Topologically Massive Gauge Theory with $O(2)$ Symmetry

Ian I. Kogan and Kai-Ming Lee

Theoretical Physics, University of Oxford,
1 Keble Road, Oxford, OX1 3NP, UK

Abstract

We discuss the structure of the vacua in $O(2)$ topologically massive gauge theory on a torus. Since $O(2)$ has two connected components, there are four classical vacua. The different vacua impose different boundary conditions on the gauge potentials. We also discuss the non-perturbative transitions between the vacua induced by vortices of the theory.

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*On leave of absence from ITEP, Moscow, 117259, Russia
1. Introduction

Topologically massive gauge theories (TMGT) [1] are of great interest for many years. They have important applications in different areas of theoretical and mathematical physics, for example quantum Hall effect [2], knot invariants and conformal field theories [3] and many other areas.

The structure of the Hilbert space in these theories may be rather unusual. For example, in abelian $U(1)$ TMGT one can see [4] that the Hilbert space of the theory is a direct product of the massive gauge particle Hilbert space and some quantum-mechanical Hilbert space which is the product of $g$ copies (for a genus $g$ Riemann surface) of the Hilbert space for the Landau problem on the torus. In the infinite mass limit all levels except the first one are decoupled as well as the massive particle Hilbert space and we then have only the first Landau level which becomes the Hilbert space of the topological Chern-Simons theory.

In this letter we shall discuss some unusual properties of the $O(2)$ theory, which can be obtained after a spontaneous breaking of the $SU(2)$ or $SO(3)$ symmetry by a Higgs field in a 5-dimensional representation [5]. We shall see that because $O(2)$, contrary to $U(1)$, has two connected components, i.e. $\pi_0(O(2)) = Z_2$, one shall get new classical vacua in $O(2)$ theory. We shall not only consider all these vacua but also find the particle spectra corresponding to each vacuum.

Let us note that the problem of this type in pure Chern-Simons theory was discussed by Moore and Seiberg [6]. They could study only the properties of the ground state, because there are no excitations in the pure Chern-Simons theory. Our results for the ground state (obtained in a full TMGT and using other methods than in a pure Chern-Simons theory) are in agreement with the results for pure Chern-Simons theory, as it must be.

We shall start with a brief review of the ordinary $U(1)$ TMGT. Then we shall go to the $O(2)$ model, which will be defined as a low-energy limit of the $SO(3)$ theory. We will also discuss the non-perturbative transitions between different vacua in the full $SO(3)$ theory. These transitions are induced by the instantons, which in this case, are the vortices associated with the non-trivial component of the $O(2)$.

2. Abelian Topologically Massive Gauge Theory
Let us consider an abelian topologically massive gauge theory [1]:

$$L_{U(1)} = -\frac{1}{4\gamma} F_{\mu\nu} F^{\mu\nu} + \frac{k}{8\pi} \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda.$$  \hspace{1cm} (2.1)

We assume that the spacetime is in the form $M \times R$ where $M$ is a Riemann surface. We choose the $A_0 = 0$ gauge to perform the canonical quantization. The corresponding constraint is

$$\frac{1}{\gamma} \partial_i \dot{A}_i + \frac{k}{8\pi} \epsilon_{ij} F_{ij} = 0. \hspace{1cm} (2.2)$$

The vector potential on a plane can be represented as $A_i = \partial_i \xi + \epsilon_{ij} \partial_j \chi$, substitute it into the constraint, one gets $\partial^2 \dot{\xi} = (k\gamma/4\pi) \partial^2 \chi$. Forget the zero modes for the moment, this is $\dot{\xi} = (k\gamma/4\pi) \chi = M \chi$. Substituting this constraint into the Lagrangian (2.1) one gets

$$L = \frac{1}{2\gamma} (\partial_i \dot{\chi})^2 - (\partial^2 \chi)^2 - M^2 \chi \partial^2 \chi$$

which is a free Lagrangian for the field $\Phi = \sqrt{\partial^2 / \gamma \chi}$

$$L = \frac{1}{2} (\dot{\Phi}^2 - (\partial_i \Phi)^2 - M^2 \Phi^2)$$

(2.4)

describing the free particle with mass $M = \gamma k/4\pi$.

