LARGE PRIME GAPS AND PROGRESSIONS WITH FEW PRIMES

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Abstract. We show that the existence of arithmetic progressions with few primes, with a quantitative bound on "few", implies the existence of larger gaps between primes less than $x$ than is currently known unconditionally. In particular, we derive this conclusion if there are certain types of exceptional zeros of Dirichlet $L$-functions.

1. INTRODUCTION

Estimation of the largest gap, $G(x)$, between consecutive primes less than $x$ is a classical problem, and the best bounds on $G(x)$ are comparatively weak. The strongest unconditional lower bound on $G(x)$ is due to Ford, Green, Konyagin, Maynard and Tao [6], who have shown that

$$G(x) \gg \frac{\log x \log_2 x \log_3 x}{\log_4 x},$$

for sufficiently large $x$, with $\log_k x$ the $k$-fold iterated natural logarithm of $x$, whereas the best unconditional upper bound is

$$G(x) \ll x^{0.525},$$

a result due to Baker, Harman and Pintz [1]. Assuming the Riemann Hypothesis, Cramér [3] showed that

$$G(x) \ll x^{1/2} \log x.$$ 

The huge distance between the lower bound (1) and upper bound (2) testifies to our ignorance about gaps between primes. Cramér [4] introduced a probabilistic model for primes and used it to conjecture that $\limsup_{x \to \infty} G(x)/\log^2 x \geq 1$; later, Shanks [16] conjectured that $G(x) \sim \log^2 x$ based on a similar model. Granville [9] modified Cramér's model and, based on analysis of the large gaps in the model, conjectured that $G(x) \geq (1 + o(1))2e^{-\gamma}(\log x)^2$. The author, together with William Banks and Terence Tao [2], has created another model of primes $\leq x$, the largest gap in the model set depending on an extremal property of the interval sieve. In particular, the existence of a certain sequence of "exceptional zeros" of Dirichlet $L$-functions (defined below) implies that the largest gap in the model set grows faster than any constant multiple of $(\log x)^2$, and suggests that the same bound holds for $G(x)$. In this paper, we show that the existence of exceptional zeros of a certain type

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implies a lower bound for \( G(x) \) which is larger than the right side of (1). We do not utilize probabilistic models of primes, but instead we argue directly. More generally, we derive a similar conclusion whenever there are arithmetic progressions containing few primes. We denote \( \pi(x;q,b) \) the number of primes \( p \leq x \) satisfying \( p \equiv b \pmod{q} \). The prime number theorem for arithmetic progressions implies that

\[
\pi(x;q,b) \sim \frac{\pi(x)}{\phi(q)}
\]

for any fixed \( q \), where \( \phi \) is Euler’s totient function and \( \pi(x) \) denotes the number of primes \( p \leq x \). It is a central problem to prove bounds on \( \pi(x;q,b) \) which are uniform in \( q \), but the best known results are only uniform for \( q \leq (\log x)^{O(1)} \); see [5] for the classical theory.

All of the methods used to prove lower bounds on \( G(x) \) utilize a simple connection between \( G(x) \) and Jacobsthal’s function \( J(u) \), the maximum gap between integers having no prime factor \( p \leq u \). A simple argument based on the prime number theorem and the Chinese Remainder Theorem implies that

\[
G(x) \geq J((1/2) \log x)
\]

if \( x \) is sufficiently large. The best bounds known today for \( J(u) \) are

\[
u(\log u) \frac{\log u}{\log_2 u} \ll J(u) \ll u^2,
\]

the lower bound proved in [6] and the upper bound due to Iwaniec [11].

**Theorem 1.1.** Suppose that \( x \) is large, \( x > q > b > 0 \) and \( \pi(x;q,b) \leq \frac{\delta x}{\phi(q)} \) with \( 0 \leq \delta \leq 1 \). Then

\[
G(e^{2u}) \geq J(u) \geq \frac{x - b}{q},
\]

where \( u \) is the smallest integer satisfying \( u > 2\sqrt{x} \) and \( \frac{u}{\log u} \geq \frac{10\delta x}{q} \).

