DIFFUSION LIMIT FOR A SLOW-FAST STANDARD MAP

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ABSTRACT. Consider the map \((x, y) \mapsto (x + \epsilon^{-\alpha} \sin(2\pi x) + \epsilon^{-1-\alpha} z, z + \epsilon \sin(2\pi x))\), which is conjugate to the Chirikov standard map with a large parameter. The parameter value \(\alpha = 1\) is related to “scattering by resonance” phenomena. For suitable \(\alpha\), we obtain a central limit theorem for the slow variable \(z\) for a (Lebesgue) random initial condition. The result is proved by conjugating to the Chirikov standard map and utilizing the formalism of standard pairs. Our techniques also yield for the Chirikov standard map a related limit theorem and a “finite-time” decay of correlations result.

1. INTRODUCTION AND STATEMENT OF RESULTS

1.1. The slow-fast standard map. Throughout, \(\alpha > 0\) is fixed. We consider the discrete-time slow fast system \(G_\epsilon\) on the cylinder \(\mathbb{T}^1 \times \mathbb{R}\) defined as follows:
\[
G_\epsilon(x, z) = (x + \epsilon^{-\alpha} \sin(2\pi x) + \epsilon^{-1-\alpha} z \pmod{1}, z + \epsilon \sin(2\pi x))
\]
This map is a composition of two maps \(G_\epsilon = S_\epsilon \circ T_\epsilon\), where the ‘tilt’ map \(T_\epsilon\) and the ‘shear’ map \(S_\epsilon\) are defined by
\[
T_\epsilon(x, z) = (x, z + \epsilon \sin(2\pi x)), \quad S_\epsilon(x, z) = (x + \epsilon^{-1-\alpha} z, z).
\]
This combination of tilting and shearing serves as a good model on many slow-fast physical systems. In particular, we are inspired by the following models.

Scattering by resonance. We only give a heuristic picture here and refer to [13], [14], [15], [8] for details. To use a specific example (see [8]), consider the slow-fast system
\[
\dot{\phi} = f(\phi, I, \theta, \epsilon), \quad \dot{I} = g(\phi, I, \theta, \epsilon), \quad \dot{\theta} = \epsilon^{-2} \omega(\phi, I, \epsilon), \quad \phi, \theta \in \mathbb{T}^1, I \in \mathbb{R}.
\]
It is assumed that the averaged system
\[
\dot{\phi} = \int_0^1 f(\phi, I, \theta, 0)d\theta = p(I), \quad \dot{I} = \int_0^1 g(\phi, I, \theta, 0)d\theta = 0
\]
is completely integrable. However, the averaging is not justified near the resonant surface \(\{\omega(\phi, I, 0) = 0\}\), since the fast variable \(\theta\) is no longer fast.

As the orbit in \((I, \phi)\) passes through the resonances, two different phenomena may happen:
- **Strong resonance**, where there is a probability of $O(\epsilon)$ for the orbit to be captured by the resonance, and stay captured for a random time before it is repelled. See [8] for a full analysis of this picture and the related limit theorems.

- **Weak resonance**, where the orbit passes through the resonance without being captured. After the passing the variable $I$ changes by order $\epsilon$, with average flux 0. This is called scattering by resonance.

As the orbit crosses a weak resonance, the change to the variables $(\theta, I)$ can be approximated by a map of the type $T_\epsilon$; while the “free flight” between two crossings of the resonance is approximated by the map $S_\epsilon(s, z) = (x + \epsilon^{-2}z, z)$. As a result, many passings of resonances can be modeled by sequential applications of maps of type $G_\epsilon$ (with $\alpha = 1$).

**Arnold diffusion in a priori unstable systems.** There is an extensive literature on Arnold diffusion in a priori unstable systems, and we refer to just a few of them ([17], [5], [10]) and refer the readers to the references therein. In these models, existence of unstable orbits that shadows a normally hyperbolic cylinder is proved. The orbits that shadow the cylinder may be modeled by the iteration of two maps: a “scattering map” which reflects the excursion along the unstable/stable manifold, and an “inner map” which is the system restricted to the cylinder. In this case the scattering map is modeled by the tilting map $T_\epsilon$, and the inner map is modeled by the shearing map $S_\epsilon$. We remark that in this case that the shearing is much weaker: in general $S_\epsilon(x, z) = (x + \ln \epsilon z, z)$, therefore it remains to be seen whether our model is a good description.

**Diffusive limit for the slow-fast system.** Writing $(x^\epsilon_n, z^\epsilon_n) = G_\epsilon^n(x_0, z_0)$ for $(x_0, z_0) \in \mathbb{T}^1 \times \mathbb{R}$ fixed, observe that

$$z^\epsilon_N = z_0 + \epsilon \sum_{n=0}^{N-1} \sin(2\pi x^\epsilon_n).$$

The $x$ coordinate is clearly ‘fast’ relative to the $z$, and so one anticipates $z^\epsilon_N$ to have a diffusion limit in the regime $N = N(\epsilon) = [\epsilon^{-2}]$, when we consider is as a random variable with respect to the initial conditions $(x_0, z_0)$. This does not follow from conventional averaging arguments, however, since, as will be explained in detail in the following sections, the fast dynamics has critical behavior at $x \approx \frac{1}{4}, \frac{3}{4}$ (the zeros of $x \mapsto 1 + 2\pi \epsilon^{-\alpha} \cos(2\pi x)$).

Our approach is to conjugate the above system to the Standard Map:

$$F_L(x, y) = (x + y + L \sin(2\pi x), y + L \sin(2\pi x))$$

by the change of variables $z = \epsilon^{-(1+\alpha)} y$; here, the parameter $L$ is defined by $L = \epsilon^{-\alpha}$. Notice that the $x$ coordinate is unchanged, and so we have that $x^\epsilon_n$ is the $x$-coordinate of $F_L^\epsilon(x_0, y_0)$ where $y_0 = \epsilon^{-(1+\alpha)} z_0$. Thus, the diffusion limit for $z^\epsilon_{N(\epsilon)}$ above is
equivalent to a central limit theorem for the sequence

\[ \left( \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} \psi \circ F_{L}^{i} \right)_{L \to \infty}. \]

Here, \( \psi(x,y) = \sin(2\pi x) \), and in the above sequence, we write \( N = N(L) = \lfloor L^{\beta} \rfloor \), \( \beta := 2/\alpha \); this scaling is equivalent to the original diffusion limit for \( z_{N(\epsilon)}^{\epsilon} \).

1.2. Statement of results.

Theorem A. Suppose \( \alpha > 8 \) and let \([a,b] \subset \mathbb{R} \) be a non-trivial interval. Let \( X, Z \) be uniformly distributed random variables on \( T \) and \([a,b]\) respectively. Define \( Z_{\epsilon}^{n} = \pi_{z} G_{\epsilon}^{n}(X,Z) \), then for \( N(\epsilon) = \lfloor \epsilon^{-2} \rfloor \), the random variable

\[ Z_{\epsilon}^{N(\epsilon)} - Z \]

converges in distribution to the centered Gaussian \( N(0, \frac{1}{2}) \), as \( \epsilon \to 0 \).

Theorem A will be deduced from the following analogous result for Standard Maps. In the following results, we regard \( F_{L} \) as a diffeomorphism of \( T^{2} \cong \mathbb{R}^{2}/\mathbb{Z}^{2} \). Let \( X, Y \) be independent random variables distributed uniformly on \( T \).

Theorem B. Let \( \phi : T^{1} \to \mathbb{R} \) be a \( C^{1} \) observable, regarded as an \( x \)-dependent observable on \( T^{2} \). Assume \( \int \phi \, dx = 0 \) and that \( \phi \) is not identically 0. Let \( N : \mathbb{R}_{>0} \to \mathbb{N} \) be an increasing function and assume

\[ N(L) \cdot L^{-\frac{1}{4}} \to 0 \quad \text{as } L \to \infty. \]

Then,

\[ \frac{1}{\sqrt{N(L)}} \sum_{i=0}^{N-1} \phi \circ F_{L}^{i}(X,Y) \]

converges in distribution to the centered Gaussian \( N(0, \sigma^{2}) \) with variance \( \sigma^{2} = \int \phi^{2} \, dx > 0 \).

As a consequence of our techniques we obtain the following result on decay of correlations, which we report here as it might be useful for future developments.

Theorem C. There exists a constant \( C > 0 \) for which the following holds for all \( L > 0 \) sufficiently large. Let \( \phi, \psi : T^{1} \to \mathbb{R} \) be \( C^{1} \) observables, each regarded as \( x \)-dependent observables on \( T^{2} \). Then, for all \( n \geq 1 \):

\[ \left| \int \psi \cdot \phi \circ F_{L}^{n} - \int \phi \int \psi \right| \leq C \| \phi \|_{C^{1}} \| \psi \|_{C^{1}} \left( (n-1)L^{-3/4} + L^{-1/2} \right). \]
Background on the proof: Standard pairs. Standard pairs are a modern tool which can be used to study statistical properties of systems with some hyperbolicity. They have been introduced by Dolgopyat in a variety of settings (see for example [6], [7], [8]) and have proved to be of invaluable help. In a nutshell, standard pairs are probability measures on the phase space which enjoy particularly good dynamical properties (see Lemmata 7–9). The main feature of such measures is that they allow to introduce a sensible notion of conditioning in the deterministic setting. In probability, conditioning is one of the most basic and useful techniques, and one would like to employ this tool also in our situation. Clearly, in deterministic settings, some care must be taken, as if one were to condition on the configuration of the system at any given time, the whole probabilistic picture would collapse (as no randomness would be present anymore). Standard pairs provide a very efficient solution to this fundamental problem.

The Standard Map. The Standard Map is a one-parameter family of area-preserving analytic diffeomorphisms of \( T^2 \). It has been the subject of extensive numerical and analytical study, starting from the pioneering work of V. Chirikov and (independently) J. B. Taylor. From the physical point of view it describes the dynamics of a mechanical system known as the kicked rotor, but it can be found in a large number of different other models. For example: it describes ground states of the Frenkel–Kontorova Model (see [9,1]), it models dynamics of particles in accelerators (see [3][11]) and dynamics of balls bouncing on periodically oscillating platform (see e.g. [16][4]). From the mathematical point of view it has been studied as a natural example of dynamical system exhibiting mixed behavior: it is conjectured\(^1\) that the phase space of the standard map has positive Lebesgue measure sets where the dynamics is hyperbolic (and enjoys strong statistical properties) and positive Lebesgue measure sets where the dynamics is regular (elliptic islands). In this respect, points belonging to the hyperbolic component of the phase space should undergo some sort of diffusion.

This article constitutes a first step in this direction, in which we consider a scaling limit in which the natural parameter of the Standard Family is increased together with the number of iterations. A first result about statistical properties of the Standard Family in this scenario can be found in [2], in which it is shown that compositions of standard maps with increasing parameter exhibit both asymptotic decay of correlations and a Central Limit Theorem with respect to Holder-continuous observables. A correlation estimate analogous to that in Theorem\(^C\) is also exhibited. Although the decay estimate in Theorem\(^C\) is better than that appearing in [2], the estimate in the present paper applies only to \( x \)-dependent observables, while that in [2] applies to all Holder-continuous observables.

