Cosmological Simulations with Self-Interacting Dark Matter I: Constant Density Cores and Substructure

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ABSTRACT
We use cosmological simulations to study the effects of self-interacting dark matter (SIDM) on the density profiles and substructure counts of dark matter halos from the scales of spiral galaxies to galaxy clusters, focusing explicitly on models with cross sections over dark matter particle mass $\sigma/m = 1$ and 0.1 cm$^2$/g. Our simulations rely on a new SIDM N-body algorithm that is derived self-consistently from the Boltzmann equation and that reproduces analytic expectations in controlled numerical experiments. We find that well-resolved SIDM halos have constant-density cores, with significantly lower central densities than their CDM counterparts. In contrast, the subhalo content of SIDM halos is only modestly reduced compared to CDM, with the suppression greatest for large hosts and small halo-centric distances. Moreover, the large-scale clustering and halo circular velocity functions in SIDM are effectively identical to CDM, meaning that all of the large-scale successes of CDM are equally well matched by SIDM. From our largest cross section runs we are able to extract scaling relations for core sizes and central densities over a range of halo sizes and find a strong correlation between the core radius of an SIDM halo and the NFW scale radius of its CDM counterpart. We construct a simple analytic model, based on CDM scaling relations, that captures all aspects of the scaling relations for SIDM halos. Our results show that halo core densities in $\sigma/m = 1$ cm$^2$/g models are too low to match observations of galaxy clusters, low surface brightness spirals (LSBs), and dwarf spheroidal galaxies. However, SIDM with $\sigma/m \approx 0.1$ cm$^2$/g appears capable of reproducing reported core sizes and central densities of dwarfs, LSBs, and galaxy clusters without the need for velocity dependence. Higher resolution simulations over a wider range of masses will be required to confirm this expectation. We discuss constraints arising from the Bullet cluster observations, measurements of dark matter density on small-scales and subhalo survival requirements, and show that SIDM models with $\sigma/m \approx 0.1$ cm$^2$/g $\approx 0.2$ barn/GeV are consistent with all observational constraints.

Key words: cosmology — dark matter — galaxies: halos — methods: numerical

1 INTRODUCTION
There is significant evidence that some form of dark matter dominates the gravitating mass in the universe and its abundance is known to great precision (Komatsu et al. 2011). The most popular candidate for dark matter is the class of weakly interacting massive particles (WIMPs), of which supersymmetric neutralinos are examples (Steigman & Turney 1983; Grier 1988; Jungman et al. 1996). WIMPs are stable, with negligible self-interactions, and are non-relativistic at decoupling ("cold"). It is important to recognize that of these characteristics, it is primarily their coldness that is well tested via its association with significant small-scale power. Indeed, WIMPs are the canonical Cold Dark Matter (CDM) candidate. Cosmological models based on CDM reproduce the spatial clustering of galaxies on large scales quite well (Reid et al. 2010) and even the clustering of galaxies on $\sim 1$ Mpc scales appears to match that expected for CDM subhalos (Kravtsov et al. 2004; Conrow et al. 2006; Trujillo-Gomez et al. 2011; Reddick et al. 2012).

Beyond the fact that the universe appears to behave as expected for CDM on large scales, we have few constraints on the microphysical parameters of the dark matter, especially those that would manifest themselves at the high densities associated with cores of galaxy halos. It is worth asking what (if anything) about vanilla CDM can change without violating observational bounds.
In this paper we use cosmological simulations to explore the observational consequences of a CDM particle that is strongly self-interacting, focusing specifically on the limiting case of velocity-independent, elastic scattering.

Dark matter particles with appreciable self-interactions have been discussed in the literature for more than two decades (Carlson et al. 1992; Machacek et al. 1993; de Laix et al. 1995; Spurge & Steinhardt 2000; Firmani et al. 2000), and are now recognized as generic consequences of hidden-sector extensions to the Standard Model (Pospelov et al. 2008; Arkani-Hamed et al. 2008; Ackerman et al. 2009; Feng et al. 2009, 2010; Loeb & Weiner 2011). Importantly, even if dark sector particles have no couplings to Standard Model particles they might experience strong interactions with themselves, mediated by dark gauge bosons (see Feng & Peter 2012 for reviews). The implication is that astrophysical constraints associated with the small-scale clustering of dark matter may be the only way to test these scenarios.

Phenomenologically, self-interacting dark matter (SIDM) is attractive because it offers a means to lower the central densities of galaxies without destroying the successes of CDM on large scales. Cosmological simulations that contain only CDM indicate that dark-matter halos should be cuspy and with (high) concentrations that correlate with the collapse time of the halo (Navarro et al. 1997; Bullock et al. 2001; Wechsler et al. 2002). This is inconsistent with observations of galaxy rotation curves, which show that galaxies are less concentrated and less cuspy than predicted in CDM simulations (e.g. Flores & Primack 1994; Simon et al. 2005; Kuzio de Naray et al. 2008; de Blok et al. 2010; Dutton et al. 2011; Kuzio de Naray & Spekkens 2011; Oh et al. 2011; Walker & Peñarrubia 2011; Salucci et al. 2012; Castignani et al. 2012). Even for clusters of galaxies, the density profiles of the host dark-matter halos appear in a number of cases to be shallower than predicted by CDM-only structure simulations, with the total (dark matter + baryons) density profile in a closer match to the CDM prediction for the dark matter alone (e.g. Sand et al. 2004, 2008; Newman et al. 2009, 2011; Coe et al. 2012; Umetzu et al. 2012).

One possible answer is feedback. In principle, the expulsion of gas from galaxies can result in lower dark matter densities compared to dissipationless simulations, and thus bring CDM models in line with observations (Governato et al. 2010; Oh et al. 2011; Pontzen & Governato 2011; Brook et al. 2012; Governato et al. 2012). However, a new level of concern exists for dwarf spheroidal galaxies (Boylan-Kolchin et al. 2011a; Ferrero et al. 2011; Boylan-Kolchin et al. 2011b). Systems with \( M_c \sim 10^8 M_\odot \) appear to be missing \( \sim 5 \times 10^8 M_\odot \) of dark matter compared to standard CDM expectations (Boylan-Kolchin et al. 2011b). It is difficult to understand how feedback from such a tiny amount of star formation could have possibly blown out enough gas to reduce the densities of dwarf spheroidal galaxies to the level required to match observations (Boylan-Kolchin et al. 2011a; Teyssier et al. 2012; Zolotov et al. 2012; Peñarrubia et al. 2012; Garrison-Kimmel et al., in preparation).

Spurge & Steinhardt (2000) were the first to discuss SIDM in the context of the central density problem (see also Firmani et al. 2000). The centers of SIDM halos are expected to have constant density isothermal cores that arise as kinetic energy is transmitted from the hot outer halo inward (Ballero et al. 2002; Colin et al. 2002; Ahn & Shapiro 2005; Koda & Shapiro 2011). This can happen if the cross section over mass of the dark matter particle, \( \sigma/m \), is large enough for there to be a relatively high probability of scattering over a time \( t_{\text{age}} \) comparable to the age of the halo:

\[
\Gamma \sim \frac{\rho(r)}{m} \sigma/m \rho_{\text{rms}}(r),
\]

where \( \Gamma \) is the scattering rate per particle. The rate will vary with local dark matter density \( \rho(r) \) as a function of radius \( r \) in a dark halo as

\[
\Gamma(r) \approx \rho(r) \sigma/m \rho_{\text{rms}}(r),
\]

up to order unity factors, where \( \rho_{\text{rms}} \) is the rms speed of dark-matter particles. Based on rough analytic arguments, Spurge & Steinhardt (2000) suggested \( \sigma/m \sim 0.1 - 100 \text{ cm}^2/\text{g} \). For the first cosmological simulations aimed at understanding dwarf densities were performed by Davé et al. (2001) who used a small volume (4h\(^{-1}\) Mpc on a side) in order to focus computational power on dwarf halos. They concluded that \( \sigma/m = 0.1 - 10 \text{ cm}^2/\text{g} \) came close to reproducing core densities of small galaxies, favoring the upper end of that range but not being able to rule out the lower end due to resolution. Almost concurrently, Yoshida et al. (2000) ran cosmological simulations focusing on the cluster-mass regime. Based on the estimated core size of cluster CL 0024+1654, they concluded that cross sections no larger than ~0.1 \text{ cm}^2/\text{g} were allowed, raising doubts that constant-cros- section SIDM models could be consistent with observations of both dwarf galaxies and clusters.

These concerns were echoed by Miralda-Escudé (2002) who suggested that SIDM halos would be significantly more spherical than observed for galaxy clusters. Similarly, Gnedin & Ostriker (2001) argued that SIDM would lead to excessive sub halo evaporation in galaxy clusters. More recently, the merging cluster system known as the Bullet Cluster has been used to derive the limits (68% C.L.) \( \sigma/m < 0.7 \text{ cm}^2/\text{g} \) (Randall et al. 2008) based on evaporation of dark matter from the subcluster and \( \sigma/m < 1.25 \text{ cm}^2/\text{g} \) (Randall et al. 2008) based on the observed lack of offset between the bullet subcluster mass peak and the galaxy light centroid. In order to relax this apparent tension between what was required to match dwarf densities and the observed properties of galaxy clusters, velocity dependent cross sections that diminish the effects of self-interaction in cluster environments have been considered (Firmani et al. 2000; Colin et al. 2002; Feng et al. 2009; Loeb & Weiner 2011; Vogelsberger et al. 2012).

There are a few new developments that motivate us to revisit constant SIDM cross sections on the order of \( \sigma/m \sim 1 \text{ cm}^2/\text{g} \). For example, the cluster (CL 0024+1654) used by Yoshida et al. (2000) to place one of the tightest limits at \( \sigma/m = 0.1 \) is now recognized as an ongoing merger along the line of sight (Czoske et al. 2001, 2002; Zhang et al. 2005; Jee et al. 2007, 2010; Umetzu et al. 2010). This calls into question its usefulness as a comparison case for non-merging cluster simulations. In a companion paper (Peter, Rocha, Bullock and Kaplinghat, 2012) we use the same simulations described here to show that published constraints on SIDM based on halo shape comparisons are significantly weaker than previously believed. Further, the results presented below clearly demonstrate that the tendency for subhalos to evaporate in SIDM models (Gnedin & Ostriker 2001) is not significant for \( \sigma/m \sim 1 \text{ cm}^2/\text{g} \).
Finally (and related to the previous point), the best numerical analysis of the Bullet Cluster [Randall et al. 2008] used initial cluster density profiles that were unmotivated cosmologically with central densities about a factor of two too high for the SIDM cross sections considered (producing a scattering rate that is inconsistently high). Based on this observation, the bullet cluster constraint based on evaporation of dark matter from the subcluster should be relaxed since the amount of subcluster mass that becomes unbound is directly proportional to the density of dark matter encountered in its orbit. Moreover, their model galaxies were placed in the cluster halo potentials without subhalos surrounding them, an assumption (based on analytic estimates for SIDM subhalo evaporation) that is not supported by our simulations. This could affect the constraints based on the (lack of) offset between dynamical mass and light. Thus we believe that the bullet cluster constraints as discussed above are likely only relevant for models with \( \sigma/m > 1 \text{ cm}^2/\text{g} \).

However, the constraints could be made significantly stronger by comparing SIDM predictions to the densities inferred from the convergence maps since the central halo densities for \( \sigma/m \approx 1 \text{ cm}^2/\text{g} \) are significantly lower than the CDM predictions, as we show later.

Given these motivations, we perform a set of cosmological simulations with both CDM and SIDM. For SIDM we ran \( \sigma/m = 1 \) and 0.1 cm\(^2\)/g models (hereafter SIDM\(_1\) and SIDM\(_{0.1}\)), i.e., models that we have argued pass the Bullet cluster tests. Our simulations provide us with a sample of halos that span a mass range much larger than either Dave et al. [2001] or Yoshida et al. [2000] both with and without self-interactions.

One of the key findings from our simulations is that the core sizes are expected to scale approximately as a fixed fraction of the NFW scale radius the halo would have in the absence of scatterings. We can see where this scaling arises from a quick look at Equation 1. This equation allows us to argue that the radius \( r_1 \) below which we expect dark matter particles (on average) to have scattered once or more is set by:

\[
\rho_s f\left(\frac{r}{r_s}\right)v_{\text{rms}} \propto \frac{V_{\text{max}}}{r_{\text{max}}}f\left(\frac{r_1}{r_s}\right)V_{\text{max}} = \text{constant},
\]

where \( f(x) \) is the functional form of the NFW (or a related) density profile. In writing the above equation we have assumed that the density profile for SIDM is not significantly different from CDM at \( r_1 \), something that we verify through our simulations. Now, since CDM enforces a \( V_{\text{max}} - r_{\text{max}} \) relation such that \( V_{\text{max}} \propto r_{\text{max}}^{-1.5} \), we see that the solution to \( r_1/r_s \) is going to be only mildly dependent on the halo properties. We develop an analytic model based on this insight later, but this is the underlying reason for why we find core sizes to be a fixed fraction of the NFW scale radius of the same halo in the absence of scatterings.

The major conclusion we reach based on the simulations and the analytic model presented here is that a self-interacting dark matter model with a cross-section over dark matter particle mass \( \sim 0.1 \text{ cm}^2/\text{g} \) would be capable of reproducing the core sizes and central densities observed in dark matter halos at all scales, from clusters to dwarf spheroidals, without the need for velocity-dependence in the cross-section.

