Abstract. I give a pedagogical review of the derivation for the effective lagrangian for nonabelian Debye screening, or hard thermal loops. Following Kelly, Liu, Lucchesi, and Manuel, I give the simplest derivation possible, using classical kinetic theory. The result is valid not just for a thermal, but for an arbitrary initial distributions. I use this to study the evolution, at short times, of a gluonic “tsunami wave”. I also suggest how classical kinetic theory may arise at one loop order. Using the wordline representation of the one loop effective action, I follow D’Hoker and Gagné to replace the Wilson line by an integral over worldline fermions. A bilinear of these worldline fermions naturally defines a nonabelian charge, whose equation of motion is Wong’s equation.

1. Introduction

In this talk I review our understanding of plasmas at very high temperatures [1]-[9]. While most of the detailed discussion is phrased in terms of a plasma near equilibrium at a temperature $T$, I show that the essential result holds not just for a thermal, but for arbitrary, initial distributions of particles. The crucial assumption is that the initial density of particles is large. For a thermal distribution, this means that the temperature is very high, much larger than any mass scale, $T \gg m$.

There are two major applications of this work. The first is to the behavior of $QCD$ in a “deconfined” phase. This is relevant to the central region of heavy ion collisions at high energies, as has been explored at the $SPS$, and will be studied at $RHIC$ and $LHC$. The second is to the electroweak
theory, in a phase in which the Higgs phase evaporates. This has become of especial interest with regards to baryogenesis at the electroweak scale.

All calculations are done using perturbation theory. At zero temperature, the normal expansion parameter is the fine structure constant, $\alpha = g^2 / 4\pi$; in QCD, if one is lucky, $\alpha$ is of order one. For systems at nonzero temperature, however, the relevant expansion parameter is not $\alpha$ but the coupling constant itself, $g$. Thus for QCD, if $\alpha \sim 1$, $g \sim 3$, and clearly a perturbative approach is invalid. It is probably a reasonable approximation in the electroweak theory, where $g \sim 3$.

My philosophy is that in order to understand any theory, one must have some controlled and calculable limit. It certainly should be able to give us some sort of picture of the important quasiparticle modes and their interactions. In a strong coupling regime the true quasiparticles may behave differently, but one must at least know what behavior that they must match onto in weak coupling.

Although it was not clear at first, the basic physical phenomenon which I will be studying is that of Debye screening. While it sounds prosaic, even to lowest nontrivial order in the coupling constant, for nonabelian gauge theories the physics is remarkably involved. For static charges, the screening is unremarkable, like that of nonrelativistic systems. It is only when one considers the screening of moving charges - that is, the dynamics of the system - that a wonderfully rich structure emerges.

This is suggestive. Numerical simulations of lattice gauge theory have clearly demonstrated their power to compute the behavior of gauge theories in equilibrium. These numerical simulations have shown that the phase structure of QCD depends upon the number and masses of the quark flavors in an unexpectedly intricate manner. At present, however, lattice gauge theory is only efficient for computing the behavior the nature of quantities in equilibrium. The present discussion shows that even perturbation theory shows that there will be many new surprises in considering truly dynamical phenomenon.

To appreciate the Debye lagrangian one really has to apply it in detailed calculations. For reasons of sloth, in these proceedings I concentrate exclusively on how to derive the effective Debye lagrangian, and not to what uses it can be put. The nonabelian Debye lagrangian was first derived by brute force, after a laborious and complicated analysis of diagrams in perturbation theory [1]. Blaizot and Iancu then derived it by a mean field approximation to the Schwinger-Dyson equations, which gives a kind of semiclassical kinetic theory [5]. My derivation herein uses classical kinetic theory, and copies that of Kelly, Liu, Lucchesi, and Manuel [6]. At one point, I use a trick of Brandt, Frenkel, and Taylor [7].

I first derive the effective lagrangian in QED, where the gauge principle
is elementary, and then that for \textit{QCD}. The beauty of classical kinetic theory is that, after invoking the nonabelian gauge symmetry, the derivation is as simple as for the abelian theory. My treatment is meant to be elementary, appropriate for those who have never had any exposure to a nonabelian gauge theory. This is only possible because of the simplicity of classical kinetic theory.

There is another advantage of classical kinetic theory: it appears to be valid not just for thermal, but for arbitrary initial distributions. In sec. 5 I use this to study what I term a gluonic “tsunami wave”.

