Center of Gravity and a Characterization of Parabolas

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ABSTRACT. Archimedes determined the center of gravity of a parabolic section as follows. For a parabolic section between a parabola and any chord $AB$ on the parabola, let us denote by $P$ the point on the parabola where the tangent is parallel to $AB$ and by $V$ the point where the line through $P$ parallel to the axis of the parabola meets the chord $AB$. Then the center $G$ of gravity of the section lies on $PV$ called the axis of the parabolic section with $PG = \frac{3}{5}PV$. In this paper, we study strictly locally convex plane curves satisfying the above center of gravity properties. As a result, we prove that among strictly locally convex plane curves, those properties characterize parabolas.

1. Introduction

Archimedes found some interesting area properties of parabolas. Consider the region bounded by a parabola and a chord $AB$. Let $P$ be the point on the parabola where the tangent is parallel to the chord $AB$. The parallel line through $P$ to the axis of the parabola meets the chord $AB$ at a point $V$. Then, he proved that the
area of the parabolic region is $4/3$ times the area of triangle $\triangle ABP$ whose base is the chord and the third vertex is $P$.

Furthermore, he showed that the center $G$ of gravity of the parabolic section lies on the segment $PV$ called the axis of the parabolic section with $PG = \frac{4}{5}PV$. For the proofs of Archimedes, see Chapter 7 of [25].

Recently, two of the present authors showed that among strictly convex plane curves, the above area properties of parabolic sections characterize parabolas. More precisely, they proved as follows ([18]).

**Proposition 1.** Let $X$ be a strictly convex curve in the plane $\mathbb{R}^2$. Then $X$ is a parabola if and only if it satisfies

$$(C) : \text{For a point } P \text{ on } X \text{ and a chord } AB \text{ of } X \text{ parallel to the tangent of } X \text{ at } P, \text{ the area of the region bounded by the curve and } AB \text{ is } 4/3 \text{ times the area of triangle } \triangle ABP.$$ 

Actually, in [18], they established five characterizations of parabolas, which are the converses of well-known properties of parabolas originally due to Archimedes ([25]). In [21], they gave some characterizations of parabolas using area of triangles associated with a plane curve, which are generalizations of some results in [23]. See also [10, 11, 12] for some generalizations of results in [21].

In [16] and [17], two of the present authors proved the higher dimensional analogues of some results in [18]. Some characteristic properties of hyperspheres, ellipsoids, elliptic hyperboloids, hypercylinders and $W$-curves in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ were given in [1, 4, 6, 7, 13, 15, 22]. In [19], some characteristics for hyperbolic spaces embedded in the Minkowski space were established.

For some characterizations of parabolas or conic sections by properties of tangent lines, see [8] and [20]. In [14], using curvature function $\kappa$ and support function $h$ of a plane curve, the first and second authors of the present paper gave a characterization of ellipses and hyperbolas centered at the origin.

Among the graphs of functions, Á. Bényi et al. proved some characterizations of parabolas ([2, 3]). B. Richmond and T. Richmond established a dozen characterizations of parabolas using elementary techniques ([24]). In their paper, parabola means the graph of a quadratic polynomial in one variable.

In this paper, we study strictly locally convex plane curves satisfying the above mentioned properties on the center of gravity. Recall that a regular plane curve $X : I \to \mathbb{R}^2$ in the plane $\mathbb{R}^2$, where $I$ is an open interval, is called convex if, for all $s \in I$ the trace $X(I)$ of $X$ lies entirely on one side of the closed half-plane determined by the tangent line at $s$ ([5]). A regular plane curve $X : I \to \mathbb{R}^2$ is called locally convex if, for each $s \in I$ there exists an open subinterval $J \subset I$ containing $s$ such that the curve $X|_J$ restricted to $J$ is a convex curve.

Hereafter, we will say that a locally convex curve $X$ in the plane $\mathbb{R}^2$ is strictly locally convex if the curve is smooth (that is, of class $C^3$) and if of positive curvature $\kappa$ with respect to the unit normal $N$ pointing to the convex side. Hence,
in this case we have \( \kappa(s) = \langle X''(s), N(X(s)) \rangle > 0 \), where \( X(s) \) is an arc-length parametrization of \( X \).

