Category Theory Using String Diagrams

Dan Marsden

January 29, 2014

Abstract

In [Fokkinga, 1992a], [Fokkinga, 1992b] and [Fokkinga and Meertens, 1994] a calculational approach to category theory is developed. The scheme has many merits, but sacrifices useful type information in the move to an equational style of reasoning. By contrast, traditional proofs by diagram pasting retain the vital type information, but poorly express the reasoning and development of categorical proofs. In order to combine the strengths of these two perspectives, we propose the use of string diagrams, common folklore in the category theory community, allowing us to retain the type information whilst pursuing a calculational form of proof. These graphical representations provide a topological perspective on categorical proofs, and silently handle functoriality and naturality conditions that require awkward bookkeeping in more traditional notation.

Our approach is to proceed primarily by example, systematically applying graphical techniques to many aspects of category theory. We develop string diagrammatic formulations of many common notions, including adjunctions, monads, Kan extensions, limits and colimits. We describe representable functors graphically, and exploit these as a uniform source of graphical calculation rules for many category theoretic concepts. We then use these graphical tools to explicitly prove many standard results in our proposed string diagram based style of proof.

1 Introduction

This work develops in some detail many aspects of basic category theory, with the aim of demonstrating the combined effectiveness of two key concepts:

1. The calculational reasoning approach to mathematics.
2. The use of string diagrams in category theory.

The calculational style of mathematics is an approach to formal proofs originating in the computer science community. The scheme is typically characterized by a goal directed systematic manipulation of equations, similar in flavour to high school arithmetic. The merits of this style of reasoning are discussed in [van Gasteren, 1990] and [Dijkstra and Scholten, 1990], and discrete mathematics is formulated in a calculational style in [Gries and Schneider, 1993].
String diagrams are a graphical formalism for working in category theory, providing a convenient tool for describing morphisms within bicategories [Bénabou, 1967], some background on string diagrams can be found in [Street, 1995]. Recent work in quantum computation and quantum foundations has successfully applied various diagrammatic calculi based on string diagrams to model quantum systems in various forms of monoidal category, see for example [Abramsky and Coecke, 2008], [Coecke and Duncan, 2011], [Baez and Stay, 2011], [Stay and Vicary, 2013], [Coecke and Duncan, 2011] and the comprehensive survey [Selinger, 2011] for more details. Further applications of string diagrams for monoidal categories in more logical settings can be found in [Melliès, 2006] and [Melliès, 2012]. Technical aspects of using string diagrams to reason in monoidal categories are detailed in [Kelly and Laplaza, 1980], [Joyal and Street, 1991], [Joyal and Street, 1995], [Joyal and Street, 1988].

Category theory was developed in a calculational style in [Fokkinga, 1992a], [Fokkinga, 1992b] and [Fokkinga and Meertens, 1994]. In these papers, we are presented with a choice of the traditional commuting diagram style of reasoning about category theory, and a calculational approach based on formal manipulation of the corresponding equations. We propose that string diagrams provide a third alternative strategy, naturally supporting calculational reasoning, but continuing to carry the important type information that is abandoned in the move to symbolic equations. Additionally, string diagram notation “does a lot of the work for free”, an important aspect in choosing notation as advocated in [Backhouse, 1989]. Issues of associativity, functorality and naturality are handled silently by the notation, allowing attention to be focused on the essential aspects of the proof.

To illustrate the difference between traditional “diagram pasting”, the calculational approach using symbolic equations as described in for example [Fokkinga, 1992a], and the string diagrammatic calculus, we will investigate a simple example.

**Example 1.1 (Algebra Homomorphisms).** Let \( C \) be a category, and \( T : C \to C \) be an endofunctor on \( C \). An algebra of \( T \) is a pair consisting of an object \( X \) of \( C \) and a \( C \) morphism of type \( T(X) \to X \). A \( T \)-algebra homomorphism of type \((X, a) \to (Y, a')\) is a \( C \) morphism \( h : X \to Y \) such that the following diagram commutes:

\[
\begin{array}{ccc}
T(X) & \xrightarrow{T(h)} & T(Y) \\
\downarrow a & & \downarrow a' \\
X & \xrightarrow{h} & Y 
\end{array}
\]

(1)

Now we will show, using the three different proof styles, that for a category \( D \), a functor \( F : C \to D \), an endofunctor \( T' : D \to D \) and natural transformation
\[ \alpha : T' \circ F \Rightarrow F \circ T, \text{ if } h : (X, a) \rightarrow (Y, a') \text{ is a } T\text{-algebra homomorphism then } Fh \text{ is a } T'\text{-algebra homomorphism of type } (X, F(a) \circ \alpha_X) \rightarrow (Y, F(a') \circ \alpha_Y). \]

Firstly we will prove this in the traditional diagram pasting manner by noting that by naturality of \(\alpha\) the following diagram commutes:

\[
\begin{array}{ccc}
T'F(X) & \xrightarrow{T'F(h)} & T'F(Y) \\
\downarrow{\alpha_X} & & \downarrow{\alpha_Y} \\
FT(X) & \xrightarrow{FT(h)} & FT(Y)
\end{array}
\] (2)

Also, as functor application preserves commuting diagrams, the following also commutes:

\[
\begin{array}{ccc}
FT(X) & \xrightarrow{FT(h)} & FT(Y) \\
\downarrow{F(a)} & & \downarrow{F(a')} \\
F(X) & \xrightarrow{F(h)} & F(Y)
\end{array}
\] (3)

We can then paste diagrams 2 and 3 together, giving:

\[
\begin{array}{ccc}
T'F(X) & \xrightarrow{T'F(h)} & T'F(Y) \\
\downarrow{\alpha_X} & & \downarrow{\alpha_Y} \\
FT(X) & \xrightarrow{FT(h)} & FT(Y) \\
\downarrow{F(a)} & & \downarrow{F(a')} \\
F(X) & \xrightarrow{F(h)} & F(Y)
\end{array}
\] (4)

The claim then follows from the outer rectangle of diagram 4.
A proof of the same claim in a calculational style similar to [Fokkinga, 1992] might proceed as follows:

\[
F(h) \circ F(a) \circ \alpha_X \\
= \{ \text{Functoriality} \} \\
F(h \circ a) \circ \alpha_X \\
= \{ h \text{ is a } T\text{-algebra homomorphism} \} \\
F(a' \circ T(h)) \circ \alpha_X \\
= \{ \text{Functoriality} \} \\
F(a') \circ FT(h) \circ \alpha_X \\
= \{ \text{Naturality} \} \\
F(a') \circ \alpha_Y \circ T'F(h)
\]

and this proves the claim.

Finally, we will sketch how such a proof will look using string diagrams, deliberately mirroring the previous calculational style of proof, but now encoding the mathematical data graphically. The precise details of this graphical approach will be developed in later sections, but hopefully the example is sufficiently straightforward for the underlying ideas to be apparent:

Note the topological nature of the proof, for example the naturality step manifests itself as sliding two morphisms past each other along the “wires” of the diagram.
Now we highlight some similarities and differences between the 3 styles of reasoning:

• Using diagram pasting, a proof may often be presented as a single large and potentially complex commuting diagram from which the required equalities can be read off. Such a diagram efficiently encodes a large number of equalities, and all the associated type information. Unfortunately, this style of presentation does not capture how the proof was constructed, it can be unclear in what order the reasoning proceeded, and constructing such a diagram is somewhat of an art form.

• The calculational approach presents a proof as a simple series of equalities and manipulations of equations. The sequence of steps and their motivation can be made very explicit, aiding both the development and later understanding of such a proof. The notation is also clearly more compact than the commuting diagrams, but at a significant cost as all the type information is lost in the conversion to equational form. This type information can help eliminate errors in reasoning, and often proofs in category theory are guided by the need to compose morphisms in a type correct manner.

• The string diagrammatic proof combines some of the benefits of both of the previous approaches. Proofs consist of equalities and manipulations of equations in a calculational style, but all the type information is retained, helping to guide and constrain the manipulations that can be formed. The approach is clearly more verbose than the traditional symbolic equations, but this additional verbosity carries useful type information, and tool support could reduce the cost of manipulating these more complex objects when developing a proof, as for example is done for a similar diagrammatic calculus used in quantum computing by the Quantomatic tool [Kissinger et al., 2013]. Also, using string diagram notation, some aspects of proofs are “handled by the notation”, for example the equational calculational proof above must explicitly invoke functoriality but in the graphical proof functoriality is implicit in the notation.

