof of Lemma 4.1 and Lemma 4.2. The strategy of our proof is: We first prove the ‘only if’ part of Lemma 4.2, then we prove Lemma 4.1. Finally we prove the ‘if’ part of Lemma 4.2.

**Proof of Lemma 4.2 (Part 1).** We are to show that if \( \omega \in B \) then its indices vanish. Take \( \alpha \in \bar{\Omega}^0_1 \) and write it uniquely as
\[
\alpha = \int \sum_{i,j} \alpha_j(i) u_j(i) \delta u_i + \delta \theta_i.
\]
then we have
\[
\tilde{D}_0 \alpha = \int \sum_{i,j} \tilde{D}_0(\alpha_j(i) u_j(i)) \delta u_i + \alpha_j(i) \delta u_i + \delta (\tilde{D}_0(u_j))
\]
Here \( \delta \theta_i \) stands for the terms that make no contribution to the index. Then from Definition 4.1 it follows that
\[
\text{ind}_{\tilde{D}_0}(\tilde{D}_0 \alpha) = 0.
\]
Similarly we have
\[
\text{ind}_{\tilde{D}_1}(\tilde{D}_1 \alpha) = 0.
\]
Hence the indices of a coboundary must vanish. \( \square \)

The above proof shows that the indices can be defined for a class \([\omega] \in Z/B\). To prove Lemma 4.1, we first choose a ‘normal form’ for every class \([\omega] \in Z/B\) that will simplify the computation.

**Lemma 4.4.** For any given class \([\sigma] \in Z/B\), there exists a unique \( \omega \in Z \) which can be represented in the form (4.1)-(4.3) and satisfies the following conditions:
1. \([\omega] = [\sigma]\);
2. \(X_j(i) = 0\) for \(j \neq i\);
3. \(Y_j(i) = 0\) for any \(i, j\);
4. \(Y_{ii}(i) = 0\).
Such a form \( \omega \) is called the normal form of the class \([\sigma]\).

**Proof.** We first take \( \omega = \sigma \), then the first condition is satisfied. We then adjust \( \omega \) by adding elements of \( B \) such that other conditions are also satisfied. Firstly, according to (4.1), we change \( \omega \) to \( \tilde{\omega} = \omega + \tilde{D}_0 \gamma \) with
\[
\gamma = \int \sum_{i,j} \frac{Y_j(i)}{f_i} u_j(i) \delta u_i,
\]
then the third condition is satisfied. As for other conditions, we want to find \( \alpha, \beta \in \bar{\Omega}^0_1 \) such that \( \tilde{\omega} + \tilde{D}_0 \alpha + \tilde{D}_1 \beta \) satisfies all the four conditions.

Let us write \( \alpha = \int \sum_i \alpha_i \delta u_i \) and \( \beta = \int \sum_i \beta_i \delta u_i \) for some \( \alpha_i, \beta_i \in \bar{A}^0_1 \). We can uniquely represent \( \tilde{D}_0 \alpha \) and \( \tilde{D}_1 \beta \) as follows:
\[
\tilde{D}_0 \alpha = \int \sum_i A_i \delta u_i + W_i \delta \theta_i; \quad \tilde{D}_1 \beta = \int \sum_i B_i \delta u_i + R_i \delta \theta_i,
\]
where the differential polynomials $W_i$ and $R_i$ are given by

$$W_i = - (\alpha_i f^i)' + \frac{1}{2} \sum_j \left( \alpha_i \partial_j f^i u^{i,1} + \alpha_j f^i \frac{\partial_j f^i}{f^i} u^{i,1} - \alpha_j f^i \frac{\partial_i f^j}{f^j} u^{j,1} \right);$$

$$R_i = - (\beta_i u^i f^i)' + \frac{1}{2} \sum_j \left( \beta_i u^i \partial_j f^i u^{j,1} + \beta_j u^j f^j \frac{\partial_j f^i}{f^i} u^{i,1} - \beta_j u^j f^j \frac{\partial_i f^j}{f^j} u^{j,1} \right) + \frac{1}{2} \beta_i f^i u^{i,1}.$$

Note that we should make sure that the third condition is satisfied, so if we further write $\alpha_i = \sum_j \alpha_j^{(i)} u^{j,1}$ and $\beta_i = \sum_j \beta_j^{(i)} u^{j,1}$, and compare the coefficients of $u^{i,2}$ of $W_i$ and $R_i$, we arrive at $u^i \beta_j^{(i)} + \alpha_j^{(i)} = 0$, hence we must take $\alpha_i = -u^i \beta_i$. Since the coefficients of $(u^{i,1})^2$ in $W_i + R_i$ are given by $\beta_j^{(i)} f^i$, we can choose suitable functions $\beta_j^{(i)}$ such that the condition (4) is satisfied. Finally we compute the coefficients of $\theta_j^2$ in $A_i + B_i$ to obtain

$$\beta_j^{(i)} (u^i - u^i) f^i,$$

so for $j \neq i$ we can choose suitable functions $\beta_j^{(i)}$ such that the condition (2) is satisfied. Thus we find a form $\omega$ which satisfy all the four conditions, and the above computation also shows that such an $\omega$ is unique. The lemma is proved.

As a byproduct of the above computation, we have the following theorem.

**Theorem 4.5.** We have $VBH^2_1(\bar{\Omega}, \bar{D}_0, \bar{D}_1) = 0$.

**Proof.** Recall that by definition $VBH^2_1(\bar{\Omega}, \bar{D}_0, \bar{D}_1) = \bar{\Omega}^2 \cap \ker \bar{D}_0 \cap \ker \bar{D}_1$. Now take any cocycle $\omega$, by using the triviality of the variational Hamiltonian cohomology we can find $\alpha, \beta \in \bar{\Omega}^0$ such that

$$\omega = \bar{D}_0 \alpha = \bar{D}_1 \beta. \tag{4.6}$$

Let us show that $\alpha = \beta = 0$. To this end we write $\alpha = \int \sum_i \alpha_i \delta u^i$ and $\beta = \int \sum_i \beta_i \delta u^i$ for some $\alpha_i, \beta_i \in \mathcal{A}_0^i$, and represent $\bar{D}_0 \alpha, \bar{D}_1 \beta$, as we do in the proof of Lemma 4.3, in the form

$$\bar{D}_0 \alpha = \int \sum_i A_i \delta u^i + W_i \delta \theta_i; \quad \bar{D}_1 \beta = \int \sum_i B_i \delta u^i + R_i \delta \theta_i. \tag{4.7}$$

From the coefficients of $u^{i,2}$ in $W_i$ and $R_i$ it follows that $\alpha_i = u^i \beta_i$, and from the coefficients of $(u^{i,1})^2$ in $W_i$ and $R_i$ we conclude that $\beta_i^{(i)} = 0$. Finally, from the coefficients of $\theta_j^2$ in $A_i$ and $B_i$ for $j \neq i$ it follows that $\beta_j^{(i)} = 0$ when $j \neq i$. Therefore $\alpha = \beta = 0$ and the theorem is proved.

Now let us come back to prepare the proof of Lemma 4.1. Take a class $[\omega] \in \mathcal{Z}/\mathcal{B}$ with $\omega$ being its normal form which can be represented as

$$\omega = \int \sum_i X^i \delta u^i + Y^i \delta \theta_i, \quad X^i \in \bar{\mathcal{A}}_2^i, \; Y^i \in \bar{\mathcal{A}}_0^i. \tag{4.8}$$
where the differential polynomials can be written in the form

\begin{align}
X^i &= X^{(i)}_i \theta^i_t + \sum_{j,k} \left( X^{(i)}_{kj} u^{j,1} \theta^1_k + Z^{(i)}_{jk} u^{k,2} \theta_j \right) + \sum_{j,k,l} Z^{(i)}_{jkl} u^{k,1} u^{l,1} \theta_j; \\
Y^i &= \sum_{j,k} Y^{(i)}_{jk} u^{j,1} u^{k,1}, \quad Y^{(i)}_{ii} = 0.
\end{align}

Further more, we require that

\begin{align}
Z_{j;kl}^{(i)} &= Z_{j;lk}^{(i)}, \quad Y_{jk}^{(i)} = Y_{kj}^{(i)}.
\end{align}

Due to the part of Lemma 4.2 that we just proved, different representatives of a class in $Z/B$ have the same indices, hence to prove Lemma 4.1 we only need to show that $\text{ind}_i(\omega) = X^{(i)}_i(f)/f_i$ is a function of $u^i$ for each $i = 1, \ldots, n$.

Since $\omega \in Z$, from the triviality of variational Hamiltonian cohomology it follows the existence of $\alpha \in \tilde{\Omega}_1^2$ such that $\tilde{D}_0 \omega = \tilde{D}_1 \alpha$. Such an $\alpha$ is unique up to the addition of an image of $\tilde{D}_1$, hence we can make a particular choice of $\alpha$ in a similar way as we choose the normal form of $\omega$. More explicitly, it is not difficult to see that there exists a unique $\alpha$ such that

\begin{align}
\alpha = \int \sum_i P^i \delta u^i + Q^i \delta \theta_i, \quad P^i \in \hat{A}_2^1, \quad Q^i \in \hat{A}_2^0,
\end{align}

where the differential polynomials can be written as

\begin{align}
P^i &= \sum_j P^{(i)}_j \theta^2_j + \sum_{j,k} \left( P^{(i)}_{kj} u^{j,1} \theta^1_k + W^{(i)}_{jk} u^{k,2} \theta_j \right) + \sum_{j,k,l} W^{(i)}_{jkl} u^{k,1} u^{l,1} \theta_j; \\
Q^i &= \sum_{j,k} Q^{(i)}_{jk} u^{j,1} u^{k,1}
\end{align}

with

\begin{align}
W^{(i)}_{j;kl} = W^{(i)}_{j;lk}, \quad Q^{(i)}_{jk} = Q^{(i)}_{kj}.
\end{align}

**Proof of Lemma 4.1** Consider a class $[\omega] \in Z/B$ where $\omega$ is given by the normal form (4.9)–(4.11). Let $\alpha \in \Omega_1^2$ be the unique element given by (4.12)–(4.15) such that $\tilde{D}_0 \omega = \tilde{D}_1 \alpha$. For the bihamiltonian structure $(P_0, P_1)$ given in (2.3), we denote

\begin{align}
a_{ij} = \frac{1}{2} \partial_i f^j, \quad b_{ij} = \frac{1}{2} f^i \partial_j f^j / f_j.
\end{align}

Let us first compute the differential polynomials $M^i, N^i, S^i, T^i$ that are defined by

\begin{align}
\tilde{D}_0 \omega = \int \sum_i M^i \delta u^i + N^i \delta \theta_i, \quad \tilde{D}_1 \alpha = \int \sum_i S^i \delta u^i + T^i \delta \theta_i.
\end{align}

In what follows of the proof, we will omit the symbol of summations and use $j, k, l$ to denote the indices that should be summed over $1, \ldots, n$. The index $i$ is a fixed index and
do not participate in the summation. It is straightforward to obtain

\[
M^i = D_0(X^i) - \partial_i f' X^j - X^k \partial_i a_{jk} u^{j,1} \theta_k + (X^j a_{ij} \theta_j)' - X^k \partial_i b_{kj} u^{j,1} \theta_j + (X^j b_{ij} \theta_j)' - X^k \partial_i a_{kj} u^{j,1} \theta_j - (X^j b_{ij} \theta_j)' - X^k \partial_i b_{kj} u^{j,1} \theta_j - (X^j b_{ij} \theta_j)'
\]

\[
S^i = D_1(P^i) - P^i \partial_i (u^j f') \theta_j + P^i \partial_i (u^j a_{jk}) u^{j,1} \theta_k + (P^i u^j a_{ij} \theta_j)'
\]

\[
N^i = D_0(Y^i) + (X^i f')' - X^i a_{ji} u^{j,1} - X^i b_{ji} u^{j,1} + X^i b_{ij} u^{j,1} - (Y^i a_{ji} \theta_i)'
\]

\[
T^i = D_1(Q^i) + (P^i u^j f')' - P^i u^j a_{ji} u^{j,1} - P^i u^j b_{ji} u^{j,1} + P^i u^j b_{ij} u^{j,1} - \frac{1}{2} P^i f' u^{j,1}
\]

By comparing the coefficients of \( \theta^3 \) of both sides of the equation \( N^i = T^i \) we obtain

\[
(4.17) \quad P^{(i)}_j = 0, \quad i \neq j; \quad X^{(i)}_i = u^i P^{(i)}_i.
\]

Comparing the coefficients of \( u^{k,3} \theta_j \) on both sides of the equation \( N^i = T^i \), we conclude that

\[
(4.18) \quad Z^{(i)}_{jk} = u^i W^{(i)}_{jk}, \quad \forall i, j, k.
\]

Next, from the coefficients of \( \theta^3 \theta^2 \) in \( M^i \) and \( S^i \) for \( j \neq i \), and from (4.17) it follows that

\[
(4.19) \quad (\partial_j P^{(i)}_k - 2P^{(i)}_j b_{ji}) (u^i - u^j) = (X^{(i)}_{ji} - P^{(i)}_j u^i) f', \quad i \neq j.
\]

Similarly, from the coefficients of \( u^{i,2} \theta^1_k \) in \( N^i \) and \( T^i \), and from the identity (4.18) it follows that

\[
(4.20) \quad X^{(i)}_{kj} = u^i P^{(i)}_{kj}, \quad \forall i, j, k.
\]
The identity (4.19) and (4.20) lead to
\[ \partial_j \frac{P_i}{f_i} = 0, \quad i \neq j. \]
Thus for each \(i\) the function
\[ \text{ind}_i(\omega) = \frac{X^{(i)}_{ji}}{f_i} = u^i \frac{P_i}{f_i}. \]
depends only on \(u^i\). The lemma is proved. \(\square\)

Finally, let us complete the proof of Lemma 4.2. Before proceeding to the proof, we first derive some relations satisfied by the coefficients in \(\omega\) and \(\alpha\). In the rest of this section, we will continue to use the notations which are used in the proof of Lemma 4.1.