It is easy to see that on the plane the spatial independent fields $A_i(x, t) = A_i(t)$ also satisfy the constraint. For these fields one gets the Landau Lagrangian [1]

$$L = \frac{1}{2\gamma} \dot{A}_i^2 - \frac{k}{8\pi} \epsilon_{ij} A_i \dot{A}_j$$

(2.5)

which describes the particle with mass $m = \gamma^{-1}$ on the plane $A_x, A_y$ in a magnetic field $B = k/4\pi$. The mass gap is $M = B/m = \gamma k/4\pi$ which is precisely the mass of the gauge particle.

Let us note that $A_x$ and $A_y$ belong to the configuration space, however if reduced to the first Landau level, which means $m = 1/\gamma \rightarrow 0$, the theory reduces to the pure Chern-Simons theory which is an exactly solvable 2 + 1 dimensional topological field theory.

For general 2-dimensional Riemann surface of genus $g$, any one-form $A$ can be uniquely decomposed according to Hodge theorem as

$$A = d\xi + \delta \chi + A, \hspace{1cm} dA = \delta A = 0$$

(2.6)

which generalizes the decomposition on the plane we have used before. The harmonic form $A$ equals

$$A = \sum_{p=1}^g (A^p \alpha_p + B^p \beta_p)$$

(2.7)
where $\alpha_p$ and $\beta_p$ are canonical harmonic 1-forms (1-cohomology) on a Riemann surface and there are precisely $2g$ harmonic 1-forms on genus $g$ Riemann surface (two in case of a torus). After diagonalization one finds that there are $g$ copies of the Landau problem and the total Hilbert space $\mathcal{H}$ of the abelian topologically massive gauge theory

$$\mathcal{H} = \mathcal{H}_\Phi \otimes \prod_{i=1}^{g} \mathcal{H}_A$$

is the product of the free massive particle Hilbert space $\mathcal{H}_\Phi$ and $g$ copies of the Landau problem’s Hilbert space $\mathcal{H}_A$.

Let us concentrate on the case of a torus. We get the Landau problem on the plane $(A_x, A_y)$. We must not forget about large gauge transformations acting on the quantum-mechanical coordinates $A_i \rightarrow A_i + 2\pi N_i$, where $N_i$ are integers. These transformations act on gauge potential because the only gauge-invariant objects one can construct for $A_i$ are Wilson lines

$$W(C) = \exp(i \int_C A_\mu dx^\mu)$$

are invariant under these transformations (we choose coordinate on a torus in a way that $x \sim x + 1$ and $y \sim y + 1$) and one can consider torus $0 \leq A_i < 2\pi$ with the area $(2\pi)^2$.

Let us note that being reduced to the first Landau level this torus becomes the phase space - thus for the consistent quantization this area must be proportional to the integer (the total number of the states must be integer). It is known that the density of states $\rho$ on Landau level equals to $B/2\pi$, where $B$ is a magnetic field. In our case the “magnetic field” in $(A_x, A_y)$ plane is equals to $B = (k/4\pi)$, thus the total number of states will be $N = (1/2\pi)(k/4\pi) \times (2\pi)^2 = k/2$. Thus, $k$ must be an even integer. In more general case we have to enlarge our phase space to have minimal possible integer number of states (for rational $k$) or even infinite number (for irrational $k$).

3. The $O(2)$ Model

From now on, we turn to the TMGT with $O(2)$ symmetry on a torus. We obtain the gauge group $O(2)$ by spontaneous breaking a non-abelian Higgs model with Chern-Simons term and gauge group $SU(2)$ or $SO(3)$. We usually suppose that the symmetry breaking scale is very large and just consider the low energy phenomena. The reduced system is a TMGT with $O(2)$ symmetry.

The full Lagrangian is

$$\mathcal{L} = \frac{1}{8\gamma} \text{Tr} F_{\mu\nu} F^{\mu\nu} + \frac{k}{16\pi} \text{Tr} (A dA + \frac{2}{3} A^3) + \frac{1}{4} \text{Tr} (\partial \phi + [A, \phi])^2 - V(\phi).$$

(3.1)
The normalization is chosen such that it will agree with the abelian theory in some special case. We choose the representation and the potential of \( \phi \) such that the unbroken group is \( O(2) \). The simplest choice is the following [5]. The Higgs field is a \( 3 \times 3 \) real symmetric traceless matrix. If \( g \in SO(3) \), the action of \( g \) on \( \phi \) is

\[
g(\phi) = g\phi g^{-1}.
\]  

