Some new inequalities for the gamma function

Xiaodong Cao

Abstract

In this paper, we present some new inequalities for the gamma function. The main tools are the multiple-correction method developed in [6, 7] and a generalized Mortici’s lemma.

1 Introduction

Duo to its importance in mathematics, the problem of finding new and sharp inequalities for the gamma function and, in particular for large values of \( x \)

\[
\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}dt, \quad x > 0,
\]

has attracted the attention of many researchers (see [2, 3, 8, 9, 12, 14, 15, 16, 17, 18] and the references therein). Let’s recall some of the classical results. Maybe one of the most well-known formula for approximation the gamma function is the Stirling’s formula

\[
\Gamma(x + 1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x, \quad x \to +\infty.
\]

(1.2)

See, e.g. [1, p. 253]. The following two formulas give slightly better estimates than Stirling’s formula,

\[
\Gamma(x + 1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \sqrt{x + \frac{1}{6}}, \quad \left(\text{Burnside [5], 1917}\right)
\]

(1.3)

\[
\Gamma(x + 1) \sim \sqrt{2\pi x} \left(\frac{x + \frac{1}{2}}{e}\right)^{x+\frac{1}{2}}, \quad \left(\text{Gosper [10], 1978}\right)
\]

(1.4)
Ramanujan [22] proposed the claim (without proof) for the gamma function

\[
\Gamma(x + 1) = \sqrt{\pi} \left( \frac{x}{e} \right)^x \left( 8x^3 + 4x^2 + x + \theta_x \right)^{\frac{1}{2}},
\]

where \( \theta_x \to 1 \) as \( x \to +\infty \) and \( \frac{3}{40} < \theta_x < 1 \). This open problem was solved by Karatsuba [13]. Thus (1.5) provides a more accurate estimate for the gamma function (see Sec. 2 below).

In this paper, we will continue the previous works [6, 7], and introduce a class of new approximations to improve these inequalities.

Throughout the paper, the notation \( \Psi(k; x) \) denotes a polynomial of degree \( k \) in \( x \) with all coefficients non-negative, which may be different at each occurrence. Let \( (a_n)_{n \geq 1} \) and \( (b_n)_{n \geq 0} \) be two sequences of real numbers with \( a_n \neq 0 \) for all \( n \in \mathbb{N} \). The generalized continued fraction

\[
\tau = b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \ddots}} = b_0 + \frac{a_1}{b_1 + a_2} \cdots = b_0 + \frac{\infty}{n=0} \frac{a_n}{b_n}
\]

is defined as the limit of the \( n \)th approximant

\[
\frac{A_n}{B_n} = b_0 + \frac{n}{k=1} \frac{a_k}{b_k}
\]

as \( n \) tends to infinity. See [2, p.105].

2 A generalized Mortici’s lemma

Mortici [14] established a very useful tool for measuring the rate of convergence, which says that a sequence \( (x_n)_{n \geq 1} \) converging to zero is the fastest possible when the difference \( (x_n - x_{n+1})_{n \geq 1} \) is the fastest possible. Since then, Mortici’s lemma has been effectively applied in many paper such as [6, 7, 17, 18]. The following lemma is a generalization of Mortici’s lemma.

Lemma 1. If \( \lim_{x \to +\infty} f(x) = 0 \), and there exists the limit

\[
\lim_{x \to +\infty} x^\lambda (f(x) - f(x + 1)) = l \in \mathbb{R},
\]

with \( \lambda > 1 \), then there exists the limit

\[
\lim_{x \to +\infty} x^{\lambda - 1} f(x) = \frac{l}{\lambda - 1}.
\]

Proof. It is not very difficult to prove that for \( x > 2 \)

\[
\frac{1}{(\lambda - 1)x^{\lambda - 1}} = \int_x^{+\infty} \frac{dt}{t^\lambda} \leq \sum_{j=0}^{\infty} \frac{1}{(x + j)^\lambda} \leq \int_{x-1}^{+\infty} \frac{dt}{t^\lambda} = \frac{1}{(\lambda - 1)(x - 1)^{\lambda - 1}}.
\]
For $\varepsilon > 0$, we assume that $l - \varepsilon \leq x^\lambda (f(x) - f(x + 1)) \leq l + \varepsilon$ for every real number $x$ greater than or equal to the rank $X_0 > 0$. By adding the inequalities of the form

