Analysis of Discrete and Hybrid Stochastic Systems by Nonlinear Contraction Theory

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Abstract—We investigate the stability properties of discrete and hybrid stochastic nonlinear dynamical systems. More precisely, we extend the stochastic contraction theorems (which were formulated for continuous systems) to the case of discrete and hybrid resetting systems. In particular, we show that the mean square distance between any two trajectories of a discrete (or hybrid resetting) contracting stochastic system is upper-bounded by a constant after exponential transients. Using these results, we study the synchronization of noisy nonlinear oscillators coupled by discrete noisy interactions.

Index Terms—Discrete systems, hybrid resetting, stochastic systems, nonlinear contraction theory, incremental stability, oscillator synchronization

I. INTRODUCTION

Contraction theory is a set of relatively recent tools that provide a systematic approach to the stability analysis of a large class of nonlinear dynamical systems [1], [2], [3], [4]. A nonlinear nonautonomous system \( \dot{x} = f(x, t) \) is contracting if the symmetric part of the Jacobian matrix of \( f \) is uniformly negative definite in some metric. Using elementary fluid dynamics techniques, it can be shown that contracting systems exponentially converge to each other [1].

From a practical viewpoint, contraction theory has been successfully applied to a number of important problems, such as mechanical observers and controllers design [5], chemical processes control [6], synchronization analysis [2], [7] or biological systems modelling [8].

Recently, contraction analysis has been extended to the case of stochastic dynamical systems governed by Itô differential equations [4]. In parallel, hybrid versions of contraction theory have also been developed [3]. A hybrid system is characterized by a continuous evolution of the system’s state, and intermittent discrete transitions. Such systems are pervasive in both artificial (e.g. analog physical processes controlled by digital devices) and natural (e.g. spiking neurons with subthreshold dynamics) environments.

The present paper benefits from these recent developments to provide an exponential stability result for discrete and hybrid systems governed by stochastic difference and differential equations. More precisely, we prove in section II and III that the mean square distance between any two trajectories of a discrete (respectively hybrid resetting) stochastic contracting system is upper-bounded by a constant after exponential transitions. This bound can be expressed as function of the noise intensities and the contraction rates of the noise-free systems. In section IV, we briefly discuss a number of theoretical issues regarding our analysis. Finally, in section V, we study, using the previously developed tools, the synchronization of noisy nonlinear oscillators which interact by discrete noisy couplings.

Notations

The symmetric part of a matrix \( A \) is defined as \( A_s = \frac{1}{2}(A + A^T) \). For a symmetric matrix \( A \), \( \lambda_{\min}(A) \) and \( \lambda_{\max}(A) \) denote respectively the smallest and the largest eigenvalue of \( A \). A set of symmetric matrices \( (A_i)_{i \in I} \) is uniformly positive definite if \( \exists \alpha > 0, \forall i \in I, \lambda_{\min}(A_i) \geq \alpha. \) Finally, for a stochastic process \( s(t) \), we note \( E_s(\cdot) \) the conditional expectation \( E(\cdot | s(0) = x) \).

II. DISCRETE SYSTEMS

We first prove a lemma that makes explicit the initial “discrete contraction” proof (see section 5 of [1]). Note that a similar proof for continuous systems can be found in [9].

Lemma 1 (and definition): Consider two metrics \( M_1 = \Theta_1^T \Theta_1 \) defined over \( \mathbb{R}^{n_1} \) \( (i = 1, 2) \) and a smooth function \( f: \mathbb{R}^{n_1} \rightarrow \mathbb{R}^{n_2} \). The generalized Jacobian of \( f \) in the metrics \( (M_1, M_2) \) is defined by

\[
F = \Theta_2 \frac{\partial f}{\partial x} \Theta_1^{-1}
\]

Assume now that \( f \) is contracting in the metrics \( (M_1, M_2) \) with rate \( \beta (0 < \beta < 1) \), i.e.

\[
\forall x \in \mathbb{R}^{n_1}, \lambda_{\max}(F(x)^T F(x)) \leq \beta
\]

Then for all \( u, v \in \mathbb{R}^n \), one has

\[
d_{M_2}(f(u), f(v))^2 \leq \beta d_{M_1}(u, v)^2
\]

where \( d_M \) denotes the distance associated with the metric \( M \) (the distance between two points is defined by the infimum of the lengths in the metric \( M \) of all continuously differentiable curves connecting these points).

