PRINCIPAL FUNDAMENTAL SYSTEM OF SOLUTIONS,
THE HARTMAN-WINTNER PROBLEM AND CORRECT SOLVABILITY
OF THE GENERAL STURM-LIOUVILLE EQUATION

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Abstract. We study the problem of correct solvability in the space $L^p(\mathbb{R})$, $p \in [1, \infty)$ of
the equation
$$-(r(x)y'(x))' + q(x)y(x) = f(x), \quad x \in \mathbb{R}$$
under the conditions
$$r > 0, \quad q \geq 0, \quad \frac{1}{r} \in L_1(\mathbb{R}), \quad q \in L_1(\mathbb{R}).$$

1. Introduction

In the present paper, we continue the study of the equation
$$-(r(x)y'(x))' + q(x)y(x) = f(x), \quad x \in \mathbb{R}, \quad f \in L^p(\mathbb{R}), \quad p \in [1, \infty) \quad (1.1)$$
developed in [3, 5, 7, 9, 4, 8]. In these papers, the problem of the existence of a unique
bounded solution of (1.1) in the space $L^p(\mathbb{R})$, $p \in [1, \infty)$ was studied under the following
two conditions:
$$r > 0, \quad q \geq 0, \quad \frac{1}{r} \in L^1_{\text{loc}}(\mathbb{R}), \quad q \in L^1_{\text{loc}}(\mathbb{R}), \quad (1.2)$$

$$\lim_{|d| \to \infty} \int_{x-d}^x \frac{dt}{r(t)} \cdot \int_{x-d}^x q(t)dt = \infty. \quad (1.3)$$

Below, in contrast with the cited papers, we study the same problem under the condition
$$r > 0, \quad q \geq 0, \quad \frac{1}{r} \in L_1(\mathbb{R}), \quad q \in L_1(\mathbb{R}). \quad (1.4)$$

To describe our work in a more conceptual way, we make some conventions and state some
definitions. In the sequel, condition (1.2) is our standing assumption and is therefore never
mentioned; the letters $c$, $c(\cdot)$ are used to denote absolute positive constants which are not
essential for exposition and may differ even within a single chain of computations. By a
solution of (1.1) we mean a function $y(\cdot)$, absolutely continuous together with the function
$r(\cdot)y'(\cdot)$ and satisfying equality (1.1) almost everywhere in $\mathbb{R}$.

We now introduce the following basic notion.

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Definition 1.1. [5, 7] We say that equation (1.1) is correctly solvable in a given space \( L^p(\mathbb{R}) \), \( p \in [1, \infty) \) if the following assertions hold:

I) for every \( f \in L^p(\mathbb{R}) \), there exists a unique solution \( y \in L^p(\mathbb{R}) \) of (1.1);

II) there exists a constant \( c(p) \in (0, \infty) \) such that regardless of the choice of \( f \in L^p(\mathbb{R}) \), the solution \( y \in L^p(\mathbb{R}) \) of (1.1) satisfies the inequality

\[
\|y\|_{L^p(\mathbb{R})} \leq c(p)\|f\|_{L^p(\mathbb{R})}.
\]  

(1.5)

If I) or II) fails to hold, we say that equation (1.1) in the space \( L^p(\mathbb{R}) \) is not correctly solvable.

We make the following conventions: for brevity, instead of repeating Definition 1.1 we say “the problem concerning I)–II)” or “the question concerning I)–II)”. We often omit the word equation before the symbol (1.1) and the word “space” before \( L^p(\mathbb{R}) \). Instead of \( L^p(\mathbb{R}) \) and \( \| \cdot \|_{L^p(\mathbb{R})} \), we write \( L^p \) and \( \| \cdot \|_p \). Finally, the symbol \( p' \) denotes the conjugate number of \( p \in (1, \infty) \).

Recall that the main goal of our work is to study the question of I)–II) in the case (1.4), and we explain now the reason for focusing on this condition. The question concerning I)–II) was posed and answered in [5] under the condition \( r \equiv 1 \), and the problem concerning I)–II) was solved in [7] under the conditions \( r \not\equiv 1 \) and (1.3). It turned out that there is an essential difference between the cases \( r \equiv 1 \) and \( r \not\equiv 1 \). Whereas in [5] for \( r \equiv 1 \) the criterion for I)–II) to hold is expressed in terms of a simple requirement for the function \( q(\cdot) \) itself, in [7] for \( r \not\equiv 1 \) the criterion for I)–II) to hold is expressed in terms of auxiliary functions (implicit function in \( r(\cdot) \) and \( q(\cdot) \)) which are only defined under condition (1.3). Thus, the question regarding I)–II) in the case (1.4) remained open.

In the present paper, we propose a new version of the method to study the problem concerning I)–II) used in [7]. Its obvious advantage compared to the approach taken in [7] can be stated as follows. In the case (1.3), combining the old and new approaches allows one to simplify, in an essential way, the solution of the problem concerning I)–II), whereas in the case of (1.4) the new approach to I)–II) guarantees a sufficiently simple solution of this problem by standard analytic techniques (see §3 and §5 below).

The structure of the paper is as follows: In §2, we collect the preliminaries used in the proofs; §3 contains the statements of our results and some additional comments; all the proofs are given in §4; in §5 we give an example and further discussion.
2. Preliminaries

Below we present the definitions and facts that are used in the proofs.

2.1. Principal fundamental system of solutions (PFSS).

Definition 2.1.1. We say that a fundamental system of solutions (FSS) \( \{u(x), v(x)\} \), \( x \in \mathbb{R} \) of the equation

\[
(r(x)z'(x)) = q(x)z(x), \quad x \in \mathbb{R}
\]  

(2.1)
is a principal fundamental system of solutions (PFSS) if the functions \( u(x) \) and \( v(x) \) possess the following properties:

\[
u(x) > 0, \quad v(x) > 0, \quad u'(x) \leq 0, \quad v'(x) \geq 0 \quad \text{for} \quad x \in \mathbb{R},
\]

(2.2)

\[
r(x)[v'(x)u(x) - u'(x)v(x)] = 1 \quad \text{for} \quad x \in \mathbb{R},
\]

(2.3)

\[
\lim_{x \to \infty} \frac{u(x)}{v(x)} = \lim_{x \to -\infty} \frac{v(x)}{u(x)} = 0.
\]

(2.4)

In connection with this definition, we refer the reader to [12]. In the sequel, the symbol \( \{u(x), v(x)\}, x \in \mathbb{R} \) stands for a PFSS of (2.1).

Theorem 2.1.2. Equation (2.1) has a PFSS if

\[
\int_{-\infty}^{0} q(t)dt > 0, \quad \int_{0}^{\infty} q(t)dt > 0.
\]

(2.5)

Theorem 2.1.3. Suppose that equation (2.1) has a PFSS. Denote

\[
\rho(x) = u(x)v(x), \quad x \in \mathbb{R}.
\]

(2.6)

Then one has the Davies-Harrell formulas:

\[
u(x) = \sqrt{\rho(x)} \exp \left(-\frac{1}{2} \int_{x_0}^{x} \frac{d\xi}{r(\xi)\rho(\xi)} \right), \quad v(x) = \sqrt{\rho(x)} \exp \left(\frac{1}{2} \int_{x_0}^{x} \frac{d\xi}{r(\xi)\rho(\xi)} \right), \quad x \in \mathbb{R},
\]

(2.7)

where \( x_0 \) is a unique root of the equation \( u(x) = v(x), \quad x \in \mathbb{R} \).

Corollary 2.1.4. We have the following relations for a PFSS of equation (2.1) and the functions \( \rho(\cdot) \) (see (2.6)):

\[
u'(x) = \frac{1 - r(x)\rho'(x)}{2r(x)\rho(x)}, \quad v'(x) = \frac{1 + r(x)\rho'(x)}{2r(x)\rho(x)}, \quad x \in \mathbb{R},
\]

(2.8)

\[
r(x)|\rho'(x)| < 1, \quad x \in \mathbb{R}.
\]

(2.9)

Remark 2.1. In a particular case, formulas (2.7) were obtained by N. Abel (see [18, 19.53]). Under the conditions (1.2) and \( r \equiv 1 \), they were proven in [11]. In the present statement, Theorem 2.1.3 was proven in [3].
2.2. Hartman-Wintner problem. In this section, we consider the equations

\[(r(x)g'(x))' = q(x)g(x), \quad x \in \mathbb{R}, \quad (r(x)z'(x))' = q_1(x)z(x), \quad x \in \mathbb{R},\]

where the functions \(r(x), q(x)\) and \(q_1(x)\) are real and continuous for \(x \in \mathbb{R}\) and, in addition, \(r(x) > 0\) for \(x \in \mathbb{R}\). We also assume that equation (2.11) has a FSS \(\{u_1(x), v_1(x)\}, x \in \mathbb{R}\).

**Definition 2.2.1.**\(^{[2]}\[^{[12]}\] We say that the Hartman-Wintner problem for equations (2.10) and (2.11) is solvable as \(x \to \infty\) (as \(x \to -\infty\)) if there is a FSS \(\{\hat{u}(x), \hat{v}(x)\}, x \in [0, \infty)\) \((\{\hat{u}(x), \hat{v}(x)\} x \in (-\infty, 0]\) of equation (2.10) such that

\[
\begin{align*}
\lim_{x \to \infty} \frac{\hat{u}(x)}{u_1(x)} &= \lim_{x \to \infty} \frac{\hat{v}(x)}{v_1(x)} = 1 \quad \left( \lim_{x \to -\infty} \frac{\hat{u}(x)}{u_1(x)} = \lim_{x \to -\infty} \frac{\hat{v}(x)}{v_1(x)} = 1 \right), \\
\left( \frac{\hat{u}'(x)}{\hat{u}(x)} - \frac{u'_1(x)}{u_1(x)} \right) &= o\left( \frac{1}{r(x)u_1(x)v_1(x)} \right) \quad \text{as} \quad x \to \infty, \\
\left( \frac{\hat{v}'(x)}{\hat{v}(x)} - \frac{v'_1(x)}{v_1(x)} \right) &= o\left( \frac{1}{r(x)u_1(x)v_1(x)} \right) \quad \text{as} \quad x \to -\infty,
\end{align*}
\]

\[
\begin{align*}
\left( \frac{\hat{u}'(x)}{\hat{u}(x)} - \frac{u'_1(x)}{u_1(x)} \right) &= o\left( \frac{1}{r(x)u_1(x)v_1(x)} \right) \quad \text{as} \quad x \to \infty, \\
\left( \frac{\hat{v}'(x)}{\hat{v}(x)} - \frac{v'_1(x)}{v_1(x)} \right) &= o\left( \frac{1}{r(x)u_1(x)v_1(x)} \right) \quad \text{as} \quad x \to -\infty.
\end{align*}
\]

**Remark 2.2.** Here, taking into account the goals of the present work, we slightly restrict the statement of the Hartman-Wintner problem compared to the original papers (see \(^{[2]},^{[12]}\)). For brevity, below we refer to the Hartman-Wintner problem as problem (2.12)-(2.14).

We need the following notation:

\[
\rho_1(x) = u_1(x)v_1(x), \quad x \in \mathbb{R}, \quad (\Delta q)(x) = q(x) - q_1(x), \quad x \in \mathbb{R},
\]

\[
I^{(-)}(x) = \int_{-\infty}^{x} (\Delta q)(t)p_1(t)dt, \quad I^{(+)}(x) = \int_{x}^{\infty} (\Delta q)(t)p_1(t)dt, \quad x \in \mathbb{R}.
\]

**Theorem 2.2.2.** \(^{[12]}\); see also \(^{[2]}\) Problem (2.12)-(2.14) for equations (2.10) and (2.11) is solvable as \(x \to \infty\) (as \(x \to -\infty\)) if the integral \(I^{(+)}(0)\) \((I^{(-)}(0))\) absolutely converges.

