Soliton appearing in boson-fermion mixture at the third order of the interaction radius

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Abstract. In this paper we consider an ultra-cold mixture of boson and fermion atoms on the basis of quantum hydrodynamics. Small one-dimensional perturbations in such systems are being analyzed. A possibility is shown for soliton solutions of a new type to appear if the third order of the interaction radius is taken into account in the analysis of interactions. A fermion-fermion interaction occurs in explicit form if this approximation is accepted. The conditions that lead to occurrence of this type of soliton in a mixture of boson and fermion atoms were investigated. Restrictions on the fermion-fermion interaction were found that are necessary for this kind of perturbations to appear in the system. Conditions determining whether perturbances would be a condensed soliton or a rarefied soliton are shown.

1 Introduction

Experimental and theoretical approaches to the physics of quantum gases have been continued to grow in recent times, and various nonlinear structures has been studied there [1–6]. Methods for producing Bose-Einstein condensate (BEC) [7] and boson-fermion mixtures [8–12] and investigating its properties have been developed. There is possibility to tune inter-particle interaction in wide range. This showed stable conditions to exist both for repulsion and attraction between the atoms, for example, in a Bose condensate of ⁷Li atoms [13]. A mixture of Bose and Fermi degenerate gases, where interaction between atoms may be tuned from repulsion to attraction, is considered in reference [8].

A lot of interest has been attracted in recent years to the investigation of solitons in quantum gases. Papers devoted to properties of different types of solitons are constantly being emerged.

In one species BEC in the absence of external field there are two well-known type of solitons. These are the bright and the dark solitons (see, for example, [14]). The bright soliton is the area of compression and the dark soliton is the area of rarefaction. These names come from studying of nonlinear light waves in medium, where increasing (decreasing) of soliton amplitude in compare with equilibrium makes it brighter (darker). The bright soliton exists in the BEC with attraction between particles. However, BEC of attracting particles is unstable to collapse in the three dimensional case. Therefore, this solution might be observed in quasi-one dimensional geometry. Whereas, the dark soliton appears in the BEC of repulsing atoms.

As to the present time a bright soliton has been produced in a Bose-Einstein condensate with a repulsive interaction between atoms. A soliton of this type, has been produced, for example, in the condensate of ⁸⁷Rb atoms in a weak periodic potential [15]. The formation of a bright soliton in the condensate of ⁷Li atoms is considered in references [13,16] for a quasi-1D case. Properties of a dark soliton have been studied in reference [17]. They show experimental generation of a dark soliton in a cigar-shaped condensate of ⁸⁷Rb atoms with the use of phase imprinting technique. The study of a dark soliton in Bose-Einstein condensate inside a vortex ring is described in reference [18]. In addition to one-dimension solitonic perturbations in a Bose-Einstein condensate, three-dimension perturbations are also attracting attention [19]. A possibility was shown for solitonic perturbations to exist in a Bose-Einstein condensate with dipole-dipole interaction [20]. In recent years predictions have been made that solitons of bright-bright, dark-bright and dark-dark types may exist in a two-component BEC [21,22].

These solutions can be described using the GP equation. It has recently theoretically been shown that BEC can revile one more soliton solution at more detailed account of interaction in compare with the GP equation. This solution was named bright-like soliton [23]. The reason for such name was that this solution is the solution of compression, but it appears in the BEC with the repulsion interaction. Soliton of compression can exist in the system of repulsing particles due to high order space derivatives in nonlinear term describing interaction appearing in the third order by the interaction radius (TOIR). In this paper
we are interested in studying influence of the interaction in the third order by the interaction radius on ultracold fermions and boson-fermion mixtures. We have described solitons in the BEC, and now we going to consider solitons in fermions, and what new soliton solutions arise at mixing of bosons and fermions in the first order by the interaction radius. Whereas main goal of the paper is to show existence of new solitons in the boson-fermion mixtures at account of interaction up to TOIR.

Superfluid Fermi gases reveals travelling of dark solitons [24,25]. Presence of relatively small number bosons can introduce nonlinearity into the fermion system, and significantly changing the properties of the system [9]. Therefore solitons in boson-fermion mixture is special and interesting item. Salerno [10] suggested that localized states in boson-fermion mixture with attractive (repulsive) Bose-Fermi interactions can be interpreted as a matter-wave realization of quantum dots. Solitons involving most of the bosons of the system and a relatively small portion of the fermions in the vicinity of the Fermi surface were studied in reference [26]. In the case of the attractive interaction between the components, the solitons may describe simultaneous increase or decrease of the densities of both the components of the mixture. In the case of the repulsive interactions between the components on the other hand, they can describe an increase of the density in one component and a decrease of the density in the other component.

