Complex q-rung orthopair fuzzy Frank aggregation operators and their application to multi-attribute decision making

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Abstract
The complex q-rung orthopair fuzzy sets (Cq-ROFSs) can serve as a generalization of q-rung orthopair fuzzy sets (q-ROFSs) and complex fuzzy sets (CFSs). Cq-ROFSs provide more freedom for people handling uncertainty and vagueness by the truth and falsity grades on the condition that the sum of the q-powers of the real part and imaginary part is within the unit interval. Further, Frank operational laws are an extended form of Archimedes’ T mode and Archimedes’ S mode and Frank aggregation operators have a certain parameter which makes them more flexible and more generalized than many other aggregation operators in the process of information fusion. The objectives of this paper are to extend the Frank operations to the complex q-rung orthopair fuzzy environment and to introduce their score function and accuracy function. Meanwhile, some complex q-rung fuzzy Frank aggregation operators are developed, such as the complex q-rung orthopair fuzzy Frank weighted averaging (Cq-ROFFWA) operator, the complex q-rung orthopair fuzzy Frank weighted geometric (Cq-ROFFWG) operator, and the complex q-rung orthopair fuzzy Frank ordered weighted averaging (Cq-ROFFOWA) operator, and their special cases are discussed. In addition, an innovative MADM method is introduced according to the propounded operators to deal with multi-attribute decision-making problems under the complex q-rung orthopair fuzzy environment. Consequently, the practicability and effectiveness of the created methods are proposed by parameter exploration and comparative analysis.

Keywords Complex q-rung orthopair fuzzy set • Aggregation operators • Frank operator • Multi-attribute decision making

1 Introduction
Multi-attribute decision-making (MADM) problems have gradually become an important area in decision science. Since the objects are usually uncertain, fuzzy numbers (Chou et al. 2006; Fan and Feng 2009) are more fit for expressing some attribute values involved in MADM problems. The fuzzy sets (FSs) (Zadeh 1965) which were defined in 1965 are suitable to express fuzzy multi-criteria decision-making problems with vagueness. Atanassov and Gargov (1989) introduced intuitionistic fuzzy sets (IFSs), which contain a membership function and a non-membership function. In the last few decades, scholars (Xu 2007; Chen et al. 2011; Zhang 2013) have introduced many relevant intuitionistic fuzzy aggregation operators. However, in some special conditions, the membership degree plus the non-membership degree may be greater than one. IFSs could not describe this situation, so Yager (2013) introduced Pythagorean fuzzy sets (PFSs). A PFS is
characterized by the sum of the squares of the membership function and the non-membership function being less than or equal to 1. The application of PFS is wider than that of IFSS in expressing the uncertainty in MADM problems. Ever since the first appearance of PFS, there are many studies (Yager 2013, 2014; Zhang and Xu 2014; Peng and Yang 2015, 2016; Zhang 2016a, 2016b; Gou et al. 2016; Mahanta and Panda 2021; Ma et al. 2021; Du et al. 2017; Akram et al. 2020; Liu et al. 2021a, 2021b, 2021c, 2021d, 2021e; Xian and Cheng 2021; Zhang and Ma 2020; Sarkar and Biswas 2020; Shakerel et al. 2020) on MADM problems under Pythagorean fuzzy circumstances.

However, in some special conditions, if a decision-maker gives 0.8 for membership degree and 0.7 for non-membership degree, then the sum of the squares of both is greater than 1. For addressing such types of difficulties, Yager (2016) explored the q-rung orthopair fuzzy set (QROFS) with a condition that the sum of the q-power of membership degree and the q-power of non-membership degree is restricted to [0, 1]. The QROFS is usually able to cope with much higher degrees of uncertainty. Since it was initiated, numerous researchers have exploited and utilized it in various areas (Wang et al. 2019; Du et al. 2021; Xing et al. 2019; Ju et al. 2019a, 2019b, 2020; Gao et al. 2019; Garg 2021a; Mahmood and Ali 2021a, 2021b; Aydemir and Yilmaz Gündüz 2020; Rawat and Komal 2022; Akram et al. 2021; Liu et al. 2021a, 2021b, 2021c, 2021d, 2021e).

According to the above prevailing studies, similar approaches are limited and are unable to represent the partial ignorance of the data and its fluctuations at a given period of time. To handle this, Ramot et al. (2002) introduced complex FS (CFS). Additionally, Alkouri and Salleh (2012) presented the theory of complex IFS (CIFS), which extended the range of the supporting grade and the supporting against grade from real numbers to complex numbers with a unit disc, and the CIFS can describe two-dimensional information.

Based on the CIFS, Ullah et al. (2020) proposed the complex PFS (CPFS), which is useful for efficiently describing uncertain and unreliable information. A characteristic of CPFS is that the sum of the square of the real part (and imaginary part) of complex membership grade and the square of the real part (and imaginary part) of complex non-membership grade is limited to [0, 1]. However, the CIFS and CPFS could not cope with information whose sum of the squares of the real part (also for the imaginary part) of both grades exceeds the unit interval. For example, some experts are asked to provide their preference about the degree of an alternative $D_i$ on a criterion $C_j$, and the complex-valued membership grade may be $0.93e^{2\pi i (0.84)}$, and the complex-valued non-membership grade may be $0.86e^{2\pi i (0.75)}$. To describe this type of information, Liu et al. (2020b, 2020c, 2020a) proposed the theory of complex QROFS (Cq-ROFS) with the condition the sum of the q-power of the real part (also for an imaginary part) of membership degree and real part (imaginary part) of the non-membership degree is belonging to [0, 1]. Since the Cq-ROFS is an extension of CIFS and CPFS to deal with real-life problems, the complex q-ROFS becomes a powerful tool for solving uncertain problems. Garg et al. (2021a) explored the aggregation operators under Cq-ROFS. Afterward, the Cq-ROFS is usually able to deal with much higher degrees of uncertainty. The application of the Cq-ROFS is more exquisitely than that of CIFS and CPFS to describe the ambiguity in real-decision activities.

Since the Cq-ROFS’ appearance, scholars and various researchers have paid great attention to multiple attribute group decision-making problems under complex q-rung orthopair fuzzy circumstances. For example, Liu et al. (2020b, 2020c, 2021a, 2021b, 2021c, 2020a) propounded distance measures and cosine similarity measures for Cq-ROFS and proposed the Schweizer–Sklar operator, Muirhead mean operator, BM operator, and Einstein operators to Cq-ROFS. Mahmood and Ali (2021a, 2021b) defined complex q-rung orthopair fuzzy Hamacher aggregation operators for Cq-ROFS. Yuan et al. (2020) propounded some fuzzy 2-tuple linguistic Maclaurin symmetric mean operators based on the Cq-ROFS. Garg et al. (2021c) developed complex interval-valued q-rung orthopair fuzzy set. Zeeshan and Mahmood (2020) extended Maclaurin symmetric mean operators in complex q-rung orthopair fuzzy environment. Garg et al. (2021b) defined complex q-rung orthopair Hamy Mean Operators.

Frank t-norm and t-conorm (1979) are generalizations of probabilistic and Lukasiewicz t-norm and t-conorm. The Frank t-norm and t-conorm are more flexible and adequate to deal with practical decision making since they have a parameter. In recent years, the Frank operator has received increased attention from the scientific community, and it has achieved a lot of researcher results on different fuzzy sets. Some operations on various fuzzy sets have been introduced based on the Frank T-norm and S-norm, such as intuitionistic Frank operations (Xia et al. 2012; Zhang et al. 2015), single-valued neutrosophic Frank operations (Garg 2016), hesitant Frank operations (Qin et al. 2016), dual-hesitant fuzzy Frank operations (Wang et al. 2016), interval intuitionistic linguistic Frank aggregation operators (Du and Hou 2018), Frank prioritized Bonferroni mean operations (Li et al. 2018), linguistic intuitionistic fuzzy Frank weighted Heronian mean operator (Peng et al. 2018), interval-valued probabilistic hesitant fuzzy aggregation operators (Yahya et al. 2021), triangular interval type-2 fuzzy Frank operations (Qin and Liu 2014), interval-valued
Pythagorean Frank power operations (Yang et al. 2018), picture fuzzy Frank weighted averaging operator (Seikh and Mandal 2021a). Some researchers also studied various mathematical properties of the Frank t-norms (Liu et al. 2018; Xing et al. 2019; Zhou et al. 2019; Seikh and Mandal 2021b).

To the best of our knowledge, Frank t-norm and t-conorm are a generalization of algebraic triangular t-norm and t-conorm and Lukasiiewicz t-norm and t-conorm. Furthermore, compared with algebraic operators, Einstein operators and many other operators, the application of Frank operator rule is more flexible and robust. The Frank t-norm and t-conorm involve a parameter, which make them more robust and resilient than other triangular norms in the procedure of fusion of information. Frank operator has less calculation quantity than many operators such as Heronian Mean (HM) operator and Bonferroni mean (BM) operator, and Frank operator is only one kind of t-norm satisfying the compatibility rule.

The primary motivations and contributions of this article are as follows:

1. Since the Cq-ROFS is an extension of CIFS and CPFS, it has more proficient, extensive and reliable than existing notions like CIFS and CPFS to deal with unpredictable information in the decision-making process. Further, up to now, there is not any research on the combination of complex q-rung orthopair fuzzy circumstance with Frank aggregation operator. So, it is very significant to extend the Frank operator to solve MCDM problems under complex q-rung orthopair fuzzy circumstance.

2. The Frank operator based on CQROFS is a meaningful concept to cope with two-dimensional information in a single set. So the goals of this article are to present complex q-rung orthopair fuzzy Frank weighted averaging (Cq-ROFFWA) operator, complex q-rung orthopair fuzzy Frank weighted geometric (Cq-ROFFWG) operator and complex q-rung orthopair fuzzy Frank ordered weighted averaging (Cq-ROFOWA) operator.

3. To discuss some properties of these operators, such as the monotonicity, idempotency and boundedness, the relationships between these operators are put forward.

4. To design two different innovative approaches based on Cq-ROFFWA operator and Cq-ROFFWG operator.

5. To propound two example of its application, which validate the feasibility and reliability of the proposed methods. Furthermore, comparing the propounded methods with the existing methods, we will conclude that the proposed methods are superior to the existing methods, and we will show that the Frank t-conorm and t-norm aggregation operators make the aggregation process more flexible.

The article is organized as below. We review some basic concepts and definitions of the existing sets in Sect. 2. In Sect. 3, we develop the operational laws of Frank t-norm and t-conorm under complex q-rung orthopair fuzzy circumstances. In Sect. 4, we propose some complex q-rung orthopair fuzzy Frank operators. In Sect. 6, we define two approaches for MADM with complex q-rung orthopair fuzzy information based on the developed operators. In Sect. 7, a numerical example is introduced to show the efficiency and a contrastive study is given to certificate the merits of the proposed method. Some conclusion remarks are listed in Sect. 8.

2 Preliminaries

Definition 1 (Liu et al. 2020a, b) Cq-ROFS is defined as follows:

\[ C = \big( (x, (u_C(x), v_C(x))) / x \in X \big), \]

where \( u_C(x) : X \rightarrow \{ w_1 : w_1 \in C, |w_1| \leq 1 \} \), \( v_C(x) : X \rightarrow \{ w_2 : w_2 \in C, |w_2| \leq 1 \} \), such that \( u_C(x) = a_1 + ib_1 \), \( v_C(x) = a_2 + ib_2 \), provided that \( 0 \leq |w_1|^{q} + |w_2|^{q} \leq 1 \), or \( u_C(x) = G_C(x)e^{i2\pi W_{G_C}(x)} \) and \( v_C(x) = H_C(x)e^{i2\pi W_{H_C}(x)} \), where \( 0 \leq G_C(x) + H_C(x) \leq 1 \) and \( 0 \leq W_{G_C}(x) + W_{H_C}(x) \leq 1 \). The complex Cq-RFN is given as \( C = (G_C(x)e^{i2\pi W_{G_C}(x)}, H_C(x)e^{i2\pi W_{H_C}(x)}) \).

