THE GROWTH SEQUENCE OF SYMPLECTOMORPHISMS
ON SYMPLECTICALLY HYPERBOLIC MANIFOLDS

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Abstract. We study the growth rate of a sequence which measures the uniform norm of the differential under the iterates of maps. On symplectically hyperbolic manifolds, we show that this sequence has at least linear growth for every non-identical symplectomorphisms which are symplectically isotopic to the identity.

1. Introduction

Let \((M, g)\) be a closed connected Riemannian manifold. Given a diffeomorphism \(\varphi\) on \(M\), its growth sequence is defined by

\[
\Gamma_n(\varphi) := \max \left( \max_{m \in M} |d\varphi^n(m)|_g, \max_{m \in M} |d\varphi^{-n}(m)|_g \right), \ n \in \mathbb{N}.
\]

Here the norm of the differential is given by the operator norm. The explicit value of the growth sequence is hard to compute. With the following terminology, however, we measure the growth rate of \(\Gamma_n(\varphi)\). Given two positive sequences \(a_n, b_n : \mathbb{N} \to [0, \infty)\), write \(a_n \preceq b_n\) if there exists \(c > 0\) such that \(a_n \leq c(b_n + 1)\) for all \(n \in \mathbb{N}\).

We investigate \(\Gamma_n(\varphi)\) for symplectomorphisms \(\varphi\) in the identity component of the group of symplectomorphisms \(\text{Symp}_0(M, \omega)\) of a symplectic manifold \((M, \omega)\). Polterovich proved in his beautiful paper \([Pol]\) many interesting results about the growth type of \(\Gamma_n(\varphi)\) for \(\varphi \in \text{Symp}_0(M, \omega) \setminus \{1\}\) on a closed symplectic manifold \((M, \omega)\) with \(\pi_2(M) = 0\). Among them, the following result is related to this article: A closed symplectic manifold \((M, \omega)\) with \(\pi_2(M) = 0\) is called symplectically hyperbolic, if the symplectic form \(\omega\) admits a bounded primitive on the universal cover \(\tilde{p} : \tilde{M} \to M\). For symplectically hyperbolic manifolds \((M, \omega)\), Polterovich showed that \(\Gamma_n(\varphi) \gtrsim n\) when \(\varphi \in \text{Symp}_0(M, \omega) \setminus \{1\}\) has a fixed point of contractible type.

The existence of fixed points with positive action difference is crucial in his proof. When \(\varphi\) is a Hamiltonian diffeomorphism, the existence of a fixed point is obtained from Floer’s proof \([Fl]\) of Arnold’s conjecture and another fixed point with different action is established in the work of Schwarz \([Sch]\) by using Floer homology. For a non-Hamiltonian symplectomorphism, the flux does not vanish, see Definition 2.7. This guarantees a second fixed point with different action.

In the 2-dimensional case, i.e. a closed oriented surface \(\Sigma_g\) of genus \(g \geq 2\), see Example 2.4 we know that there exists a fixed point of \(\varphi \in \text{Symp}_0(\Sigma_g, \omega)\). In \([Gro]\), Gromov proved that \((-1)^g \chi(M) > 0\) for a symplectically hyperbolic manifold \((M, \omega)\) which is Kähler. Using this non-vanishing of the Euler characteristic, one can also obtain a fixed point of contractible type of \(\varphi \in \text{Symp}_0(M, \omega)\) as in \([Pol\) Example 1.3.C]. In the above cases, we hence conclude \(\Gamma_n(\varphi) \gtrsim n\) for \(\varphi \in \text{Symp}_0(M, \omega) \setminus \{1\}\), but to the best of my knowledge there is no known

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result about the existence of fixed points of \( \varphi \) on symplectically hyperbolic manifolds in general. We will prove the following statement that does not depend on the existence of a fixed point.

**Main Theorem.** Let \((M, \omega)\) be a symplectically hyperbolic manifold. If \(\varphi \in \text{Symp}_0(M, \omega) \setminus \{1\}\), then \(\Gamma_n(\varphi) \gtrsim n\).

A direct consequence of Main Theorem, followed by [Pol, Corollary 1.1.D], gives us obstructions to representations of certain discrete groups such as \(\text{SL}(n, \mathbb{Z})\) with \(n \geq 3\) into \(\text{Symp}_0(M, \omega)\) of a symplectically hyperbolic manifold \((M, \omega)\). A more precise statement is the following: Let \(G\) be an irreducible non-uniform lattice in a semi-simple real Lie group of real rank at least two. Assume that the Lie group is connected, without compact factors and with finite center. Then every homomorphism \(G \to \text{Symp}_0(M, \omega)\) has finite image when \((M, \omega)\) is a symplectically hyperbolic manifold. See [Pol, Section 1.6] for details.

Our main tools are the mapping torus construction and the cofilling function of the twisted Hamiltonian structure which will be introduced in the following section. We interpret the non-trivial flux of non-Hamiltonian diffeomorphism as a certain type of cofilling function.

