FINITE AND SYMMETRIC MORDELL-TORNHEIM
MULTIPLE ZETA VALUES

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Abstract. We introduce finite and symmetric Mordell-Tornheim type of multiple zeta values and give a new approach to the Kaneko-Zagier conjecture stating that the finite and symmetric multiple zeta values satisfy the same relations.

1. Introduction and Main results

In [1] the authors described a new approach to study the relationship between finite and symmetric multiple zeta values. This was done by viewing these values as an algebraic and analytic limit of certain \( q \)-series evaluated at roots of unity. The purpose of this note is to use this approach to introduce finite and symmetric Mordell-Tornheim multiple zeta values and to give an analogue of the Kaneko-Zagier conjecture, which gives a surprising relationship between these values.

1.1. Notations. In this work, we denote by \( \mathbb{F}_p \) the finite field with \( p \) elements and by \( \mathbb{N} \) the set of positive integers. We call a tuple \( \mathbf{k} = (k_1, \ldots, k_r) \) of positive integers an index, \( wt(\mathbf{k}) = k_1 + \cdots + k_r \) the weight and \( len(\mathbf{k}) = r \) the length. An index \( \mathbf{k} = (k_1, \ldots, k_r) \) satisfying \( k_1 \geq 2 \) is said to be admissible. We regard the empty index \( \emptyset \) as admissible, and let \( wt(\emptyset) = len(\emptyset) = 0 \). We set \( F(\emptyset) \) to be a unit element for any function \( F \) on indices.

1.2. A review: the Kaneko-Zagier conjecture. Kaneko and Zagier [11] introduced the finite multiple zeta value \( \zeta_A(\mathbf{k}) \) for an index \( \mathbf{k} = (k_1, \ldots, k_r) \) as an element in the ring \( A = (\prod_{p} \mathbb{F}_p)/(\bigoplus_{p} \mathbb{F}_p) \) with \( p \) running over all primes (see Definition 2.1). Since the ring \( A \) forms a \( \mathbb{Q} \)-algebra, one can consider the \( \mathbb{Q} \)-vector subspace \( Z_k^A \) of \( A \) spanned by all finite multiple zeta values of weight \( k \). Zagier numerically observed that \( \dim_{\mathbb{Q}} Z_k^A \equiv \dim_{\mathbb{Q}} Z_k - \dim_{\mathbb{Q}} Z_{k-2} \), where \( Z_k \) denotes the \( \mathbb{Q} \)-vector
space spanned by multiple zeta values of weight $k$. Here the multiple zeta value is defined for an admissible index $k = (k_1, \ldots, k_r)$ by

$$\zeta(k) := \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}.$$  

Inspired by Kontsevich's idea, Kaneko and Zagier are led into a real counterpart of the finite multiple zeta value, called the symmetric multiple zeta value and denoted by $\zeta_S(k)$ (see Definition 2.2), in the quotient $\mathbb{Q}$-algebra $\mathbb{Z}/\zeta(2)\mathbb{Z}$, where $\mathcal{Z} = \sum_{k \geq 0} \mathcal{Z}_k$. They observed that finite and symmetric multiple zeta values satisfy the same relations over $\mathbb{Q}$, and then proposed a one-to-one correspondence between them, which we call the Kaneko-Zagier conjecture (see Conjecture 2.3).

### 1.3. Finite Mordell-Tornheim multiple zeta value.

In this paper, we wish to first mimic the above story replacing the multiple harmonic sum $H_n(k)$ defined in (2.1) with the rational number $\omega_n(k) \in \mathbb{Q}$ defined for an index $k = (k_1, \ldots, k_r)$ and $n \in \mathbb{N}$ by

$$\omega_n(k) := \sum_{m_1 + \cdots + m_r = n} \prod_{a=1}^{r} \frac{1}{m_k^a},$$

and then propose an alternative approach to the Kaneko-Zagier conjecture. Here the notion of $\omega_n(k)$ is originated from a certain truncation of generalized Mordell-Tornheim-Witten sums

$$\omega(k|l) = \sum_{m_1 + \cdots + m_r = m_1 + \cdots + n_s > 0} \prod_{a=1}^{r} \frac{1}{m_k^a} \prod_{b=1}^{s} \frac{1}{n_l^b},$$

studied by [2, 3, 4, 5]. We start by defining a finite analogue of these values.

**Definition 1.1.** For an index $k = (k_1, \ldots, k_r)$ with $r \geq 2$, we define the finite multiple omega value $\omega_A(k)$ by

$$\omega_A(k) := (\omega_p(k) \mod p)_p \in A.$$  

Denote by $\mathbb{Z}^A_{k,\omega}$ the $\mathbb{Q}$-vector space spanned by all finite multiple omega values of weight $k$. Using PARI-GP [21], we numerically computed the dimension of $\mathbb{Z}^A_{k,\omega}$. For comparison, it is listed together with the conjectural dimension of $\mathcal{Z}_k$ as follows.

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|-----|---|---|---|---|---|---|---|---|---|----|----|----|
| dim $\mathbb{Z}^A_{k,\omega}$ | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 2 | 2 | 3 | 4 | 5 |
| dim $\mathcal{Z}_k$ | 0 | 1 | 1 | 1 | 2 | 2 | 3 | 4 | 5 | 7 | 9 | 12 |
Similarly to the finite multiple zeta value, the equality \( \dim \mathbb{Q} Z^A \omega_k = \dim \mathbb{Q} Z_{k-2} \) up to \( k = 12 \) is observed. Accordingly, we may expect a real counterpart \( \omega_S(k) \) of \( \omega_A(k) \) in \( \mathbb{Z}/\zeta(2)\mathbb{Z} \).

1.4. **Symmetric Mordell-Tornheim multiple zeta value.** In order to find a proper definition of \( \omega_S(k) \), we use the same framework as in our previous work \([1]\). A crucial feature of this work was that finite and symmetric multiple zeta values are obtained from an algebraic and analytic operation for multiple harmonic \( q \)-series \( z_n(k; q) \) defined in (4.3), which is a \( q \)-analogue of \( H_n(k) \), at primitive roots of unity (see Theorem 4.3). The Kaneko-Zagier conjecture then turns out to be the statement of a relationship between ‘values’ of \( z_n(k; q) \) at \( q = \zeta_n \) with \( \zeta_n \) being a primitive \( n \)-th root of unity.

Let us begin with the \( q \)-analogue of \( \omega_n(k) \). For an index \( k = (k_1, \ldots, k_r) \), let

\[
\omega_n(k; q) := \sum_{m_1 + \cdots + m_r = n} \prod_{a=1}^r \frac{q^{(k_a-1)m_a}}{[m_a]^{k_a}},
\]

where \([m] = 1 + q + \cdots + q^{m-1} = \frac{1-q^m}{1-q}\) denotes the usual \( q \)-integer. By definition, it follows that \( \omega_n(k_{\sigma(1)}, \ldots, k_{\sigma(r)}; q) = \omega_n(k_1, \ldots, k_r; q) \) for any permutation \( \sigma \in \mathfrak{S}_r \).

If \( \text{len}(k) \geq 2 \), one can consider the value \( \omega_n(k; \zeta_n) \) in the cyclotomic field \( \mathbb{Q}(\zeta_n) \) at \( q = \zeta_n \) a primitive \( n \)-th root of unity (note that if \( \zeta_n \) is not primitive, then \( \omega_n(k; \zeta_n) \) is not well-defined). Its connection with the finite multiple omega value is as follows.

**Theorem 1.2.** Let \( \zeta_p \) be a primitive \( p \)-th root of unity. Under the identification \( \mathbb{Z}[\zeta_p]/(1-\zeta_p)\mathbb{Z}[\zeta_p] = \mathbb{F}_p \) with \( p \) being prime, for any index \( k \) with \( \text{len}(k) \geq 2 \) we have

\[
(\omega_p(k; \zeta_p) \mod (1-\zeta_p)\mathbb{Z}[\zeta_p])_p = \omega_A(k) \in \mathcal{A}.
\]

Theorem 1.2 says that the finite multiple omega value is obtained from a certain algebraic substitution \( q \rightarrow 1 \) for \( \omega_n(k; q) \). On the other hand, by an analytic limit \( q \rightarrow 1 \) along the unit circle, we are led into the definition of a real counterpart of finite multiple omega value.

**Theorem 1.3.** For any index \( k = (k_1, \ldots, k_r) \) with \( r \geq 2 \), the limit

\[
\Omega(k) := \lim_{n \to \infty} \omega_n(k; e^{2\pi i/n})
\]

exists, and it is given by

\[
\Omega(k) = \sum_{a=1}^r (-1)^{k_a} \zeta^{\text{MT}}(k_1, \ldots, k_{a-1}, k_{a+1}, \ldots, k_r; k_a),
\]

where \( \zeta^{\text{MT}} \) denotes the \( \zeta \)-function for \( \text{MT} \) series.
where $\zeta^{MT}(k_1, \ldots, k_r; l)$ is the Mordell-Tornheim multiple zeta value defined in (2.4).

Since every Mordell-Tornheim multiple zeta value can be written as $\mathbb{Q}$-linear combinations of multiple zeta values (see [6, Theorem 1.1]), the following definition makes sense.

**Definition 1.4.** For an index $k = (k_1, \ldots, k_r)$ with $r \geq 2$, we define the **symmetric multiple omega value** $\omega_S(k)$ by

$$\omega_S(k) := \Omega(k) \mod \zeta(2)\mathbb{Z}.$$

Using Mathematica [13], we also numerically checked that the identity $\dim \mathbb{Q}Z_k^{A,\omega} = \dim \mathbb{Q}Z_k - \dim \mathbb{Q}Z_{k-2}$ holds up to weight 12, where $Z_k^{A,\omega}$ denotes the $\mathbb{Q}$-vector subspace of $\mathbb{Z}/\zeta(2)\mathbb{Z}$ spanned by all symmetric multiple omega values of weight $k$. Consequently, for all $k \in \mathbb{N}$, the space $Z_k^{S,\omega}$ is expected to be isomorphic to $Z_k^{A,\omega}$ as a $\mathbb{Q}$-vector space.

### 1.5. Kaneko-Zagier conjecture revisited.

We compared all $\mathbb{Q}$-linear relations among finite and symmetric multiple omega values up to weight 12, and it is observed that they satisfy the same relations over $\mathbb{Q}$. This further observation will support the following analogous statement to the Kaneko-Zagier conjecture.

**Conjecture 1.5.** For any $k \in \mathbb{N}$, the $\mathbb{Q}$-linear map $\varphi_k : Z_k^{A,\omega} \to Z_k^{S,\omega}$ given by

$$\varphi_k(\omega_A(k)) = \omega_S(k)$$

is a well-defined isomorphism.

Another evidence of Conjecture 1.5 will be provided as follows. Let $\mathfrak{h} = \mathbb{Q}(x_0, x_1)$ and $\mathfrak{h}^1 = \mathbb{Q} + \mathfrak{h}x_1$. We write $y_k = x_0^{k_1-1}x_1$ for $k \in \mathbb{N}$ and $y_k = y_{k_1} \cdots y_{k_r}$ for an index $k = (k_1, \ldots, k_r)$. Then, the set $\{y_k | k : \text{index}\}$ is a linear basis of $\mathfrak{h}^1$. For $\bullet \in \{A, S\}$, we write $\zeta_{\bullet}(y_k) = \zeta_{\bullet}(k)$, and extend these notations to $\mathfrak{h}^1$ by $\mathbb{Q}$-linearly. Denote by $\sqcup : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$ the standard shuffle product (see (2.2)).

