On the Variance of the Area of Planar Cylinder Processes Driven by Brillinger-Mixing Point Processes

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Abstract

We study some asymptotic properties of cylinder processes in the plane defined as union sets of dilated straight lines (appearing as mutually overlapping infinitely long strips) derived from a stationary independently marked point process on the real line, where the marks describe thickness and orientation of individual cylinders. Such cylinder processes form an important class of (in general non-stationary) planar random sets. We observe the cylinder process in an unboundedly growing domain $gK$ when $g \to \infty$, where the set $K$ is compact and star-shaped w.r.t. the origin $o$ being an inner point of $K$. Provided the unmarked point process satisfies a Brillinger-type mixing condition and the thickness of the typical cylinder has a finite second moment we prove a (weak) law of large numbers as well as a formula of the asymptotic variance for the area of the cylinder process in $gK$. Due to the long-range dependencies of the cylinder process, this variance increases proportionally to $g^3$.

1 Introduction and Preliminaries

Cylinder processes in $\mathbb{R}^d$ defined as countable union of dilated affine subspaces $\mathbb{R}^k$, $k = 1, \ldots, d - 1$, are basic random set models in stochastic geometry, see e.g. [19], [25], [23], [20]. They have numerous applications (mostly for $d = 2, 3$) among others in material sciences to model materials consisting of long thick fibres, see e.g. [24]. Until now, so far as we know, asymptotic properties of cylinder processes in expanding domains were exclusively studied under Poisson assumptions, see [11], [12]. In this paper, our focus is put on planar cylinder processes which are derived from stationary independently marked point processes on $\mathbb{R}^1$. Under comparatively strong conditions on the higher-order cumulant measures of the unmarked (ground) point process we are able to prove first, a mean-square limit of the relative part of the area of an expanding star-shaped window covered by the union of cylinders, and second, we derive an explicit formula for the asymptotic variance of this area. The latter is an important first step in proving asymptotic normality of the covered area that will be carried out in a later paper. Our main results, Theorems 1 and 2, in Section 2 generalize some of the results obtained in [12] (in particular Theorem 2 in [12]) for stationary Poisson cylinder processes even under general dimensional assumptions.

Throughout in this paper, all random elements are defined on a common probability space $[\Omega, \mathcal{F}, P]$ and by $E$ resp. $\text{Var}$, we denote the expectation resp. variance w.r.t. $P$. Next we describe a

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cylinder process in $\mathbb{R}^2$ in terms of its generating stationary, independently marked point process on $\mathbb{R}^1$. For doing this, let $(\Phi_0, R_0)$ be the generic random vector taking value in the mark space $[0, \pi] \times [0, \infty)$ that describes the orientation $\Phi_0$ and the cross-section (or base) $\Xi_0 := [-R_0, R_0]$ of the typical cylinder. In addition, we assume that $R_0 \sim F$ and $\Phi_0 \sim G$ are independent, i.e. $P(R_0 \leq r, \Phi_0 \leq \varphi) = F(r) G(\varphi)$. Now we introduce a stationary independently marked point process as locally finite, simple counting measure $\Psi_{F,G} := \sum_{i \in \mathbb{Z}} \delta_{[P_i, (\Phi_i, R_i)]}$ defined on the Borel sets of $\mathbb{R}^1 \times [0, \pi] \times [0, \infty)$, whose finite-dimensional distributions are shift-invariant in the first component, see e.g. [5], [7] or [23]. The stationary unmarked (or ground) point process $\Psi = \sum_{i \in \mathbb{Z}} \delta_{P_i} \sim P$ with finite and positive intensity $\lambda = E\Psi([0,1]) > 0$ is assumed to be independent of the i.i.d. sequence $\{(\Phi_i, R_i) : i \in \mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}\}$ of mark vectors. Each triplet $[P_i, (\Phi_i, R_i)], i \in \mathbb{Z}$, determines a random cylinder $g(P_i, \Phi_i) \oplus b(o, R_i)$, where $b(o, r)$ is the circle in $\mathbb{R}^2$ with radius $r \geq 0$ and centre in the origin $o$ and $\oplus$ stands for pointwise addition (Minkowski sum) of subsets of $\mathbb{R}^2$. Here, $g(p, \varphi) := \{(x, y) \in \mathbb{R}^2 : x \cos \varphi + y \sin \varphi = p\}$ denotes the unique line with signed distance $p \in \mathbb{R}^1$ from $o$ and an angle $\varphi \in [0, \pi]$ measured anti-clockwise between the normal vector $\nu(\varphi) = (\cos \varphi, \sin \varphi)^T$ on the line with direction in the half plane not containing $o$ and the $x$-axis. The intensity measure $\Lambda_{F,G}([0, \varphi] \times [0, r]) := E\Psi_{F,G}([0, \varphi] \times [0, r])$ of $\Psi_{F,G}$ can be expressed for any $r \geq 0$ and $0 \leq \varphi \leq \pi$ as

$$\Lambda_{F,G}([0, \varphi] \times [0, r]) = E\Psi([0, \varphi]) P(\Phi_0 \leq \varphi, R_0 \leq r) = \lambda |\cdot|_1 G(\varphi) F(r) \quad \text{with} \quad \lambda > 0,$$

where $|\cdot|_k$ denotes the Lebesgue measure on $\mathbb{R}^k$. Now we are in a position to define the main subject of this paper.

**Definition 1.** A cylinder process $\Xi = \Xi_{F,G}^P$ in the Euclidean plane $\mathbb{R}^2$ derived from the stationary independently marked point process $\Psi_{F,G}^P$ is defined by random union set

$$\Xi_{F,G}^P := \bigcup_{i \in \mathbb{Z}} (g(P_i, \Phi_i) \oplus b(o, R_i)),$$

which in general is neither closed nor stationary.

For more details and a general survey on cylinder processes we refer to [25], see also the monographs [19], [20]. The aim of this paper consists first, in proving the $L^2$-convergence of the ratio $|\Xi \cap gK|_2 / |gK|_2$ to a deterministic limit as $g \to \infty$ and second, in proving the existence and determining the explicit shape of the asymptotic variance

$$\lim_{g \to \infty} \frac{\text{Var}(|\Xi \cap gK|_2)}{g^2} =: \sigma_P^2(K, F, G),$$

for some fixed compact star-shaped set $K \subset \mathbb{R}^2$ containing the origin $o$ as inner point. The limit $\sigma_P^2(K, F, G)$ is positive and finite (if $E|\Xi_0|^2 = 4E R_0^2 < \infty$) and depends on the shape of $K$, the first and second moment of $F$ and the distribution function $G$ which is assumed to be continuous (not necessarily absolutely continuous). A purely discrete distribution function $G$ yields different expressions for $\sigma_P^2(K, F, G)$ even if $\Psi \sim P = \Pi_\lambda$ is a stationary Poisson point process with intensity $\lambda > 0$, see [11],[12]. A distribution function $G$ without jumps implies that $P(\Phi_0 = \Phi_1) = 0$ if the angles $\Phi_0, \Phi_1 \sim G$ are independent.

Note that the order $g^3$ of the growth of $\text{Var}(|\Xi \cap gK|_2)$ is much faster that the growth of the area $|gK|_2 = g^3|K|_2$ which reveals a typical feature of long-range dependencies within the random set $\Xi_{F,G}^P$.  

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We recall the fact that the probability space \([\Omega, \mathcal{F}, P]\) on which the marked point process \(\Psi_m\) is defined can be chosen in such a way that the mapping \((x, \omega) \mapsto 1_{\Xi(\omega)}(x) \in \{0, 1\}\) for \((x, \omega) \in \mathbb{R}^2 \times \Omega\) is measurable w.r.t. the product \(\sigma\)-field \(\mathcal{B}(\mathbb{R}^2) \otimes \mathcal{F}\), see Appendix in [10]. This enables us to apply Fubini’s theorem to the random field of indicator variables \(\{1_\Xi(x), x \in \mathbb{R}^2\}\) and implies that the \(k\)th-order mixed moment function

\[
p^{(k)}_{\Xi}(x_1, \ldots, x_k) := E\left(\prod_{j=1}^{k} 1_{\Xi}(x_j)\right) = P(x_1 \in \Xi, \ldots, x_k \in \Xi), \quad x_1, \ldots, x_k \in \mathbb{R}^2, \quad (1.3)
\]

are \(\mathcal{B}(\mathbb{R}^{2k})\)-measurable for any \(k \in \mathbb{N} := \{1, 2, \ldots\}\).

The distribution of a random closed set \(\Xi\) is determined by its Choquet functional

\[
T_{\Xi}(X) := P(\Xi \cap X \neq \emptyset) \quad \text{for} \quad X \in K_2, \quad (1.4)
\]

where \(K_2\) denotes the family of non-empty compact sets in \(\mathbb{R}^2\). In particular, the \(k\)th order moment functions \(p^{(k)}_{\Xi}\) of the \(0-1\)-random field \(\xi(x) := 1_{\Xi}(x)\) can be expressed by \((1.3)\) and \((1.4)\) for any \(k \geq 1:\)

\[
p^{(k)}_{\Xi}(x_1, \ldots, x_k) = E\left(\prod_{j=1}^{k} \xi(x_j)\right) = P(\{x_1, \ldots, x_k\} \cap \Xi = \emptyset) = 1 - T_{\Xi}(\{x_1, \ldots, x_k\}).
\]

**Lemma 1.** For any \(X \in K_2\), we have

\[
T_{\Xi}(X) = 1 - G_P\left[1 - P(\cdot \in [-R_0, R_0] \oplus \langle v(\Phi_0), X \rangle)\right], \quad (1.5)
\]

where \(\langle v(\Phi_0), X \rangle := \bigcup_{x \in X} \langle v(\Phi_0), x \rangle\) with scalar product \(\langle \cdot, \cdot \rangle\) in \(\mathbb{R}^2\) and \(G_P[w(\cdot)]\) denotes the probability generating functional (short: pgf) of \(\Psi \sim P\) defined for Borel-measurable functions \(w : \mathbb{R}^1 \to [0, 1]\) by

\[
G_P[w(\cdot)] := E\left(\prod_{i \in \Psi(P_i) > 0} w(P_i)\right), \quad \text{where} \quad \int_{\mathbb{R}^1} (1 - w(x))dx < \infty. \quad (1.6)
\]

**Corollary 1.** For \(X = \{x_1, \ldots, x_k\}\) with pairwise distinct points \(x_1, \ldots, x_k \in \mathbb{R}^2\) we get

\[
T_{\Xi}(\{x_1, \ldots, x_k\}) = G_P\left[1 - P(\cdot \in \bigcup_{i=1}^{k}([-R_0, R_0] + \langle v(\Phi_0), x_i \rangle)\right]. \quad (1.7)
\]

**Example 1.** For a stationary Poisson process \(\Psi \sim \Pi_{\lambda}\) with intensity \(\lambda > 0\), we have \(G_{\Pi_{\lambda}}[w(\cdot)] = \exp\{\lambda \int_{\mathbb{R}^1} (w(x) - 1)dx\}\) implying that

\[
T_{\Xi}(X) = 1 - \exp\left\{-\lambda \int_{\mathbb{R}^1} P((g(p, \Phi_0) \oplus b(o, R_0)) \cap X \neq \emptyset) dp\right\}
= 1 - \exp\left\{-\lambda E\left[[-R_0, R_0] + \langle v(\Phi_0), X \rangle\right]\right\}
= 1 - \exp\left\{-\lambda \int_{0}^{\infty} \int_{-r}^{r} \left| [-r, r] \oplus \langle v(\varphi), X \rangle \right| dG(\varphi) dF(r)\right\}.
\]
Obviously, (1.9) coincides with (1.5). Hence, the proof of Lemma 1 is complete.

The last step leading to (1.9) is seen as follows:

\[ T_\Xi(\{x_1, \ldots, x_k\}) = 1 - \exp \left\{ -\lambda E \left( \sum_{i=1}^{k} \left( -R_0, R_0 \right) + x_i^{(1)} \cos(\varphi_0) + x_i^{(2)} \sin(\varphi_0) \right) \right\} \]

\[ = 1 - \exp \left\{ -\lambda \int_{-r}^{r} \int_{0}^{\pi} \left( \sum_{i=1}^{k} \left( -r, r \right) + x_i^{(1)} \cos(\varphi) + x_i^{(2)} \sin(\varphi) \right) dG(\varphi)dF(r) \right\} \]

**Proof (Lemma 1).** To prove formula (1.2), we need the orthogonal matrix

\[ O(\varphi) = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}, \]  

(1.8)

which represents an anti-clockwise rotation by the angle \( \varphi \in [0, \pi] \) so that \( O(-\varphi)v(\varphi) = (1, 0)^T \)

and \( O(\varphi)(1, 0)^T = v(\varphi) \) since it holds \( O(-\varphi) = O^T(\varphi) = O^{-1}(\varphi) \). Using the pgf (1.6) and the independence assumption in the definition of (1.1), we obtain

\[ T_\Xi(X) = 1 - P(\Xi \cap X = \emptyset) = 1 - P \left( \bigcap_{i:\Psi({\{p_i\}}) > 0} \{ (g(P, \Phi_i) + b(o, R_i)) \cap X = \emptyset \} \right) \]

\[ = 1 - E \left( \prod_{i:\Psi({\{p_i\}}) > 0} 1_{\{ (g(P, \Phi_i) + b(o, R_i)) \cap X = \emptyset \}} \right) \]

\[ = 1 - \int_{\mathbb{N}} E \left( \prod_{i:\psi({\{p_i\}}) > 0} (1 - P( (g(p_i, \Phi_i) + b(o, R_i)) \cap X = \emptyset | \Psi = \psi)) \right) P(\Psi \in d\psi) \]

\[ = 1 - \int_{\mathbb{N}} \prod_{i:\psi({\{p_i\}}) > 0} (1 - P( (g(p_i, \Phi_i) + b(o, R_i)) \cap X = \emptyset )) P(\Psi \in d\psi) \]

\[ = 1 - \int_{\mathbb{N}} \prod_{i:\psi({\{p_i\}}) > 0} (1 - P( p_i \in [-R_0, R_0] \oplus (v(\Phi_0), X)) ) P(\Psi \in d\psi), \]  

(1.9)

where \( \mathbb{N} \) denotes the set of locally finite simple counting measures on the Borel-\( \sigma \)-algebra \( B(\mathbb{R}^1) \).

