Spectrum of Perturbations in Anisotropic Inflationary Universe with Vector Hair

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We study both the background evolution and cosmological perturbations of anisotropic inflationary models supported by coupled scalar and vector fields. The models we study preserve the $U(1)$ gauge symmetry associated with the vector field, and therefore do not possess instabilities associated with longitudinal modes (which instead plague some recently proposed models of vector inflation and curvaton). We first introduce a model in which the background anisotropy slowly decreases during inflation; we then confirm the stability of the background solution by studying the quadratic action for all the perturbations of the model. We then compute the spectrum of the $h_\times$ gravitational wave polarization. The spectrum we find breaks statistical isotropy at the largest scales and reduces to the standard nearly scale invariant form at small scales. We finally discuss the possible relevance of our results to the large scale CMB anomalies.

I. INTRODUCTION

The inflationary paradigm has become a widely accepted description of the early universe, which has been successful in solving the classical problems of FRW cosmology [1, 2]. Moreover, inflation provides a mechanism for generation of nearly scale invariant spectrum of perturbations that can seed the structure we observe today [3]. Inflationary era is characterized by a quasi de-Sitter expansion with a nearly constant Hubble rate. During inflation, quantum fluctuations are generated and amplified to scales above the Hubble scale (which specifies the horizon of casual processes) where they remain frozen until they re-enter the horizon after inflation has ended. A key assumption about the quantum fluctuations during the quasi de-Sitter stage is that they are in an adiabatic vacuum stage in the deep sub-horizon (early time) regime, which results in a nearly scale invariant spectrum.

Results of the WMAP experiment [4] are in overall agreement with the predictions of inflation. However, certain features of the full sky CMB maps seem to be anomalous in the standard picture. These anomalies include the low power in the quadrupole moment [5], the alignment of the lowest multipole moments (known as the ‘axis-of-evil’) [6], and an asymmetry in power between the northern and southern ecliptic hemispheres [7]. The statistical significance of these anomalies has been extensively discussed in the literature and despite numerous efforts, an explanation due to a systematic effect or a foreground signal affecting the analysis is not forthcoming. These anomalies suggest a violation of statistical isotropy of cosmological fluctuations, which is at odds with standard mechanisms of generation of perturbations in inflationary models. Therefore, the possibility of relating the anomalies with modifications in the standard inflationary picture has been considered by numerous authors recently.

In the standard picture, $a_{lm}$ coefficients of CMB temperature anisotropies satisfy $< a_{lm} a^*_{lm'} >= C_l \delta_{ll'} \delta_{mm'}$ (so, the temperature fluctuations are said to be statistically isotropic). It has been shown in [8, 9] that an anisotropic expansion that took place at the onset of standard single field inflation, leads to a nonvanishing correlation of the $a_{lm}$ coefficients for different $l$-modes. This violation of statistical isotropy might then be related to the alignment of the lowest multipoles observed in the CMB data. Therefore, several authors considered inflationary models with an anisotropic stage. As the simplest possible modification, anisotropic initial conditions at the onset of single field inflation have been considered in references [8, 9, 10, 11]. Isotropization of the universe takes place due to the fast expansion supported by the inflaton field. Fluctuations at scales comparable to the horizon scale during the anisotropic stage of inflation are sensitive to the background evolution and thus provide breaking of statistical isotropy at those scales. If inflation had a limited duration, this could leave an imprint on the largest observable scales. On the other hand, fluctuations that leave the horizon after isotropization re-enter earlier than modes that left the horizon during the anisotropic stage, producing the standard nearly scale invariant spectrum at small scales [1]. Since in such models isotropy is reached very soon (within one Hubble time) the breaking of statistical isotropy can be visible if the total duration of inflation is tuned to only the minimal duration to solve the standard problems of FRW cosmology.

The fine-tuning can be relaxed if the anisotropic stage is prolonged, due to the existence of other sources, for example vector fields. Recently, anisotropic expansion driven by vector fields have been studied by many authors. The anisotropic expansion is achieved when the vector field acquires a nonzero vacuum expectation value (VEV) along a spatial direction. There is a range of models which differ by the way the VEV is obtained. The first anisotropic inflationary model was considered long ago in reference [12], where the $U(1)$ symmetry of the vector field action is

\[1 \text{ It was also shown in [13] that the anisotropic stage could lead to a detectable gravitational wave signal.}\]
broken by a potential term, which needs to have a negative curvature to support an anisotropic expansion. It has been shown in reference [14] that this model is plagued by a ghost instability, and its quantum theory is inconsistent. In another model, the ACW model [15], the VEV is obtained by a Lagrange multiplier field which fixes the norm of the vector field and breaks the $U(1)$ gauge symmetry. The stability analysis of the ACW model by considering the most general perturbations of the background has been performed in references [16] and [14, 17]. In the latter studies, it has been shown that the linearized perturbations diverge close to horizon crossing, indicating the instability of the background solution. Another model [19] considered a nonminimal coupling of the vector field to the scalar curvature which breaks the conformal invariance of the vector field action. The coupling with curvature allows for a slow-roll phase during the prolonged anisotropic expansion, and the universe isotropizes due to the existence of a massive scalar field, which is responsible for the overall isotropic expansion of the universe. Models of inflation with nonminimal coupling to curvature have also been considered for standard isotropic inflation [20]. Isotropy of space is realized if $3N$ mutually orthogonal vector fields have equal VEVs. Instead, for randomly oriented $N$ vector fields, one expects a statistically isotropic background with order $1/\sqrt{N}$ anisotropy. Perturbative calculations based around such a background configuration have been considered in references [21, 22, 23] and in [30]. The latter study is the only complete study linearized study of perturbations, taking into account all the physical degrees of freedom, and coupling between vector field and metric perturbations. It has been shown in [30] that the equations of motion for the linearized perturbations become singular close to horizon crossing, leading to instabilities. In all of these models (including [15, 19, 20]), instabilities are related to the existence of the longitudinal polarization of the vector field (which would otherwise be absent when the $U(1)$ symmetry is restored), which becomes a ghost close to horizon crossing (In the vector inflation model [20], ghosts also appear in the deep UV regime).

More recently, reference [24] introduced an anisotropic inflation model driven simultaneously by a vector field and a massive scalar field. The vector field is massless, but it is coupled to the scalar field through its kinetic term. Such type of coupling preserves the $U(1)$ gauge invariance of the vector field, so that the dangerous longitudinal vector polarization is absent; moreover the conformal invariance of the vector field is broken. When the vector field has a nonvanishing VEV along a spatial direction, the model possesses an anisotropically expanding attractor solution. The anisotropy in the attractor is initially small, but it increases towards the end of inflation; therefore this is a counterexample to the cosmic no hair conjecture (See [25] for a different example). The breaking of conformal invariance due to the scalar-vector coupling can also be used to generate magnetic fields from inflation, as discussed in reference [26] and more recently in [27, 28] and [24]. These models are expected to be free from ghost instabilities as long as the coupling to the scalar field remains positive. However to our knowledge, there has been no complete study of the stability of these models, which take into account all the physical degrees of freedom of the system (including gravity). A complete study of stability is therefore necessary, given the problems identified with other vector field models [14, 17, 30].

In this work, we will present a complete study of cosmological perturbations of models with scalar-vector coupling that lead to an anisotropically expanding Bianchi-I background solution. We will also consider generalizations of the original model introduced in [24], to include additional scalar fields, so that isotropization can be achieved before the end of inflation. This is required in order to obtain a scale invariant spectrum of perturbations at small scales, but the large scale spectrum is modified, which in turn can be related to the CMB anomalies. Our study has two main steps: one is to show that the model is free of ghost instabilities, and the second is to study the resulting phenomenology. We perform the first step explicitly in this paper. We develop the necessary tools to study the phenomenology of the model and as a simple exercise, we study only the spectrum of $h_{\chi}$, gravitational wave polarization in this paper. The study of the full phenomenology (including the spectrum of the curvature perturbation) will be communicated elsewhere.

We will follow the formalism developed in [3, 17] to decompose and classify the perturbations. The generic background metric we study is the Bianchi-I metric with a residual two dimensional isotropy, given by

$$ds^2 = -dt^2 + a(t)^2 dx^2 + b(t)^2 \left(dy^2 + dz^2\right)$$

We exploit the two dimensional isotropy of the $y-z$ plane to decompose the perturbations into two decoupled classes: 2d scalar modes and 2d vector modes (As discussed in [3], there is no two dimensional transverse-traceless mode). The linearized Einstein equations and the quadratic action for the two types of modes are decoupled and therefore we study them separately. We can deduce the number of degrees of freedom coming from each of the fields (gravity+vector field+scalar field) by a simple counting, which does not depend on the decomposition chosen to classify them. The metric has 10 perturbations ($\delta g_{\mu\nu}$) to start with, 4 of which can be removed by coordinate transformations. Of the

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2 One may hope that this model has a well behaved UV completion. However, inflationary predictions are sensitive to this high energy regime, where the completion is needed.
remaining 6 modes, 4 are nondynamical, which can be best understood from the ADM formalism\textsuperscript{3}. In the ADM formalism, the gravitational action is decomposed into the dynamical part containing the spatial metric $h_{ij}$ and a part containing the lapse ($N$) and shift functions ($N_i$). The lapse and shift functions have no kinetic terms in the action, and they can be integrated out by solving their equations of motion \textsuperscript{3}. This leaves only $h_{ij}$ as dynamical modes, which have only two degrees of freedom. In summary, the metric perturbations have only 2 dynamical degrees of freedom (which are the gravitational wave polarizations in the standard case) and 4 nondynamical modes. For the case of the vector field, out of 4 perturbations to start with $(\delta A_\mu)$, one perturbation can be removed by the $U(1)$ gauge transformation. Out of the remaining 3 perturbations 2 of them are dynamical and one mode is nondynamical $(\delta A_0)$. The scalar field perturbation introduces a single dynamical degree of freedom. Therefore, in total, the perturbations comprise of 5 dynamical modes, 2 of which are 2d vector and 3 are 2d scalar modes. When additional scalar fields are considered, each field introduces an extra scalar dynamical degree of freedom.

We will insert the perturbative expansion of the metric, the vector field and the scalar field(s) into the starting action and expand it at the quadratic order. We will show that the actions for the 2d vector and 2d scalar modes are decoupled, so we study them separately. We will determine the linear combinations of perturbations that canonically normalize the action, generalizing the computation of the standard isotropic case\textsuperscript{33}. The study of the quadratic action is crucial for both showing that the model is consistent (free of ghosts) and also for determining the initial conditions for the perturbations. For modes that are inside the horizon, canonical combinations of perturbations can be quantized, and initial conditions can be set by the canonical commutation relations and by the requirement that the adiabatic vacuum state has minimal energy. We then proceed by numerically integrating the evolution equations for the canonical fields, starting from adiabatic initial conditions deep inside the horizon, until the end of inflation, which gives us the primordial power spectra. We do so for the 2d vector modes in this paper (which will be related to the power spectrum of the $h_\nu$ polarization of gravitational waves) and perform the study of the spectrum of 2d scalar modes in a separate publication.

The paper is organized as follows: In section II we summarize the anisotropic inflationary background solution obtained in reference [24] and generalize the model to possess extra scalar fields. This generalization leads to isotropization before the end of inflation. In section III we discuss the perturbations around the background configuration. Specifically, in subsection III A we review and discuss the decomposition of perturbations around the Bianchi-I background solution, following the formulation of reference [9]. In subsection III B we discuss the generic properties of coupled vector-scalar models and develop tools that we will use in the following sections in order to find adiabatic initial conditions for such systems.

In subsection III C we discuss 2d vector perturbations around the background configuration. We compute the action for 2d vector modes and find the combinations of perturbations that canonically normalize the action. In section IV we compute the power spectra of the 2d vector modes numerically. This study results in the spectrum of $h_\nu$ gravitational wave mode. We show that the spectrum has angular dependence (so breaks statistical isotropy) at large scales and reduces to the standard nearly scale invariant form at small scales. We also provide a fit to the numerically obtained spectrum. Finally in section V we provide a general discussion about the results we have obtained and their possible relation to observations.