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## 2. Preliminaries

### 2.1. Cofilling function.

**Definition 2.1 ([Gro2], [Pol]).** Let \(\sigma \in \Omega^2(M)\) be a closed 2-form such that \(\tilde{\sigma} \in \Omega^2(\tilde{M})\) is exact. Let \(\mathcal{P}_\sigma\) be the space of all 1-forms on \(\tilde{M}\) whose differential is \(\tilde{\sigma}\). Pick a point \(x \in \tilde{M}\) and denote by \(B_x(s)\) the ball of radius \(s > 0\) with respect to \(\tilde{g}\) which is centered at \(x\). Then the cofilling function \(u_\sigma : [0, \infty) \to [0, \infty)\) is defined by

\[
u_\sigma(s) = u_{\sigma, g, x}(s) := \inf_{\theta \in \mathcal{P}_\sigma} \sup_{z \in B_x(s)} |\theta_z|_{\tilde{g}}.
\]

Here the norm of a differential form is

\[|\theta_z|_{\tilde{g}} := \max\{|\theta_z(v)| : \|v\|_{\tilde{g}} = 1\}.\]

Given two functions \(f, g : [0, \infty) \to [0, \infty)\), we write \(f \lesssim g\) if there exists \(c > 0\) such that \(f(s) \leq c(g(s) + 1)\) for all \(s \in [0, \infty)\), and \(f \sim g\) if \(f \lesssim g, g \lesssim f\). If we choose another Riemannian metric \(g'\) on \(M\) and a different base point \(x'\), then \(u_{\sigma, g, x} \sim u_{\sigma, g', x'}\). This means that the growth type of the cofilling function is an invariant of the closed 2-form \(\sigma\).

**Proposition 2.2.** The growth type of the cofilling function only depends on the cohomology class.
Proof. Let $\sigma$, $\sigma'$ be closed cohomologous 2-forms on a closed manifold $M$ with $\tilde{\sigma}$, $\tilde{\sigma}'$ exact. There exists a 1-form $\xi$ such that $\sigma' = \sigma + d\xi$. We fix $x \in M$ and compute

$$u_{\sigma'}(s) = \inf_{\theta \in \mathcal{P}_{\sigma}} \sup_{z \in B_x(s)} |\theta_z|_g$$

$$= \inf_{\theta \in \mathcal{P}_{\sigma}} \sup_{z \in B_x(s)} (|\theta - \tilde{\xi}|_z)_g$$

$$\leq \inf_{\theta \in \mathcal{P}_{\sigma}} \sup_{z \in B_x(s)} |\theta_z|_g + \sup_{z \in B_x(s)} |\tilde{\xi}_z|_g$$

$$\leq \inf_{\theta \in \mathcal{P}_{\sigma}} \sup_{z \in B_x(s)} |\theta_z|_g + \max_{z \in M} |\xi_z|_g$$

$$= u_{\sigma}(s) + C,$$

where $C = \max_{z \in M} |\xi_z|_g$. In a similar way we obtain $u_{\sigma}(s) \leq u_{\sigma'}(s) + C$. Hence $u_{\sigma}(s) \sim u_{\sigma'}(s)$. \hfill \Box

Example 2.3. Consider the standard symplectic torus $(\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}, \omega = \Sigma_{i=1}^n dx_i \wedge dy_i)$ with the metric induced by the Euclidean metric on $\mathbb{R}^{2n}$. Since $\tilde{\omega} = d(\sum_{i=1}^n x_i dy_i) \in \Omega^2(\mathbb{R}^{2n})$ and $\sum_{i=1}^n x_i dy_i$ has linear growth with respect to the Euclidean metric on $\mathbb{R}^{2n}$, it follows that $u_{\omega}(s) \lesssim s$.

For any primitive $\alpha \in \Omega^1(\mathbb{R}^{2n})$ of $\tilde{\omega}$, we obtain

$$\frac{\pi^n}{n!} s^{2n} = \int_{B_x(s)} \tilde{\omega} = \int_{\partial B_x(s)} \alpha \leq \sup_{z \in \partial B_x(s)} |\alpha_z| \cdot \int_{\partial B_x(s)} 1 \leq \sup_{z \in \partial B_x(s)} |\alpha_z| \cdot \frac{2\pi^n}{(n-1)!} s^{2n-1}.$$  

This implies $\sup_{z \in B_x(s)} |\alpha_z| \geq \frac{n}{2n}$ and hence we conclude that $u_{\omega}(s) \sim s$.

Example 2.4. Let $(\Sigma_g, \omega)$ be a closed oriented surface of genus $g \geq 2$ with a volume form. First represent $M$ as $\mathbb{H}/G$, where $\mathbb{H} = \{x + yi \in \mathbb{C} : y > 0\}$ is the hyperbolic upper half-plane and $G$ is a discrete group of isometries. Without loss of generality, we may assume that the lift $\tilde{\omega}$ of $\omega$ to the universal cover $\mathbb{H}$ coincides with the hyperbolic area form $\frac{1}{y} dx \wedge dy$. Note that $\tilde{\omega} = d(\frac{1}{y} dy) + \frac{1}{y} dx$ is bounded with respect to the hyperbolic metric $ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$ on $\mathbb{H}$. By the definition, $(\Sigma_g, \omega)$ is a symplectically hyperbolic manifold and every symplectically hyperbolic manifold satisfies $u_{\omega}(s) \sim 1$.

Proposition 2.5. Let $M$, $N$ be closed manifolds and let $h : N \to M$ be an immersion. Let $\sigma \in \Omega^2(M)$ be a closed 2-form which is exact on $\tilde{M}$, then $u_{h^*\sigma}(s) \lesssim u_{\sigma}(s)$.

