Supertwistor formulation for higher-dimensional superstrings

D V Uvarov
Kharkov Institute of Physics and Technology, 61108 Kharkov, Ukraine
E-mail: uvarov@kipt.kharkov.ua and d_uvarov@hotmail.com

Received 9 June 2007, in final form 3 September 2007
Published 24 October 2007
Online at stacks.iop.org/CQG/24/5383

Abstract
The formulation for the superstring action in six and ten dimensions involving supertwistor variables that appropriately generalize four-dimensional Ferber supertwistors is considered. The equations of motion and $\kappa$-symmetry transformations in terms of the supertwistors are derived. It is shown that covariant $\kappa$-symmetry gauge-fixing reduces the superstring action to the quadratic one with respect to supertwistors. Upon gauge-fixing remaining symmetries it can be cast into the form of the light-cone gauge Green–Schwarz superstring action.

PACS numbers: 11.25.−w, 11.30.Pb

1. Introduction
Twistor theory [1] and its supersymmetric generalization in four dimensions [2] are known to unveil important features of the spacetime structure and geometry and provide valuable insights into the field theory. However, until recently the application of twistor methods to string theory has been basically limited to point-like (super)particles [3–15] and the tensionless extended objects [16–20] (see, however, [21–25] for approaches to the twistor description of tensile string and brane models). Active exploration of the gauge fields/strings correspondence using spinor and twistor methods began after Witten’s construction of the topological string in the projective supertwistor space $\mathbb{C}P^{(3|4)}$ [26]. Since then other twistor superstring models [27–30] have been proposed, sharing the feature that all of them appear to be different from the conventional Green–Schwarz superstring both classically and quantum mechanically. This raises the question of examining the possibility of the twistor reformulation for the Green–Schwarz superstring and looking for adequate supertwistor variables. Such a twistor transform for the Green–Schwarz superstring could provide insights into the covariant quantization problem, as well as underline the similarities and differences as compared to known twistor string models.
In [31], such an investigation has been started by performing the supertwistor transform for the $D = 4$ superstring, using as the starting point the first-order action functional [32] that involves Lorentz harmonics [33–37] as auxiliary variables and on the classical level is equivalent to the Green–Schwarz formulation. The string action in terms of Ferber supertwistors [2] that, unlike the previously studied formulations for point-like and extended objects in terms of twistor variables [3–9, 11–24], is invariant under irreducible $\kappa$-symmetry transformations was obtained. It was shown that covariant $\kappa$-symmetry gauge-fixing results in the superstring action quadratic in the (super)twistor variables that is of the $2d$ free-field-theory-type modulo the constraints on (super)twistors necessary to ensure their incidence to the superspace with real body. The classification of the constraints and the structure of the gauge symmetries for the $\kappa$-symmetry gauge-fixed action have been considered in [38].

The present paper is devoted to the twistor transform of the superstring action in higher dimensions. It requires the proper generalization of supertwistors to dimensions $D > 4$. Such a generalization by itself appears to be rather complicated and ambiguous, as explored earlier from the geometrical and field-theoretical perspectives [39–45], as well as from the twistor description of superparticles and superstrings [11, 14, 15, 20, 24, 30]. For our consideration we use those $D = 6$ and $D = 10$ supertwistors whose projectional parts are identified with the spinor Lorentz-harmonic matrix, which allows us to establish the relation with the string momentum density represented by the Lorentz vector.

Sections 2 and 3 are devoted to the twistor transform for the $D = 6$ superstring. After defining appropriate twistor variables we present the supertwistor formulation for the superstring action, derive the corresponding equations of motion, find $\kappa$-symmetry transformation rules and consider a Lorentz-covariant $\kappa$-symmetry gauge leading to the simplification of the superstring action. Generalization of the above results to the $D = 10$ case is the subject of sections 4 and 5. Besides that, section 5 contains discussion of how the $\kappa$-symmetry gauge-fixed action can be related to the light-cone gauge formulation of the Green–Schwarz superstring.

2. Lorentz harmonics and $D = 6$ supertwistors

As the $D = 6, N = 1$ superconformal group is isomorphic to the $OSp(8^*_2)$ supergroup [46] we consider the supertwistor to realize its fundamental representation

$$Z^\Lambda^a = (\mu^a, v^a_\alpha, \eta^a). \quad (1)$$

Following the Penrose nomenclature [1] we call $\mu^a$ the primary spinor part of the supertwistor and $v^a_\alpha$ its projectional part. They are represented by the pair of $D = 6$ symplectic Majorana–Weyl spinors of opposite chiralities

$$(\mu^a)^* = C^a_\beta^\Lambda \varepsilon_{ab} h^b_\beta, \quad (v^a_\alpha)^* = C^{-\Lambda^a}_\beta^\alpha \varepsilon_{ab} v^b_\beta, \quad (2)$$

where $D = 6$ charge conjugation matrix $C^a_\Lambda$ and its inverse $C^{-\Lambda^a}_\beta$ can be expressed as

$$C^a_\Lambda = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad C^{-\Lambda^a}_\beta = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad (C^a_\Lambda)^* = -C^{-\Lambda^a}_\beta. \quad (3)$$

$\varepsilon_{ab}$ and $\varepsilon^{ab}$: $\varepsilon_{abc} = \delta^c_a$ are unit antisymmetric metric and its inverse in the fundamental representation of $SU(2)$ that will be identified below with the constituents of the $SO(4) = \Sigma^2_0$.
SU(2) × SU(2) group of rotations in the directions orthogonal to the string world-sheet. The supertwistor components are assumed to satisfy the incidence relations

\[ \mu^{\alpha a} = v^a_\alpha (x^\alpha - 2\theta^\alpha_i \theta_i^a), \quad \eta^{i a} = \bar{v}^a_\alpha g^{i a} \]  

(4)

with respect to the D = 6, N = 1 superspace coordinates \( x^{\alpha a} = x^\alpha \bar{\gamma}^{\alpha a}_\mu, \theta_i^a \), where \( \bar{\gamma}^{\alpha a}_\mu \) (\( \mu = 0, 1, \ldots, 5 \)) are 6d chiral \( \gamma \)-matrices antisymmetric in spinor indices. Superspace Grassmann coordinates \( \theta_i^a \) also obey the symplectic Majorana–Weyl condition

\[ (\theta_i^a)^* = -C^a_{\mu} e_{ij} \theta_j^i, \]  

(5)

where \( e_{ij} \) and \( e^{ij} \) represent the metric tensor and its inverse for the SU(2)R-symmetry subgroup of OSp(8⁺|2).

Being interested in the twistor description of \( D = 6 \) superstrings we identify the projectional part of the supertwistor (1) \( v^a_\alpha \) as the component of the \( D = 6 \) spinor Lorentz harmonic matrix

\[ v^{(a)}_\alpha = (v^a_\alpha, v^a_{\bar{\alpha}}) \in \text{Spin}(1, 5) \]

(6)

with the chiral spinor index in brackets realized as the aggregate of spinor indices corresponding to \( SO(1, 1) \), to be identified with the world-sheet structure group, and \( SO(4) \). Note that \( D = 6 \) spinor Lorentz harmonics [37] satisfy the reality

\[ (v^{(a)}_\alpha)^* = -C^{(a)}_{(\beta)} v^{(\beta)}_\alpha C^{-1}_{\alpha} \]

(7)

and unimodularity

\[ \det v^{(a)}_\alpha = 1 \]

(8)

constraints reducing the number of their independent components to the dimension of the Spin(1, 5) group equal to 15. Hence, we have the following pair of supertwistors associated with \( D = 6 \) superstring

\[ Z^{\lambda a} = (\mu^{\lambda a}, \nu^{\lambda a}, \eta^{i a}), \quad Z^{\lambda - i} = (\mu^{\lambda - i}, \nu^{\lambda - i}, \eta^{i a}). \]

(9)

In order to satisfy incidence relations (4) with \( v^{(a)}_\alpha \) standing for either \( v^a_\alpha \) or \( v^a_{\bar{\alpha}} \), supertwistors (9) have to be constrained by 10 relations

\[ Z^{\lambda a} G_{\lambda \Sigma} Z^{\Sigma b} = Z^{\lambda - i} G_{\lambda \Sigma} Z^{\Sigma - i} = Z^{\lambda a} G_{\lambda \Sigma} Z^{\Sigma - i} = 0 \]

(10)

ensuring the elimination of the contribution of the 3-form coordinates \( y^{\alpha a} = \gamma_{\alpha \beta \gamma} \bar{\gamma}^{\alpha a}_\mu \). In (10), the invariant scalar product of supertwistors is arranged using the OSp(8⁺|2) metric

\[ G_{\lambda \Sigma} = \begin{pmatrix} 0 & \delta_\Sigma^\lambda & 0 \\ \delta_\Sigma^\lambda & 0 & 0 \\ 0 & 0 & -i\varepsilon_{ij} \end{pmatrix}. \]

(11)

It is also worth mentioning that the OSp(8⁺|2) superalgebra can be viewed as the conformal superalgebra associated with the centrally extended D = 4, N = 2 superPoincaré algebra. The corresponding representation for the primary spinor and projectional parts of the supertwistor (1)

\[ \mu^{\alpha a} = (\mu^{\alpha a}, \bar{\mu}^\alpha_a), \quad v^{i a}_\alpha = (v^i_\alpha, \bar{v}^{i a}_\alpha) \]

(12)

is given in terms of two pairs of SL(2, \( \mathbb{C} \)) 2-component spinors. The incidence relations become

\[ \mu^{\alpha a} = \bar{v}^a_\alpha (x^{\bar{\beta} a} + 4i\theta^{\beta}_{\mu} \bar{\theta}^i_\alpha), \quad \eta^{i a} = 4(v^a_\alpha g^{i a} + \bar{v}^{i a}_\alpha) \]

(13)
where the $(4 + 2)$ splitting of $D = 6$ Minkowski coordinates $x^\mathbb{M} = (x^m, x^5)$ with $x_{a\bar{b}} = \sigma_{ma\bar{b}}x^m$ ($m = 0, 1, 2, 3$) and the complex scalar coordinate $z = x^5 + ix^4$ parameterizing bosonic body of the centrally extended $D = 4, N = 2$ superspace has been performed.

