Operation sense in algebra of junior high school students through an understanding of distributive law

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Abstract: This paper reports an investigation of students' understanding of distributive law used to facilitate students' awareness of 'the equations is an equivalent expression' where each equation can be replaced by an equivalent equation. This study used a cross-sectional approach by distributing a questionnaire to participants as many as 53 junior high school students where 45% were male students and 55% were female students who were selected using purposive sampling techniques. The instrument used in this study was a questionnaire written in Indonesian that contains five statements about distributive law, one question that is 'when is the statement true?' and three answer choices.

The results showed that the use of students' understanding of operations in understanding distributive law can be grouped into three levels, namely level 1 (pre-structural), level 2 (a combination of uni-structural and multi-structural) where level 2 is a transitional stage (ie stage pre-algebra) from level 1 to level 3 (a combination of relational and abstract expanded). Students who have been proficient in using distributive law in arithmetic, still have difficulty in applying it to algebraic manipulation. One of the causes of this difficulty is the lack of students' understanding of the equal sign as a sign of equality. In addition, the absence of parentheses also seems to be a stumbling block for students in recognizing distributive law. The reason for expanding the notion of the equal sign by using distributive property is to provide a foundation for their operation sense construction so as to provide further meaning for equations in algebra and minimize student difficulties in learning school algebra.

1. Introduction

An understanding of symbols and operations is very important in the introduction of school algebra. Symbols in algebra are related to notation and expression, usually using letters and signs of operation. The letters in algebra are commonly referred to as variables where many students think that the letters represent numbers thus school algebra is often regarded as 'general arithmetic' [1]. Students are expected to make the transition from arithmetic to algebra by accommodating the role of letters as variables in a numerical expression and equation. In the matter of equality, students mostly fail to set values on variables, lack of understanding of letters as unknown or as general numbers and are less able to make a general description of various parts of the equation including the equal sign because of poor understanding of structural aspects of algebra [2].
Operations involve various types of flexible conceptions that involve basic structures, uses, relationships with operations and other mathematical structures, and potential generalizations. According to [3] in developing a structural understanding, operations are an important 'essence of knowledge'. Therefore operation sense is very important to be developed through general awareness of the characteristics of operating groups such as commutative, associative, the existence of an identity, and distributive property, as well as awareness of the ability to reverse operations (invertibility). The distributive property provides a means of connecting two operations, such as additions and multiplications in any field. So that distributive property can be used to support development efforts of students' operation sense. This type of familiarity with the nature of surgery can be a promotion of flexibility in thinking that can lead to algebraic thinking.

Traditionally, algebra is defined as 'doing arithmetic with letters' [1]. Middle school students are in the transition from arithmetic thinking to algebraic reasoning. Students who have been able to develop awareness about generality in a variety of numerical operational strategies will be able to expand their algebraic thinking to include standard alphanumeric algebra [4]. In addition, at this stage of transition, they are in the stage between 'equal sign' as a computing sign and receive the equal sign as a symbol of equality [5]. The principle of mathematical equality serves as the main link between arithmetic and algebra [6]. Operational transformation and the meaning of the sign equal to in arithmetic as similarities can underlie the underlying "algebraic manipulation". In the case of arithmetic and algebra, these transformations can be carried out through the distributive property [7]. Explicit awareness of this principle is an important element in transforming sensory-motor action into symbolic action. Operation sense can develop the ability to link the use of operations in various different mathematical objects [1]. Therefore, operation sense is very important for middle school students who are starting their first course in algebra.

Based on this description, the purpose of this study is to describe the level of the tendency of junior high school students' responses in seeing relationships between operations through distributive law. Among the studies identified in the literature review, four reports from the following research, all of which involved research on distributive law, were very relevant to the purpose of our study. These studies have objectives including investigating students' understanding of algebraic structures that underlie distributive law [8]; investigating understanding of distributive legal structures through direct and reverse assignments to expand algebraic expressions [9]; analyzing students' flexibility in-depth about the application of distributive law [10]; comparing the role of distributive properties in arithmetic calculations and algebraic manipulation [7]. This study will illustrate the tendency of students to see the relationship between operations through distributive nature which has never been the goal of the four studies mentioned.

