VANISHING VISCOSITY LIMIT OF 1D QUASILINEAR PARABOLIC EQUATION WITH MULTIPLE BOUNDARY LAYERS

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Abstract. In this paper, we study the limiting behavior of solutions to a 1D two-point boundary value problem for viscous conservation laws with genuinely-nonlinear fluxes as \( \varepsilon \) goes to zero. We here discuss different types of non-characteristic boundary layers occurring on both sides. We first construct formally the three-term approximate solutions by using the method of matched asymptotic expansions. Next, by energy method we prove that the boundary layers are nonlinearly stable and thus it is proved the boundary layer effects are just localized near both boundaries. Consequently, the viscous solutions converge to the smooth inviscid solution uniformly away from the boundaries. The rate of convergence in viscosity is optimal.

1. Introduction and main results. In physics and fluid mechanics, it is very important to discuss the asymptotic equivalence between the viscous flows and the associated inviscid flows in the limit of small viscosity [4, 7, 8, 9, 12]. The solutions near the boundaries always exhibit very singular behavior as the viscosity is small, which is called boundary layer effects. Boundary layers appear in various physical contexts, such as the theory of rotating fluids, incompressible MHD, the inviscid limit of parabolic systems, or the inviscid limit of Navier-Stokes equations near a boundary. The rigorous mathematical justification of the asymptotic equivalence with boundary layers poses challenging problems in many important cases, see [3, 5, 6, 7] and [9]. In the case of noncharacteristic boundaries, to conquer the difficulty that systems do not admit maximum principles, in [12], the vanishing viscosity limit of one-dimensional quasilinear viscous equations in a half plane is studied by the approach based on the asymptotic analysis and a weighted energy method. Similar as noncharacteristic boundary layers, in [3], the piecewise smooth inviscid solution with a single, entropic, sufficiently weak \( p \)-shock is proved to be the limit of a

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sequence of solutions to the corresponding viscous equation away from the shock. Furthermore, the result holds as well for solutions with finitely many noninteracting entropic shocks. Here, we consider the two-point boundary value problem for the following one dimensional quasilinear equation

\[ \partial_t u^\varepsilon + \partial_x f(u^\varepsilon) = \varepsilon \partial_x^2 u^\varepsilon, \quad x \in (0, 1), \quad t > 0, \]  
(1.1)

\[ u_0(x) = u_0^\varepsilon(x), \quad x \in (0, 1), \]  
(1.2)

where \( f \) and \( u_0(x) \) are smooth, and we require that

\[ \partial_u f(u(x, t)) \neq 0, \quad \partial_x^2 f(u) \geq 0 \]  
(1.5)

in the region which we are interested in. The aim of this paper is to seek the asymptotic relations between the viscous solution \( u^\varepsilon(x, t) \), to (1.1)-(1.4) and the solution to the associated inviscid hyperbolic problem

\[ \partial_t u + \partial_x f(u) = 0, \quad x \in (0, 1), \quad 0 < t \leq T_0, \]  
(1.6)

\[ u(x, t = 0) = u_0(x), \quad x \in (0, 1), \]  
(1.7)

with suitable boundary conditions on \( \{x = 0\} \) and \( \{x = 1\} \). In view of (1.5), the boundaries are non-characteristic, and the boundary conditions for (1.6) will depend on the sign of \( \partial_u f(u) \) near the boundary. Since there may be a loss of boundary conditions when \( \varepsilon \to 0 \), hence, boundary layers may appear near \( \{x = 0\} \) or \( \{x = 1\} \). On a certain time interval, there are four cases according to the sign of \( \partial_u f(u) \) near the boundaries:

**Case I** \( \partial_u f(u) \big|_{x=0} < 0 \) and \( \partial_u f(u) \big|_{x=1} > 0 \).

By the method of characteristics, the boundary values are completely determined by the solution inside, boundary conditions are NOT needed on either boundaries for (1.6)-(1.7);

**Case II** \( \partial_u f(u) \big|_{x=0} < 0 \) and \( \partial_u f(u) \big|_{x=1} < 0 \).

In this case, only values on the left boundary is determined by the solution inside, so a suitable condition should be imposed on \( \{x = 1\} \) for (1.1)-(1.4);

**Case III** \( \partial_u f(u) \big|_{x=0} > 0 \) and \( \partial_u f(u) \big|_{x=1} > 0 \).

Similar to case ii), a boundary condition is needed on \( \{x = 0\} \) for (1.6)-(1.7);

**Case IV** \( \partial_u f(u) \big|_{x=0} > 0 \) and \( \partial_u f(u) \big|_{x=1} < 0 \).

To guarantee the well-posedness of the inviscid problem, boundary conditions should be imposed on both boundaries for (1.6)-(1.7).

It is expected that the viscous solution \( u^\varepsilon \) convergence to the inviscid solution \( u \) uniformly away from the boundaries. The leading order asymptotic behavior of the viscous solution is governed by

\[ \partial_y f(u_b^0) = \partial_x^2 u_b^0, \]

\[ u_b^0(y = 0, t) = u(t), \]

\[ u_b^0(y \to +\infty, t) = u^0(0, t), \]

and

\[ \partial_\xi f(\bar{u}_b^0) = \partial_\xi^2 \bar{u}_b^0, \]

\[ \bar{u}_b^0(1, t) = u_r(t), \]

\[ \bar{u}_b^0(\xi \to -\infty, t) = u^0(1, t), \]
where $y = \frac{x}{\varepsilon}$ and $\xi = \frac{x}{\varepsilon^2}$ are the fast variables. We define the strength of the boundary layers in the viscous flow by
\[
\delta_1(t) = \sup_{0 \leq s \leq t} |u(s) - u^0(0, s)|, \quad \delta_2(t) = \sup_{0 \leq s \leq t} |u_v(s) - u^0(1, s)|. \quad (1.8)
\]
Here, we consider the case that no interactions between the possible boundary layers. We will consider the smooth inviscid flow before shock formation. We first construct a three-term approximate solution to (1.1)–(1.4) by using the method of matched asymptotic analysis and multi-scale expansions. Then using the energy method, we prove that the boundary layers near both boundaries with various types are nonlinearly stable. The analysis depends crucially on the structure of the underlying boundary layers. The stability analysis of the boundary layers is devoted to Case I in Section 2, where the stability analysis of both an expansive boundary layer and a compressive boundary layer is not included due to the opposite sign of the $\partial_y \partial_a f$ and $\partial_x \partial_a f$ near the boundaries. Case II–Case IV are discussed in Section 3 and the corresponding inviscid problems are also stated there. The method in this paper can be similarly carried out for systems with suitable modifications, see both weak boundary layers in the two-dimensional case in [10]. Throughout this paper, we use $C$ or $O(1)$ to denote any positive bounded function which is independent of $\varepsilon$.

2. Stability analysis of both boundary layers. According to Case I, we suppose for some suitable $\nu > 0$, the initial data $u_0(x)$ satisfies
\[
\partial_a f(u_0(\nu)) < 0, \quad \text{and} \quad \partial_a f(u_0(1 - \nu)) > 0. \quad (H1)
\]
Then by continuity, $\exists x_1 > 0$ and $T > 0$ such that
\[
\partial_a f(u(\sigma, t)) < 0, \quad \text{and} \quad \partial_a f(u(\zeta, t)) > 0,
\]
where $\sigma \in [0, x_1]$, $\zeta \in [1 - x_1, 1]$ and $t \in [0, T]$. In this case, the inviscid problem (1.6)–(1.7) is well-defined without any boundary conditions. To be consistency, we assume that
\[
\partial_a f(u_1(t)) < 0, \quad t \in [0, T], \quad \partial_a f(u_v(t)) > 0, \quad t \in [0, T]. \quad (H2)
\]
In the limit of $\varepsilon \to 0$, the discrepancies of boundary conditions lead to the phenomena of boundary layers near both boundaries.

2.1. Asymptotic expansions. To prove the asymptotic equivalence between (1.1)–(1.4) and (1.6)–(1.7), we first need to construct the approximate solution through different scalings and asymptotic expansions in the regions near and away from the boundaries.

In the region away from the boundaries $\{x = 0\}$ and $\{x = 1\}$, the solution of (1.1) may be approximated by the formal series
\[
u^\varepsilon(x,t) = u^0(x,t) + \varepsilon u^1(x,t) + \varepsilon^2 u^2(x,t) + \cdots. \quad (2.1.1)
\]
Substituting this into (1.1) and equating the coefficients of different powers of $\varepsilon$ yield
\[
O(1) : \partial_t u^0 + \partial_x f(u^0) = 0, \quad (2.1.2)
\]
\[
O(\varepsilon) : \partial_t u^1 + \partial_x f(u^0)u^1 = \partial_x^2 u^0, \quad (2.1.3)
\]
\[
O(\varepsilon^2) : \partial_t u^2 + \partial_x f(u^0)u^2 = \partial_x^2 u^1 - \frac{1}{2} \partial_x(f''(u^0)(u^1)^2), \quad (2.1.4)
\]
with the following initial data for \(0 < x < 1\) that
\[
\begin{align*}
    u^0(x, t = 0) &= u_0(x), \quad (2.1.5) \\
    u^1(x, t = 0) &= 0, \quad (2.1.6) \\
    u^2(x, t = 0) &= 0. \quad (2.1.7)
\end{align*}
\]

The problem (2.1.2) and (2.1.5) for the leading order inner function is exactly the inviscid problem (1.6)-(1.7). Therefore, \(u^0(x, t)\) is the unique smooth inviscid solution to (1.6)-(1.7) with
\[
u^0(x, t) \in C^2([0, T]; H^0(0, 1)).
\]

The equation (2.1.3) is a linear hyperbolic equation, then by the method of characteristics, there exists a unique smooth solution \(u^1(x, t)\) such that
\[
u^1(x, t) \in C^1([0, T]; H^1(0, 1)).
\]

Similarly, there exists a unique smooth solution \(u^2(x, t)\) such that
\[
u^2(x, t) \in C^1([0, T]; H^2(0, 1)).
\]

Since the boundary, \(\{x = 0\}\), is non-characteristic for the inviscid hyperbolic problem (1.6)-(1.7), we will approximate the viscous solution near the boundary \(\{x = 0\}\) by the following expansion
\[
u^\varepsilon(x, t) \sim u^0_b(y, t) + \varepsilon u^1_b(y, t) + \varepsilon^2 u^2_b(y, t) + \cdots, \quad \text{with } y = \frac{x}{\varepsilon}. \quad (2.1.8)
\]

Substituting it into the equation (1.1) to get
\[
\begin{align*}
    &O\left(\frac{1}{\varepsilon}\right): \partial_y f(u^0_b) = \partial^2_y u^0_b, \quad (2.1.9) \\
    &O(1): \partial^2_y u^1_b - \partial_y (f'(u^0_b) u^1_b) = \partial_t u^1_b, \quad (2.1.10) \\
    &O(\varepsilon): \partial^2_y u^2_b - \partial_y (f'(u^0_b) u^2_b) = \partial_t u^2_b + \frac{1}{2} \partial_y (f''(u^0_b)(u^1_b)^2). \quad (2.1.11)
\end{align*}
\]

