Research Article

Blowing Up for the $p$-Laplacian Parabolic Equation with Logarithmic Nonlinearity

Asma Alharbi

Department of Mathematics, College of Sciences and Arts, ArRass, Qassim University, Saudi Arabia

Correspondence should be addressed to Asma Alharbi; ao.alharbi@qu.edu.sa

Received 9 May 2021; Accepted 17 July 2021; Published 27 July 2021

Academic Editor: Kamyar Hosseini

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In this article, we are concerned with a problem for the $p$-Laplacian parabolic equation with logarithmic nonlinearity; the blow-up result of the solution is proven. This work is completed Boulaaras’ work in Math. Methods Appl. Sci., (2020), where the author did not study the blowup of the solution.

1. Introduction

In the current manuscript, we consider the following initial-boundary value problem for a nonlinear $p$-Laplacian equation:

\[ u_t - \text{div} \left( |\nabla u|^{p-2} \nabla u \right) + |u|^{p-2} u = |u|^{p-2} u \ln |u|, \quad x \in \Omega, \quad t > 0, \]

\[ u(x, 0) = u_0(x), \quad x \in \Omega, \]

\[ u(x, t) = 0, \quad x \in \partial \Omega, \quad t \geq 0, \]

\( \tag{1} \)

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial \Omega$ and $u_0$ is the initial data $p$ satisfying

\[
\begin{align*}
2 < p < \infty, & \quad \text{if } n \leq p, \\
2 < p < \frac{np}{n-p}, & \quad \text{if } n > p.
\end{align*}
\]

The terminology of nonlinear polynomials is among the work that researchers have focused on recently. For example, it is found in edge detection and optical elasticity, materials science, engineering, physics, and photonics. In addition, many works and problems in applied sciences have been designed and proposed by means of partial differential equations, including the modeling of some dynamic systems in physics and engineering ([1–13]).

The same is said for the evolutionary partial differential equations associated with $p(x)$-Laplacian (see [8, 14, 15]).

We also note that logarithmic nonlinearity has been concerned by many scientists and researchers, and it has introduced many issues, including the wave equation (see [3, 16–18]).

And for more information on some of the other works to which this term was introduced, we refer the reader to [13, 14, 16–24].

Later on, in [25], the authors by the multiplier method gave the energy decay of the solution of the following problem:

\[ u_{tt} - \text{div} \left( |\nabla u|^{p-2} \nabla u \right) - \Delta u_t + |u|^{q-1} u = |u|^{p-1} u. \]

\( \tag{3} \)

In addition, the authors in [14] proved the decay rate of solutions (exponential and polynomial) by using the inequality of Nakao for the seminar problem (3).

On the other hand, for the Laplacian parabolic equation with the logarithmic source term in [21], Chen et al. studied the following problem:

\[ u_t - \Delta u - \Delta u_t = u \ln u. \]

\( \tag{4} \)

Then, in [23], the authors proved the global existence, the decay, and the blowup of the solutions of the problem:

\[ u_t - \text{div} \left( |\nabla u|^{p-2} \nabla u \right) - \Delta u_t = |u|^{p-2} u \ln |u|, \]

\( \tag{5} \)

where $p > 2$. 


Also, in [14], the authors established the global boundedness and the blowup of the solution of the problem (5) for \(1 < p < 2\).

Motivated by the last recent mentioned works, here, we investigated problem (1) with the nonlinear diffusion \(\Delta_p = \text{div} (|\nabla u|^{p-2} \nabla u)\) and logarithmic nonlinearity \(|u|^{p-2} u \ln |u|\) which extends problem in [14]. Our goal is to blow up solutions for problem (1) in order to put some preliminaries. More precisely, we give the blow-up result.

2. Preliminaries

As a starting point, we gave some essential definitions and lemmas.

\[
\|u\|_p = \|u\|_{L^p(\Omega)}, \|u\|_{L_p^p} = \|u\|_{W^{1,p}_0(\Omega)} = \left(\|u\|_p + \|\nabla u\|_p\right)^{1/p},
\]

for \(1 < p < \infty\), and we symbolize the positive constants by \(C_i, (i = 1, 2, \ldots)\).