2. Methods

This study uses descriptive research design and cross-sectional survey type that collect data about attitudes, beliefs, opinions that occur at this time. This study focuses on secondary school students where they have the use of distributive law that is proficient in arithmetic and has used distributive law in algebra. This research was conducted by distributing a questionnaire to participants as many as 53 junior high school students while 45% were male students and 55% were female students who were selected using purposive sampling technique. Ethics and human relations are fulfilled through permission given by research institutions and schools that are the location of research. In addition to the permits granted, researchers personally have asked students to be willing to fill out questionnaires. School names and student identities use anonymity. So that all participants involved are guaranteed the confidentiality of their identity.

The instrument that will be used in this study is a questionnaire compiled by adapting the questions submitted in the task-based interview of [8]. Mok's three interview questions were adapted and then added with two questions so that the questionnaire used contained five tasks about distributive law with one question, ‘when is the statement true?’ And three choices of answers ‘always’, ‘never’ or ‘sometimes when’ and written in Indonesian. Students are asked to give reasons for each answer they give to the column 'why?'
Clear your reason’ as shown in Figure 1. Task 1 (statement 1) aims to test students' ability to see relational relations between operations on both sides of the sign equal to, show students' intuitive knowledge of distributive law without doing calculations, and see semantic meanings from a number of syntactic composition modes namely operations and parentheses in arithmetic. Task 2 (statement 2) is still in the arithmetic domain, but inverse law. In addition, parentheses are omitted. It aims to test students' flexibility as well as whether parentheses prevent students from recognizing distributive law. According to [10], flexibility can make students continually transfer distributive law so that they are able to perceive properties in expressions in a qualitatively different way. In task 3 (statement 3), the concept of variables begin to appear with the aim of seeing the semantic meaning of a number of syntactic composition modes (operations and parentheses) and semantics (variable meanings) of algebraic expressions. Task 4 (statement 4) contains assignments with division symbols that are very helpful to test how well students understand alternative meanings embedded in symbolic strings [8]. In accordance with the objectives of [8], task 5 is designed to investigate how students consider operating options and flexible variables so that they show that distributive law can apply.

| Question | Always | Never | Sometimes | Why? |
|----------|--------|-------|-----------|------|
| $(2 \times 23 + 49) = 2 \times 23 - 2 \times 49$ | | | | |
| $37 \times 3.4 + 37 \times 532 = 37 \times 876$ | | | | |
| $15a + 5ac = a (15b - 5c)$ | | | | |
| $(a + b) - c = a - c - b - c$ | | | | |
| If $a, b, c$ are any integers, $a \text{ den} \text{ is just any operation } \div, \cdot, +$ | | | | |
| $a \text{ op} (b \text{ op} c) = a \text{ op} b \text{ op} a$ | | | | |

**Figure 1. Questionnaire Design**

The level of complexity of the question is guided by the SOLO taxonomy. The taxonomy was designed by [11] is an evaluation tool about the quality of student responses to an assignment. The taxonomy used to measure the ability and quality of students’ answers to a problem is based on the complexity of their understanding or answers to a given problem. So the SOLO taxonomy plays a role in determining the quality of student responses to the problem. Based on the quality obtained from the results of student answers, it can then be determined the quality of achievement of cognitive processes that want to be measured. Each questionnaire is presented on A4 paper size with plenty of space for students to write any additional answers and information they want. Students can ask questions to clarify questions. Students are given a maximum of 30 minutes to answer all questions. Previously the questionnaire was validated by an algebraist and selected a small group outside the research sample for reliability checks. Taxonomy is used to measure the ability and quality of student answers to a problem based on the complexity of students' understanding or answers to the problem given. So the SOLO taxonomy plays a role in determining the quality of student responses to the problem. Based on the quality obtained from the results of student answers, then the quality of achievement of the cognitive processes to be measured can be determined.

3. **Result and Discussion**

3.1 **Student Response According to Operation Sense Used**

Operation sense is defined by [1,3], namely the ability of students to use operations on at least one set of mathematical objects involving the basic structure, use, and relationship of operations with other mathematical operations and structures, as well as potential generalizations. All responses from 53 students who were classified according to SOLO taxonomy on each question are given in Figure 2 below.
Therefore, the performance of the students related to operation sense in themselves can be grouped into three levels, namely:

1. Level 1: that is pre-structural level in SOLO taxonomy, where students cannot apply distributive law in numerical situations, cannot analyze any structure and only give conclusions by guessing.