In the matching zone near \(\{x = 0\}\), both the inner and boundary layer expansions are expected to be valid, then as \(y \to +\infty\), we obtain
\[
\begin{align*}
    u^0_b(y, t) &= u^0(0, t) + o(1), \quad (2.1.12) \\
    u^1_b(y, t) &= u^1(0, t) + y \partial_x u^0(0, t) + o(1), \quad (2.1.13) \\
    u^2_b(y, t) &= u^2(0, t) + y \partial_x u^1(0, t) + \frac{1}{2} y^2 \partial^2_x u^0(0, t) + o(1). \quad (2.1.14)
\end{align*}
\]

The leading boundary function \(u^0_b(y, t)\) satisfies the second order nonlinear ODE (2.1.9) and the boundary condition (2.1.12). We impose the boundary condition on \(y = 0\) as
\[
u^0_b(y = 0, t) = u_l(t), \quad t > 0. \quad (2.1.15)
\]

Thus, the boundary condition of \(u^i_b(y, t), i = 1, 2, \) on \(y = 0\) are
\[
u^1_b(y = 0, t) = 0. \quad (2.1.16)
\]

and
\[
u^2_b(y = 0, t) = 0. \quad (2.1.17)
\]

Near the boundary \(\{x = 1\}\), we assume
\[
u^\varepsilon(x, t) \sim \bar{u}^0_b(\xi, t) + \varepsilon \tilde{u}^1_b(\xi, t) + \varepsilon^2 \tilde{u}^2_b(\xi, t) + \cdots, \quad \text{with } \xi = \frac{x - 1}{\varepsilon}. \quad (2.1.18)
\]
Substituting it into the equation (1.1) yields

\[ O(1) : \partial_\xi \left( \frac{1}{\xi} \right) f(\bar{u}_b^0) = \partial_\xi^2 \bar{u}_b^0, \quad (2.1.19) \]
\[ O(1) : \partial_\xi^2 \bar{u}_b^1 = \partial_\xi \left( f'(\bar{u}_b^0) \bar{u}_b^0 \right) = \partial_t \bar{u}_b^0, \quad (2.1.20) \]
\[ O(\varepsilon) : \partial_\xi^2 \bar{u}_b^2 - \partial_\xi \left( f'(\bar{u}_b^0) \bar{u}_b^0 \right) = \partial_t \bar{u}_b^1 + \frac{1}{2} \partial_\xi \left( f''(\bar{u}_b^0) (\bar{u}_b^0)^2 \right), \quad (2.1.21) \]

with

\[ \bar{u}_b^0(1, t) = u_r(t), \]
\[ \bar{u}_b^1(1, t) = 0, \]
\[ \bar{u}_b^2(1, t) = 0. \]

In the matching zone near \( \{ x = 1 \} \), as \( \xi \to -\infty \), there hold

\[ \bar{u}_b^0(\xi, t) = u^0(1, t) + o(1), \quad (2.1.25) \]
\[ \bar{u}_b^1(\xi, t) = \xi \partial_x u^0(1, t) + u^1(1, t) + o(1), \quad (2.1.26) \]
\[ \bar{u}_b^2(\xi, t) = \xi \partial_x u^1(1, t) + u^2(1, t) + \frac{1}{2} \xi^2 \partial_x^2 u^0(1, t) + o(1). \quad (2.1.27) \]

**Lemma 2.1.** There exists a unique bounded smooth solution \( \bar{u}_b^0(y, t) \) to the two-point boundary value problem (2.1.9), (2.1.12), (2.1.15). Furthermore, there exists a positive constant \( \alpha_0 > 0 \) such that

\[ |\partial_y \bar{u}_b^0(y, t)| \leq Ce^{-\alpha_0 y}, \quad y \geq 0. \quad (2.1.28) \]

Similarly, there exists a unique smooth solution \( \bar{u}_b^0(\xi, t) \) to the two point boundary value problem (2.1.19), (2.1.22), (2.1.25) satisfying

\[ |\partial_\xi \bar{u}_b^0(\xi, t)| \leq Ce^{\alpha_0 \xi}, \quad \alpha_0 > 0, \quad \xi \leq 0. \quad (2.1.29) \]

**Proof.** First, we consider the problem (2.1.9), (2.1.12), (2.1.15). Set

\[ u = \bar{u}_b^0(y, t) - u^0(0, t), \quad \partial_y u = v. \]

Then we have the following nonlinear ODE system

\[ \partial_y \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & \partial_u f(u^0(0, t) + u(y, t)) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \quad (2.1.30) \]

with

\[ \begin{pmatrix} u \\ v \end{pmatrix}(y = 0, t) = \begin{pmatrix} u(t) - u^0(0, t) \\ v(0) \end{pmatrix}, \quad \begin{pmatrix} u \\ v \end{pmatrix}(y \to +\infty, t) = 0. \quad (2.1.31) \]

Consider the matrix \( \begin{pmatrix} 0 & 1 \\ 0 & \partial_u f(u^0(0, t)) \end{pmatrix} \). It has eigenvalues \( \lambda_1 = 0, \quad \lambda_2 = \partial_u f(u^0(0, t)). \)

It follows from (H1) that

\[ \lambda_2 < 0, \quad \text{for} \quad t \in [0, T]. \]

Therefore, for any boundary condition \((u, v)(0, t)\) which is in a neighborhood of \(0\), the center-stable manifold theorem implies that there exits a unique bounded
smooth solution \((u, v)\) to the problem (2.1.30)–(2.1.31) with exponential decay property at infinity. Hence \(u_0^0(y, t)\) is bounded and smooth. In fact the exponential decay of \(\partial_y u_0^0(y, t)\) can also be seen by the following calculations. We rewrite (2.1.9) as
\[
\partial_y^2 u_0^0 = f'(u_0^0)\partial_y u_0^0.
\]
Let \(W = \partial_y u_0^0\), then
\[
\partial_y W = f'(u_0^0)W.
\]
Integrating over \([0, y]\) gives
\[
\partial_y u_0^0(y, t) = \partial_y u_0^0(0, t) \exp\{\int_0^y f'(u_0^0(\eta, t))d\eta\}. \tag{2.1.32}
\]
Integrating over \([0, +\infty]\) yields
\[
u^0(0, t) - u_t(t) = \partial_y u_0^0(0, t) \int_0^{+\infty} \exp\{\int_0^y f'(u_0^0(\eta, t))d\eta\}dy. \tag{2.1.33}
\]
It should be mentioned that the integrations in (2.1.32)–(2.1.33) are near the boundary \(x = 0\), and they are bounded due to (H1)–(H2) and the property of \(u_0^0\). Thus
\[
\partial_y u_0^0(y, t) = \frac{[u_t(t) - u_0^0(0, t)] \exp\{\int_0^y f'(u_0^0(\eta, t))d\eta\}}{\int_0^{+\infty} \exp\{\int_0^y f'(u_0^0(\eta, t))d\eta\}dy}, \tag{2.1.34}
\]
and similarly there exists a bounded smooth \(\bar{u}_0^0\) such that
\[
\partial_\zeta \bar{u}_0^0(\zeta, t) = \frac{[u_t(t) - u_0^0(1, t)] \exp\{\int_0^1 f'(\bar{u}_0^0(\eta, t))d\eta\}}{\int_{-\infty}^{+\infty} \exp\{\int_0^1 f'(\bar{u}_0^0(\eta, t))d\eta\}d\zeta}, \tag{2.1.35}
\]
with all the integrations involved being finite. Hence (2.1.29), (2.1.30) are proved. \(\square\)

**Remark 2.1.** It can be checked easily that
\[
|\partial_y^n u_0^0(y, t)| \leq Ce^{-\alpha_0 y}, \text{ for } n \geq 1. \tag{2.1.36}
\]
or
\[
|\partial_\zeta^n \bar{u}_0^0(\zeta, t)| \leq Ce^{\delta_0 \zeta}, \text{ for } n \geq 1. \tag{2.1.37}
\]

**Remark 2.2.** If the boundary layer is assumed to be weak, that means, there exists a small \(\delta_0 > 0\) such that \(\delta_1(t) \leq \delta_0\) or \(\delta_2(t) \leq \delta_0\), then we have
\[
|\partial_y u_0^0(y, t)| \leq C\delta_0 e^{-\alpha_0 y}, \quad y \geq 0, \tag{2.1.38}
\]
or
\[
|\partial_\zeta \bar{u}_0^0(\zeta, t)| \leq C\delta_0 e^{\delta_0 \zeta}, \quad \zeta \leq 0. \tag{2.1.39}
\]

To carry out the analysis for strong boundary layers, the structure of the underlying boundary layers should be used. It turns out the monotonicity of the wave speed in the boundary layer plays an important role in our proof. So if \(\delta_i(t)\), \(i = 1, 2\) are not required to be small, we have

**Definition 2.1.** The boundary layers can be **expansive** in the sense that
\[
\partial_y \partial_n f(u_0^0(y, t)) \geq 0, \quad \text{or} \quad \partial_\zeta \partial_n f(\bar{u}_0^0(\zeta, t)) \geq 0, \tag{2.1.40}
\]
and **compressive** boundary layers in the sense that
\[
\partial_y \partial_n f(u_0^0(y, t)) \leq 0, \quad \text{or} \quad \partial_\zeta \partial_n f(\bar{u}_0^0(\zeta, t)) \leq 0. \tag{2.1.41}
\]
Lemma 2.2. Let \( u^0_k(y, t), \bar{u}^0_k(\xi, t) \) be the leading order boundary layer function, and let \( t \in (0, T] \) be fixed. If

\[
  u(t) - u^0(0, t) \leq 0, \quad (2.1.42)
\]

then the boundary layer near \( \{ x = 0 \} \) is expansive. If

\[
  u(t) - u^0(0, t) \geq 0, \quad (2.1.43)
\]

then the boundary layer near \( \{ x = 0 \} \) is compressive. Similarly, if \( u_\tau(t) - u^0(1, t) \geq 0 \), then the boundary layer near \( \{ x = 1 \} \) is expansive. If \( u_\tau(t) - u^0(1, t) \leq 0 \), then the boundary layer near \( \{ x = 1 \} \) is compressive.

Proof. Without loss of generality, we only give the proof of the case when \( u(t) - u^0(0, t) \leq 0 \). Note that

\[
  \partial_y \partial_u f(u^0_k(y, t)) = \partial^2_u f(u^0_k(y, t)) \partial_y u^0(y, t),
\]

In view of (2.1.34) and (2.1.43), then \( \partial_y u^0 \geq 0 \). Together with the convexity condition (1.5) we can easily get

\[
  \partial_y \partial_u f(u^0_k(y, t)) \geq 0,
\]

which implies the boundary layer near \( \{ x = 0 \} \) is expansive. \( \square \)

Now, we come to the higher order boundary layer functions.

