Hydroelastic waves propagating along a frozen channel with non-uniform thickness of ice

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Abstract. Hydroelastic waves propagating along a channel covered with ice of non-uniform thickness are considered. The channel has a rectangular cross section. The fluid in the channel is inviscid and incompressible. The ice is modeled as a thin elastic plate. The ice thickness changes linearly. The problem is reduced to the problem of the wave profile across the channel, which is solved using the normal modes of an elastic beam with non-uniform thickness. It is shown that with the decrease in the change in the ice thickness, the modes approach the normal modes of an elastic beam with a constant thickness. The behavior of the dispersion relations of the hydroelastic waves depending on the parameter describing the change in the ice thickness is studied.

1. Introduction

In the last decade waves in open water became higher and could penetrate further into the ice cover. Under these conditions the propagation of waves and the interaction of such waves with engineering structures are of particular importance in both fundamental and applied research. In a lot of works authors investigate the interaction of an ice cover and structures [1-3]. Ice thickness is usually considered constant in such problems [4]. The change in the ice thickness may lead to a change in the characteristics of hydroelastic waves, propagating in ice, and increase or decrease the stresses in the ice cover and the loads on coastal structures. It is known that when external load is acting on ice, the response of the ice cover is a combination of hydroelastic waves propagating from the load. Therefore, the study of the characteristics of hydroelastic waves is an important problem.

A significant class of problems is associated with the behavior of the ice cover in the channel. The problem of periodic progressive hydroelastic waves propagating along the channel was studied in [5-6]. The dispersion relations of periodic hydroelastic waves propagating along a channel and their profiles across the channel were determined. The problem was solved using the normal modes of an elastic beam. It was shown that the hydroelastic waves in the ice cover clamped to the walls propagate faster than for the free ice cover and are of higher frequency for the same wave number.

In this article we deal with hydroelastic waves propagating along the channel with non-uniform thickness of the ice cover. Two cases are considered: the ice thickness varies linearly from the smallest value at one wall to the largest value at the other wall; the ice thickness varies linearly and symmetrically across the channel, being the smallest at the centre of the channel and the largest at the channel walls. We will investigate the dispersion relations of the hydroelastic waves and their behavior depending on the change in the ice thickness.
2. Formulation of the problem

The periodic hydoroelastic waves propagating along the frozen channel with non-uniform thickness of ice are considered. The channel has rectangular cross-section with finite depth $H$ and finite width $2b$, $(-H < z < 0, -b < y < b)$. The channel is of infinite extent in the $x$-direction. Here $(x, y, z)$ are the Cartesian coordinates. Liquid in the channel is inviscid and incompressible with constant density $\rho$. The liquid is covered with an ice cover of non-uniform thickness $h(y)$. The ice is modelled as a thin elastic plate. The shape of the ice $h(y)$ will be specified further. Other parameters of the ice are constant. The ice cover is frozen to the channel walls. The flow caused by the deflections of the ice is potential. The scheme of the channel is shown in Figure 1.

![Figure 1. The scheme of the channel.](image)

The problem is formulated within the linear theory of hydroelasticity [7] in the dimensionless variables. The length scale is $b$, the ice deflection scale is $A$, where $A$ is the amplitude of hydroelastic waves, the time scale is $1/\omega$, where $\omega$ is the frequency of the hydroelastic waves, the pressure scale is $A\rho$, the velocity potential scale is $Ab\omega$ and the scale of $h(y)$ is $\bar{h}$. The vertical displacement of the ice cover, $w(x, y, t)$, satisfies the equation of a thin elastic plate

$$
\beta \left[ h_i''(y) \left( \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) + 6h_i''(y) \frac{dh_i}{dy} \left( \frac{\partial^3 w}{\partial x^3 \partial y} + \frac{\partial^3 w}{\partial y^3} \right) + 6h_i(y) \frac{d^2 h_i}{dy^2} \left( \frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2} \right) \right] + \delta \frac{\partial^2 w}{\partial t^2} = p(x, y, 0, t),
$$

(1)

where $\beta = D_i / [\rho_i gb^4]$, $D_i(y) = D_i h_i''(y) = Eh_i''(y) / [12(1-\mu_i^2)]$ is the rigidity of the ice, $E$ is the Young modulus of the ice, $\mu$ is the Poisson's ratio of the ice, $\gamma = b\omega^2 / g$, $g$ is the gravitational acceleration, $\delta = \bar{h}\rho_i / [b\rho_f]$, $\rho_f$ is the density of ice, and $p(x, y, z, t)$ is the pressure of the liquid.

