AN EIGENVALUE ESTIMATE FOR THE \(\bar{\partial}\)-LAPLACIAN ASSOCIATED TO A NEF LINE BUNDLE

Jingcao Wu

Abstract. We study the \(\bar{\partial}\)-Laplacian on forms taking values in \(L^k\), a high power of a nef line bundle on a compact complex manifold, and give an estimate of the number of eigenvalues smaller than \(\lambda\). In particular, the \(\lambda = 0\) case gives an asymptotic estimate for the order of the cohomology groups value in nef line bundles. At last, we generalize the \(\lambda = 0\) case to a pseudo-effective line bundle.

1. Introduction

Let \(X\) be a compact complex manifold of dimension \(n\). \(L\) is a holomorphic line bundle on \(X\). Fix a Hermitian metric on \(X\) and a smooth metric on \(L\), the classic geometry theory allows us to define the adjoint operator of the \(\bar{\partial}\)-operator \(\bar{\partial}^*\) acting on \(L\)-valued forms as well as the corresponding Laplacian operator

\[
\Delta = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}.
\]

We can develop the harmonic theory starting from these operators, which (may not be a standard convention) is called \(L\)-harmonic theory. It turns out that \(L\)-harmonic theory behaves similarly with the traditional harmonic theory in the projective space as long as \(X\) is 'like' the projective space enough. The positivity of \(L\) is one way to measure the similarity between \(X\) and \(\mathbb{P}^n\). For example, if \(L\) is positive, \(X\) will be a projective manifold by the Kodaira embedding theorem, and many important properties in the harmonic theory on \(\mathbb{P}^n\) (resp. \(\mathcal{O}_{\mathbb{P}^n}(1)\)-harmonic theory in our convention) apply on \(X\) (resp. \(L\)-harmonic theory).

In this atmosphere, one naturally begins to study a more general line bundle. In this paper, we are interested in the number of eigenvalues of the Laplacian associated with a line bundle \(L\) with certain positivity. The work in this aspect dates back to [1]. Indeed, if we denote the linear space of \((n,q)\)-eigenforms of \(\Delta\), with corresponding eigenvalues smaller than or equal to \(\lambda\) by

\[
\mathcal{H}^{n,q}_{<\lambda}(X, L),
\]

it is given in [1] an asymptotic estimate for

\[
h^{n,q}_{<\lambda}(L^k \otimes E) = \dim \mathcal{H}^{n,q}_{<\lambda}(X, L^k \otimes E)
\]
in the case that $L$ and $E$ are line bundles on $X$ with $L$ semi-positive. Also it is shown in [1] that this estimate is sharp through an example (Proposition 4.2).

In this paper, we will give a similar result when $L$ is nef. We prove that

**Theorem 1.1.** Assume that $L$ is a nef line bundle on $X$, and $E$ is a vector bundle. Take $q \geq 1$. Then, if $0 \leq \lambda \leq k$,

\begin{equation}
 h^{n,q}_{\leq \lambda}(L^k \otimes E \otimes \mathcal{I}(L^k)) \leq C(\lambda + 1)^q k^{n-q}.
\end{equation}

If $1 \leq k \leq \lambda$, then

\begin{equation}
 h^{n,q}_{\leq \lambda}(L^k \otimes E \otimes \mathcal{I}(L^k)) \leq Ck^n.
\end{equation}

Here $\mathcal{I}(L)$ refers to the Nadel-type multiplier ideal sheaf [14], or we could call it a dynamic multiplier ideal sheaf.

We will explain $\mathcal{I}(L)$ and give a canonical way to define the Laplacian associated to a nef line bundle in the text. Since $E$ is allowed to be an arbitrary vector bundle, we see by substituting $E \otimes \Omega^n_X \otimes K_X^{-1}$ for $E$, that the same asymptotic estimate also holds for the numbers $h^{p,q}_{\leq \lambda}$.

This kind of estimate has an important counterpart in geometry, especially the $\lambda = 0$ case. In fact, $H^{p,q}_{\leq 0}(X, L^k \otimes E)$ is just the space of the harmonic $L^k \otimes E$-valued $(n, q)$-forms on $X$, which is isomorphic to the cohomology group

$$H^{n,q}(X, L^k \otimes E)$$

by Hodge’s theorem. In other word, we will eventually get an asymptotic estimate for $h^{n,q}(L^k \otimes E) = \dim H^{n,q}(X, L^k \otimes E)$.

The asymptotic estimate for the dimension of the cohomology group $H^{p,q}(X, L)$ where $L$ possesses a certain positivity is a complicated problem in complex geometry. There are various work in this aspect. Here we only list some literature for this type of problem. The first result is that

$$h^{0,q}(L^k) = o(k^n),$$

which is due to Siu [15, 16] when solving the Grauert–Riemenschneider conjecture [10]. Later Demailly also gives that

$$h^{0,q}(L^k \otimes E) \sim O(k^{n-q})$$

for a nef line bundle $L$ and a vector bundle $E$ on a projective manifold based on his holomorphic Moser inequality [6]. Moreover, Matsumura [11, 12] generalizes it as

$$h^{0,q}(L^k \otimes E \otimes \mathcal{J}(h^k)) \sim O(k^{n-q}),$$

where $(L, h)$ is a pseudo-effective line bundle and $E$ is a vector bundle on a projective manifold. Here $\mathcal{J}(h)$ refers to the multiplier ideal sheaf defined in [3], or we could call it a static multiplier ideal sheaf here in order to distinguish the dynamic one. Recently, this result has been generalized to a general compact complex manifold with additional
assumption that \( h \) has the algebraic singularities by [18]. We remark here that [18] also extends the estimate for \( h^{n,q}_{\leq \lambda}(L^k \otimes E) \) in [1] to the case that \( L \) is a semi-positive line bundle and \( E \) is a vector bundle.

Apply our result with \( \lambda = 0 \), we can also provide such an estimate for \( h^{0,q} \).

**Theorem 1.2.** Assume that \( L \) is a nef line bundle on \( X \), and \( E \) is a vector bundle. Take \( q \geq 1 \). Then

\[
  h^{0,q}(L^k \otimes E \otimes \mathcal{I}(L^k)) \leq Ck^{n-q}.
\]

Certainly in order to establish the relationship here between \( h^{0,q} \) and \( h^{0,q}_{\leq \lambda} \), we need a singular version of Hodge’s theorem, which will be discussed in the text (Proposition 2.1).

Then an easy application of the exact sequence, we have

**Corollary 1.1.** With the same assumptions above, assume that the dimension of the stable multiplier ideal subscheme of \( \{ V(\mathcal{I}(L^k)) \} \) is \( m \). Then for \( q > m \) and \( k \gg 1 \), we have that

\[
  h^{0,q}(L^k \otimes E) \leq Ck^{n-q}.
\]

Since \( V(\mathcal{I}(L^{k_2})) \subset V(\mathcal{I}(L^{k_1})) \) for any \( k_1 \leq k_2 \), the sequence

\[
  \{ V(\mathcal{I}(L^k)) \}
\]

will be stable at some \( k_0 \), and the stable multiplier ideal subscheme of \( \{ V(\mathcal{I}(L^k)) \} \) is then defined as \( V(\mathcal{I}(L^{k_0})) \).

Similar with [1], we can use this estimate to solve the extension problem. As a result, we can provide a more general version of the Grauert–Riemenschneider conjecture [10].