Proof

Consider a \( C^1 \) curve \( \gamma : [0, 1] \rightarrow \mathbb{R}^{n_1} \) that connects \( u \) and \( v \) (i.e. \( \gamma(0) = u \) and \( \gamma(1) = v \)). The \( M_1 \)-length of such a curve is given by

\[
L_{M_1} (\gamma) = \int_{0}^{1} \sqrt{\left( \frac{\partial \gamma}{\partial u} (u) \right)^T M_1 \left( \frac{\partial \gamma}{\partial u} (u) \right)} du
\]
Since \( f \) is a smooth function, \( f(\gamma) \) is also a \( C^1 \) curve, with
\[
L_{M_2}(f(\gamma)) = \int_0^1 \sqrt{\left( \frac{\partial f(\gamma)}{\partial u}(u) \right)^T \frac{\partial f(\gamma)}{\partial u}(u)} \, du
\]
The chain rule next implies that
\[
\frac{\partial f(\gamma)}{\partial u}(u) = \frac{\partial f}{\partial \gamma} \frac{\partial \gamma}{\partial u}(u)
\]
which leads to
\[
L_{M_2}(f(\gamma)) = \int_0^1 \sqrt{\left( \frac{\partial f}{\partial \gamma} \frac{\partial \gamma}{\partial u}(u) \right)^T \frac{\partial f}{\partial \gamma} \frac{\partial \gamma}{\partial u}(u)} \, du
\]
Choose now a sequence of curves \((\gamma_n)_{n \in \mathbb{N}}\) such that
\[
\lim_{n \to \infty} L_{M_1}(\gamma_n) = d_{M_1}(u, v).
\]
From (1), one has \( \forall n \in \mathbb{N}, L_{M_2}(f(\gamma_n)) \leq \sqrt{\beta} L_{M_1}(\gamma_n) \). By definition of distance, one then has \( \forall n \in \mathbb{N}, d_{M_2}(f(u), f(v)) \leq \sqrt{\beta} L_{M_1}(\gamma_n) \). Finally, by letting \( n \) go to infinity in the last inequality, one obtains the desired result. □

**Theorem 1 (Discrete stochastic contraction):** Consider the stochastic difference equation
\[
\begin{align*}
\begin{cases}
\mathbf{a}_{k+1} = f(\mathbf{a}_k, k) + \sigma(\mathbf{a}_k, k)w_{k+1} \\
\mathbf{a}_0 = \xi
\end{cases}
\end{align*}
\]
where \( f \) is a \( \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}^n \) function, \( \sigma \) is a \( \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}^n \) matrix-valued function, \( \{w_k, k = 1, 2, \ldots\} \) is a sequence of independent \( d \)-dimensional Gaussian noise vectors, with \( w_k \sim \mathcal{N}(0, Q_k) \) and \( \xi \) is a \( n \)-dimensional random variable independent of the \( w_k \).

Assume that the system verifies the following two hypotheses

**\( (H1) \)** the dynamics \( f(\mathbf{a}, k) \) is contracting in the metrics \( (M_k, M_{k+1}) \), with contraction rate \( \beta \) \((0 < \beta < 1)\), and the metrics \( (M_k)_{k \in \mathbb{N}} \) are uniformly positive definite.

**\( (H2) \)** the impact of noise is uniformly upper-bounded by a constant \( \sqrt{C} \) in the metrics \( M_k \)
\[
\forall \mathbf{a}, k \quad d_{M_k}(f(\mathbf{a}, k), f(\mathbf{a}, k) + \sigma(\mathbf{a}, k)w_k) \leq \sqrt{C}
\]
Let \( \mathbf{a}_k \) and \( \mathbf{b}_k \) be two trajectories whose initial conditions are given by a probability distribution \( p(\xi, \xi') \). Then for all \( k \geq 0 \)
\[
\mathbb{E}(d_{M_k}(\mathbf{a}_k, \mathbf{b}_k)) \leq \frac{2\sqrt{C}}{1 - \sqrt{\beta}} + \sqrt{\beta} \int \left[ d_{M_0}(\mathbf{a}, \mathbf{b}) - \frac{2\sqrt{C}}{1 - \sqrt{\beta}} \right]^+ \, dp(\mathbf{a}, \mathbf{b})
\]
(3)
\[
\text{where } [.]^+ = \max(0, \cdot).
\]
This implies in particular that for all \( k \geq 0 \)
\[
\mathbb{E}(d_{M_k}(\mathbf{a}_k, \mathbf{b}_k)) \leq \frac{2\sqrt{C}}{1 - \sqrt{\beta}} + \sqrt{\beta} \mathbb{E}(d_{M_0}(\xi, \xi'))
\]
(4)

