NILPOTENT CENTRALIZERS AND SPRINGER ISOMORPHISMS

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ABSTRACT. Let $G$ be a semisimple algebraic group over a field $K$ whose characteristic is very good for $G$, and let $\sigma$ be any $G$-equivariant isomorphism from the nilpotent variety to the unipotent variety; the map $\sigma$ is known as a Springer isomorphism. Let $y \in G(K)$, let $Y \in \text{Lie}(G)(K)$, and write $C_y = C_G(y)$ and $C_Y = C_G(Y)$ for the centralizers. We show that the center of $C_y$ and the center of $C_Y$ are smooth group schemes over $K$. The existence of a Springer isomorphism is used to treat the crucial cases where $y$ is unipotent and where $Y$ is nilpotent.

Now suppose $G$ to be quasisplit, and write $C$ for the centralizer of a rational regular nilpotent element. We obtain a description of the normalizer $N_G(C)$ of $C$, and we show that the automorphism of Lie$(C)$ determined by the differential of $\sigma$ at zero is a scalar multiple of the identity; these results verify observations of J-P. Serre.

CONTENTS

1. Introduction 1
2. Recollections: group schemes 2
3. Recollections: reductive groups 8
4. The center of a centralizer 13
5. Regular nilpotent elements 15
References 22

1. INTRODUCTION

Let $G$ be a reductive group over the field $K$ and suppose $G$ to be $D$-standard; this condition means that $G$ satisfies some standard hypotheses which will be described in §3.2. For now, note that a semisimple group $G$ is $D$-standard if and only if the characteristic of $K$ is very good for $G$.

Consider the closed subvariety $N$ of nilpotent elements of the Lie algebra $\mathfrak{g} = \text{Lie}(G)$ of $G$, and the closed subvariety $U$ of unipotent elements of $G$. Since $G$ is $D$-standard, one may follow the argument given by Springer and Steinberg [SS 70, 3.12] to find a $G$-equivariant isomorphism of varieties $\sigma : N \to U$. The mapping $\sigma$ is called a Springer isomorphism. There are many such maps: the Springer isomorphisms can be viewed as the points of an affine variety whose dimension is equal to the semisimple rank of $G$; see the note of Serre found in [Mc 05, Appendix] which shows that despite the abundance of such maps, each Springer isomorphism induces the same bijection between the (finite) sets of $G$-orbits in $N$ and in $U$. For some more details, see §3.3 below.

Let $y \in G(K)$ and $Y \in \mathfrak{g}(K)$. Since $G$ is $D$-standard, we observe in (3.4.1) – following Springer and Steinberg [SS 70] – that the centralizers $C_G(y)$ and $C_G(Y)$ are smooth group schemes over $K$. The first main result of this paper is as follows:

Theorem A. Let $Z_y = Z(C_G(y))$ and $Z_Y = Z(C_G(Y))$ be the centers of the centralizers.

(a) $Z_y$ and $Z_Y$ are smooth group schemes over $K$. 

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The main objects of study in this paper are group schemes over a field $K$. For the most part, we restrict our attention to affine group schemes $A$ of finite type over $K$. We begin with some general definitions.
2.1. Basic Definitions. We collect here some basic notions and definitions concerning group schemes; for a full treatment, the reader is referred to [DG 70] or to [Ja 03, part I].

For a commutative ring $\Lambda$, let us write $\text{Alg}_\Lambda$ for the category of “all” commutative $\Lambda$-algebras. We will write $\Lambda' \in \text{Alg}_\Lambda$ to mean that $\Lambda'$ is an object of this category – i.e. that $\Lambda'$ is a commutative $\Lambda$-algebra.

We are going to consider affine schemes over $\Lambda$; an affine scheme $X$ is determined by a commutative $\Lambda$-algebra $R$: the algebra $R$ determines a functor $X : \text{Alg}_\Lambda \rightarrow \text{Sets}$ by the rule

$$X(\Lambda') = \text{Hom}_{\text{Alg}}(\Lambda', R).$$

The scheme $X$ “is” this functor, and one says that $X$ is represented by the algebra $R$. One usually writes $R = \Lambda[X]$ and one says that $\Lambda[X]$ is the coordinate ring of $X$. The affine scheme $X$ has finite type over $\Lambda$ provided that $\Lambda[X]$ is a finitely generated $\Lambda$-algebra.

A group valued functor $A$ on $\text{Alg}_\Lambda$ which is an affine scheme will be called an affine group scheme. If $A$ is an affine group scheme, then $\Lambda[A]$ has the structure of a Hopf algebra over $\Lambda$.

If $\Lambda' \in \text{Alg}_\Lambda$, we write $\Lambda/\Lambda'$ for the group scheme over $\Lambda'$ obtained by base change. Thus $\Lambda/\Lambda'$ is the group scheme over $\Lambda'$ represented by the $\Lambda'$-algebra $\Lambda[A] \otimes_{\Lambda} \Lambda'$.

Let us fix an affine group scheme $A$ of finite type over the field $K$. Write $K[A]$ for the coordinate algebra of $K$, and choose an algebraic closure $\text{Alg}_K$ of $K$.

2.2. Comparison with algebraic groups. In many cases, the group schemes we consider may be identified with a corresponding algebraic group; we now describe this identification.

If the algebra $K[A]$ is geometrically reduced – i.e. is such that $K_{\text{alg}}[A] = K[A] \otimes_K K_{\text{alg}}$ has no non-zero nilpotent elements – then also $K[A]$ is reduced. The $K_{\text{alg}}$-points $A(K_{\text{alg}})$ of $A$ may be viewed as an affine variety over $K_{\text{alg}}$; since it is reduced, $K_{\text{alg}}[A]$ is the algebra of regular functions on $A(K_{\text{alg}})$. Moreover, $A(K_{\text{alg}})$ together with the $K$-algebra $K[A]$ of regular functions on $A(K_{\text{alg}})$ may be viewed as a variety defined over $K$ in the sense of [Bor 91] or [Sp 98].

Conversely, an algebraic group $B$ defined over $K$ in the sense of [Bor 91] or [Sp 98] comes equipped with a $K$-algebra $K[B]$ for which $K_{\text{alg}}[B] = K[B] \otimes_K K_{\text{alg}}$ is the algebra of regular functions on $B$. The Hopf algebra $K[B]$ represents a group scheme.

The constructions in the preceding paragraphs are inverse to one another, and these constructions permit us to identify the category of linear algebraic groups defined over $K$ with the full subcategory of the category of affine group schemes of finite type over $K$ consisting of those group schemes with geometrically reduced coordinate algebras.

There are interesting group schemes in characteristic $p > 0$ whose coordinate algebras are not reduced. Standard examples of such reduced group schemes include the group scheme $\mu_p$ represented by $K[T]/(T^p - 1)$ with co-multiplication given by $\Delta(T) = T \otimes T$, and the group scheme $\mathfrak{a}_p$ represented by $K[T]/(T^p)$ with co-multiplication given by $\Delta(T) = T \otimes 1 + 1 \otimes T$. Note that $\mu_p$ is a subgroup scheme of the multiplicative group $G_m$ and $\mathfrak{a}_p$ is a subgroup scheme of the additive group $G_a$.

2.3. Smoothness. For $\Lambda \in \text{Alg}_K$, let $\Lambda[e]$ denote the algebra of dual numbers over $\Lambda$; thus $\Lambda[e]$ is a free $\Lambda$-module of rank 2 with $\Lambda$-basis $\{1, e\}$, and $e^2 = 0$. If $A$ is a group scheme over $K$, the natural $\Lambda$-algebra homomorphisms

$$\Lambda \rightarrow \Lambda[e] \overset{\pi}{\rightarrow} \Lambda$$

yield corresponding group homomorphisms

$$A(\Lambda) \rightarrow A(\Lambda[e]) \overset{A(\pi)}{\rightarrow} A(\Lambda).$$

The Lie algebra $\text{Lie}(A)$ of $A$ is the group functor on $\text{Alg}_K$ given for $\Lambda \in \text{Alg}_K$ by

$$\text{Lie}(A)(\Lambda) = \ker(A(\Lambda[e]) \overset{A(\pi)}{\rightarrow} A(\Lambda)).$$
Abusing notation somewhat, we are going to write also \( \text{Lie}(A) \) for \( \text{Lie}(A)(K) \). We have:

(2.3.1) ([DG 70, II.4]).

(a) \( \text{Lie}(A) \) has the structure of a \( K \)-vector space, and the mapping \( \text{Lie}(A) \to \text{Lie}(A)(\Lambda) \) induces an isomorphism

\[
\text{Lie}(A)(\Lambda) \simeq \text{Lie}(A) \otimes_K \Lambda
\]

for each \( \Lambda \in \text{Alg}_K \).

(b) For \( \Lambda \in \text{Alg}_K \) and \( g \in \Lambda \), the inner automorphism \( \text{Int}(g) \) determines by restriction a \( \Lambda \)-linear automorphism \( \text{Ad}(g) \) of \( \text{Lie}(A)(\Lambda) \simeq \text{Lie}(A) \otimes_K \Lambda \); thus \( \text{Ad} : A \to \text{GL}(\text{Lie}(A)) \) is a homomorphism of group schemes over \( K \).

(2.3.2) ([DG 70, II.5.2.1, p. 238] or [KMRT, (21.8) and (21.9)]). One says that the group scheme \( A \) is smooth over \( K \) if any of the following equivalent conditions hold:

(a) \( A \) is geometrically reduced – i.e. \( A_{/\text{alg}} \) is reduced.

(b) the local ring \( K[A]_I \) is regular, where \( I \) is the maximal ideal defining the identity element of \( A \).

(c) the local ring \( K[A]_I \) is regular for each prime ideal \( I \) of \( K[A] \).

(d) \( \dim_K \text{Lie}(A) = \dim A \), where \( \dim A \) denotes the dimension of the scheme \( A \), which is equal to the Krull dimension of the ring \( K[A] \).

If \( A \) is a group scheme over \( K \), we often abbreviate the phrase “\( A \) is smooth over \( K \)” to “\( A \) is smooth”.

2.4. Reduced subgroup schemes. The following result is well known; a proof may be found in [MT 07, Lemma 3].

(2.4.1). If \( K \) is perfect, there is a unique smooth subgroup \( A_{\text{red}} \subset A \) which has the same underlying topological space as \( A \). If \( B \) is any smooth group scheme over \( K \) and \( f : B \to A \) is a morphism, then \( f \) factors in a unique way as a morphism \( B \to A_{\text{red}} \) followed by the inclusion \( A_{\text{red}} \to A \).

Note that if \( K \) is not perfect, the subgroup scheme \( (A_{/\text{alg}})_{\text{red}} \) of \( A_{/\text{alg}} \) may not arise by base change from a subgroup scheme over \( K \); see [MT 07, Example 4].

2.5. Fixed points and the center of a group scheme. For the remainder of \( \S 2 \), let us fix a group scheme \( A \) which is affine and of finite type over the field \( K \). Let \( V \) denote an affine \( K \)-scheme (of finite type) on which \( A \) acts. Define a \( K \)-subfunctor \( W \) of \( V \) as follows: for each \( \Lambda \in \text{Alg}_K \), let

\[
W(\Lambda) = \{ v \in V(\Lambda) \mid av = v \text{ for each } \Lambda' \in \text{Alg}_K \text{ and each } a \in A(\Lambda') \}.
\]

We write \( W = V^A \); it is the functor of fixed points for the action of \( A \).

In general one indeed must define the set \( W(\Lambda) \) as the fixed point set of all \( a \in A(\Lambda') \) for varying \( \Lambda' \); e.g. if \( A \) is infinitesimal, \( A(K) = \{1\} \) while \( W(K) \) is typically a proper subset of \( V(K) \).

Since \( V \) is affine – hence separated – and since \( K \) is a field so that \( K[A] \) is free over \( K \), we have:

(2.5.1) ([DG 70, II.1 Theorem 3.6] or [Ja 03, I.2.6(10)]) \( V^A \) is a closed subscheme of \( V \).

The following assertion is somewhat related to [Ja 03, I.2.7 (11) and (12)].

(2.5.2). Suppose in addition that \( A \) is smooth over \( K \). Then for any commutative \( K \)-algebra \( K' \) which is an algebraically closed field, we have \( V^A(K') = V^K A(K') \). \(^2\)

Proof. It is immediate from definitions that \( V^A(K') \subset V A(K') \). In order to prove the inclusion \( V A(K') \subset V^A(K') \), we will assume (for notational convenience) that \( K = K' \) is algebraically closed. Suppose that \( v \in V(K) \) and that \( v \) is fixed by each element of \( A(K) \).

