Modeling of multiple fractures growth during multistage hydraulic fracturing based on cell-based pseudo 3D model

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Abstract. The paper presents a mathematical model describing the process of formation of multiple fractures in inhomogeneous rock formation. The features of its numerical implementation and the results of implementation are given. Modelling of fracture growth dynamics is carried out in the frame of cell-based pseudo 3D model, taking into account the associated solution of the problems of fracture mechanics and multiphase flow of a proppant mixture acting as a proppant dispersed phase and a power law rheology fluid. To calculate the flow parameters at the points of initiation of multiple fractures for given wellhead data, the model implements the calculation of hydraulics in the wellbore based on a homogeneous multiphase model. Pathway of fracture development is determined for arbitrary field of the strained deformed state of the formation and takes into account its change caused by stress shadow effect. The developed model allows to evaluate the efficiency of the application of the multi-stage fracturing technology with several perforation clusters or hydraulic fracturing ports in a single stage, as well as to optimize the completion of a horizontal well.

1. Introduction

Receiving information on fracture growth dynamics in the rock formation is limited by a few number of indirect methods, the most common of which is micro-seismic monitoring. As no direct methods for determining the fracture geometry during its growth in the formation exists we are to appeal to results of numerical modelling. Despite a relatively long period of commercial use of hydraulic fracturing (HF) technology, there is still no generally accepted physical and mathematical model capable to describe all key processes occurring during HF treatment: rock destruction with heterogeneous mechanical (elastic-strength) properties, mass transfer of a mixture of non-Newtonian fluid and solid phase within the fracture channel.

The complexity of joint modeling of all processes is due to their strong nonlinear relation on the one hand and the need for a sufficient set of data for numerical experiments — on the other. The problem is coming less trivial while examining the case of multiple fractures growth as one needs to take into account the interaction between fractures via induced pressure and the wellbore hydraulics as well [1].
To date, a great number of works are devoted to separated problems on the topic of HF, a much smaller number of works describe the joint solution of all processes. In this regard, there is a need for a comprehensive solution to the problem of determining the dynamics of fractures growth with reasonable simplifying assumptions and at the same time sufficient for efficient planning of fracture design for tackling real industrial problems.

2. Problem statement
To tackle the problem of multi-stage HF it is first worth making a problem statement for a single fracture growth within an inhomogeneous geological formation [2]. The problem of a single fracture growth is posed as follows. For given profile of the elastic-strength properties of formation (Young’s modulus $E$, Poisson’s ratio $\nu$, the minimum horizontal stress $\sigma_{\text{min}}$, the coefficient of fracture toughness $K_{IC}$, the coefficient of leakage of $C_l$), the carrier phase properties (density $\rho_f$, the rate of non-Newtonian behavior of $n$, the index of consistency of $K$) and particles of the proppant (material density $\rho_p$, particle radius $a$), and mode download (volume flow rate of the mixture, $q_m$, the volume concentration of proppant $\beta$) we need to determine the dynamics of fracture growth, — time dependences of the fracture length $L$, the opening profile $w$, the fracture height $h_f$, and the fluid overpressure profile in the fracture channel $p_{\text{net}} = p_f - \sigma_{\text{min}}$.

While modeling numerically the HF processes, the mass transfer problem must be solved in a joint way for the ”wellbore-fracture” system by linking solutions at the bottom of the well. This is especially essential when the problem of simultaneous growth of fractures on several cuts within a single cluster of perforations in the horizontal wellbore is considered. At the same time, during the transfer of the working fluid and proppant in the wellbore there are pressure losses due mainly to viscous friction. Thus, to determine the order of initiation of cracks and the distribution of costs for perforations, it is necessary to evaluate the pressure losses in the wellbore, solving the problem associated with fractures growth.

The problem of multistage HF (MHF) can be formulated as follows. For given properties of the rock formation, fluid and proppant particles, as well as the injection mode set at the wellhead, it is a goal to determine the fracture dynamics at each stage.

