TANGENT FUNCTOR ON MICROFORMAL MORPHISMS, AND NON-LINEAR PULLBACKS FOR FORMS AND COHOMOLOGY

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Dedicated to the memory of Kirill Mackenzie (1951–2020)

Abstract. We show how the tangent functor extends from ordinary smooth maps to “microformal morphisms” (also called “thick morphisms”) of supermanifolds. Microformal morphisms generalize ordinary maps and correspond to formal canonical relations between the cotangent bundles specified by generating functions depending on position variables on the source manifold and momentum variables on the target manifold (as formal power expansions), regarded as part of the structure. Microformal morphisms act on functions by non-linear (in general) pullbacks. We obtain here non-linear pullbacks of (pseudo)differential forms and show that they respect the de Rham differentials as “non-linear chain maps” that can induce non-linear transformations of cohomology.

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1. Introduction

1.1. In the series of papers [14, 15, 16, 18, 19] (see also [3], [9]) we introduced and studied a new type of morphisms between (super)manifolds, generalizing smooth maps, which we have called microformal or thick morphisms. Their distinctive feature is that they induce non-linear, in general, pullbacks on functions. More precisely, the pullbacks are formal non-linear differential operators over ordinary maps. They are constructed by some iterative procedure. (For ordinary maps, which is a special case, the construction gives the familiar linear pullbacks.) From algebraic viewpoint, these new type of pullbacks can be described as non-linear algebra homomorphisms. By that we mean a formal map of algebras as vector spaces such that for each point the derivative (which is a linear map) is an algebra homomorphism in the usual sense.

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1.2. Below we recall thick morphisms and non-linear pullbacks induced by them. For a detailed introduction to the whole theory, we refer the reader to the above-cited papers and particularly to [18].

A thick morphism $\Phi: M_1 \rightarrow M_2$ is defined in coordinates by a generating function $S = S(x, q)$ depending on two groups of variables: local coordinates $x^a$ on the source manifold and components of momentum $q_i$ on the target manifold. Generating functions are formal power series in the momentum variables, which we write as

$$S(x, q) = S^0(x) + \varphi^i(x)q_i + \frac{1}{2} S^{ij}(x)q_iq_j + \ldots .$$

The case when $S(x, q)$ is just a linear function in the momenta, $S(x, q) = \varphi^i(x)q_i$, corresponds to an ordinary map, $y^i = \varphi^i(x)$. To a thick morphism $\Phi: M_1 \rightarrow M_2$ we assign a formal canonical relation $T^*M_1 \rightarrow T^*M_2$, between the cotangent bundles specified by $S$; with some abuse of language, one can identify a thick morphism with this relation. This is useful for intuition, but because a generating function $S$ itself is taken as part of the structure, a thick morphism contains more information than the corresponding relation, which is determined by the differential $dS$ only.

The justification of this notion is in the construction of pullback $[14]$. Heuristically, one has a canonical relation $T^*M_1 \rightarrow T^*M_2$, and, if a function $g$ on $M_2$ is given, it corresponds to a canonical relation $T^*M_1 \rightarrow \{\ast\}$; their composition will again be a canonical relation $T^*M_1 \rightarrow \{\ast\}$, corresponding to a function on $M_1$ interpreted as the desired pullback of $g$. By using the generating function $S$, this can be made fully explicit as follows.

Given a function $g = g(y)$ on $M_2$, we define its pullback $f = \Phi^*[g]$ as a function on $M_1$, $f = f(x)$, by

$$f(x) := g(y) + S(x, q) - y^iq_i ,$$

where the extra variables $q_i, y^i$ are eliminated from the right-hand side by using the equations

$$q_i = \frac{\partial g}{\partial y^i}(y), \quad y^i = (-1)^i \frac{\partial S}{\partial q_i}(x, q) .$$

The latter system gives an equation for determining $y$ as a function of $x$ (and depending on a function $g$)

$$y^i = (-1)^i \frac{\partial S}{\partial q_i}(x, \frac{\partial g}{\partial y}(y)) ,$$

of the “fixed point” type. It has a unique solution as a power series in the derivatives of $g$,

$$y^i = \varphi^i_g(x) = \varphi^i(x) + S^{ij}(x)\partial_j g(\varphi(x)) + \ldots .$$

It is a formal perturbation $\varphi_g: M_1 \rightarrow M_2$, $\varphi_g = \varphi_0 + \varphi_1 + \ldots$, of the map $\varphi: M_1 \rightarrow M_2$ given by $y^i = \varphi^i(x)$ and corresponding to $g = 0$, $\varphi = \varphi_0$. Having obtained $y = \varphi_g(x)$ as a function of $x$, we substitute it into the first equation in [3], which makes it possible to eliminate both $y$ and $q$ from [2]. For the pullback $\Phi^*[g]$, we obtain as the result a formal power-series expression in $g$ of the form

$$f(x) = S^0(x) + g(\varphi(x)) + \frac{1}{2} S^{ij}(x) \partial_j g(\varphi(x))\partial_i g(\varphi(x)) + \ldots .$$
Each term of order \( k \) in \( g \) contains the derivatives of \( g \) of orders up to \( k - 1 \) evaluated at \( y = \varphi(x) \). Thus the pullback

\[
\Phi^*: C^\infty(M_2) \to C^\infty(M_1)
\]

is a formal nonlinear differential operator over the map \( \varphi: M_1 \to M_2 \) defined by the linear term in the generating function (1).

For a linear generating function \( S(x, q) = \varphi^i(x)q_i \), the pullback defined by the above procedure coincides with be the ordinary pullback \( \varphi^* \) by the map \( \varphi \) and in particular is linear. (The corresponding canonical relation \( T^*M_1 \to T^*M_2 \) is the cotangent lift \( T^*\varphi \).

1.3. Because the pullbacks induced on smooth functions by thick morphisms are nonlinear in general, there are actually two parallel theories: of \textbf{"even"} and \textbf{"odd"} thick morphisms acting on even and odd functions ("bosonic" and "fermionic" fields in physical parlance), respectively. Above, we have described the \textbf{even} case (\( g \) should be assumed to be even); in the \textbf{odd} case, one has to use the anticotangent bundles \( \Pi T^*M_1 \) and \( \Pi T^*M_2 \) (where \( \Pi \) is the parity reversion functor) and odd generating functions \( S(x, y^*) \), where \( y^*_i \) are the antimomenta (fiber coordinates for \( \Pi T^*M_2 \)). We will use the notation \( \Psi: M_1 \Leftrightarrow M_2 \) for odd thick morphisms.

1.4. The problem that we solve in the present paper is extending the \textbf{tangent functor} \( T \) and the \textbf{antitangent functor} \( \Pi T \) from ordinary smooth maps \( \varphi: M_1 \to M_2 \) to thick morphisms of both types (even and odd).

Namely, we show that if \( \Phi: M_1 \Leftrightarrow M_2 \) is an even thick morphism, it induces an even thick morphism \( T\Phi: TM_1 \Leftrightarrow TM_2 \) and an odd thick morphism \( \Pi T\Phi: \Pi TM_1 \Leftrightarrow \Pi TM_2 \) (generalizing \( T\varphi \) and \( \Pi T\varphi \) for an ordinary map), and this is a functorial correspondence.

Likewise, if \( \Psi: M_1 \Leftrightarrow M_2 \) is an odd thick morphism, it induces an odd thick morphism \( T\Psi: TM_1 \Leftrightarrow TM_2 \) and an even thick morphism \( \Pi T\Psi: \Pi TM_1 \Leftrightarrow \Pi TM_2 \). (Note the “change of parity” from odd to even for the case of \( \Pi T\Psi \).)

The tangent and antitangent thick morphisms \( T\Phi, T\Psi, \Pi T\Phi \) and \( \Pi T\Psi \) are fiberwise in a suitable sense. (In particular, they are morphisms over an ordinary map that is the ‘core’ of a given thick morphism rather than over the thick morphism itself, see Section 3.)

Since functions in \( C^\infty(\Pi TM) \) in the case of ordinary manifolds can be identified with differential forms \( \Omega(M) \), and for supermanifolds they are by definition the Bernstein–Leites pseudodifferential forms, we in particular obtain non-linear pullbacks on such forms. We show that these non-linear maps are compatible with the de Rham differentials and induce non-linear transformations on de Rham cohomology.

1.5. Important role in our construction is played by a natural diffeomorphism involving the cotangent bundle \( T^*E \) of a vector bundle \( E \). It was discovered by W. M. Tulczyjew \[10\] for \( E = TM \) and by K. C. H. Mackenzie and P. Xu \[6\] in the general case (see also \[4\]). Their generalization to supermanifolds was established in \[12\]. Also in \[12\], we introduced an analog for the anticotangent bundle \( \Pi T^*E \).

The paper is organized as follows.

In Section 2 we elaborate the above-mentioned diffeomorphisms in an explicit form best suiting our needs, making a particular stress on the relations between tangent and

\[1\] Also considered in an unpublished text by J.-P. Dufour; I thank J. Grabowski for sharing it with me.
cotangent bundles with various parity reversed combinations (i.e., “odd analogs” of the Tulczyjew map, see subsection 2.3). The latter, to our knowledge, have not previously appeared in the literature. In central Section 3 we turn to application to thick morphisms and work out all cases of tangent and antitangent functors. As a by-product, we introduce a general concept of a vector bundle thick morphism. See also our work in preparation 2. In Section 4 we look at the consequences for forms and de Rham cohomology. On the way, we elaborate the notions of a thick \(Q\)-morphism and a “non-linear chain map”.

1.6. **Notation and terminology:** we mostly follow the usage in 18. The parity \(\mathbb{Z}_2\)-grading of an object is denoted by the tilde over the corresponding character. For local coordinates, their parities are assigned to the associated indices. We suppress the prefix ‘super-’ unless it is essential to stress the difference with the non-super case. So e.g. we speak about ‘manifolds’ meaning ‘supermanifolds’. Also, although we do not stress this specifically, but an extra \(\mathbb{Z}\)-grading or \emph{weight}, which is independent of parity, can be introduced everywhere; so our constructions hold for \emph{graded (super)manifolds}.

2. **Natural diffeomorphisms for cotangent and anticotangent bundles**

2.1. **Mackenzie–Xu isomorphism and its odd analog.** The following theorem is due to Kirill Mackenzie and Ping Xu 3 for a general vector bundle \(E\) (the special case of \(E = TM\) was obtained earlier by W. Tulczyjew 10). See also 5 Thm. 9.5.1. The original versions were for ordinary manifolds, but the statement holds true for the super and graded cases 12. We shall give a proof with explicit formulas.

To fix the notations: the canonical even symplectic form on the cotangent bundle \(T^*M\) of a supermanifold \(M\) is denoted \(\omega_{T^*M}\); it is the differential of the canonical odd 1-form on \(T^*M\) called the Liouville form, which is denoted \(\theta_M\), so \(\omega_{T^*M} = d\theta_M\). If \(x^a\) are local coordinates on \(M\) and \(p_a\) are the corresponding momenta, then \(\theta_M = p_a dx^a\) and \(\omega_{T^*M} = dp_a dx^a\). The invariance of the form \(\theta_M\) encodes the transformation law for the momentum variables. (Note that \(p_a dx^a = dx^a p_a\), because of the opposite parities of \(p_a\) and \(dx^a\).)

**Theorem 1.** For a vector bundle \(E \to M\), there is a natural diffeomorphism \(T^*(E) \cong T^*(E^*)\). It is an antisymplectomorphism of the canonical symplectic structures.

**Proof.** We shall introduce a transformation

\[
\kappa: T^*E \to T^*E^*
\]

which will be the desired diffeomorphism. Denote local coordinates on \(M\) by \(x^a\) and in the fibers of \(E\) and \(E^*\) by \(u^i\) and \(u_i\), respectively. We assume that the pairing between \(E\) and \(E^*\) is given by the invariant form \(\langle u, u^* \rangle = u^i u_i\). (We view \(u^i\) as left coordinates and \(u_i\) as right coordinates in the dual frames. The distinction between ‘left’ and ‘right’ is important in the super case.) The corresponding conjugate momenta in \(T^*E\) and \(T^*(E^*)\) will be denoted by \(p_a, p_i, p_a, p^i\). (Note the same notation \(p_a\) for coordinates on different manifolds.) We have transformation laws for \(u^i\) and \(u_i\) as \(u^i = u^i T^i_j u^j\) and \(u_i = T^i_j u^j\) with reciprocal matrices \(T^i_j\) and \(T^i_j\). Recall that the conjugate momenta transform as the partial derivatives in the respective coordinates. From here we have the transformation law for \(p_i, p_i = T^j_i p_j\), which is the same as for \(u_i\). Define \(\kappa: T^*E \to T^*E^*\) by the formulas

\[
\kappa(x^a) = x^a, \quad \kappa(u_i) = p_i, \quad \kappa(p_a) = -p_a, \quad \kappa(p^i) = (-1)^j u^i.
\]
To show that it this definition does not depend on a choice of coordinates, we use the Liouville 1-forms. On $T^*E$ we have

$$\theta_E = dx^ap_a + du^ip_i,$$

and on $E^*$ we have, respectively,

$$\theta_{E^*} = dx^ap_a + du^ip_i.$$  

If we apply $\kappa^*$ to $\theta_{E^*}$, we shall obtain

$$\kappa^*\theta_{E^*} = -dx^ap_a + (-1)^idp_iu^i = -dx^ap_a + (-1)^id(p_iu^i) - p_idu^i = -\theta_E + d(u^ip_i).$$

This proves that $\kappa$ is well-defined, and also that

$$\kappa^*\omega_{T^*E^*} = -\omega_{T^*E}. \quad (13)$$

Here $\omega_{T^*E} = d\theta_E = dp_adx^a + dp_idu^i$ and $\omega_{T^*E^*} = d\theta_{E^*} = dp_adx^a + dp_idu^i$ are the canonical symplectic forms. Hence $\kappa: T^*E \to T^*E^*$ is an antisymplectomorphism. \hfill $\square$

We shall refer to the antisymplectomorphism $\kappa: T^*E \to T^*E^*$ as the Mackenzie-Xu transformation. Its action on the Liouville forms, $\kappa^*\theta_{E^*} = d(u^ip_i) - \theta_E$, resembles Legendre transform. When we need to emphasize the bundle $E$, we write $\kappa_E$ for $\kappa$.

