A TAUBERIAN THEOREM FOR LAPLACE TRANSFORMS
WITH PSEUDOFUNCTION BOUNDARY BEHAVIOR

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To my young friend Larry Zalcman on his sixtieth birthday

Abstract. The prime number theorem provided the chief impulse for complex Tauberian theory, in which the boundary behavior of a transform in the complex plane plays a crucial role. We consider Laplace transforms of bounded functions. Our Tauberian theorem does not allow first-order poles on the imaginary axis, but any milder singularities, characterized by pseudofunction boundary behavior, are permissible. In this context we obtain a useful Tauberian theorem by exploiting Newman’s ‘contour method’.

1. Introduction

In 1980 Don Newman [19] published a beautiful proof for the prime number theorem (PNT) by complex analysis. His vehicle was an old theorem of Ingham [9] involving Dirichlet series, for which he found a clever proof by contour integration. The method is easily adapted to give Theorem 1.1 for Laplace transforms; cf. the author’s paper [14] and Zagier [22]. (Preprints of these papers circulated shortly after Newman’s article appeared.) The contour method has recently been used in numerous articles motivated by operator theory; see for example Allan, O’Farrell and Ransford [1], Arendt and Batty [2], Batty [4], and the book by Arendt, Batty, Hieber and Neubrander [3].

If one is interested only in a quick proof of the PNT, the following result will suffice:

Theorem 1.1. Let \( a(\cdot) \) be (measurable and) bounded on \([0, \infty)\), so that the Laplace transform

\[
(1.1) \quad f(z) = \mathcal{L}a(z) = \int_0^\infty a(t)e^{-zt}dt, \quad z = x + iy,
\]

is well-defined and analytic throughout the open half-plane \( \{x = \text{Re} \, z > 0\} \).

Suppose that \( f(z) \) has an analytic extension to the open interval \((-iB, iB)\) of

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\end{array}\]
the imaginary axis. Then

\[
\limsup_{T \to \infty} \left| \int_0^T a(t)dt - f(0) \right| \leq \frac{2M}{B}, \quad \text{where } M = \sup_{t>0} |a(t)|.
\]

**Corollary 1.2.** If \(a(\cdot)\) is bounded and \(f = \mathcal{L}a\) extends analytically to every point of the imaginary axis, the improper integral

\[
\int_0^{\infty -} a(t)dt = \lim_{T \to \infty} \int_0^T a(t)dt \text{ exists and equals } f(0).
\]

Here the ‘Tauberian condition’ that \(a(\cdot)\) be bounded can (in the real case) be replaced by boundedness from below. However, this makes the proof more complicated; cf. [15] (section 9). In Section 2 we sketch how to deduce the PNT.

Theorem 1.1 and Corollary 1.2 are contained in results of Karamata [10] (theorem B) and Ingham [9] (theorem III), which were obtained by Wiener’s method [21]. They did not require that \(f(z)\) can be extended analytically to every point of the imaginary axis, but could get by with weaker boundary conditions. The aim of the present paper is to reduce the boundary requirements in Theorem 1.1 to a minimum:

**Theorem 1.3.** Let \(a(\cdot)\) be bounded on \([0, \infty)\), so that the Laplace transform \(f(z) = \mathcal{L}a(z)\), \(z = x + iy\) is analytic for \(x = \text{Re } z > 0\). Suppose that \(f(x)\) tends to a limit \(f(0)\) as \(x \searrow 0\) and that the quotient

\[
q(x + iy) = \frac{f(x + iy) - f(x)}{iy}, \quad x > 0,
\]

converges in distributional sense to a pseudofunction \(q(iy)\) on the interval \(\{-B < y < B\}\) as \(x \searrow 0\). Then one has inequality (1.2).

