The Hamiltonian problem and $t$-path traceable graphs

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Abstract

The problem of characterizing maximal non-Hamiltonian graphs may be naturally extended to characterizing graphs that are maximal with respect to non-traceability and beyond that to $t$-path traceability. We show how traceability behaves with respect to disjoint union of graphs and the join with a complete graph. Our main result is a decomposition theorem that reduces the problem of characterizing maximal $t$-path traceable graphs to characterizing those that have no universal vertex. We generalize a construction of maximal non-traceable graphs by Zelinka to $t$-path traceable graphs.

1 Introduction

The motivating problem for this article is the characterization of maximal non-Hamiltonian (MNH) graphs. Skupien and co-authors give the first broad family of MNH graphs in [6] and describe all MNH graphs with 10 or fewer vertices in [2]. The latter paper also includes three constructions—types $A1$, $A2$, $A3$—with a similar structure. Zelinka gave two constructions of graphs that are maximal non-traceable; that is, they have no Hamiltonian path, but the addition of any edge gives a Hamiltonian path. The join of such a graph with a single vertex gives a MNH graph. Zelinka’s first family produces, under the join with $K_1$, the Skupien MNH graphs from [6]. Zelinka’s second family is a broad generalization of the type $A1$, $A2$, and $A3$ graphs of [2]. Bullock et al [1] provide further examples of infinite families of maximal non-traceable graphs.

In this article we work with two closely related invariants of a graph $G$, $\hat{\mu}(G)$ and $\mu(G)$. The $\mu$-invariant, introduced by Ore [5], is the maximal number of paths in $G$ required to cover the vertex set of $G$. We show that $\hat{\mu}(G) = \mu(G)$ unless $G$ is Hamiltonian, when $\hat{\mu}(G) = 0$. Maximal non-Hamiltonian graphs are maximal with respect to $\hat{\mu}(G) = 1$, and maximal non-traceable graphs are maximal with respect to $\hat{\mu}(G) = 2$. It is useful to broaden the perspective to study, for arbitrary $t$, graphs that are maximal with respect to $\hat{\mu}(G) = t$, which we call $t$-path traceable graphs.

In Section 2 we show how the $\hat{\mu}$ and $\mu$ invariants behave with respect to disjoint union of graphs and the join with a complete graph. Section 3 derives the main result, a decomposition theorem that reduces the problem of characterizing maximal $t$-path traceable to characterizing those that have no universal vertex, which we call trim. Section 4 presents a generalization of the Zelinka construction to $t$-path traceable graphs.
2 Traceability and Hamiltonicity

It will be notationally convenient to say that the complete graphs $K_1$ and $K_2$ are Hamiltonian. As justification for this view, consider an undirected graph as a directed graph with each edge having a conjugate edge in the reverse direction. This perspective does not affect the Hamiltonicity of a graph with more than 3 vertices, but it does give $K_2$ a Hamiltonian cycle. Similarly, adding loops to any graph with more than 2 vertices does not alter the Hamiltonicity of the graph, but $K_1$, with an added loop, has a Hamiltonian cycle.

Let $G$ be a graph. A vertex, $v \in V(G)$, is called a universal vertex if $\deg(v) = |V(G)| - 1$.

Lemmas 2.2 Traceability and Hamiltonicity

Let $G$ denote the graph complement of $G$, having vertex set $V(G)$ and edge set $E(K_n) \setminus E(G)$. We will use the disjoint union of two graphs, $G \sqcup H$ and the join of two graphs $G \ast H$. The latter is $G \sqcup H$ together with the edges $\{vw|v \in V(G) \text{ and } w \in V(H)\}$.

Definition 1. A set of $s$ disjoint paths in a graph $G$ that includes every vertex in $G$ is a $s$-path covering of $G$. Define the following invariants.

$$
\mu(G) := \min_{s \in \mathbb{N}} \{\exists \text{s-path covering of } G\}.
$$

$$
\tilde{\mu}(G) := \min_{l \in \mathbb{N}_0} \{K_l \ast G \text{ is Hamiltonian }\}
$$

$$
i_H(G) := \begin{cases} 
1 & \text{if } G \text{ is Hamiltonian} \\
0 & \text{otherwise}
\end{cases}
$$

We will say $G$ is $t$-path traceable when $\mu(G) = t$. A set of $t$ disjoint paths that cover a $t$-path traceable graph $G$ is a minimal path covering.