Proof. Let $g$ be a Riemannian metric on $M$, then $h^*g$ gives a Riemannian metric on $N$. We then have the following commutative diagram:

$$\begin{array}{ccc}
(N_h^*g) & \xrightarrow{h} & (\tilde{M}, \tilde{g}) \\
\downarrow_{p_N} & & \downarrow_{p_M} \\
(N, h^*g) & \xrightarrow{h} & (M, g)
\end{array}$$
where \( \tilde{\cdot} \) are the lifts of \([-\cdot]\) to the universal covers. Now choose a primitive \( \theta \in \Omega^1(\tilde{M}) \) of \( \tilde{\sigma} \), then \( h^* \theta \in \Omega^1(\tilde{N}) \) is a primitive of \( \tilde{h}^* \tilde{\sigma} \). Pick a point \( z \in \tilde{N} \), and note that
\[
|\tilde{h}^* \tilde{\sigma}|_{\tilde{h}^* \tilde{g}} = \max \{ |\tilde{h}^* \tilde{\sigma}| : \|v\|_{\tilde{h}^* \tilde{g}} = 1, v \in T_{\tilde{h}^* \tilde{N}} \} 
\leq \max \{ |\tilde{h}(z)| : \|v\|_{\tilde{g}} = 1, v \in T_{\tilde{h}(z)} \} 
\leq \sup_{\theta \in P^*} \inf_{z \in B_{\tilde{h}(z)}(s)} |\tilde{\theta}|_{\tilde{g}}.
\]
We can thus estimate
\[
u_{h^* \sigma}(s) \sim \inf_{\theta \in P^*} \sup_{z \in B_{\tilde{h}(z)}(s)} |\tilde{\theta}|_{\tilde{h}^* \tilde{g}} \]
which concludes the proof.

\[\square\]

**Corollary 2.6.** If \((M, \omega)\) is a symplectically hyperbolic manifold, then
\[
\int_{f(T^2)} \omega = 0
\]
for any immersion \( f : T^2 \to M \).

**Proof.** If the integral is non-zero, then \([f^* \omega] \neq 0\) in \( H^2(T^2, \mathbb{R}) \). By Proposition 2.2, Example 2.3 and Proposition 2.5 we deduce the following contradiction:
\[
s \sim u_{f^* \omega}(s) \lesssim u_{\omega}(s) \sim 1.
\]

Indeed, Corollary 2.6 holds true for any smooth map \( f : T^2 \to M \) and the proof is almost the same, see [Ked].

**2.2. Flux homomorphism.** Let \( \varphi \in \text{Symp}_0(M, \omega) \) and \( \{ \varphi_t \}_{t \in [0, 1]} \) a path of symplectomorphisms such that \( \varphi_0 = \mathbb{I} \) and \( \varphi_1 = \varphi \). Let \( Y_t \) be the generating vector field,
\[
\frac{d}{dt} \varphi_t = Y_t \circ \varphi_t.
\]
Since the Lie derivative \( L_{Y_t} \omega \) vanishes, we get a closed 1-form \( \iota_{Y_t} \omega \) for each \( t \in [0, 1] \).

**Definition 2.7.** The flux homomorphism \( \tilde{\text{Flux}} : \tilde{\text{Symp}}_0(M, \omega) \to H^1(M, \mathbb{R}) \) is defined as
\[
\tilde{\text{Flux}}(\{ \varphi_t \}) = \int_0^1 [\iota_{Y_t} \omega] \, dt,
\]
where \([\alpha]\) is the cohomology class of a form \( \alpha \).
The kernel of $\tilde{\text{Symp}}_0(M, \omega) \to \text{Symp}_0(M, \omega)$ can be identified with the fundamental group $\pi_1(\text{Symp}_0(M, \omega))$. We denote by $\Gamma_\omega \subset H^1(M, \mathbb{R})$ the image of $\pi_1(\text{Symp}_0(M, \omega))$ by $\text{Flux}$ and we call $\Gamma_\omega$ the flux group of $(M, \omega)$. Then $\text{Flux}$ descends to a homomorphism

$$\text{Flux}: \text{Symp}_0(M, \omega) \to H^1(M, \mathbb{R})/\Gamma_\omega.$$ 

The next well-known fact is proved in [MS].

**Theorem 2.8.** $\text{Ham}(M, \omega) = \ker \text{Flux}.$

**Proposition 2.9** ([Keil], [Pol]). Let $(M, \omega)$ be a symplectically hyperbolic manifold, then the flux group $\Gamma_\omega = 0$ in $H^1(M, \mathbb{R})$.

The atoroidal property of $\omega$, $\int_{f(T^2)} \omega = 0$ for any smooth $f : \mathbb{T}^2 \to M$, is important in the proof of Proposition 2.9. We then have

$$\text{Flux}: \text{Symp}_0(M, \omega) \to H^1(M, \mathbb{R}).$$

### 2.3. Hamiltonian structures

Let $\Sigma$ be a closed connected orientable manifold of dimension $2n+1$. A **Hamiltonian structure** on $\Sigma$ is a closed 2-form $\omega$ such that $\omega^n$ is nowhere vanishing. So its kernel $\ker \omega$ defines a 1-dimensional foliation which we call the **characteristic foliation** of $\omega$.

**Definition 2.10.** A Hamiltonian structure $(\Sigma, \omega)$ is called stable if there exists a 1-form $\lambda$ such that

$$\ker \omega \subset \ker d\lambda, \quad \lambda \wedge \omega^n > 0.$$  

We call the 1-form $\lambda$ a stabilizing 1-form. This structure defines the **Reeb vector field** $R$ by

$$\lambda(R) = 1, \quad \iota_R \omega = 0.$$

**Definition 2.11.** A Hamiltonian structure is called virtually contact if there is a covering $p : \tilde{\Sigma} \to \Sigma$ and a primitive $\lambda \in \Omega^1(\tilde{\Sigma})$ of $p^* \omega$ such that

$$\sup_{x \in \tilde{\Sigma}} |\lambda_x| \leq C < \infty, \quad \inf_{x \in \tilde{\Sigma}} \lambda(R) \geq \mu > 0,$$

where $|\cdot|$ is the lifting of a metric on $\Sigma$ and $R$ is the pullback of a non-vanishing vector field generating $\ker \omega$.

