ORBITS OF A STRONGLY SOLVABLE SPHERICAL SUBGROUP
ON THE FLAG VARIETY

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ABSTRACT. Let $G$ be a connected reductive complex algebraic group and $B$ a Borel subgroup of $G$. We consider a subgroup $H \subset B$ which acts with finitely many orbits on the flag variety $G/B$, and we classify the $H$-orbits in $G/B$ in terms of suitable combinatorial invariants. As well, we study the Weyl group action defined by Knop on the set of $H$-orbits in $G/B$, and we give a combinatorial model for this action in terms of weight polytopes.

INTRODUCTION

Let $G$ be a connected complex reductive algebraic group and let $B \subset G$ be a Borel subgroup. A subgroup $H \subset G$ is called spherical if it possesses finitely many orbits in the flag variety $G/B$. The best known example is that of $B$ itself (and more generally any parabolic subgroup of $G$), the $B$-orbits in $G/B$ are indeed the Schubert cells and they are finitely many thanks to the Bruhat decomposition. Another well studied situation is that of a symmetric subgroup $H$, i.e. the subgroup of fixed points of some algebraic involution of $G$. Especially in this case, the study of the $H$-orbits in $G/B$, their classification and the geometry of their closures are important in representation theory (see e.g. [18], [25]). An equivalent problem is studying the set $\mathcal{B}(G/H)$ of the $B$-orbits in $G/H$, and the geometry of their closures: they are fundamental objects to understand the topology of $G/H$ and of its embeddings (see [10]).

Spherical subgroups are classified in combinatorial terms, see [14], [5], [19], [15], [17] where several particular classes of subgroups are considered, and the more recent papers [16], [4], [9] where the classification is completed in full generality. Nevertheless the set $\mathcal{B}(G/H)$ is still far from being understood, essentially except for the cases where $H$ is parabolic or symmetric (the latter especially thanks to the work of Richardson and Springer [23], [24]).

The main motivation of the present paper is to explicitly understand the set $\mathcal{B}(G/H)$ in some other case, and to produce some combinatorial model for it. More precisely, we suppose that $H \subset G$ is a spherical subgroup and that it is strongly solvable, i.e. contained in a Borel subgroup of $G$. In this case the $H$-orbit closures in $G/B$ provide nice generalizations of the Schubert varieties; it was proved by Brion (see [7, Theorem 6]) that these varieties are irreducible (even though $H$ might be not connected, this also follows from our Section 1.2) and their singularities are rational.

In our main result we give an explicit parametrization of the set $\mathcal{B}(G/H)$ by attaching to every element of $\mathcal{B}(G/H)$ an element of the Weyl group $W$ of $G$ and a subroot system of the root system of $G$. We build upon known results around the classification of strongly solvable subgroups, which is available in three different forms. The first one has been given by Luna in 1993 (see [16]), the second emerges in the framework of the general classification of spherical subgroups, and a third one more explicit has been given in recent years by Avdeev in [1] (see Avdeev’s survey [2] for a comparison between the three approaches).

Our results on $\mathcal{B}(G/H)$ also provide a nice description of the action, defined by Knop in [12], of the Weyl group $W$ on $\mathcal{B}(G/H)$. This action is defined in great generality for any spherical subgroup $H \subset G$, but while the simple reflections of $W$ act in a rather explicit and simple way, proving that they glue together to an action of $W$ is quite complicated. In case $H$ is strongly solvable, we will see how this action becomes actually very simple: the fact that two $B$-orbits are $W$-conjugated will boil down to the fact that the associated subroot systems are $W$-conjugated. This will enable us to give another combinatorial model for $\mathcal{B}(G/H)$ as a $W$-set in terms of faces of weight polytopes.

We explain now with some more details our results. Fix a maximal torus $T \subset B$ and denote by $U$ the unipotent radical of $B$. We also denote by $\Phi$ the root system associated with $T$, by $\Delta \subset \Phi^+$ the basis of $\Phi$ associated with $B$ and by $W$ the Weyl group of $\Phi$. Let $H \subset B$ be a spherical subgroup of $G$. Up to
conjugating $H$ in $B$, we may assume that $T \cap H$ is a maximal diagonalizable subgroup of $H$ and $U \cap H$ is the unipotent radical of $H$. Given $\alpha \in \Phi^+$, let $U_\alpha \subset U$ be the associated unipotent one dimensional subgroup. A natural set to attach to $H$ is the set of active roots

$$\Psi = \{ \alpha \in \Phi^+ \mid U_\alpha \not\subset H \},$$

first considered by Avdeev [1]. A strongly solvable spherical subgroup is indeed uniquely determined, up to $T$-conjugation, by the subgroup $T \cap H$ and by its set of active roots $\Psi$ (see [1]).

Denote $\mathcal{P}^* = \{ \alpha \in T \cap H \mid \alpha \in \Psi \}$. From a geometrical point of view $\mathcal{P}^*$ is canonically identified with the set of $B$-stable prime divisors in $G/H$ mapped dominantly onto $G/B$ via the natural projection (see Proposition 2.5). Given $I \subset \mathcal{P}^*$, let $\Psi_I$ be the set of active roots whose restrictions are in $I$ and let

$$\Phi_I = \mathbb{Z}\Psi_I \cap \Phi.$$

This is a subroot system of $\Phi$, and we denote by $W_I \subset W$ the Weyl group of $\Phi_I$. It turns out that $\Phi_I$ is always a parabolic subroot system, although not necessarily with respect to the set of simple roots $\Delta$ (Corollary 5.10). Denote $\Phi_I^* = \Phi_I \cap \Phi^+$. Given $w \in W$ and $I \subset \mathcal{P}^*$, we say that $(w, I)$ is a reduced pair if $w(\Phi_I^*) \subset \Phi_I^*$. To understand how reduced pairs are related to $B$-orbits on $G/H$, the basic remark is that there is a natural bijection between subsets $I \subset \mathcal{P}^*$ and $T$-orbits $\mathcal{U}_I \subset B/H$. Indeed by the Bruhat decomposition any $B$-orbit is of the shape $Bw\mathcal{U}$ for some $w \in W$ and $x \in B/H$. Hence we also have $Bw\mathcal{U} = Bw\mathcal{U}_I$, where $\mathcal{U} = Tx$ is a $T$-orbit of $B/H$. On the other hand $B/H \to T/T \cap H$ is a vector bundle with fiber $U/U \cap H$, and $H$ is spherical if and only if $B/H$ contains an open $T$-orbit, and if and only if $U/U \cap H$ contains an open $(T \cap H)$-orbit. Therefore the $T$-stable prime divisors of $B/H$ correspond then to the $(T \cap H)$-stable prime divisors of $U/U \cap H$, which are easily seen to correspond to the elements of $\mathcal{P}^*$. Now, it is possible to choose a canonical representative $\mathcal{U}_I$ for any $B$-orbit $Bw\mathcal{U}_I$. Indeed, notice that given $w \in W$ and $I \subset \mathcal{P}^*$, we have $Bw\mathcal{U}_I = Bw\mathcal{U}_J$ if and only if $\mathcal{U}_J \subset (B \cap w^{-1}B)\mathcal{U}_I$. Then there exist a unique minimal and a unique maximal $J$ such that $\mathcal{U}_J \subset (B \cap w^{-1}B)\mathcal{U}_I$, and the condition that $(w, I)$ is a reduced pair translates into the fact that $\mathcal{U}_I$ is the minimal $T$-orbit in $(B \cap w^{-1}B)\mathcal{U}_I$ (See Theorem 4.3).

We come to our main results, summarized in the following theorem.

**Theorem 1** (see Corollary 4.6 Theorems 5.3 and 5.5). Let $H$ be a strongly solvable spherical subgroup. Then the map $(w, I) \mapsto Bu\mathcal{U}_I$ is a bijection between the set of reduced pairs and $\mathcal{B}(G/H)$. Denote $\mathcal{U}_{w, I} = Bu\mathcal{U}_I \in \mathcal{B}(G/H)$ and let $W$ act on $\mathcal{B}(G/H)$ via Knop’s action, then we have $\text{Stab}_W(\mathcal{U}_{w, I}) = wW_Iw^{-1}$. Moreover two orbits $\mathcal{U}_{w, I}$ and $\mathcal{U}_{v, J}$ are $W$-conjugated if and only if $I = J$, in which case we have $\mathcal{U}_{w, I} = w_{\mathcal{U}_{w, I}} = w_{\mathcal{U}_{v, J}}$.

Denote $U'$ the derived group of $U$, namely $U' = \prod_{\alpha \in \Delta} U_\alpha$, and consider the subgroup $TU'$: this is indeed a spherical subgroup which satisfies $\mathcal{P}^* = \Psi = \Delta$. In this case, the natural order on $\mathcal{B}(G/TU')$ given by the inclusion of orbit closures was studied by Timashev [26]. Using the commutation relations among root subgroups, the set $\mathcal{B}(G/TU')$ is easily seen to be in bijection with the set of faces of the weight polytope associated with a dominant regular weight of $G$, and this bijection is $W$-equivariant.

More precisely, let $\rho$ be the half-sum of the positive roots and let $P = \text{conv}(W\rho)$ the associated weight polytope in the rational vector space $\Lambda = \mathbb{Q} \otimes \Lambda$, where $\Lambda$ is the weight lattice of $G$. Since $\rho$ is a regular weight, the vertex set of $P$ is in bijection with $W$. For every $w \in W$ and for every $I \subset \Delta$ such that $w(I) \subset \Phi^+$ we define $\mathcal{F}_{w, I}$ as the face of $P$ generated in the vertex $w\rho$ by the extremal rays corresponding to $-w(I)$, and the map $(w, I) \mapsto \mathcal{F}_{w, I}$ induces a $W$-equivariant bijection between $\mathcal{B}(G/TU')$ and $\mathcal{F}(P)$.

The previous situation generalizes as follows. By a subpolytope of $P$ we mean the convex hull of a subset of vertices of $P$. Given a reduced pair $(w, I)$, we may define the subpolytope $\mathcal{F}_{w, I} = \text{conv}(wW_I\rho)$ obtained by intersecting $P$ in $w\rho$ with the subspace generated by the set of positive roots $-w(\Phi_I^*)$.

**Theorem 2** (see Theorem 5.4). Let $H$ be a strongly solvable spherical subgroup. The map $(w, I) \mapsto \mathcal{F}_{w, I}$ defines a $W$-equivariant embedding $\mathcal{F} : \mathcal{B}(G/H) \to \mathcal{F}(P)$, where $\mathcal{F}(P)$ denotes the set of the subpolytopes of $P$.

In case $H = TU'$ then is easy to see that $\mathcal{F}(\mathcal{B}(G/H)) = \mathcal{F}(P)$ coincides with the set of faces of $P$. In the general case the image of the map $\mathcal{F}$ will contains also subpolytopes that are not faces, however it is also possible to embed $W$-equivariantly $\mathcal{B}(G/H)$ into $\mathcal{F}(P)$ (see Proposition 5.10). This is essentially a consequence of the fact that $\Phi_I$ is conjugated to a parabolic subroot system of $\Phi$. In particular we get the following corollary which confirms, for strongly solvable subgroups, a conjecture of Knop on $G/TU'$. 


Corollary 1 (see Proposition 5.10). The spherical homogeneous space $G/TU'$ has maximal number of $B$-orbits among the homogeneous spaces $G/H$ with $H$ a strongly solvable spherical subgroup of $G$.

Even though $\mathcal{F}(P)$ is much smaller than $\mathcal{F}(G)$, the injection of Theorem 2 is preferable to an injection of the form $\mathcal{B}(G/H) \hookrightarrow \mathcal{F}(P)$. Indeed the first one is compatible with the Bruhat order, i.e. the inclusion relation between $B$-orbit closures, in the sense that if $(v, J)$ and $(w, I)$ are reduced couples with $\mathcal{J}_{v, J} \subset \mathcal{J}_{w, I}$, than we also have $\mathcal{O}_{v, J} \subset \mathcal{O}_{w, I}$ (see Proposition 5.12).

We point out that the converse of the above statement is false in general. A complete description of the Bruhat order based on our combinatorial models for $\mathcal{B}(G/H)$ is still a work in progress.

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1. Notations and preliminaries

In the paper all varieties and algebraic groups are defined over the base field of complex numbers. Let $G$ be a connected reductive algebraic group. Given a subgroup $K \subset G$, we denote by $\mathcal{X}(K)$ the group of characters of $K$ and by $\mathcal{K}$ the unipotent radical of $K$. The Lie algebra of $K$ will be denoted either by Lie $K$ or by the corresponding fraktur letter (here $g$). Given $g \in G$, we set $K^g = g^{-1}Kg$ and $\varphi_K = gKg^{-1}$. If $K$ acts on an algebraic variety $X$, we denote by Div$^K(X)$ the set of $K$-invariant prime divisors of $X$.

Let $B$ be a connected solvable group and let $T \subset B$ be a maximal torus. Given a $B$-variety $Z$, its weight lattice is the free lattice

$$\mathcal{X}_B(Z) = \{B\text{-weights of rational } B\text{-eigenfunctions } f \in \mathbb{C}(Z)^{(B)}\},$$

and the rank $Z$ is defined as the rank of $\mathcal{X}_B(Z)$. When the acting group is clear from the context, we will drop the subscript and write simply $\mathcal{X}(Z)$.

Given a $B$-homogeneous variety $Z$, we say that a base point $z \in Z$ is standard (with respect to $T$) if $\text{Stab}_T(z)$ is a maximal diagonalizable subgroup of $\text{Stab}_B(z)$. Notice that standard base points always exist, and that if $z \in Z$ is standard then every $z' \in Tz$ is standard as well. The proof of the following lemma is immediate.

Lemma 1.1. Let $Z$ be a $B$-homogeneous variety and let $z_0 \in Z$ be a standard base point, then $Tz_0$ is a closed $T$-orbit. If moreover $H = \text{Stab}_B(z_0)$, then $H = (T \cap H)H^u$ and the following equalities hold:

$$\mathcal{X}_B(Z) = \mathcal{X}(B)^{B \cap H} = \mathcal{X}_T(Tz_0)$$

(where $\mathcal{X}(B)^{B \cap H}$ denotes the subgroup of $\mathcal{X}(B)$ of characters that are trivial on $B \cap H$).

From now on $B$ will denote a Borel subgroup of $G$, as before $T \subset B$ denotes a maximal torus, and $U = B^u$ the unipotent radical of $B$. We also denote by $B^-$ the opposite Borel subgroup of $B$ with respect to $T$ (i.e. $T = B \cap B^-$).

Let $\Phi$ be the root system of $G$ associated with $T$, let $\Delta \subset \Phi$ be the set of simple roots associated with $B$ and $\Phi^+$ (resp. $\Phi^-$) the corresponding set of positive (resp. negative) roots. Let $W = NG(T)/T$ be the Weyl group of $G$ with respect to $T$, and let $w_0$ be its longest element associated with $B$.

If $\alpha \in \Phi$, we denote by $s_\alpha \in W$ the corresponding reflection and by $U_\alpha \subset G$ the unipotent root subgroup associated with $\alpha$. If $\alpha \in \Delta$, we denote by $P_\alpha \supset B$ the minimal parabolic subgroup associated with $\alpha$. If $w \in W$, we denote by $\Phi^+(w)$ the corresponding inversion set, namely

$$\Phi^+(w) = \{\alpha \in \Phi^+ | w(\alpha) \in \Phi^-\}.$$

From now on, if not differently stated, $H$ will denote a strongly solvable spherical subgroup of $G$, namely $H \subset G$ is a spherical subgroup contained in a Borel subgroup of $G$. Up to conjugation we may assume that $H \subset B$, that $T \cap H$ is a maximal diagonalizable subgroup of $H$ and that $H^u = U \cap H$. By Lemma 1.3 we have that $T/H \subset B/H$ is a closed $T$-orbit and $\mathcal{X}_B(B/H) = \mathcal{X}_T(T \cap H)$, in particular we get the equality rk $B/H = \text{rk } G - \text{rk } H$.

Let $\mathcal{B}(G/H)$ be the set of $B$-orbits in $G/H$. The set $\mathcal{B}(G/H)$ is naturally identified with the set of $B \times H$-orbits in $G$ and with the set of orbits of the right action of $H$ on $B \backslash G$. In case $H = B$, then we get the set of Schubert cells, which is in bijection with the Weyl group.

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1.1. The actions of $W$ and $M(W)$ on $\mathcal{B}(G/H)$. We denote by $M(W)$ the Richardson-Springer monoid, namely the monoid generated by the symbols $m(s_\alpha)$ with $\alpha \in \Delta$ with the relations $m(s_\alpha)^2 = m(s_\alpha)$ for all $\alpha \in \Delta$ and with the braid relations for all $\alpha, \beta \in \Delta$. As a set $M(W)$ is identified with $W$, and given $w \in W$ we will denote by $m(w) \in M(W)$ the corresponding element. Hence we may consider the Richardson-Springer monoid as the Weyl group with a different multiplication, defined by the following rule: if $w \in W$ and $\alpha \in \Delta$, then

$$m(s_\alpha)m(w) = \begin{cases} m(s_\alpha w) & \text{if } l(s_\alpha w) > l(w) \\ m(w) & \text{if } l(s_\alpha w) < l(w) \end{cases}$$

From a geometrical point of view, the multiplication on $M(W)$ coincides with the one defined on the Weyl group by the multiplication of Bruhat cells, namely we have $BwBw'B = Bu'Bw'$, where $w' \in W$ is the element defined by the equality $m(w') = m(w)m(w')$. Identifying $M(W)$ with $W$, we will also denote the multiplication in $M(W)$ as $w' = w * w'$.

