REALITY AND GEOMETRY OF STATES AND OBSERVABLES

IN QUANTUM THEORY

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ABSTRACT

The determination of the quantum state of a single system by protective observation is used to justify operationally a formulation of quantum theory on the quantum state space (projective Hilbert space) $P$. Protective observation is extended to a more general quantum theory in which the Schrödinger evolution is generalized so that it preserves the symplectic structure but not necessarily the metric in $P$. The relevance of this more general evolution to the apparent collapse of the state vector during the usual measurement, and its possible connection to gravity is suggested.

Some criticisms of protective observation are answered. A comparison is made between the determination of quantum states using the geometry of $P$ by protective measurements, via a reconstruction theorem, and the determination of space-time points by means of the space-time geometry, via Einstein’s hole argument. It is argued that a protective measurement may not determine a time average.

0. INTRODUCTION

Quantum mechanics was formulated using the beautiful geometry of Hilbert space [1]. However, much of this mathematical formalism is deemed unobservable. First, a state vector obtained by multiplying a given state vector by a complex number is physically indistinguishable from the given state vector. Hence, this complex number is unobservable. Second, if a particle such as an electron is in a state described by a wave function extended over a macroscopic region in space, and a measurement of the usual kind is made on the particle, e.g. let it interact with a photographic plate or cloud chamber, only a localized spot or trajectory is seen. It appears therefore that for some mysterious reason the wave function collapses to a localized wave packet. Such localized packets, which give rise to the events that constitute the world we observe, alone are observable by the usual measurements performed on a single particle.

A third aspect, which is related to the second, is that the wave function collapses to an eigenstate of the observable, represented by a Hermitian operator in the Hilbert space, in a usual measurement. Therefore, only these eigenstates are
regarded as “real” in this experiment, because a state \(|\psi\rangle\) which is not an eigenstate cannot be observed by this measurement performed on a single system. Only the probability of collapse into one of the eigenstates is predicted. The probability of transition to the state \(|\phi\rangle\) is \(|\langle \psi | \phi \rangle|^2\). This is observed and can only be observed by performing the measurement on an ensemble of identical systems. The expectation value of the observable for the state \(|\psi\rangle\) is traditionally interpreted as a statistical average of the possible outcomes of the measurement for this ensemble. Moreover, the observation of the transition probability determines the absolute value of the complex number \(\langle \psi | \phi \rangle\) but not its phase.

The first aspect of the present formulation of quantum theory implies that a physical state, or simply a state, must be represented by a ray, i.e. a one-dimensional subspace, of the Hilbert space \(\mathcal{H}\). The set of physical states, i.e. the set of rays of the Hilbert space, is called the projective Hilbert space and denoted by \(\mathcal{P}\).

Recently, it was shown that the physical state of a single particle may be observed by performing measurements of a new type called protective measurement \([2,3]\), which determines the expectation value of an observable in the given state instead of collapsing the state to one of the eigenstates of the observable as in the usual measurement. This manner of observing the wave function of a single system without changing it appreciably will be called a protective observation. It may be carried out by having the system as a non-degenerate eigenstate of the Hamiltonian and performing the measurements adiabatically. It has been extended to non-degenerate eigenstates and to many-particle systems \([4]\). This gets around the problem of collapse of the wave function in the usual measurement, which is the second aspect mentioned above. Also, it gives a new realistic meaning to the points in \(\mathcal{P}\), because it is no longer necessary to use an ensemble of identical systems in a given state in order to observe this state statistically as necessitated by the usual measurement. Instead, the state of a single system is observed.

As mentioned, a protective measurement determines the expectation value of the observable in the state being measured. It therefore extends the reality given to the eigenvalues of the observable being measured in the usual formulation of quantum theory to all expectation values of the observable. Hence, it overcomes the limitation in the observable quantities in the usual formulation mentioned as the third aspect above. Also it suggests that an observable may be represented by its expectation value, which is a real valued function of \(\mathcal{P}\), because it is this function which is determined by experiment.

Here I shall briefly review the protective measurement and describe a feasible experimental realization of it in section 1. A reconstruction theorem which assures that the expectation values contain all the kinematical and geometrical information of quantum mechanics is stated. Quantum mechanics is formulated in \(\mathcal{P}\), in section 2, using the natural symplectic structure defined by the inner product in the underlying Hilbert space. This leads to a natural generalization of quantum theory in section 3. In section 4, some criticisms of the use of protective measurement so far are shown to be not valid. It is shown, in section 5, that the determination of states by their relations to other states by means of protective measurements via the reconstruction theorem, is analogous to the determination of space-time points from a solution of gravitational field equations which are generally covariant. In section 6, I argue against the possibility that a protective measurement measures a time average of an observable of a single system, which would have enabled one to
give statistical meaning to this type of measurement.