### 3. Hamiltonian structures on mapping tori

#### 3.1. Standard and twisted Hamiltonian structures.

For a symplectically hyperbolic manifold $(M, \omega)$ and a symplectomorphism $\varphi \in \text{Symp}_0(M, \omega)$, we consider the **mapping torus** $M_\varphi$ of $M$ with respect to $\varphi$,

$$M_\varphi = \frac{M \times [0,1]}{(m,0) \sim (\varphi(m),1)}.$$  

Now we consider two Hamiltonian structures on the mapping torus $M_1, M_\varphi$. The trivial mapping torus $M_1 \cong M \times S^1$ carries the Hamiltonian structure $\omega_1 := \pi^* \omega$, where $\pi : M_1 \to M : (m, \theta) \mapsto m$. The non-trivial one $M_\varphi$ carries the Hamiltonian structure $\omega_\varphi := \pi_\varphi^* \omega$, where $\pi_\varphi : M_\varphi \to M : (m, \theta) \mapsto m$. The Hamiltonian structure $\omega_\varphi$ is a well-defined 2-form on $M_\varphi$, since $\varphi^* \omega = \omega$. Note that the kernel of both Hamiltonian structures are spanned by $\frac{\partial}{\partial \theta}$. Let us choose a path of symplectomorphisms $\{ \varphi_t \}_{t \in [0,1]}$ from $\varphi_0 = 1$ to $\varphi_1 = \varphi$. The twisting map $f : M_1 \to M_\varphi$ is defined by

$$f(m, \theta) = (\varphi_\theta(m), \theta).$$
Lemma 3.1. Let \((M_1, \omega_1), (M_\varphi, \omega_\varphi)\) and \(f : M_1 \to M_\varphi\) be as above, then we obtain

\[
f^*\omega_\varphi = \omega_1 - \pi^* (\iota_{\tilde{\gamma}_\theta} \omega) \wedge \pi^*_\theta d\theta,
\]

where \(\pi_\theta : M_1 \to S^1 : (m, \theta) \mapsto \theta\).

**Proof.** Let \((m_i, \theta_i)\) be tangent vectors in \(T_{(m, \theta)} M_1, i = 1, 2\). We identify \(T_{\theta} S^1\) with \(\mathbb{R}\), so \(\theta_i\) is considered as an element of \(\mathbb{R}\). We compute \(f^* \omega_\varphi\) as follows:

\[
(f^* \omega_\varphi)_x((m_1, \theta_1), (m_2, \theta_2)) = (\omega_\varphi)_x(f_x(m_1, \theta_1), f_x(m_2, \theta_2)) = (\omega_\varphi)_x((d\varphi_\theta(m)[m_1], \theta_1), (d\varphi_\theta(m)[m_2], \theta_2))
\]

\[
= (\omega_\varphi)_x(((\theta_1 \cdot Y_\theta[\varphi_\theta(m)], \theta_1), (d\varphi_\theta(m)[m_1], \theta_2)) + (\omega_\varphi)_x(((\theta_2 \cdot Y_\theta[\varphi_\theta(m)], \theta_1), (d\varphi_\theta(m)[m_2], \theta_2))
\]

\[
= (\omega_\varphi)_x(((\theta_1 \cdot Y_\theta[\varphi_\theta(m)], \theta_1), (d\varphi_\theta(m)[m_1], \theta_2)) + (\omega_\varphi)_x(((\theta_2 \cdot Y_\theta[\varphi_\theta(m)], \theta_1), (d\varphi_\theta(m)[m_2], \theta_2))
\]

\[
= \omega_{m}(m_1, m_2).
\]

Here \(Y_\theta\) is the vector field defined in (2.11). The third summand vanishes since \(\omega_\varphi\) is skew-symmetric. In order to simplify the \(\circ\)-term we compute

\[
\circ = (\pi^* \omega)_x((d\varphi_\theta(m)[m_1], \theta_1), (d\varphi_\theta(m)[m_2], \theta_2)) = \omega_{\varphi_\theta}(m)(d\varphi_\theta(m)[m_1], d\varphi_\theta(m)[m_2])
\]

\[
= \omega_{\varphi_\theta}(m)(m_1, m_2)
\]

where the last equality comes from the assumption that \(\varphi_\theta : M \to M\) is a symplectomorphism. We also know

\[
(\omega_1)_x((m_1, \theta_1), (m_2, \theta_2)) = (\pi^* \omega)_x((m_1, \theta_1), (m_2, \theta_2)) = \omega_{m}(m_1, m_2).
\]

By combining (3.4), (3.5) we obtain

\[
(\omega_1)_x((m_1, \theta_1), (m_2, \theta_2)) = (\omega_\varphi)_x((d\varphi_\theta(m)[m_1], \theta_1), (d\varphi_\theta(m)[m_2], \theta_2))
\]

Hence the difference between \(f^* \omega_\varphi\) and \(\omega_1\) is

\[
(f^* \omega_\varphi - \omega_1)_x((m_1, \theta_1), (m_2, \theta_2)) = (\omega_\varphi)_x((\theta_1 \cdot Y_\theta[\varphi_\theta(m)], \theta_1), (d\varphi_\theta(m)[m_2], \theta_2))
\]

\[
= (\omega_\varphi)_x((d\varphi_\theta(m)[m_1], \theta_1), (\theta_2 \cdot Y_\theta[\varphi_\theta(m)], \theta_1))
\]

\[
= \omega_{\varphi_\theta}(m)(d\varphi_\theta(m)[m_1], \theta_2 \cdot Y_\theta[\varphi_\theta(m)]) + \omega_{\varphi_\theta}(m)(\theta_1 \cdot Y_\theta[\varphi_\theta(m)], d\varphi_\theta(m)[m_2]).
\]