The potential of the Higgs field is

\[
V(\phi) = \lambda_1(\text{Tr}\phi^2 - 6v^2)^2 + \lambda_2(\det \phi + 2v^3)^2
\]

where \( \lambda_1 \) is of dimension of mass and \( \lambda_2 \) is dimensionless. They are both positive. \( V(\phi) \) is invariant under \( SO(3) \) and its value is zero if and only if the eigenvalues of \( \phi \) are 1, 1 and −2. If the vacuum expectation value of \( \phi \)

\[
\langle \phi \rangle = v \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},
\]

the group \( O(2) \) embedded in the \( SO(3) \) is then

\[
O(2) = \left\{ \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & -\cos \theta & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}.
\]

Abstractly, \( O(2) \) could be described as an \( U(1) \) with an extra element \( X \), where \( e^{i\theta}X = Xe^{-i\theta} \). In the above representation, \( X \) is

\[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

We will usually assume that the Higgs field takes its vacuum expectation values and drop the terms independent of the gauge field from the Lagrangian.

### 4. Topological Considerations

If we ignore the Higgs field and take the limit \( \gamma \to \infty \), \( \mathcal{L} \) goes to the pure Chern-Simons Lagrangian. The classical solutions of the pure Chern-Simons theory are given by

\[
\text{Hom}(\pi_1(M), G)/G
\]

where \( G \) is the gauge group and \( M \) is the space manifold.

If \( M \) is a torus, \( \pi_1(M) \) is \( Z \times Z \), the direct product of two copies of the set of all integers. We can label the classical solutions by a pair \((g_1, g_2)\) where \( g_1 \) and \( g_2 \) commute
in $G$ and we identify $(g_1, g_2)$ with $(gg_1g^{-1}, gg_2g^{-1})$. The pair describes the holonomy of the two non-trivial loops on the torus.

For $O(2)$, the solutions are one continuous family $(e^{i\theta_1}, e^{i\theta_2})$, which identifies with $(e^{-i\theta_1}, e^{-i\theta_2})$, and three discrete ones $(X, \pm 1)$, $(\pm 1, X)$ and $(X, \pm X)$. We will see that the sector corresponding to the continuous family has zero modes but the sector corresponding to the discrete solutions have none.

If we consider that the $O(2)$ group is embedded in $SO(3)$, the fundamental group of the vacuum manifold is $\pi_1(SO(3)/O(2)) = \pi_0(O(2)) = \mathbb{Z}_2$. Therefore, there are point-like topological excitations, vortices, in our theory. In $3 + 1$ dimensional spacetime, they are the cosmic strings. We will see that a pair of this vortex-anti-vortex is the instanton that mixes different classical vacua.

The second homotopy group of the vacuum manifold also has physical effect. The generator of $\pi_2(SO(3)/O(2)) = \pi_1(O(2)) = \mathbb{Z}$ is the magnetic monopole in $3 + 1$ dimensional spacetime. In this $O(2)$ theory, the monopole induces the large gauge transformation discussed in Section 2.

5. The Four Sectors

We are going to solve the equations of motion derived from (3.1) and find out the classical vacua. Define a basis of $SO(3)$ by $T^1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $T^2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $T^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$. Then, $[T^a, T^b] = \epsilon^{abc}T^c$. The gauge potential is

$$A = \begin{pmatrix} 0 & A^1 & -A^2 \\ -A^1 & 0 & A^3 \\ A^2 & -A^3 & 0 \end{pmatrix}.$$ (5.1)

We choose the $A_0 = 0$ gauge. The equation of motion of the spatial components of the gauge potential is

$$\frac{1}{\gamma}(\partial_j F_{ji} + [A_j, F_{ji}])^a - \frac{1}{2} \text{Tr}((\partial_i \phi + [A_i, \phi])[T^a, \phi]) = 0,$$ (5.2)

the equation of motion of the Higgs field is in the form

$$D_\mu D^\mu \phi - V'(\phi) = 0.$$ (5.3)
where $D_\mu$ is the covariant derivative. In addition to these two equations of motion, we also have the constraint
\begin{equation}
\frac{1}{\gamma}(\partial_i \dot{A}_i^a + e^{abc} A_i^b \dot{A}_i^c) + \frac{k}{8\pi} \epsilon^{ij} F_{ij}^a + \frac{1}{2} \text{Tr}(\partial_t \phi [T^a, \phi]) = 0. \tag{5.4}
\end{equation}