Lemma 2.3. The two-point boundary value problem (2.1.10), (2.1.13), (2.1.16) admits a unique smooth solution \( u^1_k(y, t) \) satisfying

\[
  |\partial^2_y u^1_k(y, t)| \leq Ce^{-\alpha y}, \quad \alpha_1 > 0, \ y \geq 0.
\]

Similarly, (2.1.20), (2.1.23), (2.1.26) admits a unique smooth solution \( \bar{u}^1_k(\xi, t) \) satisfying

\[
  |\partial^2_\xi \bar{u}^1_k(\xi, t)| \leq Ce^{\bar{\alpha} \xi}, \quad \bar{\alpha}_1 > 0, \ \xi \leq 0.
\]

Proof. Set

\[
  u^1_B(y, t) = u^1_k(y, t) - u^1(0, t) - y \partial_x u^0(0, t).
\]

Notice that

\[
  \partial_t u^0(0, t) + f'(u^0(0, t)) u^0(0, t) = 0.
\]

Then \( u^1_B(y, t) \) satisfies

\[
  \partial^2_y u^1_B - \partial_y (f'(u^0) u^1_B) = g(y, t), \quad (2.1.44)
\]

\[
  u^1_B(y \to +\infty) = 0, \quad (2.1.45)
\]

\[
  u^1_B(0, t) = -u^1(0, t), \quad (2.1.46)
\]

where

\[
  g(y, t) = f''(u^0) \partial_y u^0 \cdot u^1(0, t) + f''(u^0) \partial_y u^0 \cdot y \partial_x u^0(0, t)
\]

\[
  - \partial_x u^0(0, t) \int_y^{+\infty} f''(u^0) \partial_y u^0 \, d\eta - \partial_t \int_y^{+\infty} \partial_y u^0 \, d\eta.
\]

Then it follows from the exponential decay property of \( \partial_y u^0 \) that

\[
  |g(y, t)| \leq Ce^{-\beta y}, \quad \text{for some } \beta > 0. \quad (2.1.47)
\]

Then the unique solution to (2.1.44)-(2.1.46) is given explicitly by

\[
  u^1_B(y, t) = -u^1(0, t) \exp\left\{ \int_0^y f'(u^0(z, t)) \, dz \right\}
\]
In view of \((2.1.47)\) and the negativity of \(f\) near the boundary \(x = 0\), we have \(u_B^1\) exists and

\[
|\partial_y^2 u_B^1(y, t)| \leq Ce^{-\beta y}. \tag{2.1.48}
\]

This finishes the proof of Lemma 2.3. \(\square\)

Using the inner equations \((2.1.2)\) and \((2.1.3)\) and the properties of \(u_i^j(y, t), \ i = 0, 1\), we can transform \((2.1.11), (2.1.14), (2.1.17)\) into a linear second order differential equations with an exponential decay source term. Moreover, the new unknown function satisfies the zero condition at infinity, then similar as the proof of the existence of \(u_B^1(y, t)\), the unique smooth solution \(u_B^1(y, t)\) of \((2.1.11), (2.1.14), (2.1.17)\) can be obtained. It follows similarly that \((2.1.21), (2.1.24), (2.1.27)\) admits a smooth solution \(\tilde{u}_B^2(\xi, t)\). Hence, we have obtained the desired three-term asymptotic inner and boundary layer solutions.

2.2. A weak boundary layer and a strong expansive boundary layer. We discuss in this subsection the stability of the multiple boundary layers, in which one is a weak boundary layer and the other is a strong expansive boundary layer. Without loss of generality, we assume there is a weak boundary layer near \(x = 0\) and an expansive strong boundary layer near \(x = 1\), which are defined in \((1.8)\) and \((2.1.40)\). We have

**Theorem 2.1.** Assume \((H1), (H2)\) hold and the initial and boundary data in \((1.2)−(1.4)\) are compatible to any order and \(u^0(x, t) \in C^2([0, T]; H^6(0, 1))\) is the smooth solution to the inviscid problem \((1.6)\) and \((1.7)\). Furthermore, assume that there exists a \(\delta_0 > 0\) such that \(\delta_1(t) \leq \delta_0\), and \(u_\tau(t) - u^0(1, t) \geq 0\) for all \(t \in [0, T]\). Then there exists \(\varepsilon_1 > 0\), s.t. \(\forall \ 0 < \varepsilon < \varepsilon_1\), the initial boundary value problem \((1.1)−(1.4)\) has a unique smooth solution \(u^s(x, t) \in C^4([0, T]; H^2(0, 1))\) such that for \(h > 0\), there hold

\[
\sup_{0 \leq t \leq T} \| u^s(\cdot, t) - u^0(\cdot, t) \|_{L^2(0, 1)} \leq C\varepsilon, \tag{2.2.1}
\]

and

\[
\sup_{0 \leq t \leq T} \left| u^s(\cdot, t) - u^0(\cdot, t) \right| \leq C\varepsilon. \tag{2.2.2}
\]

To prove this theorem, we first patch the truncated boundary layer and inner solutions in the previous discussion to get the formal approximate solution to \((1.1)−(1.4)\). Define

\[
O(x, t) = u^0(x, t) + \varepsilon u^1(x, t) + \varepsilon^2 u^2(x, t), \tag{2.2.3}
\]

\[
I_1(x, t) = u_b^1(y, t) + \varepsilon u_{b1}^1(y, t) + \varepsilon^2 u_{b2}^1(y, t), \tag{2.2.4}
\]

\[
I_2(x, t) = \tilde{u}_b^1(\xi, t) + \varepsilon \tilde{u}_{b1}^1(\xi, t) + \varepsilon^2 \tilde{u}_{b2}^1(\xi, t). \tag{2.2.5}
\]

Let \(m(x) \in C_0^\infty(R)\) satisfy \(0 \leq m(x) \leq 1\), and

\[
m(x) = \begin{cases} 
1, & \text{if } |x| \leq 1, \\
0, & \text{if } |x| \geq 2, \\
h(x), & \text{if } 1 < |x| < 2, 
\end{cases}
\]

where \(h(x)\) is a smooth function with support in \([1, 2]\). Define

\[
\bar{u}_B^1(y, t) = m(x) \cdot u_B^1(y, t), \tag{2.2.6}
\]

\[
x = \bar{u}_B^1(y, t), \tag{2.2.7}
\]

\[
y = \bar{u}_B^1(y, t). \tag{2.2.8}
\]
where $h(x)$ is a smooth function and $0 < h(x) < 1$. Now, we define the approximate solution to (1.1) as
\[ u_a(x, t) = m_1 I_1 + m_2 I_2 + (1 - m_1 - m_2)O, \]
(2.2.6)
with $m_1 = m(x_1), m_2 = m(x_2)$, for some $\frac{11}{20} < \nu < 1$. Using the structures of the various orders of boundary layer and inner solutions, $u_a(x, t)$ satisfy the following problem
\[
\begin{align*}
\partial_t u_a + \partial_x f(u_a) - \varepsilon \partial_x^2 u_a &= -R^\varepsilon, \\
u &\leq 1, \\
&u_a(x = 0, t) = u_0(t), \\
&u_a(x = 1, t) = u_r(t), \\
&u_a(x, t = 0) = u_0(x),
\end{align*}
\]
where $-R^\varepsilon = \sum_{i=1}^{4} q_i(x, t)$ with
\[
\begin{align*}
q_1(x, t) &= (1 - m_1 - m_2)\{(f(O) - f(u^0) - \varepsilon f'(u^0)u^1
- \varepsilon^2 f''(u^0)(u^1)^2)}_x + \varepsilon^3 \partial_x^2 u^2, \\
q_2(x, t) &= m_1 \{f(I_1) - f(u^0) - \varepsilon f'(u^0)u^1 - \varepsilon^2 f'(u^0)u^2
- \varepsilon^2 f''(u^0)(u^1)^2)_x + \varepsilon^2 \partial_x u^2, \\
q_3(x, t) &= m_2 \{f(I_2) - f(u^0) - \varepsilon f'(u^0)u^1 - \varepsilon^2 f'(u^0)u^2
- \varepsilon^2 f''(u^0)(u^1)^2)_x + \varepsilon^2 \partial_x u^2, \\
q_4(x, t) &= -\varepsilon \partial_x^2 m_1(I_1 - O) - 2\varepsilon \partial_x m_1(I_1 - O)_x - \varepsilon \partial_x^2 m_2(I_2 - O)
- 2\varepsilon \partial_x m_2(I_2 - O)_x + \partial_x m_1(f(I_1) - f(O)) + \partial_x m_2(f(I_2) - f(O))
+ f(m_1 I_1 + m_2 I_2 + (1 - m_1 - m_2)O)_x
- (m_1 f(I_1) + m_2 f(I_2) + (1 - m_1 - m_2)O)_x.
\end{align*}
\]
Note that
\[\text{supp } m_1 \subseteq \{x : |x| \leq 2\varepsilon^\nu\}, \quad \text{supp } m_2 \subseteq \{x : |x - 1| \leq 2\varepsilon^\nu\}.
\]
In view of our construction, we have
(i) \[\text{supp } q_1 \subseteq \{(x, t) : \varepsilon^\nu \leq x \leq 1 - \varepsilon^\nu, 0 \leq t \leq T\},\]
and
\[\partial_x^\alpha q_1(x, t) = O(1)\varepsilon^{3-\alpha\nu}, \quad \alpha = 0, 1, 2, \quad (2.2.7)\]
(ii) \[\text{supp } q_2 \subseteq \{(x, t) : 0 \leq x \leq 2\varepsilon^\nu, 0 \leq t \leq T\}, \quad \partial_x^\alpha q_2(x, t) = O(1)\varepsilon^{2-\alpha}, \quad \alpha = 0, 1. \quad (2.2.8)\]
(iii) \[\text{supp } q_3 \subseteq \{(x, t) : 1 - 2\varepsilon^\nu \leq x \leq 1, 0 \leq t \leq T\}, \quad \partial_x^\alpha q_3(x, t) = O(1)\varepsilon^{2-\alpha}, \quad \alpha = 0, 1, 2. \quad (2.2.9)\]
(iv) $\supp q_4 \subseteq \{(x,t) : \varepsilon'_\nu \leq x \leq 2\varepsilon'_\nu, \ 1 - 2\varepsilon'_\nu \leq x \leq 1 - \varepsilon'_\nu, \ 0 \leq t \leq T\},$

$$\partial_x^\nu q_4(x,t) = O(1)\varepsilon^{(2-\alpha)\nu}, \ \alpha = 0, 1, 2. \quad (2.2.10)$$

Here, we have used

$$\partial_x^\nu (I_1 - O)(x,t) = O(1)\varepsilon^{(3-\alpha)\nu}, \ \alpha = 0, 1, 2, \quad (2.2.11)$$

for $\{(x,t) : \varepsilon'_\nu \leq |x| \leq 2\varepsilon'_\nu, t \in [0,T]\}$, and

$$\partial_x^\nu (I_2 - O)(x,t) = O(1)\varepsilon^{(3-\alpha)\nu}, \ \alpha = 0, 1, 2, \quad (2.2.12)$$

for $\{(x,t) : \varepsilon'_\nu \leq |x - 1| \leq 2\varepsilon'_\nu, t \in [0,T]\}$. Set $R^\varepsilon(x,t) = -\sum_{i=1}^{4} q_i(x,t)$, then it follows from (2.2.7)–(2.2.12) that

$$R^\varepsilon(x,t) = O(1)\varepsilon^{2\nu}, \quad \|R^\varepsilon\|_{L^2(0,1)}^2 = O(1)\varepsilon^{5\nu}, \quad (2.2.13)$$

and

$$\|\partial_t R^\varepsilon\|_{L^2(0,1)}^2 \sim O(1)\varepsilon^{5\nu}.$$  \quad (2.2.14)

