Arc spaces and DAHA representations

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Abstract

A theorem of Y. Berest, P. Etingof and V. Ginzburg states that finite dimensional irreducible representations of a type A rational Cherednik algebra are classified by one rational number \( m/n \). Every such representation is a representation of the symmetric group \( S_n \). We compare certain multiplicity spaces in its decomposition into irreducible representations of \( S_n \) with the spaces of differential forms on a zero-dimensional moduli space associated with the plane curve singularity \( x^m = y^n \).

1 Introduction

Rational double affine Hecke algebras (DAHA) were introduced by I. Cherednik ([4]) in his study of the Macdonald conjectures. Their representation theory was extensively studied by C. Dunkl ([6],[7]), Y. Berest, P. Etingof and V. Ginzburg ([1],[2]), I. Gordon and T. Stafford ([11],[12],[13]), M. Varagnolo and E. Vasserot ([21]). Their relation to the geometry of the Hilbert schemes of points and affine Springer fibers was discussed in [13],[21],[22]. We refer the reader to the lectures [8] and the references therein for more complete bibliography.

By construction, rational DAHA \( H_{n,c} \) of type \( A_{n-1} \) with parameter \( c \) has a representation \( M_c \) by Dunkl operators in the space of polynomials on the Cartan subalgebra \( V_n \). In [1] it was shown that \( H_{n,c} \) has a finite-dimensional representation if and only if \( c = m/n, \quad m \in (m,n) = 1 \). In this case there exists a unique irreducible finite-dimensional representation \( L_{m/n} \), which can be constructed as a quotient of \( M_{m/n} \) by a certain ideal \( I_{m/n} \).

By construction, \( L_{m/n} \) carry a natural representation of the symmetric group \( S_n \), so we can split it into irreducible representations of \( S_n \). The
symmetric and antisymmetric parts of $L_{m/n}$ were extensively studied in connection to the representation theory of the spherical DAHA ([1], [21]) and the geometry of the Hilbert scheme of points ([11], [12], [13], [15]). In particular, the space $L_{n+1}^m$ is related to the $q, t$-Catalan numbers introduced by A. Garsia and M. Haiman ([9]). We are interested in a slightly more general problem of describing the multiplicities of the exterior power $\Lambda^k V_n$ in $L_{m/n}$. Since $\Lambda^k V_n$ is known to be an irreducible representation of $S_n$, we can describe this multiplicity space as

$$\text{Hom}_{S_n}(\Lambda^k V_n, L_{m/n}).$$

This space is conjectured to be related to some homological invariants of torus knots ([17], [18], [14]). Since

$$\text{Hom}_{S_n}(\Lambda^k V_n, M_{m/n}) \simeq \Omega^k(V_n//S^n),$$

one can ask if the ideal $I_{m/n}$ is related to some natural ideal in $\Omega^k(V_n//S^n)$.

In [10] L. Goettsche, B. Fantechi and D. van Straten constructed a zero-dimensional moduli space $M_{m,n}$ defined by the coefficients in $z$-expansion of the equation

$$(1 + z^2 u_2 + z^3 u_3 + \ldots + z^n u_n)^m = (1 + z^2 v_2 + z^3 v_3 + \ldots + z^m v_m)^n.$$

Our main result is the following

**Theorem.** The following isomorphism holds:

$$\text{Hom}_{S_n}(\Lambda^k V_n, L_{m/n}) \simeq \Omega^k(M_{m,n}).$$

The proof is explicit: we identify $M_{m,n}$ with a subscheme in Spec $\mathbb{C}[u_2, \ldots, u_n]$, and identify $u_k$ with the $k$-th elementary symmetric polynomial on $V_n$. This allows us to embed $M_{m,n}$ into the quotient $V_n//S^n$. It rests to compare the defining ideals in both cases.

**Corollary 1.1.** The left hand side is symmetric in $m$ and $n$:

$$\text{Hom}_{S_n}(\Lambda^k V_n, L_{m/n}) = \text{Hom}_{S_m}(\Lambda^k V_m, L_{n/m}).$$

**Remark 1.2.** This relation was proved for $k = 0$ by D. Calaque, B. Enriquez and P. Etingof in [3] by different method. I. Losev announced ([16]) a generalization of their proof for higher values of $k$. 

