Complexity of Verifying Nonblockingness in Modular Supervisory Control

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Abstract—Complexity analysis becomes a common task in supervisory control. However, many results of interest are spread across different topics. The aim of this paper is to bring several interesting results from complexity theory and to illustrate their relevance to supervisory control by proving new nontrivial results concerning nonblockingness in modular supervisory control of discrete event systems modeled by finite automata.

Index Terms—Discrete event systems; Finite automata; Modular control; Complexity.

I. INTRODUCTION

Nonblockingness is an important property of discrete event systems ensuring that every task can be completed. It has therefore been intensively studied in the literature [1]. An automaton (deterministic or nondeterministic) is nonblocking if every sequence of events generated by the automaton can be extended to a marked sequence. Given a set of nonblocking automata, the modular nonblockingness problem asks whether the parallel composition of all the automata of the set results in a nonblocking automaton.

The property is easy to verify for a deterministic automaton (DFA) as we discuss in Theorem 2. However, if the automaton is nondeterministic (NFA) or a set of nonblocking DFAs is considered, the verification becomes computationally more demanding. We study the complexity in Theorems 3 and 6 respectively. A result relevant to timed discrete event systems is provided in Theorem 8.

So far, no efficient (polynomial) algorithm for verifying modular nonblockingness is known. In the light of the results of this paper, it is not surprising. The problem is complete for the complexity class for which the experts believe that no efficient algorithms exist. Therefore it is unlikely that there is an efficient algorithm solving the problem in general. However, there can still be optimization methods or algorithms working well for most of the practical cases. For instance, Malik [2] has recently shown that explicit model checking algorithms without any special data structures work well on standard computers for several practical systems with 100 million states.

The aim of this paper is to bring and apply some of the interesting results from automata and complexity theory to the nonblockingness verification problem.

II. PRELIMINARIES

An alphabet, Σ, is a finite nonempty set. The elements of an alphabet are called events. A string over Σ is a finite sequence (concatenation) of events, for example, 001 is a string over {0, 1}. Let Σ* denote the set of all finite strings over Σ; the empty string is denoted by ε.

A nondeterministic finite automaton (NFA) over an alphabet Σ is a structure A = (Q, Σ, δ, I, F), where Q is the finite nonempty set of states, I ⊆ Q is the nonempty set of initial states, F ⊆ Q is the set of accepting (marked) states, and δ : Q × Σ → 2Q is the transition function that can be extended to the domain 2Q × Σ* by induction. The language generated by A is the set L(A) = {w ∈ Σ* | δ(I, w) ≠ ∅} and the language marked by A is the set Lm(A) = {w ∈ Σ* | δ(I, w) ∩ F ≠ ∅}. Equivalently, the transition function is a relation δ ⊆ Q × Σ × Q. Then the meaning of δ(q, a) = {s, t} is that there are two transitions (q, a, s) and (q, a, t).

The prefix closure of a language L is the set L = {w ∈ Σ* | there exists u ∈ Σ* s.t. wu ∈ L}; L is prefix-closed if L = L. Obviously, Lm(A) ⊆ L(A) and L(A) is prefix-closed.

NFA A is deterministic (DFA) if it has a unique initial state, |I| = 1, and no nondeterministic transitions, |δ(q, a)| ≤ 1 for every q ∈ Q and a ∈ Σ. For DFAs, we identify singletons with their elements and simply write p instead of {p}. Specifically, we write δ(q, a) = p instead of δ(q, a) = {p}.

For every NFA A there exists a DFA B such that Lm(B) = Lm(A) and L(B) = L(A).

Let Σ and Γ be alphabets, and let f : Σ* → Γ* be a map. Then f is a morphism (for catenation) if f(xy) = f(x)f(y) for every x, y ∈ Σ*. Let Σo ⊆ Σ be alphabets. A projection P from Σ* to Σo is a morphism defined by P(a) = ε for a ∈ Σo, and P(a) = a for a ∈ Σ\o. The action of projection P on a string w ∈ Σ* is to erase all events from w that do not belong to Σo. The inverse image of P, denoted P−1, is defined as P−1(s) = {w ∈ Σ* | P(w) = s}. The definitions can readily be extended to languages.

Let L1 be a language over Σi, i = 1, . . . , n. The parallel composition of (L1)n i=1 is defined by ||i=1 L1 = ∩i=1 P−1(L1), where P1 is a projection from (∪i=1 Σi)∗ to Σ1. For i = 1, 2, let A1 = (Q1, Σ1, δ1, I1, F1) be NFA. The parallel composition of A1 and A2 is defined as the accessible part of the NFA (Q1 × Q2, Σ1 ∪ Σ2, δ, I1 × I2, F1 × F2), where

\[ δ((x, y), e) = \begin{cases} δ1(x, e) × δ2(y, e) & \text{if } e ∈ Σ1 \cap Σ2 \\ δ1(x, e) × \{y\} & \text{if } e ∈ Σ1 \setminus Σ2 \\ \{x\} × δ2(y, e) & \text{if } e ∈ Σ2 \setminus Σ1 \end{cases} \]

The parallel composition of DFAs is a DFA 3. The relationship between the definitions is L(A1 || A2) = L(A1) || L(A2) and Lm(A1 || A2) = Lm(A1) || Lm(A2).

