Phase Synchronization of non-Abelian Oscillators on Small-World Networks

Zhi-Ming Gu\textsuperscript{1}, Ming Zhao\textsuperscript{2}, Tao Zhou\textsuperscript{2}, Chen-Ping Zhu\textsuperscript{1}, and Bing-Hong Wang\textsuperscript{2}
\textsuperscript{1}College of Science, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, PR China
\textsuperscript{2}Department of Modern Physics, University of Science and Technology of China, Hefei 230026, PR China

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In this paper, by extending the concept of Kuramoto oscillator to the left-invariant flow on general Lie group, we investigate the generalized phase synchronization on networks. The analyses and simulations of some typical dynamical systems on Watts-Strogatz networks are given, including the \( n \)-dimensional torus, the identity component of 3-dimensional general linear group, the special unitary group, and the special orthogonal group. In all cases, the greater disorder of networks will predict better synchronizability, and the small-world effect ensures the global synchronization for sufficiently large coupling strength. The collective synchronized behaviors of many dynamical systems, such as the integrable systems, the two-state quantum systems and the top systems, can be described by the present phase synchronization frame. In addition, it is intuitive that the low-dimensional systems are more easily to synchronize, however, to our surprise, we found that the high-dimensional systems display obviously synchronized behaviors in regular networks, while these phenomena can not be observed in low-dimensional systems.

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I. INTRODUCTION

Synchronization is observed in many natural, social, physical and biological systems, and has found applications in a variety of fields \cite{4}. The large number of networks of coupled dynamical systems that exhibit synchronized states are subjects of great interest. As a theoretical paradigm, the Kuramoto model is usually used to investigate the phase synchronization among nonidentical oscillators \cite{5,6}. However, some real physical systems that display synchronized phenomenon (e.g. the coupled double planar pendulums, the two-state quantum system, etc.) can not be properly described by the simply Kuramoto model. Actually, the Kuramoto oscillator is equal to the left-invariant flow on the simplest nontrivial Lie group \( U(1) = \{e^{i\zeta} | \zeta \in \mathbb{R} \} \) (see the Refs. \cite{5,6} for the fundamental knowledge of Lie group). Accordingly, in this paper, by extending the concept of Kuramoto oscillator to left-invariant flow on general Lie group, we investigate the generalized phase synchronization on networks. In particular, the left-invariant flows corresponding to non-Abelian Lie groups are named non-Abelian oscillators, which can be applied on some non-Abelian dynamical systems like quantum systems.

This paper is organized as follow: In section 2, the concept of synchronization of non-Abelian oscillators is introduced. In section 3, the analyses and simulations on some typical examples are given, including the \( n \)-dimensional torus \( T^n \), the identity component of 3-dimensional general linear group \( GL(3, \mathbb{R}) \) (i.e. \( GL_0(3, \mathbb{R}) \)), the special unitary group \( SU(2) \) and the special orthogonal group \( SO(3) \). Finally, we sum up this paper and discuss the relevance of phase synchronization of non-Abelian oscillators to the real world in section 4. Some mathematic remarks are given in the Appendix.

II. SYNCHRONIZATION OF NON-ABELIAN OSCILLATORS

First of all, we review the simple oscillator:

\[
e^{i(\omega t+\theta)} = Ce^{i\omega t}, \quad C = e^{i\theta}.
\]

(1)

It is obvious that \( x(t) = \omega t + \theta \) is the solution of the equation

\[
\frac{dx}{dt} = \omega, \quad x(0) = \theta,
\]

(2)

which relates to the dynamical system

\[
\Phi(t, c) = ce^{i\omega t}
\]

(3)

on the circle \( S^1 = U(1) = \{e^{i\zeta} | \zeta \in \mathbb{R} \} \).

In terms of Lie groups, \( S^1 = U(1) \) is an abelian Lie group with its Lie algebra \( i\mathbb{R} = \{i\omega | \omega \in \mathbb{R} \} \), and the map

\[
\exp : i\mathbb{R} \rightarrow S^1, \quad \exp(i\omega) = e^{i\omega},
\]

(4)

is the exponential map of \( S^1 \). The system (3) is also called the left-invariant flow, of which the tangent vector field is called left-invariant vector field on \( S^1 \).