We consider only the low energy phenomena in this section. Equivalently, we consider the limit where $\lambda_1$, $\lambda_2$ and $v$ of (3.3) go to infinity. In this limit, the two equations of motion split into four. For example, when $\lambda_1$ and $\lambda_2$ go to infinity, the derivative term in (5.3) is negligible and we have
\begin{equation}
V'(\phi) = 0. \tag{5.5}
\end{equation}
This in turn implies that
\begin{equation}
D_\mu D^\mu \phi = 0. \tag{5.6}
\end{equation}
Similarly, when $v$ goes to infinity, we have
\begin{equation}
\text{Tr}((\partial_t \phi + [A_i, \phi])[T^a, \phi]) = 0 \tag{5.7}
\end{equation}
and
\begin{equation}
\frac{1}{\gamma}(\partial_j F_{ji} + [A_j, F_{ji}])^a = 0. \tag{5.8}
\end{equation}

One solution is that the value of the Higgs field is constant in spacetime and the gauge potential is zero up to gauge transformation. This is the trivial classical vacuum. We say that all the excitations based on this vacuum are in the trivial sector. By a suitable gauge transformation, we can assume that the Higgs field is given by (3.4).

To find out the spectrum of the trivial sector, notice that the interaction term of the Higgs and gauge potential in the Lagrangian is, in this case,
\begin{equation}
\text{Tr}(\partial \phi + [A, \phi])^2 = \text{Tr}[A, \phi]^2 = 18v^2((A^2)^2 + (A^3)^2). \tag{5.9}
\end{equation}
To consider only low energy phenomena, we set $A^2 = A^3 = 0$. The theory is reduced to the abelian TMGT. However, we also have to identify states related by gauge transformations. Now, we have one more element $X$ than the $U(1)$ case. Its action is to identify $A^1$ with $-A^1$. Thus, we have only half of the states as in the abelian case. The classical configuration space could be chosen as $0 \leq A^1_x < 1$ and $0 \leq A^1_y < \frac{1}{2}$. The total number of quantum states in the first Landau level will be $k/4$. Thus, to have a sensible quantum theory, $k$ must be a multiple of 4. In other cases we again must enlarge our phase space before quantization in the same way as in the $U(1)$ case. The Hilbert space structure is similar to that of the abelian theory.

We now consider the case that the holonomy of at least one of the non-trivial loops is in the component of $O(2)$ other than the identity component. We call it the non-trivial
sector. We already know that there are three non-trivial sectors for the $O(2)$ model on a torus.

Let $X(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix}$ where $\theta$ would be chosen such that, say, if we want to have non-trivial holonomy along the $x$-direction, $\theta(x + L, y) = \theta(x, y) + \pi$ and $\theta(x, y + L) = \theta(x, y)$. Other than this condition, $\theta$ could be any function independent of time. We will see that the low energy physics depends only on the boundary condition stated above and not on the detail form of $\theta$. Note that $X(\pi) = X$.

We will reinstall explicitly the dependence on the size of torus. We choose the torus to be the same size, $L$, in both directions. It is easy to generalize to tori of other sizes.

Now, we consider the following Higgs field and gauge potential:

$$
\phi = v X(\theta) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} X(\theta)^{-1} = v \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos^2 \theta - 2 \sin^2 \theta & -3 \sin \theta \cos \theta \\ 0 & -3 \sin \theta \cos \theta & \sin^2 \theta - 2 \cos^2 \theta \end{pmatrix}
$$

and

$$
A = -\partial X X^{-1} = -\partial \theta T^3.
$$

If $X$ was a single valued function on the torus, then these Higgs field and gauge potential would be just the gauge transformation of the corresponding vacuum fields in the trivial sector. Although $X$ is not single valued, these Higgs field and gauge potential are single valued and well defined on the torus. It is now easy to see that they satisfy equations (5.5) - (5.8) and the constraint (5.4) because those equations are gauge invariant.

Thus, the Higgs field and gauge potential in (5.10) and (5.11) define the classical vacua in the non-trivial sectors. (We can, of course, choose $\theta$ to be zero and we obtain the vacuum of the trivial sector.) We can choose any other values of the Higgs fields in the vacuum manifold as long as they give the same holonomy. Any such value of Higgs field will be gauge equivalent to this one.

