Numerical Methods for a Nonlinear BVP
Arising in Physical Oceanography

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Abstract

In this paper we report and compare the numerical results for an ocean circulation model obtained by the classical truncated boundary formulation, the free boundary approach and a quasi-uniform grid treatment of the problem. We apply a shooting method to the truncated boundary formulation and finite difference methods to both the free boundary approach and the quasi-uniform grid treatment. Using the shooting method, supplemented by the Newton’s iterations, we show that the ocean circulation model cannot be considered as a simple test case. In fact, for this method we are forced to use as initial iterate a value close to the correct missing initial condition in order to be able to get a convergent numerical solution. The reported numerical results allow us to point out how the finite difference method with a

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quasi-uniform grid is the less demanding approach and that the free boundary approach provides a more reliable formulation than the classical truncated boundary formulation.

**Key Words.** nonlinear boundary value problems, infinite intervals, shooting methods, free boundary formulation, quasi-uniform grid, finite difference methods.

**AMS Subject Classifications.** 65L10, 65L12, 34B40.

1 Introduction

Boundary value problems (BVPs) on infinite intervals arise in several branches of science. The classical numerical treatment of these problems consists in replacing the original problem by one defined on a finite interval, say \([0, x_\infty]\) where \(x_\infty\) is a truncated boundary. The oldest and simplest approach is to replace the boundary conditions at infinity by the same conditions at the value chosen values as the truncated boundary. This approach was used, for instance, by Goldstein [17, p. 136] and by Howarth [18] in 1938 to get and tabulate the numerical solution of the Blasius problem. However, in order to achieve an accurate solution, a comparison of numerical results obtained for several values of the truncated boundaries is necessary as suggested by Fox [15, p. 92], or by Collatz [6, pp. 150-151]. Moreover, in some cases accurate solutions can be found only by using very large values of the truncated boundary. This is, for instance, the case of the fourth branch of the von Karman swirling flows, where values of \(x_\infty\) up to 200 were used by Lentini and Keller [22].

To overcome the mentioned difficulties of the classical approach described above, Lentini and Keller [21] and de Hoog and Weiss [7] suggested to apply asymptotic boundary conditions (ABCs) at the truncated boundaries; see also the theoretical work of Markowich [24, 25]. Those ABCs have to be derived by a preliminary asymptotic analysis involving a Jacobian matrix of the right-hand side of the governing equations evaluated at infinity. The main idea of this ABCs ap-
proach is to project the solution into the manifold of bounded solutions. By this approach more accurate numerical solutions can be found than those obtained by the classical approach with the same values of the truncated boundaries, because the imposed conditions are obtained from the asymptotic behaviour of the solution. However, we should note that for nonlinear problems highly nonlinear ABCs may result. Moreover, it has been noticed by J. R. Ockendon that “Unfortunately the analysis is heavy and relies on much previous work, . . . ” see Math. Rev. 84c:34201. On the other hand, starting with the work by Beyn [3, 4, 5], the ABCs approach has been applied successfully to “connecting orbits” problems. Connecting orbits are of interest in the study of dynamical systems as well as of travelling wave solutions of partial differential equations of parabolic type. However, a truncated boundary allowing for a satisfactory accuracy of the numerical solution has to be determined by trial, and this seems to be the weakest point of the classical approach. Hence, a priori definition of the truncated boundary was indicated by Lentini and Keller [21] as an important area of research.

A free boundary formulation for the numerical solution of BVPs on infinite intervals was proposed in [10]. In this approach the truncated boundary can be identified as an unknown free boundary that has to be determined as part of the solution. As a consequence, the free boundary approach overcomes the need for a priori definition of the truncated boundary. This new approach has been applied to: the Blasius problem [8], the Falkner-Skan equation with relevant boundary conditions [9], a model describing the flow of an incompressible fluid over a slender parabola of revolution [10], and a model describing the deflection of a semi-infinite pile embedded in soft soil [12]. An application of the free boundary approach to a homoclinic orbit problem can be found in [11]. Moreover, a possible way to extend the free boundary formulation to problems governed by parabolic partial differential equations was proposed in [13].

