A regularized Big Bang singularity

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Abstract

Following-up on earlier work on the regularization of the singular Schwarzschild solution, we now apply the same procedure to the singular Friedmann solution. Specifically, we are able to remove the divergences of the Big Bang singularity, at the price of introducing a 3-dimensional spacetime defect where the determinant of the metric vanishes. This particular regularization also suggests the existence of a pre-Big-Bang phase.

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I. INTRODUCTION

The Friedmann solution \[1, 2\] of an expanding universe has the so-called Big Bang singularity with diverging energy density and temperature [the cosmic scale factor \(a(t)\) drops to zero at cosmic time coordinate \(t = t_{\text{BB}} = 0\)]. Quantum mechanical effects are believed to temper these divergences; see, e.g., Refs. \[3, 4\] for two general discussions from different perspectives (loop quantum gravity and string theory).

Awaiting the definitive theory of “quantum gravity,” we propose to remain within standard general relativity but to allow for degenerate metrics. In fact, a particular degenerate metric has already provided a “regularization” of the Schwarzschild singularity \[5–7\], where the divergent behavior at the center (radial coordinate \(r = 0\)) has been removed at the price of introducing a spatial 2-surface \((r = b > 0)\) with a vanishing determinant of the metric. In the present article, we propose to do something similar with the Big Bang singularity (at cosmic time coordinate \(\tilde{t} = 0\)), by the introduction of a spatial 3-surface (at \(|\tilde{t}| = b/c > 0\)) with a vanishing determinant of the metric. We emphasize, right from the start, that it is the differential structure (rather than the topology) which plays a crucial role for the regularization of the Big Bang singularity. Incidentally, the length scale \(b\) may or may not be related to the Planck length \[8\].

With a new metric for the extended cosmic time coordinate \(T \in \mathbb{R}\), we can get an odd solution for the cosmic scale factor \(a(T)\), which “jumps” over the value \(a = 0\) and, thereby, avoids the Big Bang singularity. This particular solution may be of interest to the recent proposal for a CPT-symmetric universe \[9\].

II. STANDARD FRW UNIVERSE

Let us, first, review the main points of the standard spatially-flat radiation-dominated Friedmann–Robertson–Walker (FRW) universe. Details can be found in, e.g., Ref. \[2\]. Unless stated otherwise, we set \(c = 1\) and \(\hbar = 1\).

The line-element of the spatially-flat FRW universe reads

\[
\begin{align*}
\left. ds^2 \right|_{\text{stand. FRW}} &\equiv \left. g_{\mu\nu}(x) \, dx^\mu \, dx^\nu \right|_{\text{stand. FRW}} = -dt^2 + a^2(t) \delta_{kl} \, dx^k \, dx^l, \\
t &\in (0, \infty), \\
x^k &\in (-\infty, \infty),
\end{align*}
\]

(2.1a)

(2.1b)

(2.1c)

where the restricted range of the cosmic time coordinate \(t\) will be explained shortly. The
real function \( a(t) \) corresponds to the cosmic scale factor. As far as the metric is concerned, the sign of \( a(t) \) is irrelevant.

With this metric and the energy-momentum tensor of a perfect fluid [energy density \( \rho(t) \) and pressure \( P(t) \)], the Einstein equation without cosmological constant \( \Lambda \) gives the spatially-flat Friedmann equation and the energy-conservation equation:

\[
\left( \frac{1}{a(t)} \frac{da(t)}{dt} \right)^2 = \frac{8 \pi}{3} G_N \rho(t),
\]

\[
\frac{d}{da} \left( \rho a^3 \right) = -3 P a^2,
\]

to which is joined the equation of state,

\[
P = P(\rho).
\]

Consider, for definiteness, a relativistic component,

\[
P(t) = \frac{1}{3} \rho(t),
\]

so that (2.2b) gives \( \rho \propto 1/a^4 \). The resulting cosmic scale factor from (2.2a) is then

\[
a(t) = \sqrt{t/t_0}.
\]

For the particular solution (2.4) of the cosmic scale factor, the zero point of the cosmic time coordinate \( t \) has been shifted, so that

\[
\lim_{t \to 0^+} a(t) = 0,
\]

which corresponds to the Big Bang singularity. In addition, the cosmic scale factor (2.4) has been normalized to 1 at a given time \( t = t_0 > 0 \) for which the Hubble constant is assumed to be positive, \( H_0 \equiv \left[ (da/dt)/a \right]_{t=t_0} > 0 \).

