Computing Optimal Persistent Cycles for Levelset Zigzag on Manifold-like Complexes*

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Abstract

In standard persistent homology, a persistent cycle born and dying with a persistence interval (bar) associates the bar with a concrete topological representative, which provides means to effectively navigate back from the barcode to the topological space. Among the possibly many, optimal persistent cycles bring forth further information due to having guaranteed quality. However, topological features usually go through variations in the lifecycle of a bar which a single persistent cycle may not capture. Hence, for persistent homology induced from PL functions, we propose levelset persistent cycles consisting of a sequence of cycles that depict the evolution of homological features from birth to death. Our definition is based on levelset zigzag persistence which involves four types of persistence intervals as opposed to the two types in standard persistence. For each of the four types, we present a polynomial-time algorithm computing an optimal sequence of levelset persistent $p$-cycles for the so-called weak $(p + 1)$-pseudomanifolds. Given that optimal cycle problems for homology are NP-hard in general, our results are useful in practice because weak pseudomanifolds do appear in applications. Our algorithms draw upon an idea of relating optimal cycles to min-cuts in a graph that we exploited earlier for standard persistent cycles. Note that levelset zigzag poses non-trivial challenges for the approach because a sequence of optimal cycles instead of a single one needs to be computed in this case.

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1 Introduction

Given a filtered topological space, persistent homology [16] produces a stable [7] topological signature called barcode (or persistence diagram) which has proven useful in many applications. Though being widely adopted, a persistence interval in a barcode only indicates that a certain topological feature gets born and dies with the interval but does not provide a canonical and concrete representative of the feature. In view of this, persistent cycles [9, 11] were proposed as concrete representatives for standard (i.e., non-zigzag) persistent homology, which also enables one to navigate back to the topological space from a barcode. Among the many, optimal persistent cycles (or ones with a quality measure) [11, 12, 19, 21] are of special interest due to having guaranteed quality. However, one drawback of standard persistent cycles is that only a single cycle born at the start is used, while homological features may vary continuously inside an interval. For example, in Figure 1, let the growing space be the sub-levelset filtration of a function $f$, in which $\alpha_1, \ldots, \alpha_4$ are consecutive critical values and $s_0, \ldots, s_3$ are regular values in between. If we consider the changes of homology after each critical point, then a non-trivial 1-cycle $z_1$ is first born in $f^{-1}(-\infty, \alpha_1]$ and splits into two in $f^{-1}(-\infty, s_2]$. The two separate cycles eventually shrink and die independently, generating a (standard) persistence interval $[\alpha_1, \alpha_4)$. Using standard persistent cycles, only $z_1$ would be picked as a representative for $[\alpha_1, \alpha_4)$, which fails to depict the subsequent behaviors.

In this paper, we propose alternative persistent cycles capturing the dynamic behavior shown in Figure 1. We focus on a special but important type of persistent homology – those generated by piecewise linear (PL) functions [15]. We also base our definition on an extension of standard persistence called the levelset zigzag persistence [4], which tracks the survival of homological features at and in between the critical points. Given a persistence interval from levelset zigzag, we define a sequence of cycles called levelset persistent cycles so that there is a cycle between each consecutive critical points within the interval. For example, in Figure 1, $[\alpha_1, \alpha_4)$ is also a persistence interval (i.e., a closed-open interval) in the levelset zigzag of $f$. The cycles $z_1, z_2, z_3, z_4$ forming a sequence of levelset persistent 1-cycles for $[\alpha_1, \alpha_4)$ capture all the variations across the critical points. Section 3 details the definition.

Levelset zigzag on a PL function relates to the standard sub-levelset version in the following way: finite intervals from the standard version on the original function and its negation produce closed-open and open-closed intervals in levelset zigzag, while levelset zigzag additionally provides closed-closed and open-open intervals [4]. Thus, levelset persistent cycles are oriented toward richer types of intervals (see also extended persistence [8]).

![Figure 1: Evolution of a homological feature across different critical points.](image-url)
Computationally, optimal cycle problems for homology in both persistence and non-persistence settings are NP-hard in general [5, 6, 11, 12]. Other than the optimal homology basis algorithms in dimension one [2, 13, 14], to our knowledge, all polynomial-time algorithms for such problems aim at manifold-like complexes [1, 5, 6, 12, 17]. In particular, the existing algorithms for general dimensions [6, 12] exploit the dual graph structure of given complexes and reduce the optimal cycle problem in codimension one to a minimum cut problem. In this paper, we find a way of applying this technique to computing an optimal sequence of levelset persistent cycles – one that has the minimum sum of weight. Our approach which also works for general dimensions differs from previous ones to account for the fact that a sequence of optimal cycles instead of a single one need to be computed. We assume the input to be a generalization of \((p+1)\)-manifold called weak \((p+1)\)-pseudomanifold [12]:

**Definition 1.** A weak \((p+1)\)-pseudomanifold is a simplicial complex in which each \(p\)-simplex has no more than two \((p+1)\)-cofaces.

Given an arbitrary PL function on a weak \((p+1)\)-pseudomanifold \((p \geq 1)\), we show that an optimal sequence of levelset persistent \(p\)-cycles can be computed in polynomial time for any type of levelset zigzag interval of dimension \(p\). This is in contrast to the standard persistence setting, where computing optimal persistent \(p\)-cycles for one type of intervals (the infinite intervals) is NP-hard even for weak \((p + 1)\)-pseudo-manifolds [12]. Note that among the four mentioned types of intervals in levelset zigzag, closed-open and open-closed intervals are symmetric so that everything concerning open-closed intervals can be derived directly from the closed-open case. Hence, for these two types of intervals, we address everything only for the closed-open case.

We propose three algorithms for the three types of intervals by utilizing minimum \((s,t)\)-cuts on the dual graphs. Specifically, levelset persistent \(p\)-cycles for an open-open interval have direct correspondence to \((s,t)\)-cuts on a dual graph, and so the optimal ones can be computed directly from the minimum \((s,t)\)-cut. For the remaining cases, the crux is to deal with monkey saddles and the computation spans two phases. The first phase computes minimum \(p\)-cycles in certain components of the complex; then, using minimum cuts, the second phase determines the optimal combination of the components by introducing some augmenting edges. All three algorithms run in \(O(n^2)\) time dominated by the complexity of the minimum cut computation, for which we use Orlin’s max-flow algorithm [20]. Section 4 details the computation.

2 Preliminaries

**Simplicial homology.** We only briefly review simplicial homology here; see [15] for a detailed treatment. Let \(K\) be a simplicial complex. Since coefficients for homology are in \(\mathbb{Z}_2\) in this paper, a \(p\)-chain \(A\) of \(K\) is a set of \(p\)-simplices of \(K\) and can also be expressed as the formal sum \(\sum_{\sigma \in A} \sigma\); these two forms of \(p\)-chains are used interchangeably. The sum of two \(p\)-chains is the symmetric difference of sets and is denoted as both “+” and “−” because plus and minus are the same in \(\mathbb{Z}_2\). A \(p\)-cycle is a \(p\)-chain in which any \((p−1)\)-face adjoins even number of \(p\)-simplices; a \(p\)-boundary is a \(p\)-cycle being the boundary of a \((p + 1)\)-chain. Two \(p\)-cycles \(\zeta, \zeta’\) are homologous, denoted \(\zeta \sim \zeta’\), if their sum is a \(p\)-boundary. The set of all \(p\)-cycles homologous to a fixed \(p\)-cycle \(\zeta \subseteq K\) forms a homology class \([\zeta]\), and all these homology classes form the \(p\)-th homology group \(H_p(K)\) of \(K\). Note that \(H_p(K)\) is a vector space over \(\mathbb{Z}_2\).

**Zigzag modules, barcodes, and filtrations.** A zigzag module [3] (or module for short) is a sequence of vector spaces

\[ \mathcal{M} : V_0 \leftrightarrow V_1 \leftrightarrow \cdots \leftrightarrow V_m \]
in which each $V_i \leftrightarrow V_{i+1}$ is a linear map and is either forward, i.e., $V_i \rightarrow V_{i+1}$, or backward, i.e., $V_i \leftarrow V_{i+1}$. In this paper, vector spaces are taken over $\mathbb{Z}_2$. A module $S : W_0 \leftrightarrow W_1 \leftrightarrow \cdots \leftrightarrow W_m$ is called a submodule of $\mathcal{M}$ if each $W_i$ is a subspace of $V_i$ and each map $W_i \leftrightarrow W_{i+1}$ is the restriction of $V_i \leftrightarrow V_{i+1}$. For an interval $[b, d] \subseteq [0, m]$, $S$ is called an interval submodule of $\mathcal{M}$ over $[b, d]$ if $W_i$ is one-dimensional for $i \in [b, d]$ and is trivial for $i \notin [b, d]$, and $W_i \leftrightarrow W_{i+1}$ is an isomorphism for $i \in [b, d-1]$. By the Krull-Schmidt principle and Gabriel’s theorem [3], $\mathcal{M}$ admits an interval decomposition, $\mathcal{M} = \bigoplus_{k \in \Lambda} T^{[b_k, d_k]}$, in which each $T^{[b_k, d_k]}$ is an interval submodule of $\mathcal{M}$ over $[b_k, d_k]$. We call the (multi-)set of intervals $\{[b_k, d_k] | k \in \Lambda\}$ as the zigzag barcode (or barcode for short) of $\mathcal{M}$, and denote it as $\text{Pd}(\mathcal{M})$. Each interval in a zigzag barcode is called a persistence interval.

A zigzag filtration (or filtration for short) is a sequence of simplicial complexes or general topological spaces

$$\mathcal{X} : X_0 \leftrightarrow X_1 \leftrightarrow \cdots \leftrightarrow X_m$$

in which each $X_i \leftrightarrow X_{i+1}$ is either a forward inclusion $X_i \hookrightarrow X_{i+1}$ or a backward inclusion $X_i \hookleftarrow X_{i+1}$. If not mentioned otherwise, a zigzag filtration is always assumed to be a sequence of simplicial complexes. Applying the $p$-th homology functor with $\mathbb{Z}_2$ coefficients, the $p$-th zigzag module of $\mathcal{X}$ is induced:

$$H_p(\mathcal{X}) : H_p(X_0) \leftrightarrow H_p(X_1) \leftrightarrow \cdots \leftrightarrow H_p(X_m)$$

in which each $H_p(X_i) \leftrightarrow H_p(X_{i+1})$ is the linear map induced by inclusion. The barcode of $H_p(\mathcal{X})$ is also called the $p$-th zigzag barcode of $\mathcal{X}$ and is alternatively denoted as $\text{Pd}_p(\mathcal{X})$, where each interval in $\text{Pd}_p(\mathcal{X})$ is called a $p$-th persistence interval. For an interval $[b, d] \in \text{Pd}_p(\mathcal{X})$, we also conveniently denote the interval as $[X_b, X_d] \in \text{Pd}_p(\mathcal{X})$, i.e., by its starting and ending spaces. This is specially helpful when a filtration is not naturally indexed by consecutive integers, as can be seen in Section 3. In this case, an element $X_i \in [X_b, X_d]$ is just a space in $\mathcal{X}$ with $b \leq i \leq d$.

A special type of filtration called simplex-wise filtration is frequently used in this paper, in which each forward (resp. backward) inclusion is an addition (resp. deletion) of a single simplex. Any $p$-th zigzag module induced by a simplex-wise filtration has the property of being elementary, meaning that all linear maps in the module are of the three forms: (i) an isomorphism; (ii) an injection with rank 1 cokernel; (iii) a surjection with rank 1 kernel. This property is useful for the definitions and computations.

**Graphs and $(s, t)$-cuts.** Given a graph $G = (V(G), E(G))$ and a weight function $w : E(G) \rightarrow [0, \infty]$, a cut $(S, T)$ of $G$ consists of two sets such that $S \cap T = \emptyset$ and $S \cup T = V(G)$. We define $E(G(S, T))$ as the set of all edges of $G$ connecting a vertex in $S$ and a vertex in $T$, in which each edge is said to cross the cut. The weight of the cut is defined as $w(S, T) = \sum_{e \in E(G(S, T))} w(e)$. Let $s$ and $t$ be two disjoint non-empty subsets of $V(G)$; the tuple $(G, s, t)$ is called a weighted $(s, t)$-graph, where $s$ is the set of sources and $t$ is the set of sinks. An $(s, t)$-cut $(S, T)$ of $(G, s, t)$ is a cut of $G$ such that $s \subseteq S$ and $t \subseteq T$. The minimum $(s, t)$-cut of $(G, s, t)$ is an $(s, t)$-cut with the minimum weight.

**Dual graphs for manifolds.** A manifold-like complex (e.g., a weak pseudomanifold) often has an undirected dual graph structure, which is utilized extensively in this paper. Let the complex be $(p+1)$-dimensional. Then, each $(p+1)$-simplex is dual to a vertex and each $p$-simplex is dual to an edge in the dual graph. For a $p$-simplex with two $(p+1)$-cofaces $\tau_1$ and $\tau_2$, its dual edge connects the vertex dual to $\tau_1$ and the vertex dual to $\tau_2$. For a $(p+1)$-simplex of other cases, its dual edge is problem-specific and is explained in the corresponding paragraphs.
Figure 2: A critical value $\alpha_i$ across which the 2nd homology stays the same; $f$ is defined on a 3D domain and $s_{i-1}, s_i$ are two regular values with $s_{i-1} < \alpha_i < s_i$. The levelset $f^{-1}(s_{i-1})$ is a 2-sphere where two antipodal points are getting close and eventually pinch in $f^{-1}(\alpha_i)$. Crossing the critical value, $f^{-1}(s_i)$ becomes a torus.

3 Problem statement

In this section, we develop the definitions for levelset persistent cycles and the optimal ones. Levelset persistent cycles are sometimes simply called persistent cycles for brevity, and this should cause no confusion. We begin the section by defining levelset zigzag persistence in Section 3.1, where we present an alternative version of the classical one proposed by Carlsson et al. [4]. Adopting this alternative version enables us to focus on critical values (and the changes incurred) in a specific dimension. Section 3.1 also defines a simplex-wise levelset filtration, which provides an elementary view of levelset zigzag and is helpful to our definition and computation.

Section 3.2 details the definition of levelset persistent cycles. The cycles in the middle of the sequence are the same for all types of intervals, while the cycles for the endpoints differ according to the types of ends.

Finally, in Section 3.3, we address an issue left over from Section 3.1, which is the validity of the discrete levelset filtration. The validity is found to be relying on the triangulation representing the underlying shape. We also argue that the triangulation has to be fine enough in order to obtain accurate depictions of persistence intervals by levelset persistent cycles.

3.1 $p$-th levelset zigzag persistence

Throughout the section, let $p \geq 1$, $K$ be a finite simplicial complex with underlying space $X = |K|$, and $f : X \to \mathbb{R}$ be a PL function [15] derived by interpolating values on vertices. We consider PL functions that are generic, i.e., having distinct values on the vertices. Note that the function values can be slightly perturbed to satisfy this if they are not initially. An open interval $I \subseteq \mathbb{R}$ is called regular if there exist a topological space $Y$ and a homeomorphism

$$\Phi : Y \times I \to f^{-1}(I)$$

such that $f \circ \Phi$ is the projection onto $I$ and $\Phi$ extends to a continuous function $\overline{\Phi} : Y \times \overline{I} \to f^{-1}(\overline{I})$ with $\overline{I}$ being the closure of $I$ [4]. It is known that $f$ is of Morse type [4], meaning that each levelset $f^{-1}(s)$ has finitely generated homology, and there are finitely many critical values

$$\alpha_0 = -\infty < \alpha_1 < \cdots < \alpha_n < \alpha_{n+1} = \infty$$

such that each interval $(\alpha_i, \alpha_{i+1})$ is regular. Note that critical values of $f$ can only be function values of $K$’s vertices.

As mentioned, levelset persistent cycles for a $p$-th interval should capture the changes of $p$-th homology across different critical values. However, some critical values may cause no change to the $p$-th homology. Figure 2 illustrates such a critical value for $p = 2$ around which only the 1st homology changes and the 2nd homology stays the same. Thus, to capture the most essential variation, the persistent $p$-cycles should stay
The barcode $Pd$ vertices.

Throughout this section, we let

$$\alpha_0 = -\infty < \alpha_1 < \cdots < \alpha_m < \alpha_{m+1} = \infty$$

denote all the $p$-th homologically critical values of $f$, and $\alpha_1^p, \ldots, \alpha_m^p$ denote the corresponding $p$-th critical vertices.

**Definition 2** ($p$-th homologically critical value). A critical value $\alpha_i \neq -\infty, \infty$ of $f$ is called a $p$-th homologically critical value (or $p$-th critical value for short) if one of the two linear maps induced by inclusion is not an isomorphism:

$$\begin{align*}
H_p(f^{-1}(\alpha_{i-1}, \alpha_i)) &\rightarrow H_p(f^{-1}(\alpha_{i-1}, \alpha_{i+1})) \\
H_p(f^{-1}(\alpha_{i-1}, \alpha_{i+1})) &\leftarrow H_p(f^{-1}(\alpha_i, \alpha_{i+1}))
\end{align*}$$

For convenience, we also let $-\infty, \infty$ be $p$-th critical values. Moreover, a vertex $v$ of $K$ is a $p$-th critical vertex if $f(v)$ is a $p$-th critical value.

**Remark 1.** By inspecting the (classical) levelset barcode [4] of $f$ (also see Section 5.1), it can be easily determined whether a critical value is $p$-th critical.

Throughout this section, we let

$$\alpha_0 = -\infty < \alpha_1 < \cdots < \alpha_m < \alpha_{m+1} = \infty$$

denote all the $p$-th homologically critical values of $f$, and $\alpha_1^p, \ldots, \alpha_m^p$ denote the corresponding $p$-th critical vertices.

**Definition 3** ($p$-th levelset zigzag). Denote $f^{-1}(\alpha_i^p, \alpha_j^p)$ as $X_{(i,j)}^p$ for any $i < j$. The continuous version of $p$-th levelset filtration of $f$, denoted $L_c^p(f)$, is defined as

$$L_c^p(f) : X_{(0,1)}^p \leftarrow X_{(0,2)}^p \leftarrow X_{(1,2)}^p \leftarrow X_{(1,3)}^p \leftarrow \cdots \leftarrow X_{(m-1,m+1)}^p \leftarrow X_{(m,m+1)}^p$$

The barcode $Pd_p(L_c^p(f))$ is called the $p$-th levelset barcode of $f$, in which each interval is called a $p$-th levelset persistence interval of $f$.

