Mathematical Framework for Online Social Media Regulation

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Abstract

Social media platforms (SMPs) leverage algorithmic filtering (AF) as a means of selecting the content that constitutes a user’s feed with the aim of maximizing their rewards. Selectively choosing the contents to be shown on the user’s feed may yield a certain extent of influence, either minor or major, on the user’s decision-making, compared to what it would have been under a natural/fair content selection. As we have witnessed over the past decade, algorithmic filtering can cause detrimental side effects, ranging from biasing individual decisions to shaping those of society as a whole, for example, diverting users’ attention from whether to get the COVID-19 vaccine or inducing the public to choose a presidential candidate. The government’s constant attempts to regulate the adverse effects of AF are often complicated, due to bureaucracy, legal affairs, and financial considerations. On the other hand, SMPs seek to monitor their own algorithmic activities to avoid being fined for exceeding the allowable threshold. In this paper, we mathematically formalize this framework and utilize it to construct a data-driven statistical algorithm to regulate the AF from deflecting users’ beliefs over time, along with sample and complexity guarantees. We show that our algorithm is robust against potential adversarial users. This state-of-the-art algorithm can be used either by authorities acting as external regulators or by SMPs for self-regulation.

1 Introduction

Social networking platforms (e.g., Google, Facebook, Twitter), are increasingly becoming the prevailing, most easily accessible, and most popular platforms for individual media consumption across the Western world [52]. Indeed, media platforms act as intermediaries between users and the wealth of information collected from their friends, news, opinion leaders, celebrities, politicians, and advertisers. So pervasive and eclectic is the stream of information content collected for each user at any given time that it compels social networks to filter out the most relevant information, display it in the user’s news feed, and to sort its content’s appearance. To that end, in the last decade, social platforms have been adopted various algorithmic filtering (AF) methods [16] to select and sort collections of contents to be shown on their user’s feed.

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Notwithstanding the potential of AF to provide users with a richer, more diverse, and more engaging experience, over the past two decades, these methods have been leveraged by social network platforms to selectively filter user feeds in an effort to maximize their returns (including revenue, user accumulation, popularity gain, etc.). This phenomenon has brought about harmful side effects [31, 11]. For example, an artificial comment ranking that encourage over-representation of one side’s opinion or polarization of opinions on one side [66]. Similarly, the prioritization of a specific topic contributes to the dissemination of deliberately disregarded fake news [48, 22, 25, 57], advertisements that promote products based on erroneous claims regarding the user’s interests [67, 69], leading to some information being more (or less) visible along with many others.

The foregoing examples embody the damaging fact that subjectively filtering the content to be shown on the user’s feed does not overlap with the user’s interest or the society’s interest as a whole, resulting in widespread adverse impacts on both individuals and society [57]. This, in turn, heavily impacts users’ learning, shapes their thinking as well as their decisions, and ultimately influences the way they behave as individuals or as a whole society. In particular, deliberately showcasing of polarized comments may sow hatred between groups [47], disseminating fake news may sway the presidential election results [9], intensive dietary recommendations may cause users to change their own diet [21, 41], etc.

These negative influences have led to a number of calls for regulatory action by the authorities; however, their increasing enforcement attempts encounter multiple hurdles, such as, legal barriers, cumbersome and entangled bureaucracy, high human resource costs, which usually ends with no concrete results [13, 42, 8]. The legal difficulties are mainly driven by the concern that regulations might limit free speech [42, 13], infringe on privacy by requiring content disclosure, subjectively define of what is right or wrong media behaviour [56], undermine innovation or suppress jobs and revenues (e.g. through advertising restrictions).

Meanwhile, the increasing enforcement of regulations that aims at fining violations encourages the platforms to use self-regulatory methods to prevent unintentional internal activities and avoid penalties [51]. Among others, Twitter suspends tens of thousands accounts suspected of being involved in promoting conspiracy theories [55]. Facebook has set up an independent internal team named “Oversight Board” to foster freedom of expression by making principled, independent decisions about contents [10]; YouTube has removed videos urging violence [39].

From the regulator’s point of view, the regulation should be: consistent with the relevant social and legal norms, enforceable, and flexible. By definition, any kind of regulation ultimately draws a “boundary”, on one side of which the regulated behavior is penalized. This raises two fundamental questions: where should the boundary be placed? and who makes this decision?

Indeed, the notion of regulation boundary and the investigation of some of the above questions, are by no means new, and they are being pursued and discussed in many fields, such as, in the economy literature (e.g., [35, 68, 36, 6]), as well as in the social learning literature (e.g., [46, 74, 17]). Next, we provide possible answers to the above questions, which in turn guide our mathematical formulation for a regulation. The following ideas are based on a several recent papers [34, 46, 74, 44, 50, 62], most notably, [17], and serve as a starting point for a consistent definition of regulation.
1. One straightforward option in response to the first question is to follow a hard/worst-case approach of setting the boundary globally, e.g., posts containing a certain percentage of false statements are labeled as unreliable. Obviously, this approach is highly subjective, brittle, and subject to adversarial attacks. One way to overcome these problems is for regulation to be context-dependent, namely, adaptable to different legal and social norms, platform-user interactions, etc. meaning that it is specific to the user and platform as well as mindful of relevant social and legal norms. Specifically, they would depend on the nature of the interactions between the specific user and platform under consideration. They would also depend on the norms of the society in which the regulation is applied. As described above, requiring that the regulation is context-dependent is important in ensuring that is normative, flexible, and long-lasting. For example, why did the platform show a certain advertisement to certain users but not others, and is this consistent with the social values and legal norms? Context-dependent regulations were discussed in, for example, [46, 74, 17], and in this paper we follow this approach.

2. With respect to the second question, which involves who formulates the regulatory standards, several possibilities exist. The first native option is to let the platform take the decision. Evidently, however, this would create a serious conflict of interest and extend the power of platforms over users [42, 38]. Another option is to use a set of experts (e.g., third-party organizations such as the World Economic Forum [74]), which removes the conflict of interest and provides a well-informed decision. However, it can also cause certain privacy concerns when viewed as extending the reach of outside parties into citizens’ private spheres. A third option is the user. Specifically, in order to construct a user-based regulation, we discuss the notion of consent [70, 34, 17, 59]. While a shallow interpretation of consent reflects the idea that users agree to share their private information, it can also mirror the amount of control/agency users have over the service they are being provided [70, 59], e.g., friending, liking, commenting, agreeing to ads, etc. Accordingly, the regulator can use consent as a guideline to examine whether the platform deviates significantly from the consumer-provider relationship to which the user and platform agreed. In a strike contrast to the above two options, an important implication of using consent is that it provides a flexible, user- and context-dependent reference of what the user’s would have seen over a hypothetical platform which do not filter the content in an irresponsible way, or at least one that follows the consumer-provider relationship.

The notion of an implicit agreement between users and digital platforms is far from being new [49]. This notion draws from a general implicit contract theory [43], which economists use to explain behaviors that are observed but not justified by competitive market theory. In particular, it has been invoked to explain the reason for users to keep using social media despite data privacy infractions [45, 65, 74]. It has also been advanced as a starting point for regulation [60], since it balances the interests of both parties.

As social media become increasingly popular information sources, a fundamental question remains: Is there a systematic and responsible way to regulate the effect of social media platforms on users learning and decision-making? Even though it may be possible to do so, due to the many issues raised above, and many other related ones, designing and reinforcing
a regulation is still a notoriously difficult open problem [46]. Accordingly, the challenging quest for currently a far reaching fundamental theory for systematic regulatory procedures that satisfy several social, legal, financial, and user related requirements, and its prospective practical ramifications, constitute the main impetus behind this paper.

Specifically, motivated to guarantee compliance with a consumer-provider agreement, in this paper, we propose a data-driven statistical method for efficiently applying AF regulation, namely online monitoring any adverse influences on the user learning (and thus on decision-making), while allowing real-time enforcement. Both sides, authorities, and social platforms could use the method individually or independently. One novelty of our method is that it does not require any given explicit regulation statement, such as suggested in [17], but rather moderates any detrimental influence on the user’s decision-making caused by subjectively filtering the user feed comparing to what it would have been under a natural contents selection. Therefore using our proposed data-driven algorithm could prevent undesirable negative side effects on the platforms users. Besides, it is thoroughly demonstrated that the method does not require any access to users’ personal data, thus hermetically preserving their privacy.

1.1 Related Work

Various attempts aim at regulating content moderation have been proposed over the last few years; however, all of these attempts generally focus on monitoring specific violations of the social platform-user agreement. Specifically, common methods for content moderation fall broadly into one of three categories (see, e.g., [14, 53]):

1. **Content control**, which aims at tagging or removing suspicious items. However, the ability of AI algorithms to identify such rough items grows more slowly than the ability to create them [58], and objectivity of human content control is often less trusted [4]. Content control strategies include: increasing content diversity (e.g., adding heterogeneity to recommendations [12, 37]); drawing a line in the sand (e.g., determining whether discrimination has occurred by thresholding the difference between two proportions [23]); detecting hate speech (e.g., using deep learning technique [40, 64] or NLP clustering methods [30]); or finding the origin of the content (e.g., reducing fake news by whitelisting news sources [8] or detecting the sources that generate misleading posts [61]).

2. **Transparency**, where users are required to provide lawful identification. This approach imposes a serious toll on user privacy and anonymity, while not even necessarily stopping unintended spread of misinformation.

3. **Punishment**, where the network provider or the state impose penalties for malicious spreading of fake information. This extreme approach is clearly the least desirable from both privacy and human-rights perspectives.

Most related to our work is [18], where the concept of a static time-independent “counterfactual regulations” was proposed and analyzed. Counterfactual regulations deal with regulatory statements of the form: “The platform should produce similar feeds for given users who are identical except for one single property”. The users differentiating property
could be, for example, gender, religion, left or right wing affiliation, age, among many others. This paper is the first to propose a statistical regulatory procedure testing whether regulation written in counterfactual regulations form is met. In essence, this procedure is designed as a binary hypothesis tester that, given pairs of inputs corresponding to two pairs of users who differ in a single property and a distribution model family (generating feeds), returns whether regulation is met with respect to a predetermined level of confidence (probability). Specifically, according to the model proposed in [18], the user’s decision-making process comprises three hierarchical steps: The user observes information presented in his feed, updates his internal belief and then, based on this belief, makes a decision. Then, the authors of [18] advocate the regulatory “worst-case” approach, where the regulatory test is designed to prevent violations associated with (a hypothetical) “most gullible user/best learner”, meaning the one whose decisions are most influenced by selectively filtering his feed. One can think of the level of influence of the feed on the most gullible users’ decision-making as an upper bound of the level of influence on all other users. This way, if that type of user passes regulation, then all other users will pass regulation as well. The reason given by the author for opting for the “worst-case” approach is that in order to infer the impact of a feed on users’ decision making one would need access to it, which may be unethical.

Finally, as stated in [18], research and modeling of counterfactual regulation draw parallel ideas from the differential privacy literature [32, 33], as in the case of comparing outcomes under different interventions [71]. Finally, while our paper addresses questions similar to those studied in social learning and opinion dynamics, e.g. [1, 54, 5], it is distinct from this literature in the sense that our research focuses on the question of how the flow of information, mediated by social networks, leads to undesirable biases in the way users think/learn and, consequently, to a detrimental change in their decision-making and ultimately in their actions. Furthermore, this is accomplished without the need to actually access the users’ beliefs, actions, or thoughts.

1.2 Main Contributions

Our main goal is to develop a regulatory procedure for content moderation over social networks. We split this subsection into two parts: the first focuses on our conceptual contributions to the general area of social media regulation, while the second discusses our technical contributions.

1.2.1 Conceptual Contributions

Unifying framework. We formulate a novel statistical unifying framework for online platform regulation. This framework considers the three involved parties: platform, users, and regulator, all interacting and evolving over time (see, Fig. 1). At each time point, the platform shows its users collections of content, known as “filtered feeds.” As each user in the platform browses through his own feed, he implicitly forms a belief, and ultimately modifies his actions. The regulator’s meta-objective is to moderate the effect of socially irresponsible externalities caused by the AF’s effect on user learning and decision making, either as individuals or as a society. To that end, the platform supplies the regulator with anonymous data of two types: filtered and reference. The latter is constructed by ignoring any aspect of a platform’s fiscal motivation, thus representing a natural/fair filtering of
content rather than a subjective form of filtering (see, Section 2 for a precise definition). As an illustration, it is as though the platform were a non-profit organization prioritizing the users’ experience. We show that the regulator’s task can be formulated as a certain closeness testing problem (see, e.g., [28, 15]).

**Automatic online regulatory procedure.** Our work is the first to propose a regulatory procedure that does not require any prior explicit regulation statement (such as suggested in [18]). Our regulatory procedure monitors, over a predefined adjustable time-frame, any damaging influence on the users’ decision-making, compared to what it would have been without subjective filtering of the users’ feeds, namely, under a natural/fair content filtering. This is accomplished by formulating a measure called “belief-variability”, which estimates the influence of the AF on the beliefs of all the users. Using this variability we then formulate the regulator’s objective as a sequential hypothesis testing problem. As a binary hypothesis tester, the regulator examines whether the platform exceeds a tunable threshold of acceptable values of this estimated measurement of influence, doing so over a predefined time frame with a given confidence level. The regulator outputs whether or not regulation is being complied with, meaning whether public opinion is being biased or not. For example, this regulatory procedure could easily detect the intensive promotion of a presidential candidate via posts, advertisements, the prioritization of related user comments, artificial adversarial users, or polarized recommendations. For the problem above, we devise a data-driven, statistically sound regulatory procedure which provably moderates the impact of filtering on user learning and decision-making, along with explicit sample and complexity guarantees. The following are important properties of our proposed regulations:

- Our procedure is not required to be disclosed to the internal AF mechanism used by the platform, which may not consent to be shared. This provides also a flexibility in regulating the model with no need for adaptation with respect to any future modification of the internal AF.

- Our procedure can be applied using only access to users’ observations (their feeds) in order to infer the influence of the platforms on their beliefs, decision-making, and ultimately on their actions, while having no access to their actual beliefs. It is clear that this way users’ privacy is respected.

- The “worst-case” approach proposed in [18] could be vulnerable. Indeed, in real-world SMPs, where any party is free to create a user without any supervision, this approach may be flawed, as a set of adversarial users can be created to act more credulously than the most credulous user already in existence. This way, one can clearly distort the regulator’s results. We suggest, on the other hand, a “global” approach, by averaging the influence of the platform’s AF on every user (while revealing that this can be accomplished very efficiently, in terms of the sample complexity), and we found that this approach is practically robust to adversarial users.

- As such, based on the “global” approach, which averages the influence of the platform on any pre-defined set of users, it actually makes it possible to examine bias in the opinion/decision-making of this community of users and not just compliance with
regulatory conditions about a single specific user. Thus, it is possible to examine the
effect of the platform, both on the entire community of users, on any pre-defined sub-
groups, and even on individual-specific users. According to the “worst-case” approach,
the most gullible user is the one according to which compliance with the regulation is
examined, so this is a particular case that is generalized in the regulatory procedure
based on our proposed “global” approach.

