The Canonical Transformation and Duality in the 1+1 dimensional $\phi^4$ and $\phi^6$ theory

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Abstract

We investigate the self-organizing nature of relativistic quantum field theory in terms of canonical transformation and duality presenting simple but explicit examples of $(\phi^4)_{1+1}$ and $(\phi^6)_{1+1}$ theories. Our purpose is fulfilled by applying the oscillator representation (OR) method which allows us to convert the original strong interaction theory into a weakly interacting quasiparticle theory that is equivalent to the original theory. We discuss advantages of the OR method and compare the results with what was already obtained by the method of Gaussian effective potential (GEP) and the Hartree approximation (HA). While we confirm that the GEP results are identical to the Hartree results for the ground state energy, we found that the OR method gives the quasiparticle mass different from the GEP and HA results. In our examples, the self-organizing nature is revealed by the vacuum energy density that gets lowered when the quasiparticles are formed. In the $(\phi^6)_{1+1}$ theory, we found two physically meaningful duality-related quasiparticle solutions which have different symmetry properties under the transition of quasiparticle field $\Phi \to -\Phi$. However, these two quasiparticle solutions yield the identical effective potential in the strong coupling limit of the original theory.

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I. INTRODUCTION

With the current advances of Relativistic Heavy Ion Collision (RHIC) physics, there is a growing interest in discussing the self-organizing nature of relativistic quantum field theory (RQFT). The phase transition and the spontaneous symmetry breaking anticipated to be observed in the RHIC facilities are the paramount examples of the physical phenomena due to the self-organization of quantum fields. Since these novel phenomena cannot be easily predicted in the ordinary perturbation series, they form highly nontrivial examples that ought to be analyzed by various nonperturbative methods available for the RQFTs.

Although the problem of quantization of interacting fields is not yet completely solved, it appears that the duality existing in the RQFTs is the key to handle the strong interaction theories. According to the duality, the original particle theory of strong couplings may be equivalent to the quasi-particle theory of weak couplings. As the interactions get stronger, the associate quantum fluctuations get larger and consequently the formation of a nontrivial vacuum often accompanied by the condensation of fields may be energetically more favorable than the adherence to an original trivial vacuum without any condensation. The particles moving in the ambient nontrivial vacuum condensates may be described by the quasi-particles which carry different masses compare to their original masses defined in the trivial vacuum. Then, the couplings among the quasi-particles may become weak as the couplings among the original particles grow strong. When such duality works self-consistently within the RQFT, one may be able to solve the corresponding strong interaction problem utilizing the weak coupling developed for the quasi-particles. For the phenomena involving only weak interactions, the perturbation theory may provide systematic predictions and even some physical insight how the interaction can affect the properties of the system. Thus, the duality allows us to convert the original nonperturbative problem into the quasiparticle perturbative problem. However, the key issue here is to check if the RQFTs indeed have the self-organizing characteristics that lead to the duality within themselves. To investigate such nontrivial characteristics, one cannot just rely on the perturbation series from the beginning but need to develop a nonperturbative method which may be useful to analyze the self-organizing nature of the RQFTs.

In this work, we discuss such a nonperturbative method that appears to be much simpler than other rather well-known methods such as the Hartree Approximation (HA) and the Gaussian Effective Potential (GEP) method. Following the nomenclature in the literature we call this method as the Oscillation Representation (OR) method, although in our view all of these nonperturbative methods (OR,GEP,HA) stem from the same principle of quantum effective action approach [1]. The OR method was explicitly formulated by Efimov [2] and was, in fact, used earlier by Chang [3] and Magruder [4], though in some indirect way. The method is at length described in a monograph [5]. The basic idea of the OR method is to redefine the mass of interacting field relative to the free one and simultaneously introduce a shift of the field quantization point leading to a nonzero value of its vacuum condensate. This effect is realized in the nature represented by a spontaneous symmetry breaking mechanism and thus one can expect that it may be a general feature of quantum field systems with interactions. However, the OR method has not been utilized as extensively as the methods of GEP [6–10] and HA [11,12]. Even in the 1+1 dimensional scalar field theory such as the one we present in this work, not much of the OR results distinguished from the GEP or HA
results have been known or discussed to the best of our knowledge. Although the OR results for the \((\phi^4)_{1+1}\) theory were presented in Ref. [5], the distinction between the OR and GEP (or HA) results for this simple model was not pointed out at length. Thus, for the present work, we analyze the details of the distinction in this most simple case first and extend the application to the \((\phi^6)_{1+1}\) theory:

\[
\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{g}{4!} \phi^4 - h\phi^6,
\]

where the results from the variational method of the GEP have also been studied [9]. The GEP is defined as

\[
\bar{V}_G(\phi_0) = \min_\Omega \langle \phi_0, \Omega | \mathcal{H} | \Omega, \phi_0 \rangle,
\]

where \(\mathcal{H}\) is the Hamiltonian density corresponding to Eq.(1). Here, \(\Omega\) is the mass parameter and \(|\Omega, \phi_0\rangle\) is a normalized Gaussian wave functional centered on \(\phi = \phi_0\), i.e.

\[
\langle \phi_0, \Omega | \Omega, \phi_0 \rangle = 1, \quad \langle \phi_0, \Omega | \phi | \Omega, \phi_0 \rangle = \phi_0.
\]

As we will discuss the details in the next two sections (Sections II and III), the GEP method can in principle give the two OR equations that we solve simultaneously in this work. One of them is known as the \(\bar{\Omega}\) equation in the GEP, i.e. \(\partial < \phi_0, \Omega | \mathcal{H} | \Omega, \phi_0 \rangle / \partial \Omega = 0\) at \(\Omega = \bar{\Omega}\), and the other OR equation is equivalent to \(d\bar{V}_G(\phi_0)/d\phi_0 = 0\). However, in the GEP method, these two equations are not solved simultaneously in practice. It turns out that the quasiparticle mass defined by \(d^2 \bar{V}_G(\phi_0)/d\phi_0^2\) in the GEP is not same with \(\bar{\Omega}\) which is the simultaneous solution of the quasiparticle mass from the two OR equations. In this paper, we therefore pay more attention to the distinct aspects of the OR results compared to the GEP and HA results and discuss the property of the OR results which was not mentioned in the more great detail before.

Formally, the OR is settled by requiring that the Hamiltonian operator should be written in terms of creation and annihilation operators of an oscillator basis with an appropriate frequency and in the correct form defined by

1. all field operators in the total Hamiltonian \(H = H_0 + H_I\) are written in the normal ordered product,
2. the Hamiltonian \(H_0\) is quadratic over the field operators,
3. the interaction Hamiltonian \(H_I\) contains field operators in powers more than two.

Using this method with the canonical transformation of quantum fields, one may allow to build a standard perturbation theory for the quasiparticles in the region of the coupling constants where the original theory is nonperturbative. Thus, we utilize the OR method to transform the original theory which is intrinsically nonperturbative to the equivalent theory of the quasiparticle representation that can be solved rather easily by the perturbation method.

As we will discuss, the exact OR results include several vacuum solutions yielding different mass values of the quantum field for fixed values of coupling constants. Among those multiple solutions, however, it is certainly not difficult to select the non-trivial solutions that are physically meaningful because of the duality consideration. In this work, we will thus focus only those duality-related solutions which are not hindered from the physical
interpretation. Nevertheless, we note that the OR method generates also the solutions that cannot be interpreted as straightforward as the ones we focus in this paper. In particular, we find a class of solution that doesn’t exhibit the anticipated duality property and another class of solution that we call spurious in the sense that it yields the pure imaginary value for the vacuum condensation. We do not yet know all the physical meaning of these classes of solutions, but the generation of multiple solutions are expected because the OR equations are the nonlinear algebraic equations relating the quasiparticle mass and the field condensation with the parameter of coupling constant as we will present in details for the rest of the paper.

In Section II, we present more details of the OR method using the $\phi^4$ theory which can be easily deduced from the Lagrangian density of $\phi^6$ theory by taking $h = 0$. We compare the OR results with the results obtained by the other available methods such as the GEP and the HA as well as the available lattice result. In Section III, we apply the OR method to the $(\phi^6)_{1+1}$ theory and discuss the effects from the presence of $\phi^6$ interaction to the duality-related vacuum solutions. We analyze the quasiparticle mass and the energy density in the entire domain of coupling parameter space and present the classical potential for the nontrivial solutions. We discuss two physically meaningful duality-related quasiparticle solutions with different symmetry properties and show that they however generate the identical effective potential in the strong coupling limit of the original theory. Summary and conclusions follow in Section IV. In the Appendix A, we briefly summarize some details of the HA in the $(\phi^4)_{1+1}$ theory and discuss an equivalence between the two nontrivial vacuum solutions in this approximation. In the Appendix B, we present a proof of unitary inequivalence between the bifurcated nontrivial solutions from the OR method in the infinite volume limit. In the Appendix C, we summarize the results of the spurious solutions with some discussion.

II. THE OSCILLATOR REPRESENTATION (OR) METHOD

The usual recipe to produce a quantum system comes from the correspondence principle. Namely, a quantum system can be obtained from its classical analog by assuming that the field is not a $c$-number but an operator satisfying relevant commutation relations. The oscillator representation method is a generalization of this scheme. To start with, let us consider the $\phi^4$ theory as a simple example.

