Matrix Subadditivity Inequalities and Block-Matrices

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Abstract. We give a number of subadditivity results and conjectures for symmetric norms, matrices and block-matrices. Let $A$, $B$, $Z$ be matrices of same size and suppose that $A$, $B$ are normal and $Z$ is expansive, i.e., $Z^*Z \geq I$. We conjecture that
\[
\| f(|A + B|) \| \leq \| f(|A|) + f(|B|) \| \quad \text{and} \quad \| f(|Z^*AZ|) \| \leq \| Z^* f(|A|) Z \|
\]
for all non-negative concave function $f$ on $[0, \infty)$ and all symmetric norms $\| \cdot \|$ (in particular for all Schatten $p$-norms). This would extend known results for positive operators to all normal operators. We prove these inequalities in several cases and we propose some related open questions, both in the positive and normal cases. As nice applications of subadditivity results we get some unusual estimates for partitioned matrices. For instance, for all symmetric norms and $0 \leq p \leq 1$,
\[
\left\| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\|^p \leq \| |A|^p + |B|^p + |C|^p + |D|^p \|.
\]
wherever the partitioned matrix is Hermitian or its entries are normal. We conjecture that this estimate for $f(t) = t^p$ remains true for all non-negative concave functions $f$ on the positive half-line. Some results for general block-matrices are also given.

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Introduction

This paper complements three recent works on subadditivity type inequalities for concave functions of positive operators [7], [8] and [9]. These inequalities are matrix versions of obvious scalars inequalities like
\[
f(za) \leq zf(a) \quad \text{and} \quad f(a + b) \leq f(a) + f(b)
\]
for non-negative concave functions $f$ on $[0, \infty)$ and scalars $a, b \geq 0$ and $z \geq 1$. By matrix version we mean suitable extensions where scalars $a, b, \ldots z$ are replaced by $n$-by-$n$ complex matrices, i.e., operators on an $n$-dimensional Hilbert space $\mathcal{H}$, $A, B, \ldots Z$. For instance, if $A$ is positive (semi-definite) and $Z$ is expansive we know from [6] a remarkable trace inequality, companion to one of the above scalar inequality,
\[
\text{Tr} f(Z^*AZ) \leq \text{Tr} Z^* f(A) Z.
\]
This trace inequality may be generalized by using the class of symmetric (or unitarily invariant) norms. Such norms satisfy $\|A\| = \|UAV\|$ for all $A$ and all unitaries $U, V$.
operators to the set of all normal operators. We have only partial results and some open problems are considered. A basic one would be to know if (1) can be extended to normal operators $A$ as follows:

$$\operatorname{Tr} f(|Z^*AZ|) \leq \operatorname{Tr} Z^*f(|A|)Z.$$ (2)

The main part of this paper is Section 2 where we show how some subadditivity results entail several new estimates for block matrices. A special case involving four normal operators $A, B, C, D$ is

$$\left\| \begin{pmatrix} A & D \\ C & B \end{pmatrix} \right\|^p \leq \| |A|^p + |B|^p + |C|^p + |D|^p \|$$

for all symmetric norms and $0 \leq p \leq 1$. Such an inequality also holds in the important case of a Hermitian partitioned matrix. These estimates differ from the usual ones in the literature where the norm of the full matrix is evaluated with the norms of its blocks, for instance see [2], [12], [13] and [4]. Our results start the study of the general problem of comparing an operator on $\mathcal{H} \oplus \mathcal{H}$ with an operator on $\mathcal{H}$, more precisely, comparing the block-matrix expression

$$f \left( \left\| \begin{pmatrix} A & D \\ C & B \end{pmatrix} \right\| \right)$$

with the sum

$$f(|A|) + f(|B|) + f(|C|) + f(|D|).$$

Some natural conjectures, parallel to those ones of Section 1, are naturally proposed.

The proofs use standard two by two block-matrix technics and basic facts on majorisation, log-majorisation, convex functions and norms. This background can be found in text books like [3] and will be used without further reference.

