On a $K_4$-UH self-dual 1-configuration $(102_4)_1$

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Abstract
Self-dual 1-configurations $(n_d)_1$ have the most $K_d$-separated Menger graph $Y$ for connected self-dual configurations $(n_d)$. Such $Y$ is most symmetric if it is $K_d$-ultrahomogeneous. In this work, such a graph $Y$ is presented for $(n, d) = (102, 4)$ and shown to relate $n$ copies of the cuboctahedral graph $L(Q_3)$ to the $n$ copies of $K_4$. These are shown to share each copy of $K_3$ with two copies of $L(Q_3)$. Vertices and copies of $L(Q_3)$ in $Y$ are the points and lines of a self-dual $(104_{12})_1$.

1 Introduction
Let $1 < d < n \in \mathbb{Z}$ and $1 < c < m \in \mathbb{Z}$. A configuration $R = (m_c, n_d)$ is an incidence structure of $m$ points and $n$ lines such that there are $c$ lines through each point and $d$ points on each line [8]. Thus, $cm = dn$. Let $L = L(R) = L(m_c, n_d)$ be the Levi graph of $R$, namely the bipartite graph with: (a) $m$ "black" vertices representing the points of $R$; (b) $n$ "white" vertices representing the lines of $R$; and (c) an edge between each two vertices representing a point and a line incident in $R$. To each configuration $R = (m_c, n_d)$ corresponds the dual configuration $\bar{R} = (n_d, m_c)$ by reversing the roles of points and lines in $R$. If $(m, n) = (c, d)$, then $R$ is balanced [17]. If $R$ is isomorphic to its dual $\bar{R}$, then $R = (n_d)$ is self-dual. A corresponding isomorphism is called a duality. Both $R$ and $\bar{R}$ share the same Levi graph, but the black-white coloring of their vertices is reversed. To any such configuration $(n_d)$ we can associate its Menger graph, in which the points of $(n_d)$ are represented by vertices, each two joined by an edge whenever the two corresponding points are in a common line in $(n_d)$. Let $1 \leq \lambda < d$. If any two different points of $R$ are in at most $\lambda$ lines, then $R$ is a $\lambda$-configuration $(n_d)_{\lambda}$ [15]. The 4-cube $Q_4$ is the Levi graph of the Möbius
with "white" (resp. "black") vertices being those of even (resp. odd) weight, and so on for the remaining Cox 2-configurations, in relation to the respective $d$-cube $Q_d$ [8]. Let $H$ be a connected regular graph. A graph $G$ is $C$-ultrahomogeneous [20], or $C$-UH, if every isomorphism between two induced copies of $H \in C$ in $G$ extends to an automorphism of $G$. If $C = \{H\}$ then $G$ is said to be $H$-UH.

The motivation of this paper is the study of connected Menger graphs [8] of self-dual 1-configurations $(n_d)_{1}$ [7, 15] expressible as $K_d$-ultrahomogeneous graphs [20]. The question of for which values of $n$ such graphs exist is interesting because it would yield the most symmetric, connected, edge-disjoint unions of $n$ copies of $K_d$ on $n$ vertices in which the roles of vertices and copies of $K_d$ are interchangeable. For $d = 4$, known values of $n$ are: $n = 13, 21$ (see [17, 18, 21]) and $n = 42$ [9]. It is of interest to determine the spectrum and multiplicities of the involved values of $n$. To this aim, Theorem 4.1 below contributes the value of $n = 102$. This is obtained via the Biggs-Smith association scheme [6]. This is shown in Theorem 6.1 to control attachment of 102 (cuboctahedral) copies of $L(Q_3)$ to the 102 (tetrahedral) copies of $K_4$. These copies share each (triangular) copy of $K_3$ with two copies of $L(Q_3)$. So, Theorem 7.1 guarantees the distance 3-graph of the Biggs-Smith graph $S$ [3, 5] as the Menger graph $Y$ of a self-dual 1-configuration $(102)_{1}$. On the other hand, the Möbius 2-configuration $(8)_{2}$ for example, and more generally the Cox 2-configurations $((2^{d-1})_d)_{2}$ [8], have their Menger graphs with copies of $K_4$ and $K_d$ respectively not edge-disjoint, even though these are $K_4$- and $K_d$-ultrahomogeneous graphs. Some questions arising at this level are: Are variations of the latter graphs in [21] (5.3.7) $K_d$-ultrahomogeneous? Does there exist a relation between $K_d$-ultrahomogeneous Menger graphs and geometric configurations [4]? Do there exist two different configurations with common $K_d$-ultrahomogeneous Menger graph? Must $K_d$-ultrahomogeneous duality be involutory [19, 21]?

A connected graph $G$ is an $\{H\}_{n}$-graph if it is an edge-disjoint union of $n$ induced copies of $H$ with no other copies of $H$ as subgraphs and each vertex incident to exactly $d$ copies of $H$, no two such copies sharing more than one vertex. If $H = K_r$ is the complete graph of order $r$ ($0 < r \in \mathbb{Z}$) then the vertices and copies of $H$ in $G$ can be seen as the points and lines of a 1-configuration $R_G$ with its points representing the vertices of $G$ and its lines representing the copies of $H$ in $G$. If $R_G$ is a self-dual 1-configuration, then it can be denoted $(n_d)_1$ and $G$ can be recovered as the Menger graph of $R_G = (n_d)_1$. Let us illustrate these concepts with some examples. Clearly, a connected graph $G$ is $m$-regular if and only if it is a $\{K_2\}_{[E(G)]}$-graph. In this case, $G$ is arc-transitive if and only if $G$ is $\{K_2\}$-UH. On the other hand:
(A) for $1 < r \in \mathbb{Z}$, the complete graph $K_r$ and its Cartesian powers $K_r^2 = K_r \Box K_r, K_r^3 = K_r^2 \Box K_r, \ldots, K_r^s = K_r^{s-1} \Box K_r, \ldots$ etc. are $K_r$-UH \{${K_r}$\}$^m_n$-graphs; their orders form a sequence $r, r^2, r^3, \ldots$ of integers corresponding to the respective $K_r$-UH \{${K_r}$\}$^1_n$, \{${K_r}$\}$^2_n$, \{${K_r}$\}$^3_n$, \ldots$, \{${K_r}$\}$^{s-1}_n$, \ldots$-graphs;

(B) for $3 \leq r \in \mathbb{Z}$ the line graph $L(Q_r)$ of the $r$-cube $Q_r$ is a \{${K_r}$, $K_{2.2}$\}-UH \{${K_r}$\}$^2_n$, \{${K_r}$\}$^3_n$, \{${K_r}$\}$^4_n$, \ldots$-graph. A similar argument yields a $K_r$-UH \{${K_r}$\}$^m_n$-graph out of any other regular-polytopal graph via its line graph. 

There is only one case in (A)-(B) that is Menger graph of a self-dual configuration, namely $K_2^r$ (duality sending for example the points 00, 10, 11, 01 resp. onto the lines x0, 0x, x1, 1x, where $0 \leq x \leq 1$), even though all graphs $K_r^s$ have equal numbers of vertices and of copies of $K_r$ so they are Menger graphs of balanced configurations (but not self-dual). If $r = 4$, then the orders of the $K_d$-UH \{${K_d}$\}$^m_n$-graphs in (A)-(B) are divisible by 4. Beside ours ($n = 132$), a case of even order indivisible by 4 is the one mentioned above on $n = 42$ vertices [9]. Its construction was based on the ordered pencils of the Fano plane. Extensions of that construction of [9], based on ordered pencils of binary projective spaces, are introduced in [13] which provides $K_4$-UH \{${K_4}$\}$^m_n$-graphs whose even orders are indivisible by 4, the smallest of which being 210. However, the latter graphs are not Menger graphs of self-dual configurations. A configuration \(n_4\)$ is said to be $K_d$-UH if its Menger graph is. Are there any UH-$K_4$ self-dual configurations \(n_4\) with even \(n < 42\)? Or $42 < n < 102$?

