Entanglement witness game

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Motivated by Buscemi’s semi-quantum nonlocal game [PRL 108, 200401 (2012)], We propose an entanglement witness game, a quantum game based on entanglement witness. Similar as the semi-quantum nonlocal game, the existence of entanglement shared by the players is necessary and sufficient for obtaining a positive average payoff in our entanglement witness game. Two explicit examples are constructed to demonstrate how to play our entanglement witness game and its relations with the CHSH nonlocal game.

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I. INTRODUCTION

Entanglement is a useful resource in quantum information processes. For example, quantum teleportation is impossible without shared remote entanglement. What’s the property of entanglement makes it indispensable in these tasks? It is believed that entanglement has some intriguing nonlocality, which can be revealed by violating Bell inequalities based on local Hidden variables. In 1989, however, R.F. Werner [1] discovered that there are some entanglement states satisfy all Bell inequalities, in other words, the correlations in these entangled states can be explained by the Hidden variable model.

Since Werner’s discovery, a lot of works aimed to explore the subtle differences between the nonlocality in entanglement and that revealed by violating Bell inequalities [1][2][3]. For instance, the following problem is raised [4][5]: are all the entangled states useful in quantum teleportation? This leads to the discovery of the concept of bound entanglement and the understanding of the role of entanglement played in quantum teleportation.

In another direction, the relation between nonlocal games and Bell inequalities is built: For any Bell inequality, we can build a nonlocal game [9][10] which makes the states violating Bell inequality have a positive average payoff. According to Werner’s discovery, the entanglement states satisfying all Bell inequalities will not have a positive average payoff in any such type of nonlocal game. This makes F. Buscemi put forward a revised nonlocal game [11], called semi-quantum nonlocal game, in which the questions sent to the players are allowed to be represented by nonorthogonal quantum states. In the semi-quantum nonlocal game, any entangled state can obtain a positive average payoff while all the separable states can not.

In 2013, two groups [12][13] realize that the semi-quantum nonlocal game can be used to be measurement-device-independent entanglement witness[4]. In particular, it shows how to use any entanglement witness to design a semi-quantum nonlocal game [12].

Motivated by the semi-quantum nonlocal game, we propose a simpler quantum game based on entanglement witness, in which only classical communications between the referee and the players are allowed, and the target of our game is the same as [11]: any entangled state can obtain higher payoff than all separable states.

Our paper is organized as follows. In section II we briefly review nonlocal game [10] and semi-quantum nonlocal game [11]. In section III, the main part of this paper, we propose our entanglement witness game, for simplicity, we only give the bipartite and tripartite scenario of our game, but our game can be easily extend to multipartite situations. In section IV, we give explicit two-qubit examples to show how to play our entanglement witness game, and it also reveals the relation of our entanglement witness game with the previous one. Finally, a summary is given.

II. NONLOCAL GAME AND SEMI-QUANTUM NONLOCAL GAME

Before discussing our entanglement witness game, we review the procedures of nonlocal game [10] and semi-quantum nonlocal game, which are shown in Fig. 1.

In these two games, there are two players, Alice and Bob, and one referee. The referee sends two questions, which are represented by two quantum states τs and ωi, to Alice and Bob respectively with a probability distribution II(s,t). After obtaining the questions, Alice and Bob make some local operations to give their answers x and y respectively with a conditional probability distribution V(x, y | s, t). Then the referee will pay them according to a given payoff function ϕ(x, y, s, t).

In the games, Alice and Bob share a bipartite quantum state ρAB, which can be regarded as the resource of nonlocality. In other words, Alice and Bob make use of the nonlocality of the state ρAB to get the payoff as
much as possible.

More precisely, Alice and Bob try to choose an optimal \( V(x, y|s, t) \) to maximize the average payoff:

\[
P(\rho_{AB}, G_n) = \max_{s, t, x, y} \Pi(s, t)V(x, y|s, t)\phi(x, y, s, t).
\]

There are two basic differences between nonlocal game and semi-nonlocal game. First, \( \{\tau_s\} \) and \( \{\omega_t\} \) are distinguishable for Alice and Bob, i.e., they are orthogonal in the nonlocal game; however, we require the completeness of the states in the operator space but not the orthogonality in the state space in the semi-nonlocal game.

