Global regularity for a Birkhoff-Rott-\(\alpha\) approximation of the dynamics of vortex sheets of the 2D Euler equations

Claude Bardos,1†, Jasmine S. Linshiz,2† and Edriss S. Titi2,3,‡

1Université Denis Diderot and Laboratory J.-L. Lions
Université Pierre et Marie Curie, Paris, France
2Department of Computer Science and Applied Mathematics
Weizmann Institute of Science
Rehovot 76100, Israel
3Department of Mathematics and
Department of Mechanical and Aerospace Engineering
University of California
Irvine, CA 92697-3875, USA

We present an \(\alpha\)-regularization of the Birkhoff-Rott equation, induced by the two-dimensional Euler-\(\alpha\) equations, for the vortex sheet dynamics. We show that initially smooth self-avoiding vortex sheet remains smooth for all times under the \(\alpha\)-regularized dynamics, provided the initial density of vorticity is an integrable function over the curve with respect to the arc-length measure.

PACS numbers: 47.32.C, 47.20.Ma, 47.20.Ft, 47.15.ki
Keywords: inviscid regularization of Euler equations, Birkhoff-Rott regularization, Birkhoff-Rott-\(\alpha\), vortex sheet regularization

I. INTRODUCTION

One of the novel approaches for subgrid scale modeling is the \(\alpha\)-regularizations of the Navier-Stokes equations (NSE). The inviscid Euler-\(\alpha\) model was originally introduced in the Euler-Poincaré variational framework in [1,2]. In [3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22], the corresponding Navier-Stokes-\(\alpha\) (NS-\(\alpha\)) model, also known as the viscous Camassa-Holm equations or the Lagrangian-averaged Navier-Stokes-\(\alpha\) (LANS-\(\alpha\)) model, was obtained by introducing the appropriate viscous term into the Euler-\(\alpha\) equations. The extensive research of the \(\alpha\)-models (see, e.g., [3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22]) stems from the successful comparison of their steady state solutions to empirical data, for a large range of huge Reynolds numbers, for turbulent flows in infinite channels and pipes. On the other hand, the \(\alpha\)-models can also be viewed as numerical regularizations of the original, Euler or Navier-Stokes, systems. The main practical question arising is that of the applicability of these regularizations to the correct predictions of the underlying flow phenomena.

In this paper we present some results concerning the \(\alpha\)-regularization of the two-dimensional (2D) Euler equations in the context of vortex sheet dynamics. A vortex sheet is a surface of codimension one (a curve in the plane) in inviscid incompressible flow, across which the tangential component of the velocity has a jump discontinuity, while the normal component is continuous. The evolution of the vortex sheet can be described by the Birkhoff-Rott (BR) equation [23,24,25]. This is a nonlinear singular integro-differential equation, which can be obtained formally from the Euler equations assuming that the evolution of a vortex sheet retains a curve-like structure. However, the initial data problem for the BR equation is ill-posed due to the Kelvin-Helmholtz instability [23,24]. Numerous results show that an initially real analytic vortex sheet can develop a finite time singularity in its curvature. This singularity formation was studied with asymptotic techniques in [26,29] and numerically in [30,31,32]. Specific examples of solutions were constructed in [33,34], where the development, in a finite time, of curvature singularity from initially analytic data was rigorously proved.

The problem of the evolution of a vortex sheet can also be approached, in the general framework of weak solutions (in the distributional sense) of the Euler equations, as a problem of evolution of the vorticity, which is concentrated as a measure along a surface of codimension one. The general problem of existence for mixed-sign vortex sheet initial data remains an open question. However, in 1991, Delort [35] proved a global in time existence of weak solutions of the 2D incompressible Euler equation for the vortex sheet initial data with initial vorticity being a Radon measure of a distinguished sign, see also [36,37,38,39,40,41]. This result was later obtained as an inviscid limit of the Navier-Stokes regularizations of the Euler equations [42,43], and as a limit of vortex methods [38,40]. The Delort’s result was also extended to the case of mirror-symmetric flows with distinguished sign vorticity on each side of the mirror [42]. However, the problem of uniqueness of a weak solution with a fixed sign vortex sheet initial data is still unanswered, numerical evidences of non-uniqueness can be found, e.g., in [43,44]. Furthermore, the structure of
weak solutions given by Delort’s theorem is not known, while the Birkhoff-Rott equations assume *a priori* that a vortex sheet remains a curve at a later time. A proposed criterion for the equivalence of a weak solution of the 2D Euler equations with vorticity being a Radon measure supported on a curve, and a weak solution of the Birkhoff-Rott equation can be found in [43]. Also, another definition of weak solutions of Birkhoff-Rott equation has been proposed in [46, 47]. For a recent survey of the subject, see [48].