The Hamiltonian density of the classical field is given by

$$H = \frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{g}{4} \phi^4.$$  \hspace{1cm} (4)

To plug the quantum physics in, we postulate that the field $\phi$ and the corresponding conjugate momentum $\pi$ are operators which satisfy canonical equal time commutation relations

$$[\phi(x), \pi(y)]_{x_0=y_0} = i\delta(x - y).$$  \hspace{1cm} (5)

The usual approach to quantize the field, i.e. to find the operators $\phi$ and $\pi$ which satisfy the condition (5) is to start from the free field, where $g=0$. Then the operators are
\[
\phi_m(x) = \int \frac{dk}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega_m(k)}} (a_m(k)e^{ik \cdot x} + a_m^+(k)e^{-ik \cdot x}),
\]
\[
\pi_m(x) = \frac{1}{i} \int \frac{dk}{\sqrt{2\pi}} \frac{\omega_m(k)}{2} (a_m(k)e^{ik \cdot x} - a_m^+(k)e^{-ik \cdot x}),
\]
(6)

where \( k = (\omega_m, \mathbf{k}), \omega_m = \sqrt{k^2 + m^2} \) and the operators of creation and annihilation of the particle of mass \( m \) and momentum \( \mathbf{k} \) satisfy the commutation relations

\[
[a_m(k), a_m^+(k')] = \delta(k - k'),
\]
\[
[a_m(k), a_m(k')] = [a_m^+(k), a_m^+(k')] = 0.
\]
(7)

With this definition the state \( a_m^+(k)|0\rangle \) is an eigenstate of both free Hamiltonian and momentum operators with corresponding eigenvalues \( \omega_m \) and \( k \).

In the case of interacting fields the states \( a_m^+(k)|0\rangle \) are not already eigenstates of the full Hamiltonian. Nevertheless, in the case of small interaction strength, the free representation of field operators can be a good starting point to develop a perturbation theory. The Hamiltonian density operator is then written as

\[
\mathcal{H} \rightarrow \mathcal{H}_m = \frac{1}{2} \pi_m^2 + \frac{1}{2} (\nabla \phi_m)^2 + \frac{1}{2} m^2 \phi_m^2 + \frac{g}{4} \phi_m^4.
\]
(8)

Now, we use the canonical transformation of the field operators to build the quasiparticle representation of the same Hamiltonian. We consider the essential point of the reparameterizations, \( i.e. \) the change of mass, which can affect the choice of representation for the free field quantization. For the free field operators, the change of mass \( m \rightarrow M \) of particles is described by the Bogoliubov-Valatin transformation:

\[
a_M(k) = \frac{1}{2} \left( \sqrt{\frac{\omega_m(k)}{\omega_M(k)}} + \sqrt{\frac{\omega_M(k)}{\omega_m(k)}} \right) a_m(k) - \frac{1}{2} \left( \sqrt{\frac{\omega_m(k)}{\omega_M(k)}} - \sqrt{\frac{\omega_M(k)}{\omega_m(k)}} \right) a_m^+(\mathbf{k}),
\]
\[
a_M^+(k) = \frac{1}{2} \left( \sqrt{\frac{\omega_m(k)}{\omega_M(k)}} + \sqrt{\frac{\omega_M(k)}{\omega_m(k)}} \right) a_m^+(k) - \frac{1}{2} \left( \sqrt{\frac{\omega_m(k)}{\omega_M(k)}} - \sqrt{\frac{\omega_M(k)}{\omega_m(k)}} \right) a_m(-\mathbf{k}).
\]
(9)

Here, the free field operators with mass \( M \) are expressed in terms of the field operators with mass \( m \).

A consequence of the ambiguity of the mass definition due to interactions comes from the ambiguity in the choice of the initial representation of the interacting field. This was emphasized in Coleman’s paper [13]. He also proposed a useful technique on how to redefine a normal ordered product of any number of field operators with respect to a new value of mass in the case of \((1+1)\) scalar field theory. The redefinition after the mass change is given by the formulas [3,13]:

\[
N_m \left( \frac{1}{2} \pi_m^2 + \frac{1}{2} (\nabla \phi_m)^2 \right) = N_M \left( \frac{1}{2} \pi_M^2 + \frac{1}{2} (\nabla \phi_M)^2 \right) + \frac{1}{8\pi} (M^2 - m^2), \]
(10)

\[
N_m(e^{i\beta \phi_m}) = \left( \frac{M^2}{m^2} \right)^{\beta^2/8\pi} N_M(e^{i\beta \phi_M}),
\]
(11)
where \( \beta \) is some arbitrary \( c \)-number, \( N_m(N_M) \) stands for the normal ordering of an operator with respect to the initial mass \( m \) (the new mass \( M \)), \( \phi_m \) is the free quantum field defined by Eq. (6) and \( \phi_M \) is the quantum field of independent quasi-particles for which the representation given by Eq. (6) is valid after changing \( m \rightarrow M \). The correspondence between the canonical transformation and the renormalization group equation is discussed in Ref. [5].

The expression (11) can be rewritten as

\[
N_m(\phi_m^n) = n! \sum_{j=0}^{\left[ \frac{n}{2} \right]} \left( \frac{1}{8\pi} \ln\left( \frac{M^2}{m^2} \right) \right)^j \frac{(-1)^j}{j!(n-2j)!} N_M(\phi_M^{n-2j}),
\] (12)

where \( \left[ \frac{n}{2} \right] \) is the integer part of \( \frac{n}{2} \).

The most simple illustration of nontrivial solutions in this case is the 1+1 dimensional \( \phi^4 \) theory. Rewriting the Hamiltonian (8) in terms of the quasi-particles using Coleman’s formula of normal ordering rearrangement, one gets

\[
N_m(H_m) = N_M(H_M) + \frac{1}{2}(m^2 - M^2)N_M(\phi^2_M) + \frac{1}{8\pi}(M^2 - m^2) - \frac{m^2t}{8\pi}N_M(\phi^2_M) + \frac{3g^2}{64\pi^2},
\] (13)

where \( t = \ln\left( \frac{M^2}{m^2} \right) \).

For a consistent description of the dynamics in terms of the quasi-particles one should provide the right form of the Hamiltonian, i.e. the free part should correspond to that of the field with mass \( M \) and the interacting part should contain only terms with powers of \( \phi_M \) larger than 2 [5]. As it can be seen, the expression (13) itself can not provide this requirement for the field with mass other than \( m \). Revising just the mass does not lead to a nontrivial solution and thus an additional transformation is obviously required. The latter can be done by shifting the quantization point of the field, which makes the quanta of the field to be produced around some vacuum expectation value \( \langle \phi \rangle \neq 0 \).

Thus, let us now change the field variable from \( \phi_M(x) \) to \( \Phi_M(x) \) for the inclusion of the field condensation denoted by \( b(= \langle \phi_M \rangle) \); i.e.,

\[
\phi_M(x) = \Phi_M(x) + b.
\] (14)

Then the Hamiltonian density is written as [5]:

\[
N_M(\mathcal{H}) = N_M(\mathcal{H}^\text{right}_M) + \mathcal{H}_1 + \varepsilon_M,
\] (15)

where \( \mathcal{H}^\text{right}_M \) is the right form of Hamiltonian density for the field with mass \( M \),

\[
\mathcal{H}^\text{right}_M = \frac{1}{2}\pi_M^2 + \frac{1}{2}(\nabla \Phi_M)^2 + \frac{M^2}{2}\Phi_M^2 + \frac{g}{4}\Phi_M^4 + gb\Phi_M^3,
\] (16)

\( \varepsilon_M \) is the energy density of the vacuum for the quasiparticle of mass \( M \) up to the zeroth order of the dimensionless effective coupling \( g/M^2 \),

\[
\varepsilon_M = \frac{m^2b^2}{2} + \frac{gb^4}{4} - \frac{m^2t}{8\pi} - \frac{3gb^2t}{8\pi} + \frac{3g^2}{64\pi^2} + \frac{M^2 - m^2}{8\pi},
\] (17)
and

\[ H_1 = \left\{ \frac{1}{2} (m^2 - M^2) + \frac{3}{2} gb^2 - \frac{3gt}{8\pi} \right\} \Phi^2 \]  
\[ + \left\{ m^2 b + gb^3 - \frac{3gbt}{4\pi} \right\} \Phi M \]  
(18)

is a remainder. To have the Hamiltonian in the right form the remainder must be equal to 0. Therefore, we have two equations of two variables, \( t \) and \( b \), as follows:

\[ \frac{1}{2} \left( 1 - \frac{M^2}{m^2} \right) + \frac{3}{2} Gb^2 - \frac{3Gt}{8\pi} = 0, \]  
(19)

and

\[ b \left( 1 + Gb^2 - \frac{3Gt}{4\pi} \right) = 0, \]  
(20)

where \( G \) is the dimensionless coupling of the initial theory, \( i.e. \ G = g/m^2 \).

We note here the correspondence to the GEP discussed in Section I. The energy density \( \varepsilon_M \) given by Eq.(17) corresponds to \( \langle \phi_0, \Omega | H | \Omega, \phi_0 \rangle \) presented in Eq.(2). Minimizing \( \langle \phi_0, \Omega | H | \Omega, \phi_0 \rangle \) (or \( \varepsilon_M \)) with respect to \( \Omega \) (or \( t \)), one gets the GEP given by Eq.(2) with the so-called \( \bar{\Omega} \) equation \(^1\) which is identical to Eq.(19) that is one of the OR equations. The other OR equation, Eq.(20), can be obtained by making the total (not partial) derivative of the GEP \( (i.e., \text{Eq.}(4.11) \text{ of Ref. [8]} \text{ or Eq.}(2.16b) \text{ of Ref. [10]}) \) with respect to the condensation \( \phi_0(=b) \) and setting it to zero, \( i.e. \ d\bar{V}_G(\phi_0)/d\phi_0 = 0 \). One should note, however, that the GEP is a function of the global field condensate \( \phi_0 \) not the local field \( \phi(x) \). Thus, the GEP is not a part of interaction Hamiltonian describing the field dynamics. In our work, instead of using the GEP, we will directly use the interaction Hamiltonian and the corresponding classical potential \( \text{see e.g. Eq.}(22) \).

