RSVD-CUR Decomposition
for Matrix Triplets

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Abstract
We propose a restricted SVD based CUR (RSVD-CUR) decomposition for matrix triplets
\((A, B, G)\). Given matrices \(A\), \(B\), and \(G\) of compatible dimensions, such a decomposition
provides a coordinated low-rank approximation of the three matrices using a subset of their
rows and columns. We pick the subset of rows and columns of the original matrices by ap-
plying either the discrete empirical interpolation method (DEIM) or the L-DEIM scheme on
the orthogonal and nonsingular matrices from the restricted singular value decomposition
of the matrix triplet. We investigate the connections between a DEIM type RSVD-CUR
approximation and a DEIM type CUR factorization, and a DEIM type generalized CUR
decomposition. We provide an error analysis that shows that the accuracy of the proposed
RSVD-CUR decomposition is within a factor of the approximation error of the restricted
singular value decomposition of given matrices. An RSVD-CUR factorization may be suit-
able for applications where we are interested in approximating one data matrix relative
to two other given matrices. Two applications that we discuss include multi-view/label
dimension reduction, and data perturbation problems of the form \(A_E = A + BFG\), where
\(BFG\) is a nonwhite noise matrix. In numerical experiments, we show the advantages of
the new method over the standard CUR approximation for these applications.

Keywords: Restricted SVD, low-rank approximation, CUR decomposition, interpolative
decomposition, DEIM, L-DEIM, subset selection, canonical correlation analysis, multi-view
learning, nonwhite noise, colored noise, data perturbations

1. Introduction
Identifying the underlying structure of a data matrix and extracting meaningful information
is a crucial problem in data analysis. Low-rank matrix approximation is one of the means
to achieve this. CUR factorizations and interpolative decompositions (ID) are appealing
techniques for low-rank matrix approximations, which approximate a data matrix in terms
of a subset of its columns and rows. These types of low-rank matrix factorizations have
several advantages over the ones based on orthonormal bases because they inherit properties
such as sparsity, nonnegativity, and interpretability of the original matrix. Various proposed
algorithms in the literature seek to find a representative subset of rows and or columns by
exploiting the properties of the singular vectors (Mahoney and Drineas, 2009; Sorensen and
Embree, 2016) or using a pivoted QR factorization (Voronin and Martinsson, 2017). Given
a matrix $A \in \mathbb{R}^{m \times n}$ and a target rank $k$, a rank-$k$ CUR factorization approximates $A$ as

$$A \approx C M R,$$

(1)

where $C$ and $R$ consists of $k$ columns and rows of $A$, respectively. The middle matrix $M$ can be computed as $(C^T C)^{-1} C^T A R^T (R R^T)^{-1}$; Stewart (1998) shows how this computation minimizes $\|A - CMR\|$ for specified row and column indices. Here, $\|\cdot\|$ denotes the 2-norm. To construct the factors $C$ and $R$, one can apply the discrete empirical interpolatory method (DEIM) proposed in (Chaturantabut and Sorensen, 2010) or the Q-DEIM proposed in (Drmac and Gugercin, 2016) to the leading $k$ right and left singular vectors of $A$. Gidisu and Hochstenbach (2022b) compute column and row subsets using a hybrid DEIM and leverage scores based method (L-DEIM) which may be computationally more efficient than the DEIM scheme.

In this paper, we generalize the DEIM type CUR (Sorensen and Embree, 2016) and L-DEIM type CUR methods to develop a new coordinated CUR factorization of a matrix triplet $(A, B, G)$ of compatible dimensions, based on the restricted singular value decomposition (RSVD). We call this factorization an RSVD based CUR (RSVD-CUR) factorization. We stress that this RSVD does not stand for randomized SVD (see, e.g., Halko, Martinsson, and Tropp, 2011). Both CUR decomposition and RSVD algorithms have been well studied. However, to the best of our knowledge, this work is the first to combine both methods. The RSVD has been around for over three decades now; this new method introduces a new type of exploitation of the RSVD.

In recent times, real-world data sets often comprise different representations or views, which provide information complementary to each other. Our RSVD-CUR factorization is motivated by the canonical correlation analysis (CCA) of a matrix pair $(B, G)$ (see, e.g., Golub and Zha, 1995) which is related to the RSVD of the matrix triplet $(G^T B, G^T, B)$ (see Sections 2 and 3). CCA aims to find linear combinations of $B$ and $G$ that have a maximum correlation with each other while the transformed features within each data set are linearly independent (Härdle and Simar, 2015, p. 443). Here, we aim to find subsets of columns or rows of $B$ and $G$ by exploiting the subspaces of $B$ and $G$ that maximize the pairwise correlations across the two matrices. We expect that an RSVD-CUR factorization may be useful for multi-view dimension reduction, and integration of information from multiple views in multi-view learning. This is a rapidly growing direction in machine learning which involves learning with multiple views to improve the generalization performance (see, e.g., Xu, Tao, and Xu, 2013). Analogous to CCA, an RSVD-CUR factorization as a tool for multi-view dimension reduction can cope with a two-view case. In the same context, one can use an RSVD-CUR as a supervised feature selection technique in multilabel classification problems (see Section 5).

Another motivation for an RSVD-CUR factorization stems from applications where the goal is to select a subset of rows and or columns of one data set relative to two other data sets. An example is a data perturbation problem of the form $A_E = A + BFG$ where $BFG$ is a nonwhite noise matrix (see, e.g., Hansen, 1998, p. 55) and the goal is to recover the low-rank matrix $A$ from $A_E$ given the covariance structure of $B$ and $G$. It is worth pointing out that one does not necessarily need to know the exact noise covariance matrices; the RSVD and
RSVD-CUR may still deliver good approximation results given inexact covariance matrices (see Section 5).

Over the decades, several generalizations of the singular value decomposition (SVD) corresponding to the product or quotient of two to three matrices have been proposed. The most commonly known generalization is the generalized SVD (GSVD), also referred to as the quotient SVD of a matrix pair \((A, B)\) (Chu, De Lathauwer, and De Moor, 2000), which corresponds to the SVD of \(AB^{-1}\) if \(B\) is square and nonsingular. Another generalization is the RSVD of a matrix triplet \((A, B, G)\) (Zha, 1991) which shows the SVD of \(B^{-1}AG^{-1}\) if \(B\) and \(G\) are square and nonsingular. Similarly, we have proposed generalizations of an SVD-based CUR decomposition: first, a generalized CUR (GCUR) decomposition of a matrix pair \((A, B)\) (Gidisu and Hochstenbach, 2022a); second, in this paper, an RSVD-CUR decomposition of a matrix triplet \((A, B, G)\). We emphasize that an RSVD-CUR is more general than a GCUR decomposition. One can derive a GCUR decomposition from an RSVD-CUR factorization given special choices of the matrices \(B\), \(G\) (we will see this in Proposition 1); however, we note that the converse does not hold.

Outline

A short review of CCA is provided in Section 2. Section 3 gives a brief overview of the RSVD. Section 4 introduces the new RSVD-CUR decomposition. In this section, we also discuss some error bounds. Algorithms 3 and 4 summarize the procedure of constructing a DEIM type and L-DEIM type RSVD-CUR decomposition, respectively. Results of numerical experiments using synthetic and real data sets are presented in Section 5, followed by conclusions in Section 6.

2. Canonical Correlation Analysis

This section briefly discusses CCA, one of our motivations for the proposed RSVD-CUR approximation. CCA is one of the most widely used and valuable techniques for multi-data processing. It is used to analyze the mutual relationships between two sets of variables. We assume that \(B \in \mathbb{R}^{m \times \ell}\), \(G \in \mathbb{R}^{d \times n}\) are of full column rank with \(m = d\) and \(k \leq \min(\text{rank}(G), \text{rank}(B))\). CCA seeks to find the linear combinations of the form \(Bw_i\) and \(Gz_i\) for \(i = 1, \ldots, k\) that maximize the pairwise correlations across the two matrices (Härdle and Simar, 2015, p. 443). We can define the canonical correlations \(\rho_1(B, G), \ldots, \rho_k(B, G)\) of the matrix pair \((B, G)\) as (Golub and Zha, 1995)

\[
\rho_i(B, G) = \max_{\substack{Bw_i \neq 0, Gz_i \neq 0 \\text{ with } \\
Bw_i \perp \{Bw_1, \ldots, Bw_{i-1}\} \\text{ and } \\
Gz_i \perp \{Gz_1, \ldots, Gz_{i-1}\}}} \rho(Gz_i, Bw_i) =: \rho(Gz_i, Bw_i) := \frac{z_i^T G^T B w_i}{\|Gz_i\| \|Bw_i\|}.
\]

(2)

We have that \(\rho_1(B, G) \geq \cdots \geq \rho_k(B, G)\). The vectors of unit length \(Gz_i/\|Gz_i\|\) and \(Bw_i/\|Bw_i\|\) are referred to as the canonical vectors of \((B, G)\) and the canonical weights are \(z_i/\|Gz_i\|\) and \(w_i/\|Bw_i\|\). As discussed in (Golub and Zha, 1995), there are several equivalent means to formulate CCA. We show a Lagrange multiplier formulation which is suitable for our analysis and will serve as a motivation for the proposed decomposition. The Lagrange multiplier function of the above constrained optimization problem is (Golub and
\[ f(w, z, \lambda, \mu) = z^T G^T B w - \lambda (\|Bw\|^2 - 1) - \mu (\|Gz\|^2 - 1). \]

Differentiating the above with respect to \( z, w, \mu, \) and \( \lambda \) gives

\[
\begin{align*}
G^T B w - \mu G^T G z &= 0, \\
B^T G z - \lambda B^T B w &= 0, \\
z^T G^T G z &= 1, \\
w^T B^T B w &= 1.
\end{align*}
\]

Premultiplying the first two equations in the math display above by \( z^T \) and \( w^T \), respectively, we have that \( \lambda = \mu \) and

\[
\begin{bmatrix}
B^T G \\
G^T B
\end{bmatrix}
\begin{bmatrix}
z \\
w
\end{bmatrix}
= \lambda
\begin{bmatrix}
G^T G \\
B^T B
\end{bmatrix}
\begin{bmatrix}
z \\
w
\end{bmatrix}.
\]

The canonical weights and correlations are the generalized eigenvectors and eigenvalues, respectively, of this generalized eigenvalue problem. We will show in the next section how this problem relates to the RSVD of matrix triplets which we use for our proposed RSVD-CUR factorization.

