The analytical and numerical solutions of two dimensional heat transfer equation in a multilayered composite cylinder

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Abstract. Heat transfer is one of the most observed phenomena in the fields of aerospace, industry, nuclear, power generation, automotive, etc. Currently, materials for heat transfer are made of composites of various types so that they have different properties from their constituent materials. To understand the phenomenon of heat transfer in order to save costs and risks that can arise, it can be done by simulation. In this paper, we describe analytical and numerical solutions for two dimensional (2D) heat transfer equation in a multilayered composite cylinder. In particular, we use eigen function method to find the analytical solution and finite difference method to generate the numerical solution. The simulation of heat transfer for different time frame carried out using Python 3.7 with Numpy library.

1. Introduction
The heat transfer phenomenon appears immediately after energy is produced, exchanged, or changed. Researchers are trying to understand the phenomenon of heat transfer to improve equipment’s design or on explaining the cause of errors in some devices that utilize heat transfers such as in the chemical, nuclear and power generation industries. From this understanding, the researchers are able to make some modifications on certain components in order to meet safety or cost requirements and improve the work’s efficiency [1, 2].

The experimental approach has been widely used in the past, but costs have become quite expensive when dealing with very high temperatures, pressure, speed or potentially dangerous products (chemical or nuclear industry) [3, 4]. In addition, an experimental approach in place often lacks the flexibility needed when sensitivity studies are desired. On the other hand, with strong computing facilities available on the market at affordable costs, analytical calculations and simulations with a numerical approach appear to be a very promising and accurate way to predict thermal phenomena and their effects in all situations [5].

In industry, multilayered composite materials are often used [6]. The advantages of these ingredients are combining physical, mechanical, and thermal properties of different substances. The idea of the cylindrical layer is also widely used in investigating the thermal properties of composite materials. In this case, it is assumed that the particles forming a composite matrix are cylinder in shape. The analytical and numerical methods used to investigate the heat conduction of multilayered transients are analogous to those used to investigate single-layer transient heat conduction. Among the analytical methods indicated are the variable separation method, finite difference method, Laplace-
transform method, integral method to transform, and expansion of eigen functions and Green function methods [7].

In this study heat transfer simulations were carried out on multilayered composites using the finite difference method. This simulation can be used in the industrial sector so that it can save costs and produce new composites that have better characteristics/properties.

2. Experimental
Composite models are made in cylindrical shape with a constant heat source in the center of the cylinder as shown in Figure 1.

![Figure 1. Multilayered composite design model](image)

Consider a multilayer cylindrical composite with coordinates

\[ \begin{align*}
    r_0 &\leq r \leq r_n \\
    0 &\leq \theta \leq \varphi \\
    0 &\leq z \leq L
\end{align*} \]

We assume all the layers are isotropic and the thermal contact is perfect. When \( t = 0 \) the temperature in the \( i \)-th layer is \( f_i(r, \theta, z) \). When \( t > 0 \) there are boundaries heat source \( g_i(r, \theta, z, t) \). The heat equation in cylindrical coordinates is as follows [8].

\[
\frac{dT}{dt} = \alpha \nabla^2 T = \alpha \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}
\]

In multilayered case, we have

\[
\frac{1}{\alpha_i} \frac{\partial T_i}{\partial t} = \frac{\partial T_i}{\partial r} + \frac{1}{r} \frac{\partial T_i}{\partial r} + \frac{\partial^2 T_i}{\partial \theta^2} + \frac{\partial^2 T_i}{\partial z^2} + \frac{g_i(r, \theta, z, t)}{k_i}
\] (1)

Where \( k_i \) is a dependent constant related to heat properties of corresponding material and \( T_i = T_i(r, \theta, z, t) \) is the temperature on the \( i \) layer. We also assume the following boundary for each variable.

\[ \begin{align*}
    r_0 &\leq r \leq r_n \\
    r_{i-1} &\leq r \leq r_i \\
    0 &\leq \theta \leq \varphi \\
    0 &\leq z \leq L
\end{align*} \]
\[ \varphi \leq 2\pi \]
\[ 0 \leq z \leq L \]
\[ t \geq 0 \]

With boundary condition
\[ A_{in} \frac{\partial T_i}{\partial r}(r_0, \theta, z, t) + B_{in} T_i(r_0, \theta, z, t) = C_{in} \]
\[ A_{out} \frac{\partial T_n}{\partial r}(r_n, \theta, z, t) + B_{out} T_n(r_0, \theta, z, t) = C_{out} \]

If \( \theta = 0 \), we have \( T_i = T_i(r, 0, z, t) = 0 \) or \( \frac{\partial T_i}{\partial \theta}(r, 0, z, t) = 0 \). If \( \theta = \varphi \), then \( T_i = T_i(r, \varphi, z, t) = 0 \) or \( \frac{\partial T_i}{\partial \theta}(r, \varphi, z, t) = 0 \).

