AN ALGORITHMIC APPROACH TO ANTIMAGIC LABELING OF EDGE CORONA GRAPHS

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Abstract. An antimagic labeling of a graph $G$ is a $1-1$ correspondence between the edge set $E(G)$ and \{1, 2, ..., $|E(G)|$\} in which the sum of the labels of edges incident to the distinct vertices are different. The edge corona of any two graphs $G$ and $H$, (denoted by $G \circ H$) is obtained by joining one copy of $G$ with $|E(G)|$ copies of $H$ such that the end vertices of $i^{th}$ edge of $G$ is adjacent to every vertex in the $i^{th}$ copy of $H$. In this paper, we provide an algorithm to prove that the following graphs admit an antimagic labeling:

- $n$-barbell graph $B_n$, $n \geq 3$
- edge corona of a bistar graph $B_{x,n}$ and a $k$-regular graph $H$ denoted by $B_{x,n} \circ H$, $x, n \geq 2$
- edge corona of a cycle $C_m$ and $C_n$ denoted by $C_m \circ C_n$, $m, n \geq 3$

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1. Introduction

The graphs considered in this paper are simple, finite and undirected. Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of the graph $G$ respectively. For a graph $G = (V, E)$ with $q$ edges, an antimagic labeling is a bijection $f : E(G) \rightarrow \{1, 2, ..., q\}$ such that $w(u) = \sum_{e \in E(u)} f(e)$ is distinct for all vertices $u \in V(G)$ where $E(u)$ denotes the set of edges incident on the vertex $u$. A graph is said to be antimagic if it admits an antimagic labeling. Paths, cycles, complete graphs, wheel, stars, complete bipartite graphs\cite{5}, graphs of order $n$ with maximum degree at least $n-3$\cite{9}, toroidal grid graphs\cite{8}, regular graphs\cite{3, 2}, trees with some restrictions \cite{6}, caterpillars\cite{7} are few of the graph classes proved to be antimagic in the literature.

The antimagic labeling on edge corona graphs has comparatively less results than other graph products like cartesian product, lexicographic product, corona product, etc in the literature. Also, motivated by the two conjectures proposed by Hartsfield and Ringel which is still open\cite{5}, we focus on antimagicness of $n$-barbell graph and edge corona of few graph classes in this paper.

Conjecture 1: Every connected graph other than $K_2$ is antimagic.
Conjecture 2: Every tree other than $K_2$ is antimagic.
2. Preliminaries

Let \( G \) and \( H \) be two vertex-disjoint graphs. Define \([1, n]\) = \( \{1, 2, ..., n\} \). Let \( d(v) \) denote the degree of a vertex \( v \) in the graph \( G \). We say that \( u \in V(G) \) is complete to a graph \( H \) if \( u \) is adjacent to all the vertices of \( H \). Also, an edge \( ab \in E(G) \) is complete to a graph \( H \) if the vertices \( a \) and \( b \) are adjacent to all the vertices of \( H \). The edge corona of any two graphs \( G \) and \( H \), (denoted by \( G \circ H \)) is obtained by joining one copy of \( G \) with \( |E(G)| \) copies of \( H \) such that the end vertices of \( i \)th edge of \( G \) is adjacent to every vertex in the \( i \)th copy of \( H \). The sum of the labels of all the edges incident to a vertex in a graph \( G \) is called the vertex sum (denoted by \( w(v) \), \( v \in V(G) \)). The sum of the labels of some edges (i.e., few edges remain unlabeled) in a graph \( G \) is known as the partial vertex sum (denoted by \( w'(v) \), \( v \in V(G) \)). A graph is said to be regular if the degrees of all the vertices are same. The join of two graphs \( G \) and \( H \) is obtained by making every vertex of \( G \) adjacent to all the vertices of \( H \). A bistar graph \( B_{m,s} \) is obtained by joining the apex vertices (a vertex adjacent to all the vertices of a graph) of two vertex disjoint star graphs \( K_{1,m} \) and \( K_{1,s} \) for \( m \geq 1 \) and \( s \geq 1 \) respectively. An \( n \)-barbell graph is obtained by adding an edge between two copies of \( K_n \), \( n \geq 3 \).

