Instability of Superfluid Flow in the Neutron Star Inner Crust

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ABSTRACT

Pinning of superfluid vortices to the nuclear lattice of the inner crust of a neutron star supports a velocity difference between the superfluid and the solid as the star spins down. Under the Magnus force that arises on the vortex lattice, vortices undergo vortex creep through thermal activation or quantum tunneling. We examine the hydrodynamic stability of this situation. Vortex creep introduces two low-frequency modes, one of which is unstable above a critical wavenumber for any non-zero flow velocity of the superfluid with respect to the solid. For typical pinning parameters of the inner crust, the superfluid flow is unstable over length scales $< 10$ m and over timescales as fast as months. The vortex lattice could degenerate into a tangle, and the superfluid flow could become turbulent. Unexpectedly large dissipation would suppress this instability.

Key words: hydrodynamics – turbulence – stars: neutron – pulsars: general – stars: rotation.

1 INTRODUCTION

The dynamics of the superfluid interior of a neutron star is central to understanding a variety of phenomena that includes observed spin glitches, stochastic spin variations and thermal evolution, as well as possible precession and r-modes. In this connection, the possible importance of hydrodynamic instabilities in neutron stars has become a question of considerable interest. Peralta et al. (2005, 2006) have shown that differential rotation in the core, resulting from a spin glitch or possibly causing it, drives an Ekman flow along the rotation axis than can excite a variant of the “Glaberson-Donnelly” counterflow instability in liquid helium (Glaberson et al. 1974); transitions between laminar flow and fully-developed turbulence could drive spin glitches. This instability could also be excited in precessing neutron stars (Glampedakis et al. 2008, van Hoven & Levin 2008). Unstable shear layers (Peralta & Melatos 2009) and r-mode instabilities (Glampedakis & Andersson 2009) in the outer core may also play a role in glitches.

From the standpoint of building a realistic theory of neutron star seismology with which to interpret observations, it is important to identify hydrodynamic instabilities of possible relevance. The possibility of turbulent instabilities in the neutron star inner crust, the region from the neutron drip density to about half nuclear saturation density, has received little attention in this regard. Here the vortices that thread the rotating superfluid are predicted to interact with nuclei with energies of $\sim 1-5$ MeV per nucleus in the denser regions (Alpar 1977; Epstein & Bagn 1988; Donati & Pizzochero 2006; Avogadro et al. 2007). Recent work has shown that this interaction will pin vortices to nuclei, regardless of the details of the pinning potential (Link 2009). As the star spins down, the differential velocity between the superfluid and the pinned vortices approaches the critical value at which the hydrodynamic lift force on vortices, the Magnus force, would unpin them. As suggested long ago by Anderson & Itoh (1973), the spin glitches seen in neutron stars could arise from large-scale vortex unpinning from nuclei, wherein the threshold for pinning is exceeded. Below the critical velocity, pinned vortices slowly creep through thermal activation (Alpar 1984; Link et al. 1993) or quantum tunneling (Link et al. 1993), driven by the Magnus force. Here we demonstrate the existence of a hydrodynamic instability related to the vortex creep process that could grow over timescales as short as months.

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In the next section, we describe vortex pinning in the inner crust. In §3 we give the stability analysis. In §4, we discuss hydrodynamic wave solutions in the case of no background flow. In §5, we describe the hydrodynamic instability that arises when vortices move slowly through the nuclear lattice. In §6, we calculate the vortex mobility, which we apply to obtain the growth rate of the instability. We conclude with a discussion of the possibility that the inner-crust superfluid becomes turbulent.

2  VORTEX PINNING

First we calculate the critical velocity between the superfluid and the crust that can be sustained by vortex pinning. We will use these results in §6 to calculate the vortex mobility and the growth rate of the instability.

Vortex pinning fixes the local superfluid velocity in the laboratory frame. As the crust spins down, a velocity difference \( v \) between the pinned vortices and the superfluid develops. The Magnus force per unit length of vortex is

\[ f_{\text{mag}} = \rho \kappa v, \]

where \( \rho \) is the superfluid mass density, \( \kappa \equiv h/m \) is the quantum of vorticity, and \( m \) is twice the neutron mass. Let \( F_p \) be the characteristic force of the vortex-nucleus potential. Above a critical velocity difference \( v_c \), the Magnus force will exceed the pinning force, and vortex pinning is not possible. If a vortex could bend to intersect nuclei of average spacing \( a \), the critical velocity difference \( v_c \) would be given by

\[ \rho \kappa v_c a = F_p. \]

A vortex has a large self energy (tension) that typically prevents it from bending over a length scale \( a \). If the tension were infinite, a vortex could not pin at all, since the vortex would remain straight and the forces from nuclei that surround the vortex would cancel on average. For finite tension, the vortex can bend over a length \( l_p > a \), and the critical velocity is given instead by

\[ \rho \kappa v_c l_p = F_p, \]

giving a critical velocity

\[ v_c = \frac{F_p}{\rho \kappa a} \left( \frac{a}{l_p} \right). \]