**Theorem 1.3** (Generalization of the Grauert–Riemenschneider conjecture). Let \( X \) be a compact complex manifold. Let \( L \) be a nef line bundle on \( X \). Then \( L \) is big if and only if \( L^n > 0 \).

It is well-known that \( X \) is a Moishezon manifold if and only if there exists a big line bundle on \( X \). The original Grauert–Riemenschneider conjecture says that if there is a semi-positive line bundle \( L \) on \( X \) such that the curvature \( i\Theta_L > 0 \) on an open subset, then \( X \) is Moishezon. Notice that such an \( L \) always satisfies the conditions in Theorem 1.3 (i.e. \( L \) is nef with \( L^n > 0 \)), so we actually generalize the original Grauert–Riemenschneider conjecture. We also remark here that when \( X \) is a projective manifold, Theorem 1.3 is a well-known result in algebraic geometry. Moreover, it has been extended to a Kähler manifold in [5] (Theorem 0.8) through the holomorphic Moser inequality. The method here is totally different. On the other hand, in Theorem 0.8 of [5], Demailly indeed gives three criterion for a line bundle to be big. The criterion (c) corresponds to the original Grauert–Riemenschneider conjecture. However, the precise relation of the other two criterion and Theorem 1.3 is for the moment not clear to me.
Next, we shall give a Nadel-type vanishing theorem.

**Theorem 1.4.** Let $X$ be a compact Kähler manifold, and $L$ be a nef line bundle. Suppose that the canonical pseudo-effective metric $h_0$ on $L$ is a Siu-type metric. Then we have

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(L)) = 0$$

for $q > n - \kappa(L)$.

We will explain later that what is the so-called canonical pseudo-effective metric $h_0$ for a nef line bundle and the Siu-type metric. Currently, in order to understand this result we only need to know that $\mathcal{I}(L) = \mathcal{I}(h_0)$.

Notice that in [3], it is shown another Nadel-type vanishing theorem saying that if $(L, \phi)$ is pseudo-effective, then

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(\phi)) = 0$$

for $q > n - \text{nd}(L, \phi)$. Here $\text{nd}(L, \phi)$ is the numerical dimension of $L$ associated with $\phi$ defined in [3]. Notice that we have $\kappa(L) \leq \text{nd}(L)$ in algebraic geometry, where $\text{nd}(L)$ is the numerical dimension of $L$ defined in algebraic method (of course without any specified metric). However, the example in [7] shows that there do exists the case that $\text{nd}(L) > \text{nd}(L, \phi_{\text{min}})$ with $\phi_{\text{min}}$ the minimal singular metric on $L$, so it seems to me that there is no obvious relation between $\text{nd}(L, \phi)$ and $\kappa(L)$. Therefore it is not clear currently that whether the work in [3] implies Theorem 1.4. Also we remark here that there is no obvious relationship between the work (Theorem 1.8) of [18] and Theorem 1.4.

In final, we make a discussion on the eigenform space $H^{n,q}(X, L) \leq \lambda$ with $\lambda = 0$ for a pseudo-effective line bundle $L$. More specifically, we will define $H^{n,q}(X, L)_{\leq 0}$ (Definition 5.1) and prove a singular version of Hodge’s theorem (Proposition 5.1).

The plan of this paper is as follows. In Section 2 we give a brief introduction on all the required materials including the nef line bundle, Bergman kernel for the space $\mathcal{H}^{n,q}$, Siu’s $\partial \bar{\partial}$-Bochner formula and so on. In Section 3 we prove the submeanvalue inequality for forms in $\mathcal{H}^{n,q}$ and complete the proof of Theorem 1.1. In Section 4 we relate Theorem 1.1 to the asymptotic estimate for the cohomology group and give some applications. In the final section, we consider the $\lambda = 0$ case for a pseudo-effective line bundle.

2. Preliminary

2.1. Nef line bundle. Let $L \to X$ be a line bundle on a compact complex manifold $X$. Let $S := \{h_i\}$ be a family of smooth metrics on $L$ with weight functions $\{\phi_i\}$, such that $\int_X e^{-\phi_i} \to \infty$ as $i$ tends to $\infty$, and $\int_V e^{-\phi_i} \leq C$ for some open subset $V$ of $X$. Then the Nadel-type
multiplier ideal sheaf (or dynamic multiplier ideal sheaf) at $x \in X$ can be defined as

$$\mathcal{I}(S)_x := \{f \in \mathcal{O}_{X,x}; \int_U |f|^2_{h_i} \leq C \text{ as } i \to \infty\},$$

where $U$ is a local coordinate ball of $x$.

On the other hand, if $h$ is a singular metric on $L$ with the weight function $\phi$, then its static multiplier ideal sheaf $[6]$ $\mathcal{I}(h)$ is defined as

$$\mathcal{I}(h)_x := \{f \in \mathcal{O}_{X,x}; |f|^2_h \text{ is integrable around } x\}.$$

These two type of multiplier ideal sheaves coincide well when $L$ is a nef line bundle. Indeed, if $L$ is nef, by definition there exists a family of smooth metrics $S = \{h_\varepsilon\}$ such that $i\Theta_{L,h_\varepsilon} \geq -\varepsilon \omega$ for any $\varepsilon > 0$. Here $\omega$ is a metric on $X$ fixed before. Let $\phi_\varepsilon$ be the weight function of $h_\varepsilon$, then it is quasi-plurisubharmonic. Therefore $\{\phi_\varepsilon\}$ is locally bounded in $L^1$-norm, hence relatively compact. So we can find a subsequence $\{\phi_{\varepsilon_i}\}$ converging to a limit $\phi_0$ in $L^1$-norm. Now if $f \in \mathcal{I}(S)_x$, we have

$$\int_U |f|^2 e^{-\phi_0} = \int_U \lim_{i \to \infty} |f|^2_{h_{\varepsilon_i}} = \lim_{i \to \infty} \int_U |f|^2_{h_{\varepsilon_i}} < \infty$$

by dominate convergence theorem. It means that $f \in \mathcal{I}(\phi_0)_x$. On the other hand, if $g \in \mathcal{I}(\phi_0)_x$, it is easy to see that $g \in \mathcal{I}(S)_x$ as well. In summary, we have

$$\mathcal{I}(S) = \mathcal{I}(\phi_0)$$

when $L$ is nef, and briefly denote it by $\mathcal{I}(L)$. Here we prefer to use the dynamic notation $\mathcal{I}$ in order to emphasize the role that the family $\{h_\varepsilon\}$ plays in it. It is also the start point of our work. For more information about the multiplier ideal sheaf (the dynamic one and the static one), one could refer to [6, 13].

Next we shall give a canonical way to define the Laplacian operator associated to a nef line bundle $L$. First, by definition we have a family of smooth metrics $S = \{h_\varepsilon\}$ with weight functions $\phi_\varepsilon$ discussed before. We take its convergent subsequence and still denote it by $\{h_{\varepsilon_i}\}$. In particular, we have $h_{\varepsilon_1} \leq h_{\varepsilon_2}$ for any $0 \leq \varepsilon_2 \leq \varepsilon_1$, and the $L^1$-limit is denoted by $h_0$ with weight function $\phi_0$.