In this paper, we aim to provide an introductory, and reasonably self-contained, account of the use of string diagrams to reason calculationally about category theory. We will start with the basic ideas of strings diagrams and then formulate fundamental concepts of category theory such as adjunctions, monads, Kan extensions and limits and colimits graphically. To illustrate the approach, these formulations will then be exploited to prove various standard results in the proposed graphical style. We will assume some basic familiarity with categorical concepts such as categories, functors and natural transformations, and the elementary details of adjunctions, limits and colimits, for introductory accounts see for example [Abramsky and Tzevelekos, 2011] or [Pierce, 1991], more comprehensive material can be found in the standard reference [MacLane, 1998].
As our aim to describe “ordinary” category theory, we will work exclusively with the 2-category of all categories, rather than working axiomatically with bicategories carrying sufficient structure. Many results are folklore or well documented in the literature, although our heavy emphasis on providing a graphical formulation of these results in a form suitable for calculation is certainly non-standard. In section 5 we provide a new perspective on representable functors as a source of graphical calculation rules. This perspective is then used to provide graphical rules for Kan extensions, limits and colimits in later sections.

2 An Outline of the Graphical Approach

In the string diagram formalism, a category $\mathcal{C}$ is represented as a coloured region:

Different colours are used to denote different categories within a diagram. To avoid a great deal of repetitive detail, in what follows we will not in general specify which categories correspond to which colours. This information, if needed, should be apparent from the other visual components within our diagrams. A functor $F : \mathcal{C} \to \mathcal{D}$ is represented as an edge, commonly referred to as a wire, for example:

Identity functors are often omitted and simply drawn as a region. Given a second functor $G : \mathcal{D} \to \mathcal{E}$, the composite $G \circ F$ is represented as:

So functors compose from left to right. A natural transformation $\alpha : F \Rightarrow F'$ is represented as:

6
Given a second natural transformation $\alpha' : F' \Rightarrow F''$, the vertical composite $\alpha' \circ \alpha$ is written as follows:

So **natural transformations compose vertically from bottom to top.** Now if we also have a natural transformation $\beta : G \Rightarrow G'$ we represent the horizontal composite $\beta \ast \alpha$ as:

So **natural transformations compose horizontally from left to right.** Identity natural transformations are omitted, so $1 : F \Rightarrow F$ is drawn as:

Given another natural transformation $\beta'' : G' \Rightarrow G''$, the **interchange law** holds:

$$((\beta' \circ \beta) \ast (\alpha' \circ \alpha)) = ((\beta' \ast \alpha') \circ (\beta \ast \alpha))$$  (5)

and so the following composite is well defined, corresponding to a unique natural transformation:

By substituting identities appropriately, we have the following **sliding equalities**:

$$F' \circ \alpha = F' \circ \alpha$$  (6)
Much of what can be proved with string diagram calculus flows from the fact we can slide natural transformations past each other like this, as we saw in example 1.4 in the introduction.

2.1 Some Elementary Techniques

Objects and Morphisms

Often we will want to reason using categories, functors, natural transformations and “ordinary” objects and morphisms within categories, as we did in example 1.4. To achieve this, we consider the category 1 with one object and one (identity) morphism. Functors of type 1 → C can be identified with objects of the category C. A morphism f : X → Y is then a natural transformation between two of these functors from the terminal category. If we identify the functors and the corresponding objects, then we can write f in the obvious way as:

Elements of Sets

When we work in the category Set, given a set X, we may wish to refer to an element x ∈ X. Extending the idea in section 2.1, such an element can be identified with a morphism x : * → X, where * is the one element set (the terminal object in Set). We then write this element as:

Naturality

Often given a family of morphisms:

\[(α_x : F(X) → G(X))_{X ∈ \text{obj}(C)}\]  \hspace{1cm} (7)

we wish to show that this family constitutes a natural transformation. This is equivalent to showing that for all X, Y objects in C and all f : X → Y the
following equality holds:

\[ X \xrightarrow{\alpha_X} Y \quad F \xrightarrow{\alpha_Y} G \]

Notice the similarity in the above condition with the sliding equations, effectively naturality says that the natural transformation and function \( f \) “slide past each other”, and so we can draw them as two parallel wires to illustrate this.

## 3 Adjunctions

We will now investigate some of the properties of one of the central concepts of category theory, adjunctions. As our focus is calculational proofs we will not provide concrete examples of adjunctions in the text, for many examples see [MacLane, 1998], [Borceux, 1994], [Adamek et al., 2009] or any other good book on category theory. Instead, we will formulate some of the key concepts graphically, and then illustrate the use of graphical techniques in the proof of some standard results.

### 3.1 Calculational Properties of Adjunctions

Adjunctions can be described in a myriad of different, and equivalent ways. In this section we adopt an approach that particularly suits our string diagrammatic presentation. In the various examples we will recover many of the other aspects of adjunctions. A comprehensive discussion of the different formulations of the concept of an adjunction, using an algebraic calculational style, is given in [Fokkinga and Meertens, 1994].

**Definition 3.1 (Adjunction).** An adjunction consists of functors and natural transformations:

\[
\begin{align*}
F & \quad \quad G \\
\quad & \quad \quad \bullet \\
\eta & \quad \quad \bullet \\
\quad & \quad \quad \bullet \\
\epsilon & \quad \quad \\
G & \quad \quad F
\end{align*}
\]

satisfying:

\[
\begin{align*}
\begin{array}{c}
G \\
\eta \\
F
\end{array} & \quad = \quad \begin{array}{c}
\epsilon \\
F \\
G
\end{array}
\end{align*}
\]

(9)
and:

\[ \eta \quad \quad \quad \quad \eta' \]

\[ \epsilon \quad \quad \quad \quad \epsilon' \]

Lemma 3.2 (Adjunctions Compose). Two adjunctions \( F : \mathcal{C} \to \mathcal{D} \dashv G : \mathcal{D} \to \mathcal{C} \) and \( F' : \mathcal{D} \to \mathcal{E} \dashv G' : \mathcal{E} \to \mathcal{D} \) compose to give an adjunction:

\[ F' \circ F \dashv G \circ G' \]  

(11)

Proof. Take as unit and counit the composites:

That these satisfy the adjunction axioms is then trivial to show.

We now consider in some detail how the units and counits of a pair of adjunctions relate to certain natural transformations. A lot of structure can be derived from the fact that units and counits let us “bend wires”.

Definition 3.3 (Wire Bending). We consider a situation with adjunctions \( F \dashv G \) and \( F' \dashv G' \) and functors \( H, K \), with types as in the following (not necessarily commuting) diagram:

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{H} & \mathcal{C}' \\
\downarrow F \downarrow G & & \uparrow F' \uparrow G'
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{K} & \mathcal{D}' \\
F \downarrow G & & \uparrow F' \uparrow G'
\end{array}
\]

There are then bijections between the sets of natural transformations of the four types shown below:
Graphically, these bijections are induced by “bending wires” using the adjunction axioms to move between the different types.

Starting in the top left hand corner, between each pair of types we define a function \((—)\bowtie\), referred to as “move right”, in the clockwise direction, with an inverse \((—)\bowtie\), referred to as “move left”, in the counter clockwise direction. Hopefully this overloading of names will not cause confusion, given the abundance of type information in the string diagrams we are using. Firstly:

\[
\begin{align*}
F' & \xrightarrow{\sigma} H \\
G' & \xrightarrow{\sigma'} \\
\end{align*}
\]

\[
\begin{align*}
F & \xrightarrow{\sigma} H \\
G & \xrightarrow{\sigma'} \\
\end{align*}
\]

\[
\begin{align*}
\eta' & := \sigma \\
\eta & := \sigma' \\
\end{align*}
\]

with inverse:

\[
\begin{align*}
F' & \xleftarrow{\sigma'} H \\
G' & \xleftarrow{\sigma} \\
\end{align*}
\]

\[
\begin{align*}
F & \xleftarrow{\sigma'} H \\
G & \xleftarrow{\sigma} \\
\end{align*}
\]

Secondly:

\[
\begin{align*}
F' & \xrightarrow{\sigma} H \\
G' & \xrightarrow{\sigma'} \\
\end{align*}
\]

\[
\begin{align*}
K & \xrightarrow{\sigma'} G' \\
& \xrightarrow{\sigma} \\
\end{align*}
\]

\[
\begin{align*}
\epsilon & := \sigma' \\
\epsilon' & := \sigma \\
\end{align*}
\]

with inverse:

\[
\begin{align*}
K & \xleftarrow{\sigma} G' \\
& \xleftarrow{\sigma'} \\
\end{align*}
\]

\[
\begin{align*}
F & \xleftarrow{\sigma} H \\
G & \xleftarrow{\sigma'} \\
\end{align*}
\]