It might seem that in order to face numerically a BVP defined on an infinite interval, we have to reformulate it in a way or another. However, recently, we
have found that it is also possible to apply directly to the given BVP a finite difference method defined on a quasi-uniform grid. To this end it is necessary to derive special finite difference formulae on the grid involving the given boundary conditions at infinity, but the last grid point value (infinity) is not required; see [14] for details.

In this paper, for an ocean circulation model, we report a comparison of numerical results obtained by the classical truncated boundary approach with a shooting method, those found by our free boundary approach with a finite difference method, and the ones obtained by a finite difference method with a quasi-uniform grid.

2 The physical model

A steady-state wind-driven ocean circulation model can be introduced, see Ierley and Ruehr [19], by considering the barotropic vorticity equation

\[ J(\psi, y + \gamma \nabla^2 \psi) = \kappa \gamma \nabla^4 \psi - \cos \left( \frac{\pi y}{2} \right), \tag{2.1} \]

in a region defined by \( x \in [-1, 1] \) and \( y \in [-1, 1] \) with the following boundary conditions

\[ \psi(\pm 1, y) = 0, \quad \psi(x, \pm 1) = 0, \tag{2.2} \]

and either

\[ \frac{\partial \psi}{\partial x}(\pm 1, y) = 0, \quad \frac{\partial \psi}{\partial y}(x, \pm 1) = 0, \tag{2.3} \]

known as “rigid” or no-slip boundary conditions, or

\[ \frac{\partial^2 \psi}{\partial x^2}(\pm 1, y) = 0, \quad \frac{\partial^2 \psi}{\partial y^2}(x, \pm 1) = 0, \tag{2.4} \]

known as “slippery” or stress-free boundary conditions. Equation (2.1) is written in a non-dimensional form; \( \psi(x, y) \) is the stream function; to fix a reference coordinate system is fixed with the \( x \) axis directed to the east and the \( y \) axis directed...
to the north; $J(a, b)$ is the Jacobian of the functions $a$ and $b$ with respect to $x$ and $y$, $\nabla^2$ is the Laplacian operator on the $(x, y)$ plane; the square $[-1, 1] \times [-1, 1]$ models a region of the subtropical gyre formation. Here the Jacobian represents nonlinear advection and the Laplacian the viscous drag. We assume that the curl of the wind stress in the region can be approximated by $-\cos \left( \frac{\pi y}{2} \right)$; $\gamma$ and $\kappa$, are non-dimensional parameters characterizing the widths of inertial and viscous boundary layers, respectively. We use impermeability and no-slip conditions (2.3) at the coasts and impermeability and slip conditions (2.4) at the fluid boundaries. We consider a particular solution to (2.1) of the form

$$\psi = \pi(y + 1)u(x).$$

Relation (2.5) represents the first term in the expansion of a solution of (2.1) with respect to $y$ near boundaries of the region: at $y = -1$ and at $y = 1$. Substituting (2.5) into (2.1), using a Taylor series expansion near $y = -1$ of the wind-stress term and assuming that a steady boundary-layer type solution exists, we obtain the equation for the boundary layer at the western coast, i.e. at $x = -1$,

$$\kappa \gamma \frac{d^4 u}{dx^4} = \pi \gamma \left( \frac{du}{dx} \frac{d^2 u}{dx^2} - \frac{d^3 u}{dx^3} \right) + \frac{du}{dx}, \quad x \in [0, \infty).$$

(2.6)

The parameters involved can be reduced to one if we define

$$b = \pi \left( \frac{\gamma}{\kappa^2} \right)^{1/3},$$

(2.7)

and introduce the new independent variable

$$\xi = \frac{x}{(\kappa \gamma)^{1/3}}.$$

(2.8)

