The unbearable smallness of magnetostatic QCD corrections

Chris P. Korthals Altes

National Institute for subatomic physics NIKHEF
Theory group
Science Park 105
1098 XG, Amsterdam
The Netherlands

and

Centre de Physique Thorique
Campus de Luminy, Case 907
163 Avenue de Luminy
13288 Marseille cedex 9, France

Abstract

One loop corrections to the magnetostatic QCD action are evaluated to dimension six in the magnetostatic fields. The result is remarkably simple

\[ \frac{1}{4} \int d\vec{x} (F^a_{ij})^2 - \frac{g_M^2 N}{32 \pi m_E^3} \int d\vec{x} \left( \frac{1}{60} (D^a_{mn} F^b_{mr})^2 + \frac{1}{180} f^{abc} F^a_{mn} F^b_{nr} F^c_{rm} \right) (1 + \mathcal{O}(g_M^2)). \]

In all of the deconfined phase the correction is quite small. The term of \( \mathcal{O}(g_M^2) \) is the dimension eight contribution and is only presented in a covariant derivative basis. We discuss their physical relevance for the pressure and the spatial Wilson loop.
1 Introduction

The quark gluon plasma is experimentally analyzed at RHIC and LHC. The underlying theory, thermal QCD, has made great strides in elucidating its phase structure due to lattice simulations in four dimensions.

In parallel a combination of dimensional reduction and lattice techniques has proven very useful. Dimensional reduction [1] is based on the idea to integrate out those degrees of freedom that are \( \mathcal{O}(T) \). In doing so one obtains the electrostatic QCD action (EQCD). This action is only valid if the electric screening \( m_E = \mathcal{O}(gT) \) is much smaller than the scale \( T \). It lives in three dimensions since one is integrating out the \( \mathcal{O}(T) \) degrees of freedom. All Matsubara modes except the one that is constant in Euclidean time are contributing only to the parameters in the EQCD action. So the fields in the new action have become time independent: \( A_\mu(\vec{x}), \mu = 0, 1, 2, 3 \). \( A_0 \) is now a scalar field in three dimensions. The three spatial components of the vector potential appear on a different footing. They still constitute a three dimensional Yang-Mills theory. Such a theory is not amenable to a perturbative treatment. It has a magnetic screening scale \( m_M = \mathcal{O}(g^2T) \) and Linde’s argument [2] show that to a given order depending on the dimension \( d \) of the observable in question, \( g^{2d} \) or higher, an infinite number of diagrams contributes. Therefore the lattice simulations based on dimensional reduction are three dimensional, and take apart from EQCD only the MQCD action \( \int d\vec{x} \text{ tr } F_{ij}^2 \) into account.

Corrections to the MQCD action are obtained by integrating out the scale \( m_E \). This is still a meaningful expansion, although the corrections are not anymore a series in \( g^2 \) but in \( \frac{g^2}{m_E} = \mathcal{O}(g) \). It is these corrections we calculated to one loop order. The lay-out of the paper is as follows. In section (2) notation is fixed. In the next section (3.1) we show the result for sixth and eighth order in the magnetostatic fields. The last section discusses the results and their possible use. For the necessary background the reader is invited to read the beautiful lecture notes by Laine and Vuorinen [3]. The realization that corrections to the magnetostatic action were small is old [3],[5]. What is new in this paper is the precise form of the correction.

2 Effective action and dimensional reduction

Equilibrium quantities in QCD are described by Euclidean path integrals involving the QCD action. Dimensional reduction consists of integrating out the non-zero Matsubara modes \( 2\pi nT \). The modes surviving are time independent.

