Is Complexity Always Proportional to Entanglement Entropy?

Sunandan Gangopadhyay\textsuperscript{γ}, Dharmesh Jain\textsuperscript{ψ†} and Ashis Saha\textsuperscript{σ‡}

\textsuperscript{γ,ψ} Department of Theoretical Sciences, S. N. Bose National Centre for Basic Sciences
Block–JD, Sector–III, Salt Lake City, Kolkata 700106, India

\textsuperscript{σ} Department of Physics, University of Kalyani, Kalyani 741235, India

ABSTRACT

Short answer: No. Long answer: We propose that the definition of complexity, in the “Complexity=Volume” conjecture, needs a slight modification for supergravity solutions with warped AdS factors. Such warp factors can arise due to non-trivial dilaton profile, for example, in $\text{AdS}_6$ solutions of type IIA supergravity. This modified definition ensures that the universal piece of the complexity is proportional to that of the entanglement entropy. It also means that the leading behaviour at large $N$ is the same for both these quantities, as we show for some well-known supergravity solutions (with and without warp factors) in various dimensions. We also discuss what this proportionality entails for the dual field theoretic quantities and propose some “universal” relations.
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1 Introduction

The gauge/gravity correspondence inspired holographic observations have been a matter of great interest as they provide a means to study and uncover fascinating features of strongly coupled field theories via their gravity duals [1–3]. In the landscape of quantum information theory, the holographic computations of entanglement entropy (EE) and quantum complexity (QC) of a conformal field theory (CFT) are prime applications of this correspondence [4–10].

Entanglement Entropy. The EE is a ‘good measure’ of quantum entanglement for a mixed quantum state and represents the amount of information stored in a quantum system. When a system can be divided into two subsystems \( A \) and \( B \), then the definition for EE of subsystem \( A \) (\( S_A \)) follows along the lines of von Neumann entropy:

\[
S_A = -tr[\rho_A \log \rho_A],
\]

(1.1)

where \( \rho_A = tr_B[\rho_{\text{total}}] \) is the reduced density matrix of the subsystem \( A \) obtained by tracing out the degrees of freedom of the subsystem \( B \) from the total density matrix \( \rho_{\text{total}} \) of the whole system. The holographic computation of EE of a CFT in \( d \)-dimensional spacetime is provided by the Ryu-Takayanagi prescription which is defined as [4,5]

\[
S_A = \frac{\text{Area}(\gamma_A)}{4G_N^{(d+1)}},
\]

(1.2)

where \( \gamma_A \) is a co-dimension 2 static minimal surface corresponding to the subsystem \( A \) and \( G_N^{(d+1)} \) is the \((d + 1)\)-dimensional Newton’s gravitational constant. The quantity computed
from (1.2) is usually labelled as holographic EE (or HEE, for short) and we will review its computation for several well-known supergravity solutions.

The entanglement entropy is also related to other field theoretical quantities depending on the spacetime dimensions [5,11,12]. In odd dimensions, the universal piece of entanglement entropy for a spherical subsystem is related to the sphere free energy defined to be (negative of) the logarithm of the partition function of the CFT placed on a \( d \)-sphere: \( S_A = -F_{Sd} \equiv \log |Z_{Sd}| \). In even dimensions, the universal piece of EE is related to the Weyl anomaly \( a_d \) via the relation \( S_A = (-1)^{d-1}4a_d \). This captures a part of the trace of the energy-momentum tensor \( \langle T_{\mu\nu} \rangle \sim (-1)^{\frac{d}{2}}2a_dE_d + \cdots \), where \( E_d \) is the \( d \)-dimensional Euler density.\(^1\) Both the free energy and \( a \)-anomaly are useful in the study of renormalization group flows [13–16]. The holographic computations match the corresponding field theoretical results wherever available.

**Quantum Complexity.** The QC involves minimizing the number of unitary transformations required to transform the state of a system from a reference state to a desired target state. It is a difficult concept to define in a QFT and no satisfactory field theoretical definition of complexity exists yet. But several attempts have been made in field theory to define geometric and circuit complexity [17–20], and path integral complexity [21,22].

In addition, there also have been numerous attempts to define the notion of complexity holographically. Two earlier definitions relate it to the volume of the Einstein-Rosen bridge (ERB) connecting two boundaries of the black hole [6,7], and the bulk action evaluated on its Wheeler-DeWitt patch [8,9]

\[
C_{ERB} = \frac{V_{ERB}}{8\pi L_{AdS}G^{(d+1)}_N}; \quad C_W = \frac{I_{WDW}}{\pi \hbar} .
\]  

(1.3)

Following these, there is yet another “Complexity=Volume” conjecture given in [10], which involves computing the maximal co-dimension 1 volume \( V(\gamma_A) \) enclosed by the co-dimension 2 static minimal surface \( \gamma_A \) (RT surface) foliated into the bulk.\(^2\) Explicitly, it reads

\[
C_A = \frac{V(\gamma_A)}{8\pi L_{AdS}G^{(d+1)}_N} ; 
\]  

(1.4)

where \( L_{AdS} \) is the length scale of the AdS space in consideration. This has been dubbed the holographic subregion complexity (HSC) in the literature. We will focus exclusively on this definition in this note to calculate HSC for a few well-known supergravity solutions containing \( AdS_4–AdS_7 \) spacetimes.

We study the relation between HSC and HEE for various supergravity solutions by focussing on their universal pieces\(^3\). We find that for solutions having a product geometry of pure AdS spacetime and a compact manifold, the universal piece of complexity is proportional to that of the EE. However, for solutions having a warped AdS factor (arising due to non-trivial dilaton profile), the application of (1.4) does not result in such a simple relation, which seems an

\(^1\)In \( d=2 \), the \( a \)-anomaly is related to the central charge \( c \) of the 2d CFT so the relation \( S_A \sim c \) is more common in this case.

\(^2\)The relation between the two “Complexity=Action” and “Complexity=Volume” conjectures have been explored in detail in [23].

\(^3\)The universal piece, in this context, refers to a term that does not depend on the chosen subsystem, up to a logarithmic divergence [5,12].
unlikely result. To remedy that, we propose (2.2) as a slight modification of (1.4) by arguing that the warp factor needs to be taken into account in defining the AdS length scale \( L_{\text{AdS}} \).

Most of the solutions we consider have well-known CFT duals and computation of complexity on the gravity side leads to the prediction for the associated quantity on the CFT side, as expected from AdS/CFT correspondence. This also means that the holographic relation between HEE and HSC leads to a prediction of a similar relation between the associated CFT observables. Thus, the field theoretical complexity (corresponding to the universal piece of HSC, which we will denote simply as \( C \) in the following sections) is predicted to be proportional to the \( a \)-anomaly or the sphere free energy for CFTs in even or odd dimensions, respectively. This fact has not been appreciated in the literature as far as we know, which should lead to a focussed effort in defining and computing the complexity for such dual CFTs, providing further concrete tests of the AdS/CFT correspondence.

The rest of this note is organized as follows. In Section 2, we modify the definition of HSC (1.4) to include AdS spacetimes with warp factors. In Sections 3 and 4, we consider a few well-known 10d and 11d supergravity solutions and find a simple relation between HEE and (modified) HSC. This relation leads to a prediction for field theoretical complexity in terms of either the \( a \)-anomaly or the free energy on sphere, as discussed above. In the final Section 5 we end with a summary of the results and some future directions. We also include Appendix A collecting results of straightforward application of the HSC formula (1.4), when it is different from the modified HSC we define next.

