A note on Selberg’s Lemma and negatively curved Hadamard manifolds

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Abstract

We prove that the conclusion of Selberg’s Lemma fails for discrete isometry groups of negatively curved Hadamard manifolds.

In this note we give a negative answer to the first question on Margulis’ problem list [M, pg. 27]: Margulis asked if the conclusion of Selberg’s Lemma holds for finitely generated isometry groups of Hadamard manifolds.

Theorem 1. For every \( \epsilon > 0 \) and \( n \geq 4 \) there exists an \( n \)-dimensional Hadamard manifold \( X_\epsilon \) of sectional curvature \(-1 - \epsilon \leq K_X \leq -1\) and a finitely generated discrete isometry group \( \Gamma_\epsilon \subset \text{Isom}(X_\epsilon) \) which has unbounded torsion.

The idea of the proof is simple: We start with a complete hyperbolic \( n \)-manifold \( M^n \), \( n \geq 4 \), with finitely-generated (actually, free) fundamental group and infinitely many rank one cusps \( C^n \): Such examples were constructed in [KP, K]. We then replace all but finitely many cusps \( C^n \) by metrically complete negatively curved (with pinching constants \((1 + \epsilon)^{-1}\)) orbifolds \( O_i^n \) with boundary, where \( \pi_1(O_i^n) \) is cyclic of order \( i \). The result of this “cusp closing” is a complete negatively curved orbifold \( O_\epsilon \); the action of \( \Gamma_\epsilon := \pi_1(O_\epsilon) \) on the universal cover \( X_\epsilon \) of \( O_\epsilon \) provides the required examples.

The Riemannian metrics in Theorem 1 are \( C^\infty \) but not real-analytic. It is unclear if Theorem 1 holds in the real-analytic category.

Observe that the above question has positive answer for properly discontinuous group actions in dimension 3 (and, hence, 2, although the 2-dimensional case is elementary): Given a smooth contractible 3-manifold \( X \) and a faithful properly discontinuous smooth action \( \Gamma \times X \to X \) of a finitely-generated group \( \Gamma \), there exists an orbifold analogue of the Scott compact core \( O_\epsilon \) of the orbifold \( O = X/\Gamma \); see [FM]. In particular, \( \Gamma \) is isomorphic to the fundamental group of the compact orbifold \( O_\epsilon \). According to [H] and the geometrization theorem for good compact 3-dimensional orbifolds (see [BLP] or [KL]), the orbifold \( O_\epsilon \) is \textit{very good}, i.e. \( \Gamma \) contains a torsion-free subgroup of finite index.

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1 Cusps in hyperbolic manifolds

We will use the upper half-space model \( \mathbb{H}^n = \{ x : x_n > 0 \} \) of the hyperbolic \( n \)-space. An isometry of \( \mathbb{H}^n \) is unipotent if it is conjugate to a translation \( x \mapsto x + a e_1 \) for \( a \geq 0 \). An isometry of \( \mathbb{H}^n \) is called parabolic if it has a unique fixed point in the closed ball compactification \( \mathbb{H}^n \cup \partial \mathbb{H}^n \) of \( \mathbb{H}^n \). Here \( \partial \mathbb{H}^n \) is the ideal/visual boundary of \( \mathbb{H}^n \).

We let \( \beta_\lambda : \mathbb{H}^n \to \mathbb{R} \) denote the Busemann function for the point \( \lambda \in \partial_x \mathbb{H}^n \); this function is uniquely defined up to an additive constant. Sublevel sets of Busemann functions are horoballs in \( \mathbb{H}^n \).

Throughout the paper we will be using only closed horoballs and closed metric neighborhoods.

Let \( \mathbb{H}^n \to \mathbb{H}^N \) denote an isometric totally-geodesic embedding, \( N \geq n \). This embedding is equivariant under a canonical monomorphism \( \text{Isom}(\mathbb{H}^n) \to \text{Isom}(\mathbb{H}^N) \): Each isometry \( \phi \) of \( \mathbb{H}^n \) extends to an isometry of \( \mathbb{H}^N \) acting trivially on the normal bundle of \( \mathbb{H}^n \) in \( \mathbb{H}^N \).

