Absolutely split locally free sheaves on proper $k$-schemes and Brauer–Severi varieties

Saša Novaković

January 05, 2015

Abstract. We introduce absolutely split locally free sheaves on proper $k$-schemes and analyse their structure. Furthermore, we apply the results to study the geometry of Brauer–Severi varieties.

Contents

1. Introduction 1
2. Generalities on Brauer–Severi varieties and simple algebras 2
3. Absolutely isotypical sheaves 3
4. AS-bundles on proper $k$-schemes 6
5. AS-bundles on proper $k$-schemes with cyclic Picard group 9
6. AS-bundles on Brauer–Severi varieties 12
7. AS-bundles on generalized Brauer–Severi varieties 15
References 17

1. Introduction

The goal of this paper is to describe the structure of absolutely split locally free sheaves on arbitrary proper $k$-schemes $X$. An absolutely split locally free sheaf on a proper $k$-scheme $X$ is a sheaf that splits as a direct sum of invertible sheaves after base change to the algebraic closure $\overline{k}$. We simply call such sheaves AS-bundles. It is a classical result of Grothendieck [16] that the AS-bundles on $\mathbb{P}^1$ are just the locally free sheaves. Biswas and Nagaraj [7], [8], [9] and the author [26], [27] investigated and classified AS-bundles on certain Brauer–Severi varieties. More precise, AS-bundles were completely described in dimension one and for Brauer–Severi varieties over $\mathbb{R}$. In order to analyse the structure of absolutely split locally free sheaves on arbitrary Brauer–Severi varieties $X$, we study the general case where $X$ is an arbitrary proper $k$-scheme. Our first main result is the following:

Theorem. (see Theorem 4.6) Let $X$ be a proper $k$-scheme with $H^0(X, \mathcal{O}_X) = k$. Then the $k$-rational points of the Picard scheme $\text{Pic}_{X/k}$ are in one-to-one correspondence with isomorphism classes of indecomposable AS-bundles on $X$.

In the case the proper $\overline{k}$-scheme $X \otimes_k \overline{k}$ has infinite cyclic Picard group we obtain:

Theorem. (see Theorem 5.1) Let $X$ be a proper $k$-scheme with $\text{Pic}(X \otimes_k \overline{k}) \simeq \mathbb{Z}$ and period $r$. Denote by $\mathcal{L}$ the generator of $\text{Pic}(X \otimes_k \overline{k}) \simeq \mathbb{Z}$ and suppose there is an indecomposable absolutely isotypical sheaf $\mathcal{M}_\mathcal{L}$ of type $\mathcal{L}$. Let $\mathcal{J}$ be the generator of $\text{Pic}(X)$ with $\mathcal{J} \otimes_k \overline{k} \simeq \mathcal{L} \otimes k^r$, then all indecomposable AS-bundles $\mathcal{E}$ are of the form

$$\mathcal{J} \otimes a \otimes \mathcal{M}_\mathcal{L} \otimes j$$

with unique $a \in \mathbb{Z}$ and unique $0 \leq j \leq r - 1$. 

Notice that for coherent sheaves on proper $k$-schemes the Krull–Schmidt Theorem holds (see [5]) and every such can uniquely, up to isomorphism and permutation, be decomposed as a direct sum of indecomposables. It is easy to see that all $AS$-bundles on proper $k$-schemes are obtained as a direct sum of the above indecomposable $AS$-bundles.

For a complete understanding of the $AS$-bundles one furthermore has to determine the ranks of $M_{\mathcal{L}^\otimes j}$. If $H^0(X, \mathcal{O}_X) = k$ we will see that the endomorphism algebra $\text{End}(M_{\mathcal{L}})$ is central simple and one therefore has the notion of the index of $\text{End}(M_{\mathcal{L}})$. Under the assumption that the Picard group is infinite cyclic we have the following description of the ranks of $M_{\mathcal{L}^\otimes j}$.

**Proposition.** (see Proposition 5.2) Let $X$ be a proper $k$-scheme with $H^0(X, \mathcal{O}_X) = k$ and $\text{Pic}(X \otimes_k \bar{k}) \simeq \mathbb{Z}$. Denote by $\mathcal{L}$ the generator of $\text{Pic}(X \otimes_k \bar{k})$ and suppose there is an indecomposable absolutely isotypical sheaf $M_{\mathcal{L}}$ of type $\mathcal{L}$. Then for all $j \in \mathbb{Z}$ one has

$$\text{rk}(M_{\mathcal{L}^\otimes j}) = \text{ind}(\text{End}(M_{\mathcal{L}})^{\otimes j}).$$

Applying the above results to the case where $X$ is a (generalized) Brauer–Severi variety we obtain a complete classification of $AS$-bundles on arbitrary (generalized) Brauer–Severi varieties and the results of Biswas and Nagaraj [7], [8], [9] and of the author [26], [27] as corollaries (Theorem 6.5 and Corollary 6.6). We also consider the sequence of natural numbers $(d_j)_{j \in \mathbb{Z}}$ with $d_j = \text{rk}(M_{\mathcal{L}^\otimes j})$ and in fact, for a Brauer–Severi variety of period $p$ it is enough to consider the $p + 1$-tupel $(d_0, d_1, ..., d_{p-1}, d_p)$ (see Proposition 5.3). This $p + 1$-tupel will be called the $AS$-type. We determine the $AS$-type, study the relation between the $AS$-types of Brauer equivalent and birational Brauer–Severi varieties and will show that the $AS$-type is a birational invariant (see Proposition 6.12). Finally, as consequences of the Horrocks criterion and a result of Ottaviani [29] we obtain cohomological criteria telling us exactly when a locally free sheaf on a (generalized) Brauer–Severi variety is an $AS$-bundle (Theorem 6.14 and Theorem 7.7).

**Acknowledgement.** This paper is based on the first two chapters of my Ph.D. thesis which was supervised by Stefan Schröer whom I would like to thank for a lot of comments and fruitful discussions.

2. **Generalities on Brauer–Severi varieties and simple algebras**

Throughout the paper $k$ will denote an arbitrary field, unless stated otherwise. We recall the basics of Brauer–Severi varieties and central simple algebras and refer to [4], [15], [32] and [33] for details. For the more general notions of Brauer–Severi schemes and Azumaya algebras we refer the reader to [17] and [18]. A *Brauer–Severi variety* of dimension $n$ is a scheme $X$ of finite type over $k$ such that $X \otimes_k L \simeq \mathbb{P}^n$ for a finite field extension $k \subset L$. Such a field extension $k \subset L$ is called splitting field of $X$. Clearly, the algebraic closure $\bar{k}$ is also a splitting field for any Brauer–Severi variety. One can show that a Brauer–Severi variety always splits over a finite separable field extension of $k$ (see [15], Corollary 5.1.4). By embedding the finite separable splitting field into its Galois closure, a Brauer–Severi variety splits over a finite Galois extension of the base field $k$ (see [15], Corollary 5.1.5). It follows from [19], IV, Chapter II, Theorem 2.7.1 that $X$ is projective, integral and smooth over $k$. There is a well-known one-to-one correspondence between Brauer–Severi varieties and central simple $k$-algebras. Recall, a $k$-algebra $A$ is called *central simple* if it is an associative finite-dimensional $k$-algebra that has no two-sided ideals other than 0 and $A$ and if its center equals $k$. If the algebra $A$ is a division algebra, it is called *central division algebra*. Such central simple $k$-algebras can be characterized by the following well-known fact (see [15], Theorem 2.2.1): $A$ is a central simple $k$-algebra if and only if there is a finite field extension $k \subset L$ such that $A \otimes_k L \simeq M_n(L)$ if and only if $A \otimes_k \bar{k} \simeq M_n(\bar{k})$. 
The degree of a central simple algebra $A$ is now defined to be $\deg(A) := \sqrt{\dim_k A}$. It turns out that studying central simple $k$-algebras can be reduced to studying central division algebras. Indeed, according to the Wedderburn Theorem (see [15], Theorem 2.1.3) for any central simple $k$-algebra $A$ there is an integer $n > 0$ and a division algebra $D$ such that $A \simeq M_n(D)$. The division algebra $D$ is also central and unique up to isomorphism. Now the degree of the unique central division algebra $D$ is called the index of $A$ and is denoted by $\ind(A)$. It can be shown that the index is the smallest among the degrees of finite separable splitting fields of $A$ (see [15], Corollary 4.5.9). Two central simple $k$-algebras $A \simeq M_n(D)$ and $B \simeq M_m(D')$ are called Brauer equivalent if $D \simeq D'$. Brauer equivalence is indeed an equivalence relation and one defines the Brauer group $\text{Br}(k)$ of a field $k$ as the group whose elements are equivalence classes of central simple $k$-algebras and group operation being the tensor product. It is an abelian group with inverse of the equivalence class of $A$ given by the equivalence class of $A^{op}$. The neutral element is the equivalence class of $k$. The order of a central simple $k$-algebra $A$ in $\text{Br}(k)$ is called the period of $A$ and is denoted by $\text{per}(A)$. It can be shown that the period is finite, divides the index and both period and index have the same prime factors (see [15], Proposition 4.5.13). Denoting by $\text{BS}_n(k)$ the set of all isomorphism classes of Brauer–Severi varieties of dimension $n$ and by $\text{CSA}_{n+1}(k)$ the set of all isomorphism classes of central simple $k$-algebras of degree $n + 1$, we mentioned above that one can show that there is a canonical identification

$$\text{CSA}_{n+1}(k) = \text{BS}_n(k)$$

via non-commutative Galois cohomology (see [4], [15] and [34] for details). Hence any $n$-dimensional Brauer–Severi variety $X$ corresponds to a central simple $k$-algebra of degree $n + 1$. In view of the one-to-one correspondence between Brauer–Severi varieties and central simple algebras one can also speak about the period of a Brauer–severi variety $X$. It is defined to be the period of the corresponding central simple $k$-algebra $A$.

We also need some facts concerning simple and semisimple rings in general (see [2], [11] for details). Recall, for a ring $R$ with unity, a $R$-module $M$ is called simple if $M$ has no non-trivial submodules. A $R$-module $M$ is called semisimple if $M$ is isomorphic to the direct sum of simple modules (see [2], Chapter 3, §9). Note that a ring $R$ is called simple (semisimple) if it is a simple (semisimple) left module over itself. Clearly, a simple ring $R$ is of course semisimple. The Jacobson radical of a ring $R$ is by definition the intersection of all maximal left ideals in $R$ and is denoted by $\text{rad}(R)$. For semisimple rings one has the following well-known characterization: Assume $R$ is left artinian, then $R$ is semisimple if and only if $\text{rad}(R) = 0$ (see [2], Proposition 15.16). In particular $R/\text{rad}(R)$ is semisimple. Since central simple $k$-algebras $A$ are isomorphic to $M_n(D)$, they are simple in the above sense (see [2], §13, Example 13.1) and one therefore has $\text{rad}(A) = 0$. This fact is needed in the next section (see Proposition 3.2).

3. Absolutely isotypical sheaves

In this section we investigate a certain class of locally free sheaves, called absolutely isotypical and prove some properties of the endomorphism algebra of these sheaves. In the work of Arason, Elman and Jacob [3] they are called pure and are closely related to central simple algebras. In the present work we want to call them absolutely isotypical referring to the isotypical decomposition in representation theory.