\[(l - \varepsilon) \frac{1}{x^\lambda} \leq f(x) - f(x + 1) \leq (l + \varepsilon) \frac{1}{x^\lambda},\]

we get

\[(l - \varepsilon) \sum_{j=0}^{m-1} \frac{1}{(x+j)^\lambda} \leq f(x) - f(x + m) \leq (l + \varepsilon) \sum_{j=0}^{m-1} \frac{1}{(x+j)^\lambda}\]

for every $x \geq X_0$ and $m \geq 1$. By taking the limit as $m \to \infty$, then multiplying by $x^{\lambda-1}$, we obtain

\[(l - \varepsilon)x^{\lambda-1} \sum_{j=0}^{\infty} \frac{1}{(x+j)^\lambda} \leq x^{\lambda-1}f(x) \leq (l + \varepsilon)x^{\lambda-1} \sum_{j=0}^{\infty} \frac{1}{(x+j)^\lambda}.\]

It follows from (2.3) that

\[\frac{l - \varepsilon}{\lambda - 1} \leq x^{\lambda-1}f(x) \leq \frac{l + \varepsilon}{\lambda - 1} \frac{x^{\lambda-1}}{(x - 1)^{\lambda-1}}.\]

Now by taking the limit as $x \to +\infty$, this completes the proof of the lemma at once.

**An example** Let’s consider the Ramanujan’s asymptotic formula (1.5). Let the error term $E(x)$ be defined by the following relation

\[
\Gamma(x+1) = \sqrt{\pi} \left( \frac{x}{e} \right)^x \left( 8x^3 + 4x^2 + x + \frac{1}{30} \right)^\frac{x}{2} (1 + E(x)).
\]

It follows readily from the recurrence formula $\Gamma(x+1) = x\Gamma(x)$ that

\[
\ln (1 + E(x)) - \ln (1 + E(x + 1)) = -1 + x \ln \left(1 + \frac{1}{x}\right) + \frac{1}{6} \ln \frac{8(x+1)^3 + 4(x+1)^2 + (x+1) + \frac{1}{30}}{8x^3 + 4x^2 + x + \frac{1}{30}}.
\]

By using the Mathematica software, we expand the right-hand function in the above formula as a power series in terms of $1/x$:

\[
\ln (1 + E(x)) - \ln (1 + E(x + 1)) = \frac{11}{2880x^5} + O\left(\frac{1}{x^6}\right).
\]

Thus, by Lemma 1 we have

\[
\lim_{x \to +\infty} x^4 \ln (1 + E(x)) = \frac{11}{11520}.
\]

Noting that $\lim_{u \to 0} \frac{\ln(1+u)}{u} = 1$, one get finally

\[
\lim_{x \to +\infty} x^4 E(x) = \frac{11}{11520}.
\]

**Remark** 1. Just as Motici’s lemma, Lemma 1 also provides a method for finding the limit of a function as $x$ tends to infinity.
3 Gosper-type inequalities

In this section, we use an example to illustrate the idea of this paper. To this end, we introduce some class of correction function \( \text{MC}_k(x) \) such that the relative error function \( E_k(x) \) has the fastest possible rate of convergence, which are defined by the relations

\[
\Gamma(x + 1) = \sqrt{2\pi} \left( \frac{x}{e} \right)^x \sqrt{x + \frac{1}{6} + \text{MC}_k(x) \cdot \exp(E_k(x))}.
\]

If \( \lim_{x \to +\infty} x^\mu f(x) = l \neq 0 \) with constant \( \mu > 0 \), we say that the function \( f(x) \) is order \( x^{-\mu} \), and write the exponent of convergence \( \mu = \mu(f(x)) \). Clearly if \( \mu(E_k(x)) = \mu_k \), we have the following asymptotic formula

\[
\Gamma(x + 1) = \sqrt{2\pi} \left( \frac{x}{e} \right)^x \sqrt{x + \frac{1}{6} + \text{MC}_k(x) \cdot (1 + O(x^{-\mu_k}))}, \quad x \to +\infty.
\]

Let us briefly review a so-called multiple-correction method presented in our previous paper [6, 7]. Actually, the multiple-correction method is a recursive algorithm, and one of its advantages is that by repeating correction process we always can accelerate the convergence, i.e. the sequence \( \mu(E_k(x)) \) is a strictly increasing. The key step is to find a suitable structure of \( \text{MC}_k(x) \). In general, the correction function \( \text{MC}_k(x) \) is a finite generalized continued fraction (see [7] or (3.8) below) or a hyper-power series (see [6] or (4.7) below) in \( x \).

