Criteria for regularity of Mahler power series and Becker’s conjecture

Tomasz Kisielewski

Abstract

Allouche and Shallit introduced the notion of a regular power series as a generalization of automatic sequences. Becker showed that all regular power series satisfy Mahler equations and conjectured equivalent conditions for the converse to be true. We prove a stronger form of Becker’s conjecture for a subclass of Mahler power series.

1 Introduction

We first give some historical background and motivation for the problems in this paper, starting with automatic and regular sequences and moving on to power series and Mahler equations.

1.1 Automatic and regular sequences

Fix a natural number $k \in \mathbb{N}$, $k \geq 2$. A sequence $(a_n)$ is called $k$-automatic if there exists a finite automaton which, given the base $k$ expansion of $n$, stops at a state corresponding to $a_n$. This definition of $k$-automatic sequences was introduced by Cobham [6], who also proved an alternative characterisation of these sequences, which will be much more useful for us.

Definition 1. Let $(a_i)_{i \in \mathbb{N}}$ be a sequence. The set of subsequences of this sequence $\{ (a_{k^e r+N})_{i \in \mathbb{N}} : e \in \mathbb{N}, 0 \leq r < k^e \}$ is called the $k$-kernel of the sequence.

Definition 2. A sequence $(a_i)_{i \in \mathbb{N}}$ is called $k$-automatic if its $k$-kernel is finite.

Based on this definition, Allouche and Shallit [2] defined and investigated a wider class of sequences. They were working with sequences taking values in rings, but for simplicity we will restrict our attention to a field $\mathbb{F}_k$.

Definition 3. A sequence $(a_i)_{i \in \mathbb{N}}$, $a_i \in \mathbb{F}_k$, is called $k$-regular if the vector subspace of $\mathbb{F}_k^\mathbb{N}$ generated over $\mathbb{F}_k$ by its $k$-kernel is finitely dimensional.

Among many other interesting properties, Allouche and Shallit state the following theorem.
Theorem 1. A sequence is $k$-automatic if and only if it is $k$-regular and takes only finitely many values.

This theorem fully describes the relation between regular and automatic sequences.

Automatic and regular sequences have many applications in, among others, transcendence theory, game theory, questions related to computability properties of expansions of numbers in various bases, dynamical systems, differential geometry, Fourier analysis and language theory. We refer the interested reader to the excellent book by Allouche and Shallit [4]. For examples of automatic and regular sequences we refer to the same book and also the papers [2] and [3] by the same authors.

1.2 Power series and Mahler equations

First we note that the definitions from the previous section can be naturally extended to formal power series.

Definition 4. A formal power series $\sum_{i=0}^{\infty} a_i z^i = f(z) \in \mathbb{K}[[z]]$ is called $k$-regular (resp., $k$-automatic) if the sequence of its coefficients $(a_i)_{i\in\mathbb{N}}$ is $k$-regular (resp., $k$-automatic).

This definition was also established by Allouche and Shallit [2] along with some basic results, among which was the fact that $k$-regular power series form a ring, but not a field.

In a series of papers concerning transcendence theory Mahler [8, 9, 10] considered a new type of functional equations. He proved that any power series with coefficients in a number field satisfying an equation of such a type takes a transcendental value at algebraic points in the radius of convergence. His method was later extended by many others and we refer the interested reader to Nishioka [11].

Definition 5. A formal power series $f(z) \in \mathbb{K}[[z]]$ is called $k$-Mahler if it satisfies a functional equation of the form

$$f(z) = \sum_{i=1}^{n} c_i(z) f(z^{k^i})$$

for some rational functions $c_i(z) \in \mathbb{K}(z)$. We call $n$ the order of the Mahler equation.

There are a few equivalent definitions of $k$-Mahler power series which differ slightly in the precise form of the satisfied equation. For these definitions and proofs of equivalence we refer the reader to Adamczewski and Bell [1], Becker [5] and Dumas [7].

The latter two papers also explore the relation between $k$-regular and $k$-Mahler power series. These two classes of power series are rather close. More precisely, all $k$-regular series are $k$-Mahler and the strongest known result considering the converse is the following theorem.

Theorem 2. Let the power series $f(z) \in \mathbb{K}[[z]]$ satisfy a Mahler type equation with coefficients $c_i(z) \in \mathbb{K}(z)$. If all $c_i(z)$ have poles only at 0 and roots of unity of order not coprime to $k$, then $f(z)$ is $k$-regular.
This theorem has been shown by Dumas [7, Theorem 30], while Becker [5, Theorem 2] slightly earlier showed a weaker result, with \( c_i(z) \) being polynomials. Due to that, Adamczewski and Bell used the name \( k\)-Becker for power series satisfying a Mahler equation with polynomial coefficients, so by analogy we will use the name \( k\)-Dumas for power series satisfying the conditions of Theorem 2. One of our main results is that with some additional assumptions the class of \( k\)-Dumas power series is exactly the class of \( k\)-regular power series (Theorem 6) and we conjecture that this is also the case in general (Conjecture 5).

Becker also used the fact that \( k\)-regular power series form a ring to conjecture that all \( k\)-regular power series are quotients of \( k\)-Becker power series and polynomials. We investigate this claim and show that it would follow from the converse of Theorem 2.

The relations between regular, Becker, and Mahler power series are not only interesting as a matter of mathematical curiosity. They appeared in the proof of an extended version of Cobham’s theorem by Adamczewski and Bell [1]. The original Cobham’s theorem states that a sequence that is \( k\)- and \( l\)-automatic for two multiplicatively independent integers \( k \) and \( l \) (that is \( k \) is not a power of \( l \) and \( l \) is not a power of \( k \)) is ultimately periodic. Adamczewski and Bell first extended this to the case of regular sequences and then proved that \( k\)- and \( l\)-Mahler power series for multiplicatively independent integers \( k \) and \( l \) are rational functions. The latter proof was much more complicated than the former (using methods from commutative algebra and Chebotarëv’s theorem as well as analysis), so a simpler characterisation of the relation between regular and Mahler power series might have helped. They used the characterisation given by the following lemma, which is a slightly modified version of an original result of Dumas [7, Theorem 31].

**Lemma 1.** Let \( f(z) \in \mathbb{k}[[z]] \) be a Mahler power series satisfying the equation

\[
\sum_{i=0}^{n} c_i(z) f(z^k) = 0
\]

where \( c_i(z) \in \mathbb{k}[z] \) are polynomials and \( c_0(0) = 1 \). Then there exists a Becker power series \( g(z) \) such that

\[
f(z) = \left( \prod_{i=0}^{\infty} c_0 \left( z^{k^i} \right) \right)^{-1} g(z).
\]

Note that \( h(z) = \left( \prod_{i=0}^{\infty} c_0 \left( z^{k^i} \right) \right)^{-1} \) is Mahler since it satisfies the Mahler equation

\[
h(z) = \frac{1}{c_0(z)} h(z^k).
\]

This again shows that regular and Mahler power series are quite close.

### 1.3 Plan of this work

While the results of sections 2.1 and 2.3 are well-known, most of the other definitions and results are new. The methods follow essentially Dumas [7, Proposition 54], but using the language of valuations and having some patience for computations enables us to extend the results to more general situations.
In section 2 we introduce the basic notions used in this paper. Section 3 describes the special case of Mahler equations of order one, in particular a full characterisation of their regularity. Section 4 contains one of our main results (Theorem 5) which gives a numerical criterion for precalmness, a property of coefficients in Mahler equations related to regularity. Sections 5 and 6 explore various forms of Becker’s conjecture. The former shows the equivalence of some forms, while the latter states a naive version and constructs a counterexample, which also illustrates how to apply Proposition 3 from a computational point of view. In section 7 we state and prove equivalent conditions for regularity for a subclass of Mahler power series.

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2 Preliminary definitions

This section contains most of the definitions and some basic lemmas used in later proofs. We will work with a fixed algebraically closed field \( k \) of characteristic \( \text{char } k = p \) or 0 and with a fixed integer \( k \geq 2 \). If \( \text{char } k = p > 0 \) we always assume that \( k \) is coprime to \( p \).

2.1 Valuations in the field of rational functions

We will be using the language of valuations to talk about poles and zeroes of functions. We will denote by \( v_\alpha \) the \( (z - \alpha) \)-adic valuation on \( k(z) \).

We will be using mostly basic properties of valuations, but we will actually need a more precise tool for determining the valuation of a sum of two rational functions, involving their \( (z - \alpha) \)-adic digits. We will denote the \( (z - \alpha) \)-adic digit of a rational function \( c(z) \in k(z) \) at \( (z - \alpha)^j \) by \( \text{dig}_{\alpha,j}(c(z)) \). The valuation of such a rational function is equal to the lowest index for which its \( (z - \alpha) \)-adic digit is not zero. To get the information necessary to use the above fact in our proofs, we will need a simple lemma.

**Lemma 2.** Let \( c(z), d(z) \in k(z) \) be two rational functions, \( a_0, \ldots, a_n \in k \) be pairwise distinct numbers and \( m_0, \ldots, m_n \in \mathbb{Z} \) be integers, \( m_i \geq v_{a_i}(c(z)) \). If \( v_{a_i}(d(z)) \geq v_{a_i}(c(z)) \) for every \( 0 \leq i \leq n \), then there exists a polynomial \( h(z) \in k[z] \) such that \( \text{dig}_{\alpha,j}(h(z)c(z)) = \text{dig}_{\alpha,j}(d(z)) \) for all \( 0 \leq i \leq n \) and \( v_{a_i}(c(z)) \leq j \leq m_i \).

This follows from the Approximation Lemma from [12, p. 12] applied to the single function \( \frac{d(z)}{c(z)} \), the prime ideals generated by \( (z - a_i) \) and integers \( m_i \). In fact, it is enough to choose \( h(z) \) so that

\[
 v_{a_i} \left( h(z) - \frac{d(z)}{c(z)} \right) \geq m_i - v_{a_i}(c(z)).
\]

There is one other simple property of valuations we will use.
Lemma 3. Let $i \in \mathbb{N}$ be an integer coprime to $p = \text{char} \mathbb{k}$. Assume $\alpha \neq 0$. Then

$$v_{\alpha} \left( c \left( z^i \right) \right) = v_{\alpha^i} \left( c \left( z \right) \right).$$

Proof. For any $v \in \mathbb{N}$ the polynomial $\left( \frac{(a^i - z^i)}{(a - z)} \right)^v$ has no zeroes at $\alpha$, by the assumption of $i$ being coprime to the characteristic of $\mathbb{k}$. Thus, the function $(a^i - z^i)^v c \left( z^i \right)$ has no poles at $\alpha$ if and only if it has no poles at $\alpha$ after being multiplied by this polynomial. But this means that $d \left( z \right) = (a^i - z^i)^v c \left( z^i \right)$ also has no poles at $\alpha$. Substituting the variable $t = z^i$ into $(a^i - z^i)^v c \left( z \right)$, we get $d \left( t \right)$, so the former expression has no poles at $\alpha^i$ if and only if $d \left( t \right)$ has no poles at $\alpha$. \qed

2.2 Calm numbers, calm functions and calm sequences of functions

Due to the somewhat complex criteria for poles in Theorem 2, we set up some nomenclature for numbers and functions satisfying these criteria.

Definition 6. A number $\alpha \in \mathbb{k}$ is called $k$-calm if $\alpha$ is a root of unity of order not coprime to $k$ or $\alpha = 0$. If a number is not $k$-calm, we call it $k$-anxious.

