METHOD OF NUMERICAL ANALYSIS OF THE PROBLEM OF MASS TRANSFER OF A CYLINDRICAL BODY WITH THE UNIFORM TRANSLATIONAL FLOW

Context. The problem of mass transfer of a cylindrical body with a uniform translational flow of a viscous incompressible fluid is examined in the paper.

Objective. The purpose of this work is to develop a new method for numerical analysis of the problem of mass transfer of a cylindrical body with a uniform translational flow, which is based on the joint application of the R-functions structural method and the Galerkin projection method.

Method. In general case, the problem of stationary mass transfer of a cylindrical body with a viscous incompressible fluid flow is reduced to the solution of the equation of hydrodynamic flow passing a surface and an equation for concentration with corresponding boundary conditions on the surface of the body and far away from it. The geometry of the area, and also the boundary conditions (including the condition at infinity) may be taken into account precisely by using the constructive apparatus of the R-functions theory by V. L. Rvachev, the Academician of Ukrainian National Academy of Sciences. In this study, a complete structure of the solution of a linear boundary value problem for the concentration that exactly satisfies the boundary conditions on the boundary and condition at infinity is constructed on the basis of the R-functions theory methods, and this made it possible to lead the tasks in the infinite domain to tasks in the finite domain. To solve the linear problem for concentration the numerical algorithm on the basis of Galerkin method is developed.

Results. The computational experiment for the problem of the flow past circular and elliptical cylinders at various Reynolds and Peclet numbers was carried out.

Conclusions. The conducted experiments have confirmed the efficiency of the proposed method of numerical analysis of the problem of mass transfer of a cylindrical body with a uniform translational flow, based on the joint application of the R-functions structural method and Galerkin projection method. The prospects for the further research may be to use the developed method for the implementation of iterative methods for solving the task of nonlinear mass transfer, semi-discrete and projection methods for solving the non-stationary tasks, as well as in solving the tasks of optimal management of relevant technological processes.

Keywords: flow task, stationary flow, viscous fluid, stream function, mass transfer, concentration function, R-functions method, Galerkin method.

NOMENCLATURE

c = c(r, \varphi) – a function of concentration; 
c_0 – the constant concentration given on the boundary of a streamlined body; 
\partial \Omega – the boundary of a streamlined body; 
l = 1, 2 – the number of equation; 
\mathbf{n} – the external normal to \partial \Omega; 
Pe – Peclet number; 
Re – Reynolds number; 
\Delta u – the solutions of auxiliary problems with homogeneous boundary conditions; 
U_\infty – unperturbed fluid velocity at infinity; 
\nu = Re^{-1} – the coefficient of viscosity; 
\{\varphi_k (r, \varphi)\} – a complete system of particular solutions of equation \Delta u = 0 with respect to the exterior of a cylinder of finite radius;
\{\Psi_j(r, \varphi)\} – a complete system of particular solutions of equation \(\Delta n = 0\) relative to domain \(\{\omega(r, \varphi) < M\}\);
\{\Phi_j(r, \varphi)\} – a sequence of functions that is complete with respect to the whole plane;
\Phi_1, \Phi_2 – undefined components of the solution structure;
\psi = \psi(r, \varphi) – the stream function;
\omega – a sufficiently smooth function built by using the constructive apparatus of the \(R\)-functions theory and describing the geometry of the domain \(\Omega\);
\(\omega = 0\) – a normalized equation of \(\Delta \Psi\);
\(\Omega\) – the flow domain;
\(\Omega_n\) – the bounded domain \(\{0 < \omega(r, \varphi) < M_n\}\).

INTRODUCTION

The problems of calculating viscous flows, which are complicated by mass transfer, are widely used in heat power engineering, chemical and food technologies, geo- and astrophysical studies, and environmental protection. Many processes of chemical technology are associated with fluid movement in process equipment. In the preparation of reagents and in the isolation of reaction products, such operations as leaching, absorption, extraction and distillation play an important role. The laws of hydrodynamics, heat and mass transfer are essential for the entire technological process. Processes of heat and mass transfer are also ones of the most important in the energy sector, as well as in a number of the technological processes in the metallurgical and other industries. In addition, the problems of mass transfer of bodies with a uniform viscous flow underlie the calculation of many technological processes, which are associated with dissolution, extraction, evaporation, precipitation of colloids, etc. Therefore, the development of new, as well as the improvement of existing methods of mathematical modeling and numerical analysis of external stationary problems of hydrodynamics of a viscous incompressible fluid, which take into consideration the mass transfer, is an actual scientific issue.

