ON SELMER GROUPS OF ABELIAN VARIETIES OVER ℓ-ADIC LIE EXTENSIONS OF GLOBAL FUNCTION FIELDS

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Abstract. Let $F$ be a global function field of characteristic $p > 0$ and $A/F$ an abelian variety. Let $K/F$ be an ℓ-adic Lie extension ($\ell \neq p$) unramified outside a finite set of primes $S$ and such that $\text{Gal}(K/F)$ has no elements of order $\ell$. We shall prove that, under certain conditions, $\text{Sel}_A(K)_\ell\vee$ has no nontrivial pseudo-null submodule.

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1. Introduction

Let $G$ be a compact ℓ-adic Lie group and $\Lambda(G)$ its associated Iwasawa algebra. A crucial theme in Iwasawa theory is the study of finitely generated $\Lambda(G)$-modules and their structure, up to “pseudo-isomorphism”. When $G \cong \mathbb{Z}^d_\ell$ for some integer $d \geq 1$, the structure theory for finitely generated $\Lambda(G)$-modules is well known (see, e.g., [3]). For a nonabelian $G$, which is the case we are interested in, studying this topic is possible thanks to an appropriate definition of the concept of “pseudo-null” for modules over $\Lambda(G)$ due to Venjakob (see [17]).

Let $F$ be a global function field of trascendence degree one over its constant field $\mathbb{F}_q$, where $q$ is a power of a fixed prime $p \in \mathbb{Z}$, and $K$ a Galois extension of $F$ unramified outside a finite set of primes $S$ and such that $G = \text{Gal}(K/F)$ is an infinite ℓ-adic Lie group ($\ell \in \mathbb{Z}$ a prime different from $p$). Let $A/F$ be an abelian variety: in [2], we proved that $S := \text{Sel}_A(K)_{\ell}\vee$ (the Pontrjagin dual of the Selmer group of $A$ over $K$) is finitely generated over $\Lambda(G)$ and here we shall deal with the presence of nontrivial pseudo-null submodules in $S$. For the number field setting and $K = F(A[\ell\infty])$, this issue was studied by Ochi and Venjakob ([13, Theorem 5.1]) when $A$ is an elliptic curve, and by Ochi for a general abelian variety in [12].

In Sections 2 and 3 we give a brief description of the objects we will work with and of the main tools we shall need, adapting some of the techniques of [13] to our function field setting and to a general ℓ-adic Lie extension (one of the main difference being the triviality of the image of the local Kummer maps).

In Section 4 we will prove the following

Theorem 1.1 (Theorem 4.1). Let $G = \text{Gal}(K/F)$ be an ℓ-adic Lie group without elements of order $\ell$ and of positive dimension $d \geq 3$. If $H^2(F_S/K, A[\ell\infty]) = 0$ and the map $\psi$ (induced by restriction)

\[ \text{Sel}_A(K)_{\ell} \hookrightarrow H^1(F_S/K, A[\ell\infty]) \xrightarrow{\psi} \bigoplus_S \text{Coind}^G_{G_v} H^1(K_w, A)[\ell\infty] \]

is surjective, then $\text{Sel}_A(K)_{\ell}\vee$ has no nontrivial pseudo-null submodule.

For the case $d = 2$ we need more restrictive hypotheses, in particular we have the following

Proposition 1.2 (Proposition 4.3). Let $G = \text{Gal}(K/F)$ be an ℓ-adic Lie group without elements of order $\ell$ and of dimension $d \geq 2$. If $H^2(F_S/K, A[\ell\infty]) = 0$ and $\text{cd}_{\ell}(G_v) = 2$ for any $v \in S$, then $\text{Sel}_A(K)_{\ell}\vee$ has no nontrivial pseudo-null submodule.
A few considerations and particular cases for the vanishing of $H^2(F_S/K, A[\ell^\infty])$ are included at the end of Section 4.

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2. Setting and notations

Here we fix notations and conventions that will be used throughout the paper.

2.1. Fields and extensions. Let $F$ be a global function field of trascendence degree one over its constant field $\mathbb{F}_p = \mathbb{F}_q$, where $q$ is a power of a fixed prime $p \in \mathbb{Z}$. We put $\overline{F}$ for an algebraic closure of $F$.

For any algebraic extension $L/F$, let $\mathfrak{M}_L$ be the set of places of $L$: for any $v \in \mathfrak{M}_L$ we let $L_v$ be the completion of $L$ at $v$. Let $S$ be a finite nonempty subset of $\mathfrak{M}_F$ and let $F_S$ be the maximal Galois extension of $F$ unramified outside $S$ with $G_S(F) := \text{Gal}(F_S/F)$. Put $\mathfrak{O}_{L,S}$ as the ring of $S$-integers of $L$ and $O_S^+$ as the units of $O_S = \bigcup_{L \in \mathfrak{O}_{L,S}}$. Finally, $C\ell_L(L)$ denotes the $S$-ideal class group of $\mathfrak{O}_{L,S}$: since $S$ is nonempty, $C\ell_L(L)$ is finite.

Let $\mathfrak{M}_F$ be the completion of $F$ at any place $v \in \mathfrak{M}_F$ we choose (and fix) an embedding $\overline{F} \rightarrow \overline{F}_v$, in order to get a restriction map $G_{F_v} := \text{Gal}(\overline{F}_v/F_v) \rightarrow G_F := \text{Gal}(\overline{F}/F)$.

We will deal with $\ell$-adic Lie extensions $K/F$, i.e., Galois extensions with Galois group an $\ell$-adic Lie group with $\ell \neq p$. We always assume that our extensions are unramified outside a finite set $S$ of primes of $\mathfrak{M}_F$.

In what follows $\text{Gal}(K/F)$ is an $\ell$-adic Lie group without points of order $\ell$, then it has finite $\ell$-cohomological dimension, which is equal to its dimension as an $\ell$-adic Lie group ([15, Corollaire (1) p. 413]).

2.2. Ext and duals. For any $\ell$-adic Lie group $G$ we denote by

$$\Lambda(G) = \mathbb{Z}_\ell[[G]] := \lim_{\overrightarrow{U}} \mathbb{Z}_\ell[G/U]$$

the associated Iwasawa algebra (the limit is on the open normal subgroups of $G$). From Lazard’s work (see [9]), we know that $\Lambda(G)$ is Noetherian and, if $G$ is pro-$\ell$ and has no elements of order $\ell$, then $\Lambda(G)$ is an integral domain.

For a $\Lambda(G)$-module $M$ we consider the extension groups

$$E^i(M) := \text{Ext}^i_{\Lambda(G)}(M, \Lambda(G))$$

for any integer $i$ and put $E^i(M) = 0$ for $i < 0$ by convention.