Counterfactual regulation. In addition to the above filtered vs. reference testing ap-
proach, we develop a regulatory procedure which examines whether the platform complies
with a given counterfactual regulation or not, in the course of time. Perhaps surprisingly
this procedure is constructed using a pair of parallel blocks of the filtered vs. reference regu-
latory procedure. The covert nature of this procedure provides several advantages, including
robustness against adversarial users, and its efficient sample complexity guarantees. As a
matter of fact, the only existing regulatory procedure of counterfactual regulation, introduced in [13], audits potential violations of any particular counterfactual regulation. How-
ever, it ignores violations of regulations over time, considering a static, time-independent
case of auditing a given counterfactual regulation. In “real-world” cases, this approach is
inherently problematic: First, in cases where regulations must be enforced over time, the
procedure in [13] must be repeated endlessly. Second, in the interim between these repe-
titions, all violations can never be proven to have occurred. Finally, it could be the case
that an “uncooperative” SMP choose to operate in a sophisticated/adversarial way so that
when it is being externally tested, it complies with the regulation at a specific time, and
does not at any other time. The above problems are irrelevant to our approach because our
regulatory algorithm tests compliance of a given counterfactual regulation in the course of
time.

1.2.2 Technical Contributions

In addition to formulating a mathematical model for social media regulation, our paper
contributes to the study of the closeness testing problem. The closeness testing problem
have been extensively studied in the past few years (see, e.g., [28, 7, 19, 2]), as well as its
extended version, the tolerant closeness testing problem (e.g., [29, 15]). The vanilla form of
the later is as follows.

**Problem 1:** $(\varepsilon_1, \varepsilon_2, \delta)$-i.i.d.-tolerant closeness testing: Given sample access to
distributions $P$ and $Q$ over $[n]$, and bounds $\varepsilon_2 > \varepsilon_1 \geq 0$, and $\delta > 0$, distinguish
with probability of at least $1 - \delta$ between $\|P - Q\|_1 \leq \varepsilon_1$ and $\|P - Q\|_1 \geq \varepsilon_2$,
whenever $P, Q$ satisfy one of these two inequalities.

As briefly mentioned in the previous subsection, in our paper we deal with a generalized
form of a tolerant closeness testing problem. Specifically, in our setting, samples (or, feeds)
are assumed to be generated from a certain Markovian probabilistic model (rather than
being i.i.d. as in Problem 1). Testing Markov chains is a new and active area of research
with a number of remarkable recent results, such as testing symmetric Markov chains [26],
testing ergodic Markov chains [72, 73] or testing irreducible Markov chains [20]. The general
form of these problems can be summarized as follows.
Problem 2: $(\varepsilon_1, \varepsilon_2, \delta)$-tolerant closeness testing of Markov chains: Given small constants $\varepsilon_1, \varepsilon_2, \delta \in (0, 1)$ hold $\varepsilon_1 < \varepsilon_2$, two Markovian trajectories $V_1^m = (V_0^m, \ldots, V_m^m), Y_1^m = (Y_0^m, \ldots, Y_m^m)$ from unknown Markov chains $\mathcal{M}, \mathcal{M}'$ respectively, with $n$ states each, distinguish with probability at least $1 - \delta$ between $d(\mathcal{M}, \mathcal{M}') \leq \varepsilon_1$ and $d(\mathcal{M}, \mathcal{M}') \geq \varepsilon_2$, where $d(\cdot, \cdot)$ is a given probability metric.

In our study we need to construct a method to solve a generalized form of the above two problems. As a warm-up, we start by generalizing Problem 1 to the case where one is given a set of pairs of measurements drawn from a set of pairs of probability distributions, and is tasked with deciding whether the total sum of distances between these pairs of distributions is close or far away. This problem is formulated mathematically as follows.

Problem 3: $(\varepsilon_1, \varepsilon_2, \delta)$-sum of pairs of distributions $i.i.d.$-tolerant closeness testing: Given sample access to a set of $|U|$ distributions, denoted by $P_u$ and $Q_u$ over $[n]$, for $u \in |U|$, and bounds $\varepsilon_2 > \varepsilon_1 \geq 0$, and $\delta > 0$, distinguish with probability of at least $1 - \delta$ between $\sum_{u=1}^{|U|} \|P_u - Q_u\|_1 \leq |U| \cdot \varepsilon_1$ and $\sum_{u=1}^{|U|} \|P_u - Q_u\|_1 \geq |U| \cdot \varepsilon_2$, whenever the distributions satisfy one of these two inequalities.

An algorithm to Problem 3 will serve as a building block to the actual testing problem we are after, formulated as follows: given only the trajectories of the pairs of hidden irreducible Markov chains, decides whether the total sum of distances between these hidden pairs of chains is $\varepsilon_1$-close, or $\varepsilon_2$-far away. Analytically, this problem is formulated as follows.

Problem 4: $(\varepsilon_1, \varepsilon_2, \delta)$-sum of Markov chains pairs tolerant closeness testing: Given small constants $\varepsilon_1, \varepsilon_2, \delta \in (0, 1)$ hold $\varepsilon_1 < \varepsilon_2$, set of $|U|$ corresponding pairs of Markovian trajectories $(V_1^m, Y_1^m), (V_2^m, Y_2^m), \ldots, (V_{|U|}^m, Y_{|U|}^m)$ drawn from unknown corresponding pairs of hidden Markov chains $(\mathcal{M}_1, \mathcal{M}_1'), (\mathcal{M}_2, \mathcal{M}_2'), \ldots, (\mathcal{M}_{|U|}, \mathcal{M}_{|U|}')$, with $n$ states each, distinguish with probability at least $1 - \delta$ between $\sum_{i=1}^{|U|} d(\mathcal{M}_i, \mathcal{M}_i') \leq |U| \cdot \varepsilon_1$ and $\sum_{i=1}^{|U|} d(\mathcal{M}_i, \mathcal{M}_i') \geq |U| \cdot \varepsilon_2$.

Similarly to majority of the papers mentioned above, we focus on the case where $d = \ell_\infty$, with the understanding that other metrics can be analyzed. We propose an algorithm which solves Problem 4, along with sample and complexity guarantees. It turns out that a major part of the analysis of our algorithm is related to the study of the covering time of random walks on undirected graphs [20]. Specifically, we obtain an upper bound on the time it takes for multiple parallel random walks to cover each state a given number of times. Our analysis might be of independent interest.

1.2.3 Notations

For a positive integer $m$, we denote $[m] = \{1, 2, \ldots, m\}$. The underlying space in the paper is $\mathbb{R}^n$, i.e., the space of all real-valued $n$ length column vectors endowed with the dot product $\langle x, y \rangle = x^T y$. For $p \geq 1$, The $\ell_p$-norm of a vector $x \in \mathbb{R}^n$ is given by $|x|_p = \sqrt[p]{\sum_{i=1}^n |x_i|^p}$. The $\ell_\infty$-norm of a vector $x \in \mathbb{R}^n$ is $|x|_\infty = \max_{i=1, 2, \ldots, n} |x_i|$. The $p$-norm of matrix $A$ induced by vector $p$-norms is defined by $\|A\|_p = \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$. In
the special cases of $p = 1, \infty$, the induced matrix norms can be computed or estimated by $\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{m} |a_{ij}|$, which is simply the maximum absolute column sum of the matrix; $\|A\|_{\infty} = \max_{1 \leq i \leq m} \sum_{j=1}^{n} |a_{ij}|$, which is simply the maximum absolute row sum of the matrix. $e$ is used to denote the vector of all ones and $0$ is the vector of all zeros. We denote by $|S|$ the number of element in the set $S$. The function $a : (\mathbb{N} \times \mathbb{N}) \rightarrow \{\{\mathbb{N}\}, \{\mathbb{N}\}\}$ takes a pair of elements and return set containing those two element. The function $a_f : (\mathbb{N} \times \mathbb{N}) \rightarrow \{\mathbb{N}\}$ takes a pair of elements and return the first element in the pair (first coordinate). Similarly, the function $a_s : (\mathbb{N} \times \mathbb{N}) \rightarrow \{\mathbb{N}\}$ takes a pair of elements and return the second element in the pair (first coordinate). Finally, we let $\Delta^n$ be the $n$-dimensional probability simplex.

2 Framework: Setup and Goal

2.1 The setup

Consider a system with the following three parties: a social media platform (SMP), a user, and a regulator, as illustrated in Figure 1. At each time step $t \in \mathbb{N}$, the platform shows each user a collection of contents (e.g., posts, videos, photos, ads, etc.) known as filtered feeds. We denote the filtered feed shown to user $u \in \{1, 2, \ldots, U\} \triangleq [U]$ at time $t \in \mathbb{N}$ by $X^F_u(t) \in \mathbb{Z}^F$, and assume that it consists of $M \in \mathbb{N}$ pieces of contents, namely, $X^F_u(t) = \{x^F_{1,u}(t), \ldots, x^F_{M,u}(t)\}$, where $x^F_{j,u}(t) \in \mathbb{R}^d$ denotes a piece of content, for $1 \leq j \leq M$, and some $d \in \mathbb{N}$.

Figure 1: An illustration of the interaction between the platform, the user, and the regulator.

2.1.1 User-platform relationship

Users. Next, we describe the users learning and decision-making pipeline. As users browse through their feeds, they implicitly form internal beliefs about the observed contents, and based on those beliefs they later take actions/decisions. For example, how individuals vote or the products they buy are decisions that are affected by the content they see on social
media. In addition, the decisions does not have to occur on the platform. For instance, the platform could show information on COVID-19, but the decision could be whether to get the vaccine. Let us formulate this mathematically. Let $\Omega$ be a compact metrizable set of possible states; this set can be finite, countably infinite, or continuous, and its elements $\omega \in \Omega$ can be either scalars or vectors. At each time step $t \geq 1$, each user $u \in [U]$ is associated with a belief $B^F_{u,t} \in \Delta(\Omega)$, where $\Delta(\Omega)$ is the simplex of probability distributions over the state space. At start $t = 0$, without loss of essential generality, we may assume that $B^F_{u,0} = \text{Uniform}(\Omega)$, for all $u \in [U]$. The belief $B^F_{u,t}$ is a posterior distribution on $\Omega$ conditioned on the information available to user $u$ at time $t$. This information consists of the observed feeds $\{X^F_u(\ell)\}_{\ell \leq t}$. The total history information available to user $u$ at time $t$ by $H^F_{t,u} = \{X^F_u(\ell) : \ell \leq t\}$.

Accordingly, user $u$’s belief at time $t$ is defined as $B^F_{u,t}(d\omega, h_{t,u}) \triangleq P^F_{t}(d\omega | H^F_{t,u} = h_{t,u})$, for a given sequence of feeds $h_{t,u}$. Based on the beliefs users take decisions (or, actions); each user have a set of possible actions at time $t \geq 0$. For user $u \in U$, let $A_u(t)$ denote a compact metrizable action space, and $A_{u,i}(t) \in A_u(t)$ be the $i$th action. Also, let $U_u : \Omega \times A_u \to \mathbb{R}$ be (possibly continuous) user $u$’s utility function. Consequently, for any belief $B^F_{u,t} \in \Delta(\Omega)$ and a utility function $U_u$ we define $br^F_{u,t}(h_{t,u})$ as the set of actions that maximizes user $u$’s expected utility, i.e.,

$$br^F_{u,t}(h_{t,u}) \triangleq \left\{ a \in A_u : a \in \arg\max_{b \in A_u} \int_{\Omega} U_u(\omega, b) B^F_{u,t}(d\omega, h_{t,u}) \right\}.$$

### 2.1.2 Feeds construction and regulator-platform interaction

We now switch our focus to formalize the setup for the regulator-platform interaction.

**Platform filtering.** An important component of our model is related to the question of how feeds are filtered? As mentioned before, feeds are chosen by the platform using a black-box filtering algorithm, which is utilized to maximize a certain reward function. The filtering algorithm is fed with an extensive amount of inputs that the platform uses to filter, such as, current available contents, past feeds, users interaction history, users feedback (e.g., users “sentiments” which are certain complex functions of the users beliefs), the users social network topology, and so on. The reward function reflects the platform’s objective. For example, it may balance factors like advertising revenue, personalization, user engagement (e.g., the predicted number of clicks), content novelty, acquisition of new information about users, cost of operations, or a combination of these and other factors.

We denote the platform’s reward function by $\text{Rew}^F_t : \mathcal{Z}_F \times \mathcal{P}_{t-1} \to \mathbb{R}$, where $\mathcal{P}_{t-1}$ captures the inputs mentioned above, and accordingly,

$$X^F_u(t) = \arg\max_{x \in \mathcal{Z}_F} \text{Rew}^F_t(X, \mathcal{P}_{u,t-1}),$$

where, again, $\mathcal{P}_{u,t-1}$ captures the platform external data used for filtering. For now, we leave both $\text{Rew}^F_t$ and $\mathcal{P}_{t-1}$ unspecified.

**Filtered vs. reference feeds.** The discussion in the background section about the regulation boundary and motivation suggests a neat and consistent formulation for the
regulator’s objective. Following [70, 54, 17, 59], we define a reference (or, competitive) boundary that is formed based on the users consent, and its location is determined by domain experts. Specifically, while user \( u \)'s filtered feed \( X_u^f(t) \) at time \( t \) is chosen by the platform in a certain reward-maximizing methodology, the reference feeds, on the other hand, could have hypothetically been selected by the platform, if it sticks to the consumer-provider agreement. Under this scenario, the platform would have strictly adhered the user’s interest in the current the available contents to construct the feed. In practice, the only scenario in which the platform could filter contents without expressing any subjective preference (influential) is by uniformly-random selecting contents available to construct each user’s feed. A non-uniform selection of contents would create a bias, in favor of the specific set of contents that would be shown on users feeds. This in turn may lead to a bias in the user’s decision-making process and actions, which we believe would not happen in a natural scenario. We denote these feeds by \( X_u^R(t) \), for user \( u \in \mathcal{U} \) at time \( t \). Then, in a nutshell, the regulator objective is to determine if the platform honors the consumer-provider agreement by testing whether the users beliefs implied by the filtered and reference feeds are significantly different, which necessarily leads the users to perform substantially different actions. We formulate this objective rigorously in the shortly. The following example elucidates the difference between the filtered and reference feeds.

**Example 1** (Construction of reference feeds). Suppose that the platform’s reward objective function can be written as

\[
Rew_t^F(X, P_{i,t-1}) \doteq Rew_{t,per}(X, P_{i,t-1}) + Rew_{t,rev}(X, P_{i,t-1}) + Rew_{t,self}(X, P_{i,t-1}),
\]

where \( Rew_{t,per} \) is the reward gained by those feeds which are personalized to the user, \( Rew_{t,rev} \) is the revenue-related reward gained by advertisements, and \( Rew_{t,self} \) predicts the reward associated with the information the platform would gain from platform “selfish” aspects (e.g., running a social experiment on the user). Assume that the first two types of rewards are consistent with the consumer-provider agreement, but the last one is not. Then, the reference feed could be the one that maximize the contribution of the first two types of rewards, namely,

\[
Rew_t^R(X, P_{i,t-1}) \doteq Rew_{t,per}(X, P_{i,t-1}) + Rew_{t,rev}(X, P_{i,t-1}).
\]

We can also think of \( Rew_t^R \) as the reward function of the platform, as if it is a non-profit social association, prioritize the user experience. While the above example provides a reasonable way for the construction of the reference feeds, there surely are other ways to translate consumer-provider agreements into usable forms (see, e.g., [44, 50, 62]). Finally, note that since the hypothetical reference feed \( X_u^R(t) \), of user \( u \in \mathcal{U} \) at time \( t \), is constructed by uniformly selected contents (available to user \( u \) at time \( t \)), it could be simulated easily.

**Regulator’s generative modeling.** The AF mechanism cannot and should not be disclosed to the regulator. Nonetheless, it should be clear that for the regulator to be able to inspect the SMP, something about the feeds generation process must be assumed. In this paper, we assume that from the regulator’s point of view, the feeds are generated at random,
and we denote the conditional law of the feed at time $t$ conditioned on history feeds $h_{t-1,u}$ by $P_{u,t}(h_{t-1,u} \triangleq P(X^F_{u,t}(t)|h^F_{t-1,u} = h_{t-1,u})$, for user $u$. Later on, for the framework to be mathematical tractable, we will place additional assumptions on the family of distributions.