Tension lowers the critical velocity by a factor \( 1/l_p \). To calculate \( l_p \), let \( r_v(z) \) be a vector in the \( x-y \) plane that gives the shape of the pinned vortex. The energy of a static vortex in a pinning field \( V(r_v) \), in the absence of an ambient superfluid flow, is

\[ E_v = \int dz \left[ \frac{1}{2} T_v \left( \frac{dr_v(z)}{dz} \right)^2 + V(r_v) \right], \]

where \( T_v \) is vortex tension, typically 1 MeV fm\(^{-1}\). On average, over a length \( l_p \), the vortex bends by an amount \( \delta r_v \) to intersect one nucleus in a volume \( l_p \pi (\delta r_v)^2 \). The quantities \( l_p \) and \( \delta r_v \) are therefore related by

\[ a^{-3} l_p \pi (\delta r_v)^2 = 1. \]

The energy of the vortex per unit length, from eq. (5), is approximately

\[ \frac{E_v}{l_p} \approx \frac{1}{2} T_v \frac{(\delta r_v)^2}{l_p^2} - \frac{E_p}{l_p}, \]

where \( E_p \) is the interaction energy between a vortex and a single nucleus, typically \( \sim 1 \) MeV. Contributions to the potential by nuclei that the vortex does not intersect have been ignored; these contributions will largely cancel. Minimization of \( E_v/l_p \) with respect to \( l_p \), using eq. (6), gives

\[ \frac{l_p}{a} = \left( \frac{3\alpha T_v}{2\pi E_p} \right)^{1/2}. \]

The vortex tension \( T_v \) is due mainly to the kinetic energy per unit length of vortex due to circulation about the vortex, and takes the form (Thomson 1880; Fetter 1967).

\[ T_v = \frac{\rho c^2}{4\pi} (0.116 - \ln k_v \xi), \]

where \( \xi \) is the radius of the vortex core and \( k_v \) is the characteristic bending wavenumber, \( k_v = \pi/2l_p \).

For typical conditions of the inner crust, the ratio \( l_p/a \) is much larger than unity. At a density \( \rho = 5 \times 10^{13} \) g cm\(^{-3}\) the...
lattice spacing is $a \simeq 50$ fm and the radius of the vortex core is $\xi \simeq 10$ fm. For $E_p = 1$ MeV, simultaneous solution of eqs. (8) and (10) gives $l_p \simeq 9a$. The ratio $l_p/a$ increases for weaker pinning. For example, for $E_p = 0.1$ MeV, the pinning length becomes $l_p \simeq 32a$. For $E_p = 10$ MeV, unrealistically large according to recent calculations, $l_p = 2a$.

Combining eq. (8) with eq. (10) gives the critical velocity, modified by vortex tension,

$$v_c = \frac{F_p}{\rho \kappa a} \left( \frac{a}{l_p} \right) = \frac{E_p}{\rho \kappa a^2} \left( \frac{2E_p}{3al_v} \right)^{1/2},$$

where we have taken $F_p = E_p/\xi$. Eq. (10) was found in the numerical simulations of the dynamics of an isolated vortex in a random potential (Link 2009). This equation shows that pinning is weakened by vortex tension. For $E_p = 1$ MeV and $\xi = 10$ fm, the critical velocity is $v_c \simeq 4 \times 10^5$ cm s$^{-1}$.

The corresponding differential angular velocity between the superfluid and the crust is as large as $\sim 1$ rad s$^{-1}$, but still much less than the angular velocity of the star when the superfluid condensed. The relative flow between the superfluid and the crust will thus be close to or comparable the local critical velocity in regions where there is pinning. We now examine the stability of this differentially-rotating state.

### 3 Perturbation Analysis

The problem of the coupled dynamics of the superfluid and vortex lattice can be studied using the hydrodynamic theory of Baym & Chandler (1983) which accounts for vortex degrees of freedom. The local quantities of fluid velocity, vortex density, and vortex velocity are averaged over a length scale that is large compared to the inter-vortex spacing $l_v$; the theory is valid for wavenumbers that satisfy $k l_v << 1$. We treat the superfluid as a single-component fluid at zero temperature, and ignore dissipation in the bulk fluid and the small effects of vortex inertia. These approximations are justified in a typical neutron star, for which the temperature of the inner crust is much less than the condensation temperature of the superfluid. We also treat the crust as infinitely rigid and ignore local shear deformations, an approximation that will be justified below. The motion of the superfluid does not couple to the electrons, so electron viscosity is not relevant. Magnetic fields are not relevant either, as they do not interact with the vortices of the inner crust.

We will consider only shear modes in the superfluid, so that the flow velocity $\mathbf{v}(r, t)$ is divergence-free. The rotation axis lies along $\hat{z}$, and $\mathbf{r}_v(r, t)$ denotes the continuum vortex displacement vector, with components in the $x - y$ plane only. The equations of motion in the laboratory frame are (Baym & Chandler 1983)

$$\nabla \cdot \mathbf{v} = 0$$

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla \mu - \nabla \phi - \mathbf{\sigma}_{el}/\rho + \mathbf{f}/\rho$$

$$\rho \omega \times \left( \mathbf{v} - \frac{\partial \mathbf{r}_v}{\partial t} \right) = -\mathbf{\sigma}_{el}(\mathbf{r}_v) + \mathbf{f},$$

where $\omega = \nabla \times \mathbf{v}$ is the vorticity due to the existence of vortices in the fluid, $\mu$ is the chemical potential, $\phi$ is the gravitational potential, $\mathbf{\sigma}_{el}/\rho$ is the elastic force per unit volume that arises from bending of the vortex lattice, and $\mathbf{f}/\rho$ is the force per unit volume exerted on the fluid by the normal matter. The elastic force is

