A Polynomial Time MCMC Method for Sampling from Continuous DPPs

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Abstract

We study the Gibbs sampling algorithm for continuous determinantal point processes. We show that, given a warm start, the Gibbs sampler generates a random sample from a continuous $k$-DPP defined on a $d$-dimensional domain by only taking $\text{poly}(k)$ number of steps. As an application, we design an algorithm to generate random samples from $k$-DPPs defined by a spherical Gaussian kernel on a unit sphere in $d$-dimensions, $S^{d-1}$ in time polynomial in $k, d$.

1 Introduction

Let $L \in \mathbb{R}^{n \times n}$ be a positive semi-definite (PSD) matrix. A discrete determinantal point process with kernel $L$ is a probability distribution $\mu : 2^{[n]} \to \mathbb{R}^+$ defined by

$$\mu(S) \propto \det(L_S), \forall S \subseteq [n].$$

The notion of DPP was first introduced by [Mac75] to model fermions. Since then, they have been extensively studied, and efficient algorithms have been discovered for tasks like sampling from DPPs [LJS15, DR10, AGR16], marginalization [BR05], and learning [GKFT14, UBMR17] them (in the discrete domain). In machine learning they are mainly used to solve problems where selecting a diverse set of objects is preferred since they offer negative correlation. To get intuition about why they are good models to capture diversity, suppose each row of the gram matrix associated with the kernel is a feature vector representing an item. It means the probability of a set of items is proportional to the square of the volume of the space spanned the vectors representing items. Therefore, larger volume shows those items are more spread which resembles diversity. Text summarization, pose estimation, and diverse image selection are examples of applications of DPP in this area [GKT12, KT+12, KT10, KT11]. These distributions also naturally appear in many contexts including non-intersecting random walks [Joh02], random spanning trees [BP93].

Here, we focus on the sampling problem of DPPs. In the discrete setting, the first sampling algorithms was proposed by [HKP+06]. They propose a two-step spectral algorithm which generates a random sample by running an eigen-space decomposition of the kernel and running conditional sampling on the space spanned by a randomly chosen set of the eigenvectors. Several algorithms for sampling different variation of DPPs including $k$-DPPs have been built on this idea [KT+12, DR10]. The disadvantage of spectral techniques for this problem is that they typically need the
eigen-decomposition or Cholesky decomposition of the kernel which makes them inefficient for large instances. Recently, several group of researchers studied employing the Monte Carlo Markov Chain (MCMC) technique for this task [AGR16, LSJ16, RK15]. It is shown in [AGR16] that the natural Metropolis-Hastings algorithm for $k$-DPPs gives an efficient sampling method running in time $O(n)\text{poly}(k)$, where $n$ is the number points in the underlying kernel.

In this paper, our main goal is to study the sampling problem of continuous DPPs. On a continuous domain, a DPP is defined similarly by a continuous PSD operator. For $C \subseteq \mathbb{R}^d$, and a continuous PSD kernel $L : C \times C \to \mathbb{R}$, the DPP is a distribution over finite subsets of $C$ where for every subset $S$, the probability density function at $S$, $p(S)$, satisfies

$$p(S) \propto \text{det}(L_S).$$

For an integer $k > 0$, a (continuous) $k$-DPP is the restriction of a (continuous) DPP to subsets of size exactly $k$. Continuous DPPs naturally arise in several areas of Physics, Math and Computer Science; To name a few examples, eigenvalues of random matrices [MG60, Gin65], zero-set of Gaussian analytic functions [PV05] are families of DPPs; also, see [LMR15] for applications in statistics and [BL16] for connections to repulsive systems. Recently, sampling from continuous $k$-DPP has also been used for tuning the hyper-parameters of a network [?]. Unlike the discrete setting, despite several attempts [HAFT13, SZT09, LMR12, BH16, HG16], to this date, we are not aware of any efficient sampling algorithms with provable guarantees. It remains an open problem to design an efficient sampling algorithm for continuous $k$-DPPs.

Our main contribution is to develop an algorithm to draw approximate samples from a $k$-DPP defined by a continuous kernel, and having access to a “conditional-sampling” oracle.

Unlike [AGR16], here, we analyze a different Markov chain, called the Gibbs sampler chain: Let $\pi$ be a $k$-DPP defined by a kernel $L : C \times C \to \mathbb{R}$ for $C \in \mathbb{R}^d$. Given a state $\{x_1, \ldots, x_k\}$, the Gibbs sampler $M$ moves as follows: Remove a point $x_i \in \{x_1, \ldots, x_k\}$ is chosen uniformly at random, and move to the state $\{x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_k\}$ with probability proportional to $\text{det}_L(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_k)$.

1.1 Results

We study the problem of sampling from a continuous $k$-DPP, and present the first MCMC based algorithm with provable guarantees for this problem. Our main contribution is to show that the Gibbs sampler for $k$-DPPs (see Definition 2.1) mixes rapidly, and can be simulated efficiently under some extra assumptions on the kernel. More precisely, we analyze the conductance of the Gibbs samplers $k$-DPPs (see theorems 3.1 and 4.1), and using the well-known connection between the conductance and mixing time obtain the following.

**Theorem 1.1.** Let $M$ be the Gibbs sampler for a $k$-DPP $\pi$. If we run the chain starting from an arbitrary distribution $\mu_0$, for any $\epsilon > 0$ we have

$$\tau_{\mu_0}(\epsilon) \leq O(k^4) \cdot \log \left( \frac{\text{var}_\pi(f_{\mu_0})}{\epsilon} \right).$$

In the above theorem, $f_\pi$ and $f_{\mu_0}$ refer to the probability density functions for $\pi$ and $\mu_0$, respectively. Moreover, $\tau_{\mu_0}(\epsilon)$ denotes the mixing time for the chain started from $\mu_0$, and is defined by

$$\tau_{\mu_0}(\epsilon) = \min\{t \mid d_{\text{TV}}(\mu_t, \pi) \leq \epsilon\},$$

2
where \( \mu_t \) is the distribution after \( t \) steps, and \( d_{TV} \) denotes the total variation distance. Moreover, an \( \epsilon \)-approximate sample for distribution \( \pi \) refers to a random sample from a distribution \( \mu \) where \( d_{TV}(\mu, \pi) \leq \epsilon \).

To find a “good” starting distribution \( \mu \) for which the bound in Theorem 1.1 is polynomial, we require some additional constraints on the kernel. Namely, we need to have access to conditional sampling oracles, formally defined as follows.

**Definition 1.2.** For a kernel \( L : C \times C \rightarrow \mathbb{R} \), a subset \( S \subset C \), and an integer \( j \), we define \((S, j)\)-conditional distribution of \( L \) to be a simple point process defined on \( \binom{C}{j} \) by a pdf function \( f \) satisfying

\[
\forall \{x_1, \ldots, x_j\} \subset C : f(x_1, \ldots, x_j) \propto \det(L \cup \{x_1, \ldots, x_j\}),
\]

and zero if \( S \cap \{x_1, \ldots, x_j\} \neq \emptyset \). We denote this distribution by \( CD_L(S, j) \). We say an algorithm is a \( CD_L(i, j) \) oracle for integers \( i \) and \( j \), if it returns a sample from \( CD_L(S, j) \).

It is straightforward to see that taking a step of the Gibbs sampler of the \( k \)-DPP from the state \( x_1, \ldots, x_k \) defined by \( L \) is equivalent to removing a point \( x_i \), for some \( 1 \leq i \leq k \), and generating a sample from \( CD_L(\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k\}, 1) \). We prove the following.

**Theorem 1.3** (informal). Let \( M \) be the Gibbs sampler for the \( k \)-DPP defined by a kernel \( L \). Given \( CD_L(0, 1) \) oracles for all \( 0 \leq i \leq k-1 \), we sample a starting state for \( M \) from a probability distribution \( \mu \) where

\[
\tau_{\mu}(\epsilon) \leq O(k^5 \log \frac{k}{\epsilon}).
\]

Therefore, to get a polynomial time algorithm for sampling from a \( k \)-DPPs (a CD(0, \( k \)) distribution), it is essentially enough to have efficient algorithms to sample from conditional 1-DPPs (CD(\( i \), \( 1 \)) distributions), which is a much simpler problem. As an application, we consider Gaussian kernels. For a covariance matrix \( \Sigma \in \mathbb{R}^{d \times d} \), a Gaussian kernel \( G_{\Sigma} \) is defined by \( G_{\Sigma}(x, y) = \exp(-(x - y)^{T} \Sigma^{-1}(x - y)) \). We show a simple rejection sampling can be used as conditional sampling oracles for \( G_{\sigma I} \), and obtain the following.

**Theorem 1.4** (See Theorem 5.2 for details.). Let \( d \) and \( k \) be integers. There is a randomized algorithm that for any \( \epsilon > 0 \) and \( \sigma \leq 1 \) generates an \( \epsilon \)-approximate sample from the \( k \)-DPP defined by \( G_{\sigma I} \) restricted to \( S^{d-1} \) which runs in time \( O(d \log \frac{1}{\epsilon} \cdot k^{O(1)}) \).

In the above, we are assuming a sample from the normal distribution can be generated in constant time.

### 1.2 Previous Work

In the continuous regime, the efforts have been mostly concentrated on finding the eigen-decomposition of the kernel or a low-rank approximation of it, and extending the aforementioned spectral techniques to the continuous space \([HAFT13, SZT09, LMR12, BH16]\). However, in theory, these methods does not yield provable guarantees for sampling because generally speaking, to project the DPP kernel onto a lower dimensional space, they minimize the error with respect to a matrix norm, rather than the DPP distribution. Moreover, there are two main obstacles to implement this approach: First, there is no efficient algorithm for obtaining an eigen-decomposition of a kernel defined on an
infinite space because it may have infinitely many eigenvalues. Secondly, for a general continuous eigen-decomposition, it is impossible to run a conditional sampling algorithm. For a concrete example, [LMR12] uses an orthonormal Fourier transform to find a low-rank approximation of the DPP kernel, and then proceeds with conditional sampling via rejection sampling. But, they do not provide any rigorous guarantee on the distance between the resulting distribution, and the underlying $k$-DPP distribution. A similar spectral approach is proposed in [HAFT13]. They consider the Nystöm method, and random Fourier feature as two techniques to find low-rank approximation of the kernel. The approximation scheme that they use enables them to handle a wider range of kernels. However, the first issue still remains unresolved: as the rank of the approximated kernel increases, the resulting distribution becomes closer to the initial $k$-DPP, but to the best of our knowledge there is no provable guarantee. [HAFT13] also provides empirical evidence that Gibbs sampling is efficient to generate sample from continuous $k$-DPP in many cases. However, they do not provide any rigorous justification. It is also worth mentioning that [HG16] claims to devise an algorithm to generate exact samples for specific kernels (including Gaussian), yet a careful look at their method would reveal a major flaw in their argument.