Since in our applications $G$ comes from a Galois extension, we denote with $G_v$ the decomposition group of $v \in \mathfrak{M}_F$ for some prime $w|v$, $w \in \mathfrak{M}_L$, and we use the notation

$$E^i_v(M) := \text{Ext}^i_{\Lambda(G_v)}(M, \Lambda(G_v)).$$

Let $H$ be a closed subgroup of $G$. For every $\Lambda(H)$-module $N$ we consider the $\Lambda(G)$-modules

$$\text{Coid}^i_H(N) := \text{Map}_{\Lambda(H)}(\Lambda(G), N) \quad \text{and} \quad \text{Ind}^G_H(N) := N \otimes_{\Lambda(H)} \Lambda(G).$$

For a $\Lambda(G)$-module $M$, we denote its Pontrjagin dual by $M^\vee := \text{Hom}_{\text{cont}}(M, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$. In this paper, $M$ will be a (mostly discrete) topological $\mathbb{Z}_\ell$-module, so $M^\vee$ has a natural structure of $\mathbb{Z}_\ell$-module.

1We use the notations of [13], some texts, e.g. [11], switch the definitions of $\text{Ind}^G_H(N)$ and $\text{Coid}^i_H(N)$. 
If $M$ is a discrete $G_S(F)$-module, finitely generated over $\mathbb{Z}$ and with no $p$-torsion, in duality theorems we shall use also the dual $G_S(F)$-module of $M$, i.e.,

$$M' := \text{Hom}(M, \mathcal{O}_S^\times) = \text{Hom}(M, \mu)$$

if $M$ is finite).

### 2.3. Selmer groups.

Let $A$ be an abelian variety of dimension $g$ defined over $F$: we denote by $A'$ its dual abelian variety. For any positive integer $n$ we let $A[n]$ be the scheme of $n$-torsion points and, for any prime $\ell$, we put $A[\ell^n] := \lim_{\rightarrow} A[\ell^n]$.

The local Kummer maps (for any $w \in \mathfrak{M}_L$)

$$\kappa_w : A(L_w) \otimes \mathbb{Q}_\ell / \mathbb{Z}_\ell \hookrightarrow \lim_{\rightarrow} H^1(L_w, A[\ell^n]) := H^1(L_w, A[\ell^n])$$

( ARISING FROM THE COHOMOLOGY OF THE EXACT SEQUENCE $A[\ell^n] \hookrightarrow A \twoheadrightarrow A$) enable one to define the $\ell$-part of the Selmer group of $A$ over $L$ as

$$Sel_A(L)_\ell = \text{Ker} \left\{ H^1(L, A[\ell^n]) \to \prod_{w \in \mathfrak{M}_L} H^1(L_w, A[\ell^n])/\text{Im} \kappa_w \right\}$$

(where the map is the product of the natural restrictions between cohomology groups).

For infinite extensions $L/F$ the Selmer group $Sel_A(L)_\ell$ is defined, as usual, via direct limits.

Since $\ell \neq p$, the $\text{Im} \kappa_w$ are trivial and, assuming that $S$ contains also all primes of bad reduction for $A$, we have the following equivalent

**Definition 2.1.** The $\ell$-part of the Selmer group of $A$ over $L$ is

$$Sel_A(L)_\ell = \text{Ker} \left\{ H^1(F_S/L, A[\ell^n](F_S)) \to \bigoplus_S \text{Coind}_{G_w}^G H^1(L_w, A[\ell^n]) \right\}.$$ 

Letting $L$ vary through subextensions of $K/F$, the groups $Sel_A(L)_\ell$ admit natural actions by $\mathbb{Z}_\ell$ (because of $A[\ell^n]$) and by $G = \text{Gal}(K/F)$. Hence they are modules over the Iwasawa algebra $\Lambda(G)$.

### 3. Homotopy theory and pseudo-nullity

We briefly recall the basic definitions for pseudo-null modules over a non-commutative Iwasawa algebra: a comprehensive reference is [17].

#### 3.1. Pseudo-null $\Lambda(G)$-modules.

Let $G$ be an $\ell$-adic Lie group without $\ell$-torsion, then $\Lambda(G)$ is an Auslander regular ring of finite global dimension $\delta = \text{cd}_\ell(G) + 1$ ([17, Theorem 3.26], $\text{cd}_\ell$ denotes the $\ell$-cohomological dimension).

For any finitely generated $\Lambda(G)$-module $M$, there is a canonical filtration

$$T_0(M) \subseteq T_1(M) \subseteq \cdots \subseteq T_{\delta-1}(M) \subseteq T_\delta(M) = M.$$ 

**Definition 3.1.** We say that a $\Lambda(G)$-module $M$ is pseudo-null if

$$\delta(M) := \min\{i \mid T_i(M) = M\} \leq \delta - 2.$$ 

The quantity $\delta(M)$, called the $\delta$-dimension of the $\Lambda(G)$-module $M$, is used along with the grade of $M$, that is

$$j(M) := \min\{i \mid E^i(M) \neq 0\}.$$ 

As $j(M) + \delta(M) = \delta$ ([17, Proposition 3.5 (ii)]) we have that $M$ is a pseudo-null module if and only if $E^0(M) = E^1(M) = 0$.

Since $\delta(T_i(M)) \leq i$ and every $T_i(M)$ is the maximal submodule of $M$ with $\delta$-dimension less
or equal to \( i \) ([17, Proposition 3.5 (vi) (a)]), only \( T_0(M), \ldots, T_{d-2}(M) \) can be pseudo-null. If \( T_0(M) = \cdots = T_{d-2}(M) = 0 \), \( M \) does not have any nonzero pseudo-null submodule. This is the case when \( E^iE'(M) = 0 \) for all \( i \geq 2 \) ([17, Proposition 3.5 (i) (c)]).

### 3.2. The powerful diagram and its consequences.

In [13, Lemma 4.5] Ochi and Venjakob generalized a result of Jannsen (see [8]) which is very powerful in applications (they call it “powerful diagram”). We provide here the statements we shall need later: for the missing details of the proofs the reader can consult [11, Chapter V, Section 5] and/or [13, Section 4]².

Replacing, if necessary, \( F \) by a finite extension we can (and will) assume that \( K \) is contained in the maximal pro-\( \ell \)-extension of \( F_\infty := F(A[\ell^\infty]) \) unramified outside \( S \). Then we have the following diagram:

\[
\begin{array}{ccc}
F_\infty & \xrightarrow{G} & F_

\end{array}
\]

where \( \Omega \) is the maximal pro-\( \ell \)-extension of \( F_\infty \) contained in \( F_S \). We put \( \mathcal{G} = \text{Gal}(\Omega/F) \), \( \mathcal{H} = \text{Gal}(\Omega/K) \) and \( G = \text{Gal}(K/F) \). The extension \( F_\infty/F \) will be called the trivializing extension.