Definition 7. Let $\alpha$ be a $k$-anxious number. A rational function $c \left( z \right) \in \mathbb{k} \left( z \right)$ is called $\alpha$-calm if it has no poles at $\alpha$.

Definition 8. A rational function $c \left( z \right) \in \mathbb{k} \left( z \right)$ is called $k$-calm if it has poles only at $k$-calm numbers (i.e., the function is $\alpha$-calm for all $k$-anxious $\alpha$).

Since we will be working with sequences of rational functions rather than a single function, we extend these definitions as follows.

Definition 9. Let $\alpha$ be a $k$-anxious number. A finite sequence of rational functions $(c_i \left( z \right))_{i=1}^n$, $c_i \left( z \right) \in \mathbb{k} \left( z \right)$ is called $\alpha$-calm (resp., $k$-calm) if all the functions $c_i \left( z \right)$ are $\alpha$-calm (resp., $k$-calm).

We will omit the $k$-prefixes in the notation whenever it is clear from context. In this language Theorem 2 can be stated as follows – if a power series satisfies a Mahler equation with a calm sequence of coefficients, then it is regular.

For the discussion of the Becker conjecture we will also need some further definitions.

Definition 10. We define the action of the group $\mathbb{k} \left( z \right)^* \times \mathbb{k} \left( z \right)^*$ on the set of sequences $\mathbb{k} \left( z \right)^n$. Let $(c_i \left( z \right))_{i=1}^n$, $c_i \left( z \right) \in \mathbb{k} \left( z \right)$ be a finite sequence of rational functions and $0 \neq h \left( z \right) \in \mathbb{k} \left( z \right)$ be a nonzero rational function. We define the action of the rational function $h \left( z \right)$ on the sequence $(c_i \left( z \right))_{i=1}^n$ by the formula

$$h^* \left( c_i \left( z \right) \right) = \left( h \left( z^i \right) \right) h \left( z \right)^{c_i \left( z \right)}.$$
Definition 11. Let $\alpha$ be a $k$-anxious number. A finite sequence of rational functions $(c_i(z))_{i=1}^n$, $c_i(z) \in \mathbb{K}(z)$ is called $\alpha$-precalm if there exists a polynomial $0 \neq h(z) \in \mathbb{K}[z]$ such that $h^*(c_i(z))$ is $\alpha$-calm.

Definition 12. A finite sequence of rational functions $(c_i(z))_{i=1}^n$, $c_i(z) \in \mathbb{K}(z)$ is called $k$-precalm if there exists a polynomial $0 \neq h(z) \in \mathbb{K}[z]$ such that $h^*(c_i(z))$ is $k$-calm.

It will be obvious from the proof of Theorem 5 that being $k$-precalm is equivalent to being $\alpha$-precalm for all $k$-anxious $\alpha$, but this is not completely trivial, so we only prove this later.

To prove conditions equivalent to precalmness we will also need the following quantitative measure of how far a function is from being calm.

Definition 13. For a sequence $(a_i)$ we will write $\pi_i$ for its associated sequence of partial sums $\pi_i = \sum_{j=0}^{i-1} a_j$, with $\pi_0 = 0$. For any $k$-anxious number $\alpha$ put

$$A_\alpha = \begin{cases} \{(a_i)_{i=0}^{\mu-1} : a_i \in \{1, \ldots, n\}\} & \text{if } \alpha \text{ is not a root of unity}, \\ \{(a_i)_{i=0}^{\mu-1} : a_i \in \{1, \ldots, n\}, \alpha^{\pi_\mu} = \alpha\} & \text{if } \alpha \text{ is a root of unity}. \end{cases}$$

The $k$-calmness of a finite sequence of rational functions $(c_i(z))_{i=1}^n$, $c_i(z) \in \mathbb{K}(z)$ with respect to a $k$-anxious number $\alpha$ and a sequence $(a_i) \in A_\alpha$ is

$$\text{clm}_{a_i}(c_i) = \sum_{i=1}^n v_i \pi \left( c_{a_i}(z) \right).$$

The sum is taken over the set $\{0, \ldots, \mu - 1\}$ for $\alpha$ a root of unity and over all natural numbers if it is not a root of unity. For simplicity we will usually omit $\alpha$ from the notation.

It is worth noting that this sum is always either finite or contains an infinite term. For a root of unity $\alpha$ this is obvious since the sum has finitely many terms. If $\alpha$ is not a root of unity then any nonzero rational function has only finitely many poles and zeroes, so the sum has only finitely many nonzero terms or contains at least one infinite term (which can happen only when some $c_{a_i}(z)$ is zero).

We will also use the connection between calmness and the action of rational functions on the set of sequences.

Lemma 4. Let $\alpha$ be an anxious number, $(a_i) \in A_\alpha$ and $0 \neq h(z) \in \mathbb{K}(z)$ be a nonzero rational function. Then

$$\text{clm}_{a_i} h^*(c_i) = \begin{cases} \text{clm}_{a_i}(c_i) - v_\alpha(h(z)) & \text{if } \alpha \text{ is not a root of unity}, \\ \text{clm}_{a_i}(c_i) & \text{if } \alpha \text{ is a root of unity}. \end{cases}$$
Proof.

\[
\text{clm}(a_j)h^\alpha(c_i) = \sum_i v_{a_i}^\alpha \left( \frac{h(z^{k_i})}{h(z)} c_{a_i}(z) \right) \\
= \sum_i \left( v_{a_i}^\alpha c_{a_i}(z) \right) + \sum_i v_{a_i}^\alpha \left( h(z^{k_i}) - v_{a_i}^\alpha h(z) \right) \\
= \sum_i v_{a_i}^\alpha c_{a_i}(z) + \sum_i \left( v_{a_i}^\alpha \left( h(z) \right) - v_{a_i}^\alpha h(z) \right) \\
= \begin{cases} 
\text{clm}(a_j)(c_i) - v_a(h(z)) & \text{if } a \text{ is not a root of unity,} \\
\text{clm}(a_j)(c_i) - v_a(h(z)) + v_{a_i}^\alpha h(z) = \text{clm}(a_j)(c_i) & \text{if } a \text{ is a root of unity.} 
\end{cases}
\]

In the second line, we used Lemma 3 and the definition of \( \text{clm} \), while the last equality for roots of unity holds by the definition of \( A_a \).

We will prove more results connected to calmness when discussing Becker’s conjecture. For some of that discussion, we will need one more property of calm sequences of functions.

**Lemma 5.** Let \( (c_i(z))_{i=1}^n \) be a \( k \)-calm sequence of rational functions. Then there exists a polynomial \( 0 \neq h(z) \in \mathbb{K}[z] \) such that \( h^\alpha(c_i(z)) \) is a sequence of polynomials.

**Proof.** We will first construct polynomials \( h_i'(z) \) for \( 1 \leq i \leq n \) such that

(a) \( \frac{h_i'(z^{k_j})}{h_i'(z)} \) is a polynomial for every \( 1 \leq j \leq n \).

(b) \( \frac{h_i'(z^{k_j})}{h_i'(z)} c_i'(z) \) is a polynomial.

It then remains to put \( h(z) = \prod_{i=1}^n h_i'(z) \). Property (a) ensures that \( h(z) \) will not add any poles to the terms of the new sequence and property (b) shows that \( h(z) \) will eliminate any poles that \( c_i(z) \) had before.

Let us fix \( 1 \leq i \leq n \). We construct \( h_i'(z) \). For any \( k \)-calm \( \alpha \) consider the polynomial

\[
H_\alpha(z) = \begin{cases} 
z & \text{if } \alpha = 0, \\
z^{\frac{k}{\alpha}} - 1 & \text{if } \alpha = \zeta_b \text{ is a } b^{th} \text{ primitive root of unity with } q = \gcd(k, b) \neq 0. 
\end{cases}
\]

Set

\[
h_i'(z) = \prod_{\alpha : v_\alpha(c'_i) < 0} H_\alpha(z)^{-v_\alpha(c'_i)}.
\]

It remains to prove that this function has the desired properties. Property (a) can be checked for every \( H_\alpha \) separately, since if two functions satisfy property (a), so does their product. For \( \alpha = 0 \) the property is obvious since \( \frac{H_0(z^{k_j})}{H_0(z)} = z^{k_j - 1} \in \mathbb{K}[z] \). For \( \alpha = \zeta_b \) we have

\[
\frac{H_\alpha(z^{k_j})}{H_\alpha(z)} = \frac{z^{k_j \zeta_b^{-1}} - 1}{z^{\frac{k}{\alpha}} - 1} = \frac{\left( \frac{z^k}{\zeta_b} \right)^{k_j} - 1}{\left( \frac{z^k}{\zeta_b} \right)^{\frac{k}{\alpha}} - 1} = \sum_{c=0}^{k_j-1} \left( \frac{z^k}{\zeta_b} \right)^c,
\]

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which is indeed a polynomial.

To prove that property (b) is satisfied, it suffices to show that every pole of \( c_i \) counted with multiplicity corresponds to a root of \( \frac{h_i(z)}{H(z)} \) of at least the same multiplicity. For every such pole \( \alpha \), the polynomial \( h_i \) has a corresponding factor \( H_\alpha \) and so it suffices to show that \( \frac{H_\alpha(z)}{H_\alpha(z)} \) has a root at \( \alpha \). For \( \alpha = 0 \) this is obvious. For \( \alpha = \hat{\zeta}_b \) we have

\[
(\hat{\zeta}_b)^k - 1 = \left(\frac{\hat{\zeta}_b}{\zeta_b}\right)^k - 1 = 0.
\]

However, \( \hat{\zeta}_b \) is not a root of \( \zeta_b^k - 1 \), so \( \frac{H_\alpha(z)}{H_\alpha(z)} = \frac{\hat{\zeta}_b^k - 1}{\zeta_b^k - 1} \) has a root at \( \zeta_b \).

\[\square \]

2.3 Ring of Mahler operators

We will also use a slightly different approach to Mahler equations. It was used extensively by Dumas [7] in his thesis. For more information about the notions defined in this section, we direct the reader there.

**Definition 14.** We define the operator \( \Delta_k : \mathbb{K}_\alpha((z)) \to \mathbb{K}_\alpha((z)) \) as

\[
\Delta_k f(z) = f(z^k)
\]

for any formal Laurent series \( f(z) \in \mathbb{K}_\alpha((z)) \).

We regard rational functions \( c(z) \in \mathbb{K}_\alpha(z) \) as multiplication by \( c(z) \) operators on \( \mathbb{K}_\alpha((z)) \).

**Definition 15.** We call the (non-commutative) ring \( \mathbb{K}_\alpha(z) [\Delta_k] \) the ring of \( k \)-Mahler operators. The multiplication in this ring corresponds to composition of operators and the multiplication by rational functions together with the rule

\[
\Delta_k c(z) = c(z^k) \Delta_k
\]

for any rational function \( c(z) \in \mathbb{K}_\alpha(z) \). We regard the ring \( \mathbb{K}_\alpha(z) [\Delta_k] \) as a subring of the ring of \( \mathbb{K}_\alpha(z) \to \mathbb{K}_\alpha(z) \) that are continuous in the \( z \)-adic topology.