**Lemma 2.4.** Let $u_a(x,t)$ be defined as in (2.2.6), then

$$u_a(x,t) = \begin{cases} u_0^b(y,t) + O(1)\varepsilon'_\nu, & \text{if } 0 \leq x \leq \varepsilon'_\nu, \\ u^0(x,t) + O(1)\varepsilon', & \text{if } \varepsilon'_\nu \leq x \leq 1 - \varepsilon'_\nu, \\ \bar{u}_0^b(\xi,t) + O(1)\varepsilon'_\nu, & \text{if } 1 - \varepsilon'_\nu \leq x \leq 1. \end{cases}$$

**Proof.** By construction, we have

$$u_a(x,t) = \begin{cases} I_1, & \text{if } 0 \leq x \leq \varepsilon'_\nu, \\ O + m_1(I_1 - O), & \text{if } \varepsilon'_\nu \leq x \leq 2\varepsilon'_\nu, \\ O, & \text{if } 2\varepsilon'_\nu \leq x \leq 1 - 2\varepsilon'_\nu, \\ O + m_2(I_2 - O), & \text{if } 1 - 2\varepsilon'_\nu \leq x \leq 1 - \varepsilon'_\nu, \\ I_2, & \text{if } 1 - \varepsilon'_\nu \leq x \leq 1, \end{cases}$$

and $O(x,t) = u^0(x,t) + O(1)\varepsilon$ on $|x| > \varepsilon'_\nu$, $I_1(x,t) = u_0^b(y,t) + O(1)\varepsilon'_\nu$ on $|x| \leq \varepsilon'_\nu$, $I_2(x,t) = \bar{u}_0^b(\xi,t) + O(1)\varepsilon'_\nu$ on $|x - 1| \leq \varepsilon'_\nu$. These, together with (2.1), (2.2.6) and lemma 2.1, yield (2.1). \qed

This finishes the construction of the formal approximate solution to (1.1). We now turn to the stability analysis of the multiple boundary layers. We will employ a perturbation analysis to prove Theorem 2.1. The main assumptions are the strength of the boundary layer near $\{x = 0\}$ is suitably small and the boundary layer near $\{x = 1\}$ is expansive. We decompose the exact solution $u^\varepsilon(x,t)$ as

$$u^\varepsilon(x,t) = u_a(x,t) + \varepsilon^{5/8}\varphi(x,t), \ x \in (0,1), \ t \in [0,T]. \quad (2.2.15)$$

It will be clear by our analysis that the exponent 5/8 can be chosen as any $\sigma \in (1/2,1]$. Then $\varphi$ solves

$$\partial_t \varphi - \varepsilon\partial_x^2 \varphi + \partial_x(\partial_x f(u_a)\varphi) + \varepsilon^{5/8}\partial_x(F_1\varphi^2) = \varepsilon^{-5/8}R^\varepsilon, \quad (2.2.16)$$

$$\varphi(x = 0,t) = \varphi(x = 1,t) = 0, \quad (2.2.17)$$

$$\varphi(x,t = 0) = 0, \quad (2.2.18)$$

where $F_1$ is bounded and defined by

$$\varepsilon^{5/4}F_1\varphi^2 = f(u^\varepsilon) - f(u_a) - \varepsilon^{5/8}\partial_u f(u_a)\varphi. \quad (2.2.19)$$
We thus have to find a small solution \( \varphi(x,t) \) to the problem (2.2.16)-(2.2.18). This will be given by the following result.

**Proposition 2.1.** Assume the conditions in Theorem 2.1 hold. Then the initial boundary value problem (1.1)-(1.4) has a unique solution \( u^f \in C^1([0,T]; H^2(0,1)) \) such that for \( \frac{11}{20} < \nu < 1 \), there holds

\[
\sup_{0 \leq t \leq T} \| u^f(\cdot,t) - u_a(\cdot,t) \|_{L^\infty(0,1)} \leq C \varepsilon^{5\nu/2-3/8}.
\]  

(2.2.20)

To prove this proposition, we need only to show that there exists a unique solution \( \varphi(x,t) \) to problem (2.2.16)-(2.2.18) such that \( \varphi \in C^1([0,T]; H^2(0,1)) \) and

\[
\sup_{0 \leq t \leq T} \| \varphi \|_{L^\infty(0,1)} \leq C \varepsilon^{5\nu/2-1}.
\]

(2.2.21)

This will be accomplished by some a priori estimates on the solution to (2.2.16)-(2.2.18) provided in the following several lemmas.

**Lemma 2.5.** Let \( \varphi \in C^1([0,T]; H^2(0,1)) \) be a solution to problem (2.2.16)-(2.2.18) and assume that the assumptions in Proposition 1 holds. Then there exists an absolute constant \( C \) such that

\[
\sup_{0 \leq t \leq T} \int_0^1 \varphi^2 dx + \varepsilon \int_0^T \int_0^1 |\partial_x \varphi|^2 dx dt \leq C \varepsilon^{5\nu-5/4},
\]

(2.2.22)

provided that

\[
\sup_{0 \leq t \leq T} \| \varphi \|_{L^\infty(0,1)} \leq C.
\]

(2.2.23)

**Proof.** Multiply (2.2.16) by \( \varphi \) and integrate the resulting equation over \((0,1)\) to get after integration by parts that

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 \varphi^2 dx + \varepsilon \int_0^1 |\partial_x \varphi|^2 dx + \int_0^1 \partial_x (\partial_a (f(u_a)) \varphi) dx \\
+ \varepsilon^{5/8} \int_0^1 \partial_x (F_1 \varphi^2) dx = \varepsilon^{-5/8} \int_0^1 R^e \varphi dx.
\]

(2.2.24)

Since the weak boundary layer is near \( \{x = 0\} \) and the strong expansive boundary layer is near \( \{x = 1\} \), we carry out the energy estimate by dividing the \([0,1]\) into several intervals. Then

\[
\int_0^1 \partial_x (\partial_a (f(u_a)) \varphi) dx \\
= \int_0^1 \partial_x (\partial_a (f(u_a)) \varphi) dx = \frac{1}{2} \int_0^1 \partial_x \partial_a f(u_a) \varphi^2 dx \\
= \frac{1}{2} \int_0^\nu \partial_x \partial_a f(u_a) \varphi^2 dx + \frac{1}{2} \int_\nu^{2\nu} \partial_x \partial_a f(u_a) \varphi^2 dx + \frac{1}{2} \int_{2\nu}^{1-2\nu} \partial_x \partial_a f(u_a) \varphi^2 dx \\
+ \frac{1}{2} \int_{1-2\nu}^{1-\nu} \partial_x \partial_a f(u_a) \varphi^2 dx + \frac{1}{2} \int_1^{1-\nu} \partial_x \partial_a f(u_a) \varphi^2 dx \\
= \sum_{i=1}^5 J_i,
\]

(2.2.25)
where we have used the boundary conditions (2.2.17). In view of the structure of the approximate solution \( u_a(x, t) \), we have

\[
J_1 = \frac{1}{2} \int_0^{\epsilon^\nu} \partial_x \partial_a f(I_1) \varphi^2 dx \\
= \frac{1}{2} \int_0^{\epsilon^\nu} \frac{1}{\epsilon} \partial_y \partial_a f(u_b^0 + \epsilon u_b + \epsilon^2 u_b^2) \varphi^2 dx \\
\leq C \epsilon \int_0^{\epsilon^\nu} \left| \partial_y \partial_a (u_b^0(y, t)) \right| \varphi^2 dx + C \int_0^{\epsilon^\nu} \varphi^2 dx \\
\leq C \epsilon \int_0^{\epsilon^\nu} \left| \partial_y u_b^0 \right| \cdot (\int_0^x \partial_y \varphi(\eta, t) d\eta)^2 dx + C \int_0^{\epsilon^\nu} \varphi^2 dx \\
\leq C \epsilon \int_0^{\epsilon^\nu} \left| \partial_y u_b^0 \right| \cdot x dx \cdot \| \partial_x \varphi \|_{L^2(0, 1)}^2 dx + C \int_0^{\epsilon^\nu} \varphi^2 dx \\
\leq C \epsilon \int_0^{\epsilon^\nu} \left| \partial_y u_b^0 \right| \cdot y dy \cdot \| \partial_x \varphi \|_{L^2(0, 1)}^2 dx + C \int_0^{\epsilon^\nu} \varphi^2 dx \\
\leq C \epsilon \| \partial_x \varphi \|_{L^2(0, 1)}^2 + C \int_0^{\epsilon^\nu} \varphi^2 dx,
\]

where we have used that the exponential decay property of the leading boundary layer profile (2.1.38). Using (2.2.11) and (2.1) yields

\[
J_2 = \frac{1}{2} \epsilon^{-5/8} \int_{\epsilon^\nu}^{2\epsilon^\nu} \partial_x \partial_a f(u_a) \varphi^2 dx \\
= \frac{1}{2} \epsilon^{-5/8} \int_{\epsilon^\nu}^{2\epsilon^\nu} \partial_a^2 f(u_a) \partial_x O + m_{1x} (I_1 - O) + m_1 (I_1 - O) \varphi^2 dx \\
\leq O(1) \int_{\epsilon^\nu}^{2\epsilon^\nu} \varphi^2 dx + O(1) \epsilon^{2\nu} \int_{\epsilon^\nu}^{2\epsilon^\nu} \varphi^2 dx \\
\leq O(1) \int_{\epsilon^\nu}^{2\epsilon^\nu} \varphi^2 dx.
\]

Similarly, we get

\[
J_3 = \frac{1}{2} \int_{-2\epsilon^\nu}^{1-2\epsilon^\nu} \partial_x \partial_a f(u_a) \varphi^2 dx \leq O(1) \int_{-2\epsilon^\nu}^{1-2\epsilon^\nu} \varphi^2 dx,
\]

\[
J_4 = \frac{1}{2} \int_{-1-2\epsilon^\nu}^{1-\epsilon^\nu} \partial_x \partial_a f(u_a) \varphi^2 dx \leq O(1) \int_{-1-2\epsilon^\nu}^{1-\epsilon^\nu} \varphi^2 dx.
\]

Using the expansive property \( \partial_y \partial_a f \geq 0 \) due to Lemma 2.2, we get

\[
J_5 = \frac{1}{2} \int_{-\epsilon^\nu}^{1} \partial_x \partial_a f(I_2) \varphi^2 dx \\
= \frac{1}{2} \int_{-\epsilon^\nu}^{1} \partial_x \partial_a f(u_b^0(\xi, t) + \epsilon u_b^1(\xi, t) + \epsilon^2 u_b^2(\xi, t)) \varphi^2 dx \\
= \frac{1}{2} \epsilon \int_{-\epsilon^\nu}^{1} \partial_x \partial_a f(u_b^0(\xi, t)) \varphi^2 dx + O(1) \int_{-\epsilon^\nu}^{1} \varphi^2 dx \\
= \frac{1}{2} \epsilon \int_{-\epsilon^\nu}^{1} \left| \partial_x \partial_a f(u_b^0(\xi, t)) \right| \varphi^2 dx + O(1) \int_{-\epsilon^\nu}^{1} \varphi^2 dx.
\]
In view of (2.2.19), we have
\[ \varepsilon^{5/8} \int_0^1 \partial_x(F_1 \varphi^2) \varphi dx \leq C \int_0^1 \varphi^2 dx + \eta \varepsilon \int_0^1 | \varphi_x |^2 dx. \]
By Young’s inequality and (2.2.13), we have
\[ \varepsilon^{-5/8} \int_0^1 R^c \varphi dx \leq C \varepsilon^{5/5-5/4} + C \int_0^1 | \varphi |^2 dx. \]
(2.2.26)
Collecting all the estimates, we have
\[ \frac{1}{2} \frac{d}{dt} \int_0^1 \varphi^2 dx + (1 - \eta - C \delta_0) \varepsilon \int_0^1 | \partial_x \varphi |^2 dx + \frac{1}{2 \varepsilon} \int_{1-\varepsilon}^1 | \partial_v \partial_u f(\bar{u}_a^0(\xi, t)) \varphi |^2 dx \leq C \varepsilon^{5 \nu-5/4} + C \int_0^1 \varphi^2 dx. \]
(2.2.27)
Choosing suitably small \( \eta, \delta_0 > 0 \) and using Gronwall’s inequality lead to (2.2.22).

