ANALYTICAL AND DIFFERENTIAL-ALGEBRAIC PROPERTIES OF GAMMA FUNCTION

Žarko Mijajlović¹, Branko Malešević²

¹) Faculty of Mathematics, University of Belgrade, Studentski trg 16, 11000 Belgrade, Serbia and Montenegro
²) Faculty of Electrical Engineering, University of Belgrade, Bulevar Kralja Aleksandra 73, 11000 Belgrade, Serbia and Montenegro

Abstract

In this paper we consider some analytical relations between gamma function \( \Gamma(z) \) and related functions such as the Kurepa's function \( K(z) \) and alternating Kurepa's function \( A(z) \). It is well-known in the physics that the Casimir energy is defined by the principal part of the Riemann function \( \zeta(z) \) (Blau, Visser, Wipf; Elizalde). Analogously, we consider the principal parts for functions \( \Gamma(z) \), \( K(z) \), \( A(z) \) and we also define and consider the principal part for arbitrary meromorphic functions. Next, in this paper we consider some differential-algebraic (d.a.) properties of functions \( \Gamma(z) \), \( \zeta(z) \), \( K(z) \), \( A(z) \). As it is well-known (Hölder; Ostrowski) \( \Gamma(z) \) is not a solution of any d.a. equation. It appears that this property of \( \Gamma(z) \) is universal. Namely, a large class of solutions of functional differential equations also has that property. Proof of these facts is reduced, by the use of the theory of differential algebraic fields (Ritt; Kaplansky; Kolchin), to the d.a. transcendency of \( \Gamma(z) \).

1 Analytical properties

In this section we consider analytical properties of the gamma and related functions which pertain to the principal part of a function at a point.

1.1 The principal part of the gamma function

Gamma function is defined by the integral:

\[
\Gamma(z) = \int_0^\infty e^{-t}t^{z-1} dt,
\]

¹ Email address: zarkom@eunet.yu
² Email address: malesh@eunet.yu

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which converges for \( Re(z) > 0 \). It is possible to form analytical continuation of this function over the whole set of the complex numbers \( \mathbb{C} \) except at \( z = -k \), where \( k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\} \). One approach to analytical continuation is given by:

\[
\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+z)} + \int_{1}^{\infty} e^{-t}t^{z-1} \, dt,
\]

see [2], [8]. Residue at \( z = -k, k \in \mathbb{N}_0 \), is:

\[
\text{res}_{z=-k} \Gamma(z) = \frac{(-1)^k}{k!}.
\]

It is possible to extend the domain of the gamma function to the set of all complex numbers \( \mathbb{C} \) in the sense of the principal part at a point as follows. For a meromorphic function \( f(z) \), on the basis of Cauchy’s integral formula, we define the principal part at point \( a \) (see: [3], [22]):

\[
p.p. f(z) = \lim_{\rho \to 0^+} \frac{1}{2\pi i} \oint_{|z-a| = \rho} f(z) \, dz.
\]

If the point \( a \) is regular for the function \( f(z) \) then \( p.p. f(z) = f(a) \); otherwise the principal part at the pole \( z = a \) exists as a finite complex number:

\[
p.p. f(z) = \text{res}_{z=a} \left( \frac{f(z)}{z-a} \right),
\]

as cited in [24]. Let us determine basic properties of the principal part at the point. For two meromorphic functions \( f_1(z) \) and \( f_2(z) \) additivity holds [3]:

\[
p.p. \left( f_1(z) + f_2(z) \right) = p.p. f_1(z) + p.p. f_2(z).
\]

In the paper [3] it is proved that multiplicativity of the principal part does not hold. Namely for the principal part the following statement is true.

