On the nature of steady states of spin distributions in the presence of spin-orbit interactions

Dimitrie Culcer and R. Winkler

Advanced Photon Source, Argonne National Laboratory, Argonne, IL 60439.
and
Northern Illinois University, De Kalb, IL 60115.

(Dated: February 5, 2008)

In the presence of spin-orbit interactions, the steady state established for spin distributions in an electric field is qualitatively different from the steady state for charge distributions. This is primarily because the steady state established for spin distributions involves spin precession due to spin-orbit coupling. We demonstrate in this work that the spin density matrix in an external electric field acquires two corrections with different dependencies on the characteristic momentum scattering time. One part is associated with conserved spins, diverges in the clean limit and is responsible for the establishment of a steady-state spin density in electric fields. Another part is associated with precessing spins, is finite in the clean limit and is responsible for the establishment of spin currents in electric fields. Scattering between these distributions has important consequences for spin dynamics and spin-related effects in general, and explains some recent puzzling observations, which are captured by our unified theory.

I. INTRODUCTION

Spin-orbit interactions are frequently the most important factor determining spin dynamics in solids. From a technological perspective, novel physical phenomena that may lead to improved memory devices and advances in quantum information processing have been shown to be intimately related to spin-orbit interactions. Practical applications usually rely on generating and maintaining a spin polarization, and in this context spin-orbit interactions can play a constructive or a destructive role. In an external electric field spin precession gives rise to steady-state spin densities and spin currents, and spin precession is often the leading cause of spin polarization decay. These facts suggest that spin precession plays a nontrivial role in the establishment of a steady state for spin distributions in electric fields.

Spin-orbit interactions are present in the band structure and in potentials due to impurity distributions. Spin-orbit coupling is in principle always present in impurity potentials and gives rise to skew scattering, which leads to the extrinsic anomalous Hall effect and the extrinsic spin-Hall effect. Band structure spin-orbit coupling may arise from the inversion asymmetry of the underlying crystal lattice (bulk inversion asymmetry), from the inversion asymmetry of the confining potential in two dimensions (structure inversion asymmetry), and may be present also in inversion symmetric systems. Band structure spin-orbit coupling gives rise to spin precession, which is the cause of magnetic anisotropy in magnetic semiconductors and causes spin flips even in the course of elastic scattering by spin-independent potentials. Band structure spin-orbit interactions in spin-1/2 electron systems can always be represented by a Zeeman-like Hamiltonian \( H = (1/2) \sigma \cdot \Omega_k \) describing the interaction of the spin with an effective wave vector-dependent magnetic field \( \Omega_k \). An electron spin at wave vector \( k \) precesses about this field with frequency \( \Omega_k/\hbar \equiv |\Omega_k|/\hbar \) and is scattered to a different wave vector within a characteristic momentum scattering time \( \tau_p \).

Throughout this paper we assume that \( \varepsilon_F \tau_p/\hbar \gg 1 \), where \( \varepsilon_F \) is the Fermi energy, which is tantamount to assuming that the carriers’ mean free path is much larger than their de Broglie wavelength. Within this range, the relative magnitude of the spin precession frequency \( \Omega_k \) and inverse scattering time \( 1/\tau_p \) define three qualitatively different regimes. In the ballistic (clean) regime no scattering occurs and the temperature tends to absolute zero, so that \( \varepsilon_F \tau_p \to \infty \) and \( \Omega_k \tau_p/\hbar \to \infty \). The weak scattering regime is characterized by fast spin precession and little momentum scattering.

In this paper, we will be concerned with the interplay of external fields, scattering processes, and band structure spin-orbit interactions in establishing a steady state for spin distributions. We will concentrate on electrons in uniform electric fields. In charge transport, the steady state is characterized by a nonequilibrium correction to the density matrix that is divergent in the clean limit, indicating a competition between the electric field, accelerating charge carriers, and scattering, which inhibits their forward motion. On the other hand, nonequilibrium corrections that arise as a result of band structure spin-orbit coupling in crystal Hamiltonians present a different kind of interplay between the electric field and scattering processes. Firstly, the spin-orbit splitting of the bands gives rise to spin-dependent scattering even from spin-independent potentials. Secondly, the presence of spin precession causes the steady state established for spin distributions to be highly nontrivial. We will demonstrate that the density matrix contains a contribution due to precessing spins and one due to conserved spins, and the steady states established for each of these are qualitatively different. Steady state corrections \( \propto \tau_p \), which diverge in the clean limit, are
associated with the absence of spin precession and give rise to spin densities in external fields. Steady state corrections independent of $\tau_p$, which are finite in the clean limit, are associated with spin precession and give rise to spin currents in external fields.

Scattering between these two distributions induces significant corrections to steady-state spin densities and spin currents.

In research on electrical generation of spin densities and currents two rather different approaches are employed. Linear response theories based on Green’s functions provide a diagrammatic interpretation of spin-related phenomena in electric fields. Such theories usually provide reliable results, but the physical picture is not always evident. Theories based on a kinetic equation for the density matrix tend to be transparent. However, it is often difficult, in such theories, to determine from the outset all the terms that play an important role in the processes under study. Correspondences between the two approaches mentioned were identified in a recent paper by Sinitsyn et al. The formalism used in the present paper, although relying on a density matrix, is equivalent to Kubo linear response theory.

We consider large, uniform systems, working in momentum space without making semiclassical approximations (by which we understand approximations pertaining to simultaneous consideration of a particle’s position and momentum.) The role of scattering in spin-related effects is an issue that is currently under intense investigation. In our general formalism, the terms responsible for scattering are derived rigorously beginning with the quantum Liouville equation for the density operator. One particular issue at the center of current debate is the physical interpretation of vertex corrections in spin-related phenomena. In charge transport, it is well known that vertex corrections decrease the weight of small-angle scattering. We demonstrate that, in spin transport, scattering phenomena have two main consequences. Firstly, the driving term in the equation for the spin density acquires a contribution due to spin-dependent scattering. Secondly, scattering between the distributions of conserved and precessing spins enters the equations for these two distributions. Our work suggests that both of these processes are contained in vertex corrections.

In a recent related article, we focused on the spin current response of a semiconductor to an electric field, showing that spin currents are not restricted to the spin-Hall effect and dwelling briefly on the distinction between nonequilibrium spin currents and spin densities excited by electric fields. In this article, we concentrate on the steady state and on the difference between the steady state established for spin distributions and that established for charge distributions. We investigate aspects of the steady state for spin distributions that arise as a result of spin precession and have no analog in charge distributions. We provide a detailed analysis of the distributions of conserved and precessing spins, and attempt to shed light on the physical content of vertex corrections in systems with spin-orbit interactions. In the process, we give a detailed exposition of the underlying theory, which was also employed in our recent work.

The outline of this paper is as follows. In the next section we start from the quantum Liouville equation and derive an equation describing the time evolution of the electron density matrix including the full scattering term in the first Born approximation. Section III is devoted to the complex case of steady states in the presence of band structure spin-orbit coupling, demonstrating the existence of two spin distributions and their subtle interplay. Analytical expressions are given for general scattering as far as possible, as well as exact solutions for short-range impurity scattering. We close with a summary of our findings.