Plan for the paper. The plan for the paper is as follows. In Section 2 we give some preliminaries, including the definition of standard pair and various related notions used in this paper. In Section 3 we consider the dynamics of standard pairs, prove

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\(^1\) This fact has notoriously not yet found a proof and is also widely believed to be astonishingly difficult to prove.
results on correlation decay of standard pairs, and use it to prove Theorem C. In Section 4 we prove the Central Limit Theorem as stated in Theorem B. In Section 5 we deduce Theorem A from Theorem B.

Notation and conventions.

- We parametrize the circle $\mathbb{T}^1$ by the half-open interval $[0,1)$. Additive formulas in $\mathbb{T}^1$ are always considered (mod 1), i.e., under the natural projection $\mathbb{R} \to \mathbb{T}^1 = \mathbb{R}/\mathbb{Z} \cong [0,1)$. We parametrize $\mathbb{T}^2$ by $[0,1)^2$.
- We call a continuous observable $\phi : \mathbb{T}^2 \to \mathbb{R}$ x-dependent if it can be represented as $\phi(x,y) = \hat{\phi}(x)$ for some $\hat{\phi} : \mathbb{T}^1 \to \mathbb{R}$. In this manuscript we will often use the same notation $\phi$ for both the observable on $\mathbb{T}^1$ and the corresponding $x$-dependent observable on $\mathbb{T}^2$.
- For a $C^1$ function $g$ defined on an open interval in $\mathbb{R}$ or $\mathbb{T}^1$, we write $\dot{g}$ or $\frac{d}{dx}g$ for the derivative of $g$.
- Leb refers to normalized Lebesgue measure on $\mathbb{T}^2 \cong [0,1)^2$.
- Let $G = G(L)$ denote any quantity depending on the parameter $L$. We say that another quantity $H = H(L)$ is in the class $O(G)$, written $H = O(G)$, if $\limsup_{L \to \infty} \frac{|H(L)|}{|G(L)|} < \infty$. We say $H$ is in the class $o(G)$, written $H = o(G)$, if $\lim_{L \to \infty} \frac{|H(L)|}{|G(L)|} = 0$.
- We write $G \approx H$ if $G/H = O(1)$ and $H/G = O(1)$.

2. Preliminaries

2.1. Coordinate change. Under the coordinate change $y \mapsto x - y$, the Standard map $F_L$ (defined in [1]) is conjugate to the map

$$\hat{F}_L(x,y) = (2x - y + L \sin(2\pi x) \pmod{1}, x),$$

which we regard as a map on $\mathbb{T}^2$. This change in the $y$-coordinate has no effect on the analysis of our diffusion limit, since the observable $\phi$ is $x$-dependent. This form for the Standard Map is convenient and will be used from now on. Hereafter we abuse notation and write $F = \hat{F}_L$, dropping the subscript $L$ (which is implicit throughout). Additionally, we define

$$f = f_L : \mathbb{T}^1 \to \mathbb{R}, \quad f(x) := 2x + L \sin(2\pi x),$$

so that $F = F_L$ has the form

$$F(x,y) = (f(x) - y \pmod{1}, x).$$

In all that follows, we regard $F$ as a map on the torus $\mathbb{T}^2 = \mathbb{T}^1 \times \mathbb{T}^1$. At times, it is also convenient to use instead the map $\hat{F} : \mathbb{T}^2 \to \mathbb{R} \times \mathbb{T}^1$ obtained by omitting the “$\pmod{1}$” in the $x$-coordinate.
2.2. Predominant hyperbolicity of $F$. For fixed $\eta \in (0, 1)$, define
$$S_\eta = \{(x, y) \in T^2 : |2 + 2\pi L \cos(2\pi x)| \leq 2L^\eta\} = B_\eta \times T^1$$
For all $L$ large and any $\eta \in (0, 1)$, the set $S_\eta$ consists of two small, disjoint vertical strips in $T^2$; observe that, trivially, $S_\eta \subset S_{\eta'}$ for $\eta < \eta'$. Away from the set $S_\eta$, the map $F$ is strongly expanding in the horizontal direction to order $L^\eta$; for this reason we refer to the $S_\eta$ as critical strips.

To make this picture more precise, for $\xi > 0$ let us define the horizontal cone
$$C_\xi = \{v = (u, w) \in \mathbb{R}^2 : |w| \leq \xi |u|\}.$$

**Lemma 1.** For all $L$ sufficiently large, the following holds for each $\eta \in (0, 1)$.
(a) The set $B_\eta$ is the union of two disjoint intervals, each of length $\approx L^{-1+\eta}$, containing respectively the points $1/4$ and $3/4$. In particular, the set $S_\eta$ satisfies
$$\text{Leb}(S_\eta) = O(L^{-1+\eta}).$$
(b) Let $\xi \leq L^\eta$. For all $p = (x, y) \in T^2 \setminus S_\eta$, we have that
$$dF_p C_\xi \subset C_\xi'$$
for any $\xi' \geq \frac{1}{2L^\eta} - \xi$.

**Proof.** (a) Since $S_\eta = \{|2L^{-1} + 2\pi \cos(2\pi x)| \leq 2L^{-1+\eta}\} \subset \{2\pi \cos 2\pi x \leq 4L^{-1+\eta}\}$, the estimate follows easily.

(b) Write $\Lambda(x) = 2 + \pi L \cos(2\pi x)$, then $|\Lambda(x)| > 2L^\eta$ for $x \notin S_\eta$. For a tangent vector $(1, m) \in C_\xi$, we have
$$dF_p \begin{bmatrix} 1 \\ m \end{bmatrix} = \begin{bmatrix} 2 + 2\pi L \cos(2\pi x) & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ m \end{bmatrix} = \begin{bmatrix} \Lambda(x) + m \\ -1 \end{bmatrix} = (\Lambda(x) + m) \begin{bmatrix} 1 \\ -1 \end{bmatrix} \in C_\xi'.$$

For our purposes, we usually work with the cone $C_{1/10}$, which by Lemma 1 is mapped into itself away from $S_{1/4}$, if $L$ is sufficiently large.

2.3. u-curves. We work mostly with $C^2$ curves, the tangents to which lie in the cone $C_{1/10}$. More precisely:

**Definition 2.** Let $\gamma$ be a $C^2$ embedded curve in $T^2$. We say that $\gamma$ is a u-curve if $\gamma = \{(x, h_\gamma(x)) : x \in I_\gamma\}$, where
(a) $I_\gamma \subset \mathbb{T}^1$ is an open interval; and
(b) $h_\gamma : I_\gamma \to \mathbb{T}^1$ is a $C^2$ mapping with $\|h_\gamma\|_{C^0} \leq 1/10$, $\|\dot{h_\gamma}\|_{C^0} \leq L$.

The length of a u-curve $\gamma$ is defined (with a small abuse of terminology) as the length of the interval $I_\gamma$. We call $\gamma$ a fully-crossing u-curve if $I_\gamma = (0, 1)$.

Away from the critical strips, u-curves map to u-curves, for which the following lemma is useful.

**Lemma 3.** Fix $\eta \in [1/4, 1)$. Let $\gamma$ be a u-curve with $\gamma \cap S_\eta = \emptyset$. Then, $\tilde{\gamma} := \tilde{F}(\gamma)$ is a $C^2$ curve of the form $\tilde{\gamma} = \{(x, \tilde{h}(x)) : x \in \tilde{I}\}$, where $\tilde{I} \subset \mathbb{R}$ is an interval and $\tilde{h} : \tilde{I} \to \mathbb{T}^2$ is a $C^2$ mapping with $\|\frac{d}{dx}\tilde{h}\| \leq 1/10$, $\|\frac{d^2}{dx^2}\tilde{h}\| \leq L$. 


From Lemma 3, we can represent \( F(\gamma) \) as a finite union of \( \gamma \)-curves by subdividing \( \tilde{\gamma} \) into \( \gamma \)-curves of length < 1 and then projecting \( \mathbb{R} \times \mathbb{T}^1 \rightarrow \mathbb{T}^2 \).

**Proof of Lemma 3.** Define \( f_\gamma : I_\gamma \rightarrow \mathbb{R} \) by setting
\[
f_\gamma(x) = f(x) - h_\gamma(x).
\]
Since \( \hat{f}_\gamma(x) = 2 + 2\pi L \cos(2\pi x) - \hat{h}_\gamma(x) \), we have the estimate \( |\hat{f}_\gamma(x)| \geq 2L^n - |\hat{h}_\gamma| \geq L^n \), which will be useful throughout. Let us also note \( |\hat{f}_\gamma(x)| = |4\pi^2 L \sin(2\pi x) + \hat{h}_\gamma(\gamma)| = O(L) \).

As one can check, \( \tilde{F}(x, h_\gamma(x)) = (f_\gamma(x), x) \), from which Lemma 3 follows with \( \tilde{h} := (f_\gamma)^{-1} : \tilde{I} \rightarrow \mathbb{T}^1 \), where \( \tilde{I} := f_\gamma(I) \). The estimates on \( \|d_x \tilde{h}\|, \|d^2_x \tilde{h}\| \) immediately follow from the formulæ
\[
\frac{d}{dx} \tilde{h} = \frac{1}{f_\gamma} \circ f^{-1}_\gamma, \quad \frac{d^2}{dx^2} \tilde{h} = -\frac{\hat{f}_\gamma}{(f_\gamma)^3} \circ f^{-1}_\gamma. \quad \square
\]

2.4. Standard pairs. Let \( a_0 \in (0, 1/8] \).

**Definition 4.** A measure pair is a pair \( (\gamma, \rho) \), where \( \gamma \) is a \( \gamma \)-curve and \( \rho : I_\gamma \rightarrow (0, \infty) \) is a \( C^1 \) probability density on \( I_\gamma \). We distinguish three subclasses of measure pairs:

(a) We call \( (\gamma, \rho) \) a standard pair if (i) \( |I_\gamma| > a_0 \), and (ii) \( \rho \) satisfies the distortion estimate
\[
\left\| \frac{d \log \rho}{dx} \right\| \leq 3C_0,
\]
where \( C_0 = 8\pi^2 \), \( \| \cdot \| \) denotes the uniform norm and \( a_0 > 0 \) is a small, fixed positive constant (see above). We call \( (\gamma, \rho) \) a fully-crossing standard pair if \( \gamma \) is fully-crossing.

(b) We call \( (\gamma, \rho) \) a substandard pair if (i) \( |I_\gamma| \in [L^{-\frac{1}{2}}, a_0]; \) (ii) \( \rho \) satisfies \( \|d \log \rho/dx\| \leq 2C_0L^{\frac{1}{4}} \); and (iii) \( I_\gamma \cap B_{1/2} = \emptyset \) (equivalently, \( \gamma \cap S_{1/2} = \emptyset \)).

**Remark 5.** The value \( a_0 \in (0, 1/8] \) above is fixed and independent of \( L \), although for our purposes it will be useful fix it at a sufficiently small value. This will be done by the end of Section 3 (see Remark 10). Before then, however, we include the parameter \( a_0 \) in our \( O(\cdots) \) estimates.

