VERY SMALL INTERVALS CONTAINING AT LEAST THREE PRIMES

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ABSTRACT. Let $p_n$ is the $n$-th prime. With help of the Cramér-like model, we prove that the set of intervals of the form $(2p_n, 2p_{n+1})$ containing at list 3 primes has a positive density with respect to the set of all intervals of such form.

1. Introduction

Everywhere below we understand that $p_n$ is the $n$-th prime and $\mathbb{P}$ is the class of all increasing infinite sequences of primes. If $A \in \mathbb{P}$ then we denote $\mathcal{A}$ the event that prime $p$ is in $A$. In particular, an important role in our constructions play the following sequences from $\mathbb{P}$: $A_i$ is the sequence of those primes $p_k$, for which the interval $(2p_k, 2p_{k+1})$ contains at least $i$ primes, $i = 1, 2, ...$. By $\mathcal{A}_i(n)$, we denote the event that $p_n$ is in $A_i$, $i = 1, 2, ...$

In [1] we considered the following problem. Let $p$ be an odd prime. Let, furthermore, $p_n < p/2 < p_{n+1}$. According to the Bertrand’s postulate, between $p/2$ and $p$ there exists a prime. Therefore, $p_{n+1} \leq p$. Again, by the Bertrand’s postulate, between $p$ and $2p$ there exists a prime. More subtle question is the following.

Problem 1. Consider the sequence $S$ of primes $p$ possessing the property: if $p/2$ lies in the interval $(p_n, p_{n+1})$ then there exists a prime in the interval $(p, 2p_{n+1})$. With what probability a random prime $q$ belongs to $S$ (or the event $S$ does occur)?

In this paper we prove the following theorem.

Theorem 1. The set of intervals of the form $(2p_n, 2p_{n+1})$ containing at list 3 primes has a positive density with respect to the set of all intervals of such form.

2. Criterions for R-primes, L-primes and RL-primes

In [1] we found a sieve for the separating R-primes from all primes and shown how to receive the corresponding sieve for L-primes. Now we give simple criterions for them.

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Theorem 2. 1) $p_n$ is $\mathbf{R}$-prime if and only if $\pi(\frac{p_n}{2}) = \pi(\frac{p_n+1}{2})$;
2) $p_n$ is $\mathbf{L}$-prime if and only if $\pi(\frac{p_n}{2}) = \pi(\frac{p_n-1}{2})$;
3) $p_n$ is $\mathbf{RL}$-prime if and only if $\pi(\frac{p_n-1}{2}) = \pi(\frac{p_n+1}{2})$.

Proof. 1) Let $\pi(\frac{p_n}{2}) = \pi(\frac{p_n+1}{2})$ is valid. Now if $p_k < p_n/2 < p_{k+1}$, and between $p_n/2$ and $p_{n+1}/2$ do not exist primes. Thus $p_{n+1}/2 < p_{k+1}$ as well. Therefore, we have $2p_k < p_n < p_{n+1} < 2p_{k+1}$, i.e. $p_n$ is $\mathbf{R}$-prime. Conversely, if $p_n$ is $\mathbf{R}$-prime, then $2p_k < p_n < p_{n+1} < 2p_{k+1}$, and $\pi(\frac{p_n}{2}) = \pi(\frac{p_n+1}{2})$ is valid. 2) is proved quite analogously and 3) follows from 1) and 2). ■

3. Proof of a "precise symmetry" conjecture

We start with a proof of the following conjecture [1].