**Lemma 1** [7] (logarithmic Sobolev inequality). Let \(u\) be all function \(u \in W^{1,p}_0(\mathbb{R}^n) \setminus \{0\}\). Then, for \(p > 1, \mu > 0\),

\[
p \int_{\mathbb{R}^n} u^p \ln \left(\frac{|u|}{\|u\|_{L^p(\mathbb{R}^n)}}\right) dx \leq \mu \int_{\mathbb{R}^n} |\nabla u|^p dx - \frac{n}{p} \ln \left(\frac{\mu e}{\mathcal{L}_p}\right) \int_{\mathbb{R}^n} |u|^p dx,
\]

where

\[
\mathcal{L}_p = \frac{p}{n} \left(\frac{p-1}{e}\right)^{\frac{p-1}{2}} (\pi^{1/2})^{\frac{n-1}{2}} \left[\frac{\Gamma((n/2) + 1)}{\Gamma((p-1)/p + 1)}\right]^{\frac{1}{p}}.
\]

**Remark 2.** Let \(u \in W^{1,p}_0(\Omega) \setminus \{0\}\), and by defining \(u(x) = 0\) for \(x \in \mathbb{R}^n \setminus \Omega\), we can write

\[
p \int_{\Omega} u^p \ln \left(\frac{|u|}{\|u\|_{L^p(\mathbb{R}^n)}}\right) dx \leq \mu \int_{\Omega} |\nabla u|^p dx - \frac{n}{p} \ln \left(\frac{\mu e}{\mathcal{L}_p}\right) \int_{\Omega} |u|^p dx.
\]

3. Blowup

In this third section, we gave the proof of blowup of solution of our problem.

**Theorem 3.** For any initial data \(u_0 \in \mathcal{H}\), the problem (1) has a unique weak solution:

\[
u \in C([0, T]; \mathcal{H}),
\]

for some \(T > 0\).

First, we introduce the energy functional in the following lemma.

**Lemma 4.** Let \(u(t)\) be a solution of (1), then \(E(t)\) is nonincreasing; that is,

\[
E(t) = \frac{1}{p} \|\nabla u(t)\|_p^p - \frac{1}{p} \int_{\Omega} \ln |u(t)|^p dx + \frac{p+1}{p^2} \|u(t)\|_p^p
\]

satisfies

\[
E'(t) = -\|u_t\|_2^2.
\]

**Proof.** Multiplying (1) by \(u_t\) and integrating on \(\Omega\), we have

\[
\int_{\Omega} \Delta_p u \ ln |u| u_t dx + \int_{\Omega} |\nabla u|^p u_t dx + \int_{\Omega} u_t u dx = \int_{\Omega} u^p u_t ln |u| u dx,
\]

\[
\frac{d}{dt} \left(\frac{1}{p} \|\nabla u\|_p^p + \frac{1}{p} \|u\|_p^p - \frac{1}{p} \int_{\Omega} \ln |u|^p dx + \frac{1}{p^2} \|u\|_p^p\right) = -\|u_t\|_2^2.
\]

Thus,

\[
E'(t) = -\|u_t\|_2^2.
\]

To get to our goal of proving the main result, we define the functional

\[
H(t) = -E(t) = -\frac{1}{p} \|\nabla u\|_p^p + \frac{1}{p} \int_{\Omega} \ln |u|^p dx - \frac{p+1}{p^2} \|u\|_p^p.
\]

**Theorem 5.** Assume that \(E(0) < 0\), then the solution of problem (1) blows up in finite time.

**Proof.** From (12), we have

\[
E(t) \leq E(0) \leq 0.
\]
Hence,
\[ H'(t) = -E'(t) = \|u_t\|_p^2 \geq 0, \]
\[ 0 \leq H(0) \leq H(t) \leq \frac{1}{p} \int_\Omega \ln |u|^p dx. \]  
(17)

We set
\[ {\mathcal{K}}(t) = H^{1-\alpha} + \frac{\epsilon}{2} \int_\Omega u_t^2 dx, \]
where \( \epsilon > 0 \) and
\[ 0 < \alpha < \frac{p-2}{p} < 1. \]  
(19)

Multiplying (1) by \( u \) and the derivative of (18) gives
\[ {\mathcal{K}}' - \alpha H'(t) - \epsilon \|\nabla u\|_p^p - \epsilon \|\nabla u\|_p^p + \epsilon \int_\Omega |u|^p \ln |u| dx. \]  
(20)

