1. Main result and discussion

For an open set $D \subset \mathbb{R}^n$ and $0 < \alpha < 2$ let

$$\mathcal{E}_D(u) = \frac{1}{2} \int_D \int_D \frac{(u(x) - u(y))^2}{|x - y|^{n+\alpha}} \, dx \, dy, \quad u \in L^2(D).$$

The quadratic form $\mathcal{E}$ is, up to a multiplicative constant, the Dirichlet form of the censored stable process in $D$, see [1]. It has been shown [6] that for convex, connected open sets $D$ and $1 < \alpha < 2$

$$\mathcal{E}_D(u) \geq \kappa_{n,\alpha} \int_D u^2(x) \delta_D^{-\alpha}(x) \, dx, \quad u \in C_c(D),$$

where $\delta_D(x) = \text{dist}(x, D^c)$ and

$$\kappa_{n,\alpha} = \pi^\frac{n-1}{2} \frac{\Gamma\left(\frac{1+\alpha}{2}\right) B\left(\frac{1+\alpha}{2}, \frac{2-\alpha}{2}\right)}{\Gamma\left(\frac{n+\alpha}{2}\right) \alpha 2^{\alpha}}$$

is the largest constant for which (1) holds. In (2) $B$ is the Euler beta function.

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On the other hand, if $0 < \alpha < n \land 2$, by Sobolev embedding [2, (2.3)], we have, e.g., for open convex, connected sets $D$

\[(3) \quad \mathcal{E}_D(u) + \|u\|_{L^2}^2 \geq c \left( \int_D |u(x)|^{2^*} \, dx \right)^{2/2^*}, \quad u \in C_c(D),\]

with some $c = c(D, \alpha) > 0$ and $2^* = 2n/(n - \alpha)$.

Comparing (1) and (3) an interesting question arises, whether the following Hardy-Sobolev-Maz’ya inequality

\[(4) \quad \mathcal{E}_D(u) \geq \kappa_{n,\alpha} \int_D u^2(x) \delta_D^{-\alpha}(x) \, dx + c \left( \int_D |u(x)|^{2^*} \, dx \right)^{2/2^*}, \quad u \in C_c(D),\]

holds for $1 < \alpha < n$ and convex domains $D$? A similar question was posed in [1, page 2].

The purpose of this note is to prove (4) for half-spaces and balls, see Theorem 3 and Corollary 4. We would like to note that while writing this note a paper [7] of Sloane appeared, in which the author has proved (4) for half-spaces. However, our proof is different and we also obtain (4) for balls.

2. Proofs

We denote by $B_r = \{x \in \mathbb{R}^n : |x| < r\}$ the open Euclidean ball of radius $r > 0$, we set $B = B_1$ and by $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ we denote the $(n - 1)$-dimensional unit sphere.

We define

\[L_D u(x) = \lim_{\varepsilon \to 0^+} \int_{D \cap \{|y-x|>\varepsilon\}} \frac{u(y) - u(x)}{|x-y|^{n+\alpha}} \, dy.\]

Note that $L_D$ is, up to the multiplicative constant, the regional fractional Laplacian for an open set $D$, see [5].

Let

\[(5) \quad w_n(x) = (1 - |x|^2)^{\frac{\alpha-1}{2}}, \quad x \in B \subset \mathbb{R}^n, \ n \geq 1.\]

We recall from [3, Lemma 2.1] that

\[-L_{(-1,1)} w_1(x) = \frac{(1 - x^2)^{\frac{\alpha-1}{2}}}{\alpha} \left( B \left( \frac{\alpha + 1}{2}, \frac{2 - \alpha}{2} \right) - (1 - x)^\alpha + (1 + x)^\alpha \right)\]

hence by [3, (2.3)]

\[(6) \quad -L_{(-1,1)} w_1(x) \geq c_1 (1 - x^2)^{\frac{\alpha-1}{2}} + c_2 (1 - x^2)^{\frac{\alpha+1}{2}},\]
We calculate the inner principle value integral by changing the variable $g$
Hence by (6) we have

\begin{align*}
\text{Lemma 1. We have for } w_n \text{ defined in (4) and } n \geq 2
\end{align*}

\begin{align*}
-L_B w_n(x) \geq \frac{c_1}{2} \int_{S^{n-1}} |h_n|^\alpha dh \cdot (1 - |x|^2)^{-\frac{\alpha+1}{2}} + c_2 |S^{n-1}| \cdot (1 - |x|^2)^{-\frac{\alpha-1}{2}}
\end{align*}

\text{Proof. Let } x = (0, 0, \ldots, 0, x), \ p = \frac{\alpha-1}{2}. \ We have

\begin{align*}
-L_B w_n(x) = p.v. \int_B \frac{(1 - |x|^2)^p - (1 - |y|^2)^p}{|x - y|^{n+\alpha}} dy \\
= \frac{1}{2} \int_{S^{n-1}} dh \ p.v. \int_{-xh_n+\sqrt{x^2h_n^2-x^2+1}}^{xh_n+\sqrt{x^2h_n^2-x^2+1}} \frac{(1 - |x|^2)^p - (1 - |x + ht|^2)^p}{|t|^{1+\alpha}} dt.
\end{align*}