We now calculate the potential of the gauge potential induced by the Higgs field. Since

$$
X(\theta)^{-1} A X(\theta) = \begin{pmatrix} 0 & \cos \theta A^1 + \sin \theta A^2 & \sin \theta A^1 - \cos \theta A^2 \\ -\cos \theta A^1 - \sin \theta A^2 & 0 & A^3 \\ \cos \theta A^2 - \sin \theta A^1 & -A^3 & 0 \end{pmatrix},
$$

(5.12)
\[
\text{Tr}[A, \phi]^2 = v^2 \text{Tr}[X(\theta)^{-1}AX(\theta), \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}]^2
\]
(5.13)

and
\[
2\text{Tr}\partial \phi[A, \phi] = 36v^2 A^3 \partial \theta.
\]
(5.14)

The effective potential of \(A\) is
\[
\text{Tr}(\partial \phi + [A, \phi])^2 = 18v^2((A^1 \sin \theta - A^2 \cos \theta)^2 + (A^3)^2).
\]
(5.15)

If we write the general gauge potential as the sum of the vacuum value and the fluctuations, we see that in the large \(v\) limit, \(A^3\) must be frozen to the vacuum value \(-\partial \theta\) and the fluctuations of the first two components of the gauge potential must satisfy
\[
A^1 \sin \theta = A^2 \cos \theta.
\]
Let \(A^1 = A \cos \theta\) and \(A^2 = A \sin \theta\). Here, \(A\) is a vector field with three components \((A_t, A_x, A_y)\) and not the Lie-algebra valued one form in the previous equations. Note \(A^1\) and \(A^2\) are periodic in \(x\) and \(y\) but \(A^3\) is not periodic in the non-trivial sectors. Thus \(A\) does not have the corresponding zero modes.

The components of the gauge field can be expressed by \(A\). Define \(F_{ij} = \partial_i A_j - \partial_j A_i\), we have
\[
F^1_{ij} = \partial_i A^1_j - \partial_j A^1_i + A^2_i A^3_j - A^2_j A^3_i
= F_{ij} \cos \theta,
\]
(5.16)
\[
F^2_{ij} = F_{ij} \sin \theta,
\]
(5.17)
\[
F^3_{ij} = 0.
\]
(5.18)

The Lagrangian, which was
\[
\mathcal{L} = \frac{1}{2\gamma}(\dot{A}^a_i - \partial_i A^a_i + e^{abc} A^b_i A^c_i)^2 - \frac{1}{4\gamma} F_{ij}^a F_{ij}^a - \frac{k}{8\pi} (\epsilon^{ij} A^a_i A^a_j - A^a_i \epsilon^{ij} F^a_{ij})
+ \frac{1}{4} \text{Tr}(\partial \phi + [A, \phi])^2 - V(\phi),
\]
(5.19)

is now, after neglecting the terms independent of \(A\) and in the \(A^a_i = 0\) gauge,
\[
\mathcal{L} = \frac{1}{2\gamma} \dot{A}^2_i - \frac{1}{4\gamma} F_{ij}^i F_{ij} - \frac{k}{8\pi} \epsilon^{ij} A_i \dot{A}_j.
\]
(5.20)

The constraint of the full system is
\[
\frac{1}{\gamma} (\partial_i \dot{A}^a_i + e^{abc} A^b_i \dot{A}^c_i) + \frac{k}{8\pi} \epsilon^{ij} F_{ij}^a = 0.
\]
(5.21)
Express $A_i^a$ in terms of $A_i$ and $\theta$, this is equivalent to

$$\frac{1}{\gamma} \partial_i \dot{A}_i + \frac{k}{8\pi} \epsilon^{ij} F_{ij} = 0. \quad (5.22)$$

Notice that the form of the Lagrangian and the constraint are exactly the same as the abelian theory or the trivial sector. However, recall that the boundary condition is that $A_i \cos \theta$ and $A_i \sin \theta$ are periodic. If the holonomy of the $x$-direction is non-trivial, the effective gauge potential must be anti-periodic in $x$. This distinguishes the non-trivial sectors from the trivial sector. We also see that $\theta$ does not appear in the Lagrangian or the constraint. It only imposes the boundary condition on the effective gauge potential. Hence, the detail form of $\theta$ is irrelevant.