1. PROTECTIVE OBSERVATION

The wave function undergoes two types of evolution in quantum theory, according to the usual Copenhagen interpretation. 1) It undergoes a continuous, linear, reversible, deterministic evolution according to Schrodinger’s equation when no “measurement” is being made. 2) It undergoes an apparently discontinuous, non linear, irreversible, indeterministic evolution during a “measurement”. i.e. the evolution (2) is the negation of (1) in every way. A “measurement” at present is defined only in a very vague sense as the quantum system coming into contact with a macroscopic apparatus, which is not precisely defined. This causes a “collapse” of the wave function, which is necessarily preceded by the combined system being in an entangled state as a result of the evolution (1). The outcome of this collapse can only be predicted probabilistically. This leads to the well known statistical interpretation of the wave function.

This state of affairs is unsatisfactory. But on the other hand measurement plays a very important role in physics, not only in obtaining information about the particular system being observed but also in providing operational meaning to the concepts of physics. This makes it imperative to look for a measurement that uses only the evolution (1), and avoids the collapse (2). Protective measurement, which will be described now, is such a measurement.

A protective measurement determines the expectation value of an observable for a wave function while it is prevented from collapse because of another interaction it undergoes at the same time. For example, the system may be protected by being prepared in an eigenstate of a Hamiltonian, and the measurement is made adiabatically. Then no entanglement takes place between the system and the apparatus. Therefore no collapse takes place either.

Some examples of protective observations were given in ref. 3. A protective observation may be made, in principle, on the polarization state of a photon, analogous to the protective observation of a spin 1/2 particle [3] as follows [5]: Send a circularly polarized photon through a crystal with large optical activity. This crystal has different refractive indices for the two orthogonal circularly polarized states. Make the crystal also have an elliptical birefringence which varies in space. This would have different refractive indices for the two corresponding orthogonal elliptically polarized states. Since the refractive indices vary in space, the two elliptically polarized states ordinarily get separated in space analogous to the usual Stern-Gerlach experiment. But because of the large optical activity, the circularly polarized state is protected, and this separation therefore does not take place. This is analogous to protecting the spin state of the neutron by a large magnetic field in the direction of the spin. But the trajectory of the center of mass of the photon is curved, because of the spatially varying elliptical birefringence, in a manner that is determined by its polarization state. By observing this trajectory, it would be possible to determine the polarization state of a single photon analogous to the observation of the spin state of a single neutron in the protective Stern-Gerlach experiment [3,4].

The expectation value of an observable $A$ will be denoted by $\langle A \rangle$, and its value for a given wave function $\psi$ is defined to be

$$\langle A \rangle(\psi) \equiv \frac{\langle \psi | A | \psi \rangle}{\langle \psi | \psi \rangle}.$$ (1.1)
It is clear that $\langle A \rangle$ is constant on any ray (one dimensional subspace of the Hilbert space), and therefore it may be regarded as a function of the set of rays, called the projective Hilbert space $\mathcal{P}$. By protectively measuring the expectation values of a sufficient number of such observables, the ray containing the wave function of a single system can be determined.

The term ‘expectation value’ is an unfortunate choice of terminology for (1.1) because it conveys the meaning that it is a statistical average. This was justified prior to the discovery of protective observation because the only way of giving physical meaning then to (1.1) was by making measurements on an ensemble of identical systems in the given state. But since protective measurement enables (1.1) to be determined directly for a single system, I shall refer to any particular value of (1.1) as an observed value. Also, the function of $\mathcal{P}$ defined by (1.1) will be called an observable.

It is a remarkable fact that the observables, whose values for physical states may be obtained in principle by an experimentalist via protective measurements without assuming any underlying Hilbert space structure, completely determine the Hilbert space underlying $\mathcal{P}$ and its geometry. Moreover, each observable then is given by (1.1) for a suitable Hermitian operator $A$ that acts on this Hilbert space. This follows from the following theorem [4]: From the observables given as functions on $\mathcal{P}$, regarded as an abstract set, the vector space structure and the inner product in the Hilbert space can be reconstructed. They are then respectively unique up to isomorphism and multiplication by a real positive constant. Also, after the reconstruction of the Hilbert space $\mathcal{H}$, each observable uniquely determines the corresponding Hermitian operator in $\mathcal{H}$ so that they satisfy (1.1) for each state.

