Analysis of the precision problem in numerical calculation of the reliability indices of technical systems by the topological method at application of the floating-point arithmetic

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Abstract. This paper deals with the problem of precision error arising in numerical calculation of the reliability indices of technical systems by the topological method at application of the floating-point arithmetic. The topological method for calculation of the reliability indices in accordance with the reliability models based on the Markov chains is discussed. An example of correct calculation of the reliability indices on the basis of analytical derivation of calculating formulas and substitution of the initial reliability parameters into them is given. The floating-point types and arithmetic supported in hardware by the modern computers and examples of the precision loss in floating-point arithmetic operations are also observed. Finally, an example of incorrect calculation of the reliability indices, based on the numerical solving of the system of linear algebraic equations, using the floating-point arithmetic is also given.

1. Introduction

Nowadays for reliability analysis of technical systems there are widely used such key reliability indices as: availability factor, mean time to failure and mean time to repair [1, 2]. For calculation of these reliability indices the reliability models on the basis of Markov chains and an effective topological method are often applied [3, 4]. At the same time the topological method of reliability indices calculation is based on solving of the system of linear algebraic equations, which is, as well as the solving methods, affected by the precision error arising in the case of application of numerical methods and floating-point arithmetic [5]. As a result, in the event of difference in values of initial reliability parameters on several tens or hundreds orders it is impossible to provide correct calculations of the reliability indices on computer in the case of application of the floating-point arithmetic.

This scientific paper is dedicated to the analysis of the problem of precision error arising in numerical calculation of the reliability indices of technical systems by the topological method at application of the floating-point arithmetic.

2. Topological method of calculation of the reliability indices of technical systems

In general case, Markov chain with continuous time (figure 1) in reliability models of the technical systems contains the set of states $E$, consisting of a subset of up-states $E_+$ and a subset of down-states $E_-$. The subset $E_+$, in its turn, contains a subset $H_+$ of border up-states, which have direct transition links to the downstates. Similarly, the subset $E_-$ contains a subset $H_-$ of border downstates, which have direct transition links to upstates.
Each state $i \in E$ (figure 2) can have as inbound transition links with the rates $\gamma_{ri}$ from the subset of states $\{r \in R\}$, which have transition links to the state $i$, so outbound transition links with the rates $\gamma_{is}$ into the subset of states $\{s \in S\}$, which have transition links from the state $i$.

**Figure 1.** Generalized Markov chain in reliability models of the technical systems.

![Generalized Markov chain diagram]

**Figure 2.** Inbound and outbound transitions for state $i$.

Mathematically the Markov chain model with continuous time represents the system of Kolmogorov-Chapman differential equations for each state $i \in E$. Having solved it, we can find the probabilities functions $P_i(t)$ at the given initial conditions $P_i(0)$ for all states of the technical system and, considering that the sum of probabilities of all states of the system is always equal to 1:

$$\left\{ \begin{array}{l} \sum_{i \in E} P_i(t) = 1; \\
\forall i \in E: \frac{dP_i(t)}{dt} = \sum_{r \in R_i} (\gamma_{ri} P_r(t)) - P_i(t) \sum_{s \in S_i} \gamma_{is}. \end{array} \right. \quad (1)$$
At the stationary case \( t \to \infty \), derivatives of probabilities functions tend to zero \( \frac{dP_i(t)}{dt} \to 0 \), and the system becomes the system of linear algebraic equations, having solved which it is possible to find stationary probabilities \( P_i \) for all states of the technical system:

\[
\begin{align*}
\forall i \in E: \sum_{r \in R_i} (\gamma_{ir} P_i) - P_i \sum_{s \in S_i} \gamma_{is} &= 0. 
\end{align*}
\]

(2)

According to the topological method [3, 4], by using the stationary probabilities \( P_i \) we can calculate stationary availability factor \( K \) of the technical system as the sum of stationary probabilities of all upstates of the system.

The mean time to failure \( T_F \) of the technical system can be calculated as the ratio of the sum of stationary probabilities of all upstates to the weighed sum of stationary probabilities of border upstates. The sum of rates of transition links from border state to all downstates is used as the weight for each of border upstates.

At last, it is also possible to calculate the mean time to repair \( T_R \) of the technical system using the similar formula. The only difference is that in numerator of the calculation formula it is used the sum of stationary probabilities of all downstates of the system.

\[
K = \sum_{i \in E_+} P_i; \quad T_F = \frac{\sum_{i \in E_+} P_i}{\sum_{i \in H_+} \sum_{j \in E_+} \gamma_{ij}}; \quad T_R = \frac{\sum_{i \in E_+} P_i}{\sum_{i \in H_+} \sum_{j \in E_+} \gamma_{ij}}.
\]

(3)

Example. A high-availability system with parallel structure consisting of 4 nodes with identical functionality and reliability parameters is given. The system is considered to be operable while at least one of nodes is in upstate. In figure 3 the Markov chain, which represents the reliability model of the high-availability system, is given.