3. Multi-stage hydraulic fracturing model

3.1. Fracture model
The cell-based P3D model is chosen for linking the relation between the pressure inside the fracture channel and its width profile. In the frame of P3D model the fracture is represented with hydrodynamically associated volume cells (hereinafter cells) $A_f \Delta s$, where $\Delta s$ — the length of the cell ($s$ — curvilinear coordinate that determines the direction of fracture growth), $A_f$ — cross-sectional area of the channel cracks that are defined for the specified profile disclosure $w(s, z)$ ($z$ — vertical coordinate) and fracture height $h_f(s)$ expression:

$$A_f(s) = \int_{-h_f(s)/2}^{h_f(s)/2} w(s, z)dz.$$ (1)

The dynamics of the fracture channel cross-sectional area change $A_f$ is determined by the fluid flow at the fracture inlet: continuity and momentum conservation equations are solved. The model implies the presence of a one-dimensional pressure profile inside the fracture $p(s)$, changing only along the direction of growth, i.e. along the curvilinear coordinate $s$. The fracture opening profile $W(z)$ for a given pressure in the $p$ cell is determined in the frame of the geomechanics problem.
3.2. Mass transfer model

The multiphase flow problem in the fracture channel is based on approximation of the effective medium, where the suspension is considered as a viscous incompressible fluid with a density and viscosity depending explicitly on the volume fraction of the dispersed phase. From the theory of the lubricant layer we have the following expression for the local continuity of the liquid and dispersed (proppant particles) phases in the fracture channel of variable cross-section $A_f$:

$$\frac{\partial (1 - \beta) A_f}{\partial t} + \frac{\partial (1 - \beta) q}{\partial s} + (1 - \beta) q_l = 0,$$

where $\beta$ — volumetric proppant concentration, $q$ — volumetric flow rate per unit of fracture, $\delta$ — the Kronecker’s symbol, $q_l$ — volume flow leakage from defined by Carter’s model [4] as follows

$$q_l(s, t) = \frac{2C_l h_f}{\sqrt{t - \tau(s)}},$$

where $t$ — time elapsed since the beginning of the leakage process, $\tau(s)$ - time of arrival of the fracture tip boundary to the position $s$, $C_l$ — empirical Carter’s leakage coefficient.

The relation between the mixture flow $q$ and the net pressure $p_{net}$, following from the solution of the problem of steady laminar fluid flow between two parallel planes of unit height spaced at a distance $w$, is given by the Poiseuille equation:

$$q = -\Lambda(w, p) \frac{\partial p_{net}}{\partial s},$$

where $\Lambda$ — fluid mobility operator.

As most of HF operations exploit gel-like working mixtures we consider a model of a non-Newtonian fluid whose shear stresses $\tau$ depend nonlinearly on the instantaneous shear rate $\dot{\gamma}$. The power law model is chosen as it describes the rheology of most standard HF fluids $\tau = K \dot{\gamma}^n$, where $K$ — the flow consistency index, $n$ — the flow behavior index.

For a model with power rheology, the mobility operator has the following relation:

$$\Lambda(w, p) = \frac{n}{(2n + 1)2(n+1)/n} K^{-1/n} w^{2n+1/n} \left| \frac{\partial p_{net}}{\partial s} \right|^{\frac{1-n}{n}}.$$

The influence of the solid phase on the viscosity behavior of the mixture was taken into account by using the empirical dependence of the consistency index on the volume concentration of proppant $K(\beta) = K_f (1 - \beta/\beta^*)^{5n/2}$, where $K_f$ — the consistency index of pure liquid, $\beta^*$ — the critical concentration of the solid phase.

The boundary conditions are zero volume flow (degeneration of the Stefan’s condition) and zero opening at the fracture tip ($s = L$):

$$q(t, L) = 0, w = 0, \forall t > 0,$$

as well as the conditions on the flow rate of the mixture and concentration of proppant at the fracture inlet:

$$q(0, t) = q_0(t) \quad \beta(0, t) = \beta_0(t) \quad \forall t > 0.$$

The initial condition is zero fracture opening and zero proppant concentration:

$$w(s, 0) = 0, \beta(s, 0) = 0.$$
3.3. Geomechanics problem

The geomechanics problem for each volume cell of the fracture is solved in two stages: first, the optimization problem of finding the equilibrium state (the positions of the lower and upper boundaries of the cell \(z_b, z_t\)) for a given pressure \(p\) is solved. After, the width profile \(w(z)\) is calculated for given pressure.

