SHARP ESTIMATES FOR TRILINEAR OSCILLATORY INTEGRALS AND AN ALGORITHM OF TWO-DIMENSIONAL RESOLUTION OF SINGULARITIES

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Abstract. We obtain sharp estimates for certain trilinear oscillator integrals. In particular, we extend Phong and Stein’s seminal result to a trilinear setting. This result partially answers a question raised by Christ, Li, Tao and Thiele concerning the sharp estimates for certain multilinear oscillatory integrals. The method in this paper relies on a self-contained algorithm of resolution of singularities in \( \mathbb{R}^2 \), which may be of independent interest.

1. Introduction

The purpose of this paper is to study the sharp decay estimates of the following trilinear oscillatory integrals:

\[
\Lambda_S(f_1, f_2, f_3) = \iint e^{i\lambda S(x, y)} f_1(x) f_2(y) f_3(x + y) a(x, y) \, dx \, dy,
\]

where \( a(x, y) \) is a smooth cut-off function supported in a sufficiently small neighborhood of 0, and the phase \( S(x, y) \) is a real analytic function in \( \mathbb{R}^2 \).

1.1. Backgrounds.

Consider the following oscillatory integral operator

\[
T(f)(x) = \int e^{i\lambda S(x, y)} f(y) a(x, y) \, dy,
\]

where \( S(x, y) \) is a smooth real-valued function in \( \mathbb{R}^n \times \mathbb{R}^n \) and \( a(x, y) \) is a smooth cut-off function.

One of the central topics in oscillatory integrals is the study of the asymptotic behavior of \( \|T\|_{2 \rightarrow 2} \) as \( |\lambda| \to \infty \). Equivalently, from a view of duality, one can find out the optimal \( C(\lambda) \) (up to a constant) s.t.

\[
|\Lambda_2(f, g)| \leq C(\lambda) \|f\|_2 \|g\|_2,
\]

where the bilinear form \( \Lambda_2(f, g) \) is defined as

\[
\Lambda_2(f, g) = \langle T(f), g \rangle = \iint e^{i\lambda S(x, y)} f(y) \overline{g(x)} a(x, y) \, dx \, dy.
\]

During the last decades, this problem and related topics have been extensively studied by many authors. Fruitful results have been obtained via rich techniques.
We begin with a classic result of Hörmander [10], concerning the sharp $L^2(\mathbb{R}^n)$ estimates of (1.2) when $S$ is nondegenerate:

**Theorem 1.1 (Hörmander [10]).** Assume $a(x, y)$ is a smooth cut-off function supported in a neighborhood of 0 and $S(x, y)$ is a smooth function such that

\[
\left| \det \frac{\partial^2 S}{\partial x \partial y} \right| \geq 1, \quad \text{for all } (x, y) \in \text{supp } (a).
\]

Then one has

\[
\|T(f)\|_2 \leq C|\lambda|^{-n/2}\|f\|_2.
\]

Establishing sharp estimates in a more general setting, in particular when $S(x, y)$ is degenerate, was proved to be difficult. Until the early 90s, by the seminal works of Phong and Stein [11–13], a full understanding of (1.2) was obtained when $S$ is a real analytic function of two variables. In their works, a systematic treatment was introduced to deal with the degenerate setting. The key ingredient to characterize the sharp decay rate is the geometric concept: Newton Polyhedra.

**Definition 1.1.** Let $S = \sum_{p, q} c_{p,q} x^p y^q$ be a real analytic function, where $p, q \in \mathbb{N}$. The Newton polyhedron of $S$ is defined as:

\[
\mathcal{N}(S) = \text{Conv}(\bigcup_{p,q} \{(u, v) \in \mathbb{R}^2 : u \geq p, v \geq q \text{ and } c_{p,q} \neq 0\}).
\]

Here, Conv($X$) represents the convex hull of a set $X$ in $\mathbb{R}^2$.

**Theorem 1.2 (Phong-Stein [13]).** Let $S(x, y)$ be real-analytic and assume the support of $a(x, y)$ is contained in a sufficiently small neighborhood of $0 \in \mathbb{R}^2$, then

\[
\|T(f)\|_2 \leq C|\lambda|^{-\frac{1}{2} \delta_S} \|f\|_2,
\]

where $\delta_S > 0$ is characterized in terms of the Newton polyhedron as follows

\[
\delta_S = 1 + \inf\{t \in \mathbb{R} : (t, t) \in \mathcal{N}(\partial_x \partial_y S)\}.
\]

It was shown by the classic work of Varchenko, confirming earlier hypotheses of Arnold, Newton polyhedra can be used to characterize the decay rate of the scalar oscillatory integrals [17].

### 1.2. Motivations.

In this paper, we study some trilinear analogues of the above problems. Set

\[
\Lambda_S(f_1, f_2, f_3) = \int \int e^{i\lambda S(x, y)} f_1(x) f_2(y) f_3(x + y) a(x, y) dx dy.
\]

Likewise, we want to characterize the optimal constant $\epsilon$ s.t. the following inequality is true for some constant $C$:

\[
|\Lambda_S(f_1, f_2, f_3)| \leq C|\lambda|^{-\epsilon} \|f_1\|_2 \|f_2\|_2 \|f_3\|_2.
\]

The study of the trilinear form (1.1) is motivated by the work of Christ, Li, Tao and Thiele [4], where certain multi-linear oscillatory integrals were studied in a very general setting.

To formulate the questions posed in [4], we need some preliminary notations. Let $\pi = (\pi_1, \ldots, \pi_J)$, where $\pi_j : \mathbb{R}^n \to \mathbb{R}^{n_j} \subset \mathbb{R}^n$ is a surjective linear projection. Let $S : \mathbb{R}^n \to \mathbb{R}$ be a polynomial and $a(x)$ be a smooth cut-off function supported
in a small neighborhood of $0 \in \mathbb{R}^n$. For each $j$, let $f_j : \mathbb{R}^n \to \mathbb{C}$ be a measurable function. Consider the following multilinear form:

$$
\Lambda_{S, \pi}(f_1, f_2, \ldots, f_j) = \int e^{i\lambda S(x)} a(x) \prod_{j=1}^{j} f_j \circ \pi_j(x) dx.
$$

**Q1:** For what kind of input $(S, \pi)$, the following is true

$$
|\Lambda_{S, \pi}(f_1, f_2, \ldots, f_j)| \leq C|\lambda|^{-\delta} \prod_{j=1}^{j} \|f_j\|_{p_j}
$$

for some $\delta > 0$, some $p = (p_1, \ldots, p_j) \in [1, \infty]^J$ and all $f_j \in L^{p_j}(\mathbb{R}^n)$?

**Q2:** If Q1 could be answered affirmatively, what is the optimal exponent $\delta$?

Giving a complete answer to Q1 for a most general input $(S, \pi)$ seems very difficult, not to mention the sharp estimates Q2. Still, an affirmative answer to Q1 was given in [4] under certain dimension assumptions on $\pi$. For further progress of Q1, we refer the readers to [1][3][8].

For Q2, some results were known before. For instance, when $J = 2$, $n_1 = n_2 = n/2$ (assume $n$ is even) and $S$ is smooth, Theorem 1.1 provides a sufficient characterization when the best possible decay can be obtained. Theorem 1.2 settled the case $n = J = 2$ and $S$ is an arbitrary analytic function; see [7][15][16] for $S \in C^\infty(\mathbb{R}^2)$. For $n = J \geq 2$, almost sharp estimates (probably up to a power of $\log |\lambda|$) were known, by the work of Phong, Stein and Sturm [1].

In Christ, Li, Tao and Thiele’s attempt to answer Q1, an important step is a reduction to the trilinear setting. Thus, it is crucial to fully understand the trilinear case, in particular to determine the optimal exponent in (1.10) in this setting. This motivates us to study the sharp estimate of the trilinear form (1.1), which corresponds to the case $n = 2$, $J = 3$, $S$ is analytic and $\pi = \pi_0$, where

$$
\pi_0(x, y) = (\pi_{01}(x, y), \pi_{02}(x, y), \pi_{03}(x, y)) = (x, y, x + y).
$$

Indeed, a more general setting

$$
\Lambda_{S, \pi}(f_1, f_2, f_3) = \int \int e^{i\lambda S(x, y)} a(x, y) \prod_{j=1}^{3} f_j \circ \pi_j(x, y) dxdy,
$$

can be reduced to (1.1) via an invertible linear transformation in $\mathbb{R}^2$ (see Section 2), where $\pi_j : \mathbb{R}^2 \to \mathbb{R}$ are pairwise linearly independent projections for $j = 1, 2, 3$. One necessary condition for (1.12) to possess a decay bound is that $S$ should be nondegenerate relative to $\pi$, in the sense $S(x, y)$ cannot be represented as a sum of functions of $\{\pi_j(x, y)\}$. Otherwise, if

$$
S(x, y) = \sum_{1 \leq j \leq 3} S_j \circ \pi_j(x, y),
$$

then we can incorporate each $e^{i\lambda S_j \circ \pi_j(x, y)}$ into $f_j \circ \pi_j(x, y)$ by setting

$$
\tilde{f}_j \circ \pi_j(x, y) = e^{i\lambda S_j \circ \pi_j(x, y)} f_j \circ \pi_j(x, y).
$$

Since $\|\tilde{f}_j\|_{p_j} = \|f_j\|_{p_j}$, one cannot expect any decay as in (1.10).

Let $\pi^\perp_j : \mathbb{R}^2 \to \mathbb{R} \subset \mathbb{R}^2$ be linear projections s.t. $\pi_1 \circ \pi_j^\perp = 0$ and $\|\pi_j^\perp\|_2 = 1$. Set $\pi^\perp = (\pi^\perp_1, \pi^\perp_2, \pi^\perp_3)$ and $D_{\pi^\perp} = \bigwedge_{j=1}^{3} \pi_j^\perp \cdot \nabla$. Then $S$ is called simply degenerate relative to $\pi$ if $D_{\pi^\perp} S \equiv 0$ [4], otherwise $S$ is called simply nondegenerate relative
to $\pi$. In addition, $S$ is simply degenerate at a point $(x_0, y_0)$ if $D_{\pi \perp} S(x_0, y_0) = 0$. Simply degeneracy implies degeneracy and the inverse is not true in general. But in our case, they are equivalent, see Proposition 3.1 in [4].

1.3. Results.

The following theorem, extending Theorem 1.1 to the trilinear setting when $n = 1$, states that if $S$ is simply nondegenerate everywhere in $\text{Conv}(\text{supp}(a))$, then one can obtain the optimal bound of (1.12).

**Theorem 1.3.** Assume $a(x, y)$ is a smooth cut-off function supported in a neighborhood of $0 \in \mathbb{R}^2$ and $S(x, y)$ is smooth s.t.

\[
|D_{\pi \perp} S(x, y)| \geq 1 \quad \text{for all} \quad (x, y) \in \text{Conv}(\text{supp}(a)),
\]

then

\[
\Lambda_{S, \pi}(f_1, f_2, f_3) \leq C|\lambda|^{-1/6} \prod_{j=1}^{3} \|f_j\|_2.
\]

We also extend Theorem 1.2 to the trilinear form (1.12). Different to what was expected, the characterization of the sharp exponent in this case is not the same as the one in Phong–Stein’s result. Instead, it is described by the relative multiplicity of $S$, which is an algebraic concept. Nevertheless, it can still be interpreted geometrically in terms of the Newton polyhedron of $D_{\pi} S$; see Section 4. We shall investigate such difference in Section 4. Define the multiplicity of an analytic function $S$ as

\[
\text{mult}(S) = \min\{i : S_i(x, y) \neq 0\},
\]

where $S(x, y) = \sum_i S_i(x, y)$ and $S_i(x, y) = \sum_{p+q=i} c_{p+q} x^p y^q$ are homogeneous polynomials. We also adopt the convention that $\text{mult}(S) = -\infty$ if $S \equiv 0$. The multiplicity of $S$ relative to $\pi$ is defined as

\[
\text{mult}_{\pi}(S) = \min\{i : D_{\pi \perp} S_i \neq 0\} = \text{mult}(D_{\pi \perp} S) + 3,
\]

which is the multiplicity of the quotient of $S$ by the class of degenerate analytic functions. Notice that if $S$ is simply degenerate, then $\text{mult}_{\pi}(S) = -\infty$. One of the two main results of this paper is:

**Theorem 1.4.** Assume $S(x, y)$ is a real analytic function and the support of $a(x, y)$ is sufficiently small. Then

\[
|\Lambda_{S, \pi}(f_1, f_2, f_3)| \leq C|\lambda|^{-\frac{1}{\text{mult}_{\pi}(S)}} \prod_{j=1}^{3} \|f_j\|_2.
\]

The result (1.16) is exact in the sense that if $a(0, 0) \neq 0$, then

\[
|\Lambda_{S, \pi}(f_1, f_2, f_3)| \geq C'|\lambda|^{-\frac{1}{\text{mult}_{\pi}(S)}} \prod_{j=1}^{3} \|f_j\|_2,
\]

as $|\lambda| \to \infty$, for some $C' > 0$ and some $\{f_j\}_{1 \leq j \leq 3}$.

**Remark 1.1.** The existence of a (non-sharp) decay rate in the bound of (1.16) is included in the results of [4] as a special case.
1.4. Methods.

Like Phong and Stein’s proof of Theorem 1.2, the proof of Theorem 1.4 requires elaborated analysis. There are two main ingredients in their proof:

1. The operator version of the van der Corput Lemma [12]: see Theorem 2.2 and
2. Weierstrass Preparation Theorem.

In order to extend Phong and Stein’s framework to the trilinear setting, we first establish the trilinear analogue of (1):

1’. Theorem 2.3 trilinear version of Phong–Stein’s van der Corput Lemma.

In addition, we develop

2’. a self-contained algorithm of resolution of singularities in \( \mathbb{R}^2 \), as a substitution of Weierstrass Preparation Theorem, which is our second main result:

**Theorem 1.5.** Let \( P(x,y) \) be a real analytic function in \( \mathbb{R}^2 \) and \( U = \{(x,y) : |x|, |y| < \epsilon \} \) be a neighborhood of 0, where \( \epsilon \) is sufficiently small. Then there is an algorithm, which partitions a dense open subset of \( U \) into a finite collection of regions \( \{V_k\}_{1 \leq k \leq K}, \) such that \( P \) behaves almost like a monomial in each \( V_k \) in the following sense. There is an integer \( M \in \mathbb{N} \), and for each \( k \) there is a diffeomorphism

\[
\rho_k : V_k \to \rho_k(V_k)
\]

satisfying the following properties:

\[
P(x,y) = P_k(x_k, y_k) = x_k^{p_k} y_k^{q_k} \cdot Q_k(x_k, y_k) \quad \text{for all } (x,y) \in V_k,
\]

where

1. \( (x_k, y_k) = \rho_k(x,y) \) and \( P_k = P \circ \rho_k^{-1} \);
2. \( (p_k, q_k) \) is a vertex of the Newton polyhedron of \( P_k \);
3. The function \( Q_k \) is smooth and nonvanishing near 0 in \( \rho_k(V_k) \), i.e.

\[
\lim_{(x_k, y_k) \to (0,0)} Q_k(x_k, y_k) \neq 0 \text{ inside } \rho_k(V_k);
\]
4. \( \rho_k(V_k) \) (as well as \( V_k \)) is a curved triangular region:

\[
\rho_k(V_k) = \{(x_k, y_k) : C_k' |x_k|^{m_k'} < y_k < C_k |x_k|^{m_k} \text{ and } 0 < |x_k| < \epsilon \},
\]

for some \( 0 \leq m_k \leq m_k' \leq \infty \) with \( m_k M, m_k' M \in \mathbb{N} \cup \{\infty\} \), and \( C_k, C_k' \) are constants.
5. \( \rho_k^{-1}(x_k, y_k) \) are real analytic functions of \( \{ |x_k|^{\frac{1}{m_k'}}, y_k \} \), more precisely

\[
\begin{cases}
x = x_k \\
y = \gamma_k(|x_k|^{\frac{1}{m_k'}}) + |x_k|^{\frac{m_k}{m_k'}} y_k,
\end{cases}
\]

where \( M_k \in \mathbb{N} \) and \( \gamma_k \) is a polynomial, unless \( P(x, \gamma_k(|x|^{\frac{1}{m_k'}})) = 0 \), then \( \gamma_k \) is a real analytic function.

Moreover, the constants \( m_k, m_k' \), \( (p_k, q_k) \), \( M_k/M \) and the function \( \gamma_k \) can be computed explicitly via the Newton polyhedra of \( \{P_k\}_{1 \leq k \leq K} \).

**Remark 1.2.** See Theorem 3.7 in Section 3 for a complete version.

\(^1\)In the case \( \gamma_k \) is an infinite series, we can compute any partial sum of \( \gamma_k \).
In the above theorem, almost all the important information can be computed in an explicit manner, which is the major novelty of this theorem (and the algorithm); see Section 3.

