TAKENS EMBEDDING THEOREM WITH A CONTINUOUS OBSERVABLE

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Abstract. Let \((X, T)\) be a dynamical system where \(X\) is a compact metric space and \(T : X \to X\) is continuous and invertible. Assume the Lebesgue covering dimension of \(X\) is \(d\). We show that for a generic continuous map \(h : X \to [0, 1]\), the \((2d + 1)\)-delay observation map \(x \mapsto (h(x), h(Tx), \ldots, h(T^{2d}x))\) is an embedding of \(X\) inside \([0, 1]^{2d+1}\). This is a generalization of the discrete version of the celebrated Takens embedding theorem, as proven by Sauer, Yorke and Casdagli to the setting of a continuous observable. In particular there is no assumption on the (lower) box-counting dimension of \(X\) which may be infinite.

1. Introduction

Assume a certain physical system, e.g., a certain experimental layout in a laboratory, is modeled by a dynamical system \((X, T)\) where \(T : X \to X\) represents the state of the system after a certain fixed discrete time interval has elapsed. The possible measurements performed by the experimentalist are modeled by bounded real valued functions \(f_i : X \to \mathbb{R}, i = 1, \ldots, K\) known as observables. The actual measurements are performed during a finite time at a discrete rate
\[ t = 0, 1, \ldots, N \] starting out in a finite set of initial conditions \( \{x_j\}_{j=1}^L \).

Thus the measurement may be represented by the finite collection of vectors \( (f_i(T^k x_j))_{k=0}^N \), \( i = 1, \ldots, K \), \( j = 1, \ldots, L \). The reconstruction problem facing the experimentalist is to characterize \( (X,T) \) given this data. Stated in this way the problem is in general not solvable as the obtained data is not sufficient in order to reconstruct \( (X,T) \). We thus make the unrealistic assumption the experimentalist has access to \( (f_i(T^k x))_{k=0}^N \), \( i = 1, \ldots, K \), \( x \in X \). In other words we assume the experimentalist is able to measure the observable during a finite amount of time, at a discrete rate, starting out with every single initial condition. Although this assumption is plainly unrealistic it enables one, under certain conditions, to solve the reconstruction problem and provide theoretical justification to actual (approximate) procedures used by experimentalists in real life. To first to realize this was F. Takens who proved the famous embedding theorem, now bearing his name:

\textbf{Theorem.} \textit{(Takens’ embedding theorem [Tak81, Theorem 1])} Let \( M \) be a compact manifold of dimension \( d \). For pairs \( (h,T) \), where \( T : M \to M \) is a \( C^2 \)-diffeomorphism and \( h : M \to \mathbb{R} \) a \( C^2 \)-function, it is a generic property that the \( (2d + 1) \)-delay observation map \( h_0^{2d} : M \to \mathbb{R}^{2d+1} \) given by

\begin{equation}
(1.1) \quad x \mapsto (h(x), h(Tx), \ldots, h(T^{2d}x))
\end{equation}
is an embedding, i.e. the set of pairs \((h,T)\) in \(C^2(M,\mathbb{R}) \times C^2(M, M)\) for which \((1.1)\) is an embedding is comeagre w.r.t Whitney \(C^2\)-topology\(^1\).

A key point of the theorem is the possibility to use one observable and still be able to achieve embedding through an associated delay observation map. Indeed the classical Whitney embedding theorem (see [Nar73, Section 2.15.8]) states that generically a \(C^2\)-function \(\vec{F} = (F_1, \ldots, F_{2d+1}) : M \to \mathbb{R}^{2d+1}\) is an embedding but this would correspond to the feasibility of measuring \(2d + 1\) independent observables which is unrealistic for many experimental layouts even if \(d\) is small.

