YET ANOTHER WAY OF CALCULATING MOMENTS OF THE
KESTEN’S DISTRIBUTION AND ITS CONSEQUENCES FOR
CATALAN NUMBERS AND CATALAN TRIANGLES

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ABSTRACT. We calculate moments of the so-called Kesten distribution by
means of the expansion of the denominator of the density of this distribution
and then integrate all summands with respect to the semicircle distribution.
By comparing this expression with the formulae for the moments of Kesten’s
distribution obtained by other means, we find identities involving polynomi-
als whose power coefficients are closely related to Catalan numbers, Catalan
triangles binomial coefficients.

1. INTRODUCTION

The purpose of this note is to calculate a sequence of moments of the Kesten’s
distribution and thus by comparison with the existing formulae to obtain some
polynomial type identities involving Catalan and some other sequences of numbers
related to them (see Proposition 1). In 2015 in [6] and in 2020 in [7] Szabłowski
calculated in two different ways the moments of Kesten distribution. Later T.
Hasegawa and S. Saito in [1] calculated these moments in some other ways and
equating the results obtained interesting identities involving Catalan and related
numbers. So in this note, we will calculate these moments in yet another way
and obtain some other identities, involving, surprisingly, other important numbers
sequences like Fibonacci and Lucas numbers.

2. BASIC INGREDIENTS

We start with the modified semicircle distribution i.e. distribution with the
density

\[ f_S(x) = \begin{cases} \frac{1}{2\pi} \sqrt{4-x^2} & \text{if } |x| \leq 2 \\ 0 & \text{if } |x| > 2 \end{cases} \]

It is well known that the moment sequence of \( f_S \) are the famous Catalan numbers.
More precisely, we have:

\[ \int_{-2}^{2} x^n f_S(x) dx = C_n = \frac{1}{n+1} \binom{2n}{n}. \]
We know also (see, e.g. [2] (4.8 p. 107)) that, after inserting proper values of parameters, the moment generating function see .e.g. of this distribution is equal to:
\[ g_S(z) = \frac{2}{\sqrt{1 - 4z^2 + 1}}, \]
for \(|z| \leq 1/2\). One can easily notice also that
\[ \sum_{k \geq 0} t^k C_k = \frac{2}{\sqrt{1 - 4t + 1}}, \]
for \(|t| \leq 1/4\). This is so since:
\[
\sum_{k \geq 0} \frac{2^k}{(2n)^k} \frac{C_k}{n^k} = O\left(\frac{1}{n^{1/2}}\right),
\]
for large \(n\).

The other ingredient is the definition of the Kesten distribution. It was considered in many papers including [3], [6], [7], and recently in [1] with different parametrization. Let us consider the Kesten distribution as parametrized basically as in [1] with parameter \(q\) replaced by \(r\). That is let us consider distribution with the following density
\[
f_{K}\left(x|p, r\right) = \begin{cases} 
0 & \text{if } |x| > 2\sqrt{r}, \\
\frac{p}{2\pi r} & \text{if } |x| \leq 2\sqrt{r},
\end{cases}
\]
for \(0 < p \leq 2r\).

Now notice that, if \(p = r\), then
\[ f_{K}(x|r, r) = \frac{1}{\sqrt{r}} f_{S}(x/\sqrt{r}). \]

3. Main results

Thus, we have
\[
\int_{-2\sqrt{r}}^{2\sqrt{r}} x^{2n} f_{K}(x|r, r) dx = r^{n} C_{n} = \frac{r^{n}}{n+1} \binom{2n}{n}.
\]
Now notice that for
\[
\left|\frac{p-r}{p^2}\right| x^2 \leq \left|\frac{p-r}{p^2}\right| 4r < 1,
\]
we have the following expansion:
\[
f_{K}(x|p, r) = \frac{r}{p} \sqrt{4r - x^2} \sum_{k \geq 0} \left(\frac{p-r}{p^2}\right)^k x^{2k}.
\]
Notice, that for \(|x| \leq 2\sqrt{r}\) and parameters satisfying (3.1) the convergence of the above-mentioned series, is uniform and one can integrate part by part. Then, we will get:
\[
M_{2m}(p, r) = \int_{-2\sqrt{r}}^{2\sqrt{r}} x^{2m} f_{K}(x|p, r) dx = \frac{r}{p} \sum_{k \geq 0} \left(\frac{p-r}{p^2}\right)^k r^{k+m} C_{k+m}.
\]
Let us introduce a new auxiliary variable
\[
t = \frac{r}{p}.
\]
General conditions on \( p \) and \( r \) require that \( t \geq 1/2 \). Recall that we must also have \(|t(1 - t)| < 1/4\), however taking into account (2.1) we can notice that (3.2) converges also for \(|t(1 - t)| = 1/4\). This leads to the following condition