The idea of employing resolution of singularities to investigate oscillatory integrals appeared in Varchenko’s work [17], where the deep results from Hironaka [9] played a crucial role. More recently, an algorithm of resolution of singularities in $\mathbb{R}^2$ was introduced by Greenblatt [6], where an elegant proof of Theorem 1.2 was presented based on this algorithm.

Our proof of Theorem 1.4 and the algorithm here are both inspired by the work of Greenblatt [6]. Many of the ideas inside the algorithm here are very elementary and have even been known for centuries, which may come back to Newton’s algorithm for solving $S(x, y) = 0$ by a fractional power series $y = y(x^{\frac{a}{b}})$ (Puiseux series); see [5]. The philosophy of the algorithm here is similar to that of the one in [6]. Here, we outline some of the major novelty as follows:

(I) The implicit function theorem (IFT) is not involved and the change of variables is always of the form:

$$ (x, y) = (x_1, x_1^m (r + y_1)),$$

with the possible exception at the finishing steps. In [6], the IFT plays an important role. The change of variables is of the form:

$$ (x, y) = (x_1, y_1 + q(x_1)),$$

where $q(x)$ is a Puiseux series obtained by the IFT and can be written as $q(x) = rx^m + O(x^{m+\nu})$. Thus, our change of variables is simpler and more explicit. As a result, we are able to switch variables between different stages of iterations; see (1.20) and (1.21). In addition, the $x_1^m$ factor in the 2nd coordinate of (1.22) plays an important role. Namely, it “rescales” each curved triangular region (non-standard) back into a standard non-curved region, allowing one to do iterations in the same region.

(II) Our idea of the termination of the algorithm is very natural. Only performing the form of change of variables (1.22) is not sufficient to ensure the termination of the algorithm. Greenblatt [6] had some nice observations to overcome this barrier. The key point is to invoke the IFT to find the solution of $\partial^{n-1}_y P(x, y) = 0$, which corresponds to the change of variables (1.23). Roughly speaking, each such change of variables decreases certain ‘order’ of $P(x, y)$ by at least 1, which ensures that the algorithm stops after finite steps. The cost is the resulting tail in $q_1(x)$, which is in an implicit form $^3$.

Can one retain the simplicity of the change of variables (1.22) and also ensure the termination of the algorithm? The answer is Yes. To do so, in the beginning, we assume the algorithm does not stop, which results in an infinite chain

$$ [U, P] = [U_0, P_0] \rightarrow [U_1, P_1] \rightarrow [U_2, P_2] \rightarrow \cdots \rightarrow [U_n, P_n] \rightarrow \cdots.$$ 

Each $U_n$ above can be viewed as an identical copy of $U$ in Theorem 1.5 and $P_n$ is obtained from $P_{n-1}$ via the change of variables of the form (1.22). We search for some ‘invariants’ inside this infinite chain, see Definition 3.2 and Lemma 3.5. It

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2 For example, if only performing the above form of change of variables to $P(x, y) = (y - (\sum_{i=1}^{\infty} x^i))^n$, the algorithm does not stop.

3 For instance, applying this form of change of variables to $\tilde{P}(x, y) = (y - x)^n + x^n y^{2n}$, one needs to use the IFT to solve $y - x + cx^n y^{n+1} = 0$ for $y$, whose expansion contains a tail.
turns out that these ‘invariants’ can be visualized by the Newton polyhedra: the shapes of the Newton polyhedra of $P_n$ are invariant after finite steps. Lemma 3.6 which describes such ‘invariants’ analytically, is the key observation to make the algorithm stop naturally.

The organization of this paper is as follows. In Section 2, we reduce the trilinear form $\Lambda_{S, \pi}(f_1, f_2, f_3)$ into the special case $\Lambda_S(f_1, f_2, f_3)$ as in (1.1) and then prove Theorem 1.3 and Theorem 2.1. The latter one is the trilinear version of Phong–Stein’s van der Corput Lemma. In Section 3, the algorithm of resolution of singularities in $\mathbb{R}^2$ is presented. The method is purely analytic. In Section 4, we apply the resolution algorithm and Theorem 2.1 to prove Theorem 1.4.

2. Proof of Theorem 1.3 and Theorem 2.1

In this section, we utilize the method of $TT^*$ to prove Theorem 1.3 and the following technical theorem which is needed in the proof of Theorem 1.4.

**Theorem 2.1.** Assume $a(x, y)$ is a smooth function supported in a strip of $x$-width no more than $\delta_1$ and $y$-width no more than $\delta_2$, satisfying the following derivative conditions

\[
|\partial_y a(x, y)| \lesssim \delta_2^{-1} \quad \text{and} \quad |\partial_x^2 a(x, y)| \lesssim \delta_2^{-2}.
\]

Let $\mu > 0$ and $S(x, y)$ be a smooth function s.t. for all $(x, y) \in \text{Conv(supp (a))}$:

\[
|D_{\pi_0} S(x, y)| \gtrsim \mu \quad \text{and} \quad |\partial_y^\beta D_{\pi_0} S(x, y)| \lesssim \frac{\mu}{\delta_2^\beta} \quad \text{for} \quad \beta = 1, 2
\]

then for $\Lambda_S$ defined as in (1.7), one has

\[
|\Lambda_S(f_1, f_2, f_3)| \lesssim |\lambda \mu|^{-\frac{1}{2}} \prod_{j=1}^{3} \|f_j\|_2.
\]

The above theorem can be viewed as a trilinear analogue of Phong–Stein’s operator version of van der Corput Lemma [12]:

**Theorem 2.2.** Assume $a(x, y)$ is a smooth function supported in a strip of $x$-width no more than $\delta_1$ and $y$-width no more than $\delta_2$, satisfying the following derivative conditions

\[
|\partial_y a(x, y)| \lesssim \delta_2^{-1} \quad \text{and} \quad |\partial_x^2 a(x, y)| \lesssim \delta_2^{-2}.
\]

Suppose $\mu > 0$ and $S(x, y)$ is a smooth function in $\mathbb{R}^2$ s.t. the following holds for all $(x, y) \in \text{supp (a)}$:

\[
|\partial_x \partial_y S(x, y)| \gtrsim \mu \quad \text{and} \quad |\partial_x \partial_y^\beta S(x, y)| \lesssim \frac{\mu}{\delta_2^\beta} \quad \text{for} \quad \beta = 1, 2
\]

then

\[
\|T(f)\|_2 \lesssim (\lambda \mu)^{-1/2} \|f\|_2.
\]

In both theorems above, we have adopted the notation $X \lesssim Y$ to denote $|X| \leq CY$ where $C$ can depend on $a$ and $S$, but is independent of $\delta_1$, $\delta_2$, $\mu$ and $\lambda$. It’s also worth pointing out that theorem 2.2 is not exactly the same as the one employed by Phong-Stein in [12], we have adopted a more general version from Greenblatt in [6]. For the proof of Theorem 2.2, we also refer the readers to [6].
Now we turn to the technical details. First of all, we show that (1.12) can be reduced to (1.11). Set

\[ \|A_{S,\pi}\| = \sup\{|A_{S,\pi}(f_1, f_2, f_3)|, s_j \leq 1 \quad \text{for} \quad j = 1, 2, 3 \]  

and \( \|A_S\| \) is defined similarly. We may assume \( \pi_1(x, y) = x, \pi_2(x, y) = y \) and \( \pi_3(x, y) = Ax + By \) where \( A \neq 0 \) and \( B \neq 0 \). Change variables \( u = Ax \) and \( v = By \), then

\[
A_{S,\pi}(f_1, f_2, f_3) = \int \int e^{i\lambda S(x,y)} f_1(x)f_2(y)f_3(Ax + By)dxdy
\]

\[
= \frac{1}{AB} \int \int e^{i\lambda S(u/v,B)} f_1(u/A)f_2(v/B)f_3(u + v)dudv
\]

\[
= \frac{1}{AB} \int \int e^{i\lambda S(u,v)} f_1,A(u)f_2,v,B(v)f_3(u + v)dudv
\]

where \( S_{A,B}(u,v) = S(u/A,v/B), f_1,A(u) = f_1(u/A) \) and \( f_2,B(v) = f_2(v/B) \). Notice that

\[
D_{\pi_0}S_{A,B}(u,v) = \frac{1}{AB}((\partial_u/A - \partial_v/B)\partial_u\partial_vS)(u/A,v/B)
\]

Thus \( D_{\pi_0}S_{A,B} = CD_{\pi_0}S \) for an appropriate constant \( C \). In addition \( \|f_{1,A}\|_2 = \sqrt{A}\|f_1\|_2 \) and \( \|f_{2,B}\|_2 = \sqrt{B}\|f_2\|_2 \). Finally, notice that convexity is invariance under linear transformations. Therefore, for an appropriate constant \( C \), one has

\[
\|A_{S,\pi}\| = C\|A_{S,A,B}\|.
\]

Now we turn to the proofs of Theorem 1.3 and Theorem 2.1 and only need to consider \( A_S \). For simplicity, we assume \( \|f_1\|_2 = \|f_2\|_2 = \|f_3\|_2 = 1 \). Applying change of variables \( (u,v) = (x + y, y) \) and duality, one has

\[
\|A_S(f_1, f_2, f_3)\| \leq \|B(f_1, f_2)\|_2\|f_3\|_2 = \|B(f_1, f_2)\|_2,
\]

where

\[
B(f_1, f_2)(u) = \int e^{i\lambda S(u,v)} f_1(u - v)f_2(v)a(u - v, v)dv.
\]

Employing \( TT^* \), one obtains

\[
\|B(f_1, f_2)\|_2^2 = \int \int \int e^{i\lambda S(u,v)} f_1(u - v_1)f_1(u - v_2)f_2(v_1)f_2(v_2)
\]

\[
o(u - v_1, v_1)a(u - v_2, v_2)dv_1dv_2du.
\]

Change variables: \( x = u - v_1, y = v_1 \) and \( \tau = v_2 - v_1 \) and set

\[
S_\tau(x, y) = S(x, y) - S(x - \tau, y + \tau)
\]

\[
F_\tau(x) = f_1(x)f_1(x - \tau)
\]

\[
G_\tau(y) = f_2(y)f_2(y + \tau)
\]

\[
a_\tau(x, y) = a(x, y)a(x - \tau, y + \tau)
\]

This yields

\[
\|B(f_1, f_2)\|_2^2 = \int \int \int e^{i\lambda S_\tau(x,v)} F_\tau(x)G_\tau(y)a_\tau(x, y)dxdydr.
\]

The proofs of Theorem 1.3 and Theorem 2.1 slightly diverge now and are presented in two separated subsections.
2.1. Proof of Theorem 2.3. Split $\|B(f_1, f_2)\|_2^2$ into $B_1 + B_2$ according to the value of $|\tau|$ as below:

- Case 1. $|\tau| \leq |\lambda|^{-1/3}$.
- Case 2. $|\tau| \geq |\lambda|^{-1/3}$.

In Case 1, we simply move the absolute value into the integrals, which yields

$$B_1 \leq \int_{|\tau| \leq |\lambda|^{-1/3}} \|F_\tau\|_1 \|G_\tau\|_1 d\tau \leq |\lambda|^{-1/3} \|f_1\|_2^2 \|f_2\|_2^2 = |\lambda|^{-1/3}. \tag{2.15}$$

In Case 2, in order to employ Theorem 1.1 to the inner double-integral, we assume (2.16) for a moment in the support of $a_\tau$, the following holds for some positive constant $C$:

$$|\partial_x \partial_y S_\tau(x, y)| \geq C|\tau|. \tag{2.16}$$

By Theorem 1.1, $B_2$ is dominated by

$$\int_{|\tau| \geq |\lambda|^{-1/3}} C|\lambda\tau|^{-1/2} \|F_\tau\|_2 \|G_\tau\|_2 d\tau \tag{2.17}$$

$$\leq C|\lambda|^{-1/3} \|F_\tau\|_2 \|G_\tau\|_2 d\tau \tag{2.18}$$

$$\leq C|\lambda|^{-1/3} \left( \int \|F_\tau\|_2^2 d\tau \cdot \int \|G_\tau\|_2^2 d\tau \right)^{1/2} \tag{2.19}$$

$$= C|\lambda|^{-1/3} \|f\|_2^2 \|g\|_2^2 \leq C|\lambda|^{-1/3} \tag{2.20}$$

Thus

$$\|B(f, g)\|_2^2 = B_1 + B_2 \leq C|\lambda|^{-1/3}. \tag{2.10}$$

It remains to verify (2.10) on the support of (2.13). Set $F(t) = S_{xy}(x-t, y+t)$, then

$$|F'(t)| = |(\partial_x - \partial_y)\partial_x \partial_y S(x-t, y+t)|. \tag{2.21}$$

By the mean value theorem, one has for some $t_0$ between 0 and $\tau$, s.t.

$$|\partial_x \partial_y S_\tau(x, y)| = |F(0) - F(\tau)| = |\int_0^\tau F'(t) dt| = |\tau||F'(t_0)|. \tag{2.22}$$

Notice that $(x, y) \in \text{supp}(a)$ and $(x - \tau, y + \tau) \in \text{supp}(a)$, then by convexity $(x - t_0, y + t_0) \in \text{Conv}(\text{supp}(a))$. Therefore, (2.13), (2.22) and (2.21) yield (2.16).

2.2. Proof of Theorem 2.1. Similarly, we split $\|B(f_1, f_2)\|_2^2$ into $B_1 + B_2$ according to the value of $|\tau|$ as below:

- Case 1. $|\tau| \leq |\lambda\mu|^{-1/3}$.
- Case 2. $|\tau| \geq |\lambda\mu|^{-1/3}$.

In Case 1, we simply move the absolute value into the integrals and thus

$$B_1 \leq \int_{|\tau| \leq |\lambda\mu|^{-1/3}} \|F_\tau\|_1 \|G_\tau\|_1 d\tau \leq |\lambda\mu|^{-1/3} \|f_1\|_2^2 \|f_2\|_2^2 = |\lambda\mu|^{-1/3}. \tag{2.23}$$

In Case 2, assume at a moment that (2.23) is true for $a_\tau$ and (2.4) are true for $S_\tau$ with $\mu$ replaced by $|\lambda\mu|$. Then Theorem 2.2 implies

$$B_2 \leq C \int_{|\tau| \geq |\lambda\mu|^{-1/3}} |\lambda\mu\tau|^{-1/2} \|F_\tau\|_2 \|G_\tau\|_2 d\tau \leq C|\lambda\mu|^{-1/3}. \tag{2.24}$$
It remains to verify the conditions mentioned above. Indeed (2.3) follows by
\(a(x, y) = a(x, y) + a(x, y + y')\). Theorem 1.4 satisfies the first part of (2.4) with \(\mu\) replaced by \(|\mu|\) due to (2.2), (2.21), (2.22) and the convexity assumption in theorem 2.1. If we set
\[ F_1(t) = \partial_x \partial_y^2 S(x - t, y + t) \]
and
\[ F_2(t) = \partial_x \partial_y^3 S(x - t, y + t) \]
then the second part of (2.4) (with \(\mu\) replaced by \(|\mu|\)) follows from (2.2), (2.21), (2.22) (with \(F\) replaced by \(F_1\) and \(F_2\)) and convexity.

3. An algorithm for resolution of Singularities \(\mathbb{R}^2\)

In order to employ Theorem 2.1 to attack Theorem 1.4, one needs to decompose \(\text{supp}(a)\) into regions such that \(P(x, y)\) is well-behaved, where \(P = \partial_x \partial_y (\partial_x - \partial_y) S\).

Ideally, one hopes \(P(x, y)\) to behave like a monomial with an ignorable perturbation. The algorithm is driven by this idea. In each stage of iteration, ‘good’ regions (with the desired property) are obtained via vertices and edges of the Newton polyhedron of \(P\). The vertices and the edges are called the faces of the Newton polyhedron. We use \(\mathcal{F}(P)\) to denote all the faces, including non-compact faces. The vertices and the edges are called the faces of the Newton polyhedron. We use \(\mathcal{F}(P)\) to denote all the faces, including non-compact faces.

The Euler formula gives
\[ \#\mathcal{V}(P) - \#\mathcal{E}(P) = 1. \]
For each \(E \in \mathcal{V}(P)\), define \(P_E\) as the restriction of \(P\) in \(E\):
\[ P_E(x, y) = \sum_{(p, q) \in E} c_{p, q} x^p y^q. \]
The monomial \(P_E(x, y)\) is defined similarly for \(V \in \mathcal{V}(P)\).

Choose a vertex \((p_v, q_v) = V \in \mathcal{V}(P)\), then \(V\) lies in two edges: \(E_l\) and \(E_r\), where \(E_l\) is left to \(V\) and \(E_r\) is right to \(V\). Assume the slopes of \(E_l\) and \(E_r\) are \(-1/m_l\) and \(-1/m_r\), then \(0 \leq m_l < m_r \leq \infty\). Consider the region \(|y| \sim |x|^m\) in the following three cases:

Case (1). \(m_l < m < m_r\), Case (2). \(m = m_l\) and Case (3). \(m = m_r\), which corresponds to:
(1) the vertex \(V\) ‘dominates’ \(P(x, y)\), (2) the edge \(E_l\) ‘dominates’ \(P(x, y)\) and (3) the edge \(E_r\) ‘dominates’ \(P(x, y)\) respectively.

