A note on the existence of standard splittings for conformally stationary spacetimes

Miguel Angel Javaloyes and Miguel Sánchez
Departamento de Geometría y Topología, Facultad de Ciencias, Universidad de Granada, Campus Fuentenueva s/n, 18071 Granada, Spain
E-mail: ma.javaloyes@gmail.com and sanchezm@ugr.es

Received 12 May 2008, in final form 12 June 2008
Published 5 August 2008
Online at stacks.iop.org/CQG/25/168001

Abstract
Let \((M, g)\) be a spacetime which admits a complete timelike conformal Killing vector field \(K\). We prove that \((M, g)\) splits globally as a standard conformastationary spacetime with respect to \(K\) if and only if \((M, g)\) is distinguishing (and, thus causally continuous). Causal but non-distinguishing spacetimes with complete stationary vector fields are also exhibited. For the proof, the recently solved ‘folk problems’ on smoothability of time functions (moreover, the existence of a \textit{temporal} function) are used.

PACS numbers: 04.20.Cv, 04.20.Gz, 02.40.Ma

1. Introduction

Many physically interesting spacetimes are stationary or, more generally, conformastationary. Locally, such a spacetime with a timelike conformal-Killing vector field \(K\) can be written as a \textit{standard conformastationary spacetime} with respect to \(K\), i.e., a product manifold \(M = \mathbb{R} \times S\) (\(\mathbb{R}\) real numbers, \(S\) any manifold), where the metric can be written, under natural identifications, as

\[
g_{(t,x)} = \Omega(t,x)(-\beta(x)\,dt^2 + 2\omega_x\,dt + \bar{g}_x),
\]

\((1.1)\)

\(\Omega\) being a positive function on \(M\), and \(\bar{g}, \beta, \omega\), resp., a Riemannian metric, a positive function and a 1-form, all on \(S\); the vector field \(K\) is (locally) identified with \(\partial_t\) (see e.g. \cite{23}). The case \(\Omega \equiv 1\), or independent of \(t\), corresponds to a \textit{standard stationary spacetime} (note that, in general, the function \(\beta\) can be absorbed by the conformal factor \(\Omega\)). Then, a natural question is to wonder when a spacetime admitting a (necessarily complete) conformastationary \(K\) can be written \textit{globally} as above.

The possibility of obtaining a \textit{topological} splitting was proved by Harris under the assumption of chronology (see the following section for definitions on causality):
Theorem 1.1. [10]. If a spacetime \((M, g)\) admits a complete stationary vector field \(K\) and it is chronological then it splits topologically and differentiably as a product \(M = \mathbb{R} \times Q\), where \(Q\) is the space of integral curves of \(K\), endowed with a natural manifold structure.

Moreover, for any point \(p \in M\) there is a neighborhood \(U\) such that the projection \(\pi : \mathbb{R} \times Q \to Q\) admits a local spacelike section \(U(\subseteq Q) \to \mathbb{R} \times Q, x \mapsto (t(x), x)\) and, then, \(\pi^{-1}(U)\) is standard stationary.

(The result can be easily extended to the conformastationary case, see remark 2.2.) Nevertheless, this topological splitting does not ensure the existence of the full metric splitting (1.1), except if one assumes that \(\mathbb{R} \times Q\) admits a global spacelike section \((U = Q)\). Our purpose is to give a full solution to the metric problem, by studying carefully the involved causality. Concretely:

Theorem 1.2. Let \((M, g)\) be a spacetime which admits a complete conformastationary vector field \(K\). Then, it admits a standard splitting (1.1) if and only if \((M, g)\) is distinguishing. Moreover, in this case, \((M, g)\) is causally continuous.

In the following section we will see how the causality conditions are natural and optimal for the problem, also providing some examples. In the last section, theorem 1.2 and related results are proved. We emphasize that the recent solution (see [4, 6]) on the so-called folk problems on smoothability is needed for the proof. Theorem 1.1 is sketched in the appendix for completeness.

2. Causality conditions and Killing fields

Our conventions and approach will be standard, as in the classical books [2, 11, 13, 15, 16]. Nevertheless, some folk problems on smoothability (initially suggested in [17], p 1155) will be relevant here—at least those concerning time functions. So, we also recommend the expanded discussion [20] (specially section 4.6) or the review [12] (section 3.8.3 and remark 3.77).

