ON THE DEFINING EQUATIONS OF REES ALGEBRA OF A HEIGHT TWO PERFECT IDEAL USING THE THEORY OF D-MODULES

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Abstract. Let $k$ be a field of characteristic zero, and $R = k[x_1, \ldots, x_d]$ with $d \geq 3$ be a polynomial ring in $d$ variables. Let $m = (x_1, \ldots, x_d)$ be the homogeneous maximal ideal of $R$. Let $K$ be the kernel of the canonical map $\alpha : \text{Sym}(I) \to \mathcal{R}(I)$, where $\text{Sym}(I)$ (resp. $\mathcal{R}(I)$) denotes the symmetric algebra (resp. the Rees algebra) of an ideal $I$ in $R$. We study $K$ when $I$ is a height two perfect ideal minimally generated by $d + 1$ homogeneous elements of same degree and satisfies $G_d$, that is, the minimal number of generators of the ideal $I_p$, $\mu(I_p) \leq \dim R_p$ for every $p \in V(I) \setminus \{m\}$. We show that

(i) $K$ can be described as the solution set of a system of differential equations,
(ii) the whole bigraded structure of $K$ is characterized by the integral roots of certain $b$-functions,
(iii) certain de Rham cohomology groups can give partial information about $K$.

1. Introduction

Let $R$ be a commutative Noetherian ring, and $I = (f_1, \ldots, f_n)$ be an ideal in $R$. The Rees algebra of $I$ is defined as $\mathcal{R}(I) = R[I] = \bigoplus_{i=0}^{\infty} I^i t^i$. We can see $\mathcal{R}(I)$ as a quotient of the polynomial ring $S = R[T_1, \ldots, T_n]$ via the surjective map

$$\psi : S \to \mathcal{R}(I)$$

$$T_i \mapsto f_i t.$$

The defining ideal of the Rees algebra $\mathcal{R}(I)$ is $\mathcal{I} = \ker \psi$, the kernel of the map $\psi$ and the defining equations are the generators of $\mathcal{I}$. Now the map $\theta : R^n \to I$ defined by $(r_1, \ldots, r_n) \mapsto \sum_{i=1}^{n} r_i f_i$ induces an $R$-algebra homomorphism $\beta : S = R[T_1, \ldots, T_n] \to \text{Sym}(I)$, where by $\text{Sym}(I)$ we denote the symmetric algebra of $I$. Clearly $\mathcal{L} := \ker \beta$ is generated by all linear forms $\sum_{i=1}^{n} r_i T_i$ such that $\sum_{i=1}^{n} r_i f_i = 0$. Therefore $\psi$ factors through $\text{Sym}(I)$ and we get the following diagram

$$\begin{array}{ccc}
S & \xrightarrow{\psi} & \mathcal{R}(I) \\
\downarrow{\beta} & & \downarrow{\alpha} \\
\text{Sym}(I) & \xrightarrow{\alpha} & \mathcal{R}(I)
\end{array}$$

Thus we get a relation between $\text{Sym}(I)$ and $\mathcal{R}(I)$ in the form of the following exact sequence

$$0 \to \mathcal{K} \to \text{Sym}(I) \xrightarrow{\alpha} \mathcal{R}(I) \to 0,$$

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where $\mathcal{K} = \ker \alpha$. As $\mathcal{K} \cong \mathcal{I}/\mathcal{L}$, it is enough to study $\mathcal{K}$. Determining the equations of the Rees algebra is a fundamental problem and several works are going on this topic. A brief survey in this regards can be found in [8, Introduction]. The defining ideal is known when $I$ is a linearly presented height two perfect ideal satisfying $G_d$. In this case $\mathcal{R}(I)$ is Cohen-Macaulay. So in recent years, our main aim is to investigate $\mathcal{I}$ when $I$ is a height two perfect ideal which is either not linearly presented or does not satisfy $G_d$. In this situation mostly $\mathcal{R}(I)$ is not Cohen-Macaulay. A study of Rees algebra of grade two, almost linearly presented (all but the last column of the presentation matrix $\varphi$ of the ideal are linear and the last column consists of homogeneous entries of arbitrary degree $n \geq 1$), perfect ideal which is minimally generated by homogeneous elements of the same degree, was done by A. R. Kustin, C. Polini and B. Ulrich in [8, Introduction]. The defining ideal is known when $I$ is a linearly presented height two perfect ideal satisfying $G_d$. In this case $\mathcal{R}(I)$ is Cohen-Macaulay. So in recent years, our main aim is to investigate $\mathcal{I}$ when $I$ is a height two perfect ideal which is either not linearly presented or does not satisfy $G_d$. In this situation mostly $\mathcal{R}(I)$ is not Cohen-Macaulay. A study of Rees algebra of grade two, almost linearly presented (all but the last column of the presentation matrix $\varphi$ of the ideal are linear and the last column consists of homogeneous entries of arbitrary degree $n \geq 1$), perfect ideal which is minimally generated by homogeneous elements of the same degree, was done by A. R. Kustin, C. Polini and B. Ulrich in [8] for $d = 2$ and was generalized by J. A. Boswell and V. Mukundan in [10] for $d > 2$. In [7], Yairon Cid first described $\mathcal{K}$ when $I$ is a height two ideal in $k[x_1, x_2]$, minimally generated by three homogeneous polynomials of the same degree using “$D$-module theory”. In this article we generalize his results for dimension $d \geq 3$ mostly following his path. We restrict our attention to height two perfect ideals satisfying $G_d$ and minimally generated by $d + 1$ homogeneous elements of same degree.

1.1. Let $k$ be a field of characteristic zero, $R = k[x_1, \ldots, x_d]$ with $d \geq 3$ be a polynomial ring in $d$ variables, and $m = (x_1, \ldots, x_d)$ be the homogeneous maximal ideal in $R$. Let $I = (f_1, \ldots, f_{d+1}) \subset R$ be a perfect ideal of height two minimally generated by $d + 1$ homogeneous elements of same degree $\nu$. Notice $\mathcal{R}(I) \cong S/\mathcal{I}$ where $S = R[T_1, \ldots, T_{d+1}] = k[x_1, \ldots, x_d, T_1, \ldots, T_{d+1}]$. We consider $S$ as a bigraded $k$-algebra, where $\text{bideg}(T_i) = (1, 0)$ and $\text{bideg}(x_j) = (0, 1)$. Further, if we consider $\text{bideg} t = (1, -\nu)$, then the map $\psi$ defined in (1.0.1) becomes bi-homogeneous and hence $\mathcal{I} = \ker \psi$ becomes a bi-graded $S$-module. Let $U = k[T_1, \ldots, T_{d+1}]$ be a polynomial ring. Clearly $S = R \otimes_k U$. If $M$ is a bi-graded $S$-module, we write

\[ M_{p,*} = \bigoplus_{q \in \mathbb{Z}} M_{p,q} \quad \text{and} \quad M_{*,q} = \bigoplus_{p \in \mathbb{Z}} M_{p,q}, \]

where $M_{p,*}$ is a graded $R$-module and $M_{*,q}$ is a graded $U$-module.

As $I$ is a perfect ideal, $\text{projdim}_R(R/I) = \text{grade} I = \text{ht} I = 2$ and hence $\text{projdim} I = 1$. By Hilbert-Burch theorem, $I$ has a free resolution of the form

\[ 0 \rightarrow R^d \xrightarrow{\varphi} R^{d+1} \xrightarrow{[f_1, \ldots, f_{d+1}]} I \rightarrow 0 \]

and $I \cong I_d(\varphi)$ where $I_d(\phi)$ is the ideal generated by $d \times d$-minors of $\varphi = (a_{i,j})_{(d+1) \times d}$ (see [4, Theorem 1.4.17]). If

\[ [g_1, \ldots, g_d] = [T_1, \ldots, T_{d+1}] \cdot \varphi, \]

then $\text{Sym}(I) \cong S/(g_1, \ldots, g_d)$. Thus $\mathcal{K} = \mathcal{I}/(g_1, \ldots, g_d)$. Note that

\[ g_j = a_{1,j} T_1 + \cdots + a_{d+1,j} T_{d+1} \quad \forall 1 \leq j \leq d. \]
Set \( \deg a_{i,j} = \nu_j \) for all \( 1 \leq j \leq d \) to make all \( g_j \) homogeneous with \( \text{bideg} \ g_j = (1, \nu_j) \) in \( S \). Then \( \mathcal{K} \) gets a natural structure of bi-graded \( S \)-module. Since \( I \) is generated by the maximal minors of \( \varphi \) so we get that \( \sum_{i=1}^{d} \nu_i = \nu \). We can rewrite (1.1.2) in the following way

\[
0 \to \bigoplus_{i=1}^{d} R(-\nu - \nu_i) \xrightarrow{\varphi} \bigoplus_{i=1}^{d+1} R(-\nu) \xrightarrow{[f_1, \ldots, f_{d+1}]} I \to 0,
\]

For \( 1 \leq r \leq d \), let \( \varphi_r \) be the \((d+1) \times r\) matrix consisting of the first \( r \) columns of \( \varphi \) and set \( E_r = \text{coker} \varphi_r \), so that \( \varphi_d = \varphi \) and \( E_d = I \). As \( \varphi \) is injective so is \( \varphi_r \) for all \( 1 \leq r \leq d \). Note that

\[
0 \to \bigoplus_{i=1}^{r} R(-\nu - \nu_i) \xrightarrow{\varphi_r} \bigoplus_{i=1}^{d+1} R(-\nu) \xrightarrow{[f_1, \ldots, f_{d+1}]} E_r \to 0
\]

is a presentation of \( E_r \). Now \( (g_1, \ldots, g_r) = [T_1, \ldots, T_{d+1}] \cdot \varphi_r \) implies that \( \text{Sym}(E_r) \cong S/(g_1, \ldots, g_r) \).

**Definition 1.2.** If the minimal number of generators of the ideal \( I_p, \mu(I_p) \leq \dim R_p \) for every \( p \in V(I) \setminus \{m\} \), then we say that \( I \) satisfies \( G_d \), where \( d \) is the dimension of the ring \( R \).

In our case, as \( I \) is a height two perfect ideal generated by homogeneous polynomials of the same degree \( \nu \), by [8, (3.7.3)]

\[
K' = H^0_m(\text{Sym}(I(\nu))) \iff I \text{ satisfies } G_d,
\]

where \( K' = \ker(\text{Sym}(I(\nu)) \to \mathcal{R}(I)) \). Moreover, as \( \mu(I) = d + 1 > \dim R = d \) so \( K' \neq 0 \) by [8, (3.7.2)].

**1.3.** As mentioned in [8, p. 2], the condition \( G_d \) can be interpreted in terms of the height of Fitting ideals. If \( I \) has a presentation (1.1.2), then \( I \) satisfies \( G_d \) if \( \text{ht } \text{Fitt}_i(I) = \text{ht } I_{d+1-i}(\varphi) > i \) for all \( i < d \), that is, \( \text{ht } I_i(\varphi) > d + 1 - i \) for all \( d + 1 - i < d \), i.e., \( i > 1 \). Thus \( \text{ht } I_2(\varphi) \geq d \) (as \( \text{ht } I_2(\varphi) > d + 1 - 2 = d - 1 \)). Since \( I_1(\varphi) \supseteq I_2(\varphi) \) so we get \( d \leq \text{ht } I_2(\varphi) \leq \text{ht } I_1(\varphi) \leq \dim R = d \) and hence \( \text{ht } I_1(\varphi) = d \).

**Remark 1.4.** Let \( I \) be a height two ideal in \( k[x_1, x_2] \), minimally generated by three homogeneous polynomials of the same degree. By *Auslander Buchsbaum formula* we have \( \text{projdim}_R(R/I) + \text{depth}(R/I) = \text{depth}(R) \). Since \( \dim R/I = \dim R - \text{ht } I = 0 \) so \( \text{depth}(R/I) = 0 \). Thus \( \text{projdim}_R(R/I) = 2 \) which implies that \( \text{projdim}_R(I) = 1 \). By [4, Proposition 1.4.5] we get that \( I \) has a free resolution

\[
F_* : 0 \to R^2 \xrightarrow{\varphi} R^3 \to I \to 0.
\]

So \( I \cong I_2(\varphi) \) is a perfect ideal by *Hilbert-Burch theorem*. As \( \text{ht } I_2(\varphi) = \text{ht } I = 2 \), by 1.3 it follows that \( I \) satisfies \( G_2 \).

This article contains the following four main results which are generalization of Theorem A, Theorem B, Theorem C and Theorem D in [7]. Let \( D = A_d(k) = k[x_1, \ldots, x_d, \partial_1, \ldots, \partial_d] \) be the \( d \)-th Weyl algebra over \( k \) and \( \mathcal{T} = A_d(k)[T_1, \ldots, T_{d+1}] \) be a polynomial ring over \( D \). We define \( d \) differential operators \( L_i = F(g_i) \) for all \( 1 \leq i \leq d \) by applying the Fourier transform \( F \) to \( g_i \)'s in (1.1.3). Here \( F : \mathcal{T} \to \mathcal{T} \) is an automorphism defined by \( F(x_i) = \partial_i, F(\partial_i) = -x_i \) and \( F(T_i) = T_i \). The following result says that \( \mathcal{K} \) can be described as a solution set of a system of differential equations. As an
application of it we get Corollary 2.6 which gives the highest $x$-degree for an element in the graded part $K_{p,*}$ for all $p \geq d$.

**Theorem 1.** Let $I \subset R = k[x_1, \ldots, x_d]$ with $d \geq 3$ be a perfect ideal of height two satisfying $G_d$ and minimally generated by $d+1$ homogeneous elements of same degree $\nu$, and let $L_i = F(g_i)$ be the Fourier transform of $g_i$ from (1.1.3) with bideg $g_i = (1, \nu_i)$ in $S = R[T_1, \ldots, T_{d+1}]$ for $i = 1, \ldots, d$. Then we have the following isomorphism of bigraded $S$-modules

$$K \cong \text{Sol} \left( L_1, \ldots, L_d; S \right)_{\mathcal{F}}(-d, -\nu + d),$$

where $\text{Sol}(L_1, \ldots, L_d; S) = \{ h \in S \mid L_i \cdot h = 0 \text{ for all } i = 1, \ldots, d \}$ and the subscript-$\mathcal{F}$ is used to stress the bigraded $S$-module structure induced on $\text{Sol}(L_1, \ldots, L_d; S)$ by the twisting of the Fourier transform $\mathcal{F}$ (see Section 3 or [7, Lemma 3.9]).