The limit of vanishing viscosity (small values of $\kappa$) is of particular interest. Indeed, the parameter $\gamma$ is also small, of the order of $10^{-3}$. Therefore, in terms of the new independent variable $\xi$, far from the boundaries for asymptotically matching the interior solution $\psi_I$, taken of the following form

$$\psi_I \approx (1 - x) \cos \left( \frac{\pi y}{2} \right),$$

(2.9)
we have to require that
\[ u(x) \to 1 \quad \text{as} \quad x \to \infty. \tag{2.10} \]

Our fourth order ordinary differential equation (2.6) can be integrated once, using zero boundary conditions at infinity for the second and third derivative of \( u(\xi) \), to give
\[ \frac{d^3 u}{d\xi^3} = b \left[ \left( \frac{du}{d\xi} \right)^2 - u \frac{d^2 u}{d\xi^2} \right] + u - 1, \quad \xi \in [0, \infty). \tag{2.11} \]

The boundary conditions follow from (2.2)-(2.4). In particular, we can have no-slip (or rigid) boundary data
\[ u(0) = \frac{du}{d\xi}(0) = 0, \quad u(\xi) \to 1 \quad \text{as} \quad \xi \to \infty, \tag{2.12} \]
or stress-free (or slippery) boundary conditions
\[ u(0) = \frac{d^2 u}{d\xi^2}(0) = 0, \quad u(\xi) \to 1 \quad \text{as} \quad \xi \to \infty. \tag{2.13} \]

Therefore, we get the two point BVP defined on an unbounded domain that has been investigated by Ierley and Ruehr [19], Mallier [23] or Sheremet et al. [30].

The parameter \( b \) in (2.11) can be used as a measure of the strength of the nonlinearity. In fact, for \( b = 0 \) we get the simple linear model formulated by Munk [26].

Ierley and Ruehr [19] discovered an analytical approximation for the relation between the missing initial condition and the parameter \( b \). In particular, for rigid conditions they proposed the relation
\[ \left( \frac{d^2 u}{d\xi^2}(0) \right)^2 \approx \frac{2}{1 \pm \left( 1 + \frac{4}{3} b \right)^{1/2}}, \tag{2.14} \]
and for slippery conditions they reported the relation
\[ \frac{du}{d\xi}(0) \approx \frac{2}{1 \pm \left( 1 + \frac{10}{3} b \right)^{1/2}}. \tag{2.15} \]
The approximations provided by (2.14) and (2.15) will be used for comparison with the corresponding numerical results.
In this section we present the numerical methods used in order to solve the ocean model (2.11). As a first step we rewrite the ocean equation in (2.11) as a first order system
\[
\frac{du}{d\xi} = f(\xi, u), \quad \xi \in [0, \infty),
\]
by setting
\[
u_{i+1}(\xi) = \frac{d^i u}{d\xi^i}(\xi), \quad \text{for } i = 0, 1, 2.
\]
In this way the original BVP (2.11) specializes into
\[
\begin{align*}
\frac{du_1}{d\xi} &= u_2 \\
\frac{du_2}{d\xi} &= u_3 \\
\frac{du_3}{d\xi} &= b(u_2^2 - u_1u_3) + u_1 - 1,
\end{align*}
\]
that is,
\[
u = (u_1, u_2, u_3)^T
\]
\[
f(\xi, u) = (u_2, u_3, b(u_2^2 - u_1u_3) + u_1 - 1)^T
\]
with
\[
g(u(0), u(\infty)) = (u_1(0), u_2(0), u_1(\infty) - 1)^T
\]
or
\[
g(u(0), u(\infty)) = (u_1(0), u_3(0), u_1(\infty) - 1)^T
\]
in (3.1). In the following, in order to set a specific test problem, we consider the ocean model with \(b = 2\).
3.1 The truncated boundary approach and shooting methods

It is simple to describe a classical shooting method. We set a value for the truncated boundary $\xi_\infty$. Then, we guess a value for the missing initial condition, call it $\beta$, and integrate the given problem as an initial value problem (IVP). This defines, implicitly, a nonlinear equation $F(\beta) = u(\xi_\infty; \beta) - 1$, where $\beta$ can be considered as a parameter. In order to get the correct value of $\beta$, it is possible to apply a root-finder method. We have found that the secant method is particularly suitable to this purpose. The same shooting method can be applied by using the Newton’s. This requires a more complex treatment involving a system of six differential equations. In both cases, secant or Newton’s root finder, we set $\xi_\infty = 10$.