Following the analysis of the $U(1)$ theory, it is easy to see that the non-trivial sectors can also be described by a free particle of mass $k\gamma/4\pi$. Because of the finite size of the torus, the momentum of the particle is quantized. To satisfy the anti-periodic boundary condition, only the “half integer modes” are allowed in the direction where the holonomy is non-trivial.

6. Non-perturbative Transitions Between Sectors

If we consider the full $SU(2)$ or $SO(3)$ theory, there will be transitions between different sectors. Indeed, now one can easily connect two different topological sectors by some field configuration - but to do this it is necessary to excite some heavy degrees of freedom. The instanton is one of these heavy degrees of freedom which induces the transitions. The exact instanton equation is non-linear and difficult to solve but we could easily give a qualitative picture of what will happen.

Since $O(2)$ has two connected components, there will be point-like topological defects, the vortices, in 2 + 1 dimensional spacetime [5]. It is, in fact, the same reason why there are non-trivial sectors in the theory. We will see that the instanton consists of a pair of vortex-anti-vortex winding around a non-trivial loop on the torus.

An ansatz of the vortex solution is the following: in the polar coordinates $(r, \theta)$ on a plane,

$$\phi(r, \theta) = v(r) \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos^2 \frac{\theta}{2} - 2 \sin^2 \frac{\theta}{2} & -3 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \\ 0 & -3 \sin \frac{\theta}{2} \cos \frac{\theta}{2} & \sin^2 \frac{\theta}{2} - 2 \cos^2 \frac{\theta}{2} \end{pmatrix} \quad (6.1)$$

where $v(r)$ is a function of $r$ such that $v(0) = 0$ and $v(\infty) = v$. The gauge potential is such that $A(0, \theta) = 0$ and $A(\infty, \theta) = -T^3 \frac{d\theta}{2}$. Comparing to (5.10) and (5.11), we see
that at infinity, the Higgs field and the gauge potential approach their vacuum values. At the origin there are heavy excitations in a region with size of order $1/v^2$ and the mass of the vortex is of order of magnitude $v^2$. The functional form of $v(r)$ and the gauge potential could be determined by substituting them into (3.1) and solving the differential equations derived from it. We will not do it here because they are not important to the following discussions. Notice that the low energy effective gauge potential must satisfy anti-periodic boundary condition around a vortex.

It is easier to visualize the effect of the instanton in $3+1$ dimensional spacetime. In $3+1$ dimensions, the topological defect corresponding to the vortex is the cosmic string. It can be infinite long or form a close loop. Just like the vortex, the low energy effective gauge potential must be anti-periodic around the cosmic string. If we put our torus in a three dimensional space, we could say that the anti-periodic boundary condition of the non-trivial sectors is induced by a cosmic string which goes through the “hole” of the torus. (If the boundary conditions for both loops on the torus are non-trivial, we may put a cosmic string through the “hole,” and put a close loop of string “inside” the torus. For simplicity, we consider non-trivial condition on only one boundary.)

To tunnel back to the trivial sector, the cosmic string must be pulled out of the torus. During the process, the cosmic string will in general cut the torus at two points. Since the section of a string is a vortex, the two points are exactly the positions of a pair of vortex-anti-vortex in the $2+1$ dimensional world. (In the case of $O(2)$, the anti-vortex is the vortex itself.)

If we restrict ourselves to the torus, what we see about the pulling out of the cosmic string is the instanton. The whole process of instanton is the following. If the holonomy of the $x$-direction is non-trivial, a pair of vortex-anti-vortex will nucleate, then they wind around the loop in the $y$-direction and finally, they annihilate each other. Since the gauge potential is anti-periodic around a vortex or anti-vortex, after the instanton occurs, the anti-periodic boundary condition on the $x$-loop will become periodic.

The mass of the vortex is of order $v^2$ and it must wind around the non-trivial loop of the torus. Therefore, we expect that the Euclidean action for the instanton is in the form $\exp(-\text{const } L v^2)$.

