Let $K$ be a global field and let $S$ be a finite set of primes of $K$ containing the archimedean primes. We generalize the duality theorem established in [9] for the Néron $S$-class group of an abelian variety $A$ over $K$ by removing the hypothesis that the Tate-Shafarevich group of $A$ is finite. We also derive an exact sequence that relates the indicated group associated to the Jacobian variety of a proper, smooth and geometrically connected curve $X$ over $K$ to a certain finite subquotient of the Brauer group of $X$.

0. Introduction

Let $K$ be a global field and let $S$ be a finite set of primes of $K$ containing the archimedean primes. If $v \notin S$, we will write $K_v$ for the completion of $K$ at $v$, $\mathcal{O}_v$ for the ring of integers of $K_v$, $k(v)$ for the corresponding residue field and $K_v^{un}$ for the maximal unramified extension of $K_v$ inside a fixed separable algebraic closure of $K_v$. Let $A$ be an abelian variety over $K$. The Néron $S$-class group of $A$, introduced in [9], is the finite abelian group

\begin{equation}
C_{A,S} = \text{Coker} \left[ A(K) \xrightarrow{\rho} \bigoplus_{v \notin S} \Phi_v(A)(k(v)) \right],
\end{equation}

where, for every prime $v \notin S$, $\Phi_v(A)$ is the étale $k(v)$-group scheme of connected components of the Néron model of $A_{K_v}$ over $\mathcal{O}_v$ and the $v$-component of the map $\rho$ is the canonical reduction map $A(K) \to \Phi_v(A)(k(v))$. The first objective of this paper is to extend the duality theorem established in [9] for the group (0.1) by removing from [op.cit.] the hypothesis that the Tate-Shafarevich group $\Sha_1(A)$ of $A$ is finite. To this end, we define in Section 2 a canonical map $\varphi_{A,S}: T \Sha_1(A)_{\text{div}} \to C_{A,S}$, where $T \Sha_1(A)_{\text{div}}$ is the total Tate module of the subgroup of divisible elements of $\Sha_1(A)$, and prove
Theorem 0.1. (=Corollary [2.5]) There exists a canonical perfect pairing of finite abelian groups

\[ C_{A,S}/\varphi_{A,S}(T^{1}(A)_{\text{div}}) \times C_{A',S}^{1} \to \mathbb{Q}/\mathbb{Z}, \]

where \( C_{A',S}^{1} \) is the group (1.6) attached to the abelian variety \( A' \) dual to \( A \).

The proof of Theorem 0.1 is completely different from and substantially simpler than the proof of [9, Theorem 4.9]. Its main ingredient is the generalized Cassels-Tate dual exact sequence established in [6].

In order to explain the second objective of this paper, we introduce the following notation.

Let \( X \) be a proper, smooth and geometrically connected curve over \( K \). If \( v / \in S \), let \( \delta_{v}' \) denote the period of \( X_{K_{v}} \), i.e., the least positive degree of a divisor class on \( X_{K_{v}} \), and let \( \delta_{v}^{\text{nr}} \) denote the corresponding quantity associated to \( X_{K_{v}^{nr}} \). Set

\[ d_{v} = \delta_{v}'/\delta_{v}^{\text{nr}} \]

and

\[ d = \prod_{v \in S} d_{v}. \]

The structural morphism \( X_{K_{v}} \to \text{Spec} K_{v} \) induces a pullback homomorphism between the associated Brauer groups \( \text{Br} K_{v} \to \text{Br} X_{K_{v}} \) and we write

\[ \text{Br}_{a} X_{K_{v}} = \text{Coker} \left[ \text{Br} K_{v} \to \text{Br} X_{K_{v}} \right]. \]

Next, the canonical morphism \( X_{K_{v}^{nr}} \to X_{K_{v}} \) induces a pullback homomorphism \( \text{Br}_{a} X_{K_{v}} \to \text{Br}_{a} X_{K_{v}^{nr}} \) and we define

\[ \text{Br}_{a}(X_{K_{v}^{nr}}/X_{K_{v}}) = \text{Ker} \left[ \text{Br}_{a} X_{K_{v}} \to \text{Br}_{a} X_{K_{v}^{nr}} \right]. \]

It is shown in [3] that there exists a canonical exact sequence of finite abelian groups

\[ 0 \to \text{Hom}(\text{Br}_{a}(X_{K_{v}^{nr}}/X_{K_{v}}), \mathbb{Q}/\mathbb{Z}) \to \Phi_{v}(J(k(v))) \to \mathbb{Z}/d_{v} \mathbb{Z} \to 0, \]

where \( J \) is the Jacobian variety of \( X \) and \( d_{v} \) is the integer (0.2). In this paper we establish the following global analog of the above exact sequence.

