Quantum Entanglement of Locally Excited States in Maxwell Theory

Masahiro Nozaki \textsuperscript{a} and Naoki Watamura \textsuperscript{b}

\textsuperscript{a}Kadanoff Center for Theoretical Physics, University of Chicago, Chicago, Illinois 60637, USA
\textsuperscript{b}Department of Physics Nagoya University, Nagoya 464-8602, Japan

Abstract

In 4 dimensional Maxwell gauge theory, we study the changes of (Renyi) entanglement entropy which are defined by subtracting the entropy for the ground state from the one for the locally excited states generated by acting with the gauge invariant local operators on the state. The changes for the operators which we consider in this paper reflect the electric-magnetic duality. The late-time value of changes can be interpreted in terms of electromagnetic quasi-particles. When the operator constructed of both electric and magnetic fields acts on the ground state, it shows that the operator acts on the late-time structure of quantum entanglement differently from free scalar fields.
1 Introduction and Summary

Quantum entanglement significantly distinguishes quantum states from classical states. It can characterize conformal field theories [1, 2, 3] and topological phases [4, 5, 6]. In Gauge/Gravity correspondence [7, 8, 9], the structure of quantum entanglement in quantum field theories (QFTs) living on the boundary is expected to be related to the gravity in the bulk [10, 11]. There are a lot of works done to reveal how the structure of quantum entanglement on the boundary corresponds to the geometry in the bulk [12, 13, 14, 15, 16, 17, 18]. Therefore it is important to uncover the fundamental features which quantum entanglement possesses. (Rényi) entanglement entropy is one of the useful quantities to investigate them.

However the definition of (Rényi) entanglement entropy in gauge theories has subtleties [19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34]. In gauge theories, physical states have to be gauge invariant. It obeys constraints which guarantee its gauge invariance. They make it difficult to divide the Hilbert space into subsystems A and B because the physical degrees of freedom in A depends on the freedom in B due to the constraints. Their boundary is ∂A. Then the definition of (Rényi) entanglement entropy needs the precise
method of dividing Hilbert space and defining the reduced density matrix \( \rho_A \) which is given by tracing out the degrees of freedom in \( B \),

\[
\rho_A = \text{tr}_B \rho. \tag{1.1}
\]

On the other hand, the entropy in QFTs depends on a UV cutoff (ultraviolet cutoff) \( \delta \) because by definition it has the UV divergence. It is given by a series expansion in conformal field theories. The physical degrees of freedom around \( \partial A \) have the significant effect on the terms which depend on \( \delta \). The method of dividing the Hilbert space is expect to affect the degrees of freedom around \( \partial A \) in the direct fashion. In the present paper, we study the changes of (Rényi) entanglement entropy \( \Delta S_A^{(n)} \) which is defined by subtracting the entropy for the ground state from the one for the locally excited state, which is defined by acting with a local operator on the ground state. Here we assume that the operator is located far from \( \partial A \). We will explain it more in the next section. As in \[35, 36, 37, 38, 39, 40, 41\], their changes do not possess the UV divergence. More precisely, they measure how the local operator changes the structure of quantum entanglement. Therefore they are expected to avoid the subtleties which (Rényi) entanglement entropy has.

In this paper we study \( \Delta S_A^{(n)} \) in 4d Maxwell gauge theory, which is a free CFT \[42\]. The previous works \[37, 38, 39\] show the time evolution of \( \Delta S_A^{(n)} \) can be interpreted in terms of relativistic propagation of entangled quasi-particles which are created by local operators. In the free theories, the late-time value of \( \Delta S_A^{(n)} \) is given by the constant, which depends on the operators. It comes from the quantum entanglement between quasi-particles. As in \[39\], the late-time entanglement structure depends on the kind of quasi-particles. The authors in \[40\] show that in the specific 2d CFTs, it is related to the quantum dimension of the operator which acts on the ground state. The Authors in \[41, 43\] have shown that in holographic theories the late time value of \( \Delta S_A^{(n)} \) logarithmically increases similarly to the behavior of entanglement entropy for the local quenches \[44, 45\]. \( \Delta S_A^{(n)} \) in the finite temperature system was investigated by the authors in \[46\]. There are many works done to study the fundamental properties of \( \Delta S_A^{(n)} \) \[47\]. The time evolution and late-time value of \( \Delta S_A^{(n)} \) depend on theories and the quasi-particles created by the local operator. Then we study how the structure of quantum entanglement is changed by gauge invariant local operators such as electric and magnetic fields. In particular, we study how the late-time structure of quantum entanglement depends on them. More precisely, we study how quasi-particles have the effect on the structure. We also study whether \( \Delta S_A^{(n)} \) for gauge invariant locally excited state reflects electric-magnetic duality.

**Summary**

Here we briefly summarize our results in this work. We study how they change the structure of quantum entanglement by measuring the time evolution of \( \Delta S_A^{(n)} \) for various gauge invariant local operators. We also study whether \( \Delta S_A^{(n)} \) is invariant the electric-magnetic duality transformation.
Electric-Magnetic Duality
As it will be explained later, $\Delta S_{A}^{(n)}$ for locally excited states are invariant under the transformation $E_i \rightarrow -B_i$ and $B_i \rightarrow E_i$ where $E$ and $B$ are electric and magnetic fields, respectively.

$E_i$ or $B_i$

If only $E_i$ or $B_i$ acts on the ground state, the time evolution of $\Delta S_{A}^{(n)}$ depends on the one which acts on the ground state. Because $\Delta S_{A}^{(n)}$ reflects the electric-magnetic duality, the entropy for $B_i$ is equal to that for $E_i$, which is the electric field along the same direction as that of magnetic field. $\Delta S_{A}^{(n)}$ for the electric and magnetic fields along the direction vertical to the entangling surface increases slower than those for fields along the directions parallel to the surface. However there are no difference between the effects of electromagnetic field and that of scalar one on the entanglement structure at the late time.

Composite Operators
If the composite operator such as $B^2$ acts on the ground state, they lead to the late-time structure of quantum entanglement in the same manner as a specific scalar operator. Then the late-time value of $\Delta S_{A}^{(n)}$ for that can be interpreted in terms of quasi-particles created by the scalar operator. However $\Delta S_{A}^{(n)}$ for some specific operators (e.g, $E_2B_3$) constructed of both electric and magnetic fields can be interpreted in terms of not the scalar quasi-particles but electromagnetic one, which is explained in section 4. Here $B_3$ ($E_3$) and $B_2$ ($E_2$) are the magnetic (electric) fields along the direction perpendicular to $\partial A$ as we will explain it later.

Late-time Algebra
We interpret the late-time values of $\Delta S_{A}^{(n)}$ in terms of electromagnetic quasi-particles created by an electromagnetic field, and derive a late-time algebra which they obey. There are commutation relations between the particles of the same kinds of fields. As we will mention later, there are also additional relations between $E_2$ ($E_3$) and $B_3$ ($B_2$), which are parallel to the entangling surface. They make the effect of electromagnetic fields different from that of scalar fields on the late-time structure of quantum entanglement.

Organization
This paper is organized as follows. In section 2, we will explain locally excited states and how to compute $\Delta S_{A}^{(n)}$ in the replica trick. We study the time evolution and late-time value of $\Delta S_{A}^{(n)}$ for various gauge invariant local operators in section 3. We interpret the late-time value of $\Delta S_{A}^{(n)}$ in terms of entangled quasi-particles in section 4. We study how they have the effect on the late-time structure of quantum entanglement. We finish with the conclusion, future problems and the detail of propagators is included in appendices.

\cite{footnote2}When only a component of the electric or magnetic one acts on the ground state, we do not consider the linear combination of them in this paper.
2 How to compute Excesses of (Rényi) Entanglement Entropy

By measuring the excess of (Rényi) entanglement entropies $\Delta S_A^{(n)}$, we study how local gauge-invariant operators changes the structure of quantum entanglement in the 4d Maxwell gauge theory:

$$S = -\frac{1}{4} \int d^4x F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma},$$

(2.1)

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $g^{\mu\nu} = \text{diag} (-1, 1, 1, 1)$.