It was shown that bright solitons can be generated in a Bose-Fermi mixture trapped in a three dimensional elongated harmonic potential [27]. For that the attraction between bosonic and fermionic atoms has to be strong enough changing in this way the repulsive interactions among bosons to the attractive ones. Simultaneously, the attractive forces induced in the bosonic cloud has to be weak enough to avoid the collapse.

The possibility of the formation of stable fermionic bright solitons in a mixture of a degenerate Fermi gases with a BEC in the presence of a sufficiently attractive boson-fermion interaction which can overcome the Pauli repulsion among fermions any possible repulsion in the BEC was considered [28]. At small fermionic densities a localized Bose-Fermi bright soliton may appears for \( g_{bf} < 0 \) [29]. At studying of Bose-Fermi mixture \( p \)-wave interaction between fermions was included in reference [30] at exploring bright soliton solutions. It leads to stabilization inferred by repulsive \( p \)-wave interactions. Fermionic dark solitons in a boson-fermion mixture were considered in reference [31], where was shown possibility dark soliton to undergo free expansion for a repulsive boson-fermion interaction.

In the present paper we study ultra-cold boson-fermion mixtures. Examples of such mixtures are \(^7\)Li–\(^6\)Li [32,33], \(^23\)Na–\(^9\)Li [34] and \(^{87}\)Rb–\(^{40}\)K [35]. The possibility of new solitonic perturbations in a mixture of boson and fermion atoms with short-range interaction potentials is analyzed in particular. Theoretical analysis at higher order of interaction account accuracy allows prediction of new types of soliton solutions in the BEC [36]. This approximation can be derived with the aid of quantum hydrodynamics approach, which has been developed in recent decade [36–38] and used in present paper. The set of quantum hydrodynamic equations describing an ultra-cold mixture of boson and fermion atoms was derived from the many-particle Schrödinger equation [36]. In reference [39], for example, they used this approach to study how the shape of a bright soliton changes as interactions in a Bose-Einstein condensate are considered more precisely [39]. The method of quantum hydrodynamics is very useful in different areas of physics. Namely, it was quantum plasma [37,40], plasma of particles with the own magnetic moment [41–45], relativistic quantum plasma [46], ultracold Bose and Fermi gases with nonlocal interaction [36], quantum particles with electrical [38,47] and magnetical [48] polarization, particularly being in the BEC state [38,48], the graphene electrons [49] and BEC of graphene excitons [50].

Most of theoretical papers devoted to the ultracold fermions contain a nonlinear Schrödinger equation (an analog of the Gross-Pitaevskii equation using for bosons) which does not include interaction between fermions [51–57]. It is connected with the fact that the first Born approximation gives no contribution in interaction of ultracold fermions. This equation undoubtedly accounts the Fermi pressure contribution caused by the Pauli principle. It is very interesting to understand a role of interaction in fermion systems. For example, vortices in a strongly interacting gas of fermionic atoms on the BEC- and the BCS-side of the Feshbach resonance was experimentally studied in reference [58]. Presented in our paper model based on the quantum hydrodynamics method contains fermion-fermion interaction whose form was derived directly from the Schrödinger equation [36].

In this paper we discuss the existence of new types of solitons in a mixture of boson and fermion atoms. A substantial role in our analysis is given to the fermion-fermion interaction coefficient, which appears in the analysis of the third order by the interaction radius. Obtained in this paper solution exists at repulsion between fermions \( T_{bb} < 0 \). If we have strong attraction between Bose atoms \( T_{bb} < 0 \) then we find dark soliton of bosons and, fermion perturbations would be solitons of compression (bright soliton) for \( T_{bf} > 0 \) or dark soliton for \( T_{bf} < 0 \). These are dark-bright and dark-dark solitons in a two-component system of bosons and fermions. At \( T_{bb} > 0 \) and \( |T_{bf}| < 1 \), in boson-fermion mixture we have bright-dark soliton for \( T_{bf} < 0 \), and bright-bright soliton for \( T_{bf} > 0 \). Influence of the boson-fermion interaction on the bright-like soliton solution in the BEC obtained in reference [23] is also considered here. A perturbation method introduced by Washimi and Taniuti [59], Infeld and Rowlands [60] is used here to find solitonic perturbations. It was used for a BEC studying in the TOIR approximation [23], where was shown that account of interaction up to TOIR approximation leads to the new solitons in the BEC. An analogous method was used for a spinor-1 BEC studying [61]. Application of the perturbation method to waves in the BEC studying is presented in reference [62]. We can admit that reference [62] is a brief review of application of the perturbation
method for one-dimensional the BEC in the first order by the interaction radius, for different conditions and scaling. Other methods for weak-nonlinear analysis of the BEC have been considered in literature. For example, in reference [63] the Krylov-Bogoliubov-Mitropolski method was used for nonlinear frequency shift calculation.