**Time dependent counterfactual regulations.** Above, we have focused on the “filtered vs. reference” feeds approach. We propose the following as an alternative. Let $S$ be a regulatory statement that an inspector (or, perhaps, the platform itself) wish to test. For example, $S$ could be: “The platform should produce similar feeds, in the course of a given time horizon $T$, for users who are identical except for property $P$”, where $P$ could be ethnicity, sexual orientation, gender, a combination of these factors, etc. Let $U_P \subset U \times U$ be a subset of pairs of users that comply with $P$. Then, for any pair of users $(i,j) \in U_P$, the inspector’s objective is to determine whether the platform’s filtering algorithm cause user $i$’s and user $j$’s beliefs and actions to be significantly different. We formulate this objective rigorously in the next section. We mention here that a similar approach to the above was proposed recently in [17], assuming a time-independent static model. Our study first focuses on constructing a regulation procedure given the first usable form, filtered vs. reference feeds. However, we will later reveal that a regulation procedure for the second form, counterfactual regulations, could be constructed using two parallel procedures of the first form.

**Hypothesis testing.** The regulator’s goal is to determine whether the platform upholds the consumer-provider agreement, and by doing so, to moderate intense influence on the user’s decision-making, which may be caused by observing filtered feed, compared to what would have been the user’s decision-making under the reference feed. With the model introduced above, the regulator’s task can be formulated as a hypothesis testing problem with the following two hypotheses:

- **The null hypothesis** $H_0$: the regulator (or self-regulator) decision is that the platform honors the consumer-provider agreement.
- **The alternative hypothesis** $H_1$: the regulator (or self-regulator) decision is to investigate the platform for a possible violation.

Accordingly, relying on a certain from of data, which we will specify in the sequel, the regulator’s detection problem is to determine whether $H_0$ or $H_1$ is true. We need to specify what kind of “test” is considered. Given a fixed risk $\delta \in (0,1)$, we expect the regulatory procedure to find the true one with probability $1 - \delta$, whichever it is. We call such a procedure $\delta$-correct. We consider the following notion of “frugality”, which we name batch setting: the regulator specifies in advance the number of samples needed for the test, and announce its decision just after observing the data all at once, and the sample complexity of the test is the smallest sample size of a $\delta$-correct procedure.

**Regulator’s data.** For $t \geq 1$ the regulator observes the filtered and reference feeds $\{X^F_t, X^R_t\}$, for all (or a subset of) users $u \in U$, and utilize these to test for regulation violations. There are two ways to access this data without invasions to privacy. First, under self-regulation (currently, almost all platform are entirely self-regulated [42]), the platform
obviously has access to those feeds, and therefore, there are no privacy issues. The second option is to provide anonymized data to the regulator. Indeed, both the users identities and the meaning behind the features should/can be removed since they do not affect regulation enforcement. Note that de-anonymization is not a real concern here because the anonymized datasets will not be publicly shared anyhow. Moreover, since the regulator only requires the numerical features of the feeds, rather their semantic interpretation, de-anonymization would require unreasonable significant effort that the regulator is not willing to undertake. Thus, with carefully laid out but reasonable measures, the users data would remain private and anonymous. Finally, notice that in principle the filtered and reference feeds need not necessarily correspond to real users and could represent sufficiently representative sample of hypothetical users.

2.2 Formalizing the regulator’s goal

Let \( \{X^F_u(t)\}_{t \geq 1} \) and \( \{X^R_u(t)\}_{t \geq 1} \) denote the sequences of user \( u \)'s filtered an reference feeds evolved over time, respectively. As discussed above, the users implicitly form beliefs from their feeds. With enough evidence, the users gain confidence, and then take actions. Accordingly, the corresponding user \( u \)'s beliefs and actions are denoted by \( \{B^F_{u,t}, br^F_{u,t}\}_{t \geq 1} \) and \( \{B^R_{u,t}, br^R_{u,t}\}_{t \geq 1} \), implied by the filtered and reference feeds, respectively.

**Violation.** We now define the meaning of “violation” from the regulator’s perspective. Let \( T \in \mathbb{N} \) denote the time-horizon, which determines how far into the past the regulator scrutinizes the platform’s behavior. Let \( d(\cdot, \cdot) : \Omega \times \Omega \to \mathbb{R}_{\geq 0} \) be a probability metric between two probability measures defined over \( \Omega \). Let \( \bar{U} \subseteq U \) be a certain subset of users (such representative subset of the entire set of users). Then, define the total action-variability metric as follows:

\[
V_{\text{action}}(T) \triangleq \frac{1}{T \cdot |\bar{U}|} \sum_{i \in \bar{U}} \sum_{t=1}^{T} \max_{h_{t,u}} d\left(\text{br}^F_{i,t}(h_{t,u}), \text{br}^R_{i,t}(h_{t,u})\right).
\]  

(2)

Similarly, define the total belief-variability metric as,

\[
V_{\text{belief}}(T) \triangleq \frac{1}{T \cdot |\bar{U}|} \sum_{i \in \bar{U}} \sum_{t=1}^{T} \max_{h_{t,u}} d\left(B^F_{i,t}(h_{t,u}), B^R_{i,t}(h_{t,u})\right).
\]  

(3)

Finally, recall that in Subsection 2.1.1 we also proposed a statistical model for filtering. Accordingly, as we explain below, it is beneficial to define also the total filtering-variability metric:

\[
V_{\text{filter}}(T) \triangleq \frac{1}{T \cdot |\bar{U}|} \sum_{i \in \bar{U}} \sum_{t=1}^{T} \max_{h_{t-1,u}} d\left(P^F_{i,t}(h_{t-1,u}), P^R_{i,t}(h_{t-1,u})\right).
\]  

(4)

It is useful to note that there is an analytical relationship between the above variabilities. Indeed, viewing \( \text{br}^F_{i,t} \) as a result of a probabilistic kernel that is applied on the beliefs, and assuming that the metric \( d \) satisfies the data processing inequality [24], it follows that
$V_{\text{action}}(T) \leq V_{\text{belief}}(T) \leq V_{\text{filter}}(T)$. Now, from the regulator’s perspective, violation could mean that $V_{\text{action}}(T) > \varepsilon > 0$, for some $\varepsilon > 0$ which governs the regulation strictness, i.e., higher values of $\varepsilon$ indicate greater strictness. Alternatively, violation can also be defined through the belief-variability, namely, $V_{\text{belief}}(T) > \varepsilon > 0$, for some $\varepsilon > 0$. Accordingly, depending on the regulator’s ambition, its testing/decision problem can be formulated as one of the following:

\[
\begin{align*}
H_0 : V_{\text{action}}(T) &\leq \varepsilon_1 \quad \text{vs.} \quad H_1 : V_{\text{action}}(T) \geq \varepsilon_2, \quad (5a) \\
H'_0 : V_{\text{belief}}(T) &\leq \varepsilon_1 \quad \text{vs.} \quad H'_1 : V_{\text{belief}}(T) \geq \varepsilon_2, \quad (5b) \\
H''_0 : V_{\text{filter}}(T) &\leq \varepsilon_1 \quad \text{vs.} \quad H''_1 : V_{\text{filter}}(T) \geq \varepsilon_2, \quad (5c)
\end{align*}
\]

where $\varepsilon_2 > \varepsilon_1 \geq 0$. Devising successful statistical tests which solve (5a) (or, (5b)) with high probability, guarantee that whenever the regulator decision is $H_0$ (or, $H'_0$), then the platform honors the consumer-provider agreement, since the beliefs and actions are indistinguishable under the filtered and reference feeds. Conversely, rejecting $H_0$ (or, $H'_0$) with high confidence implies that AF causes significantly different learning outcomes. Note that by the data processing inequality, accepting $H''_0$ in (5c) imply immediately that $H_0$ and $H'_0$ hold as well. Note that the general form of the hypothesis testing problems formulated in (5) reminiscent of the well-studied tolerant closeness testing problem (see, e.g., [28, 15]). In this paper, we focus on the hypothesis test in (5c).

**Testing.** Solving (5c) is mathematically intractable unless we place further assumptions on the family of distributions that generate the feeds. We assume the following (already non-trivial) quasi-Markov homogeneous model. Specifically, we assume that for a short term, but large enough for the regulator to complete performing its decision problem, the platform filtering process could be modeled as a large probabilistic state machine. From the regulator’s point of view, the platform is a rather sequentially-feeds generating system, making a probabilistic relationship of the current feed conditioned on the previous feeds. Mathematically, let $n \in \mathbb{N}$ and consider the interval $n \cdot T < t_0 < t_1 < \cdots < t_T \leq (n+1) \cdot T$. Recall that each feeds is composed of $m$ pieces of $d$-dimensional contents. In each such interval, from the regulator’s point of view, each piece of content $x_{\ell,u}(t_i)$, at time $t_i$ for $\ell \in [m]$, is drawn from a first-order hidden Markov chain, namely, $P(x_{\ell,u}(t_i)|x_{\ell,u}(t_0), \ldots, x_{\ell,u}(t_{i-1})) = P(x_{\ell,u}(t_i)|x_{\ell,u}(t_{i-1}))$, and $P(x_{\ell,u}(t_i) = s_2|x_{\ell,u}(t_{i-1}) = s_1) \equiv Q_{u,n}(s_1, s_2)$, for any two possible states $s_1, s_2 \in [L]^d$, where $L$ denotes the state support size. We denote the transition probability matrix by $Q_{u,n} = [Q_{u,n}(s_1, s_2)]_{i,j \in [L]^d}$. We assume that the $M$ Markov content trajectories are i.i.d. Note that over different intervals indexed by $n$ the filtering process could be transformed into a new state machine subjected to a different transition probabilities. For example, this transformation may occur over time when new external data incur noticeable changes in the platform’s reward. To analytically illustrate, in the $n$-th interval at each time $t_i$, the regulator observe $m$-Markovian-trajectories drawn from the chain $Q_{u,n}$, where

\[
\begin{align*}
1 : & \quad x_{1,u}(t_0), x_{1,u}(t_1), \ldots, x_{1,u}(t_{T-1}), x_{1,u}(t_T), \\
2 : & \quad x_{2,u}(t_0), x_{2,u}(t_1), \ldots, x_{2,u}(t_{T-1}), x_{2,u}(t_T), \\
& \quad \cdots \cdots \\
m : & \quad x_{m,u}(t_0), x_{m,u}(t_1), \ldots, x_{m,u}(t_{T-1}), x_{m,u}(t_T),
\end{align*}
\]
which together, at every time \( t_i \), construct a new feed

\[
\{ x_{f,u}(t_0) \}_{i=1}^m, \{ x_{f,u}(t_1) \}_{i=1}^m, \ldots, \{ x_{f,u}(t_T) \}_{i=1}^m.
\]

The above discussion is relevant to the reference feeds generation process as well; in particular, we denote by \( P_{u,n}^R \triangleq [P_{u,n}(s_1, s_2)]_{i,j \in [L^d]} \) the corresponding matrix transition probabilities.

From the regulator point of view, in terms of the reward-based platform filtering, a practical interpretation for the above modeling is as follows. At any interval \( n \), the platform generates some updated Markovian transition-matrix that is subjected to an updated Markov chain by maximizing its reward function, i.e.,

\[
Q_{u,n+1}^F = \arg \max_{Q \in \mathbb{Z}^{L^d \times L^d}} \text{Rew}_t^F(\mathcal{P}, P_{u,n})
\]

\[
\text{s.t. } \forall j \in [L^d], \quad \sum_{i=1}^{L^d} M_{i,j} = 1,
\]

where \( \mathcal{P}_{u,n} \) captures the external data and inputs to the platform used for filtering, intended for user \( u \), and was collected during the current time interval \( n \cdot T < t \leq (n+1) \cdot T \). Similarly, the reference feeds are generated by the same statistical process but by the reference-based rewards objective, i.e.,

\[
P_{u,n+1}^R = \arg \max_{M \in \mathbb{Z}^{L^d \times L^d}} \text{Rew}_t^R(M, \mathcal{P}_{u,n})
\]

\[
\text{s.t. } \forall j \in [L^d], \quad \sum_{i=1}^{L^d} M_{i,j} = 1.
\]

Throughout this paper we take the metric \( d(\cdot || \cdot) \) to be the total-variation distance \( d_{TV} \), exploiting its relation with the \( \ell_\infty \) norm, with the aim of leveraging the \( \ell_\infty \)-norm intimate connection to testing with i.i.d. samples (as shown in [19]). Then, for the above Markovian model it can be seen that the total filtering-variability metric in (4) boils down to

\[
V_{\text{filter}}(T) = \frac{1}{|\bar{U}|} \sum_{u=1}^{|\bar{U}|} \max_{i \in [L^d]} d_{TV}(P_{u,n}(i, \cdot), Q_{u,n}(i, \cdot))
\]

\[
= \frac{1}{|\bar{U}|} \sum_{u=1}^{|\bar{U}|} \max_{i \in [L^d]} \|P_{u,n}(i) - Q_{u,n}(i)\|_1 = \frac{1}{|\bar{U}|} \sum_{u=1}^{|\bar{U}|} \|P_{u,n}^R - Q_{u,n}^F\|_\infty,
\]

where \( P_{u,n}(i) \triangleq [P_{u,n}(i, j)]_{j \in [L^d]} \) and \( Q_{u,n}(i) \triangleq [Q_{u,n}(i, j)]_{j \in [L^d]} \). Finally, without loss of generality, we focus on the special case where \( U = U \), and consider a single specific interval for testing (say, \( \{t_0, t_1, \ldots, t_T\} = [T] \)); therefore, we drop the dependency of the above notations on \( n \). We assume that generative Markov chains of both types, namely the filtered and the reference, are irreducible (which analytically means that for any pairs \( i, j \in [L^d] \), there exists some \( t \in \mathbb{N} \) such that \( \{Q_{u,n}^F\}_i^t > 0, \{P_{u,n}^F\}_i^t > 0 \)). This is of course an
essential assumption, since otherwise beyond a certain time, large enough, there would be a set of states, (correspond to a pieces of contents), which will not be accessible any more.

For later use, we note that the fundamental theorem of Markov chain guarantees a unique stationary distribution for every irreducible Markov chain. According to our notation so far, we denote by \( \pi_{F_u} = (\pi_{F_u1}, \ldots, \pi_{F_uL^d}) \in \Delta_{L^d-1} \) and \( \pi_{R_u} = (\pi_{R_u1}, \ldots, \pi_{R_uL^d}) \in \Delta_{L^d-1} \) the stationary distributions of hidden content embedding Markov chains of the filtered and reference generative models, respectively, for some user \( u \in \bar{U} \).

Following that the minimum stationary probabilities are given by

\[
\pi_{F_u}^* \triangleq \min_{i \in [L^d]} \pi_{q_{u,i}}, \quad \text{and} \quad \pi_{R_u}^* \triangleq \min_{i \in [L^d]} \pi_{R_{u,i}},
\]

of the filtered and reference generative Markov models, respectively. We are now in a position to state the testing problem faced by the regulator:

Fix \( \varepsilon_1, \varepsilon_2 \in (0, 1) \) and \( \delta \in (0, 1) \) with \( \varepsilon_1 < \varepsilon_2 \). Given a set of \( t_T \) pairs of Markovian trajectories \( [(X^F_u(t_1), X^R_u(t_1)), \ldots, (X^F_u(t_T), X^R_u(t_T))] \) drawn from an unknown corresponding pair of Markov chains \( (Q^F_u, P^R_u) \), for each user \( u \in U \), an \((\varepsilon_1, \varepsilon_2, \delta)\)-sum of pairs tolerant closeness testing algorithm outputs “YES” if \( \sum_{u \in U} \| P_{u,n}^R - Q_{u,n}^F \|_{\infty} \leq U \cdot \varepsilon_1 \) and “NO” if \( \sum_{u \in U} \| P_{u,n}^R - Q_{u,n}^F \|_{\infty} \geq U \cdot \varepsilon_2 \), with probability at least \( 1 - \delta \).

