Fractional order mathematical modeling of novel coronavirus (COVID-19)

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In this manuscript, the mathematical model of COVID-19 is considered with eight different classes under the fractional-order derivative in Caputo sense. A couple of results regarding the existence and uniqueness of the solution for the proposed model is presented. Furthermore, the fractional-order Taylor’s method is used for the approximation of the solution of the concerned problem. Finally, we simulate the results for 50 days with the help of some available data for fractional differential order to display the excellency of the proposed model.

KEYWORDS
approximation, Caputo derivative, coronavirus (COVID-19), existence and uniqueness

MSC CLASSIFICATION
26A33; 65L20

1 | INTRODUCTION

Caused by coronavirus-2 SARS-COV-2, which is a serious acute respiratory syndrome. COVID-19 is also its family member and an infectious disease thought to be natural and of animal origin. The outbreak has been originated in China. A publicized report from the Chinese government highlights that on November 17, 2019, the first case of COVID-19 has been confirmed in a 55-year-old man in Hubei province in China. However, official reports say that the virus was originated from Wuhan, China, in December 2019. On October 5, 2020, World Health Organization (WHO) has published a report identifying that the worldwide infections of COVID-19 may be one in 10 people.1 While on March 11, 2020, the WHO declared COVID-19 as a global pandemic. As of October 9, 2020, around 188 countries and territories have been reported more than 36.76 million confirmed cases of COVID-19 with more than 1.06 million deaths, and 27.67 million recovered.2 According to the Worldometer, currently, in some countries like America, India, and Brazil, the pandemic has caused deaths (in millions) more than 0.21778, 0.106521, 0.149034, respectively. These figures highlight the severity and high infectivity of the COVID-19. Individuals who are infected with COVID-19 have experienced common symptoms of fever, dry cough, and tiredness. Although some less common symptoms like aches and pains, sore throat, diarrhea, conjunctivitis, headache, loss of taste, or smell a rash on the skin, or discolorization of fingers or toes. The severe symptoms include difficulty or shortness of breath, loss of movement or speech, and pressure or chest pain. Nonetheless, the symptoms or effects of COVID-19 vary from person to person based on the immune system. Individuals having a strong immune system are more likely to develop mild or moderate illness and recover without being hospitalized. Even so, some other symptoms like neurological diseases and gastroenteritis varying severity have been reported by other studies.3,4

The COVID-19 transmits mainly through aerosols or droplets when infected individual talks, sneezes, or coughs droplets or tiny particles known as aerosols channel the virus from their nose or mouth into the air. Anyone within six
feet of the infected individual can breathe the virus into their lungs. Other forms of transmissions can be an airborne transmission, surface transmission, or fecal-oral transmission. Some individuals do not even realize that they have been infected with COVID-19 and transfer the virus to others. Such a spread is called the asymptomatic spread. Also, one can pass it on without any apparent symptoms. This is called the presymptomatic. Because of the numerous means of transmission of the infection, world leaders and WHO have taken some measures to control the infection to some extent. The best possible way to control the infection is to avoid public places and the mixing of people. The intensity of the virus has attracted many researchers (see, e.g., Maier and Brockmann\(^5\) and Chen et al\(^6\) and references therein). As an immediate consequence, more and more world leaders decided to close entire cities, halt business, ban international and domestic traveling, going to market, or public places. These resulted in great economic loss for countries, societies broke down due to a huge amount of various factors like unavailability of resources, closed market places, and shortage of food. People suffer from intense psychological disorders. Because most people were forced to quit jobs, and some were restrained in their respective places far from family.

Therefore, researchers and doctors have devoted most of their time to the anti-pandemic war by conducting researches in their respective areas of expertise. The objective of such a study is to analyze COVID-19 from different angles. Such include infectious diseases, public environment, virology, microbiology, sociology, and psychology. China, America, Russia, and Germany are the leading countries on the COVID-19 study because the early outbreak drove them to start immediate and relevant study (see, for instance, other studies\(^7-9\)). Over the past decades, researchers are utilizing mathematical modeling to model dynamics of versatile diseases such as tuberculosis, influenza, human immunodeficiency virus (HIV), and malaria.\(^10-12\) Consequently, the studies resulted in significant understanding for control policies of diseases with several prevention measures (see, e.g., Owolabi and Atangana\(^15\) and Khajanchi and Nieto\(^14\) and references therein). Theoretical epidemiology has proved to be an extraordinary discipline and conceptual development. The aim of this field is not only to study and anticipate the spread of several diseases but also aims to control diseases with immediate effect.\(^15\)

The identification of those factors that increase the rate of spreading of the novel COVID-19 can drastically help to control the spreading of the pandemic. Therefore, several studies have been done from the advent of this disease to provide exact information for employing disease outbreak responses. Tang et al\(^16\) used ordinary differential equations and the Marco Monte Carlo method (MMCM) to calculate the transmission risk and implication for public health interventions. Li et al\(^17\) determined the epidemiology and found out the mean incubation period was 5.2 days from the analysis of data collected from Wuhan. Zhao et al\(^18\) have modeled the epidemic curve of COVID-19 cases, in mainland China from January 10, 2020, to January 24, 2020, by taking into account the impact of the variations in the disease. Chen et al\(^6,19\) have developed a Bats–Hosts–Reservoir–People transmission network model for stimulating potential transmission from an infectious source to a Human.

