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The seven-strand braid group is CAT(0)

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Abstract. We prove that the 7-strand braid group is CAT(0) by elaborating on the argument of Haettel, Kielak, and Schwer.

1. Introduction

In [1] Charney asked whether braid groups are CAT(0). In [2], Brady and McCammond proved that braid groups on at most 5 strands are CAT(0). They also classified finite type Artin groups on four Artin generators which are CAT(0), but they used a computer program to prove it. In [3], Haettel, Kielak, and Schwer showed that braid groups on at most 6 strands are CAT(0), without using computer programs. Brady-McCammond proved that the diagonal link of the non-crossing partition complex for 5 vertices is CAT(1) and Haettel-Kielak-Schwer proved that the diagonal link of the non-crossing partition complex for 6 vertices is CAT(1). As proved in [2][Proposition 8.3], it implies that braid groups for specific numbers of strands are CAT(0). Haettel, Kielak, and Schwer used a criterion of Gromov [4], Bowditch [5], and Charney-Davis [6] to show if non-crossing partition complex has no face satisfying certain four conditions, then the diagonal link of the non-crossing partition complex is CAT(1). In 2015, Haettel gave a conjectural classification of which Artin-Tits groups are virtually cocompactly cubulated [7]. In particular, he proved that braid groups on at least 5 strands are not virtually cocompactly cubulated, meaning that they do not act geometrically on a CAT(0) cube complex, possibly up to finite index subgroup. In this thesis, we will prove the following main theorem.

Theorem 3.11. The diagonal link of $\mathcal{NC}P_7$ endowed with the spherical orthoscheme metric is CAT(1), which implies that the Brady–McCammond complex associated to $B_7$ is CAT(0), and hence that $B_7$ is a CAT(0) group.

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2. Preliminaries

2.1. Bounded, graded poset

A poset (partially ordered set) is bounded if it has maximum and minimum. A bounded poset has rank $n$ if every chain is contained in a maximal chain with $n + 1$ elements, and a bounded poset is graded if its intervals have ranks. For an element $x$ of a bounded graded poset, the rank of $x$ is the rank of the interval from minimum to $x$.

2.2. Orthoscheme metric

For a graded poset $P$, its geometric realization $|P|$ is the simplicial complex whose vertex set is $P$ and $k$-simplices correspond to length $k$ chains of $P$. We will introduce a metric on the geometric realization $|P|$ of a bounded graded poset $P$.

We first define the $n$-orthoscheme $O_n \subset \mathbb{R}^n$ to be the $n$-simplex in $\mathbb{R}^n$ with vertex set $\{ (0, 0, 0, \ldots, 0), (1, 0, 0, \ldots, 0), (1, 1, 0, \ldots, 0), \ldots, (1, 1, \ldots, 1) \}$. We will define a metric called the orthoscheme metric on $|P|$ by letting any $n$-simplex $\{ g_0, \ldots, g_n \}$ (indices correspond to their ranks) be isometric to $O_n$ via the isometry sending $g_0$ to $(0, \ldots, 0)$, $g_1$ to $(1, 0, \ldots, 0)$, $\ldots$, $g_n$ to $(1, \ldots, 1)$ and by endowing the induced length metric. From now on, we will endow every geometric realization of a bounded graded poset with the orthoscheme metric.

2.3. Non-crossing partition complex

Consider the set of all partitions of the set $\{1, \ldots, n\}$ that is partially ordered by declaring $P \leq Q$ for partitions $P, Q$ if for any $\sigma \in P$, $\sigma \subseteq \tau$ for some $\tau \in Q$. Then $\{ \{1, \ldots, n\} \}$ is the unique maximal element and $\{ \{1\}, \ldots, \{n\} \}$ is the unique minimal element. Let $\mathcal{P}_n$ be the set of all partitions of $\{1, \ldots, n\}$ that are neither $\{ \{1, 2, \ldots, n\} \}$ nor $\{ \{1\}, \ldots, \{n\} \}$.

Sometimes it is convenient to identify $\{1, 2, \ldots, n\}$ with $U_n := \{ e^{2\pi i k n} | 1 \leq k \leq n \} \subset \mathbb{C}$. For $P \in \mathcal{P}_n$, $\sigma \in P$ is called a block if $\sigma$ has more than one element. Note that a partition is uniquely determined by its blocks. A block $\sigma$ is consecutive if $\sigma$ consists of consecutive integers modulo $n$, that is, they are consecutive in $U_n$. From now on, we assume that all computations on $\{1, 2, \ldots, n\}$ are done modulo $n$.

Blocks $\sigma$ and $\tau$ are crossing if $\sigma \cap \tau = \emptyset$ and there are $\sigma_1, \sigma_2 \in \sigma$ and $\tau_1, \tau_2 \in \tau$ such that either $\sigma_1 < \tau_1 < \sigma_2 < \tau_2$ or $\tau_1 < \sigma_1 < \tau_2 < \sigma_2$. Visually, two blocks are crossing if they are disjoint as blocks and their convex hulls in $\mathbb{C}$ intersect each other. For $P \in \mathcal{P}_n$, $P$ is non-crossing, if no two blocks of $P$ are crossing. Let $\mathcal{NCP}_n$ be the subset of $(\mathcal{P}_n, \leq)$ consisting of all non-crossing partitions. For $P, Q \in \mathcal{NCP}_n$, $P$ and $Q$ are crossing, written $P \parallel Q$, if a block of $P$ and a block of $Q$ are crossing. Otherwise, $P$ and $Q$ are non-crossing, written $P \parallel Q$.

For later use, we will declare $\mathcal{NC}P_n$ to be the poset of all non-crossing partitions (with maximum and minimum). $\mathcal{NC}P_n$ is bounded and graded. We will call $|\mathcal{NC}P_n|$ non-crossing partition complex.
2.4. Brady-McCammond complex

We will define a complex that is relevant to the braid group and is isometric to non-crossing partition complex. Let \( B_n \) be the braid group on \( n \)-strands. The band generator presentation, introduced in [8], of \( B_n \) has generators \( a_{ts}, 1 \leq s < t \leq n \) and relations \( a_{ts}a_{rq} = a_{rq}a_{ts} \) for \((t-r)(t-q)(s-r)(s-q) > 0\) and \( a_{ts}a_{sr} = a_{sr}a_{ts} \) for \( 1 \leq r < s < t \leq n \). We will use the band generator presentation for \( B_n \) to define the Brady-McCammond complex \( K_n \) associated to \( B_n \). Let \( B_n^+ \) be the monoid presented by the same generators and relations of the band generator presentation of \( B_n \). Elements of \( B_n^+ \) are called positive braids [9] and we define the prefix order on positive braids by letting for \( a, b \in B_n^+ \), \( a < b \) iff \( b = ac \) for some \( c \in B_n^+ \). We let \( K_n \) be the geometric realization of the bounded graded poset with vertex set \( \{ g \in B_n^+ | \text{id} \leq g \leq \delta \} \) with the prefix order on positive braids, where \( \delta := a_{12}a_{23} \ldots a_{(n-1)n} \in B_n^+ \). Let \( \tilde{K}_n \) be the orthoscheme complex whose vertex set is the set of elements of \( B_n \) and whose \( n \)-orthoschemes are attached to \( \text{chains} \{ kg \in B_n^+ | \text{id} \leq g \leq \delta \} \) for each \( k \in B_n \). Note that \( B_n \) acts on \( \tilde{K}_n \) simplicially by \( g \cdot \{ g_0, \ldots, g_k \} = \{ gg_0, \ldots, gg_k \} \). It is proved in [10] that the quotient complex \( \tilde{K}_n/B_n \) is compact and a \( K(B_n, 1) \) space. Therefore, to prove \( B_n \) is CAT(0), it suffices to prove that \( \tilde{K}_n/B_n \) is locally CAT(0). A Euclidean polytope is defined as the convex hull of a finite set of points in a Euclidean space. A Piecewise Euclidean complex (or PE-complex) is defined as the regular cell complex obtained from disjoint union of Euclidean polytopes by glueing via isometric identifications of their faces. It is proved in [2][Definition 8.2] that \( \tilde{K}_n/B_n \) is as a metric space, splits as a product of a PE-complex \( Y \) and the circle of length \( \sqrt{n} \) and that, the link of the unique vertex in \( Y \) is isometric to the diagonal link \( \text{lk}(e_{01}, K_n) \) of \( K_n \), where \( e_{01} \) is the edge in \( K_n \) connecting \( \text{id} \) and \( \delta \). Hence, if \( \text{lk}(e_{01}, K_n) \) is CAT(1), then \( Y \) is locally CAT(0). Therefore, \( \tilde{K}_n/B_n \) is locally CAT(0) since it is a product of two locally CAT(0) spaces.

2.5. Rank and corank of a chain

For a chain \( F = \{ P_1, \ldots, P_k \} \) of \( \mathcal{NCP}_n \), its \textit{rank} is defined to be \( \text{rk} F = \{ \text{rk} P_1, \ldots, \text{rk} P_k \} \) and its \textit{corank} is defined to be \( \text{cork} F = \{ 1, \ldots, n - 2 \} \setminus \text{rk} F \).

2.6. Duality

In this subsection, we will explain the duality of \( \mathcal{NCP}_n \), which will be used to reduce the number of cases that have to be checked in the main theorem.