The object of this study is the stationary hydrodynamic process of flow past bodies by a viscous incompressible fluid, complicated by mass transfer and described by a system of equations with respect to the stream function and concentration.

The subject of this study is a mathematical model of the stationary task of flow past a cylindrical body by a viscous incompressible fluid with allowance for mass transfer, and a method for its numerical analysis.

The purpose of the work is to develop a new method for mathematical modeling of the mass transfer of a cylindrical body with a uniform translational flow which based on the \(R\)-functions method and Galerkin method.

1 PROBLEM STATEMENT

At small Peclet numbers, to describe the mass transfer process the Oseen approximation (equation 1) is used [1]:

\[
\Delta c = \frac{\text{Pe}}{U_\infty} \left( \cos \varphi \frac{\partial c}{\partial r} - \frac{\sin \varphi}{r} \frac{\partial c}{\partial \varphi} \right) \text{in } \Omega.
\] (1)

In the general case, the mass transfer process is described by the equation for the concentration (equation 2) in the form [1, 2]:

\[
\Delta c = \frac{\text{Pe}}{U_\infty} \left( \frac{\partial}{\partial r} \left( \frac{\rho c}{r} \frac{\partial c}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 c}{\partial \varphi^2} \right) \text{in } \Omega.
\] (2)

In equations (1) and (2) \(\Delta c = 1 \frac{\partial}{\partial r} \left( \frac{\rho c}{r} \frac{\partial c}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 c}{\partial \varphi^2}\).

Equations (1) and (2) in a rectangular coordinate system have the following form:

\[
\Delta c = \frac{\text{Pe}}{U_\infty} \frac{\partial c}{\partial x} \text{ in } \Omega, \quad \Delta c = \frac{\text{Pe}}{U_\infty} \frac{\partial c}{\partial y} \text{ in } \Omega,
\] (3)

where \(\Delta c = \frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2}\).

Equations (1) and (2) should be supplemented by the boundary condition on \(\partial \Omega\) and the condition at infinity [1]:

\[
\left. c \right|_{r=\infty} = c_0,
\] (5)

\[
\lim_{r \to \infty} c = 0.
\] (6)

The stream function \(\psi(r, \varphi)\) can be found, for example, as the solution of the following nonlinear task of flow past a cylindrical body by a viscous incompressible fluid [3–6]:

\[
\nu \Delta^2 \psi = \frac{1}{r} \partial \left( \frac{\partial \psi}{\partial r} \right) - \frac{1}{r} \partial \left( \frac{\partial \psi}{\partial \varphi} \right) \text{ in } \Omega,
\] (7)

\[
\psi \left|_{r=\infty} = 0, \right. \frac{\partial \psi}{\partial \varphi} \left|_{r=\infty} = 0, \right.
\] (8)

\[
\lim_{r \to \infty} r^{-1} \psi = U_\infty \sin \varphi,
\] (9)

where \(\Delta^2 \psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \varphi^2}, \Delta^2 \psi = \Delta (\Delta \psi)\).

The task (1), (5), (6) does not depend on the stream function \(\psi(r, \varphi)\), and the solution of the task (2), (5), (6) consists of two steps:

a) the search of the stream function as a solution of the task (7)–(9);

b) the solution of the task for concentration.

2 REVIEW OF THE LITERATURE

Various tasks which are arising in the study of external viscous fluid flows can be investigated theoretically or by means of physical experiment. At present, mathematical modeling and computational experiment are increasingly
being used in the study of hydrodynamic problems. Basically, finite difference method, finite element method, boundary integral element method and others are used for the numerical analysis of such problems. These methods are easy to implement, but do not have the necessary property of universality: when moving to a new area (especially non-classical geometry), it is necessary to generate a new grid, and often to replace complex sections of the boundary with simple ones, composed of, for example, straight line segments. The use of the \( R \)-functions structural-variational method \([7, 8]\) by V. L. Rvachev, the Academician of Ukrainian National Academy of Sciences, is an alternative to the existing methods of numerical computation of hydrodynamic problems.

