Edge and vertex flippings in regular and bipartite graphs

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January 11, 2022

Abstract

This paper studies two related Markov chains on a given graph, edge flipping and its dual, vertex flipping. The edge flipping is defined by Chung and Graham in [CG12]. Its eigenvalues and stationary distributions for some cases are identified in the same paper. We further study its spectral properties to show a lower bound for the rate of convergence in the case of regular graphs. In particular, we show a cutoff of order $n^{-4\log n}$ for the edge flipping on the complete graph. Besides, some recursive formulas and explicit expressions are obtained for stationary distributions of both processes on bipartite graphs.

Keywords: Bipartite graphs, Coupling, Edge flipping, Hyperplane arrangements, Markov chains

1 Introduction

The edge flipping is a variant of the voter model, which was first defined and studied in [CG12]. It is a stochastic process on graphs where at each step an edge of a graph is randomly chosen and its both endpoints are colored either blue with probability $p$ or red with probability $q = 1 - p$. We are interested in the long-term vertex color configurations of the graph and a derived statistic, the frequency of blue vertices. In most cases, the initial configuration of vertex colors is taken to be all blue or all red.

Stationary distributions for some graph classes are already studied in [CG12, BCCG15], but various other important classes remain to be considered. We also study the stationary behavior of a closely related process called vertex flipping, which is defined in the same paper [CG12]. Taking the initial configuration to be monochromatic, a vertex is chosen uniformly at each step and its neighbors along with the vertex are colored blue with probability $p$ or red with probability $q$.

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In a broader context, the edge flipping can be considered a random walk on chambers of Boolean arrangements, which are particular cases of hyperplane arrangements. The eigenvalues of hyperplane chamber walk is obtained in [BHR99]. The transition matrix of the random walk is shown to be diagonalizable in [BD98], and its eigenvectors are studied in [Pik13, Sal12]. These provide the information to bound the rates of convergence of the process as we will see below.

We use the spectral analysis of the Markov chain and combine it with various probabilistic techniques to address the rate of convergence to its stationary distribution. Among several results obtained in Section 3 and 4, the following one is particularly concise. We refer to Section 3.3 for the definition of cutoff in Markov chains.

**Theorem 1.1** Let $K_n$ be the complete graph with $n$ vertices. The edge flipping on $K_n$ shows a cutoff at time $\frac{n \log n}{4}$ with a window of order $n$.

The organization of the paper is as follows. The next section provides background on hyperplane arrangements and the Markov chains defined on them. In Section 3 we present our results on the edge flipping process. We discuss both lower and upper bounds for convergence to the stationary distribution, and also include explicit and implicit formulas for the stationary distributions in the case of bipartite graphs. Section 4 studies the vertex flipping process. Analogous results are obtained in that case, and some similarities and differences in comparison with the edge flipping are disclosed.

## 2 Hyperplane random walks

Let $\mathcal{A} = \{H_i\}_{i=1}^n$ be a collection of hyperplanes in a real vector space $V$. The collection is called central if $\bigcap H_i \neq \emptyset$. Each $H_i$ divides $V$ into two open half-spaces. Let us denote them by $H_i^+$ and $H_i^-$, and let $H_i^0$ stand for the hyperplane itself. A face $F$ is a non-empty subset of $V$ determined by hyperplanes as

$$F = \bigcap_{i=1}^n H_i^{\sigma_i(F)}$$

where $\sigma_i(F) \in \{+, -, 0\}$. So we can represent any face by a sign sequence $\{\sigma_i(F)\}_{i=1}^n$. The faces with $\sigma_i(F) \neq 0$ for all $i$ are called chambers. The set of faces and the set of chambers are denoted by $\mathcal{F}$ and $\mathcal{C}$ respectively. Then we define a left-multiplication on $\mathcal{F}$ by

$$\sigma_i(FF') = \begin{cases} 
\sigma_i(F) & \text{if } \sigma_i(F) \neq 0, \\
\sigma_i(F') & \text{if } \sigma_i(F) = 0.
\end{cases} \quad (1)$$

The operation defined above generates a random walk on $\mathcal{C}$, which has transition probabilities

$$P(C, C') = \sum_{\substack{F \in \mathcal{F} \mid FC = C' \cap \mathcal{F} \cap \mathcal{C}}} w(F) \quad (2)$$

where $C, C' \in \mathcal{C}$. 

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To state the results on the eigenvalues of the matrix associated with the transition probabilities, we first consider a partial order on $\mathcal{A}$ which is defined via (1) as

$$F \leq F' \iff FF' = F.$$  

(3)

This implies that $F \leq F'$ if and only if either $\sigma_i(F) = \sigma_i(F')$ or $\sigma_i(F) = 0$. We can view the left-multiplication by $F$ as a projection from $\mathcal{A}$ onto

$$\mathcal{A}_{\geq F} := \{ F' \in \mathcal{A} : F \leq F' \}.$$ 

Next, we define another partial order by

$$F \preceq F' \iff F'F = F'.$$  

(4)

In other words, $F \preceq F'$ if and only if $\sigma_i(F') = 0$ implies $\sigma_i(F) = 0$. The equivalence classes of (4) give a semilattice $L$, and $L$ is a lattice if and only if $\mathcal{A}$ is central [Sta07]. The equivalence class of $F$ with respect to (4) is called the support of $F$, and is denoted by $\text{supp } F$. In lattice terminology, we have

$$\text{supp } FF' = \text{supp } F \lor \text{supp } F'.$$

The elements of $L$ are called flats. Furthermore, if $\text{supp } F = \text{supp } F'$, then the projection space $\mathcal{A}_{\geq F}$ is isomorphic to $\mathcal{A}_{\geq F'}$. So the following is well-defined;

$$c_X = |\mathcal{A}_{\geq F}|$$  

(5)

for any $F \in \mathcal{F}$ with $X = \text{supp } F$.

Observe that the support of the chambers is maximal in $L$, and the set of chambers is closed under left-multiplication. Therefore, the Markov chain is well-defined over the set of chambers. Its spectral profile is as follows.