It is quite irrefutable that after a while people will get exhausted by the virus situation. They will need to get out of their respective detained places and will yearn for their normal life. Further, the pressure on countries’ economies will also force governments to leave primary responses of epidemic prevention and take into account new control and preventive strategies.\(^6\) In such conditions, some important questions need to be answered meticulously and promptly. When the normal life of citizens can be restored? Does changing the emergency responses that individuals can relieve themselves from self-production? On what new strategies should governments prosecute to forestall outbreaks and economic collapses. The answer to the aforesaid questions was given by Yousefpur et al\(^20\) in their recent study. They have adopted the model given below:

\[
\begin{align*}
\dot{S} &= -(\beta c + cq(1 - \beta))S(I + \theta A) + \lambda S_q, \\
\dot{E} &= -\beta c(1 - q)S(I + \theta A) + \sigma E, \\
\dot{I} &= \sigma oE - (\delta I + \alpha + \gamma I)I, \\
\dot{A} &= \sigma(1 - \rho)E - \gamma AA, \\
\dot{S}_q &= (1 - \beta)cqS(I + \theta A) - \lambda S_q, \\
\dot{E}_q &= \beta cqS(I + \theta A) - \delta_q E_q, \\
\dot{H} &= \delta I + \delta_q E_q - (\alpha + \gamma_H)H, \\
\dot{R} &= \gamma_I I + \gamma_AA + \gamma_H H. 
\end{align*}
\]

In the model (1.1), they have divided the total number of population into eight epidemiological compartments: susceptible compartment \((S)\), exposed compartment \((E)\), infected compartment with symptoms \((I)\), asymptomatic compartment (pre-symptomatic) \((A)\), quarantined susceptible compartment \((S_q)\), quarantined exposed compartment \((E_q)\), hospitalized compartment with symptoms \((H)\), and the population under quarantine \((R)\).
Further, $c$ is the contact rate, $\beta$ represents the probability of transmission per contact, and $q$ is the quarantined rate of exposed individuals. Additionally, $\lambda$ indicates the rate of quarantined uninfected individuals into the wider community. $\sigma$ is the transition rate of exposure to the infected class. $\rho$ is the probability of having symptoms among infected individuals. The rate of death induced by the disease is presented by $\alpha$. $\delta_I$ stands for the transition rate of symptomatic infected individuals to quarantined the infected class. The transition rate of quarantined exposed individuals to quarantine infected individuals is expressed by $\delta_q$. While, $\gamma_I$, $\gamma_A$, and $\gamma_H$ represents the recovery rate of infected individuals, asymptomatic infected individuals, and quarantined infected individuals.

The proportion, $q$ of the exposed individuals is quarantined, with contact tracing. The quarantined individuals can either move to $S - q$ or $E_q$ base on the fact that if they are infected effectively or not. While the proportion $(1 - q)$ are individuals exposed to the virus, however, missed from the contact tracing. Therefore, they either stay in susceptible compartment ($S$) or move to exposed compartment ($E$). The infected people at the rate of $\beta c (1 - q)$ who are not quarantined will move to the compartment ($E$). The infected quarantined individuals at the rate of $\beta c q$ will move to the compartment ($E_q$). Moreover, an uninfected quarantined individual at the rate of $\beta c (1 - q)$ will move to the compartment ($S_q$). The contact rate $c$ is given by Tang et al:21

$$c = (c_0 - c_f) \exp[-r_1 t] + c_f,$$

where $c_0$ is the initial contact rate, $c_f$ is the final contact rate, $r_1$ is the exponential increasing or decreasing rate of the contact rate. In reality, $c(0) = 0$, and $\lim_{t \to \infty} c(t) = c_f$. The effects of self-isolation of all people containing susceptible people on public health intervention improvement can be measured by changing the contact rate. Yousefpour et al20 have assumed that the contacts are decreasing with time. Further, they have elaborated that the proposed model (1.1) can be supposed with minimum contact rate, and then, it will increase with time.

The function $\delta_I$ is supposed to be an increasing function with time. Therefore, it is supposed as:

$$\frac{1}{\delta_I} = \left( \frac{1}{\delta_{10}} - \frac{1}{\delta_f} \right) \exp[-r_2 t] + \frac{1}{\delta_f},$$

where the initial diagnose rate is given by $\delta_{10}$, exponential decreasing rate by $r_2$, and the fastest diagnose by $\delta_f$. $\delta_I$ is extremely dependent on government efforts and available resources.

