Coded Caching and Content Delivery with Heterogeneous Distortion Requirements

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Abstract

Cache-aided coded content delivery is studied for devices with diverse quality-of-service (QoS) requirements. The network consists of a server, which holds a database of independent contents, and users equipped with local caches of different capacities. User caches are filled by the server during a low traffic period without the knowledge of particular user demands. It is assumed that each user in the system has a distinct QoS requirement, which may be due to the viewing preference of the user, or the display capability of the device. Each user requests a single file from the database to be served at this fixed QoS level, and all the requests are satisfied simultaneously by the server over a shared error-free link. Each file is modeled as a sequence of independently and identically distributed Gaussian samples, and the QoS is measured in terms of the average squared error distortion between the original sequence and the reconstruction. Our goal in this work is to characterize the minimum delivery rate the server needs to transmit over the shared link to satisfy all possible demand combinations, both in the centralized and decentralized settings.

For a centralized caching system, the optimal coded caching scheme is characterized for the two-file two-user scenario for any pair of target distortion requirements. For the two-user scenario with more than two files, the optimal scheme is characterized when the cache capacities of the users are the same and the number of files is a multiple of 3. For the general case with arbitrary number of users and files, a layered caching and delivery scheme is proposed, which exploits the successive refinability of Gaussian sources. In the proposed layered scheme, the centralized lossy coded caching problem is divided into two subproblems: the lossless caching of each layer with heterogeneous cache sizes, and the problem of cache allocation among the layers. A delivery rate minimization problem is formulated and solved

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This paper was presented in part at the IEEE International Symposium on Information Theory, Barcelona, Spain, Jul. 2016 [30].
numerically for each layer; and two different schemes, i.e., proportional cache allocation (PCA) and ordered cache allocation (OCA), are proposed to allocate each user’s cache among layers.

The decentralized lossy caching problem is also studied, where no coordination is available for content placement. A decentralized lossy coded caching scheme is proposed, together with the delivery rate analysis. Simulation results validate the effectiveness of the proposed schemes in both settings.

I. INTRODUCTION

Consumer demand for mobile video services is growing at an unprecedented rate. This trend is expected to continue in the coming years driven by the proliferation of mobile devices with high quality display capabilities, and the explosion of high data rate multimedia contents available online. The majority of video traffic today is generated through “video surfing”, i.e., streaming of video files stored in large databases, e.g., YouTube, Hulu, Dailymotion, BBC iPlayer, etc. Such video traffic is dominated by a relatively small number of viral contents that remain popular over a certain period of time, and downloaded repeatedly by many users, sometimes connected through the same access point, creating a huge amount of repetitive traffic both on the core and the radio access networks.

Caching has long been used in the Internet to reduce traffic, as well as latency (see [1], and references therein). More recently, research on content caching has regained popularity, targeting mainly wireless networks (see [2]–[4], and references therein). While content caching at the evolved packet core, or at the radio access network, can reduce traffic and latency on the backhaul links [3], [5], caching contents directly at user devices can bring further benefits [4], [6]. The latter strategy is called proactive caching since caching at a user device requires predicting the demand, and delivering the content even before it is requested by the user. Since mobile data traffic at wireless access points exhibits a high degree of variation across time, exploiting the radio resources during low traffic periods through proactive caching will also reduce the peak traffic rates; and therefore, improve the quality-of-experience (QoE) for users, and reduce infrastructure costs for network providers. Feasibility of proactive caching in future wireless networks is further supported by the low-cost and abundance of storage space in today’s mobile devices.

In the proactive caching model considered here, users fill their caches during the off-peak traffic period, referred to as the placement phase. Each user’s cache content at the end of
the placement phase is a function of the whole database. User requests, one file per user, are revealed during the peak traffic period, and satisfied simultaneously during the delivery phase. Conventional uncoded caching schemes store contents, partially or fully, at each user’s cache, and utilize orthogonal unicast transmissions during the delivery phase. The gain from uncoded caching for each user depends only on the local cache capacity. Recently, Maddah-Ali and Niesen introduced a novel model for cache networks [6], particularly appropriate for proactive caching in wireless networks, in which the delivery phase is carried out over a shared link, modeling the broadcast nature of wireless communications. They show that centralized coded caching, in which coded bits, rather than plain bits of the available contents in the database, are cached and delivered, achieves significant gains compared to uncoded caching. This gain is realized by jointly optimizing the placement and the delivery phases to create multicasting opportunities even among distinct requests [6]. In contrast to the centralized setting in [6], where the active users are known in advance, [7] considers the so-called decentralized setting, in which, during the placement phase, the server has no prior knowledge on the number and identity of users that will participate in the delivery phase. It is shown that the multicasting opportunities still appear even if users randomly cache bits of files [7].

Numerous papers followed [6] and [7] in order to further improve the coded caching gain, and to apply it to various other network models. Chen et al. [8] achieve the optimal delivery rate for small buffer sizes by placing coded contents into users’ caches during the placement phase. When the number of users is larger than the number of files, improved delivery rates are obtained in [9], [10]. Pedarsani et al. introduce a coded least-recently sent delivery and update rule that replaces the cache content during the delivery phase for online caching systems [11]. A multi-layer caching system, in which user terminals, proxies, base stations are all equipped with cache memories, is considered in [12]. A distributed caching system is investigated in [13] with single-hop device-to-device communication, which shows that coded caching has the same scaling law with the spatial reuse of user caches. Delivery over a noisy broadcast channel is considered in [14], [15]. Similarly, delivery of contents over an interference channel is considered in [16]–[18], where both the transmitters and receivers have caches.

Common to the aforementioned works and most of the other follow-up papers in the literature, is the assumption that the files in the database have fixed sizes, and each user requests one of these files in whole. However, in practice, video contents are usually downloaded at different quality
levels by users, which may be due to their viewing preferences, or the display and processing capabilities of their devices. For example, a laptop may require high quality descriptions of requested files, whereas a mobile phone is satisfied with much lower resolution. In current video coding standards, diverse reconstruction capacities and demands of users is handled through scalable video coding (SVC). The H.264/MPEG-4 standard [19], [20] allows temporal (frame rate), spatial (picture size) or SNR/quality scalability. This is achieved by encoding the videos into multiple bit streams, i.e., substreams, such that the more substreams users receive, the higher the corresponding resolution is. In this work, we consider users with heterogeneous quality-of-service (QoS) requirements, that is, each user, instead of requesting a file in the database in full, may request a lower resolution copy. Accordingly, we exploit successive refinement [25] for the scalable compression of the files available in the database. This provides flexibility to the server not only in supporting the multiple quality levels requested by the users, but also in exploiting the different cache capacities of the users.

In particular, we consider the lossy version of the coded caching problem, such that each user has a preset average distortion requirement. This distortion target is user-dependent, and is the same for any file the user may request. In centralized caching, it is assumed that the server knows the distortion requirements of all the users in the system during the placement phase, whereas in decentralized caching only the set of possible distortion requirements is known to the server. Given the cache capacities and the distortion requirements of the users, the objective of the server is to design the placement and the delivery phases jointly, in order to minimize the delivery rate while guaranteeing that all possible demand combinations can be satisfied at the required distortion levels. For the sake of mathematical analysis, the files in the database are modeled as independent sequences of Gaussian distributed random variables. The main contributions of this paper can be summarised as follows:

- We derive a theoretical lower bound on the delivery rate based on cut-set arguments.
- For the centralized lossy caching problem, we characterize the optimal delivery rate for the two-file two-user case.
- We propose a centralized coded caching scheme for the case with two users and an arbitrary number of files, which is proven to be optimal when the cache capacities of the users are the same, and the number of files is a multiple of 3.
- For the general case, we propose a coded caching algorithm based on successive refinement
coding of the video files into as many layers as the number of different distortion requirements. We then divide the problem into two subproblems: the lossless caching problem of each layer with heterogeneous cache sizes, and the cache allocation problem among different layers. The first subproblem is formulated as an optimization problem using the achievable delivery rates available in the literature [6], [8], [9], and solved numerically. Then, two cache allocation algorithms, i.e., proportional cache allocation and ordered cache allocation, are proposed, and their performances are compared with each other and the theoretical lower bound through numerical simulations.

- We propose a coded caching scheme for the decentralized lossy caching problem, and derive its delivery rate.

The most related work to this paper is [21], in which, by exploiting decentralized coded caching scheme proposed in [22], Hassanzadeh et al. optimize allocation of cache capacities to minimize the average distortion across users, constrained by the delivery rate over the shared link and the cache capacities of the users. Successive refinement coding of the contents is also considered in [21]. In a recent work [23], Timo et al. also consider lossy caching, taking into account the correlation among the available contents, based on which the tradeoff between the compression rate, reconstruction distortion and cache capacity is characterized for a single-user scenario, as well as some special cases of the two-user scenario.

The rest of the paper is organized as follows. We present the system model and the problem formulation in Section II. Centralized coded caching is studied in Section III, and a theoretical lower bound as well as achievable schemes are proposed. Decentralized lossy coded caching is considered in Section IV. Numerical results are presented in Section V. Finally, we conclude the paper in Section VI, followed by the appendices.

II. System Model

We consider a server holding $N$ independent files, $S_{1}^{n}, ..., S_{N}^{n}$. Each file $S_{i}^{n}$ consists of $n$ independent and identically distributed (i.i.d) samples $(S_{i,1}, ..., S_{i,n})$ from a Gaussian distribution with zero-mean and variance $\sigma^2$, i.e., $S_{i}^{n} \sim \mathcal{N}(0,\sigma^2)$, for $i = 1, ..., N$. There are $K$ users in the system that may request any of the files from the server.

The operation of the caching system consists of two distinct phases: placement phase and delivery phase. In the placement phase, users pre-fetch bits from the server to fill their caches.
In centralized caching, active users that will participate in the delivery phase are known in advance during the placement phase, enabling coordination of cache placement across users. On the contrary, in decentralized caching, users fill their caches from different servers through different access points, and equivalently, the server has no prior knowledge of the active users that will participate in a particular delivery phase. Therefore, the cache placement of each user is conducted independently.

We assume that each user is equipped with a cache of size $M_k n$ bits, $k = 1, ..., K$, and denote by $Z_k$, the contents of the cache of user $k$ at the end of the placement phase. The delivery phase starts after users reveal their demands, denoted by $d \triangleq (d_1, ..., d_K)$, where $d_k \in \{1, ..., N\}$ denotes the demand of user $k$. During this phase, a single message $X^n_{(d_1, ..., d_K)}$ of size $nR$ bits is sent by the server over the shared link depending on all the users’ requests and cache contents. User $k$ reconstructs its requested file by combining $Z_k$ and $X^n_{(d_1, ..., d_K)}$.

An $(n, M_1, ..., M_K, R)$ “caching code” consists of $K$ cache placement functions:

$$f^n_k : \mathbb{R}^n \times \cdots \times \mathbb{R}^n / N \text{ files} \to \{1, ..., 2^{nM_k}\} \quad \text{for} \quad k = 1, ..., K,$$

one delivery function:

$$g^n : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{N \text{ files}} \times d_1 \times \cdots \times d_K \to \{1, ..., 2^{nR}\},$$

where $Z^n_k = f^n_k(S^n_1, ..., S^n_N), X^n_{(d_1, ..., d_K)} = g^n(S^n_1, ..., S^n_N, d_1, ..., d_K)$, and $K$ decoding functions:

$$h^n_k : \{1, ..., N\}^K \times \{1, ..., 2^{nM_k}\} \times \{1, ..., 2^{nR}\} \to \mathbb{R}^n,$$

where $\hat{S}^n_k = h^n_k(d, Z^n_k, X^n)$. Note that, in this formulation the demand vector $d$ is known by all the users.