Let

\[ B_{nr}(S, X) = \text{Ker} \left[ \text{Br}_{a} X \to \prod_{v \notin S} \text{Br}_{a} X_{K_{v}^{nr}} \times \prod_{v \in S} \text{Br}_{a} X_{K_{v}} \right], \]

where the \( v \)-component of the indicated map, for \( v \notin S \) (respectively, \( v \in S \)), is the pullback homomorphism \( \text{Br}_{a} X \to \text{Br}_{a} X_{K_{v}^{nr}} \) (respectively, \( \text{Br}_{a} X \to \text{Br}_{a} X_{K_{v}} \)) induced by the projection \( X_{K_{v}^{nr}} \to X \) (respectively, \( X_{K_{v}} \to X \)). Further, let

\[ B(X) = \text{Ker} \left[ \text{Br}_{a} X \to \prod_{\text{all } v} \text{Br}_{a} X_{K_{v}} \right]. \]

Then the following holds.
Theorem 0.2. (=Theorem 3.4) Assume that the local periods $\delta'_v$ of $X$ are pairwise coprime. Then there exists a canonical exact sequence of finite abelian groups

$$0 \rightarrow \text{Hom}(B_{nr}(S,X)/B(X), \mathbb{Q}/\mathbb{Z}) \rightarrow C_{J,S}/\varphi_{J,S}(T\text{III}^1(J)_{\text{div}}) \rightarrow \mathbb{Z}/d\mathbb{Z} \rightarrow 0,$$

where the groups $B_{nr}(S,X)$ and $B(X)$ are given by (0.5) and (0.6), respectively, $C_{J,S}$ is the Néron $S$-class group of the Jacobian variety $J$ of $X$ (0.1), $\varphi_{J,S}$ is the map (2.12) attached to $J$ and $d$ is the integer (0.3).

1. Preliminaries

If $B$ is a topological abelian group, its (Pontryagin) dual is the topological abelian group $B^* = \text{Hom}_{\text{cont.}}(B, \mathbb{Q}/\mathbb{Z})$ endowed with the compact-open topology, where $\mathbb{Q}/\mathbb{Z}$ is given the discrete topology. The dual of a morphism of topological abelian groups $f: B \rightarrow C$ will be denoted by $f^*: C^* \rightarrow B^*$. If $n$ is a positive integer, we will write $B_n$ for the $n$-torsion subgroup of $B$, $B/nB$ for $B/nB$ and $B_{\text{div}}$ for the subgroup of divisible elements of $B$. The total Tate module of $B$ is the group $TB = \lim_{\leftarrow} B_n = TB_{\text{div}}$, where the inverse limit is taken over all positive integers ordered by divisibility.

Lemma 1.1. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be morphisms in an abelian category $\mathcal{A}$. Then there exists a canonical exact sequence in $\mathcal{A}$

$$0 \rightarrow \text{Ker } f \rightarrow \text{Ker } (g \circ f) \rightarrow \text{Ker } g \rightarrow \text{Coker } f \rightarrow \text{Coker } (g \circ f) \rightarrow \text{Coker } g \rightarrow 0.$$

Proof. See, for example, [2, 1.2].

Recall from the Introduction the global field $K$ and the finite set of primes $S$. Now let $G$ be a commutative $K$-group scheme. If $L$ is a field containing $K$, we define

$$H^1(L, G) = H^1(L, G \times_K L),$$

where the right-hand group is the first fppf cohomology group of the $L$-group scheme $G \times_K L$. For every prime $v$ of $K$, let $K_v$ be the completion of $K$ at $v$. The image of a class $\xi \in H^1(K, G)$ under the restriction map $H^1(K, G) \rightarrow H^1(K_v, G)$ will be denoted by $\xi_v$. Now let $K^s$ be a fixed separable algebraic closure of $K$. For every $v \notin S$, we fix a prime $\mathfrak{v}$ of $K^s$ lying above $v$ and let $K^s_v$ be the completion of $K^s$ at $\mathfrak{v}$. Then $K^s_v$ is a separable algebraic closure of $K_v$ and we will write $K^s_{v, \text{nr}}$ for the maximal unramified extension of $K_v$ inside $K^s_v$. Identifying $\text{Gal}(K^s_{v, \text{nr}}/K)$ with a subgroup of $\text{Gal}(K^s_v/K_v)$ in the standard way, we obtain a restriction map $H^1(K_v, G) \rightarrow H^1(K^s_{v, \text{nr}}, G)$ and define

