Momenta fields and the derivative expansion

Luca Zambelli$^1$

$^1$Theoretisch-Physikalisches Institut, Friedrich-Schiller-Universität Jena, D-07743 Jena, Germany

The Polchinski exact renormalization group equation for a scalar field theory in arbitrary dimensions is translated, by means of a covariant Hamiltonian formalism, into a partial differential equation for an effective Hamiltonian density that depends on an infinite tower of momenta fields with higher spin. A natural approximation scheme is then expanding the Hamiltonian in momenta with increasing rank. The first order of this expansion, one next to the local potential approximation, is regulator-independent and already includes infinitely many derivative interactions. Further truncating this down to a quadratic dependence on the momenta leads to an alternative to the first order of the derivative expansion, which is used to compute $\eta = 0.03616(1)$ for the critical exponent of the three dimensional Ising model.

I. INTRODUCTION AND SUMMARY

This work develops a new method to construct approximate solutions of functional renormalization group (FRG) equations. The latter provide an exact representation of quantum or statistical field theory [1], and as such they can be used to describe, at least in principle, all kinds of nonperturbative phenomena. Whether this is feasible in practice solely depends on being able to truncate the FRG equations to a solvable set of equations that encode the wanted pieces of information [2]. Though it is desirable to adjust approximations to each specific problem, these should allow for some control, in the form of progressive improvement, such that it is necessary to formulate systematic strategies, and not simply ad hoc ansätze. Furthermore, since any truncation induces errors that can be hard to estimate and reduce, it is of primary importance that a rich pool of approximation schemes be available. We therefore believe that developing one more of them might be a worthy endeavor, even after forty years of FRG studies.

While dealing with infinitely many interaction terms is not a problem, as shown by the successes of the local potential approximations (LPA), which include generic functions of constant fields [3], the description of nontrivial momentum dependence of correlation functions is less straightforward and triggered the development of several approximation strategies. Historically two such schemes have been playing a prominent role in the scientific community. The first is the vertex expansion (VE), which is an expansion in field variables while retaining the full momentum dependence [4, 5]. The infinite tower of flow equations for the $n$-point functions is therefore truncated by considering only vertices up to a given number of legs. The second is the derivative expansion (DE), that is an expansion in powers of momenta while retaining the full field dependence [6]. This approximation makes use of local actions with definite given powers of field derivatives, and it relies on the assumption that the system possesses at least one mass-scale $m$, below which higher powers of $(-\partial^2/m^2)$ play a progressively less important role. As a matter of fact, however, it performs quite well also in describing conformal models and critical phenomena [6–12], that since the early history of the FRG have been a standard application and a reference point for improvements of approximation methods. For specific examples of applications of these two approximation schemes we refer the reader to the reviews [2].

As it is to be expected, several alternatives to these two complementary approaches have been developed. The method introduced by Blaizot, Mendez-Galain and Wschebor (BMW) [13, 14] allows one to keep infinitely many vertices into account as in the DE, while retaining generic functions of momenta as in the VE. This is accomplished by making use of the flow equations for the $n$-point functions, but neglecting part of the momentum dependence of some vertices. Much closer to the DE is the scaling fields expansion (SFE) [15–17], that is an expansion in eigenoperators of the linearized flow around the Gaußian FP. This method has been recently revived in [18], where it was critically compared to the DE. By making use of several features of the SFE, the same authors proposed to apply a derivative expansion to the action expressed in terms of a normal ordered basis of monomials of $\phi$ [19]. In the following we will refer to this approach by the name of normal ordered derivative expansion (NDE). Other methods have been proposed, for instance [20] and [21], but we do not aim at a comprehensive enumeration.

The construction of systematic approximations goes together with the desire to minimize problems, and not just to maximize the accessible amount of information. In fact, apart for the sometimes unknown effect of a truncation of the theory space, the FRG results are affected by other ambiguities, which can be considered as consequences of the former. Among these is the possible breaking of some wanted symmetries. A simple example is the ubiquitous symmetry under linear rescalings of the fields, which is sometimes referred to as linear reparametrization invariance. This is generically broken by the above mentioned truncation schemes, but it can be restored by an appropriate choice of regulator functions [8, 9]. Indeed, another truncation-induced ambiguity is the de-
pends on the regularization itself, which often affects quantities that should be universal. In fact, regulator dependence is present at any order of the DE beyond the LPA, as well as in the VE, in the SFE, and in the BMW method, though some approximations might result in a less severe dependence than others. Clearly, the regulator choice is not totally free and it is actually possible to optimize it for each approximation [22]. We do not know if it is possible to devise a systematic truncation strategy that preserves regularization independence of universal quantities and reparametrization invariance of the exact RG equations at any order. The goal of this work is a much easier one, yet such issues will be relevant in the present discussion.

By taking a conservative point of view, we would like to facilitate the computation of high orders of the DE. This will result in the construction of an approximation scheme that is not the DE, though it is extremely close to it. While the DE has been already pushed to the second order (\(\partial^4\)) [11, 12] and to the third order (\(\partial^6\)) [23] for a real scalar field theory, it is nowadays hard to imagine to go very far beyond these orders. The combinatoric computational difficulty in obtaining the flow equations for a high order of the DE is essentially a particular form of the combinatoric difficulty of dealing with high orders of a Taylor expansion, the variable of expansion being momentum \(p^2\). The standard way of circumventing this combinatoric problem is to compute the flow of a full function at once, which would require to keep a generic off-shell value for \(p^2\). Yet, in a DE setup one would also need to keep \(\phi\) generic and constant, in order to describe infinitely many vertices. The traditional way to achieve these conditions simultaneously is by means of a Hamiltonian formalism, whose application to the FRG is the subject of the present work.

In order to implement this idea we confine our discussion to a real \(\mathbb{Z}_2\)-symmetric scalar field theory, and we restrict ourselves to a specific class of truncations which can be essentially identified with an arbitrarily high order of the DE. The RG flow of these truncations will be analyzed in Sec. II for the specific case of the Polchinski equation [24], where we discuss how different regularizations, or coarse-graining prescriptions, lead to structurally different truncated equations. In Sec. III we apply the above mentioned idea by describing the Hamiltonian truncation of these equations, which amounts to replacing the arbitrary-order derivative

\[
\frac{d}{dx_{\mu_1}} \cdots \frac{d}{dx_{\mu_m}} \phi(x) \rightarrow \pi_{\mu_1 \cdots \mu_m}
\]

with a symmetric tensor field. An important conceptual feature of this program is that derivative terms with different tensorial signature are mapped into different momenta structures. Thus, two terms like \(\phi (\partial^2 \phi)\) and \((\partial \phi)^2)\) get translated into two very different objects, \(\varphi \pi_{\mu \nu} \delta_{\mu \nu}\) and \(\pi^\mu \pi^-\) respectively. These can be related only through canonical transformations. The systematic approximation scheme we discuss in this work breaks the latter invariance by treating these and other equivalent terms on different footing, and therefore significantly departs from the DE. In fact, we are interested in an expansion of the effective Hamiltonian in momenta fields, organized by including momenta in increasing-rank order. More precisely, only the dependence of the Hamiltonian on rank-one momenta will be explicitly addressed in this work. The corresponding truncated flow equations are derived in Sec. II and describe infinitely many derivative interactions in the form of an arbitrary function of \(\pi^\mu\). One of these equations, descending from a convenient regularization scheme, takes the simple form of a second order partial differential equation for a function of \(\phi\) and \(\pi^\mu\). Like the LPA, this is regulator independent, thanks to the freedom to rescale the additional variable \(\pi^\mu\).