\[
(3.4) \quad 1/2 \leq t \leq (1 + \sqrt{2})/2.
\]

Summarizing, we get the following result.

**Theorem 1.** Let Kesten distribution be defined by the density \( f_K \) given by (2.2). Let parameters \( p \) and \( r \) satisfy the following relationship

\[
2r \geq p \geq 2(\sqrt{2} - 1)r,
\]

then we have:

\[
(3.5) \quad M_{m}(p, r) = \frac{p^m}{(1 - t)^m}(1 - \sum_{k=0}^{m-1} t^{k+1} (1 - t)^k C_k),
\]

with \( t \) given by (3.3), with an obvious condition \(|t| \leq 1/4\) and \( t \neq 0\). For \( t = 0 \) we obviously have \( M_{m}(r, r) = r^m C_m \).

**Proof.** Keeping in mind that \( r/p = t \), let us analyze first when the series (3.2) is convergent. Namely, it is absolutely convergent if

\[
|t(1 - t)| \leq 1/4,
\]

because of (2.1), that is, when

\[
-\frac{1}{4} \leq t - t^2 \leq \frac{1}{4}.
\]

The inequality \( \frac{1}{2} - \left(\frac{1}{p}\right)^2 \leq \frac{1}{4} \) is equivalent to the following \( (\frac{1}{2} - \frac{1}{p})^2 \geq 0 \) which is true for all \( 2r \geq p \). The second one leads to inequality

\[
t^2 - t - \frac{1}{4} \leq 0,
\]

which leads to \( t < \frac{1 + \sqrt{2}}{2} \) or equivalently \( p > 2(\sqrt{2} - 1)r \) as shown.

We have also the following identity:

\[
\sum_{n \geq 0} t^n (1 - t)^n C_n = \frac{1}{t},
\]

Analyzing (3.2) we get, for \(|t| \leq 1/4\) and \( t \neq 0\).

\[
M_{2m}(p, r) = \frac{r}{p} \frac{p^{2m}}{(p - r)^m} \sum_{k \geq 0} t^{k+m} (1 - t)^{k+m} C_{k+m}
\]

\[
= \frac{r}{p} \frac{p^m}{(1 - t)^m} \left( \sum_{k \geq 0} t^k (1 - t)^k C_k - \sum_{k \neq 0} t^k (1 - t)^k C_k \right)
\]

\[
= p^m \frac{t}{(1 - t)^m} \left( \frac{1}{t} - \sum_{k \neq 0} t^k (1 - t)^k C_k \right).
\]

Now it suffices to multiply the expression in the bracket by \( t \).
Remark 1. Formula (3.3) can be derived from the unnumbered formula placed in Comment 1 of [1] in the following way (keeping in mind that \(q\) of [1] is replaced by \(r\) in this paper):

\[
M_{2k} = \frac{p}{p-r} \left( \frac{p^{2k-1}}{(p-r)^{k-1}} - \sum_{m=1}^{k} \frac{(2m-1)^{m-1} p^m}{m} \frac{p^2(k-m)}{(p-r)^{k-m}} \right) = \frac{p}{p-r} \left( \frac{p^{2k-1}}{(p-r)^{k-1}} - \sum_{j=0}^{k-1} C_j r^{j+1} \frac{p^{2(k-j-1)}}{(p-r)^{k-j-1}} \right) = \frac{p^{2k}}{(p-r)^k} \left( 1 - \sum_{j=0}^{k-1} C_j r^{j+1} \frac{1}{p} \left( \frac{p-r}{p^2} \right)^j \right).
\]

Now it remains to notice that \(\frac{1}{m} \binom{2m-1}{m-1} = C_{m-1}\) and replace \(p/r\) by \(t\). This calculation was done by the referee in his report.