Case (2) and Case (3) are exactly the same and only Case (2) is discussed here.
In Case (1), $p_v + mq_v < p + mq$ for any other $(p, q) \in \mathcal{V}(P)$. Thus in the region $|y| \sim |x|^m$ and $|x|$ sufficiently small,

$$P_V(x, y) = c_{p_v, q_v} x^{p_v} y^{q_v} \sim x^{p_v + mq_v}$$

is the dominant term in $P(x, y)$, since $P(x, y) - P_V(x, y) = O(x^{p_v + mq_v + \nu})$ has a higher $x$ degree, which can be viewed as an error term. Thus

$$(3.3) \quad P(x, y) \sim P_V(x, y) = c_{p_v, q_v} x^{p_v} y^{q_v}.$$

![Newton Polyhedron](image)

**Case (1): The vertex $V_2$ is dominant, where $1/2 \leq m \leq 2$**

$$P(x, y) = x^5 y - x^3 y^2 + xy^4$$

$$P_{V_2}(x, y) = -x^3 y^2$$

$$|x|^2 \lesssim |y| \lesssim |x|^{1/2}$$

**Figure 1.**

Case (2) $m = m_l$ is more complicated, we see $p_v + mq_v = p + mq$ for all $(p, q) \in E_l$ and $p_v + mq_v < p + mq$ for all $(p, q) \notin E_l$. Then for all $(p, q) \in E_l$, $x^p y^q \sim x^{p_v} y^{q_v}$ in the region $|y| \sim |x|^m$ and thus

$$P_{E_l}(x, y) \sim x^{p_v} y^{q_v} \sim x^{p_v + mq_v}$$

is the dominant term of $P(x, y)$, **unless** there is cancellation inside $P_{E_l}(x, y)!$ We call this is a ‘bad’ situation and it demands most of the work.
Case(2): The edge $V_1V_2$ is dominant, where $m = 1/2$

\[ P(x, y) = x^5y - x^3y^2 + x^2y^4 \]
\[ P_{V_1V_2}(x, y) = -x^3y^2 + x^2y^4 \]

|y| $\sim$ |x|^{1/2}

We shed some light on how to handle the ‘bad’ situation. Set $P_{E_i}(r) = P_{E_i}(1, r)$. Cancellation happens inside $P_{E_i}(x, y)$ if and only if $P_{E_i}(r) = 0$ has non-zero real roots. Each root $r_j$ of $P_{E_i}(r)$ corresponds to a region where $P_{E_i}(x, y)$ vanishes. Via change of variables $x = x'$ and $y = (r_j + y')x^m$, a new function $P'(x', y')$ is obtained. Previous discussion can be then repeated on $P'$. We want to emphasize two points here. Firstly, each root $r_j$ corresponds to a new branch of iteration and thus the iterations have a tree structure (not a linear structure). Secondly, the iteration ends up essentially in finite steps.

3.2. The resolution algorithm Part I: A single step of Partition.

Let $U$ be a sufficiently small neighborhood of 0 in $\mathbb{R}^2$. For simplicity, we restrict our discussion on the right half-plane $x > 0$, since the left half plane can be reduced to this case through change of variables $(x, y) \rightarrow (-x, y)$ and the $y$-axises can be ignored first. We assume $U = \{(x, y) : 0 < x < \epsilon, -\epsilon < y < \epsilon\}$ where $\epsilon$ is sufficiently small. An inductive resolution procedure will be performed on the pair $[U, P]$, where $P$ is an analytic function and $U$ defined above. Let $M \in \mathbb{N}$ be a pre-fixed large constant whose value will be chosen later.

**Definition 3.1.** Given a coordinate $(X, Y)$ and a real analytic function $Q(X, Y)$ of $(X^\frac{1}{M}, Y)$, if $W = \{(X, Y) : 0 < X < \epsilon, -\epsilon < Y < \epsilon\}$, where $\epsilon > 0$ is sufficiently small, then we call $W$ a standard region and $[W, Q]$ is a standard pair under the coordinate $(X, Y)$. In addition, we denote $\text{diam}(W) = \epsilon$, the ‘diameter’ of $W$.

By the definition, $[U, P]$ is a standard pair under the coordinate $(x, y)$ and we set $[U, P] = [U_0, P_0]$ and $(x, y) = (x_0, y_0)$ to indicate the procedure is in the starting stage (0-th stage). It worths mentioning that the algorithm will always perform on a standard region, with different analytic functions of $(x^{1/M}, y)$. Moreover, $\epsilon > 0$ denotes a sufficiently small number whose value may be varied but it is completely
harmless. Consider
\[ P_E(x, y) = \sum_{(p, q) \in E} c_{p, q} x^p y^q, \text{ for } E \in \mathcal{E}(P). \]

Let \( V_{E,l} = (p_{E,l}, q_{E,l}) \) and \( V_{E,r} = (p_{E,r}, q_{E,r}) \in \mathcal{V}(P) \) be the left and right vertices of \( E \). Set \( m_E = \frac{p_{E,l} - p_{E,r}}{q_{E,r} - q_{E,l}} \), then the slope of \( E \) is \(-1/m_E\). The constant \( m_E \) is the most important constant assigned to each edge \( E \). In addition, if we set \( e_E = p_{E,l} + m_E q_{E,l} \) then for all \((p', q') \in E\), we have \( e_E = p' + m_E q' \) and \( p'' + m_E q'' = e_E + \nu \) for some \( \nu > 0 \) and all \((p'', q'') \notin E\). In the curve \( y = rx^{m_E} \) where \( r \in \mathbb{R} \setminus \{0\} \).

\[ P_E(x, y) = x^{s_E} \sum_{(p, q) \in E} c_{p, q} r^q =: x^{s_E} P_E(r) \]

and
\[ P(x, y) - P_E(x, y) = O(x^{s_E + \nu}) \]

has a higher degree. Thus given \(|x|\) sufficiently small, \( P_E(x, y) \) dominates \( P(x, y) \), unless
\[ P_E(r) = \sum_{(p, q) \in E} c_{p, q} r^q \to 0. \]

Now it becomes clear that the nonzero roots of \( P_E(r) \) are the trouble makers and special treatment is demanded. Assume \( \{r_{E,j}\}_{1 \leq j \leq J_E} \) is the set of non-zero roots of \( P_E(r) = 0 \) of orders \( \{s_{E,j}\}_{1 \leq j \leq J_E} \). In addition, we assume the set is labeled in the increasing order. Then,
\[ J_E \leq \sum_{1 \leq j \leq J_E} s_j \leq q_{E,l} - q_{E,r}, \tag{3.4} \]

since \( P_E(r) = r^{q_{E,r}} \sum_{E} c_{p, q} r^{q - q_{E,r}} \). For simplicity, we say \( r_{E,j} \) is a root of \( E \) to represent \( r_{E,j} \) is a root of \( P_E(r) \). Let \( I_j'(E) = (r_{E,j} - \epsilon, r_{E,j} + \epsilon) \), where \( \epsilon > 0 \) is sufficiently small such that
\[ \epsilon < 2^{-10} \cdot \min\{|r_{E,j}| : E \in \mathcal{E}(P), 1 \leq j \leq J_E\}. \]

Choose two constants
\[ 0 < c_E < 2^{-10} |r_{E,j}| < 2^{10} |r_{E,j}| < C_E, \text{ for all } 1 \leq j \leq J_E \]
where \( c_E \) is sufficiently small and \( C_E \) is sufficiently large. Set
\[ I(E) = [c_E, C_E] \cup [-C_E, -c_E], \tag{3.5} \]
\[ I_{b}(E) = \bigcup_{1 \leq j \leq J_E} I_j'(E), \]
\[ I_{g}(E) = I(E) \setminus I_{b}(E). \]

Here \( I_{b}(E) \) represents the neighborhood of the roots \( \{r_{E,j}\} \) and \( I_{g}(E) \) represents the points away from the non-zero roots, 0 and \( \infty \). Then
\[ |P_E(r)| \geq 1 \quad \text{for} \quad r \in I_{g}(E). \tag{3.6} \]

Thus \( P_E(x, y) \) dominates \( P(x, y) \) if \( y = rx^{m_E} \) and \( r \in I_{g}(E) \). Let
\[ U_{0,g}(E) = \{(x, y) \in U_0 : y = rx^{m_E}, r \in I_{g}(E)\} \tag{3.7} \]
be the ‘good’ regions generated by the edge $E$. One can see that $U_{0,9}(E)$ is a disjoint union of $(J_E + 2)$ ‘good’ regions: $U_{0,9}(E, j)$. Each ‘good’ region $U_{0,9}(E, j)$ is a curved triangular region defined as

$$U_{0,9}(E, j) = \{ (x, y) \in U_0 : b_j x^{m_E} \leq y \leq B_j x^{m_E} \},$$

where $[b_j, B_j] := I_g(E, j)$ is just a connected sub-interval of $I_g(E)$ and $(J_E + 2)$ comes from the number of connected components of $I_g(E)$. In the above definition, the subindex 0 in $U_{0,9}(E, j)$ indicates the algorithm is in the 0-th stage, $g$ indicates the region is ‘good’. The ‘bad’ regions are defined as:

$$U_{0,6}(E, j) = \{ (x, y) \in U_0 : y = r x^{m_E}, r \in I_j(E) \}$$

$$= \{ (x, y) \in U_0 : (r_j - \epsilon) x^{m_E} < y < (r_j + \epsilon) x^{m_E} \},$$

for $1 \leq j \leq J_E$. If $\{r_E, j\}$ is empty, then there is no ‘bad’ region generated by this edge and the only two ‘good’ regions are

$$U_{0,9}(E) = \{ (x, y) \in U_0 : c_E x^{m_E} < |y| < C_E x^{m_E} \}.$$

The following lemma states that $P$ behaves almost like a monomial in each ‘good’ region $U_{0,9}(E, j)$.

**Lemma 3.1.** Given any positive integers $N$ and $L$, assume $|x|$ and $\epsilon$ sufficiently small (depends on $N$, $L$ and $P$), then for all $E \in \mathcal{N}(P)$ and $(x, y) \in U_{0,9}(E, j)$, one has

$$|x^{pE,i} y^{qE,i} | \sim |P_E(x, y)| \geq 2^N |P(x, y) - P_E(x, y)|.$$  

Here $(pE,i, qE,i)$ is the vertex of the edge $E$. In addition,

$$|\partial_x^\alpha \partial_y^\beta P(x, y)| < C \min \{ 1, |x^{pE,i-\alpha} y^{qE,i-\beta}| \}$$

for $0 \leq \alpha, \beta \leq L$.

**Proof.** In the region $y = r x^{m_E}$ where $r \in I(E)$,

$$|P(x, y) - P_E(x, y)| < C x^{eE+\nu},$$

where $\nu$ is a positive fraction (can be computed but not necessary). By (3.8), one has $|P_E(r)| \geq C$ for $r \in I_g(E)$, where $C = C(c_E, C_E, \epsilon, P)$ is a positive constant. Thus if $|x|$ is sufficiently small, then for all $(x, y) \in U_{0,9}(E, j)$ we have

$$|P_E(x, y)| \sim |x^{pE,i} y^{qE,i} | \sim x^{eE} \gtrsim 2^N \cdot O(x^{eE+r}) > 2^N |P(x, y) - P_E(x, y)|,$$

which proves (3.12).

Now we turn to (3.13). The bound $|\partial_x^\alpha \partial_y^\beta P(x, y)| \lesssim 1$ is trivial. In the region $y = r x^{m_E}$ where $r \in I(E)$, for $0 \leq \alpha, \beta \leq L$ and every $(p'', q'') \in E$, one has

$$|x^{pE,i-\alpha} y^{qE,i-\beta} | \sim |x|^{p''-\alpha} |y|^{q''-\beta} \sim |x|^{eE-a} \cdot \tilde{P}_E(x, y),$$

even for $p'' - \alpha < 0$ or $q'' - \beta < 0$. Notice $|y| \sim |x|^{m_E}$, then

$$|\partial_x^\alpha \partial_y^\beta (P(x, y) - P_E(x, y))| \lesssim |x|^{eE+\nu-a} \cdot \tilde{P}_E(x, y).$$

Thus given $|x|$ sufficiently small, one has

$$|\partial_x^\alpha \partial_y^\beta P(x, y)| \lesssim |x^{pE,i-\alpha} y^{qE,i-\beta}|.$$

This completes the the proof of (3.13).
The above lemma handled the case when an edge $E$ is ‘dominant’ and no cancellation inside $P_E$. Another easy case is when a vertex $V = (p_0, q_0)$ plays a dominant role. In this case, let $E_l$ and $E_r$ be the edges left and right to $V$, with slopes $-1/m_{E_l}$ and $-1/m_{E_r}$ respectively. Then $0 \leq m_{E_l} < m_{E_r} \leq \infty$. Here $m_{E_l} = 0$ means $E_l$ is the vertical non-compact edge and $m_{E_r} = \infty$ means $E_r$ is the horizontal non-compact edge. Consider the following region

$$U_{0, g}(V) = \{(x, y) \in U_0 : C_{E_r} x^{m_{E_r}} < |y| < c_{E_l} x^{m_{E_l}} \},$$

where $C_{E_r}$ and $c_{E_l}$ are constants defined above. We can always choose $\text{diam}(U_0)$ sufficiently small (i.e., $|x|$ sufficiently small) s.t. the origin $(0, 0)$ is the only interception of $y = C_{E_r} x^{m_{E_r}}$ and $y = c_{E_l} x^{m_{E_l}}$ inside $U_0$. If $m_{E_r} = \infty$, then $V$ is the most right vertex, we set

$$U_{0, g}(V) = \{(x, y) \in U_0 : |y| < c_{E_l} x^{m_{E_l}} \},$$

where the portion of the $x$-axis inside $U_0$ is included in $U_{0, g}(V)$. Similarly, if $m_{E_l} = 0$ then $V$ is the most left vertex. We replace $c_{E_l} x^{m_{E_l}}$ by $\epsilon$ in (3.14).

The following lemma is similar to Lemma 3.1. The proof is exactly the same and we omit the details.

**Lemma 3.2.** For each $V \in \mathcal{V}(P)$, suppose $C_{E_r} > 0$ is sufficiently large and $c_{E_l} > 0$ is sufficiently small. Given positive numbers $N$ and $L$, assume $|x|$ and $\epsilon$ sufficiently small (depends on $N$, $L$, $C_{E_r}$, $c_{E_l}$ and $P$), then for $(x, y) \in U_{0, g}(V)$ one has

$$|x^{p_0} y^{q_0}| \sim |P_V(x, y)| \geq 2^N |P(x, y) - P_V(x, y)|$$

and

$$|\partial_x^\alpha \partial_y^\beta P(x, y)| < C \min\{1, |x^{p_0 - \alpha} y^{q_0 - \beta}|\}$$

for $0 \leq \alpha, \beta \leq L$.

Set

$$\mathcal{G}_0(P_0) = \mathcal{G}_0(P) = \{U_{0, g}(V) : V \in \mathcal{V}(P)\} \cup \{U_{0, g}(E, j) : E \in \mathcal{E}(P) \text{ and all } j\},$$

which represents the collection of ‘good’ regions in the 0-th stage. In addition, we say $U_{0, g} \in \mathcal{G}_0(P_0)$ is defined by $(E, m_E)$ if $U_{0, g} = U_{0, g}(E, j)$ for some $E$ and $j$, where $-1/m_E$ is the slope of $E$, or defined by an edge $E$ for short. Similarly, $U_{0, g} \in \mathcal{G}_0(P_0)$ is defined by $(V, m_l, m_r)$ represents $U_{0, g} = U_{0, g}(V)$ and $-1/m_l$, $-1/m_r$ are the slopes of the edges left and right to $V$, or defined by a vertex $V$ for short.

Now we focus on the ‘bad’ regions

$$U_{0, b}(E, j) = \{(x, y) \in U_0 : (r_j - \epsilon) x^{m_E} < y < (r_j + \epsilon) x^{m_E}\}$$

and set

$$\mathcal{B}_0(P_0) = \mathcal{B}_0(P) = \{U_{0, b}(E, j) : E \in \mathcal{E}(P) \text{ and } 1 \leq j \leq J_E\}$$

to represent the collection of ‘bad’ regions in the 0-stage. If $U_{0, b} \in \mathcal{B}_0(P_0)$ has the form of (3.19), we say $U_{0, b}$ is defined by $(E, y = r_j x^{m_E})$ or defined by $y = r_j x^{m_E}$ for short. The following graph demonstrates a partition of $U$ into ‘good’ and ‘bad’ regions, according to the analytic function $P(x, y) = xy(y^2 - x(y - x^2)(y - \sum_{n=1}^{\infty} x^n)^2)$. 