For a smooth function \( f : I \to \mathbb{R} \) defined on an open interval, we will also say that \( f \) is strictly convex if the graph of \( f \) has positive curvature \( \kappa \) with respect to the upward unit normal \( N \). This condition is equivalent to the positivity of \( f''(x) \) on \( I \).

First of all, in Section 2 we prove the following:

**Theorem 2.** Let \( X \) be a strictly locally convex plane curve in the plane \( \mathbb{R}^2 \). For a fixed point \( P \) on \( X \) and a sufficiently small \( h > 0 \), we denote by \( l \) the parallel line through \( P + hN(P) \) to the tangent \( t \) of the curve \( X \) at \( P \). If we let \( d_P(h) \) the distance from the center \( G \) of gravity of the section of \( X \) cut off by \( l \) to the tangent \( t \) of the curve \( X \) at \( P \), then we have

\[
\lim_{h \to 0} \frac{d_P(h)}{h} = \frac{3}{5}.
\]

Without the help of Proposition 1, in Section 3 we prove the following characterization theorem for parabolas with axis parallel to the \( y \)-axis, that is, the graph of a quadratic function.

**Theorem 3.** Let \( X \) be the graph of a strictly convex function \( g : I \to \mathbb{R} \) in the \( uv \)-plane \( \mathbb{R}^2 \) with the upward unit normal \( N \). For a fixed point \( P = (u, g(u)) \) on \( X \) and a sufficiently small \( h > 0 \), we denote by \( l \) (resp., \( V \)) the parallel line through \( P + hN(P) \) to the tangent \( t \) of the curve \( X \) at \( P \) (resp., the point where the parallel line through \( P \) to the \( v \)-axis meets \( l \)). Then \( X \) is an open arc of a parabola with axis parallel to the \( v \)-axis if and only if it satisfies

\( (D) \) : For a fixed point \( P \) on \( X \) and a sufficiently small \( h > 0 \), the center \( G \) of gravity of the section of \( X \) cut off by \( l \) to the tangent \( t \) of the curve \( X \) at \( P \) lies on the segment \( PV \) with

\[
PG = \frac{3}{5}PV,
\]

where we denote by \( PV \) both of the segment and its length.

Note that if \( X \) is an open arc of a parabola with axis which is not parallel to the \( v \)-axis (for example, the graph of \( g \) given in (3.23) with \( \alpha \neq 0 \)), then it does not satisfy Condition \( (D) \).

Finally using Proposition 1, in Section 4 we prove the following characterization theorem for parabolas.

**Theorem 4.** Let \( X \) be a strictly locally convex plane curve in the plane \( \mathbb{R}^2 \). For a fixed point \( P \) on \( X \) and a sufficiently small \( h > 0 \), we denote by \( l \) the parallel line through \( P + hN(P) \) to the tangent \( t \) of the curve \( X \) at \( P \). We let \( d_P(h) \) the distance from the center \( G \) of gravity of the section of \( X \) cut off by \( l \) to the tangent \( t \) of the curve \( X \) at \( P \). Then \( X \) is an open arc of a parabola if and only if it satisfies
(E): For a fixed point $P$ on $X$ and a sufficiently small $h > 0$, we have

$$d_P(h) = \frac{3}{5} h.$$  

In [9], using the results in this paper and the centroids of triangles associated with a strictly convex curve, some characterizations of parabolas were established.

Throughout this article, all curves are of class $C^{(3)}$ and connected, unless otherwise mentioned.

2. Preliminaries and Theorem 2

Suppose that $X$ is a strictly locally convex curve in the plane $\mathbb{R}^2$ with the unit normal $N$ pointing to the convex side. For a fixed point $P \in X$, and for a sufficiently small $h > 0$, consider the parallel line $l$ through $P + hN(P)$ to the tangent $t$ of $X$ at $P$. Let’s denote by $A$ and $B$ the points where the line $l$ intersects the curve $X$.

