Comment on “Supersymmetry and Singular Potentials” by Das and Pernice [Nucl. Phys. B 561 (1999) 357]

Miloslav Znojil
Ústav jaderné fyziky AV ČR, 250 68 Řež, Czech Republic

Abstract

Das and Pernice [Nucl. Phys. B 561 (1999) 357 and arXiv: hep-th/0207112] proposed that the Witten’s supersymmetric quantum mechanics may incorporate potentials with strong singularities whenever one succeeds in their appropriate regularization. We conjecture that one of the most natural recipes of this type results from a detour to a non-Hermitian (usually called $\mathcal{PT}$ symmetric) intermediate Hamiltonian (with the real and discrete spectrum) obtained by an infinitesimal complex shift of the coordinate axis.

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\[\text{e-mail: znojil@ujf.cas.cz}\]
1 Introduction

Our present remark is inspired by the very recent discussion of Gangopadhyaya and Mallow [1] with Das and Pernice [2]. Although this discussion is purely technical by itself, it re-attracts attention to the singularity paradox of Jevicki and Rodriguez (JR, [3]) and to its role in Witten’s supersymmetric quantum mechanics (SUSYQM, [4]). In particular, the discussion re-opens the question of acceptability of the resolution of the above JR paradox as offered by Das and Pernice in their older and longer paper [5]. In such a setting we feel it useful to re-tell the story and to put the whole problem under a new perspective.

Let us start form the harmonic oscillator in one dimension which plays the role of one of the most elementary illustrations and realizations of the ideas of SUSYQM [6]. It comes as a definite surprise that after one restricts the same solvable Hamiltonian to the mere half-line of coordinates, the supersymmetrization immediately encounters severe technical difficulties [7]. In their above-mentioned remark, Gangopadhyaya and Mallow [1] even tried to claim that the supersymmetry (SUSY) of the latter (so called half-oscillator) model becomes spontaneously broken. In their reaction, Das and Pernice [2] resolved the puzzle by detecting a subtle flaw in the complicated construction of ref. [1]. They re-confirmed their older conjecture and belief [5] that all the paradoxes of the JR type may be resolved via a proper regularization of the singularities in the potential in question.

In constructive manner one may recollect that the standard SUSYQM considerations start from a factorizable “first” Hamiltonian $H_F = B \cdot A$, assigning to it its “second” or “supersymmetric” partner $H_S = A \cdot B$. In the case of the so called unbroken SUSY, the respective spectra are closely related, $E_{S,0} = E_{F,1}$, $E_{S,1} = E_{F,2}$ etc. This means that the “second” spectrum is obtained as an upward shift of the “first” one,

$$E_{F,n} < E_{F,n+1} = E_{S,n}, \quad n = 0, 1, \ldots.$$  \hspace{1cm} (1)

The “first” ground state $E_{F,0} = 0$ is exceptional and it does not possess any SUSY partner. The most transparent illustration of the scheme (cf. the review [8] for more details) is provided by the two harmonic oscillators $H_F^{(HO)} = p^2 + x^2 - 1$ and
\[ H^{(\text{HO})}_S = p^2 + x^2 + 1 \] in the units \( \hbar = 2m = 1 \) and with
\[
E^{(\text{HO})}_{F,0} = 0 < E^{(\text{HO})}_{F,1} = E^{(\text{HO})}_{S,0} = 4 < E^{(\text{HO})}_{F,2} = E^{(\text{HO})}_{S,1} = 8 < \ldots .
\] (2)

As we mentioned above, a nontrivial observation has been made by Jevicki and Rodrigues \[3\] who emphasized that the mathematical consistency of the theory seems to require the absence of singularities in the factors \( A \) and/or \( B \). In their singularly factorized example \( H^{(\text{sing.})}_F = p^2 + x^2 - 3 = B^{(\text{sing.})} \cdot A^{(\text{sing.})} \) one encounters a negative ground state energy in the spectrum,
\[
E^{(\text{sing.})}_{F,0} = -2, \ E^{(\text{sing.})}_{F,1} = 0, \ E^{(\text{sing.})}_{F,2} = 2, \ E^{(\text{sing.})}_{F,3} = 4, \ldots
\] (3)

and the emergence of a centrifugal-like singularity in its SUSY partner, \( H^{(\text{sing.})}_S = p^2 + x^2 - 1 + 2/x^2 = A^{(\text{sing.})} \cdot B^{(\text{sing.})} \). The main difficulty appears when we compare the “second”, \( p\)-wave spectrum
\[
E^{(\text{sing.})}_{S,0} = 4, \ E^{(\text{sing.})}_{S,1} = 8, \ E^{(\text{sing.})}_{S,2} = 12, \ldots
\] (4)

with its predecessor (3). Obviously, the characteristic SUSY isospectrality (1) is lost.