Definition 2 (Liu et al. 2020a, b) For any Cq-ROFN \( C_1 = (G_1e^{i2\pi W_{G_1}}, H_1e^{i2\pi W_{H_1}}) \), the score function and accuracy function are proposed as below:

\[ S(C_1) = \frac{1}{2} \left( G_1^{q} + W_{G_1}^{q} - H_1^{q} - W_{H_1}^{q} \right), \quad (1) \]

\[ H(C_1) = \frac{1}{2} \left( G_1^{q} + W_{G_1}^{q} + H_1^{q} + W_{H_1}^{q} \right), \quad (2) \]

where \( S(C_1) \in [-1, 1] \) and \( H(C_1) \in [0, 1] \).

Definition 3 (Liu et al. 2020a, b) For any two Cq-ROFSs \( C_1 \) and \( C_2 \), the following rules are used:

1. If \( S(C_1) > S(C_2) \), then \( C_1 > C_2 \);
2. If \( S(C_1) = S(C_2) \), then \( \left\{ \begin{array}{ll}
(1) & \text{If } H(C_1) > H(C_2), \text{ then } C_1 > C_2; \\
(2) & \text{If } H(C_1) = H(C_2), \text{ then } C_1 = C_2.
\end{array} \right. \)

Definition 4 (Liu et al. 2020a, b) For any two Cq-ROFNs \( C_1 = (G_1e^{i2\pi W_{G_1}}, H_1e^{i2\pi W_{H_1}}) \) and \( C_2 = (G_2e^{i2\pi W_{G_2}}, H_2e^{i2\pi W_{H_2}}) \), with \( \lambda \geq 1 \), then

\[ \begin{aligned}
\end{aligned} \]
Theorem 1 For any three Cq-ROFNs \( C_1 = (G_1 e^{2\pi W_{G_1}}, H_1 e^{2\pi W_{H_1}}) \), \( C_2 = (G_2 e^{2\pi W_{G_2}}, H_2 e^{2\pi W_{H_2}}) \) and \( C_3 = (G e^{2\pi W_{G}}, H e^{2\pi W_{H}}) \), with any \( k, k_1, k_2 > 0 \), then

\[
(1) \quad C_1 \oplus C_2 = (G_1^k + G_2^k - G_1^k G_2^k)^\frac{1}{k} e^{i\frac{2\pi}{k}\left(W_{G_1} W_{G_2} - W_{G_1}^\gamma W_{G_2}^\gamma + W_{G_1}^\gamma W_{G_2}^\gamma - W_{G_1}^\gamma W_{G_2}^\gamma\right)}; \\
(2) \quad C_1 \otimes C_2 = (G_1 G_2 e^{2\pi W_{G_1} W_{G_2}}, (H_1^k + H_2^k - H_1^k H_2^k)^\frac{1}{k} e^{i\frac{2\pi}{k}\left(W_{H_1}^\gamma W_{H_2}^\gamma - W_{H_1}^\gamma W_{H_2}^\gamma + W_{H_1}^\gamma W_{H_2}^\gamma - W_{H_1}^\gamma W_{H_2}^\gamma\right)}; \\
(3) \quad \lambda C_1 = \left(1 - (1 - G_1^k)^\frac{1}{k}\right) e^{i\frac{2\pi}{k}\left(W_{G_1} W_{G_1}^\gamma - W_{G_1}^\gamma W_{G_1}^\gamma + W_{G_1}^\gamma W_{G_1}^\gamma - W_{G_1}^\gamma W_{G_1}^\gamma\right)}; \\
(4) \quad C_i^k = (G_i e^{2\pi W_{G_i}}, (1 - (1 - H_i^k)^\frac{1}{k}) e^{i\frac{2\pi}{k}\left(1 - W_{H_i}^\gamma W_{H_i}^\gamma + W_{H_i}^\gamma W_{H_i}^\gamma - W_{H_i}^\gamma W_{H_i}^\gamma\right)}).
\]

The proof of Theorem 1 is given in “Appendix.”

Theorem 2 For any three Cq-ROFNs \( C_1 = (G_1 e^{2\pi W_{G_1}}, H_1 e^{2\pi W_{H_1}}) \) and \( C_2 = (G_2 e^{2\pi W_{G_2}}, H_2 e^{2\pi W_{H_2}}) \), then

\[
(1) \quad C_1 \lor C_2 = (C_1 \land C_2)^c; \\
(2) \quad C_1 \odot C_2 = (C_1 \otimes C_2)^c; \\
(3) \quad C_i \otimes C_j = (C_i \otimes C_j)^c; \\
(4) \quad (C_1 \lor C_2) \odot (C_1 \land C_2) = C_1 \odot C_2; \\
(5) \quad (C_1 \lor C_2) \otimes (C_1 \land C_2) = C_1 \otimes C_2; \\
(6) \quad (C_1 \lor C_2) \land C_3 = (C_1 \land C_3) \lor (C_1 \land C_3); \\
(7) \quad (C_1 \lor C_2) \odot C_3 = (C_1 \odot C_3) \lor (C_1 \odot C_3); \\
(8) \quad (C_1 \land C_2) \land C_3 = (C_1 \land C_3) \land (C_1 \land C_3); \\
(9) \quad (C_1 \land C_2) \odot C_3 = (C_1 \odot C_3) \land (C_1 \odot C_3).
\]

The proof of Theorem 2 can be proven from the Frank operational laws of Cq-ROFNs, so it is omitted here.

3.1 Complex q-rung orthopair fuzzy Frank aggregation operators

In this subsection, the complex q-rung orthopair fuzzy Frank aggregation operators are introduced, and the relevant properties are given.

Definition 5 Let \( C_j = (G_j e^{2\pi W_{G_j}}, H_j e^{2\pi W_{H_j}}) \) \((j = 1, 2, ..., n)\) be a family of Cq-ROFNs, \( w_j \) be the weight of \( C_j \), meeting \( w_j \in [0, 1] \), and \( \sum_{j=1}^{n} w_j = 1 \). The complex q-rung orthopair fuzzy Frank weighted averaging (Cq-ROFWA) operator is initiated by

\[
\text{Cq - ROFWA}(C_1, C_2, ..., C_n) = \bigoplus_{j=1}^{n} w_j C_j
\]

\[
= \left(1 - \log_\gamma \left(1 + \prod_{j=1}^{n} \left(\tau(H_j)^{y} - 1\right)^{w_j}\right)\right)^\frac{1}{k} e^{i\frac{2\pi}{k}\left(1 - \log_\gamma \left(1 + \prod_{j=1}^{n} \left(\tau(H_j)^{y} - 1\right)^{w_j}\right)\right)}.
\]

The aggregated value by using Definition 5 is still a Cq-ROFN.

Proof By using mathematical induction to prove Eq. (3).
Case 1: if \( n = 2 \), then

\[
\begin{align*}
\text{w}_1 C_1 &= \left(1 - \log_\tau \left(1 + \frac{\left(\frac{\tau - 2\tau^n}{\tau - 1}\right)^{w_1}}{(\tau - 1)^{w_1}} \right)\right) + 2\pi \left(1 - \log_\tau \left(1 + \frac{\left(\frac{\tau - 2\tau^n}{\tau - 1}\right)^{w_2}}{(\tau - 1)^{w_2}} \right)\right) \\
\text{w}_2 C_2 &= \left(1 - \log_\tau \left(1 + \frac{\left(\frac{\tau - 2\tau^n}{\tau - 1}\right)^{w_1}}{(\tau - 1)^{w_1}} \right)\right) + 2\pi \left(1 - \log_\tau \left(1 + \frac{\left(\frac{\tau - 2\tau^n}{\tau - 1}\right)^{w_2}}{(\tau - 1)^{w_2}} \right)\right)
\end{align*}
\]

Cq - ROFFWA\((C_1, C_2, \ldots, C_n) = w_1 C_1 + w_2 C_2\)

\[
\begin{align*}
= \left(1 - \log_\tau \left(1 + \frac{\left(\frac{\tau - 2\tau^n}{\tau - 1}\right)^{w_1}}{(\tau - 1)^{w_1}} \right)\right) + 2\pi \left(1 - \log_\tau \left(1 + \frac{\left(\frac{\tau - 2\tau^n}{\tau - 1}\right)^{w_2}}{(\tau - 1)^{w_2}} \right)\right) \\
&= \left(1 - \log_\tau \left(1 + \frac{\left(\frac{\tau - 2\tau^n}{\tau - 1}\right)^{w_1}}{(\tau - 1)^{w_1}} \right)\right) + 2\pi \left(1 - \log_\tau \left(1 + \frac{\left(\frac{\tau - 2\tau^n}{\tau - 1}\right)^{w_2}}{(\tau - 1)^{w_2}} \right)\right)
\end{align*}
\]
By Theorem 1, we prove the result of $w_1C_1 \oplus w_2C_2$ is also Cq-ROFN. For $n = 2$, the result is kept.

Case 2 Next, check for $k = m$, then Eq. (3) is hold obviously.

Cq - ROFFWA($C_1, C_2, ..., C_m$)

\[
\begin{align*}
\left( 1 - \log_\tau \left( 1 + \frac{(\tau - G_j)^{w_1} \left( \tau - G_j \right)^w}{(\tau - 1) \sum_{j=1}^m w_j} \right) \right)^{\frac{1}{\tau}} & \cdot 2 \pi \left( 1 - \log_\tau \left( 1 + \frac{(w_j)^y}{(w_j)^y} \right) \right)^{\frac{1}{\tau}} \\
= & \left( 1 - \log_\tau \left( 1 + \prod_{j=1}^m \left( \frac{\tau - G_j}{\tau - 1} \right)^{w_1} \right) \right)^{\frac{1}{\tau}} \cdot 2 \pi \left( 1 - \log_\tau \left( 1 + \prod_{j=1}^m \left( \frac{\tau - G_j}{\tau - 1} \right)^{w_1} \right) \right)^{\frac{1}{\tau}} \\
& \left( \log_\tau \left( 1 + \prod_{j=1}^m \frac{\left( \tau - G_j \right)^{w_1}}{\tau - 1} \right) \right)^{\frac{1}{\tau}} \cdot 2 \pi \left( \log_\tau \left( 1 + \prod_{j=1}^m \frac{\left( \tau - G_j \right)^{w_1}}{\tau - 1} \right) \right)^{\frac{1}{\tau}} \\
\end{align*}
\]
Case 3 lastly, we check for \( k = m + 1 \), such that

\[
Cq - \text{ROFFWA}(C_1, C_2, \ldots, C_m, C_{m+1})
\]

\[
= \left(1 - \log_{e} \left(1 + \prod_{j=1}^{m} \frac{1 - G_j^q - 1}{\frac{1}{\tau} \sum w_{j-1}}\right)\right)^{\frac{1}{q}} i2\pi \left(1 - \log_{e} \left(1 + \prod_{j=1}^{m} \frac{1 - (w_{j})^q - 1}{\frac{1}{\tau} \sum w_{j-1}}\right)\right) \frac{1}{q},
\]

\[
\left(1 - \log_{e} \left(1 + \prod_{j=1}^{m} \frac{1 - (w_{j})^q - 1}{\frac{1}{\tau} \sum w_{j-1}}\right)\right)^{\frac{1}{q}} i2\pi \left(1 - \log_{e} \left(1 + \prod_{j=1}^{m} \frac{1 - (w_{j})^q - 1}{\frac{1}{\tau} \sum w_{j-1}}\right)\right) \frac{1}{q},
\]

\[
\left(1 - \log_{e} \left(1 + \prod_{j=1}^{m} \frac{1 - (w_{j})^q - 1}{\frac{1}{\tau} \sum w_{j-1}}\right)\right)^{\frac{1}{q}} i2\pi \left(1 - \log_{e} \left(1 + \prod_{j=1}^{m} \frac{1 - (w_{j})^q - 1}{\frac{1}{\tau} \sum w_{j-1}}\right)\right) \frac{1}{q},
\]

\[
\left(1 - \log_{e} \left(1 + \prod_{j=1}^{m} \frac{1 - (w_{j})^q - 1}{\frac{1}{\tau} \sum w_{j-1}}\right)\right)^{\frac{1}{q}} i2\pi \left(1 - \log_{e} \left(1 + \prod_{j=1}^{m} \frac{1 - (w_{j})^q - 1}{\frac{1}{\tau} \sum w_{j-1}}\right)\right) \frac{1}{q}.
\]
For \( n = k + 1 \), the result is kept. Hence Eq. (3). It is kept for all \( n \).

