Non-leptonic weak decays and final state interactions in lattice QCD* †

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We show that, under a reasonable “smoothness” hypothesis, it is possible to extract informations on the amplitude and phase of two-body non-leptonic weak decay matrix elements from the study of Euclidean correlation functions in lattice QCD.

1. INTRODUCTION

Exclusive non-leptonic weak decays are a fundamental ingredient in the determination of the Cabibbo-Kobayashi-Maskawa (CKM) matrix elements and in the study of CP violation in the K-, D- and B-meson decays. However, the study of these non-leptonic decays involves non-perturbative aspects of the strong interactions, a challenging task for our still incomplete knowledge of non-perturbative dynamics. Unfortunately, a theoretical description of exclusive decays based on the fundamental theory is not possible yet. Over the years, several methods have been introduced to estimate the relevant matrix elements: vacuum saturation, bag models, quark models, QCD sum rules, $1/N_c$ expansion, chiral Lagrangians, factorization, etc. However, none of these approaches is actually based on first principles.

The lattice approach has been used to obtain results based on first principles for a wide set of relevant physical quantities such as the hadron spectrum, the meson decay constants, the form factors entering in semileptonic and radiative decays, the kaon $B$-parameter $B_K$, etc. However, any computation of exclusive non-leptonic decays performed in the Euclidean space-time suffers from severe limitations, as shown by Blok and Shifman [1] in the context of QCD sum rules, and by Maiani and Testa [2] in the framework of lattice QCD. In fact, the activity in this field has completely stopped after the publication of ref. [3].

In this paper we show that, under quite reasonable physical hypotheses, it is possible, at least in principle, to extract predictions for the relevant matrix elements in numerical simulations of lattice QCD, in spite of the difficulties due to the Maiani-Testa No-Go Theorem (MTNGT).

The MTNGT states essentially the following:

- In the calculation of a two- (many-) body decay amplitude performed in the Euclidean space-time, which is the only possibility in Monte Carlo simulations, there is no distinction between in- and out-states. As a consequence, the matrix elements that one is able to extract are real numbers resulting from the average of the two cases. This jeopardizes the possibility of any realistic prediction for the matrix elements. For example, we know from the measured $A_{1/2}$ and $A_{3/2}$ amplitudes in $D \rightarrow K\pi$ decays that there is a phase difference of about 80°.

- Matrix elements are extracted on the lattice by studying the time behaviour of appropriate correlation functions at large time distances. Maiani and Testa showed that what can be really isolated in this limit are the off-shell form factors corresponding to the final particles at rest. For kaon decays, we can use the chiral theory to extrapolate the form factor to the physical point. This is certainly not the case for $D$- and $B$-meson
decays. In the latter case it is not possible to obtain a realistic prediction for the matrix element.

We will show that both these difficulties can be overcome under a “smoothness” hypothesis, and that under this hypothesis it is possible, at least in principle, to extract the physical matrix elements, including the phase due to the strong-interaction rescattering of the final states. In the interesting case in which Final State Interaction (FSI) is dominated by the exchange of a resonance in the s-channel, it is possible to calculate the parameters of the resonance, as explained in detail in ref. [5].

Unfortunately, at present we do not know if the method we are proposing can be successfully applied in numerical simulations on currently available lattices.

2. CORRELATION FUNCTIONS IN THE EUCLIDEAN SPACE-TIME

Following ref. [2], we first examine the Euclidean three-point function $G_q(t_1, t_2)$:

$$G_q(t_1, t_2) \equiv \langle 0 | T \left[ \Pi_q(t_1) \Pi_{-q}(t_2) H(0) \right] | 0 \rangle$$

when $t_1 > t_2 > 0$. In eq. (1), $\Pi_q(t)$ is an interpolating field of the final-state particle (denoted as “pion” in the following) with a fixed spatial momentum

$$\Pi_q(t) = \int d^3x e^{-i\vec{q}\cdot\vec{x}} \Pi(\vec{x}, t);$$  

$$H(0) = H(\vec{x} = 0, t = 0)$$

is any local operator that couples to the two pions in the final state; $T[\ldots]$ represents the $T$-product of the fields and the vacuum expectation value corresponds, in a numerical simulation, to the average over the gauge field configurations.