**Definition 3.1.** A locally free sheaf $E$ of finite rank on a proper $k$-scheme $X$ is called absolutely isotypical if on $X \otimes_k \bar{k}$ there is an indecomposable locally free sheaf $W$ such that $E \otimes_{k} \bar{k} \simeq \bigoplus_{i=0}^{n} W$. The sheaf $W$ is called the type of the absolutely isotypical sheaf. If the indecomposable locally free sheaf $W$ is invertible, we say that $E$ is absolutely rank-one-isotypical.
We give a first example of such an absolutely isotypical sheaf that later on becomes important for further considerations. For this, let \( X \) be a \( n \)-dimensional Brauer–Severi variety over \( k \). We consider the Euler sequence on \( X \otimes_k \bar{k} \simeq \mathbb{P}^n \) (see [21], II Theorem 8.13):

\[
(1) \quad 0 \longrightarrow \Omega_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0.
\]

This short exact sequence does not split since \( \mathcal{O}_{\mathbb{P}^n}(-1) \) has no global sections. Applying the functor \( \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^n}, -) \) to this short exact sequence yields \( \operatorname{Ext}^1(\mathcal{O}_{\mathbb{P}^n}, \Omega_{\mathbb{P}^n}) \simeq \bar{k} \) and hence the middle term of the Euler sequence is unique up to isomorphism. Furthermore, since the sheaves \( \mathcal{O}_X \) and \( \Omega^1_{X/k} \) exist on \( X \) and \( \operatorname{Ext}^1(\mathcal{O}_X, \Omega^1_{X/k}) \simeq k \), there is also a non-split short exact sequence on \( X \)

\[
(2) \quad 0 \longrightarrow \Omega^1_{X/k} \longrightarrow \mathcal{V} \longrightarrow \mathcal{O}_X \longrightarrow 0,
\]

where the locally free sheaf \( \mathcal{V} \) is unique up to isomorphism. After base change to \( \bar{k} \) one gets back the sequence (1) on the projective space \( \mathbb{P}^n \) and therefore \( \mathcal{V} \otimes_k \bar{k} \simeq \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)} \).

Thus the locally free sheaf \( \mathcal{V} \) is absolutely isotypical of type \( \mathcal{O}_{\mathbb{P}^n}(-1) \).

The locally free sheaf \( \mathcal{V} \) from sequence (2) has an interesting property. To illustrate this, we consider the endomorphism algebra \( \operatorname{End}(\mathcal{V}) \) that is a finite-dimensional associative \( k \)-algebra. After base change to the algebraic closure \( \bar{k} \) we find \( \operatorname{End}(\mathcal{V}) \otimes_k \bar{k} \simeq \operatorname{End}(\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)}) \simeq \operatorname{End}(\mathcal{O}_{\mathbb{P}^n}) \simeq M_{n+1}(\bar{k}) \). We conclude that \( \operatorname{End}(\mathcal{V}) \) is a central simple \( k \)-algebra.

Moreover, we make the following observation.

**Proposition 3.2.** Let \( X \) be a geometrically integral and proper \( k \)-scheme and \( \mathcal{E} \) an absolutely rank-one-isotypical sheaf. Then \( \operatorname{End}(\mathcal{E}) \) is a central simple \( k \)-algebra. If furthermore \( \mathcal{E} \) is indecomposable, \( \operatorname{End}(\mathcal{E}) \) is a division algebra.

**Proof.** By definition there exists an invertible sheaf \( \mathcal{L} \) on \( X \otimes_k \bar{k} \) such that \( \mathcal{E} \otimes_k \bar{k} \simeq \bigoplus_{i=0}^n \mathcal{L} \). Writing \( X_{\bar{k}} \) for the scheme \( X \otimes_k \bar{k} \), we have \( \mathcal{E} \otimes_k \bar{k} \otimes \mathcal{L}^\vee \simeq \mathcal{O}_{X_{\bar{k}}}^{\oplus(n+1)} \). Since \( X \) is supposed to be proper, the endomorphism algebra \( \operatorname{End}(\mathcal{E}) \) is a finite-dimensional \( k \)-algebra. Furthermore, since \( X \) is geometrically integral we have \( \operatorname{End}(\mathcal{O}_{X_{\bar{k}}}) \simeq H^0(X_{\bar{k}}, \mathcal{O}_{X_{\bar{k}}}) \simeq \bar{k} \) and therefore we obtain

\[
\operatorname{End}(\mathcal{E}) \otimes_k \bar{k} \simeq \operatorname{End}\left( \bigoplus_{i=0}^n \mathcal{L} \right) \simeq \operatorname{Hom}\left( \mathcal{L} \otimes \left( \bigoplus_{i=0}^n \mathcal{O}_{X_{\bar{k}}}, \mathcal{L} \otimes \left( \bigoplus_{i=0}^n \mathcal{O}_{X_{\bar{k}}} \right) \right) \right) \simeq \operatorname{Hom}\left( \left( \bigoplus_{i=0}^n \mathcal{O}_{X_{\bar{k}}}, \mathcal{L}^\vee \otimes \mathcal{L} \otimes \left( \bigoplus_{i=0}^n \mathcal{O}_{X_{\bar{k}}} \right) \right) \right) \simeq \operatorname{End}(\mathcal{O}_{X_{\bar{k}}}^{\oplus(n+1)}) \simeq M_{n+1}(\bar{k}).
\]

This implies that \( \operatorname{End}(\mathcal{E}) \) has to be a central simple \( k \)-algebra. Now suppose that \( \mathcal{E} \) is indecomposable. According to [3], p.1324 on a proper \( k \)-scheme \( X \) a locally free sheaf \( \mathcal{E} \) is indecomposable if and only if \( \operatorname{End}(\mathcal{E})/\operatorname{rad}(\operatorname{End}(\mathcal{E})) \) is a division algebra over \( k \). Since \( \operatorname{End}(\mathcal{E}) \) is a central simple \( k \)-algebra we have \( \operatorname{rad}(\operatorname{End}(\mathcal{E})) = 0 \). Hence \( \operatorname{End}(\mathcal{E}) \) is a central division algebra.

We note that the rank of such an absolutely rank-one-isotypical sheaf on a scheme \( X \) as in Proposition 3.2 is equal to the degree of the central simple \( k \)-algebra \( \operatorname{End}(\mathcal{E}) \). Indeed, the above proof shows

\[
(3) \quad \operatorname{rk}(\mathcal{E}) = n + 1 = \deg(\operatorname{End}(\mathcal{E})).
\]
Now suppose we are given a geometrically integral and proper $k$-scheme $X$ and an absolutely rank-one-isotypical sheaf $E$. Suppose furthermore, we are given an indecomposable direct summand $F$ of $E$. We want to understand the relation between the central simple $k$-algebras $\text{End}(E)$ and $\text{End}(F)$. The following result is very useful and will be needed later on quite frequently.

**Proposition 3.3.** Let $X$ be a proper $k$-scheme and $F$ and $G$ two coherent sheaves. If $F \otimes_k \bar{k} \cong G \otimes_k \bar{k}$, then $F$ is isomorphic to $G$.

**Proof.** Wiegand [36], Lemma 2.3 proved this in the case $X$ is projective over $k$, but the proof remains valid for proper $k$-schemes. For convenience to the reader we reproduce the proof and follow exactly the lines of the proof given in [36]. Since the extension $k \subset \bar{k}$ is a direct limit of finite extensions, it suffices to prove the statement for finite field extensions. 

Now suppose $k \subset L$ is a finite field extension and $\pi : X \otimes_k L \to X$ the projection. Choose a basis $\{\alpha_1, ..., \alpha_d\}$ for $L$ over $k$. By assumption we have $\pi^*F \cong \pi^*G$. For the coherent sheaf $A = \pi_\pi^*F$ we have over any affine open set $U \subset X$, $A(U) = F(U) \otimes_k L$ and there is a unique $\mathcal{O}_X(U)$-module isomorphism from $A(U)$ to $F(U)^{\otimes d}$, assigning $m \otimes \alpha_i$ to $(0,0,...,m,...,0)$, where $m$ is located at the $i^{th}$ entry. This yields $\pi_\pi^*F \cong F^{\otimes d}$ and obviously the same holds for $\pi_\pi^*G$. Hence $\pi_\pi^*F \cong F^{\otimes d} \cong \pi_\pi^*G \cong G^{\otimes d}$ and we conclude from Krull–Schmidt Theorem that $F \cong G$. $\square$

Notice that the statement of Proposition 3.3 also holds if one considers the base change to the separable closure $k^{sep}$ instead of the base change to $\bar{k}$ (see [3], p.1325). A simple consequence of Proposition 3.3 is the following:

**Proposition 3.4.** Let $X$ be a proper and geometrically integral $k$-scheme. If $E$ and $E'$ are two indecomposable absolutely isotypical sheaves of the same type, then $E \cong E'$.

We are now able to understand the relation between the central simple $k$-algebras $\text{End}(E)$ and $\text{End}(F)$ from above.

**Proposition 3.5.** Let $X$ be a proper and geometrically integral $k$-scheme. Let $E$ be an indecomposable absolutely rank-one-isotypical sheaf and $F$ an indecomposable direct summand of $E$. Then one has $\text{End}(E) \cong M_n(\text{End}(F))$ and hence $\text{End}(E)$ and $\text{End}(F)$ are Brauer equivalent.

**Proof.** Since $X$ is geometrically integral and proper, Proposition 3.2 yields that $\text{End}(E)$ is a central simple $k$-algebra. Furthermore, since $X$ is proper we can decompose $E$ as a direct sum of indecomposable locally free sheaves according to the Krull–Schmidt Theorem. Now let $E = \bigoplus_{i=1}^n E_i$ be the Krull–Schmidt decomposition of $E$. Since $E$ is absolutely rank-one-isotypical there is an invertible sheaf $L$ such that $E \otimes_k \bar{k} \cong \bigoplus_{j=1}^r L$. Together with $E \otimes_k \bar{k} \cong \bigoplus_{i=1}^n (E_i \otimes_k \bar{k})$ we have an isomorphism

$$
\bigoplus_{j=1}^r L \cong E \otimes_k \bar{k} \cong \bigoplus_{i=1}^n (E_i \otimes_k \bar{k})
$$

and hence, by the Krull–Schmidt Theorem for locally free sheaves on $X \otimes_k \bar{k}$, we obtain that $E_i$ is also absolutely rank-one-isotypical of type $L$. Since all the locally free sheaves $E_i$ are indecomposable, Proposition 3.4 yields that they are all isomorphic. Hence $E \cong \bigoplus_{i=1}^n E_i$ and thus $F \cong E_1$ by the Krull–Schmidt Theorem. By Proposition 3.2, the endomorphism algebra $\text{End}(F)$ is a central division algebra and hence $\text{End}(E) \cong \text{End}(\bigoplus_{i=1}^n F) \cong M_n(\text{End}(F))$ what furthermore implies that $\text{End}(F)$ and $\text{End}(E)$ are Brauer equivalent. $\square$

**Remark 3.6.** The proof of the above proposition shows that all the indecomposable direct summands in the Krull–Schmidt decomposition of a absolutely rank-one-isotypical sheaf $E$ are isomorphic. In particular, $\text{End}(F)$ is isomorphic to the unique central division algebra $D$ with $M_n(D) \cong \text{End}(E)$. 

