A Family of Denominator Bounds for First Order Linear Recurrence Systems

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Abstract

For linear recurrence systems, the problem of finding rational solutions is reduced to the problem of computing polynomial solutions by computing a content bound or a denominator bound. There are several bounds in the literature. The sharpest bound [8] leads to polynomial solutions of lower degrees, but as shown in [7], this advantage need not compensate for the time spent on computing that bound.

To strike the best balance between sharpness of the bound versus CPU time spent obtaining it, we will give a family of bounds. The \( J \)th member of this family is similar to [2] when \( J = 1 \), similar to [8] when \( J \) is large, and novel for intermediate values of \( J \), which give the best balance between sharpness and CPU time.

1. Introduction

Let \( \mathbb{A} \) be a unique factorization domain and let \( \tau: \mathbb{A} \to \mathbb{A} \) be an automorphism. We denote the quotient field of \( \mathbb{A} \) by \( \mathbb{K} \) and extend \( \tau \) to \( \mathbb{K} \). This paper considers systems of the form

\[ \tau(Y) = MY \quad \text{where} \quad M \in \text{GL}_n(\mathbb{K}). \]

The goal is to reduce the problem of computing rational solutions \( Y \in \mathbb{K}^n \) of (sys) to computing polynomial solutions \( Z \in \mathbb{A}^n \) of a related system.
Definition 1 (Content Bound). We say that \(B \in \mathbb{K}\) is a global content bound for \((\text{sys})\) if all of its rational solutions \(Y \in \mathbb{K}^n\) are in \(B \cdot \mathbb{A}_n\). The denominator \(d := \text{den}(B) \in \mathbb{A} \setminus \{0\}\) is then a denominator bound, which means all rational solutions are in \(\frac{1}{d} \cdot \mathbb{A}_n\).

A vector \((B_1, \ldots, B_n) \in \mathbb{K}^n\) is a component-wise content bound if all rational solutions are in \(B \cdot \mathbb{A}_n\) where \(B = \text{diag}(B_1, \ldots, B_n)\).

Note that \(0\) is a content bound if and only if there are no non-zero rational solutions.

Content bounds and denominator bounds are found in [1], [2], [4], [3], [6], [11], or [10].

If a content bound \(B\) is an invertible scalar or matrix, then we can substitute \(Y = BZ\) in \(\tau(Y) = MY\) obtaining the equivalent system \(\tau(Z) = \tau(B^{-1})MBZ\) for which all rational solutions are in \(\mathbb{A}_n\). This way, a denominator or a content bound reduces rational solutions \(Y \in \mathbb{K}^n\) to polynomial solutions \(Z \in \mathbb{A}_n\). However, as illustrated in [7], there is tension between two goals: (1) we want a bound that can be computed quickly, and (2) want to minimize the degrees of the entries of \(Z\). The goal in this paper is to strike a good balance between these two goals.

We will formulate our bounds in a fairly general setting, see section 2 below, though the practical utility is mainly for cases that have algorithms for polynomial solutions.

2. Preliminaries

For a ring \(R\) we will use \(R^*\) to denote the group of units in \(R\). The set of \(m\)-by-\(n\) matrices with entries in \(R\) will be written as \(R^{m \times n}\). We use \(\text{GL}_n(R)\) for the set of \(n\)-by-\(n\) invertible matrices over \(R\), while \(A'\) denotes the transpose of \(A\). For \(a_1, \ldots, a_n \in R\) let \(\text{diag}(a_1, \ldots, a_n) \in R^{n \times n}\) denote the corresponding diagonal matrix.

Let \(\mathbb{A}\) be a unique factorization domain with quotient field \(\mathbb{K}\). Let \((\mathbb{A}, \tau)\) be a difference ring; that is, \(\tau: \mathbb{A} \rightarrow \mathbb{A}\) is an automorphism. Extending \(\tau\) to \(\mathbb{K}\) makes \((\mathbb{K}, \tau)\) a difference field.

Example 2. Let \(F\) be a field of characteristic \(0\). The main example is \(\mathbb{A} = F[x]\) with \(\tau\) defined by \(\tau(f(x)) := f(x + 1)\). This is called the shift case. Here \(\mathbb{K} = F(x)\).

Example 3. Similarly, if \(F\) is a field and \(q \in F^*\), we can let \(\mathbb{A} = F[x]\) and \(\tau(f(x)) := f(qx)\). This is called the \(q\)-shift case.

Example 4. Let \((\mathbb{G}, \tau)\) be a difference ring and let \(x_1, \ldots, x_s\) be indeterminates over \(\mathbb{G}\). Choose units \(a_1, \ldots, a_s, x_1, \ldots, x_s \in \mathbb{G}^*\) and \(\beta_1, \ldots, \beta_s \in \mathbb{G}\). Let \(\mathbb{A} = \mathbb{G}[x_1, \ldots, x_s]\) and extend \(\tau\) to \(\mathbb{A}\) by \(\tau(x_j) = a_jx_j + \beta_j\). If \(\tau|_{\mathbb{G}} = \text{id}\) is the identity map, then we refer to this as the multi-basic case.

