Elastic stars in general relativity: IV. Axial perturbations

Max Karlovini\textsuperscript{1} and Lars Samuelsson\textsuperscript{2}

\textsuperscript{1} Department of Theoretical Physics and Astrophysics, Masaryk University, Kotlarska 2, 611 37 Brno, Czech Republic
\textsuperscript{2} School of Mathematics, University of Southampton, Southampton SO17 1BJ, UK

E-mail: max@physics.muni.cz and lars@soton.ac.uk

Received 2 March 2007, in final form 20 April 2007
Published 12 June 2007
Online at stacks.iop.org/CQG/24/3171

Abstract
This is the fourth paper in a series that attempts to put forward a consistent framework for modelling solid regions in neutron stars. Here we turn our attention to axial perturbations of spherically symmetric spacetimes using a gauge invariant approach due to one of us. Using the formalism developed in the first paper in the series it turns out that the matter perturbations are neatly expressible in terms of a ‘metric’ tensor field depending only on the speeds of shear wave propagation along the principal directions in the solid. The results are applicable to a wide class of elastic materials and do not assume material isotropy nor quasi-Hookean behaviour. The perturbation equations are then specialized to a static background and are given by two coupled wave equations. Our formalism is thus slightly simpler than the previously existing results of Schumaker and Thorne (1983 Mon. Not. R. Astron. Soc. 203 457), where an additional initial value equation needs to be solved. The simplification is mainly due to the gauge invariance of our approach and also shows up in somewhat simpler boundary conditions. We also give a first-order formulation suitable for numerical integration of the quasi-normal mode problem of a neutron star. The relations between the gauge independent variables and the, in general, gauge-dependent perturbed metric and strain tensor are explicitly given.

PACS numbers: 04.40.Dg, 97.10.Sj, 97.60.Jd

1. Introduction

1.1. Motivation
In recent years it has become increasingly clear that, in order to extract information from observations of neutron stars, we must first understand the dynamical behaviour of matter
beyond the perfect fluid approximation. Much effort has been put into understanding, for example, the dynamics of superfluids including various viscous phenomena; see [1] for a recent review. This topic is essential in order to realistically discuss, for example, glitches and gravitational waves arising from secular instabilities. These events may also require understanding of magnetic interactions if the field is strong enough to influence the dynamics in the region where the matter currents are strong. This is certainly the case, at least in the outer layers of magnetars and the outermost crust in pulsars, but may even be important in the interior regions if the magnetic field is buried during the formation of the star. Glitches will require a thorough understanding of the behaviour of the rigid parts of the star and its interactions with normal and superfluids. Recently, a formalism was presented [4] that is suitable for handling elastic solids permeated by superfluids, including, in a MHD-like manner, magnetic fields.

The objective of this series of papers is to develop a coherent framework for purely elastic components in compact objects. In this, the fourth paper, we consider axial perturbations of spherical stars. It is worth pointing out that axial oscillations in neutron stars may already have been observed in the aftermath of giant flares in soft gamma ray repeaters, where quasi-periodic oscillations with frequencies from 18 Hz reaching into the kHz range are observed [2, 13, 31, 32, 34]. Although the enormous magnetic fields believed to be present in these objects, which are well modelled within the so-called magnetar model [10], will couple any torsional oscillations to the core within about one oscillation period via Alfvén waves [12, 19], a recent toy-model suggests [12] that the frequencies will nevertheless be close to the purely elastic ones. A slightly more general calculation [20] shows that, despite a very complex nature, the modes excited by a crust quake may well be tuned to the crustal ones. Therefore, even in these types of systems, it is important to understand the elastic oscillation spectrum. This spectrum has recently been studied, within the relativistic Cowling approximation, in [27], extending the results of [35]. The local magnetic effects (i.e. neglecting the global nature of the modes) of the torsional mode spectrum have been studied for a poloidal dipole field in the Cowling approximation [29] (see also [23]). These calculations were recently extended to include purely axial global oscillations [30], still using the Cowling approximation.

The Newtonian theory of axial perturbations of elastic bodies is well developed, see e.g. [22]. The general relativistic problem was pioneered by Schumaker and Thorne [28] (hereafter ST) who developed a detailed theory including many relevant limits. Our treatment is different from theirs in several important ways. First, we use the gauge invariant perturbation formalism due to Karlovini [14]. This formalism assumes that the perturbations are axisymmetric, but as discussed e.g. in [6], this implies no restriction for spherically symmetric backgrounds. At a practical level, the difference between ours and ST’s approach shows up in the fact that our final equations consist of just two coupled wave equations and thereby dispenses of the additional initial value equation, being traceable to the gauge choice, in ST. Moreover, the formalism applies to any spherically symmetric (static or not) background. More importantly, our treatment does not assume an isotropic background from the outset which makes it applicable to the exotic (nuclear pasta) phases believed to exist in the lower crust of the neutron star, e.g. [25]). Another minor difference is that our formalism in the main part of this work is valid for nonlinear elastic equations of state. That is, we do not assume a quasi-Hookean equation of state from the outset. Practically, this should not be very important since the behaviour of perfect elasticity (which we do assume) is likely to break down roughly when any nonlinear corrections to quasi-Hookean behaviour become important.

As mentioned above, our treatment will not be limited to isotropic backgrounds. There are two fundamentally different cases in which non-isotropy occurs, either the material is
Elastic stars in general relativity: IV. Axial perturbations

intrinsic non-isotropic (as in the pasta phases) or the star is already strained. One may think that the latter is fairly unimportant since, at any given time, the crust will be close to its unstrained state simply because it will break otherwise. However, the same limitation on the magnitude of the strain that applies to the background applies to the perturbations so that the background strain may well be of the same order as (or larger than!) the oscillations. An example where this may be important is provided by the flare model of Duncan [9]. Here the twisted magnetic field is slowly rearranging itself to a lower energy configuration. In this process, the equilibrium shape of the star changes and strains are built up in the crust until it breaks along some fault line. This initial crust quake sends seismic waves in the (now already strained) crust thereby causing it to fracture at other locations too.

The treatment relies heavily on paper I in the series [15] (hereafter paper I), where the theoretical foundations of relativistic elasticity were outlined following the framework of Carter and Quintana [3]. For convenience and in order to introduce notation, a brief review of the formalism of paper I is given below.

1.2. Relasticity

Any description of continuous media is, at least implicitly, based on the use of an abstract base manifold, the matter space, \( X \) say. This space (which in our case is three-dimensional and Riemannian) can be thought of as a book-keeping tool which assigns unique labels to each fluid element via a map \( \Psi \) that takes each flow-line on spacetime (\( M \) say) to a point in matter space. We may use this map to push forward the contravariant metric \( g^{ab} \) on \( M \) to \( X \),

\[
g^{AB} = \Psi^* g^{ab},
\]

(1)

where abstract index notation [33] with capital letters on \( X \) and lower case letters on \( M \) is adopted. We then define the tensor \( \eta^{−1AB} \) to be the value of \( g^{AB} \) that minimizes the energy density at a fixed number density \( n \). We now define \( \eta_{AB} \) through the relation \( \eta^{−1AC} \eta_{CB} = \delta^A_B \). Pulling back \( \eta_{AB} \) to \( M \) (\( \eta_{ab} = \Psi^* \eta_{AB} \)) we may define the constant volume strain tensor according to

\[
s_{ab} = \frac{1}{2} (h_{ab} - \eta_{ab}),
\]

(2)

where \( h_{ab} \) is defined by \( h_{ab} = g_{ab} + u_a u_b \), with \( u^a \) being the unit tangent vector to the flow-lines. Thus, in effect, \( s_{ab} \) measures the difference in geometry between the natural, unsheared state and the actual physical state.