Since the observed values can be measured non statistically for a single system using protective measurements, the above reconstruction theorem provides the states and observables with a new ontological meaning. This will be discussed in more detail in section 5. Historically, there has been a close affinity between such an operational approach and the geometrical approach to physics. For example, in the creation of special relativity, the realization by Einstein of the problem of simultaneity and that the $t$ and $t'$ in the Lorentz transformation which are measured by clocks in equivalent ways are on the same footing, led to Minkowski geometry. The probing of gravity by classical particles led via the equivalence principle to curved space-time geometry. So, there appears to be a metatheoretical principle that ensures that the operational approach naturally leads to a geometrical description. It should be of no surprise therefore that protective measurements, which determine the observed values as functions of $\mathcal{P}$ for a single system should lead naturally to a corresponding geometric reformulation of quantum mechanics on $\mathcal{P}$. This will be discussed in the next section.

2. QUANTUM MECHANICS IN THE QUANTUM STATE SPACE

The vector space structure of the underlying Hilbert space $\mathcal{H}$ endows the quantum state space $\mathcal{P}$ with a projective geometry. It was for this reason that $\mathcal{P}$ was called the projective Hilbert space [6]. The projective geometry of $\mathcal{P}$ may be defined, in the sense of Klein’s Erlanger program, as the set of properties that are invariant under the group of non singular linear transformations acting on $\mathcal{H}$. It may be viewed roughly as the geometry obtained from Euclidean geometry by removing the concepts of distance, angle and parallelism. For example, collinearity is a projective property.
The fundamental physical property of states which gives rise to quantum theory is the phenomenon of interference. Algebraically, this reflects the fact that a complex linear combination

$$|\psi> = c_1|\psi_1> + c_2|\psi_2>$$

(2.1)
of two state vectors $|\psi_1>$ and $|\psi_2>$ represents a new possible physical state. The relation (2.1), being invariant under linear transformations, is a projective property. The physical states, or rays to which $|\psi_1>$, $|\psi_2>$ and $|\psi>$ belong to, are said to be three collinear points in $P$, with respect to the projective geometry in $P$. The ratio $c_2/c_1$, which is not a projective invariant, may be regarded as the inhomogeneous coordinate of the physical state of $|\psi>$ in the coordinate system on this line determined by the physical states of $|\psi_1>$ and $|\psi_2>$. Conversely, given any three collinear points on $P$, each of the three states may be regarded as an appropriate superposition of the other two states. This gives a geometrical meaning to quantum interference.

Two more important geometrical structures in $P$ are determined by the inner product in $H$. These will be introduced from a physical point of view in the remainder of this section. This inner product between any two state vectors has the physical meaning of the probability amplitude for the transition between the two corresponding state vectors. Its absolute value squared has a direct physical meaning as the probability of transition when the usual measurement is performed, which is given physical meaning by means of measurements performed in an ensemble of identical systems. But the reconstruction theorem stated in section 1 shows that the entire probability amplitude, and not just its absolute value, may be given a physical meaning associated with a single system by means of protective measurements.

Another physical meaning for the inner product is provided by the geometric phase [7,6]. Suppose a state vector undergoes cyclic evolution, i.e. the initial and final state vectors are the same except for a phase. This phase contains a geometric part, the geometric phase, that depends only on the motion of the state that is given by an unparametrized closed curve $\gamma$ in $P$ and not on which of the infinite class of possible Hamiltonians actually evolves the system along this curve. Choose $|\psi>$ belonging to the points in a neighborhood containing $\gamma$ as a differentiable normalized function of this neighborhood. Then the geometric phase associated with $\gamma$ is [6]

$$\beta = \oint_\gamma i <\psi|d\psi>.$$  

(2.2)

Now $\exp(i\beta)$ is the holonomy transformation when a state vector $|\psi(s)>$ is parallel transported along $\gamma$ according to the rule that the inner product between neighboring states $<\psi(s)|\psi(s + ds)>$ is to first order real and positive [8]. For this reason it provides a fairly direct measurement of the inner product, including its phase. But $\beta$ is also the symplectic area of any surface spanned by $\gamma$ with respect to a natural symplectic structure in $P$ defined by the inner product in $H$ [9,10,11]. So, again, seeking a direct physical meaning for mathematical structures in $H$ has led us to a geometrical structure in $P$.