Second, the criteria to win the games are different. In nonlocal game, Alice and Bob win if the payoff \( P(\rho_{AB}, G_n) \) outweighs any LHV (Local Hidden Variables) states and lose otherwise. In the semi-nonlocal game, Alice and Bob win if the payoff is larger than that from any separable bipartite quantum state.

Here it is worthy to emphasize that, before the start of the games, Alice and Bob are allowed to communicate with each other to work out some strategies, then the communication is not permitted during the processes of the games.

In the two games, without using deterministic strategies, the conditional probabilities \( V(x, y|s, t) \) are given based on suitable quantum measurements.

In nonlocal game, \( V \) is

\[
V(x, y|s, t) = \text{Tr}(\rho_{AB} A_x^s \otimes B_y^t),
\]

where \( A_x^s \) and \( B_y^t \) are the measurement make by Alice and Bob according to their received labels \( s \) and \( t \) of \( \{\tau_s\} \) and \( \{\omega_t\} \). Since the construction of nonlocal game is based on Bell inequality [10], from the form of Bell inequality (like CHSH inequality) we can easily get the desired \( A_x^s \) and \( B_y^t \).

In semi-quantum nonlocal game, for any entangled state, the distribution \( V \) is written as follows

\[
V(x, y|s, t) = \text{Tr}(P_{AA_0}^x \otimes P_{BB_0}^y \tau_{s,t}^A \otimes \rho_{AB} \otimes \omega_{s,t}^B),
\]

where \( P_{AA_0}^x \) is the POVM Alice makes on the entangled state \( \rho_{AB} \) and her received state \( \tau_{s,t}^A \), and \( P_{BB_0}^y \) is Bob’s POVM. \( V \) of entangled states is out of reach for any separable states, which makes the payoff of any entangled state larger than separable states.

### III. OUR ENTANGLEMENT WITNESS GAME

In this section, we propose a quantum game to use entanglement as a resource. To simplify our notations, we assume that \( \rho_{AB} \) is a two-qubit state.

The procedure of our entanglement witness game is as follows.

First, the referee randomly sends the labels \( s, t \in \{0, 1, 2, 3\} \) to Alice and Bob with probability \( \Pi(s, t) \);

Second, according to labels \( s, t \) received from the referee, Alice and Bob do the following measurements

\[
0 \rightarrow I_{2 \times 2} \quad 1 \rightarrow \sigma_x \quad 2 \rightarrow \sigma_y \quad 3 \rightarrow \sigma_z
\]

and then Alice and Bob return the measurement results \( a, b \in \{-1, +1\} \) back to the referee;

Third, the referee pays them according to the payoff function \( \phi(a, b, s, t) = -w_{s,t} \sigma a \Pi(s, t) \), where the referee chooses \( \{w_{s,t}\} \) such that

\[
W = \sum_{s, t = 0}^{3} w_{s,t} \sigma_{s} \otimes \sigma_{t}
\]

is an entanglement witness of \( \rho_{AB} \) in the case when \( \rho_{AB} \) is entangled, where \( \sigma_0 = I_{2 \times 2} \), \( \sigma_{1,2,3} \) are Pauli matrices.

Since \( W \) is the entanglement witness of the entangled state \( \rho_{AB} \), we have

\[
\text{Tr}(\rho_{AB} W) < 0,
\]

\[
\text{Tr}(\sigma_{AB} W) \geq 0,
\]

where \( \sigma_{AB} \) is any separable state. Notice that the average payoff

\[
P = \sum_{s, t, a, b} \Pi(s, t)V(a, b|s, t)\phi(a, b, s, t)
\]

which ensures that only when \( \rho_{AB} \) is entangled the average payoff is positive. Further more, because there always exists an entanglement witness for any entangled state \( \rho_{AB} \), the payoff function based on the entanglement witness can be constructed, which makes any entanglement can be distinguished in our game.

For the referee, the process of our entanglement witness game is a stochastic tomography of a two-qubit state. When an ensemble of a two-qubit state is input, the two-qubit state can be reconstructed by the referee with the answers from Alice and Bob. The entanglement witness is involved in our game through a proper choice of the payoff function. Because the entanglement witness is relative to a given entangled state, the payoff functions may be different for different entangled states shared by
the players. In particular, for any entanglement state, we only need to adopt a proper entanglement witness to design the payoff function for our entanglement witness game.