The apparent peculiarity of classical kinetic theory is that it is necessary to introduce a nonabelian charge, \( Q \), and its attendant equation of motion, known as Wong’s equation [10]. Certainly to most high energy theorists, the introduction of \( Q \) appears \textit{ad hoc} and a little peculiar. In the final section I outline a possible way in which \( Q \) arises directly from the worldline formalism.

\section{Classical kinetic theory in QED}

The lagrangian of massless \textit{QED} is, of course,

\[ \mathcal{L} = \bar{\psi} \gamma^\alpha D_\alpha \psi + \frac{1}{4} F_{\alpha\beta}^2. \]  

(1)

The covariant derivative and field strength are

\[ D_\alpha = \partial_\alpha - ieA_\alpha, \ F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha. \]  

(2)

This is invariant under local gauge transformations,

\[ D_\alpha \to \Omega^\dagger D_\alpha \Omega, \ \psi \to \Omega^\dagger \psi. \]  

(3)

The transformation of the gauge field is more familiarly written as

\[ \Omega = e^{ie\omega}, \ A_\alpha \to A_\alpha + \partial_\alpha \omega. \]  

(4)

For the nonabelian theory, however, it will turn out that thinking about covariant derivatives, as in (3), is much more useful than in thinking about gauge potentials, as in (4). This difference isn’t apparent in the abelian theory.

We then need the classical equations of motion. To be logically complete I should start from the lagrangian and derive the appropriate equations in the limit of high density. A way in which to do this is sketched in sec. 6. For now I will take the equations of motion for granted, since they are utterly
standard: introducing the position, \( x^\alpha \), and the momentum, \( p^\alpha \), there is just the definition of the momentum,

\[
p^\alpha = \frac{dx^\alpha}{d\tau},
\]

(5)

and the Lorentz force equation,

\[
\frac{dp_\alpha}{d\tau} = e F_{\alpha\beta} p^\beta.
\]

(6)

The field is assumed to be massless, so “\( \tau \)” is not really a proper time, just an affine parameter which labels the worldline of the particle.

As typical of classical kinetic theory, I introduce the density of single particles, \( f(x, p) \). This is assumed to satisfy the Boltzmann equation in the collisionless approximation,

\[
\frac{d}{d\tau} f(x, p) = 0.
\]

(7)

Viewing \( f(x, p) \) as a density in phase space, this is a type of Liouville equation.

The only other quantity required is the current. In the classical approximation this is merely the product of the charge, the momentum, and the density in phase space, integrated over the distribution in momentum space:

\[
j_\alpha(x) = \int d^4 p \ e p^\alpha f(x, p).
\]

(8)

Beginning with this expression, one sees how integrals over the distribution in momentum space naturally arise. Also, notice that in QED, everything — \( x, p, f(x, p), \) and \( j_\alpha \) — are all gauge invariant.

These elementary equations, (5) - (8), are all that we need to solve for the effective action. Using the chain rule, the Boltzmann equation equals

\[
\frac{df}{d\tau} = \left( \frac{dx^\alpha}{d\tau} \frac{\partial}{\partial x^\alpha} + \frac{dp^\alpha}{d\tau} \frac{\partial}{\partial p^\alpha} \right) f.
\]

(9)

Plugging in the equations of motion,

\[
\frac{df}{d\tau} = p^\alpha \left( \frac{\partial}{\partial x^\alpha} - e F_{\alpha\beta} \frac{\partial}{\partial p^\beta} \right) f = 0,
\]

(10)

which is known as the Vlasov equation.

Now expand the complete distribution function about some initial value, \( f^0 \),

\[
f = f^0 + f^1 + \ldots
\]

(11)
The system must start out electrically neutral, with $f^0$ the same for positrons as for electrons, so initially there is no current. A current is induced by fluctuations, $f^1$. Solve the Boltzmann equation to first order in $f^1$:

$$p \cdot \partial f^1 = e p^\alpha F_{\alpha\beta} \frac{\partial f^0}{\partial p_\beta}.$$  

(12)

Without being too careful about what it means, we rather slopily introduce the nonlocal operator $1/p \cdot \partial$ to solve (12) for $f^1$, and obtain the induced current:

$$j^\alpha = \int d^4 p \, e p^\alpha \frac{1}{p \cdot \partial} e p^\beta F_{\beta\gamma} \frac{\partial f^0}{\partial p_\gamma}.$$  

(13)

Now I follow Brandt, Frenkel, and Taylor [7], and integrate $\partial / \partial p_\gamma$ by parts. Remember that the momentum $p$ is just a parameter of the initial (classical) distribution, so the field strength tensor is completely independent of it. It is then easy to integrate by parts,

$$j^\alpha = -e^2 \int d^4 p \, \frac{\partial}{\partial p_\gamma} \left( \frac{p^\alpha p^\beta}{p \cdot \partial} \right) F_{\beta\gamma} f^0.$$  