We also provide two extensive appendices: In appendix VI A we derive the relation between the 2d decomposed modes and the standard longitudinal mode, which has been used to determine the power spectra. In appendix VI B we perform the study of the 2d scalar modes. More precisely, we compute the quadratic action and determine the modes that canonically normalizes the action. The spectrum of scalar perturbations and the consequent phenomenological predictions (specifically, the $<a_{lm} a_{l'm'}^\ast>$ correlation) will be presented elsewhere.

\section{II. ANISOTROPIC INFLATIONARY EXPANSION DUE TO COUPLED VECTOR AND SCALAR FIELDS}

The models we discuss in this work are generically described by the following action

$$S = \int d^4x \sqrt{-g} \left[ \frac{M_p^2}{2} R - \sum_{a=1}^N \left( \frac{1}{2} \partial_\mu \phi_a \partial^\mu \phi_a + V_a(\phi_a) \right) - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) - \frac{1}{4} f^2(\phi) F_{\mu\nu} F^{\mu\nu} \right]$$ \quad (1)$$

The original discussion of these coupled vector-scalar field theories in an anisotropic inflationary background was given in [24], where there is only the single scalar field $\phi$ in the action. We assume that only one of the scalar fields

\footnote{The equations of motion derived from extremizing the gravitational action with respect to lapse and shift functions result in the momentum and hamiltonian constraints.}
(or only one linear combination) enters in the kinetic function $f$ for the vector field. We denote the remaining scalars with $\phi_a$ ($a = 1, \ldots, N$). The metric ansatz is taken to be the homogeneous but anisotropic Bianchi-I metric given by

$$ds^2 = -dt^2 + e^{2\alpha(t)} \left[ e^{-4\sigma(t)} dx^2 + e^{2\sigma(t)} (dy^2 + dz^2) \right]$$

(2)

In the above metric $\alpha$ measures the number of e-folds of average isotropic expansion of the universe and $\sigma$ measures the deviation from anisotropy. This metric also possesses a left over 2d isotropy in the $y-z$ plane (which we will make use of when decomposing the perturbations). We can also identify (2) with $ds^2 = -dt^2 + a^2 dx^2 + b^2 (dy^2 + dz^2)$, then we would have

$$a = e^{\alpha - 2\sigma} , \quad b = e^{\alpha + \sigma} , \quad H \equiv \frac{H_a + 2H_b}{3} = \dot{\alpha} , \quad h \equiv \frac{H_b - H_a}{3} = \dot{\sigma}$$

(3)

where $H_a = \dot{a}/a$, $H_b = \dot{b}/b$, $H$ is the average isotropic expansion rate, and $h$ is the anisotropic expansion rate. An overdot denotes derivative with respect to the cosmic time $t$. An isotropic flat FRW universe is the limiting case when $h \to 0$. We will make use of both representations of the Bianchi-I metric in the following discussions. The field equations obtained from the action (1) are

$$G_{\mu\nu} = \frac{1}{M_p^2} \left[ \partial_{\mu} \phi \partial_{\nu} \phi - g_{\mu\nu} \left( \frac{1}{2} \partial_{\alpha} \phi \partial^{\alpha} \phi + V(\phi) \right) + \sum_{a=1}^{N} T^{(a)}_{\mu\nu} \right]$$

$$+ f^2(\phi) F_{\mu}^{\alpha} F_{\nu \alpha} - \frac{1}{4} g_{\mu\nu} f^2(\phi) F_{\alpha \beta} F^{\alpha \beta}$$

$$- \frac{1}{\sqrt{-g}} \partial_{\mu} \left[ \sqrt{-g} g^{\mu\nu} \partial_{\nu} \phi \right] - V'(\phi) - \frac{f(\phi) f'(\phi)}{2} F_{\alpha \beta} F^{\alpha \beta} = 0$$

$$- \frac{1}{\sqrt{-g}} \partial_{\mu} \left[ \sqrt{-g} g^{\mu\nu} \partial_{\nu} \phi_a \right] - V_a'(\phi_a) = 0$$

$$- \frac{1}{\sqrt{-g}} \left[ \sqrt{-g} f^2(\phi) F^{\mu\nu} \right] = 0$$

(4)

where

$$T^{(a)}_{\mu\nu} = \partial_{\nu} \phi_a \partial_{\mu} \phi_a - g_{\mu\nu} \left( \frac{1}{2} \partial_{\alpha} \phi_a \partial^{\alpha} \phi_a + V_a(\phi_a) \right)$$

(5)

and $f'(\phi) \equiv df/d\phi$. We assume that all the scalar fields are homogeneous so that $\phi = \phi(t)$ and $\phi_a = \phi_a(t)$ for all $a$. The vector field is assumed to have a homogeneous vacuum expectation value along the $x$-direction so $A_{\mu} = (0, A_1(t), 0, 0)$ (a homogeneous $A_0$ does not enter into the field equations, and can be set to zero by using the $U(1)$ gauge invariance). With the metric and vector field ansatz, the last of (1) can be integrated to give

$$\dot{A}_1 = f^{-2}(\phi) p_A e^{-(\alpha - 4\sigma)}$$

(6)

where $p_A$ is an integration constant. We use (6) and the metric ansatz (2) in the field equations (4) to obtain the evolution equations

$$3\ddot{\alpha} - 3\dot{\sigma}^2 = \frac{1}{M_p^2} \left[ \frac{1}{2} \dot{\phi}^2 + V(\phi) + \sum_{a=1}^{N} \rho_a(t) \right] + \frac{\bar{p}_A^2}{2M_p^2} f^{-2}(\phi)$$

$$2\ddot{\alpha} + 3\dot{\sigma}^2 - 3\dot{\alpha}^2 = \frac{1}{M_p^2} \left[ - \frac{1}{2} \dot{\phi}^2 + V(\phi) - \sum_{a=1}^{N} p_a(t) \right] - \frac{\bar{p}_A^2}{6M_p^2} f^{-2}(\phi)$$

$$\ddot{\sigma} + 3\dot{\sigma} \dot{\alpha} = \frac{\bar{p}_A^2}{3M_p^2} f^{-2}(\phi)$$

$$\ddot{\phi} + 3\dot{\phi} \dot{\alpha} + V'(\phi) - \bar{p}_A^2 f^{-3}(\phi) f'(\phi) = 0$$

$$\ddot{\phi}_a + 3 \dot{\phi}_a \dot{\alpha} + V_a'(\phi_a) = 0 , \quad a = 1, \ldots, N$$

(7)

where we have defined

$$\rho_a = \frac{1}{2} \ddot{\phi}_a^2 + V_a(\phi_a) , \quad p_a = \frac{1}{2} \dot{\phi}_a^2 - V_a(\phi_a) , \quad \bar{p}_A \equiv p_A e^{-2(\alpha + \sigma)}$$

(8)
The values of the functions $\alpha$ and $\sigma$ are nonphysical; what is physical is their time derivatives. Therefore physics is the same under a constant shift $\alpha(t) \rightarrow \alpha(t) + \alpha_0$ and $\sigma(t) \rightarrow \sigma(t) + \sigma_0$ (This corresponds to a rescaling of the spatial coordinates by two constant factors.). Clearly, the value of the constant $p_A$ is nonphysical, and it changes under a coordinate transformation. We can deduce its transformation properties by imposing invariance of $F_{\alpha\beta} F^{\alpha\beta}$ which reduces to the condition

$$F_{\alpha\beta} F^{\alpha\beta} \rightarrow F_{\alpha\beta} F^{\alpha\beta} \quad \text{under} \quad \{\alpha \rightarrow \alpha + \alpha_0, \sigma \rightarrow \sigma + \sigma_0\} \Rightarrow \quad p_A \rightarrow p_A e^{2(\alpha_0 + \sigma_0)}$$

(9)

Thus, $p_A$ is a physical parameter invariant under the constant coordinate rescaling. For this reason, it is the combination which enters in the field equations (7).

We will discuss inflationary solutions of (7), with appropriate choices of $f(\phi)$ for both single field (only $\phi$) and two field models ($a=1$) respectively.

### A. Single Scalar Field Inflationary Background

In this subsection, we review and discuss the background evolution of the original model [24] with a single scalar field $\phi$ coupled to the vector field. As described in [24], we look for solutions that have a slow-roll regime and small anisotropy. Thus, we neglect $\dot{\sigma}$, $\dot{\phi}$ in the first and $\ddot{\phi}$ in the last of (7) to get

$$3\dot{\alpha}^2 \approx \frac{V(\phi)}{M_p^2} + \frac{\tilde{p}_A^2}{2M_p^2} f^{-2}(\phi), \quad 3\dot{\phi} \approx -V'(\phi) - \tilde{p}_A^2 f^{-3}(\phi) f'(\phi)$$

(10)

Up to now, $f(\phi)$ has been arbitrary. We now specify it so to obtain the desired anisotropic background solution. To do so, we first assume that the effect of the vector field is completely negligible, and the resulting equations can be solved approximately in the standard slow-roll approximation, given by

$$\alpha \approx -\frac{1}{M_p^2} \int \frac{V(\phi)}{V'(\phi)} d\phi$$

(11)

The case $\dot{\sigma} = 0$ corresponds to the isotropic FRW background. A prolonged anisotropic inflation requires $\dot{A}_1$ to be nearly constant. From equation (6) we see that this can be achieved if we choose

$$f(\phi) \approx e^{-2\alpha} \approx \exp\left[\frac{2}{M_p^2} \int \frac{V(\phi)}{V'(\phi)} d\phi\right]$$

(12)

For instance, when $V \propto \phi^n$, this becomes $f \approx e^{\phi^2/n M_p^2}$. Furthermore, one can set $f \approx e^{c \phi^2/n M_p^2}$ (where $c$ is a constant) and for $c > 1$, the anisotropy is expected to grow. From now on, we assume that the functional form of $f(\phi)$ is given by

$$f(\phi) = \exp\left[\frac{2c}{M_p^2} \int \frac{V(\phi)}{V'(\phi)} d\phi\right]$$

(13)

We now return back to equations (10) and look for solutions that have small anisotropy, but the vector field contribution is not totally negligible. We specify the scalar field potential as $V(\phi) = m^2 \phi^2/2$ and set $f = e^{c \phi^2/2M_p^2}$ from now on. The vector field modifies the evolution of the scalar field, if the terms $V'(\phi)$ and $\tilde{p}_A^2 f^{-3}(\phi) f'(\phi)$ are comparable. Namely, for our choice of the potential and $f(\phi)$

$$m^2 \sim \frac{c \tilde{p}_A^2}{M_p^2} e^{-c \phi^2/2M_p^2}$$

(14)

For such type of solutions, we can compare the energy densities of the scalar and vector fields. The ratio, (when the scalar field is still in the slow-roll regime and the anisotropy is small) is given by

$$\frac{\rho_A}{\rho_\phi} \approx \frac{\tilde{p}_A^2}{m^2} e^{-c \phi^2/2M_p^2} \approx \frac{M_p^2}{c \phi^2}$$

(15)

where, the second approximate equality is obtained from (14). This ratio is much smaller than 1 during inflation, since $\phi/M_p \gg 1$. Thus, even when the dynamics of the scalar field is modified by the action of the vector field, as
long as the anisotropy is kept small enough, the energy density of the vector field can be neglected. Therefore, we can rewrite equations (10) by neglecting the effect of the vector field only in the first equation as

$$\dot{\alpha}^2 \approx \frac{m^2 \phi^2}{6 M_p^2}, \quad 3 \dot{\alpha} \dot{\phi} \approx - m^2 \phi + \frac{c \tilde{p}_A^2}{M_p^2} \phi e^{-c \phi^2/M_p^2}$$

These equations are integrated to give

$$e^{c \phi^2/M_p^2 + 4 \alpha} = \frac{c^2 \tilde{p}_A^2}{m^2 M_p^2 (c - 1)} + D e^{-4(c-1)\alpha}$$

(17)

where $D$ is an integration constant. When the second term in the right hand side of the above equality is dominant we get

$$\frac{c \phi^2}{M_p^2} + 4 c \alpha \sim \text{constant} \rightarrow \dot{\phi} \approx - \sqrt{\frac{2}{3}} m M_p$$

