Information content and minimum-length metric: A drop of light

Alessandro Pesci

Received: 1 April 2022 / Accepted: 11 July 2022 / Published online: 22 July 2022
© The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2022

Abstract
In the vast amount of results linking gravity with thermodynamics, statistics, information, a path is described which tries to explore this connection from the point of view of (non)locality of the gravitational field. First the emphasis is put on that well-known thermodynamic results related to null hypersurfaces (i.e. to lighsheets and to generalized covariant entropy bound) can be interpreted as implying an irreducible intrinsic nonlocality of gravity. This nonlocality even if possibly concealed at ordinary scales (depending on which matter is source of the gravitational field, and which matter we use to probe the latter) unavoidably shows up at the smallest scales, read the Planck length $l_p$, whichever are the circumstances we are considering. Some consequences are then explored of this nonlocality when embodied in the fabric itself of spacetime by endowing the latter with a minimum length $L$, in particular the well-known and intriguing fact that this brings to get the field equations, and all of gravity with it, as a statistical-mechanical result. This is done here probing the neighborhood of a would-be (in ordinary spacetime) generic event through lighsheets (instead of spacelike or timelike geodesic congruences as in other accounts) from it. The tools for these derivations are nonlocal quantities, and among them the minimum-length Ricci scalar stands out both for providing micro degrees of freedom for gravity in the statistical account and for the fact that intriguingly the ordinary, or ‘classical’, Ricci scalar can not be recovered from it in the $L \to 0$ limit. Emphasis is put on that classical gravity is generically obtained this way for $\hbar \neq 0$, but not in the $h \to 0$ limit (the statistically derived field equations become singular in this limit), adding to previous results in this sense. This hints to that the geometric description of gravity we are used to is intrinsically quantum, as it requires $h \neq 0$, on top of being of statistical-mechanical origin. One would expect that this inherently non-classical nature might show up through nonlocality also at scales much larger than $l_p$ if special suitable circumstances are considered.
Keywords  Spacetime thermodynamics · Minimum-length spacetime and Horizons · Maximum entropy principle · Nonlocality · lightsheets

Contents

1 Preamble ............................................... 2
2 Spacetime and thermodynamics ................................... 3
   2.1 Thermodynamics at the centre: Jacobson’s result ........................................... 3
   2.2 Holography: Lightsheets and the (generalized) covariant entropy bound (GCE bound) .... 5
   2.3 Attaining the bound ........................................................................... 7
   2.4 Hod’s bound to relaxation times ......................................................... 8
   2.5 KSS bound to viscosity/entropy ratio ..................................................... 9
   2.6 Bekenstein’s bound to entropy ............................................................... 10
   2.7 Why can the GCE bound be actually attained? ........................................ 12
   2.8 Unavoidable nonlocality ....................................................................... 13
3 Spacetime and (a bit of) statistical mechanics ............................ 15
   3.1 (Small-scale) nonlocality built in: Minimum-length metric .................. 15
   3.2 Minimum-length metric and null separations ........................................... 19
   3.3 ‘Points’ with finite area .......................................................................... 24
   3.4 Raychaudhuri’s equation gets modified: No focal points .................... 26
   3.5 Einstein’s equations as a statistical-mechanical result ............................ 29
   3.6 On the nature of Einstein’s equations ...................................................... 33
   3.7 Classical gravity can not be considered to be the $L \to 0$ ($\hbar \to 0$) limit ....... 34
   3.8 Clues about a small-scale quantum description ...................................... 35
4 Conclusions ............................................. 37
References .............................................................................. 38

1 Preamble

Gravity and thermodynamics: It is definitely not an easy task to write on these matters, since by direct experience we all know these were topics of utmost significance for Prof. Thanu Padmanabhan (in memory of which I am writing and which I would like to call here affectionately Paddy), and we lost him.

My first acquaintance with him has been through consideration of some of his deeply inspiring works [1–3] about gravity as an emergent phenomenon, and in occasion of a paper [4] it had occurred to me to post about connections between gravity and thermodynamics, with a perspective in which—to my most complete surprise, but not such an uncommon thing as I realized afterwards for it happened to many others— he contacted me to say he saw in it something interesting. Since then, there has been a certain number of opportunities for me to discuss with him new results on these topics, almost always coming these from his side. What I would focus on in this paper however, is some (few) results from my side and the line of research they seem to suggest. I would dare to do this way in part because these are after all the things I can talk about; in part, and more to the point, also because it was Paddy himself to find (quite not convincingly) some value in them; I would feel thus as having kind of green light by him on this in a paper in his memory.

What the paper deals with is gravity and thermodynamics in the broadest sense. There is absolutely no intent (nor capability from my side) to give an exhaustive
description of such an immense field of research. What here reported is, on the contrary, a specific, particular path through the field, reflecting my personal feelings and interests often entangled with some of Paddy’s last years achievements. There will then be plenty of important results in the field not mentioned here (including many results from Paddy himself). They are of course really relevant, and this is only unavoidably due to the specificity of the path (and my lack). I apologize for this.

The plan of the paper is as follows. Section 2 deals with spacetime and field equations with an emphasis on elements which bring thermodynamics into the scene, in spite of never being called upon in the conventional derivation of field equations. As Paddy strongly emphasized (see e.g. [1–3] and [5], and references therein), this gives to gravity a thermodynamic flavor, hinting to some kind of emergent nature for it, and suggests there should be a way to look at field equations as what results from kind of max entropy procedure. No statistical mechanics can be present at this stage, since there is complete ignorance of the fundamental microscopic degrees of freedom (or dofs) from which dynamics gravity would be suspected to emerge.

Section 3 deals with the discovery [6, 7] that consistently endowing spacetime with a lower limit length is enough to provide microscopic (or mesoscopic) dofs. And how using these, a statistical-mechanical derivation of field equations becomes possible. Some intriguing aspects of the expression of the Ricci scalar in the minumum-length metric are discussed, in particular the fact that classical spacetime can not be recovered in the limit in which this minimal length is vanishingly small.

2 Spacetime and thermodynamics

2.1 Thermodynamics at the centre: Jacobson’s result

The insight of Bekenstein [8–10] of ascribing to black hole horizon an entropy, as well as the key result by Hawking [11, 12] of finding (combining quantum mechanics and general relativity) a temperature for it, brought thermodynamics in gravity at a fundamental or constitutive level. It gave the laws of black hole mechanics [13] turned into laws of thermodynamics. This was, and is, a mind-boggling feature, for Einstein’s field equations are purely geometric in their solutions, and one does not know how it is that an intrinsically geometric concept like that of a horizon of a black hole, can have an inherently statistical meaning with an associated entropy $S_H = A_H/4$ where $A_H$ is the area of the horizon in Planck units (we use these units if not explicitly stated otherwise).

Later works further strengthened the case and brought all this to a context more general than just black holes. In particular, Jacobson [14] considered a patch of Rindler horizon [15] around an arbitrary event $P$, with no expansion and shear at first order at $P$ (‘local Rindler horizon’). For it, he went to show that if we assume that any patch of the perceived horizon can be endowed, from the mere fact of having an area $A_H$, by an entropy $S_H = A_H/4$, then Einstein’s equations themselves are equivalent to thermodynamic Clausius relation as applied to the horizon. Indeed, Jacobson’s argument can be given the form

$$S_H = A_H/4$$
\[ \delta Q_H = T_{ab} k^a k^b A_H l = T_H \delta S_H = \frac{\kappa}{2\pi} \frac{1}{4} \delta A_H \]

\[ = \frac{\kappa}{2\pi} \frac{1}{4} \kappa R_{ab} k^a k^b A_H l = \frac{1}{8\pi} R_{ab} k^a k^b A_H l = \frac{1}{8\pi} R_{ab} k^a k^b V, \quad (1) \]

which stems from taking \( T_H = \kappa/(2\pi) \) [16] as horizon’s temperature with \( \kappa \) the acceleration of the observer (third equality), and from use of Raychaudhuri’s equation as applied to the local Rindler horizon (fourth equality). Here the symbol \( \delta \) denotes variations of horizon’s quantities resulting from the swallowing of matter. \( \delta Q_H \) is the total energy supplied to the horizon by the swallowing of an element of matter of small area \( A_H \) and proper thickness \( l \), as measured in matter rest frame, then of proper volume \( V = A_H l \) (Fig. 1); \( T_{ab} \) is the stress-energy tensor, \( R_{ab} \) the Ricci tensor and \( k^a = (d/d\lambda)^a \) are the generators of the Rindler horizon with the affine parameter \( \lambda \) taken to be distance or time in the local frame of matter.

**Fig. 1** Crossing of the (Rindler) horizon by a lump of matter as described in matter local frame.
2.2 Holography: Lightsheets and the (generalized) covariant entropy bound (GCE bound)

Consideration of the fact that any lump of matter/energy dumped to a black hole brings to an increase of the area of the horizon, and of the evidence that any localized amount of entropy/information has with it a finite energy (as implied e.g. by the Bekenstein bound [17]), brings generically to expect that the entropy of a black hole is actually the maximum entropy one can have inside a spherical 2-surface with area equal to black hole’ horizon [18, 19]. This has been the basis of the conjecture that all the physics in the bulk of a D-dim region, can be encoded in dofs living on the (D − 1)-dim boundary (dimensional reduction or holographic principle) [18, 19], which has found a concrete implementation in string theory in the AdS/CFT correspondence [20] and in a number of similar examples of dualities.

The generalized covariant entropy bound [21, 22]

\[ S(L) \leq \frac{A_B - A_{B'}}{4}, \tag{2} \]

can be considered as the most general formulation of the holographic principle in Einstein spacetime. Here, \( B \) is a generic spacelike 2-dim surface with area \( A_B \), from which light rays emanate orthogonally in nonexpanding directions. The rays are followed until they reach another 2-dim spacelike surface \( B' \) (area \( A_{B'} \)) orthogonal to them where they are terminated before focal points are reached (the (non-generalized) covariant entropy bound is the particular case when they are allowed to reach the focal points and \( A_{B'} = 0 \)). \( L \) is the null 3-dim submanifold swept by the light rays, and \( S(L) \) is matter system’ entropy on \( L \).

Like Einstein’s field equations, the bound relates geometry to matter dofs, but, intriguingly, the latter appear now through their possibly full information content instead of energy. One has then good reasons to suspect that a complete account of the bound, assuming it to hold true down at the fundamental level, must wait until one will have a complete theory of gravity and matter. On the positive side, any attempts to prove or check, or possibly modify, the bound at an assigned level of description might tell something about the nature of gravity, geometry and matter at a deeper level.

Even if it exhibits geometric quantities, the expression to the right in bound (2) is actually an entropy (as it ought to be, from the meaning of the left-hand side). The result [16] indeed, which the same as for black-hole horizons assigns a temperature to any local Rindler horizon, allows to extend the notion of temperature to arbitrary null hypersurfaces. A surface element of \( B \) or \( B' \) with area \( dA \) can act as a spatial section of a local Rindler horizon of an orthogonally accelerating observer, and this horizon can be assigned a temperature (dependent on the acceleration), and an entropy \( dS_H = \frac{dA}{4} \) independent of acceleration (explicitly inserting Newton’s constant \( G \), \( dS_H = \frac{dA}{4G} \) in general relativity which is where bound (2) lives, and \( dS_H = \frac{dA}{4G_{\text{eff}}} \) (Wald entropy [23–25]) in more general theories with \( G_{\text{eff}} \) an effective coupling constant replacing \( G \) and in general varying with the point (e.g. through dependence on curvature) [26]).

We can regard \( L \) in a neighborhood of \( B \) (\( B' \)) as a horizon with spatial section \( B \) (\( B' \)). This horizon can be approximated at each \( p \in B \) (\( B' \)) by a local Rindler horizon with
generators $k^a$ along the light ray at $p$. These local horizons can be chosen to have all a same temperature, and all possess entropies $dA/4$.

This shows that the fact that spacetime does have null hypersurfaces brings with it that to the latter is associated heat density, and entropy. Right this observation, we know, forms the basis of Paddy’s emergent gravity paradigm [1, 2]. In it, extending the scope of Boltzmann’s intuition to include spacetime, the idea is that if a null hypersurface can be hot it must have microscopic dofs. This suggests that spacetime as we know it might then arise from these latter with field equations being analogous to the equations of fluid mechanics.

Proofs of the bound (2) do exist [22, 27–29] within the context of classical general relativity and for matter in the hydrodynamic regime, this meaning circumstances with a classical spacetime and in which matter quantum dofs can be described in terms of continuous variables, like energy density $\rho$, entropy density $s$ with a local entropy current, pressure $p$, temperature $T$, ..., at any point of it.

Bound (2) however is found to be true also for matter considered beyond any restriction to the hydrodynamic approximation. In [30] the bound is proven indeed in the far more general context of local quantum field theories, even if restricting to the case of matter consisting of free fields, with the entropy on the lightsheet calculated as vacuum subtracted von Neumann entropy for these fields, and in the limit in which the gravitational backreaction is weak. This proof does not rely on the null energy condition, condition the latter which one might think to generically accompany any description of matter in which entropy and energy are microscopically related as in the case of hydrodynamic approximation. It accommodates then also circumstances in which the null energy condition is locally explicitly violated as for example in the case of Hawking radiation at the horizon, case in which possible violations of bound (2) have been pointed out [31]. This has shown explicitly that the bound (2) holds true as it is also in these circumstances, apparently without strictly a need for the modifications of it envisaged in [28]. The proof [30] is quite in the same vein as the local-quantum-field-theory proof [32, 33] of the closely related generalized second law (even if with the relevant entropies being however meant differently in the two contexts: entropy on the part of the horizon to the future of any given spatial section in case of [32, 33], and entropy on the lightsheet between two spatial sections in case of [30]).

Our aim here is not so much to focus on further checks of thermodynamic bound (2) or on its proofs, but, rather, on what we can infer about gravity when taking the bound as true. This would be in the spirit of the general efforts aiming to understand what is telling us holography about the nature of spacetime. While doing this we choose moreover to stay in the hydrodynamic approximation. Clearly, one would push the description, and thus likely the understanding, to a level as deep and general as possible, thus e.g. to quantum field theory description or to string theory gauge/gravity duality mentioned above. Thanks to the fact, however, that the bound keeps a same formulation both in quantum field theory and in the hydrodynamic regime, there might be a point in trying to infer something about gravity also simply staying in the latter, which presents features perhaps more easy to manage. A question we might want to try to address for example is what kind of mechanism or principle might be at work, if
any, preventing bound (2) to be violated while we keep staying in the hydrodynamic regime.

2.3 Attaining the bound

The proofs in [22, 27, 28] are in terms of some reasonable conditions, obeyed by matter entropy, which turn out to be sufficient for the validity of the bound. These conditions are considered reasonable with reference to actual, physical hydrodynamic systems. Proof [29] goes very much along the same line of effort pushing it up to the point to try to identify kind of extreme condition, the most challenging one, that the hydrodynamic matter has to satisfy, violating which (though still safely remaining within the hydrodynamic regime) would mean to violate the bound.

A convenient way to express this condition is in terms of a systems consisting of a layer of matter, i.e. occupying a volume in which two dimensions are much larger than the third one, the thickness \( l \). In case of a homogeneous system made of ultrarelativistic particles or of photons the condition takes the very simple form [29]

\[
l \geq \frac{1}{\pi T} \left[ \frac{\hbar c}{\pi k T} \right], \tag{3}
\]

with \( T \) the temperature of the system. Here (and hereafter) between square brackets we explicitly reinsert all the constants involved, namely, in present case, (reduced) Planck’s, and Boltzmann’s constants \( \hbar \) and \( k \) and the speed of light \( c \). This to explicitly remark here that this result comes about with gravity playing no role, as indicated by the absence of Newton’s constant \( G \).

In the general case with constituent particles of generic nature, one gets in an analogous manner [34]

\[
l \geq l^\ast \text{(medium, state)} \equiv \frac{1}{\pi} \frac{s}{\rho + p} = \frac{1}{\pi T} \left( 1 - \frac{\mu n}{\rho + p} \right), \tag{4}
\]

with last expression applying in case the system is homogeneous. Here \( s \) and \( n \) are respectively local entropy and number densities, and \( \rho \) is local energy density, including the rest mass of constituent particles; \( p \) is pressure, and \( \mu \) the chemical potential which also consistently includes the rest mass energy of the particles and can thus be expected to be \( \mu \geq 0 \). The equality follows from Gibbs-Duhem relation \( \rho = Ts - p + \mu n \). The length \( l^\ast \) is characteristic of the medium which the system is made of and depends moreover on the thermodynamic state corresponding to the assigned values of the thermodynamic potentials; for a photon gas at temperature \( T \), \( \mu = 0 \) and \( l^\ast = \frac{1}{\pi T} \) in agreement with (3). \( l^\ast \) sets the limit below which the generalized covariant entropy bound might ideally be violated at the assigned thermodynamic circumstances, still remaining safely within the hydrodynamic regime. More precisely, if we consider a spacelike two surface so chosen to have orthogonal rays with vanishing expansion and we put on it a layer of matter with a thickness \( l \) smaller than the \( l^\ast \) characteristic of that matter at the assigned values of thermodynamic potentials, then
if we terminate the light-sheet at the exit of the layer we would obtain a violation of bound (2).