**Theorem 2.1** ([BD98]) Let $\{w(F)\}_{F \in \mathcal{F}}$ be a probability distribution over the faces of a central hyperplane $\mathcal{A}$. The eigenvalues of the random walk given by transition probabilities (2) are indexed by the lattice $L$. For each flat $X \in L$, the associated eigenvalue is

$$\lambda_X = \sum_{\text{supp } F \subseteq X} w(F)$$

with multiplicity $m_X$ satisfying

$$\sum_{X \preceq Y} m_Y = c_X.$$  

We note that the multiplicities have the explicit expression

$$m_X = \sum_{X \preceq Y} \mu(X,Y)c_Y$$

by Möbius inversion where $\mu$ is the Möbius function of the lattice $L$. For a Boolean arrangement, the partial order (4) is the set inclusion, and its Möbius function is simply $\mu(X,Y) = (-1)^{|Y| - |X|}$ by the Inclusion-Exclusion Principle. In this case, the eigenvalues are indexed by the Boolean lattice.
Finally, we look at the edge flipping case. Let $G$ be a connected graph with $n$ vertices and $m$ edges. Denote the vertex set of $G$ by $V(G)$ and the edge set by $E(G)$. Each vertex $i$ is associated with a hyperplane $H_i$. The arrangement is central and the underlying vector space is $\mathbb{R}^n$. The faces that correspond to color configurations on subsets of vertices are the chambers of the hyperplane where the sign of $\sigma_k(C)$ gives the color of the vertex $k$ in the subset. Let us say, if $\sigma_k(F) = +(-)$, the vertex of the subset labeled by $k$ is blue (red). The faces that generate the random walk are colored edges. A blue edge connecting the vertices $i$ and $j$ is given by the face which has the sign sequence

$$
\sigma_k(F_b) = \begin{cases} 
+ , & \text{if } k \in \{i,j\}, \\
0 , & \text{otherwise}.
\end{cases}
$$

In the same way, a red edge connecting $i$ and $j$ is given by

$$
\sigma_k(F_r) = \begin{cases} 
- , & \text{if } k \in \{i,j\}, \\
0 , & \text{otherwise}.
\end{cases}
$$

The probabilities assigned to these faces are uniform for the same color, which are given by $w(F_b) = \frac{p}{m}$ and $w(F_r) = \frac{q}{m}$.

3 Edge flipping

We are ready to study the stationary distribution and the rate of convergence to it for the edge flipping. We first define the distance notion to be used. The total variation distance between $\mu$ and $\nu$ on the state space $\Omega$ is defined as

$$
\|\mu - \nu\|_{TV} = \frac{1}{2} \sum_{x \in \Omega} |\mu(x) - \nu(x)| = \max_{S \subseteq \Omega} |\mu(S) - \nu(S)|.
$$

In the context of Markov chains, we will use the following notation. For a Markov chain defined on $\Omega$, let the probability assigned to $x \in \Omega$ be $P^t(x)$ after running the chain for $t$ steps, and let it be $\pi(x)$ at the stationary distribution. A shorthand notation will be used for the total variation distance between $P^t$ and $\pi$:

$$
d_{TV}(t) = \|P^t - \pi\|_{TV}.
$$

In case that it makes difference, we denote the law of the Markov chain after $t$ steps with the initial state $z \in \Omega$ by $P^t_z$. For the rate of convergence, we define the mixing time

$$
t_{\text{mix}}(\epsilon) = \min_t \{d_{TV}(t) \leq \epsilon\}.
$$

3.1 Distance to stationary distribution

The following theorem gives an upper bound to the distance to stationary distributions in terms of the eigenvalues given in the previous section.
Theorem 3.1 ([BHR99]) Consider the edge flipping on a connected graph $G$. Let $L^*$ be the set of all flats in $L$ except the unique maximal one, and $\mu$ be the Möbius function of the lattice $L$. We have
\[ d_{TV}(t) \leq - \sum_{X \in L^*} \mu(X, V(G)) \lambda_X^t \]
where $\lambda_X$, the eigenvalue associated with flat $X$, is as defined in Theorem 2.1.

For the two main cases that we are interested in, we identify the eigenvalues and apply the theorem above.

Example 3.1 (The complete graph $K_n$) $K_n$ is the graph with $n$ vertices where every vertex is connected to every other vertex. Let $X_k$ be a flat in $L$ consisting of $k$ vertices. We have $\mu(X_k, V(G)) = (-1)^{n-k}$. By (2.1), the eigenvalue corresponding to flat $X_k$ is $\frac{k}{2} \binom{n}{2}$ and the number of flats of size $k$ is $\binom{n}{k}$. Hence,
\[ d_{TV}(t) \leq - \sum_{X \in L^*} \mu(X, V(G)) \lambda_X^t = - \sum_{k=2}^{n-1} \sum_{X_k} \mu(X_k, V(G)) \left( \frac{k}{2} \right)^t \]
\[ = - \sum_{k=2}^{n-1} (-1)^{n-k} \binom{n}{k} \left( \frac{k}{2} \right)^t \]
\[ \leq n \left( 1 - \frac{2}{n} \right)^t \text{ for large } t. \]

Example 3.2 (The complete bipartite graph $K_{m,n}$) $K_{m,n}$ is a graph whose vertex set is partitioned into two sets of sizes $m$ and $n$ where every vertex of one set is connected to every vertex of the other set. Take a flat $X_{k,l} \in L$ where $k$ is the number of vertices in $X_{k,l}$ which are in the set of size $m$ in the bipartite graph and $l$ many vertices are from the set of size $n$. By Theorem 2.1, we have $\lambda_{X_{k,l}} = \frac{kl}{mn}$, which has multiplicity $\binom{m}{k} \binom{n}{l}$. One can also show that $\mu(X_{k,l}, V(G)) = (-1)^{m+n-k-l}$. So by Theorem 3.1,
\[ d_{TV}(t) \leq - \sum_{X \in L^*} \mu(X, V(G)) \lambda_X^t \]
\[ = - \sum_{k=1}^{m-1} \sum_{l=1}^{n-1} \mu(X_{k,l}, V(G)) \left( \frac{kl}{mn} \right)^t + m \left( \frac{n(m-1)}{nm} \right)^t + n \left( \frac{(n-1)m}{nm} \right)^t \]
\[ = \sum_{k=1}^{m-1} \sum_{l=1}^{n-1} (-1)^{m+n-k-l+1} \left( \frac{kl}{mn} \right)^t + m \left( 1 - \frac{1}{m} \right)^t + n \left( 1 - \frac{1}{n} \right)^t \]
\[ \leq m \left( 1 - \frac{1}{m} \right)^t + n \left( 1 - \frac{1}{n} \right)^t \text{ for large } t. \]

A computationally better bound is obtained in [BD98] by a coarser version of the coupling argument in the proof of Theorem 3.1 in [BHR99]. It is as follows.
\[ d_{TV}(t) \leq - \sum_{X \in L^*} \mu(X, V(G)) \lambda_X^t \leq \sum_{\{M: M \text{ is co-maximal in } L\}} \lambda_M^t, \]
where $M$ is a co-maximal flat in $L$ if $M \prec X$ implies $X$ is maximal in $L$. In edge flipping, co-maximal flats are simply obtained by removing a vertex from the flat of all vertices.

The precision of the alternating bound compared to (7) is discussed in [CH14] considering various examples of random hyperplane arrangements. In the examples that we study, the latter bound is as good as the former.

Now we can provide an upper bound for the rate of convergence of the edge flipping in connected graphs.