For every hyperbolic subspace \( X' = \mathbb{H}^n \subset X = \mathbb{H}^N \) we have the orthogonal projection \( p_{X',X} : X \to X' \). Fibers of this projection are hyperbolic subspaces orthogonal to \( X' \). In the case of nested hyperbolic subspaces

\[
X'' \subset X' \subset X,
\]

we have

\[
p_{X'',X} = p_{X'',X'} \circ p_{X',X}.
\]  

(2)

Below we review the notion of cusps of hyperbolic manifolds/orbifolds; we refer to \[Bo1\, Bo2\, Ra\] for details.

Let \( \Gamma \leq \text{Isom}(\mathbb{H}^n) \) be a discrete subgroup with the limit set \( \Lambda = \Lambda(\Gamma) \subset \partial_x \mathbb{H}^n \). A parabolic limit point of \( \Gamma \) is a fixed point of a parabolic isometry \( \gamma \in \Gamma \). The \( \Gamma \)-stabilizer \( \Pi = \Gamma_\lambda < \Gamma \) of such \( \lambda \in \Lambda \) is called a maximal parabolic subgroup of \( \Gamma \). A parabolic limit point \( \lambda \) of \( \Gamma \) is called bounded (equivalently, cusped) if the quotient \( (\Lambda - \{ \lambda \})/\Gamma_\lambda \) is compact. Each bounded parabolic fixed point of \( \Gamma \) corresponds to a “cusp” of the quotient orbifold \( M = \mathbb{H}^n/\Gamma \) defined as follows. Let \( X_\lambda' \subset X = \mathbb{H}^n \) be a smallest \( \Pi \)-invariant hyperbolic subspace of \( X \) (such a subspace need not be unique). Then \( \Pi \) acts with finite covolume on the intersection \( B_\lambda \cap X_\lambda' \) for every horoball \( B_\lambda \subset \mathbb{H}^n \) centered at \( \lambda \). The virtual rank of the virtually abelian group \( \Pi \) equals \( r_\lambda = \text{dim}(X_\lambda') - 1 \).

Let \( p_\lambda := p_{X_\lambda',X} : X = \mathbb{H}^n \to X_\lambda' \) be the orthogonal projection as above. Define

\[
\tilde{C}_\lambda := B_\lambda \cap X_\lambda'
\]

and

\[
\tilde{C}_\lambda := p_\lambda^{-1}(\tilde{C}_\lambda') \subset \mathbb{H}^n
\]

(both depend on \( B_\lambda \) and \( X_\lambda' \), of course).

**Definition 1.1.** If the orbi-covering map \( \pi : \mathbb{H}^n/\Pi \to \mathbb{H}^n/\Gamma = M \) is injective on \( \tilde{C}_\lambda/\Pi \), then the image \( C_\lambda := \pi(\tilde{C}_\lambda/\Pi) \) is called a cusp neighborhood in \( M \) (or, simply, a cusp in \( M \)) corresponding to \( \Pi \). The domain \( \tilde{C}_\lambda \) is then called a cusped region of the limit point \( \lambda \in \Lambda(\Gamma) \). The number \( r_\lambda \) is the rank of the cusp \( C_\lambda \).

By abusing the notation, for a cusped region \( \tilde{C}_\lambda \), we will denote \( C_\lambda = \tilde{C}_\lambda/\Pi \) as well.

For each \( n \)-dimensional cusp \( C_\lambda \) we define its core \( C_\lambda' \subset C_\lambda \) as the quotient \( \tilde{C}_\lambda'/\Pi \). The core is unique up to an isometry \( C_\lambda \to C_\lambda \).

A parabolic limit point \( \lambda \in \Lambda(\Gamma) \) is a bounded if and only if for a sufficiently small horoball \( B_\lambda \) (depending, among other things, on the choice of \( X_\lambda' \)) \( \tilde{C}_\lambda \) is a cusped region.
Remark 1.1. Each maximal parabolic subgroup $\Gamma_\lambda < \Gamma$ (regardless of whether $\lambda$ is a bounded parabolic limit point or not) corresponds to a Margulis cusp of $M = \mathbb{H}^n/\Gamma$: It is the projection to $M$ of the region $U_\lambda \subset \mathbb{H}^n$ consisting of points $x$ such that there exists a parabolic element $\gamma \in \Gamma_\lambda$ satisfying $d(x, \gamma(x)) \leq \mu_n$, the Margulis constant of $\mathbb{H}^n$. Margulis cusps should not be confused with the cusps defined above. Margulis cusps will not be used in this paper.