**Proof** Let \( x = (\mathbf{a}, \mathbf{b})^T \in \mathbb{R}^{2n} \). We have by the triangle inequality (to avoid long formulas, we drop the second argument of \( f \) and \( \sigma \) in the following calculations)
\[
\begin{align*}
&d_{M_{k+1}}(\mathbf{a}_{k+1}, \mathbf{b}_{k+1}) \\
&\leq d_{M_{k+1}}(f(\mathbf{a}_k), f(\mathbf{b}_k)) + d_{M_{k+1}}(f(\mathbf{a}_k), f(\mathbf{a}_k) + \sigma(\mathbf{a}_k)w_{k+1}) + d_{M_{k+1}}(f(\mathbf{b}_k), f(\mathbf{b}_k) + \sigma(\mathbf{b}_k)w_{k+1})
\end{align*}
\]
Let us examine the conditional expectations of the three terms of the right hand side

- From \( (H1) \) and lemma 1 one has
\[
\mathbb{E}_x(d_{M_{k+1}}(f(\mathbf{a}_k), f(\mathbf{b}_k))) \leq \sqrt{\beta} \mathbb{E}_x(d_{M_k}(\mathbf{a}_k, \mathbf{b}_k))
\]
- Next, from \( (H2) \)
\[
\mathbb{E}_x(d_{M_{k+1}}(f(\mathbf{a}_k), f(\mathbf{a}_k) + \sigma(\mathbf{a}_k)w_{k+1})) \leq \sqrt{C}
\]
and similarly for \( d_{M_{k+1}}(f(\mathbf{b}_k), f(\mathbf{b}_k) + \sigma(\mathbf{b}_k)w_{k+1}) \).

If we now set \( u_k = \mathbb{E}_x(d_{M_k}(\mathbf{a}_k, \mathbf{b}_k)) \) then the above implies
\[
u_{k+1} \leq \sqrt{\beta} u_k + 2\sqrt{C}
\]
(5)
Define next \( v_k = u_k - 2\sqrt{C}/(1 - \sqrt{\beta}) \). Then replacing \( u_k \) by \( v_k + 2\sqrt{C}/(1 - \sqrt{\beta}) \) in (5) yields
\[
u_{k+1} \leq \sqrt{\beta} v_k
\]
This implies that \( \forall k \geq 0, \quad v_k \leq v_0 \sqrt{\beta} \leq [v_0]^+ \sqrt{\beta}^k \).
Replacing \( v_k \) by its expression in terms of \( u_k \) then yields
\[
\forall k \geq 0, \quad u_k \leq \frac{2\sqrt{C}}{1 - \sqrt{\beta}} + \sqrt{\beta} \left[ u_0 - \frac{2\sqrt{C}}{1 - \sqrt{\beta}} \right]^+
\]
which is the desired result.

Next, integrating the last inequality with respect to \( x \) leads to (3). Finally, (4) follows from (3) by remarking that
\[
\begin{align*}
&\mathbb{E}(d_{M_0}(\mathbf{a}, \mathbf{b}) - \frac{2\sqrt{C}}{1 - \sqrt{\beta}})^+ \, dp(\mathbf{a}, \mathbf{b}) \leq \\
&\int d_{M_0}(\mathbf{a}, \mathbf{b}) dp(\mathbf{a}, \mathbf{b}) = \mathbb{E}(d_{M_0}(\xi, \xi'))
\end{align*}
\]
(4)
Remark In the particular context of state-independent metrics, hypothesis \( (H2) \) is equivalent to the following simpler condition
\[
\forall \mathbf{a}, k \quad \text{tr} (\sigma(\mathbf{a}, k)^T M_{k+1} \sigma(\mathbf{a}, k) Q_k) \leq C
\]
Also, for state-independent metrics, one has
\[
d_{M_k}(\mathbf{a}_k, \mathbf{b}_k)^2 = (\mathbf{a}_k - \mathbf{b}_k)^T M_k (\mathbf{a}_k - \mathbf{b}_k) = \|\mathbf{a}_k - \mathbf{b}_k\|_{M_k}^2
\]
which leads to the following stronger result instead of (4)
\[
\mathbb{E} \left( \|\mathbf{a}_k - \mathbf{b}_k\|_{M_k}^2 \right) \leq \frac{2C}{1 - \beta} + \beta^k \mathbb{E} \left( \|\xi - \xi'\|_{M_0}^2 \right)
\]
III. HYBRID SYSTEMS

We have derived above the discrete stochastic contraction theorem for \textit{time- and state-dependent} metrics, contrary to the context of continuous systems, where the state-dependent-metrics version of the contraction theorem is still unproven [4]. We now address the case of hybrid systems, but due to the present limitations of continuous stochastic contraction, only state-independent metrics will be considered.