**Theorem 3.2** Consider the edge flipping on a connected graph $G$ with $n$ vertices and $m$ edges, and let $\delta$ be the degree of the vertex with the minimum degree in $G$. For $c > 0$, we have

$$d_{TV} \left( \frac{m}{\delta} \log n + cn \right) \leq e^{-c\delta}.$$  

**Proof:** Observe that the co-maximal flat obtained by removing the vertex with the smallest number of edges have the smallest eigenvalue among the co-maximal flats. Let $M^*$ denote this flat. Then the eigenvalue indexed by $M^*$ is

$$\lambda_{M^*} = \sum_{\text{supp } F \subseteq M^*} w(F) = \frac{m - \delta}{m} = 1 - \frac{\delta}{m}$$

by Theorem 2.1. By (7), we have

$$d_{TV}(t) \leq \sum_{\{M: M \text{ is co-maximal in } L\}} \lambda_t^M \leq n \lambda_t^{M^*} \leq n \left( 1 - \frac{\delta}{m} \right)^t \leq e^{-c\delta}$$

given that $t \geq \frac{m}{\delta} \log n + cn$. \qed

### 3.2 Rate of convergence for regular graphs

A graph $G$ is called regular if every vertex of $G$ has the same degree. It is called $k$-regular if the degree of the vertices is $k$. We show the following bounds on the mixing time of edge flipping for regular graphs, which are independent of the degree of the vertices.

**Theorem 3.3** Let $G$ be a connected, regular graph with $n$ vertices. Then for $\epsilon \in [0, 1]$, the mixing time of edge flipping satisfies

$$\frac{1}{4} n \log n - O(n) \leq t_{mix}(\epsilon) \leq \frac{1}{2} n \log n + O(n).$$

The extreme examples are $C_n$ and $K_n$. The results earlier obtained on the stationary distribution for these graphs are discussed in the next section.

In fact, both upper and lower bounds are robust under edge deletion or edge insertion. The derivation of these bounds show that as long as the number of vertices of the same degree is of order $n$, the principal terms in the bounds above remain the same.

**Proof of the upper bound in Theorem 3.3.** Suppose degree of vertices of $G$ is equal to $k$. Let $M$ denote a co-maximal flat. Then the corresponding eigenvalue

$$\lambda_M = \sum_{\text{supp } F \subseteq M} w(F) = \frac{kn \delta - k}{kn} = 1 - \frac{2}{n}$$
by Theorem 2.1. The bound (7) gives

\[ \| P^t - \pi \|_{TV} \leq n \left( 1 - \frac{2}{n} \right)^t \leq e^{-c} \]

if \( t \geq \frac{n \log n}{2} + \frac{c}{2} n. \)

For the proof of the lower bound in Theorem 3.3, we use the Wilson’s method. Wilson [Wil04] showed that an eigenvector is a good candidate for a test statistics as the variance of the associated eigenfunction can be estimated inductively from the transition probabilities of the Markov chain. Assuming the second-order estimate for the eigenfunction, a lower bound can be obtained as follows.

**Theorem 3.4 ([LPW09])** Let \( X_t \) be an irreducible, aperiodic, time-homogenous Markov chain with state space \( \Omega \). Let \( \Phi \) be an eigenfunction associated with eigenvalue \( \lambda > \frac{1}{2} \). If for all \( x \in \Omega \),

\[ E(\Phi(X_1) - \Phi(x)|X_0 = x) \leq R \]

for some \( R > 0 \), then

\[ t_{\text{mix}}(\epsilon) \geq \frac{1}{2 \log(1 - \lambda)} \left( \log \left( \frac{(1 - \lambda)\Phi(x)^2}{2R} \right) + \log \left( 1 - \frac{\epsilon}{\epsilon} \right) \right) \]

for any \( x \in \Omega \).

The maximum eigenvalue is 1 with right eigenfunction \( \phi_0(C) = 1 \) for all \( C \in C \). The eigenvectors of eigenvalues corresponding to co-maximal flats in hyperplane arrangements are identified by Pike as follows.

**Theorem 3.5 ([Pik13])** For each \( i \in \{1, \ldots, n\} \), the Markov chain defined on the chambers by the transition probabilities (2) has the right eigenfunction

\[ \phi_i(C) = \begin{cases} -\sum_{F \in F} w(F), & \sigma_i(F) = + \\ \sum_{F \in F} w(F), & \sigma_i(F) = - \end{cases} \]

for \( i \in \{1, \ldots, n\} \). Each \( i \) represents a vertex and we can read off the color of \( i \) in a chamber by the eigenfunction \( \phi_i \). These eigenfunctions are not quite distinguishing by their own. In fact, Wilson’s lemma applied to any \( \phi_i \) gives a lower bound of order \( n \) only. Nevertheless, if an eigenvalue corresponding to a co-maximal flat has algebraic multiplicity larger than one,
linear combinations of eigenvectors can be used in Wilson’s lemma. In the case of regular graphs, where each vertex has the same degree, we have the second largest eigenvalue

\[ \lambda = \sum_{\sigma_i(F)=0} w(F) = \frac{kn/2-k}{kn/2} = 1 - \frac{2}{n} \]

associated with co-maximal flat obtained by removing the vertex indexed by 1. Of course, it has multiplicity \( n \) considering all other co-maximal flats and equality of degrees. Therefore, from Theorem 3.5, the eigenspace for the second largest eigenvalue is spanned by eigenvectors (8) for \( i = 1, \ldots, n \). If we consider the linear combination \( \Phi = \sum_{i=1}^{n} \phi_i \), it is easy to see that \( \Phi(C) = \Phi(C') \) if and only if they have the same number of blue vertices. If the color configuration \( C \) has \( k \) blue vertices, then \( \Phi(C) = k - np \). So, \( \Phi \) is a statistic that counts the number of blue vertices up to an additive term which guarantees \( \mathbb{E}(\Phi(\pi)) = 0 \) where the expectation is taken with respect to the stationary distribution \( \pi \), i.e.,

\[ \mathbb{E}(\Phi(X_t|X_0 = C)) = \lambda^t \Phi(C) \to 0. \]

We can use this statistic to obtain the lower bound by Wilson’s method. Since at each step at most two vertices can change color, we have

\[ \mathbb{E}[(\Phi(X_1) - \Phi(C))^2|X_0 = C] \leq 4. \]

in Theorem 3.4.

**Proof of the lower bound in Theorem 3.3.** For the lower bound, since the second eigenvalue has multiplicity \( n \) given that the graph is regular, the sum of eigenvectors is an eigenvector of the second largest eigenvalues above. Take the initial state to be the blue monochromatic graph, for which \( \Phi(C) = n - np = nq \). Then using Theorem (3.4)

\[ t_{\text{mix}}(\epsilon) \geq \frac{n}{4} \left( \log \left( \frac{q^2 n^2}{4n} \right) + \log \left( \frac{1 - \epsilon}{\epsilon} \right) \right) \geq \frac{1}{4} n \log n - cn \]

where \( c \) is a constant depending on \( p \) and \( \epsilon \); it is greater than zero regardless of \( p \) if \( \epsilon > \frac{1}{p} \). \( \square \)

### 3.3 Exact rate of convergence for complete graphs

In Section 3.2, we established a lower bound \( \frac{n \log n}{4} \) for the mixing time of edge flipping in regular graphs with \( n \) vertices. However, the upper bound obtained from eigenvalues differs by a factor of two. So we can only give an interval for the exact rate of convergence. Yet, some earlier studies on the similar processes suggest that the lower bound captures the correct rate of convergence, see [Nes17] for instance. We can verify that in the case of complete graphs. First, we formally define what we mean by the “exact rate.”