Below we give the result for the three dimensional electrostatic action \( S_E \) obtained by the one loop approximation:

\[
S_E = \frac{1}{g_E^2} \left( \text{Tr } F_{ij}^2 + \text{Tr } (\vec{D}(A)A_0)^2 + m_E^2 \text{Tr } A_0^2 + \lambda_E (\text{Tr } A_0^2)^2 + \lambda_E (\text{Tr } A_0^4) + \delta S_E \right) . \tag{2.1}
\]

Vector potentials are denoted by e.g. \( A_0 = A_0^a \frac{A^a_0}{2} \), and field strengths \( F_{ij} \) similarly. The
$\lambda^a$ are the Gell-Mann matrices. The covariant derivative is the adjoint one

$$\bar{D}(A) = \bar{\partial} + T^b \bar{A}^b$$

$$T^b_{ac} = f_{abc}$$

$$[\lambda^a, \lambda^b] = 2i f_{abc} \lambda^c, \text{ tr}\lambda_a \lambda_b = 2\delta_{ab} \quad (2.2)$$

Tr means trace over color and integration over space, $tr$ is trace over color only. $\delta S_{EQCD}$ contains $O(A_0^6)$ terms. Taking these into account destroys the superrenormalizability of $S_{EQCD}$.

In this approach, when we integrate out hard ($\sim T$) degrees of freedom, we get for the parameters in $EQCD$:

$$g_E^{-2} = \frac{1}{g^2 M} \left( 1 + \#g^2 + \#g^4 + \#g^6 + \cdots \right)$$

$$\lambda_E(\tilde{\lambda}_E) = T \left( \#g^4 + \#g^6 + \cdots \right)$$

$$m_E^2 = \left( g^2 NT^2/3 + N_f g^2 T^2/6 \right) \left( 1 + \#g^2 + \#g^4 + \#g^6 + \cdots \right) \quad (2.3)$$

All indicated coefficients have been calculated (except for the last term in the coupling constant renormalization), for reviews see [6, 7, 8]. The running coupling $g = g(T/\Lambda_{MS})$ is determined by four dimensional methods as expounded in [13]. Four dimensional lattice methods relating $\Lambda_{MS}$ to $T_c$ are found there as well. As a result the running coupling stays $O(1)$ at $T = T_c$.

$N_f$ is the number of fermions involved. The fermion fields are all heavy so their effect appears only in the thermal mass and the couplings. From Eq. (2.1) we read of the scale in $EQCD$ to be the thermal mass $m_E = O(gT)$. In contrast the scale in MQCD is $g_M^2 = O(g^2 T)$.

### 3 Integrating out soft modes. Magnetostatic QCD

To obtain $S_M$ one integrates out degrees of freedom of the soft scale $m_E$. This can be still be done in perturbation theory. It involves the field $A_0$, and in two loop or higher order also the propagators and couplings in $F^2_{ij}$. The dimensionless expansion parameter is here $g_E^2/m_E = O(g)$.

Doing this leads to corrections to $Tr F^2_{ij}$:

$$S_M = \frac{1}{g_M^2} Tr F^2_{ij} + \Delta S_M. \quad (3.4)$$

The magnetic coupling $g_M^2$ incorporates now the well known one loop term in:

$$\frac{1}{g_M^2} = \frac{1}{g_E^2} \left[ 1 + \frac{g_E^2 N}{24\pi m_E} + \frac{19}{8} \left( \frac{g_E^2 N}{24\pi m_E} \right)^2 + \cdots \right]. \quad (3.5)$$
For completeness we have mentioned the two loop term calculated by Giovannangeli[10]. Doing so illustrates nicely the point that the expansion parameter is quite small: by choosing the expansion parameter (somewhat arbitrarily) as \( \frac{g_E^2 N}{24 \pi m_E^2} \) shows how the coefficient of the two loop term compares to that of the one loop term. Note that this expansion parameter is parametrically \( \mathcal{O}(g) \), but is numerically small, even near \( T_c \), as follows from the discussion after Eq.(2.3) and the results for the gauge coupling \( g_E \) in ref. [13].

Fluctuations on the order of the thermal mass can be integrated out. The quadratic fluctuations in EQCD are lowest order, and give according to Eq. (2.1) the determinant of the corresponding operator \(- (\bar{D}(A))^2 + m_E^2\).

### 3.1 Lowest order terms in \( S_M \)

To obtain explicitly the corrections to the classical MQCD term \( Tr F^2_{ij} \) one has to work out the fluctuation determinant involving \( A_0 \) in the electrostatic action. 

\[
\Delta S_M = \frac{1}{2} \log \det (-\bar{D}^2 + m_E^2)/(-\bar{\partial}^2 + m_E^2) + \cdots \tag{3.6}
\]

We normalize by the determinant that gives the \( \mathcal{O}(m_E^3) \) contribution to the pressure. The lowest order terms can be easily derived \(^1\)

The result is:

\[
S_M = \frac{1}{g_E^2} \left[ \frac{1}{4} \int d\vec{x}(F_{kl}^a)^2 - \frac{g_E^2}{192 \pi m_E} \text{Tr} [D_k D_l]^2 \right.