For \( P \in \mathcal{NCP}_n \), its \textit{dual} \( P^* \) is the partition containing all blocks \( \sigma = \{ i_1, \ldots, i_k \} \) written clockwise on \( U_n \) such that there exist distinct members \( \sigma_1, \ldots, \sigma_k \in \mathcal{P} \) such that \( i_1 \in \sigma_1, \ldots, i_k \in \sigma_k \) and \( i_1 + 1 \in \sigma_2, i_2 + 1 \in \sigma_3, \ldots, i_{k-1} + 1 \in \sigma_k, i_k + 1 \in \sigma_1 \). It is easy to check that \( P^* \in \mathcal{NCP}_n \) and \( P \leq Q \) iff \( Q^* \leq P^* \). Note that \( \text{rk} P^* = n - 1 - \text{rk} P \).
2.7. The four conditions

We will introduce four conditions (which are a variation of the conditions used by Haettel, Kielak, and Schwer) on a chain of $\mathcal{NCP}_n$. If no chains of $\mathcal{NCP}_n$ satisfy the four conditions, then $B_n$ is a CAT(0) group.

Let $F$ be a chain in the poset $\mathcal{NCP}_n$. For $1 \leq i \leq n$, we define $F_i$ to be the smallest subset of $\{1, 2, \ldots, n\}$ that appears as a block of a partition in $F$ containing $i$, or $F_i = \{1, 2, \ldots, n\}$ if the singleton $\{i\}$ is a member of all partitions in $F$. We remark that for a chain $F$ in $\mathcal{NCP}_n$ and $1 \leq i \leq j \leq n$, $F_j \subseteq F_i$ iff $j \in F_i$. Indeed, if $F_j \subseteq F_i$ then $F_i$ contains both $i$ and $j$. $F_j$ is the smallest block of a partition in $F$ containing $j$, so $F_j \subseteq F_i$. Set $F'_i = F_i \cap \{i - 1, i + 1\}$.

For a chain $F$ in $\mathcal{NCP}_n$, we list four conditions that $F$ may satisfy:

I. cork $F$ contains consecutive integers.
II. There exists an element $P \in F$ such that $P$ is not of the form: $P$ has exactly one block and the block is consecutive.
III. Let $F = \{P_1, \ldots, P_k\}$ so that $i < j$ if $P_i < P_j$. There are $P^+, P^- \in \mathcal{NCP}_n$ satisfying:
   (i) Either $P_i < P^\pm < P_{i+1}$ for some $1 \leq i \leq k - 1$ or $P^\pm < P_1$, or $P_k < P^\pm$;
   (ii) $P^+ \| P^-$. (We call $P^\pm$ a failing-modularity pair of $F$)

IV. There is a maximal chain $C$ in $\mathcal{NCP}_n$ such that $F'_i \cap C'_i = \emptyset$ for all $1 \leq i \leq n$.

We will prove that $B_7$ is a CAT(0) group, by proving that no faces of non-crossing partition complex on 7 points satisfy Haettel, Kielak, and Schwer’s four conditions with a slightly strengthened fourth condition. We will explain the relationship between conditions I, II, III, and IV and Haettel, Kielak, and Schwer’s four conditions in Sect. 3. Condition IV is hard to check while conditions I, II, III are easy to check. We will enumerate faces satisfying conditions I, II, III and then show no such faces satisfy condition IV. In Sect. 4, we will explain a criterion to check condition IV.

3. Proof of the main theorem

By the construction in [10], $K_n$ is isometric to $|NC P_n|$. So, to prove $B_n$ is CAT(0), it suffices to prove that $X := \text{lk}(e_{01}, |NC P_n|)$ is CAT(1). In [3], proof that $X$ is CAT(1) uses an embedding of $X$ to a spherical building. We will explain their proof.

Fix a field $\mathbb{F}$ and define $L_n$ to be the poset of all linear subspaces of $\{(x_1, \ldots, x_n) \in \mathbb{F}^n | \sum_{i=1}^n x_i = 0\}$ ordered by inclusion. Let $|L_n|$ be the geometric realization of $L_n$ with the orthoscheme metric. For an element $A \in L_n$ define its rank $\text{rk} A = \dim A$. There is a rank preserving injective poset map $\phi$ from $\text{NC P}_n$ to $L_n$. More concretely, $\phi$ sends $P \in \text{NC P}_n$ to $\phi(P) = \{(x_1, \ldots, x_n) \in \mathbb{F}^n | \sum_{j=1}^k x_{i_j} = 0$ if $\{x_{i_1}, \ldots, x_{i_k}\} \in P\} \in L_n$. For convenience, for a poset, we let 0 be the minimum element and let 1 be the maximum element. Recall that
X = \text{lk}(e_{01}, |NC\ C P_n|) and let B = \text{lk}(e_{01}, |L_n|). Since every vertex of X corresponds to a vertex in NC\ C P_n \setminus \{0, 1\}, we may think of X as a geometric realization of the poset NC\ C P_n = NC\ C P_n \setminus \{0, 1\}. Similarly, we may think B as a geometric realization of L_n \setminus \{0, 1\} and \phi_{|NC\ C P_n} induces an injective simplicial map i : X \to B.

So we can think of X as a subcomplex of B. It is proved in [11] that B is a CAT(1) space (as a spherical building [12]).

Using this embedding of X to the CAT(1) space B, if X is locally CAT(1) (see the next paragraph), then by results of Bowditch [5], there exists a short (meaning of length is less than 2\pi) local geodesic in X which is not a local geodesic in B.

From this observation, we will define the following notions. For a local geodesic path or loop l : D \to X (D = I or S^1), we say t \in D is a turning point if i \circ l : D \to B is not a local geodesic at t. If l is a local geodesic loop in X and t is a turning point, then we say a face F in X is a turning face if it is the smallest (with respect to inclusion) intersection of a chamber containing l((t, t + \epsilon)) and a chamber containing l((t - \epsilon, t)) for some \epsilon > 0. Here are some definitions used in lemmas about turning faces in [3]. We say that a face is universal if all of its vertices have exactly one block that is consecutive. We say a point is universal if it is a point in a universal face. We say x, y \in X fail modularity if x \lor y \notin X or x \land y \notin X, where \lor, \land is the join and meet in B. We will state several results about turning faces proved in [3].

Our next goal is to prove that if \text{lk}(e_{01}, |NC\ C P_m|) is CAT(1) for all 3 \leq m < n, then \text{lk}(e_{01}, |NC\ C P_n|) is locally CAT(1). By a result in [2], \text{lk}(e_{01}, |NC\ C P_n|) is locally CAT(1) iff for any nonempty cell \sigma of \text{lk}(e_{01}, |NC\ C P_n|), we have

\text{lk}(\sigma, \text{lk}(e_{01}, |NC\ C P_n|)) \simeq \text{lk}(\sigma', |NC\ C P_n|)

is not CAT(1), where \sigma' is a cell in |NC\ C P_n| containing e_{01}. Since |NC\ C P_n| has the orthoscheme metric, by a result of [2] \text{lk}(\sigma', |NC\ C P_n|) is a spherical product of diagonal link of |NC\ C P_m|’s for 3 \leq m < n. Since each factor is CAT(1) by assumption, \text{lk}(\sigma', |NC\ C P_n|) is CAT(1) as desired. From now on, for a fixed n \geq 3, we assume \text{lk}(e_{01}, |NC\ C P_m|) is CAT(1) for all 3 \leq m < n and hence \text{lk}(e_{01}, |NC\ C P_n|) is locally CAT(1).

We say a rectifiable loop l in a complete locally compact path-metric space is shrinkable if there is a rectifiable loop l’ of length shorter than the length of l and there is a homotopy between l and l’ going through length non-increasing rectifiable loops. If l is not shrinkable, then we say l is unshrinkable. Now we will state lemmas in [3] that will give a relationship between CAT(1)-ness of X and conditions in section 1.

**Lemma 3.1.** ([3],Lemma 3.12) Let l be a locally geodesic loop in X. Then for every turning point of l in B, its turning face has a corank that contains two consecutive integers.

**Lemma 3.2.** ([3],Lemma 4.8) Let x \in X be a universal point, and let l be a short loop in X through x. Then l is shrinkable in X.

**Lemma 3.3.** ([3],Lemma 3.15, Lemma 4.9) Let l : I \to X be a locally geodesic segment in X with a turning point t in B. Let E^+ (respectively E^-) be minimal faces
in $X$ containing the image under $l$ of a right (respectively left) $\epsilon$-neighbourhood of $t$ for some $\epsilon > 0$. Then there exist vertices $x^+ \in E^+$ and $x^- \in E^-$ which fail modularity.

We will identify every face of $X$ with the corresponding chain in $\mathcal{NCP}_n$. Lemma 3.1 tells us that if $F$ is a turning face, then $F$ satisfies condition I. Lemma 3.2 tells us that if $l$ is an unshrinkable short loop and $F$ is its turning face, then $F$ is not a universal face and hence $F$ satisfies condition II. Now we will give a relationship between Lemma 3.3 and condition III. There is a relation between failing-modularity with crossing-ness.

**Theorem 3.4.** If $F$ is a turning face of $X$, then $F$ satisfies condition III.

**Proof.** We will identify vertices of $X$ with corresponding elements of $\mathcal{NCP}_n$. If $F = \{x_1, \ldots, x_n\}$ is a turning face, then by Lemma 3.3, there is $x^+ \in E^+$ and $x^- \in E^-$ which fail modularity ($E^+$, $E^-$ are as in Lemma 3.3). The fact that $E^+$, $E^-$ are faces containing $x^+$, $x^-$ are adjacent to every vertex of $F$ in $X$. Since $x^+, x^-$ fail modularity, we cannot have $x^+ \leq x^-$ or $x^+ \geq x^-$. Therefore, either $x_i < x^+, x^- < x_{i+1}$ for some $1 \leq i \leq k - 1$ or $x^+, x^- < x_1$, or $x_1 < x^+, x^-$. 