What actually happens \[34\] is that making an inventory of the possible thermodynamic systems in a wide range of values of thermodynamic potentials, \(l^*\) turns out invariably to be by far smaller than the ‘size’ \(\lambda\) of constituent particles themselves, taken this size to be their intrinsic quantum indeterminacy at the thermodynamic conditions. Since the thickness \(l\) of the layer by definition can not be smaller than \(\lambda\), what we generically have is \(l \geq \lambda \gg l^*\), and the bound (2) is safe.

Looking at (4), if we want to challenge the bound we have to search for systems which have a high \(s\) gain for given \(\rho\), or high \(s/\rho\) ratio, in other words we are (of course) interested in the most entropic systems, and this means that we can not help thinking of a thermal photon gas. Indeed, if we consider a photon gas at thermal equilibrium at temperature \(T\) and assume that \(\delta p = \epsilon/2\), with \(\epsilon = 2.82T\) the peak energy of photon distribution \[35\], captures the intrinsic indeterminacy in the momentum of the photons, then from \(\delta x \delta p \geq 1/2\) with \(\delta x\) the indeterminacy in photon position, we get \(\lambda = \delta x \geq 1/\epsilon = \frac{1}{2.82T} \approx \frac{1}{\pi T} = l^*.\) Thus, ideally, a very thin layer of this gas, as thin as possible compatibly with the intrinsic spatial indeterminacy of the photons at temperature \(T\), almost does the job, it attains the bound (2) \[34\]. Layers thinner than this can be conceived, but due to uncertainty relations they necessarily would be accompanied by a temperature larger than the given \(T\). There is (almost, as we will consider later) no lower limit to the size at which we can apply our analysis, but invariably for that thinner system we just attain the bound as just showed, we can not beat it. The reason why we can not violate the bound, is that this would mean to violate the indeterminacy relations; there are no constraints instead coming from our choice to remain within the hydrodynamic regime. Quantum indeterminacy leads to (3) and (4); once this is given, then consideration of gravity (Einstein’s equations) brings to the bound (2).

### 2.4 Hod’s bound to relaxation times

Before we proceed let us pause to mention some ramifications of this result with connections to apparently far-apart topics, a fact which might deserve further investigation. One of these is the relation with the relaxation times of thermodynamic systems, namely the characteristic times for a system put slightly off equilibrium to return to the equilibrium state. For any system we can think of its relaxation time \(\tau\) as \(\tau \geq d/v_s\) where \(d\) is the size of the system and \(v_s\) is the speed of sound in it, with the limit being in principle attainable in the most favorable geometrical circumstances. Coming back to the just considered idealized system consisting of an as-thin-as-possible layer of a photon gas at temperature \(T\), namely with thickness \(l = \lambda = \frac{1}{\pi T} = l^*\), if we take as \(v_s\) the vacuum speed of light, \(v_s = 1\), we get \(\tau = \frac{1}{\pi T}\) \[36\].

Now, from thermodynamics used within quantum information theory, a universal lower limit \(\tau_{\text{min}}\) had been envisaged before precisely of the form \(\tau_{\text{min}} = \frac{1}{\pi T} \left[\frac{\hbar}{\pi kT}\right]\) \[37\]. We can see (as first noticed by Hod \[38\]) that the bound associated to this limit and bound (3) do coincide with length replacing time, this implying that the highly idealized layer we chose in order to attain the generalized entropy bound (2)
actually attains also the universal bound to relaxation times [36]. Hod’s bound is – from its derivation or looking directly at the physical constants involved in the result– independent from gravity, the same as we emphasized for (3). Still, quite intriguingly, black holes are theoretically found (from consideration of quasi-normal modes of their free oscillations when perturbed) to give relaxation times of the order of magnitude of Hod’s limit and conform to it [37]. This appears now quite confidently experimentally confirmed from the analysis [39] of the data of black hole mergers taken by LIGO-Virgo Collaboration, as it can be seen starting already from the first [40] detected event GW150914. The extremal black holes are expected to actually saturate the bound [37].

2.5 KSS bound to viscosity/entropy ratio

Another topic which the bounds (3) and (4) turn out to be related to is the so called KSS (Kovtun, Son, Starinets) bound to viscosity/entropy [41, 42], put forward in the context of AdS/CFT. In it, on the basis of the result that thermal theories with gravity dual invariably exhibit a same ratio $\eta/s = \frac{1}{4\pi}$, where $\eta$ is viscosity and $s$ entropy density, and that for thermal theories which are instead generic this ratio is in general much larger, the conjecture is made that $\eta/s \geq \frac{1}{4\pi}$ for any system which can be described through a consistent relativistic quantum field theory. Many counterexamples have been found for this bound when including higher derivative corrections, not only in the sense of lowering the factor 1/4, but also showing that the ratio $\eta/s$ can be made arbitrarily small, typically this involving systems with arbitrarily large number of species [43] (but not being restricted to this; a very recent account of possible violations of the bound in general settings can be found in [44]). However, at least in some of these cases consistency issues can then be raised, and it seems anyway that, at least when chemical potential $\mu = 0$, $\eta/s$ cannot be vanishingly small but must be bounded by some factor $\neq 0$ if not $\frac{1}{4\pi}$ [45].

As for our purposes here, we emphasize that the KSS bound can clearly be reframed as an entropy bound [46]. It is then by now no surprise that it can be related to bounds (3, 4). What happens is that our as-thin-as-possible layer above does again the job: in a back-on-the-envelope calculation it attains the KSS bound, with constant $= \frac{1}{4\pi}$; for this, we have however to think of strongly coupled massless particles, e.g. gluons, as replacing photons [47]. Indeed from $\eta \approx \frac{1}{3} t_c \rho$ [48] for radiation, with $t_c$ the collision time, we get

$$\frac{\eta}{s} \approx \frac{1}{3} t_c \frac{\rho}{s} = \frac{1}{4} t_c \frac{\rho + p}{s} = \frac{1}{4\pi} \frac{t_c}{l^*} = \frac{1}{4\pi} \frac{l_c}{l^*},$$ 

with $l_c$ the collision length [49]. Here $l_c \geq \lambda$, for in a volume $\lambda^3$ we have one radiation quantum. We can then leave $l_c$ to ideally become as small as $\lambda$, but this requires the radiation is strongly coupled.\(^\text{1}\) In fact, starting from the Thomson cross-section $\sigma_T = \frac{2}{3\pi}$ for radiation, with $t_c$ the collision time, we get

\^\text{1\hspace{1cm}}$\sigma_T = \frac{2}{3\pi}$ for radiation, with $t_c$ the collision time, we get

\[ \frac{\eta}{s} \approx \frac{1}{3} t_c \frac{\rho}{s} = \frac{1}{4} t_c \frac{\rho + p}{s} = \frac{1}{4\pi} \frac{t_c}{l^*} = \frac{1}{4\pi} \frac{l_c}{l^*}, \]

with $l_c$ the collision length [49]. Here $l_c \geq \lambda$, for in a volume $\lambda^3$ we have one radiation quantum. We can then leave $l_c$ to ideally become as small as $\lambda$, but this requires the radiation is strongly coupled.\(^\text{1}\) In fact, starting from the Thomson cross-section $\sigma_T = \frac{2}{3\pi}$ for radiation, with $t_c$ the collision time, we get

\[ \frac{\eta}{s} \approx \frac{1}{3} t_c \frac{\rho}{s} = \frac{1}{4} t_c \frac{\rho + p}{s} = \frac{1}{4\pi} \frac{t_c}{l^*} = \frac{1}{4\pi} \frac{l_c}{l^*}, \]

with $l_c$ the collision length [49]. Here $l_c \geq \lambda$, for in a volume $\lambda^3$ we have one radiation quantum. We can then leave $l_c$ to ideally become as small as $\lambda$, but this requires the radiation is strongly coupled.\(^\text{1}\) In fact, starting from the Thomson cross-section $\sigma_T = \frac{2}{3\pi}$ for radiation, with $t_c$ the collision time, we get

\[ \frac{\eta}{s} \approx \frac{1}{3} t_c \frac{\rho}{s} = \frac{1}{4} t_c \frac{\rho + p}{s} = \frac{1}{4\pi} \frac{t_c}{l^*} = \frac{1}{4\pi} \frac{l_c}{l^*}, \]

with $l_c$ the collision length [49]. Here $l_c \geq \lambda$, for in a volume $\lambda^3$ we have one radiation quantum. We can then leave $l_c$ to ideally become as small as $\lambda$, but this requires the radiation is strongly coupled.\(^\text{1}\) In fact, starting from the Thomson cross-section $\sigma_T = \frac{2}{3\pi}$ for radiation, with $t_c$ the collision time, we get

\[ \frac{\eta}{s} \approx \frac{1}{3} t_c \frac{\rho}{s} = \frac{1}{4} t_c \frac{\rho + p}{s} = \frac{1}{4\pi} \frac{t_c}{l^*} = \frac{1}{4\pi} \frac{l_c}{l^*}, \]

with $l_c$ the collision length [49]. Here $l_c \geq \lambda$, for in a volume $\lambda^3$ we have one radiation quantum. We can then leave $l_c$ to ideally become as small as $\lambda$, but this requires the radiation is strongly coupled.\(^\text{1}\) In fact, starting from the Thomson cross-section $\sigma_T = \frac{2}{3\pi}$ for radiation, with $t_c$ the collision time, we get

\[ \frac{\eta}{s} \approx \frac{1}{3} t_c \frac{\rho}{s} = \frac{1}{4} t_c \frac{\rho + p}{s} = \frac{1}{4\pi} \frac{t_c}{l^*} = \frac{1}{4\pi} \frac{l_c}{l^*}, \]

with $l_c$ the collision length [49]. Here $l_c \geq \lambda$, for in a volume $\lambda^3$ we have one radiation quantum. We can then leave $l_c$ to ideally become as small as $\lambda$, but this requires the radiation is strongly coupled.\(^\text{1}\) In fact, starting from the Thomson cross-section $\sigma_T = \frac{2}{3\pi}$ for radiation, with $t_c$ the collision time, we get

\[ \frac{\eta}{s} \approx \frac{1}{3} t_c \frac{\rho}{s} = \frac{1}{4} t_c \frac{\rho + p}{s} = \frac{1}{4\pi} \frac{t_c}{l^*} = \frac{1}{4\pi} \frac{l_c}{l^*}, \]

with $l_c$ the collision length [49]. Here $l_c \geq \lambda$, for in a volume $\lambda^3$ we have one radiation quantum. We can then leave $l_c$ to ideally become as small as $\lambda$, but this requires the radiation is strongly coupled.\(^\text{1}\) In fact, starting from the Thomson cross-section $\sigma_T = \frac{2}{3\pi}$ for radiation, with $t_c$ the collision time, we get

\[ \frac{\eta}{s} \approx \frac{1}{3} t_c \frac{\rho}{s} = \frac{1}{4} t_c \frac{\rho + p}{s} = \frac{1}{4\pi} \frac{t_c}{l^*} = \frac{1}{4\pi} \frac{l_c}{l^*}, \]

with $l_c$ the collision length [49]. Here $l_c \geq \lambda$, for in a volume $\lambda^3$ we have one radiation quantum. We can then leave $l_c$ to ideally become as small as $\lambda$, but this requires the radiation is strongly coupled.\(^\text{1}\) In fact, starting from the Thomson cross-section $\sigma_T = \frac{2}{3\pi}$ for radiation, with $t_c$ the collision time, we get

\[ \frac{\eta}{s} \approx \frac{1}{3} t_c \frac{\rho}{s} = \frac{1}{4} t_c \frac{\rho + p}{s} = \frac{1}{4\pi} \frac{t_c}{l^*} = \frac{1}{4\pi} \frac{l_c}{l^*}, \]

with $l_c$ the collision length [49]. Here $l_c \geq \lambda$, for in a volume $\lambda^3$ we have one radiation quantum. We can then leave $l_c$ to ideally become as small as $\lambda$, but this requires the radiation is strongly coupled.\(^\text{1}\) In fact, starting from the Thomson cross-section $\sigma_T = \frac{2}{3\pi}$ for radiation, with $t_c$ the collision time, we get

\[ \frac{\eta}{s} \approx \frac{1}{3} t_c \frac{\rho}{s} = \frac{1}{4} t_c \frac{\rho + p}{s} = \frac{1}{4\pi} \frac{t_c}{l^*} = \frac{1}{4\pi} \frac{l_c}{l^*}, \]
\[ \frac{8}{3\pi} \frac{\alpha^2}{m^2} = \frac{2}{3\pi} \alpha^2 \lambda_{\text{Compton}}^2 = O(1) \alpha^2 \lambda_{\text{Compton}}^2 \] of a particle with Compton wavelength \( \lambda_{\text{Compton}} \), and assuming in analogy with it the cross-section \( \sigma \) of our radiation quanta to be \( \sigma = O(1) g^2 \lambda^2 \), we have \( l_c = \frac{1}{\sigma n} = O(1) \frac{\lambda}{g^2} \). And this shows that to get on average a collision in a length \( \lambda \) we must have \( g^2 = O(1) \), [47]. A different argument leading to the KSS limit still using, among other things, the \( l^* \) concept is presented in [51, 52].

In spite of having been originally derived for quantum field theories dual to bulk stringy gravitational theories, the KSS bound has no intrinsic reference to gravity as can be envisaged from its formulation, not containing \( G \). It is then similar in this respect to Hod’s bound on relaxation times and it also exhibits, as it happens for the latter, the intriguing feature that even if the bound knows at a fundamental level nothing of gravity, black holes actually attain it. It is indeed a (much earlier than AdS/CFT) result of membrane paradigm the fact that [53, 54] the (surface) viscosity of Schwarzschild black hole is \( \eta = \frac{1}{16\pi} \left[ = \frac{1}{16\pi} \frac{c^3}{G} \right] \) which, being the (surface) entropy density \( s = \frac{1}{4} \left[ = \frac{1}{4} \frac{k^3}{\hbar G} \right] \), gives \( \frac{n}{s} = \frac{1}{4\pi} \).

### 2.6 Bekenstein’s bound to entropy

A third topic we wish to consider here in connection with bounds (3) and (4) above is the Bekenstein bound [17]. We already generically used of it while introducing the spherical entropy bound. It reads [17]

\[ S \leq 2\pi E R \left[ = 2\pi E R \frac{k}{\hbar c} \right] \]  \hspace{1cm} (6)

with \( S \) and \( E \) the entropy and the rest energy of the system and \( R \) the radius of the smallest circumscribing sphere, assuming that if spacetime nonnegligibly deviates from Minkowski, consideration is restricted to spacetimes which enjoy symmetries which allow for these quantities (or at least the combination which appears in the bound [55]) to be meaningfully defined.

Within the hydrodynamic regime and for negligible selfgravity we can express this bound in terms of densities, as

\[ s \leq 2\pi \rho R. \]  \hspace{1cm} (7)

We can recognize this has quite a resemblance to bound (4) on thermodynamic densities rewritten as

\[ s \leq \pi (\rho + p) l, \]  \hspace{1cm} (8)

except that the bound (8) is in general much stronger than (7) since we can take \( l \ll R \) [49]. Given a material medium, e.g. the quite miraculous photon gas considered above, we can imagine to make balls out of it of decreasing radius while keeping intact the values of thermodynamic potentials. Each ball can be considered as a thermodynamic
system by its own, the smaller the radius the nearer coming it to attain the Bekenstein bound (7). We can consider a radius $\bar{r}$ as small as needed to just attain the Bekenstein bound. It is $\bar{r} = \frac{1}{\pi \rho}$. As easily verified, it relates to the minimum allowed $l$ ($l = l^\ast = \frac{1}{\pi T}$) for the prescribed values of the thermodynamic potentials of the thermal photon gas as $l^\ast = \frac{3}{2} \bar{r}$. This quite nicely fits with the limiting quantum-mechanical condition (3) for the size. Indeed, a radius $r$ say with $l^\ast = 2r$ would be too small to comply with (3), for any intercept with the ball would have length $l \leq l^\ast$; a radius with $l^\ast = r$ would be a little too large than strictly needed: most of intercepts would have $l > l^\ast$, and a smaller ball could be made still on average complying with (3). It is then fair to say that the thermal photon gas adds a miracle more: quantum indeterminacy allows for it to be ideally enclosed, at the prescribed values of thermodynamic potentials, in a sphere just small enough for the Bekenstein bound to be attained.

The coefficient $C$ in the above relation $l^\ast = C \bar{r}$ is admittedly not very sharply defined; this fits with the highly idealized situation we are considering with photons in a ball with diameter just a little larger than their own wavelength. The message we get anyway, is that what at the end precludes the Bekenstein bound from being violated within the hydrodynamic regime are bounds (3), (4), which is to say quantum indeterminacy ([56, 57] and [34]). We might still contemplate smaller balls with potentials safely described by the hydrodynamic regime, in principle violating the Bekenstein bound; but in that case quantum indeterminacy would compel the gas in the balls to have values of the thermodynamic potentials different from the prescribed ones, in the manner needed to protect the bound.

