Actuator fault-based integrated control for vehicle chassis system

Jinwei Sun¹, Jingyu Cong², Weihua Zhao¹ and Yonghui Zhang¹

Abstract
An integrated fault tolerant controller is proposed for vehicle chassis system. Based on the coupled characteristics of vertical and lateral system, the fault tolerant controller mainly concentrates on the cooperative control of controllable suspension and lateral system with external disturbances and actuator faults. A nine-DOF coupled model is developed for fault reconstruction and accurate control. Firstly, a fault reconstruction mechanism based on sliding mode is introduced; when the sliding mode achieves, actuator fault signals can be observed exactly through selecting appropriate gain matrix and equivalent output injection term. Secondly, an active suspension controller, a roll moment controller and a stability controller is developed respectively; the integrated control strategy is applied to the system under different driving conditions: when the car is traveling straightly, the main purpose of the integrated strategy is to improve the vertical performance; the lateral controller including roll moment control and stability control will be triggered when there is a steering angle input. Simulations experiments verify the performance enhancement and stability of the proposed controller under three different driving conditions.

Keywords
Integrated control, fault tolerant control, sliding model observer, active suspension, stability control

Introduction
The performances of vehicles are influenced by the chassis system. Active control technology can effectively improve the riding comfort, maneuvering stability and safety characteristic of vehicles by using a variety of advanced control algorithms, such as controllable suspension system,¹² steering control systems,³⁴ electronic stability program,⁵ etc. However, most of the individual control systems were designed under special operating conditions, simple superposition of these single control systems may result in degraded or even unstable performance. To avoid mutual interference between sub-control systems, the integrated chassis control technology has been widely used.

The key issues of the vehicle chassis system are the coupling characteristics of subsystems, the interaction between tires and the road surface, and establishment of the non-linearly coupled dynamic models.⁸ Based on the coupled mechanisms, some control systems containing different subsystems have been proposed. Poussot-Vassal et al.⁷ designed a gain scheduled vehicle stability controller including steering and braking actuators, and the effectiveness of the controller was verified under critical driving situations. Yang et al.⁹ presented an optimal guaranteed cost controller to improve the stability and tracking performance of the chassis system. Mousavinejad et al.⁹ designed an integrated AFS and DYC control algorithm, so as to improve transient response of controllers. In view of the interaction between vertical and lateral dynamic of vehicles, Fergani et al.¹⁰ investigated a linear parameter varying and flatness based global chassis control of vehicles to achieve collaborative control of suspension and steering/braking systems. Rodrigue Tchamna et al.¹¹ also investigated a global control method, which combines the electronic stability control and active suspension to get superior performance of vehicles during cornering. Although the integrated control mechanism combined with multiple subsystems can achieve better performance, the number of sensors and actuators has also increased, and the structure has become increasingly complex, which may increase the probability of sensor and actuator faults.¹² To avoid these shortcomings,
fault tolerant mechanism is needed to handle system failures and maintain expected properties.

Passive fault tolerant method has been extensively used for automobile systems. Moradi and Fekih\textsuperscript{13,14} developed a passive fault tolerant mechanism to suppress the influence of actuator faults. A robust fault tolerant method was presented by Ali et al.\textsuperscript{15} to process the engine air path actuator faults. An adaptive fault tolerant compensation control and output feedback finite-time control of active suspension systems were designed by Pan and Sun\textsuperscript{16,17} for stabilizing the perturbed vehicle active suspension system to improve the suspension performance. However, the robustness based passive fault tolerant control will reduce the effect of failures at the cost of system performance and cannot detect the location of faults. Based on the fault diagnosis, active fault tolerant method can reconfigure the control system through estimating the current fault.\textsuperscript{18} Many active fault tolerant controllers have been applied to automobile systems. An Unknown Input Observer approach was used by Alain Yetendje et al.\textsuperscript{19} to detect and identify actuator fault of active suspension systems. On the basis of fault detection and diagnosis, the robust linear parameter variable control method was investigated by Peter Gáspár et al.\textsuperscript{20} to control the active suspension of heavy vehicles containing actuator faults. A fault-based fault tolerant feedback control was utilized by Oudghiri et al.\textsuperscript{21} to compensate the faults and reduce the impact of faults on the lateral stability of vehicles. The previous works on integrated chassis control such as the centralized integrated controller,\textsuperscript{7,8,10} the characteristics of each subsystem and the coupling relationship between the subsystems were not fully considered. In this paper, a hierarchical integrated controller is developed, among which suitable controllers for each subsystem are designed, and an integrated control strategy is utilized to coordinate the vertical and lateral dynamics. In the previous works on fault tolerant control for vehicle chassis systems, passive fault tolerant control or active fault tolerant control mainly concentrate on individual chassis system such as active suspension system. However, in this study, we consider the fault tolerant control for integrated vehicle model. Besides, the sliding mode-based fault observer can also observe the unmeasured states simultaneously. The necessity of providing an integrated fault tolerant control, in the presence of external road inputs and actuator faults while ensuring stability and performance of the control system, forms the focus of the work proposed in this paper.

In this study, a nine-DOF integrated model is established, and actuator faults are taken into account. A sliding model observer is employed to estimate fault signals and system states, and a hierarchical integrated mechanism is designed to improve vehicle chassis properties. The main innovations are:

1. A sliding mode fault observer is employed for actuator fault signal and state estimation of the chassis model.
2. A hierarchical mechanism is designed for the chassis model to achieve performance improvement of the integrated vertical and lateral model.

**Vehicle dynamic model**

For controller design purpose, a suspension model and a four wheels lateral model are utilized to describe the coupled dynamics of the vehicle.

**Vertical model**

A 7-DOF suspension model with heave, pitch and roll motions is used in this paper.\textsuperscript{22} as shown in Figure 1.

![Figure 1. Suspension model: (a) vertical model and (b) roll motion.](image)

The terms $\theta$ and $\varphi$ represent the pitch and roll angle. The damping and spring coefficients are indicated as $c_i$ and $k_{ii}$, and $k_{ij}$ represents tire stiffness. The external
disturbance is expressed as $z_{jl}$. $F_{ai}$ is active forces produced by the actuator.