**Remark 2.** See Figure 3 for an example of $L_c^p(f)$ and its 1st levelset barcode.

In $L_c^p(f)$, $X_{(i,i+1)}^p$ is called a $p$-th regular subspace, and a homological feature in $H_p(X_{(i,i+1)}^p)$ is alive in the entire $p$-th levelset barcode interval $(\alpha_i^p, \alpha_{i+1}^p)$; $X_{(i-1,i,i+1)}^p$ is called a $p$-th critical subspace, and a homological feature in $H_p(X_{(i-1,i,i+1)}^p)$ is alive at the critical value $\alpha_i^p$. Intervals in $Pd_p(L_c^p(f))$ can then be mapped to real-value intervals in which the homological features persist, and are classified into four types based on the open and closeness of the ends; see Table 1. From now on, levelset persistence intervals can be of the two forms shown in Table 1, which we consider as interchangeable. We postpone the justification of Definition 3 to Section 5, where we prove that the $p$-th levelset barcode in Definition 3 is equivalent to the classical one defined in [4].

Figure 3: A torus with the height function $f$ taken over the horizontal line. The 1st levelset barcode is $\{ (\alpha_1^1, \alpha_1^3) \}$. We list the first half of $L_c^1(f)$ but excluding $X_{(0,1)}^1 = \emptyset$; the remaining half is symmetric. An empty dot indicates the point is not included in the space.
We also define the subcomplex $X_L$ respectively. Then, the discrete version.

Since the optimal persistent cycles can only be computed on the discrete domain $u$ 

Note that for each $p$ of $f$ follow the order of the function values:

For the forward inclusion $K^{p}_{i,j+1} \leftrightarrow K^{p}_{i,j+2}$ in $L_p(f)$, let $u_1 = \nu^{p}_{i+1,1}, u_2, \ldots, u_k$ be all the vertices with function values in $[\alpha^{p}_{i+1}, \alpha^{p}_{i+2})$ such that $f(u_1) < f(u_2) < \cdots < f(u_k)$. Then, the lower stars [15] of $u_1, \ldots, u_k$ are added by $F_p(f)$ following the order.

Symmetrically, for the backward inclusion $K^{p}_{i,j+2} \leftrightarrow K^{p}_{i,j+1}$ in $L_p(f)$, let $u_1, u_2, \ldots, u_k = \nu^{p}_{i+1}$ be all the vertices with function values in $[\alpha^{p}_{i}, \alpha^{p}_{i+1}]$ such that $f(u_1) < f(u_2) < \cdots < f(u_k)$. Then, the upper stars of $u_1, \ldots, u_k$ are deleted by $F_p(f)$ following the order.

Note that for each $u_j \in \{u_1, \ldots, u_k\}$, we add (resp. delete) simplices inside the lower (resp. upper) star of $u_j$ in any order maintaining the condition of a filtration.

In this paper, we always assume a fixed $F_p(f)$ derived from $L_p(f)$. It is always of the form $F_p(f) : K_0 \leftrightarrow K_1 \leftrightarrow \cdots \leftrightarrow K_r$.

| closed-open: $[X^p_{(b-1,b+1)}, X^p_{(d-1,d)}]$ | $\leftrightarrow$ | $[\alpha^p_b, \alpha^p_d]$ |
| open-closed: $[X^p_{(b,b+1)}, X^p_{(d-1,d+1)}]$ | $\leftrightarrow$ | $(\alpha^p_b, \alpha^p_d)$ |
| closed-closed: $[X^p_{(b-1,b+1)}, X^p_{(d,d+1)}]$ | $\leftrightarrow$ | $[\alpha^p_b, \alpha^p_d]$ |
| open-open: $[X^p_{(b,b+1)}, X^p_{(d,d+1)}]$ | $\leftrightarrow$ | $(\alpha^p_b, \alpha^p_d)$ |

Table 1: Four types of intervals in $Pd\cdot(L^p\cdot(f))$ and their mapping to real-value intervals.
where each $K_i, K_{i+1}$ differ by a simplex denoted $\sigma_i$ and each linear map is denoted as $\phi_i : H_p(K_i) \hookrightarrow H_p(K_{i+1})$. Note that each complex in $L_p(f)$ equals a $K_j$ in $F_p(f)$, and specifically, $K_0 = \mathbb{R}^P_{(0,1)}$, $K_r = \mathbb{R}^P_{(m,m+1)}$.

**Simplex-wise intervals.** The property of zigzag persistence indicates that any interval $J$ in $P_d_p(L_p(f))$ can be considered as produced by an interval $J'$ in $P_d_p(F_p(f))$, and we call $J'$ the *simplex-wise interval* of $J$. The mapping of intervals of $P_d_p(F_p(f))$ to those of $P_d_p(L_p(f))$ has the following rule:

For any $[K_\beta, K_\delta] \in P_d_p(F_p(f))$, let $F^{[\beta,\delta]} : K_\beta \leftrightarrow K_{\beta+1} \leftrightarrow \cdots \leftrightarrow K_\delta$ be the part of $F_p(f)$ between $K_\beta$ and $K_\delta$, and let $\mathbb{R}^P_{(b,b')} \leftrightarrow \mathbb{R}^P_{(d,d')}$ respectively be the first and last complex from $L_p(f)$ which appear in $F^{[\beta,\delta]}$. Then, $[K_\beta, K_\delta]$ produces an interval $[\mathbb{R}^P_{(b,b')}, \mathbb{R}^P_{(d,d')}]$ for $P_d_p(L_p(f))$. However, if $F^{[\beta,\delta]}$ contains no complexes from $L_p(f)$, then $[K_\beta, K_\delta]$ does not produce any levelset persistence interval; such an interval in $P_d_p(F_p(f))$ is called trivial.

As can be seen later, any levelset persistent cycles in this paper are defined on *both* a levelset persistence interval and its simplex-wise interval. We further note that persistent cycles for trivial intervals in $P_d_p(F_p(f))$ are exactly the same as standard persistent cycles, and we refer to [12] for their definition and computation.

### 3.2 Definition of levelset persistent cycles

Consider a levelset persistence interval in $P_d_p(L_p(f))$ with endpoints $\alpha^P_b, \alpha^P_d$ produced by a simplex-wise interval $[K_\beta, K_\delta] \in P_d_p(F_p(f))$. The levelset persistence interval can also be denoted as $[\mathbb{R}^P_{(b,b')}, \mathbb{R}^P_{(d,d')}]$, where $b' = b$ or $b-1$, and $d' = d$ or $d+1$ (see Table 1). A sequence of levelset persistent cycles should achieve the following for the goal:

1. Reflect the changes of homological features across all $p$-th critical values between $\alpha^P_0$ and $\alpha^P_d$.
2. Capture the critical events at the birth and death points.

For the first requirement, we add to the sequence the following $p$-cycles:

$$z_i \subseteq \mathbb{R}^P_{(i,i+1)} \quad \text{for each } b \leq i < d$$

because $\mathbb{R}^P_{(i,i+1)}$ is the complex between $\alpha^P_i$ and $\alpha^P_{i+1}$. This is the same for all types of intervals. However, for the second requirement, we have to separately address the differently types of ends, and there are the following cases:

**Open birth:** The starting complex of the levelset persistence interval is $\mathbb{R}^P_{(b,b+1)}$. We require the corresponding $p$-cycle $z_b$ in $\mathbb{R}^P_{(b,b+1)}$ to become a boundary when included back into $\mathbb{R}^P_{(b-1,b+1)}$, so that it represents a new-born class in $H_p(\mathbb{R}^P_{(b,b+1)})$. In $F_p(f)$, the inclusion is further expanded as follows, where the birth happens at $K_{\beta-1} \hookrightarrow K_\beta$:

$$\mathbb{R}^P_{(b-1,b+1)} \hookrightarrow \cdots \hookrightarrow K_{\beta-1} \hookrightarrow K_\beta \hookrightarrow \cdots \hookrightarrow \mathbb{R}^P_{(b,b+1)}$$

We also consider $z_b$ as a $p$-cycle in $K_\beta$ because $\mathbb{R}^P_{(b,b+1)} \subseteq K_\beta$; then, in $F_p(f)$, $[z_b] \in H_p(K_\beta)$ should be the non-zero class in the kernel of $\varphi_{\beta-1} : H_p(K_{\beta-1}) \hookrightarrow H_p(K_\beta)$ in order to capture the birth event.

**Open death:** Symmetrically to open birth, the corresponding $p$-cycle $z_{d-1}$ in the ending complex $\mathbb{R}^P_{(d-1,d)}$ should become a boundary (i.e., die) entering into $\mathbb{R}^P_{(d-1,d+1)}$. The inclusion is further expanded as follows in the simplex-wise filtration, where the death happens at $K_\delta \hookrightarrow K_{\delta+1}$:

$$\mathbb{R}^P_{(d-1,d)} \hookrightarrow \cdots \hookrightarrow K_\delta \hookrightarrow K_{\delta+1} \hookrightarrow \cdots \hookrightarrow \mathbb{R}^P_{(d-1,d+1)}$$

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To capture the death event, \([z_{d-1}] \in \text{H}_p(K_\delta)\) should be the non-zero class in the kernel of \(\varphi_\delta\), where we also consider \(z_{d-1}\) as a \(p\)-cycle in \(K_\delta\).

**Closed birth:** The starting complex of the levelset persistence interval is \(K^p_{(b-1,b+1)}\), and the birth event happens when \(K^p_{(b-1,b)}\) is included into \(K^p_{(b-1,b+1)}\). The inclusion is further expanded as follows:

\[
K^p_{(b-1,b)} \hookrightarrow \cdots \hookrightarrow K_{b-1} \hookrightarrow K_b \hookrightarrow \cdots \hookrightarrow K^p_{(b-1,b+1)}
\]

In the simplex-wise filtration, the birth happens at the inclusion \(K_{b-1} \hookrightarrow K_b\). Since no \(z_i \subseteq \mathbb{K}_0\) for \(b \leq i < d\) can be considered as a \(p\)-cycle in \(K_\beta\) (see Proposition 1), we add to the sequence a new-born \(p\)-cycle \(z_{b-1}\) in \(K_\beta\) to capture the birth, which is equivalent to saying that \(z_{b-1}\) contains the simplex \(\sigma_{b-1}\) (note that \(\sigma_{b-1}\) is a \(p\)-simplex; see [4]).

**Closed death:** Symmetrically to closed birth, the death happens when the last complex \(K^p_{(d-1,d+1)}\) turns into \(K^p_{(d,d+1)}\) because of the deletion, which is at \(K_\delta \hookrightarrow K_{\delta+1}\) in \(F_p(f)\):

\[
K^p_{(d-1,d+1)} \hookrightarrow \cdots \hookrightarrow K_\delta \hookrightarrow K_{\delta+1} \hookrightarrow \cdots \hookrightarrow K^p_{(d,d+1)}
\]

Since no \(p\)-cycles defined above are considered to come from \(K_\delta\) (Proposition 1), we add to the sequence a \(p\)-cycle \(z_d\) in \(K_\delta \subseteq \mathbb{K}_p\) containing \(\sigma_\delta\), so that it represents a class disappearing in \(K_{\delta+1}\) (and hence in \(\mathbb{K}_p\)). Note that \(\sigma_\delta\) is a \(p\)-simplex [4].

**Proposition 1.** If the given levelset persistence interval is closed at birth end, then \(K_\beta \subseteq \mathbb{K}^p_{(b-1,b)}\) so that each \(\mathbb{K}^p_{(i,i+1)}\) for \(b \leq i < d\) is disjoint with \(K_\beta\). Similarly, if the persistence interval is closed at death end, then \(K_\delta \subseteq \mathbb{K}_p\) so that each \(\mathbb{K}^p_{(i,i+1)}\) for \(b \leq i < d\) is disjoint with \(K_\delta\).

**Remark 3.** Note that the disjointness of these complexes also makes computation of the optimal persistent cycles feasible; see Section 4.

**Proof.** See Appendix B.1.

One final thing left for the definition is to relate two consecutive \(p\)-cycles \(z_i, z_{i+1}\) in the sequence. It can be verified that both \(z_i, z_{i+1}\) reside in \(\mathbb{K}^p_{(i,i+2)}\), and hence we require them to be homologous in \(\mathbb{K}^p_{(i,i+2)}\). In this way, we have

\[
[z_i] \mapsto [z_i] = [z_{i+1}] \mapsto [z_{i+1}]
\]

under the linear maps

\[
\text{H}_p(\mathbb{K}^p_{(i,i+1)}) \to \text{H}_p(\mathbb{K}^p_{(i,i+2)}) \leftrightarrow \text{H}_p(\mathbb{K}^p_{(i+1,i+2)})
\]

so that all \(p\)-cycles in the sequence represent corresponding homology classes.

For easy reference, we formally present the definitions individually for the three types of intervals:

**Definition 5** (Open-open case). For an open-open \((\alpha^p_\delta, \alpha^p_{\delta+1}) \in \text{Pd}_p(L_p(f))\) produced by a simplex-wise interval \([K_\beta, K_\delta]\), the **levelset persistent** \(p\)-cycles is a sequence \(z_b, z_{b+1}, \ldots, z_{d-1}\) such that: (i) each \(z_i \subseteq \mathbb{K}^p_{(i,i+1)}\); (ii) \([z_0] \in \text{H}_p(K_\beta)\) is the non-zero class in the kernel of \(\varphi_{\beta-1} : \text{H}_p(K_{\beta-1}) \to \text{H}_p(K_\beta)\); (iii) \([z_{d-1}] \in \text{H}_p(K_\delta)\) is the non-zero class in the kernel of \(\varphi_{\delta} : \text{H}_p(K_\delta) \to \text{H}_p(K_{\delta+1})\); (iv) each consecutive \(z_i, z_{i+1}\) are homologous in \(\mathbb{K}^p_{(i,i+2)}\).

**Definition 6** (Closed-open case). For a closed-open \([\alpha^p_\delta, \alpha^p_{\delta+1}] \in \text{Pd}_p(L_p(f))\) produced by a simplex-wise interval \([K_\beta, K_\delta]\), the **levelset persistent** \(p\)-cycles is a sequence \(z_{b-1}, z_b, \ldots, z_{d-1}\) such that: (i) \(z_{b-1} \subseteq \mathbb{K}_\beta\) and \(\sigma_{\beta-1} \in z_{b-1}\); (ii) \(z_i \subseteq \mathbb{K}^p_{(i,i+1)}\) for each \(i \geq b\); (iii) \([z_{d-1}] \in \text{H}_p(K_\delta)\) is the non-zero class in the kernel of \(\varphi_{\beta} : \text{H}_p(K_\beta) \to \text{H}_p(K_{\beta+1})\); (iv) each consecutive \(z_i, z_{i+1}\) are homologous in \(\mathbb{K}^p_{(i,i+2)}\).
Definition 7 (Closed-closed case). For a closed-closed interval $[K_\beta, K_\delta]$, the levelset persistent $p$-cycles is a sequence $z_{b-1}, z_b, \ldots, z_d$ such that: (i) $z_{b-1} \subseteq K_\beta$ and $\sigma_{b-1} \subseteq z_{b-1}$; (ii) $z_d \subseteq K_\delta$ and $\sigma_\delta \subseteq z_d$; (iii) $z_i \subseteq \mathbb{K}_{(i,i+1)}^p$ for each $b \leq i < d$; (iv) each consecutive $z_i, z_{i+1}$ are homologous in $\mathbb{K}_{(i,i+2)}^p$.

Figure 1 illustrates a sequence of levelset persistent 1-cycles for a closed-open interval, where $z_1$ captures the birth event (created by the corresponding 1st critical vertex) and $z_2, z_3, z_4$ are the ones in the 1st regular complexes. The cycle $z_4$, which becomes a boundary when the last critical vertex is added, captures the death event. See Figure 5 and 7 in Section 4 for examples of other types of intervals.

To see that levelset persistent cycles actually “represent” an interval, we show that each such sequence induces an interval submodule so that all these interval submodules form an interval decomposition for $\mathbb{H}_p(\mathcal{L}_p(f))$. The details are provided in Section 6.

Optimal levelset persistent cycles. To define optimal cycles, we assign weights to $p$-cycles of $K$ as follows: let each $p$-simplex $\sigma$ of $K$ have a non-negative finite weight $w(\sigma)$; then, a $p$-cycle $z$ of $K$ has the weight $w(z) := \sum_{\sigma \subseteq z} w(\sigma)$.

Definition 8. For an interval of $\text{Pd}_p(\mathcal{L}_p(f))$, an optimal sequence of levelset persistent $p$-cycles is one with the minimum sum of weight.

3.3 Validity of discrete levelset filtrations

One thing left over from Section 3.1 is to justify the validity of the discrete version of $p$-th levelset filtration. It turns out that the validity depends on the triangulation of $K$. For example, let $K$ be the left complex in Figure 4; then, $\mathbb{K}_{(i,i+1)}^p$ (the blue part) is not homotopy equivalent to $\mathbb{X}_{(i,i+1)}^p$ (the part between the dashed lines), and hence $\mathcal{L}_p(f)$ is not equivalent to $\mathcal{L}_p^e(f)$. We observe that the non-equivalence is caused by the two central triangles which contain more than one critical value. A subdivision of the two central triangles on the right (so that no triangles contain more than one critical value) renders $\mathbb{X}_{(i,i+1)}^p$ deformation retracting to $\mathbb{K}_{(i,i+1)}^p$. Based on the above observation, we formulate the following property, which guarantees the equivalence of modules induced by $\mathcal{L}_p(f)$ and $\mathcal{L}_p^e(f)$:

Definition 9. The complex $K$ is said to be compatible with the $p$-th levelsets of the PL function $f$ if for any simplex $\sigma$ of $K$ and its convex hull $|\sigma|$, function values of points in $|\sigma|$ include at most one $p$-th critical value of $f$.

Proposition 2. If $K$ is compatible with the $p$-th levelsets of $f$, then $\mathbb{X}_{(i,j)}^p$ deformation retracts to $\mathbb{K}_{(i,j)}^p$ for any $i < j$, which implies that $\mathbb{H}_p(\mathcal{L}_p(f))$ and $\mathbb{H}_p(\mathcal{L}_p^e(f))$ are isomorphic.

Proof. See Appendix B.2. □

---

1In the discrete setting, $z_1$ is indeed created by an edge incident to the critical vertex.
In this paper, we only consider the situation where a complex is compatible with the $p$-th levelsets of its PL function. We regard this assumption reasonable because in the discrete optimization, the quality of computed output depends on the triangulation of underlying space. When the assumption is violated, it becomes impossible to depict certain changes of homological features on the discrete domain. Note that a complex can be refined to become compatible if it is not already.