3. Restricted SVD

The RSVD of matrix triplets as notably studied by Zha (1991) and De Moor and Golub (1991) is an essential building block for the proposed decomposition in this paper. We give a brief overview of this matrix factorization here. The RSVD may be viewed as a decomposition of a matrix relative to two other matrices of compatible dimensions. Given a matrix triplet \( A \in \mathbb{R}^{m \times n} \) (where without loss of generality \( m \geq n \)), \( B \in \mathbb{R}^{m \times \ell} \), and \( G \in \mathbb{R}^{d \times n} \), we assume that \( B \) and \( G \) are of full rank. We give an overview of the various dimensions we consider in Table 1. Following the formulation by Zha (1991), there exist orthogonal matrices \( U \in \mathbb{R}^{\ell \times \ell} \) and \( V \in \mathbb{R}^{d \times d} \) and nonsingular matrices \( Z \in \mathbb{R}^{m \times m} \) and \( W \in \mathbb{R}^{n \times n} \) such that

\[ A = Z D_A W^T, \quad B = Z D_B U^T, \quad G = V D_G W^T. \]

This means

\[
\begin{bmatrix}
A & B \\
G & \end{bmatrix}
= \begin{bmatrix}
Z & \\
V & \end{bmatrix}
\begin{bmatrix}
D_A & D_B \\
D_G & \end{bmatrix}
\begin{bmatrix}
W & \\
U & \end{bmatrix}^T,
\]

where \( D_A \in \mathbb{R}^{m \times n} \), \( D_B \in \mathbb{R}^{m \times \ell} \), and \( D_G \in \mathbb{R}^{d \times n} \) are nonnegative (possibly rectangular) diagonal matrices. Algorithms for the computation of the RSVD are still an active field of research; some recent works include (Chu, De Lathauwer, and De Moor, 2000; Zwaan, 2020). As noted by De Moor and Golub (1991), the RSVD can be implemented by a double GSVD. The following is a practical procedure to do this. For ease of presentation, we first assume that \( m = \ell \) and \( d = n \) so that \( B \) and \( G \) are square. Then we have the following
expression as the RSVD from two GSVDs:

\[
\begin{bmatrix}
A & B \\
G & &
\end{bmatrix} =
\begin{bmatrix}
U_1 & V_1 \\
& &
\end{bmatrix}
\begin{bmatrix}
\Gamma_1 & U_1^T B \\
\Sigma_1 & &
\end{bmatrix}
\begin{bmatrix}
Y_1^T \\
I &
\end{bmatrix}
\]

\[
= \begin{bmatrix}
U_1 & V_1 \\
& &
\end{bmatrix}
\begin{bmatrix}
\Gamma_1 \Sigma_1^{-1} & U_1^T B \\
I & &
\end{bmatrix}
\begin{bmatrix}
\Sigma_1 Y_1^T \\
I &
\end{bmatrix}
\]

\[
= \begin{bmatrix}
U_1 Y_2 & V_1 \\
& &
\end{bmatrix}
\begin{bmatrix}
\Sigma_2^T & \Gamma_2 \\
V_2 & &
\end{bmatrix}
\begin{bmatrix}
V_2^T \Sigma_1 Y_1^T \\
U_2^T &
\end{bmatrix}
\]

\[
= \begin{bmatrix}
U_1 Y_2 & V_1 V_2 \\
& &
\end{bmatrix}
\begin{bmatrix}
\Sigma_2^T \Gamma_G & \Gamma_2^T \\
\Gamma_G & &
\end{bmatrix}
\begin{bmatrix}
Y_1 \Sigma_1 V_2 \Gamma_G^{-1} & U_2 \\
Y_1 V_2 &
\end{bmatrix}^T
\]

The identity matrix is denoted by \( I \). In these four steps, we have first computed the GSVD of \((A, G)\),

\[
A = U_1 \Gamma_1 Y_1^T, \quad G = V_1 \Sigma_1 Y_1^T.
\] (5)

Next, we compute the GSVD of the transposes of the pair \((U_1^T B, \Gamma_1 \Sigma_1^{-1})\), so that \(U_1^T B = Y_2 \Gamma_2^T U_2^T\) and \(\Gamma_1 \Sigma_1^{-1} = Y_2 \Sigma_2^T V_2^T\). Moreover, \(\Gamma_G\) is a scaling matrix that one can freely select (see, e.g., Zwaan, 2020). In this square case we have \(\Sigma_2 = \Sigma_2\), but we keep this notation for consistency with the nonsquare case which we will discuss now.

In some of our applications of interest (see Experiments 3 and 4 in Section 5), we have that \(\ell = d > m \geq n\). In this case we get the following modifications:

\[
\begin{bmatrix}
U_1 & V_1 \\
& &
\end{bmatrix}
\begin{bmatrix}
\Gamma_1 & U_1^T B \\
\Sigma_1 & \Sigma_1 &
\end{bmatrix}
\begin{bmatrix}
Y_1^T \\
\Sigma_1 Y_1^T & &
\end{bmatrix}
\]

\[
= \begin{bmatrix}
U_1 & V_1 \\
& &
\end{bmatrix}
\begin{bmatrix}
\Gamma_1 \Sigma_1^{-1} & U_1^T B \\
I & &
\end{bmatrix}
\begin{bmatrix}
\Sigma_1 Y_1^T \\
I &
\end{bmatrix}
\]

\[
= \begin{bmatrix}
U_1 Y_2 & V_1 \\
& &
\end{bmatrix}
\begin{bmatrix}
\Sigma_2 & \Gamma_2 \\
V_2 & &
\end{bmatrix}
\begin{bmatrix}
\Sigma_2^T \Sigma_1 Y_1^T \\
U_2^T &
\end{bmatrix}
\]

\[
= \begin{bmatrix}
U_1 Y_2 & V_1 \hat{V}_2 \\
& &
\end{bmatrix}
\begin{bmatrix}
\Sigma_2^T \Gamma_G & \Gamma_2^T \\
\Gamma_G & &
\end{bmatrix}
\begin{bmatrix}
Y_1 \Sigma_1 V_2 \Gamma_G^{-1} \\
Y_1 \Sigma_1 \hat{V}_2 \Gamma_G^{-1} & U_2 \\
Y_1 V_2 &
\end{bmatrix}^T.
\]

In these steps, we have made use of \(\hat{V}_2 = \text{diag}(V_2, I_{d-n})\). In these two GSVD steps, we emphasize that the generalized singular values in both GSVDs should be maintained in the traditional nondecreasing order. That is, the diagonal entries of \(\Gamma_1\) and \(\Gamma_2\) are in nondecreasing order while those of \(\Sigma_1\) and \(\Sigma_2\) are in nonincreasing order. With reference to (4), we define \(Z := U_1 Y_2, W := Y_1 \Sigma_1 V_2 \Gamma_G^{-1}, V := V_1 \hat{V}_2, U := U_2, D_A := \Sigma_2^T \Gamma_G, D_B := \Gamma_2^T, \) and \(D_G := [\Gamma_G \quad 0_{d-n,n}]\). Denoting \(\text{diag}(D_A) = (\alpha_1, \ldots, \alpha_n)\), \(\text{diag}(D_B) = (\beta_1, \ldots, \beta_n)\), \(\text{diag}(D_G) = (\gamma_1, \ldots, \gamma_n)\) and \(\Sigma_2 = \text{diag}(\sigma_1, \ldots, \sigma_n)\), for \(i = 1, \ldots, n\), we choose \(\gamma_i = \frac{\sigma_i}{\sqrt{\sigma_i^2 + 1}}\) (note that these are ordered nonincreasingly). This implies that \(\alpha_i = \frac{\sigma_i^2}{\sqrt{\sigma_i^2 + 1}}\). Given that
\[ \beta_i^2 + \sigma_i^2 = 1 \] from the second GSVD, we have that \( \alpha_i^2 + \beta_i^2 + \gamma_i^2 = 1 \) for \( i = 1, \ldots, n \).

Note that, in view of the assumption that \( B \) and \( G \) are of full rank, \( 1 > \alpha_i \geq \alpha_{i+1} > 0, 1 > \gamma_i \geq \gamma_{i+1} > 0, \) and \( 0 < \beta_i \leq \beta_{i+1} < 1 \) and \( \frac{\alpha_i}{\beta_i \gamma_i} \geq \frac{\alpha_{i+1}}{\beta_{i+1} \gamma_{i+1}} \). The last inequality follows from the fact that \( \alpha_i/\gamma_i = \sigma_i \), which is nonincreasing.

We now state a connection of the RSVD with CCA. De Moor and Golub (1991) show a relation of the RSVD to a generalized eigenvalue problem. The related generalized eigenvalue problem of the RSVD of the matrix triplet \((G^TB, G^TB)\) with \( m = d \) as shown in (De Moor and Golub, 1991, Sec. 2.2) is

\[
\begin{bmatrix}
G^TB \\
B^T G
\end{bmatrix}
\begin{bmatrix}
z \\
w
\end{bmatrix} = \lambda
\begin{bmatrix}
G^TB \\
B^T G
\end{bmatrix}
\begin{bmatrix}
z \\
w
\end{bmatrix}.
\]

It is clear that the above problem is exactly the generalized eigenvalue problem of the \( \text{cca}(B,G) \); see (3) in Section 2. Note that matrices \( B^TB \) and \( G^TG \) are covariance matrices. In applications where these covariance matrices are (almost) singular, one may use the RSVD instead to find a solution without explicitly solving the generalized eigenvalue problem.

4. A Restricted SVD based CUR Decomposition and its Approximation Properties

In this section, we describe the proposed RSVD-CUR decomposition and provide theoretical bounds on its approximation errors. We denote the pseudoinverse of \( B \) by \( B^+ \) and use MATLAB notations to index vectors and matrices, i.e., \( A(:,p) \) denotes the \( k \) columns of \( A \) with corresponding indices in vector \( p \in \mathbb{N}_k^+ \).