**Theorem 1.1** Let \( f_1(z) \) be a holomorphic function at the point \( a \) and let \( f_2(z) \) be a meromorphic function with pole of the \( m \)-th order at the same point \( a \). Then:

\[
p.p. \left( f_1(z) \cdot f_2(z) \right) = \sum_{k=0}^{m} \frac{f_1^{(k)}(a)}{k!} \text{p.p.} \left( (z-a)^k \cdot f_2(z) \right).
\]

**Proof**  Let \( f_1(z) \) and \( f_2(z) \) be represented by the series:

\[
f_1(z) = \sum_{i=0}^{\infty} \frac{f_1^{(i)}(a)}{i!} (z-a)^i \quad \text{and} \quad f_2(z) = \sum_{j=-m}^{\infty} c_j (z-a)^j,
\]
for some $c_j \in \mathbb{C}$ ($j \geq -m$), $c_{-m} \neq 0$ and $z \neq a$. Let us notice that p.p. $f_2(z) = c_0$. Multiplying the following series:

$$\frac{f_1(z) - f_1(a)}{z - a} \cdot f_2(z) = \sum_{i=1}^{\infty} \frac{f_1^{(i)}(a)}{i!} (z - a)^{i-1} \cdot \sum_{j=-m}^{\infty} c_j (z - a)^j$$

we obtain:

$$\text{res}_{z=a} \left( \frac{f_1(z) - f_1(a)}{z - a} \cdot f_2(z) \right) = \sum_{k=1}^{m} \frac{f_1^{(k)}(a)}{k!} \cdot c_{-k}.$$ 

Hence:

$$\text{p.p.}_{z=a} \left( f_1(z) \cdot f_2(z) \right) = \text{res}_{z=a} \left( \frac{f_1(z) \cdot f_2(z)}{z - a} \right) = \text{res}_{z=a} \left( \frac{f_2(z)}{z - a} \right) \cdot f_1(a) + \text{res}_{z=a} \left( \frac{f_1(z) - f_1(a)}{z - a} \right) \cdot f_2(z) = f_1(a) \cdot c_0 + \sum_{k=1}^{m} \frac{f_1^{(k)}(a)}{k!} \cdot c_{-k} = \sum_{k=0}^{m} \frac{f_1^{(k)}(a)}{k!} \cdot \text{p.p.}_{z=a} \left( (z - a)^k \cdot f_2(z) \right). \tag{11}$$

**Remark 1.2** The phrase "function is holomorphic at the point $a" means not just function is differentiable at $a$, but differentiable everywhere within some open disk centered at $a$ in the complex plane.

**Corollary 1.3** Let $f_1(z)$ be a holomorphic function at the point $a$ and let $f_2(z)$ be a meromorphic function with simple pole at the same point $a$. Then:

$$\text{p.p.}_{z=a} \left( f_1(z) \cdot f_2(z) \right) = f_1(a) \cdot \text{p.p.}_{z=a} f_2(z) + f_1'(a) \cdot \text{res}_{z=a} f_2(z). \tag{12}$$

The previous formula, in the case of the zeta function $f_2(z) = \zeta(z)$, is also given in [15], [16].

For meromorphic function $f(z)$ with simple pole at the point $z = a$ the following formula is true [22]:

$$\text{p.p.}_{z=a} f(z) = \lim_{\varepsilon \to 0} \frac{f(a - \varepsilon) + f(a + \varepsilon)}{2}. \tag{13}$$

Especially for gamma function $\Gamma(z)$ it is true [3], [22]:

$$\text{p.p.}_{z=-n} \Gamma(z) = (-1)^n \frac{\Gamma'(n + 1)}{\Gamma(n + 1)^2} = -\gamma + \sum_{k=1}^{n} \frac{1}{k}, \tag{14}$$

where $\gamma$ is Euler’s constant and $n \in \mathbb{N}_0$. 

3
1.2 The principal part of the Kurepa’s functions

D. Kurepa introduced in paper [4] function $K(z)$ by integral:

$$K(z) = \int_0^\infty \frac{e^{-t} t^z - 1}{t - 1} \, dt,$$

which converges for $\Re(z) > 0$, and it represents one analytical extension of the sum of factorials:

$$K(n) = \sum_{i=0}^{n-1} i!.$$

For the function $K(z)$ we use the term Kurepa’s function and it is one solution of the functional equation:

$$K(z) - K(z - 1) = \Gamma(z).$$

Let us observe that it is possible to make analytical continuation of Kurepa’s function $K(z)$ for $\Re(z) \leq 0$. In that way, the Kurepa’s function $K(z)$ is a meromorphic function with simple poles at $z = -1$ and $z = -n$ ($n \geq 3$). At point $z = -2$ Kurepa’s function has a removable singularity and $K(-2) \overset{\operatorname{def}}{=} \lim_{z \to -2} K(z) = 1$. Kurepa’s function has the following residues:

$$\text{res}_{z=-1} K(z) = -1 \quad \text{and} \quad \text{res}_{z=-n} K(z) = \sum_{k=2}^{n-1} \frac{(-1)^{k-1}}{k!} \quad (n \geq 3).$$