It is not difficult to see that (3.1) is equivalent to

\[
\ln \Gamma(x + 1) = \frac{1}{2} \ln(2\pi) + x (\ln x - 1) + \frac{1}{2} \ln (x + \text{MC}_k(x)) + E_k(x).
\]

By the recurrence formula \( \Gamma(x + 1) = x\Gamma(x) \), we have for \( x > 0 \)

\[
E_k(x) - E_k(x + 1) = -1 + x \ln \left( 1 + \frac{1}{x} \right) + \frac{1}{2} \ln \frac{(x + 1)}{x + \frac{1}{6} + \text{MC}_k(x + 1)}.
\]

Now by taking the initial-correction function \( \text{MC}_0(x) = \frac{\kappa_0}{x + \lambda_0} \) and using Mathematica software, we expand \( E_k(x) - E_k(x + 1) \) into a power series in terms of \( 1/x \):

\[
E_0(x) - E_0(x + 1) = \frac{-\frac{1}{72} + \kappa_0}{x^3} + \frac{17 - 945\kappa_0 - 810\kappa_0\lambda_0}{540x^4} + \frac{-641 + 33120\kappa_0 - 12960\kappa_0^2 + 43200\kappa_0\lambda_0 + 25920\kappa_0\lambda_0^2}{12960x^5} + O \left( \frac{1}{x^6} \right).
\]

The fastest possible function \( E_0(x) - E_0(x + 1) \) is obtained when the first two coefficients in the above formula vanish. In this case, we find \( \kappa_0 = \frac{1}{1728}, \lambda_0 = \frac{31}{36} \), and

\[
E_0(x) - E_0(x + 1) = \frac{5929}{1166400x^5} + O \left( \frac{1}{x^6} \right).
\]
By Lemma 1, we can check that

\begin{equation}
\lim_{x \to +\infty} x^4 E_0(x) = \frac{5929}{4665600}.
\end{equation}

We continue the above correction process to successively determine the correction function $MC_k(x)$ until some $k^*$ you want. On one hand, to find the related coefficients, we often use an appropriate symbolic computations software because it’s huge of computations. On the other hand, the exact expressions at each occurrence also need lot of space. Hence in this paper we omit many related details. For interesting readers, see our previous paper \[6, 7\]. In fact, we can prove that for $0 \leq k \leq 3$

\begin{equation}
MC_k(x) = \sum_{j=0}^{k} \frac{\kappa_j}{x + \lambda_j},
\end{equation}

where

\[
\begin{aligned}
\kappa_0 &= \frac{1}{72}, & \lambda_0 &= \frac{31}{90}, \\
\kappa_1 &= \frac{5929}{32400}, & \lambda_1 &= \frac{481937}{3735270}, \\
\kappa_2 &= \frac{7689917249}{24803857296}, & \lambda_2 &= \frac{7745462509019287}{19149278075101482}, \\
\kappa_3 &= \frac{8515415077317527392144}{2098335745817751685364201067279071}, & \lambda_3 &= \frac{30311088872486921466334781589254970}{30311088872486921466334781589254970}.
\end{aligned}
\]

By Lemma 1 again, we get for some constant $C_k \neq 0$

\begin{equation}
\lim_{x \to +\infty} x^{2k+4} E_k(x) = C_k, \quad (k = 0, 1, 2, 3),
\end{equation}

i.e. $\mu(E_k(x)) = 2k + 4$ for $k = 0, 1, 2, 3$. Thus we obtain more accurate approximation formulas:

\begin{equation}
\Gamma(x + 1) = \sqrt{2\pi} \left( \frac{x}{e} \right)^x \sqrt{x + \frac{1}{6} + MC_k(x)} \cdot \left( 1 + O(x^{-(2k+4)}) \right), \quad x \to +\infty.
\end{equation}

It should be noted that if we rewrite $MC_k(x)$ in the form of $P_r(x)/Q_s(x)$, where $P, Q$ are polynomials with $r = k$ and $s = k + 1$, theoretically at least, for a large $x$ the above formula may reduce or eliminate numerically computations compared with the previous results, see e.g. \[9, 12\]. This is the main advantage of the multiple-correction method.