As we mentioned earlier, the testing problem above is similar to the well-studied Markov tolerant closeness testing problem (e.g., [20]). Nonetheless, the vanilla setting of this type of testing, is simpler than the one we are after, mainly because in our problem we deal with a sum of the distances between pairs of latent Markov chains, rather than a single distance, as it is in the standard setting.

![Figure 2](image.png)

**Figure 2:** An illustration of the Regulation procedure. The SMP and the uniformal filter get as an input the external data procedure, then output the filtered and the reference feeds, respectively. Both feeds are inserted into the regulator, where the last outputs “True” when the regulation is not violated, or “False” otherwise.

### 2.3 Auxiliary Definitions and Lemmas

This subsection is devoted to present several notations, definitions, and a lemma that will be needed to present our main results. As mentioned in the previous subsection, the problem
of closeness testing of a single pair of Markov chains was considered in, for example, [20]; it was shown that the testing algorithm and sample complexity depend on the random $k$-cover time and the $k$-cover time. The former is defined as the first time that a random walk has visited every state of the Markov chain at least $k$ times, while the later is the maximization of the expectation of this random variable over all initial states. As a natural generalization we define the $m$-joint-$k$-cover time, as the expected time it takes for $m \geq 1$ independent random walks to cover all states at least $k$ times. In the language of our framework, an interpretation of this $m$-joint-$k$-cover time is the expected time it takes the platform to show all users all possible contents.

**Definition 1** ($m$-joint-$k$-cover time). Let $Z^{\infty}_{1,1}, Z^{\infty}_{2,1}, \ldots, Z^{\infty}_{m,1}$ be $m$-independent infinite trajectories drawn by the same Markov chain $\mathcal{M}$. For $t \geq 1$, let $\{N_i^{Z_j}(t), \forall i \in [n]\}$ be the counting measure of states $i \in [n]$ appearing in the subtrajectory $Z^{t}_{j,1}$ up to time $t$. For any $k, m \in \mathbb{N}^+$, the random $m$-joint-$k$-cover time $\tau_{\text{cov}}^{(k)}(m; \mathcal{M})$, is the first time when all $m$ independent random walks have visited every state of $\mathcal{M}$ at least $k$ times, i.e.,

$$\tau_{\text{cov}}^{(k)}(m; \mathcal{M}) \triangleq \inf \left\{ t \geq 0 : \forall i \in [n], \sum_{j=1}^{m} N_i^{Z_j}(t) \geq k \right\}.$$  \hspace{1cm} (8)

Accordingly, the $m$-joint-$k$-cover time is given by

$$t_{\text{cov}}^{(k)}(m; \mathcal{M}) \triangleq \max_{v \in [n]^m} \mathbb{E} \left[ \tau_{\text{cov}}^{(k)}(m; \mathcal{M}) \right| Z_{1,1} = v_1, Z_{2,1} = v_2, \ldots, Z_{m,1} = v_m],  \hspace{1cm} (9)$$

where the coordinates of $v = (v_1, v_2, \ldots, v_m) \in [n]^m$ correspond to initial states.

For simplicity of notation, we denote $t_{\text{cov}} \equiv t_{\text{cov}}^{(1)}$. In addition, unless we explicitly deal with two different chains, we omit the dependency of $t_{\text{cov}}^{(k)}(m; \mathcal{M})$ on $\mathcal{M}$ and use $t_{\text{cov}}^{(k)}(m)$ instead. We denote by $\pi$ the stationary distribution of $\mathcal{M}$, and accordingly we define the mixing time as $t_{\text{mix}} \triangleq \min \{ t \geq 1 : \max_{\mu \in \Delta_{\mathcal{X}}} d_{TV}(\mu, \mu^t, \pi) \leq 1/4 \}$. The following result bounds $t_{\text{cov}}^{(k)}(m)$ by $t_{\text{cov}}$.

To date, studies have focused on one of the following two separate cases:

1. **Upper bounding the expected time required to cover all $n$ states of some general, given, irreducible Markov chain, $k$ times, with a single random walk, which given $t_{\text{cov}}^{(1)}(1)$, in the terms of $t_{\text{cov}}$.**

2. **Upper bounding the expected time required to cover all $n$ states of some general, given, irreducible Markov chain, with $m$ multiple independent random walks, which given $t_{\text{cov}}^{(1)}(m)$, in the terms of $t_{\text{cov}}$.**

The first case was studied in a recent paper [20]. The authors came up with the following tightest upper bound,

$$t_{\text{cov}}^{(k)} = \tilde{O} \left( t_{\text{cov}} + k/\pi_\ast \right).$$
The second case was studied in [3]. The authors conjecture that for any given Markov chain there exists some constant $C$, such that

$$t_{\text{cov}}^{(1)}(m) \leq \frac{t_{\text{cov}}}{C \log m}.$$ 

We combine those two approaches to upper bound $t_{\text{cov}}^{(k)}(m)$, in terms of $t_{\text{cov}}$. To that end, we start by upper bounding the expected time required to cover all Markov chain states, $k$ times, with $m$ independent random walks, by the expected time to cover all Markov chain states only one time, with $m$ independent random walks.

**Lemma 2** ($m$-joint-$k$-cover time upper bounded by 1-joint-$k$-cover time). For any $k \geq 1$ (numbers of full graph cover) and any $m \geq 1$ (number of the i.i.d random walks), the following holds

$$t_{\text{cov}}^{(k)}(m) \leq kt_{\text{cov}}^{(1)}(m). \quad (10)$$

Now, we obtain the covering time of $m$-i.i.d random walks, in which their first step drawn by the stationary distribution of the (irreducible) Markov chain.

**Lemma 3** (Covering time of stationary initialized $m$-i.i.d random walks). For any $k \geq 1$ (numbers of full graph cover) and any $m \geq 1$ (number of the i.i.d random walks), it holds that,

$$E_{V \sim \pi} \left[ \tau_{\text{joint-cov}}^{(1)}(m) \mid \forall j \in [n], Z_{j,1} = v_j \right] = O \left( \frac{t_{\text{cov}} \log n}{m} \right). \quad (11)$$

The following result combines the above bounds to upper bound $t_{\text{cov}}^{(k)}(m)$.

**Lemma 4** ($m$-joint-$k$-cover time upper bound). For any $k, m \geq 1$,

$$t_{\text{cov}}^{(k)}(m) = O \left( \frac{kt_{\text{cov}} \log n}{m} + t_{\text{mix}} \right), \quad (12)$$

and if we assume that $\frac{m}{k \log n} \leq \frac{t_{\text{cov}}}{t_{\text{mix}}}$ then, $t_{\text{cov}}^{(k)}(m) = O \left( kt_{\text{cov}} \log n/m \right)$.

Finally, following [27], we define the mapping $\psi^{(i)}_k : [n]^1 \times [n]^2 \times \cdots \times [n]^l \rightarrow [n]^1 \times [n]^2 \times \cdots \times [n]^k$, taking a Markov trajectory of length $l \in \mathbb{N}$ (where $k \leq l$) and $n$-states, that necessarily hit state $i \in [n]$ at least $k$ times, then return the first states that have immediately been observed after hitting state $i$.

### 3 Regulatory Procedure and Sample Complexity

In this section, we present our main results. Specifically, in Subsection 3.1 we start by presenting a simple algorithm, along with sample complexity guarantees, for closeness testing the sum of distances of pairs of discrete distributions using i.i.d. samples. This in turn will serve as a sub-routine in the regulation procedure we propose and analyze in Subsection 3.2. Finally, in Subsection 3.3 we analyze the counterfactual regulation approach.
3.1 Warm Up: Testing a Family of Discrete Distributions

Consider Problem 3 of testing the sum of distances of pairs of discrete distributions using i.i.d. samples. As mentioned earlier in Subsection 1.2.2, this problem is a generalization of the vanilla i.i.d. tolerant closeness testing problem (e.g., [15]), where \( U = 1 \). We propose an efficient tester solving the above testing problem along with sample complexity guarantees. We establish first a few notations. Let \( S_{u,P} \) and \( S_{u,Q} \) be two sets of \( m \in \mathbb{N} \) samples drawn from \( P_u \) and \( Q_u \), respectively, for all \( u \in [U] \), and let \( S_P = \{S_{1,P}, \ldots, S_{U,P}\} \) and \( S_Q = \{S_{1,Q}, \ldots, S_{U,Q}\} \). For every \( u \in U \), let \( V_{u,i} (Y_{u,i}) \) and \( \hat{V}_{u,i} (\hat{Y}_{u,i}) \) count the number of occurrences of symbol \( i \in [n] \), in the first and the second sets of \( m \) samples (each), sampled from \( P_u \) (\( Q_u \)), respectively. We assume that the sample sizes of \( P_u \) and \( Q_u \), for every state (symbols) \( i \in [n] \), are Poisson-distributed with mean \( m \) ("Poissonization" trick), namely, \( V_{u,i}, \hat{V}_{u,i} \sim \text{Poisson} (m \cdot P_{u,i}) \) and \( Y_{u,i}, \hat{Y}_{u,i} \sim \text{Poisson} (m \cdot Q_{u,i}) \), where \( P_{u,i} \) (\( Q_{u,i} \)) is the probability of symbol \( i \) under \( P_u \) (\( Q_u \)). Define,

\[
f_{u,i} \triangleq \begin{cases} \max \{\sqrt{mn} |P_{u,i} - Q_{u,i}|, n(P_{u,i} + Q_{u,i})\}, & \text{if } m > n, \\ \max \{m(P_{u,i} - Q_{u,i}), 1\}, & \text{otherwise}, \end{cases}
\]

where the random variables \( \hat{V}_{u,i}, \hat{Y}_{u,i} \) are used to estimate \( f_{u,i} \) with \( \hat{f}_{u,i} \), defined as,

\[
\hat{f}_{u,i} \triangleq \begin{cases} \max \left\{ \frac{|V_{u,i} - \hat{V}_{u,i}|}{\sqrt{m/n}}, \frac{\hat{V}_{u,i} + \hat{Y}_{u,i}}{m/n}, 1 \right\}, & \text{if } m > n, \\ \max \{\hat{V}_{u,i} + \hat{Y}_{u,i}, 1\}, & \text{otherwise}. \end{cases}
\]

Additionally, define \( G_{u,i} \triangleq (V_{u,i} - Y_{u,i})^2 - V_{u,i} - Y_{u,i} \), and finally,

\[
G \triangleq \sum_{u=1}^{U} \sum_{i=1}^{R} \frac{G_{u,i}}{f_{u,i}}.
\]

Consider the routine \( \text{IIDTester}(S_P, S_Q, \delta, \varepsilon_1, \varepsilon_2, m, n) \) in Algorithm 1. We have the following result.

**Theorem 5** (Sample complexity of i.i.d. tester). There exists an absolute constant \( c > 0 \) such that, for any \( 0 \leq \varepsilon_2 \leq 1 \) and \( 0 \leq \varepsilon_1 \leq c \varepsilon_2 \), given

\[
O \left( \sqrt{\frac{n}{\varepsilon_2}}, \frac{\varepsilon_2}{\varepsilon_1^2}, \frac{\varepsilon_1}{\varepsilon_2^2}, \frac{n^{2/3}}{1^{2/3}} \right)
\]

samples from each of \( \{P_u\}_{u=1}^{U} \) and \( \{Q_u\}_{u=1}^{U} \). Algorithm 1 distinguishes between \( \sum_{u=1}^{U} \|P_u - Q_u\|_1 \leq U \cdot \varepsilon_1 \) and \( \sum_{u=1}^{U} \|P_u - Q_u\|_1 \geq U \cdot \varepsilon_2 \), with probability at least \( 1 - \delta \).

**Algorithm 1:** Tolerant closeness tester for the i.i.d. case

- **Input:** \( U, n, m, 0 \leq \varepsilon_1 < \varepsilon_2, \delta \in (0,1) \), and two sets \( S_P \) and \( S_Q \) of \( \text{Poisson}(m) \) samples from \( P_u \) and \( Q_u \), for every \( u \in [U] \).

  1. Set a threshold: \( \tau \leftarrow c \min \left( \frac{m^{3/2} \varepsilon_2}{n^2}, \frac{\varepsilon_1 \varepsilon_2^2}{n} \right) \)

  2. Compute \( G \) in (15) using the sets of samples.

  3. If \( G < \tau \), then **Return** "YES".

  4. Else \( G \geq \tau \), then **Return** "NO".
Note that the sample complexity in Theorem 5 decreases as a function of the total number of users $U$, while increases as a function of the number of contents $n$.

### 3.2 Regulatory Procedure and Sample Complexity

In this subsection, we present our regulatory procedure. Let $\mathcal{T}_F$ be the set of all uses filtered feeds in the current interval, namely $\mathcal{T}_F = \{(X_1^F(t))_{t=1}^T, \ldots, (X_u^F(t))_{t=1}^T\}$ and similarly we let $\mathcal{T}_R = \{(X_1^R(t))_{t=1}^T, \ldots, (X_u^R(t))_{t=1}^T\}$ be the set of all uses reference feeds in the current interval. We denote by $m(n, \epsilon_1, \epsilon_2, \delta)$ the sample complexity of the i.i.d. tester in Algorithm 1 and assume that it satisfies the condition in Theorem 5. Consider the regulatory procedure $\text{REGULATORYTESTER}(\mathcal{T}_F, \mathcal{T}_R, \delta, T, \epsilon_1, \epsilon_2, m, n)$ in Algorithm 2.

**Algorithm 2: Regulation procedure**

**Input:** $T, n \in \mathbb{N}$ states, $0 \leq \epsilon_1 < \epsilon_2, 0 < \delta < 1$, and $\bar{m} = m(n, \epsilon_1, \epsilon_2, \delta/2n)$

**Output:** “YES” if $\sum_{u=1}^U \|P_u^R - Q_u^F\|_{\infty} \leq \epsilon_1 \cdot TU$, “NO” if $\sum_{u=1}^U \|P_u^R - Q_u^F\|_{\infty} \geq \epsilon_2 \cdot TU$.

1. For every state $i \leftarrow 1, 2, \ldots, n$,
   - Set $S^R \leftarrow \emptyset$ and $S^F \leftarrow \emptyset$
2. For every user $u \leftarrow 1, 2, \ldots, U$,
   - If $\sum_{j=1}^J X_{j,u}^F(t) < \bar{m}$ or $\sum_{j=1}^M N_{i,\delta}^F(t) < \bar{m}$,
     - Return “NO”
   - Calculate $S_u^R \leftarrow \bigcup_{j=1}^M \psi(i, j, u, \delta, \epsilon_1, \epsilon_2, \delta/2n)$ and $S_u^F \leftarrow \bigcup_{j=1}^M \psi(i, j, u, \delta, \epsilon_1, \epsilon_2, \delta/2n)$
3. Do $S^R \leftarrow S_u^R \cup S_u^F$ and $S^F \leftarrow S_u^F \cup S_u^F$
4. If $\text{IIDTESTER}(S^R, S^F, \delta, \epsilon_1 \cdot TU, \epsilon_2 \cdot TU, \bar{m}, n) = \text{“NO”}$
   - Return “NO”
5. Return “YES”

We have the following result.