In this paper, we will use the simplifying assumption that the elastic structure deforms conformally under pure compression. We believe that this is quite non-restrictive since e.g. cubic crystals and isotropic media are included in this category. As discussed in paper I this allows us to fully describe this structure by a fixed (i.e. \( n \)-independent) metric tensor field \( k_{AB} = n^{1/3} \eta_{AB} \) on \( X \). One may think of this metric as measuring the relative positions of the particles in a locally relaxed state.

Using the fact that the eigenvectors of \( k_{ab} = \Psi^* k_{AB} \) are also the eigenvectors of the stress tensor \( p_{ab} \), a preferred tetrad completed by the matter 4-velocity \( u^a \) may be introduced. Denoting the eigenvectors by \( e^a_\mu \) we use Greek indices to numerate the space-like basis vectors. The eigenvalues \( n^{1/3}_\mu \) of \( k_{ab} \) correspond, in a loose sense, to (squared) linear particle densities, whereas the eigenvalues \( p_\mu \) of the strain tensor are the pressures as measured by a comoving observer in the direction of \( e^a_\mu \).
1.3. Perturbation theory and notational remarks

We shall assume that there exists a family of solutions to Einstein’s equations, parametrized by \( \lambda \), say. We assume that the solution for a specific \( \lambda \) (equal to zero say) to be known and expand around this solution. We take \( \delta \) in front of any tensor field to represent the value of \( d/d\lambda \) on that field evaluated at \( \lambda = 0 \). We shall have occasion to define perturbed quantities without the explicit perturbation symbol \( \delta \). The indices on such quantities are raised and lowered by the unperturbed spacetime metric, whereas care must be exercised when fields have retained the \( \delta \), e.g.,

\[
\delta e^a_{\mu} = \delta (g^{ab} e_{\mu b}) \neq g^{ab} \delta e_{\mu b}. \tag{3}
\]

In order not to have too cluttered formulae we shall not notationally distinguish background fields from the full family fields but instead point out the validity of the equations in the text when confusion may arise.

In order to conform with the notation in the preceding papers in this series [15–17] as well as with the work of Karlovini [14] on which much of this paper is based, we use some non-standard notation. In particular, the perturbed metric will be denoted by \( \gamma_{ab} \) and the flow-line orthogonal piece of the metric is denoted by \( h_{ab} \), i.e. precisely the opposite of the definitions in e.g. [5].

We will use geometric units such that \( c = 1, \) \( G = 1 \) and the Einstein equations take the form

\[
Z_{ab} := R_{ab} - \kappa (T_{ab} - \frac{1}{2} T g_{ab}) = 0, \tag{4}
\]

where \( R_{ab} \) is the Ricci tensor, \( T_{ab} \) is the energy–momentum tensor and \( \kappa \) is the coupling constant which, in conventional units, takes the value \( \kappa = 8\pi Gc^{-4} \).

2. General perturbations

Perturbation theory of elastic media has already been considered by Carter in some detail in [5]. Since it seems computationally advantageous to use the eigenvalue formulation in many practical situations we shall devote this section to deriving the perturbations of the eigenvectors and from that deducing the perturbed stress–energy tensor. The derivation is performed in an arbitrary (identification) gauge and the final expression reduces to the formulae of Carter when either a Lagrangian or an Eulerian gauge (given by some displacement vector field \( \xi^a \)) is chosen. Since we are assuming a perfectly elastic conformally deforming material (so we have a fixed matter space metric \( k_{AB} \)) we shall consider the perturbed matter space and spacetime metrics to be our fundamental variables.

The perturbed eigenvectors are easily derived from the identities

\[
\begin{align*}
g_{ab} u^a u^b &= -1, \quad g_{ab} u^a e_{\mu b} = 0, \quad g_{ab} e_{\mu a} e_{\nu b} = \delta_{\mu \nu}, \\
k_{ab} u^a u^b &= 0, \quad k_{ab} u^a e_{\mu b} = 0, \quad k_{ab} e_{\mu a} e_{\nu b} = n^2 \delta_{\mu \nu}.
\end{align*}
\]

which, when perturbed, yield

\[
\begin{align*}
u^a u^b \gamma_{ab} + 2 u_\alpha \delta u^a &= 0 \quad \tag{5} \\
u^a e_{\mu b} \gamma_{ab} + e_{\mu \alpha} \delta u^a + u_\alpha \delta e_{\mu a} &= 0 \quad \tag{6} \\
e_{\mu a} e_{\nu b} \gamma_{ab} + e_{\mu \alpha} \delta e_{\nu a} + e_{\nu \alpha} \delta e_{\mu a} &= 0 \quad \tag{7}
\end{align*}
\]

3 We leave aside the technical, but important, issue of existence of such a family. On physical grounds, for the applications in mind in this paper, it is apparent that such a family should exist.
where \( \gamma_{ab} = \delta g_{ab} \). Solving for \( \delta u^a, \delta e^a_\mu \) and \( \delta (n_\mu^2) \) gives

\[
\delta u^a = \frac{1}{2} u^b u^c \gamma_{bc} u^a - \sum_{\mu=1}^3 n_\mu^{-2} u^b e^c_\mu (\delta k_{bc} - n_\mu^2 \gamma_{bc}) e^a_\mu.
\]

\[
\delta e^a_\mu = -\frac{1}{n_\mu^2} u^b e^c_\mu (\delta k_{bc} - n_\mu^2 \gamma_{bc}) u^a - \frac{1}{2} e^b_\mu e^c_\mu \gamma_{bc} e^a_\mu + \sum_{\nu \neq \mu} \frac{1}{n_\nu^2 - n_\mu^2} e^b_\nu e^c_\nu (\delta k_{bc} - n_\mu^2 \gamma_{bc}) e^a_\mu.
\]

\[
\delta (n_\mu^2) = e^a_\mu e^b_\mu (\delta k_{ab} - n_\mu^2 \gamma_{ab}).
\]

The perturbed basis 1-forms are similarly found to be

\[
\delta u_a = -\frac{1}{2} u^b u^c \gamma_{bc} u_a - \sum_{\mu=1}^3 n_\mu^{-2} u^b e^c_\mu (\delta k_{bc} - n_\mu^2 \gamma_{bc}) e_{\mu a}.
\]

\[
\delta e_{\mu a} = -n_\mu^{-2} u^b e^c_\mu (\delta k_{bc} - n_\mu^2 \gamma_{bc}) u_a + \frac{1}{2} e^b_\mu e^c_\mu \gamma_{bc} e_{\mu a} - \sum_{\nu \neq \mu} \frac{1}{n_\nu^2 - n_\mu^2} e^b_\nu e^c_\nu (\delta k_{bc} - n_\mu^2 \gamma_{bc}) e_{\nu a}.
\]