Thirdly:

\[
\begin{align*}
F' & \xrightarrow{\sigma} H \\
G' & \xrightarrow{\sigma'} \\
\end{align*}
\]

\[
\begin{align*}
K & \xrightarrow{\sigma'} G' \\
& \xrightarrow{\sigma} \\
\end{align*}
\]

\[
\begin{align*}
\epsilon' & := \sigma' \\
\epsilon & := \sigma \\
\end{align*}
\]

with inverse:

\[
\begin{align*}
K & \xleftarrow{\sigma} G' \\
& \xleftarrow{\sigma'} \\
\end{align*}
\]

\[
\begin{align*}
F' & \xleftarrow{\sigma'} H \\
G & \xleftarrow{\sigma} \\
\end{align*}
\]
Finally:

\[
\begin{array}{ccc}
  K & \sigma & H \\
  G & F' & F \\
\end{array}
\]

\[\mapsto\]

\[
\begin{array}{ccc}
  F' & K & F \\
  H & \sigma & \eta \\
\end{array}
\]

(18)

with inverse:

\[
\begin{array}{ccc}
  F & H & K \\
  G & F' & \sigma \\
\end{array}
\]

\[\mapsto\]

\[
\begin{array}{ccc}
  K & \sigma & \eta \\
  F' & H & G \\
\end{array}
\]

(19)

That each of these pairs of maps witness a bijection is easy to check by applying
the adjunction axioms.

**Lemma 3.4.** All of the mappings:

\[
\begin{array}{ll}
  \triangleright & : [C, D'](F' H, K F) \to [C, C'](H, G' K) \\
  \triangleright & : [C, C'](H, G' K) \to [D, C'](H G, G' K) \\
  \triangleright & : [D, C'](H G, G' K) \to [D, D'](F' H G, K) \\
  \triangleright & : [D, D'](F' H G, K) \to [C, D'](F' H, K F)
\end{array}
\]

(20)

(21)

(22)

(23)

and their inverses are natural in both \(H\) and \(K\).

**Proof.** This is easy, if lengthy, to check graphically and left as an exercise. \(\square\)

**Lemma 3.5.** Each of the composites \(\triangleright \triangleright \triangleright \triangleright \) and \(\triangleright \triangleright \triangleright \triangleright \) as described in definition 3.3 are equal to the identity.

**Proof.** Straightforward from expanding definitions and exploiting the adjunction axioms. \(\square\)

We will now exploit these wire bending ideas to provide a graphical proof of
a standard result about adjunctions.

**Lemma 3.6.** For every morphism \(f : X \to G(Y)\) there exists a unique morphism \(\hat{f} : F(X) \to Y\) such that:

\[
\begin{array}{ccc}
  \hat{f} & \eta & f \\
\end{array}
\]

(24)
Proof. Both existence and uniqueness follow from the following equivalences:

\[
\begin{align*}
g \eta & = f \\
\iff \{ \text{Wire bending is a bijection} \} \\
g \epsilon \eta \epsilon & = f \\
\iff \{ \text{Adjunction axioms} \}
\end{align*}
\]

Remark 3.7. Although we proved lemma 3.6 directly, we could simply have observed that equation 24 is a special case of the first bijection in definition 3.3 with the leftmost adjunction in the diagram being the trivial adjunction between the identity functors.

We now see why the suggestive names “move left” and “move right” were chosen for the mapping in definition 3.3, as they allow us to “slide” a natural transformation back and forth along the bends given by the units and counits of adjunctions, as described by the next lemma.

Lemma 3.8 (Adjunction Sliding). We have the following equalities allowing us to “slide” a natural transformation along a pair of adjunctions:

\[
\begin{align*}
\eta' & = \sigma \eta \\
\quad & = \sigma' \eta' \\
\quad & = \sigma' \sigma \eta \\
\end{align*}
\]

and

\[
\begin{align*}
\epsilon' & = \sigma' \epsilon \\
\quad & = \sigma' \epsilon' \\
\end{align*}
\]
Proof. The equalities either follow immediately from the definitions, or require a single application of one of the adjunction axioms.

Remark 3.9. By combining lemma 3.8 with the fact that the various pairs of functions \((-)^\circ\) and \((-)^\triangleright\) witness a bijection allow us to derive several similar sliding identities, involving various combinations of both \((-)^\circ\) and \((-)^\triangleright\).

Definition 3.10 (Mates Under an Adjunction). Given an adjunction \(F : C \to D \dashv G : D \to C\) and endofunctors \(H : C \to C, K : D \to D\), there is a bijection between the natural transformations of types \(HG \Rightarrow GK\) and \(FH \RightarrowKF\). This is easily seen using the operations in definition 3.3 in the special case where both adjunctions are the same, with:

\[
\begin{array}{c}
\begin{array}{c}
K \sigma \\
G \\
H
\end{array}
\end{array}
\quad \leftrightarrow \quad
\begin{array}{c}
\begin{array}{c}
F \sigma \triangleright \triangleright \\
K \\
F
\end{array}
\end{array}
\tag{27}
\]

and

\[
\begin{array}{c}
\begin{array}{c}
F \sigma \\
H \\
K
\end{array}
\end{array}
\quad \leftrightarrow \quad
\begin{array}{c}
\begin{array}{c}
G \sigma \triangleright \triangleright \\
K \\
H
\end{array}
\end{array}
\tag{28}
\]

Natural transformations related by these bijections are referred to as mates under the adjunction \(F \dashv G\).

3.2 Adjoint Squares

We will now apply our graphical approach to adjunctions in an investigation of some properties of adjoint squares.

Definition 3.11 (Adjunct Square). An adjoint square is a pair of adjunctions and two functors \(H, K\) arranged as in the following (not necessarily commuting) diagram:

\[
\begin{array}{ccc}
C & \xrightarrow{H} & C' \\
\downarrow F & & \downarrow F' \\
D & \xleftarrow{K} & D'
\end{array}
\]

and also natural transformations:
satisfying the following sliding equalities:

\[
\eta' \sigma = \eta \tau \quad (29)
\]

\[
\epsilon' \sigma = \epsilon \tau \quad (30)
\]

Such a pair of natural transformations are said to be **conjugate**. See [MacLane, 1998] exercise IV.7.4.

**Lemma 3.12.** For definitions as in definition 3.11, the conditions in equations 29 and 30 are equivalent. Furthermore, \( \sigma \) and \( \tau \) determine each other uniquely.
Proof. That the conditions are equivalent follows from:

\[
\eta' \sigma = \eta \tau \iff \{ \text{Definitions} \}
\]

\[
\begin{array}{cc}
F & G' \\
\sigma & = \sigma' \\
H & \sigma^> \sigma < \\
\end{array}
\begin{array}{cc}
F & G' \\
\tau & = \tau' \\
H & \tau^> \sigma < \\
\end{array}
\]

\[
\iff \{ \text{Applying the } \sigma < \text{ wire bending bijection twice} \}
\]

\[
\begin{array}{cc}
\sigma^> \sigma < \\
\sigma < \\
\tau^> \sigma < \\
\end{array}
\begin{array}{cc}
\sigma^> \sigma < \\
\tau^> \sigma < \\
\end{array}
\]

\[
\iff \{ \sigma^> \text{ and } \sigma < \text{ are inverses, and } \sigma < \sigma < \sigma \text{ is the identity} \}
\]

\[
\begin{array}{cc}
\sigma^> \sigma < \\
\tau^> \sigma < \\
\end{array}
\begin{array}{cc}
\sigma^> \sigma < \\
\tau^> \sigma < \\
\end{array}
\]

\[
\iff \{ \text{Definitions} \}
\]

\[
\begin{array}{cc}
\sigma^> \\
\tau^> \\
\end{array}
\begin{array}{cc}
\sigma^> \\
\tau^> \\
\end{array}
\]

That \( \sigma \) and \( \tau \) determine each other uniquely is then immediate as they are related by bijections from definition \textbf{3.3}. \qed

\textbf{Proposition 3.13.} Adjoint squares compose vertically and horizontally.