By comparing the coefficients of \(\theta \theta_j^3\), \(\theta \theta_j^1\) and \(\theta \theta_j^2\) in \(M^i\) and \(S^i\) and by using the relations (4.17) and (4.18) we obtain
\[ (4.21) \quad Z^{(i)}_{kj} = W^{(i)}_{kj} = 0, \quad i \neq j \neq k \neq i. \]
\[ (4.22) \quad P^{(i)}_{ij} = W^{(i)}_{ij} f_j, \quad i \neq j. \]
\[ (4.23) \quad P^{(i)}_{iji} = W^{(i)}_{iji} f_j, \quad i \neq j. \]

We also compare the coefficients of \(\theta \theta_j^2\), \(\theta_j^1\theta_i^2\) and \(\theta_j^2\theta_i^2\) in \(M^i\) and \(S^i\) to arrive at
\[ (4.24) \quad X^{(i)}_{ij} = P^{(i)}_{ij} u_j, \quad i \neq j \neq k \neq i. \]
\[ (4.25) \quad 3P^{(i)}_{ij}(u^i - u^j)b_{ij} = (X^{(i)}_{ij} - P^{(i)}_{ij}u^j)f_j, \quad i \neq j. \]
\[ (4.26) \quad P^{(i)}_{ij}(u^i - u^j)a_{ij} = (X^{(i)}_{ij} - P^{(i)}_{ij}u^j)f_j, \quad i \neq j. \]

By comparing the coefficients of \(u^k \theta_j^2\), \(u^j \theta_j^2\), \(u^i \theta_j^2\), \(u^i \theta_j^2\) and \(u^i \theta_j^2\) in \(N^i\) and \(T^i\), and by using the relation (4.20) we arrive at the following identities:
\[ (4.27) \quad Y^{(i)}_{jk} = Q^{(i)}_{jk} u^i, \quad i \neq j \neq k \neq i. \]
\[ (4.28) \quad 2Y^{(i)}_{jj} f_j + X^{(j)}_{ij} b_{ij} = 2Q^{(i)}_{jj} u^j f_j + P^{(j)}_{ij} u^i b_{ij}, \quad i \neq j. \]
\[ (4.29) \quad Y^{(i)}_{ji} = Q^{(i)}_{ji} u^j, \quad i \neq j. \]
\[ (4.30) \quad Y^{(i)}_{ij} = Q^{(i)}_{ij} u^i, \quad i \neq j. \]
\[ (4.31) \quad 2Y^{(i)}_{ii} = 2Q^{(i)}_{ii} u^i + \frac{1}{2} P^{(i)}_{ij} f_i. \]
Proof of Lemma 4.2 (Part 2). We need to show that a cocycle with vanishing indices must be a coboundary. Take a class $[\omega] \in Z/B$ with $\omega$ being its normal form and $\text{ind}_1(\omega) = 0$. Let $\alpha \in \Omega^1_2$ be the unique element given by \(4.12\)–\(4.15\) such that $\tilde{D}_0\omega = \tilde{D}_1\alpha$. We will prove that $\alpha = \omega = 0$.

From $\text{ind}_1(\omega) = 0$ and \(4.17\) we know that $X^{(i)}_1 = P^{(i)}_1 = 0$. Then the identities \(4.18\) and \(4.21\)–\(4.23\) show that the coefficients $W^{(i)}_{jk}$ vanish when $k \neq j$ and $Z^{(i)}_{ji} = u^iW^{(i)}_{ji}$; the identities \(4.24\)–\(4.26\) together with \(4.20\) show that the coefficients $P^{(i)}_{jk}$ vanish when $k \neq i$ and $X^{(i)}_{ji} = u^iP^{(i)}_{ji}$. Similarly, we conclude from the identities \(4.11\), \(4.15\) and \(4.27\)–\(4.31\) that the coefficients $Q^{(i)}_{jk} = 0$ when $k \neq j$ and $Y^{(i)}_{jj} = u^jQ^{(i)}_{jj}$. In particular, from \(4.11\) we know that $Q^{(i)}_{ii} = 0$.

To simplify the notations, we use $Y^{(i)}_{j}$ and $Q^{(i)}_{j}$ to denote $Y^{(i)}_{jj}$ and $Q^{(i)}_{jj}$. Therefore we can represent the coefficients of $\omega$ and $\alpha$ that are given in \(4.30\), \(4.10\) and \(4.13\), \(4.14\) as follows:

\[
\begin{align*}
X^i &= \sum_j \left( X^{(i)}_{ji} u^{i,1} \theta^1_j + Z^{(i)}_{ji} u^{i,2} \theta^2_j \right) + \sum_{j,k,l} Z^{(i)}_{ljk} u^{k,1} u^{j,1} \theta^1_l, \\
P^i &= \sum_j \left( P^{(i)}_{ji} u^{i,1} \theta^1_j + W^{(i)}_{ji} u^{i,2} \theta^2_j \right) + \sum_{j,k,l} W^{(i)}_{ljk} u^{k,1} u^{j,1} \theta^1_l, \\
Y^i &= \sum_j Y^{(i)}_{j} (u^{j,1})^2, \quad Q^i = \sum_j Q^{(i)}_{j} (u^{j,1})^2,
\end{align*}
\]

where the coefficients satisfy the conditions

\begin{align*}
X^{(i)}_{ji} &= u^i P^{(i)}_{ji}, \quad Z^{(i)}_{ji} = u^i W^{(i)}_{ji}, \\
Y^{(i)}_{j} &= u^j Q^{(i)}_{j}, \quad Y^{(i)}_{j} = Q^{(i)}_{j} = 0.
\end{align*}

Let us first compare the coefficients of $u^{i,1} u^{j,2} \theta^1_j$ in $N^i$ and $T^i$ to arrive at

\begin{equation}
2Z^{(i)}_{ji;} = 2u^i W^{(i)}_{ji;} - \frac{1}{2} W^{(i)}_{ji;} \quad \forall i, j.
\end{equation}

At the same time we compute the coefficients of $u^{i,1} \theta^1_j \theta^2_k$ in $M^i$ and $S^i$, and we obtain

\begin{equation}
-2Z^{(i)}_{ji;} \partial_i f^i - 2Z^{(i)}_{ji;} f^i = -2W^{(i)}_{ji;} \partial_i (u^i f^i) - 2W^{(i)}_{ji;} u^i f^i - \frac{1}{2} W^{(i)}_{ji;} f^i, \quad \forall i, j.
\end{equation}

Then it follows from the identities \(4.32\), \(4.34\) and \(4.35\) that

\begin{equation}
Z^{(i)}_{ji;} = W^{(i)}_{ji;} = 0, \quad \forall i, j.
\end{equation}

Next we compare the coefficients of $u^{j,1} u^{i,2} \theta^1_k$ in $N^i$ and $T^i$ for distinct $j, k$ and we obtain the relations

\begin{equation}
Z^{(i)}_{i;jk} = u^i W^{(i)}_{i;jk}, \quad j \neq k, \quad \forall i, l.
\end{equation}

For $j \neq i$, we compare the coefficients of $u^{j,1} \theta^1_k \theta^2_l$ in $M^i$ and $S^i$ and we arrive at

\begin{equation}
Z^{(i)}_{i;jk} = u^k W^{(i)}_{i;jk}, \quad i \neq j, \quad \forall k, l.
\end{equation}
Since \( Z_{i;jk} = Z_{i;kj} \), \( W_{i;jk} = W_{i;kj} \), the identities \((4.37)\) and \((4.38)\) imply that the only possibly non-vanishing coefficients are \( Z_{i;jj} \) and \( W_{i;jj} \); moreover, from the identities \((4.34)\) and \((4.36)\) we conclude that

\[
Z_{i;jj}^{(i)} = u^i W_{i;jj}^{(i)}, \quad \forall i, j, l.
\]

We proceed to compute the coefficients of \((u^{i.1})^2 \theta_j^1\) in \(N^i\) and \(T^i\). By using \((4.32)\) and \((4.33)\), it is easy to deduce that

\[
X_{ji}^{(i)} = P_{ji}^{(i)} = 0, \quad \forall i, j.
\]

Thanks to this equation, we can also compute the coefficients of \((u^{i.1})^2 \theta_j^1\) of \(N^i\) and \(T^i\) and obtain the relations

\[
f^i Z_{j;jj}^{(i)} = u^i f^i W_{j;jj}^{(i)} + 2Q_j^{(i)} f^j, \quad i \neq j.
\]

On the other hand, it follows from \((4.36)\) and the coefficients of \(u^{i.1} u^{j.2} \theta_j\) in \(N^i\) and \(T^i\) for \(i \neq j\) that

\[
2f^i Z_{j;jj}^{(i)} = 2u^i f^i W_{j;jj}^{(i)} + Q_j^{(i)} f^j, \quad i \neq j.
\]

Thus the relations \((4.41)\) and \((4.42)\) together with \((4.33)\) lead to

\[
Y^i = Q^i = 0, \quad \forall i.
\]

Now thanks to \((4.36), (4.40)\) and \((4.43)\), we are able to easily compare the coefficients of \(u^{i.1} u^{j.2} \theta_k\) and \(u^{i.1} u^{j.2} \theta_l\) in \(N^i\) and \(T^i\) respectively and we arrive at

\[
Z_{k;jj}^{(i)} = u^i W_{k;jj}^{(i)}, \quad i \neq j \neq k \neq i;
\]

\[
Z_{i;jj}^{(i)} = u^i W_{i;jj}^{(i)}, \quad i \neq j.
\]

Therefore by combining the identities \((4.39), (4.44)\) and \((4.45)\) we conclude that

\[
X^i = \sum_j Z_{j;ii}^{(i)} (u^{i.1})^2 \theta_j, \quad P^i = \sum_j W_{j;ii}^{(i)} (u^{i.1})^2 \theta_j, \quad Z_{j;ii}^{(i)} = u^i W_{i;ii}^{(i)}.
\]

and comparing the coefficients of \((u^{i.1})^3\) we deduce that \(P^i = 0\), and it follows from the above relation that \(X^i\) also vanish. The lemma is proved.

\[\square\]

5. **Vanishing theorem of the variational bihamiltonian cohomology**

5.1. **Vanishing theorem and the strategy of computation.** In this section, we compute the general variational bihamiltonian cohomology groups \(VBH_d^P(\Omega, D_0, D_1)\). The main result we are to obtain is the following theorem.

**Theorem 5.1.** The cohomology group \(H_d^P(\Omega[\lambda], \partial \lambda)\) vanishes unless the bidegree \((p,d)\) belongs to the following two cases:

- Case 1: \(d = 0, \ldots, n; \quad p = d + 1, \ldots, d + n + 1,\)
- Case 2: \(d = 2, \ldots, n + 3; \quad p = d, \ldots, d + n.\)
Proof of Theorem 2.1. Similar to Lemma 3.2, we have the following long exact sequence:

\[ \cdots \rightarrow H^p_d(\Omega^\lambda, \partial^\lambda) \rightarrow H^p_d(\bar{\Omega}^\lambda, \partial^\lambda) \rightarrow H^{p+1}_d(\Omega^\lambda, \partial^\lambda) \rightarrow \cdots. \]

Then Theorem 2.1 follows from Theorem 5.1 together with Lemma 3.5, Theorem 4.3 and Theorem 4.5. \(\square\)

The strategy to prove the above theorem is inspired by the work [5, 4], where some appropriate spectral sequences are used to compute the bihamiltonian cohomology of a bihamiltonian structure of hydrodynamic type. Our computation in this section can be viewed as a certain generalization of that of [4], therefore we will use the same notations as the ones used in [4] whenever possible.