1. Some recent and new subadditivity results

If $A$ is positive (semi-definite), resp. positive definite, we write $A \geq 0$, resp. $A > 0$. If $Z^*Z$ dominates the identity $I$, we say that $Z$ is expansive. The trace inequality (1) is a special case of the following theorem [7]:

**Theorem 1.1a.** Let $A \geq 0$ and let $Z$ be expansive. If $f : [0, \infty) \to [0, \infty)$ is concave, then, for all symmetric norms,

$$\| f(Z^*AZ) \| \leq \| Z^*f(A)Z \|.$$

This estimate for congruences have been completed in [9] with an estimate for sums:

**Theorem 1.1b.** Let $A, B \geq 0$ and let $f : [0, \infty) \to [0, \infty)$ be concave. Then, for all symmetric norms,

$$\| f(A + B) \| \leq \| f(A) + f(B) \|.$$
This result closed a long list of papers by several authors, including Ando-Zhan [1] and Kosem [14], on norm subadditivity inequalities, cf. [8], [9] and references therein. Theorems 1.1a and 1.1b are nicely compatible. There are two aspects of a unique statement [8]:

**Theorem 1.1.** Let \( \{A_i\}_{i=1}^m \) be positive, let \( \{Z_i\}_{i=1}^n \) be expansive and let \( f \) be a non-negative concave function on \([0, \infty)\). Then, for all symmetric norms,

\[
\|f \left( \sum Z_i^* A_i Z_i \right) \| \leq \left\| \sum Z_i^* f(A_i) Z_i \right\|.
\]

It would be elegant and interesting to state these theorems in the more general framework of positive linear maps \( \Phi \) between matrix algebras. Indeed the conclusions of these theorems look like statements for positive linear maps. For instance, with

\[
\Phi \left( \begin{pmatrix} S & X \\ Y & T \end{pmatrix} \right) = S + T \quad \text{and} \quad M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},
\]

Theorem 1.1b claims

\[
\|f \circ \Phi(M)\| \leq \|\Phi \circ f(M)\|.
\]

Therefore we ask two questions.

**Question 1.** Let \( \Phi : M_n \to M_k \) be a positive linear map between the algebras of matrices of size \( n \) and \( k \) and let \( \mathcal{F}_n \) and \( \mathcal{F}_k \) be the sets of non-expansive, positive operators in \( M_n \) and \( M_k \) respectively. Does

\[
\Phi^{-1}(\mathcal{F}_k) \subset \mathcal{F}_n
\]

imply

\[
\|f \circ \Phi(M)\| \leq \|\Phi \circ f(M)\|
\]

for all \( M \geq 0 \), all symmetric norms and all non-negative concave functions on the positive half line ?

**Question 2.** Let \( l : \Phi(M_n) \to M_n \) be a lifting for \( \Phi \), i.e., an injective map such that \( \Phi \circ l \) is the identity mapping of \( \Phi(M_n) \). Let \( \mathcal{E}_k \) be the set of expansive, positive operators in \( M_k \). Does

\[
\Phi \circ l(\mathcal{E}_k) \subset \mathcal{E}_k
\]

imply

\[
\|f \circ \Phi(M)\| \leq \|\Phi \circ f(M)\|
\]

for all \( M \geq 0 \), all symmetric norms and all non-negative concave functions on the positive half line ?
Now we turn to some consequences of Theorem 1.1. The usual operator norm is denoted by $\| \cdot \|_\infty$.

**Corollary 1.2.** Let $\{A_i\}_{i=1}^m$ be normal, let $\{Z_i\}_{i=1}^m$ be expansive and let $f$ be a non-negative concave function on $[0, \infty)$. Then,

$$\left\| f \left( \sum Z_i^* A_i Z_i \right) \right\|_\infty \leq \left\| \sum Z_i^* f(|A_i|) Z_i \right\|_\infty.$$ 

**Proof.** Note that

$$\begin{pmatrix} |A_i| & A_i \\ A_i^* & |A_i| \end{pmatrix} \geq 0,$$

hence

$$\begin{pmatrix} \sum Z_i^* |A_i| Z_i \\ \sum Z_i^* A_i^* Z_i \\ \sum Z_i^* A_i Z_i \\ \sum Z_i^* |A_i| Z_i \end{pmatrix} \geq 0,$$

so that

$$\left\| \sum Z_i^* A_i Z_i \right\|_\infty \leq \left\| \sum Z_i^* |A_i| Z_i \right\|_\infty.$$

Since $f$ is non-decreasing and non-negative we have

$$\left\| f \left( \sum Z_i^* A_i Z_i \right) \right\|_\infty = f \left( \left\| \sum Z_i^* A_i Z_i \right\|_\infty \right) \leq f \left( \left\| \sum Z_i^* |A_i| Z_i \right\|_\infty \right) = \left\| f \left( \sum Z_i^* |A_i| Z_i \right) \right\|_\infty.$$

We then apply Theorem 1.1 to get the conclusion. \hfill \square

We may propose:

**Conjecture 1.** Corollary 1.2 holds for all symmetric norms. In particular the trace inequality (2) holds.

A special case of Corollary 1.2 is a matrix version of an obvious inequality for complex numbers $z = a + ib$.

