Integration of Dirac-Jacobi structures

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Abstract

We study precontact groupoids whose infinitesimal counterparts are Dirac-Jacobi structures. These geometric objects generalize contact groupoids. We also explain the relationship between precontact groupoids and homogeneous presymplectic groupoids. Finally, we present some examples of precontact groupoids.

1 Introduction

Presymplectic groupoids, introduced and studied in [2], are global counterparts of Dirac structures. They allow to extend the well-known correspondence between symplectic groupoids and Poisson manifolds to the context of Dirac geometry. Moreover, they provide a framework for a unified formulation of various notions of momentum maps ([1]). On the other hand, Dirac-Jacobi structures (called $E^1(M)$-Dirac structures in [11]) include both Dirac and Jacobi structures. They naturally appeared in the geometric prequantization of Dirac manifolds [13, 14].

In this paper, our aim is to investigate the integrability problem for Dirac-Jacobi structures. This work is motivated by the fact that many Dirac manifolds can be quantized through their integrating Lie groupoids. We show that the global counterparts of Dirac-Jacobi manifolds are what we call here precontact groupoids. In particular, we recover the integrability of Jacobi structures [4]. We also prove that there is a one-to-one correspondence between precontact groupoids and homogeneous presymplectic groupoids. Moreover, the precontact groupoid $\tilde{G}$ associated with an integrable Dirac structure $L_0$ on $M$ is just the prequantization of the presymplectic groupoid $G$ associated with $L_0$ (that is, the central extension of Lie groupoids $M \times S^1 \to \tilde{G} \to G$ satisfying some compatibility conditions), provided that the canonical Dirac Jacobi structure $L$ on $M$ corresponding to $L_0$ is integrable, see Section 5.2. We should mention that M. Zambon and C. Zhu independently study the geometry of prequantization spaces [14].

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‡ The first author is partially supported by MCYT grant BFM2003-01319
Here is an outline of the paper. In Sections 2 and 3, we give some definitions and results needed to establish our results. Section 4 contains our main results (Theorems 4.2 and 4.3). In Section 5, we give some examples of precontact groupoids.

2 Basic definitions and results

2.1 Dirac structures and presymplectic groupoids

Let \( M \) be a smooth \( n \)-dimensional manifold. There is a natural symmetric pairing \( \langle \cdot , \cdot \rangle \) on the vector bundle \( TM \oplus T^*M \) given by

\[
\langle X_1 + \xi_1, X_2 + \xi_2 \rangle = \frac{1}{2} (\xi_1(X_2) + \xi_2(X_1)).
\]

Furthermore, the space of smooth sections of \( TM \oplus T^*M \) is endowed with the Courant bracket, which is defined by

\[
[X_1 + \xi_1, X_2 + \xi_2] = [X_1, X_2] + \mathcal{L}_{X_1} \xi_2 - i_{X_2} d\xi_1,
\]

for any \( X_1 + \xi_1, X_2 + \xi_2 \in \Gamma(TM \oplus T^*M) \).

Definition 2.1 \[2, 5\] A Dirac structure on a smooth manifold \( M \) is a subbundle \( L \) of \( TM \oplus T^*M \) which is maximally isotropic with respect to the symmetric pairing \( \langle \cdot , \cdot \rangle \) and whose space of sections is closed under the Courant bracket.

Let \( L_M \) and \( L_N \) be Dirac structures on \( M \) and \( N \), respectively. We say that a smooth map \( F : M \to N \) is a (forward) Dirac map if \( L_N = F_*(L_M) \), where

\[
F_*(L_M) = \{(dF)(X) + \xi \mid X + F^* \xi \in L_M\}.
\]

Recall that any Dirac structure \( L \) has an induced Lie algebroid structure: the Lie bracket on \( \Gamma(L) \) is just the restriction of the Courant bracket and the anchor map is the restriction of the first projection to \( L \), i.e. \( pr_1|_L : L \to TM \). Now, suppose that \( L \) is a Dirac structure which is isomorphic to the Lie algebroid of a Lie groupoid \( G \). Such a Lie groupoid is called an integration of the Lie algebroid \( L \). Then, there exists an induced closed 2-form on \( G \) with some additional properties. More precisely,

Definition 2.2 \[2\] A presymplectic groupoid is a pair \((G, \omega)\) which consists of a groupoid \( \overset{\alpha}{G} \rightrightarrows \overset{\beta}{M} \) such that \( \dim(G) = 2 \dim(M) \), and a 2-form \( \omega \in \Omega^2(G) \) satisfying the following conditions:

(i) \( \omega \) is closed, i.e. \( d\omega = 0 \).