Note that $K_r \ast (K_s \ast G) = K_{r+s} \ast G$. If $G$ is Hamiltonian then so is $K_r \ast G$ for $r \geq 0$. (In particular this is true for $G = K_1$ and $G = K_2$.)

We now have a series of lemmas that lead to the main result of this section, which is a formula showing how the $\mu$-invariant and $\tilde{\mu}$-invariant behave with respect to disjoint union and the join with a complete graph.

Lemma 2. $\tilde{\mu}(G) = \min_{l \in \mathbb{N}_0} \{K_l \ast G \text{ is Hamiltonian }\}$

Proof. Since $K_l \ast G$ is a subgraph of $K_l \ast G$, a Hamiltonian cycle in $K_l \ast G)$ would also be one in $K_l \ast G$.

Let $\tilde{\mu}(G) = a$. Suppose $C$ is a Hamiltonian cycle in $K_a \ast G$ and write $C$ as $v \sim P_1 \sim Q_1 \sim \ldots \sim P_s \sim Q_s \sim v$, where $v$ is a vertex in $G$ and the paths $P_i \in G$ and $Q_i \in K_a$. If any $Q_i$ contains 2 vertices or more, say $u$ and $w_1, \ldots, w_k$ with $k \geq 1$, then we may simply remove all the vertices, except $u$, and end up with a Hamiltonian graph on $K_{a-k}$. This contradicts the minimality of $a = \tilde{\mu}(G)$. Therefore, $C$ must not contain any paths of length greater than two in the subgraph $K_a$, and any Hamiltonian cycle on $K_a \ast G$ is also a Hamiltonian cycle on $K_a \ast G$. \hfill \Box

Lemma 3. $\tilde{\mu}(G) = \mu(G) - i_H(G)$

Proof. If $G$ is Hamiltonian (including $P_1$ and $P_2$) then $\tilde{\mu}(G) = 0$, $\mu(G) = 1$ so the equality holds. Suppose $G$ is non-Hamiltonian with $\mu(G) = t$ and $t$-path covering $P_1, \ldots, P_l$. Let $K_l$
have vertices $u_1, \ldots, u_t$. In the graph $K_t \ast G$, there is a Hamiltonian cycle: $v_1 \sim P_1 \sim v_2 \sim P_2 \sim \cdots \sim v_t \sim P_t \sim v_1$. Thus $\mu(G) \leq t = \mu(G)$.

Let $\tilde{\mu}(G) = a$, so there is a Hamiltonian cycle in $K_a \ast G$. Removing the vertices of $K_a$ breaks the cycle into at most $a$ disjoint paths covering $G$. Thus $\mu(G) \leq \tilde{\mu}(G)$. □

**Lemma 4.** $\mu(G \sqcup H) = \mu(G) + \mu(H)$ and $\tilde{\mu}(G \sqcup H) = \tilde{\mu}(G) + \tilde{\mu}(H) + i_H(G) + i_H(H)$.

**Proof.** A path covering of $G$ may be combined with a path covering of $H$ to create one for $G \sqcup H$. Conversely, paths in a $t$-path covering of $G \sqcup H$ can be partitioned into those contained in $G$ and those contained in $H$, giving a path covering of $G$ and one of $H$. Consequently

$$\mu(G \sqcup H) = \mu(G) + \mu(H)$$

Since $G \sqcup H$ is not Hamiltonian we have

$$\tilde{\mu}(G \sqcup H) = \mu(G \sqcup H) + i_H(G \sqcup H)$$

$$= \mu(G) + \mu(H)$$

$$= \tilde{\mu}(G) + i_H(G) + \tilde{\mu}(H) + i_H(H)$$

□

**Lemma 5.** For any graph $G$,

$$\mu(K_s \ast G) = \max \{1, \mu(G) - s\}$$

$$\tilde{\mu}(K_s \ast G) = \max \{0, \tilde{\mu}(G) - s\}$$

In particular, if $K_s \ast G$ is Hamiltonian then $\mu(K_s \ast G) = 1$ and $\tilde{\mu}(K_s \ast G) = 0$; otherwise, $\mu(K_s \ast G) = \mu(G) - s$ and $\tilde{\mu}(K_s \ast G) = \tilde{\mu}(G) - s$.