1.3 Techniques

Our first contribution is to analyze the Gibbs sampler chain in the discrete setting. We prove for a $k$-DPP defined on $n$ points, the spectral gap of the Gibbs sampler chain is a polynomial in $1/k$ and independent of $n$. So, up to logarithmic factors in $n$, the chain mixes in time polynomial in $k$. This result on its own could be of interest in designing distributed algorithms for sampling from discrete $k$-DPPs. This is because given access to $m$ processors, one can generate the next step of the Gibbs sampler in time $O(n/m)$.

Secondly, we lift the above proof to the continuous setting using a natural discretization of the underlying space. To prove the mixing time, we need to make sure that the logarithm of the variance of the starting distribution with respect to the stationary distribution of the chain, i.e., the $k$-DPP, is polynomially small in $k, d$. We use a simple randomized greedy algorithm for this task: We start from the empty set; assuming we have chosen $x_1, \ldots, x_i$ we sample $x_{i+1}$ from $CD_L(\{x_1, \ldots, x_i\}, 1)$, where as usual $L$ is the underlying kernel. We show that the distribution governing the state output by this algorithm is our desired starting distribution.

Lastly, we use our main theorem to generate samples from a $k$-DPP defined on a spherical Gaussian kernel on $S^{d-1}$. To run the above algorithm we need to construct the $CD_L(i, 1)$ for all $0 \leq i \leq k - 1$ where $L$ is the corresponding kernel. Given the point $\{x_1, \ldots, x_i\}$, we use the classical rejection sampling algorithm to choose $x_{i+1}$; namely, we generate a uniformly random point on the unit sphere and we accept it with probability $\frac{\det_L(x_1, \ldots, x_{i+1})}{\det_L(x_1, \ldots, x_i)}$. We use the distribution of the eigenvalues of the spherical Gaussian kernel [MNY06] to bound the expected number of proposals in the rejection sampler.

2 Preliminaries

Let $\mathbb{R}^d$ denote the $d$-dimensional euclidean space. Whenever, we consider $C \subset \mathbb{R}^d$ as measurable space, our measure is the standard Lebesgue measure. The $\text{vol}(C)$ denotes the $d$-dimensional volume.

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1 The distribution that they consider as the conditional distribution of the $k$-DPP is in fact equivalent to our notion of conditional distribution of the kernel (see Definition 1.2).
of C with respect to the standard measure. A function $f : C \to R$ belongs to $\ell^2(C)$, if $\int_C |f(x)|^2 dx < \infty$. For two such functions $f, g$ the standard inner product is defined by $\langle f, g \rangle = \int_C f(x)g(x)dx$. A function $L : C \times C \to R$, is a Hilbert-Schmidt kernel $\int_C \int_C |L(x,y)|dxdy < \infty$. The associated Hilbert-Schmidt integral operator is then a linear operator which for any $f \in \ell^2(C)$ is defined by $\forall x \in C : Lf(x) = \int_C L(x,y)f(y)dy$. The operator is self-adjoint if for any $f, g \in \ell^2(C)$, $\langle f, Lg \rangle = \langle Lf, g \rangle$. It is Positive Semi-Definite (PSD), if for any $f$, $\langle Lf, f \rangle \geq 0$.

**Mercer’s Conditions.** A kernel $L$ satisfies the Mercer conditions if: $L$ is symmetric, which means for any $x, y \in C$, $L(x, y) = L(y, x)$. Moreover, for any finite sequence $x_1, \ldots, x_i \in C$, the submatrix $L\{x_1, \ldots, x_i\}$ is a PSD matrix. It is known that the operators satisfying Mercer conditions are PSD. Moreover, if $L$ satisfies Mercer’s condition, for any $x \in C$, there exists a Hilbert space $H$, and a function $f_x : H \to R$, where for any $y \in C$, $L(x,y) = \langle f_x, f_y \rangle_H$. These functions are also known as feature maps. We also use the classical Mercer’s theorem which states that operators satisfying the Mercer’s condition are compact, and so have a countable system of eigen-spaces and eigenvalues. i.e. there are non-negative eigenvalues $\lambda_1, \lambda_2, \ldots$, and $\{\phi_i\}_{i=1}^\infty \subset \ell^2(C)$ where

$$L = \sum_{i=1}^\infty \lambda_i \phi_i(x)\phi_i(y).$$

In section 5, we use this result for Gaussian kernels. Throughout, the rest of the paper, whenever we say a continuous kernel, we refer to continuous Hilbert-Schmidt kernel which satisfies the Mercer’s conditions.

If $\pi$ is a probability distribution, we use $f_\pi$ to refer to the corresponding probability density function (pdf). We use bold small letters to refer to a finite set of points in $R^n$, and in particular a state of the Gibbs sampler for a $k$-DPP, e.g. $x = \{x_1, \ldots, x_k\} \subset R^n$. For $y \in R^n$, we may use $x + y$ to indicate $x \cup \{y\}$. For any $x = \{x_1, \ldots, x_k\}$, we use $\det_L(x_1, \ldots, x_k)$ and $\det_L(x)$ interchangeably to refer to determinant of the $k \times k$ submatrix where the $ij$th entry is $L(x_i, x_j)$. Whenever, the kernel is clear from the context, we may drop the subscript. For two expression $A$ and $B$, we write $A \lesssim B$ to denote $A \leq O(B)$.

### 2.1 Continuous Determinantal Point Process

A Determinantal Point Process (DPP) on a finite set, namely $[n]$ is a probability distribution $\pi$ on the subsets of $[n]$ which is defined by a PSD matrix (a.k.a kernel) $L \in R^{n \times n}$ where for every subset $S \subset [n]$,

$$\mathbb{P}(S) \propto \det(L_S)$$

where $L(S)$ is the principal submatrix of $L$ indexed by elements of $S$. For an integer $0 \leq k \leq n$, the restriction of $\pi$ to subsets of size $k$ is called a $k$-DPP defined by the kernel $L$. So support of a $k$-DPP defined on $[n]$ is $\binom{[n]}{k}$.

Similarly a continuous $k$-DPP can be defined on a continuous domain with a continuous PSD kernel $L : R^d \times R^d \to R$. The above succinct definition suffices to understand the rest of the paper. However, for completeness, we formally define them in the following. For more details about DPPs and generally point processes on continuous domains, we refer interested readers to [HKP+06].
Continuous $k$-DPP. For a kernel $L : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$, and for an integer $k$, the $k$-DPP defined by $L$ on domain $C$ is a point process with the support of subsets of $C$ of size $k$, $\binom{C}{k}$, defined as follows: For any $\{x_1, \ldots, x_k\} \subset C$ the probability density function is proportional to $\text{det}(x_1, \ldots, x_k)$. i.e. for any mutually disjoint family of subsets $D_1, \ldots, D_k \subset C$,

$$
\pi(\{\{x_1, \ldots, x_k\}|\forall i, \in D_i\}) = \frac{1}{Z} \int_{D_1} \cdots \int_{D_k} \text{det}(x_1, \ldots, x_k) dx_k \cdots dx_1,
$$

where $Z$ is the partition function $Z = \frac{1}{Z} \int_C \cdots \int_C \text{det}_L(x_1, x_2, \ldots, x_k) dx_k \cdots dx_1$. Now As alluded to before, we study the Gibbs sampling scheme for a $k$-DPP which is formally defined as follows:

**Definition 2.1** (Gibbs samplers for $k$-DPPs). Let $\pi$ be a $k$-DPP defined by a kernel $L : C \times C \rightarrow \mathbb{R}$ for $C \in \mathbb{R}^d$. The Gibbs sampler $\mathcal{M}$ for $\pi$ is a Markov chain with state space $\binom{C}{k}$ and stationary measure $\pi$ which moves as follows: Let $\{x_1, \ldots, x_k\} \subset C$ be the current state. A point $x_i \in \{x_1, \ldots, x_k\}$ is chosen uniformly at random, and the chain moves to the state $\{x_1, \ldots, x_k\} - x_i + y$ for $y \in C$ chosen by the distribution defined by the pdf function

$$
f(y) := \alpha \text{det}_L(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_k)
$$

**2.2 Markov Chains with Measurable State Space**

In this section we give a high level overview on the theory of Markov chains defined over measurable sets. We refer interested readers to [LS93] for more details. Let $(\Omega, \mathcal{B})$ be a measurable space. In the most general setting, a Markov chain is defined by the triple $(\Omega, \mathcal{B}, \{P_x\}_{x \in \Omega})$, where for every $x \in \Omega$, $P_x : \mathcal{B} \rightarrow \mathbb{R}_+$ is a probability measure on $(\Omega, \mathcal{B})$. Also, for every fixed $B \in \mathcal{B}$, $P_x(B)$ is a measurable function in terms of $x$. In this setting starting from a distribution $\mu_0$, after one step the distribution $\mu_1$ would be given by

$$
\mu_1(B) = \int_{\Omega} P_x(B) d\mu_0(x), \forall B \in \mathcal{B}.
$$

From now on, assume $\Omega \subset \mathbb{R}^k$ and $\mathcal{B}$ is the standard Borel $\sigma$-algebra. In our setting, we can assume the transition probabilities are given by a kernel transition kernel $P : \Omega \times \Omega \rightarrow \mathbb{R}_+$ where for any measurable $A \subset \Omega$, we can write

$$
P_x(A) = \int_A P(x, y) dy.
$$

In this notation, we use $P(x, B)$ and $P_x(B)$ interchangeably. $P^n(x, .)$ would also denote the probability distribution of the states after $n$ steps of the chain started at $x$. Similar to the discrete setting, we can define the stationary measure for the chain. A probability distribution $\pi$ on $\Omega$ is stationary if and only if for every measurable set $B$, we have

$$
\pi(B) = \int_{\Omega} \int_B P(x, y) dy d\pi(x).
$$

We call $\mathcal{M}$ $\phi$-irreducible for a probability measure $\phi$ if for any set $B \in \mathcal{B}$ with $\phi(B) > 0$, and any state $x$, there is $t \in \mathbb{N}$ such that $P^t(x, B) > 0$. It is called strongly $\phi$-irreducible if for any $B \subseteq \Omega$ with non-zero measure and $x \in \Omega$, there exists $t \in \mathbb{N}$ such that for any $m \geq t$, $P^m(x, B) > 0$. We say $\mathcal{M}$ is reversible with respect to a measure $\pi$ if for any two sets $A$ and $B$ we have

$$
\int_B \int_A P(y, x) dx d\pi(y) = \int_A \int_B P(x, y) dy d\pi(x).
$$
In particular, reversibility with respect to a measure, implies it is a stationary measure. Is is immediate from this to verify that for a Gibbs sampler of a $k$-DPPs $\pi$, the $\pi$ itself is the stationary measure. Moreover, if the kernel of the $k$-DPP is continuous, it is straight-forward to see that it is $\pi$-strongly irreducible. The following lemma also shows $\pi$ is the unique stationary measure, and as the number of steps increases, the chain approaches to the unique stationary measure.