The \( R \)-functions method in computational hydrodynamics was applied in \([9–14]\). The task of viscous fluid external flow around bodies of revolution in a spherical coordinate system, which is complicated by mass transfer, was solved in \([15, 16]\) with using the \( R \)-functions method.

In this study, we propose to apply the \( R \)-functions and Galerkin method for mathematical modeling the problem of mass transfer of a cylindrical body with a uniform translational flow.

### 3 MATERIALS AND METHODS

The method for solving the task for the stream function, based on the application of \( R \)-functions, successive approximations and Galerkin method, is described in \([17, 18]\). Substituting the stream function \( \psi(r, \varphi) \) so obtained into equation (2), let us solve the problems (1), (5), (6) and (2), (5), (6) by the \( R \)-functions method. To do this, using the constructive methods of the \( R \)-functions theory \([7, 8]\) let us construct the structure of the boundary value task solution, i.e. the functions bundle that exactly satisfies the boundary condition and the condition at infinity.

Let us consider a sufficiently smooth function \([19]\)

\[
f_M(x) = \begin{cases} 
1 - \exp\left(\frac{M - x}{M}\right), & 0 \leq x < M; \\
0, & x \geq M \ (M = \text{const} > 0).
\end{cases}
\]

The function (10) \( f_M(x) \in C^\infty[0, +\infty) \) and satisfies the conditions:

a) \( f_M(0) = 0 \); b) \( f_M'(0) = 1 \);

c) \( f_M(x) \geq 0 \ \forall x \geq 0 \); d) \( f_M(x) = 1 \ \forall x \geq M \).

Let us denote \( \omega_M = f_M(\omega) \), then

1) \( \omega_M > 0 \) in \( \Omega \); 2) \( \omega_M|_{\partial \Omega} = 0 \);

3) \( \frac{\partial \omega_M}{\partial n} \bigg|_{\partial \Omega} = -1 \); 4) \( \omega_M = 1 \), if \( \omega_M \geq M \).

In the problems (3), (5), (6), and (4)–(6) we make the replacement

\[ c = c_0(1 - \omega_M) + u_l. \]

The choice of such a substitution is due to the fact that the function \( c_0(1 - \omega_M) \) satisfies the boundary condition (5) and the condition at infinity (6).

Then for \( u_l, \ l = 1, 2 \), we obtain the tasks

\[
-\Delta u_l + B_l u_l = F_l \quad \text{in} \ \Omega, \quad u_l|_{\partial \Omega} = 0, \quad \lim_{r \to +\infty} u_l = 0,
\]

where

\[
B_1 = \text{Pe} U_\infty \frac{\partial u_1}{\partial x}, \quad B_2 = \text{Pe} \left( \frac{\partial y}{\partial x} \frac{\partial u_2}{\partial x} - \frac{\partial y}{\partial x} \frac{\partial u_2}{\partial y} \right),
\]

\[
F_1 = \Delta (c_0(1 - \omega_M)) - \text{Pe} U_\infty \frac{\partial (c_0(1 - \omega_M))}{\partial x},
\]

\[
F_2 = \Delta (c_0(1 - \omega_M)) - \text{Pe} \left( \frac{\partial y}{\partial x} \frac{\partial (c_0(1 - \omega_M))}{\partial x} - \frac{\partial y}{\partial x} \frac{\partial (c_0(1 - \omega_M))}{\partial y} \right).
\]

Note that \( F_1 = 0 \), \( l = 1, 2 \), in the domain \( \{\omega(x, y) \geq M\} \).

Let us find the generalized solution \( u \) of tasks (11)–(13) as the limit, when \( n \to \infty \), of solutions \( u_n \) of equations (11), which are considered in a sequence of domains \( \{\Omega_n\} \), that is a monotonic exhaustion of an infinite area \( \Omega \).

In domains \( \Omega_n \) we will consider the boundary value tasks

\[
-\Delta u_{n,l} + B_l u_{n,l} = F_l \quad \text{in} \ \Omega_n, \quad u_{n,l}|_{\partial \Omega_n} = 0,
\]

where functions \( u_{n,l} \) are continued by zero outside of \( \Omega_n \).