2 Revisiting Complexity

The complexity was defined in [10] by considering pure \( AdS_{p+1} \) spacetime. The supergravity solutions, which arise in the weak gravity limit of superstring or M-theory, are product manifolds involving AdS spacetime and a compact manifold. There can also exist non-trivial warp factors for each component of the product manifold. In general, we can consider the following metric (in Einstein frame) for the full \( (d+1) \)-dimensional spacetime \((d = 9 \) for string theory and \( d = 10 \) for M-theory):

\[
d s^2_{d+1} = L_{\text{AdS}}(x)^2 d s^2_{\text{AdS}_{p+1}} + L_X(x)^2 d s^2_{X_{d-p}},
\]

where we use a suggestive notation for the warp factors multiplying both the AdS metric\(^4\) and metric for the compact manifold \( X \). Such warp factors depending on the \( d-p \) coordinates \( \{ x \} \) of the compact manifold \( X \) can arise due to non-trivial dilaton profile, as we will see later. The existence of warp factor \( L_{\text{AdS}}(x) \) implies that the AdS radius can no longer be considered constant when we have the full supergravity solution. This leads to an ambiguity in applying (1.4) to evaluate the HSC since it is not clear which value of \( L_{\text{AdS}} \) to use. One way to resolve this ambiguity is to "integrate over all \( L_{\text{AdS}} \)'s", of course. By that, we mean a slight modification of (1.4) as follows:\(^5\)

\[
\tilde{C}_A = \frac{1}{8\pi G_N^{(d+1)}} \int_{\gamma_A} d^d x \frac{\sqrt{g^{(d)}}}{L_{\text{AdS}}(x)}. \tag{2.2}
\]

\(^4\)We will take the AdS metric to be of the form \( d s^2_{\text{AdS}_{p+1}} = \frac{1}{z^2} (-dt^2 + d\vec{x}^2 + dz^2) \), with \( d\vec{x}^2 = \sum_{i=1}^{p-1} (dx^i)^2 = dp^2 + \rho^2 d\vec{x}_{p-2}^2 \).

\(^5\)A similar modification was considered in [24] to define central charge originally defined in [25].
Here, \( g^{(d)} \) denotes the determinant of the \( d \)-dimensional metric following from (2.1) for static surfaces, i.e., \( t = 0 \) in the \( AdS_{p+1} \) metric. Also, note that \( \tilde{C}_A \equiv C_A \) as given in (1.4) when \( L_{AdS}(x) \) is constant since \( V(\gamma_A) = \int_{\gamma_A} d^d x \sqrt{g^{(d)}} \) and \( \gamma_A \) denotes the RT surface whose area computes the HEE via the relation (1.2).

One of the motivation for the modified definition (2.2) is that the universal piece of \( \tilde{C}_A \) is proportional to that of \( S_A \) whereas a direct application of (1.4) does not guarantee that (see Appendix A). Such a simple relation between \( S_A \) and \( C_A \) is implicit in [10] and has been further explored in [26–28] where AdS spacetimes are considered without any explicit embeddings in string or M-theory. We revisit those calculations both for HEE and HSC now in the context of the generic metric with warped AdS factor given in (2.1) to prove the proportionality claim.

In order to compute HEE, we consider a subsystem \( A \) realized as a round sphere: \( \rho^2 = \sum_{i=1}^{p-1}(x^i)^2 \leq R^2 \). The embedding of this static \((t = 0)\) RT surface into the bulk is specified by the profile \( \rho = \rho(z) \). The surface area of the RT surface then reads

\[
\text{Area}(\gamma_A) = \int d^{d-p}x L_{AdS}(x)^{p-1} L_X(x)^{d-p} \sqrt{g^{(d-p)}} \text{Vol}(S^{p-2}) \int_{z_0}^{R} dz \frac{\rho(z)^{p-2} \sqrt{1 + \rho'(z)^2}}{z^{p-1}}. \tag{2.3}
\]

In the above integral we have introduced a UV cut-off \( z_0 \) to regularize the area functional.\footnote{The \( z_0 \) can be related to the lattice spacing in the discretized version of the dual field theory [29].}

Solving the Euler-Lagrangian equation obtained from the above area functional we find \( \rho(z) = \sqrt{R^2 - z^2} \). This leads to the following expression for HEE [4,5]

\[
S_A = \frac{\text{Area}(\gamma_A)}{4G_N^{(d+1)}} = \frac{\text{Vol}(S^{p-2})}{4G_N^{(d+1)}} \int d^{d-p}x L_{AdS}(x)^{p-1} L_X(x)^{d-p} \sqrt{g^{(d-p)}} \int_{z_0}^{R} dz \frac{\rho(z)^{p-2} \sqrt{1 + \rho'(z)^2}}{z^{p-1}}
\]

\[
\approx \frac{\text{Vol}(S^{p-2})}{4G_N^{(d+1)}} \mathcal{I}_{(d-p)} \times \begin{cases} (-1)^{n-1} \left( n - \frac{3}{2} \right) \log \left( \frac{2R}{z_0} \right) & p = 2n \\ (-1)^n \left( n - \frac{3}{2} \right)^{-1} & p = 2n + 1 \end{cases}, \tag{2.4}
\]

where we have denoted the \((d - p)\)-dimensional integral as \( \mathcal{I}_{(d-p)} \) and kept only the universal piece of the \( z \)-integral, i.e., log term for even \( p \)-dimensional case and constant term for odd one [5,12].

Now we can compute the HSC for the RT surface specified above using (2.2)

\[
\tilde{C}_A = \frac{1}{8\pi G_N^{(d+1)}} \int_{\gamma_A} d^d x \frac{\sqrt{g^{(d)}}}{L_{AdS}(x)}
\]

\[
= \frac{1}{8\pi G_N^{(d+1)}} \int d^{d-p}x \frac{1}{L_{AdS}(x)} L_{AdS}(x)^p L_X(x)^{d-p} \sqrt{g^{(d-p)}} \text{Vol}(S^{p-2}) \int_{z_0}^{R} dz \int_0^{\sqrt{R^2 - z^2}} dp \frac{\rho(z)^{p-2}}{z^{p}}
\]

\[
\approx \frac{\text{Vol}(S^{p-2})}{8\pi G_N^{(d+1)}} \mathcal{I}_{(d-p)} \times \begin{cases} (-1)^{n-\frac{p}{2}} \pi \frac{1}{2(2n-1)} & p = 2n \\ (-1)^n \frac{\pi}{2n} \log \left( \frac{R}{z_0} \right) & p = 2n + 1 \end{cases}, \tag{2.5}
\]

where we have again kept only the universal pieces [10,28]. Note that the nature of universal pieces in (2.5) is opposite to those obtained for HEE in (2.4). It is now straightforward to show that the universal pieces of \( S_A \) and \( \tilde{C}_A \) are proportional independent of the integral over the
compact manifold by comparing (2.4) and (2.5):

\[
\tilde{C}_A = \frac{1}{4(2n-1)} \left( \frac{n-3}{n-1} \right)^{-1} \frac{S_A}{\log \left( \frac{2R}{z_0} \right)} \quad p = 2n
\]

\[
\tilde{C}_A = \frac{1}{2n \pi} \left( n - \frac{1}{n} \right) S_A \quad p = 2n + 1.
\]  

(2.6)

Since these relations are independent of \( I_{(d-p)} \) and \( G^{(d+1)} \), they do not depend on the explicit embedding in string theory or M-theory and are valid for any generic holographic CFT dual in \( p \)-dimensions. In this sense, they are “universal” relations and we will rewrite them from the CFT point of view in Section 5.