The equilibrium height is achieved in the case when the stress intensity factor (SIF) at the tip of the fracture coincides with the coefficient of fracture toughness \(K_{IC}\), i.e., \(K_\pm = K_{IC}\). In a linearly elastic medium, the SIF at the upper and lower tips of the fracture (\(K_{I+}\) and \(K_{I-}\), respectively) is defined by the following relation [3]:

\[
K_\pm = \sqrt{\frac{2}{\pi h_f}} \int_{-h_f/2}^{h_f/2} (p - \sigma(z)) \sqrt{\frac{h_f/2 \pm z}{h_f/2 \pm z} - \frac{1}{2}} dz, \quad (10)
\]

where \(\sigma(z)\) — the stress profile, \(h_f\) — the fracture height, \(p\) — pressure weighted in the vertical cross-section.

The problem of the relation between the pressure in the fracture cell \(p\) and its opening profile \(w\) is solved by finding the global minimum for the functional:

\[
\min_{z_t, z_b}\{|K_+(z_t, z_b) - K_{IC}(z_t)| + |K_-(z_t, z_b) - K_{IC}(z_b)|\}, \quad (11)
\]

by limits arising from the failure criterion

\[
K_+(z_t, z_b) \geq K_{IC}(z_t), \quad K_-(z_t, z_b) \geq K_{IC}(z_b), \quad (12)
\]

where \(K_{IC}(z)\) — a given fracture toughness profile.

The SIF is convenient to rewrite in the following form:

\[
K_\pm(z_t, z_b) = \sqrt{\frac{2}{\pi(z_t - z_b)}} \int_{z_b}^{z_t} (p - \sigma(z)) \left(\frac{z_t - z}{z_t - z_b}\right)^{\pm1/2} dz. \quad (13)
\]

The resulting functionality has a number of features. First, the functional includes the integral SIF representation for the stress profile in general case, while the required optimization parameters \(z_t, z_b\) are included both in the integral and in the integration boundaries. Second, as the stress profiles \(\sigma(z)\) and the fracture toughness coefficient \(K_{IC}(z)\) are given by piecewise constant functions, the surface of the target functional suffers a first-order discontinuity.

Non-local relationship between the pressure \(p\) and profile disclosure \(w(z)\) in a linearly elastic medium is given by relation:

\[
w(z) = \frac{4}{\pi E'} \int_{0}^{h_f} (p - \sigma(z')) K(z, z'; h_f) dz', \quad (14)
\]

where \(E' = E/(1 - \nu^2)\) — module of plane deformation, \(K(z, z'; h_f)\) — singular kernel defined by expression:

\[
K(z, z'; h_f) = \ln \left| \frac{\sqrt{h_f^2 - z^2} + \sqrt{h_f^2 - z'^2}}{\sqrt{h_f^2 - z^2} - \sqrt{h_f^2 - z'^2}} \right|. \quad (15)
\]
3.4. Join solution of multi-stage hydraulics fracturing and wellbore hydraulics

The problem of transfer of a mixture of liquid and proppant in the wellbore of an arbitrary cross section \(A_w(l)\) and an arbitrary geometric profile is considered in a non-stationary quasi-one-dimensional formulation by averaging the continuity equations and preserving the moment of pulses along the wellbore cross section: all values change only along the wellbore (along the curvilinear coordinate \(l\), measured from the wellhead to the bottom of the well).

Given the flow rate of the working mixture \(q_0(t)\) and the volume concentration of the solid phase \(\beta_0(t)\) at the wellhead (where the wellbore intersects the earth’s surface), we need to determine the same parameters at the fracture inlets, i.e. the boundary conditions for fracture inlets \(q_i(t), \beta_i(t)\), where the index \(i\) denotes \(i\)-th fracture (Figure 1).