Both $W$ and $M(W)$ act on $\mathcal{B}(G/H)$. The action of $M(W)$ on $\mathcal{B}(G/H)$ was defined by Richardson and Springer \[23\] in the case of a symmetric homogeneous space, and the definition carries over without modifications to the spherical case. Let $\alpha \in \Delta$ and $\theta \in \mathcal{B}(G/H)$, then $P_\alpha \theta$ is irreducible and decomposes into finitely many $B$-orbits. The $B$-orbit $m(s_\alpha) \cdot \theta$ is defined as the unique open $B$-orbit in $P_\alpha \theta$, and this definition extends to an action of $M(W)$ on $\mathcal{B}(G/H)$. This enables us to define the weak order on $\mathcal{B}(G/H)$, namely the partial order defined as follows:

$$\theta \preceq \theta' \text{ if and only if } \exists w \in W : \theta' = m(w) \cdot \theta$$

The action of $W$ on $\mathcal{B}(G/H)$ is much more subtle and was defined by Knop \[12\]. We recall the definition of these actions in the case of a spherical subgroup $H \subset B$, where the involved case-by-case considerations turn out to be easier. By \[12\] Lemma 3.2 together with \[7\] Lemma 5 iv] we have that $P_\alpha \theta$ is either the union of two distinct $B$-orbits or the union of three distinct $B$-orbits. More precisely we have the following possibilities:

(U) If $P_\alpha \theta = \theta \cup \theta'$, then we define $s_\alpha \cdot \theta = \theta'$. Moreover it holds $\mathcal{X}(\theta') = s_\alpha \mathcal{X}(\theta)$ and, assuming for simplicity that $m(s_\alpha) \cdot \theta = \theta'$, we have that $\dim \theta' = \dim \theta - 1$.

(T1) If $P_\alpha \theta$ is the union of three distinct orbits and if $m(s_m) \cdot \theta = \theta$, then we define $s_\alpha \cdot \theta = \theta$. Moreover, if we denote $P_\alpha \theta = \theta \cup \theta_1 \cup \theta_2$, then we have $\dim \theta_1 = \dim \theta_2 = \dim \theta - 1$ and $s_\alpha \mathcal{X}(\theta_2) = \mathcal{X}(\theta_1) \subset \mathcal{X}(\theta) = s_\alpha \mathcal{X}(\theta)$, where $\mathcal{X}(\theta)/\mathcal{X}(\theta_i) \simeq \mathbb{Z}$ for $i = 1, 2$.

(T2) If $P_\alpha \theta$ is the union of three distinct orbits and if $m(s_m) \cdot \theta \neq \theta$, then $P_\alpha \theta = m(s_\alpha) \cdot \theta \cup \theta \cup \theta'$ and we define $s_\alpha \cdot \theta = \theta'$. As in previous case, we have $\dim \theta' = \dim \theta = \dim m(s_\alpha) \cdot \theta - 1$ and $s_\alpha \mathcal{X}(\theta') = \mathcal{X}(m(s_\alpha) \cdot \theta) = s_\alpha \mathcal{X}(m(s_\alpha) \cdot \theta)$, where $\mathcal{X}(m(s_\alpha) \cdot \theta)/\mathcal{X}(\theta) \simeq \mathbb{Z}$.

By \[12\] Theorem 5.9], the $s_\alpha$-actions defined above glue together into an action of $W$ on $\mathcal{B}(G/H)$. It follows from the previous analysis that the rank of a $B$-orbit is invariant for this action.

Example 1.2. If $H = B$ is a Borel subgroup, then the unique possible rank for a $B$-orbit is zero. Indeed every Schubert cell is an orbit for a suitable unipotent group, therefore has no non-constant $B$-equivariant functions. Therefore $P_\alpha \theta$ is of type (U) for every $\alpha \in \Delta$, and $\mathcal{B}(G/B)$ is a homogeneous space under $W$.

In some special cases the rank identifies the $W$-orbit uniquely, in particular when the rank is maximal and minimal. This is summarized in the following theorem, which holds for any spherical subgroup $H$: the first part is due to Knop (see \[12\] Theorem 6.2]), the second part is due to Ressayre (see \[22\] Corollary 3.1 and Theorem 4.2]).

**Theorem 1.3.** Let $\theta \in \mathcal{B}(G/H)$.

i) $\theta$ is in the same $W$-orbit of the open $B$-orbit of $G/H$ if and only if $\dim \theta = \dim G/H$.

ii) $\theta$ is in the same $W$-orbit of any closed $B$-orbit of $G/H$ if and only if $\dim \theta = \dim G - \dim H$.

1.2. Reduction to the wonderful case. We recall some definitions and constructions of the theory of wonderful compactifications of spherical homogeneous spaces. Unless otherwise indicated, we refer for details and proofs to \[17\] and the references therein.

Let $H \subset G$ be a spherical subgroup. Then $G/H$ is called a spherical homogeneous space, and we denote $\mathcal{D} = \text{Div}^B(G/H)$ the (finite) set of its $B$-stable prime divisors, called the colors of $G/H$. If $G/H \hookrightarrow X$ is an equivariant embedding of $G/H$, we denote by $\mathcal{Y}_X = \text{Div}^B(X)$ the set of $G$-stable prime divisors of $X$. Taking closures in $X$, we identify the set of colors of $G/H$ with the set of $B$-stable prime divisors of $X$ that are not $G$-stable, hence we have a disjoint union $\text{Div}^B(X) = \mathcal{D} \cup \mathcal{Y}_X$. Every $B$-stable
prime divisor $D \in \mathcal{D} \cup \mathcal{U}_X$ induces a discrete valuation $\nu_D$ on $\mathbb{C}(G/H)$, trivial on the constant functions. Restricting such valuations on $\mathbb{C}(G/H)^{(B)}$, we get then a map

$$\rho : \mathcal{D} \cup \mathcal{U}_X \rightarrow \text{Hom}_\mathbb{Z}(\mathcal{X}(G/H), \mathbb{Z})$$

defined by $(\rho(D), \chi) = \nu_D(f_\chi)$, where $f_\chi \in \mathbb{C}(G/H)^{(B)}$ is a $B$-semiinvariant function of weight $\chi$.

The spherical subgroup $H$ is called wonderful if $G/H$ admits a wonderful compactification, namely a $G$-equivariant open embedding $G/H \hookrightarrow X$ where $X$ is complete and non-singular and with the following properties:

i) the complement of $G/H$ is the union of finitely many $G$-stable smooth prime divisors (the \emph{boundary prime divisors}),

ii) the boundary prime divisors have transversal intersection,

iii) the $G$-orbit closures in $X$ are exactly the intersections of any subset of the boundary prime divisors (any such intersection is in particular non-empty).

If such a compactification exists, it is known to be unique.

Let $H \subset G$ be a wonderful subgroup, let $X$ be the wonderful compactification of $G/H$ and let $z \in X$ be the unique point fixed by $B^-$. The weights of the $T$-action on $T_zX/T_zGz$ give a basis of the weight lattice $\mathcal{X}(G/H)$. The elements of this basis are called the \emph{spherical roots} of $G/H$. They also parametrize the boundary divisors of $X$ in the following way: for every $D \in \mathcal{U}_X$, there exists $\gamma_D \in \Sigma$ such that

$$\langle \rho(D), \gamma \rangle = \begin{cases} 0 & \text{if } \gamma \neq \gamma_D \\ -1 & \text{if } \gamma = \gamma_D \end{cases}$$

and the map $D \mapsto \gamma_D$ defines a bijection $\mathcal{U}_X \rightarrow \Sigma$. If in addition $H$ is strongly solvable, then we have $\Sigma \subset \Delta$ (see e.g. [2 Corollary 2.39]).

To any spherical subgroup $H \subset G$ one may attach a wonderful subgroup $\overline{H} \subset G$ containing $H$ as follows. The normalizer $N_G(H)$ acts naturally on $G/H$ by $G$-equivariant automorphisms, thus it acts on the set of colors of $G/H$. The spherical closure of $H$, denoted by $\overline{H}$, is defined as the kernel of the latter action, and by a theorem of Knop [13 Corollary 7.6] is a wonderful subgroup of $G$. Notice that we have $Z(G) \subset \overline{H} \subset N_G(H)$.

The purpose of this section is to compare the sets of Borel orbits $\mathcal{B}(G/H)$ and $\mathcal{B}(G/\overline{H})$. First of all, we recall the following facts from [2]:

i) If $H$ is spherical and strongly solvable, then $\overline{H}$ is also strongly solvable (see [2 Corollary 4.25])

ii) If $H$ is wonderful and strongly solvable, then $H$ is \emph{spherically closed}, i.e. $H = \overline{H}$ (see [2 Corollary 3.31]).

\textbf{Lemma 1.4.} Suppose that $G$ is adjoint and let $H \subset G$ be a strongly solvable wonderful subgroup. Then $H$ is connected.

\textbf{Proof.} The connected component $H^0$ of $H$ containing the neutral element is a spherical subgroup of $G$, and the pull-back of rational functions along the projection $G/H^0 \rightarrow G/H$ identifies the weight lattice $\mathcal{X}(G/H)$ with a sublattice of finite index of $\mathcal{X}(G/H^0)$. Since $\Sigma \subset \Delta$ we have that $\mathcal{X}(G/H)$ is a saturated sublattice of the root lattice. Since $G$ is adjoint this implies $\mathcal{X}(G/H) = \mathcal{X}(G/H^0)$, and since $H/H^0 \cong \mathcal{X}(G/H^0)/\mathcal{X}(G/H)$ (see e.g. [11 Lemma 2.4]) it follows $H = H^0$.  

\textbf{Corollary 1.5.} Suppose that $H \subset G$ is spherical and strongly solvable. Then taking images via the projection $G/H \rightarrow G/\overline{H}$ induces a bijection between $\mathcal{B}(G/H)$ and $\mathcal{B}(G/\overline{H})$.

\textbf{Proof.} Denote $H' = HZ(G)$, so that $H \subset H' \subset \overline{H}$. The projection $G/H \rightarrow G/H'$ induces a bijection between $\mathcal{B}(G/H)$ and $\mathcal{B}(G/H')$. Therefore we may replace $H$ with $H'$ and since $Z(G) \subset H'$ we may also assume that $G$ is adjoint. On the other hand the quotient $\overline{H}/H$ is connected by Lemma 1.4 hence the claim follows by [7 Lemma 3].

Corollary 1.5 reduces the study of the $B$-orbits in $G/H$ for $H$ strongly solvable and spherical to the case where $G$ is adjoint and $H$ is wonderful, or equivalently spherically closed.

\section*{2. Fibers over $T$-stable flags}

In this section we describe the set $T$-orbits of $B/H$ and begin to investigate the relationship between this set and the set of $B$-orbits of $G/H$.

From now on $G$ is adjoint and $H$ is a wonderful subgroup of $G$ contained in $B$. As before, the maximal torus $T \subset B$ is chosen in such a way that $T_H = T \cap H$ is a maximal diagonalizable subgroup of $H$. The
Lemma 2.3. Let \( w \in W \) and let \( \beta \in \mathcal{P}(G/H)_w \). The maps
\[
\beta \mapsto \beta \cap wB/H, \quad \beta \mapsto w^{-1} \beta \cap B/H
\]
induce bijections which preserve the natural order given by the inclusion of orbit closures between \( \mathcal{P}(G/H)_w \), the set of \((B \cap wB)\)-orbits in \( wB/H \), and the set of \((B \cap B^w)\)-orbits in \( B/H \).

Proof. Let \( u \in U \) be such that \( \beta = BwuH/H \), then we have \( \beta \cap wB/H = (B \cap wB)wuH/H \). The first bijection follows, and acting with \( w^{-1} \) on the left induces the second bijection. \( \square \)

Recall that \( \mathcal{P} \) is the set of \( B \)-stable prime divisors of \( G/H \), and let \( \mathcal{P}^* \subset \mathcal{P} \) be the set of those divisors which project dominantly on \( G/B \). In the case \( w = w_0 \) Lemma 2.1 gives a bijection between the \( B \)-orbits which project on \( Bw_0B/T \) and \( T \)-orbits in \( B/H \). Similarly, we have a natural bijection
\[
\mathcal{P}^* \longrightarrow \text{Div}^T(B/H)
\]
\[
D \longmapsto w_0D \cap B/H
\]
Since \( \mathcal{P}(G/H) \) is finite, it follows that \( B/H \) is a toric variety for a quotient of \( T \). If \( \mathcal{P}(B/H) \) denotes the set of the \( T \)-orbits in \( B/H \), then we have an inclusion preserving bijection
\[
\mathcal{P}(B/H) \longrightarrow \mathcal{P}(\text{Div}^T(B/H))
\]
\[
\mathcal{U} \mapsto \{D \in \text{Div}^T(B/H) \mid \mathcal{U} \subset D\},
\]
and a similar map identifies \( \mathcal{P}(G/H)_{w_0} \) with the power set \( \mathcal{P}(\mathcal{P}^*) \).

Consider the semidirect product decompositions \( B = TU \simeq T \ltimes U \) and \( H = T_HH^u \simeq T_H \ltimes H^u \). These induce a projection \( B/H \longrightarrow T/T_H \) and yield a \( T \)-equivariant isomorphism
\[
\mathcal{B}/H \simeq T \ltimes T^H \ltimes U/H^u.
\]
In other words, \( B/H \) is a homogeneous vector bundle over \( T/T_H \), whose fiber is the \( T_H \)-module \( U/H^u \). Therefore the \( T \)-orbits in \( B/H \) correspond then naturally to the \( T_H \)-orbits in \( U/H^u \), which is a module for \( T_H \) and a toric variety for a quotient of the connected component \( T^H \). Since \( T_H \) is a diagonalizable group, the \( T_H \)-module structure of \( U/U_H \) is completely determined by the \( T_H \)-weights in \( U/U_H \). This leads to the following definition.

Definition 2.2. \([1]\) A root \( \alpha \in \Phi^+ \) is called active if \( U_\alpha \not\subset H \). We denote by \( \Psi \subset \Phi^+ \) the set of active roots of \( H \).

Denote \( \tau : \mathcal{X}(T) \longrightarrow \mathcal{X}(T_H) \) the restriction. Given \( \pi \in \mathcal{X}(T_H) \), we denote by \( \mathcal{C}_\pi \) the one dimensional \( T_H \)-module where \( T_H \) acts with weight \( \pi \). Following \([20]\) Lemma 1.4], the exponential map induces a \( T_H \)-equivariant isomorphism \( \text{Lie}(U)/\text{Lie}(H^u) \longrightarrow U/H^u \), hence we get isomorphisms of \( T_H \)-modules
\[
U/H^u \simeq \text{Lie}(U)/\text{Lie}(H^u) \simeq \bigoplus_{\pi \in \tau(\Psi)} \mathcal{C}_\pi.
\]
In particular, the image \( \tau(\Psi) \) parametrizes the set \( \text{Div}^{T_H}(U/H^u) \). We state this correspondence more explicitly in the following lemma.

Lemma 2.3. There exists an ordering \( \tau(\Psi) = \{\pi_1, \ldots, \pi_m\} \) such that
\[
D'_i = T_H U_{\beta_1} \cdots U_{\beta_i} U_{\beta_{i+1}} \cdots U_{\beta_m} H^u/H^u
\]
(i.e. we drop \( U_{\beta_i} \)), where for every \( i \leq m \) we fix an active root \( \beta_i \) with \( \beta_i|_{T_H} = \pi_i \), is a \( T_H \)-stable prime divisor of \( U/H^u \). Any such prime divisor of \( U/H^u \) has this form; moreover \( D'_i \) is not \( U_{\beta_i} \)-stable, and does not depend on the choice of the representatives \( \beta_1, \ldots, \beta_m \).

Proof. Following \([3]\) 2.3.5, there exists a chain of subgroups \( H^u = K_m \subset K_{m-1} \subset \cdots \subset K_1 \subset K_0 = U \), all stable under conjugation by \( T_H \), such that \( K_i \) is normal in \( K_{i-1} \) for all \( i \in \{1, \ldots, m\} \) and \( K_i/K_{i-1} \cong \mathbb{C} \). The quotients \( \text{Lie} K_i/\text{Lie} K_{i-1} \) are exactly the \( T_H \)-modules \( \mathcal{C}_\pi \) for all \( \pi \in \tau(\Psi) \), and this induces an ordering \( \tau(\Psi) = \{\pi_1, \ldots, \pi_m\} \) such that \( \text{Lie} K_i/\text{Lie} K_{i-1} \cong \mathcal{C}_{\pi_i} \).
Then Lie $U_{\beta_i}$ is a $T_H$-stable complement of Lie $K_{i-1}$ in Lie $K_i$ for any choice of $\beta_i$ with $\tau(\beta_i) = \pi_i$. Now from the proof of [20, Lemma 1.4] it follows that the map

$$U_{\beta_1} \times \ldots \times U_{\beta_m} \rightarrow U/H^u$$

$$(u_1, \ldots, u_m) \mapsto u_1 \ldots u_m H^u$$

is a $T_H$-equivariant isomorphism of varieties. It follows that $\{D'_1, \ldots, D'_m\}$ is the set of $T_H$-stable prime divisors of $U/H^u$. Let now $u_i \in U_{\beta_i} \setminus \{e\}$, consider a point on $D'_i$ of the form $x = u_1 \ldots u_m H^u$ and notice that $u_ix = u_iu_{i+1} \ldots u_m H^u$ is not on $D'_i$. In other words $D'_i$ is not $U_{\beta_i}$-stable, and the proof is complete.