The ice deflection $w(x, y, t)$ satisfies the clamped conditions on the channel walls

$$
w = 0, \quad w_y = 0 \quad (y = \pm 1).
$$

(2)

The velocity potential of the flow, $\varphi(x, y, z, t)$, satisfies the Laplace equation

$$
\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \quad (\infty < x < \infty, -1 < y < 1, -h < z < 0),
$$

(3)

boundary conditions on the channel walls and on the bottom

$$
\frac{\partial \varphi}{\partial y} = 0 \quad (y = \pm 1), \quad \frac{\partial \varphi}{\partial z} = 0 \quad (z = -h),
$$

(4)

and linearized kinematic and dynamic conditions on the ice/liquid interface
\[ \frac{\partial \varphi}{\partial z} = \frac{\partial w}{\partial t} \quad (z = 0), \quad p(x, y, 0, t) = -\gamma \frac{\partial \varphi}{\partial t} - w \quad (z = 0). \]  

Here \( h = H / L \). Note that there are neither conditions at infinity, \( x \to \pm \infty \), nor initial conditions in the present formulation. We shall find the solution in the form of the progressive waves propagating along the channel

\[ w(x, y, t) = \text{Re}\left( F(y)e^{i(\alpha x + \xi t)}\right), \quad \varphi(x, y, t) = \text{Re}\left(i\Phi(y, z)e^{i(\alpha x + \xi t)}\right) \]

where \( \kappa = kb \) is the real positive dimensionless wave number. Substituting (6) to (1) gives

\[ \beta \left[ \frac{h_1^3(y)}{y} \left( \kappa^4 F - 2\kappa^2 F'' + F''''\right) + 6h_1^2(y)h_1' \left( -\kappa^2 F' + F'''\right) + 6h_1(y)h_1'' \left( F'' - \mu \kappa^2 F\right) - \frac{\delta \gamma}{\beta} F \right] = \xi \Phi(y, 0) - F, \]

where prime stands for the derivative with respect to \( y \). We shall find dispersion relations \( \omega(k) \) for some typical values of the parameters of the problem (1) – (7).

We deal with two cases of the ice shape across the channel, \( h_1(y) \). Case (a): the ice thickness varies linearly from the smallest value at the left wall, \( y = -b \), to the largest value at the right wall, \( y = b \); Case (b): the ice thickness varies linearly and symmetrically in \( y \), being the smallest at the channel center, \( y = 0 \), and the largest at the channel walls, \( y = \pm b \). The lowest value of the ice thickness is denoted by \( h_0 \), the largest one - by \( h_1 \) and the average one - by \( \bar{h} \). The scale \( \bar{h} \) is equal to \( h_0 \) in the case (a) and to \( h_0 \) in the case (b). The shapes are shown in Figure 2.

Figure 2. The shapes of the ice across the channel.

3. Method of the solution

The solution of equation (7) is sought in the form of the infinite series

\[ F(y) = \sum_{n=1}^{\infty} a_n \psi_n(y), \]

where \( a_n \) are the principal coordinates and \( \psi_n(y) \) are the clamped beam modes of thin elastic beam with non-uniform thickness. The modes \( \psi_n(y) \) will be determined for cases (a) and (b) separately. These functions for a beam of uniform thickness are well known, e.g., [5].

Case (a)

The modes \( \psi_n(y) \) are the solutions of the spectral problem

\[ (1 + \alpha y)^2 \psi_n^{''''} + 6\alpha (1 + \alpha y) \psi_n^{'''} + 6\alpha^2 \psi_n'' = \theta_n \psi_n \quad (-1 < y < 1), \]  

\[ \psi_n = 0, \quad \psi_n'' = 0 \quad (y = \pm 1), \]

where \( \theta_n \) are the eigenvalues of the problem (8) – (9) and \( \alpha = [h_1 - h_0] / [h_1 + h_0] \). Note that there are neither even nor odd solutions of the last problem for \( \alpha \neq 0 \). It may be shown [8] that the non-trivial solution of the problem (8) – (9) is
\[
\psi_n(y) = \frac{1}{\xi} \left[ A_n J_1(\eta_n \xi) + B_n Y_1(\eta_n \xi) + C_n I_1(\eta_n \xi) + D_n K_1(\eta_n \xi) \right], \quad n = 1, 2, 3, \ldots
\]  
(10)

where \( \eta_n = 2\theta_n / \alpha \), \( \xi = \sqrt{1 + \alpha y} \) and \( J, Y, I, K \) are the Bessel functions. Parameters \( A_n, B_n, C_n \) and \( D_n \) are solutions of the system of algebraic equations following from the boundary conditions

\[
A_n J_1(\eta_n \xi_n) + B_n Y_1(\eta_n \xi_n) + C_n I_1(\eta_n \xi_n) + D_n K_1(\eta_n \xi_n) = 0, 
\]
(11)