### 3. Results and Discussion

#### 3.1. Analytical Solution

In this part, we derive an analytical solution in two dimensional heat transfer system of equations (1). If \( z \) is fixed, then equation (1) would become

\[ \frac{1}{\alpha_i} \frac{\partial T_i}{\partial t} = \frac{\partial^2 T_i}{\partial r^2} + \frac{1}{r} \frac{\partial T_i}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T_i}{\partial \theta^2} + \frac{g_i(r, \theta, z, t)}{k_i} \]

where

\[ r_{i-1} \leq r \leq r_i \]
\[ 0 \leq \theta \leq \varphi \]
\[ \varphi \leq 2\pi \]
\[ t \geq 0 \]

In the 1st layer \((i = 1)\) we have

\[ A_{in} \frac{\partial T_1}{\partial r}(r_0, \theta, t) + B_{in} T_1(r_0, \theta, z, t) = C_{in} \]

And in the outer layer \((i = n)\), we have

\[ A_{out} \frac{\partial T_n}{\partial r}(r_n, \theta, t) + B_{out} T_n(r_0, \theta, z, t) = C_{out} \]

Moreover, for \( \theta = 0 \), we have \( T_i(r, \theta, t) = 0 \) or \( \frac{\partial T_i}{\partial \theta}(r, 0, t) = 0 \). And, for \( \theta = \varphi \), we have \( T_i(r, \varphi, t) = 0 \) or \( \frac{\partial T_i}{\partial \theta}(r, \varphi, t) = 0 \).

In the \( i \)th layer, for the inner interface of two layers, we have

\[ T_i(r_{i-1}, \theta, t) = T_{i-1}(r_{i-1}, \theta, t) \]

and

\[ k_i \frac{\partial T_i}{\partial r}(r_{i-1}, \theta, t) = k_{i-1} \frac{\partial T_{i-1}}{\partial r}(r_{i-1}, \theta, t) \]

Also, for the outer interface of two layers, we have

\[ T_i(r_1, \theta, t) = T_{i+1}(r_1, \theta, t) \]
and
\[ k_i \frac{\partial T_i}{\partial r}(r_i, \theta, t) = k_{i+1} \frac{\partial T_{i+1}}{\partial r}(r_{i+1}, \theta, t) \]

To find the analytical solution, we need to find the related eigen function \( \varphi \) of the equations. The eigen function \( \varphi \) satisfies the following equation.

\[ \nabla^2 \varphi = -\lambda^2 \varphi \quad (2) \]

In the cylindrical coordinates with fixed \( z \), we have

\[ \frac{\partial^2 \varphi_i}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi_i}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \varphi_i}{\partial \theta^2} = -\lambda^2 \varphi_i \quad (3) \]

We assume that

\[ \varphi_i(r, \theta) = R_i(r)\theta_i(\theta) \quad (4) \]

Therefore, we have

\[ \frac{\partial \varphi_i}{\partial r} = \theta_i \frac{dR_i}{dr} \]
\[ \frac{\partial^2 \varphi_i}{\partial r^2} = \theta_i \frac{d^2R_i}{dr^2} \]
\[ \frac{\partial^2 \varphi_i}{\partial \theta^2} = R_i \frac{d^2\theta_i}{d\theta^2} \]

So, (3) would become

\[ \theta_i R_i + \frac{1}{r} R_i' \theta_i + \frac{1}{r^2} R_i \theta_i = -\lambda^2 R_i \theta_i \]
\[ \frac{R_i''}{R_i} + \frac{1}{r} \frac{R_i'}{R_i} + \frac{1}{r^2} \theta_i'' = -\lambda^2 \]
\[ \frac{r^2 R_i''}{R_i} + \frac{r R_i'}{R_i} + \frac{\theta_i''}{\theta_i} = -\lambda^2 r^2 \quad (5) \]

where

\[ r^2 \frac{R_i''}{R_i} + r \frac{R_i'}{R_i} + \lambda^2 r^2 = -\frac{\theta_i''}{\theta_i} = v^2 \]
\[ r^2 R_i'' + r R_i' + R_i(\lambda^2 r^2 - v^2) = 0 \]
\[ \theta_i'' + v^2 \theta_i = 0 \quad (6) \]
\[ (7) \]

Now, let

\[ R_i(r) = \sum_{m=0}^{\infty} a_m r^m. \]

We have,

\[ R_i'(r) = \sum_{m=1}^{\infty} m a_m r^{m-1} \]

and

\[ R_i''(r) = \sum_{m=2}^{\infty} m (m - 1) a_m r^{m-2} \]