Tensoring the natural exact sequence \( I(\mathcal{G}) \hookrightarrow \Lambda(\mathcal{G}) \to Z_\ell \), one gets

\[
I(\mathcal{G}) \otimes Z_\ell A[\ell^\infty]^\vee \hookrightarrow \Lambda(\mathcal{G}) \otimes Z_\ell A[\ell^\infty]^\vee \to A[\ell^\infty]^\vee.
\]

Since the mid term is projective ([13, Lemma 4.2]), the previous sequence yields

\[
H_1(\mathcal{H}, A[\ell^\infty]^\vee) \hookrightarrow (I(\mathcal{G}) \otimes Z_\ell A[\ell^\infty]^\vee)_\mathcal{H} \to (\Lambda(\mathcal{G}) \otimes Z_\ell A[\ell^\infty]^\vee)_\mathcal{H} \to (A[\ell^\infty]^\vee)_\mathcal{H}.
\]

In order to shorten notations we put:

- \( X = H_1(\mathcal{H}, A[\ell^\infty]^\vee); \)
- \( Y = (I(\mathcal{G}) \otimes Z_\ell A[\ell^\infty]^\vee)_\mathcal{H}; \)
- \( J = \ker \{(\Lambda(\mathcal{G}) \otimes Z_\ell A[\ell^\infty]^\vee)_\mathcal{H} \to (A[\ell^\infty]^\vee)_\mathcal{H}\}. \)

So the sequence (1) becomes

\[
(2) \quad X \hookrightarrow Y \to J.
\]

For our purpose it is useful to think of \( X \) as \( H^1(F_\infty/K, A[\ell^\infty]^\vee) \) (note that \( H_1(\mathcal{H}, A[\ell^\infty]^\vee) \approx H^1(\Omega/K, A[\ell^\infty]^\vee) \approx H^1(F_\infty/K, A[\ell^\infty]^\vee) \)).

Let \( \mathcal{F}(d) \) denote a free pro-\( \ell \)-group of rank \( d = \dim \mathcal{G} \) and denote by \( N \) (resp. \( \mathcal{R} \)) the kernel of the natural map \( \mathcal{F}(d) \to \mathcal{G} \) (resp. \( \mathcal{F}(d) \to G \)). For any profinite group \( H \), we denote by \( H^{\text{ab}}(\ell) \) the maximal pro-\( \ell \)-quotient of the maximal abelian quotient of \( H \). With this notations

²Those results hold in our setting as well because we work with the \( \Lambda(G) \)-module \( A[\ell^\infty] \), with \( \ell \neq p \).
the powerful diagram reads as
\[
\begin{align*}
H^2(\mathcal{H}, A[\ell^\infty])^\vee \xrightarrow{\sim} (H^1(N^{ab}(\ell), A[\ell^\infty]))^\vee & \xrightarrow{H^1(\mathbb{R}, A[\ell^\infty])} H^1(\mathbb{R}, A[\ell^\infty]) \xrightarrow{X} \\
H^2(\mathcal{H}, A[\ell^\infty])^\vee \xrightarrow{\sim} (N^{ab}(\ell) \otimes A[\ell^\infty])^\vee_{\mathcal{H}} & \xrightarrow{\Lambda(G)^{2d}} Y \xrightarrow{J} J.
\end{align*}
\]

Moreover, since \(cd_\ell(\mathfrak{g}) \leq 2\) (just use [11, Theorem 8.3.17] and work as in [13, Lemma 4.4, (iv)])], the module \(N^{ab}(\ell) \otimes A[\ell^\infty])^\vee\) is free over \(\Lambda(\mathfrak{g})\) ([13, Lemma 4.2]), hence \((N^{ab}(\ell) \otimes A[\ell^\infty])^\vee_{\mathcal{H}}\)
is projective as a \(\Lambda(\mathfrak{g}/\mathfrak{h}) = \Lambda(G)\)-module. Therefore, if \(H^2(F_S/K, A[\ell^\infty]) = 0\), the module \(Y\) has projective dimension \(\leq 1\). Whenever this is true the definition of \(J\) provides the isomorphisms
\[
E^i(X) \simeq E^{i+1}(J) \quad \text{and} \quad E^i(J) \simeq E^{i+1}((A[\ell^\infty])^\vee_{\mathcal{H}}) \quad \forall \, i \geq 2,
\]
which will be repeatedly used in our computations.

We shall need also a “localized” version of the sequence (2). For every \(v \in S\) and a \(w \in \mathcal{M}_K\) dividing \(v\), we define
\[
X_v = H^1(K_w, A[\ell^\infty])^\vee \quad \text{and} \quad Y_v = (I(\mathfrak{g}_w) \otimes_{\mathcal{O}_w} A[\ell^\infty])^\vee_{\mathcal{H}_v}
\]
(with \(\mathfrak{g}_w\) the decomposition groups of \(v\) in \(\mathfrak{g}\) and \(\mathcal{H}_v = \mathcal{H} \cap \mathcal{S}_v\)). The exact sequence
\[
\begin{align*}
X_v \xrightarrow{\sim} Y_v \xrightarrow{\sim} J_v
\end{align*}
\]
fits into the localized version of diagram (3). If \(K_w\) is still a local field, then Tate local duality ([11, Theorem 7.2.6]) yields
\[
H^2(K_w, A[\ell^\infty]) = H^2(K_w, \text{lim} A[\ell^n]) \simeq \text{lim} H^0(K_w, A^i[\ell^n])^\vee = 0.
\]

If \(K_w\) is not local, then \(\ell^\infty\) divides the degree of the extension \(K_w/F_v\) and \(H^2(K_w, A[\ell^\infty]) = 0\) by [11, Theorem 7.1.8 (i)]. Therefore \(Y_v\) always has projective dimension \(\leq 1\) and
\[
E^i(X_v) \simeq E^{i+1}(J_v) \simeq E^{i+2}((A[\ell^\infty])^\vee_{\mathcal{H}_v}) \quad \forall \, i \geq 2.
\]

We note that, since \(\ell \neq p\), the image of the local Kummer maps is always 0, hence
\[
H^1(K_w, A[\ell^\infty])^\vee = (H^1(K_w, A[\ell^\infty])/\text{Im} \kappa_w)^\vee \simeq H^1(K_w, A)[\ell^\infty]^\vee.
\]

Then Definition 2.1 for \(L = K\) can be written as
\[
\text{Sel}_A(K)_\ell = \text{Ker} \left\{ \psi : X^\vee \longrightarrow \bigoplus_S \text{Coind}_{G_v}^G X_v^\vee \right\}
\]
and, dualizing, we get a map
\[
\psi^\vee : \bigoplus_S \text{Ind}_{G_v}^G X_v \longrightarrow X
\]
whose cokernel is exactly \(S := \text{Sel}_A(K)_\ell^\vee\).