(ii) \( \omega \) is multiplicative, that is, \( m^* \omega = pr_1^* \omega + pr_2^* \omega \).

(iii) \( \text{Ker} (\omega_x) \cap \text{Ker} (d\alpha)_x \cap \text{Ker} (d\beta)_x = \{0\} \), for all \( x \in M \).
We say that the presymplectic groupoid \((G, \omega)\) is **homogeneous** if there exists a multiplicative vector field \(Z\) such that \(L_Z\omega = \omega\).

The relationship between Dirac structures and presymplectic groupoids is provided by the following result:

**Proposition 2.3** \([2]\) Given a presymplectic groupoid \((G, \omega)\), there is a canonical Dirac structure \(L\) on \(M\) which is isomorphic to the Lie algebroid \(AG\) of \(G\), and such that the target map \(\beta : (G, L_\omega) \to (M, L)\) is a Dirac map, while the source map \(\alpha : (G, L_\omega) \to (M, L)\) is anti-Dirac.

Conversely, suppose that \(L\) is a Dirac structure on \(M\) whose associated Lie algebroid is integrable, and let \(G(L)\) be its \(\alpha\)-simply connected integration. Then, there exists a unique 2-form \(\omega\) such that \((G(L), \omega)\) is a presymplectic groupoid, the target map is a Dirac map, and the source map is anti-Dirac.

### 2.2 Dirac-Jacobi structures

Let \(M\) be a smooth \(n\)-dimensional manifold. There is a natural bilinear operation \(\langle \cdot, \cdot \rangle\) on the vector bundle \(\mathcal{E}^1(M) = (TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})\) defined by:

\[
\langle (X_1, f_1) + (\xi_1, g_1), (X_2, f_2) + (\xi_2, g_2) \rangle = \frac{1}{2}(i_{X_1}\xi_1 + i_{X_2}\xi_2 + f_1g_2 + f_2g_1),
\]

for any \((X_\ell, f_\ell) + (\xi_\ell, g_\ell) \in \Gamma(\mathcal{E}^1(M))\), with \(\ell = 1, 2\). In addition, the space of smooth sections of \(\mathcal{E}^1(M)\) is equipped with an \(\mathbb{R}\)-bilinear operation which can be viewed as an extension of the Courant bracket on \(TM \oplus T^*M\), i.e.

\[
\left[(X_1, f_1) + (\xi_1, g_1), (X_2, f_2) + (\xi_2, g_2)\right] = \left([X_1, X_2], X_1(f_2) - X_2(f_1)\right) + \left(L_{X_1}\xi_2 - L_{X_2}\xi_1 + \frac{1}{2}d(i_{X_1}\xi_1 - i_{X_2}\xi_2)\right) + f_1\xi_2 - f_2\xi_1 + \frac{1}{2}(g_2df_1 - g_1df_2 - f_1dg_2 + f_2dg_1),
\]

\[
X_1(g_2) - X_2(g_1) + \frac{1}{2}(i_{X_1}\xi_1 - i_{X_2}\xi_2 - f_2g_1 + f_1g_2),
\]

for any \((X_\ell, f_\ell) + (\xi_\ell, g_\ell) \in \Gamma(\mathcal{E}^1(M))\) with \(\ell = 1, 2\). For an alternative description of this bracket, see \([3]\).

**Definition 2.4** \([3]\) A **Dirac-Jacobi structure** is a subbundle \(L\) of \(\mathcal{E}^1(M)\) which is maximally isotropic with respect to \(\langle \cdot, \cdot \rangle\) and such that \(\Gamma(L)\) is closed under the extended Courant bracket \([\cdot, \cdot]\).