We consider first the results obtained by applying the secant method for the system of three equations (3.2). For the problem (2.11) with no-slip boundary conditions (2.12), setting $\beta_0 = 1$ and $\beta_1 = 2$, we found the missing initial condition $\beta = \frac{du}{d\xi}(0) = 0.826111$ by 12 iterations. In the second test, with slip boundary conditions (2.13), setting $\beta_0 = 0.8$ and $\beta_1 = 1$, we obtained the missing initial condition $\beta = \frac{du}{d\xi}(0) = 0.528885$ with 13 iterations.

On condition to provide a value for $F'(\beta) = \frac{\partial F}{\partial \beta}$, it is also possible to implement the Newton’s iterations. This can be done by differentiating with respect to $\beta$ the governing system. Of course, we end up to solve a system of six equations, namely the system in (3.1) with

$$u = (u_1, u_2, u_3, u_4, u_5, u_6)^T, \quad (3.3)$$

$$f(\xi, u) = \left(u_2, u_3, b(u_2^2 - u_1 u_3) + u_1 - 1, \frac{\partial f_1}{\partial \beta}, \frac{\partial f_2}{\partial \beta}, \frac{\partial f_3}{\partial \beta}\right)^T,$$

with initial conditions, for the no-slip case

$$u(0) = (0, 0, \beta, 0, 0, 1)^T, \quad (3.4)$$

and for the slip case

$$u(0) = (0, \beta, 0, 0, 1, 0)^T, \quad (3.5)$$
where \( u_4 = \frac{\partial u_1}{\partial \beta} \), \( u_5 = \frac{\partial u_2}{\partial \beta} \), \( u_6 = \frac{\partial u_3}{\partial \beta} \), and in both cases

\[
\frac{\partial f_1}{\partial \beta} = u_5, \quad \frac{\partial f_2}{\partial \beta} = u_6, \quad \frac{\partial f_3}{\partial \beta} = b(2u_2u_5 - u_3u_4 - u_1u_6) + u_4.
\]

In both cases \( \frac{\partial F}{\partial \beta} = u_4(\xi; \beta) \).

Let us report now the numerical results given by the Newton’s method with the system of six equations (3.3) with (3.4) or with (3.5). For the problem (2.11) with no-slip boundary conditions (2.12), setting \( \beta_0 = 1 \), we obtained the missing initial condition \( \beta = \frac{d^2u}{d\xi^2}(0) = 0.826111 \) by 7 iterations. In the second test, with slip boundary conditions (2.13), setting \( \beta_0 = 0.8 \), we found the missing initial condition \( \beta = \frac{du}{d\xi}(0) = 0.528910 \) with 8 iterations.

As well known, the secant and Newton’s methods are convergent provided that we use initial iterates sufficiently close to the root. Moreover, the convergence is super-linear and the order of convergence is \( (1 + \sqrt{5})/2 \) for the secant method and 2 for Newton’s one. We notice that, for the root of nonlinear equations, as reported by Gautschi [16, pp. 225-234] the secant method has an efficiency index higher than that of the Newton’s method. On the other hand, the Newton’s method might be preferable since it requires only one initial guess. The computational cost of these two shooting methods is given in table \( \Box \). As it is easily seen the shooting-Newton method is the less demanding of the two and we can conclude that in relation of the numerical solution of our BVP the Newton’s method is more efficient than the secant method.

Figure 1 and 2 display the numerical results obtained by Newton’s iterations and related, respectively, to model (2.11) with boundary conditions (2.12) and (2.13) (see also [27, pp. 50-54]).

