Universal translational velocity of vortex rings behind conical objects

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Abstract

Ring vortices are efficient at transporting fluid across long distances. They can be found in nature in various ways: they propel squids, inject blood in the heart, and entertain dolphins. These vortices are generally produced by ejecting a volume of fluid through a circular orifice. The impulse given to the vortex rings moving away results in a propulsive force on the vortex generator. Propulsive vortex rings have been widely studied and characterised. After four convective times, the vortex moves faster than the shear layer it originates from, and separates from it. This separation corresponds to a maximum circulation of the vortex and a maximal spread of the vorticity in the vortex, quantified by the non-dimensional energy. The simultaneity of these three events obfuscate the causality between them. To analyse the temporal evolution of the non-dimensional energy of ring vortices independent of their separation, we analyse the spatiotemporal development of vortices generated in the wake of cones. Cones with different apertures and diameters were accelerated from rest to produce a wide variety of vortex rings and their energy, circulation, and velocity were extracted based on time-resolved velocity field measurements. The vortex rings that form behind the cones have a self-induced velocity that cause them to follow the cone and they continue to grow as the cone travels well beyond the limiting vortex formation times scales observed for propulsive vortices. The non-dimensional circulation, based on the vortex diameter, and the non-dimensional energy of the drag vortex rings converge after three convective times to values comparable to their propulsive counterparts. This results proves that vortex pinch-off does not cause the non-dimensional energy to reach a minimum value. The limiting values of the non-dimensional circulation and energy are mostly independent of the cone geometry and translational velocity and fall within an interval of 10% around the mean value. With only 6% of variation, the velocity of the vortex is the most unifying quantity that governs the formation of vortex rings behind cones.

Introduction

Vortex rings are ubiquitous phenomena widely observed in nature. Many sea creatures produce vortex rings to propel themselves efficiently. Squids, scallops and salps eject water through a circular orifice, producing a high velocity vortex ring and thus thrust [11, 14]. Some fish release vortex rings in their wake by oscillating their tail and pectoral fins [13]. Vortex rings are also efficient at transporting fluid. The blood injected in the left ventricle of the heart forms a vortex ring, and any imperfection in the formation process can lead to severe heart disease [4]. Extinguishing powder can be transported on distances superior to 100 m to extinguish oil well fires, by shooting a vortex ring along the axis of the burning gusher [1].

Vortex rings introduced above may be classified as propulsive vortices. They are generated by ejecting fluid through a circular orifice, or around a fin, and move away from the body they originate from. The momentum given to the fluid results in a propulsive force acting on the body. A second family of vortex rings emerges from this classification. Vortices passively generated in the wake of a moving axisymmetric body. We refer to them as drag vortices. They are involved in slowing down the fall of Dandelions, improving the seeding on long distances [3]. Vortex rings also form in the wake of parachutes when they deploy and can lead to the collapse of the parachute if not properly considered [9].

Vortex rings have been studied numerically and experimentally. The classical apparatus to study propulsive vortices is to push a volume of fluid out of a cylinder with a piston. A shear layer forms at the exit of the cylinder, then rolls up to create the vortex ring. Time is measured in a non-dimensional form $T^*$ as the ratio between the length of fluid pushed by the piston and the diameter of the exit. The vortex reaches a maximum circulation $\Gamma$ at $T^* \approx 4$, also known as the vortex formation time. This timing is consistently reported for circulation based Reynolds numbers $\Gamma/\nu$ superior to 2000, and for various piston acceleration profiles [7]. When the circulation reaches a maximum, the vortex no longer accepts vorticity from the shear layer and this process is referred to as vortex separation.