Now, we need to estimate the tangential derivative of \( \varphi \). We have
\[ \text{Lemma 2.6.} \quad \text{Under the same assumptions as in Lemma 2.5, there exists an absolute positive constant } C \text{ such that} \]
\[ \sup_{0 \leq t \leq T} \int_0^1 | \partial_v \varphi(x, t) |^2 dx + \varepsilon \int_0^T \int_0^1 | \partial_t \partial_v \varphi(x, t) |^2 dx dt \leq C \varepsilon^{5 \nu-9/4}. \]
(2.2.28)
Proof. Set \( v(x, t) = \partial_v \varphi(x, t) \), then
\[ \lim_{t \to 0} v(x, t) = \lim_{t \to 0} \left( \varepsilon \partial_v^2 \varphi - \partial_x (\partial_u f(u_v) \varphi) + \varepsilon^{5/8} \partial_x (F_1 \varphi^2) + \varepsilon^{-5/8} R^c \right) = 0. \]
(2.2.29)
Thus \( v(x, t) \) solves
\[ \partial_t v - \varepsilon \partial_x^2 v + \partial_x (\partial_u f(u_v) v) + Q = \varepsilon^{-5/8} \partial_t R^c. \]
(2.2.30)
\[ v(x, t = 0) = v(x, t = 1) = 0. \]
(2.2.31)
\[ v(x, t = 0) = 0, \quad x \in (0, 1), \]
(2.2.32)
where \( Q = \partial_x (\partial_t \partial_u f(u_v) \varphi) + \varepsilon^{5/8} \partial_x (\partial_t F_1 \varphi^2) + 2 \varepsilon^{5/8} \partial_x (F_1 v \varphi) \). Multiply (2.2.30) by \( v(x, t) \) and integrate the resulting equation over \((0, 1)\) to get after integration by parts that
\[ \frac{1}{2} \frac{d}{dt} \int_0^1 v^2 dx + \varepsilon \int_0^1 | \partial_x v |^2 dx + \int_0^1 \partial_x (\partial_u f(u_v) v) v dx + 2 \varepsilon^{5/8} \int_0^1 \partial_x (F_1 v \varphi) v dx \]
\[ = \varepsilon^{-5/8} \int_0^1 \partial_t R^c v dx - \varepsilon^{5/8} \int_0^1 \partial_x (\partial_t F_1 \varphi^2) v dx - \int_0^1 \partial_x (\partial_t \partial_u f(u_v) \varphi) v dx. \]
The former five integrals can be treated similarly as in Lemma 2.5 and the last two integrals can be treated as follows.
\[ \varepsilon^{5/8} \int_0^1 \partial_x (\partial_t F_1 \varphi^2) v dx = \varepsilon^{5/8} \int_0^1 (\partial_t F_1 \varphi^2) \partial_x v dx \leq C \int_0^1 \varphi^2 dx + C \varepsilon \int_0^1 | \partial_x v |^2 dx, \]
and
\[ \int_0^1 \partial_x (\partial_t \partial_u f(u_v) \varphi) v dx = \int_0^1 \partial_t \partial_u f(u_v) \varphi \partial_x v dx \leq \frac{C_\nu}{\varepsilon} \int_0^1 \varphi^2 dx + \eta \varepsilon \int_0^1 | \partial_x v |^2 dx \]
\[ \leq \eta \varepsilon \int_0^1 \left| \partial_x \varphi \right|^2 \, dx + C \varepsilon^{5\nu-9/4}. \]

Collecting all the estimates, we obtain
\[ \frac{1}{2} \frac{d}{dt} \int_0^1 |v|^2 \, dx + \varepsilon \int_0^1 |\partial_x v|^2 \, dx \]
\[ \leq C \varepsilon \delta_0 \| \partial_x v \|_{L^2(0,1)}^2 + \varepsilon \eta \int_0^1 |\partial_x v|^2 \, dx + O(1) \int_0^1 |v|^2 \, dx + C \varepsilon^{5\nu-9/4}. \] (2.2.33)

By choosing suitably small \( \eta, \delta_0 > 0 \) and using Gronwall’s inequality, we have
\[ \sup_{0 \leq t \leq T} \int_0^1 |v|^2 \, dx + \varepsilon \int_0^T \int_0^1 |\partial_x v|^2 \, dx \, dt \leq C \varepsilon^{5\nu-9/4}. \]

This finishes the proof of Lemma 2.6.

Now, we’re ready to get the \( L^2 \) estimate of the normal derivative \( \partial_x \varphi \).

**Lemma 2.7.** Under the same assumptions as in Lemma 2.5, there holds
\[ \sup_{0 \leq t \leq T} \int_0^1 |\partial_x \varphi|^2 \, dx \leq C \varepsilon^{5\nu-11/4}. \] (2.2.34)

**Proof.** It follows from (2.2.27) that
\[ \frac{1}{2} \frac{d}{dt} \int_0^1 \varphi^2 \, dx + \varepsilon \int_0^1 |\partial_x \varphi|^2 \, dx \leq C \varepsilon^{5\nu-5/4}. \]

Then
\[ \varepsilon \int_0^1 (|\partial_x \varphi|^2) \, dx \leq -\frac{d}{dt} \int_0^1 \varphi^2 \, dx + C \varepsilon^{5\nu-5/4} \]
\[ \leq O(1) \| \varphi(x,t) \|_{L^2(0,1)} \cdot \| \partial_t \varphi(x,t) \|_{L^2(0,1)} \cdot \varepsilon \varepsilon^{5\nu-5/4} \leq C \varepsilon^{5\nu-7/4}. \]

With Lemma 2.5-2.7 at hand, one can prove Proposition 2.1 easily.

**Proof of Proposition 2.1.** To prove the existence of a solution \( \varphi \in C^1([0,T]; H^2(0,1)) \), to the problem (2.2.16)-(2.2.18), one needs only to justify the assumption (2.2.23). In fact, it follows from (2.2.22), (2.2.34), and Sobolev’s inequality that
\[ \sup_{0 \leq t \leq T} \| \varphi(x,t) \|_{L^\infty(0,1)} \leq \sqrt{2} \sup_{0 \leq t \leq T} \| \varphi(\cdot,t) \|_{L^2(0,1)}^{1/2} \cdot \sup_{0 \leq t \leq T} \| \partial_x \varphi(\cdot,t) \|_{L^2(0,1)}^{1/2} \]
\[ \leq C \varepsilon^{5\nu-5/16} \cdot \varepsilon^{5\nu-11/16} \leq C \varepsilon^{5\nu-2}. \]

This, in return, shows that there exists a unique solution \( u^\varepsilon \in C^1([0,T]; H^1(0,1)) \) to (1.1)-(1.4), which is given by the decomposition (2.2.15). Furthermore
\[ \sup_{0 \leq t \leq T} \| u^\varepsilon(\cdot,t) - u(x,\cdot,t) \|_{L^\infty(0,1)} = \sup_{0 \leq t \leq T} \| \varepsilon^{5/8} \varphi(x,t) \|_{L^\infty(0,1)} \]
\[ \leq C \varepsilon^{5/8} \cdot \varepsilon^{5\nu-2} \leq C \varepsilon^{5\nu-3/8}, \]

which verifies (2.2.20). Hence Proposition 2.1 is proved.
Proof of Theorem 2.1. First, we prove (2.2.1). For \( \nu \in (\frac{11}{20}, 1) \), in view of (2.2.15), (2.2.22) and Proposition 2.1, we obtain

\[
\sup_{0 \leq t \leq T} \| u^\varepsilon(\cdot,t) - u^0(\cdot,t) \|_{L^2(0,1)} \leq \sup_{0 \leq t \leq T} \| u^\varepsilon(\cdot,t) - u_a(\cdot,t) \|_{L^2(0,1)} + \sup_{0 \leq t \leq T} \| u_a(\cdot,t) - u^0(\cdot,t) \|_{L^2(0,1)} \leq C\varepsilon^{5/8} \| \varphi \|_{L^2(0,1)} + C\varepsilon \leq C\varepsilon.
\]

Furthermore, we have

\[
\sup_{0 \leq t \leq T, h < x < 1-h} \left| u^\varepsilon(\cdot,t) - u^0(\cdot,t) \right| \leq C\varepsilon^{5\nu/2-3/8} + C\varepsilon \leq C\varepsilon, \quad \text{for } h > 0.
\]

\[\square\]

Remark 2.3. The method in this subsection can be applied to both weak boundary layers case and both expansive strong boundary layers case. In such two cases, we have the same estimate of the error term as stated in Proposition 2.1. If the both boundary layers are strong and compressive, the analysis here can not be applied, and this case will be discussed in the following part.