With $E^{**} \cong E$, one may want to know about the transformation between the cotangent bundles going in the opposite direction. Caution is necessary in the super case because of signs. For clarity, let us express our formulas writing fiber coordinates for all vector bundles as left coordinates; in particular, for $E^*$ we will have left coordinates as $\bar{u}_i = (-1)^iu_i$ (where $u_i$ are the right coordinates used in (9)), and the invariant pairing between $E$ and $E^*$ will take the form $\langle u, u^* \rangle = u^i\bar{u}_i(-1)^i$. Therefore, in terms of left coordinates only, the Mackenzie-Xu transformation $\kappa = \kappa_E: T^*E \to T^*(E^*)$ is given by

$$\kappa_E^*\bar{u}_i = (-1)^i\bar{p}_i, \quad \kappa_E^*p_a = -p_a, \quad \kappa_E^*\bar{p}^i = u^i. \quad (14)$$

For $E^{**}$ and $E^*$, the pairing is $\langle u^*, u^{**} \rangle = \bar{u}_i\bar{u}^i(-1)^i = \bar{u}_i\bar{u}_i$, where $\bar{u}^i$ are the left coordinates in $E^{**}$. Hence the natural isomorphism of vector bundles $\sigma: E^{**} \to E$ over $M$ is given by

$$\sigma^*(u^i) := (-1)^i\bar{u}^i. \quad (15)$$

We denote by $\sigma$ also the induced diffeomorphism of the cotangent bundles, $\sigma: T^*(E^{**}) \to T^*E$, so we have for it

$$\sigma^*(u^i) = (-1)^i\bar{u}^i, \quad \sigma^*(p_i) = (-1)^i\bar{p}_i \quad (16)$$

(as well as $\sigma^*(x^a) = x^a$ and $\sigma^*(p_a) = p_a$).

Then the following holds.

**Proposition 1.** Let $\kappa_E: T^*E \to T^*(E^*)$ and $\kappa_{E^*}: T^*(E^*) \to T^*(E^{**})$ be the Mackenzie-Xu transformations for the bundles $E$ and $E^*$, and let $\sigma: T^*(E^{**}) \to T^*E$ be the natural diffeomorphism defined above. Then the diffeomorphisms $\kappa_E$ and $\sigma \circ \kappa_{E^*}$,

$$T^*E \xrightarrow{\kappa_E} T^*(E^*) \xrightarrow{\sigma \circ \kappa_{E^*}} T^*(E^{**}), \quad (17)$$

are mutually inverse.
Combining this with the diffeomorphism\( \sigma \): \( T^*(E^*) \to T^*(E^{**}) \) we have
\[
\kappa_{E^*}(\bar{u}^i) = (-1)^i \bar{p}^i, \quad \kappa_{E^*}(p_a) = -p_a, \quad \kappa_{E^*}(\bar{p}_i) = \bar{u}_i.
\] (18)
Combining this with the diffeomorphism \( \sigma \): \( T^*(E^{**}) \to T^*E \) given by (16), we obtain
\[
(\sigma \circ \kappa_{E^*})^*(u^i) = \bar{p}^i, \quad (\sigma \circ \kappa_{E^*})^*(p_a) = -p_a, \quad (\sigma \circ \kappa_{E^*})^*(\bar{p}_i) = (-1)^i \bar{u}_i,
\]
which is the inverse for (14).

Proof. Similarly with (14), for \( \kappa_{E^*}: T^*(E^*) \to T^*(E^{**}) \) we have
\[
\kappa_{E^*}(\bar{u}^i) = (-1)^i \bar{p}^i, \quad \kappa_{E^*}(p_a) = -p_a, \quad \kappa_{E^*}(\bar{p}_i) = \bar{u}_i.
\] (18)

Theorem 1 is best understood in the context of double vector bundles. As it is known [4, 5, Ch. 9], for a vector bundle \( E \to M \) the cotangent bundle \( T^*(E) \) is a double vector bundle over \( M \) with the side bundles \( E \) and \( E^* \), and a similar double vector bundle structure is for \( E^* \). Then the Mackenzie–Xu map \( \kappa \) is an isomorphism of double vector bundles:
\[
\begin{array}{ccc}
T^*E & \longrightarrow & E^* \\
\downarrow & & \kappa \downarrow \\
E & \longrightarrow & M
\end{array}
\]
\[
\begin{array}{ccc}
T^*(E^*) & \longrightarrow & E^* \\
\downarrow & & \kappa \downarrow \\
E & \longrightarrow & M
\end{array}
\]
which is identical on \( E \to M \), \( E^* \to M \) and induces “fiberwise \(-1\)” on the core \( T^*M \to M \). The second vector bundle structure in each of the double vector bundles actually encodes the Mackenzie–Xu isomorphism. This also agrees with interpretation of double vector bundles as (bi)graded manifolds [13].

Remark 1. There is some flexibility with a choice of sign in the definition of \( \kappa \). The choice used here is the same as in the original definition of Mackenzie–Xu [6], as well as [4, 5], but is different from that in [12] and [13]. Note that the equality (12) corresponds to Theorem 9.5.2 in [5]. Also, there is a “quantum analog” of the Mackenzie–Xu transformation [8] (for \( \hbar \)-differential operators).

An “odd version” of the Mackenzie–Xu diffeomorphism is as follows [12]. Recall that \( \Pi \) is the parity reversion functor (on vector spaces, modules and vector bundles). We use the terminology such as antitangent and anticotangent bundles for \( \Pi T^*M \) and \( \Pi T^*M \). Recall that the anticotangent bundle of any supermanifold \( M \) is endowed with the canonical odd symplectic form, which we denote \( \omega_{\Pi T^*M} \). There is an even 1-form on \( \Pi T^*M \) analogous to the Liouville 1-form; we denote it \( \lambda_M \), so \( \omega_{\Pi T^*M} = d\lambda_M \). In local coordinates, \( \Lambda_M = dx^a x^*_a = (-1)^{a+1} x^*_a dx^a \), where \( x^*_a \) are the antimomenta canonically conjugate to coordinates \( x^a \) on \( M \). (The variables \( x^*_a \) transform in the same way as \( p_a \), i.e., as the partial derivatives in the respective coordinates, but have the opposite parity.)

Theorem 2 (12). For a vector bundle \( E \), there is a natural diffeomorphism \( \Pi T^*E \cong \Pi T^*(\Pi E^*) \). It can be chosen as an antisymplectomorphism of the odd symplectic structures.

Proof. Denote by \( \xi_i \) fiber coordinates in \( \Pi E^* \), so that there is the odd bilinear form \( \langle u, \xi^* \rangle = u^i \xi_i \) is invariant. (That means that \( \xi_i \) are right coordinates and have transformation law \( \xi_i = T^{ij}_i \xi_j \), the same as \( u_j \). The parity of \( \xi_i \) is \( \tilde{i} + 1 \), the opposite to that of \( u^i \) or \( u_i \).) Let \( x^*_a, u^i \) denote the antimomenta conjugate to \( x^a, u^i \) on \( \Pi T^*E \), and let similarly \( x^*_a, \xi^a \) denote the antimomenta conjugate to \( x^a, \xi_i \) on \( \Pi T^*(\Pi E^*) \). Now we can introduce the desired diffeomorphism
\[
\kappa: \Pi T^*E \to \Pi T^*(\Pi E^*)
\] (20)
have

\[ \kappa^*(x^a) = x^a, \quad \kappa^*(\xi_i) = u^i, \quad \kappa^*(\bar{x}^a) = -x^a, \quad \kappa^*(\bar{\xi}_i) = u^i \]  

(21)

(note no signs depending on parities, unlike the even case [4]). To see that it is well-defined and possesses the desired properties, we use the analogs of the Liouville forms. For \( E \), we have

\[ \lambda_E = dx^a x^*_a + du^i u^*_i. \]  

Similarly, for \( \Pi E^* \) we have

\[ \lambda_{\Pi E^*} = dx^a x^*_a + d\xi_i \xi^{*i}. \]  

Then in the same way as above, we observe that from (21),

\[ \kappa^* \lambda_{\Pi E^*} = -\lambda_E + d(u^i u^*_i), \]  

and everything follows from here. \( \square \)

In the same way as above, the diffeomorphism of \( \Pi T^*E \) and \( \Pi T^*(\Pi E^*) \) is best understood as an isomorphism of the double vector bundles

\[
\begin{array}{ccc}
\Pi T^*E & \longrightarrow & \Pi E^* \\
\downarrow & & \downarrow \kappa \\
E & \longrightarrow & M
\end{array}
\quad 
\begin{array}{ccc}
\Pi T^*(\Pi E^*) & \longrightarrow & \Pi E^* \\
\downarrow & & \downarrow \\
E & \longrightarrow & M
\end{array}
\]  

(25)

There is also an analog of Proposition [1]

2.2. **Tulczyjew isomorphism.** If one specializes the identification \( T^*E \cong T^*(E^*) \) to the case \( E = TM \), this will give \( T^*(TM) \cong T^*(T^*M) \). The canonical symplectic form on \( T^*M \) makes it possible to raise and lower tensor induces and in particular to identify the cotangent bundle \( T^*(T^*M) \) and the tangent bundle \( T(T^*M) \). Combining that with the above isomorphism, we arrive at a natural diffeomorphism \( T^*(TM) \cong T(T^*M) \). Following Mackenzie [4, 5], we call it the **Tulczyjew isomorphism**. As with the Mackenzie–Xu transformation and its odd analog, the Tulczyjew isomorphism is best understood in terms of double vector bundles. One can give a direct construction for it (close to Tulczyjew’s original treatment). It is below.

**Theorem 3.** For any supermanifold \( M \), there is a natural diffeomorphism

\[ \tau: T(T^*M) \to T^*(TM), \]  

which is an isomorphism of double vector bundles:

\[
\begin{array}{ccc}
T(T^*M) & \longrightarrow & T^*M \\
\downarrow & & \downarrow \tau \\
TM & \longrightarrow & M
\end{array}
\quad 
\begin{array}{ccc}
T^*(TM) & \longrightarrow & T^*M \\
\downarrow & & \downarrow \\
TM & \longrightarrow & M
\end{array}
\]  

(27)

**Proof.** In coordinates, the double vector bundle structures in question are as follows:

\[
\begin{array}{cccc}
(x^a, p_a; \dot{x}^a, \dot{p}_a) & \longrightarrow & (x^a, p_a) & \longrightarrow & (x^a, p_a; (1), p_a(2)) \\
\downarrow & & \downarrow & & \downarrow \\
(x^a, \dot{x}^a) & \longrightarrow & (x^a) & \longrightarrow & (x^a, \dot{x}^a)
\end{array}
\]  

(28)
Each arrow is simply dropping part of the variables. Here at the right-hand side, on $T^*(TM)$ the momentum variables $p_a^{(1)}$ are canonically conjugate with $x^a$, while $p_a^{(2)}$ are canonically conjugate with $\dot{x}^a$. Note that the pullback by the horizontal projection sends the variables $p_a$ on $T^*M$ to the variables $p_a^{(2)}$ on $T^*(TM)$ (not to $p_a^{(1)}$). We define the map 

$$
\tau: T(T^*M) \to T^*(TM)
$$

by 

$$
\tau^*(x^a) = x^a, \quad \tau^*(\dot{x}^a) = \dot{x}^a, \quad \tau^*(p_a^{(1)}) = \dot{p}_a, \quad \tau^*(p_a^{(2)}) = p_a. \tag{29}
$$

On $T^*(TM)$, we have the Liouville 1-form 

$$
\theta_{TM} = dx^a p_a^{(1)} + d\dot{x}^a p_a^{(2)}. \tag{30}
$$

On the other hand, on $T(T^*M)$ we have the 1-form $\dot{\theta}_M$ obtained by formally differentiating the Liouville 1-form $\theta_M = dx^a p_a$ on $T^*M$, 

$$
\dot{\theta}_M = d\dot{x}^a p_a + dx^a \dot{p}_a = dx^a \dot{p}_a + d\dot{x}^a p_a. \tag{31}
$$

By (29), we have 

$$
\tau^*(\theta_{TM}) = \dot{\theta}_M. \tag{32}
$$

This proves that (29) is well-defined (and is a morphism of double vector bundles, i.e. commutes with the projections [29]). \qed

We can use the map $\tau$ for the identification of $T^*(TM)$ with $T(T^*M)$. In particular, we obtain a symplectic form $\omega_{T(T^*M)}$ on $T(T^*M)$, defined as $\tau^*(\omega_{T^*(TM)})$, 

$$
\omega_{T(T^*M)} := \tau^*(\omega_{T^*(TM)}) = \tau^*(d\theta_{TM}) = d(\tau^*(\theta_{TM})) = d\dot{\theta}_M = d\dot{p}_a dx^a + dp_a d\dot{x}^a. \tag{33}
$$

We observe that $d(\dot{\theta}_M) = (d\theta_M)$, so we have 

$$
\omega_{T(T^*M)} = \dot{\omega}_{T^*M}. \tag{34}
$$

There is a special class of Lagrangian submanifolds in $T(T^*M)$ given by lifts to $T(T^*M)$ of Lagrangian submanifolds of $T^*M$:

**Corollary 1.** Suppose $P \subset T^*M$ is a Lagrangian submanifold. Then $TP \subset T(T^*M)$ will be a Lagrangian submanifold of $T(T^*M)$ with respect to the symplectic structure (33), (34).