(14)

There are three terms from this derivative, which equal

$$j^\alpha = -e^2 \int d^4 p \, \frac{\partial}{\partial p_\gamma} \left( \frac{\delta^\alpha p^\beta}{p \cdot \partial} - \frac{p^\alpha p^\beta}{(p \cdot \partial)^2} \right) F_{\beta\gamma} f^0.$$  

(15)

Of the three terms, that $\sim \delta^{\beta\gamma}$ drops out, because it is contracted with the antisymmetric field strength, $F_{\beta\gamma}$.

Now, implicitly any current defines a lagrangian density through the relation

$$j^\alpha = \frac{\delta \mathcal{L}_{\text{Debye}}}{\delta A_\alpha}.$$  

(16)

Thus we can need to find the “Debye” lagrangian, $\mathcal{L}_{\text{Debye}}$, which generates the current of (15). This is easy: since $\mathcal{L}_{\text{Debye}}$ is gauge invariant, it is natural to try to use the field strength tensors, $\sim F_{\alpha\beta}$, since they are automatically gauge invariant. Without too much effort one can see that the result is

$$\mathcal{L}_{\text{Debye}} = \frac{e^2}{2} \int d^4 p \left( F_{\alpha\beta} \frac{p^\beta p^\gamma}{(p \cdot \partial)^2} F_{\alpha\gamma} \right) f^0(p).$$  

(17)
3. Classical kinetic theory in QCD

I defer any discussion of (17) to show that once one sets up the classical kinetic theory in a properly gauge invariant fashion, the nonabelian Debye lagrangian follows immediately from the abelian.

So let me start by reviewing what nonabelian gauge invariance is. The lagrangian for a massless quark coupled to a $SU(N)$ gauge field looks just like that for QED,

\[ \mathcal{L} = \overline{\psi} \gamma^\alpha D_\alpha \psi + \frac{1}{2} \text{tr}(G_{\alpha\beta}^2) , \]  

(18)

except that now the gauge potential is an $SU(N)$ matrix, $A_\alpha = A_\alpha^a t^a$, where I normalize that $SU(N)$ matrices $t^a$ as $\text{tr}(t^a t^b) = \delta^{ab}/2$, with the indices $a, b = 1...(N^2 - 1)$.

Normally, textbooks present nonabelian gauge symmetry as a direct generalization of the abelian symmetry, giving the transformation of the gauge potential, etc. This is a completely confusing way of viewing non-abelian gauge invariance. Instead of considering the gauge potential, it is much easier to concentrate on the covariant derivative, since under a local gauge transformation, it transforms homogeneously:

\[ D_\alpha = \partial_\alpha - igA_\alpha \rightarrow \Omega^\dagger D_\alpha \Omega , \quad \Omega^\dagger \Omega = 1 . \]  

(19)

The transformation of the gauge potential $A_\alpha$ can be worked out from this; its transformation is inhomogeneous so inelegant. Given (19), however, you don’t need to worry about the transformation of $A_\alpha$. For instance, if the quark fields transforms as in QED, like $\psi \rightarrow \Omega^\dagger \psi$, then obviously the quark part of the lagrangian is gauge invariant.

The field strength is constructed from the commutator of two covariant derivatives. Since $D_\alpha$ transforms homogeneously, so does its commutator:

\[ G_{\alpha\beta} = \frac{1}{-ig}[D_\alpha, D_\beta] \rightarrow \Omega^\dagger G_{\alpha\beta} \Omega . \]  

(20)

This is not obvious from the explicit expression for the field strength tensor,

\[ G_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha - ig[A_\alpha, A_\beta] , \]  

(21)

and the transformation of $A_\alpha$.

The moral of the story is that nonabelian gauge invariance can be easily ensured just by sticking to covariant derivatives everywhere. The derivation of the nonabelian Debye lagrangian below provides an striking illustration of this.