An approximate solution of tasks (14) – (15) for each \( n = 1, 2, \ldots \) according to the Bubnov-Galerkin method, will be sought in the form:

\[
u_{n,L,N} = \sum_{j=1}^{N} c_{n,l,j} \Phi_j.
\]

The numbers \( c_{n,l,j}, \ j = 1, \ldots, N \), are found from the system of linear algebraic equations

\[
\sum_{j=1}^{N} c_{n,l,j} \left\{ (\Phi_j, \Phi_l) + (K_l \Phi_j, \Phi_l) \right\} = (F_l, \Phi_l),
\]

where

\[
(F_l, \Phi_l) = - \int_{\Omega_n} c_0 \nabla (1 - \omega_M) \cdot \nabla \Phi_l d\Omega + \int_{\Omega_n} c_0 \text{Pe} U_\infty \cdot (1 - \omega_M) \frac{\partial \Phi_l}{\partial x} d\Omega,
\]

\[
(F_l, \Phi_i) = - \int_{\Omega_n} c_0 \nabla (1 - \omega_M) \cdot \nabla \Phi_i d\Omega + \int_{\Omega_n} c_0 \text{Pe} U_\infty \cdot (1 - \omega_M) \frac{\partial \Phi_i}{\partial x} d\Omega,
\]

\[
(F_l, \Phi_l) = - \int_{\Omega_n} c_0 \nabla (1 - \omega_M) \cdot \nabla \Phi_l d\Omega + \int_{\Omega_n} c_0 \text{Pe} U_\infty \cdot (1 - \omega_M) \frac{\partial \Phi_l}{\partial x} d\Omega.
\]
(F_2,\phi_j) = -\int_{\Omega_n} c_0 \nabla (1-\omega_M) \cdot \nabla \phi_j d\Omega + \int_{\Omega_n} c_0 \operatorname{Pe}(1-\omega_M) \left( \frac{\partial \psi}{\partial x} \frac{\partial \phi_j}{\partial x} - \frac{\partial \psi}{\partial y} \frac{\partial \phi_j}{\partial y} \right) d\Omega.

To construct the coordinate sequence, the complete system of particular solutions of the Laplace equation \[20\] and the R-functions method \[7, 8\] will be used.

We have proved that for any choice of sufficiently smooth functions \(\Phi_1, \Phi_2\) and at the requirement that \(\Phi_1 \to 0\) as \(r \to +\infty\) a function of the form

\[
u = \omega_M \Phi_1 + \omega_M (1-\omega_M) \Phi_2
\]

exactly satisfies the boundary conditions (12) and (13), that is, it is the structure of the solution of the boundary value problem (11)–(13).

Approximations of the functions \(\Phi_1\) and \(\Phi_2\) in the domain \(\Omega_n\) will be seek in the form

\[
\Phi_1 = \sum_{k=1}^{m_1} \alpha_k \varphi_k, \quad \Phi_2 = \sum_{j=1}^{m_2} \beta_j \tau_j,
\]

where \(\varphi_k (r, \varphi) = \left\{ \begin{array}{l} r^{-n} \cos m\varphi, \quad n = 1, 2, \ldots, \\
\sin m\varphi, \end{array} \right.\)

\(\tau_j (r, \varphi) = \left\{ \begin{array}{l} 1, r^k \cos k\varphi, \quad k = 1, 2, \ldots, \\
\sin k\varphi. \end{array} \right.\)

Then the sequence of functions \(\phi_j\) which is complete with relatively to the whole plane, has the form:

\[
\phi_j (r, \varphi) = \omega_M (r, \varphi) \varphi_k (r, \varphi), \quad \omega_M (r, \varphi)(1-\omega_M (r, \varphi)) \tau_j (r, \varphi) .
\]

(18)

The values of the coefficients \(\alpha_k (k = 1, 2, \ldots, m_1)\) and \(\beta_j (j = 1, 2, \ldots, m_2)\) in accordance with the Bubnov-Galerkin method will be found from the condition of residual orthogonality to the first \(N (N = m_1 + m_2)\) elements of the sequence (18), which leads to a system of linear algebraic equations in the form (17).

We have proved the convergence of the Galerkin approximations \(u_{n, i, N}\) of the form (16) at \(N \to \infty, M_n \to \infty\) in the energy norm to the unique generalized solutions of tasks (11)–(13).