4 Computation

In this section, given a weak $(p + 1)$-pseudomanifold with $p \geq 1$, we present algorithms that compute an optimal sequence of levelset persistent $p$-cycles for a $p$-th interval. Though the computation for all types of intervals is based on minimum cuts, we address the algorithm for each type separately in each subsection. The reasons are as follows. First, one has to choose a subcomplex to work on in order to build a dual graph for the minimum cut computation. In the open-open case, the subcomplex is always a $(p + 1)$-pseudomanifold without boundary (see Section 4.1) whose dual graph is obvious; in the other cases, however, we do not have such convenience and the dual graph construction is more involved. Also, the closed-open case has to deal with the so-called “monkey saddles” and the solution adopts a two-phase approach (see Section 4.2); in the open-open case, however, no such issues occur and the algorithm is much simpler. We also note that even for standard persistent cycles which have simpler definitions, the hardness results and the algorithms for the finite and infinite intervals are still different; see [12]. With all being said, we observe that the computation for the closed-closed case does exhibit resemblance to the closed-open case and is only described briefly; see Section 4.3.

Other than the type of persistence interval, all subsections make the same assumptions on input as the following:

- $p \geq 1$ is the dimension of interest.
- $K$ is a finite weak $(p + 1)$-pseudomanifold with a finite weight $w(\sigma) \geq 0$ for each $p$-simplex $\sigma$.
- $f : |K| \to \mathbb{R}$ is a generic PL function with $p$-th critical values $\alpha^p_0 = -\infty < \alpha^p_1 < \cdots < \alpha^p_m < \alpha^p_{m+1} = \infty$ and corresponding $p$-th critical vertices $v^p_0, \ldots, v^p_m$. We also assume that $K$ is compatible with the $p$-th levelsets of $f$.
- $F_p(f) : K_0 \leftrightarrow K_1 \leftrightarrow \cdots \leftrightarrow K_r$ is a fixed simplex-wise levelset filtration. Each $K_i, K_{i+1} \in F_p(f)$ differ by a simplex $\sigma_i$, and each linear map in $H_p(F_p(f))$ is denoted as $\varphi_i : H_p(K_i) \leftrightarrow H_p(K_{i+1})$.

4.1 Open-open case

Throughout this subsection, assume that we aim to compute the optimal persistent $p$-cycles for an open-open interval $(\alpha^p_b, \alpha^p_d)$ from $\text{Pd}_p(L_p(f))$, which is produced by a simplex-wise interval $[K_\beta, K_\delta]$ from $\text{Pd}_p(F_p(f))$. Figure 5 illustrates a sequence of persistent 1-cycles $z_1, z_2, z_3$ for an open-open interval $(\alpha^1_1, \alpha^1_2)$.

As seen from Section 3.2, the following portion of $F_p(f)$ is relevant to the definition (and hence the computation):

\[
\begin{align*}
\bK^p_{(b-1,b+1)} & \leftrightarrow K_{\beta-1} \overset{\sigma_{\beta-1}}{\longrightarrow} K_{\beta} \overset{\sigma_{\beta}}{\leftrightarrow} \bK^p_{(b,b+1)} \leftrightarrow \cdots \\
& \leftrightarrow \bK^p_{(d-1,d+1)} \leftrightarrow K_{\delta} \overset{\sigma_{\delta}}{\longrightarrow} K_{\delta+1} \leftrightarrow \bK^p_{(d-1,d+1)}
\end{align*}
\]

(2)

In the above sequence, the non-dashed hooked arrows indicate the addition or deletion of one simplex, while the dashed arrows indicate the addition or deletion of zero or more simplices. The simplices $\sigma_{\beta-1}, \sigma_{\delta}$ are the ones creating and destroying the simplex-wise interval, which are both $(p + 1)$-simplices [4]. We
Further restrict the computation to (a connected component of) $K_{p-1,d+1}$ considering that each complex in Sequence (2) is a subcomplex of $K_{p-1,d+1}$. However, instead of the usual one, we take a special type of component which considers connectedness in higher dimensions:

**Definition 10** ($q$-connected [12]). Let $\Sigma$ be a set of simplices, and let $\sigma$, $\sigma'$ be two $q$-simplices of $\Sigma$ for $q \geq 1$. A $q$-path from $\sigma$ to $\sigma'$ in $\Sigma$ is a sequence of $q$-simplices of $\Sigma$, $\tau_1, \ldots, \tau_\ell$, such that $\tau_1 = \sigma$, $\tau_\ell = \sigma'$, and each consecutive $\tau_i$, $\tau_{i+1}$ share a $(q-1)$-face in $\Sigma$. A maximal set of $q$-simplices of $\Sigma$, in which each pair is connected by a $q$-path, constitutes a $q$-connected component of $\Sigma$. We also say that $\Sigma$ is $q$-connected if it has only one $q$-connected component.

We now describe the algorithm. Since the deletion of the $(p+1)$-simplex $\sigma_{p-1}$ gives birth to the interval $[K_{p+1}, K_d]$, $\sigma_{p-1}$ must be relevant to our computation. So we let the complex being worked on, denoted $K'$, be the closure of the $(p+1)$-connected component of $K$ containing $\sigma_{p-1}$. (Note that the closure of a set of simplices consists of all faces of the simplices in the set.) Based on the property of open-open intervals, $K'$ must be a $(p+1)$-pseudomanifold without boundary (see Proposition 3, Claim 3). Letting $G$ be the dual graph of $K'$, we build a weighted $(s, t)$-graph $(G, s, t)$. To set up the sources and sinks, we define the set $K_{p-1}^p$ of simplices as follows:

$$K_{p-1}^p := K_{p-1-1,j+1}^p \setminus (K_{p-1-1,j}^p \cup K_{p-1,j+1}^p)$$

Roughly speaking, $K_{p-1}^p$ consists of simplices containing the critical value $\sigma_p^p$ (for example, the darker triangles in Figure 4 belong to $K_{p-1}^p$, and note that $K_{p-1}^p$ may not be a simplicial complex. Based on the above construction, we alternately put vertices dual to the $(p+1)$-simplices in $K_{p-1}^p, \ldots, K_{p-1}^d$ into sources and sinks. For our example in Figure 5 where $K'$ is the entire torus, the source $s$ contains vertices dual to $2$-simplices in $K_{p-1}^1 \cup K_{p-1}^3$, and the sink $t$ contains vertices dual to $2$-simplices in $K_{p-1}^1 \cup K_{p-1}^4$. Note that $K_{p-1}^1, \ldots, K_{p-1}^d$ are alternately shaded with light and dark gray in Figure 5.

The correctness of the above construction is based on the duality of the level set persistent $p$-cycles and $(s, t)$-cuts on $(G, s, t)$. To see the duality, first consider the sequence of persistent 1-cycles $z_1, z_2, z_3$ in Figure 5. By Definition 5, there exist 2-chains $A_1 \subseteq K_{p-1}^1(0,2)$, $A_2 \subseteq K_{p-1}^1(1,3)$, $A_3 \subseteq K_{p-1}^1(2,4)$, and $A_4 \subseteq K_{p-1}^1(3,5)$ as shown in the figure such that $z_1 = \partial(A_1)$, $z_1 + z_2 = \partial(A_2)$, $z_2 + z_3 = \partial(A_3)$, and $z_3 = \partial(A_4)$. Let $S$ contain the vertices dual to $A_1 + A_3$ and $T$ contain the vertices dual to $A_2 + A_4$. Then, $(S, T)$ is an $(s, t)$-cut of $(G, s, t)$. Since edges in $E(S, T)$ are dual to 1-simplices in $z_1 + z_2 + z_3$, we have that $w(S, T) = w(z_1) + w(z_2) + w(z_3)$. So we derive a cut $(S, T)$ dual to the persistent 1-cycles $z_1, z_2, z_3$. On the other hand, an $(s, t)$-cut of $(G, s, t)$ produces a sequence of persistent $p$-cycles for the given interval. For example, let $(S, T)$ be a cut where $S$ contains the graph vertices in $A_1 + A_3$ and $T$ contains the graph...
vertices in \( A_2 + A_4 \), as in Figure 5. We then take the intersection of the dual 1-simplices of \( E(S, T) \) with \( K_1^{(1,2)}, K_1^{(2,3)}, K_1^{(3,4)} \). The resulting 1-chains \( z_1, z_2, z_3 \) is a sequence of persistent 1-cycles for the interval \((\alpha^1_b, \alpha^1_d)\). Hence, by the duality, a minimum \((s, t)\)-cut of \((G, \delta, \tau)\) always produces an optimal sequence of levelset persistent \(p\)-cycles.

We formally list the pseudocode as follows:

**Algorithm 1.** Given the input as specified, the algorithm does the following:

1. Let \( K' \) be the closure of the \((p + 1)\)-connected component of \( K \) containing \( \sigma_{\beta-1} \). Note that \( K' \) is a \((p + 1)\)-pseudomanifold without boundary (see Proposition 3, Claim 3).
2. Build a weighted dual graph \( G \) of \( K' \), where \( V(G) \) corresponds to \((p + 1)\)-simplices of \( K' \) and \( E(G) \) corresponds to \(p\)-simplices of \( K' \). Let \( \theta \) denote both the bijection from the \((p + 1)\)-simplices to \( V(G) \) and the bijection from the \(p\)-simplices to \( E(G) \). For each \( e \) of \( G \), if \( \theta^{-1}(e) \in \mathbb{K}^p_{(i, i+1)} \) for \( i \) s.t. \( b \leq i < d \), then set \( w(e) \), the weight of \( e \), as \( w(\theta^{-1}(e)) \); otherwise, set \( w(e) = \infty \).
3. For each \( i \) s.t. \( b \leq i \leq d \), let \( \Delta_i \) denote the set of \((p + 1)\)-simplices in \( K' \cap \mathbb{K}^p_{(i)} \). Also, let \( L_a \) be the set of even integers in \([0, d - b] \) and \( L_o \) be the set of odd ones. Then, let \( \tau = \theta \bigl( \bigcup_{i \in L_a} \Delta_b {+} i \bigr) \), and compute the minimum \((s, t)\)-cut \((S', T')\) of \((\tau, \delta, t)\).
4. For each \( i \) s.t. \( b \leq i < d \), let \( z_i^* = \mathbb{K}^p_{(i, i+1)} \cap \theta^{-1}(E(S', T')) \). Return \( z_1^*, \ldots, z_d{-}1^* \) as an optimal sequence of levelset persistent \(p\)-cycles for the interval \((\alpha^p_b, \alpha^p_d)\).

### 4.1.1 Justification

For the correctness of Algorithm 1, we first present Proposition 3 stating several facts about the algorithm which are used to prove the two propositions (4 and 5) on the duality. Then, Proposition 4 and 5 lead to Theorem 1, which draws the conclusion.

**Proposition 3.** The following hold for Algorithm 1:

1. The simplex \( \sigma_{\delta} \) resides in \( K' \).
2. Let \( z_b, \ldots, z_{d{-}1} \) be any sequence of persistent \(p\)-cycles for \((\alpha^p_b, \alpha^p_d)\); then, there exist \((p + 1)\)-chains \( A_b \subseteq K_{\beta-1}, A_{b+1} \subseteq K^p_{(b, b+2)}, \ldots, A_{d-1} \subseteq K^p_{(d-2, d)}, A_d \subseteq K_{\delta+1} \) such that \( \sigma_{\beta-1} \) in \( A_b, \sigma_{\delta} \) in \( A_d, \)

\( z_b = \partial(A_b), z_{d-1} = \partial(A_d) \), and \( z_i = \partial(A_i) \) for each \( b < i < d \). Furthermore, let \( z_i^* = K' \cap z_i \), \( A_i' = K' \cap A_i \) for each \( i \); then, \( \sigma_{\beta-1} \) in \( A_i' \), \( \sigma_{\delta} \in A_d \), \( z_i^* = \partial(A_i'), z_i^{d-1} = \partial(A_d') \), and \( z_i^{d-1} = \partial(A_i') \) for each \( b < i < d \). Finally, one has that \( A_b' + \cdots + A_d' \) equals the set of \((p + 1)\)-simplices of \( K' \) and \( A_b', \ldots, A_d' \) are disjoint.
3. The complex \( K' \) is a \((p + 1)\)-connected \((p + 1)\)-pseudomanifold without boundary, i.e., each \(p\)-simplex has exactly two \((p + 1)\)-cofaces in \( K' \).

**Proof.** See Appendix B.3.

**Proposition 4.** Let \( z_b, \ldots, z_{d-1} \) be any sequence of levelset persistent \(p\)-cycles for \((\alpha^p_b, \alpha^p_d)\); then, there exists an \((s, t)\)-cut \((S, T)\) of \((\tau, \delta, t)\) such that \( w(S, T) \leq \sum_{i=b}^{d-1} w(z_i) \).

**Proof.** Let \( A_b', \ldots, A_d' \) and \( z_b', \ldots, z_{d-1}' \) be as specified in Claim 2 of Proposition 3 for the given \( z_b, \ldots, z_{d-1} \), and let \( S = \theta \bigl( \bigcup_{j \in L_a} A_{b+j}' \bigr), T = \theta \bigl( \bigcup_{j \in L_o} A_{b+j}' \bigr) \). We first show that for a \( \Delta_i \) such that \( i - b \) is even, \( \Delta_i \) does not intersect \( \sum_{j \in L_o} A_{b+j} \). For contradiction, suppose instead that there is a \( \sigma \) in both of them. Then,
since $\Delta_i \subseteq \mathbb{K}_p^{(i)} \subseteq \mathbb{K}_p^{(i-1,i+1)}$ and $A'_{b+j} \subseteq \mathbb{K}_p^{(b+j-1,b+j+1)}$ for each $j \in L_0$, $\sigma$ must be in $A'_{i-1} \subseteq \mathbb{K}_p^{(i-2,i)}$ or $A'_{i+1} \subseteq \mathbb{K}_p^{(i,i+2)}$ because other chains in $\{A'_{b+j} \mid j \in L_0\}$ do not intersect $\mathbb{K}_p^{(i-1,i+1)}$. So we have that $\sigma$ is in $\mathbb{K}_p^{(i-2,i)}$ or $\mathbb{K}_p^{(i,i+2)}$. The fact that $\sigma \in \Delta_i \subseteq \mathbb{K}_p^{(i-1,i+1)}$ implies that $\sigma$ is in $\mathbb{K}_p^{(i-1,i)}$ or $\mathbb{K}_p^{(i,i+1)}$, a contradiction to $\sigma \in \Delta_i \subseteq \mathbb{K}_p^{(i,i+1)} \setminus (\mathbb{K}_p^{(i-1,i)} \cup \mathbb{K}_p^{(i,i+1)})$. So $\Delta_i$ does not intersect $\sum_{j \in L_0} A'_{b+j}$.

Then, since $\sum_{j=b}^d A'_{i}$ equals the set of $(p+1)$-simplices of $K'$ by Claim 2 of Proposition 3, we have that $\Delta_i \subseteq \sum_{j \in L_0} A'_{b+j}$, i.e., $\theta(\Delta_i) \subseteq S$. This means that $\delta \subseteq S$. Similarly, we have $t \subseteq T$. Claim 2 of Proposition 3 implies that $S \cup T = V(G)$ and $S \cap T = \emptyset$, and so $(S,T)$ is an $(s,t)$-cut of $(G,s,t)$. The fact that $\sum_{i=b}^{d-1} z_i = \partial(\sum_{j \in L_0} A'_{b+j}) = \partial(\sum_{j \in L_0} A'_{b+j})$ implies that $\sum_{i=b}^{d-1} z_i = \theta^{-1}(E(S,T))$. So we have $w(S,T) = \sum_{i=b}^{d-1} w(z_i) \leq \sum_{i=b}^{d-1} w(z_i)$.

**Proposition 5.** For any $(s,t)$-cut $(S,T)$ of $(G,s,t)$ with finite weight, let $z_i = \mathbb{K}_p^{(i,i+1)} \cap \theta^{-1}(E(S,T))$ for each $i s.t. b \leq i < d$. Then, $z_0, \ldots, z_{d-1}$ is a sequence of levelset persistent $p$-cycles for $(\alpha_p, \alpha_d)$ with $\sum_{i=b}^{d-1} w(z_i) = w(S,T)$.

**Proof.** We first prove that, for any $i s.t. b < i < d$ and $i - b$ is even, $\partial(\theta^{-1}(S) \cap \mathbb{K}_p^{(i-1,i+1)}) = z_{i-1} + z_i$. To prove this, first consider any $\sigma \in \partial(\theta^{-1}(S) \cap \mathbb{K}_p^{(i-1,i+1)})$. We have that $\sigma$ is a face of only one $(p+1)$-simplex $\tau_1$ in $\theta^{-1}(S) \cap \mathbb{K}_p^{(i-1,i+1)}$. Note that $\tau_1 \in \theta^{-1}(S) \subseteq K'$. Since $K'$ is a $(p+1)$-psuedomanifold without boundary (Claim 3 of Proposition 3), $\sigma$ has another $(p+1)$-coface $\tau_2$ in $K'$. Then, it must be true that $\tau_2 \in \theta^{-1}(T)$. To see this, suppose instead that $\tau_2 \in \theta^{-1}(S)$. Note that $\tau_2 \not\subseteq \mathbb{K}_p^{(i-1,i+1)}$ because otherwise $\tau_2$ would be in $\theta^{-1}(S) \cap \mathbb{K}_p^{(i-1,i+1)}$, contradicting the fact that $\sigma$ has only one $(p+1)$-coface $\tau_1$ in $\theta^{-1}(S) \cap \mathbb{K}_p^{(i-1,i+1)}$. Also note that $\tau_2$ is not in $\mathbb{K}_p^{(i-1,i+1)}$ or $\mathbb{K}_p^{(i,i+1)}$ because if $\tau_2$ is in one of them, combining the fact that $i-1-b$ and $i+b$ are odd, we would have that $\tau_2$ is in $\Delta_{i-1}$ or $\Delta_{i+1}$ and thus $\theta(\tau_2) \in t \subseteq T$, which is a contradiction. Since $K' \subseteq \mathbb{K}_p^{(b-1,d+1)}$ and $\{\mathbb{K}_p^{(b-1,i-1,1)}, \mathbb{K}_p^{(i-1,i+1)}, \mathbb{K}_p^{(i,i+1)}, \mathbb{K}_p^{(i+1,d+1)}\}$ covers $\mathbb{K}_p^{(b-1,d+1)}$, we have that $\tau_2$ is in $\mathbb{K}_p^{(b-1,i-1)}$ or $\mathbb{K}_p^{(i+1,d+1)}$. This implies that $\sigma \subseteq \tau_2$ is in $\mathbb{K}_p^{(b-1,i-1)}$ or $\mathbb{K}_p^{(i+1,d+1)}$, contradicting that $\sigma \subseteq \tau_1 \in \mathbb{K}_p^{(i-1,i+1)}$. It is now true that $\sigma \in \theta^{-1}(E(S,T))$ because $\tau_1 \in \theta^{-1}(S)$ and $\tau_2 \in \theta^{-1}(T)$. Since $(S,T)$ has finite weight, $\sigma$ must come from a $\mathbb{K}_p^{(j,j+1)}$ for $b \leq j < d$ and thus must come from $\mathbb{K}_p^{(i-1,i)}$ or $\mathbb{K}_p^{(i,i+1)}$. Then, $\sigma$ is in $z_{i-1}$ or $z_i$. Moreover, since $z_{i-1}$ and $z_i$ are disjoint, we have $\sigma \in z_{i-1} + z_i$.