**Proof.** The formula for $\tilde{\mu}$ is immediate when $G$ is Hamiltonian since we have observed that this forces $K_s \ast G$ to be Hamiltonian. Otherwise, it follows from $K_r \ast (K_s \ast G) = K_{r+s} \ast G$: if $\tilde{\mu}(G) = a$, then $K_r \ast (K_s \ast G)$ is Hamiltonian if and only if $r + s \geq a$.

The formula for $\mu$ may be derived from the result for $\tilde{\mu}$ using Lemma 4. We may also prove it directly. Observe that it is enough to prove $\mu(K_1 \ast G) = \max \{1, \mu(G) - 1\}$. Let $u$ be the vertex of $K_1$. Let $\mu(G) = t$ and $P_1, \ldots, P_t$ a $t$-path covering of $G$. If $t = 1$ then $u$ can be connected to the initial vertex of $P_1$ to create a 1-path covering of $K_1 \ast G$. For $t \geq 2$, the path $P_1 \sim u \sim P_2$ along with $P_3, \ldots, P_t$ gives a $(t-1)$-path covering of $K_1 \ast G$. Thus for $t \geq 1$, $\mu(K_1 \ast G) \leq t - 1$. Suppose $Q_1, \ldots, Q_d$ were a minimal $d$-path covering of $K_1 \ast G$, with $u$ a vertex of $Q_1$. Removing $u$ gives at most a $(d+1)$-path covering of $G$. Thus $\mu(K_1 \ast G) + 1 \geq t$. This shows $\mu(K_1 \ast G) = \mu(G) - 1$ for $\mu(G) \geq 2$. □

The main result of this section is the following two formulas for the $\mu$ and $\tilde{\mu}$ invariants for the disjoint union of graphs, and the join with a complete graph.
Proposition 6. Let \( \{G_j\}_{j=1}^m \) be graphs.

\[
\mu\left( \bigcup_{j=1}^m G_j \right) = \sum_{j=1}^m \mu(G_j) \quad \text{and} \quad \hat{\mu}\left( \bigcup_{j=1}^m G_j \right) = \sum_{j=1}^m \hat{\mu}(G_j) + \sum_{j=1}^m i_H(G_j).
\]

Furthermore, \( \hat{\mu}\left( \bigcup_{j=1}^m G_j \right) \ast K_r = \max \{0, \sum_{j=1}^m \hat{\mu}(G_j) + \sum_{j=1}^m i_H(G_j) - r\} \).

Proof. We proceed by induction. The base case \( k = 2 \) is exactly Lemma 4. Assume the formula holds for \( k \) graphs we will prove it for \( k + 1 \) graphs.

\[
\mu\left( \bigcup_{j=1}^{k+1} G_j \right) = \mu\left( \bigcup_{j=1}^k G_j \cup G_{k+1} \right)
\]

\[
= \mu\left( \bigcup_{j=1}^k G_j \right) + \mu(G_{k+1})
\]

\[
= \sum_{j=1}^k \mu(G_j) + \mu(G_{k+1})
\]

\[
= \sum_{j=1}^{k+1} \mu(G_j)
\]

By Lemma 5 and the fact that disjoint graphs are not Hamiltonian, we have,

\[
\hat{\mu}\left( \bigcup_{j=1}^m G_j \right) = \mu\left( \bigcup_{j=1}^m G_j \right) + i_H\left( \bigcup_{j=1}^m G_j \right)
\]

\[
= \sum_{j=1}^m \mu(G_j) + 0
\]

\[
= \sum_{j=1}^m (\hat{\mu}(G_j) + i_H(G_j))
\]

\[
= \sum_{j=1}^m \hat{\mu}(G_j) + \sum_{j=1}^m i_H(G_j)
\]

Therefore, we have by Lemma 5,

\[
\hat{\mu}\left( \bigcup_{j=1}^m G_j \right) \ast K_r = \max \{0, \sum_{j=1}^m \hat{\mu}(G_j) + \sum_{j=1}^m i_H(G_j) - r\}
\]

\[
= \max \{0, \sum_{j=1}^m \hat{\mu}(G_j) + \sum_{j=1}^m i_H(G_j) - r\}
\]
The following lemma will be useful in the next section. To express it succinctly we introduce the following Boolean condition. For a graph $G$ and vertex $v \in G$, $T(v, G)$ is true if and only if $v$ is a terminal vertex in some minimal path covering of $G$.