In Sec. IV we discuss a first application of this approximation, with the only purpose to check for conceptual mistakes in the derivation of these equations and to assess whether this different treatment of the derivative sector of the effective action properly captures nonperturbative effects and universal phenomena. We conservatively turn to the study of the Wilson-Fisher fixed point (FP) in three dimensions, in a simplified set up where we neglect all the interactions that are more than quadratic in the momenta, in order to make contact with the rich literature about the first order of the DE. Since we do not aim at a comprehensive study but only at a first exploration of the properties of the truncations proposed here, we focus our attention on the computation of the critical exponent which is the most sensitive to the derivative sector, as well as the most challenging to accurately estimate by means of the FRG, namely the critical anomalous dimension \(\eta\).

In Tab. I we compare our result to some literature, especially from the state-of-the-art FRG computations. Some details are given about the specific implementations of the FRG: first of all which exact equation is used (Wilson [1], Wetterich [5, 25], Polchinski [24]), then the truncation scheme and its order. With ‘bf’ we specify the use of the background field method, while ‘implicit’ refers to the use of implicit optimization. For more details and a collection and comparison of FRG predictions for \(\eta\) at various orders of the DE we refer the reader to [12]. About the NDE reference, it should be stressed that the equations in [20] are obtained from the order (\(\partial^2\)) of the NDE by neglecting a specific term. This allows to analytically perform a choice of regulator-dependent coefficients such that linear reparametrization invariance is satisfied. We also recall some results from high-temperature expansions [27], Monte-Carlo methods [28] and the conformal bootstrap [29].

Regarding our estimate, the uncertainty is numerical, as it is explained in Sec. IV, hence it is possible to further reduce it with a more accurate analysis. Yet, this must also be interpreted as an uncertainty on the uniqueness of this result against the change of one arbitrary parameter, namely the overall normalization of the FP Hamiltonian. We did not observe any dependence of \(\eta\) on such param-
TABLE I. The critical exponent \( \eta \) for the three-dimensional Ising universality class, from the functional renormalization group, high-temperature expansions, Monte-Carlo simulations and the conformal bootstrap. For the meaning of the abbreviations see the main text.

| ref. | year | method | info | \( \eta \)  |
|------|------|--------|------|------------|
| 12   | 1984 | FRG    | Wil, SFE \( O(n_{\phi}) = 10 \) | 0.040(7) |
| 11   | 2003 | FRG    | Wet, DE \( O(\phi^2) \) | 0.033 |
| 10   | 2009 | FRG    | Pol, NDE \( O(\phi^2) \) | 0.041347 |
| 14   | 2010 | FRG    | Wet, DE \( O(\phi^2) \) bf | 0.0313 |
| 15   | 2010 | FRG    | Wet, DE \( O(\phi^2) \) bf, implicit | 0.034 |
| 15   | 2011 | FRG    | Wet, BMW \( O(s = 2) \) | 0.039 |
| 20   | 2002 | HT     | O(25) | 0.03639(15) |
| 23   | 2011 | Monte-Carlo | 0.036327(10) |
| 25   | 2015 | FRG    | this work | 0.036302(12) |

The critical exponent \( \eta \) for the three-dimensional Ising universality class, from the functional renormalization group, high-temperature expansions, Monte-Carlo simulations and the conformal bootstrap. For the meaning of the abbreviations see the main text.

RG EQUATIONS FOR LOCAL EFFECTIVE LAGRANGIANS

An operative definition of field theory needs a regularization introducing a scale \( \Lambda \). Yet, observables should be to some extent independent of such a regularization. This is made possible by the dependence of the microscopic dynamics, as encoded for example in the Wilson effective action \( S \), on the scale itself. Thus, changing the value of \( \Lambda \), must result in a change of \( S \). The standard way of interpreting such a variation is identifying it with a nonlinear redefinition of the fields, such as a change of variables in the path integral \([32]\), which therefore ensures the invariance of observables. Different choices of field redefinitions lead to different RG equations for \( S \).

In this work we will concentrate on the Polchinski equation, that reads

\[ \hat{S}[\phi] = \frac{1}{2} \int_{xy} \delta S[\phi] \hat{C}(x-y) \frac{\delta S[\phi]}{\delta \phi} - \frac{1}{2} \int_{xy} \frac{\delta}{\delta \phi(x)} \hat{C}(x-y) \frac{\delta S[\phi]}{\delta \phi(y)} \]

where dotted quantities are differentiated with respect to \( t = -\log \Lambda \). The meaning of \( \hat{C}(x-y) \) and the constraints on it are traditionally understood in terms of its Fourier transform \( \hat{C}(p^2) \). Then, one can think about \( \hat{C} \) as the rate of change \( -\Lambda \partial_\Lambda \hat{C} \) of some regularized propagator

\[ \hat{C}(p^2) = \Lambda^{2d_\phi - d} \frac{\Lambda^2}{p^2} K \left( \frac{p^2}{\Lambda^2} \right) . \]

Here \( d_\phi \) is the full (quantum) dimensionality of the field \( \phi \) and \( K \) is a cutoff function that is intended to regulate the UV and/or IR behavior of this propagator. Sticking to traditional notations, we will split the quantum dimensionality of \( \phi \) into a canonical part and an anomalous one \( d_\phi = (d - 2 + \eta)/2 \). In this work we will take \( \eta \) as independent of \( \Lambda \), since we will later address the case of theories at fixed points of the RG.

We are now interested in local truncations, which correspond to a Lagrangian density depending on generi-
cally high derivatives of the fields. For the sake of notational simplicity, let us introduce multi-indices $M \equiv (\mu_1, \ldots, \mu_m)$ with $m \in \mathbb{N}$ and denote

$$\phi_M = \phi_{\mu_1 \ldots \mu_m}(x) = \frac{d}{dx^{\mu_1}} \ldots \frac{d}{dx^{\mu_m}} \phi(x) = \frac{d}{dx^M} \phi(x).$$ \hspace{1cm} (3)

We will use a similar multi-index notation for derivatives of any other function. In formulas, we address the following truncation

$$S[\phi] = \int_x \mathcal{L}(x, \phi_M(x)).$$ \hspace{1cm} (4)