4. AS-bundles on proper $k$-schemes

In this section we prove the first main result of the present paper. We start with the main definition.

**Definition 4.1.** Let $X$ be a $k$-scheme. A locally free sheaf $\mathcal{E}$ of finite rank on $X$ is called absolutely split (separably split) if it splits after base change as a direct sum of invertible sheaves on $X \otimes_k \bar{k}$ (resp. $X \otimes_k k^{sep}$). For an absolutely split locally free sheaf we shortly write AS-bundle.

**Proposition 4.2.** Let $X$ be a proper $k$-scheme and $\mathcal{E}$ a locally free sheaf of finite rank on $X$. Then $\mathcal{E}$ is absolutely split if and only if it is separably split.

**Proof.** If $\mathcal{E}$ is separably split, then clearly it is absolutely split. For the converse, assume $\mathcal{E}$ is not separably split. Then after base change to $k^{sep}$ we obtain from the Krull–Schmidt decomposition

$$\mathcal{E} \otimes_k k^{sep} \cong \bigoplus_{i=1}^r \mathcal{E}_i$$

that at least one of the indecomposable $\mathcal{E}_i$ on $X \otimes_k k^{sep}$ has rank $> 1$. According to [30], Proposition 3.1, the locally free sheaves $\mathcal{E}_i$ remain indecomposable after base change to $\bar{k}$. Therefore, there exists at least one $\mathcal{E}_i \otimes_{k^{sep}} \bar{k}$ with $\text{rk}(\mathcal{E}_i \otimes_{k^{sep}} \bar{k}) > 1$. But this implies that

$$\mathcal{E} \otimes_k \bar{k} \cong (\mathcal{E} \otimes_k k^{sep}) \otimes_{k^{sep}} \bar{k} \cong \bigoplus_{i=1}^r (\mathcal{E}_i \otimes_{k^{sep}} \bar{k})$$

is not absolutely split. This contradicts the assumption and completes the proof. \qed

We now cite a for further investigations crucial fact that is proved in [3], Proposition 3.4.

**Proposition 4.3.** Let $X$ be a proper $k$-scheme. Suppose that $\mathcal{W}$ is an indecomposable locally free sheaf on $X \otimes_k k^{sep}$ such that the isomorphism class is $\text{Gal}(k^{sep}/k)$-invariant. Then there is an indecomposable locally free sheaf $\mathcal{E}$ on $X$ such that $\mathcal{E} \otimes_k k^{sep} \cong \mathcal{W}^{\oplus n}$. Moreover, the locally free sheaf $\mathcal{E}$ is unique up to isomorphism.

**Proof.** Let $k \subset L$ be a finite Galois extension inside of $k^{sep}$ such that $\mathcal{W} \cong \mathcal{N} \otimes_L k^{sep}$ for some locally free sheaf $\mathcal{N}$ on $X \otimes_L L$. Let $\pi_*\mathcal{N}$ be the sheaf on $X$ obtained as the direct image of the projection $\pi : X \otimes_L L \to X$. As the $\text{Gal}(L/k)$-conjugates of $\mathcal{N} \otimes_L k^{sep}$ are all isomorphic to $\mathcal{W}$, we have $\pi_*\mathcal{N} \cong \mathcal{W}^{\oplus \text{rk}(\mathcal{M})}$. Applying the Krull–Schmidt Theorem we can consider a direct summand $\mathcal{M}$ of $\pi_*\mathcal{N}$. For this locally free sheaf $\mathcal{M}$ one clearly has $\mathcal{M} \otimes_k k^{sep} \cong \mathcal{W}^{\oplus \text{rk}(\mathcal{M})}$. To prove the uniqueness, we assume that there is another indecomposable locally free sheaf $\mathcal{M}'$ with $\mathcal{M}' \otimes_k k^{sep} \cong \mathcal{W}^{\oplus \text{rk}(\mathcal{M}')}$. Let $r = \text{rk}(\mathcal{M})$ and $s = \text{rk}(\mathcal{M}')$. Then $(\mathcal{M}^{\oplus r}) \otimes_k k^{sep} \cong (\mathcal{M}'^{\oplus s}) \otimes_k k^{sep}$, what by Proposition 3.3 (and comment right afterward) implies $\mathcal{M}^{\oplus r} \cong \mathcal{M}'^{\oplus s}$. Therefore, applying Krull–Schmidt Theorem once again, we find $\mathcal{M} \cong \mathcal{M}'$. \qed

For $G = \text{Gal}(k^{sep}/k)$ we denote by $\text{Pic}^G(X \otimes_k k^{sep}) \subset \text{Pic}(X \otimes_k k^{sep})$ the subgroup of isomorphism classes of $G$-invariant invertible sheaves. As an immediate consequence of Proposition 4.3 we obtain:

**Corollary 4.4.** Let $X$ be a proper $k$-scheme. For all $\mathcal{L} \in \text{Pic}^G(X \otimes_k k^{sep})$ there is an up to isomorphism unique indecomposable $\mathcal{M}_\mathcal{L}$ on $X$ such that $\mathcal{M}_\mathcal{L} \otimes_k k \cong \mathcal{L}^{\oplus n}$.

**Theorem 4.5.** Let $X$ be a proper $k$-scheme. Then all indecomposable separably split locally free sheaves are of the form $\mathcal{M}_\mathcal{L}$ for a unique $\mathcal{L} \in \text{Pic}^G(X \otimes_k k^{sep})$. 

Proof. Let \( \mathcal{M} \) be an indecomposable separably split locally free sheaf. Then by definition we have

\[
\mathcal{M} \otimes_k k^{sep} \cong \bigoplus_{i=1}^m \mathcal{L}_i^{s_i r_i},
\]

where \( \mathcal{L}_i \) are invertible sheaves on \( X \otimes_k k^{sep} \). Since \( \mathcal{M} \otimes_k k^{sep} \) is \( G \)-invariant we have \( \mathcal{M} \otimes_k k^{sep} \cong \sigma^*(\mathcal{M} \otimes_k k^{sep}) \) what implies \( \bigoplus_{i=1}^m \sigma^*(\mathcal{L}_i^{s_i r_i}) \cong \bigoplus_{i=1}^m \mathcal{L}_i^{s_i r_i} \) for all \( \sigma \in G \). Krull–Schmidt Theorem now implies that all \( \mathcal{L}_i \) are also \( G \)-invariant. By Proposition 4.3, for all these \( \mathcal{L}_i \), there is an up to isomorphism unique indecomposable locally free sheaf \( \mathcal{M}_{\mathcal{L}_i} \) with \( \mathcal{M}_{\mathcal{L}_i} \otimes_k k^{sep} \cong \mathcal{L}_i^{s_i r_i} \), where \( s_i = \text{rk}(\mathcal{M}_{\mathcal{L}_i}) \). Now we consider the least common multiple \( d = \text{lcm}(s_1, \ldots, s_m) \) of all these \( s_i \). Then by the definition of the least common multiple there are integers \( n_i \) such that \( n_i s_i = d \). Considering the separably split locally free sheaf \( \mathcal{M}^{\oplus d} \) we find

\[
(\mathcal{M}^{\oplus d}) \otimes_k k^{sep} \cong \bigoplus_{i=1}^m \mathcal{L}_i^{r_i s_i} \cong \bigoplus_{i=1}^m \mathcal{L}_i^{r_i s_i (n_i r_i)}.
\]

Since the locally free sheaves \( \mathcal{L}_i^{s_i r_i} \) descent to \( \mathcal{M}_{\mathcal{L}_i} \), we have

\[
\left( \bigoplus_{i=1}^m \mathcal{M}_{\mathcal{L}_i}^{s_i r_i} \right) \otimes_k k^{sep} \cong (\mathcal{M}^{\oplus d}) \otimes_k k^{sep}.
\]

Applying Proposition 3.3 (especially the comment right afterward) yields

\[
\mathcal{M}^{\oplus d} \cong \bigoplus_{i=1}^m \mathcal{M}_{\mathcal{L}_i}^{s_i r_i}.
\]

Finally, applying the Krull–Schmidt Theorem to \( \mathcal{M}^{\oplus d} \) implies that \( \mathcal{M} \) has to be of the form \( \mathcal{M}_{\mathcal{L}} \) for some unique \( \mathcal{L} \in \text{Pic}^G(X \otimes_k k^{sep}) \). \( \Box \)

Theorem 4.5 has a very interesting geometric consequence that will be the content of the main theorem of this section. To prove this theorem, we first have to investigate more closely the Hochschild–Serre spectral sequence for Galois coverings (see [25] for details). In general, for a proper \( k \)-scheme \( X \) with \( H^0(X, \mathcal{O}_X) = k \) one has the four-term exact sequence for the Galois covering \( \text{Spec}(k^{sep}) \to \text{Spec}(k) \) with \( X' = X \otimes_k k^{sep} \):

\[
0 \to H^1(G, H^0(X_{et}, \mathcal{O}_m)) \to H^1(X_{et}, \mathcal{G}_m) \to H^0(G, H^1(X_{et}', \mathcal{G}_m)) \to
\]

\[
H^2(G, H^0(X_{et}', \mathcal{G}_m)).
\]

Assuming \( H^0(X, \mathcal{O}_X) = k \) one has \( H^2(G, H^0(X_{et}', \mathcal{G}_m)) = \text{Br}(k) \) and the above exact sequence becomes

\[
0 \to \text{Pic}(X) \to \text{Pic}^G(X') \to \text{Br}(k).
\]

In order to get a more geometric interpretation of the group \( \text{Pic}^G(X') \) we recall the basics of the Picard scheme. The main references for the Picard scheme are [19], [20] and [23]. For a scheme \( X \), the Picard group \( \text{Pic}(X) \) is the same as \( H^1(X, \mathcal{O}_X^*) \) (see [21], p.224). This group is also called the absolute Picard group. To get some relative version of this group we fix a \( S \)-scheme \( X \) with structural morphism \( f : X \to S \). Now for a \( S \)-scheme \( T \) one has the following base change diagram

\[
\begin{array}{ccc}
X_T = X \times_S T & \to & X \\
f_T & & \downarrow f \\
T & \to & S
\end{array}
\]

and one can form the presheaf \( T \mapsto H^1(X_T, \mathcal{O}_{X_T}) \) whose associated sheaf is \( \mathbb{R}^1 f_T^* \mathcal{O}_{X_T} \).
etale-topology, the above exact sequence becomes:

\[ f \]

Assuming relative Picard functor

One defines the

becomes

delicate problem under what kind of assumptions the Picard functor is representable in

theorem of Grothendieck about the Picard functor is that under the assumption that

what by \([20]\), p.190-216 implies

H

Picard functor simply as \(\text{Pic} \) (see \([21]\), Proposition 8.1, p.250). In the Zariski topology one then defines the

Pic

Picard functor of \( \) (\( \text{Pic} \) is representable and one has \( \text{Hom}(\text{Spec}(k), \text{Pic}(X/k)) = \text{Br}(k) \).