Definition 5 (Sharpness). Given two content bounds \(B, B'\) for the same system \(\tau(Y) = MY\), we say that \(B\) is sharper than \(B'\) if it constrains \(Y\) to a smaller set (i.e. \(B \cdot \mathbb{A}_n \subsetneq B' \cdot \mathbb{A}_n\)).

Example 6. For the shift case \(\mathbb{A} = \mathbb{Q}[x]\) and \(\tau: x \mapsto x + 1\) from Example 2, let

\[
M = \begin{pmatrix}
\frac{(x+2)^2(2x+1)}{2(x+1)(x+3)} & \frac{-1(x+2)^2}{2(x+1)(x+3)} \\
\frac{-1(x+2)^2}{2(x+1)(x+3)} & \frac{(x+2)^2(2x+1)}{2(x+1)(x+3)}
\end{pmatrix} \in \text{GL}_2(\mathbb{Q}(x)).
\]

The rational solutions of \(\tau(Y) = MY\) are

\[
V = \left\{ \begin{pmatrix} \left(\frac{(x+2)^2(2x+1)}{2(x+1)(x+3)} \right) & \left(\frac{-1(x+2)^2}{2(x+1)(x+3)} \right)
\end{pmatrix} \mid c_1, c_2 \in \mathbb{Q} \right\} \subseteq \mathbb{Q}(x)^2.
\]
Then \( V \subset B \cdot \mathbb{A}^2 \subseteq B' \cdot \mathbb{A}^2 \) where
\[
B = \frac{x + 1}{x(x + 2)} \quad \text{and} \quad B' = \frac{1}{x(x + 2)}.
\]
Here \( B \) is a sharper content bound than \( B' \). The component-wise bound
\[
C := \left( \frac{x + 1}{x(x + 2)} \right) \in \mathbb{Q}(x)^2
\]
is sharper still since \( V \subset \text{diag}(C) \mathbb{A}^2 \subseteq B \cdot \mathbb{A}^2 \).

Denominator bounds are more common than content bounds in the literature (see, for example, \([1, 2, 4, 3, 6, 11, 10]\)). If \( d \) is a denominator bound, then \( \frac{1}{d} \) is a content bound. However, Example 6 shows that a sharp global content bound \( B \) need not have that form.

3. The Exponent Function

Let \( p \in \mathbb{A} \) be a prime (\( = \) an irreducible polynomial if \( \mathbb{A} = F[x] \)) and \( a \in \mathbb{A} \). The \textit{valuation} of \( a \) at \( p \) is
\[
\nu_p(a) = \sup \{ j \mid p^j \text{ divides } a \}.
\]
Note that \( \nu_p(a) = \infty \) if and only if \( a = 0 \). We extend \( \nu_p : \mathbb{K} \to \mathbb{Z} \cup \{\infty\} \) by defining \( \nu_p(a/b) = \nu_p(a) - \nu_p(b) \) for fractions \( a/b \in \mathbb{K} \). Then
\[
\nu_p(a + b) \geq \min(\nu_p(a), \nu_p(b)) \quad \text{and} \quad \nu_p(ab) = \nu_p(a) + \nu_p(b). \quad (1)
\]
for all \( a, b \in \mathbb{K} \). For a matrix \( A \) let \( \nu_p(A) \) denote the minimum valuation of its entries. Then
\[
\nu_p(AB) \geq \nu_p(A) + \nu_p(B) \quad (2)
\]
for matrices \( A, B \) with matching sizes.

\textit{Definition} 7 (Associates and Content). Two elements \( a_1, a_2 \in \mathbb{K} \) are called \textit{associates}, denoted \( a_1 \sim a_2 \), if \( a_1 = ua_2 \) for some unit \( u \in \mathbb{A}^\times \). Just like polynomial contents in Gauss’ lemma, the \textit{content} \( \text{ct}(A) \in \mathbb{K} \) of a matrix \( A \in \mathbb{K}^{n \times m} \) is defined up to \( \sim \) by the following equivalent properties:

(a) \( A \) can be written as \( \text{ct}(A) \) times a matrix in \( \mathbb{A}^{n \times m} \) whose entries have gcd 1.

(b) \( \nu_p(\text{ct}(A)) = \nu_p(A) \) for all primes \( p \).

(c) \( \text{ct}(A) = g/d \) where \( d \) is the least common multiple of the denominators in \( A \), and \( g \) is the gcd of the entries of \( dA \).

An element \( B \in \mathbb{K} \) is a \textit{content-bound} for \( \tau(Y) = MY \) if and only if \( \nu_p(B) \leq \nu_p(Y) \) for all solutions \( Y \in \mathbb{K}^n \) and all primes \( p \) in \( \mathbb{A} \). So \textit{finding} \( B \) means \textit{finding} a lower bound for each \( \nu_p(Y) \).