$$H^1_{\text{nr}}(K_v, G) = \text{Ker}[H^1(K_v, G) \rightarrow H^1(K^s_{v, \text{nr}}, G)].$$
Next we define
\begin{align}
\Pi_{\text{nr}}^1(S, G) &= \text{Ker} \left[ H^1(K, G) \to \prod_{v \not\in S} H^1(K_v^\text{nr}, G) \times \prod_{v \in S} H^1(K_v, G) \right], \\
\Pi_S^1(G) &= \text{Ker} \left[ H^1(K, G) \to \prod_{v \in S} H^1(K_v, G) \right]
\end{align}

and let
\begin{equation}
\lambda_{G, S} : \Pi_S^1(G) \to \prod_{v \not\in S} H^1(K_v, G)
\end{equation}
be the map whose \(v\)-component, for \(v \not\in S\), is the restriction to \(\Pi_S^1(G) \subset H^1(K, G)\) of the restriction map \(H^1(K, G) \to H^1(K_v, G)\). Clearly \(\Pi_{\text{nr}}^1(S, G) \subset \Pi_S^1(G)\) and \(\text{Ker} \lambda_{G, S} = \Pi_1^1(G)\) is the Tate-Shafarevich group of \(G\). Next, set
\begin{equation}
C_{G, S}^1 = \left( \prod_{v \not\in S} H^1_{\text{nr}}(K_v, G) \right) \cap \text{Im} \lambda_{G, S} \subset \prod_{v \not\in S} H^1(K_v, G).
\end{equation}

Since, for every \(v \in S\), the restriction map \(H^1(K, G) \to H^1(K_v^\text{nr}, G)\) factors as \(H^1(K, G) \to H^1(K_v, G) \to H^1(K_v^\text{nr}, G)\), a class \(\xi \in \Pi_S^1(G)\) lies in \(\Pi_{\text{nr}}^1(S, G)\) if, and only if, \(\xi_v \in H^1_{\text{nr}}(K_v, G)\) for every \(v \in S\). Consequently, the restriction of \(\lambda_{G, S}\) to \(\Pi_{\text{nr}}^1(S, G)\) defines a surjection \(\Pi_{\text{nr}}^1(S, G) \to C_{G, S}^1\) whose kernel is \(\Pi_1^1(G)\). Consequently, the following holds

**Lemma 1.2.** The map \(\lambda_{G, S}\) \(\text{[1.5]}\) induces an isomorphism of abelian groups
\[ \Pi_{\text{nr}}^1(S, G)/\Pi_1^1(G) \cong C_{G, S}^1, \]
where the groups \(\Pi_{\text{nr}}^1(S, G)\) and \(C_{G, S}^1\) are given by \(\text{[1.3]}\) and \(\text{[1.6]}\), respectively. \(\Box\)

2. The Generalized Duality Theorem

Recall from the Introduction the abelian variety \(A\). Now write
\begin{equation}
\Psi^1(A) = \text{Coker} \left[ H^1(K, A) \to \bigoplus_{\text{all } v} H^1(K_v, A) \right]
\end{equation}
and let
\begin{equation}
\beta_{A, S} : \bigoplus_{v \not\in S} H^1_{\text{nr}}(K_v, A) \to \Psi^1(A)
\end{equation}
be the restriction to \(\bigoplus_{v \not\in S} H^1_{\text{nr}}(K_v, A) \subset \bigoplus_{\text{all } v} H^1(K_v, A)\) of the canonical projection \(\bigoplus_{\text{all } v} H^1(K_v, A) \to \Psi^1(A)\), where the groups \(H^1_{\text{nr}}(K_v, A)\) are defined by \(\text{[1.2]}\).