The suspension deflections are expressed as

$$z_{si} = z_i - z_{ai} \quad (i = 1, 2, 3, 4) \quad (1)$$

The spring and damping forces are

$$F_{si} = k_i z_{si} + k_m z_{si}^3 \quad (i = 1, 2, 3, 4)$$

$$F_{di} = c_i z_{di} \quad (i = 1, 2, 3, 4) \quad (2)$$

According to small angle assumption, the following approximate equation can be got

$$z_1 = z_i + a \sin(\theta) + c \sin \varphi,$$

$$z_2 = z_i + a \sin(\theta) - d \sin \varphi,$$

$$z_3 = z_i - b \sin(\theta) + c \sin \varphi,$$

$$z_4 = z_i - b \sin(\theta) - d \sin \varphi, \quad (3)$$

The vehicle vertical model with sprung and unsprung mass dynamics can be given as\textsuperscript{33}

$$m_i \ddot{z}_i + \sum (F_{ax} + F_{ay} - F_{di} + F_{ati1} + F_{ati2} + F_{ati3} + F_{ati4}) = 0,$$

$$I_j \ddot{\theta} + a(F_{ax} + F_{ay} + F_{di} + F_{ati1} + F_{ati2}) + F_{ati3} + F_{ati4})$$

$$- b(F_{ax} + F_{ay} + F_{di} + F_{ati1} + F_{ati2} + F_{ati3})$$

$$- F_{di} = 0,$$

$$I_l \ddot{\varphi} - d(F_{ax} + F_{ay} + F_{ati1} + F_{ati2} + F_{ati3} + F_{ati4})$$

$$+ c(F_{ax} + F_{ay} + F_{ati1} + F_{ati2} + F_{ati3})$$

$$- F_{di} + m_i \dot{\beta}(\beta + \gamma) + m_i gh \dot{\varphi} = 0,$$

$$m_{ai} \ddot{z}_{ai} - F_{di} - F_{ai} - F_{di} = 0 \quad (i = 1, 2, 3, 4) \quad (4)$$

The anti-sway bar forces are as

$$F_{ati1} = \frac{k_{af}}{c + d} \left( \varphi - \frac{z_{ai2} - z_{ai1}}{c + d} \right)$$

$$F_{ati2} = - \frac{k_{af}}{c + d} \left( \varphi - \frac{z_{ai2} - z_{ai1}}{c + d} \right)$$

$$F_{ati3} = - \frac{k_{af}}{c + d} \left( \varphi - \frac{z_{ai3} - z_{ai4}}{c + d} \right)$$

$$F_{ati4} = \frac{k_{af}}{c + d} \left( \varphi - \frac{z_{ai3} - z_{ai4}}{c + d} \right) \quad (5)$$

The meanings of specific symbols are shown in Table 1.
term with normal force is included. Then the lateral
force developed by each tire can be given by
\[ F_{yf} = C_1\alpha_f N_i + C_2\alpha_f N_i^{2/3}, \quad i = 1, 2, 3, 4 \]  
(10)
where \( \alpha_f \) and \( N_i \) are the side slip angle and the normal
force of each tire, respectively. \( C_1 \) and \( C_2 \) are
the empirical values. The front and rear tire lateral
force can be written as
\[ F_{yf} = F_{y1} + F_{y2} = \alpha_f [C_1 (N_1 + N_2) + C_2 (N_1^{2/3} + N_2^{2/3})] \]  
(11)
\[ F_{yr} = F_{y3} + F_{y4} = \alpha_f [C_1 (N_3 + N_4) + C_2 (N_3^{2/3} + N_4^{2/3})] \]  
(12)
among which
\[ \alpha_f = \delta - \beta + \frac{a y_f}{u} \]  
(13)
\[ \alpha_f = - \beta + \frac{b y_f}{u} \]  
(14)
where \( \delta \) is the front wheel angle.

When the vehicle turns, the weight of the sprung
mass shifts from the inner wheel to the outer wheel, and
the transferred weight is related to sprung mass, yaw
rate, speed and the sprung mass center position. The
transferred weight should be reacted with the suspen-
sion roll motion. Differences of the front and rear sus-
ension weight should be reacted with the suspen-
sion roll motion. Differences of the front and rear sus-
ension characteristics would produce the roll moment
distribution. By using the active control techniques, a
distribution coefficient can be assigned to
\[ F_{yr} = \alpha_f \left[ C_1 m \frac{b}{(a + b)} + \frac{1}{2} C_2 m \frac{b^2}{(a + b)^2} \right] + 2C_2 \left( \frac{m u}{c + d} \right)^2 (1 - \varepsilon)^2 \gamma^2 \]  
(20)

Fault observer design

Sliding mode observer

The state variable of the integrated model is given as
\[ X = (z_{a1} z_{a2} z_{a3} z_{a4} z_{s1} z_{s2} z_{s3} z_{s4} z \dot{z} \dot{\theta} \dot{\phi} \dot{\psi} \gamma)^T \]  
(21)
Define the road input and front wheel angle as the
external interferences input, the interference variables
are represented as
\[ w = (z_{r1} z_{r2} z_{r3} z_{r4} \delta)^T \]  
(22)
The control variables of the integrated system include
the additional yaw moment generated by the electronic
stability control system and the active suspension force,
which can be written as
\[ U = (\Delta M F_{a1} F_{a2} F_{a3} F_{a4})^T \]  
(23)
The system measurement outputs are selected as
\[ Y = (\dot{z}, \dot{\theta}, \dot{\phi}, \dot{\psi}, z_{a1} z_{a2} z_{a3} z_{a4} \gamma)^T \]  
(24)
Then, the system model can be written as
\[ \dot{X} = AX + BU + Gw \]  
\[ Y = CX \]  
(25)
\( w \) is external disturbance, and \( ||w|| \leq B_0 \). The definition
of matrices can be found in the Appendix. Partial actua-
tor failure is considered here. By letting the actuator
fault be a multiplication coefficient, the integrated model is as
\[ \dot{X} = AX + BU + MF_t + Gw \]  
\[ Y_t = CX \]  
(26)
Among which \( M = B, f_t \leq \alpha(t, U) \) represent the actuator
faults, and \( f_t = (I - f)U \). \( A \) is 16×16 state matrix, \( C \)
is 8×16 output matrix. \( B \) and \( G \) are 16×5 input matrix, respectively.
Suppose that rank \((CM) = \text{rank } (M)\), and invariant
zeros of \((A, M, C)\) is Hurwitz matrix.28 The coordinate change
\[ X \rightarrow T_0 X \] so that \((A, M, C)\) can be converted to
\[ A = \begin{bmatrix} A_{11} & A_{12} \\ A_{211} & A_{212} \\ A_{212} & A_{12} \end{bmatrix}, \quad M = \begin{bmatrix} M_2 \\ 0 \\ 0 \\ T \end{bmatrix} \]  
(27)
\( T \) is orthogonal and \( M_2 = [0 \ M_0]^T, \quad M_0 \in R^{5 \times 5}, \quad A_{11} \in R^{8 \times 8}, \quad A_{211} \in R^{8 \times 8} \).
Considering equation (26), the observer that will be considered in the paper can be written as
\[ \dot{\hat{X}} = A\hat{X} + BU - G_e\varepsilon_r(t) + G_n\varepsilon \]
(28)
\[ \dot{\hat{Y}} = C\hat{X} \]
\[ G_e \]
\[ G_n \]
\( \varepsilon_r = Y_r - \hat{Y}_r \) represents the estimation error. With equations (26) and (28), the system state error \( e(t) = \hat{X} - X \) is
\[ e(t) = A_0e(t) + G_nr - MF_t - GW \]
(30)
among which \( A_0 = A - GC \). The observer gain matrix \( G_n \) is as
\[ G_n = \begin{bmatrix} -L & T^TP_0^{-1} \\ L_p & \end{bmatrix} \]
(31)
where \( L = \begin{bmatrix} L_0 & 0 \end{bmatrix} \), \( P_0 = P_0^T \) is design matrix. The disturbance matrix can be written as
\[ G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \]
(32)
\[ G_1 \in \mathbb{R}^{8 \times 16}. \]
Suppose there is a symmetric matrix \( P \in \mathbb{R}^{16 \times 16} \) satisfies
\[ P = \begin{bmatrix} P_1 & P_1L^T & L^TP_1 & P_1L \\ P_1L^T & P_0 & T^TP_0T + L^TP_1L & \end{bmatrix} > 0 \]
(33)
where \( P_1 \in \mathbb{R}^{8 \times 8}, P_0 = P_0^T \) is design matrix. Two positive scalars are defined
\[ \mu_0 = -\lambda_{\text{max}}(PA_0 + A_0^TP_0), \mu_1 = \|PQ\| \]
(34)