**Remark 1.** The ring \( \mathbb{K}_\alpha(z) [\Delta_k] \) is a free left \( \mathbb{K}_\alpha(z) \)-module with basis \( 1, \Delta_k, \Delta_k^2, \ldots \).

**Proof.** Obvious from equation (1). \[\square \]

This definition allows us to reformulate the definition of a \( k \)-Mahler power series.

**Remark 2.** A formal power series \( f(z) \in \mathbb{K}[[z]] \) is called \( k \)-Mahler if there exists an operator \( M \in 1 + \mathbb{K}_\alpha(z) [\Delta_k] \Delta_k \) such that

\[
Mf(z) = 0.
\]
From the above it is obvious that multiplying $M$ on the left by any element $N \in 1 + \mathcal{K}(z) \Delta_k \Delta_k$ gives us a new $k$-Mahler equation satisfied by $f(z)$.

We introduce a family of operators providing us with another way of looking at regular power series. These operators have been studied by Dumas [7] and Becker [5].

**Definition 16.** We define the operator $\Lambda_{k,r} : \mathcal{K}[[z]] \to \mathcal{K}[[z]]$ for $0 \leq r < k$ by the formula

$$\Lambda_{k,r} \left( \sum_{i=0}^{\infty} a_i z^i \right) = \sum_{i=0}^{\infty} a_{ki+r} z^i.$$

These operators are sometimes called the Cartier operators.

It is quite easy to see that iterated application of the operators $\Lambda_{k,r}$ on a power series $f(z) \in \mathcal{K}[[z]]$ gives us power series corresponding to sequences in the $k$-kernel of the sequence associated to $f(z)$. This proves the following lemma, which was first noted by Becker [5, Lemma 3].

**Lemma 6.** A power series $f(z) \in \mathcal{K}[[z]]$ is $k$-regular if and only if the $k$-vector space generated by the set

$$\{ \Lambda_{k,r_0} \left( \Lambda_{k,r_0-1} \left( \cdots \left( \Lambda_{k,r_0} (f(z)) \right) \right) \right) : 0 \leq r_i < k, n \in \mathbb{N} \}$$

is finitely dimensional.

We will require a few more simple properties of the Cartier operators.

**Lemma 7.** Let $f(z), g(z) \in \mathcal{K}[[z]]$ be power series. Then

(a)

$$f(z) = \sum_{r=0}^{k-1} z^r \Lambda_{k,r} (f) \left( z^k \right)$$

and

(b)

$$\Lambda_{k,r} \left( f(z)g \left( z^k \right) \right) = \Lambda_{k,r} (f(z)) g(z).$$

**Proof.** The operator $\Lambda_{k,r}$ is $k$-linear and continuous in the $(z-a)$-adic topology, so it suffices to prove the lemma for $f(z) = z^n$ and $g(z) = z^m$. For part (a) write $n = kt + r_0, 0 \leq r_0 < k$ and note that

$$\Lambda_{k,r} (z^n) = \begin{cases} z^t & r = r_0 \\ 0 & r \neq r_0. \end{cases}$$

The proof reduces to the computation

$$\sum_{r=0}^{k-1} z^r \Lambda_{k,r} (f) \left( z^k \right) = z^{r_0} \left( z^k \right)^t = z^n$$

For part (b) note that

$$\Lambda_{k,r_0} \left( f(z)g \left( z^k \right) \right) = \Lambda_{k,r_0} \left( z^{n+km} \right) = z^{t+m} = z^t z^m = \Lambda_{k,r_0} (f(z)) g(z)$$
and
\[ \Lambda_{k,r} \left(f(z)g \left(z^k \right)\right) = \Lambda_{k,r} \left(z^{n+km} \right) = 0 = \Lambda_{k,r} \left(f(z) \right) g \left(z \right) \]
for \( r \neq r_0 \).

The following lemma will be crucial in proving the non-regularity of certain Mahler power series.

**Lemma 8.** Let \( c(z) \in k(z) \) be a rational function, \( 0 \neq \alpha \in k_\alpha \) be a nonzero number and let \( v = v_\alpha \left(c(z)\right) \). Then there exists an \( r \in \{0, \ldots, k-1\} \), such that \( v_\alpha \left(\Lambda_{k,r} \left(c\right) \left(z^k \right)\right) \leq v \).

**Proof.** First we note that multiplying by \( z^l \) has no effect on the \( p \)-adic valuation of a rational function. The lemma then follows directly from Lemma 7(a).

As before, we will write \( \Lambda_r \) for \( \Lambda_{k,r} \) whenever there is no risk of confusion.

### 3 Order one Mahler equations

In order to show the basic principle behind our methods, we start with the case of a formal power series \( f(z) \in k[[z]] \) satisfying a Mahler equation of order one, that is
\[ f(z) = c_1(z) f \left(z^k \right) \quad (2) \]
for some rational function \( c_1(z) \in k(z) \). The results of this section will not be used later, but they should help the reader understand the origin of the methods used in the following sections. Equations of order one are quite special and the criterion for regularity we provide will be a bit simpler than the criteria proven later for arbitrary order equations. Furthermore, we do not restrict the equations under consideration except for the order, which is not the case in later theorems.

#### 3.1 Calmness of a single function

We will first show how the notion of calmness defined in Definition 13 corresponds to functions being precalm. For simplicity we will say that a single function is (pre)calm if the sequence of length one containing only this function is (pre)calm. Note that for sequences of length one Definition 8 reduces to Definition 8 so saying that a function is calm is unambiguous. Furthermore, such sequences have only one nontrivial calmness for any \( \alpha \), which we will henceforth denote by \( \text{clm}_\alpha = \text{clm}_{\alpha,(1)} \).

**Lemma 9.** Given a rational function \( c(z) \in k(z) \), the following conditions are equivalent:

(i) \( \text{clm}_\alpha(c) \geq 0 \) for every anxious \( \alpha \).

(ii) \( c(z) \) is precalm.

We will later prove Theorem 5 which is an exact analogue of this lemma for sequences of functions.
Proof. (ii) ⇒ (i) In the language of valuations we can express Definition \[\text{8}\] as follows: a function is calm if and only if its valuation \(v_a\) is nonnegative for every anxious \(a\). This implies that the calmness of a calm function is nonnegative.

On the other hand, since \(c(z)\) is precalm, we get a polynomial \(0 \neq h(z) \in K[z]\), such that \(h^*(c(z)) = \frac{h(z)}{h(z)} c(z)\) is calm. By Lemma \[\text{4}\]

\[
0 \leq \begin{cases} 
\clm_a(c) - v_a(h(z)) & \text{if } a \text{ is not a root of unity,} \\
\clm_a(c) & \text{if } a \text{ is a root of unity.}
\end{cases}
\]

It remains to notice that \(v_a(h) \geq 0\) (since \(h\) is a polynomial) to get \(\clm_a(c) \geq 0\) for every anxious \(a\).

(i) ⇒ (ii) First note that if \(\frac{h_0(z)}{h(z)} c(z)\) is precalm for some polynomial \(h_0(z) \in K[z]\), then so is \(c(z)\) – we can construct the required polynomial for \(c\) by multiplying the polynomial \(h\) we get for \(\frac{h_0(z)}{h(z)} c(z)\) by \(h_0\). This simple fact allows us to construct the required polynomial step by step, in each step eliminating one anxious pole of \(c\).

First pick an anxious \(a\) at which \(c\) has a pole such that either \(a\) is a root of unity or \(c\) has no poles at \(a^{k^n}\) for \(n \geq 1\). If no such \(a\) exists, then \(c\) definitely cannot have any poles at anxious roots of unity and if it had a pole at \(\beta\) which is not a root of unity, then it would have a pole at arbitrarily large powers of \(\beta\), which is impossible for a rational function. Therefore \(c\) is calm, so also precalm, and the proof is finished.

Since \(a\) is anxious, we know from \[\text{i}\] that \(\clm_a c \geq 0\), but \(c\) has a pole at \(a\), so \(v_a(c) < 0\). This means there exists an \(n \geq 1\) such that \(v_{a^{k^n}}(c) > 0\).

We put

\[
h(z) = \prod_{i=1}^{n} \left( a^{k^i} - z \right).
\]

Now \(\frac{h(z)}{h(z)}\) has only a single pole at \(a^{k^n}\), a root at \(a\) and other, unimportant roots. In fact, the root of the \(i\)th factor of \(h(z)\) cancels out with one of the roots of the \((i + 1)\)th factor of \(h(z^k)\), while all the other roots are different from the roots of \(h(z)\). The last fact is obvious if \(a\) is not a root of unity. If \(a\) is a root of unity, then it has order coprime to \(k\) (since it is anxious) and therefore \(\zeta_k a^{k^i} \neq a^{k^i}\) (where \(\zeta_k\) is a primitive \(k\)th root of unity), because the left hand side is a root of unity of order not coprime to \(k\). This leaves only the root of the last factor of \(h(z)\) and first factor of \(h(z^k)\) corresponding to the pole and the root mentioned earlier. Therefore \(\frac{h(z)}{h(z)} c(z)\) either has no pole at \(a\) or has a pole at \(a\) of order one less that \(c\). Furthermore, the total number of its poles, counted with multiplicities, is one lower than that of \(c\), since the pole at \(a^{k^n}\) is reduced by the root at this very point. It remains to prove that the function \(\frac{h(z)}{h(z)} c(z)\) satisfies \[\text{i}\], which allows us to conclude by induction.
over the total number of anxious poles. This is a simple application of Lemma 4 – $h(z)$ has positive valuations only at numbers of the form $\alpha^k$ for $1 \leq i \leq n$, but the calmnesses $\clm_{\alpha k} c(z)$ at these points have to be positive, since $\clm_{\alpha k} c(z)$ was negative.

3.2 Regularity criterion for order one Mahler series

In this subsection we prove some equivalent conditions for the regularity of a power series satisfying an equation of the form (2).

Theorem 3. If a formal series $f \in \mathfrak{K}[[z]]$ satisfies a Mahler equation of the form

$$f(z) = c_1(z) f\left(z^k\right),$$

then the following conditions are equivalent

(i) $f$ is regular,

(ii) $c_1$ is precalm,

(iii) $\clm_{\alpha} (c_1) \geq 0$ for every anxious $\alpha$.

Before we prove this theorem it is worth noting that it provides an alternative proof of the characterisation of regular rational functions. Allouche and Shallit [2, Theorem 3.3] have shown the following.

Theorem 4. A rational function $f(z) = \frac{p(z)}{q(z)}$ is regular if and only if $q(z)$ has roots only at roots of unity.

We prove this result using Theorem 3, while the original proof used explicit computation and basic properties of regular sequences.

Proof. Note that $f(z)$ satisfies the Mahler equation

$$f(z) = c(z) f\left(z^k\right)$$

with $c(z) = \frac{p(z)q(z^k)}{q(z)p(z^k)}$. The calmness

$$\clm_{\alpha} (c(z)) = \clm_{\alpha} \frac{p(z)}{p\left(z^k\right)} + \clm_{\alpha} \frac{q(z^k)}{q(z)}$$

for any anxious $\alpha$. We can show that this expression is nonnegative for all anxious $\alpha$ if and only if $q(z)$ has roots only at roots of unity. Indeed, the first term is always nonnegative and so is the second if $q(z)$ has no roots at numbers different from roots of unity. In fact, this is a straightforward application of Lemma 3. Assume that $q(z)$ has a root at a number $\alpha$ that is not a root of unity. Then $q(z)$ has a root at $\alpha^k$, $i \geq 0$, such that $q(z)$ has no root at $\alpha_0^k$ for any $i \geq 1$. The calmness $\clm_{\alpha_0} c(z) < 0$, because the second term is negative while the first cannot be positive since $p(z)$ is coprime to $q(z)$. An application of Theorem 3 finishes the proof.