We can obtain the same type of of the instanton action considering the simpliest trial function interpolating between two vacua

$$\phi(t) = \frac{t}{T} \phi_1 + \frac{T-t}{T} \phi_2$$

(6.2)

where $\phi_1$ and $\phi_2$ are the Higgs fields in these two vacua given by (5.10), $T$ is the size (in the time axis) of the instanton and we set the gauge potential to zero. It is then straight
forward to calculate the Euclidean action for this field configuration. The result is, for example,

$$S_E(\theta = 0 \rightarrow \theta = \frac{\pi x}{L}) = \frac{9L^2v^2}{4T} + \left(\frac{3}{2}\pi^2v^2 + w\right)T$$ \hspace{1cm} (6.3)$$

where \( w = \frac{81}{80}L^2v^4(4\lambda_1 + v^2\lambda_2) \) is the contribution from the potential term. Notice that \( T \) is of order \( L \) when the Euclidean action is minimum, which is of order \( Lv^2 \), thus the transition probability will be suppressed by a factor \( \exp(-\text{const}Lv^2) \), as expected.

If the quantum corrections did not lift the degeneracy of the four classical vacua, there would be mixings between them and the true ground state will be some superposition of the classical vacua. Physical properties of this kind of mixings were considered earlier in [7]. However in this case, as we are going to argue, the quantum corrections do lift the degeneracy.

By the analysis of the previous section, in the trivial sector, the theory is free but the momentum of the effective particle must be quantized as integral multiple of \( 2\pi/L \) in both directions. Thus, the vacuum energy of the trivial sector, which is one half of the sum of frequencies of all modes, is

$$E_0 = \frac{1}{2} \sum_{n,m} ((2\pi n/L)^2 + (2\pi m/L)^2 + M^2)^{1/2}$$ \hspace{1cm} (6.4)$$

Similarly, the vacuum energies of the non-trivial sectors are

$$E_x = E_y = \frac{1}{2} \sum_{n,m} ((2\pi (n + \frac{1}{2})/L)^2 + (2\pi m/L)^2 + M^2)^{1/2}$$ \hspace{1cm} (6.5)$$

$$E_{xy} = \frac{1}{2} \sum_{n,m} ((2\pi (n + \frac{1}{2})/L)^2 + (2\pi (m + \frac{1}{2})/L)^2 + M^2)^{1/2}$$ \hspace{1cm} (6.6)$$

We will show that \( E_x > E_0 \) now. Apart from some unimportant factor, \( E_0 = \sum(n^2 + m^2 + a^2)^{1/2} \) and \( E_x = \sum((n + \frac{1}{2})^2 + m^2 + a^2)^{1/2} \). Of course, they diverge. We will use the zeta function regularization. Consider \( E_0(s) = \sum(n^2 + m^2 + a^2)^{-s} \).

$$E_0(s)\Gamma(s) = \sum_{n,m} (n^2 + m^2 + a^2)^{-s}\Gamma(s)$$

$$= \int_0^\infty t^{s-1} \sum_{n,m} e^{-(n^2+m^2+a^2)t} dt.$$ \hspace{1cm} (6.7)$$

Similarly,

$$E_x(s)\Gamma(s) = \int_0^\infty t^{s-1} \sum_{n,m} e^{-(n+\frac{1}{2})^2+m^2+a^2} t dt.$$ \hspace{1cm} (6.8)$$

The difference is

$$E_x(s) - E_0(s) = \frac{-1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_m e^{-(m^2+a^2)t} \sum_n (e^{-n^2t} - e^{-(n+\frac{1}{2})^2t}) dt$$

$$= \frac{-1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-a^2t} \partial_1(it/\pi)(\partial_1(it/\pi) - \partial_2(it/\pi)) dt.$$ \hspace{1cm} (6.9)$$
where $\vartheta_1(\tau) = \sum \exp i\pi n^2 \tau$ and $\vartheta_2(\tau) = \sum \exp i\pi (n + 1/2)^2 \tau$ are some special cases of the theta functions [8]. We need their properties that

$$
\vartheta_1(-1/\tau) = (-i\tau)^{1/2} \vartheta_1(\tau)
$$
$$
\vartheta_2(-1/\tau) = (-i\tau)^{1/2} \vartheta_3(\tau)
$$

(6.10)

where $\vartheta_3(\tau) = \sum (-1)^n \exp i\pi n^2 \tau$. For small $t$, $\vartheta_3(i\pi t)$ converges but $\vartheta_1(i\pi t)$ diverges as $1/\sqrt{\pi t}$. For large $t$, $\vartheta_1(i\pi t) = 1$. We also have

$$
\vartheta_1(i\pi t) - \vartheta_3(i\pi t) = \sum_{\text{odd } n} 2e^{-n^2 t} < 4e^{-t} \sum_n e^{-2nt} = \frac{4e^{-t}}{1-e^{-2t}}.
$$