4 EXPERIMENTS

A computational experiment was carried out for the tasks of flow past a circular cylinder \(x^2 + y^2 = 1\) and an elliptical cylinder \(\frac{x^2}{4} + \frac{y^2}{1} = 1\) at \(c_0 = 1, \ M = 5, 10, \ m_1 = 10, \ m_2 = 11\) and at various Reynolds and Peclet numbers. The double integrals in the systems for determination \(c_{n,j}\) were taken approximately by the Gauss formula with 50 nodes for each variable.

5 RESULTS

The concentration lines for a circular cylinder are shown in Fig. 1–3. Fig. 4–6 show the concentration lines for elliptical cylinder.

Figure 1 – The concentration lines for a circular cylinder

\(M = 5\) and \(\text{Pe} = 0.01\)

Figure 2 – The concentration lines for a circular cylinder at \(\text{Pe} = 10\):

\(a - \text{Re} = 0, \ b - \text{Re} = 5, \ c - \text{Re} = 15\)
Figure 3 – The concentration lines for a circular cylinder at $Pe = 20$:
   a – $Re = 0$, b – $Re = 5$, c – $Re = 15$

6 DISCUSSION
   At small Reynolds and Peclet numbers the substance is transported uniformly, dissolving into the liquid. As the Reynolds and Peclet numbers increase, the particles of substance begin to move with the flow. These results are consistent with the physics of the process. The efficiency of the proposed method for a spherical coordinate system was verified on the problem of flow past a sphere [13], for which an exact solution is known [20].

CONCLUSIONS
   The numerical method for calculating the mass transfer of a cylindrical body with a uniform translational flow, which based on the joint application of the $R$-functions method and Galerkin method, is proposed for the first time in this.
study. By using the R-functions method the structure of the solution of the problems of flow past bodies with allowance for mass transfer, which precisely satisfying the boundary condition on the boundary and the condition at infinity, was constructed, and this made it possible to lead tasks in the infinite domain to tasks in the finite domain. To approximate the uncertain components of the solution structure, the Galerkin method was applied. The stationary problem of flow past a cylindrical body in a cylindrical coordinate system for mass transfer, which precisely satisfying the boundary condition on the boundary and the condition at infinity, was solved numerically for various Reynolds and Peclet numbers.

ACKNOWLEDGMENTS

The work was carried out at the department of Applied Mathematics at Kharkiv National University of Radio Electronics and department of Advanced Mathematics at O. M. Beketov National University of Urban Economy in Kharkiv within the framework of collaborative scientific research conducted by the departments.

REFERENCES

1. Hydrodynamics, Mass and Heat Transfer in Chemical Engineering / A. D. Polyanin, A. M. Katerov, D. A. Kazenin, A. V. Vyazmin. – CRC Press, 2002. – 408 p.
2. Гупало Ю. П. Массотеплообмен реагирующих частиц с потоком / Ю. П. Гупало, А. Д. Полищук, Ю. С. Рязанцев. – М.: Наука, 1985. – 336 с.
3. Loitsyansky L. G. Mechanics of liquids and gases / L. G. Loitsiansky. – New York: Begell House, 1995. – 803 p.
4. Шкадов В. Я. Течение вязкой жидкости / В. Я. Шкадов, З. Д. Запряжко. – М.: Изд-во Моск. у-та, 1984. – 200 с.
5. Happel J. Low Reynolds number hydrodynamics / J. Happel, H. Brenner. – Netherlands: Martinus Nijhoff, The Hague, 1973. – 559 p.
6. Polyanin A. D. Handbook of Nonlinear Partial Differential Equations. Second Edition / A. D. Polyanin, V. F. Zaitsev. – CRC Press, 2011. – 1912 p.
7. Рачев В. Л. Теория R-функций и некоторые ее приложения / В. Л. Рачев. – К.: Наук. думка, 1982. – 552 c.
8. Shapiro V. Semi-analytic geometry with R-functions / V. Shapiro // Acta Numerica. – 2007. – Vol. 16. – P. 239–303. DOI: 10.1017/S096249290631001X
9. Колосова С. В. Применение проекционных методов и метода R-функций к решению краевых задач в бесконечных областях: дисс. … к. ф.-м. н. 01.01.07 – Вычислительная математика / С. В. Колосова. – Харьков: ХИРЭ, 1972. – 85 c.
10. Колосова С. В. Применение метода R-функций к расчету плоских течений вязкой жидкости / С. В. Колосова, М. В. Сидоров // Вісник Харківського національного університету. Сер. Прикл. матем. і мех. – 2003. – № 602. – С. 61–67.
11. Суворова И. Г. Компьютерное моделирование осесимметричных течений жидкости в каналах сложной формы / И. Г. Суворова // Вестник национального технического университета «ХПИ». – 2004. – № 31. – С. 141–148.
12. Тевяшев А. Д. Об одном подходе к математическому моделированию плоских стационарных течений вязкой несжимаемой жидкости в конечных одноосвязных областях / А. Д. Тевяшев, Н. В. Гибкина, М. В. Сидоров // Радиоэлектроника и информатика. – 2007. – № 2 (37). – С. 50–57.
13. Максименко-Шейко К. В. Математическое моделирование теплобмена при движении жидкости по каналам с винтовым типом симметрии методом R-функций / К. В. Максименко-Шейко // Доповіді Національної академії наук України. – 2005. – № 9. – С. 41–46.
14. Artyukh A. Mathematical modeling and numerical analysis of nonstationary plane-parallel flows of viscous incompressible fluid by R-functions and Galerkin method / A. Artyukh, M. Sidorov // Econtechmod. – 2014. – Vol. 3, No. 3. – P. 3–11.