Our paper is organized as follows. In Section 2, we present basic equation and describe using model. In Section 3, we study solitons in boson-fermion mixture which appearing due to interaction account up to the TOIR approximation. In Section 3 detailed analysis of conditions of a “fermion” soliton existence is described. In Section 4 condition of a “boson” soliton existence is described. In Section 5 brief summary of obtained results is presented.

2 The model

In this paper we are going to use the set of the QHD equations for ultracold boson-fermion mixture including interaction up to the third order by the interaction radius. This set was obtained in reference [36]. Here, we briefly describe basic steps of this derivation to explain how we get terms in the TOIR.

The QHD equations are derived from the many-particle Schrödinger equation

\[ i\hbar \partial_t \psi = \hat{H} \psi, \]  

(1)

where \( \psi(\mathbf{r}_1, \ldots, \mathbf{r}_N, t) \) is the N-particle wave function depending on 3N coordinate of N particles and time \( t \), \( \hat{H} \) is the Hamiltonian. For one species of neutral particle (bosons and fermions), it reads

\[ \hat{H} = \sum_{i=1}^{N} \left( \frac{\mathbf{p}_i^2}{2m} + V_{i,ext} \right) + \sum_{i<j} U_{ij}, \]  

(2)

where \( \mathbf{p}_i = -i\hbar \nabla_i \) is the momentum operator, \( m \) is the mass of particle, \( V_{i,ext} \) is the potential of external field, and \( U_{ij} \) is the potential of short-range inter-particle interaction.

Determining particle concentration as the quantum mechanical average of the classical microscopic concentration we get

\[ n(\mathbf{r}, t) = \langle n \rangle = \int \psi^* \sum_{i=1}^{N} \delta(\mathbf{r} - \mathbf{r}_i) \psi dR, \]  

(3)

where \( R = (\mathbf{r}_1, \ldots, \mathbf{r}_N), \ dR = \prod_{i=1}^{N} d\mathbf{r}_i \), and \( \delta(\mathbf{r}) \) is the Dirac delta function.

Differentiating the concentration (3) and using the Schrödinger equation for time derivatives of the wave function we get the continuity equation, where the particle current, usually presenting \( j = n \mathbf{v} \), appears in explicit form defined via N-particle wave function. Differentiating particles current definition with respect to time we obtain the Euler equation, where for systems of particle with the short-range interaction, appears as an expansion in series on the interaction radius. For interaction between particles of considering species we have that the force field arises as divergence of the quantum stress tensor \( T_{\alpha} = \partial_\beta \sigma^{\alpha \beta} \). For ultracold gases it is interesting to consider the zero temperature limit. In this case we have bosons in the BEC state. Account of the first term in expansion series for the force density in system of ultracold particles gives different results for bosons and fermions. For bosons it leads to the GP approximation, and for fermions it give zero force field. So, we have no interaction between fermion, in the first order by the interaction radius, laying below the Fermi energy. The second term in the expansion, as all even terms, is equal to zero for spherically symmetric potentials, considered in reference [36]. Next, the third term, gives us contribution of the short range interaction in the third order by the interaction radius. In this approximation we obtain generalization of the GP equation for Bose particles, and we get contribution of interaction in dynamics of fermions.

Let’s consider a mixture of ultra-cold boson and fermion atoms having our focus on one-dimension perturbations. Considering of the one-dimensional equations corresponds to propagation of the plane wave along \( z \)-axis in three dimensional medium. This is an ideal case. In experiments researchers have deal with quasi-one-dimensional geometries, which is more complicated case in compare with the considered in our paper. However we make an estimation of the interaction influence on properties of boson-fermion mixtures. The set of quantum hydrodynamic equations consists of the following equations: the continuity equation for bosons

\[ \frac{\partial n_b}{\partial t} + \frac{\partial (n_b \mathbf{v}_b)}{\partial x} = 0, \]  

(4)

the momentum balance equation for bosons

\[ m_b \frac{\partial \mathbf{v}_b}{\partial t} + \frac{1}{2} m_b \mathbf{v}_b \frac{\partial \mathbf{v}_b^2}{\partial x} = \frac{\hbar^2}{2m_b} \frac{\partial}{\partial x} \left( \frac{1}{\sqrt{n_b}} \frac{\partial \sqrt{n_b}}{\partial x} \right) - \frac{1}{16} \frac{T_{bf} n_f}{\mathbf{v}_b} \frac{\partial n_f^2}{\partial x^3}, \]  