We will now show a few explicit examples of the above relations in the following sections for some well-known supergravity solutions.

3 String Theory Solutions

In this section, we study the relation between HEE and HSC of the 10-dimensional supergravity solutions of the form \( AdS_5 \times X_5 \) and \( AdS_6 \times Y_4 \) and what that entails for associated field theoretical quantities.

3.1 \( AdS_5 \times X_5 \)

The \( AdS_5/CFT_4 \) is the most well-studied AdS/CFT correspondence. Many 4d \( \mathcal{N} \geq 1 \) SCFTs have been constructed that have type IIB string theory duals on \( AdS_5 \times X_5 \), where \( X_5 \) is a compact 5-dimensional Sasaki-Einstein manifold. [1, 2, 30] The 10d supergravity metric in general reads

\[
ds^2 = L^2 \left[ -dt^2 + \frac{dx^2 + dz^2}{z^2} \right] + L^2 ds^2_{X_5},
\]  

(3.1)

where \( d\tilde{x}^2 = \sum_{i=1}^{3}(dx^i)^2 = dp^2 + \rho^2 ds^2_{S^2} \) with \( \text{Vol}(S^2) = 4\pi \). The self-dual 5-form flux quantization relation is given by

\[
\frac{L^4}{l_s^4} = \frac{4\pi^4 N}{\text{Vol}(X_5)}.
\]  

(3.2)

We will also need

\[
G^{(10)}_N = \frac{(2\pi l_s)^8}{32\pi^2}
\]  

(3.3)

relating the 10d gravitational constant to string length \( l_s \).

We follow the generic calculation done in the previous section to compute HEE here. That is, we consider a spherical subsystem \( A \) given by \( \rho^2 = \sum_{i=1}^{3}(x^i)^2 \leq R^2 \), whose embedding into the bulk is \( \rho = \rho(z) = \sqrt{R^2 - z^2} \). This leads to the following expression for HEE:

\[
S_A = \frac{\text{Area}(\gamma_A)}{4G^{(10)}_N} = \frac{8\pi^2 L^8}{(2\pi l_s)^8 \text{Vol}(X_5)} \text{Vol}(S^2) \int_{z_0}^{R} dz \frac{\rho(z)^2 \sqrt{1 + \rho^2(z)^2}}{z^3} = \frac{2\pi^3 N^2}{\text{Vol}(X_5)^2} \text{Vol}(X_5) \left[ -\frac{1}{4} - \frac{1}{2} \log \left( \frac{2R}{z_0} \right) + \frac{R^2}{2z_0^2} + \mathcal{O}(z_0^3) \right]
\]
\[ \approx - \frac{\pi^3 N^2}{\text{Vol}(X_5)} \log \left( \frac{2R}{z_0} \right). \] (3.4)

The coefficient of the log term is the universal piece, which is equal to the 4d Weyl anomaly as follows
\[ \frac{S_A}{\log \left( \frac{2R}{z_0} \right)} = -4a_{4d} \quad \Rightarrow \quad a_{4d} = \frac{\pi^3 N^2}{4 \text{Vol}(X_5)}. \] (3.5)

Note the \( N^2 \) dependence and that matches the \( a \)-anomaly at large \( N \) for 4d SCFTs \([31, 32]\).

Now, we proceed to compute the volume enclosed by the embedding RT surface, which is given by
\[ V(\gamma_A) = L^9 \text{Vol}(X_5) \text{Vol}(S^2) \int_{z_0}^{R} dz \int_{z_0}^{\sqrt{R^2-z^2}} d\rho \frac{\rho^2}{z^4}. \] (3.6)

Since \( L_{\text{AdS}} = L \) is a constant, \( \tilde{C}_A = C_A \) and the HSC can be easily evaluated to be
\[ \tilde{C}_A = \frac{V(\gamma_A)}{8\pi LG_N^{(10)}} = \frac{L^9(4\pi)}{L(2\pi l_s)^8} \text{Vol}(X_5)(4\pi) \int_{z_0}^{R} dz \int_{z_0}^{\sqrt{R^2-z^2}} d\rho \frac{\rho^2}{z^4} \approx \text{Vol}(X_5) \frac{\pi^2 N^2}{\text{Vol}(X_5)^2} \left[ \frac{\pi}{6} + \frac{R^3}{9z_0^3} - \frac{R}{2z_0} + O(z_0) \right] \approx \frac{\pi^3 N^2}{6 \text{Vol}(X_5)}. \] (3.7)

We again keep only the universal piece in the last step, which is the \( R \)-independent term here. Comparing it with (3.4), we obtain a relation between the 4d Weyl anomaly and the field theoretical complexity \( C_{4d} \):
\[ \tilde{C}_A = -\frac{1}{6} \frac{S_A}{\log \left( \frac{2R}{z_0} \right)} \quad \Rightarrow \quad C_{4d} = \frac{2}{3} a_{4d}. \] (3.8)

### 3.2 \( AdS_6 \times Y_4 \)

The 5d \( \mathcal{N} = 1 \) SCFTs have seen a lot of activity recently and have been engineered in both type IIA and IIB string theory. One of the simplest class of 5d SCFTs is that of Seiberg theories whose gravity duals are given by massive type IIA string theory on \( AdS_6 \times S^4 \) \([33–35]\). The 10d supergravity metric in string frame explicitly reads
\[ ds^2 = \frac{L^2}{(\sin \alpha)^{\frac{1}{2}}} \left[ -dt^2 + dx^2 + dz^2 \right] + \frac{4L^2}{9(\sin \alpha)^{\frac{1}{2}}} \left( d\alpha^2 + \cos^2 \alpha \, ds^2_{S^3/Z_n} \right), \] (3.9)

where \( dx^2 = \sum_{i=1}^{4}(dx^i)^2 = d\rho^2 + \rho^2 ds^2_{S^3} \) with \( \text{Vol}(S^3) = 2\pi^2 \) and \( \alpha \in (0, \frac{\pi}{2}] \). The dilaton and 4-form flux quantization relation are given by
\[ e^{-2\phi} = \frac{3(8 - N_f)^{\frac{3}{2}} \sqrt{nN}}{2\sqrt{2\pi}} (\sin \alpha)^{\frac{1}{2}}, \] (3.10)

\[ \frac{L^4}{l_s^4} = \frac{18\pi^2 n N}{8 - N_f}. \] (3.11)

We again choose a spherical subsystem \( A \) and following the RT prescription, we find the
entanglement entropy\(^7\) [36]

\[
S_A = \frac{\text{Area}(\gamma_A)}{4G_N^{(10)}} = \frac{2}{(2\pi)^6 L_s^8} \int d^8 x e^{-2\phi} \sqrt{g^{(8)}}
\]

\[
= \frac{(8 - N_f)^{\frac{3}{2}} \sqrt{N}}{3^3 \sqrt{2} n^3} \left[ \frac{18\pi^2 n N}{8 - N_f} \right]^{\frac{2}{3}} \int_0^\pi d\alpha \sin^\frac{1}{3} \alpha \cos^3 \alpha \int_{z_0}^R d\rho \rho^3 \sqrt{1 + \rho'(z)^2} \frac{z^4}{z^4}
\]

\[
= \frac{3 \times 4\pi n^{\frac{3}{2}} N^{\frac{3}{2}}}{\sqrt{2}(8 - N_f)} \frac{9}{20} \left[ \frac{2}{3} - \frac{R}{z_0} + \frac{R^3}{3 z_0^3} \right]
\]

\[
\approx \frac{9\sqrt{2} n^{\frac{3}{2}} N^{\frac{3}{2}}}{5\sqrt{8 - N_f}}.
\]  

(3.12)

We again keep the universal piece in the last step. The above result satisfies the relation

\[S_A = -F_{S^5},\]

where \(F_{S^5}\) is the \(S^5\) free energy of the Seiberg theories, as shown in [36].