In the next section, we discuss our new algorithm to compute the self-interaction probability for N-body particles, derived self-consistently from the Boltzmann equation. We discuss this new algorithm in detail in Appendix A. In 2 we show how this algorithm is implemented in the publicly available code GADGET-2 [Springel 2005]. We run tests that show that our algorithm gets the correct interaction rate and post-scattering kinematics. The results of these tests are in 3. The cosmological simulations with this new algorithm are described in detail in 4. In 5, we provide some preliminary illustrations of our simulation snapshots and in 6, we demonstrate that the large-scale statistical properties of SIDM are identical to CDM. In 7, we present the properties of individual SIDM\(_1\) and SIDM\(_{0.1}\) halos and compare them to the their CDM counterparts. In 8, we discuss the subhalo mass functions in our SIDM and CDM simulations and show that SIDM\(_1\) subhalo mass functions are very close to that of CDM in the range of halo masses we can resolve. We provide scaling relations for the SIDM\(_1\) halo properties in 9 and in 10, we present an analytic model that reproduces these scaling relations as well as the absolute densities and core radii of SIDM\(_1\) halos. We use these scaling relations and the analytic model to make a broad-brush comparison to observed data in 11. We present a summary together with our final conclusions in 12.

2 SIMULATING DARK MATTER SELF INTERACTIONS

Our simulations rely on a new algorithm for modeling self-interacting dark matter with N-body simulations. Here we introduce our approach and provide a brief summary. In Appendix A, we derive the algorithm explicitly starting with the Botlzmann equation and give details for general implementation.

In N-body simulations, the simulated (macro)particles represent an ensemble of many dark-matter particles. Each simulation particle of mass \( m_p \) can be thought of as a patch of dark-matter phase-space density. In our treatment of dark matter self-scattering, the phase space patch of each particle is represented by a delta function in velocity and a spatially extended kernel \( W(r, h_{\delta}) \), smoothing out the phase space in configuration space on a self-interaction smoothing length \( h_{\delta} \). The value of \( h_{\delta} \) needs to be set by considering the physical conditions of the problem (see 3) as it specifies the range over which N-body particles can affect each other via self-interactions. In principle, \( h_{\delta} \) could be different for each particle and vary depending on the local density, but in the simulations presented here we fix \( h_{\delta} \) to be the same for all particles in a given simulation, setting the size of \( h_{\delta} \) according the lowest densities at which self-interactions are effective for a given cross section.

When two phase-space patches overlap, we need to calculate the pairwise interaction rate between them. We do so by considering the “scattering out” part of the Boltzmann collision term in Equation A1 and Eqs. A8-A13. The implied rate of scattering of an N-body particle \( j \) off of a target particle \( i \) of mass \( m_p \) is

\[
\Gamma(i|j) = (\sigma/m|n_p)(v_i - v_j|g_{ji}),
\]

where \( g_{ji} \) is a number density factor that accounts for the overlap of the two particles’ smoothing kernels:

\[
g_{ji} = \int_0^{h_{\delta}i} d^3x' W(|x'|, h_{\delta}i)W(|\delta x_{ji} + x'|, h_{\delta}i).
\]

The probability that such an interaction occurs in a time step \( \delta t \) is

\[
P(i|j) = \Gamma(i|j) \delta t,
\]

and the total probability of interaction between N-body particles \( i \) and \( j \) is

\[
P_{ij} = P(i|j) + P(j|i).
\]
Specifically, $P_{ij}$ is the probability for a macroparticle representing a patch of phase space around $(x_i, v_i)$ to interact with a target particle representing a patch of phase space around $(x_j, v_j)$ in a time $\delta t$.

We determine if particles interact by drawing a random number for each pair of particles that are close enough for the probability of interaction to be greater than zero. If a pair does scatter, we do a Monte Carlo for the new velocity directions, populating the new phase space regions, we are taking care of the “scattering in” term of the collision integral in Equation (A1). We avoid double counting by only accounting for $P_{ij} = P_{ji}$ once during a given time-step $\delta t$. In the limit of a large number of macroparticles, the total interaction probability for each particle $i$ should approach

$$P_i = \sum_j P_{ij}.$$  \hspace{1cm} (7)

We show in §3 that this approach correctly reproduces the expected number of scatterings in a idealized test case.

Our method for simulating scattering differs from previous approaches in a few key ways. It is most similar to that of Davé et al. (2001) in that we both directly consider interactions between pairs of phase-space patches and rely on a scattering rate similar in form to Equation (2). The difference is that their geometric factor $g_{ij}$ is not the same—our factor arises explicitly from the overlap in patches of phase space between neighboring macroparticles, as derived from the collision term in the Boltzmann equation (see Appendix A for details). Other authors determine the scattering rate $\Gamma$ of individual phase-space patches based on estimates of the local mass density (typically using some number of nearest neighbors or using an SPH kernel). The Monte Carlo is then based on an estimated scattering rate of an individual particle on the background, and a scattering partner is only chosen after a scattering event is determined to have occurred (Kochanek et al. 2000; Yoshida et al. 2000; Rocha et al. 2001; Randall et al. 2008). The scattering probability in this latter approach is not symmetric. For macroparticles of identical mass, $P_{ij} = P_{ji}$ explicitly in our approach, but not the other approach because the density estimated at the position of macroparticle $i$ need not be the same as that estimated at the position of particle $j$. In the future, there should be a direct comparison among these scattering algorithms to determine if they yield consistent results.

We have implemented our algorithm in the publicly available version of the cosmological simulation code GADGET-2 (Springel 2005). GADGET-2 computes the short-range gravitational interactions by means of a hierarchical multipole expansion, also known as a tree algorithm. Particles are grouped hierarchically by a repeated subdivision of space, so their gravitational contribution can be accounted by means of a single multipole force computation. A cubical root node encompasses the full mass distribution. The node is repeatedly subdivided into eight daughter nodes of half the side length each (an oct-tree) until one ends up with “leaf” nodes containing single particles. Forces for a given particle are then obtained by “walking” the tree, opening nodes that are too close for their multipole expansion to be a correct approximation to their gravitational contribution. In GADGET-2, spurious strong close encounters by particles are avoided by convolving the single point particle density distribution with a normalized spline kernel ("gravitational softening").

To implement our algorithm, we take advantage of the tree-walk already build in GADGET-2, computing self interactions during the calculation of the gravitational interactions. For this to work we have to modify the opening criterion such that nodes are opened if they are able to have particles closer than $2h_{\text{cut}}$ from a target scatterer (or $h_{\text{cut}} + h_e$ if particles have different self-interaction smoothing lengths). When computing the probability of interaction we use the same spline kernel used in GADGET-2 (Monaghan & Lattanzio 1983), defined as

$$W(r, h) = \frac{8}{\pi h^3} \begin{cases} 1 - 6 \left(\frac{r}{h}\right)^2 + 6 \left(\frac{r}{h}\right)^3, & 0 \leq \frac{r}{h} \leq 1, \\ 2 \left(1 - \frac{r}{h}\right)^3, & \frac{1}{2} \leq \frac{r}{h} \leq 1, \\ 0, & \frac{r}{h} > 1. \end{cases}$$  \hspace{1cm} (8)

If a pair interacts we give both particles a kick consistent with an elastic scattering that is isotropic in the center of mass frame. The post-scatter particle velocities are

$$v_i' = v_e + \frac{m_1}{m_0 + m_1} Ve,$$
$$v_j' = v_e - \frac{m_0}{m_0 + m_1} Ve,$$  \hspace{1cm} (9)

where $v_e$ is the center of mass velocity, $V$ is the relative speed of the particles (conserved for elastic collisions) and $e$ is a randomly chosen direction.

The time-step criterion is also modified to assure that the scattering probability for any pair of particles is small, $P = \Gamma \delta t \ll 1$. An individual particle time-step is decreased by a factor of 2 if during the last tree-walk the maximum probability of interaction for any pair involving such a particle was $P_{\text{max}} > 0.2$. Once a particle time-step is modified due to the previous restriction, if $P_{\text{max}} < 0.1$ for such a particle and its current time-step is smaller than the one given by the standard criterion on GADGET-2, we increase it by a factor of 2.
ics and number of interactions from our test simulation agrees well with the expectations from the theory as long as \( h_{\text{SIDM}} (\rho_{bg}/m_p)^{1/3} \gtrsim 0.2 \).

### 4 OVERVIEW OF COSMOLOGICAL SIMULATIONS

We initialize our cosmological simulations using the Multi-Scale Initial Conditions (MUSIC) code of Hahn & Abel (2011). We have a total of four initial condition sets, each run with both CDM and SIDM. The first two are cubic volumes of \( 25h^{-1} \) Mpc and \( 50h^{-1} \) Mpc on a side, each with \( 512^3 \) particles. As discussed below, these simulations allow us to resolve the structure of a statistical sample of group (\( \sim 10^{13}\) M\(_{\odot}\)) and cluster (\( \sim 10^{14}\) M\(_{\odot}\)) halos.

The second two initial conditions concentrate computational power on zoom regions Katz & White (1993) drawn from the \( 50h^{-1} \) Mpc box, specifically aimed at exploring the density structure of two smaller halos, one with virial mass \( M_{\text{vir}} = 7.1 \times 10^{13} h^{-1} \) M\(_{\odot}\) (Z12) and one with \( M_{\text{vir}} = 3.5 \times 10^{13} h^{-1} \) M\(_{\odot}\) (Z11). The Z12 run in particular is fairly high resolution, with more than five million particles in the virial radius. Table 1 summarizes the simulation parameters. The cosmology used is based on WMAP7 results for a ΛCDM Universe: \( h = 0.71\), \( \Omega_m = 0.266\), \( \Omega_{\Lambda} = 0.734\), \( \Omega_b = 0.0499\), \( n_s = 0.963\), \( \sigma_8 = 0.801\) (Komatsu et al. 2011).

Each of our four initial conditions has been evolved from redshift \( z = 250 \) to redshift \( z = 0 \) with collisionless dark matter (labeled CDM) and with two types of self-interacting dark matter: one with \( \sigma/m = 1 \) cm\(^2\)/g (labeled SIDM1) and another with \( \sigma/m = 0.1 \) cm\(^2\)/g (labeled SIDM0.1). We can use the same initial conditions for CDM and SIDM because at high redshift the low densities and low relative velocities of the dark matter make self-interactions insignificant. Table 2 list all the simulations used for this study and detail their force, mass, and self-interaction resolution. In addition to the simulations listed in the table, we also ran the cosmological boxes with SIDM cross sections \( \sigma/m = 0.03 \) cm\(^2\)/g. We do not present results from these low

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#### 3 TEST OF THE SIDM IMPLEMENTATION

Before performing cosmological simulations, we carried out a controlled test of the implementation in order to make sure the scattering rate and kinematics are correctly followed in the code, and to determine the optimum value of the SIDM softening kernel length \( h_{\text{SIDM}} \). The simplest and clearest scenario for testing our implementation consists of a uniform sphere of particles moving through a uniform field of stationary background particles. The coordinate system is defined such as the sphere is moving along the positive z-direction with constant velocity \( v_s \). The particles forming the sphere and the particles forming the background field are tagged as different types within the code and here we will refer to them simply as sphere (s) and background (bg) particles respectively.

We only allow scatterings involving two different types of particles (i.e. sphere-background interactions only) and turn off gravitational forces among all of the particles. Furthermore all particles have the same mass \( m_p \).

The expected number of interactions for this case is given by

\[
N_{\exp}(t) = \sum_{i \in s, j \in bg} P_{ij} = N_s (\sigma/m) \rho_{bg} v_s t \tag{10}
\]

where \( N_s \) is the total number of Sphere particles, \( \rho_{bg} \) is the density of the background field and \( t \) is the elapsed time from the beginning of the simulation. From this experiment we have found that the number of interactions computed by the code depends on the self-interaction smoothing length \( h_{\text{SIDM}} \) (see Figure 1), which is fixed to be the same for all particles in this test. The number of interactions converges to the expected value given by Equation (10) as \( h_{\text{SIDM}} \) becomes comparable to the background inter-particle separation, specifically when \( h_{\text{SIDM}} (\rho_{bg}/m_p)^{1/3} \gtrsim 0.2 \). For \( h_{\text{SIDM}} (\rho_{bg}/m_p)^{1/3} \gtrsim 0.5 \) the accuracy of the algorithm does not improve much and the time of the calculations increases rapidly, \( \propto h_{\text{SIDM}}^3 \). Apart from the expense, using larger values of \( h_{\text{SIDM}} \) would lead to increasingly non-local interactions among particles, which is inconsistent with the model under consideration.