(5)

the continuity equation for fermions

\[ \frac{\partial n_f}{\partial t} + \frac{\partial (n_f \mathbf{v}_f)}{\partial x} = 0, \]  

(6)

and the momentum balance equation for fermions

\[ m_f \frac{\partial \mathbf{v}_f}{\partial t} + \frac{1}{2} m_f \mathbf{v}_f \frac{\partial \mathbf{v}_f^2}{\partial x} = \frac{\hbar^2}{2m_f} \frac{\partial}{\partial x} \left( \frac{1}{\sqrt{n_f}} \frac{\partial \sqrt{n_f}}{\partial x^2} \right) + 3 \frac{T_{bf} n_f}{8} \frac{\partial n_f}{\partial x} + \frac{3}{8} \frac{n_f T_{bf}}{m_f} \frac{\partial n_f^3}{\partial x^3} + \frac{1}{2} (3\pi^2)^{2/3} \frac{T_{bf}}{\mathbf{v}_f} \frac{\partial n_f^{5/3}}{\partial x} \frac{\partial \mathbf{v}_f}{\partial x} + \frac{\hbar^2}{5m_f} (3\pi^2)^{2/3} \frac{\partial n_f^{5/3}}{\partial x} \frac{\partial \mathbf{v}_f}{\partial x} + \frac{1}{2} \frac{T_{bf} n_f}{\mathbf{v}_f} \frac{\partial n_f^3}{\partial x^3}. \]  

(7)
The equations given above utilize following designations: $m_b, m_f$ – masses of boson and fermion atoms, respectively; $n_b, n_f$ – concentrations of bosons and fermions; $v_b, v_f$ – respective velocity fields. $\Upsilon_{bb}, \Upsilon_{bf}$ – coefficients of boson-boson and boson-fermion interactions at the first order of the interaction range. $\Upsilon_{2bb}, \Upsilon_{2bf}, \Upsilon_{2ff}$ – coefficients of boson-boson, boson-fermion and fermion-fermion interactions at the third order of the interaction range. Interaction coefficients can be defined with following equations:

\[
\Upsilon_{ij} = \frac{4\pi}{3} \int r^2 \frac{\partial U_{ij}(r)}{\partial r} \, dr, \quad (8)
\]

\[
\Upsilon_{2ij} = \frac{4\pi}{15} \int r^3 \frac{\partial U_{ij}(r)}{\partial r} \, dr. \quad (9)
\]

Let’s discuss physical meaning of equations (4)–(7). These equations are one dimensional form of the QHD equations for an ultracold boson-fermion mixture of neutral atoms with a short range interaction, where the Bose subsystem in the BEC state. Equations (4) and (6) are the continuity equations for bosons and fermions correspondingly. The continuity equation describes conservation of a total number of particles. Equations (5) and (7) are the Euler equations. They describe evolution of particles current and momentum of particles, which are proportional to each other in a non-relativistic theory. Therefore equations (5) and (7) are the equations of the momentum evolution. The first two terms in these equations present time evolution of velocity fields. The second terms are convective part of velocity field evolution. The third terms are the quantum Bohm potential, which appears there due to the Heisenberg uncertainty principle. It gives quantum contribution in the pressure tensor. Described terms have similar structure for different species. The fourth and fifth terms in the left hand side of equation (5) cause an interaction between bosons. The fourth term appears in the first order by the interaction radius and corresponds to the nonlinear interaction term in the GP equation. The next term exists at account of a short range interaction up to the third order by the interaction radius, this approximation was developed in reference [36]. Terms in the right hand side of equation (5) describe influence of the boson-fermion interaction on evolution of the Bose subsystem. The first (second) term appears in the first (third) order by the interaction radius. The fourth-sixth terms in the left-hand side of the momentum evolution equation for fermions (7) describe a short range interaction between fermions, all these terms emerge at account of fermion-fermion interaction up to the third order by the interaction radius. In this equation there are no terms appearing in the first or second order by the interaction radius. It corresponds to the fact that scattering length equals to zero for fermions laying below the Fermi level at zero temperature. The seventh term describes contribution of the Fermi pressure. In the right hand side of equation (7) we present terms describing contribution of the boson-fermion interaction in the fermions evolution.