A first explanation to the separation is derived from the Kelvin-Benjamin variational principle: a steadily
translating vortex ring is the maximum state of kinetic energy $E$ on a iso-vortical sheet with constant impulse $I$ \cite{2}. This approach led to the computation of the energy of the vortex with respect to its impulse and circulation, $E^* = E/\sqrt{I^3}$. The vortex separates when the non-dimensional energy $E^*$ delivered by the piston falls below the energy of a steadily translating vortex ring \cite{7}. The limiting non-dimensional energy is consistently found at $0.3 \pm 15\%$ for circulation based Reynolds numbers above 2000 \cite{15}. The non-dimensional energy quantifies the vorticity distribution inside the vortex. The lower the value of $E^*$, the more uniform the vorticity distribution is. For a Hill’s spherical vortex, $E^*$ has a low value of 0.16. When the shear layer starts to roll-up, for $T^* < 1$, vorticity is concentrated near the vortex core and $E^*$ has values above one. As the piston moves, more vorticity accumulated in the vortex and spreads towards the cylindrical symmetry axis, decreasing the value of $E^*$. There is a practical limit to the spreading of the vorticity, represented by this limit of $E^* = 0.3$. A possible interpretation of the vortex separation emerges from the study of the stability of vortex rings. Dynamical system analysis \cite{22} and perturbation response of vortices from the Norbury family \cite{18} showed that vorticity close to the axis of symmetry gets shed in the tail of the vortex, where the shear layer connects to the vortex. This could prevent the accumulation of additional vorticity by the vortex.

The second explanation to the vortex separation is based on a kinematic argument. The vortex separates when it is traveling faster than the shear layer. The velocity of the shear layer is usually estimated to half of the piston velocity \cite{6}. For $E^* \approx 0.3$ the critical separation velocity was calculated at 59\% of the piston velocity \cite{23}. The translational velocity of the vortex ring depends on its non-dimensional energy. Saffman \cite{20} estimated the velocity $U_0$ of a viscous steady vortex ring by the relation

$$E = 2U_0I - \frac{3}{8}D_0\Gamma^2$$

with $D_0$ being the diameter of the vortex ring. This equation is equivalent to

$$U_0 = \frac{\Gamma}{\pi D_0} \left( E^* \sqrt{\pi} + \frac{3}{4} \right).$$

Both energetic and kinematic explanations to the vortex separation are connected and it is not obvious to assess the causality between vorticity spreading, vortex velocity and vortex separation.

The tools developed to study propulsive vortices did not encounter much success in the analysis of drag vortices. An apparent reason is that drag vortices do not separate in the same time scale. Drag vortex rings are usually studied by accelerating a disk in a fluid. During the first convective times, the vortex created in the wake of the disk develops in a similar way as propulsive vortices. Around $T^* = 4$ the circulation starts increasing at a lower rate, but does not converge, and is not followed by vortex separation \cite{10}. For diameter based Reynolds numbers ranging from 1600 to 4000, the non-dimensional energy decreases down to values between 0.28 and 0.35. Later, passed $T^* = 10$, azimuthal instabilities break the axisymmetry and lead to separation of the vortex.

We propose to experimentally extend the analysis performed on propulsive vortices to drag vortices. Cones of different apertures, diameters and velocities will be translated to produce a wide variety of vortex rings in their wake. The non-dimensional energy will be measured to assess if it converges to a lower limit when the vortex stays close to its feeding shear layer. This information will help to understand the causality between vortex separation, vortex velocity and vorticity distribution in a propulsive vortex ring. A scaling of the vortex circulation, energy and velocity will be performed to identify the most relevant parameters among the geometry and the kinematics of both the cone and the vortex.

**Experiment**

A cone is immersed in water and translated along its axis of symmetry (Figure 1a). The translation is performed by a belt driven linear actuator, powered by a NEMA 17 stepper motor. The cones are 3D printed, their diameters $D$ range from 3 cm to 9 cm and their aperture $\alpha$ from 30° to 90°. They are accelerated at 3 m s$^{-2}$ from rest, up to velocities $U$ ranging from 0.35 to 0.7 m s$^{-1}$. A series of 20 experiments was conducted, with diameter based Reynolds numbers $Re_D = UD/\nu$ comprised between $1 \times 10^4$ and $6 \times 10^4$. 


Particle image velocimetry (PIV) is carried out in a symmetry plane of the cone during its motion. The flow field is illuminated with two light emitting diodes (LED). A field of view of 18 cm × 36 cm is recorded at 1000 fps by two high speed cameras, each recording one half of the cone trajectory with a definition of 1024 px × 1024 px. The images are processed with a multi-grid algorithm and a final interrogation window size of 24 px × 24 px with an overlap of 60%, leading to a physical resolution of 1.8 mm, or 6% of the smallest cone diameter.