3. Main results

**Theorem 3.1.** The \( n \)-barbell graph is antimagic for \( n \geq 3 \).

**Proof.** The general representation of an \( n \)-barbell graph \( B_n \) is given in Figure 1. The dotted line from \( u_2 \) to \( u_{n-2} \) represent a clique \( R \) on vertices \( \{u_3, u_4, ..., u_{n-3}\} \) and the vertices \( \{u_1, u_2, u_{n-2}, u_{n-1}, u_n\} \) are complete to \( R \). Similarly, the dotted line from \( v_2 \) to \( v_{n-2} \) represent a clique \( S \) on vertices \( \{v_3, v_4, ..., v_{n-3}\} \) and the vertices \( \{v_1, v_2, v_{n-2}, v_{n-1}, v_n\} \) are complete to \( S \). Note that the vertices \( \{u_1, u_2, ..., u_{n-1}\} \) and \( \{v_1, v_2, ..., v_{n-1}\} \) need not be in cyclic order. Let \( V(B_n) = \{u_1, u_2, ..., u_n\} \cup \{v_1, v_2, ..., v_n\} \) and \( E(B_n) = \{u_i u_j \mid 1 \leq i, j \leq n, i \neq j\} \cup \{v_i v_j \mid 1 \leq i, j \leq n, i \neq j\} \cup \{uv\} \).

The graph \( B_n \) contains \( 2n \) vertices and \( n(n - 1) + 1 \) edges. The edge labels are \( \{1, 2, ..., n(n - 1) + 1\} \). Let \( A_1 \) and \( A_2 \) be the induced subgraphs of \( B_n \) such that \( A_1 = B_n[\{u_1, u_2, ..., u_{n-1}\}] \) and \( A_2 = B_n[\{v_1, v_2, ..., v_{n-1}\}] \).

**Construction of an Antimagic Labeling:**

Step 1: We label the edges of the subgraph \( A_1 \) using \( \{1, 2, ..., \frac{(n-1)(n-2)}{2}\} \).
Step 2: We label the edges of the subgraph $A_2$ using \( \left\{ \frac{(n-1)(n-2)}{2} + 1, \frac{(n-1)(n-2)}{2} + 2, ..., (n-1)(n-2) \right\} \). This labeling leads to the partial vertex sums (need not be antimagic) \( w'(u_i) \) and \( w'(v_i), 1 \leq i \leq n-1. \) The updated \( u_i \)'s are rewritten as \( a_j \)'s, \( 1 \leq i, j \leq n-1 \) such that \( w'(a_j) \leq w'(a_{j+1}), 1 \leq j \leq n-2 \) and the updated \( v_i \)'s are rewritten as \( b_j \)'s, \( 1 \leq i, j \leq n-1 \) such that \( w'(b_j) \leq w'(b_{j+1}), 1 \leq j \leq n-2. \) Also, \( w'(a_i) < w'(b_j), 1 \leq i, j \leq n-1. \)

Step 3: We label the edges \( ua_i, vb_j, uv, 1 \leq i, j \leq n-1 \) using \( \{(n-1)(n-2)+1, ..., (n-1)(n-2)+(n-1)\}, \{(n-1)(n-2)+(n-1)+1, ..., (n-1)(n-2)+(n-1)+(n-1)\} \) and \( n(n-1)+1 \) respectively such that,

\[
\begin{align*}
  f(ua_i) &= (n-1)(n-2) + i, 1 \leq i \leq n-1 \\
  f(vb_j) &= (n-1)(n-2) + (n-1) + j, 1 \leq j \leq n-1 \\
  f(uv) &= n(n-1) + 1
\end{align*}
\]

We now provide an algorithm to construct an antimagic labeling \( f \) of \( B_n \)

**Algorithm:**

**STEP 1:** Label the edges of the subgraphs \( A_1 \) and \( A_2 

1: \( f(E(A_1)) \leftarrow [1, \frac{(n-1)(n-2)}{2}] \) 

2: \( f(E(A_2)) \leftarrow [\frac{(n-1)(n-2)}{2} + 1, (n-1)(n-2)] \) 

**STEP 2:** Label the edges incident with \( a_i, 1 \leq i \leq n-1 

3: Sort the partial vertex sums \( w'(a_i), 1 \leq i \leq n-1 \) as \( w'(a_i) \leq w'(a_{i+1}), 1 \leq i \leq n-2 \) 

4: Sort the labels \( (n-1)(n-2)+1, (n-1)(n-2)+2,...,(n-1)(n-2)+(n-1) \) in an increasing order 

5: for \( i = 1, 2, ..., n-1 \) do 

6: \( f(ua_i) \leftarrow (n-1)(n-2) + i \) 

**STEP 3:** Label the edges incident with \( b_j, 1 \leq j \leq n-1 \) and an edge \( uv \) 

7: Sort the partial vertex sums \( w'(b_j), 1 \leq j \leq n-1 \) as \( w'(b_j) \leq w'(b_{j+1}), 1 \leq j \leq n-2 \) 

8: Sort the labels \( (n-1)(n-2)+(n-1)+1, (n-1)(n-2)+(n-1)+2,...,(n-1)(n-2)+(n-1)+(n-1) \) in an increasing order 

9: for \( j = 1, 2, ..., n-1 \) do 

10: \( f(vb_j) \leftarrow (n-1)(n-2) + (n-1) + j \) 