**Theorem 6** (Regulation sample complexity). *If we have an $(\epsilon_1, \epsilon_2, \delta)$ i.i.d. tolerant-closeness-tester for $n$ state distributions with the sample complexity of $m(n, \epsilon_1, \epsilon_2, \delta)$, then we can $(\epsilon_1, \epsilon_2, \delta)$ testing hypothesis using the following number of samples

$$T = O_\delta \left( \max \left\{ \frac{\epsilon_1}{\epsilon_2} \left( \frac{m(n, \epsilon_1, \epsilon_2, \delta/4n) (M; P_u^R), m(n, \epsilon_1, \epsilon_2, \delta/4n) (M; Q_u^F)}{n} \right) \right\} \right).$$

Here, $O_\delta$ hides logarithmic factors in $\delta$.

### 3.3 Time-Dependent Counterfactual Regulations

Above, we have focused on the “filtered vs. reference” feeds approach. However, it is clear that other frameworks can be formulated. Consider the following as an alternative. Let $S$
be a regulatory statement that an inspector (or, perhaps, the platform itself) wish to test. For example, $S$ could be: “The platform should produce similar articles for users who are identical except for property $\mathcal{P}$”, where $\mathcal{P}$ could be ethnicity, sexual orientation, gender, a combination of these factors, etc. Let $\mathcal{U}_P \subset \mathcal{V} \times \mathcal{V}$ be a subset of pairs of users that comply with $\mathcal{P}$. Then, for any pair of users $(i, j) \in \mathcal{U}_P$, the inspector’s objective is to determine whether the platform’s filtering algorithm cause user $i$’s and user $j$’s beliefs and actions be significantly different. A similar approach, termed “counterfactual regulation” was studied recently in [17], assuming a time-independent static model. We take into account the inherent dependency on the time dimension as in “real-world” applications regulations must be enforced over time, as explained in the introduction. Similarly to Subsection 2.2, we define the notion of counterfactual violation as follows.

**Definition 7** (Total variability for counterfactual users). Let $\mathcal{U}_P \subset \mathcal{U} \times \mathcal{U}$ be a subset of pairs of users that comply with $\mathcal{P}$. Then, for any pair of users $(i, j) \in \mathcal{U}_P$, the total variability in algorithmic filtering behavior for counterfactual users is given by

$$\bar{V}_{cu}(T, S, \mathcal{U}_P) \triangleq \frac{1}{|\mathcal{U}_P|} \sum_{(i,j) \in \mathcal{U}_P} \sum_{t=1}^{T} \max_{h_{t-1,i},h_{t-1,j}} d \left( \| P_{i,t}^F(h_{t-1,i}) \|, \| P_{j,t}^F(h_{t-1,j}) \| \right).$$

Therefore, the investigator’s task could be to test for violations in the following sense:

$$\mathcal{H}_0^S : \bar{V}_{cu}(T, S, \mathcal{U}_P) \leq \varepsilon_1 \quad \text{vs.} \quad \mathcal{H}_1^S : \bar{V}_{cu}(T, S, \mathcal{U}_P) \geq \varepsilon_2. \quad (16)$$

As before, the goal here is to construct good inspection procedures given only $S$ and a black-box access to the filtering algorithm. Note also that $\mathcal{U}_P$ need not correspond to real users and could represent hypothetical users.
Following the path taken in this paper, using \(d(\cdot|\cdot)\) as the total-variation distance \(d_{TV}\), we have

\[
\bar{V}_{cu}(T, S, U_\varphi) = \frac{1}{|U_\varphi|} \sum_{(i,j) \in U_\varphi} \max_{\ell \in [L]} d_{TV}(Q_{i,n}(\ell, \cdot), Q_{j,n}(\ell, \cdot))
= \frac{1}{|U_\varphi|} \sum_{(i,j) \in U_\varphi} \max_{\ell \in [L]} \|Q_{i,n}(\ell) - Q_{j,n}(\ell)\|_1 = \frac{1}{|U_\varphi|} \sum_{(i,j) \in U_\varphi} \|Q_{i,n}^F - Q_{j,n}^F\|_\infty.
\]

Note that by the triangle inequality we have,

\[
\sum_{(i,j) \in U_\varphi} \|P_{i,n}^R - Q_{i,n}^F\|_\infty + \sum_{(i,j) \in U_\varphi} \|P_{i,n}^R - Q_{j,n}^F\|_\infty \geq \sum_{(i,j) \in U_\varphi} \|Q_{i,n}^F - Q_{j,n}^F\|_\infty,
\]

meaning that \(V_{cu}\) could be bound by the sum of the two left side terms. Hence, if we can ensure, with a given confidence, that the sum of the two left side terms is lower than some given threshold, then we can guarantee that the platform’s AF does not cause a significant difference in the beliefs and actions of the pairs in the set of counterfactual users. This observation implies the three-blocks regulatory procedure illustrated in Fig. 3.

Let’s delve deeper into the model blocks.

- **Block 1** is identical to the filtered vs. reference regulatory procedure and illustrated in figure 2. This block sequentially receives as an input the external data corresponding to all users in the pairs \((i, j) \in U_\varphi\) and outputs “Yes” if \(\sum_{(i,j) \in U_\varphi} \|P_{i,n}^R - Q_{i,n}^F\|_\infty \leq \varepsilon_1\), or “No” if \(\sum_{(i,j) \in U_\varphi} \|P_{i,n}^R - Q_{i,n}^F\|_\infty \geq \varepsilon_2\), with a predefined certainty (probability) \(\geq 1 - \delta_{B1}\) (derived by \(\varepsilon_1\) and \(\varepsilon_2\)).

- **Block 2** is structurally identical to the filtered vs. reference regulatory procedure except that its subblock ‘SMP’ receives the external data corresponding to the \(i\) users, in the pairs of \((i, j) \in U_\varphi\), where the ‘Reference Filter’ gets as an input the external data corresponding to the \(j\) users. Consequently, the subblock ‘Social Network Regulator’ will output “Yes” if \(\sum_{(i,j) \in U_\varphi} \|P_{i,n}^R - Q_{j,n}^F\|_\infty \leq \varepsilon_1\), or “No” if \(\sum_{(i,j) \in U_\varphi} \|P_{i,n}^R - Q_{j,n}^F\|_\infty \geq \varepsilon_2\), with a predefined certainty \(\geq 1 - \delta_{B2}\).

- **Block 3** receives the outputs of block 1 and 2. If both previous block-outputs are “Yes”, meaning that both

\[
\sum_{(i,j) \in U_\varphi} \|P_{i,n}^R - Q_{i,n}^F\|_\infty \leq \varepsilon_1, \quad \text{and} \quad \sum_{(i,j) \in U_\varphi} \|P_{i,n}^R - Q_{j,n}^F\|_\infty \leq \varepsilon_1;
\]

then we have

\[
2\varepsilon_1 \geq \sum_{(i,j) \in U_\varphi} \|P_{i,n}^R - Q_{i,n}^F\|_\infty + \sum_{(i,j) \in U_\varphi} \|P_{i,n}^R - Q_{j,n}^F\|_\infty \geq \sum_{(i,j) \in U_\varphi} \|Q_{i,n}^F - Q_{j,n}^F\|_\infty,
\]

with probability at least \((1 - \delta_{B1})(1 - \delta_{B2})\). In the complementary case where at least one of block 1 or block 2 outputs “No”, meaning

\[
\sum_{(i,j) \in U_\varphi} \|P_{i,n}^R - Q_{j,n}^F\|_\infty \geq \varepsilon_2, \quad \text{or} \quad \sum_{(i,j) \in U_\varphi} \|P_{i,n}^R - Q_{i,n}^F\|_\infty \geq \varepsilon_2,
\]

with probability at least \((1 - \delta_{B1})(1 - \delta_{B2})\)
then by the triangle inequality\(^1\) we have
\[
\sum_{(i,j) \in \mathcal{U}_\mathcal{D}} \|Q_{i,n}^F - Q_{j,n}^F\|_{\infty} \geq \varepsilon_2,
\]

with probability at least \(\geq (1 - \delta_{B1})(1 - \delta_{B2})\). Hence, Block 3 returns “Yes” if both previous block-outputs are “Yes”, with probability \(\geq (1 - \delta_{B1})(1 - \delta_{B2})\), and otherwise outputs “No” with the same probability error.

The following algorithm summarizes the above discussion.

**Algorithm 3:** Time-Dependent Counterfactual Regulatory Procedure

**Input:** \(T,n,m \in \mathbb{N}, 0 \leq \varepsilon_1 < \varepsilon_2 \in \mathbb{R}, 0 < \delta_{B1}, \delta_{B2} < 1\), a set \(\mathcal{U}_\mathcal{D}\). For every user
\(u \in \bigcap_{v \in \mathcal{U}_\mathcal{D}} a(v)\), a pair of sequence of reference and filtered feeds during the batch \(\{(x_{j,u}^R)^{t_{l}}\}_{j \in [M]}\) and \(\{(x_{j,u}^F)^{t_{l}}\}_{j \in [M]}\).  

**Output:** “Yes” if \(\sum_{(i,j) \in \mathcal{U}_\mathcal{D}} \|Q_{i,n}^F - Q_{j,n}^F\|_{\infty} \leq |\mathcal{U}_\mathcal{D}|T\varepsilon_1\), or “No” if \(\sum_{(i,j) \in \mathcal{U}_\mathcal{D}} \|Q_{i,n}^F - Q_{j,n}^F\|_{\infty} \geq |\mathcal{U}_\mathcal{D}|T\varepsilon_2\).

1. Set \(E_f \leftarrow \bigcup_{v \in \mathcal{U}_\mathcal{D}} a_f(v)\)
2. Set \(E_s \leftarrow \bigcup_{v \in \mathcal{U}_\mathcal{D}} a_s(v)\)
3. \(r_1 \leftarrow \text{RegulatoryTester}(\{(x_{i,u}^F)^{t_{l}}\}_{i \in [M],u \in E_f}, \{(x_{i,u}^R)^{t_{l}}\}_{i \in [M],u \in E_f}, \delta, T, \varepsilon_1, T, m, n)\)
4. \(r_2 \leftarrow \text{RegulatoryTester}(\{(x_{i,u}^F)^{t_{l}}\}_{i \in [M],u \in E_s}, \{(x_{i,u}^R)^{t_{l}}\}_{i \in [M],u \in E_s}, \delta, T, \varepsilon_1, T, m, n)\)
5. If \(r_1 = \text{“Yes”}\) and \(r_2 = \text{“Yes”}\)  
   Return “Yes”  
6. Else  
   Return “No”

**Theorem 8** (Counterfactual regulation sample complexity). If we have an \((\varepsilon_1,\varepsilon_2,\delta)\) i.i.d. tolerant-closeness-tester for \(n\)-state distributions with sample complexity \(m(n,\varepsilon_1,\varepsilon_2,\delta)\), then we can \((\varepsilon_1,\varepsilon_2,\delta)\) testing hypothesis \((16)\) using

\[
T = O_\delta \left( \max \left\{ \max_{u \in U} \left\{ t_{\text{cov}}^{m(n-1)/2,\varepsilon_1,\varepsilon_2,\delta/4n} (M; P_u^R), t_{\text{cov}}^{m(n-1)/2,\varepsilon_1,\varepsilon_2,\delta/4n} (M; Q_u^F) \right\} \right\} \right),
\]

samples, where \(\delta = \max(\delta_{B1}, \delta_{B2})\).

Note that since the COUNTERTESTER procedure in Algorithm \[7\] employs two parallel blocks of the filtered vs. reference regulatory procedure, the only thing that differentiates the two, is the number of distinguished permutations of pairs of users which dictates the sample complexity of IIDTESTER.

4 Proofs

This section is devoted to the proofs of our results.

\(^1\)The hypotenuse is greater than each one of the perpendiculars.
4.1 Proofs of Lemmas 2–4

Below we prove Lemmas 2–4. Using these lemmas we obtain an upper bound on \( t_{\text{cov}}(k) \) in terms of the covering time \( t_{\text{cov}} \). We will use this bound to obtain an explicit sample complexity for Algorithm 2.

**Proof** [Proof of Lemma 2] Let \( Z_{1,1}, Z_{2,1}, ..., Z_{m,1} \) be \( m \)-independent infinite trajectories drawn by the same Markov chain \( \mathcal{M} \). For \( t \geq 1 \), let \( \{ N_i^Z(t), \forall i \in [n] \} \) be the counting measure of states \( i \in [n] \) appearing in the subtrajectory \( Z_{j,1}^t \) up to time \( t \). Since for any \( k \geq 1 \) and any \( m \geq 1 \),

\[
    t_{\text{cov}}(k) = t_{\text{cov}}(1) + \sum_{l=1}^{k-1} \left( t_{\text{cov}}(l+1) - t_{\text{cov}}(l) \right),
\]

so if for any \( k-1 \geq l \geq 1 \),

\[
    t_{\text{cov}}(l+1) - t_{\text{cov}}(l) \leq t_{\text{cov}}(1),
\]

then obviously,

\[
    t_{\text{cov}}(k) \leq kt_{\text{cov}}(1).
\]

Therefore, proving the lemma is equivalent of proving that for any \( k-1 \geq l \geq 1 \),

\[
    t_{\text{cov}}(l+1) - t_{\text{cov}}(l) \leq t_{\text{cov}}(1).
\]

Now, for any fixed \( k-1 \geq l \geq 1 \), and for any fixed states, at time \( l \), \( Z_{1,l}, ..., Z_{m,l} \in [n] \), let us assume the worst case scenario, where the gap \( t_{\text{cov}}(l+1) - t_{\text{cov}}(l) \) is the largest possible, which can only happens where \( \forall i \in [n], \sum_{j=1}^{m} N_i^Z = l \). In this case,

\[
    t_{\text{cov}}(l+1) - t_{\text{cov}}(l) = \mathbb{E} \left[ \inf_{t \geq 0} \{ t | \forall i \in [n], \sum_{j=1}^{m} N_i^Z(l+1-l) \geq 0 \} | Z_{1,l}, ..., Z_{m,l} \right] \\
    \leq \max_{v \in [n]^m} \mathbb{E} \left[ \inf_{t \geq 0} \{ t | \forall i \in [n], \sum_{j=1}^{m} N_i^Z(1) \geq 0 \} | Z_{1,l} = v_1, ..., Z_{m,l} = v_m \right] \\
    = t_{\text{cov}}(1). 
\]

**Proof** [Proof of Lemma 3] Let us recall the definition of \( \tau_v \) given in [63],

\[
    \tau_v = \inf_{t \geq 0} \{ t | Z_t = v \}.
\]

By the definitions of \( \tau_v \), the stationary states distribution \( \pi \), and the cover time, we have

\[
    \max_{v \in [n]} \mathbb{E}_{Z \sim \pi} \left[ \tau_v \right] = \max_{v \in [n]} \mathbb{E}_{Z \sim \pi} \left[ \inf_{t \geq 0} \{ t | Z_t = v \} \right]
\]
\[
\begin{align*}
\leq & \mathbb{E}_{Z_1 \sim \pi} \left[ \inf_{t \geq 0} \left\{ t \mid \bigcup_{i=1}^\infty \mathcal{Z}_i \supseteq [n] \right\} \mid Z_1 \right] \\
= & \mathbb{E}_{Z_1 \sim \pi} \left[ \inf_{t \geq 0} \left\{ t \mid \forall v \in [n], N_v(t) \geq 0 \right\} \mid Z_1 \right] \\
= & \mathbb{E}_{Z_1 \sim \pi} [\tau_{\text{cov}} \mid Z_1] \\
\leq & \max_{v \in [n]} \mathbb{E} [\tau_{\text{cov}} \mid Z_1 = v] = t_{\text{cov}}.
\end{align*}
\]