**Proof.** If $\omega_{T^*M}$ vanishes on $P$, then on $TP$ the condition $\omega_{T^*M} = 0$ will be satisfied. (As for dimension, $\dim TP = 2 \dim P = \dim T^*M = \frac{1}{2} \dim T(T^*M)$.) \qed

For example, such will be the submanifolds $TM \subset T(T^*M)$ given by $p_a = 0, \dot{p}_a = 0$ and $T(T^*_x M) \subset T(T^*M)$ (for any $x_0 \in M$) given by $x^a = x_0^a, \dot{x}^a = 0$.

### 2.3. Three odd analogs of the Tulczyjew isomorphism.

Consider now $E = \Pi TM$. We wish to give descriptions for the cotangent and anticotangent bundles $T^*(\Pi TM)$ and $\Pi T^*(\Pi TM)$ similar with the Tulczyjew isomorphism. Also, we shall supplement the above consideration of the Tulczyjew isomorphism by a description of $T^*(TM)$. We shall start from the latter. The statements and proofs for all three cases will go very much in parallel. As above, the main tool for us will be Liouville 1-forms. We shall use the same notation $\tau$ for all three odd analogs of the Tulczyjew map that will be introduced.

Theorem [2] gives the isomorphism $\Pi T^*(TM) \cong \Pi T^*(\Pi T^*M)$, and raising indices with the help of the odd symplectic form on $\Pi T^*M$ will give $\Pi T^*(\Pi T^*M) \cong T(\Pi T^*M)$, note
the change of parity because the form is odd; combined together, this gives $\Pi T^* (TM) \cong T(\Pi T^* M)$. Similarly with the above, we can provide a direct construction.

**Theorem 4.** For any supermanifold $M$, there is a natural diffeomorphism

$$\tau : T(\Pi T^* M) \to \Pi T^* (TM).$$

which is an isomorphism of double vector bundles:

$$
\begin{array}{ccc}
T(\Pi T^* M) & \longrightarrow & \Pi T^* M \\
\downarrow & & \downarrow \tau \\
TM & \longrightarrow & M
\end{array}
\quad
\begin{array}{ccc}
\Pi T^* (TM) & \longrightarrow & \Pi T^* M \\
\downarrow & & \downarrow \\
TM & \longrightarrow & M
\end{array}
$$

which in particular ends $T(\Pi T^* M)$ with a canonical odd symplectic structure.

**Proof.** The double vector bundle structures in (36) are given by

$$
(x^a, x^s_a; \dot{x}^a, \dot{x}^s_a) \longrightarrow (x^a, x^s_a) \\
(x^a, \dot{x}^a) \longrightarrow (x^a),
$$

similarly with (28). At the left-hand side, the variables $\dot{x}^s_a$ are understood mnemonically as $(x^s_a)$, while at the right-hand side, $x^s_a$ are conjugate to $x^a$ and $\dot{x}^s_a$ are conjugate to $\dot{x}^a$. The horizontal arrow maps $(x^a, x^s_a; \dot{x}^a, \dot{x}^s_a)$ to $(x^a, x^s_a := \dot{x}^s_a)$. Consider the even analog of the Liouville form for $\Pi T^* M$, $\lambda_M = dx^a x^s_a$. It induces an invariant even 1-form on $T(\Pi T^* M)$,

$$\dot{\lambda}_M = dx^a \dot{x}^s_a + d\dot{x}^a x^s_a.$$

At the same time, there is the form $\lambda_{TM}$ on $\Pi T^* (TM)$,

$$\lambda_{TM} = dx^a x^s_a + d\dot{x}^a x^s_a.$$

We define the map $\tau : T(\Pi T^* M) \to \Pi T^* (TM)$ by

$$
\tau^* (x^a) = x^a, \quad \tau^* (\dot{x}^a) = \dot{x}^a, \quad \tau^* (x^s_a) = \dot{x}^s_a, \quad \tau^* (\dot{x}^s_a) = x^s_a,
$$

so that

$$\tau^* (\lambda_{TM}) = \dot{\lambda}_M.$$

which in particular implies that the map $\tau$ specified by (10) is well-defined. From here we obtain a canonical odd symplectic form $\omega_{T(\Pi T^* M)}$ on $T(\Pi T^* M)$, as

$$\omega_{T(\Pi T^* M)} := d\dot{\lambda}_M = \tau^*(d\lambda_{TM}) = \tau^* (\omega_{\Pi T^* (TM)}),$$

also

$$\omega_{T(\Pi T^* M)} = (\omega_{\Pi T^* M})^\vee,$$

similarly with (34). \qed

**Corollary 2.** For a Lagrangian submanifold $P \subset \Pi T^* M$, its tangent bundle $TP \subset T(\Pi T^* M)$ will be a Lagrangian submanifold of $T(\Pi T^* M)$ with respect to the odd symplectic structure (12), (13).

**Proof.** Indeed, if $\omega_{\Pi T^* M}$ vanishes on $P$, then $\omega_{\Pi T^* M}$ vanishes on $TP$. \qed
Before we proceed to the next two theorems, describing $T^*(\Pi TM)$ and $\Pi T^*(\Pi TM)$, we need to make some convention about notation.

Functions on the antitangent bundle $\Pi TM$ are (pseudo)differential forms on a (super)manifold $M$, for any $M$; we have been normally using $dx^a$ for fiber coordinates on $\Pi TM$ induced by local coordinates $x^a$ on $M$. Now we will have to work with forms on $\Pi TM$ itself, i.e. with the iterated bundles such as $\Pi T(\Pi TM)$. A proper notation for these iterations (yielding ‘higher differential forms’ from the viewpoint of the original manifold) would be by using operators such as $d_1$, $d_2$, etc., for each successive level. Since we do not need to go beyond the second iteration and for the sake of uniformity with the other cases, we shall keep $d$ for denoting forms on $\Pi TM$ (which otherwise would be denoted using $d_2$) and to change $dx^a$ to $\partial x^a$ (which otherwise would be $d_1x^a$) for fiber coordinates on $\Pi TM$, so that functions on $\Pi TM$ will be written as functions of the variables $x^a, \partial x^a$. It is worth noting that as odd operators, $d$ and $\partial$ commute with the minus sign.

**Theorem 5.** For any supermanifold $M$, there is a natural diffeomorphism

$$\tau: \Pi T(\Pi^* M) \rightarrow T^*(\Pi TM)$$

which is an isomorphism of double vector bundles:

$$\begin{array}{ccc}
\Pi T(\Pi^* M) & \longrightarrow & \Pi T^* M \\
\downarrow & & \downarrow \\
\Pi TM & \longrightarrow & M
\end{array} \quad \begin{array}{ccc}
T^*(\Pi TM) & \longrightarrow & \Pi T^* M \\
\downarrow & & \downarrow \\
\Pi TM & \longrightarrow & M
\end{array} \quad (45)

which in particular endows $\Pi T(\Pi^* M)$ with a canonical even symplectic form.

**Proof.** Combining $T^*(\Pi TM) \cong T^*(\Pi^* M)$ with $T^*(\Pi^* M) \cong \Pi T(\Pi^* M)$, we obtain $T^*(\Pi TM) \cong \Pi T(\Pi^* M)$. To get a desired map $\tau$, we act as before. First of all, the double vector bundle structures are given in local coordinates by

$$(x^a, x^*_a; \partial x^a, \partial x^*_a) \rightarrow (x^a, x^*_a) \quad (x^a, \partial x^a; p_a, \pi_a) \rightarrow (x^a, x^*_a := \pi_a)\quad (46)$$

On $T^*(\Pi TM)$ there is the odd 1-form (analog of the Liouville form)

$$\theta_{\Pi TM} = dx^a p_a + d(\partial x^a)\pi_a, \quad (47)$$

while on $\Pi T(\Pi^* M)$ there is an odd 1-form obtained as the lift of the canonical even 1-form on $\Pi T^* M$, $\lambda_M = da^a x^*_a$:

$$\partial \lambda_M = -d(\partial a^a) x^*_a + (-1)^{\hat{a} + 1} dx^a \partial x^*_a = dx^a((-1)^{\hat{a} + 1}\partial x^*_a) + d(\partial a^a)(-x^*_a). \quad (48)$$

We define the desired diffeomorphism $\tau: \Pi T(\Pi^* M) \rightarrow T^*(\Pi TM)$ by

$$\tau^*(x^a) = x^a, \quad \tau^*(\partial x^a) = \partial x^a, \quad \tau^*(p_a) = (-1)^{\hat{a}} \partial x^*_a, \quad \tau^*(\pi_a) = x^*_a. \quad (49)$$

Then we obtain

$$\tau^*(\theta_{\Pi TM}) = -\partial \lambda_M = dx^a((-1)^{\hat{a}}\partial x^*_a) + d(\partial a^a) x^*_a. \quad (50)$$
From here we deduce that \( \tau \) is well-defined; we see that it commutes with the projections in the double vector bundle diagrams. Also, we can set

\[
\omega_{\Pi T(\Pi T^*M)} := d(-\partial \lambda_M) = \partial(d\lambda_M) = \partial \omega_{\Pi T^*M},
\]

as an even symplectic form, and we have \( \omega_{\Pi T(\Pi T^*M)} = \tau^*(\omega_{\Pi T^*M}) \).

**Corollary 3.** For a Lagrangian submanifold \( P \subset \Pi T^*M \) with respect to the odd symplectic form \( \omega_{\Pi T^*M} \), its antitangent bundle \( \Pi TP \subset \Pi T(\Pi T^*M) \) will be a Lagrangian submanifold of \( \Pi T(\Pi T^*M) \) with respect to the even symplectic form \( \omega_{\Pi T(\Pi T^*M)} \) given by \([51]\).

**Proof.** Indeed, if \( \omega_{\Pi T^*M} \) vanishes on \( P \), then \( \omega_{\Pi T(\Pi T^*M)} = \partial \omega_{\Pi T^*M} \) vanishes on \( \Pi TP \).

**Theorem 6.** For any supermanifold \( M \), there is a natural diffeomorphism

\[
\tau: \Pi T(T^*M) \to \Pi T^*(\Pi T^*M),
\]

which is an isomorphism of double vector bundles:

\[
\begin{array}{ccc}
\Pi T(T^*M) & \longrightarrow & T^*M \\
\downarrow & & \downarrow \tau \\
\Pi T^*(\Pi T^*M) & \longrightarrow & T^*M \\
\downarrow & & \downarrow \\
\Pi TM & \longrightarrow & M \\
\end{array}
\]

and in particular endows \( \Pi T(T^*M) \) with a canonical odd symplectic form.

**Proof.** Again we have the canonical diffeomorphism \( \Pi T^*(\Pi T^*M) \cong \Pi T^*(T^*M) \) and raising indices with the help of the even symplectic form on \( T^*M \) gives \( \Pi T^*(T^*M) \cong \Pi T(T^*M) \), which together yield \( \Pi T^*(\Pi T^*M) \cong \Pi T(T^*M) \). An explicit identification can be obtained as follows. The double vector bundle structures are given in local coordinates by

\[
\begin{array}{cccc}
(x^a, p_a; \partial x^a, \partial p_a) & \longrightarrow & (x^a, p_a) \\
\downarrow & & \downarrow \\
(x^a, \partial x^a) & \longrightarrow & (x^a) \\
\downarrow & & \downarrow \\
(x^a) & \longrightarrow & (x^a).
\end{array}
\]

(here \( \partial x^a \) should be understood as \( (\partial x)^a_\lambda \)). Consider now on \( \Pi T(T^*M) \) the even 1-form

\[
\partial \lambda_M = \partial(p_a dx^a) = \partial p_a dx^a + (-1)^{\hat{a}+1} p_a d(\partial x^a) = dx^a((-1)^\hat{a} \partial p_a) + d(\partial x^a)(-p_a),
\]

and on \( \Pi T^*(\Pi T^*M) \), the canonical 1-form

\[
\lambda_{\Pi TM} = dx^a x^a + d(\partial x^a) \partial x^a_\lambda.
\]

Define the desired diffeomorphism \( \tau: \Pi T(T^*M) \to \Pi T^*(\Pi T^*M) \) by

\[
\tau^*(x^a) = x^a, \quad \tau^*(\partial x^a) = \partial x^a, \quad \tau^*(x^a_\lambda) = (-1)^{\hat{a}+1} \partial p_a, \quad \tau^*(\partial x^a_\lambda) = p_a.
\]

Then we obtain

\[
\tau^*(\lambda_{\Pi TM}) = dx^a((-1)^{\hat{a}+1} \partial p_a) + d(\partial x^a) p_a = -\partial(\lambda_M).
\]

Hence \( \tau \) is well-defined and it is a double vector bundle morphism. Also, if we set

\[
\omega_{\Pi T(T^*M)} := d(-\partial \lambda_M) = \partial(d\lambda_M) = \partial \omega_{\Pi T^*M},
\]

it will be an odd symplectic form on \( \Pi T(T^*M) \), and \( \tau^*(\omega_{\Pi T^*(\Pi T^*M)}) = \omega_{\Pi T(T^*M)} \).
Corollary 4. For a Lagrangian submanifold $P \subset T^* M$ with respect to the even symplectic form $\omega_{T^* M}$, its antitangent bundle $\Pi TP \subset \Pi T(T^* M)$ will be a Lagrangian submanifold with respect to the odd symplectic form $\omega_{\Pi T(T^* M)}$.  