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Proof of Theorem 3. The equivalence of conditions (ii) and (iii) follows immediately from Lemma 9. We will again start with the simpler of the remaining implications. The opposite implication generalises the work of Dumas [7, Proposition 54].

(ii) \implies (i) Since \( c_1 \) is precalm, we have a \( h \in \kappa [z] \) such that \( \frac{h(z^k)}{h(z)} c_1 (z) \) is calm. This, together with the definition of \( f \), means that the series \( f(z) \) satisfies the Mahler equation
\[
\frac{f(z)}{h(z)} = \frac{f'(z)}{h'(z)}
\]
with \( c'_1 (z) = \frac{h(z^k)}{h(z)} c_1 (z) \) calm. From Theorem 2 we know that series satisfying such a Mahler equation are regular. But polynomials are also regular and regular power series form a ring [2, Corollary 3.2], so \( f(z) = \frac{h(z)}{h'(z)} \) is also regular.

(i) \implies (iii) We will prove that if \( \text{clm}_a (c_1) < 0 \) for some anxious \( a \), then \( f \) cannot be regular. Using the Mahler equation (2) and Lemma 7(b) we can for any number \( L + 1 \) of indices \( r_i \in \{0, \ldots, k - 1\} \) write
\[
\Lambda_{r_{L+1}} \ldots \Lambda_{r_1} f(z) = \left( \Lambda_{r_L} \ldots \Lambda_{r_1} \prod_{l=0}^L c_1 \left( z^{k^l} \right) \right) f(z). \tag{3}
\]
We will now show that the \( \kappa \)-vector space \( V \) generated by these expressions is not finitely dimensional – by Lemma 8 this implies that \( f \) is not regular.

We know that \( \text{clm}_a c_1 < 0 \) for some anxious \( a \). Let us define a set of indices \( R_a \). If \( a \) is a root of unity, then \( R_a = \{ im - 1 : i \geq 0 \} \) where \( m \) is the smallest number such that \( a^m \neq \alpha \). If \( a \) is not a root of unity, then \( R_a \) consist of all natural numbers greater than \( m \), where \( m \) is the largest number for which \( v_a (\prod_{l=0}^L c_1 \left( z^{k^l} \right)) < 0 \) for any \( L \in R_a \). The latter property holds for \( a \) not a root of unity, because the valuation is equal to \( \text{clm}_a c_1 \) and for \( a \) a root of unity because the valuation is equal to a natural multiple of \( \text{clm}_a c_1 \).

In (3) we put no restrictions on the choice of \( r_i \). Therefore, by repeated applications of Lemma 8 we can choose \( r_i \) so that
\[
v_a \left( \Lambda_{r_{L+1}} \ldots \Lambda_{r_1} \prod_{l=0}^L c_1 \left( z^{k^l+L+1} \right) \right) \leq v_a \left( \prod_{l=0}^L c_1 \left( z^{k^l} \right) \right)
\]
for \( L \in R_a \). By Lemma 8 this can be also written as
\[
v_a \left( \Lambda_{r_{L+1}} \ldots \Lambda_{r_1} \prod_{l=0}^L c_1 \left( z^{k^l} \right) \right) \leq v_a \left( \prod_{l=0}^L c_1 \left( z^{k^l} \right) \right), \tag{4}
\]
Note that \( v_\alpha \left( \prod_{l=0}^{L} c_1 \left( z^L \right) \right) < 0 \) for \( L \in R_\alpha \).

By the form of the Mahler equation (2) and Lemma 7.(b), we can see that the vector space \( V \) contains only rational functions multiplied by the power series \( f(z) \). It is therefore a subset of a one dimensional \( \mathcal{k}(z) \)-vector space, and we identify it with a subset of \( \mathcal{k}(z) \) via the map \( V \ni g(z) \mapsto g(z) \in \mathcal{k}(z) \). This allows us to talk about valuations of elements of \( V \) via the formula \( v_\alpha (g(z)f(z)) = v_\alpha (g(z)), g(z) \in \mathcal{k}(z) \).

Using equations (3) and (4) and varying \( L \in R_\alpha \) allows us to look at valuations of elements in the \( \mathcal{k} \)-vector space \( V \). There are two cases to consider:

\( \alpha \) is not a root of unity In this case there are infinitely many valuations \( v_{\alpha^{L+1}} \) for which there exist elements of \( V \) with negative valuation. However, if the vector space \( V \) was finitely dimensional, then each of its generators would have negative valuation only at finitely many points, which gives a contradiction.

\( \alpha \) is a root of unity In this case, by the form \( R_\alpha \), \( \alpha^L = \alpha \) for \( L \in R_\alpha \) and the product on the right hand side of (4) can have arbitrarily low valuation. Thus there are elements in the vector space \( V \) with arbitrarily low valuation with respect to \( v_\alpha \), so as before \( V \) cannot be finitely dimensional.

\[
4 \text{ Calmnesses and the precalmness property}
\]

In this section we explore the relation between calmnesses of a sequence of functions and its precalmness. We prove a general version of Lemma 9.

**Theorem 5.** A sequence of rational functions \( (c_i)_{i=1}^{n} \) is precalm if and only if for every anxious \( \alpha \) and for every \( (a_i) \in A_\alpha \) we have \( \clm_{(a_i)}(c_i) \geq 0 \).

We will prove this result in two steps, first considering only the case when \( \alpha \) is a root of unity and later finishing the proof for other anxious numbers.

**4.1 Eliminating poles at roots of unity**

We will first focus on roots of unity and show that under the assumption that all the calmnesses are nonnegative, we can eliminate poles at such numbers by acting on the sequence of coefficients with a certain polynomial. We introduce a few more definitions and lemmas that are required to treat this case and will not be used elsewhere.

**Definition 17.** Let \( \alpha \) be an anxious root of unity. Set \( h_\alpha (z) = z - \alpha \).

From now on, fix an anxious root of unity \( \alpha \) and let \( 0 < m \in \mathbb{N} \) be the smallest positive integer such that \( \alpha^{km} = \alpha \). Such an integer exists because anxious roots of unity are exactly roots of unity of order coprime to \( k \) and furthermore \( \left( \alpha^k \right)^m = \alpha^i \) for any \( i \in \mathbb{N} \). Let also \( A = \bigsqcup_{i=0}^{m-1} A_{\alpha^i} \) be the
disjoint union of all the sequences corresponding to calmnesses for all \(a^k_i\), \(0 \leq i \leq m - 1\).

**Lemma 10.** Let \((c_i)\) be a sequence of rational functions. Then for \((\tilde{c}_i) = \left(h_{a^k_i}\right)^* (c_i)\) we have

\[
v_{a^k_i}(\tilde{c}_i) = v_{a^k}(c_i) + \begin{cases} 
-1 & \text{if } j = j_0 \pmod{m} \text{ and } i \not\equiv (j_0 - j) \pmod{m} \\
1 & \text{if } i = (j_0 - j) \pmod{m} \text{ and } j \not\equiv j_0 \pmod{m} \\
0 & \text{otherwise.}
\end{cases}
\]

**Proof.** By Lemma 8 it suffices to check that

\[
v_{a^k_i}\left(\frac{h_{a^k_0}(z^{k_i})}{h_{a^0}(z)}\right) = v_{a^k_i}(z) - v_{a^k_i}(z - a^k_0) = \begin{cases} 
-1 & \text{if } j = j_0 \pmod{m} \text{ and } i \not\equiv (j_0 - j) \pmod{m} \\
1 & \text{if } i = (j_0 - j) \pmod{m} \text{ but } j \not\equiv j_0 \pmod{m} \\
0 & \text{otherwise.}
\end{cases}
\]

The latter equality stems from the fact that the first term is nonzero (and so equal to one) when \((j + i) = j_0 \pmod{m}\), the second term is nonzero when \(j = j_0 \pmod{m}\) and if both these conditions are true, then the terms cancel out. 

The following definition will be needed just for the next few lemmas.

**Definition 18.** For a sequence of rational functions \((c_i)\), consider some term \(v_{a^k_i}(c_b)\). Among all the calmnesses \(clm_{h_{a_i}(a_i)}(c_i)\) associated with sequences \(a_i \in A\) that contain the term \(v_{a^k_i}(c_b)\) as one of their summands (i.e., sequences \(a_i \in A_{a^k_i}\) \((0 \leq i \leq m - 1)\) such that for some \(y\), \(a^{k_p} = a^{k_i}\) and \(c_b = c_{b_a}\), there are some having minimal value. We will call the minimal calmnesses containing \(v_{a^k_i}(c_b)\) and we will denote the set of sequences \(a_i\) associated with them by \(\text{minclm}_{a_b}(c_i) \subset A\).

**Proposition 1.** Let the sequence \((c_i(z))^n_{i=1}, c_i(z) \in K(z)\) be of length \(n \geq 1\). If for every \((a_i) \in A\), \(clm_{a_i}(c_i) \geq 0\), then there exists a polynomial \(0 \neq h \in K[z]\) such that no term of the sequence \(h^* (c_i(z))\) has a pole at \(a^k_i\) for \(i \geq 0\).

We will prove this proposition in a series of lemmas.

**Lemma 11.** Let the sequence \((c_i(z))^m_{i=1}, 0 \neq c_i(z) \in K(z)\) be of length \(m \geq 1\) and assume that it contains no zero terms. In such a case, if for every \(a \geq 0\), \(b \geq 1\) and \((a_i) \in A\), \(clm_{a_i}(c_i) \geq 0\) and for every \((a_i) \in \text{minclm}_{a_i}(c_i)\) the calmness \(clm_{a_i}(c_i) = 0\), then there exists a polynomial \(0 \neq h \in K[z]\) such that the sequence \(h^* (c_i(z))\) is \(a^k_i\)-calm for any \(i \geq 0\).

**Proof.** We will construct the required polynomial in steps. In every step we will choose a polynomial to act on the sequence and substitute the result for the original sequence. We will take \(h\) to be the product of these polynomials. Every step will reduce the sum \(\sigma = -\sum_{i,j} v_{a^k_i}(c_i) < 0\). This sum is always nonnegative and if it is 0, then no term of the sequence has a pole
at \( a^k \) for \( i \geq 0 \). By Lemma 3, calmness \( \text{clm}_{(a_i)} (c_i) \) remains unchanged upon the action of \( k(z) \), and so it is enough to prove that if \( \sigma > 0 \), we can find a polynomial \( h \) such that acting by \( h \) produces a sequence with a smaller value of \( \sigma \).