**Theorem 1.6.** For any index $k = (k_1, \ldots, k_r)$ with $r \geq 2$, we have

\begin{align*}
\omega_A(k) &= (-1)^{k_r} \zeta_A(x_0^{k_r} (y_{k_1} \sqcup \cdots \sqcup y_{k_{r-1}})), \\
\omega_S(k) &= (-1)^{k_r} \zeta_S(x_0^{k_r} (y_{k_1} \sqcup \cdots \sqcup y_{k_{r-1}})).
\end{align*}

By Theorem 1.6, we see that the Kaneko-Zagier conjecture, which insists the equality $\ker \zeta_A = \ker \zeta_S$, implies Conjecture 1.5. Numerically, we also observed that all finite and symmetric multiple zeta values can be written as $\mathbb{Q}$-linear combinations.
of finite and symmetric multiple omega values, respectively. If this is true, then Conjecture 1.5 would imply the map \( \varphi_{KZ} \) defined in Conjecture 2.3 (the Kaneko-Zagier conjecture) induces a well-defined isomorphism of \( \mathbb{Q} \)-vector spaces. Thus, Conjecture 1.5 could be a new approach to the Kaneko-Zagier conjecture. An advantage of this point of view is that we do not need to regularize our values, but a weak point is that the algebraic structure of the \( \mathbb{Q} \)-vector space \( \mathcal{Z}^{\bullet, \omega} = \sum_{k \geq 0} \mathcal{Z}^{k, \omega} \) is not immediately understood (conjecturally, \( \mathcal{Z}^{\bullet, \omega} \) forms a \( \mathbb{Q} \)-algebra).

We remark that Kamano [9] introduces the finite Mordell-Tornheim multiple zeta value \( \zeta_{MT}^A(k) \) for each index \( k \). By definition, it follows that \( \omega_A(k) = (-1)^{kr} \zeta_{MT}^A(k) \) for any index \( k \). Then, Theorem 1.2 of [9] is equivalent to (1.3), but our proof is completely different (see Remark 4.5).

1.6. Contents. The organization of this paper is as follows. In Section 2, we recall some basics on the Kaneko-Zagier conjecture and Mordell-Tornheim multiple zeta values, thereby also fixing some of our notation. Section 3 is devoted to proving Theorems 1.2 and 1.3 and Section 4 gives a proof of Theorem 1.6. Our proof of (1.4) provides another expression of the limit value \( \Omega(k) \) in terms of \( \zeta^\omega(k) \)'s, which is mentioned in the end of Section 4. Sections 5 considers relations among the values \( \omega_n(k; \zeta_n) \), which is applied to the study of relations of finite and symmetric multiple omega values together with special values of them.

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2. Preliminaries

2.1. Statement of the Kaneko-Zagier conjecture. We briefly review the Kaneko-Zagier conjecture.

We define the finite multiple zeta value. It is an element in the ring \( \mathcal{A} \), which is introduced by Kontsevich [12, §2.2], defined by

\[
\mathcal{A} = \left( \prod_{p \text{ prime}} \mathbb{F}_p \right) / \left( \bigoplus_{p \text{ prime}} \mathbb{F}_p \right).
\]
Its element is denoted by \((a_p)_p\), where \(p\) runs over all primes and \(a_p \in \mathbb{F}_p\). Two elements \((a_p)_p\) and \((b_p)_p\) are identified if and only if \(a_p = b_p\) for all but finitely many primes \(p\). Rational numbers \(\mathbb{Q}\) can be embedded into \(\mathcal{A}\) as follows. For \(a \in \mathbb{Q}\), set \(a_p = 0\) if \(p\) divides the denominator of \(a\) and \(a_p = a \in \mathbb{F}_p\) otherwise. Then \((a_p)_p \in \mathcal{A}\).

In this way, the ring \(\mathcal{A}\) forms a commutative algebra over \(\mathbb{Q}\).

Let us define the multiple harmonic sum \(H_n(k) \in \mathbb{Q}\) for an index \(k = (k_1, \ldots, k_r)\) and \(n \in \mathbb{N}\) by

\[
H_n(k) = \sum_{n > m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}.
\]

**Definition 2.1.** For an index \(k = (k_1, \ldots, k_r)\), we define the finite multiple zeta value \(\zeta(\mathcal{A})(k)\) by

\[
\zeta(\mathcal{A})(k) = (H_p(k) \mod p)_p \in \mathcal{A}.
\]

Denote by \(Z^A_k\) the \(\mathbb{Q}\)-vector subspace of \(\mathcal{A}\) spanned by all finite multiple zeta values of weight \(k\) and set \(Z^A = \sum_{k \geq 0} Z^A_k\), which forms a \(\mathbb{Q}\)-algebra (see [10, 11]).

Let us turn to the symmetric multiple zeta values. We first recall the algebraic setup of multiple zeta values by Hoffman [7]. Let

\[
\mathfrak{h} = \mathbb{Q}\langle x_0, x_1 \rangle
\]

be the non-commutative polynomial ring with indeterminates \(x_0\) and \(x_1\), and set \(y_k = x_0^{k-1}x_1\) \((k \geq 1)\). For a non-empty index \(k = (k_1, \ldots, k_r)\) we set \(y_k = y_{k_1} \cdots y_{k_r}\).

We define \(y_0 = 1\) for the empty index. We also let

\[
\mathfrak{h}^1 = \mathbb{Q} + \mathfrak{h}x_1, \quad \mathfrak{h}^0 = \mathbb{Q} + x_0\mathfrak{h}x_1.
\]

The \(\mathbb{Q}\)-vector subspace \(\mathfrak{h}^1\) is the \(\mathbb{Q}\)-subalgebra freely generated by \(\{y_k\}_{k \geq 1}\), and the set of monomials \(y_k\) with admissible index \(k\) is a linear basis of \(\mathfrak{h}^0\).

Using the iterated integral expression of the multiple zeta value due to Kontsevich, one can prove that the map \(\zeta : \mathfrak{h}^0 \to \mathbb{R}\), defined by \(\zeta(y_k) = \zeta(k)\) for an admissible index \(k\), is an algebra homomorphism with respect to the shuffle product \(\sqcup : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}\) given inductively by

\[
ww \sqcup vv' = u(w \sqcup vw') + v(uw \sqcup w')
\]

for \(w, w' \in \mathfrak{h}\) and \(u, v \in \{x_0, x_1\}\), with the initial condition \(w \sqcup 1 = w = 1 \sqcup w\). Namely, \(\zeta(w_1)\zeta(w_2) = \zeta(w_1 \sqcup w_2)\) holds for any \(w_1, w_2 \in \mathfrak{h}^0\). Equipped with the shuffle product, the vector space \(\mathfrak{h}\) forms a commutative \(\mathbb{Q}\)-algebra and \(\mathfrak{h}^1, \mathfrak{h}^0\) are...
We write $h^0, h^1, h^0_{\text{sh}}$ for commutative $\mathbb{Q}$-algebras with the shuffle product. It follows that $\zeta(h^0) = \mathbb{Z}$, which is hence a $\mathbb{Q}$-subalgebra of $\mathbb{R}$.

Let us define the shuffle regularized multiple zeta values, following [8]. Recall that the algebra $h^0_{\text{sh}}$ is freely generated by $x_1$ over $h^0_{\text{sh}}$ (see [22]):

$$h^0_{\text{sh}} \cong h^0_{\text{sh}}[x_1].$$

Namely, for any word $w \in h^1$, there exist $w_i \in h^0_{\text{sh}}$ such that

$$w = w_0 + w_1 \shuffle x_1 + w_2 \shuffle x_1^2 + \cdots + w_n \shuffle x_1^n.$$

Hence, there is a unique algebra homomorphism $\zeta^w : h^1_{\text{sh}} \to \mathbb{R}[T]$ such that the map $\zeta^w$ extend $\zeta : h^0_{\text{sh}} \to \mathbb{R}$ and send $\zeta^w(x_1) = T$. Applying $\zeta^w$ to the above word, we get

$$\zeta^w(w) = \sum_{a=0}^n \zeta(w_a)T^a,$$

so by definition it follows that $\zeta^w(h^1) = \mathbb{Z}[T]$. For an index $k = (k_1, \ldots, k_r)$ we write $\zeta^w(y_k)|_{T=0} = \zeta^w(k) \in \mathbb{Z}$, which we call the shuffle regularized multiple zeta value. To define the symmetric multiple zeta value, for an index $k = (k_1, \ldots, k_r)$, let

$$\zeta_S^w(k) = \sum_{a=0}^r (-1)^{k_1+\cdots+k_a} \zeta^w(k_a, k_{a-1}, \ldots, k_1) \zeta^w(k_{a+1}, k_{a+2}, \ldots, k_r).$$

**Definition 2.2.** For an index $k = (k_1, \ldots, k_r)$, we define the symmetric multiple zeta value $\zeta_S(k)$ by

$$\zeta_S(k) := \zeta_S^w(k) \mod \zeta(2)\mathbb{Z},$$

which is an element in the quotient $\mathbb{Q}$-algebra $\mathbb{Z}/\zeta(2)\mathbb{Z}$.

We remark that Yasuda [31, Theorem 6.1] proved that the symmetric multiple zeta value $\zeta_S(k)$ spans the whole space $\mathbb{Z}/\zeta(2)\mathbb{Z}$. Remark that there is another variant of $\zeta_S^w(k)$ replacing $\zeta^w$ with $\zeta^*$ the harmonic regularized multiple zeta value in the definition, which is however known to be congruent to $\zeta_S(k)$ modulo $\zeta(2)$ (see [10, Proposition 9.1] and [11]).

The Kaneko-Zagier conjecture is stated as follows.

**Conjecture 2.3.** The $\mathbb{Q}$-linear map $\varphi_{KZ} : \mathbb{Z}^A \to \mathbb{Z}/\zeta(2)\mathbb{Z}$ given by

$$\varphi_{KZ}(\zeta_A(k)) = \zeta_S(k)$$

is a well-defined isomorphism of $\mathbb{Q}$-algebras.
2.2. Mordell-Tornheim multiple zeta value. In this subsection, we recall the Mordell-Tornheim type of multiple zeta values.

The Mordell-Tornheim multiple zeta value is defined for an index $\mathbf{k} = (k_1, \ldots, k_r)$ and $l \in \mathbb{N}$ by

$$\zeta^{MT}(\mathbf{k}; l) = \sum_{m_1, \ldots, m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r} (m_1 + \cdots + m_r)^l}.$$  

This type of sum for the case $r = 2$ was first studied by Tornheim [25] in 1950, and independently by Mordell [16] in 1958 with $k_1 = k_2 = l$, and then, rediscovered by Witten [29] in 1991 in his volume formula for certain moduli spaces related to theoretical physics (see also Zagier's number theoretical treatment [32]). The case $r \geq 3$ was first introduced by Matsumoto [14] in 2002 as a function of several complex variables, and then investigated by many authors from many point of views (see e.g. [15, 18, 20, 26, 27, 28]).

As a study of special values, it was shown by Bradley-Zhou [6, Theorem 1.1] in 2010 that every Mordell-Tornheim multiple zeta value can be written as a $\mathbb{Q}$-linear combination of multiple zeta values. Moreover, for an index $\mathbf{k} = (k_1, \ldots, k_r)$ Yamamoto [30] proved the following formula:

$$\zeta^{MT}(k_1, \ldots, k_{r-1}; k_r) = \zeta(x_0^{k_r} (y_{k_1} \sqcup \cdots \sqcup y_{k_{r-1}})).$$  

It is worth mentioning that there seems no study on the opposite implication; “Every multiple zeta value can be written as a $\mathbb{Q}$-linear combination of Mordell-Tornheim multiple zeta values”. We numerically checked this implication up to weight 12 using Mathematica:

**Conjecture 2.4.** The space $\mathcal{Z}$ is generated by Mordell-Tornheim multiple zeta values over $\mathbb{Q}$.

Remark that to check Conjecture 2.4, it is necessary to use relations among multiple zeta values. In other words, one can check that there exist a monomial $y_\mathbf{k}$ such that it cannot be written as $\mathbb{Q}$-linear combinations of $x_0^{k_r} (y_{k_1} \sqcup \cdots \sqcup y_{k_{r-1}})$'s.