Known sufficient conditions for (1.2) are uniform or \(L^1\) convergence of \(q(x + iy)\) to a limit function \(q(iy)\) on \((-B, B)\). The distributional conditions in the Theorem require two things:

(i) (convergence condition) that

\[
\int_{\mathbb{R}} q(x + iy)\phi(y)dy \quad \text{should tend to a limit } < q(iy), \phi(y) >
\]

for every \(C^\infty\) function \(\phi\) with support in \((-B, B)\);

(ii) (pseudofunction condition) that \(q(iy)\) be the restriction to \((-B, B)\) of the distributional Fourier transform of a function which tends to zero at \(\pm\infty\). Cf. Sections 4 and 5 below.

We remark that related distributional conditions received inadequate treatment in [15] (Theorem 14.6). General background material on Tauberian theory can be found in the forthcoming book [16].
2. From Corollary 1.2 to the Prime Number Theorem

Background material in number theory may be found in many books; classics are Landau [17] and Hardy and Wright [7].

To obtain the PNT from Corollary 1.2 one may take \( a(t) \) equal to

\[
(2.1) \quad b(t) = \frac{\psi(e^t) - [e^t]}{e^t} = e^{-t} \sum_{1 \leq n \leq e^t} (\Lambda(n) - 1),
\]

where \( \psi(v) = \sum_{n \leq v} \Lambda(n) \) is Chebyshev’s function. The symbol \( \Lambda(\cdot) \) stands for von Mangoldt’s function, which is given by the Dirichlet series

\[
\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^w} = -\frac{d}{dw} \log \zeta(w) = \frac{d}{dw} \sum_{p \text{ prime}} \log(1 - p^{-w}) = \sum_{p \text{ prime}} \frac{p^{-w} \log p}{1 - p^{-w}}
\]

when \( \text{Re } w > 1 \). It is elementary that \( \psi(v) = O(v) \), so that \( b(\cdot) \) is bounded.

For \( \text{Re } z > 0 \)

\[
g(z) = \mathcal{L}b(z) = \int_0^\infty \{\psi(e^t) - [e^t]\} e^{-(z+1)t} dt
\]

\[
= \int_0^\infty \{\psi(v) - [v]\} v^{-z-2} dv = \frac{1}{z+1} \int_1^\infty v^{-z-1} d\{\psi(v) - [v]\}
\]

\[
= \frac{1}{z+1} \sum_{n=1}^{\infty} \frac{\Lambda(n)-1}{n^{z+1}} = \frac{1}{z+1} \left( -\frac{\zeta'(z+1)}{\zeta(z+1)} + \zeta(z+1) \right).
\]

The function \( g(z) \) is analytic at every point of the line \( \{\text{Re } z = 0\} \). Indeed, \( \zeta(w) \) is free of zeros on the line \( \{\text{Re } w = 1\} \) and the poles of \(-(\zeta'/\zeta)(w)\) and \( \zeta(w) \) at the point \( w = 1 \) cancel each other. Conclusion:

\[
(2.3) \quad \int_0^\infty b(t) dt = \int_1^\infty \frac{\psi(v) - [v]}{v^2} dv = g(0).
\]

By the monotonicity of \( \psi \) this readily gives

\[
\psi(v) \sim v \quad \text{as } v \to \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\Lambda(n)-1}{n} = g(0).
\]

The relation \( \psi(v) \sim v \) is equivalent to the PNT:

\[
\pi(v) \sim \frac{v}{\log v} \quad \text{as } v \to \infty.
\]

3. An Auxiliary Result

We will prove Theorem 1.1 but begin with a useful preliminary form.