I shall now express this symplectic structure [12] in a language familiar to physicists [13]. Let $\tilde{\psi}_j$ be the components of $|\tilde{\psi}>$ with respect to an orthonormal basis. Then $\psi_j$ may be regarded as homogeneous coordinates in $P$. Define $Q^j = \tilde{\psi}_j$ and $P_j = i\tilde{\psi}_j$, where the * denotes complex conjugation. Then (2.2) may be rewritten as

$$\beta = \oint_\gamma P_j dQ^j = \int_\Sigma dP_j \wedge dQ^j,$$

(2.3)
where $\Sigma$ is a surface spanned by $\gamma$. Specifying a symplectic structure in $\mathcal{H}$ is equivalent to specifying a preferred class of canonical coordinates $Q^{i}$ and momenta $P_{j}$ using which the experimentally observable $\beta$ may be expressed in the form (2.3) for any cyclic evolution. The above choice is a special case for which the coordinates and momenta are complex. The 2-form $\Omega = dP_{j} \wedge dQ^{i}$ is called the symplectic 2-form which equivalently determines the symplectic structure. But (2.2) and therefore (2.3) is independent of the choice of $|\tilde{\psi}\rangle$. Therefore, we have a symplectic structure in $\mathcal{P}$ because the canonical invariant (2.3) depends only on $\mathcal{P}$. If $\mathcal{H}$ has $N+1$ complex dimension then there are $2N+2$ real coordinates in $Q^{i}$ and $P_{j}$. But the normalization of $|\tilde{\psi}\rangle$ and the choice of phase for $|\tilde{\psi}\rangle$, e.g. $\tilde{\psi}_{1}$ may be chosen to be real, implies that only $2N$ of these coordinates are independent. This makes $\mathcal{P}$, which has $N$ complex dimensions, a $2N$ real dimensional phase space.

All of quantum mechanics may be formulated entirely on $\mathcal{P}$ using the above symplectic structure. Suppose $f$ and $g$ are two functions on $\mathcal{P}$. The Poisson bracket between them is defined in the usual way:

$$\{f,g\} = \frac{\partial f}{\partial Q^{i}} \frac{\partial g}{\partial P_{j}} - \frac{\partial f}{\partial P_{j}} \frac{\partial g}{\partial Q^{i}},$$

(2.4)

with summation over repeated indices. Given two observables $<A>$ and $<B>$, which are functions of $\mathcal{P}$, formed from the operators $A$ and $B$, it is easily verified that

$$\{<A>,<B>\} = <[A,B]>.$$  

(2.5)

The Hamiltonian observable is $<H>$ which corresponds to the Hermitian Hamiltonian operator $H$. It is convenient to choose $|\psi\rangle$ in the definition of $<H>$ to be normalized. Then Schrodinger’s equation is equivalent to the Hamilton’s equations

$$\frac{dP_{i}}{dt} = -\frac{\partial <H>}{\partial Q^{i}}, \quad \frac{dQ^{i}}{dt} = \frac{\partial <H>}{\partial P_{i}}.$$  

(2.6)

For the complex $Q^{i}$ and $P_{j}$ chosen above, the two equations (2.6) are Schrodinger’s equation and its complex conjugate. Alternatively, we may choose $Q^{i} = \sqrt{2}\text{Re} \tilde{\psi}_{j}$ and $P_{j} = \sqrt{2}\text{Im} \tilde{\psi}_{j}$, which are real, in which case the two equations (2.6) are independent and represent the real and imaginary parts of Schrodinger’s equation.

On choosing local canonical coordinates $X^{A} = (P_{i},Q^{j})$ and defining $h = <H>$, (2.6) may be recast in the more compact form

$$\frac{dX^{A}}{dt} = \Omega^{AB} \frac{\partial h}{\partial X^{B}},$$  

(2.6')

where $\Omega^{AB}$ is the inverse of the symplectic 2-form $\Omega_{AB}$. The Poisson bracket (2.4) may now be rewritten as

$$\{f,g\} = \Omega^{AB} \frac{\partial f}{\partial X^{A}} \frac{\partial g}{\partial X^{B}}.$$  

(2.4')

Then, it follows from (2.6') that the time evolution of an arbitrary observable $a$ is given by

$$\frac{da}{dt} = \{a,h\}.$$  

(2.7)

An advantage of formulating quantum theory in terms of (2.7) is that it abolishes the distinction between the Schrodinger and Heisenberg pictures. Because in both pictures the
'expectation values' of the observable operators evolve in the same way, given by (2.7).