\((M, g)\) will denote an \(m\)-dimensional spacetime \((- , + , \ldots , + )\), that is assumed connected and, when a timelike vector field \(K\) is given, \((M, g)\) will be assumed future time-oriented by \(K\). A tangent vector \(v \in TM\) will be causal if it is either timelike \((g(v, v) < 0)\) or lightlike \((g(v, v) = 0 \text{ and } v \neq 0)\). A timelike vector field \(K\) is called conformastationary if it is conformal-Killing (any local flow \(\Phi_t\) of \(K\) is a conformal transformation) and stationary if \(K\) is Killing \((\Phi_1\) isometry); accordingly, \((M, g)\) will also be called conformastationary or stationary. We begin by showing that causality of conformastationary spacetimes can be reduced to the stationary case.

Lemma 2.1. Let \(K\) be a timelike vector field for \((M, g)\). Then, \(K\) is conformal-Killing for \(g\) iff \(K\) is Killing for the conformal metric \(g^\ast = -g / g(K, K)\).

Proof. Obviously, \(K\) is conformal-Killing for any metric conformal to \(g\). As \(g^\ast(K, K)\) is constant, the conformal factors for the local flows of \(K\) (which are trivially computable from the value of \(g^\ast(K, K)\) on the integral curves of \(K\)) must be equal to 1, i.e., \(K\) is Killing (see [18], lemma 2.1 for an alternative elementary reasoning or [9], theorem 1 for a general classic result).

Remark 2.2. As the metric conditions in theorem 1.2 (as well as in theorem 1.1) involve only causality, which is a conformal invariant, we can replace \(g\) by \(g^\ast\), so that we will assume without loss of generality that \(K\) is Killing in what follows (the reader can also assume \(g(K, K) \equiv -1\)).
Let us recall the basic facts about causality involved in our main result. \((M, g)\) is chronological (resp. causal) if it does not contain any closed timelike (resp. causal) curve. It is easy to construct a chronological but non-causal stationary spacetime. Consider in \(\mathbb{R}^2\) the vector fields \(X = \partial_t - \partial_x\), \(Y = \partial_t\) and let \(g_1\) be the Lorentzian metric such that \(X, Y\) are lightlike and \(g_1(X, Y) = -1\). The non-causal cylinder \(C = \mathbb{R} \times S^1\) obtained by identifying \((t, x) \sim (t, x + 1)\) admits the projection \(K\) of \(\partial_t\) as a stationary vector field.

\((M, g)\) is distinguishing if \(p \neq q\) implies both, \(I^+(p) \neq I^+(q)\) and \(I^-(p) \neq I^-(q)\) for any \(p, q \in M\). A causal non-distinguishing spacetime with a complete stationary \(K\) can be obtained from the previous example as follows. Consider the product \(\mathbb{R}^2 \times \mathbb{R}\), \(g_2 = g_1 + dy^2\), choose any irrational number \(\alpha\) and identify \((t, x, y) \sim (t, x, y + 1), (t, x, y) \sim (t, x + 1, y + \alpha)\). The quotient spacetime \((\tilde{M}, \tilde{g}_2)\) (a variant of Carter’s classical example [11], figure 39, p 195) satisfies the required properties. Note that theorem 1.1 is applicable to \((\tilde{M}, \tilde{g}_2)\) and, in fact, \(Q\) is topologically a torus. Nevertheless, \((\tilde{M}, \tilde{g}_2)\) does not split as standard stationary (apply corollary 3.2 below).

The next three steps in the causal ladder are strongly causal, stably causal and causally continuous. Remarkably, if a complete stationary vector field exists then ‘distinguishing’ implies ‘causally continuous’ (proposition 3.1). This last condition means intuitively that the sets \(I^\pm(p)\) not only characterize \(p\) (as in the distinguishing case) but also vary continuously with \(p\). There are also several characterizations (see e.g. [12], section 3.9), and we will use the following: \((M, g)\) is causally continuous iff it is reflecting and \(I^+(p) \supseteq \overline{I^-(p)} \subseteq I^-(q)\) (see e.g. [2], theorem 3.25, proposition 3.21).

