On One Aspect of Constrained Bayesian Method for Testing Directional Hypotheses

KJ Kachiashvili*

Georgian Technical University, Faculty of Informatics and Control Systems, Georgia

Received: February 23, 2018; Published: March 06, 2018

*Corresponding author: KJ Kachiashvili, Georgian Technical University, Faculty of Informatics and Control Systems, st. Kostava, Tbilisi, Georgia, Tel: 99532 223 7247. Email: k.kachiashvili@gtu.edu.ge

Abstract

The paper discusses the application of constrained Bayesian method (CBM) of testing the directional hypotheses. It is proved that decision rule of CBM restricts the mixed directional false discovery rate (mdFDR) and total Type III error rate as well.

Keywords: CBM; Hypotheses Testing; Mixed Directional False Discovery Rate; Type III Error Rate

Introduction

Directional hypotheses testing problem arises in many biomedical applications [1]. For parametrical models, the problem of testing directional hypotheses can be stated as $H_0: \theta = \theta_0$ vs. $H_1: \theta \neq \theta_0$, where $\theta$ is the parameter of the model, $\theta_0$ is known. A review of the works where the methods of testing the directional hypotheses are given can be found, for example, in [2,3]. Bayesian decision theoretical methodology for testing the directional hypotheses was developed and compared with the frequentist method in [2]. While a new approach to the statistical hypotheses testing, called Constrained Bayesian Methods (CBM), was developed and applied alongside to other types of hypotheses to the directional hypotheses in [3]. One specific aspect of CBM for testing directional hypotheses, in particular, the fact that it restricts both the mixed directional false discovery rate (mdFDR) and total Type III error rate is proved below.

Constrained Bayesian Method and Error Rates

Let’s denote: $r_0$, $r_+, r_-$ and $r_{++}$ are the regions of acceptance of the above-stated hypotheses $H_0$, $H_+$ and $H_-$ respectively, and let us consider the following losses:

$$L_i(H_0, H_i) = \begin{cases} 0 & \text{at } i = j, \\ L_i(H_0, H_i) & \text{at } i \neq j, \\ L_i(H_0, H_i) & \text{at } i = j. \end{cases}$$

Where $L_i(H_0, H_i)$ and $L_i(H_+, H_i)$ are the losses of incorrectly accepted and incorrectly rejected hypotheses. It is clear that the “0–1” loss function is a private case of the step-wise loss (1). For loss functions (1), one of possible statement of CBM takes the form [3-5].

$$r_i = \min_{p \in L_i(H_0, H_i)} \left\{ p(H) \left[ L_i(H_0, H_i) \int p(x|H_0) dx + L_i(H_0, H_i) \int p(x|H_1) dx \right] + \right.$$  
$$\left. + p(H_0) \left\{ L_i(H_0, H_i) \int p(x|H_0) dx + L_i(H_1, H_i) \int p(x|H_1) dx \right\} + \right.$$  
$$\left. + p(H_1) \left\{ L_i(H_0, H_i) \int p(x|H_0) dx + L_i(H_1, H_i) \int p(x|H_1) dx \right\} \right\}.$$  

Subject to the averaged loss of incorrectly rejected hypotheses

$$p(H \cdot L_i(H_0, H_i) \int p(x|H_0) dx \cdot L_i(H_0, H_i) \int p(x|H_1) dx + p(H_0) \left\{ L_i(H_0, H_i) \int p(x|H_0) dx \cdot L_i(H_1, H_i) \int p(x|H_1) dx \right\} +$$  
$$\geq \left\{ p(H_0) L_i(H_0, H_i) \int p(x|H_0) dx \cdot L_i(H_0, H_i) \int p(x|H_1) dx + p(H_0) L_i(H_0, H_i) \int p(x|H_0) dx \right\} \cdot\eta_i,$$  

Where $p(H_0)$ is the a priori probability of hypothesis $H_0$, $p(x|H)$ denotes the marginal density of $x$ given $H$, i.e. $p(x|H) = \int p(x|\theta)p(\theta|H) d\theta$ is a priori density with support $\Theta_0$, $i \in \{-, 0, +\}$ and $\eta_i$ is some real number determining the level of the averaged loss of incorrectly rejected hypothesis.

$$L_i(H_0, H_i) = \begin{cases} 0 & \text{at } i = j, \\ K_{i-} & \text{at } i \neq j, \\ K_{i+} & \text{at } i = j. \end{cases}$$

In this case, stated problem (2), (3) takes the following form

$$r_i = \min_{p \in L_i(H_0, H_i)} \left\{ p(H) K_i \left\{ L_i(H_0, H_i) \int p(x|H_0) dx \cdot L_i(H_0, H_i) \int p(x|H_1) dx \right\} + \right.$$  
$$\left. + p(H_0) K_i \left\{ L_i(H_0, H_i) \int p(x|H_0) dx \cdot L_i(H_1, H_i) \int p(x|H_1) dx \right\} + \right.$$  
$$\left. + p(H_1) K_i \left\{ L_i(H_0, H_i) \int p(x|H_0) dx \cdot L_i(H_1, H_i) \int p(x|H_1) dx \right\} \right\}.$$  