**Lemma 2.2 ([DF97]).** If $\pi$ is a stationary measure of $\mathcal{M}$, and $\mathcal{M}$ is strongly $\pi$-irreducible. Then for any other distribution $\mu$ which is absolutely continuous with respect to $\pi$, $\lim_{n \to \infty} |P^n(\mu, .) - \pi|_{TV} = 0$.

From now on, consider $\mathcal{M} = (\Omega, P, \pi)$ is chain with state space $\Omega$, probability transition function $P$, and a unique stationary measure $\pi$. Let us describe some results about mixing time in the Markov chains defined on continuous spaces. But before that we need to setup some notation. Consider a Hilbert space $\ell^2(\Omega, \pi)$ equipped with the following inner product.

$$\langle f, g \rangle_\pi = \int_\Omega f(x)g(x)d\pi(x).$$

$P$ defines an operator in this space where for any function $f \in \ell^2(\Omega, \pi)$ and $x \in \Omega$,

$$(Pf)(x) = \int_\Omega P(x, y)f(y)dy.$$

In particular $\mathcal{M}$ being reversible is equivalent to $P$ being self-adjoint. For a reversible chain $\mathcal{M}$ and a function $f \in \ell^2(\Omega, \pi)$, the Dirichlet form $E_P(f, f)$ is defined as

$$E_P(f, f) = \frac{1}{2} \int_\Omega \int_\Omega (f(x) - f(y))^2P(x, y)d\pi(x)dy.$$

We also define the **Variance** of $f$ with respect to $\pi$ as

$$\text{var}_\pi(f) := \int_\Omega (f(x) - \mathbb{E}_\pi(f))^2d\pi(x).$$

We may drop the subscript if the underlying stationary distribution is clear in the context. One way for upperbounding the mixing time of a chain is to use it to its spectral gap which is also known as **Poincaré Constant**.

**Definition 2.3 (Poincaré Constant).** The Poincaré constant of the chain is defined as follows,

$$\lambda := \inf_{f: \pi \to \mathbb{R}} \frac{E_P(f, f)}{\text{var}(f)},$$

where the infimum is only taken over all functions in $\ell^2(\Omega, \pi)$ with non-zero variance.

In this paper, we use the following theorem to upperbound the mixing time of the chain relevant to us.

**Theorem 2.4 ([KM12]).** For any reversible, strongly $\pi$-irreducible Markov chain $M = (\Omega, P, \pi)$, if $\lambda > 0$, then the distribution of the chain started from $\mu$ (which is absolute continuous with respect to $\pi$) is

$$\|P^t(\mu, .) - \pi\|_{TV} \leq \frac{1}{2}(1 - \lambda)^t \sqrt{\text{var}(f_\mu / f_\pi)}.$$
For the sake of completeness, we include a proof of the above theorem which is an extension of the proof of the analogous discrete result in [Fil91]. We need the following simple lemma known as Mihail’s identity.

**Lemma 2.5** (Mihail’s identity, [Fil91]). For any reversible irreducible Markov chain \( M = (\Omega, P, \pi) \), and any function \( f \) in \( L^2(\pi) \),

\[
\text{var}(f) = \text{var}(Pf) + \mathcal{E}_{P^2}(f, f).
\]

**Proof of Theorem 2.4.** First of all, one can easily verify that if a chain is lazy and irreducible, then it is strongly-irreducible. Combining it with Lemma 2.2 would guarantee the uniqueness of the stationary measure. Let \( \mu_0 = \mu \) be the starting distribution and define \( \mu_t = P^t(\mu, .) \) be the distribution at time \( t \). Set \( f_t := \frac{f_{\mu_t}}{f_\pi} \), we have

\[
(Pf_t)(x) = \int_\Omega P(x, y) \frac{f_{\mu_t}(y)}{f_\pi(y)} dy = \int_\Omega P(y, x) \frac{f_{\mu_t}(y)}{f_\pi(x)} dy = \frac{f_{\mu_{t+1}}}{f_\pi}(x) = f_{t+1}(x)
\]

which implies

\[
\text{var}(Pf_t) = \text{var}(f_{t+1}) \tag{2.2}
\]

So applying Mihail’s identity on \( \frac{f_{\mu_t}}{f_\pi} \) and using (2.2), we conclude

\[
\text{var}(f_t) = \text{var}(f_{t+1}) + \mathcal{E}_{P^2}(f_t, f_t). \tag{2.3}
\]

Now, note that \( P^2 \) has the same stationary distribution \( \pi \), so its Poincaré constant is at most

\[
\lambda(P^2) \leq \frac{\mathcal{E}_{P^2}(f_t, f_t)}{\text{var}(f_t)}.
\]

Combining this with (2.3), and using induction we can deduce

\[
\text{var}(f_t) \leq (1 - \lambda(P^2))^t \text{var}(f_0).
\]

Note that, since \( P \) is the kernel for a lazy chain, it has no negative values in its spectrum, implying \( 1 - \lambda(P^2) = (1 - \lambda(P))^2 \). So in order to complete the proof it is enough show

\[
4\|\mu_t - \pi\|_{TV}^2 \leq \text{var}(f_t).
\]

This can be seen using an application of Cauchy-Schwarz’s inequality. We have

\[
4\|\mu_t - \pi\|_{TV}^2 = \left( \int_\Omega |f_{\mu_t}(x) - f_\pi(x)| dx \right)^2
\]

\[
= \left( \int_\Omega f_\pi(x) \left| \frac{f_{\mu_t}(x)}{f_\pi(x)} - 1 \right| dx \right)^2
\]

\[
\leq \int_\Omega f_\pi(x) \left| \frac{f_{\mu_t}(x)}{f_\pi(x)} - 1 \right|^2 dx = \text{var}\left(\frac{f_{\mu_t}}{f_\pi}\right)
\]

The last identity uses that \( \mathbb{E}_\pi \frac{f_{\mu_t}}{f_\pi} = 1 \). This completes the proof. \( \square \)

In order to take advantage of Theorem 2.4, we need to lowerbound the Poicaré constant of our chain. This can be done by lowerbounding the *Ergodic Flow* of the chain.
Definition 2.6 (Ergodic Flow). For a chain $\mathcal{M} = (\Omega, P, \pi)$, the ergodic flow $Q : B \to [0, 1]$ is defined by

$$Q(B) = \int_B \int_{\Omega \setminus B} P(u, v) dv f_\pi(u) du.$$ 

The conductance of a set $B$ is defined by, $\phi(B) := \frac{Q(B)}{\pi(B)}$, and the conductance of the chain is

$$\phi(\mathcal{M}) = \min_{0 < \pi(B) \leq \frac{1}{2}} \phi(B).$$

The following theorem which is an extension of the Cheeger’s inequality for the Markov chains on a continuous space, relates the spectral gap to conductance.

**Theorem 2.7 ([LS88]).** For a chain $\mathcal{M}$ defined on a general state space with spectral gap $\lambda$ we have

$$\frac{\phi(\mathcal{M})^2}{8} \leq \lambda \leq 2\phi(\mathcal{M}).$$

3 Gibbs Sampling for Discrete $k$-DPP

In this section we prove the Gibbs sampler for a discrete $k$-DPP is an $\Omega\left(\frac{1}{k^2}\right)$-expander. Recall that the conductance of a time reversible chain $\mathcal{M} = (\Omega, P, \pi)$ is defined by

$$\Phi(\mathcal{M}) = \min_{S \subset \Omega : \pi(S) \leq \frac{1}{2}} \frac{Q(S, \overline{S})}{\pi(S)},$$

where for $x, y \in \Omega$, $Q(y, x) = Q(x, y) = \pi(x)P(x, y)$. We prove the following.

**Theorem 3.1.** Let $\mathcal{M}$ be the Gibbs sampler chain for an arbitrary discrete $k$-DPP, then for a constant $C$ we have

$$\phi(\mathcal{M}) \geq \frac{1}{Ck^2}.$$

In the rest of this section, we fix $\mathcal{M} = (\Omega, P, \pi)$ to be the Gibbs-sampler chain on a $k$-DPP defined on a set of $n$ elements.

Before discussing the details of the proof let us first fix a notation and recall fundamental properties of $k$-DPPs. For any element $1 \leq i \leq n$, define $\Omega_i, \Omega_i^c$ be the set of all states in $\Omega$ that contain, do not contain $i$, respectively. Also define

$$\pi_i := \{\pi | i \text{ is chosen}\}, \text{ i.e. } \pi_i(x) = \frac{\pi(x)}{\pi(\Omega_i)}, \forall x \in \Omega_i$$

$$\pi_i^c := \{\pi | i \text{ is not chosen}\}, \text{ i.e. } \pi_i^c(x) = \frac{\pi(x)}{\pi(\Omega_i^c)}, \forall x \in \Omega_i^c.$$ 

It follows from [AGR16] that $\pi_i, \pi_i^c$ can be identified with a $(k-1)$-DPP, $k$-DPP supported on $\Omega_i, \Omega_i^c$, respectively. We define $\mathcal{M}_i = (\Omega_i, P_i, \pi_i), \mathcal{M}_i^c = (\Omega_i^c, P_i, \pi_i^c)$ to be the restricted Gibbs samplers. So, it is straightforward to see that for any $x, y \in \Omega_i$ we get $P_i(x, y) = \frac{k}{k-1} P(x, y)$, and consequently for $Q_i$ defined as $Q$ for $\mathcal{M}_i$, we get

$$Q_i(x, y) = \frac{Q(x, y)}{\pi(\Omega_i)}.$$

(3.1)
Unlike $P_i$, $P_i^*$ is not obtained from scaling a restriction of $P$. In particular, Let $x, y \in \Omega_\pi$ so that $P_i^*(x, y) > 0$ (which implies $|x \cap y| = k - 1$). Then, setting $I = x \cap y$ and with a bit abuse of notation $\pi(I) = \sum_{j \in [n] \setminus I} \pi(I + j)$, i.e. $\pi(I) = \mathbb{P}_{z \sim \pi} [I \subset z]$, we have

$$P_i^*(x, y) = \frac{1}{k} \cdot \frac{\pi(y)}{\pi(I) - \pi(i + I)}$$

(3.2)

whereas $P(x, y) = \frac{\pi(y)}{k \cdot \pi(I)}$. For any $x \in \Omega_i$, define $N_i(x)$ be the set of its neighbours in $\Omega_i$, i.e.

$$N_i(x) = \{ y \in \Omega_i : P(x, y) > 0 \}.$$

We use the following lemma to relate $Q_i^*$ to $Q$.