![Figure 1. Multi-stage hydraulic fracturing scheme](image)

The continuity equations for the mixture of liquid and proppant are obtained in the approximation of weakly compressible fluid. Taking into account the boundary conditions at the wellhead and effluents in the perforations (in the holes of the well column, where fracture growth is initiated), the equations for the pressure of \(p\) and the proppant concentration of \(\beta\) in the wellbore are obtained:

\[
A_w c \frac{\partial p}{\partial t} + A_w v_w \frac{\partial p}{\partial l} = \delta(l)q_{M0}(t) - \sum_{I=1}^{N_f} \delta(d - l_i)q_{mi},
\]

\[
A_w \frac{\partial \beta}{\partial t} + A_w \beta v_w \frac{\partial \beta}{\partial l} = \delta(l)\beta_0(t)q_{M0}(t) - \sum_{I=1}^{N_f} \delta(d - l_i)\beta_i q_{mi},
\]

where \(N_f\) — number of fractures, \(A_w = \pi r_w^2\) — cross-sectional area of the bore of the table, \(q_{M0}\) and \(\beta_0\) — volumetric flow rate of the mixture and concentration of proppant at the wellhead, respectively, \(q_{mi}\) and \(\beta_i\) — volumetric flow rate of the mixture and the volume concentration of proppant in the \(i\)-th perforation hole (for \(i\)-th fracture), respectively, \(v_w\) — the speed of the mixture, \(c\) — the liquid compressibility coefficient.

In the approximation of the rapidly setting pressure, the expression for pressure losses in the wellbore taking into account the variable cross-sectional area of the wellbore has the form:
\[
\frac{\partial A_w \phi}{\partial l} = -2\pi r_w \tau + A_w \rho_m g \cos \phi(l), \quad \tau = \frac{f(Re) v_w |v_w|}{2},
\]  
(18)

where \( \phi(l) \) is the inclinometry angle, \( \rho_m = (1 - \beta) \rho_f + \beta \rho_p \) is the phase weighted mixture density, \( \tau \) is the tangential stresses, \( f(Re) \) is the friction coefficient depending on the flow regime.

The Reynolds number \( Re \) for fluid flow with a power-law rheology:
\[
Re = \frac{(2r_w)^n v_w^{2-n} \rho_f}{8n^{-1} (3n+1)^n K}.
\]  
(19)

The coefficient of friction \( f \) for laminar flow regime (\( Re \leq 3250 - 1150n \)) is determined by the expression \( f = 16/Re \), in turbulent flow regime we have a transcendental expression for the relation between the friction coefficient and Reynolds number for the fluid of power rheology:
\[
1/\sqrt{f} = \frac{4}{n^{3/4}} \log (Re f^{1-n/2}) - 0.395n^{-1.2}.
\]  
(20)

The system of equations is supplied with equations on mass transfer (2, 3) of geomechanics and each (11, 14) \( i \)-th fracture. The system of integral equations is closed the flow of mixture and the volume concentration:
\[
q_{M0}(t) = \sum_{I=1}^{N_f} q_{mi}, \quad \beta_0(t) = \sum_{I=1}^{N_f} \beta_I.
\]  
(21)

Boundary conditions at the wellhead are given:
\[
q_m(0,t) = q_0(t), \quad \beta(0,t) = \beta_0(t), \quad \forall t > 0.
\]  
(22)

At the initial time, the well is filled with buffer fluid with zero proppant concentration, the pressure is determined by hydrostatics:
\[
v_m(l, t = 0) = 0, \quad \beta(l, t = 0) = 0, \quad P(L, t = 0) = \rho_f g z(l), \quad \forall l \subset [0, L_w].
\]  
(23)

3.5. Stress shadow

The problem of mutual influence of fractures on each other is solved by the method of discontinuous displacements proposed by S. L. Crouch, A. M. Starfield [5], [6]. The method is based on solving the problem of an infinite plane, the displacements of which suffer a constant gap within the elementary segment. The discontinuous displacement method makes it possible to replace discontinuities of displacements continuous along the fracture with a discrete approximation, namely, to divide the fracture into a discrete set of boundary elements, considering the fracture within each element to be rectilinear, and its opening to be constant.