Moreover, in the proofs, it will be used that causal continuity implies stable causality. This condition means intuitively that \((M, g)\) not only is causal but also remains causal by opening slightly the timecones. A classical consequence of this definition (in fact, a folk characterization, [20], section 4.6) is the existence of a time function \(t\), i.e., a continuous function which is strictly increasing on any future-directed causal curve. The recent full solution of the folk problems of smoothability (see [3–5]) allows us to characterize stably causal spacetimes as those admitting a temporal function \(t\), i.e. \(t\) is smooth with timelike past-directed gradient (in particular, a time function). The existence of such a function will be essential for our proof. Also note that it implies strong causality, i.e., the absence of ‘almost closed’ causal curves.

Finally, let us point out that the two remaining steps in the ladder of causality (causal simplicity and global hyperbolicity) can be characterized in a standard stationary spacetime very accurately, in terms of Fermat metrics (work in progress, see also [7, 8]). It is also worth pointing out that globally hyperbolic spacetimes can be defined as the causal ones such that the diamonds \(J(p, q) = J^+(p) \cap J^-(q)\) are compact for all \(p, q\) (see [6]). In the previous example \((\tilde{M}, \tilde{g}_2)\) (stationary causal non-distinguishing), the closures \(\overline{J(p, q)}\) are compact, but some \(J(p, q)\) are not. For strongly causal spacetimes, the compactness of \(\overline{J(p, q)}\) suffices to ensure global hyperbolicity (see [2], lemma 4.29, [1]); in particular, any standard stationary spacetime with compact \(S\) is trivially globally hyperbolic, as \(J(p, q)\) lies in the compact region \(t^{-1}([t(p), t(q)])\).

3. Proof of the results

**Proposition 3.1.** Let \((M, g)\) be a spacetime with a stationary \(K\). If \(K\) is complete then \((M, g)\) is reflecting. Thus, if, additionally, \((M, g)\) is distinguishing, then it is causally continuous.

**Proof.** It is enough to show past reflectivity \(I^+(p) \supseteq \overline{I^+(q)} \Rightarrow I^-(p) \subseteq I^-(q)\) (the converse is analogous), and we will adapt the particular proof in ([21], theorem 3.1).
Take any \( p \neq q \) in \( M \) and let \( \Phi_t : M \to M \) the flow of \( K \) at the stage \( t \in \mathbb{R} \). Assuming the first inclusion, it is enough to prove \( p_{-\epsilon} := \Phi_{-\epsilon}(p) \in I^-(q) \), for all \( \epsilon > 0 \) (note that the relation \( \ll \) is open and, then, any \( p' \ll p \) will lie also in \( I^-(p_{-\epsilon}) \) for small \( \epsilon \)). As \( q_{\epsilon} := \Phi_{\epsilon}(q) \in I^+(p) \), there exists a future-directed timelike curve \( \gamma \) joining \( p \) and \( q_{\epsilon} \). Then, the future-directed timelike curve \( \gamma_{-\epsilon} := \Phi_{-\epsilon} \circ \gamma \) connects \( p_{-\epsilon} \) and \( q_{\epsilon} \), as required.

Note that the completeness of \( K \) is essential; otherwise, \( \Phi_{-\epsilon} \circ \gamma(s) \) may be non-defined for all \( s \). In fact, the open subset \( M \) of bidimensional Lorentz–Minkowski spacetime \( \mathbb{L}^2 \) obtained by removing a spacelike semi-axis, that is, \( M = \mathbb{L}^2 \setminus \{(0, x) \in \mathbb{L}^2 : x \leq 0 \} \), is not causally continuous (note that the past reflecting property fails for \( p = (-1, 1), q = (1, -1) \)).

Proposition 3.1 will be an essential ingredient of the proof of theorem 1.2, as it will ensure the existence of a temporal function. The following consequence shows the consistency of the results.

**Corollary 3.2.** Any standard stationary spacetime \( (M = \mathbb{R} \times S, g) \) as in (1.1) is causally continuous, and the projection \( t : M \to \mathbb{R} \) is a temporal function.