The following result will be fundamental for our computations.

**Theorem 3.2** (U. Jannsen). Let \(G\) be an \(\ell\)-adic Lie group without elements of order \(\ell\) and of dimension \(d\). Let \(M\) be a \(\Lambda(G)\)-module which is finitely generated as \(\mathbb{Z}_\ell\)-module. Then \(E^i(M)\) is a finitely generated \(\mathbb{Z}_\ell\)-module and, in particular,

1. if \(M\) is \(\mathbb{Z}_\ell\)-free, then \(E^i(M) = 0\) for any \(i \neq d\) and \(E^d(M)\) is free;
2. if \(M\) is finite, then \(E^i(M) = 0\) for any \(i \neq d + 1\) and \(E^{d+1}(M)\) is finite.
Proof. See [8, Corollary 2.6].

**Corollary 3.3.** With notations as above:

1. if $H^2(F_S/K, A[\ell^\infty]) = 0$, then, for $i \geq 2$,
   
   $$E^i(X) \text{ is } \begin{cases} 
   \text{finite} & \text{if } i = d - 1 \\
   \text{free} & \text{if } i = d - 2 \\
   0 & \text{otherwise}
   \end{cases}$$

2. $E^i_v E^j_v^{-1}(X_v) = 0$ for $i \geq 3$.

Proof. 1. The hypothesis yields the isomorphism $E^i(X) \simeq E^{i+2}((A[\ell^\infty])^\vee)$. Since
   
   $$(A[\ell^\infty])^\vee \simeq (A[\ell^\infty]^{3\ell})^\vee = A[\ell^\infty](K)^\vee \simeq Z_\ell^r \oplus \Delta$$

(with $0 \leq r \leq 2g$ and $\Delta$ a finite group) and $E^{i}(Z_\ell^r \oplus \Delta) = E^{i}(Z_\ell^r) \oplus E^{i}(\Delta)$, the claim follows from Theorem 3.2.

2. Use Theorem 3.2 and the isomorphism in (6).

**Lemma 3.4.** If $H^2(F_S/K, A[\ell^\infty]) = 0$, then there is the following commutative diagram

$$\begin{array}{ccc}
E^1(Y) & \xrightarrow{g_1} & \bigoplus_S \text{Ind}^G_{G_v} E^1_v(Y_v) \rightarrow \text{Coker}(g_1) \\
\downarrow & & \downarrow \ \\
E^1(X) & \xrightarrow{h_1} & \bigoplus_S \text{Ind}^G_{G_v} E^1_v(X_v) \rightarrow \text{Coker}(h_1) \\
\downarrow & & \downarrow f \\
E^2(J) & \xrightarrow{\bar{g}_1} & \bigoplus_S \text{Ind}^G_{G_v} E^2_v(J_v) \rightarrow \text{Coker} (\bar{g}_1). \\
\end{array}$$

Proof. The inclusions $S_v \subseteq S$ and $H_v \subseteq H$ induce the maps

$$(I(S_v) \otimes Z_\ell) A[\ell^\infty] \to (I(S) \otimes Z_\ell) A[\ell^\infty] \to (I(S) \otimes Z_\ell) A[\ell^\infty].$$

We have a homomorphism of $\Lambda(G)$-modules $g : \bigoplus_S \text{Ind}^G_{G_v} Y_v \to Y$ which, restricted to the $X_v$’s, provides the map $h : \bigoplus_S \text{Ind}^G_{G_v} X_v \to X$. So we have the following situation

$$\begin{array}{ccc}
X & \xrightarrow{h} & \bigoplus_S \text{Ind}^G_{G_v} X_v \\
\downarrow & & \downarrow \\
Y & \xrightarrow{g} & \bigoplus_S \text{Ind}^G_{G_v} Y_v \\
\downarrow & & \downarrow \bar{g} \\
J & \xrightarrow{-\bar{g}} & \bigoplus_S \text{Ind}^G_{G_v} J_v \\
\end{array}$$

where $\bar{g}$ is induced by $g$ and the diagram is obviously commutative.

Since $Y$ and the $Y_v$’s have projective dimension $\leq 1$ (i.e., $E^2(Y) = E^2(Y_v) = 0$), the lemma follows by taking Ext in diagram (7) and recalling that, for any $i \geq 0$, $E^i_v(\text{Ind}^G_{G_v}(X_v)) = \text{Ind}^G_{G_v} E^i_v(X_v)$ (see [13, Lemma 5.5]).

In the next subsection we are going to describe the structure of $\text{Coker}(g_1)$.
3.3. Homotopy theory and \( Coker(g_1) \). For every finitely generated \( \Lambda(\mathcal{G}) \)-module \( M \) choose a presentation \( P_1 \to P_0 \to M \to 0 \) of \( M \) by projectives and define the transpose functor \( DM \) by the exactness of the sequence
\[
0 \to E^0(M) \to E^0(P_0) \to E^0(P_1) \to DM \to 0.
\]
Then it can be shown that the functor \( D \) is well-defined and one has \( D^2 = Id \) (see [8]).

**Definition 3.5.** Let \( L \) be an extension of \( F \) contained in \( F_S \). Then we define
\[
Z(L) := H^0(F_S/L, \lim_{\overline{m}} D_2(A[\ell^m]))^\vee
\]
where
\[
D_2(A[\ell^m]) = \lim_{F \subseteq E \subseteq F_S} (H^2(F_S/E, A[\ell^m]))^\vee
\]
and the limit in \( \lim_{\overline{m}} D_2(A[\ell^m]) \) is taken with respect to the \( \ell \)-power map \( A[\ell^{m+1}] \to A[\ell^m] \).
In the same way we define \( Z(L) \) for any Galois extension \( L \) of \( F_v \).