This fact motivates the idea that the left-invariant flows on the general Lie groups can be considered as general oscillators. In particular, the left-invariant flows on the non-Abelian Lie groups can be considered as non-Abelian oscillators. Let \( G \) be a Lie group and \( \Gamma \) its Lie algebra. If \( v \in \Gamma \), then the left-invariant flow determined by \( v \) on the \( G \) is

\[
\Phi(t, g) = g \cdot \exp tv, \quad t \in \mathbb{R}, \quad g \in G.
\]

(5)

Different from those defined on the linear Euclidean spaces, the above dynamical system (5) is defined on the manifold \( G \).
Next we consider the synchronization of network, of which each node is located a general oscillator on the Lie group $G$ with its Lie algebra $\Gamma$. The collective synchronization behavior of the coupled general oscillators starts by randomly choosing an element $v^\alpha \neq 0$ in $\Gamma$ and an initial phase $g_0 \in G$ for each node $\alpha$. In the case of the general oscillator $v^\alpha$ and $g_0$ correspond to the frequency $\omega$ and $e = e^{i\theta}$ in Eq. (3), respectively.

It is worthwhile to note that, in general, the phase space of our general oscillator is both non-Abelian group in algebra, and non-linear space, which is a manifold, in geometry. Thus one may expect that there will be more interesting phenomena in this model.

For a dynamical system on the manifold $G$, its equations of motion should be written in a local coordinate system because, generally, there does not exist a global coordinate system on a manifold. Let $(x_1, x_2, \cdots, x_n)$ be a local coordinate system about the identity $e$ of Lie group $G$, $n = \dim G$, and $(0, \cdots, 0)$ the coordinates of $e$. If $(x_1, x_2, \cdots, x_n)$, $(y_1, y_2, \cdots, y_n)$ and $(z_1, z_2, \cdots, z_n)$ are the coordinates of $g_1, g_2 \in G$ and $g_1 g_2 \in G$ in this coordinate system respectively, then the multiplication function $f = (f_1, f_2, \cdots, f_n)$ can be defined as:

$$
\begin{align*}
    z_1 &= f_1(x_1, \cdots, x_n; y_1, \cdots, y_n) \\
    \cdots \\
    z_n &= f_n(x_1, \cdots, x_n; y_1, \cdots, y_n).
\end{align*}
$$

Suppose that $v \in \Gamma$, then the equations of the left-invariant flow determined by $v$ are

$$
\frac{dx_j(t)}{dt} = \sum_i a_i l_i^j(x(t)), \quad 1 \leq j \leq n, \tag{7}
$$

where $(a_1, a_2, \cdots, a_n)$ is the coordinates of $v$ with respect to the basis $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n})_{x=0}$, and $l = (l_i^j)$ is the matrix valued function:

$$
l_i^j(x) = \frac{\partial f_j(x, y)}{\partial y_i} \bigg|_{y=0}, \quad 1 \leq i, j \leq n. \tag{8}
$$

The set of equations (7) is the intrinsic dynamic of a single general oscillator. Note that, the multiplication functions are the local representation of the multiplication in the Lie group, and the matrix valued function (8) is the map induced by multiplying by $x$ at left. This map translates every vector in the Lie algebra $x$. Eqs. (7) mean that the left-invariant flow determined by $v$ are made from the vector field whose value at $x$ is just the vector into which translating $v$.

Similar to the case of coupled simple oscillators [12], the coupled dynamics of general oscillators are:

$$
\frac{dx_\alpha^j(t)}{dt} = \sum_i a^\alpha_i l_i^j(x^\alpha(t)) - \frac{1}{k_{\alpha}} \sum_{\beta \in \Lambda_\alpha} q_{ij} (x^\alpha(t) - x^\beta(t))
$$

$$
x^\alpha_j(0) = x^\alpha_j(0), \tag{9}
$$

where $q_{ij}$ is the coupling function assumed smooth, $\Lambda_\alpha$ is the set of $\alpha$’s neighboring nodes, and $x^\alpha_j(0)$ is the coordinates of initial phase of node $\alpha$. According to the existence and uniqueness theorem of ordinary differential equations, Eq. (9) has a unique solution. Then, by the theory of Lie groups, this solution can be extended to an one-parameter subgroup, which corresponds to a left-invariant flow.