I now turn to the classical equations of motion for a nonabelian particle. As usual the position and momentum are related as $p^\alpha = dx^\alpha/d\tau$. The
generalization of the Lorentz force equation of (6) is not trivial, though, since while \(x\) and \(p\) are gauge invariant, in a nonabelian theory (unlike the abelian case) the field strength tensor \(G_{\alpha\beta}\) is not. To form a gauge invariant equation, it is necessary to introduce a matrix valued charge, \(Q\):

\[
\frac{dp^\alpha}{d\tau} = 2g \text{tr}(QG_{\alpha\beta})p^\beta. \tag{22}
\]

This is trivially gauge invariant if, like \(G_{\alpha\beta}\), \(Q\) transforms homogeneously under a local gauge transformation:

\[
Q \rightarrow \Omega^\dagger Q \Omega. \tag{23}
\]

We now have a different problem - what is the equation of motion for \(Q\)? This can basically be guessed from gauge and lorentz invariance. We certainly want the equation of motion to be gauge covariant, which is easily ensured by taking \(D_\alpha Q\). This isn’t quite right, however, because there are then four, instead of one, equation of motion. To get one equation, we contract \(D_\alpha Q\) with the obvious vector floating about, which is the momentum. This gives us a result first obtained by S. Wong [10]:

\[
\frac{dx^\alpha}{d\tau} D_\alpha Q = 0. \tag{24}
\]

The phase space for a nonabelian particle is now \(x\), \(p\), and \(Q\); thus the single particle density is in principle a function of all three, \(f(x,p,Q)\). The classical current is again the product of the charge, \(g\) times \(Q\), the momentum, and the phase space density:

\[
j^\alpha(x) = \int d^4p \int dQ g Q p^\alpha f(x,p,Q). \tag{25}\]

I note that the equation of motion for the gauge field is

\[
D_\alpha G^{\alpha\beta} = j^\beta. \tag{26}\]

Since \(G^{\alpha\beta}\) is a commutator of \(D\)'s, \(D_\alpha D_\beta G^{\alpha\beta} = 0\), so the current is covariantly conserved,

\[
D_\alpha j^\alpha = 0. \tag{27}\]

This is only true if \(Q\) satisfies Wong’s equation.

Taking these equations for granted, solving for the nonabelian Debye action is no harder than for the abelian. The collisionless Boltzmann equation is unchanged:

\[
\frac{d}{d\tau} f(x,p,Q) = 0. \tag{28}\]
The basic point is that because $x$, $p$, and $f(x,p,Q)$ are gauge invariant, while $Q$, $j^\alpha$, and the equations of motion are gauge covariant, then the resulting $L_{\text{Debye}}$ must be gauge invariant.

Now applying the chain rule to Boltzmann’s equation gives a Vlasov equation with three terms:

$$\frac{d}{d\tau} f = p^\alpha \left( \frac{\partial}{\partial x^\alpha} - 2 g \text{tr} \left( Q G_{\alpha \beta} \right) \frac{\partial}{\partial p^\beta} + 2 g \text{tr} \left( [A_\alpha, Q] \frac{\partial}{\partial Q} \right) \right) f = 0. \quad (29)$$

The last term is from color precession of $Q$, and is where all of the complications of the nonabelian theory reside.

Suppose we only wish to compute to lowest order in the gauge potential, however; this is linear in $A_\alpha$ for the current, or quadratic in $A_\alpha$ for the lagrangian. Then the nonabelian theory is no more complicated than the abelian. Expand the distribution function about some initial value, where the initial distribution is assumed to depend only upon momentum:

$$f(x,p,Q) = f^0(p) + f^1(x,p,Q) + \ldots. \quad (30)$$

Since $f_0$ is colorless, to lowest order in $A$ we can drop the term for color precession in Boltzmann’s equation. In this case the equations abelianize. The only feature of the $Q$’s which is needed are the Casimir’s:

$$\int dQ Q^a Q^b = C \delta^{ab}. \quad (31)$$

For gluons in the adjoint representation, $C = N$, while for quarks in the fundamental representation, $C = \frac{1}{2}$.

Thus to $\sim A^2$, the nonabelian debye action is a sum over abelian actions:

$$L_{\text{Debye}}^{(2)} = g^2 \int d^4 p \text{tr} \left( F_{\alpha \beta} \frac{p^\beta p^\gamma}{(p \cdot D)^2} F_{\alpha \gamma} \right) \sum_r C_r f_r^0(p). \quad (32)$$

Here the field strength is only the abelian part, linear in $A_\alpha$: $F_{\alpha \beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$.