3. Supertwistor formulation of $D = 6$ superstring

An appropriate starting point for our consideration is the $D = 6, N = 1$ superstring in the Lorentz-harmonic formulation, whose action equals

$$S_{\text{D=6}}^{\text{LH}} = S_{\text{kin}} + S_{\text{WZ}},$$

where the kinetic term is defined as

$$S_{\text{kin}} = \frac{1}{2(a^\prime)^{1/2}} \int d^2\xi \left( e^2 n_m^{-2} - e^{-2} n_m^{+2} \right) \wedge \omega^\mathbb{M}(d) + \frac{c}{2} \int d^2\xi \ e^{-2} \wedge e^{+2}$$

and the Wess–Zumino term is given by the expression

$$S_{\text{WZ}} = \frac{ix}{c a^\prime} \int d^2\xi \omega^\mathbb{M}(d) \wedge d\theta^\mathbb{M}_{\mu\nu\rho\sigma} \theta_{\mu\nu\rho\sigma}.$$

In the superstring action (14) $\omega^\mathbb{M}(d) = dx^\mathbb{M} - id\theta^\mathbb{M}_{\mu\nu\rho\sigma} \theta_{\mu\nu\rho\sigma}$ is the $D = 6, N = 1$ supersymmetric 1-form, $e^{\pm2} (\mu = \tau, \sigma)$ are the components of the world-sheet zweibein written in the light-cone basis for the $SO(1,1)$ world-sheet structure group, $a^\prime$ is the constant of dimension $[L]^2$, $c$ is the dimensionless numerical constant, as well as $s = \pm 1$, both values of which are consistent with the $x$-invariance of the action. Action (14) also contains two light-like vectors $n_m^{\pm2}(\xi)$ from the string-adopted $D = 6$ vector harmonic matrix $n_m^{(\alpha)} = (n_m^{x5}, n_m^{x4}, n_m^{x3})$

$$n_m^{(\alpha)} n_m^{(\beta)\bar{\kappa}} = \eta^{(\alpha)(\beta)\bar{\kappa}} ;$$

$$n_m^{x5} n_m^{x5} = n_m^{x4} n_m^{x4} = 0, \quad n_m^{x5} n_m^{x4} = 2, \quad n_m^{x5} n_m^{x3} = 0, \quad n_m^{x4} n_m^{x3} = -\delta_{\alpha\beta},$$

admitting the following realization through the above-introduced spinor harmonics (6)

$$n_m^{x5} = \frac{1}{2} v_a^{x5} \gamma^{a\bar{b}} v_{\bar{b}}^{x5} e_{ab}, \quad n_m^{x4} = -\frac{1}{2} v_a^{x4} \gamma^{a\bar{b}} v_{\bar{b}}^{x5} e_{ab}, \quad n_m^{x3} = -\frac{1}{2} v_a^{x3} \gamma^{a\bar{b}} v_{\bar{b}}^{x5} e_{ab},$$

where the index $\bar{\alpha} = 1, \ldots, 4$, belongs to the vector representation of the $SO(4)$ and $\gamma_{ab}$ are the corresponding $\sigma$-matrices. Differentials/variables of the vector harmonics consistent with the normalization relations (17)

$$dn_m^{\pm2} = \mp \frac{1}{2} \Omega^{\pm2-}(d) n_m^{\pm2} + \Omega^{\pm22}(d) n_m^{\prime},$$

$$dn_m^{x5} = \frac{1}{2} \Omega^{x5}(d) n_m^{x5} + \frac{1}{2} \Omega^{x5-}(d) n_m^{x5} + \Omega^{x5\prime}(d) n_m^{\prime},$$

are expressed through the $SO(1,1) \times SO(4)$ split components of the Cartan 1-form

$$\Omega^{\mp2}(d) = \frac{1}{2} (n_m^{\alpha}) \frac{d\omega^\mathbb{M}}{dx^\mathbb{M}} - n_m^{(\alpha)} \frac{d\omega^\mathbb{M}}{dx^\mathbb{M}} ;$$

$$\Omega^{x5}(d) = \frac{1}{2} (n_m^{\alpha}) \frac{d\omega^\mathbb{M}}{dx^\mathbb{M}} - n_m^{(\alpha)} \frac{d\omega^\mathbb{M}}{dx^\mathbb{M}} ,$$

invariant under the $SO(1,5)$ transformations acting on indices without brackets. They are used to derive the equations of motion for the $D = 6$ Lorentz-harmonic superstring among which is the following nondynamical equation

$$\omega^\mathbb{M}(d) = \frac{c(a^\prime)^{1/2}}{2} (e^{x5} n_m^{-2} + e^{x4} n_m^{x3} + e^{x3} n_m^{x4} + e^{x4} n_m^{x5})$$

that defines the positioning w.r.t. to the world-sheet of the supersymmetric 1-form $\omega^\mathbb{M}(d)$. Note that light-like vectors $n_m^{\pm2}(\xi)$ can be identified with the world-sheet tangents, while other
components of the harmonic matrix \( n_{m}^{l}(\xi) \) are orthogonal to the world-sheet. Relation (21) is used to recover the Green–Schwarz action from the Lorentz-harmonic one (14)

\[
S_{\text{GS}}^{D=6} = -\frac{1}{2c' a'} \int d^{2}\xi \sqrt{-g} g^{\mu \nu} \omega_{\mu}^{m} \omega_{\nu}^{m} + \frac{i x}{c' a'} \int d^{2}\xi \bar{\theta} \gamma_{\mu} \theta^{\mu} \theta^{i},
\]

where the inverse of the world-sheet metric \( g^{\mu \nu} = \frac{1}{2}(e^{\mu \nu} e^{\nu \mu} + e^{\nu \mu} e^{\mu \nu}) \) is defined by the components of the inverse zweibein \( e^{\mu \nu} \) and \( \sqrt{-g} = e = \det(e^{\mu \nu}, e_{\nu}^{\mu}) \), thus establishing the classical equivalence of both formulations.

Twistorization of the Lorentz-harmonic superstring action (14) proceeds by representing projections of the supersymmetric 1-form \( \omega \xi(d) \) onto the tangent to the world-sheet vector harmonic components \( n_{m}^{l} \) in terms of the supertwistor variables (9)

\[
\omega^{\pm 1}(d) = n_{m}^{l} \omega \xi(d) = \frac{1}{2} \epsilon_{a b} d\xi^{a} G_{\Lambda \Sigma} \Sigma^{+a} G_{\Lambda \Sigma} \Sigma^{-b} d\xi^{b} + \frac{c}{2} \int d^{2}\xi e^{-2} \wedge e^{2} + \frac{i x}{c' a'} \int d^{2}\xi \left( \frac{1}{2} \omega^{\pm 2}(d) \wedge \phi^{\pm 2}(d) + \frac{1}{2} \omega^{\pm 2}(d) \wedge \phi^{-2}(d) - \omega^{\pm 2}(d) \wedge \phi^{\pm 2}(d) \right),
\]

Quite analogously, \( \omega \xi(d) \) projections onto the orthogonal to the world-sheet components of the vector harmonic matrix can be expressed in terms of supertwistors

\[
\omega^{i}(d) = n_{m}^{l} \omega \xi(d) = \frac{1}{2} \epsilon_{a b} d\xi^{a} G_{\Lambda \Sigma} \Sigma^{+a} G_{\Lambda \Sigma} \Sigma^{-b} d\xi^{b} \xi^{l} + \frac{C}{2} \int d^{2}\xi e^{-2} \wedge e^{2} + \frac{i x}{c' a'} \int d^{2}\xi \left( \frac{1}{2} \omega^{i+2}(d) \wedge \phi^{i+2}(d) + \frac{1}{2} \omega^{i+2}(d) \wedge \phi^{-2}(d) - \omega^{i+2}(d) \wedge \phi^{i+2}(d) \right),
\]