**Lemma 7.** Let $v \in G$ and $w \in H$.

$$\mu((G \sqcup H) + vw) = \begin{cases} 
\mu(G \sqcup H) - 1 & \text{if } T(v, G) \text{ and } T(w, H) \\
\mu(G \sqcup H) & \text{otherwise}
\end{cases}$$

**Proof.** Let $\mu(G) = c$, $\mu(H) = d$ and $\mu((G \sqcup H) + vw) = t$. Clearly, $t \leq c + d$.

Let $R_1, \ldots, R_t$ be a minimal path cover of $(G \sqcup H) + vw$. If no $R_i$ contains $vw$ then this is also a minimal path cover of $(G \sqcup H)$ so $t = c + d$. Suppose $R_1$ contains $vw$ and note that $R_1$ is the only path with vertices in both $G$ and $H$. Removing $vw$ gives two paths $P \subseteq G$ and $Q \subseteq H$. Paths $P$ and $Q$ along with $R_2, \ldots, R_t$ cover $G \sqcup H$, so $t + 1 \geq c + d$. Thus, $t$ can either be $c + d$ or $c + d - 1$.

If $t = c + d - 1$, then we have the minimal $(t + 1)$-path covering $P, Q, R_2, \ldots, R_t$ of $G \sqcup H$, as above. We note that $v$ must be a terminal point of $P$ and $w$ must be a terminal point of $Q$, by construction. This path covering may be partitioned into a $c$-path covering of $G$ containing $P$ and a $d$-path covering of $H$ containing $Q$. Thus, $T(v, G)$ and $T(w, H)$ hold.

Conversely, suppose $T(v, H)$ and $T(w, H)$ both hold. Let $P_1, \ldots, P_c$ be a minimal path of $G$ with $v$ a terminal vertex of $P_1$ and let $Q_1, \ldots, Q_d$ be a minimal path cover of $H$ with $w$ a terminal vertex of $Q_1$. The edge $vw$ knits $P_1$ and $Q_1$ into a single path and $P_1 \sim Q_1, P_1, \ldots, P_c, Q_1, \ldots, Q_d$ is a $c + d - 1$ cover of $(G \sqcup H) + vw$. Consequently, $t \leq c + d - 1$.

Thus, $T(v, G)$ and $T(w, H)$ both hold if and only if $t = c + d - 1$. Otherwise, $t = c + d$. 

**Corollary 8.** Let $v \in G$ and $w \in H$.

$$
\tilde{\mu}((G \sqcup H) + vw) = \begin{cases} 
\tilde{\mu}(G \sqcup H) - 2 & \text{if } G = H = K_1 \\
\tilde{\mu}(G \sqcup H) - 1 & \text{if } T(v, G) \text{ and } T(w, H) \\
\tilde{\mu}(G \sqcup H) & \text{Otherwise}
\end{cases}
$$

**Proof.** Let $\delta = 1$ if $T(v, G)$ and $T(w, H)$ are both true and $\delta = 0$ otherwise. Then

$$
\tilde{\mu}((G \sqcup H) + vw) = \mu((G \sqcup H) + vw) - i_H((G \sqcup H) + vw) = \mu((G \sqcup H) - \delta - i_H((G \sqcup H) + vw)
$$

The final term is $-1$ if and only if $G = H = K_1$.

### 3 Decomposing Maximal t-path traceable graphs

In this section we prove our main result, a maximal $t$-path traceable graph may be uniquely written as the join of a complete graph and a disjoint union of graphs that are also maximal...
Proof. We have already shown that $\mathcal{H}$ for $t \geq 0$ and $\mathcal{N}t$ for $t \geq 1$.

$$\mathcal{M}_t := \{G|\mu(G) = t \text{ and } \mu(G + e) < t, \forall e \in E(G)\}$$
$$\mathcal{N}_t := \{G \in \mathcal{M}_t|G \text{ is connected and has no universal vertex }\}$$

The set $\mathcal{M}_0$ is the set of complete graphs. The set $\mathcal{M}_1$ is the set of graphs with a Hamiltonian path but no Hamiltonian cycle, that is, maximal non-Hamiltonian graphs. For $t > 1$, $\mathcal{M}_t$ is also the set of graphs $G$ such $\mu(G) = t$ and $\mu(G + e) = t - 1$ for any $e \in E(G)$. We will call these maximal $t$-path traceable graphs. A graph in $\mathcal{N}_t$ will be called trim.