The question of global existence of weak solutions for the three-dimensional Euler-α equations is still an open problem. On the other hand, the 2D Euler-α equations were studied in [10], where it has been shown that there exists a unique global weak solution to the Euler-α equations with initial vorticity in the space of Radon measures on $\mathbb{R}^2$, with a unique Lagrangian flow map describing the evolution of particles. In particular, it follows that the vorticity, initially supported on a curve, remains supported on a curve for all times.

We present in this paper an analytical study of the $\alpha$-analogue of the Birkhoff-Rott equation, the Birkhoff-Rott-$\alpha$ (BR-$\alpha$) model, which is induced by the 2D Euler-α equations. The BR-$\alpha$ model was implemented computationally in [50], where a numerical comparison between the BR-$\alpha$ regularization and the existing regularizing methods, such as a vortex blob model [8, 51, 52, 54] has been performed. We remark that, unlike the vortex blob methods that regularize the singular kernel in the Birkhoff-Rott equation, the $\alpha$-model regularizes instead the Euler equations themselves to obtain a smoother kernel.

We report in Section IV our main result, which states that the initially smooth self-avoiding 2D vortex sheet, evolving under the BR-$\alpha$ equation, remains smooth for all times. In this short communication we only report the results and sketch some of their proofs, the full details will be reported in a forthcoming paper. In Section II we describe the BR-$\alpha$ equation. Section III studies the linear stability of a flat vortex sheet with uniform vorticity density for the 2D BR-$\alpha$ model. The linear stability analysis shows that the BR-$\alpha$ regularization controls the growth of high wave number perturbations, which is the reason for the well-posedness. This is unlike the case for the original BR problem that exhibits the Kelvin-Helmholtz instability, the main mechanism for its ill-posedness.

II. BIRKHOFF-ROTT-$\alpha$ EQUATION

The incompressible Euler equations in $\mathbb{R}^2$ in the vorticity form are given by

$$\begin{align*}
\frac{\partial q}{\partial t} + (v \cdot \nabla) q &= 0, \\
v &= K^\alpha * q, \\
q(x, 0) &= q^{\text{in}}(x),
\end{align*}$$

where $K(x) = \frac{1}{2\pi} \log|\cdot|$, $v$ is the fluid velocity field, $q = \text{curl} v$ is the vorticity, and $q^{\text{in}}$ is the given initial vorticity.

The 2D Euler-$\alpha$ model [1, 2, 5, 18, 49, 57] is an inviscid regularization of the Euler equations, such that the vorticity is governed by the system

$$\begin{align*}
\frac{\partial q}{\partial t} + (u \cdot \nabla) q &= 0, \\
u &= K^\alpha * q, \\
q(x, 0) &= q^{\text{in}}(x).
\end{align*}$$

Here $u$ represents the “filtered” fluid velocity, and $\alpha > 0$ is a length scale parameter, which represents the width of the filter. At the limit $\alpha = 0$, we formally obtain the Euler equations [10]. The smoothed kernel is $K^\alpha = G^\alpha * K$, where $G^\alpha$ is the Green function associated with the Helmholtz operator $(I - \alpha^2 \Delta)$, given by

$$G^\alpha(x) = \frac{1}{\alpha^2} G\left(\frac{x}{\alpha}\right) = -\frac{1}{\alpha^2} \frac{1}{2\pi} K_0\left(\frac{|x|}{\alpha}\right),$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ and $K_0$ is a modified Bessel function of the second kind [58].

Let $\mathcal{M}(\mathbb{R}^2)$ denote the space of Radon measures on $\mathbb{R}^2$; $\mathcal{G}$ denote the group of all homeomorphisms of $\mathbb{R}^2$, which preserve the Lebesgue measure; and $\eta = \eta(\cdot, t)$ denote the Lagrangian flow map induced by (2) and obeying the equation $\partial_t \eta(x,t) = u(\eta(x,t), t)$, $\eta(x, 0) = x$.

Oliver and Shkoller [49] showed global well-posedness of the Euler-$\alpha$ equations [2] with initial vorticity in $\mathcal{M}(\mathbb{R}^2)$ (which includes point-vortex data).

