BINOMIAL COEFFICIENT VIA DAMPED SINE WAVES

LORENZO DAVID

ABSTRACT. In this note I show a simple generalization of the binomial coefficient to \( \mathbb{C} \) as a linear combination of sinc functions. This allows to simplify calculations as well as having interesting applications in calculus.

The binomial coefficient \( \binom{m}{k} := \frac{m!}{k!(m-k)!} \), is extended to complex numbers via the extension of the factorial to complex numbers, i.e., by the Gamma function:

\[
\binom{w}{z} := \frac{\Gamma(w+1)}{\Gamma(z+1)\Gamma(w-z+1)}.
\]

Starting from this formula, we show how to obtain an expression of the binomial coefficient for \( w = m \) non-negative integer and \( z \) complex, involving only a finite number of sinc functions (see Theorem 1). This allows to simplify calculations, for instance, to obtain the Fourier transform of \( F_m(z) := \binom{m}{z} \). It turns out that the expression of this Fourier transform, \( \hat{F}_m(u) \), can be naturally extended to \( \hat{F}_w(u) \), with \( w \in \mathbb{C} \). Computing the antitransform of \( F_w(z) \), we obtain an infinite series of sinc functions, that we prove (see Theorem 2) to coincide with \( F_w(z) = \binom{w}{z} \). In Section 4 we introduce the D-transform \( \mathcal{D}\{f(z)\}(w) \) (see Definition 1), which, loosely speaking, tells how many binomial coefficients one needs to add in order to create a given function. Some properties of the D-transform are proved (see Theorem 3). Applications of Theorems 2 and 3 are shown at the end of this note.

1. Notations and Definitions

Along this paper we use the following notations:

- \( \text{sinc}(x) := \frac{\sin(\pi x)}{\pi x} \)
- \( \text{Si}(x) := \int_0^x \frac{\sin(t)}{t} \, dt \)
- \( G := \frac{2}{\pi} \text{Si}(\pi) \)
- \( \text{rect}(x) := \begin{cases} 1 & \text{if } |x| < 1/2 \\ 0 & \text{otherwise} \end{cases} \)
2. Preliminary Results

Theorem 1. For every non negative integer \( m \), the binomial coefficient \( \binom{m}{z} \) where \( z \) is a complex number, is equal to the following finite sum:

\[
\binom{m}{z} = \sum_{k=0}^{m} \binom{m}{k} \text{sinc}(z - k)
\]

Theorem 2. Let \( w \) be any complex number with a real part greater than -1, and \( z \) be a complex number. The binomial coefficient \( \binom{w}{z} \) is equal to the following infinite function series:

\[
\binom{w}{z} = \sum_{k=0}^{\infty} \binom{w}{k} \text{sinc}(z - k)
\]

Corollary 1.

\[
\sum_{k=0}^{\infty} \binom{w}{k} \binom{z}{k} = \binom{w}{z}
\]

3. Proofs

In order to prove Theorem 1 an important Lemma must be introduced:

Lemma 1.

\[
\prod_{k=0}^{m} \frac{1}{(z - k)} = \frac{(-1)^m}{m!} \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k}{z - k}
\]

Proof. It is possible to prove the equivalence of these two expressions by partial fraction decomposition. Since the product has exactly \( m+1 \) simple poles, for \( z_k = 0, 1, 2, \ldots, m \), and has no roots, it can be written as a sum like the following:

\[
\frac{a_0}{z} + \frac{a_1}{z - 1} + \ldots + \frac{a_m}{z - m}
\]

Multiplying both sides by \( (z - z_k) \) and taking the \( \lim \) we get an analytic expression for \( a_k \), that is

\[
\lim_{z \to z_k} \frac{(z - z_k)}{z - z_k} \ldots \frac{(z - z_k)}{z - m} = \lim_{z \to z_k} \left( \frac{a_0}{z} + \ldots + \frac{a_m}{z - m} \right) = 0 + \ldots + a_{z_k} + \ldots + 0
\]
Since the poles $z_k$ coincide with the non negative integers from 0 to $m$ we can set $z_k = k$, which implies
\[
a_k = \frac{1}{k(k-1)\ldots1(-1)\ldots(k-m)} = \frac{(-1)^{m+k}}{k!(m-k)!}
\]
Substituting the value of $a_k$ in the summation, we get eq.(4) \hfill \Box

Now that Lemma 1 has been proven, we can proceed to proving Theorem 1.