A decade after the publication of Takens’ embedding theorem it was generalized by Sauer, Yorke and Casdagli in [SYC91]. The generalization is stronger in several senses. In their theorem the dynamical system is fixed and the embedding is achieved by perturbing solely the observable. This widens the (theoretical) applicability of the theorem but necessitates some assumption about the size of the set of periodic points. Moreover they argue that the concept of (topological) genericity used by Takens is better replaced by a measurable variant of genericity they call prevalence. They also call to attention the fact that in many physical systems the experimentalist tries to characterize a finite dimensional fractal (in particular non-smooth) attractor to which the system converges to, regardless of the initial condition (for sources discussing such systems see [Hal88, Lad91, Tem97]). The key point is that although this attractor may be of low fractal dimension, say \(l\), it embeds in phase space in a high dimensional manifold of dimension,

\(^1\)In [Noa91] Noakes points out the theorem is also true in the \(C^1\)-setting and gives a alternative and more detailed proof.
say \( n >> l \). As Takens Theorem requires the phase space to be a manifold it gives the highly inflated number of required measurements \( 2n + 1 \) instead of the more plausible \( 2l + 1 \). Indeed in [SYC91] it is shown that given a \( C^1 \)-diffeomorphism \( T : U \to U \), where \( U \subset \mathbb{R}^k \) and a compact \( A \subset U \) with \textit{lower box dimension} \( d \), \( \dim_{\text{box}}(A) = d \), under some technical assumptions on points of low period, it is a prevalent property for \( h \in C^1(U, \mathbb{R}) \) that the \( (2d + 1) \)-delay observation map \( h^{2d}_0 : U \to \mathbb{R}^{2d+1} \) is a topological embedding when restricted to \( A \).

In the case of many physical systems, the underlying space in which the finite dimensional attractor arises, is infinite dimensional. In [Rob05] Robinson generalized the previous result to the infinite dimensional context and showed that given a Lipschitz map \( T : H \to H \), where \( H \) is a Hilbert space and a compact \( T \)-invariant set \( A \subset H \) with \textit{upper box dimension} \( d \), \( \overline{\dim}_{\text{box}}(A) = d \), under some technical assumptions on points of low period, and how well \( A \) can be approximated by linear subspaces, it is a prevalent property for Lipschitz maps \( h : H \to \mathbb{R} \) that the \( (2d + 1) \)-delay observation map \( h^{2d}_0 : H \to \mathbb{R}^{2d+1} \) is injective on \( A \). In this work we show that if one is allowed to use continuous

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\(^2\)Notice as pointed out in [SYC91] p. 587, it is possible that the minimal dimension of a smooth manifold containing the attractor equals the dimension of phase space.

\(^3\)Another approach for the infinite dimensional setting is given in [Gut15] with respect to a two-dimensional model of the Navier-Stokes equation. The system has (a typically infinite dimensional) compact \textit{absorbing set}, to which it reaches after a finite and calculable time (depending on the initial condition). It is shown that this set may be embedded in a cubical shift \( ([0, 1]^2 \) through a infinite-delay observation map \( x \mapsto (h(x), h(Tx), \ldots) \).
(typically non-smooth) observables then generically one needs even less measurements than previously mentioned in order to reconstruct the original dynamical system. This is achieved by using Lebesgue covering dimension instead of box dimension. We also weaken the invertibility assumption to the more realistic injectivity assumption (see discussion in [Tem97 III.6.2]). We prove:

**Theorem 1.1.** Let $X$ be a compact metric space and $T : X \to X$ an injective continuous mapping. Assume $\dim(X) = d$ and $\dim(P_n) < \frac{1}{2}n$ for all $n \leq 2d$, where $\dim(\cdot)$ refers to Lebesgue covering dimension and $P_n$ denotes the set of periodic points of period $\leq n$. Then it is a generic property that the $(2d + 1)$-delay observation map $h^{2d}_0 : X \to [0, 1]^{2d+1}$ given by

\begin{equation}
(1.2) \quad x \mapsto (h(x), h(Tx), \ldots, h(T^{2d}x))
\end{equation}

is an embedding, i.e. the set of functions in $C(X, [0, 1])$ for which (1.2) is an embedding is comeagre w.r.t supremum topology.