Let \(L_M\) (resp., \(L_N\)) be a Dirac-Jacobi structure on \(M\) (resp., \(N\)). We say that a smooth surjective map \(F : M \to N\) is a (forward) **Dirac-Jacobi map** if \(L_N = F_*(L_M)\), where

\[
F_*(L_M) = \{(df)(X, f) + (\xi, g) \mid (X, f \circ F) + (F^*\xi, g \circ F) \in L_M\}.
\]

Basic examples of Dirac-Jacobi structures are Dirac and Jacobi structures on \(M\) (this explains the terminology introduced in \([4]\)).
2.3 Action Lie algebroids and 1-cocycles

It is known that, given any a Lie algebroid \((A, [\cdot, \cdot], \rho)\) over \(M\) and any 1-cocycle \(\phi \in \Gamma(A^*)\), there is an associated Lie algebroid over \(M \times \mathbb{R}\), denoted by \((A \times_\phi \mathbb{R}, [\cdot, \cdot]^\phi, \rho^\phi)\), where the smooth sections of \(A \times_\phi \mathbb{R}\) are of the form \(\bar{X}(x, t) = X_t(x)\), with \(X_t \in \Gamma(A)\) for all \(t \in \mathbb{R}\), and

\[
\begin{align*}
[X, Y]^\phi(x, t) &= [X_t, Y_t](x) + \phi(X_t)(x)\frac{\partial Y}{\partial t} - \phi(Y_t)(x)\frac{\partial X}{\partial t}, \\
\rho^\phi(\bar{X})(x, t) &= \rho(X_t)(x) + \phi(X_t)(x)\frac{\partial}{\partial t},
\end{align*}
\]

where \(\frac{\partial \bar{X}}{\partial t} \in \Gamma(A \times_\phi \mathbb{R})\) denotes the derivative of \(\bar{X}\) with respect to \(t\).

Remark 2.5 If \(L\) is a Dirac-Jacobi structure then the restriction of the extended Courant bracket to sections of \(L\) together with the canonical projection of \(L\) onto \(TM\) make \(L\) into a Lie algebroid over \(M\). In addition, \(\phi \in \Gamma(L^*)\) defined by

\[
\phi(v) = f, \text{ for } v = (X, f) + (\xi, g) \in \Gamma(L),
\]

is a 1-cocycle for the Lie algebroid cohomology (see [8]). On the other hand, it is known that a Dirac-Jacobi structure \(L\) on \(M\) corresponds to a Dirac structure \(\tilde{L}\) on \(M \times \mathbb{R}\) given by

\[
\tilde{L} = \left\{ (X + f \frac{\partial}{\partial t}) + \left( e^t(\xi + g dt) \right) \right\} \bigg| (X, f) + (\xi, g) \in L\}.
\]

Moreover, \(\tilde{L}\) is isomorphic to \(L \times_\phi \mathbb{R}\).

2.4 Conformal classes of Dirac-Jacobi structures

Let \(L\) be a Dirac-Jacobi structure on \(M\) and let \(\varphi\) be a smooth nowhere vanishing function on \(M\). We set \(\mu = d\ln|\varphi|\). Consider the vector bundle \(L_\varphi\) over \(M\) whose space of smooth sections is given by

\[
\Gamma(L_\varphi) = \{(X, f - \mu(X)) + \varphi(\xi + g\mu, g) \mid (X, f) + (\xi, g) \in \Gamma(L)\}.
\]

One can easily check that \(L_\varphi\) is also a Dirac-Jacobi structure on \(M\). The correspondence \((L, \varphi) \mapsto L_\varphi\) is called a conformal change. For any fixed \(L\), the family of all \((L_\varphi)\) is called a conformal class of Dirac-Jacobi structures. For instance, when \(L\) comes from a presymplectic form \(\omega\) then \(L_\varphi\) is nothing but the Dirac-Jacobi structure associated with \((\varphi \omega, d\ln|\varphi|)\) (see [11, 12] for more details).