**Theorem 3** (Idempotency) Let \( C_j (j = 1, 2, \ldots, n) \) be a set of be a family of \( C_q\)-ROFNs, if all \( q_i = q_0 \)(\( j = 1, 2, \ldots, n \)), then,
\[
\text{Cq - ROFFWA}(C_1, C_2, \ldots, C_n) = C_0
\]

**Proof** We know that \( C_0 = (G_0 e^{2\pi W_{C_0}}, H_0 e^{2\pi W_{H_0}}) \)

\[
\text{Cq - ROFFWA}(C_1, C_2, \ldots, C_n) = \left( \left( 1 - \log \left( 1 + \prod_{j=1}^{n} (\tau^{1-(G_0)^{-q}} - 1)^{w_j} \right) \right)^{\frac{1}{q}} \right),
\]

\[
= \left( 1 - \log \left( 1 + \prod_{j=1}^{n} (\tau^{1-(W_{C_0})} - 1)^{w_j} \right) \right)^{\frac{1}{q}}
\]

\[
= (G_0 e^{2\pi W_{C_0}}, H_0 e^{2\pi W_{H_0}}) = C_0.
\]

---

**Theorem 4** (boundedness) Let \( C_j (j = 1, 2, \ldots, n) \) be a set of be a family of \( C_q\)-ROFNs, \( C_0 = \min \{ C_j \} \), \( C_\ast = \max \{ C_j \} \), \( H_\ast = \min \{ H_j \} \), \( H_\ast = \max \{ H_j \} \), \( W_{G_\ast} = \min \{ W_{G_j} \} \), \( W_{G_\ast} = \max \{ W_{G_j} \} \), \( W_{H_\ast} = \min \{ W_{H_j} \} \), \( W_{H_\ast} = \max \{ W_{H_j} \} \), then

\( C_\ast \leq \text{Cq - ROFFWA}(C_1, C_2, \ldots, C_n) \leq C_0 \).

**Proof** Since \( C_\ast = \min \{ C_j \} \), \( C_\ast = \max \{ C_j \} \), \( H_\ast = \min \{ H_j \} \), \( H_\ast = \max \{ H_j \} \),
\begin{align*}
W_{G_\ast} & = \min \{ W_{G_j} \}, W_{G_\ast} = \max \{ W_{G_j} \}, \\
W_{H_\ast} & = \min \{ W_{H_j} \}, W_{H_\ast} = \max \{ W_{H_j} \},
\end{align*}

Therefore, we can obtain

\[
W_{G_\ast} = \left( 1 - \log \left( 1 + \prod_{j=1}^{n} (\tau^{1-W_{G_\ast}} - 1)^{w_j} \right) \right)^{\frac{1}{q}}
\]

\[
\leq \left( 1 - \log \left( 1 + \prod_{j=1}^{n} (\tau^{1-W_{G_\ast}} - 1)^{w_j} \right) \right)^{\frac{1}{q}}
\]

\[
= W_{G_\ast}
\]

\[
H_\ast = \left( \log \left( 1 + \prod_{j=1}^{n} (\tau^{H_\ast} - 1)^{w_j} \right) \right)^{\frac{1}{q}}
\]

\[
\leq \left( \log \left( 1 + \prod_{j=1}^{n} (\tau^{H_\ast} - 1)^{w_j} \right) \right)^{\frac{1}{q}}
\]

\[
= H_\ast
\]

\[
W_{H_\ast} = \left( \log \left( 1 + \prod_{j=1}^{n} (\tau^{W_{H_\ast}} - 1)^{w_j} \right) \right)^{\frac{1}{q}}
\]

\[
\leq \left( \log \left( 1 + \prod_{j=1}^{n} (\tau^{W_{H_\ast}} - 1)^{w_j} \right) \right)^{\frac{1}{q}}
\]

\[
= W_{H_\ast}
\]
Because

\[ \text{Proof} \] Because \( \tau \to 1 \), \( \prod_{j=1}^{n} (\tau^{1-G_j^q} - 1)^{w_j} \to 0 \),

Therefore, we can obtain that

\[
\left( 1 - \frac{\ln \left( 1 + \prod_{j=1}^{n} (\tau^{1-G_j^q} - 1)^{w_j} \right)}{\ln \tau} \right)^{\frac{1}{\tau}} = \left( 1 - \frac{\prod_{j=1}^{n} (\tau^{1-G_j^q} - 1)^{w_j}}{\ln \tau} \right)^{\frac{1}{\ln \tau}}
\]

According to Taylor’s expansion, we can get

\[
\tau^{1-G_j^q} = 1 + \left( 1 - G_j^q \right) \ln \tau + \frac{1}{2} \left( 1 - G_j^q \right) \left( \ln \tau \right)^2 + \ldots
\]
\( \tau^{H^*} - 1 = H^* \ln \tau + O(\ln \tau) \)

\[
\left( \prod_{j=1}^{n} \left( \frac{\tau^{H^*_j} - 1}{\ln \tau} \right)^{w_j} \right)^{\frac{1}{\tau}} = \left( \frac{\prod_{j=1}^{n} \left( H^*_j \ln \tau \right)^{w_j}}{\ln \tau} \right)^{\frac{1}{\tau}} = \prod_{j=1}^{n} \left( H_j \right)^{w_j}
\]

Similarly, we have

\[
\tau \rightarrow 1,
\]

\[
e^{i2\pi \left( \log\left( 1 + \prod_{j=1}^{n} (\tau^{-(H^*_j)^y} - 1)^{w_j} \right) \right)} \rightarrow e^{i2\pi \sum_{j=1}^{n} (w_j)^y}.
\]

Therefore, we can obtain

\[
\lim_{\tau \rightarrow 1} \text{Cq - ROFFWA}(C_1, C_2, \ldots C_n)
\]

\[
= \lim_{\tau \rightarrow 1} \left( 1 - \log\left( 1 + \prod_{j=1}^{n} (\tau^{-(G^*_j)^y} - 1)^{w_j} \right) \right)^{\frac{i}{\tau}} e^{i2\pi \left( 1 - \log\left( 1 + \prod_{j=1}^{n} (\tau^{-(G^*_j)^y} - 1)^{w_j} \right) \right)} \frac{1}{\tau} \log\left( 1 + \prod_{j=1}^{n} (\tau^{-(G^*_j)^y} - 1)^{w_j} \right)
\]

\[
= \lim_{\tau \rightarrow 1} \left( 1 - \prod_{j=1}^{n} \frac{\left( \tau^{-(G^*_j)^y} - 1 \right)^{w_j}}{\ln \tau} \right)^{\frac{i}{\tau}} e^{i2\pi \left( 1 - \prod_{j=1}^{n} \frac{\left( \tau^{-(G^*_j)^y} - 1 \right)^{w_j}}{\ln \tau} \right)} \frac{1}{\tau} \prod_{j=1}^{n} \left( \tau^{-(H^*_j)^y} - 1 \right)^{w_j}
\]

\[
= \left( 1 - \prod_{j=1}^{n} \left( (1 - G^*_j)^y \right) \right)^{\frac{i}{\tau}} \frac{1}{\tau} \prod_{j=1}^{n} \left( (1 - W^*_j)^y \right)^{W_j} \frac{1}{\tau} \prod_{j=1}^{n} \left( H_j \right)^{W_j} \frac{1}{\tau} \prod_{j=1}^{n} \left( W_j \right)^{W_j}
\]

2. When \( \tau \rightarrow +\infty \), the Cq - ROFFWA operator is reduced to a tradition arithmetic weighted average operator, that is:

\[
\lim_{\tau \rightarrow +\infty} \text{Cq - ROFFWA}
\]

\[
= \left( \sum_{j=1}^{n} w_j H_j \right)^{\frac{i}{\tau}} e^{i2\pi \left( \sum_{j=1}^{n} w_j H_j \right)^y} \left( \sum_{j=1}^{n} w_j H_j \right)^{\frac{i}{\tau}} e^{i2\pi \left( \sum_{j=1}^{n} w_j H_j \right)^y}
\]

---

Complex q-rung orthopair fuzzy Frank aggregation operators and their application to multi-attribute...
Based on the L’Hospital’s rule, we have
Theorem 7 Let \( C_j(j = 1, 2, ..., n) \) be a set of be a family of Cq-ROFNs, \( w = (w_1, w_2, ..., w_n)^T \) is the weight vector of \( C_j(j = 1, 2, ..., n) \), and \( 0 \leq w_j \leq 1 \). At the same time, \( \sum_{j=1}^{n} w_j = 1 \), then

\[
C_q - \text{ROFFWA}(C_1 \oplus C, C_2 \oplus C, ..., C_n \oplus C) = C_q - \text{ROFFWA}(C_1, C_2, ..., C_n) + \alpha \quad (\alpha > 0)
\]

The proof of Theorem 7 is given in “Appendix.”

Theorem 8 Let \( C_j(j = 1, 2, ..., n) \) be a set of be a family of Cq-ROFNs, \( w = (w_1, w_2, ..., w_n)^T \) is the weight vector of \( C_j(j = 1, 2, ..., n) \), and \( 0 \leq w_j \leq 1 \). At the same time, \( \sum_{j=1}^{n} w_j = 1 \), then

\[
C_q - \text{ROFFWA}(k \cdot C_1 \oplus C, k \cdot C_2 \oplus C, ..., k \cdot C_n \oplus C) = k \cdot C_q - \text{ROFFWA}(C_1, C_2, ..., C_n) + \alpha \quad (\alpha > 0)
\]

The proof of Theorem 8 is given in “Appendix.”

Theorem 9 Let \( C_j(j = 1, 2, ..., n) \) be a set of be a family of Cq-ROFNs, \( w = (w_1, w_2, ..., w_n)^T \) is the weight vector of \( C_j(j = 1, 2, ..., n) \), and \( 0 \leq w_j \leq 1 \). At the same time, \( \sum_{j=1}^{n} w_j = 1 \) and \( k > 0 \), then

\[
C_q - \text{ROFFWA}(kC_1, kC_2, ..., kC_n) = k \cdot C_q - \text{ROFFWA}(C_1, C_2, ..., C_n)
\]

The proof of Theorem 9 is given in “Appendix.”