**Theorem 2.2.3.** \(^{[2]}\) Problem (2.12)-(2.14) for equations (2.10) and (2.11) is solvable as \(x \to \infty\) (as \(x \to -\infty\)) if the integral \(I^{(+)}(0)\) \((I^{(-)}(0))\) converges at least conditionally, and
the following inequality holds:

\[
\int_0^\infty \frac{I^+(x)^2}{r(x)p_1(x)} \, dx < \infty \quad \left( \int_{-\infty}^0 \frac{I^-(x)^2}{r(x)p_1(x)} \, dx < \infty \right).
\]  

(2.18)

2.3. Some facts on the correct solvability of equation (1.1).

Theorem 2.3.1. Suppose that

\[
\int_0^\infty \frac{dt}{r(t)} = \int_{-\infty}^0 \frac{dt}{r(t)} = \infty
\]  

(2.19)

and equation (1.1) is correctly solvable in \(L_p\), \(p \in [1, \infty)\). Then

\[
\int_{-\infty}^x q(t) \, dt > 0, \quad \int_x^\infty q(t) \, dt > 0 \quad \text{for every} \quad x \in \mathbb{R},
\]  

(2.20)

and therefore equation (2.1) has a PFSS (see Theorem 2.1.2).

Throughout this section, we assume that (2.20) holds. This standing assumption does not appear in the statements. From (2.20) and Theorem 2.1.2 it follows that equation (2.1) has a PFSS \(\{u(x), v(x)\} \), \(x \in \mathbb{R}\). Denote

\[
G(x, t) = \begin{cases} 
\begin{array}{ll}
u(x) \cdot v(t), & x \geq t \\
u(t) \cdot v(x), & x \leq t
\end{array}
\end{cases}, \quad x, t \in \mathbb{R};
\]  

(2.21)

\[
(Gf)(x) = \int_{-\infty}^\infty G(x, t)f(t) \, dt, \quad x \in \mathbb{R}, \ f \in L_p;
\]  

(2.22)

\[
(G_1 f)(x) = u(x) \int_{-\infty}^x v(t)f(t) \, dt, \quad (G_2 f)(x) = v(x) \int_x^\infty u(t)f(t) \, dt, \quad x \in \mathbb{R}, \ f \in L_p.
\]  

(2.23)

Here \(G(\cdot, \cdot)\) is the Green function, and \(G : L_p \to L_p\), \(p \in [1, \infty)\) is the Green operator of equation (1.1).

Lemma 2.3.2. The following relations hold:

\[
(Gf)(x) = (G_1 f)(x) + (G_2 f)(x), \quad x \in \mathbb{R}, \ f \in L_p, \ p \in [1, \infty),
\]  

(2.24)

\[
\frac{1}{2} (\|G_1\|_{p \to p} + \|G_2\|_{p \to p}) \leq \|G\|_{p \to p} \leq \|G_1\|_{p \to p} + \|G_2\|_{p \to p}, \quad p \in [1, \infty).
\]  

(2.25)

Theorem 2.3.3. Let \(p \in [1, \infty)\), and suppose that (2.20) holds. Then equation (1.1) is correctly solvable in \(L_p\) if and only if the operator \(G : L_p \to L_p\) is bounded.

Corollary 2.3.4. Let \(p \in [1, \infty)\) and suppose that \(\|G\|_{p \to p} < \infty\). Then for \(f \in L_p\), the solution \(y \in L_p\) of (1.1) is of the form

\[
y(x) = (Gf)(x) = \int_{-\infty}^\infty G(x, t)f(t) \, dt, \quad x \in \mathbb{R}.
\]  

(2.26)
Lemma 2.3.5. [3] Under condition (1.3), for any $x \in \mathbb{R}$ each of the equations
\[ \int_{x-d}^{x} \frac{dt}{r(t)} \cdot \int_{x-d}^{x} q(t) dt = 1, \quad \int_{x}^{x+d} \frac{dt}{r(t)} \cdot \int_{x}^{x+d} q(t) dt = 1 \] (2.27)
in $d \geq 0$ has a unique finite positive solution. Denote them by $d_1(x)$ and $d_2(x)$, respectively, and introduce the functions
\[ \varphi(x) = \int_{x-d_1(x)}^{x} \frac{dt}{r(t)}, \quad \psi(x) = \int_{x}^{x+d_2(x)} \frac{dt}{r(t)}, \quad x \in \mathbb{R}, \] (2.28)
\[ h(x) = \frac{\varphi(x) \cdot \psi(x)}{\varphi(x) + \psi(x)} \left( \int_{x-d_1(x)}^{x+d_2(x)} q(t) dt \right)^{-1}, \quad x \in \mathbb{R}. \] (2.29)
Further, for any $x \in \mathbb{R}$, the equation
\[ \int_{x-d}^{x+d} \frac{dt}{r(t)h(t)} = 1 \] (2.30)
in $d \geq 0$ has a unique finite positive solution. Denote it by $d(x)$, $x \in \mathbb{R}$. The function $d(x)$ is continuous for $x \in \mathbb{R}$.

Remark 2.3. From Theorem 2.1.2 it follows that under condition (2.20), equation (2.1) has a PFSS $\{u(x), v(x)\}, x \in \mathbb{R}$, and therefore the function $\rho(x), x \in \mathbb{R}$, is well-defined (see (2.6)).

Theorem 2.3.6. [3] Under condition (1.3), we have
\[ 2^{-1}h(x) \leq \rho(x) \leq 2h(x), \quad x \in \mathbb{R}. \] (2.31)

Remark 2.4. Two-sided, sharp by order, a priori estimates for the function $\rho(x), x \in \mathbb{R}$, were first obtained in [17] (under some additional requirements for $r$ and $q$) by M.O. Otelbaev. Therefore, all estimates of type (2.31) are called Otelbaev inequalities. Note that in [17] Otelbaev used a more complicated auxiliary function than $h(x), x \in \mathbb{R}$.

Theorem 2.3.7. [7, §4 (Theorem 4.1)] Let $p \in (1, \infty)$, and suppose that (1.3) holds. Then equation (1.1) is correctly solvable in $L_p$ if and only if $B < \infty$ where
\[ B = \sup_{x \in \mathbb{R}} (h(x)d(x)). \] (2.32)

2.4. Two theorems on integral operators.

Theorem 2.4.1. [13] Ch. V, §2, no.5] Let $-\infty \leq a < b \leq \infty$, let $K(s, t)$ be a continuous function in $s, t \in [a, b]$, and let $K$ be an integral operator of the form
\[ (Kf)(t) = \int_{a}^{b} K(s, t) f(s) ds, \quad t \in [a, b]. \] (2.33)
Then
\[ \|K\|_{L^1(a,b)\to L^1(a,b)} = \sup_{s \in [a,b]} \int_a^b |K(s,t)| \, dt. \] (2.34)

**Theorem 2.4.2.** [14] Let \( \mu(\cdot) \) and \( \theta(\cdot) \) be positive continuous functions in \( \mathbb{R} \), and let \( K^{(+)}(K^{(-)}) \) be an integral operator of the form
\[ (K^{(+)}f)(t) = \mu(t) \int_0^t \theta(\xi)f(\xi)d\xi, \quad t \in \mathbb{R}; \quad ((K^{(-)}f)(t) = \mu(t) \int_{-\infty}^t \theta(\xi)f(\xi)d\xi, \quad t \in \mathbb{R}). \] (2.35)
Then for \( p \in (1, \infty) \), the operator \( K^{(+)} : L_p \to L_p \) \( ((K^{(-)} : L_p \to L_p) \) is bounded if and only if \( H^{(+)}_p < \infty \) \( (H^{(-)}_p < \infty) \). Here
\[ H^{(+)}_p = \sup_{x \in \mathbb{R}} H^{(+)}_p(x) \quad (H^{(-)}_p = \sup_{x \in \mathbb{R}} H^{(-)}_p(x)), \] (2.36)
where
\[ H^{(+)}_p(x) = \left( \int_x^\infty \mu(t)^p dt \right)^{1/p} \left( \int_x^\infty \theta(t)^{p'} dt \right)^{1/p'} \] (2.37)
\[ H^{(-)}_p(x) = \left( \int_{-\infty}^x \theta(t)^{p'} dt \right)^{1/p'} \left( \int_x^\infty \mu(t)^p dt \right)^{1/p}. \]
In addition, we have the inequalities
\[ H^{(+)}_p \leq \|K^{(+)}\|_{p \to p} \leq (p)^{1/p}(p')^{1/p'} H^{(+)}_p \quad (H^{(-)}_p \leq \|K^{(-)}\|_{p \to p} \leq (p)^{1/p}(p')^{1/p} H^{(-)}_p). \] (2.38)

### 2.5. On Otelbaev’s coverings of the real semi-axis.

**Definition 2.5.1.** [3] [6] [15] Suppose we are given \( x \in \mathbb{R} \), a positive continuous function \( \varkappa(x) \), \( x \in \mathbb{R} \), and a sequence \( \{x_n\}_{n \in \mathbb{N}'} \), \( \mathbb{N}' = \{\pm 1, \pm 2, \ldots\} \). Consider the segments \( \Delta_n = x_n \pm \varkappa(x_n) \), \( n \in \mathbb{N}' \). We say that the sequence of segments \( \{\Delta_n\}_{n=1}^\infty \) \( (\{\Delta_n\}_{n=-\infty}^{-1}) \) forms an \( \mathbb{R}(x, \varkappa(\cdot)) \)-covering of \( [x, \infty) \) \( (\text{resp. an } \mathbb{R}(x, \varkappa(\cdot)) \)-covering of \( (-\infty, x] \) if the following conditions hold:

1) \( \Delta^{(+)}_n = \Delta^{(-)}_{n+1} \) for \( n \geq 1 \) \( (\text{resp. } \Delta^{(+)}_{n-1} = \Delta^{(-)}_n \text{ for } n \leq -1) \);
2) \( \Delta^{(-)}_\infty = x \) \( (\text{resp. } \Delta^{(+)}_{\infty} = x) \);
3) \( \bigcup_{n \geq 1} \Delta_n = [x, \infty) \) \( (\text{resp. } \bigcup_{n \leq -1} \Delta_n = (-\infty, x]) \).

**Lemma 2.5.2.** [6] [15] Suppose that a positive continuous function \( \varkappa(x) \), \( x \in \mathbb{R} \), satisfies the relations
\[ \lim_{x \to \infty} (x - \varkappa(x)) = \infty, \quad \lim_{x \to -\infty} (x + \varkappa(x)) = -\infty. \] (2.39)
Then for any \( x \in \mathbb{R} \) there exists an \( \mathbb{R}(x, \varkappa(\cdot)) \)-covering of \( [x, \infty) \) \( (\text{resp. an } \mathbb{R}(x, \varkappa(\cdot)) \)-covering of \( (-\infty, x]) \).
3. Results and Additional Comments

The following assertion can be viewed as a complement to Theorem 2.1.2.

**Theorem 3.1.** Let \(1/r(\cdot) \in L_1\). Then the functions
\[
u(x) = \frac{1}{\sqrt{w_0}} \int_x^\infty \frac{dt}{r(t)}, \quad v(x) = \frac{1}{\sqrt{w_0}} \int_{-\infty}^x \frac{dt}{r(t)}, \quad x \in \mathbb{R}, \quad w_0 = \int_{-\infty}^\infty \frac{dt}{r(t)}
\] (3.1)
form a PFSS \(\{u(x), v(x)\}\), \(x \in \mathbb{R}\), of the equation
\[(r(x)z'(x))' = 0, \quad x \in \mathbb{R}.\] (3.2)

Definition 2.1.1 can be complemented by Theorem 3.2 and Corollary 3.3.