- \left. \frac{g_E^2}{32 \pi^3 m_E^3} \left( \frac{1}{90} \text{Tr} [D_k D_l][D_m D_n][D_p D_q] - \frac{1}{60} \text{Tr} [D_k [D_l D_i]][D_m [D_m D_l]] \right) \right]

= \frac{1}{g_M^2} \left[ \frac{1}{4} \int d\vec{x}(F_{ij}^a)^2 - \frac{g_M^2 N}{32 \pi m_E} \int d\vec{x} \left( \frac{1}{60} \left( \frac{D_m^a F_{mr}^b}{m_E^2} \right)^2 + \frac{1}{180} f^{abc} F_{mn}^a F_{mr}^b F_{rm}^c \right) + \cdots \right] \tag{3.7}
\]

The term quadratic in the field strength just renormalizes the coupling, as in Eq. (3.5). So the relative order of \( \Delta S_M \) with respect to \( Tr F^2_{ij} \) is \( \frac{g_E^2}{32 \pi m_E} \frac{[D_i D_j]}{m_E^2} = \mathcal{O}(g^3) \). The next term with eight derivatives is \( \mathcal{O}([D_i D_j]/m_E^2) = \mathcal{O}(g^2) \) smaller, and this is the general trend as shown in the next section below Eq. (4.13).

The relative correction to the classical magnetic field density is \( \mathcal{O}(g^3) \) (because the covariant derivative in the magnetic sector \( D_i \) is \( \mathcal{O}(g^2) \)), and this correction contains only two dimension six invariants.

\(^1\)It is implicitly done by combining old work by Chapman [4] and unpublished work by Giovannangeli [5].
4 One loop correction with dimension six operators

We start with the one loop result for the magnetic action:

\[
S_M = \frac{1}{4g_E^2} \int d\vec{x} (F_{ij}^a)^2 + \frac{1}{2} \log \det(-D^2 + m_E^2)/(-\partial^2 + m^2)
\]

\[
= \frac{1}{4g_E^2} \int d\vec{x} (F_{ij}^a)^2 - \frac{1}{2} \int \frac{dt}{t} \int \frac{d\vec{p}}{(2\pi)^3} \text{Tr} \left( \exp((D^2 + 2i\vec{p} \cdot \vec{D})t) - 1 \right) \times \exp(-(p^2 + m^2)t)
\]

and \( D_i \) the adjoint covariant derivative:

\[
D_i = \partial_i 1 + T^c_i, \quad T^c_{ab} = f^{acb}.
\]

The electric mass is \( m_E^2 = g^2 N T^2 / 3 + N_f g^2 T^2 / 6 \) and

\[
\left[ D_i, D_j \right] = F_{ij}, \quad F_{ij} = T^a F^a_{ij}
\]

\[
\text{tr} T^a T^b = -N \delta_{ab}
\]

\[
\left[ T^a, T^b \right] = f^{abc} T^c
\]

and

\[
\text{tr} T^a T^b T^c = -\frac{N}{2} f_{abc} - \frac{N}{2} d^{abc}.
\]

4.1 Expanding to order \( t^3 \)

There is only one explicit scale in the determinant, \( m_E \). So taking it out gives us the factor \( m_E^3 \) familiar from the lowest order \( S_E \) contribution to the pressure.