3. PROOFS OF THEOREMS 1.2 AND 1.3

3.1. Proof of Theorem 1.2. We prove Theorem 1.2.

**Proof of Theorem 1.2.** We first notice that the value $[m]_{q = \zeta_p} = (1 - \zeta_p^m)/(1 - \zeta_p)$ for any prime $p$ is a cyclotomic unit. Hence, the value $\omega_p(\mathbf{k}; \zeta_p)$ lies in $\mathbb{Z}[\zeta_p]$; and
taking modulo the ideal \((1 - \zeta_p)\mathbb{Z}[\zeta_p]\) generated by \(1 - \zeta_p\) in \(\mathbb{Z}[\zeta_p]\) makes sense. Since \((1 - \zeta_p)\mathbb{Z}[\zeta_p]\) is a prime ideal, its residue field \(\mathbb{Z}[\zeta_p]/(1 - \zeta_p)\mathbb{Z}[\zeta_p]\) is \(\mathbb{F}_p\). It follows that \(q^{(k-1)m}/m^k|_{q=\zeta_p} \equiv m^{-k} \mod (1 - \zeta_p)\mathbb{Z}[\zeta_p]\) for all \(k \in \mathbb{N}\), and hence
\[
\omega_p(k; \zeta_p) \equiv \omega_p(k) \mod (1 - \zeta_p)\mathbb{Z}[\zeta_p].
\]
Then, the result follows from the definition of the finite multiple omega value. \(\square\)

3.2. **Proof of Theorem 1.3.** We now prove Theorem 1.3.

**Proof of Theorem 1.3.** We see that
\[
\frac{1}{[m]}|_{q=e^{2\pi i/n}} = e^{-\frac{x}{n}(m-1)} \sin \frac{x}{n} \sin \frac{m\pi}{n} \quad (n > m > 0).
\]
Therefore it holds that
\[
\omega_n(k; e^{2\pi i/n}) = \left(e^{\frac{x}{n}} \sin \frac{\pi}{n}ight)^{wt(k)} G_n(k),
\]
where the function \(G_n(k)\) is defined by
\[
G_n(k) = \frac{1}{n^{wt(k)}} \sum_{m_1, \ldots, m_r > 0 \atop m_1 + \cdots + m_r = n} \prod_{a=1}^r g_{a_k} \left(\frac{m_a}{n}\right), \quad g_k(x) = \left(\frac{\pi}{\sin \pi x}\right)^k e^{(k-2)\pi ix} \quad (k \in \mathbb{N}).
\]
It suffices to show that \(G_n(k)\) converges to the right side of (1.2) in the limit \(n \to \infty\).

We decompose the range of the sum \(I = \{(m_1, \ldots, m_r) \in \mathbb{N}^r \mid \sum_{a=1}^r m_a = n\}\) into the subsets
\[
I_0 = \{(m_1, \ldots, m_r) \in I \mid 0 < m_a \leq n/2 \quad (1 \leq \forall a \leq r)\},
\]
\[
I_j = \{(m_1, \ldots, m_r) \in I \mid m_j > n/2, 0 < m_a \leq n/2 \quad (\forall a \neq j)\}, \quad (1 \leq j \leq r).
\]
Note that \(I = \bigcup_{j=0}^r I_j\). Now we set
\[
A_n(l_1, \ldots, l_r) = \frac{1}{n^{\sum_{a=1}^r l_a}} \sum_{m_1, \ldots, m_r > 0 \atop m_1 + \cdots + m_r = n} \prod_{a=1}^r g_{a_k} \left(\frac{m_a}{n}\right),
\]
\[
B_n(l_1, \ldots, l_{r-1}; l_r) = \frac{1}{n^{\sum_{a=1}^r l_a}} \sum_{m_1, \ldots, m_r > 0 \atop m_1 + \cdots + m_r = n \atop m_1 + \cdots + m_{r-1} < n/2} \prod_{a=1}^{r-1} g_{a_k} \left(\frac{m_a}{n}\right) h_{l_r} \left(\frac{m_1 + \cdots + m_{r-1}}{n}\right),
\]
where
\[
h_l(x) = (-1)^l g_l(1 - x) = \left(\frac{\pi}{\sin \pi x}\right)^l e^{-(l-2)\pi ix}.
\]
Then it holds that
\[
\frac{1}{n^{\text{wt}(k)}} \sum_{(m_1, \ldots, m_r) \in I_0} \prod_{a=1}^r g_{k_a} \left( \frac{m_a}{n} \right) = A_n(k_1, \ldots, k_r),
\]
\[
\frac{1}{n^{\text{wt}(k)}} \sum_{(m_1, \ldots, m_r) \in I_j} \prod_{a=1}^r g_{k_a} \left( \frac{m_a}{n} \right) = (-1)^{k_j} B_n(k_1, \ldots, k_{j-1}, k_{j+1}, \ldots, k_r; k_j).
\]
Therefore it is enough to show that, for any index \( k \) and \( l \in \mathbb{N} \),
\[
A_n(k) \to 0, \quad B_n(k; l) \to \zeta^{MT}(k; l)
\]
as \( n \to \infty \).

First, we calculate the limit of \( A_n(k) \). Set
\[
\tilde{A}_n(k) = \sum_{n/2 \geq m_1, \ldots, m_r > 0} \frac{1}{m_1 \cdots m_r}.
\]
It holds that
\[
|g_k(x)| \leq \left( \frac{\pi}{2x} \right)^k \quad (0 < x \leq \frac{1}{2}).
\]
Hence \( |A_n(k)| \leq C_k \tilde{A}_n(k) \) for a constant \( C_k \) which does not depend on \( n \). We prove that the limit of \( \tilde{A}_n(k) \) as \( n \to \infty \) is equal to zero. For \( r \geq 1 \) and \( 1 \leq l \leq n \), set
\[
S_{r,l,n} = \sum_{n/2 \geq m_1, \ldots, m_r > 0} \frac{1}{m_1 \cdots m_r}.
\]
We set, by definition, \( S_{r,l,n} = 0 \) unless \( r \leq l \leq rn/2 \). Since \( 0 \leq \tilde{A}_n(k) \leq S_{r,n,n} \), it suffices to show
\[
S_{r,l,n} \leq \frac{(2(1 + \log l))^{r-1}}{l}.
\]
We prove (3.2) by induction on \( r \). When \( r = 1 \), it is trivial. For \( r \geq 2 \), setting \( k = l - m_r \), we see that
\[
S_{r,l,n} = \sum_{k=1}^{l-1} \frac{1}{l-k} S_{r-1,k,n}.
\]
By the induction hypothesis, it is estimated as
\[
\sum_{k=1}^{l-1} \frac{1}{l-k} S_{r-1,k,n} \leq \sum_{k=1}^{l-1} \frac{1}{l-k} \frac{(2(1 + \log k))^{r-2}}{k} \leq (2(1 + \log l))^{r-2} \sum_{k=1}^{l-1} \frac{1}{(l-k)k}
\]
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\[
\frac{(2(1 + \log l))^{r-2}}{l} \sum_{k=1}^{l-1} \left( \frac{1}{l-k} + \frac{1}{k} \right) = \frac{2(2(1 + \log l))^{r-2}}{l} \sum_{k=1}^{l-1} \frac{1}{k} \leq \frac{(2(1 + \log l))^{r-1}}{l}.
\]

Thus we obtain (3.2) and \( A_n(k) \to 0 \) as \( n \to \infty \).

Next we calculate the limit of \( B_n(k;l) \) for an index \( k = (k_1, \ldots, k_{r-1}) \) and a positive integer \( l \). Set

\[
\tilde{B}_n(k;l) = \sum_{m_1, \ldots, m_{r-1} > 0 \atop m_1 + \cdots + m_{r-1} < n/2} \frac{1}{m_1 k_1 \cdots m_{r-1} k_{r-1} (m_1 + \cdots + m_{r-1})!}.
\]

Since the function \( f(z) = z^{k-1}(g_k(z) - z^k) \) is regular in a neighborhood of the interval \([0, 1/2] \), there exists a positive constant \( C \) such that

\[
|g_k(x) - \frac{1}{x^k}| \leq \frac{C}{x^{k-1}} \quad (0 < x \leq 1/2).
\]

Using (3.1), (3.3) and the polynomial identity

\[
\prod_{p=1}^{r} X_p - \prod_{p=1}^{r} Y_p = \sum_{j=1}^{r-1} (\prod_{p=1}^{j} X_p)(X_j - Y_j)(\prod_{p=j+1}^{r} Y_p),
\]

we obtain the following estimation:

\[
\left| \frac{1}{n^{\sum_{a=1}^{r-1} k_a + 1}} \prod_{a=1}^{r-1} g_{k_a} \left( \frac{m_a}{n} \right) h_l \left( \frac{m_1 + \cdots + m_{r-1}}{n} \right) - \prod_{a=1}^{r-1} \frac{1}{m_a k_a} (m_1 + \cdots + m_{r-1})! \right| \\
\leq \sum_{j=1}^{r-1} \prod_{a=1}^{j-1} \left| \frac{1}{g_{k_a} \left( \frac{m_a}{n} \right)} \right| \left| \frac{1}{h_l \left( \frac{m_1 + \cdots + m_{r-1}}{n} \right)} \right| - \frac{1}{m_j k_j} \left( \frac{m_1 + \cdots + m_{r-1}}{n} \right)! \\
+ \prod_{a=1}^{r-1} \left| \frac{1}{g_{k_a} \left( \frac{m_a}{n} \right)} \right| \left| \frac{1}{h_l \left( \frac{m_1 + \cdots + m_{r-1}}{n} \right)} \right| \frac{1}{(m_1 + \cdots + m_{r-1})!} \\
\leq \frac{C'_{k,l}}{n} \sum_{j=1}^{r-1} \frac{1}{m_1^{k_1} \cdots m_j^{k_j-1} \cdots m_{r-1}^{k_{r-1}}} \left( \frac{1}{(m_1 + \cdots + m_{r-1})!} \\
+ \frac{1}{m_1^{k_1} \cdots m_j^{k_j-1} \cdots m_{r-1}^{k_{r-1}}} \left( \frac{1}{(m_1 + \cdots + m_{r-1})^{l-1}} \right) \right) \\
\leq \frac{C'_{k,l}}{n} \frac{r}{m_1 \cdots m_{r-1}}
\]


for some constant $C_{k,l}'$ which does not depend on $n$. Hence

$$|B_n(k) - \tilde{B}_n(k)| \leq \frac{C_{k,l}'}{n} \sum_{m_1,\ldots,m_{r-1}>0 \atop m_1 + \ldots + m_{r-1} < n/2} \frac{r}{m_1 \cdots m_{r-1}} \leq C_{k,l}' \frac{r}{n} \left( \sum_{n/2 > m > 0} \frac{1}{m} \right)^{r-1}$$

as $n \to \infty$. Therefore it holds that

$$\lim_{n \to \infty} B_n(k;l) = \lim_{n \to \infty} \tilde{B}_n(k;l) = \zeta^{MT}(k;l).$$

This completes the proof. \hfill $\Box$

### 4. Proof of Theorem 1.6

#### 4.1. $q$-multiple polylogarithm

In this subsection, we briefly recall the $q$-multiple polylogarithm, introduced by Zhao [33], of one variable $L_k(t)$ and its shuffle relations.

