On the intersection of solvable Hall subgroups in finite simple exceptional groups of Lie type

Evgeny. P. Vdovin

Abstract

Assume that a finite almost simple group with simple socle isomorphic to an exceptional group of Lie type possesses a solvable Hall subgroup. Then there exist four conjugates of the subgroup such that their intersection is trivial.

Keywords: almost simple group, base size, solvable Hall subgroup.

Introduction

Throughout the paper the term “group” we always use in the meaning “finite group”. We use symbols $A \leq G$ and $A \triangleleft G$ if $A$ is a subgroup of $G$ and $A$ is a normal subgroup of $G$ respectively. If $\Omega$ is a (finite) set, then by $\text{Sym}(\Omega)$ we denote the group of all permutations of $\Omega$. We also denote $\text{Sym}\{1, \ldots, n\}$ by $\text{Sym}_n$. Given $H \leq G$ by $H_G = \cap_{g \in G} H^g$ we denote the kernel of $H$.

Assume that $G$ acts on $\Omega$. An element $x \in \Omega$ is called a $G$-regular point, if $|xG| = |G|$, i.e., if the stabilizer of $x$ is trivial. We define the action of $G$ on $\Omega^k$ by

$$g : (i_1, \ldots, i_k) \mapsto (i_1g, \ldots, i_kg).$$

If $G$ acts faithfully and transitively on $\Omega$, then the minimal $k$ such that $\Omega^k$ possesses a $G$-regular point is called the base size of $G$ and is denoted by $\text{Base}(G)$. For every natural $m$ the number of $G$-regular orbits on $\Omega^m$ is denoted by $\text{Reg}(G, m)$ (this number equals 0 if $m < \text{Base}(G)$). If $H$ is a subgroup of $G$ and $G$ acts on the set $\Omega$ of right cosets of $H$ by right multiplications, then $G/H_G$ acts faithfully and transitively on $\Omega$. In this case we denote $\text{Base}(G/H_G)$ and $\text{Reg}(G/H_G, m)$ by $\text{Base}_H(G)$ and $\text{Reg}_H(G, m)$ respectively. We also say that $\text{Base}_H(G)$ is the base size of $G$ with respect to $H$. Clearly, $\text{Base}_H(G)$ is the minimal $k$ such that there exist elements $x_1, \ldots, x_k \in G$ with $H^{x_1} \cap \ldots \cap H^{x_k} = H_G$. Thus, the base size of $G$ with respect to $H$ is the minimal $k$ such that there exist $k$ conjugates of $H$ with intersection equals $H_G$.

We prove the following theorem in the paper.

Theorem 1. (Main Theorem) Let $G$ be an almost simple group with simple socle isomorphic to an exceptional group of Lie type. Assume also that $G$ possesses a solvable Hall subgroup $H$. Then $\text{Base}_H(G) \leq 4$.

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The following results were obtained in this direction. In 1966 D.S.Passman proved (see [1]) that a $p$-solvable group possesses three Sylow $p$-subgroups whose intersection equals the $p$-radical of $G$. Later in 1996 V.I.Zenkov proved (see [2]) that the same conclusion holds for arbitrary finite group $G$. In [3] S.Dolfi proved that in every $\pi$-solvable group $G$ there exist three conjugate $\pi$-Hall subgroups whose intersection equals $O_\pi(G)$ (see also [4]). Notice also that V.I.Zenkov in [5] constructed an example of a group possessing a solvable $\pi$-Hall subgroup $H$ such that the intersection of five conjugates of $H$ equals $O_\pi(G)$, while the intersection of every four conjugates of $H$ is greater than $O_\pi(G)$. In [6] Theorem 1] the following statement is proven.

**Theorem 2.** Let $G$ be a finite group possessing a solvable $\pi$-Hall subgroup $H$. Assume that for every simple component $S$ of $E(G)$ of the factor group $G = G/S(G)$, where $S(G)$ is a solvable radical of $G$, the following condition holds:

for every $L$ such that $S \leq L \leq \text{Aut}(S)$ and contains a solvable $\pi - \text{Hall subgroup} M$,

the inequalities Base$_M(L) \leq 5$ and Reg$_M(L, 5) \geq 5$ hold.