We will prove at the end of this section that $D'_i$ actually does not depend on the ordering of $\tau(\Psi)$. Similarly, if we consider instead

$$(2) \quad D_i = \frac{TU_{\beta_1} \cdots U_{\beta_i} U_{\beta_{i+1}} \cdots U_{\beta_m}}{H},$$

then we get a $T$-stable prime divisor in $B/H$, such that $D_i$ is not stable under the action of $U_i$ and does not depend on the choice of the representatives $\beta_i$, and we have $\text{Div}^T(B/H) = \{D_1, \ldots, D_m\}$.

**Definition 2.4.** We fix an ordering of $\tau(\Psi)$ as in Lemma 2.3 and let $\alpha \in \Psi$. We denote by $\delta(\alpha)$ the prime divisor $D_i$ of $B/H$, defined as in (2) where $i$ is such that $\tau(\beta_i) = \tau(\alpha)$.

We summarize the above discussion in the following proposition.

**Proposition 2.5.** The map $\alpha \mapsto \delta(\alpha)$ is a surjective map $\delta: \Psi \rightarrow \text{Div}^T(B/H)$ such that $\delta(\alpha)$ is not $U_{\alpha^*}$-stable, and $\delta(\alpha) = \delta(\beta)$ if and only if $\alpha|_{T_H} = \beta|_{T_H}$. In particular we have the identifications

$$\mathcal{D}^* \leftarrow \text{Div}^T(B/H) \leftarrow \text{Div}^{T_H}(U/H^u) \leftarrow \tau(\Psi).$$

We will often identify $\mathcal{D}^*$ and $\text{Div}^T(B/H)$, and also the maps $\delta: \Psi \rightarrow \mathcal{D}^*$ and $\tau: \Psi \rightarrow \tau(\Psi)$. Given $I \subset \mathcal{D}^* \simeq \tau(\Psi)$, we define

$$\Psi_I = \delta^{-1}(I) \cap \Psi = \{\alpha \in \Psi | \alpha|_{T_H} \in I\}.$$

Consider now the wonderful compactification $X$ of $G/H$, then the projection $p: G/H \rightarrow G/B$, and given $w \in W$ we define

$$F_w = q^{-1}(wB/B).$$

Then $wB/H$ is the open subset of $F_w$ obtained intersecting $F_w$ with $G/H$, and the $G$-action on $X$ induces an action of $B \cap wB$ on $F_w$ extending the one on $wB/H$ (cf. Lemma 2.4).

**Proposition 2.6.** Let $w \in W$, then the followings hold.

i) The variety $F_w$ is a smooth complete toric variety for a quotient of $T$, with weight lattice $w_0 \mathcal{X}_T(G/H)$.

ii) The $T$-stable prime divisors of $F_w$ are given by the intersections $w_0 \mathcal{D} \cap F_w$, with $D \in \mathcal{D}^* \cup \mathcal{Y}_X$, and the $T$-invariant valuation of $F_w$ associated with $w_0 \mathcal{D} \cap F_w$ is $(w_0)^{-1} p(D)$.

iii) $wB/H$ is a $T$-stable affine open subset of $F_w$, whose $T$-stable prime divisors are given by the intersections $w_0 \mathcal{D} \cap wB/H$ with $D \in \mathcal{D}^*$.

**Proof.** First we prove i), ii) and iii) when $w = w_0$, the general case will follow easily. This case is already considered in Luna’s preprint [16] (see also [2, §3]), for convenience we provide a direct proof.

Since $B \cap w_0 B = T$, by Lemma 2.3 we have that the $T$-orbits in $w_0 B/H$ correspond bijectively to the $B$-orbits in $G/H$ which map on the open Schubert cell $Bw_0B/B$. In particular they are finite, and being smooth and irreducible it follows that $w_0 B/H$ is a toric variety for a quotient of $T$. On the other hand the fibers of $q$ are all smooth and complete, and $w_0 B/H \subset F_{w_0}$ is an affine open subset. Therefore $F_{w_0}$ is a smooth complete toric variety for a quotient of $T$. Consider the open Schubert cell $Bw_0B/B \simeq B/T$ and its inverse image $X_0 = q^{-1}(Bw_0B/B)$, then $X_0 \simeq B \times T F_{w_0}$. It follows that the restriction to $F_{w_0}$ induces a bijection between $B$-semiinvariant rational functions on $X$ and $T$-semiinvariant rational functions on $F_{w_0}$ and being $X_T(X) = X_T(G/H)$ we get i).

The above bijection identifies the discrete valuation of $\mathcal{C}(X)$ associated with a $B$-stable prime divisor $D \subset X$ intersecting $X_0$ (hence $F_{w_0}$) with the discrete valuation of $\mathcal{C}(F_{w_0})$ associated with $D \cap F_{w_0}$. On the other hand a $B$-stable prime divisor $D$ of $X$ intersects $F_{w_0}$ if and only if $D \in \mathcal{D}^* \cup \mathcal{Y}_X$, whereas it intersects $w_0 B/H$ if and only if $D \in \mathcal{D}^*$. Therefore we get ii) and iii), and the case $w = w_0$ is proved.

Suppose now that $w \neq w_0$. Denote $F_w$, the toric variety defined by $F_{w_0}$ with the twisted action $t \cdot x = w_0 w^{-1} t w_0 x$, for all $t \in T$ and $x \in F_{w_0}$: then we have $X_T(F_w) = w_0 X_T(F_{w_0})$. On the other
hand, the multiplication by $w_0 w$ induces a $T$-equivariant isomorphism $F_{w_0}^* \to F_w$, and the proposition follows from the case $w = w_0$.  

To study $F_w$ under the action of $B \cap wB$ we use the general theory of automorphisms of toric varieties. In general, suppose that $Z$ is a toric variety for a quotient of $T$ with weight lattice $X_T(Z)$. Every $D \in \text{Div}^T(Z)$ defines a discrete valuation of $\mathbb{C}(Z)$ which is zero on the constant functions, as in the case considered at the beginning of the section we get then a map (which we still denote by $\rho$)

$$\rho : \text{Div}^T(Z) \to \text{Hom}(X_T(Z), \mathbb{Z}).$$

**Definition 2.7** (see [21 Proposition 3.13]). Let $Z$ be a smooth complete toric variety and let $\alpha \in X_T(Z)$. Then $\alpha$ is a root of $Z$ if there exists $\delta_T(\alpha) \in \text{Div}^T(Z)$ such that $\langle \rho(\delta_T(\alpha)), \alpha \rangle = -1$, and $\langle \rho(D), \alpha \rangle \geq 0$ for all $D \in \text{Div}^T(Z) \setminus \{\delta_T(\alpha)\}$. The set of the roots of $Z$ is denoted by $\text{Root}(Z)$.

The signs in the above definition differ from those in [21]. This is because in the standard theory of toric varieties the functional on $X_T(Z)$ associated with a $T$-stable prime divisor $D$ is equal to $-\rho(D)$. Correspondingly, we report the next lemma from [21] with the needed sign changes.

**Theorem 2.8** ([21 Proposition 3.14 and Demazure’s Structure Theorem]). Suppose that $Z$ is a smooth complete toric variety for a quotient of $T$ with weight lattice $X_T(Z)$. Let $\alpha \in X(T)$ and let $C_{\alpha}$ be the associated one dimensional $T$-module.

i) $\alpha \in \text{Root}(Z)$ (and in particular $\alpha \in X_T(Z)$) if and only if the $T$-action on $Z$ extends to an action $\phi_\alpha : T \times C_{\alpha} \to \text{Aut}(Z)$, where $C_{\alpha}$ acts non-trivially on $Z$ and where the extension of the $T$-action is unique up to a twist by a $T$-equivariant automorphism of $C_{\alpha}$.

ii) The connected group $\text{Aut}^0(Z)$ containing the identity is a connected linear algebraic group containing $T$ as a maximal torus, whose associated root system is $\text{Root}(Z)$. In particular $\text{Aut}^0(Z)$ is generated as a group by $T$ together with the one parameter subgroups $\phi_\alpha(C_{\alpha})$ with $\alpha \in \text{Root}(Z)$.

iii) Let $\alpha \in \text{Root}(Z)$ and let $C_{\alpha}$ be the one dimensional cone of the fan of $Z$ associated with $\delta_T(\alpha)$. Let $Z_0$ be a $T$-stable affine open subfield of $Z$ and let $C$ be the corresponding cone in the fan of $Z$. If $C_{\alpha}$ is an edge of $C$, then $Z_0$ is $T \times C_{\alpha}$-stable. If moreover $f_\beta \in \mathbb{C}(Z)^{(T)}$ is a $T$-semistable rational function of weight $\beta$ and $\xi \in C_{\alpha}$, then the rational function $f_\beta(\xi \cdot z)$ on $Z$ is given by the formula

$$(3) \quad f_\beta(\xi \cdot z) = f_\beta(z)(1 + \xi f_\beta(z))^{\langle \rho(\delta(\alpha)), \beta \rangle},$$

(all the $T$-semistable rational functions are normalized in such a way that they take value 1 on the same element in the open $T$-orbit of $Z$).

iv) In particular, if $\alpha \in \text{Root}(Z)$, then $\delta_T(\alpha)$ is the unique $T$-stable and not $C_{\alpha}$-stable prime divisor of $Z$.

*Proof.* Let $V$ be a one dimensional unipotent subgroup of the connected automorphism group $\text{Aut}^0(Z)$ of $Z$. By [21] Demazure’s Structure Theorem, part (i)], if $V$ is normalized by the image of $T$ in $\text{Aut}^0(Z)$ then $V \cong C_{\alpha}$ where $\alpha$ is a root of $Z$. Moreover, any root of $Z$ yields such a one dimensional group of automorphisms. This shows part i).

Part ii) is [21] Demazure’s Structure Theorem, part (i)]. There it is also stated that the action of $C_{\alpha}$ is given by the formula of [21 Proposition 3.14], which is the above formula (3). At this point the first assertion of part iii) is [21 Proof of Proposition 3.14, step (a)].

It remains to show part iv). Let $Z_0$ be equal to $Z \setminus \cup D$ where $D$ varies in the set of $T$-stable prime divisors of $Z$ different from $\delta(\alpha)$. Then the cone $C$ is equal to $C_{\alpha}$, and $Z_0$ is isomorphic to $(\mathbb{C}^*)^{\dim Z - 1} \times \mathbb{C}$. Apply formula (3) to the global equation $f_{-\alpha} = (f_\alpha)^{-1}$ of $Z_0 \cap \delta(\alpha)$ in $Z_0$: we have

$$f_{-\alpha}(\xi \cdot z) = f_{-\alpha}(z) + \xi$$

i.e. $Z_0 \cap \delta(\alpha)$ is not $C_{\alpha}$-stable. On the other hand $Z_0$ is $C_{\alpha}$-stable by part iii), therefore $\delta(\alpha)$ is not $C_{\alpha}$-stable. Finally, the complement $Z \setminus Z_0$ is $C_{\alpha}$-stable, and so are its irreducible components since $C_{\alpha}$ is connected. In other words all $T$-stable prime divisors of $Z$ different from $\delta(\alpha)$ are $C_{\alpha}$-stable.  

**Example 2.9.** Take $X = \mathbb{P}^2$ and $Y = \mathbb{C}^2$ given by the equation $x_0 = 1$. Let $T$ be a two dimensional torus acting on $X$ with weight $e_i$ on the coordinate $x_i$, with $i = 1, 2$. Consider the affine coordinates $x_1, x_2$ on $Y$: we have $f_{e_1} = x_1^{-1}$ and $f_{e_2} = x_2^{-1}$. The function $x_1$ on $Y$ has character $-e_1$; it vanishes with multiplicity one on the prime divisor $D = \{x_1 = 0\}$, has order $-1$ along $\{x_0 = 0\}$ and has order 0 along $\{x_2 = 0\}$. It follows that $e_1$ is a root of $X$, with $\delta(e_1) = D$. The cone of $Y$ contains the edge generated by $\rho(D)$, and the unipotent group $U_{e_1}$ acts on a point $x = (x_1, x_2) \in Y$ in the following way:

$$u_{e_1}(x_1, x_2) = (x_1(1 + \xi x_1^{-1})^{\langle \rho(D), -e_1 \rangle}, x_2(1 + \xi x_1^{-1})^{\langle \rho(D), -e_2 \rangle}) = (x_1 + \xi, x_2).$$
Let \( w \in W \) and \( \alpha \in \Phi^+ \setminus \Phi^+ (w^{-1}) \). Then we have \( U_\alpha \subset B \cap wB \), hence we get an action of \( U_\alpha \) on the smooth complete toric variety \( F_w \). Theorem 2.8 provides a necessary condition for \( U_\alpha \) to act non-trivially on \( F_w \). On the other hand, if \( U_\alpha \) acts non-trivially on \( F_w \), then by the same lemma we can recover the action of \( U_\alpha \) from the formula (3). Up to a twist by \( w \), the action of \( B \cap wB \) on \( F_w \) is the same as the action of \( B \cap B^w \) on \( F_w \), therefore we are reduced to study the fiber \( F_e \) and its open subset \( B/H \).

We begin with the following immediate consequence of the above proposition.

**Corollary 2.10.** Let \( \alpha \in \Psi \). Then \( \alpha \in \Root(F_e) \), and \( \delta(\alpha) = \delta_T(\alpha) \). In particular, \( \delta(\alpha) \) is the unique element of \( \Div^T(B/H) \) not stable under the action of \( U_\alpha \).

**Definition 2.11.** A root \( \alpha \in \Phi^+ \) is called weakly active if it is a root of the toric variety \( F_e \). We denote by \( \Psi^2 \subset \Phi^+ \) the set of the weakly active roots of \( H \).

Recall that \( \Div^T(F_e) \) is in natural bijection with \( \mathcal{D}^* \cup \mathcal{Y}_X \) after Proposition 2.6.

**Proposition 2.12.** If \( \alpha \in \Psi^2 \), then we have \( \delta_T(\alpha) \in \mathcal{D}^* \).

**Proof.** By Proposition 2.3 the \( T \)-stable prime divisors of \( F_e \) are the intersections \( w_0D \cap F_e \), where \( D \in \mathcal{D}^* \cup \mathcal{Y}_X \). Given \( D \in \mathcal{Y}_X \), by the formula (1) we have \( \langle w_0\rho(D), \alpha \rangle = \langle \rho(D), w_0\alpha \rangle \geq 0 \) for all \( \alpha \in \Sigma \). On the other hand if \( \alpha \in \Phi^+ \) is a root of \( F_e \), then we have \( \alpha \in \mathbb{N} \Delta \cap X(G/H) = \mathbb{N} \Sigma \) it follows that \( \langle w_0\rho(D), \alpha \rangle \geq 0 \) for all \( D \in \mathcal{Y}_X \), hence \( \delta_T(\alpha) \in \mathcal{D}^* \).

Therefore, if we consider \( \delta_T \) as a map \( \Root(F_e) \rightarrow \Div^T(F_e) = \mathcal{D}^* \cup \mathcal{Y}_X \), then its restriction to \( \Psi^2 \) is a map \( \Psi^2 \rightarrow \mathcal{D}^* \). Its further restriction to \( \Psi \) coincides with the map of Proposition 2.3. Therefore we unify our notations and drop the subscript "\( T \)" denoting this map simply by

\[ \delta: \Psi^2 \rightarrow \mathcal{D}^*. \]

For a positive root \( \alpha \in \Phi^+ \), the relationship between being a weakly active root and assuming certain values on \( w_0\rho(\mathcal{D}^*) \) is clear by definition. If we only consider active roots, then this relationship turns out to be even more precise and most importantly holds for all \( \alpha \in \mathbb{N}\Phi^+ \). Indeed, the following theorem holds.

**Theorem 2.13** (see [4, Theorem 4.34(b)]). An element \( \alpha \in \mathbb{N}\Phi^+ \) is an active root if and only if there exists \( D_0 \in \mathcal{D}^* \) such that

\[ \langle w_0\rho(D_0), \alpha \rangle \left\{ \begin{array}{ll} 0 & \text{if } D \in \mathcal{D}^* \setminus \{D_0\}, \\
-1 & \text{if } D = D_0 \end{array} \right. \]

in which case we have \( \delta(\alpha) = D_0 \).

Our first application of the above theorem is the following very explicit description of all \( T \)-orbits of \( B/H \).

**Definition 2.14.** Let \( I \) be a subset of \( \mathcal{D}^* \); identify it with the corresponding subset of \( \Div^T(B/H) \) and also with the corresponding subset \( \{\pi_{i_1}, \ldots, \pi_{i_k}\} \) of \( \tau(\Psi) \). For all \( D = \pi_{i_j} \in I \) choose \( \beta_j \in \Psi \) such that \( \tau(\beta_j) = \pi_{i_j} \) and choose \( u_{\beta_{i_j}} \in U_{\beta_{i_j}} \setminus \{e\} \). Then we define the \( T \)-orbit

\[ \mathcal{Y}_I = Tu_{\beta_{i_1}} \cdots u_{\beta_{i_k}} H \]

of \( B/H \).