Let us underline the important property of the sequences that we are considering, namely, that the sequences:

\[
\{M_{2m}\}_{m \geq 0} = \left\{ \frac{1}{(1-t)^m} \left( 1 - \sum_{k=0}^{m-1} t^{k+1} (1-t)^k \right) C_k \right\}_{m \geq 0}
\]

and also the sequences for \(d \in [0, 1]\)

\[
\left\{ (1-d) \sum_{k=0}^{m-1} t^{k+1} (1-t)^k C_k \right\}_{m \geq 0},
\]

are the moment sequences. The last statement follows the fact that the product of two moment sequences is another moment sequence and that the convex combination of two momentous sequences is another moment sequence.

Remark 2. To be honest, I must reveal that I was able to see the first version of the paper of Hasegawa and Saito. In this version, there were not present Comment 1 and Comment 2. It was in May and June 2021. Their paper has inspired me to write this paper. To clarify everything the final form of the paper [1] appeared at the end of October 2021. In the final form, the authors included three Comments. In the first Comment the formula from the first line of the Remark 1 appeared while in Comment 2 the expansion (3.2) appeared. In the third comment, the authors stated that these formulae are promising and that they will research further on these formulae. Anyway, the Remark 1 indicates that to get the crucial formula (3.5) one did not need to exploit the new way of calculating even moments of Kesten distribution.

Let us now compare this result with known formulae for the moments of Kesten distribution.

Let us notice that distribution given by (2.2) is in fact equal to the distribution \(f_{\text{CN}}(x|0, \rho, 0)\) considered in [6] with replacement \(\rho^2 - > 1 - p/r\). There, i.e. in [6] (Proposition 3[i]), the following result has been obtained, after a necessary change of parameters, \(\rho^2 = 1 - p/r, \ y = 0, \ q = 0\) and \(x\) replaced by \(x/\sqrt{r}\):

\[
M_{2m}(p, r) = r^m \sum_{k=0}^{m} \binom{p}{p-1}^{m-k} S_{m,k},
\]
where
\[ S_{m,k} = \binom{2m}{k} - \binom{2m}{k-1}, \]
with understanding that \( \binom{2m}{-1} = 0 \). Hence in terms of our parameter \( t \) we have:
\[
M_{2m}(p,r) = p^m \sum_{k=0}^{m} \left( -\frac{r}{p} + 1 \right)^{m-k} \binom{r}{p}^k S_{m,k}
\]
(3.6)
\[
= p^m \sum_{k=0}^{m} t^k (1-t)^{m-k} S_{m,k}.
\]
In [1] two other expressions for the moments of Kesten distribution can be found. Namely the following formulae:
\[
M_{2m}(p,r) = p^m \sum_{j=0}^{m-1} p^{m-1-j} r^j T_{m-1,j},
\]
(3.7)
\[
M_{2m}(p,r) = p^m \sum_{j=0}^{m-1} (p-r)^j r^{m-1-j} B_{m,j+1},
\]
(3.8)
where numbers \( T_{m,j} \) and \( B_{m,j} \) are called Catalan triangles, depending on the author. The first one is given by
\[
T_{m,j} = \frac{m-j+1}{m+1} \binom{m+j}{m}
\]
for integer \( m \geq j \geq 0 \) and the related sequence is A009766 in Sloane’s Encyclopedia [4], the second, introduced by L.W. Shapiro in [5] and with related sequence A039598 in OEIS [4], is given by:
\[
B_{k,j} = \frac{2k}{k-1} \binom{2k}{k-j}.
\]
For integer \( k \geq j \geq 1 \). Hence, we have the following result:

**Proposition 1.** i) For all \( m \geq 1 \) and \( t \in \mathbb{C} \) we have:
\[
(1-t)^m \sum_{k=0}^{m} S_{m,k} t^k (1-t)^{m-k} = (1-t)^m \sum_{k=0}^{m-1} T_{m-1,k} t^k =
\]
(3.9)
\[
= (1-t)^m \sum_{k=0}^{m} B_{m,k+1} (1-t)^k t^{m-1-k} = 1 - \sum_{k=0}^{m-1} C_k t^{k+1} (1-t)^k.
\]
(3.10)
ii) For all \( m \geq 1 \) and \( x \in \mathbb{C} \):
\[
\sum_{k=0}^{m} S_{m,k} x^k = (x+1) \sum_{k=0}^{m-1} B_{m,k+1} x^{m-1-k}
\]
(3.11)
\[
= \sum_{k=0}^{m-1} T_{m-1,k} x^k (x+1)^{m-k} = (x+1)^2 m - \sum_{k=0}^{m-1} C_k x^{k+1} (x+1)^{2m-2k-1}.
\]
(3.12)

**Proof.** i) Firstly, for \( t \) satisfying (3.4) we have from (3.5) and (3.6):
\[
\frac{(1-t)^m}{p^m} M_{2m} = (1-t)^m \sum_{k=0}^{m} S_{m,k} t^k (1-t)^{m-k} = 1 - \sum_{k=0}^{m-1} t^{k+1} (1-t)^k C_k.
\]
Now with parametrization $t = r/p$ formulae (3.7) and (3.8) give:

$$
\frac{(1 - t)^m}{p^m} M_{2m} = (1 - i)^m \sum_{k=0}^{m-1} T_{m-1,k} t^k
$$

$$
= (1 - t)^m \sum_{k=0}^{m-1} B_{m,k+1} t^{m-1-k}(1 - t)^k.
$$

Therefore, we obtain the chain of equalities given by (3.9) and (3.10). Finally we observe that all these equalities involve polynomials in $t$, so we extend their domain from any segment to all complex numbers and conclude that they hold for all $t \in \mathbb{C}$.

ii) Having proved i) we consider $x = t/(1-t)$, with $t \neq 1$. Then $t = x/(x+1)$ and we consider the identities (3.9) and (3.10) for $x \neq -1$. Now we multiply both sides of each of them by $(1 + x)^{2m}$. We get immediately forms (3.11) and (3.12). Now again we deal with polynomials hence we can drop assumption that $x \neq -1$. □

4. Applications

Now we can use the above-mentioned identities in different ways for different purposes concerning relationships between sets of Catalan numbers, Catalan triangles, binomial coefficients, Fibonacci and Lucas numbers.

In doing so, the following new parametrization will help. Namely, we will formulate the identities (3.9) and (3.10) under a new parametrization in the following corollary. The other corollary presents consequences of (3.11) and (3.12).

**Corollary 1.** Consider identities (3.9) and (3.10). Let us set $z = t(1 - t)$, which leads to $t = (1 + \sqrt{1 - 4z})/2$, $1 - t = (1 - \sqrt{1 - 4z})/2$ and $t/(1-t) = -1 + \frac{1}{1 + \sqrt{1 - 4z}}$. For particular values of $z$ we get:

a) for $z = -2$ we have $t = 2$, $1 - t = -1$ and $t/(1-t) = -2$ and finally we get for all $m \geq 1$ :

$$
1 - 2 \sum_{k=0}^{m-1} (-2)^k C_k = \sum_{k=0}^{m} (-2)^k S_{m,k}
$$

$$
= - \sum_{k=0}^{m-1} (-2)^{m-1-k} B_{m,k+1} = (-1)^m \sum_{k=0}^{m-1} 2^k T_{m-1,k},
$$

b) for $z = 1$ we have $t = (1 + i\sqrt{3})/2 = e^{i\pi/3}$, $1 - t = (1 - i\sqrt{3})/2 = e^{-i\pi/3}$ and $t/(1-t) = (-1 + i\sqrt{3})/2 = e^{i2\pi/3}$ and finally we get for all $m \geq 1$, when considering real parts:

$$
1 - \frac{1}{2} \sum_{k=0}^{m-1} C_k = \sum_{k=0}^{m-1} \cos((m - k)\pi/3) T_{m-1,k}
$$

$$
= \sum_{k=0}^{m} \cos(2(m - k)\pi/3) S_{m,k} = \sum_{k=0}^{m-1} \cos((2k + 1)\pi/3) B_{m,k+1},
$$

while, when considering imaginary parts, we get:

\[ \sum_{k=0}^{m-1} C_k = \frac{2\sqrt{3}}{3} \sum_{k=0}^{m-1} \sin((m - k)\pi/3)T_{m-1,k} \]