**Definition 3.1** A sequence of Markov chains shows a cutoff at the mixing time \( \{t_n\} \) with a window of size \( \{w_n\} \) if

(i) \( \lim_{n \to \infty} \frac{w_n}{t_n} = 0 \),
(ii) \( \lim_{\gamma \to -\infty} \liminf_{n \to \infty} d_{TV}(t_n + \gamma w_n) = 1 \),

(iii) \( \lim_{\gamma \to \infty} \limsup_{n \to \infty} d_{TV}(t_n + \gamma w_n) = 0 \).

We will use a coupling argument to show that the edge flipping in complete graphs shows a cutoff at time \( \frac{n \log n}{4} \) with a window of size \( n \). The coupling time of the two copies of a process is defined as

\[
\tau := \min_t \{ X_t = Y_t \}.
\]

The coupling time is related to the mixing time as follows.

\[
d(t) \leq \max_{x,y \in \Omega} P(\tau > t \mid X_0 = x, Y_0 = y). \tag{9}
\]

### 3.3.1 Proof of Theorem 1.1

Consider the linear combination of eigenvectors denoted by \( \Phi \) in Section 3.2. We already showed that it counts the number of blue vertices. Now if we view the edge flipping on \( K_n \) as a random walk on the hypercube \( \mathbb{Z}^n / 2\mathbb{Z}^n \) where at each step two coordinates are replaced either by 1’s by probability \( p \) or by 0’s with probability \( q \), then \( \Phi \) is just the Hamming weight, the sum of the 0-1 coordinates. See Section 2.3. of [LPW09] for the definition of hypercube random walk. An \( n \log n \) upper bound is obtained for lazy random walk on the hypercube by a coupling or strong stationary time argument, which can be found in Section 5.3 and Section 6.4 of [LPW09] respectively. This bound is later improved in Section 18.2 of the same book by observing that the problem can be reduced to convergence of Hamming weights, which can be considered as a lazy Ehrenfest urn problem. The same reduction argument is valid in the case of edge flipping in complete graphs, which is as follows. We claim that if the initial configuration is monochromatic, then

\[
\| P^t - \pi \|_{TV} = \| \Phi(X_t) - \pi_\Phi \|_{TV} \tag{10}
\]

where \( X_t \) is the chamber at step \( t \) and \( \pi_\Phi \) is the stationary distribution for the number of blue vertices. To argue for this, let us denote the set of chambers with \( k \) blue vertices by \( C_k = \{ C \in C : \Phi(C) = k \} \). Starting from a monochromatic configuration, any two chamber of same number of blue vertices is equally likely by symmetry. Thus, we have

\[
\sum_{C \in C_k} |P(X_t = C) - \pi(C)| = \left| \sum_{C \in C_k} P(X_t = C) - \pi(C) \right|
= \left| \mathbb{P}(\Phi(X_t) = k) - \pi_\Phi(k) \right|.
\]

The equality (10) shows that we can study the convergence of \( \Phi(X_t) \) to its stationary distribution to obtain an upper bound for the mixing time of the edge flipping on \( K_n \). We define a coupling with three stages to study the mixing time. Let \( X_t \) and \( Y_t \) be two edge flipping process where the initial states are monochromatic of opposite colors. We denote the number of blue vertices in each process by \( W_t = \Phi(X_t), Z_t = \Phi(Y_t) \), and let us take \( W_0 = 0 \) and \( Z_0 = n \). Moreover, let

\[
\Delta_t := Z_t - W_t \text{ for } t = 0, 1, \ldots
\]
For the first stage of the coupling \((W_t, Z_t)_{t=0}^\infty\), we first define the stopping time \(\tau_1\) as
\[
\tau_1 = \min \{ \Delta_t \leq \lceil \sqrt{n} \rceil \},
\]
then we define the rule of coupling up to this time. If \(t < \tau_1\), the two chains move identically in the sense that we choose the same two vertices in the underlying graphs and color them the same color. When \(t \geq \tau_1\), we choose edges independently for two graphs but color them with the same color. When the number of blue vertices agree \((\Delta_t = 0)\), \(W_t\) and \(Z_t\) move together.

The first part suggests that the process up to time \(\tau_1\) is as twice as fast as the coupon collector’s problem where you stop collecting more coupons if the remaining number of coupons is \([\sqrt{n}]\). Given that \(t \leq \tau_1\),
\[
\Delta_{t+1} = \Delta_t + \begin{cases} 
0 & \text{with prob. } \frac{(n-\Delta_t)}{2} / \binom{n}{2} \\
-1 & \text{with prob. } \Delta_t(n - \Delta_t) / \binom{n}{2} \\
-2 & \text{with prob. } \frac{\Delta_t^2}{2} / \binom{n}{2} 
\end{cases}
\]
Calculating the conditional expectation, we have
\[
E[\Delta_{t+1}|t < \tau_1, \Delta_t] = \left(1 - \frac{2}{n}\right) \Delta_t.
\]
Therefore,
\[
E[\Delta_t|t < \tau_1] = \left(1 - \frac{2}{n}\right)^t \Delta_0 \leq ne^{-\frac{2t}{n}}.
\]
By Markov’s inequality,
\[
P(\tau_1 > s) = P[\Delta_s > \sqrt{n}] \leq \sqrt{n} e^{-\frac{2s}{n}}. \tag{11}
\]
If we take \(s = \frac{1}{4} n \log n + \gamma_1 n\), we obtain
\[
P(\tau_1 > \frac{1}{4} n \log n + \gamma_1 n) < e^{-\gamma_1}. \tag{12}
\]
Note that if we continued with the first coupling until the two chains meet, this expression would give an upper bound of order \(\frac{1}{2} n \log n\), which is the same with what we obtained from eigenvalues. To improve this rate, we let the chains move according to second rule where we only match the colors of the independently chosen edges in two different graphs. We will show that it only needs \(O(n)\) steps to hit zero. Given that \(t > \tau_1\),
\[
\Delta_{t+1} = \Delta_t + \begin{cases} 
-2 & \text{with prob. } \frac{(n-W_t)(Z_t)}{2} / \binom{n}{2}^2 \\
-1 & \text{with prob. } \left[\frac{(n-W_t)}{2} Z_t(n - Z_t) + W_t(n - W_t)(Z_t)\right] / \binom{n}{2}^2 \\
1 & \text{with prob. } \left[W_t(n - W_t)\right] / \binom{n}{2}^2 \\
2 & \text{with prob. } \left(W_t\right)^2 / \binom{n}{2}^2 \\
0 & \text{otherwise.}
\end{cases}
\]
Then, examining the probabilities, we see that it is more likely for \( \{\Delta_t\}_{t>\tau_1} \) to go to left if \( \Delta_t > 0 \). Even better, we have

\[
P(\Delta_{t+1} - \Delta_t = -2) \geq P(\Delta_{t+1} - \Delta_t = 2) \quad \text{and} \quad P(\Delta_{t+1} - \Delta_t = -1) \geq P(\Delta_{t+1} - \Delta_t = 1)
\]
as \( Z_t \geq W_t \). Thus, we can compare it to following random walk, which is equally likely to go to either direction at any time.