Therefore, equation (7) can be written as
\[
    r^2 \sum_{m=2}^{\infty} m(m-1)a_m r^{m-2} + r \sum_{m=1}^{\infty} ma_m r^{m-1} + (\lambda^2 r^2 - v^2) \sum_{m=0}^{\infty} a_m r^m = 0
\]

The solution for (7) is as follows.

\[
    R_{ij}(r) = C_1 J_{ij}^{\lambda} r + C_2 Y_{ij}^{\lambda} r
\]

(9)

Where \( J_{ij} \) is the Bessel function of the 1st kind and \( Y_{ij} \) is the Bessel function of 2nd kind. Meanwhile, the solution for equation (8) is as follows.

\[
    \theta_{ij}(\theta) = C_3 \sin(v_{ij}\theta) + C_4 \cos(v_{ij}\theta) = C_3 \sin(v_{ij}\theta) + C_4 \cos(v_{ij}\theta)
\]

Therefore, we have

\[
    T_i(r, \theta, t) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} T_{ikj}(t) R_{ij}(r) \theta_k(\theta)
\]

The heat source can be written as

\[
    g_i(r, \theta, t) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} g_{ikj}(t) R_{ij}(r) \theta_k(\theta),
\]

where

\[
    g_{ikj}(t) = \frac{\int_{r_{i-1}}^{r_i} g_i(r, \theta, t) R_{ij}(r) \theta_k(\theta) r dr d\theta}{\int_{r_{i-1}}^{r_i} R_{ij}(r)^2 \theta_k^2(\theta) r^2 dr d\theta}
\]

The coefficient \( T_{ikj} \) can be written as

\[
    T_{ikj} = \frac{\alpha_i}{k_i} e^{-r_{ikj}(t)} \int_{0}^{t} g_{ikj}(t) e^{-r_{ikj}(t)} dt + \alpha_i e^{-r_{ikj}(t)}
\]

where

\[
    \alpha_i = T_{ikj}(0)
\]

\[
    a_i = \frac{\int_{r_{i-1}}^{r_i} f_i(r, \theta, z) r R_{ikj}(r) \theta_k(\theta) Z_i(z) dr d\theta dz}{\int_{r_{i-1}}^{r_i} r R_{ikj}(r)^2 \theta_k^2(\theta) Z_i^2(z) r dr d\theta}
\]

Hence, the final solution is

\[
    F_{ikj} = -\alpha_i \left( \frac{R_{ikj}''(r)}{R_{ikj}(r)} + \frac{1}{r} \frac{R_{ikj}'(r)}{R_{ikj}(r)} + \frac{1}{r^2} \frac{\theta_j''(\theta)}{\theta_j(\theta)} + \frac{Z_i''(z)}{Z_i(z)} \right)
\]

3.2. Numerical solution

In this part, we derive numerical solution of equation (1) using finite difference method. Heat equation in the \( i \)-th layer is as follows.
\[
\frac{\partial^2 T_i}{\partial r^2} + \frac{1}{r} \frac{\partial T_i}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T_i}{\partial \theta^2} + \frac{\partial^2 T_i}{\partial z^2} + \frac{g_i(r, \theta, z, t)}{K_i} = \frac{1}{\alpha_i} \frac{\partial T_i}{\partial t}
\]

Using Taylor expansion, we have

\[
T_i(x, t + \Delta t) = T_i(x, t) + \frac{\partial T_i}{\partial t} \Delta t + \frac{\partial^2 T_i}{\partial t^2} \frac{\Delta t^2}{2!} + \frac{\partial^3 T_i}{\partial t^3} \frac{\Delta t^3}{3!} + \ldots
\]

So, we have

\[
\frac{\partial T_i}{\partial t} \Delta t = T_i(x, \theta, z, t + \Delta t) - T_i(r, \theta, z, t)
\]

or

\[
\frac{\partial T_i}{\partial t} = \frac{T_i(x, \theta, z, t + \Delta t) - T_i(r, \theta, z, t)}{\Delta t}
\]

Also, we use Taylor series to get the following equations,

\[
T_i(r + \Delta r, \theta, z, t) = T_i(r, \theta, z, t) + \frac{\partial T_i}{\partial r} \Delta r + \frac{\partial^2 T_i}{\partial r^2} \frac{\Delta r^2}{2!} + \frac{\partial^3 T_i}{\partial r^3} \frac{\Delta r^3}{3!} + \ldots
\]

(10)

Also, we have

\[
\frac{\partial T_i}{\partial r} \Delta r = T_i(r + \Delta r, \theta, z, t) - T_i(r, \theta, z, t)
\]

or

\[
\frac{\partial T_i}{\partial r} = \frac{T_i(r + \Delta r, \theta, z, t) - T_i(r, \theta, z, t)}{\Delta r}
\]