**Proposition 1**: If the scalar \( \rho \) in the discontinuous switch function (29) satisfies
\[ \rho \geq 2\|P_0A_2\|\|\mu_1\beta_0/\mu_0 + \|P_0G_2\|\|\beta_0 + \|P_0M_2\|\|\alpha(t, U) + \eta_0 \]
(35)
\( \eta_0 \) denotes a positive scalar. System error \( e(t) \) in (30) is asymptotically stable in regard to the set
\[ O_\varepsilon = \{e : \|e\| < 2\mu_1\beta_0/\mu_0 + \varepsilon \} \]
(36)
\( \varepsilon > 0 \) is any small scalar.

By introducing a new coordinate transformation
\[ T_L = \begin{bmatrix} I_{6 \times 8} & L \\ 0 & T \end{bmatrix} \]
(37)
\( (A, M, C) \) can be transformed to
\[ A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, M = \begin{bmatrix} 0 \\ M_2 \end{bmatrix}, C = \begin{bmatrix} 0 & I_6 \end{bmatrix} \]
(38)
where \( A_{11} = A_{11} + L^0A_{211} \). The nonlinear gain matrix can be converted to
\[ G_n = \begin{bmatrix} 0 \\ P_0^{-1} \end{bmatrix} \]
(39)
The disturbance matrix has the form
\[ G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} G_1 + LG_2 \\ TG_2 \end{bmatrix} \]
(40)
and the Lyapunov matrix can be written as
\[ P = (T_L^{-1}PT_L^{-1} = \begin{bmatrix} P_1 & 0 \\ 0 & P_0 \end{bmatrix} \]
(41)
The estimation error in new coordinate can be partitioned as
\[ \hat{e}_1(t) = A_{11}\varepsilon_1(t) + (A_{12} - G_{i1})e_r(t) - G_{i1}W \]
\[ \hat{e}_2(t) = A_{21}\varepsilon_1(t) + (A_{22} - G_{i2})e_r(t) + P_0^{-1}\varepsilon \]
(42)
then the sliding mode will occur in finite time.

**Robust fault observer**

To estimate fault signals exactly in the presence of interferences, assume that sliding motion in 3.1 has achieved, by selecting appropriate output injection term \( r_{eq} \) and the gain matrix \( L_{eq} \), the impact of disturbances on the fault reconstruction can be minimized. The reconstruction signal can be defined as
\[ \hat{f} = W_{sc}^TP_0^{-1}r_{eq} \]
(44)
\[ W_{sc} = \begin{bmatrix} W_1 & M_{01} \end{bmatrix}, W_1 \in \mathbb{R}^{7 \times 3}. \]
Rewriting the estimation error in new coordinate and pre-multiplying \( W_{sc}^T \) implies
\[ \hat{f} = f + \hat{G}(s)W \]
(45)
where \( \hat{G}(s) \) is given as
\[ \hat{G}(s) = W_{sc}A_2(sI_8 - (A_{11} + LA_{21}))^{-1}(G_1 + G_2) + W_{sc}G_2 \]
(46)
Then the objective is to keep \( L_2 \) gain of the transfer function from the effect of external disturbance to the fault reconstruction results not exceed \( \gamma \in \mathbb{R}_+ \). Define \( D_i \in \mathbb{R}^{8 \times 8}, \gamma_0 \in \mathbb{R}_+ \), and also define the following matrices
\[ B_d = \begin{bmatrix} 0 & G \end{bmatrix}, D_f = \begin{bmatrix} D_1 & 0 \end{bmatrix}, H = \begin{bmatrix} 0 & W_{sc}G_2 \end{bmatrix} \]
\[ E = \begin{bmatrix} -W_{sc}A_2^T & E_2 \end{bmatrix} \]
(47)
The Lyapunov matrix satisfies
The stars in inequality represent the matrix is symmetric. If the gain matrices of the observer are selected as

$$P = \begin{bmatrix} P_{11}^T & P_{12}^T \\ P_{12} & P_{22} \end{bmatrix}, \quad P_{12} = [ P_{121} \ 0 ], \quad P_{121} \in \mathbb{R}^{3 \times 3}$$

(48)

and $W_1$, $E_2$ and $\gamma$ satisfy the Bounded Real Lemma

$$\begin{bmatrix} PA + A^T P - \gamma_0 C^T (D_\delta D_\delta^T)^{-1} C & -PB_3 \\ -B_3^T P & E \end{bmatrix} - \begin{bmatrix} -\gamma_1 I & H^T \\ H & -\gamma_0 I_5 \end{bmatrix} < 0$$

(49)

$$\begin{bmatrix} P_{111}A_{11} + A_{11}^T P_{11} + P_{121}A_{21} + A_{21}^T P_{12} & \star & \star \\ -P_{11}G_1 + P_{12}G_2 & \star & \star \\ -W_0A_{21} & W_0G_2 & -\gamma I_5 \end{bmatrix} < 0$$

(50)

The suspension system model can be expressed as

$$[x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6]^T = [x_s \ \dot{x}_s \ \dot{\theta} \ \varphi \ \dot{\varphi}]^T$$

