Landau meets Newton: time translation symmetry breaking in classical mechanics

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Abstract

Every classical Newtonian mechanical system can be equipped with a nonstandard Hamiltonian structure, in which the Hamiltonian is the square of the canonical Hamiltonian up to a constant shift, and the Poisson bracket is nonlinear. In such a formalism, time translation symmetry can be spontaneously broken, provided the potential function becomes negative. A nice analogy between time translation symmetry breaking and the Landau theory of second order phase transitions is established, together with several example cases illustrating time translation breaking ground states. In particular, the ΛCDM model of FRW cosmology is reformulated as the time translation symmetry breaking ground states.

Keywords: time translation symmetry breaking, Hamiltonian formalism, Landau theory of phase transition, ΛCDM model

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1 Introduction

Spontaneous symmetry breaking plays an essential role in many areas in modern theoretical physics such as the gauge theory of particle physics and the Landau-Ginzberg theory of phase transitions in condensed matter physics. Spontaneously broken symmetries can either be internal symmetry (such as gauge and chiral symmetries) or spacial symmetry (such as space translation and rotation symmetries). Shapere and Wilczek recently discovered [1] that even the time translation symmetry can be spontaneously broken in some singular Lagrangian systems. In a previous work [2], we showed that the spontaneous breaking of time translation symmetry can also be described in Hamiltonian formalism using some non-
Darboux coordinates on the phase space. We also revealed that the breaking of
time translation is always accompanied by the breaking of time reversal.

In this work, we continue the analysis on the spontaneous breaking of time
translation symmetry. We show that the phenomenon of spontaneous breaking of
time translation symmetry can be present in almost every Newtonian mechanical
system with a conservative potential, provided the potential function is shifted
enough so that its value can go negative. The symmetry breaking occurs only in
a nonstandard Hamiltonian description in which the Hamiltonian is the square of
the canonical Hamiltonian and the Poisson bracket is nonlinear. Under proper
conditions, time translation symmetry breaking can happen dynamically. This
means that for some given set of initial values, time translation can be initially
unbroken, but in the process of time, the system can evolve into symmetry breaking
phase. The mechanism of time translation symmetry breaking is quite reminiscent
to the classic Landau theory of second order phase transitions, we just need to
replace the free energy in Landau theory by the nonstandard, squared Hamiltonian,
meanwhile take the signature of the potential function as order parameter (and
time as an analogue of temperature). We shall make this analogy more concretely
in the main context. As an example, we also reformulate the ΛCDM model of
cosmology as the time symmetry breaking ground states of a mechanical system
with an upside-down harmonic potential.

2 Landau theory of second order phase transi-
tion – a mini review

Landau theory of second order phase transition consists in the analysis of stable
minima of the free energy as a function of some order parameter (e.g. total mag-
netization $\mathcal{M}$) and the temperature $T$. In the simplest version, we take the free
energy $F(T, \mathcal{M})$ to be of the following form,

$$F(T, \mathcal{M}) = F_0(T) + b(T)\mathcal{M}^2 + a\mathcal{M}^4,$$

where $a > 0$ is a constant, $b(T) = b_0(T - T_c)$ with $b_0 > 0$ being a constant, and
$F_0(T)$ is the free energy at zero magnetization.

Stable minima of the free energy must satisfy the following conditions:

$$\frac{\partial F(T, \mathcal{M})}{\partial \mathcal{M}} = 0, \quad \frac{\partial^2 F(T, \mathcal{M})}{\partial \mathcal{M}^2} > 0.$$

The first condition gives

$$2\mathcal{M}(b(T) + 2a\mathcal{M}^2) = 0,$$

from which we can get 3 distinct extrama,

$$\mathcal{M} = 0, \pm \sqrt{-\frac{b(T)}{2a}}.$$
This, combined with the second condition, gives rise to the stable minima of the free energy at

\[ \mathcal{M} = \begin{cases} 0, & (T \geq T_c), \\ \pm \sqrt{\frac{b_0(T_c - T)}{2a}}, & (T < T_c). \end{cases} \]  

At \( T \geq T_c \), the total magnetization remains zero, indicating that the system is isotropic and hence rotationally symmetric. However, as \( T \) drops continuously and reaches the regime \( T < T_c \), the total magnetization suddenly becomes nonzero. This spontaneous magnetization signifies the breaking of rotational symmetry and hence the phase transition from the paramagnetism phase to the ferromagnetism phase.