Then Base$_H(G) \leq 5$ and Reg$_H(G, 5) \geq 5$.

Moreover, at the beginning of the proof of Theorem 2 from [6] the following statement is obtained.

**Lemma 3.** If, for a group $G$ and its subgroup $H$, the inequality Base$_H(G) \leq 4$ holds, then Reg$_H(G, 5) \geq 5$.

Thus by Theorems [1] and [2] Lemma [3] and [6] Theorem 2] we immediately obtain

**Theorem 4.** Let $H$ be a solvable $\pi$-Hall subgroup of $G$. Assume that each nonabelian composition factor of the socle of $G/S(G)$, where $S(G)$ is the solvable radical of $G$, is isomorphic to either alternative, or sporadic, or exceptional group of Lie type. Then Base$_H(G) \leq 5$, i.e., there exist elements $x, y, z, t$ of $G$ such that the identity

$$H \cap H^x \cap H^y \cap H^z \cap H^t = O_\pi(G)$$

holds.

1 Notations and preliminary results

Throughout by $\pi$ a set of primes is denoted, while by $\pi'$ we denote its complement in the set of all primes. A subgroup $H$ of $G$ is called a $\pi$-Hall subgroup, if the order $|H|$ is divisible by primes from $\pi$ only, while its index $|G : H|$ is divisible by primes from $\pi'$ only. The set of all $\pi$-Hall subgroups of $G$ is denoted by Hall$_\pi(G)$. A subgroup $H$ of $G$ is called a $\text{Hall subgroup}$, if its order $|H|$ and the index $|G : H|$ are coprime. A group $G$ is called almost simple, if there exists a nonabelian simple group $S$ such that $F^*(G) = S$, where $F^*(G)$ is the generalized Fitting subgroup of $G$. In other words, $G$ is called almost simple, if there exists a simple group $S$ such that $S \cong \text{Im}(S) \leq G \leq \text{Aut}(S)$.

**Lemma 5.** [7] Lemma 1] Let $G$ be a finite group and $A$ be its normal subgroup. If $H \in \text{Hall}_\pi(G)$, then $H \cap A \in \text{Hall}_\pi(A)$ and $HA/A \in \text{Hall}_\pi(G/A)$.

**Lemma 6.** [8] Let $A$ be an abelian subgroup of a finite group $G$. Then there exists $x \in G$ such that $A \cap A^x \leq F(G)$.
Combining known results (see [11, Theorems 8.3–8.7]), we obtain the following

**Lemma 7.** Let $G$ be a simple group of Lie type over a field of characteristic $p \in \pi$ and $H$ be its solvable $\pi$-Hall subgroup. Then either $H$ is included in a Borel subgroup of $G$, or one of the following holds:

1. $G = SL_3(2)$ or $G = SL_3(3)$ and $H$ is the stabilizer of a line or of a plain in the natural 3-dimensional module, i.e., there exist two classes of conjugate $\pi$-Hall subgroups in this case.

2. $G = SL_4(2)$ or $G = PSL_4(3)$ and $H$ is the stabilizer of a two-dimensional subspace of the natural 4-dimensional module.

3. $G = SL_5(2)$ or $G = SL_5(3)$ and $H$ is the stabilizer of a chain of subspaces $V_0 < V_1 < V_2 < V_3 = V$ whose codimensions are in the set $\{1, 2\}$ (i.e., two codimensions equal 2 and one codimension equals 1). There exist three classes of conjugate $\pi$-Hall subgroups in this case.