For clarity, we assume in this paper constant dwell-times, although more elaborate conditions regarding dwell-times can be adapted from [3].

Consider the hybrid resetting stochastic dynamical system

\[
\begin{aligned}
\forall k \geq 1 & \quad a(k\tau^+ + 1) = f_d(a(k\tau^-), k) + \sigma_d(a(k\tau^-), k)w_k \\
\forall k \geq 0 & \quad \forall t \in (k\tau, (k + 1)\tau) \quad \frac{da}{dt} = f_c(a, t)dt + \sigma_c(a, t)dW \\
\end{aligned}
\]

where \( f_d, \sigma_d, f_c, \sigma_c \) are four functions of appropriate dimensions and \( \xi \) is a random variable independent of the \( w_k \) and of the process \( W \). Furthermore, \( f_d \) and \( \sigma_d \) verify suitable conditions for the existence and uniqueness of the solutions of the continuous parts (cf e.g. [4]).

All the contraction properties below will be stated with respect to a uniformly positive definite time-varying metric \( M(t) = \Theta(t)T\Theta(t) \). Furthermore, it will be assumed that for all \( k \geq 0 \), \( M \) is continuously differentiable in \( \vert k\tau, (k + 1)\tau \vert \). Finally, \( M(k\tau^-) \) and \( M(k\tau^+) \) will respectively denote the left and right limits of \( M(t) \) at \( t = k\tau \) (and similarly for \( \Theta \)).

A. The discrete and continuous parts are both contracting

\textbf{Theorem 2 (Hybrid stochastic contraction, case } \lambda > 0)\textbf{):}

Assume the following conditions

(i) For all \( k \), the discrete part is stochastically contracting at \( k\tau \) with rate \( \beta < 1 \) and bound \( C_d \), i.e.

\[
\forall a \in \mathbb{R}^n \quad \lambda_{\max}(F(k\tau)^T F(k\tau)) \leq \beta
\]

where \( F(k\tau) = \Theta(k\tau)^T \frac{\partial}{\partial a}(a, k)\Theta(k\tau^-) \), and

\[
\forall a \in \mathbb{R}^n \quad \text{tr}(\sigma_d(a, k)^T M(k\tau^+)\sigma_d(a, k)Q_k) \leq C_d
\]

(ii) For all \( k \), the continuous part is stochastically contracting in \( \vert k\tau, (k + 1)\tau \vert \) with rate \( \lambda > 0 \) and bound \( C_c \), i.e.

\[
\forall a \in \mathbb{R}^n, \forall t \in (k\tau, (k + 1)\tau],
\lambda_{\max} \left( \frac{d}{dt} \Theta(t) + \Theta(t) \frac{\partial f}{\partial a} \right) \Theta^{-1}(t) \leq -\lambda
\]

\[
\text{tr}(\sigma_c(a, t)^T M(t)\sigma_c(a, t)) \leq C_c
\]

Let \( a(t) \) and \( b(t) \) be two trajectories whose initial conditions are given by a probability distribution \( p(\xi, \xi') \). Then for all \( t \geq 0 \)

\[
\mathbb{E} \left( \Vert a(t) - b(t) \Vert_{M(t)}^2 \right) \leq C_1 + \mathbb{E} \left( \Vert \xi - \xi' \Vert_{M(0)}^2 \right) \beta^{t/\tau} e^{-2\lambda t}
\]

where \( C_1 = \frac{2C_c+2(1-\beta)(1+\beta-\gamma)C_d}{\lambda(1-\beta)(1-\gamma)} \) and \( r_1 = \beta e^{-2\lambda \tau} \).

\textbf{Proof} For all \( t \geq 0 \), let \( u(t) = \mathbb{E} \left( \Vert a(t) - b(t) \Vert_{M(t)}^2 \right) \) and let us study the evolution of \( u(t) \) between \( k\tau \) and \( (k + 1)\tau \). Condition (ii) and theorem 2 of [4] yield

\[
u((k + 1)\tau^-) \leq C_2 + u(k\tau^+) e^{-2\lambda \tau}
\]

Next, condition (i) and theorem 1 above yield

\[
u((k + 1)\tau^+) \leq \frac{2C_d}{1-\beta} + \beta u((k + 1)\tau^-)
\]

Substituting (8) into (9) leads to

\[
u((k + 1)\tau^+) \leq \frac{2C_d}{1-\beta} + \beta \left( C_2 + u(k\tau^+) e^{-2\lambda \tau} \right)
\]

\[
= \frac{2C_d}{1-\beta} + \beta C_2 + \beta u(k\tau^-) e^{-2\lambda \tau}
\]