An application of Lemma \(\text{[1.1]}\) to the pair of maps
\[ H^1(K, A) \to \bigoplus_{\text{all } v} H^1(K_v, A) \to \bigoplus_{v \not\in S} H^1(K_v^\text{nr}, A) \times \bigoplus_{v \in S} H^1(K_v, A) \]
yields an exact sequence of abelian groups

\[ (2.3) \quad 0 \to \Sha^1(A) \to \Sha_{nr}^1(S, A) \to \bigoplus_{v \notin S} H^1_{nr}(K_v, A) \xrightarrow{\beta_{A,S}} \Phi^1(A). \]

Thus the following holds

**Lemma 2.1.** There exists a canonical isomorphism of abelian groups

\[ \Sha_{nr}^1(S, A)/\Sha^1(A) \xrightarrow{\sim} \text{Ker} \left[ \bigoplus_{v \notin S} H^1_{nr}(K_v, A) \xrightarrow{\beta_{A,S}} \Phi^1(A) \right], \]

where \( \Sha_{nr}^1(S, A) \) is the group \([1.3]\) and \( \beta_{A,S} \) is the map \([2.2]\).

**Lemma 2.2.** For every \( v \notin S \), there exists a canonical isomorphism of finite abelian groups

\[ \Phi_v(A)(k(v)) \xrightarrow{\sim} H^1_{nr}(K_v, A^t)^*. \]

**Proof.** By \([11]\) Thm. 4.8, Grothendieck’s pairing \( \Phi_v(A)(k(v)^s) \times \Phi_v(A^t)(k(v)^s) \to \mathbb{Q}/\mathbb{Z} \) induces an isomorphism \( \Phi_v(A)(k(v)) \xrightarrow{\sim} H^1(k(v), \Phi_v(A^t))^* \). On the other hand, it is shown in \([14]\) proof of Proposition I.3.8, p. 47 that the reduction map \( A^t(K_v^{nr}) \to \Phi_v(A^t)(k(v)^s) \) induces an isomorphism \( H^1(\text{Gal}(K_v^{nr}/K_v), A^t(K_v^{nr})) \xrightarrow{\sim} H^1(k(v), \Phi_v(A^t)) \). The composition of the preceding map and the inverse of the isomorphism \( H^1(\text{Gal}(K_v^{nr}/K_v), A^t(K_v^{nr})) \xrightarrow{\sim} H^1_{nr}(K_v, A^t) \) induced by the inflation map \([1]\) Proposition 4, p. 100] is an isomorphism \( H^1_{nr}(K_v, A^t) \xrightarrow{\sim} H^1(k(v), \Phi_v(A^t)) \) whose dual is a map \( H^1(k(v), \Phi_v(A^t))^* \xrightarrow{\sim} H^1_{nr}(K_v, A^t)^* \). The isomorphism of the lemma is the composition \( \Phi_v(A)(k(v)) \xrightarrow{\sim} H^1(k(v), \Phi_v(A^t))^* \xrightarrow{\sim} H^1_{nr}(K_v, A^t)^* \). \(\square\)

Now let

\[ (2.4) \quad A(K)^\wedge = \varprojlim_n A(K)/n \]

be the adic (equivalently, profinite \([16][12]\)) completion of \( A(K) \) and let

\[ (2.5) \quad T\text{Sel}(A) = \varprojlim_n \text{Ker} \left[ H^1(K, A_n) \to \prod_v H^1(K_v, A_n) \right] \]

be the pro-Selmer group of \( A \). By \([6]\) p. 298], there exists a canonical exact sequence of profinite abelian groups

\[ (2.6) \quad 0 \to A(K)^\wedge \to T\text{Sel}(A) \to T\Sha_{nr}^1(A)_{\div} \to 0, \]

where the first nontrivial map is induced by the connecting homomorphisms \( A(K) \to H^1(K, A_n) \) in fppf cohomology induced by the exact sequence of abelian fppf sheaves \( 0 \to A_n \to A \xrightarrow{n} A \to 0 \). Identifying \( A(K)^\wedge \) with its image in \( T\text{Sel}(A) \) under the first
nontrivial map in (2.6), the indicated sequence induces an isomorphism of profinite abelian groups

\[ T \mathfrak{III}^1(A)_{\text{div}} \cong T \text{Sel}(A)/A(K)^\wedge. \]

Now, by [15, Proposition 3.2.5, p. 87], there exists a canonical isomorphism of finite abelian groups

\[ C_{A,S} \cong \text{Coker} \left[ A(K)^\wedge \xrightarrow{\beta_{A,S}} \bigoplus_{v \notin S} \Phi_v(A)(k(v)) \right], \]

where \( C_{A,S} \) is the group (0.1) and \( \hat{\rho} \) is the profinite completion of the map \( \rho \) in (0.1).