If $\Gamma < \text{Isom}(\mathbb{H}^n)$ is geometrically finite then every parabolic limit point of $\Gamma$ is bounded, $M$ has only finitely many cusps and, after taking sufficiently small horoballs $B_\lambda$, we can assume that these cusps are pairwise disjoint. If $n = 3$ then every finitely-generated discrete subgroup $\Gamma < \text{Isom}(\mathbb{H}^3)$ has only finitely many cusps (and Margulis cusps), but this fails in dimensions $n \geq 4$ (see [KP]); the existence of such examples is critical for the proof of Theorem 1.

We will call a cusp unipotent if its fundamental group $\Pi$ is unipotent, i.e. every element of $\Pi$ is unipotent.

For each $r > 0$ we define the $r$-collar $C_{\lambda, r} \subset C_\lambda$ of the boundary of a cusp $C_\lambda$ as the quotient by $\Pi$ of $p_\lambda^{-1}(N_r(X_\lambda' \cap \partial B_\lambda))$, where $N_r(\cdot)$ denotes the $r$-neighborhood in $X_\lambda' \cap B_\lambda$. Then the minimal distance between the boundary components of the collar $C_{\lambda, r}$ equals $r$.

Given a hyperbolic subspace $\mathbb{H}^n \subset \mathbb{H}^N$, every horoball $B_\lambda \subset \mathbb{H}^n$ properly embeds in a horoball $B_\lambda^N \subset \mathbb{H}^N$. Accordingly, each cusp $C_\lambda^r \subset \mathbb{H}^n/\Pi$ properly embeds in a cusp $C_\lambda^N \subset \mathbb{H}^N/\Pi$.

In this paper we will be only interested in rank one unipotent cusps $C_\lambda$. Such a cusp is uniquely determined (up to isometry) by its dimension $n$ and one more real parameter, the core length $\ell(C_\lambda)$, defined as the length of the boundary loop of the 2-dimensional core $C_\lambda' \subset C_\lambda$.

We will need the following example of a finitely generated discrete subgroup of $\text{Isom}(\mathbb{H}^4)$ with infinitely many cusps:

Theorem 3 ([KP, K]). There exists a discrete (geometrically infinite) subgroup $\Phi < \text{Isom}(\mathbb{H}^4)$ isomorphic to a free group $F_k$ of rank $k < \infty$, such that:

1. The quotient manifold $M^4 = \mathbb{H}^4/\Phi$ contains an infinite collection of pairwise disjoint and isometric rank 1 unipotent cusps $C_{\lambda_i}$, $i \in \mathbb{N}$.

2. $\Phi$ is a normal subgroup of a geometrically finite group $\hat{\Phi} < \text{Isom}(\mathbb{H}^4)$; every rank one cusp of $M^4$ injectively covers a rank one cusp of $\mathbb{H}^4/\hat{\Phi}$.

We let $L := \ell(C_{\lambda_i})$ denote the common core length of the cusps $C_{\lambda_i}$ of $M$.

Remark 1.2. Besides the cusps $C_{\lambda_i}$, the manifold $M^4$ also has finitely many Margulis cusps. These Margulis cusps project to rank two cusps of $\mathbb{H}^4/\hat{\Phi}$. The parabolic limit points of $\Phi$ corresponding to these Margulis cusps are not bounded.

We retain the notation $\Phi$ for the image of $\Phi$ under the embedding $\text{Isom}(\mathbb{H}^4) \to \text{Isom}(\mathbb{H}^n)$ and define $M^n := \mathbb{H}^n/\Phi$. As noted above, each cusp $C_{\lambda_i}^4$ of $M^4$ embeds properly in a cusp $C_{\lambda_i}^n \subset M^n$; these $n$-dimensional cusps are also pairwise disjoint, since there exist 1-Lipschitz retracts $M^n \to M^4$, satisfying $C_{\lambda_i}^n \to C_{\lambda_i}^4$.

2 Warped products

The “cusp closing” procedure in the proof of Theorem 1 will use warped products of negatively curved manifolds. In this section we review this basic construction.
Let \((B, ds_B^2), (F, ds_F^2)\) be Riemannian manifolds and \(f : B \to (0, \infty)\) a smooth function. The warped product of these manifolds, denoted \(W = B \times_f F\), is the product \(B \times F\) equipped with the Riemannian metric
\[
d s^2 = ds_B^2 + f^2 ds_F^2,
\]
see [BO, sect. 7] for a detailed discussion. In particular, the Riemannian manifold \(f\Gamma\)-invariant, then the product action of \(\Gamma\) on \(W\) is also isometric. In particular, the notion of warped product extends to Riemannian orbifolds.