This means that we can represent elements of \( \text{Pic}(X/k)_{\text{et}}(k) \) by invertible sheaves on \( X \) that become zero in the Brauer group \( \text{Br}(k) \). If \( X \) has a \( k \)-rational point, one can show that \( \text{Pic}(X) = \text{Pic}(X/k)_{\text{et}}(k) \) and therefore the \( k \)-rational points of \( \text{Pic}(X/k) \) are in one-to-one correspondence with invertible sheaves on \( X \). The question arises what happens if in general \( X \) does not admit a \( k \)-rational point. Comparing the two exact sequences (4) and (5) shows that the \( G \)-invariant invertible sheaves on \( X \otimes_k k^{\text{sep}} \) are in one-to-one correspondence with \( k \)-rational points in the Picard scheme \( \text{Pic}(X/k) \) since the Picard functor is representable and one has \( \text{Hom}(\text{Spec}(k), \text{Pic}(X/k)) = \text{Pic}(X/k)_{\text{et}}(k) \).

The above discussion now enables us to prove the main theorem of this section.

**Theorem 4.6.** Let \( X \) be a proper \( k \)-scheme such that \( H^0(X, \mathcal{O}_X) = k \). Then the \( k \)-rational points of \( \text{Pic}(X/k) \) are in one-to-one correspondence with isomorphism classes of indecomposable \( AS \)-bundles on \( X \).
Proof. Denoting by \( AS_X \) the set of isomorphism classes of indecomposable \( AS \)-bundles on \( X \) we will construct a map
\[
\text{Pic}_{X/k}(k) \rightarrow AS_X
\]
and show that this map is bijective. The above discussion shows that for every \( k \)-rational point \( y \in \text{Pic}_{X/k} \) we have up to isomorphism a unique \( G \)-invariant invertible sheaf \( \mathcal{L}_y \in \text{Pic}^G(X \otimes_k k^{\text{sep}}) \). According to Corollary 4.4 and Theorem 4.5, for this \( \mathcal{L}_y \) there exists a unique up to isomorphism separable isotypical sheaf \( \mathcal{M}_{\mathcal{L}_y} \). Since a separably split locally free sheaf is absolutely split (see Proposition 4.2), we define the above map by assigning to a \( k \)-rational point \( y \) the isomorphism class \([\mathcal{M}_{\mathcal{L}_y}]\). This map is well defined and according to Theorem 4.5 and Proposition 3.3 it is injective. It remains to show that this map is surjective. For this, we take an indecomposable \( AS \)-bundle \( \mathcal{E} \) on \( X \). By Proposition 4.2, \( \mathcal{E} \) is an indecomposable separably split locally free sheaf. Again by Theorem 4.5 we conclude \( \mathcal{E} \cong \mathcal{M}_L \) for a unique \( L \in \text{Pic}^G(X \otimes_k k^{\text{sep}}) \). But by the above discussion the invertible sheaf \( L \) uniquely corresponds to a \( k \)-rational point \( x \in \text{Pic}_{X/k}(k) \). Therefore the above constructed map is surjective. This completes the proof. \( \square \)

5. ASS-bundles on proper \( k \)-schemes with cyclic Picard group

In this section we prove the second main theorem of the present paper. The results of this section will be applied in Section 6 to study Brauer–Severi varieties.

We start with fixing some notation and stating some facts. From Proposition 3.3 we conclude that \( \text{Pic}(X) \) is a subgroup of \( \text{Pic}(X \otimes_k \bar{k}) \). So if \( \text{Pic}(X \otimes_k \bar{k}) \cong \mathbb{Z} \), Proposition 3.3 implies that \( \text{Pic}(X) \) is also isomorphic to \( \mathbb{Z} \). Now we fix a generator \( \mathcal{L} \) of \( \text{Pic}(X \otimes_k \bar{k}) \) and denote by \( \mathcal{J} \) the generator of \( \text{Pic}(X) \) such that \( \mathcal{J} \otimes_k \bar{k} \cong \mathcal{L}^{\otimes r} \) for some unique positive integer \( r \). This number \( r \) can be interpreted as follows: It is the smallest positive number \( l \) such that \( \mathcal{L}^{\otimes l} \) descends to an invertible sheaf on \( X \). We want to call this integer \( r \) the period of \( X \). Now suppose that for the generator \( \mathcal{L} \in \text{Pic}(X \otimes_k \bar{k}) \) there exists an indecomposable absolutely rank-one-isotypical sheaf \( \mathcal{M} \) of type \( \mathcal{L} \). By Proposition 3.4, the sheaf \( \mathcal{M} \) is unique up to isomorphism and hence we write \( \mathcal{M}_L \) for it. Clearly, if for \( \mathcal{L} \) there exists an indecomposable absolutely rank-one-isotypical sheaf \( \mathcal{M}_L \) of type \( \mathcal{L} \) then for all invertible sheaves \( \mathcal{L}^{\otimes j} \in \text{Pic}(X \otimes_k \bar{k}) \) there are indecomposable absolutely isotypical sheaves of type \( \mathcal{L}^{\otimes j} \). This is due to the following fact: Let \( s = \text{rk}(\mathcal{M}_L) \) and consider \( (\mathcal{L}^{\otimes s})^{\otimes j} \cong (\mathcal{L}^{\otimes j})^{\otimes s} \). From this one obtains \( \mathcal{M}_L^{\otimes j} \otimes_k \bar{k} \cong (\mathcal{L}^{\otimes j})^{\otimes s} \). Considering the Krull–Schmidt decomposition of \( \mathcal{M}_L^{\otimes j} \) and taking into account that all indecomposable direct summands are isomorphic (see proof of Proposition 3.5 and Remark 3.6), we get a unique indecomposable locally free sheaf \( \mathcal{M}_{L^{\otimes j}} \) such that \( \mathcal{M}_{L^{\otimes j}} \otimes_k \bar{k} \cong (\mathcal{L}^{\otimes j})^{\otimes s} \), where \( s_j \) is the rank of \( \mathcal{M}_{L^{\otimes j}} \).

With this observation and the above notation we now prove the following result:

**Theorem 5.1.** Let \( X \) be a proper \( k \)-scheme with \( \text{Pic}(X \otimes_k \bar{k}) \cong \mathbb{Z} \) and period \( r \). Denote by \( \mathcal{L} \) the generator of \( \text{Pic}(X \otimes_k \bar{k}) \) and suppose there is an indecomposable absolutely isotypical sheaf \( \mathcal{M}_L \) of type \( \mathcal{L} \). Let \( \mathcal{J} \) be the generator of \( \text{Pic}(X) \) with \( \mathcal{J} \otimes_k \bar{k} \cong \mathcal{L}^{\otimes r} \), then all indecomposable \( AS \)-bundles \( \mathcal{E} \) are of the form
\[
\mathcal{J}^{\otimes a} \otimes \mathcal{M}_{L^{\otimes j}}
\]
with unique \( a \in \mathbb{Z} \) and unique \( 0 \leq j \leq r - 1 \).

**Proof.** Let \( \mathcal{E} \) be an arbitrary, not necessarily indecomposable \( AS \)-bundle and \( \pi : X \otimes_k \bar{k} \rightarrow X \) the projection. By assumption there is an indecomposable absolutely isotypical sheaf \( \mathcal{M}_L \) of type \( \mathcal{L} \). We have shown above that there exist up to isomorphism unique indecomposable absolutely isotypical sheaves of type \( \mathcal{L}^{\otimes j} \) for all \( j \in \mathbb{Z} \). We denote these
denote by \( M \) an indecomposable absolutely isotypical sheaf. Let \( \text{Proposition 5.2.} \)

\[ M \]

be a proper \( \mathbb{A}S \)-bundle with \( \text{Pic}(\mathbb{A}S) \) the index of \( \text{End}(\mathbb{A}S) \). Since \( \mathcal{E} \) is an \( \mathbb{A}S \)-bundle, the locally free sheaf \( \mathcal{E} \) is an \( \mathbb{A}S \)-bundle too. Therefore \( \pi^*(\mathcal{E}) \) decomposes into a direct sum of invertible sheaves and we find \( \pi^*(\mathcal{E}) \) is isomorphic to

\[
\bigoplus_{i=0}^{r_0} (\mathcal{L}^{\otimes a_{i_0}})^{\otimes d_i} \oplus \bigoplus_{i=0}^{r_1} (\mathcal{L}^{\otimes a_{i_1}})^{\otimes d_i} \oplus \ldots \oplus \bigoplus_{i=0}^{r_{n-1}} (\mathcal{L}^{\otimes a_{i_{n-1}}} \otimes r_{(r-1)})^{\otimes d_i}.
\]

By the definition of \( d_i \), there are \( h_j \) such that \( h_j \cdot \text{rk}(\mathcal{M}_{\otimes j}) = d_i \) for \( 0 \leq j \leq r-1 \). Furthermore, the sheaves \( \mathcal{M}_{\otimes j} \) have the property that \( \pi^* \mathcal{M}_{\otimes j} \cong (\mathcal{L}^{\otimes j})^{\otimes d_i} \), where \( d_i = \text{rk}(\mathcal{M}_{\otimes j}) \). Now for the direct summands \( (\mathcal{L}^{\otimes a_{i_1}} \otimes \mathcal{M}_{\otimes j})^{\otimes h_j} \) we have

\[
\mathcal{L}^{\otimes a_{i_1} + \otimes} \cong (\mathcal{L}^{\otimes a_{i_1}})^{\otimes d_i} \otimes (\mathcal{L}^{\otimes a_{i_1}})^{\otimes d_i} \otimes (\mathcal{L}^{\otimes a_{i_1}})^{\otimes d_i} = \bigoplus_{i=0}^{r_{n-1}} (\mathcal{L}^{\otimes a_{i_{n-1}}} \otimes r_{(r-1)})^{\otimes d_i}.
\]

Considering \( \mathcal{L}^{\otimes a_{i_1}} \otimes \mathcal{M}_{\otimes j} \) on \( X \) we find

\[
\pi^* \left( \mathcal{L}^{\otimes a_{i_1}} \otimes \mathcal{M}_{\otimes j} \right)^{\otimes h_j} \cong (\mathcal{L}^{\otimes a_{i_1} + \otimes})^{\otimes d_i}.
\]

This implies that for the locally free sheaf

\[
\mathcal{R} = \bigoplus_{i=0}^{r_0} (\mathcal{L}^{\otimes a_{i_0}})^{\otimes d_i} \oplus \bigoplus_{i=0}^{r_1} (\mathcal{L}^{\otimes a_{i_1}} \otimes \mathcal{M}_{\otimes j})^{\otimes h_j} \oplus \ldots \oplus \bigoplus_{i=0}^{r_{n-1}} (\mathcal{L}^{\otimes a_{i_{n-1}}} \otimes \mathcal{M}_{\otimes j})^{\otimes h_j - 1}
\]

we have \( \pi^* \mathcal{R} \cong \pi^*(\mathcal{E}) \). Applying Proposition 3.3 yields that \( \mathcal{E} \) is isomorphic to \( \mathcal{R} \). And because the Krull-Schmidt Theorem holds for locally free sheaves on \( X \) we conclude that \( \mathcal{E} \) is isomorphic to the direct sum of locally free sheaves of the form \( \mathcal{J}^{\otimes a} \otimes \mathcal{M}_{\otimes j} \) with unique \( a \in \mathbb{Z} \) and \( 0 \leq j \leq r-1 \). Furthermore, since all these bundles are indecomposable by definition, we finally get that all the indecomposable \( \mathbb{A}S \)-bundles are of the form \( \mathcal{J}^{\otimes a} \otimes \mathcal{M}_{\otimes j} \) with unique \( a \in \mathbb{Z} \) and \( 0 \leq j \leq r-1 \). This completes the proof. \( \square \)

In particular, Theorem 5.1 shows that the isomorphism classes of indecomposable \( \mathbb{A}S \)-bundles do not depend on the choice of a generator of \( \text{Pic}(X \otimes \hat{k}) \). Similar results also hold for proper \( k \)-schemes with \( \text{Pic}(X \otimes \hat{k}) \cong \mathbb{Z}^{\otimes n} \) and will be the content of a further article of the author.