In what follows, we fix a complex parameter $q$ such that $|q| < 1$. Define $F_k(m)$ for positive integers $k$ and $m$ by

$$F_k(m) = \frac{q^{(k-1)m}}{[m]^k} \in \mathbb{C}.$$  

We introduce the letter $\hat{1}$ and set $\hat{N} = N \sqcup \{\hat{1}\}$. Hereafter we call a tuple of the elements of $\hat{N}$ an extended index. In order to give a closed formula for the shuffle product for the $q$-multiple polylogarithm, we also need the term

$$F_1(m) = \frac{q^m}{[m]} \in \mathbb{C}$$

for $m \in \mathbb{N}$. For a non-empty extended index $k = (k_1, \ldots, k_r) \in \hat{N}^r$ we define the $q$-multiple polylogarithm of one variable $L_k(q;t)$ by

$$L_k(t) = \sum_{m_1,\ldots,m_r>0} t^{m_1} \prod_{a=1}^r F_{k_a}(m_a)$$

as a formal power series in $\mathbb{C}[[t]]$. We set $L_\emptyset(t) = 1$.

Recall the algebraic setup (see [24, §2.2] and [34]). Let $\hbar$ be a formal variable and set $\mathcal{C} = \mathbb{Q}[\hbar, \hbar^{-1}]$. We denote the unital non-commutative polynomial ring over $\mathcal{C}$
with two indeterminates \(a\) and \(b\) by

\[ \mathfrak{H} = \mathcal{C}(a, b). \]

For \(k \in \mathbb{N}\) we define \(e_k \in \mathfrak{H}\) by

\[ e_1 = ab, \quad e_k = a^{k-1}(a + h)b \quad (k \in \mathbb{N}). \]

Let

\[ \mathfrak{H}^1 = \mathcal{C}(e_k \mid k \in \mathbb{N}) \]

be the subalgebra freely generated by the set \(\{e_k\}_{k \in \mathbb{N}}\). For a non-empty extended index \(k = (k_1, \ldots, k_r) \in \mathbb{N}^r\) we set \(e_k = e_{k_1} \cdots e_{k_r}\). For the empty index we set \(e_\emptyset = 1\). Then, the set \(\{e_k \mid k : \text{extended index}\}\) is a linear basis of \(\mathfrak{H}^1\).

We define the \(\mathbb{Q}\)-linear action of \(\mathcal{C}\) on \(\mathbb{C}[[t]]\) by \((hf)(t) = (1-q)f(t)\) for \(f \in \mathbb{C}[[t]]\) and then the \(\mathcal{C}\)-module homomorphism

\[ L : \mathfrak{H}^1 \to \mathbb{C}[[t]], \quad e_k \mapsto L_k(t). \]

We recall that the map \(L : \mathfrak{H}^1 \to \mathbb{C}[[t]]\) is viewed as an evaluation map. While the map \(\zeta : \mathfrak{h}^0 \to \mathbb{R}\) is given by an iterated integral of the integrands \(\frac{dt}{t}, \frac{dt}{1-t}\) corresponding to the indeterminates \(x_0, x_1 \in \mathfrak{h}\), we view the indeterminates \(a, b, h\) as operations on the formal power series ring \(\mathbb{C}[[t]]\) (or the ring of holomorphic functions on \(D = \{t \in \mathbb{C} : |t| < 1\}\)). They are defined for \(f(t) = \sum_{n \geq 0} c_n t^n \in \mathbb{C}[[t]]\) by

\[ (af)(t) = (1-q) \sum_{j \geq 1} f(q^j t) = \sum_{n \geq 0} c_n k^n \frac{q^n}{n}, \]

\[ (bf)(t) = \frac{t}{1-t} f(t), \quad (hf)(t) = (1-q) f(t), \]

where \(a\) is only defined for \(f\) being \(f(0) = 0\). By definition, \(h\) commutes with others. With this, for any extended index \(k \in \mathbb{N}^r\) it can be shown that

\[ L_k(t) = e_k(1). \]

For example, one can compute

\[ L_{1,1}(t) = ab(a + h)b(1) = ab(a + h) \left( \sum_{n > 0} t^n \right) = ab \left( \sum_{n > 0} t^n \frac{1}{n} \right) = a \left( \sum_{m > n > 0} t^m \frac{1}{n} \right) = \sum_{m > n > 0} t^m \frac{q^m}{m \cdot n}. \]
where for the third equality we have used
\[
q^n \left[ \frac{1}{n} \right] + (1 - q) = \frac{q^n(1 - q) + (1 - q)(1 - q^n)}{1 - q^n} = \frac{1}{n}.
\]
Thus, the map \( L : \hat{\mathcal{H}}^1 \to \mathbb{C}[t] \) is alternatively defined by \( L(e_k) = e_k(1) \).

We define the shuffle product \( \shuffle_h \) as the \( \mathbb{C} \)-bilinear map \( \shuffle_h : \hat{\mathcal{H}} \times \hat{\mathcal{H}} \to \hat{\mathcal{H}} \) by
\[
aw \shuffle_h aw' = a(aw \shuffle_h w' + w \shuffle_h aw' + h w \shuffle_h w'),
\]
\[
bw \shuffle_h w' = w \shuffle_h bw' = b(w \shuffle_h w')
\]
for \( w, w' \in \hat{\mathcal{H}} \), with the initial condition \( 1 \shuffle_h w = w \shuffle_h 1 = w \). For example, \( e_1 \shuffle_h e_1 = 2abab + abhb + 2hbab + hhbb = e_1 e_1 + e_1 e_1 \). The element \( w \shuffle_h w' \) for \( w, w' \in \hat{\mathcal{H}}^1 \) lies in \( \hat{\mathcal{H}}^1 \), but, as we see in the example, it turns out to be a \( \mathbb{Q} \)-linear combination of words in \( \hat{\mathcal{H}}^1 \):

**Lemma 4.1.** For \( w, w' \in \hat{\mathcal{H}}^1 \) we have \( w \shuffle_h w' \in \hat{\mathcal{H}}^1 \).

**Proof.** For the proof, see Proposition 12.2.20 in [34]. \( \square \)

With Lemma 4.1, we can state the shuffle relation for \( q \)-multiple polylogarithms.

**Proposition 4.2.** For \( w, w' \in \hat{\mathcal{H}}^1 \) it holds that \( L(w \shuffle_h w') = L(w)L(w') \).

**Proof.** By Lemma 4.1, the expression \( L(w \shuffle_h w') \) is well-defined. Then the desired result follows from easily checked identities \( (af) \cdot (ag) = a(f \cdot ag) + a(f \cdot g) + ha(f \cdot g) \) for \( f, g \in \mathbb{C}[t] \) with \( f(0) = g(0) = 0 \) and \( (bf) \cdot g = b(f \cdot g) = f \cdot (bg) \) for \( f, g \in \mathbb{C}[t] \), where \( f \cdot g \) means the product on \( \mathbb{C}[t] \). \( \square \)

### 4.2. Multiple harmonic \( q \)-series at primitive roots of unity

In this subsection, we recall the main results of our previous work [1].

For a word \( w \in \hat{\mathcal{H}}^1 \), we write \( u_m(w) \) the coefficient of \( t^m \) in \( L(w) \in \mathbb{C}[t] \)

\[
\sum_{m \geq 0} u_m(w) t^m := L(w)
\]

and define
\[
z_n(w) := \sum_{m=1}^{n-1} u_m(w).
\]

These are viewed as \( \mathcal{C} \)-module homomorphisms from \( \hat{\mathcal{H}}^1 \) to \( \mathbb{C} \) with the \( \mathcal{C} \)-linear action defined by \( h\alpha = (1-q)\alpha \) for \( \alpha \in \mathbb{C} \). Namely, we have \( z_n(h e_k) = (1-q)z_n(e_k) \) for
an extended index $k = (k_1, \ldots, k_r) \in \mathbb{N}^r$, where by definition

$$z_n(e_k) = \sum_{n > m_1 > \cdots > m_r > 0} \prod_{a=1}^r F_{k_a}(m_a).$$

In [1], we studied the value $z_n(e_k)$ for an index $k$ (not extended index) at $q = \zeta_n$ a primitive $n$-th root of unity. Note that the substitution $q = \zeta_n$ does not make sense if $\zeta_n$ is not primitive. For an index $k = (k_1, \ldots, k_r)$, we write

$$z_n(k; q) := z_n(e_k) = \sum_{n > m_1 > \cdots > m_r > 0} \prod_{a=1}^r \frac{q^{(k_a - 1)m_a}}{[m_a]^{k_a}}.
$$

**Theorem 4.3.** [1, Theorems 1.1 and 1.2]

(i) For any index $k \in \mathbb{N}^r$ we have

$$(z_p(k; \zeta_p) \mod (1 - \zeta_p)\mathbb{Z}[\zeta_p])_p = \zeta_A(k) \in A.$$

(ii) For any index $k = (k_1, \ldots, k_r) \in \mathbb{N}^r$ the limit

$$\xi(k) = \lim_{n \to \infty} z_n(k; e^{2\pi i/n})$$

exists, and it holds that $\text{Im}\xi(k) \in \pi\mathbb{Z}$ and

$$\text{Re}\xi(k) \equiv \zeta_S(k) \mod \zeta(2)\mathbb{Z}.$$

**4.3. Proof of (1.3).** We first prove the following theorem and then (1.3).

**Theorem 4.4.** For any index $k = (k_1, \ldots, k_r) \in \mathbb{N}^r$ with $r \geq 2$ and primitive $n$-th root of unity $\zeta_n$, it holds that

$$\omega_n(k; \zeta_n) = (-1)^{k_r} \sum_{j=1}^{k_r} \binom{k_r - 1}{j - 1} (1 - q)^{k_r - j} z_n(a^j(e_{k_1} \uplus \cdots \uplus e_{k_{r-1}})) |_{q = \zeta_n}.$$

**Proof.** It is easily seen that the value $\omega_n(k; q)$ is equal to the coefficient of $t^n$ in the product $\prod_{a=1}^r L_{k_a}(t)$ of $q$-single polylogarithms. From Proposition 4.2 it holds that

$$\prod_{a=1}^r L_{k_a}(t) = L(e_{k_r})L(e_{k_1} \uplus \cdots \uplus e_{k_{r-1}}).$$

Hence by definition (4.1) we find that

$$\omega_n(k; q) = \sum_{m=1}^{n-1} u_{n-m}(e_{k_r}) u_m(e_{k_1} \uplus \cdots \uplus e_{k_{r-1}}).$$
For $1 \leq m < n$ and $k \in \mathbb{N}$, we compute
\[
\left. u_{n-m}(e_k) \right|_{q=\zeta_n} = \left. F_k(n-m) \right|_{q=\zeta_n} = (-1)^k \left. \frac{q^m}{[m]^k} \right|_{q=\zeta_n} = (-1)^k \sum_{j=1}^{k} \binom{k-1}{j-1} (1-q)^{k-j} \left. \frac{q^j}{[m]^j} \right|_{q=\zeta_n}.
\]

Note that the element $e_{k_1} \cup_h \cdots \cup_h e_{k_r-1}$ belongs to the $C$-submodule $\sum_{l \in \mathbb{N}} e_l \mathcal{H}$. Using the equality
\[
\left. \frac{q^m}{[m]^j} F_l(m) \right|_{q=\zeta_n} = \left. F_{l+j}(m) \right|_{q=\zeta_n}
\]
and $a^j e_l = e_{l+j}$ ($j, l \in \mathbb{N}$), we see that
\[
\left. \frac{q^m}{[m]^j} u_m(e_{k_1} \cup_h \cdots \cup_h e_{k_r-1}) \right|_{q=\zeta_n} = \left. u_m(a^j(e_{k_1} \cup_h \cdots \cup_h e_{k_r-1})) \right|_{q=\zeta_n}.
\]
Therefore by definition (4.2) we obtain
\[
\omega_n(k; \zeta_n) = (-1)^{k_r} \sum_{j=1}^{k_r} \binom{k_r-1}{j-1} (1-q)^{k_r-j} \sum_{m=1}^{n-1} \left. u_m(a^j(e_{k_1} \cup_h \cdots \cup_h e_{k_r-1})) \right|_{q=\zeta_n} = (-1)^{k_r} \sum_{j=1}^{k_r} \binom{k_r-1}{j-1} (1-q)^{k_r-j} \zeta_n(a^j(e_{k_1} \cup_h \cdots \cup_h e_{k_r-1})) \big|_{q=\zeta_n}.
\]
This completes the proof. \hfill \Box

We are now in a position to prove (1.3).

