Dirac, Majorana, Weyl in 4d

L. Bonora\textsuperscript{a}, R. Soldati\textsuperscript{b} and S. Zalel\textsuperscript{c}
\textsuperscript{a} International School for Advanced Studies (SISSA), Via Bonomea 265, 34136 Trieste, Italy and INFN, Sezione di Trieste
\textsuperscript{b} Dipartimento di Fisica e Astronomia, Via Irnerio 46, 40126 Bologna, Italy and INFN, Sezione di Bologna
\textsuperscript{c} Blackett Laboratory, Imperial College London, SW7 2AZ, U.K.

Abstract. This is a review of some elementary properties of Dirac, Weyl and Majorana spinors in 4d. We focus in particular on the differences between massless Dirac and Majorana fermions, on one side, and Weyl fermions, on the other. We review in detail the definition of their effective actions, when coupled to (vector and axial) gauge fields, and revisit the corresponding anomalies using the Feynman diagram method with different regularizations. Among various well known results we stress in particular the regularization independence in perturbative approaches, while not all the regularizations fit the non-perturbative ones. As for anomalies, we highlight in particular one perhaps not so well known feature: the rigid relation between chiral and trace anomalies.

Email: bonora@sissa.it, roberto.soldati@bo.infn.it, stav.zalel11@imperial.ac.uk

1 Introduction

This paper is a review concerning the properties of Dirac, Weyl and Majorana fermions in a 4 dimensional Minkowski space-time. Fermions are unintuitive objects, thus the more fascinating. The relevant literature is enormous. Still problems that seem to be well understood, when carefully put under scrutiny, reveal sometimes unexpected aspects. The motivation for this paper is the observation that while Dirac fermions are very well known, both from the classical and the quantum points of view, Weyl and Majorana fermions are often treated as poor relatives\textsuperscript{1} of the former, and, consequently, not sufficiently studied, especially for what concerns their quantum aspects. The truth is that these three types of fermions, while similar in certain respects, behave radically differently in others. Dirac spinors belong to a reducible representation of the Lorentz group, which can be irreducibly decomposed in two different ways: the first in eigenstates of

\textsuperscript{1}Actually, the Weyl spinors are the main bricks of the original Standard Model, while the neutral Majorana spinors could probably be the basic particle bricks of Dark Matter, if any.
the charge conjugation operator (Majorana), the second in eigenstates of the chirality operator (Weyl). Weyl spinors are bound to preserve chirality, therefore do not admit a mass term in the action and are strictly massless. Dirac and Majorana fermions can be massive. In this review we will focus mostly on massless fermions, and one of the issues we wish to elaborate on is the difference between massless Dirac, Majorana and Weyl fermions.

The key problem one immediately encounters is the construction of the effective action of these fermions coupled to gauge or gravity potentials. Formally, the effective action is the product of the eigenvalues of the relevant kinetic operator. The actual calculus can be carried out either perturbatively or non-perturbatively. In the first case the main approach is by Feynman diagrams, in the other case by analytical methods, variously called Seeley-Schwinger-DeWitt or heat kernel methods. While the procedure is rather straightforward in the Dirac (and Majorana) case, the same approach in the Weyl case is strictly speaking inaccessible. In this case one has to resort to a roundabout method, the discussion of which is one of the relevant topics of this review. In order to clarify some basic concepts we carry out a few elementary Feynman diagram calculations with different regularizations (mostly Pauli-Villars and dimensional regularisation). The purpose is to justify the methods used to compute the Weyl effective action. A side bonus of this discussion is a clarification concerning the nonperturbative methods and the Pauli-Villars (PV) regularization: contrary to the dimensional regularization, the PV regularization is unfit to be extended to the heat kernel-like methods, unless one is unwisely willing to violate locality.

A second major ground on which Weyl fermions split from Dirac and Majorana fermions is the issue of anomalies. To illustrate it in a complete and exhaustive way we limit ourselves here to fermion theories coupled to external gauge potentials and, using the Feynman diagrams, we compute all the anomalies (trace and gauge) in such a background. These anomalies have been calculated elsewhere in the literature in manifold ways and since a long time, so that there is nothing new in our procedure. Our goal here is to give a panoptic view of these computations and their interrelations. The result is interesting. Not only does one get a clear vantage point on the difference between Dirac and Weyl anomalies, but, for instance, it transpires that the rigid link between chiral and trace anomalies is not a characteristic of supersymmetric theories alone, but holds in general.

The paper is organized as follows. Section 2 is devoted to basic definitions and properties of Dirac, Weyl and Majorana fermions, in particular to the differences between massless Majorana and Weyl fermions. In section 3 we discuss the problem related to the definition of a functional integral for Weyl fermions. In section 4 we introduce perturbative regularizations for Weyl fermions coupled to vector potentials and verify that the addition of a free Weyl fermions of opposite handedness allows us to define a functional integral for the system, while preserving the Weyl fermion’s chirality. In section 5 we recalculate consistent and covariant gauge anomalies for Weyl and Dirac fermions, by means of the Feynman diagram technique, and in section 7 we apply these results to the case of Majorana fermions. In section 8 we compute also the trace anomalies of Weyl fermions due to the presence of background gauge potentials and show that they are rigidly related to the previously calculated gauge anomalies. Section 9 is devoted to a summary of the results.

Historical references for this review are [1-12].
Notation

We use a metric $g_{\mu\nu}$ with mostly $(-)$ signature. The gamma matrices satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ and

$$\gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0.$$

The generators of the Lorentz group are $\Sigma_{\mu\nu} = \frac{1}{4}\{\gamma_\mu, \gamma_\nu\}$. The charge conjugation operator $C$ is defined to satisfy

$$\gamma^T_\mu = -C^{-1}\gamma_\mu C, \quad CC^* = -1, \quad CC^d = 1.$$  \hspace{1cm} (1)

For example, $C = C^d = C^{-1} = \gamma_0\gamma_2\gamma_3$ does satisfy all the above requirements. The chiral matrix $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$ has the properties

$$\gamma_5^d = \gamma_5, \quad (\gamma_5)^2 = 1, \quad C^{-1}\gamma_5 C = \gamma_5^T.$$

2 Dirac, Majorana and Weyl fermions in 4d.

Let us start from a few basic definitions and properties of spinors on a 4d Minkowski space. A 4-component Dirac fermion $\psi$ under a Lorentz transformation transforms as

$$\psi(x) \rightarrow \psi'(x') = \exp\left[\frac{-1}{2}\lambda^{\mu\nu}\Sigma_{\mu\nu}\right] \psi(x),$$  \hspace{1cm} (2)

for $x'^\mu = \Lambda^{\mu}_\nu x^\nu$. Here $\lambda^{\mu\nu} + \lambda^{\nu\mu} = 0$ are six real canonical coordinates for the Lorentz group, $\Sigma_{\mu\nu}$ are the generators in the 4d reducible representation of Dirac bispinors, while $\Lambda^{\mu}_\nu$ are the Lorentz matrices in the irreducible vector representation $D(\frac{1}{2}, \frac{1}{2})$. The invariant kinetic Lagrangian for a free Dirac field is

$$i\bar{\psi}\gamma^\mu\partial_\mu\psi.$$

It can be constructed because in the spinor space, there exists a Lorentz invariant scalar product $(\Psi_1, \Psi_2) = (\Psi_1^\dagger|\gamma^0|\Psi_2)$. So that (3) can also be written as

$$i(\psi, \gamma\cdot\partial\psi).$$  \hspace{1cm} (4)

A Dirac fermion admits a Lorentz invariant mass term $m\bar{\psi}\psi = m(\psi, \psi)$.

A Dirac bispinor can be seen as the direct sum of two Weyl spinors

$$\psi_L = P_L\psi, \quad \psi_R = P_R\psi,$$

where

$$P_L = \frac{1 - \gamma_5}{2}, \quad P_R = \frac{1 + \gamma_5}{2}$$

with opposite chiralities

$$\gamma_5\psi_L = -\psi_L, \quad \gamma_5\psi_R = \psi_R.$$

A left-handed Weyl fermion admits a Lagrangian kinetic term

$$i(\psi_L, \gamma\cdot\partial\psi_L) = i\bar{\psi}_L\gamma^\mu\partial_\mu\psi_L$$  \hspace{1cm} (5)

---

2This and the following section are mostly based on [13] and [14]
but not a mass term, because \((\psi_L, \psi_L) = 0\), since \(\gamma_5 \gamma^0 + \gamma^0 \gamma_5 = 0\). So a Weyl fermion is massless and this property is protected by its being chiral.

In order to introduce Majorana fermions we need the notion of Lorentz covariant conjugate spinor, \(\hat{\psi}\):

\[
\hat{\psi} = \gamma_0 C \psi^\ast. 
\]

(6)

It is not hard to show that if \(\psi\) transforms like (2), then

\[
\hat{\psi}(x) \to \hat{\psi}'(x') = \exp \left[ -\frac{1}{2} \lambda^{\mu \nu} \Sigma_{\mu \nu} \right] \hat{\psi}(x).
\]

(7)

Therefore it makes sense to impose on \(\psi\) the condition

\[
\psi = \hat{\psi}
\]

(8)

because both sides transform in the same way. A spinor satisfying (8) is, by definition, a Majorana spinor. A Majorana spinor admits both kinetic and mass term, which can be seen as \(\frac{1}{2}\) times those of a Dirac spinor.

It is a renowned fact that the group theoretical approach \([1]\) to Quantum Field Theory is one of the most solid and firm pillars in modern Physics. To this concern, the contributions by Eugene Paul Wigner were of invaluable importance \([2]\). In terms of Lorentz group representations we can summarize the situation as follows. \(\gamma_5\) commutes with Lorentz transformations \(\exp \left[ -\frac{1}{2} \lambda^{\mu \nu} \Sigma_{\mu \nu} \right] \). So do \(P_L\) and \(P_R\). This means that the Dirac representation is reducible and multiplying the spinors by \(P_L\) and \(P_R\) singles out irreducible representations, the Weyl ones. To be more precise, the Weyl representations are irreducible representations of the group \(SL(2, C)\), which is the covering group of the proper orthochronous Lorentz group. They are usually denoted \((\frac{1}{2}, 0)\) and \((0, \frac{1}{2})\) in the \(SU(2) \times SU(2)\) labeling of the \(SL(2, C)\) irreps. As we have seen in (7), Lorentz transformations commute also with the charge conjugation operation

\[
C \psi C^{-1} = \eta_C \gamma_0 C \psi^\ast
\]

(9)

where \(\eta_C\) is a phase which, for simplicity, we set equal to 1. This also says that Dirac spinors are reducible and suggests another way to reduce them: by imposing (8) we single out another irreducible representation, the Majorana one. The Majorana representation is the minimal irreducible representation of a (one out of eight) covering of the complete Lorentz group \([3, 8]\). It is evident, and well-known, that Majorana and Weyl representations are incompatible (in 4d).

Let us recall the properties of a Weyl fermion under charge conjugation and parity. We have

\[
C \psi_L C^{-1} = P_L \psi \psi_L C^{-1} = P_L \hat{\psi} = \hat{\psi}_L.
\]

(10)

The charge conjugate of a Majorana field is itself, by definition. While the action of a Majorana field is invariant under charge conjugation, the action of a Weyl fermion is, so to say, maximally non-invariant, for

\[
C \left( \int i \overline{\psi}_L \gamma^\mu \partial_\mu \psi_L \right) C^{-1} = \int i \overline{\psi}_L \gamma^\mu \partial_\mu \hat{\psi}_L = \int i \overline{\psi}_R \gamma^\mu \partial_\mu \psi_R.
\]

(11)

The parity operation is defined by

\[
P \psi_L(t, \vec{x}) P^{-1} = \eta_P \gamma_0 \psi_R(t, -\vec{x})
\]

(12)
where \( \eta_P \) is a phase. In terms of the action we have

\[
\mathcal{P}\left( \int \bar{\psi}_L \gamma^\mu \partial_\mu \psi_L \right) \mathcal{P}^{-1} = \int \bar{\psi}_R \gamma^\mu \partial_\mu \psi_R,
\]

while for a Majorana fermion the action is invariant under parity.

