F. Wiener’s Trick and an Extremal Problem for $H^p$

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Abstract
For $0 < p \leq \infty$, let $H^p$ denote the classical Hardy space of the unit disc. We consider the extremal problem of maximizing the modulus of the $k$th Taylor coefficient of a function $f \in H^p$ which satisfies $\|f\|_{H^p} \leq 1$ and $f(0) = t$ for some $0 \leq t \leq 1$. In particular, we provide a complete solution to this problem for $k = 1$ and $0 < p < 1$. We also study F. Wiener’s trick, which plays a crucial role in various coefficient-related extremal problems for Hardy spaces.

Keywords  Hardy spaces · Extremal problems · Coefficient estimates

Mathematics Subject Classification Primary 30H10; Secondary 42A05

1 Introduction

Let $H^p$ denote the classical Hardy space of analytic functions in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Suppose that $k$ is a positive integer. For $0 < p \leq \infty$ and $0 \leq t \leq 1$, consider the extremal problem
\[
\Phi_k(p, t) = \sup \left\{ \text{Re} \frac{f^{(k)}(0)}{k!} : \|f\|_{H^p} \leq 1 \text{ and } f(0) = t \right\}. \quad (1)
\]

By a standard normal families argument, there are extremals \( f \in H^p \) attaining the supremum in (1) for every \( k \geq 1 \) and every \( 0 \leq t \leq 1 \). A general framework for a class of extremal problems for \( H^p \) which includes (1) has been developed by Havinson [8], Kabaila [9], Macintyre and Rogosinski [11] and Rogosinski and Shapiro [14]. A particular consequence of this theory is that the structure of the extremals is well-known (see Lemma 4 below).

For our extremal problem, it can be deduced directly from Parseval’s identity that \( \Phi_1(2, t) = \sqrt{1 - t^2} \) and that the unique extremal is \( f(z) = t + \sqrt{1 - t^2} z^k \). Similarly, the Schwarz–Pick inequality (see e.g. [15, VII.17.3]) shows that \( \Phi_1(\infty, t) = 1 - t^2 \) and that the unique extremal is \( f(z) = (t + z)/(1 + tz) \). This served as the starting point for Beneteau and Korenblum [1], who studied the extremal problem (1) in the range \( 1 \leq p \leq \infty \). We will enunciate their results in Sects. 4 and 5, but for now we present a brief account of their approach.

The first step in [1] is to compute \( \Phi_1(p, t) \) and identify an extremal function. This is achieved by interpolating between the two cases \( p = 2 \) and \( p = \infty \) mentioned above, facilitated by the inner-outer factorization of \( H^p \) functions. It follows from the argument that the extremal function thusly obtained is unique.

The second step in [1] is to show that \( \Phi_k(p, t) = \Phi_1(p, t) \) for every \( k \geq 2 \) using a trick attributed to F. Wiener [2], which we shall now recall. Set \( \omega_k = \exp(2\pi i/k) \) and suppose that \( f(z) = \sum_{n \geq 0} a_n z^n \). F. Wiener’s trick is based on the transform

\[
W_k f(z) = \frac{1}{k} \sum_{j=0}^{k-1} f(\omega_k^j z) = \sum_{n=0}^{\infty} a_{kn} z^{kn}. \quad (2)
\]

The triangle inequality yields that \( \|W_k f\|_{H^p} \leq \|f\|_{H^p} \) for \( f \in H^p \) if \( 1 \leq p \leq \infty \). Hence, if \( f_1 \) is an extremal function for \( \Phi_1(p, t) \), then \( f_k(z) = f_1(z^k) \) is an extremal function for \( \Phi_k(p, t) \) and consequently \( \Phi_k(p, t) = \Phi_1(p, t) \). Note that this argument does not guarantee that the extremal \( f_k \) is unique for \( \Phi_k(p, t) \).

We are interested in the extremal problem (1) for \( 0 < p < 1 \) and whether the extremal identified using F. Wiener’s trick above for \( 1 \leq p \leq \infty \) is unique. We shall obtain the following general result, which may be of independent interest.

**Theorem 1** Fix \( k \geq 2 \) and suppose that \( 0 < p \leq \infty \). Let \( W_k \) denote the F. Wiener transform (2). The inequality

\[
\|W_k f\|_{H^p} \leq \max \left( k^{1/p - 1}, 1 \right) \|f\|_{H^p}
\]

is sharp. Moreover, equality is attained if and only if

(a) \( f \equiv 0 \) when \( 0 < p < 1 \),
(b) \( W_k f = f \) when \( 1 < p < \infty \).
The upper bound in the estimate is easily deduced from the triangle inequality. Hence, the novelty of Theorem 1 is that the inequality is sharp for $0 < p < 1$, and the statements (a) and (b). In Sect. 3, we also present examples of functions in $H^1$ and $H^\infty$ which attain equality in Theorem 1, but for which $W_k f \neq f$. However, we will conversely establish that if both $f$ and $W_k f$ are inner functions, then $f = W_k f$.

To illustrate the role played by the F. Wiener transform in various coefficient related extremal problems, we first recall that the estimate $\|W_k f\|_\infty \leq \|f\|_\infty$ was originally used by F. Wiener to resolve a problem posed by Bohr [2] and compute the so-called Bohr radius for $H^\infty$. We also know from [12, Sect. 1.7] that the Krzyż conjecture on the maximal magnitude of the $k$th coefficient in the power series expansion of a non-vanishing function with $\|f\|_\infty = 1$ is equivalent to the assertion that if $f$ is an extremal for the corresponding extremal problem, then $f = W_k f$. As far as we are aware, the Krzyż conjecture remains open for $k \geq 6$.

Theorem 1 shows that the extremal for $\Phi_k(p, t)$ is unique when $1 < p < \infty$. We shall see in Sect. 5 that the extremal problem $\Phi_k(p, t)$ with $k \geq 2$ and $1 \leq p \leq \infty$ has a unique extremal except for when $p = 1$ and $0 \leq t < 1/2$.

In the range $0 < p < 1$ with $k = 1$, the extremal problem (1) has been studied by Connelly [4, Sect. 4], who resolved the problem in the cases $0 \leq t < 2^{-1/p}$ and $2^{-1/p} \sqrt{p(2 - p)}^{1/p - 1/2} < t \leq 1$. Connelly also states conjectures on the behavior of $\Phi_1(p, t)$ in the range $2^{-1/p} \leq t \leq 2^{-1/p} \sqrt{p(2 - p)}^{1/p - 1/2}$. The conjectures are based on numerical analysis (see [4, Sect. 5]).

In Sect. 4, we will extend Connelly’s result to the full range $0 \leq t \leq 1$. Our result demonstrates that for each $0 < p < 1$ there is a unique $0 < t_p < 1/2$ such that the extremal for $\Phi_1(p, t_p)$ is not unique, thereby confirming the above-mentioned conjectures.

Brevig and Saksman [3] have recently studied the extremal problem

$$\Psi_k(p) = \sup \left\{ \text{Re} \frac{f^{(k)}(0)}{k!} : \|f\|_{H^p} \leq 1 \right\}$$

for $0 < p < 1$. It is observed in [3, Sect. 5.3] that $\Psi_k(p) = \max_{0 \leq t \leq 1} \Phi_k(p, t)$. In particular, the maxima of $\Phi_1(p, t)$ for $0 \leq t \leq 1$ is

$$\Psi_1(p) = \left( 1 - \frac{p}{2} \right)^{1/p} \frac{2}{\sqrt{p(2 - p)}}$$

and this is attained for $t = (1 - p/2)^{1/p}$. From the main result in [1], it is easy to see that $t \mapsto \Phi_1(p, t)$ is a decreasing function from $\Phi_1(p, 0) = 1$ to $\Phi_1(p, 1) = 0$ when $1 \leq p \leq \infty$. Similarly, our main result shows that $\Phi_1(p, t)$ is increasing from $\Phi_1(p, 0) = 1$ to the maxima mentioned above, then decreasing to $\Phi_1(p, 1) = 0$. Figure 1 contains the plot of $t \mapsto \Phi_1(p, t)$ for several values $0 < p \leq \infty$, which illustrates this difference between $0 < p < 1$ and $1 \leq p \leq \infty$. 
Another difference between $0 < p < 1$ and $1 ≤ p ≤ ∞$ appears when we consider $k ≥ 2$. Recall that in the latter case, we have $Φ_k(p, t) = Φ_1(p, t)$ for every $k ≥ 2$ and every $0 ≤ t ≤ 1$. In the former case, we only get from Theorem 1 that

$$Φ_1(p, t) ≤ Φ_k(p, t) ≤ k^{1/p - 1} Φ_1(p, t).$$  \hfill (3)

Theorem 1 also shows that the upper bound in (3) is attained if and only if $t = 1$, since trivially $Φ_1(p, 1) = 0$ for every $0 < p ≤ ∞$. However, by adapting an example due to Hardy and Littlewood [7], it is easy to see that if $0 < p < 1$ and $0 ≤ t < 1$ are fixed, then the exponent $1/p - 1$ in (3) cannot be improved as $k → ∞$. In the final section of the paper, we present some evidence that the lower bound in (3) can be attained for sufficiently large $t$, if $k ≥ 2$ and $0 < p < 1$ are fixed.