**Proposition 9.** For $0 \leq s < t$, $G \in \mathcal{M}_t$ if and only if $K_s \ast G \in \mathcal{M}_{t-s}$.

**Proof.** We have $\mu(K_s \ast G) = \mu(G) - s$, so we just need to show that $K_s \ast G$ is maximal if and only if $G$ is maximal. The only edges that can be added to $K_s \ast G$ are those between vertices of $G$, that is, $E(K_s + G) = E(G)$. For such an edge $e$,

$$\mu\left((K_s \ast G) + e\right) = \mu\left(K_s \ast (G + e)\right) = \mu(G + e) - s$$

Consequently, $\mu(G + e) = \mu(G) - 1$ if and only if $\mu\left((K_s \ast G) + e\right) = \mu(K_s \ast G) - 1$. $\square$

Note that the proposition is false for $s = t > 0$ since $K_s \ast G$ will not be a complete graph and $\mathcal{M}_0$ is the set of complete graphs. The proof breaks down in $\square$.

**Proposition 10.** Let $G \in \mathcal{M}_c$ and $H \in \mathcal{M}_d$. The following are equivalent.

1. $G \sqcup H \in \mathcal{M}_{c+d+i_H(G)+i_H(H)}$
2. Each of $G$ and $H$ is either complete or has no universal vertex.

**Proof.** We have already shown that $\mu(G \sqcup H) = c + d + i_H(G) + i_H(H)$. We have to consider whether adding an edge to $G \sqcup H$ reduces the $\mu$-invariant. There are three cases to consider, the extra edge may be in $E(G)$ or $E(H)$ or it may join a vertex in $G$ to one in $H$. Since $G$ is maximal, adding an edge to $G$ is either impossible, when $G$ is complete, or it reduces the $\mu$-invariant of $G$. This edge would also reduce the $\mu$-invariant of $G \sqcup H$ by Lemma. The case for adding an edge of $H$ is the same. Consider the edge $vw$ for $v \in V(G)$ and $w \in V(H)$. By Corollary the $\mu$-invariant will drop if and only if $v$ is the terminal point of a path in a minimal path covering of $G$ and similarly for $w$ in $H$, that is, $T(v, G)$ and $T(w, H)$. Clearly this holds for all vertices in a complete graph. The following lemma shows that $T(v, G)$ holds for $G \in \mathcal{M}_c$ with $c > 0$ if and only if $v$ is not a universal vertex in $G$. Thus, in order for $G \sqcup H$ to be maximal $G$ must either be complete, or be maximal itself, and have no universal vertex, and similarly for $H$. $\square$

As a key step before the main theorem, the next lemma shows that in a maximal graph, each vertex is universal, or a terminal vertex in a minimal path covering.
Lemma 11. Let $c \geq 1$ and $G \in \mathcal{M}_c$. For any two non-adjacent vertices $v, w$ in $G$ there is a $c$-path covering of $G$ in which both $v$ and $w$ are terminal points of paths. Moreover, a vertex $v \in G$ is a terminal point in some $c$-path covering if and only if $v$ is not universal.

Proof. Suppose $c > 1$ and let $v, w$ be non-adjacent in $G$. Since $G$ is maximal $G + vw$ has a $(c - 1)$-path covering, $P_1, \ldots, P_{c-1}$. The edge $vw$ must be contained in some $P_j$ because $G$ has no $(c - 1)$-path covering. Removing that edge gives a $c$-path covering of $G$ with $v$ and $w$ as terminal vertices. The special case $c = 1$ is well known, adding the edge $vw$ gives a Hamiltonian cycle, and removing it leaves a path with endpoints $v$ and $w$. A consequence is that any non-universal vertex is the terminal point of some path in a $c$-path covering.

Suppose $P_1, \ldots, P_c$ is a $c$-path covering of $G \in \mathcal{M}_c$ with $v$ a terminal point of $P_t$. Then $v$ is not adjacent to any of the terminal points of $P_j$ for $j \neq i$, for otherwise two paths could be combined into a single one. In the case $c = 1$, $v$ cannot be adjacent to the other terminal point of $P_1$, otherwise $G$ would have a Hamiltonian cycle. Consequently a universal vertex is not a terminal point in a $c$-path covering of $G$.