Moreover, for a curve \( \gamma \) we write \( \text{Leb}_\gamma \) for the (un-normalized) Lebesgue measure on \( \gamma \). Since \( x \mapsto (x, h_\gamma(x)) \) is a diffeomorphism of \( I_\gamma \) onto \( \gamma \), we identify \( \text{Leb}_\gamma \) with the corresponding measure on \( I_\gamma \) given by
\[
d \text{Leb}_\gamma(x) = \sqrt{1 + \tilde{h}^2_\gamma(x)} \, dx.
\]

Some additional conventions: we regard measure pairs \( (\gamma, \rho) \) as measures on the curve \( \gamma \) itself via the parametrization \( x \mapsto (x, h_\gamma(x)) \). In particular, \( F_*(\gamma, \rho) \) refers to the pushforward measure of \( (\gamma, \rho) \) on the image set \( F(\gamma) \) (which, we note, need not
be a u-curve). Moreover, for continuous observables \( \phi : \mathbb{T}^2 \to \mathbb{R} \) we write \( \int \phi \, d(\gamma, \rho) \) for the integral of \( \phi \) with respect to the measure \((\gamma, \rho)\) on \( \gamma \).

Before proceeding, we record the following distortion estimate, which will be used many times in the coming proofs.

**Lemma 6.** Let \( \eta \in [1/4, 1] \). Fix u-curves \( \gamma, \gamma' \subset \mathbb{T}^2 \) for which \( \gamma \cap \mathcal{S}_\eta = \emptyset \) and \( F(\gamma) \supset \gamma' \). Let \((\gamma, \rho)\) be a measure pair, and define \( \rho' \) so that \((\gamma', \rho')\) is the normalization of \( F_*(\gamma, \rho)|_{\gamma'} \). Then,

\[
\left\| \frac{d \log \rho'}{dx} \right\| \leq L^{-\eta} \left\| \frac{d \log \rho}{dx} \right\| + C_0 L^{1-2\eta},
\]

where \( C_0 := 8\pi^2 \) and \( \| \cdot \| \) refers to the uniform norm.

**Proof.** Let \( \tilde{f}_\gamma \) be as in the proof of Lemma C. Define \( I' \subset I \) to be the subinterval for which \( \gamma' = F(\text{graph}(h_{\gamma'|I'})) \), noting that \( \tilde{f}_\gamma : I' \to I_{\gamma'} \) is a \( C^2 \) diffeomorphism. Clearly,

\[
\rho' = \frac{1}{\int_{I'} \rho \, dx} \cdot \frac{\rho}{|\tilde{f}_\gamma|} \circ (f_{\gamma'|I'})^{-1}.
\]

For simplicity, assume \( \dot{\tilde{f}}_\gamma > 0 \) on \( I' \) (either this or \( \dot{\tilde{f}}_\gamma < 0 \) holds since \( \dot{\tilde{f}}_\gamma \neq 0 \) on \( I' \)); otherwise the formulas below differ by a minus sign. We compute

\[
\frac{d \log \rho'}{dx} = \frac{1}{\rho'} \frac{d \rho'}{dx} = \left( \frac{1}{\tilde{f}_\gamma} \frac{d \log \rho}{dx} - \frac{\dot{\tilde{f}}_\gamma}{(f_{\gamma})^2} \right) \circ (f_{\gamma'|I'})^{-1},
\]

from which we get the estimate

\[
\left\| \frac{d \log \rho'}{dx} \right\| \leq L^{-\eta} \left\| \frac{d \log \rho}{dx} \right\| + C_0 L^{1-2\eta},
\]

where \( C_0 = 8\pi^2 \).

### 3. Images of standard pairs and correlation decay

Our aim in this section is to describe the pushforward \( F^\eta_*(\gamma, \rho) \) of a fully-crossing standard pair. In Section 3.1, we consider pushing forward measure pairs one timestep, while in Section 3.2 we will iterate these arguments to describe \( F^\eta_*(\gamma, \rho) \). Applications to decay of correlations are derived in Section 3.3. This includes a proof of Theorem C.

#### 3.1. Pushing forward standard pairs by \( F \).

Here we describe how to push forward measure pairs of varying regularity: fully crossing, standard, and substandard.

**Notation and setup.** For a measure pair \((\gamma, \rho)\), we will describe the pushforward \( F_*(\gamma, \rho) \). Depending on the regularity (e.g., standard versus substandard) of \((\gamma, \rho)\), we will subdivide

\[
F_*(\gamma, \rho) = \mu_{\mathcal{L}(\gamma, \rho)} + \mu_{\mathcal{I}(\gamma, \rho)} + \mu_{\mathcal{J}(\gamma, \rho)} + \mu_{\mathcal{E}(\gamma, \rho)},
\]

where \( \mu_{\mathcal{L}} = \mu_{\mathcal{L}(\gamma, \rho)} \), \( \mu_{\mathcal{I}} = \mu_{\mathcal{I}(\gamma, \rho)} \), \( \mu_{\mathcal{J}} = \mu_{\mathcal{J}(\gamma, \rho)} \), and \( \mu_{\mathcal{E}} = \mu_{\mathcal{E}(\gamma, \rho)} \) are, respectively, weighted sums over collections \( \mathcal{L} = \mathcal{L}(\gamma, \rho) \), \( \mathcal{I} = \mathcal{I}(\gamma, \rho) \), \( \mathcal{J} = \mathcal{J}(\gamma, \rho) \) of measure pairs consisting, respectively,
of fully-crossing standard pairs, standard pairs, and substandard pairs. Here, $\mu_\mathcal{E} = \mu_{\mathcal{E}(\gamma, \rho)}$ is a measure corresponding to the portion of $F_*(\gamma, \rho)$ which we do not control, and is supported on a subset $\mathcal{E} = \mathcal{E}(\gamma, \rho)$ of $F(\gamma)$ for which $F_*(\gamma, \rho)|_{\mathcal{E}} = \mu_\mathcal{E}$.

**Notational remark.** Abusing notation somewhat, when it is clear from context we will use $\mathcal{L}$ to refer to (i) a collection $\{(\gamma', \rho')\}$ of fully-crossing standard pairs; (ii) a partition of a subset of $F(\gamma)$ into fully-crossing u-curves $\gamma'$; and (iii) the subset of $F(\gamma)$ itself, i.e., the union over all $\gamma' \in \mathcal{L}$. The same applies to each of $\mathcal{I}, \mathcal{J}$.

For a measure $\mu$, we write $\|\mu\|$ for the total mass of $\mu$.

We begin by describing the $\mathcal{L}, \mathcal{I}, \mathcal{J}, \mathcal{E}$ decomposition when $(\gamma, \rho)$ is a standard pair, not necessarily fully-crossing.

**Lemma 7.** Let $(\gamma, \rho)$ be a standard pair for which $\gamma$ is not necessarily fully-crossing. Then,

$$F_*(\gamma, \rho) = \mu_{\mathcal{L}} + \mu_\mathcal{E},$$

where $\|\mu_\mathcal{E}\| = O(a_0^{-1}L^{-1/2}).$

**Proof.** To start, we allocate $F(\gamma \cap S_{1/2})$ to $\mathcal{E} = \mathcal{E}(\gamma, \rho)$ and subdivide $\gamma \setminus S_{1/2}$ into at most three connected components $\tilde{\gamma}$.

For each $\tilde{\gamma}$, in the notation of Lemma 3, subdivide $\tilde{\gamma} = \tilde{F}(\tilde{\gamma})$ into pieces $\tilde{\gamma}_n = \tilde{\gamma} \cap [n, n+1), n \in \mathbb{Z}$. Of the nonempty $\tilde{\gamma}_n$, at most two have length $< 1$; these are allocated to $\mathcal{E}$, while the $\tilde{\gamma}_n$ of length 1 are are projected to $\mathbb{T}^2$ and allocated to $\mathcal{L} = \mathcal{L}(\gamma, \rho)$. Distortion is checked as in Lemma 3 with $\eta = 1/2$; details are left to the reader.

To estimate $\|\mu_\mathcal{E}\|$, we note that $(\gamma, \rho)(S_{1/2}) = O(a_0^{-1}L^{-1/2})$, while for any nonempty $\tilde{\gamma}_n$ as above, we have $(\gamma, \rho)(\tilde{F}^{-1}(\tilde{\gamma}_n)) = O(a_0^{-1}L^{-1/2})$. \hfill $\square$

Next, we consider images of substandard pairs.

**Lemma 8.** Let $(\gamma, \rho)$ be a substandard pair. Then,

$$F_*(\gamma, \rho) = \mu_{\mathcal{I}} + \mu_{\mathcal{J}} + \mu_\mathcal{E},$$

where $\|\mu_{\mathcal{J}}\| = O(a_0)$ and $\|\mu_\mathcal{E}\| = O(L^{-1/2}).$

In particular, if $a_0$ is chosen sufficiently small, we have $\|\mu_{\mathcal{J}}\| \leq 1/2$ when $(\gamma, \rho)$ is substandard.

**Proof.** Without loss of generality, let us assume that $\gamma$ has length $\in [L^{-1/2}, 2L^{-1/2}]$. If not, then subdivide $\gamma$ into pieces $\gamma_i$ with lengths $\in [L^{-1/2}, 2L^{-1/2}]$ and consider separately each $(\gamma_i, \rho_i)$, where $\rho_i$ is the renormalized restriction of the density $\rho_i := (\gamma_i, \rho)(\gamma_i)^{-1} \cdot \rho|_{\gamma_i}$. Note that by our reduction, $\sup_{x_1, x_2 \in I} |\log \rho(x_2) - \rho(x_1)| \leq \|\rho\|_I \log |I| = O(L^{1/2}L^{-1/2}) = O(1)$. Since $\rho$ is a probability density on $|I| \approx L^{-1/2}$, we have $\rho \approx L^{1/2}$.

Observe that $\tilde{\gamma} = \tilde{F}(\gamma)$ has length larger than $L^{1/2} \cdot L^{-1/2} = 1$. With $\tilde{\gamma}_n = \tilde{\gamma} \cap (\mathbb{Z} \times \mathbb{T}^1)$ as in the proof of Lemma 7, allocate all fully-crossing $\tilde{\gamma}_n$ to
\[ \mathcal{I} = \mathcal{I}(\gamma, \rho). \] At most two \( \tilde{\gamma}_n \) remain, each of length \( < 1 \). For each, we distinguish three cases: we add \( \tilde{\gamma}_n \) to

(i) \( \mathcal{I} \) if \( |I_{\tilde{\gamma}_n}| > \alpha_0 \),

(ii) \( \mathcal{J} \) if \( |I_{\tilde{\gamma}_n}| \in [L^{-1/2}, \alpha_0] \), or

(iii) \( \mathcal{E} \) if \( |I_{\tilde{\gamma}_n}| < L^{-1/2} \).

In case (ii), note that \( \tilde{\gamma}_n \cap S_{1/2} = \emptyset \) since for all \( L \) sufficiently large, the critical strips comprising \( S_{1/2} \) are a distance \( > 1/5 \) from \( \{x = 0\} \times \mathbb{T}^1 \), while \( I_{\tilde{\gamma}_n} \) has the form \([n, n+c)\) or \([n, n+1-c)\) for some \( c \leq \alpha_0 \leq 1/8 \). In order to estimate the contributions to \( \mu, \mu_\mathcal{J}, \mu_\mathcal{E} \), respectively, note that in case (ii) we have \( (\gamma, \rho)(F^{-1}(\tilde{\gamma}_n)) = O(|I_{F^{-1}(\tilde{\gamma}_n)}| \|\rho\|) = O(\alpha_0 L^{-1/2} L^{1/2} = O(\alpha_0) \), while in case (iii) we have \( (\gamma, \rho)(F^{-1}(\tilde{\gamma}_n)) = O(L^{-1/2} \cdot L^{-1/2} L^{1/2}) = O(L^{-1/2}) \).