When $t_1 \to \infty$

$$G_q(t_1, t_2) \to \sum_n \langle 0 | \Pi_q(t_1) | n \rangle \langle n | \Pi_{-q}(t_2) H(0) | 0 \rangle$$

$$\sim \sqrt{Z_{\Pi}} \frac{e^{-E_q t_1}}{2E_q} G_3(t_2),$$  

where

$$E_q = \sqrt{M^2 + \vec{q}^2}, \quad \sqrt{Z_{\Pi}} = \langle 0 | \Pi(0) | q \rangle$$

and

$$G_3(t) = \langle q | \Pi_{-q}(t) H(0) | 0 \rangle$$  

for $t > 0$.

Inserting a complete set of $out$-states $n$, we can write

$$\langle q | \Pi_{-q}(t) H(0) | 0 \rangle = \sum_n (2\pi)^3 \delta^{(3)}(P_n) \times$$

$$\langle q | \Pi_{-q}(t) H(0) | 0 \rangle e^{-(E_n - E_q) t} =$$

$$\sqrt{Z_{\Pi}} \frac{e^{-E_q t}}{2E_q} \langle q | \Pi_{-q}(t) H(0) | 0 \rangle + \sum_n (2\pi)^3 \times$$

$$\delta^{(3)}(P_n) \langle q | \Pi_{-q}(t) H(0) | 0 \rangle e^{-(E_n - E_q) t}$$

where the term proportional to $\langle q | \Pi_{-q}(t) H(0) | 0 \rangle$ on the r.h.s. is the disconnected contribution. We now define the connected matrix element of the pion field as follows:

$$\langle q | \Pi(0) | n \rangle = \frac{2\sqrt{Z_{\Pi}} M(q, -q; n)^*}{(E_n + 2E_q)(-E_n + 2E_q - i\epsilon)}$$  

The LSZ reduction formula shows that, when the four momentum of the pion field $\Pi(0)$ goes on the mass-shell ($E_n \to 2E_q$), $M$ reduces to the invariant scattering amplitude of the process $\pi(q) + \pi(-q) \to n$, cf. eq. (15) of ref. [2]:

$$\langle q | \Pi(0) | n \rangle = \frac{2\sqrt{Z_{\Pi}} M(q, -q; n)^*}{(E_n + 2E_q)(-E_n + 2E_q - i\epsilon)}$$  

In general one could write

$$\langle n | \Pi(0) | q \rangle = \sqrt{Z_{\Pi}} M(q, -q; n) \mathcal{F} \left( \frac{E_n}{2E_q} \right)$$

$$\times \left[ \frac{2}{(E + 2E_q - i\epsilon)(-E + 2E_q - i\epsilon)} \right],$$  

with the condition that the modulating factor $\mathcal{F}$ satisfies $\mathcal{F}(1) = 1$ for the on-shell pion ($E_n/2E_q = 1$). The factor in square brackets is (up to a factor $1/2E_q$) the propagator of two non-interacting pions.

Inserting the definition $[6]$ into eq. (8), and using the identity

$$\frac{1}{E - 2E_q - i\epsilon} = \mathcal{P} \left[ \frac{1}{E - 2E_q} + i\pi \delta(E - 2E_q) \right],$$  