Quadratic (squared-error) distortion is used to measure the quality of reconstruction. We assume that each user has a preset distortion requirement $D_k$, $k = 1, ..., K$. Without loss of generality, we assume that $D_1 \geq D_2 \geq \cdots \geq D_K$. We emphasize that the distortion tuple $D \triangleq (D_1, ..., D_K)$ is known in the placement phase, while the demand realization, $d$, is unknown.

**Definition 1.** A distortion tuple $D \triangleq (D_1, ..., D_K)$ is achievable if there exists a sequence of caching codes $(n, M_1, ..., M_K, R)$, such that

$$D(S^n_{d_k}, \hat{S}^n_{d_k}) \triangleq \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} (S_{d_k,j} - \hat{S}_{d_k,j})^2 \leq D_k, \quad k = 1, 2, ..., K,$$
holds for all possible request combinations \( d \).

**Definition 2.** For a given distortion tuple \( D \triangleq (D_1, ..., D_K) \), the cache capacity-delivery rate tradeoff is defined as follows:

\[
R^*(M_1, ..., M_K) \triangleq \inf \{ R : D \text{ is achievable} \}.
\]

(5)

Our goal in this paper is to characterize the cache capacity-delivery rate tradeoff for a caching system with \( N \) files and \( K \) users with arbitrary distortion requirements for both the centralized and decentralized scenarios.

**Remark 1.** We note here that, when all the users have the same distortion target, our problem reduces to the classical setting of \([6]\) and \([7]\), in which users request files in full. We can consider that files are stored in the server in their compressed form. We also highlight that, in addition to allowing lossy requests, we extend \([6]\) and \([7]\) in another direction as well, by considering different cache capacities at the users.

Following definitions will be instrumental in presenting our main results. Let \( D(r) \) denote the distortion-rate function of a Gaussian source \( S \sim N(0, \sigma^2) \) encoded at rate \( r \) bits per source sample (bpss). See \([24]\) for a rigorous definition of the distortion-rate function. We have \( D(r) \triangleq \sigma^2 2^{-2r} \). Let \( r_k \) be the minimum compression rate that achieves \( D_k \). We have

\[
r_k \triangleq R(D_k) = \frac{1}{2} \log_2 \frac{\sigma^2}{D_k}, k = 1, ..., K.
\]

(6)

This means that, to achieve the target distortion of \( D_k \), the user has to receive a minimum of \( nr_k \) bits corresponding to its desired file.

We will heavily exploit the successive refinability of a Gaussian source under squared-error distortion measure \([25]\). Successive refinement refers to compressing a sequence of source samples in multiple stages, such that the quality of reconstruction improves, i.e., distortion reduces, at every stage. A given source is said to be successively refinable under a given distortion measure if the single resolution distortion-rate function can be achieved at every stage. Successive refinement has been extensively studied in the source coding literature \([21]\).

**III. Centralized Coded Caching With Distortion Requirements**

In this section, we investigate the lossy caching problem in the centralized setting. We start by presenting a theoretical lower bound on the delivery rate of the caching system described in
Section II. We will then consider achievable lossy caching schemes, first for the two-user two-file ($N = K = 2$) scenario, then for the general two-user scenario, and finally for an arbitrary number of users and files.

A. Theoretical Lower Bound

We provide a lower bound on the delivery rate for the general setting ($K$ users and $N$ files) based on the cut-set arguments in Theorem 1.

**Theorem 1. (Cut-set Bound) For the lossy caching problem described in Section II, the optimal achievable delivery rate is lower bounded by**

$$R^\star(M_1, \ldots, M_K) \geq \max_{s \in \{1, \ldots, \min\{N, K\}\}} \max_{U \subseteq \{1, \ldots, K\}} |U| = s \left( \sum_{k \in U} r_k - \frac{\sum M_k}{\lceil N/s \rceil} \right).$$  (7)

*Proof: The proof can be found in Appendix A.  

It is known that the cut-set bound is not tight in general for the centralized coded caching problem [6]. We present another bound for the two-user case ($K = 2$), which, together with the cut-set bound, provides a tight lower bound on the delivery rate in certain scenarios.

**Theorem 2. For the lossy caching problem described in Section II with $K = 2$, we have**

$$R^\star(M_1, M_2) \geq r_1/2 + r_2 - \frac{(M_1 + M_2)}{2\lceil N/2 \rceil}.$$  (8)

*Proof: The proof can be found in Appendix B.  

B. Optimal Lossy Caching: Two Users and Two Files ($N = K = 2$)

In this section, we characterize the optimal cache capacity-delivery rate tradeoff for the lossy caching problem with two users ($K = 2$) and two files ($N = 2$). We remind the reader that the target average distortion values for user 1 and user 2 are $D_1$ and $D_2$, respectively, with $D_1 \geq D_2$, and $r_1$ and $r_2$ are the minimum compression rates that achieve $D_1$ and $D_2$, respectively, which can be obtained from [6].

We first present the lower bound on the delivery rate for given $M_1$ and $M_2$ in this particular scenario, followed by the coded caching scheme achieving this lower bound.
Corollary 1. For the lossy caching problem with $N = K = 2$, a lower bound on the cache capacity-delivery rate tradeoff is given by

$$R^*(M_1, M_2) \geq R_c(M_1, M_2) = \max\{r_1 - M_1/2, r_2 - M_2/2, r_1 + r_2 - (M_1 + M_2),
\frac{r_1}{2} + r_2 - (M_1 + M_2)/2, 0\}. \tag{9}$$

The first three terms in (9) are derived from the cut-set bound in Theorem 1, and the last one from Theorem 2.

Based on (9), we consider the following five cases depending on the cache capacities of the users, illustrated in Fig. 1:

**Case i:** $M_1 + M_2 \leq r_1$. In this case, $R_c(M_1, M_2) = r_1 + r_2 - (M_1 + M_2)$.

**Case ii:** $M_1 + M_2 > r_1, M_1 \leq r_1, M_2 \leq 2r_2 - r_1$. We have $R_c(M_1, M_2) = \frac{r_1}{2} + r_2 - \frac{M_1 + M_2}{2}$.

**Case iii:** $M_1 > r_1, M_2 \leq 2r_2, M_2 - M_1 \leq 2r_2 - 2r_1$. We have $R_c(M_1, M_2) = r_2 - \frac{M_2}{2}$.

**Case iv:** $M_1 \leq 2r_1, M_2 > 2r_2 - r_1, M_2 - M_1 > 2r_2 - 2r_1$. We have $R_c(M_1, M_2) = r_1 - \frac{M_1}{2}$.

**Case v:** $M_1 > 2r_1, M_2 > 2r_2$. We have $R_c(M_1, M_2) = 0$.

For each case, we explain the coded caching scheme that achieves the corresponding $R_c(M_1, M_2)$.

We assume that the server employs an optimal successive refinement source code, where we denote by $A(B)$ the source codeword of length $nr_2$ bits that can achieve a distortion of $D_2$ for file $S_2^n$. Thanks to the successive refinability of Gaussian sources, a receiver having received only the first $nr_1$ of these bits can achieve a distortion of $D_1$. We refer to the first $nr_1$ bits as the first layer, and the remaining $n(r_2 - r_1)$ bits as the refinement layer.
TABLE I
Illustration of Cache Placement For $N = K = 2$

| | First Layer | | Second Layer |
|---|---|---|---|
| $S_1$ | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ | $A_7$ | $A_8$ |
| $S_2$ | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ | $B_6$ | $B_7$ | $B_8$ |
| User 1 | $A_1 \oplus B_1$ | $A_3, B_3$ | $A_5, B_5$ | | | | | |
| User 2 | $A_2 \oplus B_2$ | $A_4, B_4$ | $A_5, B_5$ | | | $A_7, B_7$ | | |
| Case i | $M_1$ | $M_2$ | $0$ | $0$ | $0$ | $r_1 - M_1 - M_2$ | $0$ | $r_2 - r_1$ |
| Case ii | $M_1$ | $r_1 - M_1$ | $0$ | $0$ | $0$ | $0$ | $\frac{M_1 + M_2 - r_1}{2}$ | $r_2 - r_1 - \frac{M_1 + M_2 - r_1}{2}$ |
| Case iii | $r_1 - l_1 - 2l_2$ | $0$ | $l_2$ | $l_2$ | $l_1$ | $0$ | $l_3$ | $r_2 - r_1 - l_3$ |
| Case iv | $0$ | $r_1 - M_1$ | $M_1/2$ | $M_1/2$ | $0$ | $0$ | $r_2 - r_1$ | $0$ |
| Case v | $0$ | $0$ | $0$ | $0$ | $r_1$ | $0$ | $r_2 - r_1$ | $0$ |

In each case, we divide the first layers of codewords $A$ and $B$ into six disjoint parts denoted by $A_1, \ldots, A_6$ and $B_1, \ldots, B_6$, respectively, and the refinement layers into two disjoint parts denoted by $A_7, A_8$ and $B_7, B_8$, respectively, such that $|A_i| = |B_i|$ for $i = 1, \ldots, 8$, where $|X|$ denotes the length of the binary sequence $X$ (normalized by $n$). Table I illustrates the placement of contents in users’ caches for each case. The second and third rows illustrate how the two layers are partitioned for each file. The fourth and fifth rows indicate the cache contents of each user at the end of the placement phase. In all the cases, user 1 caches $Z_1 = (A_1 \oplus B_1, A_3, B_3, A_5, B_5)$, while user 2 caches $Z_2 = (A_2 \oplus B_2, A_4, B_4, A_5, B_5, A_7, B_7)$. The entries from the 6th row to the 10th specify the size of each portion in each case. For example, the 6th row implies that in Case i, $|A_1| = |B_1| = M_1$, $|A_2| = |B_2| = M_2$, $|A_6| = |B_6| = r_1 - M_1 - M_2$, $|A_8| = |B_8| = r_2 - r_1$, and the sizes of all other portions are equal to 0, which is equivalent to dividing $A(B)$ into four portions $A_1(B_1), A_2(B_2), A_6(B_6)$ and $A_8(B_8)$. Thus, in the placement phase, user 1 caches $Z_1 = A_1 \oplus B_1$, and user 2 caches $Z_2 = A_2 \oplus B_2$ so that $|Z_1| = M_1$ and $|Z_2| = M_2$, which meets the cache capacity constraints. The cache placements for the other 4 cases are presented in a similar manner in Table I.

Next, we focus on the delivery phase, and identify the minimum delivery rate in each case to satisfy the demands, $d = (S_1^n, S_2^n)$. Other demand combinations can be satisfied, without requiring higher delivery rates.