It remains to check distortion. For any \( (\gamma', \rho') \in \mathcal{I} \cup \mathcal{J} \), by Lemma \( \text{8} \) with \( \eta = 1/2 \) and the definition of a substandard pair we have

\[ \left\| \frac{d \log \rho'}{dx} \right\| \leq L^{-1/2} \cdot 2C_0 L^{1/2} + C_0 \leq 3C_0. \]

Finally, we consider fully-crossing standard pairs.

**Lemma 9.** Let \( (\gamma, \rho) \) be a standard pair for which \( \gamma \) is fully-crossing. Then, \( F_*(\gamma, \rho) \) admits a representation of the form

\[ F_*(\gamma, \rho) = \mu_\mathcal{L} + \mu_\mathcal{J} + \mu_\mathcal{F} + \mu_\mathcal{E}, \]

where \( \|\mu_\mathcal{L}\| = O(L^{-1/2}), \|\mu_\mathcal{J}\| = O(\alpha_0 L^{-1/2}) \) and \( \|\mu_\mathcal{E}\| = O(L^{-3/4}) \).

**Proof.** To start, \( F(\gamma \cap S_{1/4}) \) is allocated to \( \mathcal{E} \), giving an \( O(L^{-3/4}) \) contribution to the mass of \( \mu_\mathcal{E} \).

To allocate \( F(\gamma \setminus S_{1/2}) \): the set \( \gamma \setminus S_{1/2} \) has three connected components \( \tilde{\gamma} \), each of which we handle separately. Fixing a \( \tilde{\gamma} \) and setting \( \tilde{\gamma} = F(\tilde{\gamma}), \tilde{\gamma}_n = \tilde{\gamma} \cap ([n, n+1) \times \mathbb{T}^1) \), allocate all \( \tilde{\gamma}_n \) of length \( 1 \) to \( \mathcal{L} \). For the two remaining nonempty \( \tilde{\gamma}_n \), allocate to \( \mathcal{I}, \mathcal{J}, \mathcal{E} \) according to cases (i) – (iii) in the proof of Lemma \( \text{8} \). As in Lemma \( \text{8} \) in case (ii) we automatically have \( \tilde{\gamma}_n \cap S_{1/2} = \emptyset \). This step contributes \( O(L^{-1/2}) \) mass to \( \mathcal{I} \); \( O(\alpha_0 L^{-1/2}) \)-mass to \( \mathcal{J} \); and \( O(L^{-1}) \) mass to \( \mathcal{E} \). Distortion for measure pairs in \( \mathcal{L} \cup \mathcal{I} \cup \mathcal{J} \) allocated so far can be checked using Lemma \( \text{8} \) with \( \eta = 1/2 \).

For \( F(\gamma \cap (S_{1/2} \setminus S_{1/4})) \), we consider each of the four connected components \( \tilde{\gamma} \) of \( \gamma \cap (S_{1/2} \setminus S_{1/4}) \) separately. To start, observe that the length of \( \tilde{\gamma} = F(\tilde{\gamma}) \) can be estimated

\[ |I_{\tilde{\gamma}}| \approx \int_{L^{-3/4}}^{L^{-1/2}} L \cdot z \, dz \approx 1. \]

In particular, \( \tilde{\gamma} \cap S_{1/2} \) has an \( O(1) \) number of connected components. We allocate each to \( \mathcal{E} \), contributing \( O(L^{-1/2} \cdot L^{-1}) = O(L^{-3/4}) \) mass to \( \mu_\mathcal{E} \). For each connected component \( \zeta \) of \( \tilde{\gamma} \setminus S_{1/2} \), allocate \( \zeta \) to

(a) \( \mathcal{J} \) if \( \zeta \) has length \( \in [L^{-1/2}, \alpha_0] \) or

(b) \( \mathcal{E} \) if \( \zeta \) has length \( < L^{-1/2} \).
If (c) $\zeta$ has length $> a_0$, then subdivide $\zeta$ into pieces of length $[a_0/2, a_0]$ and allocate each to $\mathcal{J}$. In cases (a), (c), the contribution to $\mu_{\mathcal{J}}$ is $O(L^{-1/2})$, while in case (b) the contribution to $\mu_{\mathcal{E}}$ is $O(L^{-3/4})$.

To check distortion: for any $(\gamma', \rho') \in \mathcal{J}$ with $F^{-1}(\gamma') \subset \gamma \cap (\mathcal{S}_{1/2} \setminus \mathcal{S}_{1/4})$, we have from Lemma 6 with $\eta = 1/4$ that

$$\left\| \frac{\log d\rho'}{dx} \right\| \leq L^{-1/4} \cdot 3C_0 + C_0 L^{1/2} \leq 2C_0 L^{1/2}. \quad \Box$$

**Remark 10.** From this point on, we fix $a_0 \in (0, 1/8]$ sufficiently small so that in Lemmata 8 and 9 we have $\|\mu_{\mathcal{J}}\| \leq 1/2$. We now treat $a_0$ as a constant parameter and hereafter omit it from our $O(\cdots)$ estimates.

### 3.2. Iterated standard pairs

Fix a fully-crossing standard pair $(\gamma, \rho)$. Below, for each $n \geq 1$ we define a decomposition

$$F^n_{\gamma}(\gamma, \rho) = \mu_{\mathcal{E}}^{(\gamma, \rho)} + \mu_{\mathcal{J}}^{(\gamma, \rho)} + \mu_{\mathcal{I}}^{(\gamma, \rho)} + \mu_{\mathcal{E}}^{(\gamma, \rho)},$$

where, as in Section 3.1, each of $\mu_{\mathcal{E}} = \mu_{\mathcal{E}}^{(\gamma, \rho)}, \mu_{\mathcal{I}} = \mu_{\mathcal{I}}^{(\gamma, \rho)}, \mu_{\mathcal{J}} = \mu_{\mathcal{J}}^{(\gamma, \rho)}, \mu_{\mathcal{E}}$ is a weighted sum of measure pairs of the appropriate regularity (respectively, fully-crossing, standard, and substandard), while $\mu_{\mathcal{E}}$ is a remainder we do not otherwise control. We write $\mathcal{L}^n = \mathcal{L}_{\gamma}^{n, \rho}, \mathcal{I}^n = \mathcal{I}_{\gamma}^{n, \rho}, \mathcal{J}^n = \mathcal{J}_{\gamma}^{n, \rho}$ for the corresponding classes of, respectively, fully-crossing, standard and substandard measure pairs, and $\mathcal{E}^n = \mathcal{E}_{\gamma}^{n, \rho}$ for the corresponding remainder set.

**A**) **Constructing** $\mathcal{L}^n, \mathcal{I}^n, \mathcal{J}^n, \mathcal{E}^n$. To start, we set $\mathcal{L}^0 = \{\gamma\}, \mathcal{I}^0, \mathcal{J}^0, \mathcal{E}^0 = \{\emptyset\}$. Given $k \geq 1$, the collections $\mathcal{L}^k, \mathcal{I}^k, \mathcal{J}^k, \mathcal{E}^k$, and the measures $\mu_{\mathcal{E}}^k, \mu_{\mathcal{I}}^k, \mu_{\mathcal{J}}^k, \mu_{\mathcal{E}}^k$, we define $\mathcal{L}^{k+1}, \mathcal{I}^{k+1}, \mathcal{J}^{k+1}, \mathcal{E}^{k+1}$ as follows. Set

$$\mathcal{L}^{k+1} = \bigcup_{(\gamma_k, \rho_k) \in \mathcal{L} \cup \mathcal{I} \cup \mathcal{J}^k} \mathcal{L}_{(\gamma_k, \rho_k)}^k, \quad \mu_{\mathcal{E}}^{k+1} = \sum_{(\gamma_k, \rho_k) \in \mathcal{L} \cup \mathcal{I} \cup \mathcal{J}^k} \gamma_{\gamma_k, \rho_k} \mu_{\mathcal{E}}^{(\gamma_k, \rho_k)};$$

where $\gamma_{\gamma_k, \rho_k} = F^k_{\gamma}(\gamma, \rho)(\gamma_k)$. Here, for measure pairs $(\gamma_k, \rho_k)$, the collections $\mathcal{L}_{(\gamma_k, \rho_k)}, \mathcal{I}_{(\gamma_k, \rho_k)}, \mathcal{J}_{(\gamma_k, \rho_k)}$ are as in Lemmas 7, 8, 9. The $\mathcal{I}^{k+1}, \mathcal{J}^{k+1}, \mathcal{I}^{k+1}, \mathcal{J}^{k+1}$ are defined analogously. Finally, we define

$$\mathcal{E}^{k+1} = F(\mathcal{E}^k) \cup \bigcup_{(\gamma_k, \rho_k) \in \mathcal{L} \cup \mathcal{I} \cup \mathcal{J}^k} \mathcal{E}_{(\gamma_k, \rho_k)}^k, \quad \mu_{\mathcal{E}}^{k+1} = \sum_{(\gamma_k, \rho_k) \in \mathcal{L} \cup \mathcal{I} \cup \mathcal{J}^k} \gamma_{\gamma_k, \rho_k} \mu_{\mathcal{E}}^{(\gamma_k, \rho_k)};$$

This completes the construction.

**B**) **Estimating mass contributions.** Let us now estimate the relative sizes of the $\mu_{\mathcal{E}}^k, \mu_{\mathcal{I}}^k, \mu_{\mathcal{J}}^k, \mu_{\mathcal{E}}^k$.

**Proposition 11.** Let $(\gamma, \rho)$ be a fully-crossing standard pair, $n \geq 1$. Then,

$$F^n_{\gamma}(\gamma, \rho) = \mu_{\mathcal{E}}^n + \mu_{\mathcal{I}}^n + \mu_{\mathcal{J}}^n + \mu_{\mathcal{E}}^n,$$

where $\|\mu_{\mathcal{E}}^n\| = \|\mu_{\mathcal{I}}^n\| = O(L^{-1/2})$ and $\|\mu_{\mathcal{J}}^n\| = O(n L^{-3/4})$. 
Proof. From Lemmas 11, 12, and 13 we obtain
\[
\|\mu_{\mathcal{E}}^{k+1}\| = \|\mu_{\mathcal{E}}^k\| + O\left(\frac{L^{-1/2}}{2} \cdot \|\mu_{\mathcal{E}}^k\| + L^{-1/2} \cdot \|\mu_{\mathcal{E}}^k\| + L^{-3/4} \cdot \|\mu_{\mathcal{E}}^k\|\right)
\]
\[
\|\mu_{\mathcal{J}}^{k+1}\| \leq \frac{1}{2} \|\mu_{\mathcal{J}}^k\| + O\left(L^{-1/2} \cdot \|\mu_{\mathcal{E}}^k\|\right)
\]
\[
\|\mu_{\mathcal{I}}^{k+1}\| = O\left(\|\mu_{\mathcal{J}}^k\| + L^{-1/2} \cdot \|\mu_{\mathcal{E}}^k\|\right)
\]
\[
\|\mu_{\mathcal{L}}^{k+1}\| = (1 - O(L^{-1/2})) \|\mu_{\mathcal{J}}^k\| + (1 - O(L^{-1/2})) \|\mu_{\mathcal{E}}^k\|
\]
Proposition 14 follows by an induction argument, using the initial state \(\|\mu_{\mathcal{E}}^0\| = 1, \|\mu_{\mathcal{J}}^0\| = \|\mu_{\mathcal{I}}^0\| = \|\mu_{\mathcal{L}}^0\| = 0\). □

If, at time \(n\), we discard the curves in \(\mathcal{T}_n, \mathcal{J}_n\), we obtain the following corollary.