The inner product also gives the metric on $\mathcal{H}$

$$d\sigma^2 = \sum_k \{(dQ^k)^2 + (dP_k)^2\} = \delta_{AB} dX^A dX^B. \tag{2.8}$$

The requirement $\langle \tilde{\psi} | \tilde{\psi} \rangle = 1$, i.e.

$$\sum_k \{(Q^k)^2 + (P_k)^2\} = 1, \tag{2.9}$$

defines the unit sphere $S$ which is a submanifold of $\mathcal{H}$. The set of states $\{e^{i\theta} | \tilde{\psi} \}$ of $S$ for a given $|\tilde{\psi} \rangle$ is a point of $P$. So, $S$ is a principal fiber bundle over $P$ with structure group $U(1)$ corresponding to this multiplication by $e^{i\theta}$. A connection in this bundle may be defined by specifying the horizontal spaces to be normal to the fibers with respect to the metric (2.8).

If $\Pi$ is the projection map of the bundle $S$ into $P$, then it is natural to define a metric on $P$ by requiring that $\Pi_*$ when restricted to each horizontal space is an isometry with respect to this metric. This is a Kahler metric on $P$, called the Fubini-Study metric. More explicitly, the metric (2.8) when restricted to $S$ is [14]

$$d\sigma^2 = (d\phi - A)^2 + dS^2, \tag{2.10}$$

where $\phi$ is the phase of $\tilde{\psi}_1$, $A = i < \tilde{\psi} | d\tilde{\psi} >$ and $dS^2$ is the metric on the horizontal space, which is the same as the Fubini-Study metric on $P$ by the above construction. A horizontal vector is annihilated by the connection 1-form $A$. By integrating this equation, the geometric phase (2.2) may be obtained as the holonomy associated with the closed curve $\gamma$ of $P$.

Suppose $\tilde{\psi}_k$ are chosen to be the components with respect to an orthonormal basis of eigenstates of the Hamiltonian with eigenvalues $\omega_k$. Also, we may choose $\phi = 0$. Then $X^A$ may be regarded as local coordinates on $P$. The Hamiltonian function on $P$ is

$$h = \frac{1}{2} \sum_k \{(Q^k)^2 + (P_k)^2\} \omega_k. \tag{2.11}$$

This is like the Hamiltonian for a set of non interacting harmonic oscillators. The Schrodinger time evolution, generated by a Hermitian Hamiltonian is an isometry in $P$ with respect to the Fubini-Study metric [15]. The evolution of a given initial state is a curve in $P$. It is a remarkable fact that the time integral of the uncertainty of energy along this curve is independent of which Hamiltonian is used for this evolution. Therefore, like $\beta$, it must have a geometrical meaning but now even when the curve is not closed. It is in fact the distance along the curve measured by the Fubini-Study metric.

To conclude, the inner product in $\mathcal{H}$ gives a symplectic structure and a Kahler metric in $P$, both of which can in principle be generalized to obtain physical theories more general than the present day quantum theory.

3. POSSIBLE GENERALIZATION OF QUANTUM THEORY AND THE MEASUREMENT PROBLEM

The above geometric reformulation of quantum theory using the symplectic structure suggests a natural generalization of quantum theory [12]. We may
would allow for time evolutions which are non unitary and non linear in Hamiltonian operator. I.e. $h$ may be more general than the function (2.11). This would allow for time evolutions which are non unitary and non linear in $H$.

This generalization of the Hamiltonian evolution also leads naturally to a generalization of an observable $a$ as any real differentiable function on $P$ that need not have a representation of the form (1.1). Such an observable may in principle be observed by protective measurements as follows. Couple the system to an observable $q$ of an apparatus, for example its ‘pointer position’, so that the Hamiltonian of the combined system is

$$h = h_0 + g(t)qa + h_A,$$

(3.1)

where $h_0$ and $h_A$ are the unperturbed Hamiltonians of the system and the apparatus respectively, and $g(t)$ represents the turning on and off of the interaction between them during a time interval, say $[0,T]$. Suppose, for simplicity, that $h_0 = \langle \psi | H_0 | \psi \rangle$ and $h_A = \langle \psi | H_A | \psi \rangle$, where $H_0$ and $H_A$ are Hermitian operators, and $\langle \psi | \psi \rangle = 1$. I.e. $h_0$ and $h_A$ are like the observables in ordinary quantum theory, but $a$ is assumed to be a more general observable. Let $|n\rangle$ be the eigenstates of $H_0$ with eigenvalues $\omega_n$. An arbitrary evolution of the system may be represented by the state vector

$$|\psi(t)\rangle = \sum_j c_j(t) \exp(-i\omega_j t)|j\rangle.$$  

(3.2)