\[ \frac{2\sqrt{3}}{3} \sum_{k=0}^{m} \sin(2(m - k)\pi/3)S_{m,k} = \frac{2\sqrt{3}}{3} \sum_{k=0}^{m-1} \sin((2k + 1)\pi/3)B_{m+1,k}, \]

c) for \( z = -1 \) we have \( t = (1 + \sqrt{5})/2 \) and \( 1 - t = (1 - \sqrt{5})/2 \) and we get for all \( m \geq 1 \) the following six identities:

\[ \frac{1}{2} \sum_{k=0}^{m} S_{m,k}(L_kL_{m-k} - 5F_kF_{m-k}) = \sum_{k=0}^{m-1} T_{m-1,k}L_k \]

\[ = \frac{1}{2} \sum_{k=0}^{m-1} B_{m,k+1}(L_kL_{m-1-k} - 5F_kF_{m-1-k}) \]

\[ = (-1)^mL_m + (-1)^{m+1}L_{m+1} \sum_{k=0}^{m-1} (-1)^kC_k. \]

and

\[ \frac{1}{2} \sum_{k=0}^{m} (-1)^kS_{m,k}((-1)^mF_{2k-m} - F_{m-2k}) = \sum_{k=0}^{m-1} T_{m-1,k}F_k \]

\[ = \frac{1}{2} \sum_{k=0}^{m-1} (-1)^kB_{m,k+1}(F_{m-1-2k} + (-1)^mF_{2k-m+1}) \]

\[ = (-1)^mF_m - (-1)^{m+1}F_{m+1} \sum_{k=0}^{m-1} (-1)^kC_k. \]

where \{L_m\}_{m \geq 0} are the so-called Lucas numbers (sequence A000032 in OEIS [4]), while \{F_m\}_{m \geq 0} are the so-called Fibonacci numbers (sequence A000045 in OEIS [4]).

Proof. a) does not require justification. In the proofs of b) and c) we exploit the fact that the values of polynomials \( t^k \) and \( (1 - t)^m \) belong to quadratic fields respectively \( Q(\sqrt{-3}) \) and \( Q(\sqrt{5}) \) which are linear spaces of dimension 2. Hence, each element of these fields is a linear combination with rational coefficients of the elements of the basis of these fields. The basis of these fields are respectively: in case b) \{(1, 0), (0, \sqrt{-3})\} and \{(1, 0), (0, \sqrt{5})\}. Thus crucial are the following expansions: for the case b) \n
\[ \left( \frac{1 + \sqrt{-3}}{2} \right)^k = e^{ik\pi/3} = \cos(k\pi/3) + \sqrt{-3}(\sin(k\pi/3)/\sqrt{3}), \]

\[ \left( \frac{1 - \sqrt{-3}}{2} \right)^k = e^{-ik\pi/3} = \cos(k\pi/3) - \sqrt{-3}(\sin(k\pi/3)/\sqrt{3}), \]

\[ \left( \frac{-1 + \sqrt{-3}}{2} \right)^k = e^{2k\pi/3} = \cos(2k\pi/3) + \sqrt{-3}(\sin(2k\pi/3)/\sqrt{3}), \]
and for the case c)

\[
\left( \frac{1 + \sqrt{5}}{2} \right)^k = \frac{L_n}{2} + \frac{\sqrt{5} F_n}{2},
\]

\[
\left( \frac{1 - \sqrt{5}}{2} \right)^k = \frac{L_n}{2} + \frac{\sqrt{5} F_n}{2}.
\]

In case b) we get directly from (3.9) and (3.10):

\[
1 - e^{i\pi/3} \sum_{k=0}^{m-1} C_k = e^{-im\pi/3} \sum_{k=0}^{m-1} e^{ik\pi/3} T_{m-1,k}
\]

\[
= e^{-2im\pi/3} \sum_{k=0}^{m} e^{2ik\pi/3} S_{m,k} = e^{-i(2m-1)\pi/3} \sum_{k=0}^{m-1} e^{2(2m-1-k)\pi/3} B_{m,k+1}.
\]

Now it remains to apply (4.7), (4.8) and (4.9) and in case of imaginary parts cancel both sides by $\sqrt{3}$.