Consider now the morphism \( \phi : A \to V \) given for each \( \Lambda \in \text{Alg}_K \) and each \( a \in A(\Lambda) \) by the rule \( a \mapsto av \). The result will follow if we argue that \( \phi \) is a constant morphism. But we know that \( \phi : A(K) \to V(K) \) is constant. Since \( A \) is a reduced scheme, the morphism \( \phi \) is determined by its values on closed points; since \( K \) is algebraically closed, the closed points are in bijection with \( A(K) \); the fact that \( \phi \) is constant now follows. \( \square \)

\(^2\)Here \( V A(K') \) denotes the subset of \( V(K') \) fixed by each element of the group \( A(K') \).
Consider now the action of $A$ on itself by inner automorphisms. For any $\Lambda \in \text{Alg}_K$ and any $a \in A(\Lambda)$, let us write $\text{Int}(a)$ for the inner automorphism $x \mapsto axa^{-1}$ of the $\Lambda$-group scheme $A/\Lambda$. The fixed point subscheme for this action is by definition the center $Z$ of $A$; thus we have the following result (see also [DG 70, II.1.3.9]):

(2.5.3) The center $Z$ is a closed subgroup scheme of $A$. For any $\Lambda \in \text{Alg}_K$, we have

$$Z(\Lambda) = \{a \in A(\Lambda) \mid \text{Int}(a) \text{ is the trivial automorphism of the group scheme } A/\Lambda\}.$$

2.6. Smoothness of the center. Write $a = \text{Lie}(A)$ for the Lie algebra of $A$. Recall from (2.3.1) the adjoint action $\text{Ad}$ of $A$.

(2.6.1) Regarding $a$ as a $K$-scheme, the Lie algebra of $Z$ is the fixed point subscheme of $a$ for the adjoint action of $A$.

Proof. Since $Z$ is the fixed point subscheme of $A$ for the action of $A$ on itself by inner automorphisms, the assertion follows from [DG 70, II.4.2.5]. $\Box$

In particular, $\text{Lie}(Z)$ identifies with the $K$-points $a^{\text{Ad}(A)}(K)$ of this fixed point functor, and one recovers the fixed point functor from the $K$-points [Ja 03, I.2.10(3)]:

$$a^{\text{Ad}(A)}(\Lambda) = \text{Lie}(Z) \otimes_K \Lambda.$$

(2.6.2) The center $Z$ of $A$ is smooth over $K$ if and only if

$$\dim Z = \dim_K a^{\text{Ad}(A)}(K) = \dim_K \text{Lie}(Z).$$

Proof. Immediate from (2.3.2) and the observation (2.6.1). $\Box$

Example. Let $K$ be a perfect field of characteristic $p > 0$, and let $A$ be the smooth group scheme over $K$ for which

$$A(\Lambda) = \left\{ \begin{pmatrix} t & 0 & 0 \\ 0 & t^p & s \\ 0 & 0 & 1 \end{pmatrix} \mid t \in \Lambda^\times, s \in \Lambda \right\}$$

for each $\Lambda \in \text{Alg}_K$. The Lie algebra $a$ is spanned as a $K$-vector space by the matrices

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Write $Z = Z(A)$ for the center of $A$. Since $K$ is perfect, we may form the corresponding reduced subgroup scheme $Z_{\text{red}} \subset Z$ – see e.g. [MT 07, Lemma 3]; $Z_{\text{red}}$ is a smooth group scheme over $K$.

We are going to argue that $Z$ is not smooth – i.e. that $Z \neq Z_{\text{red}}$. Observe first that $a$ is an Abelian Lie algebra; thus its center $\z(a)$ is all of $a$.

Now, if $K_{\text{alg}}$ is an algebraic closure of $K$, it is easy to check that the center of the group $A(K_{\text{alg}})$ is trivial. It follows that the smooth group scheme $Z_{\text{red}}$ satisfies $Z_{\text{red}}(K_{\text{alg}}) = 1$; thus $Z_{\text{red}}$ is trivial and $\text{Lie}(Z_{\text{red}}) = 0$.

It is straightforward to verify that the multiples of $X$ are the only fixed points of $a$ under the adjoint action of $A$. Thus $\text{Lie}(Z) = a^{\text{Ad}(A)}$ has dimension 1 as a $K$-vector space. Since $\dim Z = \dim Z_{\text{red}} = 0$, it follows that $Z$ is not smooth.

Note that for this example, both containments in the following sequence are proper:

$$\text{Lie}(Z_{\text{red}}) \subset \text{Lie}(Z) \subset \z(a).$$
2.7. **Smoothness of certain fixed point subgroup schemes.** Recall that a group scheme $D$ over $K$ is diagonalizable if $K[D]$ is spanned as a linear space by the group of characters $X^\times(D)$. The group scheme $D$ is of multiplicative type if $D_{\text{alg}}$ is diagonalizable.

Suppose in this section that $D$ is either a group scheme of multiplicative type, or that $D$ is an étale group scheme over $K$ for which the finite group $D(K_{\text{alg}})$ has order invertible in $K$.

Assume that $D$ acts on the group scheme $A$ by group automorphisms: for any $A \in \text{Alg}_K$ and any $x \in D(A)$, the element $x$ acts on the group scheme $A/\Lambda$ as a group scheme automorphism.

The fixed points $A^D$ form a closed subgroup scheme of $A$. Moreover, we have:

(2.7.1). If $A$ is smooth over $K$, then also the fixed point subgroup scheme $A^D$ is smooth over $K$.

**Proof.** According to the “Théorème de lissité des centralisateurs” [DG 70, II.5.2.8 (p. 240)] the result will follow if we know that $H^1(D, \text{Lie}(A)) = 0$. It suffices to check this condition after extending scalars; thus we may and will suppose that $D$ is diagonalizable or that $D$ is the constant group scheme determined by a finite group whose order is invertible in $K$.

In each case, one knows that the cohomology group $H^n(D, M)$ is 0 for all $D$-modules $M$ and all $n \geq 1$; for a finite group with order invertible in $K$, this vanishing is well-known; for a diagonalizable group, see [Ja 03, I.4.3].

2.8. **Possibly disconnected groups.** Let $G$ be a smooth linear algebraic group over $K$.

(2.8.1). Suppose that $1 \to G \to G_1 \to E \to 1$ is an exact sequence, where $E$ is finite étale and $E(K_{\text{alg}})$ has order invertible in $K$. If the center of $G$ is smooth, then the center of $G_1$ is smooth.

**Proof.** Write $Z$ for the center of $G$, write $Z_1$ for the center of $G_1$. Note that $E$ acts naturally on $Z$.

There is an exact sequence of groups

$$1 \to Z^E \to Z_1 \to H \to 1$$

for a subgroup $H \subset E$. Since $Z$ is smooth, the smoothness of $Z^E$ follows from (2.7.1); since $H$ is smooth, one obtains the smoothness of $Z_1$ by applying [KMRT, Cor. (22.12)].

2.9. **Split unipotent radicals.** Fix a smooth group scheme $A$ over $K$. A smooth group scheme $B$ over $K$ is unipotent if each element of $B(K_{\text{alg}})$ is unipotent. Recall that the unipotent radical of $A/K_{\text{alg}}$ is the maximal closed, connected, smooth, normal, unipotent subgroup scheme of $A/K_{\text{alg}}$.

(2.9.1). [Sp 98, Prop. 14.4.5] If $K$ is perfect, there is a smooth subgroup scheme $R_{u} A \subset A$ such that $R_{u} A/K_{\text{alg}}$ is the unipotent radical of $A/K_{\text{alg}}$.

If $K$ is not perfect, then in general $R_{u} A/K_{\text{alg}}$ does not arise by base change from a $K$-subgroup scheme of $A$. The unipotent group $B$ is said to be split provided that there are closed subgroup schemes

$$1 = B_0 \subset B_1 \subset \cdots \subset B_n = B$$

such that $B_i/B_{i-1} \simeq G_a$ for $1 \leq i \leq n$.

**Theorem.** Let $A$ be a connected, solvable, and smooth group scheme over $K$. Let $T \subset A$ be a maximal torus, and suppose that $\phi : G_m \to T$ is a cocharacter. Write $S$ for the image of $\phi$. If $\text{Lie}(T)$ is precisely the set of fixed points $\text{Lie}(A)^S$, and if each non-zero weight $\lambda$ of $S$ on $\text{Lie}(A)$ satisfies $(\lambda, \phi) > 0$, then $R_{u} A$ is defined over $K$ and is a split unipotent group scheme.

**Proof.** Write $P = P(\phi)$ for the smooth subgroup scheme of $A$ determined by $\phi$ as in [Sp 98, §13.4]; it is the subgroup contracted by the cocharacter $\phi$. Write $M = C_A(S); M$ is connected [Sp 98, p. 110] and smooth [DG 70, p. 476, cor. 2.5]. There is a smooth, connected, normal, unipotent subgroup scheme $U(\phi) \subset P$ for which the product morphism

$$M \times U(\phi) \rightarrow P$$
is an isomorphism of varieties; [Sp 98, 13.4.2]. Moreover, since $\langle \lambda, \phi \rangle > 0$ for each weight of $S$ on $\text{Lie}(A)$, it follows that $U(-\phi)$ is trivial. Thus loc. cit. 13.4.4 shows that $A = P$.

Evidently $T \subset M$. Since $\text{Lie}(T) = \text{Lie}(M)$, it follows that $M = T$. It follows that $U(\phi)/_{K_{\text{alg}}}$ is the unipotent radical of $A/_{K_{\text{alg}}}$ as desired.

Finally, it follows from [Sp 98, 14.4.2] that $U(\phi)$ is a $K$-split unipotent group, and the proof is complete.

2.10. Torus actions on a projective space. Let $T$ be a split torus over $K$, and let $V$ be a $T$-representation. For $\lambda \in X^*(T)$, let $V_\lambda$ be the corresponding weight space; thus $T$ acts on $V_\lambda$ through the character $\lambda : T \to G_m$. There are distinct characters $\lambda_1, \ldots, \lambda_n \in X^*(T)$ such that

$$V = \bigoplus_{i=1}^n V_{\lambda_i};$$

the $\lambda_i$ are the weights of $T$ on $V$. Let us fix a vector $0 \neq v \in V_{\lambda_1}$.

Consider now the projective space $\mathbb{P}(V)$ of lines through the origin in $V$; for a non-zero vector $w \in V$, write $[w]$ for the corresponding point of $\mathbb{P}(V)$. The linear action of $T$ on $V$ induces in a natural way an action of $T$ on $\mathbb{P}(V)$.

Since $v$ is a weight vector for $T$, the point $[v] \in \mathbb{P}(V)(K)$ determined by $v$ is fixed by the action of $T$. Consider the tangent space $M = T_0[\mathbb{P}(V)]$; since $[v]$ is a fixed point of $T$, the action of $T$ on $\mathbb{P}(V)$ determines a linear representation of $T$ on $M$.

(2.10.1). The non-zero weights of $T$ on $M = T_0[\mathbb{P}(V)]$ are the characters $\lambda_i - \lambda_1$ for $1 < i \leq n$. Moreover,

$$\dim M_0 = \dim V_{\lambda_1} - 1 \quad \text{and} \quad \dim M_{\lambda_i - \lambda_1} = \dim V_{\lambda_i}, \quad 1 < i \leq n.$$

Proof. Choose a basis $S_1, S_2, \ldots, S_r$ for the dual space of $V^\vee$ for which $S_i \in V_{-\lambda_i}$ for $1 \leq i \leq r$ i.e. the vector $S_i$ has weight $-\lambda_i$ for the contragredient action of $T$ on $V^\vee$. Without loss of generality, we may and will assume that $S_1$ satisfies $S_1(v) \neq 0$ and that $S_i(v) = 0$ for $2 \leq i \leq n$.

Now consider the affine open subset $V = \mathbb{P}(V)_{S_1}$ of $\mathbb{P}(V)$ defined by the non-vanishing of $S_1$. One knows that $[v]$ is a point of $V$. Moreover, $V \simeq \text{Aff}^n$ where $r = \dim V$. Since $S_1$ is a weight vector for the action of the torus $T$, it is clear that $V$ is a $T$-stable subvariety of $\mathbb{P}(V)$. More precisely, $V$ identifies with the affine scheme $\text{Spec}(A)$ where $A$ is the $T$-stable subalgebra

$$A = k\left[\frac{S_2}{S_1}, \frac{S_3}{S_1}, \ldots, \frac{S_r}{S_1}\right]$$

of the field of rational functions $k(\mathbb{P}(V))$.