**Organization**

The present paper is organized into five additional sections and one appendix. In Sect. 2, we collect some preliminary results pertaining to $H^p$ and the structure of extremals for $Φ_k(p, t)$. Section 3 is devoted to F. Wiener’s trick and the proof of Theorem 1. A complete solution to the extremal problem $Φ_1(p, t)$ for $0 < p ≤ ∞$ and $0 ≤ t ≤ 1$ is presented in Sect. 4. In Sect. 5, we consider $Φ_k(p, t)$ for $k ≥ 2$ and $1 ≤ p ≤ ∞$ and study when the extremal is unique. Section 6 contains some remarks on $Φ_k(p, t)$ for $k ≥ 2$ and $0 < p < 1$. “Appendix A” contains the proof of a crucial lemma needed to resolve the extremal problem $Φ_1(p, t)$ for $0 < p < 1$. 

![Fig. 1 Plot of the curves $t \mapsto Φ_1(p, t)$ for $p = 1/2$, $p = 1$, $p = 2$ and $p = ∞$](image_url)
2 Preliminaries

Recall that for $0 < p < \infty$, the Hardy space $H^p$ consists of the analytic functions $f$ in $\mathbb{D}$ for which the limit of integral means

$$
\|f\|_{H^p}^p = \lim_{r \to 1^-} \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi}
$$

is finite. $H^\infty$ is the space of bounded analytic functions in $\mathbb{D}$, endowed with the norm $\|f\|_{H^\infty} = \sup_{|z| < 1} |f(z)|$. It is well-known (see e.g. [6]) that $H^p$ is a Banach space when $1 \leq p \leq \infty$ and a quasi-Banach space when $0 < p < 1$.

In the Banach space range $1 \leq p \leq \infty$, the triangle equality is

$$
\|f + g\|_{H^p} \leq \|f\|_{H^p} + \|g\|_{H^p}.
$$

(4)

The Hardy space $H^p$ is strictly convex when $1 < p < \infty$, which means that it is impossible to attain equality in (4) unless $g \equiv 0$ or $f = \lambda g$ for a non-negative constant $\lambda$. $H^p$ is not strictly convex for $p = 1$ and $p = \infty$, so in this case there are other ways to attain equality in (4). In the range $0 < p < 1$, the triangle inequality takes the form

$$
\|f + g\|_{H^p}^p \leq \|f\|_{H^p}^p + \|g\|_{H^p}^p,
$$

(5)

so here $H^p$ is not even locally convex [5]. Our first goal is to establish that the triangle inequality (5) is not attained unless $f \equiv 0$ or $g \equiv 0$. This result is probably known to experts, but we have not found it in the literature.

If $f \in H^p$ for some $0 < p \leq \infty$, then the boundary limit function

$$
f^*(e^{i\theta}) = \lim_{r \to 1^-} f(re^{i\theta})
$$

(6)

exists for almost every $\theta$. Moreover, $f^* \in L^p = L^p([0, 2\pi])$ and

$$
\|f\|_{H^p} = \|f^*\|_{L^p} = \left(\int_0^{2\pi} \left|f^*(e^{i\theta})\right|^p \frac{d\theta}{2\pi}\right)^{1/p}
$$

if $0 < p < \infty$ and $\|f\|_{H^\infty} = \text{ess sup}_\theta |f^*(e^{i\theta})|$. For simplicity, we henceforth omit the asterisk and write $f^* = f$ with the limit (6) in mind.

**Lemma 2** Fix $0 < p < 1$ and suppose that $f, g \in H^p$. If

$$
\|f + g\|_{H^p}^p = \|f\|_{H^p}^p + \|g\|_{H^p}^p
$$

then either $f \equiv 0$ or $g \equiv 0$. 
Proof We begin by looking at equality in the triangle inequality for $L^p$ in the range $0 < p < 1$. Here we have

$$\|f + g\|_{L^p}^p = \int_0^{2\pi} \left| f(e^{i\theta}) + g(e^{i\theta}) \right| \frac{d\theta}{2\pi} \leq \int_0^{2\pi} |f(e^{i\theta})|^p + |g(e^{i\theta})|^p \frac{d\theta}{2\pi} = \|f\|_{L^p}^p + \|g\|_{L^p}^p.$$ 

We used the elementary estimate $|z + w|^p \leq |z|^p + |w|^p$ for complex numbers $z, w$ and $0 < p < 1$. It is easily verified that this estimate is attained if and only if $zw = 0$. Consequently,

$$\|f + g\|_{L^p}^p = \|f\|_{L^p}^p + \|g\|_{L^p}^p$$

if and only if $f(e^{i\theta})g(e^{i\theta}) = 0$ for almost every $\theta$. It is well-known (see [6, Thm. 2.2]) that the only function $h \in H^p$ whose boundary limit function (6) vanishes on a set of positive measure is $h \equiv 0$. Hence we conclude that either $f \equiv 0$ or $g \equiv 0$. □

Let us next establish a standard result on the structure of the extremals for the extremal problem (1). The first step is the following basic result.

Lemma 3 If $f \in H^p$ is extremal for $\Phi_k(p, t)$, then $\|f\|_{H^p} = 1$.

Proof Suppose that $f \in H^p$ is extremal for $\Phi_k(p, t)$ but that $\|f\|_{H^p} < 1$. For $\varepsilon > 0$, set $g(z) = f(z) + \varepsilon z^k$. Note that $g(0) = f(0) = t$ for any $\varepsilon > 0$. If $1 \leq p \leq \infty$, then

$$\|g\|_{H^p} \leq \|f\|_{H^p} + \varepsilon < 1$$

for sufficiently small $\varepsilon > 0$. If $0 < p < 1$, then

$$\|g\|_{H^p}^p \leq \|f\|_{H^p}^p + \varepsilon^p < 1,$$

again for sufficiently small $\varepsilon > 0$, so $\|g\|_{H^p} < 1$. In both cases we find that

$$\frac{g^{(k)}(0)}{k!} = \frac{f^{(k)}(0)}{k!} + \varepsilon,$$

which contradicts the extremality of $f$ for $\Phi_k(p, t)$. □

Let $(n_j)_j$ denote a sequence of distinct non-negative integers and let $(w_j)_j$ denote a sequence of complex numbers. A special case of the Carathéodory–Fejér problem is to determine the infimum of $\|f\|_{H^p}$ over all $f \in H^p$ which satisfy

$$\frac{f^{(n_j)}}{n_j!}(0) = w_j,$$  \hspace{1cm} (7)
for \( j = 1, \ldots, k \). Set \( k = \max_{1 \leq j \leq k} n_j \). If \( f \) is an extremal for the Carathéodory–Fejér problem (7), then there are complex numbers \(|\lambda_j| \leq 1\) for \( j = 1, \ldots, k \) and a constant \( C \) such that

\[
    f(z) = C \prod_{j=1}^{l} \frac{\lambda_j - z}{1 - \lambda_j z} \prod_{j=1}^{k} (1 - \overline{\lambda_j} z)^{2/p}
\]

for some \( 0 \leq l \leq k \), and the strict inequality \(|\lambda_j| < 1\) holds for \( 0 < j \leq l \). In (8) and in similar formulas to follow, we adopt the convention that in the case \( l = 0 \) the first product is empty and considered to be equal to 1.

For \( 1 \leq p \leq \infty \), this result is independently due to Macintyre and Rogosinski [11] and Havinson [8], while in the range \( 0 < p < 1 \) the result is due to Kabaila [9]. An exposition of these results can be found in [6, Ch. 8] and [10, pp. 82–85], respectively.