Figure 3. Markov chain for the system of 4 independent nodes.

State 0 – all nodes are in upstate, 1 – one node is failed, 2 – two nodes are failed, 3 – three nodes are failed, 4 – all nodes are in downstate. Accordingly, states 0, 1, 2 and 3 are considered to be operable for the system. Initial state of the system is 0 (all nodes are in upstate).

Nodes are independent from the viewpoint of failures and repairs, and represent repairable elements with exponential distribution laws of the failure and repair time.

Nodes failure rates are \( \lambda = 10^{-20} \) hour\(^{-1} \). Node repair rates are \( \mu = 1 \) hour\(^{-1} \). Let us note that the values of the initial reliability parameters differ from each other on 20 orders.

Now, let us calculate the stationary availability factor, mean time to failure and mean time to repair of the given high-availability system by topological method. At first, we will solve the reliability task in analytic form with derivation of all the necessary calculating formulas.

In accordance with the given above Markov chain, we have the following stationary system of Kolmogorov-Chapman's equations:
By solving the system of equations in analytic form we obtain the formulas for calculation of the stationary probabilities for all states of the system:

\[
\begin{align*}
P_0 &= \frac{\mu^4}{(\mu + \lambda)^4}; & P_1 &= \frac{4\mu^3\lambda}{(\mu + \lambda)^4}; & P_2 &= \frac{6\mu^2\lambda^2}{(\mu + \lambda)^4}; & P_3 &= \frac{4\mu^3\lambda}{(\mu + \lambda)^4}; & P_4 &= \frac{\lambda^4}{(\mu + \lambda)^4}.
\end{align*}
\] (5)

Considering that the system is operable only in the states 0-3, we obtain the following formula for the stationary availability factor:

\[
K = P_0 + P_1 + P_2 + P_3 = \frac{\mu(\mu^3 + 4\mu^2\lambda + 6\mu\lambda^2 + 4\lambda^3)}{(\mu + \lambda)^4}.
\] (6)

Next, formula for the mean time to failure in accordance with the topological method and with taking into consideration the fact that the system has the only transition link with rate \(\lambda\) from upstate 3 into downstate 4:

\[
T_F = \frac{P_0 + P_1 + P_2 + P_3}{\lambda P_3} = \frac{\mu^3 + 4\mu^2\lambda + 6\mu\lambda^2 + 4\lambda^3}{4\lambda^4}.
\] (7)

Finally, formula of the mean time to repair is:

\[
T_R = \frac{P_4}{\lambda P_3} = \frac{1}{4\mu}.
\] (8)

Now, by substituting the given values of failures and repair rates into the obtained formulas, we obtain the following values of stationary probabilities for all states of the technical system:

\[
P_0 \approx 1; \quad P_1 = 4 \cdot 10^{-30}; \quad P_2 = 6 \cdot 10^{-40}; \quad P_3 = 4 \cdot 10^{-60}; \quad P_4 = 10^{-80}.
\]

Correspondingly, the stationary availability factor of the system: \(K \approx 1\).

The mean time to failure of the system: \(T_F \approx 2.5 \cdot 10^7\) hours.

The mean time to repair of the system: \(T_R = 0.25\) hour.

It should be noted, that the key step in calculations of the reliability indices of the technical system by the topological method is solving of the system of linear algebraic equations for the corresponding Markov chain in reliability model.

In particular cases, for the small number of states it is possible to derive the analytical calculation formulas and obtain proper values for the reliability indices. However, for more difficult cases – tens and hundreds of system states, we obviously have to apply the numerical methods for solving the reliability task using the modern computers. In order to perform calculations on computer with providing high performance and simplicity of software implementation of the calculation method, it is usually used floating-point arithmetic supported by modern microprocessors in hardware. But here we can meet serious «hidden pitfall», which is discussed below.

3. Calculation of the reliability indices using the floating-point arithmetic
The modern computers in the majority are based on processors with Intel® x86™ architecture [5], containing as hardware processing unit for the integer arithmetic, so the hardware processing unit for the floating point arithmetic (beginning from Intel® Pentium™ processor).

The three types of real numbers with floating point are supported in hardware: 32-bit real numbers of single precision (single), 64-bit real numbers of double precision (double) and 80-bit real numbers of extended precision (extended). It stands to reason that 80-bit type provides the broadest range of the represented real numbers and the largest precision among the aforesaid three types.