II. TIME EVOLUTION OF THE DENSITY OPERATOR

We consider a system of non-interacting spin-1/2 electrons. The electrons are represented by a one-particle density operator $\hat{\rho}$. The expectation value of an observable represented by a Hermitian operator $\hat{O}$ is given by $\text{tr}(\hat{\rho}\hat{O})$, which motivates us to study the density operator $\hat{\rho}$ in detail. The dynamics of $\hat{\rho}$ are described by the quantum Liouville equation, which is projected onto a set of states of definite wave vector (in which the matrix elements of $\hat{\rho}$ form the density matrix), and an equation is obtained for the time evolution of the density matrix. This equation is valid for any elastic scattering in the first Born approximation. In this article we discuss systems with long mean free paths and do not consider diffusion terms explicitly. Consequently, semiclassical approximations are not necessary, and the method employed in the present work is equivalent to the Kubo linear response formalism.

A. Quantum Liouville equation

The quantum Liouville equation satisfied by $\hat{\rho}$ is

$$\frac{d\hat{\rho}}{dt} + \frac{i}{\hbar}[\hat{H} + \hat{U}, \hat{\rho}] = 0. \quad (1)$$

The Hamiltonian $\hat{H}$, considered here in the framework of the envelope function approximation (EFA), contains contributions due to the kinetic energy and spin-orbit coupling. The effect of the lattice-periodic potential of the ions is taken into account through a replacement of the carrier mass by the effective mass. We shall henceforth refer to $\hat{H}$ as the EFA Hamiltonian. Scattering is introduced into the system through the potential $\hat{U}$, which may be due to impurities, phonons, surface roughness, or other perturbations. In this article we focus on impurity scattering, as the effects that are discussed are

3,5,8,39,40,41,42,43,44,45,46,47,48,49,50
frequently observed at very low temperatures, where the role of phonon scattering may be neglected.

The Liouville equation is projected onto a set of time-independent states of definite wave vector \(| ks \rangle\), which are not assumed to be eigenstates of the EFA Hamiltonian \( \hat{H} \). The matrix elements of \( \hat{\rho} \) in this basis will be written as \( \rho_{kk'} = \langle ks | \hat{\rho} | k's \rangle \), with corresponding notations for the matrix elements of \( \hat{H} \) and \( \hat{U} \). Spin indices will not be shown explicitly in our subsequent derivation, the quantities \( \rho_{kk'}, \ H_{kk'}, \) and \( \hat{U}_{kk'} \) being treated as matrices in spin space. \( \rho_{kk'} \) refers to as the density matrix. With our choice of basis functions of definite wave vector, matrix elements of the Hamiltonian \( H_{kk'} = H_k \delta_{kk'} \) are diagonal in \( k \). However, if the EFA Hamiltonian contains spin-orbit coupling terms, the matrix elements \( H_k \) are generally off-diagonal in spin space. Matrix elements of the scattering potential \( U_{kk'} \) are off-diagonal in \( k \). Matrix elements diagonal in \( k \) in the scattering potential would lead to a redefinition of \( H_k \), which is analogous, in Green’s function formalisms, to the offset introduced by the real part of the self energy. Scattering is assumed elastic and, given that we will work in the first Born approximation, the matrix \( \hat{U}_{kk'} \) is assumed diagonal in spin. (We thus do not take into account so-called skew scattering terms, which require terms of third order in \( \hat{U}_{kk'} \) as well as explicit inclusion of spin-orbit coupling in the scattering potential, and which were studied in a recent work.) As mentioned above, we focus in this article on impurity scattering. The impurities are assumed uncorrelated and our normalization is such that \( \langle ks | \hat{U} | k's \rangle \langle k's | \hat{U} | ks \rangle = n_i |U_{kk'}|^2 \delta_{ss'}, \) where \( n_i \) is the impurity density. \( \hat{U}_{kk'} \) refers therefore to the matrix elements of the potential due to a single impurity. Explicit expressions for these matrix elements for a screened Coulomb potential in two and three dimensions are given in Appendix A.

\( \rho_{kk'} \) is divided into a part diagonal in \( k \) and a part off-diagonal in \( k \), given by \( \rho_{kk'} = f_k \delta_{kk'} + g_{kk'} \), where, in \( g_{kk'} \), it is understood that \( k \neq k' \). The quantum Liouville equation can be broken down into equations for \( f_k \) and \( g_{kk'} \)

\[
\frac{df_k}{dt} + \frac{i}{\hbar} [H_k, f_k] = \frac{-i}{\hbar} [\hat{U}, \hat{g}]_{kk}, \quad (2a)
\]

\[
\frac{dg_{kk'}}{dt} + \frac{i}{\hbar} [H, \hat{g}]_{kk'} = \frac{-i}{\hbar} [\hat{U}, \hat{f} + \hat{g}]_{kk'}. \quad (2b)
\]

In the first Born approximation the solution to Eq. (2a) can be written as

\[
g_{kk'} = -\frac{i}{\hbar} \int_0^\infty dt' e^{-iHt'} \left[ \hat{U}, \hat{f}(t-t') \right] e^{iHt'}|kk'. \quad (3)
\]

In order to shorten the equations, factors of \( \hbar \) that appear in the time evolution operators will be omitted, i.e., \( e^{iHt'} / \hbar \). These factors will be restored in the final results. Since \( \varepsilon_F T_\eta / \hbar \gg 1 \), we shall expand \( \hat{f}(t-t') \) in the time integral around \( t \) and, noting that terms beyond \( \hat{f}(t) \) are of higher order in the scattering potential, we shall only retain the first term, \( \hat{f}(t) \). The equation for \( f_k \) then becomes

\[
\frac{df_k}{dt} + \frac{i}{\hbar} [H_k, f_k] + \hat{J}(f_k) = 0, \quad (4a)
\]

in which the scattering term \( \hat{J}(f_k) \) is given by

\[
\hat{J}(f_k) = \frac{1}{\hbar^2} \int_0^\infty dt' \left[ \hat{U}, e^{-iHt'} \left[ \hat{U}, \hat{f}(t) \right] e^{iHt'} \right] |kk'. \quad (4b)
\]

The integral over time in Eq. (4b) can be performed by inserting a regularization factor \( e^{-\eta t'} \) and letting \( \eta \to 0 \) subsequently. We remark that, for potentials diagonal in spin space and spin-degenerate bands, Eq. (4b) simplifies to the customary expression for Fermi’s golden rule. Therefore, Eq. (4b) can be viewed as a generalization of Fermi’s golden rule that explicitly takes into account the spin degree of freedom. The commutator present in Eq. (4a) is a commutator in spin space that represents the effect of spin precession due to spin-orbit interactions.