Using Lemma 3, we can establish that the extremals of the extremal problem \( \Phi_k(p, t) \) have to be of the same form.

Lemma 4 If \( f \in H^p \) is extremal for \( \Phi_k(p, t) \), then there are complex numbers \(|\lambda_j| \leq 1\) for \( j = 1, \ldots, k \) and a constant \( C \) such that

\[
    f(z) = C \prod_{j=1}^{l} \frac{\lambda_j - z}{1 - \lambda_j z} \prod_{j=1}^{k} (1 - \overline{\lambda_j} z)^{2/p}
\]

for some \( 0 \leq l \leq k \), and the strict inequality \(|\lambda_j| < 1\) holds for \( 0 < j \leq l \).

Proof Suppose that \( f \) is extremal for \( \Phi_k(p, t) \) and consider the Carathéodory–Fejér problem with conditions

\[
    f(0) = t \quad \text{and} \quad \frac{f^{(k)}(0)}{k!} = \Phi_k(p, t).
\]

We claim that \( f \) is an extremal for the Carathéodory–Fejér problem (9). If it is not, then there must be some \( f \in H^p \) with \( \|f\|_{H^p} < 1 \) which satisfies (9). However, this contradicts Lemma 3. Hence the extremal is of the stated form by (8).

\[\square\]

3 F. Wiener’s Trick

Recall from (2) that if \( f(z) = \sum_{n \geq 0} a_n z^n \) and \( \omega_k = \exp(2\pi i/k) \), then

\[
    W_k f(z) = \frac{1}{k} \sum_{j=0}^{k-1} f(\omega_k^j z) = \sum_{n=0}^{\infty} a_k z^{kn}.
\]

We begin by giving two examples showing that \( \|W_k f\|_{H^p} = \|f\|_{H^p} \) may occur for \( f \) such that \( W_k f \neq f \) when \( p = 1 \) or \( p = \infty \).
Example 5 Let $k \geq 2$ and consider $f(z) = (1+z)^{2k}$ in $H^1$. By the binomial theorem, we find that

$$f(z) = \sum_{n=0}^{2k} \binom{2k}{n} z^n,$$

$$W_k f(z) = 1 + \binom{2k}{k} z^k + z^{2k}.$$

Note that $f \neq W_k f$ since $k \geq 2$. By another application of the binomial theorem and a well-known identity for the central binomial coefficient, we find that

$$\|f\|_{H^1} = \|f^{1/2}\|^2_{H^2} = k \sum_{n=0}^{k} \binom{k}{n}^2 = \binom{2k}{k}.$$

Moreover,

$$\binom{2k}{k} = \int_0^{2\pi} W_k f(e^{i\theta}) \frac{e^{ik\theta}}{2\pi} d\theta \leq \|W_k f\|_{H^1}$$

by the triangle inequality. Hence

$$\binom{2k}{k} \leq \|W_k f\|_{H^1} \leq \|f\|_{H^1} = \binom{2k}{k},$$

so $\|W_k f\|_{H^1} = \|f\|_{H^1}$.

Example 6 Let $k \geq 2$ and consider $f(z) = (1+z^k)^2 - z(1-z^k)^2$ in $H^\infty$. It is clear that $W_k f(z) = (1+z^k)^2 \neq f(z)$ since $k \geq 2$. Moreover $\|W_k f\|_{H^\infty} = 4$. The supremum is attained for $z = \omega_j^k$ for $j = 0, 1, \ldots, k-1$. We next compute

$$f(e^{i\theta}) = \left(1 + e^{ik\theta}\right)^2 - e^{i\theta} \left(1 - e^{ik\theta}\right)^2 = 4e^{ik\theta} \left(\cos^2\left(\frac{k\theta}{2}\right) + e^{i\theta} \sin^2\left(\frac{k\theta}{2}\right)\right).$$

Consequently, $\|f\|_{H^\infty} = 4$ and here the supremum is attained for $z = \omega_j^{2k}$ for $j = 0, 1, \ldots, 2k-1$.

Proof of Theorem 1 It follows from the triangle inequality (4) that

$$\|W_k f\|_{H^p} \leq \|f\|_{H^p} \tag{10}$$

for every $f \in H^p$ if $1 \leq p \leq \infty$. In the range $0 < p < 1$, we get from the triangle inequality (5) the estimate

$$\|W_k f\|_{H^p} \leq k^{1/p-1} \|f\|_{H^p} \tag{11}$$
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for every $f \in H^p$. Combining (10) and (11), we have established that

$$\|W_k f\|_{H^p} \leq \max \left( k^{1/p-1}, 1 \right) \|f\|_{H^p}. $$

This is trivially attained for $f(z) = z^k$ when $1 \leq p \leq \infty$. We need to show that the upper bound $k^{1/p-1}$ cannot be improved when $0 < p < 1$ to finish proof of the first part of the theorem.

Let $\varepsilon > 0$ and consider $f_\varepsilon(z) = (z - (1 + \varepsilon))^{-1/p}$. Clearly $\|f_\varepsilon\|_{H^p} \to \infty$ as $\varepsilon \to 0^+$. Moreover

$\|f_\varepsilon\|_{H^p}^p = \int_0^{2\pi} \frac{1}{|e^{i\theta} - (1 + \varepsilon)|} \frac{d\theta}{2\pi} \leq \int_{|\theta| < \pi/k} \frac{1}{|e^{i\theta} - (1 + \varepsilon)|} \frac{d\theta}{2\pi} + \int_{|\theta| \geq \pi/k} \frac{6}{\theta^2} \frac{d\theta}{2\pi}$

from which we conclude that

$$\|f_\varepsilon\|_{H^p}^p = \int_{|\theta| < \pi/k} \frac{1}{|e^{i\theta} - (1 + \varepsilon)|} \frac{d\theta}{2\pi} + O(1). \quad \text{(12)}$$

Furthermore,

$$\|W_k f_\varepsilon\|_{H^p}^p = \sum_{j=0}^{k-1} \int_{|\theta - 2\pi j/k| < \pi/k} \left| \sum_{l=0}^{k-1} f_\varepsilon(e^{i(\theta + 2\pi l/k)}) \right|^p \frac{d\theta}{2\pi} \geq k^{-p} \sum_{j=0}^{k-1} \left( \int_{|\theta - 2\pi j/k| < \pi/k} \left| f_\varepsilon(e^{i(\theta + 2\pi j/k)}) \right|^p \frac{d\theta}{2\pi} - \frac{6k^2}{\pi^2} \right)$$

By (12) we find that

$$\lim_{\varepsilon \to 0^+} \frac{\|W_k f_\varepsilon\|_{H^p}^p}{\|f_\varepsilon\|_{H^p}^p} \geq k^{1-p}.$$ 

Hence, the constant $k^{1/p-1}$ in (11) cannot be replaced by any smaller quantity.

We next want to show that (a) and (b) holds. For a function $f \in H^p$, define $f_j(z) = f(\omega_k^j z)$ for $j = 0, 1, \ldots, k-1$ and recall that $\|f\|_{H^p} = \|f_j\|_{H^p}$.

We begin with (a). Suppose that $\|W_k f\|_{H^p} = k^{1/p-1} \|f\|_{H^p}$, which we can reformulate as

$$\|f_0 + f_1 + \cdots + f_{k-1}\|_{H^p}^p = \|f_0\|_{H^p}^p + \|f_1\|_{H^p}^p + \cdots + \|f_{k-1}\|_{H^p}^p.$$
By Lemma 2, the triangle inequality can be attained if and only if at least \( k - 1 \) of the \( k \) functions \( f_j \) are identically equal to zero. Evidently this is possible if and only if \( f \equiv 0 \).