3 Precontact groupoids

Definition 3.1 Let \(G \xrightarrow{\alpha} M\) be a Lie groupoid such that \(\dim(G) = 2\dim(M) + 1\). A precontact groupoid structure on \(G\) is given by a pair \((\eta, \sigma)\) consisting of a 1-form \(\eta\)
and a multiplicative function $\sigma$ (i.e., $\sigma(gh) = \sigma(g) + \sigma(h)$) such that

$$m^*\eta = pr_1^*\eta + pr_1^*(e^\sigma)pr_2^*\eta.$$  

(4)

$$\text{Ker}(d\eta_x) \cap \text{Ker}(\eta_x) \cap \text{Ker}(d\alpha_x) \cap \text{Ker}(d\beta_x) = \{0\}, \text{ for all } x \in M.$$  

(5)

Two precontact structures $(\eta, \sigma)$ and $(\eta', \sigma')$ on $G$ are equivalent if there exists a nowhere vanishing function $\varphi : M \to \mathbb{R}$ such that

$$\eta' = (\varphi \circ \alpha) \eta, \quad \sigma' = \sigma + \ln \left| \frac{\varphi \circ \alpha}{\varphi \circ \beta} \right|$$

Now, consider a Lie groupoid $G \xrightarrow{\alpha} M$ together with a multiplicative function $\sigma$. One can define a right action of $G$ on the canonical projection $\pi_1 : M \times \mathbb{R} \to M$ as follows:

$$(x, t) \cdot g = (\alpha(g), t + \sigma(g)), \text{ for } (x, t, g) \in M \times \mathbb{R} \times G, \text{ such that } \beta(g) = x.$$  

Therefore, we have the corresponding action groupoid $G \times \mathbb{R} \Rightarrow M \times \mathbb{R}$, denoted by $G \times \sigma \mathbb{R}$, with structural functions given by

$$\alpha_{\sigma}(g, t) = (\alpha(g), \sigma(g) + t), \quad \beta_{\sigma}(h, s) = (\beta(h), s),$$

$$m_{\sigma}((g, t), (h, s)) = (gh, t), \text{ if } \alpha_{\sigma}(g, t) = \beta_{\sigma}(h, s).$$  

(6)

We denote by $(AG, [\ [ , ] ], \rho)$ the Lie algebroid of $G$. The multiplicative function $\sigma$ induces a 1-cocycle $\phi$ on $AG$ given by

$$\phi(x)(X_x) = X_x(\sigma), \text{ for } x \in M \text{ and } X_x \in A_xG.$$  

(7)

In addition, we can identify the Lie algebroid of the Lie groupoid $G \times \sigma \mathbb{R}$ with $AG \times \sigma \mathbb{R}$. Conversely, one has the following

**Proposition 3.2** [4] Let $L$ be a Lie algebroid over $M$, $\phi$ be a 1-cocycle and $L \times _\phi \mathbb{R}$ the Lie algebroid given by Equation (7). Then, $L$ is integrable if and only $L \times _\phi \mathbb{R}$ is integrable. Moreover, if $G(L)$ (resp., $G(L \times _\phi \mathbb{R})$) is the $\alpha$-simply connected integration of $L$ (resp., $L \times _\phi \mathbb{R}$) and $\sigma$ is the multiplicative function associated with $\phi$, then $G(L \times _\phi \mathbb{R}) \cong G(L) \times _\sigma \mathbb{R}$.

There is a correspondence between precontact and presymplectic groupoids. Indeed, one has the following proposition:

**Proposition 3.3** Let $G \Rightarrow M$ be a Lie groupoid and $\sigma$ a multiplicative function on $G$. There is a one-to-one correspondence between precontact groupoids on $(G, \sigma)$ and homogeneous presymplectic groupoids on $G \times _\sigma \mathbb{R}$.  