Define \( D_1 = \frac{2C_d}{1-\beta} + \beta C_2 \) and \( u_k = u(k\tau^-) \). Then, similarly to the proof of theorem 1, we have \( v_{k+1} \leq r_1 v_k \), and then \( v_k \leq r_1^k [v_0]^k \), which implies

\[
u(k\tau^+) \leq \frac{D_1}{1-r_1} + \left[ \mathbb{E}(u(0^+) - \frac{D_1}{1-r_1}) \right]^+ r_1^k
\]

Now, for any \( t \geq 0 \), choose \( k = \lfloor t/\tau \rfloor \). Then

\[
u(t) \leq \frac{C_2}{\lambda} + u(k\tau^-) e^{-2\lambda (t-k\tau)}
\]

\[
\leq \frac{C_2}{\lambda} + D_1 e^{-2\lambda (t-k\tau)} + \mathbb{E}(u(0^+) \beta^k e^{-2\lambda t})
\]

\[
\leq \frac{C_2}{\lambda} + \frac{D_1}{1-r_1} + \mathbb{E}(u(0^+) \beta^k e^{-2\lambda t})
\]

which leads to the desired result after some algebraic manipulations. \( \square \)

B. Only the discrete part is contracting

Let us examine now the more interesting case when the continuous part is not contracting, more precisely when \( \lambda \leq 0 \) in (7). For this, we shall need to revisit the proof of theorem 2 in [4].

\textbf{Theorem 3 (Case } \lambda = 0)\textbf{): Assume all the hypotheses of theorem 2 except that } \lambda = 0 \textbf{ in (7). Then for all } t \geq 0

\[
\mathbb{E} \left( \Vert a(t) - b(t) \Vert_{M(t)}^2 \right) \leq \frac{C_2}{\lambda} + \mathbb{E} \left( \Vert \xi - \xi' \Vert_{M(0)}^2 \right) \beta^{t/\tau}
\]

where \( C_2 = \frac{2C_d+2(1-\beta)(1+\beta-\gamma)C_c}{\alpha(1-\beta)(1-\gamma)} \).

\textbf{Proof} As in the proof of theorem 2 in [4], let \( V(x, t) = V((a, b)^T, t) = (a - b)^T M(t)(a - b) \)

Lemma 1 of [4] is unchanged, yielding (see [4] for more details)

\[
\forall t \in (k\tau, (k + 1)\tau] \quad \tilde{A} V(x(t), t) \leq 2C_c
\]

where \( \tilde{A} \) is the infinitesimal operator associated with the process \( x(t) \) (see section 2.1.2 of [4] or p. 15 of [10] for more details).
By Dynkin’s formula [10], one then obtains for all \( x \in \mathbb{R}^{2n} \)
\[
\mathbb{E}_x V(x(t), t) - V(x, k\tau^+) = \mathbb{E}_x \int_{k\tau}^{t} \tilde{A}V(x(s), s)ds \\
\leq \mathbb{E}_x \int_{k\tau}^{t} 2C_c ds \\
= 2C_c(t - k\tau)
\]
Integrating the above inequality with respect to \( x \) then yields
\[
\forall t \in ]k\tau, (k + 1)\tau[ \quad u(t) \leq 2C_c(t - k\tau) + u(k\tau^+)
\]
In particular, (8) becomes
\[
u((k + 1)\tau^-) \leq 2C_c\tau + u(k\tau^+)
\]
which leads to, after substitution into (9),
\[
u((k + 1)\tau^+) \leq \frac{2C_c}{1 - \beta} + 2\beta C_c\tau + \beta u(k\tau^+)
\]
This finally implies
\[
u(k\tau^+) \leq \frac{2C_c}{1 - \beta} + 2\beta C_c\tau + u(0^+)\beta^k
\]
The remainder of the proof can be adapted from that of theorem 2. □

**Theorem 4 (Case \( \lambda < 0 \)):** Assume all the hypotheses of theorem 2 except that \( \lambda < 0 \) in (7). Let \( k = \lfloor t/\tau \rfloor \). There are two cases:
- If \( \beta < e^{-2|\lambda|\tau} \), then \( r_2 = \beta e^{-2|\lambda|\tau} < 1 \). For all \( t \geq 0 \)
\[
\mathbb{E} \left( \|a(t) - b(t)\|_{M(t)}^2 \right) \leq C_3 + \mathbb{E} \left( \|\xi - \xi'\|_{M(t)}^2 \right) e^{2|\lambda|\tau} r_2^k
\]
where \( C_3 = \frac{2|\lambda|C_x(1 - \beta)(1 + \beta - r_2)e^{2|\lambda|\tau} C_c}{|\lambda|(1 + |\lambda|)(1 - r_2)} \).
- If \( \beta \geq e^{-2|\lambda|\tau} \), then there is – in general – no finite bound on \( \mathbb{E} \left( \|a(t) - b(t)\|_{M(t)}^2 \right) \) as \( t \to +\infty \).