Definition 6 Let \( C_j = (G_j e^{j \alpha w_j}, H_j e^{j \alpha w_j}) \) \( (j = 1, 2, ..., n) \) be a family of Cq-ROFNs, \( w_j \) be the weight of \( C_j \), meeting \( w_j \in [0, 1] \), and \( \sum_{j=1}^{n} w_j = 1 \). The complex q-rung orthopair fuzzy Frank weighted geometric (Cq-ROFFWG) operator is initiated by
\[ C_q - \text{ROFFWG}(C_1, C_2, \ldots, C_n) = \bigotimes_{j=1}^{n} C_j^{w_j} \]
\[ = \left( \log \left( 1 + \prod_{j=1}^{n} \left( \left( \tau(j)_y^y - 1 \right)^{w_j} \right) \right) \right)^{\frac{1}{2} \pi i / \lambda} \left( \log \left( 1 + \prod_{j=1}^{n} \left( \left( \tau(w_j)_y^y - 1 \right)^{w_j} \right) \right) \right)^{\frac{1}{2} \pi i / \lambda} \]
\[ = \left( \log \left( 1 + \prod_{j=1}^{n} \left( \left( \tau(j)_y^y - 1 \right)^{w_j} \right) \right) \right)^{\frac{1}{2} \pi i / \lambda} \left( \log \left( 1 + \prod_{j=1}^{n} \left( \left( \tau(w_j)_y^y - 1 \right)^{w_j} \right) \right) \right)^{\frac{1}{2} \pi i / \lambda} \]

In particular, if \( w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T \), the \( C_q - \text{ROFFWG} \) operator will be simplified to \( C_q - \text{ROFFG} \) operator, that is \( C_q - \text{ROFFG}(C_1, C_2, \ldots, C_n) = \bigotimes_{j=1}^{n} C_j \)

\[ = \left( \log \left( 1 + \prod_{j=1}^{n} \left( \left( \tau(j)_y^y - 1 \right)^{w_j} \right) \right) \right)^{\frac{1}{2} \pi i / \lambda} \left( \log \left( 1 + \prod_{j=1}^{n} \left( \left( \tau(w_j)_y^y - 1 \right)^{w_j} \right) \right) \right)^{\frac{1}{2} \pi i / \lambda} \]

**Theorem 11** Let \( a_j(j = 1, 2, \ldots, n), a_j^j(j = 1, 2, \ldots, n) \) be two set of \( C_q-\text{ROFNs}, w = (w_1, w_2, \ldots, w_n)^T \) is the weight vector of \( C_j(j = 1, 2, \ldots, n) \), and \( 0 \leq w_j \leq 1 \). At the same time, \( \sum_{j=1}^{n} w_j = 1 \), then

1. \( C_q - \text{ROFFWA}(a_1^1, a_2^2, \ldots, a_n^n) = C_q - \text{ROFFWG}(a_1, a_2, \ldots, a_n)^c \)
2. \( C_q - \text{ROFFWG}(a_1^1, a_2^2, \ldots, a_n^n) = C_q - \text{ROFFWA}(a_1, a_2, \ldots, a_n)^c \)

Similar to the \( C_q - \text{ROFFWA} \) operator, the \( C_q - \text{ROFFWG} \) operator also has bounded, monotonic and idempotent properties, and we can obtain the following properties of \( C_q - \text{ROFFWG} \) operator as follows.

**Lemma 1** [73]. Let \( x_j > 0, \lambda_j > 0, j = 1, 2, \ldots, n \), and \( \sum_{j=1}^{n} \lambda_j = 1 \), then

\[ \prod_{j=1}^{n} x_j^\lambda_j \leq \prod_{j=1}^{n} \lambda_j x_j \]

with equality if and only if \( x_1 = x_2 = \cdots = x_n \).

**Theorem 12** Let \( C_j(j = 1, 2, \ldots, n) \) be a collection of \( C_q-\text{ROFNs}, w = (w_1, w_2, \ldots, w_n)^T \) is the weight vector of

\( C_j(j = 1, 2, \ldots, n) \), and \( 0 \leq w_j \leq 1 \). At the same time, \( \sum_{j=1}^{n} w_j = 1 \), then

\[ C_q - \text{ROFFWG}(C_1, C_2, \ldots, C_n) \]
\[ \leq C_q - \text{ROFFWA}(C_1, C_2, \ldots, C_n) \]

The proof can be shown in appendix.

**Theorem 13** Let \( C_j(j = 1, 2, \ldots, n) \) be a collection of \( C_q-\text{ROFNs}, \) let \( C \) is a \( C_q-\text{ROFN}, w = (w_1, w_2, \ldots, w_n)^T \) is the weight vector of \( C_j(j = 1, 2, \ldots, n) \), and \( 0 \leq w_j \leq 1 \). At the same time, \( \sum_{j=1}^{n} w_j = 1 \), then

\[ C_q - \text{ROFFWG}(C_1 \otimes C, C_2 \otimes C, \ldots, C_n \otimes C) \]
\[ = (C_q - \text{ROFFWG}(C_1, C_2, \ldots, C_n))^c \otimes C \]

**Theorem 14** Let \( C_j(j = 1, 2, \ldots, n) \) be a collection of \( C_q-\text{ROFNs}, w = (w_1, w_2, \ldots, w_n)^T \) is the weight vector of \( C_j(j = 1, 2, \ldots, n) \), and \( 0 \leq w_j \leq 1 \). At the same time, \( \sum_{j=1}^{n} w_j = 1, \lambda > 0 \), then

\[ C_q - \text{ROFFWG}(C_1^1, C_2^2, \ldots, C_n^n) \]
\[ = (C_q - \text{ROFFWG}(C_1, C_2, \ldots, C_n))^\lambda \]

**Theorem 15** Let \( C_j(j = 1, 2, \ldots, n) \) be a collection of \( C_q-\text{ROFNs}, \) and \( C \) is a \( C_q-\text{ROFN}, w = (w_1, w_2, \ldots, w_n)^T \) is the
weight vector of \( C_j(j = 1,2,\ldots,n) \), and \( 0 \leq w_j \leq 1 \). At the same time, \( \sum_{j=1}^{n} w_j = 1, \lambda > 0 \), then

\[
C_{q-ROFN} = C_{q-ROFN}(C_1 \otimes C, C_2 \otimes C, \ldots, C_n \otimes C)
\]

**Theorem 16** Let \( C_j(j = 1,2,\ldots,n) \) be a collection of \( C_{q-ROFNs} \), and \( C \) is a \( C_{q-ROFN} \), \( w = (w_1,w_2,\ldots,w_n)^T \) is the weight vector of \( C_j(j = 1,2,\ldots,n) \), and \( 0 \leq w_j \leq 1 \). At the same time, \( \sum_{j=1}^{n} w_j = 1, \lambda > 0 \), then

\[
C_{q-ROFN} = C_{q-ROFN}(C_1 \otimes C, C_2 \otimes C, \ldots, C_n \otimes C)
\]

**Theorem 17** Let \( C_j(j = 1,2,\ldots,n) \) and \( B_j(j = 1,2,\ldots,n) \) be two collections of \( C_{q-ROFNs} \), then

\[
C_{q-ROFN} = C_{q-ROFN}(C_1 \otimes B_1, C_2 \otimes B_2, \ldots, C_n \otimes B_n)
\]

By discussing different values of the parameters, the following special cases are given in the follows.

**Theorem 18** Let \( C_j(j = 1,2,\ldots,n) \) be a collection of \( C_{q-ROFNs} \), \( w = (w_1,w_2,\ldots,w_n)^T \) is the weight vector of \( C_j(j = 1,2,\ldots,n) \), and \( 0 \leq w_j \leq 1 \), then

1. If \( \tau \to 1 \), the \( C_{q-ROFN} \) operator is reduced to a complex \( q \)-rung orthopair fuzzy averaging operator, which is given as:

\[
\lim_{\tau \to 1} C_{q-ROFN} = C_{q-ROFN}(C_1, C_2, \ldots C_n)
\]

\[
= \left( \prod_{j=1}^{n} (G_j)^{w_j} e^{\frac{i\pi}{\tau} \prod_{j=1}^{n} ( \frac{1}{w_j} - 1 )^{\omega_j} \left( 1 - \prod_{j=1}^{n} \left( \frac{(1 - H_{q})^{1/\tau}}{w_j} \right) \right)^{\omega_j} } \right)^{\frac{1}{\tau}}
\]

2. If \( \tau \to +\infty \), then \( C_{q-ROFN} \) operator is reduced to the traditional arithmetic weighted average operator, which is given as:

\[
\lim_{\tau \to +\infty} C_{q-ROFN}
\]

\[
= \left( \sum_{j=1}^{n} w_j G_j^\tau \right)^{\frac{\omega_j}{\tau}} e^{\frac{i\pi}{\tau} \sum_{j=1}^{n} \left( \frac{\sum_{i=1}^{m} \omega_i}{w_j} \right)^{\omega_j} } \left( \sum_{j=1}^{n} w_j \left( H_j \right)^\tau \right)^{\frac{\omega_j}{\tau}} e^{\frac{i\pi}{\tau} \sum_{j=1}^{n} \left( \frac{\sum_{i=1}^{m} \omega_i}{w_j} \right)^{\omega_j} } \right)
\]

**3.2 Complex q-rung orthopair fuzzy Frank ordered weighted aggregation operators**

**Definition 7** Let \( C_j(j = 1,2,\ldots,n) \) be a collection of \( C_{q-ROFNs} \), and \( \omega = (\omega_1, \omega_2,\ldots,\omega_n)^T \) be the aggregation-associated vector of \( C_j(j = 1,2,\ldots,n) \), and \( \omega_i \in [0,1] \), at the same time \( \sum_{i=1}^{n} \omega_i = 1 \). Then, the complex \( q \)-rung orthopair fuzzy Frank ordered weighted average (\( C_{q-ROFFOWA} \)) operator is a mapping \( C_{q-ROFFOWA}:\Omega^n \to \Omega \), if

\[
C_{q-ROFFOWA}(C_1, C_2, \ldots C_n) = \frac{1}{n} \sum_{j=1}^{n} \omega_j C_{\sigma(j)}
\]

\[
= \left( 1 - \log_\tau \left( 1 + \prod_{j=1}^{n} \left( \frac{1 - C_{\sigma(j)}^H - 1}{\omega_j} \right) \right) \right)^{\frac{1}{\tau}} e^{\frac{i\pi}{\tau} \prod_{j=1}^{n} \left( \frac{1}{\omega_j} - 1 \right)^{\omega_j} \left( 1 - \prod_{j=1}^{n} \left( \frac{(1 - H_{q})^{1/\tau}}{\omega_j} \right) \right)^{\omega_j} }
\]

\[
= \log_\tau \left( 1 + \prod_{j=1}^{n} \left( \frac{1 - C_{\sigma(j)}^H - 1}{\omega_j} \right) \right)^{\frac{1}{\tau}} e^{\frac{i\pi}{\tau} \prod_{j=1}^{n} \left( \frac{1}{\omega_j} - 1 \right)^{\omega_j} \left( 1 - \prod_{j=1}^{n} \left( \frac{(1 - H_{q})^{1/\tau}}{\omega_j} \right) \right)^{\omega_j} }
\]
where $\Omega$ is the set of complex q-rung orthopair fuzzy variables, $C_{s(i)}$ is the $j$th largest element in $C_{j}(j = 1, 2, \ldots, n)$, and $C_{s(i-1)} \geq C_{s(i)}$; then, the aggregated value by the Cq-ROFFOWA operator is still a Cq-ROFE.

Similar to the Cq-ROFFWA operator, the Cq-ROFFOWA operator also has the properties of boundedness, idempotency and monotonicity, and the Cq-ROFFOWA operator also has the commutativity property.

**Theorem 19** (Commutativity) Let $C_{j}(j = 1, 2, \ldots, n)$ be a collection of Cq-ROFNs, and $\omega = (\omega_{1}, \omega_{2}, \ldots, \omega_{n})^{T}$ be the aggregation-associated vector of $C_{j}(j = 1, 2, \ldots, n)$, and $\omega_{i} \in [0, 1]$, at the same time $\sum_{j=1}^{n} \omega_{i} = 1$. If $(C'_{1}, C'_{2}, \ldots, C'_{n})$ is any permutation of $(C_{1}, C_{2}, \ldots, C_{n})$, then

$$C_{j} = C_{j} - \text{ROFFOWA}(C_{j1}, C_{j2}, \ldots, C_{jn}), \quad j = 1, 2, 3, \ldots, m$$

or

$$C_{j} = C_{j} - \text{ROFFWG}(C_{j1}, C_{j2}, \ldots, C_{jn}), \quad j = 1, 2, 3, \ldots, m$$

**Step 3** Use Eq. (1) to calculate the score value of $C_{j}(j = 1, 2, \ldots, m)$ obtained in Step (2).

**Step 4** Rank the feasible alternatives $A_{i}(i = 1, 2, \ldots, m)$ based on their score value. If there is equal between two score values $S(a_{i})$ and $S(a_{k})$, then we need to compute the degree of accuracy values $H(a_{i})$ and $H(a_{k})$ of the alternatives $A_{i}$ and $A_{k}$ ($i, k = 1, 2, \ldots, m$), respectively, by Eq. (2), and select the best one.