Even if –the same as for relaxation times and viscosity/entropy– the Bekenstein bound does not involve gravity, (as can be envisaged by inspecting the constants in (6)), by a historical accident we might say, its first derivation was through the mentioned consideration of processes involving black holes [17], and Schwarzschild black holes clearly attain the bound: $\frac{S}{\mathcal{E}} = \frac{\pi R^2}{R^2} = 2\pi R$. A full proof of the Bekenstein bound exists in quantum field theory [55]. It is based on a notion of entropy given by the difference of actual entropy and the entropy of vacuum state; this turns out to be finite (contrary to actual entropy and vacuum entropy taken separately). In this derivation the bound arises as the nonnegativity of relative entropy between local density matrices describing the vacuum state and the actual state reduced in the volume $V$. Similarly to the proof [30] it applies when backreaction is weak, thing that, when the circumstances are such that the only gravity potentially relevant to the case in question is the self-gravity of the body, which is the case of the Bekenstein bound, means that strictly speaking it applies to nearly Minkowski spacetime [55].

It would be interesting to have a proof of the bound for nonnegligible self-gravity (with the only constraint being that the elements which enter to define the bound can be meaningfully defined). A proof for these circumstances indeed exists [49], not in quantum field theory but working instead within the hydrodynamic approximation and using the inequality (4) above. A full proof in quantum field theory for strong self-gravity, thus transcending also in this case the need of any microscopic relation between energy and entropy densities, would clearly be highly desirable. In the meantime a quantum field theory proof of flat-spacetime relation (4) might also be welcome.
Indeed, when joined with the proof [49], it would lead to a sort of QFT-based hybrid proof of the Bekenstein bound for generic self-gravity.

The same as for the viscosity/entropy bound, also the Bekenstein bound might in principle be invalidated by a growing, ideally unboundedly, number of particle species as repeatedly noticed in literature. Point is that we can increase the number of species in a system, thus increasing its entropy, while keeping fixed its energy, and we might expect this way to break the bound. In the QFT proof [55] it is shown that with that notion of entropy this can not happen. The number $I$ of species affects indeed both the system under consideration and the vacuum and the difference of their entropies saturates for large $I$ allowing the bound to be safe.

Also in the approach we are following here the species problem seems can be avoided. This is basically the expression of nonnegativity of chemical potentials when they include the mass energy of the particles. If we go from a homogeneous system to an inhomogeneous one with $I$ species while keeping a same energy, pressure and temperature, the Gibbs-Duhem relation goes from $\rho = Ts - p + \mu n$ to $\rho = T\tilde{s} - p + \sum_{i=1}^{I} \mu_i n_i$, where $\tilde{s}$ is the entropy of the inhomogeneous system and $\mu_i$, $n_i$ are the chemical potential and number density of the species $i$ in it. Then $\tilde{s} = s + \frac{1}{T} (\mu n - \sum_{i=1}^{I} \mu_i n_i)$, and from $\mu_i \geq 0$ we get $\tilde{s}_{\text{max}} = s + \frac{\mu n}{T}$ as the maximum entropy among all the inhomogeneous systems at the assigned thermodynamic conditions.

This gives a corresponding characteristic length $\tilde{l}^*_{\text{max}} = \frac{1}{\pi} \frac{\tilde{s}_{\text{max}}}{\rho + p} = \frac{1}{\pi} \frac{1}{T}$. Now, the de Broglie wavelength $\lambda$ of a particle of mass $m$ is (with all units) $\lambda = \hbar P = \frac{c}{v} \lambda_\gamma > \lambda_\gamma$ where $\lambda_\gamma$ is the wavelength of a photon at same energy (corresponding to the assigned temperature) and $P$ and $v$ are particle’s momentum and velocity. This means that for no one of the species we have $\lambda_i < \frac{1}{\pi T}$, where $\lambda_i$ is de Broglie wavelength of species $i$ at temperature $T$. Thus, since by definition $l > \lambda_i$, $\forall i$, or at least $l > \langle \lambda_i \rangle$ where the latter is some average over the $\lambda_i$’s, we get $l > \tilde{l}^*_{\text{max}}$ and there is no violation of bound (4) relative to inhomogeneous systems (in (4), $\mu n$ is replaced by $\sum_{i=1}^{I} \mu_i n_i$) and thus of the Bekenstein bound (see also [49]).

2.7 Why can the GCE bound be actually attained?

This concludes the description of items seemingly distant at first, yet strongly connected, with conditions (3, 4). Let us come back to these conditions in their relation with the generalized covariant entropy bound. What we have seen so far is that these inequalities are safe due to quantum indeterminacy and can be actually attained by the most entropic systems (in specific geometric arrangements); and that when these conditions are then joined to Einstein’s field equations, we get the generalized covariant entropy bound (2) is preserved.

Now, two further intriguing aspects appear to be worth noticing. One (the other is left to the next subsection) is indeed the result of that paper [4] which first aroused Paddy’s curiosity. It is the fact that in so doing, namely preserving bound (2), Einstein’s equations turn out to be right what allows bound (2) to be exactly attained by the most entropic systems in the most convenient setups (namely, a layer of a thermal photon gas at temperature $T$ of thickness $l = \frac{1}{\pi T}$ with lightsheet given by orthogonal

\[ \text{Springer} \]
null geodesics emanating from one of its faces crossing it with initially vanishing expansion, i.e. a ‘plane’ layer). Point is that this is surprising because we appear to have in principle no reason for such a perfect match. Given Einstein’s field equations, the validity of the generalized bound (2) requires, in the hydrodynamic regime, a constraint in the local thermodynamic potentials, which takes the form of inequality (4); there is no surprise in this. What is surprising is that inequality (4) in specific circumstances can be attained, while generically one might had expected it to be satisfied by far in any circumstances. Indeed, once the generalized bound (2) is accepted as true (the factor 1/4 coming e.g. from the spherical entropy bound, with no hint that the gap between black hole entropy and matter entropy within an assigned spherical area can be made small, rather just the opposite), why has gravity to act in just that way that gives the generalized bound exactly attained in specific cases? The effects of gravity might give the bound satisfied by orders of magnitude to spare also for the most entropic systems and in the most favorable circumstances, why is that the bound fits in such a perfect manner instead?

There seems to be sort of coincidence or fine-tuning here which might deserve further scrutiny. It might suggest that gravity is driven by matter dofs, or more precisely by all possible matter dofs at the assigned $\rho + p$ [4], meaning this that we have to include also those that, at the assigned thermodynamic conditions, are concealed in the mass of the particles forming the system. We might take this as something hinting towards a statistical origin of gravitational dynamics in the same vein as the various derivations of field equations by thermodynamic arguments or, before that, as something further hinting to gravity as an emergent phenomenon arising in an appropriate statistical limit from a fundamental (and unknown) microscopic theory of gravity and matter, and as such prone to a thermodynamic description as in [1–3] up to the point to allow the thermodynamic language to replace the geometric one [58]. This adds some elements to the big question about the role of field equations in the process of unveiling a quantum theory for gravity: Are they a specific indication hinting to the precise underlying microscopic theory? Or would they come about the same instead, as a pure statistical effect, whichever this underlying theory is (somehow like the equation of state of a gas which happens can be calculated the same even irrespective of the fundamental description is taken to be classical or quantum mechanical)?

### 2.8 Unavoidable nonlocality

The second aspect we would like to emphasize is that the bound (2) adds a new element, not contemplated by inequalities (3, 4), and which can not be accounted for, as (3, 4) instead, by quantum uncertainty, this fact suggesting that quantum indeterminacy is definitely not the end of the story.

To see what is this about, let us take our plane layer of thermal photon gas with thickness $l = \lambda = \frac{1}{\pi T}$ between faces $B$ and $B'$. We know that the lightsheet crossing the layer with null geodesics emanating orthogonally from $B$ (thus with vanishing expansion) with area $A_B$ just attains the bound with some, in principle only very slightly smaller, exit area $A_{B'}$ (it is the geometry Fig. 1, with matter forming a very thin layer ($l$ extremely small) and the horizon playing the role of the lightsheet). Let
now slightly increase $T$, and then decrease $\lambda$ to a new value, and take as before $l = \lambda$, i.e. take a layer with thickness equal to the new $\lambda$, then comprised between $B$ and a new $B'$ nearer. For this new layer, the generalized bound (2) will be again exactly attained, but this time with an $A_{B'}$ smaller than before (this means that in a smaller thickness we get a smaller exit area; we can readily understand this if we consider that entropy density $s \sim T^3$ while the volume of the layer scales as $\lambda \sim T^{-1}$). We see, the dependencies on temperature are such that if we proceed this way ideally increasing $T$ at our wish, we eventually reach a temperature $\bar{T}$ at which the bound is still saturated by the associated layer, but the exit area has become $A_{B'} = 0$. This corresponds to a critical situation in which the entropy of the gas ‘eats’ all the entropy made available at start by the area $A_B$; any further increase in $T$, and decrease in $\lambda$, is no longer allowed as it would require the gas to have more entropy than what available. This sets an upper limit $S \leq \frac{AB'}{4}$ to the entropy $S$ of the layer of a thermal photon gas we can build on the plane surface $B$ with thickness equal to 1 photon wavelength $[\lambda]$, or equivalently the bound

$$s \leq \frac{1}{4\lambda} \left[ = \frac{1}{4\lambda} \frac{k c^3}{\hbar G} \right]$$

(9)

to entropy density, where $\tilde{\lambda}$ is (photon) wavelength corresponding to the critical temperature $\bar{T}$.

We can actually compute $\tilde{\lambda}$. Using the formula for entropy density $s$ of a thermal photon gas, $s = \frac{4 \pi^2}{45} T^3$, from (9) and $\tilde{\lambda} = 1/(\pi \bar{T})$ we get $\frac{4}{45\pi} (1/\tilde{\lambda}^3) = \frac{1}{4} (1/\tilde{\lambda})$ which gives $\tilde{\lambda} = \frac{4}{\sqrt{45\pi}} \approx 0.3 \left[ = 0.3 \sqrt{\frac{\hbar G}{c^3}} \right] = 0.3 l_p$, with $l_p = \sqrt{\frac{\hbar G}{c^3}}$ the Planck length, i.e. $\tilde{\lambda} = O(l_p)$ as we would have expected. Equation (9) is then

$$s \leq \frac{\sqrt{45\pi}}{16} \frac{k}{l_p^3} \approx 0.74 \frac{k}{l_p^3}.$$  

(10)

If the same microstates are counted in binary log (to give information in bits), we get $s^{(2)}_k \leq \frac{\sqrt{45\pi}}{16} (\log_2 e) \text{ bits}/l_p^3 \approx 1.07 \text{ bits}/l_p^3$, where $s^{(2)}_k$ is entropy in base 2.

We see what is the new element introduced with the generalized bound (2) not contained in the bounds (3, 4): it is the existence of a limit length $\tilde{\lambda} = O(l_p)$ which we cannot go below when accounting of the dofs of any physical system. This is clearly connected with the fact that bound (2) relies on and manifests gravity, as can be envisaged from (9) or from the expression of the bound (2) itself with all units set in place, $S(L) \leq \frac{AB - A_{B'}}{4} \frac{k c^3}{\hbar G}$, which explicitly exhibits $G$.

In the absence of a finite $\tilde{\lambda}$, there would be no limit on how small the length scale could be when considering inequalities (3, 4). $l^*$ for a thermal photon gas could become vanishingly small, and yet inequality (3) would always hold true thanks to quantum indeterminacy. Since in the hydrodynamic regime the Bekenstein bound follows from inequalities (3, 4) with no extra input, this means that there is no intrinsic need from it for a limit length scale, at striking variance with what happens with the generalized bound (2).
This has consequences also for the hydrodynamic approximation. As far as say inequality (3) for a thermal photon gas holds true, and can be actually attained without any limit on how small the thickness \( l \) can be, there is virtually no lower limit to the length scale of applicability of the hydrodynamic approximation and thus to the local microscopic correspondence between energy and entropy. With a finite limit length \( \bar{\lambda} \) however, we can no longer resort to (3) if we hypothetically go below a thickness \( \bar{\lambda} \), and the hydrodynamic approximation as well as the local microscopic correspondence between energy and entropy must at this scale necessarily break down. We can thus see that at length scale \( \bar{\lambda} \) nonlocality unavoidably enters the scene, even if we carefully avoided at start any circumstances requiring a nonlocal description.

Since as mentioned the limit length \( \bar{\lambda} \) appears as soon as gravity is involved, from what we just said we are lead to conclude that nonlocality irreducibly accompanies gravity. With gravity, quantum indeterminacy alone is not enough, nonlocal correlations are also unavoidably required. This brings to further appreciate how convenient are the mentioned proofs [30] of the generalized bound and [32, 33] of the generalized second law, or [55] of the Bekenstein bound, in that their scope extends to general nonlocal circumstances, explicitly refraining from any assumption about a local microscopic relation between energy and entropy. What the present discussion might add perhaps, is a way to see first hand that with gravity you necessarily get some form of nonlocality, no matter how hard you struggle to keep off it. This apparently selects nonlocality as an unavoidable ingredient in the structure of spacetime.

3 Spacetime and (a bit of) statistical mechanics

3.1 (Small-scale) nonlocality built in: Minimum-length metric

What we have seen in the above adds but a small piece to the gigantic mosaic of results which show the need for a minimal length when generically combining general relativity and quantum mechanics (a review can be found in [60, 61] and references therein). In what we described, the specific stress has been on that, based on entropy bounds, nonlocality appears as a characteristic feature accompanying gravity, shaping in particular the small-scale texture of spacetime through the existence of a minimal length. Next logical step is to try to make sense of this form of irreducible nonlocality of geometry.

A natural way to implement this is to require that no observable can distinguish between two events when their classically-expected separation is small enough; this might correspond indeed to the intrinsic impossibility to account dofs below a certain separation scale, this coming, as discussed above, from the entropy bound (2). The distance itself between events, in particular, should be such that it can not tell any better than the length scale of this unavoidable nonlocality when the two events are brought so near to each other to classically coincide.

What we ought to describe is a spacetime whose geometry generically coincides on large scales with the classical semi-Riemannian geometry of general relativity, but deviates from it in the small scale, very strongly as soon as two events \( p, P \) come to be very near to each other, giving a finite limit geodesic distance \( L \) (in principle \( O(l_p) \),
but more generally we might think also larger) between them when \( p \to P \). At an effective level, one might want this transition to be smooth.

The generic necessity of such a modified description of spacetime has been recognized long since. Yet the specific mathematical characterization of it has turned out to be not an easy task at all, not least because any metric-like tensor which converts a separation into a distance ought to be divergent at a point if it has to provide a finite result in the coincidence limit; one has also to find a way to implement the limit distance as a Lorentz invariant notion. But above all, there is no guidance (in absence of an agreed-upon quantum theory of gravity) on which physical aspect should drive the crossing of the classical-to-quantum interface in the small separations, and e.g. signature changes ought also to be contemplated possibly bringing spacetime at the smallest scales to be Euclidean (see e.g. [62]).

Only quite recently a modification of spacetime geometry along the lines above has found a solution, as something of a miracle, in [6, 63] and, after further specification, in [64]. This has been done in a framework which formally assumes that there is no change of signature when going to the smallest scales. As we will realize below, the procedure works equally well both in the Lorentzian and the Euclidean cases; therefore the results in the Euclidean case might be anyhow a useful description were the metric at the smallest scales Euclidean. What is missing is the actual handling of a possible change of signature while going to coincidence; this might be worth exploring, for example along what suggested and developed in [65–67].

In the aforementioned solution, the nonlocality intrinsic to the existence of a limit geodesic distance \( L \) is assumed to be captured by going from tensors to bitensors. Of these, the Synge’s world function \( Sy(p, P) \) [68] stands out, since when given between any two events in a geodesically convex region of spacetime it completely characterizes the metric properties of that region. The prescription is then to modify \( Sy(p, P) \), or the squared interval \( \sigma^2(p, P) = 2Sy(p, P) \), between any two assigned events (assumed to be one in a normal neighborhood of the other) to a new squared interval \( S_L \) which is assumed to depend on \( \sigma^2 \) alone, \( S_L = S_L(\sigma^2(p, P)) \). The general framework is that equigeodesic surfaces for the ordinary metric do result equigeodesic surfaces of the same nature (spacelike, timelike, null) in the new metric, down to the smallest scales.

To ensure that spacetime with the new squared interval biscalar \( S_L \) has the metric properties just mentioned, one requires that i) \( S_L(\sigma^2(p, P)) \) approaches \( \sigma^2(p, P) \) for large separations, and ii) \( S_L(\sigma^2(p, P)) \to L^2 \) when \( p \to P \) along a spacelike geodesic, and similarly \( S_L(\sigma^2(p, P)) \to -L^2 \) if along a timelike geodesic (we use mostly positive signature for the metric \( g_{ab} \)). With these positions, the null cone from \( P \) is unavoidably a discontinuity surface for \( S_L \) (we get \( L^2 \) when approaching it from one side and \(-L^2 \) from the other); the request that null equigeodesic surfaces according to the ordinary metric be null also for the qmetric fixes \( S_L = 0 \) on the null cone at any \( p \) distinct from \( P \). Notice that the rule that for any given pair of events \( S_L \) depends only on \( \sigma^2 \) applies unaltered also to the \( \sigma^2 = 0 \) case.

\( \sigma^2 \) is clearly geometrical, it is a squared distance; then, since for any assigned \((P, p)\), \( S_L \) depends only on \( \sigma^2 \), the prescription just given is also completely geometrical, independent of the charts we may use to map the manifold. This in particular implies that whenever time- or space-separated events \( P \) and \( p \) are near enough that
they can be both described by a single local Lorentz frame, based e.g. at $P$, $S_L$ is the same whichever is the actual Lorentz frame we choose, i.e. it is locally a Lorentz invariant notion.