11: \( f(uv) \leftarrow n(n-1) + 1 \) 

**Proof of Antimagicness:**

This labeling leads to the distinctness on the entire vertex sums as follows:

\[
\begin{align*}
  w(a_i) &= w'(a_i) + (n-1)(n-2) + i \\
  &< w(a_{i+1}) = w'(a_{i+1}) + (n-1)(n-2) + (i+1), 1 \leq i \leq n-2 \\
  w(b_j) &= w'(b_j) + (n-1)(n-2) + (n-1) + j \\
  &< w(b_{j+1}) = w'(b_{j+1}) + (n-1)(n-2) + (n-1) + (j+1), 1 \leq j \leq n-2
\end{align*}
\]
This clearly shows that $w(a_i) < w(b_j)$, $1 \leq i, j \leq n - 1$.

And, $w(u) = \sum_{i=1}^{n-1} f(ua_i) + n(n - 1) + 1$

$$< w(v) = \sum_{j=1}^{n-1} f(vb_j) + n(n - 1) + 1$$

since $\sum_{i=1}^{n-1} f(ua_i) < \sum_{j=1}^{n-1} f(vb_j)$.

Let us define,

set 1: \{(n-1)(n-2)/2 + 1, ..., (n-1)(n-2)\}

set 2: \{(n-1)(n-2) + (n-1) + 1, ..., (n-1)(n-2) + (n-1) + (n-1) = n(n-1)\}

set 3: \{(n-1)(n-2) + 1, ..., (n-1)(n-2) + (n-1)\}

set 4: \{n(n-1) + 1\}

The maximum of $w(b_j)$, $1 \leq j \leq n - 1$ is $w(b_{n-1})$ which is the sum of any $n - 2$ labels of set 1 and any one label of set 2; and $w(u)$ is the sum of all labels of set 3 and set 4. Observe that $d(b_j) < d(u)$, $1 \leq j \leq n - 1$. Clearly, $w(b_j) < w(u), 1 \leq j \leq n - 1$. Hence, all the vertices are distinct since $w(a_i) < w(b_j) < w(u) < w(v), 1 \leq i, j \leq n - 1$.

**Time Complexity:**

The assignments in Step 1 takes constant time. Since, the partial vertex sums are sorted in line 3 of Step 2, it requires $O(n\log n)$ time. Again, Step 3 requires the time $O(n\log n)$ due to the same fact that the partial vertex sums are sorted in line 7. Hence, the time complexity of the above algorithm is $O(n\log n)$.

Next, we illustrate the above labeling process in Figure 2 for the 4-barbell graph, $B_4$.

![Figure 2](image)

**Figure 2.** An antimagic labeling of 4-barbell graph, $B_4$.

Note that $w(a_1) = 10$, $w(a_2) = 12$, $w(a_3) = 14$, $w(u) = 37$ and $w(b_1) = 19$, $w(b_2) = 21$, $w(b_3) = 23$, $w(v) = 46$.

3.1. **Edge Corona of $B_{x,n}$ and $H$.** Let the vertices $\{l_1, l_2, ..., l_x\}$ and $\{l_{x+1}, l_{x+2}, ..., l_{x+n}\}$ be adjacent to the apex vertices $u$ and $v$ of $K_{1,x}$ and $K_{1,n}$ respectively to form a bistar graph $B_{x,n}$ with $uv \in E(B_{x,n})$ where $x, n \geq 2$. Note that the graph $B_{x,n}$ contains $x + n + 2$ vertices and $x + n + 1$ edges. To construct the graph $B_{x,n} \odot H$, we require one copy of $B_{x,n}$ and $|E(B_{x,n})|$ copies of $H$ namely $H_1, H_2, ..., H_{x+n+1}$ (each $H_i$ is a $k$-regular graph on $m$ vertices). Let the edge $l_iu$ be complete to $H_i$, $1 \leq i \leq x$ and the edge $l_i v$ be complete to $H_i$, $x + 1 \leq i \leq x + n$. And, the edge $uv$ is complete to $H_{x+n+1}$. The graph $\bigcup_{i=1}^{x+n+1} H_i$ contains $(x + n + 1)m$ vertices and $(x + n + 1)\frac{mk}{2}$ edges. Therefore, the graph $B_{x,n} \odot H$
contains \( x + n + 2 + m(x + n + 1) \) vertices and \( (x + n + 1) + 2m(x + n + 1) + \frac{mk}{2}(x + n + 1) = z \) (say) edges. For a better understanding of the graph \( B_{x,n} \circ H \), see Figure 3 (\( H \) isomorphic to \( C_4 \)). The left dotted curve represent the graph \( \bigcup_{i=2}^{x-1} (H_i + \{l_i\}) \). Similarly, the right dotted curve represent the graph \( \bigcup_{i=x+2}^{x+n-1} (H_i + \{l_i\}) \).