**Case (1) $x^+ \lor x^- \notin X$.**

Let $e_i$ be the $i$th standard generator of $\mathbb{R}^n$. As an element of $L_n \setminus \{0, 1\}$, $x^+$ is generated by $\{e_{i_1} - e_{i_2}|\{i_1, \ldots, i_k\} \subseteq x^+, 2 \leq j \leq i_k\}$ and $x^-$ is generated by $\{e_{i_1} - e_{i_2}|\{i_1, \ldots, i_k\} \subseteq x^-, 2 \leq j \leq i_k\}$. $x^+ \lor x^-$ is generated by both of them and so it is easy to see that $x^+ \lor x^-$ is generated by $\{e_{i_1} - e_{i_2}|\{i_1, \ldots, i_k\} \subseteq x^+ \lor \rho_n x^-, 2 \leq j \leq i_k\}$, where $\rho_n$ is the join in $\mathcal{P}_n$. Hence $x^+ \lor \rho_n x^- \notin \mathcal{NCP}_n$, which means that $x^+ \lor x^- \notin X$.

**Case (2) $x^+ \land x^- \notin X$.**

There is a non-zero vector $v = (v_1, \ldots, v_n) \in (x^+ \land x^-) \setminus (x^+ \land \mathcal{NCP}_n x^-)$. The fact that $v \notin 0$ and $v \notin x^+ \land \mathcal{NCP}_n x^-$ means that $\sum_{i \in \sigma} v_i = 0$ if $\sigma \in x^+ \cup x^-$. The fact that $v \neq 0$ and $v \notin x^+ \land \mathcal{NCP}_n x^-$ means that $3\sigma_1 \in x^+, 3\tau_1 \in x^-$ such that $\sigma_1 \cap \tau_1 \neq \emptyset$ and $\sum_{i \in \sigma_1 \cap \tau_1} v_i \neq 0$. Clearly, $\sigma_1 \not\subseteq \tau_1$ and $\tau_1 \not\subseteq \sigma_1$ by above conditions. There exists $\sigma_2 \in x^+$ such that $\sigma_2 \neq \sigma_1$ and $\sigma_2 \cap \tau_1 \neq \emptyset$ because if not, for each $i \in \tau_1 \setminus \sigma_1 \neq \emptyset$, we have $\{i\} \in x^+$, so $v_i = 0$. Then $\sum_{i \in \sigma_1} v_i \neq 0$, contradiction. By the same reason, actually we can choose $\sigma_2$ satisfying an additional condition $\sum_{i \in \sigma_2 \cap \tau_1} v_i \neq 0$. In the same way, we choose $\sigma_3$ and so on until the same element appears. So we get a set of distinct blocks $\sigma_1, \ldots, \sigma_k$ and $\tau_1, \ldots, \tau_k, k \geq 2$ satisfying $\sigma_i \cap \tau_i \neq \emptyset$ and $\sigma_k \cup \tau_1 \neq \emptyset$. Now, consider 4 distinct non-empty sets $\sigma_1 \cap \tau_1, \sigma_2 \cap \tau_1, \sigma_2 \cap \tau_2, \sigma_2 + 1 \cap \tau_2 (+ \text{ is done modulo } k)$. Let $y^+ = (x^+ \setminus \{\sigma_1, \sigma_2\}) \cup (\tau_1 \cap \sigma_1, \tau_2 \cap \sigma_2 + 1, (\sigma_1 \cap \tau_1) \cup (\sigma_2 \cap \tau_2))$ and $y^- = (x^- \setminus \{\tau_1, \tau_2\}) \cup (\sigma_1 \cap \tau_1, \sigma_2 \cap \tau_2, (\tau_1 \cap \sigma_2) \cup (\tau_2 \cap \sigma_2 + 1))$. Then $y_i \leq x^+ \land \mathcal{NCP}_n x^- \leq y^+ \land \mathcal{NCP}_n x^- \leq x_{i+1}$ for some $1 \leq i \leq k - 1$ or $y_i \leq x^+ \land \mathcal{NCP}_n x^- \leq x_1$ or $x_k \leq x^+ \land \mathcal{NCP}_n x^- \leq y^+ \land \mathcal{NCP}_n x^- \leq y^+ \land \mathcal{NCP}_n x^- \leq y^+$ such that $\sigma_1 \cap \tau_1 \cup (\sigma_2 \cap \tau_2) \cup (\tau_1 \cap \sigma_2) \cup (\tau_2 \cap \sigma_2 + 1) \in y^-$ as desired. 

We say that a forest in $C$ is a *non-crossing forest* if it is an embedded forest in $C$ whose vertices are elements of $U_n$, edges are straight line segments such that
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1 \leq (\text{the number of edges}) \leq n - 2$. Each non-crossing forest corresponds to a vertex of $X$ via the correspondence that $i$, $j$ are in the same block iff they are in the same connected components in the non-crossing forest.

We say that a tree in $\mathbb{C}$ is a non-crossing spanning tree if it is an embedded tree in $\mathbb{C}$ with vertices are elements of $U_n$, edges are straight line segments such that the number of edges is $n - 1$. For each non-crossing spanning tree, we associate a subcomplex $A$ of $X$ called an apartment. Apartments are geometrically realized full subcomplexes with vertices corresponding to non-crossing forests that are subsets of a fixed non-crossing spanning tree. It is known that apartments are isometric to $S^{n-3}$.

The following notion is defined in [3]. We say that a vertex $u$ of a face $F$ is dominant if, for all apartments $A$ containing $u$, $A$ contains $F$. Note that a dominant vertex of a face is unique if it exists. Suppose that $u_1 < u_2$ are dominant vertices of $F$ and a block $b_2$ of $u_2$ properly contains a block $b_1$ of $u_1$. We can construct an apartment containing $u_2$ such that the corresponding non-crossing spanning tree connects every vertex of $b_2$ by using vertices of $b_1$ only, but does not connect every vertex of $b_1$ by using vertices of $b_1$ only. Then $A$ does not contain $F$, a contradiction. If every block of $u_1$ is a block of $u_2$, then there must be a block $b_2$ of $u_2$ such that each element is a singleton in $u_1$. We can construct an apartment $A$ containing $u_1$ such that the corresponding non-crossing spanning tree connects each vertex in $b_2$ of $u_2$ to a vertex in a block of $u_1$. Then $A$ does not contain $F$, a contradiction.

Recall that $F_i = F_i \cap \{i - 1, i, i + 1\}$. It is true that if $i$ is the dominant vertex of $F$, then for all $1 \leq i \leq n$, we have $u'_i = F'_i$. Indeed, $u_i \supseteq F_i$ by definition, so $u'_i \supseteq F'_i$. Now we will show $u'_i \subseteq F'_i$. If $u_i$ is a block of $u$, then $F_i$ is also a block of a vertex of $F$. If $u_i$ properly contains $F_i$, then we can construct an apartment containing $u_i$ but not $F$ using the same argument in the last paragraph. Hence $u_i = F_i$ and so $u'_i = F'_i$. The remaining case is that $u_i = U_n$. In this case, if $u'_i \neq F'_i$, then say $i + 1 \notin F'_i$. Then we construct an apartment $A$ containing $u$ corresponding to the non-crossing spanning tree that containing the edge connecting $i$ to $i + 1$, such that $i$ is not connected with any other vertices. This can be done because $i$ is a singleton in $u$. Then $A$ does not contain $F$ since $F_i$ is a block of a vertex of $F$ not containing $i + 1$ and all non-crossing subforests of the non-crossing spanning tree cannot connect $i$ with other vertices of $F_i$ by a path avoiding $i + 1$.

To prove that turning faces have to satisfy condition IV, we will state two lemma proved in [3,5] respectively.

**Lemma 3.5.** ([5],Theorem 3.11) Let $X$ be a locally CAT(1) space. Let $x, y \in X$, consider three paths $\alpha_1, \alpha_2, \alpha_3 : [0, 1] \to X$ joining $x$ to $y$. For all $i \in \{1, 2, 3\}$, consider the loop $\gamma_i = \alpha_{i+1}^{-1} \circ \alpha_i$ based at $x$ (with indices modulo 3). Assume that for all $i \in \{1, 2, 3\}$ the loop $\gamma_i$ is short. Assume further that $\gamma_1$ and $\gamma_2$ are shrinkable, then $\gamma_3$ is shrinkable.

**Lemma 3.6.** ([3],Lemma 4.15) Let $C$ be a chamber in $X$ and $i \in U_n$, and let $i, j$ be consecutive elements of $U_n$. If $C_i$ contains $j$, then there exists an apartment in $X$ containing $C$, $v$ and $w$, where $v$ is the boundary edge having the single block $\{i, j\}$ and $w$ is the universal vertex opposite to $v$ in $B$ having the single block $\{1, \ldots, i, \ldots, n\}$.
In fact, this lemma can be strengthened so that $C$ need not be a chamber and can be any face.

**Lemma 3.7.** Let $F$ be a face in $X$ and $i \in U_n$, and let $i,j$ be consecutive elements of $U_n$. If $F_i$ contains $j$, then there exists an apartment in $X$ containing $F,v$ and $w$, where $v$ is the boundary edge having the single block $\{i,j\}$ and $w$ is the universal vertex opposite to $v$ in $B$ having the single block $\{1,\ldots,i,n\}$.