Clearly Synge’s world function is related to the metric tensor, the main tool describing the metric properties of a manifold. The relation between the two is best captured by the following formula (written here in terms of $\sigma^2(p, P)$ instead of $Sy(p, P)$) [69]

$$g^{ab}(p) \left( \partial_a |^p \sigma^2 \right) \left( \partial_b |^p \sigma^2 \right) = 4\sigma^2 = g^{ab}(P) \left( \partial_a |^p \sigma^2 \right) \left( \partial_b |^p \sigma^2 \right),$$

with the superscripts in the derivatives denoting the events at which the derivatives are taken. As said, the nonlocality associated to the new squared intervals $S_L$ is assumed to be captured by bitensors. In particular we have to think of a metric-like bitensor $q_{ab} = q_{ab}(p, P)$ (which usually goes under the name of minimum-length metric or quantum metric or simply qmetric) as replacing the metric tensor at $p$, $g_{ab}(p)$, in providing the squared interval to $p$ (which we think of as a field point) from $P$ (taken as base point). Clearly, $q_{ab}$ must be related to $S_L$ by a formula like (11), i.e.

$$q^{ab}(p, P) \left( \partial_a |^p S_L \right) \left( \partial_b |^p S_L \right) = 4S_L,$$

where partial derivatives are taken with respect to coordinates, which are the same as above.

Because the equigeodesic surfaces from $P$ in the ordinary metric act also as equigeodesic surfaces of the qmetric, any vector orthogonal to an equigeodesic surface in the ordinary metric at an event $p$, and then tangent to the geodesic from $P$ according to the same metric, is also tangent to a geodesic from $P$ according to the qmetric. This brings with it that the same set of events which act as image of a geodesic curve from $P$ according to the ordinary metric are also image of a geodesic according to the qmetric, i.e. when we go to the qmetric we map images of geodesics to images of geodesics; what changes is when a given parameterization is affine.

Our aim is now to find the expression of the qmetric in terms of quantities relative to the ordinary metric. Denoting with $t^a$ the ordinary unit tangent to the geodesics at $p$ ($t^a = dx^a/ds$ with $s$ ordinary (taken positive) geodesic distance and $x^a$ the coordinates), the requirements (i) and (ii) above lead to guess for $q^{ab}$ a form like [6, 64]

$$q^{ab} = \frac{1}{A} g^{ab} + \epsilon \left( \alpha - \frac{1}{A} \right) t^a t^b,$$

($\epsilon \equiv g_{ab} t^a t^b = \pm 1$), with $A = A (\sigma^2(p, P))$ a biscalar unspecified at this stage apart from having $A \to \infty$ in the coincidence limit $p \to P$ and $A \to 1$ for large separations, and $\alpha = \alpha (\sigma^2(p, P))$ another biscalar designed to be responsible for giving a finite limit distance for $p \to P$ and subject to the constraint $\alpha \to 1$ for diverging separations. Indeed, we see that in this way (for any non vanishing, possibly diverging $\alpha$ when $p \to P$) $q^{ab} \simeq \epsilon \alpha t^a t^b$ in the $p \to P$ limit, and $q^{ab} \simeq g^{ab}$ when separations are large. From $q^{ac} q_{cb} = \delta^a_b$, the covariant components of qmetric have the form [6, 64]
\[ q_{ab} = A g_{ab} + \epsilon \left( \frac{1}{\alpha} - A \right) t_a t_b, \]  
(14)

with \( t_a = g_{ac} t^c \).

Since (13) can be recast as \( q_{ab} = \frac{1}{\alpha} h_{ab} + \epsilon \alpha t^a t^b \), with \( h_{ab} = g_{ab} - \epsilon t^a t^b \) the metric transverse to \( t^a \), we see that \( \alpha \) alone is involved in (12) [6], for as mentioned already, since \( S_L \) depends on \( \sigma^2 \) alone, \( \partial_a S_L \) is directed as \( \partial_a (\sigma^2) \) and has then no transverse component. From

\[ q^{ab} (\partial_a S_L)(\partial_b S_L) = \epsilon \alpha (t^a \partial_a S_L)^2 = 4S_L \]  
(15)

one can get [6]

\[ \alpha = \frac{S_L}{\sigma^2} \frac{1}{\tilde{S}'_L^2}, \]  
(16)

with \( S'_L \equiv dS_L/d(\sigma^2) \). Clearly \( \alpha \to 1 \) for diverging separations, and, as also required, does not vanish (it diverges actually) for any \( S_L \) whose \( S'_L \) does not diverge when \( p \to P \).

Another input is needed to fix \( A \). Here basic results from quantum gravity come to the rescue. In particular, quantum gravity effects are expected to change the coincidence limit of the Green’s function of a free relativistic particle from \( \sim 1/\sigma^2 \) to \( 1/(\sigma^2 + \ell^2) \) with \( \ell = \mathcal{O}(l_p) \) [70, 71] (for review, [60, 61]). This, which motivated in the first place to look at squared distances (instead of e.g. to consider the metric itself) when trying to set up a description of a quantum spacetime [63], suggests to request the following: the qmetric Green’s function \( G_{(q)} \) -meant as \( G_{(q)}(\sigma^2) = G(S_L(\sigma^2)) \) where \( G(\sigma^2) \) is the ordinary Green’s function \( G \) do satisfy the same equation in terms of the qmetric d’Alembertian \( \Box_{(q)} \) as the ordinary Green’s function \( G \) does for the ordinary \( \Box \) [6, 63, 64]. Equivalently, if we instead define \( G_{(q)} \) as what satisfies the equation with \( \Box_{(q)} \), we require \( G_{(q)}(\sigma^2) = G(S_L(\sigma^2)) \). Since we are assuming that the new squared distances \( S_L \) depend only on \( \sigma^2 \), this condition has to be imposed on spacetimes in which the Green’s function \( G \) also depends only on \( \sigma^2 \), such as Minkowski spacetime or in all generality the maximally symmetric spaces [64]; clearly the qmetric we get with the \( A \) selected in this way will then apply to generic spacetimes. Adding to \( (i) \), \( (ii) \), we have thus the following requirement [64]: iii) \( \Box_{(q)} G_{(q)} = 0 \) when \( \Box G = 0 \) \( (p \neq P) \) in all maximally symmetric spaces all along the geodesic which from \( P \) goes through \( p \), with the qmetric Green’s function taken to be \( G_{(q)}(\sigma^2) = G(S_L(\sigma^2)) \). In case of a locally Euclidean metric, this is mirrored in taking the \( D \)-dimensional Laplacian as replacing the d’Alembertian.

The magic of it is that adding this third requirement completely and consistently (one gets the correct limits for \( p \to P \) and for diverging separations) specifies \( A \).

What one obtains [64] is (in general \( D \)-dim spacetime)

\[ A = \frac{S_L}{\sigma^2} \left( \frac{\Delta}{\tilde{\Delta}} \right)^{\frac{2}{\sigma^2}} \]  
(17)
where

$$\Delta(p, P) = -\frac{\det \left[ -\nabla^a \nabla^b \frac{1}{2} \sigma^2(p, P) \right]}{\sqrt{g(p)g(P)}}$$

(18)

is the van Vleck-determinant biscalar ([72–75]; see [69, 76, 77]) ($g$ is the (ordinary)
metric determinant), and the other biscalar $\tilde{\Delta}$ is $\tilde{\Delta}(p, P) \equiv \Delta(\tilde{p}, P)$ where $\tilde{p}$ is at
$\sigma^2(\tilde{p}, P) = S_L$ on the geodesic which goes through $P$ and $p$ on the same side as $p$. We can see that $A \to 1$ for diverging separations and, since $\Delta(p, P) \to 1$ when $p \to P$, $A$ diverges in the coincidence limit, as required.

Once $S_L = S_L(\sigma^2(p, P))$ is given, formulas (13) or (14) equipped with $\alpha$ and $A$ from (16) and (17), completely define the qmetric with base $P$ at any $p$ in any normal neighborhood of $P$, except at events null separated from $P$.

3.2 Minimum-length metric and null separations

Consideration of null separations appears not entirely straightforward. It is clear moreover that the examination of this case critically depends on whether the Lorentz structure is maintained at the smallest scales. In what follows, as already mentioned in the general description above, we assume this is the case.

$\sigma^2$, and $S_L$ with it, are identically vanishing along the null geodesic $\gamma$ from $P$, and equation (12), which has been key to obtaining a limit length, looses any track of the point $p$ to which it is applied; it is thus quite not immediately clear what might it imply on $\gamma$ the general mapping $\sigma^2 \rightarrow S_L$. The fact is that $\sigma^2$ and $S_L$ are both of no use as concerns the ability to pick out specific events along $\gamma$. In the impossibility to select $p$ along $\gamma$, how can we possibly give $q_{ab}(p, P)$?

Let consider at $P$ a local observer with velocity $V^a$, and (uniquely) fix an affine parameter $\lambda$ along $\gamma$, with $\lambda(P, P) = 0$, by requiring $V^a l^b g_{ab} = -1$ with $l^a = (d/d\lambda)^a$ (tangent to $\gamma$). This quantity is clearly perfectly fit for individuating events on $\gamma$ (in a Lorentz invariant way). Then the question is: does the general mapping $\sigma^2 \rightarrow S_L$ imply a $\lambda(q) = \lambda(q)(\lambda(p, P))$ to which $\lambda$ is mapped and which plays the role of affine parameterization according to the qmetric? How would it look like? What its relation with $S_L$?

Of great help for this is the fact that the given $\lambda$ can be thought of as (nonnegative) distance $l$ along the geodesic according to the observer $V^a$ [77]. Indeed, $l(p, P) = \int_{0}^{\lambda} -V^a l^b g_{ab} d\lambda' = \lambda(p, P)$, with $V^a$ parallel transported along $\gamma$. Since both the time and space quadratic distances $\sigma^2$ are sent to $S_L$, $l$ gets replaced in the qmetric by $\sqrt{S_L(l^2)}$, this actually showing that the mapping to new quadratic distances $S_L$ induces a mapping from $\lambda$ to a new parameter $\lambda(q)$, which we have for consistency to require as affine in the new metric, with

$$\frac{\lambda^2(q)}{\lambda^2} = S_L(l^2)/l^2.$$  (19)

Notice that $\lambda(p, P)$ is a biscalar strictly speaking defined in the submanifold $\Gamma$ of codimension 1 swept by all null geodesics emerging from $P$ with $V^a l^b g_{ab} = -1$. It can
however be extended infinitesimally off $\Gamma$. Any smooth extension can be described thinking to geodesics originating not only exactly from event $P$ with coordinates $X^a = (0, 0)$ in frame $V^a$ but simultaneously at time $X^0 = 0$ from a small ball $B$ of radius $r$ centered at $\vec{X} = \vec{0}$. All of them with $V^a b g_{ab} = -1$ and with $l^a$ at start at any point $\vec{X} \in B$ with spatial component in general slightly different, in an arbitrary fashion, from the direction of $\vec{X}$. We can then think of $\lambda(p, P)$, equipped with any smooth extension, as a biscalar defined in a neighborhood of $\Gamma$.

What we are doing is to take as $\lambda$ the proper time/space separations according to observer $V^a$. The requirements above of the time/space separation cases then actually translate into: $I$) $\lambda_q(p, P) \simeq \lambda(p, P)$ for $p$ diverging from $P$ along $\gamma$ (meaning, for $\lambda(p, P)$ diverging), and $II$) $\lambda_q(p, P) \to L$ for $p \to P$ along $\gamma$.

Similarly to the nonnull separation case, from these two requirements alone we can try to guess a form for $q^{ab}(p, P)$ for null separated $p, P$. As we did in that case, this turns out to be most conveniently done singling out in the expression of $g_{ab}$ the part transverse to the vector tangent to the geodesic, in present case to $l^a$. Since $l^a$ is null we need here an auxiliary null vector to characterize the transverse metric. We take it as $n^a = V^a - \frac{1}{2}l^a$. It has normalization $n^a l^a g_{ab} = -1$. In terms of it the part $h_{ab}$ of $g_{ab}$ transverse to $l^a$ reads $h_{ab} = g_{ab} + l_a n_b + n_a l_b$, and we can guess [78, 79]

$$q^{ab} = \frac{1}{A_G} g^{ab} + \left(\frac{1}{A_G} - \alpha_G\right) (l^a n^b + n^a l^b)$$

$$= \frac{1}{A_G} h^{ab} - \alpha_G (l^a n^b + n^a l^b), \tag{20}$$

where, analogously to the nonnull case, $A_G = A_G(\lambda(p, P))$ with $A_G \to \infty$ when $p \to P$ along $\gamma$ and $A_G \to 1$ when $\lambda(p, P)$ diverges, and $\alpha_G = \alpha_G(\lambda(p, P))$ responsible for giving a finite limit to $\lambda_q(p)$ for $p \to P$ on $\gamma$ and subject to the constraint $\alpha_G \to 1$ for $p$ diverging from $P$ along $\gamma$. Again from $q^{ac} q_{cb} = \delta^a_b$, the covariant components are

$$q_{ab} = A_G g_{ab} + \left(A_G - \frac{1}{\alpha_G}\right) (l_a n_b + n_a l_b). \tag{21}$$

We need an equation which replaces (12) (which keeps being true of course), in embodying the fact that geodesic distances $\sqrt{\epsilon \sigma^2}$ are mapped to new geodesics distances $\sqrt{\epsilon S_L}$, read here distances $\lambda = l$ along the null geodesic $\gamma$ according to $V^a$ are mapped to new distances $\lambda_q = \sqrt{S_L(l^2)}$. This amounts to require that in the qmetric (i.e. in the requirements ($I$) and ($II$) above) $\lambda_q$ is affine. This reads [79]

$$l^b_b \nabla_b^{(q)} l^{(q)}_a = 0, \tag{22}$$

where $\nabla_a^{(q)}$ is the covariant derivative in the qmetric $\left(\nabla_b^{(q)} l^{(q)}_a = \nabla_b l^{(q)}_a - \frac{1}{2} q^{cd} (-\nabla_d q_{ba} + 2 \nabla_{(b} q_{a)d}) l^{(q)}_c\right) [80]$, and $l^{(q)}_a = dx^a / d \lambda_q = l^a / d \lambda_q$ is the tangent
to the geodesic corresponding to the parameter $\lambda(q)$ (for which one can easily verify that $q^{ab}l^a(l_b)^{q} = 0$ in compliance with equation (12); $l^a_q = q_{ab} l^b_q$).

Doing the calculations [79] shows that equation (22) becomes

$$\frac{d\lambda}{d\lambda(q)} l_a \frac{d}{d\lambda} \left( \frac{d\lambda}{d\lambda(q)} \frac{1}{\alpha\Gamma} \right) - \left( \frac{d\lambda}{d\lambda(q)} \right)^2 \left( \frac{1}{\alpha\Gamma} - A\Gamma \right) l^b \nabla_c l_b = 0. \quad (23)$$

Here the second term of l.h.s. vanishes since $\partial_c(l^b) = 0$ whichever is the extension of a $l^a$ null off $\Gamma$. Then, the vanishing of first term requires $\alpha/\Gamma \rightarrow 1$ for $\lambda \rightarrow \infty$. The final expression is then [79]

$$\alpha/\Gamma = \frac{1}{d\lambda(q)/d\lambda}. \quad (24)$$

This can be compared with the expression we have for $\alpha$ for spacelike/timelike geodesics, Eq. (16). To this aim the latter can be conveniently recast in terms of affine parameters of the geodesics, and it then reads, as we can easily verify,

$$\alpha = \frac{1}{(ds(q)/ds)^2}, \quad (25)$$

where $s \equiv \sqrt{\epsilon\sigma^2}$ and $s(q) \equiv \sqrt{\epsilon S_L}$ are (nonnegative) geodesic distances for spacelike/timelike geodesics according respectively to the ordinary metric and the qmetric.

We see that Eq. (22), which further specifies Eq. (12) in the null case, leaves $A\Gamma$ unsettled, analogously to what happens in the nonnull case. Clearly we need an equivalent of (iii) for the null separation case. Here the problem is that the Green function $G(p, P)$ of the d’Alembertian diverges when $p$ and $P$ go to be null separated, then all along $\gamma$. How to deal with this when trying to require something like (iii)?

A way is to take $\Box$ and $G$ not exactly on $\gamma$ but at points $p'$ slightly off $\gamma$, thus time or space separated from $P$, and then consider the limit $p' \rightarrow p \in \gamma$. The fact is that at any such $p'$, in the limit $p' \rightarrow p$ the d’Alembertian of any function $f = f(\sigma^2)$ lends itself to be written as

$$\Box f = (4 + 2\lambda \nabla_a l^a) \frac{df}{d\sigma^2}, \quad (26)$$

where $\lambda$ and $\nabla_a l^a$ are taken at $p$, and analogously for the qmetric

$$(\Box f)(q) = \left( 4 + 2\lambda(q) \nabla_{q(q)} l^a_{q(q)} \right) \frac{df(q)}{dS_L(\sigma^2)}, \quad (27)$$

[78, 79], with $\sigma^2 = \sigma^2(p', P)$ along the (nonnull) geodesic $\gamma'$ through $p'$ from $P$. These relations allow the conditions on $f$ expressed in terms of $\sigma^2$ to be recast near $\gamma$ in terms of $\lambda$. 

\[ \text{Springer} \]
This is exactly what we need when trying to translate the condition \((iii)\) above for the d’Alembertian for space/time separations to the case of null separations. We take \(f = G\) and consider \(G(p', P)\) at \(p' \neq \gamma\) (thus \(G\) finite), \(p' \in \gamma'\) (nonnull), with \(p'\) going to approach \(p \in \gamma\), and require \((iii)\) along \(\gamma'\). From Eqs. (26) and (27), this translates into \(III\) \(\left(4 + 2\lambda_{(q)} \nabla^{(q)}_{a} l^{a}_{(q)}\right) \frac{dG_{(q)}}{dS_{L}(\sigma^{2})} = 0\) all along \(\gamma'\) \((p' \neq P)\), when \(4 + 2\lambda \nabla_{a} l^{a} = 0\) in the same, in all maximally symmetric spaces, with the qmetric Green’s function taken to be \(G_{(q)}(\sigma^{2}) = G(S_{L}(\sigma^{2}))\).

