Correlation Clustering Reconstruction in Semi-Adversarial Models

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Abstract

Correlation Clustering is an important clustering problem with many applications. We study the reconstruction version of this problem in which one is seeking to reconstruct a latent clustering that has been corrupted by random noise and adversarial modifications.

Concerning the latter, we study a standard “post-adversarial” model, in which adversarial modifications come after the noise, and also introduce and analyse a “pre-adversarial” model in which adversarial modifications come before the noise. Given an input coming from such a semi-adversarial generative model, the goal is to reconstruct almost perfectly and with high probability the latent clustering.

We focus on the case where the hidden clusters have equal size and show the following. In the pre-adversarial setting, spectral algorithms are optimal, in the sense that they reconstruct all the way to the information-theoretic threshold beyond which no reconstruction is possible. In contrast, in the post-adversarial setting their ability to restore the hidden clusters stops before the threshold, but the gap is optimally filled by SDP-based algorithms.

1 Introduction

The rigorous analysis of combinatorial algorithms is most often carried out as a worst-case analysis over all possible inputs. In some cases, worst-case analysis makes excessively pessimistic predictions of the running time or memory use of a given algorithm, compared to its performance on typical data. In order to achieve more predictive rigorous analyses of algorithms, there has been interest in developing data models that go “beyond worst-case analysis,” combining adversarial choices and random choices. A notable example is the framework of smoothed analysis, introduced by Spielman and Teng \cite{ST} to analyze the simplex algorithm and then extended to other numerical problems, in which a worst-case instance is perturbed using random noise. In a complementary way, several semi-random generative models have been studied in which a random instance is perturbed, in a limited way, by an adversary. The monograph by Roughgarden \cite{R} surveys this active research program.

In unsupervised machine learning, the goal is to discover structure in data that is presented in an unstructured way. Typical problems include how to infer statistical parameters of the distribution of the given data points, how to cluster data points according to similarity information, or how to discover “community” structure in networks. Since unsupervised machine learning is postulated on the existence of a “ground truth” or “latent structure” that we want to discover, the rigorous analysis of an unsupervised machine learning algorithm must be carried out according to generative models that produce both a data set and a ground truth about the data set so that one can analyze whether the algorithm is able to discover the latter from the former.
Previous work on the rigorous analysis of unsupervised machine learning algorithms has typically been done according to a fixed, purely probabilistic, generative model. This type of analysis can sometimes make excessively optimistic predictions about the performance of a given algorithm, particularly if the algorithm is “overfit” to a particular generative model. In order to study the robustness of algorithms to data coming from sources whose distribution does not perfectly fit a simple probabilistic generative model, there has been interest in going “beyond average-case” in the analysis of unsupervised machine learning algorithms, introducing semi-adversarial models that combine probabilistic generation and adversarial choices. For example, in the field of robust statistics one is interested in inferring the parameters of a distribution given a mix of samples from the distribution and of adversarially selected outliers. A number of semi-adversarial network generation models have been considered to study community detection and clustering problems (see Section 2 below for a review of such results). A common feature of such models is that one first generates a sample according to a probabilistic generative model and then allows an adversary to modify the sample in a bounded way.

There are similarities between some of the semi-adversarial models that have been developed to analyze algorithms for various computational problems and the semi-adversarial models that have been developed to analyze algorithms for unsupervised machine learning, but it is important to remark on the different uses of such models in the two settings. A computational problem always has well-defined solutions, and the goal of analysis in semi-random models is to understand whether a polynomial-time algorithm is able to find an exact or an approximate solution; in an unsupervised machine learning task, one wants to find a latent structure defined in the generative model, and it is possible for the latent structure to be information-theoretically impossible to find if the noise and/or the adversarial model are too strong. When we talk about approximation in the context of solving optimization problems in semi-random models, we refer to how close is the cost of the solution found by the algorithm to the cost of an optimal solution; in an unsupervised machine learning task, the study of approximation refers to how close is the solution itself found by the algorithm to the ground truth.

1.1 Our Setting

We are interested in studying the correlation clustering problem in a semi-adversarial generative model, as an unsupervised machine learning problem. This is an important and well-studied problem about the analysis of dense Boolean matrices.

In the correlation clustering problem, we have $n$ data items, which we identify with the integers $\{1, \ldots, n\}$, and an unknown partition $C_1, \ldots, C_k$ of the items into clusters. We are given a symmetric $n \times n$ matrix $M$, where $M_{i,j} \in \{-1, +1\}$ represents a belief about items $i$ and $j$ being in the same cluster ($M_{i,j} = +1$ represents a belief that they are in the same cluster and $M_{i,j} = -1$ represents a belief that they are in different clusters). The goal is to reconstruct the partition from $M$.

A standard probabilistic generative model for correlation clustering is to start from a random equipartition $C_1, \ldots, C_k$ of $\{1, \ldots, n\}$, consider the “zero-error” matrix $\hat{M} \in \{+1, -1\}^{n \times n}$ such that that $\hat{M}_{i,j} = 1$ if and only if $i$ and $j$ belong to the same cluster, and obtain a matrix $M$ by applying random noise to $\hat{M}$. A simple noise model is obtained by setting $M_{i,j} = \hat{M}_{i,j}$ with probability $1/2 + \epsilon$, independently for every unordered pair $\{i,j\}$, and equal to $-\hat{M}_{i,j}$ with probability $1/2 - \epsilon$, for a noise parameter $\epsilon > 0$.

For constant $k$, this model exhibits a phase transition at $\epsilon \approx 1/\sqrt{n}$. When $\epsilon = o(1/\sqrt{n})$ then it is impossible to reconstruct the partition, even in an approximate way, and when $\epsilon = \omega(1/\sqrt{n})$ it is possible to reconstruct the partition with at most $o(n)$ items being misclassified.

In the regime in which reconstruction is possible, we are interested in introducing adversarial modifications in addition to random noise.

A possible semi-adversarial model, which is analogous to problems in robust statistics and to previous work on graph partitioning in semi-adversarial models, is to allow an adversary to modify a bounded number of entries of the matrix sampled from the probabilistic model. We refer to such a model as post-adversarial, because the adversary operates after random choices have been made.

For the correlation clustering problem, moreover, it is also natural to consider a semi-adversarial model,
that we call pre-adversarial, in which the adversary is allowed to modify the zero-error matrix in a bounded number of entries, and then random noise is applied to the matrix after these adversarial modifications. This model is somewhat in the spirit of smoothed analysis, in which noise is applied after an adversarial choice.

In the regime \( \epsilon > \omega(1/\sqrt{n}) \), it is easy to see that a pre-adversary with a budget of modifying \( O(n^2) \) entries or a post-adversary with a budget of modifying \( O(\epsilon n^2) \) entries are able to force any algorithm to misclassify \( \Omega(n) \) data points. Our goal is to understand whether it is possible to reconstruct the partition when the adversary has smaller budgets and, if so, whether polynomial-time algorithms are able to recover the latent clusters.

### 1.2 Our Contribution

All our results are asymptotic in \( n \) and assume constant \( k \) number of clusters.

**Optimal Pre-Adversarial Robustness of a Spectral Algorithms.** We show that a simple spectral algorithm is able to handle any pre-adversary that makes \( o(n^2) \) changes, in the feasible noise regime \( \epsilon > \omega(1/\sqrt{n}) \), leading to a polynomial-time reconstruction of the clustering with \( o(n) \) misclassified items, with high probability.

**Sub-Optimal and Yet Non-Trivial Post-Adversarial Robustness of Spectral Algorithms.** In the post-adversarial setting, in the noise regime \( \epsilon > \omega(1/\sqrt{n}) \), the same spectral algorithm is able to handle adversaries that make \( o(\epsilon^2 n^2) \) changes, delivering, as before, with high probability a reconstructed clustering with \( o(n) \) misclassified items. This analysis is nearly tight, in that we can devise post-adversarial strategies with a budget of \( O(\epsilon^2 n^2) \) changes which, for a wide range of values for \( \epsilon \), cause the spectral algorithm to misclassify \( \Omega(n) \) items, even in the \( k = 2 \) case.

**Optimal Post-Adversarial Robustness of SDP.** We then analyze an algorithm based on semidefinite programming (SDP), and we show that, in the post-adversarial setting, in the noise regime \( \epsilon > \omega(1/\sqrt{n}) \), the algorithm reconstructs in polynomial-time the correct clustering up to \( o(n) \) misclassifications, with high probability, for all post-adversaries that have a budget of \( o(\epsilon n^2) \) changes, matching an information-theoretic lower bound.

Our spectral results are based on bounding the spectral norm of the changes caused by the adversary. An adversary that makes up to \( B \) changes to a matrix that has \( \pm 1 \) entries can make changes whose spectral norm is at most \( O(\sqrt{B}) \). If such changes are made by a pre-adversary, the spectral norm of the changes after the application of the random noise is \( O(\epsilon \sqrt{B}) \), and the spectral algorithm works well provided that this is much smaller than \( \epsilon n \), which is true if \( B = o(n^2) \). If the changes are made by a post-adversary, then we need \( O(\sqrt{B}) \) to be much smaller than \( \epsilon n \), and so we need the condition \( B = o(\epsilon^2 n^2) \).

Our analysis of the semidefinite programming algorithm relies on bounding to \( \ell_{\infty} \)-to-\( \ell_1 \) operator norm \( ||\cdot||_{\infty \to 1} \) of the changes introduced by the adversary. Such norm is \( O(B) \) for \( B \) changes, and, in the post-adversarial setting, we need this quantity to be small compared to \( \epsilon n^2 \), leading to the optimal bound. Although it is NP-hard to compute the \( ||\cdot||_{\infty \to 1} \) norm and to find the corresponding test vectors, the Grothendieck inequality tells us that semidefinite programming can deliver a polynomial-time computable constant-factor approximation, which is enough for our application.

In previous analyses of semi-adversarial settings, spectral algorithms usually performed poorly in the presence of adversaries (with some exceptions, like [33]), so it is interesting that our pre-adversarial model provides an adversarial setting in which a spectral algorithm performs well all the way to information-theoretic limits. This is perhaps our main conceptual contribution. Technically, the analysis of the spectral algorithm is able to handle any pre-adversary that makes \( O(\epsilon^2 n^2) \) changes which, for a wide range of values for \( \epsilon \), cause the spectral algorithm to misclassify \( \Omega(n) \) items, even in the \( k = 2 \) case.

\[ \text{A notable difference is that our adversary has a limit to how many entries of the zero-error matrix it can modify, while in smoothed analysis the first step is to select a completely adversarial instance. In our setting, we need a generative model that produces both an instance of the problem and a ground-truth that can be reconstructed from the instance, so it is necessary to put some constraints on the ability of the adversary to erase information from the instance.} \]
algorithm relies on recognizing that the known average-case analyses of spectral algorithms for clustering algorithms depend on bounding the spectral norm of the difference between the empirical matrix that we are given and the average matrix, using concentration results for random matrices; we are able to extend this analysis to the semi-adversarial setting by bounding the spectral norm of the difference of the matrix before and after the intervention of the adversary, which is easy to do by using Frobenius norm as an intermediate step.

Our analysis of the semidefinite programming algorithm in the case $k = 2$ has some technical similarities with previous work on the use of SDP in the stochastic block model. For larger $k$, several technical difficulties arise, which we overcome thanks to an iterative partitioning algorithm. This requires careful handling of the build-up of classification errors introduced by previous iterative steps.

We comment on an additional piece of intuition that comes out of our work. From previous work on semi-adversarial models, there is well-established evidence that spectral algorithms perform poorly on matrices that are very sparse, for example on adjacency matrices or Laplacian matrices of sparse random graphs modified by an adversary. The reason is that it is possible to change a small number of entries of a sparse matrix and create spurious large eigenvalues with localized eigenvectors, and doing so is a good adversarial strategy to make a spectral algorithm fail. In correlation clustering, the given matrix is dense, and so our analysis in the pre-adversarial setting can be seen as providing complementary intuition that spectral algorithms can be robust on dense random matrices. But where is the difference coming from between the optimal behaviour in the pre-adversarial setting and the sub-optimal behavior in the post-adversarial setting? We can think of the application of noise as the following process: each entry is left unchanged with probability $2\varepsilon$, and it is replaced with a fresh random bit with probability $1 - 2\varepsilon$. According to the above point of view, after the application of noise there is only a sparse subset of $\varepsilon n^2$ entries that give information about the clustering, while all the other entries give no information. So we can see that the pre-adversary operates on a dense matrix of entries that give information about the clustering, while the post-adversary operates, effectively, on a sparser one, explaining the sub-optimal robustness of spectral methods, and the existence of adversarial strategies to create localized eigenvectors with large eigenvalues.

1.3 Roadmap

In Section 2 we discuss relevant related work. In Section 3 we define the problem precisely, introduce the notation, and recall some of the tools from the literature that we use. In developing our solution, for the sake of the exposition, we first focus in Section 4 on the case of $k = 2$ clusters. We then generalize to the case of an arbitrary, but constant, number of clusters in Section 5. Section 6 presents our lower bounds for reconstruction. In Section 7 we conclude by discussing the limitations of spectral approaches.

2 Related Work

Semi-Adversarial Models. Semi-Random (or, Semi-Adversarial) models have been the object of intense study in the recent past — see [31] for a comprehensive introduction to the topic. The original motivation to go beyond the worst-case analysis of algorithms was to come up with fast algorithms that, with high probability over the random choice of the input, returned an (approximately) optimal solution, to avoid dealing with input substructures that make the problem hard but that might not be often found in practice. In fully random models, however, an algorithm is only required to solve instances coming from a given distribution. As a result, many optimal solutions to fully-random models are overfitted to the random model and are unlikely to behave well with real-world instances.

Researchers, then, introduced several semi-random models, Smoothed Analysis [32] being perhaps the most famous exemplar. Here, an adversary begins by providing an instance of the problem; later, Nature perturbs the adversarial instance (i.e., it adds some limited random noise to it) and gives the perturbed instance to the algorithm. For many problems, then, the algorithm has to take into account the shape of the (original) adversarial instance, to come up with a solution for the perturbed one — that is, Smoothed
Analysis makes it impossible for algorithms to just leverage on the properties of random instances. Several optimization problems have been studied under the Smoothed Analytic lens, e.g., [3, 15, 32].

In short, Smoothed Analysis adds noise — from a given error distribution — to an adversarial instance. A different type of semi-random model inverts this order. It starts by producing a random instance (according to some distribution) and, later, lets an adversary modify some parts of the instance; the modified instance is finally given to the algorithm. This random-first-adversary-second setting, then, allows for some significant generality in the choice of the error distribution — since the “noisy” step is adversarial, it can simulate many random error distributions. Several problems have been studied in this setting, e.g., planted clique [17] (whose optimal algorithm so far is based on a spectral algorithm), various of its generalizations, e.g., the Stochastic Block Model [22, 23], as well as Densest Subgraph [8] and Correlation Clustering [24]. We will later say more on some of the above works, focusing on those that are most relevant to our work.

From a technical standpoint, the algorithmic strategies required for this type of semi-random model deviate significantly from the ones that have been successfully applied to the fully-random, and to the smoothed analysis, settings. In particular, purely spectral approaches (with no regularization) work in these settings, but often fail when the adversary enters the picture [22, 23]. In this paper, we observe similar behaviors of spectral and SDP-based methods. But, as pointed out, we also observe a certain unexpected robustness of the former which is only partial in the post-adversarial setting, but optimal in the pre-adversarial one.

Possibly, the foremost difference between our work and most of the previous semi-random ones lies in its algorithmic goal: here, we are not trying to optimize an objective function over a semi-random instance — we are, rather, trying to reconstruct the unknown parameters (the unknown base clustering) of the semi-random model, given one (adversarially perturbed) sample from it. Our specific goal significantly changes the techniques employed and the overall algorithmic approach.

In particular, in the context of rigorous machine learning, it has often been observed that the max-likelihood problem, when not enough samples are available, ends up with optimal solutions that are far from the unknown model parameters (see, e.g., [31]). That is, optimizing the max-likelihood objective does not, in general, return the hidden parameters of the model. In our case, we do not optimize a particular objective function: we directly aim to reconstruct the unknown clustering, even when the adversary perturbs the random instance.