Theorem 12. For any $G \in \mathcal{M}_t$, $t > 0$, $G$ may be uniquely decomposed as $K_s \ast (G_1 \sqcup \ldots \sqcup G_r)$, where $s$ is the number of universal vertices of $G$, and each $G_j$ is either complete or $G_j \in \mathcal{N}_j$ for some $t_j > 0$. Furthermore $t = \sum_{j=1}^{r} t_j + \sum_{j=1}^{r} i_H(G_j) - s$.

Proof. Suppose $G \in \mathcal{M}_t$ and let $s$ be the number of universal vertices of $G$. Let $r$ be the number of components in the graph obtained by removing the universal vertices from $G$, let $G_1, \ldots, G_r$ be the components and let $\hat{\mu}(G_j) = t_j$.

Proposition 9 shows that $t = \sum_{j=1}^{r} t_j + \sum_{j=1}^{r} i_H(G_j) - r$. By Proposition 9 we have that $G \in \mathcal{M}_t$ if and only if $G_1 \sqcup \ldots \sqcup G_r \in \mathcal{M}_{t+s}$. Furthermore, each $G_j$ must be in $\mathcal{N}_j$ for otherwise we could. Without loss of generality if we add an edge $e$ to $G_1$, such that $\hat{\mu}(G_1 + e) < t_1$, then

$$\hat{\mu}(G + e) = \hat{\mu}(G_1 + e) + \sum_{j=2}^{r} t_j + \sum_{j=1}^{r} i_H(G_j) - s$$

$$< \sum_{j=1}^{r} t_j + \sum_{j=1}^{r} i_H(G_j) - s$$

$$= t$$

Now, we apply Proposition 10 so then $G_1 \sqcup \ldots \sqcup G_r \in \mathcal{M}_{t+s}$, where $t + s = \sum_{j=1}^{r} t_j + \sum_{j=1}^{r} i_H(G_j)$ if and only if $G_j$ is either trim or complete. In other words, $G_j \in \mathcal{N}_j$ for $t_j > 0$ or $G_j \in \mathcal{M}_0$ for $t_j = 0$. 

\[7\]
4 Trim maximal $t$-path traceable graphs

Skupien \[6\] discovered the first family of maximal non-Hamiltonian graphs, that is, graphs in $\mathcal{M}_1$. These graphs are formed by taking the join with $K_r$ of the disjoint union of $r + 1$ complete graphs. The smallest graph in $\mathcal{N}_2$ is shown in Figure 1. Chvátal identified its join with $K_1$ as the smallest maximal non-Hamiltonian graph that is not 1-tough, that is, not one of the Skupien family. Jamrozik, Kalinowski and Skupien \[2\] generalized this example to three different families.

Family $A_1$ replaces each edge $u_i v_i$ with an arbitrary complete graph containing $u_i$ and replaces the $K_3$ formed by the $u_i$ with an arbitrary complete graph. The result has four cliques, the first three disjoint from each other but each intersecting the fourth clique in a single vertex. This graph is also in $\mathcal{N}_2$ and its join with $K_1$ gives a maximal non-Hamiltonian graph. Family $A_2$ is formed by taking the join with $K_2$ of the disjoint union of a complete graph and the graph in $\mathcal{N}_2$ just described. Theorem 12 shows that the resulting graph is in $\mathcal{M}_1$. Family $A_3$ is a modification of the $A_1$ family based on the graph in Figure 2, which is in $\mathcal{N}_2$. Bullock, Frick, Singleton and van Aardt \[1\] recognized that two constructions of Zelinka \[7\] gave maximal non-traceable graphs, that is, elements of $\mathcal{N}_2$. Zelinka’s first construction is like the Skupien family: formed from $r + 1$ complete graphs followed by the join with $K_{r-1}$. The Zelinka Type II family contains graphs in $\mathcal{N}_2$ that are a significant generalization of the graphs in Figures 1 and 2. In this section we generalize this family further to get graphs in $\mathcal{N}_t$ for arbitrary $t$. Our starting point is the graph in Figure 3 which is in $\mathcal{N}_3$.

Example 13. Consider $K_m$ with $m \geq 2t - 1$ and vertices $u_1, \ldots, u_m$. Let $G$ be the graph containing $K_m$ along with vertices $v_1, \ldots, v_{2t-1}$ and edges $u_i v_i$. The case with $t = 3$ and $m = 5 = 2t - 1$ is Figure 3. We claim $G \in \mathcal{N}_t$.