5
Proof: We know that there exists a one-to-one correspondence between 1-forms on a manifold \( M \) and 2-forms on \( M \times \mathbb{R} \) homogeneous with respect to \( \frac{\partial}{\partial t} \). More precisely, if \( \eta \) is a 1-form on \( M \) then \( \omega = d(e^t \eta) \) is a homogeneous 2-form on \( M \times \mathbb{R} \). Conversely, assume that \( \omega \) is homogeneous and set \( \eta = \frac{i}{\partial_t} \omega \). One can check that \( \mathcal{L}_{\eta} \omega = \omega \). Hence \( \eta \) can be identified with a 1-form on \( M \). Using the relation \( \omega = d(e^t \eta) \), it is straightforward to prove that conditions (ii) and (iii) in Definition 2.2 are equivalent to Equations (4) and (5).

4 Integration of Dirac-Jacobi structures

In this section, we show that precontact groupoids are the global objects corresponding to Dirac-Jacobi structures. First, note that the following lemma which is an immediate consequence of Remark 2.5 and Proposition 3.2.

**Lemma 4.1** A Dirac-Jacobi structure \( L \) is integrable if and only if its associated Dirac structure \( \tilde{L} \subset T(M \times \mathbb{R}) \oplus T^*(M \times \mathbb{R}) \) is integrable.

**Theorem 4.2** Let \( L \) be an integrable Dirac-Jacobi structure on \( M \), and let \( G(L) \) be its \( \alpha \)-simply connected integration. There exists a multiplicative function \( \sigma \) and a 1-form \( \eta \) such that \( (G(L), \eta, \sigma) \) is a precontact groupoid. Furthermore, any conformal class of integrable Dirac-Jacobi structures on \( M \) induces a conformal class of precontact groupoid structures.

**Proof:** Suppose \( L \) is an integrable Dirac-Jacobi structure on \( M \). We denote by \( \phi \in \Gamma(L^*) \) the 1-cocycle defined by Equation (2). By integration, the 1-cocycle \( \phi \) induces a multiplicative function \( \sigma \) on \( G(L) \). Denote by \( \tilde{L} \) the Dirac structure on \( M \times \mathbb{R} \) associated with \( L \) given by Equation (3). Applying Proposition 2.3, one gets a presymplectic groupoid \( (G(\tilde{L}), \omega) \). Moreover, one gets from Remark 2.5 and Proposition 3.2 that

\[
G(\tilde{L}) \cong G(L \times \mathbb{R}) \cong G(L) \times \mathbb{R}.
\]

Observe that \( \omega \) is homogeneous with respect to the multiplicative vector field \( \frac{\partial}{\partial t} \). Indeed, the differentiable family of diffeomorphisms \( \psi_s : M \times \mathbb{R} \to M \times \mathbb{R} \) defined by \( \psi_s(x, t) = (x, s + t) \) consists of Dirac maps, i.e. \( \psi_s)(\tilde{L}) = \tilde{L} \). Furthermore, by integration and using the flow \( \psi_s : G(L) \times \mathbb{R} \to G(L) \times \mathbb{R} \) \( (g, t) \mapsto (g, t + s) \) of the vector field \( \frac{\partial}{\partial t} \), we get

\[
\mathcal{L}_{\psi_s} \omega = \omega.
\]

Thus, using Proposition 3.3 we deduce that there exists a 1-form \( \eta \) such that \( (G(L), \eta, \sigma) \) is precontact. Recall that, for every nowhere vanishing function \( \varphi \) on \( M \), there is an equivalent Dirac-Jacobi structure on \( M \) whose space of sections is given by

\[
\Gamma(L_\varphi) = \{(X, f - \mu(X)) + \varphi(\xi + g \mu, g) \mid (X, f) + (\xi, g) \in \Gamma(L)\}.
\]
Consider the 1-cocycle $\phi \in \Gamma(L^*_\varphi)$ defined as follows

$$\phi(e) = f - \mu(X), \quad \text{for all } e = (X, f - \mu(X)) + \varphi(\xi + g\mu, g).$$

We have a natural commutative diagram of vector bundle morphisms:

$$
\begin{array}{ccc}
L & \xrightarrow{\sim} & L_{\varphi} \\
\downarrow & & \downarrow \\
L \times_{\phi} \mathbb{R} & \xrightarrow{\sim} & L_{\varphi} \times_{\phi} \mathbb{R}
\end{array}
$$

Moreover, $L_{\varphi}$ induces a precontact structure $(\eta_{\varphi}, \sigma_{\varphi})$ on $G(L_{\varphi}) \cong G(L)$ given by

$$\eta_{\varphi} = (\varphi \circ \alpha) \eta \quad \sigma_{\varphi} = \sigma + \ln |\varphi \circ \beta|.$$

Thus, any conformal class of integrable Dirac-Jacobi structures on $M$ induces a conformal class of precontact groupoid structures.