The suspension system model can be expressed as

$$\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= f_z + G_z F_z, \\
f_z &= - \frac{1}{m_z} \left( F_{x1} + F_{x2} + F_{d1} \right), \quad G_z = \frac{1}{m_z}, \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= f_\theta + G_\theta F_\theta, \\
f_\theta &= - \frac{1}{I_y}, \\
\dot{x}_5 &= x_6, \\
\dot{x}_6 &= f_\varphi + G_\varphi F_\varphi, \\
f_\varphi &= - \frac{1}{I_x} \left[ c(F_{x1} + F_{x2} + F_{d1} + F_{d2}) - b(F_{x3} + F_{d3} + F_{d4} + F_{d5}) \right], \quad G_\varphi = \frac{1}{I_x}.
\end{align*}$$

(52)
the above functions are rewritten as

$$x(t) = A_0x(t) + B_1[U_{force} + d_j(t)]$$  \hspace{1cm} (53)

where $x(t) = [x_i(t); x_i + 1(t)]; i = 1, 3, 5; j = 1, 2, 3; U_{force} = [F_z; F_y; F_x]$ is the inner force generated by the active suspension. $A_0 = [0 \quad 1 \quad 0; 0]$; $B_1 = [0; 0]$, $B_2 = [0; G_2]$, $B_3 = [0; G_3]$; $d_1(t) = [0; \frac{q_1}{B_2}]$, $d_2(t) = [0; \frac{q_2}{B_3}]$.

$d_3(t) = [0; \frac{q_3}{B_4}]$.

Assumption 1: $d_j(t)$ satisfy $|d_j(t)| \leq d_1 + d_2|x(t)| + d_3 |U_{force}| \leq d_4$, where $d_1 \geq 0$, $d_2 \geq 0$, $d_3 > 0$, $0 \leq d_4 \leq 1$, and $d_1, d_2, d_3, d_4$ are constants.

As shown in Figure 4, a static quantizer $q_\mu(v)$ is used in the uplink channel, where $v$ is the control law designed by the quantized state, $\mu$ is a fixed number; the down channel utilized a dynamic quantizer $q_\mu(x)$ to transfer system quantized states, and $\mu$ is the quantization parameter adjusted statically. Define a function that takes the nearest integer, then the uniform quantizer with the quantization level parameter $\mu$ is as

$$q_\mu(x) \overset{\text{def}}{=} \mu \text{round}\left(\frac{x}{\mu_j}\right), \mu_j > 0; j = 1, 2, 3$$  \hspace{1cm} (54)

The uniform quantization error is

$$e_\mu \overset{\text{def}}{=} q_\mu(x) - x$$  \hspace{1cm} (55)

then

$$e_\mu \overset{\text{def}}{=} |q_\mu(x) - x| \leq \Gamma \mu_j$$  \hspace{1cm} (56)

$\Gamma = \sqrt{\rho}/2, \rho$ is the dimension of the vector $x$.

With respect to dynamic quantizer, a special quantization parameter $\mu_j = 0$ is given when the system runs to the sliding surface

$$q_\mu(x) \overset{\text{def}}{=} 0, \mu_j = 0$$  \hspace{1cm} (57)

Defined the sliding surface

$$s(x(t)) = Cx(t) = 0$$  \hspace{1cm} (58)

where $C$ is a given vector to ensure the system has stable eigenvalues, and $CB_j \neq 0$.

**Lemma 1** \hspace{1cm} $33$: There is a fixed constant $\tau > 0$, assume $\mu > 0$ which satisfies

$$\mu \leq \frac{|Cx|}{(\tau + 1)|C|^2}$$  \hspace{1cm} (59)

then the following inequation can be obtained

$$|C e_\mu| \leq |C| \mu \leq \frac{1}{\tau} |C q_\mu(x)|$$  \hspace{1cm} (60)

By considering the uplink and down channel quantize, the following control law can be designed $33$

$$r = -\frac{(CB_j)}{(\tau - \chi)}|CA q_\mu(x(t))| + |CA| \mu_j$$

$$+ |CB_j| \mu_j + |CB_j||d_1 + d_2||q_\mu(x(t))|$$

$$+ d_3 \Gamma \mu_j + d_3 \Gamma \mu_j, sgn[C q_\mu(x(t))] - \frac{\tau \rho}{\tau - \chi}(CB_j)^{-1}$$

$$sgn[C q_\mu(x(t))]|$$  \hspace{1cm} (61)

$$U_{force} = q_\mu(v)$$  \hspace{1cm} (62)

where $\chi = 1 + (\tau + 1)d_1, \tau > \frac{1 + d_1}{\tau - \chi}, \rho > 0$. The control law can keep the system states reach and maintain on the sliding surface $s(x(t)) = 0$.

A simple and efficient adjustment law is as follows $34$

$$\mu_j = \frac{|(|C x(t)|)}{(\tau + 1)|C|^2}, |Cx(t)| \geq 1$$

$$\mu_j = \frac{\alpha}{(\tau + 1)|C|^2}, 0 < |Cx(t)| < 1$$

$$\mu_j = 0, |Cx(t)| = 0$$  \hspace{1cm} (63)

where $\alpha \in (0, 1), \alpha < |Cx| < \alpha^{-1}$.

**Roll moment controller**

In order to obtain favorable handling characteristics in the stability region, the appropriate roll moment distribution coefficient $\varepsilon$ should be got. According to the input-output linearization method, $35$ the yaw rate tracking error $\varepsilon_{\text{yaw}}$ can be regarded as the output. By differentiating the tracking error, the roll moment distribution coefficient $\varepsilon$ appears explicit in the equation.

![Flow chart of the quantization system.
Figure 4.](image)
where when in the stability region, \( \Delta M = 0 \). To get the control input \( \varepsilon \), define
\[
\dot{\varepsilon}_{\text{yaw}} + K_\varepsilon \varepsilon_{\text{yaw}} = 0 \tag{65}
\]
then the tracking error of the yaw rate will converge to zero, and system is asymptotically stable. Introducing the notation \( C_f = C_{1f}m_\frac{b}{(a+b)} + \frac{1}{2} C_{2f}m^2 \frac{b^2}{(a+b)^2} \), \( C_r = 2C_2m_\frac{b}{(a+b)} \), \( C_y = C_{1y}m_\frac{b}{(a+b)} + \frac{1}{2} C_{2y}m^2 \frac{b^2}{(a+b)} \), where \( C_f \) and \( C_r \) are the linear tire stiffness. Then equation (65) can be further written as
\[
\begin{align*}
-\Delta M C_{y} \gamma^2 \left[ \delta u \right] + \gamma (a^2 + b^2) + u \beta (b-a) \\
-\frac{2C_y}{I_{u}} \gamma^2 \left[ \delta u \right] - \gamma (a^2 + b^2) - u \beta (b+a) \\
-\frac{C_r}{I_{u}} \gamma^2 \left[ \delta u \right] - \gamma (a^2 + b^2) + u \beta (b-a) \\
+ \frac{1}{I_{u}} \left[ \delta u \right] - \gamma (a^2 C_r + b^2 C_r) + u \beta (b C_r - a C_r) \\
+ \frac{u}{a + b} \delta + K \left( \frac{u}{a + b} \delta - \gamma \right) = 0
\end{align*}
\tag{66}
\]
By solving the above equation, the roll moment distribution coefficient \( \varepsilon \) can be obtained and the range of \( \varepsilon \) is set as \([-1, 1]\).