**Proposition 3.1.** Let \( \sup_{t>0} |a(t)| = M < \infty \) and let the Laplace transform

\[
(3.1) \quad f(z) = \mathcal{L}a(z), \quad z = x + iy, \quad x > 0,
\]
have an analytic extension to a neighborhood of the segment \([-iR,iR]\) where \(R > 0\). Then for every number \(T > 0\),

\[
\left| \int_0^T a(t)dt - f(0) \right| \leq \frac{2M}{R} + \frac{|f(0)|}{eRT} + \frac{1}{2\pi} \left| \int_{-R}^R \{f(iy) - f(0)\} \left( \frac{1}{iy} + \frac{iy}{R^2} \right) e^{iTy}dy \right|.
\]

**Proof.** Define

\[
f_T(z) = \int_0^T a(t)e^{-zt}dt.
\]

(i) One begins with some simple estimates. For \(x = \text{Re} z > 0\),

\[
|f_T(z) - f(z)| = \left| \int_T^\infty a(t)e^{-zt}dt \right| \leq M \int_T^\infty e^{-xt}dt = \frac{M}{x}e^{-Tx}.
\]

Similarly for \(x = \text{Re} z < 0\),

\[
|f_T(z)| = \left| \int_0^T a(t)e^{-zt}dt \right| \leq \int_0^T Me^{-xt}dt < \frac{M}{|x|}e^{-Tx}.
\]

(ii) Let \(\Gamma\) be the positively oriented circle \(C(0,R) = \{|z| = R\}\). We let \(\Gamma_1\) be the part of \(\Gamma\) in the half-plane \(\{x = \text{Re} z > 0\}\), \(\Gamma_2\) the part in the half-plane \(\{x < 0\}\). Finally, let \(\sigma\) be the oriented segment of the imaginary axis from \(+iR\) to \(-iR\) (Figure 3). Observe that for \(z = x + iy \in \Gamma\), one has

\[
\frac{1}{z} + \frac{z}{R^2} = \frac{2x}{R^2}.
\]

By the hypotheses, the quotient \(\{f(z) - f(0)\}/z\) is analytic on the segment \(\sigma\). Observe also that \(f_T(z)\) is analytic throughout the complex plane. Formulas (3.3)–(3.6) motivate the following ingenious application of Cauchy’s theorem and Cauchy’s formula due to Newman:

\[
0 = \frac{1}{2\pi i} \int_{\Gamma_{1+\sigma}} \frac{f(z) - f(0)}{z} dz
\]

\[
= \frac{1}{2\pi i} \int_{\Gamma_{1+\sigma}} \{f(z) - f(0)\} e^{Tz} \left( \frac{1}{z} + \frac{z}{R^2} \right) dz,
\]

\[
f_T(0) - f(0) = \frac{1}{2\pi i} \int_{\Gamma_{1+\sigma}} \frac{f_T(z) - f(0)}{z} dz
\]

\[
= \frac{1}{2\pi i} \int_{\Gamma_{1+\sigma}} \{f_T(z) - f(0)\} e^{Tz} \left( \frac{1}{z} + \frac{z}{R^2} \right) dz.
\]
Subtracting (3.7) from (3.8) and rearranging the result, one obtains the formula

\[ f_T(0) - f(0) = \frac{1}{2\pi i} \int_{\Gamma_1} \left( f_T(z) - f(z) \right) e^{Tz} \left( \frac{1}{z} + \frac{z}{R^2} \right) dz \]

\[ + \frac{1}{2\pi i} \int_{\Gamma_2} \left( f_T(z) - f(0) \right) e^{Tz} \left( \frac{1}{z} + \frac{z}{R^2} \right) dz \]

\[ - \frac{1}{2\pi i} \int_{\sigma} \left( f(z) - f(0) \right) e^{Tz} \left( \frac{1}{z} + \frac{z}{R^2} \right) dz \]

\[ = I_1 + I_2 + I_3, \]

say.