We also check the kinematics of the scatters in this test simulation and describe the results in Appendix B. The resulting kinematics and number of interactions from our test simulation agrees well

| Name       | Volume \( L_{\text{Box}} [h^{-1} \text{Mpc}] \) | Number of Particles \( N_p \) | Particle Mass \( m_p [h^{-1} \text{Mpc}] \) | Force Softening \( \epsilon [h^{-1} \text{kpc}] \) | Smoothing Length \( h_{\text{SIDM}} [h^{-1} \text{kpc}] \) | Cross-section \( \sigma/m [\text{cm}^2/\text{g}] \) |
|------------|-----------------------------------------------|------------------------------|--------------------------------------------|-----------------------------------------------|-----------------------------------------------|--------------------------------------------|
| CDM-50     | 50                                            | 512^3                        | 6.88 \times 10^3                           | 1.0                                           | -                                            | 0                                          |
| CDM-25     | 25                                            | 512^3                        | 8.59 \times 10^3                           | 0.4                                           | -                                            | 0                                          |
| CDM-Z11    | (3R_{\text{Vir}})^*                        | 2.5 \times 10^6             | 1.07 \times 10^6                           | 0.3                                           | -                                            | 0                                          |
| CDM-Z12    | (3R_{\text{Vir}})^*                        | 5.6 \times 10^7             | 1.34 \times 10^5                           | 0.1                                           | -                                            | 0                                          |
| SIDM0.1-50 | 50                                            | 512^3                        | 6.88 \times 10^3                           | 1.0                                           | 2.8 \epsilon                                 | 0.1                                        |
| SIDM0.1-25 | 25                                            | 512^3                        | 8.59 \times 10^3                           | 0.4                                           | 2.8 \epsilon                                 | 0.1                                        |
| SIDM0.1-Z11| (3R_{\text{Vir}})^*                        | 2.5 \times 10^6             | 1.07 \times 10^6                           | 0.3                                           | 2.8 \epsilon                                 | 0.1                                        |
| SIDM0.1-Z12| (3R_{\text{Vir}})^*                        | 5.6 \times 10^7             | 1.34 \times 10^5                           | 0.1                                           | 1.4 \epsilon                                 | 0.1                                        |
| SIDM1-50   | 50                                            | 512^3                        | 6.88 \times 10^3                           | 1.0                                           | 2.8 \epsilon                                 | 0                                          |
| SIDM1-25   | 25                                            | 512^3                        | 8.59 \times 10^3                           | 0.4                                           | 2.8 \epsilon                                 | 0                                          |
| SIDM1-Z11  | (3R_{\text{Vir}})^*                        | 2.5 \times 10^6             | 1.07 \times 10^6                           | 0.3                                           | 2.8 \epsilon                                 | 0                                          |
| SIDM1-Z12  | (3R_{\text{Vir}})^*                        | 5.6 \times 10^7             | 1.34 \times 10^5                           | 0.1                                           | 1.4 \epsilon                                 | 0                                          |

*Note: The Z11 and Z12 runs are zoom simulations with multiple particle species concentrating on halos of mass \( M_{\text{vir}} = 5 \times 10^{11} \) M\(_{\odot}\) and \( 1.0 \times 10^{12} \) M\(_{\odot}\), respectively (no \( h \)). The volumes listed refer to the number of virial radii used to find the Lagrangian volumes associated with the zoom. The particle properties listed are for the highest resolution particles only.
Figure 2. Top: Large scale structure in CDM (left) and SIDM (right) shown as a $50 \times 50 \, h^{-1} \text{Mpc}$ slice with $10 \, h^{-1} \text{Mpc}$ thickness through our cosmological simulations. Particles are colored according to their local phase-space density. There are no visible differences between the two cases. Bottom: Small scale structure in a Milky Way mass halo (Z12) simulated with CDM (left) and SIDM (right), including all particles within $200 \, h^{-1} \text{kpc}$ of the halo centers. The magnitude of the central phase-space density is lower in SIDM because the physical density is lower and the velocity dispersion is higher. The core of the SIDM halo is also slightly rounder. Note that substructure content is quite similar except in the central regions.

cross section runs here because no core density differences were resolved within the numerical convergence radii of our simulations.

As shown in §3, the self-interaction smoothing length $h_{si}$ must be larger than 20% the inter-particle separation in order to achieve convergence on the interaction rate. All the work for this paper was done with a fixed $h_{si}$ for all particles carefully chosen for each simulation so that the self-interactions are well resolved at densities a few times to an order of magnitude lower than the lowest densities for which self-interactions are significant. We have run the cosmological boxes with different choices for $h_{si}$ (changes by factors of 2 to 4) and have found that our results are unaffected. We have also run tests on isolated halos with varying smoothing lengths and again find that the effects of self-interactions are robust to reasonable changes in $h_{si}$.

All of our halo catalogs and density profiles are derived using the publicly available code Amiga Halo Finder (AHF) (Knollmann & Knebe 2009).
5 SIMULATION RESULTS

5.1 Preliminary Illustrations

Before presenting any quantitative comparisons between our CDM and SIDM runs, we provide some simulation renderings in order to help communicate the qualitative differences.

The upper panels of Figure 2 show a large-scale comparison: two (50 × 50 × 10) h⁻¹ Mpc slices from the CDM-50 and SIDM₁-50 boxes side-by-side at z = 0. The structures are color-coded by local phase-space density (∝ ρ/τₘᵦ). It is evident that there are no observable differences in the large-scale characteristics of CDM and SIDM₁. We discuss this result in more quantitative terms in §5.2 but of course this is expected. The SIDM models we explore do not have appreciable rates of interaction for densities outside the cores of dark-matter halos. The upper panels of Figure 2 provide a visual reminder that the SIDM models we consider are effectively identical to CDM on large scales.

The differences between CDM and SIDM become apparent only when one considers the internal structure of individual halos. The lower panels of Figure 2 provide side-by-side images of a Milky-Way mass halo (Z12) simulated with CDM (left) and SIDM (right). SIDM tends to make the cores of halos less dense and kinetically hotter (see §5.3) and these two differences are enhanced multiplicatively in the phase-space density renderings. The central regions of the host halo are also slightly rounder in the SIDM case (Peter et al. 2012). Importantly, the difference in substructure characteristics are minimal, especially at larger radii. We return to a quantitative description of substructure differences in §5.4.

5.2 Large Scale Structure and Halo Abundances

Figure 3 provides a quantitative comparison of both the clustering properties (left) and halo abundance evolution (right) between our full-box CDM and SIDM₁ simulations. The left panel shows the two-point function of dark matter particles in both cosmological runs for CDM and SIDM₁. There are no discernible differences between SIDM and CDM over the scales plotted, though of course the different box sizes (and associated resolutions) mean that the boxes themselves only overlap for a limited range of scales. For a given set of initial conditions, however, SIDM and CDM give identical results.

The right panel of Figure 3 shows the cumulative number density of dark-matter halos (including subhalos) as a function of their peak circular velocity (Vₘₐₓ) at any redshift. Remarkably, this comparison shows no significant difference either – indicating that SIDM with cross sections as large as the different box sizes (and associated resolutions) mean that the boxes themselves only overlap for a limited range of scales. For a given set of initial conditions, however, SIDM and CDM give identical results.

5.3 Halo Structure

Before presenting statistics on halo structure, we focus on six well resolved halos that span our full mass range Mₘₐₓ = 5 × 10¹¹ – 2 × 10¹⁴ M☉, selected from our full simulation suite, including our two zoom-simulation halos (Z12 and Z11). Figures 4 through 6 show radial profiles for the density, circular velocity and velocity dispersion for all three dark matter cases. In each figure, black circles correspond to CDM, green triangles to SIDM₀, and blue stars to SIDM₁. All profiles are shown down to the innermost resolved radius for which the average two-body relaxation time roughly matches the age of the Universe (Power et al. 2003).

We begin with the density profiles of halos shown in the six-panel Figure 4. For each halo in the CDM run we have fit an NFW profile (Navarro et al. 1997) to its radial density structure:

$$\rho_{NFW}(r) = \frac{\rho_s r_s^3}{r(r_s + r)^2},$$

and recorded its corresponding scale radius rₛ. The CDM-fit rₛ...
value for each halo is given in its associated panel along with the halo virial mass. The radial profiles for each halo (in both the CDM and SIDM runs) are normalized with respect to the CDM \( r_s \) value in the plot. This allows our full range of halo masses to be plotted on identical axes.

The SIDM versions of each halo show remarkable similarity to their CDM counterparts at large radii. However, the SIDM1 cases clearly begin to roll towards constant-density cores at small radii. The best resolved halos in the SIDM0.1 runs also demonstrate lower central densities compared to CDM, though the differences are at the factor of \( \sim 2 \) level even in our best resolved systems. Clearly, higher resolution simulations will be required in order to fully quantify the expected differences between CDM and SIDM for \( \sigma/m \sim 0.1 \text{ cm}^2/\text{g} \).

For the SIDM1 cases we can quantify the halo cores by fitting them to Burkert (1995) profiles

\[
\rho_B(r) = \frac{\rho_B r_B^3}{(r + r_B)(r^2 + r_B^2)},
\]

These Burkert fits are shown as blue dashed lines. They are good fits for radii within \( r \sim 2 - 3 r_s \), but the quality of the fits gets worse at large radii. The blue arrows in each panel show the value of the best-fit Burkert core radius for the SIDM1 halos. Note that the values are remarkably stable in proportion to the CDM \( r_s \) value at \( r_s \approx 0.7 r_s \).

As explained in \[\text{[7]}\] the fact that the SIDM1 profiles are reasonably well characterized by a single scale-radius Burkert profile may be a lucky accident, only valid for cross sections near \( 1 \text{ cm}^2/\text{g} \). It just so happens that for this cross section the radius where dark matter particles experience significant scattering sets in at \( r \sim r_s \) (see Figure \[\text{[7]}\] and related discussion). For a smaller cross section (with a correspondingly smaller core) a multiple parameter fit may be necessary. Given the beginnings of very small cores we are seeing in the SIDM0.1 runs, it would appear that we would need one scale radius to define an \( r_s \) bend and a second scale radii to define a distinct core.

Another qualitative fact worth noting is that the density profiles of the SIDM1 halos overshoot the CDM density profiles near the Burkert core radius (not as much as the Burkert fits do, but the difference in the data points is noticeable). This is due to the fact that as particles scatter in the center, those that gain energy move towards a constant density isothermal core. We will develop this model further to explain the scaling relations between core size and halo mass in \[\text{[6]}\].

The circular velocity curves for the same set of halos discussed above are shown in Figure \[\text{[5]}\]. The SIDM rotation curves rise more steeply and have a lower normalization than for CDM within the NFW scale radius \( r_s \). This brings to mind the rotation curves observed for low surface brightness galaxies and we will explore this connection later. Note though that the peak circular velocity \( V_{\text{max}} \) actually is slightly higher for the SIDM1 case because of the mass rearrangement (evident in the density profiles in Figure \[\text{[4]}\]) briefly discussed in the last paragraph. At radii well outside the core radius, the rotation curves of the CDM and SIDM1 halos converge, though this convergence occurs beyond the plot axes \( > r_s \) for most of the halos shown.

An appreciation of why the density profiles of SIDM halos become cored can be gained from studying their velocity dispersion profiles compared to their CDM counterparts, as illustrated in Figure \[\text{[6]}\]. Here \( v_{\text{rms}} \) is defined as the root-mean-square speed of all particles within radius \( r \). While the CDM halos (black) are colder in the center than in their outer parts (reflecting a cuspy density profile) the SIDM halos have hotter cores, indicative of heat transport from the outside in. Moreover, the SIDM halos are slightly colder at large radii, again reflecting a redistribution of energy. As discussed in the introduction, it is this heat transport that is the key to understanding why CDM halos differ from SIDM halos in their density structure (Balberg et al. 2002; Colín et al. 2002; Ahn & Shapiro 2005; Koda & Shapiro 2011). The added thermal pressure at small radii is what gives rise to the core. The SIDM1 simulations have sufficient interactions that they have been driven to isothermal profiles for \( r/r_s \ll 1 \) when while for SIDM0.1 the \( v_{\text{rms}} \) profiles typically begin to deviate from the CDM lines only at smaller radii, \( r/r_s \sim 0.2 \), reflecting the relatively lower scattering rate.

The deviations in the SIDM \( v_{\text{rms}} \) profiles compared to CDM appear to set in at approximately the radius where we expect every particle to have interacted once in a Hubble time. This is explored directly in Figure \[\text{[7]}\] where we present a proxy for the local scattering rate as a function of distance from the halo center:

\[
\rho(r) v_{\text{rms}}(r) \propto \Gamma(r) (\sigma/m)^{-1}.
\]

We have divided out the cross section so it is easier to compare the SIDM0.1 and SIDM1 cases. Figure \[\text{[7]}\] presents this rate proxy in units of \( 1 \text{ Gyr cm}^2/g \). For the SIDM1 case \((\sigma/m = 1 \text{ cm}^2/g)\) the radius where a typical particle will have scattered once over a 10 Gyr halo lifetime is \( \rho(r) v_{\text{rms}}(r) = 0.1 \). For the SIDM0.1 case \((\sigma/m = 0.1 \text{ cm}^2/g)\), the ordinate needs to be ten times higher \((\sim 1)\) in order to achieve the same scattering rate.

By comparing Figure \[\text{[7]}\] to Figure \[\text{[6]}\] (and to some extent to all Figures 4-6) we see that the effects of self-interactions do become evident at radii corresponding to \( \rho v_{\text{rms}} \sim 0.1 \) for SIDM1 \((r/r_s \sim 0.8)\) and \( \rho v_{\text{rms}} \sim 1 \) for SIDM0.1 \((r/r_s \sim 0.2)\). Interestingly, for the SIDM1 halos this interaction radius is fairly close to the Burkert scale radius (shown by the blue arrows). It should be kept in mind, however, that the structure of halos can be affected to larger radii because particles scattering in the inner regions can gain energy and move to larger orbits. A careful inspection of the density and rotation velocity profiles shows that this is indeed the case.

We will discuss these findings in more detail in Sections \[\text{[6]}\] and \[\text{[7]}\]. In particular, in \[\text{[7]}\] we present an analytic model aimed at understanding how the central densities and scale radii of SIDM halos are set in the context of energetics. But before moving on to those issues, we first explore halo substructure in SIDM.