**Proof of (1.3).** When $p$ is prime and $q = \zeta_p$, for $m < p$ it holds that $F_k(m) \equiv m^{-1}$ mod $(1-\zeta_p)\mathbb{Z}[\zeta_p]$ and $F_k(m) \equiv m^{-k}$ mod $(1-\zeta_p)\mathbb{Z}[\zeta_p]$ for any $k \in \mathbb{N}$. Hence, under the identification $\mathbb{Z}[\zeta_p]/(1-\zeta_p)\mathbb{Z}[\zeta_p] = \mathbb{F}_p$, for any $w \in \mathcal{H}$ it holds that
\[
\left. (z_p(w) \right|_{q=\zeta_p} \mod (1-\zeta_p)\mathbb{Z}[\zeta_p])_p = \zeta_A(\rho(w)),
\]
where $\rho : \mathcal{H}^1 \rightarrow \mathfrak{h}^1$ is the $\mathbb{Q}$-algebra homomorphism given by
\[
\rho(h) = 0, \quad \rho(e_1) = y_1, \quad \rho(e_k) = y_k \ (k \in \mathbb{N}).
\]
Note that $\mathcal{H}^1$ (resp. $\mathfrak{h}^1$) is invariant under the left multiplication with $a$ (resp. $x_0$) because $ae_1 = e_2 - he_1$, $ae_k = e_{k+1}$ and $x_0y_k = y_{k+1}$ for $k \in \mathbb{N}$. These relations imply that
\[
\rho(ae_k) = x_0\rho(e_k)
\]
for any non-empty extended index \( k \in \mathbb{N}^r \). Hence, by Theorem 4.4, for any index \( k = (k_1, \ldots, k_r) \in \mathbb{N}^r \) we obtain

\[
(\omega(k; \zeta_p) \mod (1 - \zeta_p)\mathbb{Z}[\zeta_p])_p = (-1)^{k_1} \zeta_A(x_0^{k_r} \rho(e_{k_1} \cup \cdots \cup e_{k_{r-1}})).
\]

Because of Theorem 4.3 (i), it suffices to show that

\[
\rho(w_1 \cup \cdots \cup w_2) = \rho(w_1) \cup \rho(w_2)
\]

for any \( w_1, w_2 \in \hat{\mathcal{H}}^1 \). This is done by induction on the sum of weights (degrees) of \( w_1 \) and \( w_2 \). We may assume that \( w_1 \) and \( w_2 \) are monomials in \( \{e_k\}_{k \in \mathbb{N}} \). If \( w_1 = 1 \) or \( w_2 = 1 \), it is trivial. Hence we consider the case where \( w_j = e_k \cdot w'_j \) (\( j = 1, 2 \)) for some \( k_j \in \mathbb{N} \).

First, we consider the case \( k_1 = k_2 = \hat{1} \). Then

\[
w_1 \cup \cdots \cup n \cdot w_2 = abw'_1 \cup \cdots \cup abw'_2 = a(bw'_1 \cup \cdots \cup bw'_2 + abw'_1 \cup \cdots \cup bw'_2) + e_1(w'_1 \cup \cdots \cup w'_2 + e_1 w'_1 \cup \cdots \cup w'_2) + e_1(e_1 - e_1)(w'_1 \cup \cdots \cup w'_2).
\]

Using the induction hypothesis and \( \rho(e_1) = \rho(e_1) = y_1 = x_1 \), we see that

\[
\rho(w_1 \cup \cdots \cup w_2) = x_1\rho(w'_1) \cup x_1\rho(w'_2) + x_1\rho(w'_1) \cup \rho(w'_2) = x_1\rho(w'_1) \cup \rho(e_1 w'_2) = \rho(w_1) \cup \rho(w_2).
\]

For the case \( (k_1, k_2) = (1, \hat{1}) \) or \( (1, 1) \), we obtain the desired equality in a similar way by using

\[
(e_1 w'_1) \cup \cdots \cup (e_1 w'_2) = e_1(w'_1 \cup \cdots \cup e_1 w'_2 + e_1 w'_1 \cup \cdots \cup w'_2) + e_1(e_1 - e_1)(w'_1 \cup \cdots \cup w'_2),
\]

\[
(e_1 w'_1) \cup \cdots \cup (e_1 w'_2) = e_1(w'_1 \cup \cdots \cup e_1 w'_2 + e_1 w'_1 \cup \cdots \cup w'_2) + e_1(e_1 - e_1)(w'_1 \cup \cdots \cup w'_2).
\]

Next, we consider the case where \( k_1 = \hat{1} \) and \( k_2 \geq 2 \). In this case, one can write \( w_2 = aw''_2 \) with \( w''_2 \in \hat{\mathcal{H}}^1 \). Hence we see that

\[
w_1 \cup \cdots \cup w_2 = abw'_1 \cup \cdots \cup abw''_2 = e_1 w'_1 \cup \cdots \cup w'_2 + a(w'_1 \cup \cdots \cup w''_2) + he_1(w'_1 \cup \cdots \cup w''_2).
\]

Hence the induction hypothesis and (4.4) imply that

\[
\rho(w_1 \cup \cdots \cup w_2) = x_1\rho(w'_1) \cup \rho(w'_2) + x_0\rho(w'_2) \cup \rho(w''_2) = x_1\rho(w'_1) \cup \rho(w''_2) + x_0\rho(w'_1) \cup \rho(w''_2) = x_1\rho(w'_1) \cup \rho(w''_2) = \rho(w_1) \cup \rho(w_2).
\]

The case \( k_1 = 1 \) and \( k_2 \geq 2 \) is a consequence of the above case, since

\[
e_1 w'_1 \cup \cdots \cup w'_2 = e_1 w'_1 \cup \cdots \cup w'_2 + bw''_1 \cup \cdots \cup w'_2
\]
In [9], Kamano introduced the finite Mordell-Tornheim multiple zeta
For any index the
Remark 4.5.
(4.5)
and, as a finite analogue of (2.5), proved that
This completes the proof. □
Finally, we consider the case where \( k_j \geq 2 \) \( (j = 1, 2) \). We write \( w_j = a w''_j \) with \( w''_j \in \tilde{\mathcal{S}}^1 \). Then it holds that
Using the induction hypothesis and (4.4), we see that
\[
\rho(w_1 \sqcup h w_2) = x_0(\rho(w''_1) \sqcup x_0 \rho(w''_2) + x \rho(w''_1) \sqcup \rho(w''_2)) = x_0 \rho(w''_1) \sqcup x_0 \rho(w''_2)
\]
\[
= \rho(w_1) \sqcup \rho(w_2).
\]
This completes the proof.

Remark 4.5. In [9], Kamano introduced the finite Mordell-Tornheim multiple zeta
value for an index \( \mathbf{k} = (k_1, \ldots, k_r) \in \mathbb{N}^r \) by
\[
\zeta^\text{MT}_A(\mathbf{k}) = \left( \sum_{m_1, \ldots, m_{r-1} \geq 1 \atop m_1 + \cdots + m_{r-1} < p} \frac{1}{m_1 \cdots m_{r-1} (m_1 + \cdots + m_{r-1})^{k_r}} \right) \equiv_{\mathbb{Z}/p} \in \mathcal{A}
\]
and, as a finite analogue of (2.5), proved that
\[
(4.5) \hspace{1cm} \zeta^\text{MT}_A(\mathbf{k}) = \zeta_A(x_0^{k_r}(y_{k_1} \sqcup \cdots \sqcup y_{k_{r-1}})).
\]
Since \( \omega_A(\mathbf{k}) = (-1)^{k_r} \zeta^\text{MT}_A(\mathbf{k}) \) for each index \( \mathbf{k} = (k_1, \ldots, k_r) \in \mathbb{N}^r \), Theorem 4.4 can be viewed as a lift of Kamano’s result (4.5) to the values \( \omega_n(\mathbf{k}; \zeta_n) \).

4.4. Proof of (1.4). The proof of (1.4) is based on identities of words. Let \( \phi : \mathfrak{h}^1 \to \mathbb{R} \) be the \( \mathbb{Q} \)-linear map given by \( \phi(1) = 1 \) and
\[
(4.6) \hspace{1cm} \phi(y_{k_1} \cdots y_{k_r}) = \sum_{a=0}^{r} (-1)^{k_1 + \cdots + k_a} (y_{k_a y_{k_{a-1}} \cdots y_{k_1}} \sqcup (y_{k_{a+1} y_{k_{a+2}} \cdots y_{k_r}})).
\]

Theorem 4.6. For any index \( (k_1, \ldots, k_r) \in \mathbb{N}^r \) with \( r \geq 2 \), we have
\[
\phi((-1)^{k_r} x_0^{k_r}(y_{k_1} \sqcup \cdots \sqcup y_{k_{r-1}})) = \sum_{a=1}^{r} (-1)^{k_a} x_0^{k_a}(y_{k_1} \sqcup \cdots \sqcup y_{k_{a-1}} \sqcup y_{k_{a+1}} \sqcup \cdots \sqcup y_{k_r}).
\]
First, we give a proof of (1.4), and then prove Theorem 4.6 below.

Proof of (1.4). Since $\zeta^\omega : \mathfrak{h}_1^1 \to \mathbb{R}$ is a homomorphism, it follows from the definition (2.3) that $\zeta^\omega(w) = \zeta^\omega(\phi(w))$ for any $w \in \mathfrak{h}_1^1$. By Theorem 4.6 and (2.5), we see that
\[
(-1)^k \zeta^\omega \left( x^k_0 (y_{k_1} \sqcup \cdots \sqcup y_{k_r}) \right) = \zeta^\omega \left( \phi \left( (-1)^{kr} x^k_0 (y_{k_1} \sqcup \cdots \sqcup y_{k_r}) \right) \right) \\
= \sum_{a=1}^{r} (-1)^{k_a} \zeta^\omega \left( x^{k_a}_0 (y_{k_1} \sqcup \cdots \sqcup y_{k_a-1} \sqcup y_{k_a+1} \sqcup \cdots \sqcup y_{k_r}) \right) \\
= \sum_{a=1}^{r} (-1)^{k_a} \zeta^{MT}(k_1, \ldots, k_{a-1}, k_{a+1}, \ldots, k_r; k_a),
\]
which by Theorem 1.3 is equal to $\Omega(k_1, \ldots, k_r)$. Taking modulo $\zeta(2)$, we get the desired result. \hfill $\square$

In what follows, we will prove Theorem 4.6 by calculating the generating series of both sides. For that purpose we introduce the element
\[
y(t) = \sum_{k=1}^{\infty} y_k t^{k-1} = \frac{1}{1 - x_0 t} x_1 \in \mathfrak{h}_1^1[[t]]
\]
with the indeterminate $t$. We begin with the following lemma, which is used to prove Propositions 4.8, 4.9 and 4.10 below, and then, give a proof of Theorem 4.6.

