DAMPING OF MAGNETOHYDRODYNAMIC TURBULENCE IN SOLAR FLARES

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ABSTRACT

We describe the cascade of plasma waves or turbulence injected, presumably by reconnection, at scales comparable to the size of a solar flare loop, $L \sim 10^8$ cm, to scales comparable to elementary particle gyroradii and evaluate their damping at small scales by various mechanisms. We show that the classical viscous damping valid on scales larger than the collision mean free path ($\sim 10^9$ cm) is unimportant for magnetically dominated or low-$\beta$ plasmas and the primary damping mechanism is the collisionless damping by the background particles. We show that the damping rate is proportional to the total random momentum density of the particles. For solar flare conditions this means that in most flares, except the very large ones in which essentially all background electrons are accelerated into a nonthermal distribution, the damping is dominated by thermal background electrons. In general, damping by protons is negligible compared to that of electrons except for rare proton-dominated flares with strong nuclear gamma-ray line emission and for quasi-perpendicular propagating waves. We also determine the critical scale below which the damping becomes important and the spectrum of the turbulence steepens. We show that this scale has a strong dependence on the propagation angle of the waves with respect to the background magnetic field, resulting in a highly anisotropic spectral distribution, with quasi-parallel and quasi-perpendicular waves cascading undamped to small scales.

Subject headings: acceleration of particles — plasmas — turbulence

1. INTRODUCTION

The mechanism of energy release and the process of its transfer to heating and acceleration of nonthermal particles in many magnetized astrophysical plasmas in general, and solar flares in particular, are still matters of considerable debate. Recent research shows that turbulence may play an essential role in these processes. In the case of solar flares, it is believed that the energy comes from release of stored magnetic energy via reconnection (see Priest & Forbes 2000; Lazarian et al. 2004 and discussions therein). Both the ordinary and magnetic Reynolds numbers, \( Re = lbv/\nu \) and \( Re_m = lbv/\eta \gg 1 \), respectively, so that the magnetized plasma develops turbulence. Here \( \nu \) is the velocity change across a turbulent region of scale \( l \), and \( \nu \) and \( \eta \) are the viscosity and magnetic diffusion coefficients, respectively. More importantly, recent high-resolution observations of solar flares by the Yohkoh and RHESSI satellites have provided ample evidence that, at least from the point of view of particle acceleration, plasma turbulence and plasma waves appear to be the most promising agent not only for the acceleration mechanism but also the general energizing of flare plasma (see, e.g., Petrosian & Liu 2004, hereafter PL04, and references cited therein). This may also be true in other situations (Liu et al. 2004b). These investigations go beyond assuming, e.g., a power-law electron distribution, as is commonly done, and calculate the expected spectrum based on interaction of plasma particles with turbulence. However, in such treatments the spectrum \( W(k) \) of the turbulence as a function of the wavenumber \( k \) is an input parameter rather than evaluated based on first principles. The limitations of such an approach are self-evident. The particle acceleration rate depends on the wave spectrum, and the wave damping rate is partially determined by the particle spectrum. In general, one requires a self-consistent treatment of the coupled wave-particle kinetic equations describing the generation of turbulence and its subsequent interactions with the background plasma.

An attempt to solve the problem self-consistently was undertaken by Miller et al. (1996), in which coupled equations for energetic particles and turbulence were studied. For our purposes we can rewrite these equations in the following form:

\[
\frac{\partial N}{\partial t} = \frac{\partial}{\partial E} \left( D_{EE} \frac{\partial N}{\partial E} - (A - \dot{E}_L)N \right) - \frac{N}{T_{\text{esc}}} + \dot{Q}^p,
\]

\[
\frac{\partial W}{\partial t} = \frac{\partial}{\partial k} \left( D_k \frac{\partial W}{\partial k} \right) - \Gamma(k)W - \frac{W}{T_{\text{esc}}(k)} + \dot{Q}^W. \tag{1}
\]

Here \( D_{EE}/E^2, A(E)/E, \) and \( \dot{E}_L/E \) give the diffusion, direct acceleration, and energy-loss rates of the particles, respectively, and \( D_k(k)/k^2 \) and \( \Gamma(k) \) describe the cascade and damping rates of turbulence, respectively. The \( \dot{Q} \) terms and the terms with the escape times \( T_{\text{esc}} \) describe the source and leakage, respectively, of particles and waves. Note that we use a more general form of the equation set compared to that in Miller et al. (1996). In our treatment the anisotropy of the turbulent statistics is allowed, and the leakage of the particles, as well as turbulent energy, is accounted for. In general, and particularly in the case of solar flares that we treat here, it is reasonable to assume that the turbulence is generated mainly at a large scale \( l \) comparable or somewhat smaller than the spatial extent \( L \) of the region (with initial velocity \( bV \) and magnetic field fluctuation \( \delta B \)). The above equations then determine the resulting spectrum and other characteristics of the turbulence, as it cascades to smaller scales and

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is damped by the background thermal plasma, as well as the spectrum of the accelerated nonthermal particles.

In recent years there has been a substantial progress in (1) the understanding of cascade of incompressible (Goldreich & Sridhar 1995) and compressible MHD turbulence (see Cho & Lazarian 2005 and references therein) and damping of compressible MHD turbulence (Yan & Lazarian 2004, hereafter YL04), and (2) in determination of plasma-wave-particle and MHD-turbulence-particle interaction rates (see, e.g., Dung & Petrovskii 1994; Pryadko & Petrovskii 1997, 1998, 1999; PL04; Chandran 2000; Yan & Lazarian 2002, hereafter YL02; YL04; Cho & Lazarian 2006). These advances allow a more thorough description of the coefficients involved in equation (1). In this paper we limit our attention to one aspect of this complex problem, namely, to damping of turbulence represented by the coefficients $\Gamma(k)$. After a brief review of recent progress in the understanding of the cascade process in $\S$ 2 we evaluate the damping rates due to thermal particles in the background plasma and a nonthermal population representing the accelerated spectrum $N(E)$ ($\S$ 3). In $\S$ 4 we summarize our results and discuss their use in our future work on solving the coupled wave-particle kinetic equations.

2. TURBULENCE AND ITS CASCADE

Plasma turbulence can be decomposed into many wave modes with frequencies $\omega$ extending essentially from zero frequency to the ion (in our case proton) and electron gyrofrequencies, $\Omega_p = eB/m_p c$ and $\Omega_e = \Omega_p/\delta$, respectively ($\delta = m_e/m_p$, is the electron to proton mass ratio), and wavevectors $k = 1/l$ spanning the spatial scales from the injection scale to the gyroradius of electrons obeying a complex dispersion relation determined by the values of density $n$, temperature $T$, and magnetic field $B$ of the background plasma.

When we consider turbulence injected at a scale much larger than the proton (or ion) skin depth $\sim v_A/\Omega_p = 230(10^{10} \text{ cm}^3/n)^{1/2}$ cm, where

$$\beta_A = \frac{v_A}{c} = 7 \times 10^{-3} \left(\frac{B}{100 \ \text{G}}\right) \left(\frac{10^{10} \text{ cm}^3/n}{n}\right)^{1/2} \left(\frac{\text{cm}}{\text{s}}\right)^{-1/2} \left(\frac{\text{G}}{100}\right)$$

is the Alfvén velocity in units of the speed of light $c$. Initially we deal with modes for which the plasma acts as a single fluid, and we are in the MHD regime where the dispersion relation simplifies considerably. It has been known for decades that the weak MHD perturbations can be decomposed into Alfvenic, slow, and fast waves with well-known dispersion relations (see, e.g., Sturrock 1994). However, it was also believed that such a decomposition does not make much sense for a highly nonlinear phenomenon of MHD turbulence, where the modes were believed to be strongly coupled (see Stone et al. 1998). A study of mode coupling in Cho & Lazarian (2002, 2003, hereafter CL02, CL03, respectively) has shown that the coupling is appreciable only at the injection scale, while along the cascade to smaller scales the transfer of energy between the modes is suppressed.3 This justifies the decomposition to different modes even for strong MHD turbulence (see CL02, CL03) and allows us to treat their cascade, and interactions with charged particles, of Alfvenic, fast, and slow modes separately (see YL04; Yan & Lazarian 2003, hereafter YL03).