We summarize the above discussion as follows:

**Proposition 3.3 (A Single step of Partition).**

Let $U$ be a standard region and $P$ be a real analytic function. If \( \text{diam}(U) \) is sufficiently small, then $U$ can be partitioned into two families of curved triangular regions: \( G_0(P) \) and \( B_0(P) \). For each $U_{0,g} \in G_0(P)$, $U_{0,g}$ is defined by (3.8) or (3.14). The behaviors of $P$ in $U_{0,g}$ are characterized by Lemma 3.1 or Lemma 3.2. Each $U_{0,b} \in B_0(P)$ is defined by (3.16). Finally, the cardinalities of $G_0(P)$ and $B_0(P)$ are finite, depending on $P$.

### 3.3. The resolution algorithm Part II: Iterations.

The next step is to iterate Proposition 3.3. One main problem is that $U_{0,b} \in B_0(P)$ is not a standard region. Nevertheless, this difficulty can be overcome by a “rescaling” argument. Via an appropriate change of variables, we can always turn a non-standard pair $[U_{0,b}, P_0]$ to a standard pair $[U_1, P_1]$. Here $P_0 = P$ and $P_1$ is an analytic function of $(x^{1/M}, y)$. Then Proposition 3.3 is applicable to $[U_1, P_1]$ (the arguments in the previous subsection work equally well for analytic function of $(x^{1/M}, y)$). Bad regions obtained from $[U_1, P_1]$ can be rescaled to standard regions, where Proposition 3.3 can be applied again and so on.

The following graph illustrates the main ideas of how the algorithm runs. The letter ‘g’ bellows represents a ‘good’ region while ‘b’ represents a bad region. Each
time, we pick up a ‘bad’ region, ‘rescale’ (via change of variable) it into a standard region.

Before diving into the details, we introduce the following notations to characterize some invariances inside each stage of iteration.

**Definition 3.2.** Let \((p_l, q_l)\) and \((p_r, q_r)\) be the most left and rightest vertices of \(N(P)\), the Heights of \(N(P)\) or \(P\) are defined as

\[
\text{Hght}(N(P)) = \text{Hght}(P) = q_l - q_r,
\]

\[
\text{Hght}^\ast(N(P)) = \text{Hght}^\ast(P) = q_l.
\]

For an edge \(E \in \mathcal{E}(P)\), let \((p_{E,l}, q_{E,l})\) and \((p_{E,r}, q_{E,r})\) be its left and right vertices. Then the height of this edge is defined as

\[
\text{Hght}(E) = q_{E,l} - q_{E,r}.
\]

If \(\{r_{E,j}\}_{1 \leq j \leq J_E}\) is the set of non-zero roots of \(P_E(r)\) of orders \(\{s_{E,j}\}_{1 \leq j \leq J_E}\), then we define the order of \(E\) as

\[
\text{Ord}(E) = \sum_{j=1}^{J_E} s_{E,j}.
\]

and the order of \(P\) as

\[
\text{Ord}(P) = \sum_{E \in \mathcal{E}(P)} \text{Ord}(E) = \sum_{E \in \mathcal{E}(P)} \sum_{j=1}^{J_E} s_{E,j}.
\]

Finally, we say \(r\) is a root of \(P(x,y)\) or \(N(P)\) if \(r = r_{E,j}\) for some \(E \in \mathcal{E}(P)\) and some \(1 \leq j \leq J_E\).
The above definition immediately implies

\[(3.23) \quad \text{Ord}(E) \leq \text{Hght}(E) = q_{E,l} - q_{E,r}\]

and

\[(3.24) \quad \text{Ord}(P) \leq \text{Hght}(P) = q_l - q_r \leq \text{Hght}^*(P).\]

Since \(\#B_0(P) = \sum_{E \in E(P)} J_E \leq \sum_{E \in E(P)} \text{Ord}(E) = \text{Ord}(P),\) we obtained:

**Lemma 3.4.**

\[\#B_0(P) \leq \text{Ord}(P) \leq \text{Hght}^*(P).\]

Choose a \([U_0, b] \in B_0(P_0)\) and assume it is defined by \(y = r_0 x^{m_0}\). The next step is to utilize change of variables to turn \([U_0, b], P\) into a standard pair. Adopt the previous notations \([U_0, P_0] = [U, P]\) and \((x_0, y_0) = (x, y)\) and choose \(x\) to be the principal variable which will be unchanged during the iterations, i.e. \(x = x_n\) for all \(n \in \mathbb{N}\). Change variables

\[
\begin{cases}
  x_0 = x_1 \\
  y_0 = (r_0 + y_1) x_1^{m_0}
\end{cases}
\]

and set

\[
\begin{align*}
  &P_1(x_1, y_1) = P(x_1, (r_0 + y_1) x_1^{m_0}) \\
  &U_1 = \{(x_1, y_1) : (x_0, y_0) \in U_0, b\}.
\end{align*}
\]

Notice for \((x_1, y_1) \in U_1\), one has

\[
\begin{cases}
  0 < x_1 < \epsilon \\
  -\epsilon < y_1 < \epsilon.
\end{cases}
\]

Then \([U_1, P_1]\) is a standard pair under the coordinate \((x_1, y_1)\). By applying Proposition 3.3 to \([U_1, P_1]\), a finite collection \(G_1(P_1)\) of ‘good’ regions \(U_{1,g}\)'s and a finite collection \(B_1(P_1)\) of ‘bad’ regions \(U_{1,b}\)'s are obtained. In a ‘good’ region \(U_{1,g}\), the function \(P_1(x_1, y_1)\) behaves like a monomial of \((x_1, y_1)\) and no further treatment is required. For the ‘bad’ regions, choose a \([U_1, b] \in B_1(P_1)\) and assume \(U_{1,b}\) is defined by \(y_1 = r_1 x_1^{m_1}\), i.e.

\[U_{1,b} = \{(x_1, y_1) \in U_1 : (r_1 - \epsilon) x_1^{m_1} < y_1 < (r_1 + \epsilon) x_1^{m_1}\}.
\]

Like what has been done, we perform the following change of variables

\[
\begin{cases}
  x_1 = x_2 \\
  y_1 = (r_1 + y_2) x_2^{m_1}.
\end{cases}
\]

Then a new standard pair \([U_2, P_2]\) is obtained:

\[
\begin{align*}
  &U_2 = \{(x_2, y_2) : (x_1, y_1) \in U_1, b\} \\
  &P_2(x_2, y_2) = P_1(x_2, (r_1 + y_2) x_2^{m_1}).
\end{align*}
\]

Then same procedure is repeated on \([U_2, P_2]\) and so on. A collection of standard pairs \([U_n, P_n]\) is obtained from these iterations. Here the subindex \(n\) merely represents \([U_n, P_n]\) is obtained from the \(n\)-th stage of iteration (or we say a \(n\)-th generation of \([U_0, P_0]\)). Notice that for a \(n\), there can be many \([U_n, P_n]\) and the structure of \([U_n, P_n]\) is a tree (non-linear). If we want to specify the ‘identity’ of \([U_n, P_n]\), set \([U_n, P_n] = [U_n, P_{n, \alpha}]\) where the subindex \(\alpha\) represents the ‘path’ from \([U_0, P_0]\) to \([U_n, P_{n, \alpha}]\). The subindex \(\alpha\) can also be viewed as the code that compresses the genealogy information which is needed to obtain \([U_n, \alpha, P_{n, \alpha}]\) from
[\{U_0, P_0\}] or conversely, \(P_0(x_0, y_0)\) can be ‘decoded’ from \(P_{n, \alpha}(x_n, y_n)\) by \(\alpha\). More precisely:

\((\star)\) \(\alpha\) contains the information of the changes of variables, i.e. for \(0 \leq k \leq n - 1\) the following is known if \(\alpha\) is given:

\[
\begin{align*}
x_k &= x_{k+1} \\
y_k &= (r_k + y_{k+1})x_{k+1}^{m_k}.
\end{align*}
\]

We also use \(U_{n,b,\alpha}\) and \(U_{n,g,\alpha}\) to represent an arbitrary ‘bad’ and ‘good’ regions in \(U_{n,\alpha}\). Since \(U_{n,\alpha}\) may have more than one such regions, we list them by \(U_{n,b,\alpha,j}\) and \(U_{n,g,\alpha,j}\) when necessary. The cardinality of \(j\) is uniformly bounded (see below) and there is no need to specify its range. Notice that \(U_{n,g,\alpha,j}\) is a leaf, i.e. it has no child and no further analysis is needed.

Both notations: with and without subindex \(\alpha\), are being used. In order not to confuse the reader, we follow the rules below:

1. \([U_n, P_n]\) is our priority choice, in particular for an arbitrary pair from the \(n\)-th stage of iteration.
2. \([U_{n,\alpha}, P_{n,\alpha}]\) is a secondary choice, it is often employed when at least two different pairs from the same stage of iteration appear simultaneously.

The above conventions also apply to \(U_{n,g}\)’s, \(U_{n,b}\)’s, \(U_{n,g,\alpha}\) and \(U_{n,b,\alpha}\).

**Example 1.** The following graphs demonstrate the first step of the algorithm, for the given analytic function \(P(x, y) = xy(y^2-x)(y-x^2)(y-\sum_{n=1}^{\infty} x^n)^2\). The Newton polyhedron \(N(P)\) has 3 compact edges: \(E_1, E_2\) and \(E_3\), see Figure A below.
When \( P \) is restricted in the edge \( E_1 \), \( P_{E_1} = xy(y^2 - x)y^2 \). The only non-zero root is \( y = x^\frac{1}{2} \). Change of variables: \((x, y) = (x_1, x_1^{\frac{1}{2}}(1 + y_1))\) yields \( P_1(x_1, y_1) = P(x_1, x_1^{\frac{1}{2}}(1 + y_1)) = x_1^3y_1 \cdot O(1) \). See Figure B, \( \mathcal{N}(P_1) \) has only one vertex and the algorithm stops.

![Figure B](image)

When \( P \) is restricted in the edge \( E_2 \), \( P_{E_2} = -x^2y^2(y - x)^2 \). The only non-zero root is \( y = x \). Change of variables: \((x, y) = (x_1, x_1(1 + y_1))\) yields \( P_1(x_1, y_1) = x_1^6(y_1 - \sum_{n=1}^{\infty} x^n)^2 \cdot O(1) \). See Figure C, \( \mathcal{N}(P_1) \) has one edge. The algorithm still runs. If we keep doing the change of variables \((x_{k-1}, y_{k-1}) = (x_k, x_k(1 + y_k))\), we can see that \( \mathcal{N}(P_k) = \mathcal{N}(P_{k-1}) + (2, 0) \) for \( k \geq 2 \).

![Figure C](image)
Finally, $P_E = -x^4y(y - x^2)$. The only non-zero root is $y = x^2$. Change of variables: $(x, y) = (x_1, x_1^2(1 + y_1))$ yields $P_1(x_1, y_1) = x_1^6y_1 \cdot O(1)$. See Figure D, $N(P_1)$ has only one vertex and the algorithm stops.

It is worth mentioning that, the change of variables: $(x_n, y_n) \rightarrow (x_{n+1}, y_{n+1})$ acts as a diffeomorphism from $U_{n,b}$ to $U_{n+1}$. Thus one can diffeomorphically embed $U_{n+1}$ into $U_n$ and a chain of diffeomorphic embeddings is obtained:

$$\cdots \rightarrow U_{n+2} \rightarrow U_{n+1} \rightarrow U_n \rightarrow \cdots \rightarrow U_1 \rightarrow U_0.$$

In addition, if the change of variables: $(x, y) \rightarrow (x_n, y_n)$ is specified, then we are legal to identify $(x, y) \in U_n$ (or $U_{n,b}, U_{n,g}$) with $(x_n, y_n) \in U_n$ (or $U_{n,b}, U_{n,g}$). To be precise, there is a diffeomorphism $\rho_n^{-1}$:

$$\begin{align*}
\rho_n^{-1} : U_n &\rightarrow \rho_n^{-1}(U_n) \subset U_0 \\
(x_n, y_n) &\rightarrow (x, y)
\end{align*}$$

where $(3.27)$ is defined by the composition of change of variables $(3.25)$. More precisely, $(x, y) = \rho_n^{-1}(x_n, y_n)$ means

$$\begin{align*}
x = x_n \\
y = r_0x^{m_0} + r_1x^{m_0+m_1} + \cdots + r_{n-1}x^{m_0+\cdots+m_{n-1}} + y_nx^{m_0+\cdots+m_{n-1}}.
\end{align*}$$

Under this notation, $P_n = P \circ \rho_n^{-1}$. If $U_n$ is specified to $U_{n,\alpha}$, then $\rho_n^{-1}$ is also specified to $\rho_{n,\alpha}^{-1}$. Notice that for all $j$, $U_{n,g,\alpha,j}$'s and $U_{n,b,\alpha,j}$'s are sharing the same $\rho_{n,\alpha}^{-1}$ with $U_{n,\alpha}$. In particular, $
(\rho_{n,\alpha}^{-1}(U_{n,g,\alpha,j}))_{n,\alpha,j}$ are disjoint curved triangular regions in $U_0$. It will become clear later, $(\rho_{n,\alpha}^{-1}(U_{n,g,\alpha,j}))_{n,\alpha,j}$ will form a finite disjoint partition of $U_0$. 

Figure D
The reader may now have a clear picture of how this resolution algorithm runs. Still, there are two questions need to be answer:

(i) In each stage of iteration, is the cardinality of \( \{U_{n,b,\alpha,j}\}_{\alpha,j} \) bounded above uniformly?
(ii) Does this procedure end up in a finite steps?

The answer to the first question is Yes and the upper bound can be controlled by \( \text{Ord}(P) \); to (ii), the answer is still Yes, but a refinement of change of variables is needed!

We provide the solution to (i) first.

\[ \text{Lemma 3.5. For each } n \geq 0, \text{ the cardinality of } \{U_{n,b,\alpha,j}\}_{\alpha,j} \text{ is bounded by } \text{Ord}(P). \]

\[ \text{Proof.} \text{ Indeed, there is a bijection between } \{U_{0,b}\} \text{ and the non-zero roots of } P_E(r) \text{ with } E \in \mathcal{E}(P): \text{ each } U_{0,b} = U_{0,b}(E,j) \text{ is defined by } (E, y = r_{E,j}x^{m_E}). \text{ Assume the order of } r_{E,j} \text{ is } s_{E,j}. \text{ Then } P_1(x_1, y_1) = P_{1,E,j}(x_1, y_1) \text{ is obtained by setting } P_1(x_1, y_1) = P_0(x_1, (y_1 + r_{E,j})x^{m_E}). \text{ Here, } P_{1,E,j} \text{ is used to specify that } P_1 \text{ is defined by the root } r_{E,j}. \text{ The following observation serves as an bridge between } \mathcal{N}(P_0) \text{ and } \mathcal{N}(P_1). \text{ Let } (p_{E,l}, q_{E,l}) \text{ be the left vertex of } E \text{ and } (p_{1,l}, q_{1,l}) \text{ be the most left vertex of } \mathcal{N}(P_1) \text{ then}
\]
\[ \begin{cases} p_{1,l} = p_{E,l} + m_E \cdot q_{E,l} \\ q_{1,l} = s_{E,j} \end{cases} \]

This implies the order \( s_{E,j} \) (in the 0-stage) is equal to the \( \text{Hght}^\ast(P_{1,E,j}) \) (in the 1-stage). To prove \[ (3.29) \]

\[ P_E(x, y) = P_E(x_1, (y_1 + r_{E,j})x^{m_E}) = x_1^{p_{E,l} + m_E \cdot q_{E,l}} y_1^{s_{E,j}} \cdot O(1). \]

Indeed, the fact that the degree of \( y_1 \) is \( s_{E,j} \) follows from the fact that \( r_{E,j} \) is a root of \( P_E(r) \) of order \( s_{E,j} \). Moreover, every term in

\[ P_1(x_1, y_1) - P_E(x_1, (y_1 + r_{E,j})x^{m_E}) \]

has a \( x_1 \)-degree strictly greater than \( (p_{E,l} + m_E \cdot q_{E,l}, s_{E,j}) \). Thus \( (p_{E,l} + m_E \cdot q_{E,l}, s_{E,j}) \) is the most left vertex of \( \mathcal{N}(P_1) \).

Immediately, one obtains \( \text{Ord}(P_{1,E,j}) = \text{Ord}(P_1) \leq \text{Hght}^\ast(P_1) = s_{E,j} \). Thus the number of ‘bad’ regions \( U_{1,b} \)’s coming from a single \( P_1 \) is no more than \( \text{Ord}(P_1) \leq s_{E,j} \). Counting all possible \( P_1 \) (coming from different roots of different edges), the number of all possible \( U_{1,b} \) is thus no more than

\[ \begin{align*}
(3.30) \sum_{E \in \mathcal{E}(P)} \sum_{1 \leq j \leq |E|} \text{Ord}(P_{1,E,j}) & \leq \sum_{E \in \mathcal{E}(P)} \sum_{1 \leq j \leq |E|} s_{E,j} = \text{Ord}(P).
\end{align*} \]

The cases when \( n \geq 2 \) also follow from this iteration formula \[ (3.30) \].