On the other hand, for any $\sigma \in z_{i-1} + z_i$, first assume that $\sigma \in z_{i-1} = \mathbb{K}_p^{(i-1,i)} \cap \theta^{-1}(E(S,T))$. Since $\sigma \in \theta^{-1}(E(S,T))$, $\sigma$ must be a face of a $(p+1)$-simplex $\tau$ in $\theta^{-1}(S)$ and another $(p+1)$-simplex in $\theta^{-1}(T)$. We then show that $\tau \in \mathbb{K}_p^{(i-1,i+1)}$. Suppose instead that $\tau \not\subseteq \mathbb{K}_p^{(i-1,i+1)}$, and let $v$ be the vertex belonging to $\tau$ but not $\sigma$. We have that $f(v) \not\in (\alpha_p, \alpha_{i+1})$ because if $f(v) \in (\alpha_p, \alpha_{i+1})$, the fact that $\sigma \in \mathbb{K}_p^{(i-1,i)}$ would imply that $\tau$ is in $\mathbb{K}_p^{(i-1,i+1)}$. Note that $f(v)$ cannot be greater than or equal to $\alpha_{i+1}$ because otherwise $K$ would not be compatible with the $p$-th levelsets of $f$. Therefore, $f(v) \leq \alpha_{i+1}$, and it must be true that $\tau \in \mathbb{K}_p^{(i-2,i)}$. This implies that $\tau \subseteq \mathbb{K}_p^{(i-1,i)}$. We now have that $\tau \in \Delta_{i-1}$, where $i-1-b$ is odd. Then, $\theta(\tau) \in t \subseteq T$, a contradiction to $\sigma \in \theta^{-1}(S)$. Combining the fact that $\tau \in \mathbb{K}_p^{(i-1,i+1)}$ and $\tau$ is the only $(p+1)$-coface of $\sigma$ in $\theta^{-1}(S)$, we have that $\tau$ is the only $(p+1)$-coface of $\sigma$ in $\theta^{-1}(S) \cap \mathbb{K}_p^{(i-1,i+1)}$. If $\sigma \in z_i$, we can have the same result. Therefore, $\sigma \in \theta^{-1}(S) \cap \mathbb{K}_p^{(i-1,i+1)}$, and we have proved that $\partial(\theta^{-1}(S) \cap \mathbb{K}_p^{(i-1,i+1)}) = z_{i-1} + z_i$.

Similarly, we can prove that $\partial(\theta^{-1}(T) \cap \mathbb{K}_p^{(i-1,i+1)}) = z_{i-1} + z_i$ for i.s.t. $b < i < d$ and $i - b$ is odd, $\partial(\theta^{-1}(S) \cap K_{\beta-1}) = z_b$, and $\partial(\theta^{-1}(S) \cap K_{\delta+1}) = z_{d-1}$ or $\partial(\theta^{-1}(T) \cap K_{\delta+1}) = z_{d-1}$ based on the parity of $d - b$. Since $\sigma_{\beta-1} \in K_{\beta-1} \subseteq \mathbb{K}_p^{(b,b+1)}$ and $\sigma_{\beta-1} \not\subseteq \mathbb{K}_p^{(b,b+1)}$, we have that $\sigma_{\beta-1} \in \mathbb{K}_p^{(b,b+1)}$, which means that $\theta(\sigma_{\beta-1}) \not\subseteq \Delta_i$. Therefore, $\sigma_{\beta-1} \in \theta^{-1}(S) \cap K_{\beta-1}$. Since $\partial(\theta^{-1}(S) \cap K_{\beta-1}) = z_b$, we have that $z_b \sim \partial(\sigma_{\beta-1})$ in $K_{\beta}$, i.e., $[z_b] \in H_p(K_{\beta})$ is the non-zero class in Ker($\varphi_{\beta-1}$). Analogously, $[z_{d-1}] \in H_p(K_{\delta})$.
Theorem 1. Algorithm 1 computes an optimal sequence of levelset persistent $p$-cycles for a given open-open interval.

4.2 Closed-open case

Throughout the subsection, assume that we aim to compute the optimal persistent $p$-cycles for a closed-open interval $[\alpha^p, \alpha^p]$ from $\text{Pd}_p(L_p(f))$, which is produced by a simplex-wise interval $[K_\beta, K_\delta]$ from $\text{Pd}_p(F_p(f))$. Figure 6a and 6b provide examples for $p = 1$, where $z_1'$, $z_2'$, $z_3'$ and $z_1''$, $z_2''$, $z_3''$ are two sequences of levelset persistent 1-cycles for the interval $[\alpha^1, \alpha^1]$.

Similar to the previous case, we have the following portion of $F_p(f)$ relevant to the definition and computation:

$$\mathbb{K}^p_{(b-1,b)} \hookrightarrow K_\beta \leftarrow \mathbb{K}^p_{(b-1,b+1)} \hookrightarrow K_\beta \leftarrow \mathbb{K}^p_{(b,b+1)} \hookrightarrow \mathbb{K}^p_{(d-1,d)} \leftarrow \cdots$$

(3)

The creator $\sigma_{\beta-1}$ is a $p$-simplex and the destroyer $\sigma_\beta$ is a $(p+1)$-simplex [4]. Note that we end the sequence with $\mathbb{K}^p_{(d-1,d)}$ instead of $\mathbb{K}^p_{(d-1,d+1)}$ as in the case “open death” in Section 3.2. This is valid due to the following reasons: (i) $\mathbb{K}^p_{(d-1,d)}$ is derived from $\mathbb{K}^p_{(d-1,d+1)}$ by adding the lower star of $u^p_d$ and hence must appear in $F_p(f)$ based on Definition 4; (ii) $K_{\delta+1}$ is a subcomplex of $\mathbb{K}^p_{(d-1,d)}$ and the proof is similar to that of Proposition 1. Therefore, the computation can be restricted to $\mathbb{K}^p_{(b-1,d)}$ because each complex in Sequence (3) is a subcomplex of $\mathbb{K}^p_{(b-1,d)}$.

4.2.1 Overview

To give a high-level view of our algorithm, we first use an example to illustrate several important observations. These observations provide insights into the solution and lead to the key issue. We then discuss the key issue in detail. Finally, we describe our solution in words, and postpone the formal pseudocode to Section 4.2.2.

Now consider the example in Figure 6, and let $z_1$, $z_2$, $z_3$ be a general sequence of persistent 1-cycles for $[\alpha^1_2, \alpha^1_4]$. By definition, there exist 2-chains

$$A_2 \subseteq \mathbb{K}^1_{(1,3)}, A_3 \subseteq \mathbb{K}^1_{(2,4)}, \text{ and } A_4 \subseteq \mathbb{K}^1_{(3,4)}$$

such that

$$z_1 + z_2 = \partial(A_2), z_2 + z_3 = \partial(A_3), \text{ and } z_3 = \partial(A_4)$$

Letting $A = A_2 + A_3 + A_4$, we have $\partial(A) = z_1 \subseteq K_\beta$. (We still assume that $[\alpha^1_2, \alpha^1_4]$ is produced by a simplex-wise interval denoted $[K_\beta, K_\delta]$.) One strategy we adopt is to separate $K_\beta$ from $\mathbb{K}^p_{(b-1,d)}$ and tackle $K_\beta$, $\mathbb{K}^p_{(b-1,d)} \setminus K_\beta$ independently. Note that $\mathbb{K}^p_{(b-1,d)} = \mathbb{K}^1_{(1,4)}$ in our example. Then we separate $A$ into the part that is in $K_\beta$ and the part that is not. Obviously, the part of $A$ not in $K_\beta$ comes from different 2-connected components of $\mathbb{K}^1_{(1,4)} \setminus K_\beta$, which are $C_0$, $C_1$, and $C_2$ shown in Figure 6b. We then observe that any such component intersecting $A$ must be totally included in $A$, because a 2-simplex of the component not in $A$ would cause $\partial(A)$ to contain 1-simplices not in $K_\beta$, contradicting $\partial(A) \subseteq K_\beta$. For the same reason, any component intersecting $A$ must have its boundary in $K_\beta$. For example, in Figure 6b, no 2-simplices in $C_2$ can fall in $A$, while $C_1$ can either be totally in or disjoint with $A$. The proof of Proposition 9 formally
justifies this observation. We also note that there can be only one 2-connected component of \( K^{1}_{[1,4]} \setminus K_{\beta} \) (i.e., \( C_0 \) in Figure 6b) whose boundary resides in \( K_{\beta} \) and contains \( \sigma_{\beta-1} \) (see Proposition 7). (While this is not drawn in Figure 6, we assume that \( K \) is triangulated in a way that \( \sigma_{\beta-1} \) is shared by the boundaries of \( C_0 \) and \( C_2 \).) A fact about \( C_0 \) is that it is always included in \( A \) (see the proof of Proposition 9). For the other components with boundaries in \( K_{\beta} \) (i.e., \( C_1 \) in Figure 6b), any subset of them can contribute to a certain \( A \) and take part in forming the persistent cycles. For example, in Figure 6b, only \( C_0 \) contributes to the persistent 1-cycles \( z_1'', z_2'', z_3'' \), and both \( C_0, C_1 \) contribute to \( z_1'', z_2'', z_3'' \).

The crux of the algorithm, therefore, is to determine a subset of the components along with \( C_0 \) contributing to the optimal persistent cycles (a complicated monkey saddle with multiple forks may result in many such components), because we can compute the optimal persistent cycles under a fixed choice of subset. To see this, suppose that \( z_1'', z_2'', z_3'' \) in Figure 6 are the optimal persistent 1-cycles for \( \{\alpha_0, \alpha_1\} \) under the choice of the subset \( \{C_0, C_1\} \), i.e., \( z_1'', z_2'', z_3'' \) have the minimum sum of weight among all persistent 1-cycles coming from both \( C_0 \) and \( C_1 \). We first observe that \( z_i'' \) must be the minimum 1-cycle homologous to \( \partial(C_0) + \partial(C_1) \) in \( K_{\beta} \). Such a cycle \( z_i'' \) can be computed from a minimum \((s, t)\)-cut on a dual graph of \( K_{\beta} \). Also, the set of 1-cycles \( \{z_0^0 \subseteq \mathbb{K}^1_{[2,3]} \}, z_3^0 \subseteq \mathbb{K}^1_{(3,4)} \} \) must be the one in \( C_0 \) with the minimum sum of weight such that

\[
z_0^0 \sim \partial(C_0) \text{ in } \mathbb{K}^1_{(1,3)}, z_0^0 \sim \partial(C_0) \text{ in } \mathbb{K}^1_{(2,4)}, \text{ and } z_3^0 \text{ null-homologous in } \mathbb{K}^1_{(3,4)}
\]

Additionally, \( z_2^0 \subseteq \mathbb{K}^1_{(2,3)} \) must be the minimum 1-cycle in \( C_1 \) which is homologous to \( \partial(C_1) \) in \( \mathbb{K}^1_{(1,3)} \) and is null-homologous in \( \mathbb{K}^1_{(2,3)} \). (See Step 2 of Algorithm 2 for a formal description.) To compute the minimum cycles \( \{z_2^0, z_3^0\}, z_1^0 \), we utilize an algorithm similar to Algorithm 1.

Note that a priori optimal selection of the components is not obvious; while introducing more components increases weights for cycles in the \( p \)-th regular complexes (because the components are disjoint), the cycle in \( K_{\beta} \) corresponding to this choice may have a smaller weight due to belonging to a different homology class (e.g. \( z_1'' \sim \partial(C_0) + \partial(C_1) \) may have much smaller weight than \( z_1' \sim \partial(C_0) \) in Figure 6b).

Our solution is as follows: generically, suppose that \( C_0, \ldots, C_k \) are all the \((p+1)\)-connected components of \( \mathbb{K}^p_{(b-1,d)} \setminus K_{\beta} \) with boundaries in \( K_{\beta} \), where \( C_0 \) is the one whose boundary contains \( \sigma_{\beta-1} \). We do the following:

1. For each \( j \in [0, k] \), compute the minimum (possibly empty) \( p \)-cycles \( \{z_i^j \mid b \leq i < d\} \) in \( C_j \) as described

Figure 6: (a) A complex \( K \) with all 1st critical vertices listed, in which \( v_1 \) is a monkey saddle; the direction of the height function is indicated. (b) The relevant subcomplex \( \mathbb{K}^0_{[b-1,d]} = \mathbb{K}^1_{[1,4]} \) with \( K_{\beta} \) broken from the remaining parts for a better illustration. (c) The complex \( K_{\beta} \) with boundaries filled by 2-dimensional “cells” drawn as darker regions. The blue edges are augmenting edges. Note that \( K_{\beta} \) also contains boundary 1-simplices around the critical vertex \( v_1 \), which are not drawn.
in Step 2 of Algorithm 2 (presented in Section 4.2.2). Note that for \( C_1 \) in Figure 6b, \( C_3^1 \) is empty, which makes \( C_2^1 \) null-homologous in \( K_{(2,3)}^1 \).

2. Build a dual graph \( G \) for \( K_\beta \), where dummy vertices \( \phi_0, \ldots, \phi_k \) correspond to the boundaries \( \partial(C_0), \ldots, \partial(C_k) \) and a single dummy vertex \( \overline{\phi} \) corresponds to the remaining boundary portion of \( K_\beta \). Roughly speaking, when a dummy vertex \( \phi_j \) is said to “correspond to” \( \partial(C_j) \), one can imagine that a \((p+1)\)-dimensional “cell” with boundary \( \partial(C_j) \) is added to \( K_\beta \) and \( \phi_j \) is the vertex dual to this cell. In addition to the regular dual edges, for each \( \phi_j \), we add to \( G \) an augmenting edge connecting \( \phi_j \) to \( \overline{\phi} \) and let its weight be \( \sum_{i=0}^{d-1} w(\zeta_i^j) \). See Figure 6c for an example of the dummy vertices and augmenting edges.

3. Compute the minimum \((s,t)\)-cut \((S^*, T^*)\) of \( (G, \phi_0, \overline{\phi}) \), which produces an optimal sequence of levelset persistent \( p \)-cycles for \( [\alpha_b^p, \alpha_d^p] \).

   To see the correctness of the algorithm, consider a general \((s,t)\)-cut \((S, T)\) of \( (G, \phi_0, \overline{\phi}) \). Whenever a \( \phi_j \) is in \( S \), it means that the component \( C_j \) is chosen to form the persistent cycles. Also, since the augmenting edge \( \{\phi_j, \overline{\phi}\} \) is crossing the cut, its weight records the cost of introducing \( C_j \) in forming the persistent cycles. Let \( \phi_0, \ldots, \phi_{\nu_t} \) be all the dummy vertices in \( S \). We have that the non-augmenting edges in \( E(S, T) \) produce a dual \( p \)-cycle \( \zeta_{b-1} \) in \( K_\beta \) homologous to \( \partial(C_{\nu_0}) + \cdots + \partial(C_{\nu_t}) \). Then, the \( p \)-cycle \( \zeta_{b-1} \), along with all \( \{\zeta_i^j | b \leq i < d\} \) from \( C_{\nu_0}, \ldots, C_{\nu_t} \), form a sequence of persistent \( p \)-cycles for \( [\alpha_b^p, \alpha_d^p] \) whose sum of weight equals \( \nu(S, T) \). Section 4.2.3 formally justifies the algorithm.

### 4.2.2 Pseudocode

We provide the full details of our algorithm in this subsection. For ease of exposition, so far we have let \( K_{(b-1,d)}^p \) be the complex on which we compute the optimal persistent cycles. However, this has a flaw, which can be illustrated by the example in Figure 6. Imagine that \( v_1^1 \) and \( v_2^1 \) in the figure are pinched together, so that \( K \) is not a 2-manifold anymore (but still a weak 2-pseudomanifold). The simplex-wise filtration \( F_p(f) \) can be constructed in a way that the disc around \( v_1^1 \) is formed before the disc around \( v_2^1 \); such an \( F_p(f) \) is essentially the same as the one before pinching. However, \( K_{(b-1,d)}^p \) now contains both \( v_1^1 \), \( v_2^1 \). As a consequence, Proposition 7 which is a major observation for our solution does not hold because the component containing \( v_1^1 \) (which is \( C_2 \) in Figure 6b with the right hole filled) also has its boundary containing \( \sigma_{\beta-1} \). To solve this, we make an adjustment to work on a complex \( \overline{K} \) instead of \( K_{(b-1,d)}^p \), so that Proposition 7 is still true; see Step 1 of Algorithm 2 for the definition of \( \overline{K} \). It can be easily verified that each complex in Sequence (3) (possibly excluding \( K_{(d-1,d)}^p \) which is indeed irrelevant) is a subcomplex of \( \overline{K} \).

Our exposition in Section 4.2.1 also frequently deals with the complex \( K_\beta \). However, in the pseudocode (Algorithm 2), \( K_\beta \) takes another form: we add to \( K_\beta \) some missing \((p+1)\)-simplices and denote the new complex as \( \overline{K}_\beta \); see Step 1 of the pseudocode for definition. Doing this makes our description of the components in Step 2 neater.

### Algorithm 2

Given the input as specified, the algorithm does the following:

1. Set the following:
   - \( \overline{K} = K_{(b-1,d)}^p \cup K_{\delta+1} \)
   - \( \overline{K}_\beta = K_\beta \cup \{\text{(p+1)-simplices with all p-faces in K_\beta}\} \)

2. Let \( C_0, \ldots, C_k \) be all the \((p+1)\)-connected components of \( \overline{K} \setminus \overline{K}_\beta \) such that \( \partial(C_j) \subseteq \overline{K}_\beta \) for each \( j \), where \( C_0 \) is the unique one whose boundary contains \( \sigma_{\beta-1} \). (Note that the boundary \( \partial(C_j) \) here means the boundary of the \((p+1)\)-chain \( C_j \).

For each \( C_j \), let \( M_j \) be the closure of \( C_j \), and among all sets of \( p \)-cycles of the form
\[{z_i \subseteq M_j \cap K^p_{(i,i+1)} | b \leq i < d}\]

such that

- \(z_b \sim \partial(C_j)\) in \(M_j \cap K^p_{(b-1,b+1)}\)
- \(z_{i-1} \sim z_i\) in \(M_j \cap K^p_{(i-1,i+1)}\) for each \(b < i < d\)
- \(z_{d-1}\) is null-homologous in \(M_j \cap K_{\delta + 1}\)

compute the set \(\{\zeta^i_l | b \leq i < d\}\) with the minimum sum of weight.