For (b), we suppose that \( f \in H^p \) is such that \( \|W_k f\|_{H^p} = \|f\|_{H^p} \). We need to prove that \( W_k f = f \). If \( f \equiv 0 \) there is nothing to do. As in the proof of (a), we note that \( \|W_k f\|_{H^p} = \|f\|_{H^p} \) can be reformulated as

\[
\|f_0 + f_1 + \cdots + f_{k-1}\|_{H^p} = \|f_0\|_{H^p} + \|f_1\|_{H^p} + \cdots + \|f_{k-1}\|_{H^p}.
\]

Viewing \( H^p \) as a subspace of \( L^p \), the strict convexity of the latter implies that there are non-negative constants \( \lambda_j \) for \( j = 1, 2, \ldots, k - 1 \) such that

\[
f = f_0 = \lambda_1 f_1 = \ldots = \lambda_{k-1} f_{k-1}.
\]

We shall only look at \( f = \lambda_1 f_1 \) which for \( f(z) = \sum_{n \geq 0} a_n z^n \) is equivalent to

\[
\sum_{n=0}^{\infty} a_n z^n = \lambda_1 \sum_{n=0}^{\infty} a_n \omega_k^n z^n.
\]

Using \( W_k \) on this identity we get

\[
\sum_{n=0}^{\infty} a_{kn} z^{kn} = \lambda_1 \sum_{n=0}^{\infty} a_{kn} z^{kn}.
\]

This is only possible if \( \lambda_1 = 1 \) or \( W_k f \equiv 0 \). The latter implies that \( f \equiv 0 \) since \( \|W_k f\|_{H^p} = \|f\|_{H^p} \) by assumption. Therefore we can restrict our attention to the case \( \lambda_1 = 1 \). For all integers \( n \) that are not a multiple of \( k \), we now find that

\[
a_n = \lambda_1 \omega_k^n a_n \quad \implies \quad a_n = 0,
\]

since \( \lambda_1 = 1 \) and \( \omega_k^n \neq 1 \). Hence \( W_k f = f \) as desired. \( \square \)

Recall that a function \( f \in H^p \) is called inner if \( |f(e^{i\theta})| = 1 \) for almost every \( \theta \). We shall require the following simple result later on.

**Lemma 7** If both \( f \) and \( W_k f \) are inner functions, then \( f = W_k f \).

**Proof** Since \( |W_k f(e^{i\theta})| = |f(e^{i\theta})| = 1 \) for almost every \( \theta \), we get from (2) that

\[
1 = \left| W_k f(e^{i\theta}) \right| = \left| \frac{1}{k} \sum_{j=0}^{k-1} f_j(e^{i\theta}) \right| = \frac{1}{k} \sum_{j=0}^{k-1} |f_j(e^{i\theta})|,
\]

where \( f_j(z) = f(\omega_k^j z) \). The equality on the right hand side of (13) is possible if and only if

\[
f(e^{i\theta}) = f_1(e^{i\theta}) = \ldots = f_{k-1}(e^{i\theta}) = 0.
\]
for almost every $\theta$. As in the proof of Theorem 1 (b), we find that $f = W_k f$. □

4 The Extremal Problem $\Phi_1(p, t)$ for $0 < p \leq \infty$

In the present section, we resolve the extremal problem (1) in the case $k = 1$ completely. We begin with the case $1 \leq p \leq \infty$ which has been solved by Beneteau and Korenblum [1]. We give a different proof of their result based on Lemma 4, mainly to illustrate the differences between the cases $0 < p < 1$ and $1 \leq p \leq \infty$.

**Theorem 8** (Beneteau–Korenblum) Fix $1 \leq p \leq \infty$ and consider (1) with $k = 1$.

(i) If $0 \leq t < 2^{-1/p}$, let $\alpha$ denote the unique real number in the interval $0 \leq \alpha < 1$ such that $t = \alpha(1 + \alpha^2)^{-1/p}$. Then

$$\Phi_1(p, t) = \frac{1}{(1 + \alpha^2)^{1/p}} \left( 1 + \left( \frac{2}{p} - 1 \right) \alpha^2 \right),$$

and the unique extremal is

$$f(z) = \frac{\alpha + z}{1 + \alpha z} \frac{(1 + \alpha z)^{2/p}}{(1 + \alpha^2)^{1/p}},$$

(ii) If $2^{-1/p} \leq t \leq 1$, let $\beta$ denote the unique real number in the interval $0 \leq \beta \leq 1$ such that $t = (1 + \beta^2)^{-1/p}$. Then

$$\Phi_1(p, t) = \frac{1}{(1 + \beta^2)^{1/p}} \frac{2\beta}{p},$$

and the unique extremal is

$$f(z) = \frac{(1 + \beta z)^{2/p}}{(1 + \beta^2)^{1/p}}.$$

**Proof** Note that since $k = 1$, there are only two possibilities for the extremals in Lemma 4. They are

$$f_1(z) = \frac{\alpha + z}{1 + \alpha z} \frac{(1 + \alpha z)^{2/p}}{(1 + \alpha^2)^{1/p}}, \quad 0 \leq \alpha < 1, \quad (14)$$

$$f_2(z) = \frac{(1 + \beta z)^{2/p}}{(1 + \beta^2)^{1/p}}, \quad 0 \leq \beta \leq 1. \quad (15)$$
Here we have made $\alpha, \beta \geq 0$ by rotations. Note that if $p = \infty$, then $f_2$ does not depend on $\beta$. Moreover,

$$t = f_1(0) = \frac{\alpha}{(1 + \alpha^2)^{1/p}},$$  \hfill (16)

$$t = f_2(0) = \frac{1}{(1 + \beta^2)^{1/p}}.$$  \hfill (17)

For $1 \leq p \leq \infty$ it is easy to verify that the function

$$\alpha \mapsto \frac{\alpha}{(1 + \alpha^2)^{1/p}}$$

is strictly increasing on $0 \leq \alpha < 1$ and maps $[0, 1)$ to $[0, 2^{-1/p})$. Similarly, for $1 \leq p < \infty$ we find that the function

$$\beta \mapsto \frac{1}{(1 + \beta^2)^{1/p}}$$

is strictly decreasing on $0 \leq \beta \leq 1$ and maps $[0, 1)$ to $[2^{-1/p}, 1]$. Consequently, if $0 \leq t < 2^{-1/p}$, then the unique extremal is (14) with $\alpha$ given by (16), and if $2^{-1/p} \leq t \leq 1$, then the unique extremal is (15) with $\beta$ given by (17). The proof is completed by computing

$$f'_1(0) = \frac{1}{(1 + \alpha^2)^{1/p}} \left(1 + \alpha^2 \left(\frac{2}{p} - 1\right)\right) = t \left(\frac{1}{\alpha} + \alpha \left(\frac{2}{p} - 1\right)\right),$$

$$f'_2(0) = \frac{1}{(1 + \beta^2)^{1/p}} \frac{2\beta}{p} = \frac{2\beta}{p},$$

to obtain the stated expressions for $\Phi_1(p, t)$ in (i) and (ii), respectively.

Define $\alpha$ and $\beta$ as functions of $t$ implicitly through (16) and (17). Then $\alpha$ is increasing on $0 \leq t < 2^{-1/p}$ and $\beta$ is decreasing on $2^{-1/p} \leq t \leq 1$. Inspecting the left hand side of (20) and (21), we extract the following result.

**Corollary 9**  If $1 \leq p \leq \infty$, then the function $t \mapsto \Phi_1(p, t)$ is decreasing and takes the values $[0, 1]$.

In the range $0 < p < 1$ a more careful analysis is required. This is due to the fact that the function (18) is increasing on the interval $0 \leq \alpha \leq \alpha_2$ and decreasing on the interval $\alpha_2 \leq \alpha < 1$, where

$$\alpha_2 = \sqrt{\frac{p}{2 - p}}.$$  \hfill (22)

Inspecting (16), we conclude that for each $2^{-1/p} < t < 2^{-1/p} \sqrt{p(2 - p)^{1/p - 1/2}}$ there are two possible $\alpha$-values which give the same $t = f_1(0)$. Let $\alpha_1$ denote the
unique real number in the interval $(0, 1)$ such that

$$1 + \alpha_1^2 = 2\alpha^p. \quad (23)$$

Note that $\alpha_1$ gives the value $t = 2^{-1/p}$ in (16).

**Lemma 10** If $\alpha_1 < \alpha < \alpha_2$ and $\alpha_2 < \tilde{\alpha} < 1$ produce the same $t = f_1(0)$ in (16), then the quantity $f_1'(0)$ from (20) is maximized by $\alpha$.