For a semisimple bihamiltonian structure \((P_0, P_1)\) of hydrodynamic type, we will work in its canonical coordinates \(u^1, \ldots, u^n\). Given \(\omega \in \Omega^p\), we represent it in the form

\[ \omega = \sum_{i,s \geq 0} g_{i,s} \delta u^{i,s} + h^i_s \delta \theta^s, \quad g_{i,s}, h^i_s \in \hat{A}^p, \quad h^i_s \in \hat{A}^{p-1}, \]

then by using the formula given in Example 3.2 we have

\[ \tilde{D}_a \omega = \sum_{i,s \geq 0} (D_{P_a} g_{i,s}) \delta u^{i,s} + (-1)^a g_{i,s} \delta (D_{P_a} u^{i,s}) + (D_{P_a} h^i_s) \delta \theta^s + (-1)^{p-1} h^i_s \delta (D_{P_a} \theta^s) \]

for \(a = 0, 1\). We are going to compute the cohomology of the complex \(\Omega^\lambda, \partial^\lambda = \tilde{D}_1 - \lambda \tilde{D}_0\) in what follows.

Let us define a gradation, called the \(u\)-degree, on \(\Omega\) by setting

\[ \deg_u u^{i,s+1} = 1, \quad \deg_u \delta u^{i,s} = 1, \quad i = 1, \ldots, n, \quad s \geq 0, \]

and the \(u\)-degrees of other generators are set to be 0. Let us denote the super degree of an element \(\omega\) of \(\Omega\) by \(\deg_\theta \omega\). We filtrate the complex \(\Omega[\lambda]\) by defining

\[ F^k \Omega[\lambda] = \{ \omega \in \Omega[\lambda] \mid \deg_u \omega + \deg_\theta \omega \geq k \}, \]

then we have a filtration

\[ \cdots \subset F^{k+1} \Omega[\lambda] \subset F^k \Omega[\lambda] \subset \cdots \subset F^0 \Omega[\lambda] = \Omega[\lambda]. \]

We also decompose the differential in the following way:

\[ \partial^\lambda = \Delta_{-1} + \Delta_0 + \cdots, \quad \deg_u \Delta_k = k. \]

Note that \(\deg_\theta \partial^\lambda = 1\), therefore each \(\Delta_k\) preserves the filtration. Now let \((^1E_r, d_r), r \geq 0\) be the spectral sequence induced by the filtration, then by the standard construction \((^1E_0, d_0) = (\Omega[\lambda], \Delta_{-1})\), and \(^1E_k\) is given by the cohomology of \(^1E_{k-1}\) for \(k \geq 1\). It is clear that when restricted to \(\Omega^p_d[\lambda]\), the above filtration is bounded and therefore this guarantees the convergence of the spectral sequence. Note that the differential \(d_r\) with \(r = 0\) or 1 is not the induced differential on \(\tilde{F}\) used in Sect.3.

We will compute \(^1E_1\) and \(^1E_2\) in the following subsections, to this end we need to construct some other spectral sequences as in [4]. The following standard fact is also used in the computation.
Lemma 5.2. Let \((C^\bullet, d)\) be a cochain complex in an Abelian category. Assume that for each \(p\) we have the decomposition \(C^p = A^p \oplus B^p\) with \(A^\bullet\) being a subcomplex, and \(A^\bullet\) is acyclic. Then

\[
H^p(C^\bullet, d) \cong H^p(B^\bullet, \pi_B \circ d).
\]

Here \(\pi_B\) is the projection from \(C^\bullet\) to \(B^\bullet\).

5.2. Computation of \(^1E_1\). We first write down \(\Delta_{-1}\) in the explicit form

\[
\Delta_{-1} = \sum_{s \geq 1} (-\lambda + u^i) f^i \theta^{s+1} \frac{\partial}{\partial u^{i,s}} + \sum_{s \geq 0} (-\lambda + u^i) f^i \delta \theta^{s+1} \frac{\partial}{\partial \delta u^{i,s}}
\]

\[:= \sum_i (-\lambda + u^i) f^i \delta \tilde{d}_i,
\]

here we introduce the de Rham-type differential

\[
\delta \tilde{d}_i = \sum_{s \geq 1} \theta^{s+1}_i \frac{\partial}{\partial u^{i,s}} + \sum_{s \geq 0} \delta \theta^{s+1}_i \frac{\partial}{\partial \delta u^{i,s}}.
\]

We will also use the notation

\[
\tilde{d}_i = \sum_{s \geq 1} \theta^{s+1}_i \frac{\partial}{\partial u^{i,s}}.
\]

Lemma 5.3. Each summand of the following decomposition is a cochain sub complex with respect to \(\Delta_{-1}\):

\[
\Omega[\lambda] = \tilde{A}[\lambda]\{\delta \theta_i \mid i = 1, \cdots, n\} \oplus \bigoplus_{i=1}^n \tilde{A}[\lambda]\{\delta u^{i,s}, \delta \theta^{s+1}_i \mid s \geq 0\}.
\]

Here we use the notation \(R\{g_1, g_2, \ldots\}\) to denote the free \(R\)-module generated by \(g_1, g_2, \ldots\).

In what follows, we will use the same notations as in [3] to denote the following subspaces:

\[
\tilde{C} = C^\infty(u)[\theta_1, \cdots, \theta_n, \theta_1^1, \cdots, \theta_n^1],
\]

\[
\tilde{C}_i = \tilde{C}[u^{i,s}, \theta_i^{s+1} \mid s \geq 1],
\]

and use \(\tilde{C}^nt\) to denote the subspace of \(\tilde{C}\) spanned by nontrivial monomials, i.e., all monomials that contain at least one of the variables \(u^{i,s}, \theta_i^{s+1}\) for \(s \geq 1\). We also use \(\tilde{M}\) to denote the subspace of \(\tilde{A}\) spanned by monomials which contain at least one of the mixed quadratic expressions:

\[
u^{i,s} \theta_j^t; \quad u^{i,s} \theta_i^{t+1}; \quad \theta_i^{s+1} \theta_j^t
\]

for some \(i \neq j\) and \(s, t \geq 1\). Then it is easy to see that there is a decomposition

\[
\tilde{A} = \tilde{C} \oplus \left( \bigoplus_{i=1}^n \tilde{C}^nt \right) \oplus \tilde{M},
\]

and each summand is preserved under the action of \(\tilde{d}_i\) given by (5.4).
Lemma 5.4. We have

\[ H(\mathcal{A}[\lambda]\{\delta\theta\}, \Delta_{-1}) = \tilde{C}[\lambda]\{\delta\theta\} \oplus \bigoplus_{i=1}^{n} (-\lambda + u^i)\hat{d}_i(\tilde{C}[\lambda])\{\delta\theta\}. \]

Here \{\delta\theta\} is the abbreviation of \{\delta\theta_i, i = 1, \ldots, n\}.

Proof. It is clear from the formula (5.2) that

\[ H(\mathcal{A}[\lambda]\{\delta\theta\}, \Delta_{-1}) = H(\mathcal{A}[\lambda], \Delta_{-1})\{\delta\theta\}, \]

where the action of \Delta_{-1} on \mathcal{A}[\lambda] is just \(\sum_i (-\lambda + u^i)f^i\hat{d}_i\). Hence the lemma is proved by applying the result in [5].

According to the decomposition (5.5), we still need to compute cohomology

\[ H(\mathcal{A}[\lambda]\{\delta u^{i,s}, \delta\theta_i^{s+1} | s \geq 0\}, \Delta_{-1}). \]

To this end, we construct a second spectral sequence \(2E\). For a fixed index \(i\), we define the \(u^i\)-degree by setting

\[ \text{deg}_{u^i} u^{i,s+1} = 1, \quad \text{deg}_{\delta\theta} \delta u^{i,s} = 1, \quad s \geq 0, \]

and other generators have \(u^i\)-degree zero. Accordingly, we decompose the differential \(\Delta_{-1}\) as follows

\[ \Delta_{-1} = \Delta_{-1,-1} + \Delta_{-1,0}, \quad \text{deg}_{u^i} \Delta_{-1,k} = k, \]

where

\[ \Delta_{-1,-1} = (-\lambda + u^i)f^i\delta\hat{d}_i, \quad \Delta_{-1,0} = \sum_{j \neq i} (-\lambda + u^j)f^j\delta\hat{d}_j. \]

Similar to our construction of \(1E\), we filtrate \(\mathcal{A}[\lambda]\{\delta u^{i,s}, \delta\theta_i^{s+1} | s \geq 0\}\) with \(\text{deg}_{u^i} + \text{deg}_{\theta}\), and construct the associated spectral sequence \(2E\). Then we have

\[ (2E_0, d_0) = (\mathcal{A}[\lambda]\{\delta u^{i,s}, \delta\theta_i^{s+1} | s \geq 0\}, \Delta_{-1, -1}) \]

and \(2E_1 = H(2E_0, d_0)\), with differential \(d_1 = \Delta_{-1,0}\). Since on the \(2E_2\) page the differential becomes 0, hence this spectral sequence becomes convergent on this page, i.e.

\[ H(\mathcal{A}[\lambda]\{\delta u^{i,s}, \delta\theta_i^{s+1} | s \geq 0\}, \Delta_{-1}) \cong 2E_2. \]

Let us compute \(2E_1 = H(2E_0, d_0)\). Take any element

\[ \omega = \sum_{s \geq 0} p_s \delta u^{i,s} + q_s \delta\theta_i^{s+1} \in 2E_0, \tag{5.7} \]

where we assume that \(\omega\) is homogeneous of degree \(p\) with respect to the \(\theta\)-degree. Then it is easy to see that

\[ d_0\omega = (-\lambda + u^i)f^i \sum_{s \geq 0} \hat{d}_i(p_s) \delta u^{i,s} + \left( \hat{d}_i(q_s) + (-1)^p p_s \right) \delta\theta_i^{s+1}. \]

So if \(d_0\omega = 0\), we have \(p_s = (-1)^{p+1} \hat{d}_i(q_s)\), and we can write

\[ \omega = \delta\hat{d}_i \left( \sum_{s \geq 0} (-1)^{p+1} q_s \delta u^{i,s} \right). \]
Let us denote 
\[ \tilde{B}_i = A\{\delta u^{i,s}, \ s \geq 0\}, \]
then the above computation shows that \( \ker d_0 \subset \delta \hat{d}_i(\tilde{B}_i)[\lambda] \). The inverse inclusion \( \delta \hat{d}_i(\tilde{B}_i)[\lambda] \subset \ker d_0 \) is clear, hence we have \( \ker d_0 = \delta \hat{d}_i(\tilde{B}_i)[\lambda] \).

To compute the image of the differential \( d_0 \), we take another element \( \omega \in 2E_0 \) represented in the form of (5.7). Observe that
\[
d_0 \omega = (-\lambda + u^i) f^j \sum_{s \geq 0} \hat{d}_i(p_s) \delta u^{i,s} + (\hat{d}_i(q_s) + (-1)^p p_s) \delta \theta_i^{s+1} \]
\[
= (-\lambda + u^i) f^j \sum_{s \geq 0} (p_s + (-1)p \hat{d}_i(q_s)) \delta u^{i,s},
\]
we see that \( \text{Im} \ d_0 = (-\lambda + u^i) \delta \hat{d}_i(\tilde{B}_i)[\lambda] \). Therefore we obtain the following lemma.