$$\mathbf{\sigma}_{el}/\rho = -c_T^2 \left[ 2\nabla_\perp (\nabla_\perp \cdot \mathbf{r}_v) - \nabla_\perp^2 \mathbf{r}_v \right] + c_T^2 \frac{\partial^2 \mathbf{r}_v}{\partial z^2},$$

where $\nabla_\perp$ denotes a derivative with components in the $x - y$ plane only. Here $c_T = (\hbar \Omega/4m)^{1/2}$ is the Tkachenko wave speed (Tkachenko 1966b), and $\Omega$ is the spin rate of the superfluid. The quantity $c_T^2 = (\hbar/2m) \ln(\Omega c_v/\Omega)$ is related to wave propagation along the rotation axis; $\Omega_c = \hbar/(\sqrt{3}m c_v^2)$ for a triangular vortex lattice. For a typical neutron star rotation rate of $\Omega = 100$ rad s$^{-1}$, $c_T = 0.09$ cm s$^{-1}$ and $c_T = 9 c_T$. The areal density of vortices in the $x - y$ plane is $l_v^{-2} = 2m\Omega / h$ for a uniform vortex lattice; hence, the requirement that $k l_v << 1$ is equivalent to $k c_T << \Omega$.

Eq. (13) is an expression of balance of the Magnus force, the elastic force of the deformed vortex lattice, and the force exerted on the fluid by the normal matter. If the vortex array is perfectly pinned to the normal matter of the inner crust moving at velocity $v_n$, so that $\partial \mathbf{r}_v/\partial t = v_n$, the force is

$$\mathbf{f} = \rho \omega \times (\mathbf{v} - v_n) + \mathbf{\sigma}_{el}(\mathbf{r}_v).$$

For imperfect pinning, the Magnus force and elastic force drive vortex motion with respect to the normal matter. For imperfect pinning, the force above can be generalized as

\footnote{A value of $v_c$ as large as $\sim 10^7$ cm s$^{-1}$ was estimated in Link (2009), assuming $E_p = 5$ MeV at $\rho = 10^{13}$ g cm$^{-3}$, for $\xi \simeq a \simeq 70$ fm. This number is a generous upper limit.}
The vorticity appearing in this equation is the total vorticity evaluated in the laboratory frame. For rotating with the normal matter at angular velocity \( \Omega_n \), the solid to which they are pinned can give force is then
\[
\mathbf{f} = \beta' \rho \mathbf{\omega} \times (\mathbf{v} - \mathbf{v}_n) + \beta \rho \mathbf{\omega} \times (\mathbf{\omega} \times (\mathbf{v} - \mathbf{v}_n)) + (1 - \gamma) \sigma_{\text{el}}(r_v).
\]

The first two terms of this force are present in the mutual friction force introduced by Hall & Vinen (1956). We emphasize the generality of the force law of eq. \( \text{(16)} \). The first two terms represent the force exerted on the fluid by vortices that are moving with respect to the normal matter; the first term corresponds to the force transverse to the vortex motion, while the second term corresponds to the force parallel to the vortex motion. The coefficients \( \alpha \) and \( \beta \) can be calculated using a specific theory of vortex mobility. The third term accounts for the the contribution to the force that arises from local vortex bending.

If the vortex lattice is locally undeformed \( (\sigma_{\text{el}} = 0) \), the vortex velocity from eqs. \( \text{(16)} \) and \( \text{(17)} \) is
\[
\frac{\partial r_v}{\partial t} = \mathbf{v}_n + \alpha (\mathbf{v} - \mathbf{v}_n) - \beta \mathbf{\omega} \times (\mathbf{v} - \mathbf{v}_n),
\]
where \( \alpha \equiv 1 - \beta' \). Imperfect pinning, that is, “vortex creep”, corresponds to \( \alpha < < 1 \) and \( \beta < < 1 \). We refer to \( \alpha \), \( \beta \), and \( \gamma \) as the “pinning coefficients”.

Below we estimate \( R \) so that eq. \( \text{(18)} \) requires \( \beta >> \alpha \). \( \text{(18)} \) is not true in general. The presence of non-dissipative forces between vortices and the solid to which they are pinned can give \( \beta < < \alpha \) for \( \alpha \) and \( \beta \) both small, a regime of low drag that does not follow from eq. \( \text{(18)} \) for any value of \( R \). As discussed below, it is the low-drag regime that is likely to be realized, with vortex creep being unstable in this regime. A crucial feature of our analysis is that we do not assume eq. \( \text{(18)} \).

To examine the stability of superfluid flow with imperfect pinning, we use a local plane wave analysis in the frame rotating with the normal matter at angular velocity \( \Omega_n \), in which \( \mathbf{v}_n = 0 \) and the unperturbed flow velocity arising from spin down of the crust is \( \mathbf{v}_0 \). Restricting the analysis to the regime \( k\Delta R >> 1 \), where \( \Delta R \) is the thickness of the inner crust, the background flow can be taken to be uniform and the local analysis is valid. We take the unperturbed vortex lattice to be locally undeformed \( (\sigma_{\text{el}} = 0) \). The unperturbed creep velocity in the rotating frame \( (\mathbf{v}_n = 0) \) follows from eq. \( \text{(17)} \):
\[
\frac{\partial r_{v0}}{\partial t} = \alpha \mathbf{v}_0 - \beta \mathbf{\omega} \times \mathbf{v}_0.
\]