2.3. Both strong compressive boundary layers. In this subsection, we will study the stability of both strong compressive boundary layers. We have

Theorem 2.2. Assume that

\[ u(t) - u^0(0,t) \geq 0, \tag{2.3.1} \]

and

\[ u(x) - u^0(1,t) \leq 0. \tag{2.3.2} \]

Then the viscous boundary layers are compressive on \([0,T]\), and there exits a positive constant \( \varepsilon_2 \) such that \( \forall 0 < \varepsilon \leq \varepsilon_2 \), the initial boundary value problem (1.1)-(1.4) has a unique solution \( u^\varepsilon \in C^1([0,T]; H^2(0,1)) \) such that for \( h > 0 \), there hold

\[ \sup_{0 \leq t \leq T} \| u^\varepsilon(\cdot,t) - u^0(\cdot,t) \|_{L^2(0,1)} \leq C\varepsilon, \tag{2.3.3} \]

and

\[ \sup_{h < x < 1-h} \left| u^\varepsilon(x,t) - u^0(x,t) \right| \leq C\varepsilon. \tag{2.3.4} \]

To prove the asymptotic limit, we need to find a new approximate solution to (1.1). Now, we define the approximate solution to (1.1) as

\[ \bar{u}_a(x,t) = m_1 I_1 + m_2 I_2 + (1 - m_1 - m_2) O + d(x,t) = u_a(x,t) + d(x,t), \tag{2.3.5} \]

with \( m_1 = m(\frac{1}{\nu}) \), \( m_2 = m(\frac{1}{\nu} - 1) \), for some \( \frac{19}{20} < \nu < 1 \), where \( O, I_1, I_2, m \) are defined in (2.2.3)–(2.2.5), and \( d(x,t) \) is a higher order correction term to be determined. Using the structures of the various orders of boundary layer and inner solutions, \( \bar{u}_a(x,t) \) satisfies the following problem

\[
\begin{aligned}
\partial_t \bar{u}_a &+ \partial_x f(\bar{u}_a) - \varepsilon \partial_x^2 \bar{u}_a = \sum_{i=1}^5 q_i(x,t), \\
\bar{u}_a(x = 0, t) &= u_t(t), \\
\bar{u}_a(x = 1, t) &= u_t(t), \\
\bar{u}_a(x, t = 0) &= u_0(x).
\end{aligned}
\]
Here \( q_i, i = 1, 2, 3, 4 \) are the same as in the section 2.2, and
\[
q_5(x, t) = d_t - \varepsilon d_{xx} + (f(\bar{u}_a) - f(\bar{u}_a - d))_x,
\]
with \( d(x, t) \) being the solution of the diffusion problem
\[
\begin{align*}
   \begin{cases}
      d_t = \varepsilon d_{xx} - R^e, \\
      d(x = 0, t) = d(x = 1, t) = 0, \\
      d(x, t = 0) = 0,
   \end{cases}
\end{align*}
\]
(2.3.6)
where \( -R^e = \sum_{i=1}^{4} q_i(x, t) \). Therefore, \( \bar{u}_a \) solves
\[
\begin{align*}
   \begin{cases}
      \partial_t \bar{u}_a + \partial_x f(\bar{u}_a) - \varepsilon \partial^2_x \bar{u}_a = (f(\bar{u}_a) - f(\bar{u}_a - d))_x, & x \in (0, 1), \ t > 0, \\
      \bar{u}_a(x = 0, t) = u_l(t), & \bar{u}_a(x = 1, t) = u_r(t), \ t > 0, \\
      \bar{u}_a(x, t = 0) = u_0(x), & x \in (0, 1).
   \end{cases}
\end{align*}
\]
By the standard parabolic theory, we have the following estimates for the linear diffusion wave \( d(x, t) \).

**Lemma 2.8.** Let \( d(x, t) \) be the solution of (2.3.6), the following estimates hold for all \( t \in [0, T] \) that
\[
\begin{align*}
   \sup_{0 \leq t \leq T} \int_0^1 |d(x, t)|^2 \, dx + \varepsilon \int_0^T \int_0^1 |\partial_x d(x, t)|^2 \, dxdt & \leq C\varepsilon^{5\nu}, \tag{2.3.7} \\
   \sup_{0 \leq t \leq T} \int_0^1 |\partial_t d(x, t)|^2 \, dx + \varepsilon \int_0^T \int_0^1 |\partial_x \partial_t d(x, t)|^2 \, dxdt & \leq C\varepsilon^{5\nu}, \tag{2.3.8}
\end{align*}
\]
and then
\[
\sup_{0 \leq t \leq T} \| d(x, t) \|_{L^\infty([0, 1])} \leq C\varepsilon^{5/2-1/4}. \tag{2.3.9}
\]

It follows from our construction that \( \bar{u}_a \) has the following property.

**Lemma 2.9.** Let \( \bar{u}_a(x, t) \) be defined as in (2.3.5), then
\[
\bar{u}_a(x, t) = \begin{cases}
   u_0^0(y, t) + O(1)\varepsilon^\nu, & \text{if } 0 \leq x \leq \varepsilon^\nu, \\
   u_0^0(x, t) + O(1)\varepsilon, & \text{if } \varepsilon^\nu \leq x \leq 1 - \varepsilon^\nu, \\
   u_0^0(\xi, t) + O(1)\varepsilon^\nu, & \text{if } 1 - \varepsilon^\nu \leq x \leq 1.
\end{cases}
\]
(2.3.10)

**Proof.** By construction, we have
\[
\bar{u}_a(x, t) = \begin{cases}
   I_1 + d, & \text{if } 0 \leq x \leq \varepsilon^\nu, \\
   O + m_1(I_1 - O) + d, & \text{if } \varepsilon^\nu \leq x \leq 2\varepsilon^\nu, \\
   O + m_2(I_2 - O) + d, & \text{if } 2\varepsilon^\nu \leq x \leq 1 - 2\varepsilon^\nu, \\
   I_2 + d, & \text{if } 1 - \varepsilon^\nu \leq x \leq 1,
\end{cases}
\]
and \( O(x, t) = u^0(x, t) + O(1)\varepsilon \) on \( |x| > \varepsilon^\nu \), \( I_1(x, t) = u_0^0(y, t) + O(1)\varepsilon^\nu \) on \( |x| \leq \varepsilon^\nu \), \( I_2(x, t) = u_0^0(\xi, t) + O(1)\varepsilon^\nu \) on \( |x - 1| \leq \varepsilon^\nu \). These, together with (2.2.11), (2.2.12) and Lemma 2.8, yield (2.3.10). \( \square \)

We decompose the solution as
\[
u^e(x, t) = \bar{u}_a(x, t) + \varepsilon^{5/8} \nu(x, t), \quad x \in (0, 1), \ t \in [0, T]. \tag{2.3.11}
Hence $v$ solves
\begin{align}
\partial_t v - \varepsilon \partial_x^2 v + \varepsilon^{-5/8} \partial_x \left( f(\bar{u}_a + \varepsilon^{5/8} v) - f(\bar{u}_a - d) \right) &= 0, \\
v(x = 0, t) &= v(x = 1, t) = 0, \\
v(x, t = 0) &= 0.
\end{align}

It turns out that it is more convenient to integrate this error problem once with respect to $x$. Set
\[ \varphi(x, t) = \int_0^x v(z, t) \, dz, \quad \forall x \in [0, 1], \]
then we obtain from (2.3.12)-(2.3.14) the following integrated error equation
\begin{align}
\partial_t \varphi - \varepsilon \partial_x^2 \varphi + \varepsilon^{-5/8} \left( f(\bar{u}_a + \varepsilon^{5/8} v) - f(\bar{u}_a - d) \right) &= 0, \\
\partial_x \varphi(x = 0, t) &= \partial_x \varphi(x = 1, t) = 0, \\
\varphi(x, t = 0) &= 0.
\end{align}

We thus have to find a small solution $\varphi(x, t)$ to the problem (2.3.15)-(2.3.17). This will be given by the following result.

**Proposition 2.2.** Suppose the assumptions in Theorem 2.2 hold, then $\forall 0 < \varepsilon \leq \varepsilon_2$, the initial boundary value problem (1.1)-(1.4) has a unique solution $u^\varepsilon \in C^1([0, T]; H^2(0, 1))$ such that
\begin{align}
\sup_{0 \leq t \leq T} \| u^\varepsilon(\cdot, t) - \bar{u}_a(\cdot, t) \|_{L^\infty(0, 1)} &\leq C \varepsilon^{5\nu/2 - 5/4}, \quad \text{for } \nu \in \left( \frac{19}{20}, 1 \right). \tag{2.3.18}
\end{align}

To prove this proposition 2.2, we need only to show that there exists a unique solution $\varphi(x, t)$ to problem (2.3.15)-(2.3.17) such that $\varphi \in C^1([0, T]; H^2(0, 1))$ and
\begin{align}
\sup_{0 \leq t \leq T} \| \partial_x \varphi \|_{L^\infty(0, 1)} &\leq C \varepsilon^{5\nu/2 - 15/8}. \tag{2.3.19}
\end{align}

This will be accomplished by some a priori estimates on the solution to (2.3.15)-(2.3.17) provided in the following several lemmas.

**Lemma 2.10.** Let $\varphi \in C^1([0, T]; H^2(0, 1))$ be a solution to (2.3.15)-(2.3.17) and assume that the assumptions in Theorem 2.2 holds. Then there exists an absolute constant $C$ such that
\begin{align}
\sup_{0 \leq t \leq T} \int_0^1 \varphi^2 dx + \varepsilon \int_0^T \int_0^1 |\partial_x \varphi|^2 dx dt &\leq C \varepsilon^{5\nu-5/4}, \tag{2.3.20}
\end{align}
provided that
\begin{align}
\sup_{0 \leq t \leq T} \| \partial_x \varphi \|_{L^\infty(0, 1)} &\leq C. \tag{2.3.21}
\end{align}

**Proof.** Multiply (2.3.15) by $\varphi$ and integrate the resulting equation over $(0, 1)$ to get after integration by parts that
\begin{align}
\frac{1}{2} \frac{d}{dt} \int_0^1 \varphi^2 dx + \varepsilon \int_0^1 |\partial_x \varphi|^2 dx + \varepsilon^{-5/8} \int_0^1 (f(\bar{u}_a + \varepsilon^{5/8} v) - f(\bar{u}_a - d)) \varphi dx &= 0. \tag{2.3.22}
\end{align}
The third integral on the left hand side of the above equation can be written as
\[ \varepsilon^{5/8} \int_0^1 (f(\bar{u}_a + \varepsilon^{5/8} v) - f(\bar{u}_a - d)) \varphi \, dx \]
\[ = \varepsilon^{5/8} \int_0^1 (\partial_u f(u_a)(\varepsilon^{5/8} \varphi_x + d) + O(1)(\varepsilon^{5/8} \varphi_x + d)^2) \varphi \, dx \]
\[ = \varepsilon^{5/8} \int_0^1 \partial_u f(u_a) \varphi \, dx + \int_0^1 \partial_u f(u_a) \varphi_x \cdot \varphi \, dx + O(1) \varepsilon^{-5/8} \int_0^1 (\varepsilon^{5/8} \varphi_x + d)^2 \varphi \, dx \]
\[ = \sum_{i=1}^3 A_i, \]
where we have used that \[ \partial \sum_{i=1}^3 A_i \]
\[ \leq C \int_0^1 \varphi^2 \, dx + C \varepsilon^{5/8 - 5/4}, \]
and
\[ A_2 = \int_0^1 \partial_u f(u_a) \varphi \, dx = \frac{1}{2} \int_0^1 \partial_u f(u_a)(\varphi^2) \, dx \]
\[ = \frac{1}{2} \partial_u f(u_r) \varphi^2(1) - \frac{1}{2} \partial_u f(u_l) \varphi^2(0) - \frac{1}{2} \varepsilon^{2} \int_0^1 \partial_u f(u_a) \varphi^2 \, dx \]
\[ = \frac{1}{2} \partial_u f(u_r) \varphi^2(1) + \frac{1}{2} \partial_u f(u_l) \varphi^2(0) + \sum_{i=1}^5 J_i, \]
where we have used \[ \partial_u f(u_l) < 0, \quad \partial_u f(u_r) > 0. \]