By [63, Theorem 3.2], for any \( k \geq 1 \), it holds that
\[
\mathbb{E}_{\nu \sim \pi} \left[ \tau_{\text{joint-cov}}^{(1)}(m) \mid \forall j \in [n], Z_{j,1} = v_j \right] = \mathcal{O} \left( \frac{\max_{\nu \in \mathcal{V}} \mathbb{E}_{\pi} [\tau_{\nu} \log n]}{m} \right).
\]

Now, using the last result, we have
\[
\frac{\max_{v \in \mathcal{V}} \mathbb{E}_{\pi} [\tau_{\nu} \log n]}{m} \leq \frac{t_{\text{cov}} \log n}{m},
\]

and so it is straightforward that
\[
\max_{\nu \sim \pi} \mathbb{E} \left[ \tau_{\text{joint-cov}}^{(1)}(m) \mid \forall j \in [n], Z_{j,1} = v_j \right] = \mathcal{O} \left( \frac{t_{\text{cov}} \log n}{m} \right).
\]

\[\blacksquare\]

**Proof** [Proof of Lemma 4] For some constant \( \varepsilon^* > 0 \) small enough, there exists \( t_{\text{mix}}(\varepsilon^*) \) large enough, so for any distribution \( \hat{\pi} \) in the set \( \hat{\pi} \in \arg \max \| \hat{\pi} \mathcal{M}^{t_{\text{mix}}(\varepsilon^*)} - \pi \|_{\mathcal{V}} \), it holds that
\[
\max_{\nu} \mathbb{E} \left[ \tau_{\text{joint-cov}}^{(1)}(m) \mid \forall j \in [n], Z_{j,1} = v_j \right] \leq t_{\text{mix}}(\varepsilon^*) + \mathbb{E}_{\nu \sim \hat{\pi}} \left[ \tau_{\text{joint-cov}}^{(1)}(m) \mid \forall j \in [n], Z_{j,1} = v_j \right].
\]

As \( \varepsilon^* \to 0 \), the right-hand-side of the last inequality converges to the amount of time it takes to cover the chain. Therefore,
\[
t_{\text{cov}}^{(1)}(m) \leq t_{\text{mix}}(\varepsilon^*) + \mathbb{E}_{\nu \sim \hat{\pi}} \left[ \tau_{\text{joint-cov}}^{(1)}(m) \mid \forall j \in [n], Z_{j,1} = v_j \right].
\]

Now, using Lemma 3 we have
\[
t_{\text{cov}}^{(1)}(m) \leq \mathcal{O} \left( \frac{t_{\text{cov}} \log n}{m} + t_{\text{mix}}(\varepsilon^*) \right) = \mathcal{O} \left( \frac{t_{\text{cov}} \log n}{m} + t_{\text{mix}}(\varepsilon^*) \right).
\]

Using lemma 2 we get \( t_{\text{cov}}^{(k)}(m)/k \leq t_{\text{cov}}^{(1)}(m) \), and so,
\[
t_{\text{cov}}^{(k)}(m) = \mathcal{O} \left( \frac{kt_{\text{cov}} \log n}{m} + t_{\text{mix}}(\varepsilon^*) \right).
\]

Finally, note that if \( \frac{m}{k \log n} \leq t_{\text{cov}}/t_{\text{mix}}(\varepsilon^*) \), then
\[
t_{\text{cov}}^{(k)}(m) = \mathcal{O} \left( \frac{kt_{\text{cov}} \log n}{m} \right),
\]

which concludes the proof. \[\blacksquare\]
4.2 Proof of Theorem 5

In this subsection we prove Theorem 5. To this end, we start by proving a few auxiliary results which characterize the first and second order statistics of the count in (15).

4.2.1 Auxiliary Results

**Lemma 9.** Let \( \delta_1 \in (0, 1) \), and recall the definition of \( f_{u,i} \), \( G_{u,i} \), and \( G \), in (13)–(15). Then, there exist absolute constants \( c_1, c_2, c_3 > 0 \), such that the following hold with probability at least \( 1 - \delta_1 \),

\[
\mathbb{E} \left[ G \mid f_{u,i}, u \in [U], i \in [n] \right] \geq \frac{\delta_1 m^2 \left( \sum_{u=1}^{U} \| P_u - Q_u \|_1 \right)^2}{c_1 \sum_{u=1}^{U} \sum_{i=1}^{n} f_{u,i}}, \tag{17}
\]

and

\[
\mathbb{E} \left[ G \mid f_{u,i}, u \in [U], i \in [n] \right] \leq \frac{c_2}{\delta_1} \sum_{u=1}^{U} \sum_{i=1}^{n} \frac{m^2 (P_{u,i} - Q_{u,i})^2}{f_{u,i}}, \tag{18}
\]

and

\[
\text{Var} \left[ G \mid f_{u,i}, u \in [U], i \in [n] \right] \leq \frac{c_3}{\delta_1} \sum_{u=1}^{U} \sum_{i=1}^{n} \text{Var} \left[ G_{u,i} \mid f_{u,i} \right]. \tag{19}
\]

**Proof** [Proof of Lemma 9] By standard properties of the Poisson distribution, the variables in the set of all \( G_{u,i} \) are independent. Therefore,

\[
\mathbb{E}[G_{u,i}] = \mathbb{E}[(V_{u,i} - Y_{u,i})^2 - V_{u,i} - Y_{u,i}]
\]
\[= \mathbb{E}[V_{u,i}^2] - 2\mathbb{E}[V_{u,i}]\mathbb{E}[Y_{u,i}] + \mathbb{E}[Y_{u,i}^2] - \mathbb{E}[V_{u,i}] - \mathbb{E}[Y_{u,i}]
\]
\[= (mP_{u,i})^2 + mP_{u,i} - 2m^2P_{u,i}Q_{u,i} + (mQ_{u,i})^2 + mQ_{u,i} - m^2P_{u,i} - mQ_{u,i}
\]
\[= (mP_{u,i})^2 - 2m^2P_{u,i}Q_{u,i} + (mQ_{u,i})^2 = m^2(P_{u,i} - Q_{u,i})^2
\]
\[= m^2 |P_{u,i} - Q_{u,i}|^2. \tag{20}
\]

Hence, \( G_{u,i} \) is an unbiased estimator of \( m^2|P_{u,i} - Q_{u,i}|^2 \). Similarly,

\[
\text{Var}(G_{u,i}) = \text{Var}[(V_{u,i} - Y_{u,i})^2 - V_{u,i} - Y_{u,i}]
\]
\[= \mathbb{E}[(V_{u,i} - Y_{u,i})^4 - (V_{u,i} - Y_{u,i})^2] - \mathbb{E}[(V_{u,i} - Y_{u,i})^2 - V_{u,i} - Y_{u,i}]^2
\]
\[= \mathbb{E}[(V_{u,i} - Y_{u,i})^4 - 2(V_{u,i} - Y_{u,i})^3 + (V_{u,i} - Y_{u,i})^2]
\]
\[- \mathbb{E}[(V_{u,i} - Y_{u,i})^2 - V_{u,i} - Y_{u,i}]^2
\]
\[= 4m^3(P_{u,i} - Q_{u,i})^2(P_{u,i} + Q_{u,i}) + 2m^2(P_{u,i} + Q_{u,i}). \tag{21}
\]

Next, using the fact that all the random variables \( G_{u,i} \) and \( f_{u,i} \) are independent, by the linearity of the expectation, we obtain that the conditional expectation of \( G \) is

\[
\mathbb{E} \left[ G \mid f_{u,i}, u \in [U], i \in [n] \right] = \mathbb{E} \left[ \sum_{u=1}^{U} \sum_{i=1}^{n} \frac{G_{u,i}}{f_{u,i}} \mid f_{u,i}, u \in [U], i \in [n] \right]. \tag{22}
\]
\[ E\left[ G \mid \hat{f}_{u,i}, u \in [U], i \in [n] \right] = \sum_{u=1}^{U} \sum_{i=1}^{n} \frac{\mathbb{E}[G_{u,i}]}{f_{u,i}} \]  

(23)

Similarly, the conditional variance of \( G \) is

\[
\text{Var}\left[ G \mid \hat{f}_{u,i}, u \in [U], i \in [n] \right] = \text{Var}\left[ \sum_{u=1}^{U} \sum_{i=1}^{n} \frac{G_{u,i}}{f_{u,i}} \mid \hat{f}_{u,i}, u \in [U], i \in [n] \right] = \sum_{u=1}^{U} \sum_{i=1}^{n} \text{Var}\left[ G_{u,i} \right]/f_{u,i}^2 .
\]  

(24)

(25)

Combining (20) and (23), we get

\[
\mathbb{E}\left[ G \mid \hat{f}_{u,i}, u \in [U], i \in [n] \right] = \sum_{u=1}^{U} \sum_{i=1}^{n} \frac{\mathbb{E}[G_{u,i}]}{f_{u,i}} \]

(26)

\[
= \sum_{u=1}^{U} \sum_{i=1}^{n} \frac{m^2 (P_{u,i} - Q_{u,i})^2}{f_{u,i}} \]

(27)

\[
\geq \frac{m^2 \left( \sum_{u=1}^{U} \sum_{i=1}^{n} |P_{u,i} - Q_{u,i}| \right)^2}{\sum_{u=1}^{U} \sum_{i=1}^{n} f_{u,i}} \]

(28)

\[
\geq \frac{m^2 \left( \sum_{u=1}^{U} \|P_u - Q_u\|_1 \right)^2}{\sum_{u=1}^{U} \sum_{i=1}^{n} f_{u,i}},
\]  

(29)

where the first inequality follows from the following fact that for any real \((a_i)_{i=1}^{n}\) and positive \((b_i)_{i=1}^{n}\),

\[
\sum_{i=1}^{n} \frac{a_i^2}{b_i} \geq \frac{\left( \sum_{i=1}^{n} |a_i| \right)^2}{\sum_{i=1}^{n} b_i} ,
\]  

and the last inequality follows by applying Cauchy-Schwarz to

\[
\sum_{i=1}^{n} |a_i| = \sum_{i=1}^{n} \frac{\sqrt{b_i} |a_i|}{\sqrt{b_i}} .
\]

Next, Lemma 2.5 in [15] states that there exist absolute constants \(c_1, c_2, c_3 > 0\) such that, for every \(u \in [U], i \in [n]\),

\[
\mathbb{E} \left[ \hat{f}_{u,i} \right] \leq c_1, \quad \mathbb{E} \left[ \hat{f}_{u,i}^{-1} \right] \leq \frac{c_2}{f_{u,i}}, \quad \text{and} \quad \mathbb{E} \left[ \hat{f}_{u,i}^{-2} \right] \leq \frac{c_3}{f_{u,i}^2}.
\]

Moreover, by definition, the random random variables \(\hat{f}_{u,i}\) are non-negative, and thus, applying Markov inequality we obtain that

\[
\sum_{i=1}^{n} \hat{f}_{u,i} \leq \frac{1}{\delta_1} \sum_{i=1}^{n} \mathbb{E} \left[ \hat{f}_{u,i} \right],
\]  

(27)
with probability at least $1 - \delta_1$. Combined with [15, Lemma 2.5], this means that, with probability at least $1 - \delta_1$,
\[
E \left[ G \mid \hat{f}_{u,i}, u \in [U], i \in [n] \right] \geq \frac{\delta_1 m^2 \left( \sum_{u=1}^{U} \| P_u - Q_u \|_1 \right)^2}{c_1 \sum_{u=1}^{U} \sum_{i=1}^{n} f_{u,i}}.
\] (31)

Next, applying the Markov’s inequality for the non-negative random variable,
\[
E \left[ G \mid \hat{f}_{u,i}, u \in [U], i \in [n] \right],
\]
with probability at least $1 - \delta_1$, we obtain
\[
E \left[ G \mid \hat{f}_{u,i}, u \in [U], i \in [n] \right] \leq \frac{1}{\delta_1} E \left[ E \left[ G \mid \hat{f}_{u,i}, u \in [U], i \in [n] \right] \right] \]
\[
= \frac{1}{\delta_1} \sum_{u=1}^{U} \sum_{i=1}^{n} \frac{E \left[ G_{u,i} \right]}{f_{u,i}} \]
\[
\leq c_2 \sum_{u=1}^{U} \sum_{i=1}^{n} \frac{m^2 (P_{u,i} - Q_{u,i})^2}{f_{u,i}}.
\] (34)

Similarly, with probability at least $1 - \delta_1$,
\[
\text{Var} \left[ G \mid \hat{f}_{u,i}, u \in [U], i \in [n] \right] \leq \frac{1}{\delta_1} E \left[ \text{Var} \left[ G \mid \hat{f}_{u,i}, u \in [U], i \in [n] \right] \right] \]
\[
= \frac{1}{\delta_1} \sum_{u=1}^{U} \sum_{i=1}^{n} \frac{\text{Var} \left[ G_{u,i} \right]}{f_{u,i}} \]
\[
\leq \frac{c_3}{\delta_1} \sum_{u=1}^{U} \sum_{i=1}^{n} \frac{\text{Var} \left[ G_{u,i} \right]}{f_{u,i}^2}.
\] (36)

By the union bound, we may conclude that (17)–(19), hold simultaneously with probability at least $1 - 3\delta_1$. We next bound the terms in the right-hand-side of (17)–(19). We do that separately for $m \geq n$ and $m \leq n$, respectively. To that end, we use the same ideas as in [15, Lemma 2.3], and [15, Lemma 2.4].

Lemma 10. For $m \geq n$, the following hold:

1. $\sum_{u=1}^{U} \sum_{i=1}^{n} \frac{\text{Var} \left[ G_{u,i} \right]}{f_{u,i}^2} \leq \frac{10Um^2}{n}$,

2. $\sum_{i=1}^{n} \frac{m^2 (P_{u,i} - Q_{u,i})^2}{f_{u,i}} \leq \frac{m^{3/2} \| P_u - Q_u \|_1}{n^{1/2}}$,

3. $\frac{m^2 \left( \sum_{u=1}^{U} \| P_u - Q_u \|_1 \right)^2}{\sum_{u=1}^{U} \sum_{i=1}^{n} f_{u,i}} \geq \min \left( \frac{m^{3/2} \sum_{u=1}^{U} \| P_u - Q_u \|_1}{2(Um)^{1/2}}, \frac{m^2 \left( \sum_{u=1}^{U} \| P_u - Q_u \|_1 \right)^2}{6Un} \right)$.

