The non-linear fluid dynamics of a warped accretion disc

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ABSTRACT
The dynamics of a viscous accretion disc subject to a slowly varying warp of large amplitude is considered. Attention is restricted to discs in which self-gravitation is negligible, and to the generic case in which the resonant wave propagation found in inviscid Keplerian discs does not occur. The equations of fluid dynamics are derived in a coordinate system that follows the principal warping motion of the disc. They are reduced using asymptotic methods for thin discs, and solved to extract the equation governing the warp. In general, this is a wave equation of parabolic type with non-linear dispersion and diffusion, which describes fully non-linear bending waves. This method generalizes the linear theory of Papaloizou & Pringle to allow for an arbitrary rotation law, and extends it into the non-linear domain, where it connects with a generalized version of the theory of Pringle. The astrophysical implications of this analysis are discussed briefly.

Key words: accretion, accretion discs – hydrodynamics – waves.

1 INTRODUCTION
There are many situations in astrophysics in which accretion discs are believed to be non-planar, either because their profile is observed directly (e.g. in the nucleus of the galaxy NGC 4258; Miyoshi et al. 1995) or in order to explain phenomena such as the periodic modulation of light curves (e.g. in the X-ray binary Her X-1) or the precession of jets (e.g. in the X-ray binary SS 433). In many – though not all – applications the effects of self-gravitation can be neglected and the evolution is dominated by viscous fluid dynamics. The earliest attempts to derive equations governing the time dependence of a slowly varying warp of small amplitude (Petterson 1978; Hatchett, Begelman & Sarazin 1981) suggested a simple diffusive evolution but were shown by Papaloizou & Pringle (1983) to be incorrect at a fundamental level, being inconsistent with the conservation of angular momentum.

Papaloizou & Pringle (1983) demonstrated that the problem is complicated because of the importance of a resonance in thin discs that are both Keplerian (or very nearly so) and inviscid (or very nearly so). It is therefore necessary to distinguish between the generic (non-resonant) case and the resonant case, which occurs when both

\[
\frac{\Omega^2 - \kappa^2}{\Omega^2} \approx H/r \quad \text{and} \quad \alpha \approx H/r. \tag{1}
\]

Here \(\Omega\) is the angular velocity, \(\kappa\) is the epicyclic frequency, \(H/r\) is the ratio of the semithickness of the disc to the radius, and \(\alpha\) is the dimensionless viscosity parameter (Shakura & Sunyaev 1973). Unlike most resonances in discs, which occur at specific radii, this one, which is effectively the \(m = 1\) inner Lindblad resonance, is likely to occur either everywhere in the disc or nowhere. Current estimates suggest that the resonant case is not relevant to discs in X-ray binaries or active galactic nuclei, but may apply in protostellar discs. However, the resonance is delicate and could be destroyed by the effects of self-gravitation, magnetic fields, or turbulence.

Papaloizou & Pringle (1983) derived the first consistent equation governing a warp of small amplitude. They considered the (non-resonant) case of a Keplerian disc with a significant viscosity, treating the warp as a slowly varying disturbance with azimuthal wavenumber \(m = 1\), using linear Eulerian perturbation theory. The resulting equation for the warp is a complex linear diffusion equation, which has been used in applications (e.g. Kumar & Pringle 1985).

Papaloizou & Lin (1994) considered the case of an inviscid disc. Again, linear Eulerian perturbation theory was used, and the authors assumed the warp to be a slowly varying normal mode of the disc. In the non-Keplerian (non-resonant) case the warp obeys a dispersive linear
wave equation, while in the Keplerian (resonant) case the warp obeys a non-dispersive linear wave equation. By considering the effect of a small viscosity on the inviscid modes, Papaloizou & Lin (1994) connected this theory with that of Papaloizou & Pringle (1983), showing how the transition occurs between wave-like and diffusive behaviour in Keplerian discs when $\alpha \approx H/r$. This theory has also been used in applications (e.g. Papaloizou & Terquem 1995).

The major uncertainty in these theories is whether the linear analysis is valid for warps of a sufficient amplitude to be observable. The linear Eulerian perturbation theory is formally valid only when the tilt angle $\beta(r, t)$ of the warp satisfies $|\beta| \ll H/r$. However, because of the special nature of the mode involved, which differs little locally from a rigid tilt $\beta = \text{constant}$, one suspects that the linear theory might be valid for larger warps. In fact, a more appropriate measure of the amplitude of the warp is $|\partial \beta / \partial \ln r|$. The Eulerian perturbation theory offers little information about possible non-linear effects, although it does predict that, in the resonant case, an amplitude $|\partial \beta / \partial \ln r| \approx H/r$ would result in horizontal shearing motions in the disc comparable to the sound speed, which are expected to be unstable (Kumar & Coleman 1993; Gammie, Goodman, & Ogilvie, in preparation).

Meanwhile, Pringle (1992) developed a different approach in which the forms of the equations governing a warped viscous disc are derived simply by requiring mass and angular momentum to be conserved, but without reference to the detailed internal fluid dynamics of the disc. In this scheme, neighbouring rings in the disc exchange angular momentum by means of viscous torques which are of two kinds. One kind of torque (associated with a kinematic viscosity coefficient $\nu_1$) acts on the differential rotation in the plane of the disc, and leads to accretion; the other kind (associated with $\nu_2$) acts to flatten the disc. The equations derived are

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \Sigma \vec{v}_r) = 0$$

(2)

for the surface density $\Sigma(r, t)$, and

$$\frac{\partial}{\partial t} \left( \Sigma \Omega^2 \right) + \frac{1}{r} \frac{\partial}{\partial r} \left( \Sigma \vec{v}_r \Omega^2 \right) = \frac{1}{r} \frac{\partial}{\partial r} \left( \nu_1 \Sigma r^2 \frac{d\Omega}{dr} + \frac{1}{2} \nu_2 \Sigma r^3 \Omega \frac{d\Omega}{dr} \right)$$

(3)

for the angular momentum, in the absence of external torques. Here $\vec{v}_r(r, t)$ is the mean radial velocity and $\ell(r, t)$ is the tilt vector, which is a unit vector parallel to the orbital angular momentum of the ring. From this follow the equations

$$\Sigma \vec{v}_r \frac{d(r \Omega)}{dr} \frac{\Omega}{r} = \frac{1}{r} \frac{\partial}{\partial r} \left( \nu_1 \Sigma r^2 \frac{d\Omega}{dr} - \frac{1}{2} \nu_2 \Sigma r^3 \Omega \frac{d\Omega}{dr} \right)^2$$

(4)

for the component of angular momentum parallel to $\ell$, and

$$\Sigma \frac{d^2 \ell}{dt^2} \left( \vec{v}_r - \nu_1 \frac{\partial \Omega}{\partial r} \right) \frac{\partial \ell}{dr} + \frac{1}{2} \frac{\partial}{\partial r} \left( \nu_2 \Sigma r^3 \Omega \frac{d\ell}{dr} \right) \left( \frac{1}{2} \nu_2 \Sigma r^3 \Omega \frac{d\ell}{dr} \right) \ell$$

(5)

for the tilt vector.

This approach appears to be valid for warps of large amplitude and would therefore represent an advance on the linear theory. It has been used in several applications, in particular to identify the radiation-driven instability (Pringle 1996) and to explore its linear theory (e.g. Maloney, Begelman & Pringle 1996) and non-linear evolution (e.g. Pringle 1997). However, because the equations of Pringle (1992) are derived somewhat heuristically, without reference to the detailed internal fluid dynamics of the disc, some doubts remain over the validity of this method. The obvious questions to be addressed are whether any internal degree of freedom of the rings has been neglected, whether the interaction between neighbouring rings is purely of the assumed form of viscous torques, and whether there are any non-linear fluid-dynamical effects that might limit the amplitude of the warp. It is therefore important to investigate whether the equations of Pringle (1992) can be derived ab initio from the three-dimensional fluid-dynamical equations, and to understand how they connect to the previously established linear theory. These are the aims of this paper.

The analysis is organized as follows. I first define a coordinate system that follows the principal warping motion of the disc, and derive the basic results necessary for vector calculus (Section 2). I then present the exact forms of the equations of fluid dynamics in this coordinate system (Section 3). The equations are reduced using an asymptotic analysis for thin discs (Section 4). I then apply separation of variables to solve the fully non-linear problem except for the determination of three dimensionless coefficients from the solution of a set of ordinary differential equations (ODEs) (Section 5). These are evaluated in Section 6. Readers interested only in the interpretation and application of this work may proceed to Section 7, where the principal results are summarized and discussed further.

2 DEFINITION OF WARPED COORDINATES

The non-linear fluid dynamics of a warped disc is most naturally described in the case of a spherically symmetric external potential. Then a flat disc has no preferred plane of orientation, and must possess a zero-frequency mode consisting of a rigid tilt of any amplitude. Continuous with this mode, there exist bending waves, with azimuthal wavenumber $m = 1$ in linear theory, which vary on a time-scale long compared with the orbital time-scale, and on a length-scale long compared with the thickness of the disc. This is in contrast to the other modes of the disc (Lubow & Pringle 1993) and motivates the following analysis.
2.1 Coordinates and basis vectors

Let \( x, y, z \) be Cartesian coordinates in an inertial frame of reference, and define the spherical radial coordinate \( r = (x^2 + y^2 + z^2)^{1/2} \). On each sphere \( r = \text{constant} \), define the usual angular coordinates \( \theta, \phi \), but with respect to an axis that is tilted to point in the direction of the unit vector \( \hat{r} \). It is intended that the disc matter on each sphere will lie close to \( \hat{r} \). The tilt vector can be described by Euler angles \( \beta, \gamma \):

\[
\ell = \sin \beta \cos \gamma \hat{e}_x + \sin \beta \sin \gamma \hat{e}_y + \cos \beta \hat{e}_z.
\] (6)

In detail, warped spherical polar coordinates \( (r, \theta, \phi) \) are defined by

\[
\begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix} = \mathbf{M}(r, \theta, \phi, t) \begin{bmatrix}
  r \\
  0 \\
  0
\end{bmatrix},
\] (7)

where

\[
\mathbf{M}(r, \theta, \phi, t) = \begin{bmatrix}
  \cos \gamma & -\sin \gamma & 0 \\
  \sin \gamma & \cos \gamma & 0 \\
  0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
  \cos \beta & 0 & \sin \beta \\
  0 & 1 & 0 \\
  -\sin \beta & 0 & \cos \beta
\end{bmatrix} \begin{bmatrix}
  \cos \phi & -\sin \phi & 0 \\
  \sin \phi & \cos \phi & 0 \\
  0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
  \sin \theta & \cos \theta & 0 \\
  0 & 0 & 1 \\
  \cos \theta & -\sin \theta & 0
\end{bmatrix}
\] (8)

is a composition of four orthogonal linear transformations. For the functions \( \beta(r, t) \) and \( \gamma(r, t) \), a prime and a dot will denote differentiation with respect to \( r \) and \( t \), respectively.

Although \( (r, \theta, \phi) \) are not orthogonal coordinates, it is appropriate to refer vectors to a natural orthonormal basis \( \{ \hat{e}_r, \hat{e}_\theta, \hat{e}_\phi \} \) defined by

\[
\begin{bmatrix}
  \hat{e}_r \\
  \hat{e}_\theta \\
  \hat{e}_\phi
\end{bmatrix} = \mathbf{M} \begin{bmatrix}
  \hat{e}_r \\
  \hat{e}_\theta \\
  \hat{e}_\phi
\end{bmatrix}.
\] (9)

The vector \( \hat{e}_r \) is normal to the sphere \( r = \text{constant} \), while the vectors \( \hat{e}_\theta \) and \( \hat{e}_\phi \) are tangential to it. The components of an arbitrary vector \( \mathbf{F} \) then transform according to

\[
\begin{bmatrix}
  F_r \\
  F_\theta \\
  F_\phi
\end{bmatrix} = \mathbf{M} \begin{bmatrix}
  F_r \\
  F_\theta \\
  F_\phi
\end{bmatrix}.
\] (10)

2.2 Calculus in warped coordinates

The matrix \( \mathbf{M} \) is orthogonal. Its decomposition in equation (8) makes it easy to compute the differential

\[
d\mathbf{M} = \mathbf{MA},
\] (11)

where \( \mathbf{A} \) is an antisymmetric matrix, with components given by

\[
A_{12} = -d\theta - \cos \phi d\beta - \sin \beta \sin \phi d\gamma,
\] (12)

\[
A_{13} = -\sin \theta d\phi + \cos \theta \sin \phi d\beta - (\cos \beta \sin \theta + \sin \beta \cos \theta \cos \phi) d\gamma,
\] (13)

\[
A_{23} = -\cos \theta d\phi - \sin \theta \sin \phi d\beta - (\cos \beta \cos \theta - \sin \beta \sin \theta \cos \phi) d\gamma.
\] (14)
The coordinate differentials are therefore related by

$$\begin{bmatrix} \frac{dx}{dr} \\ \frac{dy}{dr} \\ \frac{dz}{dr} \end{bmatrix} = M \begin{bmatrix} r \, d\theta + r \cos \phi \, d\beta + r \sin \beta \sin \phi \, d\gamma \\ r \sin \theta \, d\phi - r \cos \theta \sin \phi \, d\beta + r(\cos \beta \sin \theta + \sin \beta \cos \theta \cos \phi) \, d\gamma \end{bmatrix},$$

(15)

so the Jacobian matrix of the transformation is

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{bmatrix} = M \begin{bmatrix} 1 & 0 & 0 \\ r \beta' \cos \phi + r \gamma' \sin \beta \sin \phi & r & 0 \\ -r \beta' \cos \theta \sin \phi + r \gamma'(\cos \beta \sin \theta + \sin \beta \cos \theta \cos \phi) & 0 & r \sin \theta \end{bmatrix},$$

(16)

with determinant

$$J = r^2 \sin \theta,$$

(17)

as for ordinary spherical polar coordinates.