**B. Scattering term**

The scattering term \( \hat{J}(f_k) \) will now be evaluated for an electron system with spin-orbit interactions. After inserting a complete set of states, \( \hat{J}(f_k) \) becomes

\[
\hat{J}(f_k) = \frac{m}{\hbar^2} \lim_{\eta \to 0} \int_0^\infty dt' e^{-\eta t'} \sum_{k'} \left[ U_{kk'} e^{-iH_{k'} t'} (U_{k' k} f_k - f_k U_{k' k}) e^{iH_{k'} t'} - e^{-iH_{k'} t'} (U_{k' k} f_k - f_k U_{k' k}) e^{iH_{k'} t'} U_{k' k} \right].
\]

The EFA Hamiltonian contains a kinetic energy term and a spin-orbit coupling term, \( H_k = H_k^{\text{kin}} + H_k^{\text{so}} \). In spin-1/2 electron systems, band structure spin-orbit coupling can always be represented as a Zeeman-like interaction of the spin with a wave vector-dependent effective magnetic field \( \Omega_k \), thus \( H_k^{\text{so}} = (1/2) \sigma \cdot \Omega_k \). Common examples of effective fields are the Rashba spin-orbit interaction, which is often dominant in quantum wells with inversion asymmetry, and the Dresselhaus spin-orbit interaction, which is due to the inversion asymmetry of the underlying crystal lattice.
The spin-orbit interaction for electrons is usually much smaller than the kinetic energy at typical Fermi energies, with the result that terms that are second order in the ratio of the two, $H_{\text{kin}}^s/H_{\text{kin}}^t$, can usually be ignored. In the basis in spin space spanned by spin eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$ (commonly referred to as the Pauli basis), taking into account the fact that $U_{kk'}$ is diagonal in spin space, the scattering term simplifies to

$$
\hat{J}(f_k) = \frac{n_i V_d}{\hbar^2} \lim_{\eta \to 0} \int_0^\infty dt' e^{-\eta t'} \int \frac{d^dk'}{(2\pi)^d} W_{kk'} \left[ e^{-iH_{tt'}t'} (f_{k'} - f_k) e^{iH_{tt'}t'} + e^{-iH_{tt'}t'} (f_k - f_{k'}) e^{iH_{tt'}t'} \right].
$$

Here $W_{kk'} = |U_{kk'}|^2$ is the transition rate, and sums over wave vector have been converted into integrals following the standard procedure $\sum_k \to V_d \int d^4k'/(2\pi)^d$, where $d$ is the dimensionality of the system and the normalization volume $V_d$ is chosen to be the $d$-dimensional unit cell volume. The density matrix $f_k$ is decomposed into a scalar part and a spin-dependent part, $f_k = n_k |\uparrow\rangle + S_k$. Performing the time integral, after a series of lengthy but straightforward calculations, which are summarized in Appendix B, the scattering term can be expressed in the form $(\hat{J}_0 + \hat{J}_s)(n_k) + \hat{J}_0(S_k)$, with

$$
\hat{J}_0(X_k) = \frac{n_i V_d}{\hbar^2} \int \frac{d^4k'}{(2\pi)^d} W_{kk'}(X_k - X_{k'}) \left[ \delta(\epsilon_{k+} - \epsilon'_{k+}) + \delta(\epsilon_{k-} - \epsilon'_{k-}) + \delta(\epsilon_{k+} - \epsilon'_{k-}) + \delta(\epsilon_{k-} - \epsilon'_{k+}) \right],
$$

$$
\hat{J}_s(X_k) = \frac{n_i V_d}{\hbar^2} \int \frac{d^4k'}{(2\pi)^d} W_{kk'}(X_k - X_{k'}) \sigma \cdot (\Omega_k + \Omega_{k'}) \left[ \delta(\epsilon_{k+} - \epsilon'_{k+}) + \delta(\epsilon_{k-} - \epsilon'_{k-}) \right]
$$

$$
+ \frac{n_i V_d}{\hbar^2} \int \frac{d^4k'}{(2\pi)^d} W_{kk'}(X_k - X_{k'}) \sigma \cdot (\Omega_k - \Omega_{k'}) \left[ \delta(\epsilon_{k+} - \epsilon'_{k-}) - \delta(\epsilon_{k-} - \epsilon'_{k+}) \right].
$$

$X_k$ represents either $n_k$ or $S_k$, and $\Omega_k$ is a unit vector along $\Omega_k$. The energies $\epsilon_{k\pm}$ represent the eigenvalues of $H_k$, and are given by $\epsilon_{k\pm} = \epsilon_k \pm \Omega_k^2/2$, where the kinetic energy $\epsilon_k = \hbar^2 k^2/2m^*$. The term $\hat{J}_0$ in Eq. (7) illustrates the fact that, when spin-orbit interactions are present in the band structure, even a spin-independent scattering potential usually gives rise to spin-dependent scattering in the scattering integral.

### C. Time evolution of the density matrix in an external electric field

In the presence of a constant uniform electric field $E$, $k = q - eEt/\hbar$ is the gauge-invariant crystal momentum (with $q$ the canonical momentum.) The states $|ks\rangle$ are chosen to have the form $|ks\rangle = e^{i\sigma \cdot r} |u_{ks}\rangle$, where $|u_{ks}\rangle$ are lattice-periodic functions. We subdivide $f_k = f_{ok} + f_{Ek}$, where the equilibrium density matrix $f_{ok}$ is given by the Fermi-Dirac distribution, and the correction $f_{Ek}$ is due to the external field $E$. To first order in $E$, the correction $f_{Ek}$ satisfies

$$
\frac{\partial f_{Ek}}{\partial t} = \frac{i}{\hbar} [H, f_{Ek}] + \hat{J}(f_{Ek}) = \frac{eE}{\hbar} \frac{\partial f_{ok}}{\partial k}.
$$

The matrix $f_{Ek}$ is in turn divided, as above, into a scalar part and a spin-dependent part, $f_{Ek} = n_k |\uparrow\rangle + S_{Ek}$. To first order in $H_{\text{kin}}^s/H_{\text{kin}}^t$, the scattering term can be expressed as $\hat{J}(f_{Ek}) = (\hat{J}_0 + \hat{J}_s)(n_{Ek}) + \hat{J}_0(S_{Ek})$, where the scattering operators have been defined in Eq. (\[\text{Eq.}\]).

The driving term arising from the electric field in Eq. (5) is $(eE/\hbar) \cdot \partial f_{ok}/\partial k$. The equilibrium density matrix, $f_{ok}$, is subdivided as $f_{ok} = n_{ok} |\uparrow\rangle + S_{ok}$, with a corresponding subdivision for the driving term. The equation for $n_{Ek}$ is

$$
\frac{\partial n_{Ek}}{\partial t} + \hat{J}_0(n_{Ek}) = \frac{eE}{\hbar} \frac{\partial n_{ok}}{\partial k}.
$$

The solution of this equation is given by the well-known expression

$$
n_{Ek} = \frac{eE\tau_p}{\hbar} \frac{\partial n_{ok}}{\partial k},
$$

in other words, $n_{Ek}$ describes the shift of the Fermi sphere in the presence of the electric field $E$. The expression for the momentum relaxation time $\tau_p$ is a little different depending on the dimensionality of the system. In three dimensions, using $\gamma$ to denote the relative angle between $k$ and $k'$,

$$
\frac{1}{\tau_{p3}} = \frac{mkV_{d3}}{2\pi \hbar^3} \int_0^\pi d\gamma \sin \gamma W_{kk'}(1 - \cos \gamma),
$$

In two dimensions, with the same notation for $\gamma$,

$$
\frac{1}{\tau_{p2}} = \frac{mV_{d2}}{2\pi \hbar^3} \int_0^{2\pi} d\gamma W_{kk'}(1 - \cos \gamma).
$$

The spin-dependent part of the nonequilibrium correction to the density matrix $S_{Ek}$ is interpreted as the spin density induced by $E$. The equation governing the time evolution of $S_{Ek}$ is

$$
\frac{\partial S_{Ek}}{\partial t} + \frac{i}{\hbar} [H_k, S_{Ek}] + \hat{J}_0(S_{Ek}) = \frac{eE}{\hbar} \frac{\partial S_{ok}}{\partial k} - \hat{J}_s(n_{Ek}).
$$
It is seen from Eq. (12) that spin-dependent scattering gives rise to a renormalization of the driving term in the equation for $S_{E\parallel}$. Evidently, this renormalization has no analog in charge transport, see Eq. (19).