\[
S_{t+1} = S_t + \begin{cases} 
-2 & \text{with prob. } \frac{1}{16} \\
-1 & \text{with prob. } \frac{1}{4} \\
0 & \text{with prob. } \frac{3}{8} \\
1 & \text{with prob. } \frac{1}{4} \\
2 & \text{with prob. } \frac{1}{16}
\end{cases}
\]

Let us take \( S_0 = \Delta_{\tau_1} = \lceil \sqrt{n} \rceil \). We define the following stopping times

\[
\tau_2 = \min_t \{\Delta_{\tau_1+1} \in I\}, \\
\tau_S = \min_t \{S_t \in I\}
\]

where \( I = \{-1, 0, 1\} \). Comparing the transition probabilities of the two random walks, we have \( \Delta_{\tau_1+t} \leq S_t \) provided that \( Z_t \geq W_t \). Therefore, we can find a coupling for two processes such that \( \tau_2 \leq \tau_S \). Furthermore, observe that \( S_{t+1} - S_t \) is sum of the i.i.d. random variables which take values \(-1\) or \(1\) with probability \(\frac{1}{4}\) and \(0\) with probability \(\frac{1}{2}\). But then \( S_t = S'_{2t} \) where \( S' \) is none other than the lazy simple random walk on integers. Therefore, we can use the results on the latter in our case. For instance, by Lemma Corollary 2.28 of [LPW09], we have

\[
P(\tau_S > s) \leq \frac{8\sqrt{n}}{\sqrt{2s}}.
\]

So, if we take \( s = \gamma_2 n \),

\[
P(\tau_S > \gamma_2 n) \leq \frac{4\sqrt{2}}{\sqrt{\gamma_2}}.
\]

For the last stage of the coupling, let us define

\[
\tau_3 = \min_t \{\Delta_{\tau_1+\tau_2+t} = 0\},
\]

which is to account for the case that \( Z_{\tau_2} - W_{\tau_2} = 1 \). In this case, \( \Delta_t \) might not hit zero in the next step. By symmetry, we see that it is more likely for \( \{\Delta_t\}_{t>\tau_1} \) to go to right if \( \Delta_t < 0 \). So, we again can bound the stopping time by the time associated with \( S_t \). In fact, for the simple random walk, we have the probability that “the number of sign changes is equal to \( k \) up to \( t = 2n + 1 \)” is equal to “the probability that the walk is at position \( 2k + 1 \) at \( t = 2n + 1 \),” see Theorem 1 in Chapter III.5 of [Fel68]. This allows us to use the normal approximation for the sign changes (ibid., Theorem 2). Let \( F_Z \) be the distribution function of the standard normal distribution. We have

\[
P \left( \sum_{t \in \{1, 2, \ldots, n\}} : S_{\tau_2+t} \in I \right) \leq \frac{\sqrt{n}}{\gamma_3} \leq 2F_Z(a) - 1 \leq \frac{1}{\sqrt{2\pi\gamma_3}}.
\]
Therefore, $\Delta_t$ visits $I$ of order larger than $\sqrt{n}$ times with a probability bounded away from zero. At every visit to $I$, it has a positive probability to hit zero at the next stage. Let $\alpha = P(S_{t+1} = 0 | S_t = -1 \text{ or } 1)$. Thus,

$$P(\tau_3 > n) \leq (1 - \alpha)^{\frac{n}{\gamma_3}} + \frac{1}{\sqrt{2\pi\gamma_3}} < \gamma_3^{-1}. \quad (14)$$

Finally let $\tau = \tau_1 + \tau_2 + \tau_3$, which is the stopping time, and $\gamma = \gamma_1 + \gamma_2 + \gamma_3$. By (12), (13) and (14),

$$P\left(\tau > \frac{1}{4} n \log n + \gamma n\right) < 1 - \left(1 - e^{-\gamma n}\right) \left(1 - \gamma_2^{-1/2}\right) (1 - \gamma_3^{-1}).$$

Since we chose the opposite monochromatic configurations for the different initial states of the coupled chains, among all choices for the starting pair chambers the coupling time above is clearly the maximum, so is the probability in (9). Therefore,

$$d\left(\frac{1}{4} n \log n + \gamma n\right) \leq \frac{C}{\sqrt{\gamma}} \quad (15)$$

by choosing $\gamma_1, \gamma_2$ and $\gamma_3$ large enough. Combining with the lower bound in Theorem 3.3, we have $t_n = \frac{2}{3} \log n$ and $w_n = n$ in Definition (3.1).

3.4 Stationary distribution for bipartite graphs

This section studies the stationary distributions in edge flipping in bipartite graphs. We identify the stationary distribution in two simple cases and give a recursive formula for the general case. To serve as a comparison, we provide the stationary distributions of two extreme cases of regular graphs, cycle $C_n$ and complete graph $K_n$, which are studied in [CG12] and in [CG12], respectively.

**Theorem 3.6** ([CG12]) Let $C(n)$ be the probability that the cycle $C_n$ has all its vertices blue under the stationary distribution. Then

$$C(n) \sim r^n$$

where

$$r = \frac{\sqrt{pq}}{\arctan\left(\frac{q}{p}\right)}.$$ 

If $p = q = \frac{1}{2}$, then $r = \frac{2}{\pi}$. In this case, the most likely two color patterns are monochromatic ones. The least likely two color patterns for odd cycles, $C_{2n+1}$, turn out to be alternating color configuration except a pair of adjacent vertices with identical colors. While if $p > q$, it is shown that the most likely color pattern is the monochromatic one.

For the complete graph, the number of blue vertices captures all the information for the stationary distribution as each vertex is adjacent to every other.
Theorem 3.7 ([CG12]) For $0 < p < 1$, the probability that all vertices of the complete graph $K_n$ are blue is given by

$$P(n \text{ blue}) = \sqrt{\frac{2}{1 + \sqrt{p}}} \left( \frac{p + \sqrt{p}}{2} \right)^n + o(e^{-n}).$$

On the other hand, the most likely occurrence has the probability

$$P(pn \text{ blue}; qn \text{ red}) = \frac{1}{\sqrt{3pq\pi n}} + o\left(\frac{1}{\sqrt{n}}\right).$$

Now we present our results. Recall that $K_{m,n}$ is the complete bipartite graph on $m$ and $n$ vertices. The purpose of this section is to discuss the stationary distribution for $n = 1$ and $n = 2$ cases. Note that the former can be interpreted as the star graph. Let us begin with the definition of the probabilities of interest.