**Proof** One has now for all \( t \in ]k\tau, (k + 1)\tau[ \),
\[
\tilde{A}V(x(t), t) \leq 2|\lambda|V(x(t), t) + 2C_c
\]
with \( |\lambda| > 0 \). By Dynkin’s formula, one has, for all \( x \in \mathbb{R}^{2n} \)
\[
\mathbb{E}_x V(x(t), t) - V(x, k\tau^+) \leq \mathbb{E}_x \int_{k\tau}^{t} (2|\lambda|V(x(s), s) + 2C_c)ds
\]
Let now \( g(t) = \mathbb{E}_x V(x(t), t) \). The above equation then yields
\[
g(t) = V(x, k\tau^+) + 2C_c(t - k\tau) + 2|\lambda| \int_{k\tau}^{t} g(s)ds
\]
Applying the classical Gronwall’s lemma [11] to \( g(t) \) leads to
\[
g(t) \leq V(x, k\tau^+) + 2C_c(t - k\tau) + 2|\lambda| \left( \int_{k\tau}^{t} V(x, k\tau^+) + 2C_c \exp \left( \int_{k\tau}^{t} 2|\lambda|du \right)ds \right)
\]
\[
= \frac{C_c}{|\lambda|} (e^{2|\lambda|(t - k\tau)} - 1) + V(x, k\tau^+)e^{2|\lambda|(t - k\tau)}
\]
Integrating the above inequality with respect to \( x \) then yields
\[
\forall t \in ]k\tau, (k + 1)\tau[ \),
\[
u((k + 1)\tau^-) \leq \frac{C_c}{|\lambda|} (e^{2|\lambda|(t - k\tau)} - 1) + u(k\tau^+)e^{2|\lambda|(t - k\tau)}
\]
which implies
\[
u((k + 1)\tau^+) \leq D_2 + \beta e^{2|\lambda|\tau} u(k\tau^+)
\]
(10)
where \( D_2 = \frac{2C_c}{1 - \beta} + \frac{\beta C_c}{|\lambda|} (e^{2|\lambda|\tau} - 1) \).

There are three cases:
- If \( \beta < e^{-2|\lambda|\tau} \), then \( r_2 = \beta e^{-2|\lambda|\tau} < 1 \). By the same reasoning as in theorem 1, one obtains
\[
u(k\tau^+) \leq \frac{D_2}{1 - r_2} + u(0^+)r_2^k
\]
The remainder of the proof can be adapted from that of theorem 2.
- If \( \beta = e^{-2|\lambda|\tau} \), then (10) reads
\[
u((k + 1)\tau^+) \leq D_2 + u(k\tau^+)
\]
which implies \( \forall k \geq 0, (u(k\tau^+) \leq kD_2 + u(0^+) \). From this, it is clear that there is – in general – no finite bound for \( u(k\tau^+) \).
- If \( \beta > e^{-2|\lambda|\tau} \), then \( r_2 = \beta e^{-2|\lambda|\tau} > 1 \). By the same reasoning as in theorem 1, one obtains
\[
u(k\tau^+) \leq \left( u(0^+) + \frac{D_2}{r_2 - 1} \right) r_2^k - \frac{D_2}{r_2 - 1}
\]
Since \( r_2 > 1 \) in this case, it is clear that there is – in general – no finite bound for \( u(k\tau^+) \). □

**Remarks** Theorems 3 and 4 show that it is possible to stabilize an unstable system by discrete resettings. If the continuous system is indifferent \( (\lambda = 0) \), then any sequence of uniformly contracting resettings is stabilizing. However, it should be noted that the asymptotic bound \( C_2 \to \infty \) when \( \beta \to 1 \). In contrast, if the continuous system is strictly unstable \( (\lambda < 0) \), then specific contraction rates (depending on the dwell-time and the ”expansion” rate of the continuous system) of the resettings are required. Finally, note that in both cases, the asymptotic bounds \( C_2 \) and \( C_3 \) are increasing functions of the dwell-time \( \tau \).