**Theorem 1.** (Oliver and Shkoller [49]) For initial data $q^{\text{in}} \in \mathcal{M}(\mathbb{R}^2)$, there exists a unique global weak solution (in the sense of distribution) to (2) with

$$\eta \in C^1(\mathbb{R}; \mathcal{G}), u \in C(\mathbb{R}; \mathcal{G}(\mathbb{R}^2)), q \in C(\mathbb{R}; \mathcal{M}(\mathbb{R}^2)).$$

The Birkhoff-Rott-$\alpha$ equation, based on the Euler-$\alpha$ equations is derived similarly to the derivation of the original Birkhoff-Rott equation. Detailed descriptions of the Birkhoff-Rott equation as a model for the evolution of the vortex sheet can be found, e.g., in [27, 41, 50]. We remark that while the BR equations assume *a priori* that a vortex sheet remains a curve at a later time, in the 2D Euler-$\alpha$ case, if the vorticity is initially supported on a curve, then due to the existence of the unique Lagrangian flow map given by Theorem 1 it remains supported on a curve for all times. Hence the BR-$\alpha$ equation gives an equivalent description of the vortex sheet evolution, as the 2D Euler-$\alpha$ equations. It is described in the following proposition.

**Proposition 2.** Let $q$ be the solution of (2) in the sense of the Theorem 1. Assume, furthermore, that $q$ has the density $\gamma(\sigma, t)$ supported on the sheet (curve) $\Sigma(t) = \{x = x(\sigma, t) \in \mathbb{R}^2 | \sigma_0(t) \leq \sigma \leq \sigma_1(t)\}$, that is, the vorticity $q(x, t)$ satisfies

$$\int_{\mathbb{R}^2} \varphi(x) dq(x, t) = \int_{\sigma_0(t)}^{\sigma_1(t)} \varphi(x(\sigma, t)) \gamma(\sigma, t)|x_\perp(\sigma, t)| d\sigma,$$
for every $\varphi \in C_0^\infty (\mathbb{R}^2)$, $\gamma (\cdot, t) \in L^1(|x_\sigma| \, d\sigma)$. Then this sheet evolves according to the equation

$$\frac{\partial}{\partial t} x (\sigma, t) = \int_{\sigma(t)}^{\sigma(t)} K^\alpha \left( x (\sigma, t) - x (\sigma', t) \right) \gamma (\sigma', t) |x_\sigma (\sigma', t)| \, d\sigma'.$$

Additionally, if $\Gamma (\sigma, t) = \int_{\sigma_0}^{\sigma} \gamma (\sigma', t) |x_\sigma (\sigma', t)| \, d\sigma'$, where $x (\sigma^*, t)$ is some fixed reference point on $\Sigma (t)$, defines a strictly increasing function of $\sigma$ (e.g., as in the case of positive vorticity), then the evolution equation is given by the Birkhoff-Rott-$\alpha$ (BR-$\alpha$) equation

$$\frac{\partial}{\partial t} x (\Gamma, t) = \int_{\Gamma_0}^{\Gamma} K^\alpha \left( x (\Gamma, t) - x (\Gamma', t) \right) \, d\Gamma'$$

with $\gamma = 1/|x_\Gamma|$ being the vorticity density along the sheet and $-\infty < \Gamma_0 < \Gamma_1 < \infty$.

Here $\sigma_0, \sigma_1$ can represent either a finite length curve, or an infinite one. Notice that

$$K^\alpha (x) = \nabla^\perp \Psi^\alpha (|x|) = \frac{x^\perp}{|x|} D\Psi^\alpha (|x|),$$

where

$$\Psi^\alpha (r) = \frac{1}{2\pi} \left[ K_0 \left( \frac{r}{\alpha} \right) + \log r \right]$$

and

$$D\Psi^\alpha (r) = \frac{d\Psi^\alpha}{dr} (r) = \frac{1}{2\pi} \left[ -\frac{1}{\alpha} K_1 \left( \frac{r}{\alpha} \right) + \frac{1}{r} \right].$$

$K_0$ and $K_1$ denote modified Bessel functions of the second kind of orders zero and one, respectively. For details on Bessel functions, see, e.g., [58]. We remark that the smoothed kernel $K^\alpha (x)$ is a bounded continuous function, that for $|x| \to 0$ behaves as $K^\alpha (x) = \frac{1}{4\pi^2} x^\perp \log |x| + O \left( \frac{|x|}{\alpha^2} \right)$. That is, it is non-singular kernel. Since $\gamma (\cdot, t) \in L^1(|x_\sigma| \, d\sigma)$ we can show the integrability of the relevant terms, even though $|K^\alpha (x)|$ is decaying like $|x|^{-1}$ at infinity.