The following theorem tells us how to obtain sharp inequalities.

**Theorem 1.** Let $MC_k(x)$ be defined as (3.8). Let $x \geq 1$, then we have for $k = 0, 2$,

\begin{equation}
\Gamma(x + 1) > \sqrt{2\pi} \left( \frac{x}{e} \right)^x \sqrt{x + \frac{1}{6} + MC_k(x)},
\end{equation}

and for $k = 1, 3$,

\begin{equation}
\Gamma(x + 1) < \sqrt{2\pi} \left( \frac{x}{e} \right)^x \sqrt{x + \frac{1}{6} + MC_k(x)}.
\end{equation}
Proof. We let $f_k(x) = E_k(x) - E_k(x + 1)$. Clearly if $\lim_{x \to +\infty} E_k(x) = 0$, then $E_k(x) = \sum_{j=0}^{\infty} f_k(x + j)$. This transformation plays an important role in this paper (essentially, it is a difference method). Hence, in order to prove inequality $E_k(x) > 0$ (or $E_k(x) < 0$), it suffices to show that the equality $f_k(x) > 0$ (or $f_k(x) < 0$) holds under the condition $\lim_{x \to +\infty} E_k(x) = 0$. By the Stirling’s formula (1.2), we can show that the condition $\lim_{x \to +\infty} E_k(x) = 0$ always holds. In what follows, we will apply this condition many times.

By using Mathematica software, we may prove that for $x \geq 1$

\[
\begin{align*}
    f_0''(x) &= \frac{\Psi_1(8; x)}{x(1 + x)^2(31 + 90x)^2(121 + 90x)^2(77 + 552x + 1080x^2)^2(1709 + 2712x + 1080x^2)^2} > 0, \\
    f_1''(x) &= -\frac{\Psi_2(13; x)(x - 1) + 1463 \cdots 9447}{x(1 + x)^2(1359251 + 2829648x + 5976432x^2)^2\Psi_3(16; x)} < 0, \\
    f_2''(x) &= \frac{\Psi_4(20; x)}{x(1 + x)^2\Psi_5(28; x)} > 0, \\
    f_3''(x) &= \frac{\Psi_6(25; x)(x - 1) + 17135 \cdots 66999}{x(1 + x)^2\Psi_7(36; x)} < 0.
\end{align*}
\]

We only give the proof of inequalities in case $k = 3$, other may be proved similarly. In this case, we see that for $x \geq 1$ the inequality (3.12) is equivalent to $E_3(x) = 0$. As $\lim_{x \to +\infty} E_3(x) = 0$, it suffices to prove that $f_3(x) < 0$ for $x \geq 1$. Since $f_3''(x)$ is strictly decreasing, but $\lim_{x \to +\infty} f_3''(x) = 0$, so $f_3'(x) > 0$. Thus $f_3(x)$ is strictly increasing with $\lim_{x \to +\infty} f_3(x) = 0$, so $f_3(x) < 0$. This completes the proof of Theorem 1.

By the multiple-correction method, we also find another kind of inequalities.

**Theorem 2.** Let the $k$-th correction function $MC_k(x)$ be defined by

\[
MC_0(x) = \frac{K_0}{(x + \frac{23}{6})^2 + \lambda_0},
\]

\[
MC_k(x) = \frac{K_0}{(x + \frac{23}{6})^2 + \lambda_0 + \sum_{j=1}^{k} \lambda_j}, \quad (k \geq 1),
\]

where

\[
\begin{align*}
    \kappa_0 &= -\frac{1}{144}, \\
    \kappa_1 &= 4394, \\
    \kappa_2 &= 637875, \\
    \kappa_3 &= 7894414898425, \\
    \kappa_4 &= 119793516544, \\
    \kappa_5 &= 1897560849252106177858465792, \\
    \kappa_6 &= 77174813342532578267347147395, \\
    \lambda_0 &= \frac{4007}{21600}, \\
    \lambda_1 &= \frac{130311599}{15575040}, \\
    \lambda_2 &= -\frac{265702682899837009577}{34427631789478287360}, \\
    \lambda_3 &= \frac{30320380455616293004898928163131563244811979}{6134364315672065325746652708240298034227200}.
\end{align*}
\]