It is necessary to analyze mathematical models for infectious diseases for better apprehension of their evaluation like existence stability and control.22,23 Since the classical mathematical models do not ascertain the advanced degree of accuracy and efficiency. Fractional differential equations were originated to better handle these problems, having necessary applications towards, robotics, optimization problems, cosmology, medical diagnosis, and artificial intelligence. For this reason, fractional differential equations have been used over the past several years in mathematical modeling of a biological phenomenon (see, e.g., Baleanu et al24 and Lia et al25 and references therein). This is because fractional calculus can better explain and process the retention and heritage properties of numerous problems more efficiently than integer-order models.

The fractional calculus has significance over classical calculus, which extends the idea of integer-order integrals and derivatives to any real and positive order. The statement is translated as fractional-order derivatives are definite integrals providing accumulations of functions. In the overall accumulations, integer-order is a special case. Such quality of the fractional calculus helps the authors to examine the global dynamics of problems rather than local dynamics. Furthermore, the hereditary and memory phenomenon of real-world problems can be well explained by fractional-order. Recently, some researchers have considered the fractional mathematical models of COVID-19 and have produced extraordinary results, see, for example, other studies.26-30
These fascinating characteristics of fractional-order motivate us to study the model (1.1) under Caputo fractional derivative of order $\theta$ as:

\[
\begin{align*}
C D_\theta^\theta [S(t)] &= -(\beta c + cq(1 - \beta))S(I + \theta A) + \lambda S_q, \\
C D_\theta^\theta [E(t)] &= -\beta c(1 - q)S(I + \theta A) + \sigma E, \\
C D_\theta^\theta [I(t)] &= \sigma \theta E - (\delta_I + \alpha + \gamma_I)I, \\
C D_\theta^\theta [A(t)] &= \sigma (1 - q)E - \gamma_A A, \\
C D_\theta^\theta [S_q(t)] &= (1 - \beta) cqS(I + \theta A) - \lambda S_q, \\
C D_\theta^\theta [E_q(t)] &= \beta cqS(1 + \theta A) - \delta_q E_q, \\
C D_\theta^\theta [H(t)] &= \delta_I + \delta_q E_q - (\alpha + \gamma_H)H, \\
C D_\theta^\theta [H(t)] &= \gamma_I + \gamma_A A + \gamma_H H.
\end{align*}
\]

(1.2)

Along with the initial conditions:

\[
S(0) = S_0; E(0) = E_0; I(0) = I_0; A(0) = A_0; S_q(0) = S_{q0}; E_q(0) = E_{q0}; H(0) = H_0; R(0) = R_0.
\]

As it can be seen that for $\theta = 1$, the system (1.2) reduces to the system (1.1). For the solution to the concern problem, we first check the existence and uniqueness of the solution. Then, we apply the Taylor’s method for numerical simulation which is a powerful tool for numerical simulations. Thus, we, first, simulate the results with the data given in Yousefpur et al.\textsuperscript{20} Then, we compare simulated data at different fractional order.

### 2 | PRELIMINARIES

**Definition 2.1** (Podlubny\textsuperscript{31}). Let $\Delta$ be a continuous function on $L^1([0, T], \mathbb{R})$, a fractional integral in Riemann–Liouville sense is defined as follows:

\[
I^\theta \Delta(t) = \frac{1}{\Gamma(\theta)} \int_0^t (t - v)^{\theta - 1} \Delta(v)dv,
\]

(2.1)

where $\theta \in (0, 1)$.

**Definition 2.2** (Kilbas et al\textsuperscript{32}). Let $\Delta$ be a continuous function on $[0, T]$. The Caputo fractional derivative can be expressed as follows:

\[
D^\alpha \Delta(t) = \frac{1}{\Gamma(n - \alpha)} \left[ \int_0^t (t - v)^{n-\alpha-1} \frac{d^n}{dv^n} \Delta(t)(v)dv \right].
\]

where $n = [\beta] + 1$ and $[\beta]$ is the integer part of $\beta$.

### 3 | EXISTENCE THEORY

In this section, we discuss the existence and uniqueness of the solution to the fractional model (1.2). We use Banach and Schäuder fixed point theorems to obtain the required results. Considering the fractional model (1.2) and applying integral on both sides with the given initial conditions, we get:
\[
\begin{align*}
S(t) &= S_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t - \nu)^{\beta-1} \Psi_1(\nu, S, E, I, A, S_q, E_q, H, R) d\nu, \\
E(t) &= E_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t - \nu)^{\beta-1} \Psi_2(\nu, S, E, I, A, S_q, E_q, H, R) d\nu, \\
I(t) &= I_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t - \nu)^{\beta-1} \Psi_3(\nu, S, E, I, A, S_q, E_q, H, R) d\nu, \\
A(t) &= A_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t - \nu)^{\beta-1} \Psi_4(\nu, S, E, I, A, S_q, E_q, H, R) d\nu, \\
S_q(t) &= S_{q_0} + \frac{1}{\Gamma(\beta)} \int_0^t (t - \nu)^{\beta-1} \Psi_5(\nu, S, E, I, A, S_q, E_q, H, R) d\nu, \\
E_q(t) &= E_{q_0} + \frac{1}{\Gamma(\beta)} \int_0^t (t - \nu)^{\beta-1} \Psi_6(\nu, S, E, I, A, S_q, E_q, H, R) d\nu, \\
H(t) &= H_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t - \nu)^{\beta-1} \Psi_7(\nu, S, E, I, A, S_q, E_q, H, R) d\nu, \\
R(t) &= R_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t - \nu)^{\beta-1} \Psi_8(\nu, S, E, I, A, S_q, E_q, H, R) d\nu,
\end{align*}
\]