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Proof [Proof of Lemma 10] We start proving the first inequality. From (21), we get

\[
\sum_{u=1}^{U} \sum_{i=1}^{n} \frac{\text{Var} \left[ G_{u,i} \right]}{f_{u,i}^2} = \sum_{u=1}^{U} \sum_{i=1}^{n} \frac{4m^3(P_{u,i} - Q_{u,i})^2(P_{u,i} + Q_{u,i})^2 + 2m^2(P_{u,i} + Q_{u,i})^2}{f_{u,i}^2}
\]

\[
= \sum_{u=1}^{U} \sum_{i=1}^{n} \frac{4m^3(P_{u,i} - Q_{u,i})^2(P_{u,i} + Q_{u,i}) + 2m^2(P_{u,i} + Q_{u,i})^2}{\max \{ \sqrt{mn} \cdot |P_{u,i} - Q_{u,i}|, n \cdot (P_{u,i} + Q_{u,i}) \}^2} + \sum_{u=1}^{U} \sum_{i=1}^{n} \frac{2m^2}{n^2}
\]

\[
\leq \frac{8Um^2}{n} + \frac{2Um^2}{n} = \frac{10Um^2}{n}.
\]

Next, we prove the second inequality:

\[
\sum_{i=1}^{n} \frac{m^2(P_{u,i} - Q_{u,i})^2}{f_{u,i}} = \sum_{i=1}^{n} \max \{ \sqrt{mn} \cdot |P_{u,i} - Q_{u,i}|, n \cdot (P_{u,i} + Q_{u,i}) \} - \sum_{i=1}^{n} m^{3/2} \frac{|P_{u,i} - Q_{u,i}|}{n^{3/2}} = m^{3/2} \| P_u - Q_u \|_1.
\]

Finally, we prove the last inequality:

\[
\frac{m^2 \left( \sum_{u=1}^{U} \| P_u - Q_u \|_1 \right)^2}{\sum_{u=1}^{U} \sum_{i=1}^{n} f_{u,i}} \geq \frac{m^2 \left( \sum_{u=1}^{U} \| P_u - Q_u \|_1 \right)^2}{\sum_{u=1}^{U} \sum_{i=1}^{n} \max \{ \sqrt{mn} \cdot |P_{u,i} - Q_{u,i}|, n \cdot (P_{u,i} + Q_{u,i}) \}} + \frac{2m^2}{n^2}
\]

\[
\geq \frac{\sum_{u=1}^{U} \sum_{i=1}^{n} \left( \sqrt{mn} \cdot |P_{u,i} - Q_{u,i}| + n \cdot (P_{u,i} + Q_{u,i}) \right)}{\sum_{u=1}^{U} \sum_{i=1}^{n} \| P_u - Q_u \|_1 + 2Un + Un}
\]

\[
\geq \min \left\{ \frac{m^{3/2} \sum_{u=1}^{U} \| P_u - Q_u \|_1}{2U\sqrt{n}}, \frac{m^2 \left( \sum_{u=1}^{U} \| P_u - Q_u \|_1 \right)^2}{6Un} \right\}.
\]

Applying Lemma 10 on (17)–(19), we obtain the following corollary for \( m \geq n \).

Corollary 11. For \( m \geq n \), the following hold with probability at least \( 1 - \delta_1 \):

1. \( \mathbb{E} \left[ G \mid \hat{f}_{u,i}, u \in [U], i \in [n] \right] \geq \frac{\delta_1}{c_1} \min \left( \frac{m^{3/2} \sum_{u=1}^{U} \| P_u - Q_u \|_1}{2U\sqrt{n}}, \frac{m^2 \left( \sum_{u=1}^{U} \| P_u - Q_u \|_1 \right)^2}{6Un} \right) \).

2. \( \mathbb{E} \left[ G \mid \hat{f}_{u,i}, u \in [U], i \in [n] \right] \leq \frac{\delta_1}{c_1} \sum_{u=1}^{U} \frac{m^{3/2} \| P_u - Q_u \|_1}{n^{3/2}} \).
3. \( \text{Var} \left[ G \mid \hat{f}_{u,i}, u \in [U], i \in [n] \right] \leq \frac{c_1 \cdot 10 \cdot n^2}{m} \).

Next, we have the following lemma for the complementary case of \( m \leq n \).

**Lemma 12.** For \( m \leq n \), the following hold:

1. \( \sum_{i=1}^{n} \frac{\text{Var}(G_{u,i})}{f_{u,i}^2} \leq 24m \),
2. \( \sum_{i=1}^{n} \frac{m^2(P_{u,i} - Q_{u,i})^2}{f_{u,i}^2} \leq m \| P_u - Q_u \|_1 \),
3. \( \frac{m^2(\sum_{u=1}^{U} \| P_u - Q_u \|_1)^2}{\sum_{u=1}^{U} \sum_{i=1}^{n} f_{u,i}} \geq \frac{m^2(\sum_{u=1}^{U} \| P_u - Q_u \|_1)^2}{3n} \).

**Proof** [Proof of Lemma 12] As before, we start by proving the first inequality:

\[
\sum_{i=1}^{n} \frac{\text{Var}(G_{u,i})}{f_{u,i}^2} = \sum_{i=1}^{n} \frac{4m^3(P_{u,i} - Q_{u,i})^2(P_{u,i} + Q_{u,i}) + 2m^2(P_{u,i} + Q_{u,i})^2}{f_{u,i}^2} \\
= \sum_{i=1}^{n} \frac{4m^3(P_{u,i} - Q_{u,i})^2(P_{u,i} + Q_{u,i}) + 2m^2(P_{u,i} + Q_{u,i})^2}{m \cdot (P_{u,i} + Q_{u,i}) \cdot 1}^2 \\
\leq \sum_{i=1}^{n} \frac{4m^3(P_{u,i} - Q_{u,i})^2(P_{u,i} + Q_{u,i}) + 4m^2(P_{u,i} - Q_{u,i})^2 + 8m^2Q_{u,i}^2}{m \cdot (P_{u,i} + Q_{u,i}) \cdot 1}^2 \\
\leq \sum_{i=1}^{n} \frac{8m^2Q_{u,i}^2}{m \cdot (P_{u,i} + Q_{u,i}) \cdot 1} \max \left\{ m \cdot (P_{u,i} + Q_{u,i}) \cdot 1 \right\} \\
\leq 4m \sum_{i=1}^{n} |P_{u,i} - Q_{u,i}| + 4m \sum_{i=1}^{n} |P_{u,i} - Q_{u,i}| + \sum_{i=1}^{n} \frac{8m^2Q_{u,i}^2}{m \cdot (P_{u,i} + Q_{u,i}) \cdot 1} \\
\leq 8m \| P_u + Q_u \|_1 + \sum_{i=1}^{n} 8mQ_{u,i} \\
\leq 16m + 8m = 24m.
\]

Next, we prove second inequality:

\[
\sum_{i=1}^{n} \frac{m^2(P_{u,i} - Q_{u,i})^2}{f_{u,i}} = \sum_{i=1}^{n} \frac{m^2 |P_{u,i} - Q_{u,i}|^2}{\max \left\{ m \cdot (P_{u,i} + Q_{u,i}) \cdot 1 \right\}} \\
\leq \sum_{i=1}^{n} m |P_{u,i} - Q_{u,i}| = m \| P_u - Q_u \|_1.
\]

Finally, we prove the last inequality:

\[
\frac{m^2 \left( \sum_{u=1}^{U} \| P_u - Q_u \|_1 \right)^2}{\sum_{u=1}^{U} \sum_{i=1}^{n} f_{u,i}} = \frac{m^2 \left( \sum_{u=1}^{U} \| P_u - Q_u \|_1 \right)^2}{3n} \max \left\{ m \cdot (P_{u,i} + Q_{u,i}) \cdot 1 \right\}
\]

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Corollary 13. Applying Lemma 12 on (17)–(19), we obtain the following corollary for $m \leq n$.

For $m \leq n$, the following hold with probability at least $1 - \delta_1$:

1. $E \left[ G \mid \tilde{f}_{u,i}, u \in [U], i \in [n] \right] \geq \frac{\delta_1}{4} m^2 \left( \sum_{u=1}^{U} \| P_u - Q_u \|_1 \right)^2$.

2. $E \left[ G \mid \tilde{f}_{u,i}, u \in [U], i \in [n] \right] \leq \frac{\delta_1}{8} \sum_{u=1}^{U} m \| P_u - Q_u \|_1$.

3. $\text{Var} \left[ G \mid \tilde{f}_{u,i}, u \in [U], i \in [n] \right] \leq \frac{24mU\epsilon}{\delta_1}$.

4. Also,

$$\text{Var} \left[ G \mid \tilde{f}_{u,i}, u \in [U], i \in [n] \right] \leq \frac{1}{40} \left( \mathbb{E} \left[ G \mid \tilde{f}_{u,i}, u \in [U], i \in [n] \right] \right)^2 + 324 \mathbb{E} \left[ G \mid \tilde{f}_{u,i}, u \in [U], i \in [n] \right] + 648m^2 \sum_{u=1}^{U} \| Q_u \|_2^2.$$

Items 1-3 of Corollary 13 follow almost directly from Lemma 12 and we next establish the last item of Corollary 13. First, note that

$$\text{Var} \left[ G \mid \tilde{f}_{u,i} \right] = \sum_{u=1}^{U} \sum_{i=1}^{n} \text{Var} \left( \frac{G_{u,i}}{\tilde{f}_{u,i}} \right)$$

$$= \sum_{u=1}^{U} \sum_{i=1}^{n} 4m^3 \left( \frac{P_{u,i} - Q_{u,i}}{\tilde{f}_{u,i}} \right)^2 \left( \frac{P_{u,i} + Q_{u,i}}{\tilde{f}_{u,i}} \right)^2$$

$$\leq 4m^3 \left( \sum_{u=1}^{U} \sum_{i=1}^{n} \left( \frac{P_{u,i} - Q_{u,i}}{\tilde{f}_{u,i}} \right)^4 \left( \frac{P_{u,i} + Q_{u,i}}{\tilde{f}_{u,i}} \right)^2 \right)^{\frac{1}{2}}$$

$$+ \sum_{u=1}^{U} \sum_{i=1}^{n} 2m^2 \left( \frac{P_{u,i} + Q_{u,i}}{\tilde{f}_{u,i}} \right)^2 \left( \frac{P_{u,i} - Q_{u,i}}{\tilde{f}_{u,i}} \right)^2$$

$$\leq 4m^3 \left( \sum_{u=1}^{U} \sum_{i=1}^{n} \left( \frac{P_{u,i} - Q_{u,i}}{\tilde{f}_{u,i}} \right)^2 \left( \frac{P_{u,i} + Q_{u,i}}{\tilde{f}_{u,i}} \right)^2 \right)^{\frac{1}{2}}.$$
and finally (c) is because

\[ \hat{\mu} \]

We start with the case where, \( m \geq n \). Applying Chebyshev’s inequality to the conditional expectation and variance, with probability \( \geq 1 - \delta \), we have

\[
\Pr \left( |G - \mu| \leq \frac{\sigma}{\sqrt{\delta}} \mid \hat{f}_{u,i}, u \in [U], i \in [n] \right) \geq 1 - \delta,
\]

where,

\[
\mu = \mathbb{E} \left[ G \mid \hat{f}_{u,i}, u \in [U], i \in [n] \right], \quad \sigma^2 = \text{Var} \left[ G \mid \hat{f}_{u,i}, u \in [U], i \in [n] \right].
\]

4.2.2 Proof of Theorem 5

We start with the case where, \( m \geq n \). Applying Chebyshev’s inequality to the conditional expectation and variance, with probability \( \geq 1 - \delta \), we have

\[
\text{Var} \left[ G \mid \hat{f}_{u,i} \right] = 4 \left( \mathbb{E} \left[ G \mid \hat{f}_{u,i} \right] \right) \left( \sum_{u=1}^{U} \sum_{i=1}^{n} \frac{m^2 (P_{u,i} + Q_{u,i})^2}{\hat{f}_{u,i}^2} \right)^{\frac{1}{2}} + \sum_{u=1}^{U} \sum_{i=1}^{n} \frac{2m^2 (P_{u,i} + Q_{u,i})^2}{\hat{f}_{u,i}^2},
\]

where (a) follows from the Cauchy-Schwartz inequality, and (b) is due to the monotonicity of the \( \ell_p \) norm, i.e., for any vector \( u \), \( \|u\|_2 \leq \|u\|_1 \). Then,

\[
\text{Var} \left[ G \mid \hat{f}_{u,i} \right] \leq \frac{1}{40} \left( \mathbb{E} \left[ G \mid \hat{f}_{u,i} \right] \right)^2 + (160 + 2) \sum_{u=1}^{U} \sum_{i=1}^{n} \frac{m^2 (P_{u,i} + Q_{u,i})^2}{\hat{f}_{u,i}^2}
\]

\[
\leq \frac{1}{40} \left( \mathbb{E} \left[ G \mid \hat{f}_{u,i} \right] \right)^2 + 162 \sum_{u=1}^{U} \sum_{i=1}^{n} \frac{2m^2 (P_{u,i} - Q_{u,i})^2}{\hat{f}_{u,i}^2} + 162 \sum_{u=1}^{U} \sum_{i=1}^{n} \frac{4m^2 Q_{u,i}^2}{\hat{f}_{u,i}^2}
\]

\[
\leq \frac{1}{40} \left( \mathbb{E} \left[ G \mid \hat{f}_{u,i} \right] \right)^2 + 324 \sum_{u=1}^{U} \sum_{i=1}^{n} \frac{m^2 (P_{u,i} - Q_{u,i})^2}{\hat{f}_{u,i}^2} + 648 \sum_{u=1}^{U} \sum_{i=1}^{n} \frac{m^2 Q_{u,i}^2}{\hat{f}_{u,i}^2}
\]

\[
= \frac{1}{40} \left( \mathbb{E} \left[ G \mid \hat{f}_{u,i} \right] \right)^2 + 324 \mathbb{E} \left[ G \mid \hat{f}_{u,i} \right] + 648m^2 \sum_{u=1}^{U} \|Q_u\|_2
\]

\[
= \frac{1}{40} \left( \mathbb{E} \left[ G \mid \hat{f}_{u,i}, u \in [U], i \in [n] \right] \right)^2 + 324 \mathbb{E} \left[ G \mid \hat{f}_{u,i}, u \in [U], i \in [n] \right] + 648m^2 \sum_{u=1}^{U} \|Q_u\|_2
\]

where in (a) we use the fact that \( 2ab \leq a^2 + b^2 \), (b) follows from \( (a + b)^2 \leq 2(a - b)^2 + 4b^2 \), and finally (c) is because \( \hat{f}_{u,i} \geq 1 \).
Therefore, using Corollary 11 we get

\[
\frac{\delta_1}{c_1} \min \left( \frac{m^{3/2} \sum_{u=1}^{U} \|P_u - Q_u\|_1}{2U\sqrt{n}}, \frac{m^2 \left( \sum_{u=1}^{U} \|P_u - Q_u\|_1 \right)^2}{6Un} \right) - m\sqrt{\frac{10c_3U}{\delta_1 n}} \leq G, \tag{38}
\]

and

\[
G \leq \frac{c_2}{\delta_1} \sum_{u=1}^{U} \frac{m^{3/2}\|P_u - Q_u\|_1}{n^{1}} + m\sqrt{\frac{10c_3U}{\delta_1 n}}, \tag{39}
\]

with probability at least $1 - \delta$. Now, in the case where $\sum_{u=1}^{U} \|P_u - Q_u\|_1 \geq \varepsilon_2 U$, the lower bound in (38) reduces to

\[
G \geq \frac{\delta_1}{c_1} \min \left( \frac{m^{3/2}\varepsilon_2 U}{2U\sqrt{n}}, \frac{m^2\varepsilon_2^2 U^2}{6Un} \right) - m\sqrt{\frac{10c_3U}{\delta_1 n}} \geq \frac{\delta_1}{12c_1} \min \left( \frac{m^{3/2}\varepsilon_2 U}{\sqrt{n}}, \frac{m^2\varepsilon_2^2 U^2}{n} \right),
\]

where the last step holds as long as $m \geq \max \left\{ \frac{12^2 10c_3^2 U}{\delta_1^4 \varepsilon_2^4}, \frac{12c_1}{\delta_1} \sqrt{\frac{10c_1}{\delta_1} \sqrt{\frac{n}{U\delta_2^2}}} \right\}$, and so for

$m \geq C \max \left\{ \frac{\delta_1}{\varepsilon_2^4}, \sqrt{\frac{n}{U\delta_2^2}} \right\}$. Therefore, with probability at least $1 - \delta$ the tester correctly outputs $\sum_{u=1}^{U} \|P_u - Q_u\|_1 \geq \varepsilon_2$.