Case i ($M_1 + M_2 \leq r_1$): The server sends $B_1, A_2, A_6, B_6$ and $B_8$. Thus, the delivery rate is
\[ R(M_1, M_2) = r_1 + r_2 - (M_1 + M_2). \]

**Case ii** \((M_1 + M_2 > r_1, M_1 \leq r_1, M_2 \leq 2r_2 - r_1)\): The server sends \(B_1, A_2\) and \(B_8\). We have
\[ R(M_1, M_2) = \frac{r_1}{2} + r_2 - \frac{M_1 + M_2}{2}. \]

**Case iii** \((M_1 > r_1, M_2 \leq 2r_2, M_2 - M_1 \leq 2r_2 - 2r_1)\): The values of \(l_1, l_2\) and \(l_3\) in Table I are given as:
\[
\begin{align*}
l_1 &= \max\{0, \min\{M_1 - r_1, M_2/2 - (r_2 - r_1)\}\}; \\
l_2 &= \max\{0, M_2/2 - (r_2 - r_1) - l_1\}; \\
l_3 &= \min\{r_2 - r_1, M_2/2\}. \tag{10a} \\
\end{align*}
\]
The server sends \(B_1, B_3 \oplus A_4\) and \(B_8\), which results in
\[ R(M_1, M_2) = r_2 - \frac{M_2}{2}. \]

**Case iv** \((M_1 \leq 2r_1, M_2 > 2r_2 - r_1, M_2 - M_1 > 2r_2 - 2r_1)\): The server sends \(B_2, B_3 \oplus A_4\) and we have
\[ R(M_1, M_2) = r_1 - \frac{M_1}{2}. \]

**Case v** \((M_1 > 2r_1, M_2 > 2r_2)\): The cache capacities of both users are sufficient to cache the required descriptions for both files. Thus, any request can be satisfied from the local caches at the desired distortion levels, and we have
\[ R(M_1, M_2) = 0. \]

**Theorem 3.** For \(N = K = 2\), the proposed coded caching scheme meets the lower bound in Corollary 1; and hence, it is optimal, i.e., we have \(R^*(M_1, M_2) = R_c(M_1, M_2)\).

**Remark 2.** In the special case of identical distortion requirements at the two users, i.e., when \(D_1 = D_2\), Theorem 3 generalizes the optimal delivery rate result of [6] for \(N = K = 2\) to different cache capacities.

**C. Lossy Caching: Two Users and \(N\) Files \((K = 2, N > 2)\)**

Next, we investigate the more general case with two users and an arbitrary number of files, i.e., \(K = 2, N > 2\). We first present a lower bound on the delivery rate, in Lemma 1, and then present a coded caching scheme, followed by the analysis of the gap between the two.

**Lemma 1.** For the lossy caching problem with \(K = 2\) and \(N > 2\), a lower bound on the cache capacity-delivery rate tradeoff is given by
\[
R^*(M_1, M_2) \geq R_c(M_1, M_2) = \max \left\{ r_1 - \frac{M_1}{N}, r_2 - \frac{M_2}{N}, r_1 + r_2 - \frac{M_1 + M_2}{\lceil N/2 \rceil}, \right. \\
\left. \frac{r_1}{2} + r_2 - \frac{M_1 + M_2}{2\lceil N/2 \rceil}, r_1 + r_2 - \frac{M_1 + M_2}{2\lceil N/3 \rceil}, 0 \right\}. \tag{11}
\]
The first three terms in (11) follow from the cut-set bound in Theorem 1, while the forth term from Theorem 2. The proof of the fifth term can be found in Appendix C. Note that for \( N > 2 \), the fifth term in (11) is always larger than the third.

Next, we present a coded caching scheme for this scenario. Similarly to Section III-B, we consider successive refinement compression of the files. We denote by \( W_j \) the source codeword of length \( nr_2 \) bits that leads to a distortion of \( D_2 \) for file \( S_j^0 \), \( j = 1, \ldots, N \). First \( nr_1 \) bits of \( W_j \) corresponds to the first layer that would provide a distortion level of \( D_1 \) if received. We divide the first layer of \( W_j \), i.e., the first \( nr_1 \) bits, into four disjoint parts, denoted by \( W_{j1}, W_{j2}, W_{j3}, \) and \( W_{j4}, \) and the refinement layer, i.e., the remaining \( n(r_2 - r_1) \) bits, into two disjoint parts, denoted by \( W_{j5}, W_{j6}, j = 1, \ldots, N \). The size of each part is the same for all the files, e.g., \( |W_{ik}| = |W_{jk}|, \forall i, j \in \{1, \ldots, N\} \), for \( k = 1, \ldots, 6 \). Let user 1 cache \( W_{j1}, W_{j3} \), and user 2 cache \( W_{j2}, W_{j3} \) and \( W_{j5}, j = 1, \ldots, N \), i.e., \( Z_1 = \bigcup_{j=1}^{N} \{W_{j1}, W_{j2}\} \) and \( Z_2 = \bigcup_{j=1}^{N} \{W_{j2}, W_{j3}, W_{j4}\} \). During the delivery phase, for any possible demand pair \((d_1, d_2), d_1, d_2 \in \{1, \ldots, N\}\), the server sends \( W_{d13} \oplus W_{d21}, W_{d14}, W_{d24}, \) and \( W_{d6} \). Here \( \oplus \) represents the bitwise XOR operation, where the arguments are first zero-padded to the length of the longest one. It is possible to see that both users can achieve their target distortion values.

In the following, we specify the size of each partition depending on the cache capacities, \( M_1 \) and \( M_2 \), and derive the corresponding delivery rate. We consider three cases:

Case i \((M_1 + M_2 \leq Nr_1)\): We let: \( |W_{j1}| = M_1/N; |W_{j2}| = 0; |W_{j3}| = M_2/N; |W_{j4}| = r_1 - M_1/N - M_2/N; |W_{j5}| = 0; |W_{j6}| = r_2 - r_1 \). The delivery rate is \( R(M_1, M_2) = r_1 + r_2 - \frac{2(M_1+M_2)}{N} + \max\{M_1, M_2\} \).

Case ii \((M_1 + M_2 > Nr_1, M_1 \leq Nr_1)\): We let: \( |W_{j1}| = M_1/N - \max\{M_1/N + M_2/N - r_2, 0\}; |W_{j2}| = \max\{M_1/N + M_2/N - r_2, 0\}; |W_{j3}| = r_1 - M_1/N; |W_{j4}| = 0; |W_{j5}| = \min\{M_1/N + M_2/N - r_1, r_2 - r_1\}; |W_{j6}| = \max\{r_2 - (M_1/N + M_2/N), 0\} \). We have \( R(M_1, M_2) = \max\{r_2 - (M_1 + M_2)/N, 0\} + \max\{\min\{r_2 - M_2/N, M_1/N\}, r_1 - M_1/N\} \).

Case iii \((M_1 > Nr_1)\): Let \( |W_{j1}| = r_1 - \min\{M_2/N, r_1\}; |W_{j2}| = \min\{M_2/N, r_1\}; |W_{j3}| = 0; \]
\( |W_{j4}| = 0; |W_{j5}| = \max\{0, \min\{r_2, M_2/N\} - r_1\}; |W_{j6}| = \min\{r_2 - r_1, \max\{0, r_2 - M_2/N\}\} \).
It yields \( R(M_1, M_2) = \max\{0, r_2 - M_2/N\} \).

We can summarize the achievable delivery rate as follows:
\[ R(M_1, M_2) = \begin{cases} 
  r_1 + r_2 - 2M_1/N - M_2/N, & \text{if } (M_1, M_2) \in \mathcal{M}_1, \\
  r_1 + r_2 - M_1/N - 2M_2/N, & \text{if } (M_1, M_2) \in \mathcal{M}_2, \\
  r_1 - M_1/N, & \text{if } (M_1, M_2) \in \mathcal{M}_3, \\
  r_2 - M_2/N, & \text{if } (M_1, M_2) \in \mathcal{M}_4, \\
  0, & \text{if } (M_1, M_2) \in \mathcal{M}_5, 
\end{cases} \]  
(12)

where \( \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4 \) and \( \mathcal{M}_5 \), illustrated in Fig. 2, are specified as follows:

\[ \mathcal{M}_1 = \{(M_1, M_2)|0 \leq M_1 \leq Nr_1/2, M_1 + M_2 \leq Nr_2, M_1 \leq M_2\}; \]  
(13a)

\[ \mathcal{M}_2 = \{(M_1, M_2)|M_1 + M_2 \leq Nr_1, M_1 > M_2, M_2 \geq 0\}; \]  
(13b)

\[ \mathcal{M}_3 = \{(M_1, M_2)|0 \leq M_1 \leq Nr_1, M_1 + M_2 > Nr_2, M_2 - M_1 > N(r_2 - r_1)\}; \]  
(13c)

\[ \mathcal{M}_4 = \{(M_1, M_2)|M_1 > r_1N/2, 0 \leq M_2 \leq Nr_2, M_1 + M_2 > r_1N, M_2 - M_1 \leq N(r_2 - r_1)\}; \]  
(13d)

\[ \mathcal{M}_5 = \{(M_1, M_2)|M_1 > Nr_1, M_2 > Nr_2\}. \]  
(13e)

Comparing (11) with (12), we can conclude that for \((M_1, M_2) \in \mathcal{M}_3 \cup \mathcal{M}_4 \cup \mathcal{M}_5\), the achieved delivery rate meets the lower bound in Lemma 1. We analyse the optimality of the achieved delivery rate for \((M_1, M_2) \in \mathcal{M}_1 \cup \mathcal{M}_2\) in the following. Consider three cases: \(N = 3c + i\), for \(i = 0, 1, 2\) and \(c \in \mathbb{Z}^+\).
Case i ($N = 3c$): The last term of (11) can be rewritten as $r_1 + r_2 - \frac{3(M_1+M_2)}{2N}$. Hence, 
\[ \Delta R \triangleq R(M_1, M_2) - R_c(M_1, M_2) = \left| \frac{M_1-M_2}{2N} \right| \text{ for any } (M_1, M_2) \in \mathcal{M}_1 \cup \mathcal{M}_2. \]
Note that when $M_1 = M_2$, $\Delta R = 0$, that is, the achieved delivery rate meets the lower bound.

Case ii ($N = 3c + 1$): We can rewrite the last term of (11) as $r_1 + r_2 - \frac{3(M_1+M_2)}{2(N-1)}$. We have
\[ \Delta R = \left| \frac{M_1-M_2}{2(N-1)} \right| + \max \left\{ \frac{2M_1+M_2, 2M_2+M_1}{N(N-1)} \right\} \text{ for any } (M_1, M_2) \in \mathcal{M}_1 \cup \mathcal{M}_2. \]

Case iii ($N = 3c + 2$): The last term of (11) equals to $r_1 + r_2 - \frac{3(M_1+M_2)}{2(N-2)}$. We have
\[ \Delta R = \left| \frac{M_1-M_2}{2(N-2)} \right| + \max \left\{ \frac{2M_1+M_2, 2M_2+M_1}{N(N-2)} \right\} \text{ for any } (M_1, M_2) \in \mathcal{M}_1 \cup \mathcal{M}_2. \]

Corollary 2. For $K = 2$, when $N$ is divisible by 3 and the cache capacities of the users are identical, i.e., $M_1 = M_2$, the proposed coded caching scheme meets the lower bound in Lemma 1 and hence, it is optimal, i.e., we have $R^*(M_1, M_2) = R_c(M_1, M_2)$.

Remark 3. When $D_1 = D_2$, Corollary 2 generalizes the optimal delivery rate result of [6] for $N = K = 2$ to all $N$ values that are multiples of 3.

D. General Case

In this section, we tackle the general scenario with $N$ files and $K$ users. As before, we define $r_k = R(D_k)$, $k = 1, \ldots, K$, where $D_1 \geq D_2 \geq \cdots \geq D_K$. Exploiting the successive refinability of the underlying Gaussian sequences, we consider a layered structure of descriptions for each file, where the first layer, called the $r_1$-description, consists of $nr_1$ bits, and achieves distortion $D_1$ when decoded successfully. The $k$th layer, called the $(r_k - r_{k-1})$-refinement, $k = 2, \ldots, K$, consists of $n(r_k - r_{k-1})$ bits, and having received the first $k$ layers, a user achieves a distortion of $D_k$.