**Proof.** Let $F$ be a face in $X$ and $i \in U_n$, and let $i,j$ be consecutive elements of $U_n$. To prove the lemma, we will find a chamber $C$ containing $F,v$ and $w$ and hence $A$ also contains $F,v$ and $w$.

A chamber $C$ is constructed as follows. Let $F = \{x_{k_1},\ldots,x_{k_n}\}$, where indices are equal to the rank of the corresponding vertex. If $F_i = U_n$, then all vertices of $F$ do not have a block containing $i$, hence $x_{k_1} \leq \{1,\ldots,i,\ldots,n\}$. Every chamber $C$, containing $F$ and the vertex $\{\{1,\ldots,i,\ldots,n\}\}$, has the property that $j \in C_i$. Let $t$ be the minimal number such that $F_t$ is a block of a vertex $x_{k_t} = \{a_1,\ldots,a_r\}$. Say $F_i = a_p$, $1 \leq p \leq r$. For cases where $k_t = 1$ or $k_{t-1} = k_t - 1$, take any chamber $C$ containing $F_t$, then we have $j \in C_i$. For the remaining cases, let $v_{k_t-1} = \{a_1,\ldots,a_{p-1},a_p \{i\},a_{p+1},\ldots,a_r\}$. Then we have $v_{k_t-1} < v_{k_t-1} < v_{k_t}$ (or $v_{k_t-1} < v_{k_t}$ for $t = 1$ case) by minimality of $t$. Now, every chamber $C$, containing $F$ and the vertex $v_{k_t-1}$, has the property that $j \in C_i$. \hfill \Box

Two vertices in $X$ are called opposite if they are at distance $\pi$ in any apartment containing them (in fact, in $X$ too). For example, $v$ and $w$ in the above lemma are opposite. Now, we use this lemma to give a relationship between turning face and condition IV. It is essentially the same routine as in [[3], Lemma 4.16] and [[3], Theorem 4.17].

**Theorem 3.8.** If $F$ is face of $X$ and $F$ does not satisfy condition IV, then $F$ is not a turning face of a short unshrinkable loop.

**Proof.** Suppose that $F$ is a turning face of a short local geodesic loop $l$ in $X$ with length $L < 2\pi$. Let the image $x \in X$ be a turning point. Reparametrize $l$ such that $l(0) = x$. Let $y = l(L/2)$ and let $C$ be a chamber in $X$ containing $y$. Since $F$ does not satisfy condition IV, for every chamber $C'$ in $X$ and for some consecutive integers $i,j$, we have $j \in F_i \cap C'$. So especially for the chamber $C$, for some consecutive integers $i,j$, we have $j \in F_i \cap C_i$. Therefore, by lemma 3.7, applied to $F$ and $C$ independently, there are apartments $A,A'$ such that $A$ containing $F,v,w$ and $A'$ containing $C,v,w$. (Hence, $A$ contains $\{x,v,w\}$ and $A'$ contains $\{y,v,w\}$.) Now, by the fact that $v$ and $w$ are opposite, we have $d(x,v) + d(x,w) = d(y,v) + d(y,w) = \pi$. So at least one element of $\{v,w\}$, say $v$, satisfies $d(x,v) + d(v,y) \leq \pi$. Let $\alpha_1 = l|_{[0,L/2]}$ and $\alpha_2 = l|^{-1}_{[L/2,L]}$ (where, $l^{-1}$ is the reversed path of $l$) be the two subpaths of $l$ from $x$ to $y$. Let $\alpha_3 : [0,d(x,v) + d(v,y)] \to X$ denote the concatenation of the geodesic segments $[x,v]$ and $[v,y]$. Consider the loop $\alpha_3^{-1} \circ \alpha_1$. Since it is short and passes through the universal vertex $v$, by Lemma 3.2 it can be shrunk. Similarly, the loop $\alpha_3^{-1} \circ \alpha_2$ can be shrunk. Hence by Lemma 3.5, $\alpha_2^{-1} \circ \alpha_1$ is shrinkable and so $l$ is shrinkable, contradiction. \hfill \Box
Remark. In [3], their version of Theorem 3.8 is obtained by changing condition IV to the condition, say condition IV', that either $F$ does not have a dominant vertex or if $u$ is the dominant vertex of $F$, then there is a maximal chain $C$ in $\mathbb{C}P_n$ such that $u_i' \cap C_i' = \emptyset$ for all $1 \leq i \leq n$. Note that since $u_i' = F_i'$, condition IV' and condition IV are equivalent if $F$ has a dominant vertex. Hence, condition IV is stronger than condition IV'. For $n = 7$, there is a face $F$ satisfying conditions I, II, III, and IV'. For example, $F = \{\{13\}, \{13457\}\}$ satisfies conditions I, II, III, and IV' since it does not have dominant vertex so it satisfies condition IV'.

In summary, we have the following theorem.

**Theorem 3.9.** If $\text{lk}(e_{01}, |\mathbb{C}P_n|)$ is not CAT(1) and $\text{lk}(e_{01}, |\mathbb{C}P_m|)$ is CAT(1) for all $m < n$, then there is a chain $F$ in $\mathbb{C}P_n$ such that $F$ and $F^*$ satisfy conditions I, II, III, and IV.

**Proof.** Since $X = \text{lk}(e_{01}, |\mathbb{C}P_n|)$ is locally CAT(1) but not CAT(1), there is an unshrinkable short local geodesic loop $l$ in $X = \text{lk}(e_{01}, |\mathbb{C}P_n|)$ by [5]. Since $B$ is CAT(1) and $i \circ l$ is short, $i \circ l$ is not a local geodesic loop in $B$, where $i$ is the embedding from $X$ to $B$. Hence there is a turning point $t$ and let $F$ be its turning face. $F$ has to satisfy conditions I, II, III, and IV, by Theorems 3.4, 3.8 and explanations below Lemma 3.3. Consider the isometry $\phi : X \rightarrow X$ induced by the order reversing bijection of $\mathbb{C}P_n$. The unshrinkable short local geodesic loop $\phi \circ i$ has turning point $t$ again. So, its turning face is $F^*$ and it satisfies conditions I, II, III, and IV as well. \( \square \)

The next lemma is important to prove the main theorem, but its proof requires a long verification. So we will prove the lemma in Section 4.

**Lemma 3.10.** There does not exist a chain $F$ in $\mathbb{C}P_7$ such that both $F$ and its dual $F^*$ satisfy conditions I, II, III, and IV.

**Theorem 3.11.** The diagonal link of $\mathbb{C}P_7$ endowed with the spherical orthoscheme metric is CAT(1), which implies that the Brady–McCammond complex associated to $B_7$ is CAT(0), and hence that $B_7$ is a CAT(0) group.

**Proof.** It is proved in [3] that $\text{lk}(e_{01}, |\mathbb{C}P_m|)$ is CAT(1) and hence $B_m$ is CAT(0) for all $m \leq 6$. If $\text{lk}(e_{01}, |\mathbb{C}P_n|)$ is not CAT(1), then there is a chain $F$ in $\mathbb{C}P_n$ such that $F$ and $F^*$ satisfies conditions I, II, III, and IV, by Theorem 3.9. But there does not exist such chain by Lemma 3.10, a contradiction occurs. Hence $\text{lk}(e_{01}, |\mathbb{C}P_7|)$ is CAT(1) and $B_7$ is a CAT(0) group. \( \square \)

4. Proof of Lemma 3.10

We will prove Lemma 3.10: There does not exist a chain $F$ in $\mathbb{C}P_7$ such that both $F$ and its dual $F^*$ satisfy conditions I, II, III, and IV.

For the sake of clarity, we will first explain the case $n = 6$ which has far fewer cases than the case of $n = 7$. We can check conditions I and II easily. Given chain
$F$ in $\mathcal{NC}_{P_6}$, we only need to prove $F$ does not satisfy conditions I, II, III, and IV up to duality. $F$ can have rank up to duality:

\begin{align*}
\{1, 2\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 3, 4\}.
\end{align*}

But since chains of rank $\{1, 3\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 3, 4\}$ do not have consecutive integers in their corank, $F$ satisfying condition I can have rank up to duality:

\begin{align*}
\{1, 2\}, \{1, 2\}, \{1, 4\}.
\end{align*}

For convenience, we will use a simplified notation for $F$. We will omit non-block elements of partitions and write blocks in a simplified form. For example, we write $F = \{\{1, 2\}, \{3\}, \{4\}, \{5\}, \{6\}\}$, $\{\{1, 2\}, \{3\}, \{4, 5, 6\}\}$ as $F = \{12, 12, \emptyset\}$. If $\text{rk} F = 1$, up to symmetry of $U_6$, $F = \{\{12\}\}$ or $F = \{\{13\}\}$ or $F = \{\{14\}\}$, but $\{\{12\}\}$ has only one element $\{12\}$ which has exactly one block and the block is consecutive. Therefore, $\{\{12\}\}$ does not satisfy condition II, and $F = \{\{13\}\}$ or $F = \{\{14\\}$. If $\text{rk} F = 2$, up to symmetry of $U_6$, $F = \{\{123\}\}$ or $F = \{\{124\}\}$ or $F = \{\{135\}\}$ or $F = \{\{12, 34\}\}$ or $F = \{\{12, 35\}\}$ or $F = \{\{12, 36\}\}$ or $F = \{\{12, 34, 45\}\}$ or $F = \{\{13, 123\}\}$ or $F = \{\{13, 123, 45\}\}$ or $F = \{\{14, 123\}\}$, but every element of $F = \{\{12, 123\}\}$ have exactly one block and the block is consecutive. Therefore, $F = \{\{12, 123\}\}$ does not satisfy condition II, hence $F = \{\{12, 124\}\}$ or $F = \{\{12, 123\}\}$ or $F = \{\{12, 12, 34\}\}$ or $F = \{\{12, 12, 35\}\}$ or $F = \{\{12, 12, 36\}\}$ or $F = \{\{12, 12, 34, 45\}\}$ or $F = \{\{13, 123\}\}$ or $F = \{\{13, 123, 45\}\}$ or $F = \{\{13, 123, 46\}\}$ or $F = \{\{14, 123\}\}$ or $F = \{\{14, 123, 23\}\}$, by inspection, we can check whether $F$ satisfies condition III or not. For example, $F = \{\{13\}\}$ has a failing-modularity pair $P^+ = \{135\}$ and $P^- = \{13, 46\}$. $F = \{\{14\}\}$ does not have failing-modularity pairs. Similarly, $F = \{\{13, 12345\}\}$ has a failing-modularity pair $P^+ = \{135\}$ and $P^- = \{13, 46\}$. $F = \{\{12, 12345\}\}$ does not have failing-modularity pairs. Here is the list of chains $F$ satisfying conditions I, II, and III.
The seven-strand braid group is $\text{CAT}(0)$

\begin{align*}
i) & \quad \text{rk} F = \{1\} \\
& \quad F = \{(13)\}. \\
ii) & \quad \text{rk} F = \{2\} \\
& \quad F = \{(12, 34)\} \text{ or } F = \{(12, 45)\}. \\
iii) & \quad \text{rk} F = \{1, 2\} \\
& \quad F = \{(12), (12, 34)\} \text{ or } F = \{(12), (12, 45)\} \text{ or } F = \{(13), (123)\}. \\
iv) & \quad \text{rk} F = \{1, 4\} \\
& \quad F = \{(13), (13456)\}.
\end{align*}

Now we will check condition IV. Note that condition IV requires only the information of $F_i$’s, not all the information of $F$. Hence if two candidates for $F$ have the same $F_i$’s, we can ignore one of them. For example, $F = \{(12, 34)\}$ and $F = \{(12), (12, 34)\}$ have the same set of $F_i$’s. They have $F_1 = F_2 = \{1, 2\}$, $F_3 = F_4 = \{3, 4\}$, $F_5 = F_6 = U_6$. Similarly, $F = \{(13)\}$ and $F = \{(13), (13456)\}$ have the same set of $F_i$’s. They have $F_1 = F_3 = \{1, 3\}$, $F_2 = F_4 = F_5 = F_6 = U_6$.

After deleting those cases, the remaining cases are:

- $F = \{(13)\}$ or $F = \{(12, 34)\}$ or $F = \{(12, 45)\}$.

Here, we will introduce Haettel, Kielak, and Schwer’s proof that those $F$’s don’t satisfy condition IV [3][Lemma 4.16].

To prove every case at once, we use $F = \{(24)\}$ instead of $F = \{(13)\}$, so $F = \{(24)\}$ or $F = \{(12, 34)\}$ or $F = \{(12, 45)\}$.

Suppose, for a contradiction, that $C$ is a maximal chain in $\mathcal{NCP}_6$ such that for some candidate $F$, $F_i' \cap C_i' = \emptyset$ for all $1 \leq i \leq 6$. Then if $5 \in C_6$ or $1 \in C_6$, then since $F_6 = U_6$ for every case, $F_i' \cap C_i' \neq \emptyset$ we are done. Suppose that $1, 5 \notin C_6$. If $4 \in C_6$, then $4 \in C_5$ and since $4 \in F_5$ for every case, we are done. Similarly, if $2 \in C_6$, then $2 \in C_1$ and $2 \in F_1$ for every case, we are done. The remaining case is $C_6 = \{3, 6\}$. We have $6 \in C_5$ or $4 \in C_5$ and the first two cases of $F$ are done since they have $F_5 = U_6$. If $4 \notin C_5$ then $5 \in C_4$, so the third case of $F$ is done since it has $F_4 = F_5 = \{4, 5\}$.

To generalize the proof for the case $n = 7$, we have analyzed the proof and found a pattern. First, we will give some conditions of configurations of $\{F_1, \ldots, F_n\}$ that make it impossible for $C_i$ to contain some elements (Lemma 4.3). Recall that $F_i \supseteq F_j$ iff $j \in F_i$.

We say that $F$ satisfies (1, +, i) if

(1) $F_i \supseteq F_{i+1}$.

We say that $F$ satisfies (2, +, i) if either

(1) $F_i = F_{i+1}$ or
(2) $F_i \supseteq F_{i+1} \supseteq F_{i+2}$

We say that $F$ satisfies (3, +, i) if either

(1) $F_i = F_{i+1} \supseteq F_{i+2}$,
(2) $F_i = F_{i+1} \subseteq F_{i+2}$,
(3) $F_i \supseteq F_{i+1} = F_{i+2}$, or
(4) $F_i \supseteq F_{i+1} \supseteq F_{i+2} \supseteq F_{i+3}$.

We say that $F$ satisfies (4, +, i) if either
(1) \( F_i = F_{i+1} = F_{i+2} \),
(2) \( F_i = F_{i+1} \supseteq F_{i+2} \supseteq F_{i+3} \),
(3) \( F_i = F_{i+1} \subseteq F_{i+2} \supseteq F_{i+3} \),
(4) \( F_i = F_{i+1} \subseteq F_{i+2} \subseteq F_{i+3} \),
(5) \( F_i \supseteq F_{i+1} = F_{i+2} \supseteq F_{i+3} \),
(6) \( F_i \supseteq F_{i+1} = F_{i+2} \subseteq F_{i+3} \),
(7) \( F_i \supseteq F_{i+1} \supseteq F_{i+2} = F_{i+3} \), or
(8) \( F_i \supseteq F_{i+1} \supseteq F_{i+2} \supseteq F_{i+3} \supseteq F_{i+4} \).

We will also say \( F \) satisfies \((k, +, i)(m)\) if it satisfies condition (m) among conditions for \((k, +, i)\). By replacing the + sign with the − sign in the above conditions, we will define \( F \) satisfies \((k, −, i)\) and \( F \) satisfies \((k, −, i)(m)\) in the same way. For example, we say that \((F, i)\) satisfies \((2, −, i)\) if either

(1) \( F_i = F_{i-1} \) or
(2) \( F_i \supseteq F_{i-1} \supseteq F_{i-2} \).

We say that \( F \) satisfies \((4, −, i)(6)\) if

\[ F_i \supseteq F_{i-1} = F_{i-2} \subseteq F_{i-3} \]

The following lemma and corollary will be used to prove Lemma 4.3.

**Lemma 4.1.** Let \( F \) be a chain in \( NC \mathcal{P}_n \) and let \( 1 \leq i, j \leq n \). If \( F_i \nsubseteq F_j \) and \( F_j \nsubseteq F_i \), then \( F_i \cap F_j = \emptyset \) and \( F_i \parallel F_j \).

**Proof.** Suppose that \( F_i \nsubseteq F_j \) and \( F_j \nsubseteq F_i \). Then none of them are equal to \( \{1, 2, \ldots, n\} \), so \( F_i \) is a block of some \( P_1 \in F \) and \( F_j \) is a block of some \( P_2 \in F \). Without loss of generality we may assume \( P_1 \leq P_2 \) since \( F \) is a chain. Hence, by definition, for some \( \sigma \in P_2 \), we have \( \sigma \supseteq F_i \). Since we assumed \( F_i \nsubseteq F_j \), we have \( \sigma \neq \sigma \cap F_j \) (hence \( \sigma \cap F_j = \emptyset \)). So \( F_i \cap F_j \subseteq \sigma \cap F_j = \emptyset \). To prove \( F_i \parallel F_j \), suppose that \( F_i \parallel F_j \), then \( \sigma \parallel F_j \), a contradiction occurs since \( P_2 \) is a non-crossing partition.

Consider a chain \( F \) in \( NC \mathcal{P}_n \) such that \( F_1 = \{1, 4\} \). Intuitively, the fact that \( 2 \notin F_1 \) implies \( F_2 = \{2, 3\} \) or \( F_2 \supseteq F_1 \) because if \( F_2 \neq U_7, F_2 \) is a block of \( F \) and \( F_2 \) cannot cross \( F_1 \). Now we will prove this intuitive result under more general conditions. Recall that all computations on \( \{1, 2, \ldots, n\} \) are done modulo \( n \).

**Corollary 4.2.** Let \( F \) be a chain in \( NC \mathcal{P}_n \) and let \( 1 \leq i \leq n, 2 \leq k \leq n - 1, \) and \( i + 1 \leq j \leq i + k - 1 \). If \( i, i + k \in F_i \) and \( j \notin F_i \), then either \( F_i \subseteq F_j \) or \( F_j \subseteq \{i + 1, \ldots, i + k - 1\} \).