**Correlation Clustering.** Correlation clustering is a basic primitive in the machine learner’s toolkit with applications ranging in several domains, including NLP [35], social network analysis [13], and clustering aggregation [18]. Correlation Clustering was introduced by Blum et al. [6], which also presented several approximation problems and algorithms for Correlation Clustering. Currently, the most famous such problem — that of minimizing the number of “mistakes” in the output clustering, assuming that each pair of input elements is labeled as either +1 or −1 — can be approximated in polynomial time to 2.06, thanks to the LP-based algorithm of [12]; this same problem is also known to be APX-hard [11].

Interestingly, the purely-random “seed reconstruction” version of the Correlation Clustering problem had already been considered in the original Correlation Clustering paper by Blum et al. [6].

**Clustering Reconstruction.** The fully-random model closest to ours is the Stochastic Block Model. Given a seed partition of the nodes of a graph into clusters, the Stochastic Block Model samples a random graph as follows: a biased coin is flipped independently for each pair of nodes, using a different bias depending on whether the two nodes are in the same cluster or in different clusters. Any pair of nodes from the same cluster have a probability $p$ of being connected by an edge; while any pair of nodes from different clusters have a probability $q < p$ of being joined by an edge. The problem of reconstructing the seed partition starting

\[ q = 1/2 - \epsilon \quad \text{and} \quad p = 1/2 + \epsilon, \]

and in which we interpret the presence of an edge as a +1 and the absence of an edge as −1. The stochastic block model, however, is typically analyzed in settings in which $p$ and $q$ are of the order of $1/n$ or $\log n/n$, leading to very sparse graphs.

\[ 1/2 \]
from such a random graph has been studied extensively, especially in the bounded degree setting, and several spectral algorithms, as well as algorithms based on semi-definite programming and Grothendieck’s inequality \[20\], have been proposed for solving it.

Several semi-adversarial variants of the Stochastic Block Models have been studied by the community. Building on the work of Feige and Killian \[16\], Makarychev et al \[22\] (and, independently, Moitra et al \[25\]) gave algorithms to reconstruct the seed partition starting from a graph obtained by monotone modifications (plus a limited amount of adversarial ones) of a sample from the SBM. More precisely, in their semi-adversarial model, Nature first samples a SBM graph; then, an adversary — if it chooses to do so — can monotonically strengthen the random signal by adding edges (i.e., turning some \(-1\)'s into \(+1\)'s) between nodes that are part of a same cluster of the seed, and removing edges (resp., turning \(+1\)'s into \(-1\)'s) between pairs of nodes that are in different clusters of the seed. Finally, the adversary has some limited budget (i.e., sublinear in the number of nodes) for modifying the remaining edges — i.e., for removing edges inside clusters and adding edges outside of clusters. The algorithms to reconstruct the seed partition, similarly to those of \[20\], are based on SDPs.

Mathieu and Schudy \[23\] studied a different version of the semi-adversarial correlation clustering reconstruction problem: in their model, as in ours, one begins with a partition \(C_1^*, \ldots, C_k^*\) of the \(n\) nodes into clusters. Then, each pair of nodes gets corrupted iid with probability \(p\): the adversary can then change the \(+1/-1\) label of each corrupted pair however it likes. (In other words, Nature flips a random set of edges, and the adversary can choose to fix some of the flips made by Nature.) \[23\] show that, if each original cluster has size \(\Omega(\sqrt{n})\), and if \(p \leq 1/3\), then a SDP based algorithm (together with a weighted version of the randomized rounding procedure of \[2\]) reconstructs the hidden clusters — that is, \[23\] guarantees reconstructibility when at least \(2/3\) of the edges remain untouched. This result is obtained by providing a solution to the dual of the SDP driving the algorithm of \[23\] — the solution is shown to be feasible with an eigenvalue-based analysis inspired by that of \[17\]; then, \[23\] shows that the dual solution has the same value of the unknown clustering \(C_1^*, \ldots, C_k^*\) in the primal, thus proving its optimality. An important difference between the error model of \[23\] and ours is that the constraints that they put on their adversary are such that exact reconstruction is possible, while in both our pre-adversarial and post-adversarial settings our adversary is able to erase all information about a subset of vertices, and hence exact reconstruction is information-theoretically impossible. In \[23\], the authors also consider this semi-adversarial noise model from the point of view of approximation algorithms for the correlation clustering objective function.

\textbf{Spectral Algorithms.} Spectral algorithms have been extensively used for cluster reconstruction \[1,29\]. Here, we restrict our discussion to works that apply spectral algorithms applied to (semi-)random models. Spectral algorithms have also been used for the (fully-random) Stochastic Block Model reconstruction. In particular, Boppana \[9\] introduces the spectral method for the fully-random graph bisection problem; McSherry \[24\] and Coja-Oghlan \[14\] improved the method to work for more general partitions, and to work with tighter gaps between the intra-cluster, and extra-cluster, probabilities.

\section{Preliminaries}

We study Correlation Clustering Reconstruction, defined as follows. We are given a complete graph of \(n\) points \(\{1, \ldots, n\} =: [n]\) divided into \(k\) clusters, each of size \(n/k\) (which we assume to be integer). If \(i, j\) belong to the same cluster the edge \(ij\) is labeled \(+1\), otherwise the label is \(-1\). In matrix notation, we are given a matrix \(M\) such that

\[
M_{i,j} := \begin{cases} +1 & \text{if } i, j \text{ are in the same cluster;} \\ -1 & \text{otherwise.} \end{cases}
\]

The matrix \(M\) is modified by random noise and adversarially, according to the following two processes. Let \(0 \leq \epsilon \leq 1/2\) and \(0 \leq B \leq n^2\). The quantity \(B\) is an integer and referred to as the \textit{budget} of the adversary.
**Pre-Adversary.** $M$ is modified as follows. First, an adversary swaps the labels of $B$ entries of $M$. The resulting matrix $M'$ is then modified by random noise: every entry of $M'$ is swapped with probability $1/2 - \epsilon$. The resulting matrix is denoted as $M''$.

**Post-Adversary.** Here the process is inverted: first we inject random noise and then let the adversary operate. First, every element of $M$ is swapped with probability $1/2 - \epsilon$. The resulting matrix is $M'$ (same notation, but the context will disambiguate). Second, an adversary swaps the sign of $B$ elements in the matrix, giving rise to a matrix $M''$.

In both cases, the Correlation Clustering Reconstruction problem is:

given $M''$, reconstruct $M$ as accurately as possible in polynomial-time.

This reconstruction goal is different from the usual optimization point of view. It is however of fundamental concern from the machine learning perspective. Observe that in the presence of such adversarial modifications it does not make sense to ask for Maximum Likelihood Estimation recovery of the latent clusters.

Note also that asking for a high probability of perfect reconstruction is futile, for the adversary can swap the clusters of two nodes with only $B = 2n$ changes. Therefore, we focus on approximate reconstruction. Our goal is to find polynomial-time algorithms such that, with probability $1 - o(1)$, they correctly classify $n - o(n)$ vertices under the pre- and post-adversary.

### 3.1 The technical toolkit

We now describe our main technical toolkit, consisting of definitions and known facts about matrix norms, eigenvalues and eigenvectors, and concentration inequalities. The reader familiar with such background can safely skip directly to the next section.

Given a vector $x \in \mathbb{R}^n$, the euclidean norms is defined as $\|x\| := \sqrt{\sum_{i=1}^{n} x_i^2}$, the $\ell_1$ norm is defined as $\|x\| := \sum_{i=1}^{n} |x_i|$, and the $\ell_\infty$ norm is defined as $\|x\|_\infty := \max_{i \in [n]} |x_i|$. We also define the scalar product between two vectors $x, y \in \mathbb{R}^n$ as $\langle x, y \rangle = x \cdot y := \sum_{i=1}^{n} x_i y_i$.

Given a square matrix $M \in \mathbb{R}^{n \times n}$, the Frobenius norm is defined as

$$\|M\|_F := \sum_{i,j=1}^{n} M_{i,j}^2.$$  

(1)

The spectral, or Operator, norm is defined as

$$\|M\|_{op} := \max_{\|x\| = 1} \|Mx\| = \max_{\|x\| = 1} \|x\| \max_{\|y\| = 1} \|x^T A y\|.$$  

(2)

The $\ell_\infty$-to-$\ell_1$ operator norm, is defined as,

$$\|M\|_{\infty \to 1} := \max_{x, y \in \{\pm 1\}^n} |x^T M y| = \max_{\|x\|_\infty \leq 1, \|y\|_\infty \leq 1} x^T M y = \max_{\|x\|_\infty = 1} |x^T M x| = \max_{\|x\|_\infty = 1} \|Mx\|_1.$$  

(3)

Finally, the SDP-norm:

$$\|M\|_{SDP} := \max_{\|x_h\|_2 = \|y_k\|_2 = 1 \ \forall \ h,k \in [n]} \sum_{i,j=1}^{n} M_{ij} \langle x_i, y_j \rangle = \max_{\|x_h\|_2 \leq 1, \|y_k\|_2 \leq 1 \ \forall \ h,k \in [n]} \sum_{i,j=1}^{n} M_{ij} \langle x_i, y_j \rangle.$$  

(4)

Let us also recall some known relationships and inequalities about these norms. (see [9] for the proofs)

**Lemma 3.1.** If $M$ is an $n \times n$ real matrix with rank $r$, then $\|M\|_{op}^2 \leq \|M\|_{SDP}^2 \leq r \cdot \|M\|_{op}^2$

Like the operator norm, the $\ell_\infty$-to-$\ell_1$ norm is monotone with respect to inclusion.
Lemma 3.2. Let $A \in \mathbb{R}^{n \times n}$, and let $B \subseteq A$ be a square sub-matrix of $A$. Then, $\|B\|_{\infty \to 1} \leq \|A\|_{\infty \to 1}$.

Lemma 3.3. Let $M \in \mathbb{R}^{n \times n}$. Then, $\|M\|_{\infty \to 1} \leq n \cdot \|M\|_{op}$.

Theorem 3.4 (Grothendieck’s Inequality). There exists a constant $c \leq 1.8$ such that, for every matrix $M \in \mathbb{R}^{n \times n}$, it holds

$$\|M\|_{\infty \to 1} \leq \|M\|_{SDP} \leq c \cdot \|M\|_{\infty \to 1}.$$ 

From [21], $c \leq \frac{1}{2} \ln(1 + \sqrt{2}) \simeq 1.782$. We also make use of the following known facts about eigenvalues.

Lemma 3.5 (Weyl’s Inequality). Let $M, N$ be symmetric matrices in $\mathbb{R}^{n \times n}$ with eigenvalues respectively $\mu_1 \geq \ldots \geq \mu_n$ and $\nu_1 \geq \ldots \geq \nu_n$. Let $\lambda_1 \geq \ldots \geq \lambda_n$ be the eigenvalues of $M + N$. Then,

$$\lambda_k + \nu_n \leq \mu_k \leq \lambda_k + \nu_1 \ \forall \ 1 \leq k \leq n.$$ 

Corollary 3.5.1. Let $M, E$ be symmetric matrices in $\mathbb{R}^{n \times n}$, where $M$ has eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$ and $M + E$ has eigenvalues $\lambda'_1 \geq \ldots \geq \lambda'_n$. Then,

$$|\lambda_k - \lambda'_k| \leq \|E\|_{op} \ \forall \ 1 \leq k \leq n.$$ 

Our analyses study how eigenvectors are affected by perturbations of the matrix. The following result is eminently useful in this regard.

Theorem 3.6 (Davis-Kahan-Wedin). Let $M, N$ be symmetric matrices in $\mathbb{R}^{n \times n}$ such that $M$ has eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$ with corresponding orthogonal eigenvectors $v_1, \ldots, v_n$, while $N$ has eigenvalues $\lambda'_1 \geq \ldots \geq \lambda'_n$ with corresponding orthogonal eigenvectors $v'_1, \ldots, v'_n$. Let $k \leq n$, and let $V_M \in \mathbb{R}^{n \times k}$ having $v_1, \ldots, v_k$ as columns, $V_N \in \mathbb{R}^{n \times k}$ having $v'_1, \ldots, v'_k$ as columns. Also, suppose $\delta_k := \lambda_k - \lambda_{k+1} > 0$. Then,

$$\|V_M V_M^T - V_N V_N^T\|_F \leq \frac{2\|N - M\|_{op}}{\delta_k}.$$ 

Let us now recall some well-known concentration inequalities.

Theorem 3.7 (Markov’s Inequality). Let $X$ be a positive random variable with finite expectation. Then, for any $a > 0$, it holds

$$\Pr(X \geq a) \leq \frac{\mathbb{E}[X]}{a}.$$ 

Theorem 3.8 (Chernoff–Hoeffding’s Inequality). Let $X_1, \ldots, X_n$ be a sequence of scalar random variables with $X_i \in [a_i, b_i] \ \forall \ i \in [n]$. Let $\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$. Then, for any $\lambda > 0$,

$$\Pr\left(|\overline{X} - \mathbb{E}[\overline{X}]| \geq \lambda\right) \leq 2 \cdot \exp\left(-\frac{2\lambda^2 n^2}{\sum_{i=1}^{n} (b_i - a_i)^2}\right).$$ 

Theorem 3.9 (Azuma’s Inequality). Let $X_1, \ldots, X_n$ be a sequence of scalar random variables with $|X_i| \leq c_i > 0$ almost surely. Assume also that we have the martingale difference property $\mathbb{E}[X_i | X_1, \ldots, X_{i-1}] = 0$ almost surely for all $1 \leq i \leq n$. Let $S_n = \sum_{i=1}^{n} X_i$ and $\gamma := \sqrt{\sum_{i=1}^{n} c_i^2}$. Then, for any $\lambda > 0$, $S_n$ obeys the large deviation inequality

$$\Pr(|S_n| \geq \lambda) \leq 2 \cdot \exp(-2\lambda^2/\gamma^2).$$ 

Theorem 3.10. Let $M$ be a $n \times n$ real random matrix whose entries $\{M_{i,j}\}$ are independent, have all expected value 0 (\mathbb{E}[M_{i,j}] = 0 \ \forall \ i, j) and are uniformly bounded in magnitude by 1 ($|M_{i,j}| \leq 1 \ \forall \ i, j$). Then, for every $A \geq 4$,

$$\Pr(\|M\|_{op} \geq A\sqrt{n}) \leq 2^{-An}.$$
Lemma 4.1. \[ \|M' - M\|_{op} \leq 2\sqrt{B} = o(n). \]

\textbf{Proof.} Define \( E := M' - M \). By definition, \( E \) has \( B \) non-zero entries, each of which is either \(-2\) or \(2\). Therefore, \( \|E\|_{F} = \sqrt{4B} = 2\sqrt{B} \). By Lemma 3.1, this implies that \( \|E\|_{op} \leq 2\sqrt{B} \).

\textbf{Lemma 4.2.} \( \text{Pr}(\|M'' - \mathbb{E}[M'']\|_{op} \geq 16\sqrt{n}) \leq 2^{-4n}. \)

\textbf{Proof.} First, notice that \( \mathbb{E}[M''] = \left( \frac{1}{2} + \epsilon \right) M' + \left( \frac{1}{2} - \epsilon \right) (-M') = 2\epsilon \cdot M' \), so

\[ M''_{i,j} - \mathbb{E}[M''_{i,j}] = \begin{cases} (1 - 2\epsilon)M'_{i,j} & \text{w. pr. } \frac{1}{2} + \epsilon; \\ -(1 + 2\epsilon)M'_{i,j} & \text{w. pr. } \frac{1}{2} - \epsilon. \end{cases} \]

Thus \( \frac{1}{1 + 2\epsilon}(M'' - \mathbb{E}[M'']) \) has all the elements bounded by 1 in absolute value. Moreover, we can write it as the sum of its upper triangular part, name it \( N \), and its lower triangular part, \( N^T \): \( N'': \frac{1}{1 + 2\epsilon}(M'' - \mathbb{E}[M'']) = N + N^T \). The matrix \( N \) satisfies the hypothesis of Theorem 3.10, so \( \text{Pr}(\|N\|_{op} \geq 4 \cdot \sqrt{n}) \leq 2^{-4n} \). By the
Thus, by Theorem 3.6, there is such a unitary eigenvector $v_1$ with high probability at least $1 - 2^{-4n}$. By the triangle inequality, \[ \|v_1\|_2 \leq \sqrt{\|M\|_{op}} \leq \sqrt{\|M'\|_{op}}. \] Lemma 4.2, with high probability \[ \|M'\|_{op} \leq 16\sqrt{n} + 4\epsilon \cdot \sqrt{B}. \] Finally, we notice that $\|M'' - 2\epsilon \cdot M\|_{op} \leq 16\sqrt{n} + 4\epsilon \cdot \sqrt{B}$.