$$e^{-(E_n - E_q) t} =$$

$$\frac{1}{E - 2E_q - i\epsilon} =$$

$$\int \frac{d^4p}{(2\pi)^4} e^{-i\vec{p}\cdot\vec{q}} \delta^4(p) \times$$

$$\mathcal{P} \left[ \frac{1}{E - 2E_q} + i\pi \delta(E - 2E_q) \right],$$  

where

$$\mathcal{P}$$

is the Cauchy principal value.
we obtain
\begin{equation}
G_3(t) = \frac{\sqrt{Z \Pi}}{2E_q} e^{E \bar{t}} \left[ P_q(t) + \frac{1}{2} \times \right. (10)
\end{equation}
\begin{equation}
\left. \times \left( \text{out} \langle \vec{q}, -\vec{q} | H(0)|0 \rangle \text{in} \langle \vec{q}, -\vec{q} | H(0)|0 \rangle \right) \right],
\end{equation}
with
\begin{equation}
P_q(t) = (4E_q) P \sum_n e^{-(E_n - 2E_q)t} (2\pi)^3 \delta(3)(P_n)
\end{equation}
\begin{equation}
\times \frac{\mathcal{M}^*(\vec{q}, -\vec{q}; n)(n|H(0)|0)}{4E_q^2 - E_n^2}.
\end{equation}
which corresponds to eq. (21) of ref. [3]. $P_q(t)$ is a sum over off-shell amplitudes. In the limit $t \rightarrow +\infty$, it is dominated by intermediate states with energy $E_n < 2E_q$, which correspond to two-pion states with momenta $\vec{k}$ and $-\vec{k}$, with $|\vec{k}| < |\vec{q}|$. We therefore obtain, for $t \gg 0$,
\begin{equation}
P_q(t) = (4E_q) P \int \frac{d^3k_1}{(2\pi)^3 2E_{k_1}} \frac{d^3k_2}{(2\pi)^3 2E_{k_2}}
\end{equation}
\begin{equation}
\times (2\pi)^4 \delta(3)(\vec{k}_1 + \vec{k}_2) e^{-(E_{k_1} + E_{k_2} - 2E_q)t}
\end{equation}
\begin{equation}
\times \frac{\mathcal{M}^*(\vec{q}, -\vec{q}; \vec{k}_1, \vec{k}_2)(\vec{k}_1, \vec{k}_2|H(0)|0)}{4E_q^2 - (E_{k_1} + E_{k_2})^2} = (4E_q) P \int \frac{d^3k_1}{2M_*} \frac{d^3k_2}{2M_*}
\end{equation}
\begin{equation}
\times (2\pi)^4 \delta(3)(\vec{k}_1 + \vec{k}_2) \delta(E - E_{k_1} - E_{k_2})
\end{equation}
\begin{equation}
\times \mathcal{M}^*(\vec{q}, -\vec{q}; \vec{k}_1, \vec{k}_2)(\vec{k}_1, \vec{k}_2|H(0)|0) = (4E_q) P \int \frac{d^3k_1}{2M_*} \frac{d^3k_2}{2M_*}
\end{equation}
\begin{equation}
\times \frac{1}{16\pi} \sqrt{E^2 - 4M_*^2} \left[ \mathcal{M}^*(\vec{q}, -\vec{q}; \vec{k}, \vec{k})
\end{equation}
\begin{equation}
\times (\vec{k}, -\vec{k}|H(0)|0) \right]_{|k| = \sqrt{E^2 - 4M_*^2}}.
\end{equation}

In the last equality, we have only considered $s$-wave two-pion scattering, and therefore integrated over the angles in the phase space.\(^3\)

Watson theorem states that, in absence of CP violation,
\begin{equation}
\text{out} \langle \vec{k}, -\vec{k} | H(0)|0 \rangle = e^{i\delta(s)} A(s)
\end{equation}
and
\begin{equation}
\text{in} \langle \vec{k}, -\vec{k} | H(0)|0 \rangle = e^{-i\delta(s)} A(s)
\end{equation}
with $A(s)$ real. The phase $\delta$ is related to the two-pion scattering amplitude by
\begin{equation}
\text{out} \langle \vec{q}, \vec{q} | H(0)|0 \rangle = e^{2i\delta(s)}.
\end{equation}
In the case of on-shell $s$-wave $\pi - \pi$ scattering, the invariant matrix element is given in terms of the phase by
\begin{equation}
\mathcal{M}(s) = \frac{(16\pi)^\sqrt{s}}{\sqrt{s - 4M_\pi^2}} e^{i\delta(s)} \sin(\delta(s)).
\end{equation}
We now make use of our “smoothness” hypothesis to write the off-shell matrix element as
\begin{equation}
\mathcal{M}(\vec{q}, \vec{q} - \vec{k}, -\vec{k}) = \frac{(16\pi)^\sqrt{s}}{\sqrt{s - 4M_\pi^2}} e^{i\delta(s)} \sin(\delta(s)),
\end{equation}
with $s = 4E_k^2$. Inserting the definitions (13) and (17) into eqs. (14) and (12), we obtain
\begin{equation}
P_q(t) = (4E_q) P \int \frac{dE}{2M_*} \frac{e^{-(E - 2E_q)t}}{E^2 - E_q^2} A(E^2) \sin(\delta(E^2))
\end{equation}
and
\begin{equation}
G_3(t) = \frac{\sqrt{Z \Pi}}{2E_q} e^{E \bar{t}} \left[ A(s_q) \cos(\delta(s_q)) + (4E_q) \times \right.
\end{equation}
\begin{equation}
\left. \times \mathcal{P} \int \frac{dE}{2M_*} \frac{e^{-(E - 2E_q)t}}{E^2 - E_q^2} A(s) \sin(\delta(s)) \right]
\end{equation}
with $s_q = 4E_q^2$ and $s = E^2$.

