Incident and reflected wave separation on wave propagation over breakwater

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Abstract. The quantification of the reflection waves is of paramount importance in coastal engineering. The reflection wave that affects the incoming waves over time will gradually affect the calculation of the transmitted waves after passing through a submerged breakwater. In this study, the reflection waves that affect the incident waves will be separated so that the appropriate transmission coefficient is obtained. The accuracy of this separation method of the incident waves and reflections will be evaluated using a numerical scheme. The numerical scheme that we use here is a staggered finite volume method. A small error in the comparison indicates the wave separation process is good enough to produce a reflection coefficient following the numerical results.

Keyword: Incident wave, wave separation, numerical scheme

1. Introduction

When extreme waves travel from the sea towards the coast, one of the ways to reduce the damage is to build a breakwater. According to [1, 2, 3, 4], a breakwater is able to reduce the amplitude of the incident wave which propagates over it. Specifically, in research conducted by [4, 5, 6] and [7], it was found that the effectiveness of a breakwater depends on the period of the incident wave. However, once the waves propagate over a breakwater, reflection waves will be formed, as seen in [8]. Therefore, in order to give a more accurate result on the effectiveness of a breakwater to reduce the amplitude, it is necessary to take into account the existence of the reflection waves. In this paper, we use the N-gauge method to separate reflection waves and incident waves. The idea is considering the wave spectrum at N points on the line that are parallel to the direction of the waves. In the case of a physical testing, this wave spectrum is recorded through a wave probe device, where the wave probe is placed in front of the breakwater.

In particular, this method uses the frequency domain, as seen in Figure 1. Points $p_1, p_2, ..., p_N$ in front of the breakwater denote the point where the frequency is analyzed to get separated into incident and reflected wave. All in all, this paper will be divided into five sections. First, we will start with a short introduction of this research. Next, we will derive the mathematical model formulation and wave propagation simulation in the second and third section, respectively. Finally, we will discuss the result on the fourth section and conclude all of the findings in the fifth section.
2. Mathematical Model Formulation

In this section, we introduce $z(x, t)$ which is a sinusoidal function of space $x$ and time $t$. Physically, $z(x, t)$ is the wave elevation that is measured from the undisturbed water level $z = 0$. The $z(x, t)$ function is a moving sinusoidal function through the $x$ axis, also well-known as a progressive wave. The general formula of a progressive wave is given by

$$z(x, t) = a \sin \left( \pm \frac{2\pi t}{T} \pm \frac{2\pi x}{L} + \theta \right),$$

where $a$ is the wave amplitude, $T$ denotes the wave period, $L$ is the wave length in correspondent to $T$, $\theta$ is the phase introducing a random location for the $x$ and $t$ references. It can be seen from equation (1) that the progressive wave is moving in the positive direction (from the left to the right side) if the sign of two terms of sine function are different. On the contrary, if both signs are equal, the wave move in the negative direction (from the right to the left side). Here, the progressive wave moving in the positive direction can be represented as

$$z(x, t) = a \sin \left( \frac{2\pi t}{T} - \frac{2\pi x}{L} + \theta \right),$$

Progressive wave moving in the negative direction can be represented as

$$z(x, t) = a \sin \left( \frac{2\pi t}{T} + \frac{2\pi x}{L} + \phi \right),$$

where $x$ is the abscissa and $\phi$ is the wave phase different from $\theta$. In Figure 3, there are three main equipment used in the physical testing, which are wave generator, M wave probe, and breakwater. The M wave probes are placed with different distances from one to another.
In this case, the existence of breakwater results in reflecting wave with negative direction and disturb the incident wave generated by the wave generator. Therefore, it is important to separate the incident and reflected wave in order to calibrate the incoming wave. To start with, it is assumed that observed signal (the time varying wave elevation) consists of an incident and reflected signal. The incident signal represented by a summation of two sinusoidal functions of $x$ and $t$. While the observed signal can be represented as follows: At gauge-1:

$$X_1(t) = \sum_{k=1}^{N} l_k \sin \left( \frac{2\pi kt}{T} - \frac{2\pi x_1}{L_k} + \theta_k \right) + \sum_{k=1}^{N} R_k \sin \left( \frac{2\pi kt}{T} - \frac{2\pi x_1}{L_k} + \theta_k + \psi_{1k} \right),$$