Different from the traditional nonlocal game, where the player can win only when they share Bell nonlocal state, our entanglement witness game gives a positive average payoff if the players share a proper entangled state (which may not violate any Bell inequality). The reason for this is that, in the traditional game, the player can choose any two local measurements, while in our game, they are restricted to the four fixed measurement operators.

Our entanglement witness game can be easily extend to multipartite situations. We use three-qubit scenario to illustrate this. For tripartite games, there are three players, Alice, Bob and Charlie. The entanglement witness of three-qubit entangled states $\rho_{ABC}$ can be decomposed as

$$W = \sum_{i,j,k=0}^{3} w_{i,j,k} \sigma_i \otimes \sigma_j \otimes \sigma_k$$

The same as bipartite scenario, the referee send Alice, Bob and Charlie four labels $i, j, k \in \{0, 1, 2, 3\}$, and the players do the measurement of $\sigma_i \otimes \sigma_j \otimes \sigma_k$. Then they send back their binary-outcome results $a, b, c \in \{-1, +1\}$, the referee pay them according to the payoff function $-w_{i,j,k} abc/\Pi(i, j, k)$. Then the average payoff would be

$$P = -\text{Tr}(\rho_{ABC}W)$$

Only when $\rho_{ABC}$ is a tripartite entangled state, the payoff is larger than zero.

IV. EXPLICIT EXAMPLES: TWO-QUBIT CASE

In this section, we give two explicit examples to show how to play our entanglement witness game, and show that Alice and Bob are required to be honest in our game.

In the first example, Alice and Bob share the Werner state [1]:

$$\rho_z = \frac{1 - z}{4} I + z |\psi^+\rangle\langle\psi^+|,$$

where $I = \sigma_0 \otimes \sigma_0$, $|\psi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$, $z \in [0, 1]$. The entanglement witness of this Werner state is [15]

$$W = \frac{1}{\sqrt{3}} (I - \sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 - \sigma_3 \otimes \sigma_3).$$

The factor $\frac{1}{\sqrt{3}}$ in $W$ makes the projection of $W$ into the linear space expanded by $\{\sigma_i \otimes \sigma_j, i, j \in \{1, 2, 3\}\}$ is a normalized vector if we define the inner product $\langle A, B \rangle = \text{Tr}(AB)$ for any two Hermitian operators.

In fact, any two-qubit density matrix $\rho$ is an element in the linear space expanded by $\{\sigma_i \otimes \sigma_j, i, j \in \{0, 1, 2, 3\}\}$. Since $\text{Tr}(\rho) = 1$, the component of $I$ is $\frac{1}{4}$. Thus all the two-qubit density matrices form a convex body in 15 dimensional linear space, where a general entanglement witness for two-qubit states lies in according to convex analysis [16]. Usually, it is more convenient to study its subspaces, e.g., a 3 dimensional subspace, or even a 2 dimensional subspace.

Here we focus on the 3 dimensional subspace expanded by $\{\sigma_i \otimes \sigma_i, i \in \{1, 2, 3\}\}$, and the projections of two-qubit states and the entanglement witness are shown in Fig. 2. The coordinate for any two-qubit state $\rho$ in this subspace is given by $(\langle\sigma_x \otimes \sigma_x\rangle, \langle\sigma_y \otimes \sigma_y\rangle, \langle\sigma_z \otimes \sigma_z\rangle)$ with $\langle\cdot\rangle = \text{Tr}(\cdot)$. In this subspace, the numerical range [17] for all two-qubit states is the regular tetrahedron with vertexes $\{(1, -1, 1), (-1, 1, 1), (1, 1, -1), (-1, -1, -1)\}$, and the numerical range for all separable two-qubit states is the octahedron with vertexes $\{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}$. The gray hyperplane is defined by $\text{Tr}(\rho W) = 0$. The Werner state $\rho_z$ is represented by a red line, which intersects with the hyperplane at $z_c = \frac{1}{3}$.