In this section, we explain the definition of locally excited state and how to compute $\Delta S_A^{(n)}$ in the replica method.

The Definition of Locally Excited States

The locally excited state is defined by acting with a gauge invariant local operator $\mathcal{O}$ such as $F_{\mu\nu}$ on the ground state:

$$|\Psi\rangle = \mathcal{N} \mathcal{O}(-t, -l, x)|0\rangle.$$  \hspace{1cm} (2.2)

where $\mathcal{N}$ is a normalization constant and $|0\rangle$ is a gauge invariant state. As in Figure.1, $\mathcal{O}$ is located at $t = -t$, $x^1 = -l$ and $x = (x^2, x^3)$.

Subsystem

As in the previous works [37, 38, 39, 40], the subsystem $A$ is defined by $(t = 0, x^1 \geq 0)$ as in Figure.1. In free theories $\Delta S_A^{(n)}$ approaches to a constant, which comes from quantum entanglement between entangled quasi-particles. In this paper we would like to study how the constant depends on gauge invariant operators. Therefore the region in Figure.1 is chosen as $A$.

![Figure 1: The location of local gauge invariant operator in Minkowski spacetime.](image-url)
Excesses of (Rényi) Entanglement Entropy

Here we explain more about the definition of \( \Delta S_A^{(n)} \). (Rényi) entanglement entropy for the ground state is a static quantity, which does not depend on time. Then we define the excesses of (Rényi) entanglement entropy \( \Delta S_A^{(n)} \) by subtracting \( S_A^{(n)} \) for the ground state from those for locally excited states,

\[
\Delta S_A^{(n)} = S_A^{(n),EX} - S_A^{(n),G},
\]

where \( S_A^{(n),EX} \) and \( S_A^{(n),G} \), are (Rényi) entanglement entropies for the excited states in (2.2) and the ground state \( |0\rangle \), respectively. In the sense that \( \Delta S_A^{(n)} \) does not depend on \( \delta \), it is a “renormalized” (Rényi) entanglement entropy.

2.1 The Replica Trick

We would like to study the time evolution of \( \Delta S_A^{(n)} \) in 4d Minkowski spacetime. However in this paper we do not directly study the changes of entanglement structure in the spacetime. Without doing so, we compute \( \Delta S_A^{(n)} \) in Euclidean space by the replica trick. After that we perform the analytic continuation, which we will explain later. Then we compute the real time evolution of \( \Delta S_A^{(n)} \).

As in [37, 38, 39, 40], a reduced density matrix in Euclidean space is given by

\[
\rho = \mathcal{N}^2 \mathcal{O}(\tau_e, -l, x) |0\rangle \langle 0| \mathcal{O}^\dagger(\tau_1, -l, x),
\]

where \( \tau \) is Euclidean time. By introducing a polar coordinate, \((\tau_1, -l)\) is mapped to \((r_{1,2}, \theta_{1,2})\) as in Figure 2.

In the replica trick, (Rényi) entanglement entropies for (2.4) and the ground state are respectively given by 2

\[
S_A^{(n),EX} = \frac{1}{1-n} \log \left[ \frac{\int D\Phi \mathcal{O}^\dagger(r_1, \theta_1^n) \mathcal{O}(r_2, \theta_2^n) \cdots \mathcal{O}(r_1, \theta_1^1) \mathcal{O}(r_2, \theta_2^1) e^{-S_n[\Phi]}}{\left( \int D\Phi e^{-S_1[\Phi]} \right)^n} \right],
\]

\[
S_A^{(n),G} = \frac{1}{1-n} \log \left[ \frac{\int D\Phi e^{-S_n[\Phi]}}{\left( \int D\Phi e^{-S_1[\Phi]} \right)^n} \right],
\]

where \( \theta_{1,2} = \theta_{1,2} + 2(k-1)\pi \). The actions \( S_n \) and \( S_1 \) are defined on n-sheeted geometry \( \Sigma_n \) (see Figure 3) and the flat space \( \Sigma_1 \), respectively. By substituting (Rényi) entanglement entropies in (2.5) into (2.3), \( \Delta S_A^{(n)} \) is given by

\[
\Delta S_A^{(n)} = \frac{1}{1-n} \log \left[ \frac{\langle \mathcal{O}^\dagger(r_1, \theta_1^n) \mathcal{O}(r_2, \theta_2^n) \cdots \mathcal{O}^\dagger(r_1, \theta_1^1) \mathcal{O}(r_2, \theta_2^1) \rangle_{\Sigma_n}}{\langle \mathcal{O}^\dagger(r_1, \theta_1^1) \mathcal{O}(r_2, \theta_2^1) \rangle_{\Sigma_1}^n} \right].
\]

We only need to compute propagators on \( \Sigma_n \) in order to compute \( \Delta S_A^{(n)} \) in free field theories. The two point function of gauge fields \( A_a \) is defined by \(- \langle A_a(r, \theta, x) A_b(r', \theta', x') \rangle = 2The detail of this computation is explained in [37, 38, 39, 40].
Figure 2: The location of local gauge invariant operator in Euclidean space.

\[ G_{ab}(r, r', \theta - \theta', x - x'). \] If we choose a specific gauge\(^3\), their green functions obey the same equation of motion as that for 4\(d\) free massless scalar field theory,

\[
\begin{align*}
\partial^2_r G^a_{rb}(r, r', \theta - \theta', x - x') + \frac{1}{r} \partial_r G^a_{rb}(r, r', \theta - \theta', x - x') \\
+ \frac{1}{r^2} \partial^2_\theta G^a_{rb}(r, r', \theta - \theta', x - x') + \partial^2_x G^a_{rb}(r, r', \theta - \theta', x - x') = -\frac{\delta^a_b \delta(r - r') \delta(\theta - \theta') \delta^2(x - x')}{r},
\end{align*}
\]

(2.7)

where \( a = \{\tau, x^1, x^2, x^3\} \).\(^4\)

The solution of the equation is given by

\[
G_{ab}(r, r', \theta - \theta', x - x') = \frac{\delta_{ab} \sinh \left( \frac{t_0}{n} \right)}{8 \pi^2 r r' \sinh t_0 \left( \cosh \left( \frac{t_0}{n} \right) - \cos \left( \frac{\theta - \theta'}{n} \right) \right)},
\]

(2.8)

where \( t_0 \) is defined by

\[
\cosh t_0 = \frac{r^2 + r'^2 + (x^2 - x'^2)^2 + (x^3 - x'^3)^2}{2rr'},
\]

(2.9)

(2.8) has been obtained by the authors in \([37, 38, 39, 41, 48, 49]\).

**Analytic Continuation**

After computing green functions on \( \Sigma_n \) in Euclidean space, we perform the following analytic continuation,

\[
A_\tau = iA_t, \quad \partial_\tau = i\partial_t, \quad \tau_l = \epsilon - it, \quad \tau_e = -\epsilon - it,
\]

(2.10)

where \( \epsilon \) is a smearing parameter which is introduced to keep the norm of the excited state finite. Analytic-continued green functions depend on \( \epsilon \). We are interested in the behavior of \( \Delta S_A^{(n)} \) in the limit \( \epsilon \to 0 \). Their leading behavior (\( \sim \mathcal{O} \left( \frac{1}{\epsilon^4} \right) \)) are summarized in Appendix A.

---

\(^3\)The chosen gauge corresponds to Feynman gauge in Minkowski spacetime.

\(^4\)\(G^a_b \equiv \eta^{ac}G_{cb}\) where \(\eta_{ac} = \text{diag}(1, 1, 1)\)
3 Excesses of (Rényi) Entanglement Entropy

In this section, we study the time evolution and the late time value of $\Delta S_A^{(n)}$ in the following three cases.

(i) Only one electric or magnetic field $\mathcal{O} = E_i, B_i$ acts on the ground state. The time evolution of $\Delta S_A^{(n)}$ depends on the operator which acts on the ground state. If electromagnetic fields are changed by $F_{\mu\nu} \rightarrow \tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$, $\Delta S_A^{(n)}$ does not change. Here $\epsilon_{\mu\nu\rho\sigma}$ is an antisymmetric tensor. The late-time value of $\Delta S_A^{(n)}$ does not depend on the operator. It can be interpreted in terms of the quasi-particle created by a scalar operator $\phi$.

