A QUADRATIC ESTIMATION FOR THE KÜHNEL CONJECTURE ON EMBEDDINGS

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Abstract. The classical Heawood inequality states that if the complete graph $K_n$ on $n$ vertices is embeddable into the sphere with $g$ handles, then $g \geq \frac{(n-3)(n-4)}{12}$. A higher-dimensional analogue of the Heawood inequality is the Kühnel conjecture. In a simplified form it states that for every integer $k > 0$ there is $c_k > 0$ such that if the union of $k$-faces of an $n$-simplex embeds into the connected sum of $g$ copies of the Cartesian product $S^k \times S^k$ of two $k$-dimensional spheres, then $g \geq c_k n^{k+1}$. For $k > 1$ only linear estimates were known. We present a quadratic estimate $g \geq c_k n^2$. The proof is based on beautiful and fruitful interplay between geometric topology, combinatorics and linear algebra.

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1. Introduction and main result

Main result. The classical Heawood inequality states that if the complete graph $K_n$ on $n$ vertices is embeddable into the sphere with $g$ handles, then

$$g \geq \frac{(n-3)(n-4)}{12}.$$ 

Denote by

- $\Delta_n^k$ the union of $k$-dimensional faces of an $n$-dimensional simplex;

We are grateful to R. Fulek, E. Kogan, R. Karasev, S. Melikhov, R. Nikkuni, and S. Zhilina for useful discussions.
• $S_g$ the connected sum of $g$ copies of the Cartesian product $S^k \times S^k$ of two $k$-dimensional spheres.

A higher-dimensional analogue of the Heawood inequality is the Kühnel conjecture on embeddings [Ku94, Conjecture B], see. Remark 1.3.a. In a simplified form it states that for every integer $k > 0$ there is $c_k > 0$ such that if $\Delta^k_n$ embeds into $S_g$, then

$$g \geq c_k n^{k+1}.$$

We present a quadratic in $n$ estimate (Theorems 1.1 and 1.2).

**Notation and conventions.** From now on we shorten ‘$s$-dimensional’ to just ‘$s$’.

In this text a manifold may have non-empty boundary.

For a simple definition of a homology group $H_k(\cdot; \mathbb{Z}_2)$ accessible to non-specialists in topology see [IF, §2], [Sk20, §6, §10]. For a $2k$-manifold $M$ let

$$\beta_k(M) := \dim H_k(M; \mathbb{Z}_2).$$

(This is called the $k$-th mod 2 Betti number of $M$. Observe that $\beta_1(S_g) = 2g.$)

In this text references to remarks could be ignored for the first reading.

We consider only piecewise linear (PL) 2k-manifolds. Unless otherwise specified, we consider only PL maps. Thus we omit ‘PL’ from statements, definitions and proofs (except for the situations when topological maps are around). The analogues of our results are correct for topological embeddings (Remark 1.3.b), for almost embeddings (defined and discussed in Remark 1.3.c), and for $\mathbb{Z}_2$-embeddings (defined and discussed in Remark 1.9).

**Theorem 1.1** (Skeleton). *If $\Delta^k_n$ embeds into a 2k-manifold $M$, then

$$\beta_k(M) \geq \frac{n^2}{2^k(k+1)^2} \text{ as } n \to \infty \text{ for fixed } k \geq 1$$

(more precisely, $\geq \frac{(n-4k-2)^2}{2^k(k+1)^2}$ for $n \geq 5k + 3$).

Denote $[n] := \{1, \ldots, n\}$.

Let $[n]^{k+1}$ be the $k$-complex with vertex set $[k+1] \times [n]$, in which every $k+1$ vertices from different lines span a $k$-face. For $k = 1$ this is the complete bipartite graph $K_{n,n}$. For geometric interpretation see [Ma03, Proposition 4.2.4].

**Theorem 1.2** (Joinpower). *If $[n]^{k+1}$ embeds into a 2k-manifold $M$, then

$$\beta_k(M) \geq \frac{n^2}{2^k} \text{ as } n \to \infty \text{ for fixed } k \geq 1$$

(more precisely, $\geq \frac{(n-3)^2}{2^k}$ for $n \geq 4$).