An immediate corollary gives a lower bound on \( G(x) \) assuming a lower bound on \( L(q,b) \), the least prime in the progression \( b \pmod{q} \). We take \( \delta = 0 \) and \( u = \lceil 2\sqrt{x} \rceil \).

**Corollary 1.2.** Suppose that \( L(q,b) > x \). Then \( G(e^{4\sqrt{x}}) \geq \frac{x - b}{q} \).

Theorem [11] is a partial converse to a theorem of Pomerance [15, Theorem 1], which provides a lower bound on \( \max_{(b,q)=1} L(q,b) \) given a lower bound on the maximal gap between numbers coprime to \( m \), where \( (m,q) = 1 \) and \( m \leq q^{1-o(1)} \).

Linnik’s theorem [14] states that \( L(q,b) \ll q^L \) for some constant \( L \); the best quantitative result of this kind is due to Xylouris [17], who showed that the bound holds with \( L = 5.18 \). Assuming the Extended Riemann Hypothesis (ERH) for Dirichlet \( L \)-functions, we obtain a stronger bound \( L(q,b) \ll \epsilon \) \( q^{2+\epsilon} \) for every \( \epsilon > 0 \). If, for some \( c > 2 \) there are infinitely many pairs \( (q,b) \) with \( L(q,b) \geq q^c \) (a violation of ERH), then Corollary 1.2 implies that

\[
\limsup_{X \to \infty} \frac{G(X)}{(\log X)^{2-\frac{2}{\epsilon}}} > 0.
\]
It is, however, conjectured that $L(q, b) \ll \phi(q) \log^2 q$; see [13] for a precise version of this conjecture and for the best known lower bounds on $\max_{(b,q)=1} L(q, b)$.

We may also exceed the bound in (1) under the assumption that exceptional zeros of Dirichlet $L$-functions exist. Roughly speaking, an exceptional zero of $L(s, \chi)$ is a zero which is real and very close to 1. As such, their existence violates ERH for $L(s, \chi)$. Classical results (see [5, §14]) imply that if $c_0 > 0$ is small enough, and $q \geq 3$, then there is at most one character $\chi$ modulo $q$ for which $L(s, \chi)$ has a zero in the region

$$\{\sigma + it \in \mathbb{C} : \sigma \geq 1 - c_0/\log(qt)\},$$

and moreover the character is real and the zero is real. We shall refer to such zeros as “exceptional zeros” with respect to $c_0$. Moreover, by reducing $c_0$ if necessary, it is known that moduli $q$ for which an exceptional zero exists are very rare.

Siegel’s theorem [5, Sec. 21] implies that

$$\log \frac{1}{1 - \beta_q} = o(\log q) \quad (q \to \infty),$$

for (hypothetical) exceptional zeros $\beta_q$, although we cannot say any rate at which this occurs (the bound is ineffective). The exceptional zeros are also know as Siegel zeros or Landau-Siegel zeros in the literature. Their existence implies a great irregularity in the distribution of primes modulo $q$, given by Gallagher’s Prime Number Theorem [8]. Here we record an immediate corollary.

**Proposition 1.3** (Gallagher). For some absolute constant $B > 1$, we have the following. Suppose that $\chi$ is a real character with conductor $q$ and $L(1 - \delta, \chi) = 0$ for some $0 < \delta < 1$. Then, for all $b$ with $\chi_q(b) = 1$ and all $x \geq q^B$, we have

$$\pi(x; q, b) \ll \frac{\delta x}{\phi(q)},$$

One can leverage this irregularity to prove regularity results about primes that are out of reach otherwise, the most spectacular application being Heath-Brown’s [10] deduction of the twin prime conjecture from the existence of exceptional zeros (for an appropriate $c_0$). See Iwaniec’s survey article [12] for background on attempts to prove the non-existence of exceptional zeros and discussion about other applications of their existence. There are also a variety of problems where one argues in different ways depending on whether or not exceptional zeros exist, a principal example being Linnik’s Theorem on primes in arithmetic progressions (see, e.g., [7, Ch. 24]).