The examples studied in Sections III-B and III-C illustrate the complexity of the lossy content caching problem; we had to consider five cases even with two users and two files. The problem becomes intractable quickly with the increasing number of files and users. However, note that, only users $k, k+1, \ldots, K$, whose distortion requirements are lower than $D_k$, need to decode the $k$th layer for the file they request, for $k = 1, \ldots, K$. Therefore, once all the contents are compressed into $K$ layers based on the distortion requirements of the users employing an optimal successive refinement source code, we have, for each layer, a lossless caching problem. However, each user also has to decide how much of its cache capacity to allocate for each layer. Hence, the lossy
caching problem under consideration can be divided into two subproblems: the lossless caching problem of each source coding layer, and the cache allocation problem among different layers.

In general, the optimal delivery rate achieved by this layered algorithm can be found by jointly optimizing the cache capacity allocated by each user to each layer, and the caching and delivery scheme to be used for each layer to minimize the corresponding delivery rate. However, we first note that the optimal delivery rate remains an open problem even in the case of equal cache capacities. Therefore, we are bound to use achievability results. Moreover, these achievability results are often characterized as piecewise linear functions, and memory-sharing among multiple schemes may be required, further complicating the identification of the minimal delivery rate. Therefore, we are not able to provide a low complexity algorithm that can optimize all the system parameters jointly. Indeed, we will propose heuristic cache allocation schemes at the users, and provide a suboptimal caching and delivery algorithm for given cache capacities.

1) Coded Lossless Caching of Each Layer: We first assume that the cache allocation at the users for each layer is already fixed, and focus on the first subproblem of centralized lossless caching with heterogeneous cache sizes. This problem has previously been studied in [26] and [27] in the decentralized setting, while, to the best of our knowledge, it has not been considered in the centralized setting. Consider, for example, the $k$th refinement layers of all the files. There are only $L_k \triangleq K - k + 1$ users (users $k, k+1, \ldots, K$) who may request these layers. Let user $j, j \in \{k, \ldots, K\}$, allocate $M_{j,k}$ (normalized by $n$) of its cache capacity for this layer. Without loss of generality, we order users $k, \ldots, K$ according to the cache capacities they allocate, and re-index them, such that $M_{k,k} \leq M_{k+1,k} \leq \cdots \leq M_{K,k}$.

We would like to have symmetry among allocated cache capacities to enable multicasting to a group of users. Based on this intuition, we further divide layer $k$ into $L_k$ sub-layers, and let each user in $\{k, \ldots, K\}$ allocate $M^1_k = M_{k,k}$ of its cache for the first sub-layer, and each user in $\{k + i - 1, \ldots, K\}$ allocate $M^i_k = M_{k+i-1,k} - M_{k+i-2,k}$ of its cache for the $i$th sublayer, for $i = 2, \ldots, L_k$. Overall, we have $L_k$ sub-layers, and users $k + i - 1, k + i, \ldots, K$ allocate $M^i_k$ of their caches for sub-layer $i$, whereas no cache is allocated by users $k, k+1, \ldots, k+i-2$.

We denote by $r^i_k$ the size of the $i$th sub-layer of the $k$th refinement layer, and by $R(L_k, i, M^i_k, r^i_k, N)$ the minimum required delivery rate for this sub-layer. The rates, $r^i_k, i = 1, \ldots, L_k$, should be optimized jointly in order to minimize the total delivery rate for the $k$th layer. The optimization
problem can be formulated as follows:

\[
\min_{r_k^1, \ldots, r_k^{L_k}} \sum_{i=1}^{L_k} R(L_k, i, M_k^i, r_k^i, N) \tag{14a}
\]

subject to \( \sum_{i=1}^{L_k} r_k^i = r_k - r_{k-1} \). \tag{14b}

We explore the achievable \( R(L_k, i, M_k^i, r_k^i, N) \) based on the existing caching schemes in [6], [8] and [9], which are referred to as \textit{coded delivery}, \textit{coded placement} and \textit{GBC}, respectively.

We consider two cases:

Case 1) \( L_k < N \). In this case, in the worst case when users \( \{k, \ldots, K\} \) request distinct files, \textit{GBC} and \textit{coded placement} provide no caching gain; thus, we employ \textit{coded delivery}. Focus on the \( i \)th sub-layer: users \( k + i - 1, \ldots, K \) each allocate \( M_k^i \) of cache capacity, while users \( k \) to \( k + i - 2 \) allocate no cache for this sublayer. If \( r_k^i \in \mathcal{P}_{MAN} \triangleq \{0, M_k^i/N, M_k^iL_k^i/((L_k - 1)N), M_k^iL_k^i/((L_k - 2)N), \ldots, M_k^iL_k^i/N\} \), where \( L_k^i = L_k + 1 - i \), we have

\[
R_{MAN}(L_k, i, M_k^i, r_k^i, N) = (i - 1) \cdot r_k^i + r_k^iL_k^i \cdot (1 - M_k^i/r_k^iN) \cdot \frac{1}{1 + M_k^iL_k^i/r_k^iN}. \tag{15}
\]

The first term on the right hand side is due to unicasting to users \( k \) to \( k + i - 2 \), while the second term is the \textit{coded delivery} rate to users \( k + i - 1 \) to \( K \) given in [6]. Based on the memory sharing argument, any point on the line connecting two points, \( (r_1', R(L_k, i, M_k^i, r_1', N)) \) and \( (r_2', R(L_k, i, M_k^i, r_2', N)) \), is also achievable, i.e., if \( r_k^i \in [0, M_k^iL_k^i/N] \) and \( r_k^i \notin \mathcal{P}_{MAN} \), then we have

\[
R(L_k, i, M_k^i, r_k^i, N) = \min_{r_1' < r_k^i, r_2' > r_k^i} \left\{ \frac{r_k^i - r_1'}{r_2' - r_1'} R(K_k, i, M_k^i, r_1'^i, N) \right. \\
+ \frac{r_2' - r_k^i}{r_2' - r_1'} R(K_k, i, M_k^i, r_2'^i, N) \right\}, \tag{16}
\]

and if \( r_k^i > M_k^iL_k^i/N \), we have

\[
R(L_k, i, M_k^i, r_k^i, N) = (i - 1) \cdot r_k^i + \frac{M_k^iL_k^i(L_k - 1)}{2N} + r_k^iM_k^iL_k^iL_k^i/N. \tag{17}
\]

Case 2) \( L_k \geq N \). In this case, \textit{coded placement} achieves a lower delivery rate than \textit{coded delivery} when \( r_k^i \geq M_k^iL_k^i/8 \). If \( L_k^i > N \geq 3 \), \textit{GBC} outperforms \textit{coded delivery} at point \( r_k^i = M_k^iL_k^i/N \). Note that for the \( i \)th sub-layer, there are \( i - 1 \) users with no cache allocation.
If $i - 1 \geq N$, there will be no gain with any of the schemes. When $i - 1 < N$ and $r_k^i \geq M_k^i L_k^i$, the delivery rate of coded placement is

$$R_{CFL}(L_k, i, M_k^i r_k^i, N) = Nr_k^i - (N - i + 1)M_k^i. \quad (18)$$

If $i - 1 < N$, $L_k^i > N \geq 3$ and $r_k^i = M_k^i L_k^i / N$, the delivery rate provided by GBC is given by

$$R_{GBC}(L_k, i, M_k^i r_k^i, N) = (i - 1) \cdot r_k^i + N r_k^i - \frac{N(N + 1)r_k^i}{2L_k^i}. \quad (19)$$

When $0 \leq r_k^i \leq M_k^i L_k^i$, the delivery rate is given by the lower convex envelope of points $(M_k^i r_k^i, R_{CFL}(L_k, i, M_k^i, M_k^i L_k^i, N))$ given by (18), $(M_k^i L_k^i / N, R_{GBC}(L_k, i, M_k^i, M_k^i L_k^i / N, N))$ given by (19), and $(r_k^i, R_{MAN}(L_k, i, M_k^i, r_k^i, N))$; and for $r_k^i \in \mathcal{P}_{MAN} \setminus \{M_k^i L_k^i / N\}$, given by (15).

2) Allocation of Cache Capacity: Next, we propose two algorithms for cache allocation among layers: proportional cache allocation (PCA) and ordered cache allocation (OCA), which are elaborated in Algorithms 1 and 2, respectively, where $r_k^i$ is as defined earlier, and $r_0^i = 0$.

**Algorithm 1 Proportional Cache Allocation (PCA)**

1: **Require**: $r = r_1, \ldots, r_K$

2: **for** $k = 1, \ldots, K$ **do**

3: \hspace{1em} **for** $i = 1, \ldots, k$ **do**

4: \hspace{2em} user $k$ allocates $\frac{r_i - r_i - 1}{r_k^i} M_k^i$ to layer $i$

5: \hspace{1em} **end for**

6: **end for**

**Algorithm 2 Ordered Cache Allocation (OCA)**

1: **Require**: $r = r_1, \ldots, r_K$

2: **for** $k = 1, \ldots, K$ **do**

3: \hspace{1em} user $k$ allocates all of its cache to the first $i$ layers, where $r_i - 1 < \frac{M_k^i}{N} \leq r_i$

4: **end for**

PCA allocates each user’s cache among the layers it may request proportionally to the sizes of the layers, while OCA gives priority to lower layers. The server can choose the one resulting
in a lower delivery rate. Numerical comparison of these two cache allocation schemes together with the delivery scheme proposed above will be presented in Section V.

**Remark 4.** As stated earlier, when the QoS requirements of the users are identical, i.e., \( D_1 = D_2 = \cdots = D_N \), the lossy caching problem considered here is equivalent to the loseless coded caching problem with distinct cache capacities. Therefore, the centralized coded caching scheme proposed in this section is also the first centralized caching scheme, generalizing the centralized caching problem in [6] to heterogeneous cache capacities.

### IV. Decentralized Coded Caching With Distortion Requirements

In this section, we consider the lossy coded caching problem in the decentralized setting; that is, the server is assumed to have only the set of possible distortion values that may be requested by the users, but no prior knowledge of the number of users and their target distortion requirements. Accordingly, in decentralized caching, the placement phase is conducted locally and independently for each user since coordination among users is not possible.

In the placement phase of the proposed coded caching scheme, user \( k \) randomly caches \( M_k n/N \) bits of the \( r_k \)-description of each file, for \( k = 1, \ldots, K \). Recall that the \( r_k \)-description corresponds to the optimal codeword of length \( nr_k \) bits, which achieves an average distortion of \( D_k \) when decoded. As in the centralized setting, we consider successive refinement compression of the files; that is, \( r_k \)-distortion corresponds to the first \( nr_k \) bits of the \( r_{k+1} \)-description. Different layers of each file are cached by a different subset of users, and cached contents of each layer occupy different sizes of memory due to heterogeneous distortion requirements and heterogeneous cache sizes. To illustrate the achievable delivery rate, we first present an example with two users and two files \( (K = N = 2) \), and then extend our analysis to the general scenario.

**Remark 5.** Here we assume that the server has the knowledge of all possible distortion values that may be requested by the users. This is needed for the server to employ a successive refinement source code with the desired number of layers. Note that the server does not know either the number or the identity of the active users in advance; however, in practice, the number of layers, or equivalently, the number of QoS levels that can be requested will be limited either by the compression protocol employed, or due to the limited variety of devices available, and will be
TABLE II
ILLUSTRATION OF CACHE PLACEMENT

| Segment | First layer | Second layer |
|---------|-------------|--------------|
|         | $A_0^1(B_2^1)$ | $A_1^1(B_1^1)$ | $A_2^1(B_2^1)$ | $A_1^2(B_2^1)$ | $A_2^2(B_2^2)$ |
| Size    | $r_1(1-t_1)(1-t_2)$ | $r_1t_1(1-t_2)$ | $r_1t_2(1-t_1)$ | $rt_1t_2$ | $(r_2-r_1)(1-t_2)$ | $t_2(r_2-r_1)$ |

much smaller than the number of users in system.

