A Heuristic Proof Procedure for Propositional Logic

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Theorem proving is one of the oldest applications which require heuristics to prune the search space. Invertible proof procedures has been the major tool. In this paper, we present a novel and powerful heuristic called nongshim which can be seen as an underlying principle of invertible proof procedures. Using this heuristic, we derive an invertible sequent calculus [4, 6] from sequent calculus for propositional logic.

1 Introduction

Theorem proving is one of the oldest applications which require heuristics to prune the search space. One key observation about proof search is that some rules are invertible, that is, the premises are derivable whenever the conclusion is derivable. We can apply invertible rules whenever possible without losing completeness. The proof search strategy that first applies all invertible rules is called inversion. An inversion calculus has been developed for classical, intuitionistic and linear logic. However, it has the following drawbacks:

- Whenever a new logic comes along, its corresponding inversion calculus must be separately developed and proved to be correct. This is a time consuming and redundant process.
- Inversion calculus processes connectives eagerly, often processing them even when it is not necessary.

To overcome these, we propose a general heuristic called nongshim which can be universally applied to a wide class of logics, often leading to inversion calculus.

2 A Universal Heuristic for Theorem Proving

Japaridze [1, 2, 3] have used an interesting heuristic in developing his proof theory, which is closely related to the Nongshim Cup game. The Nongshim Cup is a team competition for the two Go-playing countries such as China, Korea and Japan. It has an interesting tournament format: Two countries each starts with some number (typically five each) of players. The winner of each subsequent game stays in to play the next opponent from another country. In the end the team with remaining players is the winner of the tournament.

Let us assume Korea with m korean players \{k₁, . . . , kₘ\} competes against China with n chinese players \{c₁, . . . , cₙ\}. This is written as nongshim(Go, \{k₁, . . . , kₘ\}, \{c₁, . . . , cₙ\}). The following algorithm will clarify the Cup format.

step1: set current game to nongshim(Go, \{k₁\}, \{c₁\}) % initialization

step2: if Korea wins the current game of the form nongshim(Go, \{k₁, . . . , kᵢ\}, \{c₁, . . . , cᵢ\}), then update current game to nongshim(Go, \{k₁, . . . , kᵢ\}, \{c₁, . . . , cᵢ, cᵢ₊₁\})

else update current game to nongshim(Go, \{k₁, . . . , kᵢ, kᵢ₊₁\}, \{c₁, . . . , cᵢ\})

step3: repeat step 2 until one team has no remaining player.

In this way, the decision of winning/losing can be achieved at the earliest possible time.

We are interested in applying the above idea to theorem proving. One instance of this heuristic – nongshim(literalization(F),M,O) – has been used to proof in first-order logic, where F is a first-order logic
formula, \( M \) is a set of top-level occurrences of \( \exists \)-formulas and \( O \) is a set of top-level occurrences of \( \forall \)-formulas. The process of literalization(\( F \)) transforms \( F \) to its skeleton by removing \( M \) and \( O \) in it, as we shall see below. The resulting proof procedure[4] turns out to be quite effective and performs better than the traditional inversion calculus.

In this paper, we apply this heuristic approach to propositional logic. That is, we apply

\[
\text{nongshim(literalization}(F),M,O)\]nongshim(literalization(F),M,O)

where

- \( F \) is a propositional formula,
- \( M \) is a set of top-level occurrences of \( \lor \)-formulas and
- \( O \) is a set of top-level occurrences of \( \land \)-formulas.

That is, the resulting procedure, which we call \( \text{LKg}^0 \), is a game-viewed proof which captures game-playing nature in proof search. It views

1. sequents as games between the machine and the environment,
2. proofs as a winning strategy of the machine, and
3. \( \land \) as the env’s resource and \( \lor \) as the machine’s resource.

For propositional logic, it turns out that this heuristic does not derive a new calculus. Instead, it is able to derive an existing invertible sequent calculus from sequent calculus. This is meaningful, as it provides a deeper insight into why all the rules are invertible in propositional logic[3].