Under this identification, the point $[v] \in V$ corresponds to the point $\bar{0}$ of $\text{Aff}^n$; i.e. to the maximal ideal $m = \left(\frac{S_2}{S_1}, \frac{S_3}{S_1}, \ldots, \frac{S_r}{S_1}\right) \subset A$. Now, $m$ and $m^2$ are $T$-invariant; since $\frac{S_i}{S_1}$ has weight $-\lambda_i + \lambda_1$, evidently the weights of $T$ in its representation on $m/m^2$ are of the form $-\lambda_i + \lambda_1$, and one has

$$\dim (m/m^2)_0 = \dim V_{\lambda_1} - 1 \quad \text{and} \quad \dim (m/m^2)_{-\lambda_i + \lambda_1} = \dim V_{\lambda_i}, \quad 1 < i \leq n.$$

The assertion now follows since there is a $T$-equivariant isomorphism between the tangent space to $\mathbb{P}(V)$ at $[v]$ i.e. the space $M = T_0[\mathbb{P}(V)]$ – and the contragredient representation $(m/m^2)^\vee$.

2.11. Surjective homomorphisms between group schemes; normalizers. In this section, let us fix group schemes $G_1$ and $G_2$ over $K$, and suppose that $f : G_1 \to G_2$ is a surjective homomorphism of group schemes; recall that $f$ is surjective provided that the comorphism $f^* : K[G_2] \to K[G_1]$ is injective (cf. [KMRT, Prop. 22.3]).

The mapping $f$ is said to be separable provided that $df : \text{Lie}(G_1) \to \text{Lie}(G_2)$ is surjective as well.

Let $C_2 \subset G_2$ be a subgroup scheme, and let $C_1 = f^{-1}C_2$ be the scheme-theoretic inverse image.

(2.11.1). (a) The mapping obtained by restriction $f|_{C_1} : C_1 \to C_2$ is surjective.

(b) If $C_1$ is smooth, then $C_2$ is smooth.
(c) If \(f\) is separable and \(C_2\) is smooth, then \(C_1\) is smooth.
(d) Suppose that \(f\) is separable, and that either \(C_1\) or \(C_2\) is smooth. Then both \(C_1\) and \(C_2\) are smooth, and \(f_{|C_1}\) is separable.

Proof. (a) and (b) follow from [KMRT, Prop. 22.4].

We now prove (c). Since \(f\) is separable and surjective, [KMRT, Prop. 22.13] shows that \(\ker f\) is a smooth group scheme over \(K\). Note that \(\ker f \subset C_1\). If \(C_2\) is smooth, the smoothness of \(C_1\) now follows from [KMRT, Cor. 22.12].

We finally prove (d). The smoothness assertions have already been proved. We again know \(\ker f\) to be smooth over \(K\). In particular, \(\dim \ker f = \dim \ker df\). Since \(\ker f \subset C_1\), we have

\[
\dim \text{image}(df_{|C_1}) = \dim \text{Lie}(C_1) - \dim \ker df_{|C_1} = \dim C_1 - \dim \ker f_{|C_1} = \dim C_2,
\]

where we have used [KMRT, Prop. 22.11] for the final equality; since \(C_2\) is smooth, it follows that \(df_{|C_1} : \text{Lie}(C_1) \rightarrow \text{Lie}(C_2)\) is surjective. \(\square\)

Write \(N_2 = N_{G_2}(C_2)\) for the normalizer of \(C_2\) in \(G_2\). Thus \(N_2\) is the subgroup functor given for \(\Lambda \in \text{Alg}_K\) by the rule

\[
N_2(\Lambda) = \{g \in G_2(\Lambda) \mid g \text{ normalizes the subgroup scheme } C_{2/\Lambda} \subset G_{2/\Lambda}\}
= \{g \in G_2(\Lambda) \mid gC_2(\Lambda')g^{-1} = C_2(\Lambda') \text{ for all } \Lambda' \in \text{Alg}_\Lambda\}.
\]

According to [DG 70, II.1 Theorem 3.6(b)], \(N_2\) is a closed subgroup scheme of \(G_2\).

As a consequence of (2.11.1), we find the following:

(2.11.2). Set \(N_1 = f^{-1}N_2\).
(a) \(N_1 = N_{G_1}(C_1)\).
(b) \(f_{|N_1} : N_1 \rightarrow N_2\) is surjective.
(c) If \(N_1\) is smooth, then \(N_2\) is smooth.
(d) If \(f\) is separable and \(N_2\) is smooth, then \(N_1\) is smooth.
(e) Suppose that \(f\) is separable and that either \(N_1\) or \(N_2\) is smooth. Then both \(N_1\) and \(N_2\) are smooth, and \(f_{|N_1}\) is separable.

3. Recollections: reductive groups

Let \(G\) be a connected and reductive group over \(K\). Thus \(G\) is a smooth group scheme over \(K\), or equivalently \(G\) is a linear algebraic group defined over \(K\). To say that \(G\) is reductive means that the unipotent radical of \(G/\text{K_{alg}}\) is trivial. We are going to write \(\zeta_G = \bar{Z}(G)\) for the center of \(G\).

Some results will be seen to hold for a reductive group \(G\) in case \(G\) is \(D\)-standard; in the next few sections, we explain this condition. We must first recall the notions of good and bad characteristic.

3.1. Good and very good primes. Suppose that \(H\) is a smooth group scheme over \(K\) – i.e. an algebraic group over \(K\) – for which \(H/\text{K_{alg}}\) is quasisimple; thus \(H\) is geometrically quasisimple. Write \(R\) for the root system of \(H\). The characteristic \(p\) of \(K\) is said to be a bad prime for \(R\) – equivalently, for \(H\) – in the following circumstances: \(p = 2\) is bad whenever \(R \not\equiv A_r\), \(p = 3\) is bad if \(R = G_2, F_4, E_r\), and \(p = 5\) is bad if \(R = E_8\). Otherwise, \(p\) is good.

A good prime \(p\) is very good provided that either \(R\) is not of type \(A_r\), or that \(R = A_r\) and \(r \not\equiv -1 \pmod{p}\).

If \(H\) is any reductive group, one may apply [KMRT, Theorems 26.7 and 26.8] \(^3\) to see that there is a possibly inseparable central isogeny

\[
R(H) \times \prod_{i=1}^m H_i \rightarrow H
\]

\(^3\)[KMRT] only deals with the semisimple case; the extension to a general reductive group is not difficult to handle, and an argument is sketched in the footnote found in [MT 07, §2.4].
where the radical $R(H)$ of $H$ is a torus, and where for $1 \leq i \leq m$ there is an isomorphism $H_i \simeq R_{L_i/K} J_i$ for a finite separable field extension $L_i/K$ and a geometrically quasisimple, simply connected group scheme $J_i$ over $L_i$; here, $R_{L_i/K} J_i$ denotes the “Weil restriction” – or restriction of scalars – of $J_i$ to $K$, cf. [Sp 98, §11.4]. The $H_i$ are uniquely determined by $H$ up to order of the factors. Then $p$ is good, respectively very good, for $H$ if and only if that is so for $J_i$ for every $1 \leq i \leq m$.

3.2. $D$-standard. Recall from §2.7 the notion of a diagonalizable group scheme, and of a group scheme of multiplicative type.

(3.2.1). If $D$ is subgroup scheme of $G$ of multiplicative type, the connected centralizer $C_G(D)^0$ is reductive.

When $D$ is smooth, the preceding result is well-known: the group $D$ is the direct product of a torus and a finite étale group scheme all of whose geometric points have order invertible in $K$. The centralizer of a torus is (connected and) reductive, and one is left to apply a result of Steinberg [St 68, Cor. 9.3] which asserts that the centralizer of a semisimple automorphism of a reductive group has reductive identity component. In fact, the result remains valid when $D$ is no longer smooth; a proof will appear elsewhere.

Consider reductive groups $H$ which are direct products

\[(*) \quad H = H_1 \times T\]

where $T$ is a torus, and where $H_1$ is a semisimple group for which the characteristic of $K$ is very good.

Definition. A reductive group $G$ is $D$-standard if there exists a reductive group $H$ of the form $(*)$, a subgroup $D \subset H$ such that $D$ is of multiplicative type, and a separable isogeny between $G$ and the reductive group $C_H(D)^o$. \(^4\)

(3.2.2) ([Mc 05, Remark 3]). For any $n \geq 1$, the group $GL_n$ is $D$-standard. The group $SL_n$ is $D$-standard if and only if $p$ does not divide $n$.

In order to prove (3.2.4) below, we first observe:

(3.2.3). Let $M, G_1, G_2$ be affine group schemes of finite type over $K$. Let $f : G_1 \to G_2$ be a surjective morphism of group schemes, suppose that $\ker f$ is central in $G_1$, and let $\phi : M \to G_2$ be a homomorphism of group schemes for which $\phi^{-1}(\zeta G_2)$ is central in $M$. Consider the group scheme $M$ defined by the Cartesian diagram:

\[
\begin{array}{ccc}
\tilde{M} = M \times_{G_2} G_1 & \xrightarrow{\tilde{f}} & M \\
\downarrow{\tilde{\phi}} & & \downarrow{\phi} \\
G_1 & \xrightarrow{f} & G_2
\end{array}
\]

Then

(a) $\tilde{\phi}^{-1}(\zeta G_1)$ is central in $\tilde{M}$.

(b) Suppose that $G_1, G_2$ are connected and reductive, that $f$ is a separable isogeny, and that $M$ is connected and quasisimple. Then $\tilde{M}$ is connected and quasisimple.

Proof. To prove (a), let $N = \tilde{\phi}^{-1}(\zeta G_1)$. It is enough to show that $\tilde{\phi}(N)$ is central in $G_1$ and that $\tilde{f}(N)$ is central in $M$. The first of these observations is immediate from definitions, while the second follows from assumption on the mapping $\phi : M \to G_2$ once we observe that $\tilde{f}(N) \subset \phi^{-1}(\zeta G_2)$.

For (b), we view $\tilde{f}$ as arising by base change from $f$. Then $\tilde{f}$ is an isogeny since $\ker(f)_{\alg}$ and $\ker(\tilde{f})_{\alg}$ coincide. Moreover, it follows from [Li 02, Prop 4.3.22] that $\tilde{f}$ is separable (since it is étale). Thus $\tilde{f}$ is a separable isogeny; since $\tilde{M}$ is separably isogenous to a connected quasisimple group, it is itself connected and quasisimple.

\(^4\)This definition does not require the knowledge that $C_H(D)^o$ is reductive: if there is an isogeny between $G$ and $C_H(D)^o$, then $C_H(D)^o$ is reductive.
(3.2.4). Suppose that the $D$-standard reductive group $G$ is split over $K$. There are $D$-standard reductive groups $M_1, \ldots, M_d$ together with a homomorphism $\Phi : M \to G$, where $M = \prod_{i=1}^d M_i$, such that the following hold:

(a) The derived group of $M_i$ is geometrically quasisimple for $1 \leq i \leq d$.
(b) $\Phi$ is surjective and separable.
(c) For $1 \leq i < j \leq d$, the image in $G$ of $M_i$ and $M_j$ commute.
(d) The subgroup scheme $\Phi^{-1}(\zeta_G)$ is central in $\prod_{i=1}^d M_i$.

Proof. We argue first that it suffices to prove the result after replacing $G$ by a separably isogenous group. More precisely, we prove: $(\ast)$ if $f : G_1 \to G_2$ is a separable isogeny between $D$-standard reductive groups $G_1$ and $G_2$, then (3.2.4) holds for $G_1$ if and only if it holds for $G_2$.

Suppose first that the conclusion of (3.2.4) is valid for $G_1$. If $\Phi : M \to G_1$ is a homomorphism for which (a)--(d) hold, then evidently (a)--(d) hold for $f \circ \Phi$.

Now suppose that the conclusion of (3.2.4) is valid for $G_2$, and that $\Phi : M \to G_2$ is a homomorphism for which (a)--(d) hold. For each $1 \leq j \leq d$ write $\Phi_j$ for the composite of $\Phi$ with the inclusion of $M_j$ in the product. Form the group $\tilde{M}_j = M_j \times_{G_2} G_1$ as in (3.2.3). Then by (b) of loc. cit., $\tilde{M}_j$ is quasisimple. Moreover, loc. cit. (a) shows the kernel of $\Phi_j$ it be central in $\tilde{M}_j$.

Note that the image of $\Phi_j$ is mapped to the image of $\Phi_j$ by $f$. Now, $f$ is a separable isogeny, hence in particular $f$ is central; i.e. ker $f$ is central. It follows that the image of $\Phi_j$ commutes with the image of $\Phi_j$ whenever $1 \leq i \neq j \leq n$. We can thus form the homomorphism $\tilde{\Phi} : \prod_{j=1}^d \tilde{M}_j \to G_1$ whose restriction to each $\tilde{M}_j$ is just $\Phi_j$, and it is clear that (a)--(d) hold for $\tilde{\Phi}$; this completes the proof of $(\ast)$.