(ii) Composite operators which act on the ground state are constructed of only electric or magnetic fields such as $E_2^2$ and $B_2^2$. $\Delta S_A^{(n)}$ for $E_2^2$ is equivalent to the entropy for $B_2^2$. Then $\Delta S_A^{(n)}$ for them invariant under the electric-magnetic duality transformation. There are no differences between their effect on the (Rényi) entanglement entropy. Its late-time values can be interpreted in terms of quasi-particles, which are created by the operator constructed of massless free scalar fields $\sum_{a=1}^{3} (\phi^a)^2$. Here $a$ denotes the kinds of fields.

(iii) Local operators are constructed of both electric and magnetic fields such as $E_1^2 + B_1^2$, $B_3 E_2$ and $B \cdot E$. The late-time value of $\Delta S_A^{(n)}$ shows that there is a significant difference between the effect of $E_1$ ($B_1$) and $E_{2,3}$ ($B_{2,3}$) on the late-time entanglement structure. Here $E_1$ ($B_1$) is the electric (magnetic) field along the direction vertical to the entangling surface. On the other hand, $E_{2,3}$ ($B_{2,3}$) is the electric (magnetic) field along the direction parallel to the entangling surface. As it will be explained in the next section, the difference comes from the commutation relation between electromagnetic quasi-particles created by $E_2$ ($E_3$) and the particles created by $B_3$ ($B_2$).
3.1 $\mathcal{O} = E_i$ or $B_i$

Here locally excited states are defined by acting with only $E_i$ or $B_i$ on the ground state. $\Delta S_A^{(n)}$ is given by (2.6) in the replica method with Euclidean signature. After performing the analytic continuation in (2.10) and taking the limit $\epsilon \to 0$, their time evolution is given as follows. $\Delta S_A^{(n)}$ vanishes before $t = l (> 0)$, but after $t = l$, they increase. The detail of their time evolution is summarized in Table.1. After taking the late time limit ($0 < l \ll t$), they are given by

$$\Delta S_A^{(n)} \sim \log 2.$$  \hspace{1cm} (3.1)

Their late time value is the same as that for $\phi$ in free massless scalar field theories with any spacetime dimensions. It can be interpreted as (Rényi) entanglement entropy for maximally entangled state in 2 qubit system. Therefore they do not show the difference between the effect of electromagnetic fields and that of free scalar one on the late-time structure of quantum entanglement. However time evolution of $\Delta S_A^{(n)}$ depends on the local operator which acts on the ground state as in Figure.4. Even at $t \sim l$, time evolution of $\Delta S_A^{(n)}$ depends on the one which acts on the ground state. If $\mathcal{O} = B_{2,3}$ or $E_{2,3}$ acts on the ground state, $\Delta S_A^{(n>2)}$ is given by

$$\Delta S_A^{(n)} \sim \frac{n}{n-1} \left( \frac{3(t-l)}{4l} \right) + \cdots ,$$  \hspace{1cm} (3.2)

where $\cdots$ are contributions from the higher order $\mathcal{O} \left( \left( \frac{t-l}{l} \right)^2 \right)$. $\Delta S_A^{(n>2)}$ for $\mathcal{O} = E_1$ or $B_1$ at $t \sim l$ is given by

$$\Delta S_A^{(n)} \sim \frac{n}{n-1} \left( \frac{3(t-l)^2}{4l^2} \right) + \cdots .$$  \hspace{1cm} (3.3)

$\cdots$ are contributions from the higher order $\mathcal{O} \left( \left( \frac{t-l}{l} \right)^3 \right)$. Their time evolution shows that quasi-particles created by $E_{2,3}$ ($B_{2,3}$) enter the region $A$ faster than those generated by $E_1$ ($B_1$). These behaviors seem to be natural since particles created by $E_1$ ($B_1$) do not propagate along the direction parallel to $x^1$.

$\Delta S_A^{(n)}$ in Table.1 shows that they are invariant under the transformation,

$$\Gamma_{\mu\nu} \rightarrow \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma};$$  \hspace{1cm} (3.4)

where $\epsilon_{\mu\nu\rho\sigma}$ is an anti-symmetric tensor. Under the transformation in (3.4), the local operator $E_i$ ($B_i$) changes to $-B_i$ ($E_i$). Therefore this duality changes a locally excited state to a different one.

3.2 Composite Operators Constructed of Only Electric or Magnetic Fields

The excited states which we consider here are generated by acting with the following operators: (a) $E_i E_j$ or $B_i B_j$, (b) $E^2$ or $B^2$. We study the time evolution and the late-time value of $\Delta S_A^{(n)}$ for them.
Figure 4: The time evolution of $\Delta S_A^{(2)}$ for $E_1 (B_1)$ and $E_2, 3 (B_2, 3)$. The horizontal and vertical axes correspond to time $t$ and $\Delta S_A^{(2)}$, respectively. The red and blue lines correspond to $\Delta S_A^{(2)}$ for $E_1 (B_1)$ and $E_2, 3 (B_2, 3)$, respectively.

Table 1: $\Delta S_A^{(n)}$ for $\mathcal{O}$ in the region $0 < l \leq t$

| $\mathcal{O}$     | $\Delta S_A^{(n)}$                                      |
|-------------------|--------------------------------------------------------|
| $E_1$ or $B_1$    | $\frac{1}{1-n} \log \left( \frac{-(l+t)^2(l-2t)}{4t^n} \right)^n + \frac{(l-t)^2(l+2t)}{4t^n} \right)^n$ |
| $E_{2, 3}$        | $\frac{1}{1-n} \log \left( \frac{-l^3-3lt^2+4t^3}{8t^n} \right)^n + \frac{(l^3+3lt^2+4t^3)}{8t^n} \right)^n$ |

3.2.1 $\mathcal{O} = E_i E_j$ or $B_i B_j$

First we consider $\Delta S_A^{(n)}$ for the excited states generated by acting with $E_i E_j$ or $B_i B_j$ on the ground state. When $i = j$, the late-time value of $\Delta S_A^{(n)}$ is given by

$$\Delta S_A^{(n)} = -\frac{1}{1-n} \log \frac{4^n}{2^n + 2}. \quad (3.5)$$

It is the same as that of $\Delta S_A^{(n)}$ for $\phi^2$ in the massless free scalar field theories as in [37, 38]. Therefore the late-time value of $\Delta S_A^{(n)}$ ((Rényi) entanglement entropy of operator) can be interpreted in terms of entangled quasi-particles created by $\phi^2$.

When $i \neq j$, $\Delta S_A^{(n)}$ at the late time is given by

$$\Delta S_A^{(n)} = \log 4, \quad (3.6)$$

which can be interpreted as maximum (Rényi) entanglement entropy for $\rho_A = \frac{1}{4} \text{diag}(1, 1, 1, 1)$. It is the same as $\Delta S_A^{(n)}$ for the excited state given by acting with the operator $\phi^a \phi^b$ on the ground state. Here $a, b$ denote the kind of scalar fields, and $a \neq b$. They are two kinds of massless free scalar fields. The time evolution of $\Delta S_A^{(n)}$ for them is summarized in Table 2. Table 2 shows $\Delta S_A^{(n)}$ is invariant under the transformation in [34].
Table 2: $\Delta S_{A}^{(n)} = \frac{1}{1-n} \log \left[ \frac{N_{1}+N_{2}+N_{1}}{D_{1}} \right]$ for $\mathcal{O}$ in the region $0 \leq l < t$