![FIG. 2. Entanglement witness $W$ and the projections of two-qubit states. The regular tetrahedron formed by four black lines is the numerical range of two-qubit states, and the regular octahedron formed by twelve blue lines is the numerical range for two-qubit separable states. $\text{Tr}(\rho W) = 0$ corresponds to a gray hyperplane, which is also a surface of the regular octahedron. The projection of the Werner state $\rho_z$ is denoted by a red line, which intersects with the hyperplane at $z_c = \frac{1}{3}$.](image-url)
are shown in Fig. 3. Now the numerical range for all two-qubit states is the black square, and the numerical range for all separable two-qubit states is the blue square. The hyperplanes $\text{Tr}(\rho W_{\text{CHSH}}) = 0$ is the green line, and the Werner state $\rho_z$ is denoted as the red line, which intersects with the green line at $z_1 = \frac{\sqrt{2}}{2}$.

![Entanglement witnesses](image)

In our entanglement witness game, the average payoff is

$$P = -\text{Tr}(\rho_z W_{\text{CHSH}}) = \sqrt{2}z - 1,$$

which is the geometric distance from a Werner state to the green line. When $P > 0$, we require that $z > \sqrt{2}/2$. As expected, $P(\rho_z) > 0$ implies $\rho_z$ is entangled. However, when $1/3 < z \leq \sqrt{2}/2$, $\rho_z$ is entangled but not detected in our entanglement witness game based on the CHSH inequality, which is consistent with [15][19].

In fact, in our entanglement witness game, we strengthen the CHSH entanglement witness as

$$W = \frac{1}{\sqrt{2}}(1 - \sigma_1 \otimes \sigma_1 - \sigma_3 \otimes \sigma_3),$$

which corresponds to a gray line in Fig. 3 defined by $\text{Tr}(\rho W)$. The Werner state intersect with the gray line at $z_2 = \frac{1}{2}$.

![Diagram](image)

FIG. 3. Entanglement witnesses $W_{\text{CHSH}}, \tilde{W}$ and the projections of two-qubit states. The black square is the numerical range of two-qubit states, and the blue square is the numerical range for two-qubit separable states. $\text{Tr}(\rho W_{\text{CHSH}}) = 0$ corresponds to the green line, and $\text{Tr}(\rho \tilde{W}) = 0$ corresponds to the gray line. The projection of the Werner state $\rho_z$ is denoted by a red line, which intersects with the above two lines at $z_1 = \frac{\sqrt{2}}{2}$ and $z_2 = 1/2$.
Then for the Werner state, the average payoff
\[ P = -\text{Tr}(\rho Z W) = \frac{\sqrt{2}}{2}(2z - 1). \] (23)

For \( A, A', B, B' \) given by Eq. (19), the Bell inequality as entanglement witness can be strengthened as
\[-\sqrt{2} \leq \langle A \otimes B + A' \otimes B + A \otimes B' - A' \otimes B' \rangle \leq \sqrt{2}. \] (24)

Even after the strengthen, only the Werner state for \( z > \frac{1}{4} \) is entanglement witnessed, and the entangled Werner state for \( \frac{1}{4} < z \leq \frac{1}{2} \) is not detected in our entanglement witness game.

From the above two typical examples of our entanglement witness game, we find that the higher dimension of the consider subspace, more entangled states are useful for our entanglement witness game. An interesting question to be explored in future arises: what is the lowest dimension of the subspace to make all entangled states are useful in our entanglement witness game?

V. CONCLUSION

In this paper, we proposed an entanglement witness game, which ensures any entangled state can have a positive payoff while separable states can not. The process of our entanglement witness game, in the viewpoint of the referee, is a stochastic local quantum state tomography, which makes the referee has the capacity to obtain the entangled state shared by the players asymptotically. It is worthy to point out if the players are dishonest, they may get positive average payoff without shared entanglement, which is not robust as in the semi-nonlocal game.

Compared with the nonlocal game from the Bell inequalities, any entangled state is useful in our entanglement witness game, while in the traditional one, only the entangled states violating Bell inequalities has advantage over separable states. Compared with the semi-quantum nonlocal game, we do not need quantum channels to transfer non-orthogonal quantum states from the referee to the players. In addition, the procedure is relative simpler than the previous ones, and it is more directly based on entanglement witness. Our entanglement witness game can use any entangled state as a resource by adopting its entanglement witness to design the payoff function, and it can be implemented similarly as the entanglement witness experiments [20][21].

We hope our work will increase our understandings on Bell nonlocality and entanglement through different quantum games, and promote their applications in quantum information processes.

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