Then the first-order \( D = 6 \) superstring action (14) is represented as

\[
S_{\text{tw}}^{D=6} = \frac{1}{2(c')^{1/2}} \int d^{2}\xi (e^{a b} \wedge \omega^{-2}(d) - e^{-2} \wedge \omega^{+2}(d)) + \frac{C}{2} \int d^{2}\xi e^{-2} \wedge e^{2} + \frac{i x}{c' a'} \int d^{2}\xi \left( \frac{1}{2} \omega^{i+2}(d) \wedge \phi^{i+2}(d) + \frac{1}{2} \omega^{i+2}(d) \wedge \phi^{-2}(d) - \omega^{i+2}(d) \wedge \phi^{i+2}(d) \right),
\]

where additionally the following 1-forms quadratic in the Grassmann-odd supertwistor components

\[
\psi^{+2}(d) = -\frac{1}{2} \epsilon_{a b} D\eta^{a+b}, \quad \psi^{-2}(d) = \frac{1}{4} \epsilon_{a b} D\eta^{a+b},
\]

\[
\psi^{i}(d) = \frac{1}{2}(D\eta^{a+b} \eta^{-a+b} - D\eta^{-a+b} \eta^{a+b}) \sigma^{l}_{a b},
\]

have been introduced. Covariant differentials for the Grassmann-odd supertwistor components that enter the above expressions equal

\[
D\eta^{a+b} = d\eta^{a+b} + \frac{1}{4} \Omega^{a+b} \eta^{-a-b}, \quad D\eta^{-a-b} = d\eta^{-a-b} - \frac{1}{4} \Omega^{-a-b} \eta^{a+b},
\]

where

\[
\sigma^{l}_{a b} = \frac{1}{2}(\sigma^{l}_{a b} \sigma^{l}_{a b} - \sigma^{a}_{l a} \sigma^{b}_{l b}), \quad \sigma^{l}_{a b} = \frac{1}{2}(\sigma^{l}_{a b} \sigma^{l}_{a b} - \sigma^{a}_{l a} \sigma^{b}_{l b}).
\]

Because of the presence of algebraic constraints (10) admissible differentials/variations for supertwistors should be introduced that take them into account. Those for spinor Lorentz harmonics are given by

\[
d\eta^{a+b} = -\frac{1}{4} \Omega^{a+b} \eta^{-a-b}, \quad d\eta^{-a-b} = \frac{1}{4} \Omega^{-a-b} \eta^{a+b},
\]

\[
d\eta^{i+j} = \frac{1}{2} \Omega^{i+j} \eta^{-i-j}, \quad d\eta^{-i-j} = \frac{1}{2} \Omega^{-i-j} \eta^{i+j},
\]

\[
d\eta^{i} = -\frac{1}{4} \Omega^{i} \eta^{-i}, \quad d\eta^{-i} = \frac{1}{4} \Omega^{-i} \eta^{i},
\]

\[
d\omega \xi(d) = \frac{1}{2} \Omega \omega \xi(d) + \frac{1}{2} \Omega^{+} \omega \xi(d) + \frac{1}{2} \Omega^{-} \omega \xi(d)
\]

\[
d\psi^{+2}(d) = -\frac{1}{2} \Omega^{+} \psi^{+2}(d) + \frac{1}{2} \Omega^{-} \psi^{-2}(d) + \frac{1}{2} \Omega^{+} \psi^{-2}(d) - \frac{1}{2} \Omega^{-} \psi^{+2}(d),
\]

\[
\omega^{+2}(d) = \frac{1}{2} \Omega^{+} \omega^{+2}(d) + \frac{1}{2} \Omega^{-} \omega^{+2}(d) - \frac{1}{2} \Omega^{+} \omega^{-2}(d) - \frac{1}{2} \Omega^{-} \omega^{+2}(d).
\]
Derivation coefficients (20) can equivalently be written in terms of the spinor harmonics as follows:

\[ \Omega^{2-2}(d) = v_\alpha^a d u_{\dot{a}}^a - v_\alpha^- d u_{\dot{a}}^a, \quad \Omega^{2+}(d) = v_\alpha^a d u_{\dot{a}}^a \sigma^a, \]

\[ \Omega^{-2}(d) = v_\alpha^a d u_{\dot{a}}^a \sigma_{\dot{a}a}, \quad \Omega^{2j}(d) = v_\alpha^a d u_{\dot{a}}^a \sigma^\alpha \eta^j \eta^j - v_\alpha^- d u_{\dot{a}}^a \sigma^\alpha \eta^j \eta^j. \]

(29)

In (29), also present are the components of the inverse spinor harmonic matrix

\[ v_\alpha^a = (v_\alpha^a, v_\alpha^-) : \quad v_\alpha^a v_\alpha^a = \delta^a_\alpha. \]

(30)

that can either be considered as independent variables that obey the above constraints or be directly expressed via the spinor harmonics as

\[ v_\alpha^a = \frac{1}{i} \delta^{ab} \delta_{\alpha \beta} \epsilon_{\alpha \beta \gamma} \epsilon_{\gamma \delta} v_\gamma^b v_\gamma^b v_\gamma^b. \]

(31)

Then the expressions for the admissible differentials of supertwistors respecting the constraints (10) read

\[ d\sigma^{\Lambda+a} = -\frac{1}{4} \Omega^{2-2}(d) \sigma^{\Lambda+a} + \frac{1}{4} \Omega^{2+}(d) \sigma^{\Lambda-a} - \frac{1}{2} \Omega^{2j}(d) \sigma^{\Lambda-b} \]

\[ + (\omega^2(d) \sigma^{ab} + i D \eta^{\Lambda+a} \eta^i \eta^j + \frac{1}{2} \Omega^{2j}(d) \sigma^i \eta^j \eta^j - \frac{1}{2} \Omega^{2j}(d) \sigma^i \eta^j \eta^j) \]

\[ d\sigma^{\Lambda-b} = \frac{1}{4} \Omega^{2-2}(d) \sigma^{\Lambda-b} + \frac{1}{4} \Omega^{2-2}(d) \sigma^{\Lambda-a} - \frac{1}{2} \Omega^{2j}(d) \sigma^i \eta^j \eta^j \]

\[ + (\omega^2(d) \sigma^{ab} - i D \eta^{\Lambda-b} \eta^i \eta^j + \frac{1}{2} \Omega^{2j}(d) \sigma^i \eta^j \eta^j - \frac{1}{2} \Omega^{2j}(d) \sigma^i \eta^j \eta^j) \]

\[ + \frac{1}{2} D \sigma^a - D \sigma^a \]

(32)

where

\[ V_\alpha^{\Lambda-} = (v_\alpha^- - 0, 0), \quad V_\alpha^{\Lambda+} = (v_\alpha^a + 0, 0) \]

are the supertwistors associated with inverse spinor harmonics \( v_\alpha^- \), \( v_\alpha^a \), and the matrix

\[ J_\Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \delta^a_\alpha \end{pmatrix} \]

(34)

singles out Grassmann-odd components of the supertwistors.

To derive the equations of motion for the superstring in the supertwistor formulation we note that the differentials of 1-forms entering the action (25) equal

\[ d\omega_-^2 = \frac{1}{2} \Omega^{2-2}(d) \wedge \omega^2(d) - \omega^2(d) \wedge \omega^2(d) + \frac{1}{2} D \eta^{\Lambda+a} \wedge D \eta^{\Lambda+a}, \]

\[ d\omega_-^2 = -\frac{1}{2} \Omega^{2-2}(d) \wedge \omega^2(d) - \omega^2(d) \wedge \omega^2(d) - \frac{1}{2} D \eta^{\Lambda-a} \wedge D \eta^{\Lambda-a}, \]

\[ d\omega^j = -\frac{1}{2} \Omega^{2j}(d) \wedge \omega^2(d) - \frac{1}{2} \Omega^{2j}(d) \wedge \omega^2(d) - \omega^j(d) \wedge \omega^j(d) + \frac{1}{2} D \eta^{\Lambda-a} \wedge D \eta^{\Lambda-a} \]

(35)

and

\[ d\phi_-^2 = \frac{1}{2} \Omega^{2-2}(d) \wedge \phi^2(d) - \Omega^{2j}(d) \wedge \phi^2(d) - \frac{1}{2} D \eta^{\Lambda+a} \wedge D \eta^{\Lambda+a}, \]

\[ d\phi_-^2 = -\frac{1}{2} \Omega^{2-2}(d) \wedge \phi^2(d) - \Omega^{2j}(d) \wedge \phi^2(d) + \frac{1}{2} D \eta^{\Lambda-a} \wedge D \eta^{\Lambda-a}, \]

\[ d\phi^j = -\frac{1}{2} \Omega^{2j}(d) \wedge \phi^2(d) - \frac{1}{2} \Omega^{2j}(d) \wedge \phi^2(d) - \Omega^{2j}(d) \wedge \phi^2(d) + \frac{1}{2} D \eta^{\Lambda-a} \wedge D \eta^{\Lambda-a} \]