We now estimate \( J_i, i = 1, 2, 3, 4, 5 \) as follows.
\[ J_1 = -\frac{1}{2} \int_0^{\varepsilon} \partial_u f(I_1) \varphi^2 \, dx \]
\[ = -\frac{1}{2\varepsilon} \int_0^{\varepsilon} \partial_y \partial_u f(u_0^e(y, t)) \varphi^2 \, dx + O(1) \int_0^{\varepsilon} \varphi^2 \, dx \]
\[ = \frac{1}{2\varepsilon} \int_0^{\varepsilon} |\partial_y \partial_u f(u_0^e(y, t))| \varphi^2 \, dx + O(1) \int_0^{\varepsilon} \varphi^2 \, dx, \]
where we have used that the compressibility assumption on the boundary layers that \( \partial_y \partial_u f(u_0^e) \leq 0 \). In view of (2.2.11) in the matching zone, we obtain
\[ J_2 = -\frac{1}{2} \int_{\varepsilon}^{2\varepsilon} \partial_u f(u_a) \varphi^2 \, dx \]
\[ = -\frac{1}{2} \int_{\varepsilon}^{2\varepsilon} \partial_u^2 f(u_a)(\partial_x O + m_1x(I_1 - O) + m_1(I_1 - O)_{xx}) \varphi^2 \, dx \]
\[ \leq O(1) \int_{\varepsilon}^{2\varepsilon} \varphi^2 \, dx. \]
By the structure of the approximate solution, we have
\[ J_3 = -\frac{1}{2} \int_{1-\epsilon^v}^{1-2\epsilon^v} \partial_x \partial_u f(u_a) \varphi^2 \, dx \leq O(1) \int_{2\epsilon^v}^{1-\epsilon^v} \varphi^2 \, dx, \]
\[ J_4 = -\frac{1}{2} \int_{1-2\epsilon^v}^{1-\epsilon^v} \partial_x \partial_u f(u_a) \varphi^2 \, dx \leq O(1) \int_{1-2\epsilon^v}^{1-\epsilon^v} \varphi^2 \, dx. \]

The compressibility condition on the boundary layer near \( \{x = 1\} \) gives
\[ J_5 = -\frac{1}{2} \int_{1-\epsilon^v}^{1} \partial_x \partial_u f(I_2) \varphi^2 \, dx \]
\[ = -\frac{1}{2} \int_{1-\epsilon^v}^{1} \frac{1}{\epsilon} \partial_x \partial_u f(v_b^0(\xi, t) + \epsilon u_b^1(\xi, t) + \epsilon^2 u_b^2(\xi, t)) \varphi^2 \, dx \]
\[ = \frac{1}{2\epsilon} \int_{1-\epsilon^v}^{1} |\partial_x \partial_u f(v_b^0(\xi, t))| \varphi^2 \, dx + O(1) \int_{1-\epsilon^v}^{1} \varphi^2 \, dx. \]

Next, we have
\[ |A_3| = \epsilon^{-5/8} \left| \int_0^1 (d + \epsilon^{5/8} \varphi) d \varphi \right| \]
\[ \leq C \epsilon^{-5/8} \left| \int_0^1 |d| \, d\varphi + \epsilon^{5/8} \int_0^1 (\varphi)^2 \, d\varphi \right| \]
\[ \leq C \epsilon^{-5/4} \int_0^1 d^2 \varphi + C \int_0^1 \varphi^2 \, d\varphi + C \epsilon^{1/4} \cdot \epsilon \int_0^1 |\varphi_x|^2 \, d\varphi \]
\[ \leq C \epsilon^{5\nu-5/4} + \eta \epsilon \int_0^1 |\varphi_x|^2 \, d\varphi + C \int_0^1 \varphi^2 \, d\varphi, \]
for some \( \eta > 0 \). Collecting all the estimates, we have
\[ \frac{1}{2} \frac{d}{dt} \int_0^1 \varphi^2 \, dx + (1 - \eta) \epsilon \int_0^1 |\partial_x \varphi|^2 \, dx + \frac{1}{2\epsilon} \int_0^{\epsilon^v} \left| \partial_y \partial_u f(u_b^0(y, t)) \right| \varphi^2 \, dx \]
\[ + \frac{1}{2\epsilon} \int_{1-\epsilon^v}^{1} |\partial_x \partial_u f(v_b^0(\xi, t))| \varphi^2 \, dx + \frac{1}{2} \left| \partial_y f(u_r) \right| \varphi^2(1) + \frac{1}{2} |\partial_u f(u_l)| \varphi^2(0) \]
\[ \leq C \epsilon^{5\nu-5/4} + C \int_0^1 \varphi^2 \, dx. \]

By choosing suitably small \( \eta > 0 \) and using Gronwall’s inequality, we have
\[ \sup_{0 \leq t \leq T} \int_0^1 \varphi^2 \, dx + \epsilon \int_0^T \int_0^1 |\partial_x \varphi|^2 \, dx \, dt \leq C \epsilon^{5\nu-5/4}. \]

**Lemma 2.11.** Under the same assumptions as in Lemma 2.10, there exists an absolute positive constant \( C \) such that
\[ \sup_{0 \leq t \leq T} \int_0^1 |\partial_t \varphi|^2 \, dx + \epsilon \int_0^T \int_0^1 \left| \partial_x \partial_t \varphi \right|^2 \, dx \, dt \leq C \epsilon^{5\nu-9/4}, \tag{2.3.33} \]
and
\[ \sup_{0 \leq t \leq T} \int_0^1 |\partial_x \varphi|^2 \, dx \leq C \epsilon^{5\nu-11/4}. \tag{2.3.34} \]
Proof. We first prove (2.3.33). Set $\psi = \partial_t \varphi$, then $\psi$ solves the following problem

$$\partial_t \psi - \varepsilon \partial_x^2 \psi + \varepsilon^{-5/8} \partial_t (\partial_u f(u_\varepsilon) d) + \partial_t (\partial_u f(u_\varepsilon) \partial_x \varphi) + \tilde{F} = 0,$$

where

$$\tilde{F} = \partial_t (f(u_\varepsilon + \varepsilon^{5/8} \psi + d) - f(u_\varepsilon)) - \partial_t (\partial_u f(u_\varepsilon) \partial_x \varphi) - \varepsilon^{-5/8} \partial_t (\partial_u f(u_\varepsilon) d),$$

then

$$\tilde{F} \sim O(1) \varepsilon^{-5/8} (\varepsilon^{5/8} \partial_x \varphi + d)^2.$$ 

Multiply (2.3.35) by $\psi$ and integrate the resulting equation over $(0,1)$ to obtain

$$\frac{1}{2} \frac{d}{dt} \int_0^1 \psi^2 dx + \varepsilon \int_0^1 |\partial_x \psi|^2 dx + \varepsilon^{-5/8} \int_0^1 \partial_t (\partial_u f(u_\varepsilon) d) \psi dx$$

$$+ \int_0^1 \partial_t (\partial_u f(u_\varepsilon) \partial_x \varphi) \psi dx + \int_0^1 \tilde{F} \psi dx = 0.$$ 

Similar estimates as in Lemma 2.10 leads to

$$\frac{1}{2} \frac{d}{dt} \int_0^1 \psi^2 dx + (1-\eta) \varepsilon \int_0^1 |\partial_x \psi|^2 dx + \frac{1}{2\varepsilon} \int_0^1 \partial_t \partial_u f(u_\varepsilon(y,t)) \psi^2 dx$$

$$+ \frac{1}{2\varepsilon} \int_{1-\varepsilon^2}^{1} |\partial_x \partial_u f(u_\varepsilon(\xi,t))| \psi^2 dx + \frac{1}{2} |\partial_u f(u_\varepsilon)| \psi^2(1) + \frac{1}{2} |\partial_u f(u_\varepsilon)| \psi^2(0)$$

$$\leq C \varepsilon^{5\nu-9/4} + C \int_0^1 \psi^2 dx.$$ 

By choosing suitably small $\eta > 0$ and using Gronwall’s inequality, we have

$$\sup_{0 \leq t \leq T} \| \psi \|_{L^2(0,1)}^2 + \varepsilon \int_0^T \| \partial_x \psi \|_{L^2(0,1)}^2 dt \leq C \varepsilon^{5\nu-9/4}.$$ 

This proves (2.3.33). Together with (2.1) we get

$$\varepsilon \int_0^1 |\partial_x \varphi|^2 dx \leq C \left( \int_0^1 \varphi^2 dx \right)^{1/2} \left( \int_0^1 (\partial_t \varphi)^2 dx \right)^{1/2} + C \varepsilon^{5\nu-5/4} \leq C \varepsilon^{5\nu-7/4},$$

thus

$$\sup_{0 \leq t \leq T} \int_0^1 |\partial_x \varphi|^2 dx \leq C \varepsilon^{5\nu-11/4}.$$

Finally, we sketch the $L^2$ estimate on the second order derivatives of $\varphi$.

Lemma 2.12. Under the same assumption as in Lemma 2.10, there holds the estimates

$$\sup_{0 \leq t \leq T} \int_0^1 |\partial_t \partial_x \varphi|^2 dx + \varepsilon \int_0^T \int_0^1 |\partial_x^2 \partial_t \varphi|^2 dx dt \leq C \varepsilon^{5\nu-17/4},$$

