An Aspect of Granulence in view of Multifractal Analysis

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November 14, 2018

Abstract

The probability density function of velocity fluctuations of granulence observed by Radjai and Roux in their two-dimensional simulation of a slow granular flow under homogeneous quasistatic shearing is studied by the multifractal analysis for fluid turbulence proposed by the present authors. It is shown that the system of granulence and of turbulence have indeed common scaling characteristics.

Keywords: multifractal analysis, velocity fluctuation, turbulence, granulence

1 Introduction

In this paper, we apply the multifractal analysis (MFA) [1, 2, 3, 4] of fluid turbulence to granular turbulence (granulence [5]) in order to see how far MFA works in the study of the data observed by Radjai and Roux [5] in their two-dimensional simulation of a slow granular flow subject to homogeneous quasistatic shearing. Radjai and Roux reported that there is an evident analogy between the scaling features of turbulence and of granulence in spite of the fundamentally different origins of fluctuations in these systems. MFA is a unified self-consistent approach for the systems with large deviations, which has been constructed based on the Tsallis-type distribution function [6] that provides an extremum of the extensive Rényi [7] or the non-extensive Tsallis entropy [6, 8] under appropriate constraints.
2 Multifractal Analysis

MFA of turbulence rests on the scale invariance of the Navier-Stokes equation for high Reynolds number, and on the assumption that the singularities due to the invariance distribute themselves multifractly in physical space.

The velocity fluctuation \( \delta u_n = |u(\bullet + \ell_n) - u(\bullet)| \) of the \( n \)th multifractal step satisfies the scaling law \( |u_n| \equiv |\delta u_n/\delta u_0| = \delta_n^{\alpha/3} \) with \( \delta_n = \ell_n/\ell_0 = \delta^{-n} \) \( (n = 0, 1, 2, \cdots) \). We call \( n \) the multifractal depth which can be real number in the analysis of experimental data. We will put \( \delta = 2 \) in the following in this paper that is consistent with the energy cascade model. At each step of the cascade, say at the \( n \)th step, eddies break up into two pieces producing the energy cascade with the energy-transfer rate \( \epsilon_n \) that represents the rate of transfer of energy per unit mass from eddies with diameter \( \ell_n \) to those with \( \ell_{n+1} \). Then, we see that the velocity derivative \( |u'| = \lim_{n \to \infty} u_n' \) with the \( n \)th velocity difference \( u_n' = \delta u_n/\ell_n \) for the characteristic length \( \ell_n \) diverges for \( \alpha < 3 \). The real quantity \( \alpha \) is introduced in the scale transformation \([9, 10]\) \( \vec{x} \to \vec{x}' = \lambda \vec{x}, \quad \vec{u} \to \vec{u}' = \lambda^{\alpha/3}\vec{u}, \quad t \to t' = \lambda^{1-\alpha/3}t, \quad p \to p' = \lambda^{2\alpha/3}p \) that leaves the Navier-Stokes equation \( \partial \vec{u}/\partial t + (\vec{u} \cdot \nabla)\vec{u} = -\nabla p + \nu \nabla^2 \vec{u} \) of incompressible fluid invariant for a large Reynolds number \( Re = \delta u_in/\ell_in/\nu \). Here, \( \nu \) is the kinematic viscosity, \( p = \bar{p}/\rho \) with the thermodynamical pressure \( \bar{p} \) and the mass density \( \rho \), and \( \delta u_in \) and \( \ell_in \) represent, respectively, the rotating velocity and the diameter of the largest eddies in turbulence. The largest size of eddies is, for example, about the order of mesh size of the grid, inserted in a laminar flow, which produces turbulence downstream.