In solar flares, and in many other astrophysical plasmas, one is dealing with a magnetically dominated plasma with the plasma $\beta$ parameter,

$$\beta_p = \frac{8 \pi n k T}{B^2} = 3.4 \times 10^{-2} \left(\frac{n}{10^{10} \text{ cm}^3}\right) \left(\frac{100 \ \text{G}}{B}\right)^2 \left(\frac{T}{10^8 \text{ K}}\right) \ll 1,$$

which also means that the Alfvén speed is greater than the sound speed $c_s$. In this case the dispersion relations of the three modes can be approximated as (Sturrock 1994)

$$\omega = v_A k \cos \theta, \quad \omega = c_s k \cos \theta, \quad \omega = v_A k$$

for the Alfven, slow, and fast modes, respectively, where $\theta$ is the angle of propagation of the wave with respect to the magnetic field.4

Turbulence generated at large scales can cascade to small scales by nonlinear interactions. One important characteristic of turbulence is its self-similarity. Power-law spectra were obtained numerically for Alfvénic, fast, and slow mode turbulence in CL02 and CL03 for the case when turbulent energy is injected at large scales. It has also been demonstrated that Alfvénic (and slow) modes exhibit scale-dependent anisotropy similar to that described by Goldreich & Sridhar (1995) for incompressible turbulence.

This can be understood on a qualitative level as follows. For Alfvénic turbulence, the mixing motions perpendicular to the magnetic field couple with the wavelike motions parallel to the magnetic field providing the so-called critical balance condition, $k_x v_A \sim k_z v_A$, where $k_x$ and $k_z$ are the parallel and perpendicular components of the total wavevector $k$. This, when combined with the Kolmogorov scaling for mixing motions with $v_x \approx \delta V/(k L)^{1/3}$ (see Lazarian & Vishniac 1999), yields a scale-dependent anisotropy $k, L \sim \langle \delta V/v_A \rangle (k L)^{2/3}$. Here $L$ is the injection scale. The mixing motions associated with Alfvénic turbulence induce the scale-dependent anisotropy on slow modes, which on their own would evolve on substantially longer timescales. The anisotropic spectrum of the Alfvén and slow modes can be described as (Cho et al. 2003)\(^5\)

$$W(k_x, k_z) = \frac{L^{-1/3}}{6\pi} \frac{k_z^{-10/3}}{k_x} \exp \left(-\frac{L^{1/3}}{k_x} \frac{k_x^{-10/3}}{k_z^{3/2}}\right).$$

For Alfvénic and slow modes the cascade time, like the hydrodynamic eddy turnover time, is

$$\tau_{\text{cas}} \approx \frac{L}{v_A} \approx \frac{\tau_0 (k, L)^{-2/3}}{M_A},$$

where we have defined a characteristic time and the Alfvén Mach number at injection,

$$\tau_0 = \frac{L}{v_A} = 4.5 \left(\frac{L}{10^7 \text{ cm}}\right) \left(\frac{100 \ \text{G}}{B}\right) \left(\frac{10^{10} \text{ cm}^3/n}{n}\right)^{1/2} \left(\frac{\text{cm}}{\text{s}}\right)^{-1/2} \left(\frac{\text{G}}{100}\right),$$

and $M_A \equiv \delta V/v_A$.\(^5\)

\(^3\) An intuitive insight into this process can be traced to the Goldreich & Sridhar (1995) study (see also Lithwick & Goldreich 2001; CL02).

\(^4\) In a very highly magnetized plasma $\beta_A^2 \gg 1$, $v_A$ in eq. (4) should be replaced by $v_A/k_{\text{crit}}$, and the actual dispersion relations deviate from these at shorter scales, or for $k > k_{\text{crit}} = \Omega_p/\beta A (1 + \cos^2 \theta) A(1 + \cos^2 \theta)^{1/2}$. When this scale is larger than the proton gyroradius $v_{\text{th}}/\Omega_p \sim 30 \text{ cm}$, the above dispersion relations are good approximations. As we shall see below for solar flare conditions, this is the case for most angles, except for quasi-parallel (and possibly quasi-perpendicular) propagation, in which case more accurate dispersion relations should be used.

\(^5\) Note that integrating over the parallel and perpendicular components, one gets $W(k_z) \propto k_z^{-5/3}$ and $W(k_x) \propto k_x^{-2}$, respectively.
Fast modes in low-$\beta_p$ plasma, on the other hand, develop on their own, as their phase velocity is only marginally affected by the mixing motions induced by Alfvén modes. According to CL02, fast modes follow an isotropic “acoustic” cascade with $W(k) \propto k^{-3/2}$.

For such a cascade in each wave-wave collision, a small fraction of energy equal to $\Gamma_{pb}/v_k$ is transferred to smaller scales so that the cascade timescale is characterized by (CL02)

$$\tau_{\text{cas}} = \left( \frac{\Gamma_{pb}}{v_k} \right) \left( \frac{l}{v_k} \right)^{1/2} \frac{\tau_0(kL)}{M_k^2}.$$  

(8)

Here $v_{pb} = \omega/k = v_A$ is the phase velocity of the fast mode, and we have used the scaling relation $v_k = \delta V(kL)^{-1/4}$ appropriate for fast modes.

For solar flare conditions, $\beta_p \ll 0.1$, and $\lambda_A = v_A/c \sim 10^{-2}$. Assuming $M_k \leq 1$, the above cascade times are about a few seconds at the injection scale but much shorter at shorter scales.

Collisionless thermal damping of Alfvén waves was described by Cramner & van Ballegooijen (2003) and evaluated numerically for specific parameter values. There is no discussion in the literature of damping of slow modes, although Chandran (2003) describes particle acceleration by these modes and finds them inefficient in acceleration of electrons in solar flares. Moreover, because of the anisotropic nature of their cascades described above, both these modes become inefficient in scattering of energetic particles (Chandran 2000; YL02). The fast modes, on the other hand, undergo an isotropic cascade (however, see below), and as shown in YL03, they are the dominant mechanism for scattering and acceleration of particles via resonant (and possibly nonresonant; see Cho & Lazarian 2006) interactions. Consequently, in what follows we emphasize the damping based on the dispersion relation of the fast modes. However, our general expressions for collisionless damping derived below (e.g., eq. [29]) can be applied to the slow and Alfvén modes simply by replacing $v_A$ by $c_s \cos \theta$ and $v_A \cos \theta$, respectively.

3. DAMPING RATE OF TURBULENCE

The second important process determining the spectrum of turbulence is its damping rate. Damping becomes important whenever the damping time $\Gamma^{-1}(k)$ becomes comparable to or shorter than the cascading time $\tau_{\text{cas}}(k)$. As we shall show below, for solar flare conditions, the damping time is longer than the cascade time at large scales but decreases faster with decreasing scale and becomes dominant above the critical wavevector where $\Gamma(k)\tau_{\text{cas}}(k) = 1$. In this section we derive the damping rate and the critical wavevector $k_c$. We first describe the damping by the background thermal plasma.