Theorem 1.1 (Skeleton) follows from Theorem 1.2 (Joinpower) because $\Delta^k_n \supset [s]^{k+1}$ for some $s \geq \frac{n - k + 1}{k + 1}$.

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1Observe that vice versa $K := [n] \ast \binom{k}{2} \ast \ldots \ast \binom{n}{k+1}$ contains a subdivision of $\Delta^k_n$ (here $K$ is a complex with set $1 \times [n] \cup 2 \times \binom{n}{2} \cup \ldots \cup (k+1) \times \binom{n}{k+1}$ of vertices; its $k$-faces are $\{(i, a_i)\}_{i \in [k+1]}$ for $a_i \in \binom{n}{i}$). In order to prove this take a baricentric subdivision of $\Delta^k_n$. Then every vertex $(i, a_i)$ of $K$ corresponds to the barycenter of some $i$-face of $\Delta^k_n$. 

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Relation to known results. For a $2k$-manifold $M$ into which $\Delta^n_k$ can be embedded the linear in $n$ estimate $\beta_k(M) \geq \frac{n - 2k - 1}{k + 1}$ is proved in [PT19] (after a weaker linear estimate of [GMP+]); see [KS21] for a simpler exposition.

For $k = 1$ Theorem 1.1 (Skeleton) (and the linear estimates above) follows from the Heawood inequality at the beginning of §1, and Theorem 1.2 (Joinpower) is also due to Heawood. The usual proof of the Heawood inequality via Euler inequality does not work for $k > 1$ because a $k$-sphere does not split $\mathbb{R}^{2k}$.

Remark 1.3. (a) The Kühnel conjecture on embeddings [Ku94, Conjecture B] states that if $\Delta^n_k$ embeds into a $(k-1)$-connected closed $2k$-manifold $M$, then

$$\left( \frac{2k + 1}{k + 1} \right) |\chi(M) - 2| \geq \left( \frac{n - k - 1}{k + 1} \right).$$

Different authors have considered stronger conjectures, in which $M$ is not $(k-1)$-connected, and $|\chi(M) - 2|$ is replaced either by $\beta_k(M)$ [GMP+, PT19], or by the $k$-th rational/integer Betti number

$$b_k(M) := \dim H_k(M; \mathbb{Q}) = \text{rk} H_k(M; \mathbb{Z})$$
[Ad18, Remark 4.9], see e.g. Conjecture (d). We have

- $\beta_1(M) = b_1(M) = 2 - \chi(M)$ for a closed connected 2-manifold $M$,
- $\beta_k(M) = b_k(M) = |\chi(M) - 2|$ for a closed $(k-1)$-connected $2k$-manifold $M$, and
- $\beta_k(M) \geq b_k(M)$ by the Universal Coefficients Formula, see e.g. [FF89, §15.5] [Sk20, Theorem 11.8.1].

(b) In the Kühnel conjecture on embeddings the PL and the topological embeddability are equivalent for $k \geq 3$ by the PL approximation theorem [Br72, Theorem 1] (in fact, weaker ‘metastable’ versions of this result cited in [Br72] are sufficient; recall that we consider topological embeddings into PL manifolds).

(c) For a complex $K$ and any space $M$ a map $f : K \to M$ is called an almost embedding if $f \sigma \cap f \tau = \emptyset$ for any vertex-disjoint faces $\sigma, \tau$. See some motivations in [Sk, Remark 6.7.5]. Clearly, the property of being an almost embedding is preserved under sufficiently small perturbation of the map (as opposed to the property of being an embedding). Thus by approximation of continuous maps with PL maps we observe that

- topological embeddability implies PL almost embeddability;
- PL almost embeddability is equivalent to topological almost embeddability.

For $k \geq 3$ the Kühnel conjecture on embeddings is equivalent to the analogous conjecture on almost embeddings because for $k \geq 3$ almost embeddability of a $k$-complex to a $2k$-manifold implies PL embeddability (for $\mathbb{R}^{2k}$ this is due to van Kampen-Shapiro-Wu, and the case of general $2k$-manifolds is analogous as explained in [PT19, Proposition 7 for $M = M'$, and §5, step 3 of proof of Theorem 6], [KS21e, comments after Theorem 1.3.1.a]). The analogue of the latter result for $k = 2$ is false [SSS], and for $k = 1$ is unknown, cf. [FPS] and the references therein.