It is useful to introduce a plane wave basis in the trace

\[
\int \frac{d\vec{p}}{(2\pi)^3} \exp(-i\vec{p} \cdot \vec{y}) D^2 \exp i\vec{p} \cdot \vec{x} = \int \frac{d\vec{p}}{(2\pi)^3} \exp(-i\vec{p} \cdot (\vec{y} - \vec{x})) |(D^2 + 2i\vec{p} \cdot \vec{D} - (\vec{p})^2)|.
\]

Then we let \( y \to x \) and obtain for contribution of the determinant:

\[
\Delta S_M = -\frac{1}{2} \int \frac{dt}{t} \int \frac{d\vec{p}}{(2\pi)^3} \text{Tr} \left( \exp((D^2 + 2i\vec{p} \cdot \vec{D})t) - 1 \right) \exp(-(p^2 + m_E^2)t).
\]

The legitimacy of the expansion of the determinant is based on the fact that in the magnetic sector the ratio of magnetic over electric scales appears in the covariant derivatives:

\[
D/m_E = \mathcal{O}(g).
\]
Rotational invariance allows only terms with an even number of derivatives $D$. For a given degree (starting with $D^6$) the sum can be rewritten as a sum of terms, each consisting of products of multiple commutators of $D$’s, like $[D_iD_j] = F_{ij}$ and so on.

The first step is to obtain the terms written as strings of covariant derivatives. This is simple and straightforward using the formulas in the Appendix.

The trace in Eq. (4.13) allows to perform a cyclic permutation on any given string of covariant derivatives. This limits all fourth order terms to be either $\text{Tr} D^4$ or $\text{Tr} (D_iD_j)^2$. There are five sixth order terms, shown below in Eq. (4.18).

Explicitly for terms of up and including order $t^3$ we find the following.

The terms of order $t$ in the expansion cancel.

Next are the two $t^2$ terms. The first one is obtained by expanding the exponential

$$\int \frac{d\vec{p}}{(2\pi)^3} \exp((D^2 - 2i\vec{p} \cdot \vec{D})t - p^2 t)$$

(4.15)

to $O(t^2)$, the second to $O(t^4)$ doing the $\vec{p}$ integration using Eq. (9.36): 

$$-\frac{1}{12} t^2 \text{Tr} D^4 I(t) + \frac{1}{12} t^2 \text{Tr} (D_k D_l)^2 I(t)$$

(4.16)

and the sum is

$$\frac{1}{24} t^2 \text{Tr} [D_k, D_l]^2 I(t).$$

(4.17)

So we have recovered a term proportional to the classical action. This term corresponds of course to the renormalization of the magnetic coupling, the one loop term in Eq. (3.5).

The five $t^3$ terms are, following the same lines

$$\frac{1}{180} t^3 I(t) \times \left( -2 \text{Tr} D^6 - 3 \text{Tr} D^3 D_k D^2 D_k + 9 \text{Tr} (D_k D_l)^2 D^2 \\
- \text{Tr} D_k D_l D_m D_k D_l D_m - 3 \text{Tr} D_k D_l D_m D_l D_k D_m \right)$$

(4.18)

If the covariant derivatives would commute the first three terms cancel with the last three terms in Eq.(4.18). Therefore the sum can be rearranged into terms with at least one commutator. As we expect the outcome to depend only on the gauge field configuration any term can only depend on (multiple) commutators. In this case it turns out that one can rewrite the contribution in parenthesis as the sum of only two terms: one being the square of the operator of the equation of motion, $D_k F_{kl}$ (see Appendix). The second
The term is cubic in the magnetic field strengths:

\[-2\text{Tr} \, D^6 - 3\text{Tr} \, D^2 K D^2 D_k + 9\text{Tr} \, (D_k D_l)^2 D^2 - \text{Tr} \, D_k D_l D_m D_k D_l D_m \]

\[-3\text{Tr} \, D_k D_l D_m D_k D_l D_m \]

\[= \text{Tr} \, [D_k D_l][D_l D_m][D_m D_k] - \frac{3}{2} \text{Tr} \, ([D_k[D_l D_l]])^2 \]

\[= -\frac{N}{2} f^{abc} \vec{B}^a \cdot \vec{B}^b \times \vec{B}^c + \frac{3N}{2} (\vec{B}^{ab} \times \vec{B}^{b^2})^2, \quad (4.19)\]

using Eq. (4.11) whilst the outer product

\[(\vec{x} \times \vec{y})_k \equiv \epsilon_{klm} x_l y_l, \quad \epsilon_{123} = 1. \quad (4.20)\]

This leads to the result for \( \Delta S_M \) as mentioned in Eq. (3.7).