Within MFA, it is assumed that the singularities due to the scale invariance distribute themselves, multifractly, in physical space with the Tsallis-type distribution function, i.e., the probability \( P^{(n)}(\alpha) \) to find in real space a singularity with the strength \( \alpha \) within the range \( \alpha \sim \alpha + d\alpha \) is given by \([2, 11, 12]\) \( P^{(n)}(\alpha) = (Z_\alpha^{(n)})^{-1}\{1 - [(\alpha - \alpha_0)/\Delta\alpha]^2\}^{n/(1-q)} \) with \( (\Delta\alpha)^2 = 2X/[(1-q)\ln 2] \). Here, \( q \) is the entropy index introduced in the definitions of the Rényi and the Tsallis entropies. This distribution function provides us with the multifractal spectrum \( f(\alpha) = 1 + (1-q)^{-1}\log_2[1-(\alpha-\alpha_0)^2/(\Delta\alpha)^2] \) which, then, produces the mass exponent

\[
\tau(\bar{q}) = 1 - \alpha_0\bar{q} + 2X\bar{q}^2(1 + \sqrt{C_\bar{q}})^{-1} + (1-q)^{-1}\left[1 - \log_2(1 + \sqrt{C_\bar{q}})\right]
\] (1)

with \( C_\bar{q} = 1 + 2\bar{q}^2(1-q)X\ln 2 \). The multifractal spectrum and the mass exponent are related with each other through the Legendre transformation \([10]\): \( f(\alpha) = \alpha\bar{q} + \tau(\bar{q}) \) with \( \alpha = -d\tau(\bar{q})/d\bar{q} \) and \( \bar{q} = df(\alpha)/d\alpha \).
The formula of the probability density function (PDF) $\Pi^{(n)}(u_n)$ of velocity fluctuations
is assumed to consists of two parts, i.e., $\Pi^{(n)}(u_n) = \Pi_S^{(n)}(u_n) + \Delta \Pi^{(n)}(u_n)$ where the
first term is related to $P^{(n)}(\alpha)$ by $\Pi_S^{(n)}(|u_n|)du_n \propto P^{(n)}(\alpha)da$ with the transformation
of the variables $|u_n| = \delta_n^{\alpha/3}$, and the second term is responsible to the contributions coming
from the dissipative term in the Navier-Stokes equation violating the invariance under the scale
transformation given above. Then, we have the velocity structure function (equivalently,
$\tau \alpha$ term is related to $(\alpha)$ is assumed to consists of two parts, i.e., $\Pi^{(n)}(\alpha)$
with a certain Reynolds number is settled. For smaller velocity fluctuations,
comparing with observed data is the one defined through $\hat{\Pi}^{(n)}(\xi_n)d\xi_n = \Pi^{(n)}(u_n)du_n$
with the variable $\xi_n = u_n/\langle u_n^2 \rangle^{1/2}$ scaled by the standard deviation of velocity fluctuations.
For the velocity fluctuations larger than the order of its standard deviation, $\xi^*_n \leq |\xi_n|$
(equivalently, $\alpha^* \leq |\alpha|$), the PDF is given by [3, 4]

$$\hat{\Pi}^{(n)}(\xi_n)d\xi_n = \Pi_S^{(n)}(u_n)du_n$$
$$= \Pi^{(n)}(\xi_n) \frac{\tilde{\xi}_n}{|\xi_n|} \left[ 1 - \frac{1 - q}{n} \left( \frac{3 \ln |\xi_n/\xi_{n,0}|}{2X |\ln \delta_n|} \right)^2 \right]^{\gamma_n/(1-q)} d\xi_n$$

with $\xi_{n,0} = \tilde{\xi}_n \delta_n^{\alpha^*/3 - \zeta_2/2}$ and $\Pi^{(n)} = 3(1 - 2\gamma_0^{(n)})/(2\tilde{\xi}_n \sqrt{2\pi X |\ln \delta_n|})$. This tail part
represents the large deviations, and manifests itself the multifractal distribution of the singularities
due to the scale invariance of the Navier-Stokes equation when its dissipative term can be neglected.
The entropy index $q$ should be unique once a turbulent system with a certain Reynolds number is settled. For smaller velocity fluctuations, $|\xi_n| \leq \xi^*_n$
(equivalently, $\alpha^* \leq |\alpha|$), we assume the Tsallis-type PDF of the form [3, 4]

$$\hat{\Pi}^{(n)}(\xi_n)d\xi_n = \left[ \Pi_S^{(n)}(u_n) + \Delta \Pi^{(n)}(u_n) \right] du_n$$
$$= \hat{\Pi}^{(n)} \left\{ 1 - \frac{1 - q'}{2} \left[ 1 + 3f'(\alpha^*) \left( \frac{\xi_n}{\xi_{n,0}} \right)^2 - 1 \right] \right\}^{1/(1-q')} d\xi_n$$

where a new entropy index $q'$ is introduced as an adjustable parameter. This center part
is responsible to smaller fluctuations, compared with its standard deviation, due to the dissipative term violating the scale invariance. The entropy index $q'$ can be dependent on
the distance of two measuring points.