**Lemma 3.2.** Let $A \subset \Omega_\pi$ be an arbitrary subset. For a state $x \in \Omega_i$, consider the following partitioning of $N_\pi(x)$: $N_A = N_\pi(x) \cap A$ and $N_{\pi^c} = N_\pi(x) \cap (\Omega_\pi \setminus A)$. Then we have

$$Q(x, N_A) + Q(N_A, N_{\pi^c}) \geq \pi(\Omega_\pi) \cdot Q_i^*(N_A, N_{\pi^c}).$$

(3.3)

**Proof.** Note that $x \cup N_A \cup N_{\pi^c}$ is the set of all states containing elements in $x - i$. So by definition of $Q$ and $Q_i^*$, we have

$$Q(x, N_A) + Q(N_A, N_{\pi^c}) = \frac{1}{k} \cdot \frac{\pi(x) \pi(N_A)}{\pi(x) + \pi(N_A) + \pi(N_{\pi^c})} + \frac{1}{k} \cdot \frac{\pi(N_{\pi^c}) \pi(N_A)}{\pi(x) + \pi(N_A) + \pi(N_{\pi^c})}$$

(3.4)

$$= \frac{\pi(N_A)}{k} \cdot \frac{\pi(x) + \pi(N_{\pi^c})}{\pi(x) + \pi(N_A) + \pi(N_{\pi^c})} \geq \frac{\pi(N_A)}{k} \cdot \frac{\pi(N_{\pi^c})}{\pi(N_A) + \pi(N_{\pi^c})} = \pi(\Omega_\pi) \cdot Q_i^*(N_A, N_{\pi^c})$$

(3.5)

where the inequality follows simply because $\pi(N_A) \geq 0$.

\[\square\]

Figure 1: A schematic view of the restriction chains.

yellow, red, blue, and green edges correspond to $Q(S_n, \Omega_\pi \setminus S_n), Q(S_\pi, \Omega_\pi \setminus S_\pi), Q(S_n, \Omega_{\pi^c} \setminus S_\pi)$, and $Q(S_\pi, \Omega_{\pi^c} \setminus S_\pi)$, respectively.

**High level idea of the proof of Theorem 3.1.** We follow a proof strategy similar to [Mih92], which obtains analogue of our result in an unweighted setting and for the Metropolis-Hastings samplers. We use an inductive argument to prove the theorem. We need to prove $Q(S, \overline{S}) \geq \frac{\pi(S)}{Ck^2}$ for a subset $S \in \Omega$ with $\pi(S) \leq \frac{1}{2}$. Letting $S_n = S \cap \Omega_n$ and $S_\pi = S \cap \Omega_\pi$, we have

$$Q(S, \overline{S}) = Q(S_n, \Omega_n \setminus S_n) + Q(S_\pi, \Omega_\pi \setminus S_\pi) + Q(S_n, \Omega_{\pi^c} \setminus S_\pi) + Q(S_\pi, \Omega_{\pi^c} \setminus S_\pi).$$

(3.6)
We carry out the induction step by lowerbounding the RHS of the above term by term. In order to bound $Q(S_n, \Omega_n \setminus S_n)$ we use induction hypothesis on $M_n$. To bound $Q(S_\pi, \Omega_n \setminus S_\pi)$, we combine the induction hypothesis on $M_\pi$ with Lemma 3.2. It remains to bound the other two terms which correspond to the contribution of the edge across $(\Omega_n, \Omega_\pi)$. To do that, we crucially use negative association of $\pi$. In particular, we use the following lemma (appeared before in [Mih92] in the unweighted case). For any set $A \subset \Omega_n$, let $N_\pi(A) = \{ y \in \Omega_\pi : \exists x \in A, P(x, y) > 0 \}$ denote the set of neighbors of $A$ in $\Omega_\pi$.

**Lemma 3.3 ([AGR16]).** For any subset $A \subset \Omega_n$,

$$\pi(S(A)) \geq \pi_n(A).$$

The lemma lower bounds the vertex expansion of $S_n$ in $\Omega_n$ and similarly vertex expansion of $S_\pi$ in $\Omega_n$. Later we show how to use it to bound the edge expansion which is our quantity of interest.

**Proof of Theorem 3.1.** We induct on $k + n$. So, assume, the conductance of the Gibbs sampler for any $(k - 1)$-DPP over $n - 1$ elements is at most $\frac{1}{c(k-1)^2}$ and the conductance is at most $\frac{1}{ck^2}$ for any $k$-DPP over any $n - 1$ elements.

Fix a set $S \subset \Omega$ where $\pi(S) \leq \frac{1}{2}$. We need to show $Q(S, \overline{S}) \geq \frac{\pi(S)}{ck^2}$. First, consider a simple case where $\pi_n(S) \leq \frac{1}{2}$ and $\pi_\pi(S) \leq \frac{1}{2}$. By induction hypothesis we have $Q_n(S_n, \Omega_n \setminus S_n) \geq \frac{\pi_n(S_n)}{c(k-1)^2}$. Moreover, by adding up (3.1) for the edges across the cut $(S_n, \Omega_n \setminus S_n)$, we get

$$Q(S_n, \Omega_n \setminus S_n) = \frac{(k-1)\pi_\Omega(S_n)}{k} \cdot Q_n(S_n, \Omega_n \setminus S_n).$$

So combining them we have

$$Q(S_n, \Omega_n \setminus S_n) \geq \frac{\pi(S_n)}{ck^2}. \tag{3.7}$$

Now, we use induction on $M_\pi$ along with Lemma 3.2. The induction hypothesis implies

$$Q_\pi(S_\pi, \Omega_\pi \setminus S_\pi) \geq \frac{\pi(S_\pi)}{c k^2} = \frac{\pi(S_\pi)}{\pi_\Omega(S_\pi) \cdot c k^2}$$

So to prove the theorem in this case, it is enough to show the following and add it up with (3.7).

$$Q(S_\pi, \Omega_\pi \setminus S_\pi) + Q(S_\pi, \Omega_n \setminus S_n) + Q(S_n, \Omega_n \setminus S_n) \geq \pi(S_n) \cdot Q(S_n, \Omega_n \setminus S_n). \tag{3.8}$$

To see that, it is enough to apply Lemma 3.2 and add up (3.3) for all $x \in \Omega_n$, where subset $A \subset \Omega_\pi$ in the lemma is determined as follows: if $x \in S_n$ then set $A = S_\pi$, otherwise set $A = \Omega_\pi \setminus S_n$. Note that, doing that the RHS of the result will be exactly $\pi(S_\pi) \cdot Q(S_n, \Omega_\pi \setminus S_\pi)$, because any edge $yz$ of that will only show up in (3.3) by having $x = y \cap z + n$.

So we focus on the case $\max \{ \pi_n(S_n), \pi_\pi(S_\pi) \} > \frac{1}{2}$. Since $\pi(S) \leq \frac{1}{2}$, we have $\min \{ \pi_n(S_n), \pi_\pi(S_\pi) \} \leq \frac{1}{2}$, so $\pi(S) \leq \frac{1}{2}$. Without loss of generality, perhaps by considering $S$ instead of $\overline{S}$, we may assume $\pi_n(S_n) > \frac{1}{2}$ and $\pi_\pi(S) \leq \frac{1}{2}$. Our goal is to prove

$$Q(S, \overline{S}) \geq \frac{1}{ck^2} \cdot \min \{ 1 - \pi(S), \pi(S) \} \tag{3.9}$$

For every $x \in \Omega_n$, let $N_{\pi, S}(x) := N_\pi(x) \cap S_\pi$, and $N_{\pi, \overline{S}}(x) := N_\pi(x) \cap (\Omega_\pi \setminus S_\pi)$ be a partitioning of $N_\pi(x)$, so for every subset $T \subset N_\pi(x)$ we have

$$Q(x, T) = \frac{1}{2k} \cdot \frac{\pi(x) \pi(T)}{\pi(x) + \pi(N_{\pi, S}(x)) + \pi(N_{\pi, \overline{S}}(x))} \tag{3.10}$$
Now, define \( S_{\text{leave}} \subseteq S_n \) to be
\[
S_{\text{leave}} = \{ x \in S_n : \pi(x) + \pi(N_{\pi,S}(x)) < \pi(N_{\pi,S}(x)) \},
\]
in other words, \( S_{\text{leave}} \subseteq S_n \) is the subset of states so that, if the chain takes one step from \( S_{\text{leave}} \) by removing and resampling element \( n \), then with probability at least \( \frac{1}{2} \) it leaves \( S \) and enters \( N_{\pi,S}(x) \).

We also let \( S_{\text{stay}} = S_n \setminus S_{\text{leave}} \). On the other hand, starting from \( S_{\text{stay}} \) and by resampling \( n \), the chain with probability at least half stays in \( S \). It is straight-forward to see
\[
Q(S_{\text{leave}}, \Omega_\pi \setminus S_\pi) \geq \frac{\pi(S_{\text{leave}})}{4k} \tag{3.11}
\]
To see that, note that definition of \( S_{\text{leave}} \) and setting \( T = \Omega_\pi \setminus S_\pi \) in (3.10) implies that for any \( x \in S_{\text{leave}} \), we have \( Q(x, \Omega_\pi \setminus S_\pi) \geq \frac{\pi(x)}{4k} \). To get (3.11), it suffices to sum up this over all states of \( S_{\text{leave}} \). The bound (3.11) shows that \( Q(S_{\text{leave}}, S) \geq \frac{\pi(S_{\text{leave}})}{4k} \). So roughly speaking, to prove the theorem, it suffices to show \( \phi(S_{\text{stay}} \cup S_\pi) \geq \frac{1}{Ck^2} \). Consider two cases: if \( \pi_n(S_{\text{stay}}) \leq \frac{1}{2} \), we essentially use the same argument as in the case \( \pi_n(S_n), \pi_n(S_\pi) \leq \frac{1}{2} \). Otherwise we combine the induction with Lemma 3.3 to bound the expansion.

- **Case 1**: \( \pi_n(S_{\text{stay}}) \leq \frac{1}{2} + \frac{1}{4k^2} \). We show \( Q(S, S) \geq \frac{\pi(S)}{Ck^2} \). To do that, we use the induction hypothesis on \( M_n \), and the following claim which is the stronger version of (3.8).