3.1. Thermal Damping

In fully ionized plasma, the damping can be divided into two parts: collisional and collisionless with their regimes of relevance determined by the ratio of the turbulence scale and the Coulomb collision mean free path of the background plasma (Braginskii 1965),

$$\lambda_{\text{Coul}} \sim 9 \times 10^7 \text{ cm} \left( \frac{T}{10^7 \text{ K}} \right)^2 \left( \frac{10^{10} \text{ cm}^{-3}}{n} \right).$$  

(9)

This scaling, plausible on theoretical grounds (CL03; Cho & Lazarian 2005), is supported by numerical simulations of CL03 but may still require more testing. On the other hand, in the high-$\beta_p$ case the scaling of the fast modes trivially reduces to the scaling of acoustic turbulence because the fast modes are essentially acoustic fluctuations.

Viscous damping is important for scales $l > \lambda_{\text{Coul}}$, so that it can play a role between the injection scale $< L \sim 10^8 \text{ cm}$ and $\lambda_{\text{Coul}} \sim 10^8 \text{ cm}$. For smaller scales, $k\lambda_{\text{Coul}} > 1$, the damping rate is determined by less efficient collisionless processes.

3.1.1. Viscous Damping

The viscous damping rate is derived in Appendix A, where we show that for low-$\beta$ plasma of interest, we have

$$\Gamma_{\text{vis}}(k, \theta) = 0.13 \tau_0^{-1} \beta_p^{1/2} (kL)(k\lambda_{\text{Coul}}) \sin^2 \theta, \text{ for } k\lambda_{\text{Coul}} < 1.$$  

(10)

By equating $\Gamma^{-1}(k)$ from the above equation to $\tau_{\text{cas}}$ in equation (8), we obtain the critical scale or wavevector,

$$k_c \lambda_{\text{Coul}} = 3.9 \left( \frac{M_k}{\sin \theta} \right)^{4/3} \left( \frac{\lambda_{\text{Coul}}}{L/\beta_p} \right)^{1/3}.$$  

(11)

For $M_k \sim 1$ and $\beta_p < 0.1$ the last two terms are greater than one, indicating that the critical scale is less than the Coulomb mean free path where this damping rate is not valid; $k_c \lambda_{\text{Coul}} < 1$ if $\sin \theta > 2.8M_k$, so that viscous damping of fast modes could be marginally important for $M_k \ll 0.3$. Viscous damping may be slightly more important for the other modes because of the steeper decline of their cascade time with $k$. But for low-$\beta$ plasmas the viscous damping can generally be ignored.

3.1.2. Collisionless Thermal Damping

The nature of collisionless damping is closely related to the radiation of charged particles in a magnetic field. Charged particles can emit plasma waves through acceleration (cyclotron radiation) and the Cerenkov effect and can also absorb the radiation under the same condition and cause damping of the waves (Ginzburg 1961). For example, the gyroresonance with thermal ions causes the damping of the modes with frequencies close to the ion-cyclotron frequency (Leamon et al. 1998). The particles can also be accelerated either by the parallel electric field (Landau damping) or the magnetic mirror associated with the comoving compressible modes under the Cerenkov condition $\lambda \langle v \rangle \ll \omega$, known as transit time damping (TTD). For small-amplitude waves, the above Cerenkov condition indicates that particles should have a parallel speed comparable to the wave phase velocity to be trapped in the moving mirrors. For fast modes with phase velocity $\omega/k = v_A (\beta_p < 1)$ only particles with velocities larger than $v_A$ can satisfy this condition. For a thermal plasma, this means that the fraction of particles that satisfy this condition decreases with increasing $v_A$ (or decreasing thermal speed $v_{th}$). In other words, due to this effect the damping rate decreases with decreasing $\beta_p$, exponentially as

$$\Gamma_{\text{th}}(k, \theta) = \Gamma_0 \left[ \exp \left( -\frac{\delta}{\beta_p \cos^2 \theta} \right) + \frac{5}{\sqrt{6}} \exp \left( -\frac{1}{\beta_p \cos^2 \theta} \right) \right] g(\theta),$$  

(12)

where, as stated above, $\delta = m_e/m_p$ is the electron to proton ratio and we have defined a characteristic damping rate,

$$\Gamma_0 \equiv \sqrt{\frac{4\pi e^2 \delta}{m_p}} \frac{(kL)}{v_A \cos \theta}.$$  

(13)
This damping rate, without the last term \( g(\theta) \) and valid for \( \theta \gg (\omega / \Omega_p)^{1/2} \), coincides with the one in Ginzburg (1961).\(^7\) In the brackets the first term represents the contribution from electrons, and the second term is due to protons. The function \( g(\theta) \) for \( \theta \ll 1 \) is

\[
g(\theta) = \frac{1}{2} \left( 1 + \frac{\theta^2}{\sqrt{\theta^4 + 4\omega^2 / \Omega_p^2}} \right) \begin{cases} 1, & \text{if } 1 \gg \theta \gg \sqrt{\frac{\omega}{\Omega_p}} \\ 0.5, & \text{if } \theta \ll \sqrt{\frac{\omega}{\Omega_p}} \ll 1, \end{cases}
\]

which was derived by Stepanov (1958), who extended the relation to small angles where the damping rate decreases by a factor of 2.\(^8\)

For \( \beta_p \ll 0.1 \) and sufficiently large \( \theta \), the damping due to electrons dominates, and the damping rate can be written in a simple form,

\[
\Gamma_{\text{el}}(k, \theta) = \Gamma_0 \exp \left( -\frac{\delta}{(\beta_p \cos^2 \theta)} \right), \quad \text{for } k \lambda_{\text{Coul}} > 1, \quad (15)
\]

where we have ignored the correction \( g(\theta) \) at small angles. Note the similarity of this relation to that for viscous damping; the main difference is the absence of the extra term \((k \lambda_{\text{Coul}})\) in equation (10). One can then combine the two expressions to obtain an approximate damping rate valid at all scales,

\[
\Gamma_{\text{tot}}(k, \theta) \approx \Gamma_{\text{el}}(k, \theta) \left(1 + \frac{\cosh \theta}{1 - \cosh \theta} \right), \quad \text{for } \zeta = 6.3 \cos \theta (k \lambda_{\text{Coul}}), \quad (16)
\]

where we have deleted the exponential part in equation (15), which is equal to one except in a small range of angles near \( \pi/2 \) (i.e., \( \cos \theta < 0.023 / \sqrt{\beta_p} \)). A more accurate expression is obtained if one divides \( \zeta \) by the bracketed term and \( g(\theta) \) in equation (12).

In Figure 1 we compare the cascading time with the damping time \( (\tau_d = 1/\Gamma_0) \) at different scales for \( \beta_p = 0.01 \) and different \( \theta \) values \( (\theta = 45^\circ) \) at four values of \( \beta_p = 0.001, 0.01, 0.1, \) and 1 (left), corresponding to magnetic fields of \( B = 600, 180, 60, \) and 18 G, respectively (see eq. [3]). We use typical solar flare values, temperature \( T = 10^7 \) K, density \( n = 10^{10} \) \( \text{cm}^{-3} \), and we set \( M_\lambda = (\delta V / V_\lambda) = 0.3 \). The angular dependence enters this damping by two competing factors. In general the damping increases with \( \theta \) because magnetic compression increases so that more particles can be trapped and interact with the waves. However, when \( \theta \) approaches \( 90^\circ \), i.e., for quasi-perpendicular propagation, most thermal particles will not be in resonance with the fast-mode waves in a low-\( \beta_p \) medium, which explains the decrease of damping in this regime.