5.4 Substructure

The question of halo substructure is an important one for SIDM. One of the original motivations for SIDM was to reduce the number of subhalos in the Milky-Way halo in order to match the relative dearth of observed satellite galaxies (Spergel & Steinhardt 2000). However, the over-reduction of halo substructure is now recognized as a negative feature of SIDM compared to CDM, given the clear evidence for galaxy-size subhalos throughout galaxy clusters (Natarajan et al. 2009) and the new discoveries of ultra-faint galaxies around the Milky Way (see Willman (2010) and Bullock (2010) for reviews). In fact, one of the most stringent constraints on the self-interaction cross section comes from analytic shubhalo-evaporation arguments (Gnedin & Ostriker 2001).

Figure \[\text{[8]}\] demonstrates that the effects of shubhalo evaporation...
Figure 4. Density profiles for our six example halos from our SIDM$_1$ (blue stars) and SIDM$_{0.1}$ (green triangles) simulations and their CDM counterparts. With self-interactions turned on, halo central densities decrease, forming cored density profiles. Solid lines are for the best NFW (black) and Burkert (blue) fits, with the points representing the density at each radial bin found by AHF. The arrow indicates the location of the Burkert core radius $r_b\approx 2 r_s$.

Burkert profiles provide a reasonable fit to our SIDM$_1$ halos only because $r_b\approx r_s$ for $\sigma/m = 1 \text{ cm}^2/\text{g}$, so a cored profile with a single scale radius works. As discussed in §7 this is not the case for $\sigma/m = 0.1 \text{ cm}^2/\text{g}$ and thus Burkert profiles are not a good fit to our SIDM$_{0.1}$ halos.

in SIDM are not as strong as previously suggested on analytic grounds. Here we show the cumulative number of subhalos larger than a given $V_{\text{max}}$ for a sample of well-resolved halos in our CDM (solid), SIDM$_{0.1}$ (dotted), and and SIDM$_1$ (dashed) simulations. The associated virial masses for each host halo are shown in the legend. The left panel presents the $V_{\text{max}}$ function for all subhalos within the virial radius of each host and the right panel restricts the analysis to subhalos within half of the virial radius. We see that generally the reduction in substructure counts at a fixed $V_{\text{max}}$ is small but non-zero and that the effects appear to be stronger at small radii than large. Similarly, there appears to be slightly more reduction of substructure in the SIDM cluster halos compared to the galaxy size systems.

We can understand both trends, 1) the increase in the difference between the CDM and SIDM $V_{\text{max}}$ functions as $M_{\text{vir}}$ increases and 2) the increase in the difference as one looks at the central regions of the halo, using the results from the previous section as a guide. The typical probability that particle in an SIDM subhalo will interact with a particle in the background halo is

$$P \approx \left(\rho_{\text{host}}(r)/(\sigma/m) v_{\text{orb}}(r)\right)^2 T,$$

where $v_{\text{orb}}(r)$ is the orbital speed of the subhalo at position $r$, $\rho_{\text{host}}$ is the mass density of the host halo, and $T$ is the orbital period. The typical speed of the subhalo is similar to the rms speed of the smooth component of the halo, and thus $\rho_{\text{host}}(r)/(\sigma/m) v_{\text{orb}}(r)$ should be similar to the function we show in Figure 7. At fixed $r/r_s$ we expect $P$ to scale with $V_{\text{max}}^2$, $V_{\text{max}}^2/r_{\text{max}}^2$ (given that $\rho_s \propto V_{\text{max}}^2/r_{\text{max}}^2$), which is a very mildly increasing function of $V_{\text{max}}$ over the range of halo masses we have simulated. Note though that we expect scatter at fixed halo mass because of the scatter in the $V_{\text{max}} - r_{\text{max}}$ relation (Bullock et al. 2001).

While the increase in destruction of subhalos with host halo mass is not strong, it is clear from the above arguments that subhalos in the inner parts of the halo ($r/r_s \ll 1$) should be destroyed but the bulk of the subhalos around $r/r_s \sim 1$ and beyond should survive for $\sigma/m = 1 \text{ cm}^2/\text{g}$. This effect is strengthened by the fact that subhalos in the innermost region of the halo were accreted much longer ago than subhalos in the outskirts, so they have experienced many more orbits (Rocha et al. 2011). These arguments explain the comparisons between the subhalo mass functions plotted in Figure 8. Our arguments demonstrate that a large fraction of the subhalos found in CDM halos (most of which are in the outer parts) would still survive in SIDM halos for $\sigma/m$ values around or below 1 cm$^2$/g.

Overall in the previous two sections we have seen that the effects of self-interactions between dark matter particles in cosmological simulations are primarily in the central regions of dark matter halos, leaving the large scale structure identical to our non-interacting CDM simulations. Thus we retain the desirable features of CDM on large scales while revealing different phenomenology near halo
Figure 5. Circular velocity profiles for our example selection of six well resolved halos from our CDM, SIDM\textsubscript{1} and SIDM\textsubscript{0.1} simulations. The magnitude of the circular velocity at small radii \( r \lesssim r_s \) is lowered for all halos when self-interactions are turned on. \( r_s \) is the NFW scale radius of the corresponding CDM halo density profile.

centers. In the following section we will move to explore how the properties of SIDM halos presented here scale with halo mass.

6 SCALING RELATIONS

In the previous section we saw that while SIDM preserves the CDM large scale properties of dark matter halos, self-interactions in the central regions of halos result in a decrease of central densities and the formation of cores in their density profiles. We found that the density profiles of halos from our SIDM\textsubscript{1} simulations can be relatively well fit by Burkert density profiles inside \( r \sim 2 - 3r_s \) (see Figure 4). Here we define a sample of well resolved halos from all our SIDM\textsubscript{1} simulations and use Burkert fits to their density profiles in order to quantify their central densities and core sizes. We then provide scaling relations of dark matter halo properties with maximum circular velocity \( V_{\text{max}} \).

The sample of halos used for the rest of this section consists of the two host halos in our SIDM\textsubscript{1}-Z11 and SIDM\textsubscript{1}-Z12 simulations together with the 25 most massive halos from our SIDM\textsubscript{1}-50 and the 25 most massive halos from our SIDM\textsubscript{1}-25 simulations. That gives us a total of 52 halos spanning a range \( V_{\text{max}} = 30 - 860 \) km/s or \( M_{\text{vir}} = 5 \times 10^{11} - 2 \times 10^{14} \) M\(_{\odot}\). For this set of halos the innermost resolved radius, defined by Equation 20 in Power et al. (2003), is always smaller than one third of the Burkert scale radius from which we define the sizes of cores. It is vital that we do a conservative comparison to the Power et al. (2003) radius because both gravitational scattering and self-interactions lead to the same phenomenological result of constant density cores. Most of the halos (other than the 52 we select here) do not pass this test well enough for the core set by self-interactions to be resolved with confidence. This desire to be conservative in our presentation of scaling relations forces us to find these relations from only a small sample of halos for SIDM\textsubscript{1} and leaves us with basically no halos to find scalings for SIDM\textsubscript{0.1}. Also one has to keep in mind that our SIDM\textsubscript{1} relations could be biased by selecting only the most massive halos in our full box simulations. Evidently higher resolution simulations are necessary to find definitive answers. It is reassuring however that the scaling relations derived from our analytical arguments in §7 agree so well with the ones presented here for \( \sigma/m = 0.1 \) cm\(^2\)/g.

We have checked that for all of our halos we resolve the scattering rate out to at least four times the Burkert scale radius. Outside of this point the scattering rate is underestimated because of our choice of the self-interaction smoothing length relative to the inter-particle spacing (see §3). However, the expected scattering rate is negligible with respect to the Hubble rate outside that radius (Figure 7). Moreover, we have re-run our 50h\(^{-1}\) Mpc boxes for a range of SIDM smoothing values and found identical results. Thus we consider our sample to be well resolved.

Eight halos in our sample are undergoing significant interactions and have density profiles that are clearly perturbed even in the CDM runs. We include these eight systems in all of the following plots but indicate them with open symbols. We do not use them in the best fits for the scaling relations that we provide.

We start by examining the global structure of halos as char-
Figure 6. Velocity dispersion profiles for our six example halos from our SIDM\textsubscript{1} and SIDM\textsubscript{0.1} simulations over-plotted with their CDM counterparts. The velocity dispersion is inflated at small radii and slightly suppressed at large radii. The effects set in at approximately the radius where SIDM particles experience at least one interaction on average over the lifetime of the halo (see Figure 7).

The maximum circular velocity $V_{\text{max}}$ and the radius where the rotation curve peaks, $r_{\text{max}}$, provide a simple, intermediate-scale measure of halo concentration and we aim to investigate any differences between SIDM and CDM. Figure 2 shows the $V_{\text{max}} - r_{\text{max}}$ relation for CDM (black) and SIDM\textsubscript{1} (blue) halos. We can see that small differences of about 10\% exist in both $V_{\text{max}}$ and $r_{\text{max}}$, with SIDM\textsubscript{1} halos having larger values for $V_{\text{max}}$ and smaller for $r_{\text{max}}$. This was already evident in Figure 5, where the circular velocity curves of SIDM\textsubscript{1} halos seem to peak at slightly smaller radii and slightly larger velocities than their CDM analogs, even though SIDM\textsubscript{1} curves decrease more steeply at the center.

The apparent difference is consistent with a picture where energy exchange due to scattering redistributes the SIDM dark matter particles, with many of the tightly bound particles scattered onto less bound, high apocenter orbits. Since the radius at which self-interactions are significant (see Figure 7) is smaller than (but close to) $r_s$, it is entirely reasonable that the scattered particles lead to a new $r_{\text{max}}$ for SIDM\textsubscript{1} halos that is smaller than the CDM $r_{\text{max}}$ and a $V_{\text{max}}$ that is larger. In other words, the core sizes are a fixed fraction of the CDM halo scale radius. The relation that best fits our data is given by

$$r_b = 0.71 \left( \frac{r_s}{10 \text{ kpc}} \right)^{-0.08}.$$

We note that the scaling with $V_{\text{max}}$ is close to that expected for $r_{\text{max}}$ or $r_s$. We show this explicitly by fitting for the core size of SIDM\textsubscript{1} halos $r_b$ as a function of the NFW scale radius $r_s$ of their CDM counterparts, as shown in Figure 11. We find that the ratio of the core size of a SIDM\textsubscript{1} halo to the scale radius of the corresponding CDM halo varies very mildly with $V_{\text{max}}$. In other words, the core sizes are a fixed fraction of the CDM halo scale radius. The relation that best fits our data is given by

$$\frac{r_b}{r_s} = 0.71 \left( \frac{r_s}{10 \text{ kpc}} \right)^{-0.08}.$$

This underscores the point that $r_b$ and $r_s$ are closely tied to each other and the fact that they are numerically so close to each other...
Figure 7. Estimate of the local scattering rate modulo the cross section \( \sigma v_{\text{rms}} = \Gamma (\sigma/m)^{-1} \) for six well resolved halos from our CDM, SIDM\(_{0}\), and SIDM\(_{1}\) simulations. The quantity is scaled by 1 Gyr cm\(^2\)/g, such that 1 in these units means that each particle has roughly one interaction per Gyr in SIDM\(_{1}\) and 0.1 per Gyr in SIDM\(_{0}\). Based on this argument, the effects of self-interactions in the properties of halos over \( \sim 10 \) Gyr should start to become important when the ordinate is greater than about 0.1 in SIDM\(_{1}\) \( (r/r_s \sim 0.8) \) and greater than about 1 in SIDM\(_{0}\) \( (r/r_s \sim 0.2) \). Comparisons to Figures 4-6 indicate that this is indeed the case.

In this section we have presented scaling relations for the properties of halos in our SIDM\(_{1}\) simulations. Our limited resolution allows us to use only 52 halos spanning a modest mass range, from which we throw out eight systems that are undergoing mergers. Admittedly, this sample is not large enough to be definitive, especially in regards to scatter. However, the strong correlation between the SIDM core radius \( r_b \) and the counterpart CDM scale radius \( r_s \) is clearly statistically significant and the general trends provide a useful guide for tentative observational comparisons – a subject we will return to in the final section below.

**7 ANALYTIC MODEL TO EXPLAIN THE SCALING RELATIONS**

In this section we develop a simple model to understand the scaling relations shown in Figure 7. This model is based on identifying an appropriate radius \( r_1 \) within which self-interactions are effective and demanding that the mass as well as the average velocity dispersion within this radius is set by the mass and the average velocity dispersion (within the same radius) of the same halo in the absence of self-scatterings. The mass loss due to scatterings in the core should be insignificant because particles rarely get enough energy to escape and this implies that the mass within \( r_1 \) should be close to what it would have been in the absence of self-interactions. This also implies that the potential outside \( r_1 \) is unchanged from its CDM model prediction, but tends to a constant value faster inside
Within this set of approximations, the dominant effect due to scatterings is to re-distribute kinetic energy in the core, while keeping the total kinetic energy within $r_1$ the same as it would have had before self-interactions became important. We have looked at the kinetic energy profiles in the best-resolved halos in our simulations and have confirmed that this is indeed a good approximation. Note that in this picture, there is a clear demarcation of time-scales such that the inner halo structure (say $r \lesssim r_s$) is set (the same way as in CDM model) well before self-interactions become important. For cross sections much larger than what we are interested in here, this need not hold.