Then,

\[
T_i(r - \Delta r, \theta, z, t) = T_i(r, \theta, z, t) - \frac{\partial T_i}{\partial r} \Delta r + \frac{\partial^2 T_i}{\partial r^2} \frac{\Delta r^2}{2!} - \frac{\partial^3 T_i}{\partial r^3} \frac{\Delta r^3}{3!} + \ldots
\]

(11)

By adding (10) and (11), we have

\[
\frac{\partial^2 T_i}{\partial r^2} + \theta(\Delta r^4) = \frac{T_i(r + \Delta r, \theta, z, t) - 2T_i(r, \theta, z, t) + T_i(r - \Delta r, \theta, z, t)}{\Delta r^2}
\]

or approximately,

\[
\frac{\partial^2 T_i}{\partial r^2} \approx \frac{T_i(r + \Delta r, \theta, z, t) - 2T_i(r, \theta, z, t) + T_i(r - \Delta r, \theta, z, t)}{\Delta r^2}
\]

By similar way, we have the following equation in \( \theta \) and \( z \),

\[
\frac{\partial^2 T_i}{\partial \theta^2} = \frac{T_i(r, \theta + \Delta \theta, z, t) - 2T_i(r, \theta, z, t) + T_i(r, \theta - \Delta \theta, z, t)}{\Delta \theta^2}
\]
and

\[ \frac{\partial^2 T_i}{\partial z^2} = \frac{T_i(r, \theta, z + \Delta z, t) - 2T_i(r, \theta, z, t) + T_i(r, \theta, z - \Delta z, t)}{\Delta z^2} \]

Note that, when \( z \) is fixed, \( \frac{\partial T_i}{\partial z} = 0 \) and \( \frac{\partial^2 T_i}{\partial z^2} = 0 \). Let \( l, m, \) and \( n \) are discrete indices for variables \( r, t, \) and \( \theta \), respectively. Then, we have the following equations.

\[ \frac{\partial T_i}{\partial t} = \frac{T_i(x, n, m + 1) - T_i(l, n, m)}{\Delta t} \]
\[ \frac{\partial T_i}{\partial r} = \frac{T_i(l + 1, n, m) - T_i(l, n, m)}{\Delta r} \]
\[ \frac{\partial^2 T_i}{\partial r^2} = \frac{T_i(l + 1, n, m) - 2T_i(l, n, m) + T_i(l - 1, n, m)}{\Delta r^2} \]
\[ \frac{\partial^2 T_i}{\partial \theta^2} = \frac{T_i(l, n + 1, m) - 2T_i(l, n, m) + T_i(l, n + 1, m)}{\Delta \theta^2} \]

Therefore, the final numerical solution is

\[ T_{l,n,m+1}^i = T_{l,n,m}^i + \frac{\alpha \Delta t}{\Delta r^2} (T_{l+1,n,m}^i - 2T_{l,n,m}^i + T_{l-1,n,m}^i) + \frac{\alpha \Delta t}{r_i \Delta r} (T_{l+1,n,m}^i - T_{l,n,m}^i) \]
\[ + \frac{\alpha \Delta t}{r_i \Delta \theta^2} (T_{l,n+1,m}^i - 2T_{l,n,m}^i + T_{l,n-1,m}^i) + \frac{\alpha \Delta t}{l_i} g_i(r_n, \theta_m, z, t_i) \] (12)

This numerical solution (12) then being simulated using Python 3.7 and Numpy Library with additional assumption that the heat source time independent [9, 10]. The results depicted in Figure 2 and 3. The Figure 2 shows the temperature profile for the first layer. As we can see, the temperature increases with time. This is because the heat source is constant and the heat delivered through the cylinder and meet the temperature profile in boundary condition from the second layer. When the heat delivered to the second layer, the temperature decreases as shown in Figure 3, and the curve’s negative gradient depends up on the thermal diffusivity (\( \alpha \)) of the medium.

![Figure 2](image_url)
Figure 3. The temperature distribution in the second layer for various $\alpha$, (a) $\alpha = 1.1 \times 10^{-4}$, (b) $\alpha = 1.1 \times 10^{-2}$, (c) $\alpha = 1.1 \times 10^{-0.5}$, (d) $\alpha = 1.1 \times 10^{-0.2}$, (e) $\alpha = 1.1 \times 10^{-0.05}$

4. Conclusion
The heat transfer profile in a two dimensional multilayered composite cylinder was determined using analytical and numerical method. The temperature distribution profile in different time and different thermal diffusivity for two layers was calculated using Python 3.7.

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