(36)
Then the variation of the superstring action (25) can be presented as

\[
\delta S_{\text{tw}}^{D=6} = \frac{1}{2 (\alpha')^{1/2}} \int d^2 \xi \left[ e^{*2} \wedge \left( -\frac{1}{2} \Omega^{2-2}(d) \omega^{-2}(\delta) + \frac{1}{2} \Omega^{2-2}(\delta) \omega^{-2}(d) - \Omega^{2-2}(d) \omega^{-2}(\delta) \right) \\
+ \Omega^{2-2}(\delta) \omega^{-2}(d) - i D \eta_i^{-\delta} D(\delta) \eta_i^{-s} \right) - d e^{*2} \omega^{-2}(\delta) + \delta e^{*2} \wedge \omega^{-2}(d) \\
- e^{-2} \wedge \left( \frac{1}{2} \Omega^{2-2}(d) \omega^{-2}(\delta) - \frac{1}{2} \Omega^{2-2}(\delta) \omega^{-2}(d) - \Omega^{2-2}(d) \omega^{-2}(\delta) \right) \\
+ \Omega^{2-2}(\delta) \omega^{-2}(d) + i D \eta_i^{-\delta} D(\delta) \eta_i^{-s} \right) + d e^{-2} \omega^{-2}(\delta) - \delta e^{-2} \wedge \omega^{-2}(d) \\
+ c(\alpha')^{1/2} (\delta e^{-2} \wedge e^{*2} + e^{-2} \wedge \delta e^{*2}) \right] \\
+ \frac{i s}{2 c(\alpha')} \int d^2 \xi \left\{ \omega^{*2}(d) \wedge D \eta_i^{-\delta} D(\delta) \eta_i^{-s} + \frac{1}{2} \omega^{*2}(\delta) D \eta_i^{-\delta} \wedge D \eta_i^{-s} \\
- \omega^{-2}(d) \wedge D \eta_i^{-\delta} D(\delta) \eta_i^{-s} = \frac{1}{2} \omega^{-2}(\delta) D \eta_i^{-\delta} \wedge D \eta_i^{-s} \\
- \left[ \omega^{*2}(d) \wedge (D \eta_i^{-\delta} D(\delta) \eta_i^{-s} + D \eta_i^{-\delta} D(\delta) \eta_i^{-s}) + \omega^{*2}(\delta) D \eta_i^{-\delta} \wedge D \eta_i^{-s} a_{\alpha}^I \right] \right\}.
\]

(37)

The equations of motion following from the nullification of the variation w.r.t. zweibein components and \(\Omega^{2-2j}(\delta)\) can be cast into the form

\[
\omega^{*2}(d) = c(\alpha')^{1/2} e^{*2}, \quad \omega^{*}(d) = 0
\]

(38)

and can be recognized as the supertwistor counterparts of the nondynamical equations (21). Note that the equations of motion corresponding to \(\Omega^{2-2j}(\delta)\) and \(\Omega^{2j}(\delta)\) appear to be trivial manifesting the \(SO(1, 1) \times SO(4)\) gauge invariance of the action functionals (14), (25).

Nullification of the variation w.r.t. \(\omega^{*2}(\delta)\) and \(\omega^{*}(\delta)\) yields the set of equations

\[
d e^{*2} + \frac{1}{2} e^{*2} \wedge \Omega^{2-2}(d) + \frac{i s}{2 c(\alpha')^{1/2}} D \eta_i^{-\delta} \wedge D \eta_i^{-s} = 0, \\
\frac{i s}{2 c(\alpha')^{1/2}} D \eta_i^{-\delta} \wedge D \eta_i^{-s} = 0.
\]

(39)

The first two of them are satisfied identically as the consequence of the reparameterization invariance of the action functional. Remaining fermionic equations following from the variation w.r.t. \(D(\delta) \eta_i^{-\delta} \wedge D(\delta) \eta_i^{-s}\) then acquire the form

\[
(1 + s) e^{-2} \wedge D \eta_i^{-s} = (1 - s) e^{*2} \wedge D \eta_i^{-s} = 0.
\]

(40)

The definite choice of the numerical factor \(s = \pm 1\) turns one of the above equations into identity being the manifestation of the \(\kappa\)-symmetry invariance.

Explicit form of the \(\kappa\)-symmetry transformation rules depends on the value of \(s\). When \(s = 1\) we have

\[
\delta_e Z^{\lambda+\alpha} = \frac{1}{2} \Omega^{2j}(\delta_e) \hat{D}_{\alpha} Z^{\lambda+}, \\
\delta_e Z^{\lambda-\alpha} = \frac{1}{2} \Omega^{2j}(\delta_e) \hat{D}_{\alpha} Z^{\lambda-} - \left( K^{\Sigma^{-\beta}} Z^\Sigma_{\beta} \right) V^\lambda_+ - \left( K^{\Sigma^{-\beta}} Z^\Sigma_{\beta} \right) V^\lambda_- + K^{\lambda-\alpha}, \\
\delta_e e^{*2} = 0, \quad \delta_e e^{-2} = \frac{1}{c(\alpha')^{1/2}} K^{\lambda-\alpha} D Z^{\lambda-},
\]

(41)
where
\[
\Omega^{±2i}(δ_κ) = \pm \frac{1}{c(α')}^{1/2} \mathcal{L}_{+}^{-b} e^{v±2} D_{ν} Z^a_{ν} a^f_{ab} \tag{42}
\]
and the local parameter \( κ_{i+}\xi(ρ) \) is presented in the superwistor form as \( K^{α−b} = (0, 0, κ_{i+}) \).
For the \( s = -1 \) case \( κ_{i+}(ξ) \) is presented in the superwistor form as \( K^{α−b} = (0, 0, κ_{i+}) \).

Following the lines of the discussion in [31] of the four-dimensional superstring, it is possible to simplify the action functional (25) by covariantly gauge-fixing \( κ_{i+} \). This can be achieved by substituting (38) into the WZ term

\[
S_{WZ|gf} = \frac{iε}{2(α')^{1/2}} \int d^2ξ (v^{+2}(ξ) + e^{−2} \wedge θ^{+2}(ξ)).
\tag{45}
\]

Summing up this expression with that for the kinetic term in (25) produces the action functional

\[
S_{gf}^{D=6} = \frac{1}{2(α')^{1/2}} \int d^2ξ \left[ (v^{+2}(ξ) + iεθ^{−2}(ξ)) - e^{−2} \wedge (v^{+2}(ξ) - iεθ^{−2}(ξ)) \right]
+ \frac{ε}{2} \int d^2ξ \ v^{+2} \wedge e^{−2} \tag{46}
\]

Further choosing \( s = 1 \) one obtains

\[
S_{gf}^{D=6} = \frac{1}{4(α')^{1/2}} \int d^2ξ \left[ (v^{+2}(ξ) + iεθ^{−2}(ξ)) - e^{−2} \wedge (v^{+2}(ξ) - iεθ^{−2}(ξ)) \right] + \frac{ε}{2} \int d^2ξ \ e^{−2} \wedge e^{+2},
\tag{47}
\]

and correspondingly when \( s = -1 \)

\[
S_{gf}^{D=6} = \frac{1}{4(α')^{1/2}} \int d^2ξ \left[ (v^{+2}(ξ) - iεθ^{−2}(ξ)) - e^{−2} \wedge (v^{+2}(ξ) + iεθ^{−2}(ξ)) \right] + \frac{ε}{2} \int d^2ξ \ e^{−2} \wedge e^{+2}.
\tag{48}
\]

\( z^{α+} \) and \( z^{α−} \) are bosonic \( D = 6 \) twistors that can be identified as the Spin(6, 2) = Spin(8*) symplectic Majorana–Weyl spinors

\[
(c^{α+})^* = B^a_{β} e_{ab} c^{β+}, \quad (c^{α−})^* = B^a_{β} e_{ab} c^{β−}. \tag{49}
\]

In (47), (48) it has also been assumed the following redefinition of the superwistor components:

\[
\tilde{Z}^{α+} = (\tilde{μ}^{α+}, \nu^{α+}, \tilde{η}^{α+}); \quad \tilde{μ}^{α+} = v^{α+}(\lambda^{α+} - 4iν^{0+}θ^{α+}), \quad \tilde{η}^{α+} = 4ν^{0+}θ^{α+},
\]

\[
\tilde{Z}^{α−} = (\tilde{μ}^{α−}, v^{α−}, \tilde{η}^{α−}); \quad \tilde{μ}^{α−} = v^{α−}(\lambda^{α−} - 4iν^{0−}θ^{α−}), \quad \tilde{η}^{α−} = 4ν^{0−}θ^{α−},
\tag{50}
\]

and the \( OSp(8^*|2) \) metric

\[
\tilde{G}_{αβ} = \begin{pmatrix} 0 & \delta^α_β & 0 \\ \delta^α_β & 0 & 0 \\ 0 & 0 & -\frac{1}{2} e_{ij} \end{pmatrix}.
\tag{51}
\]
Supertwistor formulation for higher-dimensional superstrings

It is readily possible to extend the above results for the case of $D = 6, N = (2, 0)$ superstring. Following the same reasoning and choosing the $\kappa$-symmetry gauge as $\eta^{1+4} = \eta^{2+3}, \eta^{1+0} = \eta^{2-3}$ we arrive at the action functional

$$S_{gf}^{D=6, N=(2,0)} = \frac{1}{4(g')^{1/2}} \int d^2 \xi \left( e^{x^2} \wedge d\tilde{Z}_\Lambda^\Lambda - \tilde{Z}_\Lambda^\Lambda + e^{-2} \wedge d\tilde{Z}_\Lambda^{\Lambda+} \tilde{Z}_{\Lambda+} + \xi \right) + \frac{c}{2} \int d^2 \xi e^{-2} \wedge e^{x^2}$$

(52)

formulated in terms of $N = 1$ supertwistors (50). It should be noted that (super)twistors entering gauge-fixed actions (47), (48), (52) satisfy algebraic constraints analogous to (10) that represent the mixture of the first-class constraints generating $SU(2) \times SU(2)$ gauge symmetry and the second-class ones.