The gradient operator takes the form

$$\nabla = e_r \mathcal{D} + e_\theta \frac{1}{r} \partial_\theta + e_\phi \frac{1}{r \sin \theta} \partial_\phi,$$

(18)

where

$$\mathcal{D} = \partial_r - (\beta' \cos \phi + \gamma' \sin \beta \sin \phi) \partial_\theta - (\beta' \cos \theta \sin \phi + \gamma'(\cos \beta \sin \theta + \sin \beta \cos \theta \cos \phi)) \frac{1}{\sin \theta} \partial_\phi$$

(19)

is a modified radial derivative.

Finally, the condition

$$\begin{bmatrix} \frac{de_r}{dr} \\ \frac{de_\theta}{dr} \\ \frac{de_\phi}{dr} \end{bmatrix} = 0$$

(20)

yields the differentials of the unit vectors in the form

$$\begin{align*}
\frac{de_r}{dr} &= e_\theta (\sin \phi \, d\beta + \cos \phi \, d\gamma) + e_\phi \left[ \sin \theta \, d\phi - \cos \theta \sin \phi \, d\beta + (\cos \beta \sin \phi + \sin \beta \cos \phi) \, d\gamma \right], \\
\frac{de_\theta}{dr} &= e_r (\cos \phi \, d\beta + \sin \phi \, d\gamma) + e_\phi \left[ \cos \theta \, d\phi + \sin \theta \sin \phi \, d\beta + (\cos \beta \cos \phi - \sin \beta \sin \phi) \, d\gamma \right], \\
\frac{de_\phi}{dr} &= -e_r (\sin \theta \, d\phi - \cos \theta \sin \phi \, d\beta + (\cos \beta \sin \phi + \sin \beta \cos \phi) \, d\gamma) \\
&\quad - e_\theta \left[ \cos \phi \, d\beta + \sin \phi \, d\gamma \right],
\end{align*}$$

(21–23)

In particular,

$$\begin{align*}
\nabla e_r &= \frac{1}{r} e_\theta e_\phi + \frac{1}{r} e_\phi e_\theta, \\
\nabla e_\theta &= -\frac{1}{r} e_\phi e_r + \cot \theta \, \frac{e_\theta e_\phi}{r} + \frac{f}{r \sin \theta} \, e_r e_\phi, \\
\nabla e_\phi &= -\frac{1}{r} e_\theta e_r - \frac{\cot \theta}{r} \, e_\phi e_\theta - \frac{f}{r \sin \theta} \, e_r e_\theta,
\end{align*}$$

(24–26)

where

$$f = r \beta' \sin \phi - r \gamma' \sin \beta \cos \phi.$$$$

(27)

The advantages of this method over that of Petterson (1978) are to be emphasized. Petterson (1978) introduced a warped coordinate system based on tilted cylinders, rather than spheres. This is less satisfactory because the resulting coordinate system has a non-trivial Jacobian determinant and, in fact, breaks down for large-amplitude warps. The present method does not suffer from these defects. Moreover, it can be understood using elementary vector calculus rather than requiring advanced differential geometry.

### 2.3 Absolute and relative velocities

Consider a fluid element with coordinates $x(t)$, $y(t)$ and $z(t)$. The velocity components are

$$\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix} = M \begin{bmatrix} u_r \\ u_\theta \\ u_\phi \end{bmatrix},$$

(28)
where
\[
\begin{bmatrix}
\frac{dr}{dt} \\
\frac{d\theta}{dt} \\
\frac{d\phi}{dt}
\end{bmatrix}
= \begin{bmatrix}
r(d\theta/dt) + r(d\beta/dt) \cos \phi + r(d\gamma/dt) \sin \beta \sin \phi \\
0 \\
r \sin \theta (d\phi/dt) - r(d\beta/dt) \cos \theta \sin \phi + r(d\gamma/dt)(\cos \beta \sin \theta + \sin \phi \cos \theta \cos \phi)
\end{bmatrix}.
\] (29)

Here, \(d/dt\) stands for the Lagrangian time-derivative
\[
D = (\partial_i)_{xyz} + u_r \partial_r + u_\theta \partial_\theta + u_\phi \partial_\phi.
\] (30)

The components \((u_r, u_\theta, u_\phi)\) are the absolute velocity components, in the sense that \(u = u_r e_r + u_\theta e_\theta + u_\phi e_\phi\) is the velocity vector as measured in the inertial frame. It is intended, however, that the warped coordinate system itself will follow the principal motion of the fluid, excluding its orbital angular velocity. The additional motion with respect to the warped coordinate system is described by the relative velocity components
\[
\begin{bmatrix}
v_r \\
v_\theta \\
v_\phi
\end{bmatrix}
= \begin{bmatrix}
\frac{dr}{dt} \\
r(d\theta/dt) \\
r \sin \theta (d\phi/dt)
\end{bmatrix}.
\] (31)

The Lagrangian time-derivative takes the form
\[
D = (\partial_i)_{xyz} + v_r \partial_r + \frac{v_\theta}{r} \partial_\theta + \frac{v_\phi}{r \sin \theta} \partial_\phi.
\] (32)

The absolute velocity may then be written as
\[
u = u_r e_r + u_\theta e_\theta + u_\phi e_\phi,
\] (33)

with
\[
u_r = v_r,
\] (34)
\[
u_\theta = v_\theta + r(D\beta) \cos \phi + r(D\gamma) \sin \beta \sin \phi,
\] (35)
\[
u_\phi = v_\phi - r(D\beta) \cos \theta \sin \phi + r(D\gamma)(\cos \beta \sin \theta + \sin \phi \cos \theta \cos \phi).
\] (36)

3 FLUID DYNAMICS IN WARPED COORDINATES

3.1 Basic equations

The equations governing the dynamics of a compressible, non-self-gravitating fluid are the equation of mass conservation,
\[
Dp = -\rho \nabla \cdot u,
\] (37)

the adiabatic condition,
\[
Dp = -\Gamma \rho \nabla \cdot u,
\] (38)

and the equation of motion,
\[
\rho Du = -\nabla \rho - \rho \nabla \Phi + \nabla \cdot [\mu \nabla u + \mu (\nabla u)^T] + \nabla \left[ (\mu_b - \frac{2}{3} \mu) \nabla \cdot u \right] + F.
\] (39)

Here \(\rho\) is the density, \(p\) is the pressure, \(\Gamma\) is the adiabatic exponent, \(\Phi\) is the external gravitational potential, \(\mu\) and \(\mu_b\) are the shear and bulk viscosities, and \(F\) is an arbitrary external force.

In using these equations, it is assumed that the agent responsible for angular momentum transport in accretion discs can be treated as an isotropic viscosity in the sense of the Navier–Stokes equation. In the case of magnetohydrodynamic turbulence, it is not known how accurate this assumption is, nor whether any alternative constitutive relation for the turbulent stress would be better. Although bulk viscosity is not usually discussed in the context of accretion discs, it is a standard component in the compressible Navier–Stokes equation and is included here for completeness. It is, however, assumed that the viscosities are only dynamically, rather than thermodynamically, important. A more sophisticated treatment would include the thermal effects of viscous dissipation and radiative transport.

Using the information given in Section 2, it is possible to derive the form of these equations in warped coordinates. Either absolute or relative velocity components are used below, whichever gives the most compact form of the equations. First, note that \(\nabla \cdot u\) may be written in

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two equivalent ways:

\[
\nabla u = \frac{1}{r^2} \mathcal{D}(r^2 u_r) + \frac{1}{r \sin \theta} \partial_\theta (u_\theta \sin \theta) + \frac{1}{r \sin \theta} \partial_\phi u_\phi \\
= \frac{1}{r} \partial_\theta (r^2 v_r) + \frac{1}{r \sin \theta} \partial_\theta (v_\theta \sin \theta) + \frac{1}{r \sin \theta} \partial_\phi v_\phi.
\]

(40)

The familiar form of the latter expression is a consequence of the fact that the Jacobian determinant of the warped coordinate system is equal to that of ordinary spherical polar coordinates. This deals with equations (37) and (38). Equation (39) requires a lengthy calculation, especially to evaluate the viscous terms, although much of their complexity is already present in ordinary spherical polar coordinates. The result is

\[
\rho \left( \mathbf{D}u_r - \frac{u_\theta^2}{r} - \frac{u_\phi^2}{r} \right) = - \nabla p - \rho \nabla \Phi + F_r
\]

\[
+ \mathcal{D} \left[ \left( u_\theta + \frac{1}{2} \mu \right) \nabla u \right] + \frac{1}{r^2} \mathcal{D}(\mu^2 \mathbf{D}u_r) + \frac{1}{r \sin \theta} \partial_\theta (u_\theta \sin \theta) \partial_\theta u_\theta + \frac{1}{r \sin \theta} \partial_\phi (u_\phi \sin \phi) - \frac{2 m \tau}{r^2} \mathcal{D}(\mathbf{r})
\]

\[
- \mathcal{D}(\mu^2) \partial_\theta (u_\theta \sin \theta) + (\partial_\mu) \mathcal{D} \left( \frac{u_\theta}{r} \right) + \frac{\partial_\mu}{r \sin \theta} \partial_\theta u_\phi + \frac{\partial_\mu}{r \sin \theta} \partial_\theta u_\phi + \frac{\partial_\mu}{r} \partial_\phi \mathcal{D} \left( \frac{u_\phi}{r} \right) + \frac{f}{r \sin \theta} \left( \frac{u_\theta}{r \sin \theta} \partial_\phi u_\theta - \partial_\phi u_\phi \right).
\]

(41)

\[
\rho \left( \mathbf{D}u_\theta + \frac{u_\phi}{r \sin \theta} \right) \left[ u_\theta \sin \theta \right] + \left( r \mathcal{D}(r \mathcal{D} u_\phi) \right) + \left( r \mathcal{D}(r \mathcal{D} u_\phi) \right) + \frac{1}{r \sin \theta} \partial_\theta (u_\theta \sin \theta) \partial_\theta u_\theta + \frac{1}{r \sin \theta} \partial_\phi (u_\phi \sin \phi) - \frac{2 m \tau}{r^2} \mathcal{D}(\mathbf{r})
\]

\[
- \mathcal{D}(\mu^2) \partial_\theta (u_\theta \sin \theta) + (\partial_\mu) \mathcal{D} \left( \frac{u_\theta}{r} \right) + \frac{\partial_\mu}{r \sin \theta} \partial_\theta u_\phi + \frac{\partial_\mu}{r \sin \theta} \partial_\theta u_\phi + \frac{\partial_\mu}{r} \partial_\phi \mathcal{D} \left( \frac{u_\phi}{r} \right) + \frac{f}{r \sin \theta} \left( \frac{u_\theta}{r \sin \theta} \partial_\phi u_\theta - \partial_\phi u_\phi \right).
\]

(42)

\[
\rho \left( \mathbf{D}u_\phi + \frac{u_\phi}{r \sin \theta} \right) \left[ u_\phi \sin \theta \right] + \left( r \mathcal{D}(r \mathcal{D} u_\phi) \right) + \left( r \mathcal{D}(r \mathcal{D} u_\phi) \right) + \frac{1}{r \sin \theta} \partial_\theta (u_\theta \sin \theta) \partial_\theta u_\theta + \frac{1}{r \sin \theta} \partial_\phi (u_\phi \sin \phi) - \frac{2 m \tau}{r^2} \mathcal{D}(\mathbf{r})
\]

\[
- \mathcal{D}(\mu^2) \partial_\theta (u_\theta \sin \theta) + (\partial_\mu) \mathcal{D} \left( \frac{u_\theta}{r} \right) + \frac{\partial_\mu}{r \sin \theta} \partial_\theta u_\phi + \frac{\partial_\mu}{r \sin \theta} \partial_\theta u_\phi + \frac{\partial_\mu}{r} \partial_\phi \mathcal{D} \left( \frac{u_\phi}{r} \right) + \frac{f}{r \sin \theta} \left( \frac{u_\theta}{r \sin \theta} \partial_\phi u_\theta - \partial_\phi u_\phi \right).
\]

(43)

These equations must be solved in conjunction with equations (34)–(36) defining the absolute velocity components and equation (32) defining the Lagrangian time-derivative.