We now compute HSC using the modified definition of complexity (2.2) here\(^8\) because as is clear from the metric (3.9) and dilaton profile (3.10), the AdS radius is not constant but depends on the \(\alpha\) coordinate of the compact manifold as follows (in Einstein frame):

\[
L_{\text{AdS}}(x) = L(\sin^\frac{1}{6} \alpha) e^{-\phi}.
\]  

(3.13)

This leads to the same large \(N\) scaling for HSC as that of HEE:

\[
\tilde{C}_A = \frac{1}{8\pi G_N^{(10)}} \int d^9 x e^{-2\phi} \sqrt{g^{(9)}} L(\sin^\frac{1}{6} \alpha) e^{-2\phi} = \frac{2}{(2\pi)^7 L_s^8} \int d^9 x e^{-2\phi} \sin^\frac{1}{3} \alpha \sqrt{g^{(9)}}
\]

\[
= \frac{(8 - N_f)^{\frac{3}{2}} \sqrt{N}}{3^3 \sqrt{2} n^4} \left[ \frac{18\pi^2 n N}{8 - N_f} \right]^{\frac{2}{3}} \int_0^\pi d\alpha \sin^\frac{1}{3} \alpha \cos^3 \alpha \int_{z_0}^R dh \int_0^\sqrt{R^2 - z^2} d\rho \rho^3
\]

\[
= \frac{6\alpha^{\frac{3}{2}} N^{\frac{3}{2}}}{\sqrt{2}(8 - N_f)} \frac{9}{20} \left[ \frac{3}{16} + \frac{1}{4} \log \left( \frac{R}{z_0} \right) - \frac{R^2}{4 z_0^2} + \frac{R^4}{16 z_0^4} \right]
\]

\[
\approx \frac{27\sqrt{2} n^{\frac{3}{2}} N^{\frac{3}{2}}}{80\sqrt{8 - N_f}} \log \left( \frac{R}{z_0} \right).
\]  

(3.14)

We have again kept the universal piece in the last step, which gives the expected relation between free energy and field theoretical complexity:

\[
\frac{\tilde{C}_A}{\log \left( \frac{R}{z_0} \right)} = \frac{3}{16\pi} S_A \quad \Rightarrow \quad C_{5d} = -\frac{3}{16\pi} F_{S^5}.
\]  

(3.15)

4 M-theory Solutions

In this section, we again verify that the HEE and HSC are proportional for SCFTs with well-known supergravity duals arising in the weak gravity limit of M-theory and discuss what that means for the corresponding field theoretical quantities.

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\(^7\)This calculation is to be done in Einstein frame, so we need to use \(g_{\mu\nu}^E \to e^{-\phi} g_{\mu\nu}^s \Rightarrow \sqrt{g^{(8),E}} \to e^{-2\phi} \sqrt{g^{(8),s}}\).

\(^8\)See Appendix A for naive application of the definition (1.4).
4.1 $AdS_4 \times Y_7$

The AdS/CFT$_3$ correspondence was put on a concrete footing after the discovery of $\mathcal{N} = 6$ ABJM theory [37] describing the low energy limit of a stack of $N$ M2-branes placed at the tip of cone over $S^7/\mathbb{Z}_k$. In the large $N$ limit, ABJM theory is dual to M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$. After this discovery, a large number of 3d $\mathcal{N} \geq 2$ SCFTs with M-theory duals have been identified by replacing $S^7/\mathbb{Z}_k$ with $Y_7$, a compact (tri-)Sasaki-Einstein 7-manifold. Following [38], we can write the general metric for the 11d supergravity solution as

$$ds^2 = \frac{L^2}{4} \left[ -dt^2 + \frac{d\vec{x}^2 + dz^2}{z^2} \right] + L^2 ds^2_{Y_7}, \tag{4.1}$$

where $d\vec{x}^2 = \sum_{i=1}^{2}(dx^i)^2 = d\rho^2 + \rho^2 d\theta^2$ with $0 \leq \theta < 2\pi$ and the 4-form flux quantization condition that relates the geometric length scale $L$ to Planck length $l_p$:

$$\frac{L^6}{l_p^6} = \frac{(2\pi)^6 N}{6 \text{Vol}(Y_7)}. \tag{4.2}$$

We will also use the relation of 11d gravitational constant to $l_p$:

$$G^{(11)}_N = \frac{(2\pi l_p)^9}{32\pi^2}. \tag{4.3}$$

Following the RT prescription for a spherical subsystem $A$, we find for the entanglement entropy

$$S_A = \frac{\text{Area}(\gamma_A)}{4G^{(11)}_N} = \frac{2}{(2\pi)^7 l_p^6} \int_{0}^{\pi} d\theta \int_{\rho(z)}^{R} d\rho \text{Vol}(Y_7) \rho(z) \sqrt{1 + \rho'(z)^2} (2\pi)^2 \rho(z)
\approx \frac{\text{Vol}(Y_7)}{2(2\pi)^6} \left[ \frac{(2\pi)^6 N}{6 \text{Vol}(Y_7)} \right]^{\frac{1}{2}} \left[ -1 + \frac{R}{z_0} \right]$$
\approx -\frac{\sqrt{2}\pi^3 N^{\frac{1}{2}}}{3\sqrt{3} \text{Vol}(Y_7)}, \tag{4.4}$$

where we keep only the universal piece ($R$-independent term) in the last step. It is a well-known fact that the HEE as given in (4.4) matches the $S^3$ free energy of the dual SCFTs in the large $N$ limit via $S_A = -F_{S^3}$ [39–41].

Now, we proceed to compute the volume enclosed by the embedding RT surface, which is given by

$$V(\gamma_A) = 2\pi L^{10} \int_{z_0}^{R} dz \int_{0}^{\sqrt{R^2 - z^2}} d\rho \text{Vol}(Y_7) \frac{\rho}{(2\pi)^3}.$$

Since $L_{\text{AdS}} = \frac{1}{2}$, we have $\tilde{C}_A = C_A$ and so the complexity turns out to be

$$\tilde{C}_A = \frac{V(\gamma_A)}{8\pi \left( \frac{L}{2} \right) G^{(11)}_N} \approx \frac{\text{Vol}(Y_7)}{2(2\pi)^7} \left[ \frac{(2\pi)^6 N}{6 \text{Vol}(Y_7)} \right]^{\frac{1}{2}} \left[ -\frac{1}{4} - \frac{1}{4} \log \left( \frac{R}{z_0} \right) + \frac{R^2}{4z_0^2} \right]$$
\approx -\frac{\sqrt{2}\pi^3 N^{\frac{1}{2}}}{12\sqrt{3} \text{Vol}(Y_7)} \log \left( \frac{R}{z_0} \right). \tag{4.6}$$
Note that \( \tilde{C}_A \) also scales as \( N^{\frac{3}{2}} \) just like \( S_A \). In this case, the universal piece is the coefficient of the logarithmic term and hence, comparing it with \( S_A \), we get the following relation:

\[
\frac{\tilde{C}_A}{\log \left( \frac{R}{z_0} \right)} = \frac{1}{4\pi} S_A \quad \Rightarrow \quad C_{3d} = -\frac{1}{4\pi} F_{S^3}. \tag{4.7}
\]

The above relation implies that in the large \( N \) limit, field theoretical complexity is proportional to the \( S^3 \) free energy for the 3d SCFTs having M-theory duals.