We denote by $S_P(h)$ (respectively, $R_P(h)$) the area of the region bounded by the curve $X$ and chord $AB$ (respectively, of the rectangle with a side $AB$ and another one on the tangent $t$ of $X$ at $P$ with height $h > 0$). We also denote by $L_P(h)$ the length of the chord $AB$. Then we have $R_P(h) = hL_P(h)$.

We may adopt a coordinate system $(x, y)$ of $\mathbb{R}^2$ in such a way that $P$ is taken to be the origin $(0, 0)$ and the $x$-axis is the tangent line of $X$ at $P$. Furthermore, we may assume that $X$ is locally the graph of a non-negative strictly convex function $f : \mathbb{R} \to \mathbb{R}$.

For a sufficiently small $h > 0$, we have

$$S_P(h) = \int_{f(x) < h} \{h - f(x)\} dx,$$

$$R_P(h) = hL_P(h) = h \int_{f(x) < h} 1 dx.$$  

The integration is taken on the interval $I_P(h) = \{x \in \mathbb{R} | f(x) < h\}$.

On the other hand, we also have

$$S_P(h) = \int_{y=0}^{h} L_P(y) dy,$$

which shows that

$$S'_P(h) = L_P(h).$$  

First of all, we need the following lemma ([18]), which is useful in this article.
Lemma 5. Suppose that $X$ is a strictly locally convex curve in the plane $\mathbb{R}^2$ with the unit normal $N$ pointing to the convex side. Then we have

$$\lim_{h \to 0} \frac{1}{\sqrt{h}} L_P(h) = \frac{2\sqrt{2}}{\sqrt{\kappa(P)}},$$

where $\kappa(P)$ is the curvature of $X$ at $P$ with respect to the unit normal $N$.

From Lemma 5, we get a geometric meaning of curvature $\kappa(P)$ of a locally strictly convex plane curve $X$ at a point $P \in M$. That is, we obtain

$$\kappa(P) = \lim_{h \to 0} \frac{8h}{L_P(h^2)}.$$

Now, we give a proof of Theorem 2.

Let us denote by $d_P(h)$ the distance from the center $G$ of gravity of the section of $X$ cut off by $l$ to the tangent $t$ of the curve $X$ at $P$. Note that the curve $X$ is of class $C^3$. If we adopt a coordinate system $(x, y)$ of $\mathbb{R}^2$ as in the beginning of this section, then the curve $X$ is locally the graph of a non-negative strictly convex $C^3$ function $f: \mathbb{R} \to \mathbb{R}$. Hence, the Taylor’s formula of $f(x)$ is given by

$$f(x) = ax^2 + f_3(x),$$

where $a = f''(0)/2$, and $f_3(x)$ is an $O(|x|^3)$ function. Since $\kappa(P) = 2a > 0$, we see that $a$ is positive.

From the definition of $d_P(h)$, we have

$$S_P(h)d_P(h) = \phi(h),$$

where we put

$$\phi(h) = \frac{1}{2} \int_{f(x) < h} \{h^2 - f(x)^2\} dx.$$

We decompose $\phi(h) = \phi_1(h) - \phi_2(h)$ as follows:

$$\phi_1(h) = \frac{1}{2} \int_{f(x) < h} h^2 dx, \quad \phi_2(h) = \frac{1}{2} \int_{f(x) < h} f(x)^2 dx.$$

It follows from the definition of $L_P(h)$ that

$$\phi_1(h) = \frac{1}{2} h^2 L_P(h).$$

Hence, Lemma 5 shows that

$$\lim_{h \to 0} \frac{\phi_1(h)}{h^2\sqrt{h}} = \frac{\sqrt{2}}{\sqrt{\kappa(P)}}.$$
Lemma 6. For the limit of $\phi_2(h)/(h^2\sqrt{h})$ as $h$ tends to 0, we get

\begin{equation}
\lim_{h \to 0} \frac{\phi_2(h)}{h^2\sqrt{h}} = \frac{\sqrt{2}}{5\sqrt{\kappa(P)}}.
\end{equation}