Lemma 4.7. For $w, w' \in \mathfrak{h}_1^1$ it holds that
\[
y(t_1)w \sqcup y(t_2)w' = y(t_1 + t_2)(w \sqcup y(t_2)w' + y(t_1)w \sqcup w').
\]

Proof. We denote the left side by $I$. Noting $y(t) = x_1 + tx_0 y(t)$, one computes
\[
I = (x_1 + t_1 x_0 y(t_1))w \sqcup (x_1 + t_2 x_0 y(t_2))w' \\
= x_1 w \sqcup x_1 w' + t_1 x_0 y(t_1)w \sqcup x_1 w' + x_1 w \sqcup t_2 x_0 y(t_2)w' + t_1 x_0 y(t_1)w \sqcup t_2 x_0 y(t_2)w' \\
= x_1 (w \sqcup x_1 w') + x_1 (x_1 w \sqcup w') \\
+ t_1 x_0(y(t_1)w \sqcup x_1 w') + t_1 x_1(x_0 y(t_1)w \sqcup w') \\
+ t_2 x_1(w \sqcup x_0 y(t_2)w') + t_2 x_0(x_1 w \sqcup y(t_2)w') \\
+ t_1 t_2 x_0(y(t_1)w \sqcup x_0 y(t_2)w') + t_1 t_2 x_0(x_0 y(t_1)w \sqcup y(t_2)w') \\
= x_0 \{ t_1 y(t_1)w \sqcup x_1 w' + x_1 w \sqcup t_2 y(t_2)w' + t_1 y(t_1)w \sqcup x_0 t_2 y(t_2)w' + t_1 x_0 y(t_1)w \sqcup t_2 y(t_2)w' \} \\
+ x_1 \{ w \sqcup x_1 w' + x_1 w \sqcup w' + x_0 t_1 y(t_1)w \sqcup w' + w \sqcup x_0 t_2 y(t_2)w' \}
\]
= x_0 (t_1 y(t_1) \shuffle y(t_2) w' + t_2 y(t_1) w \shuffle y(t_2) w') \\
+ x_1 (w \shuffle y(t_2) w' + y(t_1) w \shuffle w') \\
= (t_1 + t_2) x_0 I + x_1 (w \shuffle y(t_2) w' + y(t_1) w \shuffle w').

Hence, \( (1 - (t_1 + t_2)x_0) I = x_1 (w \shuffle y(t_2) w' + y(t_1) w \shuffle w') \), which implies the desired formula. 

From Lemma 4.7 one can compute

\[
y(t_1) \shuffle y(t_2) = y(t_1 + t_2) (y(t_2) + y(t_1))
\]

and

\[
(y(t_1) \shuffle y(t_2)) \shuffle y(t_3) = y(t_1 + t_2) (y(t_2) + y(t_1)) \shuffle y(t_3)
\]

\[
= y(t_1 + t_2) y(t_1) \shuffle y(t_3) + y(t_1 + t_2) y(t_2) \shuffle y(t_3)
\]

\[
= y(t_1 + t_2 + t_3) (y(t_1 + t_3) y(t_1) + y(t_1 + t_3) y(t_1) + y(t_1 + t_3) y(t_1))
\]

\[
+ y(t_1 + t_2 + t_3) (y(t_2 + t_3) y(t_2) + y(t_2 + t_3) y(t_2))
\]

\[
= y(t_1 + t_2 + t_3) y(t_1 + t_2) y(t_1) + y(t_1 + t_2 + t_3) y(t_1 + t_2) y(t_1)
\]

\[
+ y(t_1 + t_2 + t_3) y(t_1 + t_2) y(t_2) + y(t_1 + t_2 + t_3) y(t_1 + t_2) y(t_2)
\]

\[
+ y(t_1 + t_2 + t_3) y(t_2 + t_3) y(t_3) + y(t_1 + t_2 + t_3) y(t_1 + t_3) y(t_3).
\]

Using Lemma 4.7 repeatedly, we obtain the following formula for the shuffle product of \( y(t) \):

**Proposition 4.8.** It holds that

\[
y(t_1) \shuffle \cdots \shuffle y(t_r) = \sum_{\sigma \in S_r, 1 \leq j \leq r} \prod_{i=1}^j y \left( \sum_{i=1}^j t_{\sigma(i)} \right),
\]

where \( \prod_{m \leq j \leq n} X_j \) denotes the ordered product \( X_n X_{n-1} \cdots X_m \) for \( m \leq n \).

We denote by \([r]\) the set \( \{1, 2, \ldots, r\} \) for \( r \geq 1 \). For a subset \( I = \{p_1, \ldots, p_s\} \) of \([r]\) with \( |I| = s \), we define

\[
S_I(t_1, \ldots, t_r) = y(t_{p_1}) \shuffle \cdots \shuffle y(t_{p_s})
\]

\[
= \sum_{\sigma \in S_s, 1 \leq j \leq s} \prod_{i=1}^j y \left( \sum_{i=1}^j t_{p_{\sigma(i)}} \right).
\]
Proposition 4.9. For any subset $I = \{ p_1, \ldots, p_s \}$ of $[r]$ with $|I| = s$, it holds that
\[
\phi(y(u)S_I(t_1, \ldots, t_r)) = y(u)S_I(t_1, \ldots, t_r) - y(-u) \sqcup S_I(t_1, \ldots, t_r)
+ \sum_{b \in I} y(-t_b) \left( y(-u) \sqcup S_{I \setminus \{ b \}}(t_1, \ldots, t_r) \right).
\]

Proof. From the definition (4.6) of the map $\phi$, we see that
\[
\phi(y(s_1) \cdots y(s_r)) = \sum_{k_1, \ldots, k_r \geq 1} \phi(y_{k_1} \cdots y_{k_r}) s_1^{k_1-1} \cdots s_r^{k_r-1}
= \sum_{a=0}^{r} (-1)^{k_1+\cdots+k_a} \sum_{k_1, \ldots, k_r \geq 1} y_{k_a} \cdots y_{k_1} \sqcup y_{k_{a+1}} \cdots y_{k_r} s_1^{k_1-1} \cdots s_r^{k_r-1}
= \sum_{a=0}^{r} (-1)^a (\prod_{1 \leq j \leq a} y(-s_j)) \sqcup \left( \prod_{s+1 \leq j \leq r} y(s_j) \right).
\]

For simplicity we set $\alpha_j^\sigma = \sum_{i=1}^{j} t_{p_i(i)}$ for $1 \leq j \leq s$ and $\sigma \in \mathcal{S}_s$. Proposition 4.8 and the above formula imply that
\[
\phi(y(u)S_I(t_1, \ldots, t_r))
= \sum_{\sigma \in \mathcal{S}_s} \phi(y(u)y(\alpha_s^\sigma) \cdots y(\alpha_1^\sigma))
= \sum_{\sigma \in \mathcal{S}_s} y(u)y(\alpha_s^\sigma) \cdots y(\alpha_1^\sigma)
+ (-1)^s \sum_{\sigma \in \mathcal{S}_s} y(-u) \sqcup y(\alpha_s^\sigma) \cdots y(\alpha_1^\sigma)
+ \sum_{a=2}^{s+1} (-1)^a \sum_{\sigma \in \mathcal{S}_s} \left( \left\{ \prod_{s+2 \leq j \leq s} y(-\alpha_j^\sigma) \right\} y(-u) \sqcup \left( \prod_{1 \leq j \leq s-a+1} y(\alpha_j^\sigma) \right) \right).
\]

Applying Lemma 4.7 except for the first two terms and for the last term with $a = s+1$, we obtain
\[
= y(u)S_I(t_1, \ldots, t_r) - y(-u) \sqcup S_I(t_1, \ldots, t_r)
+ \sum_{\sigma \in \mathcal{S}_s} \sum_{a=2}^{s} (-1)^a y(-t_{p_{a+1}(a+2)}) \left( \left\{ \prod_{s+2 \leq j \leq s} y(-\alpha_j^\sigma) \right\} y(-u) \sqcup \left( \prod_{1 \leq j \leq s-a} y(\alpha_j^\sigma) \right) \right).
\]
For any subset $I$ of $[r]$, it holds that

$$\phi(S_I(t_1, \ldots, t_r)) = S_I(t_1, \ldots, t_r) - \sum_{b \in I} y(-t_b) S_{I \setminus \{b\}}(t_1, \ldots, t_r).$$

**Proof.** We use the same notation with the proof of Proposition 4.9. Replacing $y(u)$ with 1 in (4.7) and doing the same game, we get

$$\phi(S_I(t_1, \ldots, t_r))$$

$$= \sum_{\sigma \in \mathcal{S}_s} \sum_{a=0}^{s} (-1)^a \left( \prod_{s-a+1 \leq j \leq s} y(-\alpha_j^\sigma) \right) \cup \left( \prod_{1 \leq j \leq s-a+1} y(\alpha_j^\sigma) \right)$$

$$= \sum_{\sigma \in \mathcal{S}_s} \prod_{1 \leq j \leq s} y(\alpha_j^\sigma)$$

$$+ \sum_{\sigma \in \mathcal{S}_s} \sum_{a=1}^{s-1} (-1)^a y(-t_{p(a+1)}(s-a+1)) \left( \prod_{s-a+2 \leq j \leq s} y(-\alpha_j^\sigma) \right) \cup \left( \prod_{1 \leq j \leq s-a} y(\alpha_j^\sigma) \right)$$

In the third term with $3 \leq a \leq s$, change $a \to a+1$ and $\sigma \to \sigma' = \sigma(s-a+2, s-a+1)$, where $(s-a+2, s-a+1) \in \mathcal{S}_s$ is the transposition. Then, since $\alpha_j^{\sigma'} = \alpha_j^{\sigma}$ ($j \neq s-a+1$), it cancels with the second term except for the term with $a = s$. We see that the second term with $a = s$ also cancels with the last term by changing $\sigma \to \sigma(1, 2)$. For the third term with $a = 2$, decomposing the range $\mathcal{S}_s$ as $\bigcup_{l=1}^{s} \mathcal{S}^l_s$, where $\mathcal{S}^l_s = \{ \sigma \in \mathcal{S}_s \mid \sigma(s) = l \}$, we get

$$= y(u) S_I(t_1, \ldots, t_r) - y(-u) \cup S_I(t_1, \ldots, t_r)$$

$$+ \sum_{l=1}^{s} y(-t_{p(l)}(s-l+1)) \cup \sum_{\sigma \in \mathcal{S}^l_s} y(\alpha_j^{\sigma}) \cdots y(\alpha_1^{\sigma})$$

$$= y(u) S_I(t_1, \ldots, t_r) - y(-u) \cup S_I(t_1, \ldots, t_r)$$

$$+ \sum_{b \in I} y(-t_b) y(-u) \cup S_{I \setminus \{b\}}(t_1, \ldots, t_r),$$

which completes the proof. \(\square\)

In the same way as above, we obtain the following formula.

**Proposition 4.10.** For any subset $I$ of $[r]$, it holds that

$$\phi(S_I(t_1, \ldots, t_r)) = S_I(t_1, \ldots, t_r) - \sum_{b \in I} y(-t_b) S_{I \setminus \{b\}}(t_1, \ldots, t_r).$$
Let \( \{z_1, \ldots, z_n\} \subset \mathbb{C} \). From Proposition 4.8, we obtain

\[
\sum_{\sigma \in \mathfrak{S}_n} (-1)^{s} (\prod_{1 \leq j \leq n} y(\alpha_j^\sigma))
\]

Then, we have

\[
J(k_1, \ldots, k_r, t_r) := \sum_{t_1, \ldots, t_{r-1}} (-1)^{k_r} \sum_{a=1}^{r-1} (y_1 \cup \cdots \cup y_{k_r}) t_1^{k_1-1} \cdots t_{r-1}^{k_{r-1}-1} t_r^{k_r-1}.
\]

We are ready to prove Theorem 4.6.