It is then easy to make $L_{\text{Debye}}^{(2)}$ gauge invariant: replace the abelian part of the field strength by the complete field strength, and replace the ordinary by the covariant derivative, to obtain:

$$L_{\text{Debye}} = g^2 \int d^4 p \text{tr} \left( G_{\alpha \beta} \frac{p^\beta p^\gamma}{(p \cdot D)^2} G_{\alpha \gamma} \right) \sum_r C_r f_r^0(p). \quad (33)$$

The sum is over the representations of all charged fields. Since everything transforms homogeneously under a local gauge transformation, it is obviously gauge invariant. The crucial question is whether the generalization is unique. On this point I have to defer to the literature [1]-[9].
In hot QED, (17) was first derived almost 40 years ago by Silin [2], using classical kinetic theory. For an abelian theory, this is the complete result; \( \mathcal{L}_{\text{Debye}} \) is quadratic in the gauge potential, and only contributes to the photon self energy, which is how Silin wrote it. In contrast, in a nonabelian theory, because of the \( A_\alpha \) which lurks in the covariant derivative in \( 1/(p \cdot D) \), there are terms of arbitrary order in the gauge potential. This means that \( \mathcal{L}_{\text{Debye}} \) contributes not just to the gluon self energy, but to couplings between three, four, or any number of gluons. As a practical matter, this complicates using the nonabelian Debye lagrangian in real complications, although methods have been developed to deal with this [1].

The nonabelian Debye lagrangian was first derived by Taylor and Wong [3]. There are several, equivalent ways of writing it; that in (17) is handy because it is manifestly gauge invariant [4].

4. Thermal distribution

To understand the significance of the nonabelian Debye lagrangian, consider a thermal distribution at a temperature \( T \):

\[
f^0 = \delta(p^2) \times \frac{1}{e^{\frac{p^0}{T}} \pm 1},
\]

where the \(-\) is for the Bose-Einstein distribution of gluons, and the \(+\) for the Fermi-Dirac distribution of quarks. In this case it is easy to compute the nonabelian Debye lagrangian. The integral over the magnitude of \( p \) generates a factor \( \sim T^2 \), and leaves an angular integral:

\[
\mathcal{L}_{\text{Debye}} = \frac{3}{2} m_g^2 \int \frac{d\hat{p}}{4\pi} \text{tr} \left( G_{\alpha\beta} \frac{p^\beta p^\gamma}{-(p \cdot D)^2} G_{\alpha\gamma} \right).
\]

I redefine \( p \to (i, \hat{p}) \), with \( \hat{p} \) a unit spatial vector, \( \hat{p}^2 = 1 \). \( m_g \) is the gluon Debye “mass”,

\[
m_g^2 = \left( N + \frac{N_f}{2} \right) \frac{g^2 T^2}{9}.
\]

There is also a Debye lagrangian for quarks:

\[
\mathcal{L}_{\text{Debye}}^q = m_q^2 \int \frac{d\hat{p}}{4\pi} \overline{\psi} \left( \frac{p \cdot \gamma}{p \cdot D} \right) \psi,
\]

where \( m_q \) is the quark Debye “mass”,

\[
m_q^2 = \frac{N^2 - 1}{2N} \frac{g^2 T^2}{8}.
\]
The quark Debye lagrangian is chirally symmetric, like the original gauge interaction.

We can now understand the applicability of the Debye lagrangians. For a massless gas at nonzero temperature, the typical momentum is of order the temperature, which is termed “hard” [1]. The nonabelian Debye lagrangian introduces “soft” momenta, on the order of the Debye masses, $m_g \sim m_q \sim gT$. To understand the relative importance of fields, we use power counting, taking the gauge field $A_\alpha \sim T$. Thus for hard momenta, the original lagrangian $L \sim T^4$, while the nonabelian Debye term is just part of the perturbative corrections, $L_{\text{Debye}} \sim g^2 T^4$. In contrast, for soft momenta, both the original and the nonabelian Debye lagrangians are of the same order, $L \sim L_{\text{Debye}} \sim g^2 T^4$. Thus for soft momenta, since $L_{\text{Debye}}$ is as big as $L$, it is necessary to include both in an effective lagrangian.

Diagrammatically, one can show that the nonabelian Debye lagrangian only receives contributions from one loop diagrams in which the loop momentum are hard. This suggests the term “hard thermal loops”, which is commonly used in the literature. In this talk I adopt instead the more generic phrase of nonabelian Debye lagrangian, which sounds less technical. I emphasize that while the physics subsumes Debye screening, there is much more going on than just that.