**IV. Comments**

**A. Modelling issue: distinct driving noise**

In the same spirit as [4], and contrary to previous works on the stability of stochastic systems (see the references in [4]), the \( a \) and \( b \) systems considered in sections II and III are driven by distinct and independent noise processes. This approach enables us to study the stability of the system with respect to variations in initial conditions and to random perturbations: indeed, two trajectories of any real-life system are typically affected by distinct realizations of the noise. In addition, this approach leads very naturally to nice results regarding the comparison of noisy and noise-free trajectories (see section IV-B), which are particularly useful in applications (see e.g. section V).

However, because of the very fact that the two trajectories are driven by distinct noise processes, we cannot expect the influence of noise to vanish when the two trajectories get very close to each other. As a consequence, the asymptotic bounds \( 2C/(1 - \beta) \) (for discrete systems) and \( C_1, C_2, C_3 \) (for hybrid
systems) are strictly positive. These bounds are nevertheless optimal, in the sense that they can be attained (adapt the Ornstein-Uhlenbeck example in section 2.3.1 of [4]).

B. Noisy and noise-free trajectories

Instead of considering two noisy trajectories $a$ and $b$ as in theorem 1, we assume now that $a$ is noisy, while $b$ is noise-free. More precisely, for all $k \in \mathbb{N}$

$$a_{k+1} = f(a_k, k) + \sigma(a_k, k)w_{k+1}$$

$$b_{k+1} = f(b_k, k)$$

To show the exponential convergence of $a$ and $b$ to each other, one can follow the same reasoning as in the proof of theorem 1, with $C$ being replaced by $C/2$. This leads to the following result:

**Corollary 1:** Assume all the hypothesis of theorem 1 and consider a noise-free trajectory $b_k$ and a noisy trajectory $a_k$ whose initial conditions are given by a probability distribution $p(a_0)$. Then, for all $k \in \mathbb{N}$

$$\mathbb{E} \left( \| a_k - b_k \|_{\mathbb{M}_{k}} \right) \leq \frac{C}{1-\alpha} + \beta^k \int \left[ \| a - b_0 \|_{\mathbb{M}_{k}} - \frac{C}{1-\alpha} \right]^{+} dp(a)$$

**Remarks**

- The above derivation of corollary 1 is only permitted by our choice of considering distinct driving noise processes for systems $a$ and $b$ (see section IV-A).
- Based on theorems 2, 3, and 4, similar corollaries can be obtained for hybrid systems.
- These corollaries provide a robustness result for contracting discrete and hybrid systems, in the sense that any contracting system is automatically protected against noise, as quantified by (11). This robustness could be related to the exponential nature of contraction stability.

V. APPLICATION: OSCILLATOR SYNCHRONIZATION BY DISCRETE COUPLINGS

Using the above developed tools, we study in this section the synchronization of nonlinear oscillators in presence of random perturbations. The novelty here is that the interactions between the oscillators occur at discrete time instants, contrary to many previous works devoted to synchronization in the state-space\(^1\) (see [7] and references therein).

Specifically, consider the Central Pattern Generator (CPG) delivering \(2\pi/3\)-phase-locked signals of section 5.3 in [7]. This CPG consists of a network of three Andronov-Hopf oscillators $x_i = (x_i, y_i)^T$, $i = 1, 2, 3$. We construct below a discrete-couplings version of this CPG.

\(^1\)Discrete couplings are more frequent in the literature devoted to phase oscillators synchronization, where phase reduction techniques are used (see e.g. [12]). However, contrary to our approach, these techniques are only applicable in the case of weak coupling strengths and small noise intensities.

At instants $t = k\tau$, $k \in \mathbb{N}$, the three oscillators are coupled in the following way (assuming noisy measurements)

$$x_i(k\tau^+) = x_i(k\tau^-) + \gamma (R(x_{i+1}(k\tau^-)) + \frac{\sigma_d}{\sqrt{2}}w_k) - x_i(k\tau^-)$$

with $x_4 \equiv x_1$ and

$$R \equiv \begin{pmatrix} -\frac{1}{2} & -\frac{\gamma}{\sqrt{2}} \\ \frac{\gamma}{\sqrt{2}} & -\frac{1}{2} \end{pmatrix}$$

Between two interaction instants, the oscillators follow the uncoupled, noisy, dynamics

$$dx_i = f(x_i)dt + \frac{\sigma_e}{\sqrt{2}}dW$$

where

$$f(x_i) = \begin{pmatrix} x_i - y_i - x_i^3 - x_i y_i^2 \\ x_i + y_i - y_i^3 - y_i x_i^2 \end{pmatrix}$$

We apply now the projection technique developed in [7], [4]. We recommend the reader to refer to these papers for more details about the following calculations.