3. MESON DECAYS ON THE LATTICE

To fix our ideas, let us consider $D$ decays. In this case, the starting point is the four-point correlation function
\begin{equation}
G(t_1, t_2, t_D, \vec{q}, -\vec{q}; \vec{p}_D = 0) = \langle \Pi_q(t_1) \Pi_{-q}(t_2) \mathcal{H}_{\Pi}(0) D_{\vec{p}_D=0}(t_D) \rangle
\end{equation}
linear dependence. The generalization of the above formulae to the case of total momentum and angular momentum different from zero is straightforward.
in the limit $t_D \to -\infty, t_1 \to +\infty$. Here $H_W$ is the weak Hamiltonian. In this case eq. (18) becomes

$$G(t_1, t_2, t_D, \vec{q}, -\vec{q}, \vec{p}_D = 0) = \frac{Z_{II}}{4E^2_{\vec{q}} 2M_D} \times$$

$$e^{-E_0(t_1+t_2)-3M_D|t_D|} \left\{ A^W(s_q) \cos \delta(s_q) + (4E_{\vec{q}}) \times$$

$$\times \mathcal{P} \left[ \int_{2M_e}^{\infty} \frac{dE}{\pi} \frac{e^{-E-2E_{\vec{q}} t_2}}{4E^2_{\vec{q}} - E^2} A^W(s) \sin \delta(s) \right] \right\}.$$

For numerical applications, it is convenient to consider the amputated correlation function given by the ratio

$$R^{H_W}(t_2, \vec{q}) = \frac{G(t_1, t_2, t_D, \vec{q}, -\vec{q}, \vec{p}_D = 0)}{S_{II}(t_1, E_{\vec{q}}) S_{II}(t_2, E_{\vec{q}}) S_D(t_D, M_D)} ,$$

where

$$S_{II}(t_1, E_{\vec{q}}) = \frac{\sqrt{Z_{II}}}{2E_{\vec{q}}} e^{-E_{\vec{q}} t_1} ,$$

and similarly for the other meson propagators;

$$R^{H_W}(t_2, \vec{q}) = \left\{ A^W(s_q) \cos \delta(s_q) + (4E_{\vec{q}}) \times$$

$$\times \mathcal{P} \left[ \int_{2M_e}^{\infty} \frac{dE}{\pi} \frac{e^{-E-2E_{\vec{q}} t_2}}{4E^2_{\vec{q}} - E^2} A^W(s) \sin \delta(s) \right] \right\}.$$  

On a lattice of volume $L^3 \times T$, eq. (21) becomes

$$R^{H_W}(t_2, \vec{q}) = A(s_q) \cos \delta(s_q) + \left( \frac{4E_{\vec{q}}}{\pi} \right) \times$$

$$\sum_{E_i} \left[ \Delta E_i \frac{e^{-(E_i-2E_{\vec{q}} t_2)}}{4E^2_{\vec{q}} - E^2_i} A(s) \sin \delta(s) \right] ,$$

(22)

where all the quantities are given in units of the lattice spacing,

$$E_i = \sqrt{s} = 2E_{\vec{k}}, \text{ with } \vec{k} \equiv \frac{2\pi}{L} (n_x, n_y, n_z) ;$$

(23)

$s_q$ has been defined before; $n_{x,y,z} = 0,1, \ldots , L-1$ and $\sum_{E_i}$ denotes the sum over all the values of the energy corresponding to the momenta $\vec{k}$ allowed by the discretization of the space-time on a finite volume, excluding those corresponding to $E_i = 2E_{\vec{q}}$. Different combinations of momenta corresponding to the same energy should be included only once in the sum appearing in eq. (22). $\Delta E_i = E_{i+1} - E_i$ is the difference of the nearest successive allowed values of $E_i$ ($E_0 = 2M_\pi, E_1 = 2\sqrt{M^2_\pi + (2\pi/L)^2}, E_2 = 2\sqrt{M^2_\pi + 2(2\pi/L)^2}, \text{ etc.}$). One can show that the expression in eq. (24) tends to the corresponding continuum one in eq. (21) as $L \to \infty$. However, for presently available lattices, the allowed range of the momenta and of $s_q$ is limited, and therefore eq. (22) might not be a good approximation of eq. (21). One might try to improve the approximation by using $\vec{p}_D \neq 0$ to increase the allowed values of the momenta, and by correcting for the difference between the discrete phase space and the continuum one.