The following result gives us information about the lowest possible $x$-degree for an element in the graded part $K_{p,*}$ and describes the $b$-function of a family of holonomic $D$-modules which is defined in Notation 4.1.

**Theorem 2** (with hypotheses as in Theorem 1). Then for each integer $p \geq d$ there exists a nonzero $b$-function $b_p(s)$, and we have a relation between the graded structure of $K_{p,*}$ and the integral roots of $b_p(s)$ given in the following equivalence

$$K_{p,u} \neq 0 \iff b_p(-\nu + d + u) = 0.$$

Even more, we have that these are the only possible roots of $b_p(s)$, that is,

$$b_p(s) = \prod_{\{u \in \mathbb{Z} \mid K_{p,u} \neq 0\}} (s + \nu - d - u).$$

The following result shows that there is an isomorphism of graded $U$-modules between $K$ and a certain de Rham cohomology group.

**Theorem 3** (with hypotheses as in Theorem 1). Then we have the following isomorphism of graded $U = k[T_1, \ldots, T_{d+1}]$-modules

$$K \cong H^0_{dR}(Q) = \{ w \in Q \mid \partial_i \cdot w = 0 \text{ for all } i = 1, \ldots, d \},$$

where $Q$ denotes the left $T$-module $T/(L_1, \ldots, L_d)$. In particular, for any integer $p$ we have an isomorphism of $k$-vector spaces

$$K_{p,*} \cong H^0_{dR}(Q_p) = \{ w \in Q_p \mid \partial_i \cdot w = 0 \text{ for all } i = 1, \ldots, d \}.$$

**Theorem 4** (with hypotheses as in Theorem 1). Then we have the following isomorphism of bigraded $S$-modules

$$K \cong \left\{ w \in H^2_m \left( \frac{S}{(g_1, \ldots, g_{d-2})} \right)(-2, -\nu_d - \nu_{d-1}) \mid g_{d-1} \cdot w = 0 \text{ and } g_d \cdot w = 0 \right\}.$$
Remark 1.5. We put the extra conditions on \( I \) (along with \( \text{ht} I = 2 \) and \( \mu(I) = d + 1 \)), that \( I \) is a perfect ideal satisfying \( G_d \), mainly to prove Theorem 4. Although this result does not look promising, but surprisingly using this we get Theorem 1. Under these extra conditions \( g_1, \ldots, g_d \) is a regular sequence in \( S \), see Lemma 2.2. This fact helps us to prove Proposition 5.1 which is used to prove Theorem 2 and Theorem 3. Rest of the results follow by almost similar methods used in [7]. We keep the name of the sections, statements of the results and all other arrangements as it is in [7] so that readers can easily do a comparative study.

The article is organized as follows. In Section 2 we prove Theorem 4, in Section 3 we prove Theorem 1, in Section 4 we prove Theorem 2 and in Section 3 we prove Theorem 3. In Section 6, we generalize the function given in [7, Section 6] that can compute the \( b \)-function \( b_p(s) \) in Macaulay2 and give some examples to show that how it helps us to recover the bi-graded structure of \( K \). In the last section we show that the generalized version can help us to compute \( \text{reltype} F(I), \text{reg} F(I), e(F(I)) \) and \( r(I) \), where \( \text{reltype} F(I) \) denotes the relation type of \( F(I) \) (see Definition 7.2), \( \text{reg} F(I) \) denotes the regularity of \( F(I) \), \( e(F(I)) \) denotes the Hilbert-Samuel multiplicity of \( F(I) \) and \( r(I) \) denotes the reduction number of \( I \). Using this we can also get a lower bound of \( \text{reltype} R(I) \) and hence \( \text{reg} R(I) \).

2. An “Explicit” Description of the Equations

2.1. Assumptions: We will prove our results under the following assumptions:

(i) \( R = k[x_1, \ldots, x_d] \) with \( \text{char} k = 0 \) and \( d \geq 3 \),

(ii) \( I \) is a perfect ideal of height 2, satisfies \( G_d \) and \( I = (f_1, \ldots, f_{d+1}) \) with \( \text{deg} f_i = \nu \) for all \( i \).

Then we have observed in 1.1, \( I \) has a presentation \((1.1.4), \text{Sym}(I) \cong S/(g_1, \ldots, g_d) \) where \( g_1, \ldots, g_d \) as in \((1.1.3) \), and \( S = R[T_1, \ldots, T_{d+1}] \) is a bigraded \( k \)-algebra with \( \text{bideg}(T_i) = (1,0), \text{bideg}(x_j) = (0,1) \).

Lemma 2.2 (with hypotheses as in 2.1). \( g_1, \ldots, g_d \) is a regular sequence in \( S \).

Proof. By Huneke and Rossi’s result (see [9]) we have

\[ \dim \text{Sym}(I) = \max_{p \in \text{Spec}(R)} \{ \dim R/p + \mu(I_p) \}. \]

Since \( I \) satisfies \( G_d \) so for any \( p \neq m \) we have \( \mu(I_p) \leq \dim R_p = \text{ht} p \) and hence \( \dim R/p + \mu(I_p) \leq \dim R/p + \text{ht} p = \dim R = d \) (as \( R \) is Cohen-Macaulay). On the other hand, \( \dim R/m + \mu(I_m) = 0 + d + 1 = d + 1 \). It follows that \( \dim \text{Sym}(I) = d + 1 \). Therefore \( \text{ht}(g_1, \ldots, g_d) = d \) (as \( S \) is Cohen-Macaulay). Hence \( g_1, \ldots, g_d \) is a regular sequence in \( S \) (see [5, Theorem 17.4]). \( \square \)

2.3. Generalization of [7, Lemma 2.3]: We have a short exact sequence

\[ 0 \to S(-1, -\nu_1) \xrightarrow{g_1} S \to S/(g_1) \to 0 \]
which induces a long exact sequence

\[ 0 \to H^0_m(S)(-1, -\nu_1) \xrightarrow{g_1} H^0_m(S) \to H^0_m(S/(g_1)) \to \cdots \]

\[ \cdots \to H^{d-1}_m(S)(-1, -\nu_1) \xrightarrow{g_1} H^{d-1}_m(S) \to H^{d-1}_m(S/(g_1)) \to H^d_m(S) \xrightarrow{\gamma_1} H^d_m(S/(g_1)) \to H^{d+1}_m(S)(-1, -\nu_1) \xrightarrow{g_1} H^{d+1}_m(S) \to \cdots \]

Since \( R \subseteq S \) is a flat ring extension so we get that

\[ H^j_{mS}(S) = \begin{cases} 
\neq 0 & \text{if } j = d \\
0 & \text{otherwise}, 
\end{cases} \]

using the fact that

\[ H^j_m(R) = \begin{cases} 
x^{-1}_1 \cdots x^{-1}_d k[x^{-1}_1, \ldots, x^{-1}_d] & \text{if } j = d \\
0 & \text{otherwise}, 
\end{cases} \]

Thus \( H^i_m(S/(g_1)) = 0 \) for all \( 1 \leq i \leq d - 2 \), and we get an exact sequence

\[ (2.3.6) \quad 0 \to H^{d-1}_m(S/(g_1)) \xrightarrow{\gamma_1} H^d_m(S)(-1, -\nu_1) \xrightarrow{g_1} H^d_m(S) \to H^d_m(S/(g_1)) \to 0. \]

Since \( g_1, \ldots, g_d \) is a regular sequence so we have short exact sequences

\[ 0 \to S_{(g_1)}(-1, -\nu_2) \xrightarrow{g_2} S_{(g_1)} \to S_{(g_1, g_2)} = \text{Sym}(E_2) \to 0 \]

\[ 0 \to S_{(g_1, g_2)}(-1, -\nu_3) \xrightarrow{g_3} S_{(g_1, g_2)} \to S_{(g_1, g_2, g_3)} = \text{Sym}(E_3) \to 0 \]

\[ \vdots \]

\[ 0 \to S_{(g_1, \ldots, g_{d-1})}(-1, -\nu_d) \xrightarrow{g_d} S_{(g_1, \ldots, g_{d-1})} \to S_{(g_1, \ldots, g_d)} = \text{Sym}(I) \to 0 \]

From the induced long exact sequences we get

\[ H^i_m(S/(g_1, \ldots, g_j)) = 0 \] for all \( 0 \leq i \leq d - (j + 1) \) and \( 1 \leq j \leq d - 1 \)
and the following exact sequences

\[(2.3.7) \quad 0 \rightarrow H_m^{d-2}\left(\frac{S}{(g_1, g_2)}\right) \xrightarrow{\gamma_2} H_m^{d-1}\left(\frac{S}{(g_1)}\right) (-1, -\nu_2) \xrightarrow{g_2} H_m^{d-1}\left(\frac{S}{(g_1)}\right),\]

\[0 \rightarrow H_m^{d-3}\left(\frac{S}{(g_1, g_2, g_3)}\right) \xrightarrow{\gamma_3} H_m^{d-2}\left(\frac{S}{(g_1, g_2)}\right) (-1, -\nu_3) \xrightarrow{g_3} H_m^{d-2}\left(\frac{S}{(g_1, g_2)}\right),\]

\[\vdots\]

\[0 \rightarrow H_m^0\left(\frac{S}{(g_1, \ldots, g_d)}\right) \xrightarrow{\gamma_d} H_m^1\left(\frac{S}{(g_1, \ldots, g_{d-1})}\right) (-1, -\nu_d) \xrightarrow{g_d} H_m^1\left(\frac{S}{(g_1, \ldots, g_{d-1})}\right),\]

where \(\gamma_i\)'s are the connecting homomorphisms. Hence

\[(2.3.8) \quad 0 \rightarrow H_m^0(\text{Sym}(I)) \rightarrow H_m^1(\text{Sym}(E_{d-1}))(−1, −\nu_d) \xrightarrow{g_d} H_m^1(\text{Sym}(E_{d-1})),\]

\[(2.3.9) \quad 0 \rightarrow H_m^1(\text{Sym}(E_{d-1})) \rightarrow H_m^2(\text{Sym}(E_{d-2}))(−1, −\nu_{d-1}) \xrightarrow{g_{d-1}} H_m^2(\text{Sym}(E_{d-2})).\]

**Theorem 2.4** (with hypotheses as in 2.1). We have an isomorphism of bi-graded \(S\)-modules

\[\mathcal{K} \cong \left\{ w \in H_m^2\left(\frac{S}{(g_1, \ldots, g_{d-2})}\right) (-2, -\nu_d - \nu_{d-1}) \mid g_{d-1} \cdot w = 0 \text{ and } g_d \cdot w = 0 \right\}.\]

**Proof.** The commutative diagram

\[
\begin{array}{ccc}
\frac{S}{(g_1, \ldots, g_{d-2})}(-2, -\nu_{d-1} - \nu_d) & \xrightarrow{g_{d-1}} & \frac{S}{(g_1, \ldots, g_{d-2})}(-1, -\nu_d) \\
\bigg| \downarrow g_d & & \bigg| \downarrow g_d \\
\frac{S}{(g_1, \ldots, g_{d-2})}(-1, -\nu_{d-1}) & \xrightarrow{g_{d-1}} & \frac{S}{(g_1, \ldots, g_{d-2})}
\end{array}
\]

can be extended to the following commutative diagram with exact rows

\[
\begin{array}{ccc}
0 & \rightarrow & \frac{S}{(g_1, \ldots, g_{d-2})}(-2, -\nu_{d-1} - \nu_d) \\
\bigg| \downarrow g_d & & \bigg| \downarrow g_d \\
0 & \rightarrow & \frac{S}{(g_1, \ldots, g_{d-2})}(-1, -\nu_{d-1}) \\
\end{array}
\]

\[
\begin{array}{ccc}
\frac{S}{(g_1, \ldots, g_{d-2})}(-2, -\nu_{d-1} - \nu_d) & \xrightarrow{g_{d-1}} & \frac{S}{(g_1, \ldots, g_{d-2})}(-1, -\nu_d) \\
\bigg| \downarrow g_d & & \bigg| \downarrow g_d \\
\frac{S}{(g_1, \ldots, g_{d-2})}(-1, -\nu_{d-1}) & \xrightarrow{g_{d-1}} & \frac{S}{(g_1, \ldots, g_{d-2})}
\end{array}
\]

By (2.3.9) we get the following commutative diagram of exact rows

\[
\begin{array}{ccc}
0 & \rightarrow & H_m^1(\text{Sym}(E_{d-1}))(−1, −\nu_d) \\
\bigg| \downarrow g_d & & \bigg| \downarrow g_d \\
0 & \rightarrow & H_m^1(\text{Sym}(E_{d-1})) \\
\end{array}
\]

\[
\begin{array}{ccc}
H_m^2\left(\frac{S}{(g_1, \ldots, g_{d-2})}\right) (-2, -\nu_{d-1} - \nu_d) & \xrightarrow{g_{d-1}} & H_m^2\left(\frac{S}{(g_1, \ldots, g_{d-2})}\right) (-1, -\nu_d) \\
\bigg| \downarrow g_d & & \bigg| \downarrow g_d \\
H_m^2\left(\frac{S}{(g_1, \ldots, g_{d-2})}\right) (-1, -\nu_{d-1}) & \xrightarrow{g_{d-1}} & H_m^2\left(\frac{S}{(g_1, \ldots, g_{d-2})}\right)
\end{array}
\]

From the above diagram and from (2.3.8), we get the exact sequence

\[
0 \rightarrow \mathcal{K} \rightarrow \ker \left( H_m^2(\frac{S}{(g_1, \ldots, g_{d-2})})(−1, −\nu_d) \xrightarrow{g_d} H_m^2(\frac{S}{(g_1, \ldots, g_{d-2})})(−1, −\nu_{d-1}) \right) \\
\xrightarrow{g_{d-1}} \ker \left( H_m^2(\frac{S}{(g_1, \ldots, g_{d-2})})(−1, −\nu_d) \xrightarrow{g_d} H_m^2(\frac{S}{(g_1, \ldots, g_{d-2})}) \right).
\]
From this we get that

\[ \mathcal{K} \cong \{ w \in H^2_m(S/(g_1, \ldots, g_{d-2}))(\mathbb{Z}, \mathbb{Z}) | g_{d-1} \cdot w = 0 \text{ and } g_d \cdot w = 0 \} \]

\[ \cong \{ w \in H^2_m(S/(g_1, \ldots, g_{d-2}))(\mathbb{Z}, \mathbb{Z}) | g_{d-1} \cdot w = 0 \text{ and } g_d \cdot w = 0 \} \quad \text{(as } \sum_{i=1}^{d} \nu_i = \nu) \].