Let us remark that also some nonlocal actions can be rewritten in terms of Lagrangian densities of the present kind, possibly depending explicitly on the position in space, by just expanding all fields in a Taylor series around a common point $x$. The goal of this section is rewriting such a truncation of the Polchinski equation as a partial differential equation for $\mathcal{L}$. To this end we need to express the functional derivative of the action as a standard Euler-Lagrange operator acting on the Lagrangian density

$$\frac{\delta S}{\delta \phi(x)} = (-)^M \frac{d}{dx^M} \frac{\partial \mathcal{L}}{\partial \phi_M}(x)$$ \hspace{1cm} (5)

$$= \frac{\partial \mathcal{L}}{\partial \phi} - \frac{d}{dx^{\mu_1}} \frac{\partial \mathcal{L}}{\partial \phi_{\mu_1}} + \frac{d^2}{dx^{\mu_1}dx^{\mu_2}} \frac{\partial \mathcal{L}}{\partial \phi_{\mu_1\mu_2}} + \ldots$$

where we used, here and in the rest of the paper, Einstein’s summation convention for multi-indices, and we denoted $(-)^M = (-)^m$ for a multi-index of length $m$. We will write $M$ in place of its length $m$ also in other similarly situations, when this appears as an unambiguous abuse of notations. Notice that in the sum above we included the empty index with $m = 0$, corresponding to $\phi_M(x) = \phi(x)$, which sometimes we will refer to by simply writing $M = 0$. The second order functional derivative can also be rewritten in terms of infinitely many partial derivatives. First of all, it is convenient to rewrite the first order derivative as the integral of a Lagrangian density

$$\frac{\delta S}{\delta \phi(x)} = \int_y \frac{\partial \mathcal{L}}{\partial \phi_M}(y) \delta_M(y - x) = \int_y \mathcal{L}^{(1)}_x(y).$$ \hspace{1cm} (6)

Then, one can iterate the application of the Euler-Lagrange operator to get the first order functional derivative of an action with Lagrangian $\mathcal{L}^{(1)}_x$

$$\frac{\delta^2 S}{\delta \phi(y) \delta \phi(x)} = (-)^N \frac{d}{dy^N} \left[ \frac{\partial^2 \mathcal{L}}{\partial \phi_N \partial \phi_M}(y) \delta_M(y - x) \right].$$ \hspace{1cm} (7)

Applying this formula to the quantum term in the flow equation, integrating by parts and dropping the integrals of total derivatives, this can be written as

$$\int_x \mathcal{L}(x - y) \frac{\delta^2 S}{\delta \phi(y) \delta \phi(x)} = (-)^N \hat{\mathcal{C}}_{MN}(0) \int_x \frac{\partial^2 \mathcal{L}}{\partial \phi_N \partial \phi_M}(x)$$

where we assumed that the regularized propagator is an even function of the position in space $C(x) = C(-x)$. As a consequence, one can recast the Polchinski equation for the present truncation in the form

$$\int_x \mathcal{L}(x) = \frac{(-)^N}{2} \left\{ \int_{xy} \frac{\partial \mathcal{L}}{\partial \phi_M}(x) \hat{\mathcal{C}}_{MN}(x - y) \frac{\partial \mathcal{L}}{\partial \phi_N}(y) - \hat{\mathcal{C}}_{MN}(0) \int_x \frac{\partial^2 \mathcal{L}}{\partial \phi_M \partial \phi_N}(x) \right\}.$$ \hspace{1cm} (8)

This clearly shows that, despite our initial ansatz for the effective action in Eq. \[(4)\] was assuming a local effective Lagrangian, the right hand side (r.h.s.) of the flow equation generates nonlocalities through the classical term. Projecting these nonlocalities out of our truncation would be too crude an approximation, and one would miss a crucial part of the interplay between pointlike interactions and the derivative sector \[\mathcal{L}\]. We then follow the same principle inspiring the presence of infinitely many derivatives in Eq. \[(4)\], that at least part of this nonlocal structure could be rewritten as a higher-derivative local dynamics. In order to project the r.h.s. of the Polchinski equation onto a local effective Lagrangian, we then expand the integrand of the classical term about a single point

$$\frac{\partial \mathcal{L}}{\partial \phi_N}(y) = \frac{\partial \mathcal{L}}{\partial \phi_N}(x) + \frac{(y - x)^L}{L!} \frac{d}{dx^L} \frac{\partial \mathcal{L}}{\partial \phi_N}(x)$$ \hspace{1cm} (9)

where $L \neq 0$. Since $\mathcal{L}$ depends on $\phi_M$ and possibly also separately on $x$, we can further split

$$\frac{d}{dx^L} = \frac{\partial}{\partial x^L} + \phi_M \frac{\partial}{\partial \phi_M}$$ \hspace{1cm} (10)

where $\partial_x$ denotes the $x^L$-derivative at fixed $\phi_M$. Such a contribution is nonvanishing only if the couplings in $\mathcal{H}$ have an explicit space, i.e. momentum, dependence. In this paper we will restrict ourselves to the pointlike interaction limit and neglect this explicit $x$-dependence. The $L$–th derivative can then be written as

$$\frac{d}{dx^L} = \sum_{i=1}^L \phi_{(M_1 \ldots M_i)} \frac{\partial^i}{\partial \phi_{M_1} \ldots \partial \phi_{M_i}}$$ \hspace{1cm} (11)

where $\phi_{(M_1 \ldots M_i)_L}$ denotes a sum over all possible ways of distributing the indices inside $L$ on the $i$ entries $\phi_{M_1} \ldots \phi_{M_i}$, under the rules that ordering inside each entry does not matter, that permutations of $M_1 \ldots M_i$ do not matter, and that there must be at least one index out of $L$ per entry. For instance

$$\frac{d^2}{dx^2} = \phi_{M_1} \frac{\partial}{\partial \phi_{M_1}} + \phi_{M_1} \frac{\partial}{\partial \phi_{M_1}} \frac{\partial^2}{\partial \phi_{M_1} \partial \phi_{M_2}}$$

This gives rise to a factorization of regulator-dependent coefficients as in the quantum term, of the form

$$J_{L,MN} = \int_y y^L \hat{\mathcal{C}}_{MN}(y).$$ \hspace{1cm} (12)

---

1 I am grateful to T. R. Morris for pointing this out to me, and for suggesting the different treatment of this term that is described in what follows.
Whenever \( M + N > L \), integrating by parts and assuming that the regulator is such that the boundary terms vanish, one gets \( J_{L,M+N} = 0 \). For instance, for \( L = 0 \) the coefficient \( J_{0,M+N} \) is the integral of a total derivative and vanishes unless \( M = N = 0 \), in which case \( J_{0,0} = \dot{C}_A(0) \).

To sum up, the projection of the Polchinski equation on the ansatz of a local effective Lagrangian gives (the spatial integral of) the following flow equation

\[
\dot{\mathcal{L}} = \frac{1}{2} \dot{C}(0) \left( \frac{\partial \mathcal{L}}{\partial \phi} \right)^2 - \frac{(-)^N}{2} \dot{C}_{M,N}(0) \frac{\partial^2 \mathcal{L}}{\partial \phi_M \partial \phi_N} + \frac{(-)^M J_{L,M,N}}{2L!} \frac{\partial \mathcal{L}}{\partial \phi_M} \dot{\phi}_M \frac{\partial^i \phi}{\partial \phi_M} \frac{\partial^i \phi}{\partial \phi_N}
\]

where \( L \neq 0 \), while the sum over \( M_i, M, N \) includes the empty index, and for notational simplicity we dropped the summation symbol for \( i = 1 \ldots L \). The first term comes from the zeroth order \((y - x)^0\) of the Taylor expansion of the classical term, for which we assumed that only the \( M = N = 0 \) contribution is nonvanishing. This is because (and) we do not want IR divergences and thus require \( \dot{C}(0) < \infty \), which entails that any integral of a total derivative, such as \( \int_y \dot{C}_A(y) \) for \( M > 0 \), vanishes.