**Lemma 5.5.** We have
\[
2E_1 = \frac{\delta \hat{d}_i(\tilde{B}_i)[\lambda]}{(-\lambda + u^i) \delta \hat{d}_i(\tilde{B}_i)[\lambda]}.\]

Next let us compute \( 2E_2 = H(2E_1, d_1) \). We use the following identification:
\[
2E_1 = \frac{\delta \hat{d}_i(\tilde{B}_i)[\lambda]}{(-\lambda + u^i) \delta \hat{d}_i(\tilde{B}_i)[\lambda]} \cong \delta \hat{d}_i(\tilde{B}_i)
\]
by identifying \( \lambda \) with \( u^i \). Under this identification, we have
\[
d_1 = \sum_j (u^j - u^i) f^j \delta \hat{d}_j.
\]
Thus we can represent an element of \( 2E_1 \) as follows:
\[
\omega = \delta \hat{d}_i \sum_{s \geq 0} p_s \delta u^{i,s} = \sum_{s \geq 0} \hat{d}_i(p_s) \delta u^{i,s} + (-1)^p p_s \delta \theta_i^{s+1} \in 2E_1.
\]
If \( \omega \) is a cocycle, i.e. \( d_1 \omega = 0 \), by using the fact that \( \delta \hat{d}_i \) commutes with \( d_1 \) we obtain
\[
(5.8) \quad \sum_j (u^j - u^i) f^j \hat{d}_j(p_s) = 0.
\]
Let us consider the following decomposition according to (5.6) as follows:
\[
p_s = \hat{p}_s + \sum_k p^k_s + p^m_s,
\]
where \( \hat{p}_s \in \hat{C} \), \( p^k_s \in \hat{C}^m \) and \( p^m_s \in \hat{M} \). Then from the equation (5.8) and the obvious fact that \( \hat{d}_j \) annihilates \( p^k_s \) unless \( k = j \), we arrive at the following equations:
\[
(5.9) \quad (u^k - u^i) f^k \hat{d}_k(p^k_s) = 0, \quad \sum_j (u^j - u^i) f^j \hat{d}_j(p^m_s) = 0.
\]
Therefore for \( k \neq i \), we have \( \hat{d}_k(p^k_s) = 0 \) and by Poincaré Lemma (Lemma 12 in [5]), this implies \( p^k_s = \hat{d}_k(q^k_s) \) for some \( q^k_s \in \hat{C}^m \). For the term \( p^m_s \in \hat{M} \), after a rescaling by a non-zero factor, we may rewrite the second equation of (5.9) as \( \sum_{j \neq i} \hat{d}_j(p^m_s) = 0 \).
then by Proposition 11 in [4] (or a simplified version, Lemma 3.4 in [4]), we see that
\[ p_s^m = \sum_{j \neq i} \hat{d}_j(q_s^m) + h_s, \]
where \( q_s^m \in \mathcal{M} \) and \( h_s \in \hat{C}_i \). But such \( h_s \) must vanish since we have
\[ h_s = p_s^m - \sum_{j \neq i} \hat{d}_j(q_s^m) \in \hat{M}. \]
To conclude, we can rewrite \( p_s \) as
\[ p_s = \hat{p}_s + p_s^i + \sum_{j \neq i} \hat{d}_i q_s^i + d_1(q_s^m) = \hat{p}_s + p_s^i + d_1 \left( \sum_{j \neq i} \frac{q_j^i}{(w^j - u^i)f^j} + q_s^m \right). \]

Using again the fact that \( \delta \hat{d}_i \) and \( d_1 \) commutes, we see that
\[ \omega = \delta \hat{d}_i(\sum_{s \geq 0} p_s \delta u^{i,s}) = \delta \hat{d}_i(\sum_{s \geq 0} (\hat{p}_s + p_s^i) \delta u^{i,s} - d_1 \delta \hat{d}_i \sum_{s \geq 0} \sum_{j \neq i} \left( \frac{q_j^i}{(w^j - u^i)f^j} + q_s^m \right) \delta u^{i,s}). \]

So we see that for any \( \omega \in \ker d_1 \), there is \( \eta \in \hat{C}_i \{ \delta u^{i,s} \mid s \geq 0 \} \) such that \( [\omega] = [\delta \hat{d}_i(\eta)] \) in \( ^2E_2 \). Let us denote \( \hat{H}_i = \hat{C}_i \{ \delta u^{i,s} \mid s \geq 0 \} \). It is obvious that each element in \( \delta \hat{d}_i(\hat{H}_i) \) is annihilated by \( d_1 \) and different elements define different classes in \( ^2E_2 \), hence we arrive at
\[ ^2E_2 \cong \delta \hat{d}_i(\hat{H}_i). \]

To summarize all the results above, we obtain the following theorem.

**Theorem 5.6.** The first page \( ^1E_1 \) of the spectral sequence \( ^1E \) can be described as the direct sum of the following spaces:
\[ ^1E_1 \cong \hat{C}[\lambda] \{ \delta \theta \} \oplus \bigoplus_{i=1}^n \hat{d}_i(\hat{C}_i[\lambda]) \{ \delta \theta \} \oplus \bigoplus_{i=1}^n \delta \hat{d}_i(\hat{H}_i). \]

### 5.3. Computation of \( ^1E_2 \)

We will find suitable bidegrees \( (p, d) \) such that the cohomology \( ^1E_2 = H^p_d(^1E_1, \Delta_0) = 0 \), this means that the spectral sequence \( ^1E \) collapses on the second page for these bidegrees. Then we conclude that for the same bidegrees \( H^p_d(\Omega[\lambda], \partial[\lambda]) = 0 \).

We first write down explicitly the formula for \( \Delta_0 \). To avoid lengthy expressions, we will split \( \Delta_0 \) into \( \partial / \partial u^{i,s} \) part, \( \partial / \partial \delta u^{i,s} \) part, \( \partial / \partial \theta^i_s \) part and \( \partial / \partial \delta \theta^i_s \) part. In the following formulae, the index \( i \) is fixed and does not participate in the summation. These formulae are comparable to those given in [5].
The \( \partial/\partial u^{i,s} \) part of \( \Delta_0 \) reads:

\[
\sum_{s \geq 0} \sum_{t=0}^{s} \left( \begin{array}{c} s \\ t \end{array} \right) f^i s^{s-t+1} \theta_i^{s-t+1} \frac{\partial}{\partial u^{i,s}} + \sum_{s \geq 0} \sum_{t=0}^{s} \left( \begin{array}{c} s \\ t \end{array} \right) (-\lambda + u^i) \partial_{j} f^i s^{s-t+1} \theta_j^{s-t+1} \theta_i^{s-t} \frac{\partial}{\partial u^{i,s}} + \frac{1}{2} \sum_{s \geq 0} \sum_{t=0}^{s} \left( \begin{array}{c} s \\ t \end{array} \right) (-\lambda + u^i) \partial_{j} f^i s^{s-t+1} \theta_j^{s-t} \frac{\partial}{\partial u^{i,s}} + \frac{1}{2} \sum_{s \geq 0} \sum_{t=0}^{s} \left( \begin{array}{c} s \\ t \end{array} \right) (-\lambda + u^i) \partial_{j} f^i s^{s-t+1} \theta_j^{s-t} \frac{\partial}{\partial u^{i,s}}.
\]

The \( \partial/\partial \theta_i^{i,s} \) part of \( \Delta_0 \) reads:

\[
\sum_{s \geq 0} \sum_{t=0}^{s} \left( \begin{array}{c} s \\ t \end{array} \right) f^i s^{s-t+1} \theta_i^{s-t+1} \frac{\partial}{\partial u^{i,s}} + \sum_{s \geq 0} \sum_{t=0}^{s} \left( \begin{array}{c} s \\ t \end{array} \right) (-\lambda + u^i) \partial_{j} f^i s^{s-t+1} \theta_j^{s-t+1} \theta_i^{s-t} \frac{\partial}{\partial u^{i,s}} + \frac{1}{2} \sum_{s \geq 0} \sum_{t=0}^{s} \left( \begin{array}{c} s \\ t \end{array} \right) (-\lambda + u^i) \partial_{j} f^i s^{s-t+1} \theta_j^{s-t+1} \theta_i^{s-t} \frac{\partial}{\partial u^{i,s}} + \frac{1}{2} \sum_{s \geq 0} \sum_{t=0}^{s} \left( \begin{array}{c} s \\ t \end{array} \right) (-\lambda + u^i) \partial_{j} f^i s^{s-t+1} \theta_j^{s-t+1} \theta_i^{s-t} \frac{\partial}{\partial u^{i,s}}.
\]

The \( \partial/\partial \theta_i^s \) part of \( \Delta_0 \) reads:

\[
\frac{1}{2} \sum_{s \geq 0} \sum_{t=0}^{s} \left( \begin{array}{c} s \\ t \end{array} \right) f^i s^{s-t+1} \theta_i^{s-t+1} \frac{\partial}{\partial u^{i,s}} + \frac{1}{2} \sum_{s \geq 0} \sum_{t=0}^{s} \left( \begin{array}{c} s \\ t \end{array} \right) (-\lambda + u^i) \partial_{j} f^i s^{s-t+1} \theta_j^{s-t} \frac{\partial}{\partial u^{i,s}} + \frac{1}{2} \sum_{s \geq 0} \sum_{t=0}^{s} \left( \begin{array}{c} s \\ t \end{array} \right) (-\lambda + u^i) \partial_{j} f^i s^{s-t+1} \theta_j^{s-t} \frac{\partial}{\partial u^{i,s}} + \frac{1}{2} \sum_{s \geq 0} \sum_{t=0}^{s} \left( \begin{array}{c} s \\ t \end{array} \right) (-\lambda + u^i) \partial_{j} f^i s^{s-t+1} \theta_j^{s-t} \frac{\partial}{\partial u^{i,s}}.
\]
The $\partial/\partial \delta \theta_i^s$ part of $\Delta_0$ reads
\[
\frac{1}{2} \sum_{s \geq 0} \sum_{t=0}^s \left( s \right) f^i \delta(\theta_i^s \theta_i^{1+s-t}) \frac{\partial}{\partial \delta \theta_i^s} + \frac{1}{2} \sum_{s \geq 0} \sum_{j=0}^s \left( s \right) (-\lambda + u^j) \partial_j f^i \delta(\theta_j^{1+s-t}) \frac{\partial}{\partial \delta \theta_i^s} + \frac{1}{2} \sum_{s \geq 0} \sum_{j=0}^s \left( s \right) (-\lambda + u^j) f^i \delta(\theta_j^{1+s-t}) \frac{\partial}{\partial \delta \theta_i^s}.
\]

To compute the cohomology $H^1(E_1, \Delta_0)$, we introduce a third spectral sequence $^3E$ by defining the $\theta^1$-degree $\deg_{\theta^1} \theta^1_i = 1$ for $i = 1, \cdots, n$, and other generators have $\deg_{\theta^1} = 0$. Then we filtrate $^1E_1$ via
\[
F^r(^1E_1) = \{ \deg_{\theta^1} - \deg_{\theta^1} \omega \leq -r \mid \omega \in ^1E_1 \}.
\]

We also have the decomposition
\[
\Delta_0 = \Delta_{0,1} + \Delta_{0,0} + \Delta_{0,-1}, \quad \deg_{\theta^1} \Delta_{0,k} = k.
\]

The zeroth page $(^3E_0, E_0)$ is given by $(^1E_1, \Delta_{0,1})$. We want to determine the bidegrees $(p, d)$ such that $H^p_d(^3E_0, E_0) = 0$, then we conclude that for the same degrees $H^p_d(^1E_1, \Delta_0) = 0$. We first write down the explicit formula for $\Delta_{0,1}$.

The $\partial/\partial u^{i,s}$ part of $\Delta_{0,1}$ reads
\[
(-\lambda + u^i) f^i \theta_i^1 \delta u^{i,s} \frac{\partial}{\partial \delta u^{i,s}} + \frac{1}{2} \sum_{s \geq 1} \left( s \right) f^i \theta_i^1 u^{i,s} \frac{\partial}{\partial \delta u^{i,s}} + \frac{1}{2} \sum_{s \geq 1} \left( s \right) (-\lambda + u^i) \delta u^{i,s} \frac{\partial}{\partial \delta u^{i,s}}.
\]

The $\partial/\partial \delta u^{i,s}$ part of $\Delta_{0,1}$ reads
\[
\sum_{s \geq 0} f^i \theta_i^1 \delta u^{i,s} \frac{\partial}{\partial \delta u^{i,s}} + \sum_{s \geq 0} (-\lambda + u^i) \delta_j f^i \theta_j^1 \delta u^{j,s} \frac{\partial}{\partial \delta u^{i,s}}
\]

The $\partial/\partial \theta_i^1$ part of $\Delta_{0,1}$ reads
\[
\frac{1}{2} \sum_{s \geq 0} \left( s \right) (-\lambda + u^i) \partial_j f^i \left( s - 1 \right) \theta_j^1 \theta_i^s \frac{\partial}{\partial \theta_i^1} + \frac{1}{2} \sum_{s \geq 0} f^i \left( s - 1 \right) \theta_i^1 \theta_i^s \frac{\partial}{\partial \theta_i^1}.
\]
The $\partial/\partial \delta \theta^i$ part of $\Delta_0$

\[
\begin{align*}
\frac{1}{2} \sum_{s \geq 0; j} (-\lambda + u^j) \partial_i f^j s (s - 1) \theta^j \delta \theta^i \frac{\partial}{\partial \delta \theta^i} + \frac{1}{2} \sum_{s \geq 0} f^i s (s - 1) \theta^i \delta \theta^i \frac{\partial}{\partial \theta^i} \\
+ \frac{1}{2} \sum_{s \geq 0; j \neq 1} (\lambda + u^j) f^j \frac{\partial_j f^i}{f^i} (s + 1) (\delta \theta^i \theta^1 - \delta \theta^j \theta^1) \frac{\partial}{\partial \theta^i} + \sum_j (-\lambda + u^j) f^j \partial_j f^i (\lambda + u^j) \delta_{\theta^i} \frac{\partial}{\partial \theta^i}
\end{align*}
\]

To simplify the above expressions, we perform a rescaling on the generators of $\Omega$ (see \[4\]) as follows:

\[
\Psi: u^{i,s} \mapsto (f^i)^{s} u^{i,s}; \quad \theta^i \mapsto (f^i)^{s+1} \theta^i; \quad s \geq 0;
\]

\[
\Psi: \delta u^{i,s} \mapsto (f^i)^{s} \delta u^{i,s}; \quad \delta \theta^i \mapsto (f^i)^{s+1} \delta \theta^i; \quad s \geq 0.
\]

Note that this is NOT induced by a change of coordinate, but just an isomorphism of the space $\Omega$. The expression of $\Delta_{0,1}$ will be simplified after being conjugated by $\Psi$, and since $\Psi$ leaves all the decomposition of the complex invariant, this will not affect the computation of cohomology groups.