For a complete understanding of the \( \mathbb{A}S \)-bundles one furthermore has to determine the ranks of the sheaves \( \mathcal{M}_{\otimes j} \). If \( H^0(X, \mathcal{O}_X) = k \) then Proposition 3.2 shows that the endomorphism algebra \( \text{End}(\mathcal{M}_L) \) is central simple and one therefore has the notion of the index of \( \text{End}(\mathcal{M}_L) \). The purpose of the next observation is to determine the ranks of \( \mathcal{M}_{\otimes j} \).

**Proposition 5.2.** Let \( X \) be a proper \( k \)-scheme with \( H^0(X, \mathcal{O}_X) = k \) and \( \text{Pic}(X \otimes \hat{k}) \cong \mathbb{Z} \). Denote by \( L \) the generator of \( \text{Pic}(X \otimes \hat{k}) \) from above and suppose there is an indecomposable absolutely isotypical sheaf \( \mathcal{M}_L \) of type \( L \). Then for all \( j \in \mathbb{Z} \) one has

\[ \text{rk}(\mathcal{M}_{\otimes j}) = \text{ind}(\text{End}(\mathcal{M}_{\otimes j})). \]

**Proof.** Denote by \( \pi : X \otimes \hat{k} \to X \) the projection. For the locally free sheaf \( \mathcal{M}_{\otimes j} \) we have

\[ \pi^* \mathcal{M}_{\otimes j} \cong (\mathcal{L}^{\otimes \text{rk}}(\mathcal{M}_L))^{\otimes j} \cong (\mathcal{L}^{\otimes j})^{\otimes \text{rk}(\mathcal{M}_L)} \]
what implies that \( \mathcal{M}_{\mathcal{L}^\oplus} \) is absolutely isotypical of type \( \mathcal{L}^\oplus \), but in general not indecomposable. Since \( H^0(X, \mathcal{O}_X) = k \), Proposition 3.2 implies that the endomorphism algebra \( \text{End}(\mathcal{M}_{\mathcal{L}^\oplus}) \) is central simple and, since \( \mathcal{M}_{\mathcal{L}^\oplus} \) is indecomposable, it is moreover a division algebra. Applying the Krull–Schmidt Theorem for the locally free sheaf \( \mathcal{M}_{\mathcal{L}^\oplus} \) on \( X \) we have a decomposition

\[
\mathcal{M}_{\mathcal{L}^\oplus} \simeq \bigoplus_{i=1}^m \mathcal{E}_i,
\]

where all \( \mathcal{E}_i \) are indecomposable. After base change to the algebraic closure we obtain

\[
(\mathcal{L}^\oplus)^{\oplus \text{rk}(\mathcal{M}_\mathcal{L})} \simeq (\mathcal{M}_{\mathcal{L}^\oplus}) \otimes_k \bar{k} \simeq \bigoplus_{i=1}^m \mathcal{E}_i \otimes_k \bar{k}.
\]

Applying Krull–Schmidt Theorem for locally free sheaves on \( X \otimes_k \bar{k} \) yields that all the \( \mathcal{E}_i \) are absolutely isotypical of type \( \mathcal{L}^\oplus \). According to Proposition 3.4 and Remark 3.6 all \( \mathcal{E}_i \) are isomorphic to \( \mathcal{M}_{\mathcal{L}^\oplus} \). Proposition 3.5 implies that \( \text{End}(\mathcal{M}_{\mathcal{L}^\oplus}) \) is a matrix algebra over the central division algebra \( \text{End}(\mathcal{M}_{\mathcal{L}^\oplus}) \). By (3) the rank of \( \mathcal{M}_{\mathcal{L}^\oplus} \) equals the degree \( \text{deg}(\text{End}(\mathcal{M}_{\mathcal{L}^\oplus})) \). But \( \text{deg}(\text{End}(\mathcal{M}_{\mathcal{L}^\oplus})) = \text{ind}(\text{End}(\mathcal{M}_{\mathcal{L}^\oplus})) \). Finally, since \( \text{End}(\mathcal{M}_{\mathcal{L}^\oplus}) \simeq \text{End}(\mathcal{M}_{\mathcal{L}^\oplus}) \) we obtain \( \text{rk}(\mathcal{M}_{\mathcal{L}^\oplus}) = \text{deg}(\text{End}(\mathcal{M}_{\mathcal{L}^\oplus})) = \text{ind}(\text{End}(\mathcal{M}_{\mathcal{L}^\oplus})) \).

If \( X \) fulfills the assumptions stated in Theorem 5.1, one can consider the sequence \( (\text{rk}(\mathcal{M}_{\mathcal{L}^\oplus}))_{j \in \mathbb{Z}} \). Since \( \text{Pic}(X \otimes_k \bar{k}) \) is infinite cyclic one gets a periodicity in the above sequence with respect to the period \( r \) of \( X \) as will be shown in the next proposition.

**Proposition 5.3.** Let \( X \) be a proper \( k \)-scheme of period \( r \) with \( \text{Pic}(X \otimes_k \bar{k}) \simeq \mathbb{Z} \) and \( H^0(X, \mathcal{O}_X) = k \) and let \( \mathcal{L}, \mathcal{J} \) and \( \mathcal{M}_{\mathcal{L}^\oplus} \) be as in Theorem 5.1. Then for all \( j \in \mathbb{Z} \) the following hold:

1. \( \mathcal{M}_{\mathcal{L}^\oplus(j+a)} \simeq \mathcal{M}_{\mathcal{L}^\oplus(-j)} \).
2. \( \mathcal{M}_{\mathcal{L}^\oplus(j+ar)} \simeq \mathcal{J}^{\oplus a} \otimes \mathcal{M}_{\mathcal{L}^\oplus} \).

In particular one has \( \text{rk}(\mathcal{M}_{\mathcal{L}^\oplus}) = \text{rk}(\mathcal{M}_{\mathcal{L}^\oplus(-j)}) \) and \( \text{rk}(\mathcal{M}_{\mathcal{L}^\oplus(j+ar)}) = \text{rk}(\mathcal{M}_{\mathcal{L}^\oplus}) \).

**Proof.** For the invertible sheaf \( \mathcal{L} \) we have indecomposable locally free sheaves \( \mathcal{M}_\mathcal{L} \) and \( \mathcal{M}_\mathcal{L} \). Since \( \mathcal{M}_\mathcal{L} \) is an absolutely isotypical sheaf of type \( \mathcal{L} \) we conclude that \( \mathcal{M}_\mathcal{L} \) is absolutely isotypical of type \( \mathcal{L}' \). According to Proposition 3.4, \( \mathcal{M}_\mathcal{L} \) is isomorphic to \( \mathcal{M}_\mathcal{L} \). The same argument shows \( \mathcal{M}_{\mathcal{L}^\oplus} \simeq \mathcal{M}_{\mathcal{L}^\oplus(-j)} \). This proves (i). To prove (ii), we apply the same argument. For this, let \( \mathcal{J} \) be the generator of \( \text{Pic}(X) \) and note that by the definition of the period of \( X \) we have \( \mathcal{J} \otimes_k \bar{k} \simeq \mathcal{L}^\oplus \). Now consider the sheaf \( \mathcal{M}_{\mathcal{L}^\oplus(j+ar)} \) and note that it is absolutely isotypical of type \( \mathcal{L}^\oplus(j+ar) \). The locally free sheaf \( \mathcal{J}^{\oplus a} \otimes \mathcal{M}_{\mathcal{L}^\oplus} \) is indecomposable and also absolutely isotypical of type \( \mathcal{L}^\oplus(j+ar) \). Again with Proposition 3.4 we conclude \( \mathcal{M}_{\mathcal{L}^\oplus(j+ar)} \simeq \mathcal{J}^{\oplus a} \otimes \mathcal{M}_{\mathcal{L}^\oplus} \). This completes the proof.

**Remark 5.4.** Under the assumption as in the proposition above we conclude that the sequence \( (\text{rk}(\mathcal{M}_{\mathcal{L}^\oplus}))_{j \in \mathbb{Z}} \) is completely determined by the tuple \( (1, \text{rk}(\mathcal{M}_\mathcal{L}), ..., \text{rk}(\mathcal{M}_{\mathcal{L}^\oplus(-1)}), 1) \). Note that \( \text{rk}(\mathcal{M}_{\mathcal{L}^\oplus}) = \text{rk}(\mathcal{M}_{\mathcal{L}^\oplus(-1)}) = 1 \), since both the structure sheaf and \( \mathcal{L} \) descent.

In light of Proposition 5.3 and Remark 5.4 we define the \( \text{AS-type} \) of \( X \) to be the \( r+1 \)-tuple \( (1, \text{rk}(\mathcal{M}_\mathcal{L}), ..., \text{rk}(\mathcal{M}_{\mathcal{L}^\oplus(-1)}), 1) \). It would be interesting to investigate the relation between the \( \text{AS-types} \) of proper \( k \)-schemes \( X \) and \( Y \) related by certain morphisms \( \phi : X \to Y \). Furthermore, for the central simple \( k \)-algebras \( \text{End}(\mathcal{M}_\mathcal{L}) \) one also has the period as invariant and it is reasonable to study which geometric properties of \( X \) are reflected by the \( \text{AS-type} \) and the periods of \( \text{End}(\mathcal{M}_\mathcal{L}) \). Exactly this will be done for Brauer–Severi varieties in the next section.
6. **AS-bundles on Brauer–Severi varieties**

In this section we describe the structure of AS-bundles on Brauer–Severi varieties. Furthermore, we study the AS-type and show how the AS-types of Brauer equivalent and birational Brauer–Severi varieties are related.