The standard FRW spacetime manifold with metric (2.1) and cosmic scale factor (2.4) has the line-element

\[
\begin{align*}
 ds^2 \bigg|_{\text{stand. FRW}} &= -dt^2 + \sqrt{t^2/t_0^2} \delta_{kl} dx^k dx^l,
\end{align*}
\]

where the metric component \( \sqrt{t^2/t_0^2} \) can be simplified to \( t/t_0 \), because both \( t \) and \( t_0 \) are positive. The metric of this spacetime manifold solves the Einstein equation without cosmological constant \( \Lambda \), but the manifold is geodesically incomplete. Indeed, there is the Big Bang singularity at \( t = 0 \) with diverging curvature (as shown by, for example, the Kretschmann...
curvature scalar $K \equiv R_{\mu \nu \rho \sigma} R^{\mu \nu \rho \sigma}$ and diverging matter energy density $\rho$ and temperature $T$. Recall that, for the special case of relativistic matter, the Ricci curvature scalar $R \equiv g^{\mu \nu} R_{\mu \nu}$ vanishes identically. Specifically, these quantities are given by the following expressions:

$$R(t) \bigg|_{\text{stand. FRW}}^{\text{(rel-mat. sol.)}} = 6 \left[ \left( \frac{1}{a(t)} \frac{da(t)}{dt} \right)^2 + \frac{1}{a(t)} \frac{d^2 a(t)}{dt^2} \right] = 0,$$  \hspace{1cm} (2.7a)

$$K(t) \bigg|_{\text{stand. FRW}}^{\text{(rel-mat. sol.)}} = 12 \left[ \left( \frac{1}{a(t)} \frac{da(t)}{dt} \right)^4 + \left( \frac{1}{a(t)} \frac{d^2 a(t)}{dt^2} \right)^2 \right] \propto 1/t^4,$$  \hspace{1cm} (2.7b)

$$\rho(t) \bigg|_{\text{stand. FRW}}^{\text{(rel-mat. sol.)}} \propto 1/a^4(t) \propto 1/t^2,$$  \hspace{1cm} (2.7c)

$$T(t) \bigg|_{\text{stand. FRW}}^{\text{(rel-mat. sol.)}} \propto 1/\sqrt{a^4(t)} \propto 1/\sqrt{t^2},$$  \hspace{1cm} (2.7d)

where the final expression for the temperature can be simplified to $1/\sqrt{t}$, because $t$ is positive. Observe that the temperature expression $T(a) \propto [a^4]^{-1/4}$ in (2.7d) follows from considering the energy density of black-body radiation, as mentioned in the last paragraph of Box. 29.2 on p. 779 of Ref. [2].

In view of the results (2.7), it is clear that this particular solution of the Einstein equation is only well behaved if the range of the cosmic time coordinate $t$ is restricted to the open half-line $\mathbb{R}^+$.  

III. MODIFIED FRW UNIVERSE

As mentioned in Sec. I, it is possible to get a regularized version of the Schwarzschild solution by a simple procedure. The first step is to perform some surgery on the Euclidean 3-space: the interior of a ball with radius $b$ is removed and antipodal points on the boundary of the ball are identified. The resulting 3-space $\tilde{M}_3$ is topologically nontrivial (related to the 3-dimensional real-projective plane $\mathbb{R}P^3$) and a proper solution of the Einstein equation requires suitable coordinates and an appropriate Ansatz for the metric.

The obtained regularized Schwarzschild solution has a spatial 2-surface (with topology $\mathbb{R}P^2$) over which the determinant of the metric vanishes. This spatial 2-surface embedded in spacetime may be interpreted as a $(2+1)$-dimensional “defect” of spacetime with topology $\mathbb{R}P^2 \times \mathbb{R}$. The appearance of this “spacetime defect” is the price to pay for the absence of the Schwarzschild curvature singularity. The technical details of this regularization procedure
can be found in Ref. [5], with further discussion of the differential structure in Ref. [6] and of the physics of this particular type of “spacetime defect” in Ref. [7].

The idea, now, is to apply the same procedure to the FRW solution of Sec. II, where the surgery will concern the cosmic time axis. First, the cosmic time coordinate $t > 0$ is replaced by an extended coordinate $\tilde{t} \in \mathbb{R}$. Then, the surgery on this 1-space is to remove the open $\tilde{t}$ interval $(-b, b)$, for $b > 0$, and to identify the antipodal points $\tilde{t} = -b$ and $\tilde{t} = b$. In this case, the resulting 1-space $\tilde{M}_1$ is topologically trivial, $\tilde{M}_1 \simeq \mathbb{R}$ (recall that, indeed, $\mathbb{R}P^1 \simeq \mathbb{R} + \text{point}$). A sketch is given in Fig. 1 which may be considered to be the 1-dimensional analog of Fig. 1 in Ref. [5] for the 3-dimensional Schwarzschild construction.