### 3.2 Generalized form of the angular momentum equation

The equations above are capable of describing any flow, with \( \beta(r, t) \) and \( \gamma(r, t) \) being arbitrarily prescribed (differentiable) functions. In order to derive meaningful dynamical equations for \( \beta \) and \( \gamma \), one must impose suitable constraints on the flow. For a thin disc, the fluid must remain close to the surface \( \theta = \pi/2 \), and this condition is enforced by the asymptotic analysis presented in Section 4. It will be convenient to work with the dimensionless complex variable

\[
\psi = \psi_i + i \psi_v = r (\beta + i \gamma \sin \beta).
\]

(44)

Then \( |\psi| \) is a measure of the amplitude of the warp, with

\[
|\psi| = r \left| \frac{\partial \mathbf{r}}{\partial r} \right|.
\]

(45)

It will be found that, although equation (2) for the surface density can easily be derived from the equations of this section in the limit of a thin disc, it is impossible to obtain equations (4) and (5) in a similar manner. One must allow for a more general form of the torque between neighbouring rings, and attempt to derive an angular momentum equation of the form

\[
\frac{d}{dr} \left( \Sigma \Omega^2 \mathbf{r} \right) + \frac{1}{r} \frac{d}{dr} \left( \Sigma \mathbf{v} \cdot \mathbf{r} \right) \mathbf{r} = \frac{1}{r} \frac{d}{dr} \left( Q_1 \mathbf{r} \Omega^2 \Omega^2 \mathbf{r} \right) + \frac{1}{r} \frac{d}{dr} \left( Q_1 \mathbf{r} \Omega^2 \mathbf{r} \times \frac{\partial \mathbf{r}}{\partial r} \right).
\]

(46)
A dot will now denote differentiation with respect to $T$ and dynamical evolution of the warp. This is captured by the slow time coordinate $t$, so that

$$g_{ij} = \frac{1}{2\pi} \int_0^{2\pi} \ddot{\phi} \, \ddot{z} \, \dd x,$$

(47)

and is an important dynamical quantity in the theory of bending waves. The coefficients $Q_1$, $Q_2$, and $Q_3$ are dimensionless quantities to be determined, which will be found to depend on the rotation law and the shear viscosity, and also, in the non-linear theory, on the adiabatic exponent, the bulk viscosity and the amplitude of the warp. From this follow the equations

$$\Sigma \ddot{\phi} r^2 \Omega^2 \dot{r} = \frac{1}{r} \frac{\partial}{\partial r} \left[ Q_1 \ddot{r} r^2 \Omega^2 + \frac{1}{\dot{r}} \frac{\partial}{\partial \dot{r}} \left( Q_2 \ddot{r} r^2 \Omega^2 \right) + Q_3 \ddot{r} r^2 \Omega^2 \right] \dot{r}^2 + \frac{1}{r} \frac{\partial}{\partial r} \left( Q_1 \ddot{r} r^2 \Omega^2 \dot{r} \times \frac{\partial}{\partial \dot{r}} \right)$$

(48)

for the component of angular momentum parallel to $\dot{r}$, and

$$\Sigma \ddot{\Omega} \left( \ddot{r} + \ddot{\phi} \right) = Q_1 \ddot{r} \Omega^2 \ddot{r} + \frac{1}{r} \frac{\partial}{\partial r} \left[ Q_2 \ddot{r} r^2 \Omega^2 (Q_2 \dot{r} - Q_1 \dot{r} \sin \beta) \right] - \ddot{r}^2 \Omega^2 \ddot{\Omega} \cos \beta (Q_2 \dot{r} \sin \beta + Q_1 \dot{r} \beta'),$$

(49)

and

$$\Sigma \ddot{\Omega} \left( \ddot{r} + \ddot{\phi} \right) \sin \beta = \frac{Y}{r}$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left[ Q_4 \ddot{r} r^2 \Omega^2 (Q_4 \dot{r} \sin \beta + 3 \dot{r} \beta') \right] + \ddot{r}^2 \Omega^2 \ddot{\Omega} \cos \beta (Q_2 \dot{r} \beta' - Q_3 \dot{r} \sin \beta).$$

(52)

(53)

It will be useful to consider the complex combination

$$X + iY = Q_4 \ddot{r} ^2 \Omega^2 \psi + \frac{1}{r} \left[ \ddot{\theta} + \dot{r} \dot{\phi} \right] \cos \beta (Q_4 \ddot{r} r^2 \Omega^2 \psi),$$

(54)

where $Q_4 = Q_2 + iQ_3$.

4 NON-LINEAR BENDING WAVES IN A THIN DISC

4.1 Asymptotic expansions

Consider a thin disc in a spherically symmetric potential $\Phi(r)$, subject to a slowly varying, unforced warp of large amplitude. Let the small parameter $\epsilon$ be a characteristic value of the local angular semithickness of the disc. Then, to resolve the internal structure of the disc, introduce the stretched vertical coordinate $\zeta$, defined by

$$\theta = \frac{\pi}{2} - \epsilon \zeta,$$

(55)

so that $\zeta = O(1)$ within the disc. All quantities vary not on the fast, orbital time-scale, but on the slow time-scale characteristic of both viscous and dynamical evolution of the warp. This is captured by the slow time coordinate

$$T = \epsilon^2 t.$$

(56)

[It is implicit that units are chosen such that the radius of the disc and the orbital time-scale are $O(1)$.] Then set

$$\beta(r, t) \rightarrow \beta(r, T),$$

(57)

$$\gamma(r, t) \rightarrow \gamma(r, T).$$

(58)

A dot will now denote differentiation with respect to $T$. For the density and pressure, introduce the scalings

$$\rho(r, \theta, \phi, T) = \epsilon^3 \left[ \rho_0(r, \phi, \zeta, T) + \epsilon \rho_1(r, \phi, \zeta, T) + O(\epsilon^2) \right],$$

(59)

$$p(r, \theta, \phi, T) = \epsilon^4 \left[ \rho_0(r, \phi, \zeta, T) + \epsilon \rho_1(r, \phi, \zeta, T) + O(\epsilon^2) \right],$$

(60)
where $s$ is a parameter that should be positive if the self-gravitation of the disc is to be negligible, but otherwise has no effect on the following analysis (cf. Ogilvie 1997). For the velocities, set

$$v_r(r, \theta, \phi, t) = \epsilon v_{r1}(r, \phi, \xi, T) + \epsilon^2 v_{r2}(r, \phi, \xi, T) + O(\epsilon^3),$$  

(61)

$$v_{\theta}(r, \theta, \phi, t) = \epsilon v_{\theta1}(r, \phi, \xi, T) + \epsilon^2 v_{\theta2}(r, \phi, \xi, T) + O(\epsilon^3),$$  

(62)

$$v_{\phi}(r, \theta, \phi, t) = r\Omega(r) \sin \theta + \epsilon v_{\phi1}(r, \phi, \xi, T) + \epsilon^2 v_{\phi2}(r, \phi, \xi, T) + O(\epsilon^3).$$  

(63)

Finally, for the viscosities, assume

$$\mu(r, \theta, \phi, t) = \epsilon^{s+2} [\mu_0(r, \phi, \xi, T) + \epsilon \mu_1(r, \phi, \xi, T) + O(\epsilon^3)],$$  

(64)

$$\mu_0(r, \theta, \phi, t) = \epsilon^{s+2} [\mu_{00}(r, \phi, \xi, T) + \epsilon \mu_{01}(r, \phi, \xi, T) + O(\epsilon^3)].$$  

(65)

This is the correct scaling for an $\alpha$ viscosity (Shakura & Sunyaev 1973) with $\alpha = O(1)$, which includes the possibility of small or zero $\alpha$ unless resonance occurs. Note that, for this non-linear warp, all quantities other than the orbital angular velocity (and the surface density; see below) are non-axisymmetric at leading order in $\epsilon$.

These expansions may then be substituted into the dynamical equations. First, equation (41) at $O(\epsilon^4)$ yields

$$-\rho_0 \Omega^2 = -\rho_0 \Phi^2,$$  

(66)

which determines the orbital angular velocity. The epicyclic frequency $\kappa(r)$ is defined by

$$\kappa^2 = 4\Omega^2 + 2r\Omega\dot{\Omega},$$  

(67)

and the dimensionless epicyclic frequency is $\tilde{\kappa} = \kappa/\Omega$. The remaining equations may be divided into two sets.

### 4.2 Set A equations

Equation (37) at $O(\epsilon^4)$:

$$\left(\Omega \delta_0 - \frac{v_{\theta1}}{r} \partial_r \right) p_0 = \frac{\rho_0}{r} \partial_r v_{\theta1}.$$

(68)

Equation (38) at $O(\epsilon^{s+2})$:

$$\left(\Omega \delta_0 - \frac{v_{\phi1}}{r} \partial_r \right) p_0 = \frac{\Gamma_0}{r} \partial_r v_{\phi1}.$$

(69)

Equation (41) at $O(\epsilon^{s+1})$:

$$\rho_0 \left( \Omega \delta_0 - \frac{v_{\theta1}}{r} \partial_r \right) v_{\phi1} - 2\rho_0 \Omega (v_{\phi1} + rv_{r1} \gamma' \cos \beta) = - (\beta' \cos \phi + \gamma' \sin \beta \sin \phi) \partial_r \left[ \rho_0 + \left( \mu_{00} + \frac{1}{3} \mu_0 \right) \frac{1}{r} \partial_r v_{\theta1} \right]$$

$$+ \left[ \frac{1}{r} + (\beta' \cos \phi + \gamma' \sin \beta \sin \phi) \right] \partial_r \left[ \mu_0 \partial_r (v_{\theta1} + rv_{r1}) + \Omega (\beta' \cos \phi - \gamma' \sin \beta \cos \phi) \partial_r \mu_0. \right]$$

(70)

Equation (42) at $O(\epsilon^{s+2})$:

$$\rho_0 \left( \Omega \delta_0 - \frac{v_{\phi1}}{r} \partial_r \right) \left[ v_{\phi1} + rv_{r1} (\beta' \cos \phi + \gamma' \sin \beta \sin \phi) \right]$$

$$- \rho_0 \Omega \left[ v_{\phi1} + rv_{r1} (\beta' \cos \phi - \gamma' \sin \beta \sin \phi) \right] = \frac{1}{r} \partial_r \left[ \rho_0 + \left( \mu_{00} + \frac{1}{3} \mu_0 \right) \frac{1}{r} \partial_r v_{\theta1} \right]$$

$$+ \left[ \frac{1}{r} + (\beta' \cos \phi + \gamma' \sin \beta \sin \phi) \right] \partial_r \left[ \mu_0 \partial_r (v_{\phi1} + rv_{r1}) \right]$$

$$- r\Omega (\beta' \cos \phi + \gamma' \sin \beta \sin \phi) \partial_r \mu_0.$$

(71)

Equation (43) at $O(\epsilon^{s+1})$:

$$\rho_0 \left( \Omega \delta_0 - \frac{v_{\phi1}}{r} \partial_r \right) (v_{\phi1} + rv_{r1} \gamma' \cos \beta) + \frac{\rho_0 \kappa^2}{2\Omega} v_{r1} = \left[ \frac{1}{r} + (\beta' \cos \phi + \gamma' \sin \beta \sin \phi) \right] \partial_r \left[ \mu_0 \partial_r (v_{\phi1} + rv_{r1}) \right]$$

$$+ r\Omega (\beta' \cos \phi + \gamma' \sin \beta \sin \phi) \partial_r \mu_0.$$

(72)

### 4.3 Set B equations

Equation (37) at $O(\epsilon^{s+1})$:

$$\left( \Omega \delta_0 - \frac{v_{\theta1}}{r} \partial_r \right) \rho_1 + \left( v_{\phi1} \partial_r - \frac{v_{\phi1}}{r} \partial_r + \frac{v_{\phi1}}{r} \partial_r \right) \rho_0 = \frac{\rho_1}{r} \partial_r v_{\phi1} - \rho_0 \left[ \frac{1}{r^2} \partial_r (r^2 v_{r1}) - \frac{1}{r} \partial_r v_{r2} + \frac{1}{r} \partial_r v_{\phi1} \right].$$

(73)

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Equation (38) at $O(e^{s+3})$:

$$
\left(\Omega_0 - \frac{v_{q1}}{r}\right)p_1 + \left(v_1 \partial_r - \frac{v_{q2}}{r} \partial_r + \frac{v_{q3}}{r} \partial_\phi\right)p_0 = \frac{\Gamma p_1}{r} \partial_r v_{q1} - \frac{\Gamma p_0}{r} \left[\frac{1}{r} \partial_r (r^2 v_{q1}) - \frac{1}{r} \partial_r v_{q2} + \frac{1}{r} \partial_\phi v_{q1}\right].
$$  \(74\)

Equation (41) at $O(e^{s+2})$:

$$
\rho_0 \left(\Omega_0 - \frac{v_{q1}}{r}\right)v_{q2} + \rho_0 \left(v_1 \partial_r - \frac{v_{q2}}{r} \partial_r + \frac{v_{q3}}{r} \partial_\phi\right)v_{q1} + \rho_1 \left(\Omega_0 - \frac{v_{q1}}{r}\right)v_{q1} - \frac{\rho_0}{r} \left[v_{q1} + rv_{q1} (\beta' \cos \phi + \gamma' \sin \beta \sin \phi)\right]^2
$$

$$
- 2 \rho_0 \left[\frac{1}{2} \Omega v_{q2} + (r(\gamma + v_{q2}) \cos \beta - rv_{q1} (\beta' \cos \phi + \gamma' \sin \beta \sin \phi))^2\right]\partial_t \left[\rho_0 (v_1 - \gamma' \cos \beta \partial_\phi) v_{q1}\right]
$$

$$
- 2 \rho_0 \Omega (v_{q1} + rv_{q1} \gamma' \cos \beta) = -(\beta' \cos \phi + \gamma' \sin \beta \sin \phi) \partial_t \left[p_1 - \left(\mu_{b0} + \frac{1}{3} \mu_0 \right) \left[\frac{1}{r} \partial_r (r^2 v_{q1}) - \frac{1}{r} \partial_r v_{q2} + \frac{1}{r} \partial_\phi v_{q1}\right]\right]
$$

$$
+ \left(\mu_{b1} + \frac{1}{3} \mu_0 \right) \frac{1}{r} \partial_\phi v_{q1} - \left(\partial_r - \gamma' \cos \beta \partial_\phi\right) \left[p_0 (\mu_{b0} + \frac{1}{3} \mu_0) \frac{1}{r} \partial_r v_{q1}\right]
$$

$$
+ \left[\frac{1}{r} \partial_r + (\beta' \cos \phi + \gamma' \sin \beta \sin \phi) \partial_t \left[\rho_0 (v_1 - \gamma' \cos \beta \partial_\phi) v_{q1}\right]\right] - \frac{2 v_{q1}}{r} \left(\beta' \cos \phi + \gamma' \sin \beta \sin \phi\right) \partial_t v_{q1}
$$

$$
+ \frac{1}{r} \left[(\partial_r - \gamma' \cos \beta \partial_\phi) \rho_0 (\mu_{b0} r^2) \partial_t \left[v_{q1} + rv_{q1} (\beta' \cos \phi + \gamma' \sin \beta \sin \phi)\right]\right]
$$

$$
- (\partial_r \mu_0) \partial_\phi v_{q1} + v_{q1} (\beta' \cos \phi + \gamma' \sin \beta \sin \phi)
$$

$$
- \frac{1}{r} (\partial_r \mu_0) \partial_\phi \left[(\beta' \cos \phi + \gamma' \sin \beta \sin \phi) (v_{q1} + rv_{q1} \gamma' \cos \beta)\right] + \Omega \partial_\phi \mu_0
$$

$$
+ \frac{1}{r} \left(\beta' \cos \phi + \gamma' \sin \beta \sin \phi\right) \partial_\phi v_{q1} + rv_{q1} (\beta' \cos \phi + \gamma' \sin \beta \sin \phi) + \Omega (\beta' \sin \phi - \gamma' \sin \beta \cos \phi) \partial_\phi v_{q1}.
$$  \(75\)