In Section 4, the claimed Menger graph \(\mathcal{Y}\) is constructed by means of the distance-3 graphs of the 9-cycles of the Biggs-Smith graph \(\mathcal{S}\). Theorem 4.1 proves our claim about \(\mathcal{Y}\) as an application of a transformation of distance-transitive graphs into \(\mathcal{C}\)-UH graphs that took in [10] from the Coxeter graph of order 28 onto the Klein graph of order 56. A similar application allowed in [11] to confront, as digraphs, the Pappus graph of order 18 to the Desargues graph of order 20. These applications as well as [12] use the following definitions. Given a family \(\mathcal{C}\) of digraphs, a digraph \(G\) is said to be \(\mathcal{C}\)-UH if every isomorphism between two induced members of \(\mathcal{C}\) in \(G\) extends to an automorphism of \(G\). If \(\mathcal{C} = \{H\}\) then \(G\) is said to be \(H\)-UH. By removing the suffix “di” here, the definition of \(\mathcal{C}\)-UH graph is recovered. A presentation of \(\mathcal{S}\) is given in Section 2 by means of Biggs-Hoare sextets mod 17 [2] which provide a convenient notation to present \(\mathcal{Y}\) in Section 3 in preparation for Section 4.

We set one more definition to be used from Section 2 on. If \(M\) is a subgraph of \(H\) and if \(G\) is both \(M\)-UH, and \(H\)-UH, then \(G\) is an \{\(H\)\}_M-UH graph if, for each induced copy \(H_0\) of \(H\) in \(G\) containing an induced copy \(M_0\) of \(M\), there exists exactly one induced copy \(H_1 \neq H_0\) of \(H\) in \(G\) such that
\[ V(H_0) \cap V(H_1) = V(M_0) \text{ and } E(H_0) \cap E(H_1) = E(M_0). \]

2 The Biggs-Smith graph

The Biggs-Smith graph \( S \) has order \( n = 102 \), diameter \( d = 7 \), girth \( g = 9 \) and automorphism group \( A = \text{PSL}(2, 17) \) [6]. By letting \( k \) be the largest integer \( s \) such that \( S \) is \( s \)-arc transitive, it is seen that \( k = 4 \). In addition, the number \( \eta \) of 9-cycles of \( S \) is \( \eta = 136 \). Taking into account the definition in the last paragraph of Section 1 and by denoting a 3-path by \( P_4 \) and a 9-cycle by \( \gamma_9 \), the following particular case of Theorem 3 of [12] holds (which cannot be refined to a result of \( \{\bar{\gamma}_9\} \bar{P}_4 \)-UH digraphs; see (4) below):

\[ S \text{ is } \{\gamma_9\} P_4 \text{-UH}. \] (1)

Properties of \( S \) we need are presented via sextets [2], where heptadecimal notation is used to denote elements of \( GF(17) \) (for example \( g = 16 = -1 \) and \( d = 13 = -4 \)). In fact, we view \( S \) as a connected graph whose vertex set \( V(S) \) comprises 102 sextets mod 17, namely 102 unordered triples

\[ \{a_0b_0, a_1b_1, a_2b_2\} \]
composed by unordered pairs \(a_ib_i\) of points \(a_i, b_i\) of the projective line \(PG(1, 17) = GF(17) \cup \{\infty\}\) satisfying

\[(a_i - a_j)(b_i - b_j)(a_i - b_j)^{-1}(b_i - a_j)^{-1} = -1,
\]

if \(a_i \neq \infty\) and satisfying

\[(b_i - b_j)(b_i - a_j)^{-1} = -1,
\]

if \(a_i = \infty\), whenever \(i \neq j\) in \(\{0, 1, 2\}\), including the vertices

\[
\begin{align*}
A_0 &= \{2f, 5b, 6c\}, & B_0 &= \{0\infty, 2f, 89\}, & C_0 &= \{3a, 7e, 89\}, \\
D_0 &= \{5a, 7c, 4d\}, & E_0 &= \{0\infty, 1g, 4d\}, & F_0 &= \{1g, 36, be\}.
\end{align*}
\] (2)

Any two of the resulting 102 vertices are adjacent in \(S\) whenever they share one such pair \(a_ib_i\), in which case the resulting edge is labeled \(a_ib_i\). It is shown in [2] that this \(S\) is unique and that the edge labels \(a_ib_i\) are pairwise distinct, so they determine an edge labeling of \(S\) represented in Figure 1 with the following notation. The six vertices in (2) are those of a subtree \(T_0^\infty\) (of \(S\)) which is the edge-disjoint union of the paths

\[
\begin{align*}
(A_0, 2f, B_0, 89, C_0), (D_0, 4d, E_0, 1g, F_0) \text{ and } (B_0, 0\infty, E_0)
\end{align*}
\]
of lengths 3, 3 and 2, respectively. By adding to all elements of \(GF(17)\) in \(T_0^\infty\) a constant \(i \in GF(17)\), a similar tree \(T_i^\infty\) is obtained. The trees \(T_0^\infty, \ldots, T_g^\infty\), represented in Figure 1 via dark traces, are pairwise disjoint and cover \(V(S)\). The complement of their union in \(S\) is formed by 4 \(17\)-cycles

\[
\begin{align*}
A &= (A_0, 6c, A_1, \ldots, A_9, 5b), & D &= (D_0, 7c, D_2, \ldots, D_f, 5a), \\
C &= (C_0, 7e, C_4, \ldots, C_9, 3a), & F &= (F_0, be, F_8, \ldots, F_0, 36).
\end{align*}
\]

Each of these cycles \(y = A, D, C, F\) has vertices \(y_r\) with \(r \in GF(17)\) advancing in 1, 2, 4, 8 units mod 17 stepwise from left to right, respectively. Employed in [12] in proving (1) above, there is a set \(C_9\) of 136 directed 9-cycles of \(S\), of which a generating subset

\[
\{\Pi^0 = (\Pi_0^0 \Pi_1^0 \ldots \Pi_9^0) ; \Pi = S, T, \ldots, Z\}
\]
(written without commas and accompanied to the right by auxiliary permutations, as explained below) is as follows:

\[
\begin{align*}
S^0 &= (B_2 A_2 A_1 A_0 A_4 A_f B_3 C_f C_2), & s^0 &= (07c8d4865a)(\infty 8g2e3f19) \\
T^0 &= (E_2 D_2 D_1 D_0 D_2 D_3 E_1 F_3 F_2), & t^0 &= (03ac9857e)(\infty 12e9b64fg) \\
U^0 &= (B_0 C_0 C_4 C_0 C_4 C_0 C_4 B_0 A_0), & u^0 &= (06371aeab)(\infty 249e586f) \\
V^0 &= (E_1 F_0 F_3 F_0 F_3 F_0 E_1 D_1 D_0), & v^0 &= (0583f3e6c)(\infty d9pa7184) \\
W^0 &= (E_2 B_0 F_0 F_3 B_0 B_0 A_0 A_0), & w^0 &= (0ca3e986e7)(\infty df15e924) \\
X^0 &= (E_2 B_0 A_0 A_3 A_1 A_0 A_1 E_1 D_1 D_0), & x^0 &= (0ecbec91563)(08d47a2df9) \\
Y^0 &= (B_2 E_2 D_2 D_0 D_1 E_1 B_1 C_1 C_2), & y^0 &= (06e0c2f75b)(01943ed89) \\
Z^0 &= (E_3 B_2 C_4 C_0 C_4 B_3 E_1 F_1 F_0), & z^0 &= (05aed437c)(0f9b6812)
\end{align*}
\] (3)
where the permutation \( \pi^0 = (\pi^0_0, \pi^0_1, \ldots, \pi^0_8) \) of \( PG(1, 17) \) to the right of each \( \Pi^0 \) is such that: (i) the pair \( \pi^0_0, \pi^0_1 \) labels the edge \( \Pi^0_0, \Pi^0_1 \); (ii) the pair \( \pi^0_1, \pi^0_2 \) labels the only edge incident to \( \Pi^0_0 \) outside \( \Pi^0 \), where \( i = 0, \ldots, 8 \) and index addition is taken modulo 9. \( C_9 \) also contains the directed cycles \( \Pi' \) with accompanying permutations \( \pi' \) obtained from \( \Pi^0 \) and \( \pi^0 \) by uniformly adding \( r \in \mathbb{Z}_{17} \) mod 17 to all subscripts and superscripts. Observe that: (iii) passing from \( s^0 \) to \( t^0 \) to \( u^0 \) to \( v^0 \) and again to \( s^0 \), (resp. from \( w^0 \) to \( x^0 \) to \( y^0 \) to \( z^0 \) and again to \( w^0 \)) amounts to multiplying uniformly and successively the participating entries of the permutations \( \pi^0 \) by either 2 or \(-2 \) mod 17; and (iv) \( S^0, \ldots, Z^0 \) are invariant with respect to their change-of-sign involutions mod 17, with corresponding involutions on \( s^0, \ldots, z^0 \) around the initial entries of their two composing cycles, which are either 0 and \( \infty \), or \( \infty \) and 0.