**Proof.** It is enough to prove that \( t \) is a temporal function because in this case the spacetime is distinguishing, and proposition 3.1 applies. This is a well-known fact, but we sketch the proof for the sake of completeness. Let \( \gamma(s) = (t(s), x(s)) \) be a future-directed differentiable causal curve. Then, it satisfies the second-order inequality in \( \dot{i} \):

\[
g_0(\dot{x}, \dot{x}) + 2\omega(\dot{x}) \dot{i} - \beta \dot{i}^2 \leq 0. \tag{3.1}
\]

Moreover, as the Killing field \( \partial_t \equiv (1, 0) \) is future directed, we have that

\[
g((1, 0), (\dot{i}, \dot{x})) = \omega(\dot{x}) - \beta \dot{i} < 0. \tag{3.2}
\]

So, \( \dot{i} \) must remain outside the interval determined by the two roots of the equality in (3.1), and by (3.2) it must be greater than the bigger root, i.e.:

\[
\beta \dot{i} \geq \omega(\dot{x}) + \sqrt{\beta g_0(\dot{x}, \dot{x}) + \omega(\dot{x})^2} \geq 0
\]

Moreover, \( \dot{i} > 0 \)—if the equality held at some point, then \( \dot{x} = 0 \) at that point, and \( \gamma \) would not be causal. This also implies that \( g(\nabla t, \nu) > 0 \) for all future-directed causal \( \nu \) and, thus, \( \nabla t \) is past-directed timelike, as required.

The main part of the proof of theorem 1.2 is to show that the following lemma can be applied to any level \( t \equiv \) constant of any temporal function \( t \).

**Lemma 3.3.** Let \( (M, g) \) be a spacetime with a complete stationary vector field \( K \). If there exists a spacelike hypersurface \( S \) which is crossed exactly once by any integral curve of \( K \) then \( (M, g) \) is standard stationary.

**Proof.** If \( \Phi : \mathbb{R} \times M \to M \) is the globally defined flow of \( K \), its restriction \( \Phi_S : \mathbb{R} \times S \to M \) is a diffeomorphism and \( (\mathbb{R} \times S, \Phi_S^* g) \) is standard stationary.

Lemma 3.3 also shows that, as a difference with the static case (see [22]), the standard stationary splitting cannot be expected to be unique in any case. Now, we can prove our main result.
Proof of theorem 1.2. (⇒) Trivial from corollary 3.2.

(⇐). From proposition 3.1 the spacetime is (in particular) stably causal and, thus, it admits a temporal function \( t \). If, say, \( 0 \in \text{Im} \, t \), let us see that \( S = t^{-1}(0) \) satisfies the hypotheses in lemma 3.3. Note that \( S \) is spacelike due to the temporality of \( t \). Now, consider the line bundle \( \pi : \mathbb{R} \times Q \rightarrow Q \) onto the manifold of integral curves (theorem 1.1). It is enough to prove that the restriction of \( \pi \) to \( S, \pi_S : S \subset \mathbb{R} \times Q \rightarrow Q \), is bijective. In fact, \( \pi_S \) is clearly injective, as \( S \) is achronal and, thus, is not intersected twice by any integral curve. Moreover, \( \pi(S) \) is open, as \( \pi_S \) is an open map (say, \( \pi_S \) is a continuous injective map between manifolds of the same dimension; so, this is a consequence of the classical theorem of the invariance of the domain). Therefore, it is enough to prove that \( \pi(S) \) is closed.

Let \( \{ z_n = (\hat{t}_n, \hat{x}_n) \}_n \) be a sequence in \( S \subset \mathbb{R} \times Q \) with \( \{ \hat{x}_n \}_n \) converging to some \( \hat{x}_\infty \in Q \). As \( S \) is closed, if we assume by contradiction that \( \hat{x}_\infty \notin \pi(S) \), then necessarily \( \{ \hat{t}_n \}_n \) diverges up to a subsequence. We will assume \( \{ \hat{t}_n \}_n \rightarrow +\infty \), as the case \( \{ \hat{t}_n \}_n \rightarrow -\infty \) is analogous.