**Step 5 End.**

### 5 Applications based on Frank aggregation operators for Cq-ROFNs

#### 5.1 An application for emergency management system

**Numerical Example 1:** In order to minimize the impact of public health emergencies on the society and the country, strengthening emergency management capacity and effectively controlling the disaster effect of public health emergencies are important issues to all countries. How to evaluate the current public crisis emergency response capacity and optimize the emergency management system have always been the focus of various countries.

This example is based on Cq-ROFNs which is adapted from Ref Liu et al. (2021c), Du et al. (2017). There are five possible emerging technology companies in a panel; $A_{i}(i = 1, 2, 3, 4, 5)$, the experts need to give an evaluation according to four attributes: (1)$C_{1}$ is the emergency forecasting capability; (2)$C_{2}$ is the emergency process capability; (3)$C_{3}$ is the after-disaster loss evaluation capability; (4)$C_{4}$ is the emergency support capability; (5)$C_{5}$ is the after-disaster reconstruction capability. The important degree of the attribute is $\omega = (0.28, 0.35, 0.16, 0.21)^{T}$. The assessment values of each alternative are given in the form of Cq-ROFNs. Then the proposed approach is utilized in the following matrix (Table 1):

1. The method of using the Cq-ROFFWA operator in Eq. (5) to aggregate all complex q-rung orthopair fuzzy assessment values of the alternative $A_{i}(i = 1, 2, 3, 4, 5)$, and we choose $q = 1.5$ and $\tau = 3$ to study the final ranking results, which is shown in Table 2.

   Hence, the best alternative is $A_{3}$, which represents the emerging technology company (Table 3).

### 6 The method of using the Cq-ROFFWA or Cq-ROFFWG operator

Utilize the Cq-ROFFWA operator or Cq-ROFFWG operator in Eqs. (5) and (6) to aggregate all complex q-rung orthopair fuzzy assessment values of the alternative $A_{i}(i = 1, 2, 3, 4, 5)$ on the attributes $C_{j}(j = 1, 2, 3, 4)$ shown in Table 1 into the overall assessment values $h_{i}(i = 1, 2, 3, 4)$. In order to check the effect of the parameters $q$ and $\tau$ for this example, we choose different $q$...
Table 1 Complex q-rung orthopair fuzzy decision matrix $R = (a_{ij})_{n \times n} $

|   | $C_1$                          | $C_2$                          | $C_3$                          | $C_4$                          |
|---|--------------------------------|--------------------------------|--------------------------------|--------------------------------|
| $A_1$ | $(0.7e^{2e(0.65)}, 0.2e^{2e(0.3)})$ | $(0.6e^{2e(0.6)}, 0.3e^{2e(0.31)})$ | $(0.8e^{2e(0.5)}, 0.2e^{2e(0.11)})$ | $(0.73e^{2e(0.6)}, 0.25e^{2e(0.31)})$ |
| $A_2$ | $(0.6e^{2e(0.66)}, 0.21e^{2e(0.32)})$ | $(0.7e^{2e(0.71)}, 0.2e^{2e(0.33)})$ | $(0.83e^{2e(0.7)}, 0.3e^{2e(0.2)})$ | $(0.75e^{2e(0.3)}, 0.27e^{2e(0.4)})$ |
| $A_3$ | $(0.4e^{2e(0.68)}, 0.22e^{2e(0.35)})$ | $(0.41e^{2e(0.68)}, 0.3e^{2e(0.4)})$ | $(0.45e^{2e(0.8)}, 0.4e^{2e(0.5)})$ | $(0.44e^{2e(0.8)}, 0.28e^{2e(0.41)})$ |
| $A_4$ | $(0.3e^{2e(0.64)}, 0.26e^{2e(0.31)})$ | $(0.5e^{2e(0.65)}, 0.31e^{2e(0.2)})$ | $(0.6e^{2e(0.6)}, 0.3e^{2e(0.34)})$ | $(0.5e^{2e(0.8)}, 0.32e^{2e(0.4)})$ |
| $A_5$ | $(0.73e^{2e(0.67)}, 0.3e^{2e(0.35)})$ | $(0.8e^{2e(0.65)}, 0.45e^{2e(0.2)})$ | $(0.86e^{2e(0.7)}, 0.2e^{2e(0.41)})$ | $(0.76e^{2e(0.7)}, 0.29e^{2e(0.51)})$ |

Table 2 When $\tau = 3, q = 1.5$, aggregation values of the Cq-ROFNs

| Method | Getting values |
|-------|---------------|
| $A_1 = Cq$—ROFFWA(C11, C12, ..., C1n) | $(0.69529e^{2e(0.66083)}, 0.24194e^{2e(0.26178)})$ |
| $A_2 = Cq$—ROFFWA(C21, C22, ..., C2n) | $(0.71346e^{2e(0.63352)}, 0.23070e^{2e(0.51333)})$ |
| $A_3 = Cq$—ROFFWA(C31, C32, ..., C3n) | $(0.42029e^{2e(0.73058)}, 0.28455e^{2e(0.40195)})$ |
| $A_4 = Cq$—ROFFWA(C41, C42, ..., C4n) | $(0.47127e^{2e(0.68224)}, 0.29565e^{2e(0.28599)})$ |
| $A_5 = Cq$—ROFFWA(C51, C52, ..., C5n) | $(0.78165e^{2e(0.67476)}, 0.32359e^{2e(0.32251)})$ |

and $\tau$ a to study the final ranking results of this example, which is shown below.

(1) Effect of parameter $q$.

In this subsection, we examine the effect of the parameter $\delta$ on the ranking results. We obtained the ranking results for different values of the parameter $\delta$, as shown in Table 4 ($\tau=3$). As shown in Tables 4 and 5, the parameter $q$ affected the ranking results. However, the best alternative did not change.

(2) Effect of parameter $\tau$.

In this subsection, we examine the effect of the parameter $q$ on the ranking results. We obtained the ranking results for different values of $q$, as shown in Tables 6 and 7 ($q = 4$). As indicated in Tables 6 and 7, the parameter $q$ affected the ranking results. However, the best alternative did not change.

Further, when $\eta = q = 4$, we select different $\tau$ to study the ranking of enterprises, as shown in Tables 6 and 7.

From Tables 4, 5, 6 and 7, we can find that the given approaches like Cq - ROFFWA operator, Cq - ROFWG operator is giving the same ranking values, and $A_5$ is found to be the best one.

7 Comparison analyses

To elaborate the effectiveness and practicability of the created method, we give a comparative analysis of the developed operators with several previous decision methods, including Mahmood’ approach (2021) based on Hamacher aggregation operators for Cq - ROFS, Liu’ approach (2021c) is based on Einstein operator for Cq - ROFS, Liu’ approach (2020b) for Cq - ROFS and Garg’ approach (2021b) based on Hamy Mean Operators for Cq - ROFS, and we will discuss this example in the following three parts.

(1) Cq-ROFHWA operator and Cq-ROFHWG operator.

To facilitate the comparative analysis, we give the complex q-rung orthopair fuzzy Hamacher weighted average (Cq-ROFHWA) (Mahmood and Ali 2021a, 2021b) and Cq-ROFHWG operators to deal with this example. From Tables 8 and 9, it is clear that for the different values of parameter $q$, the same ranking results are obtained, and $A_5$ is found to be the best one.

(2) Cq-ROFEWA operator and Cq-ROFEWG operator.

In a comparison analysis with existing complex q-rung orthopair fuzzy Einstein weighted average (Cq-ROFEWA) operator (Liu et al. 2021c) and Cq-ROFEWG operator, we can obtain the ranking result as follows:

From Tables 10 and 11, it is obvious that the ranking results are the same for diverse parameter $q$ by utilizing Cq - ROFEWA operator or Cq - ROFEWG operator, and $A_5$ is found to be the best one.

(3) Cq-ROFWHM operator.

Compared with Cq-ROFWHM (Garg et al. 2021b) operator, the results are shown below.

(4) Cq - ROFWA operator and Cq - ROFWG operator.
Compared with Cq-ROFWA (Liu et al. 2020b) operator and Cq-ROFWG operator the results are shown in Tables 13 and 14.

From Tables 13 and 14, the ranking results are similar to our proposed method, which proves the authenticity and the rationality of the given method.

In this article, firstly, we compared the proposed methods with each other based on different parameters. Secondly, we make a comparison between the proposed and existing methods. As can be seen from Tables 4, 5, 6, 7, 8, 9, 10, 11, 12, 13 and 14, by comparing the proposed method with the Cq-ROFHWA, Cq-ROFHWG, Cq-ROFEWA, Cq-ROFHW, Cq-ROFWA, Cq-ROFWG and Cq-ROFWHM operators, the conclusion is the same and is the best desirable alternative, which shows the rationality of the defined method in this article. Thirdly, the proposed method is more flexible and universal than the methods based on the Cq-ROFWA operator and the Cq-ROFWG operator in dealing with the MADM problems according to different parameters, and the calculation amount of Mahmood’ method (Mahmood and Ali 2021a, 2021b), Liu’ methods (Liu et al. 2020b, 2021c) and Garg H’ approach (2021b) is very higher than that of our created method, and our methods can reflect the decision makers preferences more flexible based on the parameter. According to the analysis above, the proposed methods are better than the other approaches.

### 7.1 An application choosing veil for COVID-19

**Numerical Example 2:** The MCDM issue is cited from Ref Yang et al. (2021). In the current extreme situation of the spread of COVID-19, six types of veils are commonly available in the market, including clinical careful covers, particulate respirators (N95/KN95 or more), clinical defensive covers, clinical defensive covers, dispensable clinical covers, conventional non-clinical covers and gas covers. Individuals need to buy the antivirus veil from six rising star antivirus products. In addition, individuals

| Method                  | Score values |
|-------------------------|--------------|
| A1 = Cq—ROFFWA(C11, C12, ..., C1n) | S(A1) = 0.39618 |
| A2 = Cq—ROFWA(C21, C22, ..., C2n) | S(A2) = 0.4095 |
| A3 = Cq—ROFWA(C31, C32, ..., C3n) | S(A3) = 0.24515 |
| A4 = Cq—ROFWA(C41, C42, ..., C4n) | S(A4) = 0.28667 |
| A5 = Cq—ROFWA(C51, C52, ..., C5n) | S(A5) = 0.439054 |

### Table 3 Score values of the alternatives

| q   | S(α1) | S(α2) | S(α3) | S(α4) | S(α5) | Ranking |
|-----|-------|-------|-------|-------|-------|---------|
| 1.1 | 0.400405 | 0.404738 | 0.237438 | 0.287663 | 0.416741 | A5>A2>A1>A4>A3 |
| 1.5 | 0.396189 | 0.4095 | 0.245152 | 0.28667 | 0.439054 | A5>A2>A1>A4>A3 |
| 2   | 0.359788 | 0.383096 | 0.234496 | 0.262104 | 0.429349 | A5>A2>A1>A4>A3 |
| 10  | 0.021947 | 0.034676 | 0.026826 | 0.018283 | 0.056009 | A5>A2>A1>A4>A3 |
| 30  | 0.000111 | 0.00033 | 0.000232 | 0.000189 | 0.00069 | A5>A2>A1>A4>A3 |
| 50  | 0.000209 | 0.000636 | 0.000341 | 0.000352 | 0.001001 | A5>A2>A1>A4>A3 |
| 100 | 0.00000000001 | 0.000000000064 | 0.00000000003 | 0.00000000007 | 0.00000000217 | A5>A2>A1>A4>A3 |