So far, our work has not been restricted to the steady state. In this context, we remark briefly that Eq. (12) can be used to describe spin relaxation, and is valid both in the presence and in the absence of external electric fields. (In the presence of electric fields $S_{E\parallel}$ is an electric-field-induced nonequilibrium correction, whereas in their absence it is to be interpreted more generally as a nonequilibrium correction.)

III. STEADY STATES FOR CONSERVED AND PRECESSING SPINS

In the presence of band structure spin-orbit interactions, an electron spin at wave vector $k$ precesses about an effective magnetic field $\Omega_k$. The spin can be resolved into components parallel and perpendicular to $\Omega_k$. In the course of spin precession the component of the spin parallel to $\Omega_k$ is conserved, while the perpendicular component is continually changing. It will prove useful in our analysis to divide the spin distribution into a part representing conserved spin and a part representing precessing spin. This is accomplished below.

A. Distributions of conserved and precessing spins

Firstly, the effective source term, which enters the RHS of Eq. (12), is divided into two parts, $(eE/h) \cdot \dot{S}_{0k} / \partial k - \dot{J}_s(n_{E\parallel}) = \Sigma_{E\parallel} + \Sigma_{E\perp}$. Here, $\Sigma_{E\parallel}$ commutes with the spin-orbit Hamiltonian and is given by $\Sigma_{E\parallel} = \alpha_k H_{E\parallel}^{\alpha\beta}$, where

$$\alpha_k = \frac{\text{tr}\left\{\frac{eE}{h} \cdot \frac{\partial S_{0k}}{\partial k} - \dot{J}_s(n_{E\parallel}) \right\} H_{E\parallel}^{0\beta}}{\text{tr}(H_{E\parallel}^{\alpha\beta})}, \quad (13)$$

while $\Sigma_{E\perp}$ is the remainder. In matrix language $\Sigma_{E\parallel}$ is orthogonal to the spin-orbit Hamiltonian and thus $\text{tr}(\Sigma_{E\parallel} H_{E\parallel}^{\alpha\beta}) = 0$. Projections onto and orthogonal to $H_{E\parallel}^{\alpha\beta}$ are most easily carried out by defining projectors $P_{\parallel}$ and $P_{\perp}$. The actions of these projectors on the basis matrices $\sigma_i$ are given by

$$P_{\parallel} \sigma_i = \frac{2\Omega_{ki} H_{E\parallel}^{\alpha\beta}}{\Omega_k}, \quad (14a)$$
$$P_{\perp} \sigma_i = \frac{\Omega_{kx}^2 + \Omega_{ky}^2 - \Omega_{kx} \Omega_{ky} \sigma_y - \Omega_{kz} \sigma_x}{\Omega_k}, \quad (14b)$$

and the actions of $P_{\perp}$ on the remaining basis matrices are obtained by cyclic permutations.

Secondly, $S_{E\parallel}$ is likewise divided into two terms: $S_{E\parallel\parallel}$, commut ing with the spin-orbit Hamiltonian and $S_{E\parallel\perp}$, orthogonal to it. It is helpful to think of $S_{E\parallel\parallel}$ as the distribution of conserved spins. $S_{E\parallel\perp}$ can be thought of as the distribution of precessing spins. Equation (12) is divided into separate equations for $S_{E\parallel\parallel}$ and $S_{E\parallel\perp}$:

$$\frac{\partial S_{E\parallel\parallel}}{\partial t} + P_{\parallel} \dot{J}_0(S_{E\parallel}) = \Sigma_{E\parallel\parallel}, \quad (15a)$$
$$\frac{\partial S_{E\parallel\perp}}{\partial t} + \frac{i}{\hbar} [H_k, S_{E\parallel\perp}] + P_{\perp} \dot{J}_0(S_{E\parallel}) = \Sigma_{E\parallel\perp}. \quad (15b)$$

The absence of the commutator $[H_k, S_{E\parallel}] = 0$ in Eq. (15a) indicates the absence of spin precession, while the commutator $[H_k, S_{E\parallel\perp}]$ in Eq. (15b) represents spin precession. In order to solve Eqs. (15a) and (15b) for arbitrary scattering, it is necessary to expand $S_{E\parallel\parallel}$ and $S_{E\parallel\perp}$ in the transition rate $W_{kk'}$, as

$$S_{E\parallel\parallel} = S^{(1)}_{E\parallel\parallel} + S^{(0)}_{E\parallel\parallel} + \mathcal{O}(W_{kk'}^2), \quad (16a)$$
$$S_{E\parallel\perp} = S^{(1)}_{E\parallel\perp} + S^{(0)}_{E\parallel\perp} + \mathcal{O}(W_{kk'}^2). \quad (16b)$$

This expansion is indeed an expansion in the parameter $h/(\Omega_k T_p)$, a fact that can be most clearly seen by examining Eq. (16) and noting that the calculation of each term involves integration over time, which brings in a factor of $1/\Omega_k$, and the action of $\dot{J}_0$. This expansion is therefore most suited to systems in the weak scattering regime. The expansion of $S_{E\parallel\parallel}$ starts at order $-1$, a fact which can be understood by inspecting Eq. (15a). In the steady state the time derivative drops out, and the operator $\dot{J}_0$ is first order in $W_{kk'}$, while the right-hand side is independent of $W_{kk'}$. As a result, the expansion of the solution must start at order $-1$. We examine next Eq. (15b) for $S_{E\parallel\perp}$. Since $H_k$ is independent of $W_{kk'}$, and the right hand side is also independent of $W_{kk'}$, the expansion of $S_{E\parallel\perp}$ must start at order zero.

Having divided the nonequilibrium correction to the spin density matrix into a part due to conserved spin and one due to precessing spin, we wish to determine the contributions these parts make to spin densities and spin currents.\textsuperscript{13,23,24,26,28,29,30,31,32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48} and spin currents in the steady state. The steady-state spin density of spin component $\sigma$ is found by taking the trace $\text{tr}(\hat{s}\sigma S_{E\parallel})$, where $\hat{s} = (h/2)\sigma$. The steady-state spin current is found by taking the trace $\text{tr}(\hat{j}_{\sigma} S_{E\parallel})$, where $\hat{j}_{\sigma} = h\hat{k}_j s^j / m^* + (1/4h) \partial \Omega^2 / \partial k_j \mathbb{I}$. (The scalar term has zero expectation value.) Henceforth, for clarity and definiteness, integrals over wave vectors will be represented as two-dimensional. In the integrals below $\theta'$ refers to the polar angle of $k'$. The extension to three dimensions is straightforward.