The two parts of the PDF, (2) and (3), are connected at $\xi^*_n = \tilde{\xi}_n \delta_n^{\alpha^*/3 - \zeta_2/2}$ with the
conditions that they have a common value and that their slopes coincide. The value $\alpha^*$
is the smaller solution of $\zeta_2/2 - \alpha/3 + 1 - f(\alpha) = 0$. The point $\xi^*_n$ has the characteristics that the dependence of $\hat{\Pi}^{(n)}(\xi^*_n)$ on $n$ is minimum for $n \gg 1$. With the help of the second equality in (3) and (2), we obtain $\Delta \Pi^{(n)}(x_n)$, and have the analytical formula to evaluate $\gamma^{(n)}_m$. Their explicit analytical formulae and the definition of $\bar{\xi}_n$ are found in [3, 4].

3 Turbulence

![Figure 1: Analyses of the PDF's of velocity fluctuations (closed circles) and of velocity derivatives (open circles) measured in the DNS by Gotoh et al. at $R_\lambda = 380$ with the help of the present theoretical PDF's $\hat{\Pi}^{(n)}(\xi_n)$ for velocity fluctuations (solid lines) and for velocity derivatives (dashed line) are plotted on (a) log and (b) linear scales. The DNS data points are symmetrized by taking averages of the left and the right hand sides data. The measuring distances, $r/\eta = \ell_n/\eta$, for the PDF of velocity fluctuations are, from the second top to bottom: 2.38, 4.76, 9.52, 19.0, 38.1, 76.2, 152, 305, 609, 1220. For the theoretical PDF's of velocity fluctuations, $\mu = 0.240$ ($q = 0.391$), from the second top to bottom: $(n, \bar{n}, q') = (20.7, 14.6, 1.60), (19.2, 13.1, 1.60), (16.2, 10.1, 1.58), (13.6, 7.54, 1.50), (11.5, 5.44, 1.45), (9.80, 3.74, 1.40), (9.00, 2.94, 1.35), (7.90, 1.84, 1.30), (7.00, 0.94, 1.25), (6.10, 0.04, 1.20)$, and $\xi^*_n = 1.10 \sim 1.43$ ($\alpha^* = 1.07$). For the theoretical PDF of velocity derivatives, $(n, \bar{n}, q') = (22.4, 16.3, 1.55)$, and $\xi^*_n = 1.06$ ($\alpha^* = 1.07$). For better visibility, each PDF is shifted by $-1$ unit in (a) and by $-0.1$ in (b) along the vertical axis.

The dependence of the parameters $\alpha_0$, $X$ and $q$ on the intermittency exponent $\mu$ is determined, self-consistently, with the help of the three independent equations, i.e., the energy conservation: $\langle \epsilon_n/\epsilon \rangle = 1$ (equivalently, $\tau(1) = 0$), the definition of the intermittency exponent $\mu$: $\langle \epsilon_n^2/\epsilon^2 \rangle = \delta_n^{-\mu}$ (equivalently, $\mu = 1 + \tau(2)$), and the scaling relation:
\frac{1}{(1 - q)} = \frac{1}{\alpha_-} - \frac{1}{\alpha_+} \text{ with } \alpha_\pm \text{ satisfying } f(\alpha_\pm) = 0. \text{ Here, } \epsilon \text{ is the energy input rate to the largest eddies. The average } \langle \cdots \rangle \text{ is taken with } P^{(n)}(\alpha).