(18)

which is the standard slow-roll relation. The second term in the right hand side of (17) will eventually be subdominant for $c > 1$, since the universe expands and $\alpha$ grows. Therefore, the first term eventually dominates and we have

$$e^{c \phi^2/M_p^2} \approx \frac{c \tilde{p}_A^2}{m^2 M_p^2 (c - 1)} \rightarrow \dot{\phi} \approx - \sqrt{\frac{2}{3}} m M_p c$$

(19)

which has an extra $1/c$ factor compared to the standard slow-roll solution (18). Using the third of (7), for small and slowly varying anisotropy we have

$$3 \dot{\alpha} \dot{\sigma} \approx \frac{\tilde{p}_A^2}{3 M_p^2} e^{-c \phi^2/M_p^2}$$

(20)

Moreover, combining this result with (19) and the first of (13) we find that the anisotropy obeys

$$\frac{\dot{\sigma}}{\dot{\alpha}} = \frac{\tilde{p}_A^2}{9 M_p^2 \dot{\alpha}^2} \approx \frac{2}{3} \frac{M_p^2 (c - 1)}{c^2 \phi^2} \propto \frac{\rho_A}{\rho_{\phi}} \ll 1$$

(21)

which is always satisfied (either in the region described by (18) or (19)). Also, note that the anisotropy is slowly increasing with time, since $\phi$ is decreasing towards the end of inflation. The slow-roll solutions described by (18) is the standard isotropic inflationary attractor and the solutions described by (19) is the anisotropic inflationary attractor. We are interested in configurations that start close to the attractor solution (19). For illustration, we solve the system (7) numerically by setting the initial conditions on the attractor solution. Namely, we set initial conditions as

$$\dot{\phi}_{in} = - \frac{m^2}{3c \dot{\alpha}_{in}} \phi_{in}$$

$$\dot{\sigma}_{in} = \frac{m^2 (c - 1)}{9c^2 \dot{\alpha}_{in}} \phi_{in}$$

$$\frac{\tilde{p}_A^2}{M_p^2 f^2(\phi)} = \frac{m^2 (c - 1)}{c^2}$$

(22)

Inserting (22) into the constraint equation (the first of (7), we solve for $\dot{\alpha}_{in}$ to get

$$\dot{\alpha}_{in}^2 = \frac{m^2}{12} \left[ \frac{c - 1}{c^2} \phi_{in}^2 + \sqrt{\frac{25(c - 1)^2}{9c^4} + \frac{2(1 + 3c)}{3c^2} \phi_{in}^2 + \phi_{in}^4} \right]$$

(23)

4 For $c \leq 1$, the initial anisotropy will decay exponentially fast so we do not consider this possibility here. The solutions with $c \leq 1$ are disconnected from the anisotropic attractor solutions $c > 1$, that is there is no smooth limit of $c \rightarrow 1$ which takes us back to the isotropic FRW solution. We will discuss the implications of this effect on the perturbations in the next section.
FIG. 1: The left panel shows the evolution of anisotropy during inflation and the right panel after inflation (during when the scalar field is oscillating). $H \equiv \dot{\alpha}$ and $h \equiv \dot{\sigma}$. The black dashed curve represents the approximate slow-roll solution during inflation and the straight red curves the numerical solution.

FIG. 2: The left panel shows the evolution of the average Hubble rate ($H \equiv \dot{\alpha}$) and the right panel is the phase plot of the scalar field. Again, the black dashed lines are the slow-roll solutions. Time is measured in units of $m$.

Equations (22) and (23) specifies the slow-roll initial conditions, which are set on the attractor solution. The only free parameters are $\phi_{\text{in}}, \alpha_{\text{in}}$, and $\sigma_{\text{in}}$. The values of $\alpha_{\text{in}}$ and $\sigma_{\text{in}}$ are nonphysical, so we can set $\alpha_{\text{in}} = \sigma_{\text{in}} = 0$ for convenience (which is equivalent to setting $a_{\text{in}} = b_{\text{in}} = 1$). Thus, only $\phi_{\text{in}}$ needs to be specified, which determines the amount of inflation. The number of e-folds of inflation is approximately given by $\alpha(t_{\text{end}}) = N \approx c \phi_{\text{in}}^2/4M_p^2$, so for $c = 2$ we set $\phi_{\text{in}} = 11M_p$ which gives around 60 e-folds of inflation. We show in Figs. 1, 2 the numerical evolution of the system. Note that the slow-roll and the numerical solutions are in good agreement with each other. The anisotropy in the single field model is increasing towards the end of inflation and decreases after inflation (as can be seen in the right panel of Fig. 1) during when the scalar field has an effective equation of state of matter. We are however interested in cases where the anisotropic expansion takes place at the onset of inflation (for example as in [8, 9, 10, 11] and [19]). The main reason is because we only want the largest observable scales to be modified from the perturbations that left the horizon during the anisotropic initial stage. This in turn might have some relevance to the alignment of the lowest multipoles. In the current case, all observable scales are modified (indeed smaller scales are modified even more since the anisotropy increases towards the end of inflation) and this is not compatible with observations. A simple possibility to overcome this difficulty is to introduce extra scalar fields as in the starting action (1). We discuss the background evolution for the simplest possibility of a two-field modification of the original proposal of [24] in the next subsection.

B. Two Scalar Field Inflationary Background

In this section we discuss the simplest multi-field case of the action (1) with $a = 1$. In this model, the extra scalar field $\phi_1$ is not coupled to the vector field, and it is only relevant for the overall isotropic expansion of the universe. For the rest of the paper we will assume that $V(\phi) = m^2 \phi^2/2$ and $V_1(\phi_1) = m_1^2 \phi_1^2/2$. The ratio of the masses $\mu \equiv m_1/m$ is chosen to be smaller than 1. This choice leads to a two stage inflationary expansion. The first stage is driven mainly
by $\phi$, and due to its coupling with the vector field, this stage is anisotropic and it proceeds in a similar fashion as in the previous subsection. At the end of the first stage, the field $\phi_1$ takes over the expansion, while the field $\phi$ starts oscillating. Since the expansion of the universe still proceeds, the Hubble friction will drive the field $\phi$ to zero and consequently, the coupling $f(\phi)$ approaches unity. Therefore, from (16), we clearly see that the effect of the vector field quickly diminishes. Thus, after the first stage, the universe quickly isotropizes and inflation proceeds isotropically until the field $\phi_1$ starts oscillating. Problems related to an inflationary stage that has increasing anisotropy will be eliminated by the two stage modification.

As we have done in the previous section, we first study the solutions to the background evolution equations, which are given in this case as

\begin{align*}
3\ddot{\alpha} - 3\dot{\alpha}^2 &= \frac{1}{M_p^2} \left[ \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \dot{\phi}_1^2 + \frac{1}{2} m^2 \dot{\phi}^2 + \frac{1}{2} \mu^2 m^2 \dot{\phi}_1^2 \right] + \frac{\bar{p}_A^2}{2M_p^2 f^2(\phi)} \\
2\ddot{\sigma} + 3\dot{\sigma} &= \frac{\bar{p}_A^2}{3M_p^2 f^2(\phi)} \\
\ddot{\phi} + 3\dot{\phi} \dot{\sigma} + m^2 \phi - \frac{\bar{p}_A^2}{f^3(\phi)} f'(\phi) &= 0 \\
\dot{\phi}_1 + 3\dot{\phi} \dot{\phi}_1 + m^2 \mu^2 \dot{\phi}_1^2 &= 0
\end{align*}

(24)

where $f(\phi) = e^{\phi^2/2M_p^2}$ as before. We can obtain slow-roll solutions similar to (19), in the first stage which proceeds anisotropically. Equations (16) are now modified to become

\begin{align*}
\dot{\alpha}^2 &\approx \frac{m^2}{6M_p^2} \left( \phi^2 + \mu^2 \phi_1^2 \right), \quad 3\dot{\alpha} \dot{\phi} \approx -m^2 \phi + \frac{c\bar{p}_A^2}{M_p^2} \phi e^{-\phi^2/2M_p^2}, \quad 3\dot{\alpha} \dot{\phi}_1 \approx -m^2 \mu^2 \phi_1
\end{align*}

(25)

Initially, the field $\phi$ drives the expansion, so $\dot{\phi}^2 \gg \mu^2 \phi_1^2$, and the slow roll solutions for the first stage is simply given similar to (19) as

\begin{align*}
e^{\phi^2/2M_p^2} &\approx \frac{c^2 \bar{p}_A^2}{m^2 M_p^2 (c-1)} \quad \Rightarrow \quad \dot{\phi} \approx -\sqrt{\frac{2}{3}} \frac{m M_p}{c} \\
\dot{\phi}_1 &\approx -\sqrt{\frac{2}{3}} \frac{m M_p \mu}{\phi_1}
\end{align*}

(26)

and the anisotropy is given precisely as in equation (21). After this first stage ends, the isotropic expansion of the universe is driven by $\phi_1$ and this second stage is described by the following slow-roll solution

\begin{align*}
\dot{\phi}_1 &\approx -\sqrt{\frac{2}{3}} \frac{m M_p \mu}{\phi_1}
\end{align*}

(27)

We also integrate the system (24) numerically and the results are shown in Figs 3-4. As we have done in the previous section, we use the slow-roll solutions and the constraint equation (the first of 24) to determine the initial conditions $\dot{\phi}_m, \dot{\phi}_1m, \dot{\phi}_m, \dot{\phi}_1m^5$. The initial conditions for the scale factors are set as $a_{in} = 0$ and $\sigma_{in}$ chosen to satisfy $\sigma(t_{end}) = 0$ at the end of inflation. We have chosen $\phi_{in} = 5M_p, \phi_1m = 14M_p$ and $\mu = 1/10$, which gives around 60 e-folds of inflation, with the first 10 e-folds being anisotropic. In this type of background, when the anisotropic expansion takes place at the onset of inflation, we expect that only the largest scale perturbations are improved and at small scales standard results are recovered.

### III. PERTURBATIONS

This section is devoted to the study of the perturbations around the anisotropic inflationary background solutions we have discussed in the previous section. We will first discuss the decomposition of perturbations around the background

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5 In order to find the approximate slow-roll solutions, one has to ignore the effect of the second field $\phi_1$ initially. Although this approximation is fairly accurate, we let the numerical evolution to run for a few e-folds until the real attractor solution is reached.
FIG. 3: The left panel shows the evolution of the average Hubble expansion. The right panel shows the evolution of anisotropy, where \( H \equiv \dot{\alpha} \) and \( h \equiv \dot{\sigma} \).

FIG. 4: The phase portrait for the two scalar fields. The straight red line represents \( \phi \) and the dashed black line represents \( \phi_1 \). Time is measured in units of \( m \).

counting, which is different from standard scalar-vector-tensor decomposition used for isotropic backgrounds. The isotropic background solutions have a 3 dimensional rotational and translational symmetry for constant time slices, therefore small fluctuations are decomposed using the representations of the \( SO(3) \) group. In such a decomposition, one identifies decoupled classes of fluctuations and according to their transformation properties, one can identify scalar, vector (transverse) and tensor modes (transverse-traceless). Such a decomposition can still be considered in an anisotropic background, however fluctuations classified as scalar, vector and tensor would be coupled in the linearized calculation \(^6\) (see for example \([10, 11]\)). Instead, we make use of the left over 2 dimensional rotational symmetry of the \( y - z \) plane of the Bianchi-I metric \((2)\) in order to decompose small fluctuations around the background as in \([9, 17]\). In such a two dimensional decomposition, there are two decoupled sets of perturbations, the 2d scalar and 2d vector modes. We will compute the quadratic action of the small fluctuations for both type of modes and study them separately. Since the universe isotropizes either after inflation in the single field model, or after the first stage of anisotropy in the double field model, the observables today are best defined if we use standard isotropic definitions of fluctuations. Therefore, we will also find the map between the 2d decomposition and the standard 3d isotropic decomposition, using which we will determine the power spectra for the gravitational wave polarizations \( \{ h_\times, h_+ \} \) and the curvature perturbation \( \mathcal{R} \) today (which are defined with respect to an isotropic background). This procedure is described in detail in the appendix.