The length $L$ traveled by the cone is recorded and used to define the non-dimensional timing of the experiment: $T^* = L/D$. Particle image velocimetry gives access to the velocity field $(u, v)$, from which the vorticity field $\omega$ is derived. The stream function $\psi$, later used to compute the energy of the vortex, is integrated

$$u = \frac{1}{r} \frac{\partial \psi}{\partial r}, \quad v = -\frac{1}{r} \frac{\partial \psi}{\partial z}. \quad (3)$$

**Results**

**Spatial and temporal development of a drag vortex ring**

The results presented in this section focus first on the detailed analysis of a single experiment, for a cone of aperture $\alpha = 45^\circ$, diameter $D = 6$ cm and velocity $U = 0.5$ m s$^{-1}$. The corresponding Reynolds number $Re_D = 3 \times 10^4$ is in the middle range of the study. Results from all the experiments are compared in the next section.

When the cone is pulled up, fluid moves around it and high velocity gradients raise at the tip of the cone. A shear layer forms and rolls up behind the cone, creating a vortex ring in the wake. With a diameter based Reynolds number superior to $10^4$, the shear layer breaks into small scale vortices ($\sim 1$ mm) due to a Kevin-Helmholtz instability (Figure 1b). These instabilities have no effect on the accumulation of vorticity in the vortex [15, 19].

The vortex contour is determined using Lagrangian methods. Finite tine Lyapunov exponent (FTLE) has been applied to accurately delimit vortices produced by various vortex generators [8, 12, 17, 21]. The vortex is delimited by the positive FTLE ridge and the base of the cone (Figure 2). This contour does not only contain vortical fluid, but also entrained fluid of low or zero vorticity. Non-vortical fluid is pulled along in the wake of the cone during the initial acceleration from rest. This non-vortical fluid initially sits below the cone, delimited by the virtual line on Figure 2a. The integration of this line trajectory delimits the volume of non-vortical fluid from the volume of vortical fluid injected in the vortex at the tip of the cone. The non-vortical volume of fluid, indicated by light grey area on Figure 2b-f, is entrained and progressively mixed with the vortical fluid. At $T^* = 0.5$ (Figure 2b) the non-vortical fluid accounts for 38% of the vortex volume. At $T^* = 3$ (Figure 2g) there is no more non-vortical fluid. Vorticity has...
The consequences of the spreading of the vorticity is analyzed by calculating the circulation $\Gamma = \iint \omega drdz$ on the area delimited by the FTLE contour. The non-dimensional circulation $\Gamma/UD$ increases up to values around 2.3 (Figure 3e). The growth rate $\dot{\Gamma}$ of the circulation decreases progressively. Around $T^* = 1$, a maximum growth rate $\dot{\Gamma} = 1.2U^2$ is reached. After $T^* = 3$ the rate stabilizes around $\dot{\Gamma} = 0.06U^2$. The stabilization starts when the non-vortical volume of fluid $V_0$ inside the vortex volume $V$ vanishes. Similar observations where made on vortex rings generated by piston cylinders: at $T^* = 4$ the circulation reaches a maximum $\Gamma/UD \approx 2.3$ and the vorticity spreads up to the cylindrical symmetry axis. The vortex separation for propulsive vortices is attributed to the tail shedding resulting from the vorticity spread [7]. In the present experiment, no separation occurs because the vortex translational velocity is directed towards the cone. The vortex stays in the vicinity of the shear layer, keeps growing and accumulating vorticity after $T^* = 3$ (Figure 3c-d).
FIG. 3: a-d, Vorticity field and vortex boundary in the wake of a translating cone with \( \alpha = 45^\circ \), \( D = 6 \text{ cm} \) and \( U = 0.5 \text{ m s}^{-1} \). e, Temporal evolution of the vortex circulation and volume \( V_0 \) of non-vortical fluid relative to the vortex volume \( V \). Grey area delimits acceleration phase of the cone.