This also suggests a useful representation for a Majorana fermion. Let \( \psi_R = P_R \psi \) be a generic Weyl fermion. We have \( P_R \psi_R = \psi_R \) and it is easy to prove that \( P_L \psi_R = \psi_R \), i.e. \( \psi_R \) is left-handed. Therefore the sum \( \psi_M = \psi_R + \psi_R \) is a Majorana fermion because it satisfies (8). And any Majorana fermion can be represented in this way. This representation is instrumental in the calculus of anomalies, see below.

If we consider CP, the action of a Majorana fermion is obviously invariant under it. For a Weyl fermion we have

\[
\mathcal{C} \mathcal{P} \psi_L (t, \vec{x}) (\mathcal{C} \mathcal{P})^{-1} = \gamma_0 P_R \hat{\psi} (t, - \vec{x}) = \gamma_0 \hat{\psi}_R (t, - \vec{x}).
\]

Applying CP to the Weyl action one gets

\[
\mathcal{C} \mathcal{P} \left( \int i \bar{\psi}_L \gamma^\mu \partial_\mu \psi_L \right) (\mathcal{C} \mathcal{P})^{-1} = \int i \bar{\psi}_R (t, - \vec{x}) \gamma^\mu \partial_\mu \hat{\psi}_R (t, - \vec{x}) = \int i \bar{\psi}_R (t, \vec{x}) \gamma^\mu \partial_\mu \hat{\psi}_R (t, \vec{x}).
\]

But one can easily prove that

\[
\int i \bar{\psi}_R (t, \vec{x}) \gamma^\mu \partial_\mu \hat{\psi}_R (t, \vec{x}) = \int i \bar{\psi}_L (x) \gamma^\mu \partial_\mu \psi_L (x).
\]

Therefore the action for a Weyl fermion is CP invariant. It is also, separately, T invariant, and, so, CPT invariant.

Now let us go to the quantum interpretation of the field \( \psi_L \). It has the plane wave expansion

\[
\psi_L (x) = \int dp \left( a(p) u_L (p) e^{-ipx} + b^\dagger (p) v_L (p) e^{ipx} \right)
\]

where \( u_L, v_L \) are fixed and independent left-handed spinors (there are only two of them). The interpretation is: \( b^\dagger (p) \) creates a left-handed particle while \( a(p) \) destroys a left-handed particle with negative helicity (because of the opposite momentum). However eqs. (14,15) force us to identify the latter with a right-handed antiparticle: C maps particles to antiparticles, while P invert helicities, so CP maps left-handed particles to right-handed antiparticles. It goes without saying that no right-handed particles or left-handed antiparticles enter the game.

**Remark.** A mass term \( \bar{\psi} \psi \) for a Dirac spinor can also be rewritten by projecting the latter into its chiral components

\[
\bar{\psi} \psi = \bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L.
\]

If \( \psi \) is a Majorana spinor this can be written as

\[
\bar{\psi} \hat{\psi} = \bar{\psi}_L \hat{\psi}_R + \bar{\psi}_R \hat{\psi}_L,
\]
which is well defined and Lorentz invariant by construction. Now, using the Lorentz covariant conjugate we can rewrite (19) as

$$(\psi_L)^TC^{-1}\psi_L + \psi_L^T C(\psi_L)^*,$$

which is expressed only in terms of $\psi_L$, albeit it still couples both chiralities. (20) may create the illusion that there exists a mass term also for Weyl fermions. But this is not the case. If we add this term to the kinetic term (5) we obtain an action whose equation of motion is not Lorentz covariant: the kinetic and mass term in the equation of motion belong to two different representations. To be more explicit, a massive Dirac equation of motion for a Weyl fermion would be

$$i\gamma^\mu \partial_\mu \psi_L - m\psi_L = 0,$$

but this equation breaks Lorentz covariance because the first piece transforms according to the $(0, 1/2)$ representation while the second according to $(1/2, 0)$, and is not Lagrangian\textsuperscript{3}. The reason is, of course, that (20) is not expressible in the same canonical form as (5). This structure is clearly visible in the four component formalism used so far, although much less recognizable in the two-component formalism.

### 2.1 Weyl fermions and massless Majorana fermions

The fact that a massive Majorana fermion and a Weyl fermion are different objects is firmly uncontroversial. The question of whether a massless Majorana fermion is or is not the same as a Weyl fermion at both the classical level and the quantum level is, instead, not so clear in the literature. Let us consider the case in which there is no quantum number appended to the fermion. We first mention the evident differences between the two. The first, and most obvious, is the one we have already mentioned: they belong to two different representations of the Lorentz group, irreducible to each other (it is standard lore that in 4d there cannot exist a spinor that is simultaneously Majorana and Weyl). Another important difference is that the helicity of a Weyl fermion is well defined and corresponds to its chirality, while the chirality of a Majorana fermion is undefined, so that the relation with its helicity is also undefined. Next, a parity operation maps the Majorana action into itself, while it maps the Weyl action (5) into the same action for the opposite chirality. The same is true for the charge conjugation operator.

The reason why they are sometimes considered as the same object is possibly due to the fact that we can establish a one-to-one correspondence between the components of a Weyl spinor and those of a Majorana spinor in such a way that the Lagrangian, in two-component notation, looks the same. If, for instance, in the chiral representation we represent $\psi_L$ as $\left(\omega \begin{array}{c} 0 \\ \end{array} \right)$, where $\omega$ is a two component spinor, then (5) above becomes

$$i\omega^\dagger \bar{\sigma}^\mu \partial_\mu \omega$$

\footnote{Instead of the second term in the LHS of (21) one could use $mC\bar{\psi}_L^T$, which has the right Lorentz properties, but the corresponding Lagrangian term would not be self-adjoint and one would be forced to introduce the adjoint term and end up again with (20). This implies, in particular, that there does not exist such a thing as a “massive Weyl propagator”, that is a massive propagator involving only one chirality.}
which has the same form (up to an overall factor) as a massless Majorana action. Now, if the form of the classical action integral is the same for both Weyl and Majorana, how could there be differences? This would-be syllogism may cause gross misunderstandings.

In fact this argument is far from decisive. Even though numerically the actions coincide, the way they respond to a variation of the Weyl and Majorana fields is different. One leads to the Weyl equation of motion, the other to the Majorana equation of motion. The delicate point is precisely this: when we take the variation of an action with respect to a field in order to extract the equations of motion, we have to make sure that the variations do not break the symmetries or the properties we wish to be present in the equations of motion. In general, we do this automatically[4]. In this case, if we wish the equation of motion to preserve chirality we must use variations which preserve chirality, i.e., variations which are eigenfunctions of \( \gamma_5 \). If instead we wish the equation of motion to transform in the Majorana representation we have to use variations which transform suitably, i.e., which are eigenfunctions of the charge conjugation operator. If we do so we obtain two different results which are irreducible to each other no matter which action we use.

There is no room for confusing massless Majorana spinors with chiral Weyl spinors. A classical Majorana spinor is a self-conjugated bispinor, that can always be chosen to be real and always contains both chiralities in terms of four real independent component functions. It describes neutral spin 1/2 objects - not yet detected in Nature - and consequently there is no phase transformation (U(1) continuous symmetry) involving self-conjugated Majorana spinors, independently of the presence or not of a mass term. Hence, e.g., its particle states do not admit antiparticles of opposite charge, simply because charge does not exist at all for charge self-conjugated spinors (actually, this was the surprising discovery of Ettore Majorana, after the appearance of the Dirac equation and the positron detection). The general solution of the wave field equations for a free Majorana spinor always entails the presence of two polarization states with opposite helicity. On the contrary, it is well known that a chiral Weyl spinor, describing massless neutrinos in the Standard Model, admits only one polarization or helicity state, it always involves antiparticles of opposite helicity and it always carries a conserved internal quantum number such as the lepton number, which is opposite for particles and antiparticles.

Finally, and most important, in the quantum theory a crucial role is played by the functional measure, which is different for Weyl and Majorana fermions. We will shortly come back to this point. But, before that, it is useful to clarify an issue concerning the just mentioned U(1) continuous symmetry of Weyl fermions. The latter is sometime confused with the axial \( \mathbb{R} \) symmetry of Majorana fermions and assumed to justify the identification of Weyl and massless Majorana fermions. To start with let us consider a free massless Dirac fermion \( \psi \). Its free action is clearly invariant under the transformation \( \delta \psi = i(\alpha + \gamma_5\beta)\psi \), where \( \alpha \) and \( \beta \) are real numbers. This symmetry can be gauged by minimally coupling \( \psi \) to a vector potential \( V_\mu \) and an axial potential \( A_\mu \), in the combination \( V_\mu + \gamma_5 A_\mu \), so that \( \alpha \) and \( \beta \) become arbitrary real functions. For convenience let us choose the Majorana representation for gamma matrices, so that all of them, including \( \gamma_5 \), are imaginary. If we now impose \( \psi \) to be a Majorana fermion, its four component can be chosen to be real and only the symmetry parametrized by \( \beta \) makes sense in the action (let us call it \( \beta \) symmetry). If instead we impose \( \psi \) to be Weyl, say \( \psi = \psi_L \), then, since \( \gamma_5 \psi_L = \psi_L \), the symmetry transformation will be \( \delta \psi_L = i(\alpha - \beta)\psi_L \).

We believe this may be the origin of the confusion, because it looks like we can merge

---

[4]: For instance, in gravity theories, the metric variation \( \delta g_{\mu\nu} \) is generic while not ceasing to be a symmetric tensor.
the two parameters $\alpha$ and $\beta$ into a single parameter identified with the $\beta$ of the Majorana axial $\beta$ symmetry. However this is not correct because for a right handed Weyl fermion the symmetry transformation is $\delta \psi_R = i(\alpha + \beta)\psi_R$. Forgetting $\beta$, the Majorana fermion does not transform. Forgetting $\alpha$, both Weyl and Majorana fermions transform, but the Weyl fermions transform with opposite signs for opposite chiralities. This distinction will become crucial in the computation of anomalies (see below).

3 Functional integral for Dirac, Weyl and Majorana fermions

In quantum field theory there is one more reason to distinguish between massless Weyl and Majorana fermions: their functional integration measure is formally and substantially different. Although the action in the two-component formalism may take the same form (22) for both, the change of integration variable from $\psi_L$ to $\omega$ is not an innocent field redefinition because the functional integration measure changes. The purpose of this section is to illustrate this issue. To start with, let us clarify that speaking about functional integral measure is a colorful but not rigorous parlance. The real issue here is the definition of the functional determinant for a Dirac-type matrix-valued differential operator.

Let us start with some notations and basic facts. We denote by $\mathcal{D}$ the Dirac operator proper: namely, the massless matrix-valued differential operator applied in general to Dirac spinors on the 4d curved space with Minkowski signature $(+, -, -, -)$

$$\mathcal{D} = i(\partial + V)$$

where $V_\mu$ is any anti-Hermitean vector potential, including a spin connection in the presence of a non-trivial background metric. We use here the four component formalism for fermions. The functional integral, i.e. the effective action for a quantum Dirac spinor in the presence of a classical background potential

$$\mathcal{Z}[V] = \int D\psi D\bar{\psi} \, e^{\frac{i}{\hbar} \int d^4x \sqrt{g} \bar{\psi} \mathcal{D} \psi}$$

is formally understood as the determinant of $\mathcal{D}$: $\det (\mathcal{D}) = \prod_{i}^{\infty} \lambda_i$. From a concrete point of view, the latter can be operatively defined in two alternative ways: either in perturbation theory, i.e. as the sum of an infinite number of 1-loop Feynman diagrams, some of which contain UV divergences by naive power counting, or by a non-perturbative approach, i.e. as the suitably regularized infinite product of the eigenvalues of $\mathcal{D}$ by means of the analytic continuation tool. It is worthwhile to remark that, on the one hand, the perturbative approach requires some UV regulator and renormalization prescription, in order to give a meaning to a finite number of UV divergent 1-loop diagrams by naive power counting. On the other hand, in the non-perturbative framework the complex power construction and the analytic continuation tool, if available, provide by themselves the whole necessary setting up to define the infinite product of the eigenvalues of a normal operator, without need of any further regulator.