where the functions under the integral in the system (3.1), are defined as follows:

\[
\begin{align*}
\Psi_1(t, S, E, I, A, S_q, E_q, H, R) &= -(\beta c + cq(1 - \beta))S(I + \theta A) + \lambda S_q, \\
\Psi_2(t, S, E, I, A, S_q, E_q, H, R) &= -\beta c(1 - q)S(I + \theta A) + \sigma E, \\
\Psi_3(t, S, E, I, A, S_q, E_q, H, R) &= \sigma E - (\delta_I + \alpha + \gamma_I)I, \\
\Psi_4(t, S, E, I, A, S_q, E_q, H, R) &= \sigma(1 - \theta)E - \gamma_A A, \\
\Psi_5(t, S, E, I, A, S_q, E_q, H, R) &= (1 - \beta)S I q S(I + \theta A) - \lambda S_q, \\
\Psi_6(t, S, E, I, A, S_q, E_q, H, R) &= \beta c q S(1 + \theta A) - \delta_q E_q, \\
\Psi_7(t, S, E, I, A, S_q, E_q, H, R) &= \delta_I + \delta_q E_q - (\alpha + \gamma_H)H, \\
\Psi_8(t, S, E, I, A, S_q, E_q, H, R) &= \gamma_I + \gamma_A A + \gamma_H H.
\end{align*}
\]

Moreover, consider \([0, T] \) be the closed and bounded set. Suppose \(\Theta = \Theta_1 \times \Theta_2\) is a Banach space. Then, surely, the product \(\Theta\) is also a Banach space under the norm:

\[
\|\begin{align*}
S(t), & S(t), \\
E(t), & E(t), \\
I(t), & I(t), \\
A(t), & A(t), \\
S_q(t), & S_q(t), \\
E_q(t), & E_q(t), \\
H(t), & H(t), \\
R(t), & R(t)
\end{align*}\| = \sup_{t \in [0, T]} |S(t)| + \sup_{t \in [0, T]} |E(t)| + \sup_{t \in [0, T]} |I(t)| + \sup_{t \in [0, T]} |A(t)| + \\
\sup_{t \in [0, T]} |S_q(t)| + \sup_{t \in [0, T]} |E_q(t)| + \sup_{t \in [0, T]} |H(t)| + \sup_{t \in [0, T]} |R(t)|.
\]

Now, we express the system (3.1) as follows:

\[
\mathcal{Y}(t) = \mathcal{Y}_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t - \nu)^{\beta-1} \mathcal{B}(\nu, \mathcal{Y}(\nu)) d\nu,
\]

where

\[
\mathcal{Y}(t) = \begin{pmatrix}
S(t) \\
E(t) \\
I(t) \\
A(t) \\
S_q(t) \\
E_q(t) \\
H(t) \\
R(t)
\end{pmatrix}, \quad \mathcal{Y}_0 = \begin{pmatrix}
S_0(t) \\
E_0(t) \\
I_0(t) \\
A_0(t) \\
S_{q_0}(t) \\
E_{q_0}(t) \\
H_0(t) \\
R_0(t)
\end{pmatrix}, \quad \mathcal{B}(\nu, \mathcal{Y}(\nu)) = \begin{pmatrix}
\Psi_1(t, S, E, I, A, S_q, E_q, H, R) \\
\Psi_2(t, S, E, I, A, S_q, E_q, H, R) \\
\Psi_3(t, S, E, I, A, S_q, E_q, H, R) \\
\Psi_4(t, S, E, I, A, S_q, E_q, H, R) \\
\Psi_5(t, S, E, I, A, S_q, E_q, H, R) \\
\Psi_6(t, S, E, I, A, S_q, E_q, H, R) \\
\Psi_7(t, S, E, I, A, S_q, E_q, H, R) \\
\Psi_8(t, S, E, I, A, S_q, E_q, H, R)
\end{pmatrix}.
\]

For the existence and uniqueness results, consider the growth condition on function vector \(\mathcal{B} : [0, T] \times \mathbb{R}_+^8 \to \mathbb{R}_+\) as follows:
\textbf{C1} There exists a constant \( C_1 > 0 \) for each \( Y(t), \overline{Y}(t) \in \mathbb{R}^8 \), such that
\[ |\mathfrak{B}(Y(t)) - \mathfrak{B}(\overline{Y}(t))| \leq C_1|Y(t) - \overline{Y}(t)|. \]

\textbf{C2} There exist constants \( C_2 \) and \( \mathfrak{R} \), such that
\[ |\mathfrak{B}(Y(t))| \leq C_2|Y| + \mathfrak{R}. \]

Now with the help of Schäuder fixed point theorem, we prove the following result.