2. Level 2: that is a combination of uni-structural levels and multi-structural levels in SOLO taxonomy, where students can apply distributive law in numerical situations even though it is still limited to procedural processes, using computational thinking to examine using calculations when they face numerical cases, able to search at least one structure sees the results on the left = the results on the right as separate objects or stand alone. Students at this level are at the transition stage between level 1 and level 3 or can also be said as a pre-algebraic stage, where they are already proficient in arithmetic but have not been able to make an abstraction from arithmetic. So that they do not have an understanding of the generality of operations, see the essence of operations and describe and represent this essence in the symbol system.

3. Level 3: which is a combination of relational levels and expanded abstract levels in the SOLO taxonomy, where students have been able to provide overall relationships, generalize in given contexts or experience using aspects related to general distribution patterns and explain them in terms of functional symbols, can think inductively and deductive, can hold or see relationships, make hypotheses, draw conclusions and apply them to other situations, can explain different strategies and properties such as fraction representation, substitution, range of values for letters as variables, and properties of functional meaning. At this level students have been able to do algebraic reasoning so they are able to show how they understand the nature of letter variables and operating variations in algebraic expressions.

Table 1. Student's Operation Sense Percentage in Completing Tasks

| Level | Percentage |
|-------|------------|
| 1     | 44.15      |
| 2     | 49.43      |
| 3     | 6.42       |

Table 1 shows the level of operation sense used by 53 students in completing all assignments given. Based on table 1, it seems that the students' operation sense is still low. In other words, the majority of students showed their lack of understanding of the operation. Although all students have known and used distributive law when they were in elementary school, it seems that their learning so far lacked fundamental
meaning, lacked conceptualization of the basic components of the process, were not familiar with the nature of operations, lacked understanding of the relationship of operations with operations, lack of understanding of the meaning of various symbol systems related to operations, lack of familiarity with the operating context and facts of operation, lack of ability to use surgery without concrete or situational references, lack of ability that requires generalization and puts the main focus on the operation itself and lack of the ability to use operations in various different mathematical objects. The use of parentheses to the left of the equal sign is still their main focus in recognizing distributive law. For example, there are 11 students who fail to see assignment 2 as the distributive law. In fact, they have realized that the key to success in completing task 2 lies in the number 876. They realize they have to do something with this number. They have also been able to see that the number 876 has a relationship with the numbers 344 and 532. However, they fail to implement it because they have not been able to see the relationship between the equal signs with these numbers. So they have the conclusion that "product times \((37 \times 344) + (37 \times 532)\) have results that are different from \((37 \times 876)\). It can be concluded that students' appreciation of the algebraic structure that underlies distributive law is still very minimal, even among students who are already proficient in arithmetic and distributive law.

Algebraic expressions can be represented in various forms where the transformation of one expression to another can be done by following several rules, for example, distributive law. These rules can be explained in terms of syntax features, which are moving from symbols on the surface [8]. Students' responses indicate that in general students can recognize distributive law in symbolic form \(a (b + c) = a + ac\), but their understanding is limited to procedural representations where on the left have parentheses and when parentheses are present means a sign to multiply. Only a few students can understand distributive law in various contexts, such as reverse distributive law and distributive law without the presence of parentheses. The appreciation of algebraic properties such as the general rules governing numerical systems and recognizing algebraic properties in addition to syntax features is an appreciation of the systemic structure of algebraic expressions [12].

### 3.2 Overall Student Response

Overall, it seems that not many students have used operation sense in completing tasks. They still have a tendency to complete tasks with procedural methods without meaning and their lack of experience with various forms of distributive law so that they do not recognize the law other than the syntax feature. All students can recognize that \(a (b + c) = a + ac\) is distributive law, but when presented with task 1 "when is \(62 \times (23 + 49) = 62 \times 23 + 62 \times 49\) correct?" They still need to check using calculations to draw conclusions about whether the statement is "always" or "never" or maybe "sometimes when". This might be because they believe that \(62 \times (23 + 49)\) and \(62 \times 23 + 62 \times 49\) represent two different calculation procedures and thus might be able to produce different results. That is, they still have the idea that the same sign is a "signal" to do something (operator symbol).