We now show that the principal eigenvector of $M''$ is very close to that of $M$, which encodes the latent clustering, paving the way to the use of spectral algorithms.

**Lemma 4.4.** Let $v$ and $v''$ be the principal unitary eigenvectors of $M$ and $M''$, respectively. Then, with probability at least $1 - 2^{-4n} = 1 - o(1)$,

\[ \|v - v''\|_2 \leq \frac{8}{\epsilon \sqrt{n}} + \frac{2\sqrt{B}}{n} = o(1). \]

**Proof.** By what observed about the eigenvalues of $M$, $2\epsilon \cdot M$ is diagonalizable with eigenvalues $2\epsilon n$ and $0$. Thus, by Theorem 3.6, there is such a unitary eigenvector $v$ for which $\|vv^T - v''(v'')^T\|_F \leq \|M' - 2\epsilon \cdot M\|_{op}$. First, it holds $\|vv^T - v''(v'')^T\|_F^2 = \sum_{i,j=1}^n (v_i v_j - v_i' v_j'')^2 = 2 - 2\langle v, v'' \rangle^2$. Second, by Theorem 4.3 with high probability $1 - 2^{-4n} = 1 - o(1)$, it holds $\|M'' - 2\epsilon \cdot M\|_{op} \leq 16\sqrt{n} + 4\epsilon \cdot \sqrt{B} = o(\epsilon n)$, so

\[ 2 - 2\langle v, v'' \rangle^2 \leq \left( \frac{16\sqrt{n} + 4\epsilon \cdot \sqrt{B}}{2\epsilon n} \right)^2 = \left( \frac{8}{\epsilon \sqrt{n}} + \frac{2\sqrt{B}}{n} \right)^2 = o(1). \]

\[ \Rightarrow \langle v, v'' \rangle \geq \sqrt{1 - \frac{1}{2} \left( \frac{8}{\epsilon \sqrt{n}} + \frac{2\sqrt{B}}{n} \right)^2} \geq 1 - \frac{1}{2} \cdot \left( \frac{8}{\epsilon \sqrt{n}} + \frac{2\sqrt{B}}{n} \right)^2 = 1 - o(1) \]

because $\sqrt{1 - x} \geq 1 - x$ for each $x \in [0, 1]$. However, $\|v - v''\|^2 = 2 - 2\langle v, v'' \rangle$, which implies that

\[ \|v - v''\|^2 \leq \left( \frac{8}{\epsilon \sqrt{n}} + \frac{2\sqrt{B}}{n} \right)^2 = o(1). \]

Lemma 4.3 implies that a straightforward application of the well-known power method can recover the latent clusters almost perfectly (see [19]). This is spelled out in Algorithm 1.

**Algorithm 1** Input: $M''$, $n$

1. $S_1 \leftarrow \emptyset; S_2 \leftarrow \emptyset$
2. Let $v''$ be the eigenvector of the leading eigenvalue of $M''$ computed with the POWER-METHOD
3. for $i = 1, \ldots, n$ do
   4. if $v''_i > 0$ then
      5. $S_1 \leftarrow S_1 \cup \{i\}$
   6. else
      7. $S_2 \leftarrow S_2 \cup \{i\}$
8. Return $\{S_1, S_2\}$
Theorem 4.5. With probability $1 - o(1)$, Algorithm 1 correctly classifies $n - o(n)$ vertices, where the number of misclassified vertices is at most $128/\varepsilon^2 + 8B/n = o(n)$.

Proof. Let $v$ be a unitary eigenvector of the largest eigenvalue of $M$ (or, equivalently, of $2\varepsilon \cdot M$), and let $v''$ be the unitary eigenvector of the largest eigenvalue of $M''$. By Lemma 4.4 we have that, with high probability $\geq 1 - 2^{-4n} = 1 - o(1)$, it holds

$$
\|v - v''\| \leq \frac{8}{\varepsilon \sqrt{n}} + \frac{2\sqrt{B}}{n} = o(1) \tag{4.37}
$$

Now, unless we swap the clusters, we can write w.l.o.g. $v = \frac{1}{\sqrt{\alpha}} f_1 - \frac{1}{\sqrt{\alpha}} f_2$, where $f_i$ is the indicator vector of the $i$th cluster. Now, let $S_{bad} := \{ i \in [n] : v_i v_i'' < 0 \}$, i.e. the set of coordinates with swapped signs between $v$ and $v''$. Those correspond exactly to the misplaced vertices in Algorithm 1. However, if $i \in S_{bad}$, by definition of $v$, we have either $v_i = \frac{1}{\sqrt{n}}$ and $v_i'' \leq 0$ or $v_i = -\frac{1}{\sqrt{n}}$ and $v_i'' \geq 0$, implying that $|v_i - v_i''| \geq \frac{1}{\sqrt{n}}$. Thus,

$$
\left(\frac{8}{\varepsilon \sqrt{n}} + \frac{2\sqrt{B}}{n}\right)^2 \geq \|v - v''\|^2 \geq \sum_{i \in S_{bad}} |v_i - v_i''|^2 \geq \frac{|S_{bad}|}{n},
$$

implying that $|S_{bad}| \leq n \left(\frac{8}{\varepsilon \sqrt{n}} + \frac{2\sqrt{B}}{n}\right)^2 \leq n \left(\frac{128}{\varepsilon^2 n} + \frac{8B}{n^2}\right) = \frac{128}{\varepsilon^2} + \frac{8B}{n} = o(n)$, where we used that $(a + b)^2 \leq 2(a^2 + b^2)$ for each $a, b \in \mathbb{R}$. \hfill \Box

Except for the invocation of the Power-Method, the running time of Algorithm 1 is linear in $n$. The running time of the Power-Method depends on the ratio between the first eigenvalue and the gap between the first two eigenvalues of $M''$. This is $\approx 1$ because $\|M'' - 2\varepsilon \cdot M\|_{op} \leq 16\sqrt{n} + 4\varepsilon \cdot \sqrt{B} = o(cn)$ with high probability by Theorem 4.3. And so this ratio is very close to the one of $M$, which is $1$. It follows by known facts about the Power-Method that its running time is polynomial, and it computes a solution with negligible error.

4.2 Dealing with the Post-Adversary

Let us now consider the post-adversarial setting. We first show that the spectral algorithms, perhaps surprisingly, remain somewhat robust in the presence of a post-adversary. Their robustness is limited however, and unable to match the information-theoretic lower bounds we establish in Section 6. Fortunately, the gap can be filled thanks to an SDP-based approach that is optimal in this setting.

As before, we begin by proving a few useful facts about matrix norms of our matrices.

Lemma 4.6. $\Pr(\|M' - \mathbb{E}[M']\|_{1 \to 1} \geq 16n\sqrt{n}) \leq \Pr(\|M' - \mathbb{E}[M']\|_{op} \geq 16\sqrt{n}) \leq 2^{-4n}$.

Proof. First, recall that $\mathbb{E}[M'] = 2\varepsilon \cdot M$, so

$$
M', i,j : \mathbb{E}[M', i,j] = \begin{cases} 
(1 - 2\varepsilon) M_{i,j} & \text{w. pr. } \frac{1}{2} + \varepsilon; \\
-(1 + 2\varepsilon) M_{i,j} & \text{w. pr. } \frac{1}{2} - \varepsilon.
\end{cases}
$$

Thus $\frac{1}{1 + 2\varepsilon}(M' - \mathbb{E}[M'])$ has all the elements bounded by 1 in absolute value. By using the exact same argument of Lemma 4.2 we can show that $\Pr(\|M' - \mathbb{E}[M']\|_{op} \geq 16\sqrt{n}) \leq 2^{-4n}$. Now, by Lemma 4.3 it holds $\|M' - \mathbb{E}[M']\|_{op} \geq \frac{1}{\varepsilon} \cdot \|M' - \mathbb{E}[M']\|_{1 \to 1}$, so

$$
\Pr(\|M' - \mathbb{E}[M']\|_{1 \to 1} \geq 16n\sqrt{n}) \leq \Pr(\|M' - \mathbb{E}[M']\|_{op} \geq 16\sqrt{n}) \leq 2^{-4n}.
$$

Lemma 4.7. $\|M'' - M'\|_{1 \to 1} \leq 2B; \|M'' - M'\|_{op} \leq 2\sqrt{B}$. 

11
Proof. Define $E := M'' - M'$. By definition, $E$ has $B$ non-zero entries, each of which is either $-2$ or $2$. Therefore, since each one of those entries can contribute by at most $2$ to the norm of $E$, $\|E\|_{\infty \rightarrow 1} \leq 2B$. Moreover, by the same reason, $\|E\|_{2p} \leq 2\sqrt{B}$. So, by Lemma 3.3 also $\|E\|_{op} \leq 2\sqrt{B}$. 

By putting those lemmas together, we can relate the final matrix $M''$ to the initial matrix $M$.

**Theorem 4.8.** With probability at least $1 - 2^{-4n} = 1 - o(1)$,

$$\|M'' - 2\epsilon \cdot M\|_{\infty \rightarrow 1} \leq 16n\sqrt{n} + 2B; \quad \|M'' - 2\epsilon \cdot M\|_{op} \leq 16\sqrt{n} + 2\sqrt{B}$$

**Proof.** By the triangle inequality, $\|M'' - 2\epsilon \cdot M\|_{\infty \rightarrow 1} \leq \|M'' - M'\|_{\infty \rightarrow 1} + \|M' - 2\epsilon \cdot M\|_{\infty \rightarrow 1}$. First, by Lemma 4.7 it holds $\|M'' - M'\|_{\infty \rightarrow 1} \leq 2B$. Second, we have that $\mathbb{E}[M'] = 2\epsilon \cdot M$, so $\|M' - 2\epsilon \cdot M\|_{\infty \rightarrow 1} = \|M' - \mathbb{E}[M']\|_{\infty \rightarrow 1}$. Now, by Lemma 4.8 with high probability $\geq 1 - 2^{-4n} = 1 - o(1)$, it holds $\|M' - \mathbb{E}[M']\|_{\infty \rightarrow 1} \leq 16\sqrt{n}$. By putting everything together, we finally get that $\|M'' - 2\epsilon \cdot M\|_{\infty \rightarrow 1} \leq 16n\sqrt{n} + 2B$. The same exact strategy can be used to prove the bound for the operator norm too. 

### 4.2.1 Sub-optimal Robustness of the Spectral Approach

Let

$$\epsilon = \omega(n^{-1/2}) \quad \text{and} \quad B = o(\epsilon^2 n^2). \quad (6)$$

Let us show that, for these parameters, the spectral approach is robust w.r.t. a post-adversary. By Theorem 4.3 with probability at least $1 - 2^{-4n} = 1 - o(1)$, it holds $\|M'' - 2\epsilon \cdot M\|_{op} \leq 16\sqrt{n} + 2\sqrt{B} = o(en)$. Therefore, we are back to the same exact setting we analyzed with the pre-adversary: we have a matrix $M''$ such that, with high probability, $\|M'' - 2\epsilon \cdot M\|_{op} \leq o(\epsilon n)$, as in Theorem 4.3. Thus, as showed in Lemma 4.4, the eigenvector of the leading eigenvector of $M''$ is within euclidean distance $o(1)$ from the main eigenvector of the original matrix $M$, implying that it can be effectively used to approximately reconstruct the clusters. Therefore, we can rely on Algorithm 1.

As we show in Section 4, as far as $B$ is concerned this analysis is nearly tight asymptotically for a wide range of values of $\epsilon$. Indeed, when $\epsilon = \Omega((\log(n)/n)^{1/3})$, by modifying $B = \Theta(\epsilon^2 n^2)$ entries, the post-adversary can induce with high-probability a principal eigenvector that gives no information on the latent clustering. This makes the spectral algorithms like Algorithm 1 fail and creates a gap between the information-theoretic threshold, which is $B = o(\epsilon n^2)$ (see Section 4), and the reach of spectral methods. We believe that the same techniques can be used to prove that the same impossibility result holds for any $\epsilon = \omega(n^{-1/2})$, but we have not been able to prove it in this work.

In the following subsection we show that the gap can be eliminated thanks to an SDP-based algorithm.

### 4.2.2 SDP to the rescue

In this section we show that SDP can recover the cluster almost perfectly and with high probability all the way to the information-theoretic threshold against the post-adversary. Let

$$\epsilon = \omega(n^{-1/2}) \quad \text{and} \quad B = o(\epsilon n^2). \quad (7)$$

Consider the following Semidefinite Program:

$$\begin{align*}
\text{maximize} & \quad \sum_{i,j=1}^{n} M''_{ij}\langle x_i, y_j \rangle \\
\text{subject to} & \quad \|x_i\| = 1, \; x_i \in \mathbb{R}^n \quad \forall i = 1, \ldots, n \\
& \quad \|y_i\| = 1, \; y_i \in \mathbb{R}^n \quad \forall i = 1, \ldots, n
\end{align*} \quad (8)$$

Henceforth we shall refer to this program as SDP. By definition of SDP-norm, its maximum is $\|M''\|_{SDP}$. Given the optimal solution $\{x_i^*\}_{i \in [n]}, \{y_i^*\}_{i \in [n]}$, consider the matrix $X$ where $X_{ij} := \langle x_i^*, y_j^* \rangle \; \forall i, j \in [n]$. 

12
Algorithm 2: input $M''$, $n$

1: $S_1 \leftarrow \emptyset$; $S_2 \leftarrow \emptyset$
2: Let $X$ be the solution matrix of SDP obtained through SDP-Solver
3: Let $\{u^1, \ldots, u^n\}$ be an orthogonal basis of eigenvectors of $X$ with eigenvalues respectively $\{\lambda_i, i \in [n']\}$ obtained through Power-METHOD
4: Sample $u^* \in \{u^i, i \in [n]\}$ with probability distribution $\{\frac{\lambda_i}{\sum_i \lambda_i}, i \in [n]\}$
5: for $i = 1, \ldots, n$ do
6: if $u^*_i > 0$ then
7: $S_1 \leftarrow S_1 \cup \{i\}$
8: else
9: $S_2 \leftarrow S_2 \cup \{i\}$
10: Return $\{S_1, S_2\}$

Then, $\|M''\|_{SDP} = M'' \cdot X$, where $\cdot$ is the Kronecker (element-wise) product. Since $M''$ is symmetric, $X$ is symmetric too. Solving SDP is the key step in the reconstruction, carried out by Algorithm 2.

The main difference between Algorithm 2 and Algorithm 1 is how the former computes the eigenvector of interest. First, Algorithm 2 computes the matrix $X$ solution to SDP. Second, it computes an orthogonal basis of eigenvectors for $X$ with the Power-METHOD. Third, it samples an eigenvector of the basis according to an appropriate probability distribution. The use of $X$ is necessary because the main eigenvector of $M''$, as we have argued previously, could no longer be informative. Notice that $\{\lambda/\pi, i \in [n]\}$ is a probability distribution since $\sum_{i=1}^n \lambda_i = \text{tr}(X) = n$. We will show that, surprisingly, if we sample $u^* \in \{u^i, i \in [n]\}$ with probability distribution $\{\frac{\lambda_i}{\sum_i \lambda_i}, i \in [n]\}$, with high probability we get an eigenvector that is very close to a main eigenvector of the original matrix $M$, and we believe this sampling technique for the eigenvector $u^*$ to be key to the functioning of our algorithm.