### 4.2 Uplift of NATD of \( AdS_5 \times S_5 \)

Let us now consider the M-theory uplift of the solution obtained by applying nonabelian T-duality to \( AdS_5 \times S_5 \).\(^9\) The details are in \([24,42]\) and we collect here only the relevant expressions including the 11d metric

\[
ds^2 = e^{-\frac{2\phi}{\alpha'}} ds_{AdS_5}^2 + e^{\frac{4\phi}{\alpha'}} \left( dy - 2\frac{L^4 \cos^4 \alpha}{\alpha'^2} d\theta \right)^2 + e^{-\frac{2\phi}{\alpha'}} \left[ 4L^2 \left( d\alpha^2 + \sin^2 \alpha d\theta^2 \right) + \frac{\alpha'^2 d\beta^2}{L^2 \cos^2 \alpha} + \frac{e^{2\phi} L^4 \beta^2 \cos^4 \alpha \left( d\xi^2 \sin^2 \chi + d\chi^2 \right)}{\alpha'} \right], \tag{4.8}
\]

where we use the \( AdS_5 \) metric given in (3.1) and \( e^{-2\phi} = \frac{L^2}{\alpha'} \cos^2 \alpha \left( L^4 \cos^4 \alpha + \alpha'^2 \beta^2 \right) \). The flux quantization condition gives the following relation

\[
L^4 = 2^\frac{8}{3} N^{\frac{2}{9}} \alpha'^2, \tag{4.9}
\]

where \( \gamma \) is introduced by scaling the coordinate \( y \to (\frac{L^2}{\alpha'})^\gamma \sqrt{\alpha'} y \) due to an ambiguity in the uplifting procedure. We will also use (only in this subsection) \( G_N^{(11)} = \alpha'^{\frac{2}{3}} \) following [24], relating the 11d gravitational constant to string tension \( \alpha' \).

To compute HEE, we again consider a spherical subsystem and following the RT prescription, we have the surface area integral given by

\[
\text{Area}(\gamma_A) = 4L^8 \text{Vol}(S^2) \int_{z_0}^{R} dz \int_{0}^{2\pi} dy d\theta d\xi \int_{0}^{\pi} d\beta d\chi \int_{0}^{\frac{\pi}{2}} d\alpha \times \left( \frac{L^2}{\alpha'} \right)^\gamma \sqrt{\alpha'} \beta^2 \cos^3 \alpha \sin \alpha \sin \chi \frac{\rho(z)^2 \sqrt{1 + \rho'(z)^2}}{z^3}. \tag{4.10}
\]

Again, setting \( \rho(z) = \sqrt{R^2 - z^2} \) as in the previous examples, we get for HEE

\[
S_A = \frac{\text{Area}(\gamma_A)}{4G_N^{(11)}} = \frac{L^8}{\alpha'^\frac{4}{3}} \left( \frac{L^2}{\alpha'} \right)^\frac{\gamma}{3} \frac{4\pi^6}{9} \text{Vol}(S^2) \int_{z_0}^{R} dz \frac{\rho(z)^2 \sqrt{1 + \rho'(z)^2}}{z^3}
\approx \frac{2^{8(1+\frac{2}{3})} \pi^{7} N^{1+\frac{2}{3}}}{3} \left[ -\frac{1}{4} - \frac{1}{2} \log \left( \frac{2R}{z_0} \right) + \frac{R^2}{2z_0^2} + O(z_0^2) \right]
\]

\(^9\)We consider here only the case of \( S^5 \). Other cases discussed in [24] yield similar results as one can verify.
\[
\approx -\frac{2^{8(1+\frac{2}{\gamma})}\pi^7 N^{1+\frac{4}{\gamma}}}{6} \log \left( \frac{2R}{z_0} \right). \quad (4.11)
\]

We have kept the universal piece in the last step, which should equal the 4d Weyl anomaly:

\[
\frac{S_A}{\log \left( \frac{2R}{z_0} \right)} = -4a_{4d} \quad \Rightarrow \quad a_{4d} = \frac{2^{8(1+\frac{2}{\gamma})}\pi^7 N^{1+\frac{4}{\gamma}}}{24}. \quad (4.12)
\]

Note that \( a_{4d} = \frac{\pi}{8}c \), where \( c \) is the central charge for this solution obtained in [24]. For \( \gamma = 4 \), we have the usual \( N^2 \) scaling of 4d and for \( \gamma = 2 \), we have \( N^3 \) scaling reminiscent of 6d, that we will see in the next example.

Now, we compute the HSC but since the AdS radius \( L_{AdS}(x) = e^{-\frac{\Phi}{3}} \) is coordinate dependent, we use the modified complexity (2.2) to obtain\(^{10}\)

\[
\tilde{C}_A = \frac{1}{8\pi G_N^{(11)}} 4L^8 \sqrt{\alpha'} \left( L^2 \right)^\gamma \frac{16\pi^7}{3} \int_0^R dz \int_0^\sqrt{R^2-z^2} d\rho \rho^2 \cdot \left[ \frac{\pi}{6} + \frac{R^3}{9z_0^3} - \frac{R}{2z_0^2} + \mathcal{O}(z_0) \right] \]

\[
\approx \frac{2^{8(1+\frac{2}{\gamma})} \pi^6 N^{1+\frac{4}{\gamma}}}{6} \left[ \frac{\pi}{6} + \frac{R^3}{9z_0^3} - \frac{R}{2z_0^2} + \mathcal{O}(z_0) \right] \approx \frac{2^{8(1+\frac{2}{\gamma})} \pi^7 N^{1+\frac{4}{\gamma}}}{36}. \quad (4.13)
\]

We again keep only the universal piece (\( R \)-independent term) in the last step. Comparing it with (4.11), we obtain a relation between the 4d Weyl anomaly and the field theoretical complexity:

\[
C_A = -\frac{1}{6} \frac{S_A}{\log \left( \frac{2R}{z_0} \right)} \quad \Rightarrow \quad C_{4d} = \frac{2}{3}a_{4d}. \quad (4.14)
\]

This is the same relation that we got for the \( AdS_5 \times X_5 \) solution. In fact, this relation is “universal” for \( AdS_5 \) and is independent of the uplift to either string theory or M-theory as expected from the general discussion of Section 2.