### Table 4 When τ = 3, ranking results based on Cq - ROFFWA operator by using the different q

| q   | S(α1) | S(α2) | S(α3) | S(α4) | S(α5) | Ranking |
|-----|-------|-------|-------|-------|-------|---------|
| 1.1 | 0.383397 | 0.37087 | 0.228081 | 0.26568 | 0.393402 | A5>A1>A2>A4>A3 |
| 1.5 | 0.37759 | 0.369112 | 0.234322 | 0.261309 | 0.412969 | A5>A1>A2>A4>A3 |
| 2   | 0.341045 | 0.33702 | 0.222729 | 0.234767 | 0.402898 | A5>A1>A2>A4>A3 |
| 5   | 0.111429 | 0.115601 | 0.097802 | 0.073698 | 0.202971 | A5>A1>A2>A4>A3 |
| 10  | 0.014045 | 0.016209 | 0.019398 | 0.009223 | 0.050005 | A5>A1>A2>A4>A3 |
| 30  | 0.013957 | 0.016017 | 0.019076 | 0.009124 | 0.049503 | A5>A1>A2>A4>A3 |
| 50  | 0.013952 | 0.013925 | 0.018914 | 0.009002 | 0.049526 | A5>A1>A2>A4>A3 |
| 60  | 0.00000000006 | 0.00000000023 | 0.00000000165 | 0.000000000 | 0.000000013 | A5>A1>A2>A4>A3 |
consider four criteria to decide their choice: (1) $C_1$ is the spillage rate that is the adhesiveness of the veil structure configuration to cover the human face; (2) $C_2$ is the reusability; (3) $C_3$ is the nature of crude materials; (4) $C_4$ is the filtration productivity. All criteria values are benefit type. The weight vector of the criteria is $w = (0.15, 0.15, 0.2, 0.5)^T$. The five potential options are assessed regarding the four criteria by CQROFSs, and complex q-rung orthopair fuzzy decision matrix $R$ is listed in Table 15.

Because the proposed MADM based on CQROFSs with its Cq - ROFFWA operator has different q values, it generates some usefully special cases. To examine the usefulness and proficiency of the proposed approach, we next

| $q$ | $S(a_1)$ | $S(a_2)$ | $S(a_3)$ | $S(a_4)$ | $S(a_5)$ | Ranking |
|-----|----------|----------|----------|----------|----------|---------|
| 1.1 | 0.437597 | 0.434858 | 0.129117 | 0.210158 | 0.462234 | $A_5$ > $A_2$ > $A_1$ > $A_3$ > $A_4$ |
| 1.5 | 0.435419 | 0.433218 | 0.142457 | 0.220977 | 0.46033  | $A_5$ > $A_2$ > $A_1$ > $A_3$ > $A_4$ |
| 5   | 0.433038 | 0.440938 | 0.202252 | 0.278003 | 0.451915 | $A_5$ > $A_2$ > $A_1$ > $A_3$ > $A_4$ |
| 10  | 0.026639 | 0.038981 | 0.016673 | 0.011672 | 0.06849  | $A_5$ > $A_2$ > $A_1$ > $A_3$ > $A_4$ |
| 20  | 0.001668 | 0.003386 | 0.001502 | 0.001061 | 0.007026 | $A_5$ > $A_2$ > $A_1$ > $A_3$ > $A_4$ |
| 80  | 0.0000000018 | 0.000000035847 | 0.000000002179 | 0.000000003341 | 0.0000000097571 | $A_5$ > $A_2$ > $A_1$ > $A_3$ > $A_4$ |
When present ranking results based on Cq - ROFFWA operator

| $q$ | $S(a_1)$ | $S(a_2)$ | $S(a_3)$ | $S(a_4)$ | $S(a_5)$ | Ranking               |
|-----|----------|----------|----------|----------|----------|----------------------|
| 1.1 | 0.38628  | 0.37738  | 0.2993   | 0.26927  | 0.39799  | $A_3 > A_1 > A_2 > A_4$ |
| 1.5 | 0.38012  | 0.37535  | 0.23632  | 0.26455  | 0.41757  | $A_3 > A_1 > A_2 > A_4$ |
| 5   | 0.11182  | 0.11648  | 0.09826  | 0.07409  | 0.20348  | $A_3 > A_5 > A_4 > A_2 > A_1$ |
| 10  | 0.01407  | 0.01626  | 0.01944  | 0.00925  | 0.05008  | $A_3 > A_5 > A_4 > A_2 > A_1$ |
| 20  | 0.00026  | 0.00040  | 0.00075  | 0.00016  | 0.00345  | $A_3 > A_5 > A_4 > A_2 > A_1$ |
| 50  | 0.0000000027 | 0.0000000084 | 0.0000000427 | 0.0000000009 | 0.0000017088 | $A_3 > A_5 > A_4 > A_2 > A_1$ |

Table 10 When $k = 2$, ranking results based on the Cq - ROFEWA operator by using the different $q$

| $q$ | $S(a_1)$ | $S(a_2)$ | $S(a_3)$ | $S(a_4)$ | $S(a_5)$ | Ranking               |
|-----|----------|----------|----------|----------|----------|----------------------|
| 1.1 | 0.441705 | 0.439769 | 0.118027 | 0.203928 | 0.466271 | $A_3 > A_1 > A_2 > A_4$ |
| 1.5 | 0.440447 | 0.439108 | 0.129267 | 0.213466 | 0.464914 | $A_3 > A_1 > A_2 > A_4$ |
| 5   | 0.442137 | 0.447826 | 0.187336 | 0.267599 | 0.460177 | $A_3 > A_2 > A_1 > A_4$ |
| 10  | 0.029347 | 0.041919 | 0.012811 | 0.00937  | 0.074812 | $A_3 > A_2 > A_1 > A_4$ |
| 20  | 0.001872 | 0.00376  | 0.001131 | 0.000802 | 0.007875 | $A_3 > A_2 > A_1 > A_4$ |
| 80  | 0.0000000021 | 0.00000000403 | 0.0000000016 | 0.0000000025 | 0.0000001098 | $A_3 > A_2 > A_1 > A_4$ |
| 100 | 0.0000000024 | 0.00000000971 | 0.00000000019 | 0.00000000037 | 0.0000000032 | $A_3 > A_2 > A_1 > A_4$ |

Table 11 When $k = 2$, ranking results based on the Cq - ROFEWG operator by using the different $q$

| $q$ | $S(a_1)$ | $S(a_2)$ | $S(a_3)$ | $S(a_4)$ | $S(a_5)$ | Ranking               |
|-----|----------|----------|----------|----------|----------|----------------------|
| 1.1 | 0.38446  | 0.37314  | 0.22877  | 0.26691  | 0.39511  | $A_3 > A_1 > A_2 > A_4$ |
| 1.5 | 0.37848  | 0.37108  | 0.23506  | 0.26235  | 0.41470  | $A_3 > A_1 > A_2 > A_4$ |
| 5   | 0.11145  | 0.11561  | 0.09787  | 0.07373  | 0.20309  | $A_3 > A_2 > A_1 > A_4$ |
| 10  | 0.01404  | 0.01620  | 0.01939  | 0.00922  | 0.05000  | $A_3 > A_2 > A_1 > A_4$ |
| 20  | 0.00026  | 0.00040  | 0.00074  | 0.00016  | 0.00345  | $A_3 > A_2 > A_1 > A_4$ |
| 50  | 0.000000002 | 0.00000000838 | 0.00000004267 | 0.00000000095 | 0.00000170878 | $A_3 > A_2 > A_1 > A_4$ |

present ranking results based on Cq - ROFFWA operator when $\tau = 3$, six cases of $q = 1$, $q = 1.1$, $q = 2$, $q = 3$, $q = 5$, and $q = 10$. These decision matrix items are shown in Table 16.

If $q = 1$, then the Cq - ROFFWA operator is reduced to the complex intuitionistic fuzzy Frank aggregation (CIFWA) operator.

If $q = 2$, then the Cq - ROFFWA operator is reduced to the complex Pythagorean fuzzy Frank aggregation (CPFWA) operator.

If $q = 3$, then the Cq - ROFFWA operator is reduced to the complex Fermatean fuzzy Frank aggregation (CPFFA) operator.

From Table 16, we can get that $A_3$ is the best one. The prominent characteristic of the Cq - ROFFWA operator is that the decision makers can choose the appropriate parameter value $q$ and $\tau$ by their preferences. The defined method has an ideal property about the parameter value $q$ and $\tau$ which provides people to choose appropriate values on the basis of risk preferences. If the decision maker is risk averse, we can select the parameter $\tau$ as large as possible, and if the decision maker is risk loving, we can choose the parameter $\tau$ as small as possible. Since Frank $t$-norm and $t$-conorm are a generalization of algorithms such as Algebra, Einstein and Hamacher’s $t$-norm and $t$-conorm, it is more general in dealing with multi-attribute decision-making problems. Further, the complex $q$-rung orthopair fuzzy Frank aggregation operators consider the rejection level when they are compared with complex intuition fuzzy Frank aggregation operator, complex Pythagorean fuzzy Frank aggregation operator and complex $q$-rung orthopair fuzzy Frank aggregation operators.
Table 12 When \( q = 3 \), ranking results based on the Cq - ROFWHM operator by using the different \( x \)

| \( x \) | \( S(a_1) \) | \( S(a_2) \) | \( S(a_3) \) | \( S(a_4) \) | \( S(a_5) \) | Ranking |
|-------|-------|-------|-------|-------|-------|-------|
| 2     | 0.718563 | 0.721737 | 0.649679 | 0.648689 | 0.779716 | \( A_5 \rangle A_2 \rangle A_1 \rangle A_3 \rangle A_4 \) |
| 3     | 0.712878 | 0.71244 | 0.643013 | 0.639795 | 0.775243 | \( A_5 \rangle A_1 \rangle A_2 \rangle A_3 \rangle A_4 \) |
| 4     | 0.708174 | 0.705694 | 0.635857 | 0.633196 | 0.768629 | \( A_5 \rangle A_1 \rangle A_2 \rangle A_3 \rangle A_4 \) |

Table 13 Ranking results based on the Cq - ROFWA operator by using the different \( q \)

| \( q \) | \( S(a_1) \) | \( S(a_2) \) | \( S(a_3) \) | \( S(a_4) \) | \( S(a_5) \) | Ranking |
|-------|-------|-------|-------|-------|-------|-------|
| 1.1   | 0.402609 | 0.408894 | 0.238689 | 0.290738 | 0.419788 | \( A_5 \rangle A_2 \rangle A_1 \rangle A_3 \rangle A_4 \) |
| 1.5   | 0.398267 | 0.414047 | 0.246492 | 0.289777 | 0.441607 | \( A_5 \rangle A_2 \rangle A_1 \rangle A_3 \rangle A_4 \) |
| 4     | 0.184755 | 0.223394 | 0.145958 | 0.137046 | 0.282098 | \( A_5 \rangle A_2 \rangle A_1 \rangle A_3 \rangle A_4 \) |
| 10    | 0.022084 | 0.034989 | 0.027026 | 0.018523 | 0.05625 | \( A_5 \rangle A_2 \rangle A_1 \rangle A_3 \rangle A_4 \) |
| 20    | 0.001274 | 0.002706 | 0.002281 | 0.001617 | 0.005407 | \( A_5 \rangle A_2 \rangle A_1 \rangle A_3 \rangle A_4 \) |
| 50    | 0.000011 | 0.0000727 | 0.0000264 | 0.0000279 | 0.0001573 | \( A_5 \rangle A_2 \rangle A_1 \rangle A_3 \rangle A_4 \) |