### 3 Two different Hamiltonian descriptions of Newtonian mechanics

Most physicists begin their career by getting acquainted with the Newtonian equation of motion for classical mechanics. For a unit mass particle moving in a one dimensional conservative potential \( U(x) \), this reads

\[ \ddot{x} = -\frac{dU}{dx}. \]  

In the standard canonical Hamiltonian formulation of classical mechanics, this equation is obtained as the Hamiltonian flow equation arising from the canonical Hamiltonian function

\[ H_1 = \frac{1}{2}v^2 + U(x) \]  

defined on the two dimensional phase space spanned by \((x, v)\) equipped with the canonical Poisson bracket

\[ \{x, v\}_1 = 1. \]  

The flow equation actually comprises of two equations, i.e.

\[ \dot{x} = v, \quad \dot{v} = -\frac{dU}{dx}. \]  

All these are quite familiar material from every text book of classical mechanics. What may not be so familiar comes as follows. The very same Newtonian equation of motion (2), or more precisely, the canonical Hamiltonian equations of motion (5), can also arise as the Hamiltonian equations of motion with a completely different Hamiltonian structure. The unfamiliar Hamiltonian and Poisson bracket are given
by

\[ H_2 = \left( \frac{1}{2} v^2 + U(x) \right)^2 + E_0, \]  
(6)

\[ \{x, v\}_2 = \frac{1}{v^2 + 2U(x)}, \]  
(7)

where \( E_0 \) is a constant, which may be chosen arbitrarily. This system is actually the special case \( f = 1/12, g = U \) of the \( fgh \) model discussed by Shapere and Wilczek in [1], but reformulated in the Hamiltonian description given in [2]. The corresponding Lagrangian is

\[ L = \frac{1}{12} \dot{x}^4 + U \dot{x}^2 - U^2. \]

In the following, we shall be sticking exclusively to the Hamiltonian description.

Now, the same set of equations of motion (5) possesses two different Hamiltonian descriptions, \((H_1, \{\cdot, \cdot\}_1)\) and \((H_2, \{\cdot, \cdot\}_2)\). Since \( H_2 = (H_1)^2 + E_0 \), we see that \( H_1 \) and \( H_2 \) are in involution under each choices of Poisson structure. It is easy to check that an arbitrary linear combination of the above two Poisson structures with constant coefficients still makes a Poisson structure. This seems to indicate that every Newtonian mechanical system with a conservative force is a bi-Hamiltonian system. However things are not that simple. The two different Hamiltonian descriptions can have very different implications, as will be elucidated in the next section.

4 Landau meets Newton, or “phase transition” in mechanical systems

Consider the conditions for the Hamiltonians to have a local minimum (i.e. a local lowest energy state, or local ground state, which we use interchangeably below). By the word local, we mean that such minimum may not be the absolute minimum of the energy surface on the whole phase space. In particular, the energy surface may not be bounded from below even though the local minimum exists.

For the first Hamiltonian \( H_1 \), the conditions for the Hamiltonian to have a local minimum read

\[ v^2 = 0, \quad \frac{dU}{dx} = 0, \quad \frac{d^2U}{dx^2} > 0, \]  
(8)

i.e. at the local ground state, the particle must be at rest and the potential must take its minimum. If \( U(x) \) doesn’t have a minimum, then the Hamiltonian does not have a local ground state, and is unbounded from below. Whenever the Hamiltonian does have a local ground state, the state is kept by time translation symmetry (since both \( x \) and \( v \) are fixed in time in such a state). Note that
any such state must be isolated, though it may not be unique (except the most trivial case with a constant potential).