The extension of the vortex is quantified by observing the vortex center \((Z_0, R_0)\), calculated as

\[
Z_0 = \int \frac{\omega z r^2 dr dz}{\int \omega r^2 dr dz}, \quad R_0^2 = \int \frac{\omega r^2 dr dz}{\int \omega dr dz}, \quad D_0 = 2R_0. \tag{4}
\]

Positions are given relative to the base of the cone (Figure 4b). The vortex center (Figure 4a) quickly moves away from the cone and has traveled a distance of \( 0.22D \) at \( T^* = 1 \). It progressively slows down and for \( T^* > 3 \) the vortex center moves away from the cone and the axis of symmetry at a more constant velocity. A distance of \( 0.03D \) is covered between \( T^* = 4 \) and \( T^* = 5 \). The increase of the vortex diameter suggests that the cone diameter is not the best parameter for non-dimensionalisation of the circulation. Using the vortex diameter \( D_0 \), the non-dimensional circulation reaches a maximum value of 2.2 at \( T^* = 3 \) and stays constant, whereas the non-dimensional circulation based on \( D \) continues to grow (Figure 4b). The term \( \Gamma/D_0 \) has the dimension of a velocity and is featured in Equation 2. It quantifies the influence of the vortex circulation and dimension on the velocity of the ring.

The other parameter involved in the velocity of the ring is the non-dimensional energy \( E^* \) of the vortex. It represents the energy \( E \) relative to impulse \( I \) and circulation \( \Gamma \):

\[
I = \pi \int \int \omega r^2 dr dz, \quad E = \pi \int \int \psi \omega dr dz, \quad E^* = \frac{E}{\sqrt{\Gamma^3}}. \tag{5}
\]

The non-dimensional energy quantifies the distribution of the vorticity inside the vortex. It is compared to a more statistical definition of the vorticity distribution: the standard deviation of the vorticity \( \sigma_\omega \) relative to its average \( \bar{\omega} \). The evolution of \( E^* \) and \( \sigma_\omega/\bar{\omega} \) are presented in Figure 5a. The non-dimensional energy and relative vorticity distribution have a similar evolution and an empirical relation \( \sigma_\omega/\bar{\omega} = 2.3E^* + 0.5 \) can be derived. Although this relation is specific to this experiment, it confirms that \( E^* \) is a valid quantifier of vorticity distribution. During the first convective time, the shear layer starts to roll-up and vorticity is concentrated in the vortex core. Non-dimensional energy has high values around 0.7. As vorticity keeps accumulating in the vortex, \( E^* \) continuously drops towards a limiting value of 0.3, reached at \( T^* = 3 \). A minimal value between 0.27 and 0.35 was also observed for vortex rings generated by piston cylinders [5, 7, 16] and corresponds to the moment when the vortex separates from its feeding shear layer. This limit on \( E^* \) answers one of the questions that motivated this experiment: the non-dimensional energy of a vortex ring does not decrease further, even if it stays connected to its feeding shear layer. Contrary to
vortex rings generated by piston cylinders, vorticity still accumulates in the vortex at a low rate, without noticeable effect on the vorticity distribution.

The quantities $\Gamma/D_0$ and $E^*$ characterize respectively the overall vorticity present in the vortex and the distribution of the vorticity in the vortex. Both reach a limiting value at $T^* = 3$. As a consequence, the theoretical translational velocity $U_0$ of the vortex ring, derived from Saffman [20], also reaches a steady value:

$$U_0 = \frac{\Gamma}{\pi D_0} \left( E^* \sqrt{\pi} + \frac{3}{4} \right).$$  \hfill (6)

The theoretical velocity (Equation 6) is compared with the measured vortex velocity $\bar{u}$, obtained by averaging the axial velocity component inside the vortex volume. Both theoretical and measured velocities converge to a value of $0.9U$ (Figure 5). The theoretical velocity underestimates the measured velocity during the transient phase, due to the fact that Equation 6 is only valid for a steady vortex ring. Since the vortex stays attached to the cone, the velocity deficit of the vortex compared to the cone should not be interpreted as a vortex separation, but rather as an increase of the vortex volume.

**Scaling of the vortex characteristics**

The influence of the cone geometry and kinematics on the vortex circulation, non-dimensional energy and velocity are analyzed in this section. The experiments are now sorted by a circulation based Reynolds number. As seen in the previous section, the circulation does not reach a clear maximum, but $\Gamma/D_0$ does. We therefore introduce a Reynolds number $Re_\Gamma = \frac{\Gamma D}{\nu D_0}$ based on the maximum value of $\Gamma/D_0$.