Apply Proposition 1.3 with $x = q^B$. Recalling (1) we see that the quantity $u$ in Theorem 1.1 satisfies

$$u \asymp \frac{\delta x \log x}{q}$$

and consequently that $\log u \asymp \log q$. We conclude that
Theorem 1.4. Suppose that \( \chi \) is a real character with conductor \( q \) and that \( L(1 - \delta, \chi) = 0 \) for some \( 0 < \delta < 1 \). Then

\[
G(e^{2u}) \gg \frac{u}{\delta \log u},
\]

for some \( u \) satisfying \( \log u \approx \log q \).

For example, if \( k \) is fixed and there exist infinitely many exceptional zeros \( \delta = \delta_q \) satisfying \( \delta_q \leq (\log q)^{-k} \), we see that there is an unbounded set of \( X \) for which

\[
G(X) \gg_k (\log X)(\log_2 X)^{k-1}.
\]

this improves upon (1) for \( k \geq 2 \). Similarly, if there is an infinite set of \( q \) satisfying \( \delta = \delta_q = q^{-\varepsilon(q)} \), where \( \varepsilon(q) \to 0 \) very slowly, then for an unbounded set of \( X \),

\[
G(X) > X^{1+\delta(X)}
\]

with \( \delta(X) \to 0 \) very slowly.

2. Proof of Theorem 1.1

Let \( u \) be as in the theorem, and let

\[
y = \frac{x - b}{q}.
\]

To show that \( J(u) \geq y \), it suffices to find residue classes \( a_p \mod p \), one for each prime \( p \leq u \), which together cover \([0, y]\). For each prime \( p \leq u/2 \) with \( p \nmid q \), define \( a_p \) by

\[
qa_p + b \equiv 0 \mod p.
\]

Recall that \( (b, q) = 1 \). In this way, if \( 0 \leq n \leq y \) and \( n \not\equiv a_p \mod p \) for all such \( p \), then \( m = qn + b \) has no prime factor \( \leq u/2 \). Also, \( x = qy + b < (u/2)^2 \) by hypothesis, and thus \( m \) is prime. Let

\[
N = \{0 \leq n \leq y : n \not\equiv a_p \mod p, \forall p \leq u/2 \text{ with } p \nmid q\}.
\]

It follows from the hypothesis of the theorem that

\[
|N| \leq \pi(qy + b; q, b) = \pi(x; q, b) \leq \frac{\delta x}{\phi(q)}.
\]

Next, we choose residue classes \( a_p \) for primes \( p|q \) with \( p \leq u/2 \) using a greedy algorithm, successively selecting for each \( p \) a residue class \( a_p \mod p \) which covers at least a proportion \( 1/p \) of the elements remaining uncovered. As \( u > 2\sqrt{x} > 2\sqrt{q} \), there is at most one prime \( p|q \) satisfying \( p > u/2 \). Letting \( N' \) denote the set of \( n \in [0, y] \) not covered by \( \{a_p \mod p : p \leq u/2\} \), we have

\[
|N'| \leq |N| \prod_{p|q, p \leq u/2} \left(1 - \frac{1}{p}\right) \leq 2|N| \frac{\phi(q)}{q} \leq \frac{2\delta x}{q}.
\]
By hypothesis,

$$|\mathcal{N}'| \leq \frac{u}{5 \log u},$$

which, by the prime number theorem, is less than the number of primes in \((u/2, u]\) for \(u\) large enough (as \(u > \sqrt{x}\), this happens if \(x\) is large enough). Thus, we may associate each number \(n \in \mathcal{N}'\) with a distinct prime in \(p_n \in (u/2, u]\). Choosing \(a_{p_n} \equiv n \pmod{p_n}\) for each \(n \in \mathcal{N}'\) then ensures that \(\{a_p \mod p : p \leq u\}\) covers all of \([0, y]\), as desired.

\[\square\]

**Remark.** We have made no use in the proof of the estimates for numbers lacking large prime factors, a common feature in unconditional lower bounds on \(G(x)\). There does not seem to be any advantage to this in our argument.

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