III. LINEAR STABILITY OF A FLAT VORTEX SHEET WITH UNIFORM VORTICITY DENSITY FOR 2D BR-$\alpha$ MODEL

The initial data problem for the BR equation is highly unstable due to an ill-posed response to small perturbations called Kelvin-Helmholtz instability [25, 28]. The linear stability analysis of the BR-$\alpha$ equation shows that the ill-posedness of the original problem is mollified, and the Kelvin-Helmholtz instability of the original system now disappears.

When the vortex sheet can be parameterized as a graph of a function in the form $x_2 = x_2 (x_1, t)$ the BR-$\alpha$ system [4] takes the form

$$\begin{align*}
\frac{\partial x_2}{\partial t} &= \frac{\partial x_2}{\partial x_1} u_1 + u_2, \\
\frac{\partial \gamma}{\partial t} &= -\frac{\partial}{\partial x_1} (\gamma u_1),
\end{align*}$$

with velocity $u = (u_1, u_2)^t$ given by

$$u \left( x_1, t \right) = p.v. \int_{\mathbb{R}} K^\alpha (x (x_1, t) - x (x'_1, t)) \gamma (x'_1, t) \, dx'_1,$$

where $x (x_1, t) = (x_1, x_2 (x_1, t))^t$. The flat sheet $x_2^0 = 0$ with uniformly concentrated intensity $\gamma_0$ is stationary solution of [5]. By linearization about the flat sheet we obtain the following linear system

$$\begin{align*}
\frac{\partial \tilde{x}_2}{\partial t} &= \tilde{u}_2, \\
\frac{\partial \tilde{\gamma}}{\partial t} &= -\gamma_0 \frac{\partial \tilde{u}_1}{\partial x_1},
\end{align*}$$

where

$$\tilde{u}_1 (x_1, t) = -\gamma_0 \left( \operatorname{sgn} (x_1) D\Psi^\alpha (|x_1|) \right) \frac{\partial \tilde{x}_2}{\partial x_1},$$

$$\tilde{u}_2 (x_1, t) = \left( \operatorname{sgn} (x_1) D\Psi^\alpha (|x_1|) \right) \tilde{\gamma},$$

and $(\tilde{x}_2, \tilde{\gamma})$ is a small perturbation about the flat sheet.

Consequently, the equation for the Fourier modes is given by

$$\frac{d}{dt} \begin{pmatrix} \hat{x}_2 \\ \hat{\gamma} \end{pmatrix} = \begin{pmatrix} 0 & i/k \operatorname{sgn} (k) d(k) \\ -i/k^2 \operatorname{sgn} (k) d(k) & 0 \end{pmatrix} \begin{pmatrix} \hat{x}_2 \\ \hat{\gamma} \end{pmatrix},$$

where

$$d(k) = \left( 1 + \frac{1}{\alpha^2 k^2} \right)^{-1/2} - 1.$$
eigenvalues of the original BR equations \( \pm \frac{1}{2} |\gamma_0| |k| \) (see, e.g., \cite{[60]}).

For the sake of comparison, we observe that for the vortex blob regularization of Krasny \cite{[32]}, where the singular BR kernel, \( K(x) \), was replaced with the smoothed kernel

\[
K_\delta (x) = K(x) \frac{|x|^2}{|x|^2 + \delta^2} = \frac{1}{2\pi} \frac{x^\perp}{|x|^2 + \delta^2},
\]

the eigenvalues are

\[
\lambda(k) = \pm \frac{1}{2} e^{-\delta k} |\gamma_0| |k|
\]

with an exponential decay to zero, as \( k \to \infty (\delta > 0 \) is fixed). As \( \delta \to 0 \), for fixed \( k \), one recovers again the eigenvalues of the original BR equations.

The behavior of the eigenvalues of the linearized system \cite{[60]} indicates that high wave number perturbations grow exponentially in time with a rate that decays to zero, as \( k \to \infty \), which is the reason for well-posedness of the \( \alpha \)-regularized model. This is unlike the original BR problem that exhibits the Kelvin-Helmholtz instability. It is worth mentioning that the \( \alpha \)-regularization is “closer” to the original system than the vortex-blob method at the high wave numbers, due to the algebraic decay instead of exponential one in the vortex blob method. This result was also evaluated computationally in \cite{[50]}.

### IV. GLOBAL REGULARITY FOR BR-\( \alpha \) EQUATION

In this section we present the global existence and uniqueness of solutions of the BR-\( \alpha \) equation \cite{[41]} in the appropriate space of functions. We show that initially smooth solutions of \cite{[41]} remain smooth for all times.