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In sections 2 and 3 we briefly discuss the constructions of the representation $L_{m/n}$ and the moduli space $M_{m,n}$. In section 4 we compare the constructions and prove the main theorem. In section 5 we discuss the action of Dunkl operators on $\Omega^k(M_{m,n})$ and the action of the Olshanetsky-Perelomov Hamiltonians on $u_l$ and $v_l$.

The author is grateful to J. Rasmussen, A. Oblomkov, V. Shende and A. Kirillov Jr. for useful discussions. This research was partially supported by the grants RFBR-10-01-00678, NSh-8462.2010.1 and the Dynasty fellowship for young scientists.

2 Rational Cherednik algebras

Definition 2.1. ([1]) The rational Cherednik algebra $H_c$ of type $A_{n-1}$ with parameter $c$ is an associative algebra generated by $V = \mathbb{C}^{n-1}, V^*$ and $S_n$ with the following defining relations:

\[
\sigma \cdot x \cdot \sigma^{-1} = \sigma(x), \quad \sigma \cdot y \cdot \sigma^{-1} = \sigma(y), \quad \forall x \in V, y \in V^*, \sigma \in S_n
\]

\[
x_1 \cdot x_2 = x_2 \cdot x_1, \quad y_1 \cdot y_2 = y_2 \cdot y_1 \quad \forall x_1, x_2 \in V, y_1, y_2 \in V^* \quad (2.1)
\]

\[
y \cdot x - x \cdot y = \langle y, x \rangle - c \sum_{s \in S} < \alpha_s, x > \cdot < y, \alpha_s^\vee > \cdot s \quad \forall x \in V, y \in V^*,
\]

where $S \subset S_n$ is the set of all transpositions, and $\alpha_s, \alpha_s^\vee$ are the corresponding roots and coroots.

The last defining relation is motivated by the following construction of C. Dunkl ([1]).

Definition 2.2. We introduce the Dunkl operators with parameter $c$ by the formula

\[
D_i = \frac{\partial}{\partial x_i} - c \sum_{j \neq i} \frac{s_{ij} - 1}{x_i - x_j}.
\]

Proposition 2.3. Consider the space $\mathbb{C}[V]$ of polynomial functions on $V$, where the elements of $V$ act by multiplication and the basis of $V^*$ acts by Dunkl operators. This produces a representation of $H_c$, i. e. all defining relations (2.1) hold. This representation is denoted by $M_c$.

An theorem of Y. Berest, P. Etingof, and V. Ginzburg says that for $c > 0$, all finite dimensional irreducible representations of $H_c$ can be obtained from this construction:
Theorem 2.4. ([1]) $H_c$ has finite dimensional representations if and only if $c = m/n$, where $m$ is an integer and $(m,n) = 1$. In this case, $H_{m/n}$ has a unique (up to isomorphism) finite dimensional irreducible representation $L_{m/n}$. For $c = m/n > 0$, $L_c = M_c/I_c$, where $I_c$ is an ideal generated by some homogeneous polynomials of degree $m$.

Following [7] and [5], we would like to give an explicit construction of these polynomials for $c = m/n$:

$$f_i = \text{Coef}_m[(1 - zx_i)^{-1} \prod_{i=1}^n (1 - zx_i)^{m/n}].$$

Remark 2.5. It is known that $f_i$ form a regular sequence, hence $I_{m/n}$ defines a 0-dimensional complete intersection in $V$. The Bezout’s theorem implies the dimension formula (11)

$$\dim L_{m/n} = \dim \mathbb{C}[V]/I_{m/n} = m^{n-1}.$$

Let us explain the choice of the polynomials $f_i$. From now on we will assume that $c = m/n$.