Fix a smooth metric $\omega$ on $X$. Since $h_\varepsilon$ is a smooth metric on $L$, we can define the Laplacian operator $\Delta_\varepsilon$ corresponds to $\omega$ and $h_\varepsilon$. Now for any test $L$-valued $(p,q)$-form $\alpha$, we define the Laplacian operator associated to $h_0$ by

$$\Delta_0 \alpha := \lim_{\varepsilon \to 0} \Delta_\varepsilon \alpha$$

in the sense of $L^2$-topology. It is easy to verify that $\Delta_0$ is well-defined if and only if $h_\varepsilon$ converges to $h_0$ in $L^1$-norm, while the later thing has been guaranteed. $\Delta_0$ possesses some basic properties of the classic Laplacian operator, such as $\Delta_0 \bar{\partial} = \bar{\partial} \Delta_0$ and self-adjointness, i.e.
\[ <\Delta_0 \alpha, \beta >_{h_0} = <\alpha, \Delta_0 \beta >_{h_0} \] for any \( \alpha, \beta \). It is just some trivial calculation, so we omit the proof here.

There is one more issue need to be concerned. \( \Delta_0 \alpha \) may not be a smooth form even if \( \alpha \) is. So we need to clarify the definition of the eigenvalue and eigenform. First, given two \( L \)-valued smooth \((p, q)\)-forms \( \alpha, \beta \) on \( X \), we say that they are cohomological equivalent (it may not be a standard convention), if there exists an \( L \)-valued smooth \((p, q - 1)\)-form \( \gamma \) such that \( \alpha = \beta + \bar{\partial} \gamma \). It is easy to verify that it’s an equivalence relationship. We briefly denote it by \( \beta \in [\alpha] \) and vice versa. In particular, if \( \alpha \) or \( \beta \) is \( \bar{\partial} \)-closed, the cohomological equivalence just means that they belong to the same Dolbeault cohomology class.

Now we have the following definition.

**Definition 2.1.** Let \( \alpha \) be an \( L \)-valued smooth \((p, q)\)-form on \( X \). If for every \( \varepsilon \ll 1 \), there exists a cohomological equivalent representative \( \alpha_\varepsilon \in [\alpha] \) with \( \alpha_\varepsilon \rightarrow \alpha \) in \( L^2 \)-norm such that \( \Delta_\varepsilon \alpha_\varepsilon = \mu \alpha_\varepsilon \), then we call \( \alpha \) an eigenform of the Laplacian operator \( \Delta_0 \) with eigenvalue \( \mu \). We simply denote it by \( \Delta_0 \alpha = \mu \alpha \). Here we ask that \( \mu \) is independent of \( \varepsilon \).

Notice that \( h_\varepsilon \) is smooth, so it is meaningful to talk about the eigenform of \( \Delta_\varepsilon \). Then we can define the eigenform space as

\[ H^{p,q}_{\leq \lambda}(X, L, \Delta_0) := \{ \alpha \in A^{p,q}(X, L); \Delta_0 \alpha = \mu \alpha \text{ and } \mu \leq \lambda \}. \]

We are especially interested in the \( \lambda = 0 \) case since it corresponds to the Dolbeaut cohomology group. In fact, given a \( \bar{\partial} \)-closed \( L \)-valued \((p, q)\)-form \( \alpha \), it defines a Dolbeaut cohomology class \([\alpha] \in H^{p,q}(X, L)\). Then we have a unique \( \Delta_\varepsilon \)-harmonic representative \( \alpha_\varepsilon \in [\alpha] \) for every \( \varepsilon \) by Hodge’s theorem. Moreover, since the harmonic representative minimize the norm, we have \( \|\alpha_\varepsilon\|_{h_\varepsilon} \leq \|\alpha\|_{h_\varepsilon} \). Assume that \( \|\alpha\|_{h_\varepsilon} \leq C \) for all \( \varepsilon \) (which means that \( \alpha \in H^{p,q}(X, L \otimes \text{\mathcal{I}}(L)) \)). Then we can find a convergent subsequence of \( \{\alpha_\varepsilon\} \) with limit \( \tilde{\alpha} \), and \( \tilde{\alpha} \in [\alpha] \). Therefore \( \tilde{\alpha} \) is an eigenform of the Laplacian operator \( \Delta_0 \) with eigenvalue 0 by definition. In other word, we could say that \( \tilde{\alpha} \) is \( \Delta_0 \)-harmonic. On the other hand, a \( \Delta_0 \)-harmonic form \( \alpha \) must be \( \bar{\partial} \)-closed by definition, so it naturally defines a Dolbeaut cohomology class \([\alpha] \in H^{p,q}(X, L)\). Moreover, since it is a one-one correspondence between \( \alpha \) and \( \alpha_\varepsilon \) for every \( \varepsilon \), we have eventually proved a singular version of Hodge’s theorem.

**Proposition 2.1** (A singular version of Hodge’s theorem, I). Let \( X \) be a compact complex manifold. \( L \) is a nef line bundle on \( X \). Then there is a canonical Laplacian operator \( \Delta_0 \) associated to \( L \) such that the following isomorphism holds:

\[ H^{p,q}(X, L \otimes \text{\mathcal{I}}(L)) \simeq H^{p,q}_{\leq 0}(X, L \otimes \text{\mathcal{I}}(L), \Delta_0). \]

The proof of Proposition 2.1 also shows that the \( \Delta_0 \)-harmonic representative minimizes the \( L^2 \)-norm defined by \( h_0 \).
For a general $\lambda$, we will see (in the proof of the main result) that the $\alpha \in H^{p,q}_{< \lambda}$ with $\bar{\partial}\alpha = 0$ also plays an important role in the estimate of the number $h^{p,q}_{< \lambda} := \dim H^{p,q}_{< \lambda}(X, L, \Delta_0)$.

When $X$ is Kähler, one could even parallel extend the other properties in Hodge theory to this situation. However, it is not the theme of this paper, so we will not go deeper though it would be interesting.

2.2. Bergman kernel for the space $H^{n,q}_{< \lambda}$. The estimate of the numbers $h^{n,q}_{< \lambda}$ is based on an observation about the Bergman kernel. The Bergman kernel at $x \in X$ is defined as the function

$$B(x) = \sum |\alpha_j(x)|^2,$$

where $\{\alpha_j\}$ is an orthonormal basis for $H^{n,q}_{< \lambda}$, and the norm is the pointwise norm defined by the metrics on $L$ and $X$. More precisely, $B(x)$ is the pointwise trace on the diagonal of the true Bergman kernel, defined as the reproducing kernel for $H^{n,q}_{< \lambda}$.

The relevance of $B(x)$ for our problem lies in the formula

$$\int_X B(x) = h^{n,q}_{< \lambda},$$

which is evident since each term in the definition of $B$ contributes a 1 to the integral. On the other hand, $B(x)$ is intimately related to the solution of the extremal problem

$$S(x) = \frac{\sup |\alpha(x)|^2}{\|\alpha\|^2},$$

where the supremum is taken over all $\alpha$ in $H^{n,q}_{< \lambda}$. Indeed, the following lemma is classical in Bergman’s theory of reproducing kernels. It is presented as a more general version in [1]. Let $E$ be an hermitian vector bundle of rank $N$ on a manifold $X$ which is equipped with a positive measure. Let $V$ be a subspace of the space of continuous global sections of $E$ whose coefficients are in $L^2(X)$, and let $\alpha_j$ be an orthonormal basis for $V$. Define $B(x)$ and $S(x)$ with $\alpha_j$ and space $V$ same as before. Then we have

Lemma 2.1.

$$S(x) \leq B(x) \leq NS(x).$$

In particular,

$$\int_X S(x) \leq \dim(V) \leq N \int_X S(x).$$

The proof can be found in [1].