(iii) By (3.4) and (3.6) for \( z \in \Gamma_1 \), the integrand \( f^*(z) \) in \( I_1 \) can be estimated as follows:

\[ |f^*(z)| = \left| \left( f_T(z) - f(z) \right) e^{Tz} \left( \frac{1}{z} + \frac{z}{R^2} \right) \right| \leq \frac{M}{x} e^{-Tx} e^{Tx} \frac{2x}{R^2} = \frac{2M}{R^2}. \]

Thus

\[ |I_1| \leq \frac{1}{2\pi} \int_{\Gamma_1} |f^*(z)||dz| \leq \frac{1}{2\pi} \frac{2M}{R^2} \pi R = \frac{M}{R}. \]

For \( z \in \Gamma_2 \), where \( |x|e^{Tx} \leq 1/(eT) \), formulas (3.5) and (3.6) imply the estimate

\[ |I_2| = \left| \frac{1}{2\pi i} \int_{\Gamma_2} \left( f_T(z) - f(0) \right) e^{Tz} \left( \frac{1}{z} + \frac{z}{R^2} \right) dz \right| \leq \frac{M}{R} + \frac{|f(0)|}{eRT}. \]

Combination of (3.9) and (3.10)–(3.11) gives (3.2). □
Derivation of Theorem 1.1. Let $a$ and $f = L_a$ satisfy the hypotheses of Theorem 1.1. Then we can apply Proposition 3.1 for any $R \in (0, B)$. For the proof of (1.2), one has to show that for any number $\varepsilon > 0$, we can choose $T_0$ so large that the left-hand side of (3.2) is bounded by $2(M/B) + \varepsilon$ for all $T \geq T_0$. To this end, choose $R$ so close to $B$ that $2M/R < 2(M/B) + \varepsilon/2$. In order to deal with the final term in (3.2), or with

$$I_3 = \frac{1}{2\pi} \int_{-R}^{R} \left\{ f(iy) - f(0) \right\} \left( \frac{1}{iy} + \frac{iy}{R^2} \right) e^{iyT} dy,$$

one may apply integration by parts: $e^{iyT} dy = de^{iyT}/(iT)$, etc., or one may use the Riemann–Lebesgue lemma. Either method will show that for our $R$,

$$I_3 = I_3(R, T) \to 0 \quad \text{as} \quad T \to \infty.$$

We now determine $T_0$ so large that

$$|f(0)| + |I_3| < \varepsilon/2, \quad \forall \, T \geq T_0.$$

Then by (3.2)

$$\left| \int_0^T a(t) dt - f(0) \right| \leq \frac{2M}{B} + \varepsilon, \quad \forall \, T \geq T_0.$$

\[\square\]

4. Pseudofunction Boundary Behavior

The preceding results may be refined with the aid of a distributional approach. Motivated by operator theory, Katznelson and Tzafriri [12] used pseudofunctions to strengthen the following theorem of Fatou [5], [6]:

Theorem 4.1. Let the function

$$g(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |z| < 1,$$

have an analytic continuation to (a neighborhood of) the point $z = 1$ on the unit circle $C(0, 1)$. Suppose that the coefficients satisfy the ‘Tauberian condition’ $a_n \to 0$ as $n \to \infty$. Then the series $\sum_{n=0}^{\infty} a_n$ converges to $g(1)$.

The condition of analyticity at the point $z = 1$ can be relaxed in various ways. The most notable refinements in this direction are due to M. Riesz and Ingham; cf. [9], [18]; another refinement is mentioned below.

The condition $a_n \to 0$ is the signature of pseudofunction boundary behavior. In Fatou’s theorem, and for real $a_n$, it can be replaced by the one-sided condition $\liminf a_n \geq 0$; cf. [13], [15]. A $2\pi$-periodic distribution $G(t) = \sum_{n \in \mathbb{Z}} c_n e^{int}$ is called a pseudofunction if $c_n \to 0$ as $n \to \pm \infty$. The latter condition first appeared in Riemann’s localization principle [20], which Fatou used in the proof of his theorem. (A careful discussion of the localization principle may be found in [23], item (5.7) in chapter 9.)
Let $g(z)$ as in [15] be any function analytic in the unit disc. Among other things, Katznelson and Tzafriri proved that pseudofunction boundary behavior of $g$ on $C(0,1) \setminus \{z = 1\}$, together with boundedness of the sequence $\{s_n = \sum_{k=1}^{n} a_k\}$, implies that $a_n \to 0$. Their method can be used also for further relaxation of the analyticity condition at the point 1. Knowing that $a_n \to 0$, it is enough for convergence of $\sum a_n$ if $g$ in [14] is 'weakly regular' at the point 1 in the following sense. For some constant which may be called $g(1)$, the quotient
\[
\frac{g(z) - g(1)}{z - 1}
\]
have pseudofunction boundary behavior at the point $z = 1$ (more precisely, in some angle $|\arg z| < \delta$); cf. [15], [16].