| $\mathcal{O}$ | $D_{1}$ | $N_{1}$ | $N_{2}$ | $N_{3}$ |
|--------------|--------|--------|--------|--------|
| $E_{1}^{2}$ or $B_{1}^{2}$ | $(2 \mp 1)^{2}$ | $(2(f_{1})^{2})^{n}$ | $(2(f_{1})^{2})^{n}$ | $2^{2n} (f_{1}f_{2})^{n}$ |
| $E_{2,3}$ or $B_{2,3}$ | $(\mp 1)^{2}$ | $(2(f_{1})^{2})^{n}$ | $(2(f_{1})^{2})^{n}$ | $2^{2n} (f_{1}f_{2})^{n}$ |
| $E_{1}E_{2,3}$ or $E_{3}E_{2,3}$ or $E_{1}B_{2,3}$ or $B_{1}E_{2,3}$ or $B_{1}B_{2,3}$ | $(\mp 1)^{2n}$ | $(f_{1})^{2n} (f_{3})^{n}$ | $(f_{3})^{2n} (f_{1})^{n}$ | $(f_{1})^{2n} (f_{3})^{n} + (f_{3})^{2n} (f_{1})^{n}$ |
| $E_{2}B_{3}$ or $B_{2}B_{3}$ | $(\mp 1)^{2n}$ | $(f_{2})^{2n} (f_{3})^{n}$ | $(f_{3})^{2n} (f_{2})^{n}$ | $2 (f_{2})^{n} (f_{3})^{n}$ |
| $E_{1}B_{1}$ | $(\mp 1)^{2n}$ | $(f_{1})^{2n} (f_{3})^{n}$ | $(f_{3})^{2n} (f_{1})^{n}$ | $2 (f_{1})^{n} (f_{3})^{n}$ |
| Functions | $f_{1} = \frac{2^{l-20}+1}{64}$ | $f_{2} = \frac{2^{l-20}+1}{128}$ | $f_{3} = \frac{2^{l-20}+1}{128}$ | $f_{4} = \frac{2^{l-20}+1}{128}$ |

### 3.2.2 $\mathcal{O} = \text{E}^{2}$ or $\text{B}^{2}$

In order to study whether $E_{1}$ acts on the late-time structure of quantum entanglement differently from $E_{2,3}$, we study the late-time value of $\Delta S_{A}^{(n)}$ for the given locally excited state:

$$|\Psi\rangle = \mathcal{N} \text{E}^{2}(-t, -l, x) |0\rangle.$$  \hspace{1cm} (3.7)

Before studying its late-time value, we comment on its time evolution. Before $t = l$, $\Delta S_{A}^{(n)}$ for the state in (3.7) vanishes and after $t = l$, it increases. Its time evolution is summarized in Table.3.

After $t = l$, as in Table.3, $\Delta S_{A}^{(n)}$ is given by

$$\Delta S_{A}^{(n)} = \frac{1}{1-n} \log \left[ \frac{N_{1}+N_{2}+P_{1}+P_{2}+P_{3}}{D} \right],$$  \hspace{1cm} (3.8)

where $D$, $N_{i}$ and $P_{i}$ are defined in Table.3. If we take the late time limit ($0 < l \ll t$), the ratios of $P_{i}$ and $N_{i}$ to $D$ reduce to constant numbers $50$,

$$\left( \frac{N_{1}}{D} \right)^{\frac{1}{n}} = \left( \frac{N_{2}}{D} \right)^{\frac{1}{n}} = 4^{-1}, \hspace{0.5cm} \left( \frac{P_{1}}{D} \right)^{\frac{1}{n}} = \left( \frac{P_{2}}{D} \right)^{\frac{1}{n}} = \left( \frac{P_{3}}{D} \right)^{\frac{1}{n}} = \left( \frac{1}{6} \right),$$  \hspace{1cm} (3.9)

where we ignore the higher order contribution $\mathcal{O} \left( \frac{1}{t} \right)$. Amazingly, The sum of them is 1. Therefore if the effective reduced density matrix is defined by

$$\Delta S_{A}^{(n)} = \frac{1}{1-n} \log [\text{tr}_{A} (\rho_{A}^{n})],$$  \hspace{1cm} (3.10)

then the matrix is given by

$$\rho_{A}^{n} = \frac{1}{24} \text{diag}(6, 4, 4, 4, 6).$$  \hspace{1cm} (3.11)

---

5The effect of $E_{1}$ can be different from that of $E_{2,3}$ on the structure since we choose $t = 0, x^{1} \geq 0$ as the subsystem $A$. 

10
The excess of $n$–th (Rényi) entanglement entropy, entanglement entropy and Min entropy are respectively given by

\[
\Delta S_A^{(n)} = \frac{1}{n-1} \log \left( \frac{12^n}{3 \cdot 2^n + 2 \cdot 3^n} \right),
\]

\[
\Delta S_A = \frac{\log(24)}{2},
\]

\[
\Delta S_A^{(\infty)} = \log 4,
\]

which can be interpreted in terms of quasi-particles created by $(\phi^1)^2 + (\phi^2)^2 + (\phi^3)^2$, which is constructed of three kinds of free scalar fields. Therefore, there are no differences between the effect of $E_i$ and that of $E_{2,3}$ on the late-time structure of quantum entanglement. As in the Table.2, $\Delta S_A^{(n)}$ for $E^2$ is equivalent to that for $B^2$. Therefore they is the electric-magnetic duality invariant.

Table 3: $\Delta S_A^{(n)} = \frac{1}{1-n} \log \left[ \frac{N_1 + N_2 + P_1 + P_2 + P_3}{D_1} \right]$ for $O$ in the region $0 < l < t$

| $O$ | $D_1$ | $N_1$ | $N_2$ | $P_1$ | $P_2$ | $P_3$ |
|-----|-------|-------|-------|-------|-------|-------|
| $B^2 + E^2$ | $(2 \cdot 3 \left( \frac{D}{\phi} \right)^3)$ | $(2f_i^2 + 2g_i^2)^n$ | $(2f_i^2 + 2g_i^2)^n$ | $2^{2m}f_i^4f_i^4$ | $2^{2m}f_i^4f_i^4$ | $2^{2m}f_i^4f_i^4$ |
| $B^2 + E^2$ | $(2 \cdot 2 \left( \frac{D}{\phi} \right)^3)$ | $(2f_i^2 + 2g_i^2)^n$ | $(2f_i^2 + 2g_i^2)^n$ | $2^{2m}f_i^4f_i^4$ | $2^{2m}f_i^4f_i^4$ | $2^{2m}f_i^4f_i^4$ |
| $E_0 R_i$ | $(2^2 \left( \frac{D}{\phi} \right)^3)$ | $(2(2g_i)^2 + 2(g_i)^2)^n$ | $(2(2g_i)^2 + 2(g_i)^2)^n$ | $2^{2m}(g_i)^n(g_i)^n$ | $2^{2m}(g_i)^n(g_i)^n$ | $2^{2m}(g_i)^n(g_i)^n$ |
| $F_{\mu\nu}$ | $(2 \cdot 2 \left( \frac{D}{\phi} \right)^3)$ | $(2f_i^2 + 2g_i^2)^n$ | $(2f_i^2 + 2g_i^2)^n$ | $2^{2m}(f_i^2)^n(f_i^2)^n$ | $2^{2m}(f_i^2)^n(f_i^2)^n$ | $2^{2m}(f_i^2)^n(f_i^2)^n$ |
| $B_i E_0 - B_i E_j$ | $(2 \cdot 2 \left( \frac{D}{\phi} \right)^3)$ | $(2f_i^2 + 2g_i^2)^n$ | $(2f_i^2 + 2g_i^2)^n$ | $2^{2m}(g_i)^n(g_i)^n$ | $2^{2m}(g_i)^n(g_i)^n$ | $2^{2m}(g_i)^n(g_i)^n$ |

### 3.3 Composite Operators Constructed of Both Electric and Magnetic Fields

In the previous two subsection we study how the entanglement structure changes at the late time if either electric or magnetic fields act on the ground state. However we do not uncover how it changes at the late time when both of them act on the ground state. Here we study $\Delta S_A^{(n)}$ for (a) $E_1^2 + B_1^2$, (b) $E_i B_j$ and (c) $F_{\mu\nu} F_{\mu\nu}$ and $B \cdot E$, which can show that $E_i$ and $B_i$ act on the late-time structure of quantum entanglement differently from scalar fields such as $\phi^2$.