Theorem [1] therefore follows if we can prove a submeanvalue inequality that estimates the value of a form $\alpha \in H^{n,q}_{< \lambda}$ at any point $x \in X$ by its $L^2$-norm.
2.3. **Siu’s ∂¯∂-Bochner formula.** The ∂¯∂-Bochner formula for an \( L \)-valued \((n, q)\)-form was first developed by Siu in [14] on a compact Kähler manifold, then it is extended to the formula for an \( L \otimes E \)-valued \((n, q)\)-form on a general compact complex manifold in [1]. Here \( L, E \) are line bundles endowed with smooth metrics. Furthermore, it is generalized in [18] to a version suitable for a line bundle \( L \) and a vector bundle \( E \) with smooth metrics. For our purpose, we only present the latest version in [18] here.

**Proposition 2.2.** Let \( (X, \omega) \) be a compact complex manifold. Let \( E \) and \( (L, \phi) \) be holomorphic vector bundle of rank \( r \) and line bundle respectively. Let \( \alpha \) be an \( L \otimes E \)-valued \((n, q)\)-form. If \( \alpha \) is \( \bar{\partial} \)-closed, the following inequality holds:

\[
i \partial \bar{\partial} (T \alpha \wedge \omega^{q-1}) \geq (-2 \text{Re} < \Delta \alpha, \alpha > + < i \Theta_{L \otimes E} \wedge \Lambda \alpha, \alpha > -c |\alpha|^2) \omega_n.
\]

The constant \( c \) is zero if \( \bar{\partial} \omega_{q-1} = \bar{\partial} \omega_q = 0 \). Here \( T \alpha = c_{n-q}^* \alpha \wedge \ast \alpha e^{-\phi} \).

The proof can be found in [18].

If we denote \( \gamma = \ast \alpha \), \( \alpha \) can be expressed as

\[
\alpha = c_{n-q} \gamma \wedge \omega_q,
\]

and we moreover have

\[
\ast \gamma = (-1)^{n-q} c_{n-q} \gamma \wedge \omega_q.
\]

Then \( |\alpha|^2 \omega_n = T \alpha \wedge \omega_q \), so the norm of \( \alpha \) is given by the trace of \( T \alpha \).

2.4. **The Siu-type metric.** Let \( L \) be a \( \mathbb{Q} \)-effective line bundle on a compact complex manifold \( X \). In [17], Siu introduces a special singular metric \( h_{\text{siu}} \) as follows. For a basis \( \{ s^k_j \}_{j=1}^{N_k} \) of \( H^0(X, L^k) \), we define a metric \( \phi_k \) by

\[
\phi_k := \frac{1}{2k} \log \sum_{j=1}^{N_k} |s^k_j|^2.
\]

Then we take a convergent sequence \( \{ \varepsilon_k \} \), and define the metric \( \phi_{\text{siu}} \) on \( L \) by

\[
\phi_{\text{siu}} := \log \sum_{k=1}^{\infty} \varepsilon_k e^{\phi_k}.
\]

Certainly \( \phi_{\text{siu}} \) is pseudo-effective and provides an analytic Zariski decomposition.

**Proposition 2.3** (Analytic Zariski decomposition). For all \( k \geq 0 \), we have

\[
H^0(X, L^k \otimes \mathcal{O}(h_{\text{siu}}^k)) = H^0(X, L^k).
\]
3. A SUBMEANVALUE INEQUALITY FOR THE $\Delta_0$-EIGENFORMS AND THE ESTIMATE OF $h_{\leq\lambda}^{n,q}$

This section is devoted to prove a submeanvalue inequality. The argument here is borrowed from [1] with necessary adjustment. Here and in the later part of this paper, $L$ is assumed to be a nef line bundle on a compact complex manifold $X$ unless specified, and $\mathcal{H}_{\leq\lambda}^{n,q}(X, L \otimes E, \Delta_0)$ is the eigenform space defined in Section 2, which is briefly denoted by $\mathcal{H}_{\leq\lambda}^{n,q}(X, L \otimes E)$. $E$ is a vector bundle.

Fix a point $x$ in $X$ and choose local coordinates, $z = (z_1, ..., z_n)$ near $x$ such that $z(x) = 0$ and the metric form $\omega$ on $X$ satisfying

$$\omega = \frac{i}{2} \partial \bar{\partial} |z|^2 =: \beta$$

at the point $x$. The next proposition is the crucial step in this argument.

**Proposition 3.1.** With the same notations as in Section 2, let $\alpha \in \mathcal{H}_{\leq\lambda}^{n,q}(X, L^k \otimes E \otimes \mathcal{I}(L^k))$ satisfy $\bar{\partial} \alpha = 0$. Then for $r < \lambda^{-1/2}$ and $r < c_0$,

$$\int_{|z|<r} |\alpha|^2_{h_0} \leq C r^{2q}(\lambda + 1)^q \int_X |\alpha|^2_{h_0}.$$

The constants $c_0$ and $C$ are independent of $k$, $\lambda$ and the point $x$. Notice that the right hand side is finite by the condition that $\alpha \in \mathcal{H}_{\leq\lambda}^{n,q}(X, L^k \otimes E \otimes \mathcal{I}(L^k))$.

**Proof.** Since $L$ is nef, there exists a family of smooth metrics $\{h_{\varepsilon}\}$ such that $i \Theta_{L, h_{\varepsilon}} \geq -\varepsilon \omega$ for every $\varepsilon > 0$. We apply Proposition 2.2 to $(L^k, h_{k_{\varepsilon}}^k)$. The expression $< i \Theta_{L^k \otimes E} \wedge \Delta \alpha, \alpha > \omega_n$ can be estimated from below by a constant $c(1 - k\varepsilon)$ times $|\alpha|^2_{h_{\varepsilon}}$, so we get

$$i \partial \bar{\partial}(T_{\alpha} \wedge \omega_{q-1}) \geq (-2\Re < \Delta \alpha, \alpha > -c'(1 + k\varepsilon)|\alpha|^2_{h_{\varepsilon}})\omega_n. \quad (5)$$

By $T_{\alpha}$, we mean the $(n - q, n - q)$-form associated to $\alpha$ defined in Proposition 2.2. For $r$ small, put

$$\sigma(r, \varepsilon) = \int_{|z|<r} |\alpha|^2_{h_{\varepsilon}} \omega_n,$$

then it is left to prove that

$$\sigma(r, 0) \leq C r^{2q}(\lambda + 1)^q$$

if we have normalized so that the $L^2$-norm of $\alpha$ with respect to $h_0$ is equal to 1.