By equating the collisionless damping time from equation (15) with the cascade time in equation (8), we attain the critical wavevector,

\[
k_c L = \frac{4M_\lambda^4 \cos^2 \theta}{\pi \delta \beta_p \sin^4 \theta} \exp \left( \frac{2 \delta}{\beta_p \cos^2 \theta} \right). \quad (17)
\]

The variation of \( k_c \) with angle for the thermal collisionless damping using the exact expressions is shown in Figure 2. As evident
the damping scale given by equation (17) varies considerably especially when \( \theta \to 0^\circ \) and \( 90^\circ \), where it becomes smaller than the proton gyroradius (dashed line). Note also that for \( \beta_p < 0.1 \) the damping scale is larger than the collision mean free path (or \( k_c \lambda_{\text{Coll}} > 1 \), dash-dotted horizontal line), except for a few degrees around \( \theta = 85^\circ \), which is within the range of its validity. The specific range of \( \theta \) in which the relations break down depends on the plasma \( \beta_p \).

This describes the well-known fact that at large \( k \)-vectors (i.e., small scales) the turbulence will be very anisotropic. Only quasi-parallel and quasi-perpendicular modes survive at such large wavevectors (and corresponding frequencies), which are needed in the acceleration of low-energy particles. However, as turbulence undergoes cascade and/or waves propagate in a turbulent medium, the character of this anisotropy changes because the angle \( \theta \) is changing due to the randomization of the wavevector \( \mathbf{k} \) and the wandering of the magnetic field lines discussed below.

### 3.1.3. Damping Anisotropy

The damping of fast modes described above is valid for small perturbations and for a uniform background magnetic field. A more realistic setting for damping in turbulent media, which is based on a better understanding of the turbulent cascades (CL02, CL03) and magnetic field wandering (Lazarian & Vishniac 1999; Lazarian et al. 2004), is discussed in YL04 for the case of the interstellar medium. Here we apply this approach for turbulence in solar flares.

For fast-mode cascade, the nonlinear cascading occurs by interaction of wave packets that are collinear (see review by Cho & Lazarian 2004), \( \omega = \omega_1 + \omega_2 \) and \( k = k_1 + k_2 \), and the final wave is parallel to the initial ones. As has been demonstrated earlier by Cho et al. (2003), the cascading is permitted for an uncertainty in \( \delta k \) that is related to the cascading time \( \tau_{\text{cas}} \) as

\[
\tau_{\text{cas}}^{-1} \sim \tau_{\text{ph}} \delta k/k. 
\]

Combining this with equation (8), we obtain the following expression (YL04),

\[
\tan \delta \theta \simeq \frac{\delta k}{k} \simeq \left( \frac{k \omega}{\omega} \right)^{1/2} \simeq M_A (kL)^{-1/4}. \tag{18}
\]

As in the interstellar medium, the field-line wandering in the case of solar flares is mainly caused by the shearing due to Alfvén modes. According to Lazarian & Vishniac (1999),\(^9\) the field-line diffusion along and perpendicular the mean field produce angular deviations

\[
\tan \delta \theta_\parallel \simeq M_A \left( \frac{z}{27L} \right)^{1/2}, \quad \tan \delta \theta_\perp \simeq M_A \left( \frac{r}{L} \right)^{1/3}. \tag{19}
\]

where \( z \) and \( r \) are the distances along and perpendicular to the mean field direction, respectively. During one cascading time (eq. [8]), the fast modes propagate a distance \( \ell_\text{cas} \sim v_A \tau_{\text{cas}} = L/M_A^2 (kL)^{1/2} \) and see an angular deviation

\[
\tan \delta \theta \simeq \sqrt{\tan^2 \delta \theta_\parallel + \tan^2 \delta \theta_\perp} \simeq \sqrt{\frac{M_A^2 \cos \theta}{27(kL)^{1/2}} + \frac{M_A^2 \sin^2 \theta}{kL}} \tag{20}
\]

Note that at the largest scale \( kL = 1 \) the randomization is of the order of \( M_A < 1 \)\(^10\) and decreases slowly with \( k \); \( \delta \theta \) scales as \((kL)^{-1/6}\) at small angles or quasi-parallel modes and as \(- (kL)^{-1/16}\) at other angles.

While for the processes of scattering and acceleration of particles by fast modes randomization is important (YL04), it does not always result in tangible changes of the overall picture of damping. For instance, combining equations (17) and (20), we can show that for quasi-parallel modes (i.e., \( \theta \to 0^\circ \)) at the critical wavevectors \( k_c \), the randomization angle is rather small, i.e., \( \delta \theta \sim 10^{-3} M_A \beta_p^2 \). For other angles, \( \delta \theta < (10 \beta_p 30)^{1/2} \) and is still small but may not be negligible. In particular, for the quasi-perpendicular modes, we can make an estimate of the presence of field-line wandering on the damping truncation scale by evaluating the average of the damping rate in equation (12) over the small range \( \pi/2 - \delta \theta \to \pi/2 \), where from equation (20), \( \delta \theta \sim M_A^{-2} (kL)^{-1/6} \). For \( \delta \theta \ll 1 \) we can define \( \alpha = \pi/2 - \theta \) and use the approximations \( \sin \theta = 1 \) and \( \cos \theta = \alpha \) so that the average value of the damping rate given in equation (12) is roughly given by

\[
\langle \Gamma_{\text{th}} \rangle = \frac{\sqrt{\pi} \beta_p \delta \theta (kL)}{2 \pi \delta \theta} \int_0^{\delta \theta} \frac{d \alpha \exp \left[ - \delta f (\beta_p \alpha^2) \right]}{\alpha} \tag{21}
\]

Equating this with \( \tau_{\text{cas}} \) from equation (8), we get \( x^2 E_1(\alpha) = x \) with \( x = (\delta f / \beta_p^2)(kL/M_A)^{1/3} \) and \( A = (4/\pi^{1/2}) M_A^{-1} \delta^{3/2} \beta_p^{-5/2} = (0.025/\beta_p)^{-5/2} \), where we set \( M_A = 0.3 \). For \( \beta_p = 0.1 \) this relation is satisfied for \( x \sim 0.15 \), or \( kL = M_A^2 (x/\beta_p)^{6/3} \sim 20 \). This scale is shown by the horizontal dotted line in Figure 2. This means that modes in the cone near \( 90^\circ \) get damped above this scale due

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\(^9\) A typo in the numerical coefficient has been corrected.

\(^10\) It is useful, in general, to define the domain of MHD turbulence as starting from an Alfvén Mach number of unity (see discussion in Cho & Lazarian 2005). This condition is automatically satisfied for strongly magnetized turbulence in solar flares.
to randomization of $\theta$. However, because $x^2E_1(x) \approx 0.22$ there is no solution for $\beta_p < 0.05$, indicating that the field wanderings do not reduce the damping scale at $90^\circ$ for highly magnetized plasmas.

These results are only rough estimates. A more detailed study of the issue is more appropriate in the context of particle acceleration in solar flares and will be done elsewhere.

