Cavity-induced temperature control of a two-level system

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We consider a two-level atom interacting with a single mode of the electromagnetic field in a cavity within the Jaynes-Cummings model. Initially, the atom is thermal while the cavity is in a coherent state. The atom interacts with the cavity field for a fixed time. After removing the atom from the cavity and applying a laser pulse the atom will be in a thermal state again. Depending on the interaction time with the cavity field the final temperature can be varied over a large range. We discuss how this method can be used to cool the internal degrees of freedom of atoms and create heat baths suitable for studying thermodynamics at the nanoscale.

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The Jaynes-Cummings model (JCM) is a simple but powerful model describing the interaction between a two-level atom and a single mode of the radiation field. While being exactly solvable it offers a large variety of genuinely quantum phenomena like collapses and revivals in the inversion of the atom, which have been observed experimentally as well.

For a field prepared in a coherent state, the state of the atom will be almost pure at half of the revival time if the atom is initially in a pure state. However, a more realistic model would involve a thermal initial state for the atom. The thermal contribution to the initial state of the field may be neglected as long as the number of coherent photons is sufficiently larger than the number of thermal photons. Using thermal states allows for an investigation of the thermal properties of the JCM, i.e., its applicability for problems like the initial state preparation in quantum computing, cooling of atoms, or implementation of quantum thermodynamic machines.

In the following we will discuss a model where an atom in a thermal state enters a cavity prepared in a coherent state. By obtaining a closed form for the reduced density matrix for the atom we will show that after the collapse the state of the atom is independent of its initial state. After a fixed interaction time the atom is taken to leave the cavity and to interact with a laser field, which is treated as a semi-classical driver. For an appropriate laser field the final state will be thermal as well. Depending on the interaction time with the cavity the final temperature can be varied over a large range, leading to cooling or heating of the atom. We will present an expression for the minimum and maximum temperature that can be achieved. Finally, we will discuss applications of the method to cooling of the internal degrees of freedom of atoms and creating heat baths suitable for studying thermodynamics at the nanoscale.

The total system is described by the Hamiltonian

\[ \hat{H} = \hat{H}_A + \hat{H}_F + \hat{H}_I, \]

where the atomic Hamiltonian \( \hat{H}_A \) is given by

\[ \hat{H}_A = \frac{\Delta E}{2} \hat{\sigma}_z, \]

with \( \Delta E \) being the energy splitting. The field Hamiltonian \( \hat{H}_F \) is

\[ \hat{H}_F = \hbar \omega \left( \hat{a} \hat{a}^\dagger + \frac{1}{2} \right), \]

with \( \omega \) being the frequency of the single mode and \( \hat{a} \) being the annihilation operator of the field. Being in resonance, we have \( \omega = \Delta E/\hbar \). Using the electric dipole and rotating wave approximation the JCM interaction Hamiltonian is given by

\[ \hat{H}_I = g \hat{\sigma}_+ \hat{a} + g^* \hat{\sigma}_- \hat{a}^\dagger, \]

where \( g \) is the coupling constant for the atom-field interaction and \( \hat{\sigma}_\pm \) are the atomic transition operators. Particularly, the coupling constant is given by

\[ g = d \sqrt{\frac{\omega}{\hbar c_0 V}}, \]

where \( d \) is the atomic electric dipole matrix element and \( V \) is the mode volume.
We restrict ourselves to the field being initially in a coherent state $|\alpha\rangle$ and the atom being in a thermal state described by the density operator
\[ \hat{\rho}_A(0) = Z^{-1} \exp(-\beta \hat{H}_A) \equiv p_e(0) |e\rangle \langle e| + [1 - p_e(0)] |g\rangle \langle g|, \] (6)
with $Z$ being the partition function, $\beta$ the inverse temperature, $p_e$ the probability to find the atom in its excited state $|e\rangle$, and $|g\rangle$ its ground state.