4.1 A Restricted SVD based CUR decomposition

We now introduce a new RSVD-CUR decomposition of a matrix triplet \((A, B, G)\) with \( A \in \mathbb{R}^{m \times n} \) (where without loss of generality \( m \geq n \)), \( B \in \mathbb{R}^{m \times \ell} \), and \( G \in \mathbb{R}^{d \times n} \) where \( B \) and \( G \) are of a full rank. Table 1 in Section 5 gives an overview of the various sizes of \( m, n, \ell, \) and \( d \) in the applications that we consider. This RSVD-CUR factorization is guided by the knowledge of the RSVD for matrix triplets reviewed in Section 3. We now define a rank-\( k \) RSVD-CUR approximation; cf. (1).

**Definition 1** Let \( A \) be \( m \times n \), \( B \) be \( m \times \ell \), and \( G \) be \( d \times n \). A rank-\( k \) RSVD-CUR approximation of \((A, B, G)\) is defined as

\[
A \approx A_k := C_A M_A R_A := AP M_A S^T A,
B \approx B_k := C_B M_B R_B := BP_B M_B S^T B,
G \approx G_k := C_G M_G R_G := GP M_G S^T G.
\]

Here \( S \in \mathbb{R}^{m \times k}, S_G \in \mathbb{R}^{d \times k}, P \in \mathbb{R}^{n \times k}, \) and \( P_B \in \mathbb{R}^{\ell \times k} \) \((k \leq \min(m,n,d,\ell))\) are index selection matrices with some columns of the identity that select rows and columns of the respective matrices.

It is key that the same rows of \( A \) and \( B \) are picked and the same columns of \( A \) and \( G \) are selected; this gives a coupling among the decompositions. The matrices \( C_A \in \mathbb{R}^{m \times k}, \)
$C_B \in \mathbb{R}^{m \times k}$, $C_G \in \mathbb{R}^{d \times k}$, and $R_A \in \mathbb{R}^{k \times n}$, $R_B \in \mathbb{R}^{k \times \ell}$, $R_G \in \mathbb{R}^{k \times n}$ are subsets of the columns and rows, respectively, of the given matrices. Let the vectors $\mathbf{s}$, $\mathbf{s}_G$, $\mathbf{p}$, and $\mathbf{p}_B$ contain the indices of the selected rows and columns, so that $S = I(:, \mathbf{s})$, $S_G = I(:, \mathbf{s}_G)$, $P = I(:, \mathbf{p})$, and $P_B = I(:, \mathbf{p}_B)$. The choice of $\mathbf{s}$, $\mathbf{s}_G$, $\mathbf{p}$, and $\mathbf{p}_B$ is guided by the knowledge of the orthogonal and nonsingular matrices from the rank-$k$ RSVD. Given the column and row index vectors, following Sorensen and Embree (2016) and others (Mahoney and Drineas, 2009; Stewart, 1998), we compute the middle matrices as mentioned in Section 1, that is, $M_A = (C_A^T C_A)^{-1} C_A^T A R_A^T (R_A R_A^T)^{-1}$. There are several index selection strategies proposed in the literature for finding the “best” row and column indices. The two approaches we employ, namely, the DEIM algorithm (Chaturantabut and Sorensen, 2010) and the L-DEIM scheme (Gidisu and Hochstenbach, 2022b) are greedy deterministic procedures and simple to implement.

The DEIM procedure has first been introduced in the context of model reduction of nonlinear dynamical systems. It has later been used as a column and row index selection procedure for constructing a CUR factorization. To construct $C$ and $R$, Sorensen and Embree (2016) apply the DEIM scheme to the top $k$ right and left singular vectors, respectively. The DEIM procedure uses a locally optimal projection technique similar to the pivoting strategy of the LU factorization. The column and row indices are selected by processing the singular vectors sequentially as summarized in Algorithm 1.

**Algorithm 1** Discrete empirical interpolation index selection method (deim) (Chaturantabut and Sorensen, 2010)

| Input: $U \in \mathbb{R}^{m \times k}$ with $k \leq m$ (full rank) |
| Output: Indices $\mathbf{s} \in \mathbb{N}_k^+$ with non-repeating entries |

1: for $j = 1, \ldots, k$ do  
2: $s(j) = \arg \max_{1 \leq i \leq m} |(U(:, j))_i|$  
3: $U(:, j + 1) = U(:, j + 1) - U(:, 1 : j) \cdot (U(s, 1 : j) \setminus U(s, j + 1))$  
4: end for

The DEIM procedure has a striking limitation. The number of indices to be selected can only be at most the number of singular vectors available. The L-DEIM algorithm seeks to overcome this drawback. The algorithm allows for additional indices selection using the same information from the available singular vectors. The L-DEIM procedure involves two steps. Suppose there are $\hat{k} < k$ singular vectors available, and we want to select $k$ indices. First, perform DEIM to select at most $\hat{k}$ and then select the additional $k - \hat{k}$ indices using a procedure similar to the deterministic leverage scores sampling method (see Lines 2 to 4 of Algorithm 2). Compared to the DEIM scheme, the L-DEIM procedure may be computationally more efficient and requires less than $k$ input vectors to select $k$ indices. Note that the vectors in $U$ in Line 2 of Algorithm 2 are not the same as the original input vectors because of the updates done through the oblique projections in Line 1.

Both the DEIM type CUR and the L-DEIM type CUR decompositions require singular vectors. In this paper, we apply the DEIM and the L-DEIM procedures to the nonsingular and orthogonal matrices from the RSVD instead. The procedure for constructing a DEIM

1. The backslash operator used in the algorithms is a Matlab type notation for solving linear systems and least-squares problems.
Algorithm 2 L-DEIM index selection (Gidisu and Hochstenbach, 2022b)

**Input:** $U \in \mathbb{R}^{m \times \hat{k}}$ with $\hat{k} < k \leq m$ (linearly independent), target rank $k$

**Output:** Indices $s \in \mathbb{N}^{k}_{\downarrow}$ with non-repeating entries

1: $s = \text{deim}(U)$
2: Compute $\ell_i = ||U_{i,:}||$ for $i = 1, \ldots, m$; sort $\ell$ in nonincreasing order
3: Remove entries in $\ell$ corresponding to the indices in $s$
4: $s' = \hat{k} - k$ indices corresponding to $\hat{k} - k$ largest entries of $\ell$  
5: $s = [s; s']$

The L-DEIM type RSVD-CUR is summarized in Algorithm 3. In Lines 2 to 5 we perform the DEIM algorithm on the first $k$ vectors of $W$, $Z$, $U$, and $V$.

Algorithm 3 DEIM type RSVD-CUR decomposition

**Input:** $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{m \times \ell}$, $G \in \mathbb{R}^{d \times n}$, desired rank $k$

**Output:** A rank-$k$ RSVD-CUR decomposition

$A_k = A(:, p) \cdot M_A \cdot A(s,:)$, $B_k = B(:, p_B) \cdot M_B \cdot B(s,:)$, $G_k = G(:, p) \cdot M_G \cdot G(s_G,:)$

1: Compute rank-$k$ RSVD of $(A, B, G)$ to obtain $W, Z, U, V$ (see (4))
2: $p = \text{deim}(W)$ (perform DEIM on the matrices from the RSVD)
3: $s = \text{deim}(Z)$
4: $p_B = \text{deim}(U)$
5: $s_G = \text{deim}(V)$
6: $M_A = A(:, p) \setminus (A / A(s,:))$, $M_B = B(:, p_B) \setminus (B / B(s,:))$, $M_G = G(:, p) \setminus (G / G(s_G,:))$

To construct the L-DEIM type RSVD-CUR, given the target rank $k$, one can use at least the first $k/2$ vectors of $W$, $Z$, $U$, and $V$. (From experimental results in (Gidisu and Hochstenbach, 2022b, Sec. 5) given a desired rank $k$, using at least $k/2$ vectors, the L-DEIM CUR approximation quality seems as good as that of the DEIM type CUR, which uses $k$ vectors.) Algorithm 4 summarizes the procedure for the L-DEIM type RSVD-CUR decomposition.

Algorithm 4 L-DEIM type RSVD-CUR decomposition

**Input:** $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{m \times \ell}$, $G \in \mathbb{R}^{d \times n}$, desired rank $k$, $\hat{k} = k/2$

**Output:** A rank-$k$ RSVD-CUR decomposition

$A_k = A(:, p) \cdot M_A \cdot A(s,:)$, $B_k = B(:, p_B) \cdot M_B \cdot B(s,:)$, $G_k = G(:, p) \cdot M_G \cdot G(s_G,:)$

1: Compute rank-$\hat{k}$ RSVD of $(A, B, G)$ to obtain $W, Z, U, V$ (see (4))
2: $p = \text{l-deim}(W)$ (perform L-DEIM on the matrices from the RSVD)
3: $s = \text{l-deim}(Z)$
4: $p_B = \text{l-deim}(U)$
5: $s_G = \text{l-deim}(V)$
6: $M_A = A(:, p) \setminus (A / A(s,:))$, $M_B = B(:, p_B) \setminus (B / B(s,:))$, $M_G = G(:, p) \setminus (G / G(s_G,:))$
In many applications, as we will see in Section 5, one is interested in selecting only the key columns or rows and not the explicit \( A \approx C_A M_A R_A \) factorization. An interpolative decomposition aims to identify a set of skeleton columns or rows of a matrix. A CUR factorization may be computed by evaluating the ID for both the column and row spaces of a matrix simultaneously. The following are the column and row versions of an RSVD-ID factorization of a matrix triplet:

\[
A \approx C_A \tilde{M}_A, \quad B \approx C_B \tilde{M}_B, \quad G \approx C_G \tilde{M}_G, \quad \text{or} \quad A \approx \tilde{M}_A R_A, \quad B \approx \tilde{M}_B R_B, \quad G \approx \tilde{M}_G R_G.
\]

Here, \( \tilde{M}_A = C_A^+ A \) is \( k \times n \) and \( \tilde{M}_A = A R_A^+ \) is \( m \times k \); analogous remarks hold for \( \tilde{M}_B, \tilde{M}_G, \tilde{M}_B, \text{and} \tilde{M}_G \). Notice that in Algorithms 3 and 4, the key column and row indices of the various matrices are picked independently. These algorithms can therefore be restricted to select only column indices if we are interested in the column version of the RSVD-ID factorization or select only row indices if we are interested in the row version.