Presented here hydrodynamics equations correspond to the system of two nonlinear Schrödinger equations one for bosons another for fermions [36]. These nonlinear Schrödinger equations are nonlocal and integro-differential. The nonlinear Schrödinger equation for Bose particles is a generalization of the Gross-Pitaevskii equation. This generalization appears due to more detailed account of the short-range interaction up to the third order by the interaction radius. For the first time a nonlocal Gross-Pitaevskii equation were derived in references [64,65]. This equation does not contain integral terms. Brief comparison for different nonlocal generalization of the Gross-Pitaevskii equation is discussed in reference [39].

Coefficients $\Upsilon_{bb}$ and $\Upsilon_{bf}$ are related to interactions coefficients $g_{bb}$ and $g_{bf}$ of the Gross-Pitaevskii equation as follows:

\[
\Upsilon_{bb} = -g_{bb}, \quad \Upsilon_{bf} = -g_{bf}. \quad (10)
\]

It should be noted that the following relationship exists between interaction coefficients and scattering amplitude $a_{ij}$

\[
\Upsilon_{ij} = -\frac{4\pi \hbar^2 a_{ij}}{m}. \quad (11)
\]

All equations above are approximated up to the third order of the interaction range. If terms containing $\Upsilon_{2bb}, \Upsilon_{2bf}, \Upsilon_{2ff}$, are neglected then the resulting equation set corresponds to the first-order approximation (the Gross-Pitaevskii approximation) which has been used, for example, in reference [11]. It was shown in reference [36] that the approximation is possible that derives coefficients $\Upsilon_{2bb}, \Upsilon_{2bf}$ from $\Upsilon_{bb}, \Upsilon_{bf}$. The explicit form of this approximation is:

\[
\Upsilon_{2bb} \simeq \rho_0^{-2} \Upsilon_{bb}
\]

and

\[
\Upsilon_{2bf} \simeq \rho_0^{-2} \Upsilon_{bf},
\]

where $\rho_0$ – is a constant of the same order of magnitude as the atomic radius.

3 The soliton solution in a boson-fermion mixture

The equation set in question should be expected to have soliton solutions of a new type due to more accurate recognition of atomic interactions. The method of perturbations may be applied to find this soliton [59,66]. According to this method all hydrodynamic values may be represented as:

\[
n_b = n_{0b} + \varepsilon n_{1b} + \varepsilon^2 n_{2b} + \ldots, \quad (12)
\]

\[
n_f = n_{0f} + \varepsilon n_{1f} + \varepsilon^2 n_{2f} + \ldots, \quad (13)
\]

\[
v_b = \varepsilon v_{1b} + \varepsilon^2 v_{2b} + \ldots, \quad (14)
\]

\[
v_f = \varepsilon v_{1f} + \varepsilon^2 v_{2f} + \ldots \quad (15)
\]

We also performed the following “scaling” of variables:

\[
\xi = \varepsilon^{1/2}(x - Ut) \quad (16)
\]
and

$$\tau = \varepsilon^{3/2} U_1.$$  \hspace{1cm} (17)

The latter expression introduces so-called “slow” time.

The following expression of the phase velocity $U$ can be derived from equations (4)-(7) in the first order of the small parameter $\varepsilon$:

$$U^2_\pm = \frac{1}{2m_m m_f} (\Theta m_m n_{0f} - m_f n_{0b} \Upsilon_{bb})$$

$$\pm \frac{1}{2m_m m_f} (\Theta m_b n_{0f} - m_f n_{0b} \Upsilon_{bb})^2$$

$$+ 4 m_m m_f n_{0b} n_{0f} \Theta (\Upsilon_{bb} + \Upsilon^2_{ff})^{1/2},$$  \hspace{1cm} (18)

where

$$\Theta \equiv (3\pi^2)^{2/3} \frac{\eta^2}{n_{0f}} \left( \frac{\hbar^2}{3m_f} \right)^{2/3} \frac{4}{3} \Upsilon_{2f} \Theta_{n_{0f}}^{5/3}.$$  \hspace{1cm} (19)

Quantity $\Theta$ describes the contribution of fermion dynamics in boson-fermion mixture. $\Theta$ consists of two parts. The first one appears from Fermi pressure and the second term presents the fermion-fermion short-range interaction which arises at interaction account up to the third order of the interaction radius. In formula (18) the quantity under radical is positive.