\textit{Proof.} By lemma \textbf{3.12} we only need to check one of the two generalized sliding equalities holds in order to ensure our composition operations preserve conjugate pairs. The vertical composition of adjunctions will be assume to be as in lemma \textbf{3.2}. We will denote the vertical composition operation as \( \circ \), defined

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as follows:

$$\left( (P \sigma' P', J K) \sigma', (Q \tau' Q') J K \right) \circ \left( (F \sigma' F', J H) \sigma', (G \tau' G', J H) \tau' \right) = \left( (\sigma \sigma') , (\tau \tau') \right)$$

That the resulting pair of natural transformations are conjugate is trivial as the following equality holds because the component natural transformations are conjugate:

$$\eta \eta' \sigma \sigma' = \eta \eta' \sigma \sigma'$$

We will denote the horizontal composition operation $\ast$, defined as follows:

$$\left( (P \sigma' P', J K) \ast (F \sigma' F', J H) \ast (G \sigma' G', J H) \right)$$

That the resulting pair of natural transformations are conjugate can be seen from the following equalities, given by applying the generalized sliding equalities for both the component adjoint squares:

$$\sigma \sigma' = \sigma \sigma' = \sigma \sigma'$$
It is straightforward to check that the two forms of composition both give
adjoint squares the structure of a category, with an appropriate choice of iden-
tities. We also have an interchange law that holds between these two types of
composition.

**Lemma 3.14 (Interchange Law).** With the composition operations defined in
the proof of proposition 3.13, let $P, Q, R, S$ be adjoint squares such that the
required composites are well defined, then the following equality holds:

$$(S \circ Q) \star (R \circ P) = (S \star R) \circ (Q \star P) \quad (37)$$

**Proof.** By proposition 3.13 the composition operations preserve conjugate pairs,
and from lemma 3.12 the members of a conjugate pair determine each other
uniquely. We therefore only need to consider one of the component natural
transformations of the adjoint squares. We then have the following sequence of
equalities directly from the definitions (with a slight abuse of notation we omit
the second component of the adjoint squares):

\[
(\sigma_S \circ \sigma_Q) \ast (\sigma_R \circ \sigma_P) = \{\text{Definition of vertical composition of adjoint squares}\}
\]

\[
= \{\text{Definition of horizontal composition of adjoint squares}\}
\]

\[
= \{\text{Definition of vertical composition of adjoint squares}\}
\]

\[
= \{\text{Definition of horizontal composition of adjoint squares}\}
\]

Remark 3.15. In fact, the adjoint squares form a double category [Ehresmann, 1963], commonly used as a device for organizing information about adjunctions, see [Palmquist, 1971] and [Gray, 1974]. Proving the remaining details graphically is left for the interested reader.

4 Monads

Monads, also historically referred to as triples and standard constructions, are another important notion in category theory, and are also of particular interest
to computer scientists with their connections to the semantics of programming languages, see for example [Moggi, 1991] and [Wadler, 1995]. Monads are of course described in [MacLane, 1998], but there is much that is not covered, standard sources for monad theory are [Barr and Wells, 2005] and [Manes, 1976], a more modern presentation is given in the book in preparation [Jacobs, 2012].

We do not attempt to give a comprehensive treatment here, rather we aim to formulate some basic aspects of monads in graphical terms, and then provide examples of the string diagrammatic approach by proving some standard results. As with the description of adjunctions in section 3, we will not provide concrete examples of monads, leaving the interested reader to refer to the various references in the text.

The string diagram notation is particularly effective when working with monads, providing a simple perspective on many of the mathematical structures under consideration. These ideas will be highlighted in this section.

4.1 Calculational Properties of Monads

We now introduce graphical descriptions of monads and some associated mathematical structures.

Definition 4.1 (Monad). Let $\mathcal{C}$ be some category. A monad on $\mathcal{C}$, consists of an endofunctor $T$ on $\mathcal{C}$, a unit natural transformation $\eta$ and multiplication natural transformation $\mu$:

These following unit axioms are required to hold:

(38)

Also $\mu$ is required to satisfy the following associativity axiom:

(39)

The string diagram notation makes clear that a monad can be seen as a monoid in the functor category $[\mathcal{C}, \mathcal{C}]$, as suggested by the use of the names “unit” and “multiplication”.

20
There is a close relationship between monads and adjunctions, in fact every adjunction induces a monad.

**Lemma 4.2.** Every adjunction:

\[
\begin{array}{ccc}
F & G \\
\eta & & \epsilon
\end{array}
\]

induces a monad.

**Proof.** Take the endofunctor as \( G \circ F \), and the unit of the monad is also \( \eta \). Take the multiplication \( \mu \) of the monad as the composite:

\[
\begin{array}{ccc}
F & G \\
\epsilon & & \epsilon
\end{array}
\]

We must then check the required axioms hold, for the unit axioms we apply the properties of adjunctions:

\[
\begin{array}{ccc}
F & G \\
\eta & & \epsilon
\end{array}
= \begin{array}{ccc}
G & F \\
\epsilon & & \epsilon
\end{array} = \begin{array}{ccc}
F & G \\
\epsilon & & \epsilon
\end{array}
\]

(40)

and the associativity axioms follow directly from naturality:

\[
\begin{array}{ccc}
\epsilon & & \epsilon
\end{array}
= \begin{array}{ccc}
\epsilon & & \epsilon
\end{array} = \begin{array}{ccc}
\epsilon & & \epsilon
\end{array}
\]

(41)

We now introduce a suitable notion of morphism for monads on the same category. We will generalize this situation in section 4.2.

**Definition 4.3 (Monad Morphism).** For a fixed category \( \mathcal{C} \), a monad morphism \( (\mathcal{C}, T, \eta, \mu) \to (\mathcal{C}, T', \eta', \mu') \) is a natural transformation \( \sigma : T \Rightarrow T' \) such that the
following equalities hold:

\[
T T' \eta \sigma = T' \eta' \quad (42)
\]

\[
T T' \mu \sigma = T' \mu' T' T' \mu' \sigma \quad (43)
\]

Again the string notation is instructive, if we consider the two monads as monoids in \([C, C]\) then \(\sigma\) is clearly a monoid homomorphism, commuting appropriately with the unit and multiplication.

**Lemma 4.4.** Let \((C, T, \eta, \mu), (C, T', \eta', \mu')\) and \((C, T'', \eta'', \mu'')\) be monads. Also let \(\sigma : T \Rightarrow T'\) and \(\tau : T' \Rightarrow T''\) be monad morphisms. Then \(\tau \circ \sigma\) is a monad morphism. Also for any monad \((C, T, \eta, \mu)\), the identity natural transformation on \(T\) is a monad morphism.

**Proof.** For the unit we have:

\[
T'' T' \eta \sigma = T' \eta' \quad (44)
\]

Similarly for the multiplication:

\[
T'' T' \mu \sigma = T' \mu' T' T' \mu' \sigma \quad (45)
\]

The second part is trivial.

We could have seen the above lemma immediately as monoid homomorphisms compose. Clearly for a category \(C\), the monads on \(C\) and the monad morphisms form a category. We now investigate two important constructions for a given monad.

**Definition 4.5** (Eilenberg-Moore Category). The **Eilenberg-Moore category** for a monad \((C, T, \eta, \mu)\), denoted \(EM(T)\), is defined as follows:
• Objects are pairs consisting of an object $X$ of $C$ and a morphism $a : T(X) \to X$ satisfying the following equalities:

\[
X \xrightarrow{a} T \xrightarrow{\eta} X = X
\]

(46)

\[
X \xrightarrow{T \alpha} T \xrightarrow{\mu} X = X
\]

(47)

• A morphism $(X, a) \to (Y, b)$ is an algebra homomorphism, as defined in example 1.1. Composition of morphisms is as in $C$.

The graphical notation shows that if we consider the monad as a monoid in $[C, C]$ then we can view the objects in the Eilenberg-Moore category as monoid actions on objects in $C$. In fact $X$ is a constant functor, and we can generalize the above definition to arbitrary functors. We will not require this extra generality in our setting, details can be found in [Kelly and Street, 1974].

In lemma 4.2 we saw that every adjunction induces a monad. We now see that every monad arises in this way.

**Proposition 4.6** (Eilenberg and Moore [1965]). For a monad $(C, T, \eta, \mu)$ there are functors:

\[
U : EM(T) \to C
\]

and:

\[
J : C \to EM(T)
\]

with $J \dashv U$ and the monad induced as in lemma 4.2 is equal to $T$.