We recall some known technical results (see [10]). If $G$ acts transitively on the set $\Omega$, then given $x \in G$ by fpr$(x)$ we denote the fixed point ratio of $x$, i.e. fpr$(x) = |\text{fix}(x)|/|\Omega|$, where \text{fix}(x) = $\{\omega \in \Omega \mid \omega^x = \omega\}$. If $G$ acts transitively and $H$ is a point stabilizer, then the following formulae is known

\[
\text{fpr}(x) = \frac{|x^G \cap H|}{|x^G|}.
\]  

As it is noted in [11, Theorem 1.3], the base size can be bounded by using the following arguments. Assume that $G$ acts faithfully and let $Q(G, c)$ denote the probability that arbitrary chosen element of $\Omega^c$ is not a $G$-regular point. Clearly, Base$(G)$ is the minimal $c$ such that $Q(G, c) < 1$. In particular, if $Q(G, c) < 1$ then Base$(G) \leq c$. Clearly, an element of $\Omega^c$ is not a $G$-regular point if and only if it is stable under the action of an element $x$ of prime order. Notice also that the probability for arbitrary chosen element of $\Omega^c$ to be stable under $x$ is not greater than fpr$(x)^c$. Denote by $\mathcal{P}$ the set of elements of $G$ whose order is equal to a prime number. Let $x_1, \ldots, x_k$ be representatives of the conjugacy classes of elements from $\mathcal{P}$. Since $G$ acts transitively, the formulae [11] shows that fpr$(x)$ does not depend on the choice of the representative of a conjugacy class. Thus the following chain of inequalities holds.

\[
Q(G, c) \leq \sum_{x \in \mathcal{P}} \text{fpr}(x)^c = \sum_{i=1}^{k} |x_i^G| \cdot \text{fpr}(x_i)^c =: \hat{Q}(G, c).
\]  

In particular, we can use the upper bound for fpr$(x)$ in order to bound $\hat{Q}(G, c)$ and so to bound $Q(G, c)$. The following lemma is the main technical tool for this bound.

**Lemma 8.** [11, Proposition 2.3] Let $G$ be a transitive group of permutations on $\Omega$ and $H$ be a point stabilizer. Assume that $x_1, \ldots, x_k$ are representatives of distinct conjugacy classes such that the inequalities $\sum_i |x_i^G \cap H| \leq A$ and $|x_i^G| \geq B$ hold for all $i = 1, \ldots, k$. Then the inequality

\[
\sum_{i=1}^{k} |x_i^G| \cdot \text{fpr}(x_i)^c \leq B(A/B)^c
\]

holds for every $c \in \mathbb{N}$.

Notice that for every subgroup $H$ and every set $x_1, \ldots, x_k$ not containing the identity element the bound $\sum_i |x_i^G \cap H| < |H|$ holds.
2 Technical results

Our notations for groups of Lie type agree with that of [12]. In particular, for every simple group of Lie type $S$ over a field of characteristic $p$ we fix a simple algebraic group $\overline{G}$ of adjoint type and a Steinberg map $\sigma$ so that $S = O^p(\overline{G}_\sigma)$. Then $\overline{G}_\sigma$ is the group of inner-diagonal automorphisms of $S$ (we denote the group of inner-diagonal automorphisms of $S$ by $\hat{S}$). We assume that a Borel $\overline{B}$ and its maximal torus $\overline{T}$ are chosen $\sigma$-invariant, and we denote $\overline{B}_\sigma$ and $\overline{T}_\sigma$ by $B$ and $T$ respectively. Recall that if $S \in \{2A_n(q), 2D_n(q), 2E_6(q)\}$, then the definition field of $S$ equals $F_{q^2}$, if $S = 3D_4(q)$, then the definition field of $S$ equals $\mathbb{F}_{q^3}$, and the definition field of $S$ equals $\mathbb{F}_q$ in the remaining cases. For groups $2A_n(q), 2D_n(q), 2E_6(q)$ we also use the notations $A_n^- (q), D_n^- (q), E_6^- (q)$ respectively. Notice also the known fact: $Z(\overline{B}) \cap \overline{T} = Z(\overline{G}) = 1$, if $\overline{G}$ is of adjoint type and $\overline{Z}(B) \cap \overline{T} = Z(S)$ ($= 1$, if $\overline{G}$ is of adjoint type).