An NFA A is nonblocking if Lm(A) = L(A). The inclusion Lm(A) ⊆ L(A) always holds.
To show that a composition of nonblocking automata can be blocking, let $A_1$ and $A_2$ be DFAs over $\{a\}$ depicted in Fig. 1. Both automata are nonblocking but their parallel composition is blocking, because $a$ cannot be extended to a marked string.

We now briefly recall the basic notions of complexity theory. For all unexplained notions, the reader is referred to the literature \[3, 5\].

There are two complexity measures: space and time. The class $\text{NSPACE}(f(n))$ denotes the class of all problems decidable by a nondeterministic Turing Machine (TM) (a nondeterministic algorithm) in space $O(f(n))$ for an input of size $n$. The class $\text{NL} = \text{NSPACE}(\log n)$ is thus the class of all problems decidable by a nondeterministic TM in logarithmic space, and $\text{PSPACE} = \bigcup_{k \in \mathbb{N}} \text{NSPACE}(n^k)$ is the class of all problems decidable by a (nondeterministic) TM in polynomial space. The space required to store the input and output is not considered in space complexity.

The class $\text{P}$ (NP) denotes the class of all problems decidable by a (nondeterministic) TM in polynomial time.

The hierarchy of classes is $\text{NL} \subseteq \text{P} \subseteq \text{NP} \subseteq \text{PSPACE}$. Even though $\text{NL} \subseteq \text{PSPACE}$, the strictness of any other inclusion is unknown. The (non)strictness of these inclusions is the most interesting and important open problem of complexity theory.

The classes $\text{NL}$ and $\text{NP}$ are defined in terms of a nondeterministic TM (a nondeterministic algorithm). Although for every nondeterministic TM there is an equivalent deterministic TM, the difference is in complexity. A typical nondeterministic step of a nondeterministic algorithm is “choose $x \in X$”. Deterministically, one can imagine to check all the possibilities for $x$ one by one. Nondeterministically, the situation is different. There are two basic views how a nondeterministic algorithm performs a nondeterministic step. The first view is that the algorithm “guesses” the right value of $x$ that eventually leads to a success, if such a value exists. The other view is that the algorithm makes a copy of itself for every nondeterministic step with different value of $x$ in every copy. For a nondeterministic step “choose $x \in \{1, \ldots, 100\}$”, 100 copies of the algorithm would be created, where the value of $x$ in the $i$th copy is $x = i$. The nondeterministic algorithm is successful if at least one of the copies is successful. In this view, the time (space) complexity is the maximum of time (space) required by a copy.

**Example 1:** Let $G = (V, E, s, t)$ be a directed graph with $s, t \in V$ the source and target nodes. The graph reachability problem asks whether the target node $t$ is reachable from the source node $s$ in $G$. The problem belongs to $\text{NL}$ [3].

To show this, we describe a nondeterministic algorithm (Algorithm 1) that solves the graph reachability problem in logarithmic space. Algorithm 1 is nondeterministic because of the nondeterministic step on line 3. Following the first view, the algorithm correctly guesses the edges that lead from node $s$ to node $t$, if such a path exists. Following the second view, the algorithm forks for every possible edge on line 3. If any of the copies ever reaches $t$, the copy returns true, which is then the overall answer. The variable $\text{numSteps}$ counts the number of steps and terminates the cycle if it is bigger than the number of nodes. This is fine because if there is a path from $s$ to $t$, then there is a path of length at most $|V| - 1$.

**Algorithm 1:** (Graph reachability)

**Input:** A directed graph $G = (V, E, s, t)$

**Output:** true iff $t$ is reachable from $s$ in $G$

1. $k := s$; $\text{numSteps} := 0$
2. repeat
3. choose $k'$ such that $(k, k') \in E$
4. $k := k'$; $\text{numSteps} := \text{numSteps} + 1$
5. until $k = t$ or $\text{numSteps} > |V| - 1$
6. if $k = t$ then return true
7. return false

It remains to show that Algorithm 1 works in logarithmic space. Since the input is not considered in space complexity, the only space required by the algorithm is the space to store $k, k', |V| - 1$ and $\text{numSteps}$. However, $\text{numSteps}$ is a binary number, bounded by $|V|$, which requires at most $\lceil \log |V| \rceil$ digits. Similarly, $k$ is a pointer to the position in the input, where the actual value of $k$ is stored. Thus, it is again a binary number with at most $\lceil \log (|V| + |E|) \rceil$ digits. Similarly for $k'$ and $|V| - 1$.