An alternative description of the module \( Z \) is provided by the following

**Lemma 3.6.** Let \( K \) be a fixed extension of \( F \) contained in \( F_S \) and \( K_w \) its completion for some \( w|v \in S \). Then
\[
Z(K) \simeq \lim_{F \subseteq L \subseteq K} H^2(F_S/L, T_\ell(A)) \quad \text{and} \quad Z(K_w) \simeq \lim_{F \subseteq L \subseteq K_w} H^2(L, T_\ell(A)).
\]

**Proof. Global case.** For any global field \( L \), let
\[
\mathcal{III}(F_S/L, A[\ell^\infty]) := \text{Ker} \left\{ H^i(F_S/L, A[\ell^\infty]) \to \bigoplus_S H^i(L_w, A[\ell^\infty]) \right\}.
\]
We have already seen that \( H^2(L_w, A[\ell^\infty]) = 0 \), hence \( H^2(F_S/L, A[\ell^\infty]) = \mathcal{III}(F_S/L, A[\ell^\infty]). \)
Using the pairing of [10, Ch. I, Proposition 6.9], we get
\[
Z(K) = H^0(F_S/K, \lim_{\overline{m}} \mathcal{III}(F_S/L, A[\ell^m])^\vee)
\]
\[
= H^0(F_S/K, \lim_{\overline{m}} \mathcal{III}(F_S/L, A[\ell^m])^\vee)
\]
\[
= \lim_{\overline{m}} \mathcal{III}(F_S/L, A[\ell^m])^\vee
\]
\[
= \lim_{\overline{m}} H^2(F_S/L, T_\ell(A)).
\]

**Local case.** The proof is similar (using Tate local duality). \( \square \)

We recall that our group \( G \) has no elements of order \( \ell \), hence \( \Lambda(G) \) is a domain. Moreover for any open subgroup \( U \) of \( G \) we have that (see [8, Lemma 2.3])
\[
E^i(U) \simeq E^i(G) \quad \forall i \in \mathbb{Z}
\]
is an isomorphism of \( \Lambda(U) \)-modules. An \( \ell \)-adic Lie group \( G \) always contains an open pro-\( \ell \) subgroup ([6, Corollary 8.34]), so, in order to use properly the usual definitions of “torsion submodule” and “rank” for a finitely generated \( \Lambda(G) \)-module, with no loss of generality, we
will assume that $G$ is pro-$\ell$.

**Proposition 3.7.** Let $M$ be a finitely generated $\Lambda(G)$-module. Then $E_i(M)$ is a finitely generated $\Lambda(G)$-module for any $i \geq 1$.

**Proof.** Take a finite presentation $P_1 \to P_0 \to M \to 0$ with finitely generated and projective $\Lambda(G)$-modules $P_1$ and $P_0$, and the consequent exact sequence
\[
(8) \quad 0 \to R_1 \to P_0 \to M \to 0
\]
for a suitable submodule $R_1$ of $P_1$. Since $M$ and $\text{Hom}_{\Lambda(G)}(M, \Lambda(G))$ have the same $\Lambda(G)$-rank, computing ranks in the sequence coming from (8)
\[
\text{Hom}_{\Lambda(G)}(M, \Lambda(G)) \to \text{Hom}_{\Lambda(G)}(P_0, \Lambda(G)) \to \text{Hom}_{\Lambda(G)}(R_1, \Lambda(G)) \to E_1(M) \to \cdots
\]
one finds rank$_{\Lambda(G)}(E_1(M)) = 0$ for any finitely generated $\Lambda(G)$-module $M$. Therefore $E_1(R_1)$ is torsion, which yields $E_2(M) \cong E_1(R_1)$ is torsion. Iterating $E_i(M) \cong E_{i-1}(R_1)$ is $\Lambda(G)$-torsion $\forall i \geq 2$. \qed

**Lemma 3.8.** Let $F_n$ be subfields of $K$ such that $\text{Gal}(K/F) = \lim_{\overrightarrow{n}} \text{Gal}(F_n/F)$. Then
\[
H^2_{Iw}(K, T_\ell(A)) := \lim_{\overrightarrow{n,m}} H^2(F_{n,m}, A[\ell^m])
\]
is a torsion $\Lambda(G_v)$-module. If $H^2(F_S/K, A[\ell^\infty]) = 0$, then
\[
H^2_{Iw}(K, T_\ell(A)) := \lim_{\overrightarrow{n,m}} H^2(F_S/F_n, A[\ell^m])
\]
is a $\Lambda(G)$-torsion as well.

**Proof.** The proofs are identical so we only show the second statement. From the spectral sequence
\[
E_2^{p,q} = E^p(H^q(F_S/K, A[\ell^\infty])^\vee) \Rightarrow H^{p+q}_{Iw}(K, T_\ell(A))
\]
due to Jannsen (see [7]), we have a filtration for $H^2_{Iw}(K, T_\ell(A))$
\[
(9) \quad 0 = H_3^2 \subseteq H_2^2 \subseteq H_1^2 \subseteq H_0^2 = H^2_{Iw}(K, T_\ell(A)) ,
\]
which provides the following sequences:
\[
E_0^0(H^1(F_S/K, A[\ell^\infty])^\vee) \to E_2^0(H^0(F_S/K, A[\ell^\infty])^\vee) \to H_1^2
\]
\[
\to E_1^1(H^1(F_S/K, A[\ell^\infty])^\vee) \to E_3^1(H^0(F_S/K, A[\ell^\infty])^\vee)
\]
and
\[
H_1^2 \hookrightarrow H^2_{Iw}(K, T_\ell(A)) \to E_{\infty}^{0,2} .
\]
By hypothesis $E_{\infty}^{0,2} \simeq E_{\infty}^{0,2} = 0$, so $H_1^2 \simeq H^2_{Iw}(K, T_\ell(A))$.
Since $H_i(F_S/K, A[\ell^\infty])^\vee$ is a finitely generated $\Lambda(G)$-module for $i \in \{0,1\}$ (for $i = 1$ just look at $X$ in diagram (3)), Proposition 3.7 yields that the groups $E_i^1(F_S/K, A[\ell^\infty])^\vee$ and $E_1^1(H^1(F_S/K, A[\ell^\infty])^\vee)$ are $\Lambda(G)$-torsion. Hence $H_1^2$ is torsion as well. \qed

**Lemma 3.9.** With notations and hypotheses as in Lemma 3.4, $\text{Coker}(g_1)$ is finitely generated as $\mathbb{Z}_\ell$-module.
Therefore one finds

\[ \text{Proof.} \quad \text{We need to prove that} \ S \text{ is surjective, then} \]

\[ \text{Sel} \] by the Poitou-Tate sequence (see [11, 8. 6.10 p. 488]), since

\[ \text{G} \] (the same holds for the “local” modules). The map \( g_1 \) of Lemma 3.4 then reads as

\[ g_1 : \lim_n H^2(F_S/F_n, T_\ell(A)) \to \bigoplus_S \text{Ind}^G_{G_v} \lim_n H^2(F_{v_n}, T_\ell(A)) . \]

The claim follows from the Poitou-Tate sequence (see [11, 8.6.10 p. 488]), since

\[ \text{Coker}(g_1) \simeq \lim_{n,m} H^0(F_S/F_n, (A[\ell^m])'). \]

\[ \square \]

4. The main Theorem

We are now ready to prove the following

**Theorem 4.1.** Let \( G = \text{Gal}(K/F) \) be an \( \ell \)-adic Lie group without elements of order \( \ell \) and of positive dimension \( d \geq 3 \). If \( H^2(F_S/K, A[\ell^\infty]) = 0 \) and the map \( \psi \) in the sequence

\[ \text{Sel}_A(K_\ell) \to H^1(F_S/K, A[\ell^\infty]) \xrightarrow{\psi} \bigoplus_S \text{Ind}^G_{G_v} H^1(K_w, A)[\ell^\infty] \]

is surjective, then \( S := \text{Sel}_A(K_\ell) \) has no nontrivial pseudo-null submodule.