Lemma 9. Let $G$ be a group of inner-diagonal automorphisms of a finite simple group of Lie type over a field of characteristic $p$ (i.e. $G = \overline{G}_\sigma$ for some connected simple algebraic group $\overline{G}$ of adjoint type over an algebraically closed field of characteristic $p$ and a Steinberg map $\sigma$). Let $B = U \times T$ be a Borel subgroup of $G$, where $U$ is a maximal unipotent subgroup of $G$ and $T$ is a Cartan subgroup of $G$. We denote the subgroup of monomial matrices containing $T$ by $N$ so that $N/T \simeq W$ is the Weyl group of $G$. Let $w_0 \in W$ be the unique element that maps all positive roots into negatives, and $n_0$ be its preimage in $N$. Then there exists $x \in U^{n_0}$ such that $T^x \cap B = 1$. In particular, there exist $u, v \in O^p(G)$ such that $B \cap B^u \cap B^v = 1$.

Proof. Consider $B^u = U^{n_0} \times T$. The Fitting subgroup $F(U^{n_0} \times T)$ equals $U^{n_0}$ since $Z(O^p(G)) = 1$. Otherwise, since $U^{n_0}$ is a normal nilpotent subgroup of $U^{n_0} \times T$ we obtain that $U^{n_0} \leq F(U^{n_0} \times T)$. If $U^{n_0} \neq F(U^{n_0} \times T)$, then there exists $1 \neq z \in T$ centralizing $U^{n_0}$ and so lying in $Z(O^p(G)) = 1$, a contradiction. Hence $F(U^{n_0} \times T) = U^{n_0}$ and by Lemma 8 there exists $x \in U^{n_0}$ such that $T \cap T^x = 1$.

Notice that $U^{n_0} \cap B = 1$, so $(U^{n_0} \times T) \cap B = T$. Since $T^x \in U^{n_0} \times T$ we obtain

$$1 = T^x \cap T = T^x \cap ((U^{n_0} \times T) \cap B) = (T^x \cap (U^{n_0} \times T)) \cap B = B \cap B^x \cap B^v,$$

whence the main statement of the lemma follows.

Now we prove “in particular”, i.e., we show that there exist $u, v \in O^p(G)$ such that $B \cap B^u \cap B^v = 1$. By construction, $x \in U^{n_0} \leq O^p(G)$ and $1 = T^x \cap B = (B^{n_0} \cap B)^x \cap B = B \cap B^x \cap B^{n_0}$. The lemma is proven.

Let $S = O^p(\overline{G}_\sigma)$ be a finite simple nontwisted group of Lie type over a field $\mathbb{F}_q$ of characteristic $p$. A Cartan subgroup $T \cap S$ of $S$ can be obtained as $\langle h_r(\lambda) \mid r \in \Pi, \lambda \in \mathbb{F}_q^* \rangle$ (see [12], Theorem 2.4.7]), where $\Pi$ is a set of fundamental roots of the root system of $S$. Then a field automorphism $\varphi$ of $S$ can be chosen so that for every $r \in \Pi, \lambda \in \mathbb{F}_q^*$ the identity $h_r(\lambda)^\varphi = h_r(\lambda^p)$ holds. Moreover, a graph automorphism $\tau$ corresponding to the symmetry $\varphi : \Pi \to \Pi$ of the Dynkin diagram of $S$ can be chosen so that for every $r \in \Pi, \lambda \in \mathbb{F}_q^*$ the identity $(h_r(\lambda))^\tau = h_r(\bar{\lambda})$ holds, where $\bar{\lambda} = \lambda$, if all roots have the same length. Consider the subgroup $A$ generated by so chosen field automorphism and graph automorphisms (there exist several graph automorphisms for the root system $D_4$). It is well-known that $\text{Aut}(S) = \hat{S} \times A$. Moreover, $A$ normalizes a Borel subgroup $B$ containing the Cartan subgroup $T$. Since $N_S(B) = B$ we obtain that $N_{\text{Aut}(S)}(B) = B \times A$.