A problem is $\text{PSPACE}$-complete if it can be solved using only polynomial space (membership in $\text{PSPACE}$) and if every problem that can be solved in polynomial space can be reduced (transformed) to it in polynomial time ($\text{PSPACE}$-hardness). $\text{PSPACE}$-complete problems are therefore the hardest problems in $\text{PSPACE}$. Similarly for the other complexity classes, with only a different requirement on the reduction. Namely, to prove $\text{NL}$-hardness, the reduction has to be in deterministic logarithmic space, and to prove $\text{NP}$-hardness, the reduction has to be in polynomial time (as well as for $\text{PSPACE}$-hardness).

For instance, satisfiability of formulae in conjunctive normal form (3CNF) is an NP-complete problem \[6\]. Therefore, by definition, any problem in $\text{NP}$ can be reduced to $\text{3CNF}$ in polynomial time. We show in Theorem 8 that the One-shared-event DFA modular nonblockingness ($1\text{SE-DFA-MN}$) problem is in $\text{NP}$, hence reducible to $\text{3CNF}$ in polynomial time.

The membership in $\text{NP}$ gives an upper bound on the complexity of $1\text{SE-DFA-MN}$, which can still be polynomially or even trivially solvable. To rule out this possibility, we further show that $1\text{SE-DFA-MN}$ is $\text{NP}$-hard (and hence NP-complete) by reducing $\text{3CNF}$ to $1\text{SE-DFA-MN}$. Then, consequently, any problem in $\text{NP}$ can be reduced to the $1\text{SE-DFA-MN}$ problem in polynomial time. Hence, from the complexity point of view, both problems are equally difficult.

1 A (boolean) formula consists of variables, operators conjunction, disjunction and negation, and parentheses. A formula is satisfiable if there is an assigning of 1 (true) and 0 (false) to its variables making it true. A literal is a variable or its negation. A clause is a disjunction of literals. A formula is in 3-cnf if it is a conjunction of clauses, each clause with three literals. For instance, $\varphi = (x \lor y \lor z) \land (\neg x \lor y \lor z)$ is a formula in 3-cnf with two clauses $x \lor y \lor z$ and $\neg x \lor y \lor z$. Given a formula in 3-cnf, the 3CNF problem asks whether the formula is satisfiable. The formula $\varphi$ is satisfiable for, e.g., $(x, y, z) = (0, 1, 0)$. 

\[1\]
III. COMPLEXITY OF NONBLOCKINGNESS

Let $A = (Q, \Sigma, \delta, I, F)$ be an NFA. We define the size of $A$ as $|A| = |Q| + |\Sigma| + |\delta| + |I| + |F|$.

A DFA is nonblocking iff from every state a marked state is reachable (in other words, every state is reachable and co-reachable). This property can be tested in linear time using the computation of strongly connected components [4]. From the complexity point of view, under the assumption that NL \(\not=\) P, a stronger result can be shown.

Theorem 2 (DFA-nonblockingness): Given a DFA $A$, the problem whether $L_m(A) = L(A)$ is NL-complete.

Proof: The membership of DFA-nonblockingness in NL follows from Algorithm [3] below for $n = 1$.

We now show that DFA-nonblockingness is NL-hard by reducing graph non-reachability [4] to DFA-nonblockingness. Namely, let $G = (V, E, s, t)$ be a directed graph with $s, t$ in $V$. We construct a DFA $A$ from $G$ in logarithmic space such that $t$ is not reachable from $s$ if $G$ is nonblocking.

Let $A = (V \cup \{t', \Sigma, \delta, s, V\}$, where $\delta$ is defined as the relation $E$ with every transition under a unique label, and a transition under a new label is added from $t$ to the new non-marked state $t'$. This reduction (transformation) of $G$ to $A$ can be done in logarithmic space and is performed by Algorithm [4] below where $\Sigma = \{1, 2, \ldots, |E| + 1\}$. If the algorithm reads a node $v$ in $V$, it outputs state $v$. Then it prints state $t'$. After this part, it has printed the state set of $A$. It only needs to store a pointer (of logarithmic size) to the position of the input currently read. Then the algorithm counts from 1 to $|E| + 1$ and outputs the numbers, that is, it prints the alphabet of $A$. For this, two numbers, $i$ and $|E| + 1$ with at most $\log(|E| + 1)$ digits are stored. Then, it reads the input again (using the pointer as above) and uses a counter $c$ (with at most $\log(|E| + 1)$ digits) to print, for every edge $(u, v)$ in $E$, the corresponding transition $(u, c, v)$ of $\delta$. Finally, it prints the transition $(t, c, t)$, state $s$, and all $v \in V$. After this, the output contains the DFA $A$. The reduction uses logarithmic space to produce the output. Recall that the size of the input and output is not considered in space complexity.