The OR equations (Eqs.(19) and (20)) have three solutions and one of them is trivial: \( t = 0, b = 0 \). The two nontrivial mass solutions are shown in Fig.1 as solid and dashed lines. The dashed line solution on Fig.1 becomes similar to the trivial one when the dimensionless coupling becomes large: \( M_{G \rightarrow \infty} \rightarrow m, b_{G \rightarrow \infty} \rightarrow 0 \). However, the solid line solution from Fig.1 is what we are interested in because it fulfills the duality. For this case, \( M_{G \rightarrow \infty} \rightarrow \frac{3}{2\pi} m^2 G \ln G, b_{G \rightarrow \infty} \rightarrow \sqrt{\frac{3}{4\pi}} \ln G \). As one can see in this solution, when the dimensionless coupling \( G \) of the initial theory becomes large, the dimensionless couplings of the quasiparticle theory, \( \chi^{(3)} = gb/M^2 \) and \( \chi^{(4)} = g/M^2 \), get small \([5]\): \( \chi^{(3)}_{G \rightarrow \infty} \rightarrow \sqrt{\frac{3}{8\pi G \ln G}}, \) \( \chi^{(4)}_{G \rightarrow \infty} \rightarrow \frac{2\pi}{3G \ln G} \). Thus, the quasiparticle theory becomes perturbative when the initial coupling \( G \) gets very strong. The physical mass in this regime is close to \( M \approx m\sqrt{\frac{3}{2\pi} G \ln G} \), and the corrections are in the order of \( O((\chi^{(3)})^2) \). In this way, we can see the duality between the initial nonperturbative theory of unbroken \( \phi \rightarrow -\phi \) symmetry with a large dimensionless coupling \( G \) and the quasiparticle theory of broken \( \Phi \rightarrow -\Phi \) symmetry with small couplings.

\(^1\)See Eq.(4.12) of Ref. [8] which was rederived as Eq.(2.16a) of Ref. [10] using Bogoliubov transformations.
\( \chi^{(3)} \) and \( \chi^{(4)} \). This duality is exact because the canonical transformation leaves the Hamiltonian intact but changes only its representation in terms of the quasiparticle field variable \( \Phi_M(x) \).

As we discussed earlier, the multiple solutions are not unexpected because the OR equations, Eqs.(19) and (20), are algebraically nonlinear. However, it can be shown (see Appendix B) that the two vacua associated with different values of \( t \) or \( b \) are unitarily inequivalent in the infinite volume limit. Thus, the two nontrivial solutions are not connected to each other. The bifurcation of quasi-particle masses appear above the coupling \( G \approx 9.04 \). However, the correct critical coupling of the phase transition should be determined from the comparison of the energy density between the trivial and condensed vacua. In Fig.2, we present the energy density of the nontrivial solutions and find that the condensed vacuum energy density of the duality-related solution gets lower than the trivial vacuum energy density only above \( G \approx 10.21 \). This value \( G \approx 10.21 \) coincides with the critical coupling constant obtained by the GEP method since our energy density given by Eq.(17) is equal to the GEP at the minimum. This value is remarkably close to the critical coupling constant (10.24(3)) obtained by the lattice calculation [14]. Nevertheless, we note that our OR result doesn’t take into account the the higher order corrections in the quasiparticle effective coupling \( g/M^2 \) and obtain the first order phase transition instead of the second order phase transition which may be more accurate description of the phase transition in \((\phi^4)_{1+1}\) theory [11,15]. In this work, we thus focus only on the region where the original coupling is much larger than the critical coupling so that the perturbative calculation in terms of quasiparticle degrees of freedom is allowed.

Although there is a coincidence of the critical coupling, it is not clear if the values of quasi-particle masses defined in the GEP, HA and OR methods also coincide. In the GEP approach, it seems only natural to define the quasiparticle mass as the second derivative of the GEP\(^2\), \( i.e. \frac{d^2\tilde{V}_G(\phi_0)}{d\phi_0^2} \). As discussed earlier, in the GEP approach a variational method is taken to minimize the GEP satisfying just Eq.(19). Confirming the well-known equivalence between the GEP method and the HA for the ground-state energy \([1,6]\), the GEP result using this definition of quasi-particle mass, \( i.e. \) the second derivative of the GEP, indeed agrees precisely with the Hartree results (squares and circles in Fig.1)\(^3\). However, one should note that the second derivative of the GEP at the minimum \( \phi_0 = b \) does not yield the mass \( M \) of the quasi-particle precisely but somewhat modified value \( M\sqrt{\frac{4\pi M^2 - 6\lambda}{4\pi M^2 + 3g}} \), \( i.e. \) with the notations used in Eq.(4.11) of Ref. [8], the second derivative of the GEP, \( \tilde{V}_G \), is given by

\[
\frac{d^2\tilde{V}_G}{d\phi_0^2} \bigg|_{\phi_0 = \pm \sqrt{\frac{\pi x}{\pi + 3\lambda}}} = x \left( \frac{\pi x - 6\lambda}{\pi x + 3\lambda} \right) \neq x, \quad (21)
\]

\(^2\)See for example Eq.(3.2) of Ref. [8].

\(^3\)In Chang’s paper [11] he found two different types of nontrivial solutions using Hartree method (\(4\pi c^2 > 3 \) and \(4\pi c^2 < 3 \) in his notations). Nevertheless, the two type of solutions lead in fact to the same physical results as shown in Fig.1. For more details, see the Appendix A.
where \( x = \frac{M^2}{m^2}, \)  \( \hat{\lambda}_B = \frac{g}{4m^2} \) and \( \phi_0 = b \) in our notations. Removing \( t(=\ln x) \) in Eqs.(19) and (20), one can immediately get \( b = \pm \frac{M}{\sqrt{2g}} \) identical to \( \phi_0 = \pm \frac{x}{\sqrt{8\lambda_B}} \). This is the reason why the GEP result given by Eq.(21) shown by dotted line in Fig.1 doesn’t coincide either of the bifurcated OR solutions (solid and dashed lines). Of course, if both the \( \Omega \) equation and the GEP minimum condition \( dV_G(\phi_0)/d\phi_0 = 0 \) are solved simultaneously to determine \( \Omega \), then it is identical to the OR method and the value of \( \Omega \) coincides with the quasiparticle mass \( M \) defined in the OR method. However, we note once again that the GEP and HA methods are quite different from the OR procedure because the GEP and HA methods rely on the variational procedure while the OR method is explicitly solving both Eqs.(19) and (20) simultaneously.

As we have already discussed, the solid line in Fig.1 exhibits a duality between the theory with mass \( m \) based on the trivial vacuum \( b = 0 \) and the theory with mass \( M \) based on the nontrivial vacuum \( b \neq 0 \). Namely, a strong coupling theory with mass \( m \) is identical to a weak coupling theory with mass \( M \). A similar observation of duality can be found in the solutions obtained by GEP and HA (see also Appendix A), although the quasiparticle masses defined in the GEP and HA approaches do not coincide with the quasiparticle mass defined in the OR method. We also note that the GEP result (dotted line) approaches to the OR result given by the solid line in the limit \( G \to \infty \) since the terms of \( x \) overwhelm the terms of \( \hat{\lambda}_B \) in Eq.(21) for that limit.

In Ref. [10], it was argued that the method of Bogoliubov-Valatin transformations gives the same results as the GEP. In fact all of the nonperturbative methods that we discuss in this work, i.e. OR, GEP and HA, can be described formally in terms of Bogoliubov-Valatin transformations because all of them use only the change of particle mass and the field shift to get the new representation of the field variables. However, these transformations contain parameters (mass ratio and the value of the shift) which can not be determined from the transformations themselves. To fix the parameters, some additional requirement is necessary. The authors of Ref. [10] used essentially the same variational requirement as the GEP and therefore it is very natural that they found the same results as the GEP. On the other hand, in the OR method, we are not using the GEP at all but relying on the form of the Hamiltonian described in Section I.

Applying the correspondence principle to the Hamiltonian (16), we now consider the classical potential given by

\[
V^{\text{class}}(\Phi_M^{\text{class}} = \phi_M^{\text{class}} - b) = \frac{M^2}{2}(\phi_M^{\text{class}} - b)^2 + \frac{g}{4}(\phi_M^{\text{class}} - b)^4 + gb(\phi_M^{\text{class}} - b)^3, \tag{22}
\]

which is written here in terms of the classical field corresponding to the initial, unshifted field \( \phi_M \). As it can be easily seen, the potential has extremum at \( \phi_M^{\text{class}} = b \). This is provided by the demand not to have terms in Hamiltonian which are linear in \( \Phi_M \). The double derivative in this point is larger than 0 and equal to \( M^2 \), i.e. the classical potential has minimum at \( \phi_M^{\text{class}} = b \) and the mass corresponding to oscillations around the minimum is precisely \( M \). That is why the OR method provides self-consistent procedure of quantization of interacting fields and interpretation of the solutions as particle excitations is valid. (In contrast, \( \frac{d^2V_G}{d\phi_0^2} |_{\phi_0=b} \neq \frac{M^2}{m^2} \) for the GEP as shown in Eq.(21).) In addition, one can easily
prove the symmetry of $V_{\text{class}}$ under the exchange of $\phi_M^{\text{class}}$ and $-\phi_M^{\text{class}}$, i.e. $V_{\text{class}}(\phi_M^{\text{class}}) = V_{\text{class}}(-\phi_M^{\text{class}})$, utilizing the OR equations (Eqs. (19) and (20)).