In view of the definition of a $D$-standard group, we may now suppose that $G$ is the connected centralizer $C_{H_1}(D)^0$ of a diagonalizable subgroup scheme $D \subseteq H_1 = H \times S$, where $H$ is a semisimple group in very good characteristic and $S$ a torus.

We may use [Sp 98, 8.1.5] to write $G$ as a commuting product of its minimal non-trivial connected, closed, normal subgroups $T_i$ for $i = 1, 2, \ldots, n$. Fix a maximal torus $T \subset G$, so that $T_i = (T \cap T_i)^0$ is a maximal torus of $T_i$ for each $i$.

Now set $T^i = \prod_{j \neq i} T_j$; then $T^i$ is a torus in $G$. Moreover, $J_i$ is the derived subgroup of the reductive group $M_i = C_G(T_i)$.

Now, $M_i$ is the connected centralizer in $H_1$ of the diagonalizable subgroup $\langle T^i, D \rangle$; thus $M_i$ is $D$-standard.

Finally, putting $M = \prod_i M_i$, we have a natural surjective mapping $M \to G$ for which (a)-(d) hold, as required. \hfill $\Box$

3.3. Existence of Springer Isomorphisms. Let $G$ denote a $D$-standard reductive group. We write $\mathcal{N} = \mathcal{N}(G) \subset \mathfrak{g}$ for the nilpotent variety of $G$ and $\mathcal{U} = \mathcal{U}(G) \subset G$ for the unipotent variety of $G$.

By a Springer isomorphism, we mean a map

$$\sigma : \mathcal{N} \to \mathcal{U}$$

which is a $G$-equivariant isomorphism of varieties over $K$.

The first assertion of the following Theorem – the existence of a Springer isomorphism – is due essentially to Springer; see e.g. [SS 70, III.3.12] for the case of an algebraically closed field, or see [Spr69]. The second assertion was obtained by Serre and appears in the appendix to [Mc 05].

**Theorem** (Springer, Serre). (1) There is a Springer isomorphism $\sigma : \mathcal{N} \to \mathcal{U}$.

(2) Any two Springer isomorphisms induce the same mapping between the set of $G(K_{\text{alg}})$-orbits in $\mathcal{U}(K_{\text{alg}})$ and the set of $G(K_{\text{alg}})$-orbits in $\mathcal{N}(K_{\text{alg}})$, where $K_{\text{alg}}$ is an algebraic closure of $K$.

**Proof.** We sketch the argument for assertion (1) in order to point out the role of the $D$-standard assumption made on $G$.

If $G$ is semisimple in very good characteristic, the nilpotent variety $\mathcal{N}$ and the unipotent variety $\mathcal{U}$ are both normal. Indeed, for $\mathcal{U}$, one knows [SS 70, III.2.7] that $\mathcal{U}$ is normal whenever $G$ is simply connected (with no condition on $p$). Moreover, one knows that the normality of $\mathcal{U}$ is preserved by
separable isogeny \(^5\). In positive characteristic the normality of \(\mathcal{N}\) for a semisimple group \(G\) is a result of Veldkamp (for most \(p\)) and of Demazure when the characteristic is very good for \(G\); see [Ja 04, 8.5]. Using the normality of \(\mathcal{U}\) and of \(\mathcal{N}\), Springer showed that [Spr69] there is a \(G\)-equivariant isomorphism as required.

To conclude that assertion (1) is valid for any \(D\)-standard groups, it suffices to observe the following: (i) if \(\pi: G \to G_1\) is a separable isogeny, then there is a Springer isomorphism for \(G\) if and only if there is a Springer isomorphism for \(G_1\), and (ii) if \(H\) is a reductive group for which there is a Springer isomorphism, and if \(D \subset H\) is a subgroup of multiplicative type, then \(C_H^o(D)\) has a Springer isomorphism. \(\square\)

We note a related result for certain not-necessarily-connected reductive groups.

(3.3.1). Let \(G\) be a connected reductive group for which there is a Springer isomorphism \(\sigma: \mathcal{N}(G) \to \mathcal{U}(G)\). Let \(D \subset G\) be a subgroup of multiplicative type, and let \(M = C_G^o(D)\).

(a) \(\sigma\) restricts to an isomorphism \(\mathcal{N}(M) \to \mathcal{U}(M)\).

(b) The finite group \(M(K_{\text{alg}})/M^o(K_{\text{alg}})\) has order invertible in \(K\).

Proof. Assertion (a) follows from the observations: \(\mathcal{N}(M) = \mathcal{N}(G)^D\) and \(\mathcal{U}(M) = \mathcal{U}(G)^D\). To prove (b), note that \(\mathcal{N}(M) = \mathcal{N}(M^o)\) is connected, so that by (a), also \(\mathcal{U}(M)\) is connected. Thus \(\mathcal{U}(M) \subset M^o\) and (b) follows at once. \(\square\)

3.4. Smoothness of some subgroups of \(D\)-standard groups. For any algebraic group, and any element \(x \in G\), let \(C_G(x)\) denote the centralizer subgroup scheme of \(G\). Then by definition \(\text{Lie} C_G(x) = \epsilon_g(x)\), where \(\epsilon_g(x)\) denotes the centralizer of \(x\) in the Lie algebra \(g\), but since the centralizer may not be reduced, the dimension of \(\epsilon_g(x)\) may be larger than the dimension \(\dim C_G(x) = \dim C_G(x)_{\text{red}}\), where \(C_G(x)_{\text{red}}\) denotes the corresponding reduced — hence smooth — group scheme. Similar remarks hold when \(x \in G\) is replaced by an element \(X \in g\).

When \(G\) is a \(D\)-standard reductive group, this difficulty does not arise. Indeed:

(3.4.1). Let \(G\) be \(D\)-standard, let \(x \in G(K)\), and let \(X \in g = g(K)\). Then \(C_G(x)\) and \(C_G(X)\) are smooth over \(K\). In other words,

\[
\dim C_G(x) = \dim \epsilon_g(x) \quad \text{and} \quad \dim C_G(X) = \dim \epsilon_g(X).
\]

In particular,

\[
\text{Lie} C_G(x)_{\text{red}} = \epsilon_g(x) \quad \text{and} \quad \text{Lie} C_G(X)_{\text{red}} = \epsilon_g(X).
\]

Proof. When \(G\) is semisimple in very good characteristic, the result follows from [SS 70, I.5.2 and I.5.6]. The extension to \(D\)-standard groups is immediate; the verification is left to the reader. \(^6\) \(\square\)

Similar assertions holds for the center of \(G\), as follows:

(3.4.2). Let \(G\) be a \(D\)-standard reductive group. Then the center \(\zeta_G\) of \(G\) is smooth.

Proof. Indeed, for any field extension \(L\) of \(K\), the center of \(G_L\) is just the group scheme \((\zeta_G)_L\) obtained by base change. To prove that \(\zeta_G\) is smooth, it suffices to prove that \((\zeta_G)_L\) is smooth. So we may and will suppose that \(G\) is algebraically closed; in particular, \(G\) is split.

Fix a Borel subgroup \(B\) of \(G\) and fix a maximal torus \(T \subset B\). Let \(X = \sum \alpha X_\alpha \in \text{Lie}(B)\) be the sum over the simple roots \(\alpha\), where \(X_\alpha \in \text{Lie}(B)_\alpha\) is a non-zero root vector; then \(X\) is regular nilpotent.

For a root \(\beta \in X^+(T)\) of \(T\) on \(\text{Lie}(G)\), write \(\beta^\vee \in X_\alpha(T)\) for the corresponding cocharacter \(\beta^\vee: G_m \to T\), and consider the cocharacter \(\phi: G_m \to T\) given by \(\phi = \sum \beta^\vee \in X_\alpha(T)\), where the sum is over all positive roots \(\beta\). Then \(\text{Ad}(\phi(t))X = t^2X\) for each \(t \in G_m(K)\) so that the image of \(\phi\) normalizes the centralizer \(C = C_G(X)\).

\(^5\)More precisely, if \(\pi: G \to G_1\) is a separable central isogeny, the restriction of \(\pi\) determines an isomorphism between \(\mathcal{U}(G)\) and \(\mathcal{U}(G_1)\).

\(^6\)Complete details of the reduction from the case of a \(D\)-standard group to that of a semisimple group in very good characteristic can be given along the lines of the argument used in the proof of (5.4.2).
Now, $C$ is a smooth subgroup of $G$ by (3.4.1). The image of $\phi$ is a torus, hence is a diagonalizable group. So the fixed points $C^{\text{im}\phi}$ of the image of $\phi$ on $C$ form a smooth subgroup by (2.7.1).

Finally, since $X$ is contained in the dense $B$-orbit on $\text{Lie}(R_u B)$, $X$ is a distinguished nilpotent element; cf. [Ja 04, 4.10, 4.13]. So it follows from [Ja 04, Prop. 5.10], that $C^{\text{im}\phi}$ is precisely $\zeta_G$, the center of $G$. Thus indeed $\zeta_G$ is smooth. □

**Remark.** In case $G$ is semisimple in very good characteristic one can instead apply [Hum 95, 0.13] to see that the center of the Lie algebra $\text{Lie}(G)$ is trivial; this shows in this special case that $\zeta_G$ is smooth.

3.5. **The centralizer of a semisimple element of $g$.** Suppose $g$ is $D$-standard, let $X \in g = g(K)$ be a semisimple, and write $M = C_G(X)$. Recall that the closed subgroup scheme $M$ is smooth over $K$; cf. (3.4.1).

(3.5.1). (a) $X$ is tangent to a maximal torus $T$ of $G$.

(b) $M^o$ is a reductive group.

**Proof.** [Bor 91, Prop. 11.8 and Prop. 13.19]. □

Now fix a maximal torus $T$ with $X \in \text{Lie}(T)$ as in (3.5.1). Let us recall the following:

(3.5.2). If $S \subset G$ is a torus, there is a finite, separable field extension $L \supset K$ and a parabolic subgroup $P \subset G_L$ such that $C_G(S)_L$ is a Levi factor of $P$.

**Proof.** Let the finite separable field extension $L \supset K$ be a splitting field for $S$. The result then follows from [BoT 65, 4.15]. □

Suppose for the moment that the characteristic $p$ of $K$ is positive. Let $K_{\text{sep}}$ be a separable closure of $K$, and consider the (additive) subgroup $B$ of $K_{\text{sep}}$ generated by the elements $d\beta(X)$ for $\beta \in X^*(T_{K_{\text{sep}}})$; since $d\beta(X) = 0$ whenever $\beta \in pX^*(T_{K_{\text{sep}}})$, $B$ is a finite elementary Abelian $p$-group. Write $\Gamma = \text{Gal}(K_{\text{sep}}/K)$ for the Galois group; since $X \in g(K)$, the group $B$ is stable under the action of $\Gamma$.

Let $\mu = D(B)$ be the $K$-group scheme of multiplicative type determined by the $\Gamma$-module $B$. The $\Gamma$-equivariant mapping $X^*(T_{K_{\text{sep}}}) \to B$ given by $\beta \mapsto d\beta(X)$ determines an embedding of $\mu$ as a closed subgroup scheme of $T$.

(3.5.3). We have $M^o = C_G(\mu)^o$.

**Sketch.** Since $M^o$ and $C_G(\mu)^o$ are smooth groups over $K$, it suffices to give the proof after replacing $K$ by an algebraic closure. In that case $\mu$ is diagonalizable. Let $R \subset X^*(T)$ be the roots of $G$ for the torus $T$, and for $\alpha \in R$ let $U_\alpha \subset G$ be the corresponding root subgroup of $G$.

Then using the Bruhat decomposition of $G$, one finds that

$$M^o = \langle T, U_\alpha \mid d\alpha(X) = 0 \rangle = C_G(\mu)^o;$$

the required argument is essentially the same as that given in [SS 70, II.4.1] except that loc. cit. does not treat infinitesimal subgroup schemes; cf. [Mc 08a] for the details. □

**Theorem.** There is a finite separable field extension $L \supset K$ for which the connected centralizer $M^o_L = C^o_G(X)_L$ is a Levi factor of a parabolic subgroup of $G_L$.

**Proof.** Suppose first that $K$ has characteristic $p > 0$. In view of (3.5.3), the reductive group $M^o$ is $D$-standard, since $\mu$ is a group of multiplicative type. According to (3.4.2), the center $Z$ of $M^o$ is smooth. Let $S$ be a maximal torus of $Z$. We have evidently $M^o \subset C_G(S)$. It follows that $\text{Lie}(Z) = \text{Lie}(S)$. We may now use (2.6.1) to see that $X \in \text{Lie}(Z) = \text{Lie}(S)$. Thus $M^o \supset C_G(S)$.