I shall choose complex coordinates with respect to the basis $|n\rangle$, as defined in section 2. So, the evolution of the system, in terms of these coordinates, is given by $Q^j(t) = c_j(t) \exp(-i\omega_j t), P_j(t) = ic_j(t) \exp(i\omega_j t)$. Then, substituting this in the generalized Schrodinger equation (2.6′) yields

$$c_j(T) = q \int_0^T g(t) \exp(i\omega_j t) \frac{\partial a}{\partial P_j}.$$  

(3.3)

Suppose now that the interaction Hamiltonian is small compared to the protection Hamiltonian contained in $h_0$. Then first order time dependent perturbation theory may be used. This consists in approximating the $Q^j$ and $P_j$ in the right hand side of (3.3) by their unperturbed values. If the initial state vector of the system was $|s\rangle$, then these unperturbed values are $Q^j = \delta_{js} \exp(-i\omega_s t), P_j = i\delta_{js} \exp(i\omega_s t)$. Now if, as in the usual quantum mechanics, $a = \langle \psi | A | \psi \rangle = -i \sum_{j,k} P_j < j | A | k > Q^k$ for some Hermitian operator $A$, then the unperturbed value of $\frac{\partial a}{\partial P_j}$ is $-i < j | A | s > \exp(-i\omega_s t)$. Owing to the reality of $a$, the same time dependence may be assumed for the present more general case of $a$ being an arbitrary real differentiable function of $P$. Then from (3.3), $c_j(T)$ is proportional to the Fourier component of $g(t)$ corresponding to the transition frequency $\omega_j - \omega_s$.

So, if $\omega_s$ is a non degenerate eigenvalue and $g(t)$ is sufficiently slowly varying then these Fourier components for $j \neq s$ are negligible. Then the system will remain in the original state $|s\rangle$, without entanglement with the states of the apparatus. This entanglement is a prerequisite for the collapse of the state vector. Therefore, there will be no collapse in this case. But as pointed out [4], neither the non degeneracy nor the adiabaticity assumption is necessary to ensure this non entanglement. It is sufficient if $g(t)$ has negligible Fourier components corresponding
to all the transition frequencies, as shown also by the above treatment. If this is not satisfied then there would be transition to other energy eigenstates of $H_0$, which would result in entanglement.

In the present case in which there is no entanglement, a protective measurement of $a$ may be made by measuring the change in the observable $p$ of the apparatus conjugate to $q$, i.e. $\{p, q\} = -1$. The equation of motion for $p$ is given by (2.7) to be
\[
\frac{dp}{dt} = -g(t)a.
\]
Therefore, the change in $p$ is $\Delta p = -g_0 a$, where $g_0 \equiv \int_0^T g(t)dt$ is known. Hence, by measuring $\Delta p$, $a$ may be determined for a single system. So, the present generalized quantum mechanics permits the determination of the more general observables for a single system, in principle. This generalizes protective measurement to the measurement of these generalized observables.

In particular, the generalized hamiltonian $h$ in the present generalization of quantum theory may be observed, in principle, by means of protective measurements on a single system. This together with the possibility, in principle, of protectively observing the quantum state at any given time would justify regarding the state and its time evolution to be associated with a single system. The question arises if such more general time evolutions would account for the collapse of the state vector of individual systems which appears to occur in a non-linear manner.

The possible connection between this question and quantum gravity may be seen from the following consideration. From a physical point of view, the arena for the geometry of quantum theory is $P$ as I have argued in section 2. This is very different from the arena of space-time for the geometry of general relativity. But contact is made between the two geometries when a measurement is made on a particle. Because then the localization of the state vector, as it collapses, provides an approximate event which is a point of the space-time manifold that incorporates gravity. So the collapse of the wave function provides a connection between quantum theory and gravity, even at an energy scale which is far below the scale at which high energy physicists expect quantum gravitational effects to become important. This argument suggests that the scheme of Penrose [16] and others to explain the collapse of the state vector using quantum gravity is worth examining carefully.