In Fig. 3, the effective potential $V_{\text{class}}$ given by Eq. (22) is presented. As one can see in Fig. 3, the bifurcated solutions for $G > 9.04$ indeed generate two effective potentials (I,II) at a given coupling $G$, e.g. $G = 10$ or 12. Both potentials (I and II) in Fig. 3 have clear minima and the second derivative at each minimum generates the corresponding quasi-particle mass $M$ precisely. The growth of bifurcation is also evident from the comparison of the potentials I and II for $G = 10$ and 12. We note however that only the potential I provides the physically meaningful duality-related solution and the potential II doesn’t satisfy the duality that we discuss in this work.

An analysis of realistic models using the oscillator representation method is more complicated (see, for example, discussion for 3+1 dimensional $\phi^4$ theory in [5]) due to substantial sophistication of the renormalization procedure. Even the simple models however were not fully studied in the framework of OR method. As presented in this Section, the comparison of the OR predictions with the GEP and Hartree results has been done for 1+1 dimensional $\phi^4$ theory [7,11]. In higher dimensions, the variational methods such as the GEP and HA may have a rather fundamental difficulty. The reason of this problem originates from the error estimates of the approximation in the GEP and Hartree methods. For instance, the HA replaces the interaction term $\lambda \phi_n$ for $n > 2$ by the kinetic term $\Delta m^2 \phi^2 + E_0$.

The task of getting the better approximation is to minimize the positively defined measure $||\lambda \phi_n - \Delta m^2 \phi^2 - E_0||$. Nevertheless, the vacuum expectation value of the operator $\lambda \phi_n - \Delta m^2 \phi^2 - E_0$ is not positively defined. Thus, it is necessary to calculate the vacuum expectation value for its square that is always positive. In that case, even if the interaction term itself is renormalizable, its square may be not (in particular, for $D > 2$). That is why in higher dimensions the GEP and Hartree methods come across a renormalization problem in computing their error measures [16]. More discussion about the difficulty in the GEP method for the higher dimension can be found in Ref. [17]. However, this is not the case for the OR method, since it does not use any variational procedure.

Therefore, among the three (OR,GEP,Hartree), the OR method seems to be most attractive. The reasons may be summarized as follows:

i) The problems of renormalization in higher dimensions are technical, but not principal.

ii) The Hamiltonian is identically rewritten via canonical transformation and consequently the transformed Hamiltonian keeps the identical information as the original Hamiltonian.

iii) The second derivative of effective potential gives the quasi-particle mass precisely in the OR, while the others give somewhat modified values.

III. APPLICATION OF THE OR METHOD TO THE ($\phi^6$)$_{1+1}$ THEORY

Using the OR method, we now compute the parameters of different phases for the scalar field theory with both $\phi^4$ and $\phi^6$ interactions. The starting classical Hamiltonian density is

$$\mathcal{H}_m = \frac{1}{2} \pi_m^2 + \frac{1}{2} (\nabla \phi_m)^2 + \frac{m^2}{2} \phi_m^2 + \frac{g}{4} \phi_m^4 + h \phi_m^6.$$  \hspace{1cm} (23)

The way to proceed is the same as in the case of $\phi^4$ theory. Using the formulas (10) and (11) and shifting the quantization point, we get:
where the “right” Hamiltonian density $H_{M}^{\text{right}}$ is given by

$$H_{M}^{\text{right}} = \frac{1}{2} \pi_{M}^{2} + \frac{1}{2} (\nabla \Phi_{M})^{2} + \frac{M^{2}}{2} \phi_{M}^{2} + \frac{g}{4} \phi_{M}^{4} + h_{M} \phi_{M}^{6} + gb \Phi_{M}^{3}$$

$$+ 6hb \Phi_{M}^{5} + 15hb^{2} \phi_{M}^{4} + 20hb^{3} \phi_{M}^{3} - \frac{15ht}{4 \pi} \phi_{M}^{4} - \frac{15ht}{\pi} \phi_{M}^{3},$$

(25)

the remainder $H_{1}$ is given by

$$H_{1} = \left( \frac{1}{2} (m^{2} - M^{2}) + \frac{3gb^{2}}{2} - \frac{3gt}{8 \pi} + 15b^{4}h + \frac{45b^{2}ht}{2 \pi} + \frac{45ht^{2}}{16 \pi^{2}} \right) \phi_{M}^{2}$$

$$+ \left( m^{2}b + gb^{3} - \frac{3gbt}{4 \pi} + 6hb^{5} - \frac{15hb^{3}t}{\pi} + \frac{45bht^{2}}{8 \pi^{2}} \right) \phi_{M},$$

(26)

and the energy density $\varepsilon_{M}$ of the vacuum is given by

$$\varepsilon_{M} = \frac{m^{2}b^{2}}{2} + \frac{gb^{4}}{4} - \frac{m^{2}t}{8 \pi} - \frac{3gb^{2}t}{8 \pi} + \frac{3gt^{2}}{64 \pi^{2}}$$

$$+ \frac{1}{8 \pi} (M^{2} - m^{2}) + hb^{6} - \frac{15hb^{3}t}{4 \pi} + \frac{45bht^{2}}{16 \pi^{2}} - \frac{15ht^{3}}{64 \pi^{2}}.$$  

(27)

As in the last section, here we use the notation $\Phi_{M} = \phi_{M} - b$.

The requirement to have the Hamiltonian in the right form means that the remainder $H_{1}$ is equal to zero. This gives the following two equations for $t$ and $B$ ($B = b^{2}$):

$$1 - e^{t} + 3GB - \frac{3Gt}{4 \pi} + 30HB^{2} - \frac{45HBt}{\pi} + \frac{45Ht^{2}}{8 \pi^{2}} = 0,$$

(28)

$$1 + GB - \frac{3Gt}{4 \pi} + 6HB^{2} - \frac{15HBt}{\pi} + \frac{45Ht^{2}}{8 \pi^{2}} = 0,$$

(29)

where $G = g/m^{2}$ and $H = h/m^{2}$ are the dimensionless couplings. The conditions (28) and (29) again provide the minimum of the classical potential at $\phi_{M}^{\text{class}} = b$ and the right value of the double derivative around it.

Solving Eqs. (28) and (29) numerically, we found five nontrivial mass solutions shown in Figs. 4, 5 and 6. The corresponding vacuum energy densities are presented in Figs. 7, 8 and 9. In Fig.7, two identical plots are shown in two different angles of view to make clear presentation of the result. Among the five, two solutions correspond to the symmetric phase, i.e. $b = 0$. They are represented by the upper and lower branches (or sheets) of the manifold shown in Fig.4. The vacuum energy densities for these solutions are shown in Fig.7. The upper and lower branches in Fig.4 correspond to the lower and upper branches in Fig.7, respectively. The upper branch in Fig.4 is the physically meaningful duality-related solution while the lower branch in Fig.4 does not provide the duality that we are interested in this work. The duality-related solution has the lower energy density as shown in Fig.7. The dimensionless couplings which correspond to the $\Phi^{6}$ and $\Phi^{4}$ interactions of the duality-related quasiparticle theory are given by
\[ \chi_s^{(6)} = \frac{h}{M_s^2}, \quad \chi_s^{(4)} = \frac{g}{4M_s^2} - \frac{15ht}{4\pi M_s^2}. \]  

Note here that \( \chi_s^{(4)} \) includes both couplings of \( g \) and \( h \). This is due to the self-organization of fields via the quantum fluctuations in the vacuum, i.e. the original \( \phi^6 \) interaction induces the effective \( \Phi^4 \) interaction with the negative effective coupling in this case. Their dependencies on \( H \) are presented in Fig.10, where the original \( \phi^4 \) interaction coupling \( g \) is put to zero for simplicity. It can be seen from Fig.10, both \( \chi_s^{(4)} \) and \( \chi_s^{(6)} \) become small as \( H \) grows. Thus, the quasiparticle theory in the domain of large \( H \) can be solved by a standard perturbation method.