and

$$\sup_{0 \leq t \leq T} \int_0^1 |\partial_x^2 \varphi|^2 dx \leq C \varepsilon^{5\nu-19/4}.$$
Proof. Set \( \theta(x, t) = \partial_t v(x, t) = \partial_t \partial_x \varphi(x, t) \), then \( \theta \) solves
\[
\partial_t \theta - \varepsilon \partial_x^2 \theta + \varepsilon^{-5/8} \partial_t \partial_x (f(\bar{u}_a + \varepsilon^{5/8} v) - f(\bar{v}_a - d)) = 0,
\]
(2.3.40)
the initial and boundary conditions are as follows
\[
\theta(x = 0, t) = \theta(x = 1, t) = 0, \quad \theta(x, t = 0) = 0.
\]
Multiply (2.3.40) by \( \theta \) and integrate the resulting equation over \((0, 1)\) to get
\[
\frac{1}{2} \frac{d}{dt} \int_0^1 \theta^2 dx + \varepsilon \int_0^1 |\partial_x \theta|^2 dx = \varepsilon^{-5/8} \int_0^1 \partial_t (f(\bar{u}_a + \varepsilon^{5/8} v) - f(\bar{v}_a - d)) \theta dx.
\]
(2.3.41)
The right hand side of the above equation is bounded by
\[
\varepsilon^{-5/8} \left| \int_0^1 \partial_t (f(\bar{u}_a + \varepsilon^{5/8} v) - f(\bar{v}_a - d)) \theta dx \right|
\leq O(1) \varepsilon^{-5/8} \int_0^1 \left| (\varepsilon^{5/8} v + d) \theta x \right| dx + O(1) \varepsilon^{-5/8} \int_0^1 \left| (\varepsilon^{5/8} \partial_x v + \partial_t d) \theta x \right| dx
\leq O(1) \varepsilon^{-1} \int_0^1 v^2 dx + \eta \varepsilon \int_0^1 |\partial_x|^2 dx + O(1) \varepsilon^{-9/4} \int_0^1 \partial_t d^2 dx + \eta \varepsilon \int_0^1 |\partial_x|^2 dx
\]
+ O(1) \varepsilon^{-1} \int_0^1 |\partial_t v|^2 dx + \eta \varepsilon \int_0^1 |\partial_x|^2 dx + O(1) \varepsilon^{-9/4} \int_0^1 |\partial_t d|^2 dx + \eta \varepsilon \int_0^1 |\partial_x|^2 dx
\leq O(1) \varepsilon^{5\nu - 17/4} + 4\eta \varepsilon \int_0^1 |\partial_x|^2 dx,
\]
where we have used Lemma 2.8, Lemma 2.10, and Lemma 2.11. Thus
\[
\frac{1}{2} \frac{d}{dt} \int_0^1 \theta^2 dx + (1 - 4\eta) \varepsilon \int_0^1 |\partial_x|^2 dx \leq C \varepsilon^{5\nu - 17/4},
\]
choosing suitably \( \eta \) such that \( 1 - 4\eta > 0 \) and by Gronwall’s inequality, we have
\[
\sup_{0 \leq t \leq T} \int_0^1 \theta^2 dx + \varepsilon \int_0^T \int_0^1 |\partial_x \theta|^2 dx \leq C \varepsilon^{5\nu - 17/4}.
\]
This proves (2.3.38). To get the \( L^2 \) estimate of \( \partial_x^2 \varphi \), we go back to (2.3.12)-(2.3.14). Multiplying (2.3.12) by \( v \) and integrating the equation over \((0, 1)\) yield
\[
\frac{1}{2} \frac{d}{dt} \int_0^1 v^2 dx + \varepsilon \int_0^1 |\partial_x v|^2 dx - \varepsilon^{-5/8} \int_0^1 (f(\bar{u}_a + \varepsilon^{5/8} v) - f(\bar{u}_a - d)) \partial_x v dx = 0,
\]
(2.3.42)
where
\[
\varepsilon^{-5/8} \left| \int_0^1 (f(\bar{u}_a + \varepsilon^{5/8} v) - f(\bar{u}_a - d)) \partial_x v dx \right|
\leq O(1) \varepsilon^{-5/8} \int_0^1 \left| (\varepsilon^{5/8} v + d) \partial_x v dx \right| \leq C \varepsilon^{-5/8} \int_0^1 \left| \partial_x v (\varepsilon^{5/8} v + d) \right| dx
\leq \eta \varepsilon \int_0^1 |\partial_x v|^2 dx + C \varepsilon^{-9/4} \int_0^1 d^2 dx + \eta \varepsilon \int_0^1 |\partial_x v|^2 dx + C \varepsilon^{-1} \int_0^1 v^2 dx
\leq 2\eta \varepsilon \int_0^1 |\partial_x v|^2 dx + C \varepsilon^{5\nu - 15/4}.
Choosing suitably small $\eta$ such that $1 - 2\eta > 0$, and then
\[
\epsilon \int_0^1 |\partial_x v|^2 \, dx \leq O(1) \int_0^1 vv_t \, dx + C \varepsilon^{5\nu - 15/4}
\]
Thus
\[
C \int_0^1 v^2 \, dx^{1/2} \left( \int_0^1 v_t^2 \, dx \right)^{1/2} + C \varepsilon^{5\nu - 15/4} \leq C \varepsilon^{5\nu - 15/4}.
\]
Thus
\[
\int_0^1 |\partial_x v|^2 \, dx \leq C \varepsilon^{5\nu - 19/4}.
\]

With Lemma 2.10-2.12 at hand, one can prove Proposition 2.2 easily.

**Proof of Proposition 2.2.** To prove the existence of solution $\varphi \in C^1([0,T]; H^1(0,1))$, to the problem (2.3.15)-(2.3.17), one needs only to justify the assumption (2.3.21). However, it follows from (2.3.34), (2.3.39), and Sobolev’s inequality that
\[
\sup_{0 \leq t \leq T} \| \partial_x \varphi(x,t) \|_{L^\infty(0,1)} \leq \sqrt{2} \sup_{0 \leq t \leq T} \| \partial_x \varphi(\cdot,t) \|_{L^2(0,1)}^{1/2} \cdot \sup_{0 \leq t \leq T} \| \partial_x \varphi(\cdot,t) \|_{L^2(0,1)}^{1/2}
\]
\[
\leq C \varepsilon^{5\nu/4 - 11/16} \cdot \varepsilon^{-5\nu/4 - 19/16} \leq C \varepsilon^{5\nu/2 - 15/8}.
\]

This shows that there exists a unique solution $u^\varepsilon \in C^1([0,T]; H^1(0,1))$ to (1.1)-(1.4), which is given by the decomposition (2.3.11). Furthermore
\[
\sup_{0 \leq t \leq T} \| u^\varepsilon(\cdot,t) - \bar{u}_{a}(\cdot,t) \|_{L^\infty(0,1)} = \sup_{0 \leq t \leq T} \| \varepsilon^{5/8} \partial_x \varphi(x,t) \|_{L^\infty(0,1)} \leq C \varepsilon^{5\nu/2 - 5/4},
\]
which verifies (2.3.18). Hence Theorem 2.2 is proved.

**Remark 2.4.** The analysis in the previous two subsections can be applied to the case that a weak boundary layer and a strong compressive boundary layer. Without loss of generality, we may assume the weak boundary layer is near the boundary $\{ x = 0 \}$ and the strong compressive boundary layer is near the boundary $\{ x = 1 \}$. Using the same approximate solution (2.3.5), then by noticing that the term $J_1$ in Lemma 2.10 can be treated as
\[
J_1 = - \frac{1}{2} \int_0^1 \partial_x \varphi^2 \, dx - \frac{1}{2} \int_0^1 \partial_x f(u_0^0(y,t)) \varphi^2 \, dx + O(1) \int_0^{x^\nu} \varphi^2 \, dx
\]
\[
\leq \frac{C}{\varepsilon} \int_0^{x^\nu} |\partial_y f(u_0^0(y,t))\partial_y u_0^0| \varphi^2 \, dx + C \int_0^{x^\nu} \varphi^2 \, dx,
\]
\[
\leq \frac{C}{\varepsilon} \int_0^{x^\nu} |\partial_y u_0^0| \cdot \left( \int_0^{x^\nu} \partial_y \varphi(\eta,t) \, d\eta \right)^2 \, dx + C \int_0^{x^\nu} \varphi^2 \, dx,
\]
\[
\leq \frac{C}{\varepsilon} \int_0^{x^\nu} |\partial_y u_0^0| \cdot x \, dx \cdot \| \partial_x \varphi \|_{L^2(0,1)}^2 \, dx + C \int_0^{x^\nu} \varphi^2 \, dx,
\]
\[
\leq C\varepsilon \int_0^{x^\nu} |\partial_y u_0^0| \cdot y \, dy \cdot \| \partial_x \varphi \|_{L^2(0,1)}^2 \, dx + C \int_0^{x^\nu} \varphi^2 \, dx,
\]
\[
\leq C\varepsilon \delta y \cdot \| \partial_x \varphi \|_{L^2(0,1)}^2 \, dx + C \int_0^{x^\nu} \varphi^2 \, dx,
\]
where the first term can be absorbed by the viscosity term with \( \delta_0 \) being suitably small, we can finally obtain the same estimate of the error term as stated in Proposition 2.2.

3. Inviscid limit in other cases. In Section 2, we focus on Case I that \( \partial_u f(u)|_{x=0} < 0 \) and \( \partial_u f(u)|_{x=1} > 0 \). For Case II with \( \partial_u f(u)|_{x=0} < 0 \) and \( \partial_u f(u)|_{x=1} < 0 \), we need to impose a suitable boundary condition on \( \{x = 1\} \). Consider

\[
\begin{align*}
\partial_y f(u_0^y) &= \partial_y^2 u_0^y, \\
u_0^y(y = 0, t) &= u_t(t), \\
u_0^y(y = +\infty, t) &= u^0(0, t)
\end{align*}
\]

Near \( \{x = 0\} \) the solution is given by Lemma 2.1, and near the boundary \( \{x = 1\} \), we have

\[
\begin{align*}
\partial_{\xi} f(u_0^\xi) &= \partial_{\xi}^2 u_0^\xi, \\
u_0^\xi(1, t) &= u_r(t), \\
u_0^\xi(\xi \to -\infty, t) &= u^0(1, t)
\end{align*}
\]

Note that \( \partial_u f(u)|_{x=1} < 0 \). Let \( U_{bl}(\xi, t) = \tilde{u}_0(\xi, t) - u^0(1, t) \) and \( V_{bl} = \partial_\xi U_{bl} \). Then \( W_{bl} = (U_{bl}, V_{bl}) \) satisfies

\[
\begin{align*}
\partial_{\xi} W_{bl} &= \Lambda(W_{bl}) W_{bl}, \\
U_{bl}(1, t) &= u_r(t) - u^0(1, t), \\
W_{bl}(\xi \to -\infty, t) &= 0,
\end{align*}
\]

where

\[
\Lambda(W_{bl}) = \begin{pmatrix} 0 & 1 \\ \partial_u f(\tilde{u}_0^\xi) & 0 \end{pmatrix}.
\]

By continuity, we have \( \partial_u f(\tilde{u}_0^\xi) < 0 \), then the coefficient matrix \( \Lambda(W_{bl}) \) possesses two pure imaginary eigenvalues. To guarantee the asymptotic limit (3.9) as \( \xi \to -\infty \), the boundary condition (3.8) can only be chosen to satisfy \( u_r(t) - u^0(1, t) = 0 \), that is

\[
u^0(1, t) = u_r(t).
\]

In this way, the only solution of (3.4)–(3.6) is \( \tilde{u}_0^\xi = u^0(1, t) \), which is independent of \( \xi \). The corresponding inviscid hyperbolic problem is

\[
\begin{align*}
\partial_t u + \partial_x f(u) &= 0, \quad x \in (0, 1), \quad t > 0, \\
u(1, t) &= u_r(t), \quad t > 0, \\
u(x, t = 0) &= u_0(x), \quad x \in (0, 1).
\end{align*}
\]

In the process of stability analysis, noticing the term like \( J_5 \) in Lemma 2.5 vanishes, we have the same estimates of the error terms as in Proposition 2.1 and Proposition 2.2.

For Case III that \( \partial_u f(u)|_{x=0} > 0 \) and \( \partial_u f(u)|_{x=1} > 0 \). Similar to case II, the corresponding inviscid hyperbolic problem is

\[
\begin{align*}
\partial_t u + \partial_x f(u) &= 0, \quad x \in (0, 1), \quad t > 0, \\
u(0, t) &= u_t(t), \quad t > 0, \\
u(x, t = 0) &= u_0(x), \quad x \in (0, 1).
\end{align*}
\]
In the process of stability analysis, since \( u_0^0(y,t) = u^0(0,t) \), then the term like \( J_1 \) in Lemma 2.5 or Lemma 2.10 vanishes, we still have the same estimates of the error term as in Proposition 2.1 and Proposition 2.2.

For Case IV that \( \partial_n f(u) \big|_{x=0} > 0 \) and \( \partial_n f(u) \big|_{x=1} < 0 \), we have the same estimate, and the corresponding inviscid hyperbolic problem is

\[
\begin{align*}
\partial_t u + \partial_x f(u) &= 0, & x \in (0,1), \ t > 0, \\
u(0,t) &= u_l(t), & t > 0, \\
u(1,t) &= u_r(t), & t > 0, \\
u(x,t = 0) &= u_0(x), & x \in (0,1).
\end{align*}
\]

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