The last step leading to (1.9) is seen as follows:

\[ \left\{ \left( g(p, \Phi_0) + b(o, R_0) \right) \cap X \neq \emptyset \right\} = \left\{ p v(\Phi_0) \in \left( -g(0, \Phi_0) + b(o, R_0) \right) \oplus X \right\} \]

\[ = \left\{ p O(-\Phi_0) v(\Phi_0) \in \left( g(0, 0) + b(o, R_0) \right) \oplus O(-\Phi_0)X \right\} \]

\[ = \left\{ p (1, 0)^T \in \left( g(0, 0) + b(o, R_0) \right) \oplus O(-\Phi_0)X \right\} \]

\[ = \left\{ p \in [-R_0, R_0] \oplus (v(\Phi_0), X) \right\} . \]

Obviously, (1.9) coincides with (1.5). Hence, the proof of Lemma 1 is complete. \( \square \)
2 Factorial moment expansion of $E|\Xi \cap \varrho K|_2$ and $\text{Var}|\Xi \cap \varrho K|_2$

The proof of our asymptotic results relies on an expansion of the pgf (1.6) (resp. its logarithm) in terms of the factorial moment (resp. cumulant) measures, see Chapter 5.5 in [5] or [4]. To begin with, let us fix $K \subset \mathbb{R}^2$ to be a compact, star-shaped set containing the origin as an inner point. Further, let $\varrho \geq 1$ be a scaling factor tending to infinity implying that $\varrho K \uparrow \mathbb{R}^2$ as $\varrho \to \infty$. The second-order mixed moment functions (1.3) fulfill the relation

$$p^{(2)}_\Xi(x_1, x_2) - p^{(1)}_\Xi(x_1)p^{(1)}_\Xi(x_2) = p^{(2)}_\Xi(x_1, x_2) - p^{(1)}_\Xi(x_1)p^{(1)}_\Xi(x_2).$$

(2.1)

By applying Fubini’s theorem we get, together with (1.5), that

$$E|\Xi \cap \varrho K|_2 = E\int_{\mathbb{R}^2} 1_{\Xi}(x) 1_{\varrho K}(x) dx = \int_{\varrho K} p^{(1)}_\Xi(x) dx = \varrho^2 \int_{K} T_{\Xi}(\{\varrho x\}) dx.$$  

(2.2)

For the variance, we get from (1.3), (1.7) and (2.1) that

$$\text{Var}(|\Xi \cap \varrho K|_2) = \iint_{\varrho K \varrho K} (p^{(2)}_\Xi(x_1, x_2) - p^{(1)}_\Xi(x_1)p^{(1)}_\Xi(x_2)) dx_1 dx_2.$$  

Together with (1.7), we obtain to the following lemma.

**Lemma 2.** With the above notation and $\langle v(\varphi), x_i \rangle = x_i^{(1)} \cos \varphi + x_i^{(2)} \sin \varphi$ for $i = 1, 2$, we have

$$\text{Var}(|\Xi \cap \varrho K|_2) = \iint_{\varrho K \varrho K} \left( G_P \left[ 1 - P \left( \cdot \in \bigcup_{i=1}^2 \left[ -R_0, R_0 \right] + v(\Phi_0), x_i \right) \right] \right)$$

$$- \prod_{i=1}^2 G_P \left[ 1 - P \left( \cdot \in \bigcup_{i=1}^2 \left[ -R_0, R_0 \right] + v(\Phi_0), x_i \right) \right] \right) dx_1 dx_2.$$  

(2.3)

Formula (2.3) can be generalized to higher-order cumulants $\text{Cum}_k(|\Xi \cap \varrho K|_2)$ for any $k \geq 3$, where the $k$th-order cumulant $\text{Cum}_k(X)$ of a random variable $X$ can be expressed by its moments $EX, \ldots, EX^k$ as follows:

$$\text{Cum}_k(X) = \sum_{\ell=1}^k (-1)^{\ell-1}(\ell-1)! \sum_{K_1 \sqcup \ldots \sqcup K_\ell \atop \#K_i = 1, \ldots, k} \prod_{j=1}^\ell \text{EX}^{\#K_j} = k! \sum_{\ell=1}^k (-1)^{\ell-1} \frac{\ell!}{\ell} \sum_{k_1 + \ldots + k_\ell = k, \ell} \prod_{j=1}^\ell \frac{\text{EX}^{k_j}}{k_j!},$$

where the first inner sum runs over all decompositions of $\{1, \ldots, k\}$ into $\ell$ disjoint non-empty subsets $K_1, \ldots, K_\ell$ and $\#K_i$ denotes the number of elements of $K_i, i = 1, \ldots, \ell$.

Combining the latter representation and the formula
As a consequence of (2.2) and (2.6) and the definition of the factorial moment measures $\mathbb{E}|\Xi \cap \varrho K|_2 = \mathbb{E} \int \prod_{(\varrho K)^k} 1_\Xi(x_1, \ldots, x_k) d(x_1, \ldots, x_k) = \int p_{\Xi}^{(k)}(x_1, \ldots, x_k) d(x_1, \ldots, x_k)$ for $k \geq 2,$

with the $k$th-order mixed cumulant function of the random field $\{1_\Xi(x), x \in \mathbb{R}^2\}$

$$c_{\Xi}^{(k)}(x_1, \ldots, x_k) := \sum_{\ell=1}^{k} (-1)^{\ell-1} (\ell - 1)! \sum_{K_1, \ldots, K_{\ell} j=1}^{k} \prod_{i=1}^{\ell} p_{\Xi}^{(\#K_j)}(x_i : i \in K_i)$$

for $k \in \mathbb{N},$

which satisfy the identity $c_{\Xi}^{(k)}(x_1, \ldots, x_k) = (-1)^k c_{\Xi}^{(k)}(x_1, \ldots, x_k)$ for $k \geq 2,$ we arrive at

$$\text{Cum}_k(|\Xi \cap \varrho K|_2) = (-1)^k \int_{(\varrho K)^k} c_{\Xi}^{(k)}(x_1, \ldots, x_k) d(x_1, \ldots, x_k)$$

for $k \geq 2,$ (2.4)

where

$$p_{\Xi}^{(k)}(x_1, \ldots, x_k) = 1 - T_{\Xi}(|\begin{array}{c} x_1, \ldots, x_k \end{array}|) = G_P \left[1 - \mathbb{P}(\cdot \in \bigcup_{i=1}^{k}([-R_0, R_0] + \langle v(\Phi_0), x_i \rangle))\right].$$

In order to treat the moments and cumulants of $|\Xi \cap \varrho K|_2$, the following relations are useful. Let $a_1, a_2, \ldots$ be real numbers in $[0, 1]$. Then we have

$$1 - \prod_{i=1}^{n} (1 - a_i) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!} \sum_{1 \leq i_1, \ldots, i_k \leq n} a_{i_1} \cdots a_{i_k} \quad \text{for } n \geq 1. \quad (2.5)$$

Moreover, for any odd number $m < n$ (provided $n \geq 2$), the so called Bonferroni inequalities (see e.g. [S]) hold:

$$\sum_{k=1}^{m+1} \frac{(-1)^{k-1}}{k!} \sum_{1 \leq i_1, \ldots, i_k \leq n} a_{i_1} \cdots a_{i_k} \leq 1 - \prod_{i=1}^{n} (1 - a_i) \leq \sum_{k=1}^{m} \frac{(-1)^{k-1}}{k!} \sum_{1 \leq i_1, \ldots, i_k \leq n} a_{i_1} \cdots a_{i_k}. \quad (2.6)$$

**Definition 2.** To simplify the notation, we define for $k \geq 2$ (not necessarily pairwise distinct) points $x_1, \ldots, x_k \in \mathbb{R}^2$ and $\Xi_0 = [-R_0, R_0]$ the functions

$$w_{x_1, \ldots, x_k}^{(\dagger)}(p) := \mathbb{P} \left( p \in \bigcup_{i=1}^{k} (\Xi_0 + \langle v(\Phi_0), x_i \rangle) \right) \quad \text{and} \quad w_{x_1, \ldots, x_k}^{(\dagger)}(p) := \mathbb{P} \left( p \in \bigcap_{i=1}^{k} (\Xi_0 + \langle v(\Phi_0), x_i \rangle) \right).$$

For $k = 1$ we put $w_{x_1}^{(\dagger)}(p) = w_x^{(\dagger)}(p) := w_{x}(p)$. Obviously, $w_{x_1, x_2}^{(\dagger)}(p) = w_{x_1}(p) + w_{x_2}(p) - w_{x_1, x_2}^{(\dagger)}(p)$.

As a consequence of (2.2) and (2.6) and the definition of the factorial moment measures $\alpha^{(k)}(\cdot)$ of $\Psi \sim P$, we get the following series expansion

$$\mathbb{E}|\Xi \cap \varrho K|_2 = \int_{(\varrho K)^k} \left( 1 - G_P \left[1 - w_x(\cdot)\right] \right) d(x_1, \ldots, x_k) = \varrho^2 \int_{K} \left( 1 - G_P \left[1 - w_{x}(\cdot)\right] \right) dx$$

$$= \varrho^2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \int_{K} \prod_{j=1}^{k} w_{x_j}(p_j) \alpha^{(k)}(d(p_1, \ldots, p_k)) dx, \quad (2.7)$$

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provided that the infinite sum on the right hand side converges. From (2.6) we obtain immediately the estimates

$$1 - G_P (1 - w_{\text{ex}}(\cdot)) - \sum_{k=1}^{m-1} \frac{(-1)^{k-1}}{k!} \int_{\mathbb{R}^k} \prod_{j=1}^{k} w_{\text{ex}}(p_j) \alpha^{(k)}(d(p_1, \ldots, p_k)) \leq \frac{1}{m!} \int_{\mathbb{R}^m} \prod_{j=1}^{m} w_{\text{ex}}(p_j) \alpha^{(m)}(d(p_1, \ldots, p_m))$$

(2.8)

for any $m \geq 1$. It is easily seen that the right hand side of (2.7) is convergent if and only if

$$\frac{1}{m!} \int_{\mathbb{R}^m} \prod_{j=1}^{m} w_{\text{ex}}(p_j) \alpha^{(m)}(d(p_1, \ldots, p_m)) \xrightarrow{m \to \infty} 0.$$  

(2.9)

One way to show (2.9) consists of expressing $\alpha^{(m)}(\cdot)$ by factorial cumulant measures $\gamma^{(k)}(\cdot), k = 1, \ldots, m$ where $\gamma^{(1)}(B) = \alpha^{(1)}(B) = \lambda |B|_1$ and for $k \geq 2$,

$$\alpha^{(k)}(\times_{i=1}^{k} B_i) = \sum_{\ell=1}^{k} \sum_{K_1 \cup \cdots \cup K_\ell = \{1, \ldots, k\}} \prod_{j=1}^{\ell} \gamma^{(\#K_j)}(\times_{i \in K_j} B_i).$$  

(2.10)

The representation (2.10) follows by inverting the defining formula for $\gamma^{(k)}(\cdot)$ which is as follows:

$$\gamma^{(k)}(\times_{i=1}^{k} B_i) := \sum_{\ell=1}^{k} (-1)^{\ell-1} (\ell - 1)! \sum_{K_1 \cup \cdots \cup K_\ell = \{1, \ldots, k\}} \prod_{j=1}^{\ell} \alpha^{(\#K_j)}(\times_{i \in K_j} B_i).$$

The latter formula is based on the general relationship between mixed moments and mixed cumulants, see [15] or [22]. Note that $\gamma^{(k)}$ is a locally finite, signed measure on $[\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k)]$.

Due to the stationarity of $\Psi \sim P$ we can implicitly define the $k$th-order reduced cumulant measure $\gamma_{\text{red}}^{(k)}(\cdot)$ as the unique signed measure on $[\mathbb{R}^{k-1}, \mathcal{B}(\mathbb{R}^{k-1})]$ satisfying

$$\gamma^{(k)}(\times_{i=1}^{k} B_i) = \lambda \int_{B_1} \gamma_{\text{red}}^{(k)}(\times_{i=2}^{k} (B_i - p)) dp$$

for all bounded sets $B_1, \ldots, B_k \in \mathcal{B}(\mathbb{R}^1)$.