**Lemma 2.15.** Let \( D \in \Div^T(B/H) \). Then \( \mathcal{Y}_I \subseteq D \) if and only if \( D \notin I \).

**Proof.** The point \( u_{\beta_{i_1}} \cdots u_{\beta_{i_k}} H \) on the \( T \)-orbit \( \mathcal{Y}_I \) is obtained applying successively the action of non-trivial elements \( u_{\beta_{i_j}} \in U_{\beta_{i_j}} \) on the point \( eH \), which lies on the closed \( T \)-orbit \( TH/H \). The latter is contained in all elements of \( \Div^T(B/H) \). The unipotent group \( U_{\beta_{i_j}} \) stabilizes all \( T \)-stable prime divisors of \( B/H \) except for \( \delta(\beta_{i_j}) \), therefore \( \mathcal{Y}_I \) is contained in \( D \) for all \( D \notin \{\delta(\beta_{i_1}), \ldots, \delta(\beta_{i_k})\} = I \).

It remains to prove that \( \mathcal{Y}_I \) is not contained in \( \delta(\beta_{i_j}) \) for any \( j \in \{1, \ldots, k\} \). Being a smooth affine toric variety, the quotient \( B/H \) is factorial. Let \( f \in \mathbb{C}[B/H] \) be a global equation on \( B/H \) of the \( T \)-stable prime divisor \( \delta(\beta_{i_j}) \). Notice that the \( T \)-character of \( f \) is then \( -\beta_{i_j} \), thanks to Theorem 2.13.

Then by Theorem 2.3 part iii) we have

\[ f(u_{\beta_{i_j}}(x)) = f(x) + \xi \]

for all \( x \in B/H \). This implies that acting with \( u_{\beta_{i_j}} \) moves outside \( \delta(\beta_{i_j}) \) any point of \( B/H \) lying on \( \delta(\beta_{i_j}) \). On the other hand acting with \( u_{\beta_{i_j}} \) with \( i \neq j \) doesn’t move \( \delta(\beta_{i_j}) \).
To summarize: the point \( u_{\beta_{i+1}} \cdots u_{\beta_k} H \) is contained in \( \delta(\beta_i) \), the point \( u_{\beta_{i}} \cdots u_{\beta_k} H \) is not, and
so also the point \( u_{\beta_{i+1}} \cdots u_{\beta_k} H \) is not contained in \( \delta(\beta_i) \). This concludes the proof. \( \square \)

**Corollary 2.16.** The \( T \)-orbit \( \mathcal{U}_I \) depends only on \( I \), and neither on \( \beta_1, \ldots, \beta_m, u_{\beta_1}, \ldots, u_{\beta_m} \), nor on their order. Moreover \( \mathcal{U}_I \subseteq \mathcal{U}_J \) if and only if \( I \supseteq J \). This induces an isomorphisms of partially ordered sets between the set of \( T \)-orbits of \( B/H \) and the set \( \mathcal{P}(\mathcal{D}^*) \), where the first is ordered by inclusion of orbit closures, and the second by reverse inclusion.

**Proof.** A \( T \)-orbit in an affine toric \( T \)-variety is uniquely determined by the set of \( T \)-stable prime divisors containing it, its closure is the intersection of all such prime divisors, and any family of \( T \)-stable prime divisors has a non-empty intersection. The corollary follows then from Lemma 2.15. \( \square \)

### 3. Weakly active roots and root systems

In this section we study the weakly active roots of \( H \), proving several combinatorial properties and relating them to the action of the one-dimensional root subgroups of \( U \) on \( B/H \). This will provide the technical tools needed for our description of \( \mathcal{B}(G/H) \) given in the next section.

Here and in the next sections we keep the assumptions of the previous section, in particular \( H \subseteq B \) is a wonderful subgroup of \( G \), \( X \) is the wonderful compactification of \( G/H \) and \( \Sigma \subseteq \Delta \) denotes the set of spherical roots of \( X \), so that \( \mathcal{X}(G/H) = Z\Sigma \).

If \( \alpha \in \Phi^+ \) denote
\[
\Phi^+(\alpha) = (\alpha + N\Phi^+) \cap \Phi^+,
\]
\[
\Psi(\alpha) = (\alpha + N\Psi) \cap \Psi.
\]

**Lemma 3.1.**

i) Let \( \alpha \in \Phi^+ \) and \( \beta \in \Phi^+(\alpha) \setminus \{\alpha\} \). There exist \( \beta_0, \ldots, \beta_n \in \Phi^+ \) with \( \beta_0 = \alpha \), \( \beta_n = \beta \) and \( \beta_i - \beta_{i-1} \in \Phi^+ \) for all \( i \leq n \).

ii) Given \( \alpha \in \Phi^+ \), the ideal \( (u_\alpha) \) generated by \( u_\alpha \) in \( u \) is the direct sum of the root spaces \( u_\beta \) with \( \beta \in \Phi^+(\alpha) \).

**Proof.**

i) Let \( \alpha_1, \ldots, \alpha_m \in \Phi^+ \) with \( \beta = \alpha_1 + \cdots + \alpha_m \). Assume that \( m \) is minimal with this property, we prove the claim by induction on \( m \). Set \( \gamma = \beta - \alpha \). If \( m = 1 \) the claim is true. Assume \( m > 1 \), then by the minimality of \( m \) it must be \( \gamma \notin \Phi^+ \). Since \( \gamma \in N\Phi^+ \), it follows then \( \gamma \notin \Phi \), therefore we get \( (\beta, \alpha) \leq 0 \), where \((-,-)\) is an ad-invariant scalar product on \( t^* \). Hence
\[
(\alpha, \alpha) + (\gamma, \alpha) = (\alpha + \gamma, \alpha) = (\beta, \alpha) \leq 0
\]
and we get \( (\gamma, \alpha) \leq -(\alpha, \alpha) < 0 \). Up to reorder the indices, we may assume that \( (\alpha_1, \alpha) < 0 \). Therefore we have \( \alpha' = \alpha_1 + \alpha \in \Phi^+ \). On the other hand \( \beta \in \Phi^+(\alpha') \) and \( \beta = \alpha' + \alpha_2 + \cdots + \alpha_m \), therefore the claim follows by induction.

ii) Denote \( \tau = \bigoplus_{\beta \in \Phi^+(\alpha)} u_\beta \). We show that \( \tau \subseteq u \) is an ideal and that \( \tau \subseteq (u_\alpha) \), whence the lemma.

To show that \( \tau \) is an ideal, it is enough to notice that for all \( \beta \in \Phi^+(\alpha) \) and for all \( \gamma \in \Phi^+ \) the bracket \([u_\beta, u_\gamma]\) is either 0 or \( u_{\beta+\gamma} \), and that in the latter case \( \beta + \gamma \in \Phi^+(\alpha) \). It follows \( [\tau, u] \subseteq \tau \).

Let now \( \beta \in \Phi^+(\alpha) \). By i) they exist \( \beta_0, \ldots, \beta_n \in \Phi^+ \) with \( \beta_0 = \alpha \), \( \beta_n = \beta \) and \( \beta_i - \beta_{i-1} \in \Phi^+ \) for all \( i \leq n \). Denote \( \alpha_i = \beta_i - \beta_{i-1} \). Then we have
\[
[u_\alpha, u_{\alpha_1}] = u_{\beta_1} \neq 0
\]
\[
[u_{\beta_1}, u_{\alpha_2}] = u_{\beta_2} \neq 0
\]
\[
\ldots
\]
\[
[u_{\beta_{n-1}}, u_{\alpha_n}] = u_\beta \neq 0
\]
It follows that \( u_\beta \subseteq (u_\alpha) \) for all \( \beta \in \Phi^+(\alpha) \), hence \( \tau \subseteq (u_\alpha) \). \( \square \)

**Theorem 3.2.** Let \( \alpha \in \Phi^+ \). The following statements are equivalent:

i) \( \alpha \in \Psi^* \);

ii) \( U_\alpha \) acts non-trivially on \( B/H \);

iii) \( \Psi(\alpha) \cap \Psi \neq \emptyset \);

iv) \( \Phi^+(\alpha) \cap \Psi \neq \emptyset \).

If moreover \( \alpha \in \Psi^* \), then we have \( \delta(\alpha) = \delta(\beta) \) for all \( \beta \in \Psi(\alpha) \).
Given $D$ we have

$$\beta = \alpha + \sum_{D \in \mathcal{G}^\ast \setminus \{\delta(\alpha)\}} \langle w_0(\rho(D), \alpha) \rangle \beta_D.$$

By Theorem 2.13 we have $\beta \in \Psi$, and it follows $\Psi(\alpha) \neq \emptyset$.



### Table 1. Active roots

| $\text{supp}(\alpha)$ | $\alpha$ | $\pi(\alpha)$ |
|------------------------|---------|---------------|
| any of rank $n$         | $\alpha_1 + \cdots + \alpha_n$ | $\alpha_1, \cdots, \alpha_n$ |
| $B_n$                  | $\alpha_1 + \cdots + \alpha_{n-1} + 2\alpha_n$ | $\alpha_1, \cdots, \alpha_{n-1}$ |
| $C_n$                  | $2\alpha_1 + \cdots + 2\alpha_{n-1} + \alpha_n$ | $\alpha_n$ |
| $F_4$                  | $2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$ | $\alpha_3, \alpha_4$ |
| $G_2$                  | $2\alpha_1 + \alpha_2$ | $\alpha_2$ |
| $G_2$                  | $3\alpha_1 + \alpha_2$ | $\alpha_2$ |





**Proof.** i) $\Leftrightarrow$ ii) If $\alpha \in \Psi^2$, then by Proposition 2.12 there exists a $T$-stable divisor in $B/H$ which is not $U_\alpha$-stable and we get ii). If conversely $U_\alpha$ acts non-trivially on $B/H$, then it acts non-trivially on $F_4$, hence $\alpha \in \Psi^2$ by Theorem 2.13.

i) $\Rightarrow$ iii) By Proposition 2.3 the map $\delta : \Psi \to \mathcal{G}^\ast$ is surjective, hence for all $D \in \mathcal{G}^\ast$ we may fix $\beta_D \in \Psi$ with $\delta(\beta_D) = D$. Let now $\alpha \in \Psi^2$ and consider the element

$$\beta = \alpha + \sum_{D \in \mathcal{G}^\ast \setminus \{\delta(\alpha)\}} \langle w_0(\rho(D), \alpha) \rangle \beta_D.$$

Given $D \in \mathcal{G}^\ast$ we have

$$\langle w_0(\rho(D), \beta) \rangle = \begin{cases} 0 & \text{if } D \neq \delta(\alpha) \\ -1 & \text{if } D = \delta(\alpha) \end{cases}$$

By Theorem 2.13 we have $\beta \in \Psi$, and it follows $\Psi(\alpha) \neq \emptyset$.

iii) $\Rightarrow$ iv) Obvious.

iv) $\Rightarrow$ ii) Suppose that $U_\alpha$ acts trivially on $B/H$, we prove that it must be $\Phi^+(\alpha) \cap \Psi = \emptyset$. Denote $N$ the kernel of the action of $U$ on $B/H$. Given $u \in U$, we have $u \in N$ if and only if $ubH = bH$ for all $b \in B$. For all $b \in B$ we have

$$ubH = bH \iff b^{-1}ub \in H \iff u \in bHb^{-1},$$

therefore $N$ is the intersection of $U$ with the biggest normal subgroup of $B$ contained in $H$. Equivalently, $N$ is the biggest $T$-stable normal subgroup of $U$ contained in $H$. It follows that $N$ is stable under conjugation by $T$, hence it is the product of the root subgroups $U_\alpha$ which are contained in it.

By assumption we have $U_\alpha \subset N$, hence $u_\alpha \subset n$, and by Lemma 3.3(ii) the ideal $(u_\alpha)$ generated by $u_\alpha$ in $u$ is the direct sum of the $u_\beta$ with $\beta \in \Phi^+(\alpha)$. On the other hand $n$ is an ideal of $u$, and by definition $u_\alpha$ with $\alpha \in \Psi$ is contained in $\mathfrak{h}$. Therefore it must be $\Phi^+(\alpha) \cap \Psi = \emptyset$.

For the last statement, let $\alpha \in \Psi^2$ and $\beta \in \Psi(\alpha)$. By Theorem 2.13 it follows that $\langle w_0(\rho(\delta(\alpha)), \beta) \rangle \leq -1$. On the other hand $\langle w_0(\rho(D), \beta) \rangle = 0$ for all $D \in \mathcal{G}^\ast \setminus \{\delta(\beta)\}$, therefore we get $\delta(\beta) = \delta(\alpha)$. \hfill $\square$

By [2] Theorem 4.28, the set $\Sigma$ coincides with the set of simple roots which occur in the support of some active root. Therefore every spherical root is weakly active.

**Definition 3.3.** Let $I \subset \mathcal{G}^\ast$. We define $\Psi_I$ the set of active roots mapped to elements of $I$ via the map $\delta$. If $\alpha \in \Psi^2$, then we say that $\alpha$ is activated by $I$ if there exists $\beta \in \Psi_I$ such that $\alpha + \beta \in \Psi$, and that $\alpha$ stabilizes $I$ if there exists $\beta \in \mathbb{N}\Psi_I$ such that $\alpha + \beta \in \Psi_I$. We denote by $\Psi^2(I)$ the set of the roots activated by $I$, and by $\Psi_I \subset \Psi^2(I)$ the subset of the roots which stabilize $I$.

Notice that for all $I$ we have $\Psi \subset \Psi^2(I)$ and $\Psi_I \subset \Psi^2_I$. Also we have that $\alpha \in \Psi^2(I)$ stabilizes $I$ if and only if $\delta(\alpha) \in I$.

**Corollary 3.4.** Let $\alpha \in \Psi^2$. We have $\alpha \in \Psi^2(I)$ if and only if $\langle w_0(\rho(D), \alpha) \rangle \leq 0$ for all $D \in \mathcal{G}^\ast \setminus I$, whereas $\alpha \in \Psi^2_I$ if and only if $\langle w_0(\rho(D), \alpha) \rangle = 0$ for all $D \in \mathcal{G}^\ast \setminus I$. In particular we have the equality $\Psi^2_I = N\Psi^2_I \cap \Psi^2$.

**Proof.** Let $\alpha \in \Psi^2$, let $\beta \in \Psi \cap (\alpha + \mathbb{N}\Psi_I)$. If $D \in \mathcal{G}^\ast \setminus I$, then we have $\langle w_0(\rho(D), \alpha) \rangle = \langle w_0(\rho(D), \beta) \rangle \leq 0$. Conversely if such inequalities are all fulfilled, then we have $\langle w_0(\rho(D), \beta) \rangle = 0$ for all $D \in (\mathcal{G}^\ast \setminus I) \setminus \{\delta(\alpha)\}$. Therefore we may consider the element $\beta$ defined as in formula 14, and by Theorem 2.13 we get $\beta \in \Psi \cap (\alpha + \mathbb{N}\Psi_I)$. This shows the first claim. On the other hand $\alpha \in \Psi^2(I)$ stabilizes $I$ if and only if $\delta(\alpha) \in I$ if and only if $\langle w_0(\rho(D), \beta) \rangle \geq 0$ for all $D \in \mathcal{G}^\ast \setminus I$, and the second claim follows as well. \hfill $\square$

Recall the dominance order on $X(B)$, defined by $\lambda \leq \mu$ if and only if $\mu - \lambda \in \mathbb{N}_\Delta$. 


Proposition 3.5. Let $\alpha \in \Psi$ and let $\beta \in \mathbb{N}\Psi$ be such that $\beta \leq \alpha$, then $\alpha - \beta \in \Phi^\circ$. In particular if $I \subset \mathcal{D}^\circ$ then we have
\[
\Psi^\circ(I) = \{ \alpha - \beta \mid \alpha \in \Psi, \beta \in \mathbb{N}\Psi, \beta \leq \alpha \},
\]
\[
\Psi^\circ_I = \{ \alpha - \beta \mid \alpha \in \Psi_I, \beta \in \mathbb{N}\Psi, \beta \leq \alpha \}.
\]

Proof. Given $\alpha \in \Psi$ denote $F(\alpha) = \{ \beta \in \Psi \mid \beta \leq \alpha \}$. By [11] Lemma 7] for all $\beta \in F(\alpha)$ we have $\alpha - \beta \in \Phi^\circ$, let us prove the claim in the more general case $\beta \in \mathbb{N}\Psi$. Following [11] Proposition 3], for every $\alpha \in \Psi$ there exists a simple root $\pi(\alpha)$ with the following property: if $\alpha = \beta + \gamma$ for some $\beta, \gamma \in \Phi^\circ$, then $\gamma \in \Psi$ if and only if $\pi(\alpha) \not\in \text{supp}(\beta)$. Moreover by [11] Theorem 3] the pair $(\alpha, \pi(\alpha))$ appears in Table [11]. Using this fact, and the fact that the elements in $F(\alpha)$ are linearly independent (see [11] Corollary 2)), the claim may be deduced from Table [11].