4. Lorentz harmonics and $D = 10$ supertwisters

$D = 10$ supertwistor can be defined as realizing the fundamental representation of the minimal superconformal group (according to the classification developed in [47]) that contains $D = 10$ conformal group and is isomorphic to $OSp(1|32)$

$$Z^\Lambda = (\mu^\hat{\alpha}, v_{\hat{\alpha} A}, \eta)$$

(53)

The supertwistor is composed of the pair of Spin$((1, 9)$) Majorana–Weyl spinors of opposite chiralities and the Grassmann-odd scalar. The application to the $D = 10$ superstring description suggests considering the set of eight supertwistors

$$Z^\Lambda_A = (\mu^\hat{\alpha}_A, v_{\hat{\alpha} A}, \eta_A)$$

(54)

with the label $A = 1, \ldots, 8$ corresponding to $8_0$ or $8_i$ representation of the covering of transverse Lorentz group $SO(8)$ in 10d. Supertwistor components are assumed to be incident to $D = 10, N = 1$ superspace coordinates $(x^\hat{m}, \theta^\hat{m})$ as follows:

$$\mu^\hat{\alpha}_A = (x^{\hat{m}} \hat{\alpha} - 8i \theta^{\hat{m}} \hat{\alpha}) v_{\hat{\alpha} A}, \quad \eta_A = 4 v_{\hat{\alpha} A} \theta^\hat{m},$$

(55)

where $x^{\hat{m}} = x^{\hat{m}} \hat{\alpha} (\hat{m} = 0, 1, \ldots, 9)$ and symmetric matrices $\sigma_{\hat{m} \hat{m}^\prime}$ represent $D = 10$ chiral $\gamma$-matrices.

Following the consideration of section 2, we identify the projectional parts of the supertwistors (54) as the $D = 10$ spinor Lorentz harmonic matrix

$$v^{\hat{\alpha}}_A = (v^+_{\hat{\alpha} A}, v^-_{\hat{\alpha} A}) \in \text{Spin}(1, 9)$$

(56)

with the spinor index in brackets split according to the spinor representations of $SO(1, 1) \times SO(8)$. Thus we have the following sets of supertwistor variables associated with $D = 10$ superstring:

$$Z^\Lambda_A = (\mu^\hat{\alpha}_A, v^+_{\hat{\alpha} A}, \eta^+_A), \quad Z_{\Lambda}^\Lambda_A = (\mu^\hat{\alpha}_A, v^-_{\hat{\alpha} A}, \eta^-_A)$$

(57)

briefly discussed in [20]. They should obey the constraints

$$Z^\Lambda_A G_{\Lambda \Sigma} Z^\Sigma_B = Z^\Lambda_A G_{\Lambda \Sigma} Z^\Sigma_B = Z^\Lambda_A G_{\Lambda \Sigma} Z^\Sigma_B = 0,$$

(58)

which exclude the contribution to (55) of $v^{\hat{\alpha}}_A = (v^+_{\hat{\alpha} A}, v^-_{\hat{\alpha} A})$ coordinates antisymmetric in spinor indices, as well as

$$\sigma_{\hat{m} \hat{m}^\prime} (\mu^\hat{\alpha}_A v^+_{\hat{\alpha} A} + \mu^\hat{\alpha}_A v^-_{\hat{\alpha} A}) = 0$$

(59)
that remove the contribution of 5-form coordinates $\omega_5 = \omega^{ab}_{\mu
u}\sigma^a_{\mu\nu}\omega_5$ associated with tensorial central charge (TCC) generators $Z_{\mu\nu\lambda}$ of the $OSp(1|32)$ superalgebra. Equation (59) involves inverse spinor harmonics

$$v_{a\alpha}(\phi) = \left(v_a^+ \cdot v^a_{\alpha}\right): \quad v_{a\alpha}(\phi) v_{a\beta}(\phi) = \delta_{\alpha}^\beta$$

and the $OSp(1|32)$ invariant product of supertwistors is obtained via the contraction with the orthosymplectic metric

$$G_{\alpha\beta} = \begin{pmatrix} 0 & \delta_{\alpha}^\beta & 0 \\ -\delta_{\alpha}^\beta & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$  

(61)

5. Supertwistor form of $D = 10$ superstring action

Consider now the representation for $D = 10$, $N = 1$ superstring action in terms of the above-introduced $OSp(1|32)$ supertwistors. Following the discussion in lower dimensions we start with the $D = 10$ Lorentz-harmonic superstring action [32]

$$S_{\text{LH}}^{D=10} = S_{\text{kin}} + S_{WZ}$$

(62)

with

$$S_{\text{kin}} = \frac{1}{2(\alpha')^{1/2}} \int d^2 \xi \left( e^{s_2} \eta_{\hat{a}}(\phi^-) - e^{-s_2} \eta_{\hat{a}}(\phi^+) \right) \wedge \omega_{\hat{a}}(\phi) + \frac{c}{2} \int d^2 \xi \varepsilon e^{-s_2} \wedge e^{s_2},$$

(63)

and

$$S_{WZ} = \frac{ie}{c\alpha'} \int d^2 \xi \omega_{\hat{a}}(\phi) \wedge d\theta_{\hat{a}} \sigma_{\hat{a}\hat{b}} \theta_{\hat{b}},$$

(64)

where $\omega_{\hat{a}}(\phi) = d\chi_{\hat{a}} - i d\theta_{\hat{a}} \sigma_{\hat{a}\hat{b}} \theta_{\hat{b}}$ is the $D = 10$, $N = 1$ supersymmetric 1-form. The action (63) also includes $n_{\hat{a}}^{2\pm}(\phi)$ components of the string adopted $D = 10$ vector harmonic matrix $n_{\hat{a}}(\phi) = (n_{\hat{a}}^{2\pm}, n_{\hat{a}}^{\pm})$

$$n_{\hat{a}}^{2\pm}(\phi) = n_{\hat{a}}^{2\pm}, \quad n_{\hat{a}}^{\pm}(\phi) = n_{\hat{a}}^{\pm},$$

(65)

that have the following representation in terms of the spinor harmonics:

$$n_{\hat{a}}^{2\pm} = \frac{1}{2} v_{\hat{a}}^+ \sigma_{\hat{a}B} \tilde{v}_{\hat{B}}^+, \quad n_{\hat{a}}^{\pm} = \frac{1}{2} v_{\hat{a}}^+ \sigma_{\hat{a}B} \tilde{v}_{\hat{B}}^-, \quad n_{\hat{a}}^{-2} = \frac{1}{2} \tilde{v}_{\hat{a}}^- \sigma_{\hat{a}B} v_{\hat{B}}^+, \quad n_{\hat{a}}^{-2} = \frac{1}{2} \tilde{v}_{\hat{a}}^- \sigma_{\hat{a}B} v_{\hat{B}}^-, \quad \gamma_{B\bar{A}},$$

(66)

where $I, J = 1, \ldots, 8$ stand for the $SO(8)$ vector indices and $\gamma_{B\bar{A}}$ are the $8d$ analogs of the four-dimensional $\sigma$-matrices. Their differentials consistent with the constraints (65)

$$d n_{\hat{a}}^{2\pm} = \frac{1}{2} \Omega^{2\pm} (d) n_{\hat{a}}^{2\pm} + \hat{\Omega}^{2\pm} (d) n_{\hat{a}}^{2\pm},$$

$$d n_{\hat{a}}^{\pm} = \frac{1}{2} \Omega^{\pm} (d) n_{\hat{a}}^{\pm} + \hat{\Omega}^{\pm} (d) n_{\hat{a}}^{\pm},$$

(67)

depend on the $SO(1, 1) \times SO(8)$ split components of the Cartan 1-form

$$\hat{\Omega}^{(2,1)}(d) = \frac{1}{2} (n_{\hat{a}}^{2\pm} dn_{\hat{a}}^{2\pm} - n_{\hat{a}}^{-2} dn_{\hat{a}}^{-2}),$$

$$\hat{\Omega}^{(2,2)}(d) = \frac{1}{2} (n_{\hat{a}}^{2\pm} dn_{\hat{a}}^{2\pm} - n_{\hat{a}}^{2\pm} dn_{\hat{a}}^{-2}), \quad \hat{\Omega}^{(2,1)}(d) = \frac{1}{2} (n_{\hat{a}}^{2\pm} dn_{\hat{a}}^{2\pm} - n_{\hat{a}}^{2\pm} dn_{\hat{a}}^{-2}).$$

(68)
invariant under $SO(1, 9)$ Lorentz transformations acting on the indices without brackets.