**Figure 3.** A representation of \( B_{x,n} \circ C_4 \).

**Theorem 3.2.** \( B_{x,n} \circ H \) is antimagic for \( x, n \geq 2, x \leq n \) and \( H \) is a connected \( k \)-regular graph on \( m \geq 2 \) vertices.

**Proof.** Let \( \{u_1, u_2, ..., u_{mx}\} \) be the vertices of \( \bigcup_{i=1}^{x} H_i \) that are adjacent to \( u \) and \( \{v_1, v_2, ..., v_{mn}\} \) be the vertices of \( \bigcup_{i=x+1}^{x+n} H_i \) that are adjacent to \( v \). Let \( \{w_1, w_2, ..., w_m\} \) be the vertices of \( H_{x+n+1} \) that are adjacent to both \( u \) and \( v \).

**Construction of an Antimagic Labeling:**

Step 1: We label the edges of graphs \( H_1, H_2, ..., H_x \) in an order from smallest to the largest label available in the set \( \{1, 2, ..., x(\frac{mk}{2})\} \).

Step 2: We label the edges of graphs \( H_{x+1}, H_{x+2}, ..., H_{x+n} \) in an order from smallest to the largest label available in the set \( \{x(\frac{mk}{2}) + 1, x(\frac{mk}{2}) + 2, ..., x(\frac{mk}{2}) + n(\frac{mk}{2})\} \).

Step 3: We label the edges of graph \( H_{x+n+1} \) using \( \{x(\frac{mk}{2}) + n(\frac{mk}{2}) + 1, x(\frac{mk}{2}) + n(\frac{mk}{2}) + 2, ..., x(\frac{mk}{2}) + n(\frac{mk}{2}) + \frac{mk}{2} = g\} \).

Step 4: We label the set of edges incident with \( l_1, l_2, ..., l_x \) excluding the edges \( ul_1, ul_2, ..., ul_x \) in an order from smallest to the largest label available in the set \( \{g + 1, g + 2, ..., g + x\} \).
respectively.

Step 5: We label the set of edges incident with \( l_{x+1}, l_{x+2}, ..., l_{x+n} \) excluding the edges \( vl_{x+1}, vl_{x+2}, ..., vl_{x+n} \) in an order from smallest to the largest label available in the set \( \{g + xm + 1, g + xm + 2, ..., g + xm + nm\} \) respectively.

Step 6: We label the edges \( uw_1, uw_2, ..., uw_m \) using \( \{g + xm + nm + 1, g + xm + nm + 2, ..., g + xm + nm + m = h\} \). With respect to the above labeling we obtain the partial vertex sums (need not be antimagic) \( w'(u_i), 1 \leq i \leq mx, w'(v_i), 1 \leq i \leq mn, w'(w_i), 1 \leq i \leq m \) and \( w'(l_i), 1 \leq i \leq x + n \).

We exclude the partial vertex sum of the vertex \( u \). Merging all the above partial vertex sums, we update the new vertex sums as \( w'(a_i) \leq w'(a_{i+1}), 1 \leq i \leq x(m+1) + n(m+1) + m - 1 \).

Step 7: We label the edge \( uv \) as \( f(uv) = z \).

Step 8: Finally, we label the remaining edges using \( \{h + 1, h + 2, ..., h + x(m+1) + n(m+1) + m\} \) in such a way that \( f(sa_i) = h + i, 1 \leq i \leq x(m+1) + n(m+1) + m \) where \( s = u \) for all \( a_i \) that are adjacent to \( u \) and \( s = v \) for all \( a_i \) that are adjacent to \( v \).

We now provide an algorithm to construct an antimagic labeling \( f \) of \( B_{x,n} \circ H \)

**Algorithm:**

**STEP 1:** Label the edges of the graphs \( H_i, i = 1, 2, ..., x, x + 1, ..., x + n, x + n + 1 \)
1: for \( i = 1, 2, ..., x, x + 1, ..., x + n, x + n + 1 \) do
2: \( f(E(H_i)) \leftarrow [(i - 1)\frac{mk}{2} + 1, i\frac{mk}{2}] \)

**STEP 2:** Label the edges incident with \( l_1, l_2, ..., l_x, l_{x+1}, ..., l_{x+n} \) excluding the edges \( ul_1, ul_2, ..., ul_x, ul_{x+1}, ..., ul_{x+n} \) respectively
3: for \( i = 1, 2, ..., x, x + 1, ..., x + n \) do
4: \( f(E(l_i)) \leftarrow [g + (i - 1)m + 1, g + im] \) excluding \( f(ul_i) \)

**STEP 3:** Label the edges \( uw_i, 1 \leq i \leq m \) and \( uv \)
5: for \( i = 1, 2, ..., m \) do
6: \( f(uw_i) \leftarrow g + xm + nm + i \)
7: \( f(uv) \leftarrow z \)

