EARLY STAGES OF GROWTH OF QCD AND ELECTROWEAK BUBBLES\textsuperscript{a}

J. IGNATIUS

NORDITA, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark, and Department of Physics, FIN-00014 University of Helsinki, Finland.

E-mail: ignatius@iki.fi

The dynamical growth rate of bubbles nucleating in relativistic plasma in thermal first-order phase transitions is analyzed. The framework is a hydrodynamical model which consists of relativistic fluid and an order parameter field. The results of analytical approximations and numerical simulations coincide well.

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In thermal systems first-order phase transitions normally proceed via nucleation of bubbles of the new phase. Bubbles larger than a certain critical size begin to grow, whereas smaller bubbles will shrink. Langer’s formula for the nucleation rate of bubbles of the new phase is given by

\[ \Gamma = \frac{\kappa}{2\pi} \Omega_0 e^{-\Delta F/T}, \]  

where \( \Gamma \) is the probability of nucleation per volume and time, \( \kappa \) a dynamical and \( \Omega_0 \) a statistical prefactor, \( \Delta F \) the free energy difference of the system with and without the nucleating bubble, and \( T \) the temperature. The purpose of this study is to investigate the dynamical prefactor \( \kappa \).

Let the phase transition be driven by an order parameter field \( \phi(x) \). Initial growth rate of perturbations around an extremum configuration is given by the coefficient \( \kappa \):

\[ \delta \phi \propto e^{\kappa t}. \]  

Similar relations hold for perturbations of energy density and fluid velocity. A parallel definition for the growth coefficient \( \kappa \) can be expressed in terms of the radius \( R(t) \) of an expanding spherical bubble of the new phase,

\[ \frac{dR(t)}{dt} \approx \kappa [R(t) - R_{cr}], \]  

where \( R_{cr} \) is the radius of the critical or extremum bubble. This equation is valid when \( R(t) \approx R_{cr} \). In order for the bubble to grow, the initial radius \( R(0) \) must be slightly larger than that of a critical bubble.

In literature there are at least two calculations of the growth rate \( \kappa \) for relativistic plasma. Csernai and Kapusta write the free energy density as a functional of their order parameter which is just the usual internal energy density. Ruggeri and Friedman make the approximation that the interface is infinitely thin and do not need any order parameter. These two methods produce results for \( \kappa \) which disagree even qualitatively with each other.

Let us now introduce a hydrodynamical model which enables both an analytical and a numerical determination of the initial growth rate \( \kappa \). The model consists of an order parameter field \( \phi \), and perfect fluid which describes the other degrees of freedom. There are three basic locally varying quantities, namely \( \phi(x) \), fluid four-velocity \( u^\mu(x) \), and temperature \( T(x) \). Due to the small value of the baryon asymmetry all the conserved particle numbers can for the present purposes well be approximated to be zero in the early Universe. Furthermore, the expansion of the Universe can be neglected, since the whole period of nucleation, yet alone the initial growth of bubbles, is an extremely
rapid process. Thus the equations of motion can be written in the form

\[
\begin{aligned}
\partial_\mu T^{\mu\nu} &= 0 \\
\partial^2 \phi + \frac{\partial}{\partial \phi} V(\phi, T) &= -\eta u^\mu \partial_\mu \phi,
\end{aligned}
\]

(4)

where \(T^{\mu\nu} = T^{\mu\nu} \{u^\alpha(x), \partial_\beta \phi(x), \phi(x), T(x)\}\) is the energy-momentum tensor, \(V\) the potential energy density for the order parameter field, \(T\) the temperature, and \(\eta\) a phenomenological friction parameter (not to be confused with shear viscosity). The upper equation is the conservation law of total energy-momentum. The lower equation tells how energy is transported between the order parameter and the fluid through the dissipative term, proportional to \(\eta\). The same term is also responsible for the creation of entropy. In the limit of vanishing fluid velocities the lower equation gives the simple dissipative equation

\[
\frac{d\phi}{dt} = -\frac{1}{\eta} \frac{\delta S_3[\phi]}{\delta \phi},
\]

(5)

where \(S_3[\phi]\) is the usual high-temperature three-dimensional action (equalling free energy at extrema).

It is clear that a hydrodynamical description of the system cannot be complete. It is only valid at scales which are longer than particle mean free paths or interaction times. The solutions to Eqs. (4) are smoothly behaving fields, whereas in reality fields have strong thermal fluctuations on short scales. To incorporate thermal fluctuations a Langevin-type equation with a noise term would be needed. Within a purely hydrodynamical model questions about damping effects in plasma cannot be answered.

The model in Eqs. (4) can be applied to describe both electroweak and QCD phase transition. In electroweak theory the order parameter is obviously identified as the Higgs field. The true coupling term is more complicated than the frictional \(\eta\)-term of this model. But that effect should not be significant, as long as the detailed internal structure of the interface is not being discussed (something which a hydrodynamical model cannot accurately determine anyway). The value of the friction parameter \(\eta\) can be fixed by comparing with microscopic calculations. In the case of QCD the order parameter cannot be identified with a physical particle. However, one can still employ the model as a purely phenomenological construction. Interaction length or time of QCD sets the scale for the friction coefficient \(\eta\). A further complication is the macroscopic mean free path of neutrinos, but luckily the hydrodynamic energy flux is in normal cases clearly superior compared with that carried by the neutrinos.