Let
\[
D = \{ a \in \mathbb{A} \mid a \neq 0 \text{ and } \tau^k(a) \sim a \text{ for some } k \neq 0 \}.
\]
The fact that $\mathbb{A}$ is a UFD means that every non-zero $a \in \mathbb{A}$ can be written as a product of finitely many primes, unique up to $\sim$. This implies that $a \in D$ if and only if all its prime factors are in $D$.

We will only compute a lower bound for $v_p(Y)$ at primes $p \notin D$. That results in a content bound up to some factor $a \in D$. This is sufficient for the main cases including the shift case (then $D = F^*$), and the $q$-shift case when $q$ is not a root of unity (then $D = \{cx^n \mid c \in F^*, m \geq 0\}$).

Definition 8 (Exponent Function). Fix a prime $p \in \mathbb{A}$. If $c \in \mathbb{K}$ then we define its exponent function as: if $c = 0$ then $e = \infty$, otherwise $e$ is the function $e : \mathbb{Z} \to \mathbb{Z}$ with $e(k) = v_{\omega(p)}(c)$ for all $k \in \mathbb{Z}$.

We only use this for primes $p \notin D$. If $c \neq 0$ then $e$ has finite support and can be represented with a finite list containing: a lower bound $\ell$ and upper bound $m$ for the support of $e$, and the numbers $e(k)$ for $k$ from $\ell$ to $m$.

For a system $\tau(Y) = MY$ we recursively define a matrix $M_j$ such that $\tau(Y) = M_jY$, as follows: $M_0 = I$ and $M_{j+1} = \tau(Y)M_j = \tau(M)M_j$. For $j < 0$ we rewrite this as $M_j = \tau^{-1}(M^{-1})M_{j+1}$. Examples include:

$$M_1 = M, \quad M_2 = \tau(M)M, \quad M_{-1} = \tau^{-1}(M^{-1}), \quad M_{-2} = \tau^{-1}(M^{-1})\tau^{-1}(M^{-1}).$$

After selecting a prime $p \notin D$, we denote the exponent function of $e_j := \text{ct}(M_j)$ as $e_j : \mathbb{Z} \to \mathbb{Z}$.

Example 9. Let $M$ be as in Example 6, then $c_1 = \text{ct}(M_1) = x^{-1}(x+1)^{-2}(x+2)^2(x+3)^{-1}$. The matrix $M_0$ is always $I$ so $c_0 = 1$. From

$$M_{-1} = \tau^{-1}(M^{-1}) = \begin{pmatrix} \frac{(2x-1)(x+2)}{2(x+1)} & \frac{x+2}{2(x+1)^2} \\ \frac{2(x+1)(x-1)}{2(x+1)^2} & \frac{(2x-1)(x+2)}{2(x+1)} \end{pmatrix},$$

we obtain $c_{-1} = (x-1)^{-1}(x+1)^{-2}(x+2)$. After selecting $p = x$ we have

$$e_1(k) = \begin{cases} -1 & \text{if } k = 0 \\ -2 & \text{if } k = 1 \\ 2 & \text{if } k = 2 \\ -1 & \text{if } k = 3 \\ 0 & \text{otherwise} \end{cases} \quad e_0 = 0 \quad e_{-1}(k) = \begin{cases} -1 & \text{if } k = -1 \\ 0 & \text{if } k = 0 \\ -2 & \text{if } k = 1 \\ 1 & \text{if } k = 2 \\ 0 & \text{otherwise}. \end{cases}$$

4. The $J$'th global content bound

Algorithm 10 (“The global algorithm”): $J$'th global content bound).

Input $M \in \text{GL}_n(\mathbb{K})$ and an integer $J \geq 1$.

Output $B \in \mathbb{K}$ such that $\exists a \in D$ for which $aY \in B \cdot A^n$ for any rational solution $Y$. In other words, a content bound up to some factor $a \in D$. In the shift-case $a = 1$.

In the $q$-shift case if $q$ not a root of unity then $a = x^m$ for some $m$ not computed here.
Procedure

(a) Compute $M_j$ and $c_j := \text{ct}(M_j)$ for $j \in \{-J, \ldots J\}$.
(b) Let $\mathcal{P}$ be the set of prime factors in the denominators of $c_1$ and $c_{-1}$.
(c) Select one $p \in \mathcal{P}$ from each $\tau$-equivalence class, where $p_1$ is $\tau$-equivalent to $p_2$ if $\tau^k(p_1) \sim p_2$ for some $k \in \mathbb{Z}$ (recall $\sim$ from Definition 7).
   Let $\mathcal{O}$ be the resulting set of primes.
(d) Let $B := 1$.
(e) For each $p \in \mathcal{O} - D$
   (e.1) For each $j \in \{-J, \ldots J\}$ compute the exponent-function $e_j : \mathbb{Z} \to \mathbb{Z}$ of $c_j$ at $p$. Recall that $e_j$ has finite support and $e_j(k) = v_{\tau^k(p)}(c_j)$.
   (e.2) Let $f$ be the output of the local algorithm in section 5 with input $e_{-J}, \ldots, e_J$.
   (e.3) If $f = \infty$ then stop and return $B = 0$. Otherwise, $f : \mathbb{Z} \to \mathbb{Z}$ has finite support and we set $B := B \cdot \prod_{k \in \mathbb{Z}} \tau^k(p)^{f(k)}$.
(f) Return $B$.