\(^6\) We refer to both the computation of the quadratic action and linearized Einstein equations as the "linearized calculation", since the latter follows from the previous.
A. Decomposition of Perturbations

We decompose the perturbations of the metric and the vector field by exploiting the 2d isotropy of the background as

\[
\delta g_{\mu\nu} = \begin{pmatrix}
-2\Phi & a \partial_i \chi & b(\partial_i B + B_i) \\
-2a^2\Psi & a b \partial_i \left( \tilde{\partial}_i \tilde{B} + \tilde{B}_i \right) & b^2 \left( -2\Sigma \delta_{ij} - 2 \partial_i \partial_j E - \partial_i E_j - \partial_j E_i \right)
\end{pmatrix}
\]

\[
\delta A_\mu = (\alpha_0, \alpha_1, \partial_\alpha + \alpha_i)
\]

(28)

Additionally, the system of perturbations contain the fluctuations of the scalar field(s) \( \delta \phi_\alpha \). Indices \( i, j = 2, 3 \) span the isotropic 2d-plane. The perturbations \( \{ \delta \phi_\alpha, \Phi, \chi, B, \Psi, \tilde{\Sigma}, E, \alpha_0, \alpha_1, \alpha \} \) are 2d scalar perturbations, comprising one degree of freedom (d.o.f) each, and \( \{ B_i, \tilde{B}_i, E_i, \alpha_i \} \) are 2d vector modes which satisfy the transversality condition \( (\partial_i B_i = \cdots = 0) \). Therefore, the 2d vector perturbations also comprise one d.o.f. each. The 2d vector and scalar modes are decoupled at the linearized level.

We will work in the momentum space where a generic Fourier transformed mode is defined as

\[
\delta (x) = \frac{1}{(2\pi)^3} \int d^3k \delta (k) e^{-ik \cdot x} \delta k \cdot y - i k \cdot z
\]

(29)

We can use the residual 2d-isotropy to fix the comoving momentum to be aligned in the \( x - y \) plane without any loss of generality. Namely, we set \( k_{T3} = 0 \) and \( k_T = k_{T2} \) in what follows. Therefore, in Fourier space, the generic 2d decomposition can be written as (before fixing the gauge)

\[
\delta g_{\mu\nu}(k) = \begin{pmatrix}
-2\Psi & -i a k_L \chi & -i b k_T B & b_3 \\
-2a^2\Psi & -a b k_L k_T \tilde{B} & -i a b k_L \tilde{B}_3 & b^3 \\
b^2 \left( -2\Sigma + 2k_T^2 E \right) & i b^3 k_T E_3 & -2b^2 \Sigma &
\end{pmatrix}
\]

\[
\delta A_\mu(k) = (\alpha_0, \alpha_1, -i k_T \alpha, \alpha_3)
\]

(30)

Notice that, for the 2d vector modes, fixing \( k_T = k_{T2} \) resulted in \( B_2 = \tilde{B}_2 = E_2 = \alpha_2 = 0 \). Under the infinitesimal coordinate transformation \( x^\mu \rightarrow x^\mu + \epsilon^\mu \), the metric perturbations transform as

\[
\delta g_{\mu\nu} \rightarrow \delta g_{\mu\nu} - g^{(0)}_{\mu\nu,\sigma} \epsilon^\sigma - g^{(0)}_{\mu\nu} \epsilon^\sigma - g^{(0)}_{\sigma\nu} \epsilon^\sigma
\]

(31)

where we have decomposed

\[
\epsilon^\mu(k) = (\xi^0, -i k_L \xi^1, -i k_T \xi, \xi^3)
\]

(32)

We choose to fix the coordinate gauge freedom by setting \( \tilde{B} = \Sigma = E = E_3 = 0 \), namely we solve for the infinitesimal transformation to satisfy

\[
\xi^0 = -\frac{1}{H_b} \Sigma , \quad \xi = - E , \quad \xi^1 = \frac{b}{a} \left( \tilde{B} + \frac{b}{a} E \right) , \quad \xi^3 = -E_3
\]

(33)

Since the action is invariant under the \( U(1) \) gauge transformation, we can exploit this symmetry to eliminate one of the vector field perturbations. Namely, using the invariance \( \delta A_\mu \rightarrow \delta A_\mu + \partial_\mu \lambda \), we set \( \lambda = 0 \) in (28). This choice completely fixes the \( U(1) \) gauge freedom. In summary, the complete system of perturbations, after fixing the coordinate and \( U(1) \) gauge freedoms, contain three 2d vector modes \( \{ B_3, \alpha_3, B_3 \} \) and six 2d scalar modes \( \{ \Phi, \chi, B, \Psi, \alpha_0, \alpha_1 \} \).

We will also have additional scalar modes coming from the fluctuations of the scalar field(s) \( \delta \phi_\alpha \). We will show in the following section that, not all the perturbations are dynamical, namely some perturbations enter into the quadratic action without any time derivatives. Such modes are Lagrange multipliers and they can be integrated out from the action [32]. The modes \( \{ B_3, \alpha_3, \Psi, \alpha_1 \} \) and perturbations of the scalar fields are dynamical modes. On the other hand, \( \{ B_3, \Phi, \chi, B, \alpha_0 \} \) are nondynamical modes as we will demonstrate.

As we have mentioned earlier, it is useful to find the connection between the perturbations defined with respect to the 2d decomposition and perturbations defined with respect to standard isotropic decompositions. This is useful in discussing the phenomenological consequences. After the universe isotropizes, perturbations evolve in the standard way, what is different is that their solutions are nonstandard. However, it is useful to give results in terms of the modes
which are commonly used in FRW studies. For definiteness, we find the transformation between our choice of the gauge given in equation (33) and the standard longitudinal gauge. Here we present the results and leave the details to the appendix. The perturbations in the 2d decomposition are related to the perturbations of the longitudinal gauge as

\[
\dot{\bar{B}}_3 = \frac{i}{k} k_L \left( \frac{k_L^2 + b/a k_T^2}{k_T} \right) h_x
\]

\[
\Psi = \left( 1 - \frac{H_a}{H_b} \right) \psi - \left\{ \frac{H_a}{2H_b} + \frac{1}{k} \left[ \frac{1}{2} k_T^2 + \frac{b^3}{a^3} k_L^2 \left( 1 + \frac{b}{a} \frac{2k_T^2}{k_L^2} \right) \right] \right\} h_+
\]

\[
\delta \phi = \delta \phi^\perp + \frac{\dot{\phi}}{H_b} \left( \psi + \frac{1}{2} h_+ \right).
\]

In the above equations, the terms in the left hand side are perturbations defined with respect to the 2d decomposition and the terms in the right hand side are perturbations in the longitudinal gauge, where \(h_x, h_+\) are the two gravitational wave polarizations, \(\psi\) is the spatial scalar perturbation and \(\delta \phi^\perp\) is the scalar field perturbation (any generic scalar field in the model). These relations are useful in computing the power spectra of the gravitational waves and curvature perturbation.

### B. General Properties of Coupled System of Perturbations

In this subsection, we discuss the general properties of the coupled system of perturbations that arise in this study. Our main discussion is based on the quadratic action for perturbations, which we need in order to determine initial conditions by canonical quantization. We compute the action for perturbations in quadratic order by inserting the decompositions (28), \(\phi = \phi_0 + \delta \phi\) and \(\phi_a = \phi_{a0} + \delta \phi_a\) into (1), after fixing the \(U(1)\) and coordinate gauges as described in the previous subsection. We note that the 2d scalar and 2d vector perturbations decouple in the quadratic action. We will provide explicit expressions for the scalar and vector actions in the following sections. Next we integrate out the nondynamical modes from the action, to obtain an action in terms of the dynamical modes only. More specifically, we extremize the quadratic action with respect to the nondynamical modes and obtain equations of motions for them (which are actually constraint equations). Then we solve for the nondynamical modes from their equations of motion, and insert the solutions back into the starting action. The final action we obtain depends only on the dynamical modes. The next step we perform is to find linear combinations of modes that canonically normalize the quadratic action. More specifically, the quadratic action after integrating out the nondynamical modes is formally of the type

\[
S \sim \int dt d^3k \left( \delta_i K_{ij} \dot{\delta}_j + \dot{\delta}_i \Lambda_{ij} \delta_j - \delta_i \Omega_{ij} \dot{\delta}_j \right),
\]

where \(\delta_i\) denotes the dynamical modes. We find linear combinations

\[
\delta_i = R_{ij} \delta_{ij}\]

such that \(\delta_i K_{ij} \dot{\delta}_j = \frac{1}{2} \delta_{ij} \delta_{ij} + \ldots\) where the dots denote mixed terms. The modes \(\delta_{ij}\) are the canonical modes. At the moment we will study a generic action obtained after canonically normalizing the starting action, which will be relevant for both 2d vector and scalar modes. This action reads

\[
S_{can} = \frac{1}{2} \int dt d^3k \left\{ \Phi^\dagger \dot{\Phi} + \left( i \Phi^\dagger X \Phi + \text{h.c.} \right) - p^2 \Phi^\dagger \left( 1 + \frac{\omega^2}{p^2} \right) \Phi \right\}
\]

where \(\Phi\) is a column vector made from the canonically normalized perturbations, \(X\) is a symmetric matrix and \(\omega^2\) is a hermitian matrix. In the above action, \(p\) is the physical momentum with \(p^2 = p_L^2 + p_T^2\) where \(p_L = k_L/a\) and \(p_T = k_T/b\). We would like to perform a unitary transformation in the field space in order to get rid of the mixed terms \(\Phi^\dagger X \Phi\) and reduce the action to a canonical form that can be used to quantize the fields (and in turn obtain the initial conditions). This can be achieved by a unitary matrix \(U\) so that \(\Phi = U \Psi\) (The mixed terms can also be eliminated in the level of the Hamiltonian by canonical transformations. See for example [34]). We choose the matrix \(U\) to satisfy

\[
\dot{U} + i X U = 0.
\]

Inserting the transformation back into (35) and using (36) (with the unitarity condition on \(U\)), we get

\[
S = \frac{1}{2} \int dt d^3k \left[ \Phi^\dagger \dot{\Phi} - \Psi^\dagger U^\dagger \left( p^2 + \omega^2 + X^\dagger X \right) U \Psi \right].
\]

We will show in the following sections that the matrix \(X\) is of the order of the average Hubble scale \(H\) (i.e. \(X = O(H)\)) and the matrix \(\omega^2\) is of the order of Hubble scale squared (i.e. \(\omega^2 = O(H^2)\)). This means that in the deep
UV/subhorizon regime, where $p \gg H$, these matrices will become negligible compared to $p^2$. Therefore, for modes that are deep inside the horizon initially, the above action reduces to a form which has diagonal mass terms

$$S = \frac{1}{2} \int dt d^3k \left[ \hat{\Psi}^\dagger \dot{\Psi} - p^2 \Psi^\dagger \Psi + O(H^2) \right]$$  \hspace{1cm} (38)$$

In this limit when the mass terms are diagonal, the mode expansion for field $\Psi$ simply reads

$$\Psi(x) = \int \frac{d^3k}{(2\pi)^{3/2}} \left[ e^{i \vec{k} \cdot \vec{x}} v(t) \hat{a}(k) + e^{-i \vec{k} \cdot \vec{x}} v^*(t) \hat{a}^\dagger(k) \right]$$  \hspace{1cm} (39)$$

The creation and annihilation operators $\hat{a}(k), \hat{a}^\dagger(k)$ obey the standard bosonic commutation relations and the mode functions $v(t)$ satisfy

$$\ddot{v} + p^2 v = 0.$$  \hspace{1cm} (40)$$

The above equation can be solved using the WKB approximation in the adiabatic limit, which gives

$$v(t) = \frac{1}{\sqrt{2\omega}} e^{-i \int \omega dt'}$$  \hspace{1cm} (41)$$

where $\omega$ is given by

$$\omega = p - \frac{\dot{p}}{4p^2} + \frac{3}{8} \frac{p^2}{p^0} + \ldots$$

$$= p + O(H^2)$$  \hspace{1cm} (42)$$

In the above equation, $\ldots$ represent higher order corrections which are suppressed by higher powers of $H$ in the adiabatic solution. This is also consistent with the accuracy of the mode expansion (39). It is of course possible to neglect the mixed terms proportional to $\pi_1$ and $\omega^2$ from the beginning. In this way the action for the original fields $\Phi$ is canonical, however we would be working with an accuracy $O(H_{\text{in}}/p_{\text{in}})$ rather than $O(H^2_{\text{in}}/p_{\text{in}}^2)$. This situation is more difficult in terms of the numerical computation needed, since to achieve the desired accuracy, one has to start the numerical evolution much deeper inside the horizon. Moreover, initial conditions in the standard isotropic case are also given to an accuracy of $O(H^2)$. Therefore, we choose to perform the field rotation to eliminate the $X \sim O(H)$ terms and work with the fields $\Psi$. The field rotation can be performed trivially for the case of 2d vector perturbations. The case for the 2d scalar modes are more subtle, therefore we leave the discussion of scalar modes to a separate publication.