The message is clear: One may either accept the regularity rule or postulate a suitable regularization of \( A^{(\text{sing.})} \) and \( B^{(\text{sing.})} \). Das and Pernice \[3\] paid attention to the latter possibility.

2 A broader context: Harmonic oscillator in more dimensions and its textbook, “hidden” regularization

Apparently, a simple-minded resolution of the Jevicki-Rodrigues paradox is not difficult: One only has to replace the one-dimensional interpretation of \( H^{(\text{sing.})}_F = p^2 + x^2 - 3 = B^{(\text{sing.})} \cdot A^{(\text{sing.})} \) by its radial-equation version with \( x \in (0, \infty) \) (and with the vanishing angular momentum \( \ell \) of course). In this way, both the SUSY-partner Hamiltonians become defined on the same interval of coordinates and the original full-axis energies (3) become replaced by the usual \( s\)-wave spectrum
\[
E_{F,0}^{\ell=0} = 0, \ E_{F,1}^{\ell=0} = 4, \ E_{F,2}^{\ell=0} = 8, \ldots .
\] (5)
The standard SUSY pattern is restored. Unfortunately, once we move beyond the present elementary example the problem of singularities recurs [6]. For this reason, Das and Pernice [5] have proposed that within the general SYSYQM, all the singular SUSY factors $A$ and $B$ should be treated via a suitable regularization. This philosophy is generally accepted at present [8, 9].

Before we proceed further, let us return once more to the above innocent-looking $s$–wave solutions (5) where $\ell = 0$ in the radial Schrödinger equation

$$\left[-\frac{d^2}{dr^2} + r^2 + \frac{\ell(\ell + 1)}{r^2}\right]\psi^{(\ell)}(r) = E\psi^{(\ell)}(r), \quad r \in (0, \infty).$$

The explicit parabolic-cylinder [11] wave functions $\psi^{\ell=0}(r)$ of this equation are chosen as asymptotically decreasing at any real value of the energy parameter $E > 0$. In a naive but fairly popular setting (a sketchy review of which may be found elsewhere [10]) one then requires the normalizability of the wave function near the origin, i.e., its threshold behaviour $\psi^{(\ell)}(r) \sim r^{-1/2+\delta}$ with a suitable $\delta > 0$. To one’s great surprise, this condition guarantees the discrete character of the spectrum for the sufficiently large $\alpha = \ell + 1/2 > 1$ only. Thus, for our regular $V(r) = r^2$ in particular, the quantization follows from the normalizability only for $\ell = 1, 2, \ldots$ in eq. (6). The $s$–wave (or, in general, any real $\alpha = \ell + 1/2 \in (0, 1)$) is exceptional and its quantization requires an additional boundary condition. This requirement is based on the deeply physical reasoning [11] and represents, in effect, just an independent postulate of an appropriate regularization

$$\left\{ \begin{array}{l} 
\psi^{(\ell)}(0) = 0 \\
\lim_{r \to 0} \psi^{(\ell)}(r)/\sqrt{r} = 0 
\end{array} \right. \quad \text{for} \quad \left\{ \begin{array}{l} 
\alpha \in [1/2, 1) \\
\alpha \in (0, 1/2) 
\end{array} \right.$$  (7)

in quantum mechanics. This point is highly instructive and its importance is rarely emphasized in the textbooks where the independence of the regularization (7) is usually denied (for example, the Newton’s [12] proof of eq. (7) holds for $\ell = 0$ only) or disguised (for example, the Flügge’s [13] requirement of the boundedness of the kinetic energy is in effect a new, independent postulate).

In such a context it is not too surprising that the suppression (2) of the subdominant components of $\psi^{(\ell)}(r)$ near $r = 0$ is sometimes being replaced by an alternative requirement. The half-oscillator regularization of refs. [4, 11, 12] offers one of its most
characteristic examples. In a more formal mathematical setting (see, e.g., the Reed’s and Simon’s monograph [14]), one should of course use a more rigorous language and replace the word “regularization” by the phrase “selection of a suitable essentially self-adjoint extension” of the Hamiltonian operator in question.