### 3.2. Nonthermal Damping

The wave damping rate calculated above assumes that the energy lost by waves with spectrum $W(k)$ goes into heating the plasma and that the plasma maintains its Maxwellian distribution, presumably via Coulomb collisions on a timescale of $\tau_{\text{coll}} \sim \lambda_{\text{coll}}/c_s$, where $c_s$ is the sound speed. This requires a longer damping time: $\Gamma_{\text{th}}^1 > \tau_{\text{coll}}$. As stated above we are interested in a collisionless plasma with $\tau_{\text{coll}} \ll \tau_{\text{coll}}$. This combined with the fact that damping is important when $\Gamma_{\text{th}}^1 < \tau_{\text{coll}}$ implies that the above condition is not satisfied and some particles get accelerated to energies much higher than $k_B T$. Because $\tau_{\text{coll}}$ increases with energy fairly rapidly, it can be shorter than the other times at low energies, where the particle spectrum will be approximately Maxwellian. As shown in PL04, solution of the particle kinetic equation (1) with a given background thermal plasma and an assumed spectrum of turbulence does lead to a particle spectrum consisting of a quasi-thermal part and a nonthermal tail, with the dividing energy roughly where $\tau_{\text{coll}}$ is equal to the acceleration time $\tau_{\text{acc}} \approx E/A(E)$. Thus, we need to also consider damping of the waves by the nonthermal tail. As mentioned at the outset one must carry out this self-consistently by solving the coupled wave-particle kinetic equations, which is beyond the scope of this paper. Here we first derive the damping rate due to a general energy spectrum $N(E)$ of electrons and protons and apply it to the case of a power-law energy spectrum $N(E) = N_0(a-1)/E/E_0^{a-1}/E_0$ (for $E > E_0$ and total density of $N_0$) and isotropic pitch-angle distribution.

As can be surmised, the calculations of the particle diffusion coefficient $D_{\text{pp}} = (\Delta p \Delta \mu/\Delta t)$ or the acceleration rate $A(E)/E$ and the damping rate $\Gamma_{\text{nonth}}(k)$ are intimately connected. Let us represent the transition rate (integrated over the particle pitch-angle cosine $\mu$) of the interaction between a wave $k$ and a particle with energy $E$ by $\sigma(k, E)$. The Fokker-Planck diffusion coefficient $D(p)$, which is $D_{\text{pp}}$ integrated over $\mu$, is

$$D(p) = \int_{-1}^{1} d\mu D_{\text{pp}} = \int_{\theta_{\text{min}}}^{\theta_{\text{max}}} d^3k W(k)\sigma(k, E).$$

(22)

From this we can get the rate of systematic energy gain by particles (PL04):

$$A(E) = \frac{d[\mu^2 D(p)]}{4p^2 dp} = \int_{\theta_{\text{min}}}^{\theta_{\text{max}}} d^3k W(k)\Sigma(k, E),$$

(23)

which also means that

$$\Sigma(k, E) = \frac{d[\mu^2 \sigma(k, E)]}{4p^2 dp}.$$  

(24)

Because the energy lost by turbulence $V_{\text{nonth}} = \int \Gamma_{\text{nonth}}(k)W(k)d^3k$ to nonthermal particles must be equal to the energy gain by these particles, $E = \int A(E)N(E)dE$, the damping rate is given by

$$\Gamma_{\text{nonth}}(k) = \int_{E_0}^{\infty} dE N(E)\Sigma(k, E).$$

(25)

The transition rate $\sigma$ for a gyrofrequency $\Omega$ and pitch-angle cosine $\mu$ interacting with a wave of frequency $\omega$ and wavevector $k$ is determined by the resonant condition,

$$\omega - k \cos \theta \mu = \frac{n\Omega}{\gamma}, \quad n = (0, \pm 1, \pm 2, \ldots).$$

(26)

This rate can be expressed as sum over $n$ of squares of Bessel functions $J_n^2(k_1 v_1/\gamma)$ with $i = n, n-1$, or $n+1$ and $v_1 = v(1 - \mu^2)^{1/2}$. (For details see Pryadko & Petrosian [1999].) For waves propagating parallel to the $B$ field, $k_1 = 0$ and only $\Delta_0(0) \neq 0$ and $n = \pm 1 (n = 0$ term also vanishes). The resonance condition then requires $k_{\text{res}}^2 \sim \Omega/\omega \sim r_0^{-1}$. Given the parameters we adopt here $r_0 \sim 1$ and 50 cm for thermal electrons and protons, respectively, which is certainly beyond the MHD regime. Only quasi-parallel propagating waves cascade to such small scales without undergoing thermal damping and can contribute to acceleration of low-energy particles (see Fig. 2).

For obliquely propagating waves (fast or Alfvén) things are more complicated, and all $n$ values could contribute. However, these waves are damped at scales much larger than that required for the above resonant condition, except for $n = 0$, or TTD, which happens at all scales with the resonant condition $v_\mu = v_\Omega/\cos \theta$.

Thus, in what follows we consider this process for fast modes with the transition rate,

$$\sigma(k, E) = \frac{\Omega m^2}{2nmp} \int_1^1 d\mu \left(1 - \mu^2\right) \left[\frac{\tau_{\text{can}}^1}{\tau_{\text{can}}^2 + (k_1 v_\mu - \omega)^2}\right] J_n^2(x),$$

$$x = \gamma k_1 v_1 \mu \Omega,$$

(27)

where $n$ is the density of protons (which we take to be equal to that of electrons), and the resonance function in the large parentheses is produced by integration over time. In general, the width of this function $\Delta_\mu = \tau_{\text{can}}^{-1}/k_1 v = M_2^2(kL)^{-12}/v_\Omega/\cos \theta \ll 1$ because usually the Alfvén Mach number $M_2 < 1$ and damping normally becomes important at scale $k^{-1} \ll L$. (This is not true for very nearly perpendicular propagation.) Then the resonance function can be approximated by a $\delta$-function, i.e., $\tau_{\text{res}}^{-1}[\tau_{\text{can}}^{-2} + (k_1 v_\mu - \omega)^2] \approx \pi\delta(k_1 v_\mu - \omega) = \pi\delta(\mu - \mu_{\text{res}})k_1 v_\mu$, where for the fast modes $\mu_{\text{res}} = v_\Omega/\cos \theta$. In Figure 3, the integrand of $\sigma(k, E)$ is plotted versus $\mu$ for some interesting cases. As evident, when the resonance condition is satisfied, i.e., $v_\mu < v_\Omega/\cos \theta$, the transition rate $\sigma$ peaks sharply by many orders of magnitude so that it can be well represented by a $\delta$-function.

With this simplification the integration over $\mu$ can then be carried out easily. Then from the relation $J_1(J_0 - J_2)/2$ and equations (27), (25), and (24) we can obtain the damping rate,

$$\Gamma_{\text{nonth}}(k) = \pi \frac{\Omega^2 m^2}{8 \eta n_{mp} k c} \left(1 - \frac{\beta_A^2}{\eta^2}\right) \times \int_{E_0}^{\infty} dE N(E) \frac{\Theta(E - E_{\text{th}})}{\beta_\gamma} \left\{2J_n^2(x) + xJ_1(x)\left[J_0(x) - J_2(x)\right]\right\},$$

(28)

where

$$x = \beta_\gamma \eta \Omega^{-1} \sqrt{1 - \eta^2} \left[1 - \frac{\beta_A^2}{\beta_\gamma^2}\right].$$

For Alfvén and slow modes the value of the resonance pitch angle is obtained by changing $v_\Omega/\cos \theta$ to $v_\mu$ and $v_\Omega = \sqrt{1 - \beta_A^2}$). This also means that similar replacements ($\beta_A^2/\eta^2 \rightarrow \beta_A^2$ and $\beta_A^2/\beta_\gamma^2$) in the equations that follow will give the damping rates of Alfvén and slow modes, respectively.
Fig. 3.—Dependence of the diffusion coefficient on $\mu$ from eq. (27) for $\theta = 80^\circ$ (left) and $\theta = 88^\circ$ (right), and for particle kinetic energies of 10 keV (top) and 5 MeV (bottom). Most resonances can be approximated by a $\delta$-function (see text). In those cases $v_{\perp} \cos \theta < v_\lambda$, the resonance condition is not satisfied, and therefore there is a relatively very weak interaction rate.