### 4.3 \( AdS_7 \times X_4 \)

The 6d SCFTs are strongly interacting non-Lagrangian theories describing the low energy limit of \( N \) M5-branes. At large \( N \), these SCFTs are dual to M-theory on \( AdS_7 \times S^4/\Gamma \), where the compact manifold \( X_4 \) can only be an orbifold of the 4-sphere \( S^4 \) with \( \Gamma \) being a discrete subgroup of \( SU(2) \) [1, 43]. The metric of this 11d supergravity solution explicitly reads

\[
ds^2 = L^2 \left[ -dt^2 + \frac{d\vec{x}^2 + dz^2}{z^2} \right] + \frac{L^2}{4} ds_{S^4/\Gamma}^2,
\]

where \( d\vec{x}^2 = \sum_{i=1}^5 (dx^i)^2 = d\rho^2 + \rho^2 ds_{S^4}^2 \) with \( \text{Vol}(S^4) = \frac{8\pi^2}{3} \). The 4-form flux quantization relation is given by

\[
\frac{L^3}{\rho^3} = 8\pi|\Gamma|N. \quad (4.16)
\]

Similar to the previous examples, we choose a spherical geometry of the subsystem \( A \) with

\(^{10}\)See Appendix A for the naive result from the definition (1.4).
the profile of the corresponding RT surface being \( \rho(z) = \sqrt{R^2 - z^2} \), which leads to

\[
S_A = \frac{\text{Area}(\gamma_A)}{4G_N^{(11)}} = \frac{2}{(2\pi)^7} \frac{L^9}{L_p^9} \int_{z_0}^{R} dz \frac{\text{Vol}(S^4)}{z^4} \frac{\text{Vol}(S^4/\Gamma)}{z^5} \frac{\rho(z)^4 \sqrt{1 + \rho'(z)^2}}{z^5} \\
\approx \frac{1}{8(2\pi)^2} \frac{8\pi^2}{3} \frac{8\pi^2}{3|\Gamma|} \left[ \frac{9}{32} + \frac{3}{8} \log \left( \frac{2R}{z_0} \right) - \frac{3R^2}{4z_0^2} + \frac{R^4}{4z_0^4} + \mathcal{O}(z_0^2) \right] \\
\approx \frac{4N^3|\Gamma|^2}{3} \log \left( \frac{2R}{z_0} \right),
\]

where we keep only the universal piece in the last step with the famous \( N^3 \) scaling [39]. The coefficient of the log term in \( S_A \) is proportional to the 6d Weyl anomaly:

\[
\frac{S_A}{\log \left( \frac{2R}{z_0} \right)} = 4a_{6d} \quad \Rightarrow \quad a_{6d} = \frac{1}{3} N^3|\Gamma|^2.
\]

This matches the \( a \)-anomaly at large \( N \) for 6d SCFTs, at least the \( N^3|\Gamma|^2 \) factor [44–46].\footnote{The exact coefficient seems to depend on a ‘scheme-dependent’ definition of the 6d Euler density, or equivalently, the choice of renormalization of the anomaly contribution of the free \( N = (2, 0) \) tensor multiplet. We do not attempt to fix this coefficient here.}

Now, we can compute the complexity (\( C_A = C_A \) here) following steps similar to the previous examples and it reads

\[
C_A = \frac{V(\gamma_A)}{8\pi LG_N^{(11)}} = \frac{2}{(2\pi)^8} \frac{L^9}{L_p^9} \frac{8\pi^2}{3} \frac{8\pi^2}{3|\Gamma|} \frac{1}{2^4} \int_{z_0}^{R} dz \int_{z_0}^{\sqrt{R^2 - z^2}} \frac{d\rho}{z_0} \frac{\rho^4}{z_0^6} \\
\approx \frac{1}{9 \cdot 2(2\pi)^4 |\Gamma|} \left[ \frac{\pi}{10} + \frac{R^5}{25z_0^2} - \frac{R^3}{6z_0^2} + \frac{8R^4}{8z_0} + \mathcal{O}(z_0) \right] \\
\approx -\frac{8N^3|\Gamma|^2}{45}.
\]

We have again kept only the universal piece in the last step and comparing with the \( S_A \) result in (4.17), we obtain

\[
C_A = -\frac{2}{15} \frac{S_A}{\log \left( \frac{2R}{z_0} \right)} \quad \Rightarrow \quad C_{6d} = -\frac{8}{15} a_{6d}.
\]

The above relation implies that the field theoretical complexity is proportional to the 6d Weyl anomaly in the large \( N \) limit.

\section{5 Discussion}

We have computed holographic subregion complexity following the “Complexity=Volume” conjecture in \( \text{AdS}_{p+1} \) with \( p = 3, 4, 5, 6 \) for specific supergravity solutions, most of which are known to have explicit SCFT duals. We found that the universal piece of HSC is proportional to that of HEE calculated holographically via the RT-prescription for those AdS backgrounds without warp factors, as has been expected in the literature. However, we observe that in case of gravity duals with non-trivial warp factors (due to a non-trivial dilaton profile) modifying the AdS
part of the supergravity backgrounds, the expected proportionality between HSC and HEE does not hold anymore. In order to retain this simple relation, we propose a modification of the holographic formula to compute complexity as explained in Section 2. The existence of a warp factor implies that the $L_{\text{AdS}}$ should not be considered as constant anymore, leading us to the modified definition of complexity in (2.2). This simple fact drastically affects the computation of the volume enclosed by the co-dimension 2 RT surface, as one can contrast the calculations of HSC using (1.4) in the Appendix A with those using (2.2) in Sections 3 and 4.

The relation between HEE and HSC is of great importance as it enables us to predict the behavior of field theoretical analog of complexity. We find that at large $N$, in odd dimensional CFTs, the field theoretical complexity (corresponding to the universal piece of HSC) is proportional to its sphere free energy $F_{Sp}$, whereas for even dimensional CFTs, it is proportional to the Weyl $a$-anomaly. We can write a general relation for these quantities, as it straightforwardly follows from (2.6) and the relation of $S_{A}$ to $F_{Sp}$ or $a$-anomaly [12]:

$$C_{p} = \begin{cases} \frac{(-1)^{n}}{2n} \binom{n-\frac{1}{2}}{\frac{p}{n}} F_{Sp} & p = 2n + 1 \\ \frac{(-1)^{n}}{2n-1} \binom{n-\frac{1}{2}}{\frac{p}{n}-1} a_{p} & p = 2n \end{cases}$$

Note that these relations hold irrespective of the explicit nature of the dual gravity theory whether embedded in string theory or M-theory. We take this “universal” relation (for a given $p$, of course) as a justification for the modification we propose for the holographic prescription to compute HSC.

Even though, a satisfactory and universal definition of complexity in field theory is lacking at present, the definition involving path integral optimization [21,22] seems to be promising as it could lead to application of localization techniques for computing complexity. These techniques have been remarkably successful in obtaining exact results for $F$’s & $a$’s in SCFTs in various dimensions [47], which we used to compare holographic results in the large $N$ limit. Another set of “universal” relations can be obtained between complexities across dimensions by employing the results of [48]. For example, $C_{3d} = -\frac{22}{27} (g-1)C_{5d}$, given that $F_{S^3} = -\frac{2}{9} (g-1)F_{S^5}$ for 5d theories defined on $S^3 \times \Sigma_{g}$ with a topological twist on $\Sigma_{g}$ [49].

It is also worth mentioning that many proposals have been given which relate the HSC with other information theoretical quantities, like the Fisher information metric and the Bures metric (fidelity susceptibility) [50–53]. These are standard notion of distances in quantum information theory [54–56]. This in turn leads to the fact that by figuring out the correct field theoretical definition of complexity, one can relate these metrics to well-studied calculable properties (like the free energy, Weyl $a$-anomaly) of CFTs.

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A Naive Complexity Calculations

This appendix collects the computation of HSC using the expression (1.4) for the examples in Subsections 3.2 and 4.2 with non-trivial warp factors leading to different large $N$ scaling compared to HEE. This, in part, led us to modify (1.4) to the expression given in (2.2).