Algorithm 2: (Reduction of a graph to a DFA)

| Input | A directed graph $G = (V, E, s, t)$ |
|-------|-----------------------------------|
| Output | The DFA $A = (V \cup \{t'\}, \Sigma, \delta, s, V)$ |
| 1 | for each $v \in V$ do output $v$ |
| 2 | output $t'$ |
| 3 | for $i = 1, \ldots, |E| + 1$ do output $i$ |
| 4 | $c := 1$; for each $(u, v) \in E$ do {output $(u, c, v)$; $c++$} |
| 5 | output $(t, c, t')$; output $s$ |
| 6 | for each $v \in V$ do output $v$ |

It is not difficult to see that $t$ is reachable from $s$ in $G$ iff $t$ is accessible from the initial state $s$ in $A$. Namely, if $t$ is not accessible in $A$, then all accessible states are marked and the language of $A$ is nonblocking. If $t$ is accessible in $A$, then so is $t'$, which is not marked and makes thus the language of $A$ blocking; cf. Fig. [2] for an illustration.

Therefore, to check nonblockingness of a DFA is easy. This is, however, not true for NFAs. An NFA can be nonblocking even if there is a state from which no marked state is reachable, cf. Fig. [3].

Theorem 3 (NFA-nonblockingness): Given an NFA $A$, the problem whether $L_m(A) = L(A)$ is PSPACE-complete.

Proof: To show that the problem is in PSPACE, let $A = (Q, \Sigma, \delta, I, F)$ be an NFA. Let $D$ be a DFA obtained from $A$ by the standard subset construction [5]. States of $D$ are subsets of states of $A$, and $A$ is nonblocking iff $D$ is nonblocking. To check nonblockingness of $D$ in polynomial space, $D$ cannot be computed and stored, because it may require exponential space in the size of $A$. Instead, we use the on-the-fly technique that keeps only a small part of $D$ in memory and re-computes the required parts on request. Namely, for every state $X \subseteq Q$ of $D$, we check that $X$ is reachable from the initial state of $D$ (in the way depicted in Example [1]). If so, we guess a marked state $Y$ of $D$, that is, $Y \cap F \neq \emptyset$, and check that $Y$ is reachable from $X$. This principle is generalized in Algorithm [3] below. At any time during the computation, the algorithm stores only a constant number of states of $D$, which are subsets of the state set $Q$ of $A$. Therefore, the algorithm uses space polynomial in the size of $A$ and the problem is thus in PSPACE.

To show that NFA-nonblockingness is PSPACE-hard, we reduce the NFA universality problem [6] to it. NFA universality asks, given an NFA $B$ over $\Sigma$, whether $L_m(B) = \Sigma^*$. Let $B = (Q, \Sigma, \delta_B, I, F)$ be an NFA, and let $d$ be a new non-marked state. We “complete” $B$ in the sense that, if for an event $a$ in $\Sigma$, no $a$-transition is defined in a state $q$, we add the transition $(q, a, d)$ to the transition relation, see the dotted transitions in Fig. [3]. Let $x \notin \Sigma$ be a new event. State $d$ contains self-loops under all events of $\Sigma \cup \{x\}$. For each non-marked state $p$, we add the transition $(p, x, i)$ for each initial state $i$ of $I$, cf. the dotted transitions in Fig. [3]. Let $A = (Q \cup \{d\}, \Sigma \cup \{x\}, \delta_A, I, F)$ denote the resulting NFA. Notice that $L_m(B) \subseteq L_m(A)$. We now show that $B$ is universal iff $A$ is nonblocking.

If $B$ is universal, that is, $L_m(B) = \Sigma^*$, we show that $A$ is nonblocking by showing that $L_m(A) \neq L(A)$. It is sufficient to show that $L(A) \subseteq L_m(A)$. Let $w \in L(A)$. We proceed by induction on the number of occurrences of event $x$ in $w$. If $x$...
does not occur in \( w \), then \( w \in \Sigma^* = L_m(B) \subseteq L_m(A) \). Thus, assume that \( w = w_1xw_2 \) with \( w_1 \in \Sigma^* \) and \( w_2 \in (\Sigma \cup \{x\})^* \). Since \( w_1 \in \Sigma^* = L_m(B) \subseteq L_m(A) \), there is a path in \( A \) from an initial state \( i \in I \) to a marked state \( f \) in \( A \) labeled by \( w_1 \). By construction, \( I \subseteq \delta_A(i, w_1x) \), since \( \delta_A(f, x) = I \). By the induction hypothesis, \( \delta_A(I, w_2) \cap F \neq \emptyset \), hence \( w \in L_m(A) \). Thus, \( L_m(A) = L(A) \), which means that \( A \) is nonblocking.

If \( B \) is not universal, that is, \( L_m(B) \neq \Sigma^* \), then there exists a \( w \in \Sigma^* \) such that \( \delta_A(I, w) \cap F = \emptyset \), since for any \( w \) over \( \Sigma \), \( \delta_A(I, w) \cap F = \delta_A(w, w) \cap F \). By the construction, \( \delta_A(w, x) = \{d\} \), from which no marked state is reachable. Since \( A \) is complete, \( wx \) belongs to \( L(A) \), therefore \( A \) is blocking.

The situation with NFAs is even worse as shown now.

**Theorem 4 (NFA-prefix-closed):** Given an NFA \( A \), the problem whether \( L_m(A) \) is prefix-closed is PSPACED-hard.

