THRESHOLD DYNAMICS OF A TIME PERIODIC AND
TWO–GROUP EPIDEMIC MODEL WITH DISTRIBUTED DELAY

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Abstract. In this paper, a time periodic and two–group reaction–diffusion epidemic model with distributed delay is proposed and investigated. We firstly introduce the basic reproduction number \( R_0 \) for the model via the next generation operator method. We then establish the threshold dynamics of the model in terms of \( R_0 \), that is, the disease is uniformly persistent if \( R_0 > 1 \), while the disease goes to extinction if \( R_0 < 1 \). Finally, we study the global dynamics for the model in a special case when all the coefficients are independent of spatio–temporal variables.

1. Introduction. Mathematical modeling is a basic but efficient tool to study the spread mechanism of diseases, by which the future course of an outbreak can be predicted and then be controlled. In order to establish a theoretical framework for mathematical analysis of transmission of malaria, Ross [44] firstly proposed a system of ordinary differential equations which is the origin of the modern susceptible–infected–recovered (SIR) compartmental model. Since then the SIR compartmental model and many of its extensions, which are independent of the spatial variables, have been well investigated by many scholars [2, 8, 20, 40, 36]. At the same time, the heterogeneity of living environment and mobility of the host individuals play a crucial role in the geographic spread of infectious disease. In fact, there have been many articles which have analyzed mathematically the spatial dynamics of epidemic models, see [3, 16, 42, 45, 43, 46, 61, 59, 64, 65, 66] and the references therein. As reported by [29], multi–group epidemic models have been proposed to describe the spread of various infectious diseases in heterogeneous populations, such as measles, mumps, gonorrhea, and HIV/AIDS. In such models, a heterogeneous host population can be divided into several homogeneous groups according to the modes of transmission, contact patterns, or geographic distributions, so that within–group and inter–group interactions could be modeled separately. The works involved with multi-group models with or without spread diffusion can be found in [7, 13, 14, 15, 17, 21, 23, 32, 34, 50, 60, 70, 68].

Many infectious diseases, such as measles, chicken pox, cholera, influenza, HIV, SARS, etc., exhibit a latent period, namely, the infected individuals do not infect
other susceptible individuals until some time later. Meanwhile, the infected individuals may move from one spatial location to another spatial location with time, which give rise to spatial nonlocal effect. Generally, such nonlocal infection may affect the outbreak and transmission of the diseases (see, e.g. Li and Zou [27], Lou and Zhao [32], Wang and Zhao [58]). Li and Zou [28] proposed a time–delayed SIR epidemic model with nonlocal terms among n–patches in a fixed latent period, where a demographic structure is incorporated by adding recruitment (including births) and natural deaths. They found that nonlocal terms can enhance the basic reproduction number $R_0$, and thus, may leads to an otherwise dying–out disease to persist. When the habitat is a continuous domain, Guo et al. [18] derived a reaction–diffusion epidemic model with time–delay and non–locality in a fixed latent period and investigated the threshold dynamics of the epidemic model by means of the basic reproduction number $R_0$. In addition, there have been other papers studying diffusion–reaction epidemic models with fixed latent period, see [27, 32, 58, 63, 67] and the references therein.

However, it is common that the length of the latent period differs from disease to disease; even for the same disease, the length of the latent period is also different from individuals to individuals. Based on this point, instead of using the discrete (fixed) delay, we employ distributed delay to characterize the variable latency (see, e.g., van den Driessche et al. [53]). The distributed delay allows infectivity to be a function of the duration since infection, up to some maximum duration (see [38]). To characterize the distributed delay, a distribution function $p(u): [0, \infty) \rightarrow [0, \infty)$ which accounts for the variance that the infected individuals become infectious and is assumed to have compact support, $p(u) \geq 0$ and $\int_0^{\infty} p(u)du = 1$ can be used. Epidemic models with distributed delay independent of the spatial variables have been studied, see [6, 10, 22, 29, 49, 56] and the references therein.

It is well known that seasonality can impact host–pathogen interactions, including seasonal changes in host social behaviour and contact rates, variation in encounters with infective stages in the environment, annual pulses of host births and deaths and changes in host immune defences (see [1]). For an infectious disease, it is crucial and more realistic to take into account temporal heterogeneity, which gives rise to non–autonomous evolution equations. Bacaër and Guernaoui [5] defined the basic reproduction number $R_0$ in a periodic environment. For further developments, we refer to Bacaër et al. [4] and Inaba [24] and the references therein. Wang and Zhao [57] developed the basic reproduction number $R_0$ of a large class of compartmental epidemic models in periodic environments and studied the impact of periodic contacts or periodic migrations on the disease transmission by analyzing the global dynamics of a periodic epidemic model with patch structure. Peng and Zhao [41] studied the threshold dynamics of a time–periodic reaction–diffusion SIS model and showed that the persistence of the infectious disease can be enhanced by incorporating the spatial heterogeneity and temporal periodicity into the model. Recently, the theory of the basic reproduction number on the periodic and time–delayed compartmental models is established by Zhao [71] and can be applied to periodic SEIR models with incubation period. Zhang et al. [67] proposed a time–periodic reaction–diffusion epidemic model which incorporates simple demographic structure and a fixed latent period of the infectious disease, introduced the basic reproduction number $R_0$ via a next generation operator, and investigated the threshold dynamics of the epidemic model in terms of $R_0$. Some other studies on the dynamics of time heterogeneous epidemic models can be found in [33, 54, 55, 65, 68]
and the references therein. However, for such non-autonomous (even autonomous) diffusion–reaction epidemic models with distributed delays, much less is done. The purpose of this paper is to incorporate spatial diffusion, distributed latency of the disease and temporal heterogeneity into a multi-group SIR disease model and to investigate the threshold dynamics of the derived model.

The rest of this paper is organized as follows. In the next section, we derive a two–group reaction–diffusion epidemic model with seasonality and distributed delay. In section 3, we introduce the basic reproduction number $R_0$ for the system via the next generation operator method and then establish the threshold dynamics for the system in term of $R_0$, namely, the disease is uniformly persistent if $R_0 > 1$, while the disease goes to extinction if $R_0 < 1$. Section 4 is devoted to the global dynamics for the model in a special case where all the coefficients are independent of spatio–temporal variables.

2. Model formulation. Assume that an infectious disease spreads in two populations or sub-populations living in a bounded domain $\Omega \in \mathbb{R}^n$ with smooth boundary $\partial \Omega$. We always define two populations or sub-populations by the subscript 1 and 2. Without loss of generality, we divide each population/sub-population into four compartments: the susceptible compartment, the latent compartment, the infectious compartment and the removed compartment. Then we denote the densities of four compartments at time $t$ and location $x$ by $S_i(t,x)$, $L_i(t,x)$, $I_i(t,x)$ and $R_i(t,x)$, respectively, where $i = 1, 2$ and $(t,x) \in \mathbb{R}^+ \times \Omega$.

Let $E_1(t,a,x)$ and $E_2(t,a,x)$ be the densities of two exposed populations or sub-populations at time $t \geq 0$, infection age variable $a \geq 0$ and location $x \in \Omega$, respectively. Then $E_i(i = 1, 2)$ satisfy the following model

$$
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) E_i(t,a,x) = D_i \Delta_x E_i(t,a,x) \nonumber \\
- \left( \bar{D}_i(t,a,x) + M_i(t,a,x) + \alpha_i(t,x) \right) E_i(t,a,x), \quad (1)
$$

$t,a > 0, \quad x \in \Omega$

with Neumann boundary condition

$$
\frac{\partial E_i(t,a,x)}{\partial n} = 0, \quad t,a > 0, \quad x \in \partial \Omega, \quad i = 1, 2,
$$

where $n$ is the outward normal to $\partial \Omega$, $D_i$ represents the diffusion rate of the $i$–th population, $\bar{D}_i(t,a,x)$ and $M_i(t,a,x)$ mean the disease–induced mortality rate and the recovery rate of the $i$–th population which are dependent upon the infection age $a$, time $t$ and location $x$, respectively and $\alpha_i(t,x)$ denotes the natural death rate of the $i$–th population at time $t$ and location $x$ for $i = 1, 2$.

Suppose that an infectious disease has a period of latency which is not fixed. Namely, for each population, we assume that infectious individuals must have capable of infecting others after the infection age $\tau_i \in [0, \infty)$. But, between the infection age 0 and $\tau_i$, the infected individual may or may not have an infection ability. Assume that $f_i(r)dr$ denotes the probability of becoming into the individuals who are capable of infecting others between the infection ages $r$ and $r + dr$ and $F_i(a) := \int_0^a f_i(r)dr$ represents the probability of turning into the individuals with infecting others before the infection age $a$ for $i = 1, 2$. Then we have

$$
L_i(t,x) = \int_0^{\tau_i} (1 - F_i(a))E_i(t,a,x)da
$$
and

\[ I_i(t, x) = \int_0^{\tau_i} F_i(a)E_i(t, a, x)da + \int_{\tau_i}^{+\infty} E_i(t, a, x)da, \quad i = 1, 2. \]

It is clear that \( f_i(a) \geq 0 \) for \( a \in (0, \tau_i) \) and \( F_i(a) \equiv 1 \) for \( a \in [\tau_i, \infty) \) for \( i = 1, 2 \).

For the sake of simplicity, we assume that the functions \( D_i(t, a, x) \) and \( M_i(t, a, x) \) are independent of the infection age \( a \), namely,

\[ D_i(t, a, x) = \bar{D}_i(t, x), \quad M_i(t, a, x) = M_i(t, x), \quad \forall t \geq 0, \quad a \in [0, \infty), \quad x \in \Omega, \quad i = 1, 2. \]

For a convenience, we assume

\[ I_{i, 1} := \int_0^{\tau_i} F_i(a)E_i(t, a, x)da \quad \text{and} \quad I_{i, 2} := \int_{\tau_i}^{+\infty} E_i(t, a, x)da. \]

We now aim to find partial differential equations satisfied by \( L_i(t, x) \) and \( I_i(t, x) \).

Integrating (1) with respect to \( a \) and using the expressions of \( L_i(t, x) \) and \( I_i(t, x) \), one has

\[
\frac{\partial L_i}{\partial t} = D_i \Delta L_i(t, x) - \left( \bar{D}_i(t, x) + M_i(t, x) + d_i(t, x) \right) L_i(t, x)
\]

\[
- \int_0^{\tau_i} f_i(a)E_i(t, a, x)da + E_i(t, 0, x),
\]

\[
\frac{\partial I_{i, 1}}{\partial t} = D_i \Delta I_{i, 1}(t, x) - \left( \bar{D}_i(t, x) + M_i(t, x) + d_i(t, x) \right) I_{i, 1}(t, x)
\]

\[
+ \int_0^{\tau_i} f_i(a)E_i(t, a, x)da - E_i(t, \tau_i, x)
\]

and

\[
\frac{\partial I_{i, 2}(t, x)}{\partial t} = D_i \Delta I_{i, 2}(t, x) - \left( \bar{D}_i(t, x) + M_i(t, x) + d_i(t, x) \right) I_{i, 2}(t, x)
\]

\[
+ E_i(t, \tau_i, x) - E_i(t, \infty, x),
\]

where \( i = 1, 2 \). Let \( E_i(t, \infty, x) = 0(i = 1, 2) \), then we can obtain

\[
\frac{\partial I_i(t, x)}{\partial t} = D_i \Delta I_i(t, x) - \left( \bar{D}_i(t, x) + M_i(t, x) + d_i(t, x) \right) I_i(t, x)
\]

\[
+ \int_0^{\tau_i} f_i(a)E_i(t, a, x)da, \quad i = 1, 2.
\]

As the new infection individuals come from the contact of the infectious and susceptible individuals, we adopt the following form:

\[ E_i(t, 0, x) = \beta_{i1}(t, x)g_{i1}(S_i(t, x), I_1(t, x)) + \beta_{i2}(t, x)g_{i2}(S_i(t, x), I_2(t, x)), \quad i = 1, 2, \]

where \( \beta_{ij}(t, x) \geq 0 \) is called the infection rate for \( i, j = 1, 2 \). In this paper, we assume that the contacts between susceptible individuals and infectious individuals are defined by incidence functions \( g_{ij}(u, v)(i, j = 1, 2) \), which satisfy the following conditions:

**\( (H1) \)**: (i) \( g_{ij}(u, v) : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+(i, j = 1, 2) \) are continuously differentiable for all \( u, v \geq 0 \);

(ii) \( g_{ij}(0, 0) = 0 \) and \( g_{ij}(0, v) = 0 \) for all \( u, v \geq 0 \) and \( i, j = 1, 2 \);

(iii) \( \frac{\partial}{\partial u}g_{ij}(u, v) \geq 0 \) and \( \frac{\partial}{\partial v}g_{ij}(u, v) \geq 0 \) for all \( u, v \geq 0 \) and \( i, j = 1, 2 \). In particular, \( \partial_u g_{ij}(u, 0) = 0 \) and \( \partial_v g_{ij}(u, 0) \geq 0 \) for all \( u > 0 \);

(iv) there exist \( \eta_i > 0(i = 1, 2) \) such that \( g_{ij}(u, v) \leq \eta_i u \) for all \( u, v \geq 0 \);
(v) \( N_{ij}(u, v) := \frac{g_{ij}(u, v)}{v}, N_{ij}(u, v) > 0, \frac{\partial}{\partial u} N_{ij}(u, v) \geq 0 \) and \( \frac{\partial}{\partial v} N_{ij}(u, v) \leq 0 \) for all \( u, v > 0 \) and \( i, j = 1, 2 \).

Note that the class of \( g_{ij}(u, v)(i, j = 1, 2) \) satisfying (H1) include many common incidence functions such as
\[
g_{ij}(u, v) = \frac{uv}{u + v}, \quad g_{ij}(u, v) = \frac{uv}{1 + a_{ij}u + b_{ij}v + c_{ij}uv}
\]
and
\[
g_{ij}(u, v) = \frac{uv}{1 + a_{ij}u},
\]
where \( a_{ij}, b_{ij}, c_{ij} > 0 \) for \( i, j = 1, 2 \), see [39].