**Theorem 3.2.** Suppose that equation (2.1) has a PFSS \(\{u(x), v(x)\}\), \(x \in \mathbb{R}\). Then any other PFSS \(\{u_1(x), v_1(x)\}\), \(x \in \mathbb{R}\), of this equation is of the form
\[u_1(x) = \alpha u(x), \quad v_1(x) = \alpha^{-1} v(x), \quad x \in \mathbb{R}.\] (3.3)

Here \(\alpha\) is an arbitrary fixed positive constant.

**Corollary 3.3.** If equation (2.1) has a PFSS \(\{u(x), v(x)\}\), \(x \in \mathbb{R}\), then the function \(\rho(x), x \in \mathbb{R}\) (see (2.6)) does not depend on the choice of a PFSS.

**Remark 3.4.** Therefore, the function \(\rho(x), x \in \mathbb{R}\), is a well-defined implicit function in the coefficients of equation (2.1). It is instructive to compare this fact with formulas (2.7) and the Davies-Harrell representation of the Green function \(G(\cdot, \cdot)\) (see (2.21), [11] and [3]):
\[G(x,t) = \sqrt{\rho(x)\rho(t)} \exp \left(-\frac{1}{2} \left| \int_x^t \frac{d\xi}{r(\xi)\rho(\xi)} \right| \right), \quad x, t \in \mathbb{R}.\] (3.4)

Throughout the sequel, we call \(\rho(x), x \in \mathbb{R}\), the generating function of the PFSS of equation (2.1).

**Theorem 3.5.** Suppose that
\[\int_{-\infty}^0 \frac{dt}{r(t)} = \int_0^\infty \frac{dt}{r(t)} = \infty, \quad q(x) = 0 \quad \text{for} \ x \in \mathbb{R}.\] (3.5)
Then equation (3.2) has no PFSS.

In the remainder of this section, we assume that condition (1.2) holds and equation (2.1) has a PFSS \(\{u(x), v(x)\}\), \(x \in \mathbb{R}\). In the absence of a special need, we do not include these requirements in the statements. Note that we do not assume that condition (1.3) holds.
Theorem 3.6. Equation (2.1) has no solutions \( y(x) \in L_p, p \in [1, \infty) \) except for \( y(x) \equiv 0, x \in \mathbb{R} \).

Theorem 3.7. For \( p \in [1, \infty) \), equation (1.1) is correctly solvable in \( L_p \) if and only if the Green operator \( G : L_p \rightarrow L_p \) is bounded (see (2.22)).

Corollary 3.8. Let \( p \in [1, \infty) \), and suppose that equation (1.1) is correctly solvable in \( L_p \). Then for any \( f \in L_p \), the solution \( y \in L_p \) of (1.1) is of the form
\[
y(x) = (Gf)(x) = \int_{-\infty}^{\infty} G(x,t)f(t)dt, \quad x \in \mathbb{R}. \tag{3.6}
\]

Theorem 3.7 has the following advantage compared to Theorem 2.3.3: the requirement of the existence of a PFSS of equation (2.1) is weaker than condition (2.20) of Theorem 2.3.3 which guarantees this requirement. In particular, Theorem 3.7 is applicable in the case (3.1), which is not covered by Theorem 2.3.3.

Now let us introduce a new auxiliary function \( s(x), x \in \mathbb{R} \), an analogue of the function \( d(x), x \in \mathbb{R} \), (see Lemma 2.3.5). To this end, fix \( x \in \mathbb{R} \) and consider the following equation in \( s \geq 0 \):
\[
\int_{x-s}^{x+s} \frac{dt}{r(t)\rho(t)} = 1. \tag{3.7}
\]

Lemma 3.9. For any \( x \in \mathbb{R} \), there exists a unique solution of (3.7) in \( s \geq 0 \); denote it by \( s(x) \). The function \( s(x) \) is positive, continuous, and differentiable almost everywhere in \( \mathbb{R} \). In addition, the following relations hold:
\[
|s(x+t) - s(x)| \leq |t| \quad \text{for} \quad |t| \leq s(x), \quad x \in \mathbb{R}, \tag{3.8}
\]
\[
|s'(x)| < 1 \quad \text{for almost all} \quad x \in \mathbb{R}, \tag{3.9}
\]
\[
\lim_{x \to -\infty} (x+s(x)) = -\infty, \quad \lim_{x \to \infty} (x-s(x)) = \infty, \tag{3.10}
\]
\[
0 < s(x) \leq |x| \quad \text{for} \quad |x| \gg 1. \tag{3.11}
\]

Finally, there is a constant \( c \in [1, \infty) \) such that for all \( x \in \mathbb{R} \), we have the inequality
\[
s(x) \leq c(1 + |x|). \tag{3.12}
\]

Remark 3.10. Functions similar to \( s(x), x \in \mathbb{R} \) (see also the functions \( d(x), x \in \mathbb{R} \), from Lemma 2.3.5) were introduced and used in various areas of analysis by Otelbaev (see [15]). The fact that one of such functions is Lipschitz was first proved by K.T. Mynbaev (see [15]).
Lemma 3.11. For $t \in [x - s(x), x + s(x)]$, $x \in \mathbb{R}$, we have the following estimates:

$$e^{-1}\rho(x) \leq \rho(t) \leq e\rho(x), \quad e = \exp(1),$$  \hspace{1cm} (3.13)

$$e^{-1}u(x) \leq u(t) \leq eu(x), \quad e^{-1}v(x) \leq v(t) \leq ev(v).$$  \hspace{1cm} (3.14)

Theorem 3.12. For $p \in (1, \infty)$, equation (1.1) is correctly solvable in $L_p$ if and only if $\mathcal{D} < \infty$. Here, $\mathcal{D} = \sup_{x \in \mathbb{R}}(\rho(x)s(x))$. (3.15)

In particular, the following inequalities hold (see (2.22), (2.23)):

$$c^{-1}\mathcal{D} \leq \|G_1\|_{p \to p}; \quad \|G_2\|_{p \to p}; \quad \|G\|_{p \to p} \leq c\mathcal{D}.$$  \hspace{1cm} (3.16)

It is easy to see that under condition (1.3), Theorems 2.3.7 and 3.12 are equivalent. However, Theorem 3.12 is more general than Theorem 2.3.7. For example, it is applicable in the case $1/r \in L_1, q \equiv 0$ whereas Theorem 2.3.7 is obviously not applicable because of condition (2.20).

Corollary 3.13. For $p \in (1, \infty)$, equation (1.1) is correctly solvable in $L_p$ if any of the following inequalities holds:

$$\sigma_1 = \sup_{x \in \mathbb{R}}(r(x)\rho(x))^2 < \infty,$$  \hspace{1cm} (3.17)

$$\sigma_2 = \sup_{x \in \mathbb{R}}(\rho(x)|x|) < \infty,$$  \hspace{1cm} (3.18)

$$\sigma_3 = \inf_{x \in \mathbb{R}} (1/2s(x) \int_{x-s(x)}^{x+s(x)} q(\xi)d\xi) > 0,$$  \hspace{1cm} (3.19)

$$\sigma_4 = \inf_{x \in \mathbb{R}} q(x) > 0,$$  \hspace{1cm} (3.20)

$$\sigma_5 = \sup_{x \in \mathbb{R}} \left[ r(x) \left( \int_{-\infty}^{x} \frac{dt}{r(t)} \right)^2 \left( \int_{x}^{\infty} \frac{dt}{r(t)} \right)^2 \right] < \infty.$$  \hspace{1cm} (3.21)

Note that the fact that condition (3.20) is sufficient for the correct solvability of (1.1) in $L_p$ was established in [16].

Corollary 3.14. Suppose that the following two conditions hold:

1) $r_0 \overset{\text{def}}{=} \inf_{x \in \mathbb{R}} (r(x)) < \infty$;  \hspace{1cm} 2) $\lim_{|x| \to \infty} r(x)\rho(x) = \infty$. \hspace{1cm} (3.22)

Then for $p \in (1, \infty)$, equation (1.1) is not correctly solvable in $L_p$. 

In connection with Corollary 3.14, note that if the function \( q(x), x \in \mathbb{R} \) is finitary and the function \( r(x), x \in \mathbb{R} \), satisfies condition (2.19), then for \( p \in (1, \infty) \) equation (1.1) is not solvable in \( L_p \) (see Theorem 2.3.1).

**Definition 3.15.** Let functions \( f(x) \) and \( g(x) \) be defined, continuous and positive for \( x \in (a,b), -\infty \leq a < b \leq \infty \). We say that these functions are weakly equivalent for \( x \in (a,b) \) and write
\[
f(x) \equiv g(x),
\]
if there is a constant \( c \in [1, \infty) \) such that for all \( x \in (a,b) \), we have the inequalities
\[
c^{-1}f(x) \leq g(x) \leq cf(x).
\]
(3.23)

In the next definition, together with (1.1), we consider the equation
\[
-(r(x)y'(x))' + q_1(x)y(x) = f(x), \quad x \in \mathbb{R}
\]
(3.24)
under the condition
\[
r(\cdot) > 0, \quad q_1(\cdot) \geq 0, \quad \frac{1}{r} \in L^1_{\text{loc}}(\mathbb{R}), \quad q_1 \in L^1_{\text{loc}}(\mathbb{R}).
\]
(3.25)
We also assume, without mention in the statements, that equation (2.11) like equation (3.24) satisfies condition (3.25) and has a PFSS \( \{u_1(x), v_1(x)\}, x \in \mathbb{R} \).

**Definition 3.16.** We say that for equations (1.1) and (3.24) problems I)-II), \( p \in (1, \infty) \), are equivalent if from the correct solvability of one of them follows the correct solvability of the other.

**Theorem 3.17.** For equations (1.1) and (3.24), problems I)-II), \( p \in (1, \infty) \) are equivalent (not equivalent) if for \( x \in \mathbb{R} \) the functions \( \rho(x) \) and \( \rho_1(x) \), generating PFSS of equations (2.1) and (2.11) are (are not) weakly equivalent.

**Theorem 3.18.** Suppose we are given functions \( r(x), q(x), q_1(x) \), continuous for \( x \in \mathbb{R} \), such that \( r(x) > 0 \) and the following two conditions hold:

1) each of the equations (2.1) and (2.11) have a PFSS \( \{u(x), v(x)\}, x \in \mathbb{R} \), and \( \{u_1(x), v_1(x)\} \), \( x \in \mathbb{R} \), respectively;

2) problems (2.12)-(2.14) for equations (2.1) and (2.11) are solvable as \( x \to \pm \infty \).

Then for \( x \in \mathbb{R} \) the functions \( \rho(x) \) and \( \rho_1(x) \), generating PFSS of equations (2.1) and (2.11), are weakly equivalent.
Theorem 3.19. Suppose we are given functions \( r(x) \) and \( q(x) \), continuous for \( x \in \mathbb{R} \), such that \( r(x) > 0 \) and \( q(x) > 0 \) for \( x \in \mathbb{R} \) and \( 1/r \in L_1 \). Then, assuming that we have the inequalities
\[
- \int_{-\infty}^{0} q(x) \left( \int_{-\infty}^{x} \frac{dt}{r(t)} \right) dx < \infty, \quad \left( \int_{0}^{\infty} q(x) \int_{x}^{\infty} \frac{dt}{r(t)} \right) dx < \infty, \tag{3.26}
\]
Then problems I)-II), \( p \in (1, \infty) \) for equations (1.1) and
\[
(r(x)y'(x))' = f(x), \quad x \in \mathbb{R} \tag{3.27}
\]
are equivalent. In particular, these problems are equivalent if instead of (3.26) we require the inclusion \( q(\cdot) \in L_1 \).