### 4.2 Dimension eight corrections

A straightforward calculation of the eighth order polynomial using (9.36) leads to an expression with eighteen independent invariants, instead of five as for the sixth order. These invariants are written in a short hand form. This is best understood by giving an example. Take a term:

\[\text{tr} D_j D_j D_k D_l D_m D_l D_k D_m = (0312). \quad (4.21)\]

We start with the first index j. It must contract with another index j. This index is separated from the first one by no other \( D_x \), and so we start with 0. Next index is k and it contracts with another index k, that is separated by three \( D_x \), so the next number is 3. The next index l is separated by one derivative \( D_m \), and the last index m by 2 derivatives. This gives the result as in Eq. (4.21). Because of the cyclical trace it can also be written as (3120) or (2530), but we take the convention that the smallest number comes first. Because of the cyclic property any number is modulo 6, hence (2530)=(0312).

One finds by inspection that there are 18 independent terms as in (4.21). All other combinations reduce to those 18 terms, through cyclicity.

Now we can write the expansion of the determinant to eighth order.

\[
\log \det(D^2 + m_B^2)|_{8th} = \int_0^\infty \frac{dt}{t} \frac{I(t)}{7!} \left[ -10(0000) + 72(0011) - 68(0020) - 32(0121) \\
+ 72(0130) - 40(0222) - 40(0231) - 44(0312) + 32(0330) + 36(0420) - 40(0411) \\
+ 8(1111) + 16(1221) + 16(1322) + 8(1331) + 8(2332) + 4(2422) + 2(3333) \right] \quad (4.22)
\]

As for the sixth order result the coefficients in the first line and those in the second line do add up to zero. So also here the result consists of terms with at least one commutator.
factor. We have determined the minimal basis of multiple commutator terms like we did for the sixth order case. Here again ”minimal” means that any other multiple commutator term can be written through cyclicity of the trace and using the Jacobi identity as a linear combination out of the minimal set. The result is given in the Appendix.

5 Use in the EOS

Let us see what corrections the free energy picks up from the six order terms in Eq. (3.7). We treat the correction as a perturbation, so the theory stays superrenormalizable. So without the correction the free energy is of the form

$$f_M V/T = - \log \int D\tilde{A} \exp(-\frac{1}{g_E^2} \int d\bar{x} S_M) = c(g_E^2 T)^3 V/T.$$ (5.23)

Including $\Delta S_M$ we get

$$f_M V/T = - \log \int D\tilde{A} \exp(-\frac{1}{g_E^2} S_M)(1 - \Delta S_M) = - \log \int D\tilde{A} \exp(-\frac{1}{g_E^2} S_M) + \langle \Delta S_M \rangle.$$ (5.24)

The last term equals because of translation invariance

$$\langle \Delta S_M \rangle = V \langle \Delta S_M(0) \rangle.$$ (5.25)

Finally the expectation value of $\Delta S_M(0)$ has to be measured on the lattice. Because it is of sixth order in the covariant derivative we expect for the average

$$\langle trB_iB_jB_k\epsilon_{ijk} \rangle = c_1 g_M^{12}, c_1 \text{ a dimensionless lattice constant}$$ (5.26)

and similar for $tr(D_i F_{ij})^2$ with a coefficient $c_2$. The presence of the factor $\frac{1}{m_E}$ in $\Delta S_M$ renders the effect $O(g^9)$ so as already mentioned $O(g^3)$ smaller than $f_M$.

6 Use in the spatial string tension

In four dimensional QCD at a finite temperature we can define a Wilson loop in a given time slice. Such a spatial loop, $tr \exp(i \oint \tilde{A}.d\tilde{l})$, will in the Abelian case through Stokes law catch the magnetic flux in the plasma (more precisely, fluctuations thereof). However in that case there is no reason to expect any magnetic flux at all, and the loop will obey the perimeter law, reflecting the radiative corrections to the $\tilde{A}$ potential.