Here we have defined $\eta = \cos \theta$ and $E_c/m_e c^2 = [1/(1 - \beta_c^2/\eta^2)]^{1/2} - 1$, and $\Theta(x) = 1$ for $x > 1$ and zero otherwise is the Heaviside step function.

### 3.2.1. Damping by Electrons

For electrons $kc/\Omega_c \sim 10^{-8}(kL)(100 \text{ G}/B) \ll 1$ for the relevant scales $k < k_c$. Therefore, except for extreme relativistic electrons the variable $x \ll 1$, and we can use the first-order approximation $J_\nu(x) \simeq (x/2)^{\nu}/\nu!$ for the Bessel functions. Then the terms in the brackets in the above equation can be approximated as $x^2/2$ to give

$$\Gamma_{\text{nonth}}(k) = \frac{\pi}{8} \frac{\delta}{m_\lambda n_0} \left( 1 - \frac{\beta_c^2}{\eta^2} \right) \left( 1 - \frac{\nu^2}{\beta_c^2 \eta^2} \right) \int_{E_m}^{\infty} dE N(E) \Phi \left( 1 - \frac{\beta_c^2}{\beta_\lambda^2} \right),$$

(29)

where $E_m = \max(E_0, E_c)$.

We first note that if we use a nonrelativistic Maxwellian distribution, $N(E) = n(2\sqrt{\pi})(k_B T)^{-3/2} E^{1/2} \exp(-E/k_B T)$, this gives the damping rate of

$$\Gamma_{\text{th}}(k) = \Gamma_0 \exp(-\frac{\delta}{\beta_\lambda \cos^2 \theta}),$$

(30)

which is identical to the electron part of the collisionless thermal damping given in equation (12). Here we have ignored the term $(1 - \beta_\lambda^2/\eta^2)$ whose value lies between 1 and $1 - (\nu/\eta) \ll 1$.

For a nonthermal distribution we can carry out the integration that leads to a complicated expression shown in Appendix B. In the nonrelativistic and extreme relativistic cases the result simplifies considerably. For the spectral index $\alpha > 2$ most of the contribution to the integral comes from the low energies so that when $E_c < m_e c^2$ (i.e., $E_0 < m_e c^2$ and $E_c/m_e c^2 \approx \beta_c^2/(2\eta^2) \ll 1$) we can use the nonrelativistic approximation to obtain

$$\Gamma_{\text{nonth}}(k) = \Gamma_0 \left( 1 - \frac{\beta_c^2}{\eta^2} \right) \left( \frac{\sqrt{\pi}(a - 1)N_0}{4(a - 3/2)n} \right) \frac{E_c}{k_B T}^{1/2} \begin{cases} 1 - \frac{E_c}{E_0}, & \text{if } E_0 \gg E_c, \\ \frac{1}{2a - 1} \left( \frac{E_c}{E_0} \right)^{3/2 - a}, & \text{if } E_0 \ll E_c. \end{cases}$$

(31)

The second expression is valid near $\theta = \pi/2$ where $\eta \ll \beta_\lambda/E_0^{1/2}$. For the solar flare conditions we are considering here, $\beta_\lambda \sim 0.007$ and $E_0 \sim 0.02 m_e c^2 \sim 10$ keV, this expression is
applicable only for $\pi/2 - \theta < 0.05$ or within $0.3$ of the perpendicular direction. Thus, for all angles outside this range we have the first expression, which is different from the thermal damping rate of equation (30) by the presence of the terms in the brackets. The main part here is the ratio $N_0/n$ of the nonthermal to thermal particle densities and $(E_0/k_B T)^{1/2}$, i.e., the ratio of the mean momenta (or velocities) of nonthermal to thermal electrons. In general, this term is less than one, and damping by thermal electrons is dominant. However, in some large solar flares one requires acceleration of a large fraction of the background thermal particles to energies $\gg k_B T$ so that the damping by nonthermal particles could be significant.

If $E_0 \gg m_e c^2$, we can use the extreme relativistic approximation to obtain

$$\Gamma_{\text{nonth}}(k) = \Gamma_0 \left(1 - \frac{\beta^2}{\eta^2}\right)^2 \left(\frac{\pi}{32}\right)^{1/2} \left(\frac{N_0}{n}\right) \times \left(\frac{a - 1}{a - 2}\right) \left(\frac{E_0}{k_B T}\right)^{1/2} \left(\frac{E_0}{m_e c^2}\right)^{1/2},$$

(32)

which is valid for all angles, where $E_0/m_e c^2 \sim \beta^2/\eta^2 \ll E_0/m_e c^2$, so that $\Gamma_{\text{nonth}} = \Gamma_0 \gg m_e c^2$, except for a extremely narrow range of angles given by $1 - (m_e c^2/E_0)^2 \beta^2 \ll \eta^2 \ll \beta^2$. One can combine the above two expressions to obtain an approximate relation valid at all energies:

$$\Gamma_{\text{nonth}}(k) = \Gamma_{\text{nonth}} \left[1 + \frac{a - 3/2}{\sqrt{2(\alpha - 2)}} \left(\frac{E_0}{m_e c^2}\right)^{1/2}\right].$$

(33)

However, it should be noted that the extreme relativistic equation is valid only for scales $k < \Omega^{-1}/\gamma_0 \sin \theta (1 - \beta^2/\eta^2)^{1/2}$; otherwise the approximation used for the Bessel functions breaks down. For most angles, and for parameter values adapted here, this means $kL \lesssim 5 \times 10^7/\gamma_0$. But for $\theta \to 0$ or $\cos \theta \to \beta$, the above expression would be valid at much smaller scales or larger values of $kL$. These limitations are also true for the more general equation (B1) in Appendix B.

3.2.2. Damping by Ions

For protons (and other heavier ions) the condition $kL \gg \Omega^{-1}/\gamma_0 \ll 1$ is not always satisfied. Nevertheless, if we use the small-argument asymptotic expression $J_0(x) \sim (x/2)^n/n!$ for Bessel functions (as done above for electrons), we can get a similar estimate for the damping rate due to interaction with protons.

For example, for a Maxwellian proton distribution one can show that the resulting damping rate will be same as that for electrons with $m_e \to m_p$, which means setting $\delta = 1$ in equation (30). Aside from the factor of 5 this is identical to the contribution by protons to the collisionless thermal damping in equation (12).