The paper [2] gives a denominator bound that is based solely on the denominators of $M$ and $M^{-1}$. That is similar to the above algorithm with $J = 1$, and although it can be sharper with $J = 1$, see Example 14, its main novelty is when $J > 1$. Then the local algorithm uses more data, allowing it to construct a sharper bound (see the example in section 7). The goal of the local algorithm in section 5 is to obtain the sharpest content bound (up to a factor $a \in D$) that can be derived from the exponent-functions $e_{-J}, \ldots, e_J$. In the shift case, that factor $a \in D$ is simply 1.

In the $q$-shift case, if $q$ is a root of unity then $\tau$ has finite order so $D = \mathbb{A} - \{0\}$ which makes the output trivial. But the root of unity case is usually excluded. If $q$ is not a root of unity then $aY \in B \cdot F[x]^n$ for some $a = x^m$ not computed here. Then the output $B$ restricts rational solutions $Y$ not to $B \cdot F[x]^n$ but to $B \cdot F(x, 1/x)$. In the $q$-case, algorithms to bound the degree of polynomial solutions can also bound $m$ (just replace $x, q$ with $1/x, 1/q$). So in the $q$-case, finding all solutions in $F(x, 1/x)^n$ is not meaningfully harder than finding all solutions in $F[x]^n$.

In general, $B$ restricts rational solutions to $B \cdot \mathbb{A}^n$ where $\mathbb{A} := D^{-1}\mathbb{A} \subseteq \mathbb{K}$ is the localization of $\mathbb{A}$ at $D$. This reduces solutions over $\mathbb{K}$ to solutions over $\mathbb{A}$.

5. Local Bounds

Fix one prime $p \notin D$. A function $f$ is called a local content bound (for $M$ at $p$) if

$$v_{\tau^k(p)}(Y) \geq f(k) \text{ for all solutions } Y \in \mathbb{K}^n \text{ and all } k \in \mathbb{Z}. \quad (3)$$

The local algorithm below will compute such $f$ as follow: Lemma 12 below will provide an initial $f$, which is then repeatedly improved with Lemma 11.

For $j \in \{-J, \ldots, J\}$, let $e_j$ be the exponent function of the content $c_j$ of $M_j$. If $Y$ is a rational solution of $\tau(Y) = MY$ then $\tau^j(Y) = M_j Y$ and from Equation (2) we get
Lemma 11. Fix some $J > 0$. If $f$ is a local content bound then $v_{r^j(p)}(Y) \geq e_j(k+j) + f(k+j)$, so the function

$$f_{\text{new}}(k) := \max\{e_j(k+j) + f(k+j) \mid -J \leq j \leq J\} \quad (k \in \mathbb{Z})$$

is a local content bound as well.

Note that $f_{\text{new}}(k) \geq f(k)$ since $f_{\text{new}}(k)$ is the maximum of set that contains $e_0(k) + f(k) = f(k)$. The following picture illustrates for $J = 2$ how the lemma uses $2J$ neighbors of $f(k)$ to see if the current lower bound $f(k)$ for $v_{r^j(p)}(Y)$ can be improved:

```
  e_2(k+2)
   |             |
   |             |
   |             |
  e_1(k+1)    e_1(k)
   |             |
   |             |
   |             |
   f(k-3) f(k-2) f(k-1)
   |             |
   |             |
   |             |
   f(k) f(k+1) f(k+2) f(k+3)
   |             |
   |             |
   |             |
   e_{-1}(k-1)
   |             |
   |             |
   |             |
   e_{-2}(k-2)
```

The support of $f$ is the set $\text{supp}(f) = \{k \in \mathbb{Z} \mid f(k) \neq 0\}$.

Lemma 12. Take $\ell_1, m_1, \ell_{-1}, m_{-1} \in \mathbb{Z}$ such that $\text{supp}(e_1) \subseteq [\ell_1, m_1]$ and $\text{supp}(e_{-1}) \subseteq [\ell_{-1}, m_{-1}]$. For every non-zero solution $Y \in \mathbb{K}^n$ of $\tau(Y) = MY$, if $v_{r^j(p)}(Y) \neq 0$ then $k \in [\ell, m]$ where

$$\ell = \min\{\ell_1, \ell_{-1} + 1\} \quad \text{and} \quad m = \max\{m_1 - 1, m_{-1}\}.$$ 

This implies that the function $f : \mathbb{Z} \to \mathbb{Z} \cup \{-\infty\}$ defined by

$$f(k) = \begin{cases} 
-\infty & \text{if } k \in [\ell, m] \\
0 & \text{otherwise}
\end{cases}$$

is a local content bound.