Now assume that $S$ is a finite simple twisted group of Lie type distinct from a Suzuki group or a Ree group, $L$ is a nontwisted group of Lie type and $\psi$ is an automorphism
of L such that \( S = O^d(L_\psi) \). Let \( - : \Pi \to \Pi \) be the symmetry of the Dynkin diagram of a fundamental set of roots \( \Pi \) of the root system of \( L \) using for construction of \( \psi \). Then a Cartan subgroup \( T \cap L \) of \( \hat{L} \) can be written as \( \langle h_r(\lambda) \mid r \in \Pi, \lambda \in \mathbb{F}_q^* \rangle \), and a field automorphism \( \varphi \) of \( S \) can be chosen so that for every \( r \in \Pi, \lambda \in \mathbb{F}_q^* \) the equality \( (h_r(\lambda))^{\varphi} = h_r(\lambda^r) \) holds. We set \( A = \langle \varphi \rangle \), then \( \text{Aut}(S) = \hat{S} \rtimes A \), and there exists a Borel subgroup \( B \) of \( S \) such that the equality \( N_{\text{Aut}(S)}(B) = B \rtimes A \) holds.

**Lemma 10.** In the introduced notations assume that, if \( S \) is not twisted, then the order \( q \) of the definition field \( \mathbb{F}_q \) of \( S \) is greater than 2. Moreover, if \( S = D_4(q) \), assume also that \( q > 3 \). Assume also that \( S \) is neither a Suzuki group nor a Ree group. Then there exists \( x \in T \cap S \) such that \( C_A(x) = 1 \). In particular \( A \cap A^x = 1 \).

**Proof.** If \( S \) is not twisted and is distinct from \( D_4(q) \), then we can take \( x = h_r(\lambda) \), where \( r \in \Pi \) is such that \( r \neq \bar{r} \) and \( \lambda \) is a generating element of the multiplicative group of \( \mathbb{F}_q \). If \( S \) is twisted distinct from \( D_4(q) \), then we can take \( x = h_r(\lambda)h_{\bar{r}}(\lambda^q) \), where \( \lambda \) is a generating element of the multiplicative group of \( \mathbb{F}_q^2 \) and \( r \neq \bar{r} \). If \( S = D_4(q) \), then we can take \( x = h_r(\lambda)h_{\bar{r}}(\lambda^q)h_{\bar{r}}(\lambda^q) \), where \( \lambda \) is a generating element of the multiplicative group of \( \mathbb{F}_q^3 \) and \( r \neq \bar{r} \). Finally, if \( S = D_4(q) \) and \( q > 3 \), then there exist \( \lambda_1, \lambda_2 \in \mathbb{F}_q \setminus \{1\} \) such that \( \lambda_2 \notin \{\lambda_1^q, \lambda_1^q, \ldots, \lambda_1^q\} \) and \( \lambda_1 \) generates \( \mathbb{F}_q^* \). Choose fundamental roots \( r, s \) so that there exists a nontrivial symmetry of the Dynkin diagram, permuting the roots. Then we can take \( x = h_r(\lambda_1)h_s(\lambda_2) \).

**Lemma 11.** Let \( G \) be an almost simple group, whose simple socle \( S \) is a group of Lie type, satisfying the conditions of Lemma 10. Let \( B = U \rtimes T \) be a Borel subgroup of \( \hat{S} \) and \( H = N_G(B) \). Then there exist \( x, y, z \in S \), such that \( H \cap H^x \cap H^y \cap H^z = 1 \).

**Proof.** We use the notations introduced in Lemmas 9 and 10 in particular \( H \leq B \rtimes A \). It is proven in Lemma 9 that there exists \( x \in U^{\lambda_0} \leq S \) such that \( T^x \cap B = 1 \). In particular, \( B \cap B^{\lambda_0} \cap B^{-1} = 1 \). Therefore \( H \cap H^{\lambda_0} \cap H^{-1} \leq A \) and \( A \cap B = 1 \). By Lemma 10 there exists \( y \in T \cap S = (B \cap B^{\lambda_0}) \cap S \) such that \( A \cap A^y = 1 \). Thus

\[
(H \cap H^{\lambda_0} \cap H^{-1}) \cap (H \cap H^{\lambda_0} \cap H^{-1})^y = H \cap H^{\lambda_0} \cap H^{-1} \cap H^{-1} = 1,
\]

whence the lemma follows.