Suppose for instance that $\alpha = \alpha_1 + \ldots + \alpha_n$ and that $\pi(\alpha) = \alpha_k$. Then we have $F(\alpha) \supset \{ \beta_1, \ldots, \beta_n \}$, where we denote
\[
\beta_i = \begin{cases} 
\alpha_1 + \ldots + \alpha_i & \text{if } i < k \\
\alpha_1 + \ldots + \alpha_n & \text{if } i = k \\
\alpha_1 + \ldots + \alpha_n & \text{if } i > k 
\end{cases}
\]
These $n$ elements already generate $\mathbb{Z}\alpha_1 + \ldots + \mathbb{Z}\alpha_n$, which contains in any case all roots $\leq \alpha$. So $F(\alpha) = \{ \beta_1, \ldots, \beta_n \}$ since $F(\alpha)$ is linearly independent.

Therefore, if $\gamma = \gamma_1 + \ldots + \gamma_m \in \mathbb{N}\Psi$ is such that $\alpha - \gamma \in \Phi^\circ$, then it must be $m = 1$ or $m = 2$, $\gamma_1 = \beta_i$, and $\gamma_2 = \beta_j$ with $i < k < j$, and in both cases we get $\alpha - \gamma \in \Phi^\circ$. The other cases are proved reasoning in a similar manner.

Corollary 3.6. Denote $\tau : \mathcal{X}(T) \to \mathcal{X}(T_H)$ the restriction. If $\alpha \in \Psi^\circ$, then $\tau(\alpha) \not= 0$.

Proof. If $\alpha \in \Psi$, then the claim follows by [11] Theorem 1]. Let now $\alpha \in \Psi^\circ$, i.e. suppose these exist $\beta_1, \ldots, \beta_n \in \Psi$ such that $\alpha = \beta_0 - \sum_{i=1}^n a_i \beta_i$, where $a_i \in \mathbb{N}$ and $\beta_i \not\leq \beta_0$ for all $i > 0$. By [11] Lemma 5] we have $\tau(\beta_i) \neq \tau(\beta_j)$ for all $i \not= j$, hence [11] Theorem 1] shows that $\tau(\beta_0), \ldots, \tau(\beta_n)$ are linearly independent in $\mathcal{X}(T_H)$. It follows that $\tau(\alpha) = \tau(\beta_0) - \sum_{i=1}^n a_i \tau(\beta_i) \neq 0$.

Definition 3.7. Given $I \subset \mathcal{D}^\circ$ we denote
\[
\Phi_I = \mathbb{Z}\Psi_I \cap \Phi, \quad \Phi_I^\circ = \mathbb{Z}\Psi_I \cap \Phi^\circ.
\]

It is easy to check that $\Phi_I$ is a closed subroot system of $\Phi$, and that $\Phi_I^\circ \subset \Phi_I$ is a system of positive roots. We denote by $\Delta_I \subset \Phi_I^\circ$ the corresponding basis and by $W_I$ the Weyl group of $\Phi_I$ (notice that in general we have $\Delta_I \not\subseteq \Delta$).

Lemma 3.8. Let $\alpha, \beta \in \Psi^\circ$ be such that $\langle w_0 \rho(\delta(\alpha)), \beta \rangle > 0$. Then $\alpha + \beta \in \Psi^\circ$ and $\delta(\alpha + \beta) = \delta(\beta)$.

Proof. First we show that we can assume that the $T$-action on $B/H$ is faithful, namely that $\mathcal{X}(T) = \mathcal{X}(B/H) = \mathcal{X}(G/H)$. Suppose that this is not the case, then we have $\Sigma \not\subseteq \Delta$. Let $Q \supset B$ be the parabolic subgroup of $G$ associated with $\Sigma$, let $L$ be its standard Levi subgroup containing $T$, and denote $M = (L, L)$ the commutator of $L$. Then $G/H$ is a parabolic induction by means of $Q$ (see e.g. [17]): in other words we have $H = Q^\circ K$, where $K \subset M$ is a strongly solvable subgroup contained in $B_M = M \cap B$ (which is a Borel subgroup of $M$) and where the maximal torus $T_M = M \cap T$ acts faithfully on $B_M/K$. Since $H^0 \supset Q^0$ we also have that the active roots and the weakly active roots of $H$ and of $K$ coincide. Moreover, the $B$-stable prime divisors of $M/K$ are naturally identified with the subset
\[
\{ D \in \mathcal{D}^\circ \mid \exists \alpha \in \Sigma | P_\alpha D \not= D \},
\]
and in particular $\delta(\alpha)$ is one of them. Therefore we may replace $G$ with $M$ and $H$ with $K$, and we may assume that $T$ acts faithfully on $B/H$.

Let now $\alpha, \beta \in \Psi^\circ$ be such that $\langle w_0 \rho(\delta(\alpha)), \beta \rangle > 0$. Then it must be $\delta(\alpha) \neq \delta(\beta)$, and in particular we get $\langle w_0 \rho(\delta(\beta), \alpha \rangle > 0$. Suppose that $\langle w_0 \rho(\delta(\beta), \alpha \rangle > 0$, then we have $\langle w_0 \rho(D), \alpha + \beta \rangle \geq 0$ for all $D \in \mathcal{D}^\circ \cup \mathcal{X}_X$. On the other hand the valuations $w_0 \rho(D)$ generate the extremal rays of the fan of $F_\alpha$, and the latter covers $\text{Hom}(\mathcal{X}(T_F), \mathbb{Q})$ since $F_\alpha$ is complete. Moreover the action of $T$ on $B/H$ is faithful, hence $\mathcal{X}(F_\alpha) = \mathcal{X}(T)$ and it follows that $\alpha + \beta = 0$, which is absurd. Therefore it must be $\langle w_0 \rho(\delta(\beta), \alpha \rangle = 0$, thus $\alpha + \beta$ is by definition a root of $F_\alpha$ and being $\langle w_0 \rho(\delta(\beta), \alpha + \beta \rangle = -1$ we get $\delta(\alpha + \beta) = \delta(\beta)$.

To conclude the proof, we need to prove that $\alpha + \beta \in \Phi^\circ$. Consider the homomorphism of algebraic groups $\psi : B \to \text{Aut}^\circ(F_\alpha)$, and denote $V_\alpha, V_\beta, V_{\alpha + \beta} \subset \text{Aut}^\circ(F_\alpha)$ the root spaces associated to $\alpha, \beta, \alpha + \beta$.
Proposition 3.9. Given $\beta \in \text{Root}(F_\alpha)$. Since $U_\alpha$ and $U_\beta$ act non-trivially on $F_\beta$, by Theorem 2.25 we have $\psi(U_\alpha) = V_\alpha$ and $\psi(U_\beta) = V_\beta$. Notice that the claim follows if $V_\alpha$ and $V_\beta$ do not commute, since then $\text{Lie}(V_{\alpha+\beta}) = [\text{Lie}(V_\alpha), \text{Lie}(V_\beta)] = 0$ if $\psi(U_\alpha), \psi(U_\beta)$. In the notations of Theorem 2.25 let $\xi_\alpha \in C_\alpha$ and $\xi_\beta \in C_\beta$, and set $v_\alpha = \phi_\gamma(\xi_\alpha)$, $v_\beta = \phi_\gamma(\xi_\beta)$. If $z \in F_\gamma$ and $f_z \in \mathbb{C}(F_\gamma)^{(\gamma)}$ is a $T$-seminvariant function (normalized as in Theorem 2.25) of weight $\gamma$, then by formula (3) of Theorem 2.25 the action of $v_\gamma v_\alpha$ is given by

$$f_\gamma(v_\gamma v_\alpha \cdot z) = f_\gamma(z)(1 + \xi_\alpha f_\alpha(z))^{(\alpha + \beta)}(1 + \xi_\beta f_\beta(z)(1 + \xi_\alpha f_\alpha(z))^{(\alpha + \beta)}(\alpha + \beta),$$

whereas, being $\langle \alpha_0 \rho(\delta(\beta)), \alpha \rangle = 0$, the action of $v_\alpha v_\gamma$ is given by

$$f_\gamma(v_\alpha v_\beta \cdot z) = f_\gamma(z)(1 + \xi_\beta f_\beta(z))^{(\alpha + \beta)}(1 + \xi_\alpha f_\alpha(z))^{(\alpha + \beta)} \gamma.$$ 

Therefore the equality $f_\gamma(v_\gamma v_\alpha \cdot z) = f_\gamma(v_\alpha v_\beta \cdot z)$ holds if and only if

$$(1 + \xi_\beta f_\beta(z))^{(\alpha + \beta)} = (1 + \xi_\alpha f_\alpha(z))^{(\alpha + \beta)} \gamma.$$ 

On the other hand by assumption we have $\langle \alpha_0 \rho(\delta(\beta)), \beta \rangle > 0$, therefore for general $z$, $\xi_\alpha, \xi_\beta$ we have $f_\gamma(v_\gamma v_\alpha \cdot z) \neq f_\gamma(v_\beta v_\alpha \cdot z)$ and $V_\alpha$ and $V_\beta$ do not commute.

Proposition 3.9. Given $I \subset \mathcal{D}^*$, we have

$$\mathbb{N}^2 = \mathbb{Z}_I \cap \mathbb{N} = \mathbb{Q}_I \cap \mathbb{N}.$$ 

**Proof.** By [2, Theorem 4.28], $\Sigma$ coincides with the set of simple roots which occur in the support of some active root. By Theorem 3.2 we have then $\Sigma = \Delta \cap \mathbb{N} I$ and $\mathbb{Z}_I = \Sigma I$, hence $\mathbb{Z}_I \cap \mathbb{N} = \mathbb{N} I = \mathbb{N}^2$. Therefore the equalities are equivalent to

$$\mathbb{N}^2 = \mathbb{Z}_I \cap \mathbb{N}^2 = \mathbb{Q}_I \cap \mathbb{N}.$$ 

By the definition of $\Psi_I$, we have the inclusions $\mathbb{N} \Psi_I \subset \mathbb{Z}_I \cap \mathbb{N} \subset \mathbb{Z}_I \cap \mathbb{N}$. Let $\alpha \in \mathbb{Q}_I \cap \mathbb{N}^2$ and write $\alpha = \beta_1 + \ldots + \beta_n$ with $\beta_i \in \Psi^2$. In particular we have that $\langle \alpha_0 \rho(\delta(\beta_i)), \alpha \rangle = 0$ for every $D \in \mathcal{D}^* \setminus I$. Proceeding by induction on $n$, we want to show that we may choose the $\beta_i$ in $\Psi_I$. If $n = 1$ then we have $\alpha \in \Psi^2$, hence we get $\alpha \in \Psi_I$ by Corollary 3.4. Suppose that $n > 1$, then $\alpha$ may assume that some $\beta_i$ is non-zero outside of $I$, hence we also have $\langle \alpha_0 \rho(\delta(\beta_i)), \beta_j \rangle > 0$ for some $j \neq i$. By Lemma 3.8 we get then $\beta_i + \beta_j \in \Psi^2$, therefore $\alpha$ can be written as a sum of $n - 1$ weakly active roots and by the inductive hypothesis we get $\alpha \in \Psi_I$.

A subroot system $\Phi' \subset \Phi$ is called complete if $\Phi' = \mathbb{Q} \Phi' \cap \Phi$.

Corollary 3.10. Given $I \subset \mathcal{D}^*$, we have

$$\Phi_I = \mathbb{Q}_I \cap \Phi^+ = \mathbb{N}^2 \cap \Phi^+ \quad \text{and} \quad \Psi_I = \Phi_I \cap \Phi^2.$$ 

In particular $\Delta_I \subset \Psi_I$ and $\Phi_I$ is a complete subroot system of $\Phi$, and $I$ is recovered by $\Phi_I$.

**Proof.** By the previous proposition we get the equalities $\Phi^+ = \Phi^2 \cap \Phi^+ = \mathbb{N}^2 \cap \Phi^+$. In particular it follows $\Phi_I = \mathbb{Q}_I \cap \Phi$, and the inclusion $\Delta_I \subset \Psi_I^2$ also follows. Combining with Corollary 3.4 we get then $\Psi_0 = \Phi_I^2 \cap \Phi^2$, and the last claim follows by the equality $I = \delta(\Psi)^2$.

By the definition of $\Phi_I$, we have $\text{rk}(\Phi_I) = \dim(\mathbb{Q}_I)$, in some cases this rank is easy computable:

- $0 \leq \text{rk}(\Phi_I) \leq |\Sigma|$.
- $\text{rk}(\Phi_I) = 0$ if and only if $I = \emptyset$.
- $\text{rk}(\Phi_I) = |\Sigma|$ if and only if $I = \mathcal{D}^*$.
- If $I \subset D$ then $\text{rk}(\Phi_I) = |\delta^{-1}(D)|$ (see [1, Corollary 1]).
- If $T \subset H$ then $\text{rk}(\Phi_I) = |T|$ for all $I \subset \mathcal{D}^*$ (see [1, Theorem 1]).

From the above corollary it follows that $\Phi_I^2$ is a parabolic subroot system of $\Phi$, thanks to the following known fact.

**Lemma 3.11.** A subroot system $\Phi' \subset \Phi$ is parabolic if and only if it is complete.

**Proof.** A parabolic subroot system is obviously complete. Conversely, suppose that $\Phi'$ is complete. Let $V$ be the rational vector space generated by $\Phi$, let $E \subset V$ be a subspace such that $\Phi' = \Phi \cap E$. Let $\langle .., .. \rangle$ be a $W$-invariant scalar product on $V$ and let $x \in E$ be the normal vector of $E$. Given $\varepsilon > 0$, denote $B_\varepsilon$ the ball of radius $\varepsilon$ with center in $x$. If $\varepsilon$ is small enough, we may find a regular element $x_0 \in B_\varepsilon$.

We thank A. Maffei for suggesting this argument.
such that \(|(α, x_0)| < ε \) for all \(α ∈ \Phi’ \) and \(|(α, x_0)| > ε \) for all \(α ∈ \Phi \setminus \Phi’ \). Denote by \(\Phi^+_0 ⊂ \Phi \) the set of positive roots defined by \(x_0\), let \(Δ_0 ⊂ \Phi^+_0 \) be the corresponding basis and set \(Δ'_0 = Δ_0 ∩ E\). Let now \(α ∈ \Phi’ \cap \Phi^+_0 \) and write \(α = \sum_{β ∈ Δ_0} n_β β\): the previous inequalities imply then \(n_β = 0 \) for all \(β ∈ Δ_0 \setminus Δ'_0\), therefore \(α ∈ NΔ_0\), and a similar claim is proved for the negative roots lying on \(E\). It follows that \(\Phi’\) is the parabolic subroot system of \(Φ \) generated by \(Δ'_0\).

**Corollary 3.12.** For all \(I ⊂ \mathcal{D}^*\) there exists \(v \in W\) with \(v(Δ_I) ⊂ Δ\).

We get back to our study of the \(T\)-orbits in \(B/H\). Recall that these are parametrized by the subsets \(I ⊂ \mathcal{D}^*\), given such an \(I\) we denote by \(Ψ_I \subset B/H\) the corresponding \(T\)-orbit. By Proposition 2.13 we may think to this parametrization in the following two ways.

i) Let \(I ⊂ \mathcal{D}^*\), then we have \(Ψ_I = \bigcap_{D \notin I} w_D D \cap B/H\).

ii) Let \(I = \{D_1, \ldots, D_s\} \subset \mathcal{D}^*\) and let \(α_1, \ldots, α_s ∈ \Psi\) with \(δ(α_i) = D_i\), then we have

\[Ψ_I = T_{U_{α_1}} \cdots U_{α_m} H/H\]

(where we denote \(U_α = U_α \setminus \{e\}\)).

**Proposition 3.13.** Let \(I ⊂ \mathcal{D}^*\), then \(Ψ_I ⊂ X_T(Ψ_I)\).

**Proof.** Since \(ZΦ_I = ZΨ_I\), it is enough to prove that \(Ψ_I ⊂ X_T(Ψ_I)\). On the other hand if \(J \subset I\) we have \(X_T(Ψ_I) \subset X_T(Ψ_J)\), therefore it is enough to consider the case where \(I\) contains a single element.

Suppose \(I = \{D\}\), let \(β ∈ Ψ_I\) and let \(B_0 = TU_β\). Then \(Ψ_I = B_0H/H\) is homogeneous under \(B_0\), and we have an isomorphism \(Ψ_I ≃ B_0/0 B_0 \cap H\). The unipotent radical \((B_0 ∩ H)'<\) is a connected subgroup of \(B_0\), hence is trivial since \(Ψ_I \neq Ψ_I\). On the other hand by construction \(T ∩ H \subset H\) is a maximal diagonalizable subgroup, therefore \(B_0 \cap H = T \cap H\) and we get an isomorphism of \(T\)-varieties \(Ψ_I ≃ T/T ∩ H \times U_β\). This shows that \(Ψ_I\) possesses a \(T\)-eigenfunction of weight \(β\), and the claim follows. □

**Corollary 3.14.** We have

\[|I| ≤ rk Φ_I ≤ dim(T/T_H) + |I| ≤ |Σ| - |\mathcal{D}^* \setminus I|\]

**Proof.** The formula follows from the equality \(rk X_T(Ψ_I) = dim(Ψ_I) = dim(T/T_H) + |I|\) together with \(dim(T/T_H) = |Σ| - |\mathcal{D}^*|\). □

Given \(α ∈ \Psi^4\), we are now ready to describe combinatorially the action of the root subgroup \(U_α\) on the set of the \(T\)-orbits in \(B/H\).