3. Build a weighted dual graph \(G\) from \(K_\beta\) as follows:

Let each \((p+1)\)-simplex of \(K_\beta\) correspond to a vertex in \(G\), and add the dummy vertices \(\phi, \phi_0, \ldots, \phi_k\) to \(G\). Let \(\theta\) denote the bijection from the \((p+1)\)-simplices to \(V(G) \setminus \{\phi, \phi_0, \ldots, \phi_k\}\).

Let each \(p\)-simplex \(\sigma\) of \(K_\beta\) correspond to an edge \(e\) in \(G\), where the weight of \(e, w(e)\), equals the weight of \(\sigma\). There are the following cases:

- \(\sigma\) has two \((p+1)\)-cofaces in \(K_\beta\): \(e\) is the usual one.
- \(\sigma\) has one \((p+1)\)-coface \(\tau\) in \(K_\beta\): If \(\sigma \in \partial(C_j)\) for a \(C_j\), let \(e\) connect \(\theta(\tau)\) and \(\phi_j\) in \(G\); otherwise, let \(e\) connect \(\theta(\tau)\) and \(\phi\).
- \(\sigma\) has no \((p+1)\)-cofaces in \(K_\beta\): If \(\sigma\) is in the boundaries of two components \(C_i\) and \(C_j\), let \(e\) connect \(\phi_i\) and \(\phi_j\); if \(\sigma\) is in the boundary of only one component \(C_j\), let \(e\) connect \(\phi_j\) and \(\phi\); otherwise, let \(e\) connect \(\phi\) on both ends.

In addition to the above edges, add the augmenting edges with weights as described. Let \(\theta\) also denote the bijection from the \(p\)-simplices to the non-augmenting edges and let \(E'(S,T)\) denote the set of non-augmenting edges crossing a cut \((S,T)\).

4. Compute the minimum \((s,t)\)-cut \((S^*,T^*)\) of \((G, \phi_0, \phi)\). Let \(\phi_{\mu_0}, \ldots, \phi_{\mu_1}\) be all the dummy vertices in \(S^*\). Then, set

\[z^*_{b-1} = \theta^{-1}(E'(S^*, T^*))\]

\[z^*_i = \sum_{j=0}^{i} \zeta^j_i\] for each \(b < i < d\)

Return \(z^*_{b-1}, z^*_b, \ldots, z^*_{d-1}\) as an optimal sequence of levelset persistent \(p\)-cycles for \([\alpha^p_\beta, \alpha^p_0])\).

As mentioned, the minimum cycles in Step 2 can be computed using a similar approach of Algorithm 1, with a difference that Algorithm 1 works on a complex “closed at both ends” while \(M_j\) is “closed only at the right”. Therefore, we need to add a dummy vertex to the dual graph for the boundary, which is put into the source. Note that we can build a single dual graph for all the \(M_j\)’s and share the dummy vertex, so that we only need to invoke one minimum cut computation.

4.2.3 Justification

In this subsection, we prove the correctness of Algorithm 2. We first present the following proposition stating a basic fact about \(\sigma_{\beta-1}\):

Proposition 6. The \(p\)-simplex \(\sigma_{\beta-1}\) has no \((p+1)\)-cofaces in \(K_\beta\).

Proof: Supposing instead that \(\sigma_{\beta-1}\) has a \((p+1)\)-coface \(\tau\) in \(K_\beta\), then \(\partial(\tau) \subseteq K_{\beta}\). Since \(K_\beta \subseteq K^p_{(\beta-1,\beta)}\), the \(p\)-cycle \(\partial(\tau)\) created by \(\sigma_{\beta-1}\) is a boundary in \(K^p_{(\beta-1,\beta)}\). Simulating a run of Algorithm 3 (presented in Appendix A) with input \(\mathcal{F}_p(f)\), at the \((\beta - 1)\)-th iteration, we can let \(\partial(\tau)\) be the representative \(p\)-cycle at index \(\beta\) for the new interval \([\beta, \beta]\). However, since \(\partial(\tau)\) is a boundary in \(K^p_{(\beta-1,\beta)}\), the interval starting with \(\beta\) must end with an index less than \(\delta\), which is a contradiction. \(\square\)
Proposition 7 justifies the operations in Step 2:

**Proposition 7.** Among all the \((p+1)\)-connected components of \(\overline{K} \setminus K_\beta\), there is one and only one component whose boundary resides in \(K_\beta\) and contains \(\sigma_{\beta-1}\).

**Proof.** See Appendix B.4.

Finally, Proposition 8 and 9 lead to Theorem 2, which is the conclusion.

**Proposition 8.** For any \((s,t)\)-cut \((S,T)\) of \((G,\phi_0,\overline{\phi})\), let \(\phi_{v_0}, \ldots, \phi_{v_t}\) be all the dummy vertices in \(S\). Furthermore, let \(z_{b-1} = \theta^{-1}(E'(S,T))\) and \(z_i = \sum_{j=0}^{d} \zeta^{(j)}\) for each \(b \leq i < d\). Then, \(z_{b-1}, z_b, \ldots, z_{d-1}\) is a sequence of levelset persistent \(p\)-cycles for \([\alpha^p, \alpha^d]_b\) with \(\sum_{i=b-1}^{d-1} w(z_i) = w(S,T)\).

**Proof.** Note that we can also consider \((S,T)\) as an \((s,t)\)-cut of a graph derived by deleting the augmenting edges from \(G\) where the sources are \(\phi_{v_0}, \ldots, \phi_{v_t}\) and the sinks are all the other dummy vertices. This implies that \(z_{b-1} = \theta^{-1}(E'(S,T))\) is homologous to \(\partial(C_{v_0} + \cdots + C_{v_t})\) in \(K_\beta\). Since \(\phi_0\) is the source of \(G\), \(\phi_0\) must be one of \(\phi_{v_0}, \ldots, \phi_{v_t}\). Then, by Proposition 7, \(\partial(C_{v_0} + \cdots + C_{v_t})\) contains \(\sigma_{\beta-1}\). So \(z_{b-1}\) must also contain \(\sigma_{\beta-1}\) because \(z_{b-1} = \partial(C_{v_0} + \cdots + C_{v_t})\) in \(K_\beta\) and \(\sigma_{\beta-1}\) has no \((p+1)\)-coface in \(K_\beta\) (Proposition 6). Furthermore, the properties of the cycles \(\{\zeta_j\}^{d}_{\zeta_0}\) computed in Step 2 of Algorithm 2 imply that \(z_b = \zeta_0 + \cdots + \zeta_{\beta-1}\) is homologous to \(\partial(C_{v_0} + \cdots + C_{v_t})\) in \(K_\beta\). So \(z_{b-1} \sim z_b\) in \(K_\beta\).

For \(z_{b-1}, z_b, \ldots, z_{d-1}\) to be persistent \(p\)-cycles for \([\alpha^p, \alpha^d]_b\), we need to verify several other conditions in Definition 6, in which only one is non-trivial, i.e., the condition that \([z_{d-1}] \in H_p(K_\delta)\) is the non-zero class in \(\text{Ker}(\phi_\beta)\). To see this, we first note that obviously \([z_{d-1}] \in \text{Ker}(\phi_\beta)\). To prove \([z_{d-1}] \neq 0\), we use a similar approach in the proof of Proposition 3, i.e., simulate a run of Algorithm 3 for computing \(P_{\delta}(F_p(f))\) and show that \(z_{d-1} \subseteq K_\delta\) can be the representative cycle at index \(\delta\) for the interval \([\beta, \delta]\). The details are omitted.

For the weight, we have

\[
w(S,T) = \sum_{e \in E'(S,T)} w(e) + \sum_{j=0}^{\ell} w(\{\phi_{v_j}, \overline{\phi}\}) = w(z_{b-1}) + \sum_{j=0}^{d-1} \sum_{i=b}^{d-1} w(\zeta^{(j)}_i) = w(z_{b-1}) + \sum_{i=b}^{d-1} \sum_{j=0}^{d-1} w(z_i)
\]

where \(\{\phi_{v_j}, \overline{\phi}\}\) denotes the augmenting edge in \(G\) connecting \(\phi_{v_j}\) and \(\overline{\phi}\).

**Proposition 9.** Let \(z_{b-1}, z_b, \ldots, z_{d-1}\) be any sequence of levelset persistent \(p\)-cycles for \([\alpha^p, \alpha^d]_b\); then, there exists an \((s,t)\)-cut \((S,T)\) of \((G,\phi_0,\overline{\phi})\) with \(w(S,T) \leq \sum_{i=b-1}^{d-1} w(z_i)\).

**Proof.** By definition, there exist \((p+1)\)-chains \(A_0 \subseteq K^{p}_{(b-1,b+1)}, \ldots, A_{d-1} \subseteq K^{p}_{(d-2,d)}, A_d \subseteq K^{p}_{\delta+1}\) such that \(z_{b-1} + z_b = \partial(A_0), \ldots, z_{d-2} + z_{d-1} = \partial(A_{d-1}), z_{d-1} = \partial(A_d)\). Let \(A = \sum_{i=d}^{d} A_i\); then, \(\partial(A) = z_{d-1}\). Let \(C_{v_0}, \ldots, C_{v_t}\) be all the components defined in Step 2 of Algorithm 2 which intersect \(A\). We claim that each \(C_{v_j} \subseteq A\). For contradiction, suppose instead that there is a \(\sigma \in C_{v_j}\) not in \(A\). Let \(\sigma'\) be a simplex in \(A \cap C_{v_j}\). Since \(\sigma, \sigma'\) are both in \(C_{v_j}\), there must be a \((p+1)\)-path \(\tau_1, \ldots, \tau_q\) from \(\sigma\) to \(\sigma'\) in \(\overline{K} \setminus K_\beta\). Note that \(\sigma \notin A\) and \(\sigma' \in A\), and so there is an \(\epsilon\) such that \(\tau_\epsilon \notin A\) and \(\tau_{\epsilon+1} \in A\). Let \(\tau_p\) be a \(p\)-face shared by \(\tau_\epsilon\) and \(\tau_{\epsilon+1}\) in \(\overline{K} \setminus K_\beta\); then, \(\tau_p \in \partial(A)\) and \(\tau_p \notin K_\beta\). This contradicts \(\partial(A) = z_{d-1} \subseteq K_\beta\). So \(C_{v_j} \subseteq A\). We also note that \(C_{v_0}, \ldots, C_{v_t}\) are all the \((p+1)\)-connected components of \(\overline{K} \setminus K_\beta\) intersecting \(A\). The reason is that, if \(\hat{C}\) is a component intersecting \(A\) whose boundary is not completely in \(K_\beta\), then we also have \(\hat{C} \subseteq A\) and the justification is similar as above. Let \(\sigma\) be a simplex in \(\partial(\hat{C})\) but not \(K_\beta\); then,
\( \sigma \in \partial(A) \). To see this, suppose instead that \( \sigma \notin \partial(A) \). Then \( \sigma \) has a \((p+1)\)-coface \( \tau_1 \in \widehat{C} \subseteq A \) and a \((p+1)\)-coface \( \tau_2 \in A \setminus \widehat{C} \). We have \( \tau_2 \in K_\beta \) because if not, combining the fact that \( \sigma, \tau_1, \tau_2 \in K \setminus K_\beta \) and \( \tau_1 \in \widehat{C}, \tau_2 \) would be in \( \widehat{C} \). As a face of \( \tau_2 \), \( \sigma \) must also be in \( K_\beta \), which is a contradiction. So we have \( \sigma \in \partial(A) \). Note that \( \sigma \notin K_\beta \), which contradicts \( \partial(A) \subseteq K_\beta \), and hence such a \( \widehat{C} \) cannot exist. We then have
\[
\partial(A \setminus \bigcup_{j=0}^{\ell} C_{\nu_j}) = \partial(A + C_{\nu_0} + \cdots + C_{\nu_{\ell}}) = z_{b-1} + \partial(C_{\nu_{\ell}}) + \cdots + \partial(C_{\nu_0}),
\]
where \( A \setminus \bigcup_{j=0}^{\ell} C_{\nu_j} \subseteq K_\beta \).

Now \( \partial(C_{\nu_0}) + \cdots + \partial(C_{\nu_{\ell}}) \) is homologous to \( z_{b-1} \) in \( K_\beta \), which means that it must contain \( \sigma_{\beta-1} \) because \( z_{b-1} \) contains \( \sigma_{\beta-1} \) and \( \sigma_{\beta-1} \) has no \((p+1)\)-coface in \( K_\beta \) (Proposition 7). This implies that \( \{ C_{\nu_0}, \ldots, C_{\nu_{\ell}} \} \) contains \( C_0 \) by Proposition 7. Let \( S = \theta(A \setminus \bigcup_{j=0}^{\ell} C_{\nu_j} \cup \{ \phi_{\nu_0}, \ldots, \phi_{\nu_{\ell}} \} \) and \( T = V(G) \setminus S \). It can be verified that \( (S, T) \) is an \((s, t)\)-cut of \( (G, \phi_{\nu_0}, \phi_{\nu_{\ell}}) \) and \( z_{b-1} = \theta^{-1}(E'(S, T)) \).

We then prove that \( w(S, T) \leq \sum_{i=b-1}^{d-1} w(z_i) \). Let \( A_i^{\nu_j} = M_{\nu_j} \cap A_i, z_i^{\nu_j} = M_{\nu_j} \cap z_i \) for each \( i \) and \( j \).

For any \( j \), we claim the following
\[
\partial \left( \sum_{i=b+1}^{d} A_i^{\nu_j} \right) = z_b^{\nu_j}.
\]

To prove Equation (4), we first note the following
\[
\partial \left( \sum_{i=b+1}^{d} A_i^{\nu_j} \right) = \partial \left( M_{\nu_j} \cap \bigcup_{i=b+1}^{d} A_i \right), z_b^{\nu_j} = M_{\nu_j} \cap z_b = M_{\nu_j} \cap \partial \left( \sum_{i=b+1}^{d} A_i \right).
\]

So we only need to show that \( \partial(M_{\nu_j} \cap \bigcup_{i=b+1}^{d} A_i) = M_{\nu_j} \cap \partial \left( \sum_{i=b+1}^{d} A_i \right) \). Letting \( B = \sum_{i=b}^{d} A_i \), what we need to prove now becomes \( \partial \left( M_{\nu_j} \cap B \right) = M_{\nu_j} \cap \partial \left( B \right) \). Consider an arbitrary \( \sigma \in \partial \left( M_{\nu_j} \cap B \right) \).

We have that \( \sigma \) is a face of only one \((p+1)\)-simplex \( \tau \in M_{\nu_j} \cap B \). Note that \( \tau \in B \), and we show that \( \tau \) is the only \((p+1)\)-coface of \( \sigma \) in \( B \). Suppose instead that \( \sigma \) has another \((p+1)\)-coface \( \tau' \) in \( B \). Then, \( \tau' \notin M_{\nu_j} \) because \( \tau' \notin M_{\nu_j} \cap B \). Note that \( B \subseteq K_{(b,d]} \) which means that \( \tau \) is disjoint with \( K_\beta \subseteq K_{(b,1-d]} \). So \( \tau' \in B \subseteq K \setminus K_{\beta} \). It is then true that \( \sigma \in K_{\beta} \) because if not, \( \sigma \in K \setminus K_{\beta} \), then \( \tau' \) would reside in \( C_{\nu_j} \subseteq M_{\nu_j} \) (following from \( \tau \in C_{\nu_j} \)). We now have \( \tau \in B \subseteq K_{(b,d]} \) and \( \sigma \in K_{\beta} \subseteq K_{(b,1-d]} \), which implies that \( \sigma \cap \tau = \emptyset \), contradicting \( \sigma \subseteq \tau \). Therefore, \( \sigma \in \partial(B) \).

Since \( \tau \in M_{\nu_j}, \) we have \( \sigma \in M_{\nu_j} \), and so \( \sigma \in M_{\nu_j} \cap \partial(B) \). On the other hand, let \( \sigma \) be any \( p \)-simplex in \( M_{\nu_j} \cap \partial(B) \). Since \( \sigma \in \partial(B) \), \( \sigma \) is a face of only one \((p+1)\)-simplex \( \tau \) in \( B \). We then prove that \( \tau \in M_{\nu_j} \). Suppose instead that \( \tau \notin M_{\nu_j} \). Then, since \( \sigma \in M_{\nu_j}, \sigma \) must be a face of \((p+1)\)-simplex \( \tau' \in M_{\nu_j} \). It follows that \( \sigma \in K_{\beta} \), because if not, \( \tau \) and \( \tau' \) would both be in \( M_{\nu_j} \). Then we reach the contradiction that \( \sigma \cap \tau = \emptyset \) because \( \tau \in B \subseteq K_{(b,d]} \) and \( \sigma \in K_{\beta} \subseteq K_{(b,1-d]} \). Therefore, \( \sigma \) is a face of only one \((p+1)\)-simplex \( \tau \) in \( M_{\nu_j} \cap B \), which means that \( \sigma \in \partial \left( M_{\nu_j} \cap B \right) \).