Proof. If there are no non-zero solutions then there is nothing to prove. So let $Y$ be a generic non-zero solution and let $f(k) = v_{r^j(p)}(Y)$. Recall from Equation (4) that $v_{r^j(p)}(Y) \geq e_j(k+j) + v_q(Y)$ where $q$ was $r^{k+j}(p)$, in other words,

$$f(k) \geq e_j(k+j) + f(k+j).$$
Since \( e_j(k + j) = 0 \) when \( j = 1 \) and \( k + 1 > m_1 \) we find \( f(k) \geq f(k+1) \) for all \( k > m_1 - 1 \).
For such \( k \) we have \( f(k) \geq f(k+1) \geq f(k+2) \geq \cdots \geq 0 \) since \( f \) has finite support.
Since \( e_j(k + j) = 0 \) when \( j = -1 \) and \( k - 1 > m_{-1} \) we find \( f(k) \geq f(k-1) \) and thus \( f(k-1) \leq f(k) \) for all \( k - 1 > m_{-1} \). Then \( f(k) \leq f(k+1) \leq \cdots \leq 0 \) for all \( k > m_{-1} \).
Thus \( f(k) = 0 \) for all \( k > m \). The proof for \( \ell \) is similar:

\[
f(k) \geq f(k+1) \text{ for all } k + 1 < \ell_1. \text{ Then } 0 \geq \cdots \geq f(k-1) \geq f(k) \text{ for all } k < \ell_1.
f(k) \geq f(k-1) \text{ for all } k - 1 < \ell_{-1}. \text{ Then } f(k) \geq 0 \text{ for all } k < \ell_{-1} + 1. \]

\( \square \)

**Algorithm 13** (“The local algorithm”: \( J \)th local content bound).

**Input** The exponent-functions \( e_{-j}, \ldots, e_j \) from step (e.1) in the global algorithm.

**Output** A local content bound \( f : \mathbb{Z} \rightarrow \mathbb{Z} \) with respect to \( p \), or \( \infty \) if it is discovered that there can be no non-zero rational solutions.

**Procedure**

(a) Let \( \ell, m \) and \( f \) be as in Lemma 12.

(b) Repeat:

(b.1) Let \( f_{\text{new}} : \mathbb{Z} \rightarrow \mathbb{Z} \cup \{-\infty\} \) be the function given in Lemma 11.

(b.2) If \( f(k) > 0 \) for any \( k \not\in [\ell, m] \) then stop and return \( \infty \).

(b.3) If \( f = f_{\text{new}} \) then stop and return \( f \).

Otherwise set \( f := f_{\text{new}} \) and Repeat.

**Example 14.** Let \( J = 1 \) and \( p = x \). Example 9, which continued Example 6, computed

\[
\begin{array}{c|cccccccc}
  k & \ldots & -2 & -1 & 0 & 1 & 2 & 3 & 4 & \ldots \\
  e_{-1}(k) & \ldots & 0 & -1 & 0 & -2 & 1 & 0 & 0 & \ldots \\
  e_1(k) & \ldots & 0 & 0 & -1 & -2 & 2 & -1 & 0 & \ldots \\
\end{array}
\]

We do not list \( e_0 \) since that is always 0. Then.

\[ \ell_{-1} = -1, \quad \ell_1 = 0, \quad \text{and} \quad \ell = \min\{\ell_1, \ell_{-1} + 1\} = 0 \]

and

\[ m_{-1} = 2, \quad m_1 = 3, \quad \text{and} \quad m = \max\{m_1 - 1, m_{-1}\} = 2. \]

In the algorithm \( f : \mathbb{Z} \rightarrow \mathbb{Z} \cup \{-\infty\} \) successively becomes

\[
\begin{array}{c|cccccccc}
  k & \ldots & -2 & -1 & 0 & 1 & 2 & 3 & 4 & \ldots \\
  f(k) & \ldots & 0 & 0 & -\infty & -\infty & -\infty & 0 & 0 & \ldots \\
  f(k) & \ldots & 0 & 0 & -1 & -\infty & -1 & 0 & 0 & \ldots \\
  f(k) & \ldots & 0 & 0 & -1 & 1 & -1 & 0 & 0 & \ldots \\
\end{array}
\]

At that point \( f \) stabilizes (\( f_{\text{new}} = f \)) and the local algorithm returns \( f \). The global algorithm converts \( f \) to this content bound

\[
B = \frac{x + 1}{x(x + 2)}
\]

which is sharper than the denominator bound \( d = x^2(x + 1)(x + 2) \) from algorithm UniversalDenominator in Maple, which implements [2].
**Theorem 15.** Algorithm 13 is correct and terminates.