**Proof** Since $\alpha$ and $\tilde{\alpha}$ give the same $t = f_1(0)$ in (20), we only need to prove that

$$\frac{1}{\alpha} + \frac{\alpha}{\alpha_2} > \frac{1}{\tilde{\alpha}} + \frac{\tilde{\alpha}}{\alpha_2}. \quad (24)$$

Fix $\alpha_1 < \alpha < \alpha_2$. The unique number $\alpha_2 < \xi < 1$ such that

$$\frac{1}{\alpha} + \frac{\alpha}{\alpha_2^2} = \frac{1}{\xi} + \frac{\xi}{\alpha_2^2}$$

is $\xi = \alpha_2^2/\alpha$. Since the function

$$x \mapsto \frac{1}{x} + \frac{x}{\alpha_2^2}$$

is increasing for $x > \alpha_2$ it is sufficient to prove that $\xi > \tilde{\alpha}$ to obtain (24). Since

$$x \mapsto \frac{x}{(1 + x^2)^{1/p}}$$

is decreasing for $x > \alpha_2$, we see that $\xi > \tilde{\alpha}$ if and only if

$$\frac{\tilde{\alpha}}{(1 + \tilde{\alpha}^2)^{1/p}} > \frac{\xi}{(1 + \xi^2)^{1/p}} \iff \frac{\alpha}{(1 + \alpha^2)^{1/p}} > \left(1 + \left(\frac{\alpha}{\alpha_2^2}\right)^2\right)^{1/p}. \quad (25)$$

Here we used that $\alpha$ and $\tilde{\alpha}$ give the same $t = f_1(0)$ in (16) on the left hand side and the identity $\xi = \alpha_2^2/\alpha$ on the right hand side. We now substitute $\alpha = \alpha_2 \sqrt{x}$ for $0 < x < 1$ to obtain the equivalent inequality

$$\frac{x}{(1 + \alpha_2^2 x)^{1/p}} > \frac{1}{\left(1 + \frac{\alpha_2^2}{x}\right)^{1/p}}. \quad (25)$$

Actually, we only need to consider $(\alpha_1/\alpha_2)^2 < x < 1$, but the same proof works for $0 < x < 1$. We raise both sides of (25) to the power $p$, multiply by $x^{1-p}$ and rearrange.
to get the equivalent inequality $F(x) > 0$ where

$$F(x) = (x - x^{1-p}) + \alpha_2^2 \left(1 - x^{2-p}\right).$$

Recalling that $\alpha_2^2 = p/(2 - p)$, we compute

$$F'(x) = (1 - (1 - p)x^{-p}) - px^{1-p} \quad \text{and} \quad F''(x) = p(1 - p)x^{-p-1} - p(1 - p)x^{-p}. $$

Since $F(1) = F'(1) = 0$, we get from Taylor’s theorem that for every $0 < x < 1$ there is some $x < \eta < 1$ such that

$$F(x) = \frac{F''(\eta)}{2}(x - 1)^2 = \frac{p(1 - p)}{2} \eta^{-p} \left(\eta^{-1} - 1\right)(x - 1)^2 > 0,$$

which completes the proof.

By Lemma 10, we now only need to compare $f_1'(0)$ from (20) for $\alpha_1 \leq \alpha \leq \alpha_2$ with $f'_2(0)$ from (21) for $\beta$ such that $f_1(0) = t = f_2(0)$. Inspecting (16) and (17), we find that

$$\frac{\alpha}{(1 + \alpha^2)^{1/p}} = \frac{1}{(1 + \beta^2)^{1/p}} \iff \beta = \sqrt{\frac{1 + \alpha^2}{\alpha^p} - 1}. \quad (26)$$

Next, we consider the equation $f'_1(0) = f'_2(0)$ with $\beta$ as in (26). Inspecting (20) and (21) and dividing by $t$, we get the equation

$$\frac{1}{\alpha} + \alpha \left(\frac{2}{p} - 1\right) = \frac{2\beta}{p} = \frac{2}{p} \sqrt{\frac{1 + \alpha^2}{\alpha^p} - 1}. \quad (27)$$

We square both sides, multiply by $p^2$ and rearrange to find that (27) is equivalent to the equation $F_p(\alpha) = 0$, where

$$F_p(\alpha) = p^2\alpha^{-2} + 2p(2 - p) + (2 - p)^2\alpha^2 - 4\left(\alpha^{-p} + \alpha^{2-p} - 1\right). \quad (28)$$

Suppose that $\alpha_1 \leq \alpha \leq \alpha_2$. If

- $F_p(\alpha) > 0$, then $f_1$ from (14) is the unique extremal for $\Phi_1(p, t)$.
- $F_p(\alpha) = 0$, then $f_1$ from (14) and $f_2$ from (15) are extremals for $\Phi_1(p, t)$.
- $F_p(\alpha) < 0$, then $f_2$ from (15) is the unique extremal for $\Phi_1(p, t)$.

Note that any solutions of $F_p(\alpha) = 0$ with $0 < \alpha < \alpha_1$ are of no interest since this implies that $\beta > 1$ by (26). Similarly, any solutions of $F_p(\alpha) = 0$ with $\alpha_2 < \alpha < 1$ can be ignored due to Lemma 10. The following result shows that there is only one solution, which is in the pertinent range.
Lemma 11 Let $F_p$ be as in (28). The equation $F_p(\alpha) = 0$ has a unique solution, denoted $\alpha_p$, on the interval $(0, 1)$. Moreover,

(a) if $0 < \alpha < \alpha_p$, then $F_p(\alpha) > 0$.
(b) if $\alpha_p < \alpha < 1$, then $F_p(\alpha) < 0$.
(c) $\alpha_1 < \alpha_p < \alpha_2$ where $\alpha_1$ and $\alpha_2$ are from (23) and (22), respectively.

The proof of Lemma 11 is a rather laborious calculus exercise, which we postpone to “Appendix A” below. Let $\alpha_p$ be as in Lemma 11 and define

$$t_p = \frac{\alpha_p}{\left(1 + \alpha_p^2\right)^{1/p}}.$$  

Note that $2^{-1/p} < t_p < 2^{-1/p} \sqrt{p(2 - p)^{1/p} - 1/2}$ by the fact that $\alpha_1 < \alpha_p < \alpha_2$.

By the analysis above, Lemma 10 and Lemma 11, we obtain the following version of Theorem 8 in the range $0 < p < 1$.

Theorem 12 Fix $0 < p < 1$ and consider (1) with $k = 1$. Let $t_p$ be as in (29) and set $\alpha_2 = \sqrt{p/(2 - p)}$.

(i) If $0 \leq t \leq t_p$, let $\alpha$ denote the unique real number in the interval $0 \leq \alpha < \alpha_2$ such that $t = \alpha(1 + \alpha^2)^{-1/p}$. Then

$$\Phi_1(p, t) = \frac{1}{\left(1 + \alpha^2\right)^{1/p}} \left(1 + \left(\frac{2}{p} - 1\right) \alpha^2\right),$$

and an extremal is

$$f(z) = \frac{\alpha + z}{1 + \alpha z \left(1 + \alpha^2\right)^{1/p}}.$$  

(ii) If $t_p \leq t \leq 1$, let $\beta$ denote the unique real number in the interval $0 \leq \beta \leq 1$ such that $t = (1 + \beta^2)^{-1/p}$. Then

$$\Phi_1(p, t) = \frac{1}{\left(1 + \beta^2\right)^{1/p}} \frac{2\beta}{p},$$

and an extremal is

$$f(z) = \frac{(1 + \beta z)^{2/p}}{(1 + \beta^2)^{1/p}}.$$  

The extremals are unique for $0 \leq t \neq t_p \leq 1$. The only extremals for $\Phi_1(p, t_p)$ are the functions given in (i) and (ii).
Theorem 12 extends [4, Theorem 4.1] to general $0 \leq t \leq 1$. The analysis in [4] is similar to ours, and we are able to also identify the extremals in the range

$$2^{-1/p} \leq t \leq 2^{-1/p} \sqrt{p}(2 - p)^{1/p - 1/2}$$

due to Lemma 10 and Lemma 11. It is also demonstrated in [4, Thm. 4.1] that when $p = 1/2$ there must exist at least one value of $0 < t < 1$ for which the extremal is not unique. Theorem 12 shows that there is precisely one such $t$ and that this observation is not specific to $p = 1/2$, but in fact holds for any $0 < p < 1$. Figure 2 shows the value $t_p$ for which the extremal is not unique as a function of $p$.

Inspecting Theorem 12, we get the following result similarly to how we extracted Corollary 9 from Theorem 8.

**Corollary 13** If $0 < p < 1$, then the function $t \mapsto \Phi_1(p, t)$ is increasing from $\Phi_1(p, 0) = 1$ to

$$\Phi_1 \left( p, \left(1 - \frac{p}{2}\right)^{1/p} \right) = \left(1 - \frac{p}{2}\right)^{1/p} \frac{2}{\sqrt{p(2 - p)}}$$

and then decreasing to $\Phi_1(p, 1) = 0$. 