Adding and subtracting \( \epsilon \delta H(t) \) into (20) \( \delta > 0 \), we obtain
\[ {\mathcal{K}}' - \alpha H'(t) + \epsilon \left( \frac{\delta - p}{p} \right) \|\nabla u\|_p^p \]
\[ + \epsilon \left( \frac{\delta - p}{p} + \frac{1}{p^2} \right) |u|^p - \epsilon \left( \frac{\delta - p}{p} \right) \int_\Omega |u|^p dx + \epsilon \delta H(t). \]  
(21)

Applying the logarithmic Sobolev inequality gives
\[ {\mathcal{K}}' - \alpha H'(t) + \epsilon \delta H(t) + \epsilon \left( \frac{\delta - p}{p} \right) \left( 1 - \frac{H}{p} \right) \|\nabla u\|_p^p + \epsilon \left( \frac{\delta - p}{p} \right) 
\cdot \left[ 1 + \frac{\delta}{p(\delta - p)} - \ln |u|_p + \left( \frac{n}{p^2} \ln \left( \frac{pue}{n\Lambda_p} \right) \right) \right] |u|^p. \]  
(22)

Setting \( \mu = p/2 \) and taking \( \delta > p \) give
\[ \left[ 1 + \frac{\delta}{p(\delta - p)} - \ln |u|_p + \left( \frac{n}{p^2} \ln \left( \frac{pue}{2n\Lambda_p} \right) \right) \right] > 0, \]  
(23)

since
\[ |u|_p > e^{\left( 1 + \frac{\delta}{p(\delta - p)} \right) \left( \frac{p^2 e}{2n\Lambda_p} \right)} . \]  
(24)

Consequently, for some \( \beta > 0 \), inequality (25) gives
\[ {\mathcal{K}}'(t) \geq \beta \left\{ H(t) + |u|^p_p + |\nabla u|^p_p \right\}, \]  
(25)
\[ {\mathcal{K}}(t) \geq {\mathcal{K}}(0) > 0, \quad t > 0. \]  
(26)

Next, by (18), we have
\[ {\mathcal{K}}(t) = H^{1-\alpha} + \frac{\epsilon}{2} \int_\Omega u_t^2 dx \leq H^{1-\alpha} + \epsilon C |u|^2_p \]
\[ \leq H^{1-\alpha} + \epsilon C \left( |u|^p_p \right)^{2/p}. \]  
(27)

Therefore,
\[ {\mathcal{K}}^{1/1-\alpha}(t) \leq H^{1-\alpha} + \epsilon C \left( |u|^p_p \right)^{2/(p(1-\alpha))}, \]  
(28)

where \( 0 < 2/p(1-\alpha) < 1 \),
\[ \left( |u|^p_p \right)^{2/(p(1-\alpha))} \leq C \left( \left( |u|^p_p \right)^{p} + H(t) \right). \]  
(29)

Hence,
\[ {\mathcal{K}}^{1/1-\alpha}(t) \leq C_1 \left[ H(t) + |u|^p_p \right] \leq C_1 \left[ H(t) + |\nabla u|^p_p + |u|^p_p \right]. \]  
(30)

According to (25) and (30), we get
\[ {\mathcal{K}}'(t) \geq \lambda C_1^{1/1-\alpha}(t), \]  
(31)

where \( \lambda = C_1/\beta > 0 \), depending only on \( \beta \) and \( C_1 \).

Finally, by integrating (31), we obtain
\[ {\mathcal{K}}^{1/1-\alpha}(t) \geq \frac{1}{\lambda C_1^{1/1-\alpha}(0) - \lambda(\alpha/(1-\alpha))} t. \]  
(32)

Hence, \( {\mathcal{K}}(t) \) blows up in time:
\[ T \leq T^* = \frac{1 - \alpha}{\lambda \alpha C_1^{1/1-\alpha}(0)} . \]  
(33)

As a result, the proof is completed. \( \square \)

Data Availability

No data were used to support the study.

Conflicts of Interest

The author declares that he has no conflicts of interest.

Acknowledgments

The researcher would like to thank the Deanship of Scientific Research, Qassim University, for funding the publication of this project.