In many practical calculations one has to take variations of (24) with respect to $V$. In turn, any such variation requires the existence of an inverse of the kinetic operator, as follows from the abstract formula for the determinant of an operator $A$

$$\delta \det A = \det A \, \text{tr} \left( A^{-1} \delta A \right).$$
It turns out that an inverse of $D$ does exist and, if full causality is required in forwards and backwards time evolution on e.g. Minkowski space, it is the Feynman propagator or Schwinger distribution $S$, which is unique and characterized by the well-known Feynman prescription, in such a manner that

$$D_x S(x - y) = \delta(x - y), \quad DS = 1.$$  \hspace{1cm} (25)

The latter is a shortcut operator notation, which we are often going to use in the sequel.

For instance, the scheme to extract the trace of the stress-energy tensor from the functional integral is well-known. It is its response under a Weyl (or even a scale) transform $\delta_\omega g_{\mu\nu} = 2\omega g_{\mu\nu}$:

$$\delta_\omega \log Z = \int d^4 x \omega(x) \langle T^{\mu\nu}(x) \rangle$$

where $g_{\mu\nu}(x)\langle T^{\mu\nu}(x) \rangle$ is the quantum trace of the energy-momentum tensor. Analogously, the divergence of the vector current $j_\mu = \bar{\psi}\gamma_\mu \psi$ is the response of $\log Z$ under the Abelian gauge transformation $\delta_\lambda V_\mu = \partial_\mu \lambda$:

$$\delta_\lambda \log Z = -i \int d^4 x \lambda(x) \partial_\mu \langle j^\mu(x) \rangle$$

and so on. These quantities can be calculated in various ways with perturbative or non-perturbative methods. The most frequently used ones are the Feynman diagram technique and the so-called analytic functional method, respectively. The latter denomination actually includes a collection of approaches, ranging from the Schwinger’s proper-time method \cite{4} to the heat kernel method \cite{10}, the Seeley-DeWitt \cite{5,6} and the zeta-function regularization \cite{7}. The central tool in these approaches is the (full) kinetic operator of the fermion action (or the square thereof), and its inverse, the full fermion propagator. All these methods yield well-known results with no disagreement among them.

On the contrary, when one comes to Weyl fermions things drastically change. The classical action on the 4d Minkowski space for a left-handed Weyl fermion reads

$$S_L = \int d^4 x \bar{\psi}_L D_L \psi_L.$$  \hspace{1cm} (28)

The Dirac operator, acting on left-handed spinors maps them to right-handed ones. Hence, the Sturm-Liouville or eigenvalue problem itself is not well posed, so that the Weyl determinant cannot even be defined. This is reflected in the fact that the inverse of $D_L = \bar{\psi} P_L = P_R D$ does not exist, since it is the product of an invertible operator times a projector. As a consequence the full propagator of a Weyl fermion does not exist in this naive form (this problem can be circumvented in a more sophisticated approach, see below\footnote{It is incorrect to pretend that the propagator is $S_L = S P_R = P_L S$. First because such an inverse does not exist, second because, even formally, $D_L S_L = P_R$, and $S_L D_L = P_L$.}.)

\footnote{For simplicity we understand factors of $\sqrt{S}$, which should be there, see \cite{5}, but are inessential in this discussion.}

The inverse of the Weyl kinetic operator is not the inverse of the Dirac operator multiplied by a chiral projector. Therefore the propagator for a Weyl fermion is not the Feynman propagator for a Dirac fermion multiplied by the same projector.
The lack of an inverse for the chiral Dirac-Weyl kinetic term has drastic consequences even at the free non-interacting level. For instance, the evaluation of the functional integral (i.e. formally integrating out the spinor fields) involves the inverse of the kinetic operator: thus, it is clear that the corresponding formulas for the chiral Weyl quantum theory cannot exist at all, so that no Weyl effective action can be actually defined in this way even in the free non-interacting case. Let us add that considering the square of the kinetic operator, as it is often done in the literature, does not change this conclusion.

It may sound strange that the (naive) full propagator for Weyl fermions does not exist, especially if one has in mind perturbation theory in Minkowski space. In that case, in order to construct Feynman diagrams, one uses the ordinary free Feynman propagator for Dirac fermions. The reason one can do so is because the information about chirality is preserved by the fermion-boson-fermion vertex, which contains the $P_L$ projector (the use of a free Dirac propagator is formally justified, because one can add a free right-handed fermion to allow the inversion of the kinetic operator, see below). On the contrary, the full (non-perturbative) propagator is supposed to contain the full chiral information, including the information contained in the vertex, i.e. the potential, as it will be explicitly checked here below. In this problem there is no simple shortcut such as pretending to replace the full Weyl propagator with the full Dirac propagator multiplied by a chiral projector, because this would destroy any information concerning the chirality.

The remedy for the Weyl fermion disaster is to use as kinetic operator

$$i\gamma^\mu (\partial_\mu + P_L V_\mu),$$

(30)

which is invertible and in accord with the above mentioned Feynman diagram approach. It corresponds to the intuition that the free right-handed fermions added to the left-handed theory in this way do not interfere with the conservation of chirality and do not alter the left-handed nature of the theory. But it is important to explicitly check it. The next section is devoted to a close inspection of this problem and its solution.

4 Regularisations for Weyl spinors

The classical Lagrange density for a Weyl (left) spinor in the four component formalism

$$\psi(x) = \chi_L(x) = \begin{pmatrix} \chi(x) \\ 0 \end{pmatrix}$$

reads

$$K(x) = \bar{\psi}(x) i\slashed{\partial} \psi(x) = \chi_L^\dagger(x) \alpha^\nu i\partial_\nu \chi_L(x).$$

(31)

It follows that the corresponding matrix valued Weyl differential operator

$$w_L \equiv \alpha^\nu i\partial_\nu P_L$$

(32)

is singular and does not possess any rank-four inverse. After minimal coupling with a real massless vector field $A^\mu(x)$ we come to the classical Lagrangian

$$\mathcal{L} = \chi_L^\dagger \alpha^\nu i\partial_\nu \chi_L + g A^\mu \chi_L^\dagger \alpha_\nu \chi_L - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

(33)
where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). It turns out that the classical action

\[
S = \int \! d^4x \, \mathcal{L}
\]

is invariant under the Poincaré group, as well as under the internal U(1) phase transformations \( \chi_L(x) \mapsto e^{ig\theta} \chi_L(x) \). The action integral is invariant under the so called scale or dilatation transformations, viz.,

\[
x'\mu = e^{-\varrho} x^\mu \quad \chi'_L(x) = e^{\frac{3\varrho}{2}} \chi_L(e^{\varrho} x) \quad A'_\mu(x) = e^{\varrho} A^\mu(e^{\varrho} x)
\]

with \( \varrho \in \mathbb{R} \), as well as with respect to the local phase or gauge transformations

\[
\chi'_L(x) = e^{ig\theta(x)} \chi_L(x) \quad A'_\nu(x) = A_\nu(x) + \partial_\nu \theta(x)
\]

which amounts to the ordinary U(1) phase transform in the limit of constant phase. It follows therefrom that there are twelve conserved charges in this model at the classical level and, in particular, owing to scale and gauge invariance, no mass term is allowed for both spinor and vector fields. The question naturally arises if those symmetries hold true after the transition to the quantum theory and, in particular, if they are protected against loop radiative corrections within the perturbative approach. Now, as explained above, in order to develop perturbation theory, one faces the problem of the lack of an inverse for both the Weyl and gauge fields, owing to chirality and gauge invariance. In order to solve it, it is expedient to add to the Lagrangian non-interacting terms, which are fully decoupled from any physical quantity. They break chirality and gauge invariance, albeit in a harmless way, just to allow us to define a Feynman propagator, or causal Green’s functions, for both the Weyl and gauge quantum fields. The simplest choice, which preserves Poincaré and internal U(1) phase change symmetries, is provided by

\[
\mathcal{L}' = \varphi_R^\dagger \alpha^\nu i\partial_\nu \varphi_R - \frac{1}{2}(\partial \cdot A)^2
\]

where

\[
\psi(x) = \varphi_R(x) = \begin{pmatrix} 0 \\ \varphi(x) \end{pmatrix}
\]

is a left-chirality breaking right-handed Weyl spinor field. Notice in passing that the modified Lagrangian \( \mathcal{L} + \mathcal{L}' \) exhibits a further U(1) internal symmetry under the so called chiral phase transformations

\[
\psi'(x) = (\cos \theta + i\sin \theta \gamma_5)\psi(x) \quad \psi(x) = \begin{pmatrix} \chi(x) \\ \varphi(x) \end{pmatrix}
\]

so that the modified theory involves another conserved charge at the classical level. From the modified Lagrange density we get the Feynman propagators for the massless Dirac field \( \psi(x) \), as well as for the massless vector field in the so called Feynman gauge: namely,

\[
S(p) = \frac{i g^\mu}{p^2 + i \varepsilon} \quad D_{\mu\nu}(k) = \frac{-ig_{\mu\nu}}{k^2 + i \varepsilon}
\]

and the vertex \( ig\gamma^\mu \bar{P}_L \), with \( k + p - q = 0 \), which involves a vector particle of momentum \( k \) and a Weyl pair of particle and anti-particle of momenta \( p \) and \( q \) respectively and of opposite helicity.\(^7\)

\(^7\) Customarily, the on-shell 1-particle states of a left Weyl spinor field are a left-handed particle with negative helicity \(-\frac{1}{2} \hbar \) and a right-handed antiparticle of positive helicity \( \frac{1}{2} \hbar \).
Our purpose hereafter is to show that, notwithstanding the use of the non-chiral propagators \[55\], a mass in the Weyl kinetic term cannot arise as a consequence of quantum corrections. The lowest order 1-loop correction to the kinetic term \(kP_L\) is provided by the Feynman rules in Minkowski space, in the following form

\[
\Sigma_2(k) = -ig^2 \int \frac{d^4\ell}{(2\pi)^4} \gamma^\mu D_{\mu\nu}(k - \ell) S(\ell) \gamma^\nu P_L. \tag{36}
\]

A mass term in this context should be proportional to the identity matrix (in the spinor space).

By naïve power counting the above 1-loop integral turns out to be UV divergent. Hence a regularisation procedure is mandatory to give a meaning and evaluate the radiative correction \(\Sigma_2(k)\) to the Weyl kinetic operator. Here in the sequel we shall examine in detail the dimensional, Pauli-Villars and UV cut-off regularisations.

### 4.1 Dimensional, PV and cutoff regularisations

In a \(2\omega\)–dimensional space-time the radiative correction to the Weyl kinetic term takes the form

\[
\text{reg} \Sigma_2(k) = -ig^2 \mu^{2\omega} \int \frac{d^{2\omega}\ell}{(2\pi)^{2\omega}} D_{\mu\nu}(\ell) \gamma^\mu S(\ell + k) \gamma^\nu P_L \tag{37}
\]

where \(\epsilon = 2 - \omega > 0\) is the shift with respect to the physical space-time dimensions. Since the above expression is traceless and has the canonical engineering dimension of a mass in natural units, it is quite apparent that the latter cannot generate any mass term, which, as anticipated above, would be proportional to the unit matrix. Hence, mass is forbidden and it remains for us to evaluate

\[
\text{reg} \Sigma_2(k) \equiv f(k^2) kP_L \quad \text{tr}[\text{reg} \Sigma_2(k)] = \frac{1}{2} 2^{\omega} k^2 f(k^2) \tag{38}
\]

\[
\text{tr}[\text{reg} \Sigma_2(k)] = g^2 \mu^{2\omega} (2\pi)^{-2\omega} \int d^{2\omega}\ell \frac{(-i) \text{tr}\left(kS(k)\gamma^k P_L\right)}{[(\ell - k)^2 + i\varepsilon] (\ell^2 + i\varepsilon)}. \tag{39}
\]

For \(2^\omega \times 2^\omega\) \(\gamma\)–matrix traces in a \(2\omega\)–dimensional space-time with a Minkowski signature the following formulas are necessary

\[
\text{tr}(\gamma^\mu \gamma^\nu) = g^{\mu\nu} \text{tr} \mathbb{I} = 2^{\omega} g^{\mu\nu} \tag{40}
\]

\[
2^{-\omega} \text{tr}\left(\gamma^\kappa \gamma^\lambda \gamma^\mu \gamma^\nu\right) = g^{\kappa\lambda} g^{\mu\nu} - g^{\kappa\mu} g^{\lambda\nu} + g^{\kappa\nu} g^{\lambda\mu}. \tag{41}
\]