Definition 2.6. Following [7], let us introduce the formal series

$$F(z) = \prod_{i=1}^n (1 - zx_i)^{m/n}, \quad B_i(z) = \frac{1}{1 - zx_i} F(z).$$

Lemma 2.7. ([7]) The action of the Dunkl operators on $B_i(z)$ is given by the formula

$$D_s B_i(z) = \delta_{is} \left( z^2 \frac{d B_i}{d z} + (1 - m)z B_i(z) \right)$$

Corollary 2.8. ([7])

$$D_s \text{Coef}_k[B_i(z)] = \delta_{is} (k - m) \text{Coef}_{k-1}[B_i(z)].$$

In particular,

$$D_s(f_i) = D_s \text{Coef}_m[B_i(z)] = 0.$$

This explains why $f_i$ generate an ideal invariant under DAHA action.
3 Arc space on a singular curve

In [10] L. Goettsche, B. Fantechi and D. van Straten constructed a zero
dimensional quasihomogeneous complete intersection associated with a curve
\( \{x^m = y^n\} \). As before, we will assume that \((m, n) = 1\).

Consider a plane curve singularity \( C = \{x^m = y^n\} \), and its uniformization
\((x, y) = (t^n, t^m)\). Let us consider a general deformation of this parametrization:
\[
(x(t), y(t)) = (t^n + u_1 t^{n-1} + \ldots + u_n, t^m + v_1 t^{m-1} + \ldots + v_n).
\]

Consider the ideal \( I_{m,n} \) generated by the coefficients in \( t \)-expansion of the
equation
\[
(t^n + u_2 t^{n-2} + \ldots + u_n)^m - (t^m + v_2 t^{m-2} + \ldots + v_m)^n = 0. \tag{3.1}
\]

Remark 3.1. By shifting the parameter \( t \) we can annihilate the coefficient \( u_1 \).
Since
\[
x(t)^m - y(t)^n = (mu_1 - nv_1)t^{mn-1} + \text{terms of lower degree},
\]
the equation \( u_1 = 0 \) implies \( v_1 = 0 \).

Definition 3.2. The moduli scheme of arcs on \( C \) is defined as
\[
\mathcal{M}_C = \text{Spec } \mathbb{C}[u_2, \ldots, u_n, v_2, \ldots, v_m]/I_{m,n}.
\]

Lemma 3.3. ([10], Example 1) The scheme \( \mathcal{M}_C \) is a zero-dimensional complete intersection.

Proof. The equation \( x(t)^m - y(t)^n = 0 \) is equivalent to the equation
\[
mx'(t)y(t) - ny'(t)x(t) = 0. \tag{3.2}
\]
The left hand side of (3.2) is a polynomial in \( t \) of degree \( m + n - 3 \). Therefore
the ideal \( I_{m,n} \) is generated by a sequence of the \((m + n - 2)\) equations on
\((m - 1) + (n - 1) = (m + n - 2)\) variables. Since \( m \) and \( n \) are coprime, the
reduced scheme consists of one point \((x(t), y(t)) = (t^n, t^m)\).

Corollary 3.4. The algebra of algebraic differential forms on this moduli
space can be described as
\[
\Omega^*(\mathcal{M}_C) = \Omega^*(\mathbb{C}^{m+n-2})/\left(\phi \cdot \omega_1 + d\phi \wedge \omega_2 | \phi \in I_{m,n}\right).
\]
Definition 3.5. We assign the $q$-grading to $u_i$ and $v_i$ by the formula

$$q(u_i) = q(v_i) = i.$$  

Lemma 3.6. (§17) The multiplicity of $\mathcal{M}_C$ is given by the formula

$$\text{mult}(\mathcal{M}_C) = \frac{(m + n - 1)!}{m!n!}.$$  

Proof. One can check that the ideal $I_{m,n}$ is weighted homogeneous with respect to the $q$-grading, and the multiplicity of $\mathcal{M}_C$ can be computed using Bezout’s theorem:

$$\text{mult}(\mathcal{M}_C) = \frac{2 \cdot 3 \cdots (m + n - 1)}{2 \cdot 3 \cdots n \cdot 2 \cdot 3 \cdots m} = \frac{(m + n - 1)!}{m!n!}.$$