The structure of Eq. (3) has been considerably complicated by the procedure of Taylor expansion, that involves a sum of infinitely many terms. One may wonder if upon truncation of this equation by neglecting the dependence on derivatives of order bigger than some integer \( k \), i.e. the dependence of \( \mathcal{L} \) on \( \phi_M \) whenever \( M > k \), this sum gets finite. This is not the case already for \( k = 1 \). In fact, one is forced to set \( M_1 = \ldots = M_i = 0 \) but still the sum over \( i \) remains

\[
\dot{\mathcal{L}} = \frac{1}{2} \dot{C}(0) \left( \frac{\partial \mathcal{L}}{\partial \phi} \right)^2 - \frac{1}{2} \dot{C}(0) \frac{\partial^2 \mathcal{L}}{\partial \phi^2} + \frac{1}{2} \dot{C}_{\mu,0}(0) \frac{\partial^2 \mathcal{L}}{\partial \phi_{\mu} \partial \phi_{\nu}} + J_{\lambda_1, \ldots, \lambda_i, \mu, \nu} \frac{\partial \mathcal{L}}{\partial \phi_{\mu}} \frac{\partial^i \phi_{\lambda}}{\partial \phi_{\mu}} \frac{\partial^i \phi_{\nu}}{\partial \phi_{\nu}} - J_{\lambda_1, \ldots, \lambda_i, \mu, \nu} \frac{\partial \mathcal{L}}{\partial \phi_{\mu}} \frac{\partial^i \phi_{\lambda}}{\partial \phi_{\mu}} \frac{\partial^i \phi_{\nu}}{\partial \phi_{\nu}} - J_{\lambda_1, \ldots, \lambda_i, \theta} \frac{\partial \mathcal{L}}{\partial \phi_{\mu}} \frac{\partial^i \phi_{\lambda}}{\partial \phi_{\mu}} \frac{\partial^i \phi_{\nu}}{\partial \phi_{\nu}}.
\]

This is no longer the case if one further projects the flow on the sector quadratic in \( \phi_{\mu} \), which selects the following terms

\[
\dot{\mathcal{L}} = \frac{1}{2} \dot{C}(0) \left( \frac{\partial \mathcal{L}}{\partial \phi} \right)^2 - \frac{1}{2} \dot{C}(0) \frac{\partial^2 \mathcal{L}}{\partial \phi^2} + \frac{1}{2} \dot{C}_{\mu,0}(0) \frac{\partial^2 \mathcal{L}}{\partial \phi_{\mu} \partial \phi_{\nu}} + J_{\lambda_1, \ldots, \lambda_i, \mu} \frac{\partial \mathcal{L}}{\partial \phi_{\mu}} \frac{\partial^2 \mathcal{L}}{\partial \phi_{\mu} \partial \phi_{\nu}} - J_{\lambda_1, \ldots, \lambda_i, \mu} \frac{\partial \mathcal{L}}{\partial \phi_{\mu}} \frac{\partial^2 \mathcal{L}}{\partial \phi_{\mu} \partial \phi_{\nu}} \frac{\partial \mathcal{L}}{\partial \phi_{\mu}} \frac{\partial^2 \mathcal{L}}{\partial \phi_{\mu} \partial \phi_{\nu}} - J_{\lambda_1, \ldots, \lambda_i, \mu} \frac{\partial \mathcal{L}}{\partial \phi_{\mu}} \frac{\partial^2 \mathcal{L}}{\partial \phi_{\mu} \partial \phi_{\nu}}.
\]

Notice that the two terms inside the bracket in the second line are equal since they differ by a total derivative. By the same reasoning the last term can be rewritten as

\[
+ \frac{J_{\lambda_1, \ldots, \lambda_i, \mu}}{2} \frac{\partial \mathcal{L}}{\partial \phi_{\mu}} \frac{\partial^2 \mathcal{L}}{\partial \phi_{\mu} \partial \phi_{\nu}}.
\]

Later on we will change notation for the regulator-dependent terms in Eq. (13), adopting the conventions of [3] for the following positive quantities

\[
\dot{C}(0) = \Lambda^{n-2} K_0 \quad \frac{J_{\lambda_1, \ldots, \lambda_i, \mu}}{2} = -\frac{1}{2} \delta_{\lambda_1, \ldots, \lambda_i} \dot{C}(0) = \delta_{\lambda_1, \ldots, \lambda_i} \Lambda^{n-2} K_0 \quad \frac{1}{4} J_{\lambda_1, \ldots, \lambda_i} \frac{\partial^2 \dot{C}}{\partial \phi_{\mu} \partial \phi_{\nu}} = \delta_{\lambda_1, \ldots, \lambda_i} \Lambda^{n-4} K_1
\]

In obtaining Eq. (13) it was crucial to assume that the regulator \( C \) acts as a kernel in position or Fourier space. Specifically, we assumed that \( C \) is a differential operator based on a total derivative in position space, such as

\[
\dot{C}(x - y) = \dot{C} \left( -\frac{d^2}{dx^2 \partial x_{\mu}} \right) \delta(x - y).
\]

Expanding \( \dot{C} \) in series around zero and evaluating the total derivatives, leads to the infinite sum in Eq. (13), that we previously derived by the equivalent procedure of replacing a function of point \( y \) by its Taylor series in powers of \( (y - x) \). In the present context, where \( \mathcal{L} \) is treated as a function of infinitely many independent variables \( \{ \phi, \phi_{\mu}, \phi_{\mu \nu}, \ldots, \phi_M, \ldots \} \), this appears to be one among several other possible choices, corresponding to the freedom to keep some of the \( \phi_{\mu} \)'s constant while taking the spatial derivatives. This possibility, that could be unapparent in a Lagrangian formulation, will take a more familiar shape in the Hamiltonian formulation of the next section.

Since we are interested in studying truncations inspired by the derivative expansion, it would be helpful to define RG transformations that are as simple as possible in the low-momenta sectors. Hence, for definiteness, let us discuss the possibility that the undifferentiated field itself is kept constant by the coarse-graining operator \( \dot{C} \). Again this can be formalized in two equivalent ways. The first one is the procedure of Taylor expansion in powers of \( (y - x) \), in which the undifferentiated field must be considered \( y \)-independent, such that the derivatives that in Eq. (9) were total, now do not hit \( \phi \),

\[
\left. \frac{d}{dx_\lambda} \right|_\phi = \frac{\partial \phi_{\lambda}}{\partial \phi_{\mu}} + \frac{\partial \phi_{\lambda}}{\partial \phi_{\mu \nu}} + \frac{\partial \phi_{\lambda}}{\partial \phi_{\mu \nu}} + \ldots
\]

Then clearly the Taylor expansion of the nonlocality in the classical term will produce corrections that, with the only exception of the zeroth order term, do not affect a
truncation projecting on the sector of \( \phi \) and \( \phi_\mu \). As a consequence, in this case one simply needs to take the two functional derivatives in the classical term
\[
\left( - \right)^{M+N} \frac{1}{2} \int_{xy} C(x - y) \frac{d}{dx^M} \frac{d}{dy^N} \left( \frac{dL}{\partial x^\mu} \frac{dL}{\partial y^\nu} \right)(20)
\]
evaluate them at the same space-point. The remaining \( y \)-integral factorizes a \( C(0) \).