The canonical quantization procedure outlined above determines the adiabatic initial conditions for the modes $\Psi$. We are however interested in the values of the entries of $\Phi$ evaluated at the end of inflation, which are related to the primordial power spectra. We will show in the next section that the matrix $X$ along with the off-diagonal entries of $\omega^2$ is proportional to the anisotropy, so they vanish before the end of inflation in the two field model. Therefore, it is possible to choose the transformation matrix $U$ that satisfies $U(t_{\text{end}}) = 1$, which in turn implies that $U^\dagger (\omega^2 + X^\dagger X) U$ is also diagonal at $t_{\text{end}}$ ($t_{\text{end}}$ is the time when inflation ends). Then we have $\Phi(t_{\text{end}}) = \Psi(t_{\text{end}})$ and we use the following equality to compute the two point functions for the starting modes $\Phi$:

$$\langle \Psi^\dagger_a \Psi^\dagger_b \rangle |_{t_{\text{end}}} = \langle \Phi^\dagger_a \Phi^\dagger_b \rangle |_{t_{\text{end}}}$$  \hspace{1cm} (43)$$

for the components of the fields. Therefore, instead of working with the original fields $\Phi$ we work with the transformed fields $\Psi$. Notice that passing to the fields $\Psi$ was an important step, since the action is in its canonical form with diagonal mass terms $p^2$ (up to accuracy $O(H^2)$) in the deep UV regime. Therefore, this field can be quantized and $p$ represents the energy of the each particle created by $\hat{a}^\dagger(k)$. The quantization procedure also determines the mode functions $v$, which in turn determines initial conditions for the perturbations.

In the next subsection, we will discuss the evolution of the 2d vector modes, which will be related to the power spectrum of the gravitational wave polarization $h_X$.

### C. 2d Vector Perturbations

We now discuss the 2d vector perturbations of the two field inflationary background solution discussed in the previous section. As we have argued earlier, the quadratic action for the 2d vector and scalar modes decouple, so we
concentrate on the vector piece here. The action for the 2d vector perturbations in momentum space, up to a total time derivative, is calculated as

\[
S_{2dV} = \frac{M_p^2}{4} \int dt d^3k e^{3\alpha - 2\sigma} \mathcal{L}_{2dV}
\]

\[
\mathcal{L}_{2dV} = e^{2\alpha - 2\sigma} p_L^2 \left( \frac{\dot{B}_3}{p_L} \right)^2 + 2 e^{2\alpha} \frac{p^2 f(\phi)}{M_p^2} |\dot{B}_3|^2 - e^\alpha p_L^2 \left( \hat{B}_3^* B_3 + \text{h.c.} \right) + 2 e^{-\sigma} \frac{\tilde{p}_A p_L}{M_p^2} \left( i \dot{\alpha}_3 \hat{B}_3 + \text{h.c.} \right)
\]

\[
= -2 e^{-2\alpha} \frac{p_L^2}{M_p^2} f^2(\phi) \left( \alpha_3 \right)^2 + 2 e^{-\alpha + \sigma} \frac{\tilde{p}_A p_L}{M_p^2} (i \alpha_3^* B_3 + \text{h.c.}) + e^{2\sigma} p^2 |\dot{B}_3|^2
\]

where we have used the physical momenta \(p_L \equiv k_L/a, p_T = k_T/b\) and \(p = \sqrt{p_L^2 + p_T^2}\). As can be seen from (44), the mode \(B_3\) is nondynamical, so it can be integrated out from the action by solving its equation of motion. The solution for \(B_3\) is obtained as

\[
B_3 = e^{\alpha - 2\sigma} \left[ \frac{2 i p_L \tilde{p}_A}{M_p^2 p^2} e^{-\alpha + \sigma} \alpha_3 + \frac{p_L^2}{p^2} \left( \hat{B}_3 - 3 \dot{\sigma} \hat{B}_3 \right) \right]
\]

We insert the solution (45) back into the action (44) and up to a total time derivative, we obtain

\[
S_{2dV} = \frac{1}{2} \int dt d^3k \left( |\dot{H}_\times|^2 + |\Delta_-|^2 + \frac{\tilde{p}_A p_T}{\sqrt{2} M_p p f(\phi)} \left( i \dot{H}_\times \Delta_- + i H_\times \Delta_- + \text{h.c.} \right) \right)
\]

\[
- \left( \dot{H}_\times \Delta_- \right) \left( \begin{array}{cc} \Omega_{11} & \Omega_{12} \\ \Omega_{12}^* & \Omega_{22} \end{array} \right) \left( \begin{array}{c} H_\times \\ \Delta_- \end{array} \right)
\]

where the canonical modes \(H_\times\) and \(\Delta_-\) are related to the starting modes as

\[
\hat{B}_3 = \frac{\sqrt{2} \rho}{p_L p_T M_p} e^{-\frac{i}{2} \alpha + 2\sigma} H_\times, \quad \alpha_3 = \frac{e^{-\alpha/2 + \sigma}}{f(\phi)} \Delta_-
\]

and the mass terms \(\Omega^2\) are defined as

\[
\Omega_{11} = p^2 - \frac{9}{4} \dot{\alpha}^2 + \frac{3}{4M_p^2} \left( \phi^2 + \dot{\phi}_1^2 \right) + \frac{9}{2} \left( \frac{p_L^2}{p^2} + 6 \frac{\dot{p}_A^2}{p^4} - \frac{\dot{p}_L^2}{p^4} \right) \dot{\sigma}^2 + \frac{\tilde{p}_A^2}{2M_p^2 f^2(\phi)} \left( \frac{p_L^2}{p^4} - \frac{p_T^2}{p^4} \right)
\]

\[
\Omega_{12} = i \tilde{\omega}_i = \frac{i \tilde{p}_A p_T}{\sqrt{2} M_p p f(\phi)} \left( \frac{4p^2 L - 5p_T^2}{p^2} \dot{\sigma} + \dot{\alpha} \right) - \frac{f'(\phi)}{f(\phi)} \dot{\phi}_i
\]

\[
\Omega_{22} = p^2 - \frac{1}{4} \dot{\alpha}^2 + \frac{1}{4M_p^2} \left( \phi^2 + \dot{\phi}_1^2 \right) - \frac{2}{\dot{\sigma}} \dot{\phi} \dot{\phi}_i + \frac{\tilde{p}_A^2}{2M_p^2 f^2(\phi)} \left( \frac{\dot{p}_L^2}{p^2} + \frac{\dot{p}_T^2}{p^2} \right) - \frac{\tilde{p}_A^2}{f^2(\phi)} f'^2(\phi)
\]

\[
+ \left[ 2 (\dot{\sigma} + \dot{\alpha}) \phi \dot{\phi}_i + \dot{V}'(\phi) \right] \frac{f'(\phi)}{f(\phi)} - \frac{f''(\phi)}{f(\phi)} \dot{\phi}_i^2
\]

The action (46) has regular kinetic terms, and therefore there is no ghost instability in the 2d vector sector, as expected. Note that the matrix \(X\) along with the mass matrix terms \(\Omega^2\) satisfy

\[
\Omega_{12} = O(H^2), \quad \Omega_{11}^2 = \Omega_{22}^2 = p^2 + O(H^2),
\]

\[
\frac{\tilde{p}_A}{M_p f(\phi)} = \sqrt{6 \dot{\alpha}^2 - 6 \dot{\sigma}^2 - \frac{2}{M_p^2} (V(\phi) + V_1(\phi_1)) - \frac{1}{M_p^2} \left( \phi^2 + \dot{\phi}_1^2 \right)} \sim O(H)
\]

therefore, the action (46) is formally of the type given in (33) with \(\Phi\) and matrix \(X\) given by

\[
\Phi \equiv \left( \begin{array}{c} H_\times \\ \Delta_- \end{array} \right), \quad X \equiv \left( \begin{array}{cc} 0 & \lambda \\ \lambda & 0 \end{array} \right), \quad \lambda \equiv \frac{\tilde{p}_A p_T}{\sqrt{2} M_p^2 f(\phi) p}
\]

(50)
We can therefore determine the transformation matrix $U$, that eliminates the mixed terms $\dot{\Phi}^\dagger X \Phi$ by solving $\dot{U} + iXU = 0$, which can be solved by diagonalizing the matrix $X$. Notice that $R^\dagger X R = D_X$ where

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad D_X = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \quad (51)$$

Therefore (36) is solved by

$$U = R e^{-i \int^t D_X dt'} W_0 \quad (52)$$

where $W_0$ is a constant unitary matrix, which does not have any physical relevance. Therefore, we choose $W_0 = R^\dagger$, so that $U = 1$ when $\lambda = 0$ (i.e. when anisotropy vanishes). We can now determine the new mass matrix for the transformed fields $\Psi$, which are given by

$$\Omega_\Psi^2 = U^\dagger \left[ \Omega^2 + X^\dagger X \right] U \quad (53)$$

where the entries of $\Omega_\Psi^2$ are explicitly given by

$$\begin{align*}
\Omega_{\Psi 11}^2 &= \lambda^2 + \Omega_{11}^2 + \Omega_{22}^2 + \frac{\Omega_{12}^2 - \Omega_{22}^2}{2} \cos [2I(t)] + \Omega_{12}^2 \sin [2I(t)] \\
\Omega_{\Psi 22}^2 &= \lambda^2 + \Omega_{11}^2 + \Omega_{22}^2 - \frac{\Omega_{12}^2 - \Omega_{22}^2}{2} \cos [2I(t)] - \Omega_{12}^2 \sin [2I(t)] \\
\Omega_{\Psi 12}^2 &= -\Omega_{\Psi 21}^2 = i \left\{ \Omega_{12}^2 \cos [2I(t)] - \frac{\Omega_{11}^2 - \Omega_{22}^2}{2} \sin [2I(t)] \right\} \quad (54)
\end{align*}$$

where

$$I(t) \equiv \int_{t_k}^t \lambda(t') dt' = \begin{cases} \text{constant} & t > t_{iso} \\ \int_{t_k}^t \lambda(t') dt' & t < t_{iso} \end{cases} \quad (55)$$

Here, $t_{iso}$ is the time when the universe isotropizes so $\lambda(t \geq t_{iso}) = 0$. The lower limit $t_k$ in the integral is chosen for each comoving momentum $k$ such that the constant value for $t > t_{iso}$ is unity $7$. This ensures that $U(t_{iso}) = U(t_{end}) = 1$ since

$$U \equiv \begin{pmatrix} \cos [I(t)] & -i \sin [I(t)] \\ -i \sin [I(t)] & \cos [I(t)] \end{pmatrix} \rightarrow 1 \quad \text{when} \quad t \geq t_{iso} \quad (56)$$

Finally, the fields $\Psi$ satisfy the evolution equations

$$\ddot{\Psi}_a + \Omega_{\Psi ab}^2 \Psi_b = 0 \quad (57)$$

with initial conditions on the entries of $\Psi_a$ set by the form of the mode function $v(t)$ which are given deep inside the horizon as

$$\begin{align*}
\Psi_{a \text{ in}} &= \frac{1}{\sqrt{2p}} + O(H^2) \\
\dot{\Psi}_{a \text{ in}} &= \frac{1}{\sqrt{2p}} \left( -i p - \frac{\dot{p}}{2p} \right) + O(H^2) \quad (58)
\end{align*}$$

up to a nonphysical phase.