Now consider the second Hamiltonian description given by eqs. (6) and (7). The conditions for the energy to have a local minimum are

\[ v^2 = 0, \quad \frac{dU}{dx} = 0, \quad \frac{d^2U}{dx^2} > 0, \quad \text{if } U(x) \geq 0, \quad (9) \]

\[ v^2 = -2U(x) \quad \text{if } U(x) < 0. \quad (10) \]

In the former case, the local minimum of the energy is in general higher than \( E_0 \) (since \( U(x) \) needs not to be zero at its minimum), and time translation is unbroken. Such local ground states are similar in spirits to the case which we encountered in the first Hamiltonian description. In the latter case, the minimum energy is equal to \( E_0 \), the force \(-\frac{dU}{dx}\) is not necessarily zero, and time translation is broken because of the nontrivial motion in the ground state. This kind of ground states are not isolated (they actually form two smooth curves on the phase space), and they never appear in the first Hamiltonian description. Therefore, we observe significant differences between the two descriptions:

- Though we use the term “potential function” for \( U(x) \) in both Hamiltonian descriptions, the actual roles of \( U(x) \) in the two descriptions are quite different. In particular, \( U(x) \) in the first Hamiltonian has the well known interpretation as potential energy, while \( U(x) \) in the second description does not have such a simple interpretation. It does not even have the same dimensionality with the energy;

- A constant shift in the potential is physically insignificant in the first description, while it becomes significant in the second description;

- Given the same potential function \( U(x) \), the energy is not necessarily bounded from below in the first description, while it is always bounded from below in the second description. Though for classical mechanical systems, the existence of a lower energy bound looks insignificant, it immediately becomes significant when passing to quantum description is under consideration;

- When the energy possesses local minima, time translation is always preserved in the local ground states in the first description, while it can be broken in the second description as long as the potential becomes negative.

One can compare the breaking of time translation in the second description to the case of Landau theory of second order phase transitions reviewed previously. Both theories involve symmetry breaking. In Landau theory, the order parameter is the spontaneous magnetization \( M \), while in our case, the velocity \( v \) plays a similar role. Moreover, in Landau theory, the order parameter is intrinsically controlled by the temperature, while in the present case, the value of \( v \) at the ground state depends intrinsically on the value of \( U(x) \), and \( U(x) \) is controlled by \( x \) and hence indirectly by \( t \).
It is necessary to have some intuitive feelings about the time translation symmetry breaking mentioned above. For this purpose let us consider some simple example cases.

The first example case is the constant potential case $U(x) = -\frac{1}{2}u^2$, $(u > 0)$. In the symmetry breaking lowest energy state, we have $v = \pm u$. Of course, the particle can pick only one of the two values for the velocity at one time. The explicit choice would then break the time reversal.

The second example case is given by a linear potential, $U(x) = -Fx$. If the particle was initially sitting at $x = x_0 \leq 0$, it will be driven by the constant force $F$ towards the positive direction along the $x$ axis until it crosses the origin. In particular, if the particle were released from at rest at the origin, then its motion will be confined in the lowest energy states governed by the equation $v^2 = 2Fx$.

In the third example case we consider the quadratic potential. We can actually have two different choices of quadratic potentials: the usual harmonic oscillator potential with a constant shift, $U(x) = \frac{1}{2}kx^2 - U_0$, and the upside-down harmonic potential, $U(x) = U_0 - \frac{1}{2}kx^2$. For both choices the symmetry breaking lowest energy states are well defined and they all corresponds to nontrivial motion. The upside-down harmonic oscillator has appeared in the studies of matrix description of de Sitter gravity [3] and the Hamiltonian formalism of FRW cosmology [4]. We will come back to this model in Section 5.