2.5. Recall that \( H^d_m(R) \) is an \( R \) module via the action

\[ (x_1^{\alpha_1} \cdots x_d^{\alpha_d}) \cdot (x_1^{-\beta_1} \cdots x_d^{-\beta_d}) = \begin{cases} x_1^{\alpha_1-\beta_1} \cdots x_d^{\alpha_d-\beta_d} & \text{if } \alpha_i < \beta_i \text{ for all } i \\ 0 & \text{otherwise} \end{cases} \]

The following result gives us the highest possible degree for an element in the graded part \( \mathcal{K}_{p,u} \).

**Corollary 2.6** (with hypotheses as in 2.1). The following statements hold:

(i) For \( p \geq d \) the graded part \( \mathcal{K}_{p,u} \) is a finite dimensional \( k \)-vector space with \( \mathcal{K}_{p,u} = 0 \) if \( u > \nu - d \).

(ii) \( \mathcal{K}_{p,\nu-d} \neq 0 \) and \( \mathcal{K}_{u,\nu-d} \cong U(-d) \) is an isomorphism of graded \( U \)-modules.

**Proof.** (i) Fix \( p \geq d \). Set \( \mathfrak{d}(j) = \sum_{i=1}^{d} \nu_i \). If

\[ H^2_m(S/(g_1, \ldots, g_{d-2}))(\mathbb{Z}, \mathbb{Z}) = 0 \]

then by Theorem 2.4 it follows that \( \mathcal{K}_{p,u} = 0 \). From (2.3.7) and (2.3.6) we can say that \( \gamma_i \) makes a shift of bi-degree \( (-1, -\nu_i) \) for all \( 1 \leq i \leq d \) and so we get

\[ (2.6.10) \quad \begin{array}{c} 0 \rightarrow H^2_m(S/(g_1, \ldots, g_{d-2}))(\mathbb{Z}, \mathbb{Z}) \rightarrow \cdots \rightarrow H^2_m(S/(g_{d-1}, g_d))(\mathbb{Z}, \mathbb{Z}) \rightarrow H^2_m(S/(g_1, \ldots, g_d))(\mathbb{Z}, \mathbb{Z}) \rightarrow 0 \end{array} \]

Note that \( S = \bigoplus_{\beta} R \mathfrak{T}^\beta \) and \( H^d_m(S) = \bigoplus_{\beta} H^d_m(R) \mathfrak{T}^\beta \). For any \( u > \nu - d \) we have \( u - \nu > -d \). So \( H^d_m(S)(p-d,u-\nu) = 0 \) and hence \( H^2_m(S/(g_1, \ldots, g_{d-2}))(\mathbb{Z}, \mathbb{Z}) = 0 \). The result follows.

(ii) Let \( u = \nu - d \). Note that \( H^d_m(S)_{p-d,u-\nu} = \bigoplus_{|\beta|=p-d} k \cdot \frac{1}{x_1^{a_1} \cdots x_d^{a_d}} \mathfrak{T}^\beta \). Recall that \( g_1 = \sum_{i=1}^{d+1} a_i, 1 \mathfrak{T}_1 \) with \( a_{i,1} \in \mathfrak{m} \). Therefore \( a_{i,1} = x_1^{a_{i,1}} \cdots x_d^{a_{i,d}} \) with \( a_{i,j} > 0 \) for some \( j \) if \( a_{i,1} \neq 0 \). So \( a_{i,1} \cdot \frac{1}{x_1^{a_1} \cdots x_d^{a_d}} = 0 \) for all \( 1 \leq i \leq d + 1 \) and hence \( g_1 \cdot \frac{1}{x_1^{a_1} \cdots x_d^{a_d}} = 0 \). Similarly we get that \( g_1 \cdot \frac{1}{x_1^{a_1} \cdots x_d^{a_d}} = 0 \) for all \( 2 \leq i \leq d \). Thus from (2.3.6) it follows that

\[ H^d_m(S)(g_1)_{p-d+1,-d+v_1} = \ker \left( H^d_m(S)_{p-d,-d} \xrightarrow{g_1} H^d_m(S)_{p-d+1,-d+v_1} \right) = H^d_m(S)_{p-d,d} \]

Using this fact we iteratively get that

\[ H^d_m(S)(g_1, \ldots, g_i)_{p-d+i,-d+\nu(i)} \]

\[ \cong \ker \left( H^d_m(S)(g_1, \ldots, g_{i-1})_{p-d+i-1,-d+\nu(i-1)} \xrightarrow{g_i} H^d_m(S)(g_1, \ldots, g_i-1)_{p-d+i,-d+\nu(i)} \right) \]

\[ = H^d_{m-1}(S/(g_1, \ldots, g_{i-1})_{p-d+i-1,-d+\nu(i-1)} \cong H^d_m(S)_{p-d,d} \]
for all $1 \leq i \leq d - 2$. In particular, $H^2_m(S/(g_1, \ldots, g_{d-2})_{p-2,-d+\Omega(d-2)} \cong H_m^d(S)_{p-d,d-2}$. Again by the similar argument we can say that
\[
\mathcal{K}_{p,-d} \cong \{ w \in H^2_m(S/(g_1, \ldots, g_{d-2}))_{p-2,-d+\Omega(d-2)} \mid g_{d-1} \cdot w = 0 \text{ and } g_d \cdot w = 0 \}. \\
\cong \{ w \in H^d_m(S)_{p-d,d} \mid g_{d-1} \cdot w = 0 \text{ and } g_d \cdot w = 0 \}.
\]
Using $(\ast)$ we get $\mathcal{K}_{p,-d} \cong H_m^d(S)_{p-d,-d} \not\cong 0$. Note that $H_m^d(S)_{*,d} = \bigoplus k \cdot \frac{1}{x_1 \cdots x_d} \mathcal{T}^n \cong U(-d)$ as graded $U$-modules. The result follows.

\begin{remark}
Note that $H_m^d(S)$ is $\mathbb{Z}^2$-graded with $H_m^d(S)_{i,*} = 0$ for all $i < 0$. From 2.6.10 we have
\[
0 \rightarrow H^2_m(S/(g_1, \ldots, g_{d-2}))_{p-2,u-\nu+\Omega(d-2)} \xrightarrow{\gamma_1 \cdots \gamma_d-2} H^d_m(S)_{p-d,u-\nu}.
\]
Thus $H^2_m(S/(g_1, \ldots, g_{d-2}))_{p-2,u-\nu+\Omega(d-2)} = 0$ and hence $\mathcal{K}_{p,u} = 0$ for all $u \geq 0$ when $p < d$. By part (ii) of Corollary 2.6 it follows that $\text{Sym}(I)$ and $\mathcal{R}(I)$ first differ in degree $d$ for any $d \geq 3$.

3. Translation into $\mathcal{D}$-modules

We define $\mathcal{T}$ as a polynomial ring in $d + 1$ variables over the Weyl algebra $A_d(k)$, that is, $\mathcal{T} = k[x_1, \ldots, x_d] \langle \partial_1, \ldots, \partial_d \rangle [T_1, \ldots, T_{d+1}]$.

By $\mathcal{F}$ we denote the automorphism (Fourier transform) on $\mathcal{T}$ defined by
\[
\mathcal{F}(x_i) = \partial_i, \quad \mathcal{F}(\partial_i) = -x_i, \quad \mathcal{F}(T_i) = T_i.
\]

Let $S \mathcal{F}$ denote $S$ twisted by $\mathcal{F}$. Then we know that $S \mathcal{F}$ is a $\mathcal{T}$-module, where $t \ast b = \mathcal{F}(t) \cdot b$ for any $b \in S \mathcal{F}$ and $t \in \mathcal{T}$. Note that $t \ast b = \mathcal{F}^{-1}(t) \ast b = \mathcal{F}(\mathcal{F}^{-1}(t)) \cdot b = t \cdot b$ for any $b \in (S \mathcal{F})_{\mathcal{F}^{-1}}$ and $t \in \mathcal{T}$ and hence $(S \mathcal{F})_{\mathcal{F}^{-1}} = S$. Clearly $S \mathcal{F}$ is also a $S$-module with the same operation (as $S \subseteq \mathcal{T}$ is a subring). For the rest of this section we use the notations $L_i = \mathcal{F}(g_i)$ for $1 \leq i \leq d$. For all $1 \leq i_1 < \cdots < i_r \leq d$, define
\[
\text{Sol}(L_{i_1}, \ldots, L_{i_r}; S) = \{ h \in S \mid L_j \cdot h = 0 \text{ for all } j = i_1, \ldots, i_r \}.
\]

By the similar arguments as in the proof of [7, Lemma 3.9], we can show that the $k$-vector space $\text{Sol}(L_{i_1}, \ldots, L_{i_r}; S)$ has a $S$-module structure given by the twisting of the Fourier transform: if $f \in S$ and $h \in \text{Sol}(L_{i_1}, \ldots, L_{i_r}; S)$, then $f \circ h = \mathcal{F}(f) \cdot h$. Therefore
\[
\text{Sol}(L_{i_1}, \ldots, L_{i_r}; S) = \{ h \in S \mathcal{F} \mid g_j \ast h = 0 \text{ for all } j = i_1, \ldots, i_r \}
\]
is a $S$-submodule of $S \mathcal{F}$.

\begin{remark}
For any $\omega \in H^2_m(S/(g_1, \ldots, g_{d-2}))$ we have $g_i \cdot \omega = g_i \cdot \omega$ and so we get that
\[
\{ \omega \in H^2_m(S/(g_1, \ldots, g_{d-2})) \mid g_{d-1} \cdot \omega = 0 \text{ and } g_d \cdot \omega = 0 \} = \{ \omega \in H^2_m(S/(g_1, \ldots, g_{d-2})) \mid g_{d-1} \cdot \omega = 0 \text{ and } g_d \cdot \omega = 0 \}.
\]

Since $S = \bigoplus \beta R \mathcal{T}^\beta$ and $H_m^d(S) = \bigoplus \beta H_m^d(R) \mathcal{T}^\beta$ so similar way as in [7, Proposition 3.5] we can show that, the left $\mathcal{T}$-modules $H_m^d(S)$ and $S$ are cyclic with generators $\frac{1}{x_1 \cdots x_d}$ and 1 respectively. They
have the following presentations:

\[(3.1.11) \quad 0 \to T(x_1, \ldots, x_d) \to T \cdot 1 \to H^d_m(S) \to 0,\]
\[(3.1.12) \quad 0 \to T(\partial_1, \ldots, \partial_d) \to T \cdot 1 \to S \to 0.\]

(as \(x_i \cdot 1/x_1 \cdots x_d = 0\) and \(\partial_i \cdot 1/\partial_1 \cdots \partial_d = 0\) for all \(1 \leq i \leq d\)). Hence \(S \cong T/T(\partial_1, \ldots, \partial_d)\) and \(H^d_m(S) \cong T/T(x_1, \ldots, x_d)\). From the proof of [7, Theorem 3.11] (Step-1), we have an induced isomorphism of left \(T\)-modules

\[\varphi : H^d_m(S) \to S_T.\]

The isomorphism satisfies \(\varphi(\Pi_D(z)) = \Pi_D(F(z))\) for all \(z \in T\), where

\[\Pi_D : T \to T/(\partial_1, \ldots, \partial_d)(\cong S) \quad \text{and} \quad \Pi_D : T \to T/(x_1, \ldots, x_d)(\cong H^d_m(S)),\]

that is, the following diagram

\[\begin{array}{ccc}
T & \xrightarrow{F} & T \\
\Pi_D \downarrow & & \downarrow \Pi_D \\
(\partial_1, \ldots, \partial_d) & \xrightarrow{F} & (x_1, x_2, \ldots, x_d)
\end{array}\]

commutes. Since \(F\) is an isomorphism and \(z \in T(x_1, \ldots, x_d)\) if and only if \(F(z) \in T(\partial_1, \ldots, \partial_d)\) for any \(z \in T\) so \(F\) is an isomorphism. Again \(F\) is \(T\) linear which follows from the following claim.

**Claim 1.** \(F(t \cdot b) = F(t) \cdot F(b)\) for any \(t \in T\) and \(b \in T/(x_1, \ldots, x_d)(\cong H^d_m(S)).\)

Recall \(F(\Pi_D(z)) = \Pi_D(F(z))\) for all \(z \in T\). Since \(\Pi_D\) is surjective so there exists some \(c \in T\) such that \(\Pi_D(c) = b\). So \(F(t \cdot c) = F(\Pi_D(t) \cdot c) = F(\Pi_D(t) \cdot \Pi_D(c)) = F \circ \Pi_D(t \cdot c) = \Pi_D(F(t)) \cdot \Pi_D(F(c)) = F(t) \cdot F(\Pi_D(c)) = F(t) \cdot F(b) = t \cdot F(b)\). The claim follows.

Using (2.3.6) we get the following diagram

\[\begin{array}{ccc}
\cdots & \cdots & \cdots \\
0 & H^2_m(S/(g_1, \ldots, g_{d-2})) & H^{d-2}_m(S/(g_1, g_2)) \\
\varphi & H^{d-2}_m(S/(g_1, g_2)) & H^{d-2}_m(S/(g_1, g_2)) \\
\varphi & H^{d-1}_m(S/(g_1)) & H^{d-1}_m(S/(g_1)) \\
\varphi & H^d_m(S) & H^d_m(S) \\
\varphi & S_T & S_T \\
\varphi & \varphi & \varphi \\
S_T & \varphi & S_T
\end{array}\]

which induces an injective map \(\theta := \varphi \circ \gamma_1 \circ \cdots \circ \gamma_{d-2} : H^2_m(S/(g_1, \ldots, g_{d-2})) \to S_T\).

Following [7, Notation 3.10] we shall write \(S = \text{Sol}(L_1, \ldots, L_d; S)_T\) to stress the bigraded \(S\)-module structure induced on \(\text{Sol}(L_1, \ldots, L_d; S)\) by the twisting of the Fourier transform \(F\).

**Theorem 3.2** (with hypotheses as in 2.1). We have the following isomorphism of bigraded \(S\)-modules

\[\mathcal{K} \cong S(-d, -v + d),\]

where \(S = \text{Sol}(L_1, \ldots, L_d; S)_T\).