Let us make some comments on the last three assertions. First note that applying Theorem 3.17 is based on Theorem 2.3.6 in the case (1.3) and on Theorems 3.18 and 2.2.2 in the case (1.4). Furthermore, Theorem 3.17 is significant in the study of problem I)-II) for two reasons. First, it is equally relevant in both the cases (1.3) and (1.4). Second, Theorem 3.17 allows one to control the transition from the original problem I)-II) for equation (1.1) to a similar equivalent problem for the simple (model) equation (3.24). Here the relation \( \rho(x) \simeq \rho_1(x), \ x \in \mathbb{R} \) serves as a criterion for such a transition to be correct. The conditions for checking this relation are given in Theorem 2.3.6 in the case (1.3), whereas in the case (1.4) such a verification is especially simple due to Theorems 3.18 and 2.2.2 (see the proof of Theorem 3.19). Finally, we describe the conceptual difference between the case (1.3) and (1.4). It is based on the fact that in the case (1.4), for equation (1.1), regardless of its coefficient \( q(\cdot) \in L_1, \) (within the framework of problem I)-II)) one can immediately find the simplest appropriate model equation (3.27) (see Theorem 3.19), whereas (within the same problem I)-II)) in the case (1.3), the choice of an appropriate model equation (3.24) is a nontrivial problem in its own right.

To conclude this section, note that for the sake of keeping the present paper with a limited amount of pages, we do not study a priori estimates for the function \( s(x), \ x \in \mathbb{R} \) (see (3.7) and Theorem 3.12). Therefore, in the study of the example in §5 we are forced to use special trick and the assertions of Corollary 3.13. We plan to study the function \( s(x), \ x \in \mathbb{R} \) and a direct application of Theorem 3.12 in a subsequent paper.
4. Proofs and Comments

First we note some special features of this section. Our earlier study of equations (1.1) and (2.1) was based on the following (see [3, 5, 7, 9, 4, 8]):

1) Definition 2.1.1 of a PFSS of equation (2.1) and the condition for the their existence (see Theorem 2.1.2);
2) Davies-Harrell formulas (2.7) and (3.4) (see [3] for a simple proof);
3) a priori estimates for the function $\rho(x)$, $x \in \mathbb{R}$, generating a PFSS of equation (2.1) (see [1, 8]).

Items 1) and 3) require some comments (in regard to item 2), see Remark 2.1 and Theorem 2.1.3). We start with 1). The properties of solutions of equation (2.1) have been studied in many papers (see, e.g., [12]), and below one can see significant overlaps with them. However, the present paper is the first instance of viewing Definition 2.1.1 of a PFSS as a basic concept of the theory of equation (2.1) and, more significantly, it contains the first proof of all the main properties of PFSS based only on Definition 2.1.1 and Davies-Harrell formulas (2.7) (which follow from Definition 2.1.1, see [3]).

Since we can easily locate the cases where our assertions overlap with the known ones (see [12]), for the sake of brevity we do not mention or comment on them. Now we go to 3). Our main results are Theorems 3.7, 3.12, 3.17 and 3.19 and their corollaries. The proofs of Theorems 3.7 and 3.12 follow along the lines of [7]. Therefore, in the cases where the relevant part of the proof only relies on the properties of PFSS of equation (1.1) and thus mimics the corresponding part of [7], we only state the assertion and refer the reader to [7]. However, for the other theorems, we present complete new and shorter proofs than found in [7].

Proof of Theorem 3.1. A straightforward check shows that in the case (3.1) the functions $\{u(x), v(x)\}$, $x \in \mathbb{R}$, form a PFSS of equation (3.2). □

Proof of Theorem 3.2.

Lemma 4.1. [3] We have the inequalities

$$\int_{-\infty}^{0} \frac{dt}{r(t)\rho(t)} = \int_{0}^{\infty} \frac{dt}{r(t)\rho(t)} = \infty. \quad (4.1)$$
Proof. The following relations are deduced from (2.1), (2.6) and (2.7):
\[ \exp \left( - \int_{x_0}^{x} \frac{dt}{r(t)\rho(t)} \right) = \frac{u(x)}{v(x)} \to 0 \text{ as } x \to \infty \Rightarrow \int_{0}^{\infty} \frac{dt}{r(t)\rho(t)} = \infty; \]
\[ \exp \left( - \int_{x}^{x_0} \frac{dt}{r(t)\rho(t)} \right) = \frac{v(x)}{u(x)} \to 0 \text{ as } x \to -\infty \Rightarrow \int_{-\infty}^{0} \frac{dt}{r(t)\rho(t)} = \infty. \]

Lemma 4.2. For a PFSS of equation (2.1), we have the equalities
\[ u(x) = v(x) \int_{x}^{\infty} \frac{dt}{r(t)v^2(t)}, \quad v(x) = u(x) \int_{-\infty}^{x} \frac{dt}{r(t)u^2(t)}, \quad x \in \mathbb{R}. \quad (4.2) \]

Proof. Both inequalities in (4.2) are checked in the same way; using (2.7) and (4.1); for example,
\[ \int_{x}^{\infty} \frac{dt}{r(t)v^2(t)} = \int_{x}^{\infty} \frac{1}{r(t)\rho(t)} \exp \left( - \int_{x_0}^{t} \frac{d\xi}{\rho(\xi)\rho(\xi)} \right) dt = - \exp \left( - \int_{x_0}^{t} \frac{d\xi}{\rho(\xi)\rho(\xi)} \right) \bigg|_{x}^{\infty} \]
\[ = \exp \left( - \int_{x_0}^{x} \frac{d\xi}{\rho(\xi)\rho(\xi)} \right) = \frac{u(x)}{v(x)}, \quad x \in \mathbb{R}. \]

Lemma 4.3. For a PFSS of equation (2.1), we have the relations
\[ \int_{-\infty}^{0} \frac{dt}{r(t)u^2(t)} < \infty, \quad \int_{0}^{\infty} \frac{dt}{r(t)v^2(t)} < \infty; \quad (4.3) \]
\[ \int_{-\infty}^{0} \frac{dt}{r(t)u^2(t)} = \int_{0}^{\infty} \frac{dt}{r(t)u^2(t)} = \infty. \quad (4.4) \]

Proof. Both inequalities in (4.3) are checked in the same way. For example, the second inequality in (4.3) follows from (4.2) for \( x = 0 \). The equalities (4.4) are also deduced in the same way from (2.7) and (4.1); for example,
\[ \int_{x_0}^{\infty} \frac{dt}{r(t)u^2(x)} = \int_{x_0}^{\infty} \frac{1}{r(t)\rho(t)} \left( \int_{x_0}^{t} \frac{d\xi}{r(\xi)\rho(\xi)} \right) dt \geq \int_{x_0}^{\infty} \frac{dt}{r(t)\rho(t)} = \infty \Rightarrow (4.4). \]

Let us now continue with the proof of the theorem. Let \( \{u(x), v(x)\}, x \in \mathbb{R} \) be a PFSS of equation (2.1), and let \( \{u_1(x), v_1(x)\}, x \in \mathbb{R} \), be another PFSS of the same equation. Then there are constants \( \alpha \) in \( \mathbb{R} \) and \( \beta \) in \( \mathbb{R} \) such that \( 0 < |\alpha| + |\beta| < \infty \) and we have the equality
\[ u_1(x) = \alpha u(x) + \beta v(x), \quad x \in \mathbb{R}. \quad (4.5) \]

The following relations are deduced from (2.2), (2.4) and (4.5):
\[ \lim_{x \to -\infty} \frac{u_1(x)}{u(x)} = \alpha + \beta \lim_{x \to -\infty} \frac{v(x)}{u(x)} = \alpha \Rightarrow \alpha \geq 0, \quad (4.6) \]
\[
limit_{x \to \infty} \frac{u_1(x)}{v(x)} = \alpha \limit_{x \to \infty} \frac{u(x)}{v(x)} + \beta = \beta \Rightarrow \beta \geq 0. \quad (4.7)
\]

Assume that \(\beta > 0\). Denote (see (2.2))
\[
g(x) = \frac{u_1(x)}{v(x)}, \quad x \in \mathbb{R}; \quad m = \inf_{x \geq 0} g(x) \Rightarrow m \geq 0. \quad (4.8)
\]

By (2.2) and (4.8), we obtain the inequality
\[
g'(x) = \frac{u_1'(x)}{v(x)} - \frac{v'(x)}{v^2(x)} u_1(x) \leq 0, \quad x \in \mathbb{R} \Rightarrow m \in [0, \infty). \quad (4.9)
\]

Let us show that \(m = 0\). Otherwise, if \(m > 0\), we obtain (see (4.8) and (4.4)):
\[
1 \geq \frac{m}{v(x)} \geq \frac{m}{u_1(x)} \geq 0, \quad x \in \mathbb{R} \Rightarrow \int_{0}^{\infty} \frac{dt}{r(t)u_1^2(t)} = \infty,
\]

a contradiction. Hence \(m = 0\) and therefore we have (see (4.9)):
\[
\beta = \lim_{x \to \infty} \frac{u_1(x)}{v(x)} - \alpha \lim_{x \to \infty} \frac{u(x)}{v(x)} = 0 \Rightarrow \alpha > 0.
\]

From Lemma 4.3 it now follows that
\[
v_1(x) = u_1(x) \int_{-\infty}^{x} \frac{dt}{r(t)u_1^2(t)} = \alpha u(x) \int_{-\infty}^{x} \frac{dt}{r(t)\alpha^2 u^2(t)} = \frac{v(x)}{\alpha}.
\]

\[\square\]

**Proof of Corollary 3.3.** The assertion immediately follows from (3.3). \[\square\]

**Proof of Theorem 3.5.** Clearly, one can form a FSS of equation (3.2) from, say, the following functions:
\[
z_1(x) \equiv 1, \quad z_2(x) = \int_{0}^{x} \frac{dt}{r(t)}, \quad x \in \mathbb{R}. \quad (4.10)
\]

Assume to the contrary that under condition (3.5), equation (3.2) has a PFSS \(\{u(x), v(x)\}\), \(x \in \mathbb{R}\). Then there are constants \(\alpha \in \mathbb{R}\) and \(\beta \in \mathbb{R}\) \((0 < |\alpha| + |\beta| < \infty)\) such that
\[
u(x) = \alpha z_1(x) + \beta z_2(x), \quad x \in \mathbb{R}. \quad (4.11)
\]

Cases 1) \(\beta \neq 0\) and 2) \(\beta = 0\) will be considered separately. In the case 1), from (2.2) it follows that
\[
0 \geq u'(x) = \frac{\beta}{r(x)}, \quad x \in \mathbb{R} \Rightarrow \beta < 0. \quad (4.12)
\]

Then, according to (4.1) and (4.12), we get (see (3.5)):
\[
u(x) = \alpha + \beta \int_{0}^{x} \frac{dt}{r(t)} < 0, \quad \text{for} \quad x \gg 1,
\]
which contradicts (2.2), i.e., case 1) does not occur. Let us now consider case 2). Since \( \{u(x), v(x)\}, x \in \mathbb{R}, \) is a PFSS of equation (3.2), then \( \beta = 0 \) from (4.11) it follows that \( \alpha > 0 \) (see (2.2)), and therefore, by (4.2) and (3.5), we have
\[
\int_{-\infty}^{\infty} dt = \int_{-\infty}^{\infty} \frac{dt}{r(t)} = \infty.
\]

The obtained contradiction shows that case 2) also does not occur. \( \square \)

**Proof of Theorem 3.6.** Assume to the contrary that there exists a solution \( z(\cdot) \in L_p \) of (2.1) such that \( z(x) \not\equiv 0 \) for \( x \in \mathbb{R} \). Then, since \( \{u(x), v(x)\}, x \in \mathbb{R} \) is a PFSS of (2.1), there are constants \( \alpha \in \mathbb{R} \) and \( \beta \in \mathbb{R} \) \((0 < |\alpha| + |\beta| < \infty)\) such that
\[
z(x) = \alpha u(x) + \beta v(x), \quad x \in \mathbb{R}.
\]