### III. Simulation Results for Some Typical Examples

In this section, we will show some analyses and simulations on several typical examples. All the simulations are obtained based on the Watts-Strogatz (WS) network [8], which can be constructed by starting with one-dimensional lattice and randomly moving one endpoint of each edge with probability $p$. The network size

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**FIG. 1:** (Color online) Order parameters $m_1$ (a) and $m_2$ (b) vs coupling strength $K$ for different values of the rewiring probability $p$. All the simulation results are the average over 100 independent runs corresponding to the case of $n = 2$.  

[Diagram showing order parameters $m_1$ and $m_2$ as functions of $K$ for different $p$ values.]
$N = 1000$ and average degree $\langle k \rangle = 6$ are fixed. In our numerical simulation below, the coordinates $a_i$ are chosen randomly and uniformly in the range $(-0.5, 0.5)$, and the initial phases $x_{ij}^0$ in $(0, 2\pi)$. All the numerical results are obtained by integrating the dynamical equations using the Runge-Kutta method with step size 0.01. The order parameters (see below) are averaged over 2000 time steps, excluding the former 2000 time steps, to allow for relaxation to a steady state.

A. $T^n$, the $n$-dimensional torus

We firstly investigate the $n$-dimensional torus, $T^n = S^1 \times S^1 \times \cdots \times S^1$, which is a connected compact Lie group. Denote $G_1 = T^n$ and its Lie algebra $\Gamma_1 = \{ \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \}$ with $[,] \equiv 0$ (i.e. Abelian). This is a direct extension of the simple Kuramoto oscillators, and will degenerate to Karumoto oscillators when $n = 1$. It follows that an element in $\Gamma_1$, that is an intrinsic frequency, is $v = (i\omega_1, i\omega_2, \cdots, i\omega_n)$.

It is easy to see that the general oscillation determined by $v$ is periodic if $\omega_1, \omega_2, \cdots, \omega_n$ are integer-linearly dependent, and quasi-periodic if $\omega_1, \omega_2, \cdots, \omega_n$ are integer-linearly independent. For the equations of motion we have

$$\frac{d}{dt}\left(\begin{array}{c}
x_1^0 \\
\vdots \\
x_n^0
\end{array}\right) = \left(\begin{array}{c}
\omega_1^0 \\
\vdots \\
\omega_n^0
\end{array}\right) - \frac{K}{k_\alpha} \sum_{\beta \in \Lambda_n} \sin(x_1^0 - x_1^\beta)
$$

with the initial conditions $x_j^0(0) = x_{ij0}$, where $K$ is the coupling strength. To characterize the synchronized states of the $j$th dimension, we use the order parameter

$$m_j = \left\langle \left\{ \frac{1}{N} \sum_{\alpha} e^{i\alpha_j^0} \right\} \right\rangle_1, \quad j = 1, 2, \cdots, n, \quad (11)$$

where $\{ \cdot \}$ signifies the time averaging.

Fig. 1 displays the relations between the phase order parameter and the coupling strength for various values of the rewiring probability $p$, where $n = 2$. Since the dynamical behaviors in different dimensions are independent, and each one is equal to a simply Kuramoto oscillator [12]. In a word, the small-world effect ensures the global synchronization for sufficiently large coupling strength, and the higher disorder (i.e. larger $p$) will enhance the network synchronizability.

This example can be used to describe the collective behavior of the coupled double planar pendulums in a network. Furthermore, according to the Liouville-Arnold theorem, the phase space of an integrable system, if it is compact, is a $n$-torus. Therefore, this example can be widely applied to describe the synchronization phenomenon of integrable systems.