Proof. Indeed, if $\omega_{T^* M}$ vanishes on $P$, then $\omega_{\Pi T(T^* M)} = \partial \omega_{T^* M}$ vanishes on $\Pi TP$. □  

3. The tangent and antitangent morphisms for a thick morphism  

This is the main section; here we construct lifting of thick morphisms (even or odd) to tangent and antitangent bundles in a way generalizing the usual tangent map for ordinary smooth maps. Crucial role will be played by Theorems 3, 4, 5 and 6 of the previous section. There are some unexpected features making it different from ordinary maps: firstly, the tangent or antitangent thick morphisms obtained do not ‘fiber’ over the original thick morphism, rather over its core map (and there is a need for some technical assumption); secondly, the antitangent functor swaps even and odd thick morphisms (i.e., the antitangent to even will be odd, and vice versa). The whole notion of a ‘fiberwise thick morphism’ of vector bundles with different bases has to be clarified from scratch, which we do in a special subsection 3.3.  

3.1. The tangent functor $T$.  

3.1.1. Case of even thick morphisms. Our task is to define lifting of a thick morphism between manifolds to the tangent bundles. There are two kinds of thick morphisms, ‘even’ and ‘odd’ (suitable for pulling back even and odd functions, respectively). We shall start from the even case.  

Recall that a thick morphism $\Phi: M_1 \to M_2$ is described by an even generating function $S = S(x; q)$ depending on position variables $x^a$ on the source manifold and momentum variables $q_i$ on the target manifold. With respect to the momentum variables, it is a formal power series. To a thick morphism there corresponds the formal canonical relation specified by a generating function $\tilde{S}$, which the Lagrangian submanifold of the product $T^* M_2 \times (\tilde{T}^* M_1)$ given by the equation
\[
d y^i q_i - d x^a p_a = d(y^i q_i - \tilde{S}).
\] (60)

That means that
\[
y^i = (-1)^i \frac{\partial \tilde{S}}{\partial q_i}(x; q), \quad p_a = \frac{\partial \tilde{S}}{\partial x^a}(x; q).
\] (61)

We use the same letter $\Phi$ for this relation, but one should remember that the generating function contains more information than the Lagrangian submanifold (defined by $d\tilde{S}$, while the function $S$ includes a ‘constant of integration’). The relation $\Phi \subset T^* M_2 \times (\tilde{T}^* M_1)$ is formal because $S$ is a formal power series in $q_i$. One should think that we work in the formal neighborhood of the zero section in $T^* M_2$.  

In order to extend the tangent functor to thick morphisms from ordinary smooth maps, we shall first analyze the tangent map for an ordinary map $\varphi: M_1 \to M_2$ in the language of relations and generating functions, and use this as a model. A map $\varphi: M_1 \to M_2$ corresponds to a canonical relation that we shall denote $R_{\varphi}$. $R_{\varphi} \subset T^* M_2 \times (\tilde{T}^* M_1)$, specified by the generating function $S = \varphi^i(x) q_i$, if $y^i = \varphi^i(x)$ is the coordinate expression of $\varphi$. With an abuse of language we shall refer to $S$ as the generating function of $\varphi$. The tangent map $T\varphi: TM_1 \to TM_2$ is given by $y^i = \varphi^i(x), \dot{y}^i = \dot{x}^a \frac{\partial \varphi^i}{\partial x^a}(x)$. To
write down the generating function for $T\varphi$, use Theorem 3 and the diffeomorphism $\tau$ for the identification $T^*(TM) = T(T^*M)$. We can consider $T(T^*M_1)$ and $T(T^*M_2)$ as the cotangent bundles for $M_1$ and $M_2$, so that $p_a, p_a$ and $q_i, q_i$ are regarded as the conjugate momenta for $x^a, \dot{x}^a$ and $y^i, \dot{y}^i$, respectively.

**Proposition 2.** For a smooth map $\varphi: M_1 \to M_2$, the canonical relation $R_{T\varphi}$ corresponding to the tangent map $T\varphi: TM_1 \to TM_2$ is the tangent bundle $TR_{T\varphi}$ for the canonical relation $R_{T\varphi}$ corresponding to $\varphi$. Here $TR_{T\varphi}$ is regarded as a Lagrangian submanifold of $T(T^*M_2) \times (-T(T^*M_1))$. The generating function for $T\varphi$ is the symbolic time derivative of the generating function for $\varphi$,

$$\dot{S} = \dot{S}(x, \dot{x}; \dot{q}, q) = (\varphi^i(x)q_i).$$

**Proof.** The generating function for $T\varphi$ is

$$S_{T\varphi}(x, \dot{x}; \dot{q}, q) = \varphi^i(x) \dot{q}_i + \dot{x}^a \frac{\partial \varphi^i}{\partial x^a}(x)q_i = (\varphi^i(x)q_i),$$

as claimed. The Lagrangian submanifold in $T(T^*M_2) \times (-T(T^*M_1))$ it specifies is the tangent bundle $TR_{T\varphi}$ of the submanifold $R_{T\varphi} \subset T^*M_2 \times (-T^*M_1)$. □

(From Corollary 1 we already know that the tangent bundle to a Lagrangian submanifold in $T^*M$ will be a Lagrangian submanifold in $T(T^*M)$.)

This motivates the following “tautological” definition.

**Definition 1.** For a thick morphism $\Phi: M_1 \rightrightarrows M_2$ specified by a generating function $S = S(x; q)$, the tangent morphism is a thick morphism $T\Phi: TM_1 \rightrightarrows TM_2$ specified as the generating function by the formal ‘time derivative’ $\dot{S}$ of the function $S$,

$$\dot{S}(x, \dot{x}; \dot{q}, q) = \dot{x}^a \frac{\partial S}{\partial x^a}(x; q) + \dot{q}_i \frac{\partial S}{\partial q_i}(x; q).$$

(63)

Recall that with an abuse of notation we use the same symbol for a thick morphism and the corresponding relation (as a submanifold of the product of cotangent bundles). The following proposition ensures that this won’t lead us into a problem for tangent morphisms.

**Proposition 3.** The canonical relation corresponding the tangent thick morphism $T\Phi$, as a submanifold in $T(T^*M_2) \times (-T(T^*M_1))$, is the tangent bundle to the submanifold in $T^*M_2 \times (-T^*M_1)$ corresponding to a thick morphism $\Phi$.

**Proof.** Let us check directly that the generating function $\dot{S}$ given by (63) specifies the tangent bundle $T\Phi$, if $S$ specifies $\Phi \subset T^*M_2 \times (-T^*M_1)$. We have, by the definition, equation (63). By differentiating it formally with respect to $t$, we obtain

$$d\dot{y}^i q_i - d\dot{x}^a p_a + dy^i q_i - dx^a p_a = d(y^i q_i) + d\dot{x}^a \frac{\partial S}{\partial x^a} - d\dot{q}_i \frac{\partial S}{\partial q_i} - dx^a \left( \frac{\partial S}{\partial x^a} \right),$$

which is equivalent to

$$-(1)^i d\dot{q}_i y^i - d\dot{x}^a p_a - (1)^i d\dot{q}_i y^i - dx^a p_a = -d\dot{x}^a \frac{\partial S}{\partial x^a} - d\dot{q}_i \frac{\partial S}{\partial q_i} - dx^a \left( \frac{\partial S}{\partial x^a} \right),$$

or

$$d\dot{x}^a \left( p_a - \frac{\partial S}{\partial x^a} \right) + d\dot{q}_i \left( (1)^i y^i - \frac{\partial S}{\partial q_i} \right) + dx^a \left( \dot{p}_a - \left( \frac{\partial S}{\partial x^a} \right) \right) + d\dot{q}_i \left( (1)^i y^i - \left( \frac{\partial S}{\partial q_i} \right) \right) = 0.$$
The first two terms reproduce the equations specifying $\Phi$, while the other two terms give their tangent prolongation. Altogether we have arrived at equations specifying $T\Phi$ as a submanifold in $T(T^*M_2) \times T(T^*M_1)$, as expected. But at the same time, the result of the formal differentiation of equation (64) with respect to $t$ can be re-arranged differently, giving

$$d\dot{y}^i q_i + dy^i \dot{q}_i - dx^a p_a - d\dot{x}^a p_a = d \left( \dot{y}^i q_i + y^i \dot{q}_i - \dot{S} \right).$$

Since $\dot{q}_i, q_i$ are the conjugate momenta for $y^i, \dot{y}^i$ and $p_a, \dot{p}_a$ are the conjugate momenta for $x^a, \dot{x}^a$, we see it is the equation of the Lagrangian submanifold in $T(T^*M_2) \times (-T(T^*M_1))$ with the generating function $\dot{S}(x, \dot{x}; \dot{q}, q)$. Hence the claim.

We shall see that the construction of the tangent thick morphism gives the expected properties such as functoriality (Theorem 9 below); however, the property of fiberwise takes an unexpected form (see Theorem 7 and Corollary 5 below). Also, in view of the non-linearity (in general) of the pullbacks by thick morphisms, it is not a priori obvious how a vector bundle structure can be respected. The answer is given by Theorem 8.

**Theorem 7.** Suppose in the expansion $S(x, q) = S^0(x) + \varphi^i(x)q_i + \ldots$ of the generating function of a thick morphism $\Phi: M_1 \longrightarrow M_2$ there is no zero-order term, i.e., $S(x, q) = \varphi^i(x)q_i + \ldots$, where the linear term defines a map $\varphi: M_1 \rightarrow M_2$. Then there is a commutative diagram

$$
\begin{array}{ccc}
TM_1 & \overset{T\Phi}{\longrightarrow} & TM_2 \\
\pi_1 \downarrow & & \downarrow \pi_2 \\
M_1 & \overset{\varphi}{\longrightarrow} & M_2.
\end{array}
$$

**Proof.** Consider $\Phi$ first without any extra assumptions, with the generating function $S(x, q)$ of the general form

$$S(x, q) = S^0(x) + \varphi^i(x)q_i + \frac{1}{2} S^{ij}(x)q_i q_j + \ldots$$

Then $T\Phi$ is represented by the generating function $\dot{S}(x, \dot{x}; \dot{q}, q)$ given by (63). Consider diagram (64). First find the composition $\pi_2 \circ T\Phi$. For that, we represent $\pi_2$ by its generating function $S_{\pi_2}(y, \dot{y}; q) = y^i \dot{q}_i$. Now, the generating function of the composition will be

$$S_{\pi_2 \circ T\Phi}(x, \dot{x}; q) = \dot{S}(x, \dot{x}; \dot{q}, q) + S_{\pi_2}(\overline{y}, \dot{y}; q) - \overline{y}^i \dot{q}_i - \overline{y}^j q_j,$$

where the variables $\overline{y}^i, \overline{q}_i, \dot{\overline{y}}^i, \dot{\overline{q}}_i$ are determined from the equations

$$\overline{y}^i = (-1)^i \frac{\partial \dot{S}}{\partial \dot{q}_i}(x, \dot{x}; \overline{q}, q),$$

$$\dot{\overline{y}}^i = (-1)^i \frac{\partial \dot{S}}{\partial q_i}(x, \dot{x}; \overline{q}, q),$$

$$\overline{q}_i = \frac{\partial S_{\pi_2}}{\partial \overline{y}^i}(\overline{y}, \overline{q}; q),$$

$$\dot{\overline{q}}_i = \frac{\partial S_{\pi_2}}{\partial \dot{y}^i}(\overline{y}, \dot{q}; q).$$
Taking into account the explicit expressions for $\dot{S}$ and $S_{\pi_2}$ we obtain

$$S_{\pi_2 \circ \Phi}(x, \dot{x}; q) = \dot{x}^a \frac{\partial S}{\partial x^a}(x; \overline{q}) + \overline{q}_i \frac{\partial S}{\partial q_i}(x; \overline{q}) + y^i q_i - y^i \overline{q}_i,$$

where

$$\overline{y}^i = (-1)^i \frac{\partial S}{\partial q_i}(x, \overline{q}), \quad \overline{q}_i = q_i, \quad \overline{q}_i = 0,$$

hence finally

$$S_{\pi_2 \circ \Phi}(x, \dot{x}; q) = \dot{x}^a \frac{\partial S}{\partial x^a}(x, 0) + q_i \frac{\partial S}{\partial q_i}(x, 0) = \dot{x}^a \partial_a S^0(x) + \varphi^i(x) q_i.$$

Under the assumption $S^0 = 0$, this will give $S_{\pi_2 \circ \Phi}(x, \dot{x}; q) = \varphi^i(x) q_i$. This coincides with the generating function for the composition $\varphi \circ \pi_1: TM_1 \to M_2$.