Assume \( \sigma > 0 \) and pick \( 0 \leq j \leq m \) such that \( v_{\min} (j) := \min_i \{ v_{a^j} (c_i) \} \) is minimal among all \( j \). Define \( S_j := \{ 1 \leq i \leq m : v_{a^j} (c_i) = v_{\min} (j) \} \). The choice of \( j \) satisfies \( v_{\min} (j) < 0 \) since \( \sigma > 0 \). The set \( S_j \) is neither empty (by the definition of \( v_{\min} (j) \)) nor does it contain all possible indices (for example \( m \notin S_j \), because the valuation \( v_{a^j} (c_m) \) is equal to the calmness \( \text{clm}_{a^j}^\alpha(m) (c_i) \) associated with the one element sequence \( \alpha \) with inequalities

\[ v_{\min} (j) \]

\[ \text{and therefore } v_{a^j} (c_m) = 0. \]

Pick an \( i_0 \in S_j \). For any \( i_1 \in \{ 1, \ldots, m \} \backslash S_j \), we can pick a sequence \( (a_i) \in \text{minclm}_{(a_i)} (c_i) \). By assumption \( \text{clm}_{(a_i)} (c_i) = 0 \). Consider the sequence \( (b_i) \) of length equal to the length of \( (a_i) \) plus one and such that

\[
b_i = \begin{cases} 
a_i & \text{for } i < i' \ 
i_0 & \text{for } i = i' \ 
i_2 = i_1 - i_0 \pmod{m} & \text{for } i = i' + 1 \ 
a_{i-1} & \text{for } i > i' + 1 
\end{cases}
\]

with \( i' \) such that \( a_{i'} = i_1 \). If we treat \( (b_i) \) as a sequence associated with the same anxious number as \( (a_i) \), we get

\[
\text{clm}_{(b_i)} (c_i) = \text{clm}_{(a_i)} (c_i) - v_{a^j} (c_i_1) + v_{a^j} (c_{i_0}) + v_{a^j+i_0} (c_{i_2}) = -v_{a^j} (c_{i_1}) + v_{a^j} (c_{i_0}) + v_{a^j+i_0} (c_{i_2}).
\]

Again by assumption \( \text{clm}_{(b_i)} (c_i) \geq 0 \), which gives us

\[
v_{a^j+i_0} (c_{i_2}) \geq v_{a^j} (c_{i_1}) - v_{a^j} (c_{i_0}) > 0,
\]

where the last inequality holds because \( i_1 \notin S_j \). Therefore, for any \( i_1 \notin S_j \) we have \( v_{a^j+i_0} (c_{i_2}) > 0 \). On the other hand, for \( i_0 \neq m \), by applying the property \( \text{clm}_{(a_i)} (c_i) \geq 0 \) to the length two sequence \( (i_0, m-i_0) \in A_{a^j} \), we obtain \( v_{a^j+i_0} (c_{m-i_0}) > 0 \). Applying the above reasoning for all \( i_0 \in S_j \) provides us with inequalities

\[
v_{a^j+i_0} (c_{i_2}) > 0, \tag{5}
\]

where \( i_2 = i_1 - i_0 \pmod{m} \) for some \( i_1 \notin S_j \) or \( i_2 = m - i_0 \). Recall further that \( m \notin S_j \).

Take \( h = \prod_{i_0 \in S_j} a_{a^j+i_0} \). We claim that the action of \( h \) on \( (c_i) \) lowers \( \sigma \).

This is a straightforward application of Lemma 10. In fact, the valuations \( v_{a^j} (c_{i_0}) \) for \( i_0 \in S_j \) are all negative and acting on them by \( h \) adds one to each of them, so it is enough to note that this action does not cause any of the remaining valuations to drop below zero. We pick \( i_0 \in S_j \) and check that this is the case for valuations of the form \( v_{a^j+i_0} (c_{i_2}) \). The inequalities of the form (5) and the fact that acting by the polynomial \( h \) lowers valuations by at most 1 (Lemma 10) allows us to check this only for \( i_2 = i_1 - i_0 \pmod{m} \), where \( i_1 \in S_j \backslash \{i_0\} \). However, for any such \( i_2 \), the action of \( h \) does not actually change
this valuation since by Lemma 10 it is lowered by 1 by the action of $h_{a_{j+1}i_0}$ but is raised by 1 by the action of $h_{a_{j+1}i_0}$, because $j + i_1 - (j + i_0) = i_1 - i_0 = i_2 \pmod m$. This ends the step and the proof.

In the next two lemmas we will weaken the assumptions of Lemma 11.

Lemma 12. Let the sequence $(c_i(z))^m_{i=1}, 0 \neq c_i(z) \in k(z)$ be of length $m$ and contain no zero terms. In such a case, if for every $(a_i) \in A$, $\text{clm}_{(a_i)}(c_i) \geq 0$, then there exists a polynomial $0 \neq h \in k[z]$ such that the sequence $h^*(c_i(z))$ is $\alpha^{k'}$-calm for any $i \geq 0$.

Proof. Consider $(a_i) \in \text{minclm}_{a,h_k} (c_i)$. If $\text{clm}_{(a_i)}(c_i) \neq 0$, substitute the sequence $(c'_i)$ for $(c_i)$, where

$$c'_i(z) = \begin{cases} \frac{c_i(z)}{(z-a^i)^{\text{clm}(a_i)(c_i)}} & \text{for } i = b \\ c_i(z) & \text{otherwise.} \end{cases}$$

The new sequence now satisfies $\text{clm}_{(a_i)}(c'_i) = 0$ for all $a_i \in \text{minclm}_{a,h_k} (c'_i)$ while still satisfying $\text{clm}_{(a_i)}(c'_i) \geq 0$ for all $a_i \in A$. We can repeat this procedure as long as for any $a \geq 0, b \geq 1$ we have $\text{clm}_{(a_i)}(c_i) \neq 0$ for $(a_i) \in \text{minclm}_{a,h_k} (c_i)$. Since all $c_i \neq 0$, we only need to repeat this procedure finitely many times. The sequence obtained after applying these procedures satisfies the assumptions of Lemma 11. Therefore, there exists a polynomial $h$ such that $h^*(c'_i)$ is $\alpha^{k'}$-calm for any $i \geq 0$. But $\nu_{a^i}(c_i) \geq \nu_{a^i}(c'_i)$, so $h^*(c_i)$ is also $\alpha^{k'}$-calm.

Lemma 13. Let the sequence $(c_i(z))^m_{i=1}, c_i(z) \in k(z)$ be of length $m$. In such a case, if for every $(a_i) \in A$, $\text{clm}_{(a_i)}(c_i) \geq 0$, then there exists a polynomial $0 \neq h \in k[z]$ such that the sequence $h^*(c_i(z))$ is $\alpha^{k'}$-calm for any $i \geq 0$.

Proof. Let $Z = \{i: c_i(z) \neq 0\}$ and $p = \sum_{i \in Z, 0 \leq j \leq m-1} \left| \nu_{a^i}(c_i) \right|$. Consider the sequence of rational functions $(c'_i)$ where

$$c'_i(z) = \begin{cases} c_i(z) & \text{if } i \in Z \\ \prod_{j \leq i} (z - a^j)^p & \text{otherwise.} \end{cases}$$

We claim that $(c'_i)$ satisfies the assumptions of Lemma 12. Since there are no zero terms in the sequence $(c'_i)$, it is enough to show that $\text{clm}_{(a_i)}(c'_i) \geq 0$ for all $(a_i) \in A$. Assume there exists $(a_i) \in A$ such that $\text{clm}_{(a_i)}(c'_i) < 0$. We can assume there are no sequences of shorter length than $(a_i)$ satisfying this inequality. In this case, no two terms of $(\pi_i)$ are equal modulo $m$. Indeed, if there existed $i_0 \neq i_1$ such that $\pi_{i_0} = \pi_{i_1} \pmod m$ we could consider the sequence $(a_{i-i_0})_{i=0}^{i_0} \in A$ and either its calmness would be negative (contrary to the assumption that $(a_i)$ was the shortest sequence with this property) or nonnegative, but then the sequence $(a'_i)$ with

$$a'_i = \begin{cases} a_i & \text{for } i < i_0 \\ a_{i-i_0+i_1} & \text{for } i_0 \leq i \end{cases}$$
would be shorter than \((a_i)\) and also have a negative calmness. Now

\[
\text{clm}_{(a_i)} (c'_i) = \sum_{a_i \in \mathbb{Z}} v_{a_i} \left( c'_{a_i} (z) \right) + \sum_{a_i \notin \mathbb{Z}} v_{a_i} \left( c'_{a_i} (z) \right).
\]

If the latter sum contains no terms, then \(\text{clm}_{(a_i)} (c'_i)\) is nonnegative since \(\text{clm}_{(a_i)} (c_i) \geq 0\). If the latter sum contains at least one term, then the former sum is necessarily at least \(-p\) by the definition of \(p\), so adding \(p\) to it makes it nonnegative.

Therefore \((c'_i)\) satisfies the assumptions of Lemma 12 and there exists a polynomial \(h\) such that \(h^* (c'_i)\) is \(a^k\)-calm for any \(i \geq 0\), but as in the previous proof \(v_{a^k} (c_i) \geq v_{a^k} (c'_i)\), so \(h^* (c_i)\) is also \(a^k\)-calm.

\[\square\]

**Proof of Proposition 4.** Let \(s_{i,j} = \min_{r \geq 0} v_{a^k} (c_{rm+i})\). Consider the sequence of rational functions \((c'_i) = \left( \prod_{j=1}^{m} (z - a^k)^{s_{i,j}} \right)_{i=1}^{m}\). This sequence satisfies the assumptions of Lemma 13 since \(\text{clm}_{(a_i)} (c'_i) = \text{clm}_{(a_i)} (c_i) \geq 0\) for \((a'_i)\) chosen in such a way that \(a'_i = rm + a_i\) for \(r\) chosen so that \(v_{a^k} (c_{a_i}) = s_{a_i,j}\). We therefore have a polynomial \(h\) such that \(h^* (c'_i)\) is \(a^k\)-calm for any \(i \geq 0\). To see that also \(h^* (c_i)\) is \(a^k\)-calm, note that by Lemma 10 this action changes valuations at \(a^k\) of \(c_{i+rm}\) by the same amount for any \(r \in \mathbb{N}\). Since \((c'_i)\) was constructed in such a way that \(v_{a^k} (c_{rm+i}) \geq v_{a^k} (c'_i)\), this finishes the proof.

\[\square\]

### 4.2 The remaining cases

**Proof of Theorem 1.** First note that precalm sequences clearly admit only nonnegative calmnesses. Indeed, calm sequences trivially have nonnegative calmnesses, and so do precalm sequences by Lemma 4. The converse implication is slightly more challenging.

Assume \((c_i)_{i=1}^{n}\) is a sequence such that all the calmnesses \(\text{clm}_{A, (a_i)}\) are nonnegative. Then, by applying Proposition 1 to all anxious roots of unity at which terms of \((c_i)\) have poles, we find a polynomial \(0 \neq h_0 \in \mathbb{k} [z]\) such that \(h_0^* (c_i)\) has no poles at anxious roots of unity. Since \(h_0^* (c_i)\) has the same calmnesses as \((c_i)\) (by Lemma 4) we have reduced the proof to the case when \(c_i\) have no poles at anxious roots of unity. We assume this henceforth.

We will construct a polynomial \(0 \neq h \in \mathbb{k} [z]\) such that \(h^* (c_i)\) has no poles at anxious numbers \(a\) in a number of steps. At each step, we will construct a polynomial \(\tilde{h} \in \mathbb{k} [z]\) and substitute \(\tilde{h}^* (c_i)\) for the original sequence. We will show that this substitution preserves the assumptions and that the procedure finishes after a finite number of steps. We will also show that the procedure stops only when the sequence is calm. The desired polynomial \(h\) will be the product of all the polynomials \(\tilde{h}\) constructed in this way.