We now turn to the second question, which is the most crucial part of the algorithm. Assume the procedure does not stop. Thus we obtain an infinite chain of pairs:

\[ \begin{align*}
(3.31) \quad [U_0, P_0] & \rightarrow [U_1, P_1] \rightarrow [U_2, P_2] \rightarrow \cdots \rightarrow [U_n, P_n] \rightarrow [U_{n+1}, P_{n+1}] \rightarrow \cdots
\end{align*} \]

We shall find certain stable pattern inside the above chain. Specify the change of variables from \([U_n, P_n] \rightarrow [U_{n+1}, P_{n+1}] \) as

\[ \begin{align*}
\begin{cases}
 x_n = x_{n+1} \\
y_n = (r_n + y_{n+1})x^{m_n}_{n+1}
\end{cases}
\end{align*} \]
Then \( r_n \) is a root of an edge in \( \mathcal{N}(P_n) \). We assume \( s_n \) is the order of \( r_n \). Let \((p_n,l,q_n,l)\) be the most left vertex of \( \mathcal{N}(P_n) \) and \((p_n,q_n)\) be the left vertex of the edge in \( \mathcal{N}(P_n) \) that defines \([U_{n+1},P_{n+1}]\). By (3.39) one has
\[
\begin{cases}
p_{n+1} \geq p_n + l = p_n + m_n \cdot q_n \\
q_{n+1} \leq q_n + l = s_n \leq q_n
\end{cases}
\]
and thus
\[
\text{Hght}^*(P_0) \geq \text{Hght}(P_0) \geq s_0 = \text{Hght}^*(P_1) \geq \text{Hght}(P_1) \geq s_1 = \text{Hght}^*(P_2) \geq \cdots \geq s_{n-1} = \text{Hght}^*(P_n) \geq \text{Hght}(P_n) \geq s_n = \text{Hght}^*(P_{n+1}) \cdots
\]
Notice that for all \( n \), \( \text{Hght}(P_n) \) and \( s_n \) must be positive integers. Otherwise, if \( \text{Hght}(P_n) = 0 \) then \( \mathcal{N}(P_n) \) has no edge and thus no root; if \( s_n = 0 \), then \( \mathcal{N}(P_n) \) has no root. In both situations, the chain ends at the \( n \)-stage, which contradicts to our assumption.

Since (3.33) is an infinite sequence and \( \text{Hght}^*(P_0) \) is a finite positive number, there is a least integer \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \) one has
\[
\text{Hght}^*(P_n) = \text{Hght}(P_n) = s_n = \text{Hght}^*(P_{n_0}) = \text{Hght}(P_{n_0}) = s_{n_0} > 0.
\]
This implies for every \( n \geq n_0 \):
(i) \( \mathcal{N}(P_n) \) has only one compact edge \( E_n \),
(ii) in this edge \( E_n, P_n(x_n,y_n) \) has only one root \( r_n \) of order \( s_n = s_{n_0} \),
(iii) when \( P_n \) is restricted in \( E_n, P_n(x_n,y_n) = c_n(y_n - r_n x_n^{m_n})^{s_n} \), for some non-zero constant \( c_n \).
This is the exact pattern we are looking for. The following lemma shows that the chain (3.31) essentially ends up at the \((n_0 + 1)\)-stage.

**Lemma 3.6.** Assume we have an infinite chain (3.31) and \( n_0 \) is the constant defined in (3.36), then
\[
P_{n_0}(x_{n_0},y_{n_0}) = x_{n_0}^{p_{n_0}}(y_{n_0} - f(x_{n_0}))^{s_{n_0}}Q_{n_0}(x_{n_0},y_{n_0})
\]
where
\[
f(x_{n_0}) = \sum_{n=n_0}^{\infty} r_n x_{n_0}^{m_{n_0} + m_{n_0+1} + \cdots + m_n}
\]
is an analytic function of \( x_{n_0}^{1/M} \) and \( Q_{n_0}(x_{n_0},y_{n_0}) \) is an analytic function of \( (x_{n_0}^{1/M},y_{n_0}) \) with \( Q_{n_0}(0,0) \neq 0 \), where \( M \) is a large integer depending on \( P \).

**Proof.** To obtain \( P_{n_0}(x_{n_0},y_{n_0}) \) from \( P_0(x,y) \), we have only iterated finite steps. Thus \( P_{n_0}(x_{n_0},y_{n_0}) \) is a real analytic function of \((x_{n_0}^{1/M},y_{n_0})\), for some large integer \( M \). For \( n \geq n_0 \), the change of variables from \([U_n,P_n]\) to \([U_{n+1},P_{n+1}]\) is \( x_n = x_{n+1} \) and \( y_n = (y_{n+1} + r_n)x^{m_n} \). The only compact edge \( E_n \) of \( P_n \) is of the form
\[
P_{n,E_n}(x_n,y_n) = c_n x_n^{p_n}(y_n - r_n x_n^{m_n})^{s_n},
\]
where \( c_n \) is a nonzero constant and \( s_n = s_{n_0} \). Using induction, it is not difficult to prove that \( m_n M \) is an integer for all \( n \geq n_0 \). Thus \( P_n(x_n,y_n) \) is a real analytic function of \((x_n^{1/M},y_n)\) for all \( n \in \mathbb{N} \). Notice \( (p_n + s_n m_n,0) \) is the rightest vertex of \( \mathcal{N}(P_n) \), by setting \( y_n = 0 \), (3.38) yields
\[
P_n(x_n,0) = C_n x_n^{p_n + s_n m_n} + O(x_n^{p_n + s_n m_n + \nu}),
\]
where $\nu > 0$. Consider the partial sum of $f(x_{n_0})$,

$$(3.40) \quad f_k(x_{n_0}) = \sum_{n=n_0}^{k} r_n x_n^{m_{n_0}+m_{n_0+1}+\cdots+m_n}, \quad k \geq n_0.$$  

Then $y_{n_0} = y_n x_n^{m_{n_0}+m_{n_0+1}+\cdots+m_{n-1}} + f_{n-1}(x_{n_0})$, $n \geq n_0 + 1$. Notice

$$(3.41) \quad P_n(x, y) = P_{n_0}(x, y) x_{n_0}^{m_{n_0}+m_{n_0+1}+\cdots+m_{n-1}} + f_{n-1}(x_{n_0}).$$  

By (3.39), we have

$$(3.42) \quad P_{n_0}(x_{n_0}, f_{n-1}(x_{n_0})) = P_n(x, 0) = C_n x_n^{p_n+s_n m_n} + O(x_n^{p_n+s_n m_n+\nu}).$$  

Notice $m_n \geq \frac{1}{M}$ and $s_n = s_{n_0}$ is a positive integer, thus

$$p_n + s_n m_n \to \infty \quad \text{as} \quad n \to \infty,$$

and

$$(3.43) \quad P_{n_0}(x_{n_0}, f(x_{n_0})) = 0.$$  

This yields $P_{n_0}(x_{n_0}, y_{n_0})$ has a factor $(y_{n_0} - f(x_{n_0}))$. We still need to show its order $s$ is exactly $s_{n_0}$. Assume,

$$(3.44) \quad P_{n_0}(x_{n_0}, y_{n_0}) = x_{n_0}^{p_{n_0}} (y_{n_0} - f(x_{n_0}))^s Q_{n_0}(x_{n_0}, y_{n_0}),$$  

where $Q_{n_0}(x_{n_0}, f(x_{n_0})) \neq 0$. It is not difficult to see all the terms in (3.44) are analytic functions of $(x_{n_0}^{1/M}, y_{n_0})$. Notice

$$(Q_{n_0}(x_{n_0}, y_{n_0}) = Q_{n_0}(x_{n_0}, f(x_{n_0})) + \left( Q_{n_0}(x_{n_0}, y_{n_0}) - Q_{n_0}(x_{n_0}, f(x_{n_0})) \right)$$

and $(Q_{n_0}(x_{n_0}, y_{n_0}) - Q_{n_0}(x_{n_0}, f(x_{n_0})))$ is divisible by $(y_{n_0} - f(x_{n_0}))$. Assume the leading term of $Q_{n_0}(x_{n_0}, f(x_{n_0}))$ is $C x_{n_0}^A$ (the term with lowest degree). Then

$$Q_{n_0}(x_{n_0}, f_{n-1}(x_{n_0})) = C x_{n_0}^{A+\nu} + O(x_{n_0}^{A+\nu}) \quad \text{as} \quad n \to \infty.$$  

Combining (3.44), one has

$$(3.45) \quad P_{n_0}(x_{n_0}, f_{n-1}(x_{n_0})) = C x_{n_0}^{p_{n_0}+s(m_{n_0}+\cdots+m_n)+A} + O(x_{n_0}^{p_{n_0}+s(m_{n_0}+\cdots+m_n)+A+\nu}),$$

as $n \to \infty$. Notice that, for all $n > n_0$,

$$(3.46) \quad p_n = p_{n-1} + s_{n-1} m_{n-1} = p_{n-1} + s_{n_0} m_{n-1},$$

which yields

$$(3.47) \quad p_n + s_n m_n = p_{n_0} + s_{n_0} (m_{n_0} + \cdots + m_n).$$  

Comparing (3.42) and (3.45) yields

$$\begin{cases} A = 0 \\ s = s_0 \\ C \neq 0, \end{cases}$$

as desired. \qed
Based on Lemma 3.6 in the $n_0$-stage, we refine the change of variables as follows:

$$x_{n_0} = x_{n_0+1} \quad \text{and} \quad y_{n_0} = f(x_{n_0}) = y_{n_0+1}x_{n_0}^{m_{n_0}}.$$  

By (3.36) one has

$$P_{n_0+1}(x_{n_0+1}, y_{n_0+1}) = P_{n_0}(x_{n_0+1}, y_{n_0+1}x_{n_0}^{m_{n_0}} + f(x_{n_0+1}))$$

$$= x_{n_0+1}^{p_{n_0} + s_{n_0}} y_{n_0+1}^{s_{n_0}} Q_{n_0}(x_{n_0+1}, y_{n_0+1}x_{n_0}^{m_{n_0}} + f(x_{n_0+1}))$$

and $Q_{n_0+1}(0, 0) \neq 0$. This implies $N(P_{n_0+1})$ has only one vertex! Set

$$U_{n_0+1} = \{(x_{n_0+1}, y_{n_0+1}) : (x_{n_0}, y_{n_0}) \in U_{n_0, k}\},$$

then $U_{n_0+1, g} = U_{n_0+1}, U_{n_0+1, b} = \emptyset$ and the procedure ends up here. Thus if we take $N_P$ to be the maximum of all possible $n_0 + 1$, then the resolution procedure ends up at the $N_P$-stage. Set

$$G_n = \bigcup_{\alpha} \cup_j \{U_{n,g,\alpha,j} \},$$

$$V_n = \bigcup_{\alpha} V(P_{n,\alpha})$$

$$E_n = \bigcup_{\alpha} E(P_{n,\alpha})$$

which represent the ‘good’ regions, vertices and compact edges in the $n$-th stage respectively. The followings represent all the ‘good’ regions, vertices and compact edges in all stages:

$$G = \bigcup_{0 \leq n \leq N_P} G_n = \bigcup_{n} \bigcup_{\alpha} \cup_j \{U_{n,g,\alpha,j} \},$$

$$V = \bigcup_{0 \leq n \leq N_P} V_n = \bigcup_{n} \bigcup_{\alpha} V(P_{n,\alpha}),$$

$$E = \bigcup_{0 \leq n \leq N_P} E_n = \bigcup_{n} \bigcup_{\alpha} E(P_{n,\alpha}).$$

**Theorem 3.7.** Let $P$ be a real analytic function and $U$ be a sufficiently small neighborhood of 0. Then $U$ can be partitioned into a finite collection of ‘good’ regions $\{\rho_{n}^{-1}(U_{n,g}), U_{n,g} \in G\}$ where

$$\rho_{n}^{-1}(x_n, y_n) = (x, y) = (x_0, y_0)$$

is defined inductively by

$$x_k = x_{k+1}$$

$$y_k = (r_k + y_{k+1})x_{k+1}^{m_k}$$

for $k = 0, \ldots, n - 1$ via the chain

$$[U_0, P_0] \to [U_1, P_1] \to \cdots \to [U_n, P_n].$$

However, if $[U_0, P_0] \to [U_1, P_1] \to \cdots \to [U_n, P_n] \to [U_{n+1}, P_{n+1}] \to \cdots$ and $n - 1 = n_0$ for some $n_0$ defined as in (3.32), then the last step of the change of variables is redefined by

$$x_{n_0} = x_{n_0+1}$$

$$y_{n_0} - f(x_{n_0}) = y_{n_0+1}x_{n_0}^{m_{n_0}}$$
where
\[ f(x_{n_0}) = \sum_{k=n_0}^{\infty} r_k x_{n_0}^{m_0} + x_{n_0+1} + \cdots + m_k. \]

For \( 0 \leq k \leq n - 1 \), let \((p_k, q_k)\) be the left vertex of the edge where \( U_{k,b} \subset U_k \) is defined by and \((p_n, q_n)\) be defined as below: if \( U_{n,g} \) is defined by a vertex \( V \), then \((p_n, q_n) = V\); otherwise \( U_{n,g} \) is defined by an edge for some \( E \in \mathcal{E}(P_n) \), set \((p_n, q_n)\) to be the left vertex of \( E \).

Then for any given \( L \in \mathbb{N} \), for all \( 0 \leq \alpha, \beta \leq L \) and \((x, y) = \rho^{-1}_n(x_n, y_n) \in \rho^{-1}_n(U_{n,g})\) one has
\[ |P(x, y)| = |P_n(x_n, y_n)| \sim |x_n^{p_n} y_n^{q_n}| \]
\[ |\partial_x^\alpha \partial_y^\beta P_n(x_n, y_n)| \lesssim \min\{1, |x_n^{p_n-\alpha} y_n^{q_n-\beta}|\} \]
and
\[ |\partial_x^\alpha \partial_y^\beta P(x, y)| \lesssim \min\{1, |x_n^{p_n-\alpha} y_n^{q_n-\beta}|\}. \]

**Proof.** The partition in the theorem is a consequence of the algorithm. In addition, (3.63) and (3.64) come from Lemma 3.1 and Lemma 3.2. Finally, (3.65) is an outcome of the chain rule. Indeed, notice
\[ y = y_0 = r_0 x^{m_0} + r_1 x^{m_0+m_1} + \cdots + r_{n-1} x^{m_0+m_1+\cdots+m_{n-1}} + y_n x^{m_0+\cdots+m_{n-1}} \]
and
\[ \partial y / \partial y_n = x^{m_0+\cdots+m_{n-1}}. \]
Then (3.66) yields
\[ |\partial_x^\alpha \partial_y^\beta P(x, y)| \lesssim |x_n^{p_n-\alpha} y_n^{q_n-\beta}|. \]
\[ \square \]

### 3.4. A smooth partition.

In the above theorem, \( U \) is decomposed into disjoint ‘good’ regions \( \rho_n^{-1}(U_{n,g})\)’s. However, when it comes to application (in analysis), an overlap version is ofen more suitable since it provides an ‘\( \epsilon\)-room’ to fit a smooth partition of unity. Furthermore, there is extra benefit in our problem: the ‘\( \epsilon\)-room’ can help us to overcome the technical issues due to the convexity assumption in Theorem 2.1.