Previous results for Kurepa’s function are given according to [6] and [7]. The functional equation (17), besides Kurepa’s function $K(z)$, has another solution by series:

$$K_1(z) = \sum_{n=0}^\infty \Gamma(z - n),$$

which converges over the set $\mathbb{C}\setminus\mathbb{Z}$ [22].

Extension of domain of functions $K(z)$ and $K_1(z)$ in the sense of the principal part at the point is given by the following statements [7, 22].

**Lemma 1.4** Let us define $L_1 = -\sum_{n=0}^{\infty} \gamma + \sum_{n=1}^{\infty} \frac{1}{n!n} \overset{\operatorname{def}}{=} \frac{\text{Ei}(1)}{e} \approx 0.697174883$,

where Ei is function of exponential integral.
Theorem 1.5 For the functions $K(z)$ and $K_1(z)$ are true:

\begin{align}
\text{p.p. } K(z) &= -\sum_{i=0}^{n-1} \text{p.p. } \Gamma(z) = \sum_{i=0}^{n-1} (-1)^{i+1} \frac{\Gamma(i+1)}{\Gamma(i+1)^2} \quad (n \in \mathbb{N}) \\
\text{and}
\text{p.p. } K_1(z) &= \text{p.p. } K(z) - L_1 \quad (n \in \mathbb{Z}).
\end{align}

The connection between functions $K(z)$ and $K_1(z)$ is given by Slavič’s formula which is presented in the following statement [7], [12], [22].

Theorem 1.6 It is true:

\begin{align}
K(z) &= \frac{1}{e} \left(\gamma + \sum_{n=1}^{\infty} \frac{1}{n!n}\right) - \pi e \cot \pi z + \sum_{n=0}^{\infty} \Gamma(z-n),
\end{align}

where the values in the previous formula, in integer points $z$, are determined in the sense of the principal part.

Analogously to Kurepa’s function we consider the function $A(z)$ given by the integral:

\begin{align}
A(z) &= \int_0^\infty e^{-t} \frac{t^{z+1} - (-1)^{z+1}}{t+1} dt,
\end{align}

which converges for $\Re(z) > 0$ [21], and it represents one analytical extension of the alternating sum of factorials:

\begin{align}
A(n) &= \sum_{i=1}^{n} (-1)^{n-i} i!.
\end{align}

For the function $A(z)$ we use term alternating Kurepa’s function and it is one solution of the functional equation:

\begin{align}
A(z) + A(z-1) = \Gamma(z+1).
\end{align}

Let us observe that it is possible to make analytical continuation of alternating Kurepa’s function $A(z)$ for $\Re(z) \leq 0$. In that way, the alternating Kurepa’s function $A(z)$ is a meromorphic function with simple poles at $z = -n \ (n \geq 2)$. Alternating Kurepa’s function has the following residues:

\begin{align}
\text{res}_{z=-n} A(z) &= (-1)^n \sum_{k=0}^{n-2} \frac{1}{k!} \quad (n \geq 2).
\end{align}

Previous results for alternating Kurepa’s function are given according to [21].

The functional equation (26), besides alternating Kurepa’s function $A(z)$, has another solution by series:

\begin{align}
A_1(z) &= \sum_{n=0}^{\infty} (-1)^n \Gamma(z+1-n),
\end{align}

which converges over the set $\mathbb{C} \setminus \mathbb{Z}$ [24].
Extension of domain of functions $A(z)$ and $A_1(z)$ in the sense of the principal part at the point is given by following statements [24].

**Lemma 1.7** Let us define $L_2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!n} \text{p.p.} \Gamma(z)$, then:

\[
L_2 = 1 + e\gamma - e\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!n}\right) = 1 + e\text{Ei}(-1) \approx 0.403652337, \tag{29}
\]

where $\text{Ei}$ is function of exponential integral.