One may be worried that the difference between the linear number densities that appear in the denominator in expressions (12) and (15) will cause divergences when there are degenerate eigenvalues on the background. However, as we will see, for the physical quantities relevant for this work, the sums are always convergent. These relations make it a simple (but tedious) task to write down the perturbations of any tensor field in terms of the perturbed metrics. In particular, the general perturbation of the stress–energy tensor takes the form

\[
\delta T_{ab} = (\delta \rho - \rho u^c \gamma_{cd} u_a u_b) u_a u_b + \sum_{\mu=1}^3 (\delta p_\mu + p_\mu e^c_\mu \gamma_{cd}) e_{\mu a} e_{\mu b} - 2 \sum_{\mu=1}^3 u^c e^d_\mu \left[ n_\mu^{-2} (\rho + p_\mu) \delta k_{cd} - \rho \delta \gamma_{cd} \right] u_a e_{\mu b}
\]

\[
+ \sum_{\mu=1}^3 \sum_{\nu \neq \mu} e^c_\mu e^d_\nu \left[ n_\nu^{-2} (\rho + p_\nu) v_{\mu \perp \nu}^2 (\delta k_{cd} - n_\mu^2 \gamma_{cd}) + p_\nu \gamma_{cd} \right] e_{\mu a} e_{\nu b},
\]

where \( \rho \) is the energy density, \( p_\mu \) is the pressure in the \( \mu \) direction and \( v_{\mu \perp \nu} \) is the speed of shear waves along the \( \mu \) direction with polarization vector along the \( \nu \) direction,

\[
(\rho + p_\nu) v_{\mu \perp \nu}^2 = \frac{n_\nu^{-2} (p_\nu - p_\mu)}{n_\nu^2 - n_\mu^2}
\]

as derived in paper I. It is now apparent that the isotropic limit is well defined. The perturbed energy density and eigenpressures can also be expressed in terms of the perturbed metrics by
the use of (13) (remembering that the equation of state is considered to be given as a function of the linear particle densities) as

$$\delta \rho = \frac{1}{2} \sum_{\mu} (\rho + p_\mu) n_\mu n^2 (\delta k_{ab} - n_\mu^2 \gamma_{ab}) e^a_\mu e^b_\mu$$

(18)

$$\delta p_\mu = \frac{1}{2} \sum_\nu n_\nu \frac{\partial p_\mu}{\partial n_\nu} n_\nu n^2 (\delta k_{ab} - n_\nu^2 \gamma_{ab}) e^a_\nu e^b_\nu.$$

(19)

However, these will be seen to vanish for the axial case which is the main concern of this paper. One can confirm that, by comparing these relations to the expression for the elasticity tensor given in paper I, one obtains the expression given by Carter [5], a calculation that is best done in a Lagrangian gauge for which $u^a \delta k_{ab}$ vanish.

### 3. Axial Perturbations

We shall now specialize our considerations to axial (in a sense to be defined) perturbations. We use the approach of Karlovini [14]. For convenience, and to introduce notation, we briefly review this formalism below and then proceed to specialize it to perfectly elastic matter.

Consider a spacetime with a Killing vector field $\eta^a$. If this Killing vector field is nearly hyper-surface orthogonal, i.e. its twist

$$\Omega^a = -\epsilon^{abcd} \eta_b \nabla_c \eta_d$$

(20)

can be considered to be small, we may treat the deviation from hyper-surface orthogonality as a perturbation. Note that this general consideration allows, as examples, studies of axisymmetric almost-spherical systems as well as stationary almost-static systems (with no required symmetry on the space-like hyper-surfaces). The main focus in this paper is the spherical case, so that $\eta^a$ is the axisymmetry generator and reduces to one of the $SO(3)$ generators on the background. In order to make progress, we split the metric according to

$$g_{ab} = \perp_{ab} + F_{\mu a} \mu_b,$$

(21)

where

$$F = \eta^a \eta_a, \quad \mu_a = F^{-1} \eta_a, \quad \eta^a \perp_{ab} = 0.$$  

(22)

On a spherical background the squared norm of the Killing vector has the form $F = (r \sin \theta)^2$. The polar and axial perturbations of the spacetime metric are given by

$$+ \gamma_{ab} = \left( \perp_a \perp_b + \mu_a \mu_b \eta^a \eta^b \right) \gamma_{cd} = \delta \perp_{ab} + (\delta F) \mu_a \mu_b$$

(23)

$$- \gamma_{ab} = 2 \eta^c \mu_a \perp_b \gamma_{cd} = 2 \eta^c \delta \mu_b.$$  

(24)

Taking the appropriate projections of Einstein’s equations,

$$\perp_{ac} \perp_{bd} Z_{ab} = 0$$  

(25)

$$\eta^a \eta^b Z_{ab} = 0$$  

(26)

$$\perp_{ab} \eta^a Z_{ab} = 0$$  

(27)

and using a partial identification gauge-fixing amounting to

$$\delta \eta^a = 0$$

(28)

Note that the terms ‘axial’ and ‘polar’ will be inappropriate for perturbations of non-spherical spacetimes.
it is possible to show [14] that, when evaluated at \( \Omega^a = 0 \), the axial part of the metric (24) satisfies (25) and (26) identically, whereas the polar part (23) satisfies (27) identically. Hence these perturbations decouple and may be treated separately.

We henceforth restrict ourselves to the axial case whence

\[
\delta \perp_{ab} = 0, \quad \delta F = 0, \tag{29}
\]

so that

\[
\gamma_{ab} = -\gamma_{ab} = 2\eta_{(a}\delta_{b)}.
\tag{30}
\]

We next introduce the notation

\[
Q_{ab} = 2\nabla_{[a}\delta_{b]}, \quad J^a = 2\delta(\perp^{ab}\eta^c T_{bc})
\tag{31}
\]

as well as their (restricted) duals

\[
Q^a = -\frac{1}{2} F_{eabcd} \eta^b Q_{cd}, \quad J_{ab} = \epsilon_{abcd} J^c \eta^d.
\tag{32}
\]

Note that both the perturbation vectors \( Q^a \) and \( J^a \) (as well as their duals) are gauge invariant since the corresponding background quantities vanish. In terms of these fields the full perturbation equations take the Maxwell-like form [14]

\[
\nabla_b (F Q^{ab}) = \kappa J^a, \tag{33}
\]

\[
\nabla_a J^a = 0, \tag{34}
\]

or in the dual form

\[
2\nabla_{[a} Q_{b]} = -\kappa J_{ab}, \tag{35}
\]

\[
\nabla_a (F^{-2} Q^a) = 0, \tag{36}
\]

\[
\nabla_{[a} J_{bc]} = 0. \tag{37}
\]

To proceed it is evident that we must compute the axial matter current vector \( J^a \). To this end we assume that one of the eigenvectors of the matter space metric, \( e_3^a \) say, is aligned with the axisymmetry generator on the background, i.e.

\[
e_3^a = F^{-1/2} \eta^a. \tag{38}
\]

It is worth pointing out that we hereby restrict ourselves to the case when the Killing vector is space-like, but that no loss of generality is implied in the spherically symmetric background case. Up to this point the discussion has been valid (with appropriate changes of terminology) to background spacetimes admitting any hyper-surface orthogonal Killing vector.

To find \( J^a \) we project (16) with \( \perp^{ab} \eta^c \) and use the relation

\[
\delta \perp_{ab} = -\delta^{ac} \perp_{bd} \eta_{cd}, \tag{39}
\]

so that, after some algebra

\[
\eta^c T_{bc} \delta \perp^{ab} = -p_3 \eta^c \perp^{ab} \gamma_{bc}, \tag{40}
\]

\[
\perp^{ab} \eta^c \delta T_{bc} = n_3^2 (\rho + p_3) \left( -\mu^{ab} \mu^b + \sum_{\mu = 1}^{2} e^{\mu}_{e\mu} e^{b}_{e\mu} \gamma^{2}_{b} \right) (\delta k_{bc} - n_3^2 \gamma_{bc}) + p_3 \eta^c \perp^{ab} \gamma_{bc} \tag{41}
\]

\[
= (\rho + p_3) F S^{ab} K_b - \eta^c T_{bc} \delta \perp^{ab}, \tag{42}
\]
where $S^{ab}$ is the 'metric'

$$S^{ab} = -u^a u^b + \sum_{\mu=1}^{2} \epsilon_{\mu \perp 3} e_{\mu}^a e_{\mu}^b,$$

(43)

and $K_a$ is the gauge invariant (vanishing when unperturbed) quantity

$$K_a = (Fn^3)^{-1} \eta^b \eta^c k_{bc} = \left( (Fn^3)^{-1} \eta^b \eta^c (\delta k_{bc} - n_s^2 \gamma_{bc}) \right)$$

(44)

Combining terms, we finally find

$$J^a = 2(\rho + p_3)FS^{ab}K_b.$$  

(45)

The tensor $S^{ab}$ is vaguely analogous to the so-called acoustic metric (sometimes denoted $G_{ab}$) used in, for example, the study of analogues of black holes in fluid mechanics in the sense that it is related to the propagation of waves (although in this case, it is shear waves rather than sound waves).