We see this is equivalent to require that along \(\gamma'\) \((p' \neq P)\)

\[
4 + 2\lambda_{(q)} \nabla^{(q)}_{a} l^{a}_{(q)} = 0 \tag{28}
\]

when

\[
4 + 2\lambda \nabla_{a} l^{a} = 0. \tag{29}
\]

Now, computations show that Eq. (28) when expressed in terms of quantities defined in the ordinary metric becomes [79]

\[
4 + 2\lambda_{(q)} \frac{d\lambda}{d\lambda_{(q)}} \nabla_{a} l^{a} + (D - 2) \lambda_{(q)} \frac{d\lambda}{d\lambda_{(q)}} \frac{d}{d\lambda_{(q)}} \ln A_{\Gamma} = 0. \tag{30}
\]

If we then go to consider Eq. (29) in particular near \(\tilde{p} \in \gamma\) with \(\lambda(\tilde{p}, P) = \lambda_{(q)}\) (i.e. at \(\tilde{p}' \in \gamma'\) approaching \(\tilde{p}\)), we have there

\[
4 + 2\lambda_{(q)} \left(\nabla_{a} l^{a}\right)_{\lambda = \lambda_{(q)}} = 0. \tag{31}
\]

Using this in (30) we get

\[
-2 \lambda_{(q)} \left(\nabla_{a} l^{a}\right)_{\lambda = \lambda_{(q)}} + 2 \lambda_{(q)} \frac{d\lambda}{d\lambda_{(q)}} \nabla_{a} l^{a} + (D - 2) \lambda_{(q)} \frac{d\lambda}{d\lambda_{(q)}} \frac{d}{d\lambda_{(q)}} \ln A_{\Gamma} = 0. \tag{32}
\]

Using a relation which exhibits \(\Delta\) as the ratio between the actual density of geodesics and the density were spacetime flat (here in the form specific to the null-congruence case) [77]

\[
\theta = \nabla_{a} l^{a} = \frac{D - 2}{\lambda} - \frac{d}{d\lambda} \ln \Delta \tag{33}
\]

with \(\theta\) expansion, we see that Eq. (32) is equivalent to

\[
\frac{d}{d\lambda} \ln \left[ \lambda_{(q)}^{2} \left( \frac{\Delta}{\Delta_{(q)}} \right)^{2} A_{\Gamma} \right] = 0. \tag{34}
\]
with $\tilde{\Delta} \equiv \Delta(\tilde{p}, P)$. This fixes $A_\Gamma$ apart from a multiplicative constant which is determined requiring $A_\Gamma \to 1$ for $\lambda \to \infty$. The result is

$$A_\Gamma = \frac{\lambda_{(q)}^2}{\lambda^2} \left( \frac{\Delta}{\tilde{\Delta}} \right)^{\frac{2-n}{2}}. \quad (35)$$

We see that $A_\Gamma$ diverges for $\lambda \to 0$ ($\Delta(p, P) \to 1$ and $\tilde{\Delta}$ bounded for $p \to P$) coherently with the form chosen for the qmetric. Equations (20) or (21) together with $\alpha_\Gamma$ and $A_\Gamma$ as given by (24) and (35) give the qmetric along any null geodesic from $P$. $A_\Gamma$ is quite similar to $A$ of Eq. (17) for time and space separations, which we can rewrite in terms of (nonnegative) geodesic distances $s, s_{(q)}$ as

$$A = \frac{s_{(q)}^2}{s^2} \left( \frac{\Delta}{\tilde{\Delta}} \right)^{\frac{2-n}{2}}, \quad (36)$$

with $\tilde{\Delta}$ the van Vleck determinant at $\tilde{p}$ with $s(\tilde{p}, P) = s_{(q)}$.

The construction works once we are given an event $P$, a null geodesic $\gamma$ from it and an observer $V^a$. This is at variance with the case of space and time separations, for which there is no need of explicitly considering a local frame. This stems from the fact that, given any two space- or time-separated events, the squared interval along the geodesic connecting them is uniquely determined, while the affine interval between two null-separated events is not (it is defined up to a multiplicative constant). This complication seems then unavoidable, we have to live with it.

In ordinary spacetime once we are given an event $p$ it is uniquely associated to it the metric $g_{ab}$ at $p$. By contrast, in a spacetime with minimum length the geometric object playing the same role of characterizing the metric properties, the qmetric $q_{ab}$, is not defined when giving $p$ alone. We have different nonequivalent geometric objects $q_{ab}$ at $p$ depending on the choice of the base event $P$, and we need to specify also the latter to fix the ambiguity, this fact embodying nonlocality. For null separations between $P$ and $p$ this geometric object is assigned only if we further specify also an observer $V^a$, i.e. we can assign to an event $p$ the qmetric only specifying in addition to the null-separated $P$ also an observer. Strictly speaking we should then write $q_{ab} = q_{ab}(p, P, V^a)$. For example, if we consider an assigned event $p \in \gamma$ we have $q_{ab} \approx g_{ab}$ according to an observer $V^a$ at $P$ which has $p$ at large space and time distance from $P$, and $q_{ab} \approx -\alpha_\Gamma(l^a n^b + n^a l^b)$ according to another observer $\tilde{V}^a$ which has $p$ at small space and time distance from $P$. Note that we do not have an (impossible) dependence of a geometric entity on the frame, but we do have a dependence of the geometric entity we assign on the frame.

In the local frame of $V^a$ at $P$ if the affine parameter $\lambda$ is taken to be space and time from $P$, we see a certain structure in the qmetric at any given $p \in \gamma$ as given by the just derived formulas. The convenient parameterization $\lambda$ should be seen as instrumental in extracting the structure of the qmetric; and consistency demands that, once we have it, this structure must be thought as attached not to $\lambda$ but to the events $p$ to which $\lambda$ points: when $p \in \gamma$ is at a small space and time from $P$ according to
\( V^a \), we see the effects of a limit length, regardless of any parameterization we may choose on \( \gamma \).

If we change the local frame, in the new frame \( \tilde{V}^a \) at \( P \) we see that same structure, but now in terms of space and times according to \( \tilde{V}^a \), and again irrespective of any parameterization on \( \gamma \), but depending only on the events \( p \). What happens then is that whichever is the local frame we are in we observe a same structure associated to the existence of a limit length. In this sense we have Lorentz invariance of the construction: the measured local structure of the qmetric around an event \( P \) (events null-separated from \( P \) included) is the same according to every local observer at \( P \); with this we mean that the geometric object we assign to compute distances is the same, exactly as it happens in ordinary spacetime in which this same object according to any observer is invariably the metric \( g_{ab} \).

Since the use of \( q_{ab}(p, P) \) is in computing distances from \( P \), and we know from the beginning that they vanish on \( \gamma \) (keeping Lorentz intact down to the smallest scales), it might seem that all this effort about \( q_{ab} \) on \( \gamma \) null is no big deal after all. We have to consider however that the specific form of \( q_{ab} \) on \( \gamma \) null is relevant for a number of topics, like e.g. to characterize the geometry of the \((D-2)\) spatial surface to which \( \gamma \) is orthogonal, or, if \( \gamma \) is a member of a congruence of geodesics, to characterize the geometric properties of the congruence.

### 3.3 ‘Points’ with finite area

With the given specifications for the null case, what we have obtained for null separations, when joined to the nonnull-separation results, allows to characterize the qmetric along any congruence of geodesics emerging from an event \( P \) in any local frame at \( P \), being the congruence timelike, spacelike or null. This opens the way to compute possible modifications to the Raychaudhuri equation due to the existence of a minimal length. But before that, let us consider a very basic feature accompanying the qmetric concerning the metric properties of the space transverse to the direction along which the qmetric is taken. It turns out that ‘areas’ on equigeodesic surfaces at \( p \) do not vanish in the coincidence limit \( p \to P \) and approach instead a finite value [81].

One might naively think this should be expected in a spacetime endowed with a limit on distances. But, to appreciate that this might not be at all a trivial matter attention should be payed to that the areas we are talking about are taken at \( p \), not \( P \). The prescriptions we have given in (i)-(iii) and (I)-(III), and which define the qmetric, refer to affine distances from \( P \). One has no reason then e.g. to expect that the distance between two events \( p \) and \( p' \) both on the same geodesic from \( P \) do not vanish as evaluated from \( P \) (i.e. with the qmetric based at \( P \)) when both approach \( P \); one would expect this, and as a matter of fact the qmetric volumes at \( p \) do not vanish in the limit \( p \to P \) [81] (in particular how quickly the volume approaches 0 can be used to infer the dimensionality of qmetric; taking a ball of radius \( s \) in \( D \)-dimensional qmetric Euclidean space, its volume \( V_D \) approaches 0 as \( V_D \sim s^2 \) when \( s \to 0 \) showing that the Euclidean space (and, one might wonder, also the physical space) is effectively 2-dim at the smallest scales [82], a result confirming several others concerning dimensional reduction in quantum gravity starting with [83, 84]; [85, 86] for review). It is then
quite surprising and tricky that in the coincidence limit finite areas do appear; besides, they show up orthogonally to the geodesic.

To see how this comes about we might follow the original derivation [81] which was given in the Euclidean case and then continued back to Lorentz to give $D-1$ limit areas orthogonal to spacelike and timelike geodesics. But let us arrive here to the same results in a slightly different way. We choose to stay in Lorentz and to use, beside Eq. (14) for the qmetric for spacelike/timelike geodesics, Eq. (21) for null geodesics. This brings to obtain with a similar procedure the coincidence limits for orthogonal areas both in the spacelike/timelike and in the null case.

The area of a small portion at $p$ of the equigeodesic hypersurface $\Sigma_p$ of timelike or spacelike geodesics from $P$ is according to the qmetric

$$d^{D-1}a(q) = \sqrt{-h(q)} d^{D-1}a,$$

with $d^{D-1}a$ the area in the ordinary metric, and $h(q)$ the determinant of the transverse qmetric

$$h(q)_{ab} = q_{ab} - \epsilon l_a(q) n_b(q),$$

from requiring $q_{ab} l_a(q) n_b(q) = -1$.

From (14) one can verify that this gives

$$h(q)_{ab} = A h_{ab},$$

and then

$$d^{D-1}a(q) = A^{-1} d^{D-1}a$$

(37)

[80].

For light rays $\gamma$, we have

$$d^{D-2}a_{\gamma(q)} = \sqrt{-h(q)} d\lambda(q) d^{D-2}a = d^{D-2}a_{\gamma(q)} d\lambda(q).$$

Here $d^{D-2}a_{\gamma(q)}$ is the qmetric volume element of the $(D-2)$-space transverse to $\gamma$, and $d^{D-2}a_{\gamma}$ the area of the same according to the ordinary metric. The transverse metric is

$$h_{ab}^{(q)} = q_{ab} + t_a^{(q)} n_b^{(q)} + n_a^{(q)} t_b^{(q)}$$

$$n_a^{(q)} = \frac{d\lambda(q)}{d\lambda} n_a,$$

from requiring $q_{ab} l_a^{(q)} n_b^{(q)} = -1$.

From (21) one finds [79], analogously to the nonnull case, $h_{ab}^{(q)} = A_{\Gamma} h_{ab}$ and

$$d^{D-2}a_{\gamma(q)} = A_{\Gamma}^{-1} d^{D-2}a_{\gamma}$$

(38)

If we now follow towards $P$ the geodesics selected by the assigned small area, the ordinary area elements intercepted on the equigeodesic surface decrease with decreasing $s$ or $\lambda$ going to vanish in the $p \to P$ limit, while in the qmetric, from the expressions (36) and (35) for $A$ and $A_{\Gamma}$ respectively, we definitely have in the same limit

$$\left( d^{D-1}a(q) \right)_0 = \lim_{p \to P} d^{D-1}a(q)$$

$$= L^{D-1} \frac{1}{\Delta_L} d^{D-1}\eta$$

(39)

and

$$\left( d^{D-2}a_{\gamma(q)} \right)_0 = \lim_{p \to P} d^{D-2}a_{\gamma(q)}$$

$$= L^{D-2} \frac{1}{\Delta_L} d^{D-2}\theta$$

(40)
the former applying to spacelike/timelike geodesics and the latter to light rays ([81] and [79]). Here \( \eta^i, i = 1, \ldots, D - 1 \), and \( \theta^A, A = 1, \ldots, D - 2 \) are coordinates labeling the geodesics (\( d^{D-1}\eta \) and \( d^{D-2}\theta \) are then constant while \( p \) approaches \( P \)) in orthogonal directions, with \( s \, d\eta^i \) being geodesic distances, and \( \lambda \, d\theta^A \) being distances according to observer \( V^a \). \( \Delta_L \equiv \Delta(\bar{p}, P) \) with \( \bar{p} \) the event at \( s(\bar{p}, P) = L \) on timelike/spacelike geodesics and at \( \lambda(\bar{p}, P) = L \) on light rays. \( \Delta_L \) is bounded for fixed \( L \) (except for pathological circumstances; generically, \( \Delta_L \approx 1 \) for \( L \) small; we will come back to this later), and this shows that the two limiting areas (39) and (40) are not zero.

### 3.4 Raychaudhuri’s equation gets modified: No focal points

Clearly the existence of a finite limit area can be expected to impact on the behavior of congruences of geodesics when dealing with singularities or, before that, even simply with focal points. Indeed the consideration as we did above of a congruence of geodesics emerging from \( P \), selects the event \( P \) as a focal point of the congruence. With no extra effort the results can also be applied in reverse direction since reversing the sign of the affine parameters \( s \) or \( \lambda \) allows to read the results relative to a congruence emerging from \( P \) as relative to a congruence converging to \( P \). Let us consider then congruences of timelike, spacelike or null geodesics emerging from an event \( P \) and ask ourselves how their geometry appears in the qmetric, considering specifically their expansion \( \theta = \nabla_a t^a \), or \( \theta = \nabla_a l^a \) in the null case. This has been studied in [80, 87, 88]. For spacelike and timelike geodesics from the expression of the qmetric covariant derivative [80] one finds that

\[
\theta(t) \equiv \nabla_a(t) t_a^{(q)} = \sqrt{\alpha} \left[ \theta + (D - 1) \frac{d}{ds} \ln \sqrt{A} \right], \quad (41)
\]

and for light rays

\[
\theta(l) \equiv \nabla_a(l) l_a^{(q)} = \alpha \Gamma \left[ \theta + (D - 2) \frac{d}{d\lambda} \ln \sqrt{A \Gamma} \right]. \quad (42)
\]

On using the expressions for \( \alpha \Gamma, \alpha, \) and \( A \Gamma, A \), as given in terms of affine parameters by Eqs. (24, 25) and (35, 36), and exploiting relation (33) for null congruences, as well as its homologous [77]

\[
\theta = \frac{D - 1}{s} - \frac{d}{ds} \ln \Delta \quad \text{(43)}
\]
for congruences of timelike and spacelike geodesics, the expansion in the qmetric turns out to be

$$\theta(q) = \frac{D - 1}{s(q)} - \frac{d}{ds(q)} \ln \tilde{\Delta} = \theta|_{s=s(q)} \quad (44)$$

and

$$\theta(q) = \frac{D - 2}{\lambda(q)} - \frac{d}{d\lambda(q)} \ln \tilde{\Delta} = \theta|_{\lambda=\lambda(q)} \quad (45)$$

respectively for spacelike/timelike and null cases [88]. Comparing these with Eqs. (33) and (43) which connect the expansion and van Vleck determinant in the ordinary metric, we see that a most direct way to get the expansion $\theta(q)$ in the qmetric is just to replace in that equations the affine parameters $s$ and $\lambda$ with $s(q)$ and $\lambda(q)$, and the van Vleck determinant $\Delta$ with $\tilde{\Delta}$. Taking Eqs. (44) and (45) as defining the van Vleck determinant $\Delta(q)$ in the qmetric -analogously to Eqs. (33) and (43) which by integration define $\Delta$ in terms of $\theta$ and $\lambda, s$ in the ordinary metric- we get $\Delta(q) = \tilde{\Delta}$, namely

$$\Delta(q)(p, P) = -\frac{1}{\sqrt{g(p) g(P)}} \det \left[ -\nabla_a \tilde{\rho} \nabla_b \sigma^P \frac{1}{2} \sigma^2(\tilde{\rho}, P) \right], \quad (46)$$

with $\tilde{\rho}$ at $\sigma^2(\tilde{\rho}, P) = S_L$ on spacelike/timelike geodesics, and at $\lambda(\tilde{\rho}, P) = \lambda(q)$ on null geodesics.