**Theorem 3.1.** Using the continuity of \( \mathfrak{B} \) along with \( \textbf{C2} \), the fractional model (1.2) has at least one solution.

**Proof.** Suppose \( \Xi \) be a closed subset of \( \Theta \) defined as follows:
\[ \Xi = \{ Y \in \Theta : \| Y \| \leq \mathfrak{R}, \mathfrak{R} > 0 \}. \]

Further, consider the operator \( \mathfrak{X} : \Xi \to \Xi \) defined as follows:
\[ \mathfrak{X}(Y)(t) = Y_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t - v)^{\beta-1} \mathfrak{B}(v, Y(v))dv. \] (3.4)

So, for each \( Y \in \Xi \), we have
\[ |\mathfrak{X}(Y)(t)| \leq |Y_0| + \int_0^t (t - v)^{\beta-1} |\mathfrak{B}(v, Y)| dv, \]
\[ \leq |Y_0| + \int_0^t (t - v)^{\beta-1} [C_2|Y| + \mathfrak{R}] dv, \]
\[ \leq |Y_0| + \frac{\mathfrak{R}}{\Gamma(\beta + 1)} [C_2 \| Y \| + \mathfrak{R}]. \]

which further implies
\[ \| \mathfrak{X}(Y) \| \leq |Y_0| + \frac{\mathfrak{R}}{\Gamma(\beta + 1)} [C_2 \| Y \| + \mathfrak{R}] \leq \mathfrak{R}. \] (3.5)

From (3.5), it can be deduced that \( Y \in \Xi \). Therefore, \( \mathfrak{X}(\Xi) \subset \Xi \). From the given analysis, it can be concluded that the operator \( \mathfrak{X} \) is bounded.

For complete continuity, we proceed with the following result as follows:

Suppose \( t_1 < t_2 \in [0, T] \), consider
\[ |\mathfrak{X}(Y)(t_2) - \mathfrak{X}(Y)(t_1)| = \left| \frac{1}{\Gamma(\beta)} \int_0^{t_1} (t_2 - v)^{\beta-1} \mathfrak{B}(v, Y(v)) dv - \frac{1}{\Gamma(\beta)} \int_0^{t_1} (t_1 - v)^{\beta-1} \mathfrak{B}(v, Y(v)) dv \right|, \]
\[ \leq \frac{1}{\Gamma(\beta)} \left| \int_0^{t_1} [(t_1 - v)^{\beta-1} (t_2 - v)^{\beta-1}] \mathfrak{B}(v, Y(v)) dv + \int_0^{t_2} (t_2 - v)^{\beta-1} \mathfrak{B}(v, Y(v)) dv \right|, \]
\[ \leq \frac{(C_2 + \mathfrak{R} + \mathfrak{R})}{\Gamma(\beta + 1)} \left| t_2^{\beta} - t_1^{\beta} + 2(t_2 - t_1) \right|. \] (3.6)

From the inequality (3.6), it can be analyzed that if \( t_2 \to t_1 \). Then, the right side of the inequality (3.6) tends to zero. Subsequently,
\[ \| \mathfrak{X}(Y)(t_2) - \mathfrak{X}(Y)(t_1) \| \to 0, \text{ as } t_2 \to t_1. \]

Therefore, \( \mathfrak{X} \) is an equi-continuous operator. Applying the Arzelá–Ascoli theorem, the operator \( \mathfrak{X} \) is completely continuous. Consequently, by Schäuder fixed point theorem, the fractional model (1.2) has a solution. \( \square \)

For the uniqueness of the solution to the model (1.2), we obtained the following theorem.

**Theorem 3.2.** When using \( \mathbf{E1} \), the considered model has a unique solution if \( \frac{T^{\beta}}{\Gamma(\beta + 1)} C_2 < 1. \)
Proof. Consider the operator $\mathfrak{X}: \Theta \to \Theta$, defined by (3.4). Proceeding, we take $\Upsilon, \Upsilon \in \Theta$ and consider

$$
\lVert \mathfrak{X}(\Upsilon) - \mathfrak{X}(\Upsilon) \rVert = \sup_{t \in [0, T]} \left| \frac{1}{\Gamma(\vartheta)} \int_0^t (t - \nu)^{\vartheta - 1} \mathfrak{B}(\nu, \Upsilon(\nu))d\nu - \frac{1}{\Gamma(\vartheta)} \int_0^t (t - \nu)^{\vartheta - 1} \mathfrak{B}(\nu, \Upsilon(\nu))d\nu \right|,
$$

which implies

$$
\lVert \mathfrak{X}(\Upsilon) - \mathfrak{X}(\Upsilon) \rVert \leq \frac{T^\vartheta}{\Gamma(\vartheta + 1)} \mathfrak{Q}_\mathfrak{B} \lVert \Upsilon - \Upsilon \rVert, \tag{3.7}
$$

From the inequality (3.7), it can be inferred that $\mathfrak{X}$ is a contraction. So by Banach theorem, the fractional model (1.2) has a unique solution.