We still need to evaluate $K_a$. We therefore make the very natural choice that the matter space is axisymmetric$^5$ and introduce material space coordinates $(\tilde{x}^i, \tilde{\phi}), i = 1, 2$, such that

$$\eta^i \nabla_a \tilde{x}^i = 0, \quad \eta^i \nabla_a \tilde{\phi} = 1.$$  

(46)

We also assume that the pull-back of the material space metric takes the form

$$k_{ab} = \sum_{i,j} K_{ij} \nabla_a \tilde{x}^i \nabla_b \tilde{x}^j + \tilde{F} \nabla_a \tilde{\phi} \nabla_b \tilde{\phi},$$

(47)

where the metric components $K_{ij}$ and $\tilde{F}$ depend on $\tilde{x}^i$ only. Contracting this relation twice with $e_i^a$ we find that the function $\tilde{F}$ may be expressed as

$$\tilde{F} = F e_i^a e^i_{kab} = Fn^3,$$

(48)

which holds on the background. This allows us to evaluate $K_a$ to be

$$K_a = \nabla_a \delta \tilde{\phi} - \delta \mu_a.$$  

(49)

We may note in passing that an equivalent form of the perturbation equations may be found directly in terms of the gauge invariant 1-form $K_a$. The equations of motion are then

$$\nabla_b (F \nabla^{(b} K^{a)}) = \kappa (\rho + p_3) FS^{ab}K_b,$$

(50)

together with $\nabla_a J^a = 0$ with $J^a$ given in terms of $K_a$ by (45). We shall, however, not pursue this form further.

3.1. Spherically symmetric background

We shall now specialize the equations further by assuming that the background is spherically symmetric. To ease the presentation we start by introducing some notation. As much as possible we shall use the notation of paper II [17] in this series. The background metric will be decomposed as

$$g_{ab} = \tilde{g}_{ab} + I_{ab},$$

(51)

where $I_{ab}$ is the metric on the hyper-surfaces spanned by the $SO(3)$ generators and is taken to be represented by the line element

$$ds^2 = r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

(52)

$^5$ One could, in principle, consider a non-axisymmetric matter space and constrain the mapping in such a way that spacetime is still axisymmetric. This feels highly unnatural, however, and we will not consider it further.
where $r$ is the Schwarzschild radial coordinate. Introducing the shorthand notation $r^a = e^a_1$, the orthogonal two-dimensional Lorentzian metric is given by

$$j_{ab} = -u_u u_b + r_a r_b,$$  \hspace{1cm} (53)

and the associated volume form is

$$e_{ab} = -2u_u r_b.$$  \hspace{1cm} (54)

The covariant derivative operator associated with $j_{ab}$ will be denoted by $D_a$. Since the angular background eigenvectors will now correspond to degenerate eigenvalues we may further introduce the notation for the basis indices

$$\mu = 1 \rightarrow \mu = r, \quad \mu \in \{2, 3\} \rightarrow \mu = t,$$  \hspace{1cm} (55)

that is, for example,

$$p_1 = p_r, \quad p_2 = p_3 = p_t.$$  \hspace{1cm} (56)

We now proceed to separate out the angular dependence from our perturbations equations. The discussion will be very brief and the reader is again referred to [14] for details.

Using the dual form of the perturbation equations it is evident from the closedness of $J_{ab}$ expressed in (37) that we may introduce the vector potential $Y_a$ according to

$$J_{ab} = 2\nabla^a Y_b.$$  \hspace{1cm} (57)

Furthermore, this vector can be taken to be orthogonal to the $SO(3)$ generators. Proceeding to equation (35) it is seen that

$$Q_a = \nabla_a \Phi - \kappa Y_a$$  \hspace{1cm} (58)

for some axisymmetric scalar $\Phi$. Separation is achieved by putting

$$\Phi = C(\theta) r \psi, \quad Y_a = C(\theta) \epsilon_{ab} J^b.$$  \hspace{1cm} (59)

The vector $J^a$ is a two-dimensional object related to $J^a$ by

$$J^a = (r^2 \sin \theta)^{-1} [C'(\theta) J^a - C(\theta) J (\partial/\partial \theta)^\mu],$$  \hspace{1cm} (60)

for the two-scalar $J$ constrained by (34) to take the value $J = D_a J^a$. It is convenient to introduce the invariantly defined mass function $m$ through

$$1 - \frac{2m}{r} = D_a r D^a r,$$  \hspace{1cm} (61)

as well as the function $\tau$ invariantly defined on spherically symmetric spacetimes by

$$R_{ab} \eta^b = \frac{1}{2} \kappa \tau \eta^a,$$  \hspace{1cm} (62)

i.e. $\frac{1}{2} \kappa \tau$ is the eigenvalue of the Ricci tensor with respect to the eigenvector $\eta^a$. A clearer physical interpretation is seen from the background Einstein equations which implies that $\tau$ is the minus the trace of the $2 \times 2$ block of the energy–momentum tensor orthogonal to the $SO(3)$ orbits. In the present case, that is

$$\tau = \rho - p_r.$$  \hspace{1cm} (63)

Noting also the relation

$$r D^a D_a r = \frac{2m}{r} - \frac{1}{2} \kappa r^2 \tau,$$  \hspace{1cm} (64)

the perturbation equations can straightforwardly be confirmed to reduce to the two-dimensional equations

$$(D_a D^a - U) \psi = \kappa S$$  \hspace{1cm} (65)
where
\[ S = r \epsilon^{ab} \mathcal{D}_a(r^{-2} \mathcal{J}_b), \] (67)
and the potential \( U \) is given by
\[ U = \frac{1}{2} \kappa r - \frac{6m}{r^3} + \frac{l(l+1)}{r^2}, \] (68)
where \( l \) is the usual harmonic index coming from the separation of variables. The angular equation is solved by setting
\[ C(\theta) = G_{ls_2}^{-3/2}(\cos \theta), \] (69)
where \( G_{ls_2}^{-3/2}(y) \) is an ultra-spherical (or Gegenbauer) polynomial.

In our case, \( J^a \) is given in terms of background fields and \( K_a \), where, as implied by equation (49),
\[ 2 \nabla_a K_b = -2 \nabla_a \delta \mu_b = -Q_{ab}. \] (70)
This means that \( \delta \mu_a \) must be of the form
\[ \delta \mu_a = -C'(\theta) \frac{r^{-2}}{(l+2)(l-1)r^2 \sin^3 \theta} \epsilon^b_a Q_b + \nabla_a f_G \] (l \geq 2) \] (71)
\[ \delta \mu_a = r^{-2} \epsilon^b_a Q_b + \nabla_a f_G \] (l = 1), \] (72)
where
\[ Q_a = \mathcal{D}_a(r\psi) - \kappa \epsilon_{ab} J^b, \] (73)
and \( f_G \) is a free function (which we will see in section 6 corresponds to the choice of gauge).