Hence for a thick morphism $\Phi: M_1 \to M_2$, its tangent $T\Phi: TM_1 \to TM_2$ is a morphism over the ordinary map $\varphi: M_1 \to M_2$, not over the thick morphism $\Phi$ itself, as one may wish to expect following too closely the case of usual maps.

One needs the assumption $S^0 = 0$, and unless otherwise stated, we assume that in the rest of the paper.

**Corollary 5.** Under the pullback by $T\Phi$, $(T\Phi)^*: C^\infty(TM_2) \to C^\infty(TM_1)$, the functions on the base $C^\infty(M_2) \subset C^\infty(TM_2)$ are mapped to functions on the base $C^\infty(M_1) \subset C^\infty(TM_1)$, and on them the restriction of the map $(T\Phi)^*$ is an algebra homomorphism.

**Proof.** Indeed, by the commutativity of $[63]$, on $C^\infty(M_2) \subset C^\infty(TM_2)$, the pullback $(T\Phi)^*$ coincides with the usual pullback $\varphi^*$.

Note that $(T\Phi)^*$ is in general non-linear; it will be instructive, however, to check its action on the fiberwise-linear functions on the tangent bundle $TM_2$. Together with the functions from $M_2$, they generate all other functions on $TM_2$ (more precisely, those that are fiberwise polynomial). To this end, we look closer at the expression $[63]$ for the generating function of $T\Phi$ and notice that it has a general form

$$\dot{S}(x, \dot{x}; \dot{q}, q) = \dot{x}^a \left( \partial_a \varphi^i(x) q_i + \frac{1}{2} \partial_a S^{ij}(x) q_j q_i + \ldots \right) + \left( \varphi^i(x) + S^{ij}(x) q_j + \frac{1}{2} S^{ijk}(x) q_j q_k + \ldots \right) \dot{q}_i \equiv \dot{x}^a E_a(x, q) + F^i(x, q) \dot{q}_i, \quad (65)$$

where we have introduced a notation for the coefficients. Recall that $\dot{q}_i$ are conjugate to $y^i$ and $q_i$ are conjugate to $\overline{y}^i$ in $T(T^*M_2)$. Loosely, the term $F^i(x, q) \dot{q}_i$ corresponds to a map of the bases ‘with corrections’ given by the higher order terms in $q_i$ and the term $\dot{x}^a E_a(x, q)$ corresponds to thick morphisms of the fibers over each $x$ (which are “fiberwise-linear”).

**Theorem 8.** Under $(T\Phi)^*: C^\infty(TM_2) \to C^\infty(TM_1)$, the fiberwise-linear functions on $TM_2$ are mapped to fiberwise-linear functions on $TM_1$ (in general, in a non-linear fashion).
**Proof.** To calculate $(T\Phi)^*$, we write $\dot{S}$ in the form (65), i.e. as $\dot{S} = \dot{x}^a E_a(x, q) + F^i(x, q)\dot{q}_i$. A function $g \in C^\infty(TM_2)$, $g = g(y, \dot{y})$, is mapped to the function $f = f(x, \dot{x})$ on $TM_1$ given by

$$f(x, \dot{x}) = g(y, \dot{y}) + \dot{S}(x, \dot{x}; \dot{q}, q) - \dot{y}^i\dot{q}_i - y^i\dot{q}_i,$$

where $\dot{y}^i, q_i, y^i, \dot{q}_i$ are found from

$$y^i = (-1)^i \frac{\partial \dot{S}}{\partial \dot{q}_i} \equiv F^i(x, q),$$

$$\dot{y}^i = (-1)^i \frac{\partial \dot{S}}{\partial q_i} \equiv (-1)^i \frac{\partial}{\partial q_i} (\dot{x}^a E_a(x, q) + F^j(x, q)\dot{q}_j),$$

$$q_i = \frac{\partial g}{\partial y^i}(y, \dot{y}),$$

$$\dot{q}_i = \frac{\partial g}{\partial \dot{y}^i}(y, \dot{y}).$$

This gives

$$f(x, \dot{x}) = g(y, \dot{y}) + \dot{x}^a E_a(x, q, \frac{\partial g}{\partial y^i}(y, \dot{y})) - \dot{y}^i \frac{\partial g}{\partial \dot{y}^i}(y, \dot{y}),$$

where $\dot{y}^i, y^i$ are found from (66) and (67) with $q$ and $\dot{q}$ substituted from (68) and (69). Now, in the particular case of a fiberwise-linear function on $TM_2$, we have $g = \dot{y}^i g_i(y)$. Then $q_i = g_i(q)$, the first and the last terms in (70) cancel, and we obtain

$$(T\Phi)^*[\dot{y}^i g_i(y)] = \dot{x}^a E_a(x, g_i(y))$$

where $y$ is determined from the equation

$$y^i = F^i(x, g_i(y))$$

(here we are putting indices inside the arguments for the clarity of notation). We see that indeed the pullback of a fiberwise-linear function will be again fiberwise-linear, but with some nonlinear transformation of the coefficients. \qed

**Theorem 9** (functoriality). For the composition of thick morphisms,

$$T(\Phi_1 \circ \Phi_2) = T\Phi_1 \circ T\Phi_2.$$  

**Proof.** Consider $\Phi_2 : M_1 \longrightarrow M_2$ and $\Phi_1 : M_2 \longrightarrow M_3$. Denote local coordinates on $M_3$ by $z^\alpha$ and the corresponding momenta by $r_{\mu}$. Then the generating function of the composition is given by (see [15])

$$S_{\Phi_1 \circ \Phi_2}(x, r) = S_{\Phi_2}(x, q) + S_{\Phi_1}(y, r) - y^i q_i,$$

where $y^i, q_i$ are determined from the equations

$$y^i = (-1)^i \frac{\partial S_{\Phi_2}}{\partial q_i}(x, q),$$

$$q_i = \frac{\partial S_{\Phi_1}}{\partial y^i}(y, r).$$

The generating function of the tangent thick morphism $T(\Phi_1 \circ \Phi_2)$ will be the formal time derivative of $S_{\Phi_1 \circ \Phi_2}$, hence given by the formal time derivative of the above, i.e.

$$S_{T(\Phi_1 \circ \Phi_2)} = \dot{S}_{\Phi_1 \circ \Phi_2} = (S_{\Phi_2}(x, q) + (S_{\Phi_1}(y, r))') - \dot{y}^i q_i - y^i \dot{q}_i =

S_{TF_2}(x, \dot{x}, \dot{q}, q) + S_{TF_1}(y, \dot{y}, \dot{r}, r) - \dot{y}^i q_i - y^i \dot{q}_i,$$
where \( y', q_i \) are determined from (73) and \( \dot{y}', \dot{q}_i \) are determined from

\[
\dot{y}' = (-1)^i \left( \frac{\partial S_{\Phi_2}}{\partial q_i}(x, q) \right), \\
\dot{q}_i = \left( \frac{\partial S_{\Phi_1}}{\partial y'}(y, r) \right).
\]

(75) (76)

On the other hand, the composition of the tangent morphisms \( T\Phi_1 \) and \( T\Phi_2 \) will be given by the generating function

\[
S_{T\Phi_1 \circ T\Phi_2} = S_{T\Phi_1}(y, \dot{y}, r) + S_{T\Phi_2}(x, \dot{x}, \dot{q}, q) - \dot{y}'q_i - y'\dot{q}_i,
\]

(77)

which is the same as (74), and where the variables \( y', q_i \) are now determined from the system

\[
y' = (-1)^i \frac{\partial}{\partial q_i} S_{T\Phi_2} \equiv (-1)^i \frac{\partial S_{\Phi_2}}{\partial q_i}(x, q),
\]

(78)

\[
\dot{y}' = (-1)^i \frac{\partial}{\partial q_i} S_{T\Phi_2} \equiv (-1)^i \left( \dot{x}_i \frac{\partial^2 S_{\Phi_2}}{\partial x^a \partial q_i}(x, q) + \dot{q}_j \frac{\partial^2 S_{\Phi_2}}{\partial q_j \partial q_i}(x, q) \right),
\]

(79)

\[
q_i = \frac{\partial}{\partial y'} S_{T\Phi_1} \equiv \frac{\partial S_{\Phi_1}}{\partial y'}(y, r),
\]

(80)

\[
\dot{q}_i = \frac{\partial}{\partial y'} S_{T\Phi_1} \equiv \dot{y}' \frac{\partial^2 S_{\Phi_1}}{\partial y^j \partial y'}(y, r) + \dot{r}_\mu \frac{\partial^2 S_{\Phi_1}}{\partial r_\mu \partial y'}(y, r).
\]

(81)

In this formulas, (78) and (80) are the same as equations (73). It remains to observe that (79) and (81) are equivalent to (75) and (76), respectively (basically, “time differentiation” commutes with partial derivatives).

\[\square\]

3.1.2. Case of odd thick morphisms. The above construction of the tangent functor carries over to odd thick morphisms. Because of that, we only give statements and suppress proofs.

Definition 2. For an odd thick morphism \( \Psi: M_1 \leftrightarrow M_2 \) specified by an odd generating function \( S = S_\Psi \),

\[
S = S(x; y^*),
\]

(82)

the tangent morphism \( T\Psi \) is an odd thick morphism \( T\Psi: TM_1 \leftrightarrow TM_2 \) specified as the generating function \( S_{T\Psi} \) by the formal time derivative \( \dot{S} \),

\[
S_{T\Psi}(x, \dot{x}; \dot{y}^*, y^*) = \dot{S}(x, \dot{x}; \dot{y}^*, y^*) = \dot{x}^a \frac{\partial S}{\partial x^a}(x; y^*) + \dot{y}_i^* \frac{\partial S}{\partial y_i^*}(x; y^*).
\]

(83)

Here we use the canonical isomorphism \( \Pi T^*(TM) \cong T(\Pi T^* M) \) given by Theorem \[1\] and also Corollary \[2\]. The corresponding Lagrangian submanifold is the tangent bundle to the Lagrangian submanifold \( \Psi \subset \Pi T^* M_2 \times (-\Pi T^* M_1) \). (Compare with Proposition \[3\]) Note that on \( T(\Pi T^* M_j), \dot{y}_i^* \) is canonically conjugate (with respect to the odd symplectic form) to \( y_i^* \), and \( y_i^* \) is canonically conjugate to \( \dot{y}_i \).

Analogs of Theorems \[4\] and \[5\] hold without surprises:

Theorem 10. Suppose, for an odd thick morphism \( \Psi: M_1 \leftrightarrow M_2 \), in the expansion \( S(x, y^*) = S^0(x) + \varphi'(x)y_i^* + \ldots \) of the generating function there is no zero-order term, i.e., \( S(x, y^*) = \ldots \)
\[ \varphi^i(x)y^*_i + \ldots, \text{ where the linear term defines a map } \varphi: M_1 \to M_2. \] Then there is a commutative diagram

\[
\begin{array}{ccc}
TM_1 & \overset{T\Psi}{\longrightarrow} & TM_2 \\
\downarrow \pi_1 & & \downarrow \pi_2 \\
M_1 & \overset{\varphi}{\longrightarrow} & M_2.
\end{array}
\] (84)

Theorem 11 (functoriality). For odd thick morphisms \( \Psi_2: M_1 \Rightarrow M_2 \) and \( \Psi_1: M_2 \Rightarrow M_3 \), the tangent to the composition is the composition of the tangents:

\[ T(\Psi_1 \circ \Phi_2) = T\Psi_1 \circ T\Psi_2. \] (85)

Analogs of Corollary 5 and Theorem 8 can be packed into one statement:

Theorem 12. Suppose an odd thick morphism \( \Psi: M_1 \Rightarrow M_2 \) is specified by an odd generating function of the form \( S(x, y^*) = \varphi^i(x)y^*_i + \ldots, i.e. \) without the zero-order term. Then under \( (T\Psi)^*: \Pi C^\infty(TM_2) \to \Pi C^\infty(TM_1) \) the (odd) functions on the base, \( \Pi C^\infty(M_2) \subset \Pi C^\infty(TM_2) \), are mapped to functions on the base, \( \Pi C^\infty(M_1) \subset \Pi C^\infty(TM_1) \), by the algebra homomorphism \( \varphi^* \), while the odd fiberwise-linear functions on \( TM_2 \) are mapped to odd fiberwise-linear functions on \( TM_1 \) (in general, non-linearly).

3.2. The antitangent functor \( \Pi T \). Although the idea is the same, here it is where we have some surprises. Namely, the functor \( \Pi T \) swaps even and odd thick morphisms.

Recall that, by Theorems 5 and 6, there are natural isomorphisms \( T^*(\Pi T M) \cong \Pi T (T^* M) \) and \( \Pi T^*(\Pi T M) \cong \Pi T(T^* M) \), and by Corollaries 3 and 4, the antitangent bundle to a Lagrangian submanifold in \( T^*M_2 \times (-T^*M_1) \) or \( \Pi T^*M_2 \times (-\Pi T^*M_1) \) will be a Lagrangian submanifold in

\[ \Pi T(T^*M_2) \times (-\Pi T(T^*M_1)) \cong \Pi T^*(\Pi T M_2) \times (-\Pi T^*(\Pi T M_1)) \]

and

\[ \Pi T(\Pi T^*M_2) \times (-\Pi T(\Pi T^*M_1)) \cong T^*(\Pi T M_2) \times (-T^*(\Pi T M_1)), \]

respectively. Hence, for an even thick morphism

\[ \Phi: M_1 \Rightarrow M_2, \]

the antitangent morphism \( \Pi T \Phi \) will be an odd thick morphism

\[ \Pi T \Phi: \Pi T M_1 \Rightarrow \Pi T M_2, \]

and for an odd thick morphism

\[ \Psi: M_1 \Rightarrow M_2, \]

the antitangent morphism \( \Pi T \Psi \) will be an even thick morphism

\[ \Pi T \Psi: \Pi T M_1 \Rightarrow \Pi T M_2. \]

The respective generating functions are obtained from the generating functions of \( \Phi \) and \( \Psi \) by the application of the odd operator \( \partial \).