A. Two Users and Two Files ($K = N = 2$)

Here we consider the same system model as in Section III-B with two users and two files ($N = K = 2$). User 1 has a cache of size $M_1$, while user 2 has a cache of size $M_2$.

In the placement phase, user 1 randomly caches $M_1n/2$ bits from the $r_1$-description of each file; while user 2 randomly caches $M_2n/2$ bits from the $r_2$-description of each file. We denote by $A_i^j (B_i^j)$ the bits of the $i$th layer of file $S_1$ ($S_2$) that are cached exclusively by the subset $U$, where $U \subset \{1, 2\}$ and $i = 1, 2$. For example, $A_1^1$ denotes the part of the first layer of file $S_1^n$ cached by both users, while $B_2^2$ is the segment of bits from the second layer of file $S_2^n$ cached only by user 2. We list the expected size of each segment in Table II (normalized by $n$), where $t_i$ denotes the probability of any bit from the $r_i$-description of each file cached by user $i$. We have $t_i = \min\{1, \frac{M_i}{2r_i}\}, i = 1, 2$.

In the delivery phase, for demand pair $(S_1^n, S_2^n)$, the server sends $A_0^1, B_0^1, B_0^2$ and $A_2^1 \oplus B_1^1$ to satisfy both requests. Note that $A_2^1$ and $B_1^1$ may be of different sizes. We employ the bitwise XOR operation $\oplus$. Any other demand combination can be satisfied in a similar manner. Hence, the worst-case delivery rate is given by

$$R = 2r_1(1-t_1)(1-t_2) + (r_2-r_1)(1-t_2) + \max\{r_1t_1(1-t_2), r_1t_2(1-t_1)\}. \quad (20)$$

Similar to centralized coded caching, multicasting exists only in the delivery of the first layer that is requested by both users. In Fig. 3, we plot the achievable delivery rate for different cache capacity pairs $(M_1, M_2)$, and different distortion requirement pairs $(D_1, D_2)$, such that $r_1 = 1 - \alpha$ and $r_2 = 1 + \alpha$. The value of $\alpha$ varies from 0 to 1. The difference between the distortion requirements of the users becomes more significant as $\alpha$ increases, while the sum of the requested rates remains the same. When $\alpha = 0$, the users have the same distortion target,
Fig. 3. Achievable delivery rate for $r_1 = 1 - \alpha$ and $r_2 = 1 + \alpha$, $\alpha \in [0, 1]$, $N = 2$, $K = 2$.

and the delivery rate increases with the difference between the cache capacities even though the total cache capacity remains the same. We observe from the $M_1 = M_2 = 1$ curve that, a larger $\alpha$, i.e., the distortion targets of the users are more divergent, results in a higher delivery rate. We also observe that the delivery rate is smaller if the user with the lower distortion target has a larger cache capacity.

**B. General Case**

We present a decentralized coded caching scheme for the general scenario in Algorithm 3, based on the decentralized caching scheme of [7], where the parameters $r_k$ are as defined in Section III-D, and $W_{(d_s, D_s), S \setminus \{s\}}$ denotes the part of the description of the file $d_s$ requested by user $s$ at distortion level $D_s$, that is cached exclusively by the subset of users $S \setminus \{s\}$. Algorithm 3 contains two delivery procedures, DELIVERY1 and DELIVERY2, and according to the information on the active users received at the beginning of the delivery phase, the server can choose the delivery procedure with a lower delivery rate. The following lemma provides the
Algorithm 3 Decentralized Coded Caching Scheme with Lossy Distortion Requests

1: procedure PLACEMENT
2:   for k = 1, ..., K do
3:     for j = 1, ..., N do
4:       User k randomly caches $M_k n/N$ bits of the $r_k$-description of file $j$
5:     end for
6:   end for
7: end procedure
8: procedure DELIVERY1 ($d_1, ..., d_K$)
9:   for k = 1, ..., K do
10:     for $S \subset \{1, ..., K\}$: $|S| = k$ do
11:       Send $X_S = \oplus_{s \in S} W(d_s, D_s, S \setminus \{s\})$
12:     end for
13:   end for
14: end procedure
15: procedure DELIVERY2 ($d_1, ..., d_K$)
16:   for $i = 1, ..., N$ do
17:     server sends enough random linear combinations of the bits of the compressed version
18:       of file $S_i^n$ to enable all the users demanding it to decode it at their desired distortion levels.
19:   end for
20: end procedure

expected size of each segment $W(d_s, D_s, S \setminus \{s\})$.

Lemma 2. According to the placement phase of Algorithm 3, as the blocklength $n$ goes to infinity, by the law of large numbers, the size of segment $W(d_s, D_s, S \setminus \{s\})$ can be approximated as

$$|W(d_s, D_s, S \setminus \{s\})| \approx n \sum_{i=1}^{\inf S} (r_i - r_{i-1}) p^i_{(d_s, D_s, S \setminus \{s\})},$$

(21)
where \( r_0 = 0 \), and

\[
p_i^{j,(d_s,D_s),S\setminus\{s\}} = \left(1 - \frac{M_s}{N r_s}\right) \prod_{u \in S \setminus \{s\}} \frac{M_u}{N r_u} \prod_{u \in \{i,\ldots,K\} \setminus S} \left(1 - \frac{M_u}{N r_u}\right).
\]

(22)

Proof: Based on Algorithm 3, user \( k \) caches \( M_k n / N \) bits from the \( r_k \)-description of each file, which implies that each bit of the \( r_k \)-description is cached with probability \( M_k / N r_k \), and no bit from the higher layers is cached by user \( k \). We define \( m(S) \) as the user with the smallest index in set \( S \), i.e., \( m(S) = \min\{s : s \in S\} \). Then, every user \( u \) in subset \( S \) caches every bit in \( r_{m(S)} \)-description with probability \( M_u / N r_u \). Since \( D_1 \geq D_2 \geq \cdots \geq D_K \), we have \( D_{m(S)} \geq D_u \), i.e., \( r_{m(S)} \leq r_u \) for any \( u \in S \). The file segment \( W_{(d_s,D_s),S\setminus\{s\}} \) contains bits only from the \( r_{m(S)} \)-description of file \( S_{d_s} \).

Note that user \( k \) caches bits only from the first \( k \) layers, i.e., the \( k \)th layer is exclusively cached by users \( k, k+1, \ldots, K \). For the \( i \)th layer, \( i = 1, \ldots, m(S) \), which is cached by users \( i, \ldots, K \), every bit in the \( i \)th layer of file \( d_s \) is exclusively cached by users in subset \( S \setminus \{s\} \) with probability \( p_i^{j,(d_s,D_s),S\setminus\{s\}} \). Since the total rate of the \( i \)th layer is \( r_i - r_{i-1} \), the expected number of bits from the \( i \)th layer in \( W_{(d_s,D_s),S\setminus\{s\}} \) is \( n(r_i - r_{i-1}) p_i^{j,(d_s,D_s),S\setminus\{s\}} \). As \( W_{(d_s,D_s),S\setminus\{s\}} \) has bits from the first \( m(S) \) layers, we sum up the expected rate of all these layers, which yields (21).

Since \( \frac{M_k}{N r_k} \) are not identical for \( k = 1, \ldots, K \), in the delivery phase, for the multicast subset \( S \), the sizes of the corresponding segments, \( W_{(d_s,D_s),S\setminus\{s\}} \), \( s \in S \), will be different. Therefore, we apply the \( \oplus \) operation. Hence, the size of the multicasted segment for subset \( S \) is given by

\[
|X_S| = \max_{s \in S} |W_{(d_s,D_s),S\setminus\{s\}}| = |W_{(d_{r'(S)},D_{r'(S)}),S\setminus\{r'(S)\}}| = n \sum_{i=1}^{m(S)} (r_i - r_{i-1}) p_i^{j,(d_{r'(S)},D_{r'(S)}),S\setminus\{r'(S)\}}.
\]

(23)

where \( r'(S) = \arg\min_{s \in S} \frac{M_s}{N r_s} \), which is the index of the user in subset \( S \) with the smallest \( \frac{M_s}{N r_s} \). Using (23), the delivery rate of Algorithm 3 can be derived.

**Theorem 4.** For the decentralized coded caching system described above, Algorithm 3 achieves
Algorithm 4 Layered Content Delivery 1 (LCD1)

1: procedure DELIVER \((d_1, ..., d_K)\)
2:   for \(i = 1, ..., K\) do
3:     procedure DELIVERY \(i\)TH LAYER OF FILES \((d_i, ..., d_K)\) TO USERS \(\{i, ..., K\}\)
4:       for \(k = 1, ..., K + 1 - i\) do
5:         for \(S \subset \{i, ..., K\}: |S| = k\) do
6:           Send \(X^i_S = \oplus_{s \in S} W^i_{(d_s,D_s),S\{s}}\)
7:         end for
8:       end for
9:     end procedure
10: end for
11: end procedure

a delivery rate given by

\[
R(M_1, ..., M_K) = \min \left\{ \sum_{i=1}^{K} (r_i - r_{i-1}) \sum_{l=1}^{K-i+1} \prod_{k=1}^{l} (1 - t^i_k), \right. \\
\left. \quad \sum_{i=1}^{\min\{N,K\}} r_{K-i+1} - \min_{S \subset \{1,...,K\}} \sum_{|S|=\min\{N,K\}}^{\min\{N,K\}} \sum_{k \in S} \frac{M_k}{N} \right\} \tag{24}
\]

where \(\{t^1_i, t^2_i, ..., t^i_{K-i+1}\}\) is an ordered permutation of \(\left\{ \frac{M_i}{N_{r_i}}, \frac{M_{i+1}}{N_{r_{i+1}}}, ..., \frac{M_K}{N_{r_K}} \right\}\) such that \(t^1_i \leq t^2_i \leq \cdots \leq t^i_{K-i+1}, i \in \{1, ..., K\}\).

Proof: We first prove the first term in (24), which is provided by DELIVERY 1. We sum up the rates corresponding to all possible multicasting subsets, \(S \subset \{1, ..., K\}\). From (23), we have

\[
R(M_1, ..., M_K) = \sum_{k=1}^{K} \sum_{S \subset \{1, ..., K\}}^{m(S)} \sum_{i=1}^{m(S)} (r_i - r_{i-1}) p^i_{(d_{r_i(S)},D_{r_i(S)},S\{r_i(S))}} \tag{25a}
\]

\[
= \sum_{i=1}^{K} \sum_{k=1}^{K-i+1} \sum_{S \subset \{1, ..., K\}}^{m(S)} (r_i - r_{i-1}) p^i_{(d_{r_i(S)},D_{r_i(S)},S\{r_i(S))}} \tag{25b}
\]
where \( k \) denotes the cardinality of subsets, for \( k = 1, \ldots, K \). Note that \( n \cdot (r_i - r_{i-1})p^i_{(d_r(S),D_{r'},S\backslash \{r'(S)\})} \) denotes the number of bits from the \( i \)th layer in \( |X^i_S| \). The above equation implies that it is equivalent to delivering each layer separately. We present DELIVERY1 as the layered content delivery approach in Algorithm 4, where \( W^i_{(d_r(S),S\backslash \{s\})} \) denotes the set of bits from the \( i \)th layer of file \( d_s \) that are cached exclusively by users in the subset \( S \backslash \{s\} \). Hence,

\[
|X^i_S| = n \cdot (r_i - r_{i-1})p^i_{(d_r(S),D_{r'},S\backslash \{r'(S)\})}.
\]

(26)

We focus on the delivery of the \( i \)th layer to users \( \{i, \ldots, K\} \) in the following. For this layer, we order and re-index users \( \{i, \ldots, K\} \) with \( \{1, \ldots, K-i+1\} \) such that \( \frac{M_k}{N M_k} = t^i_k \), for \( k = 1, \ldots, K-i+1 \). Based on this re-indexing, we have

\[
\sum_{k=1}^{K-i+1} \sum_{S \subset \{i, \ldots, K\}} p^i_{(d_r(S),D_{r'},S\backslash \{r'(S)\})} = \sum_{l=1}^{K-i+1} \prod_{k=1}^{l} (1 - t^i_k),
\]

(27)

which yields the first term in (24). A detailed proof of (27) can be found in Appendix D.