We use the following simple demographic equation for a population \( Q(t, x) \) that admits a dynamics of global convergence to a positive periodic solution
\[
\frac{\partial Q(t, x)}{\partial t} = D_Q \Delta Q(t, x) + \mu(t, x) - d(t, x)Q(t, x),
\]
where \( \mu(t, x) \) is the recruiting rate, \( D_Q \) is the diffusion rate and \( d(t, x) \) is the natural death rate. We also assume that the disease under consideration does not transmit vertically. On the basis of the above assumptions, the disease dynamics is expressed by the following system

(2)

\[
\begin{align*}
\frac{\partial S_i(t, x)}{\partial t} &= D_S \Delta S_i(t, x) + \mu_i(t, x) - d_i(t, x)S_i(t, x) - \beta_{i1}(t, x)g_{i1}(S_i(t, x), I_1(t, x)) - \beta_{i2}(t, x)g_{i2}(S_i(t, x), I_2(t, x)), \\
\frac{\partial L_i(t, x)}{\partial t} &= D_i \Delta L_i(t, x) - \left( \bar{D}_i(t, x) + M_i(t, x) + d_i(t, x) \right)I_i(t, x) + \int_0^\infty f_i(a)E_i(t, a, x)da, \\
\frac{\partial I_i(t, x)}{\partial t} &= D_i \Delta I_i(t, x) - \left( \bar{D}_i(t, x) + M_i(t, x) + d_i(t, x) \right)I_i(t, x) + \int_0^\infty f_i(a)E_i(t, a, x)da, \\
\frac{\partial R_i(t, x)}{\partial t} &= D_{R_i} \Delta R_i(t, x) + M_i(t, x)L_i(t, x) + M_i(t, x)I_i(t, x) - d_i(t, x)R_i(t, x).
\end{align*}
\]

We make the following basic assumptions:

\textbf{(H2):} \( D_S, \) and \( D_i \) are positive constants for \( i = 1, 2; \) \( \mu_i(t, x) \) and \( \bar{D}_i(t, x) \) are Hölder continuous and nonnegative nontrivial functions on \( \mathbb{R} \times \Omega, \) and periodic in time \( t \) with the same period \( T > 0; \) \( d_i(t, x)(i = 1, 2) \) are Hölder continuous and positive functions on \( \mathbb{R} \times \Omega, \) and periodic in time \( t \) with the same period \( T > 0; \) \( \beta_{ij}(t, x)(i, j = 1, 2) \) are Hölder continuous and nonnegative nontrivial functions on \( \mathbb{R} \times \Omega, \) and periodic in time \( t \) with the same period \( T > 0. \)

The reminder is to derive functions \( E_i(t, a, x)(i = 1, 2) \) by integration along characteristics. For a convenience, let \( r_i(t, \cdot) = \bar{D}_i(t, \cdot) + M_i(t, \cdot) + d_i(t, \cdot). \) For any \( \xi \geq 0, \) we consider the solutions of (1) along the characteristic line \( t = a + \xi \) by letting \( v_i(\xi, a, x) = E_i(a + \xi, a, x)(i = 1, 2). \) Then for \( a \in (0, \tau_i], \) we have

\[
\begin{align*}
\frac{\partial v_i(\xi, a, x)}{\partial a} &= D_i \Delta v_i(\xi, a, x) - r_i(a + \xi, x)v_i(\xi, a, x), \\
v_i(\xi, 0, x) &= \beta_{i1}(\xi, x)g_{i1}(S_i(\xi, x), I_1(\xi, x)) + \beta_{i2}(\xi, x)g_{i2}(S_i(\xi, x), I_2(\xi, x))
\end{align*}
\]
where $i = 1, 2$. For the above system, we can regard $\xi$ as a parameter. Then we have
\[
v_i(\xi, a, x) = \int_\Omega \Gamma_i(\xi + a, \xi, x, y) \left( \beta_{11}(\xi, y) g_{11}(S_i(\xi, y), I_1(\xi, y)) + \beta_{12}(\xi, y) g_{12}(S_i(\xi, y), I_2(\xi, y)) \right) dy,
\]
where $\Gamma_i(t, s, x, y)$ with $t > s$ and $x, y \in \Omega$ is the fundamental solution associated with the partial differential operator $\partial_t - D_i \Delta - r_i(t, \cdot)$ and Neumann boundary condition for $i = 1, 2$. Note that $\Gamma_i(t, s, x, y) = \Gamma_i(t + T, s + T, x, y)$ for all $t > s \geq 0$ and $x, y \in \Omega$ because of $r_i(t + T, x) = r_i(t, x)$ for any $t \geq 0$. Then it follows from $E_i(t, a, x) = v_i(t - a, a, x)$ that
\[
E_i(t, a, x) = \int_\Omega \Gamma_i(t, t - a, x, y) \left( \beta_{11}(t - a, y) g_{11}(S_i(t - a, y), I_1(t - a, y)) + \beta_{12}(t - a, y) g_{12}(S_i(t - a, y), I_2(t - a, y)) \right) dy, \quad i = 1, 2. \tag{3}
\]
Substituting (3) into the second equation and the third equation of (2) respectively, and ignoring the $L_i(t, x)$ and $R_i(t, x)$ equations from (2) because they are decoupled from the $S_i(t, x)$ and $I_i(t, x)$ equations, we obtain the following system:
\[
\begin{cases}
\frac{\partial S_i(t, x)}{\partial t} = D_i \Delta S_i(t, x) + \mu_i(t, x) - d_i(t, x) S_i(t, x) - \beta_{11}(t, x) g_{11}(S_i(t, x), I_1(t, x)) - \beta_{12}(t, x) g_{12}(S_i(t, x), I_2(t, x)), \\
\frac{\partial I_i(t, x)}{\partial t} = D_i I_i(t, x) - r_i(t, x) I_i(t, x) + \int_0^{T_i} f_i(a) \int_\Omega \Gamma_i(t, t - a, x, y) \\
\quad \times \left( \beta_{11}(t - a, y) g_{11}(S_i(t - a, y), I_1(t - a, y)) + \beta_{12}(t - a, y) g_{12}(S_i(t - a, y), I_2(t - a, y)) \right) dy da, \\
\frac{\partial S_i(t, x)}{\partial n} = \frac{\partial}{\partial n} I_i(t, x) = 0, \quad t > 0, \quad x \in \partial \Omega \tag{4}
\end{cases}
\]
for $i = 1, 2$. We assume
\[
\int_0^{T_i} f_i(a) \int_\Omega \Gamma_i(t, t - a, x, y) \beta_{ij}(t - a, y) dy da > 0, \quad \forall (t, x) \in (0, +\infty) \times \Omega \tag{5}
\]
for $i, j = 1, 2$. For simplicity, letting $(u_{S_1}, u_{S_2}, u_1, u_2) = (S_1, S_2, I_1, I_2)$, we focus on the following reaction–diffusion system with Neumann boundary condition:
\[
\begin{cases}
\frac{\partial u_{S_i}(t, x)}{\partial t} = D_i \Delta u_{S_i}(t, x) + \mu_i(t, x) - d_i(t, x) u_{S_i}(t, x) - \beta_{11}(t, x) g_{11}(u_{S_i}(t, x), u_1(t, x)) - \beta_{12}(t, x) g_{12}(u_{S_i}(t, x), u_2(t, x)), \\
\frac{\partial u_i(t, x)}{\partial t} = D_i u_i(t, x) - r_i(t, x) u_i(t, x) + \int_0^{T_i} f_i(a) \int_\Omega \Gamma_i(t, t - a, x, y) \\
\quad \times \left( \beta_{11}(t - a, y) g_{11}(u_{S_i}(t - a, y), u_1(t - a, y)) + \beta_{12}(t - a, y) g_{12}(u_{S_i}(t - a, y), u_2(t - a, y)) \right) dy da, \\
\frac{\partial u_{S_i}(t, x)}{\partial n} = \frac{\partial}{\partial n} u_i(t, x) = 0, \quad t > 0, \quad x \in \partial \Omega \tag{6}
\end{cases}
\]
for $i = 1, 2$.

3. Threshold dynamics. In this section, we explore the threshold dynamics of system (6).
3.1. Global existence of solution. In this subsection, we investigate the existence and uniqueness of time-global solutions of system (6). Set $\tau = \max\{\tau_1, \tau_2\}$. Let $X := C(\Omega, \mathbb{R}^4)$ be the Banach space with the supremum norm $\|x\|$. Let $C_r := C([-\tau, 0], \mathbb{R})$ be the Banach space with the norm $\|\phi\| = \max_{\theta \in [-\tau, 0]} \|\phi(\theta)\|$, $\forall \phi \in C_r$. Define $X^+ := C(\Omega, \mathbb{R}_+^4)$ and $C_r^+ := C([-\tau, 0], X^+)$, then $(X, X^+)$ and $(C_r, C_r^+)$ are strongly ordered spaces. For $\sigma > 0$ and a given function $u_t : [-\tau, \sigma] \to X$, we denote $u_t \in C_r$ by

$$u_t(\theta) = u(t + \theta), \ \theta \in [-\tau, 0].$$

Set $Y := C(\Omega, \mathbb{R})$ and $Y^+ := C(\Omega, \mathbb{R}^+)$.

Furthermore, we consider the following system:

$$\begin{cases}
\frac{\partial w_i}{\partial t} = D_{S_1} \Delta w_i(t, x) - d_i(t, x)w_i(t, x), & t > 0, \ x \in \Omega, \ i = 1, 2, \\
\frac{\partial}{\partial n} w_i(t, x) = 0, & t > 0, \ x \in \partial \Omega, \ i = 1, 2, \\
w_i(0, x) = \phi_i(x), & x \in \Omega, \ \phi_i \in \mathbb{Y}^+, \ i = 1, 2,
\end{cases}$$

(7)

where $D_{S_1} > 0(i = 1, 2)$ and $d_i(t, x)(i = 1, 2)$ are Hölder continuous and nonnegative nontrivial functions on $\mathbb{R} \times \Omega$ and $T$-periodic in $t$. It follows from [19, Chapter II] that (7) admits an evolution operator $V_{S_1}(t, s) : \mathbb{Y}^+ \to \mathbb{Y}^+$ for $s \leq t$ satisfying $V_{S_1}(t, t) = I$, $V_{S_1}(t, s) V_{S_1}(s, \rho) = V_{S_1}(t, \rho)$ for $0 \leq \rho \leq s \leq t$ and $V_{S_1}(t, 0)(\phi_{S_1})(x) = w_i(t, x)(\phi_{S_1})$ for $t \geq 0$, $x \in \Omega$ and $\phi_{S_1} \in \mathbb{Y}^+$, where $w_i(t, x)(\phi_{S_1})$ is a solution of (7) for $i = 1, 2$. Similarly, we take into account the following system:

$$\begin{cases}
\frac{\partial \bar{w}_i}{\partial t} = D_{S_2} \Delta \bar{w}_i(t, x) - r_i(t, x)\bar{w}_i(t, x), & t > 0, \ x \in \Omega, \ i = 1, 2, \\
\frac{\partial}{\partial n} \bar{w}_i(t, x) = 0, & t > 0, \ x \in \partial \Omega, \ i = 1, 2, \\
\bar{w}_i(0, x) = \phi_i(x), & x \in \Omega, \ \phi_i \in \mathbb{Y}^+, \ i = 1, 2,
\end{cases}$$

where $D_{S_2} > 0(i = 1, 2)$ and $r_i(t, x)(i = 1, 2)$ are Hölder continuous and nonnegative nontrivial functions on $\mathbb{R} \times \Omega$ and $T$-periodic in $t$. Let $V_i(t, s)(i = 1, 2)$ be the evolution operators determined by the above system and have the similar properties as $V_{S_1}(t, s)$. Due to the periodicity of coefficients, it follows from [11, Lemma 6.1] that $V_{S_1}(t, s) = V_{S_1}(t + T, s + T)$ and $V_i(t, s) = V_i(t + T, s + T)$ hold for $(t, s) \in \mathbb{R}^2$, $t \geq s$ and $i = 1, 2$. In addition, for any $t, s \in \mathbb{R}$ and $s < t$, $V_{S_1}(t, s)$ and $V_i(t, s)$ are compact, analytic and strongly positive operators on $\mathbb{Y}^+$ for $i = 1, 2$. Together [11, Theorem 6.6] with $\alpha = 0$, we get that there exist constants $Q \geq 1$ and $c_0 \in \mathbb{R}$ such that

$$\|V_{S_1}(t, s)\|, \|V_i(t, s)\| \leq Q e^{-c_0(t-s)}, \ \forall t \geq s, \ t, s \in \mathbb{R}, \ i = 1, 2.$$
and \( U(t,s) : \mathbb{X} \to \mathbb{X} \) be an evolution operator for \((t,s) \in \mathbb{R}^2 \) with \( t \geq s \). Let
\[
A(t) := \begin{pmatrix} A_{S_1}(t) & 0 & 0 & 0 \\ 0 & A_{S_2}(t) & 0 & 0 \\ 0 & 0 & A_1(t) & 0 \\ 0 & 0 & 0 & A_2(t) \end{pmatrix},
\]
where \( A_{S_i}(t) \) and \( A_i(t) \) \((i = 1, 2)\) are defined by
\[
D(A_{S_i}(t)) = \{ \phi \in C^2(\overline{\Omega}) \mid \partial_{n}\phi = 0 \text{ on } \partial\Omega \},
\]
\[
A_{S_i}(t)\phi(x) = D_s\Delta \phi(x) - d_i(t,x)\phi(x), \forall \phi \in D(A_{S_i}(t))
\]
and
\[
D(A_i(t)) = \{ \phi \in C^2(\overline{\Omega}) \mid \partial_{n}\phi = 0 \text{ on } \partial\Omega \},
\]
\[
A_i(t)\phi(x) = D_t\Delta \phi(x) - r_i(t,x)\phi(x), \forall \phi \in D(A_i(t)),
\]
respectively. Then (6) can be written as the following Cauchy problem:
\[
\begin{aligned}
& \frac{\partial u(t,x)}{\partial t} = A(t)u(t,x) + F(t,u_t), \ t > 0, \ x \in \Omega, \\
& u(\zeta, x) = \phi(\zeta, x), \ \zeta \in [-\tau, 0], \ x \in \Omega,
\end{aligned}
\tag{8}
\]
where \( u(t,x) := (u_{S_1}(t,x), u_{S_2}(t,x), u_1(t,x), u_2(t,x)) \). Moreover, it can be rewritten as the following integral equation
\[
u(t, \phi) = U(t, 0)\phi(0) + \int_0^t U(t, s)F(s, u_s)ds, \ t \geq 0, \ \phi \in \mathbb{C}_+^\tau.
\tag{9}
\]
A solution of (9) is called a mild solution of (8).