3 Distance-3 digraphs of oriented 9-cycles

A \( k \)-arc in a (di)graph is a sequence of vertices \( v_0 v_1 \ldots v_k \) (written without parentheses or commas), where consecutive vertices are adjacent and \( v_{i-1} \neq v_{i+1} \), for \( 0 < i < k \) [14]. A \( k \)-arc can be interpreted as a directed walk of length \( k \) in which consecutive edges are distinct [16]. Thus, an arc in a (di)graph \( \Gamma \) is a 1-arc of \( \Gamma \). The form in which the directed 9-cycles \( \Pi' \) in Section 2 share 3-arcs, either oppositely oriented or not, to be used in Figure 3 below, can be encoded as in the following table that for each \( \Pi^0 \) presents details (explained below) of the 9-cycles \( \Xi_\pi \neq \Pi^0 \in C_9 \) that intersect \( \Pi^0 \) either in the succeeding 3-arcs \( \Pi^0_0, \Pi^0_1, \Pi^0_2, \Pi^0_3 \) or in their respective reversed arcs, for \( i = 0, \ldots, 8 \), with sums involving \( i \) taken mod 9:

\[
\begin{align*}
\Xi^0_{0}: & (X^0_1, S^0_1, T^0_1, U^0_1, V^0_1, W^0_1, S^0_2, Z^0_1) \\
\Xi^0_{1}: & (X^0_2, S^0_2, T^0_2, U^0_2, V^0_2, W^0_2, S^0_3, Z^0_2) \\
\Xi^0_{2}: & (X^0_3, S^0_3, T^0_3, U^0_3, V^0_3, W^0_3, S^0_4, Z^0_3) \\
\Xi^0_{3}: & (X^0_4, S^0_4, T^0_4, U^0_4, V^0_4, W^0_4, S^0_5, Z^0_4) \\
\Xi^0_{4}: & (X^0_5, S^0_5, T^0_5, U^0_5, V^0_5, W^0_5, S^0_6, Z^0_5) \\
\Xi^0_{5}: & (X^0_6, S^0_6, T^0_6, U^0_6, V^0_6, W^0_6, S^0_7, Z^0_6) \\
\Xi^0_{6}: & (X^0_7, S^0_7, T^0_7, U^0_7, V^0_7, W^0_7, S^0_8, Z^0_7) \\
\Xi^0_{7}: & (X^0_8, S^0_8, T^0_8, U^0_8, V^0_8, W^0_8, S^0_9, Z^0_8)
\end{align*}
\]

Each such \( \Xi^0_{\pi} \) has: either (I) a preceding minus sign, if the corresponding 3-arcs in \( \Pi^0 \) and \( \Xi^0_{\pi} \) are oppositely oriented, or (II) no preceding sign, otherwise. Each shown \( -\Xi^0_{j} \) (resp. \( \Xi^0_{j} \)) has a subscript \( j \) indicating the equality of initial vertices \( \Xi^0_{j} = \Pi^0_{i+3} \) (resp. \( \Xi^0_{j} = \Pi^0_{i} \)) of those 3-arcs, for \( i = 0, \ldots, 8 \).

Given a (di)graph \( \Gamma \) and a positive integer \( k \leq \text{diameter}(\Gamma) \), the distance-\( k \) (di)graph \( \Gamma_k \) of \( \Gamma \), with vertex set \( V(\Gamma_k) = V(\Gamma) \), is such that from every
$u \in V(\Gamma_k)$ an arc of $\Gamma_k$ departs to a vertex $v \neq u$ whenever there is a shortest $k$-arc of length $k$ in $\Gamma$ from $u$ to $v$. Let $(\zeta_9)_3$ be the family of distance-3 digraphs of directed 9-cycles in $\zeta_9$. On a representation of an arc $e = w_0w_1$ of a member $(\zeta_9)_3$ of $(\zeta_9)_3$, we label its tail, or initial vertex, $w_0$, its initial flag $\{w_0, e\}$, its terminal flag $\{e, w_1\}$ and its head, or terminal vertex, $w_1$, respectively by the names of the vertices $v_0, v_1, v_2, v_3$ of the 3-arc $v_0v_1v_2v_3$ in $\zeta_9$ for which $w_0w_1$ stands in $(\zeta_9)_3$. For example, if $\zeta_9 = U^3 = (B_1C_1C_2C_3C_4C_5C_6C_7C_8C_9C_0A_0A_1A_2)$, so that $(\zeta_9)_3 = (U^3)_3 = (B_1C_2C_3C_4C_5C_6C_7C_8C_9C_0A_0A_1A_2)$, then the initial flag of the arc $B_1C_2$ in $(\zeta_9)_3 = (U^3)_3$ is labeled by $C_1$, the terminal flag by $C_5$, while $B_1$ and $C_9$ are labeled exactly by $B_1$ and $C_9$, respectively. We get the labels over $(\zeta_9)_3 = (U^3)_3$ shown in Figure 2.

![Figure 2: Labels of vertices and flags of $(\zeta_9)_3 = (U^3)_3$](image)

### 4 $K_4$-UH self-dual 1-configuration $(1024)_1$

We are to fasten pairs of arcs of the digraphs $(\zeta_9)_3$ defined in Section 3 in such a way that a graph $\mathcal{Y}$ with the properties claimed in Section 1 is produced. A sequence of operations $\mathcal{S} \rightarrow \zeta_9 \rightarrow (\zeta_9)_3 \rightarrow \mathcal{Y}$ (compare with [10]) is performed in order to transform $\mathcal{S}$ into the claimed $\mathcal{Y}$. Each distance-3 digraph $(\zeta_9)_3$ of a 9-cycle $\zeta_9$ in the collection $\mathcal{C}_9$ generated via (3) is formed by 3 disjoint directed triangles. It yields a total of $3 \times 136$ directed triangles so $\mathcal{C}_9$ determines a family of 408 directed triangles in the claimed $\mathcal{Y}$ with each edge shared by exactly two such directed triangles in arcs that are either oppositely or identically oriented. It amounts to 102 copies of $K_4$; these can be subdivided into 6 subfamilies $\{\Sigma_i\}$ of 17 copies each, say with $\Sigma \in \{A, B, C, D, E, F\}$ and $i \in \{0, 1, \ldots, 16 = g\} = \mathbb{Z}_{17}$. The vertex sets $V(\Sigma)$, each followed by the set $\Lambda(\Sigma_i)$ of copies of $K_4$ containing the corresponding vertex $\Sigma_i$, can be taken as follows, showing $\mathbb{Z}_2$-symmetry produced by change of sign mod 17:

\[
\begin{align*}
V(A) &= \{C_i, \ D_i, \ E_{i+4}, E_{i-4}\}; & \Lambda(A) &= \{C^i, \ D^i, \ E^{i+7}, E^{i-7}\}; \\
V(B) &= \{D_{i+3}, D_{i-5}, E_{i+5}, E_{i-5}\}; & \Lambda(B) &= \{D^{i+2}, D^{i-2}, E^{i+8}, E^{i-8}\}; \\
V(C) &= \{A_i, \ F_i, \ E_{i+1}, E_{i-1}\}; & \Lambda(C) &= \{A^i, \ F^i, \ E^{i+6}, E^{i-6}\}; \\
V(D) &= \{A_{i+4}, A_{i-4}, A_{i+2}, A_{i-2}\}; & \Lambda(D) &= \{A^{i+1}, A^{i-1}, A^{i+3}, A^{i-3}\}; \\
V(E) &= \{C_{i+4}, C_{i-4}, A_{i+2}, A_{i-2}\}; & \Lambda(E) &= \{C^{i+1}, C^{i-1}, A^{i+3}, A^{i-3}\}; \\
V(F) &= \{C_{i+4}, C_{i-4}, B_{i+8}, B_{i-8}\}; & \Lambda(F) &= \{C^i, \ F^i, \ B^{i+5}, B^{i-5}\};
\end{align*}
\]

where $i$ varies in $\mathbb{Z}_{17}$. This reveals a duality $\phi$ from the 102 vertices of $\mathcal{S}$ onto the 102 copies of $K_4$ in $\mathcal{S}$. In fact, these copies of $K_4$ are the vertices.
of a graph $\phi(S) = S^* \equiv S$ determined by

$$
\phi(A_i) = A_i^{3i} = A_i^*, \quad \phi(B_i) = B_i^{-1} = B_i^*, \quad \phi(C_i) = C_i^{3i} = C_i^*, \quad \phi(D_i) = D_i^{3i} = D_i^*, \quad \phi(E_i) = E_i^{3i} = E_i^*, \quad \phi(F_i) = F_i^{3i} = F_i^*,
$$

(i $\in \mathbb{Z}_{17}$), with a structure similar to that of the vertices $A_1, \ldots, F_i$ of $S$, the copies of $K_4$ in $S^*$ precisely being $\Sigma_i = A_i, \ldots, F_i$ and corresponding vertex sets $\Lambda(\Sigma_i)$ as specified above. Moreover, $\phi : S \to S^*$ is a graph isomorphism, with the adjacency of $S^*$ equivalent to that of $S$.