**Proposition 3.15.** Let \(α ∈ \Phi^+\) and \(I ⊂ \mathcal{D}^*\). The orbit \(Ψ_I\) is not \(U_α\)-stable if and only if \(α ∈ Ψ^4(I)\).

**Proof.** If \(α ∉ Ψ^4\) then \(Ψ_I\) is \(U_α\)-stable, therefore we may assume that \(α ∈ Ψ^4\). In this case the action of \(U_α\) on \(B/H\) falls under the considerations of Theorem 2.8. Observe that \(T\) normalizes \(U_α\), therefore \(U_αΨ_I\) is a union of \(T\)-orbits. Moreover, as already noticed, a \(T\)-orbit in \(B/H\) is uniquely determined by the \(T\)-stable prime divisors that contain it.

The consequence is that \(Ψ_I\) is \(U_α\)-stable if and only if, for all \(x ∈ Ψ_I\) and for all \(α ∈ U_α\), the following statement holds: we have \(u_α x ∈ D\) for any \(D ∈ Div^T(B/H)\) containing \(Ψ_I\), and \(u_α x ∉ D'\) for any \(D' ∈ Div^T(B/H)\) not containing \(Ψ_I\). On the other hand \(δ(α)\) is the unique \(T\)-stable prime divisor which is not \(U_α\)-stable, therefore the above statement on \(u_α x\) is true for all \(D\) and all \(D'\) different from \(δ(α)\).

Let now \(f ∈ \mathcal{C}[B/H]\) be a global equation of \(δ(α)\), and let \(f_α ∈ \mathcal{C}(B/H)^{(T)}\) be a \(T\)-semi-invariant function of weight \(α\), normalized as in Theorem 2.8. Being \(α ∈ Ψ^4\), the function \(f_α\) on \(B/H\) has its unique pole on \(δ(α)\), hence \(F_α = f_α\) is regular on \(B/H\) and doesn’t vanish on \(δ(α)\). Let \(ξ ↦ u_α(ξ)\) be a \(T\)-equivariant parametrization of \(U_α ≃ C_α\). Then by formula (3) of Theorem 2.8 we have

\[f(u_α(ξ) · x) = f(x) + ξ F_α(x)\]

Therefore \(Ψ_I\) is \(U_α\)-stable if and only if \(x\) and \(u_α(ξ) · x\) are both zeros or non-zeros of \(f\) (for all \(ξ ∈ C_α\)), if and only if \(F_α(x) = 0\). This last statement is equivalent to the fact that \(F_α\) vanishes on some \(T\)-stable prime divisor \(E\) containing \(x\). Since \(F_α\) doesn’t vanish on \(δ(α)\) then \(E ≠ δ(α)\), and hence \(f_α\) must vanish on \(E\) since \(f\) doesn’t. In other words \(Ψ_I\) is \(U_α\)-stable if and only if \(⟨w_0 β(E), α⟩ > 0\) for some \(E\) containing \(Ψ_I\). Thanks to Corollary 3.13 the proof is complete. □
Corollary 3.16. Let \( I \subset \mathcal{D}^* \) and let \( \alpha \in \Psi^i(I) \), then we have
\[
U_\alpha \Phi_I = \begin{cases} \\
\mathcal{W}_I \cup \mathcal{W}_I(\delta(\alpha)) & \text{if } \alpha \in \Psi^i(I) \setminus \Psi^i_I \\
\mathcal{W}_I \cup \mathcal{W}_I(\delta(\alpha)) & \text{if } \alpha \in \Psi^i_I 
\end{cases}
\]
In particular, the followings hold (where \( \overline{\Phi}_I \) denotes the closure of \( \Phi_I \) in \( B/H \)):
\begin{enumerate}
  \item If \( U_\alpha \Phi_I \neq \Phi_I \), then \( U_\alpha \) acts on \( \overline{\Phi}_I \) if and only if \( \alpha \in \Psi^i_I \).
  \item If \( \alpha \in \Psi^i_I \), then \( \alpha \in \Psi^i(I \setminus \delta(\alpha)) \).
\end{enumerate}

Proof. Being \( T \)-stable, \( U_\alpha \Phi_I \) is a union of \( T \)-orbits. As in the proof of Proposition 3.13, we have that \( U_\alpha \) preserves the property of being on a \( T \)-stable prime divisor of \( B/H \) for any point \( x \in \Phi_I \) and for any \( T \)-stable prime divisor of \( B/H \) except for \( w_0 \delta(\alpha) \cap B/H \). If we only consider prime divisors different from \( w_0 \delta(\alpha) \cap B/H \), the orbits \( \Phi_I \) and \( \Phi_I(\delta(\alpha)) \) (or \( \Phi_I(\delta(\alpha)) \)), depending on whether \( \delta(\alpha) \in I \) or not) are the unique \( T \)-orbits contained in the same \( T \)-stable prime divisors containing \( \Phi_I \). Therefore \( U_\alpha \Phi_I \) is the union of these two \( T \)-orbits.

Following Definition 3.7, to every subset \( I \subset \mathcal{D}^* \) we attached a closed subroot system \( \Phi_I \subset \Phi \). Being a closed subsystem, we may attach to \( I \) also a reductive subgroup \( G_I \) of \( G \), namely the subgroup generated by \( T \) together with the subgroups \( U_\alpha \) with \( \alpha \in \Phi_I \). Note that \( B_I = G_I \cap B \) the Borel subgroup of \( G_I \) associated with \( \Phi_I \).

Corollary 3.17. Taking the closures in \( B/H \), we have the equality \( \overline{\Phi}_I = B_I H \).

Proof. By Corollary 3.16, we have that \( \overline{\Phi}_I \) is \( U_\alpha \)-stable for all \( \alpha \in \Psi^i_I \). On the other hand by Corollary 3.10 we have that every \( \alpha \in \Phi^i_I \setminus \Psi^i_I \) acts trivially on \( B/H \), so the claim follows.

4. Parametrization of the \( B \)-orbits in \( G/H \)

Denote \( r_1 = \text{rk}(G/H) = |\Sigma| \) and \( r_0 = \text{rk} G - \text{rk} H = \dim(T/T_H) = \text{rk}(B/H) \). We have \( \dim(G/H) = |\Phi^+| + |\Sigma| \) and \( \text{rk}(B/H) = |\Sigma| - |\mathcal{D}^*| \).

Lemma 4.1. The ranks \( r_1 \) and \( r_0 \) are respectively the maximum and the minimum rank of an orbit \( \mathcal{O} \in \mathcal{B}(G/H) \), and we have
\[ r_1 - r_0 = |\mathcal{D}^*| = |\Psi/\sim| = \dim(U/H^0). \]
In particular \( \dim(B/H) = \dim(T/T_H) + \dim(U/H^0) = |\Sigma| \).

Proof. The maximum of the ranks of \( B \)-orbits is attained by the open \( B \)-orbit thanks to [12, Theorem 2.2]. The rank of a \( B \)-orbit \( BgH/H \) on the other hand is the rank of the group of \( B \)-characters that vanish on \( B \cap H \); since \( H \subset B \) the rank is minimal if \( g = e \).

Let us show the equalities above. We can assume that a maximal diagonalizable subgroup \( T_H \) of \( H \) is contained in \( T \). Then the minimal rank is also attained by \( \mathcal{O} = Bw_0H/H \). On the other hand one shows as in Proposition 2.6 that any \( T \)-orbit of \( w_0B/H \) is the intersection of \( w_0B/H \) with a \( B \)-orbit, and the ranks correspond. Then the first equality is true, and the second has already been established.

Consider now the \( T \)-variety \( B/H \); it smooth affine and isomorphic to \( T \times T \times U/H^0 \), whence the number of its \( T \)-stable prime divisors is \( \dim(U/H^0) \). The last equality follows.

Definition 4.2. Given \( w \in W \) and \( I \subset \mathcal{D}^* \) we set \( \mathcal{O}_{w,1} = Bw\Phi_I \) the corresponding \( B \)-orbit in \( G/H \).

The set \( I \) is called a representative of \( \mathcal{O}_{w,1} \).

Every element in \( \mathcal{B}(G/H) \) is of the shape \( \mathcal{O}_{w,1} \) for some \( w \) and \( I \). While \( w \) is uniquely determined since the image of \( \mathcal{O}_{w,1} \) in \( G/B \) is \( BwB/B \), we may have several possible representatives \( I \) for the same \( B \)-orbit.

Theorem 4.3. 
\begin{enumerate}
  \item Every \( \mathcal{O} \in \mathcal{B}(G/H) \) admits a unique minimal representative \( m \) and a unique maximal representative \( M \), and \( I \) is a representative for \( \mathcal{O} \) if and only if \( m \subset I \subset M \).
  \item Suppose that \( \mathcal{O} = \mathcal{O}_{w,1} \) and denote
    \[ I(w) = \delta(\Psi^i(I) \setminus \Phi^+(w)) \]
    Then we have \( m = I \setminus I(w) \) and \( M = I \cup I(w) \).
  \item The following formulas hold:
    \[ \dim(\mathcal{O}) - \dim(B/H) = \ell(w) - |\mathcal{D}^* \setminus M|, \]
    \[ \text{rk}(\mathcal{O}) - \text{rk}(B/H) = |m|. \]
    More precisely, we have \( \chi_B(\mathcal{O}) = w\chi_B(\Phi_m) \), and in particular \( w(\Phi_m) \subset \chi_B(\mathcal{O}) \).
\end{enumerate}
Proof. i) Suppose \( \mathcal{O} = Bw_u H / H \). Notice that \( I \) is a representative of \( \mathcal{O} \) if and only if \( \mathcal{O} \setminus \mathcal{O} \cap B / H \). Since \( B / H \) is a toric variety and since \( w^{-1} \mathcal{O} \cap B / H \) is \( T \)-stable, the claim is equivalent to the fact that \( w^{-1} \mathcal{O} \cap B / H \) contains a unique minimal \( T \)-orbit and a unique maximal \( T \)-orbit.

If \( u \in w^{-1} \mathcal{O} \cap B / H \), then we have \( w^{-1} \mathcal{O} \cap B / H = (B \cap B^w) u H / H \). Therefore \( w^{-1} \mathcal{O} \cap B / H \) is homogeneous for a connected solvable group and it follows that it is an irreducible affine subvariety of the affine \( T \)-variety \( B / H \). It also follows that \( w^{-1} \mathcal{O} \cap B / H \) is itself a toric variety under the action of a quotient of \( T \). Being irreducible, it contains a unique maximal \( T \)-orbit, whereas being affine it has a unique closed \( T \)-orbit, which is minimal. It also follows that \( w^{-1} \mathcal{O} \cap B / H \) is open in its closure, whence it contains exactly all \( T \)-orbits whose closures contain the minimal \( T \)-orbit and are contained in the closure of the maximal one.

ii) Since \( m \) is contained in every representative of \( \mathcal{O} \), it is enough to consider the case \( I = m \). By the description of the action of \( B \cap B^w \) on \( B / H \), for every \( m \in M \setminus m \) there exists \( \beta \in \Psi \) and \( \gamma \in \Psi_m \) such that \( \beta - \gamma \in \mathbb{N} \) \( \Phi^+ \) and \( \delta(\beta) = D \). Let \( \alpha = \beta - \gamma \), then by Lemma 3.5 we have \( \alpha \in \Phi^+ \), hence \( \alpha \in \Psi_I^+(m) \). On the other hand we have \( w(\alpha) \in \mathbb{N} \Phi^+ \), hence \( \alpha \in \Phi^+ \). Therefore \( \alpha \in \Psi^+(m) \) \( \Phi^+(w) \), and being \( \delta(\alpha) = \delta(\beta) \) it follows \( D = m(w) \).

iii) For the dimension formula, notice that \( \dim(BwB/H) - \dim(\mathcal{O}) = \dim(B/H) - \dim(w^{-1} \mathcal{O} \cap B/H) \). On the other hand by the definition of \( M \) we have \( \dim(B/H) - \dim(w^{-1} \mathcal{O} \cap B/H) = |\mathcal{O} \times M| \), and the claim follows since \( \dim(BwB/H) = l(w) + \dim(B/H) \).

For the rank formula, fix a base point \( x_m \in \mathcal{O}_m \). Notice that \( x_m \) is a standard base point in \( (B \cap B^w) \mathcal{O}_m \). Indeed, if \( x \in (B \cap B^w) \mathcal{O}_m \) is a standard base point, then \( T \) is a closed \( T \)-orbit by Lemma 4.3. On the other hand \( (B \cap B^w) \mathcal{O}_m \) contains a unique closed \( T \)-orbit, namely \( \mathcal{O}_m \), therefore \( x_m \) is standard. We now show that \( x_m \in \mathcal{O} \) is also a standard base point, namely \( \text{Stab}_B(wx_m) \) is a maximal diagonalizable subgroup of \( \text{Stab}_B(wx_m) \). We have \( \text{Stab}_B(wx_m) = w \text{Stab}_{B}(x_m)w^{-1} \), and since \( \text{Stab}_{B}(x_m) \subseteq B \) we also have

\[
\text{Stab}_{B}(wx_m) = B \cap w \text{Stab}_{B}(x_m)w^{-1} = w \text{Stab}_{B}(BwB/H)(xm)w^{-1}
\]

We already noticed that \( x_m \in (B \cap B^w) \mathcal{O}_m \) is a standard base point, therefore conjugating with \( w \) we get that \( wx_m \in \mathcal{O} \) is a standard base point as well. By Lemma 1.4 we have then

\[
\text{X}_B(\mathcal{O}) = \text{X}_T(wx_m) = w \text{X}_T(\mathcal{O}_m),
\]

and by Proposition 4.3 it follows that \( w(\Phi_m) \subseteq \text{X}_B(\mathcal{O}) \).

**Definition 4.4.** Given \( w \in W \) and \( I \subset \mathcal{O} \) we denote by \( m_{w,I} \) the minimal representative of \( \mathcal{O}_{w,I} \) and by \( M_{w,I} \) the maximal representative of \( \mathcal{O}_{w,I} \). We say that \( (w,I) \) is a reduced pair if \( I = m_{w,I} \) (namely \( I \cap I(w) = \emptyset \)) and it is called an extended pair if \( I = M_{w,I} \) (namely \( I(w) \subset I \)). Given \( w \in W \) and \( I \subset \mathcal{O} \), we will call \((w, m_{w,I})\) and \((w, M_{w,I})\) respectively the reduction and the extension of \((w, I)\).

**Remark 4.5.** Given \( I \subset \mathcal{O} \), we have by definition \( \Psi_I^+ = \Psi^+(I) \cap \delta^{-1}(I) \), and Theorem 4.3 implies \( \Psi_{M_{w,I}}^+ \subseteq \Phi^+(w) \). The following characterizations of reduced pairs and of extended pairs follow:

i) The pair \((w, I)\) is reduced if and only if \( \Psi_I^+ \subseteq \Phi^+(w) \), and only if \( \Phi_I^+ \subseteq \Phi^+(w) \).

ii) The pair \((w, I)\) is extended if and only if \( \Psi_I^+ \subseteq \Phi^+(w) \).

**Corollary 4.6.** The map \((w, I) \mapsto \mathcal{O}_{w,I}\) gives a parametrization by reduced (resp. extended) pairs

\[
\{(w, I) \mid \Phi_I^+ \subseteq \Phi^+(w)\} \longleftrightarrow \mathcal{O}(G/H) \longleftrightarrow \{(w, I) \mid \Psi_I^+ \subseteq \Phi^+(w)\}.
\]

Moreover the followings hold:

i) If \((w, I)\) is reduced, then we have \( \text{rk}(\mathcal{O}_{w,I}) = \text{rk}(B/H) = |I| \).

ii) If \((w, I)\) is extended, then we have \( \text{dim}(\mathcal{O}_{w,I}) = \text{dim}(B/H) = l(w) - |\mathcal{O} \setminus I| \).

**Remark 4.7.** The following properties easily follow from Remark 4.5.

i) All the pairs of the shape \((w, \emptyset)\) are reduced, and all the pairs of the shape \((w, \mathcal{O}^*)\) are extended.

ii) All the pairs of the shape \((w_0, I)\) are both reduced and extended.

iii) If \((w, I)\) is reduced (resp. extended) and if \( J \subset I \), then \((w, J)\) is also reduced (resp. extended).

iv) If \((w, I)\) is reduced (resp. extended) and if \( v \subseteq w \) (i.e. \( w \) is a right subexpression of \( v \)), then \((w, \emptyset)\) is also reduced (resp. extended).