Proof. If we put $g(x) = f(x)^2$, then we have from (2.5)

\begin{equation}
g(x) = a^2 x^4 + f_3(x),
\end{equation}

where $f_3(x)$ is an $O(|x|^3)$ function. We let $x = \sqrt{h}\xi$. Then, together with (2.5), (2.8) gives

\begin{equation}
\phi_2(h) = \frac{1}{2h^2\sqrt{h}} \int_{f(x)<h} g(x)dx
\end{equation}

\begin{equation}
= \frac{1}{2h^2} \int_{a^2\xi^2+g_3(\sqrt{h}\xi)<1} g(\sqrt{h}\xi)d\xi,
\end{equation}

where we denote $g_3(\sqrt{h}\xi) = \frac{f_3(\sqrt{h}\xi)}{h}$.

Since $f_3(x)$ is an $O(|x|^3)$ function, we have for some constant $C_1$

\begin{equation}
|g_3(\sqrt{h}\xi)| \leq C_1\sqrt{h}|\xi|^3.
\end{equation}

We also obtain from (2.12) that

\begin{equation}
\frac{|g(\sqrt{h}\xi) - a^2 h^2\xi^4|}{h^2} \leq C_2\sqrt{h}|\xi|^4
\end{equation}

where $C_2$ is a constant.

If we let $h \to 0$, it follows from (2.13)-(2.15) that

\begin{equation}
\lim_{h \to 0} \frac{\phi_2(h)}{h^2\sqrt{h}} = \frac{1}{2} \int \frac{a^2 \xi^4 d\xi}{a}\frac{1}{5\sqrt{\kappa}}
\end{equation}

This completes the proof of Lemma 6.

Together with (2.10), Lemma 6 shows that

\begin{equation}
\lim_{h \to 0} \frac{\phi(h)}{h^2\sqrt{h}} = \frac{4\sqrt{2}}{5\sqrt{\kappa(P)}}.
\end{equation}

Since $S'_P(h) = L_P(h)$, it follows from Lemma 5 that

\begin{equation}
\lim_{h \to 0} \frac{1}{h\sqrt{h}} S_P(h) = \frac{4\sqrt{2}}{3\sqrt{\kappa(P)}}.
\end{equation}
Thus, together with (2.17) and (2.18), (2.6) completes the proof of Theorem 2.

3. Proof of Theorem 3

In this section, we give a proof of Theorem 3.

Let \( X \) be the graph of a strictly convex function \( g : I \to \mathbb{R} \) in the \( uv \)-plane \( \mathbb{R}^2 \) with the upward unit normal \( N \).

For a fixed point \( P = (b, c) \in X \) with \( c = g(b) \), we denote by \( \theta \) the angle between the normal \( N(P) \) and the positive \( v \)-axis. Then we have \( g'(b) = \tan \theta \) and \( V = (b, c + wh) \) for sufficiently small \( h > 0 \), where \( w = \sqrt{1 + g'(b)^2} = \sec \theta \).

By a change of coordinates in the plane \( \mathbb{R}^2 \) given by

\[
\begin{align*}
    u &= x \cos \theta - y \sin \theta + b, \\
    v &= x \sin \theta + y \cos \theta + c,
\end{align*}
\]

the graph \( X : v = g(u), u \in I \) is represented by \( X : y = f(x), x \in J, P \) by the origin and \( V \) by the point \((\alpha h, h)\), where \( \alpha = \tan \theta \).

Since \( f(0) = f'(0) = 0 \), the Taylor’s formula of \( f(x) \) is given by

\[
f(x) = ax^2 + f_3(x),
\]

where \( a = f''(0)/2 \), and \( f_3(x) \) is an \( O(|x|^3) \) function. Since \( \kappa(P) = 2a > 0 \), we see that \( a \) is positive.