![Figure 3](image.png)

**Figure 3:** Symmetry of edge labels in copies of $K_4$ in $\mathcal{Y}$, for $i = 0$

Figure 3 illustrates the left side of (5) for $i = 0$ in terms of edge labels, where edges of $\mathcal{Y}$ arising from pairs of 3-arcs of $S$ identically (resp. oppositely) fastened according to (1) are shown oriented (resp. unoriented) accordingly. Observe the edges oriented in

$$
A^0: D_0 C_0, C_0 E_4, C_0 E_4; \quad B^0: D_1 F_5, F_5 D_5, D_5 F_5, F_5 D_5; \quad C^0: F_3 A_0, A_0 E_1, E_3 A_0; \quad D^0: A_0 D_0, B_2 D_6, D_0 B_1; \quad E^0: A_7 C_6, A_7 C_6, C_6 A_7, C_6 A_7; \quad F^0: F_3 B_8, B_8 F_3, C_0 F_0.
$$

By uniformly adding successively $1 \in \mathbb{Z}_{17}$, each of these 6 cases yields 16 additional ones. This yields the 102 edge-labeled copies of $K_4$ in $\mathcal{Y}$. If the
two points of \(PG(1, 17)\) labeling near its center each edge \(\epsilon\) in the figure are disposed as shown, labeling the respective flags of \(\epsilon\), then the 6 cases may be indicated uniquely as \((kl, mn)(pq, rs)(xy, zw)\), where the position of the labels \(k, l, m, n, p, q, r, s, w, x, y, z\) is as in the referential depiction \(\Sigma^0\) of a copy of \(K_4\) in the lower part of the figure. Then, the flag-label triples at the upper, middle, lower-right and lower-left vertices of this depiction are respectively \(kpx, \ell rz, msy\) and \(nqw\). Moreover, the 6 points of \(PG(1, 17)\) in each of these copies of \(K_4\) not participating of its edge labeling conform a unique sextet \(\chi\) which is not a vertex of \(S\) as characterized in Section 2. However, \(\chi\) is a sextet of an alternative labeling of \(S\) happening via the remaining 102 sextets (of the total of 204). These 102 alternative sextets are the images of the 102 vertices of \(S\) via multiplication of indices in \(PG(1, 17)\) times \(3 \in GF(17)\), operation that coincides with the duality \(\phi\) expressed in (6) above. This proves the assertion in Theorem 4.1 below that the vertices and copies of \(K_4\) of \(S\) are the points and lines of a self-dual 1-configuration \((1024)_1\), which in turn has \(Y\) as its Menger graph. Correspondingly, the vertex labels in \(\Sigma^1\) are the sextets \((rz, ms, nw), (pz, nq, my)\) and \((kx, \ell z, qw)\) and \((kx, \ell r, s jewel)\).

A procedure that allows to determine which point of \(PG(1, 17)\) labels which flag in a copy of \(K_4\) as in Figure 3 is given as follows:

(i) A triangle \(\Delta\) in a copy \(\nabla\) of \(K_4\) in \(Y\), say \(\Delta = (C_0E_4D_0)\) in \(\nabla = A^0\), arises from a 9-cycle \(\Pi^j = (\Pi_0^j, \ldots, \Pi_8^j)\) in \(S\) with associated permutation \(\pi^j = (\pi^j_0, \ldots, \pi^j_8)\) as displayed in Section 2, in this case \(\Pi^0 = Y^2\) with \(\pi^0 = x^2\); and

(ii) by labeling each edge \(\Pi_i^j\Pi_{i+1}^j\) of \(\Pi^j\) just by \(\pi^j_i\), it holds that the flag label of edge \(\epsilon = \Pi_i^j\Pi_{i+1}^j\) at \(\Pi_i^j\) is \(\pi^i_{i+1}\), while the flag label of \(\epsilon\) at \(\Pi_{i+3}^j\) is \(\pi^i_{i+5}\), where \(i = 0, 3, 6\).

The distance-3 digraphs of the directed 9-cycles \(\Pi^0\) of \(S\) are composed by the following triples of disjoint directed triangles of \(Y\):

\[\begin{align*}
S^0 \rightarrow \{D^0, D_0=(B_2A_0B_1), E^0, C_3=(A_2A_0C_7), E^0, C_4=(A_1A_4C_2)\}; \\
T^0 \rightarrow \{A^0, C_5=(E_4D_0E_4), B^0, F_0=\{D_4D_2F_4\}, B^0, F_1=\{D_1D_4F_1\}\}; \\
U^0 \rightarrow \{F^0, F_1=\{B_0C_0B_5\}, E^0, A_5=\{C_0C_4A_8\}, E^0, A_2=\{C_4C_8A_9\}\}; \\
V^0 \rightarrow \{C^0, A_0=(E_2F_0E_1), B^0, D_7=(E_6F_0D_1), B^0, D_8=(F_5F_1D_3)\}; \\
W^0 \rightarrow \{F^0, C_0=(B_3F_0B_5), C^0, E_7=(E_4F_0A_4), C^0, C_5=(C_4F_0A_9)\}; \\
X^0 \rightarrow \{C^0, F_0=(E_3A_0E_1), D^0, B_7=(E_3A_1D_1), D^0, D_8=(A_4B_4D_3)\}; \\
Y^0 \rightarrow \{D^0, A_0=(B_2A_0B_1), A^0, E_5=(D_2D_7C_1), A^0, A_2=(D_2E_1C_2)\}; \\
Z^0 \rightarrow \{A^0, D_0=(E_4D_0E_4), F^0, D_5=(B_3C_4F_4), F^0, F_4=(C_1B_4F_4)d\}.
\end{align*}\]

This way, it can be seen that \(Y\) is a \(K_4\)-UH graph. However, in view of Beineke’s characterization of line graphs [1] and observing that \(Y\) contains induced copies of \(K_{1,3}\), which are forbidden for line graphs of simple graphs, we conclude that \(Y\) is non-line-graphical.
Theorem 4.1 \( \mathcal{Y} \) is both the Menger graph of a \( K_4 \)-UH self-dual 1-configuration \( (102_4)_1 \) and a non-line-graphical \( \{K_4\}_{102} \)-graph. Moreover, \( \mathcal{Y} \) is arc-transitive with regular degree 12, diameter 3, distance distribution (1, 12, 78, 11) and automorphism group \( PSL(2,17) \) of order 2448. Its associated Levi graph is a 2-arc-transitive graph with regular degree 4, diameter 6, distance distribution (1, 4, 12, 36, 78, 62, 11) and automorphism group \( SL(2,17) \) of order 4896.