Conjecture 1. Let $\mathbf{R}_n$ ($\mathbf{L}_n$) denote the $n$-th term of the sequence $\mathbf{R}$ ($\mathbf{L}$). Then we have

\begin{equation}
\mathbf{R}_1 \leq \mathbf{L}_1 \leq \mathbf{R}_2 \leq \mathbf{L}_2 \leq \ldots \leq \mathbf{R}_n \leq \mathbf{L}_n \leq \ldots
\end{equation}

Proof of Conjecture 1. It is clear that the intervals of considered form, containing not more than one prime, contain neither $\mathbf{R}$-primes nor $\mathbf{L}$-primes. Moving such intervals, consider the first from the remaining ones. The first its prime is an $\mathbf{R}$-prime ($\mathbf{R}_1$). If it has only two primes, then the second prime is an $\mathbf{L}$-prime ($\mathbf{L}_1$), and we see that ($\mathbf{R}_1$) < ($\mathbf{L}_1$); on the other hand if it has $k$ primes, then beginning with the second one and up to the $(k-1)$-th we have $\mathbf{RL}$-primes, i.e. primes which are simultaneously $\mathbf{R}$-primes and $\mathbf{L}$-primes. Thus, taking into account that the last prime is only $\mathbf{L}$-prime, we have

$\mathbf{R}_1 < \mathbf{L}_1 = \mathbf{R}_2 = \mathbf{L}_2 = \mathbf{R}_3 = \ldots = \mathbf{L}_{k-1} = \mathbf{R}_{k-1} < \mathbf{L}_k$.

The second remaining interval begins with an $\mathbf{R}$-prime and the process repeats. ■

Remark 1. Note that a corollary that "the number of $\mathbf{RL}$-primes not exceeding $x$ is not less than the number of $A_3$-primes not exceeding $x"$ is absolutely erroneously. Indeed, we should take into account that every interval of the form $\left(2p_n, 2p_{n+1}\right)$ containing $\mathbf{RL}$-prime contains at least 3 primes not exceeding $x$. A right corollary is the following. Since, by the condition of Problem 1, a prime $p$ already lies in an interval $\left(2p_n, 2p_{n+1}\right)$, then we should consider only intervals containing at least prime. Denote $\mathcal{A}_k$, $k = 1, \ldots$, the event that a random interval $\left(2p_n, 2p_{n+1}\right)$ contains at least $k$, 1, 2, ... primes. If $P(\mathcal{A}_1) = q$, then we have
(3.2) \[ P(A_k) = q^k, \quad k = 1, 2, \ldots \]

Let, furthermore, \( A^{(k)} \), \( k = 1, \ldots \), the event that a random interval \((2p_n, 2p_{n+1})\) contains exact \( k \), \( 1, 2, \ldots \) primes. Then, by (3.2),

\[ P(A^{(k)}) = P(A_k) - P(A_{k+1}) = (q - 1)q^k, \quad k = 1, 2, \ldots \]

and we have

(3.3) \[ P(\text{RL}) = (1 - q) \sum_{k \geq 3} \frac{k - 2}{k} q^{k-1} = 2 - q + 2 \frac{1 - q}{q} \ln(1 - q). \]

4. Proof of Theorem 1

The theorem immediately follows from the positivity of probability \( P(\text{RL}) \). In fact, in [1] we proved that \( q \approx 0.8010 \) and \( P(\text{RL}) \approx 0.3980 \). ■

Note that by the Cramér’s 1937 conjecture \( 2p_{n+1} - 2p_n < (2 + \varepsilon) \ln^2 n \). Thus, there exists an infinite sequence of the intervals of such small length, but having at least three primes, and, moreover, this sequence has a positive density with respect to the sequence of all intervals of the form \((2p_n, 2p_{n+1})\). By this way, in view of (3.2), it could be proved a more general result.

Theorem 3. Let \( h \) be arbitrary large but a fixed positive integer. Then the set of intervals of the form \((2p_n, 2p_{n+1})\) containing at least \( h \) primes has a positive density with respect to the set of all intervals of such form.

Quite analogously one can consider an \( m \)-generalization of Theorem 1 for every \( 1 < m < 2 \). Here the case of especial interest is the case of the values of \( m \) close to 1.

References

[1] V. Shevelev Three probabilities concerning prime gaps [http://arxiv.org/abs/0909.0715]
[2] V. Shevelev Critical small intervals containing primes [http://arxiv.org/abs/0908.2319]

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