De Moor and Golub (1991) show the relation between the RSVD and the SVD and its other generalizations. We indicate in the following proposition the corresponding connection between the DEIM type RSVD-CUR and the (generalized) CUR decomposition (Sorensen and Embree, 2016; Gidisu and Hochstenbach, 2022a).

**Proposition 1**
(i) If \( B \) and \( G \) are square and nonsingular matrices, then the selected row and column indices from a CUR decomposition of \( B^{-1}AG^{-1} \) are the same as index vectors \( p_B \) and \( s_G \), respectively, obtained from an RSVD-CUR decomposition of \( (A,B,G) \).

(ii) Moreover, if \( B \) and \( G \) are nonsquare and of full rank, we have a similar connection between the selected row and column indices from a CUR decomposition of \( B^+AG^+ \) and the index vectors \( p_B \) and \( s_G \) derived from an RSVD-CUR decomposition of \( (A,B,G) \).

(iii) Furthermore, in the particular case where \( B = I \) and \( G = I \), the RSVD-CUR decomposition of \( A \) coincides with a CUR decomposition of \( A \), in that the factors \( C \) and \( R \) of \( A \) are the same for both methods: the first line of (6) is equal to (1).

(iv) Lastly, given a special choice of \( B = I \), an RSVD-CUR decomposition of \( A \) and \( G \) coincides with the GCUR decomposition of \( (A,G) \) (see Gidisu and Hochstenbach, 2022a, Def. 4.1), in that the factors \( C_A, C_G, R_A, R_G \) of \( A \) and \( G \) are the same for both methods. In case instead of \( B, G = I \), similar remarks hold.

**Proof**
(i) We start with the RSVD (4). If \( B \) and \( G \) are square and nonsingular, then the SVD of \( B^{-1}AG^{-1} \) can be expressed in terms of the RSVD of \( (A,B,G) \), and is equal to \( U(D_B^{-1}D_AD_G^{-1})V^T \) given that \( B^{-1} = U D_B^{-1} Z^{-1} \) and \( G^{-1} = W^{-T} D_G^{-1} V^T \) (De Moor and Golub, 1991). Consequently, the row and column indices vectors from a CUR factorization of \( B^{-1}AG^{-1} \) are equal to the vectors \( s_G \) and \( p_B \), respectively, from an RSVD-CUR of \( (A,B,G) \) since they are obtained by applying DEIM to matrices \( U \) and \( V \), respectively.

(ii) If \( B \) and \( G \) are nonsquare and of full rank, then we still have a similar connection between the RSVD of \( (A,B,G) \) and the SVD of \( B^+AG^+ \) because of the following. Since the factors in the second and third lines of the RSVD (4) are assumed to be of full rank, we have \( B^+ = UD_B^{-1} Z^+ \) and \( G^+ = (W^T)^+ D_G^{-1} V^T \). For a thin matrix \( B \), the reduced RSVD of \( B \) gives a square matrix \( D_B \) and a thin matrix \( Z \). Where \( B \) is a fat matrix, we
have both $D_B$ and $Z$ being square matrices from the reduced RSVD of $B$; this implies that irrespective of the specific size of $B$, we always have $Z^+Z = I$. Following this analogy for $G$, we have that $W^T(W^T)^+ = I$ always holds. Hence, $B^+AG^+ = U(D_B^{-1}D_AD_G^{-1})V^T$ which is the SVD of $B^+AG^+$. As a result, just as in (i), the index vectors $s_G$ and $p_B$ from an RSVD-CUR of $(A, B, G)$ are equivalent to the selected column and row indices from CUR of $B^+AG^+$, respectively.

(iii) If $B = I$ and $G = I$, from (4), $I = ZD_BU^T$ and $I = VD_GW^T$ which implies $UD_B^{-1} = Z$ and $W^T = D_G^{-1}V^T$. Hence, we find that $A = UD_B^{-1}D_AD_G^{-1}V^T$ which is an SVD of $A$. Therefore the selection matrices $P$, $S$ from CUR of $A$ (1) are equal to the selection matrices $P_B$, $S_G$ from an RSVD-CUR of $(A, I, I)$ (6).

(iv) If $B = I$, again from (4), $I = ZD_BU^T$, which implies $UD_B^{-1} = Z$. Then $A = UD_B^{-1}D_AW^T$, $G = VD_GW^T$ which is (up to a diagonal scaling) the GSVD of the matrix pair $(A, G)$; see (5) (De Moor and Golub, 1991). Thus, the column and row selection matrices from GCUR of $(A, G)$ (see Gidisu and Hochstenbach, 2022a, Def. 4.1) are the same as the column and row selection matrices $P$, $S$, $S_G$ from (6), respectively. ■

4.2 Error Analysis

We begin by analyzing the error of a rank-$k$ RSVD of a matrix triplet $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{m \times \ell}$, and $G \in \mathbb{R}^{n \times n}$ with $\ell = d \geq m \geq n$. To define a rank-$k$ RSVD, let us partition the following matrices

\[
U = [U_k \hat{U}], \quad V = [V_k \hat{V}], \quad W = [W_k \hat{W}], \quad Z = [Z_k \hat{Z}],
\]

\[
D_A = \text{diag}(D_{A_k}, \hat{D}_A), \quad D_B = \text{diag}(D_{B_k}, \hat{D}_B), \quad D_G = \text{diag}(D_{G_k}, \hat{D}_G),
\]

where $\hat{D}_A \in \mathbb{R}^{(n-k) \times (n-k)}$, $\hat{D}_B \in \mathbb{R}^{(m-k) \times (\ell-k)}$, and $\hat{D}_G \in \mathbb{R}^{(d-k) \times (n-k)}$. We define a rank-$k$ RSVD of $(A, B, G)$ as

\[
A_k := Z_kD_{A_k}W_k^T, \quad B_k := Z_kD_{B_k}U_k^T, \quad G_k := V_kD_{G_k}W_k^T,
\]

where $k < n$. It follows that

\[
A - A_k = \hat{Z} \hat{D}_A \hat{W}^T, \quad B - B_k = \hat{Z} \hat{D}_B \hat{U}^T, \quad G - G_k = \hat{V} \hat{D}_G \hat{W}^T.
\]

The following statements are a stepping stone for the error bound analysis of an RSVD-CUR. Denote the singular values of matrices $A$, $Z$, and $W$ by $\psi_i(A)$, $\psi_i(Z)$, and $\psi_i(W)$, respectively. Let $A - A_k = \hat{Z} \hat{D}_A \hat{W}^T$ as in (8), then for $i = 1, \ldots, n$, $\psi_i(\hat{Z} \hat{D}_A \hat{W}^T) \leq \psi_i(\hat{D}_A) \|\hat{Z}\| \|\hat{W}\|$ (see, e.g., Horn and Johnson, 2012, p. 346). Since the diagonal elements of $\hat{D}_A$ are in nonincreasing order, we have $\|A - A_k\| \leq \psi_1(\hat{D}_A) \|\hat{Z}\| \|\hat{W}\| \leq \alpha_{k+1} \cdot \|\hat{Z}\| \|\hat{W}\|$. Similarly, we have that $\|B - B_k\| \leq \|\hat{Z}\| \|\hat{W}\|$ and $\|G - G_k\| \leq \|\hat{V}\| \|\hat{D}_G\| \|\hat{W}\| \leq \gamma_{k+1} \cdot \|\hat{W}\|$. The first inequality follows from the fact $\hat{U}$ orthonormal columns and the diagonal elements of $\hat{D}_B$ are in nondecreasing order with a maximum value of 1, so we have that $\psi_1(\hat{D}_B) = 1$ and $\|\hat{U}\| = 1$. The second equality is a result of the fact that $\hat{V}$ has orthonormal columns and the diagonal entries of $\hat{D}_G$ are in nonincreasing order, therefore, $\psi_1(\hat{D}_G) = \gamma_{k+1}$ and $\|\hat{V}\| = 1$.

We now introduce some error bounds of an RSVD-CUR decomposition in terms of the error of a rank-$k$ RSVD. The analysis closely follows the error bound analysis in (Sorensen
and Embree, 2016; Gidisu and Hochstenbach, 2022a) for the DEIM type CUR and DEIM type GCUR methods with some necessary modifications. As with the DEIM type GCUR method, here also, the lack of orthogonality of the basis vectors in \( W \) and \( Z \) from the RSVD necessitates some additional work. We take a QR factorization of \( W \) and \( Z \) to obtain an orthonormal basis to facilitate the analysis, introducing terms in the error bound associated with the triangular matrix in the QR factorization.

For the analysis, we use the following QR decomposition of the nonsingular matrices from the RSVD (see Eq. (4))

\[
\begin{bmatrix} Z_k & \hat{Z} \end{bmatrix} = Z = QZ_k T_Z = [QZ_k \hat{Q} Z] \begin{bmatrix} T_{Z_k} & T_{Z_{k2}} \\ 0 & T_{Z_{22}} \end{bmatrix} = [QZ_k T_{Z_k} \hat{Q} Z T_Z],
\]

\[
\begin{bmatrix} W_k & \hat{W} \end{bmatrix} = W = QW_k T_W = [QW_k \hat{Q} W] \begin{bmatrix} T_{W_k} & T_{W_{k2}} \\ 0 & T_{W_{22}} \end{bmatrix} = [QW_k T_{W_k} \hat{Q} W \hat{T}_W],
\]

where we have defined

\[
\hat{T}_Z := \begin{bmatrix} T_{Z_{k1}} \\ T_{Z_{22}} \end{bmatrix}, \quad \hat{T}_W := \begin{bmatrix} T_{W_{k1}} \\ T_{W_{22}} \end{bmatrix}.
\]

This implies that

\[
A = A_k + \tilde{Z} \tilde{D}_A \tilde{W}^T = Z_k D_{Ak} W_k^T + \tilde{Z} \tilde{D}_A \tilde{W}^T = QZ_k T_{Zk} D_{Ak} T_{Wk} Q_{Wk} + QZ \hat{T}_Z \tilde{D}_A \tilde{W}^T_Q,
\]

\[
B = B_k + \tilde{Z} \tilde{D}_B \tilde{U}^T = Z_k D_{Bk} U_k^T + \tilde{Z} \tilde{D}_B \tilde{U}^T = QZ_k T_{Zk} D_{Bk} U_k^T + QZ \hat{T}_Z \tilde{D}_B \tilde{U}^T,
\]

\[
G = G_k + \tilde{V} \tilde{D}_G \tilde{W}^T = V_k D_{Gk} W_k^T + \tilde{V} \tilde{D}_G \tilde{W}^T = V_k D_{Gk} T_{Wk} Q_{Wk} + V_k \tilde{D}_G \tilde{T}_W Q_{Wk}.
\]

Given an orthonormal matrix \( Q_W \in \mathbb{R}^{n \times k} \) from (Sorensen and Embree, 2016; Gidisu and Hochstenbach, 2022a) as well as here, we have that the quantity \( \|A(I - Q_W k Q_{Wk}^T)\| \) is key in the error bound analysis. Here, we have that \( \|A(1 - Q_W k Q_{Wk}^T)\| \) may not be close to \( \psi_k(A) \) since the matrix \( Q_{Wk} \) is from the RSVD, therefore we provide a bound on this quantity in terms of the error in the RSVD.