Proof. It remains to prove that \( \mathcal{Y} \) is \( K_4 \)-UH, which uses (1) and more specifically (4) above. In fact, consider an isomorphism \( \Psi : \Theta_1 \to \Theta_2 \) between copies \( \Theta_i \) of \( K_4 \) in \( \mathcal{Y} \). Each \( \Theta_i \), \( (i = 1, 2) \), arises from 4 9-cycles \( \gamma_i = 9_i \) in \( \mathcal{S} \), \( (j = 1, 2, 3, 4) \), whose union is a subgraph \( \overline{\Theta}_i \) of \( \mathcal{S} \) with 4 vertices \( v_i^j \) of degree 3 and 12 vertices of degree 2 that are the internal vertices of 6 3-paths \( P_j \) whose ends are the vertices \( v_i^j \). For example, the vertices \( v_1^1 = B_0, v_1^2 = B_1, v_1^3 = F_3, v_1^4 = C_9, v_1^5 = B_1, v_1^6 = B_2, v_1^7 = F_4, v_1^8 = C_6 \) in \( \mathcal{S} \) determine such subgraphs \( \Theta_1, \Theta_2 \) in \( \mathcal{Y} \) and \( \overline{\Theta}_1, \overline{\Theta}_2 \) in \( \mathcal{S} \). Clearly, \( \Psi \) induces an isomorphism \( \overline{\Psi} : \overline{\Theta}_1 \to \overline{\Theta}_2 \) that sends say each \( v_i^j \) onto its corresponding \( v_i^j \), \( (j = 1, 2, 3, 4) \). As an automorphism \( \overline{\Psi} \) of \( \mathcal{S} \) exists that extends \( \overline{\Psi} \), then \( \overline{\Psi} \) determines an automorphism of \( \mathcal{Y} \) that restricts to \( \Psi \), showing that \( \mathcal{Y} \) is a \( K_4 \)-UH graph. \( \square \)

5 Definitions to deal with the copies of \( L(Q_3) \)

If \( H \) is a graph with an edge partition \( \Omega = \Omega(H) \) into 2-paths, then a graph \( G \) is \( \Omega \)-preserving \( H \)-UH if every \( \Omega \)-preserving isomorphism between two induced copies of \( H \) in \( G \) extends to an automorphism of \( G \). If \( M \) is a subgraph of \( H \) and if \( G \) is both \( M \)-UH, and \( \Omega \)-preserving \( H \)-UH, then \( G \) is an \( \Omega \)-preserving \( \{H\}_M \)-UH graph if, for each induced copy \( H_0 \) of \( H \) in \( G \) containing an induced copy \( M_0 \) of \( M \), there is just one induced copy \( H_1 \neq H_0 \) of \( H \) in \( G \) such that:

(a) \( V(H_0) \cap V(H_1) = V(M_0) \);

(b) \( E(H_0) \cap E(H_1) = E(M_0) \); and

(c) the edges of \( M_0 \) are in distinct 2-paths both in \( \Omega(H_0) \) and \( \Omega(H_1) \).

A graph \( G \) is \( rK_4 \)-frequent if every edge \( e \) of \( G \) is intersection of exactly \( r \) induced copies of \( K_4 \); these copies having only \( e \) and its ends in common. For example, \( K_4 \) is \( 2K_4 \)-frequent and \( L(Q_3) \) is \( 1K_3 \)-frequent. A graph \( G \) is \( \{H_2, H_1\}_{K_4} \)-UH, where \( H_i \) is \( iK_4 \)-frequent \( (i = 1, 2) \) if:

(d) \( G \) is \( H_2 \)-UH and edge-disjoint union of induced copies of \( H_2 \).
(e) there is a partition $\Omega$ of $H_1$ into 2-paths and $G$ is $\Omega$-preserving \( \{H_1\}_{K_3}\text{-UH} \); and

(f) each induced copy of $H_2$ in $G$ has each induced copy of $K_3$ in common with exactly two induced copies of $H_1$ in $G$.

Theorem 6.1 shows that $\mathcal{Y}$ is $\{K_4, L(Q_3)\}_{K_3}\text{-UH}$. This allows to gather information on $S_2$ and $S_4$, leading to $\mathcal{Y} = S_3$ in Theorem 7.1.

6 The $K_4\text{-UH}$ graph $\mathcal{Y}$ is $\{K_4, L(Q_3)\}_{K_3}\text{-UH}$

Recall from Section 4 that each copy of $K_4$ in $\mathcal{Y}$ arises from the distance-3 digraphs of 4 directed 9-cycles of $\mathcal{S}$. The subgraph of $\mathcal{S}$ spanned by these 4 9-cycles contains 4 degree-3 vertices (which are tails and heads of corresponding 3-arcs) and 12 degree-2 vertices (internal vertices of those 3-arcs). These 12 vertices induce a copy $L$ of $L(Q_3)$ in $\mathcal{Y}$. For the copy $A_0^0$ of $K_4$ in $\mathcal{Y}$, the corresponding copy $L = a^0$ of $L(Q_3)$ in $\mathcal{Y}$ can be represented as in the big rectangle $\mathcal{R}$ in Figure 4, where:

(a) the leftmost and rightmost dashed lines of $\mathcal{R}$ are to be identified by parallel translation;

(b) each of the 8 shown triangles $\Delta$ forms part of a corresponding copy $\nabla$ of $K_4$ cited on the exterior of $\mathcal{R}$ about the horizontal edge of $\Delta$, while its 4th vertex is cited at the center of $\Delta$; and

(c) the edges are colored via a partition $\Omega$ into 2-paths $P_3$, the edges of each $P_3$ with a common color from a set of 3 colors: (i) black; (ii) light-gray; (iii) dark-gray; the 3 colors are present together in every triangle, and opposite edges in every induced 4-cycle, or 4-hole, have a common color, a total of two colors per 4-hole.

For $\sigma = a, b, c, d, e, f$, the copies $\sigma^0$ of $L(Q_3)$ are expressed by means of the data contained in Figure 4 as follows:
admits an edge partition \( \Omega = \Omega(\sigma) \) (uniformly translating all involved subscripts and superscripts). Each copy \( \sigma^i \) of \( L(Q_3) \) admits an edge partition \( \Omega = \Omega(\sigma^i) \) into \( j \)-colored 2-paths (\( j \in \{1, 2, 3\} \)) so that each (monochromatic) 2-path in an \( \Omega(\sigma^i) \) is shared only by one other copy of \( L(Q_3) \) in \( \mathcal{Y} \) (as in Theorem 6.1(3), below). We may write

\[
\sigma^i = \sigma^i_1 \cup \sigma^i_2 \cup \sigma^i_3,
\]

to stress the color partition of \( \sigma^i \) into its black, light-gray and dark-gray subgraphs, which are copies of the disconnected graph \( 4P_3 \) (formed by 4 disjoint copies of \( P_3 \)) as in Figure 4 for \( \sigma^i = a^3 \). The edge labels of \( \sigma^0 \) in Figure 4 (shown in gray type) and of all the other \( \sigma^i \)'s are taken as the flag labels for \( i = 0, \ldots, g \) in Figure 3. The relation and location of these flag labels justifies a labeling of the 12 vertices and 6 4-holes as shown with symbols \( 0, \ldots, g, \infty \) (in black type) in Figure 4, the sole edge-label notation to be used ahead.

\[
\begin{align*}
&\begin{array}{c}
\text{Figure 5: Label and vertex-tetrahedron representations of } a^0, b^0, c^0 \text{ in } Q_3 \\
\end{array}
\end{align*}
\]

The labels of the 12 vertices and 6 4-holes of each of \( \sigma^0 = a^0, \ldots, f^0 \) are depicted again on the middle thirds of Figures 5 and 6, this time on a copy
Figure 6: Label and vertex-tetrahedron representations of $d^0, e^0, f^0$ in $Q_3$

$Q_3$ of the 3-cube $Q_3$ from which a corresponding copy of $L(Q_3)$ in $\mathcal{Y}$ is obtained with its vertices taken as the middle points of the edges of $Q_3$, tracing an edge between two such vertices whenever the edges they represent have a vertex in common in $Q_3$, with the convention that labels of vertices and 4-holes of $\sigma^0$ label now respectively the corresponding edges and faces of $Q_3$. (On the bottom thirds those edges are labeled by the corresponding vertices of $\mathcal{S}$ and their vertices by the corresponding containing copies of $K_4$; on the upper thirds, 4 different cutouts of $Q_3$ are depicted to show involution symmetry around edges labeled $\infty$, where $Q_3$ is regained by identifying the upper and left sides and the lower and right sides via 90° rotations at the upper-left and lower-right corners). Opposite faces in such $\sigma^j$ determine pairs of points of $PG(1,17)$, a total of 3 such pairs leading to a unique sextet which is not a vertex of $\mathcal{S}$ but uniformly 3 times a vertex of $\mathcal{S}$. For example, these 3 pairs for $\sigma^0 = a^0$ form the sextet $\{12, 6b, fg\} = 3 \times \{6c, 2f, 5b\} = A_0$, mod 17. By denoting $a^0 = \{12, 6b, fg\}$ and so on for the 101 remaining copies of $L(Q_3)$ in $PG(1,17)$, we obtain a self-dual configuration that uses again the duality $\phi$ of Section 4, this time with points and lines taken as the vertices and copies of $L(Q_3)$ in $\mathcal{S}$. This is a self-dual 1-configuration (10231), as claimed in Theorem 6.1(8) below, depending on the facts that $L(Q_3)$ has 12 vertices and that each vertex of $\mathcal{Y}$ belongs to 12 copies of $L(Q_3)$.