The format of 80-bit real number is given below in figure 4.

| S | E | M |
|---|---|---|
| 79 | 78 | 64 63 62 | ... | 0 |

**Figure 4.** Format of 80-bit real number.

Where,
- \( S \) (sign) – bit of sign, 0 – positive number, 1 – negative.
- \( E \) (exponent) – 15-bit power of two, shifted on +16383.
- \( M \) (mantissa) – 63-bit mantissa of a fractional part of number

The actual value of the floating-point number encoded in the 80-bit format can be calculated by the next formula:

\[
X = (-1)^S \left(2^{E-16383}\right)(1+M/2^{63}).
\] (9)

The range of the nonzero numbers is the following: \(3.36\cdot10^{-4932} \leq |X| \leq 1.19\cdot10^{+4932}\). However, despite such broad range of the represented numbers, precision of numbers is quite restricted as far as mantissa contains only 63 binary bits, and actually 80-bit type of numbers in the best case can provide only 19 valid decimal digits.

Now, let us discuss the following example of 80-bit floating-point arithmetic. Let there be given a floating-point number \(X = +1.0 = (-1)^0(2^{16383-16383})(1+0/2^{63})\) (figure 5).

| S | E | M |
|---|---|---|
| 0 | 01 | 64 63 62 | ... | 0 |

**Figure 5.** Initial floating-point number +1.0 in 80-bit format.

Let us add a small amendment \(+10^{-19}\) to the initial value +1.0. In the 80-bit floating point arithmetic we receive the following result, which shows that 63-bit mantissa is still enough for representation of the sum value (figure 6):

| S | E | M |
|---|---|---|
| 0 | 01 | 64 63 62 | ... | 0 |

**Figure 6.** Result of calculation \((1.0 + 10^{-19})\) in 80-bit format.

However, it is necessary to mention, that even in this case we have precision error, as far as the number given above actually represents value \((1.0 + 2^{-63}) \approx (1.0 + 1.0842\cdot10^{-19})\), and it is obviously differing from the true sum value \((1.0 + 10^{-19})\).

Moreover, if we try to add more small amendment \(+10^{-20}\) to the initial value +1.0, then 63-bit mantissa will not be able to provide representation of sum value, and the result in 80-bit format will look as if nothing at all was added, so actually it will represent the initial value +1.0 (figure 5). So, in the 80-bit floating point arithmetic the result of addition of +1.0 and \(+10^{-20}\) is equal to +1.0.

Now, returning to the topological method of reliability indices calculation discussed above, we should note that the key stage of calculations is the solving of the linear algebraic equations system
which involves a large number of arithmetical operations including operations of addition and subtraction. It is easy to see, that if initial data, as well as the intermediate data will differ from each other on the significant number of orders (20 and above), we can face incorrect results because of precision errors in floating-point arithmetic at the stage of solving the equation system.

The authors programmed the numerical calculation of the reliability indices for the given above technical system of 4 independent elements in accordance with appropriate Markov chain (figure 3) and matrix of the transition rates $W_{ij}$ from one state to another by using of the built-in function for solving the linear algebraic equation system in the Microsoft® Excel™, based on the inverse matrix method, and obtained the following values of probabilities of all states of the system $P_j$ (figure 7).

It is rather obvious that only values of probabilities $P_0$ and $P_1$ are correct and equal to the calculated above values, and the values of other probabilities $P_2$, $P_3$ and $P_4$ are incorrect and absurd in point of fact, as the probabilities cannot be the negative by definition.

Respectively, the negative probability $P_3$ also results in absurd negative value of mean time to failure of $T_F = -2.5 \times 10^{59}$ hours after application of the corresponding topological formula (7).

![Figure 7](image_url)  
Figure 7. Example of incorrect results of numerical calculation of the reliability indices in Microsoft® Excel™ at application of the floating-point arithmetic.

4. Conclusion

Thus, the numerical calculation of the reliability indices of technical systems by the topological method, based on the solving of the system of linear algebraic equations, is affected by the precision error, if the floating-point arithmetic is used. In case of «bad» initial reliability parameters (difference of initial values among themselves on several tens orders and more), they can lead to incorrect calculation results or even impossibility of solving of the reliability task. Respectively, in such cases it is necessary to use the methods of error-free computation [6] based on using of the multi-precision rational numbers and arithmetic [7, 8], where each real value is represented by a multi-precision rational number as a relation of the two multi-precision integers, stored in memory as arrays of bytes.

Within the research work in the field of reliability models of the technical systems on basis of the Markov chains, the authors developed a software realization of the topological method for calculation of the reliability indices using the multi-precision rational numbers and arithmetic and obtained correct calculation results for a wide range of values of the initial reliability parameters for the several reliability models [9, 10]. In figure 8 is shown an example of correct results of numerical calculation of the reliability indices for the discussed above technical system of 4 independent elements in the author’s calculation software at application of the multi-precision rational numbers and arithmetic.
**Figure 8.** Example of correct results of numerical calculation of the reliability indices in the author’s software at application of the multiple precision rational numbers and arithmetic.

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