**STEP 4:** Label the remaining edges of \( B_{x,n} \circ H \)
8: Sort the partial vertex sums \( w'(a_i), 1 \leq i \leq x(m+1) + n(m+1) + m \) as \( w'(a_i) \leq w'(a_{i+1}), 1 \leq i \leq x(m+1) + n(m+1) + m - 1 \)
9: Sort the available labels \( h + 1, h + 2, ..., h + x(m+1) + n(m+1) + m \) in an increasing order
10: for \( i = 1, 2, ..., x(m+1) + n(m+1) + m \) do
11: \( f(sa_i) \leftarrow h + i, s = u \lor a_i \) adjacent to \( u \), \( s = v \lor a_i \) adjacent to \( v \)
Proof of Antimagicness:
This labeling leads to the distinctness on the entire vertex sums as follows:
\[ w(a_i) = w'(a_i) + h + i < w(a_{i+1}) = w'(a_{i+1}) + h + (i + 1) \]
for \( 1 \leq i \leq x(m + 1) + n(m + 1) + m - 1 \)

Let us define,
set 1: \( \{ g + xm + nm + 1, ..., g + xm + nm + m = h \}, \{ h + 1, h + 2, ..., h + xm \}, \{ h + xm + nm + m + 1, h + xm + nm + 2, ..., h + xm + nm + x \}, \{ z \} \)

set 2: \( \{ h + xm + 1, h + xm + 2, ..., h + xm + nm \}, \{ h + xm + nm + 1, h + xm + nm + 2, ..., h + xm + nm + m + 1, h + xm + nm + m + x + 1, h + xm + nm + m + x + 2, ..., h + xm + nm + m + x + n \}, \{ z \} \)

The edges incident with \( u \) receives the labels of set 1 and the edges incident with \( v \) receives the labels of set 2. Observe that \( d(u) \leq d(v) \). From the above, it is clear that the sum of all the labels of set 1 is less than the sum of all the labels of set 2. So, it is clearly shown that \( w(u) < w(v) \) using the labels of set 1 and set 2. And the maximum of \( w(a_i) \) is \( w(a_{x(m+1)+n(m+1)+m}) \) which is the sum of \( w'(a_{x(m+1)+n(m+1)+m}) \) and \( z - 1 \). Also,
\[ w(a_{x(m+1)+n(m+1)+m}) = w'(a_{x(m+1)+n(m+1)+m}) + (z - 1) < w(u) = w'(u) + z + \sum_{i=1}^{xm} (h + i) + \sum_{i=1}^{x} (h + xm + nm + m + i) \]
where \( w'(a_{x(m+1)+n(m+1)+m}) = \sum_{i=1}^{m} (g + xm + (n - 1)m + i) \)
and \( w'(u) = \sum_{i=1}^{m} (g + xm + nm + i) \)
Hence, \( w(a_i) < w(u) < w(v) \), \( 1 \leq i \leq x(m + 1) + n(m + 1) + m \).

Time Complexity:
The assignments in Step 1 reaches at most \( x + n + 1 \) times to label the edges of the graphs \( H_i \) and so, it takes \( O(x + n) \) time. The same \( O(x + n) \) time is required for Step 2 to label the edges incident with the vertices \( l_i \) excluding the edges \( ul_i \) respectively. An assignment in line 6 of Step 3 reaches at most \( m \) vertices and hence it require \( O(m) \) time. Line 8 in Step 4 requires \( O(\log y) \) time since the partial vertex sums are sorted where \( y = x(m + 1) + n(m + 1) + m \). Hence, the time complexity of the above algorithm is \( O(\log y) \).

Next, we illustrate the above labeling process in Figure 4 for the graph \( B_{2,3} \circ C_4 \). Note that \( w(u) = 831, w(v) = 1388 \) and \( w(a_1) = 77, w(a_2) = 81, w(a_3) = 83, w(a_4) = 87, w(a_5) = 89, w(a_6) = 97, w(a_7) = 99, w(a_8) = 102, w(a_9) = 109, w(a_{10}) = 112, w(a_{11}) = 116, w(a_{12}) = 119, w(a_{13}) = 126, w(a_{14}) = 129, w(a_{15}) = 131, w(a_{16}) = 134, w(a_{17}) = 143, w(a_{18}) = 144, w(a_{19}) = 148, w(a_{20}) = 149, w(a_{21}) = 157, w(a_{22}) = 161, w(a_{23}) = 163, w(a_{24}) = 167, w(a_{25}) = 179, w(a_{26}) = 196, w(a_{27}) = 213, w(a_{28}) = 230, w(a_{29}) = 247 \).