From (5) we see that

$$\int_{|z|<r} (r^2 - |z|^2) i \partial \bar{\partial} T_{\alpha} \wedge \omega_{q-1} \geq -c'(1 + k\varepsilon)r^2 \sigma(r) - 2r^2 \int_{|z|<r} |\Delta \alpha|_{h_{\varepsilon}} |\alpha|_{h_{\varepsilon}} \omega_n. \quad (6)$$
Put
\[ \lambda(r, \varepsilon) = \left( \int_{|z|<r} |\Delta_z \alpha|_{h_n}^2 \right)^{1/2}, \]
and use Cauchy’s inequality to obtain
\[ \int_{|z|<r} |\Delta_z \alpha|_{h_n} |\alpha|_{h_n} \omega_n \leq \lambda(r, \varepsilon) \sigma(r, \varepsilon)^{1/2}. \]

Applying Stokes’ formula to the left hand side of (6) we get, since
\[ \beta = \frac{i}{2} \partial \overline{\partial} |z|^2, \]
\[ 2 \int_{|z|<r} T_\alpha \wedge \omega_{q-1} \wedge \beta \]
\[ \leq \int_{|z|=r} T_\alpha \wedge \omega_{q-1} \wedge \partial |z|^2 + c' (1 + k\varepsilon) r^2 \sigma(r, \varepsilon) + 2r^2 \sigma(r)^{1/2} \lambda(r, \varepsilon). \]

Since \( \omega \) is smooth, by the choice of local coordinates we have,
\[ (1 - O(r)) \omega \leq \beta \leq (1 - O(r)) \omega. \]

Hence
\[ T_\alpha \wedge \omega_{q-1} \wedge \beta \geq q(1 - O(r)) |\alpha|_{h_n}^2 \omega_n. \]

Next, if \( \omega = \beta \) the boundary integral in (7) can be estimated by an integral with respect to surface measure
\[ r \int_{|z|=r} |\alpha|_{h_n}^2 dS, \]
and this implies that in our case,
\[ \int_{|z|=r} T_\alpha \wedge \omega_{q-1} \wedge \partial |z|^2 \leq r(1 - O(r)) \int_{|z|=r} |\alpha|_{h_n}^2 (\omega_n/\beta_n) dS. \]

But
\[ \int_{|z|=r} |\alpha|_{h_n}^2 (\omega_n/\beta_n) dS = \frac{d}{dr} \sigma(r, \varepsilon), \]
so if we also incorporate the term \( c' r^2 \sigma(r, \varepsilon) \) in \( O(r) \sigma(r, \varepsilon) \), we get
\[ 2q(1 - O(r)) \sigma(r, \varepsilon) \leq r \frac{d}{dr} \sigma(r, \varepsilon) + 2r^2 \sigma(r, \varepsilon)^{1/2} \lambda(r, \varepsilon) + k\varepsilon r^2 \sigma(r, \varepsilon). \]

Notice that \( \lim_{\varepsilon \to 0} \sigma(r, \varepsilon) = \sigma(r, 0) \), \( \lim_{\varepsilon \to 0} \frac{d}{dr} \sigma(r, \varepsilon) = \frac{d}{dr} \sigma(r, 0) \) and \( \lim_{\varepsilon \to 0} \lambda(r, \varepsilon) = \lambda(r, 0) \) by the fact that \( \alpha \in H^{n,q}_{\text{Loc}}(X, L^k \otimes E \otimes \mathcal{I}(L^k)) \) and the definition of the eigenform in Section 2, we have
\[ 2q(1 - O(r)) \sigma(r, 0) \leq r \frac{d}{dr} \sigma(r, 0) + 2r^2 \sigma(r, 0)^{1/2} \lambda(r, 0) \]
as \( \varepsilon \) tends to zero. Then it follows the same analytic technique as in [1], we conclude our desired result. \( \square \)

We are now ready to prove the submeanvalue inequality.
Proposition 3.2. Let \( \alpha \in \mathcal{H}^{n,q}(X, L^k \otimes E \otimes \mathcal{I}(L^k)) \) satisfy \( \bar{\partial} \alpha = 0 \). Then for any \( x \in X \)
\[
|\alpha(x)|_{h_0}^2 \leq Ck^{n-q} (\lambda + 1)^q \int_X |\alpha|_{h_0}^2 \omega_n
\]
if \( \lambda \leq k \) and
\[
|\alpha(x)|_{h_0}^2 \leq C\lambda^n \int_X |\alpha|_{h_0}^2 \omega_n
\]
if \( \lambda \geq k \geq 1 \). The constant is independent of \( k, \lambda \) and \( x \). In particular, the norm of \( \alpha \) with respect to the singular metric \( h_0 \) is finite everywhere.

Proof. The proof is mostly borrowed from [1], which is a clever application of the localization technique. Assume first \( \lambda \leq k \) and fix \( x \in X \). Choose as before local coordinates, \( z \), near \( x \) such that \( z(x) = 0 \) and \( \omega = \frac{i}{2} \bar{\partial} |z|^2 = \beta \) at the point \( x \). Choose also local trivializations of \( L \) and \( E \) near \( x \). Now we take a family of metrics \( \{ h_\epsilon \} \) with weight functions \( \phi_\epsilon \) on \( L \) as before. For any \( \epsilon > 0 \), we may assume the local trivialization is chosen so that the metric of \( L \) has the form
\[
\phi_\epsilon = \sum \mu_j |z_j|^2 + o(|z|^2).
\]
For any \( \alpha \) we express it in terms of the trivialization and local coordinates and put
\[
\alpha^{(k)}(z) = \alpha(z/\sqrt{k}),
\]
so that \( \alpha^k \) is defined for \( |z| < 1 \) if \( k \) is large enough. We also scale the Laplacian by putting
\[
k^m \Delta_\epsilon^{(k)} \alpha^{(k)} = (\Delta_\epsilon \alpha)^{(k)}.
\]
It is not hard to see that if \( \Delta_\epsilon \) is defined by the metric \( \psi \) on \( F_k \), then \( \Delta_\epsilon^{(k)} \) is associated to the line bundle metric \( \psi(z/\sqrt{k}) \). In particular, if \( F_k = L^k \) and \( \psi = k\phi_\epsilon \), then \( \Delta_\epsilon^{(k)} \) is associated to
\[
\sum \mu_j |z_j|^2 + o(1),
\]
and hence converges to a \( k \)-independent elliptic operator. Obviously, the same thing happens even if we substitute \( L^k \) by \( L^k \otimes E \) for a vector bundle \( E \). It therefore follows from Gårding’s inequality together with Sobolev estimates that
\[
|\alpha(0)|_{h_\epsilon}^2 \leq C(\int_{|z|<1} |\alpha^{(k)}|_{h_\epsilon}^2 \omega_n + \int_{|z|<1} |(\Delta_\epsilon^{(k)})^m \alpha^{(k)}|_{h_\epsilon}^2 \omega_n),
\]
if \( m > n/2 \). Now
\[
\int_{|z|<1} |\alpha^{(k)}|_{h_\epsilon}^2 \omega_n = k^n \int_{|z|<1/\sqrt{k}} |\alpha|_{h_\epsilon}^2 \omega_n
\]
and
\[
\int_{|z|<1} |\Delta_\epsilon^{(k)} \alpha^{(k)}|_{h_\epsilon}^2 \omega_n = k^{n-2m} \int_{|z|<1/\sqrt{k}} |(\Delta_\epsilon)^m \alpha|_{h_\epsilon}^2 \omega_n.
\]
So take the limit of (9) with respect to $\varepsilon$, we have
\begin{equation}
|\alpha(0)|^2_{h_0} \leq C(k^n \int_{|z| < 1/\sqrt{k}} |\alpha|^2_{h_0} \omega_n + k^{n-2m} \int_{|z| < 1/\sqrt{k}} |(\Delta_0)^m \alpha|^2_{h_0} \omega_n).
\end{equation}

Do the normalization so that the $L^2$-norm of $\alpha$ with respect to $h_0$ is one. By Proposition 3.1 we have
\begin{equation}
k^n \int_{|z| < 1/\sqrt{k}} |\alpha|^2_{h_0} \omega_n \leq C k^{-q}(\lambda + 1)^q,
\end{equation}
and
\begin{equation}
k^{n-2m} \int_{|z| < 1/\sqrt{k}} |(\Delta_0)^m \alpha|^2_{h_0} \omega_n \leq C k^{-q}(\lambda + 1)^q (\lambda/k)^{2m}
\leq C k^{-q}(\lambda + 1)^q.
\end{equation}

Combine these two inequalities with (10), we have thus proved the first part of Proposition 3.2. The second statement is much easier. We now apply (10) to the scaling $\alpha^{(\lambda)}$ instead, and get immediately that
\begin{equation}
|\alpha(0)|^2_{h_0} \leq C \lambda^n.
\end{equation}

We now have all the ingredients for the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Let first $Z_{\leq \lambda}^{n,q}$ be the subspace of $H_{\leq \lambda}^{n,q}$ consisting of all the $\bar{\partial}$-closed forms. We apply Lemma 2.1 with $L^k \otimes E$-valued $(n,q)$-forms. The estimate for $S(x)$ furnished by Proposition 3.2 together with Lemma 2.1 then immediately gives Theorem 1.1 for $Z_{\leq \lambda}^{n,q}$.