Equation (42) at $O(e^{s+2})$:

$$
\rho_0 \left(\Omega_0 - \frac{v_{q1}}{r}\right) [v_{q2} + r(\beta + v_{q2} \beta') \cos \phi + r(\gamma + v_{q2} \gamma') \sin \beta \sin \phi]
$$

$$
+ \rho_0 \left(v_1 \partial_r - \frac{v_{q2}}{r} \partial_r + \frac{v_{q3}}{r} \partial_\phi\right) [v_{q1} + rv_{q1} (\beta' \cos \phi + \gamma' \sin \beta \sin \phi)]
$$

$$
+ \rho_1 \left(\Omega_0 - \frac{v_{q1}}{r}\right) [v_{q1} + rv_{q1} (\beta' \cos \phi + \gamma' \sin \beta \sin \phi)] + \frac{\partial_t \Omega}{r} \left[v_{q1} + rv_{q1} (\beta' \cos \phi + \gamma' \sin \beta \sin \phi)\right]
$$

$$
- \rho_0 \Omega (v_{q1} + rv_{q1} \gamma' \cos \beta) + r(\beta + v_{q2} \beta') \sin \phi - r(\gamma + v_{q2} \gamma') \sin \beta \cos \phi
$$

$$
- \rho_0 \Omega (v_{q1} + rv_{q1} \gamma' \cos \beta) [r(\Omega_0) + rv_{q1} (\beta' \sin \phi - \gamma' \sin \beta \cos \phi)] - \rho_1 \Omega (r(\Omega_0) + rv_{q1} (\beta' \sin \phi - \gamma' \sin \beta \cos \phi)]
$$

$$
= \frac{1}{r} \partial_t \left[p_1 - \left(\mu_{b0} + \frac{1}{3} \mu_0 \right) \left[\frac{1}{r} \partial_r (r^2 v_{q1}) - \frac{1}{r} \partial_r v_{q2} + \frac{1}{r} \partial_\phi v_{q1}\right]\right]
$$

$$
+ \frac{1}{r} (\beta' \cos \phi + \gamma' \sin \beta \sin \phi) \partial_r \left[p_0 (\mu_{b0} + \frac{1}{3} \mu_0) \frac{1}{r} \partial_r v_{q1}\right]
$$

$$
+ \frac{1}{r} (\beta' \cos \phi + \gamma' \sin \beta \sin \phi) \partial_r \theta r [\rho_0 (v_1 - \gamma' \cos \beta \partial_\phi) v_{q1}]
$$

$$
+ \frac{1}{r} (\partial_r - \gamma' \cos \beta \partial_\phi) \left[p_0 (\mu_{b0} + \frac{1}{3} \mu_0) \frac{1}{r} \partial_r v_{q1}\right]
$$

$$
- \frac{1}{r} (\beta' \cos \phi + \gamma' \sin \beta \sin \phi) \partial_r \left[p_0 (\mu_{b0} + \frac{1}{3} \mu_0) \frac{1}{r} \partial_r v_{q1}\right]
$$

$$
- \frac{1}{r} (\partial_r \mu_0) \partial_\phi v_{q1} + \frac{1}{r} (\partial_r \mu_0) \partial_\phi r [v_{q1} + rv_{q1} \gamma' \cos \beta]
$$

$$
- \frac{1}{r} (\partial_r \mu_0) \partial_\phi r [v_{q1} + rv_{q1} \gamma' \cos \beta] - \mu_0 (\beta' \cos \phi - \gamma' \sin \beta \sin \phi) \partial_r \left[p_0 (v_{q1} + rv_{q1} \gamma' \cos \beta)\right]
$$

$$
- r \Omega (\beta' \sin \phi - \gamma' \sin \beta \cos \phi) \partial_r \left[p_0 (v_{q1} + rv_{q1} \gamma' \cos \beta)\right]
$$

$$
- \mu_0 (\beta' \cos \phi + \gamma' \sin \beta \sin \phi) \partial_r \left[p_0 (v_{q1} + rv_{q1} \gamma' \cos \beta)\right].
$$  \(76\)
Equation (43) at $O(\epsilon^{+2})$:

$$
\rho_0 \left( \Omega_0 - \frac{v_{01}}{r} \partial_r \right) \left[ v_{02} - \frac{1}{2} \Omega_0^2 + r (\gamma + v_{12} \gamma \cos \beta) - rv_{11} (\beta' \sin \phi - \gamma' \sin \beta \cos \phi) \right] \\
+ \rho_0 \left( \Omega_0 - \frac{v_{01}}{r} \partial_r \right) \left[ v_{11} + rv_{11} \gamma' \cos \beta \right] + \rho_1 \left( \Omega_0 - \frac{v_{01}}{r} \partial_r \right) (v_{12} + rv_{11} \gamma' \cos \beta) + \frac{\rho_0 k^2}{2 \Omega} v_{12} + \frac{\rho_1 k^2}{2 \Omega} v_{11} \\
+ \frac{\rho_0}{r} \left[ v_{11} + rv_{11} \gamma' \cos \beta \right] + \frac{\rho_0}{r} \left[ v_{12} + rv_{11} (\beta' \sin \phi - \gamma' \sin \beta \cos \phi) \right] \\
= - \frac{1}{r} \left[ \rho_0 + \left( \mu_0 + \frac{1}{3} \mu_0 \right) r \partial_r v_{11} \right] + r \Omega (\beta' \cos \phi + \gamma' \sin \beta \sin \phi) \partial_r \mu_1 \\
+ \frac{1}{r} \left[ v_{01} (r \Omega') - r \Omega (\partial_r \mu_0) \gamma' \cos \beta + (\beta' \cos \phi + \gamma' \sin \beta \sin \phi) \partial_r [\mu_0(\partial_r - \gamma' \cos \beta \partial_\phi)] v_{11} + rv_{11} \gamma' \cos \beta \right] \\
+ \frac{1}{r} \left[ \mu_1 (r \Omega') - \frac{1}{r} \partial_r \left[ \mu_0 (\partial_r - \gamma' \cos \beta \partial_\phi) \right] v_{11} + rv_{11} \gamma' \cos \beta \right] \\
+ \mu_1 (\partial_r - \gamma' \cos \beta \partial_\phi) [v_{01} + rv_{11} \gamma' \cos \beta] \\
+ \frac{1}{r} \partial_r \left[ \mu_0 (\partial_r - \gamma' \cos \beta \partial_\phi) \right] v_{11} + rv_{11} \gamma' \cos \beta \right] \\
+ \mu_1 (\partial_r - \gamma' \cos \beta \partial_\phi) [v_{01} + rv_{11} \gamma' \cos \beta] \\
+ \mu_1 (\partial_r - \gamma' \cos \beta \partial_\phi) [v_{01} + rv_{11} \gamma' \cos \beta] \\
+ \frac{1}{r} \partial_r \left[ \mu_0 (\partial_r - \gamma' \cos \beta \partial_\phi) \right] v_{11} + rv_{11} \gamma' \cos \beta \right] \\
+ \mu_1 (\partial_r - \gamma' \cos \beta \partial_\phi) [v_{01} + rv_{11} \gamma' \cos \beta] \\
+ \frac{1}{r} \partial_r \left[ \mu_0 (\partial_r - \gamma' \cos \beta \partial_\phi) \right] v_{11} + rv_{11} \gamma' \cos \beta \right] \\
+ \mu_1 (\partial_r - \gamma' \cos \beta \partial_\phi) [v_{01} + rv_{11} \gamma' \cos \beta] \\
+ \frac{1}{r} \partial_r \left[ \mu_0 (\partial_r - \gamma' \cos \beta \partial_\phi) \right] v_{11} + rv_{11} \gamma' \cos \beta \right] \\
+ \mu_1 (\partial_r - \gamma' \cos \beta \partial_\phi) [v_{01} + rv_{11} \gamma' \cos \beta] \\
+ \frac{1}{r} \partial_r \left[ \mu_0 (\partial_r - \gamma' \cos \beta \partial_\phi) \right] v_{11} + rv_{11} \gamma' \cos \beta \right] \\
+ \mu_1 (\partial_r - \gamma' \cos \beta \partial_\phi) [v_{01} + rv_{11} \gamma' \cos \beta].
$$

(77)

4.4 Integrated quantities

Finally, equation (37) is also required at $O(\epsilon^{+2})$, but only in its integrated form.\(^{2}\)

$$
\partial_T \left( \int \rho_0 r \, d\phi \, d\xi \right) + \frac{1}{r} \partial_r \left[ r \left( \rho_0 v_{02} + \rho_1 v_{11} \right) r \, d\phi \, d\xi \right] = 0.
$$

(78)

The surface density $\Sigma(r, T)$ at leading order in $\epsilon$ is

$$
\Sigma = \int \rho_0 r \, d\xi,
$$

(79)

and is independent of $\phi$ by virtue of equation (68). Other vertically integrated quantities are non-axisymmetric, however, and these will be written with tildes. The corresponding azimuthally averaged quantities, written without tildes, are defined by the operation

$$
\langle \cdot \rangle = \frac{1}{2\pi} \int 0 \, d\phi.
$$

(80)

Thus a suitably averaged radial velocity $\tilde{v}_r(r, T)$ may be defined by

$$
\Sigma \tilde{v}_r = \tilde{\mathcal{F}} = \langle \mathcal{F} \rangle,
$$

(81)

where

$$
\mathcal{F} = \int \left( \rho_0 v_{02} + \rho_1 v_{11} \right) r \, d\xi
$$

(82)

is the radial mass flux at $O(\epsilon^{+3})$.\(^{3}\) Then equation (78), divided by $2\pi$, reads

$$
\partial_T \Sigma + \frac{1}{r} \partial_r (r \Sigma \tilde{v}_r) = 0,
$$

(83)

which agrees with equation (2). Two more definitions will be useful. A suitably averaged kinematic viscosity $\nu(r, T)$ may be defined by

$$
\nu \Sigma = \tilde{\nu}' = \langle \tilde{\nu}' \rangle
$$

(84)

\(^{2}\)Throughout this paper, integrations with respect to $\phi$ are carried out from 0 to $2\pi$, and integrations with respect to $\xi$ are carried out over the full vertical extent of the disc.

\(^{3}\)There is a radial mass flux at $O(\epsilon^{+2})$, but its azimuthal average vanishes by virtue of equations (72) and (68).
where
\[
\hat{\mathcal{F}} = \int \mu_0 r \, d\xi
\]  
(85)

is the vertically integrated viscosity. Finally, the second vertical moment of the density is
\[
\mathcal{F} = \int \rho_0 r^2 \hat{\mathcal{F}}^2 \, r \, dr, 
\]  
(86)

and its azimuthal average is
\[
\mathcal{F} = \langle \mathcal{F} \rangle. 
\]  
(87)

### 4.5 Formal manipulations

It is clear that the problem under consideration is much more complicated than the analysis of an unwarped viscous disc. In the absence of a warp, all quantities are independent of \( \phi \) and symmetric about \( \xi = 0 \). Furthermore, the quantities \( v_{11}, v_{01}, v_{02}, v_{03} \) and \( v_{02} \) all vanish. It is worth noting that \textit{if the same assumptions could be made for a warped disc,} equations (4) and (5) with \( m_1 = m_2 = \hat{v} \) could be derived directly by integrating equations (77) and (76). However, these assumptions are inconsistent with the equations of Set A. For example, in equation (70) there is a radial force resulting from the lateral imbalance of pressure and viscous stress in a warped disc, and this drives the horizontal velocities \( v_{11} \) and \( v_{01} \).

It is important to note the formal structure of the problem. Set A consists of five coupled non-linear partial differential equations (PDEs) with two independent variables \( \{ \phi, \xi \} \) and seven dependent variables \( \{ \rho_0, \rho_0, \mu_0, \mu_0, v_{11}, v_{01}, v_{02}, v_{03} \} \). The equations must be closed by specifying suitable prescriptions for the viscosities and for the thermodynamics. In general, they must be solved numerically. Set B is a set of five coupled linear PDEs for the higher order quantities \( \{ \rho_1, \rho_1, \mu_1, \mu_1, v_{12}, v_{02}, v_{03} \} \), with coefficients that depend on the solutions of Set A and their radial derivatives. Fortunately, all the information required from Set B can be extracted by integration, so that only Set A need be solved in detail.