Among the rest three solutions, the two solutions corresponding to the broken-symmetry phase with the nonzero real condensation (\( b = \sqrt{B} \)) are shown as the two branches in Fig.5. The vacuum energy densities for these solutions are shown in Fig.8. Again the upper (lower) branch of Fig.5 corresponds to the lower (upper) branch of Fig.8. Similar to the case of symmetric phase (Fig.4), the lower branch in Fig.5 does not provide the duality that we are interested in this work. However, the upper branch in Fig.5 corresponds to the duality-related solution and gives the lower value of the vacuum energy density (the lower branch in Fig.8). For this broken-symmetry phase, the dimensionless couplings of the quasiparticle interactions are given by

\[ \chi_{bs}^{(6)} = \frac{h}{M_{bs}^2}, \quad \chi_{bs}^{(5)} = \frac{6hb}{M_{bs}^2}, \quad \chi_{bs}^{(4)} = \frac{g}{4M_{bs}^2} + \frac{15hb^2}{M_{bs}^2} - \frac{15ht}{4\pi M_{bs}^2}, \]
\[ \chi_{bs}^{(3)} = \frac{gb}{M_{bs}^2} + \frac{20hb^3}{M_{bs}^2} - \frac{15hb}{\pi M_{bs}^2}. \]  

The \( H \)-dependence of these couplings is shown in Fig.11 (again, we put \( g = 0 \) to simplify the presentation). At sufficiently large \( H \) the couplings get small and the quasiparticle theory becomes perturbative. When the dimensionless couplings of the symmetric solution go to zero, \( \chi_s^{(6)}, \chi_s^{(4)} \rightarrow 0 \), the dimensionless couplings of the broken-symmetry theory also go to zero as shown in Fig.12.

The last solution shown in Fig.6 has a pure imaginary value for the condensate \( b \) (i.e. negative \( B \)). In Fig.9, we show the corresponding vacuum energy density. Because the Hamiltonian in this case becomes non-hermitian, we call these solutions spurious. The spurious solutions appear only after the inclusion of \( \phi^6 \) interaction. Thus, they are consequences of the higher nonlinearity in the OR equations. Since they don’t seem to bear any interesting duality property that we discuss in this work, we don’t present their results in the main text but just summarize those in the Appendix C with some discussion.

The most interesting feature of the \( \phi^6 \) theory is the appearence of the two apparently different quasiparticle perturbation theories (see the upper branches of Figs. 4 and 5) which represent the initial nonperturbative theory. The first one corresponds to the symmetric phase with the dimensionless quasiparticle couplings given by Eq.(30) and the second one corresponds to the broken-symmetry phase with the dimensionless couplings given by Eq.(31). The subscripts “s” and “bs” of the coupling \( \chi \) and the mass \( M \) represent the correspondence to the symmetric and broken-symmetry phases, respectively. The two quasiparticle theories (i.e. \( \chi_s \)'s and \( \chi_{bs} \)'s) differ from each other significantly in the sense that one of
them (\(\chi_s\)’s) preserves the \(\phi \rightarrow -\phi\) symmetry of the initial Hamiltonian (23) upon changing \(\Phi \rightarrow -\Phi\), while the other (\(\chi_{bs}\)’s) does not respect the \(\Phi \rightarrow -\Phi\) symmetry. Physically, this means that in the theory of symmetric phase with the \(\chi_s\) couplings the processes involving odd number of particles are not allowed while they are allowed in the broken-symmetry theory with \(\chi_{bs}\) couplings. This is because in the theory of symmetric phase, the Hamiltonian depends on the even powers of \(\Phi\) (to provide \(\Phi \rightarrow -\Phi\) symmetry) and thus the S-matrix contains only the even number of creation and annihilation operators.

However, one may wonder if these two apparently different quasiparticle theories are in fact equivalent to each other describing the same physics with just different degrees of freedom. In this respect, we notice that the two different quasiparticle theories give the same effective potential in the limit of very large \(G\) and \(H\). In Figs.13,14 and 15 we show the classical potentials which correspond to the two quasiparticle theories at different values of the coupling \(\chi_s^{(6)}\). As it can be seen, when the quasiparticle coupling \(\chi_s^{(6)}\) becomes small (i.e. \(H\) becomes very large), the relative difference between the two quasiparticle potentials diminishes substantially. We verified that the two classical potentials are indeed identical to each other in the \(\chi^{(6)}_{s} \rightarrow 0\) limit.

In the GEP approach [9], it was also recognized that there are three solutions of symmetric phase (one of them is the trivial solution and two others are the nontrivial solutions corresponding to the ones that we showed in Fig.4). As we consider only one of the two nontrivial solutions (the upper branch in Fig.4) as the physically relevant (duality-related) solution, the GEP approach also dealt with only one (physically relevant) solution among the two nontrivial solutions. In Ref. [9] it was shown that the form of \(\Omega\) equation of the GEP approach is invariant under the exchange of \(m\) and \(M_s\) with an appropriate redefinition of effective couplings \(G' = (G - \frac{15H}{8\pi}t_0)e^{-t_0}\) and \(H' = He^{-t_0}\), as well as \(t' = t - t_0\), where \(t_0 = \ln \frac{M^2_s}{m^2}\) satisfies the constraint equation given by Eq.(29): \(1 - e^{-t_0} = \frac{3Gt_0}{4\pi} - \frac{45Ht_0^2}{8\pi^2}\) (Note \(B = 0\) for the symmetric phase). Indeed, we can show the form invariance of both OR Eqs. (28) and (29) under the same transformations.

Observing such form invariance, it was argued in the GEP approach [9] that the nontrivial symmetric phase solution is just a duplication of the trivial solution and thus should be avoided. However, as we discussed above, the duality between the trivial and nontrivial symmetric phase solutions provides the physical meaning to the form invariance of the OR equations. Because of the duality, we see the utility of both trivial and nontrivial solutions in the symmetric phase. While the trivial solution can be utilized for the perturbation theory at small \(G\) and \(H\), the nontrivial symmetric phase solution can be used for the construction of quasiparticle perturbation theory in the regime of very large \(G\) and \(H\).

Furthermore, since there exists also a duality between the trivial solution and the nontrivial broken-symmetry solution we can rather easily understand the equivalence of the classical effective potentials between the symmetric phase and the broken-symmetry phase for the very large couplings of the original theory (see Fig.15). In the limit of \(\chi_{s}^{(6)} \rightarrow 0\) (which means also \(\chi_{bs}'s \rightarrow 0\), as shown in Fig.12), the higher order corrections get negligible and thus the equivalence of the classical potentials between the symmetric and broken-symmetry phases is revealed manifestly as shown in Fig.15.
IV. SUMMARY AND CONCLUSIONS

The formation of quasiparticle in RQFT may be a good example of self-organizing nature in the nonperturbative strong interactions. This nature is realized by altering the vacuum structure and lowering the vacuum energy density as the strength of the interactions gets more enhanced. Our main goal in this paper was to provide the explicit examples that reveal such nature of self-organizing RQFT and show the utility of duality generated by the canonical transformation in solving the nonperturbative strong interaction problem. For the explicit but simple examples, we analyzed of 1+1 dimensional $\phi^4$ and $\phi^6$ theories. Our purpose was fulfilled by applying the OR method which is a generalization of the canonical quantization scheme.

The OR method uses canonical commutation relations for the free field operators and employs the canonical transformation of quantum fields represented by the change of the field mass and the condensation of the field. The idea to demand the right form of the Hamiltonian, i.e. excluding the terms linear in $\Phi$ and taking the coefficient of $\Phi^2$ as $M^2/2$ guarantees that the classical potential has the minimum at the quantization point $\Phi_{\text{class}}^\text{mass} = b$ and that the double derivative of the classical potential at this point is given by $M^2$. This assures that the OR method gives a self-consistent procedure of quantization and a valid description of the field excitations around the minimum point in terms of quasiparticles.

In contrast to the GEP and HA methods, the OR method is not based on the variational procedure so that it does not have a bottleneck of renormalization problem in higher dimensions as discussed in Section II. However, the OR method has not been utilized as extensively as the GEP and HA methods. Even in the 1+1 dimensional scalar field theory, not much of the OR results were contrasted from the GEP and HA results although the OR results for the ($\phi^4$)$_{1+1}$ theory were presented in Ref. [5]. We attempted to fill such gap in this work.

For the fixed classical Hamiltonian and renormalization procedure (fixed by the counter-terms), it is natural to expect that the OR method yields several different solutions for the quasiparticle mass and condensation because the OR equations are in general nonlinear. However, the unitary inequivalence arising from these different OR vacuum solutions prohibits the transitions among the quasiparticles with different mass and/or condensation. Although some of the solutions correspond to the heavier quasiparticles, they do not decay into the light ones due to the unitary inequivalence; i.e. each sort of the quasiparticles is stable within the OR. The unitary inequivalence between the OR solutions is shown in Appendix B. A recent application of the unitary inequivalence to the flavor mixing phenomena can be found in Ref. [18]. In this work, we selected only the duality-related OR solutions because they can provide the immediate physical interpretation. The reason why the physical interpretation is immediate for the duality-related solutions is because the strong coupling particle theory becomes solvable once it is transformed into an exactly equivalent quasiparticle theory with weak couplings.

Using the ($\phi^4$)$_{1+1}$ theory, we compared the OR results with the results obtained by GEP and HA. In Ref. [5], it was discussed that the value of the critical coupling $G \approx 10.21$ obtained by the OR method coincides with the one obtained by the GEP method. We also note that this value of the critical coupling agrees remarkably well with the lattice result [14]. Due to the equivalence between the HA and GEP methods for the ground state energy.
the agreement of the critical couplings between the OR and HA methods follows as well. Although the critical values of coupling constant $G$ turned out to be same for all three methods (OR, GEP, HA), the OR method gives the quantitatively different dependence of the nontrivial field mass on the effective coupling constant, compare to the GEP and HA results (see Fig.1 and the discussions in Section II).