Of course almost everyone is familiar with the above potential within the framework of the first Hamiltonian description. There, the second example and the model with the upside-down harmonic potential in the third example do not have a lower energy bound. The first example has a flat potential, and though the lowest energy states do exist, they do not subject to nontrivial motion. The model with the harmonic potential in the third example also possesses physically well behaved lowest energy state, which also does not subject to nontrivial motion.

Now back to the second Hamiltonian description. We can choose arbitrary, more complex potentials and see whether the time translation symmetry breaking can occur in a dynamical process. Let us consider, for instance, the potential function

$$U(x) = -x^3 + 5x^2 + 7,$$

which is chosen at random. Figure 1 gives the plot of the energy surface in the two dimensional phase space spanned by $(x, v)$ (where we have chosen $E_0 = 0$). The energy surface has an isolated local minimum (whose value is above zero) at $(x, v) = (0, 0)$, which is the time translation preserving ground state of the system, as well as a whole curve of absolute minima (whose value is exactly zero) to the right of the plot, which correspond to the time translation breaking ground states.

Now if the particle were initially sitting at the isolated, symmetry preserving, local ground state, can it evolve into the symmetry breaking ground states in the course of time? This looks impossible because the particle does not have enough energy to climb up the energy barrier between the isolated local ground state and the curve of the absolute minima of the energy surface. However, there are at
least two possible mechanisms which can make the particle to transit from the symmetry preserving ground state to the symmetry breaking ground states. The first possibility is to allow the particle to be not alone in the system. It can be part of a larger system and hence can be perturbed by other degrees of freedom in the larger system so that it gets enough energy to climb the energy barrier and then loses energy by other perturbations and stay in the symmetry breaking ground states henceforth. Another possibility is to consider quantum effect. If the particle’s motion is controlled by quantum principles, then it has some possibility of penetrating the energy barrier by quantum tunneling process. In either way the breaking of time translation symmetry can happen in a dynamical process. It is of particular interests if the particle can carry some kind of charge, because then the transition from the isolated local ground state to the continuous symmetry breaking ground states would correspond to a transition from insulating phase to a superconducting phase for the appropriate charge. In this sense, the analogy to the Landau theory of second order phase transitions may not seem to be an accident.

5 An application: ΛCDM model of the universe as symmetry breaking ground states

Perhaps the most important physical system in which time translation symmetry is explicitly broken is the Friedmann equations governing the evolution of our universe. In modern treatments, the Friedmann equations arise as components of the Einstein equation of general relativity with the insertion of the FRW metric. However, in some approximate, or effective description, they are also known to arise from Newtonian mechanics [5], as reviewed, e.g. in [6]. Below we shall try to incorporate Friedmann equations into the framework of the second Hamiltonian
description of Newtonian mechanics. In particular, the first order equation in the Friedmann equations will appear as the condition determining the time translation symmetry breaking ground states.

Recall that the Friedmann equations actually consist of two equations, i.e.

\[
\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} - \frac{\Lambda}{3} = \frac{8\pi G \rho}{3}, \tag{11}
\]

\[
\frac{\ddot{a}}{a} - \frac{\Lambda}{3} = -\frac{4\pi G}{3} (\rho + 3p), \tag{12}
\]

where \(a(t)\) is the FRW scale factor, \(G, k, \Lambda, \rho, p\) are respectively the Newtonian constant for gravitation, spacial curvature, cosmological constant, density and pressure of the ideal fluid source of the universe. The presence of \(\rho\) and \(p\) makes it difficult to think of the Friedmann equations as the equations of motion of a purely mechanical system. So, we will consider only the sourceless Friedmann equations, i.e. the \(\Lambda\)CDM model.