In the next section, we will numerically evaluate the evolution equations (57) with adiabatic initial conditions (58) for a range of comoving momenta, and determine the two point functions $<\Psi_a \Psi_b> |_{t_{end}}$ which coincides with $<\Phi_a^\dagger \Phi_b> |_{t_{end}}$ as we argued before. Finally, using (47) and (44) we deduce the power spectrum (which is related to the calculated two point functions) for the gravitational wave polarization $h_\times$. We will provide the study of the 2d scalar perturbations in the appendix and leave the study of the power spectrum for the scalar modes and the comparison with observations in a separate publication.

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7 It is possible to keep the lower limit in the integral arbitrary, since it has no physical relevance. However, for an arbitrary value, the mass terms for the fields $\Psi$ will in general be nondiagonal at the end of inflation. In this case, one has to further rotate the fields (by a constant matrix) to obtain the physical modes. We perform this step from the beginning by choosing $t_k$ correctly, so that $U = 1$ at the end of inflation.
IV. POWER SPECTRUM

This section is devoted to the study of the power spectrum of the gravitational wave polarization $h_x$. We will use the results of the previous sections to compute the evolution of the fields $\Psi$ using their equations of motion given in (57) with the elements of the mass matrices given in (54). We perform the integration for a range of co-moving momenta $k_L$ and $k_T$, which we parameterize as

$$k_L = \xi k, \quad k_T = \sqrt{1 - \xi^2} k, \quad k = \sqrt{k_L^2 + k_T^2}, \quad 0 \leq \xi \leq 1.$$  \hfill (59)

Therefore, $\xi$ is the cosine of the angle between the comoving momentum and the longitudinal component $k_L$. For each comoving mometa labelled by the two numbers $\{k, \xi\}$, initial conditions are given in equation (58). The initial time $t_{in}$ is chosen to be the moment when

$$\frac{k}{a(t_{in})} = 50 H(t_{in}) \quad \text{or} \quad k = 50 \dot{\alpha}(t_{in}) e^{\alpha(t_{in})}$$  \hfill (60)

so that the physical momentum $p$ is sufficiently larger than the Hubble scale and the adiabatic initial conditions are accurate. Since the mass matrix $\Omega_2$ depends on the background quantities, we integrate the equations (59) together with the background equations (64) with initial conditions and parameters $m_1$, $m_2$ set as explained in subsection (II B). Using (34) and (47) we can relate the gravitational wave polarization $h_x$ to the canonical field $H_x$ at the end of inflation (when $\sigma = \dot{\sigma} = 0$, so $a = b$)

$$h_x = \sqrt{2\frac{\alpha}{\dot{\alpha}}} e^{-\frac{\gamma}{\dot{\alpha}}} H_x.$$  \hfill (61)

We define the power spectrum for the gravitational wave mode $h_x$ as

$$P_x(k) \equiv \langle k^3 |h_x|^2 \rangle_{t = t_{end}}$$  \hfill (62)

At the end of inflation, the fields $\Psi$ is identical to the starting fields $\Phi$, so the power spectrum of the gravitational wave mode $h_x$ is obtained by evaluating the correlation function $\langle \Psi_1^* \Psi_1 \rangle = \langle \Phi_1^* \Phi_1 \rangle$ and using (61) which gives

$$P_x(k) = \frac{2k^3}{M_p^2} e^{-3\alpha(t_{end})} |\Psi_1(t_{end})|^2.$$  \hfill (63)

In Figures 5 and 6 we show the power spectrum obtained from the numerical integration of the background and perturbation equations. In Fig 5, parts of the power spectrum is shown for fixed values of $\xi = 0$, 0.5, 1 and $10^{-1} k_{iso} \leq k \leq 10^{1/2} k_{iso}$. We have define $k_{iso}$ to be the value of the comoving momentum, when $k = a(t_{iso}) H(t_{iso}) = \dot{\alpha}(t_{iso}) e^{\alpha(t_{iso})}$ where $t_{iso}$ is the time of isotropization (we define it to be the time when $h/H = 10^{-3}$ numerically). In Fig 6 we show contour plot of the full spectrum $P_x(k)$. At large scales for $k > k_{iso}$, there is more power and the spectrum has angular dependence. This in turn could be related to the CMB anomalies. Note that the spectrum reduces to the standard nearly scale invariant form when $k > k_{iso}$. Indeed, we have checked that for $k > k_{iso}$, the power spectrum reduces to

$$P_x(k > k_{iso}) = A_T k^{n_T}, \quad n_T \equiv -2 \epsilon = -\frac{\dot{\alpha}^2}{\alpha^2 M_p^2}$$  \hfill (64)

where $\epsilon$ is the standard slow-roll parameter for the field $\phi_1$. Indeed, the the spectrum can be written as

$$P_x(k) = \begin{cases} \alpha_T \left( \frac{k}{k_{iso}} \right)^{f(\xi, k)} n_T & k \leq k_{iso}, \\ \alpha_T \left( \frac{k}{k_{iso}} \right)^{n_T} & k > k_{iso}. \end{cases}$$  \hfill (65)

The form of the function $f(\xi, k)$ is unknown, but it is possible to consider a generic expansion of the form

$$\Log_{10} P_x = \Log_{10} \alpha_T + (c_1 + d_1 \xi^{n_1}) x + (c_2 + d_2 \xi^{n_2}) x^2 + (c_3 + d_3 \xi^{n_3}) x^3 + \ldots$$

$$x \equiv \Log_{10} \left( \frac{k}{k_{iso}} \right)$$  \hfill (66)
We fix the unknown parameters $\log_{10}aT$ (which coincides with the $\log_{10}aT$ of the spectrum for $k < k_{\text{iso}}$) and $c_1$, $c_2$, $c_3$ by fitting to the numerical spectrum when $\xi = 0$. Then, we fix the parameters $d_1$, $d_2$, $d_3$ by fitting to the numerical spectrum when $\xi = 1$ (using the previous values of $\log_{10}aT$ and $c_1$, $c_2$, $c_3$). Finally, we obtain the values of $n_1$, $n_2$, $n_3$ by fitting to a spectrum at an intermediate value of $\xi$. Increasing the number of terms in the expansion increases the accuracy of the fit. For the spectra we have obtained above, the unknown parameters are obtained as

$$\begin{align*}
\log_{10}aT &\simeq -0.43, \quad c_1 \simeq -0.23, \quad c_2 \simeq 0.83, \quad c_3 \simeq 0.38 \\
d_1 &\simeq 0.82, \quad d_2 \simeq 1.21, \quad d_3 \simeq 0.69 \\
n_1 &\simeq 3.94, \quad n_2 \simeq 3.38, \quad n_3 \simeq 3.34
\end{align*}$$

(67)

In Fig. 7 we show the fitted and numerical power spectra for $k < k_{\text{iso}}$, which are in good agreement. In Fig. 8 we show the numerical spectra for $k > k_{\text{iso}}$, which is of the standard nearly scale invariant form. The power spectrum for the gravitational wave mode $h_x$ is not of the form $P_x = P_{x,\text{iso}}(k) + \delta P_x(k,\xi)$ as one would naively expect for small anisotropy for $k < k_{\text{iso}}$, with $|\delta P_x| \ll |P_{x,\text{iso}}|$. This is due to the fact that the anisotropic background attractor is not continuously connected to standard isotropic solutions, i.e. there is no parameter with a certain limit which takes the anisotropic solution to the isotropic one. This is in contrast for example to the ACW model [15], where in
the limit \( m^2 \to 0 \) (\( m^2 \) is the norm of the VEV of the vector field in the model), the background solution reduces to an isotropic limit. For the current case, attractors with \( c > 1 \) (anisotropic) and \( c < 1 \) (isotropic) are disconnected in the limit \( c \to 1 \). Therefore, the behavior of the perturbations are dramatically different for \( k < k_{\text{iso}} \) and the spectrum is largely enhanced in this region (|\( \delta P \times P \)| is not satisfied). For phenomenological considerations, this result is not a desired one, since the CMB data would be consistent with a statistical anisotropy at large scales that would satisfy |\( \delta P \times P \)| ≪ |\( P \times P \), iso\). However, this model is the only vector field driven anisotropic inflationary model that does not suffer from any instability to our knowledge. Therefore, we believe that the results obtained in this section are still interesting.

V. CONCLUSIONS

In this work we have considered anisotropic inflationary models with coupled vector and scalar fields. The coupling of the scalar field to the vector field is chosen to preserve the \( U(1) \) gauge symmetry of the Lagrangian, therefore, the longitudinal polarization of the vector field, which has been shown to cause instabilities [14, 17, 30], does not exist. The anisotropic expansion is achieved when the vector field has a VEV along one of the spatial directions. The anisotropic solution is an attractor, with anisotropy increasing towards the end of inflation, so these types of models are counter examples to the cosmic no hair conjecture [35]. We have considered a generalization of the original model introduced in reference [24], to include additional scalar fields, and specifically concentrated on the double scalar...
field case. This modification was done in order to achieve isotropization of the universe before the end of inflation. The original model with a single scalar field coupled to the vector field has an anisotropic inflationary background which persists until the end of inflation. In this type of background, perturbations at all scales are modified. In the two field modification, the extra field is not coupled to the vector field, and it is responsible for the overall isotropic expansion of the universe only. We have chosen the ratio of the masses of the scalar fields such that the inflationary expansion takes place in two steps, the first being anisotropic and the second is isotropic. Therefore, only the largest scale perturbations are modified and at small scales the standard spectrum is recovered. As we have discussed in the introduction, vector fields with a nonvanishing VEV have been introduced since isotropization takes place very quickly in a Bianchi-I background, (when anisotropy is only coming from initial conditions) and thus leads to a fine tuning. We have introduced a two phase inflation, with the second phase being isotropic. The transition to the second isotropic phase is indeed very quick, however the fine tuning has been relaxed compared to the previous case. The second isotropic phase should still be tuned to last around 60 e-folds, but this tuning is not so strict, since it only affects the scale where the power spectrum becomes statistically isotropic. Previously, the tuning was more stringent, since inflation must last for 60 e-folds in order that the largest scales are modified from the initial anisotropic conditions. A more important drawback of the inflationary setup with only initial anisotropic conditions is that the initial singularity is very close to the time of isotropization, which is around one Hubble time. Thus, a calculation of the power spectrum shows that it becomes divergent at the largest scales

We have studied the perturbations around the background configuration in detail, taking into account all the degrees of freedom of the system. Following the formalism developed in references 8, 9, we have classified decoupled sets of perturbations around the anisotropic background and studied them separately. Furthermore, we have computed the quadratic action for perturbations and showed that the model is stable and consistent. We also found linear combinations of the perturbations which canonically normalize the action. This enabled us to quantize the system in which in turn determined the initial conditions for the perturbations in the adiabatic vacuum. We then numerically integrated the equations of motion for the 2d vector perturbations for a range of comoving momenta, which resulted in the power spectrum of the gravitational wave polarization \( h_x \). We have only computed the quadratic action and determined the canonical modes for the 2d scalar perturbations. We leave the study of the power spectra and the consequent phenomenology, which in turn is related to the curvature and \( h_+ \) gravitational wave spectrum, to a later publication.

The computed spectrum for \( h_x \) has certain interesting features, which might have some relevance to the observed CMB anomalies. First of all, the spectrum has angular dependence for \( k < k_{iso} \) which breaks the statistical isotropy at large scales, therefore leads to an alignment of the lowest multipoles as described in 8, 9, 13. Moreover, the spectrum reduces to the standard nearly scale invariant form for \( k > k_{iso} \), so the higher multipoles are not affected from the vector field. We have shown that the power spectrum is formally of the form \( P_{\delta} (k) = P_{\delta, iso}(k) + \delta P_{\delta}(k, \xi) \). We expect that this will be true also for the spectrum of scalar modes. We have also provided a functional form for \( \delta P_{\delta} \) by fitting to the numerical spectrum in equation (66). Phenomenologically acceptable modifications to the large scale spectrum should satisfy \( |\delta P| \ll |P_{iso}| \), however, we have shown that for the model under discussion, this is not the case. Although our results are obtained for the gravitational wave mode \( h_x \), we expect a similar behavior also for the curvature spectrum, which can be compared with observations. At larger scales, the power spectrum is greatly enhanced satisfying \( |\delta P_{\delta}| > |P_{\delta, iso}| \). Although the anisotropy in the background is small (in the level of a percent, as can be seen from equation (24)), the modification to the spectrum at large scales is not small. This is due to the fact that the background attractor solution is disconnected from the standard isotropic solution and therefore, the behavior of perturbations are dramatically different. There is no parameter in the attractor solution that connects it continuously to the isotropic FRW limit. The anisotropy is always small and proportional to \( 1/c^2 \) and there is no finite limit in \( c \) which can continuously deform the attractor solution to the isotropic solution. Therefore, the power spectrum cannot be approximated as to be the isotropic piece plus a small correction from anisotropy. This issue unfortunately limits the phenomenology of the model.