We now claim that
\begin{equation}
h_{\leq \lambda}^{n,q} \leq \dim Z_{\leq \lambda}^{n,q} + \dim Z_{\leq \lambda}^{n,q+1},
\end{equation}
which completes the proof since our estimate for $\dim Z_{\leq \lambda}^{n,q+1}$ is better than our desired estimate for $h_{\leq \lambda}^{n,q}$. The claim is not complicated and was first proved in [1]. We present here for readers’ convenience. First note that if $\alpha$ is an eigenform of $\Delta_0$, so that
\begin{equation}
\Delta_0 \alpha = \mu \alpha
\end{equation}
and if we decompose $\alpha = \alpha^1 + \alpha^2$ where $\alpha^1$ is $\bar{\partial}$-closed and $\alpha^2$ is orthogonal (with respect to $h_0$) to the space of $\bar{\partial}$-closed forms then the $\alpha^j$’s with $j = 1, 2$ are also eigenforms with the same eigenvalue. To see this, note that $\Delta_0$ commutes with $\bar{\partial}$, so $\bar{\partial}\Delta_0 \alpha_1 = 0$ and
\begin{equation}
< \Delta_0 \alpha^2, \eta >_{h_0} = < \alpha^2, \Delta_0 \eta >_{h_0} = 0
\end{equation}
if $\bar{\partial} \eta = 0$. Hence
\begin{equation}
\Delta_0 \alpha^j = (\Delta_0 \alpha)^j = \lambda \alpha^j
\end{equation}
for $j = 1, 2$. Now decompose
\begin{equation}
H_{\leq \lambda}^{n,q} = Z_{\leq \lambda}^{n,q} \oplus (H_{\leq \lambda}^{n,q} \ominus Z_{\leq \lambda}^{n,q}).
\end{equation}
Since $\bar{\partial}$ maps $H^{n,q}_{\leq \lambda} \to Z^{n,q}_{\leq \lambda}$ injectively into $Z^{n,q+1}_{\leq \lambda}$, (11) follows and the proof of Theorem 1.1 is complete.

Since a semi-positive line bundle will be nef, we remark here that the example (Proposition 4.2) in [1] also shows that the order of magnitude given in Theorem 1.1 cannot be improved in general.

4. AN ASYMPTOTIC ESTIMATE FOR $h^{0,q}$ AND APPLICATIONS

In this section, we shall list some applications. Firstly, we finish the proof of Theorem 1.2.

Proof of Theorem 1.2. Apply Theorem 1.1 with $\lambda = 0$ and Proposition 2.1 then substitute $E \otimes K_X^{-1}$ for $E$ to get the desired result.

Now we denote $V_k = V(I(L^k))$ the multiplier ideal subscheme. It is easy to verify that $V_{k_2} \subset V_{k_1}$ for any $k_1 \leq k_2$, and the Noetherian property implies that the sequence $\{V_k\}$ will be stable at some $V_{k_0}$. Let $m = \dim V_{k_0}$. Then we can prove Corollary 1.1.

Proof of Corollary 1.1. Take $k \geq k_0$. From the short exact sequence

$$0 \to L^k \otimes E \otimes I(L^k) \to L^k \otimes E \to L^k \otimes E \otimes (\mathcal{O}_X/I(L^k)) \to 0,$$

we get the cohomology long exact sequence

$$\cdots \to H^q(X, L^k \otimes E \otimes I(L^k)) \to H^q(X, L^k \otimes E)$$

$$\to H^q(V_{k_0}, (L^k \otimes E)|_{V_{k_0}}) \to H^{q+1}(X, L^k \otimes E \otimes I(L^k)) \to \cdots .$$

Since $\dim V_{k_0} = m$, we have $H^q(V_{k_0}, (L^k \otimes E)|_{V_{k_0}}) = 0$ when $q > m$. On the other hand, $h^{0,q}(L^k \otimes E \otimes I(L^k)) \leq Ck^{n-q}$ by Theorem 1.2. So we obtain that

$$h^{0,q}(L^k \otimes E) \leq Ck^{n-q}$$

when $q > m$ by (12).

The second application will be the extension problem of the holomorphic sections. In fact, we can prove a more general version of the Grauert–Riemenschneider conjecture.

Proof of Theorem 1.3. By Theorem 1.2 and the Riemann–Roch formula, we have

$$h^0(L^k \otimes I(L^k)) = \chi(L^k \otimes I(L^k)) + O(k^{n-1})$$

$$= \chi(L^k) - \chi(L^k|_{V(I(L^k))}) + O(k^{n-1})$$

$$= \frac{k^n L^n}{n!} - \frac{k^m L^m \cdot [V(I(L^k))]}{m!} + O(k^{n-1})$$

if $m = \dim V(I(L^k))$. Note $H^0(X, L^k \otimes I(L^k)) \subset H^0(X, L^k)$, $L$ is big if and only if $L^n > 0$. □
We prove a vanishing theorem to finish this section. First we introduce an injectivity theorem given in [11].

**Theorem 4.1** (Matsumura). Let \((L, h_L)\) and \((F, h_F)\) be two singular Hermitian line bundle on a compact Kähler manifold \(X\). Assume there are the following things:

- There exists a subvariety \(Z\) of \(X\) such that \(h_L, h_F\) are smooth on \(X - Z\).
- \(i\Theta_{L,h_L}, i\Theta_{F,h_F} \geq \gamma\) on \(X\) for some real smooth \((1,1)\)-form \(\gamma\).
- \(i\Theta_{L,h_L} \geq 0\) on \(X - Z\).
- \(i\Theta_{L,h_L} \geq \varepsilon i\Theta_{F,h_F}\) for some \(\varepsilon > 0\).

Assume that \(h^0(X, K_X \otimes L \otimes \mathcal{I}(h_L))\) is non-zero. Then we have

\[
\dim H^0_{\text{bdd}, h_F}(X, F) \leq h^q(X, K_X \otimes L \otimes F \otimes \mathcal{I}(h_L h_F)),
\]

where \(H^0_{\text{bdd}, h_F}(X, F) := \{ u \in H^0(X, F); \sup_X |u|_{h_L} < \infty \}\).