Since our mixture is comprised by two interacting subsystems, boson atoms and fermion atoms, two different values of phase velocity are obtained here. We call phase velocity $U_+$ which has “ + ” sign before the radical, the “fermion branch” of the solution. Phase velocity $U_-$ is called the “boson branch” of the solution by analogy. We introduced these terms to emphasize the fact that in the absence of boson-fermion interactions $\Upsilon_{bf} = 0$ the system can be divided in two subsystems of bosons and fermions, respectively, which do not interact. Phase velocity $U_+$ corresponds in this case to fermions

$$U^2_+ = \frac{n_{0f}}{m_f} \Theta,$$

and $U_-$ corresponds to bosons

$$U^2_- = - \frac{n_{0b}}{m_b} \Upsilon_{bb}.$$  \hspace{1cm} (20)

We can see that $U^2_- > 0$ for particles with repulsive interaction $\Upsilon_{bb} < 0$.

In the second order of the small parameter $\varepsilon$ we derive the Korteweg-de Vries equation for small perturbations of boson concentration in a boson-fermion mixture:

$$\frac{\partial n_{1b}}{\partial \tau} + q \frac{\partial^4 n_{1b}}{\partial \xi^4} + s n_{1b} \frac{\partial n_{1b}}{\partial \xi} = 0,$$  \hspace{1cm} (21)

where coefficient $p$ of the term containing slow time

$$p = 2U^2(\Theta m_b n_{0f} - \Upsilon_{bb} n_{0b} m_f - 2U^2 m_m m_f),$$  \hspace{1cm} (22)

coefficient $q$ of the term that corresponds to dispersion

$$q = - \left( \Theta n_{0f} - m_f U^2 \right)$$

$$\times \left( \frac{\hbar^2}{4m_f} + \frac{1}{8} \Upsilon_{2b} n_{0b} - \frac{1}{2} \Upsilon_{bf} n_{0f} \frac{\Upsilon_{bb}}{\Upsilon_{bf}} - \frac{1}{2} \Upsilon_{2f} \frac{U^2 m_b}{\Upsilon_{bf}} \right)$$

$$+ \left( \frac{\hbar^2}{4m_f} + \frac{1}{4} \Upsilon_{2f} n_{0f} \right) n_{0b} \Upsilon_{bb} + U^2 m_b$$

$$+ \frac{1}{2} \Upsilon_{2f} \Upsilon_{bf} n_{0b} n_{0f},$$  \hspace{1cm} (23)

and coefficient $s$ of the non-linear term

$$s = 3 \frac{U^2 m_b}{n_{0b}} (\Theta n_{0f} - U^2 m_f) + \Upsilon_{bf} (\Upsilon_{bb} n_{0b} + U^2 m_b)$$

$$+ \Upsilon_{bb} + \frac{U^2 m_b}{n_{0b}} \frac{n_{0b}}{\Upsilon_{bb}}$$

$$\times \left( \frac{2U^2 m_f}{n_{0f}} + 20 \frac{9}{(3\pi^2)^{2/3} \Upsilon_{2f} + \hbar^2 (3\pi^2)^{2/3} \Upsilon_{n_{0f}}^{3/3}} \right).$$  \hspace{1cm} (24)

The soliton solution of the Korteweg-de Vries equation (20) is well known in the form of

$$n_{1b} = \frac{3pV}{s \cosh^2 \left( \sqrt{\frac{\sqrt{24s}}{3}} \eta \right)},$$  \hspace{1cm} (25)

where $\eta = \xi - V\tau$.

Below we discuss perturbations in bosons implying that perturbations in fermions are similar to them due to the following linear relationship between $n_{1b}$ and $n_{1f}$:

$$n_{1f} = - \left( \Upsilon_{bb} + \frac{U^2 m_b}{n_{0b}} \right) n_{1b}.$$  \hspace{1cm} (26)

Soltion pairs (in bosons and fermions, respectively) of two types may evolve: dark-bright and dark-dark.

4 The fermion branch of the solution

Let’s investigate “fermion branch” of the solution, where phase velocity $U$ from equation (18) is taken with “ + ” sign.

If boson-fermion interactions are “strong” enough compared to boson-boson interactions, then $U^2 > 0$. We use apply assumption to the fermion branch below.

For numerical analysis of our results we introduce following dimensionless variables: dimensionless phase velocity

$$W_+ = m_f U_+ / (\hbar n_{0f}^{1/3}),$$

mass rate $\mu = m_f / m_b$, concentrations rate $N = \sqrt{n_{0b}/n_{0f}}$, and dimensionless interaction constants

$$\gamma_{bb} = m_f n_{0b} \Upsilon_{bb} / (n_{0f}^{1/3} \hbar^2),$$

$$\gamma_{bf} = m_f \Upsilon_{bf} n_{0f} \Upsilon_{bb} / (n_{0f}^{1/3} \hbar^2),$$

$$\gamma_{ff} = m_f n_{0f} \Upsilon_{2f} / \hbar^2, \quad B_{bb} = m_f n_{0b} \Upsilon_{2bb} / \hbar^2.$$
Expressions (24), (26) and (27) make it clear that no soliton exists in the boson-fermion mixture in the Gross-Pitaevskii approximation, as the radicand in (24) is negative.