As the distances between left and right suspensions to the vehicle centerline is equal, the force transformations of active suspension can be written as
\[
\begin{bmatrix}
F_z \\
F_\theta \\
F_\phi
\end{bmatrix} =
\begin{bmatrix}
1 & 1 & 1 & 1 \\
\frac{1}{a} & \frac{1}{a} & -\frac{1}{b} & -\frac{1}{b} \\
\frac{1}{c} & \frac{1}{c} & -\frac{1}{c} & -\frac{1}{c}
\end{bmatrix}
\begin{bmatrix}
F_{a1} \\
F_{a2} \\
F_{a3} \\
F_{a4}
\end{bmatrix}
\tag{67}
\]
where \( F_{a1}, F_{a2}, F_{a3}, F_{a4} \) are the active forces generated by the active suspensions. Through inverse matrix operation, the expression can be obtained as
\[
\begin{bmatrix}
F_{a1} \\
F_{a2} \\
F_{a3} \\
F_{a4}
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{2a + 2b} & -\frac{1}{2a + 2b} & \frac{1}{2a + 2b} & \frac{1}{2a + 2b} \\
\frac{1}{2a + 2b} & -\frac{1}{2a + 2b} & \frac{1}{2a + 2b} & \frac{1}{2a + 2b} \\
\frac{1}{2a + 2b} & -\frac{1}{2a + 2b} & \frac{1}{2a + 2b} & \frac{1}{2a + 2b}
\end{bmatrix}
\begin{bmatrix}
F_z \\
F_\theta \\
F_\phi
\end{bmatrix}
\tag{68}
\]
The active forces of front and rear axes are generated by the active suspensions, and can be given as
\[
F_y = \frac{1 + \varepsilon}{2} F_\phi, \quad F_\theta = \frac{1 - \varepsilon}{2} F_\phi
\]
The equations can then be given as
\[
\begin{bmatrix}
F_{a1} \\
F_{a2}
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{2a + 2b} & -\frac{1}{2a + 2b} & \frac{1}{2a + 2b} & \frac{1}{2a + 2b} \\
\frac{1}{2a + 2b} & -\frac{1}{2a + 2b} & \frac{1}{2a + 2b} & \frac{1}{2a + 2b}
\end{bmatrix}
\begin{bmatrix}
F_z \\
F_\theta \\
F_\phi
\end{bmatrix}
\tag{69}
\]

Control of sprung mass motions is changed into control the four subsystems by utilizing the decoupling calculation.

**ESP controller design**

The ESP controller improves the lateral dynamics through generating corrective yaw moment. A terminal sliding mode control is utilized for the ESP. According to equation (7), following equation can be got
\[
\dot{\gamma} = \gamma - \gamma_{\text{des}} \tag{71}
\]

Where \( M_z = a F_{z-y} - b F_{z-x} \). The tracking error of the yaw rate is
\[
\dot{\gamma} = \psi - \dot{\psi}_{\text{des}}, \quad \gamma = \gamma - \gamma_{\text{des}} \tag{72}
\]

Define the sliding surface as
\[
s_{\gamma} = \psi + \alpha_{\gamma} \text{sign}(\dot{\psi})(\psi) + \alpha_{\gamma_{\text{des}}} \text{sign}(\dot{\gamma})(\gamma)
\tag{73}
\]

among which, \( \alpha_{\gamma} > 0, \alpha_{\gamma_{\text{des}}} > 0, 1 < \gamma_{\text{des}} < 2, \gamma_{\text{des}} > 2 \). The deviation of the sliding surface is expressed as
\[
\dot{s}_{\gamma} = \dot{\gamma} + \alpha_{\gamma} \gamma_{\text{des}} \text{sign}(\dot{\psi})(\psi) + \alpha_{\gamma_{\text{des}}} \gamma_{\text{des}} \text{sign}(\dot{\gamma})(\gamma)
\tag{74}
\]

The control law of the ESP controller is formulated as
\[
\Delta M = I_z \left[ \frac{1}{\alpha_{\gamma} \alpha_{\gamma_{\text{des}}}} \text{sign}(s_{\gamma}) \text{sign}(\dot{s}_{\gamma})(\psi) + 1 + \alpha_{\gamma} \gamma_{\text{des}} \text{sign}(\dot{\psi})(\psi) \right]
\tag{75}
\]

where
\[
K_{\gamma_{\text{des}}} > 0
\]

Define the Lyapunov function as
\[
V_{\gamma} = \frac{1}{2} \gamma_{\text{des}}^2
\tag{76}
\]

the time derivative of (76) is as
\[
\dot{V}_{\gamma} = s_{\gamma} (\gamma - \gamma_{\text{des}}) + \alpha_{\gamma} \gamma_{\text{des}} (\psi - \dot{\psi}_{\text{des}})(\gamma - \gamma_{\text{des}})
\tag{77}
\]
\[
+ \alpha_{\gamma_{\text{des}}} \gamma_{\text{des}} (\gamma - \gamma_{\text{des}}) (\psi - \dot{\psi}_{\text{des}})(\gamma - \gamma_{\text{des}})
\tag{77}
\]
By implementing the control law (75) into (77) can obtain
\[
V_g = \frac{1}{C_0} K_g \gamma_2 g_s + \frac{1}{C_0} g_s \text{des} \quad (78)
\]
For \( V_g > 0 \), \( K_g, \gamma_2, g_s > 0 \), so it can be concluded that \( \dot{V}_g \leq 0 \). The discontinuity of the sign function can be modified as
\[
\text{sign} \gamma = \frac{\gamma}{|\gamma| + \delta} \quad (79)
\]
where \( \delta \gamma \) is an arbitrary small positive constant.