This in turn implies that \( K_a \) must be of the form
\[ K_a = \frac{C'(\theta)}{(l+2)(l-1)r^2 \sin^3 \theta} \epsilon^b_a Q_b + \frac{1}{(l+2)(l-1)} \nabla_a \sin^{-3} \theta \frac{C'(\theta)}{r^{-1} \psi} \] (l \geq 2) \] (74)
\[ K_a = r^{-2} \epsilon^b_a Q_b + \mathcal{D}_a(r^{-1} \psi) \] (l = 1), \] (75)
for some spherically symmetric scalar \( \psi \). Setting
\[ S^{ab} = -u^a u^b + v^a r^2 r^b, \] (76)
we have
\[ S^{ab} = \epsilon^{ab}_e \epsilon^{eb}_e. \] (77)
For \( l \geq 2 \), we then find
\[ J^a = 2(\rho + p_\perp) F S^{ab} K_b \]
\[ = \frac{2(\rho + p_\parallel)}{\sin \theta} \left\{ -C'(\theta) \frac{1}{(l+2)(l-1)} \epsilon^{ab}_e Q_b + \right. \] (l \geq 2) \]
\[ + \left. C(\theta) \right\} v^a \mathcal{D}_a(r^{-1} \psi) \left[ r^2 \psi (\partial/\theta)^a \right]. \] (78)
Comparing with equation (60), we obtain
\[ [\mathcal{E} S^{ab}_e + (l+2)(l-1) j^a b] \kappa \mathcal{J}^b = \mathcal{E} S^{ab} \left[ \epsilon^b c \mathcal{D}_c (r\psi) + r^2 \mathcal{D}_b (r^{-1} \psi) \right] \] (79)
\( \kappa \mathcal{J} = E v_{\perp}^{-2} r^{-1} \psi, \)  

where

\( E = 2 \kappa r^2 (\rho + p_r). \)  

Solving for \( \kappa \mathcal{J}^a, \) we find

\[ \kappa \mathcal{J}^a = (-C_0 u^a u^b + C_1 r^a r^b) \left[ e_b^c \mathcal{D}_c(r \psi) + r^2 \mathcal{D}_b(r^{-1} \psi) \right], \tag{82} \]

where

\[ C_0 = \frac{E}{E + (l + 2)(l - 1)} \]

\[ C_1 = \frac{E v_{\perp}^2}{E v_{\perp}^2 + (l + 2)(l - 1)}. \tag{84} \]

When \( l = 1, \) a similar analysis shows that \( \mathcal{J} = 0 \) and that \( \mathcal{J}^a \) is given by expression (82) with

\[ C_0 = \frac{E}{E + 1} \]

\[ C_1 = \frac{E v_{\perp}^2}{E v_{\perp}^2 + 1}. \tag{86} \]

Finally, rewriting the perturbation equations (65)–(66) in terms of the quantity \( Q_a \) defined in (73), we find

\[ r \mathcal{D}^a (r^{-2} Q_a) = \frac{(l + 2)(l - 1)}{r^2} \psi, \]  

\[ \mathcal{D}_a \mathcal{J}^a = \mathcal{J}, \]  

where \( Q_a, \mathcal{J}^a \) and \( \mathcal{J} \) are given in terms of the two-dimensional scalar fields \( \psi \) and \( \varphi \) in equations (73), (82) and (80), respectively. These equations are valid for any \( l \geq 1 \) provided one remembers to put \( \mathcal{J} = 0 \) when \( l = 1. \) The non-radiative nature of the \( l = 1 \) case is hinted by the form of the equations which are just decoupled conservation equations for the two ‘currents’ \( r^{-2} \mathcal{Q} \) and \( \mathcal{J}^a. \)

In order to better understand the content of these equations it is useful to write them out in a suitable coordinate system. For brevity, we will from now on only consider the radiative case \( l \geq 2. \) It is advantageous to use explicitly conformally flat coordinates given by

\[ u_a = -e^\nu (dt)_a, \quad r_a = e^\nu (dr_a)_a, \]  

which will reduce to the usual Regge–Wheeler radial gauge on static backgrounds. Next we introduce the notation

\[ \mathcal{X}^a = u_a \lambda^a, \quad \mathcal{X}^a = r_a \lambda^a \tag{90} \]

for any \( \lambda^a. \) Then we have

\[ \kappa \mathcal{J}_t = e^{-\nu} C_0 [(r \psi)' + r^2 (r^{-1} \varphi)] \]  

\[ \kappa \mathcal{J}_r = e^{-\nu} C_1 [(r \psi) + r^2 (r^{-1} \varphi)'] \]  

\[ Q_t = e^{-\nu} [(1 - C_1)(r \psi)' - C_1 r^2 (r^{-1} \varphi)'] \]  

\[ Q_r = e^{-\nu} [(1 - C_0)(r \psi)' - C_0 r^2 (r^{-1} \varphi)]. \tag{94} \]
where dots and primes denote derivatives with respect to $t$ and $r_*$, respectively\(^6\). Using
\[
\mathcal{W}_i = r^{-1} e^\nu Q_i, \quad \mathcal{W}_v = r^{-1} e^\phi Q_v
\]
as auxiliary variables, we can cast the perturbation equations as a first-order system
\[
-\mathcal{W}_i' + \frac{\dot{r}}{r} \mathcal{W}_v - \mathcal{W}_v' - e^{2\nu} \frac{L}{r^2} \psi = 0
\]
\[
-\mathcal{W}_v' + \frac{\dot{r}}{r} \mathcal{W}_i + \mathcal{W}_i' + e^{2\nu} \frac{\mathcal{E} v_{r_+}^2}{r^2} \psi = 0
\]
\[
-\mathcal{W}_i' - \mathcal{W}_v' + \mathcal{W}_i + \mathcal{W}_v + e^{2\nu} \frac{\mathcal{E} v_{r_+}^2}{r^2} \psi = 0
\]
\[
-\mathcal{W}_v' - \mathcal{W}_i' + \mathcal{W}_v + \mathcal{W}_i + e^{2\nu} \frac{\mathcal{E} v_{r_+}^2}{r^2} \psi = 0
\]
\[
-\mathcal{W}_i' - \mathcal{W}_v' + \mathcal{W}_v + \mathcal{W}_i + e^{2\nu} \frac{\mathcal{E} v_{r_+}^2}{r^2} \psi = 0
\]
where $L = (l + 2)(l - 1)$ and we have used the definitions of $C_0$ (83) and $C_1$ (84). We may note here that the principal part of these equations decouples into two systems, one in $\psi$ and $\phi$ which has a characteristic propagation speed $v_{r_+}^2$ and one in $\mathcal{W}_i$ and $\mathcal{W}_v$ whose speed is 1.

We may naturally interpret this as the existence of two families of modes, the shear modes and the axial gravitational $w$-modes.

This is as far as we go in the general case. Note that this system is suitable for numerical integration since it does not contain derivatives of background quantities that can be discontinuous. As we will see below, it also makes maximal use of the junction conditions so that $\mathcal{W}_i$, $\mathcal{W}_v$, and $\psi$ are all everywhere continuous.