The spatial string tension is given by a rectangular spatial Wilson loop, say in the x-y plane, of size $L \times L$:

$$W(L) = \langle tr\mathcal{P} \exp(i \oint_L \tilde{A}.d\tilde{l}) \rangle.$$ (6.27)
For definiteness we take as gauge group $SU(3)$, and $N_f \neq 0$. The average is the usual thermal path integral average. The spatial tension is then obtained from the loop average by

$$\sigma_s(T) = - \lim_{L \to \infty} \log W(L). \quad (6.28)$$

This tension has been measured on the lattice in 4d with and without fermions and in 3d pure gauge fields [9].

At very high temperature the tension will pick up only large distance contributions from the magnetic action $S_M$ after integrating out scales $T$ and $gT$.

The path integral average is now 3 dimensional and the large distance contributions are described by a non-perturbative coefficient $c_s$ multiplying $g_M^4$ for dimensional reasons,

$$\sqrt{\sigma_s(T)} = c_s g_M(T)^2. \quad (6.29)$$

If we neglect $\Delta S_M$ we have for $c_s$ from 3d lattice simulations [9]

$$c_s(0) = 0.553(1), \ N = 3. \quad (6.30)$$

Taking $\Delta S_M$ into account we expand the exponential $\exp(-S_M - \Delta S_M)$ in the path integral for the Wilson loop average

$$\frac{\int D\tilde{A}W(L) \exp(-S_M - \Delta S_M)}{\int D\tilde{A} \exp(-S_M - \Delta S_M)} = \frac{\int D\tilde{A}W(L) \exp(-S_M)(1 - \Delta S_M)}{\int D\tilde{A} \exp(-S_M)(1 - \Delta S_M)}$$

$$= \frac{\int D\tilde{A}W(L) \exp(-S_M)(1 + \langle \Delta S_M \rangle)}{\int D\tilde{A} \exp(-S_M)} - \frac{\int D\tilde{A}W(L) \Delta S_M (1 + \langle \Delta S_M \rangle) \exp(-S_M)}{\int D\tilde{A} \exp(-S_M)}$$

$$= \langle W(L) \rangle + \langle W(L) \Delta S_M \rangle - \langle W(L) \Delta S_M \rangle + O(\Delta S_M^2)$$

$$= \langle W(L) \rangle \left(1 - \frac{\langle W(L) \Delta S_M \rangle c}{\langle W(L) \rangle} \right). \quad (6.31)$$

This involves the truncated correlation of the loop with the correction to the action. From this we find easily, using Eq. (6.28) that

$$\sigma_s = \sigma_s(0) + \Delta \sigma_s, \quad (6.32)$$

with

$$\Delta \sigma = - \lim_{L \to \infty} \frac{1}{L^2} \frac{\langle W(L) \Delta S_M \rangle c}{\langle W(L) \rangle}, \quad (6.33)$$
and can only be obtained from lattice simulations\(^2\). We expect \(\mathcal{O}(g^3)\) corrections to \(c_s(0)\), with again a very small coefficient.

## 7 Conclusions

We have derived the one loop corrections to the magnetostatic action by integrating out fluctuations on the screening scale \(m_E\). They are the product of the renormalization of the electrostatic coupling to the magnetostatic coupling \(g_E^2N/(32\pi m_E)\) and a sum of dimension six, eight,....operators in units of \(m_E^2, m_E^4\). The first term in the sum is \(\mathcal{O}(g_E^2)\), the second \(\mathcal{O}(g_E^4)\) and so on. The first factor is quite small in the plasma phase\([11]\). Pressure and spatial string tension are only very little affected. For the renormalization of \(g_M\), in terms of \(g_E\) this has been analyzed and corroborated in ref.[11]. It is perhaps worthwhile to note that our Skyrme-like terms of dimension six are not generating a classical stable solution. The reason is that the dominant term \(\text{Tr} F_{ij}^2\) has qualitatively the same scaling properties as our correction terms, i.e. they scale all with a positive power. This fact excludes a non-trivial minimum.