Let us first describe the Hölder space \( C^{n, \beta} (\Sigma \subset \mathbb{R}; \mathbb{R}) \), \( 0 < \beta \leq 1 \), which is the space of functions \( x : \Sigma \subset \mathbb{R} \to \mathbb{R}^2 \), with finite norm

\[
\|x\|_{C^{n, \beta}(\Sigma)} = \sum_{k=0}^{n} \left| \frac{d^k}{d^k x} x_{\beta}(\Sigma) \right| + \left| \frac{d^n}{d^\gamma x} x_{\beta}(\Sigma) \right|,
\]

where

\[
|x|_{C^{0}(\Sigma)} = \sup_{x \in \Sigma} |x(\Gamma)|
\]

and \( |\cdot|_{\beta} \) is the Hölder semi-norm

\[
|x|_{\beta}(\Sigma) = \sup_{\Gamma, \Gamma' \in \Sigma, \Gamma \neq \Gamma'} \frac{|x(\Gamma) - x(\Gamma')|}{|\Gamma - \Gamma'|^{\beta}}.
\]

We also use the notation

\[
|x|_* = \inf_{\Gamma, \Gamma' \in \Sigma, \Gamma \neq \Gamma'} \frac{|x(\Gamma) - x(\Gamma')|}{|\Gamma - \Gamma'|}.
\]

Next we state our main result.

**Theorem 3.** Let \( n \geq 1 \), \( 0 < \beta < 1 \), \( x(\Gamma, 0) = x_0(\Gamma) \in C^{n, \beta}(\Gamma_0, \Gamma_1) \cap \{|x|_* > 0\} \), then for any \( T > 0 \) there is a unique solution

\[
x \in C^1([-T, T]; C^{n, \beta}(\Gamma_0, \Gamma_1) \cap \{|x|_* > 0\}) \text{ of } (\text{4}).
\]

In particular, if \( x_0 \in C^\infty(\Gamma_0, \Gamma_1) \cap \{|x|_* > 0\} \) then

\[
x \in C^1([-T, T]; C^\infty(\Gamma_0, \Gamma_1) \cap \{|x|_* > 0\})
\]

We remark that, although the kernel \( K^\alpha \) is a continuous bounded function, its derivatives are unbounded near the origin, and the condition \( |x|_* > 0 \), which generally means self-avoiding curves, allows us to show the integrability of the relevant terms. Furthermore, it is also worth mentioning that \( |x|_* \) being bounded away from zero is similar to the chord arc hypothesis \cite{[61]}, used later in \cite{[40], [47]}.

Now we sketch the main steps involved in the proof of Theorem 3. First, we apply the Contraction Mapping Principle to the BR-\( \alpha \) equation \cite{[41]} to prove the short time existence and uniqueness of solutions in the appropriate space of functions. We show that initially \( C^{1, \beta} \) smooth solutions of \cite{[41]} remain \( C^{1, \beta} \) smooth for a finite short time. Next, we derive an a priori bound for the controlling quantity for continuing the solution for all time. Then we extend the result for higher derivatives. The full details will be reported in a forthcoming paper.

**Sketch of the proof.** We consider the BR-\( \alpha \) equation as an evolution functional equation in the Banach space \( C^{n, \beta} \)

\[
\frac{\partial x}{\partial t}(\Gamma, t) = \int_{\Gamma_0}^{\Gamma_1} K^\alpha (x(\Gamma, t) - x(\Gamma', t)) d\Gamma',
\]

\[
x(\Gamma, 0) = x_0(\Gamma) \in C^{n, \beta} \cap \{|x|_* > 0\}
\]

with \( \gamma = 1/|x_\Gamma| \) being the vorticity density along the sheet. Notice that the initial density is well defined for the subset \( \{|x|_* > 0\} \).

**Step 1.** We show the local existence and uniqueness of solutions. To apply the Contraction Mapping Principle to the BR-\( \alpha \) equation \cite{[41]} we first prove the following proposition

**Proposition 4.** Let \( 1 < M < \infty \), \( -\infty < \Gamma_0 < \Gamma_1 \), and let \( S^M \) be the set

\[
\{ \Gamma \to x(\Gamma) \in C^{1, \beta}(\Gamma_0, \Gamma_1), |x_\Gamma| \leq M, |x|_* > \frac{1}{M} \}
\]

Then the mapping \( x(\Gamma, t) \mapsto u(x(\Gamma, t), t) = \int_{\Gamma_0}^{\Gamma_1} K^\alpha (x(\Gamma, t) - x(\Gamma', t)) d\Gamma' \)

defines a locally Lipschitz continuous map from \( S^M \) into \( C^{1, \beta} \).