**Claim 3.4.**
\[
Q(S_{\pi}, S) + Q(S_n, \Omega_\pi \setminus S_\pi) - \frac{1}{2} Q(S_{\text{leave}}, \Omega_\pi \setminus S_\pi) \geq \pi(\Omega_\pi) \cdot Q(\pi(S_\pi, \Omega_\pi \setminus S_\pi) \tag{3.12}
\]

**Proof.** The claim is implied by combining the summation of (3.13),(3.14), and (3.15) over \( \Omega_n \setminus S_n, S_{\text{stay}} \) and \( S_{\text{leave}} \), respectively. Let \( x \in \Omega_n \setminus S_n \). Then by applying Lemma 3.2 for \( x \) and \( A = S_\pi \), we get
\[
Q(N_{\pi,S}(x), \{x\} \cup N_{\pi,S}(x)) \geq \pi(\Omega_\pi) \cdot Q(\pi(N_{\pi,S}(x), N_{\pi,S}(x))) \tag{3.13}
\]
Similarly if \( x \in S_n \), by applying Lemma 3.2 for \( x \) and \( A = \Omega_\pi \setminus S_\pi \), we have
\[
Q(x \cup N_{\pi,S}(x), N_{\pi,S}(x)) \geq \pi(\Omega_\pi) \cdot Q(\pi(N_{\pi,S}(x), N_{\pi,S}(x))) \tag{3.14}
\]
Finally, for \( x \in S_{\text{leave}} \), we have
\[
Q(N_{\pi,S}(x), N_{\pi,S}(x)) + \frac{1}{2} Q(x, N_{\pi,S}(x)) = \frac{\pi(N_{\pi,S}(x))}{2k \cdot (\pi(x) + \pi(N_{\pi,S}(x)) + \pi(N_{\pi,S}(x)))} \cdot \left( \pi(N_{\pi,S}(x)) + \frac{\pi(x)}{2} \right) \\
\geq \frac{1}{2k} \cdot \pi(N_{\pi,S}(x) \cdot N_{\pi,S}(x)) + \pi(N_{\pi,S}(x)) \\
= \pi(\Omega_\pi) \cdot Q(\pi(N_{\pi,S}(x), N_{\pi,S}(x))) \tag{3.15}
\]
where the inequality follows since \( \pi(x) + \pi(N_{\pi,S}(x)) < \pi(N_{\pi,S}(x)) \) for \( x \in S_{\text{leave}} \). \( \square \)
In particular, we use the above claim to get
\[ Q(S, \overline{S}) = Q(S, \Omega_n \setminus S_n) + Q(S_n, \Omega_n \setminus \overline{S}) + Q(S_n, \Omega_n \setminus S_n) \]
\[ \geq Q(S_n, \Omega_n \setminus S_n) + \frac{1}{2} Q(S_{\text{leave}} + \Omega_n \setminus \overline{S}) + \pi(\Omega_n)Q(\overline{S_n}, \overline{S}) \quad \text{By Claim 3.4} \]
\[ \geq \frac{\pi(\Omega_n) - \pi(S_n)}{Ck(k-1)} + \frac{1}{2} Q(S_{\text{leave}}, \Omega_n \setminus \overline{S}) + \frac{\pi(S_n)}{Ck^2} \quad \text{induction Hyp. on } M_n \text{ and } M_{\overline{S}} \]
\[ \geq \frac{\pi(\Omega_n) - \pi(S_{\text{leave}}) - \pi(S_{\text{stay}}) + \pi(S_{\text{leave}})}{Ck(k-1)} + \frac{\pi(S_{\text{stay}})}{8k} + \pi(S) \quad \text{By (3.11) and } S_n = S_{\text{leave}} \cup S_{\text{stay}} \quad (3.16) \]

To finish the proof, we need to show the RHS of the above is at least \( \frac{\pi(S)}{Ck^2} \). To see that note that since \( \pi(S_{\text{leave}}) \geq \pi(S_{\text{leave}}) \cdot \left( \frac{1}{Ck} + \frac{1}{Ck(k-1)} \right) \) for sufficiently large \( k \), it suffices to show \( \frac{\pi(\Omega_n) - \pi(S_n)}{Ck(k-1)} \geq \frac{\pi(S_{\text{stay}})}{Ck^2} \), which can be directly verified for \( \pi_n(S_{\text{stay}}) \leq \frac{1}{2} + \frac{1}{10} \).

**Case 2:** \( \pi_n(S_{\text{stay}}) > \frac{1}{2} + \frac{1}{10} \). We prove
\[ Q(S, \overline{S}) \geq \frac{1 - \pi(S)}{Ck^2}. \]

Lemma 3.3 states that the vertex expansion of \( S_{\text{stay}} \) is proportional to \( \pi_n(S_{\text{stay}}) - \pi_n(S) \) (which is positive in this case by the assumption). We use it to bound \( Q(S, \overline{S}) \) by relating vertex expansion of \( S_{\text{stay}} \) to \( Q(S, \overline{S}) \). In particular, we show the following claim.

**Claim 3.5.**
\[ Q(S_{\text{stay}}, \Omega_n \setminus S_n) + Q(S_n, \Omega_n \setminus \overline{S}) \geq \frac{\pi(\Omega_n)}{2k} \cdot (\pi_n(S_{\text{stay}}) - \pi_n(S)) \]

**Proof.** Note that for any \( x \in S_{\text{stay}} \), since \( \pi(N_{\overline{S}}(x)) \leq \pi(x) + \pi(N_{\Omega_n}(x)) \), we have
\[ Q(x, N_{\Omega_n}(x)) + Q(N_{\Omega_n}(x), N_{\overline{S}}(x)) = \frac{1}{2k} \cdot \pi(N_{\Omega_n}(x)) \cdot (\pi(x) + \pi(N_{\Omega_n}(x))) \geq \frac{1}{2k} \cdot \frac{\pi(N_{\Omega_n}(x))}{2} . \]

To complete the proof, it is enough to sum up the above over \( S_{\text{stay}} \) to get the following
\[ Q(S_{\text{stay}}, \Omega_n \setminus S_n) + Q(S_n, \Omega_n \setminus \overline{S}) \geq \sum_{x \in S_{\text{stay}}} \frac{\pi(N_{\Omega_n}(x))}{4k} \geq \pi \left( \bigcup_{x \in S_{\text{stay}}} N_{\Omega_n}(x) \right) . \]
\[ \geq \pi(\Omega_n) \cdot (\pi_n(S_{\text{stay}}) - \pi_n(S)) \quad \text{By Lemma 3.3} \]

**Claim 3.5 and (3.16) implies** \( Q(S, \overline{S}) \geq \max\{L_1, L_2\} \) defined as above
\[
L_1 := \frac{\pi(S_1)}{8k} + \frac{\pi(\Omega_n) - \pi(S_{\text{leave}}) - \pi(S_{\text{stay}})}{Ck(k-1)} + \frac{\pi(S_n)}{Ck^2} \quad \text{By (3.16)}
\]
\[
L_2 := \frac{\pi(\Omega_n)}{4k} \cdot (\pi_n(S_{\text{stay}}) - \pi_n(S)) \quad \text{By Claim 3.5.}
\]

13
So we need to prove \( \max\{L_1, L_2\} \geq \frac{1 - \pi(S)}{k}. \) To prove that, we show that \( L_1 + \frac{L_2}{k-1} \geq (1 + \frac{1}{k-1}) \cdot \frac{1 - \pi(S)}{k}. \) Replacing values of \( L_1 \) and \( L_2 \) in the above and simplifying the resulting inequality, we need to show

\[
\frac{\pi(S_{\text{leave}})}{8k} + \frac{\pi(S_n)}{4k^2} + \frac{\pi(\Omega_n)}{4k(k-1)} \cdot (\pi_n(S_{\text{stay}}) - \pi_n(S_n)) \geq \frac{\pi(\Omega_n) - \pi(S_n)}{Ck(k-1)}.
\]

Ignoring the \( \frac{\pi(S_n)}{8k} \) term and rearranging the other terms, it is enough to show

\[
\frac{\pi(\Omega_n)}{4k(k-1)} \cdot (\pi_n(S_{\text{stay}}) - \pi_n(S_n)) \geq \frac{\pi(\Omega_n)}{Ck(k-1)} \cdot (1 - \frac{2k-1}{k} \cdot \pi_n(S_n)).
\]

The above can be verified for \( C > 16 \), by noting that by assumption \( \pi_n(S_{\text{stay}}) \geq \frac{1}{2} + \frac{1}{16k} \) and \( \pi_n(S_n) \leq \frac{1}{2}. \)

\( \square \)

## 4 Gibbs Sampling for Continuous \( k \)-DPP

In this section we analyze the mixing time of Gibbs samplers for continuous \( k \)-DPPs. Let \( \mathcal{M} \) be the Gibbs sampler for a \( k \)-DPP defined by a continuous kernel \( L \). In subsection 4.1, we show \( \phi(M) \geq \frac{1}{k^2}. \) Therefore, Gibbs sampling is an efficient method to generate samples from a continuous \( k \)-DPP provided that: We have access to an \( \text{CD}(1, k-1) \) oracle of \( L \) to simulate the chain, and we can find a proper starting distribution. In subsection 4.2, we show access to conditional oracles sampling is also enough to find the proper starting distributions.

As alluded to before, throughout the section \( L : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} \) is a continuous kernel which satisfies the Mercer’s condition and also \( \int \int |L(x, y)|^2 dx dy < \infty \) which also implies the partition function \( Z = \int \cdots \int \det L(x_1, \ldots, x_k) dx_k \ldots dx_1 < \infty. \)

### 4.1 Conductance of \( \mathcal{M} \)

**Theorem 4.1.** Let \( \mathcal{M} \) be the Gibbs sampler for a \( k \)-DPP defined by kernel \( L \), then

\[
\phi(\mathcal{M}) \gtrsim \frac{1}{k^2}.
\]

**Proof.** Recall that by Theorem 3.1 the conductance of a Gibbs sampler for any discrete \( k \)-DPP is at least \( \Omega(C_0/k^2) \). The key observation is that this bound is independent of the number of states. Therefore, we can obtain this bound for arbitrarily fine discretizations of \( \mathcal{M} \), and with a limiting argument extend it to \( \mathcal{M} \).

For simplicity, we assume \( d = 1 \). It is straightforward to extend the argument to higher dimensions. Let us denote the state space by \( \Omega \). Fix a measurable subset \( S \subset \Omega \) with \( \pi(S) \leq \frac{1}{2} \). Our goal is to prove \( \phi(S) = Q(S, S)/\pi(S) \geq \Omega(\frac{1}{k^2}) \). Without loss of generality, we can only consider restriction of \( \Omega \) and \( S \) to a bounded set. To see that, note that if we set \( \Omega_n = ([n/k, \infty]) \), then clearly, \( \lim_{n \rightarrow \infty} \frac{Q(S \cap \Omega_n, S \cap \Omega_n)}{\pi(S \cap \Omega_n)} = \phi(S) \), and so for large values of \( n \), \( \frac{Q(S \cap \Omega_n, S \cap \Omega_n)}{\pi(S \cap \Omega_n)} = \Theta(\phi(S)) \). So suppose that \( \Omega = ([0, 1/k]) \). For an integer \( n \), we consider a discretization \( \mathcal{M}_n \) of \( \mathcal{M} \) defined as follows. We use
\( n \) in subscript to denote quantities related to \( \mathcal{M}_n \). We partition \([0,1]\) into intervals of length \( \frac{1}{n} \), and identify each interval with an element in the ground set of \( \mathcal{M}_n \), so \( \Omega_n = \left\{ \binom{n}{k} \right\} \). \( \mathcal{M}_n \) is defined by a kernel \( L_n \) characterized below. For \( i \in [n] \) let \( I_i = \left[ \frac{i-1}{n}, \frac{i}{n} \right] \). For any \( i, j \in [n] \), we define
\[
L_n(i,j) = \int \int L(u,v) du dv,
\]
be the accumulative value of \( L \) over \( I_i \times I_j \). One can easily see \( L_n \) is a PSD matrix, as \( L \) is a PSD operator. Moreover, \( L \) and consequently \( \det L \) is a continuous function on a closed domain, so it is uniform continuous, implying for any \( \epsilon > 0 \), there exists an integer \( n(\epsilon) \) so that for all \( n > n(\epsilon) \) and any two states \( \{x_1, \ldots, x_k\} \) and \( \{y_1, \ldots, y_k\} \) with \( |y_i - x_i| \leq \frac{1}{n} \), we have
\[
|\det_L(x_1, \ldots, x_k) - \det_L(y_1, \ldots, y_k)| \leq \epsilon.
\]
Now, note that \( f_\pi(y_1, \ldots, y_k) = \frac{1}{\det \int \det_L(x_1, \ldots, x_k) dx_1 \ldots dx_k} \).