In both tests, we have used the following termination criterion

\[
\left| \frac{\beta_n - \beta_{n-1}}{\beta_n} \right| < \text{TOL} \quad \text{and} \quad \left| F(\beta_n) \right| < \text{TOL} \quad (3.7)
\]

with \( \text{TOL} = 10^{-6} \). Moreover, the numerical solutions of the IVPs were obtained
Table 1: Computational cost of the two shooting methods. Here eval stands for all function evaluations.

| BCs    | root-finder | $\beta_0$ | $\beta_1$ | steps | rejections | eval      | iterations |
|--------|-------------|-----------|-----------|-------|------------|-----------|------------|
| no-slip secant | 1          | 2         | 109111    | 142   | 327771     | 12        |
| no-slip Newton   | 1          |           | 1489      | 79    | 4711       | 7         |
| slip  secant     | 0.8        | 1         | 28461     | 208   | 86020      | 13        |
| slip  Newton     | 0.8        |           | 6263      | 114   | 19139      | 8         |

by the ODE23 solver, from the MATLAB ODE suite written by Samphine and Reichelt [28], with the accuracy and adaptivity parameters defined by default.

As far as the shooting method is concerned, it may be not suitable even for the simple truncated boundary approach. In fact, it would be possible, when a large step size is used, or always for some models, that one obtains floating-point overflows in the calculations. This is exactly the reason for the introduction of the more complex multiple shooting method (see also [1, p. 145]).

3.2 The free boundary formulation and a relaxation method

In order to introduce a free boundary formulation for our problem, we replace the far boundary condition by two boundary conditions at the free boundary $\xi_\varepsilon$

$$ u(\xi_\varepsilon) = 1 , \quad \frac{du}{dx}(\xi_\varepsilon) = \varepsilon , $$

(3.8)

where $\xi_\varepsilon$ can be considered as a truncated boundary. Then we rewrite the resulting free BVP in standard form (see Ascher and Russell [2]), defining $u_4 = \xi_\varepsilon$ and
Figure 1: Numerical solutions of the circulation model (2.11) with rigid boundary condition (2.12) by the shooting Newton’s method.

using the new independent variable

\[ z = \frac{\xi}{u_4}, \quad (3.9) \]

In general, we end up with a BVP belonging to the general class:

\[
\frac{dU}{dz} = F(z, U), \quad z \in [0, 1],
\]

\[
G(U(0), U(1)) = 0,
\quad (3.10)
\]

where

\[
U(z) \equiv (u(z), u_4)^T, \\
F(z, U) \equiv (u_4 f(z, u), 0)^T, \\
G(U(0), U(1)) \equiv (g(u(0), u(1)), h(u(1)))^T,
\quad (3.11)
\]
where, in our case, \( h(u(1)) = u_2(1) - \epsilon \). In order to simplify notation in (3.10)-(3.11) and in the following, we omit the dependence of \( u \) and \( U \) on \( \epsilon \).

In order to solve the resulting problem we apply a relaxation method. Let us introduce a mesh of points \( z_0 = 0, z_j = j\Delta z \), for \( j = 1, 2, \ldots, J \), of uniform spacing \( \Delta z \) and naturally \( z_J = 1 \). We denote by the 4—dimensional vector \( V_j \) the numerical approximation to the solution \( U(z_j) \) of (3.10) at the points of the mesh, that is for \( j = 0, 1, \ldots, J \). Keller’s box scheme for (3.10) can be written as follows:

\[
V_j - V_{j-1} - \Delta z \mathbf{F} \left( z_{j-1/2}, \frac{V_j + V_{j-1}}{2} \right) = 0 , \quad \text{for} \quad j = 1, 2, \ldots, J
\]

\[
G (V_0, V_J) = 0 ,
\]

where \( z_{j-1/2} = (z_j + z_{j-1})/2 \). It is evident that (3.12) is a nonlinear system with respect to the unknown 4(\( J + 1 \))—dimensional vector \( V = (V_0, V_1, \ldots, V_J)^T \). Foll-
lowing Keller, the classical Newton’s method, along with a suitable termination
criterion, is applied to solve (3.12).