On the other hand, we have

\[
(\pi^* (\iota_{\tilde{\gamma}_\theta} \omega) \wedge \pi^*_\theta d\theta)_x((m_1, \theta_1), (m_2, \theta_2))
\]

\[
= (\pi^* (\iota_{\tilde{\gamma}_\theta} \omega) \wedge \pi^*_\theta d\theta)_x((d\varphi_\theta(m)[m_1], \theta_1), (d\varphi_\theta(m)[m_2], \theta_2))
\]

\[
= (\iota_{\tilde{\gamma}_\theta} \omega)_{\varphi_\theta}(m)(d\varphi_\theta(m)[m_1]) \cdot (d\theta)_\theta(\theta_2)
\]

\[
- (\iota_{\tilde{\gamma}_\theta} \omega)_{\varphi_\theta}(m)(d\varphi_\theta(m)[m_2]) \cdot (d\theta)_\theta(\theta_1)
\]

\[
= \omega_{\varphi_\theta}(m)(Y_\theta[\varphi_\theta(m)], d\varphi_\theta(m)[m_1]) \cdot \theta_2
\]

\[
- \omega_{\varphi_\theta}(m)(Y_\theta[\varphi_\theta(m)], d\varphi_\theta(m)[m_2]) \cdot \theta_1.
\]
By combining (3.7) and (3.8), we conclude (3.3). This proves the lemma. \(\square\)

**Remark 3.2.** The Hamiltonian structure \(\omega_\varphi\) on \(M_\varphi\) has its kernel spanned by \(\frac{\partial}{\partial \theta}\). If there exists a closed orbit \(\gamma: S^1 \to M_\varphi\) of \(\frac{\partial}{\partial \theta}\), then its projection \(\pi \circ \gamma: S^1 \to M\) gives us a symplectic fixed point with respect to \(\varphi \in \text{Symp}_0(M, \omega)\). One can easily check that \((f^*\omega_\varphi)(\frac{\partial}{\partial \theta}) - Y_\theta) = 0\). This implies that the vector field \(\frac{\partial}{\partial \theta} - Y_\theta\) also can be interpreted as a fixed point of the flow of the vector field of \(Y_\theta\).

### 3.2. Cofilling function of Hamiltonian structures.

**Proposition 3.3.** Let \((M, \omega)\) be a symplectically hyperbolic manifold and \(\varphi \in \text{Symp}_0(M, \omega)\). If \(\text{Flux}(\varphi) \neq 0\) then the Hamiltonian structure \(f^*\omega_\varphi \in \Omega^2(M_1)\) satisfies \(u_{f^*\omega_\varphi}(s) \gtrsim s\).

**Proof.** First recall that

\[
\begin{align*}
  f^*\omega_\varphi &= \omega_1 - \pi^*(\iota_{Y_\theta}\omega) \wedge \pi^*_\theta d\theta. 
\end{align*}
\]

Since our symplectic manifold \((M, \omega)\) is symplectically hyperbolic, the standard Hamiltonian structure \(\omega_1 = \pi^*\omega \in \Omega^2(M_1)\) admits a bounded primitive \(\tilde{\pi}^*\lambda\) on \(\tilde{M} \times \mathbb{R}\), where \(\tilde{\omega} = d\lambda\) and \(\tilde{\pi}: \tilde{M} \times \mathbb{R} \to \tilde{M}\) is the lift of \(\pi\).

Now we consider the twisted term \(\pi^*(\iota_{Y_\theta}\omega)\wedge\pi^*_\theta d\theta\). Since \(\text{Flux}(\varphi)\) is nontrivial in \(H^1(M, \mathbb{R})\), there exists \(a \in \pi_1(M)\) such that \(\langle \text{Flux}(\varphi), \overline{a} \rangle \neq 0\), where \(\overline{a}\) stands for the image of \(a\) in \(H_1(M, \mathbb{Z})\) under the Hurewicz homomorphism. Choose an immersed curve \(\gamma: S^1 \to M\) such that \([\gamma] = a\).

Let us consider the induced immersion of \(\mathbb{T}^2\)

\[
  h: \mathbb{T}^2 \to M_1 = M \times S^1
\]

\[
  (t, \theta) \mapsto (\gamma(t), \theta).
\]

Then with (3.9) we calculate

\[
-\int_{h(\mathbb{T}^2)} f^*\omega_\varphi = \int_{h(\mathbb{T}^2)} \iota_{Y_\theta}\omega \wedge d\theta
\]

\[
= \int_0^1 \int_0^1 \iota_{Y_\theta}\omega \left[ \frac{d\gamma}{dt} \right] dt d\theta
\]

\[
= \int_0^1 \left( \int_0^1 \iota_{Y_\theta}\omega d\theta \right) \left[ \frac{d\gamma}{dt} \right] dt
\]

\[
= \langle \text{Flux}(\varphi), \overline{a} \rangle
\]

\[
\neq 0.
\]

This implies that \([h^* f^*\omega_\varphi] \neq 0\) in \(H^2(\mathbb{T}^2, \mathbb{R})\). From Example 3.3 and Proposition 3.5 we have

\[
  u_{f^*\omega_\varphi}(s) \gtrsim u_{h^* f^*\omega_\varphi}(s) \sim s.
\]

\(\square\)

**Remark 3.4.** Let \((M, \omega)\) be a symplectically hyperbolic manifold and \(\varphi \in \text{Symp}_0(M, \omega)\) with \(\text{Flux}(\varphi) = 0\). The induced Hamiltonian structure \((M_\varphi, \omega_\varphi)\) is then a new example of a virtually contact structure.