The aim is to extract equations resembling equations (51)–(53). To this end, multiply equation (77) by \( r \), integrate vertically and average azimuthally to obtain
\[
\Sigma_\phi \langle r^2 \bar{\Omega} \rangle = I_1, 
\]  
(88)

after an integration by parts and the use of equations (68) and (73), where
\[
I_1 = \int \left[ \frac{1}{r} \partial_r \left[ \mu_0 r^4 \Omega - \rho_0 r^3 v_{11}(v_{01} + rv_{11}) + \rho_0 r^3 (\beta' \cos \phi + \gamma' \sin \beta \sin \phi) \partial_r (v_{01} + rv_{11}) + rv_{11} \cos \beta \right] - \rho_0 \left[ v_{01} + rv_{11} (\beta' \cos \phi + \gamma' \sin \beta \sin \phi) \right] \right] \right] 
\]  
(89)

Then multiply equation (76) first by (\(-r \sin \phi\)), then by \( r \cos \phi \), in each case integrating vertically and averaging azimuthally to obtain first
\[
\Sigma_\phi \langle \hat{r} \beta + \hat{r} \beta' \rangle = I_2, 
\]  
(90)

and then
\[
\Sigma_\phi \langle \hat{r} \gamma + \hat{r} \gamma' \rangle \sin \beta = I_3, 
\]  
(91)

where
\[
I_2 = \int \left[ \rho_0 \bar{\Omega} \cos \phi \left\{ \frac{1}{r} \partial_r (r^2 \bar{v}_{11}) + \frac{1}{r} \partial_\phi (\rho_0 \bar{v}_{01}) \right\} \right] \sin \phi \, r \, dr \right] \right] 
\]  
(92)

\( \rho_0 \) vanishes. It is worth noting that \textit{if the same assumptions could be made for a warped disc,} equations (4) and (5) with \( m_1 = m_2 = \hat{v} \) could be derived directly by integrating equations (77) and (76). However, these assumptions are inconsistent with the equations of Set A. For example, in equation (70) there is a radial force resulting from the lateral imbalance of pressure and viscous stress in a warped disc, and this drives the horizontal velocities \( v_{11} \) and \( v_{01} \).

It is important to note the formal structure of the problem. Set A consists of five coupled non-linear partial differential equations (PDEs) with two independent variables \( \{ \phi, \xi \} \) and seven dependent variables \( \{ \rho_0, \rho_0, \mu_0, \mu_0, v_{11}, v_{01}, v_{02}, v_{03} \} \). The equations must be closed by specifying suitable prescriptions for the viscosities and for the thermodynamics. In general, they must be solved numerically. Set B is a set of five coupled linear PDEs for the higher order quantities \( \{ \rho_1, \rho_1, \mu_1, \mu_1, v_{12}, v_{02}, v_{03} \} \), with coefficients that depend on the solutions of Set A and their radial derivatives. Fortunately, all the information required from Set B can be extracted by integration, so that only Set A need be solved in detail.

The aim is to extract equations resembling equations (51)–(53). To this end, multiply equation (77) by \( r \), integrate vertically and average azimuthally to obtain first
\[
\Sigma_\phi \langle r^2 \bar{\Omega} \beta + r \bar{v}_{

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and
\[
I_3 = \left[ \frac{1}{r^2} \partial_r (r^2 \partial_r v_{11}) + \frac{1}{r} \partial_r (\rho_0 \Phi_{v_{11}}) \right] - \frac{\cos \phi}{r} \partial_r \left( r^2 \rho_0 \Phi_{v_{11}} \left[ v_{11} + \nu v_{11} (\beta' \cos \phi + \gamma' \sin \beta \sin \phi) \right] \right)
- \rho_0 v_{01} \sin \phi \left[ v_{11} + \nu v_{11} (\beta' \cos \phi + \gamma' \sin \beta \sin \phi) \right] + 2 \rho_0 r \Omega^2 \cos \phi (v_{01} + \nu v_{11} \gamma' \cos \beta)
+ \rho_0 \nu v_{11} \cos \phi (v_{01} + \nu v_{11} \gamma' \cos \beta) (\beta' \sin \phi - \gamma' \sin \beta \cos \phi)
+ \cos \phi \left( \beta' \cos \phi + \gamma' \sin \beta \sin \phi \right) \partial_r \left[ v_{11} + \nu v_{11} (\beta' \cos \phi + \gamma' \sin \beta \sin \phi) \right]
+ \mu_0 \cos \phi (\beta' \cos \phi + \gamma' \sin \beta \sin \phi) \partial_r \left[ v_{11} + \nu v_{11} (\beta' \cos \phi + \gamma' \sin \beta \sin \phi) \right] - \frac{\cos \phi}{r} \partial_r \left( \mu_0 r^2 \partial_r v_{11} \right) - \mu_0 \sin \phi \partial_r v_{01}
- \cos \phi \partial_r \left[ \mu_0 r^2 (\beta' \sin \phi - \gamma' \sin \beta \cos \phi) \right] + \mu_0 r^2 \gamma' \cos \phi (\gamma' \sin \beta \sin \phi)
- \mu_0 \cos \phi (\beta' \sin \phi - \gamma' \sin \beta \cos \phi) \left[ r \Omega^2 + (\beta' \cos \phi + \gamma' \sin \beta \sin \phi) \partial_r (v_{01} + \nu v_{11} \gamma' \cos \beta) \right]
\right] \, r \, d\xi. \tag{93}
\]

Again, it will be useful to consider the complex combination
\[
I_2 + iI_3 = \left[ \frac{1}{r} \partial_r + i \gamma' \cos \beta \right] \left( e^{i\phi} \{ \rho_0 r^2 \Omega^2 v_{11} - i \rho_0 r^2 v_{11} \left[ v_{11} + \nu v_{11} (\beta' \cos \phi + \gamma' \sin \beta \sin \phi) \right] \right)
+ \mu_0 r \beta' \cos \phi \left[ \beta' \cos \phi + \gamma' \sin \beta \sin \phi \right] \partial_r \left[ v_{11} + \nu v_{11} (\beta' \cos \phi + \gamma' \sin \beta \sin \phi) \right] - \mu_0 r^2 \partial_r v_{11}
- \mu_0 r \beta' \cos \phi \left[ \beta' \cos \phi + \gamma' \sin \beta \sin \phi \right] \left[ v_{11} + \nu v_{11} (\beta' \cos \phi + \gamma' \sin \beta \sin \phi) \right] + i \mu_0 r \beta' \cos \phi \partial_r \left[ v_{11} + \nu v_{11} (\beta' \cos \phi + \gamma' \sin \beta \sin \phi) \right]
- \rho_0 (v_{01} + \nu v_{11} \gamma' \cos \beta) \left[ v_{11} + \nu v_{11} (\beta' \cos \phi + \gamma' \sin \beta \sin \phi) \right] + i \mu_0 r \beta' \cos \phi \partial_r \left[ v_{01} + \nu v_{11} \gamma' \cos \beta \right]
+ \mu_0 r \beta' \cos \phi \left[ \beta' \sin \phi - \gamma' \sin \beta \cos \phi \right] \left[ r \Omega^2 + (\beta' \cos \phi + \gamma' \sin \beta \sin \phi) \partial_r (v_{01} + \nu v_{11} \gamma' \cos \beta) \right]
\right] \, r \, d\xi. \tag{94}
\]

5 SEPARATION OF VARIABLES

In order to proceed, it is helpful to make two simplifying, but physically reasonable, assumptions. First, the viscosity coefficients are assumed to be locally proportional to the pressure, so that
\[
\mu = \alpha_p \Omega, \tag{95}
\]
\[
\mu_0 = \alpha_p \Omega, \tag{96}
\]
where the dimensionless coefficients \( \alpha_p(r) \) and \( \alpha_0(r) \) are prescribed functions of radius. Secondly, the fluid is assumed to be locally polytropic, with \( \Gamma(r) > 1 \) being a prescribed function of radius. These assumptions allow the equations of Set A to be solved by separation of variables. (An isothermal assumption, \( \Gamma = 1 \), would allow a similar simplification.)

Given that the pressure and density locally satisfy a polytropic relation, \( p = K \rho^\Gamma \), introduce the enthalpy
\[
h = \left[ \frac{dp}{\rho} \right] = \left( \frac{\Gamma}{\Gamma - 1} \right) K \rho^\Gamma = \left( \frac{\Gamma}{\Gamma - 1} \right) \frac{p}{\rho}. \tag{97}
\]
Then equations (68) and (69) may be replaced by
\[
\left( \Omega \partial \left( \frac{v_{01}}{r} \right) \xi \right) \, h_0 = \left( \frac{\Gamma - 1}{\Gamma} \right) \xi \partial_r v_{01}, \tag{98}
\]
and Set A is reduced to a problem in four dependent variables \( \{ h_0, v_{01}, v_{11}, (v_{01} + \nu v_{11} \gamma' \cos \beta) \} \). Note that \( \psi = |\psi| e^{i\chi} \) occurs only in the combinations
\[
r(\beta' \cos \phi + \gamma' \sin \beta \sin \phi) = |\psi| \cos(\phi - \chi), \tag{99}
\]
\[
r(\beta' \sin \phi - \gamma' \sin \beta \cos \phi) = |\psi| \sin(\phi - \chi). \tag{100}
\]
The solution of Set A is then of the form
\[
h_0 = r^2 \Omega^2 \left[ f_1(\phi - \chi) - \frac{\xi^2}{2} \right], \tag{101}
\]
\[
v_{01} = r \Omega f_1(\phi - \chi) \xi, \tag{102}
\]
\[
v_{11} = r \Omega f_1(\phi - \chi) \xi, \tag{103}
\]
\[
v_{01} + \nu v_{11} \gamma' \cos \beta = r \Omega f_1(\phi - \chi) \xi. \tag{104}
\]
where the dimensionless functions $f_1, \ldots, f_5$ (the parametric dependence of which on $r$ and $T$ has been suppressed) satisfy the non-linear ODEs

\begin{align}
    f_1'(\phi) &= (\Gamma - 1) f_2(\phi) f_1(\phi), \\
    f_2'(\phi) &= (\Gamma + 1) f_2(\phi) f_2(\phi), \\
    f_3'(\phi) &= f_2(\phi) f_3(\phi) + 2 f_5(\phi) + \left[ 1 + \left( \alpha_0 + \frac{1}{3} \alpha \right) f_2(\phi) \right] f_3(\phi) |\psi| \cos \phi - \alpha f_2(\phi) f_3(\phi) (1 + |\psi|^2 \cos^2 \phi) - \alpha f_2(\phi) |\psi| \sin \phi, \\
    f_4'(\phi) &= -f_1(\phi) |\psi| \cos \phi + f_2(\phi) |\psi| \sin \phi + f_3(\phi) [f_4(\phi) + f_3(\phi) |\psi| \cos \phi] + 1 - \left[ 1 + \left( \alpha_0 + \frac{1}{3} \alpha \right) f_4(\phi) \right] f_2(\phi) \\
        &- \alpha f_2(\phi) [f_4(\phi) + f_3(\phi) |\psi| \cos \phi] (1 + |\psi|^2 \cos^2 \phi) + \alpha f_2(\phi) |\psi|^2 \cos \phi \sin \phi, \\
    f_5'(\phi) &= f_4(\phi) f_5(\phi) - \frac{1}{2} \left[ f_5(\phi) - \alpha f_2(\phi) f_5(\phi) (1 + |\psi|^2 \cos^2 \phi) + \frac{3}{2} (4 - \tilde{r}^2) \alpha f_2(\phi) |\psi| \cos \phi, \\
\end{align}

subject to periodic boundary conditions $f_n(2\pi) = f_n(0)$. There are five dimensionless parameters $|\psi|, \tilde{r}^2, \Gamma, \alpha, \alpha_0$. Note that equation (105) is decoupled, and that $f_1$ admits an arbitrary multiplicative constant which allows the surface density to be fitted. The (upper) surface of the disc is given at leading order by

\[ \tilde{r}^2 = f_2(\phi) \left( \frac{2 f_1(\phi) - \chi}{f_2(\phi) - \chi} \right)^{1/2}. \]

The expressions $I_n$ of Section 4.5 may now be simplified. The vertical integrals introduce the quantities $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}^*\mathcal{R}$ which are related by

\[ \tilde{\mathcal{F}}^*\mathcal{R} = \alpha f_2(\phi - \chi) \Omega \tilde{\mathcal{F}}. \]

This follows from the definition of $\tilde{\mathcal{F}}^*\mathcal{R}$ after an integration by parts. Then equations (89) and (51) can be identified provided that

\begin{align}
    Q_1 &= \left\langle 6 \left( \frac{1}{2} (4 - \tilde{r}^2) \alpha f_2 - f_3 f_5 + \alpha f_2 f_5 |\psi| \cos \phi \right) \right\rangle, \\
    Q_2 &= \frac{1}{|\psi|^2} \left( f_6 + f_3 |\psi| \cos \phi \right) (1 + f_3 |\psi| \sin \phi) + \alpha f_2 f_3 |\psi| \sin \phi - \alpha f_2 (f_4 + f_3 |\psi| \cos \phi) |\psi|^2 \cos \phi \sin \phi + \alpha f_2 |\psi|^2 \sin^2 \phi],
\end{align}

in which $f_n$ stands for $f_n(\phi)$, and

\[ f_6(\phi - \chi) = \tilde{\mathcal{F}} \tilde{\mathcal{R}} \]

contains the azimuthal dependence of $\tilde{\mathcal{F}}$. To evaluate this, note that

\[ \int \left( f_1 - \frac{1}{2} \right) \tilde{\mathcal{F}}^{1(\Gamma - 1)} \tilde{\mathcal{F}}^2 d\tilde{r}^2 = f_1^{1(\Gamma - 1)} \left( \frac{f_1}{f_2} \right)^{3/2}, \]

and so

\[ f_6(\phi) = \left[ f_1(\phi) \right]^{1(\Gamma - 1)} \left[ \frac{f_1(\phi)}{f_2(\phi)} \right]^{3/2} \left[ \left( \frac{f_1}{f_2} \right)^{3/2} \right]. \]