Corollary 12. Let \((\gamma, \rho)\) be a fully-crossing standard pair. For any \(n \geq 1\), the pushed-forward standard pair \(F_n^\gamma(\gamma, \rho)\) admits a representation of the form
\[
F_n^\gamma(\gamma, \rho) = \sum_{(\gamma_n, \rho_n) \in \mathcal{E}_n(\gamma, \rho)} c_{\gamma_n}(\gamma_n, \rho_n) + \hat{\mu}_{\mathcal{E}}^n
\]
where each \((\gamma_n, \rho_n)\) is a fully-crossing standard pair, and \(\|\hat{\mu}_{\mathcal{E}}^n\| = O(L^{-1/2} + nL^{-3/4})\).

3.3. Correlation control for \(x\)-dependent observables. We now present some consequences of the arguments in Sections 3.1, 3.2 for correlation decay. Let \(\phi : \mathbb{T}^2 \to \mathbb{R}\) be a \(C^1, x\)-dependent observable.

3.3.1. Correlation control for standard pairs.

Proposition 13 (Equidistribution). Let \((\gamma, \rho)\) be a fully-crossing standard pair and assume \(\int_{\mathbb{T}^1} \phi dx = 0\). Then, for all \(n \geq 1\) we have that
\[
\int \phi \circ F^n d(\gamma, \rho) = O\left(\|\phi\|_{C^0} \cdot ((n - 1) L^{-\frac{1}{2}} + L^{-\frac{3}{2}})\right)
\]

First, we prove a preliminary lemma.

Lemma 14 (One-step equidistribution). Let \((\gamma, \rho)\) be a fully crossing standard pair, then
\[
\int \phi \circ F d(\gamma, \rho) = O(\|\phi\|_{C^0} L^{-\frac{1}{2}}).
\]

Proof. We decompose \(\gamma \setminus \mathcal{S}_{1/2}\) into four pieces according to membership in the four regions \([0, 1/4] \times \mathbb{T}^1, [1/4, 1/2] \times \mathbb{T}^1, [1/2, 3/4] \times \mathbb{T}^1, [3/4, 1] \times \mathbb{T}^1\). For concreteness, we consider below the piece \(\tilde{\gamma} = (\gamma \setminus \mathcal{S}_{1/2}) \cap ([1/4, 1/2] \times \mathbb{T}^1)\) and will estimate \(\int_{\tilde{\gamma}} \phi \circ F d(\gamma, \rho)\). The following considerations can be straightforwardly extended to the other pieces; we leave this to the reader. Below, we write \(\tilde{\rho} : I_{\tilde{\gamma}} \to [0, \infty)\) for the density for which \((\tilde{\gamma}, \tilde{\rho})\) is the normalization of \((\gamma, \rho)|_{\tilde{\gamma}}\).
Apply Lemma \[\text{Lemma}\] to \((\bar{\gamma}, \bar{\rho})\) to obtain the collection \(\mathcal{L} = \mathcal{L}(\bar{\gamma}, \bar{\rho})\) of fully-crossing standard pairs and the remainder set \(\mathcal{E} = \mathcal{E}(\bar{\gamma}, \bar{\rho}) \subset F(\bar{\gamma})\). We have
\[
\int_{\bar{\gamma}} \phi \circ F \, d(\bar{\gamma}, \bar{\rho}) = \int \phi \, d\mu_\mathcal{L} + \int \phi \, d\mu_\mathcal{E} = \sum_{(\bar{\gamma}, \bar{\rho}) \in \mathcal{L}} c_{\bar{\gamma}} \int \phi \, d(\bar{\gamma}, \bar{\rho}) + O(\|\phi\|_{C^0} L^{-1/2}),
\]
where \(c_{\bar{\gamma}} := (\bar{\gamma}, \bar{\rho})(F^{-1}\bar{\gamma})\).

For each \((\bar{\gamma}, \bar{\rho}) \in \mathcal{L}\), we first estimate the \((\bar{\gamma}, \bar{\rho})\)-summand \(\int \phi \, d(\bar{\gamma}, \bar{\rho}) = \int_0^1 \phi \, d\bar{\rho} \, dx\). Observe from (5) that
\[
\left|\frac{d}{dx} \log d\bar{\rho}\right| = O\left(\frac{1}{|\hat{f}_\gamma(x_\bar{\gamma})|} + \frac{L}{|\hat{f}_\gamma(x_\bar{\gamma})|^2}\right),
\]
where we set \(x_\bar{\gamma}\) to be the right-endpoint of \(I_{F^{-1}\bar{\gamma}}\). Checking the simple estimate \(|\hat{f}_\gamma(x)| \approx L|x_\bar{\gamma} - \frac{1}{4}|\) on \(I_{F^{-1}\bar{\gamma}}\), it follows that \(|\bar{\rho} - 1| = O(L^{-1}|x_\bar{\gamma} - \frac{1}{4}|^{-2})\). Putting this all together,
\[
\int \phi \, d(\bar{\gamma}, \rho_{\bar{\gamma}}) = \int_0^1 \phi \, (\rho_{\bar{\gamma}} - 1) \, dx = O\left(\|\phi\|_{C^0} \cdot L^{-1} \left|x_\bar{\gamma} - \frac{1}{4}\right|^{-2}\right).
\]

Let \(E_{\bar{\gamma}} = L^{-1}|x_\bar{\gamma} - 1/4|^{-2}\); since \(x_\bar{\gamma} \in I_\gamma\) and \(I_\bar{\gamma} = [\frac{1}{4} + b_L, \frac{1}{2}]\) where \(b_L \approx L^{-1/2}\), we gather that there exists \(\hat{E} = O(1)\) so that \(E_{\bar{\gamma}} < \hat{E}\). Moreover, note that since \(c_{\bar{\gamma}} = (\gamma, \rho)(F^{-1}\bar{\gamma}) \approx |I_{F^{-1}\bar{\gamma}}|:\)
\[
\sum_{(\bar{\gamma}, \bar{\rho}) \in \mathcal{L} \text{ s.t. } E_{\bar{\gamma}} > z} c_{\bar{\gamma}} = O(L^{-1/2}z^{-1/2})\text{ for any } z > 0.
\]
Thus:
\[
\sum_{(\bar{\gamma}, \bar{\rho}) \in \mathcal{L}} c_{\bar{\gamma}} \int \phi \, d(\bar{\gamma}, \bar{\rho}) \leq C\|\phi\|_{C^0} \sum_{(\bar{\gamma}, \bar{\rho}) \in \mathcal{L}} c_{\bar{\gamma}} E_{\bar{\gamma}}
\]
\[
\leq C\|\phi\|_{C^0} \int_0^\hat{E} dz \sum_{(\bar{\gamma}, \bar{\rho}) \in \mathcal{L} \text{ s.t. } E_{\bar{\gamma}} > z} c_{\bar{\gamma}} = O(\|\phi\|_{C^0} L^{-1/2}).
\]
This completes the proof. 

**Proof of Proposition** \[\text{Proposition}\] Apply Corollary \[\text{Corollary}\] to \(F_{\gamma}^{n-1}(\gamma, \rho)\) to obtain
\[
F_{\gamma}^{n-1}(\gamma, \rho) = \sum_{(\gamma_{n-1}, \rho_{n-1}) \in \mathcal{L}_{\gamma}^{n-1}} c_{\gamma_{n-1}}(\gamma_{n-1}, \rho_{n-1}) + \bar{\mu}_{\mathcal{E}}^{n-1}.
\]
Then,
\[
\int \phi \circ F^n \, d(\gamma, \rho) = \sum_{(\gamma_{n-1}, \rho_{n-1}) \in \mathcal{L}_{\gamma}^{n-1}} c_{\gamma_{n-1}} \int \phi \circ F \, d(\gamma_{n-1}, \rho_{n-1}) + O(\|\phi\|_{C^0} \cdot ((n - 1)L^{-3/4} + L^{-1/2})).
\]
The proof is complete on applying Lemma \[\text{Lemma}\] to each summand. 

\[\square\]
3.3.2. Correlation control for Lebesgue measure. Using the equidistribution estimate for standard pairs and the machinery developed so far, we conclude this section with a decay of correlations estimate for Lebesgue measure.

**Corollary 15.** Let \( \phi, \psi : T^1 \to \mathbb{R} \) be \( C^1 \), \( x \)-dependent observables. Then,

\[
\int \psi \cdot \phi \circ F^n d\text{Leb} - \int \psi \int \phi = O \left( \|\psi\|_{C^1} \|\phi\|_{C^1} \cdot \left((n - 1)L^{-\frac{3}{4}} + L^{-\frac{1}{2}}\right) \right)
\]

**Proof.** Let \( c > 0 \) be a constant, to be specified later, and define \( \hat{\psi} = \frac{\psi + c}{\int \psi d\text{Leb} + c} \).

Define \( \hat{\phi} = \phi - \int \phi \). For each fixed \( y \in T^1 \), we intend to apply Proposition 13 to \( \int \hat{\phi} \circ F^n d(\gamma^y, \hat{\psi}) \), where \( \gamma^y := T^1 \times \{ y \} \) and we regard \( \hat{\psi} \) as a density function on \( T^1 \) as in the definition of a standard pair. To make this legitimate, the parameter \( c > 0 \) must be chosen so \( \hat{\psi} \) is (i) nonnegative and (ii) satisfies the distortion estimate (4). For this,

\[
\frac{d \log \hat{\psi}}{dx} = \frac{1}{\hat{\psi}} \frac{d\hat{\psi}}{dx} = \frac{1}{\psi + c} \frac{d\psi}{dx}
\]

hence \( \left| \frac{d \log \hat{\psi}}{dx} \right| \leq \frac{1}{c - \|\psi\|_{C^1}} \|\psi\|_{C^1} \); here \( \|\psi\|_{C^1} = \max\{ \|\psi\|_{C^0}, \|\dot{\psi}\|_{C^0} \} \). Taking \( c = 2\|\psi\|_{C^1} \) yields (i) \( \psi + c > 0 \) and (ii) \( \left| \frac{d \log \hat{\psi}}{dx} \right| \leq 1 \leq 3C_0 \), as needed.