**Corollary 3.1.** \( B_{x,n} \circ H \) admits an antimagic labeling for \( x, n \geq 2 \) and \( x > n \) where \( H \) is a connected \( k \)-regular graph on \( m \geq 2 \) vertices, \( k \geq 1 \).

**Proof.** In Theorem 3.2, we proved that \( B_{x,n} \circ H, x \leq n \) is antimagic. Obviously \( B_{x,n} \circ H, x \geq n \) is isomorphic to \( B_{x,n} \circ H, x \leq n \) and hence the result. \( \square \)
3.2. Edge Corona of $C_m$ and $C_n$. To construct the graph $C_m \circ C_n$, $m, n \geq 3$ we join one copy of $C_m$ with $|E(C_m)|$ copies of $C_n$ namely $C_{n_1}, C_{n_2}, ..., C_{n_m}$ such that the end vertices of $i^{th}$ edge of $C_m$ is adjacent to every vertex in the $i^{th}$ copy of $C_n$. The following lemma is already proved in [8]. We again prove the lemma with different labels so as to use it in Theorem 3.3.

**Lemma 3.1.** Cycle $C_m$, $m \geq 3$ is antimagic.

*Proof.* Let $V(C_m) = \{v_1, v_2, ..., v_m\}$ and $E(C_m) = \{v_1v_2, v_1v_3\} \cup \{v_iv_{i+2} | i = 2, ..., m - 2\} \cup \{v_{m-1}v_m\}$. Now, we shall show that $C_m$ is antimagic with different set of labels. In the proof of the lemma in [8], adding $3mn$ to the edge labels and $6mn$ to the vertex sums, we get $f(v_1v_2) = 3mn + 1, f(v_1v_3) = 3mn + 2, f(v_iv_{i+2}) = 3mn + i + 1$, for $2 \leq i \leq m - 2$, and $f(v_{m-1}v_m) = 3mn + m$. The edge labeling induces the following ordering on vertices as $w(v_1) < w(v_2) < ... < w(v_m)$ since the vertex sums are

$$w(v_i) = \begin{cases} 
6mn + 3 & \text{if } i = 1; \\
6mn + 2i & \text{if } i = 2, ..., m - 1; \\
6mn + 2m - 1 & \text{if } i = m.
\end{cases}$$

Hence, $C_m$ is antimagic. \hfill $\Box$

**Lemma 3.2.** Cycles $C_{nj}$, $1 \leq j \leq m$ are antimagic.

*Proof.* Let $V(C_{nj}) = \{u_1, u_2, ..., u_n\}$, $1 \leq j \leq m$ and $E(C_{nj}) = \{u_1u_2, u_1u_3\} \cup \{u_iu_{i+2} | u_i = 2, ..., n - 2\} \cup \{u_{n-1}u_n\}, 1 \leq j \leq m$. Now, we shall show that $C_{nj}$ is antimagic with different set of labels. In the proof of the lemma in [8], adding $(j - 1)n$ to the edge labels and $2(j - 1)n$ to the vertex sums, we get $f(u_1u_2) = (j - 1)n + 1, f(u_1u_3) = (j - 1)n + 2, f(u_iu_{i+2}) = (j - 1)n + i + 1$, for $2 \leq i \leq n - 2$, and $f(u_{n-1}u_n) = (j - 1)n + n$. Note that, $w(u_1) < w(u_2) < ... < w(u_n)$ since the vertex sums are

$$w(u_i) = \begin{cases} 
2(j - 1)n + 3 & \text{if } i = 1; \\
2(j - 1)n + 2i & \text{if } i = 2, ..., n - 1; \\
2(j - 1)n + 2n - 1 & \text{if } i = n.
\end{cases}$$

Hence, $C_{nj}$, $1 \leq j \leq m$ are antimagic. \hfill $\Box$

**Theorem 3.3.** $C_m \circ C_n$ is antimagic for $m, n \geq 3$

*Proof.* Let the vertex set, edge set, and an antimagic labeling for the graphs $C_m$ and $m$ copies of $C_n$ be defined as before. Rename the vertex sums $w(u_i')$, $w(v_i)$ as $w'(u_i')$ and $w'(v_i)$ respectively (the vertex sums of $C_m$ and $C_n$ are considered as partial vertex sums of $C_m \circ C_n$). The adjacency in $C_m \circ C_n$ is defined as follows:

**Case i.** when $m$ is even, $m = 2k$, $k \in \mathbb{Z}^+ - \{1\}$

- $v_1v_2$ is complete to $C_{n_1}$
- $v_1v_3$ is complete to $C_{n_2}$
- $v_2v_4$ is complete to $C_{n_3}$
- $v_3v_5$ is complete to $C_{n_4}$
- ...
- $v_{2k-2}v_{2k}$ is complete to $C_{n_{2k-1}}$
- $v_{2k}v_{2k-1}$ is complete to $C_{n_{2k}}$
Case ii. when $m$ is odd, $m = 2k + 1$, $k \in \mathbb{Z}^+$