4. REPLY TO CRITICISMS OF PROTECTIVE OBSERVATION

As originally formulated [2,3], in a protective observation the wave function of a particle is protected by an experimentalist who then gives it to another observer. This observer is only told that the wave function is protected but not what the protecting Hamiltonian is. She then determines the wave function by measurements on the given single particle. This has the novel aspect that the wave function is determined for a single particle and not an ensemble of identical particles as in all previous measurements. Nevertheless, this has been criticized by several physicists on the following grounds: (a) The experimentalist who protects the wave function has at least a partial knowledge of the wave function and therefore it cannot be said that the subsequent measurement by the observer determines a completely unknown wave function [17,18]. (b) If the wave function already comes protected, then the experimentalist performing the subsequent protective measurement is playing a passive role in that she is not providing the protection [19].

But the purpose of doing a measurement is to (i) determine the state of
the system and (ii) determine the manifestation of this state in order to infer its physical meaning. Protective measurements show the manifestation of the wave function of a single particle, which has not been done before. By this I mean that the observed (expectation) values of not necessarily commuting Hermitian operators may be measured by protective measurements on a single particle. From these measurements of sufficiently large number of observables the wave function can be reconstructed for this particle. Using the usual measurements, the expectation value could be given physical meaning only as a statistical average of measurements on an ensemble of particles. So, protective measurement, unlike the usual measurement, fulfills the goal (ii) for a single particle.

Another criticism was that the registration in the final stage of the protective measurement constitutes a measurement of the usual type involving an irreversible amplification and possibly a collapse, and therefore the protective measurement does not circumvent the measurement problem [20]. But the state being protectively measured does not become entangled with the apparatus state. Therefore, the collapse of the apparatus state does not affect the protected state. An ensemble of apparatus may be used to measure the apparatus state as in the usual measurement, but they need to be coupled only to one observed system whose state, being protected, does not collapse.

For example, in the usual Stern-Gerlach experiment of a spin 1/2 particle, quantum theory predicts an entangled state which corresponds to two spots. After the irreversible amplification by the detector D, only one spot is observed. Hence the entangled state is collapsed, in the Copenhagen interpretation, to have agreement with observation. The search for a description or explanation of the collapse is the measurement problem. But in a protective Stern-Gerlach experiment [3] only one spot appears whose position, which is between the two spots of the usual measurement, is determined by the spin state. So, there is no entanglement. The inclusion of the detector D does not change this in any way, because during the resulting irreversible amplification, the combined system is still evolving in a non entangled state corresponding to one spot only. I.e. D has only one possible “pointer reading” according to Schrödinger’s equation, and therefore there is no need to collapse the wave function. The formation of the single spot takes place exactly in the same way as the formation of either of the two possible spots in the usual Stern-Gerlach experiment, but the entanglement of the latter two spots is avoided.

The above criticisms (a) and (b) have been completely overcome by the reconstruction theorem [4], stated in section 1, that from the observed values obtained as functions of the abstract set of states $\mathcal{P}$, the Hilbert space together with the inner product can be constructed. Moreover, the operators whose observed values were experimentally obtained are then uniquely determined as Hermitian operators in this Hilbert space. Now the physical and geometric meaning for the various possible states are obtained by the observed values and the inner products between state vectors. Therefore, the same experimentalist may both protect and measure each state without knowing prior to the measurement anything about the structure of $\mathcal{P}$ or the underlying Hilbert space, or the observable operators associated with the apparatus used to make the measurements. Then the above mentioned reconstruction theorem enables the meaning of the states to be determined in a wholistic manner. Since the protective measurements may all be done on a single system, the states
and observables may therefore also be associated with a single system.

5. COMPARISON WITH EINSTEIN’S (W)HOLISTIC ARGUMENT

The wholistic manner in which protective measurements determine the state of the system, described in section 1, may be compared with the resolution of a paradox due to Einstein. In 1913, Einstein argued that if the gravitational field equations were generally covariant then for the gravitational field inside a hole of some given distribution of matter, there would be infinite number of solutions related by diffeomorphisms [21]. This is unlike in the electromagnetic case in which there is a unique solution for the electromagnetic field inside a hole surrounded by a given charge distribution. So, Einstein concluded incorrectly that the field equations for gravity cannot be generally covariant, which delayed the discovery of general relativity by two years.

In 1915, Einstein realized that this argument is not valid for the following reason: The points of space-time should be determined operationally as “spatiotemporal coincidences”. Such a specification of points, as an intersection of two world-lines, is invariant under diffeomorphisms. This implies that a space-time point may not be given reality as an independent isolated object. It may however be given reality wholistically using the geometry determined by the metric. E.g. a given point inside the hole may be determined by the distances along the geodesics joining this point to identifiable material points on the boundary of the hole. Then the infinite number of solutions related by diffeomorphisms correspond to the same objective physical geometry.