One can readily check that this graph is $t$-path covered using $v_{2i-1} \sim u_{2i-1} \sim v_{2i} \sim v_{2i+1}$ for $i = 1, \ldots, t - 1$ and $v_{2t-1} \sim u_{2t-1} \sim u_{2t} \sim \cdots \sim u_m$. We check that $G$ is maximal. By the symmetry of the graph, we need only consider the addition of the edge $v_1 u_m$ and $v_1 v_2$. In either case, the last and the first paths listed above may be combined into one, either

\[
\begin{align*}
\text{or } v_{2t-1} \sim v_{2t-1} \sim \cdots \sim u_m & \sim v_1 \sim u_1 \sim u_2 \sim v_2, \\
v_{2t-1} \sim v_{2t-1} \sim \cdots \sim u_m & \sim u_1 \sim v_1 \sim v_2.
\end{align*}
\]
Figure 2: The join of this graph with $K_1$ is the smallest graph in the $A_3$ family.

Thus, adding an edge creates a $(t - 1)$-path covered graph, proving maximality.

The next proposition shows that the previous example is the only way to have a trim maximal $t$-path covered graph with $2t - 1$ degree-one vertices. We start with a technical lemma.

**Lemma 14.** Let $G$ be a connected graph and let $u_1, v_1, v_2, v_3 \in G$ with $\text{deg}(v_i) = 1$, and $u$ adjacent to $v_1$ and $v_2$ but not $v_3$. Then $\mu(G) = \mu(G + uv_3)$.

**Proof.** Let $P_1, \ldots, P_r$ be a minimal path covering of $G + uv_3$; it is enough to show that there are $r$-paths covering $G$. If the covering doesn’t include $uv_3$, then $P_1, \ldots, P_r$ also give a minimal path covering of $G$ establishing the claim of the lemma. Otherwise, suppose $uv_3$ is an edge of $P_1$. We consider two cases.

Suppose $P_1$ contains the edge $uv_1$ (or similarly $uv_2$). Then $P_1$ has $v_1$ as a terminal point and one of the other paths, say $P_2$, must be a length-0 path containing simply $v_2$. Let $Q$ be obtained by removing $uv_1$ and $uv_3$ from $P_1$. Then $v_1 \sim u \sim v_2, Q, P_3, \ldots, P_r$, gives an $r$-path covering of $G$.

Suppose $P_1$ contains neither $uv_1$ nor $uv_2$. Then each of $v_1$ and $v_2$ must be on a length-0 path in the covering, say $P_2$ and $P_3$ are these paths. Furthermore $u$ must not be a terminal point of $P_1$, for, if were, the path could be extended to include $v_1$ or $v_2$, reducing the number of paths required to cover $G$. Removing $u$ from $P_1$ yields two paths, $Q_1, Q_2$. Then $v_1 \sim u \sim v_2, Q_1, Q_2, P_4, \ldots, P_r$ gives an $r$-path cover of $G$. This proves the lemma.

**Proposition 15.** Let $G \in \mathcal{G}_t$. The number of degree-one vertices in $G$ is at most $2t - 1$. This occurs if and only if the $2t - 1$ vertices of degree-one have distinct neighbors and removing the degree-one vertices leaves a complete graph.

**Proof.** Each degree-one vertex must be a terminal point in a path covering. So any graph $G$ covered by $t$ paths can have at most $2t$ degree-one vertices. Aside from the case $t = 1$ and...
Figure 3: Whirligig in $N_3$.

$G = K_2$, we can see that a graph with $2t$ degree-one vertices cannot be maximal $t$-path traceable as follows. It is easy to check that a $2t$ star is not $t$-path traceable (it is also not trim). A $t$-path traceable graph with $2t$ degree-one vertices must therefore have an interior vertex $w$ that is not connected to one of the degree-one vertices $v$. Such a graph is not maximal because the edge $vw$ can be added leaving $2t - 1$ degree-one vertices. This graph cannot be $(t - 1)$-path covered.

Suppose that $G \in N_t$ with $2t - 1$ degree-one vertices, $v_1, \ldots, v_{2t - 1}$. Lemma 14 shows that no two of the $v_i$ can be adjacent to the same vertex, for that would violate maximality of $G$. So, the $v_i$ have distinct neighbors. Furthermore, all the nodes except the $v_i$ can be connected to each other and a path covering will still require at least $t$ paths since there remain $2t - 1$ degree-one vertices. This proves the necessity of the structure claimed in the proposition. The previous example showed that the graph is indeed in $N_t$.