**Lemma 3.1.** For every initial value function \( \phi \in \mathbb{C}_+^\tau \), system (6) has a unique mild solution \( u(t, \phi) \) on \([0, +\infty)\) with \( u_0 = \phi \). Furthermore, system (6) generates a \( T \)-periodic semiflow \( \Phi_T(\cdot) := u_t(\cdot) : \mathbb{C}_+^\tau \to \mathbb{C}_+^\tau \), namely, \( \Phi_T(\phi)(s, x) = u_t(\phi)(s, x) = u(t + s, x; \phi) \) for each \( \phi \in \mathbb{C}_+^\tau \), \( t \geq 0, s \in [-\tau, 0], x \in \Omega \) and \( \Phi_T : \mathbb{C}_+^\tau \to \mathbb{C}_+^\tau \) has a global compact attractor in \( \mathbb{C}_+^\tau \).

**Proof.** We firstly show the local existence of the unique mild solution. It is obvious that \( F(t, \phi) \) is locally Lipschitz continuous. By Martin and Smith [37, Corollary 3] and Smith [47, Theorem 7.3.1], it is necessary to prove
\[
\lim_{k \to 0^+} \text{dist}(\phi(0) + kF(t, \phi), \mathbb{X}^+) = 0, \ \forall (t, \phi) \in [0, \infty) \times \mathbb{C}_+^\tau.
\tag{10}
\]
For any \( t \geq 0, x \in \Omega, \phi \in \mathbb{C}_+^\tau \) and \( k \geq 0 \), we have
\[
\phi(0, x) + kF(t, \phi)(x)
\]
\[
= \begin{pmatrix}
\phi_{S_1}(0, x) + k \left( \mu_1(t, x) - \left( \sum_{i=1}^{2} \beta_{1i}(t,x)g_{1i}(\phi_{S_1}, \phi_i)(0,x) \right) \right) \\
\phi_{S_2}(0, x) + k \left( \mu_2(t, x) - \left( \sum_{i=1}^{2} \beta_{2i}(t,x)g_{2i}(\phi_{S_2}, \phi_i)(0,x) \right) \right) \\
\phi_1(0, x) + k \left( f_1 \circ (\Gamma_1 \ast \left( \beta_{11}g_{11} + \beta_{12}g_{12} \right)) \right)(t,x) \\
\phi_2(0, x) + k \left( f_2 \circ (\Gamma_2 \ast \left( \beta_{21}g_{21} + \beta_{22}g_{22} \right)) \right)(t,x)
\end{pmatrix}
\geq \begin{pmatrix}
\phi_{S_1}(0, x) \left( 1 - k \sum_{i=1}^{2} \beta_{1i}(t,x) \frac{g_{1i}(\phi_{S_1}, \phi_i)(0,x)}{\phi_{S_1}(0,x)} \right) \\
\phi_{S_2}(0, x) \left( 1 - k \sum_{i=1}^{2} \beta_{2i}(t,x) \frac{g_{2i}(\phi_{S_2}, \phi_i)(0,x)}{\phi_{S_2}(0,x)} \right) \\
\phi_1(0, x) \left( 1 - k \sum_{i=1}^{2} \beta_{1i}(t,x) \frac{g_{1i}(\phi_1, \phi_i)(0,x)}{\phi_1(0,x)} \right) \\
\phi_2(0, x) \left( 1 - k \sum_{i=1}^{2} \beta_{2i}(t,x) \frac{g_{2i}(\phi_2, \phi_i)(0,x)}{\phi_2(0,x)} \right)
\end{pmatrix},
\]
where
\[
\left(f_i \circ (\Gamma_i \ast (\beta_{i1} g_{i1} + \beta_{i2} g_{i2})))\right)(t, x) = \int_0^T f_i(a) \int_{x(t, t-a, x, y)} \Gamma_i(t, t-a, x, y) \left(\beta_{i1}(t-a, y) g_{i1}(\phi_{S_i}(-a, y), \phi_1(-a, y)) + \beta_{i2}(t-a, y) g_{i2}(\phi_{S_i}(-a, y), \phi_2(-a, y))\right) d\gamma(a), \quad i = 1, 2
\]
and \(g_{i2}(\phi_{S_i}, \phi_2)(0, x) = g_{i2}(\phi_{S_i}(0, x), \phi_2(0, x))\). The above inequality implies that (10) holds when \(k\) is small enough. Consequently, by [37, Corollary 4] with \(K = \mathbb{X}^+\) and \(S(t, s) = U(t, s)\), system (6) admits a unique mild solution \(u(t, x; \phi)\) with \(u_0(\cdot; \phi) = \phi\) on its maximal interval of existence \(t \in [0, t_\phi]\), where \(t_\phi < \infty\) and \(u(t, \cdot; \phi) \in \mathbb{X}^+, \forall t \in [0, t_\phi)\). Furthermore, \(u(t, x; \phi)\) is a classic solution for \(t > \tau\) by using the analytic of \(U(t, s)\) for any \(s, t \in \mathbb{R}\) with \(s < t\).

Consider the following time–periodic reaction–diffusion equation:
\[
\begin{align*}
\frac{\partial \omega_i(t, x)}{\partial t} &= D_i \Delta \omega_i(t, x) + \mu_i(t, x) - d_i(t, x) \omega_i(t, x), \quad t > 0, \quad x \in \Omega, \\
\frac{\partial \omega_i(t, x)}{\partial n} &= 0, \quad t > 0, \quad x \in \partial \Omega,
\end{align*}
\]
where \(i = 1, 2\). It follows from [67, Lemma 2.1] that system (11) admits a unique positive \(T\)-periodic solution \(\omega_i^*(t, x)\) which is globally asymptotically stable in \(\mathbb{Y}^+\) for \(i = 1, 2\). Since the \(u_S\) \((i = 1, 2)\) equations of system (6) are dominated by (11), respectively, there exists a positive constant \(B_S\) such that for any \(\phi \in \mathbb{C}^+_\mathbb{X}\), there is a positive integer \(l_\phi = l_\phi(\phi) > 0\) such that \(u_S(t, x; \phi) \leq B_S\) for any \(t \geq l_\phi T, x \in \Omega\) and \(i = 1, 2\).

In view of (iv) of (H1), we have for \(t > 0\) and \(x \in \Omega\),
\[
\frac{\partial u_i(t, x)}{\partial t} \leq D_i \Delta u_i(t, x) - r_i(t, x) u_i(t, x) + \eta_i \int_0^T f_i(a) \int_{x(t, t-a, x, y)} \Gamma_i(t, t-a, x, y) u_S(t-a, y) \left(\beta_{i1}(t-a, y) + \beta_{i2}(t-a, y)\right) dy \, da
\]
with Neumann boundary condition
\[
\frac{\partial}{\partial n} u_i(t, x) = 0, \quad \forall x \in \partial \Omega.
\]
It follows from the comparison principle that there exists a constant \(\bar{B} > 0\) such that for any \(\phi \in \mathbb{C}^+_\mathbb{X}\), there is a positive integer \(l_\phi > l_\phi\) such that \(u_i(t, x; \phi) \leq \bar{B}\) for any \(t \geq l_\phi T, x \in \Omega\) and \(i = 1, 2\).

Define \(\Phi_t : \mathbb{C}^+_\mathbb{X} \to \mathbb{C}^+_\mathbb{X}\) by \(\Phi_t(\phi)(s, x) = u_t(\phi)(s, x) = u(t + s, x; \phi)\) for \(t > 0, s \in [-\tau, 0], x \in \Omega\) and \(\phi \in \mathbb{C}^+_\mathbb{X}\). Similar to the proof of [67, Lemma 2.1], we get that \(\{\Phi_t\}_{t \geq 0}\) is a \(T\)-periodic semiflow on \(\mathbb{C}^+_\mathbb{X}\). From the above discussion, we have that \(\Phi_t\) is point dissipative. Let \(n_0 := \min\{n \in \mathbb{N} : nT > 2\tau\}\). Then by the standard parabolic estimates, we conclude that \(\Phi_T^{n_0} = u_{n_0 T}\) is compact. Following from [35, Theorem 2.9], one has that \(\Phi_T : \mathbb{C}^+_\mathbb{X} \to \mathbb{C}^+_\mathbb{X}\) has a global compact attractor. The proof is completed.

### 3.2. Basic reproduction number.
Let \(C_T(\mathbb{R} \times \bar{\Omega}, \mathbb{R})\) be the ordered Banach space consisting of all \(\mathbb{T}\)-periodic and continuous functions from \(\mathbb{R} \times \bar{\Omega}\) to \(\mathbb{R}\), where \(|\phi|_{C_T} = \max_{t \in [0, T], x \in \bar{\Omega}} |\phi(t, x)|\) for any \(\phi \in C_T\). Denote \(C_T^+\) as the positive cone of \(C_T\), that is,
\[
C_T^+ := \{\phi \in C_T : \phi(t, x) \geq 0, \forall t \in \mathbb{R}, x \in \bar{\Omega}\}.
\]
Let $C_T(\mathbb{R} \times \bar{\Omega} , \mathbb{R} \times \mathbb{R}) = C_T(\mathbb{R} \times \bar{\Omega} , \mathbb{R}) \times C_T(\mathbb{R} \times \bar{\Omega} , \mathbb{R})$ with the norm $\| \phi \|_{C_T} = \sum_{i=1}^{2} \| \phi_i \|_{C_T}$ for any $\phi \in C_T$. Similarly, we define $C_T^+$ as the positive cone of $C_T$, namely,

$$C_T^+ := \{ \phi = (\phi_1, \phi_2) \in C_T : \phi_i(t, x) \geq 0, \forall t \in \mathbb{R}, x \in \bar{\Omega}, i = 1, 2 \}.$$

For $\tau \geq 0$, define $\mathbb{D} = C([-\tau, 0], \mathbb{Y} \times \mathbb{Y})$ with the norm $\| \phi \|_{\mathbb{D}} = \max_{\theta \in [-\tau, 0]} \| \phi(\theta) \|_{\mathbb{Y} \times \mathbb{Y}}$ and $\mathbb{D}^+ := C([-\tau, 0], \mathbb{Y}^+ \times \mathbb{Y}^+)$, then $(\mathbb{D}, \mathbb{D}^+)$ is a strongly ordered Banach space.

Setting $u_1 \equiv 0$ and $u_2 \equiv 0$, we have the following equations for the densities of the susceptible population $u_{S_i}(t, x)(i = 1, 2)$

$$\begin{aligned}
\frac{\partial u_{S_i}(t, x)}{\partial t} &= D_i \Delta u_{S_i}(t, x) + \mu_i(t, x) - d_i(t, x)u_{S_i}(t, x), \quad t > 0, \quad x \in \Omega, \quad i = 1, 2, \\
\frac{\partial u_{S_i}(t, x)}{\partial n} &= 0, \quad t > 0, \quad x \in \partial \Omega, \quad i = 1, 2,
\end{aligned}$$

respectively. It follows from [67, Lemma 2.1] that (12) admit positive solutions $u_{S_i}^*(i = 1, 2)$ which are unique, globally asymptotically stable and $T$-periodic in $t \in \mathbb{R}$, respectively. As a consequence, the function $(u_{S_1}^*, u_{S_2}^*, 0, 0)$ is called the disease–free periodic solution of (6). Linearizing the third and the forth equations of system (6) at $(u_{S_1}^*, u_{S_2}^*, 0, 0)$ and according to (iii) of (H1), we have the following system:

$$\begin{aligned}
\frac{\partial \omega_1(t, x)}{\partial t} &= D_1 \Delta \omega_1(t, x) - r_1(t, x)\omega_1(t, x) + \int_{0}^{\tau} f_1(a) \int_{\Omega} \Gamma_1(t, t - a, x, y) \\
&\quad \times \left( \beta_{11}(t - a, y) \partial_\nu g_{11}(u_{S_1}^*(t - a, y), 0) \omega_1(t - a, y) + \beta_{12}(t - a, y) \right) dy da, \quad t > 0, \quad x \in \Omega, \\
\frac{\partial \omega_2(t, x)}{\partial t} &= D_2 \Delta \omega_2(t, x) - r_2(t, x)\omega_2(t, x) + \int_{0}^{\tau} f_2(a) \int_{\Omega} \Gamma_2(t, t - a, x, y) \\
&\quad \times \left( \beta_{21}(t - a, y) \partial_\nu g_{21}(u_{S_2}^*(t - a, y), 0) \omega_1(t - a, y) + \beta_{22}(t - a, y) \right) dy da, \quad t > 0, \quad x \in \Omega,
\end{aligned}$$

$$\begin{aligned}
\omega_1(s, x) &= \phi_1(s, x), \quad i = 1, 2, \quad \phi = (\phi_1, \phi_2) \in \mathbb{D}, \quad s \in [-\tau, 0], \quad x \in \Omega, \\
\frac{\partial \omega_i(t, x)}{\partial n} &= 0, \quad t > 0, \quad x \in \partial \Omega, \quad i = 1, 2.
\end{aligned}$$

(13)

Define operators $C_{ij} : C_T(\mathbb{R} \times \bar{\Omega} , \mathbb{R}) \to C_T(\mathbb{R} \times \bar{\Omega} , \mathbb{R})(i, j = 1, 2)$ by

$$\begin{aligned}
(C_{ij}\psi_j)(t, x) &= \int_{0}^{\tau} f_i(a) \int_{\Omega} \Gamma_i(t, t - a, x, y) \beta_{ij}(t - a, y) \\
&\quad \times \partial_\nu g_{ij}(u_{S_i}(t - a, y), 0) \psi_j(t - a, y) dy da.
\end{aligned}$$