**Definition 3.1** Let us consider the edge flipping process in the $K_{m,n}$ graph. Let the vertex sets $A, B$ be a vertex partition of $K_{m,n}$ such that $|A| = m$ and $|B| = n$. We let $P_{m,n}^E(a,b)$ to be the probability that $A$ has $a$ blue vertices and $B$ has $b$ blue vertices in the stationary distribution.

Below when we write $A$ and $B$, we mean partitions with $m$ and $n$ elements respectively. Now we introduce an auxiliary graph.

**Definition 3.2** Augmented complete bipartite graph $G(|A| = m, |B| = n, k, \ell)$ is a graph such that every vertex in $A$ is connected to every vertex in $B$ by an edge, every vertex in $A$ has $k$ loops (both sides of the edge are the same vertex) and every vertex in $B$ has $\ell$ loops. The probability that all vertices are colored blue in the stationary distribution is denoted by $P(m,n,k,\ell)$.

For the probability of the blue monochromatic configuration, we have

$$P_{m,n}^E(m,n) = P(m,n,0,0). \quad (16)$$

**Lemma 3.1** The probabilities defined above satisfy the following recurrence relation.

$$P(m,n,k,\ell) = \frac{pmk}{mk + mn + n\ell} P(m - 1, n, k, \ell + 1)$$

$$+ \frac{pm\ell}{mk + mn + n\ell} P(m, n - 1, k + 1, \ell)$$

$$+ \frac{pnm}{mk + mn + n\ell} P(m - 1, n - 1, k + 1, \ell + 1). \quad (17)$$

**Proof:** According to the first selected and colored edge, three different events arise, and the sum of the probabilities in these three events gives us the formula we are willing to prove. We can choose the first edge either as a self-loop of a vertex at $A$, a self-loop of a vertex at $B$ or an edge connecting $A$ to $B$. 

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The probability that the first selected edge is a self loop of a vertex \( v \) at \( A \) is
\[
\frac{mk}{mk + mn + n\ell},
\]
and the probability that it is colored blue is \( p \). If we remove \( v \) from the graph after the first step and add one more self loop to each vertex in \( B \) replacing the edges between \( v \) and \( B \), the probability of having blue monochromatic configuration in the stationary distribution does not change. The new graph is \( G(m - 1, n, k, \ell + 1) \), and the probability that all its vertices are colored blue is \( P(m - 1, n, k, \ell + 1) \). In this way, we obtain the first term on the right-hand side of (17). Repeating the same reasoning for the other two cases and adding up the probabilities, the required result follows.

\[ \square \]

**Theorem 3.1** For the bipartite graphs \( K_{m,1} \) and \( K_{m,2} \), the probability that all vertices are blue in the stationary distribution is given by
\[
P_{m,1}^E(m, 1) = p^m
\]
and
\[
P_{m,2}^E(m, 2) = \frac{m - 1}{m + 1} p^m + \frac{2}{m + 1} p^{m+1}.
\]

**Proof:** It is straightforward that
\[
P(m, 0, k, 0) = p^m, \quad P(0, n, 0, \ell) = p^n.
\]
It follows from (16) and Lemma 3.1 that
\[
P_{m,1}^E(m, 1) = P(m, 1, 0, 0) = \frac{m}{m} P(m - 1, 0, 1, 1) = \frac{m}{m} P(m - 1, 0, 1, 0) = p^m
\]
and
\[
P_{m,2}^E(m, 2) = P(m, 2, 0, 0) = \frac{2m}{2m} P(m - 1, 1, 1, 1) = pP(m - 1, 1, 1, 1).
\]
We will now show using induction that
\[
P(m, 1, 1, \ell) = \frac{m}{m + \ell + 1} p^m + \frac{\ell + 1}{m + \ell + 1} p^{m+1},
\]
from which the result will follow. This clearly holds for \( m = 0 \) when \( \ell \geq 0 \). Suppose next that it is valid for \( m = r - 1 \), and let us verify it for \( m = r \). Via Lemma 3.1 and (18), we have
\[
P(r, 1, 1, \ell) = \frac{rp}{2r + \ell} P(r - 1, 1, 1, \ell + 1) + \frac{rp}{2r + \ell} P(r - 1, 0, 2, \ell + 1) + \frac{\ell p}{2r + \ell} P(r, 0, 2, \ell)
\]
\[
= \frac{rp}{2r + \ell} P(r - 1, 1, 1, \ell + 1) + \frac{r}{2r + \ell} p^{r+1} + \frac{\ell}{2r + \ell} p^{r+1}
\]
\[
= \frac{rp}{2r + \ell} \left( \frac{r - 1}{r + \ell} p^{r-1} + \frac{\ell + 2}{r + \ell + 1} p^r \right) + \frac{r}{2r + \ell} p^r + \frac{\ell}{2r + \ell} p^{r+1}
\]
\[
= \frac{r}{r + \ell + 1} p^r + \frac{\ell + 1}{r + \ell + 1} p^{r+1}.
\]
Considering the complete bipartite graph $K_{m,n}$, the following figure shows the probabilities of all vertices being blue in the partition with $n$ vertices for $m = 3$ and varying $n$.

![Figure 1: The probabilities of monochromatic configuration in $K_{3,n}$](image)

Finally, we note that the following recursive formula for the stationary distribution of $K_{m,n}$ can be obtained by the inductive method in Theorem 2 of [BCCG15], which the authors used to identify the stationary distribution of the edge flipping on $K_n$.

\[
P_{m,n}^E(k, l) = \frac{pkl}{ml + kn - kl} (P_{m,n}^E(k - 1, l - 1) + l(m - k)P_{m,n}^E(k - 1, l - 1) + k(n - l)P_{m,n}^E(k - 1, l)).
\]

4 Vertex flipping

In this section, we obtain upper bounds for the convergence rates in case of the vertex flipping process. The following result will be useful in the sequel.

**Theorem 4.1** ([CG12]) Let $G$ be a graph on $n$ vertices. Then the vertex flipping on $G$ has the eigenvalue

\[
\lambda_T = \frac{\delta(T)}{n}
\]

where $\delta(T)$ denotes the number of vertices $v$ with all its neighbors in $T$.