This implies the local existence and uniqueness of solutions:
Proposition 5. Given \( x_0(\Gamma) \in C^{1,\beta}(\Gamma_0,\Gamma_1) \cap \{|x| > 0\} \), there exists \( 1 < M < \infty \) and a time \( T(M) \), such that the system \([\mathbb{S}]\) has a unique local solution \( x \in C^1((-T(M), T(M)); S^M) \).

Step 2. The obtained local solutions can be continued in time provided that we have global in time, bounds on \( \frac{1}{|x(t),\partial_t|} \) and \( |x_T(\cdot,t)|_\beta \). To control these quantities we need to bound \( \int_0^T \|\nabla x u(x(\cdot,t),t)\|_{L^\infty(\Gamma_0,\Gamma_1)} dt \). We sketch the proof of this bound. We write \( \nabla x u(x(\Gamma,t),t) \) as

\[
\nabla x u (x(\Gamma,t), t) = \int_{\Gamma_0}^{\Gamma_1} \nabla x K^\alpha (x(\Gamma,t) - x(\Gamma', t)) d\Gamma' \\
= \int_{E_\varepsilon} + \int_{(\Gamma_0,\Gamma_1) \setminus E_\varepsilon} = I_1 + I_2,
\]

where

\[
E_\varepsilon = \{ \Gamma' \in (\Gamma_0, \Gamma_1) : \frac{|x(\Gamma(0),t) - x(\Gamma'(0),t)|}{\alpha} < \varepsilon \},
\]

for a fixed small \( 0 < \varepsilon < 1 \), to be further refined later. Let \( \eta \) denote the unique Lagrangian flow map given by Theorem [1]. Denote the distance between two points \( \eta(x,t) \) and \( \eta(x',t) \) by \( r(t) = |\eta(x,t) - \eta(x',t)| \), where \( r(0) = |x - x'| \).

Then, using the estimate (2.14) of [49], we have

\[
\left| \frac{d}{dt} r(t) \right| \leq \int_{\mathbb{R}^2} |K^\alpha (x, y) - K^\alpha (x', y)| |q(y,t)| dy \\
\leq C \frac{1}{\alpha} \varphi \left( \frac{r(t)}{\alpha} \right) \|q\|_{M(\mathbb{R}^2)} \\
= C \frac{1}{\alpha} \varphi \left( \frac{r(t)}{\alpha} \right) \|q^{in}\|_{M(\mathbb{R}^2)},
\]

where

\[
\varphi (r) = \begin{cases} 
0, & r = 0, \\
1 - \log r, & 0 < r < 1, \\
1, & r \geq 1.
\end{cases}
\]

By comparison with the solution of the differential equation

\[
\frac{d}{dt} r(t) = -C \frac{1}{\alpha} \varphi \left( \frac{r(t)}{\alpha} \right) \|q^{in}\|_{M(\mathbb{R}^2)},
\]

we can choose \( \varepsilon = \varepsilon \left( t, \frac{1}{\alpha} \|q^{in}\|_{M(\mathbb{R}^2)} \right) \) small enough, such that, for \( \frac{|x(\Gamma(t),t) - x(\Gamma'(t),t)|}{\alpha} < \varepsilon \),

\[
|\eta(\Gamma(0),t) - \eta(\Gamma'(0),t)| \geq \frac{\alpha}{C_1} e^{C_1} e^{1 - e^{C_1}},
\]

where \( C_1 = \sqrt{\alpha} \|q^{in}\|_{M(\mathbb{R}^2)} \). Now, using also that \( |x_0| \) is bounded away from zero, we can bound \( \frac{|x(\Gamma(t),t) - x(\Gamma'(t),t)|}{\alpha} \) from below, which in turn implies the bound

\[
I_1 \leq C \left( t, \frac{1}{\alpha} \|q^{in}\|_{M(\mathbb{R}^2)} \right) |x_0|.
\]

While to bound \( I_2 \), we use the boundness of \( \nabla x K^\alpha (x(\Gamma(t),t) - x(\Gamma'(t),t)) \) in \( \{ \Gamma' \in (\Gamma_0, \Gamma_1) : \frac{|x(\Gamma(t),t) - x(\Gamma'(t),t)|}{\alpha} \geq \varepsilon \} \). Hence

\[
\int_0^T \|\nabla x u(x(\cdot,t),t)\|_{L^\infty(\Gamma_0,\Gamma_1)} dt \leq C \left( \frac{1}{\alpha}, T, \|q^{in}\|_{M(\mathbb{R}^2)} \right) |x_0|.
\]