Let \(a \in \mathbb{k}\) be an anxious number such that there exists a \(c_{i_0}\) such that \(v_a (c_{i_0} (z)) < 0\). In particular \(a\) is not a root of unity. If no such \(a\) exists, the sequence is calm and the procedure stops. The assumptions of the theorem imply that for the constant sequence \((i_0) \in \mathbb{A}\) the calmness \(\text{clm}_{(i_0)} (c_i) \geq 0\). Therefore, since \(v_a (c_{i_0} (z)) < 0\), there exists \(j\) such that \(v_{\alpha^j} (c_{i_0} (z)) > 0\). Let \(j_0\) be the smallest such \(j\) and let \(\tilde{h}(z) = \prod_{i=1}^{j_0} (z - \alpha^{j'})\).
By Lemma 4, for any anxious $\beta$ and $(b_i) \in A_\beta$ we have $\clm_{(b_i)}(c_i) = \clm_{(b_i)}(c_i) - v_\beta(\tilde{h}(z))$. By the form of $\tilde{h}$, the term $v_\beta(\tilde{h}(z))$ is nonzero only when $\beta = \alpha^j$ for some integer $1 \leq l \leq j_0$. Let $(b'_i) \in A_\alpha$ be defined as

$$b'_i = \begin{cases} i_0 & \text{if } i \leq l, \\ b_{l-i} & \text{otherwise}. \end{cases}$$

By the choice of $j_0$, we have $\sum_{i=0}^{l-1} v_{\alpha^{j_0}}(c_i(z)) \leq -1 = -v_{\alpha^{j_0}}(\tilde{h}(z))$. Using this, it is immediate to see that

$$\clm_{(b_i)}(c_i) - v_{\alpha^{j_0}}(\tilde{h}(z)) \geq \clm_{(b'_i)}(c_i) + \sum_{i=0}^{l-1} v_{\alpha^{j_0}}(c_i(z)) = \clm_{(b'_i)}(c_i) \geq 0.$$ 

Therefore, the sequence $\tilde{h}^*(c_i)$ still satisfies the assumptions of the theorem and we can continue the procedure.

At the beginning of the procedure fix the set of anxious numbers

$$B = \left\{ \alpha_r : \exists i \geq 0 \left( \alpha_{i+1} = \alpha_{i} \Rightarrow \forall j > 0 \left( \alpha^{j} = \alpha_{r} \Rightarrow \forall j \geq 0 \left( c_{i}(z) = 0 \right) \right) \right) \right\}.$$ 

These are anxious numbers $\alpha_r$ at which at least one $c_i$ has a pole and $\alpha_r$ is not a $k^{th}$ power of another pole of some $c_i$. Since all $c_i$ are rational functions and there are only finitely many of them, this set is finite. Furthermore among the calmnesses $\clm_{(a_i)}(c_i), (a_i) \in A_{\alpha^l}, \alpha_r \in B, l \geq 0$ only finitely many are nonzero. It remains to note that each step of the procedure lowers at least one of those calmnesses by one and does not increase the remaining ones. This can only happen finitely many times, since they are all nonnegative. Specifically, the $\alpha$ chosen in each step of the procedure is equal to $\alpha^l$ for some $\alpha_r \in B$ and $l \geq 0$, so by Lemma 4 all the corresponding calmnesses are lowered.

The two above paragraphs prove that the procedure must end in finitely many steps, which ends the proof.

5 Becker’s conjecture

This whole work was inspired by a conjecture of Becker [5, Corollary 1, remark 2]. In this section we investigate some alternative formulations of that conjecture.

**Conjecture 1** (Becker). A power series $f \in \mathcal{K}[[z]]$ is $k$-regular if and only if there exists a $k$-regular rational function $0 \neq h(z) \in \mathcal{K}(z)$ such that $\frac{f(z)}{h(z)}$ satisfies a Mahler equation with polynomial coefficients.

It is useful to note that $k$-regular rational functions are exactly rational functions with poles only at 0 and roots of unity, as shown in [2, Theorem 3.3]. One implication in Conjecture 1 (namely, that if such a function exists, then $f$ is regular) has been shown by Becker [5, Corollary 1]—this is in fact what inspired the conjecture. By analogy Theorem 2 might inspire an alternative version of Becker’s conjecture.
Conjecture 2 (Becker, alternative version). A power series \( f \in \mathcal{K}[[z]] \) is \( k \)-regular if and only if there exists a polynomial \( 0 \neq h(z) \in \mathcal{K}[z] \) such that \( \frac{f(z)}{h(z)} \) satisfies a Mahler equation with a \( k \)-calm sequence of coefficients.

Again, one of the implications in this conjecture follows immediately from Theorem 2 and the fact that regular power series form a ring.

As it turns out, both these conjectures are equivalent to the following, seemingly stronger, conjecture.

Conjecture 3 (Becker, final version). A power series \( f \in \mathcal{K}[[z]] \) is \( k \)-regular if and only if there exists a polynomial \( 0 \neq h(z) \in \mathcal{K}[z] \) such that \( \frac{f(z)}{h(z)} \) satisfies a Mahler equation with polynomial coefficients.

Proof of equivalence of Conjectures 1-3. Since all the conjectures claim an equivalence of \( k \)-regularity and a certain property of a finite sequence of rational functions \( c_i(z) \), it suffices to prove the following lemma.

Lemma 14. Let \( (c_i(z))_{i=1}^n \) be a sequence of rational functions. The following conditions are equivalent:

(i) There exists a rational function \( 0 \neq h \in \mathcal{K}(z) \) with poles at most at zero and roots of unity and such that \( h^*(c_i) \) is a sequence of polynomials.

(ii) There exists a polynomial \( 0 \neq h(z) \in \mathcal{K}[z] \) such that \( h^*(c_i) \) is \( k \)-calm.

(iii) There exists a polynomial \( 0 \neq h(z) \in \mathcal{K}[z] \) such that \( h^*(c_i) \) is a sequence of polynomials.

Proof. The implications (i) \( \Rightarrow \) (ii) and (iii) \( \Rightarrow \) (ii) are obvious, since polynomials are \( k \)-calm and have no poles.

(ii) \( \Rightarrow \) (i) Choose a polynomial \( 0 \neq h(z) \in \mathcal{K}[z] \) such that the sequence \( h^*(c_i(z)) \) is \( k \)-calm. By Lemma 5 there exists a polynomial \( 0 \neq h'(z) \in \mathcal{K}[z] \) such that \( (h'h)^*(c_i(z)) = h^*(h^*(c_i(z))) \) is a sequence of polynomials.

(i) \( \Rightarrow \) (ii) Choose a rational function \( h(z) = \frac{p(z)}{q(z)} \) where \( p, q \in \mathcal{K}[z] \) are polynomials and \( q \) has roots only at zero and roots of unity and such that \( h^*(c_i) \) is a sequence of polynomials. It is sufficient to show that \( p^*(c_i) = q^*(h^*(c_i)) \) is \( k \)-precalm, because then we would have a polynomial \( h'(z) \in \mathcal{K}[z] \) such that \( h'^*(q^*(h^*(c_i))) \) is \( k \)-calm, and therefore the polynomial \( h'qh \) satisfies the condition of (ii).

To show that \( q^*(h^*(c_i)) \) is \( k \)-precalm, we note that \( h^*(c_i) \) is a sequence of polynomials, and thus all its \( k \)-calmnesses are nonnegative. By Lemma 6 the action of \( q \) on this sequence will not change the \( k \)-calmnesses, because \( q \) has nonzero valuation only at \( k \)-anxious roots of unity. Thus \( q^*(h^*(c_i)) \) is precalm by Theorem 5. \( \square \)
6 Naive version of Becker’s conjecture

After looking at Theorems 3 and 5 one might be tempted to propose the following naive version of Becker’s conjecture.

**Conjecture 4** (Becker, naive version). Let \( f (z) \in \mathbb{K}[[z]] \) be a power series satisfying the Mahler equation

\[
  f (z) = \sum_{i=1}^{n} c_i (z) f \left( z^k \right).
\]

Then \( f \) is regular if and only if the sequence of coefficients \( (c_i) \) is precalm.

Were this true, it would provide us with a simple computational criterion for regularity of a Mahler series satisfying a given equation – we could just compute the associated calmnesses and apply Theorem 5. Sadly, it is not true, and we will construct a power series satisfying a Mahler equation with a non-precalm sequence of coefficients, and which is nonetheless regular. Moreover, the Mahler equation for this series that we give is its Mahler equation of minimal degree.

**Example 1.** Let \( k \geq 3 \) and let \( \alpha \) be a \( k \)-anxious number such that \( \alpha^{k-1} \neq \pm 1 \) and \( \alpha \neq \alpha^k \left( 1 + \alpha^{1-k} \right) \) for all \( i \in \mathbb{N} \). Consider the power series \( f (z) \neq 0 \) satisfying the equation

\[
  f (z) = \frac{\left( \alpha^{k-1} - \alpha^{1-k} \right) z + \left( \alpha - \alpha^k \right) f \left( z^k \right)}{\alpha - z} f \left( z^k \right) + \frac{\alpha^k - z}{\alpha - z} f \left( z^{k^2} \right). \tag{6}
\]

Then \( f \) is \( k \)-regular even though the sequence of coefficients \( (c_i) \) is not \( k \)-precalm.

**Proof.** We first show that a power series satisfying the given equation exists. We write \( f (z) = \sum a_i z^i \). Multiplying the equation (6) by \( \alpha - z \), we get the equation

\[
  (\alpha - z) f (z) = \left( \left( \alpha^{k-1} - \alpha^{1-k} \right) z + \left( \alpha - \alpha^k \right) \right) f \left( z^k \right) + \left( \alpha^k - z \right) f \left( z^{k^2} \right).
\]

Looking at the coefficients of degree zero in this equation we get

\[
  \alpha a_0 = \left( \alpha - \alpha^k \right) a_0 + \alpha^k a_0 = \alpha a_0
\]

which is satisfied for any \( a_0 \in \mathbb{K} \). Fix a nonzero term \( 0 \neq a_0 \in \mathbb{K} \). Then the term \( a_i \) is given by the recursive relation

\[
  \alpha a_i - a_{i-1} = \begin{cases} 
    (\alpha - \alpha^k) a_q & i = kq, k \neq q, \\
    (\alpha^{k-1} - \alpha^{1-k}) a_q & i = kq + 1, k \neq q, \\
    (\alpha - \alpha^k) a_{kq} + \alpha^k a_q & i = k^2 q, \\
    (\alpha^{k-1} - \alpha^{1-k}) a_{kq} - a_q & i = k^2 q + 1, \\
    0 & \text{otherwise.}
  \end{cases}
\]


This defines a nonzero sequence of coefficients of the formal power series $f$.

We now show that the series $f(z)$ does not satisfy a Mahler equation of order lower than two.