As mentioned in Section 2, the $n$-dimensional Brauer–Severi varieties over $k$ are in one-to-one correspondence with central simple $k$-algebras of degree $n + 1$. Now for certain fields $k$ the Brauer group $\text{Br}(k)$ is trivial and hence the only element in this group is the equivalence class of $k$. This happens for instance if $k$ is finite or algebraically closed (see [32], §7). In this case the corresponding Brauer–Severi varieties are just $\mathbb{P}^n$ for suitable $n$. For $k = \mathbb{R}$ the Brauer group $\text{Br}(\mathbb{R})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ where the generator of $\text{Br}(\mathbb{R})$ is given by the equivalence class of the Hamilton quaternions. As mentioned above, the period of a Brauer–Severi variety $X$ is the order of the corresponding central simple $k$-algebra $A$ in $\text{Br}(k)$. But there is also a geometric interpretation of the period. Hochschild–Serre spectral sequence for Galois coverings applied to a Brauer–Severi variety $X$ yields the following exact sequence (see [4]):

$$0 \longrightarrow \text{Pic}(X) \xrightarrow{deg} \mathbb{Z} \longrightarrow \text{Br}(k).$$

A Theorem due to Lichtenbaum (see [15], Theorem 5.4.10.) now states that the boundary map $\delta : \mathbb{Z} \rightarrow \text{Br}(k)$ is given by sending 1 to the class of $X$ in $\text{Br}(k)$. Here the class of $X$ is that of the corresponding central simple $k$-algebra. It follows that $\text{Pic}(X)$ is identified with some subgroup $r\mathbb{Z}$ of $\mathbb{Z}$. So $r$ is the order of $A$ in $\text{Br}(k)$, where $A$ is the central simple $k$-algebra corresponding to $X$. Hence the period of $X$ can also be thought of as the smallest positive integer $r$ such that $O_X(r)$ exists in $\text{Pic}(X)$. Therefore, the period of a Brauer–Severi variety equals the period as defined in Section 5. For further investigations we need the following result contained in [32], Theorem 5.5. We denote by $(m, n)$ the greatest common divisor of the natural numbers $m$ and $n$.

**Theorem 6.1.** Let $A$ be a central simple $k$-algebra of index $i$. Then for $r \geq 0$ one has:

(i) The index of $A^\otimes r$ divides ${(i \choose r)}i$.

(ii) Suppose $i$ and $r$ are coprime. Then $A^\otimes r$ has index $i$.

(iii) Let $e$ be $(r, i)$. Then $A^\otimes r$ has index dividing $i/e$.

Notice that for Brauer–Severi varieties $X$ over $k$ one has $H^0(X, O_X) = k$ and $\text{Pic}(X) \simeq \mathbb{Z}$. Furthermore, since $X \otimes_k \bar{k} \simeq \mathbb{P}^n$ one also has $\text{Pic}(X \otimes_k \bar{k}) \simeq \mathbb{Z}$. To apply Theorem 5.1 we have to investigate if for the generator $O_{\mathbb{P}^n}(1) \in \text{Pic}(X \otimes_k \bar{k})$ there exists an indecomposable absolutely isotypical sheaf on $X$ of type $O_{\mathbb{P}^n}(1)$. To show that this is indeed the case is the content of the next proposition.

**Proposition 6.2.** Let $X$ be a $n$-dimensional Brauer–Severi variety over $k$ and $O_{\mathbb{P}^n}(1)$ the generator of $\text{Pic}(X \otimes_k \bar{k}) \simeq \mathbb{Z}$. Then there exists an indecomposable absolutely isotypical sheaf $W_1$ on $X$ of type $O_{\mathbb{P}^n}(1)$.

**Proof.** We have seen that the sheaf $\mathcal{V}$ in the exact sequence (2) for a $n$-dimensional Brauer–Severi variety $X$ has the property $\pi^*\mathcal{V} \simeq O_{\mathbb{P}^n}(-1)^{\oplus (n+1)}$, where $\pi$ is the projection $\pi : X \otimes_k \bar{k} \rightarrow X$. Hence $\pi^*\mathcal{V}'$ is isomorphic to $O_{\mathbb{P}^n}(1)^{\oplus (n+1)}$ and therefore $\mathcal{V}'$ is absolutely rank-one-isotypical of type $O_{\mathbb{P}^n}(1)$. We want to denote the sheaf $\mathcal{V}'$ by $W$. We decompose $W$ according to the Krull–Schmidt Theorem and write $W \simeq \bigoplus_{i=1}^m W_i$. Since all the $W_i$ have to be isomorphic (see Remark 3.6) we therefore have $W \simeq W_1^{\oplus m}$, where $W_1$ is unique up to isomorphism. Note that by the Krull–Schmidt Theorem $W_1$ is also absolutely isotypical of type $O_{\mathbb{P}^n}(1)$. This completes the proof. □

If the Brauer–Severi variety $X$ has period $p$ we saw that this period can be thought of as the smallest positive integer $r$ such that $O_X(r)$ exists in $\text{Pic}(X)$. Hence the period defined for Brauer–Severi varieties equals the period defined for arbitrary proper $k$-schemes in
Section 5. If we denote by $W_j$ the sheaves $M_L \otimes j$ for $L = \mathcal{O}_P(1)$ of Section 5, Theorem 5.1 applies and we obtain:

**Corollary 6.3.** Let $X$ be a Brauer–Severi variety over a field $k$ of period $p$. Then all $AS$-bundles $E$ are of the form

$$E \simeq \bigoplus_{j=0}^{p-1} \left( \bigoplus_{i=0}^{r_j} \mathcal{O}_X(a_{ij}, p) \otimes W_j \right)$$

with unique $a_{ij} \in \mathbb{Z}$ and $r_j > 0$.

As mentioned in Section 5, for a complete understanding of the $AS$-bundles on Brauer–Severi varieties one has to determine the ranks of the $W_j$. This leads us to consider the sequence of natural numbers $(d_j)_{j \in \mathbb{Z}}$ with $d_j = \text{rk}(W_j)$. Proposition 5.3 and Remark 5.4 shows that we do not have to consider the hole sequence $(d_j)_{j \in \mathbb{Z}}$. Furthermore, we note that $W_0 = \mathcal{O}_X$ and $W_p = \mathcal{O}_X(p)$, where $p$ is the period of $X$. This implies that $\text{rk}(W_0) = 1 = \text{rk}(W_p)$. Considering the locally free sheaf $W$ of the proof of Proposition 6.2, it is a matter of fact that $\text{End}(W) \simeq A$, where $A$ is the central simple algebra corresponding to $X$ (see [31], p.144 or [35], §3, 3.6). Applying Proposition 5.2 now yields:

**Corollary 6.4.** Let $X$ be a $n$-dimensional Brauer–Severi variety over a field $k$ corresponding to a central simple $k$-algebra $A$. Then for every $j \in \mathbb{Z}$ one has

$$\text{rk}(W_j) = \text{ind}(A^\otimes j).$$

Summarizing we obtain:

**Theorem 6.5.** Let $X$ be a Brauer–Severi variety over $k$ and $A$ the corresponding central simple $k$-algebra of period $p$. Then the $AS$-type of $X$ is given by $d_j = \text{ind}(A^\otimes j)$ for $0 \leq j \leq p$.

Corollary 6.4 together with Theorem 6.5 now give a complete classification of $AS$-bundles on Brauer–Severi varieties and thus the results of Biswas and Nagaraj [7], [8], [9] and the author [26], [27] as a corollary.

**Corollary 6.6.** ([27], Theorem 5.1 and [8], Theorem 1.1) Let $X$ be a $n$-dimensional Brauer–Severi variety over $k$ of index two. Then the $AS$-bundles are of the form

$$E \simeq \left( \bigoplus_{i=1}^{r} \mathcal{O}_X(2a_i) \right) \oplus \left( \bigoplus_{j=1}^{s} \mathcal{O}_X(2b_j) \otimes W_1 \right)$$

with unique $r, s, a_i$ and $b_j$ and $W_1 \otimes_k L \simeq \mathcal{O}_{P_n}(1) \otimes^2$, where $k \subset L$ is a degree two Galois extension that splits $X$.

**Proof.** Since the index of $X$ is two and the period divides the index (see [15], Proposition 4.5.13), we conclude that the period is also two. Note that the period cannot be one, since this would imply that $X$ is the projective space what contradicts the fact that the index of $X$ is two. Hence the $AS$-type of $X$ is $(1, 2, 1)$ according to Theorem 6.5. Now the assertion follows from Corollary 6.3 and [15], Corollary 4.5.9. □

Note that the results in [7], [9] and [26] concern the case of non-trivial one-dimensional Brauer–Severi varieties and therefore follow directly from Corollary 6.6 since non-split Brauer–Severi varieties of dimension one have index two.

In the case that the Brauer–Severi variety corresponds to a central simple $k$-algebra with period equals the index, the $AS$-type can be stated very explicitly. We first make a simple observation.

**Lemma 6.7.** Let $A$ be a central simple $k$-algebra of period $p$. Then one has $p/(p, r) = \text{per}(A^\otimes r)$. 

Proposition 6.8. Let $A$ be a central simple $k$-algebra with period $p$ and index $i$. Then for all $r \geq 1$ one has that $p/(p, r)$ divides $\text{ind}(A^\otimes r)$ and $\text{ind}(A^\otimes r)$ divides $i/(i, r)$. In particular one has $p/(p, r) \leq \text{ind}(A^\otimes r) \leq i/(i, r)$.

Proof. By Lemma 6.7 we have $p/(p, r) = \text{per}(A^\otimes r)$ and since the period always divides the index (see [15], Proposition 4.5.13) we have that $p/(p, r)$ divides $\text{ind}(A^\otimes r)$. The second inequality $\text{ind}(A^\otimes r) \leq i/(i, r)$ and the fact that $\text{ind}(A^\otimes r)$ divides $i/(i, r)$ is (iii) of Theorem 6.1. \hfill \Box

The last proposition enables us to calculate the AS-type of Brauer–Severi varieties in the case when the period equals the index.

Proposition 6.9. Let $X$ be a Brauer–Severi variety over $k$ corresponding to a central simple $k$-algebra $A$ such that the period $p$ equals the index $i$. Then the AS-type of $X$ is given by

$$\text{rk}(W_j) = p/(p, j)$$

for $0 \leq j \leq p$.

Proof. Since the period $p$ equals the index $i$, we conclude from Proposition 6.8 that $p/(p, j) = \text{ind}(A^\otimes j)$ for $0 \leq j \leq p$. The assertion now follows from Corollary 6.4. \hfill \Box

Remark 6.10. The problem for which fields $k$ the period equals the index is called period-index problem and is highly non-trivial. For further discussion on this problem we refer the reader to [6] and to the work of de Jong [12].

For instance if $k$ is local or global, every central division algebra over $k$ is cyclic and the period equals the index (see [32], Theorem 10.7). In this case Proposition 6.9 tells us exactly how the AS-type looks like. The next observation shows that considering central division algebras to calculate the AS-type is not a restriction.

Proposition 6.11. Let $X$ and $Y$ be Brauer–Severi varieties over $k$ that are Brauer equivalent. Then $X$ and $Y$ have the same AS-type.

Proof. Let $A$ be the central simple $k$-algebra corresponding to $X$ and $B$ the central simple $k$-algebra corresponding to $Y$. Since $A$ and $B$ are Brauer equivalent, there is a unique central division algebra $D$ such that $A \simeq M_n(D)$ and $B \simeq M_m(D)$ for suitable $n$ and $m$. Hence $\text{ind}(A^\otimes j) = \text{ind}(D^\otimes j) = \text{ind}(B^\otimes j)$ for all $j \in \mathbb{Z}$. Since Brauer equivalent Brauer–Severi varieties have the same period, Theorem 6.5 yields the assertion. \hfill \Box

Interestingly, it turns out that the AS-type is also a birational invariant for Brauer–Severi varieties.