Note that \( \sum_{i=b}^{d} A_i^{\nu_j} = M_{\nu_j} \cap A = C_{\nu_j} \) because \( C_{\nu_j} \subseteq A \). Hence, by Equation (4)
\[
z_b^{\nu_j} = \partial \left( \sum_{i=b+1}^{d} A_i^{\nu_j} \right) = \partial \left( \sum_{i=b}^{d} A_i^{\nu_j} \right) + \partial(A_b^{\nu_j}) = \partial \left( \partial \left( \sum_{i=b}^{d} A_i^{\nu_j} \right) + \partial(A_b^{\nu_j}) \right)
\]

Now we have \( z_b^{\nu_j} + \partial(C_{\nu_j}) = \partial(A_b^{\nu_j}) \), i.e., \( z_b^{\nu_j} \sim \partial(C_{\nu_j}) \) in \( M_{\nu_j} \cap K_{(b,1-d]} \). Similar to Equation (4), for each \( i \) s.t. \( b < i < d \), we have \( \partial \left( \sum_{j=1}^{d-1} A_i^{\nu_j} \right) = z_i^{\nu_j} - 1 \) and \( \partial \left( \sum_{j=1}^{d} A_i^{\nu_j} \right) = z_i^{\nu_j} \). Therefore, \( \partial(A_b^{\nu_j}) = z_i^{\nu_j} + z_i^{\nu_j} \), i.e., \( z_i^{\nu_j} \sim \partial(C_{\nu_j}) \) in \( M_{\nu_j} \cap \partial(K_{(b,1-d]} \). We also have that \( \partial \left( \sum_{j=1}^{d} A_i^{\nu_j} \right) = z_i^{\nu_j} - 1 \), i.e., \( z_i^{\nu_j} \) is null homologous in \( M_{\nu_j} \cap K_{b+1} \). So \( \{ z_i^{\nu_j} | b < i < d \} \) is a set of \( p \)-cycles satisfying the condition specified in Step 2 of Algorithm 2, which means that \( \sum_{i=b}^{d-1} w(z_i^{\nu_j}) \leq \sum_{i=b}^{d-1} w(z_i^{\nu_j}) \).
Finally, we have

\[ w(S, T) = \sum_{e \in E'(S, T)} w(e) + \sum_{j=0}^{\ell} w\{\phi_{\nu j}, \bar{\phi}\} = w(z_{b-1}) + \sum_{j=0}^{\ell} \sum_{i=b}^{d-1} w(\zeta_{i}^{j}) \]

\[ \leq w(z_{b-1}) + \sum_{i=b}^{d-1} \sum_{j=0}^{\ell} w(\zeta_{i}^{j}) = \sum_{i=b}^{d-1} w(z_{i}) \]

where \{\phi_{\nu j}, \bar{\phi}\} denotes the augmenting edge in \(G\) connecting \(\phi_{\nu j}\) and \(\bar{\phi}\).

**Theorem 2.** Algorithm 2 computes an optimal sequence of level persistent \(p\)-cycles for a given closed-open interval.

### 4.3 Closed-closed case

In the subsection, we describe the computation of the optimal persistent \(p\)-cycles for a closed-closed interval \([\alpha_{b}^{p}, \alpha_{d}^{p}]\) from \(P_d(\mathcal{L}_{p}(f))\), which is produced by a simplex-wise interval \([K_{\beta}, K_{\delta}]\) from \(P_d(\mathcal{F}_{p}(f))\). Due to the similarity to the closed-open case, we only describe the algorithm briefly. Figure 7 provides examples for \(p = 1\), in which different sequences of persistent 1-cycles are formed for the interval \([\alpha_{1}^{p}, \alpha_{3}^{p}]\), and two of them are \(z_{2}^{3} + z_{3}^{1} + z_{1}^{3} + z_{1}^{2} + z_{2}^{3} + z_{1}^{2}\) and \(z_{2}^{3} + z_{1}^{0} + z_{2}^{3} + z_{1}^{2} + z_{2}^{2}\).

Similar to the previous cases, we have the following relevant portion of \(\mathcal{F}_{p}(f)\):

\[
\begin{align*}
\mathbb{K}_{P}^{(b-1, b)} &\hookrightarrow K_{\beta-1} \overset{\beta^{-1}}{\hookleftarrow} K_{\beta} \overset{\beta}{\hookrightarrow} \mathbb{K}_{P}^{(b-1, b+1)} \overset{\beta}{\hookrightarrow} \mathbb{K}_{P}^{(b, b+1)} \overset{\beta^{-1}}{\hookrightarrow} \cdots \\
&\vdots \\
&\overset{\beta^{-1}}{\hookleftarrow} \mathbb{K}_{P}^{(d-1, d)} \overset{\beta}{\hookrightarrow} \mathbb{K}_{P}^{(d-1, d+1)} \overset{\delta}{\hookleftarrow} K_{\delta+1} \overset{\delta}{\hookrightarrow} \mathbb{K}_{P}^{(d, d+1)}
\end{align*}
\]  

(5)

Note that the creator \(\sigma_{\beta^{-1}}\) and the destroyer \(\sigma_{\delta}\) are both \(p\)-simplices [4], and the computation can be restricted to the subcomplex \(\mathbb{K}_{P}^{(b-1, d+1)}\). Roughly speaking, the algorithm for the closed-closed case resembles the one for the closed-open case in that it now performs similar operations on both \(K_{\beta}\) and \(K_{\delta}\) as Algorithm 2 does on \(K_{\beta}\). The idea is as follows:

1. First, instead of directly working on \(K_{\beta}\) and \(K_{\delta}\), we work on \(\overline{K}_{\beta}\) and \(\overline{K}_{\delta}\), which include some missing \((p+1)\)-simplices. Formally, \(\overline{K}_{\beta} = K_{\beta} \cup \{(p+1)\}-\text{simplices with all } p\text{-faces in } K_{\beta}\}\), and \(\overline{K}_{\delta}\) is defined similarly.

2. Let \(C_{0}, \ldots, C_{k}\) be all the \((p+1)\)-connected components of \(\mathbb{K}_{P}^{(b-1, d+1)} \setminus \overline{K}_{\beta} \cup \overline{K}_{\delta}\) with boundaries in \(\overline{K}_{\beta} \cup \overline{K}_{\delta}\). Then, only \(C_{0}, \ldots, C_{k}\) can be used to form the persistent \(p\)-cycles in the \(p\)-th regular complexes. Re-index these components such that \(C_{0}, \ldots, C_{h}\) are all the ones whose boundaries contain both \(\sigma_{\beta^{-1}}\) and \(\sigma_{\delta}\). We have that \(h = 0\) or \(1\) (i.e., one or two of them). If \(h = 0\), then \(C_{0}\) must take part in forming a sequence of persistent cycles for \([\alpha_{b}^{p}, \alpha_{d}^{p}]\). If \(h = 1\), then either \(C_{0}\) or \(C_{1}\) but not both must take part in forming persistent cycles for the interval.

3. Compute minimum \(p\)-cycles in the \(p\)-th regular complexes similarly as in Step 2 of Algorithm 2. For a \(C_{j}\), let \(M_{j}\) be its closure. If the boundary of \(C_{j}\) lies completely in \(\overline{K}_{\beta}\), the computed \(p\)-cycles \(\{\zeta_{i}^{j} \subseteq M_{j} \cap \mathbb{K}_{P}^{(i, i+1)} \mid b \leq i < d\}\) is the set with the minimum sum of weight satisfying the conditions as in Step 2 of Algorithm 2. If the boundary of \(C_{j}\) lies completely in \(\overline{K}_{\delta}\), the computed minimum \(p\)-cycles satisfy symmetric conditions. If the boundary of \(C_{j}\) intersects both \(\overline{K}_{\beta}\) and \(\overline{K}_{\delta}\), the computed minimum \(p\)-cycles satisfy: \(\zeta_{d}^{j} \sim \partial(C_{j}) \cap \overline{K}_{\beta}\) in \(\mathbb{K}_{P}^{(b-1, b+1)}\), \(\zeta_{d}^{j} \sim \partial(C_{j}) \cap \overline{K}_{\delta}\) in \(\mathbb{K}_{P}^{(d-1, d+1)}\), and the consecutive cycles are homologous.
values listed. (b) The relevant subcomplex $K$ in optimal combination of the components and the cycles in $K$ computes the minimum 1-cycles (e.g., $K$ Figure 7: (a) A complex $K$ with the height function $f$ taken over the horizontal line and the 1st critical
intervals to be given so that the time used for computing the levelset zigzag barcode is not included. An empty dot indicates that the point is not included in the space.

4. To compute the optimal persistent $p$-cycles, we build a dual graph for $K_\beta \cup K_\delta$, in which the boundary of each $C_j$ corresponds to a dummy vertex $\phi_j$, and the remaining boundary portion corresponds to a dummy vertex $\bar{\phi}$. We also add the augmenting edges to the dual graph and set their weights similarly to Algorithm 2. For each $i$ s.t. $0 \leq i \leq h$, we build an $(s, t)$-graph on the dual graph of $K_\beta \cup K_\delta$ with source being $\{\phi_i\}$ and sink being $\{\phi, \phi_0, \ldots, \phi_h\} \setminus \{\phi_i\}$. The minimum $(s, t)$-cut for all the $(s, t)$-graphs we build produces an optimal sequence of persistent $p$-cycles for $[\alpha_0, \alpha_d]$.

We can look at Figure 7 for intuitions of the above algorithm, where $p = 1$ and the interval of interest is $[\alpha_3, \alpha_5]$. In Figure 7b, there are four 2-connected components of $K_{(2,6)} \setminus (K_\beta \cup K_\delta)$ with boundaries in $K_\beta \cup K_\delta$, which are $C_0, C_1, C_2, \text{and } C_3$. Among them, $C_0, C_1$ are the ones whose boundaries contain both $\sigma_{\beta-1}$ and $\sigma_\delta$. The persistent 1-cycles $z_1^2 + z_2^3 + z_3^1 + z_4^3, z_5^1 + z_6^2, z_{\bar{\delta}}^1$ come from the components $C_1$ and $C_3$, in which the starting one $z_1^2 + z_2^3$ is homologous to $\partial(C_1) \cap \overline{K_\beta} + \partial(C_3) \cap \overline{K_\delta}$, and the ending one $z_5^1 + z_6^2$ is homologous to $\partial(C_1) \cap K_\beta + \partial(C_3) \cap K_\delta$. Another sequence $z_{21}^2, z_{31}^0 + z_{41}^2, z_{51}^0 + z_{61}^2$ comes from $C_0$ and $C_2$, in which the starting one $z_{21}^2$ is homologous to $\partial(C_0) \cap K_\beta$, and the ending one $z_{51}^0 + z_{61}^2$ is homologous to $\partial(C_0) \cap \overline{K_\delta} + \partial(C_2)$.

We finally note that for the degenerate case of $b = d$, since there are no $p$-th regular complexes between $K_\beta$ and $K_\delta$, the algorithm needs an adjustment: one simply does not add augmenting edges at all.

**Complexity.** Let $n$ be the number of bits used to encode $K$. Then, for the three algorithms in this section, operations other than the minimum cut computation can be done in $O(n \log n)$ time. Using the max-flow algorithm by Orlin [20], the time complexity of all three algorithms is $O(n^2)$. Note that we assume persistence intervals to be given so that the time used for computing the levelset zigzag barcode is not included.
5 Equivalence of $p$-th and classical levelset filtrations

In this section, we prove that the $p$-th levelset filtration defined in Section 3.1 and the classical one defined by Carlsson et al. [4] produce equivalent $p$-th persistence intervals. We first recall the classical definition in Section 5.1 and then prove our conclusion in Section 5.2.

5.1 Classical levelset zigzag

Throughout this section, let $K$ be a finite simplicial complex with underlying space $X = |K|$ and $f : X \rightarrow \mathbb{R}$ be a generic PL function with critical values $\alpha_0 = -\infty < \alpha_1 < \cdots < \alpha_n < \alpha_{n+1} = \infty$. The original construction [4] of levelset zigzag persistence picks regular values $s_0, s_1, \ldots, s_n$ so that each $s_i \in (\alpha_i, \alpha_{i+1})$. Then, the levelset filtration of $f$, denoted $L^c(f)$, is defined as

$$L^c(f) : f^{-1}(s_0) \hookrightarrow f^{-1}[s_0, s_1] \hookrightarrow f^{-1}[s_1, s_2] \hookrightarrow \cdots \hookrightarrow f^{-1}[s_{n-1}, s_n] \hookrightarrow f^{-1}(s_n)$$ (6)

In order to align with our constructions in Section 3.1, we adopt an alternative but equivalent definition of $L^c(f)$ as follows, where we denote $f^{-1}(\alpha_i, \alpha_j)$ as $X_{i,j}$:

$$L^c(f) : X_{(0,1)} \hookrightarrow X_{(1,2)} \hookrightarrow X_{(1,3)} \hookrightarrow \cdots \hookrightarrow X_{(n-1,n+1)} \hookrightarrow X_{(n,n+1)}$$ (7)

Note that each $X_{(i,i+1)}$ deformation retracts to $f^{-1}(s_i)$ and each $X_{(i-1,i+1)}$ deformation retracts to $f^{-1}[s_{i-1}, s_i]$, so that zigzag modules induced by the two filtrations in (6) and (7) are isomorphic.

The barcode $Pd_p(L^c(f))$ is then the classical version of $p$-th levelset barcode defined in [4]. Intervals in $Pd_p(L^c(f))$ can also be mapped to real-value intervals in which the homological features persist:

| Type          | Interval                                  | Isomorphism       |
|---------------|-------------------------------------------|-------------------|
| closed-open   | $[X_{(b-1,b+1)}, X_{(d-1,d)}]$             | $[\alpha_b, \alpha_d]$ |
| open-closed   | $[X_{(b,b+1)}, X_{(d-1,d+1)}]$             | $(\alpha_b, \alpha_d)$ |
| closed-closed | $[X_{(b-1,b+1)}, X_{(d-1,d+1)}]$           | $[\alpha_b, \alpha_d]$ |
| open-open     | $[X_{(b,b+1)}, X_{(d-1,d)}]$               | $(\alpha_b, \alpha_d)$ |

5.2 Equivalence

The following theorem is the major conclusion of this section (recall that $L^c_p(f)$ is the continuous version of $p$-th levelset filtration of $f$ as in Definition 3):

**Theorem 3.** For an arbitrary PL function $f$ as defined above, the real-value intervals in $Pd_p(L^c(f))$ and $Pd_p(L^c_p(f))$ are the same.

To prove Theorem 3, we first provide the following proposition:

**Proposition 10.** Let $\alpha_i \leq \alpha_j \leq \alpha_k$ be critical values of $f$. If $\alpha_h$ is not a $p$-th homologically critical value for each $h$ s.t. $\ell < h \leq i$ or $j \leq h < k$, then the map $H_p(X_{(i,j)}) \rightarrow H_p(X_{(\ell,k)})$ induced by inclusion is an isomorphism.

**Proof.** We first prove that the inclusion-induced map $H_p(X_{(i,j)}) \rightarrow H_p(X_{(i,k)})$ is an isomorphism. For this, we build a Mayer-Vietoris pyramid similar to the one in [4] for proving the Pyramid Theorem. Moreover, in the pyramid, let $\mathcal{D}_1$ be the filtration along the northeastbound diagonal and $\mathcal{D}_2$ be the filtration along the bottom. An example is shown in Figure 8 for $j = i + 3$, $k = i + 5$, where inclusion arrows in $\mathcal{D}_1, \mathcal{D}_2$ are solid.
This means that no interval in \( \alpha \). The fact that \( f \) is not a \( p \)-th critical value for \( f \), and let \( \alpha_1 \leq \ldots \leq \alpha_m \leq \alpha_{m+1} = \infty \) be all the \( p \)-th homologically critical values of \( f \), and let \( \alpha_i = \alpha_{\lambda_i} \) for each \( i \). Note that \( \alpha_{(i,j)} = \alpha_{(\lambda_i,\lambda_j)} \) for \( i < j \). We first show that the two zigzag modules as defined in Figure 9 are isomorphic, where the upper module is \( H_p(\mathcal{L}^c(f)) \), and the lower module is a version of \( H_p(\mathcal{L}^c(f)) \) elongated by making several copies of \( p \)-th homology groups of the regular subspaces and connecting them by identity maps. The commutativity of the diagram is easily seen because all maps are induced by inclusion. The vertical maps are isomorphisms by Proposition 10. Hence, the two modules in Figure 9 are isomorphic. This means that persistence intervals of the two modules bijectively map to each other, and we also have that their corresponding real-value intervals are the same. For example, an interval \([\alpha_{(b\lambda, a\lambda+1)}, \alpha_{(a\lambda, a\lambda+1)}]\) from \( H_p(\mathcal{L}^c(f)) \) corresponds to an interval \([\alpha_{(b\lambda, b\lambda+1)}, \alpha_{(d\lambda, d\lambda+1)}]\) from \( H_p(\mathcal{L}^c(f)) \), and they both produce the real-value interval \([\alpha_{(\lambda b), \alpha_{(\lambda d)}}]\).

6 Connection to interval decomposition

In this section, we connect levelset persistent cycles to the interval decomposition of zigzag modules. Specifically, for a generic PL function \( f \), we show that levelset persistent \( p \)-cycles induce an entire interval
The interval submodule vector space in that levelset persistent cycles also induce interval submodules. We then have the following fact: induces a sequence of representative $p$-cycles directly from the proof of Proposition 9 in [10].

\[ \bigoplus_{i,j} \mathbb{K} \]

Submodules rest being from the trivial intervals (Theorem 4).

Figure 9: Two isomorphic zigzag modules where the upper module is $H_p(\mathcal{L}^c(f))$ and the lower module is an elongated version of $H_p(\mathcal{P}^c(f))$.

decomposition for $H_p(\mathcal{L}_p(f))$ (Theorem 5), and part of an interval decomposition for $H_p(\mathcal{F}_p(f))$ with the rest being from the trivial intervals (Theorem 4).

To reach the conclusions, we first define representative cycles for a simplex-wise filtration, which generate an interval submodule in a straightforward way, i.e., picking a cycle for a homology class at each position. Note that similar definitions also appear in works [10, 18] on zigzag persistence.

**Definition 11.** Let $p \geq 0$, $\mathcal{X} : X_0 \leftrightarrow \cdots \leftrightarrow X_\ell$ be a simplex-wise filtration, and $[b, d]$ be an interval in $\text{Pd}_p(\mathcal{X})$. Denote each linear map in $H_p(\mathcal{X})$ as $\psi_i : H_p(X_j) \leftrightarrow H_p(X_{j+1})$. The representative $p$-cycles for $[b, d]$ is a sequence of $p$-cycles $\{z_i \subseteq X_j | b \leq i \leq d\}$ such that:

1. For $b > 0$, $[z_b]$ is not in $\text{Img}(\psi_{b-1})$ if $X_{b-1} \leftrightarrow X_b$ is forward, or $[z_b]$ is the non-zero class in $\text{Ker}(\psi_{b-1})$ otherwise.

2. For $d < \ell$, $[z_d]$ is not in $\text{Img}(\psi_d)$ if $X_d \leftrightarrow X_{d+1}$ is backward, or $[z_d]$ is the non-zero class in $\text{Ker}(\psi_d)$ otherwise.

3. For each $i \in [b, d-1], [z_i] \leftrightarrow [z_{i+1}]$ by $\psi_i$, i.e., $[z_i] \mapsto [z_{i+1}]$ or $[z_i] \mapsto [z_{i+1}]$.

The interval submodule $\mathcal{I}$ of $H_p(\mathcal{X})$ induced by the representative $p$-cycles is a module such that $\mathcal{I}(i)$ equals the 1-dimensional vector space generated by $[z_i]$ for $i \in [b, d]$ and equals 0 otherwise, where $\mathcal{I}(i)$ is the $i$-th vector space in $\mathcal{I}$.