Next we define a map

\[ \gamma_{A,S} : T \text{Sel}(A) \to \bigoplus_{v \notin S} \Phi_v(A)(k(v)) \]

as the composition of canonical maps

\[ T \text{Sel}(A) \cong \mathfrak{U}^1(A^i)^* \xrightarrow{\beta_{A,S}} \prod_{v \notin S} H^1_{\text{nr}}(K_v, A^i)^* = \bigoplus_{v \notin S} H^1_{\text{nr}}(K_v, A^i)^* \cong \bigoplus_{v \notin S} \Phi_v(A)(k(v)), \]

where the first map \( T \text{Sel}(A) \cong \mathfrak{U}^1(A^i)^* \) is the composition

\[ T \text{Sel}(A) \cong \text{Ker} \left[ \prod_{v \notin S} H^0(K_v, A) \to H^1(K_v, A^i)^* \right] \cong \mathfrak{U}^1(A^i)^*, \]

where the first map is defined in [8, Remark 3.4] and shown to be an isomorphism in [6, Main Theorem] and the second map is induced by the Tate local duality isomorphisms \( H^0(K_v, A) \cong H^1(K_v, A^i)^* \). See also [7, Remark 1.2] and note that \( H^0(K_v, A) = H^0(K_v, A)^\wedge \) for every \( v \) since \( H^0(K_v, A) \) is profinite (if \( v \) is archimedean, \( H^0(K_v, A) \) denotes the group of connected components of \( A(K_v) \)).

The second map in (2.10) is the dual of the map \( \beta_{A,S} \) (2.2). The last map in (2.10) is the direct sum over \( v \notin S \) of the inverses of the isomorphisms \( \Phi_v(A)(k(v)) \cong H^1_{\text{nr}}(K_v, A^i)^* \) defined explicitly in the proof of Lemma 2.2. A laborious verification, using the explicit descriptions of all the maps involved, shows that the left-hand square in the diagram

\[ \begin{array}{ccc}
A(K)^\wedge & \xrightarrow{\gamma_{A,S}} & T \text{Sel}(A) \\
\downarrow \gamma_{A,S} & & \downarrow \gamma_{A,S} \\
A(K)^\wedge & \xrightarrow{\hat{\rho}} & \bigoplus_{v \notin S} \Phi_v(A)(k(v)) \\
\downarrow \gamma_{A,S} & & \downarrow \gamma_{A,S} \\
& & \text{Coker} \hat{\rho} \\
& & \downarrow \gamma_{A,S} \\
& & 0 \\
\end{array} \]

commutes, where the first map on the top row of the above diagram is that appearing in (2.6). Consequently, there exists a canonical map \( \gamma_{A,S} : T \text{Sel}(A)/A(K)^\wedge \to \)
Coker $\hat{\rho}$ such that the full diagram (2.11) commutes. Further, as an immediate consequence of Lemma 2.1 (for $A'$) and the definition of $\gamma_{A,S}$ (2.9), we have

**Lemma 2.3.** There exists a canonical isomorphism of finite abelian groups

$$\text{Coker} \left[ T\text{Sel}(A) \xrightarrow{\gamma_{A,S}} \bigoplus_{v \notin S} \Phi_v(A)(k(v)) \right] \cong (\text{III}^1_{\text{nr}}(S, A')/\text{III}^1(A'))^*,$$

where $\text{III}^1_{\text{nr}}(S, A')$ is the group (1.3) and $\gamma_{A,S}$ is the map (2.9).

Next, let

$$\varphi_{A,S}: T\text{III}^1(A)_{\text{div}} \to C_{A,S}$$

be the composition

$$T\text{III}^1(A)_{\text{div}} \cong T\text{Sel}(A)/A(K)^\wedge \xrightarrow{\tau_{A,S}} \text{Coker} \hat{\rho} \cong C_{A,S},$$

where the first map is the isomorphism (2.7), the second map is the right-hand vertical map in diagram (2.11) and the last map is the inverse of the isomorphism (2.8). Further, let

$$\psi_{A,S}: C_{A,S} \to \text{Coker} \tau_{A,S},$$

be the composition

$$C_{A,S} \xrightarrow{\sim} \text{Coker} \hat{\rho} \to \text{Coker} \tau_{A,S},$$

where the first map is the isomorphism (2.3) and the second (canonical) map exists since $\text{Im} \hat{\rho} \subset \text{Im} \tau_{A,S}$ by the commutativity of the left-hand square in diagram (2.11). An application of Lemma 1.1 to the pair of maps

$$A(K)^\wedge \hookrightarrow T\text{Sel}(A) \xrightarrow{\tau_{A,S}} \bigoplus_{v \notin S} \Phi_v(A)(k(v)),$$