**Theorem 1.8** For the functions $A(z)$ and $A_1(z)$ we have:

\[
\begin{align*}
\text{p.p.} A(z) &= \sum_{i=0}^{n-1} (-1)^{n+1-i} \text{p.p.} \Gamma(z) = (-1)^{n+1} \left(1 - \sum_{i=1}^{n-1} \frac{\Gamma'(i)}{\Gamma(i)^2}\right) \quad (n \in \mathbb{N}) \\
\text{and} \\
\text{p.p.} A_1(z) &= (-1)^{n} L_2 + \text{p.p.} A(z) \quad (n \in \mathbb{Z}).
\end{align*}
\tag{30}
\]

The connection between functions $A(z)$ and $A_1(z)$ is given by a formula of the Slavić's type in the following statement [24].

**Theorem 1.9** It is true that:

\[
A(z) = \left(e\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!n} - 1 - e\gamma\right)(-1)^z + \frac{\pi e}{\sin \pi z} + \sum_{n=0}^{\infty} (-1)^n \Gamma(z+1-n), \tag{32}
\]

where the values in the previous formula, in integer points $z$, are determined in the sense of the principal part.

**1.3 Principal part of the zeta function and Casimir energy**

Riemann zeta function $\zeta(s)$ is a meromorphic function and it has only simple pole at $s = 1$ with the principal part:

\[
\begin{align*}
\text{p.p.} \zeta(s) &= \gamma, \quad (s = 1) \tag{33}
\end{align*}
\]

as cited in [3]. We consider the principal part of the global spectral zeta function $\zeta_L(s)$ which is a direct extension of the Riemann zeta function $\zeta(s)$ [24]. Namely, let $L$ be an elliptic differential operator of the second order acting only on the variable $x$ and let $\varphi(t, x) = e^{\pm i \omega t} \varphi_n(x)$ be a solution of the following equation:

\[
\left(L + \frac{\partial^2}{c^2 \partial t^2}\right) \varphi(t, x) = 0, \tag{34}
\]

\[a^3\text{ see Example 2.5. in this paper}\]
where $\omega$ and $c$ are constants. Let scalars $\lambda_n$ fulfill $L\varphi_n(x) = \lambda_n\varphi_n(x)$. Then we define global spectral zeta function by [9], [17], [23]:

$$\zeta_L(s) = \sum_n \lambda_n^{-s}. \tag{35}$$

Casimir energy of the field $\varphi(t, x)$ is defined by [16], [23]:

$$E_0 = \frac{1}{2} \lim_{\varepsilon \to 0} \zeta_L(-\frac{1}{2} + \varepsilon) + \zeta_L(-\frac{1}{2} - \varepsilon) = \frac{1}{2} \text{p.p.} \zeta_L(s). \tag{36}$$

In paper [23] some values of Casimir energy have been given, dependent of the fields which are considered. All computations in [23], based on paper [16], are related to the global spectral zeta functions with, as a rule, simple poles.

### 2 Differential - algebraic properties

In this section we present a method for proving that certain analytic functions are not solutions of algebraic differential equations. The method is based on model-theoretic properties of differential fields and that $\Gamma(x)$ is a transcendental differential function.

#### 2.1 Differential fields

The theory $DF_0$ of differential fields of characteristic 0 is the theory of fields with following axioms that relate to the derivative $D$:

$$D(x + y) = Dx + Dy, \quad D(xy) = xDy + yDx. \tag{37}$$

Thus, a model of $DF_0$ is a differential field $K = (K, +, \cdot, D, 0, 1)$ where $(K, +, \cdot, 0, 1)$ is a field and $D$ is a differential operator satisfying the above axioms. Abraham Robinson proved that $DF_0$ has a model completion, and then defined $DCF_0$ to be the model completion of $DF_0$. Afterwards Leonore Blum found simple axioms of $DCF_0$ not mentioning of differential polynomials in more than one variable [5]. In the following, if not otherwise stated, $F, K, L, \ldots$ will denote differential fields, $F, L, K, \ldots$ their domains while $F^*, K^*, L^*, \ldots$ will denote their field parts, i.e. $F^* = (F, +, \cdot, 0, 1)$. It is customary to denote by $L(X)$ the ring of differential polynomials over $L$ in the variable $X$, see [20]. Thus, if $f \in L\{X\}$ then for some natural number $n$, $f = f(X, DX, D^2X, \ldots, D^nX)$ where $f(x, y_1, y_2, \ldots, y_n)$ is the ordinary algebraic polynomial over $L^*$. Then the order of $f$, denoted by $\text{ord } f$, is the largest $n$ such that $D^nX$ occurs in $f$. If $f \in L$ we put $\text{ord } f = -1$ and then we write $f(a) = f$ for each $a$. For $f \in L\{X\}$ we shall write occasionally $f'$ instead of $Df$, and $f(a)$ instead of $f(a, Da, D^2a, \ldots, D^na)$ for each $a$ and $n = \text{ord } f$.