**Lemma 12.** Let \( S \) be a simple exceptional group of Lie type over a field of characteristic \( p \notin \pi \) and \( H \) be a solvable \( \pi \)-Hall subgroup of \( S \). Then one of the following holds.

1. There exists a maximal torus \( T \) of \( S \) such that \( H \leq N(S,T) \) and \( |\pi(N(S,T)/T) \cap \pi| \leq 1 \).

2. \( S = G_2(2^m+1), \pi \cap \pi(S) = \{2,7\}, |S|_{\{2,7\}} = 56, H \) is a Frobenius group of order 56.

3. \( S \in \{G_2(q), F_4(q), E_6^{-\varepsilon}(q), D_4(q)\} \), where \( \varepsilon \in \{+,-\} \) is chosen so that \( q \equiv 1 \) (mod 4); 2, 3 \( \notin \pi \), \( \pi \cap \pi(S) \leq \pi(\varepsilon-1) \), \( H \leq N(S,T) \), where \( T \) is a unique up to conjugation maximal torus such that \( N(S,T) \) contains a Sylow 2-subgroup of \( G \) and \( N(S,T)/T \) is a \{2,3\}-group. Here \( N(S,T) := N_G(T) \cap S \), where \( T = T S \cap S \) and \( S = O^d(T) \).

**Proof.** If \( 2 \notin \pi \) then by Lemmas 7–14, Theorem 3 statement (1) of the Lemma holds. If \( 2 \in \pi \) and \( 3 \notin \pi \), then by Lemma 5.1 and Theorem 5.2 (see also 9, Theorem 8.9) either statement (1) or statement (2) of the lemma holds.
Finally, if $2, 3 \in \pi$, then $S$ is neither a Suzuki group, nor a Ree group (since $p \not\in \pi$). By [15, Lemma 7.1–7.6] (see also [9, Theorem 8.15]) we have $\pi \cap \pi(S) \subseteq \pi(p - \varepsilon_1)$, $H \leq N(S, T)$, where $T$ is a unique up to conjugation maximal torus such that $N(S, T)$ contains a Sylow 2-subgroup of $S$ and either $N(S, T)/T$ is a $\{2, 3\}$-group, or $N(S, T)/T$ is a Weyl group of the root system of $S$. Since for root systems $E_6, E_7, E_8, F_4, G_2$ the Weyl groups are either $\{2, 3\}$-groups, or unsolvable, we obtain that if $S \in \{E^\pm_6(q), E_7(q), E_8(q)\}$, then $H$ is unsolvable, whence statement (3) of the lemma.

**Corollary 13.** Let $S$ be a simple exceptional group of Lie type over a field of characteristic $p \not\in \pi$, $S$ is neither a Suzuki group, nor a Ree group, and $H$ is a solvable $\pi$-Hall subgroup of $S$. Then the following statements hold.

(1) If $S = E_8(q)$, then $|H| \leq (q + 1)^8 \cdot 2^{14}$.

(2) If $S = E_7(q)$, then $|H| \leq (q + 1)^7 \cdot 2^{10}$.

(3) If $S = E_6(q)$, then $|H| \leq (q + 1)^6 \cdot 2^7$.

(4) If $S = F_4(q)$, then $|H| \leq (q + 1)^4 \cdot 2^7 \cdot 3^2$.

(5) If $S = G_2(q)$, then $|H| \leq (q + 1)^2 \cdot 12$.

(6) If $S = ^3D_4(q)$, then $|H| \leq \max\{(q^2 + q + 1)^2, (q + 1)^2 \cdot 48\}$.