(4)

at gauge-$n$:

$$X_n(t) = \sum_{k=1}^{N} l_k \sin \left( \frac{2\pi kt}{T} - \frac{2\pi (x_1 + x_n)}{L_k} + \theta_k \right) + \sum_{k=1}^{N} R_k \sin \left( \frac{2\pi kt}{T} - \frac{2\pi (x_1 - x_1n)}{L_k} + \theta_k + \psi_{1k} \right)$$

(5)

where:

- $N$ is the total number of harmonics;
- $k$ indicates the $k$th harmonics;
- $l_k, R_k, L_k, \text{ and } \theta_k$ are the incident wave amplitude, the reflected amplitude, the wave length and the wave phase of harmonic $k$, independent of wave probe, respectively;
- $x_1$ is the distance from the wave generator 1;
- $x_{1n}$ is the distance from WP1 to WP$n$;
- $\psi_{nk}$ is the wave phase shift of harmonic $k$ due to the wave travelling from WP$n$ to the breakwater and therefrom to WP$n$ again.

The first summation of Equation (4) and (5) represents the incident component of the signal and the second summation represents the reflected part. On other hand, $X_n(t)$ will be represented by the following Fourier Analysis:

$$X_n(t) = \sum_{k=-N}^{N} A_{nk} e^{\frac{2\pi i k t}{T}}.$$  

(6)

Note that it is possible that $A_{nk}$ is a complex number. Furthermore, it could be derived that $Z_{lk}$ and $Z_{Rk}$ are respectively auxiliary complex number from which $I_k$ and $R_k$ can be obtained. Hence, the complex number can be written as

$$A_{nk} = D_{nk} + i E_{nk},$$  

(7)

$$Z_{lk} = I_{lk} + i Y_{lk},$$  

(8)

$$Z_{Rk} = X_{Rk} + i Y_{Rk}.$$  

(9)

From [9], we have:

$$A_{nk} = Z_{lk} e^{-i\alpha_n} + Z_{Rk} e^{i\alpha_n}, n = 1, 2, ..., M$$

(10)

where $\alpha_n = \frac{2\pi x_{1n}}{L_k}$. 

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**Figure 3.** Sketch of the wave flume, WP: wave probe.
3. Wave Propagation Simulation

In order to simulate wave propagation over the gauges and a submerged breakwater, we use the Finite Volume Method on Staggered Grid method to solve the Linear Shallow Water Equation numerically. The accuracy of this method has been examined with several test cases as seen in [10, 11, 12, 13]. The Linear Shallow Water Equation over a bottom with depth \( h(x) \) is given by:

\[
\eta_t + (hu)_x = 0, \tag{11}
\]

\[
u_t + g\eta_x = 0, \tag{12}
\]

with \( \eta \) is the wave elevation and \( g \) as the gravity acceleration of 9.81 m/s\(^2\).

The notation \( h(x) \) is a function which denotes the depth of water from still water to the seabed. First, assuming that there is no breakwater involved, then it can be written \( h = h_0 \) where \( h_0 \) is a positive constant that states the depth of water. Consequently, if there is a trapezoidal submerged breakwater, the water depth function is given as follows:

\[
h(x) = \begin{cases} 
h_0, & x \leq x_0 \\ 
\frac{h_0}{x_0}x, & x_0 < x \leq x_1 \\ 
h_1, & x_1 < x \leq x_2 \\ 
\frac{1}{x_3 - x_2}[(h_2 - h_1)x + h_1x_3 - h_2x_2], & x_2 < x \leq x_3 \\ 
h_2, & x > x_3 
\end{cases} \tag{13}
\]