(d) The Kühnel conjecture for simplicial embeddings. If some triangulation of a $2k$-manifold $M$ has a subcomplex isomorphic to $\Delta^n_k$, then

$$\left( \frac{2k + 1}{k + 1} \right) b_k(M) \geq \left( \frac{n - k - 1}{k + 1} \right).$$

(e) Conjecture (d) is stated as a result in [Ad18, Remark 4.9]: ‘... if a complete $k$-dimensional complex on $n$ vertices embeds into $M$ sufficiently tamely (so that it extends to
a triangulation of $M$), then \((n-k-1)_{k+1} \leq (2k+1)b_k(M)\). (Here ‘$n$ vertices’ should be changed to ‘$n + 1$ vertices’, cf. (a).)

We write ‘conjecture’ not ‘theorem’ (in (d) and in Remark 4.1.a) because we do not share responsibility for the statements to be correct, see description of problems in footnote 13 and Remark 4.1.c; cf. (i). We recover ‘implies at once’ from [Ad18, Remark 4.9] for ‘Conjecture 4.1.a implies Conjecture (d)’ only for the asymptotic version of Conjecture (d), see Remarks 4.1.bc. We do not question K. Adiprasito’s priority for the implication.

(f) Conjecture. If a simplicial complex embeds into a manifold $M$, then some triangulation of $M$ has a subcomplex isomorphic to the complex.

(g) The Kühnel conjecture on embeddings (a) follows from Conjectures (d, f). Thus by (i) we think that Conjecture (f) is wrong or hard to prove. We are grateful to K. Adiprasito, S. Melikhov, P. Patak, B. Sanderson and M. Tancer, e-mail exchange with whom helped us to conclude that this conjecture is open at the moment (and so was open in 2022 when first versions of this paper appeared on arXiv).2

Let us illustrate Conjecture (f) by an example. Define the graph $X$ to be the union of the cycle on 4 vertices, and leaf edges added to each vertex of the cycle. Then $K_3$ embeds into $X$ but no subdivision of $X$ has a subgraph isomorphic to $K_3$. This example does not refute Conjecture (f) because $X$ is not a manifold.

There are complexes PL embeddable into $\mathbb{R}^d$ but not linearly embeddable into $\mathbb{R}^d$ (i.e. complexes for which there are no embedding such that the image of any face is a simplex) [vK41, PW]. It would be interesting to know if these complexes $K$ are not simplicially embeddable into $\mathbb{R}^d$ (i.e. if there are no triangulations of $\mathbb{R}^d$ having a subcomplex isomorphic to $K$), thus giving counterexamples to Conjecture (f).

2Thus the Kühnel conjecture on embeddings (a) is not proved in (or does not easily follow from) the unpublished paper [Ad18] (even assuming that Conjecture (d) is a theorem proved in [Ad18]). This contradicts a referee report we received (but this is confirmed by [PT19, Proposition 7, Theorem 1], and the quotation ‘This [an approach mentioned earlier] would be in particular interesting, if it were possible to remove the additional assumption on the embeddings in Adiprasito’s proof of the Kühnel bound’ [PT19, before Proposition 7]). In [Ad18] the Kühnel conjecture is mentioned only in [Ad18, Section 1.6, (1)] (which concerns a different Kühnel conjecture [Ku94, Conjecture C]), and in [Ad18, Remark 4.9] (which claims (d) but neither (a) nor (f); see also Remark 4.1.cd and footnote 13).

Concerning Conjecture 4.1.a at the end of [Ad18, Remark 1.1] one reads: ‘The case of arbitrary topological embeddings therefore remains open’. This phrase possibly misled a referee of our paper, because even the case of arbitrary PL embeddings remains open (indeed, [Ad18, Remark 1.1] concerns ‘embeddings as subcomplexes’ not PL embeddings; see also footnote 13).