Similarly to the lower-dimensional cases, twistorization of the action (62) proceeds by expressing $\omega^2(d)$ projections on the vector harmonics (and using (66)) through the $D = 10$ supertwistors (57)

$$\sigma^{+2}(d) = \omega^2(d)n^2_{\omega} = \frac{1}{8}dZ_A^{+}G_{\alpha\Sigma}Z_\Sigma^{+},$$
$$\sigma^{-2}(d) = \omega^2(d)n^{-2}_{\omega} = \frac{1}{8}dZ_A^{-}G_{\alpha\Sigma}Z_\Sigma^{-},$$

and

$$\sigma^I(d) = \omega^2(d)n^I_{\omega} = \frac{1}{16}\nu^I_{\alpha\beta} (dZ_A^{+}G_{\alpha\Sigma}Z_\Sigma^{+} + dZ_A^{-}G_{\alpha\Sigma}Z_\Sigma^{-}).$$

provided $Z_A^{+}$ and $Z_A^{-}$ obey the constraints (58), (59) so that the $D = 10$ superstring action (62) acquires the form

$$S_{tw}^{D=10} = \frac{1}{2(\alpha')^{1/2}} \int d^2\bar{\xi}(e^{+2} \wedge \sigma^{-2}(d) - e^{-2} \wedge \sigma^{+2}(d)) + \frac{c}{2} \int d^2\bar{\xi} e^{-2} \wedge e^{+2}$$
$$+ \frac{i\bar{\delta}}{c\alpha'} \int d^2\bar{\xi} \left( \frac{1}{2} \sigma^{-2}(d) \wedge \Phi^{+2}(d) + \frac{1}{2} \sigma^{+2}(d) \wedge \Phi^{-2}(d) - \sigma^I(d) \wedge \Phi^I(d) \right).$$

In (71), $\phi(d)$ are the 1-forms quadratic in the Grassmann-odd supertwistor components

$$\phi^{+2}(d) = \frac{1}{8}D\eta_+^A\eta_+^A,$$
$$\phi^{-2}(d) = \frac{1}{8}D\eta_-^A\eta_-^A,$$
$$\phi^I(d) = \frac{1}{16}\nu^I_{\alpha\beta} (D\eta_+^A\eta_-^A + D\eta_-^A\eta_+^A),$$

and $SO(1, 1) \times SO(8)$ covariant differentials of $\eta^A_+$ and $\eta_-$ are defined as

$$D\eta_+^A = d\eta_+^A + \frac{1}{4}\Omega^{+2}(d)\eta_+^A - \frac{1}{4}\Omega^{+2}(d)\gamma^I_{\alpha\beta}D\eta_+^A\eta_+^A,$$
$$D\eta_-^A = d\eta_-^A - \frac{1}{4}\Omega^{+2}(d)\eta_-^A - \frac{1}{4}\Omega^{+2}(d)\gamma^I_{\alpha\beta}D\eta_-^A\eta_-^A,$$

where $\gamma^I_{\alpha\beta} = \frac{1}{2}(\gamma^I_{\alpha\beta}\gamma_+^\beta - \gamma_+^\beta\gamma^I_{\alpha\beta}), \tilde{\gamma}^I_{\alpha\beta} = \frac{1}{2}(\gamma^I_{\alpha\beta}\gamma_-^\beta - \gamma_-^\beta\gamma^I_{\alpha\beta})$ are the Spin(8) generators in the $s$ and $c$ representations.

Admissible differentials of the supertwistors (57) that respect the constraints (58), (59) can be brought to the form

$$dZ_A^{\pm} = -\frac{1}{4}\Omega^{\pm2}(d)Z_A^{\pm} + \frac{1}{4}\hat{\Omega}^{\pm2}(d)\nu_+^A\nu_+^A + \nu_-^A\nu_-^A + \frac{1}{4}\hat{\Omega}^{\pm2}(d)\gamma^I_{\alpha\beta}DZ_A^{\pm};$$
$$dZ_A^{\pm} = \frac{1}{4}\Omega^{\pm2}(d)Z_A^{\pm} + \frac{1}{4}\hat{\Omega}^{\pm2}(d)\nu_-^A\nu_-^A + \nu_+^A\nu_+^A + \frac{1}{4}\hat{\Omega}^{\pm2}(d)\gamma^I_{\alpha\beta}DZ_A^{\pm};$$

where $V_A^{\pm} = (\nu_-^A, 0, 0), V_A^{\pm} = (\nu_+^A, 0, 0)$ and

$$J_\Sigma^A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$  

In deriving (74) there were used the expressions for the admissible differentials of the spinor harmonics

$$dv^I_{\alpha A} = -\frac{1}{2}\Omega^{\pm2}(d)v^I_{\alpha A} + \frac{1}{2}\hat{\Omega}^{\pm2}(d)v^I_{\alpha A} + \frac{1}{2}\hat{\Omega}^{\pm2}(d)v^I_{\alpha A}v^I_{\beta B}v^\beta_-, $$
$$dv^I_{\alpha A} = \frac{1}{2}\Omega^{\pm2}(d)v^I_{\alpha A} + \frac{1}{2}\hat{\Omega}^{\pm2}(d)v^I_{\alpha A}v^I_{\beta B}v^\beta_-. $$
The derivation coefficients (68) can equivalently be rewritten in terms of the spinor harmonics as

\[ \Omega^{+2}(d) = \frac{1}{2} (d \varphi_a^+ \bar{\nu}_A^a - d \bar{\nu}_A^a \varphi_a^+), \]
\[ \Omega^{\pm 2}(d) = \frac{1}{4} d \varphi_a^+ \gamma_A^a \bar{\nu}_A^a, \]
\[ \Omega^{-2}(d) = \frac{1}{4} d \nu_a^- \gamma_A^a \bar{\nu}_A^a, \]
\[ \Omega^{IJ}(d) = \frac{1}{8} (d \varphi_a^+ \gamma^I_A \nu_a^- + d \bar{\nu}_A^+ \gamma^I_A \bar{\nu}_A^-). \]

Then the differentials of the 1-forms (69), (70), (72) equal

\[ d\sigma^+ = -\frac{1}{2} \Omega^{+2}(d) \wedge \sigma^+ + \Omega^{+2}(d) \wedge \sigma^+(d) - \frac{1}{8} D\eta^+_A \wedge D\eta^-_A, \]
\[ d\sigma^{-} = -\frac{1}{2} \Omega^{+2}(d) \wedge \sigma^{-} + \Omega^{+2}(d) \wedge \sigma^{-}(d) - \frac{1}{8} D\eta^-_A \wedge D\eta^-_A, \]
\[ d\sigma^I = -\frac{1}{2} \Omega^{+2}(d) \wedge \sigma^I + \Omega^{+2}(d) \wedge \sigma^I(d) \]
\[ - \frac{1}{8} D\eta^+_A \wedge D\eta^-_A \gamma^I_A (78) \]

and

\[ d\phi^+ = \frac{1}{2} \Omega^{+2}(d) \wedge \phi^+ + \Omega^{+2}(d) \wedge \phi^+(d) + \frac{1}{4} D\eta^+_A \wedge D\eta^-_A, \]
\[ d\phi^- = -\frac{1}{2} \Omega^{+2}(d) \wedge \phi^- + \Omega^{+2}(d) \wedge \phi^-(d) + \frac{1}{4} D\eta^-_A \wedge D\eta^-_A, \]
\[ d\phi^I = -\frac{1}{2} \Omega^{+2}(d) \wedge \phi^I + \Omega^{+2}(d) \wedge \phi^I(d) + \frac{1}{4} D\eta^+_A \wedge D\eta^-_A \gamma^I_A, \]

so that the action (71) produces the following variation:

\[ \delta S^{D=10}_{\text{num}} = \frac{1}{2(\alpha')^{1/2}} \int d^2 \xi \left[ e^{\sigma^2} \wedge \left( -\frac{1}{2} \Omega^{+2}(\sigma) \sigma^2 - \frac{1}{2} \Omega^{+2}(\sigma) \sigma^-(\sigma) \right) \right. \]
\[ - \frac{1}{4} D\eta^+_A D(\delta) \eta^-_A \]
\[ - e^{\sigma^2} \eta^2(\delta) \eta^2(\delta) + e^{\sigma^2} \wedge \eta^2(\delta) \]
\[ - e^{\sigma^2} \eta^2(\delta) \eta^2(\delta) - \frac{1}{2} \Omega^{+2}(\sigma) \sigma^2(\sigma) + \Omega^{+2}(\sigma) \sigma^2(\sigma) - \Omega^{+2}(\sigma) \sigma^2(\sigma) \]
\[ + \frac{1}{4} D\eta^+_A D(\delta) \eta^+_A \]
\[ + e^{\sigma^2} \eta^2(\delta) \eta^2(\delta) - e^{\sigma^2} \eta^2(\delta) \eta^2(\delta) \]
\[ + c(\alpha')^{1/2}(\delta) e^{\sigma^2} \eta^2(\delta) e^{\sigma^2} \eta^2(\delta) \]
\[ + \frac{i\delta}{8c\alpha'} \int d^2 \bar{\xi} \left[ \sigma^2(\bar{\sigma}) \wedge D\eta^-_A D(\delta) \eta^-_A + \frac{1}{2} \sigma^2(\bar{\sigma}) D\eta^-_A \wedge D\eta^-_A \right. \]
\[ + \sigma^2(\bar{\sigma}) \wedge D\eta^+_A D(\delta) \eta^+_A + \frac{1}{2} \sigma^2(\bar{\sigma}) D\eta^+_A \wedge D\eta^+_A \]
\[ - \left[ \sigma^I(\delta) \wedge (D\eta^-_A D(\delta) \eta^-_A + D\eta^-_A D(\delta) \eta^-_A) + \sigma^I(\delta) D\eta^+_A \wedge D\eta^+_A \right] \gamma^I_A. \] (80)

Considering \( \sigma^{+2}(\delta), \sigma^I(\delta), D(\delta) \eta^+_A, D(\delta) \eta^-_A \) together with \( \Omega^{+2}(\delta), \Omega^{+2}(\delta), \Omega^{IJ}(\delta) \) as independent variations yields the super-twistor representation of the \( D = 10, N = 1 \) superstring equations of motion. There are present nondynamical equations

\[ \sigma^{+2}(d) = c(\alpha')^{1/2} e^{\pm 2}, \quad \sigma^I(d) = 0 \] (81)
that are the $D = 10$ analogs of the superspace formulation equations (21). Note that the equations of motion resulting from the variation w.r.t. $\hat{\Omega}^{\pm 2}(\delta)$ and $\hat{\Omega}^{IJ}(\delta)$ trivialize which is the consequence of the $SO(1, 1) \times SO(8)$ gauge invariance of the action (62), (71). Other independent bosonic equations of motion read

$$e^{-2} \wedge \hat{\Omega}^{\pm 2}(\delta) - e^{-2} \wedge \hat{\Omega}^{-2}(\delta) - \frac{ix}{4c(\alpha')^{1/2}} D\eta^*_A y_{AA}^I \wedge D\eta^-_A = 0. \tag{82}$$

Taking into account above-derived equations, one is able to present the equations of motion for the Grassmann-odd components of supertwistors in the following form:

$$(1 + s) e^{-2} \wedge D\eta^*_A = 0, \quad (1 - s) e^{-2} \wedge D\eta^-_A = 0 \tag{83}$$

similarly to the $D = 6$ superstring.