The total variation measure $|\gamma_{\text{red}}^{(k)}(\cdot)|$ is defined by $|\gamma_{\text{red}}^{(k)}(\cdot)| = (\gamma_{\text{red}}^{(k)})^+(\cdot) + (\gamma_{\text{red}}^{(k)})^-(\cdot)$, where the measures $(\gamma_{\text{red}}^{(k)})^+(\cdot)$ and $(\gamma_{\text{red}}^{(k)})^-(\cdot)$ are given by the Jordan decomposition of the signed measure $\gamma_{\text{red}}^{(k)}(\cdot) = (\gamma_{\text{red}}^{(k)})^+(\cdot) - (\gamma_{\text{red}}^{(k)})^-(\cdot)$. The total variation of $\gamma_{\text{red}}^{(k)}(\cdot)$ on $[\mathbb{R}^{k-1}, \mathcal{B}(\mathbb{R}^{k-1})]$ is defined by $||\gamma_{\text{red}}^{(k)}||_{TV} := |\gamma_{\text{red}}^{(k)}|(\mathbb{R}^{k-1})$. Furthermore, if $\gamma_{\text{red}}^{(k)}(\cdot)$ possesses a Lebesgue density $c_{\text{red}}^{(k)}(\cdot)$ on $\mathbb{R}^{k-1}$ (called $k$th-order reduced cumulative density), we define the canonical $L_q$-norm $\|c_{\text{red}}^{(k)}\|_q := \left( \int_{\mathbb{R}^{k-1}} |c_{\text{red}}^{(k)}(x)|^q dx \right)^{1/q}$ for $k \geq 2$ and the modified $L_q^*$-norm $\|c_{\text{red}}^{(k)}\|_{q}^* := \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^{k-2}} |c_{\text{red}}^{(k)}(x, p)|^q dx \right)^{1/q} dp$ for $k \geq 3$, where $1 \leq q < \infty$. Formally we may put $||\gamma_{\text{red}}^{(1)}||_{TV} := 1$ and $||c_{\text{red}}^{(1)}||_q := 1$. 

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Definition 3. A stationary point process \( \Psi \sim P \) on \([\mathbb{R}^1, \mathcal{B}(\mathbb{R}^1)]\) with intensity \( \lambda = E\Psi([0,1]) > 0 \) satisfying \( E\Psi^k([0,1]) < \infty \) for all \( k \geq 2 \), is called Brillinger-mixing if \( \|\gamma_{\text{red}}^{(k)}\|_{TV} < \infty \) for all \( k \geq 2 \). \( \Psi \sim P \) is said to be strongly Brillinger-mixing (strongly \( L_q \)-Brillinger-mixing, resp. strongly \( L_q^* \)-Brillinger-mixing for some \( q \geq 1 \)) if there are constants \( b > 0 \) and \( a \geq b^{-1} \) such that 
\[
\|\gamma_{\text{red}}^{(k)}\|_{TV} \leq a b^k k! \quad (\text{if } c_{\text{red}}^{(k)}(\cdot) \text{ exists such that } \|c_{\text{red}}^{(k)}\|_1 < \infty \text{ and } \|c_{\text{red}}^{(k)}\|_q \leq a_q (b_q)^k k! \text{ for } k \geq 2 \text{ with constants } a_q, b_q > 0 \text{ resp. } \|c_{\text{red}}^{(k)}\|_q \leq a_q^*(b_q^*)^k k! \text{ for } k \geq 3 \text{ with constants } a_q^*, b_q^* > 0).
\]

Remark. For formal reason we put \( \|\gamma_{\text{red}}^{(1)}\|_{TV} := 1 \) so that \( a \geq b^{-1} \) makes \( \|\gamma_{\text{red}}^{(1)}\|_{TV} := 1 \leq a b \). Further, note that the existence and integrability of \( c_{\text{red}}^{(k)}(\cdot) \) imply that \( \|c_{\text{red}}^{(2)}\|_1 = \|\gamma_{\text{red}}^{(2)}\|_{TV} \) and 
\[
\|c_{\text{red}}^{(k)}\|_1 = \|\gamma_{\text{red}}^{(k)}\|_1 = \|\gamma_{\text{red}}^{(k)}\|_{TV} \text{ for all } k \geq 3.
\]

Remark. In general, the Brillinger-mixing condition is formulated for stationary point processes on \( \mathbb{R}^d, d \geq 1 \). This condition expresses some kind of weak correlatedness (or asymptotic uncorrelatedness) of the numbers of points lying in bounded sets having a large (or unboundedly increasing) distance of each other. This type of weak dependence does not necessarily imply ergodicity, see [15], but allows to prove central limit theorems for various stochastic models related with point processes, e.g. in stochastic geometry, statistical physics for \( d \geq 1 \) or queueing theory for \( d = 1 \), see e.g. [9]. In [13, 16] the relations between (strong) Brillinger-mixing and classical mixing conditions are studied. Strong Brillinger-mixing requires exponential moments of the number of points in bounded sets. For any dimension \( d \geq 1 \), examples of such point processes are determinental point processes, see [14, 5]. Poisson cluster processes if the number of daughter points has an exponential moment and certain Cox processes as well as Gibbsian processes under suitable restrictions, see [21]. For \( d = 1 \), renewal processes with an exponentially decaying interrenewal density, see [9], among them the Erlang process and the Macchi process, see [5] (p. 144), are strongly Brillinger-mixing.

Lemma 3. If the stationary point process \( \Psi \sim P \) is strongly Brillinger-mixing with \( b < \frac{1}{2} \) and \( E R_0 < \infty \), then 
\[
\sum_{m=1}^{\infty} \frac{1}{m!} \int_{\mathbb{R}^m} \prod_{j=1}^{m} w_{\text{ex}}(p_j) \alpha^{(m)}(d(p_1, \ldots, p_m)) \leq \frac{b}{1 - 2b} \left( \exp\{a \lambda E|\Xi_0|\} - 1 \right), \tag{2.11}
\]
which immediately implies [24]. If \( \Psi \sim P \) is strongly \( L_q \)-Brillinger-mixing for some \( q > 1 \) such that \( b_q < \frac{1}{q} (E|\Xi_0|)^{\frac{1}{q} - 1} \), then the estimate [2.11] remains valid with \( a \) and \( b \) replaced by \( a_q (E|\Xi_0|)^{\frac{1}{q} - 1} \) and \( b_q (E|\Xi_0|)^{1 - \frac{1}{q}} \), respectively.

Proof. Using the representation [2.10], we obtain 
\[
\frac{1}{m!} \int_{\mathbb{R}^m} \prod_{j=1}^{m} w_{\text{ex}}(p_j) \alpha^{(m)}(d(p_1, \ldots, p_m))
\]
\[
= \frac{1}{m!} \sum_{\ell=1}^{m} \sum_{K_1 \cup \ldots \cup K_\ell = \{1, \ldots, m\}} \prod_{i=1}^{\ell} \int_{\mathbb{R}^{|K_j|}} \prod_{i \in K_j} w_{\text{ex}}(p_i) \gamma^{(\#K_j)}(d(p_i : i \in K_j))
\]

where

\[ f(k) := \int_{\mathbb{R}}^{k} \prod_{i=1}^{k} w_{\varphi}(p) \gamma^{(k)}(d(p_1, \ldots, p_k)) = \lambda \int_{\mathbb{R}}^{k} w_{\varphi}(p) \int_{\mathbb{R}^{k-1}}^{k} \prod_{i=2}^{k} w_{\varphi}(p_1 + p_i) \gamma^{(k)}_{\text{red}}(d(p_2, \ldots, p_k)) dp_1 \]

for \( k = 1, \ldots, m \). The equality (2.12) is justified by the invariance of \( \gamma^{(k)}(\times_{i=1}^{k} B_i) \) against permutations of the bounded sets \( B_1, \ldots, B_k \in \mathcal{B}^1 \) for each \( k \in \mathbb{N} \). We proceed with

\[ |f(k)| \leq \lambda \int_{\mathbb{R}}^{k} w_{\varphi}(p) \int_{\mathbb{R}^{k-1}}^{k} |\gamma^{(k)}_{\text{red}}(d(p_2, \ldots, p_k))| dp_1 = \lambda E|\Xi_0| \|\gamma^{(k)}_{\text{red}}\|_{TV} \leq a \lambda E|\Xi_0| b^k k!. \]

Here, we have used Fubini’s theorem combined with \( w_{\varphi}(p) \leq 1 \) for \( p \in \mathbb{R}^k \) and \( x \in \mathbb{R}^2 \) so that

\[ \int_{\mathbb{R}^k} w_{\varphi}(p) dp = \int_{\mathbb{R}^1} P(p \in \Xi_0 + g(v(\Phi_0), x)) dp = \int_{\mathbb{R}^1} P(p \in \Xi_0) dp = E|\Xi_0|_1. \]

Hence, together with the combinatorial relations

\[ \sum_{k_1 + \ldots + k_{\ell} = m \atop k_1 \geq 1, i = 1, \ldots, \ell} 1 = \binom{m-1}{\ell-1} \quad \text{and} \quad \sum_{\ell=1}^{m} \binom{m-1}{\ell-1} = 2^{m-1} \]

we arrive at

\[ \frac{1}{\ell!} \sum_{k_1 + \ldots + k_{\ell} = m \atop k_1 \geq 1, i = 1, \ldots, \ell} \prod_{i=1}^{\ell} \frac{|f(k_i)|}{k_i!} \leq b^m \sum_{\ell=1}^{m} \frac{(a \lambda E|\Xi_0|_1)^{\ell}}{\ell!} \left( \binom{m-1}{\ell-1} \right) \leq b^m 2^{m-1} \max_{1 \leq \ell \leq m} \frac{(a \lambda E|\Xi_0|_1)^{\ell}}{\ell!} \]

\[ \leq \frac{1}{2} \left( \exp\{a \lambda E|\Xi_0|_1\} - 1 \right) (2b)^m. \]  

By combining (2.12) and (2.13) with \( b < 1/2 \) the relation (2.11) follows immediately. Under the strong \( L_q \)–Brillinger-mixing condition we may rewrite \( f(k) \) for \( k \geq 2 \) as follows:

\[ f(k) = \lambda \int_{\mathbb{R}}^{k} w_{\varphi}(p) E \int_{\mathbb{R}^{k-1}}^{k} \prod_{i=2}^{k} 1_{\Xi_i + g(v(\Phi_i), x)}(p_i + p_1) c^{(k)}_{\text{red}}(p_2, \ldots, p_k) d(p_2, \ldots, p_k) dp_1, \]

where \( \Xi_i = [-R_i, R_i] \) and \( (R_2, \Phi_2), \ldots, (R_k, \Phi_k) \) are i.i.d. random vectors with same distribution as \( (R_0, \Phi_0) \). Applying Hölder’s inequality for \( q > 1 \) and \( p = q/(q-1) \), Lyapunov’s inequality \( E|\Xi_0|^{\frac{1}{p}} \leq (E|\Xi_0|)^{\frac{1}{q}} = (E|\Xi_0|)^{\frac{1}{q}} - \frac{1}{q} \) and the condition \( ||c^{(k)}_{\text{red}}||_q \leq a_q b_q^k \), we obtain that

\[ |f(k)| \leq \lambda ||c^{(k)}_{\text{red}}||_q E|\Xi_0|_1 \prod_{i=2}^{k} E|\Xi_i|_1^{\frac{1}{q}} \leq \lambda ||c^{(k)}_{\text{red}}||_q (E|\Xi_0|_1)^{1+\frac{k-1}{p}} \leq \lambda a_q (E|\Xi_0|_1)^{\frac{1}{q}} (b_q (E|\Xi_0|_1)^{1-\frac{1}{q}} k!. \]

By repeating the foregoing steps with the latter bound the proof of Lemma 3 is finished. \( \square \)
Lemma 4. Let \( \Psi \sim P \) be a stationary point process on \( \mathbb{R}^1 \) satisfying \( \max_{2 \leq k \leq m} \| \gamma_r^{(k)} \|_{TV} < \infty \) for some fixed \( m \geq 2 \). If \( E R_0 < \infty \) and \( \Phi_0 \sim G \) has a continuous distribution function \( G \) then, for \( m \geq 2 \) not necessarily distinct point \( x_1, \ldots, x_m \in \mathbb{R}^2 \setminus \{0\} \),
\[
\int \prod_{j=1}^m w_{\phi_j}(p_j) \alpha(m)(d(p_1, \ldots, p_m)) \xrightarrow{g \rightarrow \infty} \lambda^m \prod_{j=1}^m \int w_{\phi_j}(p) \, dp = (\lambda E|\Xi_0|)^m. \tag{2.14}
\]

Proof (Lemma 4). We use the representation (2.10) for \( k = m \) to rewrite the difference of left-hand and right-hand side of (2.14) as follows:
\[
\sum_{\ell=1}^{m-1} \sum_{K_j=1}^{K_1} \prod_{i=1}^{\ell} \int \prod_{i \in K_j} w_{\phi_i}(p_i) \gamma^{(\ell)}(d(p_i : i \in K_j)).
\]
Hence, the limit (2.11) is shown if and only if the finite sum in the latter line disappears as \( g \rightarrow \infty \) and this in turn follows by showing that, for \( k = 2, \ldots, m \),
\[
\int \prod_{i=1}^k w_{\phi_i}(p_i) \gamma^{(k)}(d(p_1, \ldots, p_k)) = \lambda \int w_{\phi_1}(p_1) \prod_{i=2}^k w_{\phi_i}(p_i + p_1) \gamma_r^{(k)}(d(p_2, \ldots, p_k)) \, dp_1 \xrightarrow{g \rightarrow \infty} 0.
\]
In view of \( 0 \leq w_{\phi_i}(p_i + p_1) \leq 1 \) for \( i = 3, \ldots, k \) it is sufficient to prove
\[
\int \int P(p_1 \in \Xi_0 + g(\phi(\Phi_0), x_1)) \, P(p_1 \in \Xi_0 + g(\phi(\Phi_0), x_2) - p_2) \, dp_1 \xrightarrow{g \rightarrow \infty} 0.
\]
Since the total variation measure \( \| \gamma_r^{(k)}(\cdot) \| \) is bounded on \( \mathbb{R}^{k-1} \) and the inner integral over \( \mathbb{R}^1 \) is less than or equal to \( E|\Xi_0| \), we have only to verify that the inner integral disappears as \( g \rightarrow \infty \). For this purpose, we rewrite its integrand as expectation \( E \mathbf{1}_{\Xi_0 + g(\phi(\Phi_1), x_1)}(p_1) \mathbf{1}_{\Xi_0 + g(\phi(\Phi_2), x_2) - p_2}(p_1) \), where \( \Xi_i = [-R_i, R_i] \) and \( \Phi_i \) for \( i = 1, 2 \) have the same distribution as \( \Xi_0 = [-R_0, R_0] \) and \( \Phi_0 \), respectively, and \( R_1, R_2, \Phi_1, \Phi_2 \) are independent of each other. By Fubini’s theorem and the shift-invariance of the Lebesgue measure, we arrive at
\[
\int \mathbf{1}_{\Xi_0 + g(\phi(\Phi_1), x_1)}(p_1) \mathbf{1}_{\Xi_0 + g(\phi(\Phi_2), x_2) - p_2}(p_1) \, dp_1 = E \mathbf{1}_{\Xi_1 \cap \Xi_2 - p_2 + g(\phi(\Phi_2), x_2) - g(\phi(\Phi_1), x_1))} \bigg| \xrightarrow{g \rightarrow \infty} 0.
\]