B. $GL_0(3, R)$, the identity component of 3-dimensional general linear group $GL(3, R)$

$GL_0(3, R)$, the identity component of $GL(3, R)$, is a connected Lie group of dimension 9 (Actually, $GL_0(3, R)$ is an open subset of $R^9$). Denote $G_2 = GL_0(3, R)$, and $\Gamma_2$ its Lie algebra containing all the $3 \times 3$-real matrices. The coordinates $(g_{ij})$ for all $g = (g_{ij}) \in G_2$ can be addressed on the basis $E_{ij}$ of $\Gamma_2$ (see the case (a) of Appendix A), where

$$E_{11} = \left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{array}\right), \quad \cdots, \quad E_{33} = \left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{array}\right).$$

In this circumstance the equations of motion are

$$\frac{dx_{ij}^\alpha(t)}{dt} = \sum_{p=1}^3 x_{ip}^\alpha(t)a_{pj}^{\alpha\beta} - \frac{K}{k_\alpha} \sum_{\beta \in \Lambda_n} (x_{ij}^\alpha(t) - x_{ij}^\beta(t)), \quad (12)$$

with the initial conditions $x_{ij}^\alpha(0) = x_{ij0}$, where $(a_{ij}^\alpha) \in \Gamma_2$ and $K$ is the coupling strength.

Since $G_2$ is a subset of the Euclidean space $R^9$, the coupling term of arbitrary pair of nodes $\alpha$ and $\beta$ can be directly written as $x_{ij}^\alpha - x_{ij}^\beta$. Therefore, the order parameter is defined as:

$$m = \left\langle \left\{ \frac{2}{N(N-1)} \sum_{\alpha<\beta} \exp \left[ -c \|x^\alpha - x^\beta\|^2 \right] \right\} \right\rangle_1, \quad (13)$$
where \( \tilde{x}^\alpha = (\tilde{x}_j^\alpha) = (x_j^\alpha) / M \), and \( M = \max \{\|x^\beta(t)\|\}_{1 \leq \beta \leq N} \).

In Fig. 2, we report the simulation results about the order parameter \( m \) as a function of \( K \) for different rewiring probabilities. Similar to the situation of \( T^\mu \), this system can approach to the global synchronized state for sufficiently large \( K \) when \( p \) is large enough. It is worthwhile to emphasize that the order parameter of different systems can not compare to each other directly, since their definitions are not the same.

\[ g = \exp (x_1 J_1 + x_2 J_2 + x_3 J_3), \]

which is convenient for the description of two-state quantum systems since the matrices \( J_1, J_2, \) and \( J_3 \) are relative to Pauli matrices as:

\[ -2i J_1 = \sigma_3, \quad -2i J_2 = \sigma_2, \quad -2i J_3 = \sigma_1. \]

Note that, a one-parameter subgroup \( U(t) \subset SU(2) \) can be considered as the time-evolution in the quantum mechanics. Since the subgroup \( e^{i\zeta} U(t) \), where \( \zeta \) is an arbitrary real number, represents the same time-evolution as \( U(t) \) does, we call two different nodes \( \alpha \) and \( \beta \) are synchronous if they are only differ in a phase factor \( e^{i\zeta} \), that is

\[ (w^\alpha_1 w^\beta_2 - w^\alpha_2 w^\beta_1) = 0, \]

\[ \zeta \in \mathbb{R}. \]

Therefore, the equation

\[ w^\alpha_1 w^\beta_2 = w^\alpha_2 w^\beta_1 \]

is hold for every \( 1 \leq \alpha < \beta \leq N \), the completely global synchronization is achieved. Consequently, for each pair of nodes \( \alpha \) and \( \beta \), define the dynamical “distance” between them as

\[ \mu_{\alpha \beta} = |w^\alpha_1 w^\beta_2 - w^\alpha_2 w^\beta_1|, \]

where \(| \cdot |\) signifies the modulus of the complex number. Accordingly, the order parameter is

\[ m_3 = |\exp (-\max \{\mu_{\alpha \beta} \mid 1 \leq \alpha < \beta \leq N \})|. \]

The corresponding equations of motion are

\[ \frac{dx^\alpha(t)}{dt} = \sum_{j=1}^{3} a^2 t^j (x^\alpha(t)) - K \left( \sum_{\beta \in \Lambda_n} \sin \frac{1}{2} (x^\alpha_j(t) - x^\beta_j(t)), \right) \]

with the initial condition \( x^\alpha_0(0) = x^\alpha_0, j = 1, 2, 3 \), where \( K \) is the coupling strength. It is worthwhile to emphasize that the behaviors of this dynamical system is independent to the local coordinate systems. It follows that the local coordinates (14) in \( SU(2) \) can be replaced by

\[ -2i J_1 = \sigma_3, \quad -2i J_2 = \sigma_2, \quad -2i J_3 = \sigma_1. \]

\[ \text{SU}(2), \text{ the special unitary group} \]