4.1 Upper bound for the rate of convergence

**Theorem 4.2** Let $G$ be a connected graph on $n$ vertices. Consider the vertex flipping on $G$. We have

\[
d_{TV}(t) \leq e^{-c},
\]

given that $t \geq \frac{n \log n}{2} + \frac{cn}{2}$.
**Proof:** Take a flat $T \in L$. We have

$$\lambda_T \leq 1 - \frac{|G \setminus T| + 1}{n} = \frac{|T| - 1}{n}.$$  

$T$ is a maximal flat since $|T| = n - 1$. So, by Theorem 3.1,

$$d_{TV}(t) \leq \sum_{\{M: M \text{ is co-maximal in } L\}} \lambda_M^t = \sum_{|T|=n-1} \lambda_T^t = \left(\frac{n}{n-1}\right)^t = n \left(1 - \frac{2}{n}\right)^t \leq e^{-c},$$
given that $t \geq \frac{n \log n}{2} + \frac{c n}{2}$.

$\Box$  

A finer upper bound can be obtained for regular graphs.

**Theorem 4.3** Let $G$ be a $k$-regular, connected graph on $n$ vertices. Consider the vertex flipping process on $G$. We have

$$d_{TV}(t) \leq e^{-c},$$

if $t \geq \frac{n \log n}{k+1} + \frac{c}{k+1} n$.

**Proof:** Let $M$ denote a co-maximal flat. So $|M| = n - 1$. The eigenvalues are

$$\lambda_M = 1 - \frac{k+1}{n}$$

by Theorem 4.1. Then, Theorem 3.1 gives

$$d_{TV}(t) \leq n \left(1 - \frac{k+1}{n}\right)^t \leq e^{-c}$$

if $t \geq \frac{n \log n}{k+1} + \frac{c n}{k+1}$.

$\Box$

We next consider the complete bipartite graph.

**Theorem 4.4** Let $K_{n,m}$ be a complete bipartite graph where $n$ and $m$ are the sizes of two partitions forming the bipartite graph. Consider the vertex flipping process on $K_{n,m}$. We have

$$d_{TV}(t) \leq 2e^{-c}$$

when $t \geq \frac{c m+n}{\min(m,n)}$.

**Proof:** Take a flat $X_{k,\ell} \in L$ where $k$ is the number of vertices in $X_{k,\ell}$, which are in the partition of size $n$ in the bipartite graph, and $\ell$ many vertices are from the other partition of size $m$. We can calculate eigenvalues with using Theorem 4.1. If $k < n$ and $\ell < m$, $\lambda_{X_{k,\ell}} = 0$ since $X_{k,\ell}$ does not contain any vertices with all its neighbors. The multiplicity of 0 is

$$\sum_{k=0}^{n-1} \sum_{\ell=0}^{m-1} \binom{n}{k} \binom{m}{\ell}. $$

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We also have \( \lambda_{X_{k,m}} = \frac{k}{n+m} \), which has multiplicity \( \binom{n}{k} \) since \( k \) vertices are in the partition of size \( n \) and all their neighbors are in \( X_{k,m} \). Similarly, \( \lambda_{X_{n,l}} = \frac{l}{n+m} \), which has multiplicity \( \binom{n}{l} \). Now if \( k + l = m + n - 1 \), \( X_{k,l} \) is maximal flat. So, by Theorem 3.1

\[
d_{TV}(t) \leq \sum_{\{M: M \text{ is co-maximal in } L\}} \lambda_M^t = \left( \frac{n}{n-1} \right) \left( \frac{n-1}{n+m} \right)^t + \left( \frac{m}{m-1} \right) \left( \frac{m-1}{n+m} \right)^t
\]

\[
= \frac{n(n-1)^t + m(m-1)^t}{(n+m)^t}
\]

\[
\leq 2 \left( 1 - \frac{\min(m,n)}{m+n} \right)^t
\]

\[
\leq 2e^{-c},
\]

provided that \( t \geq c \frac{m+n}{\min(m,n)} \).

\[ \square \]

### 4.2 Stationary distributions for bipartite graphs

In this subsection, we study the stationary distribution of the vertex flipping process for \( K_{1,n-1} \), which is a tree with one internal node and \( n - 1 \) leaves. Note that this is a special case of the bipartite case for which similar techniques apply - general formula for the bipartite setting is included in the Appendix. The argument below will be similar to the one in previous section.

**Definition 4.1** Let us consider the vertex flipping process on \( K_{m,n} \). Let the vertex sets \( A, B \) be the vertex partition of \( K_{m,n} \) such that \( |A| = m \) and \( |B| = n \). Let \( P_{m,n}^V(a, b) \) be the probability of having \( a \) blue vertices in set \( A \) and \( b \) blue vertices in set \( B \) in the stationary distribution.

Below, \( A, B \) refer to a partition with \( m, n \) elements, respectively. Our goal here is to compute \( P_{m,n}^V(a, b) \) values for \( K_{1,n} \) and \( K_{m,n} \).

**Theorem 4.5**

\[
P_{1,n}^V(a, b) = \begin{cases} 
\frac{1}{n+1} \left( q + q^2 \frac{1-q^{n-1}}{1-q} + q^n \right) & \text{if } a = 0, b = 0 \\
\frac{1}{n+1} \left( p^b q^2 \sum_{i=0}^{n-1-b} \binom{n-1-b}{b} q^i + pq^{n-b} \sum_{i=0}^{b} \binom{n-1-b+i}{n-1-b} p^i \right) & \text{if } a = 0, 0 < b < n \\
0 & \text{if } a = 0, b = n \\
0 & \text{if } a = 1, b = 0 \\
\frac{1}{n+1} \left( q^{n-b} p^2 \sum_{i=0}^{b-1} \binom{n-b+i}{n-b} p^i + q p^b \sum_{i=0}^{n-b} \binom{b-1+i}{b-1} q^i \right) & \text{if } a = 1, 0 < b < n \\
\frac{1}{n+1} \left( p + p^2 \frac{1-p^{n-1}}{1-p} + p^n \right) & \text{if } a = 1, b = n
\end{cases}
\]

**Proof:** We will prove for the case \( a = 0 \), and it can be proved similarly for the case \( a = 1 \). Because having one blue vertex in the set \( A \) means that we have zero red vertices in the same set. Then if we write \( q \) instead of \( p \), we get that result. There are two different ways to make the first move in the vertex flipping process. We can choose the first vertex from the set \( A \) or the set \( B \). We will calculate the probabilities for both cases and add them up.