4 | STABILITY ANALYSIS

Now, we check the stability analysis of the model (1.2), we proceed by considering a small change in $\Lambda \in [0, T]$, as $\Lambda(0) = 0$, depending only on the solution $\Upsilon$ as:

1. $|\Lambda(t)| \leq \varepsilon$, for any $\varepsilon > 0$,
2. $^c D_0^\vartheta \Upsilon(t) = \mathfrak{B}(t, \Upsilon(t)) + \Lambda(t)$.

**Lemma 4.1.** The solution of the above changed problem

$$
^c D_0^\vartheta \Upsilon(t) = \mathfrak{B}(t, \Upsilon(t)) + \Lambda(t),
$$

with respect to the initial condition $\Upsilon(0) = \Upsilon_0$ satisfies the following condition

$$
\lVert \Upsilon(t) - \left( \Upsilon_0(t) + \frac{1}{\Gamma(\vartheta)} \int_0^t (t - \nu)^{\vartheta - 1} \mathfrak{B}(\nu, \Upsilon(\nu))d\nu \right) \rVert \leq \frac{T^\vartheta}{\Gamma(\vartheta + 1)} \varepsilon = \Pi_{T, \vartheta} \varepsilon. \tag{4.1}
$$

Proof. The proof of the the Lemma 4.1 is straightforward just like the above proofs. So we omit proving the lemma.

**Theorem 4.2.** Along with the condition $C2$ together with (4.1), the solution of the integral (3.3) is Ulam–Hyers stable. Therefore, the numerical results of the considered fractional model (1.2) are Ulam-Hyers stable if the following condition holds:

$$
\mathfrak{Q} = \frac{T^\vartheta}{\Gamma(\vartheta + 1)} \mathfrak{Q}_\mathfrak{B} < 1 \tag{4.2}
$$

Proof. Consider $\Upsilon \in \Theta$, be any solution and $\overline{\Upsilon}$ be at most one solution of (3.3). Then,

$$
\lvert \Upsilon(t) - \overline{\Upsilon}(t) \rvert = \lvert \Upsilon(t) - \left( \Upsilon_0(t) + \frac{1}{\Gamma(\vartheta)} \int_0^t (t - \nu)^{\vartheta - 1} \mathfrak{B}(\nu, \overline{\Upsilon}(\nu))d\nu \right) \rvert,
$$

$$
\leq \lvert \Upsilon(t) - \left( \Upsilon_0(t) + \frac{1}{\Gamma(\vartheta)} \int_0^t (t - \nu)^{\vartheta - 1} \mathfrak{B}(\nu, \overline{\Upsilon}(\nu))d\nu \right) \rvert \tag{4.3}
$$

$$
+ \left| \frac{1}{\Gamma(\vartheta)} \int_0^t (t - \nu)^{\vartheta - 1} \mathfrak{B}(\nu, \overline{\Upsilon}(\nu))d\nu - \frac{1}{\Gamma(\vartheta)} \int_0^t (t - \nu)^{\vartheta - 1} \mathfrak{B}(\nu, \Upsilon(\nu))d\nu \right|.
$$

Now by using (4.2) and sup norm on both sides of the inequality (4.3), and rearranging the terms, we get

$$
\lVert \Upsilon - \overline{\Upsilon} \rVert \leq \frac{\Pi_{T, \vartheta}}{1 - \mathfrak{Q}}. \tag{4.4}
$$
From the inequality (4.4), we claim that the solution of the integral (3.3) is Ulam–Hyers stable. Consequently, the solution of the considered fractional model (1.2) is also Ulam–Hyers stable.

5 | NUMERICAL PROCEDURE

As discussed in Section 3, the solution of the fractional model (1.2) exists and is unique. Further, in Section 4, we have analyzed that the solution of the fractional model (1.2) is Ulam–Hyers stable. Now is the time to find the numerical solution of the model (1.2). For such achievement, Taylor’s theorem will be used on the Caputo derivative of order $\theta$. Therefore, we proceed with the first equation of the model as follows:

$$\begin{align*}
C D_t^\theta [S(t)] &= \Psi_1(t, S, E, I, A, S_q, E_q, H, R), \\
S(0) &= S_0, \ t > 0.
\end{align*}\tag{5.1}$$