We have extended the application to the 1+1 dimensional $\phi^6$ theory and compared the results of OR method with what was already obtained by the GEP. We found that the different solutions for the mass of quasi-particles between the OR and the GEP are sustained. In the OR analysis of $(\phi^6)_{1+1}$ theory we found two physically meaningful duality-related solutions (upper sheets of Figs.4 and 5), one in the symmetric phase and the other in the broken-symmetry phase. Although these two solutions have apparently different symmetry property, we find that their effective potentials do agree in the limit of very large couplings $G$ and $H$ of the initial theory, i.e. very small effective couplings $\chi_s$'s and $\chi_{bs}$'s of the quasiparticle theories. This indicates an existence of a unique effective potential to the quasiparticles regardless of their symmetry properties. The actual perturbation theory using the quasiparticles degrees of freedom with the systematic higher order corrections may deserve further investigation.

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APPENDIX A: Hartree Approximations in the $(\phi^4)_{1+1}$ Theory

The essential idea of the Hartree approximation [11] is to linearize the non-linear field equations using the mean fields, e.g. $\phi^3 \rightarrow 3\langle \phi^2 \rangle \phi - 2\langle \phi \rangle^3$, etc., splitting the fields into classical mean fields (\(\phi_c = \langle \phi \rangle\)) plus the quantum fields($\phi_q$), i.e. $\phi = \phi_c + \phi_q$. In the classical limit, we know that the ground state is given by $\phi^2 = c^2$ if $c^2$ is defined by $c^2 = -\frac{m^2}{g}$ and positive (i.e. $c^2 > 0$). The quantum fluctuations $\langle \phi_q^2(x) \rangle$ are found to be infinite. However, one can renormalize the theory using a mass counter term $\frac{1}{2}B\phi^2$, where the constant $B$ is determined by requiring that $\phi_c = c$ is a static solution (corresponding to the so-called “abnormal vacuum state” or a sort of vacuum condensate) of a Hartree equation and this leads to $B = -3g\langle \phi_q^2 \rangle_{\phi_c=c}$. After renormalization, the effective potential $V(\phi_c)$ is obtained in terms of the subtracted quantum fluctuation, i.e. $\Delta\langle \phi_q^2 \rangle = \langle \phi_q^2 \rangle_{\phi_c} - \langle \phi_q^2 \rangle_{\phi_c=c}$, which satisfies the iterative (mass-gap type) equation [11];
\[
\Delta\langle \phi_q^2 \rangle = \frac{1}{4\pi} \ln \frac{2c^2}{3\phi_c^2 - c^2 + 3\Delta\langle \phi_q^2 \rangle}.
\] (A1)

Using the solutions of Eq.(A1), one can find that $\phi_c = c$ is a local minimum (maximum) of $V(\phi_c)$ for $4\pi c^2 > 3(0 < 4\pi c^2 < 3)$ while $\phi_c = 0$ is always a local minimum. For $0 < 4\pi c^2 < 3$,
indeed identical in the sense that $g$ for $4$ and another vacuum for $0 < g < 4$ (or $0 < \phi_c = 4$) and the theory with $g$ at $4$ is dimensionless couplings; i.e.

$$m_0^2 = \frac{d^2V(\phi_c)}{d\phi_c^2} \bigg|_{\phi_c=0} = g(3\Delta\langle\phi_q^2\rangle_{\phi_c=0} - c^2),$$

$$m_c^2 = \frac{d^2V(\phi_c)}{d\phi_c^2} \bigg|_{\phi_c=c} = 2g^2 \frac{8\pi c^2 - 6}{8\pi c^2 + 3},$$

$$m_{c'}^2 = \frac{d^2V(\phi_c)}{d\phi_c^2} \bigg|_{\phi_c=c'} = 4g(c^2 - 3\Delta\langle\phi_q^2\rangle_{\phi_c=c'}) \frac{4\pi(c^2 - 3\Delta\langle\phi_q^2\rangle_{\phi_c=c'}) - 3}{8\pi(c^2 - 3\Delta\langle\phi_q^2\rangle_{\phi_c=c'}) + 3}.$$  \hspace{1cm} (A2)

Then, the intrinsic strength measured in terms of these mass parameters are given by the dimensionless couplings; i.e.

$$g_0 = \frac{g}{m_0^2}, \quad g_c = \frac{g}{m_c^2}, \quad g_{c'} = \frac{g}{m_{c'}^2}.\hspace{1cm} (A3)$$

For the entire $c^2 > 0$ region, it turns out that the nontrivial vacuum solutions exist only for $g_0 \geq g_{\text{crit}} \approx 9.045$ (note that it is not the same as the critical coupling of the phase transition $G = 10.21$ as discussed in Section II), where the critical coupling $g_0 = g_{\text{crit}}$ occurs at $4\pi c^2 = 3$. Thus, it appears that one can find the two nontrivial vacuum solutions; one for $4\pi c^2 > 3$ and the other for $0 < 4\pi c^2 < 3$. However, we find that the two solutions are indeed identical in the sense that $g_c$ and $g_{c'}$ coincide when we plot them (see Fig. 16) as a function of $g_0$, i.e. $g_c(g_0) = g_{c'}(g_0)$ for all $g_0 \geq g_{\text{crit}}$.

We also notice a duality between the theory with $g_0$ based on the trivial vacuum $\phi_c = 0$ and the theory with $g_c$ (or $g_{c'}$) based on the nontrivial vacuum $\phi_c = c$ for $4\pi c^2 > 3$ (or $\phi_c = c'$ for $0 < 4\pi c^2 < 3$). Namely, a strong coupling $g_0$ theory is identical to a weak coupling $g_c$ (or $g_{c'}$) theory.

**APPENDIX B: Unitary Inequivalence between the Nontrivial OR Solutions**

In this Appendix, we show the unitary inequivalence between a vacuum with $b_1$ and $t_1$ and another vacuum $b_2$ and $t_2$, where $t_1 \neq t_2$ or $b_1 \neq b_2$.

The canonical transformation taking a trivial vacuum (i.e. $b = 0$ and $t = 1$) into a nontrivial one (i.e. $b \neq 0$ and $t \neq 1$) has the following form (e.g. for the bosons)

1. $a_k \rightarrow a_k - 2\pi mb(k)$;
2. $a_k \rightarrow a_k \cosh(\lambda) - a_k^\dagger \sinh(\lambda)$;
3. $a_k^\dagger \rightarrow a_k^\dagger \cosh(\lambda) - a_k^\dagger \sinh(\lambda)$.

(B1)

Their corresponding operators have the form

$$U_1 = \exp \left\{ -2\pi mb(a_0 - a_0^\dagger) \right\};$$

$$U_2 = \exp \left\{ i \int d\vec{k} \lambda(k)(a_{-k}a_k - a_k^\dagger a_{-k}^\dagger) \right\}.$$  \hspace{1cm} (B2)

with $\lambda(k) = \frac{1}{2} \ln \frac{\omega(k)}{\omega(k,t)}$ and $\omega(k) = \sqrt{k^2 + t^2 m^2}$. 

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To connect the two vacua, one with \( b_1 \) and \( t_1 \) and another with \( b_2 \) and \( t_2 \), we can use the transformation given by Eq.(B2) with \( b = b_2 - b_1 \) and \( \lambda(k) = \frac{1}{2} \ln \frac{\omega_1(k)}{\omega_2(k)} \) where \( \omega_1(k) = \sqrt{k^2 + t_1^2 m^2} \). To show that, let us define \( V_i : \langle 0, 1 \rangle \rightarrow \langle b_i, t_i \rangle \) by the transformation given by Eq.(B2). Then

\[
\langle b_1, t_1 \rangle \rightarrow \langle b_2, t_2 \rangle = \langle b_1, t_1 \rangle \rightarrow \langle 0, 1 \rangle \rightarrow \langle b_2, t_2 \rangle
\]

and therefore \( V = V_2 V_1^{-1} : \langle b_1, t_1 \rangle \rightarrow \langle b_2, t_2 \rangle \). Straightforward algebraic manipulation with Eq.(B2) then results in the unitary inequivalence between the vacuum with \( \langle b_1, t_1 \rangle \) and the vacuum with \( \langle b_2, t_2 \rangle \).

In Ref. [5], it was mentioned that

\[
U_1 = e^{-\frac{1}{4}(2\pi m b)^2[a_0, a_0^\dagger]} : U_1 : ,
U_2 = e^{-\frac{V}{(2\pi)^3} \int dk \ln(\cosh|\lambda(k)|)} : U_2 : ,
\]

where \([a_0, a_0^\dagger] = \delta^{(3)}(0)\). Here, we can equate \( \delta^{(3)}(0) \) to \( \frac{V}{(2\pi)^3} \). Note also that transformation \( U_1 \) acts only on the \( k = 0 \) field operators, therefore

\[
U_2 \circ U_1 = \left( \prod_{k \neq 0} U_2(k) \right) (U_2(0) \circ U_1(0)) ,
: U_2 \circ U_1 := \left( \prod_{k \neq 0} : U_2(k) : \right) : U_2(0) \circ U_1(0) : .
\]

Hence

\[
U_2 \circ U_1 = e^{-\frac{V}{(2\pi)^3} \int \ln(\cosh|\lambda(k)|) dk} \prod_{k \neq 0} : U_2(k) : (U_2(0) \circ U_1(0)) .
\]

Then for the inner product of the vacuum \( |0\rangle_1 \) associated with \( \langle b_1, t_1 \rangle \) and \( |0\rangle_2 \) associated with \( \langle b_2, t_2 \rangle \) we have

\[
1 \langle 0|0 \rangle_2 = \prod_{1,k} \langle 0|0 \rangle_{2,k} \leq \prod_{1,k} \langle 0|0 \rangle_{2,k} = e^{-\frac{V}{(2\pi)^3} \int \ln(\cosh|\lambda(k)|) dk} \rightarrow 0 ,
\]
as

\[
V \rightarrow \infty .
\]

The two vacua therefore are unitary inequivalent if their \( t \) parameters are different. If \( t_1 = t_2 \) but \( b_1 \neq b_2 \) then \( \lambda(k) \equiv 0 \) and \( V = U_1(b_2 - b_1) \). Then

\[
1 \langle 0|0 \rangle_2 = \prod_{1,k} \langle 0|0 \rangle_{2,k} = \delta_{1,k=0} \langle 0|0 \rangle_{2,k=0} = e^{-\frac{1}{4}(2\pi m b)^2 \delta(0)} = 0 ,
\]
in the infinite volume limit.