Proposition 6.12. Let $X$ and $Y$ be two birational Brauer–Severi varieties. Then they have the same AS-type.

Proof. Let $A$ be the central simple $k$-algebra corresponding to $X$ and $B$ that corresponding to $Y$. Since $X$ and $Y$ are supposed to be birational, [15], Corollary 5.4.2 implies that $A$ and $B$ generate the same cyclic subgroup in $\text{Br}(k)$. Denoting by $i$ the index of $A$, Proposition 6.8 yields that $\text{ind}(A^\otimes r)$ divides $i/(i, r)$. Since $i/(i, r)$ divides $i$, we conclude that $\text{ind}(A^\otimes r)$ divides $i = \text{ind}(A)$. The same holds for $B$ and we have that $\text{ind}(B^\otimes r)$ divides $\text{ind}(B)$. In what follows we prove that $\text{ind}(A^\otimes r) = \text{ind}(B^\otimes r)$ for all $r$. Since $A$ and $B$ generate the same cyclic subgroup in $\text{Br}(k)$ we know that $A$ is Brauer equivalent to $B^\otimes l$ and $B$ to $A^\otimes m$ for suitable $l$ and $m$. Hence $\text{ind}(A^\otimes m) = \text{ind}(B)$ divides $\text{ind}(A)$ and $\text{ind}(B^\otimes l) = \text{ind}(A)$ divides $\text{ind}(B)$. Thus $\text{ind}(B)$ divides $\text{ind}(A)$ and vice versa and therefore they have to be equal. The same argument applied to $\text{ind}(A^\otimes r)$ and $\text{ind}(B^\otimes r)$ yields that $\text{ind}(A^\otimes r) = \text{ind}(B^\otimes r)$ divides $\text{ind}(A^\otimes r)$ and that $\text{ind}(B^\otimes r) = \text{ind}(A^\otimes r)$ divides $\text{ind}(B^\otimes r)$. This shows that $\text{ind}(A^\otimes r) = \text{ind}(B^\otimes r)$ for all $r$. As mentioned above,
A and B generate the same cyclic subgroup and therefore have the same period. Now Theorem 6.5 yields that X and Y have the same AS-type. \[\square\]

**Remark 6.13.** Note that the converse of Proposition 6.11 and 6.12 does not hold. Indeed, by [15], Theorem 1.4.2, two non-split one-dimensional Brauer–Severi varieties X and Y are birational if and only if they are isomorphic. But according to Theorem 6.5 and Corollary 6.6 every one-dimensional Brauer–Severi variety has the same AS-type and hence it is possible that Brauer–Severi varieties that are neither Brauer equivalent nor birational have same AS-type.

As a consequence of the Horrocks criterion on \(\mathbb{P}^n\) (see [28]) we can state a cohomological criterion telling us exactly when a locally free sheaf on a Brauer–Severi variety is an AS-bundle.

**Theorem 6.14.** (AS-criterion) Let X be a n-dimensional Brauer–Severi variety over k and period p. A locally free sheaf \(E\) of finite rank is an AS-bundle if and only if for \(0 < i < n\) one has

\[H^i(X, E \otimes O_X(ap) \otimes W_j) = 0\]

for every \(a \in \mathbb{Z}\) and every \(0 \leq j \leq p - 1\).

We end up this section mentioning that the duals of the indecomposable AS-bundles \(W_j\) with \(j \geq 0\) on a Brauer–Severi variety X generate the Grothendieck group \(K_0(\mathbb{X})\). This follows directly from [31], Section 8, Theorem 4.1 or [22], Theorem 3.1.

**7. AS-bundles on generalized Brauer–Severi varieties**

In this section we will see that nearly all results stated in the last section for Brauer–Severi varieties also hold for generalized Brauer–Severi varieties. Clearly, slight modifications have to be made.

Let \(A\) be a central simple \(k\)-algebra of degree \(n\) and \(1 \leq d \leq n\). Now consider the subset of \(\text{Grass}_k(d \cdot n, A)\) consisting of those subspaces of \(A\) that are left ideals \(L\) of dimension \(d \cdot n\). This subset of \(\text{Grass}_k(d \cdot n, A)\) can be given a structure of a projective scheme over \(k\), defined by the relations stating that the \(L\) are left ideals (see [10], p.100 for details). It is a closed subscheme of the Grassmannian \(\text{Grass}_k(d, n)\), called the generalized Brauer–Severi variety and is denoted by \(\text{BS}(d, A)\). One can show that for a field extension \(k \subset E\) one has \(\text{BS}(d, A \otimes_k E) \simeq \text{BS}(d, A) \otimes_k E\) (see [10]). Hence, as in the case for Brauer–Severi varieties, there is always a finite separable and thus a finite Galois extension \(k \subset K\) such that \(\text{BS}(d, A) \otimes_k K \simeq \text{Grass}_k(d, n)\). Clearly, the generalized Brauer–Severi variety becomes isomorphic to the Grassmannian after base change to the algebraic closure \(\bar{k}\). Note that for \(d = 1\) one obtains the usual Brauer–Severi variety as defined in Section 2. Unfortunately, there is no one-to-one correspondence between generalized Brauer–Severi varieties and central simple \(k\)-algebras (see [10], Theorem 1 and discussion before). Nonetheless, it is possible to describe all AS-bundles on these generalized Brauer–Severi varieties. For this, recall that on the Grassmannian \(X = \text{Grass}_k(d, n)\) there is the tautological exact sequence

\[0 \longrightarrow S \longrightarrow \mathcal{O}_X^{\oplus n} \longrightarrow \mathcal{Q} \longrightarrow 0\]

with the tautological sheaf \(S\), being a locally free sheaf of rank \(d\). On the generalized Brauer–Severi variety \(\text{BS}(d, A)\) one has also a tautological short exact sequence

\[0 \longrightarrow I \longrightarrow \mathcal{O}_{\text{BS}(d, A)}^{\oplus n^2} \longrightarrow \mathcal{R} \longrightarrow 0.\]

This short exact sequence has the property that after base change to some splitting field \(L\) of \(A\) one gets on \(\text{BS}(d, A) \otimes_k L \simeq \text{Grass}_L(d, n)\) (see [24], Section 4):

\[0 \longrightarrow S^{\oplus n} \longrightarrow \mathcal{O}_{\text{Grass}_L(d, n)}^{\oplus n^2} \longrightarrow \mathcal{Q}^{\oplus n} \longrightarrow 0.\]
The locally free sheaf \(I\) is called \textit{tautological sheaf} on \(Y = BS(d, A)\). Recall that for the Grassmannian \(Grass(d, n)\) the Picard group coincides with its first Chow group (see [14], Example 15.3.6) and by Proposition 14.6.6 in [14] we have \(\text{Pic}(Grass(d, n)) \simeq \mathbb{Z}\), where the generator can be taken to be \(L = \text{det}(S^n)\). Proposition 3.3 now implies \(\text{Pic}(BS(d, A)) \simeq \mathbb{Z}\).

In order to apply Theorem 5.1, we have to investigate if there is a up to isomorphism unique absolutely isotypical sheaf of type \(L\) on \(Y\). For this, let \(\Sigma^\lambda\) be the Schur functor associated with the partition \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)\), where \(0 \leq \lambda_i \leq n - d\) (see [1], [13] for details). Levine, Srinivas and Weyman [24], Section 4 proved that for the tautological sheaf \(S\) on \(Grass(d, n)\) the locally free sheaves \(\Sigma^\lambda(S)^{\otimes n}_{\lambda}\) descent to locally free sheaves \(N_{\lambda}\) on \(BS(d, A)\). Note that these sheaves are unique up to isomorphism by Proposition 3.3. In particular for the generator \(L\) of \(\text{Pic}(Grass(d, n))\) there is a up to isomorphism unique indecomposable locally free sheaf \(W_L\) such that \(W_L \otimes_k k^{sep} \simeq \mathcal{L}^{\oplus \text{rk}(W_L)}\). As explained in Section 5, for all \(\mathcal{L}^{\otimes j} \in \text{Pic}(Grass(d, n))\) there exist up to isomorphism unique indecomposable locally free sheaves \(W_{\mathcal{L}^{\otimes j}}\) with \(W_{\mathcal{L}^{\otimes j}} \otimes_k k^{sep} \simeq (\mathcal{L}^{\otimes j})^{\oplus \text{rk}(W_{\mathcal{L}^{\otimes j}})}\). Let us denote by \(M\) the generator of \(\text{Pic}(BS(d, A))\) with \(M \otimes_k k^{sep} \simeq \mathcal{L}^{\otimes r}\), where \(r > 0\) is the period of \(BS(d, A)\). Applying Theorem 5.1 yields:

**Theorem 7.1.** Let \(X = BS(d, A)\) be a generalized Brauer–Severi variety of period \(r\) for the central simple \(k\)-algebra \(A\) and \(M\) the generator of \(\text{Pic}(BS(d, A))\). Then all indecomposable AS-bundles of finite rank are of the form:

\[
M^{\otimes a} \otimes W_{\mathcal{L}^{\otimes j}}
\]

with unique \(a \in \mathbb{Z}\) and unique \(0 \leq j \leq r - 1\).

An immediate consequence of Proposition 5.2 is the following:

**Proposition 7.2.** Let \(X = BS(d, A)\) be a generalized Brauer–Severi variety over \(k\) for the central simple \(k\)-algebra \(A\) of degree \(n\). Then for all \(j \in \mathbb{Z}\) one has:

\[
\text{rk}(W_{\mathcal{L}^{\otimes j}}) = \text{ind}(A^{\otimes j}d).
\]

**Proof.** Let \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)\) be a partition with \(0 \leq \lambda_i \leq n - d\) and \(S\) the tautological sheaf on \(BS(d, A) \otimes_k k^{sep} \simeq Grass(d, n)\). As mentioned above, in [24], Section 4 it is proved that the locally free sheaves \(\Sigma^\lambda(S)^{\otimes n}_{\lambda}\) descend to sheaves that we denote by \(N\). Then for the endomorphism algebra of \(N^r\) one has \(\text{End}(N^r) \simeq A^{\otimes d}\) (see also [24]). Since \(N^r\) is absolutely isotypical of type \(\text{det}(S^n)\), but in general not indecomposable, we conclude that \(A^{\otimes d}\) is a matrix algebra over \(\text{End}(W_L)\) (see Proposition 3.5). Proposition 5.2 gives \(\text{rk}(W_{\mathcal{L}^{\otimes j}}) = \text{ind}(\text{End}(W_{\mathcal{L}^{\otimes j}}))\). But the index of \(\text{End}(W_L)^{\otimes j}\) is the same as the index of \(A^{\otimes j}d\). This completes the proof. \(\square\)

**Remark 7.3.** Let \(G = \text{Gal}(k^{sep}/k)\) be the absolute Galois group and \(Y\) the generalized Brauer–Severi variety \(BS(d, A)\). With a result of Blanchet [10], Theorem 7, the exact sequence

\[
0 \rightarrow \text{Pic}(Y) \rightarrow \text{Pic}(Y \otimes_k k^{sep}) \xrightarrow{\delta} \text{Br}(k) \xrightarrow{\text{res}} \text{Br}(F(Y))
\]

and the arguments of the proof of Theorem 5.4.1 of [15] one obtains that the period of \(Y\) (as defined in Section 5) is equal to the period of \(A^{\otimes d}\).