Below we estimate \( \partial r_{v0}/\partial t \sim 10^{-5} \mathbf{v}_0 \) for a typical neutron star. Linearizing eqs. \( \text{(12)} \) and \( \text{(13)} \) about \( \mathbf{v}_0 \), \( \partial r_{v0}/\partial t \), and \( \sigma_{\text{el}} = 0 \), and neglecting \( \partial r_{v0}/\partial t \) compared to \( \mathbf{v}_0 \), gives
\[
\nabla \cdot \delta \mathbf{v} = 0
\]
\[
\frac{\partial \delta \mathbf{v}}{\partial t} + \mathbf{v}_0 \cdot \nabla \delta \mathbf{v} + 2\mathbf{\Omega}_n \times \delta \mathbf{v} = -\nabla \delta \mu' - \nabla \delta \phi - \sigma_{\text{el}}/\rho + \delta \mathbf{f}/\rho
\]
\[
\rho 2\mathbf{\Omega}_n \times \left( \delta \mathbf{v} - \frac{\partial \delta r_v}{\partial t} \right) + \rho \delta \mathbf{\omega} \times \mathbf{v}_0 = -\sigma_{\text{el}} + \delta \mathbf{f}
\]
where \( \delta \) denotes a perturbed quantity, and \( \mu' \equiv \mu - \rho(\mathbf{\Omega}_n \times r)^2/2 \). We assume that \( \alpha \) and \( \beta \) are constants. The perturbed force is then
\[
\delta \mathbf{f} = (1 - \alpha) \rho \delta (\mathbf{\omega} \times \mathbf{v}) + \beta \rho \delta (\mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{v})) + (1 - \gamma) \sigma_{\text{el}}(r_v).
\]
The vorticity appearing in this equation is the total vorticity evaluated in the laboratory frame. For \( \nabla \times \mathbf{v}_0 < < 2\mathbf{\Omega}_n \), a good approximation for most neutron stars, the vorticity is
\[
\mathbf{\omega} = 2\mathbf{\Omega}_n + \nabla \times \delta \mathbf{v}.
\]

The final term in eq. \( \text{(23)} \), associated with stress in the vortex lattice, will turn out to be negligible for vortex creep driven by a flow \( \mathbf{v}_0 >> \mathbf{c} \) and \( \mathbf{v}_0 >> \mathbf{v}_v \).

We take the rotation axis to be \( \hat{z} \), with the unperturbed flow in the azimuthal direction, and along \( \hat{z} \) at some point. For simplicity, we restrict \( \mathbf{k} \) to lie in the \( x - z \) plane, with an angle \( \theta \) with respect to the rotation axis. We further restrict the analysis to the quadrant \( 0 \leq \theta \leq \pi/2 \). For shear perturbations, \( \mathbf{k} \cdot \delta \mathbf{v} = 0 \), that is, the velocity perturbations in the directions \( \hat{y} \) and \( \hat{e} \equiv -\cos \theta \hat{x} + \sin \theta \hat{z} \) are orthogonal to \( \mathbf{k} \).

We now Fourier transform \( \times e^{i(k \cdot \mathbf{r} - \omega t)} \) eqs. \( \text{(20)}-\text{(23)} \) and take the projections onto \( \hat{y} \) and \( \hat{e} \). Defining \( \sigma' \equiv \sigma - k\mathbf{v}_0 \sin \theta \), \( c \equiv \cos \theta \), and \( s \equiv \sin \theta \), we obtain the system of equations:

\[\text{eqs. (20)-(23)}\]


\[
\begin{bmatrix}
-i\sigma' + 2\Omega_n \beta - i(1 - \alpha)kv_0 s & -2\Omega_n \alpha c & \gamma(c_0^2 s^2 + c_0^2 c^2)k^2 & 0 \\
-\beta kv_0 c + 2\Omega_n \alpha c & -i\sigma' + 2\Omega_n \beta c^2 - i(1 - \alpha)kv_0 s & 0 & -\gamma(c_0^2 s^2 - c_0^2 c^2)k^2 \\
i(\sigma' + kv_0 s) & 0 & 0 & i2\Omega_n \alpha' / c \\
0 & -i(\sigma' + kv_0 s) & -i2\Omega_n \alpha \sigma' & 0
\end{bmatrix}
\begin{bmatrix}
\dot{y} \\
\dot{\delta v} \\
\dot{c} \\
\dot{\epsilon} - r_v
\end{bmatrix} = 0
\]

The resulting dispersion relation is quadratic:

\[
(\sigma')^4 + a_3(\sigma')^3 + a_2(\sigma')^2 + a_1\sigma' + a_0 = 0,
\]

where, in units with $\Omega_n = 1$,

\[
a_3 = 2(1 - \alpha)s\{kv_0\} + 2i(1 + c^2)\beta
\]

\[
a_2 = (\alpha - 1)^2 s^2\{kv_0\}^2 + \left(2i\beta s(1 + c^2) - 2i\alpha \beta s + \frac{1}{2}i\beta \gamma s^3\{kv_0\}^2 + \frac{1}{2}\beta \gamma s c^2\{kv_0\}^2\right)\{kv_0\}
\]

\[
- 4(\alpha^2 + \beta^2) c^2 - \alpha \gamma s^4\{kv_0\}^2 - \alpha \gamma c^2(1 + c^2)\{kv_0\}^2 + \frac{1}{4}\gamma^2 s^4\{kv_0\}^4 - \frac{1}{4}\gamma^2 c^4\{kv_0\}^4
\]