The incorporation can be made quite easily. We just need to rearrange the equations (11) and (12) with \(\rho = p = 0\) into the following form,

\[
\dot{a}^2 = -k + \frac{\Lambda a^2}{3}, \tag{13}
\]

\[
\ddot{a} = \frac{\Lambda a^3}{3}. \tag{14}
\]

Then, by identifying \(a(t) = x(t), v(t) = \dot{a}(t)\) and making comparison between (13) and the symmetry breaking ground state condition (10), we immediately realize that the above system is the very same model with upside-down harmonic potential in the third example of Section 4 provided \(\Lambda > 0\), which is required for a de Sitter universe. Explicitly, we now have

\[
U(a) = k - \frac{\Lambda a^2}{6},
\]

\[
H_2 = \left(\frac{\dot{a}^2}{2} + k - \frac{\Lambda a^2}{6}\right)^2,
\]

and (14) is just the Newtonian equation of motion that follows from this model.

Let us proceed to see what we can learn by reinterpreting the equations (13) and (14) respectively as the symmetry breaking ground state condition and the Hamiltonian equation of motion associated with our second Hamiltonian description of Newtonian mechanics.

The first lesson we learn from above is that although the universe could undergo perpetual time evolution (since \(\dot{a} \neq 0\)), it remains in the lowest energy state \(H_2 = 0\). The second lesson is that the spacial curvature, \(k\), has to be non-positive (and preferably be zero) if the universe began its evolution from a very small size which is close to zero. Otherwise the ground state condition, (13), will not have a solution, and our formulation of the \(\Lambda\)CDM model as the symmetry breaking
ground state collapses. The use of the second Hamiltonian description feels more natural than using the first Hamiltonian description (as was previous did in [4]), because only the zero energy condition seems to be consistent with the general relativistic Hamiltonian constraints and with the formalism of Wheeler-deWitt equation (though with a different Hamiltonian).

In closing this section, let us remark that the formalism we introduced here for describing ΛCDM model is only an effective description. The fundamental description should still be based on relativistic theory of gravity, preferably on some higher curvature generalizations of general relativity. It will be very interesting if our second Hamiltonian could arise from the Legendre transform of the reduced Lagrangian of some generalized gravity model after the insertion of FRW metric into the action.

6 Conclusions

Time translation symmetry breaking seems to happen only in systems with a higher order Lagrangian or Hamiltonian. In this work, we are particularly interested in a special kind of Hamiltonians which yield spontaneous breaking of time translation. The time evolution equations for such systems are identical to the Newtonian equation with a conservative potential. The Hamiltonian is the square (up to a constant shift) of the standard canonical Hamiltonian for Newtonian mechanical systems, which is equipped with a nonlinear Poisson structure.

For such systems, the spontaneous breaking of time translation is quite reminiscent to the mechanism of symmetry breaking appearing in Landau theory of second order phase transitions. We gave the detailed analogy in the main context. We also sketched some possible mechanisms which can make the breaking of time translation happen dynamically. Besides all these, we gave several example cases indicating the explicit breaking of time translation. In particular, the ΛCDM model of FRW cosmology is reformulated as time translation breaking ground states for a system with upside-down harmonic potential.

The works reported in [2] and in the present paper are still very preliminary. There are still a lot of open issues which need further studies. Among these, we would like to point out a few problems which we would like to study in subsequent works:

- Within the context of Hamiltonian description, the passage to the quantum description is a much needed whilst still missing piece of work. Although path integral quantization may be a possible choice [7], the analogue of canonical quantization is still needed in order to have a quantum mechanical description for the models under consideration;

- The unusual relationship between the potential function $U(x)$ and the Hamiltonian $H_2$ in the second Hamiltonian description needs further understand-
ing. It indicates that the usual, canonical way of introducing interactions is not the only possible choice;

- We have been exclusively considering the problem of time translation breaking within the scope of classical mechanics. It will of course also be interesting if similar mechanisms can also arise in field theoretic context.

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