If $b \in K$, then $L(b)$ will denote the simple differential extension of $L$ in $K$, i.e. $L(b)$ is the smallest differential subfield of $K$ containing both $L$ and $b$. Also, we shall use the following abbreviations:
d.p. is standing for differential polynomial. Thus, $f$ is a d.p. over $L$ in the variable $X$ if and only if $f \in L\{X\}$.

d.a. is standing for differential algebraic. Hence, if $L \subseteq K$ then $b \in K$ is d.a. over $L$ if and only if there is a non-zero d.p. $f$ such that $f(b, Db, D^2b, \ldots, D^n b) = 0$, otherwise $b$ is transcendental. The field $K$ is a d.a. extension of $L$ if every $b \in K$ is d.a. over $L$.

d.e. is standing for differential equation, a.d.e. is standing for algebraic differential equation. Hence, $f = 0$ is a.d.e. if $f \in L\{X\}$.

Models of DCF$_0$ are differentially closed fields. A differential field $K$ is differentially closed if, whenever $f, g \in K\{X\}$, $g$ is non-zero and ord $f >$ ord $g$, there is $a \in K$ such that $f(a) = 0$ and $g(a) \neq 0$. The theory DCF$_0$ admits elimination of quantifiers and it is submodel complete (A. Robinson): if $F \subseteq L, K$ then $K_F \equiv L_F$, i.e. $K$ and $L$ are elementary equivalent over $L$. In the following we shall use the next theorem, see [25]:

**Theorem 2.1** Suppose $F \subseteq K$ and let $L = \{b \in K: b$ is d.a. over $F\}$. Then

a. $L$ is a differential subfield of $K$ extending $F$.

b. If $K$ is d.a. closed then $L$ is d.a. closed.

Other notations, notions and results concerning differential fields that will be used corresponds to those in [3] or [20].

### 2.2 Transcendental differential functions

Suppose $L \subseteq K$. Let $R = R(x)$ be the differential field of real rational functions and $C = C(z)$ the differential field of complex rational functions. The following H"{o}lder’s famous theorem asserts the differential transcendentality of Gamma function.

**Theorem 2.2** a. $\Gamma(x)$ is not d.a. over $R(x)$. b. $\Gamma(z)$ is not d.a. over $C(z)$.

Now we shall use the transcendentality of $\Gamma(z)$ and properties of differential fields to prove differential transcendentality over $C$ of some analytic functions. Let us denote by $M_D$ the class of complex functions meromorphic on a complex domain $D$ (a connected open set in the complex $z$-plane $C$). If $D = C$ then we shall write $M$ instead of $M_D$. Then $M_D$ is differential field and $C \subseteq M$. Further, let $L = \{f \in M: f$ d.a. over $C\}$. By Theorem 2.1 $L$ is a differential subfield of $M$ extending $C$. The function $\Gamma(z)$ is meromorphic and by H"{o}lder’s theorem $\Gamma(z) \notin L$.