4. Static background

The above-derived system of equations (96)–(99) is valid for any spherically symmetric background and could therefore be applied to, for example, collapsing bodies. However, since our interest here lies on matter with nonzero shear modulus, such configurations are not realistic (since the strain would inevitably grow beyond the breaking strain of the material). One could of course study axial perturbations of a radially oscillating star, but henceforth we will restrict our attention to the case where the background is static. We can then drop all dotted background quantities. It is then possible to combine (96)–(98) to give a wave equation for $\mathcal{W}_i$ sourced by $\phi$. In addition equations (97)–(99) can be combined into a wave equation for $\phi$ with $\mathcal{W}_v$ as a source:

\[
-\ddot{\psi} + \frac{(\mathcal{E} v_{r_+}^2 \phi')'}{\mathcal{E}} = \left[ \frac{(\mathcal{E} v_{r_+}^2 \phi')'}{\mathcal{E}} + e^{2\nu} \frac{\mathcal{E} v_{r_+}^2}{r^2} (\mathcal{E} + L) \mathcal{W}_v \right] \phi = \left[ \frac{\mathcal{E} (1 - v_{r_+}^2) r \mathcal{W}_v}{\mathcal{E}} \right]' \mathcal{W}_v.
\]

Of course, many possible reformulations exist of this system of equations if new dependent variables are chosen. One such reformulation would have been to choose $\psi$ instead of $\mathcal{W}_i$ as the first dependent variable. This would have followed the general guidelines in [14] more closely, but would not have lead to simplified formulae in this case. We do not feel that any

\(^6\) This notation should not be confused with the use of dots and primes in paper II [17].
possible second-order system reformulation (that we have found) is superior in any important ways to (100)–(101). In any case, for numerical integration it is preferable to use the first-order system (96)–(99). We could also give more explicit expressions of the terms involving derivatives of background quantities by making use of the results of paper I. However, either one can then keep the elastic equation of state free in which case third-order partial derivatives of the energy per particle will appear, or one can specify, for example, a quasi-Hookean equation of state leading to rather lengthy expressions. We do not feel that this leads to any further understanding of the problem at hand. Instead we give explicit expressions below suitable for numerical integration. We shall therefore be satisfied with this form of the wave equations and pause a moment to discuss some of their properties.

Firstly, it is clear that for vacuum (101) is trivial and (100) is just the usual Regge–Wheeler equation. For this reason we may refer to (100) as the gravitational wave equation. Second, if we take the isotropic limit we have

\[ v_{r} = v_{\perp} = 2 \kappa r^{2} \hat{\mu} \mathcal{E}^{-1}, \]

so that the source terms in (100) reduce to

\[ 2 \kappa (e^{2\nu} \hat{\mu} \varphi)'. \]

so we see that, as pointed out by Dyson [11], for weak gravity, \( e^{\nu} \approx 1 \approx \) constant, gravitational waves couple only to the gradient of the shear modulus. Another important case is the perfect fluid limit. Here we see that the system reduces to

\[ -\ddot{W} + e^{2\nu} \left( 6m \frac{1}{r^{2}} - \frac{1}{2} \kappa (\rho - p) - \frac{l(l+1)}{r^{2}} \right) W = 0 \]  

\[ -r \dot{\varphi} = (r W)' . \]

The first of these equations is the standard result, see e.g. [7, 8, 18]. The second equation is now redundant since the variable \( \varphi \) does not encode any relevant physical information in this case. It is in fact easy to show, using (91)–(93) and (104), that the two-dimensional matter current \( J_{a} \) satisfies

\[ J_{r} = 0, \quad \dot{J}_{t} = 0 \]

so that the well known result that axial perturbations of perfect fluids reduce to stationary currents is evident.

4.1. Boundary conditions

The perturbation equations must be accompanied by suitable boundary conditions. These conditions were treated in detail by Karlovini for general matter sources [14] and we merely state the results here. At any boundary (including e.g. interfaces between different layers or the surface) we require that the 1-form \( Q_{a} \) as well as the scalar \( \psi \) should be continuous. These conditions follow from the requirement that the first and second fundamental forms on any hyper-surface with normal \( n_{a} \) (say) are continuous. In particular, this implies that the traction \( n^{a} T_{ab} \) is continuous as well [21]. In ST, the condition of continuous traction was enforced as an additional constraint. In view of the above, it is clear that this is only necessary as a gauge condition relating the gauges on either side of the hyper-surface. An additional boundary condition is provided by requiring regularity at the centre. Finally, if we consider an isolated object, we need to impose a condition of outgoing gravitational waves at infinity. We discuss these boundary conditions in more detail in the following section, where we give a computational recipe for quasi-normal mode solutions to the perturbation equations.

7 The extra equation corresponds in a sense to the freedom of choice of mapping between \( M \) and \( X \). For perfect fluids the labelling of fluid elements plays no role in the dynamics so there is some degree of degeneracy in defining the map.
5. Computational algorithm

We shall here present a procedure for finding quasi-normal mode solutions to the perturbation equations in situations relevant for neutron stars with a fluid core, a solid crust and, possibly, a fluid ocean. We therefore assume harmonic time dependence, i.e. that the temporal dependence of the independent variables is given by $e^{i\omega t}$. We have found it slightly preferable to present the equations in Schwarzschild coordinates, related to the Regge–Wheeler coordinates through

$$e^{\nu}d\tau = e^\lambda dr,$$

where $\lambda$ is given by

$$e^{-2\lambda} = 1 - \frac{2m}{r}. \quad (106)$$

When we wish to solve our perturbation equations numerically it is preferable that all variables scale with the same power of $r$ near the centre. To obtain this behaviour (as well as some other features), we redefine the variables in system (96)–(99) (specialized to a static background) according to

$$X_1 = r^{-l-1}\psi_i, \quad (107)$$
$$X_2 = -i\omega_0 r^{-l} \psi, \quad (108)$$
$$X_3 = -i\omega_0 e^\nu r^{-l-1} \psi, \quad (109)$$
$$X_4 = \omega_0^2 e^\nu r^{-l} \phi, \quad (110)$$

where $\omega_0 = e^{-\nu} \omega$ and $\nu_c$ is the central value of $\nu$. The reason for introducing these variables is that if one wishes to solve the background equations simultaneously with the perturbation equations using a shooting algorithm, the central value of $\nu$ is unknown since it is determined by matching the background solution to the exterior Schwarzschild solution at the surface of the star. The above scaling allows for writing the equations in terms of $\omega_0$ and $\nu_0 = \nu - \nu_c$ and, once the solution is found, scale it back to physical units.

It is easy to show that for perfect fluids the last two of the first-order equations become algebraic relations amounting to

$$X_3 = -e^{i\omega} X_1, \quad X_4 = -e^{i\nu} X_2, \quad (111)$$

whereas the other two become just

$$r \frac{dX_1}{dr} = -(l+2)X_1 - e^{i\omega} X_2 \quad (112)$$
$$r \frac{dX_2}{dr} = -e^{i\omega} \left[L e^{2\nu_0} - \alpha_0^2 r^2\right] X_1 - (l-1)X_2. \quad (113)$$