For these reasons, we slightly enlarge \( U_{n,g} \) to \( U_{n,g} \subset U_{n,g}^* \subset U_{n,b}^* \) and \( U_{n,b} \) to \( U_{n,b} \subset U_{n,b}^* \subset U_{n,b}^{**} \). The first step is to enlarge \( I(E) \), \( I_g(E) \) and \( I_b(E) \) as follows
\[ I^*(E) = [1/2 c_E, 2 c_E], \quad I^{**}(E) = [1/4 c_E, 4 c_E], \]
\[ I_g^*(E) = I^*(E) \setminus \left( \cup_{1 \leq j \leq J_E} \tilde{I}_j^*(E) \right), \quad I_g^{**}(E) = I^{**}(E) \setminus \left( \cup_{1 \leq j \leq J_E} \tilde{I}_j^{**}(E) \right) \]
and
\[ I_b^*(E) = \cup_{1 \leq j \leq J_E} \tilde{I}_j^*(E), \quad I_b^{**}(E) = \cup_{1 \leq j \leq J_E} \tilde{I}_j^{**}(E). \]

Notice that we can always choose \( \epsilon \) sufficiently small such for all \( E \), \( \{I_j^{**}(E)\} \) does not overlap. Then we can defined the enlarged ‘good’ regions as
\[ U_{0,g}^*(E) = \{ (x, y) \in I_0 : y = r x^{m_E}, r \in I_g^*(E) \} \]
The 'good' regions defined by a vertex are enlarged to:

(3.71) \[ U_{0,g}^* (E) = \{(x,y) \in U_0 : y = rx^{m_E}, r \in I_g^{**}(E)\}. \]

Both \(U_{0,g}^* (E)\) and \(U_{0,g}^{**} (E)\) consist of \((J_E + 2)\) curved triangular regions \(\{U_{0,g}^*(E,j)\}\) and \(\{U_{0,g}^{**}(E,j)\}\) respectively. In addition, one has

(3.72) \[ U_{0,g}^*(E,j) \subset U_{0,g}^*(E,j) \subset U_{0,g}^{**}(E,j). \]

The 'good' regions defined by a vertex are enlarged to:

(3.73) \[ U_{0,g}^*(V) = \{(x,y) \in U_0 : E_{x,E}^r x^{m_{E^r}} < y < 2cE_{x,E} x^{m_{E^r}}\}, \]

(3.74) \[ U_{0,g}^{**}(V) = \{(x,y) \in U_0 : E_{x,E}^r x^{m_{E^r}} < y < 4cE_{x,E} x^{m_{E^r}}\}, \]

and finally the enlarged 'bad' regions are

(3.75) \[ U_{0,b}^*(E,j) = \{(x,y) \in U_0 : (r_j - 2\epsilon)x^{m_E} < y < (r_j + 2\epsilon)x^{m_E}\}, \]

(3.76) \[ U_{0,b}^{**}(E,j) = \{(x,y) \in U_0 : (r_j - 4\epsilon)x^{m_E} < y < (r_j + 4\epsilon)x^{m_E}\}. \]

For \(n \geq 1\), \(U_{n,g}^*\)'s, \(U_{n,b}^*\)'s and \(U_{n,g}^{**}\)'s, \(U_{n,b}^{**}\)'s are defined similarly. Since \(\epsilon > 0\) can be chosen arbitrary small, the above definitions do not cause any conflict. We have:

**Corollary 3.8.** Theorem \(\gamma\) still holds with \(U_{n,g}\) replaced by \(\{U_{n,g}^{**}\}\), except \(U\) is not a disjoint union of \(\{\rho_n^{-1} (U_{n,g}^{**})\}\).

For a smooth partition, we issue some technical problems first. Let \(c\) be a positive constant such that neither \(y = cx\) nor \(y = -cx\) is a solution of \(P_E(x,y) = 0\) for any \(E \in \mathcal{E}(P)\). Then \(y = \pm cx\) divides \(U\) into four regions: \(R_1\), \(R_2\), \(R_3\) and \(R_4\), which represents the East, North, West and South regions respectively. Let \(\{\Psi_j\}_{1 \leq j \leq 4}\) be smooth functions such that

(3.77) \[ 1 = \sum_{j=1}^{4} \Psi_j(x,y), \quad (x,y) \neq (0,0). \]

Here \(\Psi_1(x,y)\) is supported in

\[ R_1^c = \{(x,y) : x > 0, -(c + \epsilon)x < y < (c + \epsilon)x\} \]

and \(\Psi_1(x,y) = 1\) if

(3.78) \[ -(c - \epsilon)x < y < (c - \epsilon)x. \]

The constant \(\epsilon\) is chosen to be sufficiently small. In addition, \(\Psi_1\) satisfies

(3.79) \[ |\partial_x^\alpha \partial_y^\beta \Psi_1(x,y)| \leq C_{\alpha,\beta} |x|^{-\alpha} |y|^{-\beta}, \quad \text{for} \quad (\alpha, \beta) \in \mathbb{N}^2. \]

The other functions \(\Psi_2, \Psi_3\) and \(\Psi_4\) are defined similarly in the other 3 regions.

Let \(W\) be a sufficiently small neighborhood of \((0,0)\) and \(\Phi(x,y)\) be a smooth function adapted to \(W\), in a sense \(\text{supp} \Phi \subset W\) and \(\Phi(x,y) = 1\) if \((2x,2y) \in W\). Then

(3.80) \[ \Phi(x,y) = \Phi(x,y) \sum_{j=1}^{4} \Psi_j(x,y), \quad \text{for} \quad (x,y) \neq (0,0). \]

We focus on \(\Phi\Psi_1\), discussions of \(\{\Phi\Psi_j\}_{2 \leq j \leq 4}\) can be reduced to this case. Let

\[ U = W \cap R_1^c \]

then \(\Phi\Psi_1\) is supported in \(U\).
For a given analytic function $P(x, y)$, applying the resolution algorithm to $P(x, y)$ in the region $U$ yields a collection of 'bad' regions $\{U^*_{n,b,\alpha,j}\}_{(n,\alpha,j)}$. For a fixed $U^*_{n,b,\alpha,j}$, $\rho_{n,\alpha}^{-1}(U^*_{n,b,\alpha,j})$ is equal to

$$U \cap \{(x, y) : r_0x^{m_0} + \cdots + r_{n-1}x^{m_0+\cdots+m_{n-1}} + (r_n - 2\epsilon)x^{m_0+\cdots+m_n} < y$$

$$< r_0x^{m_0} + \cdots + r_{n-1}x^{m_0+\cdots+m_{n-1}} + (r_n + 2\epsilon)x^{m_0+\cdots+m_n}\}.$$ 

We can then define a smooth function $\Phi_{n,b,\alpha,j}$ supported in $\rho_{n,\alpha}^{-1}(U^*_{n,b,\alpha,j})$ and $\Phi_{n,b,\alpha,j}(x, y) = 1$ if $(x, y) \in \rho_{n,\alpha}^{-1}(U^*_{n,b,\alpha,j})$. In addition, the following is true

$$|\partial_x^\alpha \partial_y^\beta \Phi_{n,b,\alpha,j}(x, y)| \leq C_{\alpha, \beta}|x|^{-\alpha-\beta(m_0+\cdots+m_n)} \quad \forall (\alpha, \beta) \in \mathbb{N}^2.$$

Then

$$\Phi(x, y)\Psi_1(x, y) \left(1 - \sum_\alpha \sum_j \Phi_{0,b,\alpha,j}(x, y)\right)$$

can be written as

$$\sum_\alpha \sum_j \Phi_{0,g,\alpha,j}(x, y)$$

where each $\Phi_{0,g,\alpha,j}(x, y)$ is supported in the 'good' region $U^*_{0,g,\alpha,j}$. Similarly,

$$\Phi(x, y)\Psi_1(x, y) \left(\sum_\alpha \sum_j \Phi_{0,b,\alpha,j}(x, y)\right) \left(1 - \sum_\alpha \sum_j \Phi_{1,b,\alpha,j}(x, y)\right)$$

can be written as

$$\sum_\alpha \sum_j \Phi_{1,g,\alpha,j}(x, y)$$

where $\Phi_{1,g,\alpha,j}$ is supported in $\rho_{1,\alpha}^{-1}(U^*_{1,g,\alpha,j})$. Then we can iterate the above procedures as in the algorithm, it ends up after finite steps. Combining (3.8), we obtain a smooth partition version of Theorem 3.7.

**Theorem 3.9.** Let $\Phi$, $\Psi_1$ and $P$ as above. Then

$$\Phi(x, y)\Psi_1(x, y) = \sum_n \sum_\alpha \sum_j \Phi_{n,g,\alpha,j}(x, y),$$

where $\Phi_{n,g,\alpha,j}(x, y)$ is a smooth function supported in $\rho_{n,\alpha}^{-1}(U^*_{n,g,\alpha,j})$, where $\{U_{n,g,\alpha,j}\}$ is the collection of 'good' regions as in Theorem 3.7. The behaviors of $P(x, y)$ in 'good' regions $\rho_{n,\alpha}^{-1}(U^*_{n,g,\alpha,j})$ and $\rho_{n,\alpha}^{-1}(U^*_{n,g,\alpha,j})$ are the same as Corollary 3.8.

Moreover, $\Phi_{n,g,\alpha,j}(x, y)$ satisfies the following derivative conditions:

1. If $U_{n,g,\alpha,j}$ is defined by an edge. We can assume $\rho_{n,\alpha}^{-1}(U_{n,g,\alpha,j})$ is contained in a curved triangular region of the form

$$|y - (r_0x^{m_0} + \cdots + r_{n-1}x^{m_0+\cdots+m_{n-1}})| \sim x^{m_0+\cdots+m_n}.$$ 

Then

$$|\partial_x^\alpha \partial_y^\beta \Phi_{n,g,\alpha,j}(x, y)| \leq C_{\alpha, \beta}|x|^{-\alpha-\beta(m_0+\cdots+m_n)} \quad \forall (\alpha, \beta) \in \mathbb{N}^2.$$

2. Otherwise, $U_{n,g,\alpha,j}$ is defined by a vertex, then $\rho_{n,\alpha}^{-1}(U^*_{n,g,\alpha,j})$ is contained in the curved triangular region of the form

$$x^{m_0+\cdots+m_{n-1}+m_n} \sim |y - (r_0x^{m_0} + \cdots + r_{n-1}x^{m_0+\cdots+m_{n-1}})| \sim x^{m_0+\cdots+m_{n-1}+m_n}.$$
where $0 \leq m_{n,l} < m_{n,r} \leq \infty$. In the upper portion of $\rho_{n,\alpha}^{-1}(U_{n,g,\alpha,j}^*) \backslash \rho_{n,\alpha}^{-1}(U_{n,g,\alpha,j})$, one has
\begin{equation}
|\partial_x^\alpha \partial_y^\beta \Phi_{n,g,\alpha,j}(x,y)| \leq C_{\alpha,\beta} |x|^{-\alpha - \beta(m_0 + \cdots + m_{n-1} + m_{n,l})}, \quad \forall (\alpha, \beta) \in \mathbb{N}^2.
\end{equation}
In the lower portion of $\rho_{n,\alpha}^{-1}(U_{n,g,\alpha,j}^*) \backslash \rho_{n,\alpha}^{-1}(U_{n,g,\alpha,j})$, if $m_{n,r} \neq \infty$ then
\begin{equation}
|\partial_x^\alpha \partial_y^\beta \Phi_{n,g,\alpha,j}(x,y)| \leq C_{\alpha,\beta} |x|^{-\alpha - \beta(m_0 + \cdots + m_{n-1} + m_{n,r})}, \quad \forall (\alpha, \beta) \in \mathbb{N}^2;
\end{equation}
otherwise $m_{n,r} = \infty$, then $U_{n,g,\alpha,j}^*$ is defined by the rightest vertex of $N(P_{n,\alpha})$ and $\rho_{n,\alpha}^{-1}(U_{n,g,\alpha,j}^*)$ can be represented as
\begin{equation}
|y - (r_0 x^{m_0} + \cdots + r_{n-1} x^{m_0 + \cdots + m_{n-1}})| \leq x^{m_0 + \cdots + m_{n-1} + m_{n,l}},
\end{equation}
one has
\begin{equation}
|\partial_x^\alpha \partial_y^\beta \Phi_{n,g,\alpha,j}(x,y)| \leq C_{\alpha,\beta} |x|^{-\alpha - \beta(m_0 + \cdots + m_{n-1} + m_{n,l})}, \quad \forall (\alpha, \beta) \in \mathbb{N}^2.
\end{equation}

4. Proof of Theorem 1.4

4.1. The exponent in the sharp bound. Before diving into the details, we give a brief exploration for the exponent appeared in the sharp decay rate of Theorem 1.4, $-1/(2 \text{mult}_{\pi_0}(S))$. Mainly, we address the following questions:
(i) Where does this exponent come from?
(ii) Why this exponent is different from the one in Phong-Stein’s result [13]?
(iii) Show that the exponent in Theorem 1.4 is sharp.

To answer the first one, set
\begin{equation}
P(x, y) = \partial_x \partial_y (\partial_x - \partial_y) S(x, y),
\end{equation}
then
\begin{equation}
\text{mult}_{\pi_0}(S) = \text{mult}(P) + 3.
\end{equation}

**Temporarily** index the vertices of $N(P)$ from left to right by $V_1 = (p_1, q_1)$, $V_2 = (p_2, q_2)$, \ldots, $V_k = (p_k, q_k)$ and all its compact edges by $E_1 = V_1 V_2$, $E_2 = V_2 V_3$, \ldots, $E_{k-1} = V_{k-1} V_k$, with slope $-\frac{1}{m_1}, \ldots, -\frac{1}{m_{k-1}}$ respectively. Then
\begin{equation}
0 = m_0 < m_1 < m_2 < \cdots < m_{k-1} < m_k = \infty,
\end{equation}
where $-1/m_0$ and $-1/m_k$ are the ‘slopes’ corresponding to the perpendicular and horizontal non-compact edges. We assign a constant $d_{E_j}$ to each $E_j$ and a constant $d_{V_j}$ to each $V_j$ below:
(1) For an edge $E_j$, $1 \leq j \leq k - 1$, let $d_{E_j,x}, d_{E_j,y}$ be the $x$-intercept, $y$-intercept of $E_j$, set
\begin{equation}
d_{E_j} = \min\{d_{E_j,x}, d_{E_j,y}\};
\end{equation}
(2) For a vertex $V_j$, $1 \leq j \leq k$, let $E$ be any line containing $V_j$ but not intersect the interior of $N(P)$. Let $d_{E,x}, d_{E,y}$ be the $x$-intercept, $y$-intercept and $d_E = \min\{d_{E,x}, d_{E,y}\}$. Then set
\begin{equation}
d_{V_j} = \sup_E d_E.
\end{equation}
We call $d_{E_j}$ and $d_{V_j}$ the decay factors corresponding to the edge $E_j$ and the vertex $V_j$. One can see that
\begin{equation}
\text{mult}(P) = \max\{d_{V_j}, d_E : \ V \in \mathcal{V}(P), \ E \in \mathcal{E}(P)\}.
\end{equation}
In addition, there is exact one vertex \( V_\ast = (p_\ast, q_\ast) \) in \( \mathcal{N}(P) \) such that
\[
m_{\ast - 1} \leq 1 < m_\ast.
\]
Then \( \text{mult}(P) = p_\ast + q_\ast \). Here \( * \) is an integer between 1 and \( k \).

If \( j < * \), \( m_j \leq 1 \), one has
\[
(4.4) \quad \text{mult}(P) = p_\ast + q_\ast \geq p_j + m_j q_j = d_{E_j} = d_{E_j,x}.
\]
The value \( |\lambda|^{-\frac{1}{2(3+d_{E_j})}} \) will correspond to the bound of \( \Lambda \) in Theorem 1.4 when restricting \((x, y)\) to the ‘good’ regions defined by \( E_j \).

Else \( j \geq * \), \( m_j \geq 1 \) then
\[
(4.5) \quad \text{mult}(P) = p_\ast + q_\ast \geq p_j/m_j + q_j = d_{E_j} = d_{E_j,y}.
\]
When restricted to the ‘good’ regions defined by \( E_j \), the corresponding bound of \( \Lambda \) will be \( |\lambda|^{-\frac{1}{2(3+d_{E_j})}} \).

Similarly, when restricted to the ‘good’ region defined by the vertex \( V_j \), the corresponding bound of \( \Lambda \) will be \( |\lambda|^{-\frac{1}{2(3+d_{E_j})}} \).

Finally, the sharp bound \( C |\lambda|^{-\frac{1}{2(3+d_{E_j})}} \) will be obtained via the vertex \( V_\ast = (p_\ast, q_\ast) \), in the region \(|x| \sim |y|\) (it may coincide with an edge).

Now we turn to the second question. One noticeable difference between the operator in Theorem 1.4 and the one in Theorem 1.2 is the extra term \( f_3(x + y) \). If \( x \) and \( y \) vary in intervals of length \( \delta_1 \) and \( \delta_2 \) respectively, then the support of \((x + y)\) is of length \( \max\{\delta_1, \delta_2\} \). If \( f_3 \) is a characteristic function supported in this interval, then \( \|f_3\|_2 \sim \max\{\delta_1^{1/2}, \delta_2^{1/2}\} \). This freezes the ratio \( \log |x|/\log |y| \) to be 1, if one wants to optimize the bound of the operator. Indeed, our example proving the sharpness of the bound in Theorem 1.4 is constructed in this favor.