### 3 Proof of the Main Theorem.

We proceed by considering distinct possible cases for the simple socle $S$ of $G$ and the structure of its $\pi$-Hall subgroup $H$. If $S$ is either a Suzuki group or a Ree group, then by [10, Tables 3 and 4] it follows that for every subgroup $H$ of $G$ the inequality $\text{Base}_{H}(G) \leq 3$ holds. So we assume later that $S$ is neither a Suzuki group, nor a Ree group.

#### 3.1 $S$ is a simple group of Lie type over a field of characteristic $p \in \pi$.

By Lemma [7] $H \cap \hat{S} \in \text{Hall}_p(\hat{S})$, so in this case for $H \cap \hat{S}$ Lemma [7] holds. Assume first that $H \cap \hat{S}$ lies in a Borel subgroup of $\hat{S}$. If $S$ is a non twisted group of Lie type over a field of order two, then $H$ is a 2-group. By [2] the inequality $\text{Base}_{H}(G) \leq 3$ holds. If $S = D_4(3)$, then $H$ is a 3-group. By [2] the inequality $\text{Base}_{H}(G) \leq 3$ holds. Assume that $S$ is not a non twisted group of Lie type over a field of two elements, and $S \not\cong D_4(3)$. Then $H \leq N_G(U) = N_G(B)$ and by Lemma [11] the inequality $\text{Base}_{H}(G) \leq 4$ holds. If one of statements (1)–(3) of Lemma [7] is satisfied, then $S$ is a classical group and calculations by using [16] show that in any case $\text{Base}_{H}(G) \leq 5$ and $\text{Reg}_{H}(G, 5) \geq 5$.

#### 3.2 $S$ is a simple exceptional group of Lie type over a field of characteristic $p \not\in \pi$.

Assume that $S = E_8(q)$. We use Lemma [8] If $x$ is a unipotent element, then $x^G \cap H = \emptyset$. If $x$ is a semisimple element from $G = \hat{G}$, then by [17, Table 2] it follows that the maximum of orders of centralizers of semisimple elements in $E_8(q)$ is not greater than $q^{64}(q^{18} - 1)(q^{14} - 1)(q^{12} - 1)(q^{10} - 1)(q^8 - 1)(q^6 - 1)(q^2 - 1)^2$. 

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whence $|x^G| > q^{112}$. Clearly, the inequality $|x^G| > q^{112}$ holds in case, when $x$ is a field automorphism. So for $c = 2$ we obtain

$$\hat{Q}(G, 2) \leq ((q + 1)^8 \cdot 2^{14})^2 / (q^{112}) < 1$$

for every $q \geq 2$. Hence, $\text{Base}_H(G) \leq 2$.

Assume that $G = E_7(q)$. We again use Lemma 8. If $x$ is a unipotent element, then $x^G \cap H = \emptyset$. If $x$ is a semisimple element from $\hat{G}$, then by [17] Table 1 it follows that the maximum of orders of centralizers of semisimple elements in $E_7(q)$ is not greater than

$$q^{31}(q^2 - 1)^2(q^4 - 1)(q^6 - 1)^2(q^8 - 1)(q^{10} - 1),$$

whence $|x^G| > (1/2)q^{64}$. Clearly, the inequality $|x^G| > (1/2)q^{64}$ holds in case, when $x$ is a field automorphism. So for $c = 2$ we obtain

$$\hat{Q}(G, 2) \leq ((q + 1)^7 \cdot 2^{20})^2 / (q^{64}) < 1$$

for every $q \geq 2$. Hence $\text{Base}_H(G) \leq 2$.