On the other hand, in the case, $\sum_{u=1}^{U} \|P_u - Q_u\|_1 \leq \varepsilon_1 U$, using Corollary 11 once again we get that

\[
G \leq \frac{c_2 m^{3/2}\varepsilon_1 U}{\delta_1 \sqrt{n}} + m\sqrt{\frac{10c_3U}{\delta_1 n}} \leq \frac{\delta_1}{24c_1} \min \left( \frac{m^{3/2}\varepsilon_2 U}{\sqrt{n}}, \frac{m^2\varepsilon_2^2 U^2}{n} \right),
\]

where the last step holds as long as $m > \max \left( \frac{\delta_1 \varepsilon_2 \sqrt{1/\delta_1^4}, \varepsilon_2^4}{c_2 \delta_1^4 \varepsilon_1^4} \right) = \tilde{C} \max \left( \frac{\varepsilon_2 \sqrt{1/\delta_1^4}}{\varepsilon_1^4}, \frac{n \varepsilon_1^2}{\varepsilon_2^4} \right)$. Therefore, with probability at least $1 - \delta$ the tester correctly outputs that $\sum_{u=1}^{U} \|P_u - Q_u\|_1 \leq \varepsilon_1$.

Finally, we turn to the case $m \leq n$. Similarly to the former case, using Corollary 13 and repeating the same process above, for the lower bound of $G$ (the case where $\sum_{u=1}^{U} \|P_u - Q_u\|_1 \geq \varepsilon_2 U$), we obtain that $m \geq \frac{n^{2/3}}{\varepsilon_2}$, while the upper bound on $G$ holds as long as $m \geq \frac{n^{2/3}}{\varepsilon_2^2}$.

\[\textbf{4.3 Proof of Theorem 6}\]

As an integral part of Procedure 2, i.i.d. samples of each Markov chain trajectories are being sent to the i.i.d. tolerant-closeness-tester (line 8 in IIDTESTER). In the following lemma, we devise a tool to pull i.i.d. samples from a given trajectory, by showing that the image coordinates of $\psi^{(i)}_k$ are i.i.d.
Lemma 14 (Independent coordinates of $\psi_k^{(i)}$ Image). For $k < T$, let $(Z_i)_{i=1}^T$ be a single trajectory, of length $T$, drawn from Markov chain $\mathcal{M}$. Given that $(Z_i)_{i=1}^T$ is large enough to contain at least $k$ hits on the $i$-th state, then the coordinates of $\psi_k^{(i)}((Z_j)_{j=1}^T)$, are independent and identically distributed defined by the $i$-th state of $\mathcal{M}$, meaning $\psi_k^{(i)}((Z_j)_{j=1}^T) \sim \mathcal{M}(i, \cdot)$.

Proof [Proof of Lemma 14] Let $(Z_i)_{i=1}^T$ be trajectory drowned by Markov chain $\mathcal{M}$, in the length of $T$. It is a consequence of the Markov property that the first succeeding states of state of $i$, in particular such $k$, be independent and identically distributed according to the conditional distribution defined by the $i$-th state of $\mathcal{M}$.

The following lemma slightly generalizes the “Exponential decay lemma” shown in [20], for a (general) case, where we have $m$-simultaneously independent random walks on the same Markovian chain (where $m > 1$). This result serves as a core tool in the proof of Theorem 6.

Lemma 15 ($m$-Joint-$k$-Cover Exponential Decay). For $m$-simultaneously independent random walks on the same $n$ states irreducible Markov chain, and for any $k, \beta \in \mathbb{N}^+$ and any initial distribution $\mathbf{q}$ over $[n]^m$, we have $\Pr\left(\tau_{\text{cov}}^{(k)}(m) \geq e\beta\tau_{\text{cov}}^{(k)}(m)\right) \leq e^{-\beta}$.

Proof [Proof of Lemma 15] Consider $\tau_{\text{cov}}^{(k)}(m)$ with any fixed starting states

$$Z_{1,1} = v_1, Z_{2,1} = v_2, \ldots, Z_{m,1} = v_m,$$

we have by Markov’s inequality and linearity of expectation that

$$\Pr(\tau_{\text{cov}}^{(k)}(m) \geq e\beta\tau_{\text{cov}}^{(k)}(m)) \leq \Pr\left(\tau_{\text{cov}}^{(k)}(m) \geq e\beta\tau_{\text{cov}}^{(k)}(m) \mid \forall j \in [m], Z_{j,1} = v_j\right) \leq 1/e. \tag{40}$$

Note that this inequality holds for any initial states $\mathbf{V} = (v_1, \ldots, v_m) \sim \mathbf{q}$, where $\mathbf{q}$ could be any discret distribution over $[n]^m$.

First, we consider the first two sub-trajectories of every Markov chain, each of length $l_j = l \triangleq e\beta\tau_{\text{cov}}^{(k)}(m)$, i.e., the chains $Z_{j,1}^l$ and $Z_{j,l+1}^{2l}$, for all $j \in [m]$. Denote the event

$$E_1 \triangleq \left\{ \text{jointly all } m \text{ sub-trajectories } \left\{ Z_{j,1}^l \right\}_{j \in m} \text{ cover the state space } k \text{ times} \right\},$$

and,

$$E_2 \triangleq \left\{ \text{jointly all } m \text{ sub-trajectories } \left\{ Z_{j,l+1}^{2l} \right\}_{j \in m} \text{ cover the state space } k \text{ times} \right\},$$

hence according to (40), we have

$$\Pr(E_1^c) = \Pr\left(\tau_{\text{cov}}^{(k)}(m) \geq e\beta\tau_{\text{cov}}^{(k)}(m)\right) \leq 1/e.$$
Denote the distribution of \( \{Z_{j,l}\}_{j \in [m]} \) conditioned on \( E_1^c \) by \( q' \), and let \( \tau_{\text{cov}}^{(k)}(m) \) be the \( k \)-cover time of \( Z_{i+1}^\infty \). Then, we have

\[
\mathbb{P}(E_2^c | E_1^c) = \mathbb{P}\left(\tau_{\text{cov}}^{(k)}(m) \geq e\gamma\tau_{\text{cov}}^{(k)}(m) | \tau_{\text{cov}}^{(k)}(m) \geq e\gamma\tau_{\text{cov}}^{(k)}(m)\right) = \mathbb{P}\left(\tau_{\text{cov}}^{(k)}(m) \geq e\gamma\tau_{\text{cov}}^{(k)}(m) \mid \{Z_{j,l}\}_{j \in [n]} \sim q'\right) \leq 1/e.
\]

Here, we used the fact that \( E_1 \) is determined by \( \{Z_{j,l}\}_{j \in [n]} \), while due to Markovian property \( E_2 \) do not depend on \( \{Z_{j,l}\}_{j \in [n]} \), but only on \( \{Z_{j,l}\}_{j \in [n]} \), that are the final states in the \( m \) sub-trajectories. Thus,

\[
\mathbb{P}(E_1^c \cap E_2^c) = \mathbb{P}(E_1^c) \mathbb{P}(E_2^c | E_1^c) \leq e^{-2}.
\]

Obviously, by induction, we can deduce that \( \mathbb{P}(\cap_{i \in [\beta]} E_i^c) \leq e^{-\beta} \). Now, since the event

\[
E \triangleq \left\{ \text{jointly all } m \text{ sub-trajectories } \left\{Z_{j,l}^{\beta}\right\}_{j \in [m]} \text{ covers the state space } k \text{ times} \right\}
\]

includes the event \( \cup_{i \in [\beta]} E_i \), we obtain that \( \mathbb{P}(E^c) \leq \mathbb{P}(\cap_{i \in [\beta]} E_i^c) \leq e^{-\beta} \).

We are now in a position to prove Theorem 6.

**Proof** [Proof of Theorem 6] For every \( u \in [U] \) and \( j \in [m] \), consider the infinite chains \( (x^{R}_{j,u})_{t_0}^\infty \) and \( (x^{F}_{j,u})_{t_0}^\infty \). Denote the events,

\[
E_{R,u} = \left\{ \sum_{j=1}^{m} N_t^{x_{j,u}^R}(l) \geq k(n, \varepsilon, \delta/4n) \mid u \in [U] \right\},
\]

and,

\[
E_{F,u} = \left\{ \sum_{j=1}^{m} N_t^{x_{j,u}^F}(l) \geq k(n, \varepsilon, \delta/4n) \mid u \in [U] \right\},
\]

and,

\[
E_i = \left\{ \text{The first } k \text{ samples succeeding state } i \text{ in } (x_{j,u}^{R})_{t_0}^\infty \text{ yields “No” during the test} \right\}.
\]

**Case 1.** \( \sum_{u=1}^{U} \|P_{u}^{R} - Q_{u}^{F}\|_\infty \leq \varepsilon_1 \)

According to Lemma 15, we have \( \mathbb{P}\left(\tau_{\text{cov}}^{(k)}(m) \geq e\beta\gamma\tau_{\text{cov}}^{(k)}(m)\right) \leq e^{-\beta} \). Then, by taking \( \beta = \ln \frac{4U}{\delta} \), we get \( \mathbb{P}\left(\tau_{\text{cov}}^{(k)}(m) \geq e\gamma\tau_{\text{cov}}^{(k)}(m) \mid \tau_{\text{cov}}^{(k)}(m) \geq e\gamma\tau_{\text{cov}}^{(k)}(m)\right) \leq \frac{\delta}{4U} \). Thus, for a length \( l = e\gamma\tau_{\text{cov}}^{(k)}(m) \ln \frac{4U}{\delta} \), we will have \( k(n, \varepsilon, \delta/4n) \) samples for each state in \([n]\) with probability \( \mathbb{P}(E_{R,u}) \geq 1 - \frac{\delta}{4U} \). Similarly, \( \mathbb{P}(E_{F,u}) \geq 1 - \frac{\delta}{4U} \). By a union bound over the two chains, the probability of passing the condition in line 6 of Algorithm 2 during the whole procedure, is \( \geq 1 - \frac{\delta}{4U} \). By the guarantee of the i.i.d tolerant-closeness-tester \( \text{IIDTESTER, we have } \mathbb{P}(E_i) \leq \frac{\delta}{4U} \), with an error probability \( \mathbb{P}(E_{R,u} \cup E_{F,u} \cup E_1 \cup \ldots \cup E_n) \leq \frac{3}{4U} \). Thus with probability at least \( \geq 1 - \delta \) the tester will return “Yes”.

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Case 2. \( \sum_{u=1}^{U} \| P_u^R - Q_u^F \|_\infty \geq \varepsilon_2 \)

The only case the algorithm outputs “Yes” is when it do not pass lines 4 and 7 in Algorithm 2 for all states, which means it will have enough samples for testing each state, and the i.i.d. tester \( T \) answers “Yes” for all sub-tests. Since \( \max_{i \in [L]} \sum_{u} \| P_{u,i} - q_{u,i} \|_1 \geq \varepsilon_2 \) implies that there exists \( i^* \in [n] \) such that \( \sum_{u} \| P_{u,i^*} - q_{u,i^*} \|_1 \geq \varepsilon_2 \), this guarantees that the sub-test for \( i^* \) will return “No” with probability \( P \left( E \right) \geq 1 - \delta/4n \). Thus the probability of the whole process answering “Yes” is \( P \left( E_{R,u} \cap E_{F,u} \cap E_1^T \ldots \cap E_u^T \right) \leq P \left( E_1^T \right) \leq \delta/4n \).

Combining both cases above, the tolerant-closeness-tester will give the correct answer with probability at least \( 1 - \delta \).

4.4 Proof of Theorem 8

Algorithm 3 runs Algorithm 2 twice independently. Specifically, in line 3 of Algorithm 3, Algorithm 2 is first called with parameters \( T, \varepsilon_1, \varepsilon_2, \delta_{B1} \) and \( m(n(n-1))/2, \varepsilon_1, \varepsilon_2, \delta_{B1}/2n \). Using Theorem 6, the sample complexity of running Algorithm 2 with the above parameters is given by

\[
O_{\delta_{B1}} \left( \max \left\{ \max_{u \in U} \left\{ m(n(n-1)/2, \varepsilon_1, \varepsilon_2, \delta_{B1}/4n) \left( M; P_u^R \right), t_{\text{cov}}(n(n-1)/2, \varepsilon_1, \varepsilon_2, \delta_{B1}/4n) \left( M; Q_u^F \right) \right\} \right\} \right),
\]

In line 4 of Algorithm 3, Algorithm 2 is called for the second time, now with parameters \( T, \varepsilon_1, \varepsilon_2, \delta_{B2} \) and \( m(n(n-1))/2, \varepsilon_1, \varepsilon_2, \delta_{B2}/2n \). Using Theorem 6, the sample complexity of running Algorithm 2 with the above parameters is given by

\[
O_{\delta_{B1}} \left( \max \left\{ \max_{u \in U} \left\{ m(n(n-1)/2, \varepsilon_1, \varepsilon_2, \delta_{B2}/4n) \left( M; P_u^R \right), t_{\text{cov}}(n(n-1)/2, \varepsilon_1, \varepsilon_2, \delta_{B2}/4n) \left( M; Q_u^F \right) \right\} \right\} \right).
\]

Hence, clearly the number of samples required is given by \( m(n, \varepsilon_1, \varepsilon_2, \delta) \), and thus we have an \( (\varepsilon_1, \varepsilon_2, \delta) \) solver for \( \{16\} \) using

\[
T = O_{\delta} \left( \max \left\{ \max_{u \in U} \left\{ t_{\text{cov}}(n(n-1)/2, \varepsilon_1, \varepsilon_2, \delta/4n) \left( M; P_u^R \right), t_{\text{cov}}(n(n-1)/2, \varepsilon_1, \varepsilon_2, \delta/4n) \left( M; Q_u^F \right) \right\} \right\} \right),
\]

sample, where \( \delta = \max(\delta_{B1}, \delta_{B2}) \).

5 Conclusion and Future Research

In this paper, we modeled the relationship between the three stakeholders: the platform, the users, and the regulator. The essence of the modeling is that from the regulator’s perspective the platform is a content-generating system formulated by a multidimensional first order Markov chain (as the fixed number of pieces of the content appearing on each feed), where at every time step the platform samples a new feed, according to the Markov transition-matrix (conditional probability). We developed a regulatory method called that tests whether there are unexpected deviations in the user’s decision-making process over a predefined time horizon. Unexpected deviations in the user’s decision-making process might be a result of the selective filtering of the contents to be shown on the user’s feed in comparison to what would be the users’ decision-making process under natural filtering. We proposed also an auditing procedure for online counterfactual regulations.
There are several exciting directions for future work, including the following. A major goal going forward is to evince our regulation procedure on real social media content. Specifically, while our work propose a theoretical framework for SMP regulation, we left several fundamental questions that revolve around implementability, such as, how do we know if the framework is effective or useful? What are the metrics that should be used? We are currently investigating these kind of questions. From the technical perspective, there are many interesting generalizations an open questions that we plan to investigate. For example, studying a sequential version of the testing problems proposed in this paper are of particular importance. Indeed, in real-world platforms decisions must be taken as quickly as possible so that proper countermeasures can be taken to suppress regulation violation. Moreover, real-world networks are gigantic and therefore it is quite important to study the performance of low-complexity algorithms. Also, it is of both theoretical and practical importance to consider more general probabilistic models which will, for example, capture the dynamic relationships between users, the varying influence of individual users within the platform, and in general weaken some of the assumptions we made about the behaviour of users, the platform/algorithm, the social relationships and dynamics. Finally, from the conceptual perspective, while our paper propose several definition for the notions of “variability” and “violation”, there probably are other possible definitions, which take into account some perspectives of responsible regulation which we ignored, and are important to investigate.

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