Figure 7 shows the complements of vertex $A_0$ in 4 of the 12 copies of $L(Q_3)$ containing $A_0$, namely $e^0, d^2, c^1, f^9$, which share the long vertical edges,
successively present in the copies $E^a, D^b, C^0, E^7$ of $K_4$, the last long vertical edge both as the leftmost and rightmost edges in the shown covering graph, say $\Upsilon_0$, of $e^b \cup d^2 \cup c^1 \cup f^9 - A_0$, where:

(a) black vertices participate of the 8 4-holes containing $A_0$, namely those labeled 5 on top and $b$ at the bottom; other labels of 4-holes internal to them, respectively;

(b) the labels $j$ of vertices $\Sigma_i$ appear as superindices, as in $\Sigma_i^j$, (with $j$ also in the citations $A_0^j$ of $A_0$ on top), or $\Sigma_i^{jj'}$, in case labels $j$ and $j'$ happen in contiguous copies of $L(Q_3)$;

(c) each triangle contains the name $\Sigma^\ell$ of the copy of $K_4$ containing it;

(d) for each $\sigma^i = e^b, d^2, c^1, f^9$, the partition $\Omega(\sigma^i)$ restricts as in the rightmost diagram, in which darts indicate the first edges of monochromatic 2-paths whose final vertex is $A_0$; as a result, the 4 mentioned long vertical edges belong each to two different monochromatic 2-paths of contiguous copies of $L(Q_3)$ in $\Upsilon$;

(e) alternate internal anti-diagonal monochromatic 2-paths (i.e. from top-right to bottom-left) coincide with directions reversed; (the middle vertices of these 4 2-paths are just two neighbors of $A_0$ in $S$, and their degree-1 vertices are at distance 2 from $A_0$ in $S$); and

(f) the rightmost diagram contains denotations $\beta_i, (i \in [0, b])$, and $\alpha_j, (j \in [0, 5])$, respectively for the vertex and 4-hole labels in their positions in the 4 copies of $L(Q_3)$.

Apart from the union $e^b \cup d^2 \cup c^1 \cup f^9$ of copies of $L(Q_3)$ sharing $A_0$ in Figure 7, there are two other unions of 4 copies of $L(Q_3)$ in $\Upsilon$ sharing $A_0$. The following display of the data in Figure 7 contains at its left the $\alpha-\beta$ denotations of (f). Moreover, the data corresponding to the 3 unions of 4 copies of $L(Q_3)$ sharing $A_0$ in $\Upsilon$ are set (or encoded) in the arrays to the right and below the $\alpha-\beta$ denotations (these solely for $e^b, d^2, c^1, f^9$, respectively), where the leftmost array summarizes $\Upsilon_0$, the two doubly repeated middle vertices in $\Upsilon_0$ (as in (e)) parenthesized to the right of $A_0$.
and the remaining data displayed in similar order, with the two rightmost arrays preceded by the first one of their 4 corresponding \(\alpha-\beta\) denotations, which condenses all needed information of \(\mathcal{Y}\) around \(A_0\):

\[
\begin{align*}
\alpha_0 \beta_0 \alpha_1 &= 52b, 56b, 5fb, 5cb & A_0(B_0 A_1) & f62 & A_0(A_1 A_0) \\
\beta_1 \alpha_1 \beta_1 &= 478, 96b, 796c, 903 & (E_4 \phi E_3 \phi E_2 D_0 E_0 C_0 E_0) & 41g & (E_7 \phi E_6 C_0 D_0 E_0 \phi E_9 D_0 E_8) \\
\beta_3 \alpha_1 \beta_1 &= e0e, 23e, 684, 6d1 & (C_4 B_3 A_3 E_4 D_0 F_2 E_1 E_9) & 5d9 & (C_1 E_0 F_2 A_3 C_7 B_0 C_6) \\
\alpha_2 \beta_0 \alpha_1 &= 4f1, 1c0, e0e, 6e4 & (B_4 B_1 E_2 C_0 F_0 A_2 C_0 E_9) & 8c0 & (D_1 A_4 C_2 B_1 C_2 B_0 A_2) \\
\beta_2 \alpha_1 \beta_2 &= e06, 4gf, 19c, a72 & (A_2 C_0 E_0 A_3 B_1 D_2 C_0 F_1) & \infty_{76} & (B_3 C_4 A_2 D_2 A_1 B_2 C_1) \\
\beta_0 \beta_0 \beta_0 &= 10d, 5b0, e0d, 9b8 & (C_1 A_5 C_4 D_4 B_2 F_0 B_0) & 3ae & (A_1 D_2 E_3 C_0 C_6 B_2 C_2) \\
\end{align*}
\]

Some edges are shared by two of these 3 unions. In fact, each of the edges bordering the central 2-paths \(\omega\) in anti-diagonal 4-paths in \(T_0\) is present also in one of the two covering graphs, say \(T_1\) and \(T_2\), corresponding to the two rightmost arrangements above, one encoded on top and the other at the bottom of the display, respectively. For example, the edge \(B_1 A_3\) of \(e^b\) on \(T_0\) appears in \(T_1\). Also, the labels \(\{\alpha_0 \alpha_4, \alpha_1 \alpha_5, \alpha_2 \alpha_3\}\) of opposite copies of \(L(Q_3)\), just sharing vertex \(A_0\), are images of vertices at distance 3 in \(\mathcal{S}\) via the duality \(\phi\) (but copies of \(L(Q_3)\) sharing a triangle containing \(A_0\) are images of vertices at distance 7). The following permutations on the set \(\{\alpha_0, \ldots, \alpha_5, \beta_0, \ldots, \beta_{11}\}\) relate the labels of the 12 copies of \(L(Q_3)\) sharing \(A_0\):

\[
\begin{align*}
e^b & \rightarrow d^2 \rightarrow e^1 \rightarrow f^0 \rightarrow e^b; \\
&= (\alpha_0)(\alpha_5)(\beta_0 \beta_4 \beta_6 \beta_8)(\beta_1 \alpha_0 \beta_9 \beta_0)(\beta_3 \beta_4 \alpha_1 \beta_5)(\beta_5 \alpha_0 \alpha_2 \beta_7) \\
e^b d^2 e^b f^0 & \rightarrow d^2 d^2 e^b f^0 e^b \rightarrow d^1 c^0 e^b f^0; \\
&= (\alpha_0 \beta_4 \beta_6)(\beta_0 \alpha_5 \beta_8)(\beta_1 \beta_3 \alpha_2)(\beta_5 \alpha_0 \alpha_3)(\alpha_1 \beta_9 \beta_0)(\beta_3 \beta_4 \beta_7).
\end{align*}
\]

The following permutations allow to relate the labels of the 12 cuboctahedral subgraphs sharing \(A_0\) to those sharing \(B_0, C_0, D_0, E_0, F_0\):

\[
\begin{align*}
A_0 & \rightarrow B_0 : (\alpha_0 \beta_3 \alpha_1 \beta_5 \alpha_4 \beta_9 \beta_6 \beta_0 \beta_3 \beta_0 \alpha_3 \beta_4 \beta_2 \beta_0 \beta_3) \\
A_0 & \rightarrow C_0 : (\alpha_0 \beta_3 \alpha_2 \beta_9 \beta_6 \beta_0 \beta_3 \beta_0 \alpha_3 \beta_4 \beta_2 \beta_0 \beta_3) \\
A_0 & \rightarrow D_0 : (\alpha_0 \beta_3 \alpha_2 \beta_9 \beta_6 \beta_0 \beta_3 \beta_0 \alpha_3 \beta_4 \beta_2 \beta_0 \beta_3) \\
A_0 & \rightarrow E_0 : (\alpha_0 \beta_3 \alpha_2 \beta_9 \beta_6 \beta_0 \beta_3 \beta_0 \alpha_3 \beta_4 \beta_2 \beta_0 \beta_3) \\
A_0 & \rightarrow F_0 : (\alpha_0 \beta_3 \alpha_2 \beta_9 \beta_6 \beta_0 \beta_3 \beta_0 \alpha_3 \beta_4 \beta_2 \beta_0 \beta_3).
\end{align*}
\]

Additions mod 17 yield the remaining information for copies of \(K_4\) and \(L(Q_3)\) neighboring each vertex of \(\mathcal{Y}\). In sum, we have the following theorem.