Assume that $G = E_6^q(q)$. As above, we obtain that $x$ is either a semisimple element from $\hat{G}$, or does not lie in $\hat{G}$. If $x$ is a semisimple element, then by [18] Table 1 and Case $E_6(q)$ it follows that the maximum of orders of centralizers of semisimple elements in $E_6^q(q)$ is not greater than

$$q^{20}(q - \epsilon 1)(q^2 - 1)(q^4 - 1)(q^6 - 1)(q^8 - 1)(q^5 - \epsilon 1),$$

whence $|x^G| > \frac{1}{8}q^{30}$. Clearly, the inequality $|x^G| > \frac{1}{8}q^{30}$ holds in case, when $x$ is either a field, or a graph-field automorphism. If $x$ is a graph automorphism, then

$$|x^G| = |E_6^q| / |F_4(q)| \geq \frac{1}{3}q^{12}(q^5 - 1)(q^9 - 1).$$

So for $c = 4$ we obtain

$$\hat{Q}(G, 2) \leq \frac{(q + 1)^{24} \cdot 2^{28} \cdot 3^3}{q^{36} \cdot (q^5 - 1)^3 \cdot (q^9 - 1)^3} < 1$$

for every $q \geq 2$. Hence $\text{Base}_H(G) \leq 4$.

Assume that $G = F_4(q)$. Again we may assume that $x$ either is a semisimple element from $G = \hat{G}$ or does not lie in $\hat{G}$. If $x$ is a semisimple element, then by [18] Table 2 it follows that the maximum of orders of centralizers of semisimple elements in $F_4(q)$ is not greater than

$$q^{16}(q^2 - 1)(q^4 - 1)(q^6 - 1)(q^8 - 1),$$

whence $|x^G| > q^{16}$. Clearly, the inequality $|x^G| > q^{16}$ holds for every $x$ not lying in $G$. So for $c = 4$ we obtain

$$\hat{Q}(G, 2) \leq \frac{(q + 1)^{16} \cdot 2^{28} \cdot 3^8}{q^{48}} < 1$$

for every $q \geq 3$. So for $q \geq 3$ the inequality $\text{Base}_H(G) \leq 4$ holds. If $q = 2$, then in view of the condition $p \notin \pi$ we obtain that the order $|H|$ is odd. By [13] Lemma 8 we obtain that either $H$ is a Sylow 3-subgroup of $G$, or $H$ is abelian. Hence the inequality $\text{Base}_H(G) \leq 3$ holds: in the first case by [2], and in the second case by Lemma 6.

Assume that $G = G_2(q)$. As above we may assume that $x$ either is a semisimple element from $G = \hat{G}$, or does not lie in $\hat{G}$. If $x$ is semisimple, then by [18] Table 4 it
follows that the maximum of orders of centralizers of semisimple elements in $F_4(q)$ is not greater than
\[ q^2(q^2 - 1)(q^3 + 1), \]
whence $|x^G| \geq q^4(q^3 - 1)$. Clearly, the inequality $|x^G| \geq q^4(q^3 - 1)$ holds for every $x$ not lying in $G$. So for $c = 4$ we obtain
\[ \hat{Q}(G, 2) \leq \frac{(q + 1)^8 \cdot 12^4}{q^{12} \cdot (q^3 - 1)^3} < 1 \]
for every $q \geq 3$. Hence for $q \geq 3$ the inequality $\text{Base}_H(G) \leq 4$ holds. If $q = 2$, then by the condition $p \not\in \pi$ we obtain that the order $|H|$ is odd. By [13] Lemma 7 we obtain that either $H$ is a Sylow 3-subgroup of $G$, or $H$ is abelian. So the inequality $\text{Base}_H(G) \leq 3$ holds: in the first case by [2], and in the second case by Lemma [3].

If $G = 3D_4(q)$, then by [18] Table 7 it is easy to get the bound $|x^G| > q^{16}$. Using the bound we obtain that for $q \geq 2$ the inequality $\text{Base}_H(G) \leq 4$ holds. The Main Theorem is proven.

Notice that for the case $p \in \pi$ we also prove the following

**Theorem 14.** Let $G$ be a finite almost simple group, whose simple socle is isomorphic to a group of Lie type over a field of characteristic $p \in \pi$. Assume that $H$ is a solvable $\pi$-Hall subgroup of $G$. Then the inequalities $\text{Base}_H(G) \leq 5$ and $\text{Reg}_H(G, 5) \geq 5$ hold.

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