Consider $[0, P]$ be the set of points on which we are willing to approximate the solution of the system (5.1). Actually, we cannot evaluate the function $S(t)$ which will be the required solution to the system (5.1). On the contrary, we generate a set of points $\{t_r, t_{r+1}\}$, for the given iterative process. So we divide the interval $[0, P]$, into $k$ subintervals $[t_r, t_{r+1}]$ of length, that is, $h = \frac{P}{k}$, by using the nodes $t_r = rh$, for $r = 0, 1, 2, \ldots, k$. Additionally, we assume that $S(t)$, $C D_t^\theta [S(t)]$, and $C D_t^{2\theta} [S(t)]$ are continuous on the interval $[0, T]$. Expanding the Taylor’s theorem at about $t = t_0$, so there is a constant $j \in [0, P]$, such that

$$S(t) = S(t_0) + C D_t^\theta [S(t)] \frac{h^\theta}{\Gamma(\theta + 1)} + C D_t^{2\theta} [S(t)] \frac{h^{2\theta}}{\Gamma(2\theta + 1)}. \tag{5.2}$$

Now substitute $C D_t^\theta [S(t)](t_0) = \Psi_1(t_0, S(t_0), E(t_0), I(t_0), A(t_0), S_q(t_0), E_q(t_0), H(t_0), R(t_0))$, and $t = t_1$ in (5.2), which implies

$$S(t_1) = S(t_0) + \Psi_1(t_0, S(t_0), E(t_0), I(t_0), A(t_0), S_q(t_0), E_q(t_0), H(t_0), R(t_0)) \frac{h^\theta}{\Gamma(\theta + 1)} +$$

$$C D_t^{2\theta} [S(t)] \frac{h^{2\theta}}{\Gamma(2\theta + 1)}. \tag{5.3}$$

If the step size, $h$ is chosen small enough. Then, we neglect the higher terms, that is, that of the second-order so, (5.3), implies

$$S(t_1) = S(t_0) + \Psi_1(t_0, S(t_0), E(t_0), I(t_0), A(t_0), S_q(t_0), E_q(t_0), H(t_0), R(t_0)) \frac{h^\theta}{\Gamma(\theta + 1)}. \tag{5.4}$$

Repeating the same fashion, a sequence of points is obtained that approximates the solution. A general formula of expanding about $t_r = t_r + h$, is

$$S(t_{r+1}) = S(t_r) + \Psi_1(t_r, S(t_r), E(t_r), I(t_r), A(t_r), S_q(t_r), E_q(t_r), H(t_r), R(t_r)) \frac{h^\theta}{\Gamma(\theta + 1)}. \tag{5.5}$$

Using the similar technique, we can obtain the same numerical scheme for the remaining compartments of the fractional model (1.2) as follows:

$$E(t_{r+1}) = E(t_r) + \Psi_1(t_r, S(t_r), E(t_r), I(t_r), A(t_r), S_q(t_r), E_q(t_r), H(t_r), R(t_r)) \frac{h^\theta}{\Gamma(\theta + 1)}, \tag{5.6}$$

$$I(t_{r+1}) = I(t_r) + \Psi_1(t_r, S(t_r), E(t_r), I(t_r), A(t_r), S_q(t_r), E_q(t_r), H(t_r), R(t_r)) \frac{h^\theta}{\Gamma(\theta + 1)}, \tag{5.7}$$

$$A(t_{r+1}) = A(t_r) + \Psi_1(t_r, S(t_r), E(t_r), I(t_r), A(t_r), S_q(t_r), E_q(t_r), H(t_r), R(t_r)) \frac{h^\theta}{\Gamma(\theta + 1)}, \tag{5.8}$$

$$S_q(t_{r+1}) = S_q(t_r) + \Psi_1(t_r, S(t_r), E(t_r), I(t_r), A(t_r), S_q(t_r), E_q(t_r), H(t_r), R(t_r)) \frac{h^\theta}{\Gamma(\theta + 1)}, \tag{5.9}$$

$$E_q(t_{r+1}) = E_q(t_r) + \Psi_1(t_r, S(t_r), E(t_r), I(t_r), A(t_r), S_q(t_r), E_q(t_r), H(t_r), R(t_r)) \frac{h^\theta}{\Gamma(\theta + 1)}. \tag{5.10}$$
\[ H(t_{r+1}) = H(t_r) + \Psi_1(t_r, S(t_r), E(t_r), I(t_r), A(t_r), S_q(t_r), E_q(t_r), H(t_r), R(t_r)) \frac{h^\theta}{\Gamma(\theta + 1)}, \]  
\[ R(t_{r+1}) = R(t_r) + \Psi_1(t_r, S(t_r), E(t_r), I(t_r), A(t_r), S_q(t_r), E_q(t_r), H(t_r), R(t_r)) \frac{h^\theta}{\Gamma(\theta + 1)}. \]  

6 | GRAPHICAL VISUALIZATION AND ANALYSIS

In the current section, we have considered the fractional model (1.2) using the parameters, defined in Table 1, and the initial values are defined in Table 2. The total population of Pakistan is nearly 221.97217 millions.

On using the estimated and available data given in Tables 1 and 2 to present the approximate solutions corresponding to different fractional-order of the model (1.2) for 50 days via the proposed numerical scheme.