It follows that $M^o = C_G(S)$, and we conclude via (3.5.2).

The situation when $K$ has characteristic zero is simpler. In that case, the center $Z$ of the reductive group $M^o$ is automatically smooth. If $S$ is a maximal torus of $Z$ then $M^o = C_G(S)$ as before. □
3.6. Borel subalgebras. Suppose that $K$ is algebraically closed. By a Borel subalgebra of $\mathfrak{g}$, we mean the Lie algebra $\mathfrak{b} = \text{Lie}(B)$ of a Borel subgroup $B \subseteq G$.

**Proposition** ([Bor 91, 14.25]). $\mathfrak{g}$ is the union of its Borel subalgebras. More precisely, for each $X \in \mathfrak{g}$, there is a Borel subalgebra $\mathfrak{b}$ with $X \in \mathfrak{b}$.

4. THE CENTER OF A CENTRALIZER

For a $D$-standard reductive group $G$ over $K$, let $x \in G(K)$ and $X \in \mathfrak{g}(K)$. We are going to consider the centralizers $C_G(X)$ and $C_G(x)$, and in particular, the centers $Z_x = Z(C_G(x))$ and $Z_X = Z(C_G(X))$ of these centralizers. As we have seen, $Z_x$ is a closed subscheme of $C_G(x)$ and $Z_X$ is a closed subscheme of $C_G(X)$. In this section, we will prove Theorem A from the introduction; namely, in §4.2, we prove that $Z_x$ and $Z_X$ are smooth. In §4.1, we establish some preliminary results under the assumption that $K$ is perfect. Since the smoothness of $Z_x$ and of $Z_X$ will follow if it is proved after base change with an algebraic closure $\text{K}_{\text{alg}}$ of $K$, this assumption on $K$ is harmless for our needs.

4.1. Unipotence of the center of the centralizer when $X$ is nilpotent. Suppose in this section that the field $K$ is perfect; thus if $A$ is a group scheme over $K$, we may speak of the reduced subgroup scheme $A_{\text{red}}$—cf. (2.4.1). We begin with the following observation which is due independently to R. Proud and G. Seitz. For completeness, we include a proof.

(4.1.1). Let $x$ be unipotent, let $X$ be nilpotent, write $C$ for one of the groups $C_G(x)$ or $C_G(X)$, and write $Z = Z(C)$; thus $Z$ is one of the groups $Z_x$ or $Z_X$.

(a) $C^o$ is not contained in a Levi factor of a proper parabolic subgroup of $G$.

(b) The quotient $(Z_{\text{red}})^o / (\zeta_G)^o$ is a unipotent group, where $Z_{\text{red}}$ is the corresponding reduced group, and $(\zeta_G)^o$ is its identity component.

(c) Let $Y \in \text{Lie}(Z)$ be semisimple. Then $Y \in \text{Lie}(\zeta_G)$.

**Proof.** It suffices to prove each of the assertions after extending scalars; thus, we may and will suppose in the proof that $K$ is algebraically closed. Moreover, if $\sigma : \mathcal{N} \to \mathcal{U}$ is a Springer isomorphism, then $C_G(X) = C_G(\sigma(X))$. Thus it suffices to give the proof for the centralizer of $X$.

We first prove (a). Suppose that $L$ is a Levi factor of a parabolic subgroup $P$, and assume that $C^o$ is a subgroup scheme of $L$. Then $C^o = C^o_L(X)$ so that $\text{Lie} \ C^o = \text{Lie} C^o_L(X)$. Since $L$ is again a $D$-standard reductive group, we see by the smoothness of centralizers that $\text{Lie} C^o_L(X)$ is the centralizer in $\text{Lie} L$ of $X$ (3.4.1); in particular, we see that every fixed point of $\text{Ad}(X)$ on $\text{Lie}(G)$ lies in $\text{Lie}(L)$. If $L$ were a proper subgroup of $G$, the nilpotent operator $\text{ad}(X)$ would have a non-zero fixed point on $\text{Lie} R_uP$; it follows that $L = G$.

We will now deduce (b) and (c) from (a). For (b), let $S \subseteq G$ be a torus. The assertion (b) will follow if we prove that $S$ is central in $G$. But $L = C_G(S)$ is a Levi factor of some parabolic subgroup $P$ of $G$ by (3.5.2), and $C^o \subseteq L$. Thus by (a) we have $P = G = L$; this shows that $S$ is central in $G$, as required.

For (c), let $Y \in \text{Lie}(Z)$ be semisimple. According to Theorem 3.5, $L = C^o_G(Y)$ is a Levi factor of some parabolic subgroup $P$, and $C^o \subseteq L$. So again (a) shows that $P = G = L$. Since $C_G(Y) = G$, it follows that $Y$ is a fixed point for the adjoint action of $G$ on $\text{Lie}(G)$. But according to (2.6.1), we have $\text{Lie}(\zeta_G) = \text{Lie}(G)^{\text{Ad}(G)}$; thus indeed $Y \in \text{Lie}(\zeta_G)$ as required. \qed

As a consequence, we deduce the following structural results:

(4.1.2). With notation and assumptions as in (4.1.1), we have:

(a) $Z_{\text{red}}$ is the internal direct product $\zeta_G \cdot R_uZ_{\text{red}}$.

(b) The set of nilpotent elements of $\text{Lie}(Z)$ forms a subalgebra $\mathfrak{u}$ for which

$$\text{Lie} Z = \text{Lie}(\zeta_G) \oplus \mathfrak{u}.$$ 

**Proof.** Note that $Z$ and also $\text{Lie}(Z)$ are commutative; since the product of two commuting unipotent elements of $G$ is unipotent and the sum of two commuting nilpotent elements of $\text{Lie}(G)$ is nilpotent, results (a) and (b) follow from (4.1.1)(b) and (c). \qed
4.2. Smoothness of the center of the centralizer. In this section, \( K \) is again arbitrary. Let \( x \in G(K) \), \( X \in g(K) \) be arbitrary, write \( C \) for one of the groups \( C_G(x) \) or \( C_G(X) \), and write \( Z = Z(C) \), so that \( Z \) is one of the groups \( Z_x \) or \( Z_X \). We are now ready to prove the following:

**Theorem.** The center \( Z = Z(C) \) is a smooth group scheme over \( K \).

**Proof.** Since a group scheme is smooth over \( K \) if and only if it is smooth upon scalar extension, we may and will suppose \( K \) to be algebraically closed (hence in particular perfect). So as in §4.1, we may speak of the reduced subgroup scheme \( A_{\text{red}} \) of a group scheme \( A \) over \( K \).

Let \( x = x_s x_u \) and \( X = X_s + X_n \) be the Jordan decompositions of the elements; thus \( x_s \in G \) and \( X_s \in g \) are semisimple, \( x_u \in G \) is unipotent, \( X_n \in g \) is nilpotent, and we have: \( x_s x_u = x_u x_s \) and \([X_s, X_u] = 0\).

Then
\[
C_G(x) = C_M(x_u) \quad \text{and} \quad C_G(X) = C_M(X_n)
\]
where \( M = C_G(x_s) \) resp. \( M = C_G(X_s) \).

Now, the Zariski closure of the group generated by \( x_s \) is a smooth diagonalizable group whose centralizer coincides with \( C_G(x_s) \). And according to §3.5 the centralizer of \( X_s \) is reductive and is the centralizer of a (non-smooth) diagonalizable group scheme. Thus in both cases, the connected component of \( M \) is itself a \( D \)-standard reductive group.

Moreover, (3.3.1) shows that \( x_u \) is a \( K \)-point of \( M^\circ \). There is an exact sequence
\[
1 \to C_{M^\circ}(x_u) \to C_M(x_u) \to E \to 1
\]
resp.
\[
1 \to C_{M^\circ}(X_n) \to C_M(X_n) \to E' \to 1
\]
for a suitable subgroup \( E \) resp. \( E' \) of \( M/M^\circ \). Since \( M/M^\circ \) has order invertible in \( K \) (3.3.1), apply (2.8.1) to see that the smoothness of \( Z \) follows from the smoothness of the center of \( C_{M^\circ}(x_u) \) resp. \( C_{M^\circ}(X_n) \); thus the proof of the theorem is reduced to the case where \( x \) is unipotent and \( X \) is nilpotent. Since in that case \( C_G(X) = C_G(\sigma(X)) \) where \( \sigma : N' \to U \) is a Springer isomorphism, we only discuss the centralizer of a nilpotent element \( X \in g \).

We must argue that \( \dim Z = \dim \text{Lie } Z \). Since it is a general fact that \( \dim \text{Lie } Z \geq \dim Z \), it suffices to show the following:

\( (*) \quad \dim Z \leq \dim \text{Lie } Z \).

By (4.1.2) we have \( \text{Lie } Z = \text{Lie}(\zeta_G) \oplus u \) where \( u \) is the set of all nilpotent \( Y \in \text{Lie } Z \). According to (3.4.2), the center \( \zeta_G \) of \( G \) is smooth. Thus \( \dim \zeta_G = \dim \text{Lie } \zeta_G \). In view of (4.1.2), the assertion (*) will follow if we prove that

\( (**) \quad \dim u \leq \dim R_u Z_{\text{red}} \).

In order to prove (**) we fix a Springer isomorphism \( \sigma : N' \to U \) — see Theorem 3.3 —, and we consider the restriction of \( \sigma \) to \( u \).

We first argue that \( \sigma \) maps \( u \) to \( R_u Z_{\text{red}} \). Since \( u \) is smooth — hence reduced — and since \( K \) is algebraically closed, it suffices to show that \( \sigma \) maps the \( K \)-points of \( u \) to \( R_u Z_{\text{red}} \). Fix \( Y \in u(K) \).

If \( g \in C_G(X)(K) \), the inner automorphism \( \text{Int}(g) \) of \( C \) is trivial on \( Z \); thus, the automorphism \( \text{Ad}(g) \) of \( C \) is trivial on \( \text{Lie } Z \). It follows that
\[
g \sigma(Y) g^{-1} = \sigma(\text{Ad}(g) Y) = \sigma(Y).
\]
Since \( K \) is algebraically closed, it now follows from (2.5.2) that
\[
\sigma(Y) \in Z(K) = C_G(X)^{\text{Int}(C_G(X))}(K).
\]
Since \( u \) is reduced, one knows \( \sigma(Y) \in Z_{\text{red}}(K) \). Since \( \sigma(Y) \) is unipotent, it follows that \( \sigma(Y) \in R_u Z_{\text{red}}(K) \).

Thus the restriction of the Springer isomorphism \( \sigma \) gives a morphism \( \sigma_{\text{red}} : u \to R_u Z_{\text{red}} \). Since \( \sigma \) is a closed morphism, it follows that the image of \( \sigma_{\text{red}} \) is a closed subvariety of \( R_u Z_{\text{red}} \) whose dimension is \( \dim u \), so that indeed (**) holds. \( \Box \)
With notation as in the preceding proof, we point out a slightly different argument. Namely, reasoning as above, one can show that the inverse isomorphism \( \tau = \sigma^{-1} : U \to N' \) maps \( R_u Z_{\text{red}} \) to \( u \). It follows that \( R_u Z_{\text{red}} \) and \( u \) are isomorphic varieties, hence they have the same dimension.

Note that we have now proved Theorem A from the introduction.

5. Regular nilpotent elements

In this section, we are going to prove Theorems B, C, and E from the introduction. We denote by \( G \) a \( D \)-standard reductive group over the field \( K \). Let \( T \subset G \) be a maximal torus, and let \( T_0 \subset T \) where \( T_0 \) is a maximal torus of the derived group \( G' = (G, G) \) of \( G \). Let us write \( r = \dim T_0 \) for the semisimple rank of \( G \). Finally, let \( W = N_G(T')/T \simeq N_G(T_0)/T_0 \) be the corresponding Weyl group.

5.1. Degrees and exponents. We give here a quick description of some well-known numerical invariants associated with the Weyl group \( W \). We suppose that the derived group \( G' \) of \( G \) is quasi-simple, and we suppose that \( T \) (and hence \( G \)) is split over \( K \).