**Proof:** Let \( A \) be an NFA. Then \( L_m(A) = L_m(A) \) iff the DFA \( D \) obtained from \( A \) by the standard subset construction has no reachable and co-reachable non-marked states. Since the class PSPACE is closed under complement, we can check the opposite – a nondeterministic algorithm guesses a subset of non-marked states of \( A \) and verifies, using the on-the-fly technique, that they form a reachable and co-reachable state in \( D \). The NFA-prefix-closed problem is thus in PSPACE.

To show PSPACE-hardness, Hunt and Rosenberg [8] have shown that a property \( R \) of languages over \( \{0,1\} \) such that \( R(\{0,1\}^*) \) is true and there exists a regular language that is not expressible as a quotient \( x | L = \{w | xw \in L\} \), for some \( L \) for which \( R(L) \) is true, is as hard as to decide “\( = \{0,1\}^* \)”. Since prefix-closedness is such a property (the class of prefix-closed languages is closed under quotient) and universality is PSPACE-hard for NFAs, the result implies that the NFA-prefix-closed problem is PSPACE-hard.

These results justify why the attention is mostly focused on DFAs rather than NFAs. In the rest of the paper, we also focus on DFAs, unless stated otherwise.

### A. Modular Nonblockingness Problem

We now focus on the modular nonblockingness problem. The simplest case is that there is no interaction between the different subsystems. The following result is well known.

**Theorem 5:** Let \( J = \{1, \ldots, n\} \), and \( A_j \) a nonblocking NFA over \( \Sigma_j \), for \( j \in J \). If the alphabets are pairwise disjoint, that is, \( \Sigma_i \cap \Sigma_j = \emptyset \), for \( i \neq j \), then the parallel composition \( L_m(\{A_{i \in J} \}) = L(\{A_{i \in J} \}) \) is nonblocking.

In many complex systems, it is however the case that there are events shared between the subsystems. In such a case, nonblockingness is in general PSPACE-complete [9]. A more fine-grained complexity can be distinguished based on the following criteria. Let \( (A_i)_{i \in J} \) be DFAs:

1. The number of DFAs is not restricted.
2. The number of DFAs is restricted by a function \( g(m) \), that is, \( n \leq g(m) \), where \( m \) is the length of the encoding of the DFAs \( A_1, A_2, \ldots, A_n \).
3. The number of DFAs is restricted by a constant \( k \), that is, \( n \leq k \).

Case 2 is the most general one and deserves a discussion. Assume, for example, that our encoding of \( A_i \) requires \( c > 1 \) bits and that the encoding of \( A_1, \ldots, A_n \), requires \( m = n \cdot c \) bits. If \( g(m) = m \), then \( n \leq g(nc) \) for every \( n \geq 1 \), which results in the non-restricted case 1. If \( g(m) = k \), for a constant \( k \), then \( n \leq g(nc) \) iff \( n \leq k \), which results in the restriction of case 3. See also Remark 7 below.

We can now prove the following result.

**Theorem 6 (DFA \( g(m) \)-bounded modular nonblockingness):** Given nonblocking DFAs \( (A_i)_{i \in J} \) with \( A_i \) over \( \Sigma_i \), \( 2 \leq n \leq g(m) \), where \( m \) is the length of an encoding of the sequence of DFAs \( A_1, A_2, \ldots, A_n \). The problem whether \( L_m(\bigwedge_{i=1}^{n} A_i) \) is NSPACE(\( g(m) \log m \))-complete.

**Algorithm 3:** Is \( A = \bigwedge_{i=1}^{n} A_i \) nonblocking?

**Input:** Encoding of \( A_1, \ldots, A_n \) of size \( m \)

**Output:** \( \text{yes} \iff \bigwedge_{i=1}^{n} A_i \) is nonblocking

1. for each \( (p_1, \ldots, p_n) \in X_1^{n} \) do
2. if \( (p_1, \ldots, p_n) \) is reachable from the init. st. of \( A \) then
3. choose \( \{s_1, \ldots, s_n\} \subseteq \bigwedge_{i=1}^{n} F_i \)
4. \( k_i := p_i \), for \( i = 1, 2, \ldots, n \)
5. repeat
6. choose \( a \in \Sigma \)
7. \( k_i := \delta(k_i, a) \), for \( i = 1, 2, \ldots, n \)
8. until \( k_i = s_i \), for \( i = 1, 2, \ldots, n \)
9. return \text{yes}

**Proof:** Let \( (A_i)_{i \in J} \) be nonblocking DFAs. Algorithm 3 solves the \( g(m) \)-bounded modular nonblockingness problem. It works as follows: for every reachable state \( (p_1, \ldots, p_n) \) of \( A \) (lines 1-2), the algorithm nondeterministically chooses a marked state \( (s_1, \ldots, s_n) \) of \( A \) (line 3) that is reachable from state \( (p_1, \ldots, p_n) \) (lines 4-8). The algorithm returns \text{yes} iff there is such a marked state for every reachable state, hence iff \( A \) is nonblocking. (Compared to Example 1, we omitted the counter numSteps for simplicity. It should be clear how the counter is introduced to make the algorithm always terminate.)