Then we have

\[
(3.13) \quad \Gamma(x + 1) < \sqrt{2\pi} \left(\frac{e}{x}\right)^x \sqrt{x + \frac{1}{6} \left(1 + MC_0(x)\right)}, \quad x \geq 13,
\]

6
\[
\Gamma(x + 1) < \sqrt{\frac{2\pi}{x}} \left( \frac{x}{e} \right)^x \sqrt{x + \frac{1}{6}} \left( 1 + MC_2(x) \right), \quad x \geq 6,
\]
and for \( k = 1, 3, \)
\[
\Gamma(x + 1) > \sqrt{\frac{2\pi}{x}} \left( \frac{x}{e} \right)^x \sqrt{x + \frac{1}{6}} \left( 1 + MC_k(x) \right), \quad x \geq 1.
\]

**Proof.** Since the proof of Theorem 2 is very similar to that of Theorem 1, here we only give the outline of the proof. First, let the relative error function \( E_k(x) \) be defined by
\[
\Gamma(x + 1) = \sqrt{\frac{2\pi}{x}} \left( \frac{x}{e} \right)^x \sqrt{x + \frac{1}{6}} \left( 1 + MC_k(x) \right) \exp(E_k(x)).
\]
Hence
\[
E_k(x) - E_k(x + 1) = -1 + x \ln \left( 1 + \frac{1}{x} \right) + \ln \frac{1 + MC_k(x + 1)}{1 + MC_k(x)}.
\]
By making use of Mathematica software and Lemma 1, we can prove
\[
\mu(E_k(x)) = 2k + 5, \quad (k = 0, 1, 2, 3).
\]
Next we let \( g_k(x) = E_k(x) - E_k(x + 1). \) By using Mathematica software, it’s not difficult to check that
\[
\begin{align*}
g_0''(x) &= \frac{\Psi_1(14; x)(x - 13) + 29707 \cdots 81369}{x(1+x)^2(1+6x)^2(7+6x)^2\Psi_2(16; x)} > 0, \quad x \geq 13, \\
g_1''(x) &= -\frac{\Psi_3(20; x)(x - 1) + 13798 \cdots 89479}{x(1+x)^2(1+6x)^2(7+6x)^2\Psi_4(24; x)} < 0, \quad x \geq 1, \\
g_2''(x) &= \frac{\Psi_5(26; x)(x - 6) + 97250 \cdots 34321}{x(1+x)^2(1+6x)^2(7+6x)^2\Psi_6(32; x)} > 0, \quad x \geq 6, \\
g_3''(x) &= -\frac{\Psi_7(32; x)(x - 1) + 836559 \cdots 37479}{x(1+x)^2(1+6x)^2(7+6x)^2\Psi_8(40; x)} < 0, \quad x \geq 1.
\end{align*}
\]
Lastly, just as the proof of Theorem 1, Theorem 2 follows from the above inequalities readily. \( \square \)

## 4 Ramanujan-type inequalities

**Theorem 3.** Let the \( k \)-th correction function \( MC_k(x) \) be defined as
\[
MC_k(x) = \sum_{j=0}^K \frac{a_j}{x + b_j},
\]

\[\text{(4.1)}\]
where
\[ a_0 = -\frac{11}{240}, \quad b_0 = \frac{79}{154}, \]
\[ a_1 = \frac{459733}{71480}, \quad b_1 = -\frac{1455925}{7079882}, \]
\[ a_2 = \frac{49600874140433}{101450127018720}, \quad b_2 = \frac{10259108965771635091}{1954564575317443762}, \]
\[ a_3 = \frac{101221579151797375403194730976}{1690853053361527131511003963}, \quad b_3 = -\frac{6141448535908002711219920016488834171}{203275987838924050801436670299517447102}. \]

Let \( x \geq 1 \), then for \( k = 0, 2, \)
\[ \Gamma(x + 1) < \sqrt{\pi} \left( \frac{x}{e} \right)^x \left( 8x^3 + 4x^2 + x + \frac{1}{30} + MC_k(x) \right)^{\frac{1}{6}}, \]
and for \( k = 1, 3, \)
\[ \Gamma(x + 1) > \sqrt{\pi} \left( \frac{x}{e} \right)^x \left( 8x^3 + 4x^2 + x + \frac{1}{30} + MC_k(x) \right)^{\frac{1}{6}}. \]