Then we get 

\[
\text{tr}\left(\gamma^\lambda \gamma^\kappa P_L\right) = 2^{\omega}(\epsilon - 1) p \cdot \ell \quad \text{and thereby}
\]

\[
k^2 f(k^2) = ig^2 \mu^{2\omega} \epsilon - \frac{1}{(2\pi)^{2\omega}} \int \frac{2p \cdot \ell \ d^{2\omega}\ell}{[(\ell - k)^2 + i\varepsilon] (\ell^2 + i\varepsilon)}. \tag{42}
\]

Turning to the Feynman parametric representation we obtain

\[
k^2 f(k^2) = ig^2 \mu^{2\omega} \epsilon - \frac{1}{(2\pi)^{2\omega}} \int_0^1 dx \int \frac{2p \cdot \ell \ d^{2\omega}\ell}{[\ell^2 - 2x k \cdot \ell + xk^2 + i\varepsilon]^2}. \tag{43}
\]

Completing the square in the denominator and after shifting the momentum \(\ell' \equiv \ell - xp\), dropping the linear term in \(\ell'\) in the numerator owing to symmetric integration, we have

\[
f(k^2) = 2ig^2 \mu^{2\omega} \epsilon - \frac{1}{(2\pi)^{2\omega}} \int_0^1 dx x \int \frac{d^{2\omega}\ell}{[\ell^2 + x(1 - x)k^2 + i\varepsilon]^2}. \tag{44}
\]
One can perform the Wick rotation and readily get the result

\[
 f(k^2) = -2 g^2 \mu^2 e^{2\epsilon} \int_0^1 dx \int_0^\infty d\tau \tau^{\epsilon-1} e^{-\tau x(1-x) k^2} \\
 = 2 \left( \frac{g}{4\pi} \right)^2 \left[ \Gamma(\epsilon) - \Gamma(1+\epsilon) \right] \left( \frac{-4\mu^2}{k^2} \right)^\epsilon B(2-\epsilon, 1-\epsilon). \tag{45}
\]

Expansion around \( \epsilon = 0 \) yields

\[
 f(k^2) = \left( \frac{g}{4\pi} \right)^2 \left[ \frac{1}{\epsilon} + 1 + 3C + \ln \left( -\frac{4\mu^2}{k^2} \right) \right] + \text{evanescent} \tag{46}
\]

where \( C \) denotes the Euler-Mascheroni constant.

Similar results are obtained with the Pauli-Villars and cut-off regularisations. In the PV case the latter is simply implemented by the following replacement of the massless Dirac propagator

\[
 \text{reg} \, \Sigma_2(k) = -ig^2 \int \frac{d^4\ell}{(2\pi)^4} \gamma^\mu D_{\mu\nu}(k - \ell) \sum_{s=0}^{S} C_s \, S(\ell, M_s) \gamma^\nu P_L
\tag{47}
\]

where \( M_0 = 0, \ C_0 = 1 \) while \( \{ M_s \equiv \lambda_s M | \lambda_s \gg 1 (s = 1, 2, \ldots, S) \} \) is a collection of very large auxiliary masses. The constants \( C_s \) are required to satisfy:

\[
 \sum_{s=1}^{S} C_s = -1 \quad \sum_{s=1}^{S} C_s \lambda_s = 0
\]

and the following identification with the divergent parameter is made

\[
 \frac{1}{\epsilon} = \sum_{s=1}^{S} C_s \ln \lambda_s.
\]

The result for \( f(k^2) \) is

\[
 f(k^2) = \left( \frac{g}{4\pi} \right)^2 \left[ \sum_{s=1}^{S} C_s \ln \lambda_s + \frac{1}{4} + \frac{1}{2} \ln \left( -\frac{M^2}{k^2} \right) \right] + \text{evanescent}. \tag{48}
\]

The same calculation can be repeated with an UV cutoff \( K \), see \[14\]. To sum up, we have verified that the 1-loop correction to the (left) Weyl spinor self-energy has the general form, which is universal, i.e. regularisation independent: namely;

\[
 \text{reg} \, \Sigma_2(k) \equiv f(k^2) \, k P_L \\
 f(k^2) := \left( \frac{g}{4\pi} \right)^2 \left[ \frac{1}{\epsilon} + 1 + 3C + \ln \left( -\frac{4\mu^2}{k^2} \right) \right] \quad (\text{DR}) \\
 := \left( \frac{g}{4\pi} \right)^2 \left[ \sum_{s=1}^{S} C_s \ln \lambda_s + \frac{1}{4} + \frac{1}{2} \ln \left( -\frac{M^2}{k^2} \right) \right] \quad (\text{PV}) \\
 := \left( \frac{g}{4\pi} \right)^2 \ln \left[ -\left( \frac{4K}{k^2} \right)^2 \right] \quad (\text{CUT-OFF})
\]

13
Remarks

1. In the present model of a left Weyl spinor minimally coupled to a gauge vector potential, **no mass term can be generated by the radiative corrections** in any regularisation scheme. The left-handed part of the classical kinetic term does renormalize, while its right-handed part does not undergo any radiative correction and keeps on being free. The latter has to be necessarily introduced in order to define a Feynman propagator for the massless spinor field, much like the gauge fixing term is introduced in order to invert the kinetic term of the gauge potential. The (one loop) renormalized Lagrangian for a Weyl fermion minimally coupled to a gauge vector potential has the universal - i.e. regularisation independent - form

\[
\mathcal{L}_{\text{ren}} = \chi_L^\dagger \alpha^\nu i \partial_\nu \chi_L + g A^\nu \chi_L^\dagger \alpha^\nu \chi_L - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \\
\varphi_R^\dagger \alpha^\nu i \partial_\nu \varphi_R - \frac{1}{4}(\partial \cdot A)^2 - (Z_3 - 1) \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \\
+ (Z_2 - 1) \chi_L^\dagger \alpha^\nu i \partial_\nu \chi_L + (Z_1 - 1) g A^\nu \chi_L^\dagger \alpha^\nu \chi_L
\]

\[(Z_2 - 1)_{\text{1-loop}} = -\left(\frac{g}{4\pi}\right)^2 \left[ \frac{1}{\epsilon} + F_2(\epsilon, k^2/\mu^2) \right]
\]

\[= -\left(\frac{g}{4\pi}\right)^2 \sum_{s=1}^S C_s \ln \lambda_s + \tilde{F}_2(\lambda_s, k^2/M_s^2) \]

\[= -\left(\frac{g}{4\pi}\right)^2 \left\{ \ln \left[-\frac{(4K)^2}{k^2} \right] + \tilde{F}_2(K_s^2/k^2) \right\}
\]

where the customary notations have been employed. Notice that the arbitrary finite parts \(F_2, \tilde{F}_2, \hat{F}_2\) of the counterterms are analytic for \(\epsilon \to 0\) and \(\lambda_s, K \to \infty\), respectively, and have to be univocally fixed by the renormalization prescription, as usual.

2. The interaction definitely preserves left chirality and scale invariance of the counterterms in the transition from the classical to the (perturbative) quantum theory: no mass coupling between the left-handed (interacting) Weyl spinor \(\chi_L\) and right-handed (free) Weyl spinor \(\varphi_R\) can be generated by radiative loop corrections.

3. While the cut-off and dimensional regularised theory does admit a local formulation in \(D = 4\) or \(D = 2\omega\) space-time dimensions, there is no such local formulation for the Pauli-Villars regularisation. The reason is that the PV spinor propagator

\[
\sum_{s=0}^S C_s S(\ell, M_s)
\]

where \(M_0 = 0, C_0 = 1\) while \(\{ M_s = \lambda_s M \mid \lambda_s \gg 1 (s = 1, 2, \ldots, S) \}\), cannot be the inverse of any local differential operator of the Calderon-Zygmund type. Hence, there is no local action involving a bilinear spinor term that can produce, after a suitable inversion, the Pauli-Villars regularised spinor propagator. So, although the Seeley-Schwinger-DeWitt method is not the main concern of this paper, there are no doubts that the Pauli-Villars regularisation cannot be applied to the construction of a regularised full kinetic operator for the Seeley-Schwinger-DeWitt method, nor, of course, to its inverse.
5 Consistent gauge anomalies for Weyl fermions

As explained in the introduction, anomalies are one of the main topics where Dirac and Weyl fermions split significantly. In the present and forthcoming sections we aim to recalculate all the anomalies (chiral and trace) of Weyl, Dirac and Majorana fermions coupled to gauge potentials, with the basic method of Feynman diagrams. Most results are supposedly well-known. Our purpose is to collect them all in order to highlight their reciprocal relations.

Let us consider the classical action integral for a right-handed Weyl fermion coupled to an external gauge field

\[ V^\mu = V^a_{\mu} T^a, \]

\( T^a \) being Hermitean generators, \([T^a, T^b] = i f^{abc} T^c\) (in the Abelian case \( T = 1, f = 0 \)) in a fundamental representation of e.g. SU(N): namely,

\[ S^{R}_{V} = \int d^4x i \overline{\psi}_R (\not{\partial} - i V) \psi_R. \] (49)

This action is invariant under the gauge transformation \( \delta V^\mu = D^\mu \lambda \equiv \partial^\mu \lambda - i [V^\mu, \lambda] \), which implies the conservation of the non-Abelian current

\[ J^a_R^\mu = \overline{\psi}_R \gamma^\mu T^a \psi_R, \]

\[ \nabla \cdot J^a_R^\mu = (\partial^\mu \delta^{ac} + f^{abc} V^b_{\mu}) J^c_R^\mu = 0. \] (50)

The quantum effective action for this theory is given by the generating functional of the connected Green functions of such currents in the presence of the source \( V^a_{\mu} \)

\[ W[V] = W[0] + \sum_{n=1}^{\infty} \frac{i^{n-1}}{n!} \int \prod_{i=1}^{n} d^4x_i V^{a_i \mu_i} (x_i) \langle 0 | T J^{a_1}_{R \mu_1} (x_1) \ldots J^{a_n}_{R \mu_n} (x_n) | 0 \rangle_c \] (51)

and the full 1-loop 1-point function of \( J^{a}_{R \mu} \) is

\[ \langle \langle J^{a}_{R \mu} (x) \rangle \rangle = \frac{\delta W[V]}{\delta V^{a \mu}(x)} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \prod_{i=1}^{n} d^4x_i V^{a_i \mu_i} (x_i) \langle 0 | T J^{a_1}_{R \mu_1} (x_1) \ldots J^{a_n}_{R \mu_n} (x_n) | 0 \rangle_c \] (52)

Our purpose here is to calculate the odd parity anomaly of the divergence \( \nabla \cdot \langle \langle J^{a}_{R \mu} \rangle \rangle \). As is well-known the first nontrivial contribution to the anomaly comes from the divergence of the three-point function in the RHS of (52). For simplicity we will denote it \( \langle \partial^\mu J_R^a J_R^b J_R^c \rangle \). Below we will evaluate it in some detail as a sample for the remaining calculations.

5.1 The calculation

Let us start with dimensional regularization. The fermion propagator is \( \frac{i}{p} \) and the vertex \( i \gamma_\mu P_R T^a \). The Fourier transform of the three currents amplitude \( \langle J_R J_R J_R \rangle \) is given by

\[ \overline{F}^{(R)}_{\mu \lambda \rho} (k_1, k_2) = \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left\{ \frac{1}{p^2 - \frac{2}{\gamma_\lambda T^b \frac{1}{p - k_1} - \frac{1}{\gamma_\rho T^c \frac{1}{p - q} - \frac{1}{2} \gamma_\mu T^a} \right\} \]

\[ \equiv \text{Tr}(T^a T^b T^c) \overline{F}^{(R)}_{\mu \lambda \rho} (k_1, k_2) \] (53)

where \( q = k_1 + k_2 \). The relevant Feynman diagram is shown in figure I.
Figure 1: The Feynman diagram corresponding to $\tilde{F}^{(R)abc}_{\mu\lambda\rho}(k_1, k_2)$.