More formally, we have the following definitions:
Definition 3. For an even thick morphism $\Phi: M_1 \rightarrow M_2$ specified by an even generating function $S = S(x; q)$, the antitangent morphism $\Pi T^e \Phi: \Pi T M_1 \rightarrow \Pi T M_2$ is an odd thick morphism specified by the odd generating function $S_{\Pi T^e \Phi} := \partial S$,

$$S_{\Pi T^e \Phi}(x, \partial x; \partial q, q) = \partial S(x, \partial x; \partial q, q) = \partial x^a \frac{\partial S}{\partial x^a}(x; q) + \partial q_i \frac{\partial S}{\partial q_i}(x; q).$$  (86)

Definition 4. For an odd thick morphism $\Psi: M_1 \Rightarrow M_2$ specified by an odd generating function $S = S(x; y^*)$, the antitangent morphism $\Pi T^o \Psi: \Pi T M_1 \rightarrow \Pi T M_2$ is an even thick morphism specified by the even generating function $S_{\Pi T^o \Psi} := \partial S$,

$$S_{\Pi T^o \Psi}(x, \partial x; \partial y^*, y^*) = \partial S(x, \partial x; \partial y^*, y^*) = \partial x^a \frac{\partial S}{\partial x^a}(x; y^*) + \partial q_i \frac{\partial S}{\partial q_i}(x; y^*).$$  (87)

Recall from subsection 2.3 that on $\Pi T(T^*M_2)$, $\partial q_i$ are conjugate to $y_i$ and $q_i$ are conjugate to $\partial y_i$. Similarly, on $\Pi T(T^*M_2)$, $\partial y^*_i$ are conjugate to $y^*_i$ and $y^*_i$ are conjugate to $\partial y^*_i$.

Analogs of Theorems 7 and 10 hold (under the same assumptions on thick morphisms):

**Theorem 13.** Suppose, for an even thick morphism $\Phi: M_1 \rightarrow M_2$, in the expansion $S(x, q) = S^0(x) + \varphi^i(x)q_i + \ldots$ of the generating function there is no zero-order term, i.e., $S(x, q) = \varphi^i(x)q_i + \ldots$, where the linear term defines a map $\varphi: M_1 \rightarrow M_2$. Then the diagram

$$\Pi T M_1 \xrightarrow{T\Phi} \Pi T M_2$$

$$\pi_1 \downarrow \quad \downarrow \pi_2$$

$$M_1 \xrightarrow{\varphi} M_2.$$  (88)

is commutative.

**Theorem 14.** Suppose, for an odd thick morphism $\Psi: M_1 \Rightarrow M_2$, in the expansion $S(x, y^*) = S^0(x) + \varphi^i(x)y^*_i + \ldots$ of the generating function there is no zero-order term, i.e., $S(x, y^*) = \varphi^i(x)y^*_i + \ldots$, where the linear term defines a map $\varphi: M_1 \rightarrow M_2$. Then the diagram

$$\Pi T M_1 \xrightarrow{T\Psi} \Pi T M_2$$

$$\pi_1 \downarrow \quad \downarrow \pi_2$$

$$M_1 \xrightarrow{\varphi} M_2.$$  (89)

is commutative.

Both theorems are proved similarly to Theorem 7 above.

Note that exactly because in the commutative diagrams (88) and (89) (as well as in (64) and (84)), the bottom arrow is the ordinary map $\varphi$ (the core of given thick morphisms) and not the given even or odd thick morphism itself, the diagrams (88) and (89) make sense. Otherwise swapping of even and odd thick morphisms by $\Pi T$ would be a problem.

Similarly to the tangent morphisms $T\Phi$ and $T\Psi$ considered above, the antitangent morphisms $\Pi T^e \Phi$ and $\Pi T^o \Psi$ share the property of being “thick morphisms of vector bundles” (see discussion of a general concept in the next subsection). Namely, we have the following analogs of Corollary 3, Theorem 8, and Theorem 12.
Theorem 15. Suppose an even thick morphism $\Phi: M_1 \leftrightarrow M_2$ is specified by an even generating function $S(x, y^*) = \varphi^i(x)y^*_i + \ldots$ without zero-order term. Then under the pullback $(\Pi T\Phi)^*: \Pi C^\infty(\Pi TM_2) \to \Pi C^\infty(\Pi TM_1)$, the functions from $\Pi C^\infty(M_2)$ are mapped to functions from $\Pi C^\infty(M_1)$ by the algebra homomorphism $\varphi^*$, and the (odd) fiberwise-linear functions on $\Pi TM_2$ are mapped to (odd) fiberwise-linear functions on $\Pi TM_1$ (in general, non-linearly).

Theorem 16. Suppose an odd thick morphism $\Psi: M_1 \leftrightarrow M_2$ is specified by an odd generating function $S(x, y^*) = \varphi^i(x)y^*_i + \ldots$ without zero-order term. Then under the pullback $(\Pi T\Psi)^*: C^\infty(\Pi TM_2) \to C^\infty(\Pi TM_1)$ the functions from $C^\infty(M_2)$ are mapped to functions from $C^\infty(M_1)$ by the algebra homomorphism $\varphi^*$, and the (even) fiberwise-linear functions on $\Pi TM_2$ are mapped to (even) fiberwise-linear functions on $\Pi TM_1$ (in general, non-linearly).

The proofs go along the same lines as for Corollary 5 and Theorem 8. See also in the next subsection.

Finally, we have the functoriality as expected:

**Theorem 17.** For even thick morphisms $\Phi_2: M_1 \leftrightarrow M_2$ and $\Phi_1: M_2 \leftrightarrow M_3$,

$$\Pi T(\Phi_1 \circ \Phi_2) = \Pi T\Phi_1 \circ \Pi T\Phi_2.$$  \hspace{1cm} (90)

Both sides of (90) are odd thick morphisms $\Pi TM_1 \leftrightarrow \Pi TM_3$.

**Theorem 18.** For odd thick morphisms $\Psi_2: M_1 \leftrightarrow M_2$ and $\Psi_1: M_2 \leftrightarrow M_3$,

$$\Pi T(\Psi_1 \circ \Psi_2) = \Pi T\Psi_1 \circ \Pi T\Psi_2.$$ \hspace{1cm} (91)

Both sides of (91) are even thick morphisms $\Pi TM_1 \leftrightarrow \Pi TM_3$.

Proofs of Theorem 17 and Theorem 18 are similar to that of Theorem 9.

### 3.3. Thick morphisms of vector bundles

In this subsection, we somewhat digress from our main subject and introduce a general notion of a “vector bundle thick morphism” modelling it on the particular examples we have encountered in this paper. More detailed analysis is given in our joint paper in preparation [2].

In the classical setup, the tangent map (for a given smooth map) is a morphism of vector bundles over the given map of bases. As we have seen above, the analog of that in microformal geometry is not straightforward. If we want to generalize, we observe, taking into account that thick morphisms are not maps of sets, that it is not so obvious what fiberwise should mean for thick morphisms of fiber bundles. Even in the example for the tangent morphism, where the bases are related by an ordinary map, we do not arrive at what one would imagine, i.e. a family of thick morphisms of the fibers of the form $E_{1x} \leftrightarrow E_{2\varphi(x)}$. So much for a “fiberwise” condition; the next question is how to approach fiberwise linearity in the case of vector bundles. The same example of tangent morphisms shows that it is possible to have linearity respected in a certain particular sense, in spite of the corresponding pullback’s being non-linear.

Below we put forward a definition that may look arbitrary, but, as we show, possesses desired properties (maybe in a modified form). We consider the case of even thick morphisms. The case of odd thick morphisms is similar.
Let $E_1$ and $E_2$ be vector bundles over bases $M_1$ and $M_2$. (As usual, everything can be in the category of supermanifolds.) Denote local coordinates on $M_1$ and $M_2$ by $x^a$ and $y^i$, respectively. Denote linear coordinates in the fibers of $E_1$ and $E_2$, by $u^α$ and $w^µ$. The corresponding conjugate momenta on $T^*E_1$ and $T^*E_2$ will be denoted by $p_a$, $p_α$, $q_i$, and $q_µ$.

**Definition 5.** A thick morphism $Φ: E_1 → E_2$ is fiberwise-linear if it is specified by a generating function of the form

$$S(x^a, u^α; q_i, q_µ) = S^i(x^a, q_µ)q_i + u^αS_α(x^a, q_µ),$$

where $S_α(x^a, 0) = 0$.

The coefficients $S^i(x^a, q_µ)$ and $S_α(x^a, q_µ)$ are formal power series in $q_µ$:

$$S^i(x^a, q_µ) = S^i(x) + S^i_µ(x)q_µ + \frac{1}{2} S^{iµν}(x)q_µq_ν + \ldots$$

and

$$S_α(x^a, q_µ) = S^µ_α(x)q_µ + \frac{1}{2} S^{µν}_α(x)q_µq_ν + \ldots$$

**Example 1.** Set

$$S(x^a, u^α; q_i, q_µ) = \varphi^i(x)q_i + u^αΦ^i_α(x)q_µ,$$

so $S^i(x^a, q_µ) = S^i(x) ≡ \varphi^i(x)$ and $S_α(x^a, q_µ) = S^µ_α(x)q_µ ≡ Φ^µ_α(x)q_µ$. This corresponds to an ordinary fiberwise map $Φ: E_1 → E_2$ linear on each fiber, over a map $φ: M_1 → M_2$, given by the formulas $y^i = \varphi^i(x)$ and $w^µ = u^αΦ^µ_α(x)$.

A general fiberwise-linear thick morphism given by (92) can be seen a ‘thickening’ of a usual fiberwise-linear map of vector bundles $E_1 → E_2$ over a map $M_1 → M_2$, as in Example 1, where $φ^i(x) ≡ S^i(x)$ of (93) and $Φ^µ_α(x) ≡ S^µ_α(x)$ of (94).

**Example 2.** Various versions of the tangent morphisms considered above, $TΦ$, $TΨ$, $ΠTΦ$, and $ΠTΨ$, all have the generating functions of the form (92) or its odd analog. In particular, for $TΦ$ we had the generating function (63), i.e.

$$\dot{S}(x, x; q_µ) = F^i(x, q)q_i + \dot{x}^αE_α(x, q)$$

where

$$F^i(x, q) = \varphi^i(x) + S^{ij}(x)q_j + \frac{1}{2} S^{ijk}(x)q_kq_j + \ldots$$

and

$$E_α(x, q) = \partial_α\varphi^i(x)q_i + \frac{1}{2} \partial_αS^{ij}(x)q_µq_i + \ldots$$

Here $\dot{x}^α$ play the role of $u^α$ and $\dot{y}^i$ play the role of $w^µ$, and recall that here $q_i$ are conjugate to $y^i$, so play the role of $q_i$ in (92), while $q_µ$ are conjugate to $\dot{y}^i$, so play the role of $q_µ$.

**Example 3** (special case of Definition 5). Suppose a thick morphism $Φ: E_1 → E_2$ has a generating function of the form

$$S(x^a, u^α; q_i, q_µ) = \varphi^i(x)q_i + u^αS_α(x^a, q_µ).$$

The difference with the general case (92) is that in (97), $S^i(x^a, q_µ) ≡ \varphi^i(x)$ do not depend on the momentum variables $q_µ$. One can see that here the thick morphism $Φ$ reduces to

\[\text{\textsuperscript{3}}\text{For clarity though at the expense of beauty, we are putting here variables with indices as arguments.}\]
a family of thick morphisms \( \Phi_x: E_{1x} \rightarrow E_{2\varphi(x)} \) of the fibers, where \( \varphi: M_1 \rightarrow M_2 \) is a map specified by \( y^i = \varphi^i(x) \). (The precise meaning is that such is the action of \( \Phi \) on functions.)

Note that the tangent morphisms do not fall under Example 3.

First we need to know that the definition of fiberwise-linear morphisms does not depend on choices of coordinates and that such morphisms are composable.

**Theorem 19.** The class of fiberwise-linear thick morphisms of vector bundles is well-defined, i.e. generating functions of the form \( S_x \) are invariant under changes of coordinates in vector bundles. It is closed under composition of thick morphisms.

(Transformation law for generating functions of thick morphisms is given in \( \text{(18)} \).)

We omit the proof of Theorem 19 (see \( \text{(2)} \)), but we will give a proof of the following theorem.