For a sufficiently small \( h > 0 \), it follows from the definition of the center \( G = (\bar{x}_P(h), \bar{y}_P(h)) \) of gravity of the section of \( X \) cut off by the parallel line \( l \) through \( P + hN(P) \) to the tangent \( t \) of \( X \) at \( P \) that

\[
\begin{align*}
    \bar{y}_P(h)S_P(h) &= \phi(h), \\
    \bar{x}_P(h)S_P(h) &= \psi(h),
\end{align*}
\]

where we put

\[
\begin{align*}
    \phi(h) &= \frac{1}{2} \int_{f(x)<h} (h^2 - f(x)^2) \, dx, \\
    \psi(h) &= \int_{f(x)<h} \{x(h - f(x))\} \, dx.
\end{align*}
\]

First of all, we get

**Lemma 7.** If we let \( I_P(h) = \{x|f(x) < h\} = (x_1(h), x_2(h)) \), then we have

\[
\phi'(h) = h \{x_2(h) - x_1(h)\} = hL_P(h)
\]
and

\[(3.7) \quad \psi'(h) = \frac{1}{2} \{x_2(h)^2 - x_1(h)^2\}.\]

Proof. If we put \( \bar{f}(x) = f(x)^2 \) and \( k = h^2 \), then we have

\[(3.8) \quad 2\phi(h) = \int_{f(x)^2 < k} \{h^2 - f(x)^2\} dx
= \int_{f(x) < k} \{k - \bar{f}(x)\} dx.\]

We denote by \( \bar{S}_P(k) \) the area of the region bounded by the graph of \( y = \bar{f}(x) \) and the line \( y = k \). Then (3.8) shows that

\[(3.9) \quad 2\phi(h) = \bar{S}_P(k).\]

It follows from (2.2) that

\[(3.10) \quad \frac{d}{dk} \bar{S}_P(k) = \bar{L}_P(k),\]

where \( \bar{L}_P(k) \) denotes the length of the interval \( \bar{I}_P(k) = \{x \in \mathbb{R} | \bar{f}(x) < k\} \).

Since \( k = h^2 \), \( \bar{I}_P(k) \) coincides with the interval \( I_P(h) = \{x \in \mathbb{R} | f(x) < h\} \). Hence we get \( \bar{L}_P(k) = L_P(h) \). This, together with (3.9) and (3.10) shows that

\[(3.11) \quad \phi'(h) = hL_P(h),\]

which completes the proof of (3.6).

Finally, note that \( \psi(h) \) is also given by

\[(3.12) \quad \psi(h) = \frac{1}{2} \int_{y=0}^h \{x_2(y)^2 - x_1(y)^2\} dy,\]

which shows that (3.7) holds.

This completes the proof of Lemma 7. \( \Box \)

We, now, suppose that \( X \) satisfies Condition (D). Then for each sufficiently small \( h > 0 \), \( V = (ah, h) \). Hence, we obtain \( G = \frac{3}{5}(ah, h) \). Therefore we get from (3.3) that

\[(3.13) \quad \frac{3}{5} hS_P(h) = \phi(h)\]

and

\[(3.14) \quad \frac{3}{5} ahS_P(h) = \psi(h).\]
It also follows from (3.13) and (3.14) that
\[(3.15) \quad \psi(h) = \alpha \phi(h).\]

By differentiating (3.13) with respect to \( h \), (3.6) shows that
\[(3.16) \quad S_P(h) = \frac{2}{3} h L_P(h).\]

Differentiating (3.16) with respect to \( h \) yields
\[(3.17) \quad 2h L_P'(h) = L_P(h).\]

Integrating (3.17) shows that
\[(3.18) \quad L_P(h) = C(P) \sqrt{h},\]
where \( C(P) \) is a constant. Hence, it follows from Lemma 5 that
\[(3.19) \quad L_P(h) = \frac{2}{\sqrt{a}} \sqrt{h},\]
from which we get
\[(3.20) \quad x_2(h) - x_1(h) = \frac{2}{\sqrt{a}} \sqrt{h}.\]

Now, differentiating (3.15) and applying Lemma 7 show that
\[(3.21) \quad x_2(h) + x_1(h) = 2 \alpha h.\]

Hence, we get from (3.20) and (3.21) that
\[(3.22) \quad x_1(h) = \alpha h - \frac{1}{\sqrt{a}} \sqrt{h}, \quad x_2(h) = \alpha h + \frac{1}{\sqrt{a}} \sqrt{h}.\]