From now on we focus on the Abelian part $\tilde{F}^{(R)}_{\mu\lambda\rho}(k_1, k_2)$. We dimensionally regularize it by introducing $\delta$ additional dimensions and corresponding momenta $\ell_\mu$, $\mu = 4, \ldots, 3 + \delta$, with the properties

$$\ell \cdot p + p \cdot \ell = 0, \quad [\ell, \gamma_5] = 0, \quad p^2 = p^2, \quad \ell^2 = -\ell^2$$

$$\text{tr}(\gamma_{\mu}\gamma_{\nu}\gamma_{\lambda}\gamma_{\rho}\gamma_5) = -2^{2+\frac{\delta}{2}} i \epsilon_{\mu\nu\lambda\rho},$$

so the relevant expression to be calculated is

$$q^\mu \tilde{F}^{(R)}_{\mu\lambda\rho}(k_1, k_2) = \int \frac{d^4pd^\delta \ell}{(2\pi)^{4+\delta}} \text{tr} \left\{ \frac{1}{p + \ell} \frac{1}{2} - \frac{\gamma_5}{\lambda} \frac{1}{p + \ell - k_1} \frac{1}{2} - \frac{\gamma_\rho}{\gamma_5} \frac{1}{p + \ell - q} \frac{1}{2} \right\}$$

$$\equiv \tilde{F}^{(R)}_{\lambda\rho}(k_1, k_2, \delta).$$

(54)

Now we focus on the odd part and work out the gamma traces:

$$\tilde{F}^{(R, odd)}_{\lambda\rho}(k_1, k_2, \delta) = -2^{1+\frac{\delta}{2}} i \epsilon_{\mu\nu\lambda\rho} \int \frac{d^4pd^\delta \ell}{(2\pi)^{4+\delta}} \left\{ \frac{\gamma^\mu}{p^2 - \ell^2} \frac{1}{2} \frac{\gamma_\lambda}{(p - k_1)^2 - \ell^2} \frac{1}{2} \frac{\gamma_\rho}{(p - q)^2 - \ell^2} \frac{1}{2} \right\}.$$ 

(55)

Let us write the numerator on the RHS as follows:

$$p^2 q^\mu + (q^2 - 2p \cdot q) p^\mu = -(p^2 - \ell^2) (p-q)^\mu + ((p-q)^2 - \ell^2) p^\mu + \ell^2 q^\mu.$$ 

(56)
Then (55) can be rewritten as

$$\tilde{F}^{(R, odd)}_{\lambda\rho}(k_1, k_2, \delta) = -2^{1+\frac{1}{2}} i \epsilon_{\mu\nu\lambda\rho} \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4\ell}{(2\pi)^4} \left\{ \ell^2 \left( \frac{q^\mu (p^\nu - k_1^\nu)}{(p^2 - \ell^2)((p - k_1)^2 - \ell^2)((p - q)^2 - \ell^2)} + \frac{(q - p)^\mu (p - k_1)^\nu}{((p - k_1)^2 - \ell^2)((p - q)^2 - \ell^2)} + \frac{p^\mu (p - k_1)^\nu}{((p - k_1)^2 - \ell^2)((p - q)^2 - \ell^2)} \right) \right\} \right.$$  

$$+ \left\{ \ell^2 \left( \frac{q^\mu (p^\nu - k_1^\nu)}{(p^2 - \ell^2)((p - k_1)^2 - \ell^2)((p - q)^2 - \ell^2)} + \frac{(q - p)^\mu (p - k_1)^\nu}{((p - k_1)^2 - \ell^2)((p - q)^2 - \ell^2)} + \frac{p^\mu (p - k_1)^\nu}{((p - k_1)^2 - \ell^2)((p - q)^2 - \ell^2)} \right) \right\} \right\} \right.$$  

$$= -2^{1+\frac{1}{2}} i \epsilon_{\mu\nu\lambda\rho} \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4\ell}{(2\pi)^4} \left\{ \ell^2 \left( \frac{q^\mu (p^\nu - k_1^\nu)}{(p^2 - \ell^2)((p - k_1)^2 - \ell^2)((p - q)^2 - \ell^2)} + \frac{(q - p)^\mu (p - k_1)^\nu}{((p - k_1)^2 - \ell^2)((p - q)^2 - \ell^2)} + \frac{p^\mu (p - k_1)^\nu}{((p - k_1)^2 - \ell^2)((p - q)^2 - \ell^2)} \right) \right\} \right.$$  

$$- \left\{ \frac{1}{p^2 - \ell^2} \right\} \right\} \right\} \right.$$  

The last two terms do not contribute because of the antisymmetric $\epsilon$ tensor, as one can easily

see by introducing a Feynman parameter. The first term can be easily evaluated by introducing
two Feynman parameters $x$ and $y$, and making the shift $p \rightarrow p + (x + y)k_1 + yk_2$.

$$\tilde{F}^{(R, odd)}_{\lambda\rho}(k_1, k_2, \delta) = -2^{1+\frac{1}{2}} i \epsilon_{\mu\nu\lambda\rho} \int_0^1 dx \int_0^{1-x} dy \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4\ell}{(2\pi)^4} \left\{ \ell^2 \left( \frac{q^\mu (p^\nu + (x + y - 1)k_1 + yk_2)^\nu}{(p^2 - \ell^2 + \Delta(x, y))^3} \right) \right.$$  

$$\right\} \right\} \right.$$  

where $\Delta = (x + y)(1 - x - y)k_1^2 + y(1 - y)k_2^2 + 2y(1 - x - y)k_1 \cdot k_2$. Now we make a Wick rotation on the integration momentum, $p^0 \rightarrow ip^0$, and the same on $k_1, k_2$ (although we stick to the same symbols).

Then, using

$$\int \frac{d^4p}{(2\pi)^4} \int \frac{d^4\ell}{(2\pi)^4} \left\{ \ell^2 \left( \frac{q^\mu (p^\nu + (x + y - 1)k_1 + yk_2)^\nu}{(p^2 - \ell^2 + \Delta(x, y))^3} \right) \right.$$  

and taking the limit $\delta \rightarrow 0$, we find

$$\tilde{F}^{(R, odd)}_{\lambda\rho}(k_1, k_2) = \frac{2}{(4\pi)^2} \epsilon_{\mu\nu\lambda\rho} k_1^\mu k_2^\nu \int_0^1 dx \int_0^{1-x} dy (1 - x) \left\{ \frac{1}{24\pi^2} \epsilon_{\mu\nu\lambda\rho} k_1^\mu k_2^\nu \right\} \right.$$  

We must add the cross term (for $\lambda \leftrightarrow \rho$ and $k_1 \leftrightarrow k_2$), so that the total result is

$$\tilde{F}^{(R, odd)}_{\lambda\rho}(k_1, k_2) + \tilde{F}^{(R, odd)}_{\rho\lambda}(k_2, k_1) = \frac{1}{12\pi^2} \epsilon_{\mu\nu\lambda\rho} k_1^\mu k_2^\nu. \right.$$  

In order to return to configuration space we have to insert this result into (52). We consider here, for simplicity, the Abelian case. We have

$$\partial^\mu \langle J_{R\mu}(x) \rangle = \int \frac{d^4q}{(2\pi)^4} e^{-iqx} \langle -i q^\mu \rangle \langle \tilde{J}_{R\mu}(q) \rangle = \frac{i}{2} \int \frac{d^4q}{(2\pi)^4} \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int d^4y d^4z \left\{ \tilde{F}^{(R, odd)}_{\lambda\rho}(k_1, k_2) + \tilde{F}^{(R, odd)}_{\rho\lambda}(k_2, k_1) \right\} V^\lambda(y) V^\rho(z). \right.$$  

After a Wick rotation we can replace (62) inside the integrals

$$\partial^\mu \langle J_{R\mu}(x) \rangle = -\frac{1}{24\pi^2} \int \frac{d^4q d^4k_1 d^4k_2}{(2\pi)^{12}} \int d^4y d^4z e^{i(qx - k_1 y - k_2 z) \delta(q - k_1 - k_2) \epsilon_{\mu\nu\lambda\rho} k_1^\mu k_2^\nu V^\lambda(y) V^\rho(z) \right.$$  

$$\right.$$  

$$= \frac{1}{24\pi^2} \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int d^4y d^4z e^{i(k_1 x - y) - i(k_2 x - z) \epsilon_{\mu\nu\lambda\rho} \partial^\mu V^\lambda(y) \partial^\nu V^\rho(z) \right.$$  

$$\right.$$  

$$= \frac{1}{24\pi^2} \epsilon_{\mu\nu\lambda\rho} \partial^\mu V^\nu(x) \partial^\lambda V^\rho(x). \right.$$  

(64)
The same result can be obtained with the Pauli-Villars regularization, see Appendix B.

5.2 Comments

Eq. (64) is the consistent gauge anomaly of a right-handed Weyl fermion coupled to an Abelian vector field $V_\mu(x)$. It is well known that the consistent anomaly [64] destroys the consistency of the Abelian gauge theory. As a matter of fact the Lorentz invariant quantum theory of a gauge vector field unavoidably involves a Fock space of states with indefinite norm. Now, in order to select a physical Hilbert subspace of the Fock space, a subsidiary condition is necessary. In the Abelian case, when the fermion current satisfies the continuity equation the equations of motion lead to $\Box(\partial \cdot V) = 0$, so that a subspace of states of non-negative norm can be selected through the auxiliary condition

$$\partial \cdot V^{(-)}(x) |_{\text{phys}} = 0$$

$V^{(-)}(x)$ being the annihilation operator, the positive frequency part of a d’Alembert quantum field. On the contrary, in the present chiral model we find

$$\Box(\partial \cdot V) = -\frac{1}{3} \left( \frac{1}{4\pi} \right)^2 F^\mu_\nu F^\nu_\mu \neq 0$$

in such a manner that nobody knows how to select a physical subspace of states with non-negative norm, if any, where a unitary restriction of the collision operator $S$ could be defined.

Another way of seeing the problem created by the consistent anomaly is to remark that, for instance, $J_{R\mu}$ couples minimally to $V^\mu$ at the fermion-fermion-gluon vertex. Unitarity and renormalizability rely on the Ward identity that guarantees current conservation at any such vertex. But this is impossible in the presence of a consistent anomaly.

The consistent anomaly in the non-Abelian case would require the calculation of at least the four current correlators, but it can be obtained in a simpler way from the Abelian case using the Wess-Zumino consistency conditions. In the non-Abelian case the three-point correlators are multiplied by

$$\text{Tr}(T^aT^bT^c) = \frac{1}{2} \text{Tr}(T^a[T^b, T^c]) + \frac{1}{2} \text{Tr}(T^a\{T^b, T^c\}) = f^{abc} + d^{abc}$$

where the normalization used is $\text{Tr}(T^aT^b) = 2\delta^{ab}$. Since the three-point function is the sum of two equal pieces with $\lambda \leftrightarrow \rho, k_1 \leftrightarrow k_2$, the first term in the RHS of (65) drops out and only the second remains. For the right-handed current $J_{R\mu}$ we have

$$\nabla \cdot \langle \langle J_{R\mu}^a \rangle \rangle = \frac{1}{24\pi^2} \varepsilon_{\mu\nu\lambda\rho} \text{Tr} \left[ T^a \partial^\mu \left( V^\nu \partial^\lambda V^\rho + \frac{i}{2} V^\nu V^\lambda V^\rho \right) \right].$$

The previous results are well-known. However they do not tell the whole story about gauge anomalies in a theory of Weyl fermions. To delve into this we have to enlarge the parameter space by coupling the fermions to an additional potential, namely to an axial vector field.

\[ \text{As is well known consistency means that the integrated } \int d^4 x \lambda^a(x) \nabla \cdot \langle \langle J_{R\mu}^a \rangle \rangle \text{ is annihilated by the BRST transformations} \]

$$\delta V_\mu = \partial_\mu \lambda + [V_\mu, \lambda], \quad \delta \lambda = -\frac{1}{2} [\lambda, \lambda], \quad \delta^2 = 0, \quad \lambda = \lambda^a(x) T^a$$

where $\lambda^a$ is an anticommuting parameter field. In the Abelian case this corresponds to ordinary gauge invariance.
6 The V – A anomalies

The action of a Dirac fermion coupled to a vector \( V_\mu \) and an axial potential \( A_\mu \) (for simplicity we consider only the Abelian case) is

\[
S[V, A] = \int d^4x \, i \bar{\psi} (\partial - iV - iA\gamma_5) \psi.
\]  

(68)