Laplace Transforms and related functions. Our aim is to prove an extension of Theorem 1.1 involving pseudofunction boundary behavior of the Laplace transform $f(z) = \mathcal{L}a(z)$. We begin with some general remarks on tempered distributions, that is, continuous linear functionals $F$ on the Schwartz space $S$. The ‘testing functions’ $\phi \in S$ include the $C^\infty$ functions with compact support. The result of applying $F$ to $\phi$ is a bilinear functional, denoted by $\langle F, \phi \rangle$. Locally integrable functions $F_x(y)$ of at most polynomial growth on $-\infty < y < \infty$ converge to a tempered distribution $F(y)$ as $x \to 0$ if
\[
\int_{\mathbb{R}} F_x(y) \phi(y) dy \to <F(y), \phi(y)>
\]
for every function $\phi \in S$.

A tempered distribution $F$ on $\mathbb{R}$ is called a pseudomeasure if it is the Fourier transform of a bounded (measurable) function; it is called a pseudofunction if it is the Fourier transform of a function which tends to zero at $\pm \infty$. Reference: Katznelson [11] (section 6.4).

By the Riemann–Lebesgue theorem, every function in $L^1(\mathbb{R})$ is a pseudofunction. A nontrivial example of a pseudomeasure on $\mathbb{R}$ is the distribution
\[
\frac{1}{y - i0} = \lim_{x \to 0} \frac{i}{x + iy} = \lim_{x \to 0} i \int_{0}^{\infty} e^{-xt} e^{-iyt} dt.
\]
It is the Fourier transform of $i$ times the Heaviside function, $1_+(t)$. Other examples are the Dirac measure and the principal-value distribution, p.v. $(1/y)$.

In the case of boundary singularities, and roughly speaking, first order poles correspond to pseudomeasures, slightly milder singularities to pseudofunctions.

Every pseudomeasure or pseudofunction $F$ on $\mathbb{R}$ can be represented in the form
\[
F(y) = \lim_{x \to 0} \int_{\mathbb{R}} e^{-x|t|} b(t) e^{-iyt} dt,
\]
where $b(\cdot)$ is a bounded function, or a function which tends to zero at $\pm \infty$, respectively. This formula can be used to justify formal inversion of the order
of integration in some situations. An important consequence is a Riemann–Lebesgue lemma for pseudofunctions $F$:

**Lemma 4.2.** For any pseudofunction $F$ on $\mathbb{R}$ and any testing function $\phi$,

$$< F(y), \phi(y)e^{iTy} > \to 0 \quad \text{as} \quad T \to \pm \infty.$$  

Indeed, by representation (4.2),

$$< F(y), \phi(y)e^{iTy} > = \int_{\mathbb{R}} b(t)dt \int_{\mathbb{R}} e^{-iyt}\phi(y)e^{iTy}dy$$

$$= \int_{\mathbb{R}} b(t)\hat{\phi}(t-T)dt \to 0 \quad \text{as} \quad T \to \pm \infty.$$

**Products.** Let $F$ be a pseudomeasure or pseudofunction as in (4.2) and let $\phi$ be a testing function. Computing the Fourier transform of $F(y)\phi(y)$, one finds that this product is the Fourier transform of the convolution

$$\int_{\mathbb{R}} b(v-u)\hat{\phi}(u)/(2\pi)du.$$  

For any other function $\Phi$ whose Fourier transform $\hat{\Phi}(u)$ is $O\{1/(u^2 + 1)\}$, the product $F\Phi$ may be defined as the Fourier transform of

$$b^*(v) = \int_{\mathbb{R}} b(v-u)\hat{\Phi}(u)/(2\pi)du.$$  

With $F$, the product $F\Phi$ is again a pseudomeasure or pseudofunction.