whose composition is the map $\hat{\rho}$ by the commutativity of the left-hand square in (2.11), yields an exact sequence of profinite abelian groups

$$T\text{III}^1(A)_{\text{div}} \cong T\text{Sel}(A)/A(K)^\wedge \xrightarrow{\tau_{A,S}} \text{Coker} \hat{\rho} \cong C_{A,S} \to 0,$$

where $\varphi_{A,S}$ and $\psi_{A,S}$ are the maps (2.12) and (2.13), respectively. Consequently, the map $\psi_{A,S}$ (2.13) induces an isomorphism of finite abelian groups

$$C_{A,S}/\varphi_{A,S}(T\text{III}^1(A)_{\text{div}}) \cong \text{Coker} \tau_{A,S}.$$  

Composing the preceding map with the isomorphism of Lemma 2.3, we obtain an isomorphism of finite abelian groups

$$C_{A,S}/\varphi_{A,S}(T\text{III}^1(A)_{\text{div}}) \cong (\text{III}^1_{\text{nr}}(S, A')/\text{III}^1(A'))^*.$$  

Thus the following holds.
Theorem 2.4. Let $A$ be an abelian variety over $K$ with dual abelian variety $A^t$. Then there exists a canonical perfect pairing of finite abelian groups
\[
C_{A,S}/\varphi_{A,S}(T\Theta^1(A)_{\text{div}}) \times \Theta^1_{\text{nr}}(S,A^t)/\Theta^1_{\text{nr}}(A^t) \rightarrow \mathbb{Q}/\mathbb{Z},
\]
where $C_{A,S}$ is the group (0.1), $\varphi_{A,S}$ is the map (2.12) and $\Theta^1_{\text{nr}}(S,A^t)$ is the group (1.3).

The following statement, which generalizes [9, Theorem 4.9], is an immediate consequence of the above theorem and Lemma 1.2.

Corollary 2.5. There exists a canonical perfect pairing of finite abelian groups
\[
C_{A,S}/\varphi_{A,S}(T\Theta^1(A)_{\text{div}}) \times C^1_{A^t,S} \rightarrow \mathbb{Q}/\mathbb{Z},
\]
where $C^1_{A^t,S}$ is the group (1.6) attached to $A^t$.

3. Jacobian Varieties

Recall from the Introduction the curve $X$ over $K$. Let
\[
P = \text{Pic}_{X/K}
\]
be the Picard scheme of $X$ over $K$. Then $P$ is a smooth and commutative $K$-group scheme whose identity component
\[
J = \text{Pic}^0_{X/K}
\]
is an abelian variety over $K$ [4, Proposition 3, p. 244]. There exists a canonical exact sequence of abelian sheaves for the étale topology on $K$
\[
0 \rightarrow J \rightarrow P^{\text{deg}} \rightarrow \mathbb{Z}_K \rightarrow 0. \tag{3.1}
\]
Now recall that the index of $X$ over $K$ is the least positive degree $\delta$ of a divisor on $X$. If $L$ is a field containing $K$, then the index (respectively, period) of $X_L = X \times_K L$ divides the index (respectively, period) of $X$. Further, $\delta' | \delta$ and (3.1) induces an exact sequence of abelian groups
\[
0 \rightarrow \mathbb{Z}/\delta'\mathbb{Z} \rightarrow H^1(K,J) \rightarrow H^1(K,P) \rightarrow 0. \tag{3.2}
\]
For every prime $v$ of $K$, we will write $\delta_v$ for the index of $X_{K_v}$. By [10, Remark 1.6, p. 249], $\delta_v = 1$ for all but finitely many primes $v$ of $K$. If $v \notin S$, $\delta^\text{nr}_v$ will denote the index of $X_{K^\text{nr}_v}$. Note that $\delta_v | \delta$ and $\delta^\text{nr}_v | \delta'$ for every $v$. Further, $\delta^\text{nr}_v | \delta_v$ and $\delta^\text{nr}_v | \delta'_v$ for every $v \notin S$.