To set up the model, we start by recalling that self-interactions work to create an isothermal core (see Figure 6) that is isotropic (both spatially and in velocity space). Using the spherical Jeans equation, one can then see that for a system with these properties

$$v_{\text{rms},0}^2 = 3\sigma_r(0)^2 = 2\pi\xi^{-1}G\rho(0)r_0^2,$$

where we have defined $r_0$ to be the expansion parameter such that $\rho(r)\sigma_r^2 = \rho(0)\sigma_r(0)^2(1 - \xi(r/r_0)^2)$ when $r \ll r_0$, and $\sigma_r$ is the radial velocity dispersion. The form of the Taylor expansion for $\rho(r)\sigma_r^2$ is dictated by the Jeans equation for density profiles that tend to a constant value, as may be readily ascertained by taking the derivative of $\rho(r)\sigma_r^2$. To fix $r_0$, we will choose it to be equivalent to the Burkert scale radius where the density is one-fourth of the central density. The parameter $\xi$ encapsulates uncertainties from the profile and velocity dispersion anisotropy in the outer parts of the halo. We test various models and find that a range of 2-3 for $\xi$ is largely consistent with most parameterizations and hence we fix $\xi = 2.5$. If we specify the central velocity dispersion, then with an additional constraint on the core region (i.e., $r_1$), we would be able to back out both the core radius and the core density.

We then set $v_{\text{rms},0}^2$ equal to the average velocity dispersion squared (i.e., two times kinetic energy divided by mass) within the region $r_1$ in the absence of self-interactions. This basically demands that the kinetic energy within $r_1$ is unchanged from the value
it would have had in the absence of self-interactions. Note, however, that we are setting the average velocity dispersion squared equal to $v_{\text{rms},0}^2$ and not the corresponding average in the SIDM halo. This is an approximation, but one that is degenerate with choosing the $\xi$ parameter.

To finish specifying this model, we need a density profile for the region inside $r_1$. A Burkert profile has a velocity dispersion profile (assuming isotropy) that asymptotes very slowly to the central dispersion. For small radii, the radial dispersion profile is slowly increasing (with radius) because of the $r/r_0$ term in the Taylor expansion for the density profile. If we want a flatter central dispersion profile (as is observed for the SIDM halos), we can fix this by either assuming an isothermal profile or something like $1/(1 + 1.52(r/r_0)^{2})^{3/2}$. The final results turn out to be qualitatively similar for these profiles. Hence we adopt a Burkert profile for ease of comparison to the fits presented here and then check the results with more appropriate profiles later. Our two constraints (on the radial velocity dispersion and mass) fully specify the density and radial scales of the Burkert profile.

In order to obtain scaling relations we need to estimate $r_1$, which demarcates the inner region where self-interactions are effective from the outer region that is mostly undisturbed by the self-interactions. In reality, this divide will not be sharp but we will see that the main features of the scaling relations are well-captured by this simple model. We define $r_1$ to be the region where each particle on average suffers one interaction. Since the region outside is assumed to be unperturbed by interactions, we may estimate $r_1$ as:

$$G(r_1)\rho(t_{\text{age}}) = 1.3\rho_{\text{CDM}}(r_1)v_{\text{rms,CDM}}(r_1)\frac{\sigma}{m}t_{\text{age}} = 1,$$  \hspace{1cm} (22)

where we set age ($t_{\text{age}}$) to be 10 Gyr for now, keeping in mind that larger halos have a shorter age and that major mergers can reset the timer. We will consider what happens when $t_{\text{age}}$ is a function of halo mass shortly. The factor 1.3 is $(v - u)/\sqrt{v^2}$ for a Maxwellian distribution where $u$ and $v$ are the velocities of the two interacting dark matter particles. We have not attempted to use a more realistic velocity distribution since the dependence of this factor on a possible high-velocity cut-off to the distribution function was found to be fairly mild.

For the density profile in the absence of self-interactions, we assume a NFW profile and to fix the velocity dispersion we use the observed fact that the phase space density is a power-law in radius (Taylor & Navarro 2001). By noting that $v_{\text{rms,CDM}}(r) = (\rho_{\text{CDM}}(r)/Q(r))^{1/3}$ and using a phase-space density profile $Q(r) = Q(r_0)(r/r_0)^{-3}$ (Taylor & Navarro 2001), Rasia et al. 2004; Ascasibar et al. 2004; Dehnen & McLaughlin 2005; Ascasibar & Gottlöber 2008, we may fully specify the dependence of $r_1$ on the cross-section and halo parameters (say $V_{\text{max}}$ and $r_{\text{max}}$). For the phase-space density profile we use a power-law index $\eta = 2$ and $Q(r_0) = 0.3/(G \rho_{\text{max}} v_{\text{rms}}^2 r_{\text{max}}^3)$ derived from jointly fitting our relaxed CDM halos; these parameters are very similar to the fits provided in Ascasibar & Gottlöber (2008).

Let us first look at how $r_1$ scales with $r_0$ in the NFW density profile. One notes that $v_0 = 1.72V_{\text{max}}/(G r_{\text{max}})$ and hence $\rho_0 V_{\text{max}} \propto V_{\text{max}}^3/r_{\text{max}}^5$ which is a very mildly increasing function of $V_{\text{max}}$ as our Equation 15 shows. Thus Equation 22 implies that $r_1/r_0$ should be roughly a constant. Numerically, we find that $r_1/r_0 \approx 0.7 - 0.8$ over the range of $V_{\text{max}}$ of interest for $\sigma/m = 1\text{cm}^2/\text{g}$.

Having now specified $r_1$, we are ready to look at the scalings of $r_0$ and $\rho_0$. For our assumed value of $\xi$, $v_{\text{rms},0} \approx 2.5G \rho_0 r_0^2$. Thus we are looking for the value of $r_0/r_s$ that solves:

$$\frac{\rho_0}{(r/r_s)(1 + r/r_s)^2} \frac{Q(r_s)}{Q(r)} = (\frac{\rho_0}{v_{\text{rms},0}})^3,$$  \hspace{1cm} (23)

with the constraint that $M_0(r_1) = M_{\text{NFW}}(r_1)$ where $M_{\text{NFW}}(r)$ and $M_0(r)$ are the masses enclosed within radius $r$ for NFW and Burkert profiles, respectively. We note that if $r_0/r_s$ is not a strong function of $V_{\text{max}}$ and since we know $r_1/r_0$ is a mild function of $V_{\text{max}}$, then the mass constraint essentially sets $\rho_0 r_s^3/(\rho_0 r_s^2)$ to be a constant or $\rho_0 r_s^3 \propto (\rho_0 r_s^2)$. This implies $v_{\text{rms},0} \rho_0 \propto V_{\text{max}}^3/r_{\text{max}}$.

Now Equation 22 sets $v_{\text{rms},0} \approx V_{\text{max}}$ because $r_1/r_0$ is a mild function of $V_{\text{max}}$ and it therefore follows that $r_0 \propto r_s$ is a consistent solution to the above equations. As a check we note that assuming $r_1/r_0 = 0.7 - 0.8$ gives $v_{\text{rms},0} \approx 1.1 V_{\text{max}}$, in reasonable agreement with our SIDM simulation results (see Figure 6). This simple model thus predicts that $r_0/r_s$ should not vary much with $V_{\text{max}}$ in agreement with the observed scaling relations from the SIDM simulation.

In detail, the model predicts that $r_0/r_s = 0.5 - 0.6$ for dwarf to cluster halos in good agreement with the fits to our SIDM halos, but about 25% smaller for $V_{\text{max}} \sim 100\text{km/s}$. It departs from the results of the simulation in predicting that $r_0/r_s$ increases gently with $V_{\text{max}}$, whereas Figure 11 predicts that this ratio should decrease gently with $V_{\text{max}}$. We find that this departure from simulations is likely related to the assumption of a constant age for all halos. To generalize our model, we use the results of Wechsler et al. 2002 who show that the virial concentrations of halos are correlated with their formation times, and in particular $v_{\text{vir}} = 4.1(1 + z_{\text{form}})$ for a particular definition of formation time. We invert this equation to derive an estimate of the halo age using $z_{\text{form}}$. With the age thus specified in Equation 22 we find that now $r_0/r_s$ decreases gently with $V_{\text{max}}$ in substantial agreement with the fit to our simulations. Thus the reason that larger halos have a smaller $r_0/r_s$ is because self-interactions have had less time to operate. We note that the values for the core radius in the analytic model with halo mass dependent $t_{\text{age}}$ are uniformly about $0.5$ to $0.6$. 

Figure 10. Burkert scale radius vs. $V_{\text{max}}$ for our combined sample of well resolved halos from our SIDM1-50 (blue circles), SIDM1-25 (green stars), SIDM1-Z12 (cyan square) and SIDM1-Z11 (red triangle) simulations. Open symbols correspond to halos that are undergoing mergers. A single power law holds along the whole range of our sample, suggesting that this dependence continues towards smaller and larger $V_{\text{max}}$ values.
8 OBSERVATIONAL COMPARISONS

The goal of this section is to discuss our results in comparison to observationally inferred properties of dark-matter density profiles. In particular, we will focus on the core densities and core sizes.

25% smaller, but this should not be a cause for concern given the approximation in denying a sharp transition at \( r_1 \).

Given the Burkert core radius \( r_b \) and the central velocity dispersion \( \sigma/m \), one can easily check that the central density \( \rho_b \) is about 0.01 \( M_\odot/pc^3 \) for \( V_{\text{max}} = 300 \text{ km/s} \) halos and 0.005 \( M_\odot/pc^3 \) for \( V_{\text{max}} = 1000 \text{ km/s} \) in this analytic model. These numbers and the scaling with \( V_{\text{max}} \) for \( r_b \) (when including the halo mass dependent \( t_{\text{age}} \)) are in good agreement with the densities in Figure 12 and the fit in Equation 19. As we have indicated before, the scaling relation for the central density should be interpreted with care given the large scatter. Given the tight correlation between core radius and \( r_s \), it is possible that the substantial scatter in the central density arises in large part due to the scatter introduced by the assembly history in the concentration-mass relation. This has important implications for fitting to the rotation velocity profiles of low-surface brightness spirals (Kuzio de Naray et al. 2010) and deserves more work.

The simple model constructed above also provides insight into the core collapse time scales. In particular, as long as the outer part (region outside \( r_1 \)) dominates the potential well and sets the average central temperature (or the total kinetic energy in the core), we do not expect core collapse. This is simply because core collapse requires uncontrolled decrease in temperature, which is prohibited here. Once \( r_1 \) moves out well beyond \( r_{\text{max}} \) or to the virial radius, there is significant loss of particles and core collapse may occur if there are no further major mergers. The time scale for this process is much longer than the age of the universe for \( \sigma/m = 1 \text{ cm}^2/\text{g} \) because the inner core is at \( r_1 < r_s \) after 10 Gyr for this self-interaction strength and we see no evidence for significant mass loss.

8.1 Predicted Core Sizes and Central Densities in SIDM

8.1.1 SIDM with \( \sigma/m = 1 \text{ cm}^2/\text{g} \).

The central properties of dark-matter halos have been inferred from observations of the Milky Way dwarf spheroidal (dSph) galaxies (\( V_{\text{max}} \leq 50 \text{ km/s} \)) to galaxy clusters (\( V_{\text{max}} \geq 1000 \text{ km/s} \)). If we extrapolate the results from our set of SIDM simulations using Eqs. (16)-(20) we predict that SIDM halos with \( \sigma/m = 1 \text{ cm}^2/\text{g} \) would have the following (Burkert) core sizes and central densities:
For galaxy clusters \((V_{\text{max}} \simeq 700 - 1000 \text{ km/s})\):
\[
r_0 \simeq (95 - 155) \text{ kpc} ; \quad \rho_0 \simeq (0.005 - 0.004) \, \text{M}_\odot \text{pc}^{-3}
\]

For low-mass spirals \((V_{\text{max}} \simeq 50 - 130 \text{ km/s})\):
\[
r_b \simeq (3 - 10) \text{ kpc} ; \quad \rho_b \simeq (0.02 - 0.01) \, \text{M}_\odot \text{pc}^{-3}
\]

For dwarf spheroidals galaxies \((V_{\text{max}} \simeq 20 - 50 \text{ km/s})\):
\[
r_b \simeq (0.9 - 3) \text{ kpc} ; \quad \rho_b \simeq (0.04 - 0.02) \, \text{M}_\odot \text{pc}^{-3}
\]

Although we cannot completely determine the scatter in our scaling relations due to low number statistics, it is important to note from Figs. 10 and 12 that a scatter of at least a factor of 2 in core sizes, and at least a factor of 3 in central densities, is expected for a given \(V_{\text{max}}\). We suspect that these differences are in large part a result of the diversity of merger histories of dark-matter halos. Note that the \(V_{\text{max}}-r_{\text{max}}\) and \(Q(r_b)\) scalings assumed in the analytic model are the median values. The strong dependence of the SIDM halo profiles on these quantities makes it clear that the scatter in these relations will introduce significant scatter in the halo core sizes and core densities. Thus the analytic model should also provide a simple way to understand (some of the) scatter seen for SIDM halo properties. In future work we will characterize the relation between the core properties and merger history in the context of a detailed discussion of the scatter in the scaling relations, especially on scales that we do not resolve with our current simulations.

### 8.1.2 SIDM with \(\sigma/m = 0.1 \text{ cm}^2/\text{g}\)

As discussed in our SIDM\(_{0.1}\) simulations are not well enough resolved to definitively measure a core radius for any of our halos, much less define scatter in that quantity. Nevertheless, our best resolved systems do demonstrate some clear deviations from CDM and allow us to cautiously estimate individual core densities. Referring back to Figure 4, we see that in our two best resolved cluster halos (at \(M_{\odot} \approx 10^{17} \text{ M}_\odot\)) the SIDM\(_{0.1}\) core densities approach \(\sim 0.01 \, \text{M}_\odot/\text{pc}^3\) – each at least a factor of \(\sim 3\) denser than their SIDM\(_1\) counterparts. Similarly, in our Z12 Milky-Way case, the SIDM\(_{0.1}\) core density appears to be approaching \(\sim 0.1 \, \text{M}_\odot/\text{pc}^3\) compared to \(\sim 0.02 \, \text{M}_\odot/\text{pc}^3\) in the SIDM\(_1\) case.