Lemma 4.9. With probability at least $1 - 2^{-4n}$, $\|M''\|_{SDP} = M'' \cdot X \geq 2\epsilon n^2 - 29n\sqrt{n} - 4B = 2\epsilon n^2 - o(\epsilon n^2)$.

Proof. First, it holds $\|M\|_{\infty-1} = n^2$. Consider the vector $x = f_1 - f_2$, having +1 for elements in the first cluster and −1 for elements in the second cluster. This satisfies $\|x\|_{\infty-1} = 1$, so $n^2 \geq \|M\|_{\infty-1} \geq |x^T M x| = n^2$. As a consequence, $\|2\epsilon \cdot M\|_{\infty-1} = 2\epsilon n^2$. Since $\|M\|_{\infty-1} \leq \|M\|_{SDP} \leq \sum_{i,j} |M_{ij}| = n^2$, we also have that $\|2\epsilon \cdot M\|_{SDP} = 2\epsilon n^2$. Second, by Lemma 4.8 with high probability $\geq 1 - 2^{-4n} = 1 - o(1)$ it holds

$$\|M'' - 2\epsilon \cdot M\|_{\infty-1} \leq 16n\sqrt{n} + 2B = o(\epsilon n^2).$$

Third, by the triangle inequality,

$$M'' \cdot X = \|M''\|_{SDP} \geq \|2\epsilon \cdot M\|_{SDP} - \|M'' - 2\epsilon \cdot M\|_{SDP}.$$

Now, by Theorem 3.4 and by what we have just proved, $\|2\epsilon \cdot M\|_{SDP} = 2\epsilon n^2$ and $\|M'' - 2\epsilon \cdot M\|_{SDP} \leq 1.8 \cdot \|M'' - 2\epsilon \cdot M\|_{\infty-1} \leq 1.8 \cdot (16n\sqrt{n} + 2B) \leq 29n\sqrt{n} + 4B$. Thus, by putting everything together, we finally get that

$$\|M''\|_{SDP} \geq 2\epsilon n^2 - 29n\sqrt{n} - 4B.$$

Corollary 4.9.1. With probability at least $1 - 2^{-4n}$, $\|M'' - 2\epsilon \cdot M\|_{SDP} \leq 29n\sqrt{n} + 4B = o(\epsilon n^2)$.

Since $X$ is symmetric, we can consider its spectral decomposition into orthogonal eigenvectors:

$$X = \sum_{i=1}^n \lambda_i u^i (u^i)^T.$$

Now, sample $u^*$ among the eigenvectors $\{u^i\}_{i \in [n]}$ of this basis, each with probability $\lambda_i / n$. We claim that $u^*$, with high probability, gives a very good clustering reconstruction, because it is very close to $u$, the main eigenvector of $M$ (or to its opposite vector).
Lemma 4.10. Let \( u^* \) be the eigenvector of the solution matrix \( X \) of SDP given by Algorithm 3 line 4. Then, with probability at least \( 1 - 2\sqrt{29/\epsilon n} + 4B/\epsilon n^2 = 1 - o(1) \), an eigenvector \( v \) of the leading eigenvalue of \( M \) satisfies
\[
\|u^* - v\|^2 \leq 2 \frac{29}{\epsilon \sqrt{n}} + \frac{4B}{\epsilon n^2} = o(1).
\]

Proof. First, we show that \( M \cdot X \geq n^2 - o(n^2) \). We have that \( M \cdot X = \frac{1}{\epsilon} (M'' \cdot X - (M'' - 2\epsilon \cdot M) \cdot X) \). Now, by Corollary 4.9.1 with high probability \( \geq 1 - 2^{-4n} = 1 - o(1) \), it holds \( \|M'' - 2\epsilon \cdot M\|_{\text{SDP}} \leq 29n/\epsilon n + 4B = o(\epsilon n^2) \), so \( (M'' - 2\epsilon \cdot M) \cdot X \leq \|M'' - 2\epsilon \cdot M\|_{\text{SDP}} \leq 29n/\epsilon n + 4B \), and by Lemma 4.10 it holds \( M'' \cdot X \geq 2\epsilon n^2 - 29n/\epsilon n - 4B \), implying that
\[
M \cdot X \geq n^2 - \frac{29n/\epsilon n + 4B}{\epsilon} = n^2 - o(n^2).
\]

Now \( X = \sum_{i=1}^{n} \lambda_i u_i(u^i)^T \) is positive semidefinite and \( \sum_{i=1}^{n} \lambda_i = \text{tr}(X) = \sum_{i=1}^{n} \|x_i\|^2 = n \). Therefore, \( \{\lambda_i/n\} \) can be seen as a probability distribution. By definition of \( X \), we can consider an orthogonal basis of eigenvectors for it, \( \{u^i\}_{i \in [n]} \), with corresponding eigenvalues \( \{\lambda_i\}_{i \in [n]} \). It holds
\[
M \cdot X = n \sum_{i=1}^{n} \frac{\lambda_i}{n} (u^i)^T M u^i.
\]

We can write \( M = n\nu\nu^T \), where \( \nu = \frac{1}{\sqrt{n}}(f_1 - f_2) \) is its main unitary eigenvector, so that \( (u^i)^T M u^i = n \langle u^i, \nu \rangle^2 \). We get that, with high probability \( \geq 1 - 2^{-4n} = 1 - o(1) \), it holds
\[
\sum_{i=1}^{n} \frac{\lambda_i}{n} \langle u^i, \nu \rangle^2 \geq 1 - \frac{29}{\epsilon \sqrt{n}} + \frac{4B}{\epsilon n^2} = 1 - o(1).
\]

We can notice that the LHS is exactly equal to \( \mathbb{E}[(u^*, \nu)^2] \). Since \( (u^*, \nu)^2 \leq \|\nu\|^2 = 1 \), we can define the positive random variable \( \chi := 1 - (u^*, \nu)^2 \). We have that \( \chi \geq 0 \) and \( \mathbb{E}[\chi] \leq \frac{29}{\epsilon \sqrt{n}} + \frac{4B}{\epsilon n^2} = o(1) \). Thus, by Theorem 3.7 we have that with high probability \( \geq 1 - \frac{29}{\epsilon \sqrt{n}} + \frac{4B}{\epsilon n^2} \), it holds \( \chi \leq \sqrt{\frac{29}{\epsilon \sqrt{n}} + \frac{4B}{\epsilon n^2}} \), implying that
\[
|\langle u^*, \nu \rangle| \geq (u^*, \nu)^2 \geq 1 - \sqrt{\frac{29}{\epsilon \sqrt{n}} + \frac{4B}{\epsilon n^2}} = 1 - o(1).
\]

We can assume w.l.o.g. that \( \langle u^*, \nu \rangle > 0 \), because we just need to take the opposite of \( \nu \) to change the sign of \( \langle u^*, \nu \rangle \). Therefore, conditioned on the previous events, it holds
\[
\|v - u^*\|^2 = \langle v - u^*, v - u^* \rangle = 2 - 2 \langle v, u^* \rangle \leq 2 \sqrt{\frac{29}{\epsilon \sqrt{n}} + \frac{4B}{\epsilon n^2}}.
\]

All the events hold together with probability \( \geq 1 - \sqrt{\frac{29}{\epsilon \sqrt{n}} + \frac{4B}{\epsilon n^2}} - 2^{-4n} \geq 1 - 2 \sqrt{\frac{29}{\epsilon \sqrt{n}} + \frac{4B}{\epsilon n^2}} \) for sufficiently large \( n \).

We are now in a very similar situation to the pre-adversarial case: we can compute an eigenvector that, with high probability, is very close to the main eigenvector of the original matrix \( M \) from which, as the next theorem shows, the latent clustering can be almost perfectly recovered.

Theorem 4.11. With probability \( 1 - o(1) \), Algorithm 3 correctly classifies \( n - o(n) \) vertices, where the number of misclassified vertices is \( \leq 2\sqrt{29n/\epsilon \sqrt{n} + 4B/\epsilon} = o(n) \).
Proof. By Lemma 4.10 $u^*$, obtained in Algorithm 2 line 4 with high probability $1 - o(1)$ satisfies $\|v - u^*\|^2 \leq 2\sqrt{\frac{29}{\epsilon \sqrt{n}} + \frac{4B}{cn^2}} = o(1)$, where $v$ is a main eigenvector of the original matrix $M$. Now, unless we swap the clusters, we can write w.l.o.g. $v = \frac{1}{\sqrt{n}} f_1 - \frac{1}{\sqrt{n}} f_2$, where $f_i$ is the indicator vector of the $i^{th}$ cluster. Now, let $S_{bad} := \{i \in [n] : v_i u_i^* < 0\}$, i.e. the set of coordinates with swapped signs between $v$ and $u^*$. Those correspond exactly the misplaced vertices in Algorithm 2. However, if $i \in S_{bad}$, since $|v_i| \geq \frac{1}{\sqrt{n}} \forall i \in [n]$, it holds $|v_i - u_i^*| \geq \frac{1}{\sqrt{n}}$. Thus,

$$2^\sqrt{\frac{29}{\epsilon \sqrt{n}} + \frac{4B}{cn^2}} \geq \|v - u^*\|^2 \geq \sum_{i \in S_{bad}} |v_i - u_i^*|^2 \geq \frac{|S_{bad}|}{n},$$

implying that $|S_{bad}| \leq n \cdot 2^\sqrt{\frac{29}{\epsilon \sqrt{n}} + \frac{4B}{cn^2}} = 2^\sqrt{29n/\epsilon + 4B/\epsilon} = o(n)$.

Let us analyse the running time of Algorithm 2. Solving SDP through SDP-Solver up to negligible error takes polynomial time. Likewise, computing an orthogonal basis of eigenvectors for $X$ up to negligible error takes polynomial time. Finally, the random sampling of $u^*$ according to a discrete distribution with support size $n$ can also be done in time polynomial in $n$ [10]. The resulting running time is thus polynomial.

5 Reconstruction for an Arbitrary, but Constant, Number of Clusters

Here we generalize the approach of Section 4 to the case in which there are $k = O(1)$ equinumerous clusters, each of size $n/k$.

5.1 Properties of the Input Matrices

Notice that $M$ has rank $k$ now, for $k > 2$, because it has $k$ distinct rows. Recall the definition of $f_i$, the characteristic vector of the $i^{th}$ cluster: there are 1’s in the positions corresponding to the elements of the cluster, and 0 everywhere else. The rows of $M$ are spanned by the vectors $\{f_i\}_{1 \leq i \leq k}$ and are linearly independent, as it can be shown by induction using the Gaussian elimination. Let us now look at the spectrum of $M$.

First, 0 is an eigenvalue for $M$ whose eigenspace, for $k > 2$, has dimension $n - k$ by the Dimension Theorem for vector spaces (it is described by a homogeneous equation whose associated matrix, $M$, has rank $k$). Second, $2 \cdot n/k$ is also an eigenvalue whose eigenspace has dimension $k - 1$. A basis for it is $\{f_i - f_{i+1}, 1 \leq i \leq n - 1\}$. If $k > 2$, we also have another eigenvalue: $-(k-2)/k \cdot n = (2/k - 1)n$, whose eigenspace has dimension 1 and is spanned by the eigenvector with all identical coordinates. There are no more eigenvectors, since the vector space $\mathbb{R}^n$ is the direct sum of these eigenspaces.

For our approach to work, we need a positive semidefinite matrix describing the clustering. Moreover, we would like the vector with all identical coordinates not to be an eigenvector with a high eigenvalue of this matrix. This is the reason why we introduce a suitable positive-semidefinite version of the input matrix.

5.1.1 A Positive Semidefinite “Correlation Matrix”

We have seen that the original matrix $M$ has a negative eigenvalue $-k-2/k \cdot n$ for $k > 2$, with corresponding unitary eigenvector $z$ whose coordinates are all equal to $1/\sqrt{n}$. This does not carry any information about the clusters. By removing it from the spectral decomposition, we can consider a different “correlation matrix”,

$$P := \frac{k}{2(k-1)} \cdot \left(M + n \left(1 - \frac{2}{k}\right)zz^T\right),$$
whose entries are:
\[ P_{i,j} := \begin{cases} 1 & \text{if } i, j \text{ are in the same cluster;} \\ -\frac{1}{k-1} & \text{otherwise.} \end{cases} \]

This matrix has rank \( k-1 \) and \( k \) distinct rows, the last of which is the opposite of the sum of the previous \( k-1 \) ones. Its spectrum consists of:

- the positive eigenvalue \( \frac{n}{k-1} \), whose eigenspace has dimension \( k-1 \), with a basis given by \( \{ f_i - f_{i+1} \}_{i \in [k-1]} \). Notice that this is also the subspace of vectors having all the same coordinates for vertices in the same cluster and having sum of coordinates equal to 0;
- 0, whose eigenspace has dimension \( n-k+1 \) and is the complementary to the previous eigenspace. This subspace is described by the equation \( Px = 0 \).

It is useful to find an orthogonal basis for the eigenspace of \( \frac{n}{k-1} \). With the Gram-Schmidt orthogonalization procedure, we can get an orthogonal basis \( v_1, \ldots, v_{k-1} \) where:
\[
v_i := \frac{1}{\sqrt{n/k}} \left( \frac{1}{\sqrt{x^2 + i}} \sum_{j=1}^{i} f_j - \frac{i}{\sqrt{x^2 + i}} f_{i+1} \right) \quad \forall i \in [k-1].
\]

Equation (9) can be shown by induction. These vectors are mutually orthogonal, have identical coordinates for vertices in the same cluster and their coordinates sum to 0. For instance, for \( n = k = 3 \), we get,
\[
v_1 = \frac{1}{\sqrt{2}} (f_1 - f_2); \quad v_2 = \frac{1}{\sqrt{6}} (f_1 + f_2) - \frac{2}{\sqrt{6}} f_3.
\]
Notice that, for any \( k \), any vector of the orthogonal basis detects at least one cluster. Moreover, any orthogonal vector in this subspace detects a bisection into disjoint clusters. We exploit this to reconstruct the original clusters iteratively.

**Lemma 5.1.** Let \( v_i \) be as in Equation (9) for \( i \in [k-1] \). And let \( x := \sum_{i=1}^{k-1} \lambda_i v_i \), where \( \|x\| = \sum_{i=1}^{k-1} \lambda_i^2 = 1 \). Then, there exists \( i \neq j \in [k] \) such that, if \( x_i \) is the coordinate of \( x \) along the vertices of the \( i \)th cluster, then
\[
|x_i - x_j| > \frac{1}{\sqrt{k} \cdot \sqrt{n}}.
\]

**Proof.** Let \( y_i := \sqrt{k} \cdot x_i \) for each \( i \in [k] \). Assume by contradiction that our statement is false, so \( |y_i - y_j| \leq \frac{1}{\sqrt{n/k}} \) for each \( i \neq j \in [k] \). First, we prove by induction that this implies
\[
\frac{|\lambda_h|}{\sqrt{h^2 + h}} \leq (1 - 2^{-h}) \frac{1}{k^{3/2}} \quad \forall h \in [k-1].
\]

**Base Case:** \( h = 1 \). By our assumption, we have that \( |y_1 - y_2| \leq \frac{1}{\sqrt{k/2}} \). However, \( |y_1 - y_2| = 2 |\lambda_1| \sqrt{2} \), thus \( \frac{|\lambda_1|}{\sqrt{2}} \leq \frac{1}{2^{1/2}} = (1 - 2^{-1}) \frac{1}{k^{3/2}} \).