In case c) we get directly from (3.9) and (3.10):

\[
\frac{L_m - F_m \sqrt{5}}{2} \sum_{k=0}^{m} S_{m,k} \frac{L_k + F_k \sqrt{5}}{2} = \frac{L_m - F_m \sqrt{5}}{2} \sum_{k=0}^{m-1} T_{m-1,k} \frac{L_k + F_k \sqrt{5}}{2} = \frac{L_m - F_m \sqrt{5}}{2} \sum_{k=0}^{m-1} B_{m,k+1} \frac{L_k - F_k \sqrt{5} F_{m-1-k} + F_{m-1-k} \sqrt{5}}{2}.
\]

Now we divide all sides by $\frac{L_m - F_m \sqrt{5}}{2}$ getting

\[
\frac{1}{\frac{L_m - F_m \sqrt{5}}{2}} = (-1)^m \frac{\left(\frac{L_m + F_m \sqrt{5}}{2}\right)}{\frac{L_m - F_m \sqrt{5}}{2}}.
\]

Then, we perform multiplications inside sums and equate firstly coefficients free of $\sqrt{5}$ the secondly with $\sqrt{5}$. When considering elements free of $\sqrt{5}$ we utilize well-known identities, for all integer $n,k$

\[
L_n F_k = F_{n+k} + (-1)^k F_{k-n}, \quad F_{-k} = (-1)^{k-1} F_k.
\]

In this way we get (4.4), (4.5) and (4.6).

□

Remark 3. The following functions of $m$: $\cos(m\pi/3)$, $\cos(2m\pi/3)$, $\frac{2\sqrt{3}}{3} \sin(m\pi/3)$ and $\frac{2\sqrt{3}}{3} \sin(2m\pi/3)$ are periodic with periods equal to 6. Moreover, we have: $\cos(m\pi/3) \in \{\pm 1/2, \pm 1\}$ and $\frac{2\sqrt{3}}{3} \sin(m\pi/3) \in \{0, \pm 1\}$.

Corollary 2. Consider identities (3.11) and (3.12). For particular values of $x$ we get:
a) when $x = 1$ we get for $m \geq 1$:
\[
\sum_{k=0}^{m} S_{m,k} = 2 \sum_{k=0}^{m-1} B_{m,k+1} = \sum_{k=0}^{m-1} T_{m-1,k} 2^{m-k}
\]
\[
= 4^m - \sum_{k=0}^{m-1} C_k 2^{2m-2k-1},
\]

b) when $x = -1/2$ we get after multiplying all sums by $2^m$ for $m \geq 1$:
\[
\sum_{k=0}^{m} (-1)^k S_{m,k} 2^{m-k} = \sum_{k=0}^{m-1} (-1)^{m-1-k} B_{m,k+1} 2^k = \sum_{k=0}^{m-1} (-1)^k T_{m-1,k}
\]
\[
= \frac{1}{2m} \sum_{k=0}^{m-1} (-1)^{k+1} C_k \frac{1}{2^{m-k}}.
\]

c) Let us divide both sides of (3.11) and (3.12) by $(x+1)$ and let us pass to the limit $x \to -1$. We get then for all $m \geq 1$
\[
\lim_{x \to -1} \left( \sum_{k=0}^{m} S_{m,k} x^k \right)/(x+1) = \sum_{k=0}^{m} (-1)^{k+1} S_{m,k} k
\]
\[
= \sum_{k=0}^{m-1} (-1)^{m-1-k} B_{m,k+1} = (-1)^{m-1} T_{m-1,m-1} = (-1)^{m-1} C_{m-1}.
\]

Proof. a) is immediate. In proving c) we used the fact that \(\lim_{x \to -1} (x^n - (-1)^n)/(x+1) = (-1)^{n+1} n\). b) requires observation that before multiplying all sums by $2^m$ the sum involving Catalan numbers was as follows
\[
2^{-2m} \sum_{k=0}^{m-1} C_k (-1)^{k+1} 2^{-k} 2^{-2m+2k+1} = 2^{-2m} - 2^{-m} \sum_{k=0}^{m-1} (-1)^{k+1} C_k 2^{-m+k}.
\]

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