However, without the term \( f_3(x+y) \), the ratio between \( \log |x| \) and \( \log |y| \) is totally free. The affect of such difference on the operators is realized by the difference of the following two Schur-type’s lemmas:

**Lemma 4.1.** Assume \( a(x, y) \) is a smooth function supported in a strip of \( x \)-width no more than \( \delta_1 \) and \( y \)-width no more than \( \delta_2 \). Assume \( \|a\|_\infty \leq 1 \), then
\[
(4.6) \quad \left| \iint f_1(x)f_2(y)a(x,y)dxdy \right| \leq C(\delta_1 \delta_2)^{1/2}\|f_1\|_2\|f_2\|_2.
\]

**Lemma 4.2.** Assume \( a(x, y) \) is a smooth function supported in a strip of \( x \)-width no more than \( \delta_1 \) and \( y \)-width no more than \( \delta_2 \). Assume \( \|a\|_\infty \leq 1 \), then
\[
(4.7) \quad \left| \iint f_1(x)f_2(y)f_3(x+y)a(x,y)dxdy \right| \leq C \min\{\delta_1^{1/2}, \delta_2^{1/2}\}\|f_1\|_2\|f_2\|_2\|f_3\|_2.
\]

Lemma 4.1 is a directly result of Schur’s Lemma, which is employed in [13] to control the norm of the operator in Theorem 1.2 when the phase fails to provide sufficient decay. Lemma 4.2 plays the same role in our proof. We provide the proof of Lemma 4.2 as an appetizer.
Indeed, if we write (4.8)
The other bound can be obtained similarly.

In addition, we assume $e$ to the equivalence between simply degeneracy and degeneracy in this case. Thus

$$L(4.9)$$

interval

interval $[A, B]$.

Proof. By the Cauchy-Schwarz Inequality and the Fubini Theorem, one has

$$\left| \int f_1(x)f_2(y)f_3(x+y)a(x,y)dxdy \right|
= \int f_1(x) \left( \int f_2(y)f_3(x+y)a(x,y)dy \right) dx
\leq \int \left( \int f_2(y)f_3(x+y)a(x,y)dy \right)^2 dx \cdot \|f_1\|_2
\leq \int \left( \int |a(x,y)|^2 dy \right) \left( \int |f_2(y)f_3(x+y)|^2 dy \right) dx \cdot \|f_1\|_2
\leq C \delta^{1/2} \|f_1\|_2 \|f_2\|_2 \|f_3\|_2.
$$

The other bound can be obtained similarly. $\square$

Finally, we provide an example to show the sharpness of the exponent. Firstly, we write $S$ as a sum of homogeneous polynomials:

$$S(x, y) = \sum_{n=n_0}^{\infty} S_n(x, y).$$

In addition, we assume

$$\partial_x \partial_y (\partial_x - \partial_y) S_{n_0} \neq 0.$$ (4.8)

Indeed, if $\partial_x \partial_y (\partial_x - \partial_y) S_n = 0$, then $S_n(x, y) = S_{n,1}(x) + S_{n,2}(y) + S_{n,3}(x+y)$ due to the equivalence between simply degeneracy and degeneracy in this case. Thus we can incorporate $e^{i S_n(x,y)}$ into the functions $f_1(x), f_2(y)$ and $f_3(x+y)$, while the $L^2$-norms of these functions are unchanged.

Let $A$ be a sufficiently large number, $f_1$ and $f_2$ be characteristic functions of the interval $I_A = [-\lambda^{-1/n_0}/A, \lambda^{-1/n_0}/A]$. Let $f_3$ be the characteristic function of the interval $[2^{-1/n_0}/A, 2\lambda^{-1/n_0}/A]$. If $A$ is sufficiently large (depending on $S$), then

$$|\lambda S(x, y)| \leq 2^{-100}, \quad \forall \quad x, y \in I_A.$$ Thus

$$\left| \int e^{i \lambda S(x,y)} a(x,y) f_1(x)f_2(y)f_3(x+y)dxdy \right| \sim |I_A| \times |I_A| \sim \lambda^{-\frac{1}{n_0}}.$$ (4.9)

Notice

$$\|f_1\|_2 \sim \|f_2\|_2 \sim \|f_3\|_2 \sim \lambda^{-\frac{1}{n_0}}.$$ (4.10)

Hence, if

$$\left| \int e^{i \lambda S(x,y)} a(x,y) f_1(x)f_2(y)f_3(x+y)dxdy \right| \leq C(\lambda) \prod_{j=1}^{3} \|f_j\|_2$$ (4.11)

then $C(\lambda) \gtrsim \lambda^{-\frac{1}{n_0}} = \lambda^{-\frac{n_0}{\sum_{j=1}^{3} n_j}}$, as desired.

Now, we come back to use the indices in section 3 and get rid of the above temporary indices.
positive constant s.t.
neither $y \in \mathcal{E}$
all (4.19)
since the other three regions can be reduced to (4.12)
for some $\alpha$
3.9, we divide $W$ into 4 regions by the lines $y = cx$ and $y = -cx$. Here $c$ is a positive constant s.t. neither $y = cx$ nor $y = -cx$ is a solution of $P_E(x, y) = 0$, for all $E \in \mathcal{E}(P)$. We restrict our discussion in the following region
(4.12) $U = W \cap \{(x, y) : x > 0, -(c + \epsilon)x < y < (c + \epsilon)x\}$, since the sum (4.18) contains only finite terms, it suffices to prove (4.18)
by an edge Proposition 4.3. If
if $\alpha$
are unimportant. Thus we drop them and use $\Lambda$
Set (4.16) $a_n, g, \alpha, j(x, y) = \Phi_{n, g, \alpha, j}(x, y)a(x, y)$.
Set (4.17) $\Lambda_{n, g, \alpha, j}(f_1, f_2, f_3) = \int \int e^{i\lambda S(x, y)} f_1(x)f_2(y)f_3(x + y)a_{n, g, \alpha, j}(x, y)dxdy$
then
(4.18) $\Lambda^2_n(f_1, f_2, f_3) = \sum_{0 \leq n \leq N_P} \sum_{\alpha} \sum_j \Lambda_{n, g, \alpha, j}(f_1, f_2, f_3)$.
Since the sum (4.18) contains only finite terms, it suffices to prove
$\|\Lambda_{n, g, \alpha, j}\| \lesssim |\lambda|^{-\frac{1}{2(\kappa + m - 1)}}$, for every $(n, g, \alpha, j)$.
In the rest of this section, we always deal with a single $\Lambda_{n, g, \alpha, j}$ and the indices $j$ and $\alpha$ are unimportant. Thus we drop them and use $\Lambda_{n, g}$ to represent $\Lambda_{n, g, \alpha, j}$ for some $\alpha$ and $j$. The proof is splitted into three cases (i) $n = 0$ and $U_{0, g}$ is defined by an edge $E$, (ii) $n = 0$ and $U_{0, g}$ is defined by a vertex $V$ and (iii) $n \geq 1$.

**Proposition 4.3.** If $U_{0, g}$ is defined by an edge $(E, m)$, where $-1/m$ is the slope of $E$, then
(4.19) $\|\Lambda_{0, g}\| \lesssim |\lambda|^{-\frac{1}{2(\kappa + m - 1)}}$. 

4.2. **Proof of Theorem 1.4** Assume $a(x, y)$ is supported in $W/100$, where $W$ is a sufficiently small neighborhood of 0. Same as what have been done in Theorem [1.9] we divide $W$ into 4 regions by the lines $y = cx$ and $y = -cx$. Here $c$ is a positive constant s.t. neither $y = cx$ nor $y = -cx$ is a solution of $P_E(x, y) = 0$, for all $E \in \mathcal{E}(P)$. We restrict our discussion in the following region
(4.12) $U = W \cap \{(x, y) : x > 0, -(c + \epsilon)x < y < (c + \epsilon)x\}$, since the other three regions can be reduced to $U$ by either changing $x$ to $-x$ or permuting $x$ and $y$ or both. Let $\Psi_j(x, y)$ and $\Phi(x, y)$ be smooth functions as in Theorem [3.9] then
(4.13) $a(x, y) = a(x, y)\Phi(x, y)\sum_{j=1}^4 \Psi_j(x, y), \text{ for } (x, y) \neq (0, 0)$
and
(4.14) $\Lambda_S(f_1, f_2, f_3) = \sum_{j=1}^4 \Lambda^2_j(f_1, f_2, f_3)$
where
(4.15) $\Lambda^2_j(f_1, f_2, f_3) = \int \int e^{i\lambda S(x, y)} a(x, y)\Phi(x, y)\Psi_j(x, y)f_1(x)f_2(y)f_3(x + y)dxdy.$
We focus only on $j = 1$. Theorem [3.9] yields:
(4.16) $a(x, y)\Phi(x, y)\Phi_1(x, y) = \sum_{0 \leq n \leq N_P} \sum_{\alpha} \sum_j a_{n, g, \alpha, j}(x, y)$
where
(4.17) $\Lambda_{n, g, \alpha, j}(f_1, f_2, f_3) = \int \int e^{i\lambda S(x, y)} f_1(x)f_2(y)f_3(x + y)a_{n, g, \alpha, j}(x, y)dxdy$
then
(4.18) $\Lambda^2_n(f_1, f_2, f_3) = \sum_{0 \leq n \leq N_P} \sum_{\alpha} \sum_j \Lambda_{n, g, \alpha, j}(f_1, f_2, f_3)$.
Since the sum (4.18) contains only finite terms, it suffices to prove
$\|\Lambda_{n, g, \alpha, j}\| \lesssim |\lambda|^{-\frac{1}{2(\kappa + m - 1)}}$, for every $(n, g, \alpha, j)$.
In the rest of this section, we always deal with a single $\Lambda_{n, g, \alpha, j}$ and the indices $j$ and $\alpha$ are unimportant. Thus we drop them and use $\Lambda_{n, g}$ to represent $\Lambda_{n, g, \alpha, j}$ for some $\alpha$ and $j$. The proof is splitted into three cases (i) $n = 0$ and $U_{0, g}$ is defined by an edge $E$, (ii) $n = 0$ and $U_{0, g}$ is defined by a vertex $V$ and (iii) $n \geq 1$.

**Proposition 4.3.** If $U_{0, g}$ is defined by an edge $(E, m)$, where $-1/m$ is the slope of $E$, then
(4.19) $\|\Lambda_{0, g}\| \lesssim |\lambda|^{-\frac{1}{2(\kappa + m - 1)}}$. 

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Proposition 4.4. If $U_{0,g}$ is defined by a vertex $V$, then
\begin{equation}
\|\Lambda_{0,g}\| \lesssim |\lambda|^{-\frac{1}{2(n+2)}}. \tag{4.20}
\end{equation}

Proposition 4.5. If $U_{n,g} \subset U_n$ comes from the following chain:
\begin{equation}
[U_0, P_0] \to [U_1, P_1] \to \cdots \to [U_n, P_n]
\end{equation}
and $[U_1, P_1]$ is obtained from $[U_0, P_0]$ by the edge $E \in \mathcal{E}(P_0)$. Then
\begin{equation}
\|\Lambda_{n,g}\| \lesssim |\lambda|^{-\frac{1}{2(n+2)}}. \tag{4.22}
\end{equation}

One can see that, Proposition 4.3, Proposition 4.4 and Proposition 4.5 imply Theorem 1.4.

Proof of Proposition 4.3
Let $0 < \sigma < 1$ be a dyadic number and $\phi_{\sigma}(x)$ be a smooth function supported in $\frac{1}{2}\sigma < |x| < 2\sigma$, such that
\begin{equation}
\sum_{0 < \sigma < 1} \phi_{\sigma}(x) = 1 \quad \text{for} \quad 0 < |x| < 1/10. \tag{4.23}
\end{equation}

Set
\begin{equation}
\Lambda_{0,g,\sigma_1,\sigma_2}(f_1, f_2, f_3) = \iint e^{i\lambda S(x,y)}f_1(x)f_2(y)f_3(x+y)a_{0,g}(x,y)\phi_{\sigma_1}(x)\phi_{\sigma_2}(y)dxdy,
\end{equation}
where $\sigma_1$ and $\sigma_2$ are dyadic numbers and $a_{0,g} = a_{0,g,\alpha,j}$ for some $(\alpha,j)$. Notice that
\begin{equation}
\text{supp}(a_{0,g}) \subset U_{0,g}^*,
\end{equation}
which is a ‘good’ region defined by $(E, m)$. Thus
\begin{equation}
|y| \sim |x|^m \quad \text{and} \quad |\sigma_2| \sim |\sigma_1|^m.
\end{equation}

This yields, for a fixed $\sigma_1$, there is only finite choices of $\sigma_2$. WLOG, we assume $\sigma_2$ is fixed given $\sigma_1$ is fixed. To employ Theorem 2.1, we need to verify its conditions. Let $K$ be a large constant, equally divide the interval $(\sigma_1/2, 2\sigma_1)$ into $K$ subintervals $\{I_k\}_{1 \leq k \leq K}$ and set
\begin{equation}
U_{0,g,k}^* = \{(x,y) \in U_{0,g}^* : x \in I_k\}.
\end{equation}

Lemma 4.6. Given $K$ large enough, depending only on $P$ and $\epsilon$, one has
\begin{equation}
\text{Conv}(U_{0,g,k}^*) \subset U_{0,g}^{**}, \tag{4.25}
\end{equation}
for all $1 \leq k \leq K$.

The proof of this lemma is postponed at the end of this section. Now let $(p_l, q_l)$ be the left vertex of $E$. Then for every $(x,y) \in \text{Conv}(U_{0,g,k}^*) \subset U_{0,g}^{**}$, Theorem 3.9 and Corollary 3.8 yield,
\begin{equation}
|P(x,y)| \gtrsim |x|^{p_l}|y|^{q_l} \sim |\sigma_1|^{p_l}|\sigma_2|^{q_l}, \tag{4.26}
\end{equation}
and for $\beta = 0, 1, 2$
\begin{equation}
|\partial_\beta P(x,y)| \lesssim |x|^{p_l}|y|^{q_l-\beta} \sim |\sigma_1|^{p_l}|\sigma_2|^{q_l-\beta}. \tag{4.27}
\end{equation}

Theorem 3.10 together with (4.16) yields
\begin{equation}
|\partial_\beta a_{0,g}(x,y)| \lesssim |\sigma_2|^{-\beta}. \tag{4.28}
\end{equation}
By invoking Theorem 2.1, one has
\[
\|\Lambda_{0,g,\sigma_1,\sigma_2,k}\| \lesssim |\lambda|^{\frac{1}{2}} |\sigma_2|^{-\frac{1}{6}},
\]
where \(\Lambda_{0,g,\sigma_1,\sigma_2,k}(f_1, f_2, f_3)\) is given by
\[
\int \int e^{i\lambda S(x,y)} f_1(x) 1_{k}(x) f_2(y) f_3(x+y) a_{0,g}(x,y) \phi_{\sigma_1}(x) \phi_{\sigma_2}(y) dx dy.
\]

Summing over \(1 \leq k \leq K\) (\(K\) is a constant) yields:
\[
\|\Lambda_{0,g,\sigma_1,\sigma_2}\| \lesssim |\lambda|^{\frac{1}{2}} |\sigma_2|^{-\frac{1}{6}}.
\]
Employing Lemma 4.2 and combining (4.30), one obtains:
\[
(4.31)
\|
\Lambda_{0,g,\sigma_1,\sigma_2}\| \lesssim \begin{cases} 
|\lambda|^{\frac{1}{2}} |\sigma_2|^{-\frac{1}{6}}, \\
\min\{\sigma_1, \sigma_2\}^{\frac{1}{2}}.
\end{cases}
\]

Notice that our assumption on \(U\) implies \(m \geq 1\) and thus \(\sigma_2 \lesssim \sigma_1\). This gives
\[
(4.32)
\|
\Lambda_{0,g,\sigma_1,\sigma_2}\| \lesssim \begin{cases} 
|\lambda|^{\frac{1}{2}} |\sigma_2|^{-\frac{1}{6}}, \\
\sigma_2^{\frac{1}{2}}.
\end{cases}
\]
Since for fixed \(\sigma_2, \sigma_1\) is fixed, summing over \(\sigma_2\) yields
\[
(4.33)
\left\| \sum_{\sigma_1,\sigma_2} \Lambda_{0,g,\sigma_1,\sigma_2} \right\| \lesssim |\lambda|^{-\frac{1}{2}} |\lambda|^{-\frac{1}{2}} = |\lambda|^{-\frac{1}{2}}.
\]
as desired.