References

[1] S. Boulaaras and Y. Bouizem, “Blow up of solutions for a nonlinear viscoelastic system with general source term,” Quaestiones Mathematicae, pp. 1–11, 2020.
[2] S. Boulaaras, A. Choucha, B. Cherif et al., "Blow up of solutions for a system of two singular nonlocal viscoelastic equations with damping, general source terms and a wide class of relaxation functions," *AIMS Mathematics*, vol. 6, no. 5, pp. 4664–4676, 2021.

[3] S. Boulaaras, "A well-posedness and exponential decay of solutions for a coupled Lamé system with viscoelastic term and logarithmic source terms," *Applicable Analysis*, vol. 100, no. 7, pp. 1514–1532, 2021.

[4] A. Choucha, D. Ouchenane, and S. Boulaaras, "Blow-up of a nonlinear viscoelastic wave equation with distributed delay combined with strong damping and source terms," *Journal of Nonlinear Functional Analysis*, vol. 2020, no. 1, 2020.

[5] A. Menaceur, S. Boulaaras, A. Makhlouf, K. Rajagobal, and M. Abdalla, "Limit cycles of a class of perturbed differential systems via the first-order averaging method," *Complexity*, vol. 2021, Article ID 5581423, 6 pages, 2021.

[6] D. Ouchenane, S. Boulaaras, A. Alharbi, and B. Cherif, "Blow up of coupled nonlinear Klein-Gordon system with distributed delay, strong damping, and source terms," *Journal of Function Spaces*, vol. 2020, Article ID 5297063, 9 pages, 2020.

[7] S. Boulaaras, "Existence of positive solutions for a new class of Kirchhoff parabolic systems," *The Rocky Mountain Journal of Mathematics*, vol. 50, no. 2, pp. 445–454, 2020.

[8] N. Ioku, "The Cauchy problem for heat equations with exponential nonlinearity," *Journal of Differential Equations*, vol. 251, no. 4–5, pp. 1172–1194, 2011.

[9] H. Yükselkaya, E. Piskin, S. M. Boulaaras, B. B. Cherif, and S. A. Zubair, "Existence, nonexistence, and stability of solutions for a delayed plate equation with the logarithmic source," *Advances in Mathematical Physics*, vol. 2021, Article ID 8561626, 11 pages, 2021.

[10] E. Piskin and H. Yükselkaya, "Local existence and blow up of solutions for a logarithmic nonlinear viscoelastic wave equation with delay," *Computational Methods for Differential Equations*, vol. 9, no. 2, pp. 623–636, 2021.

[11] H. T. Song and D. S. Xue, "Blow up in a nonlinear viscoelastic wave equation with strong damping," *Nonlinear Analysis*, vol. 109, pp. 245–251, 2014.

[12] H. T. Song and C. K. Zhong, "Blow-up of solutions of a nonlinear viscoelastic wave equation," *Nonlinear Analysis: Real World Applications*, vol. 11, no. 5, pp. 3877–3883, 2010.

[13] S. Boulaaras, A. Choucha, D. Ouchenane, and B. Cherif, "Blow up of solutions of two singular nonlinear viscoelastic equations with general source and localized fractional damping terms," *Advances in Difference Equations*, vol. 2020, no. 1, 2020.

[14] E. Piskin, S. Boulaaras, and N. Irkil, "Qualitative analysis of solutions for the p-Laplacian hyperbolic equation with logarithmic nonlinearity," *Mathematical Methods in the Applied Sciences*, vol. 44, no. 6, pp. 4654–4672, 2021.

[15] F. Ekinci, E. Piskin, S. Boulaaras, and I. Mekawy, "Global existence and general decay of solutions for a quasilinear system with degenerate damping terms," *Journal of Function Spaces*, vol. 2021, Article ID 4316238, 10 pages, 2021.

[16] I. Bialynicki-Birula and J. Mycielski, "Wave equations with logarithmic nonlinearities," *Bulletin de l’Academie Polonaise des Sciences. Serie des Sciences, Mathematiques, Astronomiques et Physiques*, vol. 23, pp. 461–466, 1975.

[17] I. Bialynicki-Birula and J. Mycielski, "Nonlinear wave mechanics," *Annals of Physics*, vol. 100, no. 1–2, pp. 62–93, 1976.