During the computation, the algorithm stores only a constant number of \( n \)-tuples of pointers \( (t_1, \ldots, t_n) \). The space used is therefore \( O(n \log m) \). Since \( n \) is bounded by \( g(m) \), the space used by Algorithm 3 is \( O(g(m) \log m) \), hence the problem is in NSPACE(\( g(m) \log m) \).

To prove hardness, we reduce the NSPACE(\( g(m) \log m) \)-complete finite DFA intersection problem (DFA-int) to our problem. DFA-int asks, given DFAs \( (B_i)_{i=1}^{n} \) with \( 2 \leq n \leq g(m) \), where \( m \) is the length of the encoding of the sequence of \( B_1, \ldots, B_n \), whether \( \bigwedge_{i=1}^{n} L(B_i) = \emptyset \). The DFAs \( B_1, \ldots, B_n \) are over a common alphabet \( \Sigma \).

We now describe a deterministic logarithmic-space reduction from DFA-int to DFA \( g(m) \)-bounded modular nonblockingness. Notice that \( n \geq 2 \). Let \( x \notin \Sigma \) be a new event.

We construct \( A_1 \) from \( B_1 \) by adding two new states \( d_1 \) and \( d_1' \) and \( x \)-transitions from every marked state of \( B_1 \) to \( d_1 \), and from \( d_1 \) to \( d_1' \), see an illustration in Fig. 5. All states of \( A_1 \), but \( d_1 \), are marked, that is, \( L_m(A_1) = L(B_1) \cup L_m(B_1) \{x, xx\} \) and \( L(A_1) = L(B_1) \cup L_m(B_1) \{x, xx\} \).
For every $i \geq 2$, we construct $A_i$ from $B_i$ by adding a new state $d_i$ and $x$-transitions from every marked state of $B_i$ to $d_i$. All states of $A_i$ are marked, hence $L_m(A_i) = L(A_i) = L(B_i) \cup L_m(B_i)$.

We thus have that $L_m(\bigcup_{i=1}^{n} A_i) = \bigcap_{i=1}^{n} L(A_i)$ and

$$L(\bigcup_{i=1}^{n} A_i) = \bigcap_{i=1}^{n} L(B_i) \cup \bigcap_{i=1}^{n} L_m(B_i).$$

We show $L_m(\bigcup_{i=1}^{n} A_i) = L(\bigcup_{i=1}^{n} A_i)$ iff $\bigcap_{i=1}^{n} L(B_i) = \emptyset$.

If $\bigcap_{i=1}^{n} L(B_i) = \emptyset$, then $L(\bigcup_{i=1}^{n} A_i) = \bigcap_{i=1}^{n} L(B_i) = L_m(\bigcup_{i=1}^{n} A_i)$.

If $\bigcap_{i=1}^{n} L(B_i) \neq \emptyset$, then there exists $w \in \bigcap_{i=1}^{n} L(B_i)$, hence $wx \in L(\bigcup_{i=1}^{n} A_i) \setminus L_m(\bigcup_{i=1}^{n} A_i)$, because $L_m(\bigcup_{i=1}^{n} A_i)$ does not contain any string with event $x$.

**Remark 7:** Let $k$ be a constant. If for every $m$, $g(m) \leq k$, then Algorithm 3 uses space $O(m)$, hence DFA $k$-bounded modular nonblockingness is in NL = NSPACE($\log m$) and it is NL-complete. If $g(m) \leq \log^k m$, then DFA $(\log^k m)$-bounded modular nonblockingness is NSPACE($\log^{k+1} m$)-complete. It is called a poly-logarithmic space complexity.

**B. One-Shared-Event Modular Nonblockingness**

We now focus on the case where exactly one event is shared. An application of this case is, for example, in the Brandin and Wonham modular framework for timed discrete event systems [11], where only one event simulating the tick of a global clock is shared and all the other events are local [12].

Unless NP = PSPACE, nonblockingness is computationally easier in this case.

Let $A$ be an NFA over $\Sigma$ and $P$ be a projection from $\Sigma^*$. Then $P(A)$ is a DFA such that $L_m(P(A)) = P(L_m(A))$ and $L(P(A)) = P(L(A))$, called an observer; cf. [3], [13] for a construction. In the worst case, $P(A)$ has exponentially many states compared to $A$ [14], [15].

**Theorem 8** (One-shared-event DFA modular nonblockingness): Given $n \geq 2$ nonblocking DFAs $(A_i)_{i=1}^{n}$ with $A_i$ over $\Sigma_i$ such that $|\bigcup_{j \neq i} (\Sigma_i \cap \Sigma_j)| = 1$. The problem to decide whether $L_m(\bigcup_{i=1}^{n} A_i) = L(\bigcup_{i=1}^{n} A_i)$ is NP-complete.