Lemma 3.7. (§17) The Hilbert series of $\mathbb{C}[u_2, \ldots, u_n, v_2, \ldots, v_n]/I_{m,n}$ with respect to the $q$-grading equals to

$$H_{m,n}(q) = \left[\frac{(m + n - 1)!}{m!n!}\right]_q = \prod_{k=2}^{n} \frac{(1 - q^{m+k-1})}{(1 - q^k)}.$$  

Proof. The ideal $I_{m,n}$ is generated by the weighted homogeneous regular sequence of equations of weights $2, 3, \ldots, (m + n - 1)$, so the proof is analogous to the Lemma 3.6.

For the further discussions we have to slightly change the notations. Let us change $t$ to $z = t^{1}$, then the equation (3.3) will have a form

$$(1 + z^2u_2 + \ldots + z^n u_n)^n = (1 + z^2v_2 + \ldots + z^m v_m)^n.$$  

(3.3)

Definition 3.8. Let $J_{m/n}$ denote the ideal in $\mathbb{C}[u_1, \ldots, u_n]$ generated by the coefficients of the series $(1 + z^2 u_2 + \ldots + z^n u_n)^{m/n}$, starting from $m + 1$-st.

Lemma 3.9. Using (3.3), one can express $v_i$ through $u_j$. The remaining equations on $u_i$ generate the ideal $J_{m/n}$.

Proof. Let us take the $n$th root of both parts of (3.3):

$$1 + z^2v_2 + \ldots + z^m v_m = (1 + z^2u_2 + \ldots + z^n u_n)^{m/n}.$$  

This allows us to express $v_i$ through $u_j$ explicitly, and the remaining equations on $u_j$ express the fact that the right hand side should be a polynomial of degree at most $m$. 

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Corollary 3.10.
\[ \mathbb{C}[u_1, \ldots, u_n]/J_{m/n} \cong \mathbb{C}[v_1, \ldots, v_m]/J_{n/m}. \]

Example 3.11. For \( n = 2 \) one can check that \( J_{(2k+1)/2} \) is generated by a single polynomial \( u_2^{k+1} \). Therefore the algebra of differential forms \( \Omega^*(\mathcal{M}_C) \) is defined by the equations
\[
  u_2^{k+1} = 0, \quad u_2^k du_2 = 0,
\]
and its basis consists of forms
\[
  1, u_2, \ldots, u_2^k, du_2, u_2^k du_2, \ldots, u_2^{k-1} du_2.
\]

Example 3.12. Let \( n = 3, m = 4 \). The ideal \( J_{4/3} \) is generated by the coefficients of the series \( (1 + u_2 z^2 + u_3 z^3)^{4/3} \), starting from 5-th, hence its generators are
\[
  \frac{4}{9} u_2 u_3, \quad \frac{2}{9} u_3^2 - \frac{4}{81} u_2^3.
\]
The basis in the quotient is presented by \( 1, u_2, u_2^2, u_2^3, u_3 \).
The algebra \( \Omega^*(\mathcal{M}_C) \) is defined by two more equations
\[
  u_2 du_3 + u_3 du_2 = 0, \quad 2u_3 du_3 - \frac{2}{3} u_2^2 du_2 = 0.
\]
Therefore we have 5 one-forms
\[
  \Omega^1(\mathcal{M}_C) = \langle du_2, du_3, u_2 du_2, u_2^2 du_3, u_2^2 du_2 \rangle
\]
and one two-form \( du_2 du_3 \).

4 A comparison

It turns out that the ideal \( I_{m/n} \) constructed in [7] is related to the moduli space \( \mathcal{M}_C \). Recall that \( V \) denotes the standard \( (n-1) \)-dimensional representation of the symmetric group \( S_n \) with the coordinates \( x_i \) modulo the relation \( \sum x_i = 0 \).