Proof of Theorem 1. Consider multiplying eq.(1) by $\Gamma(-z)\Gamma(-z)$ and applying Euler’s reflection formula. After setting the condition of $w = m$ where $m$ is a non negative integer we get the following equation:
\[
\binom{m}{z} = \frac{(-1)^mm!\sin(\pi z)}{\pi \prod_{k=0}^{m}(z-k)}
\]
Applying Lemma 1 we can simplify and rewrite the left term this way
\[
\frac{\sin(\pi z)}{\pi} \sum_{k=0}^{m} \binom{m}{k} (-1)^k \frac{1}{z-k} = \sum_{k=0}^{m} \binom{m}{k} \frac{\cos(\pi k)\sin(\pi z) - \sin(\pi k)\cos(\pi z)}{\pi(z-k)}
\]
since $\cos(\pi k) = (-1)^k$ and $\sin(\pi k) = 0$. So we get finally
\[
\binom{m}{z} = \sum_{k=0}^{m} \binom{m}{k} \text{sinc}(z-k) \hfill \Box
\]

Proof of Theorem 2. For every $w \in \mathbb{C}$, let’s consider the following complex-valued function series
\[
F_w(z) := \sum_{k=0}^{\infty} \binom{w}{k} \text{sinc}(z-k)
\]
Computing its Fourier transform, we get:
\[
\hat{F}_w(u) = \sum_{k=0}^{\infty} \binom{w}{k} \int_{-\infty}^{\infty} \text{sinc}(z-k)e^{-2\pi iuz}dz = \text{rect}(u) \sum_{k=0}^{\infty} \binom{w}{k} e^{-2\pi iuk}
\]
By the generalized binomial theorem, for $\Re(w) > -1$, the last series converges to:
\[
(1 + e^{-2\pi iu})^w \text{rect}(u) \hfill (5)
\]
In [1, Theorem 7], it is proven that, for real $v$ and $w$
\[
\int_{-\infty}^{\infty} \left( \frac{w}{z} \right) e^{ivz} dz = \begin{cases} 
(1 + e^{iv})^w & \text{if } w > -1 \text{ and } |v| < \pi \\
0 & \text{if } w > -1 \text{ and } |v| > \pi \\
0 & \text{if } w > 0 \text{ and } |v| = \pi 
\end{cases}
\]
Which, for $w > 0$ can be rewritten as
\[(1 + e^{iv})^w \text{ rect} \left( \frac{v}{2\pi} \right)\]
Substituting $v = -2\pi u$, the equivalence to formula (6) is clearly shown. Since the series and the original definition with the $\Gamma$ function are both holomorphic, we can extend the equivalence to whenever the series is convergent, i.e $\Re(w) > -1$\]

**Proof of Corollary 1.** Writing the product $\binom{k}{z} \binom{z}{k}$ by the Gamma functions, and applying Euler’s reflection formula, one obtains
\[
\binom{k}{z} \binom{z}{k} = \frac{1}{\Gamma(k + 1)\Gamma(z - k + 1)} = \frac{1}{\Gamma(k + 1)\Gamma(z - k)(z - k)}
\]
\[
= \frac{\sin(\pi z - \pi k)}{\pi} \frac{1}{(z - k)} = \text{sinc}(z - k).
\]
The claim thus follows from Theorem 2.

---

4. **The D transform**

**Definition 1.** Given a function $f(z)$ defined on $\mathbb{C}$, we denote by D transform of $f(z)$ the following integral:
\[
\mathcal{D}\{f(z)\}(w) := \int_{-\infty}^{\infty} \binom{w}{z} f(z) dz.
\]

**Theorem 3.** Let $f(z)$ be analytic on $\mathbb{R}$. The D transform is given by:
\[(6) \quad \mathcal{D}\{f(z)\}(w) = \sum_{k=0}^{\infty} \binom{w}{k} f(k)\]

**Remark 1.** With eq.(6) we can summarize [1] Theorem 4,5,6,7,8] and give an alternative definition of all identities involving a sum of binomial coefficients. A core property of the D transform is thus giving a continuous meaning to functions defined only on non-negative integers. If the inverse transform were to be found, it would be possible to extrapolate continuous functions out of non-negative integer inputs.