Applying Proposition 13 for each fixed \( y \), then integrating over \( y \in T^1 \), we have

\[
\int \hat{\psi} \cdot \hat{\phi} \circ F^n d\text{Leb} = \int \int \hat{\phi} \circ F^n d(\gamma^y, \hat{\psi}) \, dy
\]

\[
= O \left( \|\phi\|_{C^1} \left((n - 1)L^{-\frac{3}{4}} + L^{-\frac{1}{2}}\right) \right),
\]

while

\[
\int \hat{\psi} \cdot \hat{\phi} \circ F^n d\text{Leb} = \frac{1}{\hat{\psi} + c} \int (\psi + c) \cdot \phi \circ F^n d\text{Leb} - \int \phi \int (\psi + c)
\]

\[
= \frac{1}{\hat{\psi} + c} \int \psi \cdot \phi \circ F^n d\text{Leb} - \int \phi \int \psi
\]

holds since \( F \) preserves \( \text{Leb} \). This completes the proof. \( \square \)

4. Central Limit Theorem

Let \( \phi \) be a \( C^1 \), \( x \)-dependent observable. We obtain in this section a Central Limit Theorem (CLT) for sequences of the form

\[
S_{N,L} \phi := \sum_{i=0}^{N-1} \phi \circ F^i_L
\]

where \( N = N(L) \) is a suitably chosen function of \( L \geq 0 \) which increases sufficiently slowly as in the assumptions of Theorem 13.
For this, we follow the standard route of obtaining a martingale difference approximation for the sequence $S_{N,L}\phi$. The plan is as follows. In Section 3.1 we will define, for each $L$, a filtration of $T^2$ by $F_L$-preimages of fully-crossing standard pairs (plus a small remainder which we do not control). In Section 3.2, we will define a martingale difference approximation $\tilde{S}_{N,L}\phi$, and show how a CLT for the approximation implies a CLT for the original $S_{N,L}\phi$. Finally, in Section 3.3 we apply a result of McLeish (see [12]) on CLTs for martingale difference arrays to conclude the CLT for $\tilde{S}_{N,L}\phi$, thereby completing the proof of Theorem [13].

Notation for Section 4. Since this section has more of a probabilistic flavor, we will at times write $P$ for Lebesgue measure on $T^2$ and $E$ for the expectation with respect to $P$. Given a $\sigma$-algebra $F \subset \text{Bor}(T^2)$, we write $E(\cdot | F)$ for the conditional expectation w.r.t. $F$.

At times in Section 4, when $L$ is fixed or when clear from context, we will write $F = F_L$.

4.1. Filtration by u-curves. Fix $L > 0$ sufficiently large for the purposes of the results in Section 3. By the end of Section 4.1, we will have constructed a sequence of $\sigma$-algebras $U_i = U_i(L)$, $i \geq 1$, each generated by a partition of $T^2$ into fully crossing curves, plus some small remainder set, with the property that $F_L U_i \subset U_{i+1}$. As a result, the pull-backs $F_i = F_i(L) := (F_L)^{-1} U_i$ comprise a filtration on $T^2$. This is the filtration we will use to define our martingale approximation in Section 4.2.

Notation. For $y \in T^1$, let $\gamma^y := T^1 \times \{y\}$, which is clearly a fully-crossing u-curve. Writing $1 : (0,1) \to \mathbb{R}$ for the density identically equal to 1, we regard $(\gamma^y, 1)$ as a fully-crossing standard pair. Applying the machinery in Section 3.2, for $n \geq 0$ we define the collections of measure pairs

$$L^n_y = \mathcal{L}^n_{(\gamma^y,1)}, \quad T^n_y = \mathcal{T}^n_{(\gamma^y,1)}, \quad \mathcal{J}^n_y = \mathcal{J}^n_{(\gamma^y,1)},$$

and the remainder set $E^n_y = E^n_{(\gamma^y,1)}$. Define the partition $P^n_y$ of $F^n(\gamma^y)$ by

$$P^n_y = L^n_y \cup T^n_y \cup J^n_y \cup \{E^n_y\},$$

where the $L^n_y, T^n_y, J^n_y$ are treated above as collections of u-curves, and $\{E^n_y\}$ is the trivial partition on $E^n_y$.

Below, for partitions $\alpha, \beta$ on the same space, we write $\alpha \leq \beta$ if each $\alpha$-atom is a union of $\beta$-atoms (i.e. $\alpha$ is coarser than $\beta$). We write $\alpha \land \beta$ for the join of $\alpha$ and $\beta$, i.e., the partition of the form $\{C \cap D : C \in \alpha, D \in \beta\}$. Clearly, if $\alpha \leq \beta$, then $\alpha \land \beta = \beta$. Given a partition $\alpha$ we denote with $\sigma(\alpha)$ the $\sigma$-algebra generated by $\alpha$; notice that if $\alpha \leq \beta$ we have $\sigma(\alpha) \subset \sigma(\beta)$.

Construction of $U_i$. We are about to construct inductively a sequence $\Xi_i = \Xi_i(L)$ of measurable partitions of $T^2$ into (mostly) fully crossing curves with the property that $F \Xi_i \leq \Xi_{i+1}$. The $\sigma$-algebras $U_i$ will be of the form $U_i = \sigma(\Xi_i)$, and the property $F \Xi_i \subset U_{i+1}$ will follow by the remark made above.

We set $\Xi_0$ to be the partition of $T^2$ into the u-curves $\{\gamma^y\}_{y \in T^1}$. Assume by induction that we have defined the partitions $\Xi_0, \ldots, \Xi_n$, we will construct $\Xi_{n+1}$ on
we define $\Xi_{n+1}|_{F^{n+1}(\gamma^y)}$ separately for each $y \in T^1$. For fixed $y$, we set

$$
\Xi_{n+1}|_{F^{n+1}(\gamma^y)} = F(\Xi_n|_{F^n(\gamma^y)}) \lor P^{n+1}_y.
$$

Reconstituting $\Xi_{n+1}$ from its definition on each atom of $F^{n+1}(\Xi_n) = \{F^{n+1}(\gamma^y)\}_{y \in T^1}$, it is clear that $F(\Xi_n) \leq \Xi_{n+1}$, as desired.

Having constructed the $U_i$, we define the sequence of $\sigma$-algebras

$$
F_i = (F^i)^{-1}U_i, \quad i \geq 1,
$$

which is clearly seen to be an increasing filtration on $T^2$. Moreover, the partition $P^n_y$ depends measurably on $y$, (in fact, on a piecewise continuous fashion), therefore $(F^i)^{-1}U_n$ is a measurable partition of $T^2$.

**Properties of the $U_i$.** Let us record some basic facts for future use. Set $\tilde{G}^n = \tilde{G}^n(L) = \cup_y \mathcal{L}^n_y$, where $\mathcal{L}^n_y$ is regarded as a subset of $F^n(\gamma^y)$. Then, $\Gamma^n = \Gamma^n(L) := \Xi_n|_{\tilde{G}^n}$ is a partition of $\tilde{G}^n$ consisting of fully-crossing u-curves, coinciding with the union $\cup_y \mathcal{L}^n_y$ of u-curves. We continue to abuse notation and write $\Gamma^n$ for both the collection of u-curves and the corresponding collection of standard pairs $\cup_y \mathcal{L}^n_y$. We set $G^n = G^n(L) := (F^n_L)^{-1}\tilde{G}^n$ and $B^n = B^n(L) := T^2 \setminus G^n$.

**Lemma 16.** For each $n \geq 1$, the following holds.

(a) We have $\text{Leb } B^n = O(nL^{-\frac{1}{4}} + L^{-1/2})$.

(b) Restricted to the set $F^{-1}\tilde{G}^n$, the $\sigma$-algebra $F^{-1}U_n$ is generated by atoms of $F^{-1}\Gamma^n$, each of which has diameter bounded from above by $F^{-\frac{1}{4}}$.

In the coming proofs, we routinely take conditional expectations with respect to the $\sigma$-algebras $\{U_n\}$. Below we record how these computations work.

**Lemma 17.** Let $\psi : T^2 \to \mathbb{R}$ be a $C^0$ function. Then, there is a version of the conditional expectation $E(\psi|U_n)$ of $\psi$ with respect to $U_n$ with the property that for every $(\gamma_n, \rho_n) \in \Gamma^n$, we have

$$
E(\psi|U_n) = \int \psi d(\gamma_n, \rho_n) = \int \psi(x, h_{\gamma_n}(x)) \rho_n(x) dx \quad \text{on } \gamma_n.
$$

Hereafter we intentionally confuse $E(\psi|U_n)$ with the expression on the right-hand side.

**4.2. Martingale difference approximation.** From this point on, an increasing function $N : \mathbb{R}_{>0} \to \mathbb{N}$ is fixed for which the condition

$$
N(L) \cdot L^{-\frac{1}{4}} \to 0 \quad \text{as } L \to \infty,
$$

as in the hypotheses of Theorem 11 is assumed to hold. We let $\phi : T^1 \to \mathbb{R}$ be a $C^1$ observable with $\int \phi dx = 0$ and assume $\phi$ is not identically zero, hence $\int \phi^2 dx > 0$.

We intend to approximate $S_{N,L} \phi$ by

$$
\tilde{S}_{N,L} \phi = \sum_{i=1}^{N} E(\phi \circ F^{i-1}_{L}|F_{i}(L)) = \sum_{i=1}^{N} E(\phi|F^{-1}_{L}U_i(L)) \circ F^{i-1}_{L}.
$$
Lemma 18. Under condition (3), we have \( \frac{1}{\sqrt{N(L)}}|S_{N(L),L}\phi - \tilde{S}_{N(L),L}\phi| \to 0 \) in probability w.r.t. Lebesgue measure.

In particular, the convergence in distribution of \( \tilde{S}_{N(L),L}\phi / \sqrt{N(L)} \) to a centered Gaussian \( \mathcal{N}(0,\sigma^2) \) is equivalent to the convergence in distribution of \( S_{N(L),L}\phi / \sqrt{N(L)} \) to the same law \( \mathcal{N}(0,\sigma^2) \).

Proof. For the sake of readability, in the following proof we drop the ‘\( L \)’ and write \( S_N = S_{N(L),L}\phi \), \( \tilde{S}_N = \tilde{S}_{N(L),L}\phi \), \( N = N(L) \) and \( F = F_L \).

We start by examining the \( i \)-th summand of \( \tilde{S}_N \). By Lemma 17

\[
\mathbb{E}(\phi \circ F^{-1}|\mathcal{U}_i) = \int \phi \circ F^{-1}(x,h_{\gamma_i}(x)) \rho_i(x)dx
\]

holds when the left-hand side is evaluated on the (fully crossing) standard pair \((\gamma_i,\rho_i)\) \( \in \Gamma^i \). Fixing \((\gamma_i,\rho_i)\), let \((\gamma_{i-1},\rho_{i-1}) \in \Gamma^{i-1}\) be such that \( \gamma_i \subset F(\gamma_{i-1}) \).

Observe that \( f_{\gamma_{i-1}} \) maps some interval \( \tilde{I}_{\gamma_i} \) diffeomorphically onto \([0,1]\). By the change of variables formula,

\[
\int \phi \circ F^{-1}(x,h_{\gamma_i}(x)) \rho_i(x)dx = \frac{1}{\int_{\tilde{I}_{\gamma_i}} \rho_{i-1}(x)dx} \int_{\tilde{I}_{\gamma_i}} \phi(x) \rho_{i-1}(x)dx
\]

By Lemma 16(b), the length of \( \tilde{I}_{\gamma_i} \) is \( \leq L^{-1/2} \), and so for \((x,y) \in F^{-1}(\gamma_i)\) the above RHS equals \( \phi(x) + O(\|\phi\|_{C^1}L^{-1/2}) \).