We calculate the inner principle value integral by changing the variable $t = -xh_n + u\sqrt{x^2h_n^2 - x^2 + 1}$

\begin{align*}
g(x, h) := p.v. \int_{-xh_n - \sqrt{x^2h_n^2 - x^2 + 1}}^{xh_n + \sqrt{x^2h_n^2 - x^2 + 1}} \frac{(1 - |x|^2)^p - (1 - |x + ht|^2)^p}{|t|^{1+\alpha}} dt \\
= p.v. \int_{-1}^1 \frac{(1 - x^2)^p - (1 - u^2)^p(1 - x^2 + x^2h_n^2)^p}{|u - \frac{xh_n}{\sqrt{1 - x^2 + x^2h_n^2}}|^{1+\alpha}} du \\
= (1 - x^2 + x^2h_n^2)^{p-\alpha/2} p.v. \int_{-1}^1 \frac{(1 - \frac{x^2h_n^2(1 - x^2 + x^2h_n^2)^p}{1 - x^2 + x^2h_n^2})^p - (1 - u^2)^p}{|u - \frac{xh_n}{\sqrt{1 - x^2 + x^2h_n^2}}|^{1+\alpha}} du \\
= (1 - x^2 + x^2h_n^2)^{-1/2} (-L(-1, 1) w_1)(\frac{xh_n}{\sqrt{1 - x^2 + x^2h_n^2}}).
\end{align*}

Hence by (5) we have

\begin{align*}
g(x, h) \geq (1 - x^2 + x^2h_n^2)^{-1/2} \left[ c_1 \left( 1 - \frac{x^2h_n^2}{1 - x^2 + x^2h_n^2} \right)^{\frac{\alpha-1}{2}} \right]^{-\alpha} \\
+ c_2 \left( 1 - \frac{x^2h_n^2}{1 - x^2 + x^2h_n^2} \right)^{\frac{\alpha-1}{2} - 1} \\
= c_1 (1 - x^2 + x^2h_n^2)^{\alpha/2} (1 - x^2)^{-\frac{\alpha+1}{2}} + c_2 (1 - x^2 + x^2h_n^2)^{\alpha/2 - 1} (1 - x^2)^{-\frac{\alpha+1}{2}} \\
\geq c_1 |h_n|^\alpha (1 - x^2)^{-\frac{\alpha+1}{2}} + c_2 (1 - x^2)^{-\frac{\alpha-1}{2}}.
\end{align*}
Thus

\[-L_Bw_n(x) = \frac{1}{2} \int_{S^{n-1}} g(x, h)dh \geq \frac{c_1}{2} \int_{S^{n-1}} |h_n|^{\alpha}dh \cdot (1 - x^2)^{-\frac{\alpha+1}{2}} + c_2 |S^{n-1}| \cdot (1 - x^2)^{-\frac{\alpha+1}{2}}\]

and we are done. \hfill \square

**Corollary 2.** Let $1 < \alpha < 2$. Let $w_n$ be as in (5) and let $n \geq 2$. Then for every $u \in C_c(B)$,

\[
\mathcal{E}_B(u) := \frac{1}{2} \int_B \int_B \frac{(u(x) - u(y))^2}{|x-y|^{n+\alpha}} \, dx \, dy \\
\geq \frac{1}{2} \int_B \int_B \left( \frac{u(x)}{w(x)} - \frac{u(y)}{w(y)} \right)^2 \frac{w(x)w(y)}{|x-y|^{n+\alpha}} \, dx \, dy \\
+ 2^\alpha \kappa_{n,\alpha} \int_B u^2(x) (1 - |x|^2)^{-\alpha} \, dx + c_3 \int_B u^2(x) (1 - |x|^2)^{-\alpha+1} \, dx
\]

(7)

Proof. The result follows from [3, Lemma 2.2] applied to $w = w_n$, Lemma [1] and the following formula [6, (7)]

\[
\kappa_{n,\alpha} = \kappa_{1,\alpha} \cdot \frac{1}{2} \int_{S^{n-1}} |h_n|^{\alpha}dh.
\]

\hfill \square

**Theorem 3.** Let $1 < \alpha < 2$ and $n \geq 2$. There exist a constant $c = c(\alpha, n)$ such that for every $0 < r < \infty$ and $u \in C_c(B_r)$,

\[
\mathcal{E}_{B_r}(u) := \frac{1}{2} \int_{B_r} \int_{B_r} \frac{(u(x) - u(y))^2}{|x-y|^{n+\alpha}} \, dx \, dy \\
\geq 2^\alpha \kappa_{n,\alpha} \int_{B_r} u^2(x) r^\alpha (r^2 - |x|^2)^{-\alpha} \, dx + c \left( \int_{B_r} |u(x)|^{2^*} \, dx \right)^{2/2^*},
\]

(8)

where $2^* = 2n/(n - \alpha)$.