The PDF’s extracted by Gotoh et al. from their DNS data [13] at \( R_\lambda = 380 \) are shown, on log and linear scales, in Fig. 1 both for velocity fluctuations and for velocity derivatives, and are analyzed by the theoretical formulae (2) and (3) for PDF’s. We found the value \( \mu = 0.240 \) by analyzing the measured scaling exponents \( \zeta_m \) of velocity structure function with the formula given above, which leads to the values \( q = 0.391, \alpha_0 = 1.14 \) and \( X = 0.285 \). Through the analyses of the PDF’s for velocity fluctuations in Fig. 3, we extracted quite a few information of the system [14, 15, 16, 3, 4]. Among them, we only quote here the dependence of \( q' \) on \( r/\eta \): \( q' = -0.05 \log_2(r/\eta) + 1.71 \) [4].

4 Granulence

Let us now analyze the velocity fluctuations in granulence simulated by Radjai and Roux [5]. Since they observed that the fluctuations share the scaling characteristics of fluid turbulence, we try to investigate the system by means of MFA which extracted, successfully, the rich information out of turbulence as was seen in the previous section. The power

![Figure 2: Analysis of the experimental PDF of fluctuating velocities, measured in the quasistatic flow of granular media by Radjai and Roux, with the help of the present theoretical PDF \( \tilde{\Pi}^{(n)}(\xi_n) \) for velocity fluctuations (solid lines) are plotted on (a) log and (b) linear scales. The experimental data points are symmetrized by taking averages of the left and the right hand sides data. The integration time \( \tau \), normalized by a shear rate, for the experimental PDF are, from top to bottom, \( 10^{-3}, 10^{-1} \). For the theoretical PDF, \( \mu = 1.347 \) (\( q = 0.930 \)), from top to bottom: \( (n, q') = (88.0, 1.28), (40.0, 1.22), \) and \( \xi_n^* = 1.14, 1.14 (\alpha^* = 0.364) \). For better visibility, each PDF is shifted by \(-1\) unit in (a) and by \(-0.1\) in (b) along the vertical axis.]
spectrum of the fluctuating velocity field on one-dimensional cross sections exhibits a clear power-law shape with the slope $-\beta$ with $\beta \approx 1.24$ [5], which is quite similar to the power-law behavior with the slope $-5/3$ in the inertial range of the Kolmogorov spectrum [17]. However, the granular model is an assembly of frictional disks, the power-law observed in granulence does not mean the energy conservation in contrast with the case of the energy cascade model for fluid turbulence.

For the conditions to determine the parameters $\alpha_0$, $X$ and $q$, we adopt, instead of the energy conservation, the slope of the power spectrum, i.e., $\beta = 1 + \zeta_2 = 2 - \tau(2/3)$ in addition to the definition of the intermittency exponent and the scaling relation. The latter two are the same as those for turbulence. As there is no experimental data, for the present, to determine the intermittency exponent $\mu$ for granulence, we cannot have the values of the three parameters through the three conditions. Therefore, we determine the value of the intermittency exponent by adjusting the observed PDF with the theoretical formulae (2) and (3), since the accuracy of the formulae in the analysis of PDF’s for turbulence is quite high as was shown in the previous section. The best fit of the observed PDF of fluctuating velocities by the formulae (2) and (3) is shown in Fig. 4. We found the value $\mu = 1.347$ giving $q = 0.930$, $\alpha_0 = 0.377$ and $X = 0.050$. By making use of the mass exponent with these values, we have $\langle \epsilon_n / \epsilon \rangle = \delta_n^{-\tau(1)}$ with $\tau(1) = 0.648$ representing a breakdown of energy conservation. It is attractive to see that the result is quite close to $\langle \epsilon_n / \epsilon \rangle = 3/2$ which may be consistent with the coefficient of friction 0.5 for the simulation [5]. We further extract the relation between $\tau$ and $\ell_n$ as $\tau = 1.3 \delta_n^{0.131}$ by comparing the observed flatness and the one with the theoretical PDF’s (2) and (3). This relation may be a manifestation of the fact that Taylor’s frozen turbulence hypothesis does not work for granulence.

5 Prospects

We showed with the help of MFA that the system of turbulence and of granulence have, actually, common scaling feature in their velocity fluctuations as was pointed out by Radjai and Roux [5]. We expect that various observation of granulence will be reported at higher statistics, and that one can extract more information out of the data to determine the underlying dynamics for granulence in the near future.
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