Similarly, the physical states in quantum theory, which are points of the projective Hilbert space, should be determined by their relations to other points through the geometry of the projective Hilbert space. The reconstruction theorem stated in section 1 shows how the geometry of $\mathcal{P}$ may be determined by protective measurements on an individual system. Then each state acquires physical meaning through its relationship to other states which is determined by this geometry.

This way of determining the state may be compared with someone being shipwrecked in a small island. To determine where this island is it would be necessary to know its location with respect to other lands. Similarly, to know what the state is, one needs to know its relation with other states. I.e. the structure of the projective Hilbert space needs to be determined.

It is better to give a single particle interpretation right from the very beginning to the Hilbert space by means of the reconstruction theorem, rather than to first construct the Hilbert space on the basis of usual measurements which need to be done on ensembles of identical systems, and then to show how the state of a single particle may be protectively observed. Because in the latter case, which was originally done, the Hilbert space machinery used in reconstructing the state used ensembles and therefore it could be argued that the reconstructed state cannot be associated with a single system.

6. DOES PROTECTIVE MEASUREMENT DETERMINE A TIME AVERAGE?

Let us examine the possibility that in a protective measurement of the expectation value of an observable $A$ for a single system what is obtained is a time average of $A$, which is equal to the ensemble average. This would be like following
the motion of an individual molecule in a classical gas. If we follow it long enough, its time averaged properties would be like as if it takes all the positions and velocities of the molecules at a given time. This is why in classical statistical mechanics the time average equals the ensemble average. If a similar situation exists for a protective measurement, then it would not remove the statistical element of quantum theory.

It was shown in ref. 3, however, that it is not possible to regard the outcome of a protective measurement for a particle in a box as a time average of a classical variable. Consider now whether a general protective measurement of an observed value may be a time average of the quantum mechanical state treated ontologically. Then the system with state vector $|\psi>$ should pass through each of the eigenstate vectors $|\psi_i>$ of the observable $A$ spending times at these states proportional to $|<\psi|\psi_i>|^2$, in order that the time average equals the ensemble average. This would require a drastic modification of quantum theory. In any case, this possibility may be precluded by monitoring the state during the measurement and ensuring that it is in the state corresponding to $|\psi>$ and so does not pass through the states $|\psi_i>$.

But it may be argued that the observed value is the time average of possible outcomes of measurement of the state that is treated epistemologically. This means that the state may be exactly as predicted by Schrödinger’s equation, but that it does not describe a real process. Its meaning is only that it predicts the probabilities of possible outcomes of measurement. These probabilities may be given physical meaning as the average of measurements over an ensemble of identical systems having this state at a given time or as a time average of measurements performed on a single system that is protected over a long period of time.

This view is not tenable, however, for the following reason. A long period of time $T$ here means that

$$T >> \frac{\hbar}{\Delta E},$$

where $\Delta E$ is the smallest of the energy differences between the protected state and the other energy eigenstates. But there are an infinite number of Hamiltonians, with infinite values for $\Delta E$, which have the protected state as an eigenstate with the same eigenvalue. The time evolution of the protected state is the same for all these Hamiltonians. But the time scale which determines how long the measurement should be, namely $\frac{\hbar}{\Delta E}$, is different for these Hamiltonians. It is therefore not a property of the state being observed protectively. Indeed, $T$ may be made as small as one pleases, in principle, while satisfying (6.1) by making $\Delta E$ correspondingly large. Hence, the outcome of a protective measurement cannot be regarded as the time average of a property of the state.

Also, as mentioned in section 3, adiabaticity is not needed for protective measurement, provided the interaction with the apparatus does not cause transition between the original and neighboring energy eigenstates. It is not possible to claim, in such a non adiabatic protective measurement, that a time average is being measured.

This appears to undermine the usual Copenhagen probabilistic interpretation of the wave function which was developed on the basis of the usual measurement that results in a collapse of the wave function. Because probabilities can be given physical meaning only by means of an ensemble of identical systems. Whereas, protective measurements, which were unknown to the founders of quantum theory, enables the wave function of a single system to be determined up to phase. The problems associated with the usual measurement may be indicative of new physics
needed for its understanding. This new physics may then negate the meaning given to the wave function on the basis of our present understanding or lack of understanding of the usual measurement. It may therefore be safer to interpret the wave function by means of the protective measurement for which the physics is well understood.

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