Table 14 Ranking results based on the Cq - ROFWG operator by using the different \( q \)

| \( q \) | \( S(a_1) \) | \( S(a_2) \) | \( S(a_3) \) | \( S(a_4) \) | \( S(a_5) \) | Ranking |
|-------|-------|-------|-------|-------|-------|-------|
| 1.1   | 0.381670 | 0.365080 | 0.226694 | 0.262694 | 0.39520 | \( A_5 \rangle A_2 \rangle A_1 \rangle A_3 \rangle A_4 \) |
| 1.5   | 0.375824 | 0.362390 | 0.233072 | 0.258312 | 0.410543 | \( A_5 \rangle A_2 \rangle A_1 \rangle A_3 \rangle A_4 \) |
| 4     | 0.166315 | 0.167506 | 0.132392 | 0.110719 | 0.264542 | \( A_5 \rangle A_2 \rangle A_1 \rangle A_3 \rangle A_4 \) |
| 10    | 0.013957 | 0.016033 | 0.019245 | 0.009141 | 0.049770 | \( A_5 \rangle A_2 \rangle A_1 \rangle A_3 \rangle A_4 \) |
| 20    | 0.000264 | 0.000397 | 0.000744 | 0.000162 | 0.003439 | \( A_5 \rangle A_2 \rangle A_1 \rangle A_3 \rangle A_4 \) |
| 50    | 0.0000000274 | 0.00000008377 | 0.000000042674 | 0.00000000946 | 0.000001708757 | \( A_5 \rangle A_2 \rangle A_1 \rangle A_3 \rangle A_4 \) |

Table 15 Complex q-rung orthopair fuzzy decision matrix \( R = (a_{ij})_{n \times n} \)

| \( C_1 \) | \( C_2 \) | \( C_3 \) | \( C_4 \) |
|-------|-------|-------|-------|
| \( A_1 \) | \( 0.3e^{2x(0.4)} \), \( 0.2e^{2x(0.1)} \) | \( 0.31e^{2x(0.4)} \), \( 0.21e^{2x(0.1)} \) | \( 0.32e^{2x(0.4)} \), \( 0.22e^{2x(0.12)} \) | \( 0.33e^{2x(0.4)} \), \( 0.23e^{2x(0.13)} \) |
| \( A_2 \) | \( 0.5e^{2x(0.6)} \), \( 0.4e^{2x(0.3)} \) | \( 0.51e^{2x(0.6)} \), \( 0.41e^{2x(0.31)} \) | \( 0.52e^{2x(0.62)} \), \( 0.42e^{2x(0.32)} \) | \( 0.53e^{2x(0.63)} \), \( 0.43e^{2x(0.33)} \) |
| \( A_3 \) | \( 0.4e^{2x(0.68)} \), \( 0.22e^{2x(0.35)} \) | \( 0.41e^{2x(0.68)} \), \( 0.3e^{2x(0.4)} \) | \( 0.45e^{2x(0.8)} \), \( 0.4e^{2x(0.5)} \) | \( 0.44e^{2x(0.8)} \), \( 0.28e^{2x(0.41)} \) |
| \( A_4 \) | \( 0.1e^{2x(0.3)} \), \( 0.4e^{2x(0.2)} \) | \( 0.11e^{2x(0.31)} \), \( 0.41e^{2x(0.21)} \) | \( 0.12e^{2x(0.32)} \), \( 0.42e^{2x(0.22)} \) | \( 0.13e^{2x(0.33)} \), \( 0.43e^{2x(0.23)} \) |
| \( A_5 \) | \( 0.7e^{2x(0.5)} \), \( 0.1e^{2x(0.2)} \) | \( 0.71e^{2x(0.51)} \), \( 0.11e^{2x(0.21)} \) | \( 0.72e^{2x(0.52)} \), \( 0.12e^{2x(0.22)} \) | \( 0.73e^{2x(0.53)} \), \( 0.13e^{2x(0.23)} \) |
| \( A_6 \) | \( 0.8e^{2x(0.7)} \), \( 0.1e^{2x(0.1)} \) | \( 0.81e^{2x(0.73)} \), \( 0.11e^{2x(0.11)} \) | \( 0.82e^{2x(0.72)} \), \( 0.12e^{2x(0.12)} \) | \( 0.83e^{2x(0.73)} \), \( 0.13e^{2x(0.13)} \) |

To sum up, two practical examples are given in the paper to illustrate the application of the defined method and to demonstrate its effectiveness and practicality.

The key advantages of the method defined in the paper are as follows:

1. The complex q-rung orthopair fuzzy Frank aggregation operators, the complex q-rung orthopair fuzzy Einstein aggregation operators, the complex q-rung orthopair fuzzy weighted averaging operators, the complex intuitionistic fuzzy Frank aggregation operator, the complex Pythagorean fuzzy Frank aggregation operator, the complex Fermatian fuzzy Frank aggregation operator. It is clear that the Cq - ROFFWA operator in this paper has more general
and more information in terms of both membership and non-membership degrees than other above operators. Therefore, the complex q-rung orthopair fuzzy Frank aggregation operators can be applied to deal with more uncertain and complicated information of real-life decision-making problems.

2. Complex q-rung orthopair fuzzy term which can solve the uncertainty more precisely than q-rung orthopair fuzzy set, complex fuzzy set and q-rung orthopair uncertain linguistic fuzzy set in qualitative, and many fuzzy set cannot deal with MADM under complex q-rung environment such as the rough set and neutrosophic set, but the complex q-rung orthopair fuzzy number can be used to deal with such problems effectively.

3. The prominent characteristic of the proposed method is that the parameter value \( q \) and \( \tau \) can be changed by their preferences.

4. Membership and non-membership degree are complements of the complex fuzzy terms, which can show us how much degree that an attribute value belongs to or not belongs to a complex q-rung term.

8 Conclusion

In this paper, we extended the Frank operations to the complex q-rung orthopair fuzzy environment according to the definition of the complex q-rung orthopair fuzzy set and the Frank aggregation operator. To begin with, certain Frank operational laws of complex q-rung orthopair fuzzy set (Cq-ROFS) were introduced. Meanwhile, we developed a family of complex q-rung orthopair fuzzy Frank averaging operators, such as the Cq-ROFFWA operator, the Cq-ROFFWG operator and the Cq-ROFFOWA operator. Some desirable properties of the introduced operators were given. In addition, based on the defined operators, a novel approach for MADM was proposed under the complex q-rung orthopair fuzzy environment. Numerical examples were given to demonstrate the effectiveness and usefulness of the defined method by comparison with other existing methods.

The Frank aggregation operators based on the complex q-rung orthopair fuzzy set are an important complement to related research, and the method defined in this paper may add a new direction for solving MCDM problems. We have created a precedent to combine Frank aggregation operators with complex q-rung orthopair fuzzy set and make use of the complex q-rung orthopair fuzzy Frank aggregation operators to deal with decision-making problems. Since the method proposed in this paper cannot handle multi-attribute decision making in complex q-rung linguistic fuzzy environments or complex q-rung fuzzy N-soft environments, in future work, we will extend Frank aggregation operators under complex q-rung linguistic orthopair fuzzy environments, complex q-rung fuzzy N-soft environments and T-spherical fuzzy environments, or develop Frank aggregation operators to 3, 4-quasirung fuzzy sets (Seikh and Mandal, 2022). In addition, we shall further generalize these defined operators to deal with Biogas plant implementation problem (Karmakar et al. 2021), plastic ban problem (Seikh et al. 2021a), market share problem (Seikh et al. 2021b), social network analysis (Liu et al. 2022a), social trust propagation mechanism (Liu et al. 2022b) and incomplete probabilistic linguistic preference relations (Wang et al. 2021; Liu et al. 2020a, 2020b, 2020c), or extend the aggregation operators to other domains, such as pattern recognition, cluster analysis and investment decisions.

### Appendix

#### Proof of Theorem 1
We prove the parts (1, 5, 6), and the others are similar with them.

1. Let us prove part (1), we get

| \( q \) | \( S(a_1) \) | \( S(a_2) \) | \( S(a_3) \) | \( S(a_4) \) | \( S(a_5) \) | \( S(a_6) \) | Ranking |
|-------|-------|-------|-------|-------|-------|-------|-------|
| 1     | 0.1994 | 0.2003 | 0.2457 | −0.099 | 0.4505 | 0.6508 | \( A_6 \gg A_5 \gg A_3 \gg A_2 \gg A_1 \gg A_4 \) |
| 1.1   | 0.1917 | 0.2040 | 0.2528 | −0.095 | 0.4493 | 0.6818 | \( A_6 \gg A_5 \gg A_3 \gg A_2 \gg A_1 \gg A_4 \) |
| 2     | 0.1076 | 0.1885 | 0.2601 | −0.053 | 0.3638 | 0.5822 | \( A_6 \gg A_5 \gg A_3 \gg A_2 \gg A_1 \gg A_4 \) |
| 3     | 0.0471 | 0.1366 | 0.2208 | −0.0250 | 0.2516 | 0.4621 | \( A_6 \gg A_5 \gg A_3 \gg A_2 \gg A_1 \gg A_4 \) |
| 5     | 0.0079 | 0.0571 | 0.1383 | −0.0051 | 0.1163 | 0.2837 | \( A_6 \gg A_5 \gg A_3 \gg A_2 \gg A_1 \gg A_4 \) |
| 10    | 0.00009 | 0.0049 | 0.0410 | −0.00008 | 0.01979 | 0.0888 | \( A_6 \gg A_5 \gg A_3 \gg A_2 \gg A_1 \gg A_4 \) |
Complex q-rung orthopair fuzzy Frank aggregation operators and their application to multi-attribute selection problems

\[\begin{align*}
C_1 \oplus C_2 &= \left(1 - \log_\tau \left(1 + \frac{(1-G_1^2) - 1}{(\tau - 1)} \right)\right) \cdot \frac{\tau}{\pi} \left(1 - \log_\tau \left(1 + \frac{(1-G_2^2) - 1}{(\tau - 1)} \right)\right) \cdot \frac{\tau}{\pi} \\
&= C_2 \oplus C_1.
\end{align*}\]

2. Let us prove part (5), we get

\[\begin{align*}
k \cdot (C_1 \oplus C_2) &= kC_1 \oplus kC_2; \\
C_1 \oplus C_2 &= \left(1 - \log_\tau \left(1 + \frac{(1-G_1^2) - 1}{(\tau - 1)} \right)\right) \cdot \frac{\tau}{\pi} \left(1 - \log_\tau \left(1 + \frac{(1-G_2^2) - 1}{(\tau - 1)} \right)\right) \cdot \frac{\tau}{\pi} \\
kC_1 &= \left(1 - \log_\tau \left(1 + \frac{(1-G_1^2) - 1}{(\tau - 1)^{k-1}} \right)\right) \cdot \frac{\tau}{\pi} \left(1 - \log_\tau \left(1 + \frac{(1-G_2^2) - 1}{(\tau - 1)^{k-1}} \right)\right) \cdot \frac{\tau}{\pi} \\
kC_2 &= \left(1 - \log_\tau \left(1 + \frac{(1-G_1^2) - 1}{(\tau - 1)^{k-1}} \right)\right) \cdot \frac{\tau}{\pi} \left(1 - \log_\tau \left(1 + \frac{(1-G_2^2) - 1}{(\tau - 1)^{k-1}} \right)\right) \cdot \frac{\tau}{\pi} \left(1 - \log_\tau \left(1 + \frac{(1-G_2^2) - 1}{(\tau - 1)^{k-1}} \right)\right) \cdot \frac{\tau}{\pi}.
Then, we have

$$kC_1 \oplus kC_2 =$$

\[
\begin{pmatrix}
1 - \log \left( 1 + \frac{(\tau^{1-G_1^k} - 1)^k}{(\tau - 1)^{2k-1}} \right)
\end{pmatrix} e^{\frac{i}{2} \pi} \left( \begin{pmatrix}
\text{log (1)} - \log \left( 1 + \frac{(\tau^{1-G_1^k} - 1)^k}{(\tau - 1)^{2k-1}} \right)
\end{pmatrix} \right)
\]
So, $k(C_1 \oplus C_2) = kC_1 \oplus kC_2$.