We can take $\theta(q)$ from Eqs. (44) and (45) and consider the limit $(\theta(q))_0$ we get when $p \to P$. We have

$$(\theta(q))_0 = \frac{D - \delta}{L} - \frac{d}{dL} \ln \Delta_L, \quad (47)$$

with $\delta = 1 \ (2)$ for spacelike/timelike (null) geodesics.

To make further statements about $(\theta(q))_0$ we need knowledge of $\Delta_L$. Expressions for $\Delta(p, P)$ are known through expansions in powers of affine intervals $s, \lambda$ [76]. One finds that [69, 77]

$$\Delta(p, P) = 1 + \frac{1}{6} \ell^2 R_{ab} v^a v^b + \mathcal{O}(\ell^3), \quad (48)$$

with $\ell = s, \lambda$ and $v^a = t^a, l^a$. The part $\mathcal{O}(\ell^3)$ contains, among others, terms with any power of the quantity $\ell^2 R_{ab} v^a v^b$ and derivatives of any order of it. The possibility for it to be really negligible depends critically on the value of $\ell^2 R_{ab} v^a v^b$ and its derivatives. From the mentioned geometrical meaning of van Vleck determinant as ratio between the actual density of geodesics to that were spacetime flat [77], we have that the sum in (48) is not diverging provided that $p$ is close enough to $P$ to be not
possible to be its conjugate. At $\ell = L$ Eq. (48) gives

$$\Delta_L = 1 + \frac{1}{6} L^2 R_{ab} v^a v^b + O(L^3),$$

from which,

$$\left(\theta(q)\right)_0 = \theta(\ell = L) = \frac{D - \delta}{L} - \frac{1}{3} L R_{ab} v^a v^b + O(L^2)$$

[88].

At generic conditions $L^2 R_{ab} v^a v^b \ll 1$. This gives $\Delta_L \approx 1$ and $(\theta(q))_0 \approx \frac{D - \delta}{L}$ finite. Generically then, caustics can not be formed in a qmetric spacetime, whichever is the geometry of the congruence [88]. However, at increasing curvature, meaning at increasing $R_{ab} v^a v^b$, $\theta(\ell = L)$ decreases and $\Delta_L$ has then to increase, and actually diverges for $R_{ab} v^a v^b$ large enough to have the event $\tilde{p}$ (which is defined by $\ell(\tilde{p}, P) = L$) conjugate of $P$, i.e. $\theta(\ell = L) = 0$. At these so huge values of $R_{ab} v^a v^b$, which we might think can be found near singularities of the ordinary metric, $\Delta_L$ diverges and $(dD - 1 a_{(q)})_0$ and $(dD - 2 a_{y(q)})_0$ of Eqs. (39) and (40) can become in principle $0$. In these circumstances the construction of the qmetric would hardly be applicable since we can no longer have the crucial prerequisite of the points $p$ and $\tilde{p}$ to be in a normal neighborhood of $P$ (and thus have a unique geodesic connecting $P$ and $p$).

There is something artificial however in such unboundedly high values of $L^2 R_{ab} v^a v^b$, in the sense that if we are really given a spacetime which endows a lower limit length $L$, $R_{ab} v^a v^b$ can arguably never diverge. Indeed, let us consider circumstances in which in ordinary spacetime unboundedly high values of $R_{ab} v^a v^b$ actually do develop. As a prototypical example we may take the collapse towards $P$ of a thin shell of matter with exact spherical symmetry on an otherwise flat Minkowski background. It is clear that, as soon as $p$ taken on the shell becomes nearer and nearer to $P$, arbitrarily high values of $R_{ab} v^a v^b$ can be reached; indeed these are circumstances of a classically blatant singularity formation. Yet, all along the path to $P$, never $p$ becomes conjugate of $P$ nor such becomes $\tilde{p}$ at $\ell(\tilde{p}, P) = \sqrt{\epsilon S_L}$, and the qmetric construction does apply all the way. Now, in the considered circumstances the qmetric prescriptions demand that for $p$ at any $\ell$ we consider $\tilde{p}$ on the same geodesic at the less evolved spacetime $\ell(\tilde{p}, P) = \sqrt{\epsilon S_L}$. This means that at coincidence, i.e. at the most extreme conditions, we have to look at the slightly antecedent situation corresponding to $\ell(\tilde{p}, P) = L$, at which everything is regular and the stress energy tensor, and thus the Ricci tensor, is finite. We see that in all this, beside a bounded Ricci tensor, we have avoidance of classically-blatant singularity formation [88]; further investigation might be worth doing, exploring singularity formation avoidance of the qmetric in more general settings.

Of help for this should be the qmetric Raychaudhuri equation

$$\frac{d\theta(q)}{d\ell(q)} = -\frac{1}{D - \delta} \theta(q)^2 - \sigma_{ab}(q) \sigma^{ab} - R_{ab} v^a v^b,$$
\( l(q) = s(q) \), \( \lambda(q) \) (\( \sigma_{ab} \) is shear; no twist, from hypersurface orthogonality of the congruence of geodesics emerging from \( P \)) [88]. In view of Eqs. (44–45) it takes the form

\[
-\frac{D - \delta}{\ell_{(q)}} - \frac{d^2}{d\ell_{(q)}^2} \ln \Delta = -\frac{1}{D - \delta} \left( \frac{D - \delta}{\ell_{(q)}} - \frac{d}{d\ell_{(q)}} \ln \Delta \right)^2 - \sigma_{ab}^{(q)} \sigma_{ab}^{(q)} - R_{ab}^{(q)} v_{(q)}^a v_{(q)}^b,
\]

\[d^2 \ln \Delta + \frac{2}{\ell_{(q)}} \frac{d}{d\ell_{(q)}} \ln \Delta - \frac{1}{D - \delta} \left( \frac{d}{d\ell_{(q)}} \ln \Delta \right)^2 = \sigma_{ab}^{(q)} \sigma_{ab}^{(q)} + R_{ab}^{(q)} v_{(q)}^a v_{(q)}^b, \tag{52}\]

from which the coincidence limit of the r.h.s. can be extracted [88].

Again concerning singularity formation, an interesting line of research put forward recently is their consideration in field-space rather than solely in spacetime [89, 90]. It has been shown indeed (explicitly in a cosmological setting) that certain spacetime singularities are not such in field-space and give place to observables which are well-defined [91, 92]. One aspect worth investigating at this regard might be if and how the fields are affected by the existence of a limit length on the geometric side.

### 3.5 Einstein’s equations as a statistical-mechanical result

The existence of a finite area at coincidence [81], is a key prediction of the qmetric (Eqs. (39) and (40)). It describes spacetime at the smallest scales as possessing a structure, and tells something about the latter; in particular it is suggestive of elementary areas as what are made up events. This structure hints to dofs associated with it, which clearly we ought to think of as proper of spacetime itself. This can tentatively offer a way to capture, in a very basic manner, some features of the microscopics of the gravitational field, this providing a basis for some statistical-mechanical description of the latter, even though in absence of a fully convincing quantum theory of gravity and, moreover, irrespective of the specific form it might or will take. The meaning would be to get some specific results from the mere existence of a minimal length (a generic prediction, the latter, shared by many quantum approaches to gravity); these results are generic, they do not select any theory in particular, but act as constraints on viable theories. In addition they might offer some hints on how to conveniently shape or think of a quantum theory of gravity.

Generically, when considering the states we can assign to two ‘adjacent’ (in some sense better defined below) independent events \( P \) and \( P' \), we expect their total number is the product of the states constituting \( P \) and \( P' \) separately. Looking at Eqs. (39), (40), we see that the micro-based entities that are associated to events \( P \) and \( P' \) sum up instead, since the total limit area corresponding to the two is clearly the sum of the two areas. This consideration suggests to regard the quantities to the r.h.s.’s of (39) and (40) as expressing numbers of dofs (kind of elementary dofs or ‘atoms’ of spacetime [81, 93, 94]), not numbers of states. Inspecting the form of the r.h.s.’s we are led moreover to interpret the quantity \( 1/\Delta_L \) as (proportional to) the number \( N_G \) of gravitational
dofs building up the event $P$ in the direction $v^a$, to mean the number of dofs in area $L^{D-1}$ (area $L^{D-2}$ for $v^a = l^a$) transverse to $v^a$. We can write

$$N_G = N_G(x, v) = (N_G)_0 \frac{1}{\Delta L} = C \frac{1}{\Delta L},$$  

(53)

[81], where $x, v$ are short for coordinates $x^a$ and tangent vector $v^a$ at $P$, and $(N_G)_0$ is the same as $N_G$ but for Minkowski. $(N_G)_0$ can not depend on $x, v$ due to the symmetries of Minkowski, and to simplify notation in the last equality we denote it as $(N_G)_0 \equiv C$ with $C$ being a positive number, independently defined (and thus different) in each set of geodesics from $P$, i.e. if timelike, spacelike or null. Using the series expansion (49) for $\Delta L$ this can be expressed as

$$N_G = C \left(1 - \frac{1}{6} L^2 R_{ab} v^a v^b\right) + \mathcal{O}(L^3).$$  

(54)

From the mentioned geometrical meaning of van Vleck determinant as ratio of densities of geodesics emanating from an event, we can geometrically view $N_G$ as $C ((N_G)_0)$ times the ratio between the density of geodesics emerging from $P$ were spacetime flat and the density in actual spacetime [81].

Having this, we can consider beside gravitational dofs also matter dofs $N_m$ in $L^{D-1}$ (or $L^{D-2}$ for $v^a = l^a$) associated to matter entropy at the coincidence limit. With both, we could then proceed to describe the equilibrium configuration they eventually form as what we get from extremizing total entropy or total number of microscopic configurations [81] according to a principle of maximum entropy [95, 96]. This has been investigated in [81, 93, 94, 97, 98] (for review, [99]), in particular concerning the possibility to arrive this way to field equations. It turns out that this is indeed possible, and in the limit $L^2 R_{ab} v^a v^b \ll 1$ one gets Einstein’s field equations with cosmological constant, the latter arising as an integration constant.

In $D = 4$ the extremization of entropy leads to the condition

$$-\frac{1}{6} C L^2 R_{ab} v^a v^b + L^4 T_{ab} v^a v^b = \lambda(x) g_{ab} v^a v^b,$$  

(55)

where the second term in the l.h.s. comes from matter dofs $(N_m = L^4 T_{ab} v^a v^b [= L^4 \frac{1}{\hbar c} T_{ab} v^a v^b])$; this, in case e.g. of $v^a$ timelike, from $N_m = \delta Q / T$ with energy $\delta Q$ in $L^3$ given by $\delta Q = L^3 T_{ab} v^a v^b$ as probed along the geodesic, at temperature $T \approx 1/L$, this coming from $\delta Q$ having to be in the volume $L^3$ [98]) and the extremization is taken indifferently in the sets of spacelike or timelike or null tangent vectors $v^a$ at $P$. $\lambda$ is arbitrary, function of $x$ only not of $v$. In the case of light rays, $v^a = l^a$, this gives

$$-\frac{1}{6} C L^2 R_{ab} l^a l^b + L^4 T_{ab} l^a l^b = 0,$$  

(56)
which implies

\[ -\frac{1}{6} C L^2 R_{ab} + L^4 T_{ab} = f(x) g_{ab}, \]  

(57)

with \( f \) function of \( x \). Here, \( \nabla_b G^b_a = 0 \) joined with \( \nabla_b T^b_a = 0 \) implies

\[ -\frac{1}{12} C L^2 \partial_a R = \partial_a f, \]

which gives \( f = -\frac{1}{12} C L^2 R + \text{const} \). Writing \( f + \frac{1}{12} C L^2 R = \text{const} \equiv \frac{1}{6} C L^2 \Lambda \) with \( \Lambda \) independent from \( x \), one easily verifies that Eq. (57) becomes

\[ \frac{1}{6} C L^2 \left( R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} \right) = L^4 T_{ab}, \]

that is

\[ G_{ab} + \Lambda g_{ab} = \frac{6}{C} L^2 T_{ab}, \]

\[ \left[ \frac{6}{C} L^2 \frac{1}{\hbar c} T_{ab} \right], \]

(58)

with the constant \( \Lambda \), we see, actually playing the role of the cosmological constant.

Now, since Einstein’s field equations read

\[ G_{ab} + \Lambda g_{ab} = 8\pi T_{ab} \]

(58) is actually Einstein’s equations if \( l_p^2 = L^2 / \left( \frac{4}{3} \pi C \right) \). Using this in the extremality condition (56) we get

\[ -\frac{1}{6} C L^2 \frac{8\pi l_p^2}{8\pi l_p^2} \frac{R_{ab} l^a l^b}{8\pi l_p^2} + L^4 T_{ab} l^a l^b = 0 \]

\[ L^4 \left( -\frac{R_{ab} l^a l^b}{8\pi l_p^2} \right) + L^4 T_{ab} l^a l^b = 0 \]

\[ H_G + H_m = 0 \]  

(59)

[81, 93, 94, 97], with \( H_G \equiv -R_{ab} l^a l^b / (8\pi l_p^2) = \left[ -\frac{1}{8\pi} \frac{\hbar c}{l_p^2} R_{ab} l^a l^b \right] \) and \( H_m \equiv T_{ab} l^a l^b \).

Coming back to the specifications just below Eq. (1) (with \( k^a = l^a \)), we see that \( H_m = \delta Q_H / (A_H l) \), is the heat flux brought by matter when crossed by the horizon in time \( dt = l \). Indeed, \( H_m \) is energy (in matter’s frame) supplied by matter per unit (engulfed) proper volume, or also matter energy per unit cross-section area of horizon per unit (matter) crossing time.

The \( H_G \) term can also be a heat flux, but relative to spacetime instead of matter. It can be interpreted as the heat generated by spacetime-induced viscous stresses on our congruence considered as a null fluid, in reaction to the presence of matter [101, 102]. This can be understood as follows. We start from the identity (see e.g. [100], equation (A.56) there)

\[ \nabla_a (\theta l^a) + \nabla_a l^b \nabla_b l^a - \theta^2 = -R_{ab} l^a l^b, \]

valid for a generic null congruence, which we apply to our affine congruence (in general non-affine parameterization \( \theta = \nabla_a l^a + \kappa \), with \( \kappa \) defined by \( l^b \nabla_b l^a = \kappa l^a \); in our affine parameterization \( \theta = \nabla_a l^a \)). From \( \nabla_a l^b \nabla_b l^a = \sigma_{ab} \sigma^{ab} + \frac{1}{2} \theta^2 \) in our
case (as can be easily worked out, starting e.g. from [103]), Eq. (60) becomes

\[ \nabla_a (\theta l^a) + \sigma_{ab} \sigma^{ab} - \frac{1}{2} \theta^2 = -R_{ab} l^a l^b, \quad (61) \]

or

\[ \nabla_a (\theta l^a) + \sigma_{ab} \sigma^{ab} - \frac{1}{2} \theta^2 = 0 \quad (62) \]

in vacuum.

On the other hand, the heat flux \( H \) generated by viscous stresses in a fluid can be expressed as [104, 105]

\[ H = 2 \eta \sigma_{ab} \sigma^{ab} + \zeta \theta^2, \quad (63) \]

with \( \eta \) and \( \zeta \) coefficients of shear and bulk viscosity respectively. At equilibrium, one expects no entropy generation by viscous heating, and thus, in case the viscous stresses are the only source of entropy,

\[ 2 \eta \sigma_{ab} \sigma^{ab} + \zeta \theta^2 = 0. \quad (64) \]

Now, from general results concerning null fluids and black hole membrane paradigm [53, 54, 106] one has

\[ \eta = \frac{1}{16\pi}, \quad \zeta = -\frac{1}{16\pi}, \quad (65) \]

This implies that at equilibrium we must have \( \sigma_{ab} \sigma^{ab} = \frac{1}{2} \theta^2 \). Using this in (62) we see that the equilibrium condition corresponds to \( \nabla_a (\theta l^a) = 0 \). For null hypersurfaces generic we have in general no reason to expect \( \nabla_a (\theta l^a) = 0 \), but based on the above it seems we can say that a null hypersurface is actually a description of a null fluid at equilibrium when \( \nabla_a (\theta l^a) = 0 \).

The introduction of matter brings additional stresses in the null fluid as can be inferred from looking at (61). We have a new source of heating and then of entropy. These are stresses induced by spacetime (from the presence of matter) and the heat flow they induce can be read in (61) (using (63) and (65)) as \( \frac{R_{ab} \rho_j}{8\pi} \). A new equilibrium is reached when the total entropy generation, including that sourced by spacetime-induced heat, vanishes. This corresponds to \( 2 \sigma_{ab} \sigma^{ab} - \theta^2 + 2 R_{ab} l^a l^b = 0 \) and again \( \nabla_a (\theta l^a) = 0 \). We thus have equilibrium when

\[ -R_{ab} l^a l^b = \sigma_{ab} \sigma^{ab} - \frac{1}{2} \theta^2, \quad (66) \]
and then when the viscous heating in presence of matter, which is $H_G = 2 \eta \sigma_{ab} \sigma^{ab} + 
abla \theta^2 = \frac{1}{8\pi} \sigma_{ab} \sigma^{ab} - \frac{1}{16\pi} \theta^2 \neq 0$, is given by

$$2 \eta \sigma_{ab} \sigma^{ab} + \nabla \theta^2 = -\frac{R_{ab} \Gamma^{ab}}{8\pi} = -\frac{\hbar c}{l_p^2} \frac{R_{ab} \Gamma^{ab}}{8\pi}$$

(67)

[101, 102, 107].