**Corollary 4.8.** Suppose that \((w, I)\) is reduced. Then \( \mathcal{O}_{w,I} \) is closed in \( G/H \) if and only if the following conditions hold:

i) \( I = \emptyset \);
Lemma 5.1.

i) \( G/H \) and \( B \) all + + 

Example then we may simplify the dimension formula as follows:

\[ \dim(\mathcal{O}_{w,I}) = l(w) - |\Psi \cap \Phi^+(w)| = \dim(B/H) + |\Phi^+(w) \setminus \Psi|. \]

\[ \dim(\mathcal{O}_{w,I}) = \dim(B/H) + l(w) - |\Psi \cap \Phi^+(w)| = \dim(B/H) + |\Phi^+(w) \setminus \Psi|. \]

Given a reduced pair \( (w,I) \), the description of the maximal representative of \( \mathcal{O}_{w,I} \) reduces to

\[ M_{w,J} = I \cup \{ \beta \in \Psi | \beta - \alpha \in N\Psi, \exists \alpha \in \Phi^+ \setminus \Phi(w) \}. \]

Notice that \( \Psi \setminus M_{w,J} \subset \Phi^+(w) \). Indeed if \( I \subset J \subset \mathcal{O} \) then it also holds \( M_{w,J} \subset M_{w,I} \), therefore we get \( \Phi^+(w) = M_{w,J} \subset M_{w,I} \). Denote

\[ \Phi^+(w,I) = \Phi^+(w) \setminus (\Psi \setminus M_{w,I}), \]

then we may simplify the dimension formula as follows:

\[ \dim(\mathcal{O}_{w,I}) = \dim(B/H) + |\Phi^+(w,I)|. \]

Notice that we have \( \Phi^+_J \subset \Phi^+(w,I) \). Also, by the above inclusions we have \( M_{w,I} = \Psi \setminus (\Phi^+(w) \setminus \Phi^+(w,I)) \), so that the orbit \( \mathcal{O}_{w,I} \) is uniquely determined by \( w \) together with the set \( \Phi^+(w,I) \).

5. Stabilizers and weight polytopes

Since \( (w_0,I) \) is reduced for every \( I \), we have exactly \( 2^{r_1-r_0} \) orbits of this shape and these are all the orbits which project on the open Schubert cell via the projection \( G/H \rightarrow G/B \). Since \( I(w_0) = \emptyset \) for all \( I \) we have \( r(\mathcal{O}_{w,I}) = |I| + r_0 \) and \( \dim(\mathcal{O}_{w,I}) = \dim(G/H) - |\mathcal{O}^+ \setminus I| \).

Notice that, if \( (w,I) \) is a reduced pair, the inclusion \( \Phi^+_I \subset \Phi^+(w) \) implies the equalities

\[ w^{-1}(\Delta^-) \cap \Phi^+_I = w^{-1}(\Delta^-) \cap \Psi^I = w^{-1}(\Delta^-) \cap \Delta. \]

In order to study the actions of \( W \) and of \( M(W) \) on \( B(G/H) \) in terms of reduced and extended pairs, we turn to a closer look to the possible cases arising for the \( B \)-stable subsets of the shape \( P_\alpha \mathcal{O}_{w,I} \), where \( \mathcal{O}_{w,I} \in B(G/H) \) and \( \alpha \in \Delta \).

**Lemma 5.1.**

i) Let \( (w,I) \) be a reduced pair, let \( \alpha \in \Delta \) and denote \( \beta = -w^{-1}(\alpha) \). Then we have

\[ s_\alpha \cdot \mathcal{O}_{w,I} = \begin{cases} \mathcal{O}_{w,I} & \text{if } \beta \in \Psi^I \\ \mathcal{O}_{s_\alpha w,I} & \text{otherwise} \end{cases} \]

where all the orbits above are expressed in terms of reduced pairs.

ii) Let \( (w,I) \) be an extended pair, let \( \alpha \in \Delta \) and denote \( \beta = -w^{-1}(\alpha) \). Then we have

\[ m(s_\alpha) \cdot \mathcal{O}_{w,I} = \begin{cases} \mathcal{O}_{s_\alpha w,I} & \text{if } \beta \in \Psi^I(I) \\ \mathcal{O}_{s_\alpha w,I} & \text{if } \beta \notin \Psi^I(I) \end{cases} \]

where all the orbits above are expressed in terms of extended pairs. In particular, we have

\[ m(s_\alpha) \cdot \mathcal{O}_{w,I} = \mathcal{O}_{w,I} \text{ if and only if } \beta \in \Phi^+(w) \setminus (\Psi^I(I) \setminus \Psi^I). \]
Proof. Let \( w \in W \) and \( I \subset \mathcal{P}^* \). By the analysis in Section 1.1, the \( B \)-stable subset \( P_\alpha \mathcal{O}_{w,I} = P_\alpha w \mathcal{U}_I \) decomposes in the union of two \( B \)-orbits which are permuted by the action of \( s_\alpha \), or in the union of three \( B \)-orbits, an open one fixed by \( s_\alpha \) and two of codimension one which are permuted by \( s_\alpha \).

Suppose that \( w^{-1}(\alpha) \in \Phi^+ \) and consider the decomposition

\[
P_\alpha = Bs_\alpha \cup Bs_\alpha Bs_\alpha = Bs_\alpha \cup BU_{-\alpha}.
\]

Denote \( v = s_\alpha w \) and \( \beta = -w^{-1}(\alpha) \), then \( P_\alpha \mathcal{O}_{w,I} = Bv \mathcal{U}_I \cup Bw \mathcal{U}_I \). By Proposition 3.15 and by Corollary 3.16 we have

\[
U_\beta \mathcal{U}_I = \begin{cases} 
\mathcal{U}_I & \text{if } \beta \in \Phi^+ \setminus \Psi^f(I) \\
\mathcal{U}_I \cup \mathcal{U}_{1\cup \{\delta(\beta)\}} & \text{if } \beta \in \Psi^f(I) \setminus \Psi^f_I \\
\mathcal{U}_I \cup \mathcal{U}_{1\cup \{\delta(\beta)\}} & \text{if } \beta \in \Psi^f_I
\end{cases}
\]

Correspondingly, we have the following possibilities, where we denote \( m = m_{w,I} \) and \( M = M_{w,I} \):

\( \text{U) Suppose } \beta \in \Phi^+ \setminus \Psi^f(m) \text{ or } \beta \in \Psi^f(m) \cap \Psi^f_M \setminus \Psi^f_{m} \).

We have in this case \( U_\beta \mathcal{U}_I \subset (B \cap B^w) \mathcal{U}_I \), hence \( BwU_\beta \mathcal{U}_I = \mathcal{O}_{w,I} \). Therefore \( P_\alpha \mathcal{O}_{w,I} = \mathcal{O}_{v,I} \cup \mathcal{O}_{w,I} \) decomposes in the union of two orbits as represented in the following diagram, where we have

- \( \dim \mathcal{O}_{v,I} = \dim \mathcal{O}_{w,I} - 1; \)
- \( \text{rk} \mathcal{O}_{v,I} = \text{rk} \mathcal{O}_{w,I}; \)
- \( M_{v,I} = M; \)
- \( m_{v,I} = m. \)

\[
\begin{array}{c}
\mathcal{O}_{w,m} = \mathcal{O}_{w,M} \\
\mathcal{O}_{v,m} = \mathcal{O}_{v,M}
\end{array}
\]

T1) Suppose that \( \beta \in \Psi^f_{m} \).

We have in this case \( U_\beta \mathcal{U}_I = \mathcal{U}_I \cup \mathcal{U}_{I'} \), where we set \( I' = I \setminus \{\delta(\beta)\} \), and by Theorem 4.3 we have \( \mathcal{U}_{I'} \not\subset (B \cap B^w) \mathcal{U}_I \). Therefore \( P_\alpha \mathcal{O}_{w,I} = \mathcal{O}_{v,I} \cup \mathcal{O}_{w,I} \cup \mathcal{O}_{w,I'} \) decomposes in the union of three orbits as represented by the following diagram, where we have

- \( \dim \mathcal{O}_{v,I} = \dim \mathcal{O}_{w,I} = \dim \mathcal{O}_{w,I'} - 1; \)
- \( \text{rk} \mathcal{O}_{v,I} = \text{rk} \mathcal{O}_{w,I} = \text{rk} \mathcal{O}_{w,I'} - 1; \)
- \( M_{v,I} = M, M_{w,I'} = M \setminus \{\delta(\beta)\}; \)
- \( m_{v,I} = m, m_{w,I'} = m \setminus \{\delta(\beta)\}. \)

\[
\begin{array}{c}
\mathcal{O}_{w,m \setminus \{\delta(\beta)\}} = \mathcal{O}_{v,M} \\
\mathcal{O}_{w,m \setminus \{\delta(\beta)\}} = \mathcal{O}_{w,M \setminus \{\delta(\beta)\}}
\end{array}
\]

T2) Suppose \( \beta \in \Psi^f_{M} \).

Then \( U_\beta \mathcal{U}_I = \mathcal{U}_I \cup \mathcal{U}_{I'} \), where we set \( I' = I \setminus \{\delta(\beta)\} \), and by Theorem 4.3 we have \( \mathcal{U}_{I'} \not\subset (B \cap B^w) \mathcal{U}_I \). Therefore \( P_\alpha \mathcal{O}_{w,I} = \mathcal{O}_{v,I} \cup \mathcal{O}_{w,I} \cup \mathcal{O}_{w,I'} \) decomposes in the union of three orbits as represented by the following diagram, where we have

- \( \dim \mathcal{O}_{v,I} = \dim \mathcal{O}_{w,I} = \dim \mathcal{O}_{w,I'} - 1; \)
- \( \text{rk} \mathcal{O}_{v,I} = \text{rk} \mathcal{O}_{w,I} = \text{rk} \mathcal{O}_{w,I'} - 1; \)
- \( M_{v,I} = M, M_{w,I'} = M \cup \{\delta(\beta)\}; \)
- \( m_{v,I} = m, m_{w,I'} = m \cup \{\delta(\beta)\}. \)

\[
\begin{array}{c}
\mathcal{O}_{w,m \cup \{\delta(\beta)\}} = \mathcal{O}_{w,M \cup \{\delta(\beta)\}} \\
\mathcal{O}_{v,m} = \mathcal{O}_{v,M \cup \{\delta(\beta)\}} \\
\mathcal{O}_{v,m} = \mathcal{O}_{v,M}
\end{array}
\]

Suppose now that \( w^{-1}(\alpha) \in \Phi^+ \) and consider the decomposition

\[
P_\alpha = B \cup Bs_\alpha B = B \cup Bs_\alpha U_{-\alpha}.
\]
Denote $v = s_\alpha w$ and $\beta = w^{-1}(\alpha)$, then $P_\alpha \mathcal{G}_{w,I} = Bw_\beta \mathcal{U} \cup BvU_\beta \mathcal{U}_I$. Since $(w,I)$ is reduced we have $\beta \notin \Psi^+_I$ and $\beta \notin \Psi^+(I) \setminus \Psi^+_M$. Reasoning as before, we get the following possibilities, where we denote $m = m_{w,I}, M = M_{w,I}, M' = M_{v,I}$:

U) Suppose $\beta \in \Phi^+ \setminus \Psi^2(m)$ or $\beta \in \Psi^2(m) \setminus \Psi^2_M$. Then we have $U_\beta \mathcal{U} \subset (B \cap B^*) \mathcal{U}_I$, hence $BvU_\beta \mathcal{U}_I = \mathcal{O}_{v,I}$. Therefore $P_\alpha \mathcal{O}_{w,I} = \mathcal{O}_{w,I} \cup \mathcal{O}_{v,I}$ decomposes in the union of two orbits as represented in the following diagram, where we have:

- $\dim \mathcal{O}_{w,I} = \dim \mathcal{O}_{v,I} + 1$;
- $\text{rk} \mathcal{O}_{w,I} = \text{rk} \mathcal{O}_{v,I}$;
- $M_{v,I} = M$;
- $m_{w,I} = m$.

\[ \begin{array}{c}
\mathcal{O}_{w,m} = \mathcal{O}_{v,M} \\
\mathcal{O}_{w,m} = \mathcal{O}_{w,M}
\end{array} \]

T) Suppose $\beta \in \Psi^2(m) \setminus \Psi^2_M$. Then we have $U_\beta \mathcal{U} = \mathcal{U}_I \cup \mathcal{U}_I$, where we set $I' = I \cup \{\delta(\beta)\}$, and by Theorem 5.3 we have $\mathcal{U}_I \not\subset (B \cap B^*) \mathcal{U}_I$. Therefore $P_\alpha \mathcal{O}_{w,I} = \mathcal{O}_{w,I} \cup \mathcal{O}_{v,I} \cup \mathcal{O}_{v,I'}$ decomposes in the union of three orbits as represented by the following diagram, where we have:

- $\dim \mathcal{O}_{v,I} = \dim \mathcal{O}_{w,I} = \dim \mathcal{O}_{v,I'} - 1$;
- $\text{rk} \mathcal{O}_{v,I} = \text{rk} \mathcal{O}_{w,I} = \text{rk} \mathcal{O}_{v,I'} - 1$;
- $M_{v,I} = M \setminus \{\delta(\beta)\}, M_{v,I'} = M$;
- $m_{w,I} = m, m_{v,I'} = m \cup \{\delta(\beta)\}$.

\[ \begin{array}{c}
\mathcal{O}_{v,m,\cup(\delta(\beta))} = \mathcal{O}_{v,M} \\
\mathcal{O}_{w,m} = \mathcal{O}_{w,M} \\
\mathcal{O}_{v,m} = \mathcal{O}_{v,M \setminus \{\delta(\beta)\}}
\end{array} \]

The claims follow now by the definitions of the actions of $s_\alpha$ and of $m(\alpha)$.

\[ \square \]

**Corollary 5.2.** Let $(w,I)$ be an extended pair. The orbit $\mathcal{O}_{w,I}$ is minimal with respect to the weak order on $\mathcal{B}(G/H)$ if and only if

\[ w^{-1}(\Delta^-) \cap \Phi^+ \subset \Psi^+(I) \setminus \Psi^+_I. \]

We now focus on the action of $W$ on $\mathcal{B}(G/H)$. First we show that the minimal representative of an orbit is a complete invariant for the action of $W$. Then we will describe the stabilizers of the actions in terms of reduced pairs.

**Theorem 5.3.** Let $(w,I), (v,J)$ be reduced pairs. $\mathcal{O}_{w,I}$ and $\mathcal{O}_{v,J}$ are in the same $W$-orbit if and only if $I = J$, in which case we have $\mathcal{O}_{v,I} = vw^{-1} \cdot \mathcal{O}_{w,I}$.

**Proof.** By Lemma 5.4 it follows that the minimal representative $I$ is an invariant for the action of $W$. To show it is a complete invariant we show that, whenever $(w,I)$ is reduced, we have $\mathcal{O}_{w,I} = w_0 w^{-1} \cdot \mathcal{O}_{w,I}$, so that every $W$-orbit in $\mathcal{B}(G/H)$ contains a unique orbit which projects dominantly on $G/B$.

Let $w_0w^{-1} = s_{\alpha_1} \cdots s_{\alpha_n}$ be a reduced expression such that, for every $i$, it holds $w_i^{-1}(\alpha_{i-1}) \in \Phi^+$, where we denote

\[ w_i = \begin{cases} w & \text{if } i = n \\ s_{\alpha_i+1} \cdots s_{\alpha_n}w & \text{if } 0 < i < n \end{cases} \]

Therefore, for every $i \leq n$, we have $\Phi^+(w_{i-1}) = \Phi^+(w_i) \cup \{w_i^{-1}(\alpha_{i-1})\}$. Thus $(w,I)$ is reduced, and we have $\mathcal{O}_{w_{i-1},I} = s_{i-1} \cdot \mathcal{O}_{w,I}$. Combining all the steps we get that $\mathcal{O}_{w,I} = w_0 w^{-1} \cdot \mathcal{O}_{w,I}$, and the last claim also follows.

\[ \square \]

**Corollary 5.4.** The Weyl group $W$ has $2^{r_+ - r_0}$ orbits in $\mathcal{B}(G/H)$, and a complete set of representatives is given by the orbits which project dominantly on $G/B$, namely by the subsets of $\varPhi^*$.

We now turn to the description of the stabilizers for the action of $W$ on $\mathcal{B}(G/H)$, and we prove the following theorem.

**Theorem 5.5.** Let $(w,I)$ be a reduced pair. Then $\text{Stab}_W(\mathcal{O}_{w,I}) = wW_I w^{-1}$.  

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We proceed by steps, proving first one of the two inclusions.

**Lemma 5.6.** Let \((w,I)\) be a reduced pair. Then \(\text{Stab}_W(\mathcal{O}_{w,I}) \subset wW_Iw^{-1}\).

**Proof.** We first show that \(\text{Stab}_W(\mathcal{O}_{w,I})\) is generated by reflections of the shape \(s_{w(\beta)}\) with \(\beta \in \Psi^+_I\), which implies the inclusion \(\text{Stab}_W(\mathcal{O}_{w,I}) \subset wW_Iw^{-1}\). Let \(v \in \text{Stab}_W(\mathcal{O}_{w,I})\), we proceed by induction on the length of \(v\). If \(l(v) = 1\), then by Lemma 5.1 we have \(v = s_\alpha\) for some \(\alpha \in w(\Psi^+_I)\), hence we assume \(l(v) > 1\).

Let \(v = s_{\alpha_m} \cdots s_{\alpha_1}\) be a reduced expression, for \(i \leq n\) we set \(v_i = s_{\alpha_i} \cdots s_{\alpha_1}\) and \(\mathcal{O}_i = v_i \cdot \mathcal{O}_{w,I}\). If \(\mathcal{O}_i \neq \mathcal{O}_{i-1}\) for all \(i \leq n\), then it would be \(v \cdot \mathcal{O}_{w,I} = \mathcal{O}_{wv,I}\), hence \(v = e\). Therefore there exists \(n \leq m\) such that \(\mathcal{O}_n = \mathcal{O}_{n-1}\), and we take \(n\) minimal with this property. If \(n = 1\) then the claim follows by the inductive hypothesis, so we assume that \(n > 1\). Denote \(\alpha = s_{\alpha_1} \cdots s_{\alpha_n} (\alpha_n)\): then \(v = v's_{\alpha_n}\), where \(v' = s_{\alpha_m} \cdots s_{\alpha_1}\).