**Proof:** Let $(A_i)_{i=1}^{n}$ be nonblocking DFAs, and let $a$ be the only event such that $\Sigma_i \cap \Sigma_j = \{a\}$, for every $i \neq j$. Let $P$ be a projection from $(\bigcup_{i=1}^{n} \Sigma_i)$ to $\{a\}^*$. Let $m$ be the maximum number of states of all $A_i$’s. Then $P(A_i)$ is a DFA with at most $2^m$ states, hence the composition $A = \bigcup_{i=1}^{n} P(A_i)$ has at most $2^{mn}$ states, each of the form $(X_1, \ldots, X_n)$, where $X_i$ is a subset of states of $A_i$.

Let $\delta$ denote the transition function of $A$ and $q_0$ its initial state. Notice that $A$ is a sequence of transitions possibly with a cycle at the end. Then $A$ is nonblocking if there exist $k \leq 2^m$ and $\ell < 2^{m+1}$ such that $\delta(q_0, a^k)$ is an accepting state of $A$ and either $\delta(q_0, a^{k+1})$ is not defined or $\delta(q_0, a^k) \neq \delta(q_0, a^\ell)$. We now show how to check this property in nondeterministic polynomial time.

The nondeterministic algorithm guesses $k$ and $\ell$ in binary, requiring at most $mn+1$ digits each. To compute the states $\delta(q_0, a^k)$ and $\delta(q_0, a^\ell)$ in polynomial time, the algorithm proceeds as follows.

Let $A_i'$ denote the NFA obtained from $A_i$ by replacing each transition $(s, b, t)$ with the transition $(s, P(b), t)$, and by eliminating the $\varepsilon$-transitions afterwards. This can be computed in polynomial time [16] and is often used as the middle step in the computation of the observer; namely, it preserves the languages. Then $A_i'$ is over $\{a\}$ and has the same states as $A_i$.

Let $J_i$ denote the set of all initial states of $A_i'$; it is computed in polynomial time as the set of all states of $A_i'$ reachable under $\Sigma \setminus \{a\}$ from the initial state of $A_i$ (it is also the initial state of $P(A_i)$, that is, $q_0 = (J_1, \ldots, J_n)$).

The transition relation of $A_i'$ can be represented as a binary matrix $M_i$, where for states $s, t$ of $A_i'$, $M_i[s, t] = 1$ iff $(s, a, t)$ is a transition in $A_i'$. For $k \geq 2$, let $M_i^k$ be the multiplication of $M_i$ with itself $k$ times. Then $M_i^k[s, t]$ is the number of paths of length $k$ from $s$ to $t$ in $A_i'$ [17]. Let $\delta_A'$ denote the transition function of $A_i'$. Then $\delta_A'(q_i, a^k) = \{t \mid M_i^k[q_i, t] > 0\}$ (if it is empty, the transition is undefined). The size of matrix $M_i^k$ is polynomial in the number of states of $A_i'$ and can be computed in time logarithmic in $k$ by fast matrix multiplication: $M_i^1 = M_i \times M_i$, $M_i^2 = M_i^1 \times M_i$, $M_i^3 = M_i^2 \times M_i$, $M_i^4 = M_i^3 \times M_i$, $\ldots$.

To compute $\delta(q_0, a^k)$, we compute $M_i^k$, for $i = 1, \ldots, n$, in polynomial time. Then the state $\delta(q_0, a^k) = (\delta_A'(J_1, a^k), \ldots, \delta_A'(J_n, a^k))$ and it is marked iff every $\delta_A'(J_i, a^k)$ contains a marked state of $A_i$. It should now be clear how to check, in polynomial time, that either $\delta(q_0, a^{k+1})$ is not defined or $\delta(q_0, a^k) = \delta(q_0, a^\ell)$, cf. also Example 10 and Remark 11.

To show NP-hardness, we reduce 3CNF to our problem and use the construction of [18]. Let $\varphi$ be a formula in 3CNF (see footnote 1 on page 2 with $n$ distinct variables and $m$ clauses, and let $C_k$ be the set of literals in the $k$th clause, $1 \leq k \leq m$. The assignment to the variables is represented as a binary vector of length $n$. Let $p_1, \ldots, p_n$ denote the first $n$ prime numbers. For a natural number $z$ congruent with 0 or 1 modulo $p_i$, for all $i = 1, \ldots, n$, $z$ satisfies $\varphi$ if the assignment $(z \mod p_1, \ldots, z \mod p_n)$ satisfies $\varphi$.

For $u = 1, \ldots, n$ and $j = 2, \ldots, p_u - 1$, let $B_{u,j}$ denote a nonblocking DFA such that

$$L(B_{u,j}) = \{0^j \cdot (0^{p_u})^*\}.$$

Then $\bigcup_{u=1}^{n} \bigcup_{j=1}^{p_u-1} L_m(B_{u,j}) = \{0^u \mid \exists u \leq n, z \neq 0 \mod p_u \land z \neq 1 \mod p_u\}$ is the set of all natural numbers that do not encode an assignment to the variables.