\[
A_n J_0(\eta_n \xi_n) + B_n Y_0(\eta_n \xi_n) + C_n I_0(\eta_n \xi_n) - D_n K_0(\eta_n \xi_n) = 0, 
\]
(12)

where \( \xi_n = \sqrt{1 \pm \alpha} \). The system of equations (11) – (12) can be written in a matrix form. The eigenvalues \( \eta_n \) are the solutions of the equation \( \det(A) = 0 \), where \( A \) is the matrix of the system (11) – (12). Parameters \( A_n, B_n, C_n, D_n \) are the coordinates of the corresponding eigenvectors. There will be infinitely many solutions of the system (11) – (12). The functions \( \psi_n(y) \) are orthogonal [9]

\[
\int_{-1}^{1} (1 + \alpha y) \psi_n(y) \psi_m(y) dy = 0 \quad (n \neq m) 
\]
(13)

and \( A_n, B_n, C_n, D_n \) are normalized in such a way that the integral in (13) is equal to 1 for \( n = m \).

**Case (b)**

Within the second case, equation (8) is separated and \( \psi_n(y) \) satisfy one of the two equations depending on the sign of \( y \)

\[
(1 + \alpha y)^2 \psi_n''' + 6\alpha (1 + \alpha y) \psi_n'' + 6\alpha^2 \psi_n' = \theta_n^2 \psi_n, \quad (0 \leq y < 1), 
\]
(14)

\[
(1 - \alpha y)^2 \psi_n''' - 6\alpha (1 - \alpha y) \psi_n'' - 6\alpha^2 \psi_n' = \theta_n^2 \psi_n, \quad (-1 < y < 0), 
\]
(15)

where \( \alpha = (h_1 - h_0) / h_0 \). Both even and odd functions may be solutions of the system (14) – (15). The even and odd solutions are considered separately. We determine \( \psi_n \) as the solution of equation (14) for positive \( y \) and continue the solution to negative \( y \), using even/odd properties of the described functions. The odd modes \( \psi_n^e(y) \) are the solutions of the spectral problem described by the equation (14), boundary conditions at \( y = 1 \) (11) – (12) for \( \xi = \xi_n \) and boundary conditions at \( y = 0 \)

\[
A_n J_1(\eta_n \xi_0) + B_n Y_1(\eta_n \xi_0) + C_n I_1(\eta_n \xi_0) + D_n K_1(\eta_n \xi_0) = 0, 
\]
(16)

\[
A_n J_0(\eta_n \xi_0) + B_n Y_0(\eta_n \xi_0) + C_n I_0(\eta_n \xi_0) - D_n K_0(\eta_n \xi_0) - \eta \psi_{ln}(\xi_0) = 0, 
\]
(17)

Here \( \xi_0 = 1 \). The conditions (16) – (17) follow from zero ice deflections and bending moment at the centre of the beam. The even modes \( \psi_n^e(y) \) are the solutions of the same spectral problem with conditions (16) – (17) replaced by

\[
A_n J_1(\eta_n \xi_0) + B_n Y_1(\eta_n \xi_0) + C_n I_1(\eta_n \xi_0) + D_n K_1(\eta_n \xi_0) - \eta \psi_{ln}(\xi_0) / 2 = 0, 
\]
(18)

\[
\psi_{ln}(\xi_0) - \eta \psi_{2n}(\xi_0) / 2 = 0, 
\]
(19)

The conditions (18) – (19) follow from zero slope and transverse shears of a beam with non-uniform thickness at the centre of the beam. Both odd and even mods are orthonormalized.
One can estimate that $\psi_n(y)$ tend to normal modes of the thin elastic beam with uniform thickness $h_0$ when $\alpha \to 0$. The calculations show that dispersion relations for both cases of the non-uniform beam approximate dispersion relations of the uniform beam, $h_i = 10$ cm, with visual accuracy for $h_i = 10$ cm, $h_0 = 0.95$ cm and $h_i = 1.05$ cm in both cases. These values correspond to $\alpha = 0.05$ in the case (a) and $\alpha \approx 0.105$ in the case (b).