Indeed, consider the textbook example of hot QED. While the photon self energy is treated in the textbooks [2], that of hot fermions is not. The Debye lagrangian for fermions in hot QED is very similar to that of quarks in hot QCD, (37); except for a change in the Debye quark mass, $m_q$, the only other change is to use the abelian covariant derivative. As noted, in hot QED the abelian Debye lagrangian just contributes to the photon self energy, nothing more. But the fermion Debye lagrangian is as complicated in hot QED as in hot QCD: the nonlocal factor of $1/p \cdot D$ generates not just a contribution to the fermion self energy, but as well an infinite series of couplings between a fermion anti-fermion pair and any number of photons. Thus the textbook treatment of hot QED is seriously incomplete. There is a good reason for this: the fermion Debye lagrangian does not appear to be easily derivable by the standard form of classical kinetic theory. It has been derived in perturbation theory, and from the semiclassical kinetic theory of Blaizot and Iancu [1]; perhaps it could be derived using the approach outlined in sec. 6.

One novel aspect of the Debye lagrangians is that they are nonlocal, because of the the factors of $1/p \cdot \partial$. In a thermal distribution, this gives rise to discontinuities for spacelike momenta, which can be understood as a relativistic generalization of Landau damping.

There is a more general lesson. At zero temperature, a standard assumption in constructing any effective lagrangian is that all terms must
be local. The nonabelian Debye lagrangian shows that this is no longer true at nonzero temperature, although the nonlocality which enters is of an extremely specific form.

The nonlocality can be made to (apparently) disappear by choosing the gauge $p \cdot A = 0$. In that case, the entire lagrangian collapses to a mass term for the gauge field:

$$L_{\text{Debye}} = g^2 \int d^4p \ tr \left( A_\alpha^2 \right) \sum_r C_r f^0_r(p).$$  \hspace{1cm} (39)

This form was first noted by Frenkel and Taylor [1], [7]. Elmflors and Hansson have shown how it can be used to provide a direct functional derivation of $L_{\text{Debye}}$ from the background field method [8]. Similarly, in this gauge the quark Debye lagrangian also collapses to just a self energy. For a general distribution this gauge choice can only be imposed after generalizing the gauge potential to be a function not just of spacetime, but also of the momentum of the initial distribution, $p$.

5. Tsunami wave distribution

In the above I have made no reference to the initial distribution, $f^0$. It may be thermal, but need not be; all that is necessary is that the change from the initial distribution is small, $f^1 \ll f^0$. For a thermal distribution, a small perturbation will automatically return to thermal, but this is certainly not true for an arbitrary initial distribution.

For a general distribution, the nonabelian Debye lagrangian is presumably valid not just at weak coupling, but only for small times; at long times, what $f^0$ evolves into depends on the detailed dynamics. I now apply these results to another distribution, admittedly cooked up.

In cascade models of heavy ion collisions, the two colliding nuclei are modeled by two pancakes, where in each pancake, all partons move in lock-step with the same momentum. Now take one pancake away, and enlarge the other pancake until it fills up space uniformly.

This leads to the “tsunami wave” problem: at time $t = 0$, assume that one has a large and spatially constant density of particles, all moving together with the same momentum. Into what state does this evolve at infinite time?

The particles must be bosons, since I have assumed that the initial density is high. If the particles have nonzero mass, then one can transform into their rest frame, in which case the particles are just a condensate (of some sort) at zero momentum. So assume that they are massless, and move on the light cone.

The initial state is a system with nonzero energy and momentum density. Thus one natural guess for the state at infinite time is that of a boosted
thermal distribution, since that also has nonzero energy and momentum density. That the system thermalizes is not obvious; assuming that it does, the relevant question then is, over what time scales?

This is a very difficult question which I could not attempt to solve analytically. For small times, however, a perturbative analysis should be a reasonable approximation. I define the tsunami wave distribution as:

\[ f^0(p) \sim \rho_0 \, \delta(p^2) \, |p_0| \, \delta^3(\vec{p} - \vec{p}_0). \]  

(40)

Here \( \rho_0 \) is a parameter proportional to the density, and \( p_0 = (i|p_0|, \vec{p}_0) \), \( p_0^2 = 0 \), is the momentum of the particles in the tsunami wave.

We now have a problem in which there is a preferred four vector, \( p_0 \). For this problem the obvious choice of gauge is \( p_0 \cdot A = 0 \). Consider the gluon self-energy: it is a function of the four momentum, \( k \), and also of this vector \( p_0 \). In general four functions enter into the self energy:

\[ \Pi^{\alpha\beta}(k) = \Pi_t \delta^{\alpha\beta} + \Pi_\ell k^\alpha k^\beta + \Pi_3(k^\alpha p_0^\beta + p_0^\alpha k^\beta) + \Pi_4 p_0^\alpha p_0^\beta. \]  

(41)

This is analogous to the situation at nonzero temperature, where the rest frame of the thermal bath provides a preferred four vector. For the thermal case, the preferred vector is timelike; here it is null. Also as at nonzero temperature, the four functions in \( \Pi^{\alpha\beta} \) are related by a Ward identity, but that doesn’t matter here.