Let us recall now the main properties of the box scheme proved by Keller in the
main theorem of [20]. Under the assumption that $U(z)$ and $F(z, U)$ are sufficiently
smooth, each isolated solution of (3.10) is approximated by a difference solution
of (3.12) which can be computed by Newton’s method, provided that a sufficiently
fine mesh and an accurate initial guess for the Newton’s method are used. As far as
the accuracy issue is concerned, the truncation error has an asymptotic expansion
in powers of $(Δz)^2$.

For the Newton’s method we used the simple termination criterion

$$\frac{1}{4(J+1)} \sum_{\ell=1}^{4} \sum_{j=0}^{J} |ΔV_{j\ell}| \leq TOL ,$$

(3.13)

where $ΔV_{j\ell}$, $j = 0,1,\ldots,J$ and $\ell = 1,2,3,4$, is the difference between two suc-
cessive iterate components and TOL is a fixed tolerance. The key point for the
numerical solution of the nonlinear system is that Newton’s method converges
only locally. Therefore, some preliminary numerical experiments may be helpful
and worth of consideration. However, for the results reported, our initial guess to
start the iterations was always as follows:

$$u_1(z) = z , \quad u_2(z) = 0.5 z , \quad u_3(z) = 1-z , \quad u_4(z) = 2 .$$

(3.14)

In Figures 3 and 4 and in Tables 2 and 3 we report some of the numerical
results, obtained with the free boundary approach, related to different values of $ε$
obtained by setting $J = 2000$ and $TOL = 1E−6$. Here and in the following $1E−k$
is the standard notation for $10^{−k}$ in simple precision arithmetic.

We have several reasons to consider a free boundary formulation more effec-
tive than the simple truncated boundary approach. First of all, the free boundary
conditions are less suitable for the applications of a shooting method forcing us to
use a more suitable relaxation (finite difference) method. Furthermore, we know
a priori that the free boundary has to be an increasing function of $ε$. Moreover, $ε$
Figure 3: Numerical solution of the BVP (2.11) with no slip conditions (2.12) by the free boundary approach.
Figure 4: Numerical solution of the BVP (2.11) with slip conditions (2.13) by the free boundary approach.
Table 2: Free boundary formulation for the BVP (2.11) with (2.12).

| $\varepsilon$ | $\xi_{\varepsilon}$ | iter | $\frac{d^2u}{d\xi^2}(0)$ |
|--------------|---------------------|------|-------------------------|
| $1\times10^{-2}$ | 6.485761           | 7    | 0.826184               |
| $1\times10^{-3}$ | 8.792991           | 8    | 0.826141               |
| $1\times10^{-4}$ | 11.098635          | 10   | 0.826141               |
| $1\times10^{-5}$ | 13.402219          | 11   | 0.826142               |

Table 3: Free boundary formulation for the BVP (2.11) with (2.13).

| $\varepsilon$ | $\xi_{\varepsilon}$ | iter | $\frac{du}{d\xi}(0)$ |
|--------------|---------------------|------|----------------------|
| $1\times10^{-2}$ | 5.828307           | 7    | 0.528970              |
| $1\times10^{-3}$ | 8.132813           | 8    | 0.528922              |
| $1\times10^{-4}$ | 10.437875          | 9    | 0.528921              |
| $1\times10^{-5}$ | 12.741323          | 11   | 0.528921              |

itself can be consider as a continuation parameter. This means that the numerical results obtained for a value $\varepsilon$ can be used as the initial guess for Newton’s method for the next value of $\varepsilon$. Consequently, except for the first value of $\varepsilon$, for both kind of boundary conditions the number of iterations of the relaxation method can be reduced to 7, 6, 6, 6.