Suppose that $f$ satisfies an order one Mahler equation

$$f(z) = c(z) f(z^k)$$

with $c(z) = \frac{p(z)}{q(z)}$; $p, q \in \mathbb{K}[z]$. Substituting $f(z^k) = \frac{1}{c(z)}f(z)$ and $f(z^{k^2}) = \frac{1}{c(z)\cdot c(z)}f(z)$ to the original equation, dividing it by $f(z)$ and multiplying by the denominators, we get

$$
(a - z) p(z) p(z^k) = \left( (a^{k-1} - a^{1-k}) z + (a - a^k) \right) q(z) p(z^k) + (a^k - z) q(z) q(z^k).
$$

Set $P = \deg p(z)$, $Q = \deg q(z)$. Then the left hand side has degree $P + kP + 1$ and the right hand side at most degree $\max \{Q + kP + 1, Q + kQ + 1\}$ and the degree is in fact equal to $\max \{Q + kP + 1, Q + kQ + 1\}$ if $P \neq Q$. If $P > Q$, then we get a contradiction, since then $P + kP + 1 > Q + kP + 1 > Q + kQ + 1$. If $Q > P$, then we also have a contradiction: $Q + kQ + 1 > Q + kP + 1 > P + kP + 1$. Therefore $Q = P$. Since $p(z^k)$ divides two of the three terms, and is coprime to $q(z^k)$, we have $p(z^k) \mid (a^k - z) q(z)$. This gives us the inequality $kP \leq Q + 1 = P + 1$, but this is a contradiction if $P \neq 0$ since $k > 2$.

If $P = Q = 0$, then Adamczewski and Bell [11] Lemma 7.1] have shown that $f$ has to be constant, which is not the case. The argument is even simpler in this case, so we recall it. Let $f(z)$ satisfy the Mahler equation

$$f(z) = cf(z^k)$$

for some constant $c \in \mathbb{K}$. If $f(z) = \sum_{i=1}^{\infty} a_i z^i$ is not constant, then there exists the lowest integer $1 \leq i$ such that $a_i \neq 0$. However, on the right side of the equation the coefficient of $z^i$ is 0, which is a contradiction. By the above argument $f$ satisfies no Mahler equations of order less than two.

Let $c_1(z) = \frac{a^{k-1} - a^{1-k}}{a - z}$ and $c_2(z) = \frac{a^k - z}{a - z}$. For any sequence $(a_i) \in \mathbb{A}$ with $a_0 = a_1 = 1$ the calmness $\text{clm}_{\text{c}_{c_i(a_i)}}(c_i) = -1$ is negative. By Theorem 5 this sequence cannot be precalm.

Consider the operator $1 - c_1(z) \Delta_k - c_2(z) \Delta_k^2 \in \mathbb{K}(z)[\Delta_k]$ corresponding to the equation satisfied by $f$. This operator annihilates $f$ and hence so does the operator obtained from it after multiplying it on the left by $1 + \psi(z) \Delta_k$ with $\psi(z) = -a^{1-k} \frac{a^{k-1} - a^{1-k}}{a - z}$. We have

$$
(1 + \psi(z) \Delta_k) \left( 1 - c_1(z) \Delta_k - c_2(z) \Delta_k^2 \right) =
1 - (c_1(z) - \psi(z)) \Delta_k - \left( c_2(z) + \psi(z) c_1(z) \Delta_k^2 - \psi(z) c_2(z) \Delta_k^3 \right) \Delta_k \Delta_k^3
$$

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This corresponds to a Mahler equation of order three satisfied by \( f \). The coefficients of this equation are polynomials. The coefficient at \( \Delta_k \) is

\[
-(c_1(z) - \psi(z)) = \frac{(a^{k-1} - a^{-1-k})z + (\alpha - a^k) + a^{1-k}(\alpha - z^k)}{\alpha - z}
\]

and it is a polynomial, because the numerator has a zero at \( \alpha \). The coefficient at \( \Delta_2^k \) is

\[
-(c_2(z) + \psi(z)c_1(z^k)) = -\frac{\alpha^k - z}{\alpha - z} + \frac{a^{1-k}(\alpha - z^k)(a^{k-1} - a^{-1-k})z^k + (\alpha - a^k)}{(\alpha - z)(\alpha - z^k)}
\]

\[
= -\frac{(\alpha^{2-2k} - 1)z^k - z + a^k - a^{2-k} + \alpha}{\alpha - z}
\]

and is a polynomial for the same reason. So is the coefficient at \( \Delta_3^k \)

\[
-\psi(z)c_2(z^k) = \frac{a^{1-k}(\alpha - z^k)(\alpha^k - z^k)}{(\alpha - z)(\alpha - z^k)} = \frac{a^{1-k}(\alpha^k - z^k)}{\alpha - z}.
\]

By Theorem 2, this implies that \( f \) is regular.

### 7 The criterion for regularity

In this final section we prove the following theorem.

**Theorem 6.** Let \( f(z) \in \mathbb{K}[[z]] \) be a Mahler power series. Assume that the coefficients of the minimal order Mahler equation satisfied by \( f \) have no poles at roots of unity. Then \( f \) is regular if and only if it satisfies a Mahler equation

\[
f(z) = \sum_{i=1}^{n} c_i(z) f(z^k)\]

with \( (c_i)_{i=1}^{n} \) a calm sequence of rational functions.

We also conjecture that the assumptions on coefficients of the minimal equation are unnecessary.

**Conjecture 5.** A power series \( f(z) \in \mathbb{K}[[z]] \) is regular if and only if it satisfies a Mahler equation

\[
f(z) = \sum_{i=1}^{n} c_i(z) f(z^k)\]

with \( (c_i)_{i=1}^{n} \) a calm sequence of rational functions.

Conjecture 5 trivially implies Conjecture 2 and therefore it also implies the original Becker’s conjecture (see the discussion of various versions of Becker’s conjecture in Section 5). By the same arguments the following weaker version of Becker’s conjecture is also implied by Theorem 6.
Proposition 2. Let \( f(z) \in \mathbb{K}[[z]] \) be a Mahler power series. Assume that the coefficients of the minimal order Mahler equation satisfied by \( f \) have no poles at roots of unity. Then \( f \) is \( k \)-regular if and only if there exists a \( k \)-regular rational function \( h(z) \in \mathbb{K}(z) \) such that \( \frac{f(z)}{h(z)} \) satisfies a Mahler equation with polynomial coefficients.

The statement of Theorem 6 is purely existential. However, in the process of proving the theorem we will explicitly give a computational criterion involving coefficients of the minimal order Mahler equation satisfied by \( f \).

7.1 Generating higher order Mahler equations

Consider a power series \( f(z) \in \mathbb{K}[[z]] \) satisfying the Mahler equation

\[
f(z) = \sum_{i=1}^{n} c_i(z) f(z^k).
\]

This equation corresponds to the operator

\[
1 - \sum_{i=1}^{n} c_i(z) \Delta_i^k
\]

that annihilates \( f \). Consider a sequence of rational functions \( (\psi_i(z)) \) and consider the following operators

\[
\Gamma_m(\psi_i) = \left(1 + \sum_{i=1}^{m} \psi_i(z) \Delta_i^k\right) \left(1 - \sum_{i=1}^{n} c_i(z) \Delta_i^k\right) = \left(1 - \sum_{i=1}^{n+m} d_i(z) \Delta_i^k\right).
\]

These operators also annihilate \( f \) and we have the following recursive formulas for \( d_i \):

\[
d_{i,0}(z) = c_i(z),
\]

\[
d_{i,m}(z) = \begin{cases} 
  d_{i,m-1}(z) & \text{for } 0 \leq i < m \\
  d_{i,m-1}(z) - \psi_m(z) & \text{for } i = m \\
  d_{i,m-1}(z) + \psi_m(z) c_{i-m}(z^m) & \text{for } m < i < m+n \\
  \psi_m(z) c_{i-m}(z^m) & \text{for } i = m + n.
\end{cases}
\]
These formulas can be proven by a simple induction. The case \( m = 0 \) is trivial. Assume the claim is true for \( m - 1 \). Then we get

\[
\left( 1 + \sum_{i=1}^{m} \psi_i(z) \Delta^i_k \right) \left( 1 - \sum_{i=1}^{n} c_i(z) \Delta^i_k \right) = \\
\left( 1 + \sum_{i=1}^{m-1} \psi_i(z) \Delta^i_k \right) \left( 1 - \sum_{i=1}^{n} c_i(z) \Delta^i_k \right) + \psi_m(z) \Delta^m_k \left( 1 - \sum_{i=1}^{n} c_i(z) \Delta^i_k \right) = \\
\left( 1 - \sum_{i=1}^{n+m-1} d_{i,m-1}(z) \Delta^i_k \right) + \left( \psi_m(z) \Delta^m_k - \sum_{i=1}^{n} \psi_m(z) c_i \left( z^k \right) \Delta^{m+i}_k \right) = \\
1 - \left( \sum_{i=1}^{m-1} d_{i,m-1}(z) \Delta^i_k \right) - (d_{m,m-1}(z) - \psi_m(z)) \\
- \left( \sum_{i=m+1}^{n+m-1} d_{i,m-1}(z) + \psi_m(z) c_{i-m} \left( z^k \right) \right) - \psi_m(z) c_m \left( z^k \right).
\]

We now define a particularly interesting family of operators of the above type.

**Definition 19.** Let the sequence of rational functions \( \psi_i(z) \) be defined recursively by the formula

\[
\psi_m(z) = d_{m,m-1}(z).
\]

This definition is not circular, since the value of \( d_{m,m-1}(z) \) depends only on \( \psi_i(z) \) for \( 1 \leq i < m \). We obtain in this manner a sequence of operators \( \Gamma_m \) annihilating \( f \). We call these operators the special operators for \( f \).

It is easy to see from the recursive formulas for \( d_{i,m}(z) \) that the special operators of \( f \) have the form

\[
\Gamma_m = 1 - \sum_{i=1}^{n} d_{m+i,m}(z) \Delta^{m+i}_k
\]

and correspond to Mahler equations satisfied by \( f \) of the form

\[
f(z) = \sum_{i=1}^{n} d_{m+i,m}(z) f \left( z^{k+m} \right).
\]

Defining the special operators for \( f \) allows us to state a more precise version of Theorem 6.

**Proposition 3.** Let \( f(z) \in \mathbb{A}[[z]] \) be a power series satisfying the Mahler equation

\[
f(z) = \sum_{i=1}^{n} c_i(z) f \left( z^k \right)
\]

and satisfying no Mahler equations of order lower than \( n \). Define the set

\[
\mathcal{V} = \{ \alpha \in \mathbb{A} : \alpha \text{ is anxious and for some } 1 \leq i \leq n, v_{\alpha} (c_i(z)) < 0 \}.
\]

If \( \mathcal{V} \) contains no roots of unity, then the following conditions are equivalent
(i) For every \( \alpha \in \mathcal{V} \) there exists a natural number \( m \) such that \( v_\alpha (d_{i,m}(z)) \geq 0 \) for all \( m + 1 \leq i \leq m + n \) and \( v_{\alpha^{(i)}} (c_i(z)) \geq 0 \) for all \( j > m, 1 \leq i \leq n \) (note that the latter is automatically satisfied for \( m \) large enough).

(ii) \( f \) satisfies a Mahler equation with a calm sequence of coefficients.

(iii) \( f \) is regular.

In condition (i), \( d_{i,m}(z) \) are the coefficients of the special operators for \( f \).

Theorem 6 states the equivalence of conditions (ii) and (iii) in this proposition.

The criterion stated in Proposition 3 is effective and, given a Mahler equation for \( f \) of minimal order, allows to check whether \( f \) is regular. In fact, rational functions have only finitely many poles and roots, so at some point they stop influencing the valuations of \( d_{i,m}(z) \).

7.2 Proof of the criterion

Proof. The implication (ii) \( \Rightarrow \) (iii) follows immediately from Theorem 2.