\( SU(2) = S^3 \), the special unitary group, is a connected compact Lie group consisted of all anti-Hermitian traceless matrices. Denote \( G_3 = SU(2) \), and \( \Gamma_3 \) its Lie algebra having the following basis

\[ J_1 = \frac{1}{2} \left( \begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right), \quad J_2 = \frac{1}{2} \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), \quad J_3 = \frac{1}{2} \left( \begin{array}{cc} 0 & i \\ i & 0 \end{array} \right). \]

Give an arbitrary element \( g \in SU(2) \), its local coordinates \((x_1, x_2, x_3)\) can be found from:

\[ g = \left( \begin{array}{cc} w_1 & w_2 \\ -\bar{w}_2 & \bar{w}_1 \end{array} \right) = (\exp x_1 J_1)(\exp x_2 J_2)(\exp x_3 J_3), \]

and

\[ w_1 \bar{w}_1 + w_2 \bar{w}_2 = 1. \]
IV. CONCLUSION

In this paper, we extended the concept of Kuramoto oscillator to the left-invariant flow on general Lie group, and studied the generalized phase synchronization on small-world networks. The left-invariant flow on Lie group can be used to represent many significant physical systems, such as the integrable systems (e.g. the double planar pendulums), the two-state quantum systems (e.g. Ising model), the motions of top, and so on. In particular, the dynamics of two-state quantum systems on complex networks are extensively studied recently [11, 12, 13], the present work can provide us a theoretical frame to investigate their collective synchronized behaviors.

It is intuitive that the low-dimensional systems are more easily to synchronize. However, one may note that in the regular networks (i.e. \( p = 0 \)), the system \( T^2 \) does not exhibit synchronized behavior while the obvious synchronization is observed in the higher dimensional systems \( GL_0(3, \mathbb{R}) \) and \( SU(2) \) when the coupling strength gets large. This is an interesting phenomenon that worths a further study in the future.

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APPENDIX A: HOW TO CHOOSE THE COORDINATE SYSTEM

In order to investigate the collective synchronization behaviors of the coupled non-Abelian oscillators, we have to integrate Eq. (9) numerically. Because of the using of local coordinate system, a question rises: Does the Eq. (9) express the behavior of the system in the whole manifold \( G \)? The answer is as follows.

Case (a) Manifold \( G \) is a connected open subset of Euclidean space \( \mathbb{R}^n \). In this case, we can directly use the natural coordinate system of \( \mathbb{R}^n \) as the local coordinate system of \( G \).

Case (b) \( G \) is a connected compact Lie group. In this case, we have \( \exp(\Gamma) = G \), so we can choose the coordinate system of \( \Gamma \) as the local coordinate system of \( G \).

APPENDIX B: RELEVANCE OF THE DYNAMICAL SYSTEM \( SU(2) \)

According to the theory of fiber bundles, the condition (20) is equivalent to that \( g^p(t) \) and \( g^q(t) \) is all in the same fiber of the Hopf bundle \( S^3 \rightarrow S^2 \rightarrow \mathbb{R}^2 \). In order to interpret the map \( p \) we consider the 2-sphere \( S^2 \) as the complex projective line \( \mathbb{C}P^1 \) in which a point is written as \([w_1, w_2]\), where \( w_1 \) and \( w_2 \) are two complex numbers and not all zero. Here, the point \([w_1, w_2]\) is identified with \([\lambda w_1, \lambda w_2]\) for any nonzero complex number \( \lambda \). Then the map \( p \) can be written as

\[
p(w_1, w_2) = [w_1, w_2].
\]

It is easy to show that the inverse image \( p^{-1}[w_1, w_2] \) is the sphere \( S^1 \). This fact tells us that, in the sense of our definition, the collective synchronization behaviors of the coupled non-Abelian oscillators \( SU(2) \) can represent the two-state quantum system.

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