**Proof of Theorem 4.6.** Let

\[
J(t_1, \ldots, t_{r-1}, t_r) := \sum_{k_1, \ldots, k_r \geq 1} (-1)^{k_r} x_0^{k_r} (y_{k_1} \cup \cdots \cup y_{k_r}) t_1^{k_1-1} \cdots t_{r-1}^{k_{r-1}-1} t_r^{k_r-1}.
\]

Then, we have

\[
\sum_{k_1, \ldots, k_r \geq 1} \sum_{a=1}^{r} (-1)^{k_a} x_0^{k_a} (y_{k_{a-1}} \cup \cdots \cup y_{k_a}) t_1^{k_1-1} \cdots t_{r-1}^{k_{r-1}-1} t_r^{k_r-1}
\]

Therefore, for \( r \geq 2 \), it suffices to show that

\[
\phi(J(t_1, \ldots, t_r)) = \sum_{a=1}^{r} J(t_1, \ldots, t_{a-1}, t_{a+1}, \ldots, t_r, t_a).
\]

For simplicity we set

\[
\alpha_j = \sum_{i=1}^{k} t_i \ (1 \leq j \leq r) \quad \text{and} \quad \alpha_j^\sigma = \sum_{i=1}^{k} t_{\sigma(i)} \ (1 \leq j \leq r-1, \sigma \in \mathfrak{S}_{r-1}).
\]

From Proposition 4.8, we obtain

\[
J(t_1, \ldots, t_r) = -\frac{x_0}{1 + x_0 t_r} (y(t_1) \cup \cdots \cup y(t_{r-1}))
\]

\[
= -\frac{x_0}{1 + x_0 t_r} \sum_{\sigma \in \mathfrak{S}_{r-1}} y(\alpha_{r-1}^\sigma) \cdots y(\alpha_2^\sigma) y(\alpha_1^\sigma)
\]
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again by Proposition 4.8 it can be reduced to

\[
\sum_{\sigma \in \mathfrak{S}_{r-1}} \frac{x_0}{1 + x_0 t_r} y(\alpha_{r-1}) y(\alpha_{r-2}) \cdots y(\alpha_2) y(\alpha_1).
\]

Since

\[
\frac{x_0}{1 + x_0 t_r} y(\alpha_{r-1}) = -\frac{1}{\alpha_r} \left( \frac{1}{1 + x_0 t_r} x_1 - \frac{1}{1 - x_0 \alpha_{r-1}} x_1 \right)
\]

again by Proposition 4.8 it can be reduced to

\[
\sum_{\sigma \in \mathfrak{S}_{r-1}} \frac{x_0}{1 + x_0 t_r} y(\alpha_{r-1}) y(\alpha_{r-2}) \cdots y(\alpha_2) y(\alpha_1)
\]

Likewise, Proposition 4.10 for the case \(\phi(S_{[r]})(t_1, \ldots, t_r)\) gives

\[
\phi \left( y(-t_r) S_{[r]} \right) = \bigcup_{a \in [r-1]} y(t_r) S_{[a]}(t_1, \ldots, t_{r-1})
\]

Applying \(\phi\), Proposition 4.9 for the case \(u = -t_r\) and \(I = [r] \setminus \{a, r\}\) gives

\[
\phi \left( y(-t_r) S_{[r]} \right) = \bigcup_{a \in [r-1]} y(t_r) S_{[a]}(t_1, \ldots, t_{r-1}) + \bigcup_{b \in \{r\}} y(t_r) S_{[b]}(t_1, \ldots, t_r)
\]

Likewise, Proposition 4.10 for the case \(I = [r] \setminus \{r\}\) shows

\[
\phi \left( S_{[r]} \right) = \bigcup_{a \in [r-1]} y(t_r) S_{[a]}(t_1, \ldots, t_{r-1}) + \bigcup_{b \in \{r\}} y(t_r) S_{[b]}(t_1, \ldots, t_r).
\]
Therefore we have
\begin{align*}
\phi(J(t_1, \ldots, t_r)) &= \frac{1}{\alpha_r} \sum_{a \in [r-1]} \left( \sum_{b \in [r] \setminus \{a\}} y(-t_b) S_{[r] \setminus \{a,b\}}(t_1, \ldots, t_r) - S_{[r] \setminus \{a\}}(t_1, \ldots, t_r) \right) \\
&\quad - \frac{1}{\alpha_r} \left( S_{[r] \setminus \{r\}}(t_1, \ldots, t_r) - \sum_{b \in [r] \setminus \{r\}} y(-t_b) S_{[r] \setminus \{b,r\}}(t_1, \ldots, t_r) \right) \\
&= \frac{1}{\alpha_r} \sum_{a \in [r]} \left( \sum_{b \in [r] \setminus \{a\}} y(-t_b) S_{[r] \setminus \{a,b\}}(t_1, \ldots, t_r) - S_{[r] \setminus \{a\}}(t_1, \ldots, t_r) \right) \\
&= \sum_{a=1}^r J(t_1, \ldots, t_{a-1}, \underbrace{t_{a+1}, \ldots, t_r}_{r-a}, t_a),
\end{align*}
which proves (4.8). We complete the proof. \(\square\)

4.5. **Remark on the value** \(\Omega(k)\). In the proof of (1.4), we actually proved the equality
\begin{equation}
\Omega(k) = (-1)^k \zeta_S^w(x_0^{k_r} (y_{k_1} \sqcup \cdots \sqcup y_{k_{r-1}}))
\end{equation}
for any index \(k = (k_1, \ldots, k_r) \in \mathbb{N}^r\). Accordingly, relations of \(\Omega(k)\) give rise to relations of \(\zeta_S^w(k)\) (without taking modulo \(\zeta(2)\)). For example, from (1.2) and \([16, (5)]\), we have
\begin{equation}
\Omega(\{1\}^k) = -k \zeta^{MT}(\{1\}^{k-1}; 1) = -k! \zeta(k),
\end{equation}
where \(\{1\}^k\) means a sequence of 1 repeated \(k\) times. On the other hand, it follows that
\begin{equation*}
\zeta_S^w(x_0(y_1 \sqcup \cdots \sqcup y_{k_1})) = (k - 1)! \zeta_S^w(2, \{1\}^{k-2}).
\end{equation*}
Therefore, we obtain
\begin{equation*}
\zeta_S^w(2, \{1\}^{k-2}) = k \zeta(k).
\end{equation*}

Similarly to Conjecture 2.4, we observed that all multiple zeta value up to weight 12 can be written as \(\mathbb{Q}\)-linear combinations of \(\Omega(k)\)'s.

**Conjecture 4.11.** *The space \(\mathcal{Z}\) is generated by the set \(\{\Omega(k) \mid k : \text{index}\}\).*

We further remark that Ono, Seki and Yamamoto [19] introduced the ‘\(t\)-adic’ symmetric Mordell-Tornheim multiple zeta value as a counterpart of Kamano’s finite Mordell-Tornheim multiple zeta value. It is defined for an index \(k = (k_1, \ldots, k_r, k_{r+1})\)
by
\[ \zeta_{S}^{MT}(k; t) = \lim_{M \to \infty} \sum_{i=1}^{r+1} \zeta_{S,M}(i; k; t), \]
where for \(1 \leq i \leq r\) we set
\[ \zeta_{S,M}(i; k; t) = \sum_{m_1, \ldots, m_{i-1} > 0, m_{i+1}, \ldots, m_{r+1} > 0} \frac{1}{(t - M_{i-1} - M_{i+1, r+1})^{k_i} (t - m_{r+1})^{k_{r+1}}} \prod_{j=1, j \neq i}^{r} \frac{1}{m_j^{k_j}} \]
with \(M_0 = 0, M_i = m_1 + \cdots + m_i\) for \(1 \leq i \leq r + 1\), \(M_{i,j} = m_i + m_{i+1} + \cdots + m_j\) for \(i \leq j\) and \(M_{i,j} = 0\) if \(i > j\), and also let
\[ \zeta_{S,M}(r+1; k; t) = \sum_{m_1, \ldots, m_r > 0, M_r < M} \frac{1}{m_1^{k_1} \cdots m_r^{k_r} M_r^{k_{r+1}}}. \]

Letting \(t \to 0\) (or taking the constant term), we see that
\[ \zeta_{S}(k; 0) = (-1)^{k_{r+1}} \sum_{a=1}^{r+1} (-1)^{k_a} \zeta_{S,M}(k_1, \ldots, k_{a-1}, k_a, k_{a+1}, \ldots, k_{r+1}; k_a). \]

Combining this with (1.2), we obtain another expression of \(\Omega(k)\) as follows:
\[ \Omega(k_1, \ldots, k_r) = \lim_{M \to \infty} \sum_{i=1}^{r} (-1)^{k_i} \zeta_{S,M}(i; k_1, \ldots, k_r; 0). \]

The authors were informed that in [19], they prove a truncated version of the formula
\[ \sum_{a=1}^{r} (-1)^{k_a} \zeta_{S}(k_1, \ldots, k_{a-1}, k_{a+1}, \ldots, k_r; k_a) = (-1)^{k_r} \zeta_{S}(x_0^{k_r} (y_{k_1} \sqcup \cdots \sqcup y_{k_{r-1}})), \]
which provides an alternative proof of (4.9).

5. Further remarks

5.1. Relations of \(\omega_n(k; \zeta_n)\). In this subsection, we consider relations among the values \(\omega_n(k; \zeta_n)\) which are valid for all \(n\), as well as its application to relations among finite and symmetric multiple omega values.

For that purpose we consider the \(\mathbb{Q}\)-algebra
\[ \mathcal{R} = \prod_{n \geq 1} \mathbb{Q}(\zeta_n). \]
In this $\mathbb{Q}$-algebra, we will count the number of linearly independent relations of the form

$$\sum_{k:\text{index}} a_k(m) (1 - \zeta_n)^m \omega_n(k; \zeta_n) = 0 \quad (a_k(m) \in \mathbb{Q})$$

which holds for all $n$. It is apparent that the relation (5.1) induces relations

$$\sum_{k:\text{index}} a_k(0) \omega(k) = 0 \quad \text{and} \quad \sum_{k:\text{index}} a_k(0) \Omega(k) = 0,$$

by applying Theorems 1.2 and 1.3. Hence, finding relations of the form (5.1) will be a first step toward to capture all relations among finite and symmetric multiple omega values (note that the coefficient $a_k(m)$ could be a polynomial in $n$ over $\mathbb{Q}$, but we get rid of such relations below). For an index $k$, we then consider a universal object

$$\omega(k) = (\omega_n(k; e^{\frac{2\pi i}{m}}))_{n \geq 1} \in \mathcal{R},$$

which we call a cyclotomic multiple omega value.

We give a family of relations among the cyclotomic multiple omega values. Set $\zeta = (\zeta_n)_{n \geq 1} \in \mathcal{R}$.