Figure 6 – The concentration lines for elliptical cylinder at Pe = 20:

a – Re = 5, b – Re = 15, c – Re = 30
МЕТОД ЧИСЛЕНИКОГО АНАЛІЗУ ЗАДАЧІ МАСООБМІНУ ЦИЛІНДРИЧНОГО ТІЛА З РІВНОМІРНИМ ПОСТУПАЛЬНИМ ПОТОКОМ

Актуальність. У статті розглядається задача масообміну циліндричного тіла з рівномірним поступальним потоком в'язкої нестисливої рідини.

Метод роботи – розробка нового методу численного аналізу задачі масообміну циліндричного тіла з рівномірним поступальним потоком, заснованого на сумісному застосуванні структурного методу R-функцій і проекційного методу Гальоркіна.

Метод. У загальному випадку задача про стаціонарний масообмін циліндричного тіла з потоком в'язкої нестисливої рідини зводиться до розв'язання рівняння гідродинамічного обгінання поверхні рівня і рівняння для концентрації з відповідними крайовими умовами на поверхні тіла і далеко від неї. Точно врахувати геометрию області, а також крайові умови (в т.ч. і умову на несміченості), можна, скориставшись конструктивним апаратом теорії R-функцій акад. НААН України В. Л. Рвачова. У роботі на основі методів теорії R-функцій побудована відповідна система численного алгоритму для розв'язання задачі концентрації, яка точно задовольняє крайовим умовам на межі тіла і умову на несміченості, що дозволило звести задачи в несміченій області до задач в скінченої області. Для розв'язання двох типів задач для концентрації розроблено чисельний алгоритм на основі методу Гальоркіна.

Результати. Обчислювальний експеримент проведений для задач обгінання кругового і еліптичного циліндрів для різних чисел Рейнольдса і Пекле.
Выводы. Проведенные эксперименты подтвердили работоспособность предложенного метода численного анализа задачи массообмена цилиндрического тела с равномерным поступательным потоком, основанного на совместном применении структурного метода $R$-функций и проекционного метода Галеркина. Перспективы дальнейших исследований могут заключаться в использовании разработанного метода при реализации итерационных методов решения задачи нелинейного массообмена, полуискривленных и проекционных методов решения неstationарных задач, а также при решении задач оптимального управления соответствующими технологическими процессами.

Ключевые слова: задача обтекания, стационарное течение, вязкая жидкость, функция тока, массообмен, функция концентрации, метод $R$-функций, метод Галеркина.