B. Steady state for conserved spins

We have shown that the expansion of $S_{E\parallel\parallel}$ begins at order $-1$, and we wish to find the first term in this expansion. In the steady state, Eq. (15a) for the first term
in this expansion can be written as

\[ P_\parallel \hat{J}_0 (S_{E_k}^{(-1)}) = \Sigma_{E_k \parallel} \]  

This equation can be recast as

\[ S_{E_k \parallel}^{(-1)} - P_\parallel \hat{J}_0 (S_{E_k \parallel}^{(-1)}) = \Sigma_{E_k \parallel} \]  

where we have introduced

\[ \tau_0 = \frac{m^*}{2\pi\hbar^3} \int d\theta W_{kk'}, \]  

\[ \hat{J}_0 S_{E_k \parallel}^{(-1)} = \frac{m^*}{2\pi\hbar^3} \int d\theta W_{kk'} S_{E_k \parallel}^{(-1)} \]  

\[ \tau_0 \]  

is the quantum lifetime of the charge carriers, i.e., the time between two consecutive scattering events. It differs from the momentum scattering time of Eqs. (11) because, for nonisotropic scattering mechanisms, the information about the initial momentum is not lost after time \( \tau_0 \). Equation (13) can be solved iteratively for any scattering

\[ S_{E_k \parallel}^{(-1)} = \Sigma_{E_k \parallel} \tau_0 + P_\parallel \hat{J}_0 (\Sigma_{E_k \parallel}) \tau_0^2 + \ldots \]  

(20)

The equations for higher orders in \( W_{kk'} \) are easily deduced. However, the term of order \(-1\) is by far the dominant one in the weak momentum scattering regime and is expected to be dominant over a wide range of strengths of the scattering potential.

We examine more closely the nature of the steady state established for conserved spins. It is evident that this steady state involves no spin precession, and that the correction \( S_{E_k \parallel} \) depends explicitly on the nonequilibrium shift in the Fermi surface and diverges in the ballistic regime, as \( \tau_0 \to \infty \). In addition, it is important to note that scattering terms contain only the even function \( W_{kk'} \). As a result, the correction \( S_{E_k \parallel} \) does not give rise to a spin current. Inspection of Eq. (20) shows that integrals of the form

\[ \int d\theta \hat{\mathcal{J}}_i^\sigma S_{E_k \parallel} \]  

contain an odd number of powers of \( k \) and are therefore zero. Consequently, in the absence of impurity spin-orbit interactions, the distribution of conserved spins can give no spin current. It can, however, give rise to a nonequilibrium spin density \( S_{E_k \parallel} \), since integrals of the form

\[ \int d\theta \hat{s}^\sigma S_{E_k \parallel} \]  

contain an even number of powers of \( k \) and may be nonzero.

C. Steady state for precessing spins

The equations for the contributions to \( S_{E_k \perp} \) of orders zero and one in \( W_{kk'} \) are

\[ \frac{\partial S_{E_k \perp}^{(0)}}{\partial t} + \frac{i}{\hbar} [H_k, S_{E_k \perp}^{(0)}] = \Sigma_{E_k \perp} + P_\perp \hat{J}_0 (S_{E_k \parallel}) \]  

(23a)

\[ \frac{\partial S_{E_k \perp}^{(1)}}{\partial t} + \frac{i}{\hbar} [H_k, S_{E_k \perp}^{(1)}] = P_\perp \hat{J}_0 (S_{E_k \parallel}) \]  

(23b)

To solve the equation for \( S_{E_k \perp}^{(0)} \), it is easiest to go into the interaction picture, obtain an expression for \( S_{E_k \perp}^{(0)} \), and then transform back to the Schrödinger picture. This procedure yields for \( S_{E_k \perp}^{(0)} \)

\[ S_{E_k \perp}^{(0)} = \frac{1}{2} \frac{\Omega_k \times [\Sigma_{E_k \perp} + P_\perp \hat{J}_0 (S_{E_k \parallel})] \cdot \sigma}{\Omega_k / \hbar} \]  

(24)

where we have written \( \Sigma_{E_k \perp} = (1/2) \Sigma_{E_k \perp} \cdot \sigma \) and \( S_{E_k \parallel} = (1/2) S_{E_k \parallel} \cdot \sigma \). The result expressed by Eq. (24) is valid for any elastic scattering. Since it does not depend explicitly on the form of the impurity potential, this term is usually regarded as intrinsic. Terms of higher order in \( W_{kk'} \) are regarded as extrinsic because they depend explicitly on the form of the impurity potential. Nevertheless, it is evident from our work that, if \( \Sigma_{E_k \perp} + P_\perp \hat{J}_0 (S_{E_k \parallel}) \) vanishes, then \( S_{E_k \perp}^{(0)} \) vanishes and all the terms in \( S_{E_k} \) of higher order in \( W_{kk'} \) also vanish.

The steady state established for precessing spins is finite in the clean limit. An argument similar to that given above for the distribution of conserved spin \( S_{E_k \parallel} \) shows that \( S_{E_k \perp}^{(0)} \) cannot lead to a nonequilibrium spin density (although, as will be shown below, higher-order terms in \( S_{E_k \perp} \) can contribute to the spin density). For, taking the expectation value of the spin operator, one arrives at integrals of the form \( \int d\theta \hat{s}^\sigma S_{E_k \perp}^{(0)} \), which involve odd numbers of powers of \( k \) and are therefore zero. This term in the distribution of precessing spin does, however, give rise to nonzero spin currents, since integrals if the form \( \int d\theta \hat{\mathcal{J}}_i^\sigma S_{E_k \perp} \) contain an even numbers of powers of \( k \) and may be nonzero. Consequently, in the absence of spin-orbit coupling in the scattering potential, nonequilibrium spin currents arise from spin precession.

In order to investigate terms of higher order in \( W_{kk'} \), it is easiest to examine a concrete case. Although the general conclusions apply to any elastic, spin-independent scattering, this analysis will be done in the next section in the context of short-range impurity scattering, where an exact solution is possible, which reveals an interesting physical picture.