Given the lack of well-resolved halo profiles, it is worth appealing to the analytic model presented in 13 to estimate core radii for SIDM\(_{0.1}\). Using exactly the same arguments (including the halo mass dependent age), we find that \(r_1/r_s \simeq 0.05 - 0.12\) in the 100-1000 km/s \(V_{\text{max}}\) range and a corresponding Burkert core radius \(r_b / r_s \simeq 0.09 - 0.17\). We note that the Burkert radius is close to but slightly larger than \(r_1\). It is important to keep in mind that in this analytic model we are only explicitly fitting the inner “self-interaction zone” of \(r < r_1\). This does not imply that the entire halo has to be well-fit by the Burkert profile. Recall that a single-scale Burkert profile only works as well as it does for \(\sigma/m = 1 \text{ cm}^2/\text{g}\) because \(r_b \approx r_s\), such that to a good approximation there is only one relevant length scale. For the smaller cross section that we are now considering we expect the core and NFW scale radii to be widely separated, suggesting that a generic functional form for SIDM halos should have two scale radii. A wide separation between the SIDM\(_{0.1}\) core and \(r_s\) does appear to be consistent with the highest resolved halos presented in Figure 4. However, we note that given the strong correlation between \(r_b/r_s\), we still expect a one-parameter family of models for a given \(\sigma/m\).

To see how dependent our results are on the shape of the inner halo profile, we modify the analytic model to include a density profile that decreases with radius as \(1/(1 + (r/r_s)^2)^{\alpha/2}\). For this density profile, the velocity dispersion profile has the right form to match our simulation results. The price we pay is the introduction of a new parameter \(\alpha\). We set this parameter alpha by additionally demanding that the slope of the mass profile (i.e., density) is continuous at \(r_1\), so that the mass profile joins smoothly with the NFW mass profile. This picks out a narrow range \(\alpha = 5.5 - 7.0\) as the solution over most of the \(V_{\text{max}}\) range of interest (with smaller values corresponding to lower \(V_{\text{max}}\)). Interestingly, this implies that at \(r_1\), the slope of the density profile is very close to \(-2\) for the entire range of \(V_{\text{max}}\) values of interest. Note that while the mass profile is continuous, the slope of the density profile is not matched smoothly at \(r_1\) (since the slope of the NFW profile would be closer to \(-1\) at \(r_1 \ll r_b\)). This probably signals that if the matching were not done sharply (at \(r_1\)), the density profile of SIDM would overshoot that of CDM and catch up at some radius beyond \(r_1\) (as is seen in the comparison of SIDM\(_1\) and CDM density profiles).

As a check we apply this \(\alpha\)-model to \(\sigma/m = 1 \text{ cm}^2/\text{g}\) case and find that the results are qualitatively the same as the model with the Burkert profile. The quantitative differences are at 20% level with the densities being smaller and inferred Burkert core radii (where density is 1/4 of the central density) larger compared to the Burkert profile model. The predicted slope of the density profile at \(r_1\) is close to \(-2.5\) implying a smoother transition to the NFW profile (since \(r_1 \sim r_b\) for \(\sigma/m = 1 \text{ cm}^2/\text{g}\)), as is seen Figure 4.

For the \(\sigma/m = 0.1 \text{ cm}^2/\text{g}\) case, we obtain \(r_b/r_s = 0.08 - 0.17\) and an equivalent Burkert core radius (where the density is one-fourth of the central density) \(r_{b1}/r_s = 0.06 - 0.14\), in substantial agreement with the results we obtained using the Burkert profile. Thus our analysis would suggest that core sizes \(\sim 0.1 r_s\) for \(\sigma/m = 0.1 \text{ cm}^2/\text{g}\). The results from the analytic model for \(\sigma/m = 0.1 \text{ cm}^2/\text{g}\) also seem consistent with our simulation results; see Figure 6 where the \(V_{\text{max}}\) profiles for SIDM\(_{0.1}\) start to deviate from CDM at \(\sim 0.2 r_s\).

Based on the discussion above we conclude that for \(\sigma/m = 0.1 \text{ cm}^2/\text{g}\) we expect:

For galaxy clusters \((V_{\text{max}} \simeq 700 - 1000 \text{ km/s})\):
\[
r_b \sim (16 - 20) \text{ kpc} ; \quad \rho_b \sim 0.04 \, \text{M}_\odot \text{pc}^{-3}
\]

For low-mass spirals \((V_{\text{max}} \simeq 50 - 130 \text{ km/s})\):
\[
r_b \sim (0.6 - 2.5) \text{ kpc} ; \quad \rho_b \sim 0.2 - 0.1 \, \text{M}_\odot \text{pc}^{-3}
\]

For dwarf spheroidals galaxies \((V_{\text{max}} \simeq 20 - 50 \text{ km/s})\):
\[
r_b \sim (0.2 - 0.6) \text{ kpc} ; \quad \rho_b \sim 0.5 - 0.2 \, \text{M}_\odot \text{pc}^{-3}
\]

These values do not include the scatter from mass assembly history. It is probably reasonable to assume a factor of 2 scatter for both core radii and core densities based on what we see in SIDM\(_1\). It is also possible that the core densities are \(\sim 50\%\) smaller than what we would see in simulations, given that the SIDM\(_1\) simulations have core densities that are somewhat larger than the predictions from the analytic model. For the dwarf spheroidal galaxies, the values should be interpreted with caution as it is the prediction for field halos with \(V_{\text{max}}\) range \(20 - 50 \text{ km/s}\).

While these values are somewhat tentative compared to those presented above for SIDM\(_1\) (given our lack of direct simulation fits), two factors are reassuring. First, the analytic model is based on the simple assumption that scattering redistributes kinetic energy within the inner halo and the non-trivial aspect of the model is defining this “inner halo” region. There is no reason to suspect that...
this assumption or the prescription breaks down for SIDM\(_{0.1}\) halos when it works so well in describing the SIDM\(_{1}\) halos. The predicted densities are in line with those inferred for the best resolved halos in our SIDM\(_{0.1}\) simulations (shown in Figure 4 and discussed above). For the core radii, we reiterate that the label “\(r_\text{c}\)” should be interpreted (according to its definition in the analytic model) as the radius where the density reaches one-fourth the asymptotic core density. The overall profile of a halo with such a small core compared to \(r_\text{c}\) will not be fit by the Burkert form. Note that the strong correlations we predict between the core radius and the NFW scale radius raise the intriguing possibility that the SIDM halos may be also well fit (modulo scatter) by a single parameter profile as is the case for CDM.

Next, we compare our predictions for SIDM core properties against data and show that the core radii and densities appear to be consistent with that seen in real data, motivating future simulations with high enough resolution to resolve cores in SIDM\(_{0.1}\) halos.

8.2 Observed Core Sizes and Central Densities vs. SIDM

In this section, we explore the predictions for the properties of density profiles with SIDM in the context of observational constraints on density profiles. We also revisit previous constraints on SIDM from observations in light of our simulation suite.

8.2.1 Clusters

One of the tightest SIDM constraints from the first generation of SIDM studies emerged from one cluster simulation and one observed galaxy cluster. Specifically, Yoshida et al. (2000) simulated an individual galaxy cluster with different SIDM cross sections. When comparing the core size of this simulated cluster to the core size estimated by Tyson et al. (1998) for CL 0024+1654, they found that the observed core in CL 0024+1654 would be consistent with SIDM only if \(\sigma/m \lesssim 0.1 \text{ cm}^2/\text{g}\). Since that time, evidence has emerged that this particular cluster is undergoing a merger along the line of sight (Bosak et al. 2001, 2003, Zhang et al. 2003, Lee et al. 2007, Jed 2010, Umetsu et al. 2010). Thus, this cluster is not the ideal candidate for SIDM constraints based on the properties of relaxed halos, and the Yoshida et al. (2000) constraint is not valid in this context.

Using X-ray emission, weak lensing, strong lensing, stellar kinematics of the brightest cluster galaxy (BCG) or some combination thereof, the mass distributions within a number of galaxy clusters have been mapped in the past decade. Abadij et al. (2002) placed a conservative upper limit of 75 kpc on the size of any constant-density core, and an average density within the inner 50 kpc of \(\sim 0.025 \text{ M}_\odot/\text{pc}^3\) for an halo with an estimated mass \(M \sim 4 \times 10^{14} \text{ M}_\odot\). Sand et al. (2004), Sand et al. (2008), Newman et al. (2009), and Newman et al. (2011) all find central density profiles in clusters shallower than the NFW CDM prediction. The difference in the work between these authors and others is that they use stellar kinematics of the BCG to constrain the density profile of the cluster dark-matter halo on small scales. While this probe of the density profile is more sensitive on small scales than strong lensing is, proper inference of the dark halo properties depends on accurate modeling of the BGC density profile and equilibrium structure. They have typically assumed a “gNFW” profile in order to constrain the central densities: \(\rho(r) \propto 1/(x/(1+x)^{3-\beta})\) with \(x = r/r_\text{s}\) and the NFW form obtained when \(\beta = 1\). The Newman et al. (2009) and Newman et al. (2011) mass models of \(M \sim 10^{15} \text{ M}_\odot\) clusters show average dark matter central densities within 10 kpc of \(\sim 0.03 - 0.06 \text{ M}_\odot/\text{pc}^3\) and \(r_\text{s}\) of order 100 kpc. Note that 10 kpc is typically the smallest radii our simulations can resolve.

Saha et al. (2006) and Saha & Read (2009) studied the mass structure of 3 cluster halos from gravitational lensing and obtained density profiles that are consistent with \(\rho \propto r^{-1}\) outside the inner 10 – 20 kpc regions. Similarly Morandi et al. (2010) and Morandi & Limousin (2012) find that the radial mass distribution of cluster dark-matter halos are consistent with NFW predictions outside 30 kpc in projection. The CLASH multi-cycle treasury program on the Hubble Space Telescope is finding many new strongly lensed galaxies in about a set of 25 massive clusters (Postman et al. 2012). Initial results from this program show that the total density profile of these clusters (or total density minus the brightest cluster galaxy), if modeled as spherically symmetric, are consistent with NFW predictions for the halo alone if the gNFW functional form is used in the fit (Zitrin et al. 2011; Coe et al. 2012; Umetsu et al. 2012). However, Morandi et al. (2010) and Morandi et al. (2011) find that spherical mass modeling of galaxy clusters typically results in an overestimate of the the cuspiness of the density profile, although axially-symmetric modeling is found to lead to underestimates (Meneghetti et al. 2007). Thus, the present status of the density profiles of the CLASH clusters is unclear and clearly an interesting data set to look forward to.

We note here a complexity involved in using the lensing results to constrain SIDM models. Lensing provides mass in cylinders along the line of sight and this 2D mass profile is sensitive to mass from a large range of radii. As an example, lets consider mass within 30 kpc in projection. If we were to do something extreme and create a zero density core inside 30 kpc sphere, the differences in the 2D mass profile would be less than a factor of 2 for clusters in the \(10^{14-15} \text{ M}_\odot\) mass range. For SIDM\(_{0.1}\), the differences are comparatively benign. Our analytic model predicts that differences relative to CDM at about 0.1\(r_\text{s}\) (which is 10 – 40 kpc for \(10^{14-15} \text{ M}_\odot\) virial mass range) are 20-30%, which implies SIDM\(_{0.1}\) surface mass density profiles are very similar to CDM on these scales. But for SIDM\(_{1}\) the expected differences would be measurably large.

On a related technical note, we discourage the use of the gNFW functional form when thinking about models that deviate from the CDM paradigm. In the SIDM case, for \(\sigma/m < 1 \text{ cm}^2/\text{g}\), there will generically be two scale radii: one is the NFW-like scale radius which is the result of hierarchical structure formation (Lithwick & Dalal 2011), and the second is the core radius from dark-matter self-scattering. For \(\sigma/m = 1 \text{ cm}^2/\text{g}\), as we explained in detail in §7 the two scales are about the same. If most of the cluster data constrain the density profile beyond a SIDM core, as they may for weak lensing and X-ray studies, the gNFW or NFW fit is dominated by those data, and a core will not be “detected” in the fit. In future work, we will simulate halos with a broader range of \(\sigma/m\) and provide SIDM-inspired density profiles to the community.

The results discussed above seem to suggest that the density profile beyond about 25 kpc should be close to the predictions from the NFW profile. To test this we plot the average physical density within 25 kpc for well-resolved halos in our CDM (black), SIDM\(_{0.1}\) (green), and SIDM\(_{1}\) (blue) simulations in Figure 13. We see that for the most massive halos, the \(\sigma/m = 1 \text{ cm}^2/\text{g}\) run produces densities at 25 kpc that are \(\sim 2 - 3\) times lower than their CDM counterparts. Thus it seems like the measured densities in clusters rule out \(\sigma/m = 1 \text{ cm}^2/\text{g}\) SIDM model. At the same time the \(\sigma/m = 0.1 \text{ cm}^2/\text{g}\) simulations are quite similar to CDM at
that SIDM cluster scales. For the SIDM range 50 will provide essential tests of interesting SIDM models. We conclude that estimates of the central dark matter density in clusters appear on cluster scales. Thus observations of central densities in clusters with cross sections in the vicinity of 0.1 cm²/g is likely to provide the most promising avenue to look for signatures of SIDM that would be compatible with SIDM0.1. Given this outlook, it is reasonable to conclude that estimates of the central dark matter density in clusters will provide essential tests of interesting SIDM models.