Indeed, a covering of \(M_\varphi\) is given by

\[
\tilde{M}_\varphi := \frac{\tilde{M} \times [0, 1]}{(\tilde{m}, 0) \sim (\varphi(\tilde{m}), 1)}
\]
with a covering map \( \tilde{p} : \tilde{M}_\varphi \to M_\varphi : (\tilde{m}, \theta) \mapsto (m, \theta) \) where \( \tilde{\varphi} : \tilde{M} \to \tilde{M} \) is the lift of \( \varphi : M \to M \). Note that \( \tilde{M}_1 = \tilde{M} \times S^1 \). To obtain a primitive of the Hamiltonian structure, we need the following notations:

\[
\tilde{f} : \tilde{M}_1 \to \tilde{M}_\varphi : (\tilde{m}, \theta) \mapsto (\tilde{\varphi}(\tilde{m}), \theta); \\
\tilde{\pi} : \tilde{M}_1 \to \tilde{M} : (\tilde{m}, \theta) \mapsto \tilde{m}.
\]

With this covering we obtain a primitive of \((\tilde{M}_\varphi, \tilde{p}^* \omega_\varphi) \cong (\tilde{M}_1, \tilde{f}^* \tilde{p}^* \omega_\varphi)\) as follows:

\[
\tilde{f}^* \tilde{p}^* \omega_\varphi = \tilde{p}^* f^* \omega_\varphi = \tilde{p}^* (\omega_1 - \pi^* (\tilde{\iota}_{\tilde{Y}_\theta} \omega) \wedge \pi^* d\theta) = d(\tilde{\pi}^* \lambda - \tilde{p}^* h_\theta d\theta + K d\theta).
\]

Here \( \lambda \) is a bounded primitive of \( \tilde{\omega} \in \Omega^2(\tilde{M}) \), \( h_\theta \in C^\infty(M) \) is a time-dependent Hamiltonian function corresponding to the Hamiltonian diffeomorphism \( \varphi_\theta \) and \( K \) is a constant fixed later on.

Since \( \lambda, h_\theta \) and \( K \) are bounded, the primitive \( \tilde{\lambda} \) is obviously bounded. To verify the second condition of a virtually contact structure in \( (2.3) \), let \( R \) be the lifted vector field of \( \frac{\partial}{\partial \theta} - \tilde{Y}_\theta \) on \( \tilde{M}_1 \). Then we have

\[
\tilde{\lambda}(R) = K - \lambda(\tilde{Y}_\theta) - \tilde{h}_\theta,
\]

where \( \tilde{Y}_\theta, \tilde{h}_\theta \) are the lifts of \( Y_\theta, h_\theta \) to the universal cover \( \tilde{M} \). We guarantee \( \tilde{\lambda}(R) \geq \mu \) by taking

\[
K = \|\lambda_x\|_\infty \max_{\theta \in S^1} \|Y_\theta\|_\infty + \max_{\theta \in S^1} \|h_\theta\|_\infty + \mu.
\]

**Corollary 3.5.** Let \((M, \omega)\) be a symplectically hyperbolic manifold and \((M_\varphi, \omega_\varphi)\) be the mapping torus with the induced Hamiltonian structure. If \( \varphi \) is not Hamiltonian, then \((M_\varphi, \omega_\varphi)\) admits a stable Hamiltonian structure but no virtually contact structure.

**Proof.** Let \( f : M_1 \to M_\varphi \) be the twisting map defined in \( (3.2) \). As mentioned in Remark \( (3.2) \), \( \ker(f^* \omega_\varphi) \) is spanned by the vector field \( \frac{\partial}{\partial \theta} - \tilde{Y}_\theta \). In order to define a stable structure on \((M_\varphi, \omega_\varphi) \cong (M \times S^1, f^* \omega_\varphi)\), we choose the stabilizing 1-form \( \lambda \) in Definition \( (2.10) \) as \( \pi^*_g d\theta \).

Since \( \lambda = \pi^*_g d\theta \) is closed, the first condition in \( (2.2) \) holds trivially. Using \( (3.9) \) one verifies that \( \lambda \wedge (f^* \omega_\varphi)^n = \pi^*_g d\theta \wedge \pi^* \omega^n \), \( 2n = \dim M \), and this form vanishes nowhere, which implies that the second condition in \( (2.2) \) also holds true. Thus \((M_\varphi, \omega_\varphi)\) admits a stable Hamiltonian structure with a stabilizing 1-form \( \pi^*_g d\theta \). By Proposition \( (3.3) \), the Hamiltonian structure \( \omega_\varphi \) has a cofilling function of at least linear type. This means that there is no bounded primitive of \( \omega_\varphi \) even in the universal cover and hence \((M_\varphi, \omega_\varphi)\) cannot be a virtually contact structure. \( \square \)

### 3.3. Metrics on Hamiltonian structures.

In order to obtain primitives of the Hamiltonian structures \((M_1, \omega_1), (M_\varphi, \omega_\varphi)\), we now consider the lifted structures on the universal covers.

The lifted Hamiltonian structure \((\tilde{M}_1, \tilde{\omega}_1)\) is clearly isomorphic to \((\tilde{M} \times \mathbb{R}, \tilde{\pi}^* \tilde{\omega})\), where \( \tilde{\pi} : \tilde{M} \times \mathbb{R} \to \tilde{M} : (\tilde{m}, r) \mapsto \tilde{m} \). Note that \( \tilde{M}_\varphi = \tilde{M} \times \mathbb{R} \) and \( \tilde{\omega}_\varphi = \tilde{\pi}^* \tilde{\omega} \), where \( \tilde{\pi} : \tilde{M}_\varphi \to \tilde{M} \) is the lift of \( \pi_\varphi : M_\varphi \to M \). Since \( \tilde{\pi} = \tilde{\pi}_\varphi \), we have \((\tilde{M}_\varphi, \tilde{\omega}_\varphi) \cong (\tilde{M} \times \mathbb{R}, \tilde{\pi}^* \tilde{\omega})\).