Different cone aperture were used, from $\alpha = 30^\circ$ to $\alpha = 90^\circ$ (flat disk). The trajectory of the vortex center
is extracted for four cones of different aperture and identical diameter \( D = 6 \) cm and velocity \( U = 0.5 \) m s\(^{-1}\) (Figure 6). The four trajectories show an increase of the vortex diameter passed \( T^* = 2 \). Larger aperture leads to larger diameter because the cone deviates the flow more in the radial direction. At \( T^* = 3 \) the vortex radius ranges from \( D_0 = 0.88D \) at \( \alpha = 30^\circ \) to \( D_0 = 1.1D \) at \( \alpha = 90^\circ \). The maximum non-dimensional circulation \( \Gamma/UD_0 \) is calculated for all cases and presented in Figure 6b. The average value is 2.26, with variations of \( \pm 9\% \). Cones with an aperture of 90\(^\circ\), or disks, exhibit lower non-dimensional circulations. For structural reason they are not strictly disks: they have a thickness of \( 0.06D \) and a reverse sweep of 30\(^\circ\). It is believed to be responsible for the lower non-dimensional circulation of the disk compared to the cones. The effect is reversed for the non-dimensional energy. Disks have slightly higher values (Figure 6c). Non-dimensional energy converges in average to a minimum of \( 0.3 \), with variations of \( \pm 12\% \).

The non-dimensional circulation and energy are quite insensitive to the Reynolds number. Similar behavior is observed for vortices generated by piston cylinders. For various simulations with \( Re = \Gamma \) \( > 2000 \), non-dimensional circulation and energy were recorded to have variations of respectively \( \pm 10\% \) and \( \pm 15\% \) [15].

From the circulation and the non dimensional energy, the theoretical velocity \( U_0 \) of the vortex ring is computed (Equation 6). The relative velocity \( U_0/U \) has an average of 0.93, within an interval of \( \pm 6\% \) (Figure 6d). The interval of variation is smaller than the ones of \( \Gamma/UD_0 \) (\( \pm 9\% \)) and \( E^* \) (\( \pm 12\% \)). The variations in non-dimensional circulation and non-dimensional energy compensate to produce a regular relative velocity. In particular the disks, which have higher non-dimensional energy and lower circulation, have a relative velocity equivalent to the cones. This suggests that the cone’s velocity is a better scaling parameter than the non-dimensional circulation or energy.

### Conclusion

Vortex rings produced by piston cylinders separate or pinch-off when the vortex reaches its minimal non-dimensional energy and when the vortex moves faster than its feeding shear layer. The simultaneity of these three events obfuscate the causality between them. To analyse the temporal evolution of the non-dimensional energy of ring vortices independent of their pinch-off, we focussed on vortices generated in the wake of cones. Cones with different apertures and diameters were accelerated from rest to produce a wide variety of vortex rings. The initial development and growth of these vortex rings were studied experimentally using time-resolved velocity field measurements.

The vortex rings that form behind the cones have a self-induced velocity that cause them to follow the cone beyond the typical vortex formation time scales observed for vortex rings emanating from a piston cylinder apparatus. Another difference is that propulsive vortex rings reach a maximum in circulation but the circulation of drag vortex rings keeps increasing and does not reach a clear plateau. However, for \( T^* > 3 \), the circulation of drag vortices...
increases proportionally to the vortex size and the circulation non-dimensionalised by the vortex diameter $\Gamma/UD_0$ converges to a limiting value between 2.05 and 2.45. The non-dimensional energy is linearly related to the relative standard deviation of the vorticity demonstrating that $E^*$ is a measure of vorticity distribution inside the vortex ring. At the start of the vortex formation, vorticity is concentrated near the vortex core and $E^*$ is at its highest. In time, the vortex grows and the vorticity distribution spreads, which is reflected by a decrease in $E^*$. The non-dimensional energy converges around $T^* = 3$ to a minimum value between 0.27 and 0.35. Similar values of non-dimensional circulation and energy were observed for vortices produced by piston cylinders. This results proves that vortex pinch-off does not cause the non-dimensional energy to converge to a minimum value.

The non-dimensionalised energy, circulation, and velocity of the ring vortices reach constant values independent of the cone diameter, aperture angle, and translational velocity when scaled based on the vortex diameter instead of the cone diameter. The limiting values of the circulation and the energy display variations of 9% and 12% and there variations compensate each other to produce a constant vortex velocity of $U_0 = 0.93U \pm 6\%$. The different between the vortex velocity and the cone velocity does not indicate vortex separation but is the result of the spatial growth of the vortex. The vortex velocity is the most unifying quantity to scale and predict the development of vortex rings behind various cone geometries.

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