**Proof.** We need to prove that

\[ E^iE^i(S) = 0 \quad \forall \ i \geq 2, \]

and we consider two cases.

**Case i = 2.** Let \( \mathcal{D} := g_1(E^2(J_v)) \). Then

\[ \text{Coker}(\bar{g}_1) = \bigoplus_S \text{Ind}^G_{G_v} E^2(J_v)/\mathcal{D}. \]

Observe that \( \mathcal{D} \simeq g_1(E^3(A[\ell^\infty]_{3,v})) \) is a finitely generated \( \mathbb{Z}_\ell \)-module (it is zero if \( d \neq 3 \) and free as \( \mathbb{Z}_\ell \)-module if \( d = 3 \)), so \( E^1(\mathcal{D}) = 0 \). Even if the theorem is limited to \( d \geq 3 \) we remark here that, for \( d = 2 \), \( \mathcal{D} \) is finite and, for \( d = 1 \), \( \mathcal{D} = 0 \): hence \( E^1(\mathcal{D}) = 0 \) in any case.

Moreover

\[ E^2(\bigoplus_S \text{Ind}^G_{G_v} E^2(J_v)) = E^2(\bigoplus_S \text{Ind}^G_{G_v} E^3(A[\ell^\infty]_{3,v})) \]

\[ = \bigoplus_S \text{Ind}^G_{G_v} E^2E^3(A[\ell^\infty]_{3,v}) = 0, \]

so, taking Ext in the sequence,

\[ \mathcal{D} \to \bigoplus_S \text{Ind}^G_{G_v} E^2(J_v) \to \bigoplus_S \text{Ind}^G_{G_v} E^2(J_v)/\mathcal{D}, \]

one finds

\[ E^1(\mathcal{D}) \to E^2(\bigoplus_S \text{Ind}^G_{G_v} E^2(J_v)/\mathcal{D}) \to E^2(\bigoplus_S \text{Ind}^G_{G_v} E^2(J_v)). \]

Therefore

\[ E^2(\bigoplus_S \text{Ind}^G_{G_v} E^2(J_v)/\mathcal{D}) = 0. \]
Recall the sequences
\[(13) \quad \bigoplus_S \text{Ind}_{G_v}^G X_v \hookrightarrow X \twoheadrightarrow S\]
\[(14) \quad \text{Ker}(f) \hookrightarrow \text{Coker}(h_1) \twoheadrightarrow \text{Coker}(\bar{g}_1)\]

provided (respectively) by the hypothesis on \(\psi\) and by Lemma 3.4. Take Ext on (13) to get
\[E^1(X) \xrightarrow{h_1} E^1\left(\bigoplus_S \text{Ind}_{G_v}^G X_v\right) \to E^2(S) \to E^2(X) .\]

If \(d \geq 5\), then \(E^2(X) \simeq E^3(J) \simeq E^4(A[\ell^\infty]) = 0\). When this is the case \(\text{Coker}(h_1) \simeq E^2(S)\) and sequence (14) becomes
\[\text{Ker}(f) \hookrightarrow E^2(S) \to \bigoplus_S \text{Ind}_{G_v}^G E^2(J_v)/\mathcal{D} .\]

By Lemma 3.9, \(\text{Ker}(f)\) is a finitely generated \(\mathbb{Z}_d\)-module. Taking Ext, one has
\[E^2\left(\bigoplus_S \text{Ind}_{G_v}^G E^2(J_v)/\mathcal{D}\right) \to E^2E^2(S) \to E^2(\text{Ker}(f)) ,\]
where the first and third term are trivial, so \(E^2E^2(S) = 0\) as well.

We are left with \(d = 3, 4\). We know that \(E^1(A[\ell^\infty]) = E^2(X)\) is free over \(\mathbb{Z}_d\) if \(d = 4\) or finite if \(d = 3\) (again we remark it is 0 if \(d = 1, 2\)). Anyway \(E^2E^2(X) = 0\) in all cases. From the sequence
\[\text{Coker}(h_1) \hookrightarrow E^2(S) \twoheadrightarrow E^2(X)\]
one writes
\[(15) \quad \text{Coker}(h_1) \hookrightarrow E^2(S) \twoheadrightarrow \text{Im}(\eta)\]
where \(\text{Im}(\eta)\) is free over \(\mathbb{Z}_d\) if \(d = 4\) or finite if \(d = 3\).

Taking Ext in (14) one has
\[E^2(\text{Coker}(\bar{g}_1)) \to E^2(\text{Coker}(h_1)) \to E^2(\text{Ker}(f)) \]
with the first (see equation (12)) and third term equal to zero, so \(E^2(\text{Coker}(h_1)) = 0\). This fact in sequence (15) implies
\[0 = E^2(\text{Im}(\eta)) \to E^2E^2(S) \to E^2(\text{Coker}(h_1)) = 0 ,\]
so \(E^2E^2(S) = 0\).

**Case** \(i \geq 3\). From sequence (13) we get the following
\[(16) \quad E^{i+1}(A[\ell^\infty]) \simeq E^{i-1}(X) \to \bigoplus_S \text{Ind}_{G_v}^G E^{i-1}(X_v) \to E^i(S) \to E^i(X) \simeq E^{i+2}(A[\ell^\infty]) .\]

We have four cases, depending on whether \(E^{i-1}(X)\) and \(E^i(X)\) are trivial or not.