The Lebesgue covering dimension of a compact metric space is always smaller or equal to the lower box-counting dimension (See [Rob11 Equation 9.1]) and it is not hard to construct compact metric spaces for which the Lebesgue covering dimension is strictly less than the (lower) box-counting dimension, e.g. if $C$ is the Cantor set then the box dimension of $C^N$ is infinite whereas the covering dimension is zero. Thus from a theoretical point of view this enables one to reconstruct (using typically a non-smooth observable) dynamical systems with less measurements than were known to suffice previously. Moreover this can be
used when the goal of the experiment is to calculate a topological invariant such a topological entropy. However I am not certain this result has a bearing on actual experiments. Indeed it has been pointed out to me by physicists that modelling measurements in the lab by smooth functions is realistic, thus non smooth observables are “non-accessible” for the experimentalist.

Our result is closely related to a result we published in [Gut12]. In that article it was shown, among other things, that given a finite dimensional topological dynamical system $(X,T)$, where $X$ is a compact metric space with $\dim(X) = d < \infty$ and $T$ is a homeomorphism $T : X \to X$, such that $\dim(P_n) < \frac{1}{2}n$ for all $n \leq 2d$, then $(X,T)$ embeds in the cubical shift $([0,1])^Z$, $(X,T) \leftrightarrow (([0,1])^Z, \sigma - \text{shift})$, where the shift action $\sigma$ is given by $\sigma(x_i)_{i \in Z} = (x_{i+1})_{i \in Z}$. It is not hard to conclude this result from Theorem 1.1 but we are interested in the reverse direction. It would have been possible to rewrite [Gut12] in such a way so that Theorem 1.1 follows, however at the time of its writing we were not aware of the connection to Takens Theorem. Unfortunately a specific part of the proof in [Gut12] uses the fact that $([0,1])^Z$ is infinite dimensional and therefore is not straightforwardly adaptable to a proof of Theorem 1.1. In this work we give an alternate and detailed proof of this specific part which is suitable for Theorem 1.1 and indicate how the other parts directly follow from [Gut12]. As mentioned before we only assume $T : X \to X$ is injective and not necessarily a homeomorphism such as in [Gut12]. Following Takens we will only deal with the case of one observable. The case of several observables follows similarly.
Remark 1.2. Let \((X, (T_t)_{t \in \mathbb{R}})\) be a flow on a compact metric space \(X \subset \mathbb{R}^k\) with \(\dim(X) = d\), arising from an ordinary differential equation \(x = \dot{F}(x)\) where the function \(F : X \to \mathbb{R}^k\) obeys the Lipschitz condition \(\|F(x) - F(y)\| \leq L\|x - y\|\). By a theorem of Yorke ([Yor69]) for any \(0 < t < \frac{\pi}{Ld}\) the dynamical system \((X, T_t)\) has no periodic points of order less than \(2d + 1\) and therefore satisfies the assumptions of Theorem 1.1.

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2. Preliminaries

2.1. Dimension. Let \(C\) denote the collection of open (finite) covers of \(X\). Given an open cover \(\alpha \in C\) and a point \(x \in X\) we may count the number of elements in \(\alpha\) to which \(x\) belongs, i.e. \(|\{i \mid x \in U_i\}| = \sum_{U \in \alpha} 1_U(x)\). The order of \(\alpha\) is essentially defined by maximizing this quantity: \(\text{ord}(\alpha) = -1 + \max_{x \in X} \sum_{U \in \alpha} 1_U(x)\). Alternatively the order of \(\alpha\) is the minimal integer \(n\) for which any distinct \(U_1, U_2, \ldots, U_{n+2} \in \alpha\) obey \(\bigcap_{i=1}^{n+2} U_i = \emptyset\). Let \(D(\alpha) = \min_{\beta \succ \alpha} \text{ord}(\beta)\) (where \(\beta\) refines \(\alpha\), \(\beta \succ \alpha\), if for every \(V \in \beta\), there is \(U \in \alpha\) so that \(V \subset U\)). The Lebesgue covering dimension is defined by \(\dim(X) = \sup_{\alpha \in C} D(\alpha)\).

2.2. Period. For an injective map \(T : X \to X\) we define the period of \(x \in X\) to be the minimal \(p \geq 1\) so that \(T^p x = x\). If no such \(p\) exists the period is said to be \(\infty\). If the period of \(x\) is finite we say \(x\) is periodic.
We denote the set of periodic points in $X$ by $P$. As $T$ is injective any
preimage of a periodic point is periodic of the same period. Indeed $T|_P$, $T$ restricted to $P$, is invertible.