Now, by Grönwall inequality the bound \([10]\) provides bounds on \( \frac{1}{|x_T(\cdot,t)|_\beta} \) and \( |x_T(\cdot,t)|_{\beta} \) on \( [0, T] \). The bound on \( |x_T(\cdot,t)|_\beta \) on \( [0, T] \) is a consequence of

\[
\frac{d}{dt} |x_T(\cdot,t)| = \nabla x u(x(\cdot,t),t) \cdot \frac{\partial x_T(\cdot,t)}{\partial \Gamma},
\]

\[
|\nabla x u(x(\cdot,t),t)|_\beta \leq C \left( \frac{1}{\alpha}, |x_T(\cdot,t)|, |x(\cdot,t)|_{\Gamma_1 - \Gamma_0} \right),
\]

\([10]\) and the Grönwall inequality.

This yields global in time existence and uniqueness of \( C^{1,\beta} \) solutions of \([\mathbb{S}]\).

Step 3. To provide an a priori bound for higher derivatives in terms of lower ones, we show that for \( x \in S^M \cap C^{n,\beta}(\Gamma_0, \Gamma_1) \),

\[
|u(x(\cdot,t),t)|_{n,\beta} \leq C \left( \frac{1}{\alpha}, M, |x(\cdot,t)|_{n-1,\beta} \right) |x(\cdot,t)|_{n,\beta},
\]

hence by Grönwall inequality and the induction argument, it is enough to control \( |x(\cdot,t)|_\beta \) and \( |x_T(\cdot,t)|_\beta \), to guarantee that \( x(\Gamma(t),t) \in C^{n,\beta}(\Gamma_0, \Gamma_1) \), for all \( n \geq 1 \), (and consequently in \( C^{\infty}(\Gamma_0, \Gamma_1) \), whenever \( x_0 \in C^{\infty}(\Gamma_0, \Gamma_1) \cap \{|x| > 0\} \).}

V. CONCLUSIONS

The 2D Euler-\( \alpha \) model \([1, 2, 5, 53, 56, 57]\) is an inviscid regularization of the Euler equations. In \([49]\) it has been shown the existence of a unique global weak solution of 2D Euler-\( \alpha \) equations, when the initial vorticity is in the space of Radon measures on \( \mathbb{R}^2 \). The Birkhoff-Rott-\( \alpha \) equation for the evolution of the 2D vortex sheet is induced by the 2D Euler-\( \alpha \) equations, and it is an \( \alpha \)-analogue of the Birkhoff-Rott equation, induced by the 2D Euler equations.

The structure of weak solutions of 2D Euler equations, for the vortex sheet initial data with initial vorticity being a Radon measure of a distinguished sign, given by Delort.
is not known, yet the BR equations assume \( \alpha \) that a vortex sheet remains a curve at a later time. On the contrary, in the 2D Euler-\( \alpha \) case, if the vorticity is initially supported on a curve, it remains supported on a curve for all times, hence the BR-\( \alpha \) equation gives an equivalent description of the vortex sheet evolution, as the 2D Euler-\( \alpha \) sheet evolution, as the 2D Euler-

In this paper we report the global regularity of the BR-\( \alpha \) approximation for the 2D vortex sheet evolution. We show that initially smooth self-avoiding vortex sheet remains smooth for all times, under the condition that the initial density is an integrable function of the vortex curve with respect to the arc-length measure.

Unlike the original BR problem that exhibits the Kelvin-Helmholtz instability, the linearized, about the flat solution, BR-\( \alpha \) model has growth rates that decay to zero for large wave numbers, larger than \( O(\alpha) \). This, in turn, is also an indication of the role that the parameter \( \alpha \) plays in slowing the process of formation of scales smaller than \( \alpha \). Another indication that \( \alpha \) controls the development of small scales, smaller than \( \alpha \), arises from the Lagrangian description of the flow. The lower bound \( \alpha \) implies that the evolution of small scales, relative to \( \alpha \), at each instant of time, is controlled from below by the initial ratio. That is, for any finite time, the spatial scales smaller than \( \alpha \) develop at a controlled rate.

The linear stability analysis also implicates that the BR-\( \alpha \) approximation could be closer to the original BR equation than the existing regularizing methods, such as vortex blob model, due to the less regular kernel. A numerical study comparing the \( \alpha \) and the vortex blob regularizations for planar and axisymmetric vortex filaments and sheets is reported in [50].

The full details of the results reported in this paper will be presented in a forthcoming paper.

