Consensus control for stochastic delayed multi-agent systems with disturbances and switching topologies

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Abstract. This paper is devoted to the study of consensus of a class of stochastic delayed multi-agent systems with disturbances and switching topologies. The aim is to design an output-feedback controller so that the system under consideration reaches L2-L∞ consensus in mean square. By applying a model transformation, the closed-loop system is transformed into a reduced-order system. Next, a sufficient condition concerning the mean-square stability and L2-L∞ performance of the reduced-order system is established. With the aid of the condition, a method for the output-feedback controller design is developed in terms of linear matrix inequalities. Finally, an example is provided to illustrate the applicability of the present method.

1. Introduction

During the past twenty years, consensus control for delayed multi-agent systems has attracted more and more attention in various fields such as power systems and robot systems [1]. One of the most significant issues of consensus control is to design a controller to ensure the consensus of the delayed multi-agent systems with disturbances and switching topologies. This had led to an interest of L2-L∞ consensus control for such delayed multi-agent systems. Under undirected graphs, a method for the L2-L∞ consensus controller design was presented in [2]. For the case of directed interaction graphs, sufficient conditions for ensuring the existence of an L2-L∞ consensus controller were given in [3]. Note that the controllers proposed in [2, 3] are based on full-state feedback, which is often unavailable in practice [4]. A question is raised naturally concerning the possibility of designing output feedback controllers. It is also worth remarking that the effects of stochastic noises on each agent are not considered in [2, 3], whereas, as is known, stochastic noises are often encountered in a real physical system and can influence the system performance considerably. Motivated by the above discussions, the paper is devoted to the study of the consensus of a class of stochastic delayed multi-agent systems with disturbances and switching topologies. The purpose is to develop a method for the design of an output-feedback controller so that the system under consideration reaches L2-L∞ consensus in mean square. By applying a model transformation, the closed-loop system is transformed into a reduced-order system. Then, a sufficient condition concerning the mean-square stability and L2-L∞ performance of the reduced-order system is established. With the aid of such a condition, a method for the output-feedback controller design is developed in terms of linear matrix inequalities (LMIs). Finally, an example is provided to illustrate the applicability of our method.
2. Preliminaries
Consider a stochastic delayed multi-agent system consisting of n agents
\[
\begin{align*}
    dx(t) &= \left[A_{c}x(t) + A_{w}(t - \tau) + Bu(t) + D_{i}v_{i}(t)\right]dt + Hx(t)d\omega(t), \\
    y_{i}(t) &= C_{w}x(t) + D_{2}v(t), \\
    z(t) &= x(t) - \frac{1}{n} \sum_{j=1}^{n} x_{j}(t), i = 1, ..., n,
\end{align*}
\]
where \(A, A_{c}, B, C, D_{i}, D_{2}, H\) are real constant matrices; \(x(t) \in \mathbb{R}^{m}, y_{i}(t) \in \mathbb{R}^{p}, z(t) \in \mathbb{R}^{m}\), and \(u(t) \in \mathbb{R}^{m_{2}}\) are respectively, the state, the measured output, the controlled output, and the control input; \(v_{i}(t) \in \mathbb{R}^{m_{1}}\) stands for the disturbance input which belongs to \(L_{2}[0, \infty)\), \(\omega(t)\) represents a scalar Brownian motion. Throughout this paper, it is assumed that the interaction topologies switch arbitrarily between directed and balanced graphs \(G_{1}, \cdots, G_{N}\). If not explicitly stated, the notations used in this paper are the same as in [5].

The following output feedback controller is used
\[
u_{i}(t) = K_{c} \sum_{j \in N_{i}(t)} a_{ij}(t) (y_{j}(t) - y_{i}(t)).
\]

Substituting controller (2) into system(1) gives
\[
\begin{align*}
    dx(t) &= \left[I_{n} \otimes A + L_{\sigma(t)} \otimes B_{1}KC\right]x(t)dt + \left(I_{n} \otimes A_{c}\right)x(t - \tau)dt \\
    &\quad + \left(I_{n} \otimes D_{1} + L_{\sigma(t)} \otimes B_{1}KD_{2}\right)v(t)dt + \left(I_{n} \otimes H\right)x(t)d\omega(t), \\
    z(t) &= \left(L_{c} \otimes I_{m}\right)x(t),
\end{align*}
\]
where \(x(t) = [x_{1}^{T}(t) ... x_{n}^{T}(t)]^{T}, v(t) = [v_{1}^{T}(t) ... v_{n}^{T}(t)]^{T}, z(t) = [z_{1}^{T}(t) ... z_{n}^{T}(t)]^{T}\).

Definition 1. For a prescribed positive scalar \(\gamma\), system (1) is said to reach \(L_{2}-L_{\infty}\) consensus in mean square, if, \(\lim_{t \to \infty} \mathbb{E}\left\{x(t) - x(t)\right\} = 0\) is satisfied when \(v(t) \equiv 0\) and \(\|z(t)\|_{E_{\infty}} - \gamma \|v(t)\|_{E_{2}} \leq 0\) holds for any non-zero \(v(t) \in L_{2}[0, \infty)\) under the zero initial condition.