**Proof.** (Step-1:) Set \(\gamma := \gamma_1 \circ \cdots \circ \gamma_{d-2}\). As each \(\gamma_i\) is an injective map so is \(\gamma\). For any \(w \in H^2_m(S/(g_1, \ldots, g_{d-2}))\) we have \(\gamma(w) \in H^d_m(S)\) and hence \(\gamma(w) = \Pi_D(z)\) for some \(z \in T\) (as \(\Pi_D\) is a
surjective map). Now \( g_i \cdot \Pi_0(z) = g_i \cdot \Pi_0(z) = g_i \cdot \gamma(w) = \gamma(g_i \cdot w) \) (as \( \gamma \) is a \( S \)-linear map). Since \( \gamma \) is an injective map so we get that \( g_i \cdot \Pi_0(z) = 0 \) if and only if \( g_i \cdot w = g_i \cdot w = 0 \). Thus for any \( z \in \mathcal{T} \) we get the following equivalences

\[
\begin{align*}
( g_{d-1} \cdot \Pi_0(z) = 0 ) & \quad \iff \quad ( g_{d-1} z \in \mathcal{T}(x_1, \ldots, x_d) ) \quad \iff \quad ( \mathcal{F}(g_{d-1}) \mathcal{F}(z) \in \mathcal{T}(\partial_1, \ldots, \partial_d) ) \\
& \quad \iff \quad ( L_{d-1} \cdot \Pi_0(\mathcal{F}(z)) = 0 ) \quad \iff \quad ( L_{d-1} \cdot \mathcal{F}(\Pi_0(z)) = 0 ).
\end{align*}
\]

Recall that \( \theta := \mathcal{F} \circ \gamma : H^2_m(S/(g_1, \ldots, g_{d-2})) \to S_F \) is an injective map. Since \( \gamma \) is \( S \)-linear and \( \mathcal{F} \) is \( \mathcal{T} \)-linear so \( \theta \) is \( S \)-linear and hence image \( \theta \) is a \( S \)-submodule of \( S_F \). Due to the above observation \( \theta \) induces an isomorphism of \( S \)-modules

\[
\{ \omega \in H^2_m(S/(g_1, \ldots, g_{d-2})) \mid g_{d-1} \cdot \omega = 0 \text{ and } g_d \cdot \omega = 0 \}
\]

\[
\cong S \{ b \in \text{image } \theta \mid g_{d-1} \ast b = 0, g_d \ast b = 0 \}.
\]

(as \( b = \theta(\omega) \) for some \( \omega \in H^2_m(S/(g_1)) \)). We claim that image \( \theta = (0 : S_F (g_1, \ldots, g_{d-2})) \). Note that \( g_i \cdot (\omega) = \gamma_1(\cdots i \cdot \gamma_1(i \cdot \gamma_1(\omega \cdots )) = 0 \) for all \( 2 \leq i \leq d - 2 \) and \( g_i \cdot \gamma_1(\omega_1) = 0 \) where \( w_i = \gamma_{i+1} \circ \cdots \circ \gamma_{d-2}(\omega) \) for all \( 1 \leq i \leq d - 2 \) (as \( \gamma_i \) is \( S \)-linear, \( g_i \in S \) and \( g_i \cdot \gamma_i = 0 \)). Thus for any \( \omega \in H^2_m(S/(g_1, \ldots, g_{d-2})) \) we get that \( g_i \ast (\theta(\omega)) = \mathcal{F}(g_i \cdot \gamma_1(\omega)) = 0 \) for all \( 1 \leq i \leq d - 2 \) and hence \( (g_1, \ldots, g_{d-2}) \subseteq (0 : S H^2_m(S/(g_1, \ldots, g_{d-2}))) \). So \( (0 : S \text{ image } \theta) \supseteq \mathcal{F}(g_1, \ldots, g_{d-2}) \) = \( (S \mathcal{F}(g_1), \ldots, \mathcal{F}(g_{d-2})) = (L_1, \ldots, L_{d-2}) \). Thus image \( \theta \subseteq (0 : S \text{ image } (g_1, \ldots, g_{d-2})) \). On the other hand, for any \( \mathcal{F}(g_1, \ldots, g_{d-2})) \subseteq S_F \), there exists some \( a \in H^2_m(S) \) such that \( \mathcal{F}(a) = y \) (as \( \mathcal{F} \) is onto). Note that \( 0 = g_1 \ast y = \mathcal{F}(g_1) \ast \mathcal{F}(a) = \mathcal{F}(g_1 \cdot a) \) (by Claim 1) and hence \( g_1 \cdot a = 0 \) (as \( \mathcal{F} \) is one-one). So \( a \in \ker(H^2_m(S) \xrightarrow{g_1} H^2_m(S)) = \text{image } \gamma_1 = \gamma_1(H^d_m(S/(g_1))) \). Let \( a = \gamma_1(a_1) \) for some \( a_1 \in H^d_m(S/(g_1)) \). Now \( 0 = g_2 \ast y = \mathcal{F}(g_2) \ast \mathcal{F}(a) = \mathcal{F}(g_2) \ast \mathcal{F}(\gamma_1(a_1)) = \mathcal{F}(g_2 \cdot \gamma_1(a_1)) = \mathcal{F} \circ \gamma_1(g_2, a_1) \) (as \( \gamma_1 \) is \( S \)-linear). Since both \( \mathcal{F} \) and \( \gamma_1 \) are one-one so we have \( g_2 \cdot a_1 = 0 \). Thus \( a_1 \in \ker(H^d_m(S/(g_1)) \xrightarrow{g_2} H^d_m(S/(g_1))) = \text{image } \gamma_2 = \gamma_2(H^d_m(S/(g_1, g_2))) \). Therefore \( a \in \text{image } \gamma_1 \circ \gamma_2 \). Proceeding in this way we get that \( a \in \text{image } \gamma \) and hence \( y \in \text{image } \theta \). Thus \( (0 : S \text{ image } (g_1, \ldots, g_{d-2})) \subseteq \text{image } \theta \). The claim follows.

Hence from (3.2.13) it follows that

\[
\{ \omega \in H^2_m(S/(g_1, \ldots, g_{d-2})) \mid g_{d-1} \cdot \omega = 0 \text{ and } g_d \cdot \omega = 0 \}
\]

\[
\cong S \{ b \in S_F \mid g_i \ast b = 0 \text{ for all } i = 1, \ldots, d \}
\]

\[
= \{ b \in S \mid L_i \ast b = 0 \text{ for all } i = 1, \ldots, d \} = \text{Sol}(L_1, \ldots, L_d; S_F).
\]

(Step-2:) From the definition of \( \mathcal{F} \) we have that \( \mathcal{F} \) is homogeneous of degree 0 on \( T_i \)'s. Since

\[
\frac{1}{x_1 \cdots x_d} = (-1)^{a_1 + \cdots + a_d} \frac{(a_1 - 1)! \cdots (a_d - 1)!}{x_1^{a_1} \cdots x_d^{a_d}}
\]
so we have
\[ \frac{1}{x_1^{\alpha_1} \cdots x_d^{\alpha_d}} = (-1)^{\alpha_1+\cdots+\alpha_d} \frac{\partial_1^{\alpha_1-1} \cdots \partial_d^{\alpha_d-1}}{\alpha_1! \cdots (\alpha_d-1)!} \cdot \frac{1}{x_1 \cdots x_d} \in H_d^0(R). \]
and makes a shift degree of \( d \) in the \( x_i \)'s. By (3.1.11) it follows that the map \( \Pi_\theta : T \to T/(x_1, \ldots, x_d) \) is defined by \( z \mapsto z \cdot \frac{1}{x_1 \cdots x_d} \). Thus we get that
\[
\mathcal{F}\left( \frac{1}{x_1^{\alpha_1} \cdots x_d^{\alpha_d}} \right) = \mathcal{F}\left( \Pi_\theta \left( (-1)^{\alpha_1+\cdots+\alpha_d} \frac{\partial_1^{\alpha_1-1} \cdots \partial_d^{\alpha_d-1}}{\alpha_1! \cdots (\alpha_d-1)!} \right) \right)
\]
\[
= (-1)^{\alpha_1+\cdots+\alpha_d} \frac{x_1^{\alpha_1-1} \cdots x_d^{\alpha_d-1}}{\alpha_1! \cdots (\alpha_d-1)!} \in R.
\]
Hence we have that \( \mathcal{F} \) makes a shift degree of \( d \) in \( x_i \)'s. Moreover, from (2.3.6) we can say that \( \gamma_i \) makes a shift of degree \(-1\) in \( T_j \)'s and of degree \(-\nu_i\) in \( x_j \)'s for all \( i = 1, \ldots, d \). Therefore \( \theta = \mathcal{F} \circ \gamma \) makes a shift of degree \(-\sum_{i=1}^{d-2} \nu_i + d\) in \( x_j \)'s and of degree \(-d-2\) in \( T_j \)'s. Adding the shift degree \((-d+2, -\sum_{i=1}^{d-2} \nu_i + d)\) to Theorem 2.4 we get the result. \( \square \)

Cid proved [7, Proposition 3.13] observing that the way used in [2, Chapter 6, Theorem 1.2] can be used to prove his result. He also used [7, Subsection 3.1]. Both of them are independent of the number of variables and number of equations in a system of differential equations. So in the same way we get the following result.

**Proposition 3.3** (with hypotheses as in 2.1). We have the following isomorphism of graded \( U \)-modules
\[
\mathcal{K} \cong H^* \left( \mathcal{T}/(T(L_1, \ldots, L_d), S) \right)(-d).
\]

4. The bigraded structure of \( \mathcal{K} \) and its relation with \( b \) functions

**Notation 4.1.** Fix the integers \( p \geq d, m = \binom{p}{d} \) and \( n = \binom{p+1}{d} \), where \( \dim R = d \geq 3 \). Note that if \( p = d \), then \( m = 1, n = d+1 \), i.e., \( m = \# \{ T^\beta \mid |\beta| = 0 \} \) and \( n = \# \{ T^\beta \mid |\beta| = 1 \} \); if \( p = d+1 \), then \( m = d+1, n = \binom{d+1}{d+1-1} = \binom{d+1}{d-1} \), i.e., \( m = \# \{ T^\beta \mid |\beta| = 1 \} \) and \( n = \# \{ T^\beta \mid |\beta| = 2 \} \).

The graded part \( S_{p-d, *} \) is given as the solution set of the system of differential equations
\[
V = \{ h = (h_1, \ldots, h_m) \in R^m \mid [L_i] \bullet h = 0 \text{ for all } i = 1, \ldots, d \},
\]
where \([L_i] \in D^{n \times m}\) is an \( n \times m \) matrix with entries in \( D \) and induced by restricting \( L_i \) to the monomials \( T^\beta \) of degree \( |\beta| = p-d \). We define a new matrix \( H \in D^{m \times m} \) defined by
\[
H = \begin{pmatrix}
L_1 \\
L_2 \\
\vdots \\
L_d
\end{pmatrix}.
\]
Then we can write $V = \{ h = (h_1, \ldots, h_m) \in R^m \mid H \cdot h = 0 \}$. Set $N = D^{dn} \cdot H$. Clearly $N \subset D^m$ is a left $D$-module and image of the map $\phi : D^{dn} \to D^m$ induced by $H$. Set $M = D^m/N$. Following the same way as in [2, Theorem 1.2] we can show that $V \cong \text{Hom}_D(M, R^m)$ as $k$ vector spaces.

Example 4.2. For $d = 3$ we have $m = (p/3)$ and $n = (p+1)/3$. In this case we compute the system of differential equations when $p = 2$ and $p = 3$. Suppose
\[
L_1 = a_1 T_1 + a_2 + T_2 + a_3 T_3 + a_4 T_4, \\
L_2 = b_1 T_1 + b_2 + T_2 + b_3 T_3 + b_4 T_4, \\
L_3 = c_1 T_1 + c_2 + T_2 + c_3 T_3 + c_4 T_4.
\]
For $p = 3$, we have $h = (h_1) \in S_0 = R$, and the equations $L_1 \cdot h = 0, L_2 \cdot h = 0$ and $L_3 \cdot h = 0$ can be expressed as
\[
\begin{pmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4 \\
\end{pmatrix} \cdot (h_1) = 0, \quad \begin{pmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4 \\
\end{pmatrix} \cdot (h_1) = 0 \quad \text{and} \quad \begin{pmatrix}
c_1 \\
c_2 \\
c_3 \\
c_4 \\
\end{pmatrix} \cdot (h_1) = 0.
\]
So in this case, $N = D(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4)$.

For $p = 4$, we have $h = (h_1, h_2, h_3, h_4) \in S_1 = RT_1 + RT_2 + RT_3 + RT_4$. Note that $f = h_1 T_2 + h_2 T_2 + h_3 T_3 + h_4 T_4 \in S_1$, if $L_i \cdot f = 0$ for all $i = 1, 2, 3$. Now $L_1 \cdot f = 0$ implies that $(\sum_{i=1}^4 a_i T_i) \cdot (\sum_{j=1}^4 h_j T_j) = 0$, that is, $(a_1 h_1) T_1^2 + (a_2 h_1 + a_1 h_2) T_1 T_2 + (a_3 h_1 + a_1 h_3) T_1 T_3 + \cdots + (a_4 h_4) T_4^2 = 0$.
Hence $a_1 h_1 = 0$, $a_2 h_1 + a_1 h_2 = 0$, $a_3 h_1 + a_1 h_3 = 0, \ldots, a_4 h_4 = 0$. Thus shorting the monomials $T^\beta$ in lexicographical order, the equations $L_1 \cdot h = 0$ can be expressed as
\[
\begin{pmatrix}
T_1^2 \\
T_1 T_2 \\
T_1 T_3 \\
T_1 T_4 \\
T_2^2 \\
T_2 T_3 \\
T_2 T_4 \\
T_3^2 \\
T_3 T_4 \\
T_4^2 \\
\end{pmatrix} \begin{pmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4 \\
0 \\
a_3 \\
a_4 \\
0 \\
0 \\
0 \\
\end{pmatrix} \cdot \begin{pmatrix}
h_1 \\
h_2 \\
h_3 \\
h_4 \\
\end{pmatrix} = 0.
\]
Replacing $a_i$ by $b_i$ (resp. $c_i$) we get the expression for the equation $L_2 \cdot h = 0$ (resp. $L_3 \cdot h = 0$).