**Theorem 6.1** In addition to Theorem 4.1, the following properties of \(\mathcal{Y}\) hold:

1. \(\mathcal{Y}\) is a connected union of 102 copies \(\sigma\) of \(L(Q_3)\), each with an edge partition \(\Omega(\sigma)\) into 2-paths;
(2) each edge in $\mathcal{Y}$ is shared exactly by 4 copies of $L(Q_3)$ in $\mathcal{Y}$;

(3) each copy $\Delta$ of $K_3$ (resp. each 2-path $\omega \in \Omega(\sigma)$) in a copy $\sigma$ of $L(Q_3)$ in $\mathcal{Y}$ is shared exactly by two copies $\sigma, \sigma'$ of $L(Q_3)$ in $\mathcal{Y}$;

(4) Each two copies of $L(Q_3)$ sharing a copy $\Delta$ of $K_3$ in $\mathcal{Y}$ share $\Delta$ with exactly one copy of $K_3$ in $\mathcal{Y}$;

(5) each 4-hole in $\mathcal{Y}$ happens in just one copy of $L(Q_3)$ in $\mathcal{Y}$;

(6) $\mathcal{Y}$ is an $\Omega$-preserving $\{L(Q_3)\}_L$-UH graph;

(7) $\mathcal{Y}$ is $(K_4, L(Q_3))_L$-UH;

(8) the vertices and copies of $L(Q_3)$ in $\mathcal{Y}$ are the points and lines of a self-dual 1-configuration $(102_{12})_1$.

In Theorem 6.1(3), for each triangle $\Delta$ in $\sigma$, the copies $\sigma, \sigma'$ of $L(Q_3)$ intersect exactly in $\Delta$, while for each 2-path $\omega \in \Omega(\sigma)$ in $\sigma$, not only $\omega$ is shared by $\sigma, \sigma'$, but these also share a vertex at distance 2 from the ends of $\omega$. This common distance, 2, is realized by 2-paths in the other two colors distinct from the color of $\omega$, in each of $\sigma$ and $\sigma'$, as in Figure 4, where for example the dark-gray-colored 2-path $F_1D_2B_1$ (present both in $a^0$ and $c^3$) is at distance 2 from vertex $D_4$ (also present in $a^0$ and $c^3$) via the black-colored path $B_1F_2D_4$ and the light-gray-colored path $F_1C_4D_4$.

**Proof.** It only remains to prove item (6). We explain how a monochromatic 2-path-preserving isomorphism $\Psi' : \sigma'_1 \rightarrow \sigma'_2$ between two copies of $L(Q_3)$ $\sigma'_1, \sigma'_2$ in $\mathcal{Y}$ extends to an automorphism of $S$. Both $\sigma'_1$ and $\sigma'_2$ are colored as in Figure 4 with $\Psi'$ respecting the color structure, thus inducing a 1-1 correspondence between the color classes of $\sigma'_1$ and $\sigma'_2$. In each copy of $L(Q_3)$ in $\mathcal{Y}$ there are exactly 12 monochromatic 2-paths, 4 in each of the 3 colors, and exactly 12 dichromatic 2-paths not contained in any triangle, a total of 24 2-paths not contained in any triangle. A $\Psi' : \sigma'_1 \rightarrow \sigma'_2$ as mentioned can be extended to an automorphism of $\mathcal{Y}$ because the information gathered in $\sigma'_1$ comes via sextets from corresponding information in a subgraph $\overline{\sigma}'_i$ of $S$, $i = 1, 2$, so that $\Psi'$ arises from an isomorphism $\overline{\Psi} : \overline{\sigma}'_1 \rightarrow \overline{\sigma}'_2$. However, $\overline{\sigma}'_i = \overline{\sigma}_j$, $i = 1, 2$, for a corresponding copy $\sigma_i$ of $L(Q_3)$ in $\mathcal{Y}$, but while the vertices of $\sigma'_i$ are denoted like the degree-2 vertices of $\overline{\sigma}'_i = \overline{\sigma}_i$, the vertices of $\sigma_i$ are denoted like the degree-3 vertices of $\overline{\sigma}_i = \overline{\sigma}'_i$. Here the pairs $(\sigma_i, \sigma'_j)$ are of the form $(\Sigma', \sigma')$, where $(\Sigma, \sigma) \in \{(A, a), (B, b), (C, c), (D, d), (E, e), (F, f)\}$ and $j \in Z_{17}$. Then $\overline{\Psi} = \overline{\Psi} : \sigma_1 \rightarrow \sigma_2$ is a corresponding map as in the proof of Theorem 4.1. But now $\overline{\Psi} = \overline{\Psi}$ extends to an automorphism of $S$. This takes us to an automorphism of $\mathcal{Y}$ that extends $\Psi'$, as claimed above.

For example, the black 2-path $B_4F_2D_4$ in the copy $a^0$ of $L(Q_3)$ in $\mathcal{Y}$ rep-
resented in Figure 4 arise from the 3-paths $B_4E_4F_4D_4$ and $F_4D_4E_4D_4$ in $\mathcal{S}$, which share the 2-path $F_4E_4D_4$ and differ otherwise, so their union
$(B_4E_4F_4) \cup (F_4D_4E_4D_4)$ is realized by a tree $T_1$ with just one vertex of degree 3, namely $E_4$, from which two 1-paths and one 2-path depart.
A similar tree $T_2$ is obtained from the black 2-path $D_4F_4B_4$ in Figure 4. However $T_1 \cap T_2 = F_4D_4$, a terminal 1-path of $T_1$ on its 2-path departing from $t_i$, for both $i = 1, 2$, where $t_1 = E_4$ and $t_2 = E_d$, the vertex of degree 3 in $T_2$. The other two black 2-paths in Figure 4 behave similarly, leading to trees $T_3$ and $T_4$ intersecting at the 1-path $B_0E_0$. Similar behavior holds for the dark gray and the light gray quadruples of 2-paths in Figure 4, leading to pairs of trees that intersect respectively at the 1-paths $D_4D_2$, $B_dC_d$ and the 1-paths $B_3C_4$, $D_fD_d$. Thus, if $\sigma_i'$ is this copy of $L(Q_3)$ in $\mathcal{Y}$, then $\mathcal{S}_i'$ coincides with $\mathcal{S}_1$, where $\sigma_1 = A^0$. □

7 Using the Biggs-Smith association scheme

The 2-paths $\omega$ of Theorem 6.1(3) rearrange into an edge partition $\mathcal{I}$ of $\mathcal{Y}$ into 102 4-holes. In fact, each 4-hole in $\mathcal{I}$ is the union of 4 successive 2-paths $\omega_0, \omega_1, \omega_2, \omega_3$ from 4 respective partitions $\Omega(\sigma^0), \Omega(\sigma^1), \Omega(\sigma^2), \Omega(\sigma^3)$ of $L(Q_3)$ into 2-paths, with each two successive 2-paths $\omega_i, \omega_{i+1}$ here overlapping in just one edge, (subindex addition taken mod 4).

$\mathcal{I}$ can be reconstructed by adding $r \in \mathbb{Z}_{17}$ uniformly mod 17 to all indexes in the following generating-set table of its member 4-holes, from those 4-holes shown in the left column of the table. In each line of the table, the 4 pairs of copies $\sigma_i^j$ of the disconnected graph $4P_3$ shown to the right (as in (7) above) overlap at succeeding pairs of 2-paths of the 4-hole shown on their left. This is continued to its right by the citation of two vertices that alternatively are at distance 2 from the ends of those composing 2-paths:

| (A_2B_2B_1A_0) A_0A_1 | (c_1^3 c_2^3) | (c_1^2 c_2^2) | (d_1^3 d_2^3) | (c_1^2 d_1^2) |
|---------------|-------------|-------------|-------------|-------------|
| (C_2A_4A_2A_1) A_0B_0 | (d_1^2 f_1^2) | (c_1^3 d_1^3) | (c_1^2 d_1^2) | (d_1^3 f_1^3) |
| (C_2E_4C_0A_0) B_0C_0 | (c_2^3 e_1^3) | (f_2^3 d_1^3) | (c_1^2 e_1^2) | (d_2^3 f_2^3) |
| (D_0A_0F_0G_0) B_0E_0 | (c_2^3 c_2^3) | (f_2^3 f_2^3) | (a_2^3 a_2^3) | (d_1^3 d_1^3) |
| (C_2B_2B_1C_0) C_0C_2 | (c_2^3 e_2^3) | (c_1^3 a_2^3) | (f_2^3 d_2^3) | (d_2^3 f_1^3) |
| (D_0D_4E_2E_0) D_0D_2 | (c_2^3 b_1^3) | (b_2^3 d_2^3) | (d_2^3 b_1^3) | (c_2^3 b_2^3) |
| (F_0D_4D_2B_1E_1) D_0E_0 | (c_2^3 c_2^3) | (a_2^3 c_2^3) | (b_2^3 c_2^3) | (c_2^3 c_2^3) |
| (F_0D_4D_2D_1) D_0D_0 | (c_2^3 f_2^3) | (c_2^3 a_3^3) | (b_2^3 b_2^3) | (a_2^3 c_3^3) |
| (F_0D_4D_2F_0) E_0D_0 | (c_2^3 f_2^3) | (c_2^3 a_3^3) | (b_2^3 b_2^3) | (a_2^3 c_3^3) |
| (E_0E_0F_0F_0) F_0E_0 | (c_2^3 f_2^3) | (c_2^3 a_3^3) | (b_2^3 b_2^3) | (a_2^3 c_3^3) |