For solving Eqs. (4) the potential \(V(\phi, T)\) must be known. Here the usual quartic potential has been employed. By fixing the parameters of it in a suitable manner, the desired values for the surface tension and latent heat of the
transition will be reproduced. The coordinate system is spherically symmetric 1+3 dimensional space-time.

In order to create the initial configuration the critical bubble solution must be known with high accuracy. Going closer to the thin-wall limit, that is, using larger critical bubbles, has the advantage that numerical errors in the determination of growth coefficient $\kappa$ decrease. But in this limit the field equation cannot be directly integrated numerically to produce the critical bubble. Instead, the following Ansatz is used:

$$\phi_A(r) = \frac{\phi_{\text{min}}}{2} \left\{ 1 - \tanh \left( \frac{r - R_{\text{cr}}}{2\xi_{\text{cr}}} \right) \right\}. \quad (6)$$

Here $R_{\text{cr}}$, $\xi_{\text{cr}}$ are unknown parameters, and $\phi_{\text{min}}$ is the position of the new minimum of the potential. In this two-dimensional subspace set by the Ansatz the extremum of the action becomes saddle point of an ordinary function, $S_3[\phi_A] = S_{3A}(R_{\text{cr}}, \xi_{\text{cr}})$. This saddle point can then be located numerically. For larger bubbles this method produces quite accurate results, which was actually unexpected. Fixing $R_{\text{cr}}$ naively by Laplace’s relation, $R_{\text{cr}} = 2\sigma/\Delta p$, where $\sigma$ is surface tension and $\Delta p$ pressure difference between the two phases, would lead to huge inaccuracies. However, the correct value of $R_{\text{cr}}$ can be found directly by employing curvature-dependent surface tension, $\sigma(R)$.

Analytically the initial growth rate $\kappa$ can be determined as follows. Expand the low-velocity dissipation equation (5) around the critical bubble $\bar{\phi}$ by making the substitution $\phi(t, x) = \bar{\phi}(r) + \varphi(t, x)$. The resulting equation for fluctuations is

$$\frac{d\varphi}{dt} = -\frac{1}{\eta} \frac{\delta^2 S_3[\bar{\phi}]}{\delta \bar{\phi}^2} \varphi. \quad (7)$$

Next insert the unstable growth mode, Eq. (2). The thin-wall result for the negative eigenvalue of the fluctuation operator, $\lambda^- = 2/\lambda^2_{\text{cr}}$, has been observed to hold surprisingly well in the general case, too. This gives the approximate solution for the initial growth rate in the hydrodynamical model:

$$\kappa \approx \frac{2}{\eta R_{\text{cr}}^2}. \quad (8)$$

Radius of the critical bubble, $R_{\text{cr}}$, depends on one hand on the cooling rate of the system—in cosmology on the strength of gravitational interaction—and on the other hand on the thermodynamical properties of the phase transition, especially on the values of latent heat and surface tension.

A more straightforward method is to integrate the dissipation equation (5) directly in the thin-wall limit. The result for the initial acceleration of the
Figure 1: Numerical determination of the initial growth rate \( \kappa \). Horizontal axis is time in units of inverse thermodynamical phase transition temperature, \( T_c^{-1} \), and vertical axis is \( \ln([R(t) - R_{cr}]/[R(0) - R_{cr}]) \equiv y \). Initial growth rate is given by \( \kappa = \lim_{t \to 0} \frac{d}{dt}(\ln((R(t) - R_{cr})/(R(0) - R_{cr}))) \) (when \( R(0) \to R_{cr} \)). In the figure \( \eta = 1T_c, \xi_{cr} = 1.07243T_c^{-1}, R_{cr} = 19.695T_c^{-1} \), and \( R(0)/R_{cr} = 1.01 \).

Bubble radius is

\[
\frac{dR}{dt} \approx \frac{2}{\eta R_{cr}^2} \left\{ \frac{d-1}{2} R - (d-2)R_{cr} \right\}, \tag{9}
\]

where the dimensionality of space, \( d \), is explicitly visible. In the case \( d = 3 \) Eq. (8) follows from this by comparing with Eq. (3). Eq. (8) states clearly how in the real world the initial growth of bubbles is qualitatively different, much slower, than in one spatial dimension.

The opposite approach is to let the bubble to expand in a hydrodynamical computer simulation, and to measure the initial growth rate numerically. Bubble radius is defined to be the distance where the tension or gradient energy has the maximum. The value of \( \kappa \) can be read from Fig. 1 as the slope of the curve at origin. In the example case this gives \( \kappa = (0.0052 \pm 0.0001)T_c \), which coincides well with the analytical estimate from Eq. (8), \( \kappa \approx 0.00516T_c \).

These results can be applied in the analysis of thermal cosmological phase
transitions. In the case of relativistic heavy-ion collisions there is severe doubt on the validity of the general framework, nucleation in thermal systems.

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