The time evolution of the full system is then given by
\[ \hat{\rho}(t) = p_e(0) \hat{U} |e, \alpha\rangle \langle e, \alpha| \hat{U}^\dagger + [1 - p_e(0)] \hat{U} |g, \alpha\rangle \langle g, \alpha| \hat{U}^\dagger \equiv p_e(t) |\psi_e(t)\rangle \langle \psi_e(t)| + [1 - p_e(t)] |\psi_g(t)\rangle \langle \psi_g(t)|, \] (7)
where $\hat{U}$ is the time evolution operator of the full system.

In order to obtain the effective time evolution for the atom alone, the degrees of freedom corresponding to the field have to be traced out \[14\]. Here, the partial trace over the field is given by
\[ \hat{\rho}_A(t) = \text{Tr}_F \{ |\psi(t)\rangle \langle \psi(t)| \} = \sum_n \langle n |\psi(t)\rangle \langle \psi(t)| n \rangle. \] (8)

Since $\hat{\rho}_A$ is Hermitian and has unit trace, the atom is effectively described by the diagonal element $\rho_{11}$ and the off-diagonal element $\rho_{01}$.

In the following we first consider the case where the initial state is $|e, \alpha\rangle$. Then, the full time evolution is given by (see, e.g., \[13\])
\[ |\psi_e(t)\rangle = \sum_n \left( e^{-i\Omega_n t/2} |n+1\rangle - e^{i\Omega_n t/2} |n-1\rangle \right) \times \frac{e^{-|\alpha|^2/2}}{\sqrt{2}} \frac{\alpha^n}{\sqrt{n!}} e^{-i(n+1/2)\omega t}, \] (9)
where the $n$-photon Rabi frequency $\Omega_n = g \sqrt{n}$ and the $n$-photon eigenstates of the atom-field system,
\[ |\pm n\rangle = \frac{1}{\sqrt{2}} (|g, n + 1\rangle \pm |e, n\rangle), \] (10)
have been used. The time evolution of the reduced density matrix element $\rho_{11}$ before the revival time have been studied extensively (see, e.g., \[13\]) and is given by
\[ \rho_{11}(t) = \frac{1}{2} + \frac{1}{2} \cos(2gt) \exp \left( -\frac{t^2}{\tau_C^2} \right), \] (11)
with $\tau_C$ being the collapse time $\tau_C = \sqrt{2}/g$. For an atom initially in $|g\rangle$ the result is
\[ \rho_{11}(t) = \frac{1}{2} - \frac{1}{2} \cos(2gt) \exp \left( -\frac{t^2}{\tau_C^2} \right). \] (12)

Therefore, after the collapse the diagonal elements are constant and $\rho_{ii} = 1/2$.

The off-diagonal element $\rho_{01}$ (again, first for the atom initially in $|e\rangle$) is given by
\[ \rho_{01}(t) = \sum_n \langle \psi_e(t)| n, g \rangle \langle n, e| \psi_e(t)\rangle. \] (13)

Evaluating the summands $\rho_{01}^{(n)}$ using Eq. (9) and Eq. (10) leads to
\[ \rho_{01}(t) = i w(n) \frac{\sqrt{n}}{2\alpha} e^{-i\omega t} \left\{ \sin \left[ \left( \Omega_{n+1} + \Omega_n \right) t \right] \frac{t}{2} - \sin \left( \Omega_{n+1} - \Omega_n \right) t \right\}, \] (14)
with $w(n)$ being the Poisson distribution. The first term inside the square brackets oscillates at a much higher frequency than the second and results only in a random phase, which vanishes after summation. In the high-photon limit $\sqrt{n}$ may be approximated by (see \[6\])
\[ \sqrt{n} \approx \sqrt{n^2 + n - \tilde{n}}. \] (15)