Let \( V = X^* (T_0) \otimes_{\mathbb{Z}} \mathbb{Q} \) and note that the action of the Weyl group \( W \) on \( T_0 \) determines a linear representation \( (\rho, V) \) of \( W \). The algebra of polynomials (regular functions) on \( V \) may be graded by assigning the degree 1 to each element of the dual space \( V^* \subset \mathbb{Q}[V] \). The action via \( \rho \) of \( W \) on \( V \) determines an action of \( W \) on \( \mathbb{Q}[V] \) by algebra automorphisms, and it is known that the algebra \( \mathbb{Q}[V]^W \) of \( W \)-invariant polynomials on \( V \) is generated as a \( \mathbb{Q} \)-algebra by \( r \) algebraically independent homogeneous elements of positive degree [Bou 02, V.5.3 Theorem 3]. The degrees of \( W \) are the degrees \( d_1, d_2, \ldots, d_r \) of a system of homogeneous generators for \( \mathbb{Q}[V]^W \). The degrees depend – up to order – only on \( W \); see [Bou 02, V.5.1]. The exponents of \( W \) are the numbers \( k_1, k_2, \ldots, k_r \), where \( k_i = d_i - 1 \) for \( 1 \leq i \leq r \).

Recall that the “exponents” earn their name as follows. Let \( c \in W \) be a Coxeter element [Bou 02, V.6.1], and write \( h \) for the order of \( c \). If \( E \) is a field of characteristic 0 containing a primitive \( h \)-th root of unity \( \omega \in E^\times \), then [Bou 02, V.6.2 Prop. 3] the eigenvalues of \( \rho(c) \) on \( V \otimes_{\mathbb{Q}} E \) are the values \( \omega^{k_1}, \omega^{k_2}, \ldots, \omega^{k_r} \).

The exponents and degrees are known explicitly; cf. [Bou 02, Plate I – IX].

5.2. The centralizer of a regular nilpotent element. In this section, \( G \) is again a \( D \)-standard reductive group (whose derived group is not required to be quasi-simple) which we assume to be quasisplit over \( K \).

If \( \phi : G_m \to G \) is a cocharacter and \( i \in \mathbb{Z} \), we write \( g(\phi; i) \) for the \( i \)-weight space of the action of \( \phi(G_m) \) on \( g \) under the adjoint action of \( \phi(G_m) \); thus

\[
g(\phi; i) = \{ Y \in g \mid \text{Ad}(\phi(t)) Y = t^i Y \quad \forall t \in K^\times \} \}.
\]

Any cocharacter \( \phi \) determines a unique parabolic subgroup \( P = P(\phi) \) whose \( K_{\text{alg}} \) points are given by:

\[
P(K_{\text{alg}}) = \{ g \in G(K_{\text{alg}}) \mid \lim_{t \to 0} \text{Int}(\phi(t)) g \text{ exists} \}.
\]

One knows that \( p = \text{Lie}(P) = \sum_{i \geq 0} g(\phi; i) \).

Let \( X \in g(K) \) be nilpotent. Following [Ja 04, §5.3], we say that a cocharacter \( \psi : G_m \to G \) is said to be associated to a nilpotent element \( X \) in case (i) \( X \in g(\psi; 2) \), and (ii) there is a maximal torus \( S \) of the centralizer \( C_G(X) \) such that the image of \( \psi \) lies in \( (L, L) \), where \( L = C_G(S) \).

(5.2.1).

(a) There are cocharacters associated to \( X \).

(b) If \( \phi \) and \( \phi' \) are cocharacters associated to \( X \), then \( P(\phi) = P(\phi') \).

(c) The centralizer \( C_G(X) \) is contained in \( P = P(\phi) \) for a cocharacter \( \phi \) associated to \( X \).

(d) The unipotent radical \( R_{C_G(X) / K_{\text{alg}}} \) is defined over \( K \) and is a \( K \)-split unipotent group.

(e) Any two cocharacters associated to \( X \) are conjugate by a unique element of \( R(K) \).
Proof. In the geometric setting, these assertions may be found in [Ja 04]; the existence of an associated cocharacter is an essential part of the Bala-Carter, a conceptual proof of which may be found [Pr 03]. Over the ground field $K$, (a) and (c) follow from [Mc 04, Theorem 26 and Theorem 28]. (b) follows since associated cocharacters are optimal for the unstable vector $X$ in the sense of Kempf; see [Pr 03]. Finally, (d) and (e) follow from [Mc 05, Prop/Defn 21].

Finally, recall that a nilpotent element $X \in g$ is distinguished provided that a maximal torus of the centralizer $C_G(X)$ is central in $G$.

(5.2.2). Let $X \in g$ be nilpotent. The following are equivalent:

(a) $X$ is regular – i.e. $\dim C_G(X)$ is equal to the rank of $G$.
(b) $X \in \text{Lie}(B)$ for precisely one Borel subgroup of $G$.

Moreover, if $X$ is regular then $X$ is distinguished, and if $\phi$ is a cocharacter associated with $X$, then $B = P(\phi)$ is the unique Borel subgroup with $X \in \text{Lie}(B)$.

Proof. The equivalence of (a) and (b) can be found in [Ja 04, Cor. 6.8]. Note that in loc. cit. it is assumed that $K$ is algebraically closed. But, it suffices to prove that (b) implies (a) after replacing $K$ by an extension field. It remains to argue that (a) implies (b). But given (a), one knows there to be a unique Borel subgroup $B \subset G_{\text{alg}}$ with $X \in \text{Lie}(B)$, where $K_{\text{alg}}$ is an algebraic closure of $K$. It now follows from [Mc 04, Prop. 27] that $B$ is a parabolic subgroup of $G$ [i.e. that $B$ is defined over $K$], and (b) follows.

That a regular element is distinguished follows from the Bala-Carter theorem; it can be seen perhaps more directly just by observing that $B$ is a distinguished parabolic subgroup, so that an element of the dense orbit of $B$ on $\text{Lie}(R_uB)$ is distinguished by [Ca 93, 5.8.7].

Finally, write $P = P(\phi)$. It follows from [Ja 04, 5.9] that $X$ is in the dense $P$-orbit on $\text{Lie}(R_uP)$ and that $C_P(X) = C_G(X)$; thus $\dim \text{Ad}(G)X = 2 \dim R_uP$ so that indeed $P$ must be a Borel subgroup. 

Since $G$ is assumed to be quasisplit, we have

(5.2.3) ([Mc 05, Theorem 54]). There is a regular nilpotent element $X \in g(K)$.

We fix now a regular nilpotent element $X$. Let $C = C_G(X)$ be the centralizer of $X$, and write $\zeta_G$ for the center of $G$.

(5.2.4). For the group $C = C_G(X)$ we have:

(a) The maximal torus of $C$ is the identity component of the center $\zeta_G$ of $G$.
(b) $C = \zeta_G \cdot R_u(C)$.
(c) $C$ is commutative.

Proof. Assertions (a) and (b) follow from [Ja 04, §4.10, §4.13] precisely as in the proof of (3.4.2).

For (c), use a Springer isomorphism $\sigma : N \to U$, to see that $C$ is the centralizer of the regular unipotent element $u = \sigma(X)$. Then the commutativity of $C$ follows from a result of Springer – see [Hum 95, Theorem 1.14] – which implies that the centralizer of $u$ contains a commutative subgroup of dimension equal to the rank of $G$. This shows that the identity component of $C$ is commutative.

Since $R_uC$ is connected and since $C = \zeta_G R_uC$, the group $C$ is itself commutative.

We now fix a cocharacter $\phi$ of $(G, G)$ associated to $X$.

(5.2.5). The image $\phi$ normalizes $C$. Suppose that the derived group of $G$ is quasisimple. We have

(a)

$$\text{Lie}(R_uC) = \bigoplus_{i=1}^r \text{Lie}(C)(\phi; 2k_i)$$

where $1 = k_1 \leq \cdots \leq k_r$ are the exponents of the Weyl group of $G$.

(b) $\dim \text{Lie}(R_uC)(\phi; 2) = 1$. 

Proof. First suppose that $K$ has characteristic $0$. In that case, the assertions are a consequence of results of [Ko 59]. One deduces (a) immediately from [Ko 59, §6.7]. For (b), one knows that the integers $2k_i$ are the highest weights for the action of a principal $sl_2$ on $g$. Examining the roots of $g$, one knows that the largest weight $2k_r$ occurs precisely once; thus $\dim V(\phi; 2k_r) = 1$.

Now the duality of the exponents [Ko 59, Theorem 6.7] shows that
\[
\dim \underline{V}(\phi; 2) = \dim \underline{V}(\phi; 2k_1) = \dim \underline{V}(\phi; 2k_r) = 1
\]
as required.

For general $K$, consider a discrete valuation ring $A$ whose residue field is $K$ and whose field of fractions $L$ has characteristic 0, and denote by $\mathcal{G}$ a split reductive group scheme over $A$ such that upon base change with $K$ one has $\mathcal{G} / K \simeq G$. Of course, the Weyl groups of $\mathcal{G} / K$ and of $\mathcal{G} / L$ are isomorphic.

According to [Mc 08, Theorems 5.4 and 5.7] we may find a suitable such $A$ for which there is a nilpotent section $X_0 \in \text{Lie}(\mathcal{G})(A)$ and a homomorphism of $A$-group schemes $\phi : \mathbb{G}_m \to \mathcal{G}$ with the following properties:

(i) the image $X_0(K)$ of $X_0$ in $g = \text{Lie}(G)$ is an isomorphism between $\mathbb{G}_m$ and $\mathcal{G}$ coincides with $X$,

(ii) the image $X_0(L)$ of $X_0$ in $\text{Lie}(G)$ is regular nilpotent,

(iii) the cocharacter $\phi_K$ of $G = \mathcal{G} / K$ is associated to $X = X_0(K)$, and

(iv) the cocharacter $\phi_L$ of $\mathcal{G} / L$ is associated to $X_0(L)$.

Moreover, it follows from [Mc 08, Prop. 5.2] that the centralizer subgroup scheme $C_G(X_0)$ is smooth. In particular, $\text{Lie}(C_G(X_0))$ is free as an $A$-module, and $\text{Lie}(C) = \text{Lie}(C_G(X_0)) \otimes_A K$. We may regard $\text{Lie}(C_G(X_0))$ as a representation for the diagonalizable $A$-group scheme $\mathbb{G}_m$ via $\text{Ad} \circ \phi$. Decompose this representation as a sum of its weight subspaces:
\[
\text{Lie}(C_G(X_0)) = \bigoplus_{i \in \mathbb{Z}} \text{Lie}(C_G(X_0))(\phi; i).
\]
Extending scalars to $L$, one sees that $\text{Lie}(C_G(X_0))(\phi; i)$ is non-zero if and only if $i/2$ is one of the exponents of the Weyl group of $G$, and $\text{Lie}(C_G(X_0))(\phi; 2)$ has rank 1. The assertions (a) and (b) now follow by base change with $K$. \hfill $\Box$

5.3. Lifting regular nilpotent elements.

(5.3.1). Let $f : G \to H$ be a homomorphism between reductive groups such that $f$ is surjective and central – i.e. the subgroup scheme $\ker f$ is contained in the center of $G$. Then $f$ restricts to a surjective homomorphism $\tilde{f} : \mathcal{G} \to \mathcal{H}$.

Proof. The assertion is geometric, so we may and will suppose the field $K$ to be algebraically closed. Since $f$ is central, the pre-image of each maximal torus $S$ of $H$ is a maximal torus $T$ of $G$. Then $\tilde{f} : T \to S$ is surjective. The result now follows because $\zeta_G$ is the (scheme theoretic) intersection of all maximal tori in $G$, and $\zeta_H$ is the intersection of all maximal tori in $H$; see [SGA3, Exp. XII Prop. 4.10]. \hfill $\Box$

Suppose now that $G_1$ and $G_2$ are $D$-standard reductive groups, and that $f : G_1 \to G_2$ is a separable surjective homomorphism of reductive groups which is central, as before. Recall that the separability of $f$ means that the tangent mapping $df$ is surjective.

(5.3.2). (a) Suppose that $X_2 \in \text{Lie}(G_2)(K)$ is regular nilpotent. There is a nilpotent element $X_1 \in \text{Lie}(G_1)(K)$ for which $df(X_1) = X_2$.

(b) If $df(Y_1) = Y_2$ for nilpotent elements $Y_1 \in \text{Lie}(G_1)$, then $Y_1$ is regular if and only if $Y_2$ is regular.