Subject to

$$p(H \cdot L_i(H_0, H_i) \int p(x|H_0) dx + p(H_0) L_i(H_0, H_i) \int p(x|H_0) dx + p(H_1) L_i(H_0, H_i) \int p(x|H_0) dx \geq \frac{\eta_i}{K_i}.$$  

By solving constrained optimization problem (5), (6), using the Lagrange multiplier method, we obtain...
\[ \Gamma = \{ x : K \cdot \{ p(H_0)p(x | H_0) + p(H_1)p(x | H_1) \} < \lambda \cdot K_0 \cdot p(H_0)p(x | H_0) \}, \]
\[ \Gamma = \{ x : K \cdot \{ p(H_0)p(x | H_0) + p(H_1)p(x | H_1) \} < \lambda \cdot K_0 \cdot p(H_0)p(x | H_0) \}, \quad (7) \]
\[ \Gamma = \{ x : K \cdot \{ p(H_0)p(x | H_0) + p(H_1)p(x | H_1) \} < \lambda \cdot K_0 \cdot p(H_0)p(x | H_0) \}. \]

Where Lagrange multiplier \( \lambda \) is determined so that in (6) equality takes place. For optimality of testing directional hypotheses, different concepts such as: mixed directional false discovery rate (mdFDR), directional false discovery rate (DFDR) and the Type III errors are offered in [1,6-10]. Let us consider mdFDR which is the expected proportion of falsely selecting \( H \) or \( H \). In our case, it has the following form

\[
\text{mdFDR} = \mathbb{P}(x \in \Gamma_+ | H_0) + \mathbb{P}(x \in \Gamma_- | H_0) + \mathbb{P}(x \in \Gamma_+ | H_0) + \mathbb{P}(x \in \Gamma_- | H_0) = \int \int p(x | H_0) dx + \int \int p(x | H_0) dx + \int \int p(x | H_0) dx + \int \int p(x | H_0) dx. \quad (8)
\]

According to [6,9], The Type III error rate is

\[
\text{Type – III error rate} = \mathbb{P}(x \in \Gamma_- | H_0) + \mathbb{P}(x \in \Gamma_+ | H_0). \quad (9)
\]

But in [10], It is defined as

\[
\text{Type – III error rate} = \mathbb{P}(x \in \Gamma_- | H_0) + \mathbb{P}(x \in \Gamma_+ | H_0). \quad (10)
\]

Let us denote the Type III error rate (9) as \( \text{ERR}_{m} \) and Type III error rate (4) as \( \text{ERR}^r_{m, I} \).

From (8),(9) and (10), it follows that

\[
\text{mdFDR} = \text{ERR}^r_{m, I} + \text{ERR}^r_{m}. \quad (11)
\]

**Theorem**

When satisfying the condition \( \frac{1}{1 - K_0} \leq \alpha \), where \( \alpha = \min \{p(H_0) | H_0 \}, \)\( \min \{p(H_1) | H_1 \} \), CBM with restriction level of (6) ensures a decision rule with mixed directional false discovery rate or with total Type III error rate less or equal to \( \alpha \), i.e. with \( \text{mdFDR} = \text{ERR}^r_{m, I} + \text{ERR}^r_{m} \leq \alpha \).

**Proof**: Because of specificity of decision rules of CBM, alongside of hypotheses acceptance regions, the regions of impossibility of making a decision exist. Therefore, instead of the condition

\[
\int \int p(x | H_0) dx + \int \int p(x | H_0) dx + \int \int p(x | H_0) dx + \int \int p(x | H_0) dx = 1, \quad \{ x \in (-H_0, +) \}
\]

Of classical decision rules, the following condition is fulfilled in CBM

\[
\int \int p(x | H_0) dx + \int \int p(x | H_0) dx + \int \int p(x | H_0) dx + \int \int p(x | H_0) dx = 1, \quad \{ x \in (-H_0, +) \}
\]

Where \( P(\text{imd} | H) \) is the probability of impossibility of making a decision [3,11,12].