Suppose that $\phi(s, x) := (\phi_1(s, x), \phi_2(s, x))$ is the initial distribution of infectious individuals at time $s \in \mathbb{R}$ and the spatial location $x \in \bar{\Omega}$. Given $t \in \mathbb{R}$. Due to the synthetical influence of mobility, mortality and recovery, $\left( V_i(t - a, s) \phi_i(s) \right) (x)$, where $s < t - a$ represents the density distribution at location $x$ of those infective individuals who were infected at time $s$ and remained infective at time $t - a$ when time evolved from $s$ to $t - a$ for $a \in [0, \tau]$. Furthermore, $\int_{-\infty}^{t-a} (V_i(t - a, s) \phi_i(s)) (x) ds$ denotes the density distribution of the accumulative infective individuals of the $i$-th group at locations $x$ and time $t - a$ for all previous time $s < t - a$ when time evolved from the previous time $s$ to $t - a$. After that, the term

$$\begin{aligned}
\int_{0}^{\tau} f_i(a) \int_{\Omega} \Gamma_i(t, t - a, x, y) \left\{ \beta_{i1}(t - a, y) \partial_\nu g_{i1}(u_{S_i}^*(t - a, y), 0) \\
&\quad + \beta_{i2}(t - a, y) \partial_\nu g_{i2}(u_{S_i}^*(t - a, y), 0) \right\} dy da.
\end{aligned}$$
\[ L x \text{ represents the distribution of new infected individuals of the } i \text{-th group at location } x \text{ and time } t \text{ for } i = 1, 2. \] As a consequence, we can define the next generation infection operator \( \mathcal{L} \) as
\[ \mathcal{L}(\phi)(t, x) := (\mathcal{L}_1(\phi)(t, x), \mathcal{L}_2(\phi)(t, x)), \]
where
\[ \mathcal{L}_i(\phi)(t, x) = \int_0^\infty f_i(a) \int_{\Omega} \Gamma_i(t, t-a, x, y) \left\{ \sum_{j=1}^2 \beta_{ij}(t-a, y) \partial_s g_{ij}(u_{S_i}^*(t-a, y), 0) \right\} \times \int_0^{+\infty} (V_j(t-a, t-a-s)\phi_j(t-a-s))(y)ds \right\} dyda \]
for \( i = 1, 2. \) It is obvious that \( \mathcal{L} \) is a positive and bounded linear operator on \( C_T. \)
Let \( r(\mathcal{L}) \) be the spectral radius of \( \mathcal{L}. \) Similar to [5, 12, 57, 71, 67], denote the spectral radius of \( \mathcal{L} \) as the basic reproduction number \( R_0 \) of model (6), that is,
\[ R_0 := r(\mathcal{L}). \]
Next, we define an operator \( \hat{\mathcal{L}}(\phi)(t, x) : C_T \to C_T \) by
\[ \hat{\mathcal{L}}(\phi)(t, x) := (\hat{\mathcal{L}}_1(\phi)(t, x), \hat{\mathcal{L}}_2(\phi)(t, x)), \]
where
\[ \hat{\mathcal{L}}_i(\phi)(t, x) = \int_0^\infty \sum_{j=1}^2 \left( V_i(t, t-s) (C_{ij} \phi_j)(t-s) \right)(x)ds, \text{ } t \in \mathbb{R}, \text{ } s \geq 0, \text{ } i = 1, 2. \]
Clearly, \( \hat{\mathcal{L}} \) is a compact, positive and bounded linear operator on \( C_T. \) Let
\[ \mathcal{A} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}, \quad \tilde{V}(t, s) = \begin{pmatrix} V_1(t, s) & 0 \\ 0 & V_2(t, s) \end{pmatrix} \]
and
\[ A(\phi)(t, x) = \mathcal{A}(\phi)(t, x), \quad B(\phi)(t, x) = \int_0^\infty (\tilde{V}(t, t-s)\phi(t-s))(x)ds. \]
Then one has \( \mathcal{L} = AB \) and \( \hat{\mathcal{L}} = BA. \) It follows that \( R_0 = r(\mathcal{L}) = r(\hat{\mathcal{L}}), \) where \( r(\hat{\mathcal{L}}) \) is the spectral radius of the operator \( \hat{\mathcal{L}}. \)
As the previous discussion, there exist constants \( \mathcal{Q} > 1 \) and \( c_i \in \mathbb{R} \) such that
\[ \|V_i(t, s)\| \leq \mathcal{Q} e^{c_i(t-s)}, \forall t \geq s, \text{ } t, s \in \mathbb{R}, \text{ } i = 1, 2. \]
It follows that \( c_i^* := \bar{\omega}(V_i) \leq c_i, \) where
\[ \bar{\omega}(V_i) = \inf\{\omega \mid \exists M \geq 1, \forall s \in \mathbb{R}, \text{ } t \geq 0 : \|V_i(t + s, s)\| \leq Me^{\omega s}\} \]
is the exponent growth bound of the evolution operator $V_i(t,s)$. Define $c^* = \max\{c_1^*, r_2^*\}$. \(r(V_i(T,0))\) is defined as the spectral radius of $V_i(T,0)$ for $i = 1, 2$. In addition, $V_i(t,0)$ is compact and strongly positive on $\mathbb{Y}$ for any $t > 0$ and $i = 1, 2$. By the Krein–Rutman theorem [19, Theorem 7.2], we have $r(V_i(T,0)) > 0$ for $i = 1, 2$. It further follows from [19, Lemma 14.2] that $r(V_i(T,0)) < 1$ for $i = 1, 2$. According to [51, Proposition 5.6] with $s = 0$, one has $c_1^* < 0$ for $i = 1, 2$.

For any given $\lambda \in (c^*, \infty)$, we introduce an operator $\hat{L}_\lambda$ on $\mathbb{C}T$

$$
\hat{L}_\lambda(\phi)(t,x) := \int_0^\infty e^{-\lambda s} \left( \hat{V}(t,t-s)(\mathcal{L}(\phi))(t-s) \right)(x)ds.
$$

Clearly, $\hat{L}_0 = \hat{L}$. It follows that the operator $\hat{L}_\lambda$ is bounded for $\lambda \in (c^*, \infty)$. Moreover, the compactness of $V_i(t,s)(i = 1, 2)$, $t > s$, implies that $\hat{L}_\lambda$ is compact. Denote $\rho(\lambda)$ as the spectral radius of $\hat{L}_\lambda$ for $\lambda \in (c^*, \infty)$. It is easy to see that $R_0 = r(\mathcal{L}) = r(\hat{L}) = \rho(0)$. Similar to the arguments in [4, Lemma 1] and [67, Lemma 3.2], we can show the following properties of the function $\rho(\lambda)$.

**Lemma 3.2.** For $\lambda \in (c^*, \infty)$, the following statements are true for $\rho(\lambda)$

(i): $\rho(\lambda)$ is continuous and non-increasing;

(ii): $\rho(\infty) = 0$;

(iii): $\rho(\lambda) = 1$ has at most one solution; $\rho$ is either strictly decreasing in $\lambda \in (c^*, \infty)$, or strictly decreasing in $\lambda \in (c^*, b)$ for some $b > c^*$, and $\rho(\lambda) = 0$ in $\lambda \in [b, \infty)$.

Let $\epsilon$ be a positive parameter. Consider the following periodic time-delayed nonlinear equations:

$$
\begin{align*}
\frac{\partial \omega_i^*(t,x)}{\partial t} & = D_1 \Delta \omega_i^*(t,x) - r_1(t,x)\omega_i^*(t,x) + \int_0^{\tau_1} f_1(a) \int_0^{\tau_1} \Gamma_1(t,t-a,x,y) \\
& \quad \times \left\{ \left[ \beta_{11}(t-a,y) + \epsilon \right] \partial_6 g_{11}(u_{S_1}^*(t-a,y),0) \omega_i^*(t-a,y) \\
& \quad + \left[ \beta_{12}(t-a,y) + \epsilon \right] \partial_6 g_{12}(u_{S_1}^*(t-a,y),0) \omega_i^*(t-a,y) \right\}dyda, \\
& \quad t > 0, \quad x \in \Omega, \\
\frac{\partial \omega_i^*(t,x)}{\partial t} & = D_2 \Delta \omega_i^*(t,x) - r_2(t,x)\omega_i^*(t,x) + \int_0^{\tau_2} f_2(a) \int_0^{\tau_2} \Gamma_2(t,t-a,x,y) \\
& \quad \times \left\{ \left[ \beta_{21}(t-a,y) + \epsilon \right] \partial_6 g_{21}(u_{S_2}^*(t-a,y),0) \omega_i^*(t-a,y) \\
& \quad + \left[ \beta_{22}(t-a,y) + \epsilon \right] \partial_6 g_{22}(u_{S_2}^*(t-a,y),0) \omega_i^*(t-a,y) \right\}dyda, \\
& \quad t > 0, \quad x \in \Omega, \\
\omega_i^*(s,x) & = \psi_i(s,x), \quad \psi_i(\psi_1, \psi_2) \in \mathbb{D}, \quad s \in [-\tau, 0], \quad x \in \Omega, \quad i = 1, 2,
\end{align*}
$$

Define the Poincaré map of (14) $\mathcal{P}^\epsilon : \mathbb{D} \rightarrow \mathbb{D}$ by $\mathcal{P}^\epsilon(\psi) = \omega_1^*(\psi)$ for all $\psi \in \mathbb{D}$, where

$$
\omega_1^*(\psi)(s,x) = \omega^*(s+T, x; \psi) = (\omega_1^*(s+T, x; \psi), \omega_2^*(s+T, x; \psi))
$$

for all $(s,x) \in [-\tau, 0] \times \Omega$ (\(\tau := \max\{\tau_1, \tau_2\}\), and $\omega_t^*$ is the solution map of (14). Let $n_0 := \min\{n \in \mathbb{N} : nT > 2\tau\}$. $\mathcal{P}^\epsilon(n_0) : \mathbb{D} \rightarrow \mathbb{D}$ is denoted by $\mathcal{P}^\epsilon(n_0) = \omega^*(n_0T + s, x; \phi)$ for all $(s,x) \in [-\tau, 0] \times \Omega$. Define $\tau_0^*$ as the spectral radius of $\mathcal{P}^\epsilon$. Without loss of generality, we replace $\mathcal{P}$ and $r_0^*$ with $\mathcal{P}^0$ and $r_0^0$, respectively. It follows from [25, Section 3] (see also [47, Section 5.3]) that $\omega^*(t,x; \phi) > 0$ for $t > \tau$, $x \in \Omega$, $\phi \in \mathbb{D}^+$ with $\phi \neq 0$, and $\omega_t^*(\cdot, \phi)$ is strongly positive for $t > 2\tau$. Moreover, $\omega_t^*$ is compact on $\mathbb{D}^+$ for all $t > 2\tau$. Hence, $(\mathcal{P}^\epsilon)^{n_0} = \omega_{n_0T}^*(\cdot)$ is compact and
strongly positive. By [30, Lemma 3.1], \( r^*_0 \) is a simple eigenvalue of \( P^r \) having a strong positive eigenvector \( \psi \in D^+ \) and the modulus of any other eigenvalue is less than \( r^*_0 \). Assume that \( \omega^r(t, x; \psi) \) is the solution of system (14) with \( \omega^r(s, x; \psi) = \psi(s, x) \) for all \( s \in [-\tau, 0] \), \( x \in \Omega \). We can conclude from the strong positivity of \( \psi \) that \( \omega^r(\cdot, \cdot; \psi) \geq 0 \). Let \( \mu^r = \frac{\ln r^*_0}{T} \) and \( \nu^r(t, x) = e^{-\mu^r t} \omega^r(t, x; \psi) \) for all \( t > \tau \) and \( x \in \Omega \).

By arguments similar to those in [25, Lemma 3.2] and [62, Theorem 2.1], we have \( \omega^r \). Assume that \( t > \tau \) and \( x \in \Omega \).

By virtue of [51, Proposition A.2], we have \( \omega^r \). Assume that \( t > \tau \) and \( x \in \Omega \).

Proof. If \( r^*_0 > r(V_i(T, 0)) \) for \( i = 1, 2 \), then \( \rho(\mu) = 1 \).

\( \mu = \frac{\ln r^*_0}{T} \). If \( r^*_0 > r(V_i(T, 0)) \) for \( i = 1, 2 \), then \( \rho(\mu) = 1 \).

Let \( \epsilon_n = \frac{1}{n} \) for \( n \geq 1 \) such that \( r^*_0 > r(V_i(T, 0)) \). We define

\[
(C_{ij}^n \psi_j)(t, x) := \int_{\Omega} f_i(a) \int_{\Omega} \Gamma_i(t, t-a, x, y) \left( \beta_{ij}(t-a, y) + c_n \right) \\
\times \partial_t g_{ij}(u^*_x(t-a, y), 0) \psi_j(t-a, y) dy dy,
\]

\[
\hat{L}^\epsilon_n(\psi)(t, x) = (\hat{L}_1^\epsilon_n(\psi)(t, x), \hat{L}_2^\epsilon_n(\psi)(t, x)),
\]

\[
\hat{L}_i^\epsilon_n(\psi)(t, x) = \int_{0}^{\infty} e^{-\lambda s} \left\{ V_i(t, t-s) \left( (C_{i1}^n \phi_1)(t-s) + (C_{i2}^n \phi_2)(t-s) \right) \right\}(x) ds,
\]

for \( i = 1, 2 \).