The limit in the last line can verified as follows: We fix two points \( x_i = \| x_i \| (\cos(\alpha_i), \sin(\alpha_i)) \in \mathbb{R}^2, i = 1, 2 \), and two points \( v(\varphi_i) = (\cos(\varphi_i), \sin(\varphi_i)) \), \( i = 1, 2 \), on the unit circle line. It is easily seen that the equation \( \langle v(\varphi_1), x_1 \rangle = \langle v(\varphi_2), x_2 \rangle \), i.e. \( \| x_1 \| \cos(\varphi_1 - \alpha_1) = \| x_2 \| \cos(\varphi_2 - \alpha_2) \) holds for at most a finite number of pairs \( \varphi_1, \varphi_2 \in [0, \pi] \). Hence, for two independent random angles \( \Phi_1, \Phi_2 \) with common atomless distribution function \( G(\cdot) \) we have
\[
P(\langle v(\Phi_1), x_1 \rangle \neq \langle v(\Phi_2), x_2 \rangle) = 1 \quad \text{for any two points} \quad x_1, x_2 \in \mathbb{R}^2 \quad \text{with} \quad \| x_1 \| + \| x_2 \| > 0.
\]
From Lemma 4 and (2.8) we obtain the behaviour of the expectation of $|\Xi \cap qK|_2$ as $q \to \infty$.

**Corollary 2.** Let $\Psi \sim P$ be a Brillinger-mixing point process $\mathbb{R}^1$. If $E R_0 < \infty$ and $\Phi_0 \sim G$ has a continuous distribution function $G$ then

$$E|\Xi \cap qK|_2 = \frac{1}{|qK|_2} \int_\mathbb{K} (1 - G_P[1 - w_{ex}(\cdot)])dx \xrightarrow{e \to \infty} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(\lambda E|\Xi_0|_1)^k}{k!}$$

Proof (Corollary 2). An application of (2.14) for $x_1 = \cdots = x_m = x \neq o$ to the inequality (2.8) yields

$$\left| \lim_{e \to \infty} (1 - G_P[1 - w_{ex}(\cdot)]) - \sum_{k=1}^{m-1} \frac{(-1)^{k-1}(\lambda E|\Xi_0|_1)^k}{k!} \right| \leq \frac{(\lambda E|\Xi_0|_1)^m}{m!}$$

for any $m \geq 1$. (2.15)

Combining this with (2.7) leads to

$$E|\Xi \cap qK|_2 = \frac{1}{|qK|_2} \int_\mathbb{K} (1 - G_P[1 - w_{ex}(\cdot)])dx \xrightarrow{e \to \infty} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(\lambda E|\Xi_0|_1)^k}{k!}$$

which immediately gives the assertion of Corollary 2.

### 3 Main Results

The first result can be considered as a planar mean-square ergodic theorem which implies a weak law of large numbers for $\Xi \cap qK$ in the Euclidean plane $\mathbb{R}^2$.

**Theorem 1.** Assume that the stationary point process $\Psi \sim P$ on $\mathbb{R}^1$ is Brillinger-mixing. Further suppose that $E R_0 < \infty$ and $\Phi_0 \sim G$ has a continuous distribution function $G$. Then

$$\frac{E|\Xi \cap qK|_2}{|qK|_2} - \left(1 - \exp\{-\lambda E|\Xi_0|_1\}\right)^2 \xrightarrow{e \to \infty} 0 \quad \text{with} \quad \Xi_0 := [-R_0, R_0]. \quad (3.1)$$

Our second result provides the exact asymptotic behavior of the variance of the area of the cylinder process (1.1) that is contained in a star-shaped set $qK$ which is growing unboundedly in all directions. For this purpose, in comparison with Theorem 1, we need a strengthening and quantification of the classical Brillinger-mixing condition.

**Theorem 2.** Assume that the stationary point process $\Psi \sim P$ on $\mathbb{R}^1$ is either strongly Brillinger-mixing with $b < 1/2$ or strongly $L_q$-Brillinger-mixing with $(E|\Xi_0|_1)^{1-\frac{1}{q}} b_q < 1/2$ and strongly $L_q^*$-Brillinger-mixing with $(E|\Xi_0|_1)^{1-\frac{1}{q}} b_q^* < 1/2$ for some $q > 1$, where $\Xi_0 := [-R_0, R_0]$. Further suppose that $E R_0^2 < \infty$ and $\Phi_0 \sim G$ has a continuous distribution function $G$. Then
\[
\lim_{\theta \to \infty} \text{Var}\left(\frac{\Xi \cap gK}{g^3}\right) = \lambda e^{-2\lambda \mathbf{E}|\Xi_0|} \left( \left( \mathbf{E}|\Xi_0|_1 \right)^2 \gamma_{\text{red}}^{(2)}(\mathbb{R}^1) C_1^{G,K} + 2 \mathbf{E}|\Xi_0|^2 C_2^{G,K} \right),
\]

where

\[
C_1^{G,K} := \int_{\mathbb{R}^1} (\mathbf{E}|g(p, \Phi_0) \cap K|_1)^2 \, dp \quad \text{and} \quad C_2^{G,K} := \int_0^\pi \int_0 \sum s v(\varphi \pm \pi/2) |2| \, ds \, dG(\varphi).
\]

Remark. In the special case \( K = b(o, 1) \), one can show that \( C_1^{G,K} = \frac{16}{3} \) and \( C_2^{G,K} = \frac{8}{3} \) are independent of the distribution function \( G \). If \( \Phi_0 \) is uniformly distributed on \([0, \pi]\), then we get

\[
C_1^{G,K} = \frac{1}{\pi^2} \int_{-\infty}^\infty \left( \int_0^\pi |g(p, \varphi) \cap K|_1 \, d\varphi \right)^2 \, dp,
\]

\[
C_2^{G,K} = \frac{1}{2\pi} \int_0^{2\pi} \int_0 |K \cap (K + s \varphi \pm \pi/2)| \, ds \, d\varphi = \frac{1}{2\pi} \int_{\mathbb{R}^2} |K \cap (K + x)| \, dx \frac{dx}{\|x\|} = \frac{1}{2\pi} \int_K \int_K \frac{dx \, dy}{\|x - y\|}.
\]

The latter double integral is known as second-order chord power integral of \( K \), see e.g. \cite{12}, p. 327, and \cite{23}, Chapt. 7, for integral geometric background.

The following two lemmas are essential for the calculation of the right-hand side of (3.2). Interestingly, the assumptions to prove these lemmas are rather mild in comparison with the Brillinger-mixing-type conditions in the Theorems 1 and 2.

Lemma 5. Let \( \Psi \sim P \) be a second-order stationary point process on \( \mathbb{R}^1 \) satisfying \( \|\gamma_{\text{red}}^{(2)}\|_{TV} < \infty \). Further, suppose that \( \mathbf{E}R_0 < \infty \) and \( \Phi_0 \sim G \) with a not necessarily continuous distribution function \( G \). Then

\[
\theta \int_K \int_K w_{\varphi_1}(p_1) w_{\varphi_2}(p_2) \gamma^{(2)}(d(p_1, p_2)) \, dx_1 \, dx_2 \quad \to \quad \int_{\mathbb{R}^1} \lambda (\mathbf{E}|\Xi_0|_1)^2 \gamma_{\text{red}}^{(2)}(\mathbb{R}^1) \left( \mathbf{E}|g(p, \Phi_0) \cap K|_1 \right)^2 \, dp.
\]
Lemma 6. Assume that $E R_0^2 < \infty$ and $\Phi_0 \sim G$ with a not necessarily continuous distribution function $G$. Then

$$
\varrho \int \int \int \frac{w_{\varrho x_1, \varrho x_2}(p)}{R^2} \varrho \mathrm{d}x_1 \mathrm{d}x_2 \int_0^\infty \int_0^\pi \varrho K (\varphi \pm |\varrho|) \mathrm{d}s \mathrm{d}G(\varphi)
$$

with $r_K(\psi) := \max\{r \geq 0 : rv(\psi) \in K \oplus (-K)\}$. Obviously, it holds $r_K(\psi) = r_K(\psi \pm \pi)$.

Remark. Note that in Theorem 1 and 2 the interval $\Xi_0 := [-R_0, R_0]$ with $E R_0^k < \infty$ can be replaced by a finite union of random closed intervals $\Xi_0 \subset \mathbb{R}^1$ satisfying $\inf \Xi_0 \leq 0 \leq \sup \Xi_0$ and $E|\Xi_0|^k < \infty$ for $k = 1$ or $k = 2$, respectively. This restriction is based on the definition of a process of cylinders with non-convex bases, see e.g. [24]. In Lemma 5 and 6 the cross section (or base) $\Xi_0$ of the typical cylinder can be chosen as random compact set satisfying $0 < E|\Xi_0|^1 < \infty$ or $E|\Xi_0|^2 < \infty$, respectively.

4 Proofs of the Main Results

Proof (Theorem 1). The expectation on the left-hand side of (3.1) can be expressed as follows:

$$
\frac{\text{Var}(|\Xi \cap \varrho K|^2)}{|\varrho K|^2} + \left( \frac{E|\Xi \cap \varrho K|^2}{|\varrho K|^2} - \left(1 - \exp\{-\lambda E|\Xi_0|^1\}\right)^2 \right).
$$

In view of Corollary 2 it remains to prove that $\varrho^{-4} \text{Var}(|\Xi \cap \varrho K|^2) \int_0^\infty$. Using Lemma 2 and the notation introduced in Section 2 we get

$$
\varrho^{-4} \text{Var}(|\Xi \cap \varrho K|^2) = \varrho^{-4} \int \int \int \left( G_P[1 - w_{x_1,x_2}^j] - \prod_{i=1}^2 G_P[1 - w_{x_i}^j]\right) \mathrm{d}x_1 \mathrm{d}x_2
$$

$$
= \int \int \left( G_P[1 - w_{x_1,x_2}^j] - G_P[1 - w_{x_1}^j] \right) \mathrm{d}x_1 \mathrm{d}x_2.
$$

Thus, we just have to show that the integrand disappears as $\varrho \to \infty$ for distinct points $x_1, x_2 \in K \setminus \{o\}$, that is,

$$
G_P[1 - w_{x_1,x_2}^j] - G_P[1 - w_{x_1}^j] \to 0.
$$

(4.1)

We make use of the finite expansion (2.8) of the pgf $G_P[1 - w_{x_1}^j]$ with remainder term, where $w_{x_1}$ can be replaced by any Borel-measurable function $w : \mathbb{R}^1 \to [0,1]$. For brevity, we put

$$
S_m(w) := \sum_{k=0}^{m-1} (-1)^k T_k(w) \text{ with } T_0(w) := 1 \text{ and } T_k(w) := \frac{1}{k!} \int \prod_{j=1}^k w(p_j) \alpha(k)(d(p_1, \ldots, p_k)).
$$
for $1 \leq k \leq m \in \mathbb{N}$. Hence, (2.12) reads as $|G_P[1 - w(\cdot)] - S_m(w)| \leq T_m(w)$ which leads us to the following estimate

$$\left| G_P[1 - w^{ij}_{\bar{g}x_1, \bar{g}x_2}(\cdot)] - G_P[1 - w_{\bar{g}x_1}(\cdot)] G_P[1 - w_{\bar{g}x_2}(\cdot)] - \left( S_m(w^{ij}_{\bar{g}x_1, \bar{g}x_2}) - S_m(w_{\bar{g}x_1}) S_m(w_{\bar{g}x_2}) \right) \right|$$

$$\leq \left| G_P[1 - w^{ij}_{\bar{g}x_1, \bar{g}x_2}(\cdot)] - S_m(w^{ij}_{\bar{g}x_1, \bar{g}x_2}) \right| + \left| G_P[1 - w_{\bar{g}x_1}(\cdot)] - S_m(w_{\bar{g}x_1}) \right| \left| G_P[1 - w_{\bar{g}x_2}(\cdot)] - S_m(w_{\bar{g}x_2}) \right|$$

$$\leq T_m(w^{ij}_{\bar{g}x_1, \bar{g}x_2}) + T_m(w_{\bar{g}x_1}) + T_m(w_{\bar{g}x_2}) + T_m(w_{\bar{g}x_1}) T_m(w_{\bar{g}x_2}) \quad \text{for } m \geq 2. \quad (4.2)$$

Here, we have additionally used that $G_P[1 - w(\cdot)] \leq 1$ and $|S_m(w)| \leq G_P[1 - w(\cdot)] + T_m(w)$. We are now in a position to apply the limit (2.12) under the assumptions of Lemma 4. This yields for $i = 1, 2$ and $m \in \mathbb{N}$

$$T_m(w_{\bar{g}x_i}) \xrightarrow{\theta \to \infty} \frac{(\Lambda E|\Xi_0|_1)^m}{m!} \quad \text{and} \quad S_m(w_{\bar{g}x_i}) \xrightarrow{\theta \to \infty} \sum_{k=0}^{m-1} \frac{(-\Lambda E|\Xi_0|_1)^k}{k!} = e^{-\Lambda E|\Xi_0|_1} + \theta_1 \frac{(\Lambda E|\Xi_0|_1)^m}{m!}$$

for some $\theta_1 \in [-1, 1]$ in accordance with $e^{-x} - \sum_{k=0}^{m-1} \frac{(-x)^k}{k!} \leq \frac{\theta_1}{m!}$ for any $m \in \mathbb{N}$ and $x \geq 0$.