Fig. 2  Plot of the curve $p \mapsto t_p$. Points $(p, t)$ above and below the curve correspond to the cases (i) and (ii) of Theorem 12, respectively. The estimates $2^{-1/p} < t_p < 2^{-1/p} \sqrt{p}(2 - p)^{1/p - 1/2}$ are represented by dotted curves. In the shaded area and in the range $1/2 \leq t \leq 1$, Theorem 12 is originally due to Connelly [4].
5 The Extremal Problem $\Phi_k(p, t)$ for $k \geq 2$ and $1 \leq p \leq \infty$

We begin by recalling how F. Wiener’s trick was used in [1] to obtain the solution to the extremal problem $\Phi_k(p, t)$ for $k \geq 2$ from Theorem 8.

**Theorem 14** (Benetau–Korenblum) Let $k \geq 2$ be an integer. For every $1 \leq p \leq \infty$ and every $0 \leq t \leq 1$,

$$\Phi_k(p, t) = \Phi_1(p, t).$$

If $f_1$ is the extremal function for $\Phi_1(p, t)$, then $f_k(z) = f_1(z^k)$ is an extremal function for $\Phi_k(p, t)$.

**Proof** Suppose that $f$ is an extremal for $\Phi_k(p, t)$. Since $\|W_k f\|_{H^p} \leq \|f\|_{H^p}$,

$$f(0) = W_k f(0) \quad \text{and} \quad \frac{f^{(k)}(0)}{k!} = \frac{(W_k f)^{(k)}(0)}{k!},$$

we conclude that $W_k f$ is also an extremal for $\Phi_k(p, t)$. Thus we may restrict our attention to extremals $\tilde{f}_k$ of the form $\tilde{f}_k(z) = \tilde{f}(z^k)$ for $\tilde{f} \in H^p$. The stated claims now follow at once from Theorem 8, since $\|\tilde{f}_k\|_{H^p} = \|\tilde{f}\|_{H^p}$. \qed

The purpose of the present section is to answer the following question. For which trios $k \geq 2$, $1 \leq p \leq \infty$ and $0 \leq t \leq 1$ is the extremal for $\Phi_k(p, t)$ unique? Note that while Theorem 14 provides an extremal $f_k(z) = f_1(z^k)$ where $f_1$ denotes the extremal from (the statement of) Theorem 8, it might not be unique.

In the case $1 < p < \infty$ it follows at once from Theorem 1 (b) that this extremal is unique, although it is perhaps easier to use the strict convexity of $H^p$ and Lemma 3 directly. Since $H^p$ is not strictly convex for $p = 1$ and $p = \infty$, these cases require further analysis. Note that the case (a) below is certainly known to experts as a consequence of the general theory developed in [8, 11, 14].

**Theorem 15** Consider the extremal problem (1) for $k \geq 2$ and $1 \leq p \leq \infty$.

(a) If $1 < p \leq \infty$, then the unique extremal is $f_k(z) = f_1(z^k)$.

(b) If $p = 1$ and $1/2 \leq t \leq 1$, then the unique extremal is $f_k(z) = f_1(z^k)$.

(c) If $p = 1$ and $0 \leq t < 1/2$, then the extremals are the functions of the form

$$f(z) = C \prod_{j=1}^{k} \left( \lambda_j - z \right) \left( 1 - \overline{\lambda_j z} \right)$$

with $|\lambda_j| \leq 1$ such that $\|f\|_{H^1} = 1$, $f(0) = t$ and $f^{(k)}(0) > 0$.

**Proof of Theorem 15(a)** In view of the discussion above, we need only consider the case $p = \infty$. By Lemma 4, we know that any extremal must be of the form

$$f(z) = e^{i\theta} \prod_{j=1}^{l} \frac{\lambda_j - z}{1 - \lambda_j z}$$

(30)
for some $0 \leq l \leq k$, constants $\lambda_j \in \mathbb{D}$ and $\theta \in \mathbb{R}$. If $f$ is extremal for $\Phi_k(\infty, t)$, then so is $W_k f$ by Theorem 14. Consequently, $W_k f$ is also of the form (30). In particular, since both $f$ and $W_k f$ are inner, we get from Lemma 7 that $f = W_k f$. From the definition of $W_k$, we know that $f(z) = W_k f(z) = g(z^k)$ for some analytic function $g$. This shows that the only possibility in (30) is

$$f(z) = e^{i\theta} \frac{\lambda - z^k}{1 - \lambda z^k}$$

for some $\lambda \in \mathbb{D}$ and $\theta \in \mathbb{R}$. The unique extremal has $\theta = \pi$ and $\lambda = -t$. $\square$

**Proof of Theorem 15(b)** Suppose that $f$ is extremal for $\Phi_k(1, t)$. By rotations, we extend our scope to functions $f$ such that $|f(0)| = t$. In this case, we can use Lemma 4 and write $f = gh$ for

$$g(z) = C \prod_{j=1}^{l} (z + \alpha_j) \prod_{j=l+1}^{k} (1 + \overline{\alpha}_j z),$$

$$h(z) = C \prod_{j=1}^{k} (1 + \overline{\alpha}_j z).$$

The constant $C > 0$ satisfies

$$\frac{1}{C^2} = \sum_{j=0}^{k} \left| \sum_{j_1 + j_2 + \ldots + j_k = j} \alpha_1^{j_1} \alpha_2^{j_2} \cdots \alpha_k^{j_k} \right|^2,$$

where $j_1, j_2, \ldots, j_k$ take only the values 0 and 1. Evidently $\|g\|_{H^2} = \|h\|_{H^2} = 1$. Set $A_l = |\alpha_1 \cdots \alpha_l|$ and $B_l = |\alpha_{l+1} \cdots \alpha_k|$. By keeping only the terms $j = 0$ and $j = k$ we obtain the trivial estimate

$$\frac{1}{C^2} \geq 1 + |\alpha_1 \alpha_2 \cdots \alpha_k|^2 = 1 + A_l^2 B_l^2. \quad (31)$$

We will adapt an argument due to F. Riesz [13] to get some additional information on the relationship between $g$ and $h$. Write

$$f(z) = \sum_{j=0}^{2k} a_j z^j, \quad g(z) = \sum_{j=0}^{k} b_j z^j \quad \text{and} \quad h(z) = \sum_{j=0}^{k} c_j z^j$$

and note that $|b_0| = t/|c_0| = t/C$. By the Cauchy product formula we find that

$$a_k = \sum_{j=0}^{k} b_j c_{k-j} = \frac{t}{C} \frac{c_k b_0}{|b_0|} + \sum_{j=1}^{k} b_j c_{k-j}. \quad (32)$$
Suppose that \( \tilde{g} \in H^2 \) satisfies \( |\tilde{g}(0)| = t/C \) and \( \|\tilde{g}\|_{H^2} \leq 1 \). Define \( \tilde{f} = \tilde{g}h \). The Cauchy–Schwarz inequality shows that \( \|\tilde{f}\|_{H^1} \leq 1 \), so the extremality of \( f \) implies that \( |\tilde{a}_k| \leq |a_k| \). Inspecting (32) and using the Cauchy–Schwarz inequality, we find that the optimal \( g \) must therefore satisfy

\[
g(z) = \frac{t}{C} \frac{c_k}{|c_k|} + \sqrt{1 - \frac{t^2}{C^2}} \sum_{j=1}^k \frac{c_{k-j}}{C^2} z^j,
\]

where we used that \( \|h\|_{H^2} = 1 \). Using that \( c_0 = C \), we compare the coefficients for \( z^k \) in (33) with the definition of \( g \), to find that

\[
\sqrt{1 - \frac{t^2}{C^2}} C = C \prod_{j=l+1}^k \alpha_j \quad \implies \quad \frac{1 - \frac{t^2}{C^2}}{1 - |c_k|^2} = B_l^2.
\]

Next we insert \( t = C^2 A_l \) from the definition of \( f = gh \) and \( |c_k|^2 = C^2 A_l^2 B_l^2 \) from the definition of \( h \) to obtain

\[
\frac{1 - C^2 A_l^2}{1 - C^2 A_l^2 B_l^2} = B_l^2 \quad \iff \quad \frac{(1 - B_l^2)(1 - C^2 A_l^2(1 + B_l^2))}{1 - C^2 A_l^2 B_l^2} = 0.
\]

The additional information we require is encoded in the equation on the right hand side of (34).