4.3. Let $A \in D^{r \times s}$ be an $r \times s$ matrix with entries in $D$. Multiplying with $A$ gives us a map $D^r \xrightarrow{A} D^s$ of left $D$-modules. Applying $\text{Hom}_D(\cdot, D)$ to this map induces the map $(D^s)^T \xrightarrow{A^T} (D^r)^T$ of right $D$-modules.
We have an equivalence between the category of left $D$-modules and the category of right $D$-modules, given by the algebra involution $\tau : D \to D$ defined by $f \partial^a \mapsto (-1)^{|a|} \partial^a f$, where $f \in R$. The map $\tau$ is called the standard transposition.

Given a left $D$-module $D^r/M_0$, it can be checked that

$$\tau\left(\frac{D^r}{M_0}\right) = \frac{D^r}{\tau(M_0)},$$

where $\tau(M_0) = \{ \tau(L) \mid L \in M_0 \}$.

**Proposition 4.4.** The left $D$-module $M = D^m/N$ is holonomic.

**Proof.** Consider $\mathcal{T}$ as standard graded with $\deg r = 0$ for all $r \in D$ and $\deg T_i = 1$ for all $i$. From the graded part $p$ of (5.1.17) we get the following exact sequence of the left $D$-modules,

$$(4.4.14) \quad 0 \to \mathcal{T}_{p-d} \xrightarrow{A} \mathcal{T}_{p-d-1} \to \cdots \to \mathcal{T}_{p-1} \to \mathcal{T}_p \to Q_p \to 0,$$

where $A = ((-1)^{d-1}[L_d]^T | \cdots | -[L_2]^T | [L_1]^T)$ and $[L_i]^T$ denotes the transpose of the matrix $[L_i]$ defined in Notation 4.1. Applying $\text{Hom}_D(-, D)$ to (4.4.14) we get the following complex of right $D$-modules

$$0 \to (\mathcal{T}_p)^T \to (\mathcal{T}_{p-1})^T \to \cdots \to (\mathcal{T}_{p-d+1})^T \xrightarrow{A^T} (\mathcal{T}_{p-d})^T \to 0,$$

where the cokernel of the last map $A$: is $\text{Ext}^d_D(Q_p, D)$, that is, $\text{Ext}^d_D(Q_p, D) \cong \tau(M)$ (see Example 4.5 for clarification). By [1, Lemma 7.3] we have $d(\text{Ext}^d_D(Q_p, D)) \leq 2d - d = d$. Since for any finitely generated $D$-module $N$, $d \leq d(N) \leq 2d$ so we get that $d(\text{Ext}^d_D(Q_p, D)) = d$ and hence it is a holonomic right $D$-module. The result follows. \qed

**Example 4.5.** Let $d = 3, p = 3$ and $m, n, L_1, L_2, L_3$ as in Example 4.2. Then

$$A = ([L_3]^T | -[L_2]^T | [L_1]^T)$$

$$= \begin{pmatrix} c_1 & c_2 & c_3 & c_4 & -b_1 & -b_2 & -b_3 & -b_4 & a_1 & a_2 & a_3 & a_4 \end{pmatrix} \in D^{1\times 12}.$$

Note that $\mathcal{T}_{q-3} \cong D$ and $\mathcal{T}_{q-2} = DT_1 + DT_2 + DT_3 + DT_4 \cong D^4$. Hence $(\mathcal{T}_{p-2}^3)^T \xrightarrow{A^T} (\mathcal{T}_{p-3})^T$ induces a map $((D^4)^3)^T \xrightarrow{A} (D)^3$. Now $\tau(M) = \tau(D^{12} / D \cdot A) = D^{12} / \tau(D \cdot A)$. So $\text{Ext}^d_D(Q_p, D) \cong \tau(M)$.

**4.6.** Let $\ell = \sum_{\alpha, \beta} C_{\alpha, \beta} x^\alpha \partial^\beta$ be an element in $D = k[x_1, \ldots, x_d][\partial_1, \ldots, \partial_d]$, where $x^\alpha = x_1^{a_1} \cdots x_d^{a_d}$ and $\partial^\beta = \partial_1^{b_1} \cdots \partial_d^{b_d}$. Let $w = (w_1, \ldots, w_d)$ be a given generic weight vector. Then the initial form of $\ell$ with respect to $w$ is denoted by $\text{in}_{(-w, w)}(\ell)$ and defined as

$$\text{in}_{(-w, w)}(\ell) = \sum_{-\alpha \cdot w + \beta \cdot w \text{ is maximum}} C_{\alpha, \beta} x^\alpha \partial^\beta.$$

To make $\deg(x_i) = 1$ and $\deg(\partial_i) = -1$ we take the weight vector $(-1, \ldots, -1) \in \mathbb{Z}^d$.

**Definition 4.7.** [7, Definition 4.4] Let $J \subset D$ be a left ideal, then the $k$-vector space

$$\text{in}(J) = k \cdot \left\{ \text{in}(\ell) \mid \ell \in J \right\}$$
is a left ideal in $D$ and it is called the initial ideal of $J$.

**Definition 4.8.** [7, Definition 4.5] Let $J \subset D$ be a holonomic left ideal. The elimination ideal

$$\text{in}(J) \cap k[-\sum_{i=1}^{d} x_i \partial_i]$$

is principal in the univariate polynomial ring $k[s]$, where $s = -\sum_{i=1}^{d} x_i \partial_i$ and the generator $b_J(s)$ of this ideal is called the $b$-function of $J$.

In [7], Cid gave the following definition for the $b$-function of a left $D$-module.

**Definition 4.9.** [7, Definition 4.6] Let $M'$ be a holonomic left $D$-module given as the quotient module $M' = D'/N'$. For each $i = 1, \ldots, r$ with the canonical projection $\pi_i : D^r \to D$ of $D^r$ onto the $i$-th component $e_i$, we define a left $D$-ideal

$$J_i = \pi_i(N' \cap D \cdot e_i) = \{ \ell \in D \mid (0, \ldots, \ell, \ldots, 0) \in N' \}.$$  

Then the $b$-function of $M'$ is given as the least common multiple of the $b$-functions of the $D$-ideals $J_i$, that is,

$$b_{M'}(s) = \text{LCM}_{i=1,\ldots,r} (b_{J_i}(s)).$$

For each $i = 1, \ldots, r$, the canonical injection $D/J_i \hookrightarrow D'/N'$ implies that each ideal $J_i$ is holonomic. Thus $b_{J_i}(s)$ is a non-zero polynomial by [6, Theorem 5.1.2] and hence the $b$-function of a holonomic module is a non-zero polynomial.

**Theorem 4.10.** Consider the $b$-function $b_M(s)$ of the holonomic $D$-module $M$ defined in Notation 4.1. For any integer $u$, if $b_M(-\nu + d + u) \neq 0$ then we have $K_{p,u} = 0$.

**Proof.** We prove by contradiction. Suppose $K_{p,u} \neq 0$. Then by Theorem 3.2 there exists $0 \neq h \in S_{p_d-k}$, where $-k = -\nu + d + u$. Following Notation 4.1 we can write $h = (h_1, \ldots, h_m) \in V$ with $\deg h_i = k$. Let $b_{J_i}(s)$ be the $b$-function corresponding to the left $D$-ideal

$$J_i = \pi_i(N \cap D \cdot e_i) = \{ \ell \in D \mid (0, \ldots, \ell, \ldots, 0) \in N \}.$$  

Thus $b_{J_i}(s) \cdot e_i \cdot h = 0$, which implies that $b_{J_i}(s) \cdot h_i = 0$ and hence $b_{J_i}(-k)h_i = 0$. As $b_M(-k) \neq 0$ so $b_{J_i}(-k) \neq 0$ (as $b_M(s) = \text{LCM}_{i=1,\ldots,m}(b_{J_i}(s))$ by the result generalizing [7, Definition 4.6] in our case). Therefore $h_i = 0$ for all $i$, a contradiction. \hfill $\square$

**Corollary 4.11** (with hypotheses as in 2.1). Let $u$ be the lowest possible $x$-degree for an element in the graded part $K_{p,s}$, that is, $K_{p,u} \neq 0$ and $K_{p,u-1} = 0$. Then the polynomial $s(s+1) \cdots (s+\nu-d-u)$ divides the $b$-function $b_M(s)$.

**Proof.** Since $K_{p,v} \neq 0$ for all $v \geq u$ so by Theorem 4.10 it follows that $b_M(-\nu+d+v) = 0$ for all $v \geq u$. Hence $s(s+1) \cdots s(\nu-d-u)$ divides $b_M(s)$. \hfill $\square$
We have the following result from [7, Lemma 4.10].

Lemma 4.12. For any \( k \geq 0 \) we have the identity

\[
(4.12.15) \quad s(s + 1) \cdots (s + k) = (-1)^{k+1} \sum_{|\alpha|=k+1} \frac{(k+1)!}{\alpha_1! \alpha_2! \cdots \alpha_d!} x^\alpha \partial^\alpha.
\]

Thus, we have that

(i) \( s(s + 1) \cdots (s + k) \in D(\partial_1, \ldots, \partial_d)^{k+1} \), where \( D(\partial_1, \ldots, \partial_d)^{k+1} \) denotes the left \( D \)-ideal generated by the elements \( \{\partial_i^{\alpha_i} \cdots \partial_d^{\alpha_d} | \zeta_1 + \cdots + \zeta_d = k + 1 \} \);

(ii) \( s(s + 1) \cdots (s + k) \) is homogeneous, that is,

\[
\text{in } (s(s + 1) \cdots (s + k)) = s(s + 1) \cdots (s + k).
\]

Proof. From the proof of [7, Lemma 4.10] we have

\[
x_i^{\beta_i} \partial_i^{\beta_i}(x_i \partial_i - \beta_i) = x_i^{\beta_i+1} \partial_i^{\beta_i+1}
\]

for all \( \beta_i \geq 0 \) and \( 1 \leq i, j \leq d \). For all \( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d \) we have

\[
x^{\alpha+e_i} \partial^{\alpha+e_i} = x_1^{\alpha_1} \cdots x_i^{\alpha_i+1} x_i^{\alpha_i+1} \cdots x_i^{\alpha_i} \partial_i^{\alpha_i} \cdots x_i^{\alpha_i+1} \partial_i^{\alpha_i+1} \cdots \partial_d^{\alpha_d}
\]

\[
= x_1^{\alpha_1} \cdots x_i^{\alpha_i+1} x_i^{\alpha_i+1} \cdots x_i^{\alpha_i} \partial_i^{\alpha_i} \cdots x_i^{\alpha_i+1} \partial_i^{\alpha_i+1} \cdots \partial_d^{\alpha_d} (x_i^{\alpha_i+1} \partial_i^{\alpha_i+1})
\]

\[
= x_1^{\alpha_1} \cdots x_i^{\alpha_i+1} x_i^{\alpha_i+1} \cdots x_i^{\alpha_i} \partial_i^{\alpha_i} \cdots x_i^{\alpha_i+1} \partial_i^{\alpha_i+1} \cdots \partial_d^{\alpha_d} (x_i^{\alpha_i} \partial_i^i (x_i \partial_i - \alpha_i))
\]

\[
= x^\alpha \partial^\alpha (x_i \partial_i - \alpha_i)
\]

(as \( x_i, \partial_j \) commute with each other when \( i \neq j \)). Thus we get

\[
(-1)^d \sum_{i=1}^d x^{\alpha+e_i} \partial^{\alpha+e_i} = (-1)^d \sum_{i=1}^d x^\alpha \partial^\alpha (x_i \partial_i - \alpha_i) = x^\alpha \partial^\alpha \sum_{i=1}^d (-x_i \partial_i + \alpha_i)
\]

\[
= x^\alpha \partial^\alpha (-\sum_{i=1}^d x_i \partial_i + \sum_{i=1}^d \alpha_i) = x^\alpha \partial^\alpha (s + k + 1),
\]

where \( |\alpha| = \alpha_1 + \cdots + \alpha_d = k + 1 \).

To prove the result we use induction hypothesis on \( k \). For \( k = 0 \), the result is trivially true (take \( \alpha = e_1, \ldots, e_d \)). Now we show that the result is true for \( k + 1 \) if it holds true for \( k \). Then

\[
s(s + 1) \cdots (s + k)(s + k + 1) = (-1)^{k+1} \sum_{|\alpha|=k+1} \frac{(k+1)!}{\alpha_1! \alpha_2! \cdots \alpha_d!} x^\alpha \partial^\alpha (s + k + 1)
\]

\[
= (-1)^{k+2} \sum_{|\alpha|=k+1} \frac{(k+1)!}{\alpha_1! \alpha_2! \cdots \alpha_d!} \sum_{i=1}^d x^{\alpha+e_i} \partial^{\alpha+e_i}.
\]

Note that for any \( \beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}^d \) with \( |\beta| = k + 2 \), we can write \( \beta = \alpha + e_i \) for some \( i \) and \( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d \) with \( |\alpha| = k + 1 \). Again \( \alpha + e_i = \alpha' + e_j \), where \( \alpha' = (\alpha_1, \ldots, \alpha_j - 1, \ldots, \alpha_i + 1, \ldots, \alpha_d) \) for all \( j \neq i \) (position of \( i, j \) may vary). Choose any \( i \in \{1, \ldots, d\} \) and fix it. Then for any
\[ \frac{(k + 1)!}{\alpha_1! \cdots \alpha_d!} + \sum_{j=1}^{d} \frac{(k + 1)!}{\alpha_1! \cdots (\alpha_j - 1)! \cdots (\alpha_i + 1)! \cdots \alpha_d!} = \frac{(k + 2)!}{\alpha_1! \cdots (\alpha_i + 1)! \cdots \alpha_d!}. \]

We prove the sub-claim by induction on \( d \). We know \( \binom{k+1}{j} + \binom{k+1}{j-1} = \binom{k+2}{j} \), that is,
\[
\frac{(k + 1)!}{j! \, i!} + \frac{(k + 1)!}{(j - 1)! \, (i + 1)!} = \frac{(k + 2)!}{j! \, (i + 1)!},
\]
where \( i + j = k \). This takes care of the base case. Let the sub-claim is true for any \( r < d \). Let us assume that \( i \neq 1 \). Since \( k + 1 - \alpha_1 \leq k + 1 \), so we have \( K_1 := (k + 1)!/(k + 1 - \alpha_1)! \in \mathbb{N} \). Now
\[
\frac{(k + 1)!}{\alpha_1! \cdots \alpha_d!} + \sum_{j=1}^{d} \frac{(k + 1)!}{\alpha_2! \cdots (\alpha_j - 1)! \cdots (\alpha_i + 1)! \cdots \alpha_d!} = \frac{K_1}{\alpha_1!} \cdot \frac{(k + 2 - \alpha_1)!}{\alpha_2! \cdots (\alpha_i + 1)! \cdots \alpha_d!} + \frac{(k + 1)!}{\alpha_1! \cdots (\alpha_i + 1)! \cdots \alpha_d!} \cdot \alpha_1 \cdot \alpha_1 = \frac{K_1}{\alpha_1!} + \frac{(k + 2 - \alpha_1 + \alpha_1)!}{\alpha_1! \cdots (\alpha_i + 1)! \cdots \alpha_d!} \times (K_1 \cdot (k + 2 - \alpha_1)! = (k + 2 - \alpha_1)(k + 1)!)
\]
\[
= \frac{(k + 2)!}{\alpha_1! \cdots (\alpha_i + 1)! \cdots \alpha_d!},
\]
Thus the sub-claim is true. From (4.12.16) we get that
\[
s(s + 1) \cdots (s + k)(s + k + 1) = (-1)^{k+2} \sum_{|\beta|=k+1} \frac{(k + 2)!}{\beta_1! \beta_2! \cdots \beta_d!} x^\beta \partial^\beta.
\]
The result follows.