Since \( I_P(h) = (x_1(h), x_2(h)) \), we obtain from (3.22) that the graph \( X : y = f(x) \) is given by
\[(3.23) \quad f(x) = \begin{cases} \frac{1}{2a} \{2a \alpha x + 1 - \sqrt{4a^2 \alpha x + 1}\}, & \text{if } \alpha \neq 0, \\ ax^2, & \text{if } \alpha = 0. \end{cases}\]

It follows from (3.23) that \( X \) is an open arc of the parabola defined by
\[(3.24) \quad ax^2 - 2a \alpha xy + a \alpha^2 y^2 - y = 0.\]

Note that if \( \alpha \neq 0 \), the function \( f(x) \) in (3.23) is defined on an interval \( J \) such that \( J \subset (-\infty, -1/(4a\alpha)) \) or \( J \subset (-1/(4a\alpha), \infty) \) according to the sign of \( \alpha \).
Finally, we use the following coordinate change from (3.1):

\begin{align}
    x &= u \cos \theta + v \sin \theta - b \cos \theta - c \sin \theta, \\
    y &= -u \sin \theta + v \cos \theta + b \sin \theta - c \cos \theta.
\end{align}

Then, after a long calculation we see that the curve \( X : v = g(u) \) is an open arc of the parabola determined by the following quadratic polynomial

\begin{align}
    g(u) &= \begin{cases} 
        aw^3(u - b)^2 + \alpha(u - b) + c, & \text{if } \alpha \neq 0, \\
        a(u - b)^2 + c, & \text{if } \alpha = 0.
    \end{cases}
\end{align}

Note that \( g(b) = c, g'(b) = \alpha \) and \( \kappa(P) = 2a \). This completes the proof of the if part of Theorem 3.

It is elementary to show the only if part of Theorem 3, or see Chapter 7 of [25], which is originally due to Archimedes. This completes the proof of Theorem 3.

\section{Proof of Theorem 4}

In this section, using Proposition 1, we give the proof of Theorem 4.

Let \( X \) be a strictly locally convex plane curve in the plane \( \mathbb{R}^2 \) with the unit normal \( N \) pointing to the convex side. For a fixed point \( P \) on \( X \) and a sufficiently small \( h > 0 \), we denote by \( l \) the parallel line through \( P + hN(P) \) to the tangent \( t \) of the curve \( X \) at \( P \). We let \( d_P(h) \) the distance from the center \( G \) of gravity of the section of \( X \) cut off by \( l \) to the tangent \( t \) of the curve \( X \) at \( P \).

First, suppose that \( X \) satisfies Condition (E). For a fixed point \( P \in X \), we adopt a coordinate system \((x, y)\) of \( \mathbb{R}^2 \) as in the beginning of Section 2. Then the curve \( X \) is locally the graph of a non-negative strictly convex \( C^3 \) function \( f : \mathbb{R} \rightarrow \mathbb{R} \). Hence, the Taylor’s formula of \( f(x) \) is given by

\begin{align}
    f(x) &= ax^2 + f_3(x),
\end{align}

where \( a = f''(0)/2 \), and \( f_3(x) \) is an \( O(|x|^3) \) function. Since \( \kappa(P) = 2a > 0 \), we see that \( a \) is positive.

It follows from (E) and the definition of \( d_P(h) \) that

\begin{align}
    3 \int_{f(x) < h} (h^2 - f(x)^2) \, dx.
\end{align}

If we differentiate \( \phi(h) \) with respect to \( h \), then Lemma 7 shows that

\begin{align}
    \phi'(h) = hL_P(h).
\end{align}
By differentiating both sides of (4.2) with respect to $h$, we get from (4.4) and (2.2) that

\begin{equation}
S_P(h) = \frac{2}{3} h L_P(h),
\end{equation}

which shows that the curve $X$ satisfies Condition (C) in Proposition 1. Note that the argument in the proof of Proposition 1 given by [18] can be applied even if the curve $X$ is a strictly locally convex plane curve. This completes the proof of the if part of Theorem 4.

For a proof of the only if part of Theorem 4, see Chapter 7 of [25], which is originally due to Archimedes. This completes the proof of Theorem 4.

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