We will need the sectional curvature formula for the warped product in the case of \((B, ds_B^2) = (\mathbb{R}, dt^2)\), given in [BO, pg. 27]:
\[
K(\Pi) = -\frac{f''(t)}{f(t)} ||x||^2 + \frac{L(v, w) - (f'(t))^2}{f^2(t)} ||v||^2.
\]
Here \(\Pi\) is plane in \(T_{(t, q)} W\) with the orthonormal basis \(\{x + v, w\}\), where \(v, w \in T_q F\), \(x\) is a horizontal tangent vector (thus, \(||x||^2 + ||v||^2 = ||x||^2 + f^2||v||_F^2 = 1\)) and \(L(v, w)\) is the sectional curvature of \((F, ds_F^2)\) at \(q\) on the plane spanned by the vectors \(v, w\). (Note that [BO, pg. 26] also contains the sectional curvature formula for general warped products).

In particular, if \(F\) is negatively curved with sectional curvature \(-1 - \epsilon \leq L \leq -1\) and \(f(t) = \cosh(t)\), then
\[-1 - \epsilon \leq K(\Pi) \leq -1\]
as well.

The hyperbolic space \(\mathbb{H}^n\) is isometric to a warped product \(\mathbb{H}^{n-2} \times_f \mathbb{H}^2\) with \(f : \mathbb{H}^{n-2} \to \mathbb{R}_+, \quad f(p) = \cosh(d(o, p))\), where \(o \in \mathbb{H}^{n-2}\) is a basepoint. This warped product decomposition can be realized as follows. We let \(\mathbb{H}^2\) be embedded in \(\mathbb{H}^n\) as
\[
\{(x_1, 0, 0, ..., 0, x_n) : x_n > 0\}.
\]
Horizontal (totally-geodesic) leaves of the warped product \(\mathbb{H}^{n-2} \times_f \mathbb{H}^2\) correspond to codimension two hyperbolic subspaces in \(\mathbb{H}^n\) orthogonal to \(\mathbb{H}^2\), while vertical leaves are obtained from \(\mathbb{H}^2\) by rotating it via elements of \(SO(n - 1) < SO(n)\) fixing pointwise the coordinate line
\[
\mathbb{R}e_n = \{(x_1, 0, ..., 0, 0)\}.
\]
The vertical projection \(\mathbb{H}^{n-2} \times_f \mathbb{H}^2 \to \mathbb{H}^2\) is just the orthogonal projection \(p_{\mathbb{H}^2, \mathbb{H}^n}\).

Yes another way to realize this decomposition of \(\mathbb{H}^n\) is as the iterated warped product
\[
\mathbb{H}^n = \mathbb{R} \times_{f_{-2}} \mathbb{H}^2,
\]
\[
\mathbb{H}^3 = \mathbb{R} \times_{f} \mathbb{H}^2, \mathbb{H}^4 = \mathbb{R} \times_{f} \mathbb{H}^3, ..., \mathbb{H}^n = \mathbb{R} \times_{f} \mathbb{H}^{n-1},
\]
where \(f(t) = \cosh(t)\). The orthogonal projection \(p_{\mathbb{H}^2, \mathbb{H}^n}\) equals the vertical projection \(\eta : \mathbb{R} \times_{f_{-2}} \mathbb{H}^2 \to \mathbb{H}^2\) given by iterating vertical projections in the warped product decompositions \(\mathbb{H}^k = \mathbb{R} \times_{f} \mathbb{H}^{k-1}\), cf. (2).