**Theorem 1 (Generalization of (Sorensen and Embree, 2016, Theorem 4.1) and (Gidisu and Hochstenbach, 2022a, Theorem 4.8))** Given \( A, B, \) and \( G \) as in Definition 1 and \( Z_k \in \mathbb{R}^{m \times k}, W_k \in \mathbb{R}^{n \times k}, U_k \in \mathbb{R}^{e \times k}, \) and \( V_k \in \mathbb{R}^{d \times k} \) from (7), let \( Q_{Zk} \in \mathbb{R}^{m \times k}, Q_{Wk} \in \mathbb{R}^{n \times k} \) be the Q-factors of \( Z_k, W_k, \) respectively, and \( \hat{T}_Z, T_{Z_{k2}}, \hat{T}_W, \) and \( T_{W_{k2}} \) as in (9)–(10). Suppose \( Q_{Wk}^T P, U_k^T P_B, S_G^T V_k, \) and \( S^T Q_{Zk} \) are nonsingular, then with the error constants

\[
\eta_p := \|(Q_{Wk}^T P)^{-1}\|, \quad \eta_s := \|(S^T Q_{Zk})^{-1}\|, \quad \eta_{BP} := \|(U_k^T P_B)^{-1}\|, \quad \eta_{SG} := \|(S_G^T V_k)^{-1}\|,
\]

we have

\[
\begin{align*}
\|A - C_A M_A R_A\| & \leq \alpha_{k+1} \cdot (\eta_p \cdot \|\hat{T}_Z\| \|T_{W_{k2}}\| + \eta_s \cdot \|T_{Z_{k2}}\| \|\hat{T}_W\|), \\
& \leq \alpha_{k+1} \cdot (\eta_p + \eta_s) \cdot \|\hat{T}_W\| \|\hat{T}_Z\|, \\
\|B - C_B M_B R_B\| & \leq \eta_{BP} \cdot \|T_{Z_{k2}}\| + \eta_s \cdot \|\hat{T}_Z\| \leq (\eta_{BP} + \eta_s) \cdot \|\hat{T}_Z\|, \\
\|G - C_G M_G R_G\| & \leq \gamma_{k+1} \cdot (\eta_p \cdot \|\hat{T}_W\| + \eta_{SG} \cdot \|T_{W_{k2}}\|) \leq \gamma_{k+1} \cdot (\eta_p + \eta_{SG}) \cdot \|\hat{T}_W\|.
\end{align*}
\]
We will prove the result for \( \| A - C_A M_A R_A \| \) in Theorem 1; the results for \( \| B - C_B M_B R_B \| \) and \( \| G - C_G M_G R_G \| \) follow similarly. We first show the bounds on the errors between \( A \) and its interpolatory projections \( PA \) and \( AS \), i.e., the selected rows and columns. Then, using the fact these bounds also apply to the orthogonal projections of \( A \) on to the same columns and rows spaces (Sorensen and Embree, 2016, Lemma 4.2), we prove the bound on the approximation of \( A \) by an RSVD-CUR.

Given \( P \), an index selection matrix derived from performing the DEIM scheme on matrix \( W_k \). Suppose \( Q W_k \) is an orthonormal basis for \( \text{Range}(W_k) \), with \( W^T_k \) and \( Q_W^T_k P \) being nonsingular, we have the interpolatory projector \( P(W_k^T P)^{-1} W_k^T = P(Q_W^T_k P)^{-1} Q_W^T_k \) (see Chaturantabut and Sorensen, 2010, Def. 3.1, Eq. 3.6). With this equality, to exploit the special properties of an orthogonal matrix, we use the orthonormal basis of the nonsingular matrices from the RSVD instead.

Let \( Q W_k^T P \) be nonsingular so that \( P = P(Q W_k^T P)^{-1} Q W_k^T \); an oblique projector. We have that \( Q W_k^T P = Q W_k^T P(Q W_k^T P)^{-1} Q W_k^T = Q W_k^T \), which implies \( Q W_k^T (I - P) = 0 \). Therefore the error in the oblique projection of \( A \) is (cf. (Sorensen and Embree, 2016, Lemma 4.1))

\[
\| A - A P \| = \| A(I - P) \| = \| A(I - Q W_k Q W_k^T) (I - P) \| \\
\leq \| A(I - Q W_k Q W_k^T) \| \| I - P \| .
\]

Note that, since \( k < n \), it is well known that \( P \neq 0 \) and \( P \neq I \), hence (see, e.g., Szyld, 2006)

\[
\| I - P \| = \| P \| = \| P \| W_k^T -1 \| .
\]

Using the partitioning of \( A \) in (11), we have

\[
A Q W_k Q W_k^T = [Q W_k \tilde{Q} Z T Z_k T Z_{21} D_{A_k} 0 \tilde{D} A T W_k^T 0 T W_{12} T W_{22}] [I_k Q W_k^T
\]

and hence

\[
A(I - Q W_k Q W_k^T) = (A - A_k) - Q Z \hat{T} Z \hat{D} A T W_{12} Q W_k^T
\]

\[
= Q Z \hat{T} Z \hat{D} A \hat{T} W Q W_k^T - Q Z \hat{T} Z \hat{D} A T W_{12} Q W_k^T = Q Z \hat{T} Z \hat{D} A T W_{22} \hat{Q} W_k^T
\]

This implies

\[
\| A(I - Q W_k Q W_k^T) \| \leq \| \hat{D} A \| \| T Z \| \| T W_{22} \| \leq \alpha_{k+1} \cdot \| T Z \| \| T W_{22} \|
\]

and

\[
\| A(I - P) \| \leq \alpha_{k+1} \cdot \| (Q W_k^T P)^{-1} \| \| T Z \| \| T W_{22} \|
\]

Let us now consider the operation on the left-hand side of \( A \). Given that \( S^T Q Z_k \) is nonsingular, we have the DEIM interpolatory projector \( S = Q Z_k(S^T Q Z_k)^{-1} S^T \). It is well known that (Sorensen and Embree, 2016, Lemma 4.1)

\[
\| A - S A \| = \| (I - S) A \| = \| (I - S)(I - Q Z_k Q Z_k^T) A \|
\]

\[
\leq \| (I - S) \| \| (I - Q Z_k Q Z_k^T) A \|
\]
Similar to before, since \( k < m \), we know that \( S \neq 0 \) and \( S \neq I \) hence

\[
\|I - S\| = \|S\| = \|(S^T Q Z_k)^{-1}\|. 
\]

In the same setting as earlier, we have the following expansion

\[
Q Z_k Q_z^T A = [Q Z_k] \begin{bmatrix} T_{Z_k} & T_{Z_{12}} \\ 0 & T_{Z_{22}} \end{bmatrix} \begin{bmatrix} D_A & 0 \\ 0 & \hat{D}_A \end{bmatrix} \begin{bmatrix} T_{W_k}^T & 0 \\ T_{W_{12}}^T & T_{W_{22}}^T \end{bmatrix} [Q W_k]^T.
\]

We observe that

\[
(I - Q Z_k Q_z^T) A = (A - A_k) - Q Z_k T_{Z_{12}} \hat{D}_A \hat{T}_W^T Q_W^T
\]

\[
= Q Z \hat{T}_Z \hat{D}_A \hat{T}_W^T Q_W^T - Q Z_k T_{Z_{12}} \hat{D}_A \hat{T}_W^T Q_W^T = \hat{Q} Z T_{Z_{22}} \hat{D}_A \hat{T}_W^T Q_W^T.
\]

Consequently,

\[
\|(I - Q Z_k Q_z^T) A\| = \|\hat{Q} Z T_{Z_{22}} \hat{D}_A \hat{T}_W^T Q_W^T\| \leq \|\hat{D}_A\|\|T_{Z_{22}}\|\|T_W\|
\]

\[
\leq \alpha_{k+1} \cdot \|T_{Z_{22}}\|\|\hat{T}_W\|,
\]

and

\[
\|(I - S) A\| \leq \alpha_{k+1} \cdot \|(S^T Q Z_k)^{-1}\|\|T_{Z_{22}}\|\|\hat{T}_W\|.
\]

Suppose that \( C_A \) and \( R_A \) are of full rank, given the orthogonal projectors \( C_A C_A^+ \) and \( R_A R_A^+ \) and computing \( M_A \) as \((C_A^T C_A)^{-1} C_A^T R_A^+ (R_A R_A^+)^{-1} = C_A^+ R_A^+\), we have (see Mahoney and Drineas, 2009, Eq. 6)

\[
A - C_A M_A R_A = A - C_A C_A^+ R_A R_A^+ = (I - C_A C_A^+) A + C_A C_A^+ A (I - R_A R_A^+).
\]

Using the triangle inequality, it follows that (Sorensen and Embree, 2016, Lemma 4.2)

\[
\|A - C_A M_A R_A\| = \|A - C_A C_A^+ R_A R_A^+\| \leq \|(I - C_A C_A^+) A\| + \|C_A C_A^+\|\|A (I - R_A R_A^+)\|.
\]

Leveraging the fact that (Sorensen and Embree, 2016, Lemma 4.2)

\[
\|(I - C_A C_A^+) A\| \leq \|A (I - P)\|, \quad \|A (I - R_A R_A^+)\| \leq \|(I - S) A\|,
\]

and \( C_A C_A^+ \) is an orthogonal projector, \( \|C_A C_A^+\| = 1 \), as a small variant of (Gidisu and Hochstenbach, 2022a, Theorem 4.8) we have

\[
\|A - C_A M_A R_A\| \leq \alpha_{k+1} \cdot (\|\hat{T}_Z\|\|T_{W_{22}}\|\|(Q W_k P)^{-1}\| \|T_{Z_{22}}\|\|\hat{T}_W\|) + \|S^T Q Z_k)^{-1}\|\|T_{Z_{22}}\|\|\hat{T}_W\|
\]

\[
\leq \alpha_{k+1} \cdot (\|S^T Q Z_k)^{-1}\| + \|(S^T Q Z_k)^{-1}\|\|\hat{T}_Z\|\|\hat{T}_W\|.
\]

The last stated inequality follows directly from the fact that the norms of the submatrices of \( \hat{T}_Z \) and \( \hat{T}_W \) are at most \( \|\hat{T}_Z\| \) and \( \|\hat{T}_W\| \), respectively.