So, we took into account the third order by the interaction radius.

Now the coefficient \( q \) has the following form:

\[
q = \beta \left( \frac{\alpha}{4m_b} + \frac{\hbar^2}{4m_f} + \frac{1}{4} \gamma_{bf} \right) + \ldots \tag{30}
\]

It means that if the repulsion of fermions is strong enough (i.e. the coefficient of fermion-fermion interaction \( \gamma_{bf} < 0 \)) the \( q \) value would be negative. Strong fermion-fermion interactions in the boson-fermion mixture make possible perturbations of a new type, which do not occur in the first order of the interaction range.

In the case the considered perturbations of concentration are small, solitons of two kinds may arise: solitons of fermion rarefication and of fermion compression. The soliton type is determined by the sign of its dimensionless amplitude of bosons

\[
\Xi_b = \frac{3pV}{\hbar m_b},
\]

and fermions \( \Xi_f \) which connected with the \( \Xi_b \) by formula (25).

The dependence between amplitude of soliton in the boson subsystem and the coefficient of boson-fermion interaction \( \gamma_{bf} \) is shown in Figures 2-4. This solution exists at repulsion between bosons (\( \gamma_{bb} < 0 \)). If \( \gamma_{bf} \) coefficient value is high positive (it corresponds to a strong attraction between boson and fermion atoms) then boson perturbations would be solitons of rarefication (dark soliton, it is shown in Fig. 2) and, as it follows from (25), fermion perturbations would be solitons of compression (bright soliton) for \( \gamma_{bf} > 0 \) or soliton of rarefication (dark soliton) for \( \gamma_{bf} < 0 \) (it is shown in Figs. 3 and 4). These are dark-bright and dark-dark solitons in a two-component system of bosons and fermions. So, a strong repulsion between bosons and fermions (i.e. negative values of the \( \gamma_{bf} \) coefficient) would lead to solitons of compression of bosons and fermions, which correspond to a dark-dark soliton.
The dimensionless soliton width $D$ can be defined by the following expression:

$$D = \sqrt{\frac{4\eta^2}{Vp}}.$$  \hfill (32)

It follows from (21) and (30) that soliton width depends on the fermion-fermion interaction. This dependence is plotted in Figure 5. Note that the surface’s shape does not change along with the sign of $\gamma_{bf}$ as the expression for the soliton width $D$ contains the coefficient of boson-fermion interaction in a squared form.

There is interesting behavior of this solution for attractive boson-boson interaction. In this case soliton exist in two areas of system parameters. For the large $|\gamma_{bf}|$ ($|\gamma_{bf}| > 10$), function $D(\gamma_{ff}, \gamma_{bf})$ is analogous to the same presented in Figure 5. But for $\gamma_{bb} > 0$ there is an area at the small $|\gamma_{bf}|$ where exist soliton solution. In this area soliton width is large, much more than 1. It is presented in Figures 6–8.

In the case $\gamma_{bb} > 0$ and $|\gamma_{bf}| < 1$ amplitude of bosons is positive $\Xi_b > 0$. So, we have bright soliton in subsystem of bosons Figure 6. From Figure 7, we can see that if $\gamma_{bf} < 0$ amplitude of soliton in Fermi subsystem is positive. Thus, we have found bright soliton in Fermi subsystem. If $\gamma_{bf} > 0$ we have dark soliton in Fermi subsystem. In the result in boson-fermion mixture we have bright-dark soliton for $\gamma_{bf} < 0$, and bright-bright soliton for $\gamma_{bf} > 0$.

Considered in the paper values of interaction parameters may be obtained using Feshbach resonance [67,68]. It follows from (8) and (9), that changing potential of atomic interaction changes both $T_{ij}$ and $T_{2ij}$ coefficients. Note that soliton of this type occurs due to boson-fermion interactions.

5 The boson branch of the solution

To obtain the boson branch phase velocity $U$ from equation (18) should be taken with “…” sign.
when soliton corresponding to boson branch exists at more large formation is possible, changes. In boson-fermion mixture a the acceptable range of physical values, where the soliton cause soliton formation in a subsystem of boson atoms.