**Proof.** Fix \( i, j, k \) as in the statement. Suppose that \( i, i + k \in F_i \) and \( j \notin F_i \). We have \( F_j \nsubseteq F_i \) since \( j \notin F_i \). Then by Lemma 4.1, \( F_i \subseteq F_j \) or \( (F_i \cap F_j = \emptyset \) and \( F_i \parallel F_j \). By definition, the latter case implies that there does not exist \( a, c \in F_i \) and \( b, d \in F_j \) such that \( a < b < c < d \) or \( b < c < d < a \). If \( F_j \nsubseteq \{i + 1, \ldots, i + k - 1\} \), choose \( l \in F_j \setminus \{i + 1, \ldots, i + k - 1\} \). Then \( i, k \in F_i \) and \( j, l \in F_j \) such that \( i < j < k < l \) or \( j < k < l < i \), a contradiction occurs. Hence, \( F_i \cap F_j = \emptyset \) and \( F_i \parallel F_j \) implies that \( F_j \subseteq \{i + 1, \ldots, i + k - 1\} \) as desired.
The seven-strand braid group is $\text{CAT}(0)$. Now we will prove a lemma that explaining why the condition $(k, +, i)$ on a chain $F \in \mathcal{NC}_{\mathcal{P}}$ is so named.

**Lemma 4.3.** Let $F$ be a chain in $\mathcal{NC}_{\mathcal{P}}$. If $F$ satisfies $(k, +, i)$ for some $1 \leq i \leq n$, then for all maximal chains $C$ in $\mathcal{NC}_{\mathcal{P}}$ satisfying $F_j' \cap C_j' = \emptyset$ for all $1 \leq j \leq n$, we have $i+1, \ldots, i+k \notin C_i (i-k, \ldots, i-1 \notin C_i$, respectively).

**Proof.** Clearly, it suffices to prove that $(k, +, i)$ cases only. Observe that if $F$ satisfies $(k, +, i)$, then $F$ satisfies $(k'; +, i)$ for all $1 \leq k' < k$. For example, if $F$ satisfies $(4, +, i)$, then $F$ satisfies $(4, +, i)(m)$ for some $1 \leq m \leq 8$. Say $F$ satisfies $(4, +, i)(6)$, then $F_i \supseteq F_{i+1} = F_{i+2} \subseteq F_{i+3}$. Hence $F_i \supseteq F_{i+1} = F_{i+2}$ and $F$ satisfies $(3, +, i)(3)$. Therefore $F$ satisfies $(3, +, i)$. Other cases can be checked in a similar way. Choose a maximal chain $C$ satisfying $F_j' \cap C_j' = \emptyset$ for all $1 \leq j \leq n$.

If $F$ satisfies $(1, +, i)$, then $i + 1 \in F_i$ and $F_i' \cap C_i' = \emptyset$, so $i + 1 \notin C_i$.

If $F$ satisfies $(2, +, i)$, then we have $i + 1 \notin C_i$ by the above observation. Assume that $i + 2 \in C_i$, then by Corollary 4.2 (with $k = 2$ and $j = i + 1$), $C_i \subseteq C_{i+1}$. Hence $i, i + 3 \in C_i'$. But $i \in F_{i+1}'$ (if $F$ satisfies $(2, +, i)(1)$) or $i + 2 \in F_{i+1}'$ (if $F$ satisfies $(2, +, i)(2)$), contradiction since $F_{i+1}' \cap C_{i+1}' = \emptyset$.

If $F$ satisfies $(3, +, i)$, then we have $i + 1, i + 2 \notin C_i$. Assume that $i + 3 \in C_i$ then by Corollary 4.2, either $i, i + 3 \in C_i' \cup C_{i+1}' = \{i + 1, i + 2\}$. For $(3, +, i)(1)$ ($F_i = F_{i+1} \supseteq F_{i+2}$) is excluded since $\{i, i + 2\} \not\subseteq F_{i+1}$. For the remaining cases, suppose that $i, i + 3 \in C_i$, then $i + 2 \notin C_{i+1}$ since $F_{i+1}'$ is nonempty. Since $i + 1, i + 3 \in C_{i+1}$ and $i + 2 \notin C_{i+1}$, by Corollary 4.2 we have $i + 1, i + 3 \in C_{i+2}'$ and a contradiction occurs because $F_{i+2}'$ is nonempty in $(3, +, i)(2), (3, +, i)(3)$, and $(3, +, i)(4)$. Suppose that $C_{i+1} = \{i + 1, i + 2\}$, then since there is a block $\{i + 1, i + 2\}$ of some partition $P \in C$, we have $C_{i+2} = \{i + 1, i + 2\}$ and a contradiction occurs since $i + 2 \in F_{i+1}'$ or $i + 1 \in F_{i+1}'$ in every case.

If $F$ satisfies $(4, +, i)$, then we have $i + 1, i + 2, i + 3 \notin C_i$. Assume that $i + 4 \in C_i$, then by Corollary 4.2, either $C_i \subseteq C_{i+1}$ or $C_{i+1} \subseteq \{i + 1, i + 2, i + 3\}$.

**Claim.** $C_i \not\subseteq C_{i+1}$

If $C_i \subseteq C_{i+1}$, then $i, i + 4 \in C_{i+1}$ and $(4, +, i)(1), (4, +, i)(2), (4, +, i)(3)$, and $(4, +, i)(4)$ are excluded. Since $F_{i+1}' \neq \emptyset$ for every case, we have $i + 2 \notin C_{i+1}$. Hence, $i + 1, i + 4 \in C_{i+2}$ or $C_{i+2} = \{i + 2, i + 3\}$. Suppose $i + 1, i + 4 \in C_{i+2}$ then $C_{i+2} \neq \emptyset$ (5) and $(4, +, i)(6)$ are excluded. Since $F_{i+2}' \neq \emptyset$ in every case, $i + 3 \notin C_{i+2}$. Hence, $i + 2, i + 4 \in C_{i+4}$ by Corollary 4.2 and $F_{i+4}' \neq \emptyset$ for $(4, +, i)(7)$ and $(4, +, i)(8)$, a contradiction occurs. Suppose $C_{i+2} = \{i + 2, i + 3\}$, then $C_{i+3} = \{i + 2, i + 3\}$ and a contradiction occurs because $i + 3 \in F_{i+2}'$ or $i + 2 \in F_{i+3}'$ in $(4, +, i)(5), (4, +, i)(6), (4, +, i)(7)$, and $(4, +, i)(8)$. So the claim is proved.

Hence $C_{i+1} \subseteq \{i + 1, i + 2, i + 3\}$. Which means that either $C_{i+1} = \{i + 1, i + 2\}$, $C_{i+1} = \{i + 1, i + 3\}$, or $C_{i+1} = \{i + 1, i + 2, i + 3\}$. Suppose that $C_{i+1} = \{i + 1, i + 2\}$, then $C_{i+2} = \{i + 1, i + 2\}$ and a contradiction occurs since $i + 2 \in F_{i+1}'$ or $i + 1 \in F_{i+2}'$ in every case. Suppose that $C_{i+1} = \{i + 1, i + 3\}$, then by Corollary 4.2, we have $i + 1, i + 3 \in C_{i+2}'$ and a contradiction occurs since $F_{i+2}' \neq \emptyset$ in every case. So we have $C_{i+1} = \{i + 1, i + 2, i + 3\}$ and then $\{i + 1, i + 2, i + 3\}$ is a block of some partition $P \in C$. Since $C$ is a maximal chain, there is a partition $P' \in C$.
such that $P' < P$ and $P'$ has a block $\sigma$ properly contained in $\{i + 1, i + 2, i + 3\}$. We know that $i + 1 \notin \sigma$ by definition of $C_{i+1}$. So $\sigma = \{i + 2, i + 3\}$ and hence $C_{i+2} = \{i + 2, i + 3\}$ and $C_{i+3} = \{i + 2, i + 3\}$. A contradiction occurs since $i + 2 \in F'_{i+1} \lor i + 3 \in F'_{i+2} \lor i + 2 \in F'_{i+3}$ in every case. □

**Corollary 4.4.** Let $F$ be a chain in $NC_P_n$. If for some $1 \leq i \leq n$, $F$ satisfies condition $(k, +, i)$ and $(l, −, i)$ and $k + l \geq n − 1$, then $F$ does not satisfy condition IV.

**Proof.** If $F$ satisfies condition IV, then we can choose a maximal chain $C$ in $NC_P_n$ such that for all $1 \leq i \leq n$, $F'_i \cap C'_i = \emptyset$. By Lemma 4.3, $i − l, \ldots, i − 1 \notin C_i$ and $i + 1, \ldots, i + k \notin C_i$. Since $k + l \geq n − 1$, we have $C_i \subseteq \{i\}$. A contradiction occurs because $C_i$ must contain $i$ properly. □

Recall that when $n = 6$, we had to check condition IV for cases

- $F = \{\{13\}\}$ or $F = \{\{12, 34\}\}$ or $F = \{\{12, 45\}\}$.

- $F = \{\{13\}\}$ satisfies $(2, +, 5)(1)$, which is $F_5 = F_6$, and satisfies $(3, −, 5)(1)$, which is $F_5 = F_4 \supseteq F_3$. Hence by Corollary 4.4, $F$ does not satisfy condition IV. Similarly, $F = \{\{12, 34\}\}$ satisfies $(3, +, 6)(3)$, which is $F_6 \supseteq F_1 = F_2$, and satisfies $(2, −, 6)(1)$, which is $F_5 = F_6$. Hence by corollary 4.4, $F$ does not satisfy condition IV. Lastly, $F = \{\{12, 45\}\}$ satisfies $(3, +, 6)(3)$, which is $F_6 \supseteq F_1 = F_2$, and satisfies $(3, −, 6)(3)$, which is $F_6 \supseteq F_5 = F_4$. Hence by Corollary 4.4, $F$ does not satisfy condition IV.

Now we are ready to prove Lemma 3.10.