**Proof.** We define:

\[
U : EM(T) \to C
\]

(50)

(51)

(52)

and:

\[
J : C \to EM(T)
\]

(53)

(54)

(55)

That these are valid functors is easy to check. Now we require a bijection:
\[ J(X) \to (Y, a : T(Y) \to Y) \]
\[ X \to U(Y, a : T(Y) \to Y) \]

Expanding definitions this becomes:

We define a map in the downward direction as follows:

\[ X \to Y \]

In the upward direction we define the map:

That this results in a valid algebra homomorphism follows immediately as \( a \) is an Eilenberg-Moore algebra and so:

We now check these maps constitute a bijection. Firstly, as \( a \) is an Eilenberg-Moore algebra homomorphism we immediately have:
In the opposite direction:

\[ \eta h = \{ h \text{ is an } \mathcal{EM}(T) \text{ morphism with domain } J(X) = \mu_X \} \]

\[ \mu \eta = \{ \text{Unit axiom} \} \]

Checking the naturality of these mappings is left as an exercise. That \( T \) is the monad induced by this adjunction follows from the construction in lemma \ref{lemma:adjunction-induced-monad}.

**Proposition 4.7.** Every monad morphism:

\[ \sigma : (\mathcal{C}, T, \eta, \mu) \rightarrow (\mathcal{C}, T', \eta', \mu') \]  

induces a functor:

\[ \mathcal{EM}(\sigma) : \mathcal{EM}(T') \rightarrow \mathcal{EM}(T) \]

**Proof.** On objects, a suitable functor acts as follows:

\[ \begin{array}{ccc}
X & \xrightarrow{T^*} & X \\
T & \searrow & T' \\
\downarrow & \downarrow & \downarrow \\
X & \sigma & X \\
\end{array} \]

On morphisms, the functor acts as the identity. That this is functorial is easy to show by applying a special case of example \ref{example:monad-morphism}. It remains to check the unit.
and multiplication axioms, the unit axiom follows from the equalities:

\[
\begin{align*}
X & \xrightarrow{\sigma} T & X & \xrightarrow{\eta'} = X \\
X & \xrightarrow{\mu} T & X & \xrightarrow{\mu} T & \xrightarrow{\eta} = X \\
\end{align*}
\]

That the multiplication axiom is satisfied is shown as follows:

\[
\begin{align*}
X & \xrightarrow{\sigma} T & X & \xrightarrow{\mu} T & \xrightarrow{\eta} = X \\
X & \xrightarrow{\mu} T & X & \xrightarrow{\mu} T & \xrightarrow{\eta} = X \\
\end{align*}
\]

**Definition 4.8 (Kleisli Category).** For a monad \((C, T, \eta, \mu)\) the **Kleisli category** \(K\ell(T)\) is defined as follows:

- **Objects** are the objects of \(C\)
- A morphism of type \(f : X \to Y\) in \(K\ell(T)\) is a \(C\) morphism \(f : X \to T(Y)\).

Given another morphism \(g : Y \to Z\) in \(K\ell(T)\), their (Kleisli) composite, written \(g \cdot f\) is given by the following composite in \(C\):

The identity morphism on \(X\) in \(K\ell(T)\) is given by \(\eta_X\), that this is a valid identity morphism follows immediately from the monad axioms.

We now see that the Kleisli category can be used to give an alternative construction of an adjunction inducing a particular monad, as was done with the Eilenberg-Moore category in proposition 4.6.
Proposition 4.9 (Kleisli [1965]). For a monad \((C, T, \eta, \mu)\) there are functors:

\[ V : \mathcal{K}\ell(T) \to C \tag{65} \]

and:

\[ H : C \to \mathcal{K}\ell(T) \tag{66} \]

with \(H \dashv V\) and the monad induced as in lemma 4.2 is equal to \(T\).

Proof. We define:

\[ V : \mathcal{K}\ell(T) \to C \tag{67} \]

\[ X \mapsto T(X) \tag{68} \]

\[ f : X \to T(X) \mapsto \mu_Y \circ T(f) \tag{69} \]

That this preserves identities is easy to check. For functoriality of composition,
we have:

\[ V( Y \circ Z T \mu ) = \{ \text{Definition of Kleisli composition} \} \]

\[ V( f \circ g ( Y \circ Z T )\mu ) = \{ \text{Definition of } V \} \]

\[ ( Y \circ Z T )\mu \circ ( f \circ g ) = \{ \text{Associativity axiom} \} \]

\[ V( Y \circ Z T \mu ) = \{ \text{Definition of } V \} \]
We also define:

\[ H : \mathcal{C} \to \mathcal{K}\ell(T) \]  
\[ X \mapsto X \]  
\[ f : X \to Y \mapsto \eta_Y \circ f \]  

Again that this preserves identities is easy to check. For functoriality of composition we have:

\[
H(Y) \circ H(X) = \{ \text{Definition of } H \}
\]
\[
H(Y) \circ T \circ \eta \circ T \circ \eta = \{ \text{Definition of Kleisli composition} \}
\]
\[
\mu \circ \eta \circ \eta = \{ \text{Monad axioms} \}
\]
\[
H(X) \circ X = \{ \text{Definition of } H \}
\]

Now we require a bijection:

\[
\begin{align*}
H(X) & \to Y \\
X & \to V(Y)
\end{align*}
\]

Expanding definitions, and noting that a Kleisli morphism \( X \to Y \) is a \( \mathcal{C} \) morphism \( X \to T(Y) \) by definition, the required bijection reduces to:
\[
\begin{align*}
X &\rightarrow T(Y), \\
X &\rightarrow T(Y)
\end{align*}
\]

Naturality of this bijection under the two forms of composition is easy to check. That we recover the original monad from this adjunction follows from the construction in lemma 4.2.

**Proposition 4.10.** Every monad morphism:

\[ \sigma : (C, T, \eta, \mu) \rightarrow (\mathcal{C}, T', \eta', \mu') \tag{73} \]

induces a functor:

\[ \mathcal{K}(\sigma) : \mathcal{K}(T) \rightarrow \mathcal{K}(T') \tag{74} \]

**Proof.** On objects, the functor acts as the identity. The action on morphisms is defined in \( \mathcal{C} \) as follows:

\[ X \rightarrow T(Y) \rightarrow T'(Y) \]

That this preserves identities is immediate from equation 42 as:

\[ \eta = \eta' \tag{76} \]

Similarly, composition follows from equation 43 as:

\[ \mu = \mu' \tag{77} \]

\[ \square \]

30
4.2 Categories of Monads

We now consider some appropriate types of morphisms between monads on different base categories.

**Definition 4.11.** The category *Monad* is defined as having:

- **Objects:** Monads

- **Morphisms:** A morphism of type \((C, T, \eta, \mu) \rightarrow (C', T', \eta', \mu')\) is a pair consisting of a functor \(F : C \rightarrow C'\) and a natural transformation \(f : T'F \Rightarrow FT\) satisfying the following equations:

\[
\eta' = f \eta \tag{78}
\]

\[
\mu' = \mu F \tag{79}
\]

**Proposition 4.12.** Every *Monad* morphism:

\[(C, T, \eta, \mu) \rightarrow (C', T', \eta', \mu')\] 

induces a functor:

\[\mathcal{E}M(T) \rightarrow \mathcal{E}M(T')\] 

**Proof.** For *Monad* morphism \((F, f)\) the action on objects of our functor is given by:

\[(a, x, T) \mapsto (a, x, T')\] 

That this extends to a functor to the category of \(T'\)-algebras follows from example 1.1. We must then confirm that the resulting algebra is in \(\mathcal{E}M(T')\). For
the unit axiom we have:

\[ a \xrightarrow{\eta} F \]

\[ = \{ \text{Monad morphism axioms} \} \]

\[ = \{ \text{Monad unit axiom} \} \]
For the multiplication axiom:

\[
\begin{align*}
X \times F & = \{ \text{Monad morphism axioms} \} \\
X \times X F & = \{ \text{Monad multiplication axiom} \} \\
F & = \{ \text{Monad multiplication axiom} \}
\end{align*}
\]

\[\square\]

Remark 4.13. Every monad morphism (definition 4.3) is an endomorphism in Monad with the functor part the identity. Proposition 4.7 is then a special case of proposition 4.12.