3.3 Finite difference method on a quasi-uniform grid

Let us consider the smooth strict monotone quasi-uniform map $\xi = \xi(\eta)$, the so-called grid generating function,

$$\xi = -c \cdot \ln(1 - \eta),$$ (3.15)
where \( \eta \in [0, 1], \xi \in [0, \infty] \), and \( c > 0 \) is a control parameter. We notice that for (3.15) \( \xi_{J-1} = c \ln J \). The problem under consideration can be discretized by introducing a uniform grid \( \eta_j \) of \( J + 1 \) nodes on \([0, 1]\) with \( \eta_0 = 0 \) and \( \eta_{J+1} = \eta_j + h \) with \( h = 1/J \), so that \( \xi_j \) defines a quasi-uniform grid on \([0, \infty]\). The last interval in (3.15), namely \([\xi_{J-1}, \xi_J]\), is infinite but the point \( \xi_{J-1}/2 \) is finite, because the non integer nodes are defined by

\[
\xi_{j+\alpha} = \xi \left( \eta = \frac{j+\alpha}{J} \right),
\]

with \( j \in \{0, 1, \ldots, J-1\} \) and \( 0 < \alpha < 1 \). The map allows us to describe the infinite domain by a finite number of intervals. The last node of such grid is placed on infinity so that right boundary condition is taken into account correctly.

For the sake of simplicity we consider here the simple scalar case. The finite difference formulae can be applied component-wise to a system of differential equations. We can define the values of \( u(\xi) \) on the middle-points of the grid

\[
u_{j+1/2} \approx \frac{\xi_{j+1} - \xi_{j+1/2}}{\xi_{j+1} - \xi_j} u_j + \frac{\xi_{j+1/2} - \xi_j}{\xi_{j+1} - \xi_j} u_{j+1}.
\]

As far as the first derivative is concerned we can apply the following approximation

\[
\frac{du}{d\xi} \bigg|_{j+1/2} \approx \frac{u_{j+1} - u_j}{2(\xi_{j+3/4} - \xi_{j+1/4})}.
\]

These formulae use the value \( u_J = u_\infty \), but not \( \xi_J = \infty \). Both finite difference approximations (3.17) and (3.18) have order of accuracy \( O(J^{-2}) \).

A finite difference scheme on a quasi-uniform mesh for the class of BVPs (3.1) can be defined by using the approximations given by (3.17) and (3.18). We denote by the 3—dimensional vector \( U_j \) the numerical approximation to the solution \( u(\xi_j) \) of (3.1) at the points of the mesh, that is for \( j = 0, 1, \ldots, J \). We can define a second order finite difference scheme for (3.1) as

\[
U_{j+1} - U_j - a_{j+1/2}f(x_{j+1/2}, b_{j+1/2}U_{j+1} + c_{j+1/2}U_j) = 0,
\]

\[
g(U_0, U_J) = 0,
\]

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for \( j = 0, 1, \ldots, J - 1 \), where

\[
\begin{align*}
\alpha_{j+1/2} &= 2 \left( \xi_{j+3/4} - \xi_{j+1/4} \right), \\
\beta_{j+1/2} &= \frac{\xi_{j+1/2} - \xi_j}{\xi_{j+1} - \xi_j}, \\
\gamma_{j+1/2} &= \frac{\xi_{j+1} - \xi_{j+1/2}}{\xi_{j+1} - \xi_j}.
\end{align*}
\]

It is evident that (3.19) is a nonlinear system with respect to the unknown

\( 3(J + 1) \)-dimensional vector \( \mathbf{U} = (U_0, U_1, \ldots, U_J)^T \). We notice that \( b_{j+1/2} \approx \gamma_{j+1/2} \approx 1/2 \) for all \( j = 0, 1, \ldots, J - 2 \), but when \( j = J - 1 \), then \( b_{J-1/2} = 0 \) and \( \gamma_{J-1/2} = 1 \). On the contrary, we choose to set \( b_{J-1/2} = b_{J-3/2} \) and \( \gamma_{J-1/2} = \gamma_{J-3/2} \) in order to avoid a suddenly jump for the coefficients of (3.19). This produces a much smaller error in the numerical solution of the system at \( \xi_J \).