In order to avoid confusion, here we present our April 14, 2024 letter to K. Adiprasito.

Dear Karim,

We wish you all the best for proving Conjecture 4.1.a (for $2k$-manifolds $M$ non-embeddable into $\mathbb{R}^{2k+1}$) and Conjecture 1.3.f from our paper attached. This would be an outstanding result of yours, because this would imply the Kühnel conjecture on embeddings (except topological embeddings for $k = 2$). We would be glad to refer in our paper to arXiv update of [Ad18], or to a new arXiv paper. We encourage you to put your paper on arXiv whenever you feel your text is ready for praise and for criticism.

In our opinion, making a claim for Conjecture 1.3.f upon the text you sent us on April 6, 2024 will jeopardize your reputation. You will presumably realize this by critical reading of your text, so there is no need to send you our specific critical remarks (also, your letter does not ask for them). However, we would be glad to present critical remarks to (or praise) any text publicly available on arXiv, and relevant to our paper. ArXiv publication (which could never be completely removed) allows one to bear responsibility for a claim, which is necessary for development of mathematics. So in order to avoid confusion, unfortunately we would have to delete without reading your letters making a claim for Conjecture 1.3.f. But we would feel obliged to publicly react to an arXiv update of [Ad18], or to a new arXiv paper making such a claim.

Best wishes, Arkadiy, Slava.
(h) For a short description of references on embeddability of $k$-complexes into $2k$-manifolds see [KS21e, Remark 1.1.1.b]. There are algebraic criteria for such embeddability, due to Harris-Krushkal-Johnson-Paták-Tancer-Kogan-Skopenkov, see [KS21e, §1] and the references therein. Theorem 1.1 (and Theorem 1.2) is non-trivial in spite of the existence of these criteria and Remark 1.10.b. The criterion of [KS21e] shows that such embeddability is closely related to the low rank matrix completion problem [FK19, KS21e] (and thus to the Netflix problem from machine learning). This is the problem of minimizing the rank of a matrix, of whose entries some are fixed, and the other can be changed (see [DGN+] for an introduction accessible to students). Our proof of Theorem 1.2 (Joinpower) is also related to this problem. We study a more general problem, in which instead of knowing specific matrix elements, we know linear relations on such elements. We estimate the minimal rank of matrices with such relations (Theorem 1.5).

(i) We believe that even the asymptotic version

$$\beta_k(M) \gtrsim c_k n^{k+1} \text{ as } n \to \infty \text{ for fixed } k > 1 \text{ and some } c_k > 0$$

of the Kühnel conjecture on embeddings (a) is wrong or hard to prove.

Obstructions to embeddability constructed in this paper are presumably complete, see [SS23, Conjecture 1.6.b]. Then (dis)proof of the Kühnel conjecture on embeddings would require algebraic technique very different from the one used for simplicial embeddings of (d,e).

The Heawood inequality has the following equivalent restatement: if the sphere with $g$ handles has a triangulation on $n$ vertices, then $g \geq \frac{(n-3)(n-4)}{12}$. The following generalization is the Kühnel conjecture on triangulations: if a closed $2k$-manifold $M$ has a triangulation on $n+1$ vertices, then the opposite to the inequality of Remark 1.3.a holds [Ku94, Conjecture C], [Ku95, Conjecture B]. This is proved in [NS09, Theorem 4.4 and inequality (11) before].

For a related Kühnel conjecture on triangulations see [Ku95, Conjecture C]. The latter conjecture implies that if there is an embedding $\Delta^k_n \to M$ extendable to a triangulation of $M$ without adding new vertices, then the opposite to the inequality of Remark 1.3.a holds. (Indeed, then $M$ embeds into $\Delta^k_n$, so [Ku94, Conjecture C] can be applied.) This is a version of the Kühnel conjecture on embeddings under the stronger restriction, and with the opposite inequality in the conclusion.

For yet another Kühnel conjectures on tight polyhedral $2k$-submanifolds of $\mathbb{R}^n$ see [Ku94, Conjecture A], [Ku95, Conjecture A].