We generalize this iterated warped product as follows. We let \(\tilde{F}\) be a simply-connected complete negatively curved surface with sectional curvature in the interval \([-1 - \epsilon, -1]\). Define the iterated warped product
\[
\tilde{W} = \mathbb{R} \times_{f_{-2}} \tilde{F}.
\]
It follows that \(\tilde{W}\) is still a simply-connected complete negatively curved manifold with sectional curvature in \([-1 - \epsilon, -1]\). We let \(\tilde{\eta} : \tilde{W} \to \tilde{F}\) denote the vertical projection. Then for an open subset \(\tilde{U} \subset \tilde{F}\), the preimage \(\tilde{\eta}^{-1}(\tilde{U})\) is an iterated warped product \(\mathbb{R} \times_{f_{-2}} \tilde{U}\). In particular, if \(\tilde{U}\) has constant curvature \(-1\), so does \(\tilde{\eta}^{-1}(\tilde{U})\).
3 Closing rank one cusps

We will apply iterated warped products \( \square \) to surfaces \( \tilde{F} \) (and their Riemannian orbifold quotients), constructed by splicing quotients of \( \mathbb{H}^2 \) by cyclic parabolic and by finite cyclic groups. The goal is “close” \( n \)-dimensional rank one unipotent cusps \( C^u_\lambda \), converting them to orbifolds of variable negative curvature with finite cyclic fundamental groups, while leaving the Riemannian metric on a suitable \( r \)-collar of \( C_\lambda \) unchanged. The cusp-closing is a rather standard procedure, we describe it here in detail for the sake of completeness.

We start by describing cusp-closing in dimension 2. Let \( \Sigma_0 < \text{Isom}(\mathbb{H}^2) \) be a cyclic parabolic subgroup; the surface \( T_0 := \mathbb{H}^2/\Sigma_0 \) is foliated by projections of \( \Sigma_0 \)-invariant horocycles in \( \mathbb{H}^2 \). Let \( c_0 \in T_0 \) be the (unique) leaf of length \( a > 0 \). (The number \( a \) will be specified later on.)

Similarly, let \( \Sigma_i < \text{Isom}(\mathbb{H}^2) \) be a finite cyclic subgroup of order \( i \geq 2 \). The quotient-orbifold \( T_i = \mathbb{H}^2/\Sigma_i \) is foliated by projections of \( \Sigma_i \)-invariant circles in \( \mathbb{H}^2 \). Let \( c_i \in T_i \) be the (unique) leaf of the same length \( a \) as above. The hyperbolic surfaces/orbifolds \( T_0 \) and \( T_i \) admit isometric \( U(1) \)-actions whose orbits are leaves of the above foliations. The distance from \( c_i \) to the singular point of \( T_i \) equals \( R_i = \arcsinh(\frac{ai}{2r}) \).

We let \( T_0' \) denote the closure of the infinite area component in \( T_0 - c_0 \) and let \( T''_i \) denote the closure of the bounded component of \( T_i - c_i \). Gluing \( T_0', T''_i \) via an isometry of their boundaries results in a metric orbifold \( S_i \); the metric on \( S_i \) is, of course, smooth away from \( \bar{c}_i := c_0 \equiv c_i \) and is singular along that curve. (The group \( U(1) \) still acts isometrically on \( S_i \).) Below we smooth out the metric on \( S_i \) by modifying it near \( \bar{c}_i \), so that the new metric has negative curvature with small pinching constant when \( i \) is large.

Fix \( r > 0 \) and let the nested annuli \( A_{i,r} \subset A_{i,2r} \) denote the \( r \)- and \( 2r \)-neighborhoods of \( \bar{c}_i \) in \( S_i \) with respect to the singular metric on \( S_i \). We will take \( r \) such that

\[
r < \frac{1}{2} \arcsinh(a/\pi) \leq R_i,
\]

hence, the annulus \( A_{i,2r} \) is disjoint from the singular point of the orbifold \( S_i \).

The next lemma follows from the geometric convergence of suitable conjugates of subgroups \( \Sigma_i < \text{Isom}(\mathbb{H}^2) \) to \( \Sigma_0 < \text{Isom}(\mathbb{H}^2) \), since the latter implies \( C^\infty \) Gromov–Hausdorff convergence

\[ (T_i, t_i) \rightarrow (T_0, t_0), \]

where \( t_i \in c_i, t_0 \in c_0 \); see [BP Ch. E].

**Lemma 4.** For each \( \epsilon > 0 \) there is \( i_\epsilon \) such that for all \( i \geq i_\epsilon \) there exist \( U(1) \)-invariant Riemannian metrics \( g_i \) on the orbifolds \( S_i \) satisfying:

1. \( g_i \) equals the restrictions of the metrics of \( T_0 \) and \( T_i \) respectively on the unbounded/bounded components of \( S_i - A_{i,r} \).
2. The curvature of \( g_i \) lies in the interval \([−1 − \epsilon, −1]\).