Assume that \( \beta \neq 0 \). Then there is \( x_1 \gg 1 \) such that the following inequality holds (see (2.4)):
\[
|\frac{\alpha}{\beta}| \frac{u(x)}{v(x)} \leq \frac{1}{2} \quad \text{for} \quad x \geq x_1;
\]

and therefore, according to (2.2), we have
\[
\infty > \|z\|_{L_p} = \int_{-\infty}^{\infty} |z(x)|^p dx \geq \int_{x_1}^{\infty} |\alpha u(x) + \beta v(x)|^p dx
\]
\[
\geq |\beta|^p v(x_1)^p \int_{x_1}^{\infty} \left( 1 - \left| \frac{\alpha}{\beta} \frac{u(x)}{v(x)} \right| \right)^p dx \geq |\beta|^p v(x_1)^p \int_{x_1}^{\infty} \frac{dx}{2} = \infty.
\]

The obtained contradiction shows that \( \beta = 0 \) and therefore \( \alpha \neq 0 \) (see (4.13)). Once again, we apply (2.2) and get
\[
\infty > \|z\|_{L_p} = |\alpha|^p \int_{-\infty}^{\infty} u(x)^p dx \geq |\alpha|^p \int_{-\infty}^{0} u(x)^p dx \geq |\alpha|^p u(0)^p \int_{-\infty}^{0} 1 dx = \infty,
\]
a contradiction. Hence \( \alpha = 0 \) and therefore \( z(x) \equiv 0, \ x \in \mathbb{R} \), contradicting the original assumption. \( \square \)

**Lemma 4.4.** \([7]\) For the norms of the integral operators \( G_1, G_2 \) and \( G \) (see (2.21), (2.22) and (2.23)), we have the following relations:

\[
G = G_1 + G_2; \quad \tag{4.14}
\]
\[
\frac{|G_1|_{p \rightarrow p} + |G_2|_{p \rightarrow p}}{2} \leq \|G\|_{p \rightarrow p} \leq |G_1|_{p \rightarrow p} + |G_2|_{p \rightarrow p}, \quad p \in [1, \infty). \tag{4.15}
\]

**Proof.** Equality (4.14) is obvious; the proof of (4.15) given in [7] relies only on (2.2) and (2.21) and is therefore omitted. \( \square \)
From (1.2), (2.2), (2.6), (4.1) and (4.16), it follows that

For a given

Proof of Lemma 3.9. □

and Corollary 1.5 in [7] coincide and are therefore omitted.

(2.1) are a priori assumptions, and therefore the proofs of these assertions and Theorem 1.4 and Corollary 3.8, the validity of (1.2) and the existence of a PFSS of equation (2.1). In Theorem 3.7 and Corollary 3.8, the required to guarantee the existence of a PFSS of equation (2.1) (according to Theorem 2.1.2) and all remaining arguments in the proof of Theorem 1.4 and Corollary 1.5 in [7] only rely on condition (1.2) and properties (2.2), (2.3) and (2.4) of the PFSS equation (2.1). In Theorem 3.7 and Corollary 3.8, the assertions were obtained in [7] under conditions (1.2) and properties (2.2), (2.3) and (2.4) of the PFSS equation (2.1). In Theorem 3.7 and Corollary 3.8, the validity of (1.2) and the existence of a PFSS of equation (2.1) are a priori assumptions, and therefore the proofs of these assertions and Theorem 1.4 and Corollary 1.5 in [7] coincide and are therefore omitted.

Proof of Lemma 3.9. For a given $x \in \mathbb{R}$, consider the following function in $s \geq 0$

$$F(s) = \int_{x-s}^{x+s} \frac{dt}{r(t)\rho(t)}, \quad s \in [0, \infty].$$

(4.16)

From (1.2), (2.2), (2.6), (4.1) and (4.16), it follows that

$$F(0) = 0, \quad F(s) > 0 \quad \text{for} \quad s > 0; \quad F(\infty) = \infty.$$  

In addition, $F(s)$ is monotone increasing and continuous for all $s \in [0, \infty)$. Hence, equation (3.7) has a unique solution $s(x)$, and it is clear that the function $s(x)$, $x \in \mathbb{R}$ is well-defined. Let us check (3.8). Let $t \in [0, s(x)]$, $x \in \mathbb{R}$ (the case $t \in [-s(x), 0]$, $x \in \mathbb{R}$ is treated in a similar way).

We have the following obvious equations:

$$(x + t) - (t + s(x)) \leq x - s(x) \leq x + s(x) \leq (x + t) + (t + s(x)),$$

$$x - s(x) \leq x + t - (s(x) - t) \leq x + t + (s(x) - t) \leq x + s(x).$$

Together with the definition of $s(x)$, $x \in \mathbb{R}$, this implies

$$1 = \int_{x-s(x)}^{x+s(x)} \frac{d\xi}{r(\xi)\rho(\xi)} \leq \int_{(x+t)-(t+s(x))}^{(x+t)+(t+s(x))} \Rightarrow s(x + t) \leq s(x) + t = s(x) + |t|,$$

$$1 = \int_{x-s(x)}^{x+s(x)} \frac{d\xi}{r(\xi)\rho(\xi)} \geq \int_{(x+t)-(s(x)-t)}^{(x+t)+(s(x)-t)} \Rightarrow s(x + t) \geq s(x) - t = s(x) - |t|.$$  

The obtained inequalities imply (3.8). Furthermore, since the function $s(x)$, $x \in \mathbb{R}$ satisfies Lipschutz’s condition (3.8), it has a bounded derivative almost everywhere (see [7] Ch.9,§10]).

Setting $s = s(x)$, $x \in \mathbb{R}$ in (3.7) and differentiating for almost all $x \in \mathbb{R}$, we obtain

$$|s'(x)| = \frac{|r(x + s(x))\rho(x + s(x)) - r(x - s(x))\rho(x - s(x))|}{r(x + s(x))\rho(x + s(x)) + r(x - s(x))\rho(x - s(x))} < 1 \Rightarrow (3.9).$$

Let us now consider (3.10). Since both equalities are checked in the same way, we consider the second one. Assume to the contrary that $x - s(x) \not\to \infty$ as $x \to \infty$. Then there exist a
constant $c \in [1, \infty)$ and a sequence \( \{x_n\}_{n=1}^{\infty} \) such that

\[
x_n \geq c \quad \text{for} \quad n \geq 1, \quad x_n \to \infty \quad \text{as} \quad n \to \infty, \quad \text{for} \quad x_n - s(x_n) \leq c.
\]

Then from (3.7) (for $s = s(x_n)$, $n \geq 1$) and (4.1), it follows that

\[
1 = \int_{x_n - s(x_n)}^{x_n + s(x_n)} \frac{d\xi}{r(\xi)\rho(\xi)} \geq \int_{c}^{x_n} \frac{d\xi}{r(\xi)\rho(\xi)} \to \infty \quad \text{as} \quad n \to \infty.
\]

This yields a contradiction, showing (3.10) and hence (3.11). To prove (3.12), define

\[
g(x) = s(x)(1 + |x|)^{-1}, \quad x \in \mathbb{R}.
\]

According to (3.11), there is $x_0 \gg 1$ such that $g(x) \leq 1$ for all $x \geq x_0$. The function is continuous (see (3.8)) and positive on the finite interval $[-x_0, x_0]$. Hence it is bounded by a constant $c \geq 1$, so that $g(x) \leq c$ for $x \in [-x_0, x_0]$, giving (3.12). $\square$

**Proof of Lemma 3.11.** Let $x \in \mathbb{R}$, $\omega(x) = [x - s(x), x + s(x)]$, $\xi \in \omega(x)$, $t \in \omega(x)$. From (2.9), it follows that

\[
-1 = -\int_{\omega(x)} \frac{d\xi}{r(\xi)\rho(\xi)} \leq -\left| \int_{x}^{t} \frac{d\xi}{r(\xi)\rho(\xi)} \right| \leq -\int_{x}^{t} \frac{d\xi}{r(\xi)\rho(\xi)} \leq \ln \frac{\rho(t)}{\rho(x)}
\]

\[
\leq \int_{x}^{t} \frac{d\xi}{r(\xi)\rho(\xi)} \leq \left| \int_{x}^{t} \frac{d\xi}{r(\xi)\rho(\xi)} \right| \leq \int_{\omega(x)} \frac{d\xi}{r(\xi)\rho(\xi)} = 1 \quad \Rightarrow (3.13)
\]

Now, for $t \in \omega(x)$, from (2.7), (3.13) and the definition of $s(x)$, $x \in \mathbb{R}$, it follows that

\[
\frac{v(t)}{v(x)} = \sqrt{\frac{\rho(t)}{\rho(x)}} \exp \left( \frac{1}{2} \int_{x}^{t} \frac{d\xi}{r(\xi)\rho(\xi)} \right) \leq \sqrt{e} \exp \left( \frac{1}{2} \left| \int_{x}^{t} \frac{d\xi}{r(\xi)\rho(\xi)} \right| \right)
\]

\[
\leq \sqrt{e} \exp \left( \frac{1}{2} \int_{\omega(x)} \frac{d\xi}{r(\xi)\rho(\xi)} \right) = e,
\]

\[
\frac{v(t)}{v(x)} = \sqrt{\frac{\rho(t)}{\rho(x)}} \exp \left( \frac{1}{2} \int_{x}^{t} \frac{d\xi}{r(\xi)\rho(\xi)} \right) \geq \frac{1}{\sqrt{e}} \exp \left( -\frac{1}{2} \left| \int_{x}^{t} \frac{d\xi}{r(\xi)\rho(\xi)} \right| \right)
\]

\[
\geq \frac{1}{\sqrt{e}} \exp \left( -\frac{1}{2} \int_{\omega(x)} \frac{d\xi}{r(\xi)\rho(\xi)} \right) = \frac{1}{e}.\]

Inequalities (3.14) for $u(\cdot)$ are checked in a similar way. $\square$
**Proof of Theorem 3.12. Necessity.** Since equation (1.1) is correctly solvable in $L_p$, $p \in (1, \infty)$, we have $\|G\|_{p \rightarrow p} < \infty$ (see Theorem 3.7). Below we successively use (4.15), (2.38), Lemmas 3.9 and 3.11 and (2.6):

$$\|G\|_{p \rightarrow p} \geq \sup_{x \in \mathbb{R}} \left( \int_{-\infty}^{x} v(t)^p \, dt \right)^{1/p} \cdot \left( \int_{x}^{\infty} u(t)^{p'} \, dt \right)^{1/p'}$$

$$\geq \sup_{x \in \mathbb{R}} \left[ v(x)^p \int_{x-s(x)}^{x} \left( \frac{v(t)}{v(x)} \right)^p \, dt \right]^{1/p} \cdot \left[ u(x)^{p'} \int_{x}^{x+s(x)} \left( \frac{u(t)}{u(x)} \right)^{p'} \, dt \right]^{1/p}$$

$$\geq e^{-2} \sup_{x \in \mathbb{R}} \left( \rho(x) \cdot s(x)^{\frac{1}{p} + \frac{1}{p'}} \right) = e^{-2} D \Rightarrow D < \infty.$$

**Proof of Theorem 3.12. Sufficiency.** We need the following lemmas.

**Lemma 4.5.** For any $x \in \mathbb{R}$, the semi-axes $(-\infty, x]$ and $[x, \infty)$ admit $\mathbb{R}(x, s(\cdot))$-coverings (see Definition 2.5.2, Lemma 3.9 and Lemma 2.5.2).