**Integrated control strategy**

When the car travels straightly, the active suspension controller generates active control force to minimizing sprung mass motion; the roll moment controller and ESP controller do not work. As the brake-based ESP controller is not suitable for the vehicle in the stable area for the ESP controller directly affects the longitudinal motion of the vehicle. When there is a steering angle input, coordination of roll moment controller and ESP controller determines which controller is used. The boundary of the phase plane is defined as
\[
\Omega = |2.5 \dot{\beta} + 10 \beta| < 1 \quad (80)
\]
In the stability area, the roll moment controller forces the yaw rate to track desired yaw rate through controlling lateral tire forces. When the system exceeded the stable lines, ESP controller affords additional yaw moment to keep the car running in stability region through the differential braking system. The switching between roll moment controller and ESP controller can be expressed as
\[
M_{yaw} = \rho^* M_z + (1 - \rho^*) \Delta M \quad (81)
\]
where \( \rho^* \) is the switching gain which is varying according to the stability region, and \( \rho^* \) can be expressed as
\[
\rho^* = \begin{cases} 
1, & |\Omega| \leq 0.8 \\
-5|\Omega| + 5, & 0.8 < |\Omega| < 1 \\
0, & |\Omega| \geq 1
\end{cases} \quad (82)
\]

**Simulation results**

The control goal of the designed controller is to improve the chassis characteristics. In this section, the effect of the fault tolerant mechanism is validated through three different operating conditions. The front wheel inputs are shown in Figure 5, and the external road disturbances of four wheels are given in Figure 6. The simulation parameters are listed in Table 1.

**Step steering input**

The step steering input plotted in Figure 5(a) is considered in this section, and the vehicle travels at a speed of 60 km/h.
The observed results under step steering are given in Figure 7(a) to (d). The figures show that the designed observer can track the system states exactly.

The responses of the nominal model to the controller are given in Figure 8(a) to (e). Figure 8(a) indicates that the controller can suppress the sprung mass motion when there is no steering input, and the vibration performance of the sprung mass has been effectively improved; when there is a step steering input, the suspension performance is degraded. When there is steering angle input, the suspension and tire deflections are decreased simultaneously, as shown in Figure 8(b) and (c). Figure 8(d) and (e) indicate that during step steering, the yaw rate and lateral acceleration are reduced significantly with the integrated controller. The results are in line with the design goals.

**SLC steering input**

This section mainly discusses the control results of the fault tolerant controller under SLC steering input condition.

Figure 9(a) to (d) shows the observed results of unmeasured states under SLC steering input. The results show that the designed method can follow system states exactly.

The time responses of the vertical and lateral dynamics to the integrated controller are shown in Figure 10(a) to (e). Figure 10(a) shows that the vibration performance improves significantly when there is no steering input, and the performance closes to the passive system with steering input. Obviously, the suspension and tire deflections in Figure 10(b) and (c) are almost consistent with no steering input. The lateral dynamics are given in Figure 10(d) and (e), and the lateral performance has been improved greatly under the SLC input.

**Actuator fault**

Loss of actuator effectiveness is discussed in this section. Assume that there is a virtual actuator that produces a yaw moment, so the actuator fault can be taken into account. The failure of four suspension actuators
and one DYC actuator is analyzed here, and the considered faults are defined as follows:

\[
\eta_{fl} = \begin{cases} 
  1 & \text{if } t < 2 \\
  0.25t + 1.5 & \text{if } 2 \leq t < 4 \\
  0.5 & \text{if } 4 \leq t < 6 \\
  1 & \text{if } t \geq 6 
\end{cases}
\]

\[
\eta_{fr} = \begin{cases} 
  1 & \text{if } t < 4 \\
  0.5 & \text{if } 4 \leq t < 6 \\
  1 & \text{if } t \geq 6 
\end{cases}
\]

\[
\eta_{rl} = \begin{cases} 
  1 & \text{if } t < 4 \\
  0.5 & \text{if } 4 \leq t < 6 \\
  1 & \text{if } t \geq 6 
\end{cases}
\]

\[
\eta_{rr} = \begin{cases} 
  1 & \text{if } t < 4 \\
  0.5 & \text{if } 4 \leq t < 6 \\
  1 & \text{if } t \geq 6 
\end{cases}
\]

\[
\eta_v = \begin{cases} 
  1 & \text{if } 0 \leq t < 2 \\
  0.25t + 1.5 & \text{if } 2 \leq t < 4 \\
  0.5 & \text{if } 4 \leq t < 5 \\
  1 & \text{if } t \geq 5 
\end{cases}
\]

The simulation is carried out under a step input. The real fault signals and estimated fault signals of active suspension and DYC actuators are given in Figure 11(a) to (e). The estimation results indicate that the designed observer can reconstruct actuator fault signals exactly when the actuator faults occur, and the estimated values of the actuators are basically consistent with the real values.

To evaluate the control performance with actuator faults, the RMS values under step steering and SLC input are presented in Tables 2 and 3, respectively. It can be deduced from tables that the proposed active fault tolerant controller provides efficient performance under fault conditions.

Comparisons between $H_\infty$ fault tolerant control and the designed controller are presented to further
illustrate the effectiveness of the proposed method, and the simulation was carried out under straight driving condition. In the presence of actuator faults, the dynamic response of the vertical system under and the designed controller is plotted in Figure 12(a) to (c). It can be seen from Figure 12(a) that the vertical, pitch and roll accelerations with the designed controller can be well controlled in the interval of actuator faults, while the $H_\infty$ controller is greatly affected by the faults, as shown by the blue dotted line. Similarly, from Figure 12(b) and (c), it can be found that in the actuator fault interval, the suspension deflection and tire deflection performances with the designed controller are better than those of the $H_\infty$ fault tolerant controller.

**Conclusions**

This study presented a fault observer-based hierarchical integrated control algorithm to process the actuator faults and coupling properties of vehicle chassis model. The observer was designed for the integrated model with respect to unmeasured states and actuator faults, to provide accurate control parameters and improve the reliability of the system. Then, a hierarchical
An integrated control mechanism was proposed for the integrated model to improve the chassis performance with interferences and failures. Simulation results indicate that the controller can perfectly estimate actuator fault signals and effectively improve the vehicle chassis performance. Conclusions of the study are:

(1) The proposed observer can track the system states and estimate the actuator faults accurately;

(2) The hierarchical control strategy can promote the vehicle performance with different steering input;

(3) The fault tolerant control mechanism can provide excellent characteristics for vehicle chassis system.

The active fault tolerant method was verified under different steering inputs and actuator faults. Future work would possibly concern the following aspects:

Figure 10. Control results under SLC input: (a) sprung mass accelerations, (b) suspension deflections, (c) tire deflections, (d) lateral acceleration, and (e) yaw rate.
Figure 11. Actuator faults and estimations: (a) FL ASS actuator fault, (b) FR ASS actuator fault, (c) RL ASS actuator fault, (d) RR ASS actuator fault, and (e) DYC actuator fault.
1. Chassis system modeling with consideration of component nonlinearities and longitude dynamic study for further potential exploration of the integrated control system.

2. Multiple sensor and actuator faults should be considered simultaneously.

3. More advanced fault tolerant control strategies that could be suitable for the complete failure condition being required for vehicle performance improvement.