The generating functional of the connected Green functions is

\[
W[V, A] = W[0, 0] + \sum_{n,m=0}^{\infty} \frac{i^{n+m}}{n!m!} \int \prod_{i=1}^{n} d^4x_i \, V^{\mu_i}(x_i) \prod_{j=1}^{m} d^4y_j \, A^{\nu_j}(y_j)
\]

\[
\times \langle 0 | \mathcal{T} J_{\mu_1}(x_1) \cdots J_{\mu_n}(x_n) J_{5\nu_1}(y_1) \cdots J_{5\nu_m}(x_m) | 0 \rangle_c.
\]

(69)

We can extract the full one-loop one-point function for two currents: the vector current \( J_\mu = \bar{\psi} \gamma_\mu \psi \)

\[
\langle J_\mu(x) \rangle = \frac{\delta W[V, A]}{\delta V^{\mu}(x)} = \sum_{n,m=0}^{\infty} \frac{i^{n+m}}{n!m!} \int \prod_{i=1}^{n} d^4x_i \, V^{\mu_i}(x_i) \prod_{j=1}^{m} d^4y_j \, A^{\nu_j}(y_j)
\]

\[
\times \langle 0 | \mathcal{T} J_{\mu_1}(x_1) \cdots J_{\mu_n}(x_n) J_{5\nu_1}(y_1) \cdots J_{5\nu_m}(x_m) | 0 \rangle_c.
\]

(70)

and the axial current \( J_\mu = \bar{\psi} \gamma_\mu \gamma_5 \psi \)

\[
\langle J_{5\mu}(x) \rangle = \frac{\delta W[V, A]}{\delta A^{\mu}(x)} = \sum_{n,m=0}^{\infty} \frac{i^{n+m}}{n!m!} \int \prod_{i=1}^{n} d^4x_i \, V^{\mu_i}(x_i) \prod_{j=1}^{m} d^4y_j \, A^{\nu_j}(y_j)
\]

\[
\times \langle 0 | \mathcal{T} J_{5\mu_1}(x_1) \cdots J_{5\mu_n}(x_n) J_{5\nu_1}(y_1) \cdots J_{5\nu_m}(x_m) | 0 \rangle_c.
\]

(71)

These currents are conserved except for possible anomaly contributions. The aim of this section is to study the continuity equations for these currents, that is to compute the 4-divergences of the correlators on the RHS of (70) and (71). For the same reason explained above we focus on the three current correlators: they are all we need in the Abelian case (and the starting point to compute the full anomaly expression by means of the Wess-Zumino consistency conditions in the non-Abelian case). So for \( \partial \cdot J(x) \) the first relevant contributions are

\[
\partial^\mu \langle J_\mu(x) \rangle = -\left( \frac{1}{2} \right) \int d^4x_1 d^4x_2 \, V^{\mu_1}(x_1) V^{\mu_2}(x_2) \partial^\mu \langle 0 | \mathcal{T} J_{\mu_1}(x_1) J_{\mu_2}(x_2) | 0 \rangle
\]

\[
+ \int d^4x_1 d^4y_1 \, V^{\mu_1}(x_1) A^{\nu_1}(y_1) \partial^\mu \langle 0 | \mathcal{T} J_{\mu_1}(x_1) J_{5\nu_1}(y_1) | 0 \rangle
\]

\[
+ \frac{1}{2} \int d^4y_1 d^4y_2 A^{\nu_1}(y_1) A^{\nu_2}(y_2) \partial^\mu \langle 0 | \mathcal{T} J_{\mu_1}(x_1) J_{5\nu_1}(y_1) J_{5\nu_2}(y_2) | 0 \rangle.
\]

(72)

and for \( \partial^\mu J_{5\mu}(x) \)

\[
\partial^\mu \langle J_{5\mu}(x) \rangle = -\left( \frac{1}{2} \right) \int d^4x_1 d^4x_2 \, V^{\mu_1}(x_1) V^{\mu_2}(x_2) \partial^\mu \langle 0 | \mathcal{T} J_{5\mu_1}(x_1) J_{\mu_2}(x_2) | 0 \rangle
\]

\[
+ \int d^4x_1 d^4y_1 \, V^{\mu_1}(x_1) A^{\nu_1}(y_1) \partial^\mu \langle 0 | \mathcal{T} J_{5\mu_1}(x_1) J_{\mu_1}(y_1) | 0 \rangle
\]

\[
+ \frac{1}{2} \int d^4y_1 d^4y_2 A^{\nu_1}(y_1) A^{\nu_2}(y_2) \partial^\mu \langle 0 | \mathcal{T} J_{5\mu_1}(x_1) J_{5\nu_1}(y_1) J_{5\nu_2}(y_2) | 0 \rangle.
\]

(73)
Since we are interested in odd parity anomalies, the only possible contribution to (72) is from the term in the second line, which we denote concisely $\langle \partial \cdot J J J \rangle$. As for (73) there are two possible contributions from the first and third lines, i.e. $\langle \partial \cdot J J J \rangle$ and $\langle \partial \cdot J J J \rangle$. Below we report the results for the corresponding amplitudes, obtained with dimensional regularization.

The amplitude for $\langle \partial \cdot J J J \rangle$ is

$$q^\mu F^{(5)}_{\mu \lambda \rho}(k_1, k_2) = \int \frac{d^4p d^4\ell}{(2\pi)^{4+\delta}} \text{tr} \left\{ \frac{1}{p + \ell \gamma_\lambda} \frac{1}{p + \ell - k_1 \gamma_\rho} \frac{1}{p + \ell - q \gamma_5} \right\}. \quad (74)$$

The relevant Feynman diagram is shown in figure 2.

![Figure 2: The Feynman diagram corresponding to $F^{(5)}_{\mu \lambda \rho}(k_1, k_2)$.](image)

Adding the cross contribution one gets

$$q^\mu \left( F^{(5)}_{\mu \lambda \rho}(k_1, k_2) + T^{(5)}_{\mu \lambda \rho}(k_2, k_1) \right) = \frac{1}{2\pi^2} \epsilon_{\mu \nu \lambda \rho} k_1^\mu k_2^\nu. \quad (75)$$

The amplitude for $\langle \partial \cdot J J J \rangle$ is given by

$$q^\mu F^{(555)}_{\mu \nu \rho}(k_1, k_2) = \int \frac{d^4p d^4\ell}{(2\pi)^{4+\delta}} \text{tr} \left\{ \frac{1}{p + \ell \gamma_\lambda} \frac{1}{p + \ell - k_1 \gamma_\rho} \frac{1}{p + \ell - q \gamma_5} \right\} \quad (76)$$

The first line in the last expression, after introducing the Feynman parameters $x$ and $y$ and shifting $p$ as usual, yields a factor $\int_0^1 dx \int_0^{1-x} dy (1 - 3x) = 0$, so it vanishes. The last line is $2 \times \tilde{F}^{(R, odd)}_{\lambda \rho}(k_1, k_2, \delta)$, cf. (54, 55). Therefore, using (62), we get

$$q^\mu \left( F^{(555)}_{\mu \nu \rho}(k_1, k_2) + \tilde{F}^{(555)}_{\nu \rho \lambda}(k_2, k_1) \right) = \frac{1}{6\pi^2} \epsilon_{\mu \nu \lambda \rho} k_1^\mu k_2^\nu. \quad (77)$$
Finally the amplitude for $\langle \partial \cdot J J J \rangle$ is

$$q^\mu F^{(5)}_{\mu\lambda\rho}(k_1, k_2) = \int \frac{d^4 p d^4 \ell}{(2\pi)^{4+\delta}} \text{tr} \left\{ \frac{1}{\not p + \not q} \gamma^\lambda \frac{1}{\not p + \not \ell - \not k_1} \gamma^\rho \gamma^5 \frac{1}{\not p + \not \ell - \not q} \right\} = 0. \tag{78}$$

All the above results have been obtained also with PV regularization.

Plugging in these results in (70) and (71) we find

$$\partial^\mu \langle \langle J_\mu(x) \rangle \rangle = 0 \tag{79}$$

and

$$\partial^\mu \langle \langle J_{5\mu}(x) \rangle \rangle = \frac{1}{24\pi^2} \epsilon_{\mu\nu\lambda\rho} \partial^\mu V^\nu(x) \partial^\lambda V^\rho(x) + \frac{1}{3} \partial^\mu A^\nu(x) \partial^\lambda A^\rho(x) \tag{80}$$

which is Bardeen’s result, [15], in the Abelian case. From (80) we can derive the covariant chiral anomaly by setting $A_\mu = 0$, then

$$\partial^\mu \langle \langle J_{5\mu}(x) \rangle \rangle = \frac{1}{24\pi^2} \epsilon_{\mu\nu\lambda\rho} \partial^\mu V^\nu(x) \partial^\lambda V^\rho(x). \tag{81}$$

Of course this is nothing but (74). For the $J_{5\mu}(x)$ current is obtained by differentiating the action with respect to $A_\mu(x)$ and its divergence leads to the covariant anomaly.

### 6.1 Some conclusions

Let us recall that in the collapsing limit $V \rightarrow V/2, A \rightarrow V/2$ in the action (68) we recover the theory of a right-handed Weyl fermion (with the addition of a free left-handed part, as explained at length above). Now $J_\mu(x) = J_{R\mu}(x) + J_{L\mu}(x)$ and $J_{5\mu}(x) = J_{R\mu}(x) - J_{L\mu}(x)$. In the collapsing limit we find

$$\partial^\mu \langle \langle J_{R\mu}^{(cs)}(x) \rangle \rangle = \frac{1}{24\pi^2} \epsilon_{\mu\nu\lambda\rho} \partial^\mu V^\nu(x) \partial^\lambda V^\rho(x). \tag{82}$$

Similarly

$$\partial^\mu \langle \langle J_{L\mu}^{(cs)}(x) \rangle \rangle = -\frac{1}{24\pi^2} \epsilon_{\mu\nu\lambda\rho} \partial^\mu V^\nu(x) \partial^\lambda V^\rho(x). \tag{83}$$

These are the consistent right and left gauge anomalies - the label $^{(cs)}$ stands for consistent, to be distinguished from the covariant anomaly. As a matter of fact, application of the same chiral current splitting to the covariant anomaly of eq. (109) yields instead

$$\partial^\mu \langle \langle J_{R\mu}^{(cv)}(x) \rangle \rangle = \frac{1}{8\pi^2} \epsilon_{\mu\nu\lambda\rho} \partial^\mu V^\nu(x) \partial^\lambda V^\rho(x) \tag{84}$$

and

$$\partial^\mu \langle \langle J_{L\mu}^{(cv)}(x) \rangle \rangle = -\frac{1}{8\pi^2} \epsilon_{\mu\nu\lambda\rho} \partial^\mu V^\nu(x) \partial^\lambda V^\rho(x). \tag{85}$$
The label \((cv)\) stands for covariant, and it is in order to tell apart these anomalies from the previous consistent ones. The two cases should not be confused: the consistent anomalies appear in the divergence of a current minimally coupled in the action to the vector potential \(V_\mu\). They represent the response of the effective action under a gauge transform of \(V_\mu\), which is supposed to propagate in the internal lines of the corresponding gauge theory. The covariant anomalies represent the response of the effective action under a gauge transform of the external axial current \(A_\mu\).

It goes without saying that, both for right and left currents in the collapsing limit, in the non-Abelian case the consistent anomaly takes the form (66), while the covariant one reads
\[
\nabla \cdot \langle\langle J^\mu_R(x) \rangle\rangle = \frac{1}{32\pi^2} \epsilon_{\mu\nu\lambda\rho} \text{Tr} \left( T^a F^{\mu\nu}(x) F^{\lambda\rho}(x) \right)
\]
(86)

where \(F^{\mu\nu}(x) = F^a_{\mu\nu}(x) T^a\) denotes the usual non-Abelian field strength. At first sight the above distinction between covariant and consistent anomalies for Weyl fermion may appear to be academic. After all, if a theory has a consistent anomaly it is ill-defined and the existence of a covariant anomaly may sound irrelevant. However this distinction becomes interesting in some non-Abelian cases since the non-Abelian consistent anomaly is proportional to the tensor \(d_{abc}\).

Now for most simple gauge groups (except \(SU(N)\) for \(N \geq 3\)) this tensor vanishes identically. In such cases the consistent anomaly is absent and so the covariant anomaly becomes significant.