The following identities are easy to verify and helpful for our computation of the conjugated operator $\tilde{\Delta}_{0,1} = \Psi^{-1} \Delta_{0,1} \Psi$:

\[
\begin{align*}
\Psi^{-1} u^{i,s} \Psi &= (f^i)^{s} u^{i,s}; \quad \Psi^{-1} \theta^i \Psi = (f^i)^{s} \theta^i; \\
\Psi^{-1} \delta u^{i,s} \Psi &= (f^i)^{s} \delta u^{i,s}; \quad \Psi^{-1} \delta \theta^i \Psi = (f^i)^{s+1} \delta \theta^i; \\
\Psi^{-1} \frac{\partial}{\partial u^{i,s}} \Psi &= (f^i)^{s} \frac{\partial}{\partial u^{i,s}}, s \geq 1; \quad \Psi^{-1} \frac{\partial}{\partial \theta^i} \Psi = (f^i)^{s+1} \frac{\partial}{\partial \theta^i}, s \geq 0; \\
\Psi^{-1} \frac{\partial}{\partial \delta u^{i,s}} \Psi &= (f^i)^{s} \frac{\partial}{\partial \delta u^{i,s}}, s \geq 0; \quad \Psi^{-1} \frac{\partial}{\partial \delta \theta^i} \Psi = (f^i)^{s+1} \frac{\partial}{\partial \delta \theta^i}, s \geq 0; \\
\Psi^{-1} \frac{\partial}{\partial u^i} \Psi &= \frac{\partial}{\partial u^i} + \sum_{s \geq 0; j} \frac{\partial f^j}{f^j} f^j \left( \frac{s}{2} u^{i,s} \frac{\partial}{\partial u^{i,s}} + \frac{s + 1}{2} \theta^i \frac{\partial}{\partial \theta^i} \right).
\end{align*}
\]

Then by using the rotation coefficients

\[
\gamma_{ij} = \frac{1}{2} \left( \frac{f^i}{f^j} \right)^{1/2} \frac{\partial f^j}{f^j}
\]

defined for the diagonal metric $(f^1, \cdots, f^n)$, we can represent $\tilde{\Delta}_{0,1}$ in the form

\[
\tilde{\Delta}_{0,1} = \phi_1 + \phi_2 + \phi_3,
\]
where

\[ \phi_1 = - \sum_{s \geq 1; i, j} (-\lambda + u^i) \left( \frac{f_i}{f_j} \right)^{s+1} ((s + 2)\gamma_{ji}\theta_i^1 + s\gamma_{ij}\theta_j^1) u^{j,s} \frac{\partial}{\partial u^{j,s}} \]

\[ - \sum_{s \geq 0; i, j} (-\lambda + u^i) \left( \frac{f_i}{f_j} \right)^{s+1} ((s + 2)\gamma_{ji}\theta_i^1 + s\gamma_{ij}\theta_j^1) \delta u^{j,s} \frac{\partial}{\partial \delta u^{j,s}} \]

\[ + \sum_{s \geq 2; i, j} (-\lambda + u^i) \left( \frac{f_i}{f_j} \right)^{s+1} ((1 - s)\gamma_{ij}\theta_j^1 - (1 + s)\gamma_{ji}\theta_i^1) \delta \theta^s \frac{\partial}{\partial \delta \theta^s} \]

\[ \phi_2 = \sum_{s \geq 2; i, j} (-\lambda + u^i) \left( \frac{f_i}{f_j} \right)^{1+s} ((1 - s)\gamma_{ij}\theta_j^1 - (1 + s)\gamma_{ji}\theta_i^1) \delta \theta^s \frac{\partial}{\partial \delta \theta^s} \]

\[ + \sum_{i, j} (-\lambda + u^j) \gamma_{ij}\theta_j^1 \delta \theta^s \frac{\partial}{\partial \delta \theta^s} \]

\[ \phi_3 = \frac{1}{2} \sum_{s \geq 0; i, j} (-\lambda + u^i) \gamma_{ij}\theta_j^1 \delta \theta^s \frac{\partial}{\partial \delta \theta^s} \]

\[ + \sum_{i, j} \theta_i^1 \mathcal{E}_i + \sum_i (-\lambda + u^i) \theta_i^1 \frac{\partial}{\partial \theta^i} \]

here \( \mathcal{E}_i \) is an Euler-type vector field given by

\[ \mathcal{E}_i = \sum_{s \geq 1} \left( \frac{s+1}{2} \right) u^{i,s} \frac{\partial}{\partial u^{i,s}} + \sum_{s \geq 0} \left( \frac{s+1}{2} \right) \delta u^{i,s} \frac{\partial}{\partial \delta u^{i,s}} + \sum_{s \geq 0} \frac{s-1}{2} \left( \theta_i^1 \frac{\partial}{\partial \theta_i^s} + \delta \theta_i^s \frac{\partial}{\partial \delta \theta_i^s} \right) \]

We first simplify \( \tilde{\Delta}_{0,1} \) by the following observation (which is parallel to Lemma 3.6 of [1]).

**Lemma 5.7.** Both \( \phi_1 \) and \( \phi_2 \) act trivially on \( {}^1E_1 \).

**Proof.** We first recall from Theorem 5.6 that

\[ {}^1E_1 \cong \hat{\mathcal{C}}[\lambda] \{ \delta \theta \} \oplus \bigoplus_{i=1}^n \frac{\hat{d}_i(\hat{\mathcal{C}}[\lambda])}{(-\lambda + u^i)\hat{d}_i(\hat{\mathcal{C}}[\lambda])} \{ \delta \theta \} \oplus \bigoplus_{i=1}^n \hat{d}_i(\hat{\mathcal{H}}_i). \]

The vanishing of the action of \( \phi_1 \) on the cohomology

\[ \hat{\mathcal{C}}[\lambda] \{ \delta \theta \} \oplus \bigoplus_{i=1}^N \frac{\hat{d}_i(\hat{\mathcal{C}}[\lambda])}{(-\lambda + u^i)\hat{d}_i(\hat{\mathcal{C}}[\lambda])} \{ \delta \theta \} \]

of \( \hat{\mathcal{A}}[\lambda] \{ \delta \theta \} \) is a direct consequence of Lemma 3.6 of [1], and the vanishing of that of \( \phi_2 \) is obvious.

Next we consider action of \( \phi_1 \) and \( \phi_2 \) on

\[ H(\hat{\mathcal{A}}[\lambda] \{ \delta u^{i,s}, \delta \theta^{i,s+1} \}, \Delta_{-1}) = \delta \hat{d}_i(\hat{\mathcal{H}}_i) \]
for a fixed index $i$. By identifying $\lambda$ with $u^i$, we can represent $\omega$ in the form

$$\omega = \delta \hat{d}_i(\sum_{s \geq 0} p_s \delta u^{i,s}) = \sum_{s \geq 0} \hat{d}_i(p_s) \delta u^{i,s} + (-1)^{p_s} \delta \theta^{s+1}_i, \quad p_s \in \hat{C}_i.$$  

Under such an identification, the action of $\phi_1$ can be represented as

$$\phi_1 = - \sum_{s \geq 1; k, j} (-u^i + u^k) \left( \frac{f_k}{f_j} \right) \left( (s + 2) \gamma_{jk} \theta^i_k + s \gamma_{kj} \theta^i_j \right) u^{j,s} \frac{\partial}{\partial u^{k,s}}$$

$$- \sum_{s \geq 0; k, j} (-u^i + u^k) \left( \frac{f_k}{f_j} \right) \left( (s + 2) \gamma_{jk} \theta^i_k + s \gamma_{kj} \theta^i_j \right) \delta u^{j,s} \frac{\partial}{\partial \delta u^{k,s}}$$

$$+ \sum_{s \geq 2; k, j} (-u^i + u^j) \left( \frac{f_k}{f_j} \right) \left( (1 - s) \gamma_{kj} \theta^i_j - (1 + s) \gamma_{jk} \theta^i_k \right) \theta^s \frac{\partial}{\partial \theta^s}.$$  

Hence it is clear that

$$\phi_1 \omega \in \bigoplus_{j \neq i} \hat{C}_j \{ \delta u^{i,s}, \delta \theta^{s+1}_i | s \geq 0 \}.$$  

For the action of $\phi_2$ on $\omega$, we first regard $\omega$ as an element of $\delta \hat{d}_i(\hat{H}_i)[\lambda]$, then we observe that

$$\phi_2 \omega \in \bigoplus_j (-\lambda + u^j) \hat{C}_i[\lambda] \{ \delta \theta^{s+1}_j | s \geq 0 \}.$$  

We further make the decomposition $\phi_2 \omega = \alpha_1 + \alpha_2$, where

$$\alpha_1 \in \bigoplus_{j \neq i} (-\lambda + u^j) \hat{C}_i[\lambda] \{ \delta \theta^{s+1}_j | s \geq 0 \}, \quad \alpha_2 \in (-\lambda + u^j) \hat{C}_i[\lambda] \{ \delta \theta^{s+1}_i | s \geq 0 \}.$$  

Finally it is easy to see that from the definition of $\phi_3$ that

$$\phi_3 \omega \in \hat{C}_i[\lambda] \{ \delta u^{i,s}, \delta \theta^{s+1}_i | s \geq 0 \}.$$  

It follows from

$$\Delta_{0,1} \Delta_{-1} + \Delta_{-1} \Delta_{0,1} = 0$$  

that $\hat{\Delta}_{0,1} \omega = (\phi_1 + \phi_2 + \phi_3) \omega$ still lies in the cohomology group. Since the subspaces

$$\bigoplus_{j \neq i} \hat{C}_j \{ \delta u^{i,s}, \delta \theta^{s+1}_i | s \geq 0 \},$$

$$\bigoplus_{j \neq i} (-\lambda + u^j) \hat{C}_i[\lambda] \{ \delta \theta^{s+1}_j | s \geq 0 \},$$

$$\hat{C}_i[\lambda] \{ \delta u^{i,s}, \delta \theta^{s+1}_i | s \geq 0 \}$$

are disjoint and they are all invariant under the action $\Delta_{-1}$, we conclude that $\phi_1 \omega$, $\alpha_1$ and $\alpha_2 + \phi_3 \omega$ lie in the cohomology group.

From our computation of $\delta E_1$ given in (5.2) it follows that terms in the subspace

$$\bigoplus_{j \neq i} \hat{C}_j \{ \delta u^{i,s}, \delta \theta^{s+1}_i | s \geq 0 \}$$

vanish in $^1E_1$ and hence action of $\phi_1$ vanishes. Similarly, from the computation of the spectral sequence $^2E$ given in [5.2] it follows that the terms in the subspace
\[
\bigoplus_{j \neq 1} (-\lambda + u^j)\hat{C}_i[\lambda]\{\delta \theta_j^{s+1} | s \geq 0\}
\]
also vanish, hence $\alpha_1 = 0$. For the same reason, the multiples of $(-\lambda + u^i)$ in the subspace $\hat{C}_i[\lambda]\{\delta u^{i,s}, \delta \theta_j^{s+1} | s \geq 0\}$ are trivial in the cohomology as well, hence in particular $\alpha_2 = 0$ which implies the vanishing of $\phi_2$. The lemma is proved.

This above lemma shows that $\hat{\Delta}_{0,1} = \phi_3$ and therefore each summand in the decomposition (5.10) of $^1E_1$ is preserved by $\hat{\Delta}_{0,1}$. In what follows, we will show that for some bidegrees $(p,d)$ the action of $\hat{\Delta}_{0,1}$ is acyclic on each summand when restricted to the elements with super degree $p$ and differential degree $d$.