where \( x_i, i = 1, 2, 3 \) is the division of the horizontal domain area and \( h_1 \) is the distance of the trapezoid peaks to the undisturbed water surface. For the mass conservation equation given in Equation (11), variables will be calculated in half grid \([x_{j-1/2}, x_{j+1/2}]\) while the momentum conservation equation calculated on the full grid of \([x_j, x_{j+1}]\). Applying the Finite Volume on a Staggered Grid method to Equation (11) and (12), we have the following numerical scheme:

\[
\frac{\eta_{j+1/2,n+1} - \eta_{j,n}}{\Delta t} + \frac{(\ast hu)_{j+1/2,n} - (\ast hu)_{j-1/2,n}}{\Delta x} = 0, \tag{14}
\]

\[
\frac{u_{j+1/2,n+1} - u_{j+1/2,n}}{\Delta t} + g\frac{\eta_{j+1,n+1} - \eta_{j,n+1}}{\Delta x} = 0, \tag{15}
\]

The h notation is marked \( \ast \) because at the point \( j + 1/2 \), the value of \( h \) is undefined. Therefore, to obtain the value of \( h(x) \) in the grid \( 1/2 \), we use the upwind method, as follows:

\[
\ast h_{j+1/2} = \begin{cases} 
h_j, & u_{j+1/2} \geq 0 \\ 
h_{j+1}, & u_{j+1/2} < 0 
\end{cases}
\]

wave elevation \( \eta(x_j, t_n), j = 1, 2, 3, ..., N_x \) and \( u_{j+1/2,n} = u(x_{j+1/2}, t_n), j = 0, 1, 2, ..., N_x \), for \( n = 0, 1, 2, ..., N_t \) with \( N_t = \left\lfloor \frac{T}{\Delta t} \right\rfloor + 1, T \) is the observation time. To simplify writing, equation (14) and (15) can be written as follow

\[
\eta_{j,n+1} = \eta_{j,n} - \frac{\Delta t}{\Delta x}((\ast hu)_{j+1/2,n} - (\ast hu)_{j-1/2,n}), \tag{16}
\]

\[
u_{j+1/2,n+1} = u_{j+1/2,n} - g\frac{\Delta t}{\Delta x}(\eta_{j+1,n+1} - \eta_{j,n+1}), \tag{17}
\]
This model is good for simulating wave propagation because it is stable by Von Neumann Stability Analysis and has proven in [14]. Next, the numerical scheme in Equation (16) and Equation (17) is used to simulate wave propagation over a trapezoidal submerged breakwater and the gauges numerically. From the model, we could derive the reflection coefficient and compare the results with the reflection coefficient from Equation (5).

4. Mathematical Model Formulation

Based on the mathematical model and numerical simulation from the Section 4, we then compare the reflection coefficient from the numerical simulation with the wave elevation equation in Section 3. In the numerical simulation, we record the wave elevation value in each point of the $x$ axis at every value of $t$. Thus, the recorded reflected wave from the numerical simulation can be compared with the reflected wave recorded at every gauges. The reflection coefficient recorded in several gauges can be assumed to remain constant over time and determined from the inverse of Equation (5) as follows:

$$R_k = \frac{X_n(t) - \sum_{k=1}^{N} I_k \sin \left( \frac{2\pi kt}{T} - \frac{2\pi(x_1 + x_n)}{L_k} + \theta_k \right)}{\sum_{k=1}^{N} \sin \left( \frac{2\pi kt}{T} - \frac{2\pi(x_1 - x_{1n})}{L_k} + \theta_k + \psi_{1k} \right)},$$

where $X_n(t)$ is the wave elevation in gauge-$n$ at every time ($t$) recorded from generated wave propagation with finite volume method. Here, we have the domain length of 30m, a breakwater which is placed in $x = 12m$ to $x = 20m$, and 5 wave gauges placed between $x = 0m$ and $x = 12m$. The amplitude and wave period used in this simulation is 2m and 1s, respectively. The reflection coefficients from the simulation which are recorded at first gauge, are shown in figure 4.