The supertwistor representation (71) of the $D = 10$ Lorentz-harmonic superstring action (62) is thus invariant under the $\kappa$-symmetry transformations. When $s = 1$ the explicit transformation laws are

$$\delta_s Z^+_A = \frac{1}{2} \hat{\Omega}^{\pm 2}(\delta_s) y_{AA}^I Z^+_A,$$

$$\delta_s Z^-_A = \frac{1}{2} \hat{\Omega}^{-2}(\delta_s) y_{AA}^I Z^-_A - \left(K^+_A Z^-_A - (K^+_A Z^-_B) - (K^+_A Z^+_B) V^{A+} - (K^-_A Z^+_B) V^{A-} + K^+_A Z^+_A, \tag{84}$$

$$\delta_s e^{+2} = 0, \quad \delta_s e^{-2} = \frac{1}{4c(\alpha')^{1/2}} DZ^+_A K^-_A,$$

where

$$\hat{\Omega}^{\pm 2}(\delta_s) = \pm \frac{1}{8c(\alpha')^{1/2}} e^{\mu \pm 2} D_{\mu} Z^+_A \eta^I_{AA}, \tag{85}$$

and $K^+_A = (0, 0, \kappa^+_A)$ is the supertwistor realization of the gauge parameter $\kappa^+_A(\xi)$. Correspondingly when $s = -1$ we obtain

$$\delta_s Z^+_A = \frac{1}{2} \hat{\Omega}^{\pm 2}(\delta_s) y_{AA}^I Z^+_A - \left(K^+_A Z^-_A - (K^+_A Z^-_B) - (K^+_A Z^+_B) V^{A+} - (K^-_A Z^+_B) V^{A-} + K^+_A Z^+_A, \tag{86}$$

$$\delta_s e^{+2} = \frac{1}{4c(\alpha')^{1/2}} DZ^+_A K^+_A, \quad \delta_s e^{-2} = 0,$$

where

$$\hat{\Omega}^{\pm 2}(\delta_s) = \pm \frac{1}{8c(\alpha')^{1/2}} e^{\mu \pm 2} K^+_A D_{\mu} Z^-_A \eta^I_{AA}, \tag{87}$$

with $K^+_A = (0, 0, \kappa^+_A)$.

Quite analogously to the $D = 4$ and $D = 6$ cases, the supertwistor action (71) can be simplified if the gauge freedom related to $\kappa$-symmetry is fixed. The gauge-fixed action form depends on the value of $s$ and is given by the expressions

$$S^{D=10}_{s=10} = \frac{1}{16(\alpha')^{1/2}} \int d^2 \xi \left(e^{2} \wedge d \bar{Z}^+_A \bar{Z}^-_A - e^{-2} \wedge d \bar{Z}^+_A \bar{Z}^-_A \right) + \frac{c}{2} \int d^2 \xi \ e^{2} \wedge e^{2} \tag{88}$$

or

$$S^{D=10}_{s=-10} = \frac{1}{16(\alpha')^{1/2}} \int d^2 \xi \left(e^{2} \wedge d \bar{Z}^+_A \bar{Z}^-_A - e^{-2} \wedge d \bar{Z}^+_A \bar{Z}^-_A \right) + \frac{c}{2} \int d^2 \xi \ e^{2} \wedge e^{2}. \tag{89}$$
In (88), (89) \( z^A_+, z^A_- \) stand for bosonic \( Sp(32) \) twistors and the following redefinition of the supertwistor components has been assumed:

\[
\begin{align*}
\hat Z^A_+ &= (\hat \mu^\alpha_A, \nu^{\hat \beta, A}_+, \eta^\alpha_A) : \\
\hat Z^A_- &= (\hat \mu^\alpha_A, \nu^{\hat \beta, A}_-, \tilde \eta^\alpha_A) :
\end{align*}
\]

\[
\hat \mu^\alpha_A = (\lambda \hat \alpha^\alpha_A - 16 i \theta^\alpha \theta^\beta) \nu^{\hat \beta, A}_+ , \quad \eta^\alpha_A = 8 \nu^{\hat \alpha, A}_+ \theta^\beta , \\
\hat \mu^\alpha_A = (\lambda \hat \alpha^\alpha_A - 16 i \theta^\alpha \theta^\beta) \nu^{\hat \beta, A}_- , \quad \tilde \eta^\alpha_A = 8 \nu^{\hat \alpha, A}_- \theta^\beta ,
\]

(90)

and the \( OSp(32|1) \) metric

\[
\hat G_{AB} = \begin{pmatrix} 0 & \delta_{\hat \alpha}^\beta & 0 \\
-\delta_{\hat \beta}^\alpha & 0 & 0 \\
0 & 0 & -\frac{i}{2} \end{pmatrix} .
\]

(91)

Note that the above (super)twistors still have to satisfy the constraints analogous to (58) and (59) to be incident to \( D = 10 \) vector coordinates \( x^\mu \).

The above consideration of the twistor transform for \( D = 10 \), \( N = 1 \) superstring in the Lorentz-harmonic formulation can be generalized to include type II case as well. In particular, for the type IIB superstring choosing the \( \kappa \)-symmetry gauge conditions \( \eta^\alpha_{I A} = \eta^{A}_+, \eta^\alpha_{I A} = \eta^A_- \) results in the action functional with quadratic dependence on the \( N = 1 \) supertwistor variables (90)

\[
S^{D=10, IIB}_{\hat t} = \frac{1}{16 (\alpha')^{1/2}} \int d^2 \xi (e^{+2} \wedge d\hat Z^A_+ \hat G_{AB} \hat Z^B_+ - e^{-2} \wedge d\hat Z^A_- \hat G_{AB} \hat Z^B_- ) \\
+ \frac{c}{2} \int d^2 \xi \ e^{-2} \wedge e^{+2} .
\]

(92)

In the conclusion, let us discuss the relation between the proposed supertwistor formulation of the \( D = 10 \) Lorentz-harmonic superstrings and the light-cone gauge formulation of the Green–Schwarz superstrings on example of the twistor transformed type IIB superstring action (92). To this end it is convenient to introduce Lorentz-harmonics normalized up to a scale factor

\[
v^{(\hat \alpha)}_{\mu, A} = n\delta^{(\hat \alpha)}_{\mu, A} , \quad n^{(\hat \alpha) \mu}_{\hat \mu} (\hat \xi) = n^3 \Lambda (\hat \xi) .
\]

(93)

The corresponding modification of the action (92) affects only the last term

\[
\frac{c}{2} \int d^2 \xi \ e^{-2} \wedge e^{+2} \rightarrow \frac{c}{2} \int d^2 \xi n^2 e^{-2} \wedge e^{+2} .
\]

(94)

Its advantage is that the action functional becomes invariant under the gauge Weyl scalings

\[
e^{-\lambda} e^{+2} \rightarrow e^{-\lambda^\prime} e^{+2} , \quad \hat Z^A_+ \rightarrow e^{\lambda/2} \hat Z^A_+ , \quad \hat Z^A_- \rightarrow e^{\lambda/2} \hat Z^A_-
\]

(95)

that allows us to gauge away all the zweibein components, e.g., as

\[
e^{\lambda} = (\alpha')^{-1/2} \delta^\mu_\mu^\mu , \quad e^{\lambda} = (\alpha')^{-1/2} \delta^\mu_\mu^\mu .
\]

(96)

Then consider the expansion of the supertwistor primary spinor components over the basis of inverse spinor harmonics

\[
\mu^\alpha_A = \frac{1}{n} \left[ \left( x^{+2} \delta_{AB} + \frac{i}{4} \eta^\alpha_A \eta^\beta_B \right) v^{\hat \beta -}_{AB} + \left( x^{fl} \delta_{AB} + \frac{i}{4} \eta^\alpha_A \eta^\beta_B \right) v^{\hat \beta +}_{AB} \right] ,
\]

\[
\mu^\alpha_A = \frac{1}{n} \left[ \left( x^{-2} \delta_{AB} + \frac{i}{4} \eta^\alpha_A \eta^\beta_B \right) v^{\hat \beta -}_{AB} + \left( x^{fl} \delta_{AB} + \frac{i}{4} \eta^\alpha_A \eta^\beta_B \right) v^{\hat \beta +}_{AB} \right] ,
\]

(97)

where

\[
x^{\pm 2} = x^{\pm 2} n^{\pm 2}_D , \quad x^{fl} = x^{fl} n^{fl}_D .
\]