**Lemma 5.8.** We have $H^p_0(\hat{C}[\lambda]\{\delta \theta\}, \hat{\Delta}_{0,1}) = 0$ unless
\[
d = 0, \cdots, n; \quad p = d + 1, \cdots, d + n + 1.
\]

**Proof.** Indeed, possible bidegrees $(p,d)$ of elements of in $\hat{C}[\lambda]\{\delta \theta\}$ are precisely those excluded in the lemma. The lemma is proved. \(\square\)

To compute the cohomology of $\hat{\Delta}_{0,1}$ on the space
\[
\frac{\hat{d}_i(\hat{C}_i[\lambda])}{(-\lambda + u^i)d_i(\hat{C}_i[\lambda])}\{\delta \theta\},
\]
let us first identify this space with $\hat{d}_i(\hat{C}_i)\{\delta \theta\}$ by sending $\lambda$ to $u^i$. After this identification, the action of $\hat{\Delta}_{0,1}$ reads (we keep the same notation)
\[
\hat{\Delta}_{0,1} = \sum_j \theta^1_j \mathcal{E}_j - \frac{1}{2} \sum_{j,k} (-u^i + u^j) \frac{\partial f^k}{f^j} \theta^1_j \delta \theta_k \frac{\partial}{\partial \delta \theta_k} + \sum_j (-u^i + u^j) f^j \theta^1_j \frac{\partial}{\partial u^j}
\]

\[
+ \sum_{j,k} (-u^i + u^j) (\gamma_{kj} \theta^1_j - \gamma_{jk} \theta^1_k) \left( \theta_j \frac{\partial}{\partial \theta_k} + \delta \theta_j \frac{\partial}{\partial \delta \theta_k} \right).
\]

**Lemma 5.9.** We have $H^p_4(\hat{d}_i(\hat{C}_i)\{\delta \theta\}, \hat{\Delta}_{0,1}) = 0$ unless
\[
d = 2, \cdots, n + 3; \quad p = d, \cdots, d + n.
\]

**Proof.** To compute the cohomology, we introduce a forth spectral sequence $^4E$ given by a filtration of $\hat{d}_i(\hat{C}_i)\{\delta \theta\}$ using the $\theta^1_i$-degree, which is defined by $\deg_{\theta^1_i} \theta^1_i = 1$ and by setting the degrees of other generators to be zero. By decomposing the differential $\hat{\Delta}_{0,1}$ with respect to the $\theta^1_i$-degree, we conclude that the zeroth page of this spectral sequence is given by $^4E_0 = \hat{d}_i(\hat{C}_i)\{\delta \theta\}$ with the differential $\mathcal{D}_i$ given by
\[
\mathcal{D}_i = \theta^1_i \mathcal{E}_i + \sum_j (u^i - u^j) \gamma_{ji} \theta^1_j \left( \theta_j \frac{\partial}{\partial \theta_i} + \delta \theta_j \frac{\partial}{\partial \delta \theta_i} \right).
\]
The idea to compute \( H(4E_0, D_i) \) is as follows. We first make the decomposition

\[
d_i(\mathring{C}_i)\{\delta \theta_i\} = d_i(\mathring{C}_i)\{\delta \theta_i\} \oplus \bigoplus_{j \neq i} d_i(\mathring{C}_i)\{\delta \theta_j\}.
\]

Note that \( \bigoplus_{j \neq i} d_i(\mathring{C}_i)\{\delta \theta_j\} \) is an invariant subspace of \( D_i \) while \( d_i(\mathring{C}_i)\{\delta \theta_i\} \) is not. Nevertheless, if we can find proper bidegrees \((p, d)\) such that \( D_i \) is acyclic on \( \bigoplus_{j \neq i} d_i(\mathring{C}_i)\{\delta \theta_j\} \cap \Omega_d^p \), then from Lemma 5.2 we know that the cohomology of \( d_i(\mathring{C}_i)\{\delta \theta_i\} \cap \Omega_d^p \) is given by the cohomology of the space \( d_i(\mathring{C}_i)\{\delta \theta_i\} \cap \Omega_d^p \) with the differential given by the projection of \( D_i \).

Take any monomial \( m \) in \( u^{i_n}, \theta_i^{s+1} \) for \( s \geq 1 \) and any monomial \( g \in \mathring{C} \). For \( j \neq i \), we have

\[
D_i(\mathring{g}\mathring{d}_i(m)\delta \theta_j) = \theta_i^1 \left( \sum_j (u^j - u^j) \gamma_{ji} \theta_j \frac{\partial g}{\partial \theta_i} + (w_i(g) + w_i(m) - 1) g \right) d_i(m)\delta \theta_j,
\]

here for a monomial \( \omega \in \Omega \), the rational number \( w_i(\omega) \) is defined by \( \mathring{C}_i \omega = w_i(\omega) \omega \).

So for any fixed \( m \) and \( j \neq i \), the subspace \( \mathring{C}\mathring{d}_i(m)\{\delta \theta_j\} \) is invariant under the action of \( D_i \), and we can concentrate first on computing the cohomology of this subspace. The following argument is basically the same as that of Lemma 3.10 of [4], nonetheless we still write it down for the convenience of readers. Let us decompose \( \mathring{C} \) in the form

\[
\mathring{C} = \mathring{C}_0^i \oplus \theta_i \mathring{C}_0^i,
\]

where \( \mathring{C}_0^i \) is the subspace of \( \mathring{C} \) spanned by monomials that do not contain \( \theta_i \). For \( g \in \mathring{C}_0^i \), the action of \( D_i \) is given by

\[
(5.12) \quad D_i(\mathring{g}\mathring{d}_i(m)\delta \theta_j) = \theta_i^1 (w_i(m) - 1) \mathring{d}_i(m)\delta \theta_j,
\]

therefore the subspace \( \mathring{C}_0^i \mathring{d}_i(m)\{\delta \theta_j\} \) is acyclic. Indeed, since \( w_i(m) \geq \frac{3}{4} \) if \( \mathring{d}_i(m) \neq 0 \), so a nonzero cocycle \( \mathring{C}^i \mathring{d}_i(m)\{\delta \theta_j\} \) must contain \( \theta_i^1 \). Assume we have a cocycle \( \theta_i^1 h \) for some \( h \), then it follows from \( 5.12 \) the existence of a suitable constant \( c \) such that \( \theta_i^1 h = c D_i(h) \).

Due to Lemma 5.2 in order to compute the cohomology of \( \mathring{C}\mathring{d}_i(m)\{\delta \theta_j\} \) we only need to compute the one for the subspace \( \theta_i \mathring{C}_0^i \{\delta \theta_j\} \). The projection of \( D_i \) on \( \theta_i \mathring{C}_0^i \mathring{d}_i(m) \) is just a multiplication by \( \theta_i^1 (w_i(m) - \frac{3}{4}) \), which is acyclic if \( w_i(m) \neq \frac{3}{4} \). So the nontrivial cocycles are given by elements of \( \theta_i \mathring{C}_0^i \mathring{d}_i(u^{i-1})\delta \theta_j = \mathring{C}_0^i \theta_i \theta_i^2 \delta \theta_j \), which have the following possible bidegrees:

\[
d = 2, \ldots, 2 + n; \quad p = d + 1, \ldots, d + n.
\]

Thus unless a bidegree \((p, d)\) takes the values given above, the subcomplex

\[
\bigoplus_{j \neq i} d_i(\mathring{C}_i)\{\delta \theta_j\} \cap \Omega_d^p
\]

is acyclic.

Thus to compute the cohomology of \( 4E_0 \), we only need to consider the cohomology of the space \( d_i(\mathring{C}_i)\{\delta \theta_i\} \) due to Lemma 5.2. The differential is the projection of \( D_i \) which can be represented as

\[
\theta_i^1 \mathring{E}_i + \sum_j (u^j - u^j) \gamma_{ji} \theta_i^1 \theta_j \frac{\partial}{\partial \theta_i}.
\]
Now by repeating the argument above we see that the nontrivial cocycles are of the form \( \hat{C}_0^i \delta \theta_i \), or \( \hat{C}_0^i \theta_i^2 \delta \theta_i \). By counting the possible bidegrees of these elements, we complete the proof of the lemma. \( \square \)

**Remark 5.1.** By a more careful analysis, we can prove that actually
\[
H_{n+3}^{2n+3}(\delta \hat{d}_i(\hat{C}_i)^{(\delta \theta)}, \hat{\Delta}_{0,1}) = 0.
\]
But this is not important for our consideration of the deformation problem.

Finally we are to compute the cohomology of \( \hat{\Delta}_{0,1} \) on the space \( \delta \hat{d}_i(\mathcal{H}_i) \). Recall that on this space we have to identify \( \lambda \) with \( u^i \), and therefore the differential reads
\[
\hat{\Delta}_{0,1} = -\frac{1}{2} \sum_{s \geq 0, k,j} (-u^i + u^j) \partial f^k_j \theta^1 \left( s \delta^k u^{i,s} \frac{\partial}{\partial u^{k,s}} + (s + 1) \delta u^{k,s} \right) + \sum_{k,j} (-u^i + u^j)(\gamma_{kj} \theta_j^1 - \gamma_{jk} \theta_k^1) \partial \frac{\partial}{\partial \theta_k} + \sum_j (u^i - u^j) \theta_j^1 \partial \frac{\partial}{\partial u^j}.
\]

**Lemma 5.10.** We have \( H_d^p(\delta \hat{d}_i(\mathcal{H}_i), \hat{\Delta}_{0,1}) = 0 \) unless
\[
d = 3, \ldots, n + 3; \quad p = d, \ldots, d + n - 1.
\]

**Proof.** The idea is very similar to the proof of Lemma 5.9. We introduce another spectral sequence by filtrating \( \delta \hat{d}_i(\mathcal{H}_i) \) using \( \deg \theta_i \). The differential on the zeroth page reads
\[
D_i = \theta_i^1 \mathcal{E}_i + \sum_j (u^i - u^j) \gamma_{ij} \theta_j^1 \theta_i \frac{\partial}{\partial \theta_i}.
\]

Let us denote \( \psi = \sum_j (u^i - u^j) \gamma_{ij} \theta_j^1 \theta_i \frac{\partial}{\partial \theta_i} \). For any monomial \( m \) in \( u^{i,s}, \theta_i^{s+1} \) with \( s \geq 1 \), and any monomial \( g \in \hat{C} \), we have
\[
D_i \delta \hat{d}_i(g \delta u^{i,s}) = D_i(g \hat{d}_i(m) \delta u^{i,s} + (-1)^p g \delta \theta_i^{s+1})
\]
\[
= \theta_i^1 \left( w_i(g) + w_i(m) - 1 + \frac{s}{2} + 1 \right) g \hat{d}_i(m) \delta u^{i,s}
\]
\[
+ (-1)^p \theta_i^1 \left( w_i(g) + w_i(m) + \frac{s}{2} \right) g m \delta \theta_i^{s+1}
\]
\[
+ \theta_i^1 \left( \psi(g) \hat{d}_i(m) \delta u^{i,s} + (-1)^p \psi(g) m \delta \theta_i^{s+1} \right)
\]
\[
= - \delta \hat{d}_i \left( \theta_i^1 \psi(g) m \delta u^{i,s} + \theta_i^1 \left( w_i(g) + w_i(m) + \frac{s}{2} \right) g m \delta u^{i,s} \right).
\]

So the subspace \( \hat{C} \delta \hat{d}_i(m \delta u^{i,s}) \) is invariant under the action of \( D_i \). Similarly, we make the decomposition \( \hat{C} = \hat{C}_0 \oplus \theta_i \hat{C}_0 \). For \( g \in \hat{C}_0 \), it is easy to see that
\[
D_i \delta \hat{d}_i(g \delta u^{i,s}) = - \delta \hat{d}_i \left( \theta_i^1 \left( w_i(m) + \frac{s}{2} \right) g m \delta u^{i,s} \right).
\]

On the other hand, for a monomial in \( u^{i,s}, \theta_i^{s+1} \) with \( s \geq 1 \), we must have that \( w_i(m) \geq \frac{1}{2} \), thus the subspace \( \hat{C}_0 \delta \hat{d}_i(m \delta u^{i,s}) \) is acyclic. By applying Lemma 5.2 again, we know that
we only need to consider the cohomology of $\theta_i \hat{\Omega}_1^d \delta \hat{\omega}_i (m \delta u^{i,s})$ with the differential being the projection of $D_i$ given by the multiplication of

$$-\theta_i^1 \left( w_i(m) + \frac{s}{2} - \frac{1}{2} \right).$$

So the only possible nontrivial cocycle is given by the case when $s = 0$ and $w_i(m) = \frac{1}{2}$, i.e. when the monomial $m$ is of the form

$$\theta_i \hat{\Omega}_1^d \delta \hat{\omega}_i (\theta_i^2 \delta u^i) = \hat{\Omega}_1^d \theta_i \theta_i^2 \delta \theta_i^1.$$

By counting the possible bidegrees of such elements, we complete the proof of the lemma.