We can then read off the result from (39). Ignoring inessential constants,

\[ \Pi_t \sim +g^2 N \rho_0, \quad \Pi_\ell = \Pi_3 = \Pi_4 = 0. \]  

(42)

This is not the whole story, however; it is still necessary to work out the effective propagator, including this self energy. This is a straightforward if tedious exercise. In \( p_0 \cdot A = 0 \) gauge, the result is

\[ \Delta^{\alpha\beta} = \frac{1}{k^2 + \Pi_\ell} \left( \delta^{\alpha\beta} - \frac{p_0^\alpha k^\beta + k^\alpha p_0^\beta}{p_0 \cdot k} \right) - \frac{k^2}{k^2 - \Pi_\ell} \left( \Pi_t + \Pi_\ell \right) \frac{p_0^\alpha p_0^\beta}{(p_0 \cdot k)^2}. \]  

(43)

Thus we see that in a tsunami wave, the two, spatially transverse modes which one expects for a spin-one field are screened. If \( \Pi_\ell \) were nonzero, it would represent a collective mode, analogous to the plasmon of a thermal distribution. Because \( \Pi_\ell = 0 \), there is no plasmon for this type of tsunami wave.

Thus at short times, interactions between gluons in a dense tsunami act to screen the usual transverse modes of the gluon. What happens at longer times is an open question. This can be studied numerically in a scalar theory in the limit of a large number of components [11].
6. Where does $Q$ come from?

While classical kinetic theory is often used in nonrelativistic systems, its use in a nonabelian gauge theory is unfamiliar. What are the original degrees of freedom in the gauge theory which conspire, in the classical limit, to generate the nonabelian charge $Q$?

Let me start with something familiar, the lagrangian for a charged scalar field, $\phi$, in the presence of a background gauge field:

$$\mathcal{L} = \phi^\dagger (-D^2) \phi .$$

(Integrating over $\phi$ gives the effective action

$$S_{\text{eff}} = \text{tr} \log \left( -D^2 \right) .$$

Now I use the usual trick of Feynman and Schwinger, to turn this $S_{\text{eff}}$ into the path integral for a particle. First introduce a parameter, $\tau$, which will become like a proper time:

$$S_{\text{eff}} = \int_0^\infty d\tau \frac{d\tau}{\tau} \text{tr} \left( e^{-\frac{1}{2} \tau (-D^2)} \right) .$$

Going to momentum space,

$$S_{\text{eff}} = \int_0^\infty d\tau \int \frac{d^4p}{(2\pi)^4} \frac{d^4p}{\tau} \text{tr} \left( e^{-\frac{1}{2} \tau (p-gA)^2} \right) .$$

The remaining trace is only for the $SU(N)$ matrix $A_\alpha$. This is then converting into a sum over paths:

$$S_{\text{eff}} \sim \int_0^\infty d\tau \int D\alpha Dp \text{tr} \left( e^{-\int_0^\tau d\tau' \mathcal{L}(p)} \right) ,$$

$$\mathcal{L}(p) = -ip \cdot \dot{x} + \frac{1}{2} (p-gA)^2 .$$

where $\dot{x}^\alpha = dx^\alpha/d\tau$. The integral over the momentum is trivial,

$$S_{\text{eff}} \sim \int_0^\infty d\tau \int D\alpha Dx \text{tr} \left( e^{-\int_0^\tau d\tau' \mathcal{L}_A} \right) ,$$

$$\mathcal{L}_A = \frac{\dot{x}^2}{2} - igA \cdot \dot{x} ,$$

subject to the boundary conditions $x^\alpha(\tau) = +x^\alpha(0)$

This form of the one loop effective action has been studied by many authors [12]. This representation has been shown to be a powerful way
of reorganizing perturbation theory. From (50), this is just a matter of expanding the exponent in powers of the gauge field.

In a more general context, however, (50) is incomplete. In (51), the “lagrangian” $L$ is not really that at all: it is a matrix, as in (50) there is still a color trace left to do. What we want instead is a form where the lagrangian is a true scalar in color space.

This can be done by replacing the Wilson line by an integral over worldline fermions, $\lambda(\tau)$ [13, 14]:

$$\lambda = \lambda^\dagger (\dot{x} \cdot D) \lambda .$$  

(53)

The $\lambda(\tau)$ lie in the fundamental representation of $SU(N)$ color, and satisfy antiperiodic boundary conditions, $\psi(\tau) = -\psi(0)$. Path ordering is denoted by $\mathcal{P}$, and is automatic in the path integral.