**Theorem 20.** Let \( \Phi: E_1 \rightarrow E_2 \) be a fiberwise-linear thick morphism of vector bundles with a generating function \( S_x \). Then:

1. the diagram

\[
\begin{array}{ccc}
E_1 & \xrightarrow{\Phi} & E_2 \\
\pi_1 \downarrow & & \downarrow \pi_2 \\
M_1 & \xrightarrow{\varphi_0} & M_2
\end{array}
\]  

(98)

is commutative, where the \( \varphi_0: M_1 \rightarrow M_2 \) is given by \( y^i = \varphi^i(x) \equiv S^i(x) \) from \( \text{(93)} \); and

2. for a function of the form \( g(y, w) = g_0(y) + w^\mu g_\mu(y) \) on \( E_2 \) ("fiberwise-affine"), its pullback by \( \Phi \) will be a function on \( E_1 \) of the same form:

\[
\Phi^*[g] = f_0(x) + u^a f_\alpha(x) = g_0(y) + u^\alpha S_\alpha(x; g_\mu(y)),
\]

(99)

where in \( \text{(99)} \), \( y \) is the function of \( x \) found from the equations

\[
y^i = S^i(x; g_\mu(y)).
\]

(100)

In particular, for functions on the base, \( g = g(y) \), there will be \( \Phi^*[g] = \varphi^*(g) \); and for fiberwise-linear functions, \( g = w^\mu g_\mu(y) \), there will be \( \Phi^*[g] = u^\alpha S_\alpha(x; g_\mu(y)) \), where \( y \) is found from \( \text{(100)} \).

**Proof.** To prove part (1) of the theorem, consider the generating functions for all the arrows in diagram \( \text{(18)} \). For the top, we have \( \text{(92)} \). For the bottom, it is \( S_{\varphi_0}(x; q_i) = \varphi^i(x)q_i \). For the vertical arrows, we have \( S_{\pi_1}(x^a; u^\alpha; \bar{q}_a) = x^a \bar{p}_a \) and \( S_{\pi_2}(y^i; w^\mu; \bar{q}_i) = y^i \bar{q}_i \). Therefore, we obtain the following for the compositions:

\[
S_{\pi_2 \circ \Phi}(x^a; u^\alpha; q_i) = S(x^a; u^\alpha; q_i \equiv \bar{q}_i \equiv \bar{q}_\mu) + S_{\pi_2}(\bar{y}^i, \bar{w}^\mu; q_i) - \bar{y}^i \bar{q}_i - \bar{w}^\mu \bar{q}_\mu = S^i(x^a, \bar{q}_\mu) \bar{q}_i + u^\alpha S_\alpha(x^a, \bar{q}_\mu) + \bar{y}^i \bar{q}_i - \bar{y}^i \bar{q}_i - \bar{w}^\mu \bar{q}_\mu,
\]

\[
\frac{\partial}{\partial x^a} S = \frac{\partial}{\partial x^a} \left( f_0(x) + u^a f_\alpha(x) \right) = \frac{\partial}{\partial x^a} f_0(x) + \frac{\partial}{\partial x^a} u^a f_\alpha(x).
\]
where \( \bar{q}_i, \bar{q}_\mu, \bar{y}^i, \) and \( \bar{\omega}^\mu \) are determined from the equations

\[
\bar{q}_i = \frac{\partial}{\partial y^i} S_{\pi_2}(\bar{y}^i, \bar{\omega}^\mu; q_i) \equiv q_i,
\]

\[
\bar{q}_\mu = \frac{\partial}{\partial \bar{\omega}^\mu} S_{\pi_2}(\bar{y}^i, \bar{\omega}^\mu; q_i) \equiv 0,
\]

\[
\bar{y}^i = (-1)^i \frac{\partial}{\partial \bar{q}_i} S(x^a, u^\alpha; \bar{q}_i, \bar{q}_\mu) \equiv S^i(x^a, \bar{q}_\mu),
\]

\[
\bar{\omega}^\mu = (-1)^\mu \frac{\partial}{\partial \bar{q}_\mu} S(x^a, u^\alpha; \bar{q}_i, \bar{q}_\mu),
\]

hence we have

\[
S_{\pi_2 \Phi}(x^a, u^\alpha; q_i) = S^i(x^a, 0) q_i + u^\alpha S_a(x^a, 0) = S^i(x) q_i + 0 = \varphi^i(x) q_i,
\]

from (92) and (93), and similarly

\[
S_{\varphi_0 \pi_1}(x^a, u^\alpha; q_i) = S_{\pi_1}(x^a, u^\alpha; \bar{p}_a) + S_{\varphi_0}(x^a, q_i) - x^a \bar{p}_a = x^a \bar{p}_a + \varphi^i(x) q_i - x^a \bar{p}_a,
\]

where \( \bar{x}^a \) and \( \bar{p}_a \) are determined from

\[
\bar{x}^a = (-1)^i \frac{\partial}{\partial x^a} S_{\pi_1}(x^a, u^\alpha, \bar{p}_a) \equiv x^a,
\]

\[
\bar{p}_a = \frac{\partial}{\partial \bar{x}^a} S_{\varphi_0}(\bar{x}^a; q_i),
\]

hence

\[
S_{\varphi_0 \pi_1}(x^a, u^\alpha; q_i) = \varphi^i(x) q_i \equiv S_{\pi_2 \Phi}(x^a, u^\alpha; q_i),
\]

which proves the commutativity of (98). (Note that we substantially relied on the condition \( S_a(x^a, 0) = 0 \), which is part of Definition 5; otherwise in the final formula for \( S_{\pi_2 \Phi}(x^a, u^\alpha; q_i) \) there would be an extra term obstructing the commutativity.)

To prove part (2), consider an even function on \( E_2 \) of the form \( g(y^i, u^\mu) = g_0(y) + u^\mu g_\mu(y) \).

Its fullback by \( \Phi \) will be the function \( f(x^a, u^\alpha) \) on \( E_1 \) given by

\[
f(x^a, u^\alpha) = S(x^a, u^\alpha; \bar{q}_i, \bar{q}_\mu) + g(\bar{y}^i, \bar{\omega}^\mu) - \bar{y}^i \bar{q}_i - \bar{\omega}^\mu \bar{q}_\mu =
\]

\[
S^i(x^a, \bar{q}_i) q_i + u^\alpha S_a(x^a, \bar{q}_\mu) + g_0(\bar{y}) + \bar{\omega}^\mu g_\mu(\bar{y}) - \bar{y}^i \bar{q}_i - \bar{\omega}^\mu \bar{q}_\mu,
\]

where \( \bar{q}_i, \bar{q}_\mu, \bar{y}^i, \) and \( \bar{\omega}^\mu \) are determined from

\[
\bar{q}_i = \frac{\partial}{\partial y^i} g(\bar{y}^i, \bar{\omega}^\mu),
\]

\[
\bar{q}_\mu = \frac{\partial}{\partial \bar{\omega}^\mu} g(\bar{y}^i, \bar{\omega}^\mu) \equiv g_\mu(\bar{y}),
\]

\[
\bar{y}^i = (-1)^i \frac{\partial}{\partial \bar{q}_i} S(x^a, u^\alpha; \bar{q}_i, \bar{q}_\mu) \equiv S^i(x^a, \bar{q}_\mu),
\]

\[
\bar{\omega}^\mu = (-1)^\mu \frac{\partial}{\partial \bar{q}_\mu} S(x^a, u^\alpha; \bar{q}_i, \bar{q}_\mu),
\]

hence we obtain that, for our \( g_0 \),

\[
\Phi^*[g] = f(x, u) = u^\alpha S_a(x^a, q_\mu) + g_0(y)
\]
(we were able to get rid of bars over the variables, now redundant) where \( q_\mu = g_\mu(y) \) and \( y \) is determined from the equations \( y^i = S^i(x; g_\mu(y)) \). This is what is claimed in (99) and (100). In particular, when \( g_\mu(y) \equiv 0 \), i.e. we have a function on the base \( g = g_0(y) \), then \( y \) is obtained explicitly as \( y = S^i(x, 0) = \varphi^i(x) \), hence \( \Phi^*[g] = \Phi^*[g_0] \) is the ordinary pullback \( \varphi^*_0(g_0) \). (This also follows from part (1).) On the other hand, when \( g_0(x) \equiv 0 \), so we have a fiberwise-linear function on \( E_2 \), \( g = w^\mu g_\mu(y) \), we arrive at the function

\[
\Phi[w^\mu g_\mu(y)] = u^\alpha f_\alpha(x) \equiv u^\alpha S_\alpha(x; g_\mu(y)) ,
\]

where \( y \) is given as the solution of \( y^i = S^i(x; g_\mu(y)) \), i.e. at a fiberwise-linear function on \( E_1 \), as claimed (whose coefficients \( f_\alpha \) depend non-linearly on the coefficients \( g_\mu \) of \( g \)).

Theorem 20 justifies the name “fiberwise-linear” for the class of thick morphisms of vector bundles introduced in Definition 5: the corresponding pullbacks map functions on the base to functions and fiberwise-linear functions to fiberwise-linear functions. (The latter, in spite of the non-linearity of the pullbacks as such.)

**Remark 2.** If for a vector bundle \( E_1 \), we introduce weights, i.e. a \( \mathbb{Z}_\alpha \)-grading, by setting \( \mathbf{w}(x^a) = 0 \) for base coordinates and \( \mathbf{w}(u^\alpha) = +1 \) for fiber coordinates, and do the same for \( E_2 \), and consider the induced grading for the cotangent bundles, i.e. require that the weights of the canonically conjugate variables be opposite (e.g. \( \mathbf{w}(p_\alpha) = -\mathbf{w}(x^a) = 0 \) and \( \mathbf{w}(p_\alpha) = -\mathbf{w}(u^\alpha) = -1 \) ), then one can characterize generating functions of the form (92) by the condition

\[
(\mathbf{w} + \text{deg})(S) = +1 ,
\]

where \( \text{deg} = \#p_\alpha + \#p_\alpha + \#q_i + \#q_\mu \) is the degree in the momentum variables. Indeed, for such “total weight” \( \mathbf{w} + \text{deg} \) we have

\[
\mathbf{w} + \text{deg} = (\#u^\alpha - \#p_\alpha + \#u^\mu - \#q_\mu) + (\#p_\alpha + \#p_\alpha + \#q_i + \#q_\mu) = \#p_\alpha + \#u^\alpha + \#q_i + \#u^\mu ,
\]

and clearly a function \( S(x^a, u^\alpha; q_i, q_\mu) \) satisfies (101) if and only if has the form (92).

**Remark 3.** Above, we have considered the case of even fiberwise-linear thick morphisms of vector bundles. Odd fiberwise-linear thick morphisms are defined in the same way, and Theorems 19 and 20 hold mutatis mutandis.

### 4. Thick \( Q \)-morphisms. Consequences for forms and cohomology

In the previous section, we have established that any thick morphism of (super)manifolds, even \( M_1 \hookrightarrow M_2 \) or odd \( M_1 \hookrightarrow M_2 \), induces a thick morphism of the antitangent bundles, of the opposite parity: \( \Pi TM_1 \hookrightarrow \Pi TM_2 \) and \( \Pi TM_1 \hookrightarrow \Pi TM_2 \), respectively. Since functions on \( \Pi TM \) are the same as differential forms on \( M \) when \( M \) is an ordinary manifold and for a supermanifold, functions on \( \Pi TM \) are by definition the Bernstein–Leites pseudodifferential forms on \( M \). De Rham differential commutes with ordinary pullbacks of forms (induced by usual smooth maps). What about pullbacks by thick morphisms? In this section, we find the answer. The problem we have to deal with, is the non-linearity of pullbacks. It is useful to look at the case of general \( Q \)-manifolds.
4.1. Thick morphisms of $Q$-manifolds. Recall that a $Q$-manifold is a supermanifold equipped with a homological vector field, i.e. an odd vector field $Q$ satisfying $Q^2 = 0$. (This notion is due to A. S. Schwarz \[7\]. See also \[1\]) In particular, the algebra of functions on a $Q$-manifold is a chain complex. (Here by a “chain complex” we mean just a super vector space equipped with an odd differential.)

A morphism of $Q$-manifolds or, shortly, a $Q$-morphism, is a smooth supermanifold map that intertwines the homological vector fields. (As it is known, such are the supergeometric descriptions of the $L_\infty$-morphisms of $L_\infty$-algebras and the Lie algebroid morphisms.) We shall extend this notion to thick morphisms.

Suppose $M_1$ and $M_2$ are $Q$-manifolds with the homological vector fields $Q_1$ and $Q_2$. In coordinates, $Q_1 = Q^a(x)\partial/\partial x^a$ and $Q_1 = Q^i(y)\partial/\partial y^i$.

**Definition 6.** An even thick morphism $\Phi: M_1 \rightarrow M_2$ is a thick $Q$-morphism if the Hamiltonians corresponding to $Q_1$ and $Q_2$ are $\Phi$-related, i.e. are equal on the Lagrangian submanifold in $T^*M_2 \times (-T^*M_1)$ given by $\Phi$. In coordinates,

$$Q_1^a(x^n)\frac{\partial S}{\partial x^n} = Q_2^i((-1)^i)\frac{\partial S}{\partial q_i} q_i,$$

where $S = S(x^n; q_i)$ is the (even) generating function of $\Phi$.

Likewise, an odd thick morphism $\Psi: M_1 \rightarrow M_2$ is a thick $Q$-morphism if the multivector fields corresponding to $Q_1$ and $Q_2$ are $\Psi$-related, i.e. are equal on the Lagrangian submanifold in $\Pi T^*M_2 \times (-\Pi T^*M_1)$ given by $\Psi$. In coordinates,

$$Q_1^a(x^n)\frac{\partial S}{\partial x^n} = Q_2^i(\frac{\partial S}{\partial q_i}) y^i,$$

where $S = S(x^n; y^i)$ is the (odd) generating function of $\Psi$.

Definition 6 can be seen as a special case of the notions of $S_\infty$ and $P_\infty$-thick morphisms introduced in [14] if one notes that a $Q$-structure can be regarded both as an $S_\infty$-structure and a $P_\infty$-structure with a single odd unary bracket given by $Q$ as an operator on functions.