A.1 $AdS_6 \times Y_4$

Here is the result one would get by naively using the formula (1.4) to compute the HSC:

\[
C_A = \frac{V(\gamma_A)}{8\pi LG_N^{(10)}} = \frac{2}{(2\pi)^2 L_5^8} \int d^8 x e^{-\frac{\phi}{4\alpha'}} \sqrt{g^{(9)}}
\]

\[
= \frac{(8 - N_f) \frac{2^\gamma}{\pi} N \frac{n_0}{n_f}}{18 \times 2 \frac{\pi}{\alpha} \times 3 \frac{\pi}{\alpha} \frac{n_0}{n_f}} \left[ \frac{18\pi^2 n N}{8 - N_f} \right]^2 \int_0^{\frac{\pi}{2}} d\alpha \sin^2 \alpha \cos^3 \alpha \int_0^R dz \int_0^{\sqrt{R^2 - z^2}} d\rho \rho^3
\]

\[
= \frac{2\pi^2 3^\frac{\gamma}{n} N \frac{n_0}{n_f}}{\pi^2 (8 - N_f) \frac{n_0}{n_f}} \frac{128}{297} \left[ 3 \frac{16}{16} + \frac{1}{4} \log \left( \frac{R}{z_0} \right) - \frac{R^2}{4z_0^2} + \frac{R^4}{16z_0^4} \right]
\]

\[
\approx \frac{32 \times 2 \frac{\pi}{\alpha} \frac{n_0}{n_f} N \frac{n_0}{n_f}}{33 \times 3 \frac{\pi}{\alpha} (8 - N_f) \frac{n_0}{n_f}} \log \left( \frac{R}{z_0} \right) . \quad (A.1)
\]

We kept the universal piece in the last line, which has a different large $N$ scaling as compared to $S_A$ in (3.12).

A.2 Uplift of NATD of $AdS_5 \times S_5$

Following is the result obtained by naively using the formula (1.4) to compute the HSC:

\[
C_A = \frac{V(\gamma_A)}{8\pi LG_N^{(11)}} = \frac{L^8 \sqrt{\alpha'}}{2\pi^2 \alpha^{\frac{3}{2}}} \left( \frac{L^2}{\alpha'} \right)^{\gamma} (4\pi) \int_0^{2\pi} dy d\theta d\xi \int_0^\pi d\beta d\chi
\]

\[
\times \int_0^{\frac{\pi}{2}} d\alpha e^{-\frac{\phi}{\alpha} \beta^2 \cos^3 \alpha \sin \alpha \sin \chi} \int_0^R dz \int_0^{\sqrt{R^2 - z^2}} d\rho \rho^2
\]

\[
\approx 2 \times 2^{8(1 + \frac{3}{7})\pi^2 N^{1 + \frac{2}{7}}} \left( 2^{\frac{3}{2} N \frac{h}{2}} \right)^{\frac{1}{2}} \int_0^{\frac{\pi}{2}} d\alpha \int_0^\pi d\beta \beta^2 \cos^4 \alpha \sin \alpha \left[ \frac{\pi}{6} + \frac{R^3}{9z_0} - \frac{R}{2z_0} + O(z_0) \right]
\]

\[
\approx \frac{2^{2(4 + \frac{3}{7})\pi^2 N^{1 + \frac{2}{7}}}}{45} . \quad (A.2)
\]

We again have the universal piece in the last line with a different large $N$ scaling when compared to $S_A$ in (4.11).
References

[1] J. M. Maldacena, “The Large N Limit of Superconformal Field Theories and Supergravity”, *Int. J. Theor. Phys.* **38** (1999) 1113, arXiv:hep-th/9711200.

[2] E. Witten, “Anti-de Sitter Space and Holography”, *Adv. Theor. Math. Phys.* **2** (1998) 253, arXiv:hep-th/9802150.

[3] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, “Large N Field Theories, String Theory and Gravity”, *Phys. Rept.* **323** (2000) 183, arXiv:hep-th/9905111.

[4] S. Ryu and T. Takayanagi, “Holographic Derivation of Entanglement Entropy from AdS/CFT”, *Phys. Rev. Lett.* **96** (2006) 181602, arXiv:hep-th/0603001.

[5] S. Ryu and T. Takayanagi, “Aspects of Holographic Entanglement Entropy”, *JHEP* **08** (2006) 045, arXiv:hep-th/0605073.

[6] L. Susskind, “Computational Complexity and Black Hole Horizons”, *Fortsch. Phys.* **64** (2016) 24, [Addendum: Fortsch.Phys. 64 (2016) 44], arXiv:1403.5695 [hep-th].

[7] D. Stanford and L. Susskind, “Complexity and Shock Wave Geometries”, *Phys. Rev. D* **90[12]** (2014) 126007, arXiv:1406.2678 [hep-th].

[8] A. R. Brown, D. A. Roberts, L. Susskind, B. Swingle and Y. Zhao, “Holographic Complexity Equals Bulk Action?”, *Phys. Rev. Lett.* **116[19]** (2016) 191301, arXiv:1509.07876 [hep-th].

[9] A. R. Brown, D. A. Roberts, L. Susskind, B. Swingle and Y. Zhao, “Complexity, Action, and Black Holes”, *Phys. Rev. D* **93[8]** (2016) 086006, arXiv:1512.04993 [hep-th].

[10] M. Alishahiha, “Holographic Complexity”, *Phys. Rev. D92[12]* (2015) 126009, arXiv:1509.06614 [hep-th].

[11] R. C. Myers and A. Sinha, “Holographic c-theorems in Arbitrary Dimensions”, *JHEP* **01** (2011) 125, arXiv:1011.5819 [hep-th].

[12] H. Casini, M. Huerta and R. C. Myers, “Towards a Derivation of Holographic Entanglement Entropy”, *JHEP* **05** (2011) 036, arXiv:1102.0440 [hep-th].

[13] A. Zamolodchikov, “”Irreversibility” of the Flux of the Renormalization Group in a 2D Field Theory”, *JETP Lett.* **43** (1986) 730.

[14] J. L. Cardy, “Is There a c Theorem in Four-Dimensions?”, *Phys. Lett. B* **215** (1988) 749.

[15] Z. Komargodski and A. Schwimmer, “On Renormalization Group Flows in Four Dimensions”, *JHEP* **12** (2011) 099, arXiv:1107.3987 [hep-th].

[16] D. L. Jafferis, I. R. Klebanov, S. S. Pufu and B. R. Safdi, “Towards the $F$-Theorem: $\mathcal{N} = 2$ Field Theories on the Three-Sphere”, *JHEP* **06** (2011) 102, arXiv:1103.1181 [hep-th].

[17] M. A. Nielsen, M. R. Dowling, M. Gu and A. C. Doherty, “Quantum Computation as Geometry”, *Science* **311** (2006) 1133, arXiv:quant-ph/0603161.
[18] M. R. Dowling and M. A. Nielsen, “The Geometry of Quantum Computation”, *Quantum Info. Comput.* 8[10] (2008) 861, arXiv:quant-ph/0701004.

[19] S. Chapman, M. P. Heller, H. Marrochio and F. Pastawski, “Toward a Definition of Complexity for Quantum Field Theory States”, *Phys. Rev. Lett.* 120[12] (2018) 121602, arXiv:1707.08582 [hep-th].

[20] R. Jefferson and R. C. Myers, “Circuit Complexity in Quantum Field Theory”, *JHEP* 10 (2017) 107, arXiv:1707.08570 [hep-th].

[21] P. Caputa, N. Kundu, M. Miyaji, T. Takayanagi and K. Watanabe, “Liouville Action as Path-Integral Complexity: From Continuous Tensor Networks to AdS/CFT”, *JHEP* 11 (2017) 097, arXiv:1706.07056 [hep-th].

[22] A. Bhattacharyya, P. Caputa, S. R. Das, N. Kundu, M. Miyaji and T. Takayanagi, “Path-Integral Complexity for Perturbed CFTs”, *JHEP* 07 (2018) 086, arXiv:1804.01999 [hep-th].