\[
a_1 = \frac{1}{2}i\beta \gamma(\{kv_0\}^2 s^4 + \{kv_0\}^2 c^2 s^4)\{kv_0\}^2 - \left(\alpha \gamma(\{kv_0\}^2 s^4 + \{kv_0\}^2 s c(1 + c^2)) - \frac{1}{4}\gamma^2(\{kv_0\}^4 s^4 - \{kv_0\}^4 s c^4)\right)\{kv_0\}
\]

\[
a_0 = \frac{1}{4}\gamma^2(\{kv_0\}^4 s^6 - \{kv_0\}^4 s c^4)\{kv_0\}^2,
\]

where, $\{kv_0\} \equiv kv_0 / \Omega_n$, $\{kv_0\} \equiv kcv / \Omega_n$, and $\{kv_0\} \equiv kcv / \Omega_n$.

4 WAVESOLUTIONS WITHOUT FLOW

Before turning to the full problem with non-zero $v_0$, we consider the limit of $v_0 = 0$ for the two cases of zero pinning and imperfect pinning. The dispersion relation is quadratic

\[
\sigma^2 + 2i\beta(1 + c^2)\sigma - 4c^2(\alpha^2 + \beta^2) + \gamma c_0^2 k^2 s^4 \left(-\alpha + \frac{1}{4}\gamma c_0^2 k^2\right) - \gamma c_0^2 k^2 c^2 \left(\alpha(1 + c^2) + \frac{1}{4}\gamma c_0^2 k^2 c^2\right) = 0
\]

For zero pinning force ($\alpha = \gamma = 1$, $\beta = 0$), the dispersion relation to order $c_0^2 k^2$ and $c_0^2 k^2$ becomes

\[
\sigma^2 = (2\Omega_n \cos \theta)^2 + c_0^2 (k^2 \cos^2 \theta)(1 + \cos^2 \theta) + c_0^2 k^2 \sin^4 \theta,
\]

as found by Baym & Chandler (1983). The fluid supports Tkachenko modes for $\theta = \pi/2$, and axial modes (modified inertial modes) for $\theta = 0$. In the limit $ct = cv = 0$, the system supports only ordinary inertial modes.

The role of pinning can be seen by considering axial modes for the case $\gamma c_0^2 k^2 << \alpha \Omega_n^2$. The solutions to eq. (31) in this limit are

\[
\sigma_{\pm} = 2i\beta \Omega_n \pm \left(\alpha \Omega_n + \frac{1}{4}\gamma c_0^2 k^2\right)
\]

which shows the damping effect of $\beta$. Pinning strongly suppresses the axial mode given by eq. (32), eliminating it entirely for perfect pinning. The waves are underdamped for $\beta < \alpha$, which defines the regime of low drag that we will study further.

5 INSTABILITY

We now show that a non-zero background flow $v_0$ drives a hydrodynamic instability if the vortices are imperfectly pinned ($\alpha << 1$, $\beta << 1$, $\gamma << 1$). We are interested in flow velocities of $v_0 \sim 10^5$ cm s$^{-1}$. By comparison,

\[
cv \sim 10ct \sim 10^{-5} v_0
\]

We will find that there is an instability for wavenumbers $k \gtrsim \Omega_n / v_0$. The hydrodynamic limit imposes the restriction $kcv / \Omega_n << 1$. The regime of interest is thus,

\[
\Omega_n / v_0 < k << \Omega_n / cv.
\]

We will estimate below that $\alpha \sim 10 \beta \sim 10^{-10}$. We assume that $\gamma$ is similarly small.

We will not present here an analysis of the full mode structure of the system, but focus on two low-frequency modes that appear for imperfect pinning. We simplify the problem by proceeding to linear order in the small quantities $kcv / \Omega_n$ and $kcv / \Omega_n$. At this level of approximation:

\[
a_3 = 2(1 - \alpha)s\{kv_0\} + 2i(1 + c^2)\beta
\]
that is, the vortex lattice exerts no stresses on the fluid to first order in \( c_T/v_0 \) and \( c_V/v_0 \). The dispersion relation eq. now simplifies to:

\[
\left( \sigma' \right)^2 \left\{ \left( \sigma' \right)^2 + 2\left( 1 - \alpha \right)kv_0s + i\beta \left( 1 + c^2 \right) \right\} \sigma' + (1 - \alpha)^2 k^2 v_0^2 s^2 - 4c^2 (\alpha^2 + \beta^2) + 2i\beta kv_0 s (1 + c^2 - \alpha) = 0.
\]

The two degenerate solutions \( \sigma'^2 = 0 \) correspond to \( \sigma = kv_0 \sin \theta = k \cdot v_0 \), the frequency associated with translation of the wave pattern at velocity \( v_0 \). The other two other solutions are, switching from \( \sigma' \to \sigma \) and restoring \( \Omega_n \),

\[
\sigma_{\pm} = \pm \alpha kv_0 \sin \theta - i\Omega_n (1 + \cos^2 \theta) \beta \pm \left( 4\Omega_n^2 \alpha^2 \cos^2 \theta - \Omega_n^2 \beta^2 \sin^4 \theta - 2i\alpha \beta \Omega_n k v_0 \cos^2 \theta \sin \theta \right)^{1/2}.
\]

For \( \beta < \alpha \) and low wavenumber \( kv_0 < \Omega_n \), there are two damped modes

\[
\sigma_{\pm} \approx \alpha (kv_0 \sin \theta \pm 2\Omega_n \cos \theta) - i\beta \left( \Omega_n \left( 1 + \cos^2 \theta \right) \pm \frac{1}{2}kv_0 \cos \theta \sin \theta \right).
\]

Slow vortex motion has introduced two low-frequency modes to the system. Removing pinning and drag (\( \alpha = 1, \beta = 0 \)) and taking \( k = 0 \), we recover the ordinary inertial modes \( \sigma_{\pm} = \pm 2\Omega_n \cos \theta \).