Therefore \( 1 \langle 0|0 \rangle_2 = 0 \) in the infinite volume limit whenever \( t_1 \neq t_2 \) or \( b_1 \neq b_2 \).

**APPENDIX C: Spurious Solutions in the \((\phi^6)_{1+1}\) Theory**
Among the solutions of OR Eqs.(28) and (29) for 1+1 dimensional $\phi^6$ theory there is one with negative $B$ that gives pure imaginary condensate $b = i\sqrt{|B|}$. This solution is shown in Fig.6 and the corresponding vacuum energy density is shown in Fig.9. In this case the Hamiltonian becomes non-hermitian because it contains anti-hermitian terms in odd powers of $\Phi^M$. When one calculates average values of such operators in any state with definite number of particles, these terms drop out, because odd number of creation and annihilation operators can not leave the number of particles unchanged. However, the physical states are complicate and contain indefinite number of particles so that the antihermitean terms would contribute to the physical observables and give them imaginary expectation values. Moreover, our starting Hamiltonian density given by Eq.(23) involves only the real (not complex as in the O(2) symmetric theory) scalar field $\phi_m$ and even after shifting the quantization point the field $\Phi^M$ is required to remain as real not complex. For these reasons, we call the imaginary $b$ solution spurious. In principle, we can throw away the spurious solution and do not consider the imaginary $b$ phase in the simple one-component model.

Nevertheless, in the history of many-body analysis sometimes the occurrence of such spurious solution (forbidden in principle) led to interesting physics. For example, the occurrence of imaginary solution has been previously observed in the analysis of many-body system when the method gets more accurate but the number of degrees of freedom is not sufficient. Such example may be illustrated in the random phase approximation (RPA) calculation of a fermion system [19] where the imaginary energy poles are found if the fermion interaction is attractive and weak below a certain critical temperature. These imaginary poles were interpreted [19] as the mathematical manifestation of the instability which leads to the superconducting state. Such imaginary poles were however not found in the less accurate method such as the Tamm-Dancoff approximation (TDA). Thus, if there is any possibility that the imaginary solution could actually mean something interesting in the many-body analysis, we should not overlook such a possibility.

Due to such a tantalizing possibility that the imaginary solution could occur indicating an existence of a new phase, we note a few more particular features of the spurious solutions instead of simply throwing those solutions away. In particular, it may be interesting to note that the corresponding quasi-particle mass values become quite small ($M << m$) when the imaginary $b$ solutions have the least vacuum energy density as shown in Figs.6 and 9. We do not know yet, however, whether these spurious solutions are anything to do with the condensation of (zero-mass) Goldstone-bosons if the degrees of freedom are added in such a way that the fields become complex and a continuous symmetry (see however the discussion for O(2) symmetry near the end of this Appendix) is restored. With the limited degrees of freedom at hands, the theory we consider in the present work does not lead to an appearance of some effective charge in the theory. For the spurious solutions, the point of quantization is fixed as pure imaginary and can not be rotated by any arbitrary phase. Also, in the effective Hamiltonian given by Eq.(25), $b$ can be thought as a part of effective coupling. For example, consider the term $g b \Phi^3_M$ in Eq.(25) and build up an effective potential between two $\Phi^M$ particles intermediating another $\Phi^M$ as an exchange-boson. Then, the coefficient $gb$ of $\Phi^3_M$ is an effective coupling of the triple vertices and the square of the coefficient would determine an overall sign of effective potential. The detailed form of the potential is of course determined only after taking into account the propagator of exchange-boson. However, the overall sign of effective potential determines whether the force between the two $\Phi^M$ particles
is attractive or repulsive. Due to the sign change of the potential, if some bound states may not be formed (although they could be formed for the real $b$), then the corresponding energy eigenvalues would appear as complex values indicating the instability of those states. In some sense, the imaginary $b$ resembles a sort of optical potential which has been used rather frequently in the phenomenology of many-body nuclear physics. A recent work showing an example of how the imaginary coupling can be used in practice was presented in Ref. [20].

To show an overall picture of the effective potential in $(\phi^b)_{1+1}$ theory, we present the dimensionless effective potential $V(\phi) = \left[ V_{\text{class}}(\phi_M^{\text{class}}) + \varepsilon_M \right]/m^2$ for the broken-symmetry solution and the spurious solution with $H/G = -0.209(\alpha = -\beta$ in notations of Ref. [9]) in Fig.17 for several different $G$ values. For simplicity, we didn’t plot $V(\phi)$ for the symmetric phase ($b = 0$) solution. In Fig.17, $\phi$ denotes $\phi_M^{\text{class}}$ and the symmetry $V_{\text{class}}(\phi_M^{\text{class}}) = V_{\text{class}}(-\phi_M^{\text{class}})$ is manifest for the OR solutions with $B = b^2 > 0$, i.e. $\text{Im}(\phi) = 0$. Indeed, no matter what the sign of $B$ is, we can in general prove the symmetry $V(\phi) = V(-\phi)$ using the OR equations given by Eqs.(19) and (20). Thus, for the spurious solutions drawn off from $\text{Im}(\phi) = 0$ plane, one can realize the symmetry $V(\phi) = V(-\phi)$ by considering both planes of $\text{Im}(\phi) = \pm b$ although only the $\text{Im}(\phi) = +|b|$ solutions are shown in Fig.17. The solid, long-dashed, short-dashed, dotted and dot-dashed lines stand for $G=-0.838, -1.68, -2.51, -3.35$ and $-4.19$ $(\alpha=-0.1, -0.2, -0.3, -0.4$ and $-0.5$ in notations of Ref. [9]), respectively. For the coupling constants $G = -2.51$ (short-dashed), -3.35 (dotted) and -4.19 (dot-dashed), the effective potentials corresponding to the three nontrivial solutions including a spurious solution with the imaginary $b$ are shown in Fig.17: e.g. three dot-dashed lines (the spurious one is on the plane $\text{Im}(\phi) \neq 0$ and the other two real broken-symmetry solutions are on the plane $\text{Im}(\phi) = 0$) are shown for the single value of $G = -4.19$. For the comparison with the trivial solutions, we added the solid line for $G = -0.838$ and the long-dashed line for $G = -1.68$ which have the minimum only at $\text{Re}(\phi) = \text{Im}(\phi) = 0$.

In Figs.18-20, the effective potentials for the spurious solutions are shown. Since the spurious solutions have $\text{Im}(\phi) \neq 0$, the corresponding effective potentials $V(\phi)$ are complex. The real parts of effective potentials for the spurious solutions of $G = -0.838$(solid line), $-1.68$(long-dashed line), $-2.51$(short-dashed line), $-3.35$(dotted line) and $-4.19$(dot-dashed line) are shown in Figs.18 and 20 while the imaginary parts for the same coupling constants are shown in Fig.19. Also, the horizontal axes of Figs.18-20 are given by the real field $\Phi_M^{\text{class}} = \phi_M^{\text{class}} - b$ rather than the complex field $\phi_M^{\text{class}}$ due to the imaginary $b$. However, the effective potentials for $G = -2.51$, -3.35 and -4.19 in Figs.18-20 correspond to the ones drawn off from the plane $\text{Im}(\phi) = 0$ in Fig.17. As shown in Fig.9, the energy density of the spurious solution gets the lowest among all the OR solutions for the small coupling such as $G = -0.838$ and -1.68. Especially, the quasi-particle mass gets close to zero as $G$ becomes close to zero. This behavior may be noticed from Fig.18 where the second derivative of $\text{Re}(V_{\text{class}})$ near $\phi_M^{\text{class}} = b$ is the smallest for $G = -0.838$. The corresponding behaviors of $\text{Im}(V_{\text{class}})$ are shown in Fig.19. It is manifest from Fig.19 that $\text{Im}(V_{\text{class}})$ contains only the odd powers of the real field $\Phi_M^{\text{class}}(= \phi_M^{\text{class}} - b)$, which can also be explicitly shown from our effective Hamiltonian given by Eq.(25). The large field behaviors of $\text{Re}(V_{\text{class}})$ are shown in Fig.20. We can see that the minima of $\text{Re}(V_{\text{class}})$ appear in the large real field $\Phi_M^{\text{class}}$ for the spurious solutions, which one can also notice from the effective Hamiltonian (Eq.(25)) by keeping only the even powers of $\Phi_M$. 