Consider first the (linear) subspace $M$ of the global state space (the global state is defined by $\vec{x} = (x_1, x_2, x_3)^T$) where the oscillators are $2\pi/3$-phase-locked

$$M = \{ (R^2(x), R(x), x)^T : x \in \mathbb{R}^2 \}$$

Let $V$ and $U$ be two orthonormal projections on $M$ and $M$ respectively and consider $\tilde{y} = VX$. Since the mapping is linear, using Ito differentiation rule yields the following dynamics for $\tilde{y}$

\begin{align*}
\forall k \in \mathbb{N} & \quad \tilde{y}(k\tau^+) = g_d(\tilde{y}(k\tau^-)) + \gamma \frac{\sigma_d}{\sqrt{2}}w_k \\
\forall t \in]k\tau, (k+1)\tau[ & \quad d\tilde{y} = g_e(\tilde{y})dt + \frac{\sigma_e}{\sqrt{2}}dW
\end{align*}

with

$$g_d(\tilde{y}) = VLV\tilde{x} = VLV(V^T\tilde{y} + U^TU\tilde{x}) = VLV^T\tilde{y}$$

$$g_e(\tilde{y}) = \tilde{f}(V^T\tilde{y} + U^TU\tilde{x})$$

where

$$L = \begin{pmatrix} (1 - \gamma)I_2 & \gamma R & 0 \\
0 & (1 - \gamma)I_2 & \gamma R \\
\gamma R & 0 & (1 - \gamma)I_2 \end{pmatrix}$$

$$\tilde{f}(\vec{x}) = (f(x_1), f(x_2), f(x_3))^T$$

Remark that $g_d(0) = 0$ and $g_e(0) = 0$ (the last equality holds because of the symmetry of $f$: $f(2x) = R(f(x))$). Thus, $0$ is a particular solution to the noise-free version of the hybrid stochastic system (12,13).

Let us now examine the contraction properties of equations (12) and (13).

We have first

$$\frac{\partial g_d}{\partial \tilde{y}} = VLV^T VLV^T = (3\gamma^2 - 3\gamma + 1)I_4$$
so that \( \lambda_{\text{max}} \left( \frac{\partial g_c}{\partial y}^T \frac{\partial g_c}{\partial y} \right) = 3\gamma^2 - 3\gamma + 1 < 1 \) (for \( 0 < \gamma < 1 \)).

Second,
\[
\frac{\partial g_c}{\partial y} = V \frac{\partial f}{\partial x} V^T = V \begin{pmatrix} \frac{\partial f}{\partial x}(x_1) & 0 & 0 \\ 0 & \frac{\partial f}{\partial x}(x_2) & 0 \\ 0 & 0 & \frac{\partial f}{\partial x}(x_3) \end{pmatrix} V^T
\]

Now observe that \( \lambda_{\text{max}} \left( \frac{\partial f}{\partial x} \right) = 1 - x^2 - y^2 \leq 1 \). Since \( V \) is an orthonormal projection, one then has \( \lambda_{\text{max}} \left( \frac{\partial g_c}{\partial y} \right) \leq 1 \).

Therefore, if
\[
3\gamma^2 - 3\gamma + 1 < e^{-2\tau}
\]
then theorem 4 together with the corollaries of section IV-B imply that, after exponential transients,
\[
\mathbb{E}(\|\tilde{y}\|^2) \leq \frac{2\gamma^2 \sigma_\epsilon^2 + (1 - \beta)(1 + \beta - \beta e^{2\tau}) e^{2\tau} \sigma_\epsilon^2}{2(1 - \beta)(1 - \beta e^{2\tau})}
\]

where \( \beta = 3\gamma^2 - 3\gamma + 1 \).

To conclude, observe that
\[
\|\tilde{y}\|^2 = \|V\tilde{x}\|^2 = \frac{1}{3} \sum_{i=1}^{3} \|Rx_{i+1} - x_i\|^2
\]
Define the phase-locking quality \( \delta \) by
\[
\delta = \sum_{i=1}^{3} \|Rx_{i+1} - x_i\|^2
\]
then one finally obtains
\[
\mathbb{E}(\delta) \leq \frac{6\gamma^2 \sigma_\epsilon^2 + 3(1 - \beta)(1 + \beta - \beta e^{2\tau}) e^{2\tau} \sigma_\epsilon^2}{2(1 - \beta)(1 - \beta e^{2\tau})}
\]

after exponential transients.

A numerical simulation is provided in Fig. 1.

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