![Reflection coefficient comparison of two method at the first gauge](image)

**Figure 4.** Reflection coefficient comparison of two method at the first gauge

From Figure 4, it can be seen that the reflection coefficient obtained from numerical and analytical method are similar. Thus, we can hypothesize that the analytical solution can describe how reflected wave phenomenon occurs due to the existence of breakwater. To further validate the result, the reflection coefficient error from the two methods are calculated at five gauges, as seen in Table 1.
Based on the errors which are presented in Table 1, we can see that the errors from the two methods are below 1%.

| \( t(s) \) | \( x(m) \) | 2.4m | 4.8m | 7.2m | 9.6m | 12m |
|----------|----------|------|------|------|------|------|
| 1.1      | 0.0070   | 0.0053 | 0.0335 | 0.0178 | 0.0002 |
| 1.2      | 0.0042   | 0.0042 | 0.0047 | 0.0091 | 0.0021 |
| 1.3      | 0.0005   | 0.0077 | 0.0052 | 0.0014 | 0.0001 |
| 1.4      | 0.0038   | 0.0030 | 0.0133 | 0.0001 | 0.0048 |
| 1.5      | 0.0032   | 0.0045 | 0.0106 | 0.0146 | 0.0046 |
| 1.6      | 0.0011   | 0.0371 | 0.0116 | 0.0104 | 0.0025 |
| 1.7      | 0.0014   | 0.0145 | 0.0019 | 0.0086 | 0.0040 |
| 1.8      | 0.0148   | 0.0137 | 0.0019 | 0.0025 | 0.0027 |
| 1.9      | 0.0060   | 0.0031 | 0.0012 | 0.0089 | 0.0031 |
| 2.0      | 0.0123   | 0.0004 | 0.0085 | 0.0067 | 0.0009 |
| 2.1      | 0.0038   | 0.0065 | 0.0014 | 0.0146 | 0.0006 |
| 2.2      | 0.0019   | 0.0095 | 0.0094 | 0.0035 | 0.0022 |
| 2.3      | 0.0110   | 0.0097 | 0.0017 | 0.0020 | 0.0050 |
| 2.4      | 0.0109   | 0.0029 | 0.0002 | 0.0068 | 0.0014 |
| 2.5      | 0.0003   | 0.0048 | 0.0082 | 0.0003 | 0.0024 |
| 2.6      | 0.0100   | 0.0076 | 0.0021 | 0.0008 | 0.0047 |
| 2.7      | 0.0006   | 0.0076 | 0.0097 | 0.0023 | 0.0011 |
| 2.8      | 0.0041   | 0.0037 | 0.0034 | 0.0006 | 0.0014 |
| 2.9      | 0.0063   | 0.0013 | 0.0010 | 0.0022 | 0.0038 |
| 3.0      | 0.0010   | 0.0019 | 0.0165 | 0.0200 | 0.0022 |

Hence, the errors from both methods are very small and our initial hypothesis is confirmed. Therefore, we can conclude that the numerical and analytical method are in a good agreement. Furthermore, from Table 1, we can also see that that the average error is relatively lower at the first and last gauge. It is because at gauge 2, 3, and 4, there is a lot more wave scattering process gathered from wave that reflected from the breakwater and wave generator. Thus, this process caused more complicated numerical approach calculation for reflected wave, considering that we use the Linear Shallow Water Equation.

5. Conclusion
In this paper, we successfully derived an analytical solution that is able to separate the incident waves and reflected waves due to the existence of a breakwater. This method produces a reflection coefficient value with a small error compare to the numerical results in all gauges. It is also shown that the reflected wave obtained from the analytical solution can describe how incident wave is separated. This results in a more precise transmission coefficient of breakwater. Therefore, we believe that the result from this paper can be utilised to observe the effectiveness of breakwater better.

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