#### 3.3.1 $E_1^2 + B_1^2$

Here in order to study whether there are differences between the effects of electric and magnetic fields on the late-time structure of quantum entanglement, we study $\Delta S_A^{(n)}$ for the
following excited state:
\[ |\Psi\rangle = N (E_2^2 + B_2^2) (-t, -l, x) |0\rangle. \tag{3.13} \]

Before investing the late time value of \( \Delta S_A^{(n)} \), let’s study the time evolution of \( \Delta S_A^{(n)} \). \( \Delta S_A^{(n)} \) vanishes before \( t = l \). After \( t = l \), its time evolution is summarized in Table 3. If you take the late time limit \( t \to \infty \), the late time value of \( \Delta S_A^{(n)} \) reduces to the (Rényi) entanglement entropy whose effective reduced density matrix is given by
\[ \rho_A^\epsilon = \frac{1}{4} \text{diag}(1, 1, 1, 1). \tag{3.14} \]

Its entropies are given by
\[ \Delta S_A^{(n)} = \Delta S_A = \Delta S_A^{(\infty)} = \log 4. \tag{3.15} \]

It shows there are no differences between the effects of electric and magnetic fields on the late-time structure.

### 3.3.2 \( E_i B_j \)

Here let’s find out how the operators constructed of both electric and magnetic fields, \( E_i B_j \), affect the late-time structure of quantum entanglement. The late-time values of \( \Delta S_A^{(n)} \) for \( E_i B_j \) except for \( E_2 B_3 \) and \( E_3 B_2 \) are the same as \( (3.16) \). Their time evolution is summarized in Table 2.

On the other hand, after \( t = l \) the time evolution of \( \Delta S_A^{(n)} \) for \( E_2 B_3 \) or \( E_3 B_2 \) is summarized in Table 3. We can see that it has the electric-magnetic duality from the Table 3. The late-time value of \( \Delta S_A^{(n)} \) is given by (Rényi) entanglement entropy whose reduced density matrix is given by
\[ \rho_A^\epsilon = \frac{1}{64} \text{diag}(25, 7, 7, 25). \tag{3.16} \]

Its entropies are given by
\[ \Delta S_A^{(n)} = - \frac{\log (2^{1-6n} (7^n + 25^n))}{n - 1}, \]
\[ \Delta S_A = \log \left( \frac{64}{5 \cdot 59/16^{7/32}} \right), \tag{3.17} \]
\[ \Delta S_A^{(\infty)} = 2 \log \left( \frac{8}{5} \right). \]

It shows how different the effect of \( E_1 \) (\( B_1 \)) is from that of \( E_2,3 \) (\( B_{2,3} \)) on the structure. The value can not be interpreted in terms of quasi-particles created by scalar fields such as \( \phi^a \phi^b \). As we will explain later, in the entangled quasi-particle interpretation, there is a commutation relation between the quasi-particle created by \( E_2 \) (\( B_2 \)) and that by \( B_3 \) (\( E_3 \)).
3.3.3 \( \mathbf{B} \cdot \mathbf{E} \) and \( F_{\mu \nu} F^{\mu \nu} \) and \( B_2 E_3 - B_3 E_2 \)

We finally study \( \Delta S_A^{(n)} \) for more complicated operators, \( \mathbf{B} \cdot \mathbf{E} \), \( F_{\mu \nu} F^{\mu \nu} \) and \( B_2 E_3 - B_3 E_2 \). Before \( t = l \), \( \Delta S_A^{(n)} \) for them vanish, but after \( t = l \), they increases. The detail of them is summarized in Table 3.6. It shows that \( \Delta S_A^{(n)} \) for \( \mathbf{B} \cdot \mathbf{E} \) is the same as that for \( F_{\mu \nu} F^{\mu \nu} \).

The effective reduced density matrices for \( \mathbf{B} \cdot \mathbf{E} \) (or \( F_{\mu \nu} F^{\mu \nu} \)) and \( B_2 E_3 - B_3 E_2 \) are given by

\[
\rho_A^e = \frac{1}{192} \text{diag} (30, 30, 16, 16, 49, 49, 1, 1), \quad \text{for } \mathcal{O} = \mathbf{B} \cdot \mathbf{E} (F_{\mu \nu} F^{\mu \nu}),
\]

\[
\rho_A^e = \frac{1}{128} \text{diag} (50, 50, 7, 7, 7, 7), \quad \text{for } \mathcal{O} = B_2 E_3 - B_3 E_2.
\]

Entropies are respectively given by

\[
\Delta S_A^{(n)} = \frac{1}{n-1} \log \left( \frac{2^{6n-3n}}{16^n + 30^n + 49^n + 1} \right),
\]

\[
\Delta S_A = \log \left( \frac{32 \cdot 3^{11/16} \cdot 7^{3/16}}{7 \cdot 5^{9/16}} \right),
\]

\[
\Delta S_A^{(\infty)} = \log \left( \frac{192}{49} \right),
\]

which are for \( \mathcal{O} = \mathbf{B} \cdot \mathbf{E} \) (or \( F_{\mu \nu} F^{\mu \nu} \)), and

\[
\Delta S_A^{(n)} = - \log \left( \frac{2^{1-7n} (2 \cdot 7^n + 50^n)}{n-1} \right),
\]

\[
\Delta S_A = \log \left( \frac{64 \left( \frac{7}{5} \right)^{7/32}}{5^{9/16}} \right),
\]

\[
\Delta S_A^{(\infty)} = 2 \log \left( \frac{8}{5} \right),
\]

which are for \( \mathcal{O} = B_2 E_3 - B_3 E_2 \). As we will explain in the next section, they can be reproduced by using a late-time algebra which electromagnetic quasi-particles obey.

4 \( \) A Late-time Algebra

We interpret the late-time value of \( \Delta S_A^{(n)} \) in terms of quasi-particles. More precisely, let’s interpret the effective reduced density matrix in (3.10) in terms of quasi-particles. The effective reduced density matrix for the excited state generated by a composite operator \( \mathcal{O}(-t, -l, x) \) is defined by

\[
\Delta S_A^{(n)} = \frac{1}{1 - n} \log \left[ \text{tr} \left( \rho_A^e \right)^n \right] = \frac{1}{1 - n} \log \left[ \text{tr} \left( \hat{N}^2 \mathcal{O} |0\rangle \langle 0| \mathcal{O}^\dagger \right)^n \right],
\]

\( \Delta S_A^{(n)} \) is commuted by the green functions in Appendix B.
where \( \hat{N} \) is a normalization constant. The operator \( O \) is assumed to be constructed of electric and magnetic fields.\(^7\) As in [37, 38, 39, 41], these fields can be decomposed into left moving and right moving electromagnetic quasi-particles as follows,

\[
\begin{align*}
E_i &= E_i^{L\dagger} + E_i^{R\dagger} + E_i^{L} + E_i^{R}, \\
B_i &= B_i^{L\dagger} + B_i^{R\dagger} + B_i^{L} + B_i^{R},
\end{align*}
\]

(4.2)

where since we take \( x^1 \geq 0 \) as \( A \) in this paper, left-moving and right-moving quasi-particles correspond to particles included in \( B \) and \( A \) at late time, respectively. The ground state for them is defined by

\[
E_i^{L,R} |0\rangle_{L,R} = B_i^{L,R} |0\rangle_{L,R} = 0,
\]

\[
|0\rangle = |0\rangle_L \otimes |0\rangle_R.
\]

(4.3)

The late-time algebra which quasi-particles obey is given by

\[
\begin{align*}
\left[ E_i^{L,R}, E_j^{L,R\dagger} \right] &= C\delta_{ij}, \\
\left[ B_i^{L,R}, B_j^{L,R\dagger} \right] &= C\delta_{ij},
\end{align*}
\]

(4.4)

which is obtained so that the results by the replica trick are reproduced. Here \( C \) is a real number.\(^8\) In the gauge theory in addition to (4.2), we need the following commutation relation for different particles:

\[
\begin{align*}
\left[ E_3^{L,R}, B_2^{L,R}\dagger \right] &= X_{R,L}, \\
\left[ E_2^{L,R}, B_3^{L,R}\dagger \right] &= Y_{R,L},
\end{align*}
\]

(4.5)

where \( X_{R,L} \) and \( Y_{R,L} \) are given by

\[
\begin{align*}
X_R &= -X_L = Y_L = -Y_R, \\
X_{R,L}^2 &= Y_{R,L}^2 = \frac{9}{16} C^2.
\end{align*}
\]

(4.6)

Here \( X_{R,L} \) and \( Y_{R,L} \) are real numbers.\(^9\) The commutation relation between electric (magnetic) quasi-particles is determined so that the effective density matrices computed by (4.4) are consistent with (3.11) respectively. The relation for the quasi-particles by \( E_1 \) should be the same as that for \( B_1 \) so that the effective density matrix in (4.1) reproduces \( \Delta S^{(n)}_A \) for the matrix in (3.14). That between quasi-particles generated by \( E_2 \) (\( E_3 \)) and those by \( B_3 \) (\( B_2 \)) reproduces the matrix in (3.16). We also check that \( \Delta S^{(n)}_A \) for \( O = B \cdot E \left( F_{\mu\nu} F^{\mu\nu} \right) \), \( B_3 E_2 - B_2 E_3 \) are reproduced by using the commutation relation in (4.4) and (4.5).