Even though both lifted structures \((\tilde{M}_1, \tilde{\omega}_1), (\tilde{M}_\varphi, \tilde{\omega}_\varphi)\) are isomorphic to \((\tilde{M} \times \mathbb{R}, \tilde{\pi}^* \tilde{\omega})\), they have different deck transformations as follows: An element \( n \in \mathbb{Z} \cong \pi_1(S^1) \mapsto \pi_1(M_1) \) induces
a translation \((\tilde{m}, r) \mapsto (\tilde{m}, r + n)\) on the universal cover \(\tilde{M}_1\), while \(n \in \mathbb{Z} \cong \pi_1(S^1) \hookrightarrow \pi_1(M_\varphi)\) act by \((\tilde{m}, r) \mapsto (\varphi^n(\tilde{m}), r + n)\) on \(\tilde{M}_\varphi\). Here \(\varphi^n\) is the \(n\)-th iterate of \(\varphi\).

We next consider the lift \(\tilde{f} : \tilde{M}_1 \to \tilde{M}_\varphi\) of \(f : M_1 \to M_\varphi\). Since \(\tilde{M}_1 = \tilde{M} \times \mathbb{R} = \tilde{M}_\varphi\), it suffices to define
\[
\tilde{f} : \tilde{M} \times \mathbb{R} \to \tilde{M} \times \mathbb{R} \quad (\tilde{m}, r) \mapsto (\varphi^n(\tilde{m}), r).
\]

Here \(\tilde{\varphi}_r = \tilde{\varphi}_{r-[r]} \circ \tilde{\varphi}^{|r|}\), where \([r]\) is the largest integer not greater than \(r\) and \(\tilde{\varphi}_\theta\) is the lift of \(\varphi\theta\) for \(0 \leq \theta < 1\). We summarize the Hamiltonian structures, maps and their lifts in the following diagram:
\[
\begin{align*}
(M, \omega) & \xleftarrow{\tilde{\pi}} (\tilde{M}_1, \tilde{\omega}_1) & \xrightarrow{\tilde{f}} & (\tilde{M}_\varphi, \tilde{\omega}_\varphi) \\
(M, \omega) & \xleftarrow{\pi} (M_1, \omega_1) & \xrightarrow{f} & (M_\varphi, \omega_\varphi)
\end{align*}
\]

(3.10)

Here \(p_1, p_\varphi\) are the natural projections.

Now we consider Riemannian metrics on the above spaces. Let \(g\) be a Riemannian metric on \(M\), \(g_\theta\) be the standard metric on \(S^1 = \mathbb{R}/\mathbb{Z}\) which is induced by the Euclidean metric on \(\mathbb{R}\). We consider a product Riemannian metric \(\tilde{g}_1 := g \oplus g_\theta\) on \(M_1 = M \times \mathbb{S}^1\) and the lifts \(\tilde{g}\), \(\tilde{g}_1\) to the corresponding universal covers \(\tilde{M}, \tilde{M}_1\times \mathbb{R}\).

The lifted Hamiltonian structure \((\tilde{M}_\varphi, \tilde{\omega}_\varphi) \cong (\tilde{M} \times \mathbb{R}, \tilde{\varphi}^* \tilde{\omega})\) admits a bounded primitive \(\tilde{\varphi}^* \lambda \in \Omega^1(\tilde{M}_\varphi)\), i.e. \(d(\tilde{\varphi}^* \lambda) = \tilde{\varphi}^* \tilde{\omega}\), with respect to \(\tilde{g}_1\). Here the primitive 1-form \(\lambda \in \Omega^1(\tilde{M})\) exists and is bounded with respect to \(\tilde{g}\), since our manifold \((M, \omega)\) is symplectically hyperbolic. This immediately implies that the pull-back \(\tilde{f}^* \tilde{\omega}_\varphi\) also has a bounded primitive with respect to the pull-back metric \(\tilde{f}^* \tilde{g}_1\). By Proposition 3.3, however, \(\tilde{f}^* \tilde{\omega}_\varphi\) never admits a bounded primitive with respect to the metric \(\tilde{g}_1\), when \(\varphi\) is a non-Hamiltonian symplectomorphism. Note in particular that \(\tilde{f}^* \tilde{g}_1\) cannot be expressed as a lift of a Riemannian metric on \(M_1\).

We now investigate the pull-back metric \(\tilde{f}^* \tilde{g}_1\) on \(\tilde{M}_1\). For \(n \in \mathbb{Z} \subset \mathbb{R}\), \((\tilde{m}, n) \in \tilde{M}_1\) and \((\tilde{m}_i, 0) \in T(\tilde{m}, n) \tilde{M}_1\), we have
\[
\begin{align*}
(\tilde{f}^* \tilde{g}_1)(\tilde{m}_1, 0), (\tilde{m}_2, 0) = (\tilde{g}_1)(\tilde{f}^*(\tilde{m}_1, 0), \tilde{f}^*(\tilde{m}_2, 0))
= (\tilde{g}_1)(\tilde{f}(\tilde{m}, n), (d\varphi^n(\tilde{m})[\tilde{m}_1], 0), (d\varphi^n(\tilde{m})[\tilde{m}_2], 0))
= (\tilde{g} \varphi^n(\tilde{m}))(d\varphi^n(\tilde{m})[\tilde{m}_1], d\varphi^n(\tilde{m})[\tilde{m}_2])
= g \varphi^n(m)(d\varphi^n(m)[m_1], d\varphi^n(m)[m_2])
\end{align*}
\]

(3.11)

where \(m = p(\tilde{m})\) and \(m_i = p_\varphi(\tilde{m}_i)\).