According to Lemma 3.3, there is a positive periodic function \( \nu^\epsilon_n(t, x) \) such that \( \omega^\epsilon_n(t, x) = e^{\mu^\epsilon_n t} \nu^\epsilon_n(t, x) \) is a solution of (14). That is, it satisfies for \( t \geq s \) and \( s \in \mathbb{R} \),

\[
\omega_1^\epsilon_n(t, \cdot) = V_1(t, s) \omega_1^\epsilon_n(s) + \int_{s}^{t} V_1(t, \eta) \left( (C_{11}^\epsilon_n \omega_1^\epsilon_n)(\eta) + (C_{12}^\epsilon_n \omega_2^\epsilon_n)(\eta) \right) d\eta,
\]

\[
\omega_2^\epsilon_n(t, \cdot) = V_2(t, s) \omega_2^\epsilon_n(s) + \int_{s}^{t} V_2(t, \eta) \left( (C_{21}^\epsilon_n \omega_1^\epsilon_n)(\eta) + (C_{22}^\epsilon_n \omega_2^\epsilon_n)(\eta) \right) d\eta,
\]

which implies that
\[
\begin{align*}
\begin{cases}
  e^{t^n - t}V^n_1(t, \cdot) = V_1(t, s) \left( e^{t^n - s}V^n_1(s) \right) + \int_s^t e^{t^n - \tau}V_1(t, \eta) \left( (C_{11} \nu_1^n) \right) (\eta) \\
  + (C_{12} \nu_2^n) (\eta) \rangle d\eta,
  \\
  e^{t^n - t}V^n_2(t, \cdot) = V_2(t, s) \left( e^{t^n - s}V^n_2(s) \right) + \int_s^t e^{t^n - \tau}V_2(t, \eta) \left( (C_{21} \nu_1^n) \right) (\eta) \\
  + (C_{22} \nu_2^n) (\eta) \rangle d\eta,
\end{cases}
\end{align*}
\]

where \( t \geq s \) and \( s \in \mathbb{R} \). Since \( r^n \rightarrow r(V(t, 0)) \), then we have \( \mu^n \coloneqq \frac{\ln r^n}{n} > c_i^* \) and \( [V_1(t, s)(e^{t^n - s}V^n_1(s))] \rightarrow 0 \) as \( s \rightarrow -\infty \). Letting \( s \rightarrow -\infty \) in the first equation of (15), we get

\[
V^n_1(t, \cdot) = \int_{-\infty}^t e^{-\mu^n(t-\eta)}V_1(t, \eta) \left( (C_{11} \nu_1^n) \right) (\eta) \rangle d\eta
= \int_0^t e^{-\mu^n(t-s)}V_1(t, t-s) \left( (C_{11} \nu_1^n) \right) (t-s) \rangle ds
\]

Similarly, one has

\[
V^n_2(t, \cdot) = \int_0^{+\infty} e^{-\mu^n(s-t)}V_2(t, s) \left( (C_{21} \nu_1^n) \right) (s-t) \rangle ds,
\]

which implies that \( \hat{L}_{\mu^n}(V^n)(t) = V^n(t, \cdot) \). Denote \( \rho^n(\lambda) \) as the spectral radius of \( \hat{L}_{\lambda} \) for \( \lambda \in (c^*, \infty) \). Since \( \beta_2(t, x) + c_0 > 0 \), it follows that \( \hat{L}_{\lambda} : C_T \rightarrow C_T \) is continuous, compact and strongly positive, and hence, the Krein–Rutmann theorem associated with the strongly positivity of \( V^n \) imply that \( \rho^n(\mu^n) = 1 \). It is easy to see that \( \hat{L}_{\lambda}^n \psi \geq \hat{L}_{\lambda}^{n+1} \psi \) for all \( \psi \in C_T \). Let \( f_n(\lambda) = \rho^n(\lambda) \). It then follows from [9, Theorem 1.1] that the sequence \( \{f_n\}_{n \geq 1} \) is non–increasing. Similarly, according to the upper semi–continuity of the spectral [26, Section IV 3.1] and the continuity of a finite system of eigenvalues [26, Section IV.3.5], one has \( \lim_{n \rightarrow \infty} \rho^n(\lambda) = \rho(\lambda) \) for any fixed \( \lambda \in [a, b] \subset (c^*, \infty) \). Hence, Dini’s theorem implies that \( \lim_{n \rightarrow \infty} \rho^n(\lambda) = \rho(\lambda) \) uniformly for \( \lambda \in [a, b] \). Choose a sufficient small \( \delta > 0 \) such that \( \mu - \delta > c^* \).

By the above analysis, it follows that there exists a constant \( N_1 = N_1(\delta) \geq 1 \) such that for any \( n \geq N_1 \),

\[
\mu - \delta \leq \mu^n \leq \mu + \delta.
\]

On the one hand, we obtain from the continuity of \( \rho(\lambda) \) for \( \lambda \in (c^*, \infty) \) that for any \( \eta > 0 \), there exists an \( N_2 \in \mathbb{N} \) such that,

\[
|\rho(\mu^n) - \rho(\mu)| < \frac{\eta}{2}.
\]

On the other hand, for any \( \eta > 0 \), there is an \( N_3 \geq 1 \) such that for \( n \geq N_3 \),

\[
|\rho^n(\mu^n) - \rho(\mu^n)| < \frac{\eta}{2}.
\]

Thus, for any \( \eta > 0 \), we have

\[
|\rho^n(\mu^n) - \rho(\mu)| \leq |\rho^n(\mu^n) - \rho(\mu^n)| + |\rho(\mu^n) - \rho(\mu)| < \frac{\eta}{2} + \frac{\eta}{2} = \eta,
\]

when \( n \geq N \coloneqq \max\{N_1, N_2, N_3\} \). Letting \( n \rightarrow \infty \), we have \( \rho^n(\mu^n) \rightarrow \rho(\mu) \). Therefore, \( \rho(\mu) = 1 \). This completes the proof.

Next, we state the main result of this subsection.

**Theorem 3.5.** one has:

(i): \( R_0 > 1 \) if and only if \( r_0 > 1 \);
(ii): $R_0 = 1$ if and only if $r_0 = 1$;
(iii): $R_0 < 1$ if and only if $r_0 < 1$.

Proof. (i) Assume $R_0 > 1$. In this case one has $\rho(0) > 1$. In view of Lemma 3.2, there exists a constant $\lambda_0 > 0$ such that $\rho(\lambda_0) = 1$. Since $\hat{L}_{\lambda_0}$ is compact and positive on $C_T$, it follows from the Krein–Rutmann theorem [19, Theorem 7.1] that $\rho(\lambda_0)$ is an eigenvalue of $\hat{L}_{\lambda_0}$ with a positive eigenfunction $\psi^* \in C_T$, that is, $\hat{L}_{\lambda_0}\psi^* = \psi^*$. Since $V_i(t, s) = V_i(t, r)V_i(r, s)$ for $t \geq r \geq s$, we have

\[
\hat{L}_{1\lambda_0}(\psi^*)(t, x)
= \int_0^\infty e^{-\lambda_0 s} \left\{ V_1(t, t-s) \left( (C_{11}\psi^*_1)(t-s) + (C_{12}\psi^*_2)(t-s) \right) \right\} (x) ds
= \int_{-\infty}^t e^{-\lambda_0(t-s)} \left\{ V_1(t, s) \left( (C_{11}\psi^*_1)(s) + (C_{12}\psi^*_2)(s) \right) \right\} (x) ds
= \int_{-\infty}^m e^{-\lambda_0(t-s)} \left( V_1(t, s) \left( (C_{11}\psi^*_1)(s) + (C_{12}\psi^*_2)(s) \right) \right) (x) ds
+ \int_m^t e^{-\lambda_0(t-s)} \left( V_1(t, s) \left( (C_{11}\psi^*_1)(s) + (C_{12}\psi^*_2)(s) \right) \right) (x) ds
= e^{-\lambda_0(t-m)} V_1(t, m) \int_{-\infty}^m e^{-\lambda_0(m-s)} \left( V_1(m, s) \sum_{i=1}^2 (C_{1i}\psi^*_i)(s) \right) (x) ds
+ e^{-\lambda_0(t-m)} \left( V_1(t, m) \psi^*_1(m) \right) (x)
+ e^{-\lambda_0 t} \int_m^t V_1(t, s) \sum_{j=1}^2 C_{1j} \left( e^{\lambda_0 s} \psi^*_j(s) \right) (x) ds,
\]

namely,

\[
e^{\lambda_0 t} \psi^*_1(t, x)
= (V_1(t, m) \left( e^{\lambda_0 m} \psi^*_1(m) \right))(x) + \int_m^t (V_1(t, s) \sum_{j=1}^2 C_{1j} \left( e^{\lambda_0 s} \psi^*_j \right)(s) )(x) ds.
\]

Similarly,

\[
e^{\lambda_0 t} \psi^*_2(t, x)
= (V_2(t, m) \left( e^{\lambda_0 m} \psi^*_2(m) \right))(x) + \int_m^t (V_2(t, s) \sum_{j=1}^2 C_{2j} \left( e^{\lambda_0 s} \psi^*_j \right)(s) )(x) ds.
\]

Set $\psi^*_i(\theta, x) = \psi^*_i(t + \theta, x)$, $\forall \theta \in [-\tau, 0]$. It is obvious that

\[
\omega(t, x) := (e^{\lambda_0 t} \psi^*_1(t, x), e^{\lambda_0 t} \psi^*_2(t, x))
\]

is a solution of (13) with $\omega_0 := e^{\lambda_0 \tau} \psi^*$. Note that

\[
\omega_i(\theta, x) = (e^{\lambda_0(t+\theta)} \psi^*_1(t + \theta, x), e^{\lambda_0(t+\theta)} \psi^*_2(t + \theta, x))
= e^{\lambda_0 t} \left( e^{\lambda_0 \theta} \psi^*_1(t, x), e^{\lambda_0 \theta} \psi^*_2(t, x) \right)
\]
Theorem 3.7. Let $u(t, x; \phi)$ be the solution of (6) with $u_0 = \phi \in \mathbb{C}_T^+$, then the following two statements are valid:

(i): If $R_0 < 1$, then the disease free $T$-periodic solution $(u_{S_1}^*, u_{S_2}^*, 0, 0)$ is globally attractive in $\mathbb{C}_T^+$.

(ii): If $R_0 > 1$, then there exists an $\eta > 0$ such that for any $\phi \in \mathbb{C}_T^+$ with $\phi_1(0, \cdot) \not\equiv 0$ or $\phi_2(0, \cdot) \not\equiv 0$, one has

$$\lim_{t \to \infty} \inf_{x} u_i(t, x) \geq \eta, \quad i = 1, 2$$

uniformly for all $x \in \bar{\Omega}$.
Proof. (i) Assume that $R_0 < 1$. It follows from Theorem 3.5 that $r_0 < 1$. Consider the following system with parameter $\epsilon > 0$:

$\frac{\partial u^1(t,x)}{\partial t} = D_1 \Delta u^1(t,x) - r_1(t,x)u^1(t,x) + \int_0^{\tau_1} f_1(a) \int_\Omega \Gamma_1(t,t-a,x,y) \times \left\{ \beta_{11}(t-a,y) + \epsilon \right\} \partial_v g_{11}(u_{s_1}^*(t-a,y) + \epsilon,0) \omega^1(t-a,y) dyda,$

$\frac{\partial u^2(t,x)}{\partial t} = D_2 \Delta u^2(t,x) - r_2(t,x)u^2(t,x) + \int_0^{\tau_2} f_2(a) \int_\Omega \Gamma_2(t,t-a,x,y) \times \left\{ \beta_{21}(t-a,y) + \epsilon \right\} \partial_v g_{21}(u_{s_2}^*(t-a,y) + \epsilon,0) \omega^2(t-a,y) dyda,$

$t \geq kT, \ x \in \Omega,$

where $k$ is an integer determined later. Define the Poincare map of (16) $T^\epsilon : \mathbb{D} \rightarrow \mathbb{D}$ by

$T^\epsilon(\phi) = \mathcal{P}_T^\epsilon(\phi), \ \forall \phi \in \mathbb{D},$

where

$\mathcal{P}_T^\epsilon(s,x) = \mathcal{P}(s + T, x; \phi), \ \forall (s,x) \in [-\tau,0] \times \bar{\Omega}$

and $\mathcal{P}(t,x; \phi)$ is the solution of (16) with $\mathcal{P}(s,x) = \phi(s,x)$ for all $s \in [-\tau,0], x \in \Omega$. Let $\mathfrak{m}^\epsilon$ be the spectral radius of $T^\epsilon$. Since $r_0 < 1$, then there exists a positive constant $\epsilon_0$ such that $\mathfrak{m}^\epsilon < 1$ for any $\epsilon \in [0, \epsilon_0]$. Fix $\epsilon \in [0, \epsilon_0]$. Then, one has $\mathfrak{m}^\epsilon := \frac{\ln \mathfrak{m}^\epsilon}{T} < 0$. By Lemma 3.3, there is a positive $T$–periodic function $(\mathcal{V}_1^\epsilon(t,x), \mathcal{V}_2^\epsilon(t,x))$ such that $(\mathcal{V}_1^\epsilon(t,x), \mathcal{V}_2^\epsilon(t,x)) = (e^{\mathfrak{m}^\epsilon} \mathcal{V}_1^\epsilon(t,x), e^{\mathfrak{m}^\epsilon} \mathcal{V}_2^\epsilon(t,x))$ is a solution of (16).

Since the $u_{s_i}(i = 1, 2)$ equations of (6) are dominated by (11), respectively, we obtain that there exists an integer $k > 0$ such that $u_{s_i}(t,x) \leq u_{s_i}^*(t,x) + \epsilon$ for any $t \geq kT, \ x \in \bar{\Omega}$ and $i = 1, 2$. According to (v) of (H1), for all $t \geq kT$ and $x \in \Omega$, we have

$\frac{\partial u_1(t,x)}{\partial t} \leq D_1 \Delta u_1(t,x) - r_1(t,x)u_1(t,x) + \int_0^{\tau_1} f_1(a) \int_\Omega \Gamma_1(t,t-a,x,y) \times \left\{ \beta_{11}(t-a,y) + \epsilon \right\} \partial_v g_{11}(u_{s_1}^*(t-a,y) + \epsilon,0) u_1(t-a,y) dyda,$

$\frac{\partial u_2(t,x)}{\partial t} \leq D_2 \Delta u_2(t,x) - r_2(t,x)u_2(t,x) + \int_0^{\tau_2} f_2(a) \int_\Omega \Gamma_2(t,t-a,x,y) \times \left\{ \beta_{21}(t-a,y) + \epsilon \right\} \partial_v g_{21}(u_{s_2}^*(t-a,y) + \epsilon,0) u_2(t-a,y) dyda,$

For any given $\phi \in \mathbb{C}_1^\epsilon$, since $u_i(t,x; \phi)(i = 1, 2)$ are globally bounded, there exists some $\alpha > 0$ such that $(u_1(t,x; \phi), u_2(t,x; \phi)) \leq \alpha(e^{\mathfrak{m}^\epsilon} \mathcal{V}_1^\epsilon(t,x), e^{\mathfrak{m}^\epsilon} \mathcal{V}_2^\epsilon(t,x))$ for any $t \in [kT, kT + \tau]$ and $x \in \bar{\Omega}$. By the similar arguments in [25, Section 2] and using the comparison theorem for the abstract functional differential equation [37, Proposition 3], one has $(u_1(t,x; \phi), u_2(t,x; \phi)) \leq \alpha(e^{\mathfrak{m}^\epsilon} \mathcal{V}_1^\epsilon(t,x), e^{\mathfrak{m}^\epsilon} \mathcal{V}_2^\epsilon(t,x))$ for any $t \geq kT$ and $x \in \bar{\Omega}$. It then follows from $\mathfrak{m}^\epsilon < 0$ that $u_i(t,x; \phi) \rightarrow 0$ as $t \rightarrow \infty$ uniformly $x \in \Omega$. In addition, the equations $u_{s_i}(i = 1, 2)$ in system (6) are asymptotic to system (11). By [67, Lemma 2.1], we get that $u_{s_i}^*(i = 1, 2)$ are global attractive solutions of (11).
Next, similar to the proof of [67, Theorem 4.3 (i)], we get
\[ \lim_{t \to \infty} \left( u_{S_i}(t, x; \phi) - u_{S_i}^*(t, x) \right) = 0 \]
uniformly for \( x \in \bar{\Omega} \).