In what follows, we equip the orbifolds \( S_i \) with the above metrics \( g_i \) and denote the resulting Riemannian orbifold \( F_i \). We let \( \tilde{F}_i \rightarrow F_i \) denote the (degree \( i \)) universal cover of the orbifold \( F_i \); then \( \tilde{F}_i \) is a simply-connected negatively curved complete Riemannian surface. We let \( O^2_{i,2r} \subset \tilde{F}_i \) denote the union of the annulus \( A_{i,2r} \subset \tilde{F}_i \) and the suborbifold \( T''_i \subset F_i \) (equipped with the restriction of the metric \( g_i \), of course). The boundary curve of \( O^2_{i,2r} \) has length \( ae^{2r} \).

Before extending this construction to higher dimensions we describe cusp-closing for hyperbolic surfaces. Let \( M^2 \) be a complete hyperbolic surface (possibly of infinite area) and \( C_{\lambda_i} \subset M^2 \) be pairwise disjoint cusps with equal core lengths \( = L > a \). Each \( C_{\lambda_i} \), of course
embeds isometrically in the surface \( T_0 \) as above; we let \( 2r \) denote the distance between the boundary curve of \( C_\lambda \) and the loop \( c_0 \subset T_0 \). Thus, \( L = ae^{2r} \).

The \( r \)-collar \( C_{\lambda,r} \subset C_\lambda \) is isometric to the \( r \)-neighborhood \( E_{i,r} = N_r(\partial O_{i,2r}^2) \) of the boundary of \( O_{i,2r}^2 \) (\( E_{i,r} \) is a component of \( A_{i,2r} - A_{i,r} \) and has constant negative curvature). Hence, we can replace each cusp \( C_\lambda \subset M^2 \) with a Riemannian orbifold \( O_{i,2r}^2 \) by first removing \( C_{\lambda_i} - C_{\lambda_i,r} \) and then gluing \( O_{i,2r}^2 \) via an isometry \( E_{i,r} \to C_{\lambda_i,r} \). The resulting Riemannian orbifold \( O^2 \) is said to be obtained from \( M^2 \) by \textit{cusp-closing}.

**Remark 4.1.** The Riemannian orbifold \( O^2 \) is complete and has negative curvature, which equals \(-1\) except for the annuli \( A_{i,r} \) where the curvature is variable. In dimension 2 (at least if the group \( \pi_1(M^2) \) is finitely generated) one can also accomplish cusp-closing via a metric of constant negative curvature (by perturbing the hyperbolic metric of \( M^2 \) globally rather than inside of the cusps \( C_\lambda \)), but this is not what we are interested in. See also the [K, Corollary 2] in the setting of hyperbolic 4-manifolds/orbifolds.

We now proceed with cusp-closing in higher dimensions.

Applying the iterated warped product construction to the Riemannian surfaces \( \tilde{F} = \tilde{F}_i \) as described above, we obtain \( n \)-dimensional Hadamard manifolds \( \tilde{W}_i^n \) equipped with isometric \( \Sigma_i \)-actions; let \( W_i^n := \tilde{W}_i^n / \Sigma_i \) be the Riemannian quotient-orbifolds. Thus, each \( W_i^n \) is the \( n - 2 \)-fold iterated warped product

\[
W_i^n = \mathbb{R} \times_{\cosh} F_i
\]

with the vertical projection \( \eta_i : W_i \to F_i \). As noted above, \( W_i \) has constant curvature \(-1\) away from \( \eta^{-1}(A_{i,r}) \).

As with the 2-dimensional cusp closing, we will only need the parts

\[
O_{i,2r}^n := \eta^{-1}(O_{i,2r}^2) \subset W_i^n
\]

of the orbifolds \( W_i^n \). Each \( O_{i,2r}^n \) can be regarded as an \( \mathbb{H}^{n-2} \)-bundle over the orbifold \( O_{i,2r}^2 \). The orbifolds \( O_{i,2r}^n \) will be replacing rank one unipotent cusps of a hyperbolic \( n \)-manifold. This will be accomplished by gluing along constant curvature boundary collars

\[
E_{i,r}^n := \eta^{-1}(E_{i,r}) \subset O_{i,2r}^n.
\]