**Proof of Lemma 3.10.** We will enumerate all chains satisfying conditions I, II, and III up to duality, then show none of them satisfies condition IV.

Chains satisfying condition I have rank up to duality:

- $\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 5\}$.

Note that conditions I, II, III, and IV are invariant under symmetries of $U_7$. So we will enumerate chains satisfying conditions I, II, and III up to symmetries of $U_7$. Recall that we used a simplified notation for $F$. We will omit non-block elements of partitions and write block in a simplified form. For example, we write $F = \{\{1, 2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}\}$ as $F = \{\{12\}, \{12, 456\}\}$.

Also, we will omit cases that have the same set of $F'_i$’s as a case that previously appeared, because condition IV is a condition for $F'_i$’s only. For example, we will omit $F = \{\{12\}, \{12, 34\}\}$ in $rkF = \{1, 2\}$ case because it has the same set of $F'_i$’s as the case $F = \{\{12, 34\}\}$ in $rkF = \{2\}$ case.

Checking condition II is straightforward. Also, we can check condition III by inspection.

Here is an enumeration of chains satisfying conditions I, II, and III up to duality and symmetries of $U_7$.

(i) $rkF = \{1\}$

$F = \{\{13\}\}$ or $F = \{\{14\}\}$. 

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(ii) \( \text{rk} F = \{2\} \)
\( F = \{(124)\} \) or \( F = \{(12, 34)\} \) or \( F = \{(12, 35)\} \) or \( F = \{(12, 37)\} \)
or \( F = \{(12, 45)\} \) or \( F = \{(12, 46)\} \).

(iii) \( \text{rk} F = \{3\} \)
\( F = \{(1235)\} \) or \( F = \{(1245)\} \) or \( F = \{(1246)\} \) or \( F = \{(123, 45)\} \)
or \( F = \{(123, 46)\} \) or \( F = \{(12, 34, 56)\} \).

(iv) \( \text{rk} F = \{1, 2\} \)
\( F = \{(13), (123)\} \) or \( F = \{(12), (124)\} \) or \( F = \{(14), (124)\} \)
or \( F = \{(24), (124)\} \).

(v) \( \text{rk} F = \{1, 3\} \)
\( F = \{(13), (1234)\} \) or \( F = \{(14), (1234)\} \) or \( F = \{(12), (123, 45)\} \)
or \( F = \{(13), (123, 45)\} \) or \( F = \{(23), (123, 45)\} \) or \( F = \{(12), (123, 56)\} \)
or \( F = \{(13), (123, 56)\} \).

(vi) \( \text{rk} F = \{1, 4\} \)
\( F = \{(15), (12345)\} \) or \( F = \{(12), (12346)\} \) or \( F = \{(16), (12346)\} \)
or \( F = \{(23), (12346)\} \) or \( F = \{(12), (12356)\} \) or \( F = \{(16), (12356)\} \) or \( F = \{(56), (12356)\} \) or \( F = \{(56), (1234, 56)\} \) or \( F = \{(57), (1234, 57)\} \)
or \( F = \{(67), (1235, 67)\} \) or \( F = \{(67), (1245, 67)\} \).

(vii) \( \text{rk} F = \{1, 5\} \)
\( F = \{(13), (123456)\} \) or \( F = \{(16), (123456)\} \) or \( F = \{(24), (123456)\} \)
or \( F = \{(26), (123456)\} \) or \( F = \{(12), (12345, 67)\} \) or \( F = \{(15), (12345, 67)\} \) or \( F = \{(23), (12345, 67)\} \) or \( F = \{(67), (12345, 67)\} \) or \( F = \{(56), (1234, 567)\} \) or \( F = \{(57), (1234, 567)\} \).

(viii) \( \text{rk} F = \{2, 3\} \)
\( F = \{(124), (1234)\} \).

(ix) \( \text{rk} F = \{1, 2, 3\} \)
\( F = \{(13), (123), (1234)\} \) or \( F = \{(14), (124), (1234)\} \)
or \( F = \{(24), (124), (1234)\} \).

(x) \( \text{rk} F = \{1, 2, 5\} \)
\( F = \{(12), (126), (123456)\} \) or \( F = \{(16), (126), (123456)\} \)
or \( F = \{(12), (12, 34), (123456)\} \) or \( F = \{(12), (12, 45), (123456)\} \)
or \( F = \{(12), (12, 56), (123456)\} \) or \( F = \{(16), (16, 23), (123456)\} \)
or \( F = \{(16), (16, 34), (123456)\} \) or \( F = \{(23), (23, 45), (123456)\} \).

Now we will prove that these chains do not satisfy condition IV by using Lemma 4.3 and Corollary 4.4. For a contradiction, suppose that for each chain \( F \) is the list, let \( C \in \mathcal{NCP}_7 \) be a maximal chain satisfying \( F_i ^j \cap C_i ^j = \emptyset \) for all \( 1 \leq i \leq n \).

(1) \( F = \{(13)\} \) or \( F = \{(14)\} \) or \( F = \{(124)\} \) or \( F = \{(12, 34)\} \)
\( F \) satisfies (3, +, 6)(1) and (3, −, 6)(1) (and apply Corollary 4.4).

(2) \( F = \{(12, 35)\} \)
\( F \) satisfies (3, +, 7)(3) and (3, −, 7)(1).

(3) \( F = \{(12, 37)\} \)
\( F \) satisfies (3, +, 5)(1) and (3, −, 5)(1).

(4) \( F = \{(12, 45)\} \)
\( F \) satisfies (3, +, 7)(3) and (3, −, 7)(1).
(5) \( F = \{12, 46\} \)

\( F \) satisfies \((1, +, 3)(1)\) and \((4, -, 3)(6)\), so \( C_3 = \{3, 5\} \) (since by Lemma 4.3, \((1, +, 3)(1)\) implies \( 4 \notin C_3 \) and \((4, -, 3)(6)\) implies \( 6, 7, 1, 2 \notin C_3 \) and \( C_3 \) properly contains \( 3 \)) and hence \( C_5 = \{3, 5\} \) (since by definition, if \( C_3 = \{3, 5\} \), then \( C_5 = \{3, 5\} \)) but \( F \) satisfies \((4, +, 7)(6)\) and \((1, -, 7)(1)\), so \( C_7 = \{5, 7\} \) and hence \( C_5 = \{5, 7\} = \{3, 5\} \), a contradiction.

(6) \( F = \{1235\} \) or \( F = \{1245\} \)

\( F \) satisfies \((3, +, 7)(3)\) and \((3, -, 7)(1)\)

(7) \( F = \{1246\} \)

\( F \) satisfies \((1, +, 3)(1)\) and \((4, -, 3)(6)\), so \( C_3 = \{3, 5\} \) and hence \( C_5 = \{3, 5\} \) but \( F \) satisfies \((4, +, 7)(6)\) and \((1, -, 7)(1)\), so \( C_7 = \{5, 7\} \) and hence \( C_5 = \{5, 7\} = \{3, 5\} \), a contradiction.

(8) \( F = \{123, 45\} \)

\( F \) satisfies \((3, +, 7)(3)\) and \((3, -, 7)(1)\)

(9) \( F = \{123, 46\} \) or \( F = \{12, 34, 56\} \)

\( F \) satisfies \((3, +, 7)(3)\) and \((3, -, 7)(3)\)

(10) \( F = \{13, 123\} \) or \( F = \{12, 124\} \) or \( F = \{14, 124\} \)

or \( F = \{124, 124\} \) or \( F = \{13, 1234\} \) or \( F = \{14, 1234\} \)

\( F \) satisfies \((3, +, 6)(1)\) and \((3, -, 6)(1)\).

(11) \( F = \{12, 123, 45\} \) or \( F = \{13, 123, 45\} \) or \( F = \{23, 123, 45\} \)

\( F \) satisfies \((3, +, 6)(1)\) and \((3, -, 6)(3)\).

(12) \( F = \{12, 123, 56\} \) or \( F = \{13, 123, 56\} \)

\( F \) satisfies \((4, +, 4)(6)\) and \((1, -, 4)(1)\) and so \( C_4 = \{2, 4\} \) and hence \( C_2 = \{2, 4\} \) but \( F \) satisfies \((1, +, 7)(1)\) and \((4, -, 7)(6)\), so \( C_7 = \{2, 7\} \) and hence \( C_2 = \{2, 7\} = \{2, 4\} \), a contradiction.

(13) \( F = \{15, 12345\} \)

\( F \) satisfies \((3, +, 3)(1)\) and \((3, -, 3)(1)\).

(14) \( F = \{12, 12346\} \)

\( F \) satisfies \((2, +, 3)(1)\) and \((4, -, 3)(6)\).

(15) \( F = \{16, 12346\} \)

\( F \) satisfies \((4, +, 2)(6)\) and \((1, -, 2)(1)\), so \( C_2 = \{2, 7\} \) and hence \( C_7 = \{2, 7\} \) but \( F \) satisfies \((1, +, 5)(1)\) and \((4, -, 5)(6)\), so \( C_5 = \{5, 7\} \) and hence \( C_7 = \{5, 7\} = \{2, 7\} \), a contradiction.

(16) \( F = \{23, 12346\} \)

\( F \) satisfies \((1, +, 5)(1)\) and \((4, -, 5)(7)\), so \( C_5 = \{5, 7\} \). \( F \) satisfies \((4, -, 4)(6)\), so \( 7, 1, 2, 3 \notin C_4 \) and since \( C_5 = \{5, 7\} \), we have \( 5 \notin C_4 \) (if not, \( 7 \in C_5 \subseteq C_4 \)). Hence \( C_4 = \{4, 6\} \), a contradiction since \( C_5 \) and \( C_4 \) are crossing.