Theorem 1 suggests that to keep the approximation errors as small as possible, a good index selection procedure that provides smaller quantities \( \|U_k^T P_B\|^{-1}\), \( \|(S_k^T V_k)^{-1}\|\), \( \|(Q W_k P)^{-1}\| \), and \( \|(S^T Q Z_k)^{-1}\| \) would be ideal. The DEIM procedure may be seen as an
attempt to attain exactly that. Meanwhile, the quantity \( \alpha_{k+1} \cdot \| \hat{T}_Z \| \| \hat{T}_W \| \) is a result of the error of the rank-\( k \) RSVD.

Comparing the results of the decomposition of \( A \) in Theorem 1 to (Sorensen and Embree, 2016, Theorem 4.1), we have that the \( \sigma_{k+1} \) in (Sorensen and Embree, 2016, Theorem 4.1) is replaced by the error in the RSVD through \( \| (I - Q Z_k^T Q Z_k^T) A \| \) and \( \| A (I - W_k Q W_k^T) \| \). Here, \( \| (Q W_k^T P)^{-1} \| \), and \( \| (S^T Q Z_k)^{-1} \| \) are computed using the orthonormal basis of the nonsingular matrices from the RSVD of \( A \) rather than the singular vectors. Compared with the results in (Gidisu and Hochstenbach, 2022a, Theorem 4.8) where all quantities are a result of the GSVD, in Theorem 1 we have an additional \( \| \hat{T}_Z \| \), and all quantities are a result of the RSVD.

One key assumption of Theorem 1 is that \( Q W_k^T P, U_k^T P B, S G V_k, \) and \( S^T Q Z_k \) are nonsingular. This is not applicable for the L-DEIM procedure since these resulting matrices are no longer square because we oversample. Let \( r \) be the number of columns and rows indices selected where, \( k < r \leq 2k \), so that we have the following index selection matrices \( \bar{P} \in R^{n \times r}, \bar{S} G \in R^{d \times r}, \bar{P} B \in R^{r \times 1}, \) and \( \bar{S} \in R^{n \times r} \) and \( C A = A P, C B = B \bar{P} B, C G = G \bar{P}, R A = S^T A, R B = S^T B, \) and \( \bar{R} _ G = S^T G \) are of a full rank. We now state the error bound for RSVD-CUR approximation using the L-DEIM index selection procedure; see Algorithm 4.

**Proposition 2** (Generalization of (Sorensen and Embree, 2016, Theorem 4.1) and (Hendryx, Rivière, and Rusin, 2021, Lemma 2 and 3)) Given \( A, B, \) and \( G \) as in Definition 1 and \( Z_k \in R^{n \times k}, W_k \in R^{n \times k}, U_k \in R^{k \times k}, \) and \( V_k \in R^{d \times k} \) from (7), let \( Q Z_k \in R^{n \times k}, Q W_k \in R^{n \times k} \) be the \( Q \)-factors of \( Z_k W_k, \) respectively, and \( \hat{T}_Z, T Z_k, \hat{T}_W, \) and \( T W_k \) as in (9)–(10). Suppose \( Q W_k^T \bar{P}, U_k^T \bar{P} B, \bar{S} G V_k, \) and \( S^T Q Z_k \) are of a full column rank and the error constants

\[
\bar{\eta}_p := \| (Q W_k^T \bar{P})^+ \|, \quad \bar{\eta}_b := \| (S^T Q Z_k)^+ \|, \quad \bar{\eta}_p B := \| (U_k^T \bar{P} B)^+ \|, \quad \bar{\eta}_s G := \| (S^T G V_k)^+ \|
\]

are finite, we have the equations as in (12) except here, we replace \( \bar{\eta}_p, \bar{\eta}_b, \) and \( \bar{\eta}_s G \) by their respective tilde variants.

**Proof** Since in Theorem 1 we have shown the proof for the error bound of an RSVD-CUR approximation of \( A \), here we shall prove the error bound of \( \| G - C G M G R G \| \), the results of \( \| A - C A M A R A \| \) and \( \| B - C B M B R B \| \) follow similarly. The proof is very similar to the proof of Theorem 1 so we only show the differences. First, note that \( \bar{P} (W_k^T \bar{P})^+ W_k^T = \bar{P} (Q W_k^T \bar{P})^+ Q W_k^T \) assuming \( Q W_k \) is an orthonormal basis for Range\((W_k)\) and \( W_k^T \bar{P}, Q W_k^T \bar{P} \) are of a full column rank. Let \( \bar{P} = \bar{P} (Q W_k^T \bar{P})^+ Q W_k^T \) and \( \bar{S} = V_k (S^T G V_k)^+ S^T G \) be interpolatory projectors. Contrary to the DEIM interpolatory projector which interpolates any vector \( x \), because we use the pseudo-inverse to form the projectors, \( \bar{S} \) and \( \bar{P} \) only maintain their interpolatory nature for vectors in Range\((W_k)\) and Range\((V_k)\). That is, if \( y \in \text{Range}(V_k) \), there exists \( x \in R^k \) so that \( y = V_k x \) and \( (S y)(\bar{s}) = S^T G V_k x = S^T G W_k \bar{x} = y(\bar{s}) \) (Hendryx, Rivière, and Rusin, 2021). We still have the nice properties of an oblique projector, i.e., \( Q W_k^T \bar{P} = Q W_k^T \) hence \( Q W_k^T (I - \bar{P}) = 0 \). Using the results of (Hendryx, Rivière, and Rusin, 2021, Lemma 2) we have these inequalities for the error in
the interpolatory projections of $G$,
\[
\|G - \tilde{S}G\| = \|(I - \tilde{S})G\| = \|(I - \tilde{S})(I - V_k V_k^T)G\|
\leq \|(I - \tilde{S})\| \|(I - V_k V_k^T)G\|,
\|G - \tilde{P}\| = \|G(I - \tilde{P})\| = \|G(I - Q_{W_k} Q_{W_k}^T)(I - \tilde{P})\|
\leq \|G(I - Q_{W_k} Q_{W_k}^T)\| \|I - \tilde{P}\|.
\]
Since $k < r$, we know that $\tilde{P} \neq 0$, $\tilde{P} \neq I$, $\tilde{S} \neq 0$, and $\tilde{S} \neq I$ hence (see, e.g., Szyld, 2006)
\[
\|I - \tilde{P}\| = \|	ilde{P}\| = \|(Q_{W_k}^T \tilde{P})^+\|, \quad \|I - \tilde{S}\| = \|	ilde{S}\| = \|(S_k^T V_k)^+\|.
\]
Using the matrix partitioning in (9)–(10), we have that
\[
G Q_{W_k} Q_{W_k}^T = [V_k \quad \tilde{V}] \begin{bmatrix} D_{G_k} & 0 \\ 0 & \tilde{D}_G \end{bmatrix} \begin{bmatrix} T_{W_k}^T & 0 \\ T_{W_{22}}^T & T_{W_{12}}^T \end{bmatrix} \begin{bmatrix} I_k \\ 0 \end{bmatrix} Q_{W_k}^T
= V_k D_{G_k} T_{W_k}^T Q_{W_k}^T + \tilde{V} \tilde{D}_G T_{W_{12}}^T Q_{W_k}^T,
\]
where $D_{G_k} \in \mathbb{R}^{k \times k}$ and $\tilde{D}_G \in \mathbb{R}^{(d-k) \times (n-k)}$. Hence
\[
G(I - Q_{W_k} Q_{W_k}^T) = (G - G_k) - \tilde{V} \tilde{D}_G T_{12}^T Q_{W_k}^T
= \tilde{V} \tilde{D}_G \tilde{T}^T Q^T - \tilde{V} \tilde{D}_G T_{W_{12}}^T Q_{W_k}^T = \tilde{V} \tilde{D}_G T_{W_{22}}^T Q_{W_k}^T.
\]
This implies
\[
\|G(I - Q_{W_k} Q_{W_k}^T)\| \leq \gamma_{k+1} \cdot \|T_{W_{22}}\|.
\]
Similarly we have that
\[
(I - V_k V_k^T)G = G - V_k D_{G_k} W_k^T = \tilde{V} \tilde{D}_G \tilde{W}^T = \tilde{V} \tilde{D}_G T_{W_k}^T Q_{W_k}^T,
\]
consequently
\[
\|(I - V_k V_k^T)G\| = \|G - G_k\| \leq \gamma_{k+1} \cdot \|\tilde{T}_W\|.
\]
Given that $M_G = C_G^+ GR_G^+$ and (see Mahoney and Drineas, 2009, Eq. 6)
\[
G - C_G M_G R_G = G - C_G C_G^+ G R_G^+ R_G = (I - C_G C_G^+) G + C_G C_G^+ (I - R_G^+ R_G),
\]
we leverage that fact that (see Hendryx, Rivi`ere, and Rusin, 2021, Lemma 3)
\[
\|(I - C_G C_G^+) G\| \leq \|G(I - \tilde{P})\|, \quad \|G(I - R_G^+ R_G)\| \leq \|(I - \tilde{S})G\|,
\]
and $C_G C_G^+$ is an orthogonal projector, $\|C_G C_G^+\| = 1$ so that
\[
\|G - C_G M_G R_G\| \leq \gamma_{k+1} \cdot (\tilde{\eta}_p \cdot \|\tilde{T}_W\| + \tilde{\eta}_{sc} \cdot \|T_{W_{22}}\|) \leq \gamma_{k+1} \cdot (\tilde{\eta}_p + \tilde{\eta}_{sc}) \cdot \|\tilde{T}_W\|. \quad \blacksquare
\]
The final inequality follows directly from the fact that the norm of any submatrix of $\tilde{T}_W$ is at most $\|\tilde{T}_W\|$.
Notice that the difference between the error bounds in Theorem 1 and Proposition 2 is the Lebesgue constant for the discrete interpolation, i.e., \( \tilde{\eta}_p \) and \( \eta_p \) for the L-DEIM and the DEIM scheme, respectively. In (Sorensen and Embree, 2016, Lemma 4.4), the authors present an upper bound on the DEIM selection scheme error constants. We show here that this bound holds for the error constants from the L-DEIM selection scheme. Given a matrix with orthogonal columns \( V_k \in \mathbb{R}^{d \times k} \), we define the selection of \( k \) and \( r \) rows of \( V_k \) (where \( r > k \)) by the DEIM and L-DEIM scheme, respectively, as