By construction we have \(s_{\alpha_n} \in \text{Stab}_W(\mathcal{O}_{n-1})\), hence \(s_{\alpha_n} \in \text{Stab}_W(\mathcal{O}_{w,I})\), and by Lemma 5.1 it follows that \(\alpha = s_{\alpha_1} \cdots s_{\alpha_n} (\alpha_n) \in w(\Psi^+_I)\), is of the desired shape. Moreover we have \(v' \in \text{Stab}_W(\mathcal{O}_{w,I})\) and \(l(v') < l(v)\), therefore we may apply induction and we get \(v' \in wW_Iw^{-1}\).

By Theorem 5.3 every \(W\)-orbit contains a distinguished element: indeed, for every reduced \((w,I)\), we have that \(w^{-1} \cdot \mathcal{O}_{w,I}\) depends only on \(I\). We denote such an element by \(\mathcal{O}_I^w\).

**Corollary 5.7.** Let \(w_I \in W_I\) be the longest element. Then we have \(\mathcal{O}_I^w = \mathcal{O}_{w,I}^w\).

**Proof.** Let \(w \in W\) be such that \(\mathcal{O}_I^w = \mathcal{O}_{w,I}^w\), then by the property of \(\mathcal{O}_I^w\) we have \(\mathcal{O}_I^w = w^{-1} \cdot \mathcal{O}_{w,I}^w\), hence \(w \in \text{Stab}_W(\mathcal{O}_{w,I})\), and by Lemma 5.7 we get \(w \in W_I\). On the other hand \(W_I\) contains a unique element such that \((w,I)\) in reduced, which is \(w_I\).

**Proof of Theorem 5.7.** We have already shown in Lemma 5.6 that \(\text{Stab}_W(\mathcal{O}_{w,I}) \subset wW_Iw^{-1}\), we show the other inclusion. Since \(\mathcal{O}_I = w \cdot \mathcal{O}_I^w\) and since by previous corollary \(\mathcal{O}_I = \mathcal{O}_{w,I}\), it is enough to show that \(\text{Stab}_W(\mathcal{O}_I^w) = W_I\). Let \(v \in W_I\), we proceed by induction on the length \(l(v)\), where we regard \(v\) as an element of \(W\). If \(l(v) = 1\), then we have \(v = s_{\alpha}\) for some \(\alpha \in \Delta \cap \Phi_I^+ \subset \Delta_I\), hence \(\alpha = \Delta_I^+ (w_I)\) for some other \(\beta \in \Delta_I\). On the other hand \(\Delta_I \subset \Phi_I^+\), hence by Lemma 5.1 we get that \(s_{\alpha} \in \text{Stab}_W(\mathcal{O}_I^w)\).

Suppose now that \(l(v) > 1\), and let \(v = s_{\alpha_n} \cdots s_{\alpha_1}\) be a reduced expression as an element of \(W\). If \(i \leq n\), we denote \(v_i = s_{\alpha_i} \cdots s_{\alpha_1}\) and \(\mathcal{O}_i = v_i \cdot \mathcal{O}_I^w\). Since \(w_I \in W_I\) is the unique element such that \(\Phi_I^+ \subset \Phi^+(w_I)\), by the definition of the action it must be \(s_i \in \text{Stab}_W(\mathcal{O}_{i-1})\) for some \(i \leq n\). If we take \(k\) minimal with this property, then we have

\[
\mathcal{O}_{k-1} = v_{k-1} \cdot \mathcal{O}_I^w = \mathcal{O}_{v_{k-1}w_I,I}.
\]

By construction we have \(s_{\alpha_k} \in \text{Stab}_W(\mathcal{O}_{k-1})\), hence by Lemma 5.1 it follows \(s_{\alpha_k} = \Delta_I^+ (w_I) \in \Psi^+_I\). Therefore, if we denote \(\gamma = v_{k-1}^{-1}(\alpha_k)\), then we have \(\gamma \in \Phi_I^+\) and \(\gamma \in \text{Stab}_W(\mathcal{O}_I^w)\). On the other hand

\[
s_{\gamma} = s_{\alpha_n} \cdots s_{\alpha_k} = s_{\alpha_n} \cdots s_{\alpha_k} v_{k-1} = s_{\alpha_n} \cdots s_{\alpha_k} s_k \cdots s_{\alpha_1},
\]

therefore \(s_{\alpha_n} \cdots s_{\alpha_k} \cdots s_{\alpha_1} \in W_I\) and by induction on the length we get \(s_{\alpha_n} \cdots s_{\alpha_k} \cdots s_{\alpha_1} \in \text{Stab}_W(\mathcal{O}_I^w)\), and it follows \(v \in \text{Stab}_W(\mathcal{O}_I^w)\) as well.

Using the previous theorem, we now produce a combinatorial model for the action of \(W\) on \(\mathcal{B}(G/H)\) in terms of weight polytopes.

**Lemma 5.8.** Let \((w,I)\) be a reduced pair and denote \(\leq_I\) the Bruhat order on \(W_I\) defined by \(\Phi_I^+\). If \(v_1, v_2 \in W_I\) satisfy \(v_1 \leq_I v_2\), then it holds also \(w_2 \leq_I w_1\). In particular, the left coset \(wW_I\) possesses a unique minimal element and a unique maximal element with respect to the Bruhat order, namely \(ww_I\) and \(w\).

**Proof.** Denote \(G_I \subset G\) the reductive subgroup generated by \(T\) together with the root spaces \(U_\alpha\) with \(\alpha \in \Phi_I\) and denote \(B_I \subset G\) the Borel subgroup associated with \(\Phi_I^+\), then we have an embedding \(G_I/B_I \subset G/B\). In particular, being \(w_1v_2 \leq_I w_1v_1\), it follows that \(w_1v_2 \in B_Iw_1v_1B/B\).

Since \(\Phi_I^+ \subset \Phi^+(w) \cap \Phi^+(w_I)\), we have that \(\Phi^+(w) \cap \Phi_I^+ = \emptyset\), hence \(Bw_1 = Bw_Iw_1B\). Therefore we get

\[
w_2 \in Bw_I(w_1v_2)B/B \subset w_1Bw_Iv_1B/B \subset Bw_1B/B
\]

and the claim follows.

\[\square\]
Let $\lambda$ be a regular dominant weight and denote $P = \conv(W\lambda)$ its weight polytope, then the vertices of $P$ correspond bijectively to the elements of $W$. By a subpolytope of $P$ we mean the convex hull of a subset of vertices of $P$. Denote $\mathcal{I}(P)$ the set of the subpolytopes of $P$, then the Weyl group acts naturally on $\mathcal{I}(P)$. Given $I \subset \mathcal{D}^*$ denote $\mathcal{I}_I = \conv(W^I \lambda)$, and for a reduced pair $(w, I)$ we set

$$\mathcal{I}_{w,I} = \ker(w \mathcal{I}_I) = \conv(\{w\mathcal{I}_I \mid v \in \text{stab}_W(\mathcal{I}_{w,I})\}),$$

where the last equality follows from Theorem 5.5. Denote moreover

$$\mathcal{E}_{w,I} = \conv(-w(\Psi^+_I)) = \conv(-w(\Phi^+_I)) = -w(\Omega \Phi^+_I \cap Q_{\geq 0} \Delta),$$

Theorem 5.9. The map $\mathcal{E}_{w,I} \mapsto \mathcal{I}_{w,I}$ is a $W$-equivariant embedding of $\mathcal{H}(G/H)$ into $\mathcal{I}(P)$. Moreover we have $\mathcal{I}_{w,I} = P \cap (\mathcal{E}_{w,I} + w\lambda)$, so that $\dim(\mathcal{I}_{w,I}) = \text{rk}(\Phi_I)$.

Proof. Let $\mathcal{H}(G/H) \subset \mathcal{H}(G/H)$ be the $W$-orbit determined by $I$. We claim that the map $\mathcal{E}_{w,I} \mapsto \mathcal{I}_{w,I}$, seen as a map $\mathcal{H}(G/H) \mapsto \mathcal{I}(P)$, is $W$-equivariant and injective. Indeed if $(w, I)$ is reduced then by construction we have $\text{stab}_W(\mathcal{E}_{w,I}) = \text{stab}_W(\mathcal{I}_{w,I})$. In addition, this equality also implies that the map $\mathcal{E}_{w,I} \mapsto \mathcal{I}_{w,I}$ is $W$-equivariant, because together with Lemma 5.1 it assures that the actions of the simple reflections on $\mathcal{E}_{w,I}$ and on $\mathcal{I}_{w,I}$ correspond.

In particular the map $\mathcal{H}(G/H) \mapsto \mathcal{I}(P)$ is $W$-equivariant and we need only to show the injectivity, namely that $I$ is determined by $\mathcal{I}_{w,I}$. First of all $w$ is determined by $\mathcal{I}_{w,I}$. Indeed, by Lemma 5.8 the element $w$ is maximal in $w\mathcal{I}_I$ with respect to the Bruhat order and is unique with this property. Then $w\lambda$ is the unique minimal vertex of $\mathcal{I}_{w,I}$ with respect to the dominance order, and $w$ is uniquely determined by $\mathcal{I}_{w,I}$. Similarly $ww\lambda$ is the unique maximal vertex of $\mathcal{I}_{w,I}$ with respect to the dominance order. Then equality $\mathcal{I}_{w,I} = \conv(w\mathcal{I}_I)$ implies that the set of differences $\{w\lambda - w\lambda \mid v\lambda \in \mathcal{I}_{w,I}\}$ generates the semigroup $w(N\Phi^+_I)$, hence following Corollary 5.10 we may recover $I$ from $w$ and $\mathcal{I}_{w,I}$.

The last claim also follows. $\square$

Let $\mathcal{I}(P) \subset \mathcal{I}(P)$ be the set of faces of $P$, which still carries an action of $W$. In case $H = T(U, U)$ is the spherical subgroup of Example 4.9, then the previous embedding induce a $W$-equivariant bijection between $\mathcal{H}(G/H)$ and $\mathcal{I}(P)$ (see also 26 Proposition 3.5). Thanks to the following proposition, we may describe in this way a $W$-equivariant embedding of $\mathcal{H}(G/H)$ into $\mathcal{I}(P)$ for any strongly solvable spherical subgroup $H$.

Proposition 5.10. Let $H_0 = T(U, U)$ be the spherical subgroup of Example 4.9. Then there exists a $W$-equivariant embedding $\mathcal{H}(G/H) \hookrightarrow \mathcal{H}(G/H_0)$. In particular, we get a $W$-equivariant embedding of $\mathcal{H}(G/H) \hookrightarrow \mathcal{I}(P)$.

Proof. By Corollary 3.12 for every $I \subset \mathcal{D}^*$ there exists an element $v \in V$ such that $v(\Delta_I) \subset \Delta$. Fix such an element $v_I$ for every $I$, and consider the pair $(wv_I^{-1}, v_I(\Delta_I))$: since $wv_I^{-1}(\Delta_I) = -\Delta_I$, this is a reduced pair for $G/H_0$. Denote $\mathcal{Y}_I \subset \mathcal{H}(G/H_0)$ the orbit which corresponds to the pair $(wv_I^{-1}, v_I(\Delta_I))$, then by previous theorem we have $\text{stab}_W(\mathcal{Y}_I) = \text{stab}_W(\mathcal{I}_I) = W_I$. Therefore we get a $W$-equivariant map $\mathcal{H}(G/H) \mapsto \mathcal{H}(G/H_0)$, defined by $\mathcal{I}_I \mapsto \mathcal{Y}_I$ and extending equivariantly. On the other hand we have $wv_I^{-1}(v_I(\Delta_I)) = -\Delta_I$, and by Corollary 3.11 we may recover $I$ from $\mathcal{Y}_I$. Therefore the map $\mathcal{H}(G/H) \mapsto \mathcal{H}(G/H_0)$ separates $W$-orbits, and since the stabilizers of the base points coincide the injectivity follows. $\square$

Remark 5.11. Knop conjectured (see e.g. 26 [0.1]) that the variety $G/K$ of Example 4.9 has the largest number of $B$-orbits among all the homogeneous spherical $G$-varieties. The previous theorem confirms the conjecture in the case of a solvable spherical subgroup.

The parametrization of orbits via polytopes is compatible with the Bruhat order on $\mathcal{H}(G/H)$ in the following sense.

Proposition 5.12. Let $(w, I)$ and $(v, J)$ be reduced pairs and suppose that $\mathcal{I}_{v,J} \subset \mathcal{I}_{w,I}$. Then we have $\mathcal{I}_{v,J} \subset \mathcal{E}_{w,I}$.

The proposition will follow easily from the following lemma.

Lemma 5.13. Let $(w, I)$ be a reduced pair and let $J \subset M_{w,I}$. If $v \in wW_I$, then $\mathcal{I}_{v,J} \subset \mathcal{E}_{w,I}$.

Proof. Recall first the following fact, which follows by the description of the $T$-stable curves in $G/B$ (see 8 Proposition 3.9): if $x \in W$ and if $a \in \Phi^+(x)$, then it holds $xa \in BxU_a$. 21

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Let now $\alpha_1, \ldots, \alpha_m \in \Delta_I$ with $w^{-1}v = s_{\alpha_1} \cdots s_{\alpha_m}$. If $i \leq m$ we denote $w_i = ws_{\alpha_1} \cdots s_{\alpha_i}$, then by Lemma 5.8 we have the following chain in the Bruhat order
\[ v = w_n < \ldots < w_i < \ldots < w_1 < w. \]
Denote $M = M_{w, I}$. Since $\mathcal{O}_{v, J} \subset \mathcal{O}_{v, M}$ we may assume $J = M$, and in particular that $I \subset J$. Therefore we have $\Delta_I \subset \Psi^+_J \subset \Psi^+_I$, and by Corollary 5.16 it follows that $\mathcal{O}_{J}$ is $U_{\alpha_i}$-stable for all $i$. In particular we have that $U_{\alpha_i}U_{\alpha_1} \cdots U_{\alpha_i} \mathcal{O}_{J} \subset \mathcal{O}_{J}$, and by the remark at the beginning of the proof we get
\[ \mathcal{O}_{v, J} = Bw\mathcal{O}_{J} = BuU_{\alpha_1} \cdots U_{\alpha_i} \mathcal{O}_{J} \supset BwU_{\alpha_2} \cdots U_{\alpha_i} \mathcal{O}_{J} \supset \cdots \supset Bw_{n} \mathcal{O}_{J} \supset Bv\mathcal{O}_{J}. \]

Proof of Proposition 5.14. Notice that $w^{-1}v \in W_I$, hence $W_J \subset W_I$. In particular it follows that $\Phi_J^+ \subset N\Phi_I^+ \cap \Phi^+ = \Phi_I^+$. Therefore we get
\[ J = \delta(\Psi^+_J) = \delta(\Phi^+_J \cap \Psi^+) \subset \delta(\Phi^+_I \cap \Psi^+) = \delta(\Psi^+_I) = I, \]
and the claim follows by Lemma 5.13.

More generally, we can find analogue inclusions as follows, by considering the maximal representative of an orbit instead of its minimal representative.

Given $I \subset \mathscr{D}^+$ denote $\leq_I$ the Bruhat order on $W_I$. As in the case of a standard parabolic subgroup of $W$, denote $W^I = \{ w \in W | \Phi^+_I(w) = \emptyset \} = \{ w \in W | l(ws_{\alpha}) > l(w) \ \forall \alpha \in \Phi^+_I \}$.

Proposition 5.14. Let $w \in W$ and $I \subset \mathscr{D}^+$, let $v \in wW_I$ be the representative of minimal length. Then we have
\[ Bw_{B_I} = \bigcup_{y \in W_I, \ y \leq_I w^{-1}w} Bv_{B_I}. \]

Proof. Suppose first that $w \in W^I$, namely that $\Phi^+_I(w) = \emptyset$. Notice that $B \setminus BwG_I$ is isomorphic to the flag variety of $G_I$; indeed we have that $wB_Iw^{-1} = B \cap wG_Iw^{-1}$ is a Borel subgroup in $wG_Iw^{-1}$, hence $B \setminus BwG_I \simeq B \setminus BwG_Iw^{-1} \simeq wB_Iw^{-1} \setminus wG_Iw^{-1}$ is a translate of the flag variety of $wG_Iw^{-1}$.

Suppose now that $w \notin W^I$ and let $v \in wW_I$ be the representative of minimal length. Then $v \in W^I$ and $w^{-1}w \in W_I \subset G_I$, hence
\[ Bw_{B_I} = BwB_I = Bv_{B_I}w^{-1}wB_I \subset Bv_{B_I}. \]

Therefore $B \setminus Bw_{B_I} = B \setminus Bv_{B_I}w^{-1}wB_I$ and the claim follows.

Corollary 5.15. Let $(w, I)$ be reduced, let $M = M_{w, I}$ and let $v \in wW_M$ be the representative of minimal length. If $y \in W_M$ and $y \leq_M w^{-1}w$, then we have $Bv_{B_M} \subset Bw_{B_I}$.

Proof. This stems from Corollary 5.17 and the above proposition.

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