Definition 4.1. Let us introduce the elementary symmetric polynomials \( u_i(x_1, \ldots, x_n) \in \mathbb{C}[V]^{S_n} \) by the formula
\[
  U(z) = \prod_{i=1}^n (1 - zx_i) = 1 + \sum_{i=2}^n z^i u_i(x).
\]
Lemma 4.2. Let us introduce the power sums
\[ p_i = x_1^i + \ldots + x_n^i. \]
Then
\[ \frac{d}{dz} \ln U(z) = -\sum_{i=0}^{\infty} p_{i+1}z^i. \]

Theorem 4.3. For \( c = m/n \) one have
\[ (I_c)^{S_n} = J_{m/n}. \]

Proof. By construction, the set of generators of the ideal \( I_{m/n} \) form a standard representation of \( S_n \) (cf. Remark 4.4). Therefore to construct the generators of the symmetric part of \( I_{m/n} \), we have to compute
\[ [V \otimes \mathbb{C}[V]]^{S_n} / \mathbb{C}[V]_+^{S_n} = \text{Hom}_{S_n}(V, \mathbb{C}[V] / \mathbb{C}[V]_+^{S_n}), \]
where \( \mathbb{C}[V]_+^{S_n} \) denotes the ideal generated by the symmetric polynomials of positive degree. It is well known that \( \mathbb{C}[V] / \mathbb{C}[V]_+^{S_n} \) is isomorphic to the regular representation of \( S_n \), so there are \( (n-1) \) different copies of \( V \) in it, generated by \( x_i^k \) for \( 1 \leq k \leq n-1 \). This means that the symmetric part of the ideal \( I_{m/n} \) is generated by \( (n-1) \) polynomials:
\[ (I_c)^{S_n} = \langle \sum_i x_i^k f_i | 1 \leq k \leq n-1 \rangle. \]

Let us compute these generators.

Recall that
\[ F(z) = \prod_{i=1}^{n} (1 - zx_i)^{m/n} = \sum_{k=0}^{\infty} v_k(x) z^k. \]

By Lemma 4.2
\[ \frac{d}{dz} F(z) = \frac{m}{n} F(z) \frac{d}{dz} \ln U(z) = -\frac{m}{n} F(z) \cdot \sum_{i=0}^{\infty} p_{i+1}z^i, \]
Therefore
\[ (k+1)v_{k+1} = -\frac{m}{n} \sum_{i=0}^{k} v_{k-i}p_{i+1}. \] (4.1)
On the other hand,
\[ \sum_i f_i x_i^k = \sum_i \text{Coef}_m \frac{x_i^k}{1 - zx_i} F(z) = \sum_{j=0}^m v_{m-j}p_{j+k}. \]

Therefore by (4.1)
\[ \sum_i f_i x_i = \sum_{j=0}^m v_{m-j}p_{j+1} = -\frac{n}{m} (m+1)v_{m+1}, \]
\[ \sum_i f_i x_i^2 = \sum_{j=0}^m v_{m-j}p_{j+2} = -\frac{n}{m} (m+2)v_{m+2} - v_{m+1}p_1, \ldots \]
\[ \sum_i f_i x_i^k = \sum_{j=0}^m v_{m-j}p_{j+k} = -\frac{n}{m} (m+k)v_{m+k} - \sum_{j=1}^{k-1} v_{m+j}p_{k-j}, \]
and \( v_{m+1}, \ldots, v_{m+n-1} \) can be obtained from \( \sum_i f_i x_i^k \) with a triangular change of variables.

It rests to note that by Lemma 3.9 the polynomials \( v_{m+1}, \ldots, v_{m+n-1} \) generate the ideal \( J_{m/n} \).

\[ \square \]

Remark 4.4. If we plug in \( k = m - 1 \) in (4.1), we will get
\[ mv_m = -\frac{m}{n} \sum_{i=0}^{m-1} v_{k-i}p_{i+1}, \]
\[ n v_m + \sum_{i=0}^{m-1} v_{k-i}p_{i+1} = 0, \]
and
\[ \sum_j f_j = \sum_j \text{Coef}_m[(1 - zx_j)^{-1} F(z)] = n v_m + \sum_{i=1}^m v_{m-i}p_i = 0. \]

Corollary 4.5. \( \text{Coef}_k[B_i(z)] \in I_{m/n} \) for \( k \geq m \).