So, using the simple fact that for any two sequences of numbers \( \{a_n\} \) and \( \{b_n\} \),
\[
\left( \lim_{n \to \infty} a_n = a \right) \land \left( \lim_{n \to \infty} b_n = b \neq 0 \right) \implies \lim_{n \to \infty} \frac{a_n}{b_n} = \frac{a}{b} \quad (4.2)
\]
we get that for any \( \epsilon > 0 \), there exists an integer \( m(\epsilon) \), where \( m(\epsilon) \) depends on \( n(\epsilon) \), such that
\[
\forall n \geq m(\epsilon), \forall \{t_1, \ldots, t_k\} \in \left( \binom{n}{k} \right) : \left| \pi_n(t_1, \ldots, t_k) - \pi \left( \prod_{i=1}^{k} I_t \right) \right| \leq \frac{\epsilon}{nk} \quad (4.3)
\]

We define a set \( S_n \subset \Omega_n \) corresponding to \( S \) for any \( n \), so that
\[
\lim_{n \to \infty} \phi_n(S_n) = \phi(S). \quad (4.4)
\]
Clearly, the above proves the theorem as by **Theorem 3.1**, we know that \( \phi_n(S_n) \geq \frac{1}{S} \) for any \( n \). In what follows, we use \( A \subset B \) to denote both of \( A - B \) and \( B - A \) have Lebesgue measure zero. Also, define
\[
S_n = \left\{ \{t_1, \ldots, t_k\} \in \left( \binom{n}{k} \right) \mid I_{t_1} \times \cdots \times I_{t_k} \subset S \right\}.
\]
Following (4.2), to prove (4.4), it is enough to argue that \( \lim_{n \to \infty} Q_n(S_n, \overline{S_n}) = Q(S, \overline{S}) \), and \( \lim_{n \to \infty} \pi_n(S_n) = \pi(S) \). We first show the latter. This follows by (4.3) and that
\[
\lim_{n \to \infty} \mu \left( \bigcup_{\{t_1, \ldots, t_k\} \in S_n} \prod_{i=1}^{k} I_{t_i} \right) = \mu(S) \quad (4.5)
\]
for \( \mu \) being the Lebesgue measure.

It remains to see \( \lim_{n \to \infty} Q_n(S_n, \overline{S_n}) = Q(S, \overline{S}) \). First, note that \([0,1]^{k-1}\) is a closed set, so for any \( \delta > 0 \) and \( \epsilon > 0 \), there exists an integer \( n(\delta, \epsilon) \) so that for any \( n > n(\delta, \epsilon) \), and points \( x_1, \ldots, x_k, x_{k+1} \) and \( y_1, \ldots, y_k, y_{k+1} \) with \( |x_i - y_i| \leq \frac{1}{n} \), and \( \int_0^1 \det L(x_1, \ldots, x_{k-1}, \tau) d\tau \geq \delta \), we have
\[
\left| \frac{\det L(x_1, \ldots, x_k) \det L(x_1, \ldots, x_{k-1}, x_{k+1})}{\int_0^1 \det L(x_1, \ldots, x_{k-1}, \tau) d\tau} - \frac{\det L(y_1, \ldots, y_k) \det L(y_1, \ldots, y_{k-1}, y_{k+1})}{\int_0^1 \det L(y_1, \ldots, y_{k-1}, \tau) d\tau} \right| \leq \epsilon.
\]
Therefore, similar to the case for \( \pi_n \), it follows that for any \( \epsilon, \delta > 0 \), there exists integer \( m(\delta, \epsilon) \) depending on \( n(\delta, \epsilon) \) so that for any \( n \geq m(\delta, \epsilon) \) and for all \( t_1, \ldots, t_{k-1}, s, t \in \left( \binom{n}{k+1} \right) \) with \( \sum_{i=1}^{k} \pi_n(t_1, \ldots, t_i) \geq \frac{\delta}{n^{k-1}} \),
\[
\left| Q_n(t_1, \ldots, t_{k-1}, t), \{t_1, \ldots, t_{k-1}, s\} - Q(I_t \times I_{t_1} \times I_s \times \prod_{i=1}^{k-1} I_{t_i}) \right| \leq \frac{\epsilon}{n^{k+1}}. \quad (4.6)
\]
Now, combining the above equation with (4.5), and noting \( \epsilon \) and \( \delta \) can be chosen arbitrary close to zero, we obtain \( \lim_{n \to \infty} Q_n(S_n, \overline{S_n}) = Q(S, \overline{S}) \), which completes the proof.

Combining the theorem with Theorem 2.7, we get that \( \lambda_\mathcal{M} \gtrsim \frac{1}{k^4} \), where \( \lambda_\mathcal{M} \) is the poincare constant of \( \mathcal{M} \). Moreover, clearly the above argument implies the chain is \( \pi \)-strongly irreducible as well. So we can apply Theorem 2.4 to obtain the following corollary.

**Corollary 4.2.** Let \( \pi \) be the \( k \)-DPP defined by \( L \). If \( \mu \) is an arbitrary starting distribution, then

\[
\tau_\mu(\epsilon) \leq O(k^4) \cdot \log \left( \frac{\text{var}_\pi(\frac{f_\mu}{f_\pi})}{\epsilon} \right).
\]

**4.2 Finding a Starting**

In this subsection, we prove the following theorem, which shows that if we have access to CD\((i, 1)\) oracles of the kernel for any \( 0 \leq i \leq k - 1 \), then a proper starting distribution for the associated Gibbs sampler can be found.

**Theorem 4.3.** Let \( \mathcal{M} \) be the Gibbs sampler for the \( k \)-DPP defined by kernel \( L : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \). There is a polynomial time algorithm which given access to CD\((i, 1)\) oracles all \( 0 \leq i \leq k - 1 \), returns a state of \( \mathcal{M} \) from a distribution \( \mu \) where

\[
\tau_\mu(\epsilon) \leq O(k^5 \log \frac{k}{\epsilon}).
\]

Moreover, the algorithm only uses \( k \) oracle accesses.

To prove the above theorem, and generate a sample from such a distribution \( \mu \), we use Algorithm 1 which is the continuous analog of a greedy algorithm analyzed at [DV07] as approximate volume sampling. In particular, we crucially use the following lemma which directly follows [DV07]. As always, \( \pi \) denotes our \( k \)-DPP.

**Lemma 4.4.** Let \( \nu \) be the probability distribution of the output of Algorithm 1. Then, for any \( \{x_1, \ldots, x_k\} \subset \mathbb{R}^d \),

\[
f_\nu(\{x_1, \ldots, x_k\}) \leq (k!)^2 f_\pi(\{x_1, \ldots, x_k\}).
\]

We include the proof in the appendix for the sake of completeness.

**Algorithm 1** Choosing a starting state for the Gibbs sampler

**Input:** A kernel \( L \) and CD\((i, 1)\) oracles for \( 0 \leq i \leq k - 1 \).

1. Let \( \mathbf{x} = \{\} \).
2. **for** \( i \) from 0 to \( k - 1 \) **do**
3. \hspace{5mm} Use the CD\((i, 1)\) oracle to generate a sample \( x_i \) and add \( x_i \) to \( \mathbf{x} \).
4. **end for**

return \( \mathbf{x} \)

**Proof of Theorem 4.3.** First of all, clearly the algorithm use each CD\((i, 1)\) oracle for \( 1 \leq i \leq k - 1 \) once. So letting \( \mu \) be the distribution of the output of the algorithm, it suffices to show (4.7).
Applying Corollary 4.2, it is equivalent to show \( \text{var}_\pi(\frac{f_\mu}{f_\pi}) \leq O(k \log k) \). It straight-forwardly follows by applying Lemma A.2. More precisely,
\[
\text{var}_\pi(\frac{f_\mu}{f_\pi}) = \mathbb{E}_\pi \left( \frac{f_\mu(x)}{f_\pi(x)} \right)^2 - 1 \leq (k!)^4 \cdot \mathbb{E}_\pi 1 = (k!)^4
\]
which completes the proof.

\[\square\]

Remark 4.5. It is straight-forward to use a similar discretization argument to prove Theorem 4.1, and consequently Theorem 4.3 when the domain of the kernel is restricted to a closed subset \( C \subset \mathbb{R}^d \) which can be nicely discretized as in Theorem 4.1. In particular, we assume \( C \) is an sphere in the next section. More precisely, \( C \) could be any closed subset which its interior has also the same measure.

5 Applications for Sampling from Gaussian \( k \)-DPP’s

As pointed out before, to find a proper starting state, and simulate the Gibbs sampler for a \( k \)-DPPs, we need to have access to CD(\( i, 1 \)) \((0 \leq k - 1)\) sampling oracles of the kernel. In this section, we study the problem for Gaussian kernels, and as a special case argue a simple rejection sampling algorithm is an efficient CD(\( i, 1 \)) oracle, when restricting the kernel to the unit sphere.

In particular, fix \( \mathcal{G}_\sigma : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} \) to denote the Gaussian kernel with covariance matrix \( \sigma I \), \( \mathcal{G}_\sigma(x, y) = \exp(-\frac{\|x-y\|^2}{2\sigma^2}) \). Also let \( \mathbb{S}^{d-1} = \{x \in \mathbb{R}^d \mid \|x\| = 1\} \) denote the unit sphere. We prove the following.