4.13. Using notations as in Notation 4.1, define the matrix \( F = \mathcal{F}(H) = (\mathcal{F}(H_{i,j})) \in \mathbb{R}^{dn \times m} \), the \( dn \times m \) matrix with entries in \( R \) obtained after applying the Fourier transform to each entry of the matrix \( H \). As we have defined \( M = D^m/N \), in the similar way we define the graded \( R \)-module \( L = R^m/(R^{dn} \cdot F) \). Notice the rows of \( F \) are homogeneous of degree \( \nu_i \) for some \( 1 \leq i \leq d \).

Consider \( S \) as standard graded with \( \deg r = 0 \) for all \( r \in R \) and \( \deg T_i = 1 \) for all \( i \). Since \( \{g_1, \ldots, g_d\} \) is a regular sequence in \( S \) so the Koszul complex
\[
\mathbb{K}_*(g_1, \ldots, g_d) : \quad 0 \to S(-d) \xrightarrow{\cdot \, (-1)^{d-1}g_1 \ldots \cdot \, -g_2 \, g_1} S(-d+1)^d \to \cdots \to s(-1)^d \xrightarrow{\beta \, g_1} S \to S/(g_1, \ldots, g_d) \to 0.
\]
gives a free resolution of \( \text{Sym}(I) \cong S/(g_1, \ldots, g_d) \). Recall that \( \text{bideg} \, g_i = (1, \nu_i) \) for all \( i = 1, 2, 3 \). Now restricting \( \mathbb{K}_*(g_1, \ldots, g_d) \) to the graded part \( p \), we get a free resolution of \( \text{Sym}_p(I) \) as a \( R \)-module
0 \to R(-\nu)^{(\nu)} \to \bigoplus_{i=0}^{d-1} R(-\nu + \nu_{d-i})^{(p+1)_{d-i}} \to \cdots \to \bigoplus_{i=1}^{d} R(-\nu + \sum_{j=1}^{d-1} \nu_j)^{(p+d-1)_{d-i}} \to R^{(p+d)} \to \text{Sym}_R(I) \to 0

(as \#S_p = \binom{p+(d+1)-1}{d+1} = \binom{p+d}{d} \implies S_p \cong R^{(p+d)} \text{ and so on}). Applying \text{Hom}_R(\cdot, R) \text{ we get the complex}

0 \to R^{(p+d)} \to \bigoplus_{i=1}^{d} R(\nu - \sum_{j=1}^{d-1} \nu_j)^{(p+d-1)_{d-i}} \to \cdots \to \bigoplus_{i=0}^{d-1} R(\nu - \nu_{d-i})^{(p+1)_{d-i}} \to R(\nu)^{(\nu)} \to 0,

where the cokernel of the map on the right is the graded \text{R}-module \ast \text{Ext}^d_R(\text{Sym}_p(I), R). Let \( L \) denote the cokernel of the map on the right of the new complex induced from the above complex by degree \(-\nu\) shifting of each modules appearing in the complex. Therefore \( L(\nu) \cong \ast \text{Ext}^d_R(\text{Sym}_p(I), R) \) as graded \( R \)-modules.

**Lemma 4.14.** \( L \) is a finite length module.

**Proof.** We have the following commutative diagram

\[
\begin{array}{ccc}
R^{dn} & \xrightarrow{F} & R^m \\
\downarrow \phi & & \downarrow \theta \\
D^{dn} \xrightarrow{\partial} D^m & & 0
\end{array}
\]

Since \( \mathcal{F}|_{R^m}(R^{dn} \cdot F) \subset D^{dn} \cdot H \), so it induces a map \( \theta : L = R^m/(R^{dn} \cdot F) \to D^m/(D^{dn} \cdot H) = M \). We claim that \( \theta \) is injective. Let \( a \in \ker \theta \). Since \( \phi \) is surjective so there exists \( b \in R^m \) such that \( \phi(b) = a \). Let \( \mathcal{F}|_{R^m}(b) = c \). Since \( \psi(c) = \psi \circ \mathcal{F}|_{R^m}(b) = 0, \) so \( c \in \text{image}(\cdot H) \), that is, there exists some \( e \in D^{dn} \) such that \( e \cdot H = c \). Notice \( e \) is a polynomial in \( \partial_i \)'s (as all entries of \( H \) are in \( \partial_i \)'s). Thus we can construct \( e' \in R \) (polynomial in \( x_i \)'s) such that \( \mathcal{F}|_{R^m}(e') = e \). Clearly \( \mathcal{F}(e' \cdot F|_{R^m}(e') \cdot F) = e \cdot H = c \). Since \( \mathcal{F}|_{R^m} \) is injective so we have \( \mathcal{F}|_{R^m}(e' \cdot F) = b \) and hence \( a = 0 \). The claim follows.

Thus \( L \) is isomorphic to a submodule of \( M \). Now \( M \) is holonomic by Proposition 4.4 and so it’s length is finite by [2, 2.3, p. 89]. Hence length of \( L \) is also finite.

The following result will show that the approximation given in Corollary 4.11 is actually strict. While proving this result we denote the \( i \)-th component of the free \( R \)-module \( R^m \) by \( e_i^R \) and the \( i \)-th component of the free \( D \)-module \( D^m \) by \( e_i^D \) to avoid confusion.

**Theorem 4.15** (with hypotheses as in 2.1). Let \( b_M(s) \) be the \( b \)-function of the holonomic module \( M \) defined in Notation 4.1 and let \( u \) be the lowest possible \( x \)-degree for an element in the graded part \( K_{p, \ast} \). Then

\[
b_M(s) = s(s + 1) \cdots (s + \nu - d - u).
\]
Proof. From Theorem 4.11 we have \( s(s+1) \cdots (s+\nu-d-u) \mid b_M(s) \). Recall that \( \text{in}(J_i) \cap k[s] = (b_{J_i}(s)) \) and \( b_M(s) = \text{LCM}_{i=1}^{m}(b_{J_i}(s)) \). So if we prove that for each \( i = 1, \ldots, m \) we have
\[
s(s+1) \cdots (s+\nu-d-u) \in \text{in}(J_i) \cap k[s],
\]
where \( J_i = \pi_i(N \cap D \cdot e_i^D) \), then \( b_{J_i}(s) \mid (s(s+1) \cdots (s+\nu-d-u) \) and hence \( b_M(s) \mid (s(s+1) \cdots (s+\nu-d-u) \).

The result follows.

Let \( a = \text{end}(L) = \max \{k \mid L_k \neq 0\} \) (since \( L \) is a finite length module), then for any \( x_1^{\alpha_1} \cdots x_d^{\alpha_d} \) with \( \alpha_1 + \cdots + \alpha_d = a + 1 \) we have
\[
x_1^{\alpha_1} \cdots x_d^{\alpha_d} e_i^R = (0, \ldots, x_1^{\alpha_1} \cdots x_d^{\alpha_d}, \ldots, 0) \in R^{dn} \cdot F,
\]
where \( i = 1, \ldots, m \). Applying the inverse of the Fourier transform we get that
\[
\partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} e_i^D = (0, \ldots, \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}, \ldots, 0) \in D^{dn} \cdot H = N,
\]
Since \( \text{in}(s(s+1) \cdots (s+a)) = s(s+1) \cdots (s+a) \in D(\partial_1, \ldots, \partial_d)^{a+1} \) by Lemma 4.12 so it follows that
\[
s(s+1) \cdots (s+a) \in \text{in}(J_i) \cap k[s]
\]
for each \( i = 1, \ldots, m \). By the local duality theorem for graded modules (see [4, Theorem 3.6.19]) we get
\[
K_{p,*} = H_m^0(\text{Sym}_p(I)) \cong ^* \text{Hom}_k \left( ^* \text{Ext}_R^d(\text{Sym}_p(I), R(-d)), k \right) \cong ^* \text{Hom}_k \left( L(\nu - d), k \right).
\]
Since the grading of \( ^* \text{Hom}_k \left( L(\nu - d), k \right) \) is given by
\[
^* \text{Hom}_k \left( L(\nu - d), k \right) = \text{Hom}_i \left( L(\nu - d), k \right) = \text{Hom}_k \left( L(\nu - d - i), k \right),
\]
(using notations as in [4, p. 33]) we have that \( a = \nu - d - u \), and so the statement of theorem follows. \( \square \)

5. Computing Hom with duality

For any \( i \geq 0 \), we define the \( k \)-vector space
\[
F_i = \{ x_1^{\alpha_1} \cdots x_d^{\alpha_d} \partial_1^{\zeta_1} \cdots \partial_d^{\zeta_d} T_1^{\beta_1} \cdots T_{d+1}^{\beta_{d+1}} \mid |\alpha| + |\zeta| + |\beta| \leq i \}
\]
with \( F_{-1} = 0 \). Then \( \text{gr}(T) \cong T := k[x_1, \ldots, x_d, \delta_1, \ldots, \delta_d, T_1, \ldots, T_{d+1}] \) and we get a canonical map \( \sigma : T \to T \) given by \( \sigma(x_i) = x_i, \sigma(\partial_i) = \delta_i \) and \( \sigma(T_j) = T_j \). Since \( q_1, \ldots, q_d \) are linear on the \( T_i \)'s and have degree \( \nu_1, \ldots, \nu_d \) on the \( \partial_i \)'s with \( \sum_{i=1}^d \nu_i = \nu \), so they are homogeneous polynomials in \( T \) having degree \( \nu_1 + 1, \ldots, \nu_d + 1 \) respectively.

Notice that [7, Proposition 5.2, Proposition 5.3] holds in this case also. Since the Hilbert-Samuel function of \( T \) is given by \( \frac{(1+3d+1)}{3d+1} \), thus we have \( d(T) = 3d + 1 \). Set \( Q = T / T(L_1, \ldots, L_d) \).

Proposition 5.1. The following statements hold.

(i) The dimension of \( Q \) is \( d(Q) = 2d + 1 \).
(ii) The following Koszul complex in $\mathcal{M}_T^I(\mathcal{T})$ is exact

\begin{equation}
A: \quad 0 \to \mathcal{T}(-d) \xrightarrow{\cdot \{(-1)^{d-1}L_d \cdots -L_2 L_1\}} \mathcal{T}(-d+1)^d \to \cdots \to \mathcal{T}(-1)^d \xrightarrow{\cdot L_d} \mathcal{T} \to Q \to 0.
\end{equation}

**Note.** For $d = 3$,

\[ A: \quad 0 \to \mathcal{T}(-3) \xrightarrow{\cdot \{L_3 - L_2 L_1\}} \mathcal{T}(-2)^3 \xrightarrow{\cdot \{L_3 - L_2 0 L_1\}} \mathcal{T}(-1)^3 \xrightarrow{\cdot \{L_1\}} \mathcal{T} \to Q \to 0. \]

**Proof.** (i) Set $\Gamma = \{F_i\}_{i \geq 0}$. Notice $\Gamma' = \{F_i \cap \mathcal{T}(L_1, \ldots, L_d)\}_{i \geq 0}$ and $\Gamma'' = \{F_i/(F_i \cap \mathcal{T}(L_1, \ldots, L_d))\}_{i \geq 0}$ are natural good filtrations for $\mathcal{T}(L_1, \ldots, L_d)$ and $Q$ respectively. By [2, Lemma 5.1] we have the following exact sequence

\[ 0 \to \text{gr}^{\Gamma''}(\mathcal{T}(L_1, \ldots, L_d)) \to \text{gr}^{\Gamma}(\mathcal{T}) \to \text{gr}^{\Gamma''}(Q) \to 0. \]

Hence $\text{gr}^{\Gamma''}(Q) = \bigoplus_{i \geq 0} F_i/(F_{i-1} + F_i \cap \mathcal{T}(L_1, \ldots, L_d)) \cong T/(q_1, \ldots, q_d)$.