Proof. We have
\[ \text{Coef}_k[B_i(z)] = \text{Coef}_k[\frac{F(z)}{1 - zx_i}] = \sum_{a=0}^k x_i^a v_{k-a} = \]
\[ \sum_{a=0}^{k-m-1} x_i^a v_{k-a} + \sum_{a=k-m}^k x_i^a v_{k-a}. \]
By Theorem 4.3 the first sum belongs to $I_{m/n}$, and the second sum can be rewritten as
\[
\sum_{a=0}^{m} x_i^{k-a} v_{m-a} = x_i^{k-m} \sum_{a=0}^{m} x_i^a v_{m-a} = x_i^{k-m} \text{Coef}_m[B_i(z)] \in I_{m/n}.
\]

\begin{proof}
One can present $h$ as a linear combination of some powers of $x_1$ multiplied by some symmetric polynomials in $x_1, \ldots, x_n$. Therefore it is sufficient to prove (4.2) for $h = x_1^k$.

Since
\[
(1 - z x_1) \frac{\partial F(z)}{\partial x_1} = -\frac{m}{n} z F(z),
\]
one has
\[
x_1 \frac{\partial v_k}{\partial x_1} = \frac{\partial v_{k+1}}{\partial x_1} + \frac{m}{n} v_k.
\]
Using this equation, one can express
\[
x_1^k f_1 = -\frac{n}{m} x_1^k \frac{\partial v_{m+1}}{\partial x_1}
\]
via $v_k$ and $\frac{\partial v_k}{\partial x_1}$ with $k > m$.
\end{proof}

**Theorem 4.7.**
\[
\text{Hom}_{S_n}(\Lambda^k V, L_{m/n}) \simeq \Omega^k(M_{m,n}).
\]

Remark that
\[
\text{Hom}_{S_n}(\Lambda^k V, M_{m/n}) = \text{Hom}_{S_n}(\Lambda^k V, \mathbb{C}[V]) = \Omega^k(V)^{S_n} \simeq \Omega^k(V//S_n),
\]
where the last isomorphism follows from the results of [20]: every $S_n$-invariant differential form on $V$ can be obtained as a pullback of a form on $V//S_n$. To fix the notation, we give the following
Definition 4.8. We introduce a map 
\[ \lambda : \Omega^k(V//S_n) \to \text{Hom}_{S_n}(\Lambda^k V, \mathbb{C}[V]) \]
by the formula 
\[ \lambda_{i_1,\ldots,i_k}(\omega) = \pi^*\omega(\frac{\partial}{\partial x_{i_1}}, \ldots, \frac{\partial}{\partial x_{i_k}}), \]
where \( \pi : V \to V//S_n \) denotes the natural projection.

Proposition 4.9. The following equation holds:
\[ \lambda_{i_1,\ldots,i_s}(du_{k_1} \wedge \ldots \wedge du_{k_s}) = \left| \frac{\partial u_{k_s}}{\partial x_{i_h}} \right|. \]

Proof of Theorem 4.7. Let us check that the map \( \lambda \) sends the defining equations of \( \Omega^* (\mathcal{M}_C) \) inside the ideal \( I_{m/n} \).

Recall that
\[ F(z) = \prod_{i=1}^{n} (1 - zx_i)^{m/n} = \sum_{k=0}^{\infty} v_k(u) z^k, \]
and the defining ideal of \( \Omega^* (\mathcal{M}_C) \) is generated by the equations
\[ v_k(u) = dv_k(u) = 0 \quad \text{for} \quad k > m. \]

We checked in Theorem 4.3 that \( \lambda(v_k) \in I_{m/n} \), let us check that \( \lambda(dv_k) \in I_{m/n} \).

Remark that
\[ \lambda_i(dv_k) = \frac{\partial v_k}{\partial x_i} = \text{Coef}_k \frac{\partial F(z)}{\partial x_1} = -\frac{m}{n} \text{Coef}_{k-1} \frac{F(z)}{1 - z x_i}. \]
This coefficient belongs to \( I_{m/n} \) by Corollary 4.5.