Let us summarize all the results obtained above. We first construct a spectral sequence $^1E$ to compute the cohomology group $H^p_\theta(\Omega[\lambda], \partial_\lambda)$. Then we transform the computation to finding suitable bidegrees $(p, d)$ such that the second page $^1E_2 = 0$ when restricted to $\Omega^p_{d^*}$. To this end we introduce a third spectral sequence $^3E$, and we conclude that the first page $^3E_1$ vanishes for suitable bidegrees in Lemmas 5.8–Lemma 5.10, thus $^3E$ converges to $^1E_2$ and so $^1E_2$ vanishes. Consequently $^1E$ converges to $H^p_\theta(\Omega[\lambda], \partial_\lambda)$. In this way, we prove Theorem 5.1.

The following proposition is an illustration of applications of the variational bihamiltonian cohomology.

**Proposition 5.11.** Let $(P_0, P_1)$ be a semisimple bihamiltonian structure of hydrodynamic type, and let $X \in \text{Der}(A)^0_1$ commute with $\partial_x$, $D_{P_0}$ and $D_{P_1}$. Then for any deformation $(\bar{P}_0, \bar{P}_1)$ of $(P_0, P_1)$, there exists a unique $\bar{X} \in \text{Der}(A)^0_{k+1}$ with leading term given by $X$ such that $\bar{X}$ commutes with $\partial_x$, $D_{\bar{P}_0}$ and $D_{\bar{P}_1}$.

**Proof.** Let us decompose $\bar{X}$ and $\bar{P}_a$ according to the differential degree as follows:

$$\bar{X} = X^{[0]} + \sum_{k \geq 1} X^{[k]}, \quad X^{[k]} \in \text{Der}(A)^0_{k+1}, \quad X^{[0]} = X;$$

$$\bar{P}_a = P_a + \sum_{k \geq 1} P_a^{[k]}, \quad P_a^{[k]} \in \hat{F}^2_{k+1}, \quad a = 0, 1.$$

The condition that $\bar{X}$ commutes with $D_{\bar{P}_a}$ is equivalent to the following equations:

$$(5.13) \quad \left[ D_{P_a}, X^{[k]} \right] + \sum_{i=1}^k \left[ D_{P_{a}^{[i]}}, X^{[k-i]} \right] = 0, \quad k \geq 1, \quad a = 0, 1.$$

To prove the uniqueness of $\bar{X}$, we only need to show that if $X = 0$ then $\bar{X} = 0$. Indeed, if $X = 0$, then we have the following equation for $X^{[1]}$:

$$\left[ D_{P_a}, X^{[1]} \right] = 0,$$

which is equivalent to $D_a X^{[1]} = 0$. Here $X^{[k]} \in \Omega^1_{k+1}$ is the 1-form corresponds to $X^{[k]}$. Recall that

$$V BH^1_d(\bar{Q}, \bar{D}_0, \bar{D}_1) = \bar{\Omega}_1^d \cap \ker \bar{D}_0 \cap \ker \bar{D}_1,$$
then by the vanishing of $\text{VBH}_{2}^{\Omega}(\tilde{\Omega}, \tilde{D}_{0}, \tilde{D}_{1})$ given by Theorem 6.3, we conclude that $X[1] = 0$. In a similar way, we can prove recursively that $X[k] = 0$ and the uniqueness is proved.

The proof of the existence is very similar to the one given in Sect. 6.2 and we omit the details here. For a sketch of the proof, we may first assume $P_{a}[1] = 0$ and hence we can choose $X[1] = 0$, then by using $\text{VBH}_{2}^{\Omega}(\tilde{\Omega}, \tilde{D}_{0}, \tilde{D}_{1}) = 0$, we can recursively solve the equations (6.1) to obtain $X[\geq 2]$. The proposition is proved.

**Remark 5.2.** This result is a generalization of the fact that a bihamiltonian vector filed is uniquely determined by its leading term. This generalized version can be applied to the case when the flows are not in the space $\text{Der}(\tilde{\mathcal{A}})^{D}$. Typical examples of such kind of flows are given by Virasoro symmetries.

### 6. Conformal bihamiltonian structures

#### 6.1. Conformal Bihamiltonian Structures of hydrodynamic type.

This section is devoted to the proof of the Theorem 6.2. To this end, we consider a semisimple bihamiltonian structure $(P_{0}, P_{1})$ of hydrodynamic type which is represented as in (2.4).

Assume that the bihamiltonian structure $(P_{0}, P_{1})$ is conformal, we are going to find a derivation $E$ such that it satisfied the equation (2.5). We first make the following decomposition

$$E = E^{[0]} + E^{[\geq 1]}, \quad E^{[0]} \in \text{Der}(\tilde{\mathcal{A}})^{0}, \quad E^{[\geq 1]} \in \text{Der}(\tilde{\mathcal{A}})^{\geq 1}.$$

According to the equation (2.5), it is easy to see that $E^{[\geq 1]}$ is a vector field that commutes with $D_{P_{0}}$ and $D_{P_{1}}$ and thus it is an element of $\text{VBH}_{\geq 1}(\tilde{\Omega}, \tilde{D}_{0}, \tilde{D}_{1})$. In what follows we assume that $E \in \text{Der}(\tilde{\mathcal{A}})^{0}$.

Denote $E(u^{i}) = F^{i}$ and $E(\theta_{j}) = \sum_{j} G_{j}^{i} \theta_{j}$, where $F^{i}$ and $G_{j}^{i}$ are some smooth functions of $u^{1}, \ldots, u^{n}$. We are to solve these functions from the equations given in (2.5).

Firstly, from the equation $[E, \partial_{e}] = \mu \partial_{e}$ it follows that

$$E(u^{i+s}) = \partial_{s}^{e} G_{e}^{i} + s \mu u^{i+s}, \quad E(\theta_{j}) = \sum_{j} \partial_{s}^{e} (G_{j}^{i} \theta_{j}) + s \mu \theta_{j}^{s}.$$

By comparing the $\theta_{j}^{i}$ coefficients ($j \neq i$) of both sides of the equation (6.1)

$$[E, D_{P_{a}}]u^{i} = \lambda_{a} D_{P_{a}}(u^{i})$$

for $a = 0, 1$ we obtain the following equations:

$$f^{i} G_{j}^{i} - (\partial_{j} F^{i}) f^{j} = 0, \quad j \neq i$$

$$u^{i} f^{i} G_{j}^{i} - (\partial_{j} F^{i}) u^{j} f^{i} = 0, \quad j \neq i.$$

From these equations it follows that $G_{j}^{i} = 0$ for $j \neq i$. We also compare the $\theta_{j}^{i}$ coefficients of both sides of the equation (6.1) to obtain

$$\sum_{j} \partial_{j} f^{i} F^{j} + f^{i} (G_{i}^{i} + \mu) - \partial_{i} F^{i} f^{i} = \lambda_{0} f^{i},$$

$$\sum_{j} \partial_{j} (u^{i} F^{j}) F^{j} + u^{i} f^{i} (G_{i}^{i} + \mu) - \partial_{i} F^{i} u^{i} f^{i} = \lambda_{1} u^{i} f^{i}.$$
We solve these equations to arrive at

\[ F^i = (\lambda_1 - \lambda_0)u^i, \quad G^i = \lambda_1 - \mu - \frac{1}{f^i} \sum_j (\lambda_1 - \lambda_0)u^j \partial_j f^i. \]

Together with the fact that \( G^i = 0 \) for \( j \neq i \), we conclude that

\[ E(u^i) = (\lambda_1 - \lambda_0)u^i, \quad E(\theta_i) = \left( \lambda_1 - \mu - \frac{1}{f^i}E(f^i) \right) \theta_i. \]

For simplicity, we will denote \( E(\theta_i) = G^i \theta_i \).

Next we compare the \( \theta_i \) coefficients of the equation

\[ [E, D_{P_0}]u^i = \lambda_0 D_{P_0}(u^i) \]

to arrive at the following equation:

\[
\begin{align*}
\sum_{j,k} f^i \partial_j \partial_k f^i - \partial_j f^i \partial_k f^i &= \frac{(\lambda_0 - \lambda_1)u^k u^{j,i} + \sum_j \partial_j f^i (\lambda_0 - \lambda_1)u^{j,i}}{f^i} + \frac{1}{2} \sum_j \partial_k \partial_j f^i (\lambda_1 - \lambda_0)u^{j,i} + \frac{1}{2} \sum_j \partial_j f^i u^{j,i} G^i = \frac{1}{2} \sum_j (\lambda_0 - \mu) \partial_j f^i u^{j,i}. \\
\end{align*}
\]

By further comparing the \( u^{j,i} \) coefficients for both sides of the above equation, one can show that the above equation is actually equivalent to

\[ E \left( \frac{\partial_j f^i}{f^i} \right) = (\lambda_0 - \lambda_1) \left( \frac{\partial_j f^i}{f^i} \right), \quad \forall i, j. \]

For \( j \neq i \), compare the coefficients of \( u^{j,i} \theta_j \) and \( u^{j,i} \theta_j \) on both sides of the equation (6.2), one obtains the following equations:

\[ E \left( f^j f^i \frac{\partial_j f^i}{f^i} \right) + f^i \partial_j f^i \partial_j G^j = (\lambda_0 - \mu) f^i \frac{\partial_j f^i}{f^i} \partial_j \theta_i, \quad j \neq i. \]

Finally let us compare the coefficients on both sides of the equation

\[ [E, D_{P_0}]\theta_i = \lambda_0 D_{P_0}(\theta_i). \]

For \( j \neq i \), we consider first the coefficients of \( \theta_i \theta_j \) and obtain that

\[ E \left( f^j f^i \frac{\partial_j f^i}{f^i} \right) + f^j \frac{\partial_j f^i}{f^i} G^j = (\lambda_0 - \mu) f^j \frac{\partial_j f^i}{f^i} - \partial_j G^i f^j, \quad j \neq i. \]

Therefore it is easy to conclude that \( \partial_j G^i = 0 \) for \( j \neq i \) by comparing (6.6) and (6.5).

We also compare the coefficients of \( \theta_i \theta_1 \), after a straightforward computation, we arrive at an equation

\[ E \left( \frac{\partial_j f^i}{f^i} \right) = (\lambda_0 - \lambda_1) \left( \frac{\partial_j f^i}{f^i} \right) - \partial_j G^i. \]

Hence we conclude that \( \partial_j G^i = 0 \) by comparing (6.7) with (6.3), and \( G^i \) must be a constant. This means that there exists real numbers \( \alpha^i \) such that \( E(f^i) = \alpha_i f^i \), which
gives the condition \(2.6\) with \(d^i\) determined by \(\alpha^i = (\lambda_1 - \lambda_0)d^i\). Substitute the expression for \(G^j\) into \(6.4\), we obtain the condition \(2.7\), and hence the ‘only if’ part of the Theorem 2.2 is proved.

The ‘if’ part of the Theorem 2.2 can be checked easily by a straightforward computation and hence the theorem is proved.