---

\(^7\)Here \( \rho_A^\phi \) is not the same as the reduced density matrix for the locally excited state. It is for a “effective” state \( \hat{N}O |0\rangle \). It is different form the “original” locally excited state.

\(^8\)The redefinition of quasi-particles can absorb the constant \( C \).

\(^9\)We find the correspondence between propagators and commutation relations. The commutations can be defined by the late time limit of propagators. We will discuss the detail of the correspondence in [50]. When we use this correspondence, \( X_L = -\frac{3}{4} C \).
The relation in (4.5) shows that the effect of fields along the direction vertical to ∂A is significantly different from that along the direction parallel to ∂A on the late-time structure. It makes the effects of electromagnetic fields different from that of free scalar fields on the late-time structure of quantum entanglement.

5 Conclusion and Future Problems

We also studied how gauge invariant operators such as $E_i$, $B_i$ and the composite operators constructed of them changes the structure of quantum entanglement by studying $\Delta S_A^{(n)}$. We studied whether $\Delta S_A^{(n)}$ for locally excited states created by gauge invariant local operators reflects the electric magnetic duality. $\Delta S_A^{(n)}$, which we studied in this paper, is invariant under the duality transformation. If only $E_i$ or $B_i$ acts on the ground state, without taking the late time limit, the time evolution of $\Delta S_A^{(n)}$ depends on them. Due to the duality, $\Delta S_A^{(n)}$ for $E_i$ is equal to that for $B_i$. Around $t = l$, $\Delta S_A^{(n)}$ for $E_{2,3}$ ($B_{2,3}$) increases slower than that for $E_1$ ($B_1$). However they can not show the difference between the effects of electromagnetic fields and that of scalar fields on the late-time structure because the late-time values of $\Delta S_A^{(n)}$ for them can be interpreted in terms of quasi-particle created by scalar fields.

On the other hand, the late-time values of $\Delta S_A^{(n)}$ for the specific operators constructed of both electric and magnetic fields can not be interpreted in terms of quasi-particles by scalar fields. They show that there are differences between the effects of electromagnetic and that of scalar fields on the late-time structure of quantum entanglement. If their late-time values are interpreted in terms of electromagnetic quasi-particles in (4.2), there are commutation relations between $E_2$ ($E_3$) and $B_3$ ($B_2$), which make the effect of electromagnetic field significantly different from that of scalar fields on the late-time structure. The effect of $E_1$ and $B_1$ on the late-time structure is different from that of $E_{2,3}$ and $B_{2,3}$.

Future Problem

We finish with comments on a few of future problems:

- In this paper we only consider 4d Maxwell gauge theory which has conformal symmetry. $D(\neq 4)$ dimensional Maxwell gauge theory is not a CFT. Therefore it is interesting to generalize the analysis in 4d Maxwell theory to that in the theories with general dimensions.

- We expect that the structure of the late-time algebra depends on the spacetime dimension $D$. Then it is also interesting to study it in general dimensions.

Acknowledgments

MN thanks Tadashi Takayanagi for useful discussions and comments on this paper. MN and NW thank Pawel Caputa, Tokiro Numasawa, Shunji Matsuura and Akinori Tanaka for useful
comments on this work.

A Green Functions

The relation between $E_i$, $B_i$ and field strengths which are defined in Euclidean space is given by

$$E_i = -iF_{\tau i}, \quad B_1 = -F_{23}, \quad B_2 = F_{13}, \quad B_3 = -F_{12}. \quad (A.1)$$

The analytic continued green functions are defined by

$$\langle E_1(\theta)E_1(\theta') \rangle = F_{E1E1}(\theta - \theta'),$$
$$\langle E_2(\theta)E_2(\theta') \rangle = \langle E_3(\theta)E_3(\theta') \rangle = F_{E2E2}(\theta - \theta'),$$
$$\langle B_1(\theta)B_1(\theta') \rangle = F_{B1B1}(\theta - \theta'),$$
$$\langle B_2(\theta)B_2(\theta') \rangle = \langle B_3(\theta)B_3(\theta') \rangle = F_{B2B2}(\theta - \theta'),$$
$$\langle E_2(\theta)B_3(\theta') \rangle = F_{E2B3}(\theta - \theta'),$$
$$\langle B_2(\theta)E_3(\theta') \rangle = F_{B2E3}(\theta - \theta'). \quad \text{(A.2)}$$

If the limit $\epsilon \to 0$ is taken, their leading terms for $n = 1$ are given by

$$F_{E1E1}(\theta_1 - \theta_2) \sim \frac{1}{16\pi^2\epsilon^4},$$
$$F_{E2E2}(\theta_1 - \theta_2) \sim \frac{1}{16\pi^2\epsilon^4},$$
$$F_{B1B1}(\theta_1 - \theta_2) \sim \frac{1}{16\pi^2\epsilon^4},$$
$$F_{B2B2}(\theta_1 - \theta_2) \sim \frac{1}{16\pi^2\epsilon^4}. \quad \text{(A.3)}$$

That for the arbitrary $n$ in $0 < t < l$ at are given by (A.3)
That for the arbitrary n in 0 < l ≤ t at the are given by

\[ F_{E1E1}(\theta_1 - \theta_2) = F_{E1E1}(\theta_2 - \theta_1) \sim -\frac{(l-2t)(l+t)^2}{64\pi^2\theta^4}, \]

\[ F_{E2E2}(\theta_1 - \theta_2) = F_{E2E2}(\theta_2 - \theta_1) \sim \frac{l^3 + 3lt^2 + 4t^3}{128\pi^2\theta^4}. \]

\[ F_{B1B1}(\theta_1 - \theta_2) = F_{B1B1}(\theta_2 - \theta_1) \sim -\frac{(l-2t)(l+t)^2}{64\pi^2\theta^4}, \]

\[ F_{B1B1}(\theta_1 - \theta_2) = F_{B1B1}(\theta_2 - \theta_1) \sim -\frac{(l-2t)(l+t)^2}{64\pi^2\theta^4}, \]

\[ F_{B2B2}(\theta_1 - \theta_2) = F_{B2B2}(\theta_2 - \theta_1) \sim \frac{l^3 + 3lt^2 + 4t^3}{128\pi^2\theta^4}, \]

\[ F_{E2B2}(\theta_1 - \theta_2) = F_{E2B2}(\theta_2 - \theta_1) \sim \frac{3(l-t)(l+t)}{128\pi^2\theta^4}, \]

\[ F_{B3B2}(\theta_1 - \theta_2) = F_{B3B2}(\theta_2 - \theta_1) \sim \frac{3(l-t)(l+t)}{128\pi^2\theta^4}, \]

\[ F_{E3B2}(\theta_1 - \theta_2) = F_{E3B2}(\theta_2 - \theta_1) \sim \frac{3(l-t)(l+t)}{128\pi^2\theta^4}, \]

\[ F_{E2E3}(\theta_1 - \theta_2) = F_{E2E3}(\theta_2 - \theta_1) \sim \frac{3(l-t)(l+t)}{128\pi^2\theta^4}, \]

\[ F_{E1E1}(\theta_1 - \theta_2 + 2\pi) = F_{E1E1}(\theta_2 - \theta_1 - 2\pi). \]
The contribution of the other propagators is much smaller than them in (A.4).