Now, we continue with analysis of the DELIVERY2 procedure in Algorithm 3. Note that the message sent at step 17 of Algorithm 3 is targeted at a group of users requesting file \( S^n_i \), for \( i = 1, \ldots, N \). Let \( S_i \) denote the set of users requesting file \( S^n_i \), \( i = 1, \ldots, N \). Since each user \( k \in S_i \) has \( n M_k / N \) bits of file \( S^n_i \) stored in its caches, with the fact stated in [7, Appendix A], at most

\[
n \left\{ \max_{k \in S_i} r_k - \min_{k \in S_i} \frac{M_k}{N} \right\}
\]

(28)

random linear combinations need to be sent for all those users in \( S_i \) to decode file \( S^n_i \) at their desired distortion values. If \( N \geq K \), the worst case demand combination corresponds to each user requesting a distinct file. There are at most \( K \) files requested, which yields a delivery rate of

\[
\sum_{i=1}^{K} \left\{ r_i - \frac{M_i}{N} \right\}
\]

(29)

for the DELIVERY2 procedure.

If \( N > K \), the worst case demand combination occurs when the \( N \) users with the lowest distortion requirements, i.e., the \( N \) largest \( r_k \) values, \( k = 1, \ldots, K \), and the \( N \) users with the smallest cache capacities are in different groups \( S_i \), \( i = 1, \ldots, N \), that is, they request \( N \) distinct
files. Since $D_1 \geq D_2 \geq \cdots \geq D_k$, i.e., $r_1 \leq r_2 \leq \cdots \leq r_K$, for this case, the DELIVERY2 procedure achieves a delivery rate of

$$
\sum_{i=1}^{N} r_{K-i+1} - \min_{|S|=N} \sum_{k \in S} \frac{M_k}{N}.
$$

(30)

Combining (29) and (30), we have the second term in (24). This completes the proof.

We remark that Eqn. (27) denotes the probability of any bit of the $i$th layer to be sent in the delivery phase. Note that (27) has a similar form to the expression of the achievable delivery rate in the lossless coded caching problem with heterogeneous cache sizes presented in [26, Theorem 3]. This implies that, with the placement phase carried out as in Algorithm 3, the delivery rate is equivalent to the lossless coded caching scheme proposed in [26] for each layer.

Algorithm 5 Layered Content Delivery 2 (LCD2)

1: procedure DELIVER $(d_1, \ldots, d_K)$
2: for $i = 1, \ldots, K$ do
3: procedure DELIVERY $i$TH LAYER OF FILES $(d_i, \ldots, d_K)$ TO USERS $\{i, \ldots, K\}$
4: Define $K_j$ as the number of users in $\{i, \ldots, K\}$ that requests file $j$, for $j = 1, \ldots, N$;
5: Reorder and reindex users $\{i, \ldots, K\}$ such that $d_k = j$, for $j = 1, \ldots, N$, and $k = S_{j-1} + 1, \ldots, S_j$ where $S_0 = i - 1$ and $S_j = \sum_{l=1}^{j} K_l$;
6: Execute Algorithm 1 in [27];
7: end procedure
8: end for
9: end procedure

For the decentralized lossless coded caching problem with distinct cache capacities, [27] further exploits the multicasting gain among users requesting the same file, which achieves a lower delivery rate when the number of users is larger than the number of files, compared to [26]. Here, in Algorithm 5, we employ a caching strategy similar to [27, Algorithm 1]. Based on [27, Theorem 1], we have Theorem 4 specifying the delivery rate achieved by Algorithm 5.
Theorem 5. For the decentralized coded caching system described above, Algorithm 5 achieves a delivery rate given by

\[ R(M_1, ..., M_K) = \sum_{i=1}^{K} (r_i - r_{i-1}) \left( \sum_{l=1}^{K-i} \prod_{k=1}^{l} (1 - t_k^i) - \Delta R_1^i - \Delta R_2^i \right), \]  

where \( t^i_k \) is defined as in Theorem 1, for \( k \in \{1, ..., K - i + 1\} \) and \( i \in \{1, ..., K\} \), and

\[ \Delta R_1^i = \begin{cases} (L_i - N) \prod_{l=1}^{L_i} (1 - t_l^i), & \text{if } L_i > N \\ 0, & \text{if } L_i \leq N \end{cases} \]  

\[ \Delta R_2^i = \begin{cases} \sum_{k=1}^{L_i-N} \left( \frac{(k-1)t^i_{k+N}}{1-t^i_{k+N}} \right) \prod_{l=1}^{L_i} (1 - t_l^i), & \text{if } L_i > N \\ 0, & \text{if } L_i \leq N \end{cases} \]  

and \( L_k = K - i + 1 \).

We can see that if \( K \leq N, L_i \leq N \) holds for \( i = 1, ..., K \), thus (31) is equivalent to (24), that is, LCD2 has the same performance with LCD1. For the case \( K > N \), we have \( L_i > N \) for \( i = 1, ..., K - N \), and LCD2 results in a reduction in the delivery rate for these layers, quantified by \( \Delta R_1^i \) and \( \Delta R_2^i \), in (32a) and (32b), respectively. Numerical comparison of the performance of these two content delivery algorithms, LCD1 and LCD2, is presented in Section V.

V. NUMERICAL RESULTS

In this section, we numerically compare the delivery rates of the proposed centralized and decentralized coded lossy caching schemes. We particularly consider two different cases depending on the relative numbers of users and files in the system, i.e., \( N \geq K \) and \( N < K \), as the proposed caching schemes exhibit different behaviors in these cases.

In the first scenario, we consider a caching system with a server containing \( N = 10 \) files serving \( K = 10 \) users. The target distortion levels of the users are given by \((D_1, D_2, ..., D_{10})\), such that \((r_1, r_2, ..., r_{10}) = (1, 2, ..., 10)\). We consider two cases for the cache sizes: the first one with identical cache capacities, i.e., \( M_1 = M_2 = \cdots = M_{10} = M \), and the second one with heterogeneous cache capacities, where \( M_k = 0.2kM \), for \( k = 1, ..., 10 \). The results for these two scenarios for both the centralized and decentralized caching are plotted in Fig. 4 and Fig. 5, respectively.
Fig. 4. Comparison of the achievable delivery rates with identical cache capacities, i.e., $M_1 = M_2 = \ldots = M_K$, for $N = 10$, $K = 10$.

In Fig. 4, we observe that the centralized coded caching scheme with OCA achieves the best delivery rate when the cache capacities are very small but its performance approaches that of the uncoded caching scheme as $M$ increases. This implies that when the cache capacity is small, it should be allocated mainly to the first layer, which is requested by all the users. Since the cache capacities are identical, coded caching of the same layer across users creates more multicasting opportunities, which better exploits the limited cache capacity. On the other hand, for medium to large cache capacities, PCA significantly outperforms OCA. This is because, when there is sufficient cache capacity, caching higher layers creates new multicasting opportunities, which further reduces the delivery rate. The black line in the figure is achievable by memory sharing between the two caching schemes, OCA and PCA, further reducing the delivery rate for moderate cache capacities. In Fig. 5, a network with heterogeneous cache capacities is considered, and it is observed that PCA outperforms OCA for all values of $M$, since PCA is capable of exploiting the
additional cache capacity of users to meet the requirements of reduced distortion target, retaining the symmetry among the amount of cache allocated to each layer across different users.

We remark that, since $N = K$, decentralized caching with LCD1 and LCD2 have the same performance as characterised in Theorem 4. For very small cache capacities, the decentralized scheme achieves almost the same performance as the centralized scheme with PCA. This is because, when cache capacities are very small, users will cache distinct bits from their required layers with high probability despite random placement, which is similar to the cache placement phase used by the centralized scheme with PCA. The performance improvement of centralized caching becomes more pronounced as $M$ increases since the collaboration between users during the placement phase will fully exploit the cache capacities to create the maximum number of multicasting opportunities. We observe that, despite lack of cache coordination across users, decentralized caching still achieves a performance not far from the best centralized caching.
scheme.

In both Fig. 4 and Fig. 5, it is observed that a significant improvement can be achieved by coded caching. We also remark that the total cache capacity across the whole network is $10M$ and $11M$ for the settings considered in Fig. 4 and Fig. 5, respectively. However, we can notice that the delivery rate achieved in Fig. 4 is significantly higher in both the centralized and decentralized scenarios. This is due to the distribution of the cache capacity across the users. In the latter scenario, the users with lower distortion requirements have larger cache capacities, allowing them to achieve their desired QoS targets without increasing the delivery rate.

![Comparison of the achievable delivery rates with identical cache capacities, i.e., $M_1 = M_2 = \ldots = M_K$, for $N = 10$, $K = 15$.](image)

In the second scenario, we consider a server with $N = 10$ files serving $K = 15$ users. The target distortion levels are such that $(r_1, r_2, \ldots, r_{15}) = (1, 2, \ldots, 15)$. As in the first scenario, we consider two cases: identical cache sizes, i.e., $M_1 = M_2 = \cdots = M_{15} = M$, plotted in Fig. 6; and heterogeneous cache sizes, i.e., $M_k = 0.125kM$, for $k = 1, \ldots, 15$, plotted in Fig. 7, such that the total cache capacity across the network for both scenarios is $15M$. 
We observe that, for centralized coded caching, PCA outperforms OCA in both Fig. 6 and Fig. 7. This implies that it is always better for each user to distribute its cache capacity to all the layers it may require, rather than to allocate it only to the first layer. Since $N < K$, the number of users requesting the first layer is larger than the number of files. There are at least two users requesting the same file, which creates multicasting opportunities even with uncoded caching, reducing the gain of coded caching over uncoded caching. Allocating the cache capacities to other layers will better exploit coded caching. In decentralized caching, we see that, LCD1 has the same performance as uncoded caching for small cache capacities. However, Fig. 6 shows a larger range of cache capacities where LCD1 and uncoded caching have the same performance, since when the cache capacities are identical, for each layer, the expected number of bits cached by each user is distinct, which reduces the gain from coded delivery. It is observed in both figures, LCD2 greatly outperforms LCD1, but the gap between two schemes reduces with the cache
capacity. This is because, the improvement of LCD2 over LCD1 derives from more effectively delivering the bits that are cached by at most one user. When the cache capacity increases, the number of such bits reduces, hence, the performance of LCD2 approaches that of LCD1. Although the total cache capacity across the network is the same for the scenarios considered in Fig. 6 and Fig. 7, it is observed that in both the centralized and decentralized settings, the rates achieved in Fig. 7 are significantly lower, similar to the $N = K$ scenario, since the larger cache capacities can be exploited to improve the QoS of users by the proposed layered caching approach.

We also observe that in all the figures, the gain of coded caching, both in centralized and decentralized settings, becomes more significant as the cache capacity, $M$, increases. We also note that there is a relatively large gap between the best achievable delivery rates and the cut-set lower bound, but part of this gap is potentially due to the looseness of the cut-set bound, as also suggested in [6].