(2) Assume \( R_0 > 1 \). In the case, one has \( r_0 > 1 \). Let
\[ \mathcal{W}_0 = \{ \phi \in C_+^1 : \phi_1(0, \cdot) \neq 0 \text{ or } \phi_2(0, \cdot) \neq 0 \} \]
and
\[ \partial \mathcal{W}_0 := C_+^1 \setminus \mathcal{W}_0 = \{ \phi \in C_+^1 : \phi_1(0, \cdot) \equiv 0 \text{ and } \phi_2(0, \cdot) \equiv 0 \} \].
If \( \phi \in \mathcal{W}_0 \) with \( \phi_1(0, \cdot) \neq 0 \), then Lemma 3.6 implies that \( u_1(t, x; \phi) > 0 \) for any \( x \in \bar{\Omega} \) and \( t > 0 \). Thus, by (5), one gets \( \int_0^{\tau_2} f_2(a) \int_{\Omega} \Gamma_2(t, a, x, y) \beta_{21}(t-a, y)g_2(u_{S_i}, u_1)(t-a, y)dyda > 0 \) for \( t > \tau_2 \), which yields \( u_2(t, x; \phi) > 0 \) for \( t > \tau_2 \) and \( x \in \bar{\Omega} \). Similarly, if \( \phi \in \mathcal{W}_0 \) with \( \phi_2 \neq 0 \), then one has \( u_i(t, x; \phi) > 0 \) for \( i = 1, 2 \) for any \( t > \tau_i \) and \( x \in \bar{\Omega} \). Thus, there exists \( k_0 \in \mathbb{N} \) such that \( \Phi^k_T(\mathcal{W}_0) \subseteq \mathcal{W}_0 \) for each \( k > k_0 \), where \( \Phi^k_T(\phi) = u_{kT}(s, x; \phi) \) for \( s \in [-\tau_0, 0] \times \bar{\Omega} \) and \( \phi \in \mathcal{W}_0 \) and \( u(t, x; \phi) \) is a solution of system (6) for \( t > 0 \), \( x \in \bar{\Omega} \) and \( \phi \in \mathcal{W}_0 \).

Define
\[ M_0 := \{ \phi \in \partial \mathcal{W}_0 : \Phi^k_T(\phi) \in \partial \mathcal{W}_0, \forall k \in \mathbb{N} \} \].

Let \( \omega(\phi) \) be the omega limit set of the orbit \( \gamma^+ := \{ \Phi^k_T(\phi) : \forall k \in \mathbb{N} \} \) and \( M := (u_{S_i}^{0,0}, u_{S_i}^{0,0}, 0, \bigemptyset) \), where \( 0 \) is the constant function and identical to zero. For any given \( \phi \in M_0 \), we have \( \Phi^k_T(\phi) \in \partial \mathcal{W}_0 \), \( \forall k \in \mathbb{N} \). Thus, one has \( u_i(t, x; \phi) \equiv 0 \) for \( t > 0 \), \( x \in \bar{\Omega} \), \( \phi \in M_0 \) and \( i = 1, 2 \). It follows from [67, Lemma 2.1] that
\[ \lim_{t \to \infty} \left( u_{S_i}(t, x; \phi) - u_{S_i}^*(t, x) \right) = 0 \]
uniformly for \( x \in \bar{\Omega} \) and \( i = 1, 2 \). Consequently, we have \( \omega(\phi) = \{ M \}, \forall \phi \in M_0 \).

Next, we consider the following linear system with parameter \( \theta > 0 \):

\[
\begin{cases}
\frac{\partial v_1^0(t,x)}{\partial t} = D_1 \Delta v_1^0(t,x) - r_1(t,x)v_1^0(t,x) + \int_0^{\tau_1} f_1(a) \int_{\Omega} \Gamma_1(t, a, x, y) \\
\quad \times \left( \beta_{11}(t-a, y)N_{11}^0(u_{S_i}^0(t-a, y) - \theta, \theta)v_1^0(t-a, y) \\
\quad + \beta_{12}(t-a, y)N_{12}^0(u_{S_i}^0(t-a, y) - \theta, \theta)v_2^0(t-a, y) \right)dyda,
\end{cases}
\]
\[ t > 0, \ x \in \bar{\Omega}, \]
\[
\begin{cases}
\frac{\partial v_2^0(t,x)}{\partial t} = D_2 \Delta v_2^0(t,x) - \int_0^{\tau_2} f_2(a) \int_{\Omega} \Gamma_2(t, a, x, y) \\
\quad \times \left( \beta_{21}(t-a, y)N_{21}^0(u_{S_i}^0(t-a, y) - \theta, \theta)v_1^0(t-a, y) \\
\quad + \beta_{22}(t-a, y)N_{22}^0(u_{S_i}^0(t-a, y) - \theta, \theta)v_2^0(t-a, y) \right)dyda,
\end{cases}
\]
\[ t > 0, \ x \in \Omega, \]
\[
\begin{cases}
\frac{\partial v_i^0(t,x)}{\partial t} = \frac{\partial v_i^0(t,x)}{\partial \Omega} = 0, \ t > 0, \ x \in \partial \Omega, \\
v_i^0(s,x) = \phi_i(s,x), \ s \in [-\tau_0, 0], \ x \in \bar{\Omega}, \ \phi = (\phi_1, \phi_2) \in D, \ i = 1, 2.
\end{cases}
\]

Let \( n_0 := \min \{ n \in \mathbb{N} : nT > 2\tau \} \). Define the Poincaré map of (17) \( \mathcal{E}_\theta^{n_0} : D \to D \) by \( \mathcal{E}_\theta^{n_0}(\phi) = v_1^{n_0}(\phi), \mathcal{E}_\theta^{n_0}(\phi) = (v_1^{n_0}, v_2^{n_0}) \), where \( v_i^{n_0, n_0 T}(\phi)(s,x) = v_i^0(s+n_0 T, x; \phi) \)
for \( (s,x) \in [-\tau_0, 0] \times \bar{\Omega} \), and \( v_i^0(t,x; \phi) \) is the solution of (17) with \( v_i^0(s,x) = \phi(s,x) = (\phi_1(s,x), \phi_2(s,x)) \) for all \( (s,x) \in [-\tau_0, 0] \times \bar{\Omega} \). Let \( r_\theta^{n_0} \) be the spectral radius of \( \mathcal{E}_\theta^{n_0} \). It is obvious that \( \mathcal{E}_\theta^{n_0} \) is compact and positive. The Krein–Rutman theorem [19, Theorem 7.1] implies that there exist a positive eigenvalue \( r_\theta^{n_0} > 0 \) and a positive eigenfunction \( \tilde{\phi} = (\tilde{\phi}_1, \tilde{\phi}_2) \in D \) such that \( \mathcal{E}_\theta^{n_0}(\tilde{\phi}) = r_\theta^{n_0} \tilde{\phi} \). Denote \( r_0^{n_0} \) as the spectral radius of \( \Phi_T^{n_0} \) and \( \Phi_T^{n_0} \) is defined as in Lemma 3.1. Since \( r_0 > 1 \) which
implies that $r_0^{n_0} > 1$, there exists a parameter $\theta_0 > 0$ small enough such that $r_\theta^{n_0} > 1$ for $\theta \in [0, \theta_0)$. Fix $\theta \in [0, \theta_0)$.

**Claim.** $M$ is a uniform weak repeller for $W_0$ in the sense that

$$\limsup_{k \to \infty} \|\Phi_k^T(\phi) - M\| \geq \theta, \forall \phi \in W_0.$$ 

Applying a contradiction way, suppose that $\limsup_{k \to \infty} \|\Phi_k^T(\phi) - M\| < \theta$ for some $\phi_0 \in W_0$. Namely, there exists $k_1 > 0$ large enough such that $0 < u_1(t, x; \phi_0) < \theta$ and $0 < u_2(t, x; \phi_0) < \theta$, $u_{S_1}(t, x; \phi_0) > u_{S_1}^*(t, x; \phi_0) - \theta$ and $u_{S_2}(t, x; \phi_0) > u_{S_2}^*(t, x; \phi_0) - \theta$ for any $t \geq k_1T$ and $x \in \Omega$. Furthermore, we select $K = \max\{n_0, k_1\}$. According to (v) of (HI), for any $t \geq KT$ and $x \in \Omega$, $u_1(t, x; \phi_0)$ and $u_2(t, x; \phi_0)$ satisfy

$$\begin{cases}
\frac{\partial}{\partial t} u_1(t, x) \geq D_1 \Delta u_1(t, x) - r_1(t, x) u_1(t, x) + \int_0^{r_1} f_1(a) \int_{\Omega} \Gamma_1(t, t-a, x, y) \\
\times (\beta_{11}(t-a, y) N_{11}(u_{S_1}^*(t-a, y) - \theta) u_1(t-a, y) \\
+ \beta_{12}(t-a, y) N_{12}(u_{S_1}^*(t-a, y) - \theta) u_2(t-a, y)) dy \, da,
\end{cases}
\frac{\partial}{\partial t} u_2(t, x) \geq D_2 \Delta u_2(t, x) - r_2(t, x) u_2(t, x) + \int_0^{r_2} f_2(a) \int_{\Omega} \Gamma_2(t, t-a, x, y) \\
\times (\beta_{21}(t-a, y) N_{21}(u_{S_2}^*(t-a, y) - \theta) u_1(t-a, y) \\
+ \beta_{22}(t-a, y) N_{22}(u_{S_2}^*(t-a, y) - \theta) u_2(t-a, y)) dy \, da.\tag{18}
$$

Since $u_i(t, x; \phi_0) > 0(i = 1, 2)$ for $t > \tau$ and $x \in \bar{\Omega}$, there exists a constant $\kappa > 0$ such that $u_i((K + 1)T + s, x; \phi_0) \geq \kappa \hat{\varphi}_i(s, x), \ s \in [-\tau, 0], \ x \in \bar{\Omega}, \ i = 1, 2.$

Due to (17), (18) and the comparison principle, there exists $\kappa > 0$ such that

$$(u_1(t, x; \phi_0), u_2(t, x; \phi_0))^T \\
\geq \kappa (v_1^0(t - (K + 1)T, x; \hat{\varphi}), \ v_2^0(t - (K + 1)T, x; \hat{\varphi})), \ \forall t \geq (K + 1)T, \ x \in \bar{\Omega}.\tag{19}
$$

Therefore, one has

$$u_i(KT, x; \phi_0) \geq \kappa v_i^0((K - K - 1)T, x; \hat{\varphi}) = \kappa (r_0^{m_i})^m \hat{\varphi}_i(s, x),$$

where we select $K = (K + 1) + mn_0$ and $i = 1, 2$. Since $\hat{\varphi}_i(s, x)(i = 1, 2)$ are positive, there exist $s_i \in [-\tau, 0]$ and $x_i \in \bar{\Omega}$ such that $\hat{\varphi}_i(s, x_i) > 0(i = 1, 2)$. Thus, it follows from (19) that $u_i(KT, x; \phi_0) \to +\infty$ as $K \to \infty$ (namely, $m \to \infty$) which contradicts the boundedness of $u_i(t, x; \phi)(i = 1, 2)$. The claim is proved.

It follows from the above claim that $M$ is an isolated invariant set for $\Phi_T$ in $W_0$, and $W^s(M) \cap W_0 = \emptyset$, where $W^s(M)$ is the stable set of $M$. According to the acyclicity theorem on uniform persistence for maps (see [71, Theorem 1.3.1 and Remark 1.3.1]), one has that $\Phi_T : C_+^+ \to C_+^+$ is uniformly persistence with respect to $(W_0, \partial W_0)$, namely, there exists a $\delta > 0$ such that

$$\liminf_{k \to \infty} d(\Phi_k^T, \partial W_0) \geq \delta, \forall \phi \in W_0.\tag{17}
$$

It then follows from [71, Theorem 3.1.1] that the periodic semiflow $\Phi : C_+^+ \to C_+^+$ is also uniformly persistent with respect to $(W_0, \partial W_0)$. It is easy to see that $\Phi_T^{n_0}$ is compact and point dissipative on $W_0$. Therefore, according to [35, Theorem 2.9], one obtains that $\Phi_T^{n_0} : W_0 \to W_0$ has a global attractor $Z_0$. 