Taking into account (12), condition (6) can be rewritten as follows

\[
p(H_0) \int \int p(x | H_0) dx + p(H_1) \int \int p(x | H_0) dx + p(H_0) \int \int p(x | H_0) dx = p(H_0) \int \int p(x | H_0) dx - \int \int p(x | H_0) dx - P(\text{imd} | H) =
\]

\[
= p(H_0) \left[ \int \int p(x | H_0) dx - \int \int p(x | H_0) dx - p(H_0) \int \int p(x | H_0) dx - P(\text{imd} | H) \right] +
\]

\[
+ p(H_1) \left[ \int \int p(x | H_1) dx - \int \int p(x | H_1) dx - P(\text{imd} | H_1) \right] +
\]

\[
+ p(H_0) \left[ \int \int p(x | H_0) dx - \int \int p(x | H_0) dx - P(\text{imd} | H_0) \right] =
\]

\[
= p(H_0) \left[ \int \int p(x | H_0) dx - \int \int p(x | H_0) dx - p(\text{imd} | H_0) \right] -
\]

\[
- p(H_1) \left[ \int \int p(x | H_1) dx - \int \int p(x | H_1) dx + p(\text{imd} | H_1) \right] -
\]

\[
- p(H_0) \left[ \int \int p(x | H_0) dx - \int \int p(x | H_0) dx + p(\text{imd} | H_0) \right] -
\]

\[
\text{mdFDR} = \min \{p(H_0), p(H_1), p(H_0)\}. \quad (13)
\]

Let us denote \( p_{\text{min}} = \min \{p(H_0), p(H_1), p(H_0)\} \).

Then, from (13), we have

\[
\int \int p(x | H_0) dx + \int \int p(x | H_0) dx + \int \int p(\text{imd} | H_0) +
\]

\[
\int \int p(x | H_0) dx + \int \int p(x | H_0) dx + \int \int p(\text{imd} | H_0) +\]

\[
\int \int p(x | H_0) dx + \int \int p(x | H_0) dx + \int \int p(\text{imd} | H_0) \leq \frac{1}{P_{\text{min}}} \frac{r}{k_0}. \quad (14)
\]

Taking into account (8), we write

\[
\text{mdFDR} + \int \int p(x | H_0) dx + \int \int p(x | H_0) dx + \int \int p(\text{imd} | H_0) +
\]

\[
+ P(\text{imd} | H_0) + P(\text{imd} | H_0) \leq \frac{1}{P_{\text{min}}} \frac{r}{k_0}.
\]

This proves the statement of the theorem.

**Conclusion**

One more property of the optimality of CBM when testing the directional hypotheses is shown. In particular, when testing the directional hypotheses, the optimal decision rule of CBM restricts both the mixed directional false discovery rate (mdFDR) and total Type III error rate.

**References**

1. Bansal NK, Hamedani GG, Maadoolat M (2015) Testing Multiple Hypotheses with Skewed Alternatives. Biometrics 72(2): 494-502.
2. Bansal NK, Sheng R (2010) Bayesian Decision Theoretic Approach to Hypothesis Problems with Skewed Alternatives. Journal of Statistical Planning and Inference 140: 2894-2903.
3. Kachiashvili KJ (2018) Constrained Bayesian Methods of Hypotheses Testing: A New Philosophy of Hypotheses Testing in Parallel and Sequential Experiments. Nova Science Publishers Inc p. 456.
4. Kachiashvili KJ, Hashmi MA, Mueed A (2012) Sensitivity Analysis of Classical and Conditional Bayesian Problems of Many Hypotheses Testing. Communications in Statistics-Theory and Methods 41(4): 591-605.
5. Kachiashvili KJ (2011) Investigation and Computation of Unconditional and Conditional Bayesian Problems of Hypothesis Testing. J of Statistical Planning and Inference 141(4): 591-605.
6. Shaffer JP (2002) Multiplicity, directional (Type III) errors, and the null hypothesis. Psychological Methods 7(3): 356-369.
7. Benjamin Y, Yekutieli D (2005) False discovery rate: Adjusted multiple confidence intervals for selecting parameters. Journal of the American Statistical Association 100: 71-93.
8. Bansal NK, Mueed A (2013) A Bayesian decision theoretic approach to directional multiple hypotheses problems. J of Multivariate Analysis 120: 205-215.
9. Jones LJ, Tukey JW (2000) A sensible formulation of the significance test. Psychological Methods 5(4): 411-414.
10. Kaiser HF (1960) Directional statistical decisions. Psychological Review 67: 160-167.
11. Kachiashvili GK, Kachiashvili KJ, Mueed A (2012) Specific Features of Regions of Acceptance of Hypotheses in Conditional Bayesian Problems
1. Kachiashvili KJ (2011) Conditional Bayesian Task of Testing Many Hypotheses. Statistics 47(2): 274-293.

2. Kachiashvili KJ, Mueed A (2011) Conditional Bayesian Task of Testing Many Hypotheses. Statistics 47(2): 274-293.

Submission Link: https://biomedres.us/submit-manuscript.php

Assets of Publishing with us
- Global archiving of articles
- Immediate, unrestricted online access
- Rigorous Peer Review Process
- Authors Retain Copyrights
- Unique DOI for all articles

http://biomedres.us/

This work is licensed under Creative Commons Attribution 4.0 License