Suppose that \( l \geq 1 \). Evidently \( A_l < 1 \), since \( |\alpha_j| < 1 \) for \( j = 1, \ldots, l \) by Lemma 4. It follows that the second factor on the right hand side of (34) can never be 0, since the trivial estimate (31) implies that

\[
C^2 \leq \frac{1}{1 + A_l^2 B_l^2} < \frac{1}{A_l^2 (1 + B_l^2)}.
\]

From the right hand side of (34) we thus find that \( B_l = 1 \), which shows that \( C^2 < 1/(2 A_l^2) \) by (35). Since \( t = C^2 A_l \), we conclude that \( 0 \leq t < 1/2 \).

By the contrapositive, we have established that if \( 1/2 \leq t \leq 1 \), then the extremal for \( \Phi_k(1, t) \) has \( l = 0 \). In this case \( A_0 = 1 \) by definition, which shows that \( C = \sqrt{t} \). The right hand side of (34) becomes

\[
\frac{(1 - B_0^2)(1 - t(1 + B_0^2))}{1 - t B_0^2} = 0,
\]

so either \( B_0 = 1 \) or \( B_0^2 = 1/t - 1 \). Returning to the definition of \( h \) we find that \( |c_0|^2 = t \) and \( |c_k|^2 = t B_0^2 \). Consequently,

\[
1 = \|h\|_{H^2}^2 = t(1 + B_0^2) + \sum_{j=1}^{k-1} |c_j|^2.
\]
Since $1/2 \leq t \leq 1$, we find that both $B_0 = 1$ and $B_0^2 = 1/t - 1$ will imply that $c_j = 0$ for $j = 1, \ldots, k - 1$. Thus $h(z) = \sqrt{t} + \sqrt{1 - t}z^k$. When $l = 0$ we have $g = h$, which shows that the unique extremal is

$$f(z) = \left(\sqrt{t} + \sqrt{1 - t}z^k\right)^2,$$

which is of the form $f_k(z) = f_1(z^k)$ as claimed. \hfill \Box

**Proof of Theorem 15(c)** In the case $0 \leq t < 1/2$, we know from Theorem 8 and Theorem 14 that $\Phi_k(1, t) = 1$. See also Figure 1. The stated claim follows from Exercise 3 on page 143 of [6] by scaling and rotating the function

$$f(z) = C \prod_{j=1}^k (\lambda_j - z)(1 - \overline{\lambda}_j z)$$

to satisfy the conditions $\|f\|_{H^1} = 1$, $f(0) > 0$ and $f^{(k)}(0) > 0$. If the resulting function satisfies $f(0) = t$, then it is an extremal for $\Phi_k(p, t)$ and every extremal is obtained in this way. (This can be established similarly to the case (b) above.) \hfill \Box

### 6 The Extremal Problem $\Phi_k(p, t)$ for $k \geq 2$ and $0 < p < 1$

The purpose of this final section is to record some observations pertaining to the extremal problem (1) in the unresolved case $k \geq 2$ and $0 < p < 1$.

Suppose that $k \geq 0$ and consider the related extremal problem

$$\Psi_k(p) = \sup \left\{ \frac{\operatorname{Re} f^{(k)}(0)}{k!} : \|f\|_{H^p} \leq 1 \right\}.$$

Evidently, $\Psi_0(p) = 1$ for every $0 < p \leq \infty$ and the unique extremal is $f(z) = 1$. Recall (from [3] or [9]) that the extremals for $\Psi_k$ satisfy a structure result identical to Lemma 4. Note that the parameter $l$ in Lemma 4 describes the number of zeroes of the extremal in $\mathbb{D}$. Conjecture 1 from [3, Sect. 5] states that the extremal for $\Psi_k(p)$ does not vanish in $\mathbb{D}$ when $0 < p < 1$. The conjecture has been verified in the cases $k = 0, 1, 2$ and for $(k, p) = (3, 2/3)$.

Let us now suppose that $k \geq 1$. There are two obvious connections between the extremal problems $\Phi_k$ and $\Psi_k$. Namely,

$$\Phi_k(p, 0) = \Psi_{k-1}(p) \quad \text{and} \quad \max_{0 \leq t \leq 1} \Phi_k(p, t) = \Psi_k(p).$$

Assume that the above-mentioned conjecture from [3] holds. This assumption yields that the extremal for $\Phi_k(p, 0)$ has precisely one zero in $\mathbb{D}$ and the extremal for the $t$ which maximizes $\Phi_k(p, t)$ does not vanish in $\mathbb{D}$. Note that the extremal for $\Phi_k(p, 1)$, which is $f(z) = 1$, does not vanish in $\mathbb{D}$.
Question 1 Suppose that $0 < p < 1$. Is it true that the extremal for $\Phi_k(p, t)$ has at most one zero in $D$?

We have verified numerically that the question has an affirmative answer for $k = 2$. Note that for $1 < p \leq \infty$, the extremal for $\Phi_k(p, t)$ either has 0 or $k$ zeroes in $D$ by Theorem 15 (a). In the case $p = 1$, the extremal may have anywhere from 0 to $k$ zeroes by Theorem 15 (b) and (c).

As mentioned in the introduction, Theorem 1 yields the estimates

$$\Phi_1(p, t) \leq \Phi_k(p, t) \leq \frac{k}{1/p - 1} \Phi_1(p, t).$$

The upper bound is only attained if $\Phi_1(p, t) = 0$ which happens if and only if $t = 1$. Of course, since $\Phi_1(p, 1) = 0$ the lower bound is also attained.

Question 2 Fix $k \geq 2$ and $0 < p < 1$. Is there some $t_0$ such that $\Phi_k(p, t) = \Phi_1(p, t)$ holds for every $t_0 \leq t \leq 1$?

By a combination of numerical and analytical computations, we have strong evidence that the question has an affirmative answer for $k = 2$ and that in this case

$$t_0 = \left(1 + \left(\frac{p}{2 - p}\right)^2\right)^{1/p}.$$

Let us close by briefly explaining our reasoning. We began by considering the case $l = 0$ in Lemma 4. Setting

$$\tilde{f} = \tilde{g} h^{2/p - 1}$$

and arguing as in the proof of Theorem 15 (b) (see also [3]), we found if $t \geq t_0$, then the only possible extremal for $\Phi_2(p, t)$ with $l = 0$ is of the form $f_2(z) = f_1(z^2)$ where $f_1$ is the corresponding extremal for $\Phi_1(p, t)$. Next, if $l = 2$ then (as in the case $k = 1$) we can only obtain $t$-values in the range $0 \leq t \leq 2^{-1/p} \sqrt{p(2 - p)^{1/p - 1/2}}$. However, since

$$2^{-1/p} \sqrt{p(2 - p)^{1/p - 1/2}} < t_0$$

for $0 < p < 1$ we can ignore the case $l = 2$. The case $l = 1$ was excluded by numerical computations.

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Appendix A: Proof of Lemma 11

We will frequently appeal to the following corollary of Rolle’s theorem: Suppose that \( f \) is continuously differentiable on \([a, b]\) and that \( f'(x) = 0 \) has precisely \( n \) solutions on \((a, b)\). Then \( f(x) = 0 \) can have at most \( n+1 \) solutions on \([a, b]\).

We are interested in solutions of the equation \( F_p(\alpha) = 0 \) on the interval \((0, 1)\), where we recall from (28) that

\[
F_p(\alpha) = p^2\alpha^{-2} + 2p(2-p) + (2-p)^2\alpha^2 - 4\left(\alpha^{-p} + \alpha^{2-p} - 1\right).
\]

The initial step in the proof of Lemma 11 is to identify the critical points of \( F_p \) on the interval \( 0 < \alpha < 1 \). It turns out that there is only one.

**Lemma 16** Fix \( 0 < p < 1 \) and let \( F_p \) be as in (28). The equation \( F'_p(\alpha) = 0 \) has the unique solution

\[
\alpha = \alpha_2 = \sqrt{\frac{p}{2-p}}
\]

on \( 0 < \alpha < 1 \).