3 The logic \( \text{LKg}^0 \)

The formulas are the standard classical propositional formulas, with the features that (a) \( \top, \bot \) are added, and (b) \( \neg \) is only allowed to be applied to atomic formulas. Thus we assume that formulas are in negation normal form.

The deductive system \( \text{LKg}^0 \) below axiomatizes the set of valid propositional formulas. \( \text{LKg}^0 \) is a one-sided sequent calculus system, where a sequent is a multiset of formulas. Our presentation follows the one in [1].

First, we need to define some terminology.

1. A surface occurrence of a subformula is an occurrence that is not in the scope of any connectives (\( \land \) and/or \( \lor \)).
2. A sequent is literal iff all of its formulas are so.
3. The literalization \( \|F\| \) of a formula \( F \) is the result of replacing in \( F \) every surface occurrence of \( \lor \)-subformulas by \( \bot \), and every surface occurrence of \( \land \)-subformulas by \( \top \).
   
   The literalization \( \|F_1, \ldots, F_n\| \) of a sequent \( F_1, \ldots, F_n \) is the propositional formula \( \|F_1\| \lor \cdots \lor \|F_n\| \).
4. A sequent is said to be stable iff its literalization is classically valid; otherwise it is unstable.

THE RULES OF \( \text{LKg}^0 \)

\( \text{LKg}^0 \) has the four rules listed below where \( \Gamma \) is a multiset of formulas and \( F \) is a formula.

The deductive system \( \text{LKg}^0 \) is shown below. Below, \( X:\text{stable} \) means that \( X \) must be stable. Similarly for \( X: \text{unstable} \). The Fail rule reads: an unstable sequent \( X \) containing no surface occurrences of \( H_0 \lor H_1 \) is not derivable.
A \textbf{L Kg}⁰-proof of a sequent $X$ is a sequence $X_1, \ldots, X_n$ of sequents, with $X_n = X$, $X_1 = \top$ such that, each $X_i$ follows by one of the rules of \textbf{L Kg}⁰ from $X_{i-1}$.

Below we describe some examples.

**Example 3.1** The formula $p(a) \land p(b), \neg p(a) \lor \neg p(b)$ is provable in \textbf{L Kg}⁰ as follows:

1. $p(b), \neg p(a), \neg p(b)$ \hspace{1em} \textit{Succ}
2. $p(a), \neg p(a), \neg p(b)$ \hspace{1em} \textit{Succ}
3. $p(b), \neg p(a) \lor \neg p(b)$ \hspace{1em} \lor \text{ from 1}
4. $p(a), \neg p(a) \lor \neg p(b)$ \hspace{1em} \lor \text{ from 2}
5. $p(a) \land p(b), \neg p(a) \lor \neg p(b)$ \hspace{1em} \land \text{ from 3, 4}



4 \hspace{1em} \textbf{The soundness and completeness of L Kg}⁰

We now present the soundness and completeness of \textbf{L Kg}⁰.

**Theorem 4.1** 1. If \textbf{L Kg}⁰ terminates with success for $X$, then $X$ is valid.

2. If \textbf{L Kg}⁰ terminates with failure for $X$, then $X$ is invalid.

**Proof.** Consider an arbitrary sequent $X$.

\textit{Soundness}: Induction on the length of derivations.

\textit{Case 1:} $X$ is derived from $Y$ and $Z$ by \land-rule. By the induction hypothesis, both $Y$ and $Z$ are valid, which implies that $X$ is valid.
Case 2: $X$ is derived from $Y$ by $\lor$-rule. By the induction hypothesis, $Y$ is valid, which implies that $X$ is valid.

Case 3: $X$ is derived from $Y$ by Succ.