**B Other Bases**

We introduce new bases \( \mathcal{O}_{1,2} = \frac{E_{2}+B_{1}}{2} \), \( \mathcal{O}_{3,4} = \frac{E_{2}+B_{2}}{2} \). Their green functions in \((0 < t < l)\) are given by

\[
\begin{align*}
\langle O_1 O_1 \rangle (\theta_1 - \theta_2) &\sim \frac{1}{4 \cdot 8\pi^2 \epsilon^4}, \\
\langle O_2 O_2 \rangle (\theta_1 - \theta_2) &\sim \frac{1}{4 \cdot 8\pi^2 \epsilon^4}, \\
\langle O_3 O_3 \rangle (\theta_1 - \theta_2) &\sim \frac{1}{4 \cdot 8\pi^2 \epsilon^4}, \\
\langle O_4 O_4 \rangle (\theta_1 - \theta_2) &\sim \frac{1}{4 \cdot 8\pi^2 \epsilon^4},
\end{align*}
\]

and those in \((0 < l \leq t)\) are given by

\[
\begin{align*}
\langle O_1 O_1 \rangle (\theta_1 - \theta_2) &= \langle O_1 O_1 \rangle (\theta_2 - \theta_1) \sim \frac{(l + t)(l^2 - 4lt + 7l^2)}{4 \cdot 64\pi^2 l^3 \epsilon^4}, \\
\langle O_2 O_2 \rangle (\theta_1 - \theta_2) &= \langle O_2 O_2 \rangle (\theta_2 - \theta_1) \sim \frac{(l + t)^3}{4 \cdot 64\pi^2 l^3 \epsilon^4}, \\
\langle O_3 O_3 \rangle (\theta_1 - \theta_2) &= \langle O_3 O_3 \rangle (\theta_2 - \theta_1) \sim \frac{(l + t)^3}{4 \cdot 64\pi^2 l^3 \epsilon^4}, \\
\langle O_4 O_4 \rangle (\theta_1 - \theta_2) &= \langle O_4 O_4 \rangle (\theta_2 - \theta_1) \sim \frac{(l + t)(l^2 - 4lt + 7l^2)}{4 \cdot 64\pi^2 l^3 \epsilon^4}, \\
\langle O_1 O_1 \rangle (\theta_1 - \theta_2 - 2\pi) &= \langle O_1 O_1 \rangle (\theta_2 - \theta_1 - 2\pi) \\
&\sim \frac{(l - t)^3}{4 \cdot 64\pi^2 l^3 \epsilon^4}, \\
\langle O_2 O_2 \rangle (\theta_1 - \theta_2 - 2\pi) &= \langle O_2 O_2 \rangle (\theta_2 - \theta_1 - 2\pi) \\
&\sim \frac{(l - t)(l^2 + 4lt + 7l^2)}{4 \cdot 64\pi^2 l^3 \epsilon^4}, \\
\langle O_3 O_3 \rangle (\theta_1 - \theta_2 - 2\pi) &= \langle O_3 O_3 \rangle (\theta_2 - \theta_1 - 2\pi) \\
&\sim \frac{(l - t)(l^2 + 4lt + 7l^2)}{4 \cdot 64\pi^2 l^3 \epsilon^4}, \\
\langle O_4 O_4 \rangle (\theta_1 - \theta_2 - 2\pi) &= \langle O_4 O_4 \rangle (\theta_2 - \theta_1 - 2\pi) \\
&\sim \frac{(l - t)^3}{4 \cdot 64\pi^2 l^3 \epsilon^4}, \\
\text{and those in (0 < l ≤ t) are given by}
\end{align*}
\]

The green functions \( \langle O_i O_j \rangle \) vanish.
References

[1] C. Holzhey, F. Larsen, and F. Wilczek, “Geometric and Renormalized Entropy in Conformal Field Theory,” Nucl. Phys. B 424, 443 (1994) [hep-th/9403108]

[2] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, “Entanglement in quantum critical phenomena,” Phys. Rev. Lett. 90, 227902 (2003) [quant-ph/0211074].
J. I. Latorre, E. Rico, and G. Vidal, “Ground state entanglement in quantum spin chains,” Quant. Inf. and Comp. 4, 048 (2004) [quant-ph/0304098]

[3] P. Calabrese and J. Cardy, “Entanglement entropy and quantum field theory,” J. Stat. Mech. P06002 (2004) [hep-th/0405152]

[4] P. Calabrese, and A. Lefevre “Entanglement spectrum in one-dimensional systems,” Phys. Rev. A 78, 032329 (2008) [arXiv:0806.3059 [cond-mat.str-el]].

[5] A. Kitaev and J. Preskill, “Topological entanglement entropy,” Phys. Rev. Lett. 96, 110404 (2006) [hep-th/0510092].

[6] M. Levin, X.-G. Wen “Detecting topological order in a ground state wave function” Phys. Rev. Lett., 96, 110405 (2006) [arXiv:cond-mat/0510613 [cond-mat.str-el]]

[7] J. M. Maldacena, “The Large N limit of superconformal field theories and supergravity,” Int. J. Theor. Phys. 38, 1113 (1999) [Adv. Theor. Math. Phys. 2, 231 (1998)] [hep-th/9711200].

[8] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2, 253 (1998) [hep-th/9802150].

[9] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” Phys. Lett. B 428, 105 (1998) [hep-th/9802109].

[10] M. Van Raamsdonk, “Building up spacetime with quantum entanglement,” Gen. Rel. Grav. 42, 2323 (2010) [Int. J. Mod. Phys. D 19, 2429 (2010)] [arXiv:1005.3035 [hep-th]]; M. Van Raamsdonk, “Comments on quantum gravity and entanglement,” [arXiv:0907.2939 [hep-th]].

[11] S. Ryu and T. Takayanagi, “Aspects of Holographic Entanglement Entropy,” JHEP 0608, 045 (2006) [hep-th/0605073]; S. Ryu and T. Takayanagi, “Holographic derivation of entanglement entropy from AdS/CFT,” Phys. Rev. Lett. 96, 181602 (2006) [hep-th/0603001].

[12] B. Swingle, “Entanglement Renormalization and Holography,” Phys. Rev. D 86, 065007 (2012) [arXiv:0905.1317 [cond-mat.str-el]].
[13] B. Swingle, “Constructing holographic spacetimes using entanglement renormalization,” arXiv:1209.3304 [hep-th].

[14] M. Nozaki, S. Ryu and T. Takayanagi, “Holographic Geometry of Entanglement Renormalization in Quantum Field Theories,” JHEP 1210, 193 (2012) arXiv:1208.3469 [hep-th];

[15] T. Faulkner, M. Guica, T. Hartman, R. C. Myers and M. Van Raamsdonk, “Gravitation from Entanglement in Holographic CFTs,” JHEP 1403 (2014) 051 [arXiv:1312.7856 [hep-th]]; N. Lashkari, M. B. McDermott and M. Van Raamsdonk, “Gravitational dynamics from entanglement ‘thermodynamics’,” JHEP 1404, 195 (2014) doi:10.1007/JHEP04(2014)195 [arXiv:1308.3716 [hep-th]].

[16] A. Almheiri, X. Dong and D. Harlow, “Bulk Locality and Quantum Error Correction in AdS/CFT,” JHEP 1504, 163 (2015) [arXiv:1411.7041 [hep-th]]; A. Almheiri, X. Dong and D. Harlow, “Bulk Locality and Quantum Error Correction in AdS/CFT,” JHEP 1504, 163 (2015) [arXiv:1411.7041 [hep-th]]; X. Dong, D. Harlow and A. C. Wall, “Bulk Reconstruction in the Entanglement Wedge in AdS/CFT,” arXiv:1601.05416 [hep-th];

[17] M. Miyaji and T. Takayanagi, “Surface/State Correspondence as a Generalized Holography,” PTEP 2015, no. 7, 073B03 (2015) [arXiv:1503.03542 [hep-th]]; M. Miyaji, T. Numasawa, N. Shiba, T. Takayanagi and K. Watanabe, “Continuous Multiscale Entanglement Renormalization Ansatz as Holographic Surface-State Correspondence,” Phys. Rev. Lett. 115, no. 17, 171602 (2015) doi:10.1103/PhysRevLett.115.171602 [arXiv:1506.01353 [hep-th]]; M. Miyaji, S. Ryu, T. Takayanagi and X. Wen, “Boundary States as Holographic Duals of Trivial Spacetimes,” JHEP 1505, 152 (2015) doi:10.1007/JHEP05(2015)152 [arXiv:1412.6226 [hep-th]].