REFERENCES

1. Polyanin A. D., Kuptepov A. M., Kazenin D. A., Vyazmin A. V. Hydrodynamics, Mass and Heat Transfer in Chemical Engineering. CRC Press, 2002, 408 p.
2. Gupalov Yu. P. Polyanin A. D., Rzazcev Yu. S. Massoteploobmen reagiruyushchih chastic s potokom. Moscow, Nauka, 1985, 336 p.
3. Loitsyansky L. G. Mechanics of liquids and gases. New York, Begell House, 1995, 803 p.
4. Shkadov V. Ja, Zaprjanov Z. D. Techenija vjazkoj zhizhki. Moscow, Izd-vo Mosk. u-ta, 1984, 200 p.
5. Hapfel J., Brenner H. Low Reynolds number hydrodynamics. Netherlands: Martinus Nijhoff, The Hague, 1973, 559 p.
6. Polyanin A. D., Zaitsev V. F. Handbook of Nonlinear Partial Differential Equations. Second Edition. CRC Press, 2011, 1912 p.
7. Rvachev V. L. Teoriya R-funkcij i nekotorye ee prilozhenija. Kiev, Nauk. dumka, 1982, 552 p.
8. Shapiro V. Semi-analytic geometry with R-functions, Acta Numerica, 2007, V. 16, pp. 239–303. DOI: 10.1017/S096249290631001X
9. Kolosova S. V. Primenenie proekcionskih metodov i metoda R-funkcij k resheniju kraevyh zadach v beskonechnyh oblastjah: diss. … k. f.-m. n.: 01.01.07. Vychislitel'naja matematika. Har'kov, HIRJe, 1972, 85 p.
10. Kolosova S. V., Sidorov M. V. Primenenie metoda R-funkcij k raschetu ploskich techenij vjazkoj zhizhki, Visnyk Harkiv's'kogo nacional'nogo novo universyetyt. Ser. Prykl. matem. i meh., 2003, No. 602, pp. 61–67.
11. Suvorova I. G. Komp'juternoe modelirovanie oseismetricnych techenij zhizhki v kanalakh slozhnoj formy, Vestnik nacional'nogo novo tehnicheskogo universiteta «HPI», 2004, No. 31, pp. 141–148.
12. Tcjashev A. D., Gibkina N. V., Sidorov M. V. Ob odnom podhode k matematicheskomu modelirovaniju ploskih stacionarnykh techenij vjazkoj nezhizmaemoj zhizhki v konechnyh odnosvayznyh oblastjah, Radioelektronika i informatika, 2007, No. 2 (37), pp. 50–57.
13. Maksimenko-Shejko K. V. Matematicheskoe modelirovanie teploobmena pri dvizhenii zhizhki po kanal s vintovym tipom simmetrii metodom R-funkcij, Dopovidi Nacional'noi' akademii' nauk Ukrai'ny, 2005, № 9, pp. 41–46.
14. Artyukh A., Sidorov M. Mathematical modeling and numerical analysis of nonstationary plane-parallel flows of viscous incompressible fluid by $R$-functions and Galerkin method, EconTechmod, 2014, Vol. 3, No 3, pp. 3–11.
15. Kolosova S. V., Lamtyugova S. N., Sidorov M. V. Ob odnom metode chislennogo analiza vjazkij techenij, uslozhnennyh massoobobnom (zadacha obtekanija), Radioelektronika i informatika, 2014, No. 1 (64), pp. 25–30.
16. Lamtyugova S. N. The $R$-functions method application to solving mass transfer problems, Theoretical and applied aspects of cybernetics. Proceedings of the 2nd international scientific conference of students and young scientists. Kyiv, Bukrek, 2012, pp. 108–111.
17. Lamtyugova S. N., Sidorov M. V. Matematicheskoe modelirovanie zadach obtekanija v cilindricheskoj sisteme koordinat, Visnyk HNU im. V. N. Karazina. Ser. Matematychne modeljuvannja. Informacijni tehnologii'. Avtomatyzovani systemy upravlinnja, 2014, №1105, vyp. 24, pp. 111–121.
18. Lamtyugova S. N., Sidorov M. V. Numerical analysis of the problem of flow past a cylindrical body applying the $R$-functions method and the Galerkin method, EconTechmod, 2014, Vol. 3, No. 3, pp. 43–50.
19. Strel'chenko A. J., Kolosova S. V., Rvachov V. L. Pro odyn metod rozv'jazuvannja kraevyh zadach, Dop. AN URSR, ser. A, 1972, No. 9, pp. 837–839.
20. Polyanin A. D., Nazaiinskiikkei V. E. Handbook of Linear Partial Differential Equations for Engineers and Scientists. Second Edition. CRC Press, 2016, 1632 p.