(6.11)

Taking all these properties into account, it is easy to see that

$$
E_x(s) - E_0(s) = -\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-a^2 t} \vartheta_1(it/\pi)(\vartheta_1(it/\pi) - \vartheta_2(it/\pi))dt
$$
$$
= -\frac{\pi}{\Gamma(s)} \int_0^\infty t^{-s} e^{-a^2/2t} \vartheta_1(i\pi t)(\vartheta_1(i\pi t) - \vartheta_3(i\pi t))dt
$$

(6.12)

converges for $s = -1/2$ for both small and large $t$. By (6.11), for large $a$, the integrand is in the form $\exp(1/2 \log t - a^2/t - t)$. The minimum of the exponent is around $t \approx a = ML/\pi$. Thus, the splitting of vacuum energies is of order $\exp(-\text{const } ML)$. Notice that both the integral and $-\pi/\Gamma(-1/2) = \sqrt{\pi}/2$ are positive. We have $E_x > E_0$. Similarly, $E_{xy} > E_x$. In conclusion, there is no mixing between different classical vacua and the vacuum of the trivial sector is the true vacuum after quantum correction. The vacua of the non-trivial sectors are metastable and will decay into the vacuum of the trivial sector through the vortex type instanton transitions discussed above. The transition probability is suppressed as $\exp(-\text{const } L\nu^2)$, where the action is the double instanton action corresponding to the bounce solution.

7. Conclusion

We have discussed the topologically massive gauge theory with $O(2)$ symmetry on a torus. We found that there are four different sectors corresponding to the different holonomy on the non-trivial loops. All four sectors can be described by a free particle of mass $k\gamma/4\pi$. However, its momentum in the direction of non-trivial holonomy is restricted to half integral multiple of some fundamental unit determined by the size of the torus, whereas the momentum in other direction is whole integral multiple. In addition to the restriction of momenta, the number of quantum states of the first Landau level in the trivial sector is only half of that of the abelian theory. We have also discussed the transitions between
different vacua induced by the instantons. A very clear picture on effect of the instanton in term of cosmic string in $3 + 1$ dimensions is given. The transition probability is estimated to be suppressed by an exponential factor proportional to the size of the torus. After including the quantum effect, the true vacuum is the vacuum of the trivial sector and other classical vacua are metastable.

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References

[1] J.F. Schonfeld, Nucl. Phys. B185 (1981) 157;
S. Deser, R. Jackiw and S. Templeton, Phys. Rev. Lett. 48 (1982) 975;
Ann. Phys. (N.Y.) 140 (1982) 372.

[2] R.E. Prange and S.M. Girvin, eds., The Quantum Hall Effect (Springer, New York, 1986).

[3] E. Witten, Comm. Math. Phys. 121 (1989) 377.

[4] I.I. Kogan and A.Yu. Morozov, Sov. Phys. JETP 61 (1985) 1;
I.I. Kogan, Comm. Nucl. Part. Phys. 19 (1990) 305;
I.I. Kogan, Int. J. Mod. Phys. A9 (1994) 3887;
G. Dunne, R. Jackiw and C.A. Trugenberger, Phys. Rev. D 41 (1990) 661;
A.P. Polychronakos, Ann. Phys. (N.Y.) 203 (1990) 231;
X.G. Wen, Int. J. Mod. Phys. B 4 (1990) 239.

[5] A.S. Schwarz, Nucl. Phys. B208 (1982) 141;
A.S. Schwarz and Y.S. Tyupkin, Nucl. Phys. B209 (1982) 427;
M. Alford, K. Benson, S. Coleman, J. March-Russell and F. Wilczek, Phys. Rev. Lett. 64 (1990) 1632, [Erratum: 65 (1990) 668]; Nucl. Phys. B349 (1991) 414;
M. Bucher, H.-K. Lo and J. Preskill, Nucl. Phys. B386 (1992) 3.

[6] G. Moore and N. Seiberg, Phys. Lett. B220 (1989) 422.

[7] M. Bucher, Nucl. Phys. B350 163 (1991);
H.-K. Lo, K.-M. Lee and J. Preskill, Phys. Lett. B 318, 287 (1993);
K.-M. Lee, Phys. Rev. D49 2030 (1994).
[8] See, for example, David Mumford, *Tata Lectures on Theta I* (Birkhäuser, Boston, 1983).