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In the following, we further prove the persistence stated in (ii). Define a continuous function \( p : \mathbb{C}^+ \to \mathbb{R}_+ \) by (similar to [31, Theorem 4.1])
\[
p(\phi) = \min\{\min_{x \in \Omega} \phi_1(0, x), \min_{x \in \Omega} \phi_2(0, x)\}, \quad \forall \phi \in \mathbb{C}^+.
\]
Since \( Z_0 = \Phi_T^n(Z_0) \), we have that
\[
\phi_i(0, \cdot) > 0, \quad \phi \in Z_0, \quad i = 1, 2.
\]
Let \( B_0 := \bigcup_{t \in [0, n_0 T]} \Phi_t(Z_0) \). It then follows that \( B_0 \subset \mathcal{W}_0 \) and
\[
\lim_{t \to \infty} d(\Phi_t(\phi), B_0) = 0
\]
for all \( \phi \in \mathcal{W}_0 \). Since \( B_0 \) is a compact subset of \( \mathcal{W}_0 \), we have \( \min_{\phi \in \mathcal{B}_0} p(\phi) > 0 \). Thus, by Lemma 3.6, there exists a \( \delta^* > 0 \) such that \( \liminf_{t \to \infty} u_i(t, \cdot, \phi) \geq \delta^* (i = 1, 2) \). Furthermore, there exists \( 0 < \delta < \delta^* \) such that
\[
\liminf_{t \to \infty} u_i(t, \cdot, \phi) \geq \delta, \quad \phi \in \mathcal{W}_0, \quad i = 1, 2.
\]
The proof is completed. \( \square \)

4. A special case. In this section, we investigate the special case where all the coefficients in (4) are independent of the time variable \( t \) and the spatial variable \( x \). That is,
\[
\mu_i(t, x) \equiv \mu_i, \quad d_i(t, x) \equiv d_i, \quad \beta_{ij}(t, x) \equiv \beta_{ij}, \quad r_i(t, x) \equiv r_i, \quad \Gamma_i(t, t-a, x, y) \equiv \Gamma_i(a, x, y), \quad t > 0, \quad x \in \Omega, \quad i, j = 1, 2.
\]
In addition, \( g_{ij}(u, v) \equiv p_i(u)q_j(v) \) and \( p_i(u) \) and \( q_i(v) \) satisfy
\begin{itemize}
\item[(A1)] (i): \( p_i(u), q_i(v) : \mathbb{R}^+ \to \mathbb{R}^+ (i, j = 1, 2) \) can be continuously differentiable for all \( u, v \geq 0 \);
\item[(ii)] \( q_i(0) = 0 \) and \( p_i(0) = 0 \) for \( i, j = 1, 2 \). Furthermore, \( p_i(u) > 0 \) for \( u > 0 \);
\item[(iii)] \( p_i'(u) \geq 0 \) and \( q_i'(v) \geq 0 \) for all \( u, v \geq 0 \) and \( i, j = 1, 2 \). In particular, \( q_i'(0) > 0 \) for \( i = 1, 2 \);
\item[(iv)] there exist \( \eta_i > 0 (i = 1, 2) \) such that \( p_i(u)q_i(v) \leq \eta_i u v \) for all \( u, v \geq 0 \).
\end{itemize}

\[
\mathcal{N}_j(v) := \frac{q_j(v)}{v} > 0, \quad \mathcal{N}_j'(v) \leq 0 \quad \text{for all} \quad v > 0 \quad \text{and} \quad j = 1, 2.
\]
In short, we consider the following spatio-temporally homogeneous reaction–diffusion epidemic model with Neumann boundary condition:
\[
\begin{aligned}
\frac{\partial S_i(t, x)}{\partial t} &= D_i \Delta S_i(t, x) + \mu_i - d_i S_i(t, x) - \beta_{i1} p_i(S_i(t, x)) q_1(I_1(t, x)) - \beta_{i2} p_i(S_i(t, x)) q_2(I_2(t, x)), \quad t > 0, \quad x \in \Omega, \\
\frac{\partial I_i(t, x)}{\partial t} &= D_i \Delta I_i(t, x) - r_i I_i(t, x) + \int_0^\tau f_i(a) \int_\Omega \Gamma_i(a, x, y) \\
&\quad \times \left( \beta_{i1} p_i(S_i(t-a, y)) q_1(I_1(t-a, y)) + \beta_{i2} p_i(S_i(t-a, y)) q_2(I_2(t-a, y)) \right) dyda, \quad t > 0, \quad x \in \Omega, \\
\frac{\partial S_i(t, x)}{\partial n} &= \frac{\partial I_i(t, x)}{\partial n} = 0, \quad t > 0, \quad x \in \partial \Omega.
\end{aligned}
\]
By a straightforward computation, one has
\[
S_i^0(t, x) \equiv \frac{S_i(x)}{d_i}, \quad t \geq 0, \quad x \in \Omega.
\]
Next, we give the explicit expression with the basic reproduction number \( R_0 \). Let \( \Phi = (\phi_1, \phi_2)^T \) be the initial distribution of infective individuals such that
\[ \int_{\Omega} \phi_i(x) dx = 1 \text{ for } i = 1, 2. \] Then \( T_i(t) \phi_i \) represents the solution of the following system
\[
\begin{cases}
\frac{\partial u_i(t, x)}{\partial t} = D_i \Delta u_i(t, x) - r_i u_i(t, x), & t > 0, \ x \in \Omega, \\
\frac{\partial u_i(t, x)}{\partial \eta} = 0, & t > 0, \ x \in \partial \Omega, \\
u_i(0, x) = \phi_i(x), & x \in \Omega
\end{cases}
\]
for \( i = 1, 2 \). Thus, \( T_i(t) \phi_i \) can be given by
\[
(T_i(t) \phi_i)(x) = \int_{\Omega} \tilde{\Gamma}_i(t, x, z) \phi_i(z) dz,
\]
where \( \tilde{\Gamma}_i(t, x, z) \) for \( t > 0 \) and \( x, z \in \Omega \) is the fundamental solution associated with the partial differential operator \( \partial_t - D_i \Delta - r_i \) and Neumann boundary condition for \( i = 1, 2 \) and \( \int_{\Omega} \tilde{\Gamma}_i(t, x, z) dz = \int_{\Omega} \tilde{\Gamma}_i(t, x, z) dx = e^{-r_i t} \) for \( i = 1, 2 \). Let \( T(t) \phi = (T_1(t) \phi_1, T_2(t) \phi_2)^T \) be the remaining distribution of infective individuals at time \( t \). Also in this case, \( V \) is the positive linear operator on \( C(\Omega, \mathbb{R} \times \mathbb{R}) \) defined by
\[
V(\phi)(x) = \begin{pmatrix} V_{11}(\phi)(x) & V_{12}(\phi)(x) \\ V_{21}(\phi)(x) & V_{22}(\phi)(x) \end{pmatrix}, \forall \phi \in C(\Omega, \mathbb{R} \times \mathbb{R}), \ x \in \Omega,
\]
where \( V_{ij}(\phi)(x) = \beta_{ij} p_i(S_i^0(q_j^0(0)) \int_0^r f_i(a) \int_{\Omega} \Gamma_i(a, x, y) \phi_j(y) dy da, \ i, j = 1, 2. \) Thus, \( V(T(t) \phi) \) is the distribution of newly infected individuals at time \( t \). Thus, the next generate operator can be represented by
\[
L(\phi) := \int_0^\infty V(T(t) \phi) dt = V \left( \int_0^\infty T(t) \phi dt \right).
\]
Furthermore, the total number of infectious individuals is given by
\[
\mathcal{L} := \int_{\Omega} L(\phi)(x) dx.
\]
Let \( \vartheta_1 := \int_0^{r_1} f_1(a) \int_{\Omega} \Gamma_1(a, x, y) dx da. \) According to (v) of (A1), we can obtain
\[
\int_0^\infty \int_{\Omega} V_{11}(T_1(t) \phi_1)(x) dt dx = \int_\Omega \beta_{11} p_1(S_1^0) q_1'(0) \int_0^\infty \int_0^{r_1} f_1(a) \int_{\Omega} \int_{\Omega} \Gamma_1(a, x, y) \tilde{\Gamma}_1(t, y, z) \phi_1(z) dz dy da dt dx
\]
\[
= \int_\Omega \beta_{11} p_1(S_1^0) q_1'(0) \int_0^\infty e^{-r_1 t} \int_{\Omega} \phi_1(z) dz dt
\]
\[
= \frac{\vartheta_1 \beta_{11} p_1(S_1^0) q_1'(0)}{r_1}.
\]
By the same methods, one has
\[
\int_\Omega \int_0^\infty V_1(T_2(t) \phi_2)(x) dt dx = \frac{\vartheta_1 \beta_{12} p_1(S_1^0) q_2'(0)}{r_2},
\]
\[
\int_\Omega \int_0^\infty V_2(T_1(t) \phi_1)(x) dt dx = \frac{\vartheta_2 \beta_{21} p_2(S_2^0) q_1'(0)}{r_1}, \ j = 1, 2.
\]
As a consequence, it follows that
\[
\mathcal{L} = \begin{pmatrix} \frac{\vartheta_1 \beta_{11} p_1(S_1^0) q_1'(0)}{r_1} & \frac{\vartheta_1 \beta_{12} p_1(S_1^0) q_2'(0)}{r_2} \\ \frac{\vartheta_2 \beta_{21} p_2(S_2^0) q_1'(0)}{r_1} & \frac{\vartheta_2 \beta_{22} p_2(S_2^0) q_2'(0)}{r_2} \end{pmatrix}.
\]
Let $r(\mathcal{L})$ be the spectral radius of $\mathcal{L}$. Finally, we can define the spectral radius of $\mathcal{L}$ as the basic reproduction number $R_0$, that is,

$$R_0 = r(\mathcal{L}).$$

We are in a position to show the main results in this section.

**Theorem 4.1.** Let $u(t, x, \phi)$ be the solution of (21) with $u_0 = \phi \in \mathbb{C}_+^d$, then the following two statements are valid:

1. If $R_0 < 1$, then the disease free equilibrium $(\frac{u_1}{a_1}, \frac{u_2}{a_2}, 0, 0)$ is globally attractive.
2. If $R_0 > 1$, then the system (21) has a positive constant steady state $u^* = (S_1^*, S_2^*, I_1^*, I_2^*)$ which is globally attractive.

**Proof.** The conclusion of (1) follows from Theorem 3.7(i).

In the following, we prove the conclusion (2) by using a Volterra like Lyapunov functional. Similarly to the proof of [48, Theorem 2.1], we can show that the corresponding ordinary differential equations of (21):

\[
\begin{align*}
\frac{dS_i(t)}{dt} &= \mu_i - d_iS_i(t) - \beta_{ij}p_i(S_i(t))q_j(I_j(t)) - \beta_{12}p_i(S_i(t))q_2(I_2(t)), \\
& \quad t > 0, \ i = 1, 2, \\
\frac{dI_i(t)}{dt} &= -r_iI_i(t) + \beta_{ij}p_i(S_i(t))q_j(I_j(t)) + \beta_{12}p_i(S_i(t))q_2(I_2(t)), \\
& \quad t > 0, \ i = 1, 2
\end{align*}
\]

admits at least one endemic equilibrium $u^* = (S_1^*, S_2^*, I_1^*, I_2^*)$ such that $R_i^* > 0$, which is also a positive constant steady state of (21).

Next, we are ready to prove the global attractivity of the endemic equilibrium $u^*$ and hence, the endemic equilibrium $u^*$ is unique. Set $V(x) = x - 1 - \ln x$. Then we define

$$W(t) = \int_\Omega \left\{ \frac{\mathcal{W}_1(t, x)}{\beta_2p_1(S_1^*)q_2(I_2^*)} + \frac{\mathcal{W}_2(t, x)}{\beta_2p_2(S_2^*)q_1(I_1^*)} \right\} dx,$$

where

$$\mathcal{W}_i(t, x) = \Phi_{S_i}(t, x) + \frac{1}{\vartheta_i} \Phi_{I_i}(t, x) + \frac{1}{\vartheta_i} \sum_{j=1}^2 \beta_{ij}p_i(S_i^*)q_j(I_j^*)Q(S_{i,t}, I_{j,t}),$$

$$\Phi_{S_i}(t, x) = \int_{S_i^*}^{S_i(t, x)} \frac{p_i(s) - p_i(S_i^*)}{p_i(s)} ds, \quad \Phi_{I_i} = \int_{I_i^*}^{I_i(t, x)} \frac{q_i(s) - q_i(I_i^*)}{q_i(s)} ds$$

and

$$Q(S_{i,t}, I_{j,t}) = \int_0^{\tau_t} f_i(a) \int_\Omega \Gamma_i(a, x, y) \int_0^\infty V \left( \frac{p_i(S_i(t - \sigma, y))q_j(I_j(t - \sigma, y))}{f_i(S_i^*)q_j(I_j^*)} \right) d\sigma dy da$$

for $i, j = 1, 2$. Let $\phi = (\phi_{S_1}, \phi_{S_2}, \phi_1, \phi_2) \in \mathcal{Z}_0$ ($\mathcal{Z}_0$ is defined as Theorem 3.7(ii)). Since $\mathcal{Z}_0$ is invariant, there exists a non–negative solution $(S_1, S_2, I_1, I_2)$ of (21) that is defined for all $t \in \mathbb{R}$, takes all its value in $\mathcal{Z}_0$ and satisfies $S_i(0, x) = \phi_{S_i}(x)$ and $I_i(0, x) = \phi_i(x)$ for all $x \in \Omega$. It further follows from Lemma 3.6 and (20) that
Moreover, one has

\[ \frac{\partial \Phi_i(t,x)}{\partial t} = D_t \left( \frac{p_i(S_i(t,x)) - p_i(S_i^*)}{p_i(S_i(t,x))} \right) \Delta S_i(t,x) + F_i(t,x) \]

\[ + \sum_{j=1}^{2} \beta_{ij} p_i(S_i^*) q_j(I_j^*) H_{ij}(t,x), \]

\[ \frac{\partial \Phi_1(t,x)}{\partial t} = D_1 \left( \frac{q_1(I_1(t,x)) - q_1(I_1^*)}{q_1(I_1(t,x))} \right) \Delta I_1(t,x) + J_1(t,x) + \sum_{j=1}^{2} \beta_{1j} p_1(S_1^*) q_j(I_j^*) \]

\[ \int_0^{\tau_1} f_1(a) \int_\Omega \Gamma_1(a,x,y) T_{1j}(t,a,x,y) dy da, \]

\[ \frac{\partial \Phi_2(t,x)}{\partial t} = D_2 \left( \frac{p_2(S_2(t,x)) - p_2(S_2^*)}{p_2(S_2(t,x))} \right) \Delta S_2(t,x) + F_2(t,x) \]

\[ + \sum_{j=1}^{2} \beta_{2j} p_2(S_2^*) q_j(I_j^*) H_{2j}(t,x) \]

and

\[ \frac{\partial \Phi_2(t,x)}{\partial t} = D_2 \left( 1 - \frac{q_2(I_2^*)}{q_2(I_2(t,x))} \right) \Delta I_2(t,x) + J_2(t,x) \]

\[ + \sum_{j=1}^{2} \beta_{2j} p_2(S_2^*) q_j(I_j^*) \int_0^{\tau_2} f_2(a) \int_\Omega \Gamma_2(a,x,y) T_{2j}(t,a,x,y) dy da, \]

where

\[ F_i(t,x) = -d_i \int_\Omega \frac{(p_i(S_i(t,x)) - p_i(S_i^*))(S_i(t,x) - S_i^*)}{S_i(t,x)} dx, \quad i = 1, 2, \]

\[ J_i(t,x) = r_i \frac{I_i(t,x) I_i^*}{q_i(I_i(t,x)) q_i(I_i^*)} \left( q_i(I_i(t,x)) - q_i(I_i^*) \right) \left( N_i(I_i(t,x)) - N_i(I_i^*) \right), \quad i = 1, 2, \]

\[ H_{ij}(t,x) = 1 - \frac{p_i(S_i^*)}{p_i(S_i(t,x))} + \frac{q_i(I_j(t,x))}{q_i(I_j^*)} - \frac{p_i(S_i(t,x)) q_j(I_j(t,x))}{p_i(S_i^*) q_j(I_j^*)}, \quad i, j = 1, 2 \]

and

\[ T_{ij}(t,a,x,y) = 1 + \frac{p_i(S_i(t-a,y)) q_j(I_j(t-a,y))}{p_i(S_i^*) q_j(I_j^*)} - \frac{q_i(I_i(t,x))}{q_i(I_i^*)} \]

\[ - \frac{p_i(S_i(t-a,y)) q_j(I_j(t-a,y)) q_i(I_i^*)}{p_i(S_i^*) q_j(I_j^*) q_i(I_i(t,x))}, \quad i, j = 1, 2. \]