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In Fig.21, we show the domain of nontrivial solutions that have the vacuum energy density less than zero in the coupling space of $G$ and $H$. Without the spurious solutions, the domain of nontrivial OR solutions coincides with the GEP result (Fig.1 in Ref. [9]) denoted by the solid line of boundary. Taking into account the spurious solutions, however, the domain of trivial solutions reduces to the shaded region of the phase diagram shown in Fig.21. The appearance of the spurious phase ($B = b^2 < 0$) in the domain where couplings are small may be interesting because it is quite opposite to the appearance of the normal nontrivial phase ($B > 0$), for which one would expect that the interaction term that leads to the formation of vacuum condensate is dominant at the larger coupling and therefore the nontrivial phases appear in the domain of large couplings. The presence of the nontrivial solution for any values of the couplings is also interesting in connection with the Haag’s theorem [21], according to which, if the vacuum state of a Lorentz invariant theory is unique, then all of its observables are identical with those of the free theory. Even very small admixture of $\phi^6$ interaction to the $\phi^4$ theory allows to have a set of vacuum states for any value of the $\phi^4$ coupling and thus to overcome the conditions of the theorem.

In order to explore whether the spurious solution can be considered as an evidence of some real solution by adding more degrees of freedom, we applied the OR method to the two-component ($\phi^6_{1+1}$) theory. In this model, the imaginary condensation of one component could be represented by a real condensation of the other component. However, we found that the imaginary-$b$ solution of the one-component model corresponds to imaginary-$b$ solution of two-component $O(2)$ model, but not to the real solution. The matching was done by solving an arbitrary two-component model

$$
\mathcal{H} = N_m \left( \frac{1}{2} (\pi_1^2 + \pi_2^2) + \frac{1}{2} ((\nabla \phi_1)^2 + (\nabla \phi_2)^2) + \frac{m^2}{2} (\phi_1^2 + \phi_2^2) + g \frac{\phi_1^4 + \phi_2^4}{4} + h (\phi_1^6 + \phi_2^6) + \beta \frac{g}{2} \phi_1^2 \phi_2^2 + 3 h \phi_1^4 \phi_2^2 + 3 h \phi_1^2 \phi_2^4 \right).
$$

(B1)

For $\beta = 0$ and 1, Eq.(B1) corresponds to $O(1) \times O(1)$ and $O(2)$ models, respectively. The evolution of $b^2$ for a “typical” value of $G$ and $H$ (we took $G = 1$, $H = 3$) is shown on Fig.22. As it can be seen, the negative $b^2$ solution in $O(1) \times O(1)$ never becomes positive as $\beta$ gets increased and even more spurious solutions appear in the limit of $O(2)$. However, there is one-to-one correspondence between positive $b^2$ solutions in $O(1) \times O(1)$ and $O(2)$. We found that the same is true for any other $G$ and $H$. Thus, our most immediate exploration didn’t realize the anticipated possibility that the negative $b^2$ solution in $O(1) \times O(1)$ theory corresponds to a positive $b^2$ solution in $O(2)$ theory. This may be a symptom of nonexistence of Goldstone mode in $1 + 1$ dimension known as the Coleman’s theorem [22]. Thus, further exploration including the higher dimension may be necessary for the better understanding of the spurious solution.

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4In Ref. [9], Stevenson and Roditi used somewhat different notations for couplings. Their couplings $\alpha$ and $\beta$ are related to our $G$ and $H$ as $\alpha = \frac{3G}{8\pi}$ and $\beta = \frac{45H}{8\pi}$, respectively. The boundary of allowed $\alpha$ and $\beta$ regions for the GEP were found as a curve (see Fig.1 of Ref. [9]) that can be drawn by the following formulas: $\alpha = \frac{(2z + 3 + (z - 3)e^z)}{z^2}$, $\beta = \frac{3(z + 2 + (z - 2)e^z)}{z^3}$ with $0 < z < \infty$.
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FIG. 1. The nontrivial mass solutions for OR (solid and dashed lines), GEP (dotted line) and Hartree methods (squares correspond to the $4\pi c^2 < 3$ solution, circles to the $4\pi c^2 > 3$ one). Here, $G = g/m^2$ and for $G < 9.04$ there are no nontrivial solutions.

FIG. 2. Vacuum energy densities of the nontrivial OR solutions (solid and dashed lines), GEP (dotted line) and Hartree methods (squares correspond to the $4\pi c^2 < 3$ solution, circles to the $4\pi c^2 > 3$ one). Here, $G = g/m^2$ and for $G < 9.04$ there are no nontrivial solutions.

FIG. 3. The effective classical potential corresponding to Fig. 1 with $G = 8$, 10 and 12. Although $G = 8$ has only a trivial effective potential denoted by a solid line, both $G = 10$ and 12 have two nontrivial effective potentials denoted by I and II.
FIG. 4. Nontrivial solutions of the OR equations for $\phi^4$ and $\phi^6$ theory, symmetric phase.

FIG. 5. Nontrivial solutions of the OR equations for $\phi^4$ and $\phi^6$ theory, broken-symmetry phase.

FIG. 6. Nontrivial solutions of the OR equations for $\phi^4$ and $\phi^6$ theory, imaginary $b$ phase.

FIG. 7. Vacuum energy density of nontrivial solutions of the OR equations for $\phi^4$ and $\phi^6$ theory, symmetric phase. The two plots (right and left) are identical but presented in two different angles of view. The $90^\circ$ clockwise rotation of the left plot around the axis of $\varepsilon M/m^2$ is identical to the right plot.
FIG. 8. Vacuum energy density of non-trivial solutions of the OR equations for $\phi^4$ and $\phi^6$ theory, broken-symmetry phase.

FIG. 9. Vacuum energy density of non-trivial solutions of the OR equations for $\phi^4$ and $\phi^6$ theory, imaginary $b$ phase.

FIG. 10. Dimensionless couplings of the symmetric perturbative solution for $G = 0$.

FIG. 11. Dimensionless couplings of the broken-symmetry perturbative solution for $G = 0$.

FIG. 12. Dimensionless couplings of the broken-symmetry perturbative solution versus the dimensionless coupling of the symmetric solution ($G = 0$).

FIG. 13. Classical potential of broken-symmetry (BS) and symmetric (S) theories at $\chi_s^{(6)} = 5.85 \cdot 10^{-3}$ ($H = 100, G = 0$).
FIG. 14. Classical potential of broken-symmetry (BS) and symmetric (S) theories at $\chi_s^{(6)} = 1.33 \cdot 10^{-3}$ ($H = 10^6, G = 0$).

FIG. 15. Classical potential of broken-symmetry (BS) and symmetric (S) theories at $\chi_s^{(6)} = 1.02 \cdot 10^{-3}$ ($H = 10^{15}, G = 0$).

FIG. 16. (a) $x$ axis is $g_0$ and $y$ axis is $g_c$ (solid line) and $g_c'$ (dot). (b) $x$ axis is $g_0$ and $y$ axis is $\frac{g_0}{g_c}$ (solid line) and $\frac{g_0}{g_c'}$ (dot).
FIG. 17. The dimensionless effective potential $V(\phi) = \left[ V_{\text{class}} + \varepsilon_M \right]/m^2$ for $H/G = -0.209 (\alpha = -\beta$ in notations of Ref. [9]), where $\phi$ means $\phi_{\text{class}}^M$. $x$ axis is $\text{Re}(\phi)$, $y$ axis is $\text{Im}(\phi)$ and $z$ axis is $V(\phi)$. In general, for the complex $\phi$ the potential $V(\phi)$ is complex. For $G = -0.838$ (solid line) and $-1.68$ (long-dashed line), we show only the trivial solutions ($t = 0$ and $b = 0$) for the comparison with the nontrivial solutions for $G = -2.51$ (short-dashed line), $-3.35$ (dotted line) and $-4.19$ (dot-dashed line). For each coupling constant, $G = -2.51$ (short-dashed line), $-3.35$ (dotted line) and $-4.19$ (dot-dashed line), there are two real $b$ broken-symmetry solutions and one pure imaginary $b$ solution. The two real $b$ broken-symmetry solutions for each coupling constant ($G = -2.51, -3.35$ and $-4.19$) are shown on the $\text{Im}(\phi) = 0$ plane. For the pure imaginary $b$ solutions, the potential graphs are drawn on the planes off from the $\text{Im}(\phi) = 0$ plane with the different imaginary $b$ values depending on the $G$ values.
FIG. 18. Real part of effective potentials for the spurious solutions of $G = -0.838$ (solid line), $-1.68$ (long-dashed line), $-2.51$ (short-dashed line), $-3.35$ (dotted line) and $-4.19$ (dot-dashed line).

FIG. 19. Imaginary part of effective potentials for the spurious solutions of $G = -0.838$ (solid line), $-1.68$ (long-dashed line), $-2.51$ (short-dashed line), $-3.35$ (dotted line) and $-4.19$ (dot-dashed line).

FIG. 20. Real part of effective potentials for the spurious solutions of $G = -0.838$ (solid line), $-1.68$ (long-dashed line), $-2.51$ (short-dashed line), $-3.35$ (dotted line) and $-4.19$ (dot-dashed line) in the larger scale of classical fields. The blow-up of the results in a tiny region of $\phi_M^{\text{class}} - b < 0.025$ is shown in Fig. 18.
FIG. 21. The domain of trivial solutions in $G, H$ parameter space (shadowed area). If the spurious solution shown in Fig. 6 is not taken into account, then the domain enlarges to the inside of the entire area bounded by the external solid line, which is equivalent to the GEP result shown in Fig. 1 of Ref. [9].

FIG. 22. Correspondence between solutions of OR equations for $O(1) \times O(1)$ and $O(2)$ ($\phi^6$)$_{1+1}$ theories at $G = 1.0, H = 3.0$. 

\[ \beta \]