Disregarding the singular solution scaling as $r^{-l-1}$ near $r = 0$ it is easy to see that the variables leave the centre according to

$$X_1 = \hat{X}_1 \left\{ 1 - \frac{\alpha_0^2 + \frac{\xi}{6} (l+2)(3p_c - (2l+1)\rho_c)}{2(2l+3)} r^2 + O(r^4) \right\} \quad (114)$$
$$X_2 = \hat{X}_1 \left\{ -(l+2) + \frac{(l+4)\alpha_0^2 - \frac{\xi}{6} (l+2)(l-1)[3p_c + (2l+7)\rho_c]}{2(2l+3)} r^2 + O(r^4) \right\}, \quad (115)$$
where $\hat{X}_1$ is an arbitrary constant and $p_c$ and $\rho_c$ are the central values of the pressure and density, respectively. The equations in a solid becomes

$$r \frac{dX_1}{dr} = -(l + 2)X_1 - e^{\lambda_0}X_2 - e^{\lambda_0} \frac{Ev_{\perp}^2}{r^2 \omega_0^2} X_4$$  \hspace{1cm} (116)

$$r \frac{dX_2}{dr} = e^{\lambda_0} \frac{Ev + L}{E} X_2 - (l - 1)X_2 + e^{\lambda_0} L X_3$$  \hspace{1cm} (117)

$$r \frac{dX_3}{dr} = e^{\lambda_0} \frac{Ev + L}{E} X_3 + \left[ r \nu_r - (l + 2) \right] X_3 + e^{\lambda_0} \frac{Ev^2}{r} X_4$$  \hspace{1cm} (118)

$$Ev_{\perp}^2 r \frac{dX_4}{dr} = -e^{\lambda_0} \omega_0^2 \left[ \frac{Ev_{\perp}^2 + L}{E} X_1 - e^{\lambda_0} \omega_0^2 L X_3 + Ev_{\perp}^2 \left[ r \nu_r - (l + 1) \right] X_4, \right.$$  \hspace{1cm} (119)

where $\nu_r = \frac{d\nu}{dr} = m + \frac{1}{2} kr^3 \frac{p_r}{r(r - 2m)}$  \hspace{1cm} (120)

and the shear wave speeds depend on the equation of state according to (17). For the quasi-Hookean equation of state discussed in paper I they take the form

$$Ev_{\perp}^2 = k \mu r^2 \left[ 1 + n^2 \frac{r^2}{n^2} \right]$$  \hspace{1cm} (121)

$$Ev_{\perp}^2 = 2k \mu r^2 \left[ \frac{n^2}{r^2} + \frac{n^2}{r^2} - 1 \right]$$  \hspace{1cm} (122)

In vacuum the equations are

$$X_1 = -e^{\lambda_0} X_1, \hspace{1cm} X_4 = \text{unconstrained}$$ \hspace{1cm} (123)

$$r \frac{dX_1}{dr} = -(l + 2)X_1 - \frac{r e^{\lambda_0}}{r - 2M} X_2$$ \hspace{1cm} (124)

$$r \frac{dX_2}{dr} = -e^{-\nu_r} \left[ L - \frac{e^{2\nu_r} \omega_0^2 r^3}{r - 2M} \right] X_1 - (l - 1)X_2,$$  \hspace{1cm} (125)

where $M$ is the total mass of the star.

Using expansion (114)–(115), we may now integrate the fluid equations to the crust core interface, where we need to impose the boundary conditions. In the present variables these are just the continuity of $X_1$, $X_2$ and $X_3$. Note that $X_3$ is free at the boundary and can in principle be set to any value. However, as we proceed with the integration in the solid we eventually encounter the next boundary. If this boundary is an interface to a fluid phase or vacuum we have two conditions on the variable $X_3$, (i) it must be continuous and (ii) it has to satisfy the constraint (111) (which is identical to (123) in the vacuum case). In general, these will not be satisfied, signalling the wrong choice of the value of $X_3$ at the previous boundary. The remedy to this situation is to find a second independent solution to the equation in the solid phase and then to take the linear combination of the two solutions which satisfy all the boundary conditions. The simplest way to find the second solution is to start with $X_1 = X_2 = X_3 = 0$ and $X_4$ any value. Clearly, adding any multiple of this solution to the original one will not violate the initial boundary conditions, so it is a simple task of solving an algebraic equation to find the correct total solution in this part of the star. It should be evident how to work out the entire solution to the problem using this algorithm regardless of the number of interfaces in the star. Finally, the quasi-normal mode frequencies $\omega$ are determined by making sure that the vacuum solution is described by outgoing gravitational waves. This specifies the solutions up to a scale.
6. Identifying the metric and the strain

Since neither the metric nor the strain on a strained background is gauge invariant, we need to specify the gauge in order to evaluate them. We have already used some of our gauge freedom by setting \( \delta \eta^a = 0 \), so any further gauge transformation has to be generated by a vector field, \( \xi^a \) say, that is Lie-dragged by the axisymmetry generator,

\[
\mathcal{L}_\eta \xi^a = -\mathcal{L}_\xi \eta^a = 0. \tag{126}
\]

We may decompose \( \xi^a \) according to

\[
\xi^a = \xi^a_\perp + f_G \eta^a, \quad \xi^a_\perp \eta_a = 0, \quad \eta^a \nabla_a f_G = 0. \tag{127}
\]

Now, since

\[
\mathcal{L}_\xi \mu_a = 0 \tag{128}
\]
due to the fact that \( \eta^a \) is a hyper-surface orthogonal Killing vector on the background, it is clear that gauge transformations generated by a vector field orthogonal to, and Lie-dragged by, \( \eta^a \) do not affect the axial perturbations. Likewise, it is easy to show that a gauge transformation of the type \( \xi^a = f_G \eta^a \) does not affect the polar perturbations. Hence we need only consider gauge transformations of the latter type. Under such a transformation we have

\[
\delta \mu_a \rightarrow \delta \mu_a + \nabla_a f_G, \tag{129}
\]
\[
\delta \bar{\phi} \rightarrow \delta \bar{\phi} + f_G. \tag{130}
\]

Since \( \eta^a \nabla_a \delta \bar{\phi} = 0 \), we may use this freedom to set \( \delta \bar{\phi} = 0 \). This gauge choice is of a comoving, or Lagrangian type and is the natural way to measure the strain since, for example, the breaking strain will invariably be calculated in a comoving frame. We thus find that in a Lagrangian gauge we have \( \Delta \phi = 0 \) so that \( f_G = -\delta \bar{\phi} \) and \( \Delta \mu_a = -K_a \), where we use \( \Delta \) to indicate that a special gauge has been chosen. Using these relations, we find

\[
\Delta s_{ab} = -\eta(a)K_b + u_{(a} \Delta u_{b)} = -\eta_{(a} h_{b)}^c K_c. \tag{131}
\]

The perturbed metric is just

\[
\Delta g_{ab} = -2\eta_{(a} K_{b)} \tag{132}
\]

and

\[
\Delta k^a_{\ b} = 2k_{bc} \eta^c K^a. \tag{133}
\]

Usually estimates of the breaking, or yield, strain are given as a dimensionless strain angle, \( \Theta_{\text{max}} \), say. The precise relation to the components of the strain tensor depends on the type of deformation and the microscopic structure of the solid material. However, for simple (e.g. isotropic or cubic) structures under simple deformations (e.g. pure twist or shear) the nonzero components of \( s_{ab} \) have the form

\[
s_{ab} \approx \frac{1}{2} \Theta. \tag{134}
\]

Given the uncertainty in the literature on the value of \( \Theta_{\text{max}} \), it does not feel meaningful to digress too deeply into the subject of breakdown of elasticity. It is clear that the approximation of perfect elasticity will break down before the material actually cracks, so for all purposes of this paper we may assume that something catastrophic happens when any component of \( s_{ab} \) exceeds some value of the order of \( \frac{1}{2} \Theta_{\text{max}} \).