Proof. Let $B \subset G_2$ be a Borel subgroup with $X \in \text{Lie}(B)(K)$. The inverse image $B_1$ of $B$ in $G_1$ is a parabolic subgroup [Bor 91, 22.6]; since $B_1$ is evidently solvable, $B_1$ is a Borel subgroup of $G_1$. Thus $f$ induces a morphism $\tilde{f} : B_1 = G_1 / B_1 \to G_2 / B$, and it is clear that the tangent map at the point $B_1$ of $B_1$ is an isomorphism. It follows from [Sp 98, Theorem 5.3.2(iii)] that $\tilde{f}$ is an isomorphism between the flag varieties.
Write $u_1 = \text{Lie } R_u B_1$ and $u = \text{Lie } R_u B$. According to [Bor 91, 22.5], $f$ induces a bijection between the roots of $G_1$ (with respect to some maximal torus) and the roots of $G$ (with respect to a compatible maximal torus). In particular, $\dim R_u B_1 = \dim R_u B$. Since $\ker f$ is central in $G$, $\ker df$ is contained in $\text{Lie}(T)$ for each maximal torus $T$. It follows that the restriction of $df$ to $u_1$ is injective, so that $df(u_1) = u$. Since $X \in \text{Lie}(B)$ is nilpotent, we have $X \in u$. It follows that there is a – necessarily nilpotent – element $X_1 \in u_1$ with $df(X_1) = X$. This proves (a).

Now, $f$ induces a bijection between the varieties $B_1Y_1$ and $B_2Y_2$, where $B_iY_i$ consists of those Borel subgroups $B$ with $Y_i \in \text{Lie}(B)$. Assertion (b) now follows from (5.2.2).

(5.3.3). Suppose that the elements $X_i \in \text{Lie}(G_i)$ are nilpotent for $i = 1, 2$, that $df(X_1) = X_2$, and that $X_1$ is regular, equivalently that $X_2$ is regular. If $C_1 = G_{G_1}(X_1)$ and $C = G_{G_2}(X_2)$, then $C_1 = f^{-1}C$. In particular, $f$ restricts to a surjective separable mapping $\phi_{G_1} : C_1 \rightarrow C$.

Proof. As before, the assertion is geometric; thus we may and will suppose that $K$ is algebraically closed for the proof. We only must argue that (*) $C_1 = f^{-1}C$. Indeed, the remaining assertions follow from (*) by using (2.11.1)(d) and the smoothness of $C_1$ (3.4.1).

We will argue that $f|_{G_1} : C_1 \rightarrow C$ is surjective; assertion (*) will then follow since $\ker f$ is central in $G_1$. Recall that $C_1 = \xi_{G_1} : R_u C_1$ and $C = \xi_{G_2} : R_u C$. The restriction $f|_{G_1} : \xi_{G_1} \rightarrow \xi_{G_2}$ is surjective (5.3.1).

It remains to argue that $f|_{R_u C_1}$ yields a surjective mapping $R_u C_1 \rightarrow R_u C$. Since $G_1$ and $G_2$ are $D$-standard, the centralizers $C_1$ and $C$ are smooth by (3.4.1). Thus the unipotent radicals of $C_1$ and of $C$ are smooth group schemes over $K$. So the surjectivity of $f|_{R_u C_1} : R_u C_1 \rightarrow R_u C$ will follow if we only prove that $df : \text{Lie}(R_u C_1) \rightarrow \text{Lie}(R_u C)$ is surjective.

But $df|_{\text{Lie}(R_u C_1)}$ is injective since $\ker df$ is central. Moreover, $\dim R_u C_1$ is the semisimple rank of $G_1$, and $\dim R_u C$ is the semisimple rank of $G_2$. Since $f$ is surjective with central kernel, the semisimple ranks of $G_1$ and $G_2$ coincide. Thus $df|_{\text{Lie}(R_u C_1)}$ is bijective and the assertion follows.

5.4. The normalizer of $C$. Let us again fix a regular nilpotent element $X$ together with a cocharacter $\phi$ associated to $X$. Let $N = N_G(C)$ be the normalizer of $C$.

We will argue in (5.4.2) below that $N$ is a smooth group scheme over $K$. Meanwhile, we consider in the next assertion the $N$-orbit of $X$. Viewing this orbit as a subspace of $\text{Lie}(R_u C)$, we may consider its closure; that closure has a unique structure of reduced subscheme [Li 02, Prop. 2.4.2]. Since the orbit of $X$ is open in its closure, that orbit inherits a structure as a reduced subscheme.

The following argument essentially just records observations made by Serre in his note found in [Mc 05, Appendix].

(5.4.1). (a) The $N$-orbit of $X$ is the open subset of $\text{Lie}(R_u C)$ consisting of the regular elements; i.e.

\[ \text{Ad}(N)X = \text{Lie}(R_u C)_{\text{reg}} \]

(b) The group $N/C$ is connected and has dimension equal to the semisimple rank $r$ of $G$.

(c) In particular, $\dim N = 2r + \dim \xi_G$.

Proof. Before giving the proof, we recall that (*) $C = C^0 \cdot \xi_G$ where $\xi_G$ is the center of $G$; see (5.2.4).

For the proof of (a), we have evidently $\text{Ad}(N)X \subset \text{Lie}(R_u C)_{\text{reg}}$. Since $\text{Ad}(N)X$ is a reduced scheme, to prove equality it suffices to show that any closed point of $\text{Lie}(R_u C)_{\text{reg}}$ is contained in this orbit. If $K_{\text{alg}}$ is an algebraic closure of $K$ and $Y \in \text{Lie}(R_u C)_{\text{reg}}(K_{\text{alg}})$, then $Y$ is a Richardson element for $B$, where $B$ is the Borel subgroup as in (5.2.2). Since the Richardson elements form a single orbit under $B$, there is $x \in B(K_{\text{alg}})$ for which $\text{Ad}(x)Y = X$. Since $C$ is commutative, a dimension argument shows that $C^0_Y = C^0$. Since also $C_G(Y) = C^0_C(Y) \cdot \xi_G$; it follows from (*) that $C = C_G(Y)$. Since

\[ xCx^{-1} = xC_G(Y)x^{-1} = C_G(\text{Ad}(x)Y) = C_G(X) = C, \]

one sees that $x \in N(K_{\text{alg}})$. It follows that $\text{Ad}(N)X = \text{Lie}(R_u C)_{\text{reg}}$. 

(5.4.2). Suppose that the elements $X_i \in \text{Lie}(G_i)$ are nilpotent for $i = 1, 2$, that $df(X_1) = X_2$, and that $X_1$ is regular, equivalently that $X_2$ is regular. If $C_1 = G_{G_1}(X_1)$ and $C = G_{G_2}(X_2)$, then $C_1 = f^{-1}C$. In particular, $f$ restricts to a surjective separable mapping $\phi_{G_1} : C_1 \rightarrow C$.
For (b), first suppose that $K = K_{\text{alg}}$ is algebraically closed. By (a), $(N/C)_{\text{red}}$ identifies with $\text{Lie}(R_u C)_{\text{reg}}$, an open subvariety of the affine space $\text{Lie}(R_u C)$. It follows that $(N/C)_{\text{red}}$ is an irreducible variety; thus the variety $N/C$ is connected.

But then relaxing the assumption on $K$, it follows that $N/C$ is connected in general. Since $\text{Lie}(R_u C)$ has dimension equal to $r$, conclude that $\dim N/C = r$.

Finally, (c) follows since $\dim C = r + \dim \zeta_G$. \hfill \Box

We can now prove:

(5.4.2). $N$ is a smooth subgroup scheme of $G$.

Proof. The statement is geometric; thus we may and will suppose $K$ to be algebraically closed. Let $f : G_1 \to G_2$ be a surjective separable morphism with central kernel, and suppose that $G$ is one of the groups $G_1$ or $G_2$.

If $G = G_1$, write $X_1$ for $X$ and set $X_2 = df(X_1)$. If $G = G_2$, write $X_2$ for $X$ and use (5.3.2) to find a regular nilpotent $X_1 \in \text{Lie}(G_1)$ for which $df(X_1) = X_2$.

Now write $C_i = C_{G_i}(X_i)$. It follows from (5.3.3) that $C_1 = f^{-1}C_2$, so we may apply (2.11.2) to see that

\[ (\ast) \quad N_{\text{G}_1}(C_1) \text{ is smooth over } K \text{ if and only if } N_{\text{G}_2}(C_2) \text{ is smooth over } K. \]

We are now going to argue: it suffices to prove the result when $G$ is quasisimple in very good characteristic.

Well, if the result is known for quasisimple $G$ in very good characteristic, it follows easily for any semisimple, simply connected group scheme in very good characteristic (since any such is a direct product of simply connected quasisimple groups). But any semisimple group in very good characteristic is separably isogenous to a simply connected one, so (\ast) then permits us to deduce the result for any semisimple $G$ in very good characteristic.

For a general $D$-standard group $G$, we must consider a reductive group $H$ of the form $H = H_1 \times T$ where $H_1$ is semisimple in very good characteristic, together with a diagonalizable subgroup scheme $D \subset H$. We suppose that $G$ is separable isogenous to $C_H(D)^\circ$. The above arguments show that the desired result holds for $H$, and we want to deduce the result for $G$. Again using (\ast), we may suppose that $G = C_H(D)^\circ$.

But if $N = N_G(C)$, we see that $N = N_H(C_H(X))D$. Our assumption means that $N_H(C_H(X))$ is smooth. But then [SGA3, Exp. XI, Cor. 5.3] shows that $N = N_H(C_H(X))D$ is smooth, as required.

Thus, we now suppose $G$ to be quasisimple in very good characteristic. Now, $\dim N = 2r$ by (5.4.1), where $r$ is the rank of $G$. Thus to show that $N$ is smooth, we must show that $2r = \dim \text{Lie}(N)$.

Since one has always $\dim \text{Lie}(N) \geq \dim N$, it is enough to argue that $\dim \text{Lie}(N) \leq 2r$.

Write $n = \{ Y \in g \mid [Y, \text{Lie} C] \subset \text{Lie} C \}$ for the normalizer in $g$ of $\text{Lie} C$. Evidently $\text{Lie}(N) \subset n$; it therefore suffices to show that $\dim n \leq 2r$.

Suppose that $Y \in n$. Since $C$ is commutative, evidently $[[Y, X], X] = 0$, so that $Y \in \ker(\text{ad}(X)^2)$. Thus, it suffices to show that

\[ (\ast) \quad \dim \ker(\text{ad}(X)^2) = 2r. \]

But in view of our assumptions on the characteristic of $K$, (\ast) follows from [Spr 66, Cor. 2.5 and Theorem 2.6]. \hfill \Box

(5.4.3). $N$ is a solvable group.

Proof. Let $B$ be the unique Borel subgroup of $G$ with $X \in \text{Lie}(B)$ as in (5.2.2). Since $B$ is solvable, the result will follow if we argue that $N \subset B$.

Since $N$ is smooth – in particular, reduced – it suffices to argue that $B$ contains each closed point of $N$. Thus, it is enough to suppose that $K$ is algebraically closed and prove that $N(K) \subset B(K)$.

Recall first that according to (5.2.1)(c), we have $C \subset B$. If $y \in N(K)$ it follows that $\text{Int}(y)B$ contains $C$, hence $\text{Lie}(\text{Int}(y)B)$ contains $X$. This proves that $\text{Int}(y)B = B$, so $y$ normalizes $B$. Since Borel subgroups are self normalizing, we deduce $N(K) \subset B(K)$, and the result follows. \hfill \Box
(5.4.4). Write $S$ for the image of $\phi$ and write $\xi^0_G$ for the connected center of $G$. Then $S \cdot \xi^0_G$ is a maximal torus of $N$.

Proof. Let $T \subset N$ be any maximal torus of $N$ containing $S$. Since $T$ commutes with the image of $\phi$, it follows that the space $\text{Lie}(C)(\phi; 2)$ is stable under $T$. But that space is one dimensional (5.2.5) and has $X$ as a basis vector; it follows that $X$ is a weight vector for $T$ so that $T$ lies in the stabilizer in $G$ of the line $[X] \in \mathbb{P}(\text{Lie}(G))$. We know by (5.2.4) that $\xi^0_G$ is a maximal torus of $C$; applying [Ja 04, 2.10 Lemma and Remark], one deduces that $S \cdot \xi^0_G$ is a maximal torus of that stabilizer, which completes the proof. □

Note that together (5.4.1), (5.4.3), and (5.4.4) yield Theorem B from the introduction.

(5.4.5). Consider the line $[X] \in \mathbb{P}(\text{Lie}(R_u C))$ and let $O$ be the $N$-orbit of $[X]$.

(a) The orbit mapping $(a \mapsto [\text{Ad}(a)X]): N \to O$ is smooth.
(b) The stabilizer $\text{Stab}_N([X])$ of $[X]$ in $N$ is smooth and is equal to $S \cdot C$.
(c) The $N$-orbit of $[X]$ is open and dense in $\mathbb{P}(\text{Lie}(R_u C))$.