Similarly to the proof of Theorem 4.3, one can check that every element of \( \text{Hom}_{S_n}(\Lambda^k V, I_{m/n}) \) can be presented as a combination of the determinants of the form
\[ \begin{vmatrix} f_{a_1} h_1 & f_{a_2} h_2 & \cdots & f_{a_k} h_k \\ \frac{\partial u_{\beta_1}}{\partial x_{a_1}} & \frac{\partial u_{\beta_1}}{\partial x_{a_2}} & \cdots & \frac{\partial u_{\beta_1}}{\partial x_{a_k}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u_{\beta_{k-1}}}{\partial x_{a_1}} & \frac{\partial u_{\beta_{k-1}}}{\partial x_{a_2}} & \cdots & \frac{\partial u_{\beta_{k-1}}}{\partial x_{a_k}} \end{vmatrix} \]
with symmetric coefficients, where $h_i$ are symmetric in all variables but $x_{\alpha_i}$. By Lemma 4.6 we can present (modulo $J_{m/n}$) every such determinant as a combination of the expressions

$$\lambda_{\alpha_1, \ldots, \alpha_k}(dv_s \wedge du_{\beta_1} \wedge \ldots \wedge du_{\beta_{k-1}}), \quad s > m$$

with symmetric coefficients. It rests to note that the form

$$dv_s \wedge du_{\beta_1} \wedge \ldots \wedge du_{\beta_{k-1}} = dv_s \wedge \omega$$

belongs to the defining ideal of $\Omega^*(\mathcal{M}_C)$. □

5 Action of Dunkl operators

We start with the following reformulation of the Corollary 2.8.

Lemma 5.1.

$$D_s \lambda_i(dv_k) = \delta_{is}(k - 1 - m)\lambda_i(dv_{k-1}).$$

Proof. Remark that

$$\lambda_i(dv_k) = \frac{\partial v_k}{\partial x_i} = -\frac{m}{n}\text{Coef}_{k-1}[B_i(z)],$$

by Corollary 2.8

$$D_s\text{Coef}_{k-1}[B_i(z)] = \delta_{is}(k - 1 - m)\text{Coef}_{k-2}[B_i(z)].$$

hence

$$D_s \lambda_i(dv_k) = \delta_{is}(k - 1 - m)\lambda_i(dv_{k-1}).$$

□

Lemma 5.2. ([7]) The following product rule holds for Dunkl operators:

$$D_i(fg) = D_i(f)g + D_i(g)f + \frac{m}{n} \sum_{i \neq j} \frac{(f - s_{ij}(f))(g - s_{ij}(g))}{x_i - x_j}.$$ 

Corollary 5.3. If $g$ is a symmetric polynomial then

$$D_i(fg) = D_i(f)g + D_i(g)f.$$
Lemma 5.4. (7) Suppose $f_1, \ldots, f_m$ are polynomials satisfying $(ij)f_i = f_i$ if $l \notin \{i, j\}$. Then

$$D_i(f_1 \cdots f_m) = \sum_{l=1}^m (D_if_l) \prod_{s \neq i} f_s + \frac{m}{n} \sum_{l \neq i} \frac{(f_i - s_l(f_i))(f_i - s_l(f_i))}{x_i - x_l} \prod_{s \neq l, i} f_s.$$  

(5.1)

Lemma 5.5. Suppose that the functions $a_j, 1 \leq j \leq k$ are symmetric with respect to all variables but $x_1$. Consider a matrix $M = (s_1(a_j))_{j=1}^k$. Then

$$D_i \det(M) = \sum_l \det(M_{i,l}),$$  

(5.2)

where $M_{s,l}$ denotes the matrix $M$ where the entries in the $l$'th row are replaced by their images under $D_i$.

Remark 5.6. This lemma shows that although $D_i$ is not a first order differential operator, it acts on these determinants as a first order differential operator would act.