The following proposition connects representative cycles to the interval decomposition:

**Proposition 11.** Let $p \geq 0$, $\mathcal{X} : X_0 \leftrightarrow \cdots \leftrightarrow X_\ell$ be a simplex-wise filtration with $H_p(X_0) = 0$, and $\text{Pd}_p(\mathcal{X}) = \{[b_\alpha, d_\alpha] | \alpha \in \mathcal{A}\}$ be indexed by a set $\mathcal{A}$. One has that $H_p(\mathcal{X})$ is equal to a direct sum of interval submodules $\bigoplus_{\alpha \in \mathcal{A}} \mathcal{I}^{[b_\alpha, d_\alpha]}$ if and only if for each $\alpha$, $\mathcal{I}^{[b_\alpha, d_\alpha]}$ is induced by a sequence of representative $p$-cycles for $[b_\alpha, d_\alpha]$.

**Proof.** Suppose that $H_p(\mathcal{X}) = \bigoplus_{\alpha \in \mathcal{A}} \mathcal{I}^{[b_\alpha, d_\alpha]}$ is an interval decomposition. For each $\alpha$, define a sequence of representative $p$-cycles $\{z_i^\alpha | b_\alpha \leq i \leq d_\alpha\}$ for $[b_\alpha, d_\alpha]$ by letting $z_i^\alpha$ be an arbitrary cycle in the non-zero class of the $i$-th vector space of $\mathcal{I}^{[b_\alpha, d_\alpha]}$. It can be verified that $\{z_i^\alpha | b_\alpha \leq i \leq d_\alpha\}$ are valid representative $p$-cycles for $[b_\alpha, d_\alpha]$ inducing $\mathcal{I}^{[b_\alpha, d_\alpha]}$. This finishes the “only if” part of the proof. The “if” part follows directly from the proof of Proposition 9 in [10].

Now consider a generic PL function $f : [K] \to \mathbb{R}$ on a finite simplicial complex $K$ and a non-trivial interval $[K_\beta, K_\delta]$ of $\text{Pd}_p(\mathcal{F}_p(f))$ for $p \geq 1$. A sequence of levelset persistent $p$-cycles $\{z_i\}$ for $[K_\beta, K_\delta]$ induces a sequence of representative $p$-cycles $\{\zeta_j | \beta \leq j \leq \delta\}$ for this interval as follows: for any $K_j \in [K_\beta, K_\delta]$, we can always find a $z_i$ satisfying $z_i \subseteq K_j$, i.e., the complex that $z_i$ originally comes from is included in $K_j$; then, set $\zeta_j = z_i$. It can be verified that the induced representative $p$-cycles are valid so that levelset persistent cycles also induce interval submodules. We then have the following fact:
Theorem 4. For any non-trivial interval $J$ of $\text{Pd}_p(\mathcal{F}_p(f))$, a sequence of levelset persistent $p$-cycles for $J$ induces an interval submodule of $\text{H}_p(\mathcal{F}_p(f))$ over $J$. These induced interval submodules constitute part of an interval decomposition for $\text{H}_p(\mathcal{F}_p(f))$, where the remaining parts are from the trivial intervals.

Proof. This follows from Proposition 11. Note that in order to apply Proposition 11, $\text{H}_p(\mathbb{K}^p_{(0,1)})$ has to be trivial, where $\mathbb{K}^p_{(0,1)}$ is the starting complex of $\mathcal{F}_p(f)$. If the minimum value of $f$ is $p$-th critical, then $\mathbb{K}^p_{(0,1)} = \emptyset$, and so $\text{H}_p(\mathbb{K}^p_{(0,1)})$ is trivial. Otherwise, since $\text{H}_p(\mathbb{K}^p_{(0,1)}) = \text{H}_p(\mathbb{K}^p_{(0,2)})$ (Proposition 10) and $\mathbb{K}^p_{(0,2)}$ deformation retracts to a point, we have that $\text{H}_p(\mathbb{K}^p_{(0,1)})$ is trivial.

Similarly as for $\text{H}_p(\mathcal{F}_p(f))$, levelset persistent $p$-cycles can also induce interval submodules for $\text{H}_p(\mathcal{L}_p(f))$, the details of which are omitted. The following fact follows:

Theorem 5. Let $\text{Pd}_p(\mathcal{L}_p(f)) = \{J_k \mid k \in \Lambda\}$ be indexed by a set $\Lambda$. For any interval $J_k$ of $\text{Pd}_p(\mathcal{L}_p(f))$, a sequence of levelset persistent $p$-cycles for $J_k$ induces an interval submodule $I_k$ of $\text{H}_p(\mathcal{L}_p(f))$ over $J_k$. Combining all the modules, one derives an interval decomposition $\text{H}_p(\mathcal{L}_p(f)) = \bigoplus_{k \in \Lambda} I_k$.

Proof. This follows from Theorem 4. Note that $\text{H}_p(\mathcal{L}_p(f))$ can be viewed as being “contracted” from $\text{H}_p(\mathcal{F}_p(f))$. While in Theorem 4, the induced interval submodules form only part of the interval decomposition of $\text{H}_p(\mathcal{F}_p(f))$, the remaining submodules from the trivial intervals disappear in the interval decomposition of $\text{H}_p(\mathcal{L}_p(f))$.

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Appendix

A An abstract algorithm for zigzag persistence

We introduce an abstract algorithm for zigzag persistence that helps us prove some results in the appendix. Given $p \geq 0$ and a simplex-wise filtration $\mathcal{X} : \emptyset = X_0 \leftrightarrow \cdots \leftrightarrow X_\ell$ starting with an empty complex, the algorithm computes the $p$-th zigzag persistence intervals and their representative $p$-cycles for $\mathcal{X}$. Each linear map in $H_p(\mathcal{X})$ is denoted as $\psi_i : H_p(X_i) \leftrightarrow H_p(X_{i+1})$. Also, for any $i$ s.t. $0 \leq i \leq \ell$, $\mathcal{X}^i$ denotes
the filtration $X_0 \leftrightarrow X_1 \leftrightarrow \cdots \leftrightarrow X_i$, which is a prefix of $X$. Inspired by the algorithm by Maria and Oudot [18], the idea is to directly compute an interval decomposition by maintaining representative cycles for all intervals:

**Algorithm 3 (Zigzag persistence algorithm).** First set $Pd_p(X^0) = \emptyset$. The algorithm then iterates for $i \leftarrow 0, \ldots, \ell - 1$. At the beginning of the $i$-th iteration, the intervals and their representative cycles for $H_p(X^i)$ have already been computed. The aim of the $i$-th iteration is to compute these for $H_p(X^{i+1})$. For describing the $i$-th iteration, let $Pd_p(X^i) = \{[b_\alpha, d_\alpha] \mid \alpha \in A^i\}$ be indexed by a set $A^i$, and let $\{z_k^\alpha \subseteq X_k \mid b_\alpha \leq k \leq d_\alpha\}$ be a sequence of representative p-cycles for each $[b_\alpha, d_\alpha]$. For ease of presentation, we also let $z_k^\alpha = 0$ for each $\alpha \in A^i$ and each $k \in [0, i] \setminus [b_\alpha, d_\alpha]$. We call intervals of $Pd_p(X^i)$ ending with $i$ as surviving intervals at index $i$. Each non-surviving interval of $Pd_p(X^{i+1})$ is directly included in $Pd_p(X^{i+1})$ and its representative cycles stay the same. For surviving intervals of $Pd_p(X^i)$, the $i$-th iteration proceeds with the following cases:

**ψ is an isomorphism:** In this case, no intervals are created or cease to persist. For each surviving interval $[b_\alpha, d_\alpha]$ in $Pd_p(X^i)$, $[b_\alpha, i]$ now corresponds to an interval $[b_\alpha, i + 1]$ in $Pd_p(X^{i+1})$. The representative cycles for $[b_\alpha, i + 1]$ are set by the following rule:

Trivial setting rule of representative cycles: For each $j$ with $b_\alpha \leq j \leq i$, the representative cycle for $[b_\alpha, i + 1]$ at index $j$ stays the same. The representative cycle for $[b_\alpha, i + 1]$ at $i + 1$ is set to a $z_{i+1}^\alpha \subseteq X_{i+1}$ such that $[z_i^\alpha] \leftrightarrow [z_{i+1}^\alpha]$ by $\psi_i$ (i.e., $[z_i^\alpha] \mapsto [z_{i+1}^\alpha]$ or $[z_i^\alpha] \leftarrow [z_{i+1}^\alpha]$).

**ψ is forward with non-trivial cokernel:** A new interval $[i + 1, i + 1]$ is added to $Pd_p(X^{i+1})$ and its representative cycle at $i + 1$ is set to a $p$-cycle in $X_{i+1}$ containing $\sigma_i$ ($\sigma_i$ is a $p$-simplex). All surviving intervals of $Pd_p(X^i)$ persist to index $i + 1$ and are automatically added to $Pd_p(X^{i+1})$; their representative cycles are set by the trivial setting rule.

**ψ is backward with non-trivial kernel:** A new interval $[i + 1, i + 1]$ is added to $Pd_p(X^{i+1})$ and its representative cycle at $i + 1$ is set to a $p$-cycle homologous to $\partial(\sigma_i)$ in $X_{i+1}$ ($\sigma_i$ is a $(p + 1)$-simplex). All surviving intervals of $Pd_p(X^i)$ persist to index $i + 1$ and their representative cycles are set by the trivial setting rule.

**ψ is forward with non-trivial cokernel:** A surviving interval of $Pd_p(X^i)$ does not persist to $i + 1$. Let $B^i \subseteq A^i$ consist of indices of all surviving intervals. We have that $\{[z_i^\alpha] \mid \alpha \in B^i\}$ forms a basis of $H_p(X_i)$. Suppose that $\psi_i([z_i^{\alpha_1}] + \cdots + [z_i^{\alpha_h}]) = 0$, where $\alpha_1, \ldots, \alpha_h \in B^i$. We can rearrange the indices such that $b_{\alpha_1} < b_{\alpha_2} < \cdots < b_{\alpha_h}$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_h$. Let $\lambda$ be $\alpha_1$ if $\psi_{b_{\alpha_1} - 1}$ is backward for every $\alpha \in \{\alpha_1, \ldots, \alpha_h\}$ and otherwise be the largest $\alpha \in \{\alpha_1, \ldots, \alpha_h\}$ such that $\psi_{b_{\alpha_1} - 1}$ is forward. Then, $[b_\lambda, i]$ forms an interval of $Pd_p(X^{i+1})$. For each $k \in [b_\lambda, i]$, let $z_k^\alpha = z_{k-1}^{\alpha_1} + \cdots + z_{k-1}^{\alpha_h}$; then, $\{z_k^\alpha \mid b_\lambda \leq k \leq i\}$ is a sequence of representative cycles for $[b_\lambda, i]$. All the other surviving intervals of $Pd_p(X^i)$ persist to $i + 1$ and their representative cycles are set by the trivial setting rule.

**ψ is backward with non-trivial cokernel:** A surviving interval of $Pd_p(X^i)$ does not persist to $i + 1$. Let $B^i \subseteq A^i$ consist of indices of all surviving intervals, and let $z_i^{\alpha_1}, \ldots, z_i^{\alpha_h}$ be the cycles in $\{z_i^\alpha \mid \alpha \in B^i\}$ containing $\sigma_i$ ($\sigma_i$ is a $p$-simplex). We can rearrange the indices such that $b_{\alpha_1} < b_{\alpha_2} < \cdots < b_{\alpha_h}$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_h$. Let $\lambda$ be $\alpha_1$ if $\psi_{b_{\alpha_1} - 1}$ is forward for every $\alpha \in \{\alpha_1, \ldots, \alpha_h\}$ and otherwise be the largest $\alpha \in \{\alpha_1, \ldots, \alpha_h\}$ such that $\psi_{b_{\alpha_1} - 1}$ is backward. Then, $[b_\lambda, i]$ forms an interval of $Pd_p(X^{i+1})$ and the representative cycles for $[b_\lambda, i]$ stay the same. For each $\alpha \in \{\alpha_1, \ldots, \alpha_h\}$, let $z_k^\alpha = z_k^\alpha + z_k^\lambda$ for each $k$ s.t. $b_\alpha \leq k \leq i$, and let $z_{i+1}^\alpha = z_i^\alpha$; then, $\{z_k^\alpha \mid b_\alpha \leq k \leq i + 1\}$ is a sequence of representative cycles for $[b_\alpha, i + 1]$. For the other surviving intervals, the setting of representative cycles follows the trivial setting rule.
We now have that $i$ an $a$ valid interval decomposition of $\psi \in \cal{L}$ for $\alpha_i' \leftrightarrow \lambda \alpha_i$ instead that $\psi < b \alpha_i$. To show that Algorithm 3 is correct, we prove by induction on each $i \in [0, \ell]$ that the algorithm computes a valid interval decomposition of $H_p(X^i)$. For $i = 0$, this is trivially true. Now suppose that this is true for an $i \in [0, \ell - 1]$, i.e., before the $i$-th iteration, what the algorithm computes are valid. In the $i$-th iteration, for the case that $\psi_i$ is isomorphic, the case that $\psi_i$ is forward with non-trivial cokernel, or the case that $\psi_i$ is backward with non-trivial cokernel, the proof is similar to what is done in the proof of Proposition 11 and is omitted.

For the case that $\psi_i$ is forward with non-trivial cokernel, we first verify that $\{z_k^i \mid b_\lambda \leq k \leq i\}$ computed in the $i$-th iteration is a valid sequence of representative cycles for $[b_\lambda, i] \in \cal{P}d_p(X^i)$. Condition 2 of Definition 11 is trivially satisfied. Suppose that $\psi_{b_\lambda - 1}$ is forward. Then, for each $\alpha \in \{\alpha_1, \ldots, \alpha_h\} \setminus \{\lambda\}$ s.t. $z_{b_\lambda}^\alpha \neq 0$, we must have $b_\lambda < b_\lambda$. Therefore, $z_{b_\lambda}^\alpha$ is in the image of $\psi_{b_\lambda - 1}$. Since $z_{b_\lambda}^\alpha$ is not in the image of $\psi_{b_\lambda - 1}$, so is not $z_{b_\lambda - 1}^i = z_{b_\lambda}^{\alpha_1} + \cdots + z_{b_\lambda}^{\alpha_h}$, and Condition 1 is satisfied. For $k \in [b_\lambda, i - 1]$ s.t. $k + 1 \neq b_\lambda$ for any $\alpha \in \{\alpha_1, \ldots, \alpha_h\}$, we have that $[z_k^i] \leftrightarrow [z_{k+1}^i]$ by $\psi_k$ for each $\alpha \in \{\alpha_1, \ldots, \alpha_h\}$. This implies that $[z_k^i] \leftrightarrow [z_k^i]$ by $\psi_k$. For $k \in [b_\lambda, i - 1]$ s.t. $k + 1 = b_\lambda$ for a $\beta \in \{\alpha_1, \ldots, \alpha_h\}$, we have that $\psi_{b_\lambda - 1} = \psi_k$ is backward because $\beta$ is the largest $\alpha \in \{\alpha_1, \ldots, \alpha_h\}$ such that $\psi_{b_\lambda - 1}$ is forward. We then have $0 \leftrightarrow [z_{b_\lambda}^i]$ by $\psi_k$, which implies that $[z_k^i] \leftrightarrow [z_{k+1}^i]$ by $\psi_k$. Hence, Condition 3 is verified. Now suppose that $\psi_{b_\lambda - 1}$ is backward. Under this situation, every $\alpha \in \{\alpha_1, \ldots, \alpha_h\}$ has $\psi_{b_\lambda - 1}$ being backward where $b_\lambda$ is the smallest birth index. In $z_{b_\lambda}^i = z_{b_\lambda}^{\alpha_1} + \cdots + z_{b_\lambda}^{\alpha_h}$, $z_{b_\lambda}^\lambda$ is the only non-zero cycle. So we have $\psi_{b_\lambda - 1}([z_{b_\lambda}^i]) = \psi_{b_\lambda - 1}([z_{b_\lambda}^i]) = 0$, and Condition 1 is satisfied. It can also be verified that Condition 3 is satisfied. We now have that $\{z_k^i \mid b_\lambda \leq k \leq i\}$ is a valid sequence of representative cycles for $[b_\lambda, i] \in \cal{P}d_p(X^i)$ with $\psi_i([z_k^i]) = 0$. It is then not hard to verify that the $i$-th iteration computes a valid interval decomposition for $H_p(X_i^i)$.

For the case that $\psi_i$ is backward with non-trivial cokernel, we first verify that for any $\alpha \in \{\alpha_1, \ldots, \alpha_h\} \setminus \{\lambda\}$, the sequence $\{z_k^i \mid b_\lambda \leq k \leq i\}$ computed in the $i$-th iteration provides valid representative cycles for $[b_\alpha, i] \in \cal{P}d_p(X^i)$. Condition 2 of Definition 11 is trivially satisfied. Suppose that $\psi_{b_\alpha - 1}$ is backward and $b_\alpha < b_\lambda$. Then, $z_{b_\lambda}^i = z_{b_\lambda}^\alpha$, and Condition 1 is satisfied. For $k \in [b_\lambda, i - 1]$ s.t. $k + 1 \neq b_\alpha$, it is obvious that $[z_k^i] \leftrightarrow [z_{k+1}^i]$ by $\psi_k$. Since $\psi_{b_\lambda - 1}$ is backward, we have $\psi_{b_\lambda - 1}([z_{b_\lambda}^i]) = 0$, and so $\psi_{b_\lambda - 1}([z_{b_\lambda}^i]) = [z_{b_\lambda}^i]$. Hence, Condition 3 is satisfied. Suppose that $\psi_{b_\lambda - 1}$ is backward and $b_\alpha > b_\lambda$. Under this situation, $\psi_{b_\lambda - 1}$ must be forward because $\lambda$ is the largest $\alpha \in \{\alpha_1, \ldots, \alpha_h\}$ such that $\psi_{b_\lambda - 1}$ is backward. We then have that $z_{b_\lambda}^\alpha$ is outside the image of $\psi_{b_\lambda - 1}$ and $z_{b_\lambda}^\alpha$ is in, which implies that $[z_{b_\lambda}^i]$ is outside the image of $\psi_{b_\lambda - 1}$. Therefore, Condition 1 is satisfied. It is also not hard to see that Condition 3 is satisfied. Now suppose that $\psi_{b_\lambda - 1}$ is forward. Under this situation, every $\beta \in \{\alpha_1, \ldots, \alpha_h\}$ has $\psi_{b_\lambda - 1}$ being forward where $b_\lambda$ is the smallest birth index. Therefore, $\psi_{b_\lambda - 1}$ is forward and $b_\alpha > b_\lambda$. Condition 1 and 3 can be similarly verified. We now have that $\{z_k^i \mid b_\lambda \leq k \leq i\}$ is a valid sequence of representative cycles for $[b_\lambda, i] \in \cal{P}d_p(X^i)$ with $[z_k^i] \in \text{Im}(\psi_i)$. It is then not hard to verify that the $i$-th iteration computes a valid interval decomposition for $H_p(X_i^{i+1})$.