\( \mathbf{1} \Rightarrow \mathbf{11} \). First note that if condition \( \mathbf{1} \) is satisfied for a number \( m \), then it is also satisfied for any number bigger than \( m \). Indeed, since \( v_{\alpha^{(i)}} (c_i(z)) \geq 0 \) for all \( j > m, 1 \leq i \leq n \), the functions \( c_i(z) \) will not lower any valuations of \( d_{i,m'}(z) \) for \( m' \geq m \) and sums of terms with nonnegative valuation cannot have negative valuation.

Since \( \mathcal{V} \) is a finite set, we can pick a number \( m_0 \) such that property \( \mathbf{1} \) is satisfied for all \( \alpha \in \mathcal{V} \). We therefore have that \( d_{i,m_0}(z) \) have no poles at \( \alpha \in \mathcal{V} \). We will construct a sequence of rational functions \( (\psi'_i)_{i=1}^{m_0} \) such that \( \Gamma_m (\psi'_i) = 1 - \sum_{j=1}^{n+m} d_{i,m}(z) \Delta_k \psi \) for \( 1 \leq m \leq m_0 \) and such that the coefficients \( d_{i,m_0}(z) \) are calm.

The choice of \( \psi'_1, \ldots, \psi'_{m_0} \) determines \( d_{i,m_0-1} \) and these values will be used in turn to construct \( \psi'_{m_0} \). Note that \( d'_{i,0}(z) = c_i(z) \) does not depend on the choice of \( \psi'_1 \). Consider the number

\[
\nu = -\min_{\psi \in \mathcal{V} \setminus \mathcal{V}_0} \{ v_\alpha (d_{i,m}(z)) \}.
\]

Note that \( \nu > 0 \) since \( \alpha \in \mathcal{V} \). Define the set

\[
B_m = \{ \alpha \in \mathcal{K} : v_{\alpha^{(i)}} (c_j(z)) < 0 \text{ for some } 1 \leq j \leq n \} \setminus \mathcal{V}.
\]

Set

\[
\psi'_m(z) = h(z) d'_{m,m-1}(z) \prod_{\alpha \in B_m} (\alpha - z)^{-\min_{\psi \in \mathcal{V} \setminus \mathcal{V}_0} \{ v_\alpha (c_j(z)) \}},
\]

where \( h(z) \in \mathcal{K}[z] \) is chosen by Lemma 2 so that \( \psi'_m \) has the same \( \nu \) first \((z-\alpha)\)-adic digits as \( d'_{m,m-1}(z) \) for any \( \alpha \in \mathcal{V} \). Such a choice of \( h \) is
possible since, as we will show shortly, \( d_{m,m-1}(z) \) has the same \((z-\alpha)\)-adic valuation as \( d'_{m,m-1}(z) \) for \( \alpha \in \mathcal{V}' \).

We first show that \( d'_{i,m}(z) \) are \( \alpha \)-calm for \( \alpha \in \mathcal{V}' \). It is sufficient to show that \( d'_{i,m}(z) \) is \( \alpha \)-calm for \( 1 \leq i \leq m \) and has the same \( v \) first \((z-\alpha)\)-adic digits as \( d_{i,m}(z) \) for all \( m+1 \leq i \leq m+n, 0 \leq m \leq m_0 \). (Here we say that the power series have the same first \( s \) digits if their valuations have the same value (say \( w \)) and the digits at \((z-\alpha)^j\) are the same for \( w \leq j \leq w + s - 1 \).) Indeed, in this case we know that \( d'_{i,m} \) and \( d_{i,m} \) have the same \((z-\alpha)\)-adic valuation and that \( d_{i,m_0}(z) \) has positive valuation at all \( \alpha \in \mathcal{V}' \) by condition \( \emptyset \). We prove this statement inductively on \( m \).

The functions \( d'_{i,0}(z) = c_i(z) = d_{i,0}(z) \), so the condition is satisfied. Assume that \( d'_{i,m-1}(z) \) is \( \alpha \)-calm for all \( 1 \leq i \leq m - 1 \) and has the same first \((z-\alpha)\)-adic digit as \( d_{i,m-1}(z) \) for \( m \leq i \leq m + n - 1 \). Recall the formulas for the functions

\[
d'_{i,m}(z) = \begin{cases} 
    d'_{i,m-1}(z) & \text{for } 0 \leq i < m \\
    d'_{i,m-1}(z) - \psi'_{m}(z) & \text{for } i = m \\
    d'_{i,m-1}(z) + \psi'_{m}(z) c_{i-m} \left( z^{k^m} \right) & \text{for } m < i < m + n \\
    \psi'_{m}(z) c_{i-m} \left( z^{k^m} \right) & \text{for } i = m + n.
\end{cases}
\]

For \( 0 \leq i \leq m - 1 \), \( d'_{i,m}(z) \) is \( \alpha \)-calm since \( d'_{i,m-1}(z) \) is \( \alpha \)-calm for all \( \alpha \in \mathcal{V}' \). For \( i = m \) note that \( d'_{i,m-1}(z) \) has the same \( v \) first \((z-\alpha)\)-adic digits as \( d_{i,m-1}(z) \), so also as \( \psi'_{m}(z) \) by construction. Therefore, by the choice of \( v \), \( d'_{i,m} = d'_{i,m-1}(z) - \psi'_{m}(z) \) has positive \((z-\alpha)\)-adic valuation, so is \( \alpha \)-calm.

For \( m + 1 \leq i \leq m + n \) again \( d'_{i,m-1}(z) \) has the same \( v \) first \((z-\alpha)\)-adic digits as \( d_{i,m-1}(z) \) and \( \psi'_{m}(z) \) has the same \( v \) first \((z-\alpha)\)-adic digits as \( d_{m,m-1}(z) = \psi_{m}(z) \). Therefore all the terms in the formula for \( d'_{i,m}(z) \) have the same \( v \) first \((z-\alpha)\)-adic digits as the terms in the formula for \( d_{i,m}(z) \), so these functions also have the same \( v \) first \((z-\alpha)\)-adic digits finishing the proof by induction.

Now we show that \( d'_{i,m}(z) \) are \( \alpha \)-calm for all \( \alpha \notin \mathcal{V}' \) and \( 0 \leq m \leq m_0 \).

We again show this inductively. For \( m = 0 \) the function \( d'_{i,0}(z) = c_i(z) \) is \( \alpha \)-calm for \( \alpha \notin \mathcal{V}' \) by the definition of \( \mathcal{V}' \). Assume that \( d'_{i,m-1}(z) \) are \( \alpha \)-calm for \( \alpha \notin \mathcal{V}' \). Note that then \( \psi'_{m}(z) \) is also \( \alpha \)-calm. Indeed,

\[
\psi'_{m}(z) = h(z) d'_{m,m-1}(z) \prod_{a \in B_m} (a - z)^{-\min_{i \leq s \leq m} v_{i}^{a}(c_{j}(z))},
\]

where \( h(z) \) and the product \( \prod_{a \in B_m} (a - z)^{-\min_{i \leq s \leq m} v_{i}^{a}(c_{j}(z))} \) are polynomials and \( d'_{m,m-1}(z) \) is \( \alpha \)-calm. Therefore, \( d'_{m,m}(z) \) is \( \alpha \)-calm as a sum of two \( \alpha \)-calm functions. The function \( d'_{i,m}(z) \), \( m + 1 \leq i \leq m + n \) is also a sum of \( \alpha \)-calm functions. Indeed, if \( \alpha \notin B_m \), then \( c_{i-m}(z^{k^m}) \) is \( \alpha \)-calm.
and if $a \in B_m$, then the valuation
\[
\nu_a \left( \frac{\partial^n}{\partial x^n} \left( z^{m(i)} \right) \right) =
\nu_a \left( h(z) \right) + \nu_a \left( \prod_{a \in B_m} (a - z)^{-\min_{1 \leq j \leq n} \nu_{a^{i}}(c_j(z))} \right) +
\nu_a \left( d_{m,m-1}^{i}(z) \right) + \nu_a \left( c_{i-m} \left( z^{k^m} \right) \right) \\
\geq 0 - \min_{1 \leq j \leq n} \nu_{a^{i}}(c_j(z)) + \nu_{a^{i}}(c_{i-m}(z)) \geq 0.
\]
This finishes the proof by induction. Therefore, we constructed an operator corresponding to a Mahler equation of order $n + m_0$ satisfied by $f(z)$ with a calm sequence of coefficients $(d_{i,m_0})_{i=1}^{n+m_0}$.

\(\text{(iii) } \Rightarrow \text{(iv)}\)  As in Theorem 3 we follow the idea of the proof in Dumas [7 Proposition 54]. Consider the equations corresponding to the special operators for $f$. We can apply $m + 1$ arbitrary Cartier operators to the equation $\Gamma_m f = 0$, which gives us
\[
\Lambda_{r_n} \left( (\Lambda_{r_0} (f(z))) \right) = \sum_{i=1}^{n} \Lambda_{r_n} \left( (\Lambda_{r_0} (d_{m+i,m}(z))) \right) f \left( z^{k^{i-1}} \right). \tag{7}
\]
However, by assumption that $f$ satisfies no Mahler equations of order less than $n$, the terms $f \left( z^{k^{i-1}} \right)$ on the right hand side are $k(z)$-linearly independent. If the $k$-vector space $V$ spanned by $\Lambda_{r_n} \left( (\Lambda_{r_0} (f(z))) \right)$ was finitely dimensional, then its generators would be of the form
\[
\sum_{i=1}^{n} h_i(z) f \left( z^{k^{i-1}} \right)
\]
for some rational functions $h_i(z) \in k(z)$. Since $h_i$ have only finitely many poles of finite order and by the $k(z)$-linear independence of $f \left( z^{k^{i-1}} \right)$, $V \subset h(z) \sum_{i=1}^{n} k(z) f \left( z^{k^{i-1}} \right)$ for some $0 \neq h \in k(z)$. It remains to prove that if condition (iii) is not satisfied, then we can find elements of $V$ for which this is not true.

If condition (iii) is not satisfied, then we can pick arbitrarily large $m$ such that a certain $d_{i,m}(z)$ has a pole at some $a \in p'$. Then, by a repeated application of Lemma 3 we can choose $r_i (0 \leq i \leq m)$ so that
\[
0 > \nu_a \left( \Lambda_{r_n} \left( (\Lambda_{r_0} (d_{i,m}(z))) \right) \left( z^{k^{m+1}} \right) \right).
\]
However,
\[
\nu_a \left( \Lambda_{r_n} \left( (\Lambda_{r_0} (d_{i,m}(z))) \right) \left( z^{k^{m+1}} \right) \right) = \nu_{a^{k^{m+1}}} \left( \Lambda_{r_n} \left( (\Lambda_{r_0} (d_{i,m}(z))) \right) \right).
\]
Therefore, we see by (7) that $h(z)$ has a pole at $\alpha^{k^{m+1}}$ for arbitrarily large $m$. Since $a$ is not a root of unity this shows that $h(z)$ has infinitely many poles which yields a contradiction, since $h$ is a rational function. $\square$
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Tomasz Kisielewski,
Institute of Mathematics,
Jagiellonian University,
ul. prof. Stanisława Łojasiewicza 6,
30-348 Kraków,
Poland,
e-mail: tomasz.kisielewski@uj.edu.pl