**Proof of Theorem 4.1** Suppose that \(h^q(X, K_X \otimes L \otimes \mathcal{I}(L))\) is non-zero for \(q > n - \kappa(L)\). We have

\[
h^0(X, L^{k-1}) = h^0_{\text{bdd}, h^{k-1}_0}(X, L^{k-1}) \leq h^q(X, K_X \otimes L^k \otimes \mathcal{I}(h^k_0)).
\]

The first equality comes from the assumption that \(h_0\) is a Siu-type metric and the second inequality is due to Theorem 4.1. By the definition of Iitaka dimension \(\kappa(L)\), we have

\[
\limsup_{k \to \infty} \frac{h^0(X, L^{k-1})}{(k - 1)\kappa(L)} > 0.
\]

It means that

\[
\limsup_{k \to \infty} \frac{h^q(X, K_X \otimes L^k \otimes \mathcal{I}(h^k_0))}{(k - 1)\kappa(L)} > 0,
\]

which is a contradiction to \(q > n - \kappa(L)\). In fact, using the same notations as before, since

\[
\mathcal{I}(L) = \mathcal{I}(h_0),
\]

we have \(h^q(X, K_X \otimes L^k \otimes \mathcal{I}(h^k_0)) \leq C k^{a-q}\) by Theorem 1.2. It completes the proof of Theorem 4.1. \(\square\)

5. **The pseudo-effective case**

In this section, we will discuss the situation when \(L\) is merely pseudo-effective. As we have shown before, the ingredient to define a Laplacian operator as well as the associated eigenvalue for a singular metric \(\phi\) is to approximate it by a family of smooth metrics \(\{\phi_\varepsilon\}\). The problem for a pseudo-effective metric is that we can do such an approximation by [8] only on an open subvariety \(Y = X - Z\). At this time, certainly we could formally define the similar Laplacian operator on \(Y\) as before,
but it seems to be not a good definition. In fact, $h^{p,q}_{<\lambda}(Y, L, \Delta_{\phi})$ is usually bigger than $h^{p,q}_{<\lambda}(X, L, \Delta_{\tilde{\phi}})$ for a smooth $\phi$. It means that such a definition for Laplacian doesn’t coincide with the smooth one. It is not clear currently how to define the Laplacian operator associated to a pseudo-effective metric, nor the corresponding space $\mathcal{H}^{p,q}_{\leq \lambda}$.

On the other hand, it is still possible to talk about the harmonic $(n, q)$-form, i.e. the element of $\mathcal{H}^{n,q}_{<\lambda}$. Now assume that $(L, \phi)$ is a pseudo-effective line bundle on a compact complex manifold $X$. In particular, we ask that there exits a holomorphic section $s$ of $L$ for some integer $k_0$, such that $\sup_{X} |s|_k < \infty$. Fix a Hermitian metric $\omega$ on $X$. Then by Demailly’s approximation, we can find a family of metrics $\{\phi_{\varepsilon}\}$ on $L$ with the following properties:

(a) $\phi_{\varepsilon}$ is smooth on $X - Z_{\varepsilon}$ for a subvariety $Z_{\varepsilon}$;
(b) $\phi_{\varepsilon_1} \leq \phi_{\varepsilon_2} \leq \phi$ holds for any $0 < \varepsilon_1 \leq \varepsilon_2$;
(c) $\mathcal{J}(\phi) = \mathcal{J}(\phi_{\varepsilon})$; and
(d) $\Theta_{L,\phi_{\varepsilon}} \geq -\varepsilon \omega$.

Thanks to the proof of the openness conjecture by Berndtsson [2, 4], one can arrange $h_{\varepsilon}$ with logarithmic poles along $Z_{\varepsilon}$ according to the remark in [3]. Moreover, since the norm $|s|_k$ is bounded on $X$, the set $\{x \in X | \nu(\phi_{\varepsilon}, x) > 0\}$ for every $\varepsilon > 0$ is contained in the subvariety $Z := \{x | s(x) = 0\}$ by property (b). Here $\nu(\phi_{\varepsilon}, x)$ refers to the Lelong number of $\phi_{\varepsilon}$ at $x$. Hence, instead of (a), we can assume that

(a') $\phi_{\varepsilon}$ is smooth on $X - Z$, where $Z$ is a subvariety of $X$ independent of $\varepsilon$.

Now let $Y = X - Z$. We can use the method in [4] to construct a complete Hermitian metric on $Y$ as follows. Since $Y$ is weakly pseudoeconvex, we can take a smooth plurisubharmonic exhaustion function $\psi$ on $X$. Define $\tilde{\psi} = \omega + \frac{1}{l} i \partial \bar{\partial} \psi^2$ for $l > 0$. It is easy to verify that $\tilde{\psi}$ is a complete Hermitian metric on $Y$ and $\tilde{\psi} \geq \frac{1}{l} \omega$.

Let $L^{n,q}_{(2)}(Y, L)_{\phi_{\varepsilon}} \tilde{\psi}$ be the $L^2$-space of $L$-valued $(n, q)$-forms $u$ on $Y$ with respect to the inner product given by $\phi_{\varepsilon}, \tilde{\psi}$. Then we have the orthogonal decomposition

$$L^{n,q}_{(2)}(Y, L)_{\phi_{\varepsilon}} \tilde{\psi} = \text{Im} \tilde{\partial} \bigoplus \mathcal{H}^{n,q}_{\phi_{\varepsilon}, \tilde{\psi}}(L) \bigoplus \text{Im} \tilde{\partial}^*_{\phi_{\varepsilon}}$$

where

$$\mathcal{H}^{n,q}_{\phi_{\varepsilon}, \tilde{\psi}}(L) = \{\alpha | \tilde{\partial} \alpha = 0, \tilde{\partial}^*_{\phi_{\varepsilon}} \alpha = 0\}.$$

We give a brief explanation for decomposition (13). Usually $\text{Im} \tilde{\partial}$ is not closed in the $L^2$-space of a noncompact manifold even if the metric is complete. However, in the situation we consider here, $Y$ has the compactification $X$ and the forms are bounded in $L^2$-norms. Such a form will have certain extension properties. Therefore the space $L^{n,q}_{(2)}(Y, L)_{\phi_{\varepsilon}} \tilde{\psi} \cap \text{Im} \tilde{\partial}$ behaves much like the space $\text{Im} \tilde{\partial}$ on $X$. The complete explanation can be found in [9, 19].
Now we have all the ingredients for the definition of $\Delta_{\phi}$-harmonic forms. We denote the Laplacian operator on $Y$ associated to $\tilde{\omega}$ and $\phi_\varepsilon$ by $\Delta_\varepsilon$.

**Definition 5.1.** Let $\alpha$ be a $L$-valued smooth $(n, q)$-form on $X$ with bounded $L^2$-norm with respect to $\omega, \phi$. If for every $\varepsilon \ll 1$, there exists a cohomological equivalent class $\alpha_\varepsilon \in [\alpha|_Y]$ such that $\alpha_\varepsilon \rightarrow \alpha|_Y$ in $L^2$-norm and $\Delta_\varepsilon \alpha_\varepsilon = 0$ on $Y$, then we call $\alpha$ a $\Delta_\phi$-harmonic form. The space of all the $\Delta_\phi$-harmonic forms is denoted by $\mathcal{H}^{n,q}_{\leq 0}(X, L \otimes \mathcal{I}(\phi), \Delta_\phi)$.

We will show that Definition 5.1 is compatible with the usual definition of $\Delta_\phi$-harmonic forms if $\phi$ is smooth by proving the following Hodge-type isomorphism.