For the solution of (3.19) we can apply the classical Newton’s method along with the simple termination criterion

\[
\frac{1}{3(J + 1)} \sum_{\ell=1}^{3} \sum_{j=0}^{J} |\Delta U_{j\ell}| \leq \text{TOL},
\]

where \( \Delta U_{j\ell} \), \( j = 0, 1, \ldots, J \) and \( \ell = 1, 2, 3 \), is the difference between two successive iterate components and \( \text{TOL} \) is a fixed tolerance. The results listed in the next section were computed by setting \( \text{TOL} = 1 \times 10^{-6} \).

Figures 5 and 6 show the numerical solution of ocean model (2.11) obtained setting the initial iterate

\[
\begin{align*}
u_1(\xi) &= 0.1, \quad \nu_2(\xi) = \nu_3(\xi) = 0.1.
\end{align*}
\]

From these figures we notice how the grid is denser close to the origin in comparison with the side of the far boundary at infinity.

4 Final remarks and conclusions

In this paper we describe several methods for the numerical solution a simple wind driven circulation model arising in physical oceanography. Our final aim is
Figure 5: Numerical solution for the BVP (2.11) with no slip boundary conditions (2.12) obtained with the map (3.15) and $c = 5$ for $J = 200$. We found a missing value of $\beta = 0.826180$.

the comparison of numerical results. This is provided in tables 4–5 where we used FBF (free boundary formulation) and QUG (quasi-uniform grid). For the sake of simplicity we limited ourself to compare the computed values of the missing initial condition $\beta$. We applied the simple shooting method to the truncated boundary formulation and a finite difference method to both the free boundary approach and the quasi-uniform grid treatment. For the sake of comparison, all numerical methods used in this study were second order methods. The reported numerical results allow us to point out that the finite difference method with a quasi-uniform grid is the less demanding approach and that the free boundary approach provides a more reliable formulation than the classical truncated boundary.

The shooting method supplemented by the Newton’s iterations shows that the
Figure 6: Numerical solution for the BVP (2.11) with slip boundary conditions (2.13) obtained with the map (3.15) and $c = 5$ for $J = 200$. We found a missing value of $\beta = 0.528927$.

ocean circulation model cannot be considered as a simple test case. In fact, for this method we are forced to use as initial iterate a value close to the correct missing initial condition in order to be able to get a convergent numerical solution.

As a last remark, we want to mention that for the numerical solution of BVPs on unbounded domains it is also possible to consider spectral methods that use mapped Jacobi, Laguerre and Hermite functions (see Shen and Wang [29] for a review on this topic).

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Table 4: Comparison of numerical results for the ocean model (2.11) with no slip boundary conditions (2.12). Equation (2.14) gives $\beta = 0.828336$.

| Numerical Method | grid-points | iter | $\frac{d^2u}{d\xi^2}(0)$ |
|------------------|-------------|------|--------------------------|
| Shooting-secant  | $\xi_\infty = 10$ | 12   | 0.826111                 |
| Shooting-Newton  | $\xi_\infty = 10$ | 7    | 0.826111                 |
| FBF              | $\xi_e = 13.402219$ | 2000 | 11 | 0.826142 |
| FBF              | $\xi_e = 13.402251$ | 4000 | 11 | 0.826140 |
| QUG              | $\xi_J = \infty$ | 200  | 5 | 0.826180 |
| QUG              | $\xi_J = \infty$ | 400  | 5 | 0.826150 |
Table 5: Comparison of numerical results for the ocean model (2.11) with slip boundary conditions (2.13). Equation (2.15) gives $\beta = 0.530662$.

| Numerical Method  | grid-points | iter | $\frac{du}{d\xi}(0)$ |
|-------------------|-------------|------|----------------------|
| Shooting-secant   | $\xi_\infty = 10$  | 13 | 0.528885         |
| Shooting-Newton   | $\xi_\infty = 10$  | 8  | 0.528910          |
| FBF               | $\xi_e = 12.741323$ | 2000 | 11 | 0.528921        |
| FBF               | $\xi_e = 12.741353$ | 4000 | 11 | 0.528921        |
| QUM               | $\xi_J = \infty$  | 200 | 4 | 0.528927        |
| QUM               | $\xi_J = \infty$  | 400 | 4 | 0.528922        |
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