We now explain the strategy to extract the matrix element of $H_W$ from $R^{H_W}(t_2, \vec{q})$. To fix our ideas, we consider the case $\vec{p}_D = 0$. The procedure is the following:

1. Compute $R^{H_W}(t_2, \vec{q} = 0)$. The phase vanishes at threshold, and therefore we have

$$R^{H_W}(t_2, \vec{q} = 0) = A^W(4M^2_{\pi}) ,$$

(24)

and we can extract $A^W(4M^2_{\pi})$.

2. Compute $R^{H_W}(t_2, \vec{q}_1)$ for the first allowed non-zero value of the momentum, $|\vec{q}_1| = (2\pi)/L$. The only value of $k$ that might contribute in the sum in eq. (24), apart from exponentially suppressed terms, is $k = 0$, but at threshold the phase vanishes and therefore there is no contribution from the sum over $k$. Thus we have

$$R^{H_W}(t_2, \vec{q}_1) = A^W(4E^2_1) \cos \delta(4E^2_1) ,$$

(25)

with $E_1 = \sqrt{M^2_{\pi} + (2\pi)^2/(La)^2}$.

3. Compute $R^{H_W}(t_2, \vec{q}_2)$, with $|\vec{q}_2| = \sqrt{(2\pi)/L}$. In this case, the term in the sum over $k$ corresponding to $|k| = (2\pi)/L$ gives an exponentially increasing contribution, and we get

$$R^{H_W}(t_2, \vec{q}_2) = A^W(4E^2_2) \cos \delta(4E^2_2) + \frac{E_2}{\pi} \times$$

$$(E_2 - E_1) \frac{e^{2(E_2-E_1)t_2}}{E^2_2 - E^2_1} A^W(4E^2_1) \sin \delta(4E^2_1),$$
where $E_2 = \sqrt{M_\sigma^2 + 2(2\pi)^2/(La)^2}$.

It is straightforward to derive the expression of $R^{H\Pi}$ for the next steps. In this way, we can extract

$$\tan \delta(s) = \frac{A_s}{A_c} \quad \text{and} \quad |A(s)| = \sqrt{A_c^2 + A_s^2}$$

(26)

as a function of the centre-of-mass energy, where $A_c = A(s) \cos \delta(s)$ and $A_s = A(s) \sin \delta(s)$.

An interesting case, which has been discussed in detail in ref. [5], is the one of FSI’s dominated by the exchange of a resonance $\sigma$ in the $s$-channel. In this case, we can express the matrix elements of $H$ and $\Pi$ in terms of the parameters of the resonance by the

$$\langle \vec{q}, -\vec{q} | H(0) | 0 \rangle_{\text{out}} = \frac{g(s_q)}{M_\sigma^2 - s_q - iX(s_q)}|s_q = p_H^2|,$$

$$\langle \vec{q} | \Pi(0) | \vec{k}, -\vec{k} \rangle_{\text{out}} = \left[ \frac{|V(s)|^2}{M_\sigma^2 - s - iX(s)} \right] \times$$

$$\times \frac{2\sqrt{Z_H}}{(E + 2E_q - i\epsilon)(-E + 2E_q - i\epsilon)}.$$ (27)

The phase shift and the amplitude are given in terms of the parameters of the resonance by the following relations:

$$\frac{g(s)}{M_\sigma^2 - s - iX(s)} = A(s)e^{i\delta(s)}$$

(28)

and

$$\cos \delta(s) = \frac{M_\sigma^2 - s}{\sqrt{(M_\sigma^2 - s)^2 + X(s)^2}}$$

(29)

$$\sin \delta(s) = \frac{X(s)}{\sqrt{(M_\sigma^2 - s)^2 + X(s)^2}}.$$ (30)

By studying the three-point $H - \Pi - \Pi$ correlator $G_q(t_1, t_2)$, we can extract the parameters of the resonance and exploit this information when analyzing the four-point function.

4In ref. [3], the result corresponding to eq. (26) of the present work has been derived in the case of FSI’s dominated by the exchange of a resonance. However, as we have shown above, the result (26) holds under a more general “smoothness” hypothesis.

4. CONCLUSIONS

We have shown that, in spite of the MTNGT, it is in principle possible to extract the amplitude and phase of two-body non-leptonic decay matrix elements, under a reasonable “smoothness” hypothesis. We have sketched the strategy to extract the relevant information from Euclidean correlators computed numerically in lattice QCD. In the case of FSI’s dominated by a resonance, it is possible to extract the parameters of the resonance from the study of a three-point correlation function. A feasibility study of the method we propose is currently under way on the APE machine.

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