Proof. Recall that a mapping $f: X \to Y$ between smooth varieties over $K$ is smooth if the tangent map $df_x$ is surjective for all closed points of $X$. If $X$ and $Y$ are homogeneous spaces for an algebraic group, it suffices to check that $df_x$ is surjective for one point $x$ of $X$.

Moreover, it follows from [Sp 98, Prop. 12.1.2] that if an algebraic group $H$ acts on a variety $X$, and if $x \in X$ is a closed point, then the stabilizer $\text{Stab}_H(x)$ is a smooth subgroup scheme if and only if the orbit mapping $H \to Hx$ determined by $x$ is a smooth morphism.

Now, assertion (a) is the content of [Mc 04, Lemma 23] As to (b), first note that the fact that the orbit mapping $N \to O$ is smooth shows that stabilizer $\text{Stab}_N([X])$ is smooth over $K$. Now, according to [Ja 04, 2.10] the stabilizer in $G$ of the line $[X]$ is $S \cdot C$. Since $S \cdot C$ is a closed subgroup of $N$, the remaining assertion of (b) follows.

For (c), notice that $\dim N/(S \cdot C) = \dim N/C - 1 = r - 1$ by (5.4.1). Since we have also $\dim \mathbb{P}(\text{Lie}(R_u C)) = r - 1$, it follows that the $N$-orbit of $[X]$ is open and dense in $\mathbb{P}(\text{Lie}(R_u C))$, as required. □

Let us write $D = \text{Stab}_N([X]) = S \cdot C$, and let $1$ be the closed point of $N/D$ determined by the trivial coset of $D$ in $N$. From the adjoint action of the torus $S$ on $\text{Lie}(N)$ one deduces an action of $S$ on the tangent space $T_1(N/D)$; thus one may speak of the weight spaces $T_1(N/D)(\phi; j)$ for $j \in \mathbb{Z}$.

(5.4.6). Assume that the derived group of $G$ is quasi-simple, and let the positive integers $k_1, k_2, \ldots, k_r$ be as in 5.1. Then we have the following:

$$T_1(N/D) = \bigoplus_{i=2}^r T_1(N/D)(\phi; 2k_i - 2)$$

Proof. Let $O \subset \mathbb{P}(\text{Lie}(R_u C))$ be the $N$-orbit of $[X]$. By (5.4.5)(c), one knows that $O$ is an open subset of $\mathbb{P}(\text{Lie}(R_u C))$; in particular, $T_{[X]}O = T_{[X]}\mathbb{P}(\text{Lie}(R_u C))$. Also by (5.4.5)(c), one knows that the orbit mapping $a: N \to O$ given by $a(y) = [\text{Ad}(y)X]$ induces an $S$-equivariant isomorphism $\tilde{a}: N/D \to O$. Since $\tilde{a}(1) = [X]$, the tangent map to $\tilde{a}$ at $1$ yields an $S$-isomorphism between $T_1(N/D)$ and $T_{[X]}O = T_{[X]}\mathbb{P}(\text{Lie}(R_u C))$. The assertion now follows from (5.2.5) and the description of the $S$-module structure on the tangent space $T_{[X]}\mathbb{P}(\text{Lie}(R_u C))$ given in (2.10.1). □

We can now complete the proofs of Theorems C and D from the introduction.

Proof of Theorem C. Consider the quotient morphism

$$\Phi: N/C \to N/(S \cdot C) = N/D$$

Alternatively, one can argue as follows. Write $\mathcal{L}$ for the tautological line bundle – corresponding to the invertible sheaf $O_{\mathbb{P}(\text{Lie}(R_u C))}(-1)$ – over $\mathbb{P}(\text{Lie}(R_u C))$. Then $\text{Lie}(R_u C) - \{0\}$ identifies with the total space of $\mathcal{L}$ with the zero-section removed. It follows that the natural mapping $(\text{Lie}(R_u C) - \{0\}) \to \mathbb{P}(\text{Lie}(R_u C))$ is flat and hence open.
and again write 1 for the closed point of $N/C$ determined by the trivial coset, and 1 for the closed point of $N/D$ determined by the trivial coset. Then differentiating $\Phi$ gives an $S$-equivariant mapping

$$d\Phi_1 : T_1(N/C) \to T_1(N/D).$$

Evidently the kernel of $d\Phi_1$ is the image of Lie($S$) in $T_1((N/C)$. Regard $T_1(N/C)$ as an $S$-module; by complete reducibility one can find an $S$-subrepresentation $V \subset T_1(N/C)$ which is a complement to ker $d\Phi_1$. Then evidently $d\Phi_1$ yields an isomorphism between $V$ and $T_1(N/D)$, and the assertion of Theorem C follows.

Proof of Theorem D. We must argue that $R_uN$ is defined over $K$ and split. Keep the preceding notations of this section; in particular, $S$ is the image of the cocharacter $\phi$ associated to the regular nilpotent element $X \in \text{Lie}(G)$. According to Theorem 2.9, it will suffice to show that Lie($S$) = Lie($N$)$^S$ and that each non-0 weight of $S$ on Lie($N$) is positive. It suffices to prove these statements after extending scalars; thus we may and will suppose that $K$ is algebraically closed.

If $G$ is any $D$-standard reductive group, we may find $D$-standard groups $M_1, \ldots, M_d$ together with a homomorphism $\Phi : M \to G$ where $M = \prod_{i=1}^d M_i$, satisfying (a)--(d) of (3.2.4).

Using (5.3.3) we may find a regular nilpotent element $X_1 \in \text{Lie}(M)$ such that -- writing $C_1 = C_M(X_1)$ -- the restriction $\Phi|_{C_1} : C_1 \to C = C_G(X)$ is surjective (and separable). Moreover, we may choose a cocharacter $\phi_1 : G_m \to M$ associated with $X_1$ such that $\phi = \Phi \circ \phi_1$ is associated with $X$. Write $S_1 \subset M$ for the image of $\phi_1$ and $S \subset G$ for the image of $\phi$.

Now, by (3.2.4)(a) each $M_i$ has quasisimple derived group. In the case where $M$ itself has quasisimple derived group -- i.e. if $M = M_1$ -- one uses (5.2.5) and Theorem C to deduce that

(i) $\text{Lie}(S_1) = \text{Lie}(N_1)^{S_1}$, and

(ii) the non-zero weights of $S_1$ on Lie($N_1$) are positive,

where we have written $N_1 = N_M(C_1)$. Since in general $M$ is a direct product of reductive groups each having quasisimple derived group, one sees readily that (i) and (ii) hold for $M$.

The normalizer $N_1 = N_M(C_1)$ is smooth by Theorem B. Since $\Phi$ is separable, it follows from (2.11.2) that $\Phi|_{N_1} : N_1 \to N$ is surjective and separable -- i.e. $d\Phi|_{N_1} : \text{Lie}(N_1) \to \text{Lie}(N)$ is surjective. Using the fact that (i) and (ii) hold together with the surjectivity of $d\Phi|_{N_1}$, one sees that Lie($S$) = Lie($N$)$^S$ and that the non-zero weights of $S$ on Lie($N$) are positive, and the proof is complete.

5.5. The tangent map to a Springer isomorphism. In this section, we give the proof of Theorem E. Thus we suppose in this section that the derived group of $G$ is quasisimple. We fix a Springer isomorphism $\sigma : N \overset{\sim}{\to} U$, and we write $u = \sigma(X)$ where $u \in G$ is regular unipotent and $X \in g$ is regular nilpotent.

Since $\sigma$ is $G$-equivariant, one knows that $C = C_G(X) = C_G(u)$.

(5.5.1). The restriction of $\sigma$ to Lie $R_uC$ determines an isomorphism $\gamma : \text{Lie } R_uC \overset{\sim}{\to} R_uC$. In particular, the tangent mapping $d\gamma = (d\gamma)_0$ determines an isomorphism $d\gamma : \text{Lie } R_uC \overset{\sim}{\to} \text{Lie } R_uC$.

Proof. Indeed, recall that $C$ is a smooth group scheme, and that $C = C_G : R_uC$ by (5.2.4), so that $R_uC$ is the space of fixed points of $\text{Int}(u)$ on $U$ and Lie $R_uC$ is the space of fixed points of $\text{Ad}(u)$ on $N$; the assertion is now immediate.

Write $V = \text{Lie } R_uC$. Then $d\gamma$ is an endomorphism of $V$ as an $N$-module, where $N$ is the normalizer in $G$ of $C$. As in §5.4, we fix a cocharacter $\phi$ associated to $X$; write $S \subset N$ for the image of $\phi$. We now give the

Proof of Theorem E. For (1), note first that the mapping $\gamma$ is in particular an $S$-module endomorphism of $V$. Since dim $V(\phi; 2) = 1$ by Theorem (5.2.5), one knows that $X$ spans $V(\phi; 2)$. It follows that $d\gamma(X) = ax$ for some $a \in K^\times$.

If now $Y \in V_{\text{reg}} = (\text{Lie } R_u(C))_{\text{reg}}$, there is an element $g \in N$ with $\text{Ad}(g)X = Y$; cf. (5.4.1). Then

$$d\gamma(Y) = d\gamma(\text{Ad}(g)X) = \text{Ad}(g)d\gamma(X) = ax\text{ Ad}(g)X = ax.$$
It follows that \( d\gamma \) and \( \alpha \cdot 1_V \) agree on the dense subset \( (\text{Lie}(U_\alpha))_{\text{reg}} \subset \text{Lie}(U_\alpha) \) so that indeed \( d\gamma = \alpha \cdot 1_V \).

For (2), recall that \( B \) is a Borel subgroup of \( G \) with unipotent radical \( U \). That \( \sigma|_{\text{Lie}U} \) is an isomorphism onto \( U \) follows from [Mc 05, Remark 10].

Now fix a Richardson element \( X \in \text{Lie}(U)(K) \); then \( X \) is a regular nilpotent element of \( g \), and part (1) shows that \( d\sigma|_{\text{Lie}U}(X) = \alpha X \) for some \( \alpha \in K^\times \). If \( Y = \text{Ad}(g)X \) for \( g \in B_{\text{alg}}(K) \), and it is then clear by the equivariance of \( d(\sigma|_{\text{Lie}U})_0(Y) \) that \( d(\sigma|_{\text{Lie}U})_0(Y) = \alpha Y \). Since the Richardson elements are dense in \( \text{Lie} U \), the result follows.

Note that Theorem E need not hold when the derived group of \( G \) fails to be quasi-simple. Indeed, take for \( G \) the \( D \)-standard group \( G = GL_n \times GL_m \) where \( n, m \geq 2 \). Then \( g = gl_n \oplus gl_m \), and the mapping

\[
(X, Y) \mapsto (1 + \alpha X, 1 + \beta Y)
\]

defines a Springer isomorphism \( \sigma \) for any \( \alpha, \beta \in K^\times \). If \( X_0 \in \text{gl}_n \) and \( Y_0 \in \text{gl}_m \) are regular nilpotent, then \( X = (X_0, Y_0) \in g \) is regular nilpotent; the mapping \( d\sigma \) has eigenvalues \( \alpha \) and \( \beta \) on \( \text{Lie}(U_\alpha) \) and hence is not a multiple of the identity if \( \alpha \neq \beta \).

We finally conclude with an argument which gives an alternate proof of (b) of Theorem A in case \( G \) has quasi-simple derived group. This argument does not rely on the fact that \( Z(C_1) \) is smooth; on the other hand, in order to make sense of \( Z(C_1)_{\text{red}} \), we are forced to assume \( K \) to be perfect.

(5.5.2). Let \( K \) be perfect, let \( X_1 \in \text{gl}(K) \) be nilpotent, and let \( C_1 = C_G(X_1) \) be its centralizer. Then the rule \( t \mapsto \sigma(tX_1) \) defines a mapping \( \Phi: \text{Aff}^1 \to Z(C_1)_{\text{red}} \), and \( X_1 = c \cdot d\Phi_0(1) \in \text{Lie}(Z(C_1)_{\text{red}}) \) for some \( c \in K^\times \).

Proof. Let \( u = \sigma(X_1) \) and observe that \( C_1 = C_G(u) \) by the \( G \)-equivariance of \( \sigma \), so in particular, \( u \in C_1 \). Then for each \( t \in \text{Aff}^1 \), and for each \( g \in C_1 \), we have

\[
g \cdot \sigma(tX_1) \cdot g^{-1} = \sigma(t \text{Ad}(g)X_1) = \sigma(tX_1).
\]

Since \( \text{Aff}^1 \) is reduced, it follows that \( \sigma(tX_1) \) indeed lies in \( Z(C_1)_{\text{red}} \).

The formula for the tangent mapping of \( \Phi \) is now immediate from Theorem E. \( \square \)

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