Conversely, we have the following result.

**Theorem 4.3** Let $(G, \eta, \sigma)$ be a precontact groupoid over $M$. Then, there exists a canonical Dirac-Jacobi structure $L_M$ on $M$ which is isomorphic to the Lie algebroid $AG$. Moreover, $\beta : G \to M$ is a Dirac-Jacobi map.

**Proof:** By Proposition 3.3, one has the presymplectic groupoid $(G \times_{\sigma} \mathbb{R}, \omega = d(e^t \eta))$ over $M \times \mathbb{R}$. Then, using Proposition 2.3, one gets a Dirac structure $L_{M \times \mathbb{R}}$ on $M \times \mathbb{R}$ such that $\beta_{\sigma}$ is a Dirac map. Thus,

$$L_{M \times \mathbb{R}} = \left\{ \left. \left( d\beta_{\sigma}(X + F\frac{\partial}{\partial t}), \xi + Gdt \right) \right| i_{(X + F\frac{\partial}{\partial t})}\omega = \beta^*_{\sigma}(\xi + Gdt) \right\}$$

$$= \left\{ \left. \left( d\beta_{\sigma}(X + F\frac{\partial}{\partial t}), \xi + Gdt \right) \right| \beta^*_{\sigma}(\xi) = e^t(i_X d\eta + F \eta), \ G = -e^t \eta(X) \right\}.$$ 

Therefore, one can write $L_{M \times \mathbb{R}} \equiv \overline{L}_M$, where $L_M$ is the Dirac-Jacobi structure given by

$$L_M = \left\{ \left. \left( (d\beta)(X), F + \xi, G \right) \right| \beta^*(\xi) = i_X d\eta + F \eta, \ G = -\eta(X) \right\}.$$ 

By construction, $\beta : G \to M$ is a Dirac-Jacobi map. There follows the result.

**5 Examples**

In this section, we will give some examples of Dirac-Jacobi structures and describe their corresponding precontact groupoids.
5.1 Precontact structures

A **precontact structure** on a manifold $M$ is just a 1-form $\theta$ on $M$. A precontact structure $\theta$ on $M$ induces a Dirac-Jacobi structure $L_\theta$ whose space of smooth sections is

$$\Gamma(L_\theta) = \{(X, f) + (i_X d\theta + f \theta, -i_X \theta) \mid (X, f) \in \mathfrak{X}(M) \times C^\infty(M)\}.$$  

We observe that the Lie algebroids $L_\theta$ and $(TM \times \mathbb{R}, [\cdot, \cdot], \pi)$ are isomorphic, where $\pi : TM \times \mathbb{R} \to TM$ is the canonical projection over the first factor and $[\cdot, \cdot]$ is given by

$$[(X, f), (Y, g)] = (\lbrack X, Y \rbrack, X(g) - Y(f)),$$

for $(X, f), (Y, g) \in \mathfrak{X}(M) \times C^\infty(M)$. Moreover, under the isomorphism between $L_\theta$ and $TM \times \mathbb{R}$, the 1-cocycle $\phi$ is the pair $(0, 1) \in \Omega^1(M) \times C^\infty(M)$.

On the other hand, consider the product $G = M \times M \times \mathbb{R}$ of the pair groupoid with $\mathbb{R}$. The function $\sigma : G \to \mathbb{R}$, $(x, y, t) \mapsto t$, is trivially multiplicative. In addition, if $\theta$ is a 1-form on $M$ then one can define the 1-form $\eta$ on $G$ given by

$$\eta = \pi_1^* \theta - e^\sigma \pi_2^* \theta,$$

where $\pi_i, i \in \{1, 2\}$, is the projection on the $i$-th component. Then $(G, \eta, \sigma)$ is a precontact groupoid and, moreover, the corresponding Dirac-Jacobi structure on $M$ is just $L_\theta$.