Note that we will use the same symbol $b$ for the regulator of the Schwarzschild solution and the one of the Friedmann solution, but these length scales can, in principle, be different. Remember that we have set $c = 1$.

Next, define a suitable cosmic time coordinate $T$ (not to be confused with the temperature $\mathcal{T}$ of the matter):

$$T \equiv \begin{cases} +\sqrt[4]{\tilde{t}^4 - b^4}, & \text{for } \tilde{t} \geq b, \\ -\sqrt[4]{\tilde{t}^4 - b^4}, & \text{for } \tilde{t} \leq -b, \end{cases}$$

$$\tilde{t} \in (-\infty, -b] \cup [b, \infty).$$

(3.1a)

(3.1b)

The coordinate $T$ covers the real line $\mathbb{R}$, with a unique value of $T$ for each point of the line. Furthermore, the expression (3.1a) has the same mathematical structure as the Schwarzschild-construction expression (2.26) in Ref. [6], where $y$ must be replaced by $T$ and $\pm r$ by $\tilde{t}$ and where, following Endnote 18 of Ref. [6], the powers 2 are replaced by powers 4 and the square roots by fourth roots.

![FIG. 1. Cosmic time axis $\tilde{t} \in (-\infty, -b] \cup [b, \infty)$, where the points $\tilde{t} = -b$ and $\tilde{t} = b$ are identified (as indicated by the dots).](image-url)
With this new cosmic time coordinate $T$, we make the following Ansatz for the metric:

$$ds^2 \bigg|_{\text{mod. FRW}} = -\frac{T^6}{(b^4 + T^4)^{3/2}} dT^2 + a^2 \left( \tilde{t} \right) \delta_{kl} dx^k dx^l,$$

(3.2a)

$$a^2 \left( \tilde{t} \right) \bigg|_{\tilde{t} = -b} = a^2 \left( \tilde{t} \right) \bigg|_{\tilde{t} = b},$$

(3.2b)

$$T \in (-\infty, \infty),$$

(3.2c)

$$x^k \in (-\infty, \infty),$$

(3.2d)

$$\tilde{t}(T) = \begin{cases} +\sqrt{b^4 + T^4}, & \text{for } T \geq 0, \\ -\sqrt{b^4 + T^4}, & \text{for } T \leq 0, \end{cases}$$

(3.2e)

where the coordinates $\tilde{t} = -b$ and $\tilde{t} = b$ correspond to the single point $T = 0$ on the cosmic time axis (cf. Fig. 1). Remark that, even for $a \left( \tilde{t} \right) \neq 0$, the metric from (3.2) is degenerate: $\det g_{\mu\nu} = 0$ at $T = 0$. The corresponding $T = 0$ spacetime slice may be interpreted as a 3-dimensional “defect” of spacetime with topology $\mathbb{R}^3$. The degeneracy issue will be discussed further in Sec. IV.

It is straightforward to calculate the dynamic equations by inserting this new metric for the coordinates $\{T, x^1, x^2, x^3\}$ and the energy-momentum tensor of a perfect fluid [energy density $\rho(T)$ and pressure $P(T) = \rho(T)/3$ for the relativistic-matter case] into the Einstein equation without cosmological constant $\Lambda$. But the result can also be obtained from (2.2) by the observation that the new metric (3.2a) written in terms of $\tilde{t}$ takes the same form as the standard metric (2.1a) written in terms of $t$. The dynamic equations are then as follows:

$$\left( \frac{1}{a \left( \tilde{t} \right)} \frac{da \left( \tilde{t} \right)}{dt} \right)^2 = \frac{8\pi}{3} G_N \rho \left( \tilde{t} \right),$$

(3.3a)

$$\frac{d}{da} \left( \rho a^3 \right) = -3P a^2,$$

(3.3b)

$$P \left( \tilde{t} \right) = \frac{1}{3} \rho \left( \tilde{t} \right),$$

(3.3c)

where $\tilde{t} = \tilde{t}(T)$ has been defined by (3.2e) and where we have, again, considered relativistic matter.