**Corollary 1.3.** Let $Z = A + iB$ be a decomposition in real and imaginary parts, and let $f$ be a non-negative concave function on $[0, \infty)$. Then,

$$\left\| f(|Z|) \right\|_\infty \leq \left\| f(|A|) + f(|B|) \right\|_\infty.$$

The next application of Theorem 1.1 gives a partial answer to Conjecture 1. We first give a simple definition.
**Definition.** A function $f(t)$ on $[0, \infty)$ is **e-convex** if $f(e^t)$ is convex on $(-\infty, \infty)$. Some non-negative concave but e-convex functions are $f(t) = t^p$, $0 \leq p \leq 1$, and $f(t) = \log(1 + t)$.

**Corollary 1.4.** Let $\{A_i\}_{i=1}^m$ be normal, let $\{Z_i\}_{i=1}^m$ be expansive and let $f$ be a non-negative concave and e-convex function on $[0, \infty)$. Then, for all symmetric norms,

$$\left\| f\left(\sum Z_i^* A_i Z_i\right) \right\| \leq \left\| \sum Z_i^* f(|A_i|) Z_i \right\|.$$

Thus we have the following two special cases:

**Corollary 1.5.** Let $A, B$ be normal and let $f$ be a non-negative concave and e-convex function on $[0, \infty)$. Then, for all symmetric norms,

$$\| f(|A + B|) \| \leq \| f(|A|) + f(|B|) \|.$$

**Corollary 1.6.** Let $A, B$ be normal. Then, for all symmetric norms and $0 \leq p \leq 1$,

$$\| |A + B|^p \| \leq \| |A|^p + |B|^p \|.$$

We turn to the proof of Corollary 1.4. Given $A, B \geq 0$ we use the following notations. The weak-log majorisation relation

$$A \prec_{wlog} B$$

means that for all $k = 1, 2, \ldots$ we have

$$\prod_{j=1}^k \lambda_j(A) \leq \prod_{j=1}^k \lambda_j(B)$$

where $\lambda_j(\cdot)$ are the eigenvalues arranged in decreasing order and repeated according their multiplicities. The weak majorisation relation

$$A \prec_{w} B$$

means that for all $k = 1, 2, \ldots$ we have

$$\sum_{j=1}^k \lambda_j(A) \leq \sum_{j=1}^k \lambda_j(B).$$

**Proof of Corollary 1.4.** As in the proof of Corollary 1.2, we have

$$\left( \frac{\sum Z_i^* |A_i| Z_i}{\sum Z_i^* A_i Z_i} \frac{\sum Z_i^* A_i Z_i}{\sum Z_i^* |A_i| Z_i} \right) \geq 0.$$
so that for some contraction $K$

$$\sum Z_i^* A_i Z_i = \left( \sum Z_i^* |A_i| Z_i \right)^{1/2} K \left( \sum Z_i^* |A_i| Z_i \right)^{1/2}$$

which implies via Horn’s inequalities

$$\left| \sum Z_i^* A_i Z_i \right| \preceq_{wlog} \sum Z_i^* |A_i| Z_i.$$

Therefore, since a non-decreasing convex function preserves weak-majorisation, the $e$-convexity property of $f$ entails

$$f \left( \left| \sum Z_i^* A_i Z_i \right| \right) \preceq_w f \left( \sum Z_i^* |A_i| Z_i \right).$$

Since $f$ is non-negative, this statement is equivalent to

$$\left\| f \left( \left| \sum Z_i^* A_i Z_i \right| \right) \right\| \leq \left\| f \left( \sum Z_i^* |A_i| Z_i \right) \right\|$$

for all symmetric norms. Hence, Theorem 1.1 shows the result. \qed

Theorem 1.1 also yields some results for non-negative convex functions vanishing at 0, see [8]. For instance:

**Corollary 1.7.** Let $\{A_i\}_{i=1}^m$ be positive, let $\{Z_i\}_{i=1}^m$ be expansive and let $p \geq 1$. Then, for all symmetric norms,

$$\left\| \sum Z_i^* A_i^p Z_i \right\| \leq \left\| \left( \sum Z_i^* A_i Z_i \right)^p \right\|.$$  

The important special case

$$\left\| A^p + B^p \right\| \leq \left\| (A + B)^p \right\|$$

for positive $A$, $B$ being due to Ando-Zhan [1] and to Bhatia-Kittaneh [5] for integer exponents. This inequality raises some questions.

**Question 3.** Given $A, B \geq 0$ and $p, q \geq 0$, is it true that

$$\left\| A^{p+q} + B^{p+q} \right\| \leq \left\| (A^p + B^p)(A^q + B^q) \right\| ?$$

More generally, let $p = \sum p_i$ and $q = \sum q_i$ where $p_i, q_i \geq 0, i = 1, \ldots, m$. Does

$$\left\| A^p + B^p \right\| \leq \left\| (A^{p_1} + B^{q_1}) \cdots (A^{p_m} + B^{q_m}) \right\|$$

hold? One may also ask whether stronger inequalities like

$$\left\| A^{p+q} + B^{p+q} \right\| \leq \left\| (A^p + B^p)^{1/2}(A^q + B^q)(A^p + B^p)^{1/2} \right\|$$

hold true.

**Question 4.** Given $A, B \geq 0$ and $p, q \geq 0$, is it true that

$$\left\| A^p B^q + B^p A^q \right\| \leq \left\| A^{p+q} + B^{p+q} \right\| ?$$
Note that the similar inequality for the Heinz means
\[ \|A^p B^q + A^q B^p\| \leq \|A^{p+q} + B^{p+q}\| \]
is well-known. See [11] for much more on matrix means.

2. Applications to block-matrices

Let \( A = [A_{i,j}] \) be an arbitrary block-matrix and let \( \|\cdot\|_1 \) stand for the trace norm. Then, for all non-negative concave function \( f \) on \([0, \infty)\),
\[ \|f(|A|)\|_1 \leq \sum \|f(|A_{i,j}|)\|_1. \] (3)
Indeed, this follows from Rotfeld’s inequality [15]: For arbitrary operators \( X, Y \), we have \( \text{Tr} f(|X + Y|) \leq \text{Tr} f(|X|) + \text{Tr} f(|Y|) \). As a consequence of Theorem 1.1b we state a result for all symmetric norms:

**Theorem 2.1.** Let \( A = [A_{i,j}] \) be a block matrix with normal entries and let \( f \) be a non-negative concave, e-convex function on \([0, \infty)\). Then, for all symmetric norms,
\[ \|f(|A|)\| \leq \left\| \sum f(|A_{i,j}|) \right\|, \]

For \( f(t) = t \) we recapture (with a simpler proof) an observation from [9],
\[ \|A\| \leq \left\| \sum |A_{i,j}| \right\|. \]
Applying a non-negative concave function \( f(t) \) to both sides in the operator norm case and using Theorem 1.1b we then obtain:

**Corollary 2.2.** Let \( A = [A_{i,j}] \) be a block matrix with normal entries and let \( f \) be a non-negative concave function on \([0, \infty)\). Then,
\[ \|f(|A|)\|_\infty \leq \left\| \sum f(|A_{i,j}|) \right\|_\infty. \]

This result combined with (3) strongly suggests:

**Conjecture 2.** Corollary 2.2 holds for all symmetric norms.

Theorem 2.1 partially answers this conjecture. We turn to the proof of Theorem 2.1. We start with two elementary, well-known lemmas.