“do not feel” the presence of fermions, divides into two subsystems that do not interact, bosons taken into account. If the mixture of bosons and fermions mixtures. It can be done only when the third order is allow to predict solitonic perturbations in boson-fermion systems. In our case we applied the QHD approach to the ultra-cold mixture of boson and fermion atoms with short-range interaction potential in order to analyze the possibility of small-scale solitonic perturbations. We show that solitons of two novel types may occur, which are related to fermion and boson branches of the solution, respectively.

In the fermion branch the soliton of a new type occurs due to boson-fermion interactions at the first order of the interaction range. Fermion-fermion interactions also play substantial role in formation of these perturbations. The type of soliton formed in the system (compressed or rarified) is determined by the sign of boson-fermion interaction, i.e. repulsion or attraction, respectively. The dependence of fermion concentration on system characteristics was successfully derived. Consideration of the third order of the interaction range is the key factor that allowed prediction of a novel type of soliton solution. It also helped us to find new acceptable ranges of physical values, where the existence of soliton is possible for the boson branch of the solution in the boson-fermion mixture, compared to the system comprised of bosons only.

6 Conclusion

The quantum hydrodynamics approach is well suitable to derive equations that describe multiparticle quantum systems. In our case we applied the QHD approach to the ultra-cold mixture of boson and fermion atoms with short-range interaction potential in order to analyze the possibility of small-scale solitonic perturbations. We show that solitons of two novel types may occur, which are related to fermion and boson branches of the solution, respectively.

If the interaction of bosons and fermions occurs, then the acceptable range of physical values, where the soliton formation is possible, changes. In boson-fermion mixture a soliton corresponding to boson branch exists at more large strength of boson-boson interaction, in comparison with the case when \( \Delta_{bf} = 0 \). We have bright soliton solution \((\Xi_b > 0)\) in Bose subsystem. In Fermi subsystem we find dark \( \Xi_f < 0 \) (bright \( \Xi_f > 0 \)) soliton for repulsion (attraction) between Bose and Fermi particles \( \Delta_{bf} < 0 \) \((\Delta_{bf} > 0)\).

We have got it because quantity \(-\Delta_{bf} + m_b U^2 / m_a\) presented by formula (25) is positive. Consequently, \( \Xi_f \) has the same sign as \( \Delta_{bf} \). So, in a mixture of bosons and fermions this type of soliton solution may exist.

Finally, we can note that two soliton solutions exists in boson-fermion mixture due to account of short range interaction up to the third order of the interaction radius.

Fig. 7. The dependence curve of the soliton width \( D \) (blue-top surface) and amplitude for bosons \( \Xi_b \) (red-middle surface) on the coefficient of boson-fermion \( \gamma_{bf} \) and fermion-fermion \( \gamma_{ff} \) interaction for the fermion branch. Soliton exist if its width \( D \) is positive. We present soliton width \( D \) and amplitude for bosons \( \Xi_b \) on the same plot to show the value of amplitude in the area where soliton exist. Green plane-lower surface present zero level. It show us that amplitude is positive. The plot is built with following parameters: \( \gamma_{bf} = 10^{-2}, \mu = 1, N = 1, B_{bf} = 10^{-4}, \) and \( B_{bb} = -10^{-4} \).

Fig. 8. The dependence curve of the soliton width \( D \) the coefficient of boson-fermion \( \gamma_{bf} \) and fermion-fermion \( \gamma_{ff} \) interaction for the fermion branch. The plot is built with following parameters: \( \gamma_{bf} = 10^{-2}, \mu = 1, N = 1, B_{bf} = 10^{-4}, \) and \( B_{bb} = -10^{-4} \).

Using the first order of the interaction radius does not allow to predict solitonic perturbations in boson-fermion mixtures. It can be done only when the third order is taken into account. If the mixture of bosons and fermions divides into two subsystems that do not interact, bosons “do not feel” the presence of fermions, \( \Delta_{bf} = 0 \), and the coefficient \( q \approx -h^2 / 4m_b n_{bf} \tau_{2bf} / 8 \) \[23\]. This means that repulsion of bosons \( (\Delta_{bf} < 0) \) must be strong enough to cause soliton formation in a subsystem of boson atoms.

If the interaction of bosons and fermions occurs, the acceptable range of physical values, where the soliton formation is possible, changes. In boson-fermion mixture a soliton corresponding to boson branch exists at more large strength of boson-boson interaction, in comparison with the case when \( \Delta_{bf} = 0 \). We have bright soliton solution \((\Xi_b > 0)\) in Bose subsystem. In Fermi subsystem we find dark \( \Xi_f < 0 \) (bright \( \Xi_f > 0 \)) soliton for repulsion (attraction) between Bose and Fermi particles \( \Delta_{bf} < 0 \) \((\Delta_{bf} > 0)\).

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