*Proof.* Throughout the algorithm $f$ is a local content bound by Lemmas 11 and 12. If step (b.2) returns $\infty$ then this is correct by Lemma 12. Otherwise the support of $f$ stays inside a finite range $[\ell, m]$. As long as $f(k) = -\infty$ for some $k$ we get $f_{\text{new}} \neq f$. So all $f(k)$ are in $\mathbb{Z}$ before the algorithm can terminate in step (b.3). Since no $f(k)$ ever decreases and the support is bounded, it follows that either (a) the algorithm terminates after finitely many steps, or (b) some $f(k)$ grows without bound. Option (b) leads to a contradiction, because if $f(k)$ grows without bound, then so does $f(k + 1)$ since $f_{\text{new}}(k + 1) \geq e_{\ell - 1}(k) + f(k)$. Then $f(k + 1), f(k + 2), \ldots$ must also grow without bound, which contradicts the fact that the support of $f$ stays inside $[\ell, m]$. \qed

The global algorithm only needs to consider primes in $c_1$ or $c_{-1}$, otherwise $f$ in Lemma 12 would be 0. Correctness of the global algorithm follows from Theorem 15.

6. Component-wise Bounds

We give $\hat{\mathbb{Z}} := \mathbb{Z} \cup \{\infty\}$ the structure of a *tropical semi-ring* $(\hat{\mathbb{Z}}, \oplus, \otimes)$ with $\oplus = \min$ and $\otimes = +$. We extend this to matrices. If $A \in \hat{\mathbb{Z}}^{m \times n}$ and $B \in \hat{\mathbb{Z}}^{n \times \ell}$ then the $ij$'th entry of $A \otimes B$ is

$$(A \otimes B)_{ij} := \bigoplus_{k=1}^{n} A_{ik} \otimes B_{kj} := \min\{A_{ik} + B_{kj} | 1 \leq k \leq n\}.$$

If $p$ is a prime and $A \in \mathbb{K}^{m \times n}$ then $V_p(A) \in \hat{\mathbb{Z}}^{m \times n}$ denotes the matrix whose $ij$'th entry is $v_p(A_{ij})$. The smallest entry is $v_p(A)$. Equation (1) implies:

$$V_p(AB) \geq V_p(A) \otimes V_p(B)$$

for all $A \in \mathbb{K}^{m \times n}$ and $B \in \mathbb{K}^{n \times \ell}$, where the inequality is interpreted for each entry separately.

**Example 16.** Let $\mathbb{A} = \mathbb{Q}[x]$, $p = x$ and

$$M = \begin{pmatrix} -1 + x^3 & -x^2 + x^3 & x \\ 0 & x & 1 \\ x + x^2 & x & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} x \\ -x \\ 1 \end{pmatrix}.$$ 

Then

$$V_p(M) = \begin{pmatrix} 0 & 2 & 1 \\ \infty & 1 & 0 \\ 1 & 1 & \infty \end{pmatrix} \quad \text{and} \quad V_p(Y) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$ 

Let’s check Equation (5) for $M$ and $Y$:

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix} = V_p\left( \begin{pmatrix} x^3 \\ x^3 \end{pmatrix} \right) = V_p(\mathbb{Y}) \geq V_p(M) \otimes V_p(Y) = \begin{pmatrix} \min(0 + 1, 2 + 1, 1 + 0) \\ \min(\infty + 1, 1 + 1, 0 + 0) \\ \min(1 + 1, 1 + 1, \infty + 0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$
Algorithm 17 (J’th component-wise content bound).

Input $M \in \text{GL}_n(\mathbb{K})$ and $J \geq 1$.

Output $B \in \mathbb{K}^n$ such that $\exists a \in D$ with $aY \in \text{diag}(B)\mathbb{A}^n$ for any rational solution $Y$.

Procedure

(a) Compute $M_j$ for $j \in [-J, \ldots J]$.
(b) Let $\mathcal{P}$ be the set of prime factors in the denominators in $M$ and $M_{-1}$.
(c) $O := \{p \in \mathcal{P}$ from each $\tau$-equivalence class.
(d) Let $B_i := 1$ for $i \in \{1, \ldots, n\}$.
(e) For each $p \in O$.
   (e.1) For $j \in [-J, \ldots J]$, compute the \textit{exponent-function} $E_j$ of $M_j$ at $p$, which is a function $E_j : \mathbb{Z} \to \mathbb{Z}_{\text{p}}^{\inf}$ where $E_j(k) := V_{\tau(p)}(M_j)$.
   (e.2) Call the \textit{local algorithm} below with input $E_{-J}, \ldots E_J$.
   (e.3) It returned a function $F : \mathbb{Z} \to \hat{\mathbb{Z}}^n$. For $i \in \{1 \ldots n\}$: If $F_i$ (the $i$th component of $F$) is $\infty$ then $B_i := 0$, otherwise $B_i := B_i \cdot \prod_{k \in \mathbb{Z}} t^k \tau(p)^{F_i(k)}$.
(f) Return $(B_1, \ldots, B_n)$.