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ORCID iD
Jinwei Sun https://orcid.org/0000-0003-4971-2327

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### Appendix

\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{51} & 0 & 0 & 0 & a_{55} & 0 & 0 & a_{59} & a_{510} & a_{511} & a_{512} & a_{513} & a_{514} & 0 & 0 \\
a_{62} & 0 & 0 & 0 & a_{66} & 0 & 0 & a_{69} & a_{610} & a_{611} & a_{612} & a_{613} & a_{614} & 0 & 0 \\
0 & 0 & a_{73} & 0 & 0 & 0 & a_{77} & 0 & a_{79} & a_{710} & a_{711} & a_{712} & a_{713} & a_{714} & 0 & 0 \\
0 & 0 & 0 & a_{84} & 0 & 0 & 0 & a_{88} & a_{89} & a_{810} & a_{811} & a_{812} & a_{813} & a_{814} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{101} & a_{102} & a_{103} & a_{104} & a_{105} & a_{106} & a_{107} & a_{108} & a_{109} & a_{1010} & a_{1011} & a_{1012} & a_{1013} & a_{1014} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{121} & a_{122} & a_{123} & a_{124} & a_{125} & a_{126} & a_{127} & a_{128} & a_{129} & a_{1210} & a_{1211} & a_{1212} & a_{1213} & a_{1214} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{141} & a_{142} & a_{143} & a_{144} & a_{145} & a_{146} & a_{147} & a_{148} & a_{149} & a_{1410} & a_{1411} & a_{1412} & a_{1413} & a_{1414} & a_{1415} & a_{1416} \\
a_{151} & a_{152} & a_{153} & a_{154} & a_{155} & a_{156} & a_{157} & a_{158} & a_{159} & a_{1510} & a_{1511} & a_{1512} & a_{1513} & a_{1514} & a_{1515} & a_{1516} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{1615} & a_{1616}
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b_{52} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & b_{63} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & b_{74} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & b_{85} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & b_{102} & b_{103} & b_{104} & b_{105} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & b_{122} & b_{123} & b_{124} & b_{125} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{142} & b_{143} & b_{144} & b_{145} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{152} & b_{153} & b_{154} & b_{155} & 0 & 0 & 0 & 0 & 0 \\
b_{161} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
G = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
g_{51} & 0 & 0 & 0 & 0 \\
0 & g_{62} & 0 & 0 & 0 \\
0 & 0 & g_{73} & 0 & 0 \\
0 & 0 & 0 & g_{84} & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
\[ C = \begin{bmatrix}
    a_{101} & a_{102} & a_{103} & a_{104} & a_{105} & a_{106} & a_{107} & a_{108} & a_{109} & a_{110} & a_{111} & a_{112} & a_{113} & a_{114} & 0 & 0 \\
    a_{121} & a_{122} & a_{123} & a_{124} & a_{125} & a_{126} & a_{127} & a_{128} & a_{129} & a_{130} & a_{131} & a_{132} & a_{133} & a_{134} & 0 & 0 \\
    a_{141} & a_{142} & a_{143} & a_{144} & a_{145} & a_{146} & a_{147} & a_{148} & a_{149} & a_{150} & a_{151} & a_{152} & a_{153} & a_{154} & a_{155} & a_{156} \\
    -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \]

\[ a_{51} = -k_1 + k_4; \quad a_{55} = -c_1 \frac{m_1}{m_a}; \quad a_{59} = k_1 \frac{m_1}{m_a}; \quad a_{511} = a_{513} = c_k \frac{m_1}{m_a}; \quad a_{510} = c_1 \frac{m_1}{m_a}; \quad a_{512} = a_c \frac{m_1}{m_a}; \quad a_{514} = c_c \frac{m_1}{m_a}; \]

\[ b_{52} = a_c \frac{1}{m_a}; \quad g_{51} = k_1 \frac{1}{m_a}; \]

\[ a_{62} = -k_2 + k_4 \frac{m_2}{m_a}; \quad a_{66} = -c_2 \frac{m_2}{m_a}; \quad a_{69} = k_2 \frac{m_2}{m_a}; \quad a_{611} = a_{613} = c_k \frac{m_2}{m_a}; \quad a_{610} = c_2 \frac{m_2}{m_a}; \quad a_{612} = a_c \frac{m_2}{m_a}; \]

\[ a_{614} = a_c \frac{d_c}{m_a}; \quad b_{63} = a_c \frac{1}{m_a}; \quad g_{62} = k_2 \frac{1}{m_a}; \]

\[ a_{73} = -k_3 + k_5 \frac{m_3}{m_a}; \quad a_{77} = -c_3 \frac{m_3}{m_a}; \quad a_{79} = k_3 \frac{m_3}{m_a}; \quad a_{711} = a_{713} = c_k \frac{m_3}{m_a}; \quad a_{710} = c_3 \frac{m_3}{m_a}; \quad a_{712} = b_c \frac{m_3}{m_a}; \quad a_{734} = c_c \frac{m_3}{m_a}; \]

\[ b_{74} = a_c \frac{1}{m_a}; \quad g_{73} = k_3 \frac{1}{m_a}; \]

\[ a_{84} = -k_4 + k_5 \frac{m_4}{m_a}; \quad a_{88} = -c_4 \frac{m_4}{m_a}; \quad a_{89} = k_4 \frac{m_4}{m_a}; \quad a_{811} = a_{813} = c_k \frac{m_4}{m_a}; \quad a_{810} = c_4 \frac{m_4}{m_a}; \quad a_{812} = b_c \frac{m_4}{m_a}; \]

\[ a_{814} = a_c \frac{d_c}{m_a}; \quad b_{85} = a_c \frac{1}{m_a}; \quad g_{84} = k_4 \frac{1}{m_a}; \]

\[ a_{101} = a_{102} = a_{103} = a_{104} = k_5 \frac{m_5}{m_a}; \quad a_{105} = c_5 \frac{m_5}{m_a}; \quad a_{106} = c_5 \frac{m_5}{m_a}; \quad a_{107} = c_5 \frac{m_5}{m_a}; \quad a_{108} = c_5 \frac{m_5}{m_a}; \quad a_{109} = a_{1010} = \frac{c_3 a + c_3 b + c_3 c + c_3 d}{m_s}; \]

\[ a_{1010} = a_{1012} = a_{1014} = -c_1 a + c_2 b + c_3 c + c_4 d; \]

\[ a_{121} = a_{122} = a_{123} = a_{124} = a_{125} = a_{126} = a_{127} = a_{128} = -b_c \frac{1}{I_y}; \]

\[ a_{129} = a_c \frac{1}{I_y}; \]

\[ a_{1211} = -a^2 k_1 - a^2 k_2 - b^2 k_3 - b^2 k_4; \quad a_{1212} = -a c k_1 + a d k_2 + b c k_3 - b d k_4; \]