Definition 4.14. The categoryMonad\textsuperscript{*} is defined as having:

- **Objects**: Monads

- **Morphism**: A morphism of type \((C, T, \eta, \mu) \rightarrow (C', T', \eta', \mu')\) is a pair consisting of a functor \(F : C \rightarrow C'\) and a natural transformation \(f : FT \Rightarrow T'F\) satisfying the following equations:

\[
\begin{align*}
\eta' & = f \circ \eta \\
\eta & = \mu \circ f
\end{align*}
\]  

(83)
Proposition 4.15. Every Monad\(^\ast\) morphism:
\[(C, T, \eta, \mu) \to (C', T', \eta', \mu')\] (85)

induces a functor:
\[\mathcal{K}\ell(T) \to \mathcal{K}\ell(T')\] (86)

Proof. On objects, the induced functor maps \(X\) to \(F(X)\). On morphisms the action is:
\[
\begin{align*}
\text{That this preserves identities follows immediately from the Monad}\(^\ast\) axioms as: \quad &f = f
\end{align*}
\] (88)

Functoriality of composition also follows immediately from the Monad\(^\ast\) axioms as we have:
\[
\begin{align*}
\text{as we have:} \quad &f = f
\end{align*}
\] (89)
Remark 4.16. Every monad morphism (definition 4.3) is an endomorphism in Monad$^*$ with the functor part the identity. Proposition 4.10 is then a special case of proposition 4.15.

Remark 4.17. The categories Monad and Monad$^*$ can be described more generally for an arbitrary 2-category, and in fact can be given the structure of 2-categories themselves, although we will not require this additional structure for our examples. For more details, see Street, 1972 and Street and Lack, 2002 and also the excellent exposition in the early parts of Power and Watanabe, 2002.

4.3 Distributive Laws

In lemma 3.2 we saw that adjunctions compose in a straightforward manner. Monads do not in general compose, but they can be composed in the presence of a suitable mediating natural transformation, referred to as a distributive law.

Definition 4.18 (Distributive Law). For monads $(C, T, \eta, \mu)$ and $(C, T', \eta', \mu')$ a distributive law [Beck, 1969] is a natural transformation:

\[
\begin{array}{c}
T \\
\downarrow \delta \\
T'
\end{array}
\quad \quad
\begin{array}{c}
T' \\
\downarrow \delta' \\
T
\end{array}
\]

satisfying the following equations:

\[
\begin{align*}
\eta' \delta &= \eta' (90) \\
\eta \delta &= \eta (91) \\
\mu' \delta &= \delta \mu (92)
\end{align*}
\]
Remark 4.19 (Artistic Values). The axioms presented for a distributive law in definition 4.18 perhaps best demonstrate the importance of how we choose to draw our string diagrams. In the axioms of equations 90, 91, 92 and 93 we choose to draw the natural transformation $\delta$ in different ways. The different choices of how $\delta$ is drawn allow us to emphasise the nature of the transformations as “sliding” various $\eta$ and $\epsilon$ natural transformations across an appropriate line in the diagram. A less careful choice of layout would obscure the visual intuition for how these axioms behave.

Proposition 4.20. Given monads $(\mathcal{C}, T, \eta, \mu)$ and $(\mathcal{C}, T', \eta', \mu')$ and a distributive law as in definition 4.18, then $T' \circ T$ gives a monad on $\mathcal{C}$ with unit:

and multiplication:
Proof. For the first of the unit axioms, we have the following equalities:

\[
\begin{align*}
\mu \delta \eta \eta' & = \begin{cases} \text{Distributive law axioms} \\ \mu \eta \eta' \end{cases} \\
\mu \delta \eta \eta' & = \begin{cases} \text{Monad axioms in each of the component monads} \\ T T' \end{cases}
\end{align*}
\]
The second unit axiom follows dually. For the associativity axiom:

\[
\begin{align*}
\mu \delta \mu' & = \{ \text{Distributive law axioms} \} \\
\mu \delta \mu' & = \{ \text{Associativity for } \mu \text{ and } \mu' \} \\
\mu \delta \mu' & = \{ \text{Distributive law axioms} \}
\end{align*}
\]

**Proposition 4.21.** Given monads \((C, T, \eta, \mu)\) and \((C, T', \eta', \mu')\) and a distributive law as in definition 4.18, then \(T'\) induces a monad on \(\mathcal{E}M(T)\).
Proof. We first note that a distributive law is a Monad morphism, and so $T'$ lifts to an endofunctor on $\mathcal{EM}(T)$ by proposition 4.12. We take $\eta'$ and $\mu'$ as the unit and multiplication. That $\eta$ gives a natural transformation in $\mathcal{EM}(T)$ follow from:

$$\eta_a f = \eta_{a f}$$ (94)

Similarly that $\mu'$ is a natural transformation in $\mathcal{EM}(T)$ follows from:

$$\mu_a f = \mu_{a f} f$$ (95)

That $\eta'$ and $\mu'$ satisfying the monad axioms in then obvious as morphisms in $\mathcal{EM}(T)$ compose exactly as in $\mathcal{C}$. \qed

5 Representable Functors

We now investigate the important concept of representable functors. Many concepts in category theory can be phrased as requiring that a certain functor is representable, and examined from the string diagram perspective, representability provides a standard framework for generating calculation rules in a uniform manner. These calculation rules can then be specialized to provide graphical rules for limits, colimits, adjunctions, and left and right Kan extensions with little effort, unifying how these concepts can be approached graphically. Our use of representability bears some resemblance to the use of initiality to provide calculation rules in [Fokkinga, 1992b].

5.1 Covariant Representable Functors

Definition 5.1 (Covariant Representable Functor). A covariant functor $G : \mathcal{C} \to \text{Set}$ is said to be representable if there is an object $S$ of $\mathcal{C}$ such that there is a natural isomorphism:

$$G \cong \mathcal{C}(S, -)$$ (96)

We will identify the elements of a set with morphisms from the one element set, as discussed in section 2.1. We will use a box notation to represent the mappings of the natural isomorphism, similar to the box notation for monoidal
functors introduced in [Cockett and Seely, 1999], [Blute et al., 2002] and described in detail in [Mellies, 2006]. For an object $X$ in $\mathcal{C}$ the mapping from left to right in equation 96 can be drawn graphically as:

$$G(X) \rightarrow \mathcal{C}(S, X) \quad (97)$$

$$\begin{array}{c}
\xymatrix{
X & \ar[l]_r & G \\
& * & \\
S & & \\
}\end{array} \quad \begin{array}{c}
\xymatrix{
X & \ar[l]_r & G \\
& * & \\
S & & \\
}\end{array} \quad (98)$$

In the other direction, we can write the mapping as:

$$\mathcal{C}(S, X) \rightarrow G(X) \quad (99)$$

$$\begin{array}{c}
\xymatrix{
X & \ar[l]_k & S \\
& & \\
} \end{array} \quad \begin{array}{c}
\xymatrix{
X & \ar[l]_k & S \\
& & \\
} \end{array} \quad (100)$$

As these operations correspond to isomorphisms, the following two conditions hold:

$$\begin{array}{c}
\xymatrix{
X & \ar[l]_r & G \\
& * & \\
S & & \\
} \end{array} \quad = \quad \begin{array}{c}
\xymatrix{
X & \ar[l]_r & G \\
& * & \\
S & & \\
} \end{array} \quad (101)$$

$$\begin{array}{c}
\xymatrix{
X & \ar[l]_k & S \\
& & \\
} \end{array} \quad = \quad \begin{array}{c}
\xymatrix{
X & \ar[l]_k & S \\
& & \\
} \end{array} \quad (102)$$
The important aspect of representability is that the isomorphism is natural. This can be seen as requiring that the following two calculation rules hold, allowing us to “push and pop” morphisms in and out of the box notation.

\[
Y^\ast X^\ast G \mapsto S 
\]

\[
X^\ast Y^\ast G \mapsto K 
\]

The next lemma shows that to prove representability it is sufficient to show a family of bijections satisfy just one of the two equations above.

**Lemma 5.2.** For a given family of isomorphims of hom sets:

\[
\theta_X : F(X) \cong \mathcal{C}(S, X) \tag{105}
\]

The conditions in equations 103 and 104 are equivalent.

**Proof.** It is easy to show that if every component of a natural transformation has an inverse, then the inverses form a natural transformation. The claim then follows immediately.

**Remark 5.3.** Given a representable functor:

\[
G \cong \mathcal{C}(S, -) \tag{106}
\]

Generally we will know more about the structure of the left hand side than just that it is a functor with codomain Set. It will be common for \( G \) to involve a hom bifunctor \( \mathcal{C}(-, -) \) in some way, and this structure can be used to put the above equations into more convenient form than reasoning in terms of elements of an abstract set. This will be seen in the examples in section 5.4.
5.2 Contravariant Representable Functors

**Definition 5.4** (Contravariant Representable Functor). A contravariant functor $F : \mathcal{C}^{\text{op}} \to \text{Set}$ is said to be representable if there is an object $R$ of $\mathcal{C}$ such that there is a natural isomorphism:

$$F \cong \mathcal{C}(\text{−}, R) \quad (107)$$

It is obvious from equation 107 we also have:

$$F \cong \mathcal{C}^{\text{op}}(R, \text{−}) \quad (108)$$

This is now expressed as in equation 96 for the covariant case, and so we can use identical definitions to those in section 5.1, but replacing $\mathcal{C}$ with $\mathcal{C}^{\text{op}}$ everywhere. Alternatively, we can use both $\mathcal{C}$ and $\mathcal{C}^{\text{op}}$ in our diagrams. The first mapping then becomes:

$$F(X) \to \mathcal{C}(X, R) \quad (109)$$

The other direction is then written as:

$$\mathcal{C}(X, R) \to F(X) \quad (111)$$

The calculation rules given in equations 103 and 104 can also be adapted in a similar manner.