**Case 1.** Assume \(E^{i-1}(X) = E^i(X) = 0\).
From (16) we obtain the isomorphism
\[\bigoplus_S \text{Ind}_{G_v}^G E^{i-1}(X_v) \simeq E^i(S) ,\]
so
\[\bigoplus_S \text{Ind}_{G_v}^G E^{i-1}(X_v) \simeq E^i(S) = 0\]
thanks to Corollary 3.3 part 2. We remark that this is the only case to consider when \(d = 1, 2\).
**Case 2.** Assume $E^{i-1}(X) = 0$ and $E^i(X) \neq 0$.
This happens when $i = d - 2$ or $i = d - 1$ and $A[\ell^\infty]_{\mathcal{S}_t}$ is finite. From (16) we have
\[
\bigoplus_S \text{Ind}_{G_v}^G E^{d-3}_v E^{d-2}_v \rightarrow E^{d-2}(S) \rightarrow N
\]
( resp. \[
\bigoplus_S \text{Ind}_{G_v}^G E^{d-2}_v \rightarrow E^{d-1}(S) \rightarrow N
\])
where $N$ is a submodule of the free module $E^{d-2}(X)$ (resp. of the finite module $E^{d-1}(X)$).
Therefore $E^{d-2}(N) = 0$ (resp. $E^{d-1}(N) = 0$) and, moreover, $E^{d-2}_v E^{d-3}_v (X_v) = 0$ (resp. $E^{d-1}_v E^{d-2}_v (X_v) = 0$) by Corollary 3.3 part 2. Hence $E^{d-2} E^{d-2}(S) = 0$ (resp. $E^{d-1} E^{d-1}(S) = 0$).

**Case 3.** Assume $E^{i-1}(X) \neq 0$ and $E^i(X) = 0$.
This happens when $i = d$ or $i = d - 1$ and $A[\ell^\infty]_{\mathcal{S}_t}$ is free. The sequence (16) gives
\[
N \rightarrow \bigoplus_S \text{Ind}_{G_v}^G E^{d-1}_v(X_v) \rightarrow E^d(S)
\]
( resp. \[
N \rightarrow \bigoplus_S \text{Ind}_{G_v}^G E^{d-2}_v(X_v) \rightarrow E^{d-1}(S)
\])
where now $N$ is a quotient of the finite module $E^{d-1}(X)$ (resp. of the free module $E^{d-2}(X)$).
Then $E^d(N) = 0$ (resp. $E^{d-1}(N) = 0$) and
\[
\bigoplus_S \text{Ind}_{G_v}^G E^{d-1}_v E^{d-1}_v(X_v) \simeq E^d E^d(S) = 0
\]
( resp. \[
\bigoplus_S \text{Ind}_{G_v}^G E^{d-1}_v E^{d-2}_v(X_v) \simeq E^{d-1} E^{d-1}(S) = 0
\]).

**Case 4.** Assume $E^{i-1}(X) \neq 0$ and $E^i(X) \neq 0$.
This happens when $i = d - 1$ and $A[\ell^\infty]_{\mathcal{S}_t}$ has nontrivial rank and torsion. From sequence (16) we have
\[
E^{d-2}(X) \rightarrow \bigoplus_S \text{Ind}_{G_v}^G E^{d-2}_v(X_v) \rightarrow E^{d-1}(S) \rightarrow E^{d-1}(X).
\]
Let $N_1, N_2$ and $N_3$ be modules such that:
- $N_1$ is a quotient of $E^{d-2}(X)$ (which is torsion free so that $E^{d-2}(N_1) = 0$);
- $N_2$ is a submodule of $E^{d-1}(X)$ (which is finite so that $E^{d-1}(N_2) = 0$);
- $N_3$ is a module such that the sequences

\[
N_1 \hookrightarrow \bigoplus_S \text{Ind}_{G_v}^G E^{d-2}_v(X_v) \rightarrow N_3 \quad \text{and} \quad N_3 \hookrightarrow E^{d-1}(S) \rightarrow N_2
\]

are exact.
Applying the functor $\text{Ext}$ we find
\[
E^{d-2}(N_1) \rightarrow E^{d-1}(N_3) \rightarrow \bigoplus_S \text{Ind}_{G_v}^G E^{d-1}_v E^{d-2}_v(X_v)
\]
(which yields $E^{d-1}(N_3) = 0$), and
\[
E^{d-1}(N_2) \rightarrow E^{d-1} E^{d-1}(S) \rightarrow E^{d-1}(N_3)
\]
which proves $E^{d-1} E^{d-1}(S) = 0$. \[\square\]

**Remark 4.2.** As pointed out in various steps of the previous proof, most of the statements still hold for $d = 1, 2$. The only missing part is $E^2(Ker(f)) = 0$ for $i = 2$, in that case only our calculations to get $E^2 E^2(S) = 0$ fail. In particular the same proof shows that $E^2 E^2(S) = 0$ when $Ker(f)$ is free and $d = 1$ or when $Ker(f)$ is finite and $d = 2$ or, obviously, for any $d$ if $f$ is injective.
We can extend the previous result to the \(d \geq 2\) case with some extra assumptions.

**Proposition 4.3.** Let \(G = \text{Gal}(K/F)\) be an \(\ell\)-adic Lie group without elements of order \(\ell\) and of dimension \(d \geq 2\). If \(H^2(F_S/K, A[\ell^\infty]) = 0\) and \(\text{cd}_\ell(G_v) = 2\) for any \(v \in S\), then \(\text{Sel}_A(K)_\ell\) has no nontrivial pseudo-null submodule.

**Proof.** Since \(\text{cd}_\ell(F_v) = 2\) (by [11, Theorem 7.1.8]), our hypothesis implies that \(\text{Gal}(F_v/K_w)\) has no elements of order \(l\) (see also [11, Theorem 7.5.3]). Hence \(H^1(K_w, A[\ell^\infty])^\vee = 0\) and \(\text{Sel}_A(K)_\ell \simeq X\) embeds in \(Y\). Now \(H^2(F_S/K, A[\ell^\infty]) = 0\) yields \(Y\) has projective dimension \(\leq 1\), so \(Y\) has no nontrivial pseudo-null submodule (by [13, Proposition 2.5]). \(\Box\)

4.1. **The hypotheses on** \(H^2(F_S/K, A[\ell^\infty])\) **and** \(\psi\). Let \(F_m\) be extensions of \(F\) such that \(\text{Gal}(K/F) \simeq \lim_{m} \text{Gal}(F_m/F)\). To provide some cases in which the main hypotheses hold we consider the Poitou-Tate sequence for the module \(A[\ell^n]\), from which one can extract the sequence

\[
\begin{align*}
0 \rightarrow \text{Ker}(\psi_{m,n}) & \rightarrow H^1(F_S/F_m, A[\ell^n]) \rightarrow \prod_{v \in S} H^1(F_v, A[\ell^n]) \\
& \rightarrow H^2(F_S/F_m, A[\ell^n]) \rightarrow \text{Ker}(\psi_{m,n})^\vee \\
& \rightarrow H^0(F_S/F_m, A[\ell^n])^\vee \rightarrow 0
\end{align*}
\]