For each clause $C_k$, we construct a nonblocking DFA $B_k'$ such that if $0^u \in L_m(B_k')$ and $z$ is an assignment, then $z$ does not assign value 1 to any literal in $C_k$. For example, if $C_k = \{x_r, \neg x_s, x_t\}$, for $1 \leq r, s, t \leq n$ and $r, s, t$ distinct, let $z_k$ be the unique integer such that $0 \leq z_k < p_r p_s p_t$, $z_k \equiv 0 \mod p_r$, $z_k \equiv 1 \mod p_s$, and $z_k \equiv 0 \mod p_t$. Then

$$L(B_k') = 0^{z_k} \cdot (0^{p_r p_s p_t})^*.$$

Let $B_1, \ldots, B_t$ denote all the DFAs $B_{u,j}$ and $B_k'$ constructed above, and let $A_i$ denote $B_i$ with the sets of marked and non-marked states exchanged, that is, $L_m(A_i) = 0^u \setminus L_m(B_i)$. Note that all $B_i$ and $A_i$ are nonblocking and their generated languages are $0^*$. Now, $\varphi$ is satisfiable if and only if there exists $z$ such that $z$ encodes an assignment to $\varphi$, i.e., $0^u \notin \bigcup_{u=1}^{n} \bigcup_{j=1}^{p_u-1} L_m(B_{u,j})$, and $z$ satisfies every clause $C_k$, that is, $0^u \notin L_m(B_k')$ for all $1 = k, \ldots, m$. This is iff $0^u \in \bigcap_{i=1}^{n} L_m(A_i) = L_m(\bigcap_{i=1}^{n} A_i)$. 


We show that $\|\ell_{=1} A_i$ is nonblocking iff $L_m(\|\ell_{=1} A_i) \neq \emptyset$. If $L_m(\|\ell_{=1} A_i) = \emptyset$, then $\|\ell_{=1} A_i$ is blocking, because $\varepsilon \in L(\|\ell_{=1} A_i)$.

If $0^s \in L_m(\|\ell_{=1} A_i)$, then $z$ satisfies $\varphi$. For a natural number $c$, the number $z + c \cdot \Pi_{=1}^n p_i$, also satisfies $\varphi$: indeed, if $z \equiv x_i \mod p_i$, then $(z + c \cdot \Pi_{=1}^n p_i) \equiv x_i \mod p_i$, for all $i$. Thus, $0^s ((0^{|1^m-1| p_i}|)^s \subseteq L_m(\|\ell_{=1} A_i)$). Since for every $0^s \in L(\|\ell_{=1} A_i)$, there exists $c$ such that $s \leq z + c \cdot \Pi_{=1}^n p_i$, we have that $\|\ell_{=1} A_i$ is nonblocking.

**Remark 9:** If the number of DFAs in Theorem 8 is at most $k$, for a constant $k$, the problem is NL-complete. The membership in NL is by Theorem 8 and NL-hardness by Theorem 2.

**Example 10:** We illustrate the polynomial computation used in the proof of Theorem 8 for $n = 1$. Its generalization to $n > 1$ is straightforward. Let $A_1 = \{(1, 2, 3, 4), \{a, b\}, \{(1, 2), (2, a, 1), (2, b, 3), (3, a, 4), (4, a, 1), 1, \{1\}\}$ be a DFA, and let $A'_1$, depicted in Fig. 5 be the NFA obtained from $A_1$ by renaming $b$-transitions to $\varepsilon$-transitions, and by the elimination of $\varepsilon$-transitions afterwards. Then the $4 \times 4$ transition matrix $M_1$ and its $4$th power $M_1^4$ are

$$M_1 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad M_1^4 = \begin{pmatrix} 1 & 2 & 2 & 2 \\ 3 & 1 & 1 & 2 \\ 1 & 0 & 0 & 2 \\ 2 & 1 & 1 & 0 \end{pmatrix}.$$ 

The reader may verify that $\delta(\{1\}, a^4) = \{1, 2, 3, 4\}$, where $\delta$ is the transition relation of the observer $P(A_1)$.

**Remark 11:** The number $M_1^k[s, t]$ represents the number of paths from state $s$ to state $t$ of length $k$. This information is not important for us. The information we need is whether there is a path, i.e., $M_1^k[s, t] > 0$, or not, i.e., $M_1^k[s, t] = 0$. The numbers $M_1^k[s, t]$ may become large and affect thus the complexity. To keep the complexity polynomial (the numbers small), the + operation in the definition of matrix multiplication is replaced by max operation. This minor trick keeps the matrices $M_1^k$ binary, while providing the same information.

**IV. CONCLUSION**

The theoretical results do not seem very optimistic. However, there are techniques to reduce the size of an automaton, which allows to handle large automata that appear in practical applications. A well-known technique is the BDD diagrams [19]. Another technique is the state-tree structures [20] or the method using extended finite-state machines and abstractions [21].

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