**Lemma 1.** Let \( A, B, X, Y \geq 0 \) such that \( B \prec_w Y \) and \( A \prec_w X \). Then,
\[ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \prec_w \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}. \]
Proof. We have
\[
\sum_{j=1}^{k} \lambda_j(A \oplus B) = \max_{s+t=k} \left\{ \sum_{j=1}^{s} \lambda_j(A) + \sum_{j=1}^{t} \lambda_j(B) \right\}.
\]
Combining this with
\[
\sum_{j=1}^{s} \lambda_j(A) + \sum_{j=1}^{t} \lambda_j(B) \leq \sum_{j=1}^{s} \lambda_j(X) + \sum_{j=1}^{t} \lambda_j(Y) \leq k \sum_{j=1}^{k} \lambda_j(X \oplus Y)
\]
ends the proof. \(\square\)

Lemma 2. Let \(A, B \geq 0\). Then,
\[
\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \prec_w \begin{pmatrix} A + B & 0 \\ 0 & 0 \end{pmatrix}.
\]

Proof. Note that
\[
\begin{pmatrix} A + B & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A^{1/2} & B^{1/2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A^{1/2} & 0 \\ B^{1/2} & 0 \end{pmatrix}
\]
so that
\[
\begin{pmatrix} A + B & 0 \\ 0 & 0 \end{pmatrix} \simeq \begin{pmatrix} A & A^{1/2}B^{1/2} \\ B^{1/2}A^{1/2} & B \end{pmatrix} \simeq \begin{pmatrix} A & -A^{1/2}B^{1/2} \\ -B^{1/2}A^{1/2} & B \end{pmatrix}
\]
where \(\simeq\) means unitarily congruent. Combining with
\[
\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \frac{1}{2} \begin{pmatrix} A & A^{1/2}B^{1/2} \\ B^{1/2}A^{1/2} & B \end{pmatrix} + \frac{1}{2} \begin{pmatrix} A & -A^{1/2}B^{1/2} \\ -B^{1/2}A^{1/2} & B \end{pmatrix}
\]
gives the lemma. \(\square\)

Proof of Theorem 2.1. We prove the theorem for a partition in four normal blocks
\[
\tilde{A} = \begin{pmatrix} S & R \\ T & Q \end{pmatrix}
\]
the proof of a more general partition being quite similar. Let
\[
\tilde{\tilde{A}} = \begin{pmatrix} 0 & \tilde{A} \\ \tilde{A}^* & 0 \end{pmatrix}
\]
and note that
\[
|\tilde{\tilde{A}}| = \begin{pmatrix} |\tilde{A}|^* & 0 \\ 0 & |\tilde{A}| \end{pmatrix}
\]
so that
\[
|\tilde{\tilde{A}}| \simeq \begin{pmatrix} |\tilde{A}| & 0 \\ 0 & |\tilde{A}| \end{pmatrix}
\] (4)
where the symbol $\sim$ stands for unitarily equivalent. On the other hand

$$\tilde{A} = \tilde{S} + \tilde{T} + \tilde{R} + \tilde{Q}$$

where

$$\tilde{S} = \begin{pmatrix} 0 & 0 & S & 0 \\ 0 & 0 & 0 & 0 \\ S^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \tilde{T} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & T & 0 \\ 0 & T^* & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\tilde{R} = \begin{pmatrix} 0 & 0 & 0 & R \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ R^* & 0 & 0 & 0 \end{pmatrix} \quad \tilde{Q} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & Q^* & 0 & 0 \end{pmatrix}$$

are Hermitian. Therefore, as in the proof of Corollary 1.4,

$$|\tilde{A}| \sim_{w} |\tilde{S}| + |\tilde{T}| + |\tilde{R}| + |\tilde{Q}|,$$

that is

$$|\tilde{A}| \sim_{w} \begin{pmatrix} |S^*| + |R^*| & 0 & 0 & 0 \\ 0 & |T^*| + |Q^*| & 0 & 0 \\ 0 & 0 & |S| + |T| & 0 \\ 0 & 0 & 0 & |R| + |Q| \end{pmatrix}.$$

Since $f$ is non-decreasing and e-convex we infer

$$f(|\tilde{A}|) \sim_{w} \begin{pmatrix} f(|S^*| + |R^*|) & 0 & 0 & 0 \\ 0 & f(|T^*| + |Q^*|) & 0 & 0 \\ 0 & 0 & f(|S| + |T|) & 0 \\ 0 & 0 & 0 & f(|R| + |Q|) \end{pmatrix}.$$

By Theorem 1.1b and Lemma 1 we then get

$$f(|\tilde{A}|) \sim_{w} \begin{pmatrix} f(|S^*|) + f(|R^*|) & 0 & 0 & 0 \\ 0 & f(|T^*|) + f(|Q^*|) & 0 & 0 \\ 0 & 0 & f(|S|) + f(|T|) & 0 \\ 0 & 0 & 0 & f(|R|) + f(|Q|) \end{pmatrix}.$$