(17) \( F = \{12, 12356\} \)

\( F \) satisfies \((4, +, 4)(6)\) and \((1, -, 4)(1)\), so \( C_4 = \{2, 4\} \) and hence \( C_2 = \{2, 4\} \) but \( F \) satisfies \((1, +, 7)(1)\) and \((4, -, 7)(6)\), so \( C_7 = \{2, 7\} \) and hence \( C_2 = \{2, 7\} = \{2, 4\} \), a contradiction.

(18) \( F = \{16, 12356\} \)

\( F \) satisfies \((2, +, 4)(2)\) and \((4, -, 4)(5)\).

(19) \( F = \{56, 12356\} \)

\( F \) satisfies \((4, +, 4)(6)\) and \((1, -, 4)(1)\), so \( C_4 = \{2, 4\} \) and hence \( C_2 =
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\[ \{2, 4\} \text{ but } F \text{ satisfies } (1, +, 7)(1) \text{ and } (4, -, 7)(6), \text{ so } C_7 = \{2, 7\} \text{ and hence } C_2 = \{2, 7\} = \{2, 4\}, \text{ a contradiction.} \]

(20) \[ F = \{56\}, \{1234, 56\} \text{ or } F = \{57\}, \{1234, 57\} \]

\[ F \text{ satisfies } (4, +, 2)(1) \text{ and } (2, -, 2)(1). \]

(21) \[ F = \{67\}, \{1235, 67\} \]

\[ F \text{ satisfies } (4, +, 2)(3) \text{ and } (2, -, 2)(1). \]

(22) \[ F = \{67\}, \{1245, 67\} \]

\[ F \text{ satisfies } (3, +, 3)(3) \text{ and } (3, -, 3)(3). \]

(23) \[ F = \{13\}, \{123456\} \]

\[ F \text{ satisfies } (4, +, 5)(3) \text{ and } (3, -, 5)(1). \]

(24) \[ F = \{16\}, \{123456\} \]

\[ F \text{ satisfies } (3, +, 3)(1) \text{ and } (3, -, 3)(1). \]

(25) \[ F = \{24\}, \{123456\} \]

\[ F \text{ satisfies } (2, +, 7)(2) \text{ and } (4, -, 7)(5). \]

(26) \[ F = \{26\}, \{123456\} \]

\[ F \text{ satisfies } (3, +, 4)(1) \text{ and } (3, -, 4)(1). \]

(27) \[ F = \{12\}, \{12345, 67\} \]

\[ F \text{ satisfies } (3, +, 3)(1) \text{ and } (3, -, 3)(3). \]

(28) \[ F = \{15\}, \{12345, 67\} \]

\[ F \text{ satisfies } (3, +, 3)(1) \text{ and } (3, -, 3)(1). \]

(29) \[ F = \{23\}, \{12345, 67\} \]

\[ F \text{ satisfies } (2, +, 4)(1) \text{ and } (4, -, 4)(6). \]

(30) \[ F = \{67\}, \{12345, 67\} \]

\[ F \text{ satisfies } (3, +, 3)(1) \text{ and } (3, -, 3)(1). \]

(31) \[ F = \{56\}, \{1234, 567\} \text{ or } \{57\}, \{1234, 567\} \]

\[ F \text{ satisfies } (4, +, 2)(1) \text{ and } (2, -, 2)(1). \]

(32) \[ F = \{124\}, \{1234\} \text{ or } F = \{13\}, \{123\}, \{1234\} \text{ or } F = \{14\}, \{124\}, \{1234\} \]

\[ F \text{ satisfies } (3, +, 5)(1) \text{ and } (3, -, 5)(1). \]

(33) \[ F = \{12\}, \{126\}, \{123456\} \text{ or } \{16\}, \{126\}, \{123456\} \]

\[ F \text{ satisfies } (3, +, 4)(1) \text{ and } (3, -, 4)(1). \]

(34) \[ F = \{12\}, \{12, 34\}, \{123456\} \]

\[ F \text{ satisfies } (3, +, 7)(3) \text{ and } (3, -, 7)(3). \]

(35) \[ F = \{12\}, \{12, 45\}, \{123456\} \]

\[ F \text{ satisfies } (3, +, 3)(3) \text{ and } (3, -, 3)(3). \]

(36) \[ F = \{12\}, \{12, 56\}, \{123456\} \]

\[ F \text{ satisfies } (3, +, 3)(1) \text{ and } (3, -, 3)(3). \]

(37) \[ F = \{16\}, \{16, 23\}, \{123456\} \]

\[ F \text{ satisfies } (3, +, 4)(1) \text{ and } (3, -, 4)(3). \]

(38) \[ F = \{16\}, \{16, 34\}, \{123456\} \]

\[ F \text{ satisfies } (4, +, 2)(6) \text{ and } (1, -, 2)(1), \text{ so } C_2 = \{2, 7\} \text{ and hence } C_7 = \{2, 7\} \text{ but } F \text{ satisfies } (1, +, 5)(1) \text{ and } (4, -, 5)(6), \text{ so } C_5 = \{5, 7\} \text{ and hence } C_7 = \{5, 7\}, \text{ a contradiction.} \]

(39) \[ F = \{23\}, \{23, 45\}, \{123456\} \]

\[ F \text{ satisfies } (4, +, 7)(7) \text{ and } (4, -, 7)(7). \]

Hence there are no chains satisfying conditions I, II, III, and IV. \[ \square \]
Although Lemma 3.10 was proven, many cases had to be checked by hand. So, for the sake of accuracy, the author wrote and ran a C++ code. Here is a summary of the code. We use the following method to enumerate non-crossing partitions. Start with the rank 0 partition whose elements are singletons. For every partition, order its members by the smallest element in members. Pick 1 \( \leq i, j \leq 7 \) and merge \( i \)th and \( j \)th members of the rank 0 partition to construct a rank 1 partition. Pick 1 \( \leq i, j \leq 6 \) and merge \( i \)th and \( j \)th members of the rank 1 partition to construct a rank 2 partition. At each stage, we check whether the new partition is non-crossing or not. In this way, we can construct all possible maximal chains \( C \) in \( NC\mathcal{P}_n \). Note that when we take \((i, j)\) to construct a rank 1 partition, even if we choose only within \((1, 2), (1, 3), \) and \((1, 4)\), it is the same as considering every case by the symmetry of \( U_7 \). Then we take faces as subsets of \( C \) so that it satisfies condition I. Checking condition II is straightforward. To check condition III, we have to check whether there are failing modularity pairs \( P^+ \) and \( P^- \) or not. Suppose that \( F = \{P_1, \ldots, P_k\} \) and \( P_i < P^\pm < P_{i+1} \) for some \( 1 \leq i \leq k - 1 \), it is clear that a necessary condition for \( P^\pm \) to crossing is that there are at least 4 distinct members \( a, b, c, d \) of \( P_i \) such that \( a, b, c, d \) are contained in a single member of \( P_{i+1} \). (Otherwise, there are no places for crossing.) If such \( i \) exists, and \( P_i < P^\pm < P_{i+1} \) and \( P^\pm \) are crossing then we can choose minimal \( P^\mp \) such that for some distinct members \( a, b, c, d \) of \( P_i \), \( P^+_i \) is obtained from \( P_i \) by merging \( a, c \) into one block and \( P^-_i \) is obtained from \( P_i \) by merging \( b, d \) into one block. (Such minimal \( P^\pm \) can be obtained by excluding merging irrelevant to crossing.) For each such \( i \), the code searches through distinct members \( a, b, c, d \) in \( P_i \) such that \( P^+_i \) is obtained from \( P_i \) by merging \( a, c \) into one block and \( P^-_i \) is obtained from \( P_i \) by merging \( b, d \) into one block are non-crossing partitions and \( P^\pm \) are crossing. The cases \( P^\pm < P_1 \) and \( P_k < P^\pm \) are similar. In this way, the code can check condition III. We checked condition IV by enumerating all maximal chains \( C \) and calculate \( F_i \) and \( C_i \) to check whether \( F_i' \cap C_i' = \emptyset \) for all \( 1 \leq i \leq n \) directly, so we didn’t use Corollary 4.4. The enumeration of chains of non-crossing partition we used is simple, but there are many repeating elements. (because subchains of two distinct maximal chains can be the same.) As a result, there are 7138 faces satisfying conditions I, II, and III. Just as we have proven, there are no faces satisfying condition IV among those faces. It took 2496 seconds to run the code.

**Remark.** It is natural to ask whether Lemma 3.10 holds for \( NC\mathcal{P}_n, n > 7 \). Unfortunately, there is a chain \( F \) satisfying conditions I, II, and IV as follows.

Consider the chain (with one element) \( F \) in \( NC\mathcal{P}_8 \) such that \( F = \{P_1\} \) with \( P_1 = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}\} \), then \( F \) satisfies conditions I, II, and III. \( F \) satisfies condition IV because for any maximal chain \( C \) containing the element \( \{\{2, 3\}, \{4, 5\}, \{6, 7\}, \{8, 1\}\} \), then for any consecutive integers \( 1 \leq i, j \leq 8 \), we have \( j \notin F_i \cap C_i \) (for example, \( F_2 = \{1, 2\}, C_2 = \{2, 3\} \), so \( 1, 3 \notin F_2 \cap C_2 = \{2\} \)).

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