The second equivalent way of reaching this conclusion is directly replacing \( \dot{C}(x - y) \) in the last equation with the expression in Eq. (19), where the Laplacian now is understood as a differentiation at constant \( \phi \) as in Eq. (19), and then expanding \( \dot{C} \) around zero argument. Since the Laplacian at fixed \( \phi \) will produce only terms with momenta of rank bigger than one, only the zeroth order term will contribute to the selected truncation. Hence, in both cases, one obtains a simpler alternative to Eq. (14), namely
\[
\dot{L} = 2 \dot{C}(0) \left( \frac{\partial L}{\partial \phi_\mu} \right)^2 - 2 \dot{C}(0) \frac{\partial^2 L}{\partial \phi_\mu \partial \phi_\lambda} + \frac{1}{2} \frac{\partial^2 L}{\partial \phi_\mu \partial \phi_\nu}.
\]

which, when further truncated to the subspace quadratic in \( \phi_\mu \), coincides with the \( J_{\lambda \mu \nu} = 0 \) version of Eq. (15).

### III. HAMILTONIAN REPRESENTATION

The standard way of dealing with flow equations like the spatial integrals of Eq. (14) and Eq. (21) is to project them on progressive powers of the Fourier momenta with the help of functional derivatives in the field, i.e. a DE. We suggest here a different treatment. If we consider \( \phi_M(x) \) to be \( x \)-independent, we are forced to have all the \( \phi_N(x) \), with \( N \neq M \), \( x \)-dependent accordingly. In particular, setting \( \phi(x) \) at a constant value forces \( \phi_N(x) = 0 \), \( \forall x \), \( \forall N \neq 0 \). In order to overcome this Hamiltonian formalism it makes sense to consider \( \phi(x) \) and \( \pi(x) = \frac{\partial L}{\partial \phi(x)} \) to be simultaneously constant, we want to employ a generalized covariant Hamiltonian formalism that enables us to consider \( \phi(x) \) and \( \pi^M(x) = \frac{\partial L}{\partial \phi_M(x)} \), \( M \neq 0 \), simultaneously constant. This can be understood as a covariant version of the Ostrogradsky formalism. One goes to ‘phase space’ by means of the generalized transform
\[
H(x, \phi, \pi^M) = \text{ext}_{\phi_M} \left\{ i\pi^M \phi_M + L(x, \phi, \phi_M) \right\} (22)
\]
and here and in the following the assumed sums over repeated multi-indices do not include the empty index, i.e. \( L, M, N \neq 0 \). Thus
\[
\pi^M(x) = i \frac{\partial L}{\partial \phi_M}(x), \quad \phi_M(x) = -i \frac{\partial H}{\partial \phi}(x)
\]
\[
\frac{\partial H}{\partial \phi}(x) = \frac{\partial L}{\partial \phi}(x), \quad \frac{\partial H}{\partial \phi^\mu}(x) = \frac{\partial L}{\partial \phi^\mu}(x) \tag{23}
\]
and also
\[
\frac{\partial \pi^L}{\partial \phi_M}(x) = i \left( \frac{\partial^2 L}{\partial \phi \partial \phi}(x) \right)^{-1} \frac{\partial^2 H}{\partial \phi \partial \phi}(x), \quad \frac{\partial \pi^L}{\partial \phi^\mu}(x) = i \left( \frac{\partial^2 H}{\partial \phi^\mu \partial \phi}(x) \right)^{-1} \frac{\partial^2 L}{\partial \phi^\mu \partial \phi}(x) \tag{24}
\]
By means of these formulas one can translate the flow equations for \( L \) into flow equations for \( H \).

The equations of motion in the Hamiltonian form read
\[
i(-)^M \frac{d\pi^M}{dx^M}(x) = \frac{\partial H}{\partial \phi}(x), \quad \phi_M(x) = -i \frac{\partial H}{\partial \phi_M}(x). \tag{23}
\]
Notice that the momenta are related to the derivatives of \( \phi \) only on-shell, that is on the stationarity trajectories in phase-space, while off-shell they are independent of them. This enables us to set \( \phi \) and all \( \pi \)'s equal to different and arbitrary constant values. These constants would become tied to each other if we tried to satisfy the equations of motion by means of constant \( \pi^M \) and \( \phi \), in which case we would find the requirement that they correspond to stationarity points of \( H \). The stationarity condition for the momenta, which corresponds to homogeneous \( \phi \) configurations, is expected to be usually solved by setting all the momenta to zero: \( \pi^M = 0 \), purely on the grounds of rotational symmetry that enforces contraction of indices. Nevertheless, this trivial stationarity point might or might not correspond to an absolute minimum. In case \( H \) has an absolute minimum for non-vanishing values of some momenta, one faces a situation where rotational symmetry is spontaneously broken [33].

Now we want to discuss possible approximations of the flow equation for \( H \). If we neglect the explicit spacetime, i.e. momentum, dependence of the couplings, then \( H \) depends on the position \( x \) only through the fields \( \phi(x) \) and \( \pi^M(x) \), and we can study the flow of the Hamiltonian density by setting both fields to constant values. Under this approximation, the RG flow of the effective Hamiltonian density is encoded in a partial differential equation for a function of infinitely many fields. These fields are symmetric tensors with arbitrarily high rank, and therefore contain higher-spin components. Here one could consider a further systematic approximation scheme, that arises by neglecting the dynamics of momenta with rank \( M > k \), where \( k \) is some chosen positive integer. That is, in practice, one considers \( H \) as independent of these fields. Other kinds of truncations of the full function \( H \) are clearly possible, but will not be discussed in this work.

The zeroth order of such an approximation consists in
dropping all the momenta. This is the same as the LPA. The first order of this expansion originates from keeping

\[ \mathcal{H} = \Lambda^{n-2} K_0 \left[ -\left( \frac{\partial H}{\partial \phi} \right)^2 + \frac{\partial H}{\partial \phi} \frac{\partial H}{\partial \pi^\mu} \frac{\partial^2 H}{\partial \pi^\nu \partial \phi} \left( \frac{\partial^2 H}{\partial \pi^\nu \partial \pi^\mu} \right) + \Lambda \right] + \Lambda^{d-2+n} I_0 \frac{\partial^2 H}{\partial \phi^2} + \Lambda^{d+n} \frac{d}{d} I_1 \delta_{\mu\nu} \left( \frac{\partial^2 H}{\partial \pi^\mu \partial \pi^\nu} \right)^{-1} \nu^\mu \]

where \( H \) depends on \( \phi \) and \( \pi^\mu \) only, i.e. any other momentum with higher rank is neglected. The Hamiltonian translation of Eq. (15) instead would miss the last term inside the square bracket of the first line, and would contain the additional term \( +\Lambda^{n-4} K_1 \frac{\partial H}{\partial \phi} \left( \frac{\partial^2 H}{\partial \phi^2} \right)^2 \).

Notice that by appropriate rescalings of \( H, \varpi, \phi \), one can absorb (i.e. set equal to one) the three regulator-dependent parameters \( K_0, I_0, I_1 \), and get a regulator-independent flow equation. In details, the rescaling is

\[ H \to a H \, , \quad \varpi \to b \varpi \, , \quad \phi \to c \phi \, , \quad a = \frac{I_0}{K_0} \, , \quad b = \frac{I_0^2}{K_0^2 I_1} \, , \quad c = \sqrt{I_0} \, . \]

This is not possible for Eq. (15) where the last term \( +\Lambda^{n-4} K_1 \frac{\partial H}{\partial \phi} \left( \frac{\partial^2 H}{\partial \phi^2} \right)^2 \) introduces an extra regulator-dependent coefficient that, after the previous rescalings, provides an essentially arbitrary parameter \( B > 0 \) multiplying \( 2 \varpi \left( H^{(10)} H^{(02)} \right)^2 \).