Analogously, the difference of the Rabi frequencies can be expressed as
\[ \Omega_{n+1} - \Omega_n = 2g \left( \sqrt{n+1} - \sqrt{n} \right) \approx 2g \left( \frac{1}{2\sqrt{n}} - \frac{1}{8n^{3/2}} - \frac{n - \tilde{n}}{4n^{3/2}} \right). \] (17)

Plugging only the leading order into Eq. (14) and replacing the sum in Eq. (15) by an integral over a Gaussian distribution leads to
\[ \rho_{01}(t) = -\frac{i}{2} \exp [i(\omega t + \phi)] \sin \frac{gt}{2\sqrt{n}}, \] (18)
where $\phi$ is the initial phase of the radiation field. Using the same approximations for the atom initially in its ground state yields the same result for $\rho_{01}(t)$. Therefore, after the collapse the atom evolves totally independent from its initial state. A comparison of Eq. (18) with the numerical solution of the full time-dependent Schrödinger equation is shown in Fig. 2. Apart from the collapse and revival phase there is excellent agreement. This further shows that the random phase approximation applied to Eq. (14) was perfectly justified.

Since the diagonal elements of $\hat{\rho}_A$ are both at $\frac{1}{2}$ the Bloch vector only moves within $x - y$ plane of the Bloch sphere. Furthermore, the phase oscillates at $\omega$, i.e., in the rotating frame only the initial phase $\phi$ is relevant [see Eq. (18)]. Therefore, in order to obtain a thermal state one always has to apply a $\pi/2$ pulse to the system (see Fig. 3). Since the pulse diagonalizes $\hat{\rho}_A$, the probability to find the atom in its excited state after the pulse $p_e(t)$ is given by the smallest eigenvalue of $\hat{\rho}_A$. Computation of $p_e(t)$ yields
\[ p_e(t) = \frac{1}{2} \left( 1 - \sin \frac{gt}{2\sqrt{n}} \right). \] (19)
This can also be expressed as a temperature using
\[ T = -\frac{\Delta E}{k_B \log \left( \frac{p_e}{1-p_e} \right)}. \] (20)
This temperature should be considered as a parameter characterizing the mixedness of the output state rather than as an indication for a stable thermal state proper. Quantum objects prepared like this (or ensembles thereof) would then constitute resources for further applications like quantum information processing.

Equation (19) suggests that at half of the revival time the atom will be in its ground state (i.e., \( T = 0 \)). However, this minimum temperature would only be reached for infinitely large \( \bar{n} \). For which it would take an infinitely long time to reach this state. In order to determine the actual minimum temperature a correction for finite \( \bar{n} \) is required. A correction to Eq. (19) can be obtained by including the next order in Eq. (17). Close to half of the revival time the sine in Eq. (11) is near its maximum and can be approximated by a second order Taylor expansion, which leads to a final result of

\[
 p_c \left( \frac{\tau R}{2} \right) = \frac{\pi^2}{32 \bar{n}}. \tag{21}
\]

Using the next order in Eq. (15) as well leads to an additional correction in \( O(1/\bar{n}^2) \). Putting this \( p_c \) into Eq. (20) gives the minimum temperature \( T_{\text{min}}(\bar{n}) \) as shown in Fig. 4. Temperatures as low as 0.2 \( \Delta E/k_B \) can be obtained, which correspond to an occupation probability of the excited state of the order of \( 10^{-3} \).

In order to determine the maximum temperature that can be reached we require that the collapse must have taken place [i.e., the difference in the occupation probabilities Eqs. (11) and (12) is negligible compared to the difference induced by the laser]. Requiring the former to be smaller by a factor of 10, this can be expressed as

\[
10 \cos(2gt) \exp \left( -\frac{t^2}{\tau C} \right) = \sin \frac{gt}{2\sqrt{\bar{n}}}. \tag{22}
\]