D. Short-range impurities

An enlightening closed-form solution can be found for \( S_{E_k \parallel} \) and \( S_{E_k \perp} \) for short-range impurities. In this case
\( W_{\mathbf{k}k'} \equiv W \) is a constant and \( \mathbf{j}_0 \cdot \mathbf{S}_{\mathbf{k}} = (S_{\mathbf{k}} - \bar{S}_{\mathbf{k}}) / \tau_p \), where the scattering time \( \tau_p = \hbar^3 / (mW) \) and the bar represents averaging over directions in \( \mathbf{k} \). Equation (20) yields a closed-form solution for \( S_{\mathbf{E}||k} \)

\[
S_{\mathbf{E}||k} = \Sigma_{\mathbf{E}||k} \tau_p + P_\perp (1 - P_\parallel)^{-1} \Sigma_{\mathbf{E}||k} \tau_p.
\]  

(25)

This solution enters Eq. (24) for \( S_{\mathbf{E}\perp k} \). The equations for the contributions to \( S_{\mathbf{E}\perp k} \) of higher orders in \( W \) can be easily determined

\[
S_{\mathbf{E}\perp k}^{(1)} = \frac{1}{2} \bar{\mathbf{S}}_k \times \{ \mathbf{S}_{\mathbf{E}\perp k} - \mathbf{j}_0 \} \cdot \mathbf{\sigma}_k \tau_p / \hbar^2,
\]

(26a)

\[
S_{\mathbf{E}\perp k}^{(2)} = \frac{1}{2} \bar{\mathbf{S}}_k \times \{ \mathbf{S}_{\mathbf{E}\perp k} - \mathbf{j}_0 \} \cdot \mathbf{\sigma}_k \tau_p^2 / \hbar^3.
\]

(26b)

and so on. Since \( \Sigma_{\mathbf{E}\perp k} \) is orthogonal to the spin-orbit Hamiltonian \( H^S_p \), we have \( \text{tr}(\Sigma_{\mathbf{E}\perp k} H^S_p) = 0 \), which tells us immediately that \( \Sigma_{\mathbf{E}\perp k} \perp \mathbf{k} \). As a result, \( \bar{\mathbf{S}}_k \times \mathbf{\Omega}_k \times \Sigma_{\mathbf{E}\perp k} = -\Sigma_{\mathbf{E}\perp k} \) and

\[
S_{\mathbf{E}\perp k}^{(1)} = -\frac{1}{2} \mathbf{\Omega}_k \times \{ \mathbf{S}_{\mathbf{E}\perp k} - \mathbf{j}_0 \} \cdot \mathbf{\sigma}_k \tau_p / \hbar^2,
\]

(27a)

\[
S_{\mathbf{E}\perp k}^{(2)} = -\frac{1}{2} \mathbf{\Omega}_k \times \{ \mathbf{S}_{\mathbf{E}\perp k} - \mathbf{j}_0 \} \cdot \mathbf{\sigma}_k \tau_p^2 / \hbar^3.
\]

(27b)

It is evident that the terms \( S_{\mathbf{E}\perp k}^{(odd)} \) and \( S_{\mathbf{E}\perp k}^{(even)} \) give two separate geometric progressions. These progressions are easily summed to give for \( S_{\mathbf{E}\perp k} \)

\[
S_{\mathbf{E}\perp k} = \frac{\mathbf{\Omega}_k \times \{ \mathbf{S}_{\mathbf{E}\perp k} \tau_p + P_\perp \bar{S}_{\mathbf{E}||k} \} \cdot \mathbf{\sigma}_k \tau_p \hbar^2}{2h(1 + \mathbf{\Omega}_k^2 \tau_p^2 / \hbar^2)}
\]

\[
-\frac{(\mathbf{S}_{\mathbf{E}\perp k} \tau_p + P_\perp \bar{S}_{\mathbf{E}||k} \tau_p)}{1 + \mathbf{\Omega}_k^2 \tau_p^2 / \hbar^2}.
\]

(28)

Once again, if \( \Sigma_{\mathbf{E}\perp k} \tau_p + P_\perp \bar{S}_{\mathbf{E}||k} \) vanishes, then all the corrections to \( S_{\mathbf{E}\perp k} \) of order zero and higher also vanish.

Let us analyze the two terms in \( S_{\mathbf{E}\perp k} \). By identifying terms in \( S_{\mathbf{E}\perp k} \) even and odd in \( \mathbf{k} \), as was done above, it is evident that the first term on the RHS of Eq. (28) leads to a spin current, but no spin density. The second term does not lead to a spin current, but it does produce a spin density. Nevertheless, this term tends to zero in the ballistic regime as well as in the strong momentum scattering regime, and it is not expected to be dominant.

The closed-form solution found in this section shows that, in the absence of spin-orbit interactions in the impurity potential, there is only one spin current, which in the weak momentum-scattering limit is independent of \( \tau_p \) and in the strong momentum-scattering limit is \( \propto \tau_p^2 \).

Bearing in mind that if \( S^{(0)}_{\mathbf{E}||k} \) vanishes all corrections of higher order also vanish, we conclude that, in the absence of spin-orbit interactions in the impurity potential, the distinction between intrinsic (disorder-independent) and extrinsic (disorder-dependent) spin currents is not useful.

### E. Steady state spin densities and currents

Our work helps to understand the origins of nonequilibrium spin densities and spin currents in electric fields. The preceding sections illustrate the fact that nonequilibrium spin densities have two origins. The first, giving the dominant contribution, arises from the absence of precession. As charge carriers are accelerated, a fraction of their spin is conserved and produces a steady-state spin density, with a process which has no analog in charge transport. Thus the dominant contribution to the nonequilibrium spin density in an electric field exists because in the course of spin precession a component of each individual spin is preserved. For an electron with wave vector \( \mathbf{k} \), this spin component is parallel to \( \mathbf{\Omega}_k \). In equilibrium the average of these conserved components is zero. However, when an electric field is applied, the Fermi surface is shifted, and the average of the conserved spin components may be nonzero, as illustrated in Fig. 1. This intuitive physical argument, to our knowledge absent from earlier work, demonstrates the imbalance of spin structure in electric fields which requires scattering to balance the drift of the Fermi surface. It is interesting to note, also, that, although spin densities in electric fields require the presence of band structure spin-orbit interactions and therefore spin precession, the dominant contribution arises as a result of the absence of spin precession.

An additional contribution, which vanishes in both the ballistic and the weak momentum scattering regime, is associated with spin precession. This contribution arises from the terms on the last line of Eq. (28). The origin of this term can be understood by noting that, in the presence of an electric field, the effective magnetic field about which a spin precesses changes slowly, \( \propto \tau_p^2 \) (This is true in between scattering events.) This change is contained in the gauge-invariant crystal wave vector \( \mathbf{k} = \mathbf{q} - eE \mathbf{t} / \hbar \). The fact that the effective magnetic field is changing slowly causes the spin to acquire a small component in the direction in which the effective magnetic field is changing. This component is proportional to the rate of change of the effective magnetic field and therefore, in our case, to \( E \). It is associated with the ‘flow’ of \( \mathbf{\Omega}_k \) around the Fermi surface discussed by Shytov et al. This argument explains why this term in the spin density vanishes in the clean limit as well as in the strong momentum scattering limit. For, in the absence of scattering, as \( \mathbf{k} \) changes, the effective magnetic field will circle around the Fermi surface, and the component of the spin following it will average to zero. In the strong momentum scattering limit, on the other hand, the spin will not have time to acquire a component in the direction in which the effective magnetic field is changing, due to the high frequency of scattering events.