Within a similar accuracy we can also estimate the damping rate due to nonthermal protons. Ignoring angles near $\pi/2$ for the moment, from equations (28) or (29) we note that $\Gamma_{\text{nonth}} \propto \langle p \rangle N_0$ so the relative importance of protons and electrons depend on their total number ratios and their mean momenta, which will be same as the ratio of the momenta $p_0$ at the low end of the spectrum. In solar flares we deal with nonrelativistic values of $E_0 = (2m_p p_0)^{1/2}$ and much fewer number of accelerated protons compared to electrons. In a majority of flares the ratio of the total energies $R = (N_0 E_{0,e})/(N_0 E_{0,p})$ (in the observable range $E_{0,e} > 10$ keV for electrons and $E_{0,p} > 10$ MeV for protons) is much less than one and varies from 0.01 to 10 in flares with detectable gamma-ray line emission produced by the accelerated protons and ions (Miller et al. 1997, attributed to R. Ramaty & N. Mandzavidze). Thus, $\Gamma_{\text{nonth, p}} \Gamma_{\text{nonth, e}} = R |E_{0,e}(E_{0,p})|^{1/2} \sim R$ for the above mentioned energies. In summary, usually one can ignore the nonthermal damping due to protons relative to electrons (as was the case for the collisionless thermal damping) in most solar flares. But nonthermal damping by protons could be more important than that of nonthermal electrons in flares with strong gamma-ray line emission.

4. SUMMARY AND DISCUSSION

The study of interactions of plasma waves and turbulence with particles of magnetized plasma is a complex process and requires an accurate formulation of the cascade of the turbulence from a large injection scale to smaller scales and the damping of the waves by the background thermal particles and those accelerated into a superthermal power-law tail arising from these interactions. Equipped with this knowledge one can then determine the evolution of the spectrum and angular characteristics of the turbulence and particles by solving the coupled kinetic equations (see eq. [1]). There has been considerable progress in the understanding of the cascade of turbulence from an injected scale $L$ to the lower scale wavevector $k$ and its expected spectral and angular characteristics. We briefly review these for magnetically dominated or low-$\beta$ plasmas ($\beta_p < 1$), such as those envisioned for solar flares, and indicate that of the three MHD modes, Alfvěn, slow, and fast, the latter can play a dominant role in heating and acceleration of plasma particles. The aspect of this process most relevant to our goals in this paper is the rate of cascade or cascade time as a function of scale of the turbulence. In general, the cascade time $\tau_{\text{can}}$ for all these modes is of order the Alfvěn injection scale $\tau_{\text{inj}} = L/e_A$ and decreases as $(kL)^{-1/2}$.

The spectral and angular distribution of the turbulence is further modified at smaller scales when the damping rate becomes comparable to and larger than the cascade rate. The main goal of this paper is to give a complete description of the damping process. We review the basic processes involved here and present equations describing the damping rate of the turbulence due to different mechanisms. We first consider viscous damping valid on scales larger than the collision mean free path, or for $kL_{\text{coll}} < 1$, with the damping timescale $\tau_d \sim T_{\text{obs}} \lesssim (kL_{\text{coll}})^{-1} \propto k^{-2}$. Because of this rapid decline this damping can become quickly important and stop the cascade process. This would be the case for high-$\beta$ plasmas, but for solar flare conditions, this damping mechanism is applicable in the narrow range of scales between the injection scale $L \sim 10^9$ and $L_{\text{coll}} \sim 10^8$ cm, where because of small values of $\beta_p$, $\tau_d > \tau_{\text{can}}$ and can be neglected. For smaller scales the damping is produced by collisionless processes. Here we have described the damping due to thermal (Maxwellian) and a nonthermal (power-law) distributions separately.

The thermal damping is dominated by electrons and for most practical purposes can be approximated as $\tau_d \sim T_{\text{obs}} \lesssim (\sqrt{\beta kL})^{-1} \propto k^{-1}$. The proton contribution to this process can be important for $\beta_p \cos^2 \theta > 0.18$, which will not be the case for low-$\beta$ plasmas $\beta_p < 0.1$ under consideration here. Here $\theta$ is the angle between the magnetic field and the propagation direction. The same is true for other ions. We combine the collisionless damping valid for $kL_{\text{coll}} > 1$ with that for the viscous damping and give a simple expression valid approximately at all scales. Equating the damping and cascade times, we determine the wavevector $k_c$ above which the damping becomes dominant and would cause the spectrum of the turbulence to steepen. The damping is highly anisotropic and the critical wavevector varies considerably with angle $\theta$, being much larger for quasi-parallel and quasi-perpendicular
propagations. However, we show that this anisotropy around the perpendicular direction is smoothed out by magnetic field wanderings caused by the shearing due to Alfvén modes for $\beta_p < 0.05$. The quasi-parallel waves, on the other hand, are not affected by this process and can survive without damping to scales as small as the particle gyroradii where the MHD regime breaks down and other plasma and kinetic effects become important. These parallel propagating waves may be then the most important agents for acceleration of low-energy electrons, protons, and other ions (see PL04; Liu et al. 2004a, 2006).

We have also evaluated the collisionless damping rate due to a population of nonthermal electrons and protons. We have argued that the most important process here is the TTD mechanism, and show that this process gives a damping rate very similar to that obtained for a thermal distribution. In general, the damping rate is essentially proportional to the mean momentum times the number of the particles. Thus, the relative importance of thermal and nonthermal populations depends on the product of the ratios of their densities and average momenta. In most cases, except for extremely hard nonthermal tails (electron index $a > -1.5$ or $-2$, for nonrelativistic and extreme relativistic cases, respectively), this ratio will be less than the energy content of the two populations. In particular, this will be true for most solar flares, except for the strongest bursts, which require acceleration of all the available background electrons. This behavior also indicates that, as is the case of thermal damping, here also the contribution of protons relative to electrons can be neglected except for very rare flares with strong nuclear gamma-ray line emission, which require more energy for accelerated protons than electrons. Most flares, however, are electron dominated, and the contribution of nonthermal electron will increase the damping rate by the above basic ratio at all $k$ and $\theta$ and decrease the critical wavevector but not affect its anisotropy.

MHD turbulence that we considered was balanced in the sense that the equal flux of energy was assumed in every possible direction. In the solar corona we expect the energy injection to be localized both in space and in time. As a result, turbulent energy propagates from such sources, e.g., the reconnection region, creating an imbalanced cascade. The properties of imbalanced turbulence (see Maron & Goldreich 2001; Cho et al. 2002; Lithwick & Goldreich 2003), in particular, its damping time and scaling, can be very different from the balanced one. The Alfvén cascade is being strongly modified by imbalance, and this may result in much lower rates of cascading, if the imbalance is strong. However, variations in Alfvén speed that are present in the solar corona are likely to result in reflecting Alfvén perturbations. These reflections mitigate the imbalance, and therefore, we believe that the effects of imbalance will not be substantial. A more detailed study of the issue will be presented elsewhere.

We have limited our considerations to scales above the particle gyroradii, where the MHD approximation is valid. For shorter scales one must consider mechanisms of damping or acceleration other than the TTD. In particular gyroresonance scattering must be included with more realistic dispersion relations than those given in equation (4). We intend to address these extensions of the current results in future publications.

Transfer of turbulent energy from large to small scales and its damping is a general process that can be important for heating and particle acceleration in various environments other than solar flares, such as in gamma-ray bursts (see Lazarian et al. 2004) and accretion around black holes (Liu et al. 2004b).

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**APPENDIX A**

**DAMPING DUE TO ION VISCOSITY**

Viscous damping is not important unless there is compression. Therefore, it only influences compressible modes and has marginal effects on Alfvén modes. In a strong magnetic field, the proton gyrofrequency is much larger than the proton collisional frequency $\tau_{\text{coll}} \propto cs/2C_{\text{coll}}$, $[\Omega_p^2\tau_{\text{coll}} = 5 \times 10^{14} (B/100 \text{G}) (10^{10} \text{ cm}^{-3}/n)(T/10^7 \text{ K})^{1/2},]$ and the transport of transverse momentum is prohibited by the magnetic field. Thus, the transverse viscosity coefficient $\eta_{\perp} \sim \eta_{\parallel}/(\Omega_p^2\tau_{\text{coll}})^2$ is much smaller than longitudinal viscosity coefficient $\eta_{\parallel} = 0.96n_{\text{kg}}T_{\text{Col}}$ (see, e.g., Braginskii 1965).