Above a critical wavenumber \( k_c \), the the solution with eigenvalue \( \sigma_- \) is unstable:

\[
k > k_c \equiv \frac{1}{v_0} \frac{\Omega_n (\beta^2 + \alpha^2)^{1/2}}{\alpha} \frac{1 + \cos^2 \theta}{\sin \theta \cos \theta}.
\]

Numerical solution of the full dispersion relation, eq. (25), for reasonable values of \( c_T, c_V, \) and \( v_0 \), confirms that there are no other instabilities. The critical wavenumber \( k_c \) is minimized for \( \theta = \tan^{-1}(\sqrt{2}) \). For \( k >> k_c \), we have the approximate solutions

\[
\sigma_{\pm} \approx \alpha kv_0 \sin \theta \mp i(\alpha \beta \Omega_n kv_0 \cos^2 \theta \sin \theta)^{1/2}.
\]

The instability arises from coupling between velocity and vorticity through the first two terms of eq. (25). Dissipation damps perturbations for \( k < k_c \), but for \( k > k_c \), the finite vortex mobility gives rise to growing perturbations under the Magnus force. For \( k >> k_c \), the growth rate scales as \( (\alpha \beta v_0)^{1/2} \). For \( \beta << \alpha \), \( k_c \) takes a constant value, but the growth rate of the mode becomes small, going to zero as \( \beta \) goes to zero. In the highly-damped regime, \( \beta >> \alpha \), damping restricts the unstable mode to large \( k \), generally stabilizing the system. There are no unstable modes for either \( \alpha = 0 \) or \( \beta = 0 \); the instability occurs only if the vortices move with respect to the crust, both along the flow and transverse to the flow.

We now show that our neglect of shear deformations of the crust is a good approximation. The modes we are studying are in the regime \( kv_0 > \Omega_n \). In this limit, the dominant contribution to the shear force per unit volume in the fluid is (see eq.

\[
\delta f/\rho \sim kv_0 \delta v.
\]

Because the vortices are nearly perfectly pinned, this shear force creates a strain field in the solid with a shear force per unit volume of

\[
\delta f_s/\rho \sim c^2_\delta k^2 \delta u,
\]

where \( c_\delta \) is the shear speed of the solid and \( \delta u \) is the characteristic displacement. The speed of a mass element in the solid is \( \delta v_n \sim \text{Re}(\sigma_{\pm}) \delta u \). Equating \( \delta f \) and \( \delta f_s \), and using eq. (13) for the limit of large \( k \), gives

\[
\frac{\delta v_n}{\delta v} \sim \alpha \left( \frac{v_0}{c_\delta} \right)^2.
\]

The values \( \alpha = 10^{-10}, v_0 = 10^5 \text{ cm s}^{-1} \) and \( c_\delta = 10^8 \text{ cm s}^{-1} \), give \( \delta v_n/\delta v \sim 10^{-16} \). The displacement of the solid is very small for two reasons: i) the solid is very rigid compared to the vortex lattice, and, ii) the vortex creep modes are of very low frequency, proportional to \( \alpha \ll 1 \).

We have restricted the analysis to shear waves (\( \nabla \cdot \delta u = 0 \)). Because these waves to not perturb the density, we do not expect the finite compressibility of the matter to change our results. Compressibility will introduce new modes (Haskell 2011), an effect that merits further study in the context of imperfect pinning.

6 ESTIMATES

To obtain the growth rate of the instability, we now estimate the pinning parameters \( \alpha \) and \( \beta \) for the vortex creep process. To make these estimates, we regard the process of vortex creep as consisting of two distinct states of motion for a given vortex segment. Most of the time, the vortex segment is pinned. A small fraction of the time, the vortex segment is translating
against a drag force to a new pinning configuration. The mutual friction force we are using (eq. 10) is, ignoring the small force from the vortex lattice,

\[ f / \rho = \omega \times v - \alpha \omega \times v + \beta \omega \times (\omega \times v). \]  

This force represents the average force exerted on the neutron fluid by the vortex array. The first term is the Magnus force for perfect pinning, while the remaining terms give the contribution to the force due to vortex motion. For those vortex segments that are unpinned and moving against drag, we take the force to have the same form, but with different coefficients:

\[ f_0 / \rho = \omega \times v - \alpha_0 \omega \times v + \beta_0 \omega \times (\omega \times v). \]  

An unpinned vortex segment remains unpinned for a time \( t_0 \sim d/v_0 \), where \( d \) is the distance the segment moves before repinning. This distance is comparable to the distance between pinning sites [Link et al. 1993], roughly ten times the unit cell size, giving \( t_0 \sim 10^{-15} \) s, much shorter than the hydrodynamic timescales of interest. Suppose that at any instant, the fraction of vortex length that is unpinned is \( f_v < 1 \). We now average \( f_0 \) over a volume that contains many vortices, and over a time long compared to \( t_0 \) but short compared to hydrodynamic timescales, to obtain

\[ \langle f_0 / \rho \rangle = \omega \times v - f_v \alpha_0 \omega \times v + f_v \beta_0 \omega \times (\omega \times v). \]  