Even though the difficulties involved, anisotropic backgrounds still have potentially interesting phenomenological consequences. Off-diagonal correlations of the \( a_{lm} \) coefficients of CMB temperature fluctuations arise in such backgrounds and can have relevance to the alignment of lowest multipoles. Moreover, at the largest scales, non standard scalar to tensor ratio is obtained, which can be tested by upcoming CMB experiments. The gravitational wave modes \( h_+ \) and \( h_x \) behave differently at large scales, therefore a future detection of gravitational waves might be tested against the consequences of an early stage of anisotropic inflation. In this paper, we have provided a solid example of anisotropic inflationary calculation of perturbations and computed the spectrum of \( h_x \) polarization of gravitational

\[ \frac{16}{7} \] This does not mean that the model is unstable, it reflects that nonlinear effects become important, and one has to go beyond linear perturbation theory.
waves. Although the obtained spectrum has phenomenologically limited implications, we believe that our results are still relevant, since it is the first complete study of power spectrum in an anisotropic inflationary model driven by a vector field to our knowledge. In a further publication, we will study the spectrum of the curvature $R$ perturbation, which could then be compared with observational data.

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VI. APPENDICES

A. Appendix I

In this appendix we derive the relations (34), which are used in the main text to compute the power spectrum of the gravitational wave mode $h_\times$. These relations are also useful when the curvature and $h_+^2$ power spectra are calculated. For this calculation, one needs to analyze the 2$\text{d}$ scalar perturbations, which is done in the next appendix. We will study the power spectrum for the scalar modes in a separate publication.

We start with the decomposition of perturbations in the longitudinal gauge, when the universe is isotropic, which is given by

$$ds^2 = -(1 + 2\phi) \, dt^2 + 2 \, a \, V_1 \, dt \, dx^i + a^2 \, [(1 - 2\psi) \, \delta_{ij} + h_{ij}] \, dx^i \, dx^j$$

where $i = 1, 2, 3$. $V_1$’s are vector modes satisfying $\partial_i \, V_1 = 0$ and $h_{ij}$’s are transverse-traceless modes satisfying $\delta_{ij} \, h_{ij} = h_{ii} = 0$. Since the spatial slices are isotropic, we can fix the comoving momentum to lie entirely along the $x$-direction $\vec{k} = (k, 0, 0)$, without any loss of generality. In this case, the longitudinal gauge can be expressed in Fourier space as

$$\delta g_{\mu\nu}(k) = \begin{pmatrix} -2\phi & 0 & V_2 & V_3 \\ -2a^2 \psi & 0 & 0 & 0 \\ a^2 \left(-2\psi + h_+\right) & -a^2 h_\times & -a^2 h_\times & a^2 \left(-2\psi - h_+\right) \end{pmatrix}$$

We can now rotate the momentum from the $x$-direction, to $x - y$ plane in order to make the comparison with the 2$\text{d}$ decomposition. Namely, we perform the following rotation over the comoving momentum:

$$\begin{pmatrix} k_L \\ k_T \end{pmatrix} = \frac{1}{k} \begin{pmatrix} k_L & -k_T \\ k_T & k_L \end{pmatrix} \begin{pmatrix} k \\ 0 \end{pmatrix}$$

where $k = \sqrt{k_L^2 + k_T^2}$. The matrix appearing in (70) also determines the Jacobian matrix which fixes the coordinate transformation rule from the metric with $\vec{k} = (k, 0, 0)$ (69) to a metric with $\vec{k} = (k_L, k_T, 0)$. The resulting decomposition will determine the 2$\text{d}$-decomposed longitudinal gauge. Using the appropriate scale factors $a$ and $b$ where necessary to take into account the anisotropy of the universe, the 2$\text{d}$ decomposed longitudinal gauge becomes

$$\delta g^{L\mu\nu}(k) = \begin{pmatrix} -2\phi & -a \frac{k_T}{k} V_2 & b k_T \frac{k_L}{k} V_2 & b V_3 \\ a^2 \left(-2\psi + \frac{k_T^2}{k} h_+\right) & -a b \frac{k_T}{k} \frac{k_T}{k} h_+ & a b \frac{k_T}{k} h_\times & a b \frac{k_T}{k} h_\times \\ b^2 \left(-2\psi + \frac{k_T^2}{k} h_\times\right) & b^2 \left(-2\psi - h_\times\right) & b^2 \left(-2\psi - h_\times\right) & b^2 \left(-2\psi - h_\times\right) \end{pmatrix}$$

where the superscript ”$L$” indicates that longitudinal gauge is used. Comparing (71) with (30), we identify:

$$\Phi_L = \phi \ , \ \chi_L = \frac{k_T}{i \, k \, k_L} \, V_2 \ , \ B_L = \frac{k_L}{i \, k \, k_T} \, V_2 \ , \ B_{3L} = V_3 \ , \ \hat{B}_{3L} = -\frac{k_T}{i \, k \, k_L} h_\times \ , \ E_{3L} = -\frac{k_L}{i \, k \, k_T} h_\times$$

$$\Sigma_L = \psi + \frac{1}{2} \, h_+ \ , \ \hat{B}_L = \frac{1}{k^2} \, h_+ \ , \ E_L = \frac{2k_T^2 + k_L^2}{2k^2 k_T} \, h_+$$

(72)
We can now find the infinitesimal transformation between our gauge choice of $\tilde{B} = \Sigma = E = E_3 = 0$ and the longitudinal gauge using

$$\delta g_{\mu\nu}^{(\text{our})} = \delta g_{\mu\nu}^L - g_{\mu\nu,\sigma}^{(0)} \epsilon^\sigma - g_{\mu\sigma}^{(0)} \epsilon^{\sigma}_{,\nu} - g_{\sigma\nu}^{(0)} \epsilon_{,\mu}$$

(73)

where

$$\epsilon^\mu(k) = \left( \xi_0^0, -i k L \xi_1^1, -i k T \xi_1^1, \xi_3^3 \right)$$

(74)

Finally, the infinitesimal transformation between the two gauges are determined by

$$\xi_0^0 = -\frac{1}{H_b} \left( \psi + \frac{1}{2} h_+ \right), \xi_L = -\frac{2k_L^2 + b^2}{2k_L^2 k_T} h_+$$

$$\xi_1^1 = \frac{k}{a k_T^2} \left( k_L^2 + b/a k_L^2 \right) \frac{1}{a + \frac{b}{2} k_L^2} h_+, \xi_3^3 = \frac{k_L}{i k k_L} h_\times$$

(75)

Using these transformations, we can relate the power spectra calculated in the 2d decomposition with the standard isotropic definitions of power. For instance, using (73) and (75) the dynamical modes of the metric perturbations of the 2d decomposition are related to the gravitational wave modes $h_+, h_\times$ and $\psi$ (related to curvature perturbation) as

$$\tilde{B}_3 = \frac{i}{k k_L} \left( k_T^2 + b/a k_L^2 \right) h_\times$$

$$\Psi = \left( 1 - \frac{H_a}{H_b} \right) \psi - \left\{ \frac{H_a}{2H_b} + \frac{1}{k^2} \left[ \frac{k_T^2}{2} + \frac{b^3}{a^3} k_L^2 \left( 1 + \frac{b}{2} \frac{2k_L^2 + k_T^2}{k_T^2} \right) \right] \right\} h_+.$$

Moreover, using the infinitesimal transformation $\delta \phi = \delta \phi^L - \dot{\phi}^0 \xi_0^0$, the fluctuation of any scalar field in our gauge is related to the one in longitudinal gauge as

$$\delta \phi = \delta \phi^L + \frac{\dot{\phi}}{H_b} \left( \psi + \frac{1}{2} h_+ \right).$$

**B. Appendix II**

In this appendix, we study the 2d scalar perturbations. As we have done for the case of the 2d vector perturbations, we insert the metric and vector field decompositions (28) and $\phi = \phi_0(t) + \delta \phi, \phi_1 = \phi_1(t) + \delta \phi_1$ into the action (I) (with $a = 1$), after fixing the $U(1)$ and general coordinate invariance as described previously. The quadratic action in
momentum space up to a total time derivative is given by

$$S_{2dS} = \frac{1}{2} \int dt \, d^3k \, e^{3\alpha} \mathcal{L}_{2dS}$$

$$\mathcal{L}_{2dS} = \{ \delta \dot{\phi} \}^2 + \{ \dot{\delta} \}^2 + e^{-2\alpha + 4\sigma} f^2(\phi) |\dot{\alpha}_1|^2 - \dot{\phi} \left( \delta \dot{\phi}^* \Psi + \text{h.c.} \right) - \dot{\phi} \left( \delta \dot{\phi}^* \Phi + \text{h.c.} \right)$$

$$- \dot{\phi}_1 \left( \delta \dot{\phi}_1^* \Psi + \text{h.c.} \right) - \dot{\phi}_1 \left( \delta \dot{\phi}_1^* \Phi + \text{h.c.} \right) - 2M_p^2 (\dot{\alpha} + \dot{\sigma}) \left( \dot{\Psi}^* \Phi + \text{h.c.} \right)$$

$$+ 2e^{-\alpha + 2\sigma} \frac{\tilde{p}_A f'(\phi)}{f(\phi)} \left( \dot{\alpha}_1^* \delta \phi + \text{h.c.} \right) + e^{-\alpha + 2\sigma} \tilde{p}_A \left( \dot{\alpha}_1^* \Psi + \text{h.c.} \right) - e^{-\alpha + 2\sigma} \tilde{p}_A \left( \dot{\alpha}_1^* \Phi + \text{h.c.} \right)$$

$$+ e^{-\alpha + 2\sigma} p_L f^2(\phi) (i \, \dot{\alpha}_1^* a_0 + \text{h.c.}) - \left[ p^2 + V''(\phi) - \tilde{p}_A^2 \left( \frac{f'(\phi)^2}{f(\phi)^2} + \frac{f''(\phi)}{f(\phi)^2} \right) \right] |\delta \phi|^2$$

$$- (p^2 + V''(\phi_1)) |\dot{\delta} \phi_1|^2 + \left[ \frac{\tilde{p}_A^2 f'(\phi)}{f(\phi)} + V'(\phi) \right] (\delta \phi^* \Psi + \text{h.c.})$$

$$- \left[ \frac{\tilde{p}_A^2 f'(\phi)}{f(\phi)} + V'(\phi) \right] (\delta \phi^* \Phi + \text{h.c.}) - V'(\phi_1) (\delta \dot{\phi}_1^* \Psi + \text{h.c.}) - V'(\phi_1) (\delta \dot{\phi}_1^* \Phi + \text{h.c.})$$

$$- e^{-2\alpha + 2\sigma} \tilde{p}_L^2 \dot{\phi} (\delta \dot{\phi}^* \chi + \text{h.c.}) - e^{\alpha + \sigma} \tilde{p}_L^2 \dot{\phi} (\delta \dot{\phi}^* B + \text{h.c.}) - e^{-2\alpha - 2\sigma} \tilde{p}_L^2 \dot{\phi}_1 (\delta \dot{\phi}_1^* \chi + \text{h.c.})$$

$$- e^{\alpha + \sigma} \tilde{p}_L^2 \dot{\phi}_1 (\delta \dot{\phi}_1^* B + \text{h.c.}) + \frac{2 \tilde{p}_A p_L f'(\phi)}{f(\phi)} \left( i \delta \phi^* a_0 + \text{h.c.} \right) + \frac{\tilde{p}_A^2}{f(\phi)^2} |\Psi|^2$$