**Example 2.3** As we have seen in the first part, Kurepa’s function $K(z)$ can be continued meromorphically to whole complex plane. Therefore, $K(z - 1)$ is meromorphic either. Also, as we have seen in the first part, Kurepa’s function satisfies the recurrence relation

\[(38) \quad K(z) - K(z - 1) = \Gamma(z),\]
Now, suppose that $K(z)$ belongs to $\mathcal{L}$. Then $K(z)$ satisfies an a.d.e.

\begin{equation}
\tag{39}
f(z, y, Dy, D^2y, \ldots, D^n y) = 0
\end{equation}

where $f(z, y_1, y_2, \ldots, y_n) \in \mathcal{C}[x, y_1, y_2, \ldots, y_n]$. Then $K(z+1)$ satisfies the a.d.e.

\begin{equation}
\tag{40}
f(z + 1, y, Dy, D^2y, \ldots, D^n y) = 0
\end{equation}

so $K(z+1)$ belongs to $\mathcal{L}$. As $\mathcal{L}$ is a field, by (38) it follows that $\Gamma(z)$ belongs to $\mathcal{L}$, what yields a contradiction. Hence, $K(z)$ is a transcendental differential function.

Using previous method we can conclude that each meromorphic solution of a functional equation (38) is transcendental differential function over the field $\mathcal{C}$. For example, another solution of this functional equation is series (19).

Therefore, $K_1(z)$ is a transcendental differential function too.

In a similar way, one can prove that alternating functions $A(z)$ and $A_1(z)$ are differentially transcendental too over $\mathcal{C}$.

The following general proposition concerning transcendental differential functions holds.

**Theorem 2.4** Let $a(z)$ be a meromorphic differentially transcendental function over $\mathcal{C}$ and $f(z, u_0, u_1, \ldots, u_m, y_1, \ldots, y_n) = a(z)$, where $E_i f(z) = \alpha_i z + \beta_i, \alpha_i, \beta_i \in \mathcal{C}$, then $b$ is differentially transcendental over $\mathcal{C}$.

**Proof** Suppose that $b$ is d.a. over $\mathcal{C}$, i.e. that $b \in \mathcal{L}$. Then $Db, \ldots, D^n b$ belong to $\mathcal{L}$. Further, there is a.d.e. $f(z, y, Dy, \ldots, D^k y) = 0$ satisfied by $b$, so $E_i b$ satisfies

\begin{equation}
\tag{41}
f(\alpha_i z + \beta_i, y, \alpha_i^{-1} Dy, \ldots, \alpha_k^{-1} D^k y) = 0,
\end{equation}

i.e. $E_i b \in \mathcal{L}$, too. Therefore, $g(z, b, E_1 b, \ldots, E_m b, Db, \ldots, D^n b) \in \mathcal{L}$, so $a(z)$ belongs to $\mathcal{L}$, a contradiction.

**Example 2.5** The Riemann zeta function defined by

\begin{equation}
\tag{42}
\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}, \quad \text{Re}(s) > 1,
\end{equation}

is differentially transcendental over $\mathcal{C}$ (Hilbert). First we observe that $\zeta(s)$ can be continued meromorphically to whole complex plane with a simple pole at $s = 1$ and that $\zeta(s)$ satisfies the well-known functional equation (13):

\begin{equation}
\tag{43}
\zeta(s) = \chi(s)\zeta(1-s), \quad \text{where} \quad \chi(s) = \frac{(2\pi)^s}{2\Gamma(s) \cos\left(\frac{\pi s}{2}\right)}.
\end{equation}

Now, suppose that $\zeta(s)$ is d.a. over $\mathcal{C}$, i.e. that $\zeta(s) \in \mathcal{L}$. Then $\chi(1-s)$ and $\zeta(s)/\zeta(1-s)$ belong to $\mathcal{L}$, too, so $\chi(s)$ belongs to $\mathcal{L}$. The elementary functions...
(2\pi)^s$, and \(\cos(\frac{\pi}{2}s)\) obviously are d.a. over \(C\) i.e. they belong to \(L\). As \(L\) is a field, it follows that \(\Gamma(z)\) belong to \(L\), too. But this yield a contradiction, therefore \(\zeta(s)\) is differentially transcendental function over \(C\). Generally, Dirichlet \(L\)-series

\[
L_k(s) = \sum_{n=1}^{\infty} \kappa_k(n) \frac{1}{n^s} \quad (k \in \mathbb{Z}),
\]

where \(\kappa_k(n)\) is Dirichlet character \([11]\), is differentially transcendental function over \(C\). This follows from a well-known functional equations

\[
L_{-k}(s) = 2^s\pi^{s-1}k^{-s+\frac{1}{2}}\Gamma(1-s)\cos\left(\frac{\pi s}{2}\right)L_{-k}(1-s)
\]

and

\[
L_{+k}(s) = 2^s\pi^{s-1}k^{-s+\frac{1}{2}}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)L_{+k}(1-s).
\]