![Comparison of achievable delivery rates versus $\alpha$, $\alpha \in [0, 1]$, $M_1 = M_2 = \cdots = M_{10} = 5$, for $N = 10$, $K = 10$.](image)

Next, we consider a server with $N = 10$ files serving $K = 10$ users, with identical cache capacities, i.e., $M_1 = M_2 = \cdots = M_{10} = 5$. The target distortion levels of the users,
\((D_1, D_2, \ldots, D_{10})\), are given by \(r_k = 5 + (k - 5.5)\alpha\), for \(k = 1, \ldots, 10\) and \(\alpha \in [0, 1]\). In this scenario, the average value of \(r_k\) is 5, i.e., \(\frac{10}{k=1}\sum r_k/10 = 5\), independent of the value of \(\alpha\). As \(\alpha\) increases, the distortion requirements of users become more diverse. The delivery rates of the proposed caching schemes, in centralized and decentralized setting, are shown in Fig. 8, compared with uncoded caching and cut-set bound. We emphasize that the delivery rate of the centralized caching scheme in Fig. 8 is the lower one of the delivery rates achieved by PCA and OCA. We observe that, while the delivery rate of uncoded caching remain the same, the delivery rates of both centralized and decentralized coded caching schemes increase with \(\alpha\), which indicates the loss of coded caching gain due to the diversity of distortion requirements.

Finally, we consider a server with \(N = 10\) files, and \(K = 15\) users with identical QoS requirements, i.e., \(D_1 = D_2 = \cdots = D_{15}\), such that \(r_1 = r_2 = \cdots = r_{15} = 2\). The cache size of each user is given by \(M_k = 8 + (k - 8)\beta\), for \(k = 1, \ldots, 15\) and \(\beta \in [0, 1]\). The larger \(\beta\), the more skewed the cache size distribution is. This setting is equivalent to a lossless coded caching problem with distinct cache capacities. The state-of-art decentralized coded caching scheme for this setting is presented in [27]. In Fig. 9, we compare the best achievable rates.
by the proposed centralized and decentralized coded caching schemes with the one proposed in [27]. Our decentralized coded caching scheme is shown to achieve the same performance as the scheme proposed in [27]. This is because we adopt the scheme proposed in [27] for each layer. Since there is only one layer in this scenario, it is equivalent to applying the scheme proposed in [27] to this layer. For all the three schemes, the gain from coded caching becomes more pronounced as $\beta$ becomes small, i.e., as the cache capacities of users become more similar.

**VI. Conclusions**

We have considered coded caching and delivery of contents in wireless networks, taking into account the heterogeneous distortion requirements of users, in both centralized and decentralized settings. The caching and delivery schemes considered here exploit the specific properties of lossy reconstruction of contents by users. In particular, we have exploited successive refinement source coding, which allowed us to cache contents incrementally across users depending on their cache capacities and distortion requirements. In the centralized setting, for a simple setting of two users and two files, we have derived the optimal coded caching scheme that achieves the information-theoretic lower bound by explicitly considering all possible caching options. We have further explored the case with two users and an arbitrary number of files. The proposed coded caching scheme was proven to be optimal when the cache capacities of the two users are the same and the number of files in the database is divisible by 3. Then, we tackled the general case with $K$ users and $N$ files in two steps: delivery rate minimization, which finds the minimum delivery rate for each layer separately, and cache capacity allocation among layers. We proposed two algorithms for the latter, namely, PCA and OCA.

In the decentralized setting, since the number and identity of the active users are not known during the placement phase, we have employed random cache placement. We have proved that a layered delivery scheme, which delivers each layer separately, is without loss of optimality, compared to joint delivery across layers. We have applied the existing coded caching scheme for the lossless caching problem with heterogeneous cache capacities to the delivery of each layer, together with the analysis on the achievable delivery rate. We have validated the improvement of the proposed coded caching schemes compared to uncoded caching through numerical results. However, there is still a remarkable gap between the best achieved delivery rate and the cut-set lower bound in all the scenarios considered in Section V. We have derived a tighter bound for
a special scenario with two users, which was then used to prove the optimality of the proposed coded caching scheme for this scenario. Extending the tighter lower bounds for the lossless caching problem proposed in [28], [29] to the lossy caching problem studied here is currently under consideration to better understand the optimal performance.

REFERENCES

[1] L. Fan, P. Cai, J. Almeida and A. Z. Broder, “Summary Cache: a scalable wide-area web cache sharing protocol,” IEEE/ACM Trans. Networking, vol. 8, no. 3, pp. 281-293, Jun. 2000.
[2] K. C. Almeroth and M. H. Ammar, “The use of multicast delivery to provide a scalable and interactive video-on-demand service,” IEEE J. Sel. Areas Commun., vol. 14, no. 6, pp. 1110-1122, Aug. 1996.
[3] K. Shanmugam, N. Golrezaei, A. G. Dimakis, A. F. Molisch and G. Caire, “Femtocaching: wireless video content delivery through distributed caching helpers,” arXiv:1109.4179v4 [cs.NI], Sep. 2013.
[4] M. Gregori, J. Gomez-Vilardebo, J. Matamoros and D. Gündüz, “Wireless content caching for small cell and D2D networks,” to appear, IEEE J. Sel. Areas Commun., 2016.
[5] X. Wang, M. Chen, T. Taleb, A. Ksentini and V. C. M. Leung, “Cache in the air: Exploiting content caching and delivery techniques for 5G systems,” IEEE Commun. Mag., vol. 52, no. 2, pp. 131-139, Feb. 2014.
[6] M. Maddah-Ali and U. Niesen, “Fundamental limits of caching,” IEEE Trans. Inform. Theory, vol. 60, no. 5, pp. 2856-2867, May 2014.
[7] M. A. Maddah-Ali and U. Niesen, “Decentralized coded caching attains order-optimal memory-rate tradeoff,” IEEE/ACM Trans. Networking, vol. 23, no. 4, pp. 1029-1040, Apr. 2014.
[8] Z. Chen, P. Fan and K. B. Letaief, “Fundamental limits of caching: Improved bounds for small buffer users,” arXiv:1407.1935v2 [cs.IT], Nov. 2015.
[9] M. Mohammadi Amiri, Q. Yang and D. Gündüz, “Coded caching for a large number Of users,” arXiv:1605.01993v1 [cs.IT], May 2016.
[10] K. Wan, D. Tuninetti and P. Piantanida, “On caching with more users than files,” in Proc. IEEE Int’l Symp. on Inform. Theory (ISIT), Barcelona, Spain, Jul. 2016, pp. 135–139.
[11] R. Pedarsani, M. Maddah-Ali and U. Niesen, “Online coded caching,” arXiv:1311.3646v1 [cs.IT], Nov. 2013.
[12] N. Karamchandani, U. Niesen, M. Maddah-Ali, and S. Diggavi “Hierarchical coded caching,” arXiv:1403.7007v2 [cs.IT], Jun. 2014.
[13] M. Ji, G. Caire and A. F. Molisch, “Fundamental limits of distributed caching in D2D wireless networks,” in Proc. IEEE Inform. Theory Workshop (ITW), Jeju Island, Korea, Oct. 2015, pp. 1–5.
[14] R. Timo and M. Wigger, “Joint cache-channel coding over erasure broadcast channels,” arXiv:1505.01016v1 [cs.IT], May. 2015.
[15] J. Zhang and P. Elia, “Fundamental limits of cache-aided wireless BC: interplay of coded-caching and CSIT feedback,” arXiv:1511.03961v2 [cs.IT], Apr. 2016.
[16] M. Maddah-Ali and U. Niesen, “Cache-aided interference channels,” arXiv:1510.06121v1 [cs.IT], Oct. 2015.
[17] A. Sengupta, R. Tandon and O. Simeone, “Cache aided wireless networks: tradeoffs between storage and latency,” arXiv:1512.07856v1 [cs.IT], Dec. 2015.
Appendix A

Proof of Theorem 1

Let $s \in \{1, \ldots, \min\{N, K\}\}$, and consider a set of users $\mathcal{U}$, such that $|\mathcal{U}| = s$. There exists a demand combination and a corresponding message over the shared link, say $X_1^n$, such that $X_1^n$ and $\{Z_k^n \mid k \in \mathcal{U}\}$ allow the reconstruction of files $S_1^n, \ldots, S_s^n$, each at the required distortion level from $\{D_k \mid k \in \mathcal{U}\}$. Similarly, there exists an input to the shared link, say $X_2^n$, such that $X_2^n$ and $\{Z_k^n \mid k \in \mathcal{U}\}$ allow the reconstruction of files $S_{s+1}^n, \ldots, S_{2s}^n$, each at the required distortion level from $\{D_k \mid k \in \mathcal{U}\}$, and so on so forth. We can continue in the same manner considering messages $X_3^n, \ldots, X_{\lfloor N/s \rfloor}^n$ for further demand combinations. Hence, with $X_1^n, \ldots, X_{\lfloor N/s \rfloor}^n$ and $\{Z_k^n \mid k \in \mathcal{U}\}$, each user $k \in \mathcal{U}$ should be able to reconstruct a distinct set of $\lfloor N/s \rfloor$ files at the
corresponding distortion level \( D_k \). Therefore, we have

\[
[N/s] \tilde{R}^c(M_1, ..., M_K) + \sum_{k \in \mathcal{U}} M_k \geq [N/s] \sum_{k \in \mathcal{U}} R(D_k). \tag{33}
\]

Since this inequality must hold for all possible choices of \( s \), and corresponding subset of users \( \mathcal{U} \) with \( |\mathcal{U}| = s \), we can obtain the following cut-set bound by substituting \( r_k = R(D_k) \):

\[
\tilde{R}^c(M_1, ..., M_K) \geq R_C(M_1, ..., M_K) \triangleq \max_{s \in \{1, \ldots, \min\{N,K\}\}} \max_{\mathcal{U} \subseteq \{1, \ldots, K\}, |\mathcal{U}| = s} \left( \sum_{k \in \mathcal{U}} r_k - \frac{\sum_{k \in \mathcal{U}} M_k}{[N/s]} \right).
\tag{34}
\]

**APPENDIX B**

**PROOF OF THEOREM 2**

We consider two groups of demands, i.e., \( \{(2i + 1, 2i + 2)\mid i = 0, 1, \ldots, [N/2] - 1\} \) and \( \{(2i + 2, 2i + 1)\mid i = 0, 1, \ldots, [N/2] - 1\} \). We define the vector of channel inputs corresponding to these demand combinations as follows

\[
\tilde{X}_{12}^n = \bigcup_{i=0}^{[N/2]-1} X_{2(i+1),2i+2}^n, \quad \text{and} \quad \tilde{X}_{21}^n = \bigcup_{i=0}^{[N/2]-1} X_{2i+2,2i+1}^n.
\]