6.2. Deformed Conformal Bihamiltonian Structures. In this section, we will use the theory of variational bihamiltonian cohomology to prove Theorem 2.3. Let \((\tilde{P}_0, \tilde{P}_1)\) be a conformal semisimple bihamiltonian structure of hydrodynamic type as described in Theorem 2.2 and \((\tilde{P}_0, \tilde{P}_1)\) be any of its deformation. We are going to find a deformation \(\tilde{E} \in \text{Der}(\hat{A})^0\), such that

\[
(6.8) \quad \left[\tilde{E}, \partial_x\right] = \mu \partial_x; \quad \left[\tilde{E}, D_{\tilde{P}_a}\right] = \lambda a D_{\tilde{P}_a}, \quad a = 0, 1.
\]

We first decompose \(\tilde{P}_a\) and \(\tilde{E}\) as follows:

\[
\tilde{P}_a = \sum_{k \geq 0} \tilde{P}_a^{[k]}, \quad \tilde{P}_a^{[k]} \in \tilde{F}_{k+1}; \quad \tilde{E} = \sum_{k \geq 0} E^{[k]}, \quad E^{[k]} \in \text{Der}(\hat{A})^0_k.
\]

We also make the same decomposition for both sides of the equation \(\left[\tilde{E}, D_{\tilde{P}_a}\right] = \lambda a D_{\tilde{P}_a}\) to obtain

\[
(6.9) \quad \sum_{i=0}^{l-1} \left[ E^{[i]}, D_{\tilde{P}_a^{[l-i]}}, a = 0, 1; \quad l \geq 1.
\]

According to the result of bihamiltonian cohomology, we can assume (by doing a Miura transformation if necessary) that \(\tilde{P}_a^{[1]} = 0\), hence we can choose \(E^{[1]} = 0\). Let us proceed to find the deformations \(\tilde{E}^{\geq 2}\). Let us denote

\[
W_{a}^{[l]} = \sum_{i=0}^{l-1} \left[ E^{[i]}, D_{\tilde{P}_a^{[l-i]}}, a = 0, 1; \quad l \geq 1.
\]

then we can recursively determine the deformation by solving the equation

\[
\left[ D_{\tilde{P}_a^{[0]}}, E^{[l]}\right] = W_{a}^{[l]}.
\]

Lemma 6.1. Assume that we have found \(E^{[i]}\) for all \(i < k\) such that the equations in \(6.3\) are satisfied for \(l < k\). Then we have the following identity:

\[
\left[ W_{b}^{[k]}, D_{\tilde{P}_a^{[0]}}\right] + \left[ W_{a}^{[k]}, D_{\tilde{P}_a^{[0]}}\right] = 0, \quad a, b = 0, 1.
\]

Proof. The proof is a straightforward computation by using the equations

\[
\left[ E^{[i]}, D_{\tilde{P}_a^{[0]}}\right] = W_{a}^{[i]}, \quad i < k; \quad \left[ D_{\tilde{P}_a}, D_{\tilde{P}_a}\right] = 0,
\]

and the (graded) Jacobi identities. \(\square\)

In order to apply the theory of variational bihamiltonian cohomology, we need the following fact.

Lemma 6.2. We have \(\left[ E^{[l]}, \partial_x\right] = 0\) and \(\left[ W_{a}^{[l]}, \partial_x\right] = 0\) for \(l \geq 1\).
Proof. This follows directly from the assumption \([\tilde{E}, \partial_x] = \mu \partial_x\).

As a consequence of the above lemma, we can regard the vector fields \(E[[l]]\) and \(W_a[[l]]\) as elements of the space \(\tilde{\Omega}\). We denote the corresponding 1-forms by \(E[[l]]\) and \(W_a[[l]]\) respectively. More precisely, we have

\[
\mathcal{E}[[l]] \in \tilde{\Omega}_1^1, \quad W_a[[l]] \in \tilde{\Omega}_{l+1}^2.
\]

Now the equations we are to solve can be rewritten in the form

\[
(6.10) \quad \tilde{D}_a E[[l]] = W_a[[l]], \quad a = 0, 1, \quad l \geq 2,
\]

and Lemma 6.1 gives the following conditions

\[
\tilde{D}_a W_a[[l]] = 0, \quad a = 0, 1.
\]

Thus by using the triviality of the variational Hamiltonian cohomology we can find an element \(\gamma[[l]] \in \tilde{\Omega}_1^1\) such that

\[
\tilde{D}_0 \gamma[[l]] = W_a[[l]].
\]

Then the solution of \(\mathcal{E}[[l]]\) can be represented by

\[
\mathcal{E}[[l]] = \gamma[[l]] + \tilde{D}_0 \alpha[[l]], \quad \alpha[[l]] \in \Omega_{l-1}^0.
\]

The 1-form \(\alpha[[l]]\) should be determined by the equation

\[
\tilde{D}_1 \mathcal{E}[[l]] = \tilde{D}_1 \left( \gamma[[l]] + \tilde{D}_0 \alpha[[l]] \right) = W_a[[l]].
\]

By using Lemma 6.1 again, we see that \(W_a[[l]] - \tilde{D}_1 \gamma[[l]]\) lies in \(\ker \tilde{D}_0 \cap \ker \tilde{D}_1\). Therefore from the fact that \(VBH_{2,4}(\tilde{\Omega}, D_0, \tilde{D}_1) = 0\) it follows that we can always solve the above equation to obtain \(\alpha[[l]]\) for \(l \geq 3\).

Now let us try to find \(\mathcal{E}[[2]]\) by solving the equations in (6.10) for \(l = 2\). We will work in the canonical coordinates \(u^1, \ldots, u^n\) of \((P_0, P_1)\). According to the results in [31], we can choose a Miura type transformation such that \(\tilde{P}_0 = P_0, \tilde{P}_1[[1]] = 0\) and the derivation \(D_{P_1[[2]]}\) is given by the 1-form \(\tilde{D}_0 \mathcal{T}\), where \(\mathcal{T}\) is given by

\[
\mathcal{T} = \int \left( D_{P_2} \sum_i c_i(u^i) u^{i,1} \log u^{i,1} - D_{P_0} \sum_i u^i c_i(u^i) u^{i,1} \log u^{i,1} \right).
\]

Here the functions \(c_i(u^i)\) are the central invariants of the deformed bihamiltonian structure \((\tilde{P}_0, \tilde{P}_1)\). Since \(D_0[[2]] = 0\), we can choose \(\gamma[[2]] = 0\). Then we have

\[
W_1[[2]] = \left[ E[[0]], D_{P_1[[2]]} \right] - \lambda_1 D_{P_1[[2]]},
\]
here $E^{[0]} = E$ is described as in Theorem 4.2. Let us use $T$ to denote the derivation given by the 1-form $\mathcal{T}$, then we have
\[
W_i^{[2]} = \left[ E^{[0]}, [D_{P_0}, T] \right] - \lambda_1 [D_{P_0}, T]
= - \left[ D_{P_0}, \left[ T, E^{[0]} \right] \right] - \left[ T, \left[ E^{[0]}, D_{P_0} \right] \right] - \lambda_1 [D_{P_0}, T]
= - \left[ D_{P_0}, \left[ T, E^{[0]} \right] \right] + \lambda_0 [D_{P_0}, T] - \lambda_1 [D_{P_0}, T]
= \left[ D_{P_0}, (\lambda_0 - \lambda_1)T - \left[ T, E^{[0]} \right] \right].
\]

Denote by $\beta \in \tilde{\Omega}_2$ the 1-form corresponding to the derivation $(\lambda_0 - \lambda_1)T - \left[ T, E^{[0]} \right]$, then we see that
\[
\tilde{D}_1 \mathcal{E}^{[2]} = \tilde{D}_1 \tilde{D}_0 \mathcal{E}^{[2]} = W_i^{[2]} = \tilde{D}_0 \beta.
\]

To solve this equation for $\mathcal{E}^{[2]}$, we need to check that $[\tilde{D}_0 \beta] \in VBH^2_3(\tilde{\Omega}, \tilde{D}_0, \tilde{D}_1)$ is trivial. According to Lemma 4.2 we only need to check that the indices $\text{ind}_i(\beta)$ for $i = 1, \cdots, n$ vanish.

We first note that
\[
\text{ind}_i(T) = -3c_i(u^i).
\]

In another word, if we represent $T$ in the form
\[
T = \int \sum_i X^i \delta u^i + Y^i \delta \theta_i, \quad X^i \in \tilde{A}_2^1, \quad Y^i \in \tilde{A}_2^0,
\]
where $X^i$ and $Y^i$ are given by
\[
X^i = \sum_j X_j^{(i)} \theta_j^2 + \sum_{j,k} \left( X_j^{(i)} u^j \theta_k^1 + Z_j^{(i)} u^k \theta_j \right) + \sum_{j,k,l} Z_j^{(i)} u^k \theta_l,
\]
\[
Y^i = \sum_j Y_j^{(i)} u^j + \sum_{j,k} Y_j^{(i)} u^j \theta_k + \sum_{j,k,l} \bar{Z}_j^{(i)} u^k \theta_l,
\]
then we must have $X_j^{(i)} + Y_j^{(i)} = -3c_i(u^i)f^i$. From the explicit formula (2.8) for $E^{[0]}$ it follows that
\[
\left[ T, E^{[0]} \right] u^i = (\lambda_1 - \lambda_0) X_j^{(i)} u^i \theta_j^2 - E^{[0]} \left( X_j^{(i)} \right) u^i \theta_j^2
= - \left( \lambda_1 - \lambda_0 + 2\mu \right) u^i \theta_j^2 + \cdots
= -2\mu X_j^{(i)} u^i \theta_j^2 - E^{[0]} \left( X_j^{(i)} \right) u^i \theta_j^2 + \cdots;
\]
\[
\left[ T, E^{[0]} \right] \theta_i = - (\lambda_1 - (\lambda_1 - \lambda_0) d^i - \mu) Y_j^{(i)} \theta_j^2 + E^{[0]} \left( Y_j^{(i)} \right) \theta_j^2
+ (\lambda_1 - (\lambda_1 - \lambda_0) d^i + \mu) Y_j^{(i)} \theta_j^2 + \cdots
= 2\mu Y_j^{(i)} \theta_j^2 + E^{[0]} \left( Y_j^{(i)} \right) \theta_j^2 + \cdots.
\]

Here we omit all the terms that do not contribute to the computation of indices. Then by the definition of the index we conclude that
\[
\text{ind}_i(\beta) = 3 \left( \lambda_1 - \lambda_0 - 2\mu - (\lambda_1 - \lambda_0) d^i \right) c_i(u^i) - 3E^{[0]} \left( c_i(u^i) \right).
\]
The equation \( \text{ind}_i(\beta) = 0 \) is actually an ODE for \( c_i(u^t) \), which can be easily solved to give the solution

\[
c_i(u^t) = C_i(u^t)^{m_i}, \quad m_i = \frac{\lambda_1 - \lambda_0 - 2\mu - (\lambda_1 - \lambda_0)d}{\lambda_1 - \lambda_0},
\]

where \( C_i \) are arbitrary constants. Thus we prove the Theorem 2.3.

7. Conclusion

In this paper, we propose a generalization of the bihamiltonian cohomology, called the variational bihamiltonian cohomology, to deal with more general bihamiltonian flows. The eventual goal for developing the theory of variational bihamiltonian cohomology is to prove the following conjectures.

**Conjecture 7.1.** For any deformation of the bihamiltonian structure of the Principal Hierarchy associated to a semisimple Frobenius manifold with constant central invariants, the Virasoro symmetries \( \frac{\partial}{\partial s} \) can be deformed to be symmetries of the deformed integrable hierarchy.

The above conjecture is verified for the case \( m = -1 \) in [11]. We will study this conjecture in detail and give a proof of it in the second paper [30] of this series of papers. The proof is based on the variational bihamiltonian cohomology theory established in this paper and the construction of the super tau-covers for any tau-symmetric bihamiltonian integrable hierarchies which are generalizations of the results given in [29].

Among all the possible deformations of the bihamiltonian structure with constant central invariants, there is a particular deformation with all the central invariants being \( \frac{1}{24} \). Such a deformation is called the topological deformation, and it is conjectured that:

**Conjecture 7.2.** When the central invariants of the deformation of the bihamiltonian structure of the Principal Hierarchy are all equal to \( \frac{1}{24} \), the corresponding deformation of Virasoro symmetries can be represented by linear actions on the tau function.

Apart from the above-mentioned goal, there are some other interesting problems concerning the variational bihamiltonian cohomology itself. For example, the natural map \( \delta : \tilde{\mathcal{F}} \to \tilde{\Omega} \) induces an isomorphism from the bihamiltonian cohomology to the variational bihamiltonian cohomology for most of the bidegrees \( (p, d) \), in particular, for the bidegree where both the bihamiltonian cohomology and the variational bihamiltonian cohomology vanish and for the bidegree \( (p, d) = (2, 3) \). Then it is natural to ask if this map is indeed a quasi-isomorphism of the complex, and if so can we obtain a homotopy inverse? Also, we can generalize the notion of variational bihamiltonian cohomology to include all the possible differential forms on \( J^\infty(\hat{M}) \), then we have:

\[
\tilde{\mathcal{F}} \xrightarrow{\delta} \tilde{\mathcal{E}}^1 \xrightarrow{\delta} \tilde{\mathcal{E}}^2 \xrightarrow{\delta} \tilde{\mathcal{E}}^3 \xrightarrow{\delta} \cdots
\]

It is then interesting to consider the corresponding bihamiltonian cohomology on each \( \tilde{\mathcal{E}}^k \) and to ask if each \( \delta \) is an quasi-isomorphism. We conjecture that the cohomology on \( \tilde{\mathcal{E}}^k \) is related to the general \( \mathcal{W} \)-symmetry of order \( k + 1 \). In another word, the cohomology
on $\hat{F}$ controls the flows that do not explicitly contain time variables, and the cohomology on $\hat{E}^k$ controls the flows that explicitly depend on at most $k$ time variables.

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