Quantities related to the flow are unchanged by the averaging procedure since the superfluid flow velocity is independent of vortex length that is unpinned is \( f_v << 1 \). We now average \( f_0 \) over a volume that contains many vortices, and over a time long compared to \( t_0 \) but short compared to hydrodynamic timescales, to obtain

\[ \langle f_0 / \rho \rangle = \omega \times v - f_v \alpha_0 \omega \times v + f_v \beta_0 \omega \times (\omega \times v). \]  

The force of eq. (47), which is appropriate for vortex creep, must equal the average force \( \langle f_0 / \rho \rangle \), giving the following relationships:

\[ \alpha = f_v \alpha_0 \quad \text{and} \quad \beta = f_v \beta_0 \quad \Rightarrow \quad \frac{\beta}{\alpha} = \frac{\beta_0}{\alpha_0} \]  

We now use estimates of \( \beta_0 / \alpha_0 \) to obtain the ratio \( \beta / \alpha \).

The dominant drag process on unpinned vortex segments considered so far arises from the excitation of Kelvin modes as the vortex moves past nuclei. Calculations of dissipation by Kelvin phonon production on a long vortex with periodic boundary conditions for \( v_0 \sim 10^7 \) cm s\(^{-1}\) give typical values of \( \beta_0 / \alpha_0 = 0.1 \) and \( \alpha_0 \sim 1 \) [Enstein & Baym 1992]. Pinning occurs for \( v_0 \lesssim 10^5 \) cm s\(^{-1}\), and \( \beta_0 / \alpha_0 = 10 \) is likely to be significantly smaller in this velocity regime due to strong suppression of Kelvin phonon production [Jones 1992]. Vortex creep is therefore a low-drag process if Kelvin phonon production is the dominant dissipative mechanism. We fix \( \beta / \alpha = 0.1 \) for illustration in the following, which we consider to be an upper limit; we expect typical values to be smaller.

We now estimate \( \beta \). We adopt polar coordinates \( (r, \phi, z) \), with the unperturbed vorticity along \( \hat{z} \) and the unperturbed flow \( v_0 \) along \( \hat{\phi} \), and take the unperturbed flow and vortex velocity field to be axisymmetric. In the rotating frame, the unperturbed vortex velocity from eq. (19) is

\[ \frac{\partial r_{\alpha\phi}}{\partial t} = \alpha v_0 \hat{\phi} + \beta v_0 \hat{r} = \hat{\eta} \frac{\partial r_{\alpha\phi}}{\partial t} \]  

where \( \hat{\eta} \) is the average direction of vortex motion.

For steady spin down of the star, the inner crust superfluid and the crust are spinning down at the same rate for a local differential velocity \( v_0 \). The creep velocity in this steady state is related to the spin-down rate by [Alpar et al. 1984; Link et al. 1993]

\[ \Omega = -\frac{\Omega}{r} \frac{\partial r_{\alpha\phi}}{\partial t} \cdot \hat{r} = -\frac{\Omega}{r} v_0 \beta = \tilde{\Omega}_0, \]  

where \( \Omega \) is the spin rate of the superfluid, \( \tilde{\Omega}_0 \) is the observed spin down rate of the crust, and \( r \) is approximately the stellar radius \( R \). We arrive at the estimate

\[ \beta \approx \frac{R}{4 v_0 t_{\text{age}}} \approx 10^{-11} \left( \frac{v_0}{10^5 \text{ cm s}^{-1}} \right)^{-1} \left( \frac{t_{\text{age}}}{10^4 \text{ yr}} \right)^{-1}. \]  

where \( \Omega \approx \Omega_0 \) is assumed, and \( t_{\text{age}} \equiv \Omega_0 / 2 |\tilde{\Omega}_0| \) is the spin-down age. Eq. (53), with \( \beta = 0.1 \alpha \), gives the fiducial value \( \alpha \beta = 10^{-21} \). For this value, we deduce \( f_v \approx (\alpha \beta / \alpha_0 \beta_0)^{1/2} \approx 10^{-11} \), that is, most of the vortex length is pinned at any instant. The unperturbed vortex creep speed, from eq. (51), is \( \sim \alpha v_0 \sim 10^{-5} \) cm s\(^{-1}\) << \( v_0 \), justifying the neglect of \( \partial r_{\alpha\phi} / \partial t \) compared to \( v_0 \) in the stability analysis.

We can now proceed with estimates of the instability length scale and growth rate. For \( \beta < \alpha \) and \( \theta = \tan^{-1}(\sqrt{2}) \) in eq. (12), the critical wavenumber is

\[ k_c \approx 6 \frac{\Omega}{v_0} = 6 \times 10^{-3} \left( \frac{\Omega}{100 \text{ rad s}^{-1}} \right) \left( \frac{v_0}{10^5 \text{ cm s}^{-1}} \right)^{-1} \text{ cm}^{-1}, \]  

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corresponding to a wavelength $\lambda = 2\pi/k \simeq 10$ m. For $k >> k_c$, the growth rate from eq. (53) is

$$
\frac{1}{2\pi} \text{Im}(\sigma) \simeq 0.6 \left( \frac{\alpha \beta}{10^{-21}} \right)^{1/2} \left( \frac{\Omega}{100 \text{ rad s}^{-1}} \right)^{1/2} \left( \frac{v_0}{10^5 \text{ cm s}^{-1}} \right)^{1/2} \left( \frac{\lambda}{1 \text{ cm}} \right)^{-1/2} \text{yr}^{-1}.
$$