Next, we have to find the limit of $T_m(w^{ij}_{\bar{g}x_1, \bar{g}x_2})$ as $\theta \to \infty$. Using the relation $w^{ij}_{\bar{g}x_1, \bar{g}x_2}(p) = w_{\bar{g}x_1}(p) + w_{\bar{g}x_2}(p) - u^{\bar{g}x_1, \bar{g}x_2}(p)$ and taking into account that the factorial moment measure $\alpha^{(m)}$ is invariant under permutation of its $m$ components, we may write

$$T_m(w^{ij}_{\bar{g}x_1, \bar{g}x_2}) = \frac{1}{m!} \int_{\mathbb{R}^m} \prod_{j=1}^{m} \left( w_{\bar{g}x_1}(p_j) + w_{\bar{g}x_2}(p_j) - w^{\bar{g}x_1, \bar{g}x_2}(p_j) \right) \alpha^{(m)}(dp_1, \ldots, dp_m)$$

$$= \frac{1}{m!} \int_{\mathbb{R}^m} \prod_{j=1}^{m} \left( w_{\bar{g}x_1}(p_j) + w_{\bar{g}x_2}(p_j) \right) \alpha^{(m)}(dp_1, \ldots, dp_m)$$

$$+ \frac{1}{m!} \sum_{\ell=1}^{m} \left( m \right)_\ell \int_{\mathbb{R}^m} \prod_{i=1}^{\ell} w^{\bar{g}x_1, \bar{g}x_2}(p_i) \prod_{j=\ell+1}^{m} \left( w_{\bar{g}x_1}(p_j) + w_{\bar{g}x_2}(p_j) \right) \alpha^{(m)}(dp_1, \ldots, dp_m). \quad (4.3)$$

There is at least one term $w^{\bar{g}x_1, \bar{g}x_2}(p_i) = P(p_i \in (\Xi_0 + \varrho(v(\Phi_0), x_1)) \cap (\Xi_0 + \varrho(v(\Phi_0), x_2)))$ in each summand of the last line which will be integrated over $\mathbb{R}^1$ w.r.t. $dp_i$ so that after expressing $\alpha^{(m)}$ by cumulant measures, see (2.10), the expectation $E[\Xi_0 \cap (\Xi_0 + \varrho(v(\Phi_0), x_2 - x_1))|1]$ emerges and disappears as $\varrho \to \infty$ if $x_1 \neq x_2$. Thus, the last line disappears completely as $\varrho \to \infty$, whereas the line (4.3) converges to the limit $(2\Lambda E|\Xi_0|_1)^m/m!$ as $\varrho \to \infty$ by applying the limit (2.13) once more. Therefore, we obtain for any $m \in \mathbb{N}$ that $T_m(w^{ij}_{\bar{g}x_1, \bar{g}x_2}) \xrightarrow{\theta \to \infty} (2\Lambda E|\Xi_0|_1)^m/m!$ and

$$S_m(w^{ij}_{\bar{g}x_1, \bar{g}x_2}) \xrightarrow{\theta \to \infty} \sum_{k=0}^{m-1} \frac{(2\Lambda E|\Xi_0|_1)^k}{k!} = e^{-2\Lambda E|\Xi_0|_1} + \theta_2 \frac{(2\Lambda E|\Xi_0|_1)^m}{m!} \quad \text{for some } \theta_2 \in [-1, 1].$$

The latter limit combined with above limits of $S_m(w_{\bar{g}x_i})$ for $i = 1, 2$ leads to

$$\lim_{\theta \to \infty} |S_m(w^{ij}_{\bar{g}x_1, \bar{g}x_2}) - S_m(w_{\bar{g}x_1}) S_m(w_{\bar{g}x_2})| \leq \frac{(2\Lambda E|\Xi_0|_1)^m}{m!} + 2 \frac{(\Lambda E|\Xi_0|_1)^m}{m!} + \frac{(\Lambda E|\Xi_0|_1)^{2m}}{(m!)^2}.$$
For any given $\varepsilon \in (0, 1]$ we find some $m(\varepsilon)$ such that $\frac{(2 \lambda \text{E}(\Xi_2))_m}{m!} \leq \varepsilon$ for all $m \geq m(\varepsilon)$.

Thus, the right-hand side of the last inequality does not exceed $2\varepsilon + \varepsilon^2$ for sufficiently large $m$. The same bound can be obtained for the limit (as $\varrho \to \infty$) of the four summands in line (4.2).

Finally, after summarizing all $\varepsilon$-bounds of the above limiting terms we arrive at

$$\lim_{\varrho \to \infty} \left| G_P[1 - w^{ij}_{\varrho x_1, \varrho x_2}(\cdot)] - G_P[1 - w_{\varrho x_1}(\cdot)] G_P[1 - w_{\varrho x_2}(\cdot)] \right| \leq 2(2\varepsilon + \varepsilon^2) \leq 6\varepsilon.$$

This implies (4.1) completing the proof of Theorem 1.

Proof (Lemma 5). By the stationarity of $\Psi \sim P$ we may write $\gamma^{(2)}(d(p_1, p_2)) = \lambda \gamma^{(2)}(dp_2 - p_1) dp_1$ which gives

$$\varrho \int \int \int \frac{w_{\varrho x}(p_1) w_{\varrho y}(p_2)}{p_1} dx dy \gamma^{(2)}(d(p_1, p_2)) = \varrho \lambda \int \int \int \frac{w_{\varrho x}(p_1) w_{\varrho y}(p_2 + p_1)}{p_1} dx dy \gamma^{(2)}(dp_2 dp_1)$$

To determine the limit of the right hand side as $\varrho \to \infty$, we rewrite the probabilities $w_{\varrho x}(p_1) = \mathbb{P}(p_1 \in \{ \cdots \})$ and $w_{\varrho y}(p_2 + p_1) = \mathbb{P}(p_2 + p_1 \in \{ \cdots \})$ by means of the expectation (as integral over the product of probability measures) over the corresponding indicator function $\mathbb{I}_{\{ \cdots \}}$. We fix $\Xi_i = \xi_i$ (compact sets in $\mathbb{R}^1$) and $\Phi_i = \varphi_i$ (angles in [0, $\pi$]) for $i = 1, 2$ and omit the expectation which stands in front of all other integrals due to Fubini’s theorem. The intensity $\lambda$ will be suppressed. Further, we write $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Thus, we only treat the integral

$$\varrho \int \int \int \frac{1_{\xi_1 + \varrho(x_1 \cos \varphi_1 + x_2 \sin \varphi_1)}(p_1) 1_{\xi_2 + \varrho(y_1 \cos \varphi_2 + y_2 \sin \varphi_2)}(p_2 + p_1)}{p_1} dx_1 dx_2 dy_1 dy_2 \gamma^{(2)}(dp_2 dp_1)$$

$$= \varrho \int \int \int 1_K(x_1, x_2) 1_K(y_1, y_2) 1_{\xi_1 + \varrho(x_1 \cos \varphi_1 + x_2 \sin \varphi_1)}(p_1) 1_{\xi_2 + \varrho(y_1 \cos \varphi_2 + y_2 \sin \varphi_2)}(p_2 + p_1)$$

$$\times \frac{d(x_1, x_2) d(y_1, y_2) \gamma^{(2)}(dp_2 dp_1)}{p_1} = : J_{\varrho}(K, \xi_1, \varphi_1, \xi_2, \varphi_2). \quad (4.4)$$

Now, we substitute $(x_1, x_2)^T = O(\varphi_1)(u_1, u_2)^T, (y_1, y_2)^T = O(\varphi_2)(v_1, v_2)^T$, where $O(\varphi_1)$ and $O(\varphi_2)$ are defined by (1.5). Then $x_1 = u_1 \cos \varphi_1 - u_2 \sin \varphi_1, x_2 = u_1 \sin \varphi_1 + u_2 \cos \varphi_1$ and $y_1 = v_1 \cos \varphi_2 - v_2 \sin \varphi_2, y_2 = v_1 \sin \varphi_2 + v_2 \cos \varphi_2$. Hence, since $O(\varphi_i)^{-1} = O(-\varphi_i)$ for $i = 1, 2$, the integral $J_{\varrho}(K, \xi_1, \varphi_1, \xi_2, \varphi_2)$ in (4.4) takes on the form

$$\varrho \int \int \int 1_{O(\varphi_1)K}(u_1, u_2) 1_{O(\varphi_2)K}(v_1, v_2) 1_{\xi_1 + \varrho u_1}(p_1) 1_{\xi_2 + \varrho u_2}(p_2 + p_1) d(u_1, u_2) d(v_1, v_2)$$

$$\times \gamma^{(2)}(dp_2 dp_1)$$

$$= \varrho \int \int \int 1_{O(\varphi_1)K}(u_1, u_2) 1_{O(\varphi_2)K}(v_1, v_2) 1_{\xi_1 + \varrho(u_1 - v_1)}(p_1) 1_{\xi_2 + \varrho u_2}(p_2 + p_1) d(u_1, u_2) d(v_1, v_2)$$

$$\times \gamma^{(2)}(dp_2 dp_1).$$
It is easy to see that the invariance properties of the one-dimensional Hausdorff measure on \( \mathbb{R}^2 \) (also denoted by \( | \cdot |_1 \) ) yield

\[
\int_{\mathbb{R}^1} 1_{O(-\varphi_1)K}(u_1, u_2) \, du_2 = |g(u_1, 0) \cap O(-\varphi_1)K|_1 = |O(\varphi_1)g(u_1, 0) \cap K|_1 = |g(u_1, \varphi_1) \cap K|_1.
\]

and likewise \( \int_{\mathbb{R}^1} 1_{O(-\varphi_2)K}(v_1, v_2) \, dv_2 = |g(v_1, \varphi_2) \cap K|_1 \).

Therefore, the integral \( J_e(K, \xi_1, \varphi_1, \xi_2, \varphi_2) \) is equal to

\[
\varrho \int_{\mathbb{R}^2} \int_{\mathbb{R}^1} |g(u, \varphi_1) \cap K|_1 |g(v, \varphi_2) \cap K|_1 1_{\xi_1 + g(u-v)}(p_1) 1_{\xi_2}(p_2 + p_1) \, du \, dv \, \gamma^{(2)}_{red}(dp_2) \, dp_1
\]

\[
= \varrho \int_{\mathbb{R}^2} \int_{\mathbb{R}^1} |g(w + v, \varphi_1) \cap K|_1 |g(v, \varphi_2) \cap K|_1 1_{\xi_1 + \varrho w}(p_1) 1_{\xi_2}(p_2 + p_1) \, dw \, dv \, \gamma^{(2)}_{red}(dp_2) \, dp_1
\]

\[
= \lim_{\varrho \to \infty} \int_{\mathbb{R}^2} \int_{\mathbb{R}^1} |g(v, \varphi_1) \cap K|_1 |g(v, \varphi_2) \cap K|_1 1_{-\xi_1 + \varrho v}(w) 1_{\xi_2}(p_2 + p_1) \, dw \, dv \, \gamma^{(2)}_{red}(dp_2) \, dp_1
\]

\[
= |\xi_1|_1 |\xi_2|_1 \gamma^{(2)}_{red}(\mathbb{R}^1) \int_{\mathbb{R}^1} 1_{[\ell(\varphi_1, K), r(\varphi_1, K)]}(v) |g(v, \varphi_1) \cap K|_1 1_{[\ell(\varphi_2, K), r(\varphi_2, K)]}(v) |g(v, \varphi_2) \cap K|_1 \, dv,
\]

where the interval \([\ell(\varphi_i, K), r(\varphi_i, K)] = \{ v \in \mathbb{R}^1 : g(v, \varphi_i) \cap K \neq \emptyset \} \) coincides with the orthogonal projection of \( O(-\varphi_1)K \) on the \( v \)-axis for \( i = 1, 2 \). To justify the above limit we have used that \( |g(w/\varrho + v, \varphi_1) \cap K|_1 \leq \text{diam}(K) \) so that Lebesgue’s dominated convergence theorem can be applied. Furthermore, it is easily seen that

\[
|J_e(K, \xi_1, \varphi_1, \xi_2, \varphi_2)| \leq \text{diam}(K) |K|_2 |\xi_1|_1 |\xi_2|_1 \|\gamma^{(2)}_{red}\|_{TV}.
\]

Hence, the limit of \( \mathbb{E}J_e(K, \Xi_1, \Phi_1, \Xi_2, \Phi_2) \) as \( \varrho \to \infty \), exists and can be expressed by using the independence assumptions as follows:

\[
\lambda(\mathbb{E}1_{|\ell(\Phi_0, K), r(\Phi_0, K)]}(v) |g(v, \Phi_0) \cap K|_1)^2 \, dv.
\]

Note that the indicator function \( 1_{[\ell(\Phi_0, K), r(\Phi_0, K)]}(\cdot) \) can be omitted since the range of integration w.r.t. \( v \) is well-defined.