3. The proof of part (2,4,5) is straightforward. So, it has been omitted here.

4. Let us consider part (6), we have
\[
C_1 \otimes C_2 = \left( \log_\tau \left( 1 + \frac{(\tau^G - 1)(\tau^{G_1} - 1)}{(\tau - 1)^{k_1}} \right) \right)^{\frac{1}{k_1}} e^{\frac{1}{\tau} \log \left( 1 + \frac{w^G_{G_1 - 1} w^G_{G_2 - 1}}{(\tau - 1)^{k_1}} \right)}
\]

\[
(C_1 \otimes C_2)^k = \left( \log_\tau \left( 1 + \frac{(\tau^G - 1)(\tau^{G_1} - 1)}{(\tau - 1)^{k_1}} \right) \right)^{\frac{1}{k_1}} e^{\frac{1}{\tau} \log \left( 1 + \frac{w^G_{G_1 - 1} w^G_{G_2 - 1}}{(\tau - 1)^{k_1}} \right)}
\]

\[
= \left( \log_\tau \left( 1 + \frac{(\tau^G - 1)(\tau^{G_1} - 1)}{(\tau - 1)^{2k_1 - 1}} \right) \right)^{\frac{1}{k_1}} e^{\frac{1}{\tau} \log \left( 1 + \frac{w^G_{G_1 - 1} w^G_{G_2 - 1}}{(\tau - 1)^{2k_1 - 1}} \right)}
\]
According to the definition, we have

\[ C_1^t = \left\{ \begin{array}{l} \left( \log_t \left( 1 + \left( \frac{x \tau q - 1}{(\tau - 1)^{t-1}} \right)^{1/\tau} \right) \right)^{1/\tau} e^{\frac{1}{\tau} \left( \log_t \left( 1 + \left( \frac{x \tau q - 1}{(\tau - 1)^{t-1}} \right)^{1/\tau} \right) \right)^{1/\tau}} \\
1 - \log_t \left( 1 + \left( \frac{x \tau q - 1}{(\tau - 1)^{t-1}} \right)^{1/\tau} \right)^{1/\tau} e^{\frac{1}{\tau} \left( \log_t \left( 1 + \left( \frac{x \tau q - 1}{(\tau - 1)^{t-1}} \right)^{1/\tau} \right) \right)^{1/\tau}} \end{array} \right. \]

\[ C_2^t = \left\{ \begin{array}{l} \left( \log_t \left( 1 + \left( \frac{x \tau q - 1}{(\tau - 1)^{t-1}} \right)^{1/\tau} \right) \right)^{1/\tau} e^{\frac{1}{\tau} \left( \log_t \left( 1 + \left( \frac{x \tau q - 1}{(\tau - 1)^{t-1}} \right)^{1/\tau} \right) \right)^{1/\tau}} \\
1 - \log_t \left( 1 + \left( \frac{x \tau q - 1}{(\tau - 1)^{t-1}} \right)^{1/\tau} \right)^{1/\tau} e^{\frac{1}{\tau} \left( \log_t \left( 1 + \left( \frac{x \tau q - 1}{(\tau - 1)^{t-1}} \right)^{1/\tau} \right) \right)^{1/\tau}} \end{array} \right. \]

Then, we have

\[ C_1^t \otimes C_2^t = \left\{ \begin{array}{l} \left( \log_t \frac{1}{1 + \left( \frac{\tau \log_t \left( 1 + \left( \frac{x \tau q - 1}{(\tau - 1)^{t-1}} \right)^{1/\tau} \right) \right)^{1/\tau}} - 1 \right) \left( \log_t \frac{1}{1 + \left( \frac{\tau \log_t \left( 1 + \left( \frac{x \tau q - 1}{(\tau - 1)^{t-1}} \right)^{1/\tau} \right) \right)^{1/\tau}} - 1 \right)^{1/\tau} \\
1 - \log_t \frac{1}{1 + \left( \frac{\tau \log_t \left( 1 + \left( \frac{x \tau q - 1}{(\tau - 1)^{t-1}} \right)^{1/\tau} \right) \right)^{1/\tau}} - 1 \right) \left( \log_t \frac{1}{1 + \left( \frac{\tau \log_t \left( 1 + \left( \frac{x \tau q - 1}{(\tau - 1)^{t-1}} \right)^{1/\tau} \right) \right)^{1/\tau}} - 1 \right)^{1/\tau} \end{array} \right. \]

\[ e^{-\frac{1}{r} \left( \log_t \frac{1}{1 + \left( \frac{\tau \log_t \left( 1 + \left( \frac{x \tau q - 1}{(\tau - 1)^{t-1}} \right)^{1/\tau} \right) \right)^{1/\tau}} - 1 \right) \left( \log_t \frac{1}{1 + \left( \frac{\tau \log_t \left( 1 + \left( \frac{x \tau q - 1}{(\tau - 1)^{t-1}} \right)^{1/\tau} \right) \right)^{1/\tau}} - 1 \right)^{1/\tau} + \frac{1}{r} \left( \log_t \frac{1}{1 + \left( \frac{\tau \log_t \left( 1 + \left( \frac{x \tau q - 1}{(\tau - 1)^{t-1}} \right)^{1/\tau} \right) \right)^{1/\tau}} - 1 \right) \left( \log_t \frac{1}{1 + \left( \frac{\tau \log_t \left( 1 + \left( \frac{x \tau q - 1}{(\tau - 1)^{t-1}} \right)^{1/\tau} \right) \right)^{1/\tau}} - 1 \right)^{1/\tau} \right) \]
Proof of Theorem 8  Since

\[
(C_1 \otimes C_2)^k.
\]

So, we have
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Cq - ROFFWA\(C_1 \oplus C, C_2 \oplus C, \ldots, C_n \oplus C\)

\[
\left(1 - \log_\tau \left(1 + \prod_{j=1}^{n} \left(\frac{\left(\tau^{1-G_j'}-1\right)}{\left(\tau^{1-G_j'}-1\right)}\right)^{w_j}\right)\right) e^{\frac{i}{\pi} \text{arctan} \left(1 + \prod_{j=1}^{n} \left(\frac{\left(\tau^{1-G_j'}-1\right)}{\left(\tau^{1-G_j'}-1\right)}\right)^{w_j}\right)\sqrt{\left(1 + \prod_{j=1}^{n} \left(\frac{\left(\tau^{1-G_j'}-1\right)}{\left(\tau^{1-G_j'}-1\right)}\right)^{w_j}\right)}}
\]

Furthermore,

Cq - ROFFWA\((C_1, C_2, \ldots, C_n) \oplus C\)

\[
\left(1 - \log_\tau \left(1 + \prod_{j=1}^{n} \left(\frac{\left(\tau^{1-G_j'}-1\right)}{\left(\tau^{1-G_j'}-1\right)}\right)^{w_j}\right)\right) e^{\frac{i}{\pi} \text{arctan} \left(1 + \prod_{j=1}^{n} \left(\frac{\left(\tau^{1-G_j'}-1\right)}{\left(\tau^{1-G_j'}-1\right)}\right)^{w_j}\right)\sqrt{\left(1 + \prod_{j=1}^{n} \left(\frac{\left(\tau^{1-G_j'}-1\right)}{\left(\tau^{1-G_j'}-1\right)}\right)^{w_j}\right)}}
\]
So, we have

\[
\text{Cq - ROFFWA}(C_1 \oplus C, C_2 \oplus C, \ldots, C_n \oplus C) = \text{Cq - ROFFWA}(C_1, C_2, \ldots, C_n) \oplus C.
\]

**Proof of Theorem 9** Since

\[
k_{C_j} = \left(1 - \log_{\tau} \left(1 + \frac{\left(1-G_j^k\right)}{\tau-1} \right)\right) e^{\frac{1}{\tau} i \log_{\tau} \left(1 + \frac{\left(1-G_j^k\right)}{\tau-1} \right)}
\]

Furthermore,

\[
k_{C_j} \oplus C = \left(1 - \log_{\tau} \left(1 + \frac{\left(1-G_j^k\right)}{\tau-1} \right)\right) e^{\frac{1}{\tau} i \log_{\tau} \left(1 + \frac{\left(1-G_j^k\right)}{\tau-1} \right)}
\]

Therefore, we have

\[
\text{Cq - ROFFWA}(k \cdot C_1 \oplus C, k \cdot C_2 \oplus C, \ldots, k \cdot C_n \oplus C)
\]
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Furthermore,

\[ k \cdot C_q \cdot \text{ROFWA}(C_1, C_2, \ldots, C_n) \]

Then,
\[ kCq - ROFFWA(C_1, C_2, \ldots, C_n) \oplus C \]

\[
\begin{align*}
\left( 1 - \log_\tau \left( 1 + \frac{\prod_{j=1}^{n} \left( \tau^{1-G_j'} - 1 \right)^{kw_j}}{(\tau - 1)^k} \right) \right)^{\frac{1}{\tau}} e^{i2\pi \left( 1 - \log_\tau \left( \prod_{j=1}^{n} \left( \frac{1-w_j^\tau}{\tau - 1} \right)^{kw_j} \right) \right)^{\frac{1}{\tau}}}, \\
= \left( 1 - \log_\tau \left( 1 + \frac{\prod_{j=1}^{n} \left( \tau^{1-G_j'} - 1 \right)^{kw_j}}{(\tau - 1)^k} \right) \right)^{\frac{1}{\tau}} e^{i2\pi \left( 1 - \log_\tau \left( \prod_{j=1}^{n} \left( \frac{1-w_j^\tau}{\tau - 1} \right)^{kw_j} \right) \right)^{\frac{1}{\tau}}}, \\
= \left( \frac{1}{\tau} \log_\tau \left( 1 + \frac{\prod_{j=1}^{n} \left( \tau^{1-G_j'} - 1 \right)^{kw_j}}{(\tau - 1)^k} \right) \right)^{\frac{1}{\tau}} e^{i2\pi \left( \frac{1}{\tau} \log_\tau \left( \prod_{j=1}^{n} \left( \frac{1-w_j^\tau}{\tau - 1} \right)^{kw_j} \right) \right)^{\frac{1}{\tau}}}, \\
\end{align*}
\]

Then, we can get
\[ Cq - ROFFWA(k \cdot C_1 \oplus C, k \cdot C_2 \oplus C, \ldots, k \cdot C_n \oplus C). \]

\[ C_n \oplus C = kCq - ROFFWA(C_1, C_2, \ldots, C_n) \oplus C. \]

**Proof of Theorem 10** For the left-hand side of Theorem 10, we can have
\[
kC_1 = \left( \log_\tau \left( 1 + \left( \frac{\tau H_i - 1}{(\tau - 1)^{k-1}} \right)^{w_j} \right) \right)^\frac{1}{\beta} e^{\frac{1}{\beta} \log_\tau \left( 1 + \left( \frac{\tau H_i - 1}{(\tau - 1)^{k-1}} \right)^{w_j} \right) \frac{1}{\beta}}
\]

Cq - ROFFWA(C1, C2, ..., Cn)

\[
kC_1 = \left( \log_\tau \left( 1 + \prod_{j=1}^{k} \left( \frac{\tau G_i - 1}{(\tau - 1)^{k-1}} \right)^{w_j} \right) \right)^\frac{1}{\beta} e^{\frac{1}{\beta} \log_\tau \left( 1 + \prod_{j=1}^{k} \left( \frac{\tau G_i - 1}{(\tau - 1)^{k-1}} \right)^{w_j} \right) \frac{1}{\beta}}
\]

Therefore, we can get

\[
kC_1 = \left( \log_\tau \left( 1 + \prod_{j=1}^{k} \left( \frac{\tau H_i - 1}{(\tau - 1)^{k-1}} \right)^{w_j} \right) \right)^\frac{1}{\beta} e^{\frac{1}{\beta} \log_\tau \left( 1 + \prod_{j=1}^{k} \left( \frac{\tau H_i - 1}{(\tau - 1)^{k-1}} \right)^{w_j} \right) \frac{1}{\beta}}
\]
Enquiries about data availability should be directed to the authors.

Data Availability

Enquiries about data availability should be directed to the authors.

Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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