(55)

The hydrodynamic treatment is restricted to $c_T k_{\text{max}} << \Omega$. To estimate how high the growth rate could be, we consider a maximum wavenumber defined by $c_T k_{\text{max}} = 0.1 \Omega$, where $c_T \simeq 10^{-1} (\Omega/100 \text{ rad s}^{-1})^{1/2} \text{ cm s}^{-1}$. The growth rate at this wavenumber, from eq. (53), is

$$
\frac{1}{2\pi} \text{Im}(\sigma(k_{\text{max}})) \simeq 3 \left( \frac{\alpha \beta}{10^{-21}} \right)^{1/2} \left( \frac{\Omega}{100 \text{ rad s}^{-1}} \right)^{3/4} \left( \frac{v_0}{10^5 \text{ cm s}^{-1}} \right)^{1/2} \text{yr}^{-1},
$$

(56)

For $\Omega = 100$ rad s$^{-1}$, the corresponding wavenumber is $k_{\text{max}} \simeq 100$ cm$^{-1}$. Eq. (56) does not represent a physical limit, but only the restrictions of the hydrodynamic treatment; the instability could continue to exist also for wavenumbers in the regime $c_T k > \Omega$.

If vortex creep is in the strongly-damped regime $\beta >> \alpha$, contrary to the estimates here, there is still a broad window for instability. Requiring $k_c < k_{\text{max}}$ gives

$$
\beta < 2 \times 10^4 \left( \frac{v_0}{10^5 \text{ cm s}^{-1}} \right) \left( \frac{\Omega}{100 \text{ rad s}^{-1}} \right)^{-1/2} \alpha,
$$

(57)

and the star will be unstable at some wavenumber that is consistent with the hydrodynamic regime $c_T k \ll \Omega$.

7 DISCUSSION AND CONCLUSIONS

We have identified a dissipation-driven instability that could operate in the fluid of the neutron star crust over length scales shorter than ~ 10 m and over timescales as fast as months. This instability is different than other superfluid instabilities already considered in two-component systems. In the case of liquid helium, the Glaberson-Donnelly instability arises when the fluid normal component has a component of flow along the rotation axis of the system (Glaberson et al. 1974). That instability occurs even if the mutual friction is zero. A variant of the Glaberson-Donnelly instability has been studied in the mixture of superfluid neutrons and superconducting protons of the neutron star core, and instability was found to occur under the assumption that the vortices are perfectly pinned to the flux tubes that penetrate the superconducting proton fluid (Glampedakis et al. 2008; van Hoven & Levin 2008). Glampedakis & Andersson (2009) have identified a two-stream instability that might occur in the neutron-proton mixture of the core, again assuming perfect pinning of vortices to flux tubes and neglecting magnetic stresses. By contrast, the instability described here exists in a single-component fluid, and occurs because the vortices can move with respect to the solid. The component of the motion that is transverse to the flow, the component related to $\beta$ in eq. (10), though dissipative, is essential for the instability to occur.

To illustrate the basic instability, we have taken the pinning coefficients $\alpha$ and $\beta$ to be constants. For thermally-activated vortex creep, these coefficients will have exponential dependence on the velocity difference between the superfluid and the crust (Alpar et al. 1984; Link et al. 1993). We expect that this strong velocity dependence will significantly enhance the growth rate of the instability. The instability of the system will be determined by four coefficients: $\alpha(v_0)$, $\beta(v_0)$, and the derivatives $d\alpha/dv$ and $d\beta/dv$ evaluated at $v_0$. Further work is needed to calculate these coefficients and to incorporate them in the instability analysis. We restricted the analysis to wavevectors that are co-planar with the rotation axis and the unperturbed flow; more general perturbations should be studied, including waves with compressive components.

The flow of the inner crust superfluid could become turbulent. If the system evolves into a state of fully-developed superfluid turbulence with a vortex tangle, the friction force in the fluid would be better described by an isotropic (Gorter & Mellink 1949) or polarized (Andersson et al. 2007) form, rather than the anisotropic form of eq. (10) that is appropriate to a regular vortex array in the initial stages of the instability. The friction force for the tangle could be larger or smaller than the force given by eq. (12). On the one hand, the tangle has more vortex length per unit volume to interact with the solid, which tends to increase the force for a given value of the velocity. On the other hand, if the vortex distribution becomes highly tangled the momentum transfer from different regions will cancel to at least some extent, decreasing the friction force compared to that of a straight vortex array. If the force increases, so does the effective value of $\beta$, and the average value of the equilibrium differential velocity will decrease (see eq. 53). This effect could spell trouble for inner-crust models of glitches. By contrast, if the friction is decreased by turbulence, there will be more excess angular momentum in the superfluid available to drive glitches.

The instability results from the forcing of the vortex lattice through the solid lattice. If fully-developed turbulence results, the closest experimental analogue might be grid turbulence that has been well studied in superfluid helium (Smith et al. 1993). Further work is needed to determine the observational consequences of this instability, or if it is damped by some mechanism that has been overlooked here. We have shown elsewhere that a similar instability occurs in the mixture of superfluid neutrons and protons of the outer core (Link 2012).
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