\[
\begin{bmatrix}
S_D^T \\
S_L^T
\end{bmatrix}
V_k =: \begin{bmatrix}
Z_D \\
Z_L
\end{bmatrix} =: Z,
\]

where \( S_D \) selects \( k \) rows and \( S_L \) selects additional \( (r-k) \) rows. The error constant of the DEIM procedure is \( \sigma_{\min}^{-1}(Z_D) \) while that of the L-DEIM algorithm is \( \sigma_{\min}^{-1}(Z) \leq \sigma_{\min}^{-1}(Z_D) \) (see Experiment 1 for an illustration). This implies that any upper bound for \( \sigma_{\min}^{-1}(Z_D) \) is also an upper bound for \( \sigma_{\min}^{-1}(Z) \). Therefore, given that \( k < r < \min(d, \ell, m, n) \), we have

\[
\begin{align*}
\tilde{\eta}_p & := \|(Q^TW_k \tilde{P})^+\| < \sqrt{\frac{nk}{3}} 2^k, \\
\tilde{\eta}_p_B & := \|(U_k^TP_B)^+\| < \sqrt{\frac{nk}{4}} 2^k, \\
\tilde{\eta}_s & := \|(\tilde{S}_TQZ_k)^+\| < \sqrt{\frac{mk}{3}} 2^k, \\
\tilde{\eta}_s_G & := \|(\tilde{S}_TGV_k)^+\| < \sqrt{\frac{dk}{4}} 2^k.
\end{align*}
\]

For constructive proofs and further explanation of the bound on the Lebesgue constant for the discrete interpolation, we refer the reader to (Sorensen and Embree, 2016, Lemma 4.4).

5. Numerical Experiments

In this section, we first evaluate the performance of the proposed RSVD-CUR decomposition for reconstructing a data matrix perturbed with nonwhite white. We then show how the proposed factorization can be used for feature selection in multi-view/label classification problems. An overview of the various examples and sizes for \( A \in \mathbb{R}^{m \times n} \), \( B \in \mathbb{R}^{m \times \ell} \), and \( G \in \mathbb{R}^{d \times n} \) is provided in Table 1.

| Exp. | Problem | Relations | \( m \) | \( n \) | \( \ell \) | \( d \) |
|------|---------|-----------|--------|--------|--------|--------|
| 1–2  | Perturbation | \( m = \ell, n = d \) | 10000 | 10000 | 10000 | 1000 |
| 3    | Multiview  | \( \ell = d > m > n \) | 240 | 76 | 2000 | 2000 |
|      |          |           | 76 | 64 | 2000 | 2000 |
|      |          |           | 928 | 512 | 2386 | 2386 |
| 4    | Multilabel | \( \ell = d > m > n \) | 1836 | 159 | 7395 | 7395 |
|      |          |           | 103 | 14 | 2417 | 2417 |
|      |          |           | 2150 | 208 | 87856 | 87856 |

**Experiment 1** For our first experiment, we consider a matrix perturbation problem of the form \( A_E = A + BFG \) where \( F \) is a random matrix and \( B,G \) are the Cholesky factors of two non-diagonal covariance matrices. The goal is to reconstruct a low-rank matrix \( A \) from \( A_E \) assuming that at least the structure of the covariance matrices is known. We consider two popular covariance structures (Wicklin, 2013): (i) a compound symmetry structure, which
means the covariance matrix has constant diagonal and constant off-diagonal entries; (ii) a first-order autoregressive structure, which means the matrix has a constant diagonal and the off-diagonal entries are related to each other by a multiplicative factor. We evaluate and compare a rank-$k$ RSVD-CUR and CUR decomposition of $A_E$ in terms of reconstructing matrix $A$. The approximation quality of each decomposition is assessed by the relative matrix approximation error, i.e., $\|A - \tilde{A}\|/\|A\|$, where $\tilde{A}$ is the reconstructed low-rank matrix. As an adaptation of the first experiment in (Sorensen and Embree, 2016, Ex. 6.1) we generate a rank-100 sparse nonnegative matrix $A \in \mathbb{R}^{10000 \times 1000}$ of the form

$$A = \sum_{j=1}^{10} \frac{2}{j} x_j y_j^T + \sum_{j=11}^{100} \frac{1}{j} x_j y_j^T,$$

where $x_j \in \mathbb{R}^{10000}$ and $y_j \in \mathbb{R}^{1000}$ are random sparse vectors with nonnegative entries. We then perturb $A$ with a nonwhite noise matrix $BFG$ (see, e.g., Hansen, 1998, p. 55). The matrix $F \in \mathbb{R}^{10000 \times 1000}$ is random Gaussian noise. We assume that $B \in \mathbb{R}^{10000 \times 10000}$ is the Cholesky factor of a symmetric positive definite covariance matrix with compound symmetry structure, and $G \in \mathbb{R}^{1000 \times 1000}$ is the Cholesky factor of a symmetric positive definite covariance matrix with first-order autoregressive structure. The resulting perturbed matrix we use is of the form $A_E = A + \varepsilon \|A\|/\|BFG\| BFG$, where $\varepsilon$ is the noise level. Given a noise level, we compute the SVD of $A_E$ and the RSVD of $(A_E, B, G)$ to obtain the input matrices for a CUR and an RSVD-CUR decomposition, respectively. Figure 1 summarizes the results of three noise levels (0.1, 0.15, 0.2). Given each noise level, we generate ten random cases and take the average of the relative errors for varying $k$ values.

We observe that to approximate $A$, the RSVD-CUR factorization enjoys a considerably lower average approximation error than a CUR decomposition irrespective of the index selection procedure used (whether the DEIM or the L-DEIM scheme). Meanwhile, the average relative error of an RSVD-CUR approximation unlike that of the RSVD (monotonically decreasing) approaches $\varepsilon$ after a certain value of $k$. This situation is natural because the RSVD-CUR routine picks actual columns and rows of $A_E$ so, the relative error is likely to be saturated by the noise. The rank-$k$ SVD of $A_E$ fails in approximating $A$ for the given values of $k$. Its average relative error rapidly approaches $\varepsilon$; this is expected since the covariance of the noise is not a multiple of the identity. It is worth noting that the improved performance of an RSVD-CUR approximation compared to a CUR factorization is particularly more attractive for higher noise levels with modest $k$ values, i.e., $k$ is significantly less than rank($A$).

Using 100 different test cases of $A$, Fig. 2 illustrates that given $k$ orthogonal vectors, the average error constants $\tilde{\eta}_p$ and $\tilde{\eta}_s$ (see Proposition 2) from the L-DEIM procedure cannot be larger than $\eta_p$ and $\eta_s$ (see Theorem 1), respectively, from the DEIM scheme. For varying $k$ vectors, we pick $2k$ indices, which results in $2k \times k$ matrices for the L-DEIM as compared to $k \times k$ matrices for the DEIM.

In Fig. 3, using an RSVD-CUR decomposition of $A_E$, we show the various quantities in Theorem 1. We observe that the upper bound in Theorem 1 may be a rather crude bound on the true RSVD-CUR error. As in (Sorensen and Embree, 2016, Fig. 4), the magnitude of the quantities $\eta_s$ and $\eta_p$ may vary. We see that $\|\hat{T}_W\|$ and $\|\hat{T}_Z\|$ seem to stabilize as $k$ increases.
Figure 1: The approximation quality of RSVD-CUR approximations compared with CUR approximations in recovering a sparse, nonnegative matrix $A$ perturbed with a nonwhite noise. The average relative errors $\|A - \tilde{A}_k\|/\|A\|$ (on the vertical axis) as a function of rank $k$ (on the horizontal axis) for $\varepsilon = 0.2, 0.15, 0.1,$ respectively.

Figure 2: Comparison of the average error constants $\eta_p = \|(Q_W^T P)^{-1}\|$ (blue solid) and $\eta_s = \|(S^T Q_Z)^{-1}\|$ (red solid) of the $k \times k$ matrices from DEIM, and $\tilde{\eta}_p = \|(Q_W^T \tilde{P})^+\|$ (blue dashed) and $\tilde{\eta}_s = \|\tilde{(S^T Q_Z)}^+\|$ (red dashed) of the $2k \times k$ matrices from L-DEIM, for varying $k$ available vectors.