The solutions $a(T)$ of (3.3) can be even or odd in $T$. In view of the recent interest [9] in a $T$-odd solution, we give explicitly our regularized $T$-odd relativistic-matter solution $a(T)$ from (3.3),

$$a(T) \bigg|_{\text{mod. FRW}} \bigg|_{\text{(rel-mat. sol.)}} = \begin{cases} +\sqrt{(b^4 + T^4)/(b^4 + t_0^4)}, & \text{for } T \geq 0, \\ -\sqrt{(b^4 + T^4)/(b^4 + t_0^4)}, & \text{for } T < 0, \end{cases}$$

(3.4)
which is discontinuous at $T = 0$ but has a monotonic behavior, $da(T)/dT \geq 0$. Observe that $d^2a(T)/dT^2$ from (3.4) is continuous at $T = 0$, which would not be the case if the $T$ definition (3.1a) were replaced by $\pm \sqrt{t^2 - b^2}$. Note also that the solution (3.4) for $T > 0$ reproduces, in the limit $b \to 0^+$, the positive-$t$ branch of the standard solution (2.4).

Taking a nonvanishing regulator $b$, the solution (3.4) gives finite values at $T = 0$ for the Ricci curvature scalar $R$ [identically zero, in fact] and the Kretschmann curvature scalar $K$,

$$R(T) \bigg|_{\text{mod. FRW}}^{(\text{rel-mat. sol.)}} = 0, \quad (3.5a)$$

$$K(T) \bigg|_{\text{mod. FRW}}^{(\text{rel-mat. sol.)}} = \frac{3}{2} \frac{1}{b^4 + T^4}, \quad (3.5b)$$

and also for the matter energy density $\rho$ and the temperature $\mathcal{T}$,

$$\rho(T) \bigg|_{\text{mod. FRW}}^{(\text{rel-mat. sol.)}} = \rho_0 \sqrt{\frac{b^4 + t_0^4}{b^4 + T^4}}, \quad (3.5c)$$

$$\mathcal{T}(T) \bigg|_{\text{mod. FRW}}^{(\text{rel-mat. sol.)}} = \mathcal{T}_0 \sqrt{s \frac{b^4 + t_0^4}{b^4 + T^4}}, \quad (3.5d)$$

with finite boundary conditions $\rho_0 > 0$ and $\mathcal{T}_0 > 0$. For the last result (3.5d), the relation $\mathcal{T}(a) \propto [a^4]^{-1/4}$ was used, as explained in the sentence below (2.7d). For completeness, we mention that the $T$-even solution has the same eighth roots as in (3.4) but now with equal over-all plus signs. The results (3.5) hold also for the $T$-even solution.

In view of the results (3.5), the solution (3.4) can be interpreted as a regularized version of the function used in Ref. [9], which corresponds to

$$a(T) \bigg|_{\text{stand. FRW}}^{(\text{rel-mat. sol.)}} = \text{sgn}(T) \sqrt{T^2/t_0^2}, \quad (3.6)$$

for an extended cosmic time coordinate $T \in \mathbb{R}$. In fact, the quantities $K(T), \rho(T)$, and $\mathcal{T}(T)$ obtained from (3.6) diverge as $T \to 0$ and are given by (3.5b), (3.5c), and (3.5d) with $b$ set to zero.

**IV. DISCUSSION**

The regularized FRW spacetime manifold with metric (3.2) and cosmic scale factor solution (3.4) has the following line-element:

$$ds^2 \bigg|_{\text{mod. FRW}}^{(\text{rel-mat. sol.)}} = -\frac{T^6}{(b^4 + T^4)^{3/2}} dT^2 + \sqrt{\frac{b^4 + T^4}{b^4 + t_0^4}} \delta_{kl} dx^k dx^l, \quad (4.1)$$
with all spacetime coordinates \( \{T, x^1, x^2, x^3\} \) ranging over \( \mathbb{R} \). The standard FRW spacetime manifold with metric (2.1) for an extended cosmic time coordinate \( T \in \mathbb{R} \) and cosmic scale factor solution (3.6) has the line-element

\[
ds^2_{\text{stand. FRW}}^{(\text{rel-mat. sol.) }} = -dT^2 + \sqrt{\frac{T^2}{t_0^2}} \delta_{kl} dx^k dx^l. \tag{4.2}
\]

Formally, the metric in (4.2) results from the one in (4.1) by setting \( b \) to zero.

Both of the above metrics are degenerate, \( \det g_{\mu\nu} = 0 \) at \( T = 0 \), but the metric from (4.1) with nonzero \( b \) has only a single vanishing eigenvalue, whereas the metric from (4.2) has three vanishing eigenvalues. In fact, the Kretschmann curvature scalar \( K \), as given by (3.5b), is finite at \( T = 0 \) for the metric (4.1) with nonzero \( b \) but singular at \( T = 0 \) for the metric (4.2). In this way, the spacetime manifold (4.1), with a nonzero length parameter \( b \) and a particular differential structure, may be considered to be a regularized version of the spacetime manifold (4.2). In App. A, we give the corresponding results for a modified FRW universe with a nonrelativistic matter component.