**Lemma 4.6.** Let $x \in \mathbb{R}$, and let $\{\Delta_n\}_{n=-\infty}^{-1}$ and $\{\Delta_n\}_{n=1}^{\infty}$ be $\mathbb{R}(x, s(\cdot))$-coverings of the semi-axes $(-\infty, x]$ and $[x, \infty)$, respectively. Then we have the equalities

$$\int_{\Delta_n^{(-)}} \frac{d\xi}{r(\xi)\rho(\xi)} = |n| - 1 \quad \text{for} \quad n \leq -1 \quad \int_{\Delta_n^{(-)}} \frac{d\xi}{r(\xi)\rho(\xi)} = n - 1 \quad \text{for} \quad n \geq 1. \quad (4.17)$$

**Remark 4.7.** Lemma 4.5 immediately follows from Lemma 2.5.2, (3.8) and (3.10) (see [6]).

The proof of Lemma 4.6 is similar to the proof of equalities (3.32) in [7] because the function $s(\cdot)$ is a ‘twin sister’ of the function $d(\cdot)$ (see Lemma 2.3.5, (2.30) and Lemma 3.9).

The proof of the sufficiency of the conditions of Theorem 3.12 is similar to the proof of Theorem 1.8 in [7] because the latter proof only relies on the properties of the PFSS of equation (2.1), Lemmas 4.4, 4.5, 4.6, Theorem 2.4.2 and consists of checking the upper estimates in (3.16).

**Proof of Corollary 3.13.** Below we show that each of the relations (3.17), (3.18), (3.19), (3.20), (3.21) implies the inequality $D < \infty$ (see (3.15)), which guarantees the correct solvability of equation (1.1) in $L_p$, $p \in (1, \infty)$. So we proceed case by case.

1) Let $\sigma_1 < \infty$. Below, for $x \in \mathbb{R}$, we use the definition of the function $s(\cdot)$ (see (3.7) and (3.13)):

$$1 = \int_{x-s(x)}^{x+s(x)} \frac{dt}{r(t)\rho(t)} = \rho(x) \int_{x-s(x)}^{x+s(x)} \left( \frac{\rho(t)}{\rho(x)} \right) \frac{dt}{r(t)\rho^2(t)} \geq 2 \frac{1}{e\sigma_1} \rho(x)s(x) \Rightarrow D < \infty.$$
2) Let \( \sigma_2 < \infty \). By (3.11), there is \( x_0 \gg 1 \) such that the following inequalities hold:

\[
\rho(x)s(x) \leq |\rho(x)|x \leq 2\sigma_2 \quad \text{for} \quad |x| \geq x_0. \tag{4.18}
\]

Furthermore, the function \((\rho(x)s(x))\) is non-negative and continuous on the finite interval \([-x_0, x_0]\) (see (2.6) and (3.8)) and is therefore bounded there. Together with (4.18), this gives the inequality \( \mathcal{D} < \infty \).

3) Let \( x \in \mathbb{R} \). From (2.2), (2.3) and (2.21), it follows that

\[
\int_{-\infty}^{\infty} q(t)G(x, t)dt = u(x) \int_{-\infty}^{x} q(t)v(t)dt + v(x) \int_{x}^{\infty} q(t)u(t)dt \tag{4.19}
\]

\[
= u(x) \int_{-\infty}^{x} (r(t)v'(t))'dt + v(x) \int_{x}^{\infty} (r(x)u'(t))'dt
\]

\[
= u(x) \left( r(t)v'(t) \bigg|_{-\infty}^{x} \right) + v(x) \left( r(x)u'(t) \bigg|_{x}^{\infty} \right) \leq r(x)(v'(x)u(x) - u'(x)v(x)) = 1.
\]

Below, for \( x \in \mathbb{R} \), we use relations (4.19), (3.14) and the definition of the function \( s(x) \), \( x \in \mathbb{R} \):

\[
1 \geq \int_{-\infty}^{\infty} q(t)G(x, t)dt \geq \int_{x-s(x)}^{x+s(x)} q(t)G(x, t)dt
\]

\[
= u(x)v(x) \int_{x-s(x)}^{x} q(t) (\frac{v(t)}{v(x)}) dt + u(x)v(x) \int_{x}^{x+s(x)} q(t) (\frac{u(t)}{u(x)}) dt
\]

\[
\geq \frac{\rho(x)}{e} \int_{x-s(x)}^{x+s(x)} q(t)dt = \frac{1}{e} (\rho(x) \cdot s(x)) \left[ \frac{1}{2s(x)} \int_{x-s(x)}^{x+s(x)} q(t)dt \right]
\]

\[
\geq \frac{2\sigma_3}{e} (\rho(x)s(x)) \Rightarrow \mathcal{D} < \infty.
\]

4) The assertion follows from criterion (3.19):

\[
\sigma_3 = \inf_{x \in \mathbb{R}} \left[ \frac{1}{2s(x)} \int_{x-s(x)}^{x+s(x)} q(t)dt \right] \geq \sigma_4 \inf_{x \in \mathbb{R}} \left[ \frac{1}{2s(x)} \int_{x-s(x)}^{x+s(x)} 1dt \right] = \sigma_4 > 0 \Rightarrow \mathcal{D} < \infty.
\]

5) From (3.21), it follows that \( \frac{1}{r} \in L_1 \).

Furthermore, since (2.1) has a PFSS \( \{u(x), v(x)\}, x \in \mathbb{R} \), from (2.2) and (4.2) we obtain the inequalities:

\[
v(x) = u(x) \int_{-\infty}^{x} \frac{dt}{r(t)u^2(t)} \leq \frac{1}{u(x)} \int_{-\infty}^{x} \frac{dt}{r(t)}, \quad x \in \mathbb{R};
\]

\[
u(x) = v(x) \int_{x}^{\infty} \frac{dt}{r(t)v^2(t)} \leq \frac{1}{v(x)} \int_{x}^{\infty} \frac{dt}{r(t)}, \quad x \in \mathbb{R}.
\]
These inequalities imply the estimates:

\[ \rho(x) = u(x) \int_{-\infty}^{x} \frac{dt}{r(t)} \leq \left( \int_{0}^{\infty} \frac{dt}{r(t)} \right)^{-1} \cdot \int_{-\infty}^{x} \frac{dt}{r(t)} \cdot \int_{x}^{\infty} \frac{dt}{r(t)} \text{ for } x \leq 0, \]

\[ \rho(x) = u(x)v(x) \leq \int_{x}^{\infty} \frac{dt}{r(t)} \leq \left( \int_{-\infty}^{0} \frac{dt}{r(t)} \right)^{-1} \cdot \int_{-\infty}^{x} \frac{dt}{r(t)} \cdot \int_{x}^{\infty} \frac{dt}{r(t)} \text{ for } x \geq 0. \]

Hence,

\[ \rho(x) \leq \tau \int_{-\infty}^{x} \frac{dt}{r(t)} \cdot \int_{x}^{\infty} \frac{dt}{r(t)}, \quad x \in \mathbb{R}, \tag{4.20} \]

where

\[ \tau = \max \left\{ \left( \int_{-\infty}^{0} \frac{dt}{r(t)} \right)^{-1}, \left( \int_{0}^{\infty} \frac{dt}{r(t)} \right)^{-1} \right\}. \]

The assertion now follows from (4.20) and criterion (3.17). □

**Proof of Corollary 3.14.** By 2) (see (4.22)), for any \( n \geq 1 \), there is \( x_0(u) \gg 1 \) such that \( r(x)\rho(x) \geq n \) for \( |x| \geq x_0(n) \). Furthermore, according to (3.10), there is \( x_1(n) \gg 1 \) such that

\[ [x - s(x), x + s(x)] \cap [-x_0(n), x_0(n)] = \emptyset \quad \text{for} \quad |x| \geq x_1(n). \]

Set \( x_2(n) = \max\{x_0(n), x_1(n)\} \). Then for \( |x| \geq x_2(n) \), from 2) and Lemma 3.9 it follows that

\[ 1 = \int_{x-s(x)}^{x+s(x)} \frac{dt}{r(t)\rho(t)} \leq \frac{2}{n} s(x) \Rightarrow s(x) \geq \frac{n}{2} \text{ for } |x| \geq x_2(n), \]

i.e., \( s(x) \to \infty \) as \( |x| \to \infty \). Besides, from Lemmas 3.9 and 3.11 we obtain

\[ 1 = \int_{x-s(x)}^{x+s(x)} \frac{dt}{r(t)\rho(t)} \geq \frac{1}{c} \frac{1}{\rho(x)} \int_{x-s(x)}^{x+s(x)} \frac{dt}{r(t)} \geq \frac{2}{c r_0} \frac{s(x)}{\rho(x)} \tag{see 1 in (4.22)} \Rightarrow \]

\[ \rho(x) \geq c^{-1}s(x) \quad \text{for} \quad |x| \geq x_2(n), \quad c \in [1, \infty) \Rightarrow \]

\[ \rho(x)s(x) \geq c^{-1}s^2(x) \to \infty \quad \text{as} \quad |x| \to \infty \Rightarrow D = \infty. \]

It remains to refer to Theorem 3.12. □

**Proof of Theorem 3.17.** Below, we assume that all conditions in Theorem 3.17 are satisfied and do not quote them. To prove the theorem, we establish the inequalities

\[ c^{-1}D_1 \leq D \leq c(p)D_1, \quad p \in (1, \infty); \quad c, c(p) \in [1, \infty). \tag{4.21} \]

Here

\[ D = \sup_{x \in \mathbb{R}} (\rho(x)s(x)), \quad D_1 = \sup_{x \in \mathbb{R}} (\rho_1(x)s_1(x)), \tag{4.22} \]

\( \rho(\cdot), \rho_1(\cdot) \) are the functions generating the PFSS of equations (2.11) and (2.12), respectively (under conditions (1.2) and (3.25)); \( s(x) \) and \( s_1(x) \) are the solutions in \( s \geq 0 \) (for a fixed
\( x \in \mathbb{R} \) of the equations

\[
\int_{x-s}^{x+s} \frac{dt}{r(t)\rho(t)} = 1, \quad \int_{x-s}^{x+s} \frac{dt}{r(t)\rho_1(t)} = 1,
\]

respectively (see Lemma 3.9).

Clearly, Theorem 3.17 follows from (4.21) and Theorem 3.12. So let us go to (4.21). We need the inequalities (see (2.23)):

\[
\|G_2\|_{p \to p} \leq c(p) \sup_{x \in \mathbb{R}} \left[ \int_{-\infty}^{x} \rho(t) \exp \left( -(p - 1) \int_{t}^{x} \frac{d\xi}{r(\xi)\rho(\xi)} \right) dt \right]^{1/p} \]

\[
\cdot \left[ \int_{x}^{\infty} \rho(t) \exp \left( -\int_{x}^{t} \frac{d\xi}{r(\xi)\rho(\xi)} \right) dt \right]^{1/p'} \quad \text{for} \quad p \in (1, 2],
\]

\[
\|G_2\|_{p \to p} \leq c(p) \sup_{x \in \mathbb{R}} \left[ \int_{-\infty}^{x} \rho(t) \exp \left( -\int_{t}^{x} \frac{d\xi}{r(\xi)\rho(\xi)} \right) dt \right]^{1/p} \]

\[
\cdot \left[ \int_{x}^{\infty} \rho(t) \exp \left( -(p' - 1) \int_{x}^{t} \frac{d\xi}{r(\xi)\rho(\xi)} \right) dt \right]^{1/p'} \quad \text{for} \quad p \in (2, \infty),
\]

\[
\|G_2\|_{p \to p} \geq \sup_{x \in \mathbb{R}} \left[ \int_{-\infty}^{x} v(t)^p dt \right]^{1/p} \cdot \left[ \int_{x}^{\infty} u(t)^{p'} dt \right]^{1/p'} \quad \text{for} \quad p \in (1, \infty)
\]

(see relations (3.30), (3.34) and (3.35) in [7]). We can use these inequalities in our proof because they were obtained in [7] from the properties of the PFSS of equation (2.1) (see Definition 2.1.1 and Theorems 2.4.2 and 2.1.3).