### IV. 3D ISING CRITICAL \( \eta \)

We now want to start testing the quality of the truncation encoded in Eq. (26). Such an approximation cannot be expected to perform equally well for any kind of observable. In this section we specifically ask how does this approximation perform in the task of describing the critical properties of a three dimensional theory. The motivation for starting with this application is that such a critical behavior, namely the possible set of conformal theories at the phase transitions and the corresponding critical exponents, is well known and provide a traditional benchmark for any nonperturbative tool in statistical field theory. To address this question, the first step to take is shifting our attention from the dimensionful fields and interactions to the renormalized dimensionless ones. In other words, a conformal behavior of the system is expected to reveal itself by means of self-similarity, such that every dimensionful quantity at criticality should scale with \( \Lambda \) according to its full (quantum) dimensionality. We called \( d_{\varpi} \) and \( d_{\phi} \) such dimensions for \( \phi \) and \( \pi^\mu \) respectively; it is therefore convenient to rescale

\[ H \to \Lambda^d H \, , \quad \varpi \to \Lambda^{2-d} \varpi \, , \quad \phi \to \Lambda^{d_{\phi}} \phi \] (27)

because then the new quantities can be considered \( \Lambda \)-independent at criticality. The full dimensionality of \( \pi^\mu \) can be fixed by asking that the Legendre transform term \( \pi^\mu \phi_\mu \) have dimension \( d \). This is equivalent to demanding that \( (d-d_{\varpi}-1) \) be equal to the full dimension of \( \pi_\mu \), as one would expect by performing the Legendre transform of a simple Langrangian with a kinetic term of the form \( \phi^\mu \phi_\mu /2 \). As a consequence we set \( d_\varpi = (d-\eta)/2 \) and \( d_{\phi} = (d-2+\eta)/2 \).

To sum up, the critical theories will be independent of the scale \( \Lambda \) and therefore correspond to FPs of the RG flow, that is, to solutions of the previous equations where one sets \( H = 0 \), \( \Lambda = 1 \) and adds the canonical rescaling terms: \( d \mathcal{H} - (d-\eta) \varpi \mathcal{H}^{(1.0)} - (d-2+\eta)/2 \phi \mathcal{H}^{(0.1)} \), on the r.h.s.. We are interested in studying the simple truncation that arises by projecting the flow equations on the ansatz

\[ \mathcal{H}(\varpi, \phi) = \varpi/Z(\phi) + V(\phi) \] (28)
which corresponds to the Legendre transform of
\[ \mathcal{L}(\phi_\mu, \phi) = Z(\phi) \frac{\phi_\mu \phi_\mu}{2} + V(\phi). \]  (29)

Then Eq. (20) provides
\[ \dot{V} = dV - \frac{d + \eta}{2} \phi V' - (V')^2 + V'' + Z \]  (30)
\[ \dot{Z} = -\eta Z - \frac{d + \eta}{2} \phi Z' - 2ZV'' + Z'' - 2(\phi')^2. \]  (31)

The Hamiltonian translation of Eq. (15) would lead to the same equation for \( \dot{V} \) and it would add a term \(+2B(V''/Z)^2\) on the r.h.s of \( \dot{Z} \). These equations differ from the ones obtained by a first order of the DE in several respects. The equation for \( \dot{V} \) is essentially the same, apart for the fact that in the DE there is a regulator in several respects. The equation for \( \dot{Z} \) to the same equation for \( \dot{Z} \) and +2

For instance, upon inclusion of \( Z(\phi)^{-1} = \zeta_0 (1 + \zeta_1 \phi^2 / 2) \), one finds
\[ \eta = -2V''(0) \frac{2 + \zeta_0}{2 - \zeta_0} \]  (33)

and a similar parametric dependence survives also including one more coupling in \( Z \). Since it does not seem straightforward to go to higher orders of this kind of successive approximations, let us turn to the task of solving the full system of Eqs. (30,31) at a FP.

We choose the method of shooting from large field values. To this end we use the large-field asymptotics of the FP solution, as parametrized by \( \eta \) itself and other two free parameters \( A_V \) and \( A_Z \). The reader can find it in Appendix A. These three parameters are not completely free since the FP solutions we are after enjoy \( \mathbb{Z}_2 \)-symmetry, that is \( V'(0) = Z'(0) = 0 \). Thus, we need numerical solutions that interpolate the right field asymptotics, say at \( \phi = 1 \), with the needed behavior at \( \phi = 0 \). Unfortunately, regardless the use of high-order large field asymptotic expansions (ten terms for each function), the numerical integration does not always reach \( \phi = 0 \). In general, one gets a solution that extends till \( \phi = 0 \) only if the corresponding \( V'(0) \) and \( Z'(0) \) would be small enough. Then, one could trade in the parameter \( A_Z \) for \( Z(0) \), even if this is not necessary. At fixed generic values of \( \eta \) and \( A_V \), \( Z(0) \) is bigger if \( A_Z \) is closer to zero. The two parameters have the same sign and roughly their order of magnitude is related by \( Z(0) \sim (10A_Z)^{-1} \).

The strategy we are going to follow for constructing the FP solutions is the following. The symmetry conditions at the origin can be used to fix \( A_V \) and \( \eta \), namely by locating a discrete set of points in the \((A_V, \eta)\)-plane where both \( V'(0) \) and \( Z'(0) \) vanish. In so doing \( A_Z \) remains undetermined. Thus, by variation of \( A_Z \) and relocation of the zeros in \((A_V, \eta)\) one can construct a discrete set of lines of FPs. For practical reasons, it is mandatory to analyze the \( A_Z \)-dependence of \( V'(0) \) and \( Z'(0) \) even before extracting their common zeros in the \((A_V, \eta)\)-plane. This is because the number of their zeros changes as \( A_Z \) is changed.

Then we start by fixing an arbitrary initial \( \eta \), for instance \( \eta = 1.2 \times 10^{-2} \), and plot \( V'(0) \) and \( Z'(0) \) as functions of \( A_V \), for several values of \( A_Z \). Because of the numerical shooting procedure, the resolution of these plots is limited, a fact that represents the main source of uncertainties in the final estimate of \( \eta \). We first analyze \( V'(0) \), which is shown in Fig. 1. The numerical integration is successful in reaching the origin when \( A_V \) is close to \(-2.5\), where \( V'(0) \) shows two zeros. The left one corresponds to a quadratic potential (far enough from the matching with the asymptotic expansion), therefore we expect it to be connected to the high-temperature FP, while at the right one \( V(\phi) \) has the right qualitative shape for a Wilson-Fisher FP. Unfortunately, the latter is much harder to locate due to the fact that the curve is very steep close to this zero, such that a high-resolution plot is needed to reveal it. For instance, in the first (upper left) panel of...
Fig. 1. $V'(0)$ as a function of $A_V$, at fixed $\eta = 1.2 \times 10^{-2}$ and for various values of $A_Z \in \{1, 10^{-1}, 10^{-2}, 10^{-3}, -10^{-3}, -1\}$ (from left to right in each row, and from top row to bottom row). These are discrete plots with step-size $\Delta A_V = 10^{-2}$. Where the curve is missing, the numerical integration (starting from $\phi = 1$) does not reach $\phi = 0$.