**Proposition 5.1** (A singular version of Hodge’s theorem, II). Let $X$ be a compact complex manifold. $(L, \phi)$ is a pseudo-effective line bundle on $X$. In particular, there exists a section $s$ of some multiple $L^k$ such that $\sup_X |s|_{\phi^k} < \infty$. Then the following isomorphism holds:

\[
H^{n,q}(L \otimes \mathcal{I}(\phi)) \cong H^{n,q}_{\leq 0}(X, L \otimes \mathcal{I}(\phi), \Delta_\phi).
\]

In particular, when $\phi$ is smooth, $\alpha \in H^{n,q}_{\leq 0}(X, L, \Delta_\phi)$ if and only if $\alpha$ is $\Delta_\phi$-harmonic in the usual sense.

**Proof.** We use the de Rham–Weil isomorphism

\[
H^{n,q}(L \otimes \mathcal{I}(\phi)) \cong \frac{\text{Ker} \bar{\partial} \cap L^{n,q}_{(2)}(X, L_{\phi^k})}{\text{Im} \partial}
\]

to represent a given cohomology class $[\alpha] \in H^{n,q}(L \otimes \mathcal{I}(\phi))$ by a $\bar{\partial}$-closed $L$-valued $(n, q)$-form $\omega$ with $\|\alpha\|_{\phi, \omega} < \infty$. We denote $\alpha|_Y$ simply by $\alpha_Y$. Since $\tilde{\omega} \geq \frac{1}{k}\omega$, it is easy to verify that $|\alpha_Y|^2_{\phi_\varepsilon, \omega} dV_{\omega} \leq C|\alpha|^2_{\phi_\varepsilon, \omega} dV_{\omega}$, which leads to inequality $\|\alpha_Y\|_{\phi_\varepsilon, \omega} \leq C\|\alpha\|_{\phi, \omega}$ with $L^2$-norms. Here $C$ is a constant used in a generic sense. Hence by property (b), we have $|\alpha_Y|_{\phi_\varepsilon, \omega} \leq C|\alpha|_{\phi, \omega}$ which implies $\alpha_Y \in L_{(2)}^{n,q}(Y, L)_{\phi_\varepsilon, \omega}$. By decomposition (13), we have a harmonic representative $\alpha_\varepsilon$ in $\mathcal{H}^{n,q}_{\phi_\varepsilon, \omega}(L)$, which means that $\Delta_\varepsilon \alpha_\varepsilon = 0$ on $Y$ all $\varepsilon$. Moreover, since a harmonic representative minimize the $L^2$-norm, we have

\[
\|\alpha_\varepsilon\|_{\phi_\varepsilon, \omega} \leq \|\alpha_Y\|_{\phi_\varepsilon, \omega} \leq C\|\alpha\|_{\phi, \omega}.
\]

So we can take the limit $\tilde{\alpha}$ of (a subsequence of) $\{\alpha_\varepsilon\}$ such that $\tilde{\alpha} \in [\alpha_Y]$. In particular,

\[
\tilde{\alpha} \in \mathcal{H}^{n,q}_{\leq 0}(Y, L \otimes \mathcal{I}(\phi), \Delta_\phi)
\]

by definition. So it is left to extend it to $X$. Indeed, by (the proof of) Proposition 2.1 in [19], there is an injective morphism, which maps $\tilde{\alpha}$ to a $\bar{\partial}$-closed $L$-valued $(n - q, 0)$-form, which can be formally denoted by $*\tilde{\alpha}$, on $Y$ with bounded $L^2$-norm. So the canonical extension theorem applies here and $*\tilde{\alpha}$ extends to a $\bar{\partial}$-closed $L$-valued $(n - q, 0)$-form on
Furthermore, it is shown by Proposition 2.2 in \[19\] that $c_{n-q} \omega_q \wedge *\tilde{\alpha}$ is an $L$-valued $(n, q)$-form with

$$c_{n-q} \omega_q \wedge *\tilde{\alpha}|_Y = \tilde{\alpha}.$$ 

Therefore we finally get the extension $\hat{\alpha}$ of $\tilde{\alpha}$. In other words, $\hat{\alpha} \in H^{n,q}(X, L \otimes I(\phi, \Delta))$. We denote this morphism by $i(\alpha) = \hat{\alpha}$.

On the other hand, for a given $\alpha \in H^{n,q}(X, L \otimes I(\phi, \Delta))$, by definition there exists an $\alpha_\varepsilon \in \mathcal{H}^{n,q}_{\phi_\varepsilon}(L)$ for every $\varepsilon$. In particular, $\partial \alpha_\varepsilon = 0$. So all of the $\alpha_\varepsilon$ together with $\alpha_Y$ define a common cohomology class $[\alpha_Y]$ in $H^{n,q}(Y, L \otimes I(\phi))$. So it is left to extend this class to $X$. Indeed, by (the proof of) Proposition 2.1 in \[19\], there is an injective morphism $S^q$, which maps $[\alpha_Y]$ to $S^q(\alpha_Y) \in H^0(X, \Omega^{n-q}_X \otimes L \otimes I(\phi))$. Furthermore,

$$c_{n-q} \omega_q \wedge S^q(\alpha_Y) \in H^{n,q}(X, L \otimes I(\phi))$$

with $[(c_{n-q} \omega_q \wedge S^q(\alpha_Y))|_Y] = [\alpha_Y]$. We denote this morphism by $j(\alpha) = [c_{n-q} \omega_q \wedge S^q(\alpha_Y)]$. It is easy to verify that $i \circ j = \text{id}$ and $j \circ i = \text{id}$. The proof is finished. \hfill \Box

We remark that the estimate for $h^{n,q}_{\leq 0}(L^k \otimes E)$ is enough to get the ones for other $h^{p,q}_{\leq 0}(L^k \otimes E)$ since we can substitute $E$ by $E \otimes \Omega^{1}_{X} \otimes K_{X}^{-1}$. So it remains to prove an estimate for $h^{n,q}_{\leq 0}(L^k \otimes E)$.

Unfortunately we cannot go any deeper at this moment since our method fails. Our method highly depends on a submeanvalue inequality (Proposition 3.2). However, there is a gap when proving such an inequality on an open subset.

In fact, the proof of Proposition 3.2 involves the localization technique to give a Sobolev-type estimate (the formula (9)). If we want to get a similar inequality on $Y$, first we need to equip $Y$ with a complete Hermitian metric $\tilde{\omega}$. At this time, (9) is sort of like to estimate

$$\int_{|z|<1} |F| |\nabla h_\varepsilon|^2 \tilde{\omega}_n.$$ 

Here $F$ is the multiplier and describes the local rescalings of infinitesimally small coordinate charts. When the first derivative $\nabla h_\varepsilon$ becomes large as the point approaching $Z$ and $\varepsilon$ tending zero, to make the $L^2$-norm bounded, we have to enlarge the coordinate in that direction at that point. It is the same as collapsing the manifold along that direction at that point. When we fix our sight on the manifold, $\nabla h_\varepsilon$ blows up, but when we fix our sight on $\nabla h_\varepsilon$, the manifold collapses. It is the basic reason that we need a multiplier ideal sheaf in the statement of Theorem 1.1. Unfortunately in the pseudo-effective situation it seems to be less helpful. So it is still an open question to get an estimate for $h^{n,q}_{\leq 0}(L^k \otimes E)$. 17
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Current address: School of Mathematical Sciences, Fudan University, Shanghai 200433, People’s Republic of China.
E-mail address: jingcaowu08@gmail.com, jingcaowu13@fudan.edu.cn