In this case, we know that there is no surface occurrence of $F \land G$ in $X$ and $\|X\|$ is classically valid. It is then easy to see that, reversing the literalization of $\|X\|$ (replacing $\bot$ by any formula of the form $F \lor G$) preserves validity. For example, if $X$ is $p \rightarrow q, q \lor r$, then $\|X\|$ is valid and $X$ is valid as well.

Completeness: Assume LKg$^0$ terminates with failure.

We proceed by induction on the length of derivations.

If $X$ is stable, then there should be a LKg$^0$-unprovable sequent $Y$ with the following condition.

Case 1: $\land$: $X$ has the form $\Gamma, F \land G$, and $Y$ is either $\Gamma, F$ or $\Gamma, G$. Suppose $\Gamma, F$ is LKg$^0$-unprovable. By the induction hypothesis, $\Gamma, F$ is invalid and $X$ is invalid as well. Similarly for $\Gamma, G$.

Next, we consider the cases when $X$ is not stable. Then there are two cases to consider.

Case 2.1: $\lor$: In this case, there is no surface occurrence of $F \lor G$ and the algorithm terminates with failure. As $X$ is not stable, $\|X\|$ is not classically valid. If we reverse the propositionalization of $\|X\|$ by replacing $\top$ by any formula with some surface occurrence of $F \land G$, we observe that invalidity is preserved. Therefore, $X$ is not valid.

Case 2.2: $\lor$: In this case, $X$ has the form $\Gamma, F \lor G$ and $Y$ is $\Gamma, F, G$. In this case, $Y$ is a LKg$^0$-unprovable sequent. By the induction hypothesis, $Y$ is invalid. Therefore $X$ is not valid.

A simplified LKg$^0$

LKg$^0$ in the previous section can be simplified by observing the following:

A sequent $X$ is stable iff $X$ is either $\Gamma, \top$ or $\Gamma, p, \neg p$ or $\Gamma, F \land G$.

This observation leads to a simplified version of LKg$^0$, as shown below:

Procedure $pv(X)$:

if $X$ is $\Gamma, p, \neg p$ then return Yes.

elsif $X$ is $\Gamma, F \land G$ then return $pv(\Gamma, F)$ and $pv(\Gamma, G)$.

elsif $X$ is $\Gamma, F \lor G$ then return $pv(\Gamma, F, G)$.

otherwise return No.

Of course, we can speed up the above procedure by processing $\lor, \land$ in parallel.

Procedure $pv(X)$:

if $X$ is $\Gamma, p, \neg p$ then return Yes.

elsif $X$ is $\Gamma, F_1 \land G_1, \ldots, F_n \land G_n$ then return

$pv(\Gamma, F_1, \ldots, F_n)$ and ... and $pv(\Gamma, G_1, \ldots, G_n)$.

% total $2^n$ combinations above.

elsif $X$ is $\Gamma, F_1 \lor G_1, \ldots, F_n \lor G_n$ then return

$pv(\Gamma, F_1, G_1, \ldots, F_n, G_n)$.

otherwise return No.
References

[1] G. Japaridze. *Introduction to computability logic*. Annals of Pure and Applied Logic 123 (2003), No.1-3, pp. 1-99.

[2] G. Japaridze. *From truth to computability I*. Theoretical Computer Science 357 (2006), No.1-3, pp. 100-135.

[3] G. Japaridze. *Computability logic: a formal theory of interaction*. In: *Interactive Computation: The New Paradigm*. D. Goldin, S. Smolka and P. Wegner, eds. Springer 2006, pp. 183-223.

[4] O. Ketonen. *Untersuchungen zum Prädikatenkalkül*. Annales Academiae scientiarum fennicae, Ser. A.I. 23, Helsinki, 1944.

[5] Keehang Kwon. *A Heuristic Proof Procedure for First-order Logic*. IEICE Transactions on Information and Systems 2020 Volume E103.D, Issue 3, Pages 549-552.

[6] A. Troelstra and H. Schwichtenberg. *Basic Proof Theory*. Cambridge University Press, 2nd edition, 2000.