Now we can state versions of Proposition 6.11 and 6.12 for generalized Brauer–Severi varieties.

**Proposition 7.4.** Let \(D\) be a central division algebra over \(k\) of degree \(n\) and \(A = M_m(D)\). Then \(BS(d, D)\) and \(BS(d, A)\) have the same AS-type.

**Proof.** Since \(D\) and \(A\) are Brauer equivalent we conclude that \(\text{ind}(D^{\otimes i}) = \text{ind}(A^{\otimes i})\) for all \(i \in \mathbb{Z}\). Thus \(\text{ind}(D^{\otimes d-i}) = \text{ind}(A^{\otimes d-i})\) for all \(i \in \mathbb{Z}\). Furthermore, the period of \(D^{\otimes d}\) equals the period of \(A^{\otimes d}\). Applying Proposition 7.2 and Theorem 7.1 yields the assertion. \(\square\)
Proposition 7.5. Let $X = BS(d, A)$ and $Y = BS(d, B)$ be two birational generalized Brauer–Severi varieties over $k$, then they have the same $AS$-type.

Proof. As a consequence of [10], Theorem 7, the period of $A^d$ equals the period of $B^d$ and hence the period of $X$ equals the period of $Y$. Now the same arguments as in the proof of Proposition 6.12 show that $\text{ind}(A^d) = \text{ind}(B^d)$. From Theorem 7.1 and Proposition 7.2 we conclude that both have the same $AS$-type. \qed

Ottaviani [28], Theorem 2.1 proved a splitting criterion for locally free sheaves on $\text{Grass}(d, n)$, at least if $\text{char}(k) = 0$. As a consequence of this result we obtain a criterion for a locally free sheaf on $BS(d, A)$ to be an $AS$-bundle. We first cite Ottaviani’s result.

Theorem 7.6. Let $k$ be a field of characteristic zero and $n \geq 3$. Then a locally free sheaf $E$ of finite rank on $X = \text{Grass}(d, n)$ splits as a direct sum of invertible sheaves if and only if for $0 < r < \dim(X)$ and all $t \in \mathbb{Z}$ one has

$$H^r(X, \bigwedge^{i_1} (Q^\nu) \otimes \cdots \otimes \bigwedge^{i_s} (Q^\nu) \otimes E(t)) = 0$$

for all $i_1, ..., i_s$ such that $0 \leq i_1, ..., i_s \leq n - d$ and $s \leq d$.

Now denote by $\mathcal{M}$ the generator of $\text{Pic}(BS(d, A))$. Considering the above $i_1, ..., i_s$, we note that for a fixed $s$-tuple $i_1, ..., i_s$ the $i_s$ can be ordered in a weakly decreasing way. We denote the reordering by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s$. So in this way we get a partition $\lambda = (\lambda_1, ..., \lambda_s)$ and we can associate a Young diagram to it with at most $d$ rows and $n - d$ columns. Now let $\mu'$ be the conjugate of the partition $\mu = (\lambda_i)$ (we have exactly $\lambda_i$ boxes).

Then we have $\Sigma^\mu(Q^\nu) = \bigwedge^{\lambda_i}(Q^\nu)$ on $BS(d, A) \otimes_k \mathbb{k} \simeq \text{Grass}(d, n)$. In [24], Section 4 it is shown that $(\Sigma^\mu(Q^\nu))^{\oplus n(\mu')} \rightarrow \mathcal{P}_{\lambda_1}$ descends to a locally free sheaf $\mathcal{P}_{\lambda_1}$ on $BS(d, A)$. By Proposition 3.3 we conclude that these locally free sheaves are unique up to isomorphism. With the generator $\mathcal{M} \in \text{Pic}(BS(d, A))$ we simply write $\mathcal{F}(m)$ for $\mathcal{F} \otimes \mathcal{M}^{\oplus m}$. With this notation we have the following result:

Theorem 7.7. (AS-criterion) Let $BS(d, A)$ be the generalized Brauer–Severi variety of period $r$ for the central simple $k$-algebra $A$ of degree $n \geq 3$ and $\mathcal{P}_{\lambda_1}$ the locally free sheaves from above. A locally free sheaf $E$ of finite rank is an $AS$-bundle if and only if for $0 < r < \dim(BS(d, A))$ and all $t \in \mathbb{Z}$ one has

$$H^r(BS(d, A), \mathcal{P}_{\lambda_1} \otimes \cdots \otimes \mathcal{P}_{\lambda_s} \otimes \mathcal{E}(t)) = 0$$

for all $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s$ such that $0 \leq \lambda_1, ..., \lambda_s \leq n - d$ and $s \leq d$.

Finally, we note that Levine, Srinivas and Weyman [24] investigated the $K$-theory of generalized Brauer–Severi varieties and showed that the Grothendieck group is generated by $\mathcal{N}_{\lambda}$, where $\mathcal{N}_{\lambda}$ are the locally free sheaves obtained by descent from $\Sigma^\lambda(S)^{\oplus n(\lambda)}$. Therefore, as distinguished from the case of Brauer–Severi varieties, the $AS$-bundles on generalized Brauer–Severi varieties do not generate the Grothendieck group of $BS(d, A)$ for $d > 1$.

References

[1] K. Akin, D.A. Buchsbaum and J. Weyman: Schur Functors and Schur Complexes. Adv. in Math. Vol. 44 (1982), 207-278.
[2] F.W. Anderson and K.R. Fuller: Rings and Categories of modules. Springer-Verlag, New York (1974).
[3] J.K. Arason, R. Elman and B. Jacob: On indecomposable vector bundles. Comm. in Alg. Vol. 20 (1992), 1323-1351.
[4] M. Artin: Brauer-Severi varieties. Brauer groups in ring theory and algebraic geometry, Lecture Notes in Math. 917, Notes by A. Verschoren, Berlin, New York: Springer-Verlag (1982), 194-110.
[5] M. Atiyah: On the Krull-Schmidt Theorem with application to sheaves. Bull. Soc. Math. France Vol. 84 (1956), 307-317.
[6] A. Auel, E. Brussel, S. Garibaldi and U. Vishne: Open Problems on Central Simple Algebras. arXiv:1006.3304 [math.RA] (2010).
[7] I. Biswas and D. Nagaraj: Classification of real algebraic vector bundles over the real anisotropic conic. Int. J. Math. Vol. 16 (2005), 1207-1220.
[8] I. Biswas and D. Nagaraj: Absolutely split real algebraic vector bundles over a real form of projective space. Bull. Sci. Math. Vol. 131 (2007), 686-696.
[9] I. Biswas and D. Nagaraj: Vector bundles over a nondegenerate conic. J. Aust. Math. Soc. Vol. 86 (2009), 145-154.
[10] A. Blanchet: Function Fields of Generalized Brauer–Severi Varieties. Communications in Algebra. Vol. 19 (1991), 97-118.
[11] N. Bourbaki: Algèbre, Chapitre X, Masson. (1980).
[12] A.D. de Jong: The period-index problem for the Brauer group of an algebraic surface. Duke Math. J. 123 No. 1 (2004), 71-94.
[13] W. Fulton: Young Tableaux, with Applications to Representation Theory and Geometry. Cambridge University Press (1997).
[14] W. Fulton: Intersection Theory. Ergebnisse der Mathematik und ihre Grenzgebiete, Springer, second edition (1998).
[15] P. Gille and T. Szamuely: Central Simple Algebras and Galois Cohomology. Cambridge Studies in advanced Mathematics. 101. Cambridge University Press. (2006)
[16] A. Grothendieck: Sur la classification des fibré holomorphes sur la sphére de Riemann. Amer. J. Math. Vol. 79 (1957), 121-138.
[17] A. Grothendieck: Le group de Brauer I: Algebras d’Azumaya et interpretations diverses, Seminaire Bourbaki. No. 290 (1964).
[18] A. Grothendieck: Le group de Brauer II: Theorie cohomologique, Seminaire Bourbaki. No. 297 (1965).
[19] A. Grothendieck et J. Dieudonne: Elements de Géométrie Algébrique. Publ. Math. IHES. (1960-68).
[20] A. Grothendieck: Fondements de la Géométrie Algébrique [Extraits du Séminaire Bourbaki 1957-1962], Secrétariat mathématique (1962).
[21] R. Hartshorne: Algebraic Geometry. Springer-Verlag, New York, Berlin, Heidelberg (1977).
[22] N. Karpenko: Codimension 2 Cycles on Severi–Brauer Varieties. K-Theory 13 No. 4 (1998), 305-330.
[23] S. L. Kleiman: The Picard scheme: arXiv:math/0504020 [math.AG] (2005).
[24] M. Levine, V. Srinivas and J. Weyman: K-Theory of Twisted Grassmannians. K-Theory Vol. 3 (1989), 99-121.
[25] J. Milne: Étale Cohomology. Princeton Mathematical Series. Vol. 33, Princeton University Press (1980).
[26] S. Novaković: Der Spaltungssatz für Quadriken. Diploma Thesis, Heinrich-Heine Universität Düsseldorf (2009).
[27] S. Novaković: Absolutely split locally free sheaves on Brauer-Severi varieties of index two. Bull. d. Sci. Math. Vol. 136 (2012), 413-422.
[28] C. Okonek, M. Schneider and H. Spindler: Vector Bundles on Complex Projective Space. Progress in Mathematics. 3. Birkhäuser, Boston (1980).
[29] G. Ottaviani: Some extensions of the Horrocks criterion to vector bundles on grassmannians and quadrics. Annali di Matem. Vol. 155 (1989), 317-341.
[30] S. Pumpkin: Vector bundles and symmetric bilinear forms over curves of genus 1 and arbitrary index. Mathematische Zeitschrift Vol. 246 (2004), 563-602
[31] D. Quillen: Higher algebraic K-theory. Algebraic K-theory I, Lecture Notes in Math. 341, Springer (1979), 85-147.
[32] D. Saltman: Lectures on division algebras. Vol. 94 of CBMS Regional conference Series in Mathematics. American Math. Soc. (1999).
[33] J.-P. Serre: Local Fields. Springer-Verlag, New York, Berlin (1980).
[34] J.-P. Serre: Cohomologie Galoisienne. Lecture Notes in Mathematics 5, Springer-Verlag Berlin (1994); English translation: Galois Cohomology, Springer-Verlag Berlin (2002).
[35] V.E. Voskresenskii: Algebraic Groups and Their Birational Invariants. Translated by Boris Kunyavski, Translations of Math. Monographs AMS Vol. 179 (2000).
[36] R. Wiegand: Torsion in Picard Groups of Affine Rings. Contemp. Math. Vol 159 (1994), 433-444.

MATHEMATISCHES INSTITUT, HEINRICH-HEINE-UNIVERSITÄT 40225 DÜSSELDORF, GERMANY
E-mail adress: novakovic@math.uni-duesseldorf.de