Then it follows from equations (105) and (106) that

\[ f_6(\phi) = -2 f_4(\phi) f_6(\phi), \]

while the definition of $f_6$ requires $\langle f_6 \rangle = 1$. Note that, for any function $F$,

\[ \langle e^{i \phi} F(\phi - \chi) \rangle = \langle e^{i \phi + \chi} F(\phi) \rangle = \frac{\psi}{|\psi|^2} \langle e^{i \phi} F(\phi) \rangle. \]

Thus $I_2 + i I_3$ can be identified with $X + i Y$, provided that

\begin{align}
    Q_1 &= \frac{1}{|\psi|^2} \left( e^{i \phi} f_6 \left( - \frac{1}{2} \tilde{r}^2 f_3 - f_3 f_5 + f_3 |\psi| \cos \phi \right) + i f_2 f_3 |\psi| \sin \phi - \alpha f_2 f_5 - i \alpha f_2 \left[ \frac{3}{2} (4 - \tilde{r}^2) f_3 |\psi| \cos \phi \right] |\psi| \sin \phi \right), \\
    Q_2 &= \frac{1}{|\psi|^2} \left( e^{i \phi} f_6 [f_3 - i f_3 f_5 |\psi| \cos \phi] + i \alpha f_2 f_3 + f_3 |\psi| \cos \phi \right) |\psi| \cos \phi - i \alpha f_2 f_5 - i \alpha f_2 |\psi| \sin \phi \right),
\end{align}

These correspondences are equivalent: equations (112) and (119), and equations (113) and (120), can be shown to agree by using equations (109) and (117). Therefore equations (112) and (120) for $Q_1$ and $Q_4$ are sufficient and will be taken as definitive. This completes the
demonstration that the fully non-linear problem can be reduced to one-dimensional conservation equations for mass and angular momentum as anticipated in Section 3.2.

In the inviscid case \( \alpha = \alpha_0 = 0 \), the functions \( f_1, f_2, f_5 \) and \( f_6 \) are even, while \( f_3 \) and \( f_4 \) are odd. Then \( Q_1 = Q_2 = 0 \), i.e. \( Q_3 \) is purely imaginary.

6 EVALUATION OF THE COEFFICIENTS

In principle, equations (105)–(109) and (117) can be expanded in powers of \( |\psi| \) to determine the dynamics to any desired order. In the Appendix, truncated Taylor series for the coefficients \( Q_1, Q_2 \) and \( Q_3 \) are developed in this way. This generalizes the linear theory of Papaloizou & Pringle (1983) to allow for an arbitrary rotation law, and extends it into the weakly non-linear domain. These series can be used to estimate at which amplitude, and in what way, the linear theory breaks down. This behaviour is interpreted further in Section 7. The expansion fails only when both \( k^2 \rightarrow 1 \) and \( \alpha \rightarrow 0 \); this is the previously identified resonant case, which cannot be described using this method.

It is also straightforward to integrate equations (106)–(109) and (117) numerically, imposing periodic boundary conditions and the normalization condition \( \langle f_6 \rangle = 1 \). The coefficients \( Q_1, Q_2 \) and \( Q_3 \) can then be determined directly for any amplitude of warp, resulting in a fully non-linear theory. Two important cases have been investigated in this way.

6.1 Inviscid, non-Keplerian disc

The first case is that of an inviscid disc with \( \Gamma = 5/3 \) and \( \alpha = \alpha_0 = 0 \). The remaining parameters are \( |\psi| \) and \( k^2 \). A contour plot of the only non-vanishing coefficient, \( Q_3 \), is shown in Fig. 2. For reasonably small values of \( |\psi| \) there is good agreement with the truncated Taylor series. Where \( k^2 > 1 \), \( Q_3 \) is negative and the solution can be continued to large values of \( |\psi| \), although the disc becomes highly non-axisymmetric in this limit. Where \( k^2 < 1 \), \( Q_3 \) is positive and the solution exists only for sufficiently small \( |\psi| \). The solution terminates when \( f_2 \) becomes zero at some point, with the result that the disc can no longer contain itself vertically and ruptures.

6.2 Viscous, Keplerian disc

The second case is that of a viscous, Keplerian disc with \( \Gamma = 5/3 \) and \( \alpha_0 = 0 \). The remaining parameters are \( |\psi| \) and \( \alpha \). Contour plots of the three coefficients are shown in Figs 3, 4 and 5. Again, for reasonably small values of \( |\psi| \) there is good agreement with the truncated Taylor series, except when \( \alpha \) is small. The solution can be continued to large values of \( |\psi| \) for any value of \( \alpha \), and again the disc becomes highly non-axisymmetric in this limit. There are several features to note. First, as predicted by the Taylor series, \( Q_1 \) increases with increasing \( |\psi| \) and even becomes positive for sufficiently small \( \alpha \). Secondly, \( Q_2 \) has a strong peak at \( |\psi| = \alpha = 0 \) where the resonance is located, but the resonance does not extend to large values of \( |\psi| \). Thirdly, \( Q_3 \) is typically much smaller in magnitude than \( Q_2 \).

7 SUMMARY AND INTERPRETATION

7.1 General remarks

In this paper, the non-linear fluid dynamics of a warped accretion disc has been investigated by developing a theory of fully non-linear bending waves for a thin, viscous disc in a spherically symmetric external potential. It has been found that the dynamics is described by an equation for
the surface density,
\[ \frac{\partial \Sigma}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \Sigma \bar{v}_r) = 0, \tag{121} \]
and an equation for the angular momentum,
\[ \frac{\partial}{\partial t} (\Sigma r^2 \Omega \ell) + \frac{1}{r} \frac{\partial}{\partial r} (\Sigma \bar{v}_r r^2 \Omega \ell) = \frac{1}{r} \frac{\partial}{\partial r} \left( Q_1 \bar{J} r^2 \Omega^2 \ell \right) + \frac{1}{r} \frac{\partial}{\partial r} \left( Q_2 \bar{J} r^3 \Omega^2 \frac{\partial \ell}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left( Q_3 \bar{J} r^3 \Omega^2 \ell \frac{\partial \ell}{\partial r} \right) + T. \tag{122} \]

Here \( \Sigma(r,t) \) is the surface density, \( \bar{v}_r(r,t) \) is the mean radial velocity, \( \Omega(r) \) is the orbital angular velocity, \( \ell(r,t) \) is the tilt vector, \( \bar{J}(r,t) \) is the azimuthally averaged second vertical moment of the density (cf. equation 47), and a new term \( T \) represents any additional torque resulting from self-gravitation, radiation forces, tidal forcing, etc. The dimensionless coefficients \( Q_1, Q_2 \) and \( Q_3 \) depend on the rotation law and the shear viscosity, and also, in the non-linear theory, on the adiabatic exponent, the bulk viscosity and the amplitude of the warp. They have been determined both analytically, as truncated Taylor series in the amplitude of the warp, and numerically, by solving a set of ordinary differential equations. The angular momentum equation can be decomposed into an equation.

Figure 3. Contour plot of the coefficient \( Q_1 \) for a viscous, Keplerian disc with \( \Gamma = 5/3 \) and \( \alpha_b = 0 \). The horizontal coordinate is the amplitude of the warp. The vertical coordinate is the dimensionless viscosity parameter.

Figure 4. Contour plot of the coefficient \( Q_2 \) for a viscous, Keplerian disc with \( \Gamma = 5/3 \) and \( \alpha_b = 0 \).
for the component parallel to $\ell$, 
\[
\Sigma v_r \frac{d}{dr} \left( r^2 \Omega_1 \right) = \frac{1}{r} \frac{d}{dr} \left( Q_1 r^2 \Omega^2 \right) - Q_2 r^2 \Omega \frac{d|\ell|^2}{dr} + \ell \cdot T 
\]
and an equation for the tilt vector,
\[
\Sigma r^2 \Omega \left( \frac{\partial \ell}{\partial t} + \ell \cdot \frac{\partial \ell}{\partial r} \right) = Q_1 r^2 \Omega^2 \frac{\partial \ell}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} \left( Q_2 r^2 \Omega \frac{\partial \ell}{\partial r} \right) + Q_2 r^2 \Omega \frac{d|\ell|^2}{dr} \ell + \frac{1}{r} \frac{\partial}{\partial r} \left( Q_3 r^2 \Omega \ell \times \frac{\partial \ell}{\partial r} \right) - \ell \times (\ell \times T). 
\]

This scheme is a generalization of the form proposed by Pringle (1992), and is equally suitable for numerical implementation. The internal torques between one ring in the disc and its neighbours are of three kinds. Coefficient $Q_1$ represents a torque tending to spin up (or down) the ring. This would be the usual viscous torque proportional to $d\Omega/dr$ in a flat disc, but in a warped disc there is an additional contribution, not proportional to $d\Omega/dr$, resulting from a correlation between the radial and azimuthal velocities induced by the warp; this vanishes in an inviscid disc because the radial and azimuthal velocities are perfectly out of phase. Coefficient $Q_2$ represents a torque tending to align the ring with its neighbours, which acts to flatten the disc; this also vanishes in an inviscid disc. Coefficient $Q_3$ represents a torque tending to make the ring precess if it is misaligned with its neighbours; this leads to the dispersive wave-like propagation of the warp. It is likely that this form of angular momentum equation applies in much more general circumstances than those considered in this paper. However, the present analysis provides a consistent evaluation of the three coefficients under a number of simplifying assumptions.

There are essentially four different assumptions or approximations involved in this analysis. The first assumption is that the disc is thin ($H/r \ll 1$), which is essential for the asymptotic expansions in Section 4. The second assumption is that the fluid obeys the compressible Navier–Stokes equation even though, in reality, the turbulent stresses in an accretion disc are likely to be anisotropic and may not be purely viscous. The third assumption is that the fluid is locally polytropic and has viscosity coefficients locally proportional to the pressure. These are the simplest possible closure relations in the absence of a full treatment of thermal and radiative physics, and greatly simplify the problem by allowing separation of variables (Section 5). In a fully self-consistent treatment, the thickness of the disc, and therefore the relation between $\mathcal{f}$ and $\Sigma$, would be determined by the local viscous dissipation of energy. The final assumption is that the conditions for resonance,
\[
\frac{\Omega^2 - \kappa^2}{\Omega^2} \approx H/r \quad \text{and} \quad \alpha \approx H/r 
\]
are not simultaneously satisfied. (It has been shown in Section 6.2 that a further condition for resonance is that the amplitude of the warp be small.) The asymptotic expansions formally break down as the resonance is approached. The nature of the approximation is as follows. In equations (70) and (72) governing the horizontal velocities induced by the warp, the time-derivatives of the velocities (in the inertial frame) do not appear because those terms are of higher order in the assumed ordering scheme. The horizontal velocities are in a ‘geostrophic’ balance in which the driving forces are matched instantaneously by inertial and viscous forces, and for this reason they do not constitute additional dynamical degrees of freedom. In the resonant case, this approximation breaks down and the time-derivatives must be restored. This results in a

It is to be expected that $Q_2$ will normally be positive, as it represents a diffusion coefficient. The linear theory (see the Appendix) suggests that this is true except possibly for unphysical rotation laws ($\kappa^2 > 8$).
different ordering scheme and makes the problem hyperbolic rather than parabolic (Papaloizou & Lin 1994). The non-linear dynamics of the resonant case is likely to be very different.

Two important cases merit further interpretation.

### 7.2 Inviscid, non-Keplerian disc

In the inviscid case, the radial velocity vanishes, the surface density is independent of time, and the dynamics is described by an equation for the tilt vector of the form

$$\Sigma^2 \frac{\partial \ell}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( Q_3 \Omega^2 \ell \times \frac{\partial \ell}{\partial r} \right).$$  \hspace{1cm} (126)

where $Q_3$ is plotted in Fig. 2 for the case $\Gamma = 5/3$. It has the Taylor series (see the Appendix)

$$Q_3 = \frac{1}{2(1 - \kappa^2)} + \frac{(6 + \Gamma)}{4(3 - \Gamma)(1 - \kappa^2)^2} |\psi|^2 + O(|\psi|^3),$$  \hspace{1cm} (127)

where

$$|\psi| = r \frac{\partial \ell}{\partial r}$$  \hspace{1cm} (128)

is the amplitude of the warp. Equivalently,

$$\Sigma^2 \frac{\partial \ell}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( \mathcal{S}^2 \frac{\partial \ell}{\partial r} \right)$$

To make the connection with the linear theory of bending waves, suppose that the disc is initially flat, with $\ell = e_z$, and is then perturbed weakly, with $|\ell_\ell|, |\ell_z| \ll 1$. Then

$$\Sigma^2 \frac{\partial W}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( \mathcal{S}^2 \frac{\partial W}{\partial r} \right)$$

may be compared with equation (12) of Papaloizou & Lin (1994). Taking the low-frequency, non-resonant limit of their equation, setting the precession frequency to zero for a spherical potential, restoring the time derivative and identifying $W \rightarrow \hat{\Omega}^2$, one obtains

$$\Sigma^2 \frac{\partial W}{\partial t} = \frac{\partial W}{\partial r} \left( \frac{\partial W}{\partial r} \right).$$  \hspace{1cm} (131)

The only discrepancy concerns whether a factor $\mathcal{S}^2 \hat{\Omega}^2$ should appear inside or outside the radial derivative. This is of little significance for nearly Keplerian discs in which $\hat{\Omega}^2$ is approximately constant, and may be attributable to a failure of the approximations adopted in the passage from equation (11) to equation (12) of that paper. It is evident that equation (131) is of the correct form to ensure the conservation of angular momentum.