Thus

\[
\mathbb{E}(\phi \circ F^{-1}|\mathcal{U}_i) = \phi(x) + O(\|\phi\|_{C^1}L^{-1/2}) \quad \text{on } \tilde{G}^i.
\]

We conclude that \( \frac{|\tilde{S}_N - S_N|}{\sqrt{N}} \leq \sqrt{NL}^{-1/2}\|\phi\|_{C^1} \) holds on \( \cap_{n=1}^N G^n \). By (6), this right-hand quantity goes to 0 as \( L \to \infty \).

To complete the proof of convergence in probability, it suffices to show that \( \mathbb{P}(\cup_{n=1}^N B^n) \to 0 \) as \( L \to \infty \). For this, from the estimate in Lemma 16 (a) we have \( \mathbb{P}(\cup_{n=1}^N B^n) = O(N^2L^{-3/4} + NL^{-1/2}) \), which also goes to 0 as \( L \to \infty \) under (6).

4.2.1. Representation of \( \tilde{S}_{N,L} \) as a sum of martingale differences. In the next Lemma, we represent \( \tilde{S}_N \) as a sum of the form \( \tilde{S}_N = \sum_{i=1}^N U_i \), where the \( U_i = U_i(L) \) are martingale differences with respect to the filtration \( (\mathcal{F}_i(L))_i \). Below, we use the convention \( \mathcal{F}_0 = \{\mathbb{T}^2,\emptyset\} \).

Lemma 19. Fix \( L \) and define

\[
U_i = \sum_{m=i}^N \left( \mathbb{E}(\phi \circ F^{m-1}|\mathcal{F}_i) - \mathbb{E}(\phi \circ F^{m-1}|\mathcal{F}_{i-1}) \right).
\]

(a) The sequence \( (U_i)_{i=1}^N \) is a martingale difference, i.e., each \( U_i \) is \( \mathcal{F}_i \)-measurable and \( \mathbb{E}(U_i|\mathcal{F}_{i-1}) = 0 \) for all \( 1 \leq i \leq N \); and

(b) we have \( \tilde{S}_{N,L} = \sum_{i=1}^N U_i \).
Proof. Item (a) is obvious. For (b), we compute:

\[
\sum_{i=1}^{N} U_i = \sum_{i=1}^{N} \sum_{m=i+1}^{N} \left( \mathbb{E}(\phi \circ F^{m-1}|\mathcal{F}_i) - \mathbb{E}(\phi \circ F^{m-1}|\mathcal{F}_{i-1}) \right)
\]

\[
= \sum_{i=1}^{N} \mathbb{E}(\phi \circ F^{i-1}|\mathcal{F}_i) + \sum_{i=1}^{N} \sum_{m=i+1}^{N} \mathbb{E}(\phi \circ F^{m-1}|\mathcal{F}_i) - \sum_{i=1}^{N} \sum_{m=i}^{N} \mathbb{E}(\phi \circ F^{m-1}|\mathcal{F}_{i-1})
\]

For the \(I\) term, the \(i = N\) summand is empty, and so

\[
I = \sum_{i=1}^{N-1} \sum_{m=i+1}^{N} \mathbb{E}(\phi \circ F^{m-1}|\mathcal{F}_i)
\]

For the \(II\) term, the \(i = 1\) summand is zero since \(\mathcal{F}_0\) is the trivial \(\sigma\)-algebra. On replacing \(i \mapsto i + 1\),

\[
II = \sum_{i=2}^{N} \sum_{m=i}^{N} \mathbb{E}(\phi \circ F^{m-1}|\mathcal{F}_i) = \sum_{i=1}^{N-1} \sum_{m=i}^{N} \mathbb{E}(\phi \circ F^{m-1}|\mathcal{F}_i)
\]

and so \(I = II\). We conclude \(\sum_{i=1}^{N} U_i = \tilde{S}_N\). \(\square\)

4.2.2. Asymptotic estimate for \(U_i\). Before continuing, we give the following asymptotic estimate on the \(U_i\).

**Proposition 20.** For each \(1 \leq i \leq N\), the function

\[
V_i = U_i - \phi \circ F^{i-1}
\]

satisfies \(V_i = O(N\|\phi\|_{C^0})\) and \(\mathbb{E}|V_i| = O(\|\phi\|_{C^1} NL^{-\frac{1}{2}})\).

Proof. We expand

\[
V_i = \underbrace{\mathbb{E}(\phi|F^{-1}\mathcal{U}_i)}_{(a)} \circ F^{i-1} - \phi \circ F^{i-1} - \underbrace{\mathbb{E}(\phi|\mathcal{U}_{i-1})}_{(b)} \circ F^{i-1} + \underbrace{\mathbb{E}(\phi|\mathcal{U}_i)}_{(c)} \circ F^i
\]

\[
+ \sum_{j=1}^{N-i-1} \mathbb{E}(\phi \circ F^j|\mathcal{U}_i) \circ F^i - \sum_{j=1}^{N-i} \mathbb{E}(\phi \circ F^j|\mathcal{U}_{i-1}) \circ F^{i-1}
\]

Clearly \(|V_i| = |U_i - \phi \circ F^{i-1}| = O(N\|\phi\|_{C^0})\), and so we only need to estimate the expectation.

In the estimates below, we make liberal use of the fact that under (6), we have \(NL^{-3/4} = o(L^{-1/2})\), hence the term \(O(NL^{-3/4} + L^{-1/2})\) appearing in the error estimate for Proposition 20 can be written \(O(L^{-1/2})\).

**Term (a):** From (7),

\[
|\phi \circ F^{i-1} - \mathbb{E}(\phi \circ F^{i-1}|\mathcal{F}_i)| = O(\|\phi\|_{C^1} L^{-1/2}) \text{ holds on } G^i.
\]
The component on $B^i$ has expectation $O(\|\phi\|_{C^0} L^{-\frac{1}{2}})$, since $\text{Leb}(B^i) = O(L^{-\frac{1}{2}})$. In total, $\mathbb{E}|(a)| = O(\|\phi\|_{C^0} L^{-1/2})$.

**The (b) terms:** Evaluating at $(\gamma_i, \rho_i) \in \Gamma^i$, we have

$$\mathbb{E}(\phi|\mathcal{U}_i) = \int_0^1 \phi \cdot \rho_i \, dx = O(\|\phi\|_{C^0} L^{-1/2})$$

by Proposition 13. Similarly, when evaluated at $\gamma_{i-1} \in \Gamma_i$, we have

$$\mathbb{E}(\phi|\mathcal{U}_{i-1}) = O(\|\phi\|_{C^0} L^{-1/2}) .$$

Since $\mathbb{P}(B^i), \mathbb{P}(B^{i-1})$ are each $O(\|\phi\|_{C^0} L^{-\frac{1}{2}})$, we obtain $\mathbb{E}|(b)| = O(\|\phi\|_{C^0} L^{-1/2})$.

**The (c) terms:** Evaluating at $(\gamma_i, \rho_i) \in \Gamma^i$, we have

$$\mathbb{E}(\phi \circ F^i|\mathcal{U}_i) = \int \phi \circ F^i \, d(\gamma_i, \rho_i) = O(\|\phi\|_{C^0} L^{-\frac{1}{2}})$$

by Lemma 17 and Proposition 13. Similarly, when evaluated at $\gamma_{i-1} \in \Gamma_i$,

$$\mathbb{E}(\phi \circ F^{i-1}|\mathcal{U}_{i-1}) = O(\|\phi\|_{C^0} L^{-\frac{1}{2}}) .$$

The expectations on the bad sets $B^i, B^{i-1}$ are again $O(\|\phi\|_{C^0} L^{-\frac{1}{2}})$. Since there are at most $N$ such terms, we have $\mathbb{E}|(c)| = O(\|\phi\|_{C^0} NL^{-\frac{1}{2}})$. Summing (a), (b), (c) completes the proof.

**Corollary 21.** For $1 \leq i \leq N$, the function

$$W_i = U_i^2 - \phi^2 \circ F^{i-1}$$

satisfies $W_i = O(N^2\|\phi\|^2_{C^0})$ and $\mathbb{E}|W_i| = O(\|\phi\|^2_{C^1} NL^{-\frac{1}{2}})$.

**Proof.** The estimate $W_i = O(N^2)$ is straightforward and left to the reader. For $\mathbb{E}|W_i|$, we have

$$W_i = 2(\phi \circ F^{i-1})V_i + V_i^2 .$$

From Proposition 20 we estimate $\mathbb{E}(2(\phi \circ F^{i-1})V_i) \leq 2\|\phi\|_{C^0} \mathbb{E}(|V_i|) = O(\|\phi\|^2_{C^1} NL^{-\frac{1}{2}})$, and $\mathbb{E}(V_i^2) \leq \sup(|V_i|)\mathbb{E}(|V_i|) = O(\|\phi\|^2_{C^1} N^2 L^{-\frac{1}{2}}) . \quad \Box$

### 4.3. CLT for the martingale approximation.

Lemma 18 reduces Theorem B to verifying the same CLT for $\tilde{S}_{N(L)}^2 \phi/\sqrt{N(L)}$ as $L \to \infty$. We will deduce this using the following result due to McLeish.

**Theorem 22 (12).** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\{k_n\}_{n \geq 1}$, be an increasing sequence of whole numbers tending to infinity, and for each $n \geq 1$, let $\mathcal{F}_{1,n} \subset \mathcal{F}_{2,n} \subset \cdots \subset \mathcal{F}_{k_n,n} \subset \mathcal{F}$ be an increasing sequence of sub-$\sigma$ algebras. For each such $n, i$, let $X_{i,n}$ be a random variable, measurable with respect to $\mathcal{F}_{i,n}$, for which $\mathbb{E}(X_{i,n}|\mathcal{F}_{i-1,n}) = 0$, and write $Z_n = \sum_{1 \leq i \leq k_n} X_{i,n}$. Assume

1. $\max_{1 \leq i \leq k_n} |X_{i,n}|$ is uniformly bounded, in $n$, in the $L^2$ norm,
2. $\max_{1 \leq i \leq k_n} |X_{i,n}| \to 0$ in probability as $n \to \infty$, and
3. $\sum_i X_{i,n}^2 \to 1$ in probability as $n \to \infty$.
Then, $Z_n$ converges weakly to a standard Gaussian.

Given an arbitrary increasing sequence $L_n \to \infty$, we intend to apply this theorem to the array

$$X_{i,n} := \frac{U_i(L_n)}{\sqrt{\sum_{i=1}^n E(U_i(L_n))^2}}, \quad F_{i,n} = F_i(L_n), \quad k_n = N(L_n).$$

Assuming this can be done, we will have proved that

$$\tilde{S}_{N(L_n), L_n} \phi \sqrt{\sum_{i=1}^n E(U_i(L_n))^2}$$

converges to a standard Gaussian $\mathcal{N}(0, 1)$. Afterwards, Theorem \[\text{B}\] easily follows from the asymptotic estimate for $\sum_i E(U_i(L_n))^2$ given below.