Proof. By scaling, we may and do assume that $r = 1$, that is we consider only the unit ball $B = B_1 \subset \mathbb{R}^n$. Recall (5) and let $v = u/w_n$, $a_k = 1 - 2^{-k}$ and $B_k = B(0, a_k)$. For $x, y \in B_{k_0}$ we have

\[
w_n(x)w_n(y) = w_1(|x|)w_1(|y|) \geq \sum_{k=k_0}^{\infty} (w_1^2(a_k) - w_1^2(a_{k+1})).
\]
thus for any $x, y \in B$

$$w_n(x)w_n(y) \geq \sum_{k=1}^{\infty} \left( \left( \frac{3}{2} \right)^{\alpha-1} - 1 \right) 2^{-k(\alpha-1)} 1_{B_k}(x)1_{B_k}(y)$$

$$\geq \sum_{k=1}^{\infty} \frac{\alpha - 1}{2} 2^{-k(\alpha-1)} 1_{B_k}(x)1_{B_k}(y).$$

It follows that

$$\int_B \int_B \left( \frac{u(x)}{w_n(x)} - \frac{u(y)}{w_n(y)} \right)^2 \frac{w_n(x)w_n(y)}{|x-y|^{n+\alpha}} \, dx \, dy + \int_B \nu^2 \, dx \geq \frac{\alpha - 1}{2} \sum_{k=1}^{\infty} 2^{-k(\alpha-1)} \left( \int_{B_k} \int_{B_k} \left( \frac{v(x) - v(y)}{|x-y|^{n+\alpha}} \right)^2 \, dx \, dy + \int_{B_k} \nu^2 \, dx \right).$$

(9)

We write Sobolev inequality (3) for $D = B_k$ and a function $v$

$$\int_{B_k} \int_{B_k} \frac{(v(x) - v(y))^2}{|x-y|^{n+\alpha}} \, dx \, dy + \int_{B_k} \nu^2 \, dx \geq c \left( \int_{B_k} |v(x)|^{2^*} \, dx \right)^{2/2^*}.$$

(10)

The constant $c = c(\alpha, n)$ in (10) may be chosen such that it does not depend on $k$, because the radii $a_k$ of $B_k$ satisfy $1/2 \leq a_k \leq 1$. By (9) and (10) we obtain

$$\int_B \int_B \left( \frac{u(x)}{w_n(x)} - \frac{u(y)}{w_n(y)} \right)^2 \frac{w_n(x)w_n(y)}{|x-y|^{n+\alpha}} \, dx \, dy + \int_B \nu^2 \, dx \geq c \sum_{k=1}^{\infty} 2^{-k(\alpha-1)} \left( \int_{B_k} |v(x)|^{2^*} \, dx \right)^{2/2^*}$$

$$\geq c \left( \sum_{k=1}^{\infty} \int_{B_k} 2^{-2^* k(\alpha-1)} |v(x)|^{2^*} \, dx \right)^{2/2^*}$$

$$\geq c' \left( \int_B w_n(x)^{2^*} |v(x)|^{2^*} \, dx \right)^{2/2^*} \geq c' \left( \int_B |u(x)|^{2^*} \, dx \right)^{2/2^*}.$$

By this and Corollary 2 we obtain (8). \qed
Corollary 4. Let $1 < \alpha < 2$, $n \geq 2$ and $\Pi = \mathbb{R}^{n-1} \times (0, \infty)$. There exist a constant $c = c(\alpha, n)$ such that for every $u \in C_c(B_r)$

$$
\mathcal{E}_\Pi(u) := \frac{1}{2} \int_\Pi \int_\Pi \frac{(u(x) - u(y))^2}{|x - y|^{n+\alpha}} \, dx \, dy
\geq \kappa_{n,\alpha} \int_\Pi u^2(x) x_n^{-\alpha} \, dx + c \left( \int_\Pi |u(x)|^{2^*} \, dx \right)^{2/2^*},
$$

where $2^* = 2n/(n - \alpha)$.

Proof. By Theorem 3

$$
\mathcal{E}_{B_r}(u) \geq \kappa_{n,\alpha} \int_{B_r} u^2(x) \delta_{B_r}(x)^{-\alpha} \, dx + c \left( \int_{B_r} |u(x)|^{2^*} \, dx \right)^{2/2^*},
$$

where $\delta_{B_r}(x) = \text{dist}(x, B_r^c)$. Let $x_r = (0, \ldots, 0, r) \in \Pi$, by translation and inequality $\delta_{B(x_r,r)}(x) \leq x_n$ we obtain

$$
\mathcal{E}_{B(x_r,r)}(u) \geq \kappa_{n,\alpha} \int_{B(x_r,r)} u^2(x) x_n^{-\alpha} \, dx + c \left( \int_{B(x_r,r)} |u(x)|^{2^*} \, dx \right)^{2/2^*}.
$$

The corollary follows by letting $r \to \infty$. \qed

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