**Proof (Lemma 5).** With the abbreviation \( \Xi_0 = [-R_0, R_0] \) we obtain that

\[
J_e(K) := \varrho \int_K \int_K w_{ex, ey}(p) \, dp \, dx = \varrho \int_K \int_K P(p \in \Xi_0 \cap (\Xi_0 + g(v(\Phi_0), y - x))) \, dp \, dx \, dy
\]
where we have substituted $y = s v(\psi)$ with $v(\psi) = (\cos \psi, \sin \psi)^T$ and with $r_K(\psi) = \max\{s \geq 0 : s v(\psi) \in K \oplus (-K)\} \ (= r_K(\psi \pm \pi) \text{ due to symmetry reasons})$

$$= \varrho \int_{-\Phi_0}^{2\pi r_K(\psi + \Phi_0)} \int_{0}^{2\pi r_K(\psi)} |K \cap (K - s v(\psi + \Phi_0))|_2 \mathbf{E}[\Xi_0 \cap (\Xi_0 + \varrho s \cos(\Phi_0 - \psi))]_1 s \, ds \, d\psi,$$

where we have used $\int_{-\Phi_0}^{\Phi_0} (\cdots) \, d\psi = \int_{-\Phi_0}^{2\pi - \Phi_0} (\cdots) \, d\psi$ due to $v(\psi) = v(\psi + 2\pi)$

$$= 2 \varrho \mathbf{E} \int_{-\Phi_0}^{\Phi_0} \int_{0}^{\pi} |K \cap (K + s v(\psi + \Phi_0))|_2 \mathbf{E}[\Xi_0 \cap (\Xi_0 + \varrho s \cos(\psi))]_1 s \, ds \, d\psi,$$

where we have used $v(\psi + \pi) = -v(\psi)$ and the shift-invariance of $| \cdot |_1$ as well as the motion-invariance of $| \cdot |_2$.

By definition of $r_K(\psi)$ we have $s > r_K(\psi)$ iff $s v(\psi) \notin K \oplus (-K)$ iff $K \cap (K + s v(\psi)) = \emptyset$. Thus, the inner integral $\int_{0}^{r_K(\psi + \Phi_0)}$ in the above double integral can be replaced by $\int_{0}^{\infty}$ showing that

$$J_{\varrho}(K) = 2 \varrho \mathbf{E} \int_{-\Phi_0}^{\Phi_0} \int_{0}^{\pi} |K \cap (K + s v(\psi + \Phi_0))|_2 \mathbf{E}[\Xi_0 \cap (\Xi_0 + \varrho s \cos(\psi))]_1 s \, ds \, d\psi$$

$$= 2 \varrho \int_{1}^{\infty} \mathbf{E} \int_{0}^{1} |K \cap (K + s v(\arccos(y) + \Phi_0))|_2 \mathbf{E}[\Xi_0 \cap (\Xi_0 + \varrho s y)]_1 s \, ds \, \frac{(-1) \, dy}{\sqrt{1 - y^2}}$$

by substituting $y = \cos(\psi) \in [-1, 1]$ so that $\psi = \arccos(y)$, $(\arccos(y))' = -\frac{1}{\sqrt{1 - y^2}}$. 

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Interchanging again the integration over $J_4 \leq \|z\|_{\mathbb{R}}$ where $0$, further we have used the continuity of the function $z_{sv}$ combined with a multiple application of Fubini’s theorem. Finally, we arrive at

$$J_\Phi(K) = 2 \int_0^1 \int_{|u| \leq \varphi} |K \cap (K + h(s, z, \Phi_0))|_2 E[\Xi_0 \cap (\Xi_0 + s)]_1 \frac{s \, ds \, dy}{\sqrt{1 - y^2}}$$

by substituting $z = y \in [-s, s]$ so that $y = z/s$ and changing the order of integration.

In the last line, we could apply Lebesgue’s dominated convergence theorem since

$$\int_{|u| \leq \varphi} |K \cap (K + h(s, u, \Phi_0))|_2 E[\Xi_0 \cap (\Xi_0 + u)]_1 \frac{s \, ds \, du}{\sqrt{s^2 - (u/\varphi)^2}} \to 2 \int_0^1 \int_{|u| \leq \varphi} |K \cap (K + s \, v(\Phi_0 + \pi/2))|_2 E[\Xi_0 \cap (\Xi_0 + u)]_1 \, ds \, du.$$

In the last line, we could apply Lebesgue’s dominated convergence theorem since

$$\int_{|u| \leq \varphi} |K \cap (K + h(s, u, \Phi_0))|_2 \frac{s \, ds}{\sqrt{s^2 - (u/\varphi)^2}} \leq |K|_2 \frac{1}{2} \int_0^{r/K - \varphi^2} \frac{dt}{\sqrt{t}} \leq |K|_2 \text{diam}(K). \quad (4.6)$$

Further, we have used the continuity of the function $z \mapsto h(s, z, \varphi)$, arcsin$(0) = \pi/2$ and $h(s, 0, \varphi) = s \, v(\varphi + \pi/2) = s (-\sin \varphi, \cos \varphi)^T$ ($= -s \, v(\varphi - \pi/2)$) and the relation $\int_{\mathbb{R}} |\Xi_0 \cap (\Xi_0 + u)|_1 \, du = |\Xi_0|_1^2 = 4 \, R_0^2$ combined with a multiple application of Fubini’s theorem. Finally, we arrive at

$$J_\Phi(K) = \int_{K \in \mathbb{R}} \int_{K \in \mathbb{R}} \int_{|u| \leq \varphi} w_{\phi x_1, x_2}(p) \, dp \, dx_1 \, dx_2 \to 2 \int_0^1 |K \cap (K + s \, v(\phi \pm \pi/2))|_2 \, ds \, dG(\phi).$$
Proof (Theorem 2). In view of Lemma 2 and Definition 2 we can state the equality
\[ \vartheta^{-3} \text{Var}(| \Xi \cap \varrho K |_2) = \int K \int K \vartheta \left( \left( G_P[1 - w^\uparoondown_{\varrho x_1, \varrho x_2}(\cdot)] - G_P[1 - w_{\varrho x_1}(\cdot)]G_P[1 - w_{\varrho x_2}(\cdot)] \right) \right) dx_1 dx_2. \]

Instead to use the factorial moment expansion of the pgf's \( G_P[1 - w^\uparoondown_{\varrho x_1, \varrho x_2}], G_P[1 - \varrho x_1] \) and \( G_P[1 - \varrho x_2] \) as in (2.7) and (2.8), we first rewrite the integrand of the right-hand side of the foregoing equality as follows:
\[
\vartheta \left( G_P[1 - w^\uparoondown_{\varrho x_1, \varrho x_2}(\cdot)] - G_P[1 - \varrho x_1(\cdot)]G_P[1 - \varrho x_2(\cdot)] \right) = G_P[1 - \varrho x_1(\cdot)]G_P[1 - \varrho x_2(\cdot)]
\times \vartheta \left( \exp \left\{ \log G_P[1 - w^\uparoondown_{\varrho x_1, \varrho x_2}(\cdot)] - \log G_P[1 - \varrho x_1(\cdot)] - \log G_P[1 - \varrho x_2(\cdot)] \right\} - 1 \right). \tag{4.7}
\]

In order to evaluate the exponent in line (4.7) we use an expansion of \( \log G_P[1 - \varrho(\cdot)] \) in terms of the factorial cumulant measures \( \gamma^{(k)} \) of \( \Psi \sim P \), see (2.10), which is as follows:
\[
\log G_P[1 - \varrho(\cdot)] = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{\mathbb{R}^k} \prod_{j=1}^k w(p_j) \gamma^{(k)}(d(p_1, \ldots, p_k)), \quad \text{see [5], p.146}, \tag{4.8}
\]
provided the sum in (4.8) is convergent. In what follows we will show that
\[
\lim_{\varrho \to \infty} \frac{1}{K} \int K \log G_P[1 - w^\uparoondown_{\varrho x_1, \varrho x_2}(\cdot)] - \log G_P[1 - \varrho x_1(\cdot)] - \log G_P[1 - \varrho x_2(\cdot)] \, dx_1 dx_2 < \infty. \tag{4.9}
\]
Before proving this, we note that the relation (2.15) implies that
\[
\lim_{\varrho \to \infty} G_P[1 - \varrho x(\cdot)] = \sum_{k=0}^{m-1} \frac{(-1)^k (\lambda E|\Xi_0|_1)^k}{k!} + \frac{\theta (\lambda E|\Xi_0|_1)^m}{m!} \quad \text{as} \quad \frac{m}{m \to \infty} \exp\{-\lambda E|\Xi_0|_1\}, \tag{4.10}
\]
for some \( \theta \in [-1, 1] \) uniformly for all \( x \neq 0 \). Furthermore, it is rapidly seen that the limit (4.11) (which has been proved under the assumptions of Theorem 1) holds if and only if
\[
\lim_{\varrho \to \infty} \left\{ \log G_P[1 - w^\uparoondown_{\varrho x_1, \varrho x_2}(\cdot)] - \log G_P[1 - \varrho x_1(\cdot)] - \log G_P[1 - \varrho x_2(\cdot)] \right\} = 0
\]
for distinct points \( x_1, x_2 \in K \setminus \{0\} \). Finally, the latter limit combined with (4.9) proves the equality
\[
\lim_{\varrho \to \infty} \frac{1}{K} \int K \left( \exp \left\{ \log G_P[1 - w^\uparoondown_{\varrho x_1, \varrho x_2}(\cdot)] - \log G_P[1 - \varrho x_1(\cdot)] - \log G_P[1 - \varrho x_2(\cdot)] \right\} - 1 \right) dx_1 dx_2
= \lim_{\varrho \to \infty} \frac{1}{K} \int K \log G_P[1 - w^\uparoondown_{\varrho x_1, \varrho x_2}(\cdot)] - \log G_P[1 - \varrho x_1(\cdot)] - \log G_P[1 - \varrho x_2(\cdot)] \, dx_1 dx_2.
\]
The equality of both limits results from the inequality \(|e^x - 1 - x| \leq \frac{x^2}{2} e^{\max(x,0)}\) and Lebesgue's dominated convergence theorem.
Combining the latter equality with the (4.7), (4.9), (4.10) and the integral representation of $\varrho^{-3} \mathbb{V}ar(\{\Xi \cap \varrho K\}_{2})$ at the very beginning of the proof of Theorem 2 we can state the relation

$$\lim_{\varrho \to \infty} \varrho^{-3} \mathbb{V}ar(\{\Xi \cap \varrho K\}_{2}) = e^{-2\lambda \mathbb{E}[\Xi_0]} \times \lim_{\varrho \to \infty} \int \int_k \varrho \left( \log G_P[1 - w_{\varrho x, \varrho y}(\cdot)] - \log G_P[1 - w_{\varrho x}(\cdot)] - \log G_P[1 - w_{\varrho y}(\cdot)] \right) dx \, dy.$$  (4.11)

By using the expansion (4.8) the double integral on the right-hand side of (4.11) takes the form

$$\int \int_k \varrho \left( \log G_P[1 - w_{\varrho x, \varrho y}(\cdot)] - \log G_P[1 - w_{\varrho x}(\cdot)] - \log G_P[1 - w_{\varrho y}(\cdot)] \right) dx \, dy = \sum_{n=1}^{\infty} \frac{(-1)^n T_n^{(\varrho)}(K)}{n!},$$

where $T_n^{(\varrho)}(K)$ for $n \in \mathbb{N}$ is defined by

$$T_n^{(\varrho)}(K) := \int_k \int_k \varrho \left( \prod_{j=1}^{n} w_{\varrho x, \varrho y}(p_j) \right) \gamma^{(n)}(d(p_1, \ldots, p_n)) \, dx \, dy.$$  (4.12)

Since $\gamma^{(1)}(dp) = \lambda dp$ and $w_{\varrho x, \varrho y}(p) - w_{\varrho x}(p) - w_{\varrho y}(p) = -w_{\varrho x, \varrho y}(p)$, we get

$$-T_{1}^{(\varrho)}(K) = \lambda \int_k \int_k \varrho w_{\varrho x, \varrho y}(p) \, dp \, dx \, dy = \lambda J_\varrho(K) \underset{m \to \infty}{\longrightarrow} 2\lambda \mathbb{E}[\Xi_0^2] C_{2}^{G,K},$$

where the limit is just the assertion of Lemma 3. The above proof of Lemma 3 reveals that $|T_{n}^{(\varrho)}(K)| \leq \lambda J_\varrho(K) \leq 2\lambda \mathbb{E}[\Xi_0^2] |K|_2 \text{diam}(K)$. In the next step we derive a uniform bound of $T_{2}^{(\varrho)}(K)$ as well as its the limit as $\varrho \to \infty$. For doing this, we rewrite

$$\prod_{j=1}^{2} w_{\varrho x, \varrho y}(p_j) - \prod_{j=1}^{2} w_{\varrho x}(p_j) - \prod_{j=1}^{2} w_{\varrho y}(p_j) = w_{\varrho x}(p_1)w_{\varrho y}(p_2) + w_{\varrho y}(p_1)w_{\varrho x}(p_2) - w_{\varrho x, \varrho y}(p_1)w_{\varrho x, \varrho y}(p_2)$$

and by regarding the symmetry in $x, y$ and $p_1, p_2$ we get

$$T_{2}^{(\varrho)}(K) = \varrho \int_k \int_k \left( \prod_{j=1}^{2} w_{\varrho x, \varrho y}(p_j) - \prod_{j=1}^{2} w_{\varrho x}(p_j) - \prod_{j=1}^{2} w_{\varrho y}(p_j) \right) \gamma^{(2)}(d(p_1, p_2)) \, dx \, dy$$

$$= \varrho \int_k \int_k \left( 2w_{\varrho x}(p_1)w_{\varrho y}(p_2) - (w_{\varrho x, \varrho y}(p_2) + 2w_{\varrho x}(p_2)w_{\varrho y}(p_1)) \right) \gamma^{(2)}(d(p_1, p_2)) \, dx \, dy$$