Proof. We define the relative error function \( E_k(x) \) by the relation
\[ \Gamma(x + 1) = \sqrt{\pi} \left( \frac{x}{e} \right)^x \left( 8x^3 + 4x^2 + x + \frac{1}{30} + MC_k(x) \right)^{\frac{1}{6}} \exp(E_k(x)). \]

Thus
\[ E_k(x) - E_k(x + 1) = -1 + x \ln \left( 1 + \frac{1}{x} \right) + \frac{1}{6} \ln \frac{8(x + 1)^3 + 4(x + 1)^2 + (x + 1) + \frac{1}{30} + MC_k(x + 1)}{8x^3 + 4x^2 + x + \frac{1}{30} + MC_k(x)}. \]

By using Mathematica software and Lemma 1, we can check
\[ \mu(E_k(x)) = 2k + 6, \quad (k = 0, 1, 2, 3). \]

We let \( U_k(x) = E_k(x) - E_k(x + 1) \). By making use of Mathematica software again, we can prove
\[ U_0''(x) = \frac{\Psi_1(13; x)(x - 1) + 41683855850929775426164731715717}{3x(1 + x)^2(79 + 154x)^2(233 + 154x)^2(\Psi_{21}(3; x)(x - 1) + 363565)^2 \Psi_{22}(4; x)} < 0, \]
\[ U_1''(x) = \frac{\Psi_3(19; x)(x - 1) + 85653 \cdots 25001}{x(1 + x)^2 \Psi_4(28; x)} > 0, \]
\[ U_2''(x) = -\frac{\Psi_5(25; x)(x - 1) + 32968 \cdots 13479}{x(1 + x)^2 \Psi_6(36; x)} < 0, \]
\[ U_3''(x) = \frac{\Psi_7(31; x)(x - 1) + 17145 \cdots 57723}{3x(1 + x)^2 \Psi_8(44; x)} > 0. \]

Similar to the proof of Theorem 1, we can get the desired assertions from the above inequalities.
Theorem 4. Let the first-correction function $MC_1^*(x)$ be defined by

\begin{equation}
MC_1^*(x) = \frac{\kappa_0}{x + \lambda_0} + \frac{\kappa_1}{x^3 + \lambda_{10}x^2 + \lambda_{11}x + \lambda_{12}}, \tag{4.7}
\end{equation}

where

\begin{align*}
\kappa_0 &= \frac{-11}{240}, \quad \lambda_0 = \frac{79}{154}, \\
\kappa_1 &= \frac{15523200}{717183502490887}, \quad \lambda_{10} = \frac{71181889}{70798882}, \\
\lambda_{11} &= \frac{520777318696096}{1118629052995381153799}, \quad \lambda_{12} = \frac{195887879227282473920}{195887879227282473920}.
\end{align*}

Then for $x \geq 1$, the following inequality holds true

\begin{equation}
\Gamma(x + 1) < \sqrt{\pi} \left( \frac{x}{e} \right)^x \left( 8x^3 + 4x^2 + x + \frac{1}{30} + MC_1^*(x) \right)^\frac{1}{6}. \tag{4.8}
\end{equation}

Proof. Let the first-correction error function $E_1^*(x)$ be defined by

\begin{equation}
\Gamma(x + 1) = \sqrt{\pi} \left( \frac{x}{e} \right)^x \left( 8x^3 + 4x^2 + x + \frac{1}{30} + MC_1^*(x) \right)^\frac{1}{6} \exp(E_1^*(x)). \tag{4.9}
\end{equation}

Hence

\begin{equation}
E_1^*(x) - E_1^*(x + 1) = -1 + x \ln \left( 1 + \frac{1}{x} \right) + \frac{1}{6} \ln \frac{8(x + 1)^3 + 4(x + 1)^2 + (x + 1) + \frac{1}{30} + MC_1^*(x + 1)}{8x^3 + 4x^2 + x + \frac{1}{30} + MC_1^*(x)}. \tag{4.10}
\end{equation}

By using Mathematica software and Lemma 1, we have

\begin{equation}
\mu(E_1^*(x)) = 10. \tag{4.11}
\end{equation}

Now we let $V(x) = E_1^*(x) - E_1^*(x + 1)$. By using Mathematica again, we have

\begin{equation}
V_1''(x) = -\frac{\Psi_1(33;x)(x-1) + 96057 \cdots 27429}{3x (\Psi_2(3;x))(\Psi_3(12;x) (\Psi_4(6;x)(x-1) + 2169 \cdots 3461)^2 \Psi_5(14;x))} < 0. \tag{4.12}
\end{equation}

By the same approach as the proof of Theorem 1, the inequality (4.8) follows from the (4.12). □

Remark 2. It is an interesting question whether our method may be used to obtain some sharp bounds for the ratio of the gamma functions, see e.g. [11, 19, 20, 21].
References

[1] M. Abramowitz, I. A. Stegun (Editors), Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables, Applied Mathematics Series 55, ninth printing, National Bureau of Standards, Washington D.C., 1972.