5. **Proof of Theorem 1.3**

Let $a(\cdot)$ and $f = La$ satisfy the hypotheses of the Theorem. It is convenient to set $a(t) = 0$ for $t < 0$. Denoting $\sup_{t>0} |a(t)|$ by $M$, taking $\varepsilon > 0$ and $0 < R < B$, we now apply Proposition 3.1 to $a(t)e^{-\varepsilon t}$ and $f(\varepsilon + z)$ instead of $a(t)$ and $f(z)$. Thus we obtain the inequality

$$\left| \int_0^T a(t)e^{-\varepsilon t}dt - f(\varepsilon) \right| \leq \frac{2M}{R} + \frac{|f(\varepsilon)|}{\varepsilon RT} + \frac{1}{2\pi} \left| \int_{-R}^R \left\{ f(\varepsilon + iy) - f(\varepsilon) \right\} \left( \frac{1}{iy} + \frac{iR}{R^2} \right)e^{iTy}dy \right|.$$  

To treat the final integral we set

$$\left\{ f(\varepsilon + iy) - f(\varepsilon) \right\} \left( \frac{1}{iy} + \frac{iR}{R^2} \right) = g_\varepsilon(y).$$

Let $\chi_R$ denote the characteristic function of the interval $[-R, R]$. For any number $\lambda > 0$ we let $\tau_\lambda$ denote a ‘trapezoidal’ testing function, that is, a $C^\infty$ function which is equal to 1 on $[-\lambda, \lambda]$ and equal to 0 outside a suitable neighborhood of $[-\lambda, \lambda]$. The last integral in (5.1) may then be written in distributional notation as

$$I(T, \varepsilon) = < g_\varepsilon(y)\tau_R(y)\chi_R(y), e^{iTy}\tau_R(y) >.$$
Here we take the support of $\tau_R$ inside $(-B, B)$. Then by the hypotheses, $g_\varepsilon(y)\tau_R(y)$ tends to the pseudofunction
\[
g_0(y)\tau_R(y) = q(iy)(1 - y^2/R^2)\tau_R(y)
\]as $\varepsilon \searrow 0$; cf. (1.4). The question is whether the integral $I(T, \varepsilon)$ tends to the formal limit $I(T, 0)$. Multiplication by the cut-off function $\chi_R(y)$ in (5.3) may cause problems!

One may get around this difficulty by splitting the integral $I(T, \varepsilon)$. Choosing a trapezoidal function $\tau_\mu$ with support in $(-R, R)$, we first consider the relation
\[
<g_\varepsilon(y)\tau_\mu(y), e^{iTy}\tau_R(y)> \to <g_0(y)\tau_\mu(y), e^{iTy}\tau_R(y)> \quad \text{as} \quad \varepsilon \searrow 0.
\]

By our Riemann–Lebesgue lemma 4.2, the final expression tends to zero as $T \to \infty$.