$$= 2\varrho \int_k \int_k \left( w_{\varrho x}(p_1)w_{\varrho y}(p_2) \right) \gamma^{(2)}(d(p_1, p_2)) \, dx \, dy + \tilde{T}_{2}^{(\varrho)}(K),$$  (4.13)

where

$$|\tilde{T}_{2}^{(\varrho)}(K)| \leq 3\lambda \varrho \int_k \int_k \int_{\mathbb{R}^2} w_{\varrho x, \varrho y}(p_1)w_{\varrho x}(p_2 + p_1) \left| \gamma^{(2)}_{\text{red}} \right| (dp_2) \, dp_1 \, dx \, dy$$

$$= 3\lambda \varrho \int_k \int_k \int_{\mathbb{R}^2} w_{\varrho x, \varrho y}(p_1) \mathbb{E} \left| \gamma^{(2)}_{\text{red}} |(\Xi_0 + \varrho \langle v(\Phi_0), x \rangle - p_1) \right| dp_1 \, dx \, dy.$$  (4.14)
Clearly, we have $\infty > |\gamma^{(2)}_{\text{red}}(\mathbb{R}^1)| \geq \mathbb{E}|\gamma^{(2)}_{\text{red}}(\Xi_0 + \rho(\Phi_0, x) - p_1)\rightarrow 0$ for $x \neq 0$. Together with the arguments used in the proof of Lemma 6, among them the uniform estimate $J_\rho(K) \leq 2\mathbb{E}|\Xi_0|^2|K|_2 \text{ diam}(K)$, it follows that $\overline{T}_2^{(\rho)}(K) \underset{\rho \rightarrow \infty}{\rightarrow} 0$. Finally, Lemma 5 and (4.13) show that

$$
\frac{T_2^{(\rho)}(K)}{2} \underset{\rho \rightarrow \infty}{\rightarrow} \lambda(\mathbb{E}|\Xi_0|_1)^2 \gamma^{(2)}_{\text{red}}(\mathbb{R}^1) \int_{\mathbb{R}^1} (\mathbb{E}|g(p, \Phi_0) \cap K|_1)^2 dp = \lambda(\mathbb{E}|\Xi_0|_1)^2 \gamma^{(2)}_{\text{red}}(\mathbb{R}^1) C_1^{G,K}.
$$

In addition, we can derive a uniform bound of $T_2^{(\rho)}(K)$. From (4.11) and the above bound of $T_1^{(\rho)}(K)$ we get that $|\overline{T}_2^{(\rho)}(K)| \leq 3|\gamma^{(2)}_{\text{red}}|_{TV}|T_1^{(\rho)}(K)| \leq 6 \lambda |K|_2 \text{ diam}(K) |\gamma^{(2)}_{\text{red}}|_{TV} \mathbb{E}|\Xi_0|^2$. Hence, we see from (4.5) and (4.12) that, for two independent pairs $(\Xi_i, \Phi_i), i = 1, 2$, with the same distribution as $(\Xi_0, \Phi_0)$, the following estimate holds:

$$
|T_2^{(\rho)}(K)| \leq 2 \lambda |J_\rho(K, \Xi_1, \Phi_1, \Xi_2, \Phi_2)| + \overline{T}_2^{(\rho)}(K) \leq 8 \lambda |K|_2 \text{ diam}(K) \mathbb{E}|\Xi_0|^2 |\gamma^{(2)}_{\text{red}}|_{TV}.
$$

Obviously, the limit (3.2) coincides with $\lim_{\rho \rightarrow \infty}(-T_1^{(\rho)}(K) + \frac{1}{2}T_2^{(\rho)}(K))$. Thus, the proof of Theorem 2 is accomplished if we show that

$$
\lim_{\rho \rightarrow \infty} T_2^{(\rho)}(K) = 0 \quad \text{and} \quad \sup_{\rho \geq 1} \frac{|T_2^{(\rho)}(K)|}{n!} \leq C_n^K \quad \text{for} \quad n \geq 3 \quad \text{such that} \quad \sum_{n \geq 3} C_n^K < \infty. \quad (4.15)
$$

This means we have to find suitable upper bounds of the integrals (4.12) for each $n \geq 3$ which are uniform w.r.t. $\rho$ and disappear as $\rho \rightarrow \infty$. Using the reduced factorial cumulant measures $\gamma^{(n)}_{\text{red}}$ defined (in differential notation) by $\gamma^{(n)}_{\text{red}}(d(p_1, \ldots, p_n)) = \lambda \gamma^{(n)}_{\text{red}}((dp_i - p_j : i \neq j)) dp_j$ for any $j = 1, \ldots, n$, the boundedness of the total variation measure $|\gamma^{(n)}_{\text{red}}|_{TV}$ on $\mathbb{R}^{n-1}$ and obvious relations

$$
\prod_{i=1}^n (w_{ex}(p_i) + w_{\ell y}(p_i)) - \prod_{i=1}^n w_{ex}(p_i) - \prod_{i=1}^n w_{\ell y}(p_i) = \sum_{k=1}^{n-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{i=1}^k w_{ex}(p_{i}) \prod_{j \neq 1, \ldots, i_k}^n w_{\ell y}(p_j)
$$

$$
\prod_{i=1}^n (w_{ex}(p_i) + w_{\ell y}(p_i)) - \prod_{i=1}^n w_{ex,\ell y}(p_i) = \sum_{k=1}^{n} \prod_{i=1}^k w_{ex,\ell y}(p_k) \prod_{j=k+1}^n (w_{ex}(p_j) + w_{\ell y}(p_j)) \leq \sum_{k=1}^{n} w_{ex,\ell y}(p_k) \prod_{j=k+1}^n (w_{ex}(p_j) + w_{\ell y}(p_j)),
$$

we obtain the following estimates

$$
\left| \int K \int K \int_{\mathbb{R}^n} \rho \left( \prod_{i=1}^n (w_{ex}(p_i) + w_{\ell y}(p_i)) - \prod_{i=1}^n w_{ex}(p_i) - \prod_{i=1}^n w_{\ell y}(p_i) \right) \gamma^{(n)}(d(p_1, \ldots, p_n)) dx dy \right|
$$

$$
= \left| \sum_{k=1}^{n-1} \binom{n}{k} \int K \int K \int_{\mathbb{R}^n} \rho \prod_{i=1}^k w_{ex}(p_i) \prod_{j=k+1}^n w_{\ell y}(p_j) \gamma^{(n)}(d(p_1, \ldots, p_n)) dx dy \right|
$$

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\[ T_{n,1}^{\varphi}(K) := \lambda \sum_{k=1}^{n-1} \binom{n}{k} \int_{K} \int_{K} \int_{\mathbb{R}^1} q \, w_{\varphi x}(p_1) \prod_{j=k+1}^{n} w_{\varphi y}(p_j + p_1) \prod_{i=2}^{k} w_{\varphi x}(p_i + p_1) \prod_{j=2}^{n} w_{\varphi y}(p_j + p_1) \times |^{(n)}(\gamma_{\text{red}})| \, dp_1 \, dx \, dy \] (4.16)

and

\[ \left| \int_{K} \int_{K} \int_{\mathbb{R}^n} q \left( \prod_{i=1}^{n} (w_{\varphi x}(p_i) + w_{\varphi y}(p_i)) - \prod_{i=1}^{n} w_{\varphi x,y}(p_i) \right) \gamma^{(n)}(dp_1, \ldots, dp_n) \, dx \, dy \right| \]

\[ \leq \lambda n \int_{K} \int_{K} \int_{\mathbb{R}^n} q \, w_{\varphi x,y}(p_1) \prod_{j=2}^{n} \left( w_{\varphi x}(p_j + p_1) + w_{\varphi y}(p_j + p_1) \right) |^{(n)}(\gamma_{\text{red}})| \, dp_1 \, dx \, dy \]

\[ \leq T_{n,2}^{\varphi}(K) := \lambda n \sum_{k=1}^{n} \binom{n-1}{k-1} \int_{K} \int_{K} \int_{\mathbb{R}^n} q \, w_{\varphi x,y}(p_1) \prod_{i=2}^{k} w_{\varphi x}(p_i + p_1) \prod_{j=k+1}^{n} w_{\varphi y}(p_j + p_1) \times |^{(n)}(\gamma_{\text{red}})| \, dp_1 \, dx \, dy \] (4.17)

Obviously, we have \(|T_{n,1}^{\varphi}(K)| \leq T_{n,1}^{\varphi}(K) + T_{n,2}^{\varphi}(K)\) for \(n \geq 3\). Let us first, rewrite the integral terms in (4.16). For this purpose we introduce the abbreviation

\[ I_{n,k}^{\varphi}(K) := \int_{K} \int_{K} \int_{\mathbb{R}^1} q \, w_{\varphi x}(p_1) \prod_{i=2}^{k} w_{\varphi x}(p_i + p_1) \prod_{j=k+1}^{n} w_{\varphi y}(p_j + p_1) |^{(n)}(\gamma_{\text{red}})| \, dp_1 \, dx \, dy \]

for \(k = 2, \ldots, n-1\).

As in (4.14) we substitute \(x = O(\Phi_1) u\) and \(y = O(\Phi_n) w\) with \(O(\cdot)\) as defined in (1.8). Since \(O^{-1}(\varphi) = O(-\varphi)\) and \(\det(O(\varphi)) = 1\) it follows that \(u = O(-\Phi_1) x, w = O(-\Phi_n) y\) and \(\langle v(\Phi_1), x \rangle = \langle v(\Phi_1), u \rangle = \langle v(\Phi_1) - \Phi_1, u \rangle\) for \(i = 1, \ldots, k\) and \(\langle v(\Phi_j), y \rangle = \langle v(\Phi_j - \Phi_n), w \rangle\) for \(j = k + 1, \ldots, n\). Note that \(\langle v(\Phi_1), x \rangle = u_1\) and \(\langle v(\Phi_n), y \rangle = w_1\) for \(u = (u_1, u_2)^T\) and \(w = (w_1, w_2)^T\), respectively.

Similarly as in the proof of Lemma 3 we introduce independent copies \((R_1, \Phi_1), \ldots, (R_n, \Phi_n)\) of the random vector \((R_0, \Phi_0)\) and independent copies \(\Xi_1, \ldots, \Xi_n\) of the random interval \(\Xi_0 = [-R_0, R_0]\).

Then the product \(w_{\varphi x}(p) \prod_{i=2}^{k} w_{\varphi x}(p_i + p_1) \prod_{j=k+1}^{n} w_{\varphi y}(p_j + p_1)\) can be expressed as expectation

\[ \mathbb{E}\left( \mathbf{1}_{\Xi_1 + \varphi(\Phi_1)x}(p_1) \prod_{i=2}^{k} \mathbf{1}_{\Xi_1 + \varphi(v(\Phi_i), x)}(p_i + p_1) \prod_{j=k+1}^{n} \mathbf{1}_{\Xi_j + \varphi(v(\Phi_j), y)}(p_j + p_1) \right) \]

which together with the above transformations of \(x, y \in \mathbb{R}^2\) and Fubini’s theorem allows us to write \(I_{n,k}^{\varphi}(K)\) in the form
Replacing the two products of indicator functions in (4.18) by 1 leads to the following bound of $I_{n,k}^\varrho(K)$ provided that $|\gamma_{\text{red}}^{(n)}|(\mathbb{R}^{n-1}) < \infty$:

$$I_{n,k}^\varrho(K) \leq \mathbb{E} \int \int \int \left( 1_{\Xi_1+p_1(z_1)} 1_{\Xi_n-p_n(p_n)} \right) \left| \gamma_{\text{red}}^{(n)} \right|(d(p_2, \ldots, p_n)) dp_1 \times 1_{O(-\Phi_1)K}\left(\left(\frac{z_1}{q} + w_1, z_2\right)\right) 1_{O(-\Phi_n)K}\left(\left(w_1, w_2\right)\right) d(z_1, z_2) d(w_1, w_2).$$
By Hölder’s inequality as follows:

\[
\begin{align*}
= \mathbb{E} \int \int \int \int \left( \mathbf{1}_{-\Xi_1 + p_1 (z_1)} \mathbf{1}_{\Xi_n - p_n (p_1)} \right) \gamma^{(n)}_{red} (d(p_2, \ldots, p_n)) \, dp_1 \\
\times |g (\bar{z}_1 / q + w_1, \Phi_1) \cap K|_1 |g (w_1, \Phi_n) \cap K|_1 \, dz_1 \, dw_1 \\
\leq \text{diam}(K) \mathbb{E} \int |g (w_1, \Phi_n) \cap K|_1 \, dw_1 \int \int \mathbf{1}_{-\Xi_1 + p_1 (z_1)} \mathbf{1}_{\Xi_n - p_n (p_1)} \, dz_1 \, dp_1 |\gamma^{(n)}_{red}| (\mathbb{R}^{n-1}) \\
= \text{diam}(K) |K|_2 \mathbb{E} |\Xi_1|_1 \mathbb{E} |\Xi_n|_1 |\gamma^{(n)}_{red}| (\mathbb{R}^{n-1}) = \text{diam}(K) |K|_2 (\mathbb{E} |\Xi_0|_1)^2 ||\gamma^{(n)}_{red}||_{TV} .
\end{align*}
\]

Here, we have used arguments which have already been applied to prove (1.15). On the other hand, the product of the indicator functions in the first line of (4.18) disappears as \( q \to \infty \) P-almost surely and for almost all \((w_1, w_2), (z_1, z_2), p_1, (p_2, \ldots, p_n) \in \mathbb{R}^{n+2} \) w.r.t. the corresponding product measure. Therefore, again by Lebesgue’s dominated convergence theorem,

\[
\lim_{q \to \infty} I^{(q)}_{n,k} (K) = 0 \quad \text{for} \quad k = 2, \ldots, n , \ n \geq 3 .
\]