**Theorem 5.1.** Let \( \mathcal{G}_\sigma\big|_{\mathbb{S}^{d-1}} \) denote the restriction of \( \mathcal{G}_\sigma \) to the unit sphere. For any integer \( k \) and any \( x_1, \ldots, x_k \in \mathbb{S}^{d-1} \), Algorithm 2 returns a sample from the conditional distribution \( \text{CD}(\{x_1, \ldots, x_k\}, 1) \) associated with \( \mathcal{G}_\sigma\big|_{\mathbb{S}^{d-1}} \). If \( k \leq \exp(d/4) \) and \( t \) is the smallest integer that \( \lfloor \frac{d}{t} \rfloor \geq k \), then the algorithm queries at most \( e^{\frac{d}{2t}} \cdot \sigma^{2t} \cdot t! \) uniform samples from the sphere in expectation. Moreover, if \( \sigma \lesssim \frac{1}{\sqrt{\log k}} \), the algorithm uses \( O(1) \) samples in expectation.

Then, combining with Theorem 4.3, and assuming generating a sample from the normal distribution can be done in constant time, we get our main theorem for sampling from Gaussian \( k \)-DPPs.

**Theorem 5.2.** Let \( d, k \leq e^{d^{1-\delta}} \) for some \( 0 < \delta < 1 \) be two integers. There is a randomized algorithm that for any \( \epsilon > 0 \) and \( \sigma > 0 \), generates an \( \epsilon \)-approximate sample from the \( k \)-DPP defined by \( \mathcal{G}_\sigma \) on \( \mathbb{S}^{d-1} \) which runs in time \( O(d \log \frac{1}{\epsilon} \cdot k^{O(\frac{1}{\epsilon})} \sigma^{2t}) \) if \( \sigma \lesssim \frac{1}{\sqrt{\log k}} \), and for the larger values of \( \sigma \) the running time is bounded by
\[
O(d \log \frac{1}{\epsilon} \cdot k^{O(\frac{1}{\epsilon})} \sigma^{2t}).
\]
where \( t = \min \{ t \in \mathbb{N} \mid \frac{d}{t} \geq k \} \).

**Proof.** Theorem 5.1 states that Algorithm 2 gives CD(\( i, 1 \)) oracles for any \( i \). Having these oracles, we simulate the corresponding Gibbs sampler starting from the distribution \( \mu \) given by Theorem 4.3. Let \( T \) denote the cost of a single step of the chain. So, the running time of the algorithm is \( \tau_{\mu}(\epsilon) \cdot T \).
By Theorem 4.3, \( \tau_\mu(\epsilon) \leq O(\sqrt{d} \log \frac{1}{\epsilon}) \). So we only need to analyze \( \mathcal{T} \). Note that a uniform sample from the sphere can be generated by using \( d \) samples from the normal distribution, so \( \mathcal{T} \) is at most a factor of \( d \) of the bound of Theorem 5.1. By Theorem 5.1, if \( \sigma \lesssim \frac{1}{\sqrt{\log k}} \), then \( \mathcal{T} = O(d) \), so \( \tau_\mu(\epsilon) \cdot \mathcal{T} \leq \tilde{O}(dk^5 \log \frac{1}{\epsilon}) \) which completes the proof in this case. For larger \( \sigma \) we get that

\[
\tau_\mu(\epsilon) \cdot \mathcal{T} \leq \tilde{O}(dk^5 \log \frac{k}{\epsilon}) \cdot (e^{\frac{2}{d} t!} \sigma^2 t),
\]

To complete the proof note that \( e^{\frac{2}{d} t!} \leq k^{O(1)} \), and moreover the definition of \( t \) and noting that \( k \leq \exp(d^{1-\delta}) \) implies \( t! \leq k^{O(\frac{1}{d})} \).

We remark that in the above algorithm if \( k = \text{poly}(d) \), then \( t = O(1) \), and so the running time is polynomial in terms of \( d, k, \sigma \). Moreover, one can see that the same holds if \( \sigma = O(1) \), as \( t \leq \log k \).

Algorithm 2 Rejection Sampling for sampling from the conditional distribution

**Input:** A Gaussian kernel \( \mathcal{G} \) restricted to \( C \subset \mathbb{R}^d \), and \( k \) points \( x_1, \ldots, x_k \in C \).

**Output:** A sample from the \( \text{CD}_C(\{x_1, \ldots, x_k\}, 1) \).

1. Draw a uniform sample \( x \) from \( C \).
2. Draw a uniform number \( u \) from \([0, 1]\).
3. If \( u \leq \frac{\det_\mu(x_1, \ldots, x_k, x)}{\det_\mu(x_1, \ldots, x_k)} \), accept and return \( x \). Otherwise go to line 1.

We conclude the section by proving Theorem 5.1.

### 5.1 Analysis of Algorithm 2 and Proof of Theorem 5.1

**Correctness.** One can show that for any set of points \( x = \{x_1, \ldots, x_k\} \), and any kernel \( L : C \times C \to \mathbb{R} \) such that for all \( z \in C \), \( L(z, z) \leq 1 \), the output has \( \text{CD}_C(x, 1) \) distribution. Clearly, any Gaussian kernel has this property. To see that, let \( y \) be the point uniformly selected from \( C \). The algorithm returns \( y \) with probability \( \frac{\det_\mu(x + y)}{\det_\mu(x)} \), where we are using the fact that this number is at most \( L(y, y) \), and \( L(y, y) \leq 1 \) by the assumption. Therefore, if \( \phi \) denotes the distribution of the output,

\[
f_\phi(y) = \frac{1}{\text{vol}(C)} \cdot \frac{\det_\mu(x + y)}{\det_\mu(x)} \propto \det_\mu(x + y),
\]

which implies the output has the desired distribution.

From now on, fix a kernel \( \mathcal{G}_\sigma|_{\mathbb{S}^{d-1}} \) to be the input kernel, and let \( T \) denote the number of the steps (samples generated from the sphere) until the algorithm terminates. So we only need to analyze \( E[T] \). Let \( \mu \) be the uniform distribution on \( \mathbb{S}^{d-1} \). The probability that the algorithm accepts and outputs the sample generated in the current step is

\[
P_{\mu, y \sim \mu} \left[ u \leq \frac{\det_\mu(x + y)}{\det_\mu(x)} \right] = E_{y \sim \mu} \frac{\det(x + y)}{\det_\mu(x + y)}.
\]

So \( T \) forms a geometric distribution and \( E[T] = \frac{\det(x)}{E_{y \sim \mu} \det(x + y)} \). The following lemma concludes the proof of Theorem 5.1.
Lemma 5.3. For any parameter σ ≥ 0, any integer k ≤ exp(d/4) and any set of points x_1, ..., x_k ∈ S^{d−1}, if we set μ to be the uniform distribution on S^{d−1} and t to be the smallest number such that \( \frac{d}{t} \geq 2k \), then

\[
\mathbb{E}_{y \sim \mu} \frac{\det \mathcal{G}_\sigma(x + y)}{\det \mathcal{G}_\sigma(x)} \gtrsim \frac{e^{-\frac{1}{2}}}{{t!} \cdot \sigma^t} \tag{5.1}
\]

Moreover, if σ ≤ \( \frac{1}{\sqrt{\log k}} \), then the bound can be improved to Ω(1).

To prove the lemma, we relate the quantity \( \mathbb{E}_{y \sim \mu} \frac{\det \mathcal{G}_\sigma(x + y)}{\det \mathcal{G}_\sigma(x)} \), to eigenvalues of \( \mathcal{G}_\sigma \mid_{S^{d−1}} \) and use the work of [MNY06] who studied eigenvalues, and eigenspaces of Gaussian kernels. Set \( \tilde{\mathcal{G}} = \frac{\mathcal{G}_\sigma \mid_{S^{d−1}}}{\text{vol}(S^{d−1})} \) be the kernel normalized with the uniform measure. In particular we use the following theorem.

Theorem 5.4 ([MNY06]). For any integer \( \ell \geq 0 \), \( \tilde{\mathcal{G}} \) has an eigenvalue \( \mu_\ell \) with multiplicity \( N(d, \ell) = \frac{(2d+2-2\ell)(2\ell+1)}{\ell(d-2)!} \) where

\[
\mu_\ell = e^{-\frac{1}{\sigma^2}} \frac{d^2 - 2}{d^2 + (2\ell + 1)^2} \frac{\sigma^d}{2} \Gamma\left(\frac{d}{2}\right),
\]

and \( I \) denotes the modified Bessel function of the first kind, defined by \( I_\nu(z) = \sum_{i=0}^{\infty} \frac{1}{i!(i+\nu+1)!} \left(\frac{z}{2}\right)^{\nu+2i} \).

Also, for any integer \( \ell \), \( \mu_\ell \) satisfies the following.

\[
\left(\frac{2e}{\sigma^2}\right)^{\ell} \cdot \frac{A_1}{(2\ell + d - 2)^{\ell + \frac{1}{2}}} \leq \mu_\ell \leq \left(\frac{2e}{\sigma^2}\right)^{\ell} \cdot \frac{A_2}{(2\ell + d - 2)^{\ell + \frac{1}{2}}},
\]

where \( A_1 = e^{-\frac{1}{2}} - \frac{1}{2\sqrt{\pi}} (2e)^{\frac{1}{2}} \Gamma\left(\frac{d}{2}\right) \) and \( A_2 = A_1 \cdot e^{\frac{d}{2} + \frac{1}{2}} \).

Proof of Lemma 5.3. Since \( \mathcal{G}_\sigma \) is a PSD operator, for any \( x \in S^{d−1} \), there exists function (feature map) \( f_x : S^{d−1} \to \mathbb{R} \) such that for any \( y \in S^{d−1} \), \( \mathcal{G}_\sigma(x, y) = \langle f_x, f_y \rangle \). For any \( y \in S^{d−1} \), define \( \mathcal{E}(y) = \prod_{i=1}^{k} (f_{x_i + y})^2 \), be the projection of \( f_y \) onto the space orthogonal to vectors corresponding to \( x_1, ..., x_k \). Then, by definition \( \frac{\det(x + y)}{\det(x)} = \|\mathcal{E}(y)\|^2 \), where recall that \( x = \{x_1, ..., x_k\} \). It implies

\[
\mathbb{E}_{y \sim \mu} \frac{\det(x + y)}{\det(x)} = \mathbb{E}_{y \sim \mu} \|\mathcal{E}(y)\|^2 = \frac{\text{tr} (\mathcal{E})}{\text{vol}(S^{d−1})} \tag{5.3}
\]

for the kernel \( \mathcal{E} : S^{d−1} \times S^{d−1} \to \mathbb{R} \) defined by \( \mathcal{E}(x, y) = \langle \mathcal{E}(x), \mathcal{E}(y) \rangle \). We further simplify this by noting that \( \mathcal{E} \) satisfies Mercer’s condition, as \( \mathcal{E}(., .) \) is a PSD kernel. It implies \( \text{tr}(\mathcal{E}) = \sum_{i=1}^{\infty} \lambda_i(\mathcal{E}) \).