3 Regularization recipe of Das and Pernice

The necessity of a restoration of SUSY in eq. (6) (let us just take now $\ell(\ell + 1) = 0$ for simplicity) has led the authors of ref. [5] to an artificial extension of the range of the coordinates to the whole axis, $r \in (-\infty, \infty)$. This was compensated by an introduction of a “very high” barrier to the left, $V(x) = c^2 \gg 1$ for $r \in (-\infty, 0)$. Under this assumption they succeeded in a reconstruction of SUSY partnership between the two harmonic-oscillator-like potentials

$$V_F(x) = (x^2 - 1) \theta(x) + c^2 \theta(-x) - c \delta(x),$$
$$V_S(x) = (x^2 + 1) \theta(x) + c^2 \theta(-x) + c \delta(x).$$

(8)

Both of them depend on the sign of $x$ (via Heavyside functions $\theta$) and contain Dirac delta-functions multiplied by the above-mentioned large constant $c \gg 1$.

The above-mentioned, asymptotically correct parabolic-cylinder wave functions $\psi^{(\ell=0)}(x)$ still satisfy the corresponding Schrödinger equation whenever $x > 0$. The price to be paid for the presence of the delta-function in eq. (8) lies in a violation of the boundary condition (7). In place of the elementary $\psi^{(\ell=0)}(0) = 0$ we now have the two separate and more complicated requirements

$$\left\{ \begin{array}{l}
\mu_F(E, c) \psi_F^{(\ell=0)}(0) + \nu_F(E, c) \psi_F'^{(\ell=0)}(0) = 0 \\
\mu_S(E, c) \psi_S^{(\ell=0)}(0) + \nu_S(E, c) \psi_S'^{(\ell=0)}(0) = 0
\end{array} \right..$$

(9)

They mix the values $\psi(0)$ and derivatives $\psi'(0)$ of the wave function in the origin. The explicit form of the coefficients $\mu$ and $\nu$ has been given in refs. [1, 2] as well as [5]. In the special case of the half-oscillator limit $c \to \infty$, equation (9) degenerates to the much simpler rule

$$\left\{ \begin{array}{l}
\psi_F'^{(\ell=0)}(0) = 0 \\
\psi_S'^{(\ell=0)}(0) = 0
\end{array} \right..$$

(10)
We may summarize that in our above notation and units, the $c \to \infty$ recipe is based, simply, on a sophisticated replacement of the full-line spectrum (3) by its subset selected by the Neumann boundary condition (10) in the origin,

$$E^{(DP)}_{F,0} = 0, \quad E^{(DP)}_{F,1} = 4, \quad E^{(DP)}_{F,2} = 8, \ldots .$$  (11)

As long as the “second” spectrum (4) remains unchanged by the Dirichletian second line in eq. (4), the two SUSY partner spectra obey, by construction, the unbroken SUSY and its isospectrality rule (1) even in the limit $c \to \infty$. The mathematics has got simplified – all we need are just the adapted boundary conditions (10). Of course, within the textbook quantum mechanics their intuitive physical acceptability is less obvious due to their manifest disagreement with the much more common suppression (7) of dominant terms.

4 SUSY via analytic continuation

Buslaev and Grecchi [15] were probably the first who understood that a complex shift of coordinates may leave the spectrum (or at least its part) in many radial Schrödinger equations unchanged. In their spirit we introduced the complexified radial oscillator

$$\left[ -\frac{d^2}{dx^2} + (x - i\varepsilon)^2 + \frac{\ell(\ell + 1)}{(x - i\varepsilon)^2} \right] \psi^{(\ell,\varepsilon)}[r(x)] = E \psi^{(\ell,\varepsilon)}[r(x)]$$  (12)

with $x \in (-\infty, \infty)$ and analyzed its spectrum under the standard pair of the asymptotic boundary conditions

$$\lim_{X \to \pm\infty} \psi^{(\ell,\varepsilon)}[r(X)] = 0.$$  (13)

The work has been done in ref. [16] and its result may be perceived, in SUSY context, as a regularization recipe. It is worth noticing that the new recipe is different from the very specific and closely SUSY-related technique of preceding section.