2.3. **Supremum topology.** One defines on $C(X, [0, 1])$ the supremum metric $\| \cdot \|_{\infty}$ given by $\| f - g \|_{\infty} \triangleq \max_{x \in X} |f(x) - g(x)|$.

3. **Proof of the theorem**

In this section we prove Theorem 1.1. The proof is closely related to the proof of [Gut12, Theorem 8.1] but unfortunately does not follow directly from it. We thus supply the necessary details.

3.1. **The Baire Category Theorem Framework.** The main tool of the proof is the Baire category theorem. We start with several definitions:

**Definition 3.1.** A **Baire space** is a topological space where the intersection of countably many dense open sets is dense. By the Baire category theorem, $(C(X, [0, 1]), \| \cdot \|_{\infty})$, is a Baire space. A set in a topological space is said to be **comeagre** or **generic** if it is the complement of a countable union of nowhere dense sets. A set is said to be $G_\delta$ if it is the countable intersection of open sets. Thus in a Baire space a dense $G_\delta$ set is comeagre.

**Definition 3.2.** Let $K \subset (X \times X) \setminus \Delta$ be a compact set, where $\Delta = \{(x,x) | x \in X \}$ is the diagonal of $X \times X$ and suppose $h \in C(X, [0, 1])$. Denote $h_0^{2d}(x) \triangleq (h(x), h(Tx), \ldots, h(T^{2d}x))$. We say that $h_0^{2d}$ is $K$-**compatible** if for every $(x,y) \in K$, $h_0^{2d}(x) \neq h_0^{2d}(y)$, or equivalently if
for every $(x, y) \in K$, there exists $n \in \{0, 1, \ldots, 2d\}$ so that $h(T^n x) \neq h(T^n y)$. Define:

$$D_K = \{ h \in C(X, [0, 1]) | h_0^{2d} \text{ is } K \text{-compatible} \}$$

In the next subsection we prove the following key lemma:

**Lemma 3.3. (Main Lemma)** One can represent $(X \times X) \setminus \Delta$ as a countable union of compact sets $K_1, K_2, \ldots$ such that for all $i$ $D_{K_i}$ is open and dense in $(C(X, [0, 1]), \| \cdot \|_\infty)$.

**Proof.** [Proof of Theorem 1.1 using Lemma 3.3] As for all $i$, $D_{K_i}$ is open and dense in $(C(X, [0, 1]), \| \cdot \|_\infty)$, we have that $\bigcap_{i=1}^\infty D_{K_i}$ is dense in $(C(X, [0, 1]), \| \cdot \|_\infty)$. Any $h \in \bigcap_{i=1}^\infty D_{K_i}$ is $K_i$-compatible for all $i$ simultaneously and therefore realizes an embedding $h_0^{2d} : (X, T) \hookrightarrow [0, 1]^{2d+1}$. As a dense $G_\delta$ set is comeagre, the above argument shows that the set $\mathcal{A} \subset C(X, [0, 1])$ for which $h_0^{2d} : (X, T) \hookrightarrow [0, 1]^{2d+1}$ is an embedding is comeagre, or equivalently, that the fact of $h_0^{2d}$ being an embedding is generic in $(C(X, [0, 1]), \| \cdot \|_\infty)$. \[\square\]

It is not hard to see that for every compact $K \subset (X \times X) \setminus \Delta$, $D_K$ is open in $(C(X, [0, 1]), \| \cdot \|_\infty)$ (see [Gut12 Lemma A.2]).

### 3.2. Proof of the main lemma.

We write $(X \times X) \setminus \Delta$ as the union of the following three sets: $C_1 = (X \times X) \setminus (\Delta \cup (P \times X) \cup (X \times P))$, $C_2 = (P \times P) \setminus \Delta$, $C_3 = ((X \setminus P) \times P) \cup (X \times (X \setminus P))$. In words $(x, y)$ (where $x \neq y$) belong to the first, second, third set if both $x, y$ are not periodic, both $x, y$ are periodic, either $x$ or $y$ are periodic but not both respectively. We then cover each of these sets, $j = 1, 2, 3$ by a
countable union of compact sets $K^{(j)}$, $K^{(j)}$, $\ldots$ such that for all $i$, $D_{K_i}^{(j)}$ is open and dense in $(C(X, [0, 1]), \| \cdot \|_{\infty})$.