In a short-wavelength (WKB) limit, equation (131) has wave-like solutions of the form

$$W(r, t) = \tilde{W}(r) \exp \left[ -i\omega t + i \int k(r) dr \right].$$  \hspace{1cm} (133)

where $\omega$ is the frequency, $k(r)$ is the radial wavenumber (satisfying $|kr| \gg 1$) and $\tilde{W}(r)$ is a slowly varying function. The non-linear WKB dispersion relation is then

$$\omega = \pm \frac{1}{2} \left( \frac{\Omega^2}{\Omega^2 - \kappa^2} \right) \mathcal{G} \kappa^2 \left[ 1 + \frac{(6 + \Gamma)}{2(3 - \Gamma)} \left( \frac{\Omega^2}{\Omega^2 - \kappa^2} \right)^2 \right] + O(\tilde{W}^4),$$  \hspace{1cm} (134)

the $\pm$ arising since the complex conjugate solution is equally valid but is physically distinct. In the linear approximation, this can be shown to agree with the analytic WKB dispersion relation for isothermal discs found by Lubow & Pringle (1993). To make the connection, note that their $K_i$ is our $kH$, where $H$ is the isothermal scaleheight of the disc, while their $F$ is our $(\omega / \Omega) / \Omega$ for an $n = \pm 1$ mode. Expanding their equation (54) about the point $K_i = 0$, $F = \pm 1$, and assuming a non-Keplerian disc, one obtains for the ‘$n = 0$’ mode

$$\omega = \pm \frac{1}{2} \left( \frac{\Omega^2}{\Omega^2 - \kappa^2} \right) (kH)^2 + O[(kH)^4].$$  \hspace{1cm} (135)

For an isothermal disc $\mathcal{G} = \Sigma H^2$ and the two agree.

The interpretation of the non-linear term in the dispersion relation is that the dispersion of waves either increases or decreases as the amplitude increases, depending on the signs of $(\Omega^2 - \kappa^2)$ and of $(3 - \Gamma)$. The cubic non-linearity arises through a three-mode coupling.
involving (i) the ‘tilt’ mode or f mode \((m = 1, \text{ of odd symmetry})\), consisting locally of a uniform vertical translation of the disc, (ii) an inertial or r mode \((m = 1, \text{ of odd symmetry})\), consisting of a horizontal epicyclic motion proportional to \(\xi\), and (iii) an acoustic or p mode \((m = 2, \text{ of even symmetry})\), consisting of a vertical motion proportional to \(\xi\). The magnitude of the coupling depends on the adiabatic exponent because of the compressive nature of the p mode, which has a natural frequency of \((1 + \Gamma)^{1/2}\Omega\) in the comoving frame. Indeed, in the unlikely case \(\Gamma = 3\), the p mode would be driven at resonance and a large response would result.

Fig. 2 shows that, when \(\Gamma = 5/3\), the non-linear behaviour is very different depending on the sign of \((\Omega^2 - \kappa^2)\). If \(\kappa^2 > \Omega^2\) the disc can support warps of large amplitude and the dispersion coefficient becomes smaller in magnitude as the amplitude increases. However, if \(\kappa^2 < \Omega^2\), the dispersion coefficient increases with increasing amplitude and the disc eventually ruptures.

7.3 Viscous, Keplerian disc

In the viscous, Keplerian case, the three coefficients are plotted in Figs 3, 4 and 5 (with \(\Gamma = 5/3\) and \(\alpha_b = 0\)). They have Taylor series (see the Appendix)

\[
Q_1 = -\frac{3\alpha}{2} + \left[\frac{1 - 17\alpha^2 + 21\alpha^4}{4\alpha(4 + \alpha^2)}\right] \hat{\psi}^2 + O(\hat{\psi}^4),
\]

\[
Q_2 + iQ_3 = \frac{1 + 2i\alpha + 6\alpha^2}{2\alpha(2 + i\alpha)} + \frac{\tilde{a} + \tilde{b}\Gamma + \tilde{c}[\Gamma - 2i(\alpha_b + \frac{1}{2}\alpha)]}{4\alpha[(3 - \Gamma) + 2i(\alpha_b + \frac{1}{2}\alpha)](2i - \alpha)(2i + \alpha)} |\hat{\psi}|^2 + O(\hat{\psi}^4),
\]

where

\[
\tilde{a} = 12 + 115i\alpha + 49\alpha^2 + 109i\alpha^3 - 408\alpha^4 - 41i\alpha^5 + 2\alpha^6 + 4i\alpha_0^7,
\]

\[
\tilde{b} = 18 + 87i\alpha - 87\alpha^2 - 24\alpha^4 - 6i\alpha^5,
\]

\[
\tilde{c} = 12 + 25i\alpha - 36\alpha^2 - 11i\alpha^3 + 140\alpha^4 + 21\alpha^5 + 2\alpha^6.
\]

In the important limit \(\alpha \ll 1, \alpha_b \ll 1\) (but requiring \(\alpha \simeq H/r\) to avoid the resonant case),

\[
Q_1 = -\frac{3\alpha}{2} + \frac{1}{16\alpha} |\hat{\psi}|^2 + O(\hat{\psi}^4),
\]

\[
Q_2 = \frac{1}{4\alpha} + O(\hat{\psi}^2),
\]

\[
Q_3 = \frac{3}{8} + O(\hat{\psi}^2).
\]

This correctly predicts the behaviour seen in Fig. 3, in which the usual viscous torque parallel to \(\ell\) can be reversed by a warp of amplitude \(|\hat{\psi}| \gtrsim \sqrt{2/\alpha}\). This occurs because the usual viscous torque is small (proportional to \(\alpha\)) and is easily overwhelmed by the torque resulting from the correlation of the radial and azimuthal velocities, since these are almost resonantly driven. It also shows that the diffusion of the warp is more important than any dispersive wave propagation in this limit. Unfortunately, the truncated Taylor series for \(Q_2\) is inaccurate even for reasonably small values of \(|\hat{\psi}|\) in this limit, and does not predict that it ultimately decreases with increasing \(|\hat{\psi}|\), as seen in Fig. 4. Therefore a numerical solution is required in this case.

To the extent that the dispersion coefficient \(Q_3\) can be neglected, the original equations of Pringle (1992) are formally valid, but with the following caveats concerning the viscosities \(\nu_1\) and \(\nu_2\). For small-amplitude warps, the approximations

\[
\nu_1 \approx \tilde{p}
\]

and

\[
\nu_2 \approx \frac{2(1 + 7\alpha^2)}{\alpha^2(4 + \alpha^2)} \tilde{p} \approx \frac{\tilde{p}}{2\alpha^2} \quad \text{for} \quad \alpha \ll 1
\]

hold (cf. Papaloizou & Pringle 1983). These are the approximations relevant for deciding whether a flat disc is unstable to the radiation-driven instability (Pringle 1996). The fact that \(\nu_2\) is potentially much larger than \(\nu_1\) means that the radius outside which the instability sets is likely to be much larger than would be estimated on the assumption that \(\nu_2 = \nu_1\). If the instability does proceed, then \(\nu_1\) and \(\nu_2\) are subject to non-linear corrections depending on the amplitude of the warp. In the case investigated in this paper, both \(\nu_1\) and \(\nu_2\) typically decrease with increasing amplitude, with \(\nu_1\) even becoming negative if \(\alpha\) is sufficiently small. Although this means that the usual accretion torque is reversed, the disc so strongly resists being warped in this limit that this situation may not persist for long unless the warp is strongly forced.

7.4 Outlook

This work has shown that, with the exception of discs that are both accurately Keplerian and almost inviscid, the dynamics of a warped accretion disc can be reduced to simple one-dimensional conservation equations for mass and angular momentum even if the warp is non-linear. This generally supports the approach adopted by Pringle (1992), although, for completeness, it requires an additional type of internal
torque to be included. Based on the assumption of an isotropic underlying viscous process, it also predicts the values of the coefficients in the equations and their variation with local parameters of the disc and with the amplitude of the warp.

Some features of the behaviour of the system can be estimated by inspection of the equations and the variation of the coefficients. Non-linear effects are strongest in the case of most interest, that of a Keplerian disc with $H/r \approx \alpha \ll 1$. However, the detailed application of this work to astrophysical situations properly requires a numerical solution of the equations. As shown by Pringle (1992, 1996, 1997), the reduction of the problem to one-dimensional equations which do not require the fast orbital time-scale to be followed offers very significant practical advantages over three-dimensional numerical simulations.

At the same time, there are two important ways in which this analysis should be improved in future. First, a better analytical modelling of the turbulent stress tensor in accretion discs is desirable. It should be possible, using local numerical simulations, to measure the response of magnetohydrodynamic turbulence to imposed motions such as those induced by a warp (Torkelsson et al., in preparation) and to use this to calibrate an analytical model. Secondly, the effect of increased dissipation caused by the warp on the thickness of the disc, and therefore on the torques between neighbouring rings, ought to be taken into account. Unfortunately, to do this properly requires the solution of an energy equation including radiative transport, which is unlikely to be amenable to the technique of separation of variables used in this paper. However, rather than invalidating the form of the equations derived here, these developments would be expected only to provide improved values for the coefficients.

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REFERENCES

Hatchett S. P., Begelman M. C., Sarazin C. L., 1981, ApJ, 247, 677
Kumar S., Coleman C. S., 1993, MNRAS 260, 523
Kumar S., Pringle J. E., 1985, MNRAS, 213, 435
Lubow S. H., Pringle J. E., 1993, ApJ, 409, 360
Maloney P. R., Begelman M. C., Pringle J. E., 1996, ApJ, 472, 582
Miyoshi M., Moran J., Herrnstein J., Greenhill L., Nakai N., Diamond P., Inoue M., 1995, Nat, 373, 127
Ogilvie G. I., 1997, MNRAS, 288, 63
Papaloizou J. C. B., Lin D. N. C., 1994, in Duschl W. J., Frank J., Meyer F., Meyer-Hofmeister E., Tscharnuter W. M., eds, Theory of Accretion Disks – 2. Kluwer, Dordrecht, p. 329
Papaloizou J. C. B., Lin D. N. C., 1995, ApJ, 438, 841
Papaloizou J. C. B., Pringle J. E., 1983, MNRAS, 202, 1181
Papaloizou J. C. B., Terquem C., 1995, MNRAS, 274, 987
Petterson J. A., 1978, ApJ, 226, 253
Pringle J. E., 1992, MNRAS, 258, 811
Pringle J. E., 1996, MNRAS, 281, 357
Pringle J. E., 1997, MNRAS, 292, 136
Shakura N. I., Sunyaev R. A., 1973, A&A, 24, 337

APPENDIX A: TRUNCATED NON-LINEAR EQUATIONS

The aim of this section is to derive truncated Taylor series for the coefficients $Q_i$ and $Q_i = Q_+ + iQ_-$, which will provide an equation for the tilt vector correct to cubic order. To achieve this, introduce the expansions

$$f_1(\phi) = f_0 + |\psi|^2f_{12}(\phi) + O(|\psi|^4),$$

$$f_2(\phi) = f_0 + |\psi|^2f_{23}(\phi) + O(|\psi|^4),$$

$$f_3(\phi) = |\psi|^2f_{31}(\phi) + |\psi|^3f_{32}(\phi) + O(|\psi|^4),$$

$$f_4(\phi) = |\psi|^2f_{42}(\phi) + O(|\psi|^4),$$

$$f_5(\phi) = |\psi|^2f_{53}(\phi) + |\psi|^3f_{51}(\phi) + O(|\psi|^4),$$

$$f_6(\phi) = f_{60} + |\psi|^2f_{62}(\phi) + O(|\psi|^4).$$

Note that each function is either even or odd in $|\psi|$, and that all terms with the scaling $|\psi|^0$ are axisymmetric, because they represent an unwarped disc.