If an entry of $M_j$ is zero, then the corresponding entry of $E_j(k)$ is $\infty$ for all $k \in \mathbb{Z}$. To obtain a finite “support”, we define $\text{supp}(E_j)$ as the set of all $k \in \mathbb{Z}$ for which $E_j(k) \notin [0, \infty)^{\text{p}}$. This way we can represent $E_j$ in finite terms with: integers $\ell_j, m_j$ such that $\text{supp}(E_j) \subseteq [\ell_j, m_j]$, matrices $E_j(k) \in \mathbb{Z}_{\text{p}}^{\inf}$ for $k \in [\ell_j, m_j]$, and a matrix we denote as $E_j(\infty) = \prod_{k \in \mathbb{Z}} t^k \tau(p)^{E_j(k)}$ for all $k \notin [\ell_j, m_j]$.

Algorithm 18 (Jth local component-wise content bound).

Input: $E_{-J}, \ldots, E_J$.

Output: $F : \mathbb{Z} \to \hat{\mathbb{Z}}^n$ such that $F(k) \leq V_{\tau(p)}(Y)$ for all $k \in \mathbb{Z}$ and rational solutions $Y$.

Procedure:

(a) Let $\ell, m$ be as in Lemma 12, let $c = 0$ and let $F : \mathbb{Z} \to (\hat{\mathbb{Z}} \cup \{-\infty\})^n$

\[F(k) := \begin{cases} 
(-\infty, \ldots, -\infty)^t & \text{if } \ell \leq k \leq m \\
(0, \ldots, 0)^t & \text{otherwise.}
\end{cases}\]

(b) Repeat:
   (b.1) $F_{\text{new}}(k) := \max\{E_j(k + j) \otimes F(k + j) \mid -J \leq j \leq J\}$ (for all $k \in \mathbb{Z}$) where the maxima are taken component-wise.
   (b.2) If $F_{\text{new}} = F$ then stop and return $F$.
   (b.3) If all negative entries of $F$ and $F_{\text{new}}$ are the same, then $c := c + 1$.
   If $c > 10$ then return $F_{\text{new}}$. (For alternatives see subsection 6.1.)
   (b.4) Let $F := F_{\text{new}}$ and Repeat.
Theorem 19. Algorithm 18 is correct and terminates.

Proof. As in section 5, entries can not decrease and the algorithm does not stop if any entries \( = -\infty \) remain. Apart from replacing scalars with matrices and vectors, correctness is proved in the same way as well. As for termination, negative entries can only increase finitely many times, which makes \( c \) in step (b.3) a simple termination mechanism. For more sophisticated versions, see subsection 6.1 below. \( \square \)

6.1. Alternatives to an arbitrary cut-off

The question in this subsection is how to ensure termination without an arbitrary cut-off counter \( c \) in step (b.3). We sketch one approach with an example, and an alternative that is easier to implement.

Let \( M = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \), take \( p = x \), and let \( P_n = \tau^{-1}(p) \cdots \tau^{-n}(p) = (x-1) \cdots (x-n) \). Up to constants, the only rational solution of \( \tau(Y) = MY \) is \((0,1)\). Now \((p_n,1)^t\) is a valid content bound for any \( n \) since \( Y_1 = 0 \) is divisible by any \( P_n \). In every loop, Algorithm 18 constructs an \( F_{\text{new}} \) that is strictly sharper than \( F \) (if \( F \) encodes \((p_n,1)^t\) then \( F_{\text{new}} \) encodes \((p_{n+1},1)^t\)). So if we remove step (b.3) without implementing an alternative, then the algorithm will not terminate for \( M \).

During the computation \( F \) looks as follows. Since \( M \) is a 2 by 2 matrix, \( F \) has two components \( F_1 \) and \( F_2 \), each of which is a function \( \mathbb{Z} \to \mathbb{Z} \cup \{-\infty\} \). After the first loop \( F_2 \) is identically 0, while \( F_1 \) looks like this: \( 0, 0, 1, 0, 0, \ldots \) which encodes \( P_n \) where \( n \) is the number of 1’s. This \( n \) increases by 1 in each loop.

We now sketch the first approach to ensure termination without an arbitrary cut-off. Outside a finite range of \( k \)’s, the matrices \( E_j(k) \) are constant (recall \( E_j(\infty) \) right before Algorithm 18). If a sufficiently long repeating pattern of positive entries in \( F(k) \)’s outside of this range forms during the computation, then, since the \( E_j(k) \) are constant here, it is not hard for the algorithm to prove that this pattern will continue indefinitely. In the example, when at least \( n = 1 \) positive entries have formed outside this finite range, then one can immediately deduce from \( E_1(\infty) \) that this pattern can only grow in each loop. But that means that \( Y_1 \), the first entry of \( Y \), must be divisible by a polynomial \( P_n \) whose degree keeps increasing. That implies \( Y_1 = 0 \), so we can replace \( F_1 \) by the function that is identically \( +\infty \). With this strategy, only finitely many entries \( \notin [0,\infty) \) can occur, because if more than a bounded number appear, the algorithm can construct a proof from \( E_{-j}(\infty) \cdots E_j(\infty) \) that the pattern will continue, allowing it to replace a component of \( F \) by \( +\infty \).