The purpose of this paper is to develop a method for the design of controller (2) so that system (1) reaches \(L_{2}-L_{\infty}\) consensus in mean square. In this end, we use a model transformation by defining
\[
\left(U_{1}^{T} L_{c} \otimes I_{m}\right)x(t) = \xi(t), \quad \left(U_{1}^{T} L_{c} \otimes I_{m}\right)v(t) = \vartheta(t).
\]

Then, one can obtain a reduced-order system
\[
\begin{align*}
    d\xi(t) &= \left(\Lambda_{1}(\sigma(t))\xi(t) + \Lambda_{2}\xi(t - \tau) + \Lambda_{3}\vartheta(t)\right)dt + \Lambda_{4}\xi(t)d\omega(t), \\
    z(t) &= \Lambda_{5}\xi(t),
\end{align*}
\]
where \(\Lambda_{1}(\sigma(t)) = I_{n} - 1 \otimes A + L_{\sigma(t)} \otimes B_{1}KC\), \(\Lambda_{2} = I_{n} - 1 \otimes A_{c}\), \(\Lambda_{3}(\sigma(t)) = I_{n} - 1 \otimes D_{1} + L_{\sigma(t)} \otimes B_{1}KD_{2}\), \(\Lambda_{4} = I_{n} - 1 \otimes H\), \(\Lambda_{5} = U_{1} \otimes I_{m}\).

3. Main Result

Theorem 1. Let \(\gamma > 0\) be a given scalar. Then system (3) is mean-square stable and has an \(L_{2}-L_{\infty}\) performance, if there exist positive-definite symmetric matrices \(P_{i} \in R^{m_{i}m_{i}} (i = 1, 2), Q \in R^{m_{2}m_{2}}\) such that
\[
\Delta_{i_{1}} = \begin{bmatrix}
He(\bar{P}_{\Lambda_{1i}}) + \bar{Q} + I_{n-1} \otimes H^{T} P_{H} \\
* & -\bar{Q} \\
* & -\bar{Y}
\end{bmatrix} < 0,
\]

(4)

\[
\Delta_{i_{2}} = I_{m} - \gamma \bar{P}_{i} < 0, \quad i = 1, 2,
\]

(5)

where \( \bar{P}_{i} = I_{n-1} \otimes P_{i}, \quad \bar{Q} = I_{n-1} \otimes Q. \)

**Proof.** Choose a Lyapunov functional candidate as follows

\[
V_{i}(t) = \xi^{T}(t) \bar{P}_{i} \xi(t) + \int_{t-\tau}^{t} \xi^{T}(s) \bar{Q} \xi(s) ds.
\]

Then one has

\[
LV_{i}(t) = 2 \xi^{T}(t) \bar{P}_{i} \Lambda_{1i} \xi(t) + 2 \xi^{T}(t) \bar{P}_{i} \Lambda_{2i} \xi(t) - \tau + 2 \xi^{T}(t) \bar{P}_{i} \Lambda_{3i} \vartheta(t)
\]

\[
+ \xi^{T}(t) \left( I_{n-1} \otimes H^{T} P_{H} \right) \xi(t) + \xi^{T}(t) \bar{Q} \xi(t) - \xi^{T}(t-\tau) \bar{Q} \xi(t-\tau).
\]

In the case when \( \vartheta(t) \equiv 0 \), one can obtain

\[
LV_{i}(t) = \begin{bmatrix}
H_{e}(\bar{P}_{i} \Lambda_{1i}) + \bar{Q} + I_{n-1} \otimes H^{T} P_{H} \\
* & -\bar{Q}
\end{bmatrix} \begin{bmatrix}
\xi^{T}(t) \\
\xi^{T}(t-\tau)
\end{bmatrix}.
\]

It follows by (4) that \( LV_{i}(t) < 0 \) for any \( \xi(t) \neq 0 \), which ensures the mean-square stability of system (3).

To study the L2-L∞ performance of system (3), let us define

\[
J(t) = \mathbb{E}\left[ V(t) - \gamma \int_{0}^{t} \vartheta^{T}(s) \vartheta(s) ds \right].
\]

Due to the zero initial conditions, by Ito’s formula one can write

\[
J(t) = \mathbb{E}\left[ \int_{0}^{t} (LV(s)) ds \right]
\]

\[
= \mathbb{E}\left[ \int_{0}^{t} \begin{bmatrix}
\xi^{T}(s) \\
\xi^{T}(s-\tau)
\end{bmatrix} \Lambda_{1i} \begin{bmatrix}
\xi(s) \\
\xi(s-\tau)
\end{bmatrix} + \begin{bmatrix}
\xi^{T}(s) \\
\xi^{T}(s-\tau)
\end{bmatrix} \begin{bmatrix}
\vartheta^{T}(s) \\
\vartheta^{T}(s-\tau)
\end{bmatrix} ds \right].
\]

It follows by (4) that \( J(t) \leq 0 \), which means that

\[
\mathbb{E}(V(t)) \leq \gamma \mathbb{E}\left[ \int_{0}^{t} \vartheta^{T}(s) \vartheta(s) ds \right].
\]

Using (5), one can get

\[
\mathbb{E}(\pi(t)^{T} z(t)) \leq \gamma \mathbb{E}(V(t)) \leq \gamma^{2} \mathbb{E}\left[ \int_{0}^{t} \vartheta^{T}(s) \vartheta(s) ds \right].
\]

Taking the maximum value of time \( t \) gives

\[
\|z(s)\|_{L_{\infty}}^{2} \leq \gamma \|\vartheta(s)\|_{L_{2}}^{2}.
\]

Thus system (3) has an L2-L∞ performance.