**Proof** We begin by computing

\[
F'_p(\alpha) = -2p^2\alpha^{-3} + 2(2-p)^2\alpha + 4p\alpha^{-p-1} - 4(2-p)\alpha^{1-p}.
\]

The solutions of the equation \( F'_p(\alpha) = 0 \) on \( 0 < \alpha < 1 \) do not change if we multiply both sides by \( \alpha^{1+p}/(4-2p) \). Hence, we consider the equation \( G_p(\alpha) = 0 \), where

\[
G_p(\alpha) = \frac{\alpha^{1+p}}{2(2-p)} F'_p(\alpha) = -\frac{p^2}{2-p} \alpha^{-p-2} + (2-p)\alpha^{2+p} + \frac{2p}{2-p} - 2\alpha^2.
\]

Evidently,

\[
G'_p(\alpha) = \alpha \left( p^2\alpha^{p-4} + (4-p^2)\alpha^p - 4 \right),
\]

and the sign of \( G'_p(\alpha) \) is the same as the sign of \( p^2\alpha^{p-4} + (4-p^2)\alpha^p - 4 \). Since

\[
\frac{d}{d\alpha} \left( p^2\alpha^{p-4} + (4-p^2)\alpha^p - 4 \right) = 0 \quad \iff \quad \alpha = \sqrt[4]{\frac{4p - p^2}{4-p^2}},
\]

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and since \( G'_p(1) = 0 \), we conclude that \( G'_p \) changes sign at most once on \( 0 < \alpha < 1 \). Since \( G_p(0) = -\infty \), this means that \( G_p(\alpha) = 0 \) can have at most two solutions on \( (0, 1] \). Hence \( F'_p(\alpha) = 0 \) can have at most two solutions on \( (0, 1] \). It is easy to verify that these solutions are

\[
\alpha = \sqrt{\frac{p}{2 - p}} \quad \text{and} \quad \alpha = 1,
\]

and hence the proof is complete. \( \square \)

We next want to demonstrate that \( F_\alpha(\alpha_1) > 0 \) and \( F_\alpha(\alpha_2) < 0 \) where \( \alpha_1 \) and \( \alpha_2 \) are from (23) and (22), respectively.

**Lemma 17** Fix \( 0 < p < 1 \). If \( \alpha_2 = \sqrt{p/(2 - p)} \), then \( F_p(\alpha_2) < 0 \).

**Proof** We begin reformulating the inequality \( F_p(\alpha_2) < 0 \) as \( H(p) > 0 \), for

\[
H(p) = -\frac{2 - p}{4} \alpha_2^p F_p(\alpha_2) = 2 - \left(1 + 2p - p^2\right) p^{p/2} (2 - p)^{(2-p)/2}.
\]

Since we have \( H(0) = H(1) = 0 \), it is sufficient to prove that the function \( H \) has precisely one critical point on \( 0 < p < 1 \) and that it is strictly positive for some \( 0 < p < 1 \). We first check that

\[
H(1/2) = \frac{16 - 7 \cdot 3^{3/4}}{8} > 0.
\]

We then compute

\[
H'(p) = -p^{p/2} (2 - p)^{(2-p)/2} \left(2(1 - p) + \frac{1 + 2p - p^2}{2} \log \left(\frac{p}{2 - p}\right)\right).
\]

The first factor is non-zero, so we therefore need to check that the equation \( I(p) = 0 \) has only one solution on \( 0 < p < 1 \), where

\[
I(p) = \frac{4(1 - p)}{1 + 2p - p^2} + \log \left(\frac{p}{2 - p}\right).
\]

We compute

\[
I'(p) = \frac{-4(3 - 2p + p^2)}{(1 + 2p - p^2)^2} + \frac{2}{p(2 - p)} = \frac{2(1 - p)^2 (3p^2 - 6p + 1)}{p(2 - p) (1 + 2p - p^2)^2}.
\]

Hence \( I'(p) = 0 \) has the unique solution \( p_0 = 1 - \sqrt{2/3} \) on the interval \( 0 < p < 1 \). Noting that \( I(0) = -\infty \) and \( I(1) = 0 \), we conclude by verifying that

\[
I(p_0) = \sqrt{6} + \log \left(5 - 2\sqrt{6}\right) > 0
\]
which demonstrates that $I(p) = 0$ has a unique solution on $0 < p < 1$. \qed

**Lemma 18** Fix $0 < p < 1$. Let $\alpha_1$ denote the unique solution of the equation $1 - 2\alpha_1^p + \alpha_1^2 = 0$ on the interval $(0, 1)$. Then $F_p(\alpha_1) > 0$.

**Proof** Using the equation defining $\alpha_1$, we see that

$$F_p(\alpha_1) = \frac{p^2}{\alpha_1^2} + 2p(2 - p) + (2 - p)^2 \alpha_1^2 - 4$$

$$= \left(\frac{p}{\alpha_1} + \alpha_1(2 - p) + 2\right)\left(\frac{1}{\alpha_1} - 1\right)\left(p - \alpha_1(2 - p)\right).$$

The first two factors are strictly positive for every $0 < \alpha_1 < 1$ and every $0 < p < 1$. Consequently, $F_p(\alpha_1) > 0$ if and only if $\alpha_1 < p/(2 - p)$. The function

$$J_p(\alpha) = 1 - 2\alpha^p + \alpha^2$$

satisfies $J_p(0) = 1$ and $J_p(1) = 0$. Moreover, $J_p$ is strictly decreasing on $(0, p^{2-p})$ and strictly increasing on $(p^{2-p}, 1)$. Since $\alpha_1$ is the unique solution to $J_p(\alpha) = 0$ for $0 < \alpha < 1$, the desired inequality $\alpha_1 < p/(2 - p)$ is equivalent to

$$0 > J_p\left(\frac{p}{2 - p}\right) = 1 - 2\left(\frac{p}{2 - p}\right)^p + \left(\frac{p}{2 - p}\right)^2.$$

In order to establish this inequality, we multiply by $(2 - p)^2/2$ on both sides to get the equivalent inequality $K(p) < 0$, where

$$K(p) = 2 - 2p + p^2 - p^p(2 - p)^{2-p}.$$ 

Our plan is to use Taylor’s theorem to write

$$K(p) = K(1) + K'(1)(p - 1) + \frac{K''(\eta)}{2}(p - 1)^2,$$

where $0 < p < \eta < 1$. The claim will follow if we can prove that $K(1) = K'(1) = 0$ and $K''(p) < 0$ for $0 < p < 1$. Hence we compute

$$K'(p) = -2 + 2p - p^p(2 - p)^{2-p}\log\left(\frac{p}{2 - p}\right),$$

$$K''(p) = 2 - p^p(2 - p)^{2-p}\left(\log^2\left(\frac{p}{2 - p}\right) + \frac{2}{p(2 - p)}\right).$$

Evidently, $K(1) = K'(1) = K''(1) = 0$. Hence we are done if we can prove that $K''$ is strictly increasing on $0 < p < 1$. This will follow once we verify that both

$$p^p(2 - p)^{2-p} \quad \text{and} \quad \log^2\left(\frac{p}{2 - p}\right) + \frac{2}{p(2 - p)}$$
are strictly positive and strictly decreasing on $0 < p < 1$. Strict positivity is obvious. The first function is strictly decreasing since

$$\frac{d}{dp} \left( p^p (2 - p)^{2-p} \right) = p^p (2 - p)^{2-p} \log \left( \frac{p}{2 - p} \right)$$

and $\log(p/(2 - p)) < 0$ for $0 < p < 1$. For the second function, we check that

$$\frac{d}{dp} \left( \log^2 \left( \frac{p}{2 - p} \right) + 2 \frac{p}{p(2-p)} \right) = \frac{4}{p^2} \left( \frac{p}{2 - p} \log \left( \frac{p}{2 - p} \right) + \frac{p - 1}{(2 - p)^2} \right) < 0,$$

where for the final inequality we have again used that $\log(p/(2 - p)) < 0$.

We can finally wrap up the proof of Lemma 11.

**Proof of Lemma 11** By Lemma 16 we know that $F_p'(\alpha) = 0$ has precisely one solution for $0 < \alpha < 1$. Since $F_p(0) = \infty$ and $F_p(1) = 0$, this implies that the equation $F_p(\alpha) = 0$ can have at most one solution on the interval $(0, 1)$. Lemma 17 shows that there is exactly one solution, since $F_p(\alpha_2) < 0$. Let $\alpha_p$ denote this solution. Inspecting the endpoints again, we find that $F_p(\alpha) > 0$ for $0 < \alpha < \alpha_p$ and $F_p(\alpha) < 0$ for $\alpha_p < \alpha < 1$. Using Lemma 17 again we conclude that $\alpha_p < \alpha_2$, while the inequality $\alpha_1 < \alpha_p$ follows similarly from Lemma 18.

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