Given the meaning of $H_G$ and of $H_m$ the extremality condition (59) acquires an interpretation as total heat flux in the null hypersurface = 0 [81, 93, 94, 97]. That is: the heat flux brought in by matter crossing the null hypersurface, call the latter a horizon, plus the heat flux associated to the internal stresses of the horizon must vanish, a result which is known as ‘dissipation without dissipation’ or also as ‘zero-dissipation principle’ [58, 93, 94, 97]. But, we showed that (provided we set a suitable link between the constant $C$ and $l_p$) the extremality condition means Einstein’s equations (with cosmological constant). The statistical extremization procedure we have described then shows that what Einstein’s equations really are is the statement that total heat flux is 0 [81, 93, 94, 97].

3.6 On the nature of Einstein’s equations

The importance of this result, by Paddy, can hardly be overstated. This way to look at Einstein’s equations indeed results in something far more satisfactory than the conventional one. Indeed the equality $G_{ab} = 8\pi T_{ab}$ (ignoring here the cosmological term) is between two quantities which could not be more different in their own nature: on one side exact pure geometry, on the other all the stuff related to matter; we might call it an equality between marble (geometry) and wood (matter quantum fields) following [108], or between apples and oranges in Paddy’s own words [99], and this without any link to all the thermodynamic meanings found for gravity in the many years from when these equations were first formulated.

Precisely this unsatisfactory element of the conventional perspective lead Paddy to look for a description of gravity embodying thermodynamics since start, seeking for a picture of field equations as what comes from the extremization of a suitable entropy functional capable to include matter and gravity. Well before the minimum-length spacetime entered the scene, he and collaborators found this functional, and the extremization turned out to consist in imposing an equality between gravitational and matter entropy variations on null hypersurfaces, and resulted equivalent to field equations (with cosmological constant) [109, 110] (see also [5]).

With the advent of the minimum-length description of spacetime this functional nicely was found to coincide, up to a total divergence which can be ignored in the extremization procedure, with $N_G + N_m$, i.e. the dofs of spacetime and matter described above [99]. Thus, even in the absence of a full theory of quantum gravity and then of an account of gravity intrinsic dofs, the approach with minimum-length spacetime is capable of giving to that thermodynamic description of field equations a statistical-mechanical basis, sort of a hint towards a possible statistical origin of gravity along what sought for from entropy bounds (cf. the paragraph just above that of Eq. (9)).
Looking at Eq. (1) describing Jacobson’s result [14], we recognize in it the terms $H_m$ and $H_G$ (with $k^a = l^a$), and that equation too can actually be read as a total heat flux = 0. But, building on Jacobson’s, Paddy’s result opens new avenues of research by going definitely beyond it in at least two respects.

One is the fact that in Jacobson’s result [14] the heat flux $H_m$ is balanced by the heat flux associated to entropy variation of the horizon assuming this entropy is proportional to horizon’s area. In Paddy’s derivation on the contrary, there is no such assumption: the heat flux $H_G$ is that computable from the viscous stresses of the null surface making up the horizon, in reaction to the presence of matter.

The other one is the fact that Jacobson’s is a thermodynamic balance equation; Paddy’s is instead a statistical-mechanical one. As such it arises from a description in terms of micro dofs. In this respect Paddy’s result goes beyond Jacobson’s in the same sense that statistical mechanics goes beyond thermodynamics. It provides indeed a more accurate picture of the physical world, in that it takes note, and then uses in the mathematical description, kind of particle nature of the constituents. With Paddy we can talk of particle-like constituents for spacetime, meaning with this what is responsible for its micro dofs.

3.7 Classical gravity can not be considered to be the $L \rightarrow 0 (\hbar \rightarrow 0)$ limit

The derivation of field equations is universally done from extremization of a suitable action with respect to the field, and gravity is no exception with the metric being the field. One would like to know how to reconnect an extremization of this kind with finding the field equations as an extremization of entropy along the lines described above.

In case of Einstein’s equations the Lagrangian in the action is the Ricci scalar $R$. Considering things at thermodynamic level (as opposed to a statistical-mechanical one), the relation between entropy extremization and derivation from variation of the action has been discussed in [111]. What has been found is that the thermodynamic extremization can be seen as mathematically equivalent to varying a modified, suitable action (the ‘Augmented variational principle’ of [112]) with respect to the field.

One would however also investigate which kind of connection exists at the statistical-mechanical level, meaning with this what we obtained with the minimum-length description of spacetime. Crucial for this is to find out the expression $R(q)$ of the Ricci scalar in the qmetric.

The calculation of $R(q)$ is not an easy task since the qmetric covariant derivative has terms additional to ordinary covariant derivative (see right below Eq. (22)) and with further manipulations the algebraic complications diverge. Fortunately, a relatively quick way to compute it has been devised. The basic idea is to avoid the calculation of the components of the Ricci tensor (going then to take the trace) and to calculate instead directly the Ricci scalar resorting to Gauss-Codazzi relations for the equigeodesic hypersurfaces [6, 64, 80].

The nonlocal Ricci scalar $R(q) = R(q)(p, P)$ one thus obtains has the property that if we take the coincidence limit $p \rightarrow P$ and then consider the limit $L \rightarrow 0$ we get (in D-dim spacetime)
\[ R(q)(p, P) \rightarrow \epsilon \, D \, R_{ab} t^a t^b \]  

(68)

for spacelike/timelike geodesics with (unit) tangent \( t^a \) \([6, 64]\). This has been extended to include the case of null geodesics \([113]\) (from Gauss-Codazzi as applied to null hypersurfaces \([114, 115]\)) getting

\[ R(q)(p, P) \rightarrow (D - 1) \, R_{ab} l^a l^b, \]  

(69)

with \( l^a \) the null tangent vector.

A most intriguing feature of these results is that the coincidence-limit qmetric Ricci scalar does not tend to \( R \) when \( L \rightarrow 0 \) as one would have instead naively expected. We do not recover ordinary spacetime when letting \( L \rightarrow 0 \). If we think of the minimal length as \( L = C \, l_p \) with \( C \) a constant, this means that we do not get classical spacetime in the \( \hbar \rightarrow 0 \) limit. Even more so in the general situation in which \( L \) may not vanish with \( \hbar \).

Moreover, the limiting values are proportional to terms like \( R_{ab} t^a t^b \) and \( R_{ab} l^a l^b \), namely right the terms we get in Eq. (54) above when counting the micro dofs of gravity. Things go as if endowing spacetime with a minimum length, read giving spacetime quantum characteristics, turns the Lagrangian into counting gravitational dofs, and the extremization through a variational principle into an extremization of entropy. For any nonvanishing \( L \) this statistical-mechanical machinery works, providing the field equations of classical gravity. From inspection of Eq. (58), note that the field equations become singular in the \( \hbar \rightarrow 0 \) limit. This shows that we can not get classical gravity when \( \hbar \rightarrow 0 \); to have classical gravity (the field Eq. (58)) we definitely need \( \hbar \neq 0 \) (cf. \([81]\)).

On top of this, since this correspondence \{action extremization\} \rightarrow \{entropy extremization\} remains there no matter how small \( L \) is, the terms \( R_{ab} t^a t^b \) and \( R_{ab} l^a l^b \), with all their thermodynamic significance, are sort of echo and witness of the underlying quantum structure \([6]\), with the latter made this way visible even when \( L \) (read the Planck length) is hopelessly (for direct experimental detection) small.

### 3.8 Clues about a small-scale quantum description

We see in the formulas above that the \( L \rightarrow 0 \) value of the qmetric Ricci scalar depends on the direction of approach to \( P \). Since the Ricci scalar ought to be considered as a quantity given with the manifold and \( P \), this of the dependence on the direction is a peculiar feature that calls for further understanding (a proposed interpretation is in \([116]\)). The ordinary Ricci scalar \( R \) is recovered (in the limits above) when consistently averaging over all geodesics from \( P \) \([117]\). All this might hint to that what the minimum-length metric (to some extent) captures is a quantum structure for spacetime at \( P \), and that it is by averaging, or taking expectation values we would say, that we can reconnect \( R(q) \) with the ordinary \( R \). This structure would be encoded and witnessed in the terms \( R_{ab} t^a t^b \) and \( R_{ab} l^a l^b \) and their thermodynamic baggage. One can also try to characterize this local quantum structure in an operational way, thinking of the event \( P \) as a coincidence between a quantum reference body (\( B \))
and a test particle (\(T\)). In this case, pure states for the system \(B \otimes T\) (corresponding to definite directions of approach to \(P\)) turn out to be mixed for system \(B\) alone (due to the nonvanishing limiting transverse area which gives an irreducible finite probability for \(B\) to wrongly guess the arrival direction of \(T\)). Considering photons as \(T\) the quantity \(R_{ab} I^a I^b\) emerges once more, this time in the average information gain \(I\) associated to \(B\) finding the photon on nominal geodesics (tangent \(I^a\)) when measuring along \(I^a\); one finds \(I \propto L^2 R_{ab} I^a I^b\) [118].

These quantum features would stay there no matter the smallness of \(L\) (read \(\hbar\) if we think of the Planck length \(l_p\) for \(L\)), this reminding of Bell inequalities, whose violation certifies nonlocality, but with the size of the violation being unrelated to the value of the Planck constant. This offers in principle a hope to test quantum features of gravity also in circumstances in which any effect proportional to \(l_p\), or its powers, is hopelessly small as is surely the case in the lab, thing that intriguingly resonates with recent new proposals to check the non-classicality of gravity in the lab [119–122].

As for possible signatures of a minimum-length metric, we have to consider that these come with the fact that the latter is a way to coherently embody the metric description of spacetime with a limit length, plausibly of quantum origin, in the small scale. Then any circumstances in which quantum effects are expected to induce deviations from the classical metric are suitable test-beds for the minimum-length metric, and the use of the latter in place of ordinary metric should affect, and hopefully refine the theoretical predictions. (Very early) cosmology and extreme astrophysical events might be thought then as the main topics where to find signatures or make convenient use of a modification of the ordinary metric to a minimum-length metric.

In particular, on the basis of the back-on-the-envelope arguments mentioned above concerning singularity-formation avoidance (the two paragraphs right after Eq. (50)), one can expect that in the minimum-length metric the prototypical Oppenheimer-Snyder black-hole collapse brings to shrink matter to a finite limit area and not to form a singularity (this is what expected on the basis of those arguments, but might deserve a full-fledged proof). Evaporation will then shrink the horizon and the process might be expected to effectively halt when the horizon area becomes small enough to be comparable with the limit area; this might likely correspond to a black hole remnant and the minimum-length metric might be a tool for the description of its metric. This could be possibly helpful when coming e.g. to study dark matter along the well-known idea that it might be explained in terms Planck-size black-hole remnants (problematic idea in principle [123, 124], but possibly viable, according to successive accounts [125–127]).

On the same grounds, following back in time the evolution of a LFRW universe as traced by worldlines of particles comoving with the universe, if we use the minimum-length metric we may expect to reach a limit configuration corresponding to the limit ‘area’ (spatial volume actually) (39). As in the collapse case described above, this would hint to circumstances in which the universe can not effectively shrink (back in time) any longer, and the minimum-length metric might be useful in describing its metric properties at these and (not too much) later times when typical distances between inhomogeneities would be strongly affected by the deviation from classicality of the metric. This would correspond to the first stages of the very early universe, right after the Planck epoch, if the limit length \(L\) is the Planck length \(l_p\); but it could be
relevant for much later stages (still in the very-early-universe epoch) if \( L \) is significantly larger than \( l_p \). Detailed calculations with the minimum-length metric might be worth pursuing within the LFRW models.

**4 Conclusions**

What we have done in the paper has been to go through a series of results connecting gravity and thermodynamics/statistics/quantum information starting from the fundamental result [14] by Jacobson. The path chosen reflects the personal line of research of the author entangled in multiple ways with (small) part of own research by Paddy in the last decade.

In the first part, specifically of thermodynamic flavor, the focus has been on that even if we try to keep nonlocality away as long we can in the consideration of gravity, resorting in particular to matter in the hydrodynamic approximation (thus with energy and entropy microscopically related), we end up unavoidably with a nonlocal picture, thus kind of a nonlocal spacetime, at least at the smallest scales (the Planck length). The stress is on that gravity exhibits an irreducibly nonlocal character. This has been discussed building basically on entropy bounds, with null hypersurfaces and photon gases playing a major role.

The second part has been devoted to describe the explicit implementation of this nonlocality in the description of spacetime, with the so called minimum-length metric or qmetric [6, 63, 64]. We have seen how this is accomplished through use of bitensors, thus objects depending on two separated events, in particular through a metric bitensor \( q_{ab} \) which gives that in the modified (or enriched with a microstructure) spacetime there is a lower limit \( L \) to the separation distance between two events in their coincidence limit. Several interesting things do happen in the qmetric spacetime. Among them one stands out (and has been given specific attention in the paper): The existence of nonvanishing limit areas attached to every event in spacetime. Among them one stands out (and has been given specific attention in the paper): The existence of nonvanishing limit areas attached to every event in spacetime. This is what brings to introduce micro dofs for gravity and enables a statistical-mechanical description of matter+gravity system with a max-statistical entropy account of field equations; this has been spelled out in some detail in the paper for the case of null separated events. Very intriguingly, this statistical description keeps staying there also in the \( L \to 0 \) limit.

The gross picture we can get from this is that the field equations of gravity might be in essence a statistical-mechanical result, and as such ought to be expressible completely using statistical-mechanical, not geometric, concepts. Spacetime and the geometric description of gravity we are used to, emerge when circumstances are such that what we observe is due to cumulative effects over many micro dofs and a continuous description becomes viable, what we call the thermodynamic limit (cf. [128] and references therein). Key to this picture would be a clear understanding of the emergence of time; we believe that something so much intriguing/promising regarding this and possibly work on is Paddy’s recent work [129].

On top of this, gravity appears to be inherently quantum (as hinted to by the persistence of the statistical-mechanical construct in the \( L \to 0 \) limit; the ‘classical description’ does not come up in the \( \hbar \to 0 \) limit: it is there for (any) \( \hbar \neq 0 \) but we
do not have it any longer (the field equations become singular) in the $\hbar \to 0$ limit).
And even though nonlocality unavoidably appears (in whichever circumstances) in the small scale limit, this does not preclude it to come up at more mundane or, also, lab scales (as mentioned at the end of previous section) if suitable circumstances can be considered. “Gravity is quantum mechanical at all scales” [81] after all, using again Paddy’s words, enlightening as ever.

Acknowledgements I thank Francesco Anselmo for drawing attention to one of the references. This work was supported in part by INFN grant FLaG.

Data Availability All data generated or analysed during this study are included in this published article.