Based on Theorem 1, we can propose a method for the design of controller (2), which is given in the following theorem.

**Theorem 2.** Let \( \gamma > 0 \) and \( \eta > 0 \) be given scalars. Then, system (1) reaches L2-L∞ consensus in mean square under controller (2), if, there exist matrices \( X \in R^{m \times n+1}, Y \in R^{m \times p}, \) and positive-definite symmetric matrices \( P_{i} \in R^{m \times m} (i = 1, 2), Q \in R^{n \times n} \) such that (5) and
hold for $i = 1, 2$, where $\Lambda_{6i} = I_{n-1} \otimes P_iA + L_{ii} \otimes B_iYD_2$, $\Lambda_{7i} = I_{n-1} \otimes (P_i B_1 - B_i X) + \eta L_{ii}^T \otimes (YD_2^T)^T$, $\Lambda_{8i} = He(I_{n-1} \otimes P_i A + L_{ii} \otimes B_i YC) + Q + I_{n-1} \otimes H^T P_i H$. The desired gain of controller (2) is given by $K = X^{-1}Y$.

**Proof.** Noting that
\[
\begin{aligned}
\overline{P}_i\Lambda_{6i} &= I_{n-1} \otimes P_iA + L_{ii} \otimes (P_i B_1 - B_i X)X^{-1}YC + L_{ii} \otimes B_i YC, \\
\overline{P}_i\Lambda_{7i} &= I_{n-1} \otimes P_iD_i + L_{ii} \otimes (P_i B_1 - B_i X)X^{-1}YD_2 + L_{ii} \otimes B_i YD_2,
\end{aligned}
\]
inequality (4) can be re-organized as
\[
\Theta_i + He[U_i X^{-1}V_i^T] < 0, \tag{7}
\]
where $U_i = [I_{n-1} \otimes (P_i B_1 - B_i X)]^T 0 0^T, V_i = [L_{ii} \otimes YC 0 L_{ii} \otimes YD_2]^T$, and
\[
\Theta_i = 
\begin{bmatrix}
He(I_{n-1} \otimes P_i A + L_{ii} \otimes B_i YC) + Q + I_{n-1} \otimes H^T P_i H & I_{n-1} \otimes PA_i & \Lambda_{6i} \\
* & -Q & 0 \\
* & * & -\eta
\end{bmatrix}.
\]
By Lemma 4 of [5], (7) can be guaranteed by (6). Thus, the proof of this theorem is completed.

**4. Example**

**Example 1.** Consider system (1) of 5 agents with the following parameters:
\[
A = \begin{bmatrix}
0 & -1 \\
2 & 1
\end{bmatrix}, \quad A_\tau = \begin{bmatrix}
0.1 & 0 \\
0 & 0.1
\end{bmatrix}, \quad B_1 = \begin{bmatrix}
1 & 0 \\
1 & 1
\end{bmatrix}, \quad H = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \quad D_1 = \begin{bmatrix}
1 & 0 \\
0 & 2
\end{bmatrix}, \quad C = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \quad D_2 = \begin{bmatrix}
1 & 2 \\
0 & 1
\end{bmatrix}.
\]

Figures 1, 2 depict the interaction topologies of the multi-agent system. Set $\gamma = 1, \eta = 0.8$. Then, by Theorem 2, the system can reach mean-square L2-L\infty-consensus under controller (2) with $K = [-1.0088, -0.0198, 0.2751, -2.3831]$. In the simulations, the disturbance inputs are chosen to be
\[ v_i(t) = 0.5[-\sin(t) \quad -\cos(t)]^T, \quad i = 1, \ldots, 5. \] The system states response with controller (2) and the curve of \( \gamma(t) \) under zero initial condition are given in Figures 3 and 4, respectively.

![Figure 3. States response with controller](image1)

![Figure 4. Evolution of \( \gamma(t) \)](image2)
From these figures, one can see that the agents can quickly reach consensus under control and the maximum value is $\gamma_{\max}(t) = 0.5402$ is less than the prescribed $L_2 - L_\infty$ performance index $\gamma$. In this way, the applicability of the present method is verified.

5. Conclusion
The issue of consensus of a class of stochastic delayed multi-agent systems with disturbances and switching topologies has been considered. The closed-loop system has been transformed into a reduced-order system by means of a model transformation. Then, a sufficient condition concerning the mean-square stability and $L_2-L_\infty$ performance of the reduced-order system has been established. With the aid of the condition, a method for the output-feedback controller design has been developed in terms of LMIs. Finally, an example has been used to illustrate the applicability of the present method.

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