8.2.2 Low-Mass Spirals

For low-mass spirals with maximum circular velocities in the range 50 – 130 km/s, constant-density cores with sizes of ~ 0.5 – 8 kpc and central densities of approximately ~ 0.01 – 0.5 M⊙/pc³ have been observed (de Blok et al. 2001; Simon et al. 2005; Sánchez-Salcedo 2005; Kuzio de Naray et al. 2008, 2010; Oh et al. 2011; Salucci et al. 2012). Similar to what we found for clusters scales, SIDM with σ/m = 1 cm²/g would be able to reproduce the largest core sizes observed in low-mass galaxies but it predicts central densities that are too low. SIDM with σ/m = 0.1 cm²/g would be much more consistent. Moreover, the predicted log-slope of the density profile at 500 parsecs for σ/m = 0.1 cm²/g halos in the 50-130 km/s range is -0.5 to 0, both facts consistent with results from THINGS (Oh et al. 2011). Note that the slope at 500 pc for the σ/m = 1 cm²/g model is 0 in the same Vmax range, which is not consistent with the scatter seen in the data.

We conclude, as before, that the observed densities and core radii are not consistent with SIDM but are fairly well reproduced in SIDM models with σ/m ≃ 0.1 cm²/g.

8.2.3 Dwarf Spheroidals in the Milky Way Halo

The least massive and most dark-matter-dominated galaxies provide an excellent setting to confront the predictions of different dark matter models with observations. Recent work by Boylan-Kolchin et al. (2011a,b) has found that the estimated central densities of the bright Milky Way dwarf spheroidal satellites are lower than the densities of the massive subhalos in dark-matter-only simulations. SIDM offers a way to solve this problem because it reduces the central density of halos. Thus in SIDM, the massive subhalos do host the luminous dSph but have shallower density profiles than predicted in CDM simulations. This has recently been demonstrated by Vogelsberger et al. (2012). We do not directly compare to Vogelsberger et al. (2012) because their work is focused on the subhalos of the Milky Way and the velocity-independent cross section that they simulate (σ/m = 10 cm²/g) is larger than the cross sections considered in our work.

Regardless of whether Milky Way dSphs have cuspy or cored dark-matter halos, we may estimate the enclosed mass, and hence average density, around the half-light radius of the stellar distribution. Mass estimates within 300pc and mass profile modellings using stellar kinematics together with chemo-dynamically distinct stellar subcomponents of Milky Way dwarf spheroidal galaxies suggest central densities of approximately ~ 0.1 M⊙/pc³ (Strigari et al. 2008; Wolf et al. 2010; Walker & Peñarrubia 2011; Amorisco & Evans 2012; Wolf & Bullock 2012). For the faintest dSph Segue 1, the density within the half-light radius (about 40 pc) is measured to be about 2.5–4.1 M⊙/pc³ (Simon et al. 2011; Martinez et al. 2011). The errors on Segue 1 density are large but it is clear that if SIDM is to accommodate this result, it must allow for large scatter in the core sizes and densities for small Vmax halos. With a factor of 2-3 scatter in the densities quoted earlier for SIDM0.1 halos, Segue 1 would appear to be compatible with SIDM0.1 if its Vmax value is towards the lower end of the 20 – 50 km/s range in Vmax.

For the two dSph galaxies that appear to have cored density profiles (Fornax and Sculptor), the cores sizes must be of order ~ 0.2 – 1 kpc (Walker & Peñarrubia 2011). For small halos with circular velocities in the 20 – 50 km/s range, which is close to the expected peak circular velocities of dwarf spheroidal halos before infall into the Milky Way host halo, an SIDM with σ/m = 1 cm²/g predicts core sizes in the order of ~ 0.8 – 3.0 kpc, with central densities of about ~ 0.02 – 0.04 M⊙/pc³. Therefore, we find again that σ/m = 1 cm²/g cannot reproduce the observed high central densities. On the other hand, our estimates suggest that an SIDM model with σ/m = 0.1 cm²/g would produce central densities and core sizes consistent with the Milky Way dSph.

In this last section we have used the analytic results that explain the scaling relations for the core sizes and central densities of halos in our SIDM and SIDM0.1 simulations, to extrapolate our results to scales ranging from galaxy clusters to dwarf spheroidal galaxies and to lower cross sections. We have found that σ/m = 1 cm²/g would be unable to reproduce the observed high central densities. Remarkably, we find that the observations should be consistent with the predictions of a self-interacting dark matter with cross section in the ballpark of σ/m = 0.1 cm²/g. These expectations are based on the scaling relations seen in SIDM1 simulations and our analytic model, which is consistent with the results from our direct σ/m = 0.1 cm²/g simulations at the radii where we can trust our simulations. This deserves further study both in terms of simulations with SIDM cross section values smaller than 1 cm²/g and more detailed comparisons to observations. Our current look at the global data does not suggest a need for a velocity dependent cross section as has been previously suggested. In the companion paper we will use the analytic results that explain the scaling relations for the core sizes and central densities of halos in our SIDM and SIDM0.1 simulations, to extrapolate our results to scales ranging from galaxy clusters to dwarf spheroidal galaxies and to lower cross sections. We have found that σ/m = 1 cm²/g would be unable to reproduce the observed high central densities. Remarkably, we find that the observations should be consistent with the predictions of a self-interacting dark matter with cross section in the ballpark of σ/m = 0.1 cm²/g. These expectations are based on the scaling relations seen in SIDM1 simulations and our analytic model, which is consistent with the results from our direct σ/m = 0.1 cm²/g simulations at the radii where we can trust our simulations. This deserves further study both in terms of simulations with SIDM cross section values smaller than 1 cm²/g and more detailed comparisons to observations. Our current look at the global data does not suggest a need for a velocity dependent cross section as has been previously suggested. In the companion paper

Figure 13. Dark matter average density within 25 kpc vs. Vmax for resolved halos in our CDM, SIDM0.1, and SIDM1 simulations. It is clear that SIDM1 has significantly lower densities than CDM halos at group and cluster scales. For the SIDM0.1 model, the differences are muted and only appear on cluster scales. Thus observations of central densities in clusters likely provide the most promising avenue to look for signatures of SIDM with cross sections in the vicinity of 0.1 cm²/g.
8.3 Observed Substructure vs. SIDM

In Figure 8 we showed that the number of subhalos for $\sigma/m = 1 \text{ cm}^2/\text{g}$ is not significantly different from CDM predictions, especially in galaxy-scale halos. This is interesting because it means that SIDM fails to deliver on one of the original motivations for considering this model of dark matter. Recall that Spergel & Steinhardt (2000) originally promoted SIDM as a solution to the missing satellites problem (Klypin et al. 1999, Moore et al. 1999), stating that many subhalos would be evaporated by interactions with the background halo. Given the new discoveries of ultra-faint galaxies around the Milky Way and the high likelihood of many more discoveries from surveys like LSST (Willman 2010, Bullock et al. 2010), a significant reduction in substructure counts may very well be a negative characteristic of any non-CDM model (Tollerud et al. 2008).

However, in Milky Way mass halos, SIDM with $\sigma/m = 1 \text{ cm}^2/\text{g}$ will yield a significant probability for subhalo particle scattering only for systems that pass within ~ 10 kpc of the host halo center. Thus, for this cross section, we can form interesting-sized cores but largely leave the subhalo mass function unaffected in Milky Way-mass halos. For smaller cross sections, the differences between SIDM and CDM subhalo mass functions will be even smaller. We note that we are not the first to find that SIDM can form cores but not solve the missing satellites problem; it was first discussed in D’Onghia & Burkert (2003).

This finding is also interesting in the context of other alternatives to CDM. Warm dark matter (WDM) models, for which the outstanding difference from CDM is that dark-matter particles have high speeds at matter-radiation equality and a related free-streaming cutoff in the matter power spectrum, predict a suppression in the halo (and subhalo) mass function at small scales. Otherwise, the abundance and structure of halos and subhalos is nearly indistinguishable from CDM (Villaescusa-Navarro & Dalal 2011, Maccio’ et al. 2012). WDM halos may be less concentrated than CDM halos on scales not much larger than the free-streaming scale, but are still cusped. They are only significantly cored right at the free-streaming scale, at which the halo and subhalo abundance is highly suppressed. Thus, each of the two leading modifications to CDM can solve only one of the two historical motivations for looking beyond the CDM paradigm.

The lack of subhalo suppression for $\sigma/m \lesssim 1 \text{ cm}^2/\text{g}$ has implications for another of the SIDM halo constraints from a decade ago. Gnedin & Ostriker (2001) set a constraint excluding the range of $0.3 < \sigma/m < 10^4 \text{ cm}^2/\text{g}$ based on the fundamental plane of elliptical galaxies. The argument rests on the observation that there are not significant differences in the fundamental plane of field ellipticals and cluster ellipticals (e.g., Kochanek et al. 2004; Bernardi et al. 2003; La Barbera et al. 2010). Elliptical galaxies have a significant amount of dark matter within their half-light radii, with more massive ellipticals having larger mass-to-light ratios, either caused by varying stellar mass-to-light ratios or varying dark matter content (Padmanabhan et al. 2004, Tollerud et al. 2011, Conroy & van Dokkum 2012). Gnedin & Ostriker (2001) argue that elliptical galaxies falling into cluster-mass halos should have dark matter evaporated from their centers if $\sigma/m \neq 0$, which would cause the stars in the elliptical galaxy to adiabatically expand and hence move the galaxy off the fundamental plane.

However, in our simulations, we find that few subhalos are fully evaporated, and that the subhalo $V_{\text{max}}$ function is not greatly different for $\sigma/m = 1 \text{ cm}^2/\text{g}$ from CDM. In addition, our analytic arguments show that the trend with (host) halo mass for the evaporation of subhalos at fixed $r/r_s$ is mild. This suggests that the SIDM constraints are overly conservative even at the $\sigma/m \simeq 1 \text{ cm}^2/\text{g}$ level. The main caveats are that the suppression of the subhalo $V_{\text{max}}$ function is higher in more massive clusters and that the suppression is highest at the center of the cluster halo. It would also be interesting to see if there are any differences in the fundamental plane as a function of projected distance in the cluster, both observationally and in simulations. For all of these reasons, it would be worthwhile to perform simulations of elliptical galaxies in clusters with SIDM and explore the fundamental-plane constraints in more depth.

To summarize, although we have not fully resolved the cores of $\sigma/m = 0.1 \text{ cm}^2/\text{g}$ SIDM halos, the intuition gleaned from our analytic model (tested against the SIDM$_1$ results) and our moderately-resolved simulation results suggest that $\sigma/m = 0.1 \text{ cm}^2/\text{g}$ is an excellent fit to the data across the range of halo masses from dwarf satellites of the Milky Way to clusters of galaxies. Values of cross section over dark matter particle mass in this range are fully consistent with the published Bullet cluster constraints (cf. 1), measurements of dark matter density on small-scales and subhalo survival requirements. In a companion paper (Peter, Rocha, Bullock and Kaplinghat 2012), we show that this model is also consistent with halo shape estimates. It is therefore important to simulate galaxy and cluster halos with cross sections in the $0.1 \text{ cm}^2/\text{g}$ range.

9 SUMMARY AND CONCLUSIONS

We have presented a new algorithm to include elastic self-scattering of dark matter particles in N-body codes and used it to study the structure of self-interacting dark matter (SIDM) halos simulated in a full cosmological context. Our suite of simulations (summarized in Table 1) rely on identical initial conditions to explore SIDM models with velocity-independent cross sections (Peter, Rocha, Bullock and Kaplinghat 2012), we show that this model is consistent with SIDM halos simulated in a full cosmological context. Our suite of simulations (summarized in Table 1) rely on identical initial conditions to explore SIDM models with velocity-independent cross sections ($\sigma/m = 1 \text{ cm}^2/\text{g}$ and $\sigma/m = 0.1 \text{ cm}^2/\text{g}$) as well as a comparison set of standard CDM simulations (with $\sigma/m = 0$).

Our primary conclusion is that while SIDM looks identical to CDM on large scales, SIDM halos have constant density cores, with core radii that scale in proportion to the standard CDM scale radius ($r_{\text{core}} \approx \epsilon r_s$). The relative size of the core increases with increasing cross section ($\epsilon \simeq 0.7$ for $\sigma/m = 1$ and $\epsilon \sim 0.2$ for $\sigma/m = 0.1 \text{ cm}^2/\text{g}$). Correspondingly, at fixed halo mass, core densities decrease with increasing SIDM cross section. For both core radii and core densities, there is significant scatter about the scaling with $V_{\text{max}}$ of the halo. The scaling relationship is strong enough that measurements of dark matter densities in the cores of dark matter dominated galaxies and large galaxy clusters likely provide the most robust constraints on the dark matter cross section at this time. In a companion paper (Peter, Rocha, Bullock and Kaplinghat 2012) we demonstrate, contrary to previous claims, that SIDM constraints from halo shape measurements may be less restrictive than (or at least similar to) those from measurements of absolute core densities alone.

Based on our simulation results we conclude that the dark matter self-scattering cross section must be smaller than $1 \text{ cm}^2/\text{g}$ in order to avoid under-predicting the observed core densities in galaxy clusters, low surface brightness spirals (LSBs), and dwarf...
spheroidal galaxies. However, an SIDM model with a velocity-independent cross section of about $\sigma/m = 0.1 \, \text{cm}^2/\text{g}$ appears capable of reproducing reported core sizes and central densities of dwarfs, LSBs, and galaxy clusters. Higher resolution simulations with better statistics will be needed to confirm this expectation.