**Theorem 5.1.** Suppose that $k_1, \ldots, k_r \geq 2$. Then it holds that

$$\sum_{j=1}^{r} \sum_{l_1, \ldots, l_{j-1} \geq 2} \sum_{l_j, \ldots, l_r \geq 1} \left\{ \prod_{p=1}^{j-1} \left( \frac{k_p - 2}{l_p - 2} \right) \right\} \left( \frac{k_j - 2}{l_j - 1} \right) \left\{ \prod_{p=j+1}^{r} \left( \frac{k_p - 1}{l_p - 1} \right) \right\}$$

$$\times (1 - \zeta) \sum_{p=1}^{r} (k_p - l_p)^{-1} \omega(l_1, \ldots, l_r) = 0.$$

**Proof.** We show that for all $n \geq 1$ and primitive roots of unity $\zeta_n$ we have

$$\sum_{j=1}^{r} \sum_{l_1, \ldots, l_{j-1} \geq 2} \sum_{l_j, \ldots, l_r \geq 1} \left\{ \prod_{p=1}^{j-1} \left( \frac{k_p - 2}{l_p - 2} \right) \right\} \left( \frac{k_j - 2}{l_j - 1} \right) \left\{ \prod_{p=j+1}^{r} \left( \frac{k_p - 1}{l_p - 1} \right) \right\}$$

$$\times (1 - \zeta) \sum_{p=1}^{r} (k_p - l_p)^{-1} \omega_n(l_1, \ldots, l_r; \zeta_n) = 0.$$

For $m_1, \ldots, m_r \geq 1$ it holds that

$$\sum_{j=1}^{r} q^{m_1 + \cdots + m_{j-1}} [m_j] = [m_1 + \cdots + m_r].$$
Suppose that $\sum_{j=1}^{r} m_j = n$. Divide both sides with $\prod_{p=1}^{r} [m_p]^{k_p}$ and set $q = \zeta_n$. Then we have
\[
\sum_{j=1}^{r} \prod_{p=1}^{j-1} \frac{q^{m_p}}{[m_p]^{k_p}} \prod_{p=j+1}^{r} \frac{1}{[m_p]^{k_p}} \bigg|_{q=\zeta_n} = 0.
\]
Now the desired equality follows from
\[
\frac{q^m}{[m]^k} = \sum_{l \geq 2} \binom{k-2}{l-2} (1-q)^{k-l} F_l(m) \quad (k \geq 2),
\]
\[
\frac{1}{[m]^k} = \sum_{l \geq 1} \binom{k-1}{l-1} (1-q)^{k-l} F_l(m) \quad (k \geq 1).
\]
We complete the proof. \qed

For example, taking $k_1 = k_2 = \cdots = k_r = 2$ we obtain for all $r \geq 2$
\[
\sum_{j=1}^{r} \binom{r}{j} (1-\zeta)^{j-1} \omega(\{2\}^{r-j}, \{1\}^j) = 0. \tag{5.3}
\]
Using Theorems 1.2 and 1.3, we obtain the following.

**Corollary 5.2.** Suppose that $k_1, \ldots, k_r \geq 2$. Then we have
\[
\sum_{j=1}^{r} \omega_A(k_1, \ldots, k_j - 1, \ldots, k_r) = 0,
\]
\[
\sum_{j=1}^{r} \Omega(k_1, \ldots, k_j - 1, \ldots, k_r) = 0. \tag{5.4}
\]

*Proof.* The first equation is an immediate from (5.2) and Theorem 1.2. For the second equation, set $\zeta_n = e^{2\pi i/n}$ in (5.2), and take the limit $n \to 0$. Then the terms which satisfy $l_p = k_p (p \neq j)$ and $l_j = k_j - 1$ only remain, and by Theorem 1.3 we get the desired formula. \qed

Remark that the relation (5.4) can be also deduced from the following relations of the Mordell-Tornheim multiple zeta values. For $k_1, \ldots, k_r, l \geq 2$ we have
\[
\zeta^{MT}(k_1, \ldots, k_r; l-1) = \sum_{m_1, \ldots, m_r \geq 1} \frac{m_1 + \cdots + m_r}{m_1^{k_1} \cdots m_r^{k_r} (m_1 + \cdots + m_r)^l}
\]
\[
= \sum_{a=1}^{r} \zeta^{MT}(k_1, \ldots, k_a - 1, \ldots, k_r; l),
\]
which is found in the literature (see e.g. the proof of [20, Proposition 4.16]).
5.2. **Observation on cyclotomic multiple omega values.** In this subsection, we work on the \( \mathbb{Q} \)-vector subspace \( \mathcal{Z}_k^\omega \) of \( \mathcal{R} \) defined for \( k \geq 1 \) by

\[
\mathcal{Z}_k^\omega = \langle (1 - \zeta)^m \omega(k_1, \ldots, k_r) \mid 0 \leq m < k, 2 \leq r \leq k, k_1 + \cdots + k_r + m = k \rangle_{\mathbb{Q}}.
\]

In the following we discuss the relations among the generators of \( \mathcal{Z}_k^\omega \) and its dimension. From Theorems 1.2 and 1.3, one obtains two well-defined \( \mathbb{Q} \)-linear maps \( \varphi_k^\bullet: \mathcal{Z}_k^\omega \to \mathcal{Z}_k^{\bullet, \omega} \) given for \( \bullet \in \{ A, S \} \) and each generator by

\[
\varphi_k^\bullet((1 - \zeta)^m \omega(k)) = \begin{cases} 
\omega_\bullet(k) & \text{if } m = 0 \\
0 & \text{if } m > 0
\end{cases}.
\]

By definition these maps are surjective, so we get

\[
\mathcal{Z}_k^{A, \omega} \xrightarrow{\varphi_k^A} \mathcal{Z}_k^\omega \xrightarrow{\varphi_k^S} \mathcal{Z}_k^{S, \omega},
\]

where the map \( \varphi_k: \mathcal{Z}_k^{A, \omega} \to \mathcal{Z}_k^{S, \omega} \) denotes the conjectured isomorphism of Conjecture 1.5. This picture suggests that \( \ker \varphi_k^A \cong \ker \varphi_k^S \). Clearly, we have \( (1 - \zeta)\mathcal{Z}_k^{\omega, -1} \subset \ker \varphi_k^\bullet \) for \( \bullet \in \{ A, S \} \), but, besides these, there are more elements in \( \ker \varphi_k^\bullet \), since Theorem 5.1 gives relations among elements in \( \mathcal{Z}_k^\omega \). Numerically, we get the following relations.

**Weight 3:**

\[
\omega(2, 1) = -\frac{1}{2}(1 - \zeta)\omega(1, 1).
\]

**Weight 4:**

\[
\begin{align*}
\omega(3, 1) & \overset{?}{=} \omega(2, 1, 1) + \frac{1}{2}(1 - \zeta)\omega(1, 1, 1) + \frac{1}{4}(1 - \zeta)^2\omega(1, 1), \\
\omega(2, 2) & \overset{?}{=} -\omega(2, 1, 1) - \frac{1}{2}(1 - \zeta)\omega(1, 1, 1) + \frac{1}{4}(1 - \zeta)^2\omega(1, 1).
\end{align*}
\]

**Weight 5:**

\[
\begin{align*}
\omega(4, 1) & \overset{?}{=} -\frac{3}{2}(1 - \zeta)\omega(2, 1, 1) - \frac{3}{4}(1 - \zeta)^2\omega(1, 1, 1) - \frac{1}{8}(1 - \zeta)^3\omega(1, 1), \\
\omega(3, 2) & \overset{?}{=} \frac{1}{2}(1 - \zeta)\omega(2, 1, 1) + \frac{1}{4}(1 - \zeta)^2\omega(1, 1, 1) - \frac{1}{8}(1 - \zeta)^3\omega(1, 1), \\
\omega(2, 2, 1) & = -(1 - \zeta)\omega(2, 1, 1) - \frac{1}{3}(1 - \zeta)^2\omega(1, 1, 1).
\end{align*}
\]
Notice that the first and the last relation are consequences of (5.3). Remember that 
\[ \omega(k_{\sigma(1)}, \ldots, k_{\sigma(r)}) = \omega(k_1, \ldots, k_r) \] holds for any permutation \( \sigma \in \mathcal{S}_r \). The above relations conjecturally give all relations in each weight. Doing numerical calculation until weight 11, we get the following table:

| \( k \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|----------|---|---|---|---|---|---|---|---|---|----|----|
| \( \dim \mathcal{Z}_k^{\omega} \) | 0 | 1 | 2 | 4 | 7 | 12 | 19 | 30 | 45 | 68 | 99 |
| \( \dim \mathcal{Z}_k^{\omega}/(1-\zeta) \mathcal{Z}_k^{\omega} \) | 0 | 1 | 1 | 2 | 3 | 5 | 7 | 11 | 15 | 23 | 31 |

Notice that the numbers in the second row are given by the number of partitions of \( k \) until weight \( k = 9 \). Until this weight it seems that the elements of the form \( (1-\zeta)^m \omega(k_1, \ldots, k_r, 1, 1) \) with \( m \geq 0, r \geq 0, k_1 \geq \ldots \geq k_r \geq 1 \) and \( k_1 + \cdots + k_r + m = k - 2 \) form a basis. But starting in weight 10 there seem to be elements, which are not linear combinations of these.

In contrast to the \( z_n(k; q) \) introduced in [1] (see (4.3)) the algebraic structure of the cyclotomic multiple omega values is not known yet. We are expecting that \( \mathcal{Z}_k^{\omega} \mathcal{Z}_k^{\omega} \subseteq \mathcal{Z}_{k+k_2}^{\omega} \) holds for \( k_1, k_2 \geq 2 \). Namely, the space \( \mathcal{Z}_k^{\omega} = \sum_{k \geq 0} \mathcal{Z}_k^{\omega} \) forms a \( \mathbb{Q} \)-subalgebra of \( \mathcal{R} \), where \( \mathcal{Z}_0^{\omega} = \mathbb{Q} \). For example, one can check numerically that

\[
\omega(1, 1)^2 = -5\omega(2, 1, 1) - \frac{5}{2}(1-\zeta)\omega(1, 1, 1) + \frac{1}{4}(1-\zeta)^2\omega(1, 1),
\]

\[
(5.5) \quad \omega(1, 1)\omega(1, 1, 1) = -2\omega(2, 1, 1, 1) - 3\omega(3, 1, 1) - (1-\zeta)\omega(1, 1, 1, 1) - 3(1-\zeta)\omega(2, 1, 1) - \frac{1}{3}(1-\zeta)^2\omega(1, 1, 1).
\]

Since \( \varphi^\bullet_{wn}(\omega(k)\omega(1)) = \omega^\bullet(k)\omega^\bullet(1) \) and \( \omega^\bullet(1, 1) = 0 \) for \( \bullet \in \{\mathcal{A}, \mathcal{S}\} \) (see Section 5.3 below), the equations (5.5) would imply the relations

\[
\omega^\bullet(2, 1, 1) = 0, \quad 2\omega^\bullet(2, 1, 1, 1) + 3\omega^\bullet(3, 1, 1) = 0.
\]

5.3. **Special values.** In this subsection, we make a list of known values of finite and symmetric multiple omega values.

(i) From Theorem 1.6, for \( \bullet \in \{\mathcal{A}, \mathcal{S}\} \) and \( k_1, k_2 \geq 1 \) we obtain \( \omega^\bullet(k_1, k_2) = (-1)^{k_2} \zeta^\bullet(k_1 + k_2) \). Since \( \zeta^\bullet(k) = 0 \), we see that

\[
\omega^\bullet(k_1, k_2) = 0.
\]

(ii) Letting \( k_1 = \cdots = k_r = k \geq 2 \) in Corollary 5.2, we obtain

\[
\omega^\bullet(\{k\}^{r-1}, k-1) = 0.
\]
(iii) By (4.10), for $k \geq 2$ we see that
\[ \omega_S(\{1\}^k) \equiv -k! \zeta(k) \mod \zeta(2) Z. \]
This also follows from (1.4) and a special case of the sum formula for symmetric multiple zeta values by Murahara [17, Theorem 1.2]. Likewise, from (1.3) and a special case of the sum formula for finite multiple zeta values by Saito and Wakabayashi [23], for $k \geq 2$ we have
\[ \omega_A(\{1\}^k) = -k! Z(k), \]
where $Z(k)$ is the ‘true’ analogue of $\zeta(k)$ in $A$ defined for $k \geq 1$ by
\[ Z(k) = \left( \frac{B_{p-k}}{k} \mod p \right)_p \in A \quad (B_{p-k} : \text{Bernoulli number}). \]
So far, we were not able to find explicit formulas for $\omega_\bullet(\{a\}^k)$ (note that $\zeta_\bullet(\{a\}^k) = 0$ for any $a, k \in \mathbb{N}$ and $\bullet \in \{A, S\}$).

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