Assume $(x, y) \in C_3$, w.l.o.g $y \in P$ and $x \notin P$. Denote the period of $y$ by $n < \infty$. Let $t_y = \min\{n - 1, 2d\}$. Let $H_n$ be the set of $z \in X$, whose period is $n$. In other words $H_n = P_n \setminus P_{n-1}$. Notice $H_n$ is open in $P_n$ and $T$-invariant. Let $U_y$ be an open set in $H_n$ (but not necessarily open in $X$) so that $y \in U_y \subset \overline{U}_y \subset H_n$ and $\overline{U}_y \cap T^{l} \overline{U}_y = \emptyset$ for $l = 1, 2, \ldots, t_y$. E.g. if $d(y, P_{n-1}) = r > 0$, let $0 < \epsilon < r$ small enough so that $U_y = B_\epsilon(y) \cap H_n$ and $\overline{U}_y = \overline{B}_\epsilon(y) \cap P_n \subset \overline{B}_\epsilon(y) \cap H_n$. As $x \notin P$, the forward orbit $\{T^k y\}_{k \geq 0}$ of $x$ is disjoint from $P_n$. In particular we may choose an open set $U_x$ such that $x \in U_x \subset X \setminus P_n$ (note $X \setminus P_n$ is a $T$-invariant open set) such that, setting $t_x = 2d$, $U_x, TU_x, \ldots, T^{t_x} U_x, \overline{U}_x, T^{t_x} \overline{U}_x, \ldots, T^{t_y} \overline{U}_y$ are pairwise disjoint. We now define $K_{(x, y)} = \overline{U}_x \times \overline{U}_y$. As $X$ is second-countable, every subspace is a Lindelöf space, i.e every open cover has a countable subcover. For every $n = 1, 2, \ldots$, $H_n$ can be covered by a countable number of sets of the form $U_y$. Similarly $X \setminus P$ can be covered by countable number of sets of the form $U_x$. We can thus choose a countable cover of $C_3$ by sets of the form $K_{(x, y)}$. We are left with the task of showing $D_{K_{(x, y)}}$ is dense in $(C(X, [0, 1]), \| \cdot \|_{\infty})$. Let $\epsilon > 0$. Let $\tilde{f} : X \to [0, 1]$ be a continuous function. We will show that there exists a continuous function $f : X \to [0, 1]$ so that $\| f - \tilde{f} \|_{\infty} < \epsilon$ and $f^{2d}_{0}$ is $K_{(x, y)}$-compatible. Let $\alpha_x$ and $\alpha_y$ be open covers of $\overline{U}_x$ and $\overline{U}_y$ respectively such that it holds for $j = x, y$ $\max_{W \in \alpha_j, k \in \{0, 1, \ldots, t_j\}} \operatorname{diam}(\tilde{f}(T^k W)) < \frac{\epsilon}{4}$
For $\alpha_x$ this amounts to $\text{ord}(\alpha_x) \leq d$ which is possible as $\dim(X) = d$ (recall $t_x = 2d$). The same is true for $\alpha_y$ if $t_y \geq 2d$. If $t_y < 2d$, this is possible as by assumption $\dim(\overline{U}_y) \leq \dim(P_{t_y+1}) < \frac{t_y+1}{2}$. For each $W \in \alpha_j$ choose $q_W \in W$ so that $\{q_W\}_{W \in \alpha_j}$ is a collection of distinct points in $X$. Define $\tilde{v}_W = (\tilde{f}(T^k q_W))_{k=0}^{t_j}$. Notice $t_x \geq t_y$. By Lemma \[\text{Gut12 Lemma A.9}\], as (3.1) holds, one can find for $j = x, y$ continuous functions $F_j : \overline{U}_j \to [0, 1]^{t_j+1}$, with the following properties:

1. $\forall W \in \alpha_j, \|F(q_W) - \tilde{v}_W\|_\infty < \frac{\epsilon}{2}$,
2. $\forall z \in \overline{U}_x \cup \overline{U}_y, F_j(z) \in \text{co}\{F_j(q_W)|z \in W \in \alpha_j\}$, where $\text{co}\{\{v_1, \ldots, v_m\} \triangleq \{\sum_{i=1}^{m} \lambda_i v_i | \sum_{i=1}^{m} \lambda_i = 1, \lambda_i \geq 0\}$,
3. If $x' \in \overline{U}_x$ and $y' \in \overline{U}_y$ then $F_x(x') \neq F_y(y')^{(2d+1)}$, where $F_y(y')^{(2d+1)} : \overline{U}_y \to [0, 1]^{2d+1}$ is the function given by the formula $[F_y(y')^{(2d+1)}](k) \triangleq [F_y(y')](k \mod (t_y + 1)), k = 0, 1, \ldots, 2d$.

Let $A = \bigcup_{j=x,y} \bigcup_{k=0}^{t_j} T^k \overline{U}_j$. Define $f' : A \to [0, 1] (j = x, y)$ by:

$$f'_{T^k \overline{U}_j}(T^k z) = [F_j(z)](k)$$

Fix $z \in \overline{U}_j$ and $k \in \{0, 1, \ldots, t_j\}$. As by property (2), $f'(T^k z) = [F_j(z)](k) \in \text{co}\{[F_j(q_W)](k) | z \in W \in \alpha_j\}$, we have $|f'(T^k z) - \tilde{f}(T^k z)| \leq \max_{z \in W \in \alpha_j} |[F_j(q_W)](k) - \tilde{f}(T^k z)|$. Fix $W \in \alpha_j$ and $z \in W$. Note $|[F_j(q_W)](k) - \tilde{f}(T^k z)| \leq |[F_j(q_W)](k) - [\tilde{v}_W](k)| + |[\tilde{v}_W](k) - \tilde{f}(T^k z)|$. The first term on the right-hand side is bounded by $\frac{\epsilon}{2}$ by property (1).

As $\text{diam}((\tilde{f}(T^k W))) < \frac{\epsilon}{2}$ and $[\tilde{v}_W](k) = \tilde{f}(T^k q_W)$ we have $|\tilde{f}(T^k q_W) - \tilde{f}(T^k W)| < \frac{\epsilon}{2}$. Thus $\forall j \in \{x, y\}$ we have $\max_{z \in W \in \alpha_j} |[F_j(q_W)](k) - \tilde{f}(T^k z)| < \frac{\epsilon}{2}$.
\[ f(T^k z) < \frac{\epsilon}{2} \]. We finally conclude \[ \|f' - \tilde{f}_A\|_\infty < \epsilon \]. By an easy application of the Tietze Extension Theorem (see \[ \text{[Gut12, Lemma A.5]} \]) there is \( f : X \to [0, 1] \) so that \( f|_A = f' \) and \( \|f - \tilde{f}\|_\infty < \epsilon \). Assume for a contradiction \( f^2 \circ d_0(x') = f^2 \circ d_0(y') \) for some \((x', y') \in K(x, y)\). This implies
\[
 F_x(x') = (f(x'), \ldots, f(T^{2d}x')) = (f(y'), \ldots, f(T^{2d}y')) = (F_y(y'))^{\oplus(2d+1)}
\]
which is a contradiction to property (3).

Unlike the previous case which differs in its treatment from the corresponding case in \[ \text{[Gut12, Theorem 8.1]} \], the cases \((x, y) \in C_1, (x, y) \in C_2\) follow quite straightforwardly. Indeed if \((x, y) \in C_1\) (both \(x\) and \(y\) are not periodic) and in addition the forward orbits of \(x\) and \(y\) are disjoint then we can use almost verbatim the case \((x, y) \in C_3\). The same is true if \((x, y) \in C_1\) and in addition \(y\) belongs to the forward orbit of \(x\), i.e. \(y = T^lx\), and \(l > 2d\). If \((x, y) \in C_1\), \(y = T^lx\) and \(l \leq 2d\) then one continues exactly as in Case 2 of \[ \text{[Gut12, Proposition 4.2]} \]. For \((x, y) \in C_2\) one uses \[ \text{[Gut12, Theorem 4.1]} \].

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