$$- M_p^2 \left( \frac{\tilde{p}_A^2}{M_p^2 f'(\phi)} \right) (\Psi^* \Phi + \text{h.c.}) - 3e^{\alpha + \sigma} M_p^2 p_L^2 \dot{\sigma} (\Psi^* B + \text{h.c.}) + \tilde{p}_A p_L (\dot{\Phi}^* a_0 + \text{h.c.})$$

$$- e^{-2\alpha + 4\sigma} \tilde{p}_L^2 f^2(\phi) |\alpha_1|^2 - e^{3\alpha} \tilde{p}_L^2 p_L^2 (\alpha_1^* B + \text{h.c.})$$

$$+ M_p^2 \left[ \frac{\tilde{p}_A^2}{M_p^2 f'(\phi)^2} + \frac{\dot{\phi}_1^2}{M_p^2} + \frac{\dot{\phi}^2}{M_p^2} + 6(\dot{\phi}^2 - \dot{\phi}_1^2) \right] |\Phi|^2 + 2e^{-2\alpha} M_p^2 p_L^2 (\dot{\alpha} + \dot{\sigma}) (\Phi^* \chi + \text{h.c.})$$

$$+ e^{\alpha + 2\sigma} M_p^2 p_L^2 p_T^2 (\chi^* B + \text{h.c.}) - \tilde{p}_A p_L (\dot{\Phi}^* a_0 + \text{h.c.}) + \frac{1}{2} e^{2\alpha - 4\sigma} M_p^2 p_L^2 p_T^2 |\chi|^2$$

$$- \frac{1}{2} e^{2\alpha - 2\sigma} M_p^2 p_L^2 p_T^2 \left( \chi^* B + \text{h.c.} \right) + \frac{1}{2} e^{2\alpha + 2\sigma} M_p^2 p_L^2 p_T^2 |B|^2 + p^2 f^2(\phi) |a_0|^2$$

$$\text{(76)}$$

We can see from the above action that the modes \{\Phi, \chi, B, a_0\} are nondynamical, and can be integrated out. More precisely, we extremize the action (76) with respect to the nondynamical modes to obtain constraint equations, which we solve for the values of the nondynamical modes. We insert the solutions back into the action (76) to obtain an action purely in terms of the dynamical modes \{\delta \dot{\phi}, \delta \dot{\phi}_1, \Psi, \alpha_1\} up to a total time derivative. This action reads

$$S_{2dS} = \frac{1}{2} \int dt \, d^3k \left\{ |\dot{\mathcal{V}}|^2 + |\mathcal{V}_1|^2 + |\dot{\mathcal{H}}_+|^2 + |\dot{\Delta}_+|^2 + \mathcal{F} \left( i \, \dot{\mathcal{V}}^* \Delta_+ + i \, \mathcal{V} \dot{\Delta}_+ \text{h.c.} \right) + \mathcal{G} \left( i \, \dot{\mathcal{H}}_+ \Delta_+ + i \, \mathcal{H}_+ \dot{\Delta}_+ \text{h.c.} \right) \right\}$$

$$\text{(77)}$$

where \mathcal{F} and \mathcal{G} are given by

$$\mathcal{F} = \frac{\tilde{p}_A f'(\phi) \, p_T}{f(\phi)^2} \, , \, \mathcal{G} = \frac{\tilde{p}_A \sqrt{2} M_p f(\phi) \, p_T}{p}$$

$$\text{(78)}$$

with \( f(\phi_1) = \text{Exp} \left[ \frac{4}{M_p^2} \int \frac{V_1(\phi_1)}{V_1(\phi)} \, d\phi_1 \right] \). Notice that the action (77) has regular kinetic terms, therefore the 2d scalar sector is free from ghost instabilities as in the case of 2d vector modes. The elements of the hermitian matrix \( \Omega^2 \) are
\[ \Omega_{11}^2 = p^2 - \frac{9}{4} \alpha^2 + 15 \phi_1^2 + 3 \phi_2^2 + \frac{9}{4} \dot{\phi}^2 + 2 \phi_1 V'(\phi) + V''(\phi) - 2 \frac{p^4}{D^2} \phi_1^4 - 2 \frac{p^4}{D^2} \phi_2^2 \]
\[ + \frac{2}{D^2} \frac{p_2^2 - 2p_1^2}{M_p^2} \frac{\phi_1}{\alpha} - 3 \left( 4 \frac{p_1^2 - p_2^2}{D^2} \alpha + \frac{8p_1^4 - 8p_1^2 p_2^2 + 5p_2^4}{D^2} \sigma \right) \frac{\phi_2^2}{M_p^2} \]
\[ + \frac{p_2^2}{2M_p^2 f(\phi)^2} \left[ 1 + \frac{2M_p^2 (3p_1^2 - p_2^2) f'(\phi)^2}{D^2} - 4 \frac{p_1^2 p_2^2}{D^2} \phi_2^2 - 8 \frac{p_2^4}{D^2} \frac{f'(\phi)}{f(\phi)} - 2M_p^2 f''(\phi) \right] \]
\[ \Omega_{12}^2 = 2 \dot{\sigma} \frac{(p_1^2 - p_2^2)}{D^2} \phi_2^2 \left[ \frac{\phi_1 V'(\phi)}{M_p^2} + \frac{\phi_1 V'(\phi)}{M_p^2} - 6 \frac{p_1^4 - p_1^2 p_2^2 + p_2^4}{(2p_1^2 - p_2^2) p^2} \sigma \right] \]
\[ - 2 \frac{p_1^4}{D} \left[ \frac{\phi_3}{M_p^2} - 2 \frac{\phi_1 V'(\phi)}{M_p^2} \sigma + \frac{\phi}{M_p} \left( \frac{\phi_3}{M_p^2} - 2 \frac{V_1(\phi)}{M_p} \sigma - 6 \frac{\phi_1}{M_p} \sigma^2 \right) \right] \]
\[ - \frac{p_2^2}{D^2} \frac{\phi_1}{M_p} \left[ \frac{f'(\phi)}{M_p} \frac{\phi_1}{M_p} \sigma + \frac{f'(\phi)}{M_p} \frac{\phi_1}{M_p} \sigma \left( 4p_2^2 \sigma + 2 \left( 7p_2^2 - 2p_1^2 \right) \alpha \sigma \right) \right] \]
\[ + \left( 2p_2^2 - p_2^2 \right) \frac{2 \alpha + 2p_2^2 - p_2^2}{p^2} \sigma \]
\[ \Omega_{24}^2 = \frac{-2 i \dot{p}_A}{M_p f(\phi)} \frac{p_T p}{\mathcal{D} \mathcal{M}_p} \left[ \frac{p^2}{M_p^2} \frac{\phi_A}{f(\phi)} - \frac{\phi_A'}{f(\phi)} \right] \]

\[ \Omega_3^2 = p^2 - \frac{1}{4} \dot{\alpha}^2 - \frac{3}{6} \left( \ddot{\phi} + \ddot{\phi}^2 \right) - \frac{8}{4 M_p^2} \frac{p_T^2}{\mathcal{D}^2} \dot{\alpha}^4 - \frac{8}{4 \phi_A} \phi_A' \ddot{\phi} + \frac{p_T^2}{M_p^2} \frac{\phi_A'}{f(\phi)} - \frac{8}{2} \left( \frac{p_T^2 - \phi_A^2}{\mathcal{D}^2} \right) \frac{p^2}{\alpha^3} \dot{\alpha}^3 \]

\[ \Omega_3^4 = \frac{\phi_A p_T}{\sqrt{2} M_p f(\phi)} \left[ -\frac{6 \phi_A^2}{M_p^2 f(\phi)^2} \frac{M_p^2}{p_T^2} \dot{\phi}^2 + \frac{f'(\phi)}{f(\phi)} - \frac{4 p^4}{\mathcal{D}^2} \alpha^3 \right] \]

\[ \Omega_{44}^2 = p^2 + \frac{p^2}{2M_p^2} \frac{\phi_A}{f(\phi)} \left[ V' + 2 \phi \frac{p_T^2 - \phi_A^2}{p_T^2} \right] \left( \alpha - \frac{\alpha^4}{p_T^2} \right) \frac{p^4}{p_T^2} \frac{\phi_A}{f(\phi)} \]

\[ \mathcal{D} = 2 \alpha^3 + \frac{p_T^2 - \phi_A^2}{\alpha^3} \]

The canonically normalized modes \( \{ V, V_1, H_+, \Delta_+ \} \) are related to the starting dynamical modes as

\[ V = e^{3\alpha/2} \left[ \delta \phi + \frac{p_T^2 \phi}{\mathcal{D}} \Psi \right] \]

\[ V_1 = e^{3\alpha/2} \left[ \delta \phi_1 + \frac{p_T^2 \phi_1}{\mathcal{D}} \Psi \right] \]

\[ H_+ = e^{3\alpha/2} \frac{\sqrt{2} M_p p_T^2}{\mathcal{D}} (\dot{\alpha} + \dot{\sigma}) \Psi \]

\[ \Delta_+ = e^{3\alpha_2} \frac{p_T}{\mathcal{D}} \left[ \frac{\phi_A}{f(\phi)} \frac{p_T^2}{\mathcal{D}} \Psi + e^{-\alpha + 2\sigma} f(\phi) \alpha_1 \right] \]

When the universe becomes isotropic at the end of the first stage, we have \( \dot{\sigma} = 0 \) (we also rescaled the initial scale factors such that \( \sigma = 0 \) when the universe isotropizes). Using the results of the equations (71/75/33), the canonical modes in the isotropic limit is related to the perturbations defined in the longitudinal gauge as

\[ h_+ = -\frac{\sqrt{2}}{M_p} e^{-3\alpha/2} H_+ \]

\[ V = v = e^{3\alpha/2} \left[ \left( \phi \left( 1 - \frac{\phi_A}{f(\phi)} \right) \right) \Psi \right], \quad V_1 = v_1 = e^{3\alpha/2} \left( \frac{\phi_A}{\dot{\phi}_1} + \frac{\dot{\phi}_1}{\alpha_1} \Psi \right) \]
where $v, v_1$ are the standard Mukhanov-Sasaki variables \[33\] (defined with respect to the action in the cosmic time). All the above quantities are gauge invariant. These relations will be useful when computing the curvature ($\mathcal{R}$) and $h_+$ gravitational wave polarization spectrum.

Note that the final action (77) is of the generic form \[33\], with the corresponding matrix $X$ given by

$$
X = \begin{pmatrix}
0 & 0 & \mathcal{F} \\
0 & 0 & \mathcal{G} \\
\mathcal{F} & \mathcal{G} & 0
\end{pmatrix}
$$

(83)

In this case, the solution of the equation $\dot{U} + iXU = 0$ is more challenging, since the matrix that diagonalizes $X$ is given by

$$
R = \frac{1}{\sqrt{\mathcal{F}^2 + \mathcal{G}^2}} \begin{pmatrix}
\mathcal{G} & \mathcal{F}/\sqrt{2} & -\mathcal{F}/\sqrt{2} \\
-\mathcal{F}/\sqrt{2} & \mathcal{G}/\sqrt{2} & -\mathcal{G}/\sqrt{2} \\
0 & \sqrt{\mathcal{F}^2 + \mathcal{G}^2}/\sqrt{2} & \sqrt{\mathcal{F}^2 + \mathcal{G}^2}/\sqrt{2}
\end{pmatrix},
$$

$$
D_X = \begin{pmatrix}
0 & 0 & 0 \\
0 & \sqrt{\mathcal{F}^2 + \mathcal{G}^2} & 0 \\
0 & 0 & -\sqrt{\mathcal{F}^2 + \mathcal{G}^2}
\end{pmatrix}
$$

(84)

which is not a constant matrix, unlike the case for the 2d vector modes. However, the time derivatives of $R$ are suppressed by slow-roll parameters, therefore an approximate solution for $U$ can be found. We will discuss the details and the computation of the power spectra of the curvature perturbation $\mathcal{R}$ and $h_+$ in a separate publication.

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