First, let us solve for the case \( b \neq 0 \). Since \( a = 0 \), we cannot choose the first vertex from \( A \) if \( b \neq 0 \). We have to choose the first vertex from \( B \) and color it red. The probability of
choosing the vertex in $A$ in step $t \in \{1, 2, \cdots, n + 1\}$ is $\frac{1}{n+1}$. Since all uncolored vertices will be colored in the same color after this vertex is chosen, all blues or reds in $B$ must be colored before choosing this vertex. If we chose the vertex in $A$ in step $t$, in the $t-1$ vertices chosen before this vertex, there must be either $b$ blue or $n-b$ red vertices. If there are $b$ blue vertices, we color the vertex in $A$ red, otherwise we color it blue. We know that the first vertex should be colored red, so at the end of the process the probability that there will be $b$ blue vertices in $B$ is

$$\frac{1}{n+1} \left( q \left( \begin{array}{c} t-2 \\ b \end{array} \right) p^b q^{t-2-b} + q \left( \begin{array}{c} t-2 \\ n-1-b \end{array} \right) p^{t-1-n+b} q^{n-1-b} \right).$$

(19)

If we sum from $t = 2$ to $n+1$, we will get $P^V_{1,n}(0,b)$.

$$P^V_{1,n}(0,b) = \sum_{t=2}^{n+1} \frac{1}{n+1} \left( q \left( \begin{array}{c} t-2 \\ b \end{array} \right) p^b q^{t-2-b} + q \left( \begin{array}{c} t-2 \\ n-1-b \end{array} \right) p^{t-1-n+b} q^{n-1-b} \right) = 
\sum_{t=2}^{n+1} \frac{1}{n+1} \left( \left( \begin{array}{c} t-2 \\ b \end{array} \right) p^b q^{t-b} + \left( \begin{array}{c} t-2 \\ n-1-b \end{array} \right) p^{t-n+b} q^{n-b} \right).

$$

Arranging the summation, we obtain

$$P^V_{1,n}(0,b) = \sum_{t=2}^{n+1} \frac{1}{n+1} \left( \left( \begin{array}{c} t-2 \\ b \end{array} \right) p^b q^{t-b} + \left( \begin{array}{c} t-2 \\ n-1-b \end{array} \right) p^{t-n+b} q^{n-b} \right) 
= \frac{1}{n+1} \left( \sum_{i=0}^{n-1-b} \left( \begin{array}{c} b+i \\ b \end{array} \right) p^b q^{i+2} q.i \sum_{i=0}^{b} \left( \begin{array}{c} n-1-b+i \\ n-1-b \end{array} \right) p^{i+1} q^{n-b} \right) 
= \frac{1}{n+1} \left( p^b q^2 \sum_{i=0}^{n-1-b} \left( \begin{array}{c} b+i \\ b \end{array} \right) q^i + pq^{n-b} \sum_{i=0}^{b} \left( \begin{array}{c} n-1-b+i \\ n-1-b \end{array} \right) p^i \right).

$$

Now let us calculate the probabilities for $b = 0$. In this case, we should also consider the case $t = 1$. We first calculate the probability for the $t = 1$ then sum (19) over $t \in \{2, 3, \cdots, n+1\}$. The probability of $t = 1$, $a = 0$ and $b = 0$ is $\frac{q}{n+1}$. Finally, we add them up to have

$$P^V_{1,n}(0,0) = \frac{1}{n+1} \left( q + pq^n + q^2 \sum_{i=0}^{n-1} q^i \right) 
= \frac{1}{n+1} \left( q + q^n + q^2 \sum_{i=0}^{n-2} q^i \right) 
= \frac{1}{n+1} \left( q + q^2 \frac{1-q^{n-1}}{1-q} + q^n \right).

$$

This gives the result to be proven. □

As noted earlier, the formula for general bipartite graphs is given in the Appendix.

**Acknowledgments:** The second author is supported partially by BAP grant 20B06P1. The second and the third authors would like to thank Jason Fulman for the suggestion of this topic of study.
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A Vertex flipping stationary distribution for bipartite graphs

The following theorem can be proven along the same lines in Section 4.2.

Theorem A.1

\[ \begin{align*}
    \mathbb{P}_{m,n}^V(a,b) &= \begin{cases}
        \sum_{i=2}^{m} \left( \prod_{j=0}^{i-2} \frac{m-j}{m+n-j} \right) \frac{n}{m+n-i+1} p^i + \left( \prod_{j=0}^{m-1} \frac{m-j}{m+n-j} \right) p^m + \\
        \sum_{i=2}^{n} \left( \prod_{j=0}^{i-2} \frac{n-j}{m+n-j} \right) \frac{m}{m+n-i+1} p^i + \left( \prod_{j=0}^{n-1} \frac{n-j}{m+n-j} \right) p^n, & \text{if } a = 0, b = 0 \\
        \sum_{i=n-b+1}^{n+1} \left( \prod_{j=0}^{i-2} \frac{m-j}{m+n-j} \right) \frac{n}{m+n-i+1} \left( \frac{i-2}{n-b} \right) p^{i-n+b} q^{n-b} + \\
        \sum_{i=b+1}^{n+1} \left( \prod_{j=0}^{i-2} \frac{m-j}{m+n-j} \right) \frac{n}{m+n-i+1} \left( \frac{i-2}{b} \right) p^{i-b} q^{b-i}, & \text{if } a = 0, 0 < b < n \\
        \sum_{i=m-a+1}^{m+1} \left( \prod_{j=0}^{i-2} \frac{n-j}{m+n-j} \right) \frac{m}{m+n-i+1} \left( \frac{i-2}{m-a} \right) p^{i-m+a} q^{m-a} + \\
        \sum_{i=a+1}^{m+1} \left( \prod_{j=0}^{i-2} \frac{n-j}{m+n-j} \right) \frac{m}{m+n-i+1} \left( \frac{i-2}{a} \right) p^{i-a} q^{a-i}, & \text{if } 0 < a < m, b = 0 \\
        \sum_{i=m-a+1}^{m+1} \left( \prod_{j=0}^{i-2} \frac{n-j}{m+n-j} \right) \frac{m}{m+n-i+1} \left( \frac{i-2}{m-a} \right) q^{m-a} p^{i-m+a}, & \text{if } 0 < a < m, b = n \\
        \sum_{i=a+1}^{m+1} \left( \prod_{j=0}^{i-2} \frac{n-j}{m+n-j} \right) \frac{m}{m+n-i+1} \left( \frac{i-2}{a} \right) q^{a-i} p^{i-a}, & \text{if } a = m, b = 0 \\
        \sum_{i=b+1}^{n+1} \left( \prod_{j=0}^{i-2} \frac{m-j}{m+n-j} \right) \frac{n}{m+n-i+1} \left( \frac{i-2}{b} \right) q^{i-b} p^{b-i} + \\
        \sum_{i=n-b+1}^{n+1} \left( \prod_{j=0}^{i-2} \frac{m-j}{m+n-j} \right) \frac{n}{m+n-i+1} \left( \frac{i-2}{n-b} \right) q^{n-b} p^{i-n+b}, & \text{if } a = m, 0 < b < n \\
        \sum_{i=2}^{m} \left( \prod_{j=0}^{i-2} \frac{m-j}{m+n-j} \right) \frac{n}{m+n-i+1} q^i + \left( \prod_{j=0}^{m-1} \frac{m-j}{m+n-j} \right) q^m + \\
        \sum_{i=2}^{n} \left( \prod_{j=0}^{i-2} \frac{n-j}{m+n-j} \right) \frac{m}{m+n-i+1} q^i + \left( \prod_{j=0}^{n-1} \frac{n-j}{m+n-j} \right) q^n, & \text{if } a = m, b = n
    \end{cases}
\]