Fig. 2. $Z'(0)$ as a function of $A_V$, at fixed $\eta = 1.2 \times 10^{-2}$ and for various values of $A_Z \in \{1, 10^{-1}, 10^{-2}, 10^{-3}, -3 \times 10^{-2}, -10^{-3}, -1\}$ (from left to right in each row, and from top row to bottom row). These are discrete plots with step-size $\Delta A_V = 5 \times 10^{-4}$. Where the curve is missing the numerical integration (starting from $\phi = 1$) does not reach $\phi = 0$.

In Fig. 1 where $A_Z = 1$ the resolution of the plot is too low to show this zero. If we increase $A_Z$ the slope further increases, making this practical problem more severe. One the other hand, decreasing $A_Z$, i.e. moving to the following panels of Fig. 1, makes the curve less steep and the location of the rightmost zero easier. However, the number of the zeros and their qualitative position does not change. We can even lower $A_Z$ to negative values, as in the last two (lower) panels of Fig. 1, and the very same two zeros remain visible. The fact that the zeros of $V'(0)$ are two is however not generic. For different values of $\eta$ we observe more than these two zeros, but we interpret the fact that these additional zeros are not present for all values of $A_Z$ as a manifestation of their spurious nature.

Still at $\eta = 1.2 \times 10^{-2}$, the picture for $Z'(0)$ as a function of $A_V$ is more complicated than the one for $V'(0)$, and it is shown in Fig. 2. At $A_Z = 1$, in the upper left panel, there seems to be no zero. As we move to $A_Z = 10^{-1}$, in the upper right panel, a zero becomes visible. Lowering further $A_Z$ more zeros show up, as in the third and fourth panel of Fig. 2 (mid row), revealing that $Z'(0)$ is wildly oscillating close to the value of $A_V$ beyond which the integration is no longer reaching $\phi = 0$. This observation, together with the previous study of $V'(0)$, suggests that even at $A_Z = 1$ there can be zeros which are hard to reveal because $Z'(0)$ is too steep in their neighborhood. Thus, expecting that the slope of the curve be again a growing function of $A_Z$, we are lead towards lowering the latter parameter, even below zero. For a negative $A_Z$ close enough to zero, like $A_Z = 10^{-1}$ in the lower left panel, there still are oscillations in an inner-region of $A_V$, let us say roughly on the right of the point that for positive $A_Z$ was the end of successful numerical integration. On the left of such a point, instead, there are only two zeros, which seem to remain isolated and clearly distinguishable independently of $A_Z$. If $A_Z$ is further lowered, as in the last (lower right) panel where $A_Z = -1$, the inner oscillations disappear, the curve becomes less steep, and only the latter two zeros survive. On the basis of all these facts, we assume that these two zeros exist for any $A_Z$, such that when $A_Z$ is increased from $-1$ towards $+1$ they must get closer and closer, while the function $Z'(0)$ itself becomes steeper and steeper in their vicinity. By inspecting the shape of the corresponding solutions we can again discard one of these two zeros, the one on the right. In fact, the corresponding $V'(\phi)$ is positive all over its domain, and again linear far enough from the asymptotic large field behavior. At the left zero instead, $V(\phi)$ has the right shape for a
Wilson-Fisher FP.

To sum up, at the starting value $\eta = 1.2 \times 10^{-2}$, by requiring existence for any $A_Z$ and a proper shape for the potential $V(\phi)$, compatible with the properties of the Wilson-Fisher FP, it is possible to select one zero for $V'(0)$ and one for $Z'(0)$. Then, the construction of the true FP proceeds by merging these zeros by tuning $\eta$. Having observed that locating the above mentioned zeros is easier for lower $A_Z$, especially for negative values, we do this at $A_Z = -1$. When $\eta$ increases, both zeros move towards less negative values. However, they do so at a different speed, such that they get closer and closer. Since the zero of $V'(0)$ is always on the right of that of $Z'(0)$, let us define $\Delta A_V$ as the position of the former minus the one of the latter. This quantity approaches zero with exponential rate in $\eta$, as can be guessed by inspecting the upper panel of Fig. 3. We can model this by looking at the function $F(\eta) = -(\log_{10} \Delta A_V)^{-1}$, that is $\Delta A_V = 10^{-1/F(\eta)}$, such that if $F(\eta) = 0$ then $\Delta A_V$ vanishes exponentially when $\eta \to \eta_s$. This is indeed the case, as the lower panel of Fig. 3 reveals.

The last point in this list, at which we stopped the merging procedure, is $\eta_{st} = 0.03615$. At this value of $\eta$ the position of the zero of $V'(0)$ is $A_{V_{st}} = -2.17999226178876724872935061$ while there is still a difference of approximately $\Delta A_{V_{st}} = 2.4 \times 10^{-25}$. For completeness in Fig. 4 we also show how the position of the zero of $V'(0)$ on the $A_V$ axis, as a function of $\eta$ at fixed $A_Z = -1$, in linear (upper panel) and double-logarithmic (lower panel) scale. In the latter case $\eta_{st}$ and $A_{V_{st}}$ are the values corresponding to the last point on the lists, i.e. the closest to the physical fixed point.

Though the difference between the location of the two zeros is tiny if quantified in terms of $\Delta A_V$, it is less satisfactory if quantified in terms of the actual values of $V'(0)$ and $Z'(0)$ at a random point in between these two zeros. Take for instance $\eta = \eta_{st}$ and inspect the solution at the zero of $V'(0)$: the $Z_2$ symmetry of $Z$ is severely violated by $Z'(0) = -234$. Vice versa, at the zero of $Z'(0)$ one has $V'(0) = -0.114$. Thus, in order to construct a good numerical approximation of the FP functions it would be necessary to decrease $\Delta A_V$ much beyond the point where we stopped our analysis. Yet, we can get a qualitative portrait of these functions by observing how they evolve.
along the curves of the corresponding zeros parameterized by $\eta$. That is, in Fig. 5 we show the scalar potential $V(\phi)$ (left panel) and its derivative $V'(\phi)$ (lower panel) at the values of $A_V$ corresponding to the zeros of $V'(0)$, for $A_Z = -1$ and for $\eta \in \{0.03600, 0.03605, 0.03610, 0.03612\}$ from red (deeper) to black (shallower). All these plots are obtained by numerical integration from $\phi = 1$ to $\phi = 0$ and are extended beyond $\phi = 1$ by means of the large field asymptotic behavior $V_{\infty}$.