Proposition 23. Under condition \[\text{(6)}\], we have for all $L$ sufficiently large that

$$\sum_{i=1}^{N(L)} E(U_i(L))^2 = N(L) \int \phi^2 + o(N(L)).$$

Proof. Dropping the ‘$L$’ and using Corollary \[\text{21}\] we estimate

$$\sum_{i=1}^{N} E(U_i^2) = \int \phi^2 \circ F^{i-1} + \sum_{i=1}^{N} E(W_i) = N \int \phi^2 + O(\|\phi\|_{C^1}^2 N^3 L^{-1}) = N \int \phi^2 + o(\|\phi\|_{C^1}^2 N).$$

It remains to verify the hypotheses (M1) – (M3) in Theorem \[\text{22}\] for our choice of $X_{i,n}$. In the following estimates, we write $L = L_n$ and otherwise drop the ‘$L$’ from our notation where possible. Moreover, to improve readability we will drop $\|\phi\|_{C^1}$ terms from our estimates, absorbing them into the $O(\cdot \cdot \cdot), o(\cdot \cdot \cdot)$ notation.

Conditions (M1) and (M2) in Theorem \[\text{22}\] We will prove

$$\int \max_{i \leq k_n} X_{i,n}^2 \to 0 \quad \text{as } n \to \infty,$$

which implies both (M1) and (M2). Using Proposition \[\text{23}\] we estimate

$$\int \max_{i \leq k_n} X_{i,n}^2 = \int \frac{\max_{i \leq N} U_i^2}{N \int \phi^2 + o(1)} \leq \int \frac{\max_i \phi^2 \circ F^{i-1} + \max_i |W_i|}{N \int \phi^2 + o(1)}$$

$$\leq \frac{O(1) + \sum_i E|W_i|}{N \int \phi^2 + o(1)} = \frac{O(1) + O(N^3 L^{-\frac{1}{2}})}{N \int \phi^2 + o(1)},$$

which, under \[\text{6}\], goes to 0 as $L = L_n \to \infty$. 
Condition (M3). We write
\[ \sum_i X_i^2 - 1 = \frac{\sum_i U_i^2 - \sum_i \mathbb{E}(U_i^2)}{\sum_i \mathbb{E}(U_i^2)} = \frac{\sum_i \phi^2 \circ F^{i-1} - N \int \phi^2 \circ F^{i-1}}{\sum_i \mathbb{E}(U_i^2)} + \frac{\sum_i W_i - \sum_i \mathbb{E}|W_i|}{\sum_i \mathbb{E}(U_i^2)}. \]

By Corollary 21,
\[ \mathbb{E} \left| \sum_i W_i - \sum_i \mathbb{E}|W_i| \right| = \frac{O(N^3 L^{-\frac{3}{2}})}{N(\int \phi^2 + o(1))} = O(N^2 L^{-\frac{3}{2}}) \to 0, \]
That is term (II) above converges to 0 in L^1, hence in probability, as \( L = L_n \to \infty \).

On the other hand, we will prove (I) converges to 0 in L^2, hence in probability. For this,
\[ \mathbb{E} \left( \sum_i \phi^2 \circ F^{i-1} - N \int \phi^2 \right)^2 = \sum_{i,j} \left( \int (\phi^2 \circ F^{i-1})(\phi^2 \circ F^{j-1}) - (\int \phi^2)^2 \right) \]
\[ = N \left( \int \phi^4 - (\int \phi^2)^2 \right) + 2 \sum_{1 \leq i < j \leq N} \left( \int (\phi^2 \circ F^{i-1})(\phi^2 \circ F^{j-1}) - (\int \phi^2)^2 \right). \]
The first term on the RHS, corresponding to the diagonal summation \( i = j \), is clearly \( O(N) \). For each off-diagonal summand \( 1 \leq i < j \leq N \), we apply Corollary 15 with the replacements \( \phi, \psi \mapsto \phi^2 \) to get
\[ \int (\phi^2 \circ F^{i-1})(\phi^2 \circ F^{j-1}) - (\int \phi^2)^2 = \int \phi^2 \cdot \phi^2 \circ F^{i-1} - (\int \phi^2)^2 \]
\[ = O((j - i)L^{-\frac{3}{2}} + L^{-\frac{1}{2}}) \]
\[ = O(L^{-1/2}), \]
using (6). Therefore
\[ 2 \sum_{i<j} \left( \int (\phi^2 \circ F^{i-1})(\phi^2 \circ F^{j-1}) - (\int \phi^2)^2 \right) = O(N^2 L^{-\frac{3}{2}}) = o(N^2), \]
again using (6). As a result,
\[ \mathbb{E} \left( \frac{\sum_i \phi^2 \circ F^{i-1} - N \int \phi^2}{(\sum_i \mathbb{E}(U_i^2))^{\frac{1}{2}}} \right)^2 = \frac{O(N) + o(N^2)}{N^2(\int \phi^2 + o(1))} \to 0. \]
Thus (I) \( \to 0 \) in L^2, hence in probability, as claimed. This completes the verification of property (M3), hence the proof of Theorem 13.

5. Diffusive limit for the slow-fast system

We prove Theorem 12 in this section. Set \( L = \epsilon^{-\alpha} \) and \( N(\epsilon) = N(L) = \lfloor L^{2/\alpha} \rfloor = \lfloor \epsilon^{-2} \rfloor \), since \( \alpha > 8 \), we have \( N(L)L^{\frac{2}{\alpha}} \to 0 \) as \( L \to \infty \), therefore Theorem 13 applies.
Let \( X, Y \) be independent uniformly distributed random variables on \([0, 1]\), using the law of large numbers, we conclude that

\[
\pi_x G_\epsilon(x, \epsilon^{1+\alpha} y) = \pi_x F_L(x, y),
\]

have

\[
(8) \\
\pi_x G^{N(e)}_\epsilon(X, \epsilon^{1+\alpha} Y) - \epsilon^{1+\alpha} Y = \sum_{i=0}^{N(e)-1} \epsilon \sin \left( \pi_x G^{i}_{\epsilon}(X, \epsilon^{1+\alpha} Y) \right) \\
\frac{1}{\sqrt{N(e)}} \sum_{i=0}^{N(e)-1} \sin \left( \pi_x F_L(X, Y) \right) \to N(0, \frac{1}{2})
\]

in distribution as \( \epsilon \to 0 \).

Recall that \( Z \) is a uniformly distributed random variable on \([a, b]\). We define

\[
A(\epsilon) = \epsilon^{1+\alpha} \left( \int [A(\epsilon) - b^\alpha] \right) + 1 \) and \( B(\epsilon) = \epsilon^{1+\alpha} \left( \int [A(\epsilon) - b^\alpha] \right), \) and let \( Z_\epsilon(\epsilon) \) be uniformly distributed on the interval \([A(\epsilon), B(\epsilon)]\).

Note that for each \( i \in \mathbb{Z} \), the random variables

\[
\pi_x G^{N(e)}_\epsilon(X, \epsilon^{1+\alpha} (i + Y)) - \epsilon^{1+\alpha} (i + Y)
\]

are all identically distributed, as a result, the random variables

\[
\pi_x G^{N(e)}_\epsilon(X, \epsilon^{1+\alpha} Y) - \epsilon^{1+\alpha} Y \text{ and } \pi_x G^{N(e)}_\epsilon(X, Z_\epsilon(\epsilon)) - Z_\epsilon(\epsilon)
\]

are identically distributed. Moreover, for \( t \in \mathbb{R} \), we have

\[
P \left( \pi_x G^{N(e)}_\epsilon(X, Z) - Z < t \right) \\
\frac{1}{\sqrt{N(e)}} \sum_{i=0}^{N(e)-1} \sin \left( \pi_x F_L(X, Y) \right) \to N(0, \frac{1}{2})
\]

we conclude that

\[
\pi_x G^{N(e)}_\epsilon(X, Z) - Z, \pi_x G^{N(e)}_\epsilon(X, Z_\epsilon(\epsilon)) - Z_\epsilon(\epsilon), \pi_x G^{N(e)}_\epsilon(X, \epsilon^{1+\alpha} Y) - \epsilon^{1+\alpha} Y
\]

all have the same distributional limit as \( \epsilon \to 0 \). Theorem A follows from (8).

References

[1] S. Aubry and P. Y. Le Daeron. The discrete Frenkel-Kontorova model and its extensions. I. Exact results for the ground-states. Phys. D, 8(3):381–422, 1983.

[2] Alex Blumenthal. Statistical properties for compositions of standard maps with increasing coefficient. 2017.

[3] Boris V. Chirikov. A universal instability of many-dimensional oscillator systems. Phys. Rep., 52(5):264–379, 1979.

[4] Jacopo De Simoi. Stability and instability results in a model of Fermi acceleration. Discrete Contin. Dyn. Syst., 25(3):719–750, 2009.

[5] Amadeu Delshams, Rafael De la Llave, and Tere M Seara. A Geometric Mechanism for Diffusion in Hamiltonian Systems Overcoming the Large Gap Problem: Heuristics and Rigorous Verification on a Model: Heuristics and Rigorous Verification on a Model, volume 179. American Mathematical Soc., 2006.

[6] Dmitry Dolgopyat. Limit theorems for partially hyperbolic systems. Transactions of the American Mathematical Society, 356(4):1637–1689, 2004.
[7] Dmitry Dolgopyat. On differentiability of SRB states for partially hyperbolic systems. *Inventiones Mathematicae*, 155(2):389–449, 2004.

[8] Dmitry Dolgopyat. *Repulsion from resonances*. Société Mathématique de France, 2012.

[9] J. Frenkel and T. Kontorova. On the theory of plastic deformation and twinning. *Acad. Sci. U.S.S.R. J. Phys.*, 1:137–149, 1939.

[10] Vassili Gelfreich and Dmitry Turaev. Arnold diffusion in a priori chaotic symplectic maps. *Communications in Mathematical Physics*, 353(2):507–547, Jul 2017.

[11] F. M. Izraelev. Nearly linear mappings and their applications. *Phys. D*, 1(3):243–266, 1980.

[12] Donald L McLeish. Dependent central limit theorems and invariance principles. *the Annals of Probability*, pages 620–628, 1974.

[13] Anatoly I Neishtadt. Passage through a separatrix in a resonance problem with a slowly-varying parameter: Pmm vol. 39, n 4, 1975, pp. 621–632. *Journal of Applied Mathematics and Mechanics*, 39(4):594–605, 1975.

[14] Anatoly I Neishtadt. Averaging and passage through resonances. In *Proceedings of the International Congress of Mathematicians*, Kyoto, Japan, pages 1271–1283, 1990.

[15] Anatoly I Neishtadt. Scattering by resonances. *Celestial Mechanics and Dynamical Astronomy*, 65(1-2):1–20, 1996.

[16] L. D. Pustylnikov. Stable and oscillating motions in nonautonomous dynamical systems. A generalization of C. L. Siegel’s theorem to the nonautonomous case. *Mat. Sb. (N.S.*), 94(136):407–429, 495, 1974.

[17] D Treschev. Evolution of slow variables in a priori unstable Hamiltonian systems. *Nonlinearity*, 17(5):1803, 2004.

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