We can now generalize the Zelinka family.

Construction 16. Let $U_0, U_1, \ldots, U_{2t - 1}$ be disjoint sets and $U = \bigcup_{i=0}^{2t-1} U_i$. Let $m_i = |U_i|$ and assume that for $i > 0$ the $U_i$ are non-empty, so $m_i > 0$. For $i = 1, \ldots, 2t - 1$ (but not $i = 0$) and $j = 1, \ldots, m_i$, let $V_{ij}$ be disjoint from each other and from $U$. Form the graph with vertex set $U \cup \left( \bigcup_{i=1}^{2t-1} \left( \bigcup_{j=1}^{m_i} V_{ij} \right) \right)$ and edges $uu'$ for $u, u' \in U$ and $uv$ for any $u \in U_i$ and $v \in V_{ij}$ with $i = 1, \ldots, 2t - 1$ and $j = 1, \ldots, m_i$. The cliques of this graph are $K_U$ and $K_{U \cup V_{ij}}$ for each $i = 1, \ldots, 2t - 1$ and $j = 1, \ldots, m_i$.

The graph in Figure 2 has $m_0 = 0$, $m_1 = m_2 = 1$ and $m_3 = 2$, and the graph in Figure 4 indicates the general construction.
Theorem 17. The graph $W$ in Construction 16 is a trim, maximal $t$-path traceable graph.

Proof. We must show that $W$ is $t$-path covered and not $(t - 1)$-path covered, and that the addition of any edge yields a $(t - 1)$-path covered graph. The argument is analogous to the one in Example 13.

Let $R$ be a Hamiltonian path in $U_0$. For each $i = 1, \ldots, 2t - 1$ and $j = 1, \ldots, m_i$ let $Q_{ij}$ be a Hamiltonian path in $K_{V_{ij}}$. Let $P_i$ be the path

$$P_i : Q_{i1} \sim u_{i1} \sim \cdots \sim Q_{im_i} \sim u_{im_i}$$

and let $P_i^{-1}$ be the reversal of $P_i$.

Since there is an edge $u_{im_i}u_{j_{m_j}}$ there is a path $P_i \sim P_j$ for any $i \neq j \in \{1, \ldots, 2t - 1\}$. Therefore the graph $W$ has a $t$-path covering $P_{2i-1} \sim P_{2i}$ for $i = 1, \ldots, (t - 1)$, along with $P_{2t-1} \sim R$. We leave to the reader the argument that there is no $(t - 1)$-path cover.

To show $W$ is maximal we show that after adding an edge $e$, we can join two paths in the $t$-path cover above, with a bit of rearrangement. There are three types of edges to consider, the edge $e$ might join $V_{ij}$ to $U_{i'}$ for $i \neq i'$; or $V_{ij}$ to $V_{ij'}'$ for $j \neq j'$; or $V_{ij}$ to $V_{i'j'}'$ for $i \neq i'$. Because of the symmetry of $W$, we may assume $i = 1$ and $j = 1$ and that the vertex chosen from $V_{ij}$ is the initial vertex of $Q_{i1}$. Other simplifications due to symmetry will be evident in what follows.

In the first case there are two subcases—determined by $i' \geq 2t$ or not—and after permutation, we may consider the edge $e$ from the initial vertex of $Q_{11}$ to the terminal vertex of $R$, or to the terminal vertex of $P_{2t-1}$. We can then join two paths in the $t$-path cover: either $P_{2i-1} \sim R \sim P_1 \sim P_2$ or $P_2 \sim P_1 \sim P_{2t-1} \sim R$. 

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Suppose next that we join the initial vertex of $Q_{11}$ with the terminal vertex of $Q_{12}$. We then rearrange $P_1$ and join two path in the $t$-path cover to get

$$P_{2t-1} \sim R \sim u_{11} \sim Q_{11} \sim Q_{12} \sim u_{12} \sim \cdots \sim Q_{1m_1} \sim u_{1m_1} \sim \overleftarrow{P_2}$$

Finally, suppose that we join the initial vertex of $Q_{11}$ with the initial vertex of $Q_{2t-1,1}$. Then we rearrange to

$$R \sim \overleftarrow{P}_{2t-1} \sim P_1 \sim \overleftarrow{P}_2.$$

\[\square\]

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