**Experiment 2** In this experiment, we examine the case of inexact Cholesky factors $\tilde{B}$ and $\tilde{G}$. Suppose that given $A_E$, the exact noise covariance matrices $B^T B$ and $G^T G$ are unknown;
we now investigate the approximation quality of an RSVD-CUR decomposition compared with a CUR factorization in reconstructing $A$ from $A_E$. We derive the inexact Cholesky factors $\hat{B}$ and $\hat{G}$ by multiplying all off-diagonal elements of the exact Cholesky factors $B$ and $G$ by uniform random numbers from the interval $[0.9, 1.1]$. Aside from the Cholesky factors, we maintain the experimental setup described in Experiment 1 using noise levels $\varepsilon = 0.1, 0.2$; the difference here is we compute the RSVD of $(A_E, \hat{B}, \hat{G})$ to get the input matrices for an RSVD-CUR decomposition. Figures 4a and 4b show that, when we use inexact Cholesky factors, the RSVD and an RSVD-CUR factorization still deliver good approximation results, which may imply that we may not necessarily need the exact noise covariance.

Figure 4: The approximation quality of RSVD-CUR factorizations using inexact Cholesky factors of the noise covariances compared with CUR decompositions in recovering a sparse, nonnegative matrix $A$ perturbed with a nonwhite noise. The average relative errors $\|A - \hat{A}_k\|/\|A\|$ (on the vertical axis) as a function of rank $k$ (on the horizontal axis) for $\varepsilon = 0.2, 0.1$, respectively.
Experiment 3 In this experiment, we demonstrate the effectiveness of an RSVD-CUR in discovering the underlying class structure shared by two views of data. We show that classification accuracy can be improved using an RSVD-CUR decomposition as a feature selection method in a multi-view classification problem. We compare the classification accuracy results of the DEIM type RSVD-CUR with that of the DEIM type CUR decomposition. Let the first view of the data be matrix $B$, the second view be matrix $G$, and a column concatenation of both views be $BG$, we compare the following results:

(i) CUR on single views (CUR-$B$ and CUR-$G$); run DEIM type CUR on each view to select a subset of the original features.

(ii) Fused-CUR; concatenate the features selected in (i), i.e., [CUR-$B$, CUR-$G$].

(iii) Concat-CUR; run DEIM type CUR on the concatenated views $BG$, i.e., CUR-$BG$.

(iv) Run RSVD-CUR on the two views, thus, RSVD-CUR of $(B^TG, G^TB, B)$ so that RSVD-CUR-$B$, RSVD-CUR-$G$ are the RSVD-CUR selected features of $B$, $G$, respectively, and Fused-RSVD-CUR is [RSVD-CUR-$B$, RSVD-CUR-$G$].

We use the handwritten digits from the UCI repository for our first two experiments. This data set comprises 2000 instances and three views, which we combine to form two experiments on multi-view classification. The first experiment has the pixel averages in $2 \times 3$ windows (pix) as the first view and the Fourier coefficients of the character shapes (fou) as the second view. In the second experiment, we take the Fourier coefficients of the character shapes (fou) and the Karhunen-Love coefficients (kar) as view-1 and view-2, respectively. Our third experiment uses the Caltech-20, a subset of Caltech-101 data set containing 20 classes. The data set consists of 2386 samples, with view-1 being the local binary patterns (lbp) feature and view-2 being the gist feature. Table 2 summarizes the basic traits of the various data sets. We normalize all the data sets to have zero center and a standard deviation of 1.

Table 2: Summary characteristics of multiview data sets used in the experimentation

| Data set            | Samples | View 1 ($B$) | View 2 ($G$) |
|---------------------|---------|--------------|--------------|
| Digits (pix vs. fou)| 2000    | 240          | 76           |
| Digits (fou vs. kar)| 2000    | 76           | 64           |
| Caltech-20 (lbp vs. gist) | 2386  | 928          | 512          |

For each experiment, we randomly split the normalized data into train and test data of ratio 75:25. For randomization of the experiments, using the default $k$-nearest neighbor ($k$-NN) classifier in MATLAB, we conduct 20 runs with different random seeds and Fig. 5 reports the average classification accuracy for varying reduced dimensions.

From Fig. 5, we observe that an RSVD-CUR method consistently performs better than a CUR scheme. In particular, from the classification results using single views, the RSVD-CUR significantly improves the worse CUR single view results, as seen in the first column plots of Fig. 5. We notice that using information from multiple views indeed improves
classification accuracy. Furthermore, feature fusion from an RSVD-CUR approximation gives the best classification accuracy rate compared with concatenating all views into a single matrix and performing a CUR decomposition on this matrix or performing a CUR factorization on each feature set and merging the reduced data.

Figure 5: The average classification accuracy over 20 different random train-test splits of each data set using CUR and RSVD-CUR as a feature selection method for a k-nearest neighbor classifier in Experiment 3. The average classification accuracy (on the vertical axis) as a function of reduced dimensions $k$ (on the horizontal axis). The $k$ values of the combined views (second column) are twice that of the single views (first column).
Experiment 4  Our final experiment illustrates how an RSVD-CUR decomposition may be viewed as a supervised feature selection technique for multi-label classification problems. Traditional supervised feature selection techniques are usually for one output class. They may be suitable for multi-label data sets after transforming the data set by using some transformation methods such as label powerset and binary relevance (Herrera et al., 2016). An RSVD-CUR decomposition provides an alternative way that does not require transforming the data set. This decomposition exploits the subspace that maximizes the cross-correlation across the feature and output spaces. Here we take the feature space as the matrix $B$ and the response space as matrix $G$. Note that one can reduce the dimension of the feature space to at most the number of class labels (that is if the number of features is greater than the number of class labels).

Using three benchmark multi-label data sets from the Mulan database\(^3\), we investigate the performance of the L-DEIM type RSVD-CUR (which incorporates information from the output space) as a feature selection technique for multi-label classification problems compared with the L-DEIM type CUR. The characteristics of the data sets are summarized in Table 3. The bibtex and bookmarks data contains the metadata for bibliographic entries and bookmarks shared by users, respectively; the origin of these data sets is Bibsonomy\(^4\). The yeast data set contains micro-array expression data and phylogenetic profiles. A subset of 14 functional classes from the comprehensive yeast genome database\(^5\) is used as the labels. Each gene can be associated with more than one functional class.

We train two multi-label classifiers with logistic regression classifier as the baseline using both the original and reduced feature spaces. That is, the classifier chains (CC) developed by Read et al. (2011) and the binary relevance (BR) proposed by Godbole and Sarawagi (2004). We use the default arguments of the various classifiers as implemented in the sklearn package. For convenience, we first randomly select 50\% of the samples in the bookmarks data set and then split that into 80\% train set and 20\% test set. For the other two data sets, we use the train and test data sets as provided in the Mulan database. To assess the performance of the classifiers after the feature selection process, we employ the micro average F1 score (a label-based metric) and the hamming loss (a sample-based metric) (Herrera et al., 2016). For the F1 score, a higher value indicates good performance. On the other hand, for hamming loss, a lower value implies good performance. We emphasize that the goal is to evaluate which reduced feature set produce a better classification result (we are not concerned about which classifier is the best).

Table 3: Basic traits of multilabel data sets used in the experimentation

| Data set   | Domain | Samples | Features ($B$) | Class labels ($G$) |
|------------|--------|---------|----------------|-------------------|
| bibtex     | text   | 7395    | 1836           | 159               |
| yeast      | biology| 2417    | 103            | 14                |
| bookmarks  | text   | 87856   | 2150           | 208               |

3. http://mulan.sourceforge.net/datasets-mlc.html
4. https://www.bibsonomy.org/
5. https://pubmed.ncbi.nlm.nih.gov/15608217/
In Fig. 6, we observe that incorporating information from the output space in selecting the features, i.e., an RSVD-CUR method yields better classification results than a CUR method irrespective of the classifier used. For the bookmarks data set, we see that the features selected by the RSVD-CUR scheme produced a slight improvement over those picked by the CUR method for both metrics. For the other two data sets, the improved results are very apparent (we include the performance of the classifiers using all the original features as reference only).

Figure 6: The hamming loss and micro average F1 measure of each data set using CUR and RSVD-CUR as a feature selection method for classifier chains (CC) and binary relevance (BR) multilabel classifiers in Experiment 4. The evaluation metrics (on the vertical axis) as a function of reduced dimensions $k$ (on the horizontal axis).
6. Conclusions

In this paper, we propose a new low-rank matrix decomposition, an RSVD-CUR factorization with pseudocode in Algorithms 3 and 4. This factorization is an extension of a CUR decomposition for matrix triplets. We use the DEIM and the L-DEIM index selection procedures to construct $C$ and $R$ factors of this factorization. Note that other than these two index selection schemes, one may use alternative selection methods such as column-pivoting QR decomposition (Drmac and Gugercin, 2016) or maximum volume algorithm (Goreinov et al., 2010) on the matrices from the RSVD. We have discussed the connection between the DEIM type RSVD-CUR of $(A, B, G)$ and the DEIM type CUR of $B^{-1}AG^{-1}$ for a square and nonsingular $B, G$ as well as $B^+AG^+$ for a nonsquare but full-rank $B, G$. In a particular case where $B = I$ and $G = I$, the RSVD-CUR decomposition of $A$ coincides with a CUR decomposition of $A$, in that the factors $C$ and $R$ of $A$ are the same for both methods: the first line of (6) is equal to (1). We have also pointed out the connection between the DEIM type RSVD-CUR of $(A, B, G)$ and the DEIM type GCUR of $(A, G)$ or the transpose of $(A, B)$ when $B = I$ or $G = I$, respectively.

An RSVD-CUR factorization may be suitable for feature fusion and applications where one is interested in selecting a subset of features in one data set relative to two other data sets. For subset selection in a multi-view classification problem where two feature sets are available, the new method may yield better classification accuracy than the standard CUR decomposition as shown in the numerical experiments. The proposed method may also be used as a supervised feature selection technique in multilabel classification problems. An RSVD-CUR approximation may also be useful in data perturbation problems of the form $A_E = A + BFG$, where $BFG$ is a nonwhite noise matrix and $B, G$ are matrices of a known covariance structure. An RSVD-CUR decomposition can provide more accurate approximation results compared to a CUR factorization when reconstructing a low-rank matrix from a data matrix perturbed with nonwhite noise. As shown in Section 5 we do not necessarily need to know the exact noise covariance matrices; the RSVD and RSVD-CUR may still deliver good approximation results given inexact Cholesky factors $B$ and $G$ of the noise covariance matrices.

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