Furthermore, in the absence of spin-orbit interactions in the impurity potentials, spin currents are associated with displacement of spins. The relation between spin currents and spin precession was made explicit in the
F. Interplay of conserved and precessing spin densities

It is enlightening to compare the results obtained in the absence of scattering (i.e., the clean limit) with the results obtained when scattering is present. The aim is to obtain an understanding of the way scattering processes affect steady-state spin distributions in electric fields. This is done by comparing results obtained using the approach outlined in this paper with results obtained previously using Green’s functions approaches, both for the case when scattering is not included and for the case in which scattering is taken into account. This process will aid us in identifying the information contained in vertex corrections to spin-related quantities in the framework of Green’s functions-based theories of systems with spin-orbit interactions. The nature of this information is by no means obvious, and we will show that it has no analog in charge transport.

In the absence of scattering, Eq. (12) takes the form

$$\frac{\partial S_{E k}}{\partial t} + \frac{i}{\hbar} [H_k, S_{E k}] = \frac{eE}{\hbar} \cdot \frac{\partial S_{0k}}{\partial k}. \tag{29}$$

Comparison of Eqs. (12) and (29) shows that the driving term in the equation for the nonequilibrium spin distribution $S_{E k}$ is renormalized by the term $J_s(n_{E k})$, which accounts for spin-dependent scattering.

In addition, Eq. (25a) shows that scattering mixes the distributions of conserved and precessing spins. This is so because when one spin at wave vector $k$ and precessing about $\Omega_k$ is scattered to wave vector $k'$ and precesses about $\Omega_k'$, its conserved component changes, a process which alters the distributions of conserved and precessing spin. Consequently, scattering processes in systems with spin-orbit interactions cause a renormalization of the driving term for the spin distribution, contained in $\Sigma_{E k\perp}$, as well as scattering between the conserved and precessing spin distributions, described by $P_\perp J'_s(S_{E k\parallel})$. Our analysis suggests that contributions due to these two processes are contained in vertex corrections to spin-dependent quantities found in Green’s functions formalisms.

In two dimensions, for Hamiltonians linear in wave vector, we find that the renormalization term $J_s(n_{E k})$ does not contribute to the spin current for any elastic scattering. Therefore, the vertex correction to spin currents, found in other work, represents only scattering between conserved and precessing spin distributions. By noting that in Eq. (28) the correction to the source term in the equation for the precessing spin distribution has the form $P_\perp S_{E k\parallel}$ and thus depends on the steady state spin density, it becomes evident that the existence of a nonzero nonequilibrium spin density does affect the spin current.

Furthermore, in three dimensions, for electrons in zincblende crystals, which are described by the $k^3$-Dresselhaus model, we find that, for short range impurities, the spin current obtained after inclusion of scattering is the same as when scattering is not included. It is known that the vertex correction to spin currents also vanishes in these systems. Noting that the steady-state spin density in zincblende crystals vanishes by symmetry, and more generally vanishes in any non-gyrotropic medium, these observations reinforce our conclusion relating to the connection between vertex corrections to spin currents and the presence of a steady-state spin density. We therefore expect vertex corrections to spin currents to vanish in non-gyrotropic materials, in which no steady-state spin density is possible in an electric field.\textsuperscript{2}

G. Comparison with previous work

Our calculations for known cases give results in agreement with previous work. Firstly, our results agree with previous calculations of nonequilibrium spin densities.\textsuperscript{4,7} Furthermore, in two dimensions, for Hamiltonians linear in wave vector, the spin current vanishes for short-range impurities,\textsuperscript{7,34,35,39,40,46} as well as for small-angle scattering.\textsuperscript{40,41} (We find that in fact it vanishes for any elastic scattering.)\textsuperscript{46} For spin-orbit Hamiltonians characterized solely by one angular Fourier component $N$ (Ref. 41) the spin current $\propto N$.

Spin currents in systems in which band structure spin-orbit interactions are negligible were studied by Engel et al.,\textsuperscript{32} who demonstrated that spin currents in those circumstances are due to skew scattering. Skew scattering appears as a term of third order in the scattering potential $U_{kk'}$ (which must include spin-orbit coupling explicitly), whereas in this work we have restricted our discussion to terms of second order in $U_{kk'}$, and we have not considered higher-order skew-scattering effects.
H. Observable effects

Spin densities and spin currents excited by electric fields give rise to observable effects. In materials in which a nonequilibrium spin density is excited by an electric field, the presence of this spin density can be observed, for example, by means of magnetic circular dichroism, which has long been used as a characterization tool for magnetic materials. Similarly, a spin current flowing transversely to the direction of the charge current (i.e., the spin-Hall effect) will give rise to a spin accumulation at the edge of the sample. A spin current flowing parallel to the direction of the charge current could be used as a means of spin injection from one semiconductor into another, as discussed in our recent work.

Spin accumulation as a result of a spin-Hall current, or a spin density injected by means of a longitudinal spin current, can in turn be measured using magnetic circular dichroism techniques developed recently. We note that alternative techniques can be used to observe nonequilibrium spin densities, spin currents and edge spin accumulations.

IV. SUMMARY

We have demonstrated that, in the presence of band structure spin-orbit interactions, the steady state established for the carrier spin distribution contains two qualitatively distinct contributions, corresponding to conserved and precessing spin. The distribution of conserved spin acquires a nonequilibrium correction that diverges in the ballistic regime. This correction is responsible for the establishment of the dominant nonequilibrium spin densities in electric fields. The distribution of precessing spin acquires a nonequilibrium correction that is finite in the ballistic regime. This correction is responsible for the establishment of nonequilibrium spin currents in electric fields and a small nonequilibrium spin polarization, which vanishes in the ballistic and strong momentum scattering regimes. We have demonstrated that, when spin-orbit interactions are present in the band structure and absent from the impurity potential, there is only one contribution to the spin current, which appears independent of disorder in the ballistic regime but dependent on disorder in the strong momentum scattering regime. Moreover, we have also shown that scattering processes in systems with spin-orbit interactions give rise to a renormalization of the driving term in the equation for the spin distribution, as well as scattering between the conserved and precessing spin distributions, which sheds light on the nature of vertex corrections in these systems.

The authors would like to acknowledge enlightening discussions with E. Rashba, Q. Niu, A. H. MacDonal, J. Sinova, D. L. Smith, J. Shi, H. A. Engel, and R. A Duine. The research at Argonne National Laboratory was supported by the US Department of Energy, Office of Science, Office of Basic Energy Sciences, under Contract No. DE-AC02-06CH11357.