**Inductive Step:** \( (h-1) \rightarrow h \). By our assumption, we have that \( |y_{h-1} - y_{h+1}| \leq \frac{1}{\sqrt{k/2}} \). However, \( |y_{h} - y_{h+1}| = 2 \frac{|\lambda_{h-1}|}{\sqrt{h^2 + h}} - \frac{|\lambda_h|}{\sqrt{h^2 + h}} \leq 2 \frac{|\lambda_{h-1}|}{\sqrt{h^2 + h}} - \frac{|\lambda_h|}{\sqrt{h^2 + h}} \) by the triangle inequality. Moreover, by the inductive hypothesis, \( \frac{|\lambda_{h-1}|}{\sqrt{h^2 + h}} \leq (1 - 2^{-(h-1)}) \frac{1}{k^{3/2}} \). Thus, \( \frac{|\lambda_h|}{\sqrt{h^2 + h}} \leq (1 - 2^{-h}) \frac{1}{k^{3/2}} \).
As a consequence, we also get that
\[
|\lambda_h| \leq \frac{\sqrt{h^2 + h}}{k^{3/2}} \quad \forall h \in [k-1].
\]
However, by hypothesis, \( \sum_{h=1}^{k-1} |\lambda_h|^2 = 1 \). Therefore,

\[
1 = \sum_{h=1}^{k-1} |\lambda_h|^2 \leq \frac{1}{k^3} \sum_{h=1}^{k-1} (h^2 + h) \leq \frac{1}{k^3} \sum_{h=1}^{k-1} k(k - 1) \leq \frac{(k - 1)^2 k}{k^3} < 1,
\]

which is a contradiction. \( \square \)

It is also useful to find the value of some norms for the matrix \( P \).

**Lemma 5.2.** We have that \( \|P\|_F = \frac{n}{\sqrt{k-1}} \), \( \|P\|_{op} = \frac{n}{k-1} \), and \( \|P\|_{SDP} = \frac{n^2}{k-1} \). Moreover, when \( k \) is even \( \|P\|_{\infty \rightarrow 1} = \frac{n^2}{k-1} \), while when \( k \) is odd, \( \|P\|_{\infty \rightarrow 1} = \frac{n^2(k+1)}{k^2} \).

**Proof.** The first two equations follow by what we have just said on the spectrum of \( P \). As for the third one, it is easy to observe that the \( \pm 1 \) values in the corresponding norm should be symmetric. Let \( \alpha \) be the number of \( +1 \) and \( \beta = k - \alpha \) be the number of \( -1 \) in the optimal solution for the case \( n = k \) (when \( k < n \), it just suffices to multiply everything by \( \frac{n^2}{k^2} \)). We have that

\[
\|M\|_{\infty \rightarrow 1} = 1 \cdot k + \frac{1}{k-1} \cdot [\alpha \cdot (\beta - \alpha + 1) + \beta \cdot (\alpha - \beta + 1)] = k + \frac{\alpha + \beta - (\alpha - \beta)^2}{k-1}.
\]

Now, since \( \alpha + \beta = k \), we get \( k + 1 + \frac{1}{k-1} \cdot \frac{(2\alpha - k)^2}{k-1} \), which is maximized when \( \alpha \) is as close as possible to \( k/2 \), yielding different values for \( k \) even/odd, respectively \( k + 1 + \frac{1}{k-1} = \frac{k^2}{k-1} \) and \( k + 1 \). The SDP norm of \( P \) is the maximum of its Frobenius scalar product with a set of positive semidefinite matrices which contains \( P \) itself, so it is

\[
\|P\|_{SDP} = P \cdot P = k + \frac{k^2}{(k-1)^2} = k + 1 + \frac{1}{k-1} = \frac{k^2}{k-1}.
\]

We finally need to assess how \( P \) changes after a random perturbation as the one described in the random model for the original matrix \( M \). We define

\[
P'_{i,j} := \begin{cases} 
M'_{i,j} & \text{if } M'_{i,j} > 0; \\
\frac{M_{i,j}}{k-1} & \text{otherwise.}
\end{cases}
\]

Equivalently, we can write

\[
P'_{i,j} = \begin{cases} 
P_{i,j} & \text{w. pr. } \frac{1}{2} + \epsilon; \\
-P_{i,j} + \left(1 - \frac{1}{k-1}\right) & \text{w. pr. } \frac{1}{2} - \epsilon.
\end{cases}
\]

We can see that this turns 1 into \(-\frac{1}{k-1}\) w. pr. \( \frac{1}{2} - \epsilon \) and vice versa. Observe that:

\[
\mathbb{E}[P'] = \left(\frac{1}{2} - \epsilon\right) \left(1 - \frac{1}{k-1}\right) \cdot 1 + 2\epsilon \cdot P,
\]

where \( \mathbb{1} \) is the \( n \times n \) matrix with all entries equal to 1. We can now define \( Q' := P' - \left(\frac{1}{2} - \epsilon\right) \left(1 - \frac{1}{k-1}\right) \cdot \mathbb{1} \).

By what we have said, this gives \( \mathbb{E}[Q'] = 2\epsilon \cdot P \), so \( Q' \) can be used as a random perturbation of the matrix \( 2\epsilon \cdot P \).
5.2 Pre-Adversarial Model

Here we generalize the approach used for \( k = 2 \) clusters: instead of recovering an approximation of the main eigenvector of the original input matrix, we recover an approximation of a basis of the eigenspace of the \( k \) leading eigenvalues of the modified matrix \( P \). This will reveal completely the actual clustering. We recall that \( \epsilon = \omega(n^{-1/2}) \), and \( B = o(n^2) \) (Eq. [5]).

Consider the modified positive semidefinite “correlation matrix” \( P \), defined as

\[
P = \frac{k}{2(k-1)} \cdot \left( M + n \left( 1 - \frac{2}{k} \right) zz^T \right).
\]

Analogously, define \( P' \) from the intermediate matrix \( M' \), obtained after the adversarial modifications, and \( P'' \) from the final matrix \( M'' \), obtained after the random perturbation too. Finally, define

\[
Q := P'' - \left(\frac{1}{2} - \epsilon\right) \left(1 - \frac{1}{k-1}\right) \cdot I.
\]

We will recover the \( k-1 \) orthogonal eigenvectors associated to the \( k-1 \) largest eigenvalues of the random matrix \( Q \), and then show that their eigenspace is very close to the one of the main eigenvectors of \( P \). Therefore, we will use those vectors to reconstruct the \( k \) clusters.

First, we can extend Theorem 5.3 to this setting. The following results can be proved analogously.

**Theorem 5.3.** With high probability \( \geq 1 - 2^{-4n} = 1 - o(1) \), it holds

\[
||Q - 2\epsilon \cdot P||_{op} \leq 16\sqrt{n} + 4\epsilon \cdot \sqrt{B} = o(\epsilon n).
\]

**Proof.** First, analogously to Lemma 4.1 it holds \( ||P' - P||_{op} \leq 2\sqrt{B} \), because there are \( \leq B \) deterministic changes from \( P \) to \( P' \), each of amount \( 1 + \frac{1}{k} \leq 2 \). Second, analogously to Lemma 4.2, with high probability \( \geq 1 - 2^{-4n} = 1 - o(1) \), it holds \( ||P'' - E[P'']||_{op} \leq 16\sqrt{n} \). However,

\[
E[P''] = \left(\frac{1}{2} - \epsilon\right) \left(1 - \frac{1}{k-1}\right) \cdot I + 2\epsilon \cdot P' .
\]

Moreover, by definition of \( Q \), we finally get that \( Q - 2\epsilon \cdot P' = P'' - E[P''] \). Third, by the triangle inequality, \( ||Q - 2\epsilon \cdot P||_{op} \leq ||Q - 2\epsilon \cdot P'||_{op} + 2\epsilon ||P' - P||_{op} \). By putting everything together, we finally get that \( ||Q - 2\epsilon \cdot P||_{op} \leq 16\sqrt{n} + 4\epsilon \cdot \sqrt{B} \). Finally, we notice that \( 16\sqrt{n} + 4\epsilon \cdot \sqrt{B} = o(\epsilon n) \) by Eq. [5].

We can now proceed by showing that the eigenspace of the \( k-1 \) leading eigenvalues of \( Q \) is very close to the one of \( P \).

**Lemma 5.4.** Let \( \nu_{k-1}'', \nu_{k-1}'' \) be unitary eigenvectors of the largest \( k-1 \) eigenvalues of \( Q \), and let \( \nu_1, \ldots, \nu_{k-1} \) be an orthogonal basis of the largest eigenvalue of \( P \). Let \( V \in \mathbb{R}^{n,k-1} \) with \( \nu_1, \ldots, \nu_{k-1} \) as columns, and \( V'' \in \mathbb{R}^{n,k-1} \) with \( \nu_{k-1}'', \nu_{k-1}'' \) as columns. Then, with high probability \( \geq 1 - 2^{-4n} = 1 - o(1) \), it holds

\[
||VV^T - V''(V'')^T||_F \leq \frac{8k}{\epsilon \sqrt{n}} \cdot \frac{2\sqrt{B}}{n} = o(1).
\]

**Proof.** By what previously observed, \( 2\epsilon \cdot P \) is diagonalizable with eigenvalues \( \frac{2}{k-1} \cdot \epsilon n \) and \( 0 \), with the first having an eigenspace of dimension \( k-1 \). Thus, by Theorem 5.3, for any orthogonal basis of eigenvectors \( \nu_1, \ldots, \nu_{k-1} \) of the largest eigenvalue of \( P \), it holds

\[
||VV^T - V''(V'')^T||_F \leq \frac{||Q - 2\epsilon \cdot P||_{op}}{\frac{2}{k-1} \cdot \epsilon n}.
\]
Now, by Theorem 5.3 with high probability $\geq 1 - 2^{-4n} = 1 - o(1)$, it holds $\|Q - 2\epsilon \cdot P\|_{op} \leq 16\sqrt{n} + 4\epsilon \cdot \sqrt{B} = o(\epsilon n)$, so

$$\|VV^T - V''(V'')^T\|_F \leq \frac{16\sqrt{n} + 4\epsilon \cdot \sqrt{B}}{2^{1-2}} < \frac{8k}{\epsilon\sqrt{n}} + \frac{2k\sqrt{B}}{n} = o(1).$$

Thanks to this result, we can generalize our spectral approach. By recovering the eigenvectors of the $k - 1$ largest eigenvalues of $Q$, we get a very good approximation of a basis of the eigenspace of the leading eigenvector of $P$, which can be used to set the clusters apart.

Algorithm 3 : input $Q$, $n$, $k$

1: $S \leftarrow \emptyset$
2: $U \leftarrow [n]$
3: Let $\{v_1', \ldots, v_{k-1}'\}$ be the $k - 1$ main eigenvectors of $Q$ obtained through Power-Method
4: $\{S_1, \ldots, S_n\} \leftarrow$ Get-Clusters($\{v_1'', \ldots, v_{k-1}''\}$)
5: for $\ell = 1, \ldots, k - 1$ do
6: repeat
7: Sample $i \in U$ U.A.R.
8: until $|S_i \cap U| \geq \frac{2k}{2k} n$
9: $S \leftarrow S \cup \{S_i \cap U\}$
10: $U \leftarrow U \setminus S_i$
11: $S \leftarrow S \cup \{U\}$
12: return $S$

Algorithm 4 Get-Clusters

1: Procedure Get-Clusters($\{v_1', \ldots, v_{k-1}'\}$)
2: for $i = 1, \ldots, n$ do
3: for $j = 1, \ldots, n$ do
4: $S_{ij} \leftarrow \emptyset$
5: for $h = 1, \ldots, k - 1$ do
6: if $|(v_h')_i - (v_h')_j| \geq \frac{1}{2\sqrt{2n}}$ then
7: $S_{ij} \leftarrow S_{ij} \cup \{h\}$
8: $S_i \leftarrow \{j \in [n] : S_{ij} = \emptyset\}$
9: return $\{S_1, \ldots, S_n\}$

In Algorithm 3 we first obtain the eigenvectors of the $k - 1$ leading eigenvalues of $Q$. After that, we use those to retrieve the cluster each element belongs to in procedure Get-Clusters(Algorithm 4): for each $i \in [n]$, this procedure computes a tentative cluster $S_i$ for it, which contains all the indices whose corresponding elements in every generated eigenvector are close to the $i$th element. We will show that most of these tentative clusters are correct, meaning that they approximately reconstruct the cluster the element belongs to. Therefore, we can sample $k - 1$ distinct approximate clusters and build the $k$th cluster with the remaining elements. With high probability, this procedure returns $k$ almost correct clusters.

First, we can use the previous results to show that the eigenspace of the obtained eigenvectors of $Q$ is “close” to the one of the leading eigenvectors of $P$.

Lemma 5.5. Let $\{v_1'', \ldots, v_{k-1}''\}$ be the unitary eigenvectors of the largest $k - 1$ eigenvalues of $Q$, as returned in line 3 of Algorithm 3 and let $v_1, \ldots, v_{k-1}$ be an orthogonal basis of the largest eigenvalue of $P$. Then, with high probability $\geq 1 - 2^{-4n}$, for each $v_h, h \in [k - 1]$, it holds

$$\sum_{\ell=1}^{k-1} \langle v_h, v_\ell'' \rangle^2 \geq 1 - \frac{64k^2}{\epsilon^2 n} - \frac{4k^2 B}{n^2} = 1 - o(1).$$
Analogously, for each $v''_m, m \in [k-1]$, it holds
\[ \sum_{h=1}^{k-1} \langle v_h, v''_m \rangle^2 \geq 1 - \frac{64k^2}{\epsilon^2 n} - \frac{4k^2 B}{n^2} = 1 - o(1). \]

**Proof.** With high probability $1 - 2^{-4n} = 1 - o(1)$, by Lemma 5.4, it holds
\[ \|VV^T - V''(V'')^T\|_F \leq \frac{8k}{\epsilon \sqrt{n}} + \frac{2k\sqrt{B}}{n}. \]

Now, we can notice that
\[ \|VV^T - V''(V'')^T\|_F^2 = \sum_{i,j=1}^n \left( \sum_{h=1}^{k-1} (\langle v_h)_i (v_h)_j - (v''_h)_i (v''_h)_j) \right) \]
\[ = 2 \left( k - 1 - \sum_{\ell=1}^{k-1} \sum_{h=1}^{k-1} \langle v_h, v''_{h} \rangle^2 \right). \]

Now fix a generic $h \in [k-1]$. For each $h' \in [k-1] \setminus \{h\}$, it holds $\sum_{\ell=1}^{k-1} \langle v_{h'}, v''_{h} \rangle^2 \leq \|v_{h'}\|^2 = 1$, because it is the sum of the projections of orthogonal vectors onto $v_{h'}$. Therefore, $\|VV^T - V''(V'')^T\|_F^2 \geq 2 \left( 1 - \sum_{\ell=1}^{k-1} \langle v_h, v''_{h} \rangle^2 \right)$. Moreover, we have shown that with high probability
\[ \|VV^T - V''(V'')^T\|_F^2 \leq \left( \frac{8k}{\epsilon \sqrt{n}} + \frac{2k\sqrt{B}}{n} \right)^2 \leq \frac{128k^2}{\epsilon^2 n} + \frac{8k^2 B}{n^2} = o(1) \]
by using $(a + b)^2 \leq 2(a^2 + b^2) \forall a, b \in \mathbb{R}$. Therefore, we have that
\[ \sum_{\ell=1}^{k-1} \langle v_h, v''_{h} \rangle^2 \geq 1 - \frac{64k^2}{\epsilon^2 n} - \frac{4k^2 B}{n^2} = 1 - o(1). \]

Since everything is symmetric, the symmetric version of this inequality follows analogously. 

**Theorem 5.6.** With high probability $1 - o(1)$, Algorithm 5 is well-defined and misplaces at most $o(n)$ vertices, where the number of misplaced vertices is at most $\frac{248k^4}{\epsilon^2 n} + \frac{128k^2 B}{n^2} = o(n)$.

**Proof.** We show that, with high probability $1 - o(1)$, the Algorithm 5 is well-defined, so it always succeeds in finding $i \in \mathcal{U}$ satisfying the condition of line 8 and that our solution is an approximate reconstruction of the original clusters, i.e., that there is a bijection $\varphi$ from the returned sets $S_i$ to the original clusters such that $|S_i \Delta \varphi(S_i)| = o(1)$.