Remark that, in general, a regularized theory may temporarily lose certain desirable properties, which are only recovered as the regulator is removed. An example is given by the lattice regularization of flat-spacetime quantum field theory, where the full Poincaré-invariance group is recovered in the continuum limit as the lattice spacing is taken to zero. Our regularized Friedmann solution is also far from perfect, as the “spacetime defect” at \( T = 0 \) makes clear. There is, for example, the \( T = 0 \) discontinuity in the \( T \)-odd cosmic scale factor solution (3.4). This discontinuity disappears in the corresponding metric (4.1), so that scalar, vector, and tensor fields are unaffected by the \( a(T) \) discontinuity at \( T = 0 \). The spinor-field boundary conditions at \( T = 0^+ \) and \( T = 0^- \) may require a further \( \mathbb{C}P \) transformation. In any case, the metric (4.1) provides a spacetime manifold without curvature singularities, which allows for a meaningful study of the behavior of relativistic matter.

It may be that the length parameter \( b \) entering the metric (4.1) is not just a mathematical artifact (“regulator”) but that it traces back to the underlying theory of “quantum spacetime.” Still, it is unclear whether or not the length scale \( b \) is then determined by the Planck length \( l_{\text{Planck}} \equiv \sqrt{\hbar G_N/c^3} \approx 1.62 \times 10^{-35} \text{ m} \), as the definitive quantum-spacetime theory has not yet been established.

Leaving aside a possible physical origin of the length parameter \( b \) in the metric (4.1), we observe that the corresponding spacetime manifold is geodesically complete, as long as the cosmic time coordinate \( T \) has an extended range, \( T \in \mathbb{R} \). This manifold, then, has a
pre-Big-Bang phase \( (T < 0) \), which may or may not have produced relics in the present \( (T > 0) \) universe.

**Appendix A: Modified FRW universe with nonrelativistic matter**

In this appendix, we give some results for the spatially-flat modified FRW universe with nonrelativistic matter instead of relativistic matter. Specifically, the equation of state \( (3.3c) \) is replaced by

\[
P(T) = 0, \tag{A1}
\]

where \( T \) is the cosmic time coordinate from \( (3.1) \).

The regularized \( T \)-odd nonrelativistic-matter solution \( a(T) \) from \( (3.3a), (3.3b) \), and \( (A1) \) is given by

\[
a(T) \bigg|_{\text{mod. FRW}}^{(\text{nonrel-mat. sol.})} = \begin{cases} 
+ \sqrt[3]{\frac{b^4 + T^4}{b^4 + t_0^4}}, & \text{for } T \geq 0, \\
- \sqrt[3]{\frac{b^4 + T^4}{b^4 + t_0^4}}, & \text{for } T < 0.
\end{cases} \tag{A2}
\]

The corresponding expressions for the Ricci curvature scalar \( R \) and the Kretschmann curvature scalar \( K \) are

\[
R(T) \bigg|_{\text{mod. FRW}}^{(\text{nonrel-mat. sol.})} = \frac{4}{3} \sqrt[3]{\frac{1}{b^4 + T^4}}, \tag{A3a}
\]

\[
K(T) \bigg|_{\text{mod. FRW}}^{(\text{nonrel-mat. sol.})} = \frac{80}{27} \frac{1}{b^4 + T^4}. \tag{A3b}
\]

Both curvature scalars are nonsingular at \( T = 0 \) for \( b \neq 0 \) and singular at \( T = 0 \) for \( b \to 0 \).

Finally, the regularized FRW spacetime manifold with metric \( (3.2) \) and cosmic scale factor solution \( (A2) \) has the following line-element:

\[
ds^2 \bigg|_{\text{mod. FRW}}^{(\text{nonrel-mat. sol.})} = -\frac{T^6}{(b^4 + T^4)^{3/2}} dT^2 + \left[ \frac{b^4 + T^4}{b^4 + t_0^4} \right]^{1/3} \delta_{kl} dx^k dx^l, \tag{A4}
\]

with all spacetime coordinates ranging over \( \mathbb{R} \). For the record, the \( T \)-even solution has the same sixth roots as in \( (A2) \) but now with equal over-all plus signs. The results \( (A3) \) and \( (A4) \) hold also for the \( T \)-even solution.

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