We decided not to spell out the details of this approach, because there is a simpler approach which accomplishes a similar outcome. Let the degree of a rational function be the degree of the numerator minus the degree of the denominator. To compute rational solutions \( Y \), one needs to compute a degree-bound for the entries of \( Y \). For instance, if \( Y_1 \) is a polynomial of degree \( \leq 3 \), then the information that \( Y_1 \) is divisible by \( P_3 \) is equivalent to the “sharper” bound that \( Y_1 \) is divisible by \( P_{10} \), since both imply \( Y_1 = 0 \). So one can design a version of Algorithm 18 where the arbitrary cut-off \( c > 10 \) is replaced with a cut-off informed by a degree-bound.

Among these alternatives, while the arbitrary cut-off approach is the least elegant, we presented it as the default because it takes the least amount of implementation effort,
and its practical performance, except in very rare cases, will likely be the same as the alternatives sketched in this subsection.

7. Example, an eigenring system

To factor an operator \( L = \tau^2 + a_1 \tau + a_0 \in \mathbb{Q}(\tau) \) with the eigenring [12, 4] method we need rational solutions for the system

\[
\tau(Y) = MY \quad \text{where} \quad M = \begin{pmatrix}
0 & 0 & 1 \\
0 & -b & -a_1 b \\
a_0 b & a_0 a_1 b & a_1 b + a_1^2 q
\end{pmatrix}
\]

with \( b = \frac{1}{\tau(a_0)} \).

For our example\(^1\) let

\[
a_0 = \frac{x^2(x + 3)(x^2 + 5x + 5)}{(x + 2)(x - 1)(x^2 + 3x + 1)} \quad \text{and} \quad a_1 = \frac{-(x + 1)(x^3 + 7x^2 + 11x^2 - 4x - 4)}{(x + 2)(x - 1)(x^2 + 3x + 1)}.
\]

The global content bounds for \( J \leq 4 \) are:

\[
\begin{align*}
B_{j=1}^{\text{global}} &= \frac{1}{(x - 1)x^4(x + 1)^3(x + 2)(x + 3) pq} \\
B_{j=2}^{\text{global}} &= \frac{1}{(x - 1)x^2(x + 1)(x + 2)(x + 3) pq} \\
B_{j=3}^{\text{global}} &= \frac{1}{(x - 1)x^2(x + 2)(x + 3) pq} \\
B_{j=4}^{\text{global}} &= \frac{1}{(x - 1)x^2(x + 3) pq}
\end{align*}
\]

where \( p = x^2 + 3x + 1 \) and \( q = \tau(p) \). The bound from [2] (Maple’s UniversalDenominator) is the same as \( B_{j=1}^{\text{global}} \). Among global content bounds, \( B_{j=4}^{\text{global}} \) is sharp (it equals the content of the set of all entries of all rational solutions). But our component-wise content bounds are sharper still. For \( J = 1 \) and \( J = 2 \) they are:

\[
\left( \frac{1}{(x - 1)x^2(x + 2)p}, \frac{1}{x^2(x + 1)(x + 3)q} \right)^t
\]

The \( J = 2 \) component-wise bound is much sharper than the sharpest global bound. In fact, even the \( J = 1 \) component-wise bound is better than the sharpest global bound (compare the degree of its denominators with that of \( B_{j=4}^{\text{global}} \)).

---

\(^1\)If an operator \( L \) is the LCLM (Least Common Left Multiple) of smaller operators then \( L \) can be factored with the eigenring method. This example was constructed as LCLM(\( \tau - x(x + 3)/(x + 1), \tau - (x + 1)/x \)). This construction ensures that \( M \) will have at least two (exactly two here) independent rational solutions.
The component-wise bound for \( J = 1 \) involves computing \( M^{-1} \) but this is done in all variations. For \( J = 2 \) we also have to compute two matrix products \( M_2 \) and \( M_{-2} \). After that, we have to compute valuations of their entries, as these valuations form the entries of the exponent-functions \( E_j \). If \( Q \in \mathbb{Q}(x) \) is an entry of \( M_j \), then a full factorization of \( Q \) immediately gives its valuation at every prime \( q \in \mathbb{Q}[x] \). However, a full factorization also computes information we do not need, since the only valuations we use are at primes \( q \) of the form \( \tau^k(p) \) with \( p \) as in the algorithm.

We need to compute valuations rapidly in order for the component-wise algorithm to be quick. With modular techniques one can quickly compute an upper bound for the valuation of a rational function \( Q \) at any \( q \), correctness can then be proved with a trial division.

For special matrices such as the example \( M \) above, only a few rational functions need to be factored for the \( J = 1 \) component-wise algorithm, namely \( a_0 \) and \( a_1 \) (then use that \( b \) is a shift of \( 1/a_0 \)). The same is true for “exterior power systems” which are used [5] to factor general difference operators (the eigenring method only factors special cases, in particular LCLM’s). The first author’s factoring implementation [9] is set up in a way where rational solutions are already polynomials, but the implementation still computes a component-wise content bound because it significantly reduces the degrees of the polynomials that the algorithm has to find.

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