First, for each eigenvector $v''_{\ell}, \ell \in [k-1]$, we can define $\tilde{v}_\ell := \sum_{h=1}^{k-1} \langle v_h, v''_{\ell} \rangle v_h$, which is the projection of $v''_{\ell}$ onto the eigenspace spanned by $\{v_1, \ldots, v_{k-1}\}$. Now, for each cluster $C$, we can define $\lambda_{\ell,C}$ as the coordinate of the vertices belonging to cluster $C$ in vector $\tilde{v}_\ell$, which is well-defined by what said about the spectrum of the input matrix. We can also define $S_{\text{bad}}^{\ell,C} := \{ i \in C : |(v''_{\ell})_i - \lambda_{\ell,C}| > \frac{1}{4\sqrt{2n}} \}$, which is the set of indices of $C$ which have been moved far from $\lambda_C$ in $v''_{\ell}$. By Lemma 5.5 with high probability $\geq 1 - 2^{-4n}$ it holds $\|v''_{\ell} - \tilde{v}_\ell\|^2 \leq \frac{64k^2}{\epsilon^2 n} + \frac{4k^2 B}{n^2} = o(1)$, thus
\[ \frac{64k^2}{\epsilon^2 n} + \frac{4k^2 B}{n^2} \geq \|v''_{\ell} - \tilde{v}_\ell\|^2 \geq \sum_{i \in S_{\text{bad}}^{\ell,C}} |(v''_{\ell})_i - \lambda_{\ell,C}|^2 > \frac{|S_{\text{bad}}^{\ell,C}|}{32n}. \]
which implies that \( |S_{bad}^{C_i}| \leq \frac{2048k^2}{\epsilon^2} + \frac{128k^2B}{n} = o(n) \). Therefore, for each cluster, all but \( o(n) \) vertices have coordinates close to \( \lambda_{\ell,C} \) in \( v^\ell_i \).

Second, we show that, for each pair of different clusters \( C_1, C_2 \), there exists \( v^\ell_i, \ell \in [k-1] \), such that \( |\lambda_{\ell,C_1} - \lambda_{\ell,C_2}| > \frac{1}{\sqrt{2n}} \). Consider the orthogonal basis of the eigenspace of the main eigenvalue of \( P \), as defined at the beginning of this section. By its definition, for each \( v^\ell_i, \ell \in [k-1] \), it holds \( |\lambda_{\ell,C_1} - \lambda_{\ell,C_2}| = |\langle v_1, v^\ell_i \rangle| \sqrt{\frac{2k}{n}} \), because the coordinates of clusters \( C_1, C_2 \) only differ in vector \( v_1 \), and by an amount of \( \sqrt{\frac{2k}{n}} \) (they are \( \pm \sqrt{\frac{k}{2n}} \) and \( -\sqrt{\frac{k}{2n}} \)). By Lemma 5.3, it holds \( \sum_{\ell=1}^{k-1} \langle v_1, v^\ell_i \rangle^2 \geq 1 - \frac{64k^2}{\epsilon^2 n} - \frac{4k^2B}{n} = 1 - o(1) \), so there exists \( v^\ell_i, \ell \in [k-1] \), such that \( \langle v_1, v^\ell_i \rangle^2 > \frac{1}{\ell} \), implying that \( |\lambda_{\ell,C_1} - \lambda_{\ell,C_2}| = |\langle v_1, v^\ell_i \rangle| \sqrt{\frac{2k}{n}} > \sqrt{\frac{2}{n}} > \frac{1}{\sqrt{2n}} \).

Third, consider \( S_{bad} := \bigcup_{\ell \in [k-1], C} S^{C_i}_{bad} \). By the union bound and by what just proven,\n\[
|S_{bad}| \leq k \cdot \left( \frac{2048k^2}{\epsilon^2} + \frac{128k^2B}{n} \right) = \frac{2048k^4}{\epsilon^2} + \frac{128k^4B}{n} = o(n).
\]

Moreover, for each \( i \notin S_{bad} \), we have that:

- if \( j \notin S_{bad} \) belongs to the same cluster \( C_1 \) of \( i \), then for each \( v^\ell_i, \ell \in [k-1] \) it holds \( \|v^\ell_i - \lambda_{\ell,C_1}v^\ell_j\| \leq \frac{1}{\sqrt{2n}} \) by the triangle inequality;

- if \( j \notin S_{bad} \) belongs to a different cluster \( C_2 \) from \( i \), then there exists \( v^\ell_i, \ell \in [k-1] \), such that \( \|\lambda_{\ell,C_1} - \lambda_{\ell,C_2}\| \geq \frac{1}{\sqrt{2n}} \) implying that \( |\langle v^\ell_i, v^\ell_j \rangle| \geq |\lambda_{\ell,C_1} - \lambda_{\ell,C_2}| \geq |\langle v^\ell_i, v^\ell_j \rangle| \geq \frac{1}{\sqrt{2n}} \) by the triangle inequality.

We have shown that \( \frac{1}{\sqrt{2n}} \) is an appropriate distance threshold to separate elements in different clusters that do not belong to \( S_{bad} \).

Finally, by summing up, since \( |S_{bad}| = o(n) \), at each time step with high probability \( \geq 1 - o(1) \) we select \( i \notin S_{bad} \) from line 7. In this case, \( S_i \) has \( \frac{2}{\epsilon^2} + o(n) \) elements, which are the elements in its cluster plus/minus eventual elements of \( S_{bad} \). The elements of \( S_{bad} \) could be wrongly added to \( S_i \), or wrongly removed and associated to a different set of those. Since \( k = O(1) \), with high probability \( \geq 1 - o(1) \) this happens for \( k \) straight times. Under all these assumptions, only the elements in \( S_{bad} \) can be classified incorrectly, but they are at most \( \frac{2048k^4}{\epsilon^2} + \frac{128k^4B}{n} = o(n) \) by Eq. 5. \( \square \)

**Polynomial Running Time.** If we neglect the cost of the Power-Method and of procedure Get-Clusters, Algorithm 3 has a linear cost in \( n \). Exactly as shown for the case \( k = 2 \), the running time of the Power-Method is \( poly(n) \) and it computes the eigenvectors with negligible error. It is also easy to observe that procedure Get-Clusters has cost \( O(n^2) \) because \( k = O(1) \). Therefore, the total running time is still \( poly(n) \).

### 5.3 Post-Adversarial Model

Here we generalize the approach used for \( k = 2 \) clusters: instead of solving a single SDP and extracting a good eigenvector from the solution matrix, we do this process iteratively, because each eigenvector only partitions the set of vertices in two. The errors made during these bisections are under control and sum up to \( o(n) \). We recall that \( \epsilon = \omega(n^{-1/2}) \), and \( B = o(n^2) \) (Eq. 7).

**Extending the SDP-Based Approach.** Here we use the positive semidefinite matrix \( P \) instead of \( M \), with its random perturbation \( P' \) and its post-adversarial perturbation \( P'' \), which can be obtained from \( M'' \) by turning its negative entries to \( \frac{1}{k-1} \). We also recall that
\[
E[P'] = 2\epsilon \cdot P + \left( \frac{1}{2} - \epsilon \right) \left( 1 - \frac{1}{k-1} \right) \cdot I.
\]
We can define
\[ Q = P'' - \left( \frac{1}{2} - \epsilon \right) \left( 1 - \frac{1}{k-1} \right) \cdot 1. \]
By doing so, \( Q \) can be seen as a perturbation of \( 2\epsilon \cdot P \), which is a positive semidefinite matrix. Thus, it can be effectively used as input matrix for an SDP that aims to reconstruct the clusters.

**The Need for Recursion.** Algorithm 3 avoids recursion by obtaining an approximation of the entire basis of eigenvectors of the leading eigenvalue of the original matrix, and we wondered if we could do the same in the post-adversarial model. However, we believe recursion to be difficult to bypass. We defined \( P \) so that \( Q \sim 2\epsilon \cdot P \). Then, in our SDP, we look for a positive semidefinite matrix \( Q \) maximizing \( P \cdot X \). Since \( P \cdot X \) is maximized at \( X = P \), by maximizing \( Q \cdot X \sim 2\epsilon \cdot P \cdot X \) we hoped to recover a perturbated form of the matrix \( P \), i.e. we hoped that \( X \sim P \). However, we will see why this could not happen. Take \( k \) even and consider a partition of the clusters into two groups with \( k/2 \) clusters each. Let \( B \) be the corresponding bisecting matrix: it has \( B_{ij} = 1 \) if and only if \( i, j \) belong to clusters in the same group (half), \( B_{ij} = -1 \) otherwise. It is clear that the matrix \( B \) only gives information about a bisection of the vertices, but it does not allow a complete clustering reconstruction, while \( P \) does. We can notice that
\[ P \cdot P = P \cdot B = \frac{n^2}{k-1}. \]
As a consequence, \( P \cdot X \) is maximized both at \( X = B \) and \( X = P \). Therefore, \( Q \cdot P \) could be maximized by a matrix \( X \sim B \) as well as a matrix \( X \sim P \). In other words, the matrix \( X \) could only give information about a bisection, instead of allowing to reconstruct all of the clusters simultaneously. This is why, with our current method, we cannot hope to avoid a recurring procedure. We do not yet know if adding extra constraints to the SDP can rule out undesired solutions — we believe this to be an interesting question for future work.

### 5.3.1 General Properties

Analogously to what has been done with \( k = 2 \) clusters, we can prove the following norm inequalities.

**Lemma 5.7.**
\[ \Pr(\|P' - \mathbb{E}[P']\|_{\infty \rightarrow 1} \geq 16n\sqrt{n}) \leq 2^{-4n}. \]

**Proof.** First, it holds
\[ P'_{ij} - \mathbb{E}[P'_{ij}] = \begin{cases} (1 - 2\epsilon)P_{ij} - \left( \frac{1}{2} - \epsilon \right) \left( 1 - \frac{1}{k-1} \right), & \text{w. pr. } \frac{1}{2} + \epsilon; \\ -(1 + 2\epsilon)P_{ij} + \left( \frac{1}{2} + \epsilon \right) \left( 1 - \frac{1}{k-1} \right), & \text{w. pr. } \frac{1}{2} - \epsilon. \end{cases} \]

Thus \( \frac{1}{1+2\epsilon}(P' - \mathbb{E}[P']) \) has all the elements bounded by 1 in absolute value. By using the exact same argument of Lemma 4.2, we can show that \( \Pr(\|P' - \mathbb{E}[P']\|_{\text{op}} \geq 16\sqrt{n}) \leq 2^{-4n} \). Now, by Lemma 5.3 it holds \( \|P' - \mathbb{E}[P']\|_{\text{op}} \geq \frac{1}{n} \cdot \|P' - \mathbb{E}[P']\|_{\infty \rightarrow 1} \), so
\[ \Pr(\|P' - \mathbb{E}[P']\|_{\infty \rightarrow 1} \geq 16n\sqrt{n}) = \Pr(\|P' - \mathbb{E}[P']\|_{\text{op}} \geq 16\sqrt{n}) \leq 2^{-4n}. \]

We can also bound the norm displacement after the post-adversary intervention.

**Lemma 5.8.**
\[ \|P'' - P'\|_{\infty \rightarrow 1} \leq 2B = o(en^2). \]

**Proof.** Let \( P'' = P' + E \), where \( E \), the matrix of adversarial changes, has \( B = o(en^2) \) non-zero entries, all with absolute value \( 1 + 1/k-1 \leq 2 \). Therefore, \( \|E\|_{\infty \rightarrow 1} \leq 2B \).
Consider the auxiliary matrix $Q$, defined as:

$$Q := P'' - \left(\frac{1}{2} - \epsilon\right) \left(1 - \frac{1}{k - 1}\right) \cdot \mathbb{I}.$$ 

**Lemma 5.9.** With high probability $\geq 1 - 2^{-4n} = 1 - o(1)$, it holds

$$\|Q - 2\epsilon \cdot P\|_{\infty \rightarrow 1} \leq 16n\sqrt{n} + 2B = o(\epsilon n^2).$$

**Proof.** By definition of $Q$ and $P'$, we get that

$$Q - 2\epsilon \cdot P = (P'' - P') + (P' - \mathbb{E}[P']).$$

By Lemma 5.10 with probability $\geq 1 - 2^{-4n}$ it holds $\|P' - \mathbb{E}[P']\|_{\infty \rightarrow 1} \leq 16n\sqrt{n}$, by Lemma 5.8 it holds $\|P'' - P'\|_{\infty \rightarrow 1} \leq 2B$. By putting everything together and using the triangle inequality, we finally get that with high probability ($\geq 1 - 2^{-4n}$)

$$\|Q - 2\epsilon \cdot P\|_{\infty \rightarrow 1} \leq 16n\sqrt{n} + 2B = o(\epsilon n^2).$$

\[\square\]

### 5.3.2 A Recursive SDP-Based Approach

Consider the following SDP:

$$\text{maximize} \quad \sum_{j=1}^{n} Q_{ij} \langle x_i, y_j \rangle$$

subject to \hspace{1cm} $\|x_i\| = 1, x_i \in \mathbb{R}^n$ \hspace{0.5cm} $i = 1, \ldots, n$

\hspace{1cm} $\|y_i\| = 1, y_i \in \mathbb{R}^n$ \hspace{0.5cm} $i = 1, \ldots, n$ \hspace{1cm} (11)

The maximum of this SDP is equal to $\|Q\|_{SDP}$. Given the optimal solution $\{x_i^\ast\}_{i \in [n]}, \{y_i^\ast\}_{i \in [n]}$, consider the matrix $X$ where $X_{ij} := \langle x_i^\ast, y_j^\ast \rangle \forall i, j \in [n]$. Then, $\|Q\|_{SDP} = Q \cdot X$, where $\cdot$ represents the Kronecker (element-wise) product. Since $Q$ is symmetric, by a well-known characteristic of SDPs, $X$ is symmetric too. The following lemma holds.

**Lemma 5.10.** With high probability ($\geq 1 - 2^{-4n}$), it holds

$$\|Q\|_{SDP} = Q \cdot X \geq \frac{2}{k - 1} \cdot \epsilon n^2 - 29n\sqrt{n} - 4B = \frac{2}{k - 1} \cdot \epsilon n^2 - o(\epsilon n^2).$$

**Proof.** By the triangle inequality, we have that

$$\|Q\|_{SDP} \geq 2\epsilon \cdot \|P\|_{SDP} - \|Q - 2\epsilon \cdot P\|_{SDP}.$$  

Now, by Lemma 5.9 with high probability $\geq 1 - 2^{-4n} = 1 - o(1)$, it holds $\|Q - 2\epsilon \cdot P\|_{\infty \rightarrow 1} \leq 16n\sqrt{n} + 2B = o(\epsilon n^2)$ so, by Theorem 3.4 we get that

$$\|Q - 2\epsilon \cdot P\|_{SDP} \leq 1.8 \cdot (16n\sqrt{n} + 2B) \leq 29n\sqrt{n} + 4B = o(\epsilon n^2).$$

Moreover, by Lemma 5.2 we get that $\|P\|_{SDP} = \frac{n^2}{k - 1}$. By substituting these above, and exploiting Eq. 7 we finally get that

$$\|Q\|_{SDP} \geq \frac{2}{k - 1} \cdot \epsilon n^2 - 29n\sqrt{n} - 4B = \frac{2}{k - 1} \cdot \epsilon n^2 - o(\epsilon n^2).$$

\[\square\]
Since $X$ is symmetric, we can consider its spectral decomposition into orthogonal eigenvectors:

$$X = \sum_{i=1}^{n} \lambda_i u_i^{(i)} (u_i^{(i)})^T.$$ 

Now, consider $u^*$ picked randomly in $\{u_i, i \in [n]\}$, where $u_i$ is picked with probability proportional to $\lambda_i$. We will show that, with high probability, $u^*$ gives a separation of the vertices into two sets, each containing at least one original cluster, and putting almost always together vertices belonging to the same cluster. This random sampling technique for an eigenvector is the same used for $k = 2$ clusters.