$v_1v_2$ is complete to $C_{n_1}$
$v_1v_3$ is complete to $C_{n_2}$
$v_2v_4$ is complete to $C_{n_3}$
$v_3v_5$ is complete to $C_{n_4}$

$\vdots$

$v_{2k-1}v_{2k+1}$ is complete to $C_{n_{2k}}$
$v_{2k+1}v_{2k}$ is complete to $C_{n_{2k+1}}$

Construction of an Antimagic Labeling:

The edge labels of $C_m \circ C_n$ are $\{1, 2, \ldots, 3mn + m\}$. As in the above lemma 3.1, 3.2 we assign the labels $\{1, 2, \ldots, mn\}$ to the edges of $C_{n_j}$, $1 \leq j \leq m$ and the labels $\{3mn + 1, \ldots, 3mn + m\}$ to the edges of $C_m$. Note that the induced edges between $C_m$ and $C_{n_j}$, $1 \leq j \leq m$ are to be labelled with $\{mn + 1, mn + 2, \ldots, 3mn\}$ such that,

$$f(v_1u_1^1) = mn + i, 1 \leq i \leq n$$
$$f(v_1u_1^2) = mn + n + i, 1 \leq i \leq n$$
$$f(v_2u_1^1) = mn + 2n + i, 1 \leq i \leq n$$
$$f(v_2u_1^2) = mn + 3n + i, 1 \leq i \leq n$$
$$f(v_3u_1^1) = mn + 4n + i, 1 \leq i \leq n$$
$$f(v_3u_1^2) = mn + 5n + i, 1 \leq i \leq n$$
$$f(v_4u_1^1) = mn + 6n + i, 1 \leq i \leq n$$
$$f(v_4u_1^2) = mn + 7n + i, 1 \leq i \leq n$$
$$f(v_5u_1^1) = mn + 8n + i, 1 \leq i \leq n$$
$$f(v_5u_1^2) = mn + 9n + i, 1 \leq i \leq n$$

\vdots

$$f(v_mu_1^{m-1}) = mn + (m-1)2n + i, 1 \leq i \leq n$$
$$f(v_mu_1^m) = 3mn - n + i, 1 \leq i \leq n$$

Algorithm:

**STEP 1:** Label the edges of $C_m$

1. $f(v_1v_2) \leftarrow 3mn + 1$
2. $f(v_1v_3) \leftarrow 3mn + 2$
3. **for** $i = 2$ to $m - 2$ **do**
4. $f(v_iv_{i+2}) \leftarrow 3mn + i + 1$
5. $f(v_{m-1}v_m) \leftarrow 3mn + m$

**STEP 2:** Label the edges of $C_{n_j}$, $1 \leq j \leq m$

6. **for** $j = 1, 2, \ldots, m$, $i = 2$ to $n - 2$ **do**
7. $f(u_1^1u_2^1) \leftarrow (j-1)n + 1$
8. $f(u_1^1u_2^2) \leftarrow (j-1)n + 2$
9. $f(u_1^1u_{i+2}) \leftarrow (j-1)n + i + 1$
10: \( f(u^j_{n-1} u^j_n) \leftarrow (j - 1)n + n \)

**STEP 3:** Label the induced edges between \( C_{n_j}, 1 \leq j \leq m \) and \( C_m \\
11: \text{for } i = 1 \text{ to } n \text{ do} \\
12: \( f(v_1 u^1_i) \leftarrow mn + i \) \\
\( f(v_1 u^2_i) \leftarrow mn + n + i \) \\
\( f(v_2 u^1_i) \leftarrow mn + 2n + i \) \\
\( f(v_2 u^2_i) \leftarrow mn + 3n + i \) \\
\( f(v_3 u^1_i) \leftarrow mn + 4n + i \) \\
\( f(v_3 u^2_i) \leftarrow mn + 5n + i \) \\
\( f(v_4 u^1_i) \leftarrow mn + 6n + i \) \\
\( f(v_4 u^2_i) \leftarrow mn + 7n + i \) \\
\( f(v_5 u^1_i) \leftarrow mn + 8n + i \) \\
\( f(v_5 u^2_i) \leftarrow mn + 9n + i \) \\
\vdots \\
\( f(v_m u^m_{i-1}) \leftarrow mn + (m - 1)2n + i, 1 \leq i \leq n \) \\
\( f(v_m u^m_i) \leftarrow 3mn - n + i, 1 \leq i \leq n \)

