Inversion Formula

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Abstract

This work introduces a new inversion formula for analytical functions. It is simple, generally applicable and straightforward to use both in hand calculations and for symbolic machine processing. It is easier to apply than the traditional Lagrange-Bürmann formula since no taking limits is required. This formula is important for inverting functions in physical and mathematical problems.∗

0.1 Keywords
inversion of functions, Taylor series, Lagrange-Bürmann inversion formula, reversion of series

0.2 Mathematical Classification
Mathematics Subject Classification 2010: 11A25, 40E99, 32H02

1 Introduction

1.1 General
The inversion of an analytic function \( f(z) \) with \( z, u \in \mathbb{C} \)

\[ f(z) = u \]  

is defined as

\[ z = g(u) \]  

There is no general simple method known to determine \( g(u) \) unless the variable \( z \) can be readily solved from \( f(z) \). Lagrange [1] was the first to find a useful series expansion. Bürmann [2] and [3] generalized it to the Lagrange-Bürmann formula.

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Good \[3\] extended the Lagrange-Bürmann formula to multiple variables. His formula is known as the Lagrange-Good formula and Hofbauer \[4\] supplied the proof. A number of investigations has been published over the Lagrange-Bürmann formula for various applications, like Zhao \[5\] and Merlini et al. \[6\]. Sokal \[7\] recently introduced a new generalization of the Lagrange-Bürmann formula. We first express the Lagrange-Bürmann inversion formula which is the present standard method for calculating the inverse. The new inversion formula is derived next.

1.2 The Lagrange-Bürmann Inversion Formula

Lagrange \[1\] and Bürmann \[2\] introduced an inversion formula for a function \( f(z) \) of a complex variable \( z \).

\[
f(z) = u
\]

with \( f \) being analytic at some point \( z_0 \) and the first derivative at \( z_0 \), is required to be nonzero.

\[
\left. \frac{df(z)}{dz} \right|_{z_0} \neq 0
\]

\( f(z) \) has a value \( u_0 \) at \( z_0 \). The inverse function is \( g(u) \)

\[
z = g(u) = g(f(z))
\]

The Lagrange inversion formula or the Lagrange-Bürmann formula is a Taylor series as follows.

\[
z = z_0 + \sum_{n=1}^{\infty} \frac{(u - u_0)^n}{n!} \lim_{z \to z_0} \left[ \frac{d^{n-1}}{dz^{n-1}} \left( \frac{z - z_0}{f(z) - u_0} \right)^n \right]
\]

Proof of this formula can be found in \[1\] and \[2\]. Taking limits in terms in equation \[6\] usually requires lengthy calculations and a repeated use of L’Hospital’s rule to get rid of the singularity. All terms belonging to a certain coefficient need to be kept together to determine the limit properly. This may be a very laborious task in hand calculations.

2 The Inversion Formula

Using the annotation of the preceding chapter, let

\[
u = f(z) \quad z, u \in C
\]

and \( f(z) \) be analytic over the interior of a circle

\[
r = |z - z_0|
\]

Let the inverse function \( g(u) \) be analytic over the interior of a circle \( R_0 \) at \( u_0 \)

\[
R_0 = |u - u_0|
\]
We have a Taylor series
\[ z = z_0 + \sum_{n=1}^{\infty} \frac{(u - u_0)^n}{n!} \left[ \frac{d^n}{du^n} g(u) \right]_{u_0} \] (10)

This series converges over the circle \( R_1 \) (as in equation (9)). The equation (10) is very difficult to be used any further as such. Higher derivatives of \( g(u) \) are requested and to get them, one would need the \( g(u) \). We can use derivatives of \( f(z) \) instead of \( g(u) \). In order to circumvent the generation of progressively complicated terms, we proceed as follows. Differentiate equation (11) below.

\[ z = g(u) \] (11)

to obtain
\[ \frac{d}{dz} z = 1 = \left( \frac{d}{du} g \right) \left( \frac{d}{dz} u \right) = \left( \frac{d}{du} g(u) \right) \left( \frac{d}{dz} f(z) \right) \] (12)

and solve it as
\[ \frac{d}{du} g(u) = \frac{1}{\left( \frac{d}{dz} f(z) \right)} \] (13)

Differentiate (13) further and solve it for
\[ \frac{d^2}{du^2} g(u) = \frac{1}{\left( \frac{d}{dz} f(z) \right)^2} \frac{1}{\left( \frac{d}{dz} f(z) \right)} \] (14)

In the same manner the \( n \)'th derivative would be solved as
\[ \frac{d^n}{du^n} g(u) = \frac{1}{\left( \frac{d}{dz} f(z) \right)^n} \left[ \frac{d}{dz} \left( \frac{1}{\frac{d}{dz} f(z)} \right) \right] \] (15)

having \( n - 1 \) derivatives acting on the right side in addition to the bracketed derivatives acting on \( f(z) \) alone. We can rearrange the brackets yielding
\[ \frac{d^n}{du^n} g(u) = \frac{1}{\left( \frac{d}{dz} f(z) \right)^n} \frac{d}{dz} \left[ \frac{1}{\left( \frac{d}{dz} f(z) \right)} \right] \] (16)

The multiplying factor is a differential operator acting on all terms to the right containing any dependence on \( z \). Placing this result to equation (10) yields the simplified inversion formula
\[ z = z_0 + \sum_{n=1}^{\infty} \frac{(u - u_0)^n}{n!} \left[ \frac{1}{\left( \frac{d}{dz} f(z) \right)^n} \frac{d}{dz} \left[ \frac{1}{\left( \frac{d}{dz} f(z) \right)} \right] \right]_{u_0} \] (17)

The necessary, but not sufficient, condition for the new inversion formula to converge is that the first derivative of \( f(z) \) must be nonzero at \( z_0 \). The radius of convergence \( R_1 \) must be evaluated for each resulting series. If a singularity would appear at \( z_0 \), a translation to a nearby point should be made.
3 Conclusions

The equation (17) represents a simple alternative to the Lagrange-Bürmann formula (equation (6)). The Lagrange-Bürmann formula requires taking limits and repeated use of L’Hospital’s rule to remove the singularity. The new formula requires only elementary differentiation and evaluation at $z_0$.

Comparison of coefficients in each term between the two formulas is not possible since the expansions are based on polynomials of $u$. A special case appears when $u_0 = 0$ making the expansions powers of $u$. This leads to equalities but not directly. One has to approach the limit ($z \to 0$) in equation (6) finally reaching terms identical with equation (17). Working in the opposite way is not possible.

In spite of its simplicity, this inversion formula can be applied generally. It can be used for inversion of functions and polynomials and for reversion of series. It is valid also for real variables. It is useful for estimating the behavior of the inverse function at some point with a few beginning terms. The radius of convergence needs to be studied for each new series.

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