We can observe the dynamical behaviors of the model (1.2) at various fractional-order in the figures below. Figure 1 shows the decline of the susceptible population in Pakistan. As currently, the people have not taken the situation seriously. Therefore, the infection again has started to increase so, the susceptibility decreasing at a faster rate at lower fractional order. As the fractional order is enlarging, the decay becomes slow.

From Figure 2, we can observe that, as the decline occurs in susceptibility as a consequence the population is exposing to the infection. The concerned growth in exposed class is different at various fractional order. The growth rate at small fractional order is slow as the order increase the corresponding process becomes faster.

From Figure 3, an increase in the exposed class yields an increase in the infection for the coming 50 days in Pakistan if proper protection is not taken by the people. The dynamical behavior is like the behavior of exposed class upon various fractional order.

On the other hand, the asymptomatic class will rise quite rapidly, as shown in Figure 4. The corresponding dynamical behavior of the mentioned class is also raising slightly at a higher rate on greater fractional-order while using smaller fractional-order, the growth will be slow.

It can be concluded that the quarantined susceptible population will be decreasing rapidly for almost 50 days. The dynamics of the mentioned class at different fractional-order is shown in Figure 5.

Contrast to Figure 5, the density of quarantined exposed people will increase in the next 50 days. The concerned dynamical behavior of the quarantined exposed class is shown in Figure 6.

### Table 1

| Parameters | Estimated mean value | Source       |
|------------|----------------------|--------------|
| \( \beta \) | \( 3.11 \times 10^{-5} \) | Assumed      |
| \( \sigma \) | \( 6.2 \times 10^{-4} \) | Assumed      |
| \( \lambda \) | \( 1.03 \times 10^{-5} \) | Assumed      |
| \( \rho \) | \( 6.217 \times 10^{-2} \) | Assumed      |
| \( \delta_q \) | \( 4.1 \times 10^{-2} \) | Assumed      |
| \( \delta_1 \) | \( 5.508 \times 10^{-2} \) | Assumed      |
| \( \gamma_I \) | \( 0.12365 \) | Assumed      |
| \( \gamma_A \) | \( 0.15621 \) | Assumed      |
| \( \gamma_H \) | \( 0.13899 \) | Assumed      |
| \( \alpha \) | \( 0.02 \) | Worldometer\(^{33}\) |
| \( \theta \) | \( 0.01 \) | Assumed      |
| \( c \) | \( 3.01 \times 10^{-4} \) | Assumed      |
| \( q \) | \( 0.716666 \) | Assumed      |

### Table 2

| Initial values | Values | Source                        |
|----------------|--------|-------------------------------|
| \( S(0) \)    | 220 millions | Worldometer\(^{33}\)        |
| \( S_q(0) \)  | 0.455085 millions | Assumed                  |
| \( E(0) \)    | 0 millions | Assumed                  |
| \( E_q(0) \)  | 0 millions | Assumed                  |
| \( I(0) \)    | 0.475085 millions | Worldometer\(^{33}\)        |
| \( H(0) \)    | 0.435085 millions | Worldometer\(^{33}\)        |
| \( A(0) \)    | 0 millions | Assumed                  |
| \( R(0) \)    | 0.424494 millions | Worldometer\(^{33}\)        |
In the next 50 days, the density of hospitalized people again will rise as shown in Figure 7. This is due to the ignorance of the public who do not follow the preventive measures against COVID-19. The aforesaid class will be highly increasing on larger fractional-order as compared with small fractional order.
The recovery rate is also in the progress. In Pakistan, the death rate due to corona is 0.02%. So the recovered class is also rising due to the increase in death as well as recovery of people from infection. The growth in recovered class is different at different fractional-order as shown in Figure 8. It is faster at lower fractional-order as compared to larger fractional-order.
From all of the above Figures 1–8, using the fractional derivative, one can observe different dynamical behavior of the population dynamics, where the decline is faster at lower fractional-order and slower as the order increases. On the other hand, if the fractional-order is larger then the growth rate will be high as compared to the lower fractional order. Hence, the fractional calculus approach provides a better understanding of global dynamics of real-world phenomenon and process.

7 Conclusion

In this study, we have investigated the fractional-order dynamical model for COVID-19. We have used nonlinear analysis to show the existence and uniqueness of the model (1.2). Further, numerical analysis has been used to obtain an approximate solution for the proposed model through the fractional Taylor’s method. In this study, we have observed that the fractional differential equations provide global dynamics of the aforesaid model. Additionally, from the study, we have also analyzed that at smaller fractional-order the decay is faster while growth is slower. As the order increases, the decay slows down while the growth gets faster. The second wave of spreading of COVID-19 has been predicted in this model for the next 50 days. This study is just an indication that to observe the transmission dynamics of COVID-19 in Pakistan.

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AUTHOR CONTRIBUTIONS

All authors have read and agreed to the published version of the manuscript.

CONFLICTS OF INTEREST

The authors declare no conflict of interest.

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