[18] Y. Nakayama and H. Ooguri, “Bulk Locality and Boundary Creating Operators,” JHEP 1510, 114 (2015) [arXiv:1507.04130 [hep-th]].

[19] D. N. Kabat, “Black hole entropy and entropy of entanglement,” Nucl. Phys. B 453, 281 (1995) [hep-th/9503016].

[20] C. Eling, Y. Oz and S. Theisen, “Entanglement and Thermal Entropy of Gauge Fields,” JHEP 1311, 019 (2013) [arXiv:1308.4964 [hep-th]].

[21] W. Donnelly and A. C. Wall, “Do gauge fields really contribute negatively to black hole entropy?,” Phys. Rev. D 86, 064042 (2012) [arXiv:1206.5831 [hep-th]].

[22] W. Donnelly and A. C. Wall, “Entanglement entropy of electromagnetic edge modes,” Phys. Rev. Lett. 114, no. 11, 111603 (2015) [arXiv:1412.1895 [hep-th]].

[23] K. Ohmori and Y. Tachikawa, “Physics at the entangling surface,” J. Stat. Mech. 1504, P04010 (2015) [arXiv:1406.4167 [hep-th]].
[24] D. Radicevic, “Notes on Entanglement in Abelian Gauge Theories,” arXiv:1404.1391 [hep-th].

[25] D. Radicevic, “Entanglement in Weakly Coupled Lattice Gauge Theories,” arXiv:1509.08478 [hep-th].

[26] H. Casini, M. Huerta and J. A. Rosabal, “Remarks on entanglement entropy for gauge fields,” Phys. Rev. D 89, no. 8, 085012 (2014) arXiv:1312.1183 [hep-th].

[27] H. Casini and M. Huerta, “Entanglement entropy for a Maxwell field: Numerical calculation on a two dimensional lattice,” Phys. Rev. D 90, no. 10, 105013 (2014) arXiv:1406.2991 [hep-th].

[28] S. Aoki, T. Iritani, M. Nozaki, T. Numasawa, N. Shiba and H. Tasaki, “On the definition of entanglement entropy in lattice gauge theories,” JHEP 1506, 187 (2015) arXiv:1502.04267 [hep-th].

[29] W. Donnelly, “Decomposition of entanglement entropy in lattice gauge theory,” Phys. Rev. D 85, 085004 (2012) arXiv:1109.0036 [hep-th].

[30] W. Donnelly, “Entanglement entropy and nonabelian gauge symmetry,” Class. Quant. Grav. 31, no. 21, 214003 (2014) arXiv:1406.7304 [hep-th].

[31] S. Ghosh, R. M. Soni and S. P. Trivedi, “On The Entanglement Entropy For Gauge Theories,” JHEP 1509, 069 (2015) arXiv:1501.02593 [hep-th].

[32] R. M. Soni and S. P. Trivedi, “Aspects of Entanglement Entropy for Gauge Theories,” JHEP 1601, 136 (2016) arXiv:1510.07455 [hep-th].

[33] C. T. Ma, “Entanglement with Centers,” JHEP 1601 (2016) 070 arXiv:1511.02671 [hep-th].

[34] K. W. Huang, “Central Charge and Entangled Gauge Fields,” Phys. Rev. D 92 (2015) no.2, 025010 arXiv:1412.2730 [hep-th].

[35] F. C. Alcaraz, M. I. Berganza and G. Sierra, “Entanglement of low-energy excitations in Conformal Field Theory,” Phys. Rev. Lett. 106, 201601 (2011) arXiv:1101.2881 [cond-mat.stat-mech].

[36] M. I. Berganza, F. C. Alcaraz and G. Sierra, “Entanglement of excited states in critical spin chains,” J. Stat. Mech. 1201, P01016 (2012) arXiv:1109.5673 [cond-mat.stat-mech].

[37] M. Nozaki, T. Numasawa and T. Takayanagi, “Quantum Entanglement of Local Operators in Conformal Field Theories,” Phys. Rev. Lett. 112, 111602 (2014) arXiv:1401.0539 [hep-th].
[38] M. Nozaki, “Notes on Quantum Entanglement of Local Operators,” JHEP 1410, 147 (2014) [arXiv:1405.5875 [hep-th]].

[39] M. Nozaki, T. Numasawa and S. Matsuura, “Quantum Entanglement of Fermionic Local Operators,” arXiv:1507.04352 [hep-th].

[40] S. He, T. Numasawa, T. Takayanagi and K. Watanabe, “Quantum dimension as entanglement entropy in two dimensional conformal field theories,” Phys. Rev. D 90, no. 4, 041701 (2014) [arXiv:1403.0702 [hep-th]].

[41] P. Caputa, M. Nozaki and T. Takayanagi, “Entanglement of local operators in large-N conformal field theories,” PTEP 2014, 093B06 (2014) [arXiv:1405.5946 [hep-th]].

[42] Y. Nakayama, “Scale invariance vs conformal invariance,” Phys. Rept. 569, 1 (2015) [arXiv:1302.0884 [hep-th]].

[43] C. T. Asplund, A. Bernamonti, F. Galli and T. Hartman, “Holographic Entanglement Entropy from 2d CFT: Heavy States and Local Quenches,” JHEP 1502, 171 (2015) [arXiv:1410.1392 [hep-th]].

[44] P. Calabrese and J. L. Cardy, “Entanglement and correlation functions following a local quench: a conformal field theory J. Stat. Mech. 0710 P10004, [arXiv:0708.3750].

[45] M. Nozaki, T. Numasawa and T. Takayanagi, “Holographic Local Quenches and Entanglement Density,” JHEP 1305, 080 (2013) [arXiv:1302.5703 [hep-th]].

[46] P. Caputa, J. Simn, A. tikonas and T. Takayanagi, “Quantum Entanglement of Localized Excited States at Finite Temperature,” JHEP 1501, 102 (2015) [arXiv:1410.2287 [hep-th]].

[47] A. Sivaramakrishnan, “Localized Excitations from Localized Unitary Operators,” arXiv:1604.00965 [hep-th]; B. Chen, W. Z. Guo, S. He and J. q. Wu, “Entanglement Entropy for Descendent Local Operators in 2D CFTs,” JHEP 1510, 173 (2015) [arXiv:1507.01157 [hep-th]]; P. Caputa and A. Veliz-Osorio, “Entanglement constant for conformal families,” Phys. Rev. D 92 (2015) no.6, 065010 [arXiv:1507.00582 [hep-th]]; W. Z. Guo and S. He, “Rnyi entropy of locally excited states with thermal and boundary effect in 2D CFTs,” JHEP 1504, 099 (2015) [arXiv:1501.00757 [hep-th]]; P. Caputa, T. Numasawa and A. Veliz-Osorio, “Scrambling without chaos in RCFT,” arXiv:1602.06542 [hep-th]; M. M. Sheikh-Jabbari and H. Yavartanoo, “Excitation Entanglement Entropy in 2d Conformal Field Theories,” arXiv:1605.00341 [hep-th].

[48] Metlitski, Max A. Fuertes, Carlos A. and Sachdev, Subir “Entanglement entropy in the O(N) model” Phys.Rev.B.80.115122(2009) [hep-th/0904.4477].

[49] M. E. X. Guimaraes and B. Linet, “Scalar Green’s functions in an Euclidean space with a conical-type line singularity,” Commun. Math. Phys. 165, 297 (1994).
We prepare the paper for more precise relation between the effective reduced density matrix and diagrams.