Gathering the two first lines, and the two last ones, we have via Lemmas 2 and 1

$$f(|\tilde{A}|) \sim_{w} \begin{pmatrix} f(|S^*|) + f(|T^*|) + f(|R^*|) + f(|Q^*|) \\ 0 & f(|S|) + f(|T|) + f(|R|) + f(|Q|) \end{pmatrix}.$$

By using (4) we then obtain, using normality of $S, T, R, Q,$

$$f(|\tilde{A}|) \sim_{w} f(|S|) + f(|T|) + f(|R|) + f(|Q|)$$

which is equivalent to inequalities for symmetric norms. \(\Box\)

Let us point out two variations of Theorem 2.1 in which some operators are not necessarily normal.
Corollary 2.3. Let $T$ be a triangular block-matrix
\[ T = \begin{pmatrix} A & N \\ 0 & B \end{pmatrix}, \]
in which $N$ is normal. Let $f : [0, \infty) \to [0, \infty)$ be concave and e-convex. Then, for all symmetric norms,
\[ \| f(|T|) \| \leq \| f(|A^*|) + f(|N|) + f(|B|) \|. \]

Proof. Consider the polar decompositions $A = |A^*|U$ and $B = V|B|$, note that
\[ \begin{pmatrix} A & N \\ 0 & B \end{pmatrix} \simeq \begin{pmatrix} I & 0 \\ 0 & V^* \end{pmatrix} \begin{pmatrix} A & N \\ 0 & B \end{pmatrix} \begin{pmatrix} U^* & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} |A^*| & N \\ 0 & |B| \end{pmatrix}, \]
and apply Theorem 2.1. \qed

Using polar decompositions as above we may also obtain:

Corollary 2.4. Let $S$ be a block-matrix of the form
\[ S = \begin{pmatrix} A & I \\ I & B \end{pmatrix}, \]
in which $I$ is the identity. Let $f : [0, \infty) \to [0, \infty)$ be concave and e-convex. Then, for all symmetric norms,
\[ \| f(|S|) \| \leq \| f(|A^*|) + 2f(I) + f(|B|) \|. \]

The assumption in Theorem 2.1 requiring normality of each block is rather special. The next Theorem meets the simple requirement that the full matrix is Hermitian.

Theorem 2.5. Let $A = [A_{i,j}]$ be a Hermitian matrix partitioned in blocks of same size and let $f$ be a non-negative concave, e-convex function on $[0, \infty)$. Then, for all symmetric norms,
\[ \| f(|A|) \| \leq \left\| \sum f(|A_{i,j}|) \right\|. \]

Proof. The proof of Theorem 2.1 actually shows that for a general block-matrix $A = (A_{i,j})$ partitioned in blocks of same size, we have
\[ \begin{pmatrix} f(|A|) & 0 \\ 0 & f(|A|) \end{pmatrix} \wprel \begin{pmatrix} \sum f(|A_{i,j}|) & 0 \\ 0 & \sum f(|A_{i,j}|) \end{pmatrix}. \]
for all non-negative concave, e-convex function $f$. Therefore
\[ \| f(|A|) \| \leq \max \left\{ \left\| \sum f(|A_{i,j}|) \right\| ; \left\| \sum f(|A_{i,j}|) \right\| \right\}. \]
Assuming $A$ Hermitian, we have $A_{i,j}^* = A_{j,i}$ and Theorem 2.5 follows. \hfill $\square$

Thus we have

**Corollary 2.6.** For all Hermitian matrices partitioned in four blocks of same size, all symmetric norms and $0 \leq p \leq 1$,
\[
\left\| \begin{pmatrix} A & B^* \\ B & C \end{pmatrix} \right\|^p \leq \| A^p + |B|^p + |B^*|^p + |C|^p \|.
\]

It seems natural to propose:

**Conjecture 3.** Theorem 2.5 holds for all symmetric norms.

as well as a possible stronger statement

**Conjecture 4.** Theorem 2.5 holds for all normal matrices partitioned in blocks of same size and all symmetric norms.

Now we state a result for a general full matrix. However we have to confine to the trace norm, and the result can be stated as a trace inequality improving (3).