For an ordinary $Q$-morphism $\phi: M_1 \rightarrow M_2$, the pullback $\phi^*: C^\infty(M_2) \rightarrow C^\infty(M_1)$ is by the definition a chain map (w.r.t. the differentials $Q_1$ and $Q_2$). What about pullbacks by thick $Q$-morphisms? Anticipating the answer, since these pullbacks are in general non-linear, we need to explain what would be a “non-linear chain map” of complexes. Actually, this is nothing but an $L_\infty$-morphism of complexes, which can be elaborated as follows.

Let $V$ and $W$ be complexes, i.e. (super) vector spaces equipped with odd operators $d_1$ and $d_2$ of square zero. We can consider them as supermanifolds and the linear operators $d_1$ and $d_2$, as vector fields, which will be homological. By parity reversion, the same works for $\Pi V$ and $\Pi W$ (with the operators $d_1^H$ and $d_2^H$, which for simplicity of notation, we will denote simply by $d_1$ and $d_2$).

**Definition 7.** A *non-linear chain map* $f: V \rightarrow W$ is a formal map $f: V \rightarrow W$, with $V$ and $W$ regarded as supermanifolds\(^4\), which is a $Q$-morphism w.r.t. $d_1$ and $d_2$.

\(^1\)Q-manifolds are sometimes referred to as “DG-manifolds”, but the definition does not require a $\mathbb{Z}$-grading and interesting examples can have none or several, so we prefer the original terminology.

\(^2\)with zero as a base point.
In the same way we can speak about non-linear chain maps $f : \Pi V \to \Pi W$.
(With an abuse of language, we can refer to the corresponding formal supermanifold maps $f : V \to W$ or $f : \Pi V \to \Pi W$ also as non-linear chain maps.)

If one expands a non-linear chain map $f : V \to W$ into the Taylor series, its terms will give even symmetric multilinear maps of super vector spaces

$$f_k : V \times \ldots \times V \to W,$$

$k = 0, 1, 2, \ldots$ (equivalently, even linear maps $S^k V \to W$) satisfying the relations

$$d(f_0) = 0,$$

$$d(f_1(v)) = f_1(d(v)),$$

$$d(f_2(v_1, v_2)) = f_2(d(v_1), v_2) + (-1)^{\tilde{v}_1} f_2(v_1, d(v_2)),$$

$$d(f_3(v_1, v_2, v_3)) = f_3(d(v_1), v_2, v_3) + (-1)^{\tilde{v}_1} f_3(v_1, d(v_2), v_3) + (-1)^{\tilde{v}_1 + \tilde{v}_2} f_3(v_1, v_2, d(v_3)),$$

... where we have put $d$ for both $d_1$ and $d_2$; i.e. the element $f_0 \in W$ is closed, the linear map $f_1 : V \to W$ is a usual chain map, and the Leibniz rule is satisfied for each $f_k$, $k \geq 2$.

These are just the relations for an $L_\infty$-morphism specialized for our case (see e.g. [19]).

In the same way, a non-linear chain map $f : \Pi V \to \Pi W$ expands into the Taylor terms, equivalent to a sequence of antisymmetric multilinear maps from $V$ to $W$ of alternating parities (or linear maps $\Lambda^k V \to W$ of parities $k - 1$) satisfying relations similar to above.

The chain map $f_1 : V \to W$ induces as usual an even linear map $f_1^* : H(V, d_1) \to H(W, d_2)$.

Together with it, a standard argument based on the Leibniz rule applied to “higher” $f_k$ gives the following statement:

**Proposition 4.** A non-linear chain map $f : V \to W$ induces a sequence of even symmetric multilinear maps of cohomology

$$f_k^* : H(V, d_1) \times \ldots \times H(V, d_1) \to H(W, d_2),$$

$k = 0, 1, 2, \ldots$, which can be assembled into a single formal map

$$f^* : H(V, d_1) \to H(W, d_2),$$

(104)

of the cohomology spaces regarded as supermanifolds, where 0 is mapped to $[f_0] \in H(W, d_2)$.

Similar statement holds for non-linear chain maps $\Pi V \to \Pi W$.

Now we can return to thick $Q$-morphisms.

**Theorem 21.** Let $\Phi : M_1 \Rightarrow M_2$ be an even thick $Q$-morphism. Then the pullback

$$\Phi^* : C^\infty(M_2) \to C^\infty(M_1)$$

is a non-linear chain map.

Similarly, let $\Psi : M_1 \Leftrightarrow M_2$ be an odd thick $Q$-morphism. Then the pullback

$$\Psi^* : \Pi C^\infty(M_2) \to \Pi C^\infty(M_1)$$

is a non-linear chain map.
Proof. Consider the case of an even thick morphism \( \Phi: M_1 \Rightarrow M_2 \). (The odd case can be proved in the same way.) Suppose the generating function of \( \Phi \) satisfies (102). We need to show that the pullback \( \Phi^* \colon C^\infty(M_2) \rightarrow C^\infty(M_1) \) is a non-linear chain map with respect to the \( Q_2 \) and \( Q_1 \) as the differentials. According to Definition 7 this would mean the pullback \( \Phi^* \) intertwining the corresponding linear vector fields \( \hat{Q}_2 \in \text{Vect}(C^\infty(M_2)) \) and \( \hat{Q}_1 \in \text{Vect}(C^\infty(M_1)) \) on the spaces of functions. This is a special case of the statement for \( S_\infty \)-structure (see Theorem 6 in [14] or Theorem 4.4 in [19]). For completeness, we provide the argument, which is as follows. We need to show that

\[
(id + \varepsilon \hat{Q}_1) \circ \Phi^* = \Phi^* \circ (id + \varepsilon \hat{Q}_2)
\]

where \( \varepsilon^2 = 0 \). Note that, for \( Q_1 \), \( (id + \varepsilon \hat{Q}_1)f(x) = f(x) + \varepsilon Q^a(x) \frac{\partial f}{\partial x^a}(x) \), and the same for \( Q_2 \). So if we start from some \( g \in C^\infty(M_2) \), we have \( \Phi^*[g] = f \), where

\[
f(x) = S(x, q) + g(y) - y^i q_i.
\]

and \( y \) and \( q \) are defined from the equations

\[
q_i = \frac{\partial y}{\partial y}(y),
\]

\[
y^i = (-1)^i \frac{\partial S}{\partial q_i}(x, q).
\]

We have

\[
\frac{\partial f}{\partial x^a}(x) = \frac{\partial S}{\partial x^a} + \frac{\partial q_i}{\partial x^a} \frac{\partial S}{\partial q_i} + \frac{\partial y^i}{\partial x^a} \frac{\partial q_i}{\partial y^i} - \frac{\partial y^i}{\partial x^a} q_i - (-1)^i \frac{\partial q_i}{\partial x^a} y^i = \frac{\partial S}{\partial x^a}(x, \frac{\partial g}{\partial y}(y)),
\]

where \( y \) is found from

\[
y^i = (-1)^i \frac{\partial S}{\partial q_i}(x, \frac{\partial g}{\partial y}(y)).
\]

Hence

\[
((id + \varepsilon \hat{Q}_1) \circ \Phi^*)[g] = f(x) + \varepsilon Q^a(x) \frac{\partial S}{\partial x^a}(x, \frac{\partial g}{\partial y}(y)).
\]

On the other hand, we know that the derivative of \( \Phi^* \) at \( g \in C^\infty(M_2) \) is the usual pullback \( \varphi_g^* \), i.e. the substitution \( y = y(x) \) defined by (106) (by Theorem 2 in [14]). Hence

\[
(\Phi^* \circ (id + \varepsilon \hat{Q}_2))[g] = \Phi^*[g + \varepsilon \hat{Q}_2 g] = \Phi^*[g] + \varepsilon \varphi_g^*(\hat{Q}_2 g) = f(x) + \varepsilon Q^i(y) \frac{\partial g}{\partial y^i}(y),
\]

where \( y \) is determined from (106). From (102) and (106), we obtain

\[
Q^a(x) \frac{\partial S}{\partial x^a}(x, \frac{\partial g}{\partial y}(y)) = Q^i(y) \frac{\partial g}{\partial y^i}(y),
\]

hence the right-hand sides of (107) and (108) coincide, and we have the equality (105) as claimed.

\[\square\]

Corollary 6. An even thick \( Q \)-morphism \( \Phi: M_1 \Rightarrow M_2 \) induces a non-linear formal map on cohomology

\[
\Phi^* : H(C^\infty(M_2), Q_2) \rightarrow H(C^\infty(M_1), Q_1)
\]
regarded as supermanifolds. Similarly, an odd thick $Q$-morphism $\Phi: M_1 \leftrightarrow M_2$ induces a non-linear formal map on cohomology with reversed parity
\[
\Phi^*: \Pi H(C^\infty(M_2), Q_2) \to \Pi H(C^\infty(M_1), Q_1)
\]
regarded as supermanifolds.

4.2. **Action of thick morphisms on forms and de Rham cohomology.** Let us return to the setup of Section 3. Consider supermanifolds $M_1$ and $M_2$ and a thick morphism $\Phi: M_1 \hookrightarrow M_2$. (The case of odd thick morphisms $M_1 \leftrightarrow M_2$ can be treated similarly.)

From Section 3, we know that there is the induced antitangent odd thick morphism
\[
\Pi T\Phi: \Pi T M_1 \hookrightarrow \Pi T M_2
\]
and hence we have the non-linear pullback on odd forms
\[
\Phi^*: \Pi \Omega(M_2) \to \Pi \Omega(M_1)
\]
Here by “forms” we mean pseudodifferential forms and use the notation $\Omega(M) := C^\infty(\Pi T M)$.

**Theorem 22.** The antitangent morphism (111) is a thick $Q$-morphism (with respect to the de Rham differentials).

**Proof.** Let $S = S(x; q)$ be the generating function for $\Phi$. In the description of $\Pi T\Phi$ we use the identification $\Pi T\left(\Pi T M\right) \cong \Pi T(T^* M)$ established in Section 3. We change back the notation and use $d$ instead of $\partial$ because here we will not need forms on the antitangent bundles. In order to prove that $\Pi T\Phi$ is a thick $Q$-morphism, we need to show that multivector fields corresponding to $d$ on $M_1$ and $M_2$, i.e. $dx^a x^*_a$ and $dy^i y^*_i$, are equal on $\Pi T\Phi$ (see (103) for the abstract case). Taking into account formulas (57), we need to show that
\[
(-1)^{\tilde{a}+1} dp_a = dy^i (-1)^{\tilde{i}+1} dq_i,
\]
if we have the equations for $p_a$ and $y^i$:
\[
p_a = \frac{\partial S}{\partial x^a},
\]
\[
y^i = (-1)^{\tilde{i}} \frac{\partial S}{\partial q_i}
\]
and the corresponding equations for $dp_a$ and $dy^i$ obtained by applying $d$:
\[
dp_a = dx^a \frac{\partial^2 S}{\partial x^b \partial x^a} + dq_i \frac{\partial^2 S}{\partial q_i \partial x^a},
\]
\[
dy^i = (-1)^{\tilde{i}} \left(dx^a \frac{\partial^2 S}{\partial x^a \partial q_i} + dq_j \frac{\partial^2 S}{\partial q_j \partial q_i}\right).
\]
Hence we have for the left-hand side of (113):
\[
(-1)^{\tilde{a}+1} dx^a dp_a = (-1)^{\tilde{a}+1} dx^a dx^b \frac{\partial^2 S}{\partial x^b \partial x^a} + (-1)^{\tilde{a}+1} dx^a dq_i \frac{\partial^2 S}{\partial q_i \partial x^a}
\]
(114)
and for the right-hand side of (113):

\[(−1)^{j+1}dy^i dq_i = −dx^a \frac{∂^2 S}{∂x^a ∂q_i} dq_i − dq_j \frac{∂^2 S}{∂q_j ∂q_i} dq_i.\]  

(115)

The terms with $dx^a dx^b$ and with $dq_i dq_j$ in (114) and (115) identically vanish and we can see that the remaining terms with $dx^a dq_i$ are equal. Hence (113) holds, and the theorem is proved. □

**Corollary 7.** The pullback on odd forms (112) is a non-linear chain map, inducing a non-linear formal map

$$\Psi^*: \Pi \mathcal{H}^*(M_2) \rightarrow \Pi \mathcal{H}^*(M_1).$$

(116)

Here we have used boldface to emphasize that we consider vector supermanifolds corresponding to the de Rham cohomology spaces. (Recall that the cohomology of pseudo-differential forms is isomorphic to the ordinary cohomology of the underlying topological space, see e.g. [11].)

Mutatis mutandis, the same holds for an odd thick morphism $\Psi: M_1 \Rightarrow M_2$. The anti-tangent morphism

$$\Pi T \Psi: \Pi T M_1 \Rightarrow \Pi T M_2$$

(117)

will be a thick $Q$-morphism, and hence the pullback on even forms,

$$\Psi^*: \Omega(M_2) \rightarrow \Omega(M_1),$$

(118)

will be a non-linear chain map, inducing a non-linear formal map of de Rham cohomology:

$$\Psi^*: \mathbf{H}^*(M_2) \rightarrow \mathbf{H}^*(M_1).$$

(119)

Further work is required to see if the higher terms of the maps (116) and (119) can be non-trivial, and explore significance of such non-linear cohomology transformations.

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