To prove the upper estimate in (4.24), denote by \( \{\Delta\}_{n=-\infty}^{1} \) and \( \{\Delta_n\}_{n=1}^{\infty} \) the segments of \( \mathbb{R}(x, s(x))-\text{coverings of the semi-axes } (-\infty, x] \) and \([x, \infty)\), respectively (see Definition 2.5.1 Lemma 2.5.2 and (3.10)), constructed for the function \( \rho_1(\cdot) \); by the letter “a” \((a \in [1, \infty))\), for the sake of clarity, we denote below the constant for which, by the conditions of the theorem, the following inequalities hold:

\[
a^{-1} \rho(\xi) \leq \rho_1(\xi) \leq ap(\xi) \quad \xi \in \mathbb{R}.
\]
Below, for \( p \in (1, 2) \), we successively use relations (3.15), (3.16), (3.13), (4.24) and (4.17):

\[
c^{-1} \mathcal{D} \leq \| G_2 \|_{p \to p} \leq c(p) \sup_{x \in \mathbb{R}} \left[ \int_{-\infty}^{x} \rho(t) \exp \left( - \left( p - 1 \right) \int_{x}^{t} \frac{d \xi}{r(\xi) \rho(\xi)} \right) dt \right]^{1/p} \\
\cdot \left[ \int_{x}^{\infty} \rho(t) \exp \left( - \int_{x}^{t} \frac{d \xi}{r(\xi) \rho(\xi)} \right) dt \right]^{1/p'} \\
\leq c(p) \sup_{x \in \mathbb{R}} \left[ a \int_{-\infty}^{x} \rho_1(t) \exp \left( - \frac{p - 1}{a} \int_{x}^{t} \frac{d \xi}{r(\xi) \rho_1(\xi)} \right) dt \right]^{1/p} \\
\cdot \left[ a \int_{x}^{\infty} \rho_1(t) \exp \left( - \frac{1}{a} \int_{x}^{t} \frac{d \xi}{r(\xi) \rho_1(\xi)} \right) dt \right]^{1/p'} \\
\leq ac(p) \sup_{x \in \mathbb{R}} \left[ \sum_{n=-\infty}^{-1} \int_{\Delta_n} \rho_1(t) \exp \left( - \frac{p - 1}{a} \int_{\Delta_n}^{t} \frac{d \xi}{r(\xi) \rho_1(\xi)} \right) dt \right]^{1/p} \\
\cdot \left[ \sum_{n=1}^{\infty} \int_{\Delta_n} \rho_1(t) \exp \left( - \frac{1}{a} \int_{\Delta_n}^{t} \frac{d \xi}{r(\xi) \rho_1(\xi)} \right) dt \right]^{1/p'} \\
\leq ac(p) \sup_{x \in \mathbb{R}} \left[ \sum_{n=-\infty}^{-1} 2 e \rho_1(x_n) s_1(x_n) \exp \left( - \frac{p - 1}{a} \int_{\Delta_n}^{t} \frac{d \xi}{r(\xi) \rho_1(\xi)} \right) \right]^{1/p} \\
\cdot \left[ \sum_{n=1}^{\infty} 2 e \rho_1(x_n) s_1(x_n) \exp \left( - \frac{1}{a} \int_{\Delta_n}^{t} \frac{d \xi}{r(\xi) \rho_1(\xi)} \right) \right]^{1/p'} \\
\leq 2ac(p) \left[ \mathcal{D}_1 \sum_{n=1}^{\infty} \exp \left( - \frac{p - 1}{a} (n - 1) \right) \right]^{1/p} \cdot \left[ \mathcal{D}_1 \sum_{n=1}^{\infty} \exp \left( - \frac{n - 1}{2} \right) \right]^{1/p'} = c(p) \mathcal{D}_1,
\]

i.e., \( \mathcal{D} \leq c(p) \mathcal{D}_1 \), as required. The upper estimate in (4.21) for \( p \in (2, \infty) \) is proved in a similar way, with the help of (4.25) instead of (4.24).
Let us go to the lower estimate in (4.21). Below, for \( p \in (1, \infty) \), we successively use (3.16), Theorem 2.4.2, (4.26) and Lemmas 3.9 and 3.11:

\[
\begin{align*}
&c(p)D \geq \|G_2\|_{p-p} \geq \sup_{x \in \mathbb{R}} \left[ \int_{-\infty}^{x} v(t)p \, dt \right]^{1/p} \cdot \left[ \int_{x}^{\infty} u(t)p^{-1} \, dt \right]^{1/p'} \\
&= \sup_{x \in \mathbb{R}} \left[ \int_{-\infty}^{x} \rho(t)^{p/2} \exp \left( \frac{p}{2} \int_{x}^{t} \frac{d\xi}{r(\xi)\rho(\xi)} \right) \, dt \right]^{1/p} \cdot \left[ \int_{x}^{\infty} \rho(t)^{-p/2} \exp \left( -\frac{p'}{2} \int_{x}^{t} \frac{d\xi}{r(\xi)\rho(\xi)} \right) \, dt \right]^{1/p'} \\
&= \sup_{x \in \mathbb{R}} \left[ \int_{-\infty}^{x} \rho(t)^{p/2} \exp \left( -\frac{p}{2} \int_{x}^{t} \frac{d\xi}{r'(\xi)\rho(\xi)} \right) \, dt \right]^{1/p} \cdot \left[ \int_{x}^{\infty} \rho(t)^{-p/2} \exp \left( -\frac{p'}{2} \int_{x}^{t} \frac{d\xi}{r(\xi)\rho(\xi)} \right) \, dt \right]^{1/p'} \\
&\geq \sup_{x \in \mathbb{R}} \left[ a^{-p/2}e^{-p/2} \rho_{1}(x)^{p/2} \exp \left( -\frac{ap}{2} \int_{x-s_{1}(x)}^{x} \frac{d\xi}{r(\xi)\rho_{1}(\xi)} \right) s_{1}(x) \right]^{1/p} \\
&\quad \cdot \left[ a^{-p'/2}e^{-p'/2} \rho_{1}(x)^{-p'/2} \exp \left( -\frac{ap'}{2} \int_{x}^{x+s_{1}(x)} \frac{d\xi}{r(\xi)\rho_{1}(\xi)} \right) s_{1}(x) \right]^{1/p'} \\
&= (ae)^{-1} \exp \left( -\frac{a}{2} \sup_{x \in \mathbb{R}} (\rho_{1}(x)s_{1}(x)) \right) = c^{-1}D_{1} \Rightarrow (4.21).
\end{align*}
\]

\[\square\]

**Proof of Theorem 3.18**. The relations \( \rho(x) \asymp \rho_{1}(x) \) for \( x \in (-\infty, 0] \) and \( [0, \infty) \) are proved in the same way. They imply the assertion of the theorem. Therefore, below we only consider the case \( [x, \infty) \). Since for equations (2.1) and (2.11) the Hartman-Wintner problem for \( x \to \infty \) is solvable, there exists a FSS \( \{ \hat{u}(x), \hat{v}(x) \} \), \( x \in \mathbb{R} \), of equation (2.1) for which equalities (2.12) - (2.14) hold as \( x \to \infty \). In addition, since \( \{ u(x), v(x) \} \), \( x \in \mathbb{R} \), is a PFSS of equation (2.1), there are constants \( \alpha \) and \( \beta \) (\( 0 < |\alpha| + |\beta| < \infty \)) such that

\[
\hat{u}(x) = \alpha u(x) + \beta v(x), \quad x \in \mathbb{R}.
\]

Assume that in (4.28) we have \( \beta \neq 0 \). By (2.12), there is \( x_{1} \gg 1 \) such that

\[
2^{-1}u_{1}(x) \leq \hat{u}(x) \leq 2u_{1}(x) \quad \text{for} \quad x \geq x_{1}.
\]

Then from (4.4), (4.28) and (4.29), it follows that

\[
\int_{x_{1}}^{\infty} \frac{dt}{r(t)\hat{u}(x)^{2}} = \int_{x_{1}}^{\infty} \frac{1}{r(t)u_{1}^{2}(t)} \left( \frac{u_{1}(t)}{\hat{u}(t)} \right)^{2} \, dt \geq \frac{1}{4} \int_{x_{1}}^{\infty} \frac{dt}{r(t)u_{1}^{2}(t)} = \infty.
\]
Furthermore, according to (2.1), by enlarging $x_1$ if needed, we get
$$\left| \frac{\alpha}{\beta} \right| \frac{u(x)}{v(x)} \leq \frac{1}{2} \quad \text{for} \quad x \geq x_1. \tag{4.31}$$

Now, from (4.28), (4.30), (4.31) and (4.4), we obtain
$$\infty = \int_{x_1}^{\infty} \frac{dt}{r(t)u^2(t)} = \int_{x_1}^{\infty} \frac{dt}{r(t)(\alpha u(t) + \beta v(t))^2} \leq \int_{x_1}^{\infty} \frac{dt}{r(t)v^2(t)} \left(1 - \left| \frac{\alpha}{\beta} \right| \frac{u(t)}{v(t)} \right)^2$$
$$\leq \frac{4}{\beta^2} \int_{x_1}^{\infty} \frac{dt}{r(t)v^2(t)} < \infty.$$  

We get a contradiction. Hence, $\beta = 0$ and therefore $\alpha > 0$ (see (4.28), (4.29), (2.12), and (2.2)). Therefore, from (4.28) (recall that $\beta = 0$) and (2.12), we obtain
$$1 = \lim_{x \to \infty} \frac{\dot{u}(x)}{u_1(x)} = \alpha \lim_{x \to \infty} \frac{u(x)}{u_1(x)} \Rightarrow \lim_{x \to \infty} \frac{u(x)}{u_1(x)} = \frac{1}{\alpha}, \quad \alpha \in (0, \infty). \tag{4.32}$$

Note that since $\{u(x), v(x)\}, \ x \in \mathbb{R}, \text{ and } \{u_1(x), v_1(x)\}, \ x \in \mathbb{R}, \text{ are PFSS of equations (2.1) and (2.11), respectively, we obtain from Definition 2.1.1,}$
$$\left( \frac{v(x)}{u(x)} \right)' = \frac{1}{r(x)u^2(x)}, \quad x \in \mathbb{R}; \quad \left( \frac{v_1(x)}{u_1(x)} \right)' = \frac{1}{r(x)u_1^2(x)}, \quad x \in \mathbb{R}. \tag{4.33}$$

This implies that
$$v(x) = \frac{v(o)}{u(o)} u(x) + u(x) \int_0^x \frac{dt}{r(t)u^2(t)}, \quad x \geq 0, \tag{4.34}$$
$$v_1(x) = \frac{v_1(o)}{u_1(o)} u_1(x) + u_1(x) \int_0^x \frac{dt}{r(t)u_1^2(t)}, \quad x \geq 0. \tag{4.35}$$

Below we use (4.32), (4.33), (4.35) and L’Hôpital’s rule (see (4.1)):
$$\lim_{x \to \infty} \frac{v(x)}{v_1(x)} = \lim_{x \to \infty} \frac{u(x)}{u_1(x)} \frac{\frac{v(o)}{u(o)}}{\frac{v_1(o)}{u_1(o)}} + \int_0^x \frac{dt}{r(t)u^2(t)} = \alpha \lim_{x \to \infty} \frac{r(x)u_1^2(x)}{r(x)u^2(x)} = \alpha. \tag{4.36}$$

By (4.32) and (4.36), we thus get
$$\lim_{x \to \infty} \frac{\rho(x)}{\rho_1(x)} = \lim_{x \to \infty} \frac{u(x)}{u_1(x)} \cdot \frac{v(x)}{v_1(x)} = 1. \tag{4.37}$$

In particular, according to (4.37), there is $x_2 \gg 1$ such that
$$2^{-1} \rho_1(x) \leq \rho(x) \leq 2 \rho(x) \quad \text{for} \quad x \geq x_1. \tag{4.38}$$

Since the function
$$f(x) = \frac{\rho(x)}{\rho_1(x)}, \quad \text{for} \quad x \in [0, x_2]$$
is continuous and positive for $x \in [0, x_2]$, there is $c \geq 2$ such that
$$c^{-1} \leq f(x) \leq c \quad \text{for} \quad x \in [0, x_1], \tag{4.39}$$
and by (4.38) and (4.39), we obtain the relation $\rho(x) \asymp \rho_1(x), \ x \in [0, \infty), \text{ as required} \quad \Box$
Proof of Theorem 3.19. First note that equation (2.1) has a PFSS \( \{u(x), v(x)\}, x \in \mathbb{R} \) due to Theorem 2.1.2. Furthermore, since \( \frac{1}{r} \in L_1 \), equation (3.2) has a PFSS \( \{u_1(x), v_1(x)\}, x \in \mathbb{R} \), of the form (3.1), and we have the inequalities (see (3.1)):

\[
\rho_1(x) = u_1(x)v_1(x) \leq \frac{1}{w_0} \int_{-\infty}^{x} dt \int_{x}^{\infty} \frac{dr(t)}{r(t)}, \quad x \leq 0
\]

\[
\leq \int_{-\infty}^{x} \frac{dr(t)}{r(t)}, \quad x \geq 0.
\]

(4.40)

Then, from (3.26), (4.40) and Theorem 2.2.2, it follows that the Hartman-Wintner problems for equations (2.1) and (3.2), for \( x \to -\infty \) and \( x \to \infty \), are solvable, and therefore the functions \( \rho(x) \) and \( \rho_1(x), x \in \mathbb{R} \), generating the PFSS of equations (2.1) and (3.2), are weakly equivalent. It remains to refer to Theorem 3.17.