It has been known for some time that worldline fermions are required to describe finite dimensional representations of nonabelian charge [13]. To understand (52), notice that in the absence of a gauge field, the propagator for a fermion in one dimension is a step function: the solution to $\partial_\tau \Delta_\lambda = \delta(\tau)$ is $\Delta_\lambda \sim \theta(\tau - \tau')$. Then, since propagation is in one dimension, the complete propagator is naturally an exponential, where the step function provides the path ordering.

The complete form in (52), including the sum over $k$, was derived by D’Hoker and Gagné, eq. (5.2) of [14]. The sum over $k$ is required in order to project upon states with occupation number one.

The integral over worldline fermions then replaces the color sum, and gives us a true lagrangian. We can then use this to go back, reintroduce the momentum conjugate to the position $x$ (this is not necessary for the worldline fermions, since $\psi^\dagger$ is already conjugate to $\psi$), to obtain

$$S_{\text{eff}} \sim \int_0^\infty \frac{d\tau}{\tau} \int DxdpD\lambda D\lambda^\dagger \sum_{k=1}^N e^{ik(\lambda^\dagger \lambda + N/2 - 1)} \left(\frac{\pi}{\tau}\right)^N e^{-\int_0^\tau d\tau' L} ,$$  

(54)

$$L = -ip \cdot \dot{x} - \lambda^\dagger \dot{\lambda} + \frac{1}{2} (p - 2g \text{tr}(QA))^2 .$$  

(55)

The nonabelian charge $Q$ is just

$$Q^a = \lambda^\dagger t^a \lambda .$$  

(56)

The lagrangian $L$ of (55) is similar to the original $L(p)$ of (49), but now the color trace downstairs is replaced by an integral over the worldline fermions.
Moreover, the nonabelian charge now enters in a most natural manner, as a bilinear in the worldline fermions.

It is easy to check that $\mathcal{L}$ gives the correct equations of motion for a classical, nonabelian particle. For example, evidently the equation of motion for the worldline fermion, $\dot{x} \cdot D\lambda = 0$, gives Wong's equation for $Q$, (24).

A lagrangian very similar to $\mathcal{L}$ was proposed by Brandt, Frenkel, and Taylor [7] to generate the nonabelian Debye lagrangian. (They used worldline scalars instead of fermions, but this difference doesn't matter in the classical limit [13].) My contribution is to note that the introduction of the worldline fermions is not just a trick to get the correct equations of motion, but is a systematic approximation to the correct effective action.

Given this lagrangian, we then adopt a classical approximation. Instead of an integral over the nonabelian charge $Q$, one should integrate over the worldline fermions. Since all we needed before was the Casimir of the representation, this doesn't matter in the classical limit.

Other approximation schemes have been developed to analyze worldline path integrals at nonzero temperature. For instance, one can sum over paths which wind around in the imaginary time direction [15]. The resulting expressions are not especially simple, though; classical kinetic theory appears to be more useful.

However, this begs the question of how the collisionless Boltzmann equation arises. I conclude with a suggestion. In the absence of interactions, at nonzero temperature the propagator is directly proportional to the statistical distribution function:

\[ \text{tr} \left( \frac{1}{-D^2} \right) \sim n(p) . \]  

(57)

In the presence of a background gauge field, the propagator can be written as a sum over paths:

\[ \text{tr} \left( \frac{1}{-D^2} \right) \sim \int_0^\infty d\tau \int Dx Dp D\lambda D\lambda^\dagger \sum_{k=1}^N e^{ik(\lambda^\dagger + N/2 - 1)} \left( \frac{\pi}{T} \right)^N e^{-\int_0^\tau d\tau' \mathcal{L}} . \]

(58)

From this, define the “single particle” density, $f$, as

\[ \text{tr} \left( \frac{1}{-D^2} \right) \sim \int_0^\infty d\tau \int Dx Dp D\lambda D\lambda^\dagger f(x, p, Q) . \]

(59)

Perhaps in the classical approximation, the dominant term is that where $f$ is stationary with respect to $\tau$; this would then give the collisionless Boltzmann equation. The expression for the current, (25), follows by differentiation with respect to the gauge field. At two loop order, the effective
action is not given by (45), and is how collisions enter into the Boltzmann equation.

7. Acknowledgements

I thank M. H. G. Tytgat for discussions, and especially for bringing (52) to my attention.

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