Let \( \tilde{V} \) be some neighborhood of \( \hat{x}_\infty \) such that \( \pi^{-1}(\tilde{V}) = \mathbb{R} \times \tilde{V} \) is standard stationary. Choosing such a standard splitting, we can write \( \pi^{-1}(\tilde{V}) = \mathbb{R} \times V \) for some spacelike hypersurface \( V \subset \pi^{-1}(\tilde{V}) \) and, consistently: \( \{ z_n = (t_n, x_n) \}_n \) with \( \pi(z_n) = \hat{x}_n \) and \( \{ x_n \}_n \rightarrow x_\infty \in V \) with \( \pi((0, x_\infty)) = \hat{x}_\infty \) (the latter because \( V \) is obtained as a section on \( \tilde{V} \), according to theorem 1.1).

Let \( W \subset V \) be a compact neighborhood of \( x_\infty \). Note that there is a constant \( T_W > 0 \) such that \( \{ t_0 + T_W, +\infty \} \times W \subset \pi^{-1}(W) \) for any \( t_0 \in \pi^{-1}(W) \). In fact, consider the future Fermat arrival function\(^1\) \( T : V \times V \rightarrow [0, +\infty) \), i.e. \( T(x_1, x_2) \) is the infimum of the \( t \geq 0 \) such that \( (x_1, 0) \leq (x_2, t) \). As \( T \) is continuous (see [19], proposition 2.2) one can take any \( T_W \geq T(W, W) \).

Thus, fix some \( n_0 \in \mathbb{N} \) so that \( x_{n_0} \in W \). For large \( n \), \( (t_n, x_n) \in [t_{n_0} + T_W, +\infty) \times W \subset \pi^{-1}(W) \), in contradiction with the achronality of \( S \).

\[ \square \]

Acknowledgments

The readings by Professors E Caponio and JMM Senovilla are warmly acknowledged. Both authors are partially supported by Regional J Andalucía Grant P06-FQM-01951. MAJ was also partially supported by Spanish MEC Grant MTM2007-64504 and MS by MEC-FEDER Grant MTM2007-60731.

Appendix A

As commented above, Harris’ theorem 1.1 (which is stated in a somewhat more general form in [10]) can be extended to the conformal case by using remark 2.2. For the sake of completeness, we sketch the proof, readapting Harris’ arguments. First, the following general result is needed ([10], theorem 2):

**Theorem A.1.** Let \( (M, g) \) be a chronological spacetime with a complete timelike vector field \( K \). Then \( M \) is naturally a principal line bundle over the space \( Q \) of integral curves of \( K \), and \( Q \) is a near-manifold.

Here, a near-manifold is a topological space which satisfies all the axioms of a smooth manifold (including paracompactness) except at most to be Hausdorff. The proof is subtle,
and uses general properties of actions of groups by Palais (see [14]). Now, theorem 1.1 follows directly from:

**Lemma A.2.** In the hypotheses of theorem A.1, if \( K \) is Killing then \( Q \) is Hausdorff.

**Proof.** Note first that \( K \) is also Killing for the Riemannian metric \( g_R \) obtained by reversing the sign of \( K \), i.e.:

\[
g_R(v, v) = g(v, v) - \frac{2}{g(K, K)} g(K, v)^2, \quad \forall v \in TM,
\]

and the problem becomes purely Riemannian. Assume that the projections of \( x, y \in M \) on \( Q \) are different (\( \pi(x) \neq \pi(y) \)) and cannot be separated by disjoint neighborhoods. Then, for any \( \epsilon > 0 \) the \( g_R \)-balls \( B(x, \epsilon/2), B(y, \epsilon/2) \) have non-disjoint projections, and there exists some \( t_\epsilon \in \mathbb{R} \) such that \( \Phi_{t_\epsilon} (B(x, \epsilon/2)) \cap B(y, \epsilon/2) \neq \emptyset \). That is, there exists some integral curve \( \gamma \) of \( K \) such that \( \gamma(0) \in B(x, \epsilon/2) \) and \( \gamma(t_\epsilon) \in B(y, \epsilon/2) \). Therefore, as \( \Phi_{t_\epsilon} (B(x, \epsilon/2)) = B(\Phi_{t_\epsilon}(x), \epsilon/2) \), we have that \( y \in B(\Phi_{t_\epsilon}(x), \epsilon) \), and this implies that \( \pi(y) \in \pi(B(\Phi_{t_\epsilon}(x), \epsilon)) = \pi(B(x, \epsilon)) \) for all \( \epsilon > 0 \). As a consequence, \( Q \) is not a \( T_1 \) topological space, in contradiction with theorem A.1. \( \square \)