Besides Riemann zeta function \(\zeta(s) = L_{+1}(s)\), Dirichlet eta function

\[
\eta(s) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^s} = (1 - 2^{1-s})L_{+1}(s)
\]

and Dirichlet beta function

\[
\beta(s) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)^s} = L_{-4}(s)
\]

are transcendental differential functions as examples of Dirichlet series \([14]\). ■

**Example 2.6** The meromorphic function

\[
H_1(z) = \sum_{n=0}^{\infty} \frac{1}{(n + z)^2}
\]

is differentially transcendental over \(C\). Really, \(D^2 \ln(\Gamma(z)) = H_1(z)\), i.e. \(\Gamma(z)\) satisfies the a.d.e. \((D^2\Gamma)\Gamma-(D\Gamma)^2-H_1\Gamma^2 = 0\) over \(C(H_1)\). Thus, if \(H_1\) would be d.a. over \(C\), then by Theorem 2.4 \(\Gamma\) would be too, what yields a contradiction. Hence, \(H_1(z)\) is differentially transcendental function over \(C\). **Remark 2.7** We see that in Example 2.6 functions \(\Gamma(z)\) and \(H_1(z)\) are differentially algebraically dependent, i.e. \(g(\Gamma, H_1) = 0\), where \(g(x, y) = x''x-(x')^2-xy^2\). We do not know if similar dependencies exist for pairs \((K, \Gamma)\) and \((\zeta, \Gamma)\). It is very likely that these pairs are in fact differentially transcendental.
2.3 Operator \( \delta \)

Let \( F_D = (F, +, \cdot, D, 0, 1) \) be a differential field and \( \theta \in F \). We can introduce a new differential operator \( \delta = \theta \cdot D \), i.e. by \( \delta(x) = \theta \cdot D(x) \), \( x \in F \). Then \( F_\delta = (F, +, \cdot, \delta, 0, 1) \) becomes a new differential filed. Let \( \overline{F}_D \) and \( \overline{F}_\delta \) denote differential closures of fields \( F_D \) and \( F_\delta \) respectively.

**Proposition 2.8** Domains of fields \( \overline{F}_D \) and \( \overline{F}_\delta \) are same, i.e. \( \overline{F}_D = \overline{F}_\delta \).

**Proof** If \( a \in \overline{F}_\delta \) then \( a \) is a solution of an a.d.e. \( \mathcal{E}(\delta) \) in respect to the operator \( \delta \). We can substitute in this equation operator \( \delta \) with \( \theta \cdot D \), and we shall obtain again an a.d.e. \( \mathcal{E}'(D) \) but now in respect to \( D \). Then \( a \) is a solution of this equation, hence \( a \in \overline{F}_D \). So we proved that \( \overline{F}_\delta \subseteq \overline{F}_D \). On the other hand, \( D = \theta^{-1} \delta \), so we may apply a symmetrical argument, hence \( a \in \overline{F}_D \) implies \( a \in \overline{F}_\delta \), i.e. \( \overline{F}_D \subseteq \overline{F}_\delta \). Therefore, we proved \( \overline{F}_D = \overline{F}_\delta \). \( \square \)

We can ask the natural question if fields \( \overline{F}_D, \overline{F}_\delta \) are isomorphic. We observe that it is not necessary \( \overline{F}_D \cong \overline{F}_\delta \). For example, if \( F = \mathbb{R}(x) \), \( D \) is the ordinary differentiation operator and \( \delta = xD \), then the equation \( \delta y = y \) has a solution in \( \overline{F}_\delta \), \( y = x \), while the equation \( Dy = y \) has no solution in \( \overline{F}_D \). Hence \( \overline{F}_D \not\cong \overline{F}_\delta \). Let us remind that \( \tau : \overline{F}_\delta \rightarrow \overline{F}_D \) is an isomorphism if \( \tau \) satisfies:

\[
\tau(x + y) = \tau x + \tau y, \quad \tau(xy) = \tau x \tau y, \quad \tau(\delta x) = D\tau(x), \quad \tau(0) = 0, \quad \tau(1) = 1.
\]