Looking at (5.3), it remains to consider the ‘inner product’
\[
<g_\varepsilon(y)\tau_R(y)\{1 - \tau_\mu(y)\}\chi_R(y), e^{iTy}\tau_R(y)>
\]
As $\varepsilon \searrow 0$, the part of this expression which comes from $f(\varepsilon)$ tends to a trigonometric integral of an integrable function,
\[
\int_{-R}^{R} f(0) \left( \frac{1}{iy} + \frac{iy}{R^2} \right) \{1 - \tau_\mu(y)\} e^{iTy}dy.
\]
The latter tends to zero as $T \to \infty$. From here on, we focus on the constituent of the first factor in (5.5) which involves $f(\varepsilon + iy)$:
\[
f(\varepsilon + iy)\tau_R(y) \cdot \left( \frac{1}{iy} + \frac{iy}{R^2} \right) \{1 - \tau_\mu(y)\} \chi_R(y).
\]
The functions $f(\varepsilon + iy)$ tend to the pseudomeasure $f(iy) = \hat{a}(y)$ as $\varepsilon \searrow 0$, and by the hypothesis about the quotient in (1.4), the restriction of $f(iy)$ to $(-B, B)$ is equal to a pseudofunction. Hence the product $f(iy)\tau_R(y)$, which by (1.4) is the Fourier transform of
\[
\frac{1}{2\pi} \int_R a(\nu - u)\tau_R(u)du,
\]
is a pseudofunction.

The functions $f(\varepsilon + iy)\tau_R(y)$ are the Fourier transforms of the functions $a(t)e^{-\varepsilon t}$, which form a uniformly bounded family. The factor
\[
\Phi(y) = \left( \frac{1}{iy} + \frac{iy}{R^2} \right) \{1 - \tau_\mu(y)\} \chi_R(y),
\]
which vanishes for $|y| \leq \mu$ and for $|y| \geq R$, has Fourier transform $\hat{\Phi}(t) = O(1/(t^2 + 1))$. It follows that the functions in (5.6) are distributionally convergent. The limit $f(iy)\tau_R(y)\Phi(y)$ is a pseudofunction; cf. (1.3). The same will then be true for the limit
\[
g_0(y)\tau_R(y)\{1 - \tau_\mu(y)\}\chi_R(y) = \lim_{\varepsilon \searrow 0} g_\varepsilon(y)\tau_R(y)\{1 - \tau_\mu(y)\}\chi_R(y)
\]
of the functions in the first member of (5.5). Combining the results, one concludes that the limit \( I(T, 0) \) of \( I(T, \varepsilon) \) can be written as an inner product

\[
I(T, 0) = \langle H(y), e^{iTy} \tau_R(y) \rangle
\]

involving a pseudofunction \( H \), so that \( I(T, 0) \to 0 \) as \( T \to \infty \).

To complete the proof of Theorem 1.3 we return to inequality (5.1). Letting \( \varepsilon \) go to zero one finds that

\[
(5.7) \quad \left| \int_0^T a(t)dt - f(0) \right| \leq \frac{2M}{R} + \frac{|f(0)|}{erT} + \frac{1}{2\pi} |I(T, 0)|.
\]

Finally taking \( T \) large and \( R \) close to \( B \), one obtains the desired inequality (1.2).

**Remark 5.1.** Related considerations show that one can introduce pseudo-function boundary behavior in the statement of the Wiener–Ikehara theorem [8], [21]. One thus obtains

**Theorem 5.2.** Let \( S(t) \) vanish for \( t < 0 \), be nondecreasing, continuous from the right and such that the Laplace–Stieltjes transform

\[
(5.8) \quad f(z) = \mathcal{L}dS(z) = \int_0^\infty e^{-zt}dS(t) = z \int_0^\infty S(t)e^{-zt}dt, \quad z = x + iy,
\]

exists for \( \text{Re} \, z = x > 1 \). Suppose that for some constant \( A \), the analytic function

\[
(5.9) \quad g(x + iy) = f(x + iy) - \frac{A}{x + iy - 1}, \quad x > 1,
\]

converges distributionally to a pseudofunction \( g(1+iy) \) on every finite interval \( -B < y < B \) as \( x \searrow 1 \). Then

\[
(5.10) \quad e^{-t}S(t) \to A \quad \text{as} \quad t \to \infty.
\]

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