(4.20)

Next, we derive a further bound of \( I^{(q)}_{n,k} (K) \) that depends more on the mean thickness \( \mathbb{E} |\Xi_0|_1 \) of the typical cylinder. For this, we need the Radon-Nikodym density \( c^{(n)}_{red} (p_2, \ldots, p_n) \) of \( \gamma^{(n)}_{red} (\cdot) \) w.r.t. to Lebesgue measure on \( \mathbb{R}^{n-1} \). Hence, by Fubini’s theorem, we replace the integral (4.19) over \( \mathbb{R}^{n-1} \) by two iterated integrals. The first integral over \((p_2, \ldots, p_{n-1}) \in \mathbb{R}^{n-2}\) can be estimated by Hölder’s inequality as follows:

\[
\int_{\mathbb{R}^{n-2}} \prod_{i=2}^{k} \mathbf{1}_{\Xi_i + \langle \nu (\Phi_i - \Phi_1), (z_1 + \varrho w_1, \xi \varrho z_2) \rangle - \varrho w_1 - p_i (p_i)} \prod_{j=k+1}^{n-1} \mathbf{1}_{\Xi_j + \langle \nu (\Phi_j - \Phi_n), w \rangle - \varrho w_1 - p_1 (p_j)} \\
\times |c^{(n)}_{red} (p_2, \ldots, p_{n-1}, p_n)| \, d(p_2, \ldots, p_{n-1}) \leq \\
\left( \int_{\mathbb{R}^{n-2}} \prod_{i=2}^{k} \mathbf{1}_{\Xi_i + \langle \nu (\Phi_i - \Phi_1), (z_1 + \varrho w_1, \xi \varrho z_2) \rangle - \varrho w_1 - p_i (p_i)} \prod_{j=k+1}^{n-1} \mathbf{1}_{\Xi_j + \langle \nu (\Phi_j - \Phi_n), w \rangle - \varrho w_1 - p_1 (p_j)} \, d(p_2, \ldots, p_{n-1}) \right)^{\frac{q-1}{q}} \\
\times \left( \int_{\mathbb{R}^{n-1}} |c^{(n)}_{red} (p_2, \ldots, p_{n-1}, p_n)|^q \, d(p_2, \ldots, p_{n-1}) \right)^{\frac{1}{q}} = \left( \prod_{i=2}^{n-1} |\Xi_i|_1 \right)^{\frac{q-1}{q}} ||c^{(n)}_{red} (\cdot, p_n)||_q
\]

(4.21)

for any \( q > 1 \), where \( ||c^{(n)}_{red} (\cdot, p_n)||_q \) coincides with the term in front of the equal sign in (4.21). Combining the estimates (1.16) and (4.21) with \( |g (p, \varphi) \cap K|_1 \leq \text{diam}(K) \) for \((p, \varphi) \in \mathbb{R}^1 \times [0, \pi] \),

\[
\int_{\mathbb{R}^1} |g (p, \varphi) \cap K|_1 \, dp = |K|_2 , \text{switching the order of integration and finally applying Lyapunov’s inequality we arrive at}
\]

\[
\]
\[ J^{(0)}_{n,k}(K) \leq \mathbb{E} \int \int \int \left( \prod_{i=2}^{n-1} |\Xi_i| \right)^{2-1} \| c^{(n)}_{red}(\cdot, p_n) \|_q 1_{-\Xi_1+\Phi_1} 1_{\Xi_2-p_n(p_1)} \, dp_1 \, dp_n \]

\[ \times |g(z_1 + w_1, \Phi_1) \cap K|1 |g(w_1, \Phi_n) \cap K|1 \, dz_1 \, dw_1 \]

\[ \leq \text{diam}(K) |K|_2 \mathbb{E} \left( \prod_{i=2}^{n-1} |\Xi_i| \right)^{2-1} \int \int \int \| c^{(n)}_{red}(\cdot, p_n) \|_q 1_{-\Xi_1+\Phi_1} 1_{\Xi_2-p_n(p_1)} \, dz_1 \, dp_1 \, dp_n \]

\[ = \text{diam}(K) |K|_2 \int \| c^{(n)}_{red}(\cdot, p) \|_q \mathbb{E}[\Xi_0] \frac{n(q-1)+2}{q}. \]

Applying the same arguments as above, the estimate \((4.20)\) reveals that \((4.20)\) remains true if, instead of \(\| \gamma_{red}^{(n)} \|_{TV} < \infty\), the \(L^q_{\phi}\)-norm \(c^{(n)}_{red}\) is finite for some \(q > 1\) and \(n \geq 3\). Hence, we have

\[ T^{(q)}_{n,1}(K) = \frac{1}{n} \prod_{k=1}^{n-1} \left( \frac{n}{k} \right)^{2-1} \mathbb{E}[\Xi_0] \frac{n(q-1)+2}{q} \| c^{(n)}_{red}\|_q. \]

Together with the strong \(L^q_{\phi}\)-Brillinger mixing condition with \(b^*_q(\mathbb{E}[\Xi_0])^{1-\frac{1}{q}} < 1/2\) we get

\[ \sum_{n \geq 3} \frac{T^{(q)}_{n,1}(K)}{n!} \leq \frac{1}{n} \frac{\text{diam}(K) |K|_2}{\prod_{n \geq 3} \left( 2 b^*_q(\mathbb{E}[\Xi_0])^{1-\frac{1}{q}} \right)^n} \leq \frac{\lambda a^*_q(\mathbb{E}[\Xi_0])^{2-\frac{1}{q}} \text{diam}(K) |K|_2}{1 - 2 b^*_q(\mathbb{E}[\Xi_0])^{1-\frac{1}{q}}}. \]

Next, we derive two different bounds for the sum \(T^{(q)}_{n,2}(K)\) defined in \((4.17)\). For this purpose, in analogy to \(I^{(q)}_{n,k}(K)\), we need uniform bounds of

\[ J^{(q)}_{n,k}(p) := \int_{\mathbb{R}^{n-1}} \prod_{j=2}^{k} w_{\phi x_j}(p_1 + p_j) \prod_{j=k+1}^{n} w_{\phi y_j}(p_j + p) \| \gamma_{red}^{(n)}(d(p_2, \ldots, p_n)) \|_q. \]

It is easily seen that

\[ J^{(q)}_{n,k}(p) = \mathbb{E} \int_{\mathbb{R}^{n-1}} \prod_{j=2}^{k} \Xi_{i}^{\phi(v(\Phi_i), x) - p_i(p_i)} \prod_{j=k+1}^{n} \Xi_{j}^{\phi(v(\Phi_i), y) - p_j(p_j)} \| \gamma_{red}^{(n)}(d(p_2, \ldots, p_n)) \|_q \leq \| \gamma_{red}^{(n)}(\mathbb{R}^{n-1}) \|_q. \]

and, for any \(q > 1\) such that \(\| c^{(n)}_{red}\|_q < \infty\),

\[ J^{(q)}_{n,k}(p) = \mathbb{E} \int_{\mathbb{R}^{n-1}} \prod_{j=2}^{k} \Xi_{i}^{\phi(v(\Phi_i), x) - p_i(p_i)} \prod_{j=k+1}^{n} \Xi_{j}^{\phi(v(\Phi_i), y) - p_j(p_j)} c^{(n)}_{red}(p_2, \ldots, p_n - 1) d(p_2, \ldots, p_n - 1) \]

\[ \leq \mathbb{E} \prod_{j=2}^{n} |\Xi_j|^{\frac{1}{q}} \left( \int_{\mathbb{R}^{n-1}} \| c^{(n)}_{red}(p_2, \ldots, p_n - 1) \|_q d(p_2, \ldots, p_n - 1) \right)^{\frac{1}{q}} \leq (\mathbb{E}[\Xi_0])^{(n-1)\frac{1}{q}} \| c^{(n)}_{red}\|_q. \]
The foregoing estimates show that
\[
\lim_{q \to \infty} J_{n,k}^{(q)}(p) = 0 \quad \text{for} \quad k = 2, \ldots, n, \; n \geq 3. \tag{4.22}
\]

Further, from the definition of \(T_{n,2}^{(q)}(K)\), see (4.17), and the integral \(J_{n}^{(q)}(K)\) introduced and estimated in the proof of Lemma 6 with the uniform upper bound \(2 \text{diam}(K) |K|_{2} \mathbb{E}|\Xi|_{1}^{2}\), we see that

\[
T_{n,2}^{(q)}(K) \leq \lambda n \sum_{k=1}^{n} \binom{n-1}{k-1} \int_{K} \int_{K} \int_{\mathbb{R}^1} g^{\ast} w_{\eta x,\eta y}(p_{1}) \, dp_{1} \, dx \, dy \max_{2 \leq k \leq n \; p \in \mathbb{R}^1} J_{n,k}^{(q)}(p)
\]

\[
= \lambda n 2^{n-1} J_{n}(K) \max_{2 \leq k \leq n \; p \in \mathbb{R}^1} J_{n,k}^{(q)}(p) \leq \lambda n 2^{n} \text{diam}(K) |K|_{2} \mathbb{E}|\Xi|_{1}^{2} \max_{2 \leq k \leq n \; p \in \mathbb{R}^1} J_{n,k}
\]

Under the assumption that \(\Psi \sim P\) is either strongly Brillinger-mixing with \(b < 1/2\) or strongly \(L_q\)-Brillinger-mixing with \(b_q (\mathbb{E}|\Xi|_{1})^{1-\frac{1}{q}} < 1/2\) we obtain the inequalities

\[
\sum_{n \geq 3} \frac{T_{n,2}^{(q)}(K)}{n!} \leq 2 \lambda a b \mathbb{E}|\Xi|_{1}^{2} \text{diam}(K) |K|_{2} \sum_{n \geq 3} n(2b)^{n-1} \leq \frac{2 \lambda a b \mathbb{E}|\Xi|_{1}^{2} \text{diam}(K) |K|_{2}}{(1-2b)^{2}}
\]

and

\[
\sum_{n \geq 3} \frac{T_{n,2}^{(q)}(K)}{n!} \leq 2 \lambda a q b_q \mathbb{E}|\Xi|_{1}^{2} \text{diam}(K) |K|_{2} \sum_{n \geq 3} n(2b)^{n-1} (\mathbb{E}|\Xi|_{1})^{1-\frac{1}{q}} \leq \frac{2 \lambda a q b_q \mathbb{E}|\Xi|_{1}^{2} \text{diam}(K) |K|_{2}}{(1-2b)^{2}}.
\]

Finally, summarizing the above-proved relations (4.20), (4.22) and the convergence of the series \(\sum_{n \geq 3} T^{(q)}_{n,i}(K)/n!\) for \(i = 1, 2\) shows the validity of (4.15) which in turn implies (4.9). Thus, the proof of Theorem 2 is complete.

**Remark.** Strong Brillinger-mixing with \(b < 1/2\) is a rather restrictive condition for the one-dimensional point process \(\Psi \sim P\). Equivalently formulated, the power series \(\sum_{n=2}^{\infty} \frac{z^{n}}{n!} |\gamma^{(n)}(\mathbb{R}^{n-1})|\) is analytic in the interior of the disk \(b(0,2)\) in the complex plane. For example, the condition has been used for statistical analysis of point processes in [6]. The Gauss-Poisson process, see [5], Poisson cluster processes with a finite number of non-vanishing cumulant measures, see [11, 2], and certain Neyman-Scott processes satisfy this condition. In the case that \(\Psi \sim P\) is strongly \(L_q\)-resp. strongly \(L^*_q\)-Brillinger-mixing for some \(q > 1\) with \(b_q > 0\) resp. \(b^*_q > 0\) we can choose \(\mathbb{E}|\Xi|\) sufficiently small to fulfill the assumptions of Theorem 2 which greatly expands its applicability.

In a separate paper we will study the asymptotic normality of \(\theta^{-3/2} (|\Xi \cap \theta K|_{2} - \mathbb{E}|\Xi \cap \theta K|_{2})\) as \(\theta \to \infty\). To achieve this goal we have to find the conditions which allow to verify that

\[
\theta^{-3/2} \; \text{Cum}_k (|\Xi \cap \theta K|_{2}) \xrightarrow{\theta \to \infty} 0 \quad \text{for} \quad k \geq 3, \tag{4.23}
\]
where with the notation and the formulas of Chapter 2 we can use the following representation of the $k$th-order cumulant $\text{Cum}_k(\Xi \cap \theta K)$:

$$
(-1)^k \sum_{\ell=1}^{k} (-1)^{\ell-1}(\ell - 1)! \sum_{K_1 \cup \cdots \cup K_\ell = \{1, \ldots, k\} \cap (\theta K)^k} \int \prod_{j=1}^{\ell} G_P \left[ 1 - P \left( (\cdot) \in \bigcup_{i \in K_j} (\Xi_0 + \langle v(\Phi_0), x_i \rangle) \right) \right] d(x_1, \ldots, x_k).
$$

From the latter formula it is easily seen that (4.23) is equivalent to

$$
\theta^{k/2} \sum_{\ell=1}^{k} \frac{(-1)^{\ell-1}}{\ell} \sum_{k_1 + \cdots + k_\ell = k \atop k_i \geq 1, i = 1, \ldots, \ell} \frac{k!}{k_1! \cdots k_\ell!} \prod_{j=1}^{\ell} \prod_{K_j} G_P \left[ 1 - w_{x_1, \ldots, x_{k_j}}^{j} (\cdot) \right] d(x_1, \ldots, x_{k_j}) \xrightarrow{\theta \to \infty} 0
$$

for any $k \geq 3$. A modification of a recursive proving technique developed in Chapter 2 of [11] to treat the analogous problem for Poisson cylinder processes could be useful.

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