Proof. Let us expand $\det(M)$ and apply the equation (5.1). We have to show that the ”correction terms” with divided differences will cancel out. These terms are labelled by the pairs $(\sigma, l)$ where $l \neq i$ and $\sigma \in S_k$, and the terms corresponding to $(\sigma, l)$ and $((il)\sigma, l)$ have opposite sign but same value

$$\frac{1}{x_i - x_l}(s_{1i}a_{\sigma(i)} - s_{il}s_{1i}a_{\sigma(i)})(s_{1i}a_{\sigma(l)} - s_{il}s_{1i}a_{\sigma(l)}) =$$

$$\frac{1}{x_i - x_l}(s_{1i}a_{\sigma(i)} - s_{1il}a_{\sigma(i)})(s_{1il}a_{\sigma(l)} - s_{1il}a_{\sigma(l)}).$$

We are ready to describe the action of Dunkl operators on the image of the map $\lambda$. By Lemma 5.3 and it is sufficient to compute the action of $D_i$ on the components of the differential form $dv_{\alpha_1} \wedge \ldots \wedge dv_{\alpha_k}$.

Theorem 5.7. Suppose that $\beta_1 < \ldots < \beta_j$. If $\beta_1 > 1$, then

$$D_1 \frac{\partial v_{\beta_i}}{\partial x_{\beta_j}} = 0.$$
If $\beta_1 = 1$, then

$$D_1 \left| \frac{\partial v_{\alpha_i}}{\partial x_{\beta_j}} \right| = \begin{vmatrix} (\alpha_1 - 1 - m) \frac{\partial v_{\alpha_1}}{\partial x_1} & \ldots & (\alpha_k - 1 - m) \frac{\partial v_{\alpha_k}}{\partial x_1} \\ \frac{\partial v_{\alpha_1}}{\partial x_{\beta_2}} & \ldots & \frac{\partial v_{\alpha_k}}{\partial x_{\beta_2}} \\ \vdots & \ddots & \vdots \\ \frac{\partial v_{\alpha_1}}{\partial x_{\beta_k}} & \ldots & \frac{\partial v_{\alpha_k}}{\partial x_{\beta_k}} \end{vmatrix}$$

Proof. Follows from Lemma 5.5 and Lemma 5.1.

**Definition 5.8.** ([19],[8]) The quantum Olshanetsky-Perelomov Hamiltonians are defined as

$$H_k = \sum_{s=1}^{n} D_s^k.$$ 

**Lemma 5.9.**

$$\sum_i \lambda_i dv_k = (k - 1 - m)v_{k-1}$$

Proof.

$$\sum_i \lambda_i dv_k = \sum_i \frac{\partial F(z)}{\partial x_i} = -\frac{m}{n} z F(z) \sum_i \frac{1}{1 - zx_i} = -nzF(z) - \frac{m}{n} z^2 F(z) \sum_i \frac{x_i}{1 -zx_i} = -nzF(z) + z^2 \frac{dF(z)}{dz}.$$ 

Theorem 5.10. The action of Hamiltonians on $u_l$ and $v_l$ has the following form:

$$H_k(u_l) = (l - 1 - m) \cdots (l - k + m)u_{l-k},$$

$$H_k(u_l) = (\frac{m}{n})^{k-1}(l - 1 - n) \cdots (l - k - n)u_{l-k} \quad (5.3)$$

Proof. Since $u_l$ is symmetric, $D_i(u_l) = \frac{\partial u_l}{\partial x_i}$, and one can check that

$$D_i^2(u_l) = \frac{m}{n}(l - 1 - n) \frac{\partial u_{l-1}}{\partial x_i}, \quad \sum_i \lambda_i (du_l) = (l - 1 - n)u_{l-1}$$
hence

\[ H_k(u_l) = \left(\frac{m}{n}\right)^{k-1}(l-1-n) \cdots (l-k+1-n) \sum_i \frac{\partial u_{l-k+1}}{\partial x_i} = \]

\[ \left(\frac{m}{n}\right)^{k-1}(l-1-n) \cdots (l-k+1-n)(l-k-n)u_{l-k}. \]

The proof for \( v_l \) is similar – it follows from Lemma 5.1 and Lemma 5.9. \( \square \)

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