\[ a_{1210} = a c_2 + b c_3 + b c_4; \quad a_{1214} = a c_1 + a d c_2 + b c c_3 - b d c_4; \]

\[ a_{1214} = a c c_1 + a d c_2 + b c c_3 - b d c_4; \quad b_{121} = a \frac{1}{I_y}; \quad b_{123} = a \frac{1}{I_y}; \quad b_{124} = b \frac{1}{I_y}; \quad b_{125} = -b \frac{1}{I_y}; \]
\[ a_{141} = \left( \frac{mv}{mv I_x + m_2^2 v h^2} \right) \left( c k_1 + \frac{d k_{af} - c k_{af}}{(e + d)^2} \right); a_{142} = \left( \frac{mv}{mv I_x + m_2^2 v h^2} \right) \left( -d k_2 + \frac{d k_{af} + c k_{af}}{(e + d)^2} \right); \]

\[ a_{143} = \frac{mv}{mv I_x + m_2^2 v h^2} \left( c k_3 + \frac{d k_{af} - c k_{af}}{(e + d)^2} \right); \]

\[ a_{144} = \frac{mv}{mv I_x + m_2^2 v h^2} \left( -d k_4 + \frac{d k_{af} + c k_{af}}{(e + d)^2} \right); a_{145} = \frac{mv c c_1}{mv I_x + m_2^2 v h^2} ; a_{146} = -\frac{m v d c_2}{mv I_x + m_2^2 v h^2}; \]

\[ a_{147} = \frac{mv c c_3}{mv I_x + m_2^2 v h^2} ; a_{148} = -\frac{m v d c_4}{mv I_x + m_2^2 v h^2}; \]

\[ a_{149} = mv \left( -c k_1 + d k_2 - c k_3 + d k_4 \right); a_{1411} = mv \left( -a c k + a d k_2 + b c k_3 - b d k_4 \right); \]

\[ a_{1413} = \frac{mv}{mv I_x + m_2^2 v h^2} \left( -c^2 k_1 - d^2 k_2 - c^2 k_3 - d^2 k_4 + \frac{-d k_{af} - c k_{af} + c k_{af} + d k_{af} - m_1 g h}{(e + d)} \right); \]

\[ a_{1410} = \frac{mv}{mv I_x + m_2^2 v h^2} \left( -c c_1 + d c_2 - c c_3 + d c_4 \right); a_{1412} = mv \left( -a c c_1 + a d c_2 + b c c_3 - b d c_4 \right); \]

\[ a_{1414} = \frac{mv}{mv I_x + m_2^2 v h^2} \left( -\frac{-c^2 c_1 - d^2 c_2 - c^2 c_3 - d^2 c_4}{I_2(e + d)} \right); \]

\[ a_{1415} = \frac{vm h (2 K_f + 2 K_r)}{mv I_x + m_2^2 v h^2} ; a_{1416} = \frac{m h (2 a K_f - 2 b K_r)}{mv I_x + m_2^2 v h^2} + \frac{m m v^2 h}{mv I_x + m_2^2 v h^2}; \]

\[ b_{142} = \frac{mv c}{mv I_x + m_2^2 v h^2} ; b_{143} = -\frac{m v d}{mv I_x + m_2^2 v h^2}; \]

\[ b_{144} = \frac{mv c}{mv I_x + m_2^2 v h^2} ; b_{145} = \frac{g_{145} = -\frac{vm h 2 K_f}{mv I_x + m_2^2 v h^2}}{mv I_x + m_2^2 v h^2}. \]

\[ a_{151} = \frac{m m v h}{mv I_x + m_2^2 v h^2} \left( c k_1 + \frac{d k_{af} - c k_{af}}{(e + d)^2} \right); \]

\[ a_{152} = \frac{m m v h}{mv I_x + m_2^2 v h^2} \left( -d k_2 + \frac{d k_{af} + c k_{af}}{(e + d)^2} \right); a_{153} = \left( \frac{m m v h}{mv I_x + m_2^2 v h^2} \right) \left( c k_3 + \frac{d k_{af} - c k_{af}}{(e + d)^2} \right); \]

\[ a_{154} = \frac{m m v h}{mv I_x + m_2^2 v h^2} \left( -d k_4 + \frac{d k_{af} + c k_{af}}{(e + d)^2} \right); \]

\[ a_{155} = \frac{m m v h}{mv I_x + m_2^2 v h^2} \left( -c k_1 + d k_2 - c k_3 + d k_4 \right); a_{1511} = \frac{m m v h}{mv I_x + m_2^2 v h^2} \left( -a c k + a d k_2 + b c k_3 - b d k_4 \right); \]

\[ a_{1513} = \frac{m m v h}{mv I_x + m_2^2 v h^2} \left( -c^2 k_1 - d^2 k_2 - c^2 k_3 - d^2 k_4 + \frac{-d k_{af} - c k_{af} + c k_{af} + d k_{af} - m_1 g h}{(e + d)} \right); \]

\[ a_{1510} = \frac{m m v h}{mv I_x + m_2^2 v h^2} \left( -c c_1 + d c_2 - c c_3 + d c_4 \right); a_{1512} = \frac{m m v h}{mv I_x + m_2^2 v h^2} \left( -a c c_1 + a d c_2 + b c c_3 - b d c_4 \right); \]

\[ a_{1514} = \frac{m m v h}{mv I_x + m_2^2 v h^2} \left( -c^2 c_1 - d^2 c_2 - c^2 c_3 - d^2 c_4 \right); \]

\[ a_{1515} = \frac{vm h^2 (2 K_f + 2 K_r)}{mv I_x + m_2^2 v h^2} + \frac{-2 K_f - 2 K_r}{mv} ; a_{1516} = \frac{m v h^2 (2 a K_f - 2 b K_r)}{mv I_x + m_2^2 v h^2} + \frac{m m v^2 h}{mv I_x + m_2^2 v h^2} + \frac{-2 a K_f + 2 b K_r}{mv^2} - 1; \]

\[ g_{155} = \frac{2 K_f}{mv} \frac{vm h^2 2 K_f}{mv I_x + m_2^2 v h^2} ; b_{152} = \frac{mv h c}{mv I_x + m_2^2 v h^2} ; b_{153} = -\frac{m m h d}{mv I_x + m_2^2 v h^2} ; b_{154} = \frac{mv h c}{mv I_x + m_2^2 v h^2} ; b_{155} = -\frac{m m h d}{mv I_x + m_2^2 v h^2} ; \]

\[ a_{1615} = \frac{-2 a K_f + 2 b K_r}{I_z} ; a_{1616} = \frac{-2 a^2 K_f - 2 b^2 K_r}{I_z v} ; b_{161} = \frac{1}{I_z g_{165}} ; g_{165} = \frac{2 a K_f}{I_z}. \]