5.3 Universality from Representables

Much of category theory is concerned with the important notion of a universal property. We now show how the representability of a functor leads to a particular universal property. When more structure is known about the representable functor, we can recover well known universal properties such as those of adjunctions, limits, colimits and Kan extensions, as will be discussed in later sections and examples.
Definition 5.5 (Counit). For a representable functor:

\[ G \cong C(R, -) \quad (113) \]

the counit is the element of \( G(R) \) that is the image of \( 1_R \) under the natural isomorphism.

Proposition 5.6. For a representable functor:

\[ G \cong C(R, -) \quad (114) \]

and \( X \) an object of \( C \), for every element \( r \in G(X) \), there exists a unique \( f \) such that \( r \) can be written in the form:

\[ r = G(f)(c) \quad (115) \]

where \( c \) is the counit of the representable functor.

Proof. For an arbitrary \( r \in F(X) \) we have the following sequence of equalities:
Now assume $r = G(f)(c)$ and $r = G(g)(c)$, then:

\[
\begin{align*}
X @> f >> G & \equiv & X @> g >> G \\
R @> * >> & \equiv & R @> * >> \\
\Leftrightarrow & \{ \text{ push / pop equations } \} \\
X @> f >> R @> * >> & \equiv & X @> g >> R @> * >> \\
\Leftrightarrow & \{ \text{ isomorphism } \}
\end{align*}
\]

Remark 5.7. A dual universality result can be given for contravariant representable functors, the details are left to the interested reader.

5.4 Examples of Representable Functors and their Calculation Rules

In this section we will consider two important applications of the representability based approach of section 5, adjunctions and Kan extensions. Another important source of examples are limits and colimits, these will be discussed in detail later in section 6.

Example 5.8 (Adjunctions). For $F \dashv G$, for each $X$ and $Y$ we have mappings:

\[
\begin{align*}
X @> f >> F @< X >> & \equiv & X @> f >> F @< X >> \\
Y @> G >> & \equiv & Y @> G >> \\
\Leftrightarrow & \{ \text{ push / pop equations } \} \\
Y @> f >> X @> F >> & := & X @> F >> \\
\Leftrightarrow & \{ \text{ isomorphism } \}
\end{align*}
\]

$\square$
and:

\[
\begin{array}{c}
\begin{array}{c}
G \\
g \\
Y \\
X
\end{array} \\
\rightarrow \\
\begin{array}{c}
G \\
g \\
Y \\
F \\
X
\end{array} :=
\end{array}
\]

(117)

It is immediate from the adjunction axioms that these maps are mutually inverse. Naturality of the first mapping follows easily from the following equalities:

\[
\begin{array}{c}
\begin{array}{c}
Z \\
h \\
Y \\
X
\end{array} \\
\rightarrow \\
\begin{array}{c}
Y \\
h \\
F \\
X \\
\eta
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
Z \\
h \\
Y \\
X \\
\eta \\
\eta
\end{array} \end{array} = \begin{array}{c}
\begin{array}{c}
Z \\
h \\
Y \\
X \\
F
\end{array} \\
\rightarrow \\
\begin{array}{c}
Z \\
h \\
Y \\
X \\
F
\end{array}
\end{array}
\]

(118)

Naturality of the second map then follows from lemma 5.2. We remark that the usual bijection, natural in \(X\) and \(Y\) induced by an adjunction:

\[
F(X) \rightarrow Y \\
X \rightarrow G(Y)
\]

can also be seen as an immediate corollary of lemma 5.4.

**Example 5.9 (Kan Extensions).** A novel graphical notation for Kan extensions was introduced in [Hinze, 2012]. The paper then develops many proofs using both a traditional symbolic style and a string diagram based graphical approach, illustrating the compactness and efficiency of the later.

We will examine Kan extensions as an example application of the representability based results we have developed in earlier sections. Given functors \(J : B \rightarrow A\) and \(F : B \rightarrow C\), \(F\) is said to have a left Kan extension along \(J\) if the functor \([B, C](F, (-) \circ J)\) is representable. Then we have:

\[
[B, C](F, (-) \circ J) \cong [A, C](\text{Lan}_J(F), -)
\]

(119)

Dually, \(F\) has a right Kan extension along \(J\) if we have the following natural isomorphism:

\[
[B, C]((-) \circ J, F) \cong [A, C](-, \text{Ran}_J(F))
\]

(120)

If we denote the counit as \(\epsilon\), in the case of left Kan extensions, the universality results of section 5.3 specialize to the following axiom, giving the usual
unique factorization property of left Kan extensions:

\[
\text{Lan}_J(F) \cong \text{Lan}_J(F),
\]

We recover the calculation laws of Hinze, 2012, as follows:

- The computation law follows immediately from the universal property above.
- The reflection law follows directly from the definition of the counit.
- The fusion law is an instance of the “push / pop” identity in equation 103.

6 Limits and colimits

Limits and colimits are key notions in category theory. They can be approached from the perspective of representability introduced in section 5. We will start with the general setting, and then provide specialized results and notation for some common cases.

6.1 Arbitrary Limits and Colimits

We first define a standard functor used when reasoning about limits and colimits.

**Definition 6.1.** Let \( C, D \) be categories, then we define the functor \( \Delta : C \to [D, C] \) as taking objects to the corresponding constant functor, and morphism \( f \) to the natural transformation with all components equal to \( f \).

Now we define limits and colimits in terms of representability of appropriate functors.

**Definition 6.2.** Let \( C, D \) be categories, and the functor \( D : D \to C \) a diagram in \( C \). The limit of \( D \) exists if the functor \( [D, C](\Delta(-), D) \) is representable, i.e.

\[
[D, C](\Delta(-), D) \cong C(-, \lim D) \tag{122}
\]

The limiting cone then corresponds to the identity morphism on \( \lim D \).

Dually, the colimit of \( D \) exists if the functor \( [D, C](D, \Delta(-)) \) is representable, i.e.

\[
[D, C](D, \Delta(-)) \cong C(\colim D, -) \tag{123}
\]
As the existence of limits and colimits corresponds to representability of certain functors, we can use the calculation rules in section 5. We can use our additional knowledge of the structure of the functor on the left hand side to phrase the calculation laws in a more convenient form. For limits we have:

\[
\lim D_X \Delta Y = \lim D_{X Y} \Delta D_h \lambda k \tag{124}
\]

\[
\lim D_X \Delta Y \Delta D_h = \lim D_X X Y \Delta D_\lambda \tag{125}
\]

For colimits the following equations hold:

\[
\colim D_X \Delta Y = \colim D_{X Y} \Delta D_h \lambda \tag{126}
\]

\[
\colim D_{X Y} \Delta \lambda k = \colim D_{X Y} \Delta \tag{127}
\]

We will specialize the universality result described in section 5.3 when we discuss some specific limits and colimits in later sections.
6.2 Specific Limits and Colimits

We now examine a few specific limits and colimits. In these concrete cases we can specialize the results in section 6.1 and provide more convenient notation and calculation rules for some common cases.

Initial Objects

The initial object will be denoted 0 and we will use ! : 0 \to X to denote the unique morphism from the initial object to an object X. The universal property for initial objects can be expressed by the following relationship, for each object X in \mathcal{C}:

\[
0 \xrightarrow{f} X \quad \iff \quad 0 \xrightarrow{!} X
\]

(128)

We will now further specialize this rule for initial algebras, and as an example calculation apply the rule to provide a graphical proof of Lambek’s lemma. Initial algebras, and their dual notion terminal coalgebras, are of interest in modelling of datatypes [Goguen et al., 1975], [Goguen et al., 1977], [Hagino, 1987a], [Hagino, 1987b], [Malcolm, 1990a] and [Malcolm, 1990b].