---

8 Note that Schumaker and Thorne use the gauge freedom to set the \( \theta \) component of \( \delta \mu_a = 0 \). This would correspond to setting \( f_G = \text{constant} \). It is also worth pointing out that, since these authors consider isotropic backgrounds, their perturbed strain tensor is gauge invariant.
An interesting observation is that, even though the first-order perturbations of the linear particle densities are all zero, so that the strain scalar is also unperturbed to first order, we may still estimate the second-order contribution to the strain scalar and thereby estimate the energy stored in the elastic material. We may namely compute the strain scalar by first evaluating the total

\[ k^a_b = 0k^a_b + \Delta k^a_b + \Delta^2 k^a_b \]  (135)

for some unspecified second-order perturbation \( \Delta^2 k^a_b \). Noting that the non-orthogonal pieces will not alter the strain scalar (all such pieces are contracted with orthogonal tensors) and using bold face letters to denote the \( 3 \times 3 \) matrices \( k = h^{a'b'} k^{b'}_{\, \, b} \) we may write (see paper I)

\[ s^2 = \frac{1}{36 \det k} \left[ (\text{Tr} k)^3 - \text{Tr}(k^3) - 24 \det k \right]. \]  (136)

It then turns out (after some straightforward but tedious algebra) that the second-order perturbation terms only come in multiplied by the background strain. These terms will therefore always (due to the small numerical value of the breaking strain) be much smaller than other second-order terms. Neglecting them leads to an expression of the form

\[ s^2 = s^2_0 + \frac{1}{2} F h^{ab} K_a K_b + \cdots, \]  (137)

where \( s^2_0 \) is the background value.

In summary, once a solution to the perturbation equations is found using the procedure described in section 5 it is easy to find the metric and stress tensor perturbations from equations (132) and (131), respectively. Since the solutions will only be defined up to a scale, one may then set this scale such that the largest component of \( s^{ab} + \Delta s^{ab} \) is smaller than some maximum value of order \( \frac{1}{2} \Theta_{\text{max}} \), and find the maximally allowed amplitude of the perturbations consistent with elastic response. It is also straightforward to estimate the maximal stored elastic energy density by the formula

\[ \rho_{\text{elastic}} = \mu s^2, \]  (138)

where \( \mu \) is the shear modulus and \( s^2 \) is given by (137).

7. Conclusion

We have developed the general relativistic theory of torsional oscillations in elastic matter. In the light of the recent very exciting observations of quasi-periodic oscillations in the tail of giant flares in soft gamma-ray repeaters, with frequencies matching well the expected spectrum of such modes, we believe that it is important to have a well founded theory for these modes. We have improved on the previously existing theory (mainly ST) in several respects. The equations presented here are gauge invariant and are valid to second order in strain for any equation of state as long as the material is of a conformally deforming type. Various gauge issues have been resolved and resulted in simplified equations and boundary conditions compared to ST. The elastic response to perturbations is conveniently parametrized in terms of the shear wave velocities and should be straightforward to apply to the anisotropic ‘pasta’ phases near the crust-core boundary of neutron stars once the relevant elastic parameters have been estimated.

The results of this paper have in fact already been applied in the relativistic Cowling approximation [27]. Although it is expected that the spectrum should not change substantially when gravitational degrees of freedom are included (see ST), it is nevertheless important to check this numerically and work in that direction is in progress.
Acknowledgments

LS gratefully acknowledges the support by the Marie Curie Intra-European Fellowship, contract number MEIF-CT-2005-009366. Parts of this work were carried out when the authors were at Stockholm University. We are grateful to Nils Andersson and Brandon Carter for discussion and insightful comments. In the course of this work, extensive use was made of the computer algebra package GRTensorII [24] running within Maple. We also acknowledge support from the EU-network ILIAS in providing opportunities for valuable discussions with our European colleagues.

References

[1] Andersson N and Comer G L 2007 Relativistic fluid dynamics: physics for many different scales Living Rev. Rel. 10 1 http://www.livingreviews.org/lrr-2007-1 (Preprint gr-qc/0605010)
[2] Barat C, Hayles R I, Hurley K, Niel M, Vedrenne G, Desai U, Kurt V G, Zhenchen V M and Estulin I V 1983 Fine time structure in the 1979 March 5 gamma ray burst Astron. Astrophys. 126 400–2
[3] Carter B and Quintana H 1972 Foundations of general relativistic high pressure elasticity theory Proc. R. Soc. A 331 57
[4] Carter B and Samuelsson L 2006 Relativistic mechanics of neutron superfluid in (magneto) elastic star crust Class. Quantum Grav. 23 5367–88 (Preprint gr-qc/0605024)
[5] Carter B 1973 Elastic perturbation theory in general relativity and a variation principle for a rotating solid star Commun. Math. Phys. 30 261–86
[6] Chandrasekhar S 1992 The Mathematical Theory of Black Holes (Oxford: Clarendon)
[7] Chandrasekhar S and Ferrari V 1991 On the non-radial oscillations of a star Proc. R. Soc. A 432 247
[8] Glampedakis K, Samuelsson L and Andersson N 2006 Elastic or magnetic? A toy model for global magnetar oscillations with implications for quasi-periodic oscillations during flares Mon. Not. R. Astron. Soc. 371 L74–L77 (Preprint astro-ph/0605461)
[9] Israel G L, Belloni T, Stella L, Rephaeli Y, Gruber D E, Casella P, Dall’Osso S, Rea N, Persic M and Rothschild R E 2005 The discovery of rapid x-ray oscillations in the tail of the SGR 1806-20 hyperflare Astrophys. J. Lett. 628 L53–L55 (Preprint astro-ph/0505255)
[10] Karlovini M 2002 Axial perturbations of general spherically symmetric spacetimes Class. Quantum Grav. 19 2125–40 (Preprint gr-qc/0111066)
[11] Karlovini M and Samuelsson L 2003 Elastic stars in general relativity: I. Foundations and equilibrium models Class. Quantum Grav. 20 3613–48 (Preprint gr-qc/0211026)
[12] Levin Y 2007 On the theory of magnetar QPOs Mon. Not. R. Astron. Soc. 377 159–67 (Preprint astro-ph/0612725)
[25] Pethick C J and Ravenhall D G 1995 Matter at large neutron excess and the physics of neutron-star crusts Annu. Rev. Nucl. Part. Sci. 45 429–84
[26] Ruderman M 1991 Neutron star crustal plate tectonics: III. Cracking, glitches, and gamma-ray bursts Astrophys. J. 382 587–93
[27] Samuelsson L and Andersson N 2007 Neutron star asteroseismology. Axial crust oscillations in the Cowling approximation Mon. Not. R. Astron. Soc. 374 256–68 (Preprint astro-ph/069265)
[28] Schumaker B L and Thorne K S 1983 Torsional modes of neutron stars Mon. Not. R. Astron. Soc. 203 457
[29] Sotani H, Kokkotas K D and Stergioulas N 2007 Torsional oscillations of relativistic stars with dipole magnetic fields Mon. Not. R. Astron. Soc. 375 261–77 (Preprint astro-ph/0608626)
[30] Sotani H, Kokkotas K D, Stergioulas N and Vavoulidis M 2006 Torsional oscillations of relativistic stars with dipole magnetic fields. II. Global Alfvén modes Preprint astro-ph/0611666
[31] Strohmayer T E and Watts A L 2005 Discovery of fast x-ray oscillations during the 1998 giant flare from SGR 1900+14 Astrophys. J. Lett. 632 L111–L114 (Preprint astro-ph/0508206)
[32] Strohmayer T E and Watts A L 2006 The 2004 Hyperflare from SGR 1806-20: further evidence for global torsional vibrations Astrophys. J. 653 593–601 (Preprint astro-ph/0608463)
[33] Wald R M 1984 General Relativity (Chicago, IL: The University of Chicago Press)
[34] Watts A L and Strohmayer T E 2006 Detection with RHESSI of high-frequency x-ray oscillations in the tail of the 2004 hyperflare from SGR 1806-20 Astrophys. J. Lett. 637 L117–L120 (Preprint astro-ph/0512630)
[35] Yoshida S and Lee U 2002 Nonradial oscillations of neutron stars with a solid crust, analysis in the relativistic cowling approximation Astron. Astrophys. 395 201 (Preprint gr-qc/0210594)