(98)
Expansion (97) is used to represent 1-forms quadratic in the supertwistors that enter the action (92) as
\[
\frac{1}{8} d\tilde{Z}_A^\dagger \tilde{G}_{\Lambda \Sigma} \tilde{Z}_A^\Sigma = \text{dx}^{-2} - \left( \frac{d\Omega^2 - 2}{2} \tilde{\Omega}^{-2} \right) x^{-2} - \tilde{\Omega}^{-2}(d) \left( x'^I + i \tilde{\eta}^\gamma \tilde{\gamma}^J \tilde{\eta}^J \right)
\]
\[
\frac{1}{8} d\tilde{Z}_A^\dagger \tilde{G}_{\Lambda \Sigma} \tilde{Z}_A^\Sigma = \text{dx}^{-2} - \left( \frac{d\Omega^2 - 2}{2} \tilde{\Omega}^{-2} \right) x^{-2} - \tilde{\Omega}^{-2}(d) \left( x'^I + i \tilde{\eta}^\gamma \tilde{\gamma}^J \tilde{\eta}^J \right)
\]
\[
\frac{1}{64} \tilde{\Omega}^{IJ}(d) \eta^{-\alpha} \tilde{\gamma}^{J\beta} \tilde{\eta}^J, \quad \text{(99)}
\]
so that the action (92) acquires the form
\[
S_{\text{gt}}^{D=10,IIIB} = \frac{1}{2(\alpha')^{1/2}} \int d^2 \xi \left[ \frac{1}{64} \tilde{\Omega}^{IJ}(d) \eta^{-\alpha} \tilde{\gamma}^{J\beta} \tilde{\eta}^J \right] + \frac{c}{2} \int d^2 \xi \eta^{-2} \wedge e^2
\]
\[
\text{depending on the components of the Cartan form (77) and } D = 10, N = 1 \text{ superspace coordinates contracted with the vector harmonics}^2.
\]
Consider now the representation for the Cartan 1-form components defined by the pair of unconstrained 8-vectors \( p^{\pm2} \)
\[
\tilde{\Omega}^{2}(d) = 2(dp^{21}p^{-21} - p^{21} dp^{p21}), \quad \tilde{\Omega}^{2}(d) = 2dp^{p21},
\]
\[
\tilde{\Omega}^{J}(d) = p^{p21} dp^{p21} - dp^{p21} p^{p21} + dp^{p21} p^{-21} + dp^{p21} d p^{p21}
\]
\[
\text{that up to quadratic terms satisfy Maurer–Cartan equations}
\]
\[
d\tilde{\Omega}^{2}(d) \wedge \tilde{\Omega}^{-2}(d) = 0,
\]
\[
d\tilde{\Omega}^{21} + \frac{1}{2} \tilde{\Omega}^{-2} \wedge \tilde{\Omega}^{21} \wedge \tilde{\Omega}^{IJ}(d) \wedge \tilde{\Omega}^{IJ}(d) = 0,
\]
\[
dx^{I} + \left( \frac{1}{2} \tilde{\Omega}^{21}(d) \wedge \tilde{\Omega}^{-2} \wedge \tilde{\Omega}^{21}(d) \wedge \tilde{\Omega}^{IJ}(d) \wedge \tilde{\Omega}^{K}(d) \wedge \tilde{\Omega}^{IJ}(d) \right) = 0.
\]
Upon substituting (102) into (101), choosing the conformal gauge for the zweibein (96) and concentrating on the quadratic terms one arrives at the following action:
\[
S_{\text{c}}^{IIIB} = -\frac{2}{\alpha'} \int d^2 \xi (p^{x21} \partial_{x2} x^I + p^{x21} \partial_{x2} x^I - 2 cp^{x21} p^{-21})
\]
\[
+ \frac{i}{16 \alpha'} \int d^2 \xi \left( \partial_{x2} \tilde{\eta}^{\alpha} \tilde{\gamma}^{J} \tilde{\eta}^J + \partial_{x2} \tilde{\eta}^{\alpha} \tilde{\gamma}^{J} \tilde{\eta}^J \right)
\]
\[
\text{that has to be supplemented by the Virasoro constraints}
\]
\[
\partial_{x2} \eta^{x2} + 2 p^{-21} \partial_{x2} x^I - \frac{i}{16} \partial_{x2} \tilde{\eta}^{\alpha} \tilde{\gamma}^{J} \tilde{\eta}^J = 0,
\]
\[
\partial_{x2} \eta^{x2} + 2 p^{x21} \partial_{x2} x^I - \frac{i}{16} \partial_{x2} \tilde{\eta}^{\alpha} \tilde{\gamma}^{J} \tilde{\eta}^J = 0
\]
\[
\text{2 A similar form of the } \kappa \text{-symmetry gauge-fixed Lorentz-harmonic superstring action was previously considered in [48].}
that determine $\lambda^{\pm 2}$ variables. It should be noted that the degrees of freedom number for the action (104) precisely coincides with that for the supertwistor action (92) it was derived from, provided the constraints on harmonics and supertwistor variables are taken into account. Varying (104) w.r.t. $p^{\pm 2 I}$ yields

$$p^{\pm 2 I} = \frac{1}{2c} \partial_{\pm 2 x^I}.$$  \hspace{1cm} (106)

so that $p^{\pm 2 I}$ admit interpretation as the linear combinations of momenta conjugate to $\partial_\tau x^I$ and $\partial_\sigma x^I$. Substituting (106) back into (104) gives the light-cone gauge action for the IIB superstring

$$S_{IIB}^{lc} = -\frac{1}{c\alpha'} \int d^2 \xi \left( \partial_+ 2 x^I \partial_- x^I + \frac{i}{16} \tilde{\eta}_+^{A} \partial_- \tilde{\eta}_-^{A} + \frac{i}{16} \tilde{\eta}_-^{A} \partial_+ \tilde{\eta}_+^{A} \right).$$  \hspace{1cm} (107)

6. Conclusion

The reformulation in terms of supertwistor variables of the first-order Lorentz-harmonic superstring action in six and ten dimensions has been considered. Relevant higher-dimensional supertwistors realize the fundamental representations of the respective minimal superconformal groups and include spinor Lorentz-harmonics as their projectional parts to match with the string momentum density representation, and satisfy the set of algebraic constraints, whose solution can be cast into the form of higher-dimensional generalization of the Penrose–Ferber incidence relations with the conventional superspace coordinates. The expressions for the admissible variations/differentials of the supertwistor variables that respect these constraints have been obtained. The derivation coefficients then appear as the independent parameters in the action functional variation yielding the set of superstring equations of motion in the supertwistor form. Analogously to the considered earlier twistor transformed $D = 4$ superstring, the proposed supertwistor formulation for the superstring in $D = 6, 10$ dimensions is invariant under the irreducible $\kappa$-symmetry. The corresponding transformation rules for supertwistors and auxiliary variables have been explicitly given. There has been an analysis of the possibility of simplification of the supertwistor formulation by covariantly gauge-fixing $\kappa$-symmetry leading to the quadratic superstring action that is of the two-dimensional free-field-theory-type modulo the constraints on supertwistors. It was also shown that by fixing remaining gauge freedom the action can be reduced to that for the Green–Schwarz superstring in the light-cone gauge. Though we have started with the Lorentz-harmonic superstring action that is classically equivalent to the Green–Schwarz one, and in the appropriate gauge proposed supertwistor formulation matches the light-cone gauge superstring action, the issue of quantization of the considered supertwistor representation for the superstring awaits its solution.

Acknowledgments

The author is indebted to A A Zheltukhin for valuable discussions and the Abdus Salam ICTP, where part of this work was done, for the warm hospitality.

References

[1] Penrose R 1967 J. Math. Phys. 8 345
Penrose R and MacCallum M A H 1972 Phys. Rept. 6 241
Penrose R and Rindler W 1986 Spinors and Space-time (Cambridge: Cambridge University Press)
Hughston L P 1987 Applications of \(SO(8)\) spinors Gravitation and Geometry (Naples: Bibliopolis)
Hughston L P 1990 The wave equation in even dimensions Further Advances in Twistor Theory vol 1 (Pitman Research Notes in Mathematics Series vol 231) (London: Longman) p 26
Hughston L P 1990 A remarkable connection between the wave equation and pure spinors in higher dimensions Further Advances in Twistor Theory vol 1 (Pitman Research Notes in Mathematics Series vol 231) (London: Longman) p 37

Ward R S 1984 Nucl. Phys. B 236 381
Witten E 1986 Nucl. Phys. B 266 245
Harnad J and Shnider S 1992 J. Math. Phys. 33 3197
Harnad J and Shnider S 1995 J. Math. Phys. 36 1945
Gindikin S 1998 J. Geom. Phys. 26 26
Krasnov K 2003 Twistors, CFT and holography Preprint hep-th/0311162
Berkovits N and Cherkis S 2004 J. High Energy Phys. JHEP12/2004049 (Preprint hep-th/0409243)
Claus P, Kallosh R and van Proeyen A 1997 Nucl. Phys. B 518 117 (Preprint hep-th/9711161)
von Holten J and van Proeyen A 1982 J. Phys. A: Math. Gen. 15 3763
Uvarov D V 2000 Phys. Lett. B 493 421 (Preprint hep-th/0006185)
Uvarov D V 2001 Nucl. Phys. B (Proc. Suppl.) 102–103 120 (Preprint hep-th/0104235)