In addition, the presence of parentheses is the main marker for students about the validity of distributive law. The presence of parentheses means the procedure for 'rainbow multiplication' applies. For example, In assignment 1, students show a successful numerical application when they can say that \(62 \times (23 + 49) = 62 \times 23 + 62 \times 49\) is correct without doing calculations [8]. Of the 17 students who are at the uni-structural level, they have achieved this success. The main focus of their success is parentheses by stating that "because parentheses can be interpreted as times so they can use the rainbow multiplication" as shown in Figure 3. Another example, in task 2 some students have the conclusion that the statement "never" is correct because the results of times \(37 \times 344 + 37 \times 532\) have different results with \(37 \times 876\). In fact, \(37 \times 876 = 37 \times (344 + 532)\). In addition, most students give "always" correct responses to task 5 on the grounds that "distributive properties always apply". In fact, parentheses are not always a sign for "multiplying" but grouping. The rest, students may have answered that the statement on assignment 5 is "sometimes" true, but unable to give a reason. There is only 1 student who succeeds in giving correct reasoning that the statement is "sometimes"
true because distributive law does not apply to expressions $a \div (b + c)$. Thus, it seems that the absence of parentheses is a stumbling block for students to recognize distributive law.

**Figure 3.** 'The Rainbow Multiplication' Concept that is owned by Students

How students understand the nature of variables and operating variations in algebraic expressions is examined in assignments 3, 4 and 5. In assignments 4 and 5 for example, there are some students who are able to write in the form of fraction representations. However, they have not considered if $a = 0, b = 0, c \neq 0$ or vice versa.

### 3.3 Building Previous Findings

As mentioned in the introduction section, there are four relevant previous research reports on distributive law in high school algebra. And two papers on operation sense belonging to [1, 13]. So that there are six previous research reports that will be reviewed in order to guarantee the renewability and urgency of the topics discussed in this study.

The findings from [8] show a lack of student appreciation of the algebraic structure that underlies distributive law, especially among students who are already proficient in arithmetic and distributive law. Our research did adopt three questions from [8], but they did not involve inverse distributive law applied in our five questions. The inverse distributive law aims to see students' flexibility in recognizing and implementing distributive law. In addition, reverse distributive law according to [9] gives students an understanding of distributive legal structures that can be applied to expand or determine algebraic expressions. this is because this method can open the dimensions of the variation of the same sign that gives students the opportunity to see the similarities that form a direct and meaningful relationship between direct and inverse distributive law.

In-depth student flexibility regarding the application of distributive law has been the focus of research reported by [10]. Deep student flexibility is defined as the ability of students to transfer distributive law to unknown expressions, where the ability translates into the ability to reconstruct distributive property in expression through relevant structural features (p.2729). According to him, deep flexibility is a prerequisite needed to work conceptually interpreting algebraic expressions (p. 2730).

Comparing the role of distributive property in arithmetic calculations and in algebraic manipulation is the main topic discussed in [7]. This is done to provide an overview of the importance of understanding numerical structures as a prerequisite for understanding algebraic structures, in that framework, then Vermeulen, et al. designing learning activities such as solving concrete problems and decontextualized problems and making predictions without counting two equivalent number expressions. The results show that students in grades 6 and 7 experienced a substantial increase in their awareness of distributive property.

The paper [13] discusses the operation sense of students in relation to associative law understanding, commutative law, and addition and division as a general process that is believed to help a successful transition from arithmetic to algebra. It is clear, that the paper has not discussed the operation sense of students in relation to the understanding of distributive law which is the novelty of our research. The work of [1] is used as a theoretical basis for our research which is useful in discussing mathematical operations and exploring students 'understanding of students' initial abilities in numeracy can be seen as the root of the next form of algebraic thinking.
4. Conclusion

Students' understanding of distributive law in this paper has provided a clear picture of student operation sense. The three levels of operation sense in completing distributive tasks not only help to show the development of students' reasoning from arithmetic, pre-algebra, and algebra but also at the same time help to make meaningful steps to assist in this development. The main focus in the step in question is on the same sign as a symbol of equality by adapting [5] about the expansion of the sign equal to. According to [5], a comparison between the left and right sides of the equal sign shows that the equality symbol seen at this stage is more of a relational symbol than as a signal of doing something. The right side does not have to contain an answer, but it can be an expression that has the same value as the left side. This step is given in Figure 4 as follows.

![Figure 4](image)

**Figure 4.** Steps for Expanding the Meaning of The Equal Sign in Distributive Law

Each symbol $\Box$, $\ast$, and $\Delta$ represents numbers that are not hidden. Then the symbols are replaced with letters that hide the number. The letter that is closely related to the idea of hidden numbers is then referred to as an unknown term. The same letter can be used in equations more than once, as long as it is used to hide the same number. That is, if the numbers hidden are different, such as 2, 3 and 7 in figure 4, they must use two different letters, $a$, $b$, and $c$.

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