**Lemma 5.11.** Consider $Q \in \mathbb{R}^{n \times n}$ such that $\|Q - 2\epsilon \cdot P\|_{\infty} \leq f(n, B, \epsilon) = o(e^2)$. Let $X$ be the (symmetric positive semidefinite) solution matrix of SDP (11) w.r.t. $Q$, let $\{u_i\}_{i \in [n]}$ an orthogonal basis of eigenvectors for $X$ with eigenvalues respectively $\{\lambda_i\}_{i \in [n]}$. Pick $u^* \in \{u_i\}_{i \in [n]}$ randomly, where each $u_i$ is chosen with probability $\frac{\lambda_i}{\sum \lambda_i}$. Then, with high probability $\geq 1 - 2k \cdot \sqrt{\frac{f(n, B, \epsilon)}{\epsilon^2}} 2^{-4n} = 1 - o(1)$, there exists $v$ eigenvector of $P$ with eigenvalue $\frac{\lambda_i}{n}$ such that

$$\|u^* - v\|^2 \leq 4k \cdot \sqrt{\frac{f(n, B, \epsilon)}{\epsilon^2}} = o(1).$$

**Proof.** First, by definition of $X$ it holds $\|Q\|_{SDP} = Q \cdot X$. Now, by the triangle inequality, $\|Q\|_{SDP} \leq \|Q - 2\epsilon \cdot P\|_{SDP}$, however, by Theorem 3.3 $\|Q - 2\epsilon \cdot P\|_{SDP} \leq 1.8 \cdot \|Q - 2\epsilon \cdot P\|_{\infty} \leq 1.8 \cdot f(n, B, \epsilon) = o(e^2)$. So, by Lemma 5.2 we get that $\|Q - 2\epsilon \cdot P\|_{SDP} \leq \|Q - 2\epsilon \cdot P\|_{SDP} \leq \frac{2}{\kappa_1} \cdot e^2 - 1.8 \cdot f(n, B, \epsilon) = \frac{2}{\kappa_1} \cdot e^2 - o(e^2)$. Now, recall that $\|Q - 2\epsilon \cdot P\|_{SDP} \leq 1.8 \cdot f(n, B, \epsilon)$, so $\|Q - 2\epsilon \cdot P\|_{SDP} \leq 1.8 \cdot f(n, B, \epsilon) = o(e^2)$, i.e. that

$$P \cdot X \geq \frac{n^2}{k-1} - \frac{1.8}{\epsilon} \cdot f(n, B, \epsilon) \geq \frac{n^2}{k-1} - \frac{2}{\epsilon} \cdot f(n, B, \epsilon) = \frac{n^2}{k-1} - o(n^2).$$

Now, we can use the spectral decomposition $X = \sum_{i=1}^{n} \lambda_i u_i^{(i)} (u_i^{(i)})^T$ is positive semidefinite, and $\sum_{i=1}^{n} \lambda_i = \text{tr}(X) = \sum_{i=1}^{n} \|x_i\|^2 = n$. Therefore, $\{\lambda_i/n\}_i$ can be seen as a probability distribution. So it holds

$$P \cdot X = n \sum_{i=1}^{n} \frac{\lambda_i}{n} (u_i^{(i)})^T P u_i.$$ 

However, $P = \frac{n}{k-1} \sum_{j=1}^{k-1} v_j v_j^{T}$ and it is positive semidefinite, so $\|u_i^{(i)} P u_i^{(i)}\| = \frac{n}{k-1} \sum_{j=1}^{k-1} \langle v_j, u_i^{(i)} \rangle^2$ for each vector $u_i$, implying that

$$\sum_{i=1}^{n} \frac{\lambda_i}{n} \sum_{j=1}^{k-1} \langle v_j, u_i^{(i)} \rangle^2 = \frac{k-1}{n^2} \cdot P \cdot X \geq 1 - \frac{2(k-1)}{\epsilon n^2} \cdot f(n, B, \epsilon) \geq 1 - \frac{2k}{\epsilon n^2} \cdot f(n, B, \epsilon).$$

We can notice that the LHS is exactly $\mathbb{E} \left[ \sum_{j=1}^{k-1} \langle v_j, u^* \rangle^2 \right]$. So, we have shown that

$$\mathbb{E} \left[ \sum_{j=1}^{k-1} \langle v_j, u^* \rangle^2 \right] = \sum_{i=1}^{n} \frac{\lambda_i}{n} \sum_{j=1}^{k-1} \langle v_j, u_i^{(i)} \rangle^2 \geq 1 - \frac{2k}{\epsilon n^2} \cdot f(n, B, \epsilon) = 1 - o(1).$$

Moreover, for each vector $u$, the quantity $\sum_{j=1}^{k-1} \langle v_j, u \rangle^2$ is the squared norm of its projection onto the eigenspace of the eigenvalue $\frac{\lambda_i}{k-1}$ of $P$, so it is always a quantity in $[0, 1]$. Therefore, we can define the positive random variable $\chi := 1 - \sum_{j=1}^{k-1} \langle v_j, u^* \rangle^2$. We have that $\chi \geq 0$ and $\mathbb{E}[\chi] \leq \frac{2k}{\epsilon n^2} \cdot f(n, B, \epsilon) = o(1)$. 

24
Thus, by Theorem 3.7 we have that with high probability $\geq 1 - \frac{2k}{en^2} \cdot f(n, B, \epsilon) \geq 1 - 2k \frac{f(n, B, \epsilon)}{en^2}$, it holds $\chi \leq \sqrt{\frac{2k}{en^2} \cdot f(n, B, \epsilon)}$, implying that

$$\sum_{j=1}^{k-1} \langle v_j, u^* \rangle^2 \geq 1 - \sqrt{\frac{2k}{en^2} \cdot f(n, B, \epsilon)} = 1 - o(1).$$

Now, let

$$v' := \sum_{j=1}^{k-1} \langle v_j, u^* \rangle v_j; \quad v := \frac{v'}{\|v'\|}$$

be the normalized projection of $u^*$ onto the eigenspace of the eigenvalue $\frac{n}{k}$ of $P$. It holds (using that $\sqrt{1-x} \geq 1 - x \forall x \in [0,1]$)

$$\|v - u^*\|^2 = \langle v - u^*, v - u^* \rangle = 2 - 2\langle v, u^* \rangle = 2 - \frac{2}{\|v'\|} \langle v', u^* \rangle = 2 - 2\|v'\| \leq$$

$$2 - 2\sqrt{1 - \sqrt{\frac{4k}{en^2} \cdot f(n, B, \epsilon)}} \leq 2 - 2\sqrt{\frac{2k}{en^2} \cdot f(n, B, \epsilon)} \leq 4k \cdot \sqrt{\frac{f(n, B, \epsilon)}{en^2}} = o(1).$$

As a consequence of Lemma 5.11, Lemma 5.9 and Lemma 5.1 we can use $u^*$ to separate $[n]$ into two smaller sets with minimal separation of vertices in the same cluster and with at least one cluster on each side. We see how through Algorithm RECURSIVE-CLUST($[n], k, f, 1$) where $f = f(n, B, \epsilon) = 16n\sqrt{n} + 2B = o(n^2)$. Before that, we need a formal definition.

**Definition 5.1.** Let $P \in \mathbb{R}^{n \times n}$ matrix and let $S_1, S_2 \subseteq [n]$. We define $P^{S_1, S_2} \in \mathbb{R}^{|S_1| \times |S_2|}$ be the sub-matrix of $P$ restricted only to the rows in $S_1$ and to the columns in $S_2$.

Each iterative step of Algorithm 5 is similar to Algorithm 2 but there are some main differences.

First, we need to sample an appropriate threshold value for the eigenvector to separate the vertices (line 9), because we are getting an approximate eigenvector of the eigenspace of the leading vector of $P$, but we do not know exactly which approximate eigenvector we are getting. This is done by procedure GET-THRESHOLD. The threshold could be at any point in between the maximum and the minimum value of an eigenvector of $P$. However, since we only get an approximate eigenvector, we need to consider more robust order statistics to establish the feasible range for thresholds.

Second, we need to fix the cardinality of the bisection (line 14) because we want the size of each partition to be an integer multiple of $n/k$.

Third, we need to carry the information about the number of clusters in each partition: this will be used to scale the negative elements of the input matrix $Q'$ of SDP (line 5), so that this input matrix is always positive semidefinite.

We also need to carry an estimate $f'$ of the distance in norm $\ell_\infty$-to-$\ell_1$ between the scaled matrix $Q'$ and the scaled original matrix $2\epsilon \cdot P$, whose negative entries are scaled like the ones of $Q'$.

We show that Algorithm 5 with high probability, always splits the solution into two “smaller” solutions that we can solve recursively, i.e. that the bisection of the input set $S$ satisfies the condition of line 12. Moreover, we also show that the produced solutions only mislabel $o(n)$ vertices at each step. Before that, we prove an auxiliary lemma.

**Lemma 5.12.** Let $u, v \in \mathbb{R}^n$ s.t. $\|u - v\|^2 \leq \delta$. Suppose that the coordinates of $v$ can be partitioned into $k$ groups $P_1, \ldots, P_k$ of $\frac{n}{k}$ elements each, such that all the coordinates in the same group $P_i$ are equal. Then, for each group $P_i$, there cannot be $\geq \delta^{1/3} \cdot n$ elements $j \in P_i$ such that $|u_j - v_j| > \delta^{1/3} \sqrt{n}$. 25
Algorithm 5 Recursive-Clust($S$, $k'$, $f$, $\gamma$): input $S$ set of indices, $k'$ number of clusters in $S$, $f = f(n, B, \epsilon) = o(en^2)$ such that $\|Q^{S,S} - 2\epsilon \cdot P^{S,S}\|_{\infty \rightarrow 1} \leq f$ (after having their negative entries multiplied by $\gamma$), $\gamma$ rescaling factor for the negative entries of $Q$. Global variables $n$, $k$, $Q$.

1: $\delta \leftarrow 4k \cdot \sqrt{\frac{1}{f \cdot n'}}$
2: $n' \leftarrow |S|$
3: if $n' = n/k$ then
4: return $\{S\}$
5: Let $Q_{k'}$ be the matrix obtained from $Q^S$ by multiplying its negative coordinates by $\gamma$
6: Let $X$ be the solution matrix of SDP [11] for $Q_{k'}$ obtained through SDP-Solver
7: Let $\{u^1, \ldots, u^n\}$ be an orthogonal basis of eigenvectors of $X$ with eigenvalues respectively $\{\lambda_i, i \in [n']\}$ obtained through Power-Method
8: Sample $u^* \in \{u^i, i \in [n']\}$ with probability distribution $\{\frac{\lambda_i}{\sum_{i=1}^{n'} \lambda_i}, i \in [n']\}$
9: $t \leftarrow \text{Get-Threshold}(u^*, n', \delta)$ is the separating threshold according to vector $u^*$
10: Let $S_1 := \{i \in S : u^*_i < t\}$
11: $k'' := \lfloor \frac{|S_1|}{n/k} \rfloor$ (closest integer function $\lfloor \cdot \rfloor$)
12: if $k'' \in \{0, k'\}$ then
13: abort (the algorithm failed)
14: $S' \leftarrow \{i \in [n'] : u^*_i \text{ is among the } k'' \cdot \frac{2}{\delta} \text{ smallest coordinates of } u^*_i \text{ (ties broken arbitrarily)}\}$
15: $f' \leftarrow k \cdot f + 4k\delta^{1/3} \cdot (n')^2 = o(en^2)$
16: $\gamma' \leftarrow \frac{k''-1}{k-k''-1}$ scaling factor for $S'$ because it now contains $k''$ clusters instead of $k'$
17: $\gamma'' \leftarrow \frac{k-1}{k-k''-1}$ scaling factor for $S \setminus S'$ because it contains the remaining $k' - k''$ clusters
18: $C_1 \leftarrow \text{Recursive-Clust}(S', k'', f', \gamma')$
19: $C_2 \leftarrow \text{Recursive-Clust}(S \setminus S', k' - k'', f', \gamma'')$
20: return $C_1 \cup C_2$

Algorithm 6 Get-Threshold

1: Procedure Get-Threshold($u$, $n'$, $\delta$)
2: Let $\pi$ be the ordering permutation of vector $u$, i.e. the permutation on $[n']$ s.t. $u_{\pi(i)} \leq u_{\pi(j)} \forall 1 \leq i \leq j \leq n'$
3: $t_{\min} \leftarrow u_{\pi([n'/3, n'])}$
4: $t_{\max} \leftarrow u_{\pi([2n'/3, n'])}$
5: Pick $t \in [t_{\min}, t_{\max}]$ Uniformly At Random as the separating threshold for vector $u$
6: return $t$
Proof. Assume there is $P'_t \subseteq P_t$ such that $|P'_t| \geq \delta^{1/3} \cdot n$ and $|u_j - v_j| > \frac{\delta^{1/3}}{\sqrt{n}} \forall j \in P'_t$. Then,

$$\delta \geq \|u - u\|^2 \geq \sum_{j \in P'_t} (u_j - v_j)^2 > (\delta^{1/3} \cdot n) \cdot \left(\frac{\delta^{1/3}}{\sqrt{n}}\right)^2 \geq \delta,$$

which is a contradiction.

In other words, in $u$, all but $\leq \delta^{1/3} \cdot n$ elements of a group are within a distance $\leq \frac{\delta^{1/3}}{\sqrt{n}}$ from their coordinate in $v$. The previous lemma is used to show that almost all the coordinates of approximate eigenvectors of $P$ are very close to the coordinates of the actual eigenvector of $P$. In order to get closer to the proof of the effectiveness of Algorithm 5, we state precise guarantees on what happens in the first round. To extend this to the recursive sub-problems, we need to make some adjustments to take into account the previous classification errors too.

**Lemma 5.13.** Consider the invocation of **RECURSIVE-CLUST**([n], k, f, 1) where $f = f(n, B, \epsilon) = 16n\sqrt{n} + 2B = o(en^2)$, and let $\delta := 4k \cdot \sqrt{\frac{\epsilon}{cn}} = 4k \cdot \sqrt{\frac{16}{\epsilon \cdot n} + \frac{2B}{\epsilon n}} = o(1)$. With high probability $1 - 2^{-4n} - 6k^2 \cdot \delta^{1/3} = 1 - o(1)$, the first sampled threshold $t$ does not satisfy the condition of line 12 in Algorithm 5, so the algorithm does not fail. Moreover:

- for each cluster, either $S'$ or $S \setminus S'$ contains $\leq 2\delta^{1/3} \cdot n = o(n)$ of its vertices, meaning that there are $\leq 2k\delta^{1/3} \cdot n = o(n)$ misplaced vertices overall in the first recursive step;
- let $A$ be one of the sub-sets on which the algorithm is applied recursively (the same holds for the other subset), and let $A^*$ be the union of the $k$ clusters having $\geq \frac{\delta^{1/3}}{k} - 2\delta^{1/3} \cdot n$ elements in $A$. Let $Q_{kA}$ be the matrix obtained from $Q$ by multiplying the negative entries by $\frac{kA - 1}{kA}$ and let $P_{kA}$ be the analogous matrix obtained from $P$. Then,

$$\|Q_{kA}A - \epsilon \cdot P_{kA}A\|_{\infty \rightarrow 1} \leq f' := k \cdot f(n, B, \epsilon) + 4k\delta^{1/3} \cdot en^2 = o(en^2).$$

**Proof.** By Lemma 5.9, with high probability $1 - 2^{-4n}$ it holds $\|Q - 2\epsilon \cdot P\|_{\infty \rightarrow 1} \leq f(n, B, \epsilon) = 16n\sqrt{n} + 2B = o(en^2)$, and we consider this to be true from now on (by the union bound, the small probability of this to be false will sum up with the other encountered small probabilities). Therefore, by Lemma 5.12, for each set of indices $S \subseteq [n]$, it also holds $\|Q^{S,S} - 2\epsilon \cdot P^{S,S}\|_{\infty \rightarrow 1} \leq 16n\sqrt{n} + 2B = o(en^2)$. Now, by Lemma 5.11 with high probability $1 - \delta/2 = 1 - o(1)$, there exists an eigenvector $v$ of the leading eigenvalue $\frac{\epsilon}{kA - 1}$ of $P$ such that $\|v - u^*\|^2 \leq \delta = o(1)$. Now, let $v_{\text{max}} := \max_{i \in [n]} v_i$ and $v_{\text{min}} := \min_{i \in [n]} v_i$. By Lemma 5.1 it holds $|v_{\text{max}} - v_{\text{min}}| > \frac{1}{\sqrt{kA}}$. By Lemma 5.12 it follows that $|v_{\text{max}} - v_{\text{min}}| \leq \frac{\delta^{1/3}}{\sqrt{n}}$. As a consequence, we have that

$$\Pr \left( t \in \left[ v_{\text{min}} + \frac{\delta^{1/3}}{\sqrt{n}}, v_{\text{max}} - \frac{\delta^{1/3}}{\sqrt{n}} \right] \right) \geq \frac{|v_{\text{max}} - v_{\text{min}}| - 2\frac{\delta^{1/3}}{\sqrt{n}}}{|v_{\text{max}} - v_{\text{min}}| + 2\frac{\delta^{1/3}}{\sqrt{n}}} \geq 1 - \frac{4\delta^{1/3}}{kA \sqrt{n}} + 2\frac{\delta^{1/3}}{\sqrt{n}} \geq 1 - 4k \cdot \delta^{1/3} = 1 - o(1).$$

However, if $t \in \left[ v_{\text{min}} + \frac{\delta^{1/3}}{\sqrt{n}}, v_{\text{max}} - \frac{\delta^{1/3}}{\sqrt{n}} \right]$, by Lemma 5.12 we separate almost exactly the clusters corresponding to the largest and the smallest coordinate of $v$: $\geq \frac{\epsilon}{k} - \delta^{1/3} \cdot n$ elements of each cluster are split correctly according to the threshold, which makes the condition of line 12 not satisfied and the algorithm does not fail. Now, we need to show that, with high probability $1 - \delta/2 = 1 - o(1)$, by Lemma 5.12 for each cluster $C$ with coordinate $v_C$ in $v$,
We show the theorem by induction on the number of recursive calls each invocation of the recursive cluster.