We assume that the fracture opening width is equal to the normal displacement rupture, there is no shear stress on the fracture surface.

Induced stresses acting on the fracture element in the presence of discontinuous displacements in the computational domain are expressed explicitly:
\[
\sigma_n^i = \sum_{j=1}^{N} A^{ij} C_{ns}^{ij} D^{ij}_s + \sum_{j=1}^{N} A^{ij} C_{ni}^{ij} D^{ij}_n, \quad \sigma_s^i = \sum_{j=1}^{N} A^{ij} C_{ss}^{ij} D^{ij}_s + \sum_{j=1}^{N} A^{ij} C_{si}^{ij} D^{ij}_n,
\]  
(24)

where \( D^{ij}_s \) and \( D^{ij}_n \) — tangential and normal components of displacement breaks the \( j \)-th item and \( C^{ij} \) — boundary influence coefficient of the gap offsets the \( j \)-th element of tension in the \( i \)-th element of \( A^{ij} \) — coefficients proposed by D. Olson [7] that account for the finite height of fractures.
4. Results

This section presents an example of a multi-stage fracturing simulation based on real data. In the example, a horizontal well with 3 stages of fracturing with a different number of ports for each stage is considered (1st stage - 1 port, 2 stage - 2 ports, 3 stage - 3 ports). Figure 2 shows the well trajectory (top view) and the port positions in the stages of the MHF.

![Figure 2. Results of MHF modeling based on real data. Fracture pathawys.](image1.png)

The values of the geomechanical properties for the depth of the target interval (minimum horizontal stresses, Young’s modulus, Poisson’s ratio) are shown in the Figure 3. The value of toughness is equal to 10 atm m$^{1/2}$. The leakage coefficient throughout the entire section is equal to $6 \times 10^{-5}$ m sec$^{1/2}$.

In each stage of MHF a mixture of liquid with power-law rheology ($K=$0.1 Pa sec$^{n}$, $n=0.7$) and proppant 0.5 mm in diameter was pumped at a flow rate of 3.5 m$^3$/min. During the first 22 minutes of the stage pure fluid was pumped, then the proppant concentration increased from 125 kg/m$^3$ to 275 kg/m$^3$. For each stage: total injection time 38 min, total mass of proppant 9 800 kg, total fluid volume 133 m$^3$. The simulation results are shown in Figures 4, 5.

![Figure 3. Mechanical properties profiles.](image2.png)

![Figure 4. Width profile of hydraulic fracture at stage 1.](image3.png)

The results show an expected difference in the total length of fractures which observed in the case of development of 1, 2 and 3 fractures simultaneously, caused by a decrease in the amount...
of fluid and proppant per stage. At a given distance between ports (more than 70 meters), an insignificant change in the path of the development of fractures is observed. Calculation of the proppant deposition after stopping the injection made it possible to estimate the conductivity of the fixed fractures. For example, for the first stage, the half-length of the fracture is reduced from 400 to 240 m, and the height of the conductive part of the fracture after depositing proppant was reduced from 30 to 10 m.

5. Conclusions
A mathematical model for multistage hydraulic fracturing in the inhomogeneous rock formation is given.

The constructed numerical solution allows separately and in aggregate way to carry out: analysis of the influence of the tip screen-out effect on the profile of the proppant-fixed fracture, analysis of the stress shadow effect on the fixed fracture profile and its propagation path in the rock, injection modes, properties of the carrier and dispersed phases on the dynamics of fracture growth.

On the basis of developed approach to MHF modeling it is possible to evaluate the efficiency of the application of MHF technology in the horizontal well, in particular to specify an injection rate at the wellhead, various types of proppant and liquid, a geomechanical section with an arbitrary number of layers, and a spatial profile of the well with an arbitrary number of stages of the MHRP and fracture initiation points in each of them.

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