[2] H. Alzer, On Ramanujan’s double inequality for the gamma function, Bull. London Math. Soc. 35 (2003), no. 5, 601–607.

[3] N. Batir, Very accurate approximations for the factorial function. J. Math. Inequal. 4(3)(2010), 335–344.

[4] B.C. Berndt, Ramanujan’s Notebooks, Part II, Springer-Verlag, 1989.

[5] W. Burnside, A rapidly convergent series for log N!. Messenger Math. 46(1917), 157–159.

[6] X.D. Cao, H.M. Xu and X. You, Multiple-correction and faster approximation, J. Number Theory 149(2015), 327–350. Available at: http://dx.doi.org/10.1016/j.jnt.2014.10.016.

[7] X.D. Cao, Multiple-Correction and Continued Fraction Approximation, J. Math. Anal. Appl. 424(2015)1425–1446. Available at: http://dx.doi.org/10.1016/j.jmaa.2014.12.014.

[8] C.-P. Chen and L. Lin, Remarks on asymptotic expansions for the gamma function. Appl. Math. Lett. 25(2012), 2322–2326.

[9] Chao-Ping Chen and Jing-Yun Liu, Inequalities and asymptotic expansions for the gamma function, Journal of Number Theory 149(2015), 313–326.

[10] R.W. Gosper, Decision procedure for indefinite hypergeometric summation. Proc. Natl. Acad. Sci. USA 75(1978), 40–42.

[11] S. Guo, J. Xu and F. Qi, Some exact constants for the approximation of the quantity in the Wallis’ formula, Journal of Inequalities and Applications 2013, 2013:67, 7 pp.

[12] M.D. Hirschhorn and M.B. Villarino, A refinement of Ramanujan’s factorial approximation, Ramanujan J. 34(2014),73–81.

[13] E.A. Karatsuba, On the asymptotic representation of the Euler gamma function by Ramanujan, J. Comput. Appl. Math. 135 (2001), no. 2, 225–240.

[14] C. Mortici, New approximations of the gamma function in terms of the digamma function, Applied Mathematics Letters, 23 (2010) 97–100.

[15] C. Mortici, On Ramanujan’s large argument formula for the gamma function, Ramanujan J. 26 (2011), no. 2, 185-192.

[16] C. Mortici, Ramanujan’s estimate for the gamma function via monotonicity arguments, Ramanujan J. 25 (2011), no. 2, 149–154.
[17] C. Mortici, A new fast asymptotic series for the gamma function, Ramanujan J. DOI 10.1007/s11139-041-9589-0.

[18] C. Mortici, Sharp bounds for gamma function in terms of $x^{x-1}$, Applied Mathematics and Computation, 249(2015),278–285.

[19] Feng Qi, Bounds for the Ratio of Two Gamma Functions, Journal of Inequalities and Applications, Volume 2010, Article ID 493058, 84 pp.

[20] Feng Qi, Integral representations and complete monotonicity related to the remainder of Burnside’s formula for the gamma function, Journal of Computational and Applied Mathematics, 268 (2014), 155–167.

[21] Feng Qi and Qiu-Ming Luo, Bounds for the ratio of two gamma functions: from Wendel’s asymptotic relation to Elezović-Giordano-Pečarić’s theorem, Journal of Inequalities and Applications 2013, 2013:542, 20 pp.

[22] S. Ramanujan, The Lost Notebook and Other Unpublished Papers. Narosa, Springer, New Delhi, Berlin (1988). Intr. by G.E. Andrews.

Xiaodong Cao
Department of Mathematics and Physics,
Beijing Institute of Petro-Chemical Technology,
Beijing, 102617, P. R. China
E-mail: caoxiaodong@bipt.edu.cn