By Lemma 5.9, we have that under the previously mentioned events holding with high probability, the recursive cluster invocation of $\|A\|_t$ to the threshold bisection at $|A|$ holds with probability $\geq \delta_2$ for each clusters, for a total of $k\delta^{1/3} \cdot n$ total misplaced vertices according to the threshold bisection at $t$. Finally, the process of line 13 can bring other $k\delta^{1/3} \cdot n$ mistakes (extra $\delta^{1/3} \cdot n$ for each clusters), for a total of $2k\delta^{1/3} \cdot n = o(n)$ misplaced vertices. By the union bound, everything holds with probability $\geq 1 - 2^{-4n} - \frac{\delta}{2} - 2(k + 2) \cdot \delta^{1/3} \geq 1 - 2^{-4n} - 6k^3 \cdot \delta^{1/3} \cdot k$ (for sufficiently small $\delta$).

We now focus on the correctness of the estimate $f'$ of the $\ell_{\infty}$-to-$\ell_1$ norm of the generated subsets. First, we notice that $|A| = |A^*| = \frac{\delta}{k}$. Now, by what just proved, we can assume that $A \Delta A^* \leq 2k\delta^{1/3} \cdot n = o(n)$. By the triangle inequality

$$\|2\epsilon \cdot P_{k,a}^{A,A} - Q_{k,a}^{A^*,A^*}\|_{\infty \to 1} \leq \|2\epsilon \cdot P_{k,a}^{A,A} - 2\epsilon \cdot P_{k,a}^{A^*,A^*}\|_{\infty \to 1} + \|2\epsilon \cdot P_{k,a}^{A^*,A^*} - Q_{k,a}^{A^*,A^*}\|_{\infty \to 1}.$$ 

By Lemma 5.10, we have that, under the previously mentioned events holding with high probability, $\|Q - 2\epsilon \cdot P\|_{\infty \to 1} \leq 16n\sqrt{n} + 2B = o(cn^2)$. Therefore, by Lemma 5.12, it follows that

$$\|2\epsilon \cdot P_{k,a}^{A^*,A^*} - Q_{k,a}^{A^*,A^*}\|_{\infty \to 1} \leq \frac{k - 1}{k} \cdot \|2\epsilon \cdot P_{k,a}^{A^*,A^*} - Q_{k,a}^{A^*,A^*}\|_{\infty \to 1} \leq 16k \cdot n\sqrt{n} + 2k \cdot B = o(cn^2).$$

Moreover,

$$\|2\epsilon \cdot P_{k,a}^{A^*,A^*} - Q_{k,a}^{A^*,A^*}\|_{\infty \to 1} = 2\epsilon \cdot \|P_{k,a}^{A^*,A^*}\|_{\infty \to 1}.$$ 

Since $A \Delta A^* \leq 2k\delta^{1/3} \cdot n = o(n)$ and the entries of $P_{k,a}$ are bounded in absolute value by 1, we get that $\|P_{k,a}^{A^*,A^*}\|_{\infty \to 1} \leq (2k\delta^{1/3} \cdot n) \cdot (2n) = 4k\delta^{1/3} \cdot n^2 = o(n^2)$. By putting everything together, we finally get that

$$\|Q_{k,a}^{A^*,A^*} - 2\epsilon \cdot P_{k,a}^{A^*,A^*}\|_{\infty \to 1} \leq k \cdot f + 4k\delta^{1/3} \cdot n^2 \leq 16k \cdot n\sqrt{n} + 2k \cdot B + 4k\delta^{1/3} \cdot n^2 \cdot cn^2 = o(cn^2).$$

We are now ready to extend the previous lemma to any recursive invocation of the recursive cluster.

**Theorem 5.14.** Consider a generic invocation of $\text{Recursive-Cluster}(S, k', f, \gamma)$ originated from the first invocation of $\text{Recursive-Cluster}(\|S\|, k, 16n\sqrt{n} + 2B, 1)$, and let $\delta := 4k \cdot \sqrt{\frac{n}{c}}$. With high probability $\geq 1 - o(1)$:

- for each cluster $C$, either $|C \cap S| \leq o(n)$ or $|C \cap S| \geq \frac{n}{k} - o(n)$, and there are exactly $k'$ clusters satisfying the second condition;
- let $S^*$ be the union of the $k'$ clusters having $\geq \frac{n}{2k}$ elements in $S$, let $Q_{k'}$ be the matrix obtained from $Q$ by multiplying the negative entries by $\gamma$, and let $P_{k'}$ be the analogous matrix obtained from $P$. Then

$$\|Q_{k'}^{S^*,S^*} - 2\epsilon \cdot P_{k'}^{S^*,S^*}\|_{\infty \to 1} \leq f = o(cn^2).$$

- if $k' > 1$, the first sampled threshold $t$ does not satisfy the condition of line 13 in Algorithm 3 so the algorithm does not fail.

**Proof.** We show the theorem by induction on the number of recursive calls each invocation of the recursive cluster comes from.
**Base Case.** We start with the first invocation, i.e. **Recursive-Clust**([n], k, 16n√n + 2B, 1). First, each one of the k clusters has \( \frac{n}{k} \) elements in common with [n]. Second, by Lemma 6.9 it holds \( \|Q - 2\varepsilon \cdot P\|_{\infty} \leq 16n\sqrt{n} + 2B = o(n^2) \), as desired. Finally, by Lemma 5.13 with high probability 1 – o(1) the algorithm samples an appropriate threshold \( t \) and it does not fail. Thus, everything holds in the first invocation of **Recursive-Clust**.

**Inductive Step:** from \((S, k', f, \gamma)\) to \((S', k'', f', \gamma')\). This follows the exact same steps of the proof of Lemma 5.13 which can also be seen as a special case, proving the inductive step from the first invocation of **Recursive-Clust** to its direct calls. We quickly go through these steps. Let \( n' := |S| \). We start from the fact that \( Q_{k'} \), thanks to the scaling by \( \gamma' \), becomes positive semidefinite. Moreover, let \( S^* \) be defined as in the statement of the lemma. By inductive hypothesis we get that, with high probability, \( \|Q_{k'}^S S^* - 2\varepsilon \cdot P_{k'}^S S^*\|_{\infty} \leq f = o(n^2) \). This is the only necessary ingredient to show that, by Lemma 5.11 with high probability there exists \( v \) eigenvector of the leading eigenvalue of \( P_{k'} \) such that

\[
\|u^* - v\|^2 \leq 4k' \cdot \sqrt{\frac{f}{\epsilon (n')^2}} = o(1).
\]

We now notice that \( P_{k'} \) is the positive semidefinite “correlation matrix” of a set of \( k' \) clusters with size \( \frac{n}{k} \). Apart from a normalization factor that depends on \( k, k' \), it has the same eigenvectors and eigenvalues of the positive semidefinite “correlation matrix” \( P' \) of a set of \( k' \) clusters with size \( \frac{n}{k'} \). Thus, we can proceed as before, ignoring these \( \Theta(k) = \Theta(1) \) normalization factors. From now on, we can proceed exactly as in the proof of Lemma 5.13 first, we can use \( u^* \) to effectively proceed with the recursive calls to the sub-problems. Let \((S', k'', f', \gamma')\) be the input of one of these sub-problems. In the exact same way of Lemma 5.13 we get that \( \|Q_{k''}^{S'} S'' - 2\varepsilon \cdot P_{k''}^{S'} S''\|_{\infty} \leq f' = o(n^2) \), where we have used a coherent notation on the sub-problem. The remaining properties follow exactly as in the proof of Lemma 5.13. Notice that each recursive call comes from at most \( k \) chained invocations of **Recursive-Clust**, so all the estimates about the norms and the small probabilities (e.g., of failure of the algorithm) can be affected by a factor of \( poly(k) = \Theta(1) \), which does not affect the asymptotic estimates.

We can conclude that with high probability \( \geq 1 - o(1) \) all the recursive calls are successful and that the total number of misplaced nodes is \( o(n) \), achieving the desired result.

**Theorem 5.15.** With probability \( 1 - o(1) \), Algorithm 4 correctly classifies \( n - o(n) \) vertices.

Let us analyse the running time of Algorithm 5. Solving SDP through SDP-SOLVER up to negligible error takes polynomial time. Likewise, computing an orthogonal basis of eigenvectors for \( X \) up to negligible error takes polynomial time. Procedure GET-THRESHOLD takes polynomial time too, like all the remaining operations. The resulting running time is thus polynomial.

### 6 Information-Theoretic Lower Bounds

This section is dedicated to the analysis of the information-theoretic lower bounds for our semi-adversarial settings. We remark that we are not interested in recovering the exact thresholds of efficient solvability of the problems, but only in the asymptotic ones.

In the special case \( B = 0 \), the pre-adversarial model and the post-adversarial model are the same, and they are equal to the well-known random model (see [1] for a comprehensive survey). By the results of [26] approximate reconstruction is not solvable information-theoretically for \( k = 2 \) clusters if \( \epsilon \leq \frac{1}{\sqrt{2n}} \). If \( k \geq 2 \) is a constant, Banks et al. [7] showed that approximate reconstruction is not possible in an information-theoretic setting if \( \epsilon \leq \frac{c}{\sqrt{n}} \), where \( c \) is a constant eventually depending on \( k \). Therefore, we cannot hope to solve our problem if \( \epsilon = o(n^{-1/2}) \), or if \( \epsilon \leq \frac{c}{\sqrt{n}} \). Since we are not interested in recovering the exact thresholds of efficient solvability of the problems, but only asymptotic ones, we focus only on \( \epsilon = \omega(n^{-1/2}) \). Now, we focus on the budget parameter \( B \) in the two different semi-adversarial models.
Pre-Adversarial Model. We have seen that $\epsilon = \omega(n^{-1/2})$. We prove that if $B \geq \Omega(n^2)$, then the adversary could change the clusters of a constant fraction of all the nodes, therefore making it impossible to approximately reconstruct the clusters. More precisely, for each constant $\delta > 0$, we can take $B \leq \delta \cdot n^2$ and the adversary could randomize the information for $\delta \cdot n = \Theta(n)$ vertices, making approximate reconstruction impossible. For this reason, we can only consider $B = o(n^2)$.

Post-Adversarial Model. We have seen that $\epsilon = \omega(n^{-1/2})$. We prove that if $B \geq \Omega(cn^2)$, then the adversary could make it impossible to approximately reconstruct the clusters. We have seen that the random perturbation is equivalent to the following operation:

$$M'_{i,j} := \begin{cases} 
\text{U.A.R. in } \{−1, 1\} & \text{w. pr. } 1 - 2\epsilon; \\
M_{i,j} & \text{w. pr. } 2\epsilon
\end{cases}$$

therefore, the information about each pair of edges is turned into uniformly random in $\{−1, 1\}$ with probability $1 - 2\epsilon$, and preserved otherwise. By Lemma 3.8, this means that, with high probability $\geq 1 - e^{-\Theta(cn^2)} = 1 - o(1)$, for each vertex $i \in [n]$ we have between $cn$ and $3cn$ corresponding entries among $\{M'_{i,j}\}_{j \in [n]}$ preserving the original information. Therefore, for a post-adversary, it would suffice to turn those $\Theta(cn)$ entries of $M'$ into random for a constant fraction $\Theta(n)$ of the vertices to disrupt the original information for them. The final matrix $M''$ will only have completely random entries in $\{−1, 1\}$ for those $\Theta(n)$ elements, making approximate reconstruction impossible. The total number of post-adversarial changes needed would be $\Theta(cn^2)$. Notice that for any constant $\delta > 0$, we can take $B \leq \delta \cdot cn^2$ and disrupt the information for $\geq \frac{1}{\delta} \cdot n = \Theta(n)$ vertices, making approximate reconstruction impossible. Thus, we can only have $B = o(cn^2)$.

7 Limitations of a Spectral Approach

We show that while spectral methods can withstand pre-adversaries, they falter against post-adversaries. Consider the setting with 2 equinumerous clusters. And recall that $M'$ is the resulting matrix after the noise is injected. Let $\epsilon = \Omega(\log(n)^{1/3}n^{-1/3})$. We show that if the post-adversary can modify $B = \Theta(\epsilon^2 n^2)$ entries of $M'$, it can create a large eigenvalue whose corresponding eigenvector carries no information about the original clusters. Here is a post-adversarial strategy that achieves this.

Take a set $S$ of $4cn$ vertices with $2cn$ from each cluster and consider the induced minor in $M'$. Change all elements in the $4cn \times 4cn$ minor to 0. By Lemma 3.8, with probability $\geq 1 - e^{\Theta(cn^2)} = 1 - o(1)$, these are $8\epsilon^2 n^2 \leq B \leq 16\epsilon^2 n^2$ many changes. Consider now the set of columns of the elements in $S$. This contains $n - 4cn$ sub-rows of elements outside $S$, each with $4cn$ elements. By Lemma 3.9, the absolute value of the sum of the elements in each of these sub-rows is $\leq 2\sqrt{cn \log(n)}$ with probability $\geq 1 - \frac{1}{\sqrt{n}}$. Therefore, by the union bound, with probability $1 - \frac{1}{\sqrt{n}} = 1 - o(1)$, for each sub-row we can change $\leq 2\sqrt{cn \log(n)}$ elements in such a way as to ensure that the sum of the elements is 0. If we do this for each sub-row, we just need $\leq 2n\sqrt{cn \log(n)}$ additional changes. These adversarial changes turn the matrix $M'$ into the final matrix $M''$. After these changes, consider the vector $v_S$ having $\frac{1}{\sqrt{cn}}$ for indices of elements in $S$ and 0 everywhere else. Because of our changes, we have that

$$M''v_S = 4cn \cdot v_S.$$ 

Thus, $v_S$ is an eigenvector for $M''$ with eigenvalue $4cn$. The total number of changes, with high probability, has been

$$8\epsilon^2 n^2 \leq B \leq 16\epsilon^2 n^2 + 2\epsilon^{1/2}n^{3/2} \log^{1/2}(n) = O(\epsilon^2 n^2)$$

because, by hypothesis, $\epsilon = \Omega(\log(n)^{1/3}n^{-1/3})$. Therefore, with only $B = \Theta(\epsilon^2 n^2)$ post-adversarial changes, we can create an eigenvector $v_S$ with eigenvalue $\Theta(cn)$. Now, by Theorem 1.8, $\|M''\|_op = \Theta(cn)$, so our new eigenvalues is asymptotically of the same order of magnitude of the largest eigenvalue of $M''$. This vector could become the eigenvector of the largest eigenvalue of $M''$, even if it does not tell anything about the original clustering, making a simple spectral approach fail. Notice that $\epsilon^2 n^2 = o(cn^2)$, so the number of changes is within the information-theoretic feasibility range.
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