## 8 Acknowledgements

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## 9 Appendix: Independent six and eighth order operators

\(^2\)On the lattice the strong coupling expansion produces indeed an area law for the right hand side of Eq. (6.33). For the operator \(\text{tr} F^3\) the strong coupling series starts with a term \(\beta_L^3\) in the lattice coupling.
\( a_2 \delta_{ij} \equiv t(2i)^2 < p_ip_j >, \ a_2 = -2I(t) \)
\( a_4(\delta_{ij}\delta_{kl} + 2 \text{ perm}) \equiv t^2(2i)^4 < p_ip_jp kp l >, \ a_4 = 4I(t) \)
\( a_6(\delta_{ij}\delta_{kl}\delta_{mn} + 14 \text{ perm}) \equiv t^3(2i)^6 < p_ip_jp k p m p n >, \ a_6 = -8I(t) \)
\( a_8(\delta_{ij}\delta_{kl}\delta_{mn}\delta_{rs} + 104 \text{ perm}) \equiv t^4(2i)^8 < p_ip_jp k p m p n p r p s >, \ a_8 = 16I(t) \)

(9.36)

\[ e_{i_1,\ldots,i_{2n}}^2 = 2n - 1!!2n + 1!! \]
\[ (2i)^{2n} e_{i_1,\ldots,i_{2n}} \langle p_{i_1} \cdots p_{i_{2n}} \rangle = (-2)^n2n - 1!!2n + 1!!I(t) \]
\[ \langle \vec{p}^{2n} \rangle = (-\frac{d}{dt})^nI(t) = \frac{(2n + 1)!!I(t)}{2^n t^n} \]

(9.37)

The result of the effective action is written in strings of covariant derivatives as Eq.(4.18) and (4.22). But it is ultimately in terms of field strengths and their covariant derivatives. Therefore the result can be written in terms of factors, each of which is a multiple commutator in terms of covariant derivatives. The overall color trace necessitates at least two of such factors. The contraction of indices should give a scalar, we have cyclic permutation inside the trace, and the Jacobi identity:

\[ [D_i[D_jD_k]] = [[D_iD_j]D_k] + [D_j[D_iD_k]] \]
\[ [D_i[D_j[D_kD_l]]] = [[D_iD_j][D_kD_l]] + [D_j[[D_iD_k]D_l]] + [D_j[D_k[D_iD_l]]]. \]  

(9.38)

To order four we only one possibility:

\( \text{• } Tr[D_iD_j]^2. \)

To order six we have three structures possible:

\( \text{• } C_1 = Tr[D_iD_j][D_i[D_k[D_jD_k]]] \)
\( \text{• } C_2 = Tr[D_k[D_kD_i]]^2 \)
\( \text{• } C_3 = Tr[D_i[D_jD_k]]][D_i[D_jD_k]] \)
\( \text{• } C_4 = Tr[D_iD_j][D_kD_i][D_jD_k][ \).

\( C_3 \) can be trivially rewritten as a \( C_1 \) or \( C_2 \) type. Only the second and the fourth contribute to the determinant.

To order eight the following structures are possible-writing for short \( [D_iD_j] = F_{ij} \):

\( \text{• } O_1 = TrF_{ij}F_{jk}F_{kl}F_{li} \text{ and } TrF_{ij}F_{ji}F_{kl}F_{lk} \)  

(2)
• $O_2 = Tr F_{ij} F_{kl} [D_i [D_j F_{kl}]]$  (1)
• $O_3 = Tr F_{ij} F_{jl} [D_i [D_k F_{kl}]]$  (2)
• $O_4 = Tr F_{ij} [D_i F_{j}] [D_m F_{mj}]$  (1)
• $O_5 = Tr F_{ij} [D_i F_{lm}] [D_m F_{ij}]$  (2)
• $O_6 = Tr [D_i [D_j F_{kl}]] [D_i [D_j F_{kl}]]$  (6)
• $O_7 = Tr [D_i [D_k F_{kj}]] [D_i [D_l F_{lj}]]$  (2)
• $O_8 = Tr [D_i [D_k F_{kj}]] [D_l [D_l F_{ij}]]$  (1)

In $O_2$ and $O_6$ all permutations of the 4 indices in say the righthand factor are understood.

We see that transposition of the indices $i$ and $j$ in $O_6$ leads to an operator of type $O_2$ and $O_4$. And obviously transposition of $k$ and $l$ just flips the sign. So out of the 24 permutations that constitute $O_6$ only 6 are independent. $O_1$ has only two inequivalent contractions. $O_4$ is unique.

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