An accounting of our results are as follows:

- Outside of the central regions of dark matter halos ($r \gtrsim 0.5 R_{\text{vir}}$) the large scale properties of SIDM cosmological simulations are effectively identical to CDM simulations. This implies that all of the large-scale confirmations of the CDM theory apply to SIDM as well.
- The subhalo $V_{\text{max}}$ function in SIDM with $\sigma/m = 1 \, \text{cm}^2/\text{g}$ differs by less than $\sim 30\%$ compared to CDM across the mass range $5 \times 10^{13} M_\odot - 2 \times 10^{14} M_\odot$ studied directly with our simulations. Differences in the $V_{\text{max}}$ function with respect to CDM are only apparent deep within the centers of large dark-matter halos. Thus, although possible, it will be difficult to constrain SIDM models based on the effects subhalo evaporation.
- SIDM produces halos with constant density cores, with correspondingly lower central densities than CDM halos of the same mass. For $\sigma/m = 1 \, \text{cm}^2/\text{g}$, our simulated halo density structure is reasonably well characterized by a Burkert (1995) profile fit with a core size $r_s \lesssim 0.7 r_s$, where $r_s$ is the NFW scale radius of the same halo in the absence of self-interactions. Core densities tend to increase with decreasing halo mass ($\rho_0 \propto M_{\text{vir}}^{-0.2}$) but demonstrate about a factor of $\sim 3$ scatter at fixed mass (likely owing to the intrinsic scatter in dark matter halo concentrations).
- SIDM halo core sizes, central densities, and associated scaling relations can be understood in the context of a simple analytic model. The model treats the SIDM halo as consisting of a core region, where self-interactions have redistributed kinetic energy to create an approximately isothermal cored density profile; and an outer region, where self-interactions are not effective. The transition between these regions is set by the strength of the self-interactions and this model allows us to make quantitative predictions for smaller cross sections where the cores are not resolved by our simulations. Based on this model and a few of our best resolved simulated halos we find core sizes $\sim 0.1 r_s$ for $\sigma/m = 0.1 \, \text{cm}^2/\text{g}$.
- Halo core densities over the mass range $10^{15} - 10^{10} M_\odot$ in SIDM with $\sigma/m = 1 \, \text{cm}^2/\text{g}$ are too low ($\sim 0.005 - 0.04 \, M_\odot/\text{pc}^3$) to match observed central densities in galaxy clusters ($\sim 0.03 M_\odot/\text{pc}$) and dwarf spheroidals ($\sim 0.1 M_\odot/\text{pc}^3$).
- Halo core central densities in SIDM with $\sigma/m = 0.1 \, \text{cm}^2/\text{g}$ are in line with those observed from galaxy clusters to tiny dwarfs ($0.02 - 0.5 M_\odot/\text{pc}^3$) without the need for any velocity dependence. The densities are more consistent with observations than those predicted in dissipationless CDM simulations, which are generically too high. SIDM models with this cross section over dark matter particle mass value are consistent with Bullet cluster observations, subhalo survival requirements and, as we show in a companion paper (Peter, Rocha, Bullock and Kaplinghat, 2012), measurements of dark matter halo shapes.

Future work is necessary to expand both the dynamic range of our simulations in halo mass and resolution as well as the dynamic range in cross sections. These simulations are necessary in order to make detailed comparisons with observations given the exciting possibility that dark matter self-interaction with $\sigma/m$ in the ballpark of $0.1 \, \text{cm}^2/\text{g}$ could be an excellent fit to the central densities of halos over 4-5 orders of magnitude in mass.

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APPENDIX A: DERIVATION OF THE HARD-SPHERE INTERACTION RATE IN N-BODY SIMULATIONS

The challenge is to represent a microphysical scattering process in a macroscopic context in which neither a fluid nor collisionless treatment is appropriate. In order to develop a Lagrangian technique in which to represent the scattering process, we start with the Boltzmann equation. Particles with mass \( m \), a hard-sphere scattering cross section \( d\sigma/d\Omega \) (as a function of center-of-mass scattering angle), and a distribution function \( f(x, v, t) \) evolve as

\[
\frac{Df(x, v, t)}{Dt} = \Gamma[f, \sigma] = \int d^3v_1 \int d\Omega \frac{d\sigma}{d\Omega} |v - v_1| \left[ f(x, v', t)f(x, v'_1, t) - f(x, v, t)f(x, v, t) \right].
\]  

(A1)

Here, \( D/Dt \) is a Lagrangian time derivative and \( \Gamma[f, \sigma] \) is the collision operator. If the particles were collisionless, the Lagrangian time-derivative of the distribution function would be zero; the phase-space density of particles would be conserved. The left-hand expression in the brackets in Equation (A2) represents scattering of particles into a small patch of phase space centered on \( (x, v) \), and the right-hand expression (after the \(-\) sign) represents scattering out of that patch of phase space. If \( v \) and \( v_1 \) represent the initial velocities of the primary and target particles, then \( v' \) and \( v'_1 \) are their post-scatter velocities, which are related to the initial velocities by the center-of-mass scattering angle \( \Omega \).

The key step in being able to represent the scattering process in a simulation is the ansatz that the evolution of the coarse-grained distribution function \( \bar{f} \) (the distribution function averaged over several times the interparticle spacing) is a good representation of the evolution of the fine-grained distribution function \( f \). In other words, the ansatz is that the solution to

\[
\frac{D\bar{f}}{Dt} = \int d^3v_1 \int d\Omega \frac{d\sigma}{d\Omega} |v - v_1| \left[ \bar{f}(x, v', \tau)f(x, v'_1, \tau) - \bar{f}(x, v, \tau)f(x, v, \tau) \right]
\]

(A3)

is the same as the solution for \( f \) in Equation (A2) averaged over a patch of phase space. If this is the case, our next step is to discretize Equation (A3) such that we can solve the Boltzmann equation by Monte Carlo N-body methods.

To discretize Equation (A3), we consider a particle-based Lagrangian method in which each particle in the N-body simulation represents a patch of phase space. In the absence of collisions, the simulation particles trace out geodesics in the gravitational field of the particles. When we discretize the phase space, we do it as follows:

\[
\bar{f}(x, v, \tau) = \sum_i (M_i/m)W((x - x_i); h_i)\delta^3(v - v_i).
\]

(A4)

Here, \( i \) labels a discrete macro particle representing a patch of phase space that has mass \( M_i \); thus each macro particle represents a patch of phase space inhabited by \( M_i/m \) of the true particles. We assume a delta-function form for the velocity distribution because each macro particle travels at only one speed. We treat each macro particle as being smoothed out in configuration space with a smoothing kernel \( W \) with smoothing length \( h_i \). The reason for treating each macro particle as inhabiting a finite region of configuration space is that we want the local estimate of the density

\[
n(x) = \int d^3v \bar{f}
\]

(A5)

to be smooth. Preliminary tests show that smoothness is necessary to properly estimate the collision term of the Boltzmann equation. Note that in the main text we use \( M_i = m_i \) and \( h_i = h_{\text{d}} \) for all \( i \). This is because all of the N-body particles have the same mass in our simulations and we have fixed \( h_{\text{d}} \) to be constant for all particles in the simulations we present.

In the particle-based discretization of the Boltzmann equation, the fact that each particle represents a patch of phase space means that we must discretize the collision operator; we must integrate the collision operator over the patch of phase space inhabited by a single particle. Thus, if a specific particle represents a patch of phase space of size \( \delta x_p \delta v_p \), we must calculate

\[
\int \int \frac{D\bar{f}}{Dt} = \int \int \int \int \int \int \int |v - v_1| \left[ \bar{f}(x, v', \tau)f(x, v'_1, \tau) - \bar{f}(x, v, \tau)f(x, v, \tau) \right]
\]

(A6)

Thus, our approach to estimating the collision term and the Boltzmann equation are as follows. To find the collision rate for the region of phase space associated with a particle \( j \), we divide Equation (A5) through by \( M_j/m \) (so that we are calculating the scattering probability for a single macro particle \( j \), and we consider only the "scattering out" part of the collision operator. We consider the pairwise rate \( \Gamma_{ij} \) for particle \( j \) to scatter off any of the other \( i \) particles. We do a Monte Carlo simulation of the scatters; if a pair of particles is allowed to scatter in a given small timestep, we calculate the macro particles’ post-scatter velocity using the center-of-mass scattering angle \( \Omega \). This latter step is our approximation to the "scatter out" term of the Boltzmann collision operator.

The pairwise collision operator is

\[
\Gamma_{pq} = \frac{\Gamma(p|q) + \Gamma(q|p)}{2},
\]

(A7)

where the conditional probability of scattering a specific particle \( p \) off a target particle \( q \) is \( \Gamma(q|p) \), which is determined by the collision term...
of the Boltzmann equation. This collision term is derived from Equation (A6), such that

$$
\Gamma(p) = \int_{\delta x_p} d^3x \int_{\delta v_p} d^3v \int d^3v_1 \int d\Omega \frac{d\sigma}{d\Omega} (v - v_1) (M_p/m)^{-1} \hat{f}(x, v, t) \hat{f}(x, v_1, t) \nonumber
$$

$$
= \int_{\delta x_p} d^3x \int_{\delta v_p} d^3v \int d^3v_1 \int d\Omega \frac{d\sigma}{d\Omega} (v - v_1) (M_p/m)^{-1} \{ \sum_j (M_j/m) W(|x - x_j|; h_j) \delta^3(v - v_j) 
\nonumber
\sum_q (M_q/m) W(|x - x_q|; h_q) \delta^3(v_1 - v_q) \} \nonumber
$$

$$
= \sum_q \int d\Omega \frac{d\sigma}{d\Omega} \frac{M_q}{m} |v_q - v_p| \int_{\delta x_p} d^3x W(|x - x_p|; h_p) W(|x - x_q|; h_q) \nonumber
$$

$$
= \sum_q \Gamma(q|p). \nonumber
$$

(A8)

(A9)

(A10)

(A11)

(A12)

(A13)

We note the appearance of the term \(\sigma/m\), which is the scattering cross section per unit mass. The kernel \(g\) is defined as

$$
g_{pq} = \int_0^{\max(h_p, h_q)} d^3x' W(|x'|, h_p) W(|\delta x_{pq} + x'|, h_q). \nonumber
$$

(A14)

Using these sets of equations, we calculate \(\Gamma_{pq}\) for each pair of particles whose configuration-space patches overlap at each time step \(\delta t\), making sure to keep the time steps small enough that \(\Gamma_{pq} \delta t \ll 1\) for each time step.

**APPENDIX B: TEST FOR THE SCATTERING KINEMATICS**

We use the same setup described in [1] to test our implementation against the expected kinematics. For this we looked at the distributions of the post-scatter velocity magnitudes and directions for both the Sphere and Background particles. For the distributions of the velocity directions we looked at the inclination and azimuthal angles of the post-scatter velocity vectors. The angles are defined such as the line of interaction is along the \(\theta = 0\) direction (i.e the z-axis) and \(\phi\) is the azimuthal angle about which the experiment is symmetric. The distributions resulting from our test simulation are compared to those obtained from the transformation of a uniform isotropic distribution in the center of mass frame to the simulation/lab frame; the results are shown in Figures [51],[53], Figure [51] shows that the distributions of the velocity magnitudes rise linearly from \(v = 0\) to \(v = v_s\), followed by a sharp cutoff at \(v = v_s\), where \(v_s\) is the relative speed between the sphere and the background. From conservation of energy it is only possible to have particles with \(v > v_s\) if they have interacted multiple times. Multiple interactions are not considered in our calculations of the theoretical distributions but they are allowed in our simulation, hence, in Figure [51] one can observe a tail for velocities \(> v_s\) on the distributions of both types of particle velocities, but not on the theoretical distribution. Looking at Figure [52] one can see that most of the particles are scattered towards the \(\theta = 45^\circ\) directions, i.e. forming a 45° angle with respect to \(v_s\). It is visible that the distributions resulting from the simulation in the left panel of Figure [52] are higher than expected for \(\theta < \sim 20^\circ\). This is again from the fact that multiple scatters are possible in the simulation and this is not included in the calculations of the theoretical histograms. We demonstrate that this is the case by showing in the right panel of Figure [52] the distributions obtained from the simulation when we exclude any particles with \(v > v_s\), excluding that way any particles that we know have interacted multiple times, as one can see doing this brings the distributions from the simulation to a better match with the theory. Figure [53] shows the distributions of the velocities as a function of \(\phi\), these are flat as expected due to the symmetry of the experiment.
Figure B1. Distribution of the post-scatter velocity magnitudes. From conservation of energy it is only possible to have particles with velocities $> v_s$ if they have interacted multiple times, this is not included in our calculation of the theoretical distribution but it is allowed in our simulation, hence one can observe a tail for velocities $> v_s$ on the distributions of both types of particle velocities, but not on the theoretical distribution.

Figure B2. Distributions of the post-scatter velocities along the $\theta$-directions. It is evident that most of the particles are scattered towards the $\theta = 45^\circ$ directions, i.e. forming a $45^\circ$ angle with $v_s$. Note that the distributions resulting from the simulation in the left panel are higher than expected for $\theta < \sim 20^\circ$. This is because multiple scatters are possible in the simulation and they are not considered in the calculations of the theoretical histograms. We demonstrate this by showing in the right panel the distributions from the simulation when we exclude any particles with $v > v_s$, excluding that way any particles that we know have interacted multiple times and bringing the distributions from the simulation to a better agreement with the theory.
Figure B3. Distributions of the velocities along the $\phi$-directions. The flat distributions show that the results are symmetric about the direction of motion, i.e. the $z$-axis.