1. I. Žutić et al., Rev. Mod. Phys. 76, 323 (2004).
2. E. L. Ivchenko and G. E. Pikus, JETP Lett. 27, 604 (1978).
3. L. S. Levitov et al., Sov. Phys. JETP 61 (1), 133 (1985).
4. V. M. Edelstein, Solid State Comm. 73, 233 (1990).
5. A. G. Aronov et al., Sov. Phys. JETP 58, 537 (1991).
6. L. I. Magarill et al., Semiconductors 35 (9), 1081 (2001).
7. J. Inoue, G. E. W. Bauer, and L. W. Molenkamp, Phys. Rev. B 67, 033104 (2003).
8. H.-A. Engel, E. I. Rashba, B. I. Halperin, Phys. Rev. Lett. 98, 036602 (2007).
9. L. E. Vorob'ev et al., JETP Lett. 29, 441 (1979).
10. Y. Kato et al., Phys. Rev. Lett. 93, 176601 (2004).
11. S. D. Ganichev, S. N. Danilov, P. Schneider, V. V. Bel'kov, L. E. Golub, W. Wegscheider, D. Weiss, and W. Prettl, cond-mat/0403641 (2004).
12. A. Y. Silov, P. A. Blajnov, J. H. Wolter, R. Hey, K. H. Ploog, and N. S. Averkiev, Appl. Phys. Lett. 85, 5929 (2004).
13. M. I. D'yakonov and V. I. Perel’, Zh. Eksp. Teor. Fiz 60, 1954 (1971).
14. J. E. Hirsch, Phys. Rev. Lett. 83, 1834 (1999).
15. S. Murakami et al., Science 301, 1348 (2003).
16. Y. K. Kato et al., Science 306, 1910 (2004).
17. J. Wunderlich et al., Phys. Rev. Lett. 94, 047204 (2005).
18. V. Sih et al., Nature Phys. 1, 31-35 (2005).
19. S. O. Valenzuela and M. Tinkham, Nature 442, 176 (2006).
20. B. Liu et al., cond-mat/0610150 (2006).
21. N. P. Stern et al., Phys. Rev. Lett. 97, 126603 (2006).
22. S. D. Ganichev et al., Phys. Rev. Lett. 86, 004358 (2001).
23. J. Sinova et al., Phys. Rev. Lett. 92, 126603 (2004).
24. H. A. Engel et al., cond-mat/0603306 (2006).
25. R. Winkler, cond-mat/0605390 (2006).
26. I. Adagideli and G. E. W. Bauer, Phys. Rev. Lett. 95, 256602 (2005); X.-D. Cui et al., cond-mat/0608546 (2006).
27. J. Shi et al., Phys. Rev. Lett. 96, 076604 (2006).
28. E. I. Rashba, Phys. Rev. B 70, 201309 (2004).
29. S. Zhang and Z. Yang, Phys. Rev. Lett. 84, 066602 (2005).
30. Y. Wang et al., Phys. Rev. Lett. 96, 066601 (2006).
31. L. Sheng et al., Phys. Rev. Lett. 95, 136602 (2005).
32. T. Damker et al., Phys. Rev. B 69, 205327 (2004).
33. E. M. Hankiewicz and G. Vignale, Phys. Rev. B 73, 115339 (2006).
34. J. Inoue, G. E. W. Bauer, and L. W. Molenkamp, Phys. Rev. B 70, 041303 (2004).
35. O. V. Dimitrova, Phys. Rev. B 71, 245327 (2005).
36. A. G. Mal’shukov and K. A. Chao, Phys. Rev. B 71, 121308 (2005).
37. B. K. Nikolic et al., Phys. Rev. B 73, 075303 (2006).
38. O. Bleibaum and S. Wachsmuth, Phys. Rev. B 74, 195330 (2006).
39. E. G. Mishchenko et al., Phys. Rev. Lett. 93, 226602 (2004).
APPENDIX A: MATRICES ELEMENTS OF A SCREENED COULOMB POTENTIAL

In two dimensions, the matrix element $U_{kk'}$ of a screened Coulomb potential between plane waves is given by

$$U_{kk'} = -\frac{Ze^2}{\epsilon_0 V_{d=2}} \frac{1}{\sqrt{|k-k'|^2 + 1/L_s^2}}.$$  \hfill (A1)

where $Z$ is the ionic charge, $V_{d=2}$ corresponds to the unit cell area, and $L_s$ is the screening length. The corresponding expression in three dimensions is

$$U_{kk'} = -\frac{Ze^2}{\epsilon_0 V_{d=3}} \frac{1}{|k-k'|^2 + 1/L_s^2},$$  \hfill (A2)

where now $V_{d=3}$ is the unit cell volume.

APPENDIX B: INTEGRALS OVER TIME

For a Hamiltonian given by $H_k = \varepsilon_k \mathbb{1} + (1/2) \sigma \cdot \Omega_k$, the product of two time evolution operators $e^{-iH_{k'}t'}e^{iH_k t}$ can be written as

$$e^{-iH_{k'}t'}e^{iH_k t} = e^{i(\varepsilon_0 - \varepsilon_0') t'} \left[ \cos \frac{\Omega_{k'} t'}{2} \cos \frac{\Omega_k t}{2} - i \sigma \cdot \Omega_{k'} \cos \frac{\Omega'_{k'} t'}{2} \sin \frac{\Omega_k t}{2} + i \sigma \cdot \Omega_k \sin \frac{\Omega'_{k'} t'}{2} \cos \frac{\Omega_k t}{2} \right] + \Omega_k \cdot \Omega_{k'} + i \sigma \cdot (\Omega_k \times \Omega_{k'}) \sin \frac{\Omega'_{k'} t'}{2} \sin \frac{\Omega_k t}{2}.$$

with a similar expression holding for the Hermitian conjugate of this product. In the time integrals the trigonometric functions are expressed as complex exponentials, the integrals are evaluated, and the results are replaced by their principal parts, yielding

$$\int_0^\infty dt' e^{i(\varepsilon_0 - \varepsilon_0') t'} \cos \frac{\Omega_{k'} t'}{2} \cos \frac{\Omega_k t}{2} = \frac{\pi \hbar}{4} \left[ \delta(\varepsilon_k - \varepsilon_{k'}) + \delta(\varepsilon_k - \varepsilon_{k'}) - \delta(\varepsilon_k + \varepsilon_{k'}) - \delta(\varepsilon_k + \varepsilon_{k'}) \right].$$  \hfill (B2a)

$$\int_0^\infty dt' e^{i(\varepsilon_0 - \varepsilon_0') t'} \cos \frac{\Omega_{k'} t'}{2} \sin \frac{\Omega_k t}{2} = \frac{\pi \hbar}{4i} \left[ \delta(\varepsilon_k - \varepsilon_{k'}) + \delta(\varepsilon_k - \varepsilon_{k'}) - \delta(\varepsilon_k + \varepsilon_{k'}) - \delta(\varepsilon_k + \varepsilon_{k'}) \right].$$  \hfill (B2b)

$$\int_0^\infty dt' e^{i(\varepsilon_0 - \varepsilon_0') t'} \sin \frac{\Omega_{k'} t'}{2} \cos \frac{\Omega_k t}{2} = \frac{\pi \hbar}{4i} \left[ \delta(\varepsilon_k - \varepsilon_{k'}) + \delta(\varepsilon_k - \varepsilon_{k'}) - \delta(\varepsilon_k + \varepsilon_{k'}) - \delta(\varepsilon_k + \varepsilon_{k'}) \right].$$  \hfill (B2c)

$$\int_0^\infty dt' e^{i(\varepsilon_0 - \varepsilon_0') t'} \sin \frac{\Omega_{k'} t'}{2} \sin \frac{\Omega_k t}{2} = \frac{\pi \hbar}{4} \left[ \delta(\varepsilon_k - \varepsilon_{k'}) + \delta(\varepsilon_k - \varepsilon_{k'}) - \delta(\varepsilon_k + \varepsilon_{k'}) - \delta(\varepsilon_k + \varepsilon_{k'}) \right].$$  \hfill (B2d)

Evaluation of all time integrals in this manner leads to Eq. (7) for the scattering term.