Eventually one needs to consider again the effect of changing $A_Z$. On the basis of our previous discussion, we expect that the zeros of $V'(0)$ and $Z'(0)$ that we have analyzed at $A_Z = -1$ continue to exist also at any other value of $A_Z$, while their position should smoothly depend on the latter parameter. Indeed, it can be immediately observed, for instance by comparing the plots in the last line of Fig. 4 to those in the last line of Fig. 2, that at $\eta = 0.012$ the distance $\Delta A_V$ does indeed change when we shift $A_Z$ from $-1$ to $-10^{-1}$. Therefore the curves in Fig. 4 must change. Yet, this does not prove that the position where these curves meet the horizontal axis is shifting. To assess whether this is the case or not, one should produce new curves at several other values of $A_Z$, and extrapolate the position of the critical $\eta$. This analysis has not been performed so far. What we tried instead is a simpler check, that can only give us a bound on the $A_Z$-dependence of the critical $\eta$. This is related to the linear extrapolation method that we adopted for the estimate of $\eta$ at $A_Z = -1$. We focus on two values of $\eta$ that for $A_Z = -1$ are close to the critical one, such that we can use the corresponding $\Delta A_V$ to linearly extrapolate a best value and an error bar for $\eta$. We then analyze the $A_Z$ dependence of $\Delta A_V$ at these two points only. If there is any, by extrapolation we can estimate the change in the critical $\eta$. In order to be more precise in revealing possible changes, we need to be as close as possible to the critical point. Therefore we chose these two points to be $\eta_{ul} = 0.03615$ and $\eta = 0.03614$. Since the changes are expected to be more pronounced far from the critical point, especially the value of $\Delta A_V$ at the latter $\eta$ has been under our focus. For $A_Z = -1$ this is approximately $\Delta A_V = 3 \times 10^{-15}$. We then followed the position of the zeros of $V'(0)$ and $Z'(0)$ changing $A_Z$ in small discrete steps. By gradually increasing the step-size we sampled the interval $A_Z \in [-1, +10^{-3}]$. We never witnessed any change in $\Delta A_V$. This leads us to the conclusion that, if the critical $\eta$ undergoes a variation when $A_Z$ is changed inside $[-1, +10^{-3}]$, this variation must be smaller than the numerical uncertainty of our estimate at $A_Z = -1$. 

FIG. 5. The potential $V(\phi)$ (upper panel) and its first derivative $V'(\phi)$ (lower panel) at the values of $A_V$ corresponding to the zeros of $V'(0)$, for $A_Z = -1$ and for $\eta \in \{0.03600, 0.03605, 0.03610, 0.03612\}$ from red (deeper) to black (shallower). All these plots are obtained by numerical integration from $\phi = 1$ to $\phi = 0$ and are extended beyond $\phi = 1$ by means of the large field asymptotic behavior $V_{\infty}$.

FIG. 6. The function $Z(\phi)$, normalized to one at the origin, at the values of $A_V$ corresponding to the zeros of $Z'(0)$, for $A_Z = -1$ and for $\eta \in \{0.03600, 0.03605, 0.03610, 0.03612\}$ from red (smoother) to black (sharper). The true overall scales are $Z(0) \in \{-37.6, -53.4, -96.3, -146.8\}$ correspondingly. All these plots are obtained by numerical integration from $\phi = 1$ to $\phi = 0$ and are extended beyond $\phi = 1$ by means of the large field asymptotics $Z_{\infty}$. In the lower panels it is shown the behavior in the inner region (left) and close to the matching with the outer region (right).
Unfortunately this analysis has been affected by all the problematic features of the pattern of zeros that we outlined while commenting Figs. 1 and 2. Specifically, we again observed that the shapes of $V'(0)$ and $Z'(0)$ as functions of $A_V$ become steeper and steeper close to their zeros, as $A_Z$ increases from $-1$ towards $0$ and then to positive values. This makes the numerical location of these zeros harder and harder, such that from some values of $A_Z$ on (around $A_Z = -0.5$) we simply identified the position of the zeros with the location of the end of successful numerical integration, as in the upper-left panel of Figs. 1 and 2. A further unpleasant ambiguity affects the study of positive $A_Z$ values, since in this case there appear wild oscillations of these functions, similar to those that were depicted in Figs. 1 and 2, which often bring corresponding additional zeros. We then stuck to our assumption that the zeros connected to those at $A_Z = -1$ occur indefinitely close to the ending point of successful numerical integration, and we simply computed $\Delta A_Z$ by locating the latter.

**ACKNOWLEDGMENTS**

I am grateful to H. Gies and O. Zanusso for precious discussions and advices. I would like to thank T. R. Morris for correcting a mistake in a former version of this work. It is a pleasure to stress that large parts of this research has been inspired by collaboration with G. P. Vacca on related projects. I acknowledge support by the DFG under grant GRK1523/2.

**Appendix A: Large-field asymptotic expansion**

We made use of the following truncated expansions

$$V_{as}(\phi) = \left(1 - \frac{\eta}{2}\phi^2\right) + A_V\phi^{\frac{d}{2}+\eta} + \sum_{i=1}^{7} c_{V,i}(d, \eta, A_V)\phi^{c_{V,i}(d, \eta)}$$

$$Z_{as}(\phi)^{-1} = A_Z\phi^{\frac{d}{2}+\eta} + \sum_{i=1}^{9} c_{Z,i}(d, \eta, A_V, A_Z)\phi^{c_{Z,i}(d, \eta)}$$

Here

$$c_{V,1}(d, \eta) = 2 \frac{d - 2 + \eta}{d + 2 - \eta}$$
$$c_{V,2}(d, \eta) = 0$$
$$c_{V,3}(d, \eta) = -4 + \frac{6d}{d + 2 - \eta}$$
$$c_{V,4}(d, \eta) = -2 + \frac{2d}{d + 2 - \eta}$$

and

$$c_{Z,1}(d, \eta) = -2 + \frac{2d}{d + 2 - \eta} + \frac{4}{d - 2 + \eta}$$
$$c_{Z,2}(d, \eta) = -4 + \frac{4d}{d + 2 - \eta} + \frac{4}{d - 2 + \eta}$$
$$c_{Z,3}(d, \eta) = -2 + \frac{4}{d - 2 + \eta}$$
$$c_{Z,4}(d, \eta) = -6 + \frac{4d}{d + 2 - \eta} + \frac{4}{d - 2 + \eta}$$

Notice that the last order we took into account, for $d = 3$ and small enough $\eta > 0$, corresponds to the first negative power of $\phi$ inside $Z^{-1}$ (in the limiting case $\eta \to 0$ it gives $\phi^0$). The first few coefficients in $V$ are

$$c_{V,1}(d, \eta, A_V) = -\frac{4d^2 A_V^2}{(d + 2 - \eta)^2(-2 + \eta)}$$
$$c_{V,2}(d, \eta, A_V) = \frac{-2 + \eta}{2d}$$
$$c_{V,3}(d, \eta, A_V) = \frac{-16d^3 A_V^3(d - 2 + \eta)}{(d + 2 - \eta)^4(-2 + \eta)^2}$$
$$c_{V,4}(d, \eta, A_V) = -2dA_V(d - 2 + \eta)^3.$$ 

Up to the computed order these depend only on $A_V$, while the coefficient for $Z$

$$c_{Z,1}(d, \eta, A_V, A_Z) = \frac{4dA_V A_Z}{(d + 2 - \eta)(-2 + \eta)}$$
$$c_{Z,2}(d, \eta, A_V, A_Z) = -\frac{32d^2 A_V^2 A_Z}{(d + 2 - \eta)^3(-2 + \eta)}$$
$$c_{Z,3}(d, \eta, A_V, A_Z) = \frac{4A_Z}{(d - 2 + \eta)^3}$$
$$c_{Z,4}(d, \eta, A_V, A_Z) = \frac{64d^3 A_V A_Z(d - 10 + 2\eta)}{(d + 2 - \eta)^5(-2 + \eta)^2}.$$ 

involve also $A_Z$. The remaining coefficients are too long to appear here.