Under some circumstances the isomorphism exists between fields \( \overline{F}_D, \overline{F}_\delta \), or between certain intermediate fields. For example, the conditions will be fulfilled if these fields have functional representation and a particular differential equation has a solution. Let \( \mathcal{L}_D = \{ f \in \mathcal{M} : f \text{ d.a. over } C \text{ in respect to the operator } D \} \) and \( \mathcal{L}_\delta = \{ f \in \mathcal{M} : f \text{ d.a. over } C \text{ in respect to the operator } \delta \} \).

**Theorem 2.9** If \( \theta \in \mathcal{L}_D \) is non-constant and \( g \) is a non-constant solution of a.d.e. \( Dx = \theta \cdot x \) then \( \mathcal{L}_D \cong \mathcal{L}_\delta \).

**Proof** First, we observe, using the argument as in the proof of the above proposition, that domains of \( \mathcal{L}_D \) and \( \mathcal{L}_\delta \) are same. Let \( \tau : \mathcal{L}_\delta \rightarrow \mathcal{L}_D \) be defined by \( \tau(x) = x \circ g \), where \( \circ \) is the composition operator. We see that \( g \) is meromorphic and is a.d. over \( C \), therefore \( g \in \mathcal{L}_D \). \( \tau \) is well defined since \( \mathcal{L}_D \) is closed under composition. Obviously it satisfies \( \tau(x + y) = \tau x + \tau y \), \( \tau(xy) = \tau x \tau y \). Further, as \( Dg = \theta \circ g \),

\[
(51) \quad \tau(\delta x) = (\theta Dx) \circ g = (\theta \circ g)((Dx) \circ g) = Dg((Dx) \circ g) = D(x \circ g) = D(\tau x).
\]

\( \tau \) is 1–1 function, since \( g \) takes infinitely many values over a bounded region. Therefore, \( \tau : \mathcal{L}_D \cong \mathcal{L}_\delta \). \( \square \)

In the case of \( \theta = x \), \( x \) here denotes a variable (i.e. the polynomial of the degree one), we can produce an explicit isomorphism \( \tau : \mathcal{L}_D \cong \mathcal{L}_\delta \). We can define \( \tau \) by \( \tau : f \rightarrow f \circ g, \quad f \in \mathcal{L}_D \), where \( g(x) = e^x \). Observe that this isomorphism corresponds to the transformation \( x = e^x \) in the algorithm of solving of Euler
Example 2.10 Solve $x^3y''' + 3x^2y'' - 2xy' + 2y = 0$.

Solution This equation is equivalent to

\[(\delta(\delta - 1)(\delta - 2) + 3\delta(\delta - 1) - 2\delta + 2)y = 0,\]

i.e. to the equation $(\delta^3 - 3\delta + 2)y = 0$ in $\mathcal{L}_\delta$. The corresponding equation $(D^3 - 3D + 2)y = 0$ in $\mathcal{L}_D$ has general solution $c_1h_1 + c_2h_2 + c_3h_3$, where $h_1(x) = e^x$, $h_2(x) = xe^x$, $h_3(x) = e^{-2x}$. As $\tau: \mathcal{L}_D \cong \mathcal{L}_\delta$, and $\tau^{-1}$ is given by $\tau^{-1}: f \to f \circ g^{-1}$ (here $g^{-1}(x) = \ln x$), it follows that

\[
\tau^{-1}(c_1h_1 + c_2h_2 + c_3h_3) = c_1h_1 \circ g^{-1} + c_2h_2 \circ g^{-1} + c_3h_3 \circ g^{-1}
\]

is the general solution of $(\delta^3 - 3\delta + 2)y = 0$ in $\mathcal{L}_\delta$, and so the solution of the starting equation is $y = c_1x + c_2x\ln x + c_3x^{-2}$.

Hence, one should expect that standard methods of solving differential equations which are done by "properly chosen transformations of the independent variable" correspond in fact to constructions of an isomorphism between $\mathcal{L}_D$ and $\mathcal{L}_\delta$, or some other intermediate fields, for properly chosen differential operators $\delta$.

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