Example 6.3 (Initial Algebras and Lambek’s Lemma). For endofunctor T : \mathcal{C} \to \mathcal{C}, if T has an initial algebra ((\mu(T), \text{in}_T) then we can rewrite the initiality condition in a more useful form, with the left hand equation in \mathcal{C} and the right hand equality in T-\text{Alg}. We also adopt some standard notation in this context and denote the unique morphism from the initial algebra to an algebra a as fold_T a.

\[
\begin{align*}
\text{fold}_T a &= 1_{\mu T} \\
\text{fold}_T a &= \text{fold}_T a
\end{align*}
\]

(129)

It is immediately obvious that \text{fold}_T \text{in}_T = 1_{\mu T}. We can then prove Lambek’s
lemma, that the initial algebra is an isomorphism. We have:

\[ \mu(T) \]

\[ \Rightarrow \{ \text{ initial algebra law and } \text{fold}_T \text{ in}_T = 1_{\mu(T)} \} \]

\[ \text{true} \]

For the other direction we have:

\[ \text{fold}_T(T \text{ in}_T) \]

\[ \Rightarrow \{ \text{ fold}_T(T \text{ in}_T) \text{ is an algebra homomorphism } \} \]
Binary Products

The binary product of objects $Y$ and $Z$ will be written $Y \times Z$, in the usual way usual. The representability condition gives a bijective correspondence between morphisms $X \rightarrow Y \times Z$ and pairs of morphisms $X \rightarrow Y$ and $X \rightarrow Z$. To aid calculations we will introduce three different maps:

\[
\begin{align*}
Y \times Z & \quad h \quad \mapsto \quad X \\
X & \end{align*}
\]

(130)

\[
\begin{align*}
Y \times Z & \quad h \quad \mapsto \quad X \\
X & \end{align*}
\]

(131)

\[
\begin{align*}
(Y \times Z, f, g) & \mapsto \quad X \\
X & \end{align*}
\]

(132)

We then define the usual notation for the two components of the unit:

\[
\begin{align*}
Y \times Z & \quad \pi_1 \quad = \quad Y \\
Y \times Z & \end{align*}
\]

(133)
Using the material in section 5.3 we can then write the universal property of products as:

\[ Y \times Z \xrightarrow{\pi_1} Y \times Z \quad \wedge \quad Y \times Z \xrightarrow{\pi_2} Y \times Z \]

\[ (134) \]

The “push / pop” equations 103 and 104 then lead to the following three equalities:

\[ (135) \]

\[ (136) \]
That these maps witness a bijection leads to the following three of equalities:

\[(137)\]

\[(138)\]

\[(139)\]

\[(140)\]
Readers familiar with more standard introductions to category theory will hopefully recognise many standard exercises in the properties of binary products are given in graphical form by the various equations above.

Terminal Objects

The terminal object will be denoted 1, and ! : X → 1 will denote the unique morphism from an object X to the terminal object. The universal property of the terminal object can then be written as for each object X in C:

\[ X \simeq 1 \implies X \simeq 1 \]  \[(141)\]

**Example 6.4** (Terminal Coalgebras and the Fusion Law). Coalgebras are the dual notion to algebras, a standard reference is [Rutten, 2000]. For an endofuncto

\[ T \colon \mathcal{C} \to \mathcal{C} \]  \[\text{a coalgebra is a pair consisting of an object } X \text{ and a morphism } X \to T(X). \text{ A coalgebra morphism } (X, a) \to (Y, b) \text{ is a } \mathcal{C} \text{ morphism } h : X \to Y \text{ satisfying:} \]

\[ T a h = T b h \]  \[(142)\]

Dually to the situation with algebras in example 6.3, if an endofunctor T has a terminal coalgebra (ν(T), out_T), we can rewrite the terminal object condition for coalgebras in a more convenient form. We will adopt the standard notation of unfold_T a for the unique morphism from an algebra a to the terminal coalgebra.

\[ \text{out}_T h = \text{out}_T a \iff T h a = \text{unfold}_T a \]  \[(143)\]

We will now prove the important (strong) fusion law for terminal coalgebras.
We have the following chain of equivalences:

\[
\begin{align*}
\text{h} & \text{unfold}_T f = \text{unfold}_T g \\
\Leftrightarrow & \{ \text{universal property of terminal coalgebra} \} \\
\text{out}_T \text{unfold}_T f & = \text{unfold}_T f \\
\Leftrightarrow & \{ \text{unfold}_T f \text{ is a coalgebra homomorphism} \} \\
\text{unfold}_T f & = \text{unfold}_T g 
\end{align*}
\]

The intuitive explanation of the fusion law is that if the final equality is satisfied, then we can “fuse” the morphism \( h \) with \( \text{unfold}_T f \) to give a single unfold morphism \( \text{unfold}_T g \). As the precondition for fusion is rather cumbersome, it is often easier to apply the weak fusion law which follows as an immediate corollary via Liebniz:

\[
\text{h} f = g \Rightarrow \text{h} \text{unfold}_T f = \text{unfold}_T g (144)
\]

**Binary Coproducts**

Binary coproducts follow the dual pattern to the description of binary products in section 6.2. We will write the binary coproduct of \( X \) and \( Y \) as \( X + Y \) in the usual way. The representability condition gives a bijective correspondence between morphisms \( X + Y \to Z \) and pairs of morphisms \( X \to Z \) and \( Y \to Z \). As with binary products, to aid calculations we will introduce three different
We then define the usual notation for the two components of the counit:

\[ X + Y \xrightarrow{\kappa_1} X + Y \xrightarrow{\Delta} X \]

\[ X + Y \xrightarrow{\kappa_2} X + Y \xrightarrow{\Delta} Y \]
Using the material in section 5.3 we can then write the universal property of coproducts as:

\[
\begin{align*}
Z \xrightarrow{h} X + Y \xleftarrow{\kappa_1} X & \quad = \quad Z \xrightarrow{h} X \wedge X + Y \xleftarrow{\kappa_2} Y \quad = \quad Z \xrightarrow{g} Y \\
\Leftrightarrow \\
Z \xrightarrow{h} X + Y & \quad = \quad Z \xrightarrow{f} X \wedge Z \xrightarrow{g} Y \\
\end{align*}
\]

The “push / pop” equations 103 and 104 lead to the following three equalities:

\[
\begin{align*}
\text{X} \ \text{Y} & \quad f \quad Z \quad g \quad X + Y \\
\text{X} \ \text{Y} & \quad h \quad Z \quad Z' \quad v \\
\text{X} \ \text{Y} & \quad \triangleleft \quad Z' \quad Z \quad \triangleright \quad Z \\
\text{X} \ \text{Y} & \quad \triangleleft \quad Z \quad Z \quad \triangleright \quad Z
\end{align*}
\]
That these maps witness a bijection leads to the following three identities:

\[ X + Y \vdash f \sigma = g \tau \ (153) \]

\[ X + Y \vdash f \sigma = g \tau \ (154) \]

\[ X + Y \vdash f \sigma = g \tau \ (155) \]

Remark 6.5. Throughout this section there has been no attempt to present a minimal set of equations. Many of the equalities above are interderivable, but we have aimed to give explicit statements of many useful properties, and to exploit results about of representable functors to prove standard identities from general principles.

7 Conclusion

We have shown in the previous sections how string diagrams can be used to perform calculational proofs whilst retaining the vital type information. Although we could not hope to provide comprehensive coverage, for example we have not discussed dual adjunctions, equivalences, exponentials or many common types
of limits and colimits, hopefully we have provided sufficient background to make
extending the graphical proof style to further topics straightforward.

Tool support for developing and documenting proofs in this diagrammatic
style would be tremendously useful, and is a direction for further investigation.
The proofs we develop, although conceptually compact, often physically take
up a large amount of page space and the proof style also makes essential use of
colour. These attributes are not particularly “publication friendly”, tool support
could also aim to provide (preferably bidirectional) translations between string
based proofs and more traditional formats.

Although we do not claim that string diagrammatic proofs are always the
most suitable approach to any category theoretic question, many problems
where there is a lot of “bookkeeping” involving functoriality and naturality con-
ditions can be greatly simplified. String diagrams also provide a different, more
visual and topological intuition for category theoretic concepts and proofs. It is
therefore hard to explain the absence of string diagrammatic tools from intro-
ductive courses and texts on category theory and the relegation of this material
to folklore in the community. Teaching experience introducing these tools earlier
in category theory courses would be also therefore be of interest.

Acknowledgements

The author would like to thank Samson Abramsky, Andreas Döring, Ralf Hinze,
Peter Selinger, Pawel Sobocinski and Brendan Fong for useful feedback and
encouragement.

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