Acknowledgments

C.B. would like to thank the Faculty of Mathematics and Computer Science at the Weizmann Institute of Science for the kind hospitality where this work was initiated. This work was supported in part by the BSF grant no. 2004271, the ISF grant no. 120/06, and the NSF grants no. DMS-0504619 and no. DMS-0708832.

[1] D. D. Holm, J. E. Marsden, T. S. Ratiu, The Euler-Poincaré equations and semidirect products with applications to continuum theories, Adv. Math. 137 (1) (1998) 1–81.
[2] D. D. Holm, J. E. Marsden, T. S. Ratiu, Euler-Poincaré models of ideal fluids with nonlinear dispersion, Phys. Rev. Lett. 80 (19) (1998) 4173–4176.
[3] S. Chen, C. Foias, D. D. Holm, E. Olson, E. S. Titi, S. Wynne, Camassa-Holm equations as a closure model for turbulent channel and pipe flow, Phys. Rev. Lett. 81 (24) (1998) 5338–5341.
[4] S. Chen, C. Foias, D. D. Holm, E. Olson, E. S. Titi, S. Wynne, The Camassa-Holm equations and turbulence, Phys. D 133 (1-4) (1999) 49–65.
[5] S. Chen, C. Foias, D. D. Holm, E. Olson, E. S. Titi, S. Wynne, A connection between the Camassa-Holm equations and turbulent flows in channels and pipes, Phys. Fluids 11 (8) (1999) 2343–2353, the International Conference on Turbulence (Los Alamos, NM, 1998).
[6] C. Foias, D. D. Holm, E. S. Titi, The Navier-Stokes-alpha model of fluid turbulence, Phys. D 152/153 (2001) 505–519, advances in nonlinear mathematics and science.
[7] C. Foias, D. D. Holm, E. S. Titi, The three dimensional viscous Camassa-Holm equations, and their relation to the Navier-Stokes equations and turbulence theory, J. Dynam. Differential Equations 14 (1) (2002) 1–35.
[8] D. D. Holm, E. S. Titi, Computational models of turbulence: the LANS-alpha model and the role of global analysis, SIAM News 38 (7).
[9] A. Cheskidov, D. D. Holm, E. Olson, E. S. Titi, On a Leray-\( \alpha \) model of turbulence, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 461 (2055) (2005) 629–649.
[10] A. A. Ilyin, E. Lunasin, E. S. Titi, A Modified-Leray-\( \alpha \) subgrid scale model of turbulence, Nonlinearity 19 (4) (2006) 879–897.
[11] M. I. Vishik, E. S. Titi, V. V. Chepyzhov, Trajectory attractor approximations of the 3D Navier-Stokes system by a Leray-\( \alpha \) model, Russian Mathematical Dokladi (Translated from Russian) 71 (2005) 92–95.
[12] V. V. Chepyzhov, E. S. Titi, M. I. Vishik, On the convergence of solutions of the Leray-\( \alpha \) model to the trajectory attractor of the 3D Navier-Stokes system, Discrete Contin. Dyn. Syst. 17 (3) (2007) 481–500.
[13] K. Mohseni, B. Kosović, S. Shkoller, J. E. Marsden, Numerical simulations of the Lagrangian averaged Navier-Stokes equations for homogeneous isotropic turbulence, Phys. Fluids 15 (2) (2003) 524–544.
[14] W. Layton, R. Lewandowski, On a well-posed turbulence model, Discrete Contin. Dyn. Syst. Ser. B 6 (1) (2006) 111–128.
[15] W. Layton, R. Lewandowski, A simple and stable scale-similarity model for large eddy simulation: energy balance and existence of weak solutions, Appl. Math. Lett. 16 (8) (2003) 1205–1209.
[16] R. Lewandowski, Vorticities in a LES model for 3D periodic turbulent flows, Journ. Math. Fluid. Mech. 8 (2006) 879–897.
[17] B. J. Geurts, D. D. Holm, Leray and LANS-\( \alpha \) models for large-eddy simulation: energy balance and existence of weak solutions, Appl. Math. Lett. 16 (8) (2003) 1205–1209.
[18] B. J. Geurts, D. D. Holm, Leray and LANS-\( \alpha \) modelling of turbulent mixing, J. Turbul. 7 (10) (2006) 1–33.
[19] Y. Cao, E. Lunasin, E. S. Titi, Global well-posedness of the three-dimensional viscous and inviscid simplified Bardina turbulence models, Commun. Math. Sci. 4 (4) (2006) 823–848.
[20] J. Bardina, J. H. Ferziger, W. C. Reynolds, Improved subgrid-scale models for large-eddy simulation, Am. Inst.