7 The case of Majorana fermions

As we have seen above, Majorana fermions are defined by the condition
\[
\Psi = \hat{\Psi}, \quad \text{where} \quad \hat{\Psi} = \gamma_0 C \Psi^*.
\]
(87)

Let \(\psi_R = P_R \psi\) be a generic Weyl fermion. We have
\[
P_R \psi_R = \psi_R \quad P_L \psi_R = \psi_R
\]
i.e. \(\hat{\psi}_R\) is left-handed. We have already remarked that \(\psi_M = \psi_R + \psi_R\) is a Majorana fermion and any Majorana fermion can be represented in this way. Using this correspondence one can transfer the results for Weyl fermions to Majorana fermions. The vector current is defined by \(J^\mu_M = \psi_M \gamma^\mu \psi_M\) and the axial current by \(J^\mu_5_M = \psi_M \gamma^\mu \gamma_5 \psi_M\). We can write
\[
J^\mu_M(x) = \overline{\psi_R}(x) \gamma^\mu \psi_R(x) + \overline{\psi_R}(x) \gamma^\mu \hat{\psi}_R(x) \equiv J^\mu_R(x) + J^\mu_L(x)
\]
(88)

and
\[
J^\mu_5_M(x) = \overline{\psi_R}(x) \gamma^\mu \psi_R(x) - \overline{\psi_R}(x) \gamma^\mu \hat{\psi}_R(x) \equiv J^\mu_R(x) - J^\mu_L(x).
\]
(89)

It should be remarked that \(J^\mu_R(x)\) and \(J^\mu_L(x)\) are not independent currents, since they are built out of the very same field degrees of freedom.

Using (82) and (83) one concludes that, as far as the consistent anomaly is concerned,
\[
\partial_\mu \langle\langle J^\mu_M(x) \rangle\rangle = 0
\]
(90)

while for the axial current we have
\[
\partial_\mu \langle\langle J^\mu_5_M(x) \rangle\rangle = \frac{1}{8\pi^2} \epsilon_{\mu\nu\lambda\rho} \partial^\mu V^\nu(x) \partial^\lambda V^\rho(x)
\]
(91)

where the naive sum has been divided by 2, because the two contributions come from the same degrees of freedom (which are half those of a Dirac fermion).
8 Relation between chiral and trace gauge anomalies

There exists a strict relation between chiral gauge anomalies and trace anomalies in a theory of
fermions coupled to a vector (and axial) gauge potential. This section is devoted to analysing
this relation. When the background fields are not only $V^\mu$ and $A^\mu$, but also a non-trivial tensor-
axial metric $G_{\mu\nu} = g_{\mu\nu} + \gamma^5 f_{\mu\nu}$, see [13], the generating function must include two e.m. tensors,
which in the flat-space limit take the Belinfante-Rosenfeld symmetric form

$$T^{\mu\nu} = -\frac{i}{4} \left( \overline{\psi} \gamma^\mu \partial^\nu \psi + \mu \leftrightarrow \nu \right),$$

and

$$T_5^{\mu\nu} = \frac{i}{4} \left( \overline{\psi} \gamma_5 \gamma^\mu \partial^\nu \psi + \mu \leftrightarrow \nu \right).$$

The quantities we are interested in here are, in particular, the 1-loop VEVs $\langle \langle T_{\mu\nu}(x) \rangle \rangle$ and $\langle \langle T_5^{\mu\nu}(x) \rangle \rangle$ when $h_{\mu\nu} = f_{\mu\nu} = 0$ : namely,

$$\langle \langle T_{\mu\nu}(x) \rangle \rangle = \sum_{r,s=0}^{\infty} \frac{i^{r+s}}{2^r r! s!} \int \prod_{l=1}^{r} d^4 x_l V_{\sigma_l}(x_l) \prod_{k=1}^{s} d^4 y_k A_{\tau_k}(y_k) \langle 0 | T_{\mu\nu}(x) J_{\sigma_1}(x_1) \cdots J_{\sigma_r}(x_r) J_{5\tau_1}(y_1) \cdots J_{5\tau_s}(y_s) | 0 \rangle \tag{94}$$

and

$$\langle \langle T_5^{\mu\nu}(x) \rangle \rangle = \sum_{r,s=0}^{\infty} \frac{i^{r+s}}{2^r r! s!} \int \prod_{l=1}^{r} d^4 x_l V_{\sigma_l}(x_l) \prod_{k=1}^{s} d^4 y_k A_{\tau_k}(y_k) \langle 0 | T_5^{\mu\nu}(x) J_{\sigma_1}(x_1) \cdots J_{\sigma_r}(x_r) J_{5\tau_1}(y_1) \cdots J_{5\tau_s}(y_s) | 0 \rangle \tag{95}$$

of which we will compute the trace, i.e. contraction, over the indices $\mu$ and $\nu$. Since we are
interested in odd parity anomalies, the first nontrivial contributions come from the three-point
correlators (i.e. $r + s = 2$). Denoting by $t, t_5$ the traces of $T_{\mu\nu}, T_5^{\mu\nu}$, the relevant correlators
are $\langle t J J_5 \rangle$ for (94), and $\langle t_5 J J_5 \rangle$ for (95). We claim that they are simply related to
$\langle \partial \cdot J J_5 \rangle$, $\langle \partial \cdot J_5 J J \rangle$ and $\langle \partial \cdot J_5 J_5 J_5 \rangle$, respectively.

We will also need

$$T_R^{\mu\nu}(x) = \frac{1}{2} \left[ T^{\mu\nu}(x) + T_5^{\mu\nu}(x) \right]$$

$$T_L^{\mu\nu}(x) = \frac{1}{2} \left[ T^{\mu\nu}(x) - T_5^{\mu\nu}(x) \right] \tag{96}$$

together with the one-loop 1-point function

$$\langle \langle T_{R,L}^{\mu\nu}(x) \rangle \rangle = \sum_{r=0}^{\infty} \frac{i^{r}}{r!} \int \prod_{l=1}^{r} d^4 x_l V_{\sigma_l}(x_l) \langle 0 | T_{R,L}^{\mu\nu}(x) J_{R,L\sigma_l}(x_l) | 0 \rangle. \tag{97}$$
Figure 3: The Feynman diagram corresponding to $\tilde{F}^{(R)\mu}(k_1, k_2)$.

### 8.1 Difference between gauge and trace anomaly

Let us start from the case of the right-handed fermion. The correlator is, symbolically, $\langle t_R J_R J_R \rangle$, i.e. $\langle 0 | T T^{\mu}_R(x) J_{R\lambda(y)} J_{R\rho(z)} | 0 \rangle$, its Fourier transform being given by

$$\tilde{F}^{(R)\mu}_{\mu\lambda\rho}(k_1, k_2) = \frac{1}{4 \sqrt{\ell^2}} \int d^4p \delta(k_1 - k_2) \left( \frac{1}{p - \frac{1}{2} \gamma_5} \frac{1}{p - k_1} - \frac{1}{p - k_2} \right).$$

The relevant Feynman diagram is shown in figure [3]. The difference with respect to the Fourier transform of $\langle \partial - J_R J_R J_R \rangle$ - see eq. (54) - apart from the factor $\frac{1}{4}$, is the $(2p - q)$ factor in the RHS, instead of $q$. The relevant difference is therefore twice

$$\Delta \tilde{F}^{(R)\mu}_{\mu\lambda\rho}(k_1, k_2) = \frac{1}{4} \int d^4p \delta(\ell) \left( \frac{1}{p - \frac{1}{2} \gamma_5} \frac{1}{p - k_1} - \frac{1}{p - k_2} \right).$$

We can now replace $p \rightarrow p + k_1$

$$\Delta \tilde{F}^{(R)\mu}_{\mu\lambda\rho}(k_1, k_2) = \frac{i}{4} \int d^4p d^4\ell \delta(\ell) \left( \frac{1}{p - k_1} - \frac{1}{p - k_2} \right).$$

The odd part vanishes by symmetry.

If we consider instead the amplitude for $\langle \partial - J_5 J J \rangle$, the result does not change. In that
and for the same reason as above. In the same way one can easily prove that

\[ \Delta F^{(5)}_{\mu \lambda \rho}(k_1, k_2) \sim \int \frac{d^4 p d^4 \ell}{(2\pi)^{4+\delta}} \text{tr} \left( \gamma_\lambda \frac{p + \ell - k_1}{(p - k_1)^2 - \ell^2} \gamma_\rho \frac{p + \ell - q}{(p - q)^2 - \ell^2} \gamma_5 \right) \]

\[ = \int \frac{d^4 p d^4 \ell}{(2\pi)^{4+\delta}} \left( -\ell^2 \text{tr}(\gamma_\lambda \gamma_\rho \gamma_5) + \text{tr} \left( \gamma_\lambda (p - k_1) \gamma_\rho (p - q) \gamma_5 \right) \right). \quad (102) \]

The first term in the numerator vanishes. The rest can be rewritten as

\[ \Delta F^{(5)}_{\mu \lambda \rho}(k_1, k_2) \sim \int \frac{d^4 p d^4 \ell}{(2\pi)^{4+\delta}} \frac{\text{tr} \left( \gamma_\lambda \gamma_\rho \gamma_5 \right)}{(p - k_1)^2 - \ell^2} \left( p - q \right)^2 - \ell^2 \right) = 0 \]

for the same reason as above. In the same way one can easily prove that

\[ \Delta F^{(5)}_{\mu \lambda \rho}(k_1, k_2) = \Delta F^{(5')}_{\mu \lambda \rho}(k_1, k_2) = 0. \quad (104) \]

In conclusion, the amplitude for the chiral anomalies and those for the trace anomalies due to couplings with gauge fields are rigidly related, the corresponding coefficients exhibiting a fixed ratio, i.e. the former are minus four times the latter.

### 8.2 Trace anomalies due to a gauge field

Using the above results, which say that the difference between \((-4) \times [\text{the divergence}]\) and the trace anomaly is null, we can immediately deduce the corresponding consistent gauge trace anomalies, using (94,95) and (97), viz.,

\[ \langle \langle T^{(cs)}_{\mu \lambda \rho}(x) \rangle \rangle = -\frac{1}{48\pi^2} \epsilon_{\mu \nu \lambda \rho} \partial^\mu V^\nu(x) \partial^\lambda V^\rho(x). \quad (105) \]

As for \(T_{L\mu}(x)\), it carries the consistent anomaly

\[ \langle \langle T^{(cs)}_{L\mu}(x) \rangle \rangle = \frac{1}{48\pi^2} \epsilon_{\mu \nu \lambda \rho} \partial^\mu V^\nu(x) \partial^\lambda V^\rho(x). \quad (106) \]

On the other hand in the \(V - A\) framework we find

\[ \langle \langle T^{(\mu)}_{\mu}(x) \rangle \rangle = 0 \quad (107) \]

and

\[ \partial_\mu \langle \langle T^{(\mu)}_{\mu}(x) \rangle \rangle = -\frac{1}{16\pi^2} \epsilon_{\mu \nu \lambda \rho} \left( \partial^\mu V^\nu(x) \partial^\lambda V^\rho(x) + \frac{1}{3} \partial^\mu A^\nu(x) \partial^\lambda A^\rho(x) \right). \quad (108) \]

From (108) we can derive the covariant chiral anomaly for a Dirac fermion by setting \(A_\mu = 0\), then

\[ \langle \langle T^{(cv)}_{5\mu}(x) \rangle \rangle = \frac{1}{16\pi^2} \epsilon_{\mu \nu \lambda \rho} \partial^\mu V^\nu(x) \partial^\lambda V^\rho(x). \quad (109) \]

From this we can derive the covariant (invariant) trace anomaly for a right-handed

\[ \langle \langle T^{(cv)}_{R\mu}(x) \rangle \rangle = \frac{1}{16\pi^2} \epsilon_{\mu \nu \lambda \rho} \partial^\mu V^\nu(x) \partial^\lambda V^\rho(x) \quad (110) \]

and left-handed Weyl fermion

\[ \langle \langle T^{(cv)}_{L\mu}(x) \rangle \rangle = \frac{1}{16\pi^2} \epsilon_{\mu \nu \lambda \rho} \partial^\mu V^\nu(x) \partial^\lambda V^\rho(x). \quad (111) \]

25
8.3 Gauge anomalies and diffeomorphisms

In this review we have not considered diffeomorphisms. Nevertheless a devil’s accountant could argue that there might be violation of diffeomorphism invariance in a fermionic system coupled to gauge fields, due to the presence of the gauge fields themselves. In order to see this one has to consider three point correlators involving the divergence of the energy momentum tensor and two currents. More precisely, odd parity anomalies could appear in the following amplitudes: \( \langle \partial \cdot T_R J_R J_R \rangle \), in the right-handed fermion case, or \( \langle \partial \cdot T_5 J J \rangle \), \( \langle \partial \cdot T_5 J_5 J \rangle \), \( \langle \partial \cdot T_5 J_5 J_5 \rangle \) in the V − A case. They can be computed with the same methods as above, and here, for brevity, we limit ourselves to record the final results: they all vanish.