One of the most distinguished features of the use of the complex shift $\varepsilon > 0$ in eqs. (12) and (13) is that it produces the discrete energies

$$E^{(\pm)}_{BG,N}(\alpha) = 4N + 2 \mp 2\alpha$$  (14)
which are all real and numbered by the integers $N = 0, 1, \ldots$ and by the (super-scripted) quasiparities $Q = \pm$. In this way, the complexification reproduces the one-dimensional harmonic-oscillator spectrum at $\ell = \alpha - 1/2 = 0$ and extends it in a certain non-empty vicinity of $\alpha = 1/2$ in an entirely smooth manner.

One may use the similar analytic continuation technique within the SUSY QM context as well. As a universal regularization recipe, it has been proposed in ref. [9]. Its consequences proved extremely satisfactory in the domain of $\varepsilon > 0$ (we may refer to loc. cit. for more details).

What happens when we perform the backward limiting transition $\varepsilon \to 0$? Firstly, the de-complexification of the coordinate $r(x)$ splits the whole real line of $x$ in two (viz., positive and negative) half-axes of $r$ which become completely separated [10]. Fortunately, there occur only marginal discontinuities in the spectrum itself. Manifestly, this is illustrated in Figure 1 where

- at a fixed $N$, the ordering of the levels
  
  \[ E_F^{(+)} \leq E_S^{(+)} \leq E_F^{(-)} \leq E_S^{(-)} \]

  is preserved for all the values of an auxiliary real $\gamma \in \mathbb{R}$ which parametrizes both $\alpha_F = |\gamma|$ and $\alpha_S = |\gamma + 1|$;

- each of the above four energies is a piecewise linear function of $\gamma$ which changes its slope at a single point;

- the domain of $E_F^{(+)}$ or $E_S^{(+)}$ is a finite interval, out of which the related wave function $\psi$ ceases to belong to the Hilbert space in the limit $\varepsilon \to 0$.

The complete isospectrality of $H_F$ and $H_S$ occurs far to the left and right in the picture,

\[
\begin{align*}
E_F^{(-)}(N) &= E_S^{(-)}(N), & \gamma \in (-\infty, -2), \\
E_F^{(-)}(N+1) &= E_S^{(-)}(N), & \gamma \in (1, \infty), & N \geq 0.
\end{align*}
\]

In the domain of the upper line the ground-state energy is positive and all the spectrum is $\alpha-$dependent and equidistant. In contrast, in the domain of the lower line, the SUSY is unbroken and the spectrum is $\alpha-$independent.
In the non-central interior intervals, the SUSY-type isospectrality is destroyed by the new levels which start to exist. The SUSY interpretation of the partners is lost in the intervals $\gamma \in (-2, -1)$ and $\gamma \in (0, 1)$. A manifest breakdown of SUSY is encountered in this regime.

The most interesting pattern is obtained in the central interval of $\gamma \in (-1, 0)$. The completely standard SUSY behaviour is revealed there for all $\alpha_F = 1 - \alpha_S$. The SUSY-related isospectrality holds there, with the exceptional ground state $E_F^{(+)}(0)$. At a fixed main quantum number $N$ we have

$$E_F^{(+)} < E_S^{(+)} = E_F^{(-)} < E_S^{(-)}$$

and return to eq. (2) at $\alpha = 1/2$. The SUSY is unbroken even though the spectrum itself ceases to be equidistant at $\alpha \neq 1/2$.

5 Conclusions

We may summarize that the SUSY regularization via an intermediate complexification of ref. [9] represents an improved implementation of the universal idea of ref. [5]. First of all, it avoids the incompleteness of the supersymmetrization performed in the spirit of section 3 which works, as a rule, just with a subset of the whole one-dimensional spectrum. In contrast, the complexification recipe (with its subsequent $\varepsilon \to 0$ return to the real axis) works with the complete one-dimensional spectrum. Moreover, it leads to a consequent and smooth $\ell \neq 0$ generalization of the one-dimensional SUSY scheme. In the other words, the regularization of section 4 remains applicable within all the interval of $\ell \in (-1/2, 1/2)$ where our “spiked” harmonic oscillator potentials (with various interesting applications [17]) are strongly singular.

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Figure captions

Figure 1. Degeneracy of the spectrum and its $\gamma$–dependence after the SUSY regularization of section 4.