**References**

[1] Beem J K and Ehrlich P E 1979 The space-time cut locus *Gen. Rel. Grav.* 11 89–103
[2] Beem J K, Ehrlich P E and Easley K L 1996 Global Lorentzian Geometry Monographs Textbooks Pure Appl. Math. vol 202 (New York: Dekker)
[3] Bernal A N and Sánchez M 2003 On smooth Cauchy hypersurfaces and Geroch’s splitting theorem *Commun. Math. Phys.* 243 461–70
[4] Bernal A N and Sánchez M 2005 Smoothness of time functions and the metric splitting of globally hyperbolic spacetimes *Commun. Math. Phys.* 257 43–50
[5] Bernal A N and Sánchez M 2006 Further results on the smoothability of Cauchy hypersurfaces and Cauchy time functions *Lett. Math. Phys.* 77 183–97
[6] Bernal A N and Sánchez M 2007 Globally hyperbolic spacetimes can be defined as ‘causal’ instead of ‘strongly causal’ *Class. Quantum Grav.* 24 745–50
[7] Caponio E and Javaloyes M A 2007 Cauchy surfaces in conformally stationary spacetimes *Preprint* arXiv:0709.2437
[8] Caponio E, Javaloyes M A and Masiello A 2007 Variational properties of geodesics in non-reversible Finsler manifolds and applications *Preprint* math0702323v2
[9] Defrise-Carter L 1975 Conformal groups and conformally equivalent isometry groups *Commun. Math. Phys.* 40 273–82
[10] Harris S G 1992 Conformally stationary spacetimes *Class. Quantum Grav.* 9 1823–7
[11] Hawking S W and Ellis G F R 1973 The Large Scale Structure of Space-Time (Cambridge: Cambridge University Press)
[12] Minguzzi E and Sánchez M 2008 The causal hierarchy of spacetimes *Recent Developments in Pseudo-Riemannian Geometry* (ESI Lect. Math. Phys.) (Zurich: Eur. Math. Soc.) pp 299–358 (*Preprint* gr-qc/0609119) (in progress)
[13] O’Neill B 1983 Semi-Riemannian Geometry with Applications to Relativity (New York: Academic)
[14] Palais R 1961 On the existence of slices for actions of non-compact Lie groups *Ann. Math.* 73 295–323
[15] Penrose R 1972 Techniques of Differential Topology in Relativity (Philadelphia, PA: SIAM)
[16] Sachs R K and Wu H 1977 General Relativity for Mathematicians (Graduate Texts in Mathematics) (New York: Springer)
[17] Sachs R K and Wu H 1977 General relativity and cosmology *Bull. Am. Math. Soc.* 83 1101–64
[18] Sánchez M 1997 Structure of Lorentzian tori with a Killing vector field *Trans. Am. Math. Soc.* 349 1063–80
[19] Sánchez M 1999 Timelike periodic trajectories in spatially compact Lorentzian manifolds *Proc. Am. Math. Soc.* 127 3057–66
[20] Sánchez M 2005 Causal hierarchy of spacetimes, temporal functions and smoothness of Geroch’s splitting. A revision *Contemp. Math.* 28 127–55
[21] Sánchez M 2005 On the geometry of static space-times *Nonlinear Anal.* 63 e455–63
[22] Sánchez M and Senovilla J M M 2007 A note on the uniqueness of global static decompositions *Class. Quantum Grav.* **24** 6121–6

[23] Stephani H, Kramer D, MacCallum M, Hoenselaers C and Herlt E 2003 *Exact Solutions of Einstein’s Field Equations (Cambridge Monographs on Mathematical Physics)* 2nd edn (Cambridge: Cambridge University Press)