Let $B = k[x_1, \ldots, x_d, \delta_1, \ldots, \delta_{d+1}]$, $h_i = \sigma(F(f_i)) \in B$ and $J = (h_1, \ldots, h_{d+1})$ be an ideal in $B$. Note that $\sigma \circ F|_R : R = k[x_1, \ldots, x_d] \to k[\delta_1, \ldots, \delta_d] = A$ defined by $x_i \mapsto \partial_i \mapsto \delta_i$ is an isomorphism. So applying $\sigma \circ F|_R$ to (1.1.2) we get

\begin{equation}
0 \to A^d \xrightarrow{\varphi'} A^{d+1} \xrightarrow{[h_1, \ldots, h_{d+1}]} J' \to 0,
\end{equation}

where $J' = (h_1, \ldots, h_{d+1})$ an ideal in $A$. Note that $B = A[x_1, \ldots, x_d]$ is a flat extension of $A$. So applying $B \otimes \mathcal{I}$ to (5.1.18) we get a resolution of $J$,

\[ 0 \to B^d \xrightarrow{\varphi'} B^{d+1} \xrightarrow{[h_1, \ldots, h_{d+1}]} J \to 0 \]

(as $B$ is a flat $A$-module so by [5, Theorem 7.7] we have $J' \otimes_A B = J'B = J$). If $\varphi = (a_{ij})(d+1) \times d$ for some $a_{ij} \in R$, then $\varphi' = (\sigma(F|_R(a_{ij})))_{(d+1) \times d}$. Now

\[ \sum_{i=1}^{d+1} \sigma \circ F|_S(a_{ij})T_i = \sigma \circ F|_S \left( \sum_{i=1}^{d+1} a_{ij}T_i \right) = \sigma \circ F|_S(g_i) = q_j \]

for all $1 \leq j \leq d$, where $\sigma \circ F|_S : S = k[x_1, \ldots, x_d, T_1, \ldots, T_{d+1}] \to k[\delta_1, \ldots, \delta_d, T_1, \ldots, T_{d+1}] = U$ is an isomorphism defined by $x_i \mapsto \delta_i$ and $T_i \mapsto T_i$. So $[T_1, \ldots, T_{d+1}] \cdot \varphi' = [q_1, \ldots, q_d]$ and hence $\text{Sym}(J) \cong T/(q_1, \ldots, q_d)$, where $T = B[T_1, \ldots, T_{d+1}]$. Since $g_1, \ldots, g_d$ is a regular sequence in $S$ so $q_1, \ldots, q_d$ is a regular sequence in $U$ and hence $q_1, \ldots, q_d$ is a regular sequence in $U[x_1, \ldots, x_d] = T$ (as $T$ is a free $U$-module). Hence $\dim T/(q_1, \ldots, q_d) = \dim T - \text{ht}(q_1, \ldots, q_d) = (3d + 1) - d = 2d + 1$. By [5, Theorem 13.4] it follows that $d(Q) = \dim T/(q_1, \ldots, q_d) = 2d + 1$.

(ii) Since $L_i$'s are linear on the $T_j$'s so the shifting of degrees in (5.1.17) are clear. Recall that $A_p \cong \sum_{r=1}^p L_i e_{i_1 \cdots i_p}$ is a free $\mathcal{T}$-module of rank $(d^p)$ with basis $\{e_{i_1 \cdots i_p} | 1 \leq i_1 < i_2 \cdots < i_p \leq d\}$ and the differential $d : K_p \to K_{p-1}$ is defined by $d(e_{i_1 \cdots i_p}) = \sum_{i=1}^p (-1)^{r-1} e_{i_1 \cdots i_r \cdots i_{r+1}}$ (for $p = 1$; $d(e_i) = L_i$).
Thus
\[
d^2(e_{i_1\ldots i_{p+1}}) = d\left( \sum_{i=1}^{r} (-1)^{r-1} L_{i_r} e_{i_1\ldots i_{r} \ldots i_{p+1}} \right) = \sum_{r=1}^{p+1} (-1)^{r-1} L_{i_r} \left( \sum_{s=1}^{p+1} (-1)^{s-1} L_{i_s} e_{i_1\ldots i_{s} \ldots i_{p+1}} \right) = \sum_{r=1}^{p+1} \sum_{s=1, s \neq r}^{p+1} (-1)^{r+s-2} L_{i_r} L_{i_s} e_{i_1\ldots i_{r} \ldots i_{p+1}} \]

(position of \( r \) and \( s \) may vary). Without loss of generality we may assume that \( s < r \). Then coefficient of \( e_{i_1\ldots i_{p+1}} \) in the above expression is \((-1)^{r+s-2} L_{i_r} L_{i_s} + (-1)^{s-1} (-1)^{r-2} L_{i_s} L_{i_r} = (-1)^{r+s-3}(L_{i_s} L_{i_r} - L_{i_r} L_{i_s})\). Although \( \mathcal{T} \) is non-commutative, as \( L_1, \ldots, L_d \) are defined in the \( \partial_r \) and \( T_r \) variables so \( L_i L_j - L_j L_i = 0 \) for all \( 1 \leq i, j \leq d \) with \( i \neq j \) and hence \( d^2(e_{i_1\ldots i_{p+1}}) = 0 \) for all \( 1 \leq p \leq d \). It follows that (5.1.17) is a complex. So it is enough to prove exactness in the category \( \mathcal{T} \). Now (5.1.17) induces the following graded Koszul complex in \( T \),

\[
\mathcal{A}': \quad 0 \to T(-d) \xrightarrow{[(-1)^{d-1}q_d \ldots -q_2 q_1]} T(-d+1)^d \to \cdots \to T(-1)^d \xrightarrow{q_d} T \to \frac{T}{(q_1, \ldots, q_d)} \to 0.
\]

As \( q_1, \ldots, q_3 \) is a regular sequence, the above complex is exact. Hence by [1, Lemma 3.13] we get that (5.1.17) is exact. 

\[\square\]

**Corollary 5.2.** For any \( j \neq d \) we have \(* \text{Ext}_T^j(Q, \mathcal{T}) = 0\), and \(* \text{Ext}_T^d(Q, \mathcal{T}) \neq 0\).

**Proof.** Since (5.1.17) is a free resolution of \( Q \) so we have \(* \text{Ext}_T^j(Q, \mathcal{T}) = 0\) for \( j > d \). From [1, Theorem 7.1, p 73] we have \( j(Q) + d(Q) = 3d + 1 \), where \( j(Q) = \inf \{ k \mid * \text{Ext}_T^k(Q, \mathcal{T}) \neq 0 \} \). Again \( d(Q) = 2d + 1 \) by Proposition 5.1. Hence \( j(Q) = d \). The result follows. 

\[\square\]

**Theorem 5.3.** For any \( i \) we have the following isomorphism of graded \( U \)-modules (with hypotheses as in 2.1)

\[ * \text{Ext}_T^i(Q, S) \cong * \text{Tor}_{d-1}^T( * \text{Ext}_T^d(Q, \mathcal{T}), S). \]

**Proof.** We have \( S \cong \mathcal{T}/(\partial_1, \ldots, \partial_d) \). Since \( \partial_1, \ldots, \partial_d \) is a regular sequence in \( \mathcal{T} \) (considering \( \mathcal{T} \) as left ring), so a resolution of \( S \) in \( \mathcal{M}_U^T(\mathcal{T}) \) is given by the Koszul complex

\[
(5.3.19) \quad B := \mathbb{K}(\partial_1, \ldots, \partial_d): \quad 0 \to \mathcal{T} \to \mathcal{T}^d \to \cdots \to \mathcal{T}^d \to \mathcal{T} \to S \to 0.
\]

We define the following third quadrant double complex \(* \text{Hom}_\mathcal{T}(A, \mathcal{T}) \otimes_\mathcal{T} B, \)
Note that this is a double complex in the category of graded $U$-modules, that is, all its elements are graded $U$-modules and all its maps are homogeneous homomorphisms of graded $U$-modules (due to the construction of $\mathcal{M}_{U}(\mathcal{T})$ and $\mathcal{M}_{U}^{r}(\mathcal{T})$).

Since each $\ast \text{Hom}_{T}(\mathcal{A}_{j}, \mathcal{T}) \in \mathcal{M}_{U}^{r}(\mathcal{T})$ is a free module then by computing homology on each column we get that the only the last row does not vanish On the other hand, by Theorem 5.2 when we compute homology on each row only the leftmost column does not vanish.

Thus the spectral sequence determined by the first filtration is given by

$$\ast E_{2}^{p,q} = \begin{cases} \ast \text{Ext}_{T}^{p}(Q, S) & \text{if } q = d, \\ 0 & \text{otherwise}, \end{cases}$$

and the spectral sequence determined by the second filtration is given by

$$\Pi E_{2}^{p,q} = \begin{cases} \ast \text{Tor}_{d-q}^{d}(\ast \text{Ext}_{T}^{d}(Q, \mathcal{T}), S) & \text{if } p = d, \\ 0 & \text{otherwise}. \end{cases}$$

Since both spectral sequences collapse so we get the following isomorphisms of graded $U$-modules

$$\ast E_{2}^{i,d} \cong H_{dR}^{i+d}(\text{Tot}(\ast \text{Hom}_{T}(\mathcal{A}, \mathcal{T}) \otimes \mathcal{B})) \cong \Pi E_{2}^{d,i},$$

The result follows.

**Theorem 5.4.** [with hypotheses as in 2.1] Then we have the following isomorphism of graded $U$-modules

$$\mathcal{K} \cong H_{dR}^{0}(Q) = \{ w \in Q \mid \partial_{i} \bullet w = 0 \text{ for all } 1 \leq i \leq d \}.$$ 

In particular, for any integer $p$ we have an isomorphism of $k$-vector spaces

$$\mathcal{K}_{p,*} \cong H_{dR}^{0}(Q_{p}) = \{ w \in Q_{p} \mid \partial_{i} \bullet w = 0 \text{ for all } 1 \leq i \leq d \}.$$ 

**Proof.** From the resolution (5.1.17) of $Q$ we get the following complex in

$$\ast \text{Hom}_{T}(\mathcal{A}, \mathcal{T}) : \quad 0 \to \mathcal{T} \xrightarrow{L_{1}} \mathcal{T}(1) \xrightarrow{L_{2}} \cdots \xrightarrow{L_{d}} \mathcal{T}(d-1) \xrightarrow{(-1)^{d-1}L_{d} \cdots -L_{2} L_{1}} \mathcal{T}(d) \to 0.$$
Computing the $d$-th cohomology of this complex we get $\Ext^d_T(Q, T) \cong (\mathcal{T}/(L_1, \ldots, L_d)\mathcal{T})(d)$, where $\mathcal{T}/(L_1, \ldots, L_d)\mathcal{T} = \tau(Q)$ is the standard transposition of $Q$.

Note that the Koszul complex (5.3.19) is a resolution of $S$. Computing the third homology of the Koszul complex $\tau(Q)(d) \otimes T$ we get the following isomorphisms of graded $U$-modules

$$\ast \Tor^T_d(\ast \Ext^d_T(Q, T), S) \cong H_d(\tau(Q)(d) \otimes T) \cong \{ w \in \tau(Q)(d) \mid w \cdot \partial_i = 0 \text{ for all } 1 \leq i \leq d \}.$$

Since $\tau(T_i) = T_i$ so we have an isomorphism of graded $U$-modules

$$\{ w \in \tau(Q)(d) \mid w \cdot \partial_i = 0 \text{ for all } 1 \leq i \leq d \} \cong \{ w \in Q(d) \mid w \cdot \partial_i = 0 \text{ for all } 1 \leq i \leq d \}.$$

From Proposition 3.3 and Theorem 5.3 we get the following isomorphisms of graded $U$-modules

$$\mathcal{K} \cong \ast \Hom_T(Q, S)(-d) \cong \ast \Tor^T_d(\ast \Ext^d_T(Q, T), S)(-d) \cong \{ w \in Q \mid w \cdot \partial_i = 0 \text{ for all } 1 \leq i \leq d \},$$

The result follows. \Box

**Note.** For $d = 3$,

$$\ast \Hom_T(A, T) : \begin{array}{c}
0 \\
\mathcal{T} \xrightarrow{L_1}
\mathcal{T} \xrightarrow{L_2}
\mathcal{T} \xrightarrow{L_3}
\mathcal{T} \xrightarrow{L_2}
\mathcal{T} \xrightarrow{L_1}
\mathcal{T} \rightarrow 0.
\end{array}$$

6. Examples and Computations

The following function can be used to compute the $b$-function of each $D$-module $M$ from Notation 4.1 using Macaulay2.

```plaintext
needsPackage "Dmodules"

bFunctionRees = (I, p, d) -> (
  R := ring I;
  W := makeWeylAlgebra R;
  e := d+1;
  V1 := apply(0..(d-1),i->((vars W)_(0,i)));
  D1 := toList V1;
  T := W[S_1..S_e], U := QQ[Z_1..Z_e];
  V2 := apply(1..e,i->S_i);
  D2 := toList V2;
  A := Fourier (map(W, R, D1)) (res I).dd_2;
  L := matrix(D2) * A;
  src := flatten entries (map(T, U, D2)) basis(p - d, U);
  dest := flatten entries (map(T, U, D2)) basis(p - d+1, U);
  m := #src, n := #dest;
  H := mutableMatrix(W, m, d * n);
  V3 := apply(1..e,i->1);
  D3 := toList V3;
  V4 := apply(1..d,i->-1);
  D4 := toList V4;
  for i from 0 to m - 1 do (}
```
\begin{verbatim}
M=(i,k)->src#i * L_(0, k-1);
for j from 0 to n - 1 do (  
    for l from 1 to d do (  
        N=(i,j,k)->M(i,k)// gens ideal(dest#j);
        F := N(i,j,l);
        H_(i, j + (l-1)*n) = (map(W, T, D3)) F_(0, 0);
    );
);
)

bM := bFunction(coker matrix H, D4, toList(m:0));
use R;
bM

Example 6.1. Let \( R = k[x, y, z] \), \( S = R[a, b, c, d] \). Consider a matrix

\[
\varphi = \begin{bmatrix}
x & 0 & 0 \\
y & x & 0 \\
z & y & x^2 \\
0 & z & z^2 \\
\end{bmatrix}
\]

In [10, Example 4.10] it was checked that there exists a height two perfect ideal \( I \) satisfying \( G_3 \) with presentation matrix \( \varphi \). It was also shown that \( \mathcal{I} = \mathcal{L} + I_3(B_2(\varphi)) \), where Sym(\( I \)) = \( S/\mathcal{L} \) and

\[
B_2(\varphi) = \begin{bmatrix}
a & b & c & (c^2 - bd) \\
b & c & 0 & 0 \\
c & d & dz & d(-b^2 + ac) \\
\end{bmatrix}
\]

Thus \( \mathcal{K} = \mathcal{I}/\mathcal{L} = I_3(B_2(\varphi)) \). Computing in Macaulay2 we get \( I_3(\varphi) = (x^4, x^2z^2, -x^3z + xyz^2, -x^2yz + y^2z^2 - xz^3) \) and \( \mathcal{K} = (c^3x + b^2dz - acdz - bcdx, c^5 - b^4d + 2ab^2cd - a^2c^2d - 2bc^3d + b^2cd^2, b^3c^2dx + c^4dz + ab^2d^2z - a^2c^2d^2z - abcdx - bc^2d^2z, b^3cdx - abc^2dx + bc^3dz - b^2cd^2z) \). Notice in this case \( \delta = 4 \).