Let \( C_\lambda^n \) be an \( n \)-dimensional rank one unipotent cusp of core length \( \ell(C_\lambda^n) = L = ae^{2r} \). In view of the iterated warped product decomposition

\[
\mathbb{H}^n = \mathbb{R} \times_{\cosh} \mathbb{H}^2,
\]

the cusp \( C_\lambda^n \) also decomposes as the iterated warped product

\[
C_\lambda^n = \mathbb{R} \times_{\cosh} C_\lambda^2,
\]

where \( C_\lambda^2 \subset C_\lambda^n \) is a 2-dimensional core of \( C_\lambda^n \), with the projection \( \eta : C_\lambda^n \to C_\lambda^2 \). The boundary \( r \)-collar \( C_{\lambda,r}^n \subset C_\lambda^n \) equals the preimage

\[
\eta^{-1}(C_{\lambda,r}^2)
\]

of the \( r \)-collar of the core cusp \( C_{\lambda}^2 \).

Thus we obtain isometries of boundary collars

\[
C_{\lambda,r}^n \to E_{i,r}^n,
\]

from the boundary collar of a cusp \( C_\lambda^n \) to the boundary collars of the orbifolds \( O_{i,2r}^n \).
5 Proof of Theorem \[ \] 

We construct a Riemannian orbifold \( O^\epsilon \) as follows. Given \( \epsilon > 0 \), we let \( i_\epsilon \in \mathbb{N} \) be as in Lemma \[ \] Recall that \( L = \ell(C^4_{\lambda i}) \) is the common core length of the cusps \( C^4_{\lambda i} \) of the hyperbolic 4-manifold \( M^4 \) from Theorem \[ \] We then let \( r > 0 \) and \( a > 0 \) be such that 

\[
L = e^{2r}a, \quad r < \frac{1}{2} \arcsinh(a/\pi),
\]

which can be always accomplished by taking \( r \) to be sufficiently small.

From each cusp \( C^n_{\lambda i}, i \geq i_\epsilon \) of the hyperbolic \( n \)-manifold \( M^n = \mathbb{H}^n/\Phi \) we remove the complement to the boundary \( r \)-collar \( C^n_{\lambda i, r} \); let \( M' \) denote the remaining manifold:

\[
M' := M^n - \bigcup_{i \geq i_\epsilon} (C^n_{\lambda i} - C^n_{\lambda i, r}).
\]

Then for each \( i \geq i_\epsilon \) we glue to \( M' \) the Riemannian orbifold \( O_i^{n, 2r} \) via an isometry of the collars 

\[
C^n_{\lambda i, r} \to E^n_{i, r}.
\]

The result is an \( n \)-dimensional Riemannian orbifold \( O_{\epsilon} := O^n_{\epsilon} \).

Remark 5.1. Since \( \pi_1(M^n) \cong \pi_1(M^4) \cong \Phi \) is free of rank \( k \), the fundamental group \( \Gamma_{\epsilon} = \pi_1(O_{\epsilon}) \) has the presentation 

\[
\langle s_1, \ldots, s_k | w_i^i, i \geq i_\epsilon \rangle,
\]

where the words \( w_i \) represent generators of fundamental groups of the cusps \( C^4_{\lambda i} \subset M^4 \).

By the construction, the sectional curvature of \( O_{\epsilon} \) lies in the interval \([-1 - \epsilon, -1]\). Since \( M' \) and all orbifolds \( O_i^{n, 2r} \) are metrically complete and the minimal distance between the boundary components of each collar \( C^n_{\lambda i, r} \) equals \( r > 0 \), it follows that the Riemannian orbifold \( O_{\epsilon} \) is also complete.

Since the orbifold \( O_{\epsilon} \) is complete and negatively curved, it is good (developable); see \[ BH \] pg. 603, Theorem 2.15] and also \[ Ra \] Ch. 13]. Hence, the universal cover of \( O_{\epsilon} \) is a \( n \)-dimensional Hadamard manifold \( X = X_{\epsilon} \) of curvature \(-1 - \epsilon_m \leq K_X \leq -1\). The fundamental group \( \Gamma_{\epsilon} \) acts on \( X_{\epsilon} \) faithfully, properly discontinuously and isometrically with \( O_{\epsilon} \cong X_{\epsilon}/\Gamma_{\epsilon} \). In particular, \( \Gamma_{\epsilon} \) has unbounded torsion: The fundamental group (cyclic group of order \( i \)) of each orbifold \( O_i^{n, 2r} \ (i \geq i_\epsilon) \) embeds in \( \Gamma_{\epsilon} \).

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