Next let
\[
D : \mathbb{Z}/\delta'\mathbb{Z} \rightarrow \prod_{v} \mathbb{Z}/\delta'_v\mathbb{Z} \tag{3.3}
\]
be the natural diagonal map and set
\[
\Delta' = \text{l.c.m.}\{\delta'_v\}. \tag{3.4}
\]
Then $\text{Ker } D = \Delta' \mathbb{Z}/\delta' \mathbb{Z}$ and $\text{Coker } D$ is a finite abelian group of order

$$|\text{Coker } D| = \left( \prod \delta_v' \right)/\Delta'.$$

An application of the snake lemma to the exact and commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \mathbb{Z}/\delta' \mathbb{Z} & \to & H^1(K, J) & \to & H^1(K, P) & \to & 0 \\
\downarrow D & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \prod_{v \not\in S} \mathbb{Z}/\delta_v' \mathbb{Z} & \to & \prod_{v \not\in S} H^1(K_v, J) & \to & \prod_{v \not\in S} H^1(K_v, P) & \to & 0,
\end{array}$$

whose rows are induced by (3.2), yields an exact sequence of abelian groups

$$0 \to \Delta' \mathbb{Z}/\delta' \mathbb{Z} \to \Pi^1(J) \to \Pi^1(P) \to \text{Coker } D.$$

Similarly, let

$$D_{\text{nr}}^S: \mathbb{Z}/\delta' \mathbb{Z} \to \prod_{v \not\in S} \mathbb{Z}/\delta_v^\text{nr} \mathbb{Z} \times \prod_{v \in S} \mathbb{Z}/\delta_v' \mathbb{Z}$$

be the natural diagonal map and let

$$\Delta'' = \text{l.c.m.}\{\delta_v^\text{nr}, v \not\in S, \delta_v', v \in S\}.$$

Then there exists a canonical exact sequence of abelian groups

$$0 \to \Delta'' \mathbb{Z}/\delta' \mathbb{Z} \to \Pi_{\text{nr}}^1(S, J) \to \Pi_{\text{nr}}^1(S, P) \to \text{Coker } D_{\text{nr}}^S,$$

where the groups $\Pi_{\text{nr}}^1(S, J)$ and $\Pi_{\text{nr}}^1(S, P)$ are given by (1.3).

The groups $\text{Coker } D$ and $\text{Coker } D_{\text{nr}}^S$ are related as follows.

**Lemma 3.1.** There exists a canonical exact sequence of finite abelian groups

$$0 \to \Delta'' \mathbb{Z}/\Delta' \mathbb{Z} \to \prod_{v \not\in S} \mathbb{Z}/d_v \mathbb{Z} \to \text{Coker } D \to \text{Coker } D_{\text{nr}}^S \to 0,$$

where the integers $\Delta''$, $\Delta'$ and $d_v$ are given by (3.4), (3.9) and (0.2), respectively, and the maps $D$ and $D_{\text{nr}}^S$ are given by (3.3) and (3.8), respectively.

**Proof.** This follows by applying Lemma 1.1 to the pair of maps

$$\mathbb{Z}/\delta' \mathbb{Z} \to \prod_{v \not\in S} \mathbb{Z}/\delta_v' \mathbb{Z} \to \prod_{v \not\in S} \mathbb{Z}/\delta_v^\text{nr} \mathbb{Z} \times \prod_{v \in S} \mathbb{Z}/\delta_v' \mathbb{Z}$$

whose composition is the map $D_{\text{nr}}^S$. \qed
**Proposition 3.2.** Assume that the integers $\delta'_v$ are pairwise coprime. Then there exists a canonical exact sequence of finite abelian groups

$$0 \to \mathbb{Z}/d\mathbb{Z} \to \Pi_{\text{nr}}^1(S, J)/\Pi_{\text{nr}}^1(J) \to \Pi_{\text{nr}}^1(S, P)/\Pi_{\text{nr}}^1(P) \to 0,$$

where $d$ is the integer (0.3) and the groups $\Pi_{\text{nr}}^1(S, J)$ and $\Pi_{\text{nr}}^1(S, P)$ are given by (1.3).

**Proof.** The hypothesis and (3.5) show that $\text{Coker } D = 0$. Now Lemma 3.1 shows that $\text{Coker } D^S = 0$ as well. Thus there exists a canonical exact and commutative diagram of abelian groups

$$
\begin{array}{cccccc}
0 & \to & \Delta'\mathbb{Z}/\delta'\mathbb{Z} & \to & \Pi^1(J) & \to & \Pi^1(P) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Delta''\mathbb{Z}/\delta''\mathbb{Z} & \to & \Pi_{\text{nr}}^1(S, J) & \to & \Pi_{\text{nr}}^1(S, P) & \to & 0 \\
\end{array}
$$

whose top and bottom rows are the sequences (3.7) and (3.10), respectively. An application of the snake lemma to the above diagram yields an exact sequence of finite abelian groups

$$0 \to \Delta''\mathbb{Z}/\Delta'\mathbb{Z} \to \Pi_{\text{nr}}^1(S, J)/\Pi^1(J) \to \Pi_{\text{nr}}^1(S, P)/\Pi^1(P) \to 0.$$