9 Conclusion

The purpose of this review was to highlight some subtle aspects of the physics of Weyl fermions, as opposed in particular to massless Majorana spinors. To this end we have decided not to resort to powerful non-perturbative methods, like the Seeley-Schwinger-DeWitt method, which would require a demanding introduction. Rather, we have used the simple Feynman diagram technique. In doing so we have focused on two aims. The first one is to justify the method of computing the effective action for a Weyl fermion coupled to gauge potentials, which requires the presence of free fermions of opposite chirality, in such a way as to produce the effective kinetic operator of eq. 30. We have shown that, notwithstanding the presence of fermions of both chiralities, no mass term can arise as a consequence of quantum corrections. As a by-product we were led to the conclusion that, while the Pauli-Villars regularization is a perfectly available and useful tool for perturbative calculations, it does not fit at all in the case of non-perturbative heat kernel-like methods.

Our second aim was to compute all the anomalies (trace and chiral) of Weyl, Dirac and Majorana fermions coupled to gauge potentials. The calculations are actually standard, but, once juxtaposed, they reveal a perhaps previously unremarked property: the chiral and trace anomalies due to a gauge background are rigidly linked.

Acknowledgement: S.Z. thanks SISSA for hospitality while this work was being completed. L.B. would like to thank his former collaborators A. Andrianov, M. Cvitan, P. Dominis-Prester, A. Duarte Pereira, S. Giaccari and T. Štembergera, whom part of the material of this paper was elaborated or discussed.

Appendix. Consistent gauge anomaly with PV regularization

To implement a PV regularization we replace \( \tilde{F}^{(R)}_{\mu \nu \lambda}(k_1, k_2) \) with

\[
\tilde{F}^{(R)}_{\mu \nu \lambda}(k_1, k_2) = \int \frac{d^4p}{(2\pi)^4} \text{tr} \left\{ \frac{1}{p + m} \frac{1 - \gamma_5}{2 \gamma_\nu} \frac{1}{p - k_\parallel + m} \frac{1 - \gamma_5}{2 \gamma_\lambda} \frac{1}{p - q + m} \frac{1 - \gamma_5}{2 \gamma_\mu} - \frac{1}{p + M} \frac{1 - \gamma_5}{2 \gamma_\nu} \frac{1}{p - k_\parallel + M} \frac{1 - \gamma_5}{2 \gamma_\lambda} \frac{1}{p - q + M} \frac{1 - \gamma_5}{2 \gamma_\mu} \right\} \tag{112}
\]

where \( \gamma_\mu \) is the gamma matrix, and \( \text{tr} \) is the trace of gamma matrices.
Contracting with $q^\mu$ and working out the traces one gets

\[
q^\mu \mathcal{F}^{(R)}_{\mu \nu \rho \lambda}(k_1, k_2) = -2i \epsilon_{\mu \nu \rho \lambda} \int \frac{d^4p}{(2\pi)^4} (2p \cdot q - p^2 q^\mu - q^2 p^\mu) (p - k_1)^\rho \left( \frac{1}{\Delta_{m^2}} - \frac{1}{\Delta_{M^2}} \right) \tag{113}
\]

where

\[
\Delta_{m^2} = (p^2 - m^2)((p - k_1)^2 - m^2)((p - q)^2 - m^2),
\]

\[
\Delta_{M^2} = (p^2 - M^2)((p - k_1)^2 - M^2)((p - q)^2 - M^2)
\]

For later use we introduce also

\[
\Omega_{m^2} = ((p - k_1)^2 - m^2)((p - q)^2 - m^2), \quad \Lambda_{m^2} = (p^2 - m^2)((p - k_1)^2 - m^2), \tag{114}
\]

\[
\Omega_{M^2} = ((p - k_1)^2 - M^2)((p - q)^2 - M^2), \quad \Lambda_{M^2} = (p^2 - M^2)((p - k_1)^2 - M^2). \tag{115}
\]

Now all the integrals are convergent because the divergent terms have been subtracted away. Let us proceed

\[
q^\mu \mathcal{F}^{(R)}_{\mu \nu \rho \lambda}(k_1, k_2) = -2i \epsilon_{\mu \nu \rho \lambda} \int \frac{d^4p}{(2\pi)^4} \left\{ \left( \frac{-k_2^\mu (p - k_1)^\rho}{\Omega_{m^2} \Delta_{M^2}} + \frac{p^\mu k_1^\rho}{\Lambda_{m^2} \Delta_{M^2}} \right) \right. \tag{116}
\]

\[
\left( m^6 - M^6 + (m^4 - m^4) (p^2 + (p - k_1)^2 + (p - q)^2) \right.
\]

\[
+ (m^2 - M^2) ((p - k_1)^2 (p - q)^2 + (p - k_1)^2 p^2 + p^2 (p - q)^2) \right.
\]

\[
+ m^2 (p^\mu k_1^\rho - q^\mu (p - k_1)^\rho) \left( \frac{1}{\Delta_{m^2}} - \frac{1}{\Delta_{M^2}} \right) \} \}
\]

The last line does not contribute, for the integrals converge (separately) and give a finite result, but since they are multiplied by $m^2$ they vanish in the limit $m \to 0$. So the last line can be dropped.

Now the strategy consists in simplifying separately each monomials in the numerator with a corresponding term in the denominator. For instance, if in a term of order $M^*$ there is the ratio $p^2/(p^2 - m^2)$, write $p^2$ as $p^2 - m^2 + m^2$. The $p^2 - m^2$ can be simplified with a corresponding term in the denominator. If $p^2 - m^2$ in the denominator is missing, there will be $p^2 - M^2$. So we write $p^2$ as $p^2 - M^2 + M^2$, and $p^2 + M^2$ can be simplified, while the term proportional to $M^2$ remains and contributes to the term of order $M^{*+2}$. Proceed in the same way also with $(p - q)^2$ and $(p - k_1)^2$. Many terms (such as those of order $M^6$) cancel out. What remains is

\[
q^\mu \mathcal{F}^{(R)}_{\mu \nu \rho \lambda}(k_1, k_2) = -2i \epsilon_{\mu \nu \rho \lambda} \int \frac{d^4p}{(2\pi)^4} \left\{ \right. \tag{117}
\]

\[
p^\mu k_1^\rho \left[ \frac{(M^2 - m^2)^2}{\Lambda_{m^2} \Delta_{M^2}} - \frac{(M^2 - m^2)}{\Lambda_{M^2}} \left( \frac{1}{p^2 - m^2} + \frac{1}{(p - k_1)^2 - m^2} \right) - \frac{M^2 - m^2}{\Delta_{M^2}} \right] \right.
\]

\[
-k_2^\rho (p - k_1)^\rho \left[ \frac{(M^2 - m^2)^2}{\Omega_{m^2} \Omega_{M^2}} - \frac{(M^2 - m^2)}{\Omega_{M^2}} \left( \frac{1}{(p - k_1)^2 - m^2} + \frac{1}{(p - q)^2 - m^2} \right) - \frac{M^2 - m^2}{\Delta_{M^2}} \right] \}
\]

It is easy too verify that, after introducing the relevant Feynman parameters, most of the terms vanish either because there is only one $p$ in the numerator or because of the anti-symmetry of
the $\epsilon$ tensor. Only the last term in each line remains, so that:

$$q^\mu \tilde{F}^{(R)}_{\mu\nu\lambda}(k_1, k_2) = 2i M^2 \epsilon_{\mu\nu\rho\lambda} \int \frac{d^4 p}{(2\pi)^4} \frac{p^\mu k_1^\rho - k_2^\mu (p - k_1)^\rho}{\Delta M^2}$$

(118)

Next we introduce two Feynman parameters $x$ and $y$, shift $p$ like in section 5.1 and make a Wick rotation on the momenta. Then (117) becomes

$$q^\mu \tilde{F}^{(R)}_{\mu\nu\lambda}(k_1, k_2) = -4 M^2 \epsilon_{\mu\nu\rho\lambda} \int_0^1 dx \int_0^{1-x} dy \int \frac{d^4 p}{(2\pi)^4} \frac{(1-x)k_1^\mu k_2^\rho}{(p^2 + M^2 + A(x,y))^3}$$

(119)

$$= -\frac{1}{8\pi^2} M^2 \epsilon_{\mu\nu\rho\lambda} \int_0^1 dx \int_0^{1-x} dy \frac{(1-x)k_1^\mu k_2^\rho}{M^2 + A(x,y)}$$

$$= -\frac{1}{24\pi^2} \epsilon_{\mu\nu\rho\lambda} k_1^\mu k_2^\rho$$

Adding the cross term we get

$$q^\mu \tilde{T}^{(R)}_{\mu\nu\lambda}(k_1, k_2) = -\frac{1}{12\pi^2} \epsilon_{\mu\nu\rho\lambda} k_1^\mu k_2^\rho$$

(120)

which is the same result as in section 5.1.

References

[1] Hermann Klaus Hugo Weyl (Elmshorn, Germany, 9.11.1885 – Zurich, CH, 8.12.1955) *Gruppentheorie und Quantenmechanik*, Leipzig, Hirzel (1928) VIII, 288; I.M. Gel’fand and M.A. Naimark, *Unitary representations of the Lorentz group* Izv. Akad. Nauk. SSSR, matem. 11, 411, 1947.

[2] Eugene Paul Wigner (Budapest 17.11.1902 – Princeton 1.1.1995) *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren*, Fredrick Vieweg und Sohn, Braunschweig, Deutschland, 1931, pp. 251–254; *Group Theory and its Application to the Quantum Theory of Atomic Spectra*, Academic Press Inc., New York, 1959, pp. 233–236.

[3] G. Racah, *On the symmetry of particles and antiparticles*, Nuovo Cimento, 14 (1936) 322.

[4] J. Schwinger, *On gauge invariance and vacuum polarization*, Phys. Rev. 82 (1951) 664-679.

[5] B. S. DeWitt, *The dynamical theory of groups and fields*, Gordon & Breach, New York (1965).

[6] R.T. Seeley, *Complex powers of an elliptic operator*, Proc. Sympos. Pure Math. 10, Amer. Math. Soc. (1967) 288-307; *The resolvent of an elliptic boundary value problem*, Am. J. Math. 91 (1969) 889-920.

[7] S.W. Hawking, *Zeta function regularization of path integrals in curved space-time*, Commun. Math. Phys. 55 (1977) 133-162.

[8] J.F. Cornwell, *Group theory in physics*, vol.II, Ch.17, (1984) Academic Press, London.

[9] R. A. Bertlmann, *Anomalies in Quantum Field Theory*, Oxford Univ. Press, Oxford, 1996.
[10] K. Kirsten, *Heat kernel asymptotics: more special case calculations*, Nuclear Physics B (Proc. Suppl.) **104** (2002) 119-126; D.V. Vassilevich, *Heat kernel expansion: users manual*, Phys. Rep. **388** (2003) 279-360.

[11] B.S. DeWitt, *Global approach to quantum field theory*, Oxford University Press (2003) vol. I and II.

[12] P. B. Pal, *Dirac, Majorana and Weyl fermions*, arXiv:1006.1718 [hep-th].

[13] L. Bonora, M. Cvitan, P. Dominis Prester, A. Duarte Pereira, S. Giaccari and T. Štemberga, *Axial gravity, massless fermions and trace anomalies*, Eur. Phys. J. C **77** (2017) 511 [arXiv:1703.10473 [hep-th]].

[14] L. Bonora and R. Soldati, *On the trace anomaly for Weyl fermions* [arXiv:arXiv:1909.11991[hep-th]]

[15] W. A. Bardeen, *Anomalous Ward Identities in Spinor Field Theories*, Phys. Rev. **184** (1969) 1848.