**Theorem 2.7.** Let $A = [A_{i,j}]$ be a matrix partitioned in blocks of arbitrary size and let $f$ be a continuous function on $[0, \infty)$ with $f(0) \geq 0$. Then, if $f(\sqrt{t})$ is concave,
\[
\operatorname{Tr} f(|A|) \leq \sum \operatorname{Tr} f(|A_{i,j}|).
\]

A special case of this theorem, with $f(t) = t^p$, $2 \geq p$, is a simple inequality for the Schatten $p$-norms, $2 \geq p$, of a square matrix $A = (a_{i,j})$,
\[
\| A \|_p \leq \left( \sum |a_{i,j}|^p \right)^{1/p}.
\]

For $p > 2$, the reverse inequality holds. These inequalities go back to [10] if not sooner. In contrast with Theorem 2.7, we cannot extend Corollary 2.6 for $p$ running over $[0,2]$. Let us give a simple example for a partitioned positive matrix, $p = 2$ and the operator norm.

**Example 2.8.** Let
\[
Z = \begin{pmatrix} A & B \\ B & C \end{pmatrix}
\]
where
\[
A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix}.
\]

Then $\|Z^2\|_\infty = 6 + \sqrt{35} > 9 = \|A^2 + 2B^2 + C^2\|_\infty$. 

We turn to the proof of Theorem 2.7. We start with an elementary well-known Lemma. For \( A, B \geq 0 \) the majorisation relation \( A \prec B \), or \( B \succ A \), means \( A \prec_w B \) and \( \text{Tr} \, A = \text{Tr} \, B \). If \( f(t) \) is a convex function, not necessarily increasing, \( A \prec B \Rightarrow f(A) \prec_w f(B) \). This property is crucial for the proof of Theorem 2.7.

**Lemma 3.** Let \( B = [B_{i,j}] \) be a positive block-matrix, with \( 1 \leq i, j \leq m \). Then
\[
B \succ \begin{pmatrix}
B_{1,1} & & \\
& \ddots & \\
& & B_{m,m}
\end{pmatrix}.
\]

**Proof.** Let
\[
B = \begin{pmatrix}
B_{1,1} & C \\
C^* & D
\end{pmatrix},
\]
and note that
\[
\begin{pmatrix}
B_{1,1} & 0 \\
0 & D
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
B_{1,1} & C \\
C^* & D
\end{pmatrix} + \frac{1}{2} \begin{pmatrix}
B_{1,1} & -C \\
-C^* & D
\end{pmatrix}.
\]
Hence,
\[
B \succ \begin{pmatrix}
B_{1,1} & 0 \\
0 & D
\end{pmatrix}.
\]
Repeating this process with \( D \), we see that the block-diagonal part of \( B \) lies in the convex hull of matrices unitarily congruent to \( B \). This implies the majorisation relation. \( \square \)

**Proof of Theorem 2.7.** We have \( |A|^2 = A^*A \), hence Lemma 3 yields
\[
|A|^2 \succ \begin{pmatrix}
\sum_j |A_{j,1}|^2 & & \\
& \ddots & \\
& & \sum_j |A_{j,m}|^2
\end{pmatrix}.
\]
Therefore, given any convex function \( g(t) \) on \([0,\infty)\),
\[
g(|A|^2) \succ_w \begin{pmatrix}
g \left( \sum_j |A_{j,1}|^2 \right) & & \\
& \ddots & \\
& & g \left( \sum_j |A_{j,m}|^2 \right)
\end{pmatrix}
\]
so that
\[
\text{Tr} \, g \left( |A|^2 \right) \geq \text{Tr} \, g \left( \sum_j |A_{j,1}|^2 \right) + \cdots + \text{Tr} \, g \left( \sum_j |A_{j,m}|^2 \right) \quad (5)
\]
Now, if we assume furthermore that \( g(0) \leq 0 \), we have
\[
\text{Tr} \, g(A + B) \geq \text{Tr} \, g(A) + \text{Tr} \, g(B) \quad (6)
\]
for all $A, B \geq 0$, as a consequence of Theorem 1.1b (or of Rotfeld’s inequality).
Combining (5) and (6) with $g(t) = -f(\sqrt{t})$ ends the proof. \hfill \Box

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