We also get \( b_M(s) = s \) if \( p = 3, 4 \) and \( b_M(s) = s(s+1) \) if \( p = 5 \). Comparing with Theorem 4.15 we get that \( s + \delta - 3 - u = s \), i.e., \( u = 1 \) if \( p = 3, 4 \) and \( s + \delta - 3 - u = s + 1 \), i.e., \( u = 0 \) if \( p = 5 \). Comparing with generators of \( \mathcal{K} \) we can say that Theorem 4.15 holds true in this case. Note that \( \mathcal{K}_{s,1} \neq 0 \) which satisfies Corollary 2.6.

Computations in Macaulay 2:

\begin{verbatim}
i1 : R = QQ[x,y]
o1 = R
o1 : PolynomialRing
i2 : load "bFunctionRees.m2"
i3 : I = ideal(x^4,x^2z^2,-x^3z+xyz^2,-x^2yz+y^2z^2-xz^3)
o3 = ideal(x ,x z , -x z+xyz , -x yz+y z -xz )
o3 : Ideal of R
i4 : for p from 3 to 5 do << factorBFunction bFunctionRees(I, p, 3) << endl;
\end{verbatim}
(s)
(s)
(s)(s + 1)

**Remark 6.2.** Note that in Example 6.1, $\mathcal{K}$ is generated by elements of degree $(3,1), (5,0), (5,1),$ so the value of $u$, the lowest possible $x$-degree for an element in the graded part $\mathcal{K}_{p,*}$ will be 1 for $p = 3, 4$ and 0 for $p = 5$ onwards which is clear from the output in Macaulay2. Observing the change of the value of $u$ in the output we can also claim that $\mathcal{K}$ has generators of degrees $(3,1), (5,0)$ which is true.

**Example 6.3.** Let $R = k[x, y, z], S = R[K_1, K_2, K_3, K_4]$. Consider a matrix

$$
\varphi = \begin{bmatrix}
x & 0 & 0 \\
y & x & 0 \\
z & y & x^5 \\
0 & z & z^5
\end{bmatrix}.
$$

Then $I_3(\varphi) = (x^7, x^2z^5, -x^6z + xyz^5, -x^5yz + y^2z^5 - xz^6)$. Since grade $I_3(\varphi) \geq 2$, so by the Hilbert-Burch Theorem (see [4, Theorem 1.4.17]), $I = I_3(\varphi)$ has the free resolution $0 \to R^3 \to R^4 \to I \to 0$ (by the converse part) and $I$ is perfect of grade 2. As grade $I_1(\varphi) = 3$ grade $I_2(\varphi) \geq 3$ and grade $I_3(\varphi) \geq 2$, by 1.3 it follows that $I$ satisfies $G_3$. Again by [10, Theorem 2.1] we have $I = \mathcal{L} : (x, y, z)^6$. So $I = (xK_2 + yK_3 + zK_4, xK_1 + yK_2 + zK_3, x^5K_3 + z^5K_4, x^4K_3^3 + z^4K_2^3K_4 - z^4K_1K_3K_4 + x^3yK_2^3K_4 + x^3zK_3K_4^2, x^3K_3^5 + z^3K_2^3K_4^2 - 2z^3K_1K_3K_4 + z^3K_2^2K_3K_4^2 + 2x^2yzK_2^3K_4 + x^2K_3^7 + z^2K_2^6K_4 - 3z^2K_1K_3K_4 + 3z^2K_2^2K_3^2K_4^2 - z^2K_1^2K_3^3K_4 + 3xyK_3K_4 + (3y^2 + 3xz)K_3^3K_4^2 - y^2K_2K_3^3K_4^3 - 2yK_2K_3^2K_4^4 + 3zK_3^3K_4^4 - z^2K_2K_3K_4^n, \ldots)$ (computed in Macaulay2). Thus $\mathcal{K} = (x^4K_3^3 + z^4K_2^3K_4 - z^4K_1K_3K_4 + x^3yK_2^3K_4 + x^3zK_3K_4^2, x^3K_3^5 + z^3K_2^3K_4 - 2z^3K_1K_3K_4 + z^3K_2K_3K_4^2 + 2x^2yzK_2^3K_4 + (x^2 + 2z^2)K_2^3K_4^3 + 2xyK_2^3K_3K_4 + x^2K_3^7 + z^2K_2^6K_4 - 3z^2K_1K_3K_4^2 + 3z^2K_2^2K_3^2K_4^2 - z^2K_1^2K_3^3K_4 + 3xyK_3K_4 + (3y^2 + 3xz)K_3^3K_4^2 - y^2K_2K_3^3K_4^3 - 2yK_2K_3^2K_4^4 + 3zK_3^3K_4^4 - z^2K_2K_3^n, \ldots)$ (omitting generators which are linear in $K_i$’s).

Note that $\delta = 7$. Computing in Macaulay2 we get $b_M(s) = s$ if $p = 3, 4$; $b_M(s) = s(s + 1)$ if $p = 5, 6$; $b_M(s) = s(s + 1)(s + 2)$ if $p = 7, 8$; and $b_M(s) = s(s + 1)(s + 1)(s + 3)$ if $p = 9, 10$ and $b_M(s) = s(s + 1)(s + 2)(s + 3)(s + 4)$ if $p = 11$. Comparing with Theorem 4.15 we get that $u = 4$ if $p = 3, 4$; $u = 3$ if $p = 5, 6$; $u = 2$ if $p = 7, 8$; $u = 1$ if $p = 9, 10$ and $u = 0$ if $p = 11$ (so onwards). From the observation in Remark 6.2 we can also say that $\mathcal{K}$ has generators of degrees $(3,4), (5,3), (7,2), (9,1), (11,0)$ which is in fact true.

Computations in Macaulay2 (a little time consuming, but finally we get the result with less effort):

```
i1 : R = QQ[x,y]
o1 = R
o1 : PolynomialRing
i2 : load "bFunctionRees.m2"
i3 : I = ideal(x^7,x^2z^5,-x^6z+xyz^5,-x^5yz+y^2z^5-xz^6)
```
7. Applications and Observations

7.1. Let $S = \bigoplus_{n \geq 0} S_n$ be a finitely generated standard graded ring over a Noetherian commutative ring $S_0$. Consider $S$ as a factor ring $S_0[T]/J$ of a polynomial ring corresponding to a minimal generating set of $S_1$.

**Definition 7.2.** The maximal degree occurring in a homogeneous minimal generating set of $J$ is called the relation type of $S$ and is denoted by reltype $S$.

It is well-known that reltype $S \leq \text{reg}_T S + 1$ as an $S_0[T]$-module, see [11, p. 1]. Consider $S$ as standard graded with $\text{deg} r = 0$ for all $r \in R$ and $\text{deg} T_i = 1$ for all $i$. Then we can define relation type for $R(I) = S/I = R[T_1, \ldots , T_{d+1}]/I$. Recall that $L = SI_1$ and $K_p \cong I_p/(S_{p-1}I_1)$ for all $p \geq 1$ (as $K \cong I/L$). Thus if reltype $R(I) > 1$, then we can say that reltype $R(I)$ maximal degree occurring in a homogeneous minimal generating set of $K$.

In Example 6.3, reltype $R(I) \geq 11$ (in fact equal as we get comparing with the output in Macaulay2 using general method). So in this case we can say that $\text{reg}_T R(I) \geq 11 - 1 = 10$. On the other hand, we can sometime compute the exact value of reltype $R(I)$ using our function when reltype $R(I)$ is known.

**Remark 7.3.** Let $R = k[x_1, \ldots , x_n]$ be a polynomial ring, and $m = (x_1, \ldots , x_n)$ be the homogeneous maximal ideal of $R$. Let $I$ be an ideal in $R$ generated by homogeneous polynomials of the same degree $\nu$. Let $I_m$ be the image of $I$ in the local ring $(R_m, mR_m)$. As $I$ is a homogeneous ideal, $I$ and $I_m$ have same generating set. Now $C = R \setminus m$ is a multiplicative set in $R(I)$ (as $R \subseteq R(I)$ is a sub-ring). Note that $R(I_m) \cong R(I) \otimes_R R_m \cong S/I \otimes_R R_m \cong C^{-1}S/C^{-1}I \cong R_m[T_1, \ldots , T_{d+1}]/C^{-1}I$. So

$$F(I_m) = R(I_m) \otimes_{R_m} R_m/mR_m \cong R(I) \otimes_R R_m \otimes_{R_m} R_m/mR_m \cong R(I) \otimes_R R/m = F(I),$$
where $F(I)$ is the special fiber ring of $I$. In view of these relations we can use some properties of $\mathcal{R}(I)$ and $F(I)$ which are known when $I$ is an ideal in a local ring $(R, \mathfrak{m})$.

**7.4.** Take $R$ and $I$ as in assumption 2.1. Note that in this case second analytic deviation $\mu(I) - \ell(I)$ one is equivalent to the fact that $\ell(I) = d$, where $\ell(I) = \dim F(I)$ is the analytic spread of $I$.

**Definition 7.5.** Let $I = (f_1, \ldots, f_n) = (\underline{f})$ be an ideal of a local ring $R$ of dimension $d$. By $H_i(\underline{f})$ we denote the $i$-th homology of the Koszul complex $\mathbb{K}_*(\underline{f})$. We say $I$ satisfies sliding depth if

$$\text{depth} H_i(\underline{f}) \geq d - n + i \quad \text{for all } i \geq 0.$$ 

**Definition 7.6.** We say $I$ is strongly Cohen-Macaulay if the Koszul homology modules of $I$ with respect to one (and then to any) generating set are Cohen-Macaulay.

Since grade two perfect ideals are in the linkage class of a complete intersection so always strongly Cohen-Macaulay if the ring $R$ is Gorenstein local, see [14, Theorem 1.14, Example 2.1]. Clearly if $I$ is strongly Cohen-Macaulay, then $I$ satisfies sliding depth. By [12, Proposition 2.4] we have $\ell(I) = d$ when $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring of dimension $d$, $I$ satisfies $G_d$, sliding depth, $\mu(I) = d + 1$ and $\text{ht} I \geq 1$. Thus if $(R, \mathfrak{m})$ is a Gorenstein local ring of dimension $d$, $I$ is a grade two perfect ideal satisfying $G_d$ and $\mu(I) = d + 1$, then $\ell(I) = d$ and hence $I$ has second analytic deviation one.

Note that

$$F(I) = \frac{\mathcal{R}(I)}{m\mathcal{R}(I)} \cong \mathcal{R}(I)_{*,0} \cong \left( \frac{S}{\mathcal{I}} \right)_{*,0} = \frac{S_{*,0}}{\mathcal{I}_{*,0}} = \frac{k[T_1, \ldots, T_{d+1}]}{\mathcal{I}_{*,0}} = \frac{U}{\mathcal{I}_{*,0}}.$$ 

In [8, 3.3], it is observed that $F(I) = k[f_1 t, \ldots, f_{d+1} t] \cong k[f_1, \ldots, f_{d+1}] \subset R$ (this isomorphism not necessarily homogeneous). Thus $F(I)$ is a domain (as $R$ is so). Moreover, if we assume that $\dim F(I) = \ell(I) = d$, then $\mathcal{I}_{*,0}$ is a height one prime ideal in $U$ and hence it is principle (as $U$ is a UFD). This fact implies that $F(I)$ is Cohen-Macaulay.

Clearly $\mathcal{I}_{1,0} = 0$. As otherwise, if $a_1 T_1 + \cdots + a_{d+1} T_{d+1} \in \mathcal{I}_{1,0}$ with $a_i \in k$ such that not all are zero, then $\sum_{i=1}^{d+1} a_i (f_i t) = 0$, that is, $\left( \sum_{i=1}^{d+1} a_i f_i \right) t = 0$ in $R[t]$. Hence $\sum_{i=1}^{d+1} a_i f_i = 0$ in $R[t]$ (as $R[t]$ is a domain) and hence in $R$. This contradicts the fact that $\{f_1, \ldots, f_{d+1}\}$ is a minimal generating set of $I$. Thus reltype $F(I) = p_0 > 1$ and $\mathcal{I}_{*,0}$ will be generated by an element $u$ having bi-degree $(p_0, 0)$ as an ideal in $U$. Notice $p_0$ is the minimum value such that $\mathcal{I}_{p_0,0} \neq 0$. Now $K_{p,0} \cong \mathcal{I}_{p,0}/(S_{p-1,0} \mathcal{I}_{1,0})$ for all $p \geq 1$ and hence $K_{*,0} \cong \mathcal{I}_{*,0}$. Clearly image of $u$ in $\text{Sym}(I)$ will generate $K_{*,0}$ and will belong to any minimal generation set of $K$ (as no other element having bi-degree $(p, q)$ with $q > 0$ can generate $K_{*,0}$). Moreover, if $\dim F(I) = d$, then by [13, Lemma 5.2] we have reltype $F(I) = r(I) + 1$, where $r(I)$ denotes the reduction number of $I$. Also $\deg u = p_0 = e(F(I))$ (when we consider $S$ is graded) where $e(F(I))$ denotes the Hilbert-Samuel multiplicity of $F(I)$. Since $F(I) \cong k[T_1, \ldots, T_{d+1}]/(u)$ is a hypersurface, so

$$\mathbb{K}_*(u) : 0 \to U(-p_0) \xrightarrow{u} U \to 0$$
is a minimal free resolution of $F(I)$ and hence $\text{reg } F(I) = p_0 - 1 = \text{reltype } F(I) - 1$, where $\text{reg } F(I)$ denotes the regularity of $F(I)$. Thus from the output of our function ($b\text{FunctionRees}(I, p, d)$) in Macaulay2, we get reltype $F(I)$ from which we get $e(F(I)), \text{reg } F(I)$ and $r(I)$.

For Examples 6.1 and 6.3 we get that reltype $F(I) = 5, 11$ respectively.

**Remark 7.7.** Let $\nu = d$. Since $\sum_{i=1}^d \nu_i = \nu$ so in this case $\nu_i = 1$ for all $i$, that is, $I$ has a linear presentation. By Corollary 2.6 it follows that for all $p \geq d$, $K_{p,u} = 0$ for all $u > 0$ and $K_{p,0} \neq 0$. Again by Remark 2.7 we have $K_{p,0} = 0$ for all $p < d$. So by the above observation we get that reltype $F(I) = d$.

From the Sub-section 7.1 we also get that reltype $R(I) = d$.

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