Then, for any \((n, M_1, M_2, R)\) caching code that achieves the distortion tuple \((D_1, D_2)\), we have

\[
2[N/2]nR + nM_1 + nM_2 \geq H(\tilde{X}_{12}^n, \tilde{Z}_1^n) + H(\tilde{X}_{21}^n, \tilde{Z}_2^n) \tag{35a}
\]

\[
= H(\tilde{X}_{12}^n, \tilde{Z}_1^n | \tilde{S}_1^n) + H(\tilde{X}_{21}^n, \tilde{Z}_2^n | \tilde{S}_2^n) + I(\tilde{S}_1^n; \tilde{X}_{12}^n, \tilde{Z}_1^n) + I(\tilde{S}_2^n; \tilde{X}_{21}^n, \tilde{Z}_2^n) \tag{35b}
\]

\[
\geq H(\tilde{X}_{12}^n, \tilde{Z}_1^n | \tilde{S}_1^n) + H(\tilde{X}_{21}^n, \tilde{Z}_2^n | \tilde{S}_2^n) + I(\tilde{S}_1^n; \tilde{X}_{12}^n, \tilde{Z}_1^n) + I(\tilde{S}_2^n; \tilde{X}_{21}^n, \tilde{Z}_2^n) \tag{35c}
\]

\[
\geq I(\tilde{S}_1^n; \tilde{X}_{12}^n, \tilde{Z}_1^n, \tilde{S}_2^n; \tilde{X}_{21}^n, \tilde{Z}_2^n) + I(\tilde{S}_1^n; \tilde{X}_{12}^n, \tilde{Z}_1^n, \tilde{S}_2^n; \tilde{X}_{21}^n, \tilde{Z}_2^n) \tag{35d}
\]

where \( \tilde{S}_1^n = \bigcup_{i=0}^{[N/2]-1} S_{2i+1}^n \) and \( \tilde{S}_2^n = \bigcup_{i=0}^{[N/2]-1} S_{2i+2}^n \). We denote by \( \hat{S}_{i,k}^n \) the reconstruction of \( S_{i,k}^n \) at user \( k \), for \( i = 1, \ldots, N \) and \( k = 1, 2 \). Let \( \tilde{S}_{11}^n = \bigcup_{i=0}^{[N/2]-1} \hat{S}_{2i+1,1}^n, \tilde{S}_{12}^n = \bigcup_{i=0}^{[N/2]-1} \hat{S}_{2i+1,2}^n \) and \( \tilde{S}_{22}^n = \bigcup_{i=0}^{[N/2]-1} \hat{S}_{2i+2,2}^n \). We have

\[
I(\tilde{S}_1^n; \tilde{X}_{12}^n, \tilde{Z}_1^n) \overset{(a)}{\geq} I(\tilde{S}_1^n; \tilde{S}_{11}^n) \overset{(b)}{\geq} \sum_{i=0}^{[N/2]-1} I(S_{2i+1}^n; \tilde{S}_{11}^n) \overset{(c)}{\geq} \sum_{i=0}^{[N/2]-1} I(S_{2i+1}^n; \hat{S}_{2i+1,1}^n) \overset{(c)}{\geq} [N/2]nR(D_1),
\tag{36}
\]

where the inequality (a) follows from the data processing inequality since \( \tilde{S}_1^n = (X_{12}^n, Z_1^n) \) forms a Markov chain; (b) holds due to the independence of files. Since the reconstruction of
\( S_{2i+1} \) at user 1, i.e., \( \hat{S}_{2i+1,1} \), is required to be within distortion \( D_1 \), the inequality (c) follows from the definition of the rate distortion function [25].

Similarly, we have

\[
I(S_1^n; \hat{S}_{12}^n) \geq \sum_{i=0}^{\lfloor N/2 \rfloor - 1} I(S_{2i+1}^n; \hat{S}_{2i+1,2}^n) \geq \lfloor N/2 \rfloor - 1 \sum_{i=0}^{\lfloor N/2 \rfloor - 1} I(S_{2i+1}^n; \hat{S}_{2i+1,2}^n) \geq \lfloor N/2 \rfloor nR(D_2).
\]

(37)

Finally, also using the data processing inequality due to the Markov chain, \((\bar{S}_1^n, \bar{S}_2^n) - (\hat{X}_{12}^n, Z_2^n) - \hat{S}_{22}^n\), we can write

\[
I(\bar{S}_2^n; \hat{S}_{22}^n | \bar{S}_1^n) \geq I(\bar{S}_2^n; \hat{S}_{22}^n | \bar{S}_1^n)
\]

(38a)

\[
= I(\bar{S}_2^n; \hat{S}_{22}^n | \bar{S}_1^n)
\]

(38b)

\[
= I(\bar{S}_2^n; \hat{S}_{22}^n | \bar{S}_1^n)
\]

(38c)

\[
= I(\bar{S}_2^n; \hat{S}_{22}^n) \geq \lfloor N/2 \rfloor nR(D_2),
\]

(38d)

where (38c) holds since \( \bar{S}_2^n \) is independent \( \bar{S}_1^n \); and (e) follows from the definition of the rate-distortion function.

Substituting inequalities (36), (37), (38d) into (35d) and replacing \( R(D_k) \) with \( r_k, k = 1, 2 \), we obtain

\[
R \geq r_1/2 + r_2 - \frac{(M_1 + M_2)}{2\lfloor N/2 \rfloor}.
\]

(39)

**APPENDIX C**

**LOWER BOUND OF LEMMA 2**

We consider two groups of \( \lfloor N/3 \rfloor \) demands, i.e., \( \{(3i + 1, 3i + 2) | i = 0, 1, ..., \lfloor N/3 \rfloor - 1\} \) and \( \{(3i + 2, 3i + 3) | i = 0, 1, ..., \lfloor N/3 \rfloor - 1\} \). Similarly to the proof of Theorem 2, we define the vector of channel inputs corresponding to these demand combination as \( \hat{X}_{12}^n = \bigcup_{i=0}^{\lfloor N/3 \rfloor - 1} X_{(3i+1,3i+2)}^n \), and \( \hat{X}_{23}^n = \bigcup_{i=0}^{\lfloor N/3 \rfloor - 1} X_{(3i+2,3i+3)}^n \). Then for any \((n, M_1, M_2, R)\) caching code that achieves the distortion
tuple \((D_1, D_2)\), we have

\[
2\lfloor N/3\rfloor nR + nM_1 + nM_2 \geq H(\bar{X}_{12}^n, Z_2^n) + H(\bar{X}_{23}^n, Z_1^n)
\]

\[
= H(\bar{X}_{12}^n, Z_2^n | S_2^n) + H(\bar{X}_{23}^n, Z_1^n | S_2^n) + I(S_2^n; \bar{X}_{12}^n, Z_2^n) + I(S_2^n; \bar{X}_{23}^n, Z_1^n) \tag{40a}
\]

\[
\geq H(\bar{X}_{12}^n, Z_2^n, \bar{X}_{23}^n, Z_1^n | S_2^n) + I(S_2^n; \bar{X}_{12}^n, Z_2^n) + I(S_2^n; \bar{X}_{23}^n, Z_1^n) \tag{40b}
\]

\[
\geq I(S_1^n, \bar{S}_3^n, \bar{X}_{12}^n, Z_2^n, \bar{X}_{23}^n, Z_1^n | S_2^n) + I(S_2^n; \bar{X}_{12}^n, Z_2^n) + I(S_2^n; \bar{X}_{23}^n, Z_1^n) \tag{40c}
\]

\[
\geq I(S_1^n, \bar{S}_3^n, \bar{X}_{12}^n, Z_2^n, \bar{X}_{23}^n, Z_1^n | S_2^n) + I(S_2^n; \bar{X}_{12}^n, Z_2^n) + I(S_2^n; \bar{X}_{23}^n, Z_1^n) \tag{40d}
\]

where \(S_1^n = \bigcup_{i=0}^{\lfloor N/3\rfloor-1} S_{3i+1}^n, \bar{S}_2^n = \bigcup_{i=0}^{\lfloor N/3\rfloor-1} S_{3i+2}^n, \text{ and } S_3^n = \bigcup_{i=0}^{\lfloor N/3\rfloor-1} S_{3i+3}^n\). We remind that \(\hat{S}_{i,k}^n\) denotes the reconstruction of \(S_i^n\) at user \(k\), for \(i = 1, ..., N\) and \(k = 1, 2\). We define \(\hat{S}_{11}^n = 3_i \bigcup_{i=0}^{\lfloor N/3\rfloor-1} \hat{S}_{i+1}^n, \hat{S}_{21}^n = \bigcup_{i=0}^{\lfloor N/3\rfloor-1} \hat{S}_{3i+1}^n, \hat{S}_{22}^n = \bigcup_{i=0}^{\lfloor N/3\rfloor-1} \hat{S}_{3i+2}^n, \text{ and } \hat{S}_{32}^n = \bigcup_{i=0}^{\lfloor N/3\rfloor-1} \hat{S}_{3i+3}^n\). Following the similar arguments as in (36) and (37), we have

\[
I(\hat{S}_2^n; \bar{X}_{12}, Z_2^n) \geq n\lfloor N/3\rfloor R(D_2) = n\lfloor N/3\rfloor r_2, \tag{41a}
\]

\[
I(\hat{S}_2^n; \bar{X}_{23}, Z_1^n) \geq n\lfloor N/3\rfloor R(D_1) = n\lfloor N/3\rfloor r_1. \tag{41b}
\]

We also have

\[
I(S_1^n, S_3^n, \bar{X}_{12}^n, Z_2^n, \bar{X}_{23}^n, Z_1^n | S_2^n) \geq I(S_1^n, S_3^n, \bar{X}_{12}^n, Z_2^n, \bar{X}_{23}^n, Z_1^n | S_2^n) - I(S_1^n, S_3^n, \bar{S}_2^n) \tag{42a}
\]

\[
\geq I(S_1^n; \bar{X}_{12}^n, Z_2^n, \bar{X}_{23}^n, Z_1^n) + I(S_3^n; \bar{X}_{12}^n, Z_2^n, \bar{X}_{23}^n, Z_1^n | S_1^n) \tag{42b}
\]

\[
\geq I(S_1^n; S_3^n) + I(S_3^n; \bar{S}_1^n) \tag{42c}
\]

\[
\geq n\lfloor N/3\rfloor R(D_1) + n\lfloor N/3\rfloor R(D_2) \tag{42d}
\]

\[
= n\lfloor N/3\rfloor r_1 + n\lfloor N/3\rfloor r_2, \tag{42e}
\]

where (42b) follows since \(S_1^n, S_2^n\) and \(S_3^n\) are independent; (42c) follows due to the nonnegativity of mutual information; and inequality (42d) is due to the data processing inequality and the fact that \(S_1^n - (\bar{X}_{12}^n, Z_2^n) - \hat{S}_{11}^n\) and \(S_3^n - (\bar{X}_{23}^n, Z_2^n) - \hat{S}_{32}^n\) are both Markov chains. Substituting inequalities (41a), (41b) and (42f) into (40d), we obtain

\[
R \geq r_1 + r_2 - \frac{(M_1 + M_2)}{2\lfloor N/3\rfloor} \tag{43}
\]
Here, we prove Eqn. (27). We have
\[
\frac{M_k}{N_{tk}} = t^i_k \quad \text{for} \quad k = 1, \ldots, K+1-i; \quad \text{equality (44b) comes from iterating the smallest index in the multicasting set from 1 to } K-i-k+2, \text{which is the maximum that the value of the smallest index could be in a multicasting set with cardinality equals to } k; \quad \text{equality (44c) is derived by switching the order of summations regarding to } l \text{ and } k; \quad \text{equality (44d) is obtained by expressing } p^i(t, D) \text{ as a function of } t^i_s; \quad \text{since } \prod_{s=1}^{l} (1 - t^i_s) \text{ is independent of the value of } k \text{ and subset } S, \text{equality (44e) holds. Since } \sum_{j=1}^{K+1-i} \sum_{S \subseteq \{1, \ldots, K+1-i\}} \prod_{s \in \{l+1, \ldots, K+1-i\} \setminus S} (1 - t^i_s) \prod_{s \in S} t^i_s = 1, \text{ we have}
\]

\[
\sum_{j=1}^{K+1-i} \sum_{S \subseteq \{1, \ldots, K\}} p^i(t, D) \times \sum_{S \subseteq \{1, \ldots, K\}} p^i(t, D) \times \left( \prod_{s=1}^{l} (1 - t^i_s) \right) = \sum_{l=1}^{K+1-i} \prod_{s=1}^{l} (1 - t^i_s) = \sum_{l=1}^{K+1-i} \prod_{k=1}^{l} (1 - t^i_k),
\]