Now, since $\text{Coker } D = 0$, Lemma 3.1 shows that $\Delta''\mathbb{Z}/\Delta'\mathbb{Z}$ is canonically isomorphic to $\prod_{v\in S} \mathbb{Z}/d_v\mathbb{Z}$. Further, since the integers $\delta'_v$ are pairwise coprime, so also are the integers $d_v$ (0.2). Thus the Chinese Remainder Theorem yields a canonical isomorphism $\prod_{v\in S} \mathbb{Z}/d_v\mathbb{Z} \simeq \mathbb{Z}/d\mathbb{Z}$, whence the proposition follows. \qed

Next, there exists a canonical exact sequence of abelian groups (see [5, pp. 400-401])

$$0 \to \text{Pic } X \to P(K) \to \text{Br } K \to \text{Br } X \to H^1(K, P) \to 0.$$

The above sequence induces a functorial isomorphism of abelian groups

(3.11) $\text{Br}_a X \xrightarrow{\sim} H^1(K, P),$

where $\text{Br}_a X$ is the group (0.4) over $K$. If $L = K_v$ or $K_v^{\text{nr}}$, where $v$ is a prime of $K$, there exists a commutative diagram of abelian groups

(3.12) $\text{Br}_a X \xrightarrow{\sim} H^1(K, P) \xrightarrow{\text{Br}_a X_L} H^1(L, P),$

where the horizontal arrows are the maps (3.11) over $K$ and over $L$, the left-hand vertical arrow is induced by the pullback homomorphisms $\text{Br } K \to \text{Br } L$ and $\text{Br } X \to \text{Br } X_L$ and the right-hand vertical map is the restriction map in Galois cohomology.
By the commutativity of diagram (3.12), the map (3.11) and its analogs over $K_v$ for every $v$ and over $K_{nr}^v$ for every $v \notin S$ yield isomorphisms of abelian groups
\begin{align}
\mathfrak{B}_{nr}(S, X) &\xrightarrow{\sim} \Sha_1^{nr}(S, P) \\
\mathfrak{B}(X) &\xrightarrow{\sim} \Sha_1(P),
\end{align}
where the groups $\mathfrak{B}_{nr}(S, X)$ and $\mathfrak{B}(X)$ are given by (0.5) and (0.6), respectively, and the group $\Sha_1^{nr}(S, P)$ is defined by (1.3).

**Proposition 3.3.** Assume that the integers $\delta'_v$ are pairwise coprime. Then there exists a canonical exact sequence of finite abelian groups
$$0 \to \mathbb{Z}/d\mathbb{Z} \to \Sha_1^{nr}(S, J)/\Sha_1(J) \to \mathfrak{B}_{nr}(S, X)/\mathfrak{B}(X) \to 0,$$
where $d$ is the integer (0.3), $\Sha_1^{nr}(S, J)$ is given by (1.3) and the groups $\mathfrak{B}_{nr}(S, X)$ and $\mathfrak{B}(X)$ are given by (0.5) and (0.6), respectively.

**Proof.** This is immediate from Proposition 3.2 using the isomorphisms (3.13) and (3.14). \qed

Via the autoduality of the Jacobian [13, Theorem 6.6], the preceding proposition and Theorem 2.4 yield the second main result of this paper.

**Theorem 3.4.** Assume that the integers $\delta'_v$ are pairwise coprime. Then there exists a canonical exact sequence of finite abelian groups
$$0 \to \text{Hom}(\mathfrak{B}_{nr}(S, X)/\mathfrak{B}(X), \mathbb{Q}/\mathbb{Z}) \to C_{J,S}/\varphi_{J,S}(T\Sha_1^{nr}(J)_{\text{div}}) \to \mathbb{Z}/d\mathbb{Z} \to 0,$$
where $C_{J,S}$ is the Néron $S$-class group of $J$ (0.1), $\varphi_{J,S}$ is the map (2.12) and $d$ is the integer (0.3).

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Departamento de Matemáticas, Universidad de La Serena, Cisternas 1200, La Serena 1700000, Chile

E-mail address: cgonzalez@userena.cl