Moreover, it follows from the definition of \( \mathcal{E} \), that \( \mathcal{G}_\sigma \mid_{S^{d−1}} - \mathcal{E} \) is an operator of rank at most \( k \). So \( \sum_{i=1}^{k} \lambda_i(\mathcal{E}) \geq \sum_{i=k+1}^{\infty} \lambda_j(\mathcal{G}_\sigma \mid_{S^{d−1}}) \). So recalling \( \tilde{\mathcal{G}} = \mathcal{G}_\sigma \mid_{S^{d−1}} \), and using (5.3), we get \( \mathbb{E}_{y \sim \mu} \frac{\det(x + y)}{\det(x)} \geq \sum_{j=k+1}^{\infty} \lambda_j(\tilde{\mathcal{G}}) \). We first prove, if \( \sigma \leq \frac{1}{\sqrt{2 \log k}} \), then \( \sum_{j=k+1}^{\infty} \lambda_j(\tilde{\mathcal{G}}) \geq \Omega(1) \). Using the Cauchy-Schwarz inequality we have

\[
k \cdot \sum_{i=1}^{k} \lambda_i(\tilde{\mathcal{G}})^2 \geq \left( \sum_{i=1}^{k} \lambda_i(\tilde{\mathcal{G}}) \right)^2 = \left( 1 - \sum_{i=k+1}^{\infty} \lambda_i(\tilde{\mathcal{G}}) \right)^2.
\]

We show \( \sum_{i=1}^{k} \lambda_i(\tilde{\mathcal{G}})^2 \leq \frac{1}{k^2} \) which implies \( \sum_{i=k+1}^{\infty} \lambda_i(\tilde{\mathcal{G}}) \geq (1 - 1/\sqrt{k}) \) which completes the proof. To see that, note that \( \sum_{i=1}^{k} \lambda_i(\tilde{\mathcal{G}})^2 = \sum_{i=1}^{k} \lambda_i(\tilde{\mathcal{G}})^2 \leq \text{tr}(\tilde{\mathcal{G}}^2) \) and \( \text{tr}(\tilde{\mathcal{G}}^2) = \langle \tilde{\mathcal{G}}, \tilde{\mathcal{G}} \rangle = \sum_{x, y \sim \mu} e^{-\|x - y\|^2/2\sigma^2} \).
where recall $\mu$ is the uniform measure on the sphere. Fix $x \in S^{d-1}$. It follows from basic concentration inequalities for Gaussian measures that $E_{y \sim \mu} e^{-\|x-y\|^2/2\sigma^2} \leq e^{-1/2\sigma^2}$. So $\text{tr}(\mathcal{G}^2) \leq e^{-1/2\sigma^2}$ which is at most $\frac{1}{k^2}$ for $\sigma^2 \leq \frac{1}{4 \log k}$, and completes the proof.

So from now on, we only need to prove for any $\sigma$

$$\sum_{i=k+1}^{\infty} \lambda_i(\mathcal{G}) \gtrsim \frac{e^{\frac{-\sigma^2}{2t}}}{t! \cdot \sigma^{2t}}. \tag{5.4}$$

For any integer $\ell \geq 0$, let $\mu_\ell$ be the eigenvalue of $\tilde{G}$ with multiplicity $n_\ell = N(\ell, d)$ given by Theorem 5.4. It suffices to show $n_t \mu_t \gtrsim \frac{e^{\frac{-\sigma^2}{2t}}}{t! \cdot \sigma^{2t}}$ where we are using the fact that for any $\ell$, $n_\ell \geq \frac{d^\ell}{t!}$, and so $n_t \geq 2k$. Now using $n_t \geq \frac{d^t}{t!}$, and the bound on $\mu_t$ by Theorem 5.4, we get

$$n_t \mu_t \gtrsim \frac{d^t}{t!} \cdot \frac{e^{\frac{-\sigma^2}{2t}}}{(2t + d)^{t + \frac{d+1}{2}}} \gtrsim \frac{d^t}{t!} \cdot \frac{e^{\frac{-\sigma^2}{2t}}}{\sigma^{2t} \cdot (1 + \frac{2t}{d})^{t + \frac{d+1}{2}}},$$

Sterling’s approximation

$$\gtrsim \frac{e^{\frac{-\sigma^2}{2t}}}{\sigma^{2t} \cdot t! \cdot \frac{2t}{d} \cdot \frac{2t}{d} \cdot (1 + \frac{2t}{d})^{t + \frac{d+1}{2}}},$$

by $(1 + 2t/d) \leq (2t/d)$. Noting that $k \leq \exp(d/4)$ implies $t \leq \frac{d}{4}$ and $\exp(2t/d) \leq 2$, completes the proof of (5.1).

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21
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A Missing Proofs

For a vector $v$, and a linear subspace $H$, we use $d(v, H)$ to denote the distance of $v$ from $H$.

Proof of Lemma A.2. For any $x \in \mathbb{R}^d$, let $f_x$ be the corresponding feature map, i.e. $f_x : \mathcal{H} \to \mathbb{R}$ for some Hilbert space $\mathcal{H}$ and for any $x, y \in \mathbb{R}^d$, $L(x, y) = \langle f_x, f_y \rangle$. Fix $x = \{x_1, \ldots, x_k\}$, and let $S_k$ be the set of all permutations of $\{x_1, \ldots, x_k\}$. Also, for any $\sigma \in S_k$ and for any $1 \leq i \leq k - 1$, define $H^{\sigma} = \langle f_{\sigma(1)}, \ldots, f_{\sigma(i)} \rangle$. In the above the range of all integrals is $\mathbb{R}^d$. We have

$$f_\nu(x) = \sum_{\sigma \in S_k} \left[ \frac{\|f_{\sigma(1)}\|^2}{\int \|f_y\|^2 dy} \cdot \frac{d(f_{\sigma(2)}, H^{\sigma}_1)^2}{\int d(f_y, H^{\sigma}_1)^2 dx} \cdots \frac{d(f_{\sigma(k)}, H^{\sigma}_{k-1})^2}{\int d(f_y, H^{\sigma}_{k-1})^2 dy} \right].$$

Note that the above integrals are well-defined since our kernel is continuous. For any $1 \leq i \leq k - 1$, let $H^i = \arg\min_{H=\{f_{y_1}, \ldots, f_{y_i}\}} \int d(f_y, H)^2 dy$, where $y_1, \ldots, y_i$ range over $\mathbb{R}^d$. Note that, the minimum of the quantity is defined since $L$ is continuous on a closed set. Combining with the above, and noting that for any $\sigma$, $\det(x_1, \ldots, x_k) = \|f_{\sigma(1)}\|^2 \cdot d(f_{\sigma(2)}, H^2)^2 \cdots d(f_{\sigma(k)}, H^k)^2$, we obtain

$$f_\nu(x) \leq k! \cdot \frac{\det(x_1, \ldots, x_k)}{\int \|f_y\|^2 dy \cdot \int d(f_y, H^1)^2 dy \cdot \int d(f_y, H^k)^2 dy} \leq k! \cdot \frac{\int \cdots \int \det(y_1, \ldots, y_k) dy_k \cdots dy_1}{\int \|f_y\|^2 dy \cdot \int d(f_y, H^1)^2 dx \cdots \int d(f_y, H^k)^2 dy} \leq (k!)^2.$$  \hspace{1cm} (A.1)

So, rearranging the above to show $\frac{f_\nu(x)}{f_{\nu}(x)} \leq (k!)^2$, it suffices to show

$$\frac{\int \cdots \int \det(y_1, \ldots, y_k) dy_k \cdots dy_1}{\int \|f_y\|^2 dy \cdot \int d(f_y, H^1)^2 dx \cdots \int d(f_y, H^k)^2 dy} \leq (k!)^2.$$  \hspace{1cm} (A.1)

To proof the above, we use induction on $k$. For $k = 1$, the statement is obvious as for any $y \in \mathbb{R}^d$, $\det(y) = L(y, y) = \|f_y\|^2$. It is straight-forward to see, applying the above claim will prove the induction step, and completes the proof.

Claim A.1.

$$\int \cdots \int \det(y_1, \ldots, y_k) dy_k \cdots dy_1 \leq k^2 \left( \int d(f_y, H^k)^2 dy \right) \left( \int \cdots \int \det(y_1, \ldots, y_{k-1}) dy_{k-1} \cdots dy_1 \right)$$  \hspace{1cm} (A.2)

Proof of Claim A.1. For any $y = \{y_1, \ldots, y_k\} \subset \mathbb{R}^d$, let $G_y$ be a $(k - 1)$-dimensional linear subspace of $\langle f_{y_1}, \ldots, f_{y_k} \rangle$ which contains the projection of $H^{(k-1)}$ onto $\langle f_{y_1}, \ldots, f_{y_k} \rangle$. Now, for any $y$, using Lemma A.2, we get

$$\det(y) \leq \left( \sum_{i=1}^k d(f_y, G_y) \sqrt{\det(y - y_i)} \right)^2 \leq k \left( \sum_{i=1}^k d(f_y, G_y)^2 \det(y - y_i) \right) \text{ Cauchy-Schwarz Inequality.}$$

23
By integrating the above, we get

$$\int \ldots \int \det(y)dy \leq k \int \ldots \int \sum_{i=1}^{k} d(f_{y_i}, G_{y})^2 \det(y - y_i)dy$$

$$\leq k^2 \int_{y \in \mathbb{R}^d} \int_{z_1 \in \mathbb{R}^d} \ldots \int_{z_{k-1} \in \mathbb{R}^d} d(f_{y}, G_{z+y})^2 \det(z)dzdy \quad \text{(setting } z = \{z_1, \ldots, z_{k-1}\})$$

$$\leq \int_{y \in \mathbb{R}^d} \int_{z_1 \in \mathbb{R}^d} \ldots \int_{z_{k-1} \in \mathbb{R}^d} d(f_{y}, H_{*}^{k-1})^2 \det(z)dzdy$$

$$= \left( \int d(f_{y}, H_{*}^{k-1})^2 dy \right) \left( \int_{z_1 \in \mathbb{R}^d} \ldots \int_{z_{k-1} \in \mathbb{R}^d} \det(z)dz \right),$$

where in the third inequality, the fact $d(f_{y}, G_{z+y}) \leq d(f_{y}, H_{*}^{k-1})$ holds because $f_{y} \in \langle f_{z_1}, \ldots, f_{z_{k-1}}, f_{y} \rangle$, and $G_{z+y}$ contains the projection of $H_{*}^{k-1}$ onto this space. Thus, the proof of the claim and the theorem is complete.

Lemma A.2 (Lemma 2 of [DV07]). Let $S$ be a set of $k$ vectors, and $H$ be any $(k - 1)$-dimensional subspace of $\langle S \rangle$. Then

$$\text{vol}(S) \leq \sum_{v \in S} d(v, H) \text{vol}(S - v),$$

where volume of a set of vectors, refer to the volume of the parallelepiped spanned by them.