**Corollary 2.** Setting $w = 0$ and making a change of coordinates, we have the surprising equation,
\[
\int_{-\infty}^{\infty} \text{sinc}(z - k) f(z) dz = f(k)
\]
Remark 2. This is a very important result because it implies that a continuous function behaves like a distribution. In fact if we were to substitute sinc\((z - k)\) with \(\delta(z - k)\) the integral would be the same.

To prove the above theorem it is convenient to introduce another Lemma, namely:

**Lemma 2.**

\[
\int_{-\infty}^{\infty} \left( \frac{w}{z} \right)^n dz = \sum_{k=0}^{\infty} \binom{w}{k} k^n
\]

**Proof.** It was previously shown that

\[
\int_{-\infty}^{\infty} \left( \frac{w}{z} \right) e^{-2\pi iuz} dz = (1 + e^{-2\pi iu}) w \text{rect}(u)
\]

In the following step, we won’t consider the \(\text{rect}\) function because it doesn’t affect the derivative at \(u = 0\).

The application of the generalized binomial theorem and the \(n^{th}\) iteration of Leibniz’s integral rule yields:

\[
\int_{-\infty}^{\infty} \left( \frac{w}{z} \right) (-2\pi iuz)^n e^{-2\pi iuz} dz = \sum_{k=0}^{\infty} \binom{w}{k} (-2\pi iuk)^n e^{-2\pi iuk}
\]

To continue, we must get rid of the exponential functions. That can be achieved by setting \(u = 0\),

\[
\left. \int_{-\infty}^{\infty} \left( \frac{w}{z} \right) z^n e^{-2\pi iuz} dz \right|_{u=0} = \sum_{k=0}^{\infty} \binom{w}{k} k^n e^{-2\pi iuk} \left|_{u=0} \right.
\]

Which simplifies to:

\[
\int_{-\infty}^{\infty} \left( \frac{w}{z} \right) z^n dz = \sum_{k=0}^{\infty} \binom{w}{k} k^n
\]

**Proof of Theorem 3.** Since \(f(z)\) is analytic, it can be defined by a Maclaurin series. Thus we can write its D transform likewise:

\[
\mathcal{D}\{f(z)\}(w) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \int_{-\infty}^{\infty} \left( \frac{w}{z} \right) z^n dz
\]

By Lemma 2,

\[
\mathcal{D}\{f(z)\}(w) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \sum_{k=0}^{\infty} \binom{w}{k} k^n
\]

Applying Weierstrass’s double series theorem we can then simplify the last formula to:

\[
\int_{-\infty}^{\infty} \left( \frac{w}{z} \right) f(z) dz = \sum_{k=0}^{\infty} \binom{w}{k} f(k)
\]
5. Applications

Using Theorem 2, it is possible to write the indefinite integral of \( \binom{w}{z} \), as the following function series

\[
\int \binom{w}{z} \, dz = \frac{1}{\pi} \sum_{k=0}^{\infty} \binom{w}{k} \text{Si}(\pi z - \pi k) + C
\]

Notice that two interesting definite integrals follow from eq.(8)

\[
\int_{-\infty}^{\infty} \sin(\pi w) \binom{w}{z} \, dw = \frac{\sin \pi z}{2z+1}
\]

The latter can be solved with the identity \( \sin(\pi x)(x - y - 1) = \sin(\pi y)(x - y - 1) \) and the application of Lemma 2. As follows from Theorem 2, it converges for \( \Re(z) < 0 \)

\[
\int_{0}^{1} \frac{1}{z} \, dz = G
\]

Applying Theorem 3, Corollary 1 can be rewritten likewise,

\[
\int_{-\infty}^{\infty} \binom{w}{t} \binom{t}{z} \, dt = \binom{w}{z}
\]

Some D transforms: (assuming \( \Re(w) > -1 \))

\[
\mathcal{D}\{\sin(z)\}(w) = \frac{(1 + e^i)w - (1 + e^{-i})w}{2i}
\]

\[
\mathcal{D}\{\cos(z)\}(w) = \frac{(1 + e^i)w + (1 + e^{-i})w}{2}
\]

\[
\mathcal{D}\left\{\frac{1}{1 + z}\right\}(w) = \frac{2^{w+1} - 1}{w + 1}
\]

References

[1] David Salwinski (2018) The Continuous Binomial Coefficient: An Elementary Approach, The American Mathematical Monthly, 125:3, 231-244, DOI: 10.1080/00029890.2017.1409570

LICEO CASSINI, GENOVA, ITALY.

E-mail address: lorydavid03@gmail.com