**Proof of Antimagicness:** This labeling leads to the distinctness on the entire vertex sums as follows:

\[
\begin{align*}
w(u^1_1) &= w'(u^1_1) + f(v_1 u^1_1) + f(v_2 u^1_1) \\
&< w(u^2_1) = w'(u^2_1) + f(v_1 u^2_1) + f(v_3 u^2_1) \\
&< w(u^3_1) = w'(u^3_1) + f(v_2 u^3_1) + f(v_4 u^3_1) \\
&< w(u^4_1) = w'(u^4_1) + f(v_3 u^4_1) + f(v_5 u^4_1) \\
&\vdots \\
&< w(u^{m-1}_{i_{m-1}}) = w'(u^{m-1}_{i_{m-1}}) + f(v_{m-2} u^{m-1}_{i_{m-1}}) + f(v_m u^{m-1}_{i_{m-1}}) \\
&< w(u^m_{i_m}) = w'(u^m_{i_m}) + f(v_m u^m_{i_m}) + f(v_{m-1} u^m_{i_m}), 1 \leq i_1, i_2, \ldots, i_m \leq n
\end{align*}
\]

And,

\[
\begin{align*}
w(u^1_{i_1}) &= w'(u^1_{i_1}) + f(v_1 u^1_{i_1}) + f(v_2 u^1_{i_1}) \\
&< w(u^1_{i_1+1}) = w'(u^1_{i_1+1}) + f(v_1 u^1_{i_1+1}) + f(v_2 u^1_{i_1+1}) \\
w(u^2_{i_2}) &= w'(u^2_{i_2}) + f(v_1 u^2_{i_2}) + f(v_3 u^2_{i_2}) \\
&< w(u^2_{i_2+1}) = w'(u^2_{i_2+1}) + f(v_1 u^2_{i_2+1}) + f(v_3 u^2_{i_2+1}) \\
w(u^3_{i_3}) &= w'(u^3_{i_3}) + f(v_2 u^3_{i_3}) + f(v_4 u^3_{i_3}) \\
&< w(u^3_{i_3+1}) = w'(u^3_{i_3+1}) + f(v_2 u^3_{i_3+1}) + f(v_4 u^3_{i_3+1}) \\
&\vdots \\
w(u^m_{i_m}) &= w'(u^m_{i_m}) + f(v_m u^m_{i_m}) + f(v_{m-1} u^m_{i_m}) \\
&< w(u^m_{i_m+1}) = w'(u^m_{i_m+1}) + f(v_m u^m_{i_m+1}) + f(v_{m-1} u^m_{i_m+1}), 1 \leq i_1, i_2, \ldots, i_m \leq n - 1
\end{align*}
\]
Also,

\[
    w(v) = w'(v) + \sum_{j=1}^{2n} (mn + (i-1)2n + j)
\]

\[
    < w(v_{i+1}) = w'(v_{i+1}) + \sum_{j=1}^{2n} (mn + (i)2n + j), 1 \leq i \leq m - 1
\]

Let us define,

set 1: \{1, 2, ..., mn\}

set 2: \{mn + 1, ..., 3mn\}

set 3: \{3mn + 1, ..., 3mn + m\}

Observe that \(w(u^i_j) < w(v_s), 1 \leq i, s \leq m, 1 \leq j \leq n\). Since, \(w(u^i_j)\) is the sum of any two labels of the set 1 and any two labels of the set 2; \(w(v_s)\) is the sum of any two labels of set 3 and any \(2n\) labels of set 2, \(w(u^i_j) < w(v_s), 1 \leq i, s \leq m, 1 \leq j \leq n\). Hence the vertex sums of the graph \(C_m \odot C_n\) are distinct.

**Time Complexity:**

Step 1 takes \(O(m)\) time since the line 4 iterates at most \(m - 3\) times. Step 2 requires \(O(mn)\) time since the line 9 iterates for \(m(n - 3)\) times. Finally, the step 3 takes \(O(mn)\) time since it iterates \(2mn\) times. Hence, the time complexity of the above algorithm is \(O(mn)\).

We illustrate the above labeling process in Figure 5 for the graph \(C_4 \odot C_4\). Note that \(w(u^1_1) = 45, w(u^2_1) = 48, w(u^3_1) = 52, w(u^4_1) = 55, w(u^2_1) = 65, w(u^2_2) = 68, w(u^3_2) = 72, w(u^2_3) = 75, w(u^2_4) = 89, w(u^3_3) = 92, w(u^3_4) = 96, w(u^4_3) = 99, w(u^4_4) = 109, w(u^1_4) = 112, w(u^2_4) = 116, w(v_1) = 119\) and \(w(v_1) = 263, w(v_2) = 328, w(v_3) = 394, w(v_4) = 459\).

![Figure 4](image-url)
4. Conclusions

As the conjecture due to Hartsfield and Ringel remains open for all these years we have investigated the antimagic labeling of the barbell graph and edge corona of some classes of graphs. It is also interesting to work on antimagic labeling of the generalized edge corona of graphs.

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