Considering only the zeroth-order terms due to longitudinal viscosity, the viscosity tensors are $\tau_{xx} = \tau_{yy} = -\eta_{\parallel}(W_{xx} + W_{yy})/2$ and $\tau_{zz} = -\eta_{\parallel}W_{zz}$, where $W_{jk} \equiv \partial v_j/\partial x_k + \partial v_k/\partial x_j - (2/3)\delta_{jk}\nabla \cdot v$ is the rate-of-strain tensor and $v$ is the fluid velocity (Sigmar 2002). Here, the $z$-axis is defined by the magnetic field. Heat generated by the viscosity is

$$Q_{\text{vis}} = \pi : \nabla v = -\tau_{xx} \partial v_x/\partial x - \tau_{xx} \partial v_x/\partial y - \tau_{zz} \partial v_z/\partial z = \eta_{\parallel} (\partial v_x/\partial x + \partial v_y/\partial y - 2\partial v_z/\partial z)^2/3. \quad (A1)$$

Dividing this by the total energy associated with the fast modes, we obtain the damping rate $\Gamma_{\text{vis}}$. While the damping due to compression along the magnetic fields (the third term) can be easily understood, it is somewhat counterintuitive that the compression perpendicular to magnetic field also results in damping through longitudinal viscosity. However, the origin of this viscosity can be easily traced (see Braginskii 1965). Indeed, for motions perpendicular to the magnetic field $B$, $\nabla \cdot v = n/\eta_{\parallel} \sim B/B$, implies the transverse energy of the ions increases due to the adiabatic invariant $v^2/B$. If the rate of compression is faster than that of collisions, the ion distribution in the momentum space will become distorted away from the isotropic Maxwellian sphere to an oblate spheroid with the long axis perpendicular to the magnetic field. As a result, the transverse pressure becomes greater than the longitudinal pressure by a factor $\tau_{\text{coll}}h/n$, resulting in a stress $-P_{\text{coll}}h/n = \eta_{\parallel}\nabla \cdot v$, where $P = nk_BT$ is the longitudinal pressure. The restoration of the equilibrium increases the entropy and causes the dissipation of energy. In a low-$\beta_p$ medium, compressions are perpendicular to the magnetic field; thus, $\Gamma_{\text{vis}} = k^2\eta_{\parallel}/3nm$. In a high-$\beta_p$ medium, as pointed out in $\frac{3}{2}$, the velocity perturbations are radial. Thus, according to equation (A1), the corresponding damping rate $\Gamma_{\text{vis}} = k^2\eta_{\parallel}(1 - 3\cos^2\theta)^2/(3nm)$. 

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Putting all these together for \( m_i = m_p \), we get
\[
\Gamma_{\text{vis}} = 0.13 \left( \frac{\beta_p^{1/2}}{\gamma_0} \right) (kL)(k\lambda_{\text{Coul}}) \left\{ \sin^2 \theta, \quad \text{for } \beta_p \ll 1, \right.
\]
\[
(1 - 3 \cos^2 \theta)^2, \quad \text{for } \beta_p \gg 1,
\]
(A2)
or in terms of physical parameters appropriate for solar flares,
\[
\Gamma_{\text{vis}} = 0.05 \, \text{s}^{-1}(kL)^2 \left( \frac{10^8 \text{ cm}}{L} \right)^2 \left( \frac{T}{10^7 \text{ K}} \right)^{2.5} \left( \frac{10^{10} \text{ cm}^{-3}}{n} \right)^2 \sin^2 \theta, \quad \text{for } \beta_p \ll 1,
\]
\[
(1 - 3 \cos^2 \theta)^2, \quad \text{for } \beta_p \gg 1.
\]
This damping becomes important at a scale where \( \Gamma_{\text{vis}} \tau_{\text{cas}} = 1 \). From the above and equation (8), we find
\[
k_L = \left( \frac{L \beta_p}{18} \right)^{-1/3} \left( \frac{\lambda_{\text{Coul}} \sin^2 \theta}{M_A} \right)^{-2/3} = \frac{4}{\beta_p^{1/3}} \left( \frac{M_A}{\sin \theta} \right)^{4/3} \left( \frac{L}{10^8 \text{ cm}} \right)^{2/3} \left( \frac{10^{10} \text{ cm}^{-3}}{n} \right)^{-2/3} \left( \frac{10^7 \text{ K}}{T} \right).
\]
(A4)

APPENDIX B

GENERAL NONTHERMAL DAMPING RATES

The nonthermal electron damping expression (29) can be integrated for a power-law distribution of electrons. The general expression is
\[
\Gamma_{\text{norm}}(k) = \frac{\pi \delta}{8 \beta_A \gamma_0} \left( \frac{1 - \eta^2}{\eta} \right) \left( 1 - \frac{\beta_A^2}{\eta^2} \right) \left( \frac{a - 1}{a} \right) \int_0^{\gamma_0} \frac{d\gamma}{\sqrt{\gamma^2 - 1}}
\]
\[
\times \left\{ \begin{align*}
\frac{1}{32} & \left[ \sqrt{E_m + 2} \left( 32 - \frac{\beta_A^2}{\eta^2} \left( 26 + 8 \frac{\gamma m}{E_m} \right) \right) + \left( 16 - 19 \frac{\gamma^2}{\eta^2} \left( 2\gamma + 6 \right) \right) \right] \sqrt{2 \ln \left( \sqrt{E_m + 2} + \sqrt{E_m + 2} \right)}, \quad \text{for } \alpha = 2.5, \\
\frac{1}{15} & \left[ -5 + \frac{7 \gamma m}{\eta^2} \right] + \frac{\gamma m \beta m}{E_m} \left[ 5(\gamma m \beta m)^2 - \frac{\beta_A^2}{\eta^2} \left( 7\gamma^2 - 6\gamma + 2 \right) \right], \quad \text{for } \alpha = 3, \\
\frac{1}{384} & \left[ \sqrt{E_m + 2} \left( 48 + \frac{192}{E_m} - \frac{\beta_A^2}{\eta^2} \left( 78 + 152 \frac{E_m}{E_p^2} \right) \right) - \left( 24 - 39 \frac{\beta_A^2}{\eta^2} \right) \right] \sqrt{2 \ln \left( \sqrt{E_m + 2} + \sqrt{E_m + 2} \right)}, \quad \text{for } \alpha = 3.5, \\
\frac{1}{105} & \left[ -7 + \frac{3 \gamma m}{\eta^2} \right] - \frac{\gamma m \beta m}{E_m} \left[ 7(\gamma m \beta m)^2 - \frac{\beta_A^2}{\eta^2} \left( 13\gamma^2 - 52\gamma + 32\gamma - 8 \right) \right], \quad \text{for } \alpha = 4,
\end{align*} \right.
\]
where \( \gamma_m \) and \( \beta_m \) are the Lorentz factor and velocity, respectively, corresponding to the kinetic energy \( E_m \), which is now expressed in units of \( mc^2 \). In the solar case, \( E_m \ll 1 \) and the above expression simplifies to equation (31) in § 3.2.1. Similarly we can get the extreme relativistic limit (\( E_m \gg 1 \)) of equation (32).

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