4. Dispersion relations

Principal coordinates $a_n$ are the solutions of the infinite system of equations

$$\sum_{n=1}^{\infty} \left\{ \beta(1 + \alpha y)\theta_n + \kappa^2 \beta(1 + \alpha y)^3\psi_n - 2\kappa^2 \beta \left( (1 + \alpha y)^3\psi_n^+ + 3\alpha(1 + \alpha y)^2\psi_n^+ \right) - \kappa^2 \tilde{a} \beta(1 + \alpha y)\psi_n - \delta y(1 + \alpha y)\psi_n - \Phi_n(y,0) + \psi_n \right\} = 0,$$

$$\Phi(y,z) = \sum_{n=1}^{\infty} a_n \Phi_n(y,z), \quad \frac{\partial^2 \Phi_n}{\partial y^2} + \frac{\partial^2 \Phi_n}{\partial z^2} = \kappa^2 \Phi_n,$$

$$\frac{\partial \Phi_n}{\partial y} = 0 \quad (y = \pm 1), \quad \frac{\partial \Phi_n}{\partial z} = 0 \quad (z = -h), \quad \frac{\partial \Phi_n}{\partial z} = \psi_n \quad (z = 0).$$

Multiplying both sides of equation (20) by $\psi_n(y)$, integrating the result from $-1$ to $1$ in $y$ and reducing the number of equations to finite value we arrive at the matrix problem. Frequencies $\omega$ are calculated for each $\kappa$ when the determinant of this matrix equals 0. There will be an infinite number of such relations, $\omega_n(k)$. They are numbered, being the smallest for $n = 1$ and increasing with the number $n$. Each dispersion relation corresponds to the mode of ice oscillations in the channel with unique profile across the channel. The solution method is described in detail in [5].

Numerical calculations of dispersion relations $\omega_n(k)$ for linear hydroelastic waves in a channel covered with an ice cover of non-uniform thickness are performed for a freshwater ice with density $\rho_i = 917$ kg/m$^3$, Young’s modulus $E = 4.2 \times 10^6$ N/m$^2$ and Poisson’s ratio $\mu = 0.3$. The depth of the channel $H$ is 2 m and the width $2L$ is 20 m.

The dispersion relations of the first 5 modes are shown in Figure 3a for non-uniform beam with $h_i = 1.0$ cm, $h_h = 0.5$ cm and $h_i = 1.5$ cm (solid lines) and for the beam with uniform thickness $h_i = 10$ cm (markers) in the case (a). The dispersion relations for the cases (a) (solid lines) and (b) (dashed lines) are shown in Figure 3b for the same parameters. The frequencies of the short waves are higher in the case (a) and the frequencies of the long waves are higher in the case (b) for the same parameters of the ice thickness.
Figure 3. The dispersion relations of the first 5 modes for non-uniform beam with \( h_c = 1.0 \) cm, \( h_0 = 0.5 \) cm and \( h_1 = 1.5 \) cm in the first case (solid lines) and for the beam with uniform thickness \( h_c = 10 \) cm (markers) (a). The dispersion relations for the same parameters (b), solid lines show the first case, dashed lines show the second case.

The dispersion relations of the first mode are shown in Figure 4 for non-uniform beam in the case (a) (dashed line), in the case (b) (markers) and for the beam with uniform thickness \( h_c = 15 \) cm, \( h_0 = 10 \) cm and \( h_1 = 5 \) cm (solid lines). Results for non-uniform beam with \( h_c = 1.0 \) cm, \( h_0 = 0.5 \) cm and \( h_1 = 1.5 \) cm are shown in Figure 4a. The same results with \( h_c = 1.0 \) cm, \( h_0 = 0.625 \) cm and \( h_1 = 1.375 \) in the case (b) are shown in Figure 4b. The dispersion relation of the first mode coincides in both cases, if the jump in the plate thickness is smaller in the second case.

Figure 4. The dispersion relations of the first mode (a) for non-uniform beam in the first case (dashed line), in the second case (markers) and for the beam with uniform thickness (solid lines) for \( h_c = 15 \) cm, \( h_0 = 10 \) cm and \( h_1 = 5 \) cm.

The same results with \( h_c = 1.0 \) cm, \( h_0 = 0.625 \) cm and \( h_1 = 1.375 \) cm in the second case (b).

Conclusions
Hydroelastic waves propagating along a frozen channel with non-uniform thickness of ice and their dispersion relations have been studied. Two different cases of the shape of the ice thickness across the channel have been investigated. In both cases, the ice thickness changes linearly. In the first case, minimum and maximum values of the thickness are at the opposite walls of the channel, and in the second case, minimum is at the centre line of the channel and maximum is at the walls. The is problem solved using the normal modes of the non-uniform beam. These modes are a sum of the Bessel functions. There are infinitely many dispersion relations, which are numbered starting from the smallest frequency for the same wave number. Each dispersion relation corresponds to the mode of ice oscillations in the channel with unique profile across the channel. It is shown that the frequencies of the short waves are higher in the first case and the frequencies of the long waves are higher in the
second case for the same parameters of the ice thickness. The dispersion relation of the first mode coincides in both cases, if the jump in the ice thickness is smaller in the second case.

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