References

1. Padmanabhan, T.: A dialogue on the nature of gravity, arXiv:0910.0839 (2009)
2. Padmanabhan, T.: Thermodynamical aspects of gravity: new insights. Rept. Prog. Phys. 73, 046901 (2010). arXiv:0911.5004
3. Padmanabhan, T.: Equipartition of energy in the horizon degrees of freedom and the emergence of gravity. Mod. Phys. Lett. A 25, 1129 (2010). arXiv:0912.3165
4. Pesci, A.: Gravity from the entropy of light. Class. Quantum Grav. 28, 045001 (2011). arXiv:1002.1257
5. Padmanabhan, T.: Gravitation: Foundations and frontiers. Cambridge University Press, Cambridge (2010)
6. Kothawala, D., Padmanabhan, T.: Grin of the Cheshire cat: Entropy density of spacetime as a relic from quantum gravity. Phys. Rev. D 90, 124060 (2014). arXiv:1405.4967
7. Kothawala, D., Padmanabhan, T.: Entropy density of spacetime from the zero point length. Phys. Lett. B 748, 67 (2015). arXiv:1408.3963
8. Bekenstein, J.D.: Black holes and the second law. Nuovo Cim. Lett. 4, 737 (1972)
9. Bekenstein, J.D.: Black holes and entropy. Phys. Rev. D 7, 2333 (1973)
10. Bekenstein, J.D.: Generalized second law of thermodynamics in black-hole physics. Phys. Rev. D 9, 3292 (1974)
11. Hawking, S.W.: Black hole explosions? Nature 248, 30 (1974)
12. Hawking, S.W.: Particle creation by black holes. Commun. Math. Phys. 43, 199 (1975)
13. Bardeen, J.M., Carter, B., Hawking, S.W.: The four laws of black hole mechanics. Commun. Math. Phys. 31, 161 (1973)
14. Jacobson, T.: Thermodynamics of spacetime: the Einstein equation of state. Phys. Rev. Lett. 75, 1260 (1995). (gr-qc/9504004)
15. Rindler, W.: Relativity: Special, general, and cosmological. Oxford University Press, Oxford (2006)
16. Unruh, W.G.: Notes on black-hole evaporation. Phys. Rev. D 14, 870 (1976)
17. Bekenstein, J.D.: Universal upper bound on the entropy-to-energy ratio for bounded systems. Phys. Rev. D 23, 287 (1981)
18. ’t Hooft, G.: Dimensional reduction in quantum gravity, essay dedicated to Abdus Salam, published in Salamfest 0284 (1993) arXiv:gr-qc/9310026
19. Susskind, L.: The world as a hologram. J. Math. Phys. 36, 6377 (1995). hep-th/9409089
20. Maldacena, J.M.: The large N limit of superconformal field theories and supergravity. Adv. Theor. Math. Phys. 2, 231 (1998). arXiv:hep-th/9711200
21. Bousso, R.: A covariant entropy conjecture. JHEP 07, 004 (1999). arXiv:hep-th/9905177
22. Flanagan, É.É., Marolf, D., Wald, R.M.: Proof of classical versions of the Bousso entropy bound and of the generalized second law. Phys. Rev. D 62, 084035 (2000). arXiv:hep-th/9908070
23. Wald, R.M.: Black hole entropy is Noether charge. Phys. Rev. D 48, 3427 (1993). arXiv:gr-qc/9307038
24. Jacobson, T., Kang, G., Myers, R.C.: On black hole entropy. Phys. Rev. D 49, 6587 (1994). arXiv:gr-qc/9312023
25. Iyer, V., Wald, R.M.: A comparison of Noether charge and Euclidean methods for computing the entropy of stationary black holes. Phys. Rev. D 52, 4430 (1995). arXiv:gr-qc/9503052
26. Brustein, R., Gorbonos, D., Hadad, M.: Wald’s entropy is equal to a quarter of the horizon area in units of the effective gravitational coupling. Phys. Rev. D 79, 044025 (2009). arXiv:0712.3206
27. Bousso, R., Flanagan, É.E., Marolf, D.: Simple sufficient conditions for the generalized covariant entropy bound. Phys. Rev. D 70, 044001 (2003). arXiv:hep-th/0305149
28. Strominger, A., Thompson, D.M.: Wald Bousso bound. Phys. Rev. D 70, 044007 (2004). arXiv:hep-th/0303067
29. Pesci, A.: From Unruh temperature to the generalized Bousso bound. Class. Quantum Grav. 24, 6219 (2007). arXiv:0708.3729
30. Bousso, R., Casini, H., Fisher, Z., Maldacena, J.: Proof of a quantum Bousso bound. Phys. Rev. D 90, 044002 (2014). arXiv:1404.5635
31. Lowe, D.A.: Comments on a covariant entropy conjecture. JHEP 10, 026 (1999). arXiv:hep-th/9907062
32. Wall, A.C.: A proof of the generalized second law for rapidly-evolving Rindler horizons. Phys. Rev. D 82, 124019 (2010). arXiv:1007.1493
33. Wall, A.C.: A proof of the generalized second law for rapidly changing fields and arbitrary horizon slices. Phys. Rev. D 85, 104049 (2012). arXiv:1105.3445
34. Pesci, A.: On the statistical-mechanical meaning of the Bousso bound. Class. Quantum Grav. 25, 125005 (2008). arXiv:0803.2642
35. Kittel, C., Kroemer, H.: Thermal physics. W.H. Freeman & Co., San Francisco (1980)
36. Pesci, A.: A note on the connection between the universal relaxation bound and the covariant entropy bound. Int. J. Mod. Phys. D 18, 831 (2009). arXiv:0807.0300
37. Hod, S.: Universal bound on dynamical relaxation times and black-hole quasinormal ringing. Phys. Rev. D 75, 064013 (2007). arXiv:gr-qc/0611004
38. Hod, S.: private communication
39. Carullo, G., Laghi, D., Veitch, J., Del Pozzo, W.: The Bekenstein-Hod universal bound on information emission rate is obeyed by LIGO-Virgo binary black hole remnants. Phys. Rev. Lett. 126, 161102 (2021). arXiv:2103.06167 (2021)
40. Abbott, B.P., et al. (LIGO Scientific Collaboration and Virgo Collaboration), Observation of gravitational waves from a binary black hole merger. Phys. Rev. Lett. 116, 061102 (2016). arXiv:1602.03837
41. Kovtun, P., Son, D.T., Starinets, A.O.: Holography and hydrodynamics: Diffusion on stretched horizons. JHEP 03(10), 064 (2003). arXiv:hep-th/0309213
42. Kovtun, P., Son, D.T., Starinets, A.O.: Viscosity in strongly interacting quantum field theories from black hole physics. Phys. Rev. Lett. 94, 116601 (2005). arXiv:hep-th/0405231
43. Cremonini, S.: The shear viscosity to entropy ratio: A status report. Mod. Phys. Lett. B 25, 1867 (2011). arXiv:1108.0677
44. Meert P.: Transport coefficients associated to black holes on the brane: analysis of the shear viscosity-to-entropy ratio. arXiv:2206.14650
45. Lawrence, S.: Resurrecting the strong KSS conjecture, arXiv:2111.08158 (2021)
46. Fouxon, I., Betschart, G., Bekenstein, J.D.: The bound on viscosity and the generalized second law of thermodynamics. Phys. Rev. D 77, 024016 (2008). arXiv:0710.1429
47. Pesci, A.: A semiclassical approach to eta/s bound through holography, In: Proc. of the 12th Marcel Grossman Meeting on General Relativity (July 12-18, 2009, Paris, France), eds. T. Damour, R.T. Jantzen and R. Ruffini (World Scientific, 2011) 2324, arXiv:0910.0766 (2009)
48. Misner, C.W.: The isotropy of the universe. Ap. J. 151, 431 (1968)
49. Pesci, A.: A proof of the Bekenstein bound for any strength of gravity through holography. Class. Quantum Grav. 27, 165006 (2010). arXiv:0903.0319
50. Son, D.T., Starinets, A.O.: Viscosity, black holes, and quantum field theory. Ann. Rev. Nucl. Part. Sci. 57, 95 (2007). arXiv:0704.0240
51. Hod, S.: Gravitation, thermodynamics, and the bound on viscosity. Gen. Relativ. Gravit. 41, 2295 (2009). arXiv:0905.4113
52. Hod, S.: From thermodynamics to the bound on viscosity. Nucl. Phys. B 819, 177 (2009). arXiv:0907.1144
53. Damour, T.: Quelques proprietes mecaniques, electromagnetiques, thermodynamiques et quantiques des trous noirs, Thèse de doctorat d’État, Université Paris, http://www.ihes.fr/~damour/Articles/these1.pdf (1979)
54. Damour, T.: Surface effects in black hole physics, Proc. of the 2nd Marcel Grossmann Meeting on General Relativity, ed. R. Ruffini (North Holland, Amsterdam, 1982) 587
55. Casini, H.: Relative entropy and the Bekenstein bound. Class. Quantum Grav. 25, 205021 (2008). arXiv:0804.2182
56. Ivanov, M.G., Volovich, I.V.: Entropy bounds, holographic principle and uncertainty relation. Entropy 3, 66 (2001). arXiv:gr-qc/9908047
57. Bousso, R.: Flat space physics from holography. JHEP (05)2004, 050 (2004). arXiv:hep-th/0402058
58. Padmanabhan, T.: Exploring the nature of gravity, arXiv:1602.01474 (2016)
59. Pesci, A.: The existence of a minimum wavelength for photons, arXiv:1108.5066 (2011)
60. Karas, L.J.: Quantum gravity and minimum length. Int. J. Mod. Phys. A 10, 145 (1995). arXiv:gr-qc/9403008
61. Hossenfelder, S.: Minimal length scale scenarios for quantum gravity. Liv. Rev. Rel. 16, 2 (2013). arXiv:1203.6191
62. White, A., Weinertner, S., Visser, M.: Signature change events: A challenge for quantum gravity? Class. Quantum Grav. 27, 045007 (2010). arXiv:0812.3744
63. Kothawala, D.: Minimal length and small scale structure of spacetime. Phys. Rev. D 88, 104029 (2013). arXiv:1307.5618
64. Jaffino Stargen, D., Kothawala, D.: Small scale structure of spacetime: van Vleck determinant and equi-geodesic surfaces. Phys. Rev. D 92, 024046 (2015). arXiv:1503.03793
65. Kothawala, D.: Action and observer dependence in Euclidean quantum gravity. Class. Quantum Grav. 35, 03LT01 (2018). arXiv:1705.02504
66. Kothawala, D.: Euclidean action and the Einstein tensor. Phys. Rev. D 97, 124062 (2018). arXiv:1802.07055
67. Singh, R., Kothawala, D.: Geometric aspects of covariant Wick rotation, arXiv:2010.01822 (2020)
68. Synge, J.L.: Relativity: The general theory. North-Holland, Amsterdam (1960)
69. Born, M., Jordan, P.: Dynamical Theory of Groups and Fields. Gordon and Breach, New York (1965)
70. Christensen, S.M.: Vacuum expectation value of the stress tensor in an arbitrary curved background: The covariant point-separation method. Phys. Rev. D 14, 2490 (1976)
71. Visser, M.: van Vleck determinants: geodesic focussing and defocussing in Lorentzian spacetimes. Phys. Rev. D 47, 2395 (1993). hep-th/9303020
72. Pesci, A.: Looking at spacetime atoms from within the Lorentz sector, arXiv:1803.05726
73. Carlip, S.: Dimension and dimensional reduction in quantum gravity. Class. Quantum Grav. 36, 075009 (2019). arXiv:1812.01275
74. Kothawala, D.: Intrinsic and extrinsic curvatures in Finsler esque spaces. Gen. Relativ. Gravit. 46, 1836 (2014). arXiv:1406.2672
75. Padmanabhan, T.: Distribution function of the atoms of spacetime and the nature of gravity. Entropy 17, 7420 (2015). arXiv:1508.06286
76. Padmanabhan, T., Chakraborty, S., Kothawala, D.: Spacetime with zero point length is two-dimensional at the Planck scale. Gen. Relativ. Gravit. 48, 55 (2016). arXiv:1507.05669
77. Ambjørn, J., Jurkiewicz, J., Loll, R.: Spectral dimension of the universe. Phys. Rev. Lett. 95, 171301 (2005). hep-th/0505113
78. Carlip, S.: Dimension and dimensional reduction in quantum gravity. Class. Quantum Grav. 34, 193001 (2017). arXiv:1705.05417
79. Carlip, S.: Dimension and dimensional reduction in quantum gravity. Universe 5, 83 (2019). arXiv:1904.04379
80. Pesci, A.: Effective null Raychaudhuri equation. Particles 1, 230 (2018). arXiv:1809.08007
88. Chakraborty, S., Kothawala, D., Pesci, A.: Raychaudhuri equation with zero point length. Phys. Lett. B 797, 134877 (2019). arXiv:1904.09053
89. Casadio, R., Kamenshchik, A., Kuntz, I.: Absence of covariant singularities in pure gravity. Int. J. Mod. Phys. D 31, 2150130 (2022). arXiv:2008.09387
90. Casadio, R., Kamenshchik, A., Kuntz, I.: Covariant singularities in quantum field theory and quantum gravity. Nucl. Phys. B 971, 115496 (2021). arXiv:2102.10688
91. Kamenshchik, A.Yu., Pozdeeva, E.O., Vernov, S.Yu., Tronconi, A., Venturi, G.: Transformations between Jordan and Einstein frames: Bounces, antigravity, and crossing singularities. Phys. Rev. D 94, 063510 (2016). arXiv:1602.07192
92. Casadio, R., Kamenshchik, A., Kuntz, I.: Covariant singularities: a brief review, arXiv.org:2203.11259 (2022)
93. Padmanabhan, T.: The atoms of space, gravity and the cosmological constant. Int. J. Mod. Phys. D 25, 1630020 (2016). arXiv:1603.08658
94. Padmanabhan, T.: The atoms of spacetime and the cosmological constant. J. Phys. Conf. Ser. 880, 012008 (2017). arXiv:1702.06136
95. Jaynes, E.T.: Information theory and statistical mechanics. Phys. Rev. 106, 620 (1957)
96. Jaynes, E.T.: Information theory and statistical mechanics II. Phys. Rev. 108, 171 (1957)
97. Padmanabhan, T.: The kinetic theory of the mesoscopic spacetime. Int. J. Mod. Phys. D 27, 1846004 (2018). arXiv:1805.07218
98. Pesci, A.: Spacetime atoms and extrinsic curvature of equi-geodesic surfaces. Eur. Phys. J. Plus 134, 374 (2019). arXiv:1511.08665
99. Padmanabhan, T.: Gravity and quantum theory: Domains of conflict and contact. Int. J. Mod. Phys. D 29, 2030001 (2020). arXiv:1909.02015
100. Padmanabhan, T.: General relativity from a thermodynamic perspective. Gen. Relativ. Gravit. 46, 1673 (2014). arXiv:1311.3253
101. Padmanabhan, T.: Entropy density of spacetime and the Navier-Stokes fluid dynamics of null surfaces. Phys. Rev. D 83, 044048 (2011). arXiv:1012.0119
102. Kolekar, S., Padmanabhan, T.: Action principle for the Fluid-Gravity correspondence and emergent gravity. Phys. Rev. D 85, 024004 (2011). arXiv:1109.5353
103. Poisson, E.: A relativist’s toolkit. Cambridge University Press, Cambridge (2004)
104. Landau, L.D., Lifshitz, E.M.: Fluid mechanics, 2nd edn. Pergamon Press, Oxford (1987)
105. Misner, C.W., Thorne, K.S., Wheeler, J.A.: Gravitation, Princeton University Press edition Princeton University Press, Princeton NJ (2017)
106. Thorne, Kip S., Price, R. H., MacDonald, D. A.: (eds.), Black holes: The membrane paradigm (Yale University Press, New Haven CT, 1986)
107. Kaku, M.: Hyperspace. Oxford University Press, Oxford (1994)
108. Padmanabhan, T., Chakraborty S.: Microscopic origin of Einstein’s field equations and the raison d’être for a positive cosmological constant. Phys. Lett. B 824, 136828 (2022). arXiv:2112.0944
109. Padmanabhan, T., Paranjape, A.: Entropy of null surfaces and dynamics of spacetime. Phys. Rev. D 75, 064004 (2007). (gr-qc/0701003)
110. Padmanabhan, T.: Gravity: The inside story. Gen. Relativ. Gravit. 40, 2031 (2008)
111. Tuveri, M., Fatibene, L., Ferraris, M.: Emergent gravity from an Augmented Variational Principle arXiv:1604.08067
112. Fatibene, L., Ferraris, M., Francaviglia, M.: Augmented variational principles and relative conservation laws in classical field theory. Int. J. Geom. Meth. Mod. Phys. 2, 373 (2005). arXiv:math-ph/0411029
113. Pesci, A.: Minimum-length Ricci scalar for null separated events. Phys. Rev. D 102, 124057 (2020). arXiv:1911.04135
114. Gemelli, G.: Observer-dependent Gauss-Codazzi formalism for null hypersurfaces in the space-time. J. Geom. Phys. 43, 371 (2002)
115. Chakraborty, S., Parattu, K.: Null boundary terms for Lanczos-Lovelock gravity. Gen. Relativ. Gravit. 51, 23 (2019). arXiv:1806.08823
116. Pesci, A.: Zero-point gravitational field equations. Class. Quantum Grav. 38, 145007 (2021). arXiv:2005.03258
117. Pesci, A.: Expectation values of minimum-length Ricci scalar. Int. J. Mod. Phys. D 31, 2250007 (2022). arXiv:2010.10063
118. Pesci, A.: Quantum states for a minimum-length spacetime, arXiv:2105.07764 (2021)
119. Bose, S., Mazumdar, A., Morley, G.W., Ulbricht, H., Toroš, M., Paternostro, M., Geraci, A., Barker, P., Kim, M.S., Milburn, G.: A spin entanglement witness for quantum gravity. Phys. Rev. Lett. 119, 240401 (2017). arXiv:1707.06050

120. Marletto, C., Vedral, V.: Gravitationally-induced entanglement between two massive particles is sufficient evidence of quantum effects in gravity. Phys. Rev. Lett. 119, 240402 (2017). arXiv:1707.06036

121. Christodoulou, M., Rovelli, C.: On the possibility of laboratory evidence for quantum superposition of geometries. Phys. Lett. B 792, 64 (2019). arXiv:1808.05842

122. Marshman, R.J., Mazumdar, A., Bose, S.: Locality & entanglement in table-top testing of the quantum nature of linearized gravity. Phys. Rev. A 101, 052110 (2020). arXiv:1907.01568

123. Susskind, L.: Trouble for remnants, arXiv:hep-th/9501106 (1995)

124. Hawking, S.W.: Information preservation and weather forecasting for black holes, arXiv:1401.5761 (2014)

125. Calmet, X.: Virtual black holes, remnants and the information paradox. Class. Quantum Grav. 32, 045007 (2015). arXiv.org:1412.6270

126. Bianchi, E., Christodoulou, M., D’Ambrosio, F., Haggard, H.M., Rovelli, C.: White holes as remnants: A surprising scenario for the end of a black hole. Class. Quantum Grav. 35, 225003 (2018). arXiv:1802.04264

127. Rovelli, C., Vidotto, F.: Small black/white hole stability and dark matter. Universe 4(11), 127 (2018). arXiv:1805.03872

128. Chakraborty, S., Padmanabhan, T.: Thermodynamical interpretation of the geometrical variables associated with null surfaces. Phys. Rev. D 92, 104011 (2015). arXiv:1508.04060

129. Padmanabhan, T.: Probing the Planck scale: The modification of the time evolution operator due to the quantum structure of spacetime. JHEP 11, 13 (2020). arXiv:2006.06701

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.