5.2 Dirac structures

Let $L_0$ be a vector subbundle of $TM \oplus T^*M$ and consider the vector subbundle $L$ of $\mathcal{E}^1(M)$ whose sections are

$$\Gamma(L) = \{(X, 0) + (\alpha, f) \mid X + \alpha \in \Gamma(L_0), f \in C^\infty(M)\}.$$  

Then, $L_0$ is a Dirac structure on $M$ if and only if $L$ is a Dirac-Jacobi structure.

If we denote by $L_0$ (resp., $L$) the Lie algebroid associated with the Dirac structure (resp., the Dirac-Jacobi structure), then we have that $L = L_0 \times \mathbb{R}$. Moreover, a direct computation shows that the bracket on $\Gamma(L)$ is given by

$$\mathrm{[[}(X_1, f_1), (X_2, f_2)]_L = \left(\left[[X_1, X_2]_{L_0}, \rho_{L_0}(X_1)(f_2) - \rho_{L_0}(X_2)(f_1) + \Omega_{L_0}(X_1, X_2)\right]\right),$$

for $(X_1, f_1), (X_2, f_2) \in \Gamma(L)$, and where $(\mathcal{[}, \mathcal{]}_{L_0}, \rho_{L_0})$ (resp., $(\mathcal{[}, \mathcal{]}_L, \rho_L)$) denotes the Lie algebroid structure on $L_0$ (resp., $L$) and $\Omega_{L_0} \in \Gamma(\wedge^2 L_0^*)$ is the closed 2-section given by

$$\Omega_{L_0}(X_1, X_2) = \frac{1}{2} \left(\xi_1(X_2) - \xi_2(X_1)\right),$$

for $X_i = X_i + \xi_i \in \Gamma(L_0)$.

Therefore, we have that the Lie algebroid structure on $L$ is just the central extension of the Lie algebroid $L_0$ by the closed 2-section $\Omega_{L_0}$. On the other hand, from Equation 2, one
sees that the 1-cocycle $\phi$ identically vanishes. If $L_0$ as well as $L$ are integrable and $(G, \omega)$ is the presymplectic groupoid associated with $L_0$ then one obtains a prequantization of $(G, \omega)$, that is, a central extension of Lie groupoids

$$M \times S^1 \to \tilde{G} \to G,$$

and a multiplicative 1-form $\eta \in \Omega^1(\tilde{G})$ ($m^*\eta = pr_1^*\eta + pr_2^*\eta$) which is a connection 1-form for the principal $S^1$-bundle $\pi : \tilde{G} \to G$ and which satisfies $d\eta = \pi^*\omega$ (see [3]).

5.3 Jacobi manifolds

Let $(\Lambda, E)$ be Jacobi manifold and $L_{(\Lambda,E)}$ the corresponding Dirac-Jacobi structure

$$L_{(\Lambda,E)} = \{ (\Lambda^x(\alpha_x) + \lambda E_x, -\alpha_x(E_x)) + (\alpha_x, \lambda) | (\alpha_x, \lambda) \in T^*_xM \times \mathbb{R}, x \in M \}.$$

In this case, the corresponding Dirac structure on $M \times \mathbb{R}$ is the one coming from the Poissonization of the Jacobi structure $(\Lambda, E)$, i.e. $\Pi = e^{-t}(\Lambda + \frac{\partial}{\partial t} \wedge E)$. Thus, the presymplectic groupoid is a honest symplectic groupoid (see [2]), and therefore, the precontact groupoid structure integrating $L_{(\Lambda,E)}$ is a contact groupoid, i.e., the 1-form defines a contact structure on $G$. This result was first proved in [6] (see also, [4, 10]).

Acknowledgments:

D. Iglesias wishes to thank the Spanish Ministry of Education and Culture and Fulbright program for a MECD/Fulbright postdoctoral grant. A. Wade would like to thank the MSRI for hospitality while this paper was being prepared.

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