A1 Zeroth-order solution

At zeroth order, the solution is that of an unwarped disc. For the vertical equilibrium, equation (108) at $O(1)$ yields simply

$$f_{20} = 1.$$

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Also
\[ f_{00} = 1, \]  
\[ \text{because } (f_0) = 1 \text{ by definition.} \]

### A2 First-order solution

The horizontal velocities at first order are determined by equation (107) at \( O(\psi) \),

\[ f_{31}(\phi) + \alpha f_{31}(\phi) - 2f_{31}\phi) = \cos \phi - \alpha \sin \phi, \quad \text{(A9)} \]

and equation (109) at \( O(\psi) \),

\[ f_{31}(\phi) + \alpha f_{31}(\phi) + \frac{1}{\kappa^2}f_{31}(\phi) = \frac{1}{\kappa}(4 - \kappa^2)\alpha \cos \phi. \quad \text{(A10)} \]

The solution is of the form

\[ f_{31}(\phi) = C_{r1} \cos \phi + S_{r1} \sin \phi, \quad \text{(A11)} \]

\[ f_{31}(\phi) = C_{\phi1} \cos \phi + S_{\phi1} \sin \phi, \quad \text{(A12)} \]

with

\[
\begin{bmatrix}
\alpha & 1 & -2 & 0 \\
-1 & \alpha & 0 & -2 \\
\frac{1}{\kappa^2} & 0 & \alpha & 1 \\
0 & \frac{1}{\kappa^2} & -1 & \alpha
\end{bmatrix}
\begin{bmatrix}
C_{r1} \\
S_{r1} \\
C_{\phi1} \\
S_{\phi1}
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
-\alpha \\
\frac{1}{\kappa}(4 - \kappa^2)\alpha
\end{bmatrix}.
\]

This may be expressed more compactly using a complex notation, with (generically)

\[ Z = C + iS, \]

so that

\[
\begin{bmatrix}
\alpha - i & -2 \\
\frac{1}{\kappa^2} & \alpha - i
\end{bmatrix}
\begin{bmatrix}
Z_{r1} \\
Z_{\phi1}
\end{bmatrix}
= 
\begin{bmatrix}
1 - i\alpha \\
\frac{1}{\kappa}(4 - \kappa^2)\alpha
\end{bmatrix}.
\]

The determinant of this matrix is

\[ -(1 - \kappa^2) - 2i\alpha + \alpha^2, \]

and so a solution exists unless the disc is both Keplerian \((\kappa^2 = 1)\) and inviscid \((\alpha = 0)\) to the accuracy of these scalings. (As anticipated, the resonant case cannot be described using the present expansions.) The solution follows by inversion of the matrix. In detail,

\[ Z_{r1} = \frac{i - (4 - \kappa^2)\alpha + i\alpha^2}{(1 - \kappa^2) - 2i\alpha - \alpha^2}, \quad \text{(A17)} \]

\[ Z_{\phi1} = \frac{1}{2} \left[ \frac{\kappa^2 + 2i(2 - \kappa^2)\alpha - (4 - \kappa^2)\alpha^2}{(1 - \kappa^2) - 2i\alpha - \alpha^2} \right]. \quad \text{(A18)} \]

### A3 Second-order solution

The enthalpy and vertical velocity at second order are determined by equation (105) at \( O(\psi^2) \),

\[ f_{12}(\phi) = (\Gamma - 1)f_{42}(\phi)f_{10}, \quad \text{(A19)} \]

equation (106) at \( O(\psi^2) \),

\[ f_{22}(\phi) = (\Gamma + 1)f_{42}(\phi), \quad \text{(A20)} \]

and equation (108) at \( O(\psi^2) \),

\[ f_{42}(\phi) + (\alpha_0 + \frac{1}{\kappa^2})f_{42}(\phi) = -f_{32}(\phi) \cos \phi + (2 \sin \phi - \alpha \cos \phi)f_{31}(\phi) - f_{22}(\phi) + \alpha \cos \phi \sin \phi. \quad \text{(A21)} \]

These may be combined to give

\[ f_{42}(\phi) + (\alpha_0 + \frac{1}{\kappa^2})f_{42}(\phi) + (\Gamma + 1)f_{42}(\phi) = (3C_{r1} - \alpha S_{r1} + \alpha) \cos 2\phi + (3S_{r1} + \alpha C_{r1}) \sin 2\phi. \quad \text{(A22)} \]

The solution is of the form

\[ f_{42}(\phi) = C_{r2} \cos 2\phi + S_{r2} \sin 2\phi, \quad \text{(A23)} \]

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with
\[
\begin{bmatrix}
-(3 - \Gamma) & 2(\alpha_\nu + \frac{2}{3}\alpha) \\
-2(\alpha_\nu + \frac{2}{3}\alpha) & -(3 - \Gamma)
\end{bmatrix}
\begin{bmatrix}
C_{s2} \\
S_{s2}
\end{bmatrix}
= \begin{bmatrix}
3C_{s1} - \alpha S_{s1} + \alpha \\
3S_{s1} + \alpha C_{s1}
\end{bmatrix}.
\] (A24)

In a complex notation, the solution is
\[
Z_{s2} = \frac{-(3 + i\alpha) Z_{s1} - \alpha}{(3 - \Gamma) + 2i(\alpha_\nu + \frac{2}{3}\alpha)}
= \frac{-3i + 2(6 - \kappa^2)\alpha - (1 + \kappa^2)i\alpha^2 + 2\alpha^3}{(3 - \Gamma) + 2i(\alpha_\nu + \frac{2}{3}\alpha)} \left(1 - \kappa^2 + 2i\alpha - \alpha^2\right)
\] (A25)

It then follows that
\[
f_{s2}(\phi) = -\frac{1}{2}(\Gamma - 1)f_{s0}(\cos 2\phi - C_{s2} \sin 2\phi),
\] (A26)
\[
f_{s3}(\phi) = -\frac{1}{2}(\Gamma + 1)(S_{s2} \cos 2\phi - C_{s2} \sin 2\phi) + \frac{1}{2}S_{s1} - \frac{1}{2}\alpha C_{s1},
\] (A27)
\[
f_{s2}(\phi) = S_{s2} \cos 2\phi - C_{s2} \sin 2\phi.
\] (A28)

A4 Third-order solution

The horizontal velocities at third order are determined by equation (107) at \(O(|\psi|^3)\).
\[
f_{s31}(\phi) + \alpha f_{s31}(\phi) - 2f_{s3}(\phi) = f_{s2}(\phi)f_{s31}(\phi) + \left( \alpha_\nu + \frac{1}{3}\alpha \right)f_{s2}(\phi) \cos \phi - \alpha [f_{s2}(\phi) + \cos^2 \phi]f_{s31}(\phi) - \alpha \sin \phi f_{s2}(\phi),
\] (A29)

and equation (109) at \(O(|\psi|^3)\).
\[
f_{s3}(\phi) + \alpha f_{s3}(\phi) + \frac{1}{2}\alpha^2 f_{s3}(\phi) = f_{s2}(\phi)f_{s3}(\phi) - \alpha [f_{s2}(\phi) + \cos^2 \phi]f_{s3}(\phi) + \frac{1}{2}(4 - \kappa^2)\alpha \cos \phi f_{s3}(\phi).
\] (A30)

The solution is of the form
\[
f_{s31}(\phi) = C_{s3} \cos \phi + S_{s3} \sin \phi + \{m=3 \text{ terms}\},
\] (A31)
\[
f_{s3}(\phi) = C_{s3} \cos \phi + S_{s3} \sin \phi + \{m=3 \text{ terms}\},
\] (A32)

with
\[
\begin{bmatrix}
\alpha - i & -2 \\
\frac{1}{2}\alpha & \alpha - i
\end{bmatrix}
\begin{bmatrix}
Z_{s1} \\
Z_{s3}
\end{bmatrix}
= \begin{bmatrix}
R_1 \\
R_2
\end{bmatrix},
\] (A33)

where
\[
R_1 = \left[ \frac{1}{2} - \frac{1}{4}(\Gamma + 1)\alpha \right]Z_{s0}Z_{s1}^* + \left[ \frac{1}{4}(\Gamma + 1)(\Gamma - \alpha) + \frac{1}{2}\left( \alpha_\nu + \frac{1}{3}\alpha \right) \right]Z_{s0} - \frac{3}{4}\alpha Z_{s1} - \frac{1}{2}(S_{s1} - \alpha C_{s1}) \left[ \alpha Z_{s1} - (1 - i\alpha) \right] + \frac{1}{2}i\alpha S_{s1},
\] (A34)
\[
R_2 = \left[ \frac{1}{2} - \frac{1}{4}(\Gamma + 1)\alpha \right]Z_{s0}Z_{s3}^* + \frac{1}{8}(4 - \kappa^2)(\Gamma + 1)i\alpha Z_{s0} - \frac{3}{4}\alpha Z_{s1} - \frac{1}{2}\alpha (S_{s1} - \alpha C_{s1}) \left[ Z_{s1} - \frac{1}{2}(4 - \kappa^2) \right] + \frac{1}{2}i\alpha S_{s3}.
\] (A35)

The \(m = 3\) terms will not be required.

A5 Evaluation of the coefficients

The coefficients \(Q_1\) and \(Q_4\) are even in \(|\psi|\) and have expansions
\[
Q_1 = Q_{10} + |\psi|^2 Q_{12} + O(|\psi|^4),
\] (A36)
\[
Q_4 = Q_{40} + |\psi|^2 Q_{42} + O(|\psi|^4).
\] (A37)

At leading order,
\[
Q_{10} = -\frac{1}{2}(4 - \kappa^2)\alpha
\] (A38)
and
\[ Q_{40} = \langle e^{i\phi} [(1 - i\alpha)f_{31}(\phi) - i\alpha \sin \phi] \rangle \]
\[ = \frac{1}{2} \frac{Z_r}{(1 - \kappa^2)} \frac{1}{2}(1 - i\alpha)Z_r \]
\[ = \frac{1}{2} \left[ \frac{1}{2} - \frac{1}{2} \left( \frac{7 - \kappa^2}{(1 - \kappa^2)^2 + 2\alpha^2} \right) \right]. \quad (A39) \]

At second order,
\[ Q_{12} = \frac{1}{4}(4 - \kappa^2)\alpha(S_r - \alpha C_r) - \frac{1}{2}(C_r C_{\phi 1} + S_r S_{\phi 1}) + \frac{1}{2} \alpha C_{\phi 1} \]
\[ = \frac{1}{4} \left[ -(8 - \kappa^2)(28 - 12\kappa^2 + 2\kappa^2)\alpha^2 \right. \]
\[ \left. + (8 - \kappa^2)(4 - \kappa^2)\alpha^4 \right] \quad (A40) \]
and
\[ Q_{42} = \langle e^{i\phi} \left\{ (1 - i\alpha)f_{31}(\phi) - i[f_{31}(\phi) - \alpha \cos \phi] \left[ f_{42}(\phi) + f_{31}(\phi) \cos \phi \right] \right. \]
\[ \left. \right. \left. - i\alpha f_{22}(\phi) \right\} \left\{ (1 - i\alpha)f_{31}(\phi) - i\alpha \sin \phi \right\} \rangle \]
\[ = \frac{1}{2} \left( (1 - i\alpha)Z_r \right. \frac{1}{2} \left. \right. \left[ \frac{1}{2} \left[ \frac{1}{2} + \frac{1}{4}(3 - \Gamma) \right] \right. \]
\[ \left. \right. \left. \left[ \frac{1}{4}(3 - \Gamma) \right] \frac{1}{8}(3 - \Gamma)\alpha Z_{\phi 2} - \frac{1}{4}i\alpha Z_{\phi 1} - \frac{1}{8}i\alpha Z_{\phi 2} - \frac{3}{8}i\alpha Z_{\phi 1} \right] \right] \]
\[ = \frac{a + b\Gamma + c \left[ \Gamma - 2i(\alpha_0 + \frac{1}{4}\alpha) \right]}{4[(3 - \Gamma) + 2i(\alpha_0 + \frac{1}{4}\alpha)] [(1 - \kappa^2) + 2(1 - \alpha^2)]^3 [1 - 2(1 - \alpha^2)} \right], \quad (A41) \]

where
\[ a = 6i(1 - \kappa^2)^2 - 3(1 - \kappa^2)(8 - 7\kappa^2 + 2\kappa^4)\alpha + 3(1 - \kappa^2)(28 - 32\kappa^2 + 2\kappa^4 + \kappa^6)\alpha^2 \]
\[ - (258 - 337\kappa^2 + 103\kappa^4 - 15\kappa^6 + 3\kappa^8)\alpha^3 + (224 - 750\kappa^2 + 111\kappa^4 - 106\kappa^6 + 6\kappa^8)\alpha^4 \]
\[ - (442 - 511\kappa^2 + 116\kappa^4 + 2\kappa^6)\alpha^5 - (552 - 560\kappa^2 + 124\kappa^4 - 7\kappa^6)\alpha^6 + (562 - 153\kappa^2 - \kappa^4)\alpha^7 \]
\[ + (66 - 26\kappa^2 + \kappa^4)\alpha^8 + 2(1 - \kappa^2)\alpha^9 - 4i\alpha^{10}, \quad (A42) \]
\[ b = -4(\kappa^3)\alpha \left[ (1 - \kappa^2) + 2i\alpha - \alpha^2 \right] \left[ (1 - \kappa^2) - (2 - \kappa^2)\alpha + \alpha^2 \right] \left[ 3 + 2(6 - \kappa^2)\alpha + (1 + \kappa^2)\alpha^2 + 2i\alpha^3 \right], \quad (A43) \]
\[ c = i(1 - \kappa^2)^2 - 2(1 - \kappa^2)(1 - \kappa^2)\alpha - (1 - \kappa^2)(2 + 9\kappa^2 - 3\kappa^4)\alpha^2 - 2(25 - 40\kappa^2 + 2\kappa^4 - 3\kappa^6)\alpha^3 \]
\[ - (262 - 428\kappa^2 + 233\kappa^4 - 44\kappa^6 + 2\kappa^8)\alpha^4 + 2(115 - 19\kappa^2 + 23\kappa^4 - \kappa^6)\alpha^5 \]
\[ + (164 - 196\kappa^2 + 45\kappa^4 - 2\kappa^6)\alpha^6 - 2(87 - 18\kappa^2 + \kappa^4)\alpha^7 - 3(9 - 2\kappa^2)\alpha^8 - 2\alpha^9. \quad (A44) \]

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