B Missing proofs

B.1 Proof of Proposition 1

We only prove that $K_\beta \subseteq \cal{K}_p(b-1,b]$ because the proof for $K_\delta \subseteq \cal{K}_p(d,d+1)$ is similar. For contradiction, assume instead that $K_\beta \nsubseteq \cal{K}_p(b-1,b]$. Note that from $\cal{K}_p(b-1,b]$ to $\cal{K}_p(b-1,b+1)$, we are not crossing any $p$-th critical values, and so the linear map $H_p(\cal{K}_p(b-1,b]) \to H_p(\cal{K}_p(b-1,b+1))$ is an isomorphism (see Proposition 10). Since $\cal{K}_p(b-1,b]$ appears between $\cal{K}_p(b-1,b)$ and $\cal{K}_p(b-1,b+1)$ in $\cal{F}_p(f)$ (see Definition 4), we have the following subsequence in $\cal{F}_p(f)$:

$$
\cal{K}_p(b-1,b) \hookrightarrow \cdots \hookrightarrow \cal{K}_p(b-1,b] \hookrightarrow \cdots \hookrightarrow K_\beta \hookrightarrow \cdots \hookrightarrow \cal{K}_p(b-1,b+1) \hookrightarrow \cdots \hookrightarrow K_\delta
$$

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The fact that $[K_{\beta}, K_{\delta}]$ forms an interval in $\mathcal{P}_{p}(F_{p}(f))$ indicates that a $p$-th homology class is born (and persists) when $\mathbb{K}_{p}(b_{-1}, b)$ is included into $\mathbb{K}_{p}(b_{-1}, b+1)$, contradicting the fact that $H_{p}(\mathbb{K}_{p}(b_{-1}, b)) \rightarrow H_{p}(\mathbb{K}_{p}(b_{-1}, b+1))$ is an isomorphism.

### B.2 Proof of Proposition 2

Let $S$ consist of simplices of $K$ not in $\mathbb{K}_{i,j}^{p}$ whose interiors intersect $\mathbb{X}_{i,j}^{p}$. Then, let $\sigma$ be a simplex of $S$ with no proper cofaces in $S$. We have that there exists a $u \in \sigma$ with $f(u) \in (\alpha_{i}^{p}, \alpha_{j}^{p})$ and a $w \in \sigma$ with $f(w) \not\in (\alpha_{i}^{p}, \alpha_{j}^{p})$. If $f(w) \leq \alpha_{i}^{p}$, then all vertices in $\sigma$ must have the function values falling in $(\alpha_{i}^{p}-1, \alpha_{j}^{p}+1)$ because $K$ is compatible with the $p$-th levelsets of $f$. We then have that $|\sigma| \cap \mathbb{X}_{i,j}^{p}$ deformation retracts to $\partial|\sigma| \cap \mathbb{X}_{i,j}^{p}$, where $\partial|\sigma|$ denotes the boundary of the topological disc $|\sigma|$. This implies that $\mathbb{X}_{i,j}^{p}$ deformation retracts to $\mathbb{X}_{i,j}^{p} \backslash \text{Int}(\sigma)$, where $\text{Int}(\sigma)$ denotes the interior of $|\sigma|$. If $f(w) \geq \alpha_{j}^{p}$, the result is similar. After doing the above for the all such $\sigma$ in $S$, we have that $\mathbb{X}_{i,j}^{p}$ deformation retracts to $\mathbb{X}_{i,j}^{p} \backslash \bigcup_{\sigma \in S} \text{Int}(\sigma)$. Note that $\mathbb{X}_{i,j}^{p} \backslash \bigcup_{\sigma \in S} \text{Int}(\sigma) = |\mathbb{K}_{i,j}^{p}|$, and so the proof is done.

### B.3 Proof of Proposition 3

For the proof, we first observe the following fact which follows immediately from Proposition 11:

**Proposition 12.** Let $p \geq 0$, $X : X_{0} \leftrightarrow \cdots \leftrightarrow X_{t}$ be a simplex-wise filtration with $H_{p}(X_{0}) = 0$, $[\beta', \delta']$ be an interval in $\mathcal{P}_{p}(X)$, and $\zeta_{p}, \ldots, \zeta_{p}'$ be a sequence of representative $p$-cycles for $[\beta', \delta']$. One has that $\zeta_{i}$ is not a boundary in $X_{i}$ for each $\beta' \leq i \leq \delta'$.

The following fact is also helpful to our proof:

**Proposition 13.** Let $X$ be a simplicial complex, $A$ be a $q$-chain of $X$ where $q \geq 1$, and $X'$ be the closure of a $q$-connected component of $X$; one has that $X' \cap \partial(A) = \partial(X' \cap A)$.

**Proof.** First, let $B$ be an arbitrary $q$-chain of $X$ and $\sigma^{q-1}$ be an arbitrary $(q-1)$-simplex in $X$. Define $\text{cof}_{q}(B, \sigma^{q-1})$ as the set of $q$-simplices in $B$ having $\sigma^{q-1}$ as a face. It can be verified that $\text{cof}_{q}(B, \sigma^{q-1}) = \text{cof}_{q}(X' \cap B, \sigma^{q-1})$ if $\sigma^{q-1} \in X'$.

To prove the proposition, let $\sigma^{q-1}$ be an arbitrary $(q-1)$-simplex in $X' \cap \partial(A)$. Since $\sigma^{q-1} \in \partial(A)$, we have that $|\text{cof}_{q}(A, \sigma^{q-1})|$ is an odd number. Since $\sigma^{q-1} \in X'$, the fact in the previous paragraph implies that $|\text{cof}_{q}(X' \cap A, \sigma^{q-1})| = |\text{cof}_{q}(A, \sigma^{q-1})|$ is also an odd number. Therefore, $\sigma^{q-1} \in \partial(X' \cap A)$. On the other hand, let $\sigma^{q-1}$ be an arbitrary $(q-1)$-simplex in $\partial(X' \cap A)$. Then, $|\text{cof}_{q}(X' \cap A, \sigma^{q-1})|$ is an odd number. Since $\sigma^{q-1}$ is a face of a $q$-simplex in $X'$, we have that $\sigma^{q-1} \in X'$. Therefore, $|\text{cof}_{q}(A, \sigma^{q-1})| = |\text{cof}_{q}(X' \cap A, \sigma^{q-1})|$ is an odd number. So we have that $\sigma^{q-1} \in \partial(A)$ and then $\sigma^{q-1} \in X' \cap \partial(A)$. □

Now we prove Proposition 3. Let $z_{b}, \ldots, z_{d-1}$ be a sequence of persistent $p$-cycles for $(\alpha_{i}^{p}, \alpha_{j}^{p})$ as claimed. Note that $|\partial(\sigma_{b-1})|$ is the non-zero class in $\text{Ker}(\beta_{b-1})$. Therefore, by Definition 5, $\partial(\sigma_{b-1}) \sim z_{b}$ in $K_{\beta}$. This means that there exists a $(p+1)$-chain $C \subseteq K_{\beta}$ such that $z_{b} + \partial(\sigma_{b-1}) = \partial(C)$. Let $A_{b} = C + \sigma_{b-1}$; then, $z_{b} = \partial(A_{b})$ where $A_{b}$ is a $(p+1)$-chain in $K_{\beta-1}$ containing $\sigma_{b-1}$. Similarly, we have that $z_{d-1} = \partial(A_{d})$ for a $(p+1)$-chain $A_{d} \subseteq K_{\beta+1}$ containing $\sigma_{d}$. By Definition 5, there exists a $(p+1)$-chain $A_{i} \subseteq \mathbb{K}_{i,i+1}$ for each $b < i < d$ such that $z_{i-1} + \partial(z_{i}) = \partial(A_{i})$. Thus, $A_{b}, \ldots, A_{d}$ are the $(p+1)$-chains satisfying the condition in Claim 2. Let $z_{i}' = K' \cap z_{i}$ and $A_{i}' = K' \cap A_{i}$ for each $i$. By Proposition 13, $z_{i}' = \partial(A_{i}')$. Since $A_{i}'$ contains $\sigma_{b-1}$, it follows that $z_{b}' + \partial(\sigma_{b-1}) = \partial(A_{b}' \setminus \{\sigma_{b-1}\})$, where $A_{b}' \setminus \{\sigma_{b-1}\} \subseteq K_{\beta}$. It is then true that $z_{b}' \sim \partial(\sigma_{b-1})$ in $K_{\beta}$. Now we simulate a run of Algorithm 3 for
computing $\text{Pd}_p(\mathcal{F}_p(f))$. Then, at the $(\beta - 1)$-th iteration of the run, we can let $z'_b \subseteq K_\beta$ be the representative cycle at index $\beta$ for the new interval $[\beta, \beta]$.

Let $\lambda$ be the index of the complex $\mathbb{K}^p_{(b,b+2)}$ in $\mathcal{F}_p(f)$, i.e., $K_\lambda = \mathbb{K}^p_{(b,b+2)}$. In the run of Algorithm 3, the interval starting with $\beta$ must persist to $\lambda$ because this interval ends with $\delta$. At any $j$-th iteration for $\beta \leq j \leq \lambda - 2$, other than the case that $\varphi_j$ is backward with a non-trivial cokernel, the setting of representative cycles for all intervals persisting through follows the trivial setting rule. For the case that $\varphi_j$ is backward with a non-trivial cokernel, since $z'_b \subseteq K_{j+1}$, the setting of the representative cycles for the interval $[\beta, j + 1]$ must also follow the trivial setting rule. Hence, at the end of the $(\lambda - 2)$-th iteration, $z'_b \subseteq K_{\lambda - 1}$ can be the representative cycle at index $\lambda - 1$ for the interval $[\beta, \lambda - 1]$. Meanwhile, it is true that $K' \cap (z_b + z_{b+1}) = K' \cap z_b + K' \cap z_{b+1}$. So $z'_b + z'_{b+1} = K' \cap \partial(A_{b+1}) = \partial(K' \cap A_{b+1}) = \partial(A'_{b+1})$, which means that $z'_b \sim z'_{b+1}$ in $\mathbb{K}^p_{(b,b+2)} = K_\lambda$. Therefore, $[z'_b] \mapsto [z'_{b+1}]$ by $\varphi_{\lambda - 1}$, which means that $z'_{b+1} \subseteq K_\lambda$ can be the representative cycle at index $\lambda$ for the interval $[\beta, \lambda]$. By repeating the above arguments on each $z'_{i}$ that follows, we have that $z'_{i-1} \subseteq K_\delta$ can be the representative cycle at index $\delta$ for the interval $[\beta, \delta]$. Finally, for contradiction, assume instead that $\sigma_\delta \not\subseteq K'$. This means that $\sigma_\delta \not\subseteq A'_d$, and hence $A'_d \subseteq K_\delta$. Since $z'_{d-1} = \partial(A'_d)$, we then have that $z'_{d-1}$ is a boundary in $K_\delta$. However, by Proposition 12, $z'_{d-1}$ cannot be a boundary in $K_\delta$, which is a contradiction. Therefore, Claim 1 is proved. Furthermore, we have that $z'_0, \ldots, z'_{d-1}$ and $A'_0, \ldots, A'_d$ satisfy the condition in Claim 2.

To prove the last statement of Claim 2, first note that $\partial(\sum_{i=b}^d A'_i) = 0$. Let $A' = \sum_{i=b}^d A'_i$. Since $\sigma_{\beta - 1} \not\subseteq \mathbb{K}^p_{(b,b+1+1)}$ and $\sigma_{\beta - 1} \not\subseteq \mathbb{K}^p_{(b,b+1)}$, there must be a vertex in $\sigma_{\beta - 1}$ with function value in $(\alpha'_{b-1}, \alpha'_{b})$. So $\sigma_{\beta - 1} \not\subseteq \mathbb{K}^p_{(b,b+1)}$, which means that $\sigma_{\beta - 1} \not\subseteq A'_i$ for any $b < i \leq d$. We also have that $\sigma_{\beta - 1} \not\subseteq A'_{b}$, and hence $\sigma_{\beta - 1} \subseteq A'$. We then show that $A'$ equals the set of $(p + 1)$-simplices of $K'$. First note that $A' \subseteq K'$. Then, for contradiction, suppose that there is a $(p + 1)$-simplex $\sigma \subseteq K'$ not in $A'$. Since $\sigma \subseteq K'$, there is a $(p + 1)$-path $\tau_0, \ldots, \tau_{l}$ from $\sigma$ to $\sigma_{\beta - 1}$ in $K'$. Since $\sigma \not\subseteq A'$ and $\sigma_{\beta - 1} \subseteq A'$, there must be a $j$ such that $\tau_j \not\subseteq A'$ and $\tau_{j+1} \subseteq A'$. Let $\tau_j$ and $\tau_{j+1}$ share a $p$-face $\tau^p$; then, $\tau^p \in \partial(A')$, contradicting the fact that $\partial(A') = 0$. For the disjointness of $A'_0, \ldots, A'_d$, suppose instead that there is a $\sigma$ residing in more than one of $A'_0, \ldots, A'_d$. Then, $\sigma$ can only reside in two consecutive chains $A'_i$ and $A'_{i+1}$, because pairs of chains of other kinds are disjoint. This implies that $\sigma \not\subseteq A'$, contradicting the fact that $A'$ contains all $(p + 1)$-simplices of $K'$. Thus, Claim 2 is proved.

Combining the fact that $\partial(A') = 0$, $K'$ is a pure weak $(p + 1)$-pseudomanifold, and Claim 2, we can reach Claim 3.

### B.4 Proof of Proposition 7

We first show that there is at least one such component. Let $z_{b-1}, z_b, \ldots, z_{d-1}$ be a sequence of persistent $p$-cycles for $[\alpha^p_0, \alpha^p_d]$. Then, by definition, there exist $(p + 1)$-chains $A_b \subseteq \mathbb{K}^p_{(b-1,b+1)}$, $A_{b+1} \subseteq \mathbb{K}^p_{(b,b+2)}$, $A_d \subseteq \mathbb{K}^p_{(d-2,d)}$ such that $z_{b-1} + z_b = \partial(A_b), \ldots, z_{d-2} + z_{d-1} = \partial(A_{d-1}), z_{d-1} = \partial(A_d)$. Let $A = \sum_{i=b}^d A_i$; then, $\partial(A) = z_{b-1} \subseteq \overline{K}_\beta$. Note that $\sigma_{\beta - 1} \subseteq z_{b-1}$ by definition, which implies that $\sigma_{\beta - 1}$ is a face of only one $(p + 1)$-simplex $\tau \in A$. Note that $\tau \not\subseteq \overline{K}_\beta$ by Proposition 6, which means that $\tau \in \overline{K} \setminus \overline{K}_\beta$. Let $\mathcal{C}$ be the $(p + 1)$-connected component of $\overline{K} \setminus \overline{K}_\beta$ containing $\tau$. We show that $\mathcal{C} \subseteq A$. For contradiction, suppose instead that there is a $\tau' \in \mathcal{C}$ which is not in $A$. Since $\tau, \tau' \in C$, there is a $(p + 1)$-path $\tau_0, \ldots, \tau_{l}$ from $\tau$ to $\tau'$ in $\overline{K} \setminus \overline{K}_\beta$. Also since $\tau_1 \in A$ and $\tau_{l} \not\subseteq A$, there must be an $i$ such that $\tau_i \in A$ and $\tau_{i+1} \not\subseteq A$. Let $\tau^p$ be a $p$-face shared by $\tau_i$ and $\tau_{i+1}$ in $\overline{K} \setminus \overline{K}_\beta$; then, $\tau^p \in \partial(A)$ and $\tau^p \not\subseteq \overline{K}_\beta$. This contradicts $\partial(A) \subseteq \overline{K}_\beta$. Since $\mathcal{C} \subseteq A$, we have that $\tau$ is the only $(p + 1)$-coface of $\sigma_{\beta - 1}$ in $C$, which means that $\sigma_{\beta - 1} \in \partial(C)$. We then show that $\partial(\overline{C}) \subseteq \overline{K}_\beta$. For contradiction, suppose instead that there is a $\sigma \in \partial(C)$ which is not in $\overline{K}_\beta$, and let $\sigma'$ be the only $(p + 1)$-coface of $\sigma$ in $C$. If $\sigma$ has only one $(p + 1)$-coface in $\overline{K}$, the fact that $\mathcal{C} \subseteq A$ implies that $\tau'$ is the only $(p + 1)$-coface of $\sigma$ in $A$. Hence, $\sigma \in \partial(A)$, contradicting $\partial(A) \subseteq \overline{K}_\beta$. If $\sigma$ has
another \((p + 1)\)-coface \(\tau''\) in \(\bar{K}\), then \(\tau''\) must not be in \(\bar{K}_\beta\) because the \(p\)-face \(\sigma\) of \(\tau''\) is not in \(\bar{K}_\beta\). So \(\tau'' \in \bar{K} \setminus \bar{K}_\beta\). Then, \(\tau'' \in \mathcal{C}\) because it shares a \(p\)-face \(\sigma \in \bar{K} \setminus \bar{K}_\beta\) with \(\tau' \in \mathcal{C}\), contradicting the fact that \(\tau'\) is the only \((p + 1)\)-coface of \(\sigma\) in \(\mathcal{C}\). Now we have constructed a \((p + 1)\)-connected component \(\mathcal{C}\) of \(\bar{K} \setminus \bar{K}_\beta\) whose boundary resides in \(\bar{K}_\beta\) and contains \(\sigma_{\beta-1}\).

We then prove that there is only one such component. For contradiction, suppose that there are two components \(\mathcal{C}_l, \mathcal{C}_j\) among \(\mathcal{C}_0, \ldots, \mathcal{C}_k\) whose boundaries contain \(\sigma_{\beta-1}\). Then, at least one of \(\mathcal{C}_l, \mathcal{C}_j\) does not contain \(\sigma_\delta\). Let \(\mathcal{C}_j\) be the one not containing \(\sigma_\delta\). Note that the set \(\{\zeta^j_l \mid b \leq i < d\}\) computed in Step 2 of Algorithm 2 satisfies that \(\zeta^j_{d-1}\) is null-homologous in \(M_j \cap K_{\delta+1}\). The fact that \(\sigma_\delta \notin M_j\) implies that \(\zeta^j_{d-1}\) is also null-homologous in \(K_\delta\). Then, similar to the proof for Claim 1 of Proposition 3, we can derive a representative cycle \(\zeta^j_{d-1}\) for the interval \([\beta, \delta]\) at index \(\delta\) which is a boundary, and thus a contradiction.