Moreover, one has

\[ \frac{\partial Q(S_{i,t}, I_{j,t})}{\partial t} = \int_0^{\tau_i} f_i(a) \int_\Omega \Gamma_i(a,x,y) \left( V \left( \frac{p_i(S_i(t,y)) q_j(I_j(t,y))}{p_i(S_i^*) q_j(I_j^*)} \right) \right. \]

\[ - V \left( \frac{p_i(S_i(t-a,y)) q_j(I_j(t-a,y))}{p_i(S_i^*) q_j(I_j^*)} \right) \left. \right) dy da \]
for $i, j = 1, 2$. Secondly, let

$$W_i(t) = \int_{\Omega} W_i(t,x)dx, \ i = 1, 2.$$ 

By a straightforward computation, we can get

$$\frac{dW_1(t)}{dt} = D_{S_1} \int_{\Omega} \left(1 - \frac{p_1(S_1^*)}{p_1(S_1(t,x))}\right) \Delta S_1(t,x)dx$$

$$+ \int_{\Omega} D_1 \left(1 - \frac{q_1(I_1^*)}{q_1(I_1(t,x))}\right) \Delta I_1(t,x)dx + \int_{\Omega} F_1(t,x)dx$$

$$+ \frac{1}{\vartheta_1} \int_{\Omega} J_1(t,x)dx - \frac{\beta_1}{\vartheta_1} p_1(S_1^*) q_1(I_1^*) \int_{\Omega} \int_{\Gamma_1(a,x,y)} f_1(a) \int_{\Omega} \Gamma_1(a,x,y)$$

$$\left(V \left(\frac{p_1(S_1^*)}{p_1(S_1(t,x))}\right) \right) V \left(\frac{p_1(S_1(t-a,y)) q_1(I_1(t-a,y))}{p_1(S_1^*) q_1(I_1(t,x))}\right) dydx da$$

$$+ \frac{\beta_2}{\vartheta_1} p_1(S_1^*) q_2(I_2^*) \int_{\Omega} \int_{\Omega} f_1(a) \int_{\Omega} \Gamma_1(a,x,y)$$

$$\left(-V \left(\frac{p_1(S_1(t-a,y)) q_1(I_1(t-a,y)) q_2(I_2)}{p_1(S_1^*) q_1(I_1(t,x))}\right) + V \left(\frac{p_2(S_2(t-a,y)) q_1(I_1(t-a,y)) q_2(I_2)}{p_2(S_2^*) q_1(I_1(t,x)) q_2(I_2(t,x))}\right)\right)$$

$$+ V \left(\frac{q_1(I_1(t,x))}{q_1(I_1^*)}\right) - V \left(\frac{q_2(I_2(t,x))}{q_2(I_2^*)}\right) dydx da$$

and

$$\frac{dW_2(t)}{dt} = D_{S_2} \int_{\Omega} \left(1 - \frac{p_2(S_2^*)}{p_2(S_2(t,x))}\right) \Delta S_2(t,x)dx$$

$$+ \int_{\Omega} D_2 \left(1 - \frac{q_2(I_2^*)}{q_2(I_2(t,x))}\right) \Delta I_2(t,x)dx + \int_{\Omega} F_2(t,x)dx$$

$$+ \frac{1}{\vartheta_2} \int_{\Omega} J_2(t,x)dx + \frac{\beta_1}{\vartheta_2} p_2(S_2^*) q_1(I_1^*) \int_{\Omega} \int_{\Omega} f_2(a) \int_{\Omega} \Gamma_2(a,x,y)$$

$$\left(-V \left(\frac{p_2(S_2(t-a,y)) q_1(I_1(t-a,y)) q_2(I_2)}{p_2(S_2^*) q_1(I_1(t,x)) q_2(I_2(t,x))}\right) + V \left(\frac{p_2(S_2(t-a,y)) q_1(I_1(t-a,y)) q_2(I_2)}{p_2(S_2^*) q_1(I_1(t,x)) q_2(I_2(t,x))}\right)\right)$$

$$+ V \left(\frac{q_1(I_1(t,x))}{q_1(I_1^*)}\right) - V \left(\frac{q_2(I_2(t,x))}{q_2(I_2^*)}\right) dydx da$$

In addition, there holds

$$D_{S_j} \int_{\Omega} \Delta S_j(t,x) \left(1 - \frac{p_j(S_j^*)}{p_j(S_j(t,x))}\right) dx$$

$$= D_{S_j} \int_{\Omega} \left(1 - \frac{p_j(S_j^*)}{p_j(S_j(t,x))}\right) \frac{\partial}{\partial n} S_j(t,x)dx - D_{S_j} \int_{\Omega} \frac{p_j(S_j) p_j(S_j^*) (\nabla S_j(t,x))^2}{p_j^2(S_j(t,x))} dx$$

$$= - D_{S_j} \int_{\Omega} \frac{p_j(S_j) p_j(S_j^*) (\nabla S_j(t,x))^2}{p_j^2(S_j(t,x))} dx, \ j = 1, 2.$$
According (iii) of (A1), one has 

\[-D_S \int_\Omega \frac{q_i(J^*_i)}{q_i(J(t,x))} \left( 1 - \frac{q_j(I^*_j)}{q_j(I(t,x))} \right) dx = -D_j \int_\Omega \frac{q_j(I^*_j)q'_j(I_j)(\nabla I_j(t,x))^2}{q^2_j(I_j(t,x))} dx \leq 0 \]

for \( j = 1, 2 \). By virtue of (iii) of (A1), one has

\[ \int_\Omega F_i(t,x) dx = - \int_\Omega \frac{(p_i(S_i(t,x)) - p_i(S^*_i))(S_i(t,x) - S^*_i)}{p_i(S_i(t,x))} dx \leq 0, \quad i = 1, 2. \]  

(22)

Furthermore, in view of (iii) and (v) of (A1), it follows that

\[ \int_\Omega J_i(t,x) dx \leq 0, \quad i = 1, 2. \]  

(23)

As a consequence,

\[
\frac{dW_1(t)}{dt} = -D_S \int_\Omega \frac{p_1(S^*_1)p'_1(S_1)(\nabla S_1(t,x))^2}{p^2_1(S_1(t,x))} dx + \int_\Omega F_1(t,x) dx \\
- \frac{D_1}{q_1} \int_\Omega \frac{q_1(I^*_1)q'_1(I_1)(\nabla I_1(t,x))^2}{q^2_1(I_1(t,x))} dx + \frac{1}{q_1} \int_\Omega J_1(t,x) dx \\
+ \frac{\beta_{11}p_1(S^*_1)q_1(I_1)}{q_1} \int_\Omega \int_\Omega f_1(a) \int_\Omega f_1(a) \int_\Omega \Gamma_1(a,x,y) \left( -V \left( \frac{p_1(S^*_1)}{p_1(S_1(t,x))} \right) \right) dadydx \\
+ \frac{\beta_{12}p_1(S^*_1)q_2(I_2)}{q_2} \int_\Omega \int_\Omega f_1(a) \int_\Omega f_1(a) \int_\Omega \Gamma_1(a,x,y) \left( -V \left( \frac{p_1(S^*_1)}{p_1(S_1(t,x))} \right) \right) dadydx \\
- V \left( \frac{q_1(I_1(t,x))}{q_1(I^*_1)} \right) dadydx
\]

and

\[
\frac{dW_2(t)}{dt} = -D_S \int_\Omega \frac{p_2(S^*_2)p'_2(S_2)(\nabla S_2(t,x))^2}{p^2_2(S_2(t,x))} dx + \int_\Omega F_2(t,x) dx \\
+ \frac{D_2}{q_2} \int_\Omega \frac{q_2(I^*_2)q'_2(I_2)(\nabla I_2(t,x))^2}{q^2_2(I_2(t,x))} dx + \frac{1}{q_2} \int_\Omega J_2(t,x) dx \\
+ \frac{\beta_{22}p_2(S^*_2)q_2(I_2)}{q_2} \int_\Omega \int_\Omega f_2(a) \int_\Omega f_2(a) \int_\Omega \Gamma_2(a,x,y) \left( -V \left( \frac{p_2(S^*_2)}{p_2(S_2(t,x))} \right) \right) dydadydx \\
+ \frac{\beta_{21}p_2(S^*_2)q_1(I_1)}{q_2} \int_\Omega \int_\Omega f_2(a) \int_\Omega f_2(a) \int_\Omega \Gamma_2(a,x,y) \left( -V \left( \frac{p_2(S^*_2)}{p_2(S_2(t,x))} \right) \right) dydadydx \\
+ V \left( \frac{q_2(I_2(t,x))}{q_2(I^*_2)} \right) - V \left( \frac{p_2(S^*_2)}{p_2(S_2(t,x))} \right) dydadydx.
\]
Set

\[ W = \frac{1}{\beta_1 p_1(S_1^*)} W_1 + \frac{1}{\beta_2 p_2(S_2^*)} W_2. \]

Then we have

\[
\frac{dW}{dt} = -\frac{D_1}{\beta_1 p_1(S_1^*)} p_1(S_1^*) (\nabla S_1(t,x))^2 \int_\Omega p_1(S_1^*) (\nabla S_1(t,x)) \, dx
\]

\[
-\frac{D_2}{\beta_1 p_2(S_1^*)} p_1(S_1^*) (\nabla S_2(t,x))^2 \int_\Omega p_1(S_1^*) (\nabla S_2(t,x)) \, dx
\]

\[
-\frac{D_2}{\beta_2 p_1(S_2^*)} q_1(I_1^*) \int_\Omega q_2(I_1^*) (\nabla I_2(t,x))^2 \, dx
\]

\[
-\frac{D_2}{\beta_2 p_2(S_2^*)} q_1(I_1^*) \int_\Omega q_2(I_1^*) (\nabla I_2(t,x))^2 \, dx
\]

\[
+ \int_\Omega \frac{J_1(t,x)}{\beta_1 p_1(S_1^*)} q_1(I_1^*) \, dx
\]

\[
+ \int_\Omega \frac{J_2(t,x)}{\beta_1 p_1(S_1^*)} q_1(I_1^*) \, dx
\]

\[
+ \int_\Omega \frac{\beta_1 q_1(I_1^*)}{\beta_2 q_2(I_2^*)} \int_0^t f_1(a) \int_\Omega \Gamma_1(a,x,y) \, dydadx
\]

\[
\left( -V \left( \frac{p_1(S_1^*)}{p_1(S_1^*)} \right) - V \left( \frac{p_1(S_1(t-a,y)) q_1(I_1(t-a,y))}{p_1(S_1(t,x)) q_1(I_1(t,x))} \right) \right) \, dydadx
\]

\[
- \frac{1}{\beta_1} \int_\Omega \int_0^t f_1(a) \int_\Omega \Gamma_1(a,x,y) \, dydadx
\]

\[
\left( V \left( \frac{p_1(S_1(t,x))}{p_1(S_1(t,x))} \right) + V \left( \frac{p_1(S_1(t-a,y)) q_2(I_2(t-a,y)) q_1(I_1(t,x))}{p_1(S_1(t,x)) q_2(I_2(t,x))} \right) \right) \, dydadx
\]

\[
- \frac{\beta_2 q_2(I_2^*)}{\beta_1 q_1(I_1^*)} f_2(a) \int_\Omega \Gamma_2(a,x,y) \, dydadx
\]

\[
\left( V \left( \frac{p_2(S_2(t,x))}{p_2(S_2(t,x))} \right) + V \left( \frac{p_2(S_2(t-a,y)) q_2(I_2(t-a,y))}{p_2(S_2(t,x)) q_2(I_2(t,x))} \right) \right) \, dydadx
\]

\[
- \frac{1}{\beta_2} \int_\Omega \int_0^t f_2(a) \int_\Omega \Gamma_2(a,x,y) \, dydadx
\]

\[
\left( V \left( \frac{p_2(S_2(t,x))}{p_2(S_2(t,x))} \right) + V \left( \frac{p_2(S_2(t-a,y)) q_1(I_1(t-a,y)) q_2(I_2(t,x))}{p_2(S_2(t,x)) q_1(I_1(t,x)) q_2(I_2(t,x))} \right) \right) \, dydadx
\]

On the basis of (22) and (23), we have \( \frac{dW}{dt} \leq 0 \). By using the similar arguments introduced in [52, Theorem 12.1], we can prove that the attractor \( Z_0 \) in Theorem 3.7 is a singleton set which is formed by the endemic equilibrium \( u^* \). This completes the proof. \( \Box \)

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