4. Proof of Main Theorem

When \(\varphi\) is Hamiltonian diffeomorphism on \((M, \omega)\), as mentioned in the introduction, Polterovich’s result implies \(\Gamma_n(\varphi) \geq n\). So we only need to consider a symplectomorphism \(\varphi \in \text{Symp}_0(M, \omega)\) with a non-vanishing flux.

A crucial observation in this article is the following. We can choose a primitive \(\tilde{\varphi}^* \lambda \in \Omega^1(\tilde{M}_1)\) of \(\tilde{f}^* \tilde{\omega}_\varphi\) which is bounded with respect to the twisted metric \(\tilde{f}^* \tilde{g}_1\). But \(\tilde{f}^* \tilde{\varphi}^* \lambda\) has
at least linear growth with respect to the standard metric $\overline{g}_1$ by Proposition 3.3. Now we interpret the difference between $\tilde{f}^*\overline{g}_1$ and $f^*\tilde{g}_1$ as a lower bound for the growth rate of $\Gamma_n(\varphi)$.

Let us fix a primitive
\[
\tilde{f}^*\overline{\pi}^* \lambda = \overline{\pi}^* \lambda + r \cdot \tilde{\pi}^*(\pi_\nu \omega) \tag{4.1}
\]

of $\tilde{f}^*\overline{\omega}_\varphi$ in Lemma 3.1. Here $r$ is the coordinate for $\mathbb{R} = \mathbb{S}^1$ and $Y_r$ is the lift of $Y_\theta$ to $\tilde{M}_1$. Without loss of generality, we may assume that $[(\gamma_n \omega)] \in H^1(M, \mathbb{R})$ is non-trivial for $t = 0$ which implies that max $z \in \overline{M} |(tY_n \omega)|_{\overline{g}}$ is positive for all $n \in \mathbb{N} \subset \mathbb{R}$. Now we pick a point $m \in M$ such that
\[
|\langle \pi_\nu \omega \rangle_m|_{\overline{g}} = |\langle \pi_\nu \omega \rangle_{\overline{g}}| \geq 0, \quad \forall n \in \mathbb{N} \subset \mathbb{R}.
\]

Since $|\tilde{\pi}^* \lambda|_{\overline{g}_1}$ is bounded,
\[
|\langle \tilde{f}^*\tilde{\pi}^* \lambda \rangle_{(\overline{m}, n)}|_{\overline{g}_1} \sim n \cdot |\langle \tilde{\pi}_\nu \omega \rangle_{\overline{g}}|, \quad \forall n \in \mathbb{N} \subset \mathbb{R}.
\]

By definition of the norm, there is a sequence of tangent vectors $\{X_n\}_{n \in \mathbb{N}}$, $X_n \in T_{(\overline{m}, n)}\tilde{M}_1$ which meets the following conditions:

- $\|X_n\|_{\overline{g}_1} = \|\tilde{\pi}_\nu X_n\|_{\overline{g}} = \|p_* \tilde{\pi}_\nu X_n\|_{\overline{g}} = 1$, \quad $\forall n \in \mathbb{N};$
- $\langle \tilde{f}^*\tilde{\pi}^* \lambda \rangle_{(\overline{m}, n)}(X_n) \sim n,$

where $\tilde{\pi} : \tilde{M}_1 \to \tilde{M}$, $p : \tilde{M} \to M$ are as in (3.11). We can assume that $X_n$ has no $r$-component, because $\tilde{f}^*\tilde{\pi}^* \lambda$ in (4.1) has no $dr$-part.

Now we change the metric $\overline{g}_1$ to $\tilde{f}^*\tilde{g}_1$ on $\tilde{M}_1$
\[
n \sim \langle \tilde{f}^*\tilde{\pi}^* \lambda \rangle_{(\overline{m}, n)}(X_n)
\]
\[
\leq \sup_z \langle \tilde{f}^*\tilde{\pi}^* \lambda \rangle_z \cdot \|X_n\|_{\tilde{f}^*\tilde{g}_1}
\]
\[
= \sup_z \langle \tilde{\pi}^* \lambda \rangle_z \cdot \|X_n\|_{\tilde{f}^*\tilde{g}_1}
\]
\[
\leq C \cdot \|X_n\|_{\tilde{f}^*\tilde{g}_1},
\]

where the constant $C > 0$ comes from the fact that $\tilde{\pi}^* \lambda$ is bounded with respect to the metric $\overline{g}_1$. Since the tangent vector $X_n$ has no $r$-direction, we can use (3.11) to obtain
\[
n^2 \lesssim \|X_n\|_{\tilde{f}^*\tilde{g}_1}^2
\]
\[
= \langle \tilde{f}^*\tilde{g}_1 \rangle_{(\overline{m}, n)}(X_n, X_n)
\]
\[
= g_{\varphi^n(m)}(d\varphi^n(m)[p_* \tilde{\pi}_\nu X_n], d\varphi^n(m)[p_* \tilde{\pi}_\nu X_n])
\]
\[
\leq \max_{m \in M} |d\varphi^n(m)|_g^2 \cdot g(p_* \tilde{\pi}_\nu X_n, p_* \tilde{\pi}_\nu X_n)
\]
\[
= \max_{m \in M} |d\varphi^n(m)|_g^2.
\]

Hence we conclude that $\max_{m \in M} |d\varphi^n(m)|_g$ has at least linear growth.

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