\(\square\)

**Proof of Proposition 4.4.**

Like the proof of Proposition 4.3, insert the smooth support \(\phi_{\sigma_1}(x) \phi_{\sigma_2}(y)\) into \(\Lambda_{0,g}\). Set \(V = (p, q)\) and assume \(-1/m_l\) and \(-1/m_r\) be the slopes of the edges left and right to \(V\). Due to the following assumption on \(U\):
\[
(4.34) \quad U = W \cap \{(x, y) : x > 0, -(c + \epsilon)x < y < (c + \epsilon)x\},
\]
we may replace \(m_l\) with 1 if \(m_l < 1\). Thus
\[
(4.35) \quad \infty \geq m_r > m_l \geq 1.
\]
Notice that
\[
\sigma_2^{\frac{1}{m_r}} \geq \sigma_1 \geq \sigma_2^{\frac{1}{m_l}} \geq \sigma_2.
\]
Consider all \((\sigma_1, \sigma_2)\) with \(\sigma_2 \lesssim \lambda_2 := |\lambda|^{-\frac{1}{2}}|\frac{1}{m_l + m_r}| = |\lambda|^{-\frac{1}{2}}|\frac{1}{m_l + m_r}|\). By Lemma 4.2 we have
\[
(4.36) \quad \left\| \sum_{\sigma_2 \lesssim \lambda_2} \left( \sum_{\sigma_1} \Lambda_{0,g,\sigma_1,\sigma_2} \right) \right\| \lesssim \sum_{\sigma_2 \lesssim \lambda_2} |\sigma_2|^{\frac{1}{2}} \lesssim |\lambda_2|^{\frac{1}{2}} = |\lambda|^{-\frac{1}{2}}.
Now we assume $\sigma_2 \gtrsim \lambda_2$. Theorem 2.1 yields

$$\|A_{0,g,\sigma_1,\sigma_2}\| \lesssim |\lambda \sigma_1^p \sigma_2^q|^{-1/6},$$

Since $\sigma_1 \gtrsim \sigma_2^{1/m_1}$, thus

$$\sum_{\sigma_1} \|A_{0,g,\sigma_1,\sigma_2}\| \lesssim |\lambda \sigma_2^{p/m_1+q}|^{-1/6}.$$  

Summing all $\sigma_2 \gtrsim \lambda_2$ we obtain the same bound as (4.36).

Proof of Proposition 4.5

For $0 \leq k \leq n-1$, assume the change of variables is

$$\begin{cases}
x_k = x_{k+1} \\
y_k = (r_k + y_{k+1})x^{m_k}.
\end{cases}$$

If (4.21) is a subchain of an infinite chain as (3.31) and $n = n_0 + 1$ ($n_0$ defined in (3.35) and hence there is no $n > n_0 + 1$), then the last step of change of variables is replaced by (4.45):

$$\begin{cases}
x_{n_0} = x_{n_0+1} \\
y_{n_0} - f(x_{n_0}) = y_{n_0+1}x^{m_{n_0}}.
\end{cases}$$

The behavior of $P_n(x, y)$ in $U_{n,g}$ (and $U_{n,g}^*$) is either dominant by a vertex or an edge; if by a vertex, let $(p_n, q_n)$ be that vertex; else let $(p_n, q_n)$ be the left vertex of that edge. In addition, let $s_k$ be the order of $r_k$, $(p_k, q_k)$ be the left vertex of the edge $E_k$, where $E_k$ is the edge corresponding to $y_k = r_k x^{m_k}$ in $N(P_k)$. Theorem 4.7 and Corollary 3.8 yield

$$|P(x, y)| = |P_n(x, y)| \sim |x^{p_n} y^{q_n}|$$

for all $(x, y) \in U_{n,g}^{**}$. Since $U_{n,g}^{**} \subset U_{n,g}^*$ is a ‘good’ region, we can find $m'_n$ and $m_n$ s.t.

$$|x_n|^{m'_n} \lesssim |y_n| \lesssim |x_n|^{m_n},$$

where $0 \leq m_n \leq m'_n \leq \infty$. In addition, if $m_n = m'_n$, then $U_{n,g}^{**}$ is defined by an edge; otherwise by a vertex.

Dyadically decompose $(x, y)$ as

$$\begin{cases}
|x| \sim \sigma_1 \\
|y| \sim \sigma_2 x^{m_n} \sim \sigma_2 \sigma_1^{m_n}
\end{cases}$$

and let $A_{0,g,\sigma_1,\sigma_2}$ denote the operator $A_{0,g}$ when $(x, y)$ is restricted in this region. Different to the case when $n = 0$, when $n \geq 1$ the ‘almost orthogonality’ plays a crucial role, which comes from the diagonal distribution of $U_{n,g}$. In fact, from the change of variables, we have

$$y = y_n = r_0 x^{m_0} + r_1 x^{m_0+m_1} + \cdots + r_{n-1} x^{m_0+\cdots+m_{n-1}} + y_n x^{m_0+\cdots+m_{n-1}}.$$  

In addition, the assumption on $U$ ensures $m_0 \geq 1$.

---

4Here we have invoked the same trick to meet the convexity condition of Theorem 2.1 as in the proof of Proposition 4.3 splitting $A_{0,g,\sigma_1,\sigma_2}$ into the sum of $A_{0,g,\sigma_1,\sigma_2,k}$, applying Theorem 2.1 to each $A_{0,g,\sigma_1,\sigma_2,k}$ and summing them together.
Notice that $y_n \sim \sigma_2 \sigma_1^{mn}$ and $|x| \sim \sigma_1$, thus in this region the width of $y$: $\triangle y$ is bounded by

$$|\triangle y| \lesssim \sigma_2 \sigma_1^{m_0 + \cdots + m_n}.$$  

(4.42) In addition, (4.41) yields

$$\triangle y \sim \triangle x \frac{dy}{dx} \sim \triangle x \cdot \sigma_1^{m_0-1}$$

and thus

$$|\triangle x| \lesssim \sigma_2 \sigma_1^{m_0 + \cdots + m_n - (m_0-1)}.$$  

(4.43)

Based on the above analysis, we divide the interval $(\sigma_1/2, 2\sigma_1)$ equally into $H$ subintervals $\{I_h\}_{1 \leq h \leq H}$, where

$$H = K \cdot \sigma_1 \left( \sigma_2 \sigma_1^{m_0 + \cdots + m_n - (m_0-1)} \right)^{-1}.$$  

Here $K$ is a large constant designed merely to treat the convexity condition in Theorem 2.1. Set

$$U_{n,g,h}^* = \{ (x, y) \in U_{n,g}^* : x \in I_h \}$$

(4.44) and

$$Y(I_h) = \{ y(x) : x \in I_h \text{ and } y_n \sim \sigma_2 \sigma_1^{mn} \}$$

where $y(x)$ is defined in (4.41). Then $\Lambda_{n,g,\sigma_1,\sigma_2}$ can be further decomposed into $\{\Lambda_{n,g,\sigma_1,\sigma_2,h}\}_{1 \leq h \leq H}$ by restricting $x \in I_h$. By (4.41), $y = y(x)$ is monotone given $|x|$ sufficiently small. Hence, given $L = L(P, \epsilon, K)$ large enough, by (4.42) and (4.43) one has

$$Y(I_h) \cap Y(I_{h'}) = \emptyset \text{ if } |h-h'| \geq L,$$

(4.45) which implies the following ‘almost orthogonality’ principle:

Claim 1. If there is a constant $A$ s.t.

$$\| \Lambda_{n,g,\sigma_1,\sigma_2,h} \| \leq A,$$

then

$$\| \Lambda_{n,g,\sigma_1,\sigma_2} \| \leq LA$$

(4.47)

Proof of Claim 7.

Consider the congruence classes modulo $L$ in $H$: let $0 \leq \ell < L$ and

$$H_{\ell} = \{ 1 \leq h \leq H : h \equiv \ell \mod L \}.$$  

Then

$$\left\| \sum_{1 \leq h \leq H} \Lambda_{n,g,\sigma_1,\sigma_2,h} \right\| \leq L \sup_{0 \leq \ell < L} \left\| \sum_{h \in H_{\ell}} \Lambda_{n,g,\sigma_1,\sigma_2,h} \right\|.$$  

In addition, notice that

$$\Lambda_{n,g,\sigma_1,\sigma_2,h}(f_1,f_2,f_3) = \Lambda_{n,g,\sigma_1,\sigma_2,h}(f_11_{I_h}, f_21_{Y(I_h)}, f_3).$$
The Cauchy-Schwartz inequality and (4.46) yield
\[
\left| \sum_{h \in H} \Lambda_{n,g,\sigma_1,\sigma_2,\lambda}(f_1, f_2, f_3) \right| \leq \sum_{h \in H} \left| \Lambda_{n,g,\sigma_1,\sigma_2,\lambda}(f_1 1_{I_h}, f_2 1_{Y(I_h)}, f_3) \right|
\]
\[
\leq A \sum_{h \in H} \|f_1 1_{I_h}\|_2 \|f_2 1_{Y(I_h)}\|_2 \|f_3\|_2
\]
\[
\leq A \|f_1\|_2 \|f_2\|_2 \|f_3\|_2
\]
which is controlled by \( A \|f_1\|_2 \|f_2\|_2 \|f_3\|_2 \) due to (4.45). Thus (4.47) is true as desired.

To prove (4.46), we also need the following lemma which is similar to Lemma 4.6 whose proof can be found in the end of this section.

**Lemma 4.7.** If \( K = (P, \epsilon) \) large enough, then for \( 1 \leq h \leq H \), one has
\[
\text{Conv}(\rho_n^{-1}(U_{n,g,h}^*))) \subset \rho_n^{-1}(U_{n,g}^{**}).
\]

Invoking Lemma 4.2, Theorem 3.7, Corollary 3.8 and Theorem 2.1 yields
\[
\|\Lambda_{n,g,\sigma_1,\sigma_2,\lambda}\| \lesssim \begin{cases} \lambda \sigma_1^{n + q_n m_n} \sigma_2^q \cdot 1/6 \\ \sigma_1^{m_0 + \cdots + m_n} 1/2, \end{cases}
\]
for every \( 1 \leq h \leq H \). By Claim 1 summing over \( h \) yields
\[
\|\Lambda_{n,g,\sigma_1,\sigma_2}\| \lesssim \begin{cases} \lambda \sigma_1^{n + q_n m_n} \sigma_2^q \cdot 1/6 \\ \sigma_1^{m_0 + \cdots + m_n} 1/2. \end{cases}
\]
Summing over \( \sigma_1 \) yields
\[
\sum_{\sigma_1} \|\Lambda_{n,g,\sigma_1,\sigma_2}\| \lesssim \|\lambda\|^{n + q_n m_n} \sigma_2^{1/6} \sigma_1^{m_0 + \cdots + m_n} \text{ for } p_n + m_n q_n - q_n (m_0 + \cdots + m_n) > 0\]
and obtain
\[
\sum_{\sigma_1} \sum_{\sigma_2} \|\Lambda_{n,g,\sigma_1,\sigma_2}\| \lesssim \|\lambda\|^{n + q_n m_n} \sigma_2^{1/6} \sigma_1^{m_0 + \cdots + m_n} \}
where the latter inequality will be proved in a moment.

Otherwise we assume
\[
p_n + m_n q_n - q_n (m_0 + \cdots + m_n) \leq 0
\]
Again, by tracking back to the change of variables (see (3.32)), one can see that for \( 1 \leq k \leq n \)
\[
p_k \geq p_{k-1} + m_{k-1} q_{k-1},
\]
and
\[
q_0 \geq q_1 \geq q_2 \geq \cdots \geq q_{n-1} \geq q_n.
\]
Hence, the only possibility (4.52) holds is when \( p_0 = 0 \) and
\[
q_0 = q_1 = q_2 = \cdots = q_{n-1} = q_n.
\]
Then (4.49) becomes:

\[ \| \Lambda_{n,g,\sigma_1,\sigma_2} \| \lesssim \left\{ \begin{array}{ll} |\lambda\sigma_1^{q_0(m_0+\cdots+m_n)}\sigma_2^{q_1}|^{-1/6} \\ (\sigma_2\sigma_1^{m_0+\cdots+m_n})^{1/2}. \end{array} \right. \]

Summing over \( \sigma_2 \) yields

\[ \| \Lambda_{n,g,\sigma_1} \| = \| \sum_{\sigma_2} \Lambda_{n,g,\sigma_1,\sigma_2} \| \leq \sum_{\sigma_2} \| \Lambda_{n,g,\sigma_1,\sigma_2} \| \lesssim |\lambda|^{-\frac{1}{3(q_0+q_1)}}. \]

By the Cauchy-Schwarz Inequality, one has

\[ \sum_{\sigma_1} \| \Lambda_{n,g,\sigma_1} \| \lesssim \sup_{\sigma_1} \| \Lambda_{n,g,\sigma_1} \|, \]

where we have employed the same almost orthogonality trick as in the proof of Claim 1 Indeed, by (4.41) there is a constant \( L \) s.t. if \( \sigma_1 > 2^2\sigma_1', \) then \( \Lambda_{n,g,\sigma_1,\sigma_2} \) and \( \Lambda_{n,g,\sigma_1',\sigma_2} \) integrate in distinct \( x \)-region and distinct \( y \)-region, which allows us to invoke Cauchy-Schwartz Inequality to the functions \( f_1 \) and \( f_2 \).

It remains to show \( q_0 \leq d_E \) and \( (p_n + q_n m_n)/(m_0 + \cdots + m_n) \leq d_E \).

For the former, notice \( p_0 = 0 \) and \( m_0 \geq 1 \), hence \( \text{mult}(P) = d_E = q_0 \).

For the latter, \( m_0 \geq 1 \) implies

\[ d_E = \frac{p_0 + m_0 q_0}{m_0} \geq q_0. \]

Therefore

\[ \frac{p_n + q_n m_n}{m_0 + \cdots + m_n} \leq \frac{(p_0 + m_0 q_0) + q_0 (m_1 + \cdots + m_n)}{m_0 + (m_1 + \cdots + m_n)} \leq \frac{p_0 + m_0 q_0}{m_0} = d_E, \]

as desired.

4.3. Verification of Lemma 4.6 and Lemma 4.7. We only provide the proof of Lemma 4.7 and the proof of Lemma 4.6 is similar and somewhat easier.

Firstly, notice that the upper and the lower boundaries of \( U_{n,g,h}^* \) can be represented by two curves:

\[ \gamma_1(x) = r_0 x^{m_0} + r_1 x^{m_0+m_1} + \cdots + r_{n-1} x^{m_0+\cdots+m_{n-1}+m_n} + r_{n,1} x^{m_0+\cdots+m_{n-1}+m_n} \]

\[ \gamma_2(x) = r_0 x^{m_0} + r_1 x^{m_0+m_1} + \cdots + r_{n-1} x^{m_0+\cdots+m_{n-1}+m_n} + r_{n,2} x^{m_0+\cdots+m_{n-1}+m_n}, \]

and the upper and the lower boundaries of \( U_{n,g,h}^{**} \) by

\[ \gamma_1^*(x) = r_0 x^{m_0} + r_1 x^{m_0+m_1} + \cdots + r_{n-1} x^{m_0+\cdots+m_{n-1}+m_n} + r_{n,1} x^{m_0+\cdots+m_{n-1}+m_n} \]

\[ \gamma_2^*(x) = r_0 x^{m_0} + r_1 x^{m_0+m_1} + \cdots + r_{n-1} x^{m_0+\cdots+m_{n-1}+m_n} + r_{n,2} x^{m_0+\cdots+m_{n-1}+m_n}, \]

where \( r_{n,1} < r_{n,1}' \), \( r_{n,2} > r_{n,2}' \), and \( 0 \leq m_n \leq m_n' \), if they are equal, then \( r_{n,1} > r_{n,2} \).

WLOG, we assume \( r_0 > 0 \) and thus all the curves above are increasing functions of \( x \). By our assumption, \( m_0 \geq 1 \) and thus we only need to take care of the upper boundary of \( U_{n,g,h}^{**} \). Let \( \sigma_{1,h} \) be the left end point of the interval \( I_h \). By the definition of convexity, one needs to verify that if \( K \) sufficiently large, then

\[ \gamma_1(t \sigma_{1,h} + (1-t) \sigma_{1,h+1}) < t \gamma_1^*(\sigma_{1,h}) + (1-t) \gamma_1^*(\sigma_{1,h+1}). \]
for all $0 \leq t \leq 1$ and $0 < \sigma_1 < 1$. Since both $\gamma_1$ and $\gamma_1^*$ are increasing, it suffices to show

\begin{equation}
\gamma_1(\sigma_{1,h+1}) < \gamma_1^*(\sigma_{1,h}).
\end{equation}

By the Mean Value Theorem, there is a $\sigma \in I_h$ s.t.

\begin{equation}
\gamma_1(\sigma_{1,h+1}) - \gamma_1(\sigma_{1,h}) = \frac{3}{2K} \sigma_2 \sigma_1^{m_0 + \cdots + m_n - (m_0 - 1)} \gamma_1'(\sigma).
\end{equation}

Since $\sigma_1/2 \leq \sigma \leq 2\sigma_1$, then $|\gamma_1'(\sigma)| \leq C \sigma_1^{m_0 - 1}$, where $C$ is a constant. Thus

\begin{equation}
\gamma_1(\sigma_{1,h+1}) - \gamma_1(\sigma_{1,h}) \leq C \sigma_2 \sigma_1^{m_0 + \cdots + m_n}.
\end{equation}

On the other hand

\begin{equation}
\gamma_1^*(\sigma_{1,h+1}) - \gamma_1^*(\sigma_{1,h}) = (r_{1,n}^* - r_{1,n}) \sigma_1^{m_0 + \cdots + m_n} \geq (r_{1,n}^* - r_{1,n})(\sigma_1/2)^{m_0 + \cdots + m_n}
\end{equation}

Notice that $\sigma_2 < 1$, thus by choosing $K$ sufficiently large, \ref{eq:4.58} and \ref{eq:4.59} yield \ref{eq:4.56}, as desired.

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