5. Example

First recall that our goal is the study of problem I)-II), \( p \in (1, \infty) \) within the framework of (1.4). Therefore, although some of our statements are relevant also in the case (1.3) along with (1.4) (see, e.g., Theorem 3.12), we restrict our attention to the case (1.4) and do not give any example of the study of the question on I)-II) under the condition (1.3) (the latter case is considered in [3, 5, 7, 9, 4, 8]).

Below we consider the equation

\[
-(r(x)y'(x))' + q(x)y(x) = f(x), \quad x \in \mathbb{R}.
\]

(5.1)

Here and in the sequel, \( f(\cdot) \in L_p, p \in (1, \infty) \) and

\[
 r(x) = (1 + x^2)\alpha, \quad \alpha > \frac{1}{2}; \quad q(x) = \frac{1}{(1 + x^2)^\beta}, \quad \beta > \frac{1}{2}, \quad x \in \mathbb{R}.
\]

(5.2)

The question on I)-II) \( p \in (1, \infty) \) is denoted below as problem (5.1)-(5.2). Our study of (5.1)-(5.2) is based on Theorems 3.12, 3.19 and Corollary 3.13. For the reader’s convenience, we divide the exposition into stages 1), 2) and 3) and comment on each stage separately.

1) Homogeneous equations

Below, we use the fact that by (5.2), each of the equations

\[
(r(x)z'(x))' = q(x)y(x), \quad x \in \mathbb{R},
\]

(5.3)

\[
(r(x)z'(x))' = 0, \quad x \in \mathbb{R}
\]

(5.4)

has a PFSS (see Theorems 2.1.2 and 3.1). Below we do not refer to this fact.

2) Solving problem (5.1)-(5.2) for \( \alpha \geq 1 \)

If \( \frac{1}{r} \in L_1 \), one can apply criterion (3.21) for the correct solvability of problem I)-II), which is convenient because this allows one to avoid the proofs of estimates for the functions \( \rho(x), \)

\( x \in \mathbb{R} \), and \( s(x), x \in \mathbb{R} \) (cf. Theorem 3.12). In our case, the functions in (5.2) are even, and therefore we have the following relations (see (3.21)):

\[
\sigma_5 = \sup_{x \in \mathbb{R}} \left[ r(x) \left( \int_{-\infty}^{x} \frac{dt}{r(t)} \right)^2 \left( \int_{x}^{\infty} \frac{dt}{r(t)} \right)^2 \right] \leq c \sup_{x \geq 0} \left[ r(x) \left( \int_{x}^{\infty} \frac{dt}{r(t)} \right)^2 \right].
\]

Denote (see (5.2)):

\[
F(x) = (1 + x^2)^\alpha \left( \int_{x}^{\infty} \frac{dt}{(1 + t^2)^\alpha} \right)^2, \quad x \geq 0.
\]  (5.5)

L’Hôpital’s rule implies that

\[
\lim_{x \to \infty} \sqrt{F(x)} = \begin{cases} 
1, & \text{if } \alpha = 1 \\
0, & \text{if } \alpha > 1.
\end{cases}
\]  (5.6)

By (5.6), there is \( x_0 \gg 1 \) such that

\[
0 < F(x) \leq 2 \quad \text{for } x \geq x_0.
\]  (5.7)

Since the continuous positive function \( F(x) \) is bounded on \([0, x_0]\), by (5.7) \( F(\cdot) \) is absolutely bounded on the semi-axis \([0, \infty)\). This implies (see (5.5)) that \( \sigma_5 < \infty \), i.e., for \( \alpha \geq 1 \) problem (5.1)-(5.2) is correctly solvable in \( L_p, p \in (1, \infty) \).

3) Problem I)-II), \( p \in (1, \infty) \) for the model equation (3.17)

Thus, in the case \( \alpha \in \left( \frac{1}{2}, 1 \right) \), Corollary 3.13 does not give an answer to the question on I)-II), \( p \in (1, \infty) \). Therefore, to complete our study of (5.1)-(5.2), we have to first study problem I)-II), \( p \in (1, \infty) \), for the model equation (3.27) and then, applying Theorem 3.19, get information on problem (5.1)-(5.2). By Theorem 3.1, the function \( \rho(x), x \in \mathbb{R} \), generating the PFSS of equation (3.1), is of the form

\[
\rho(x) = \frac{1}{w_0} \int_{-\infty}^{x} \frac{dt}{(1 + t^2)^\alpha} \cdot \int_{x}^{\infty} \frac{dt}{(1 + t^2)^\alpha}, \quad x \in \mathbb{R}, \quad w_0 = \int_{-\infty}^{\infty} \frac{dt}{(1 + t^2)^\alpha}.
\]  (5.8)

L’Hôpital’s rule implies that

\[
\lim_{|x| \to \infty} \rho(x)|x|^{2\alpha - 1} = \frac{1}{2\alpha - 1}, \quad \alpha \in \left( \frac{1}{2}, 1 \right)
\]  (5.9)

which gives an asymptotic formula for \(|x| \to \infty\):

\[
r(x)\rho(x) = \frac{1 + \delta(x)}{2\alpha - 1}|x|, \quad \lim_{|x| \to \infty} \delta(x) = 0.
\]  (5.10)
To estimate \( s(x) \) for \( x \gg 1 \) (see Lemma 3.9), we use equation (3.7) and relations (3.10) and (5.10):

\[
1 = \int_{x-s(x)}^{x+s(x)} \frac{dt}{r(t)s(t)} = \int_{x-s(x)}^{x+s(x)} \frac{2\alpha - 1}{1 + \delta(t)} \frac{dt}{t} \leq 2(2\alpha - 1) \int_{x-s(x)}^{x+s(x)} \frac{dt}{t} \Rightarrow \\
\forall (\alpha) := \frac{1}{2(2\alpha - 1)} \leq \ln \frac{x + s(x)}{x - s(x)}, \quad x \gg 1 \Rightarrow s(x) \geq \frac{e^{\forall (\alpha)} - 1}{e^{\forall (\alpha)} + 1} x, \quad x \gg 1. \quad (5.11)
\]

Thus, for the model equation (3.27), problem I)-II), \( p \in (1, \infty) \) for \( \alpha \in \left( \frac{1}{2}, 1 \right) \) is not correctly solvable in light of Theorem 3.12 because (see (3.15), (5.10) and (5.11))

\[
\mathcal{D} = \sup_{x \in \mathbb{R}} (\rho(x)s(x)) \geq \sup_{x \gg 1} (\rho(x)s(x)) \geq \frac{e^{\forall (\alpha)} - 1}{e^{\forall (\alpha)} + 1} \sup_{x \gg 1} \frac{1 + \delta(x)}{2\alpha - 1} \frac{x^2}{(1 + x^2)^{\alpha}} \\
\geq \frac{1}{e} \sup_{x \gg 1} x^{2(1-\alpha)} = \infty.
\]

Together with Theorem 3.19, this implies that problem (5.1)-(5.2), \( p \in (1, \infty) \), for \( \alpha \in \left( \frac{1}{2}, 1 \right) \) is also not correctly solvable.

References

[1] P.S. Areksandrov and A.N. Kolmogorov, An Introduction to the Theory of Functions of a Real Variable, 3rd ed., Moscow-Leningrad, 1938, no. 11 (in Russian).
[2] N. Chernyavskaya and L. Shuster, Necessary and sufficient conditions for the solvability of a problem of Hartman and Wintner, Proc. Amer. Math. Soc. 125 (1997), no. 11, 3113-3228.
[3] N. Chernyavskaya and L. Shuster, Estimates for the Green function of a general Sturm-Liouville operator and their applications, Proc. Amer. Math. Soc. 127 (1999), 1413-1426.
[4] N. Chernyavskaya and L. Shuster, Regularity of the inversion problem for a Sturm-Liouville equation in \( L_p(\mathbb{R}) \), Methods Appl. Anal. 7 (2000), no. 1, 65-84.
[5] N. Chernyavskaya and L. Shuster, A criterion for correct solvability of the Sturm-Liouville equation in the space \( L_p(\mathbb{R}) \), Proc. Amer. Math. Soc 130 (2001), 1043-1054.
[6] N. Chernyavskaya and L. Shuster, Conditions for correct solvability of a simplest singular boundary value problem of general form, I, Zeitschrift für Analysis und ihre Anwendungen 25 (2006), 2005-2035.
[7] N. Chernyavskaya and L. Shuster, A criterion for correct solvability in \( L_p(\mathbb{R}) \) of a general Sturm-Liouville equation, J. London Math. Soc. (2) 80 (2009), 99-120.
[8] N. Chernyavskaya and L. Shuster, Methods of analysis of the condition for correct solvability in \( L_p(\mathbb{R}) \) of general Sturm-Liouville equations, Czechoslovak Math. J. 64(139) (2014), no. 4, 1067-1098.
[9] N. Chernyavskaya and L. Shuster, Criterion for correct solvability of a general Sturm-Liouville equation in the space \( L_1(\mathbb{R}) \). Bull. Unione. Math. Ital. 11 (2018), no. 4, 417-443.
[10] R. Courant, Differential and Integral Calculus, Vol. 1, 2nd edition, Blackie & Son Ltd., 1945.
[11] E.B. Davies and E.M. Harrell, Conformally flat Riemannian metrics, Schrödinger operators and semi-classical approximations. J. Diff Eq. 66 (1987), 165-168.
[12] P. Hartman, Ordinary Differential Equations, John Wiley & Sons, 1964.
[13] L.W. Kantorovich and G.P. Akilov, Functional Analysis, Nauka, Moscow, 1977.
[14] A. Kufner and L.E. Persson, Weighted Inequalities of Hardy Type, World Scientific Publishing Co., 2003.
[15] K.T. Mynbaev and M.O. Otelbaev, Weighted Function Spaces and the Spectrum of Differential Operators, Nauka, Moscow, 1988.
[16] R. Oinarov, Properties of Sturm-Liouville operator in \( L_p \), Izv. Akad. Nauk Kaz. SSR 1 (1990), 43-47.
[17] M. Otelbaev, A criterion for the resolvent of a Sturm-Liouville operator to be a kernel, Math. Notes 25 (1979), 296-297 (translation of Mat. Zametki).
[18] E.T. Whittaker and G.N. Watson, A Course of Modern Analysis, Cambridge, 1958.