[23] D. Carmi, R. C. Myers and P. Rath, “Comments on Holographic Complexity”, *JHEP* 03 (2017) 118, arXiv:1612.00433 [hep-th].

[24] N. T. Macpherson, C. Núñez, L. A. Pando Zayas, V. G. J. Rodgers and C. A. Whiting, “Type IIB Supergravity Solutions with $AdS_5$ from Abelian and Non-Abelian T-dualities”, *JHEP* 02 (2015) 040, arXiv:1410.2650 [hep-th].

[25] I. R. Klebanov, D. Kutasov and A. Murugan, “Entanglement as a Probe of Confinement”, *Nucl. Phys. B* 796 (2008) 274, arXiv:0709.2140 [hep-th].

[26] D. Momeni, M. Faizal and R. Myrzakulov, “Holographic Cavalieri Principle as a Universal Relation between Holographic Complexity and Holographic Entanglement Entropy”, *Int. J. Mod. Phys. D* 27[09] (2018) 1850103, arXiv:1703.01337 [hep-th].

[27] A. Bhattacharyya, K. T. Grosvenor and S. Roy, “Entanglement Entropy and Subregion Complexity in Thermal Perturbations around Pure-AdS Spacetime”, *Phys. Rev. D* 100[12] (2019) 126004, arXiv:1905.02220 [hep-th].

[28] O. Ben-Ami and D. Carmi, “On Volumes of Subregions in Holography and Complexity”, *JHEP* 11 (2016) 129, arXiv:1609.02514 [hep-th].

[29] L. Susskind and E. Witten, “The Holographic Bound in Anti-de Sitter Space”, arXiv:hep-th/9805114.

[30] I. R. Klebanov and E. Witten, “Superconformal Field Theory on Three-branes at a Calabi-Yau Singularity”, *Nucl. Phys. B* 536 (1998) 199, arXiv:hep-th/9807080.

[31] S. S. Gubser, “Einstein Manifolds and Conformal Field Theories”, *Phys. Rev. D* 59 (1999) 025006, arXiv:hep-th/9807164.

[32] S. S. Gubser and I. R. Klebanov, “Baryons and Domain Walls in an $\mathcal{N} = 1$ Superconformal Gauge Theory”, *Phys. Rev. D* 58 (1998) 125025, arXiv:hep-th/9808075.
[33] N. Seiberg, “Five-dimensional SUSY Field Theories, Nontrivial Fixed Points and String Dynamics”, Phys. Lett. B 388 (1996) 753, arXiv:hep-th/9608111.

[34] A. Brandhuber and Y. Oz, “The D4-D8 Brane System and Five-dimensional Fixed Points”, Phys. Lett. B 460 (1999) 307, arXiv:hep-th/9905148.

[35] O. Bergman and D. Rodriguez-Gomez, “5d Quivers and Their AdS$_6$ Duals”, JHEP 07 (2012) 171, arXiv:1206.3503 [hep-th].

[36] D. L. Jafferis and S. S. Pufu, “Exact Results for Five-dimensional Superconformal Field Theories with Gravity Duals”, JHEP 05 (2014) 032, arXiv:1207.4359 [hep-th].

[37] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, “$N = 6$ Superconformal Chern-Simons-Matter Theories, M2-branes and Their Gravity Duals”, JHEP 10 (2008) 091, arXiv:0806.1218 [hep-th].

[38] M. Marino, “Lectures on Localization and Matrix Models in Supersymmetric Chern-Simons-matter Theories”, J. Phys. A 44 (2011) 463001, arXiv:1104.0783 [hep-th].

[39] I. R. Klebanov and A. A. Tseytlin, “Entropy of Near Extremal Black $p$-branes”, Nucl. Phys. B 475 (1996) 164, arXiv:hep-th/9604089.

[40] N. Drukker, M. Mariño and P. Putrov, “From Weak to Strong Coupling in ABJM Theory”, Commun. Math. Phys. 306 (2011) 511, arXiv:1007.3837 [hep-th].

[41] C. P. Herzog, I. R. Klebanov, S. Pufu and T. Tesileanu, “Multi-Matrix Models and Tri-Sasaki Einstein Spaces”, Phys. Rev. D83 (2011) 046001, arXiv:1011.5487 [hep-th].

[42] K. Sfetsos and D. C. Thompson, “On Non-abelian T-dual Geometries with Ramond Fluxes”, Nucl. Phys. B 846 (2011) 21, arXiv:1012.1320 [hep-th].

[43] S. Ferrara, A. Kehagias, H. Partouche and A. Zaffaroni, “Membranes and Five-branes with Lower Supersymmetry and Their AdS Supergravity Duals”, Phys. Lett. B 431 (1998) 42, arXiv:hep-th/9803109.

[44] M. Henningson and K. Skenderis, “The Holographic Weyl Anomaly”, JHEP 07 (1998) 023, arXiv:hep-th/9806087.

[45] F. Bastianelli, S. Frolov and A. A. Tseytlin, “Conformal Anomaly of (2,0) Tensor Multiplet in Six-dimensions and AdS/CFT Correspondence”, JHEP 02 (2000) 013, arXiv:hep-th/0001041.

[46] C. Córdova, T. T. Dumitrescu and K. Intriligator, “Anomalies, Renormalization Group Flows, and the $\alpha$-Theorem in Six-dimensional (1,0) Theories”, JHEP 10 (2016) 080, arXiv:1506.03807 [hep-th].

[47] V. Pestun et al., “Localization Techniques in Quantum Field Theories”, J. Phys. A 50[44] (2017) 440301, arXiv:1608.02952 [hep-th].

[48] N. Bobev and P. M. Crichigno, “Universal RG Flows Across Dimensions and Holography”, JHEP 12 (2017) 065, arXiv:1708.05052 [hep-th].
[49] P. M. Crichigno, D. Jain and B. Willett, “5d Partition Functions with A Twist”, *JHEP* **11** (2018) 058, arXiv:1808.06744 [hep-th].

[50] M. Miyaji, T. Numasawa, N. Shiba, T. Takayanagi and K. Watanabe, “Distance between Quantum States and Gauge-Gravity Duality”, *Phys. Rev. Lett.* **115[26]** (2015) 261602, arXiv:1507.07555 [hep-th].

[51] M. Alishahiha and A. Faraji Astaneh, “Holographic Fidelity Susceptibility”, *Phys. Rev. D* **96[8]** (2017) 086004, arXiv:1705.01834 [hep-th].

[52] S. Banerjee, J. Erdmenger and D. Sarkar, “Connecting Fisher Information to Bulk Entanglement in Holography”, *JHEP* **08** (2018) 001, arXiv:1701.02319 [hep-th].

[53] S. Karar, R. Mishra and S. Gangopadhyay, “Holographic Complexity of Boosted Black Brane and Fisher Information”, *Phys. Rev. D* **100[2]** (2019) 026006, arXiv:1904.13090 [hep-th].

[54] S. L. Braunstein and C. M. Caves, “Statistical Distance and the Geometry of Quantum States”, *Phys. Rev. Lett.* **72** (1994) 3439.

[55] W. Wootters, “Statistical Distance and Hilbert Space”, *Phys. Rev. D* **23** (1981) 357.

[56] D. Šafránek, “Discontinuities of the Quantum Fisher Information and the Bures Metric”, *Phys. Rev. A* **95[5]** (2017) 052320, arXiv:1612.04581 [quant-ph].