The cosine on the left hand side may be replaced by unity without violating the above requirement. For large \( \bar{n} \) the right hand side can be approximated linearly in \( t \), resulting in

\[
10 \exp \left( -\frac{t^2}{\tau_C^2} \right) = \frac{gt}{2\sqrt{\bar{n}}}. \tag{23}
\]

Solving for the appropriate cavity interaction time \( t \) and using Eqs. (19) and (20) leads to a maximum temperature \( T_{\text{max}} \) of

\[
T_{\text{max}} = \frac{\Delta E}{k_B \log \frac{4\sqrt{n}+\sqrt{W(400n)}}{4\sqrt{n}-\sqrt{W(400n)}}}, \tag{24}
\]

where \( W(\cdot) \) denotes the Lambert \( W \) function, i.e., the inverse function of \( f(x) = xe^x \). Figure 5 shows the dependence of \( T_{\text{max}} \) on \( \bar{n} \).

These results show that the temperature can be tuned over a large range, which depends only on the average photon number \( \bar{n} \), the coupling time \( t \), and the energy splitting \( \Delta E \). However, there are some other applications for this procedure, which are realizable within present experimental setups. A rather obvious one is the cooling of the internal degrees of freedom of atoms. However, an implementation using a cavity would be extremely difficult as the frequencies relevant for cooling are in the MHz range, where the coupling constant \( g \) is much too small to observe any effects [due to the \( \omega \) dependence in Eq. (5)]. A much more promising implementation could be realized

\[
\begin{align*}
\text{FIG. 2: Comparison of the real and imaginary part of Eq. (18)} & \text{(solid lines)} & \text{and the solution of the full time-dependent Schrödinger equation. Initial states for the atom were } \left| g \right\rangle \text{ (Re} \rho_{01}: \text{crosses), } \left| e \right\rangle \text{ (Re} \rho_{01}: \text{boxes}) & \text{and } \left| e \right\rangle \text{ (Re} \rho_{01}: \text{diamonds, Im} \rho_{01}: \text{triangles). } (\bar{n} = 36, g = \Delta E, & \text{and } \phi = 0) \\
\text{FIG. 3: Illustration of the } \frac{\pi}{2} \text{ pulse acting on the Bloch vector of the atom.} \\
\text{FIG. 4: Minimum temperature } T_{\text{min}} \text{ over average photon number } \bar{n}. \\
\end{align*}
\]
using circuit quantum electrodynamics (QED) \cite{16,17}, in which the atom is replaced by a Cooper-pair box and the cavity is implemented by a one-dimensional resonator. There, the coupling constant is sufficiently large even in the relevant frequency range. Although dephasing plays an important role in circuit QED, the dephasing time is of the order of several hundred nanoseconds \cite{18}, while the required time for reaching the minimum temperature is about one order of magnitude smaller for experimentally feasible parameters. Using our procedure thus might lead to lower temperatures than currently employed techniques. Besides circuit QED, other implementations involving a Jaynes-Cummings Hamiltonian with a tunable coupling constant may prove useful as well.

Another interesting application of this procedure could be the realization of tiny local baths. Local baths are an important ingredient in nonequilibrium quantum thermodynamics \cite{13}, where it is necessary to create and control a temperature gradient on a nanoscopic scale. This could be used to investigate transport behavior \cite{19,20} or quantum thermodynamic machines \cite{21}. Using our framework to repeatedly set a temperature of a single two-level system could act as such a local bath as long as the cavity is reset after each step and the temperature control happens on a much smaller timescale than the other processes within the system (i.e., strong bath coupling).

In summary, we have shown that the temperature of a two-level atom could be efficiently controlled via a resonant interaction with a cavity. Depending only on the interaction time with the cavity, it should be possible to tune the final temperature over a large range. The expression for the reduced density matrix of the atom has been obtained in the high photon limit using a systematic series expansion and has been verified by comparison with the solution of the full time-dependent Schrödinger equation. Besides temperature control our procedure may prove useful for cooling various microscopic systems or realizing local baths in nanothermodynamics.

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