(17) where \(\psi_{m,n}^t\) is the analogue of \(\psi_{m,n}\) for the dual abelian variety \(A^t\), i.e., their kernels represent the Selmer groups over \(F_m\) for the modules \(A^t[\ell^n]\) and \(A[\ell^n]\) respectively. Taking direct limits on \(n\) and recalling that \(H^2(F_v, A[\ell^\infty]) = 0\), the sequence (17) becomes

\[
\begin{align*}
0 \rightarrow \text{Sel}_A(F_m)_\ell & \rightarrow H^1(F_S/F_m, A[\ell^\infty]) \\
& \rightarrow \prod_{v \in S} H^1(F_v, A[\ell^\infty]) \\
& \rightarrow H^2(F_S/F_m, A[\ell^\infty]) \leftarrow \left(\lim_{n} \text{Ker}(\psi_{m,n}^t)\right)^\vee
\end{align*}
\]

(18) (for more details one can consult [5, Chapter 1]). One way to prove that \(H^2(F_S/K, A[\ell^\infty]) = 0\) and \(\psi\) is surjective is to show that \(\left(\lim_{n} \text{Ker}(\psi_{m,n}^t)\right)^\vee = 0\) for any \(m\). We mention here two cases in which the hypothesis on the vanishing of \(H^2(F_S/K, A[\ell^\infty])\) is verified. The following is basically [5, Proposition 1.9].

**Proposition 4.4.** Let \(F_m\) be as above and assume that \(\text{Sel}_A(F_m)_\ell \subset \infty\) for any \(m\), then \(H^2(F_S/K, A[\ell^\infty]) = 0\).

**Proof.** From [10, Chapter I Remark 3.6] we have the isomorphism

\[A^t(F_v)^* \simeq H^1(F_v, A[\ell^\infty])^\vee,\]

where \(A^t(F_v)^* \simeq \lim_n A^t(F_v)/\ell^n A^t(F_v)\).

Taking inverse limits on \(n\) in the exact sequence

\[A^t(F_m)/\ell^n A^t(F_m) \rightarrow \text{Ker}(\psi_{m,n}^t) \rightarrow \text{III}(A^t/F_m)[\ell^n],\]

and noting that \(\text{III}(A^t/F_m)[\ell^n]) < \infty\) yields \(T_\ell(\text{III}(A^t/F_m)) = 0\), we find

\[A^t(F_m)^* \simeq \lim_n \text{Ker}(\psi_{m,n}^t)\].
Therefore (18) becomes

$$0 \longrightarrow Sel_A(F_m) \longrightarrow H^1(F_S/F_m, A[\ell^\infty]) \longrightarrow \prod_{v \in S} (A'(F_{v_m})^r) \phi \longrightarrow 0$$

By hypothesis $A'(F_m)^r$ is finite, therefore $H^2(F_S/F_m, A[\ell^\infty])$ is finite as well. From the cohomology of the sequence

$$A[\ell] \hookrightarrow A[\ell^\infty] \longrightarrow A[\ell^\infty]$$

(and the fact that $H^3(F_S/F_m, A[\ell]) = 0$, because $cd_{\ell}(\text{Gal}(F_S/F_m)) = 2$), one finds

$$H^2(F_S/F_m, A[\ell^\infty]) \longrightarrow H^2(F_S/F_m, A[\ell^\infty]),$$

i.e., $H^2(F_S/F_m, A[\ell^\infty])$ is divisible. Being divisible and finite $H^2(F_S/F_m, A[\ell^\infty])$ must be 0 for any $m$ and the claim follows.

We can also prove the vanishing of $H^2(F_S/K, A[\ell^\infty])$ for the extension $K = F(A[\ell^\infty])$.

**Proposition 4.5.** If $K = F(A[\ell^\infty])$, then $H^2(F_S/K, A[\ell^\infty]) = 0$.

**Proof.** $\text{Gal}(F_S/K)$ has trivial action on $A[\ell^\infty]$ and (by the Weil pairing) on $\mu_{\ell^\infty}$, so

$$H^2(F_S/K, A[\ell^\infty]) \simeq H^2(F_S/K, (\mathbb{Q}_{\ell}/\mathbb{Z}_{\ell})^{2g}) \simeq H^2(F_S/K, (\mu_{\ell^\infty})^{2g}).$$

Let $F_n = F(A[\ell^m])$, using the notations of Lemma 3.6, Poitou-Tate duality ([11, Theorem 8.6.7]) and the isomorphism $\prod_1^1(F_S/F_n, \mathbb{Z}/\ell^m\mathbb{Z}) \simeq \text{Hom}(\text{C}(F_n), \mathbb{Z}/\ell^m\mathbb{Z})$ ([11, Lemma 8.6.3]), one has

$$H^2(F_S/K, A[\ell^\infty]) \simeq \prod_1^2(F_S/K, A[\ell^\infty]) \simeq \lim_{n,m} \prod_1^1(F_S/F_n, A[\ell^m])$$

$$\simeq \lim_{n,m} \prod_1^1(F_S/F_n, A[\ell^m])^r \simeq \lim_{n,m} \prod_1^1(F_S/F_n, \mathbb{Z}/\ell^m\mathbb{Z})^r$$

$$\simeq \lim_{n,m} \text{Hom}(\text{C}(F_n), \mathbb{Z}/\ell^m\mathbb{Z})^r \simeq \lim_{n,m} \text{C}(F_n)/\ell^m$$

$$\simeq \lim_{n,m} \text{C}(F_n) \otimes_{\mathbb{Z}} \mathbb{Q}_{\ell} = 0$$

since $\text{C}(F_n)$ is finite.

**Remark 4.6.** The above proposition works in the same way for a general $\ell$-adic Lie extensions, unramified outside $S$, which contains the trivializing extension.

**Example 4.7.** Let $A$ be an abelian variety without complex multiplication: by Proposition 4.5, the extension $K = F(A[\ell^\infty])$ realizes the hypothesis of Proposition 4.3 when every bad reduction prime is of split multiplicative reduction (in order to have $cd_{\ell}(G_v) = 2$) and $\ell > 2g + 1$ (by [16] and the embedding $\text{Gal}(K/F) \hookrightarrow \text{GL}_{2g}(\mathbb{Z}_{\ell})$). Therefore $Sel_A(K)^r$ has no nontrivial pseudo-null submodule. When $A = E$ is an elliptic curve (using Igusa’s theorem, see, e.g., [1]) one can prove that $\dim \text{Gal}(K/F) = 4$ and also the surjectivity of the map $\psi$ (which, in this case, is not needed to prove the absence of pseudo-null submodules): more details can be found in [14].

The same problem over number fields cannot (in general) be addressed in the same way and one needs the surjectivity of the map $\psi$. The topic is treated (for example) in [4, Section 4.2].
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