The vertices of each such 4-hole coincide in notation with the degree-1 vertices of a tree $T$ in $\mathcal{S}$ isomorphic to $T_0^\infty$, (itself present in the 4th row of this table), with the two vertices that follow each 4-hole being the vertices of degree 3 in $T$. These data insure that $\mathcal{Y}$ is $I$-UH.
Of the 24 2-paths in a copy $\sigma^i$ of $L(Q_3)$ in $\mathcal{Y}$, 12 are in the partition $\Omega(\sigma^i)$ of $\sigma^i$. The other 12 form a different edge partition $\Omega'(\sigma^i) \neq \Omega(\sigma^i)$ of $\sigma^i$. The family of 2-paths in all of the $\Omega'(\sigma^i)$ s reassembles, by means of unions of those of its members having a common degree-2 vertex, as a family $\mathcal{J}$ of 306 copies of $K_{1,4}$.

A generating-set table for $\mathcal{J}$ representing 18 copies of $K_{1,4}$ is shown subsequently, with the remaining copies of $K_{1,4}$ obtained from those 18 by uniform addition of $r \in \mathbb{Z}_{17}$ to all indexes $i \in \mathbb{Z}_{17}$ of vertices $\Sigma_i$ and subgraphs $\sigma_j^i$, where $j = 1, 2, 3$ stands for black, dark gray and light gray, respectively. This generating-set table has each entry starting with a vertex $\Sigma_0$ of degree 4 in a copy of $K_{1,4}$ in $\mathcal{J}$ followed by 4 parenthesized expressions, each containing as its central entry a neighbor $\Sigma'$ of $\Sigma_0$ flanked by two subgraphs $\sigma_j^i$ to which the edge $\Sigma_0\Sigma'$ belongs, so that each participating $\sigma^i$ appears repeated twice — with 2 different colors $j,j'$, as $\sigma_j^i$ and $\sigma_{j'}^i$, once before a right parenthesis and once after the subsequent left parenthesis, the first of the 4 left parentheses considered subsequent to the last right parenthesis, in a mod 4 fashion:

```
A_0 (c_3^6 A_3 d_2^6) (d_1^6 E_1 c_1^6) (e_1^6 B_2 e_3^6) (e_1^6 C_1 e_1^6)
A_0 (f_3^6 C_4 d_2^6) (d_1^6 D_{04} d_4^6) (d_1^6 C_4 f_2^6) (f_3^6 F_0 f_2^6)
A_0 (d_1^6 A_1 e_2^6) (e_1^6 C_2 e_2^6) (c_2^6 B_2 c_3^6) (c_2^6 E_2 d_1^6)
B_0 (c_3^6 B_6 a_2^6) (a_2^6 B_3 a_1^6) (c_1^6 B_3 f_1^6) (c_1^6 C_4 e_2^6)
B_0 (e_1^6 A_1 d_2^6) (d_2^6 A_2 c_3^6) (c_2^6 B_6 c_1^6) (c_2^6 B_1 e_2^6)
B_0 (a_1^6 D_2 d_2^6) (d_2^6 A_1 e_2^6) (c_2^6 F_0 a_1^6) (c_2^6 F_3 a_2^6)
C_0 (d_1^6 D_0 d_2^6) (d_2^6 A_1 f_1^6) (f_3^6 F_0 f_2^6) (f_1^6 A_3 d_2^6)
C_0 (e_1^6 A_2 e_2^6) (c_2^6 B_6 a_2^6) (c_1^6 E_3 a_1^6) (f_2^6 C_4 e_2^6)
C_0 (d_1^6 B_6 a_2^6) (a_1^6 E_3 f_1^6) (f_2^6 C_4 e_2^6) (c_2^6 A_3 d_2^6)
D_0 (b_1^6 F_3 b_1^6) (b_2^6 E_3 d_3^6) (d_1^6 B_3 f_1^6) (a_2^6 D_3 f_1^6)
D_0 (a_2^6 F_3 a_2^6) (c_2^6 A_3 a_2^6) (d_2^6 E_3 b_2^6) (b_2^6 F_6 b_2^6)
E_0 (b_2^6 E_3 b_2^6) (b_2^6 E_3 b_2^6) (d_2^6 E_3 b_2^6) (b_2^6 D_4 a_3^6)
E_0 (b_2^6 F_3 b_2^6) (c_2^6 F_3 b_2^6) (b_2^6 F_3 b_2^6) (f_3^6 E_3 b_2^6)
E_0 (f_2^6 A_3 d_2^6) (d_2^6 C_4 f_2^6) (f_1^6 A_3 d_2^6) (d_1^6 C_4 f_2^6)
F_0 (c_2^6 A_0 c_1^6) (c_2^6 D_2 a_2^6) (c_1^6 C_0 a_1^6) (a_2^6 D_1 a_2^6)
F_0 (c_2^6 D_0 b_2^6) (b_2^6 F_2 c_2^6) (c_2^6 D_3 b_2^6) (f_2^6 E_2 b_2^6)
F_0 (f_2^6 E_1 b_2^6) (b_2^6 D_0 b_2^6) (b_2^6 E_4 b_2^6) (c_2^6 B_6 f_2^6)
```

Here, a copy of $K_{1,4}$ with degree-4 vertex $\Sigma_i$ has its degree-1 vertices as those of a binary tree of $\mathcal{S}$ with depth 2 and whose root is one of the 3 neighbors of $\Sigma_i$. Thus, there are 3 such copies of $K_{1,4}$. As a result, in contrast to the fact mentioned above that $\mathcal{Y}$ is $\mathcal{I}$-UH, now any homomorphism between members of $\mathcal{J}$ preserving the order of presentation of the degree-1 vertices in corresponding copies of $K_{1,4}$, as in the table above (with the expressed parenthetical behavior with respect to the $\sigma_j^i$ s), extends to an automorphism of $\mathcal{Y}$. On the other hand, each copy $\sigma$ of $L(Q_3)$ in $\mathcal{Y}$ intersects 8 other copies of $L(Q_3)$ in a triangle each, and 12 other copies of $L(Q_3)$, each in a 2-path of $\Omega(\sigma)$ and one more vertex at distance 2 from the ends of the 2-path.
The graph $I'$ generated by the (diagonal) chords of the 4-cycles of $I$ coincides with $S_2$. On the other hand, by expressing the copies of $K_{1,4}$ in $\mathcal{J}$ as $u(v)(w)(x)(y)$, (for example the copy of $K_4$ in the first line of the last table as $A_0(A_3)(E_1)(B_2)(C_1)$), we consider the graph $\mathcal{J}'$ generated by the corresponding 4-cycles $(v, w, x, y)$. Then $\mathcal{J}'$ coincides with $S_4$. We obtain the following final result.

**Theorem 7.1** $\mathcal{Y} = S_3$.

**Proof.** This is obtained from the Biggs-Smith association scheme, as follows. As $I' = S_2$ and $\mathcal{J}' = S_4$, and because $S$ has girth 9 and $\mathcal{Y}$ was constructed from the family $(C_9)_3$ of distance-3 digraphs of directed 9-cycles in the set $C_9$ of 136 directed 9-cycles in Section 3, taking into account the discussion previous to the statement, we arrive at

$$K_{102} = S \cup S_2 \cup S_3 \cup S_4 = S \cup I' \cup \mathcal{Y} \cup \mathcal{J'},$$

and so $\mathcal{Y} = S_3$. $\square$

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