**Topological and linear algebraic parts.** Our theoretical achievement allowing to prove Theorems 1.1 and 1.2 is to fit what we can prove in topology to what is sufficient for algebra. Thus our main idea is the notion of an $(n, k)$-matrix, whose definition is postponed until after Remark 1.6. Before we introduce the definition, we show how it works. Theorem 1.2 (Joinpower) is implied by the following Theorems 1.4 (Embeddability) and 1.5 (Low Rank). Thus the proof is split into two independent parts.

**Theorem 1.4 (Embeddability).** If $[n]^{k+1}$ embeds into a $2k$-manifold $M$, then there is an $(n, k)$-matrix of rank at most $\beta_k(M)$.

**Theorem 1.5 (Low Rank; proved in §2).** For $n \geq 4$ the rank of any $(n, k)$-matrix is at least $(n-3)^2/2^k$. 
Denote by $\cap_M$ the mod 2 algebraic intersection of $k$-cycles on a $2k$-manifold $M$; for a simple definition accessible to non-specialists in topology see [IF, §2], [Sk20, §10], [KS21e, §1.2].

Denote by $\oplus$ the mod 2 sum of sets.

**Remark 1.6.** Here we motivate by low-dimensional examples the definition of an $(n, 1)$-matrix (to be introduced later), and Theorem 1.4 (Embeddability).

Let $M$ be a 2-manifold, and $f: [n]^2 \to K_{n,n} \to M$ a map. For $a \in [n]$ let $a, a'$ be vertices of $K_{n,n}$ from different parts. For 2-element subsets $P_1 = \{a, b\}$ and $P_2 = \{u, v\}$ of $[n]$ denote by $P = P_1 \ast P_2 := au'bv'$ the (set of edges of the) cycle of length 4 in $K_{n,n}$. For such cycles $P, Q$ denote

$$A_{P,Q} = A(f)_{P,Q} := fP \cap_M fQ \in \mathbb{Z}_2.$$

The obtained square matrix $A$ is symmetric. The matrix $A$ is the Gram matrix (with respect to $\cap_M$) of some homology classes in $H_1(M; \mathbb{Z}_2)$. Hence $\text{dim} \, H_1(M; \mathbb{Z}_2) \geq \text{rk} \, A$.

If $f$ is an embedding then the following properties hold for any cycles $P, Q \subset K_{n,n}$ of length 4 (for additivity it is not even required that $f$ is an embedding):

1. (independence) $A_{P,Q} = 0$ if $P$ and $Q$ are vertex-disjoint;
2. (additivity) $A_{P,Q} = A_{X,Q} + A_{Y,Q}$ if $X, Y \subset K_{n,n}$ are cycles of length 4 and $P = X \oplus Y$;
3. (non-triviality) if $\{P, Q\}, \{P', Q'\}$ are the two different unordered pairs of cycles of length 4 in $K_{3,3} \subset K_{n,n}$ such that $P \cap Q = P' \cap Q'$ is the edge 11, then $SA := A_{P,Q} + A_{P',Q'} = 1$; in other words,

$$SA = A_{\{1,2\},\{1,2\}} + A_{\{1,2\},\{1,3\}} + A_{\{1,2\},\{1,3\}} = A_{11'22',11'33'} + A_{11'23',11'32'} = 1.$$

Independence and additivity clearly follow from properties of the mod 2 algebraic intersection of 1-cycles. Non-triviality is a reformulation of [FK19, Lemma 17], and is a version of the following result:

"for any general position map of $K_{3,3}$ in the plane there is an odd number of intersection points of images of vertex-disjoint edges" (cf. [KS21, Remark 1.3]).

This result is proved in [vK32, Satz 5] (for more general case; see an alternative proof as proof of Lemma 4.2), and is rediscovered in the Kleitman 1976 paper cited in [FK19, §5].

A symmetric, independent, additive, non-trivial matrix, whose rows correspond to cycles of length 4 in $K_{n,n}$, is called an $(n, 1)$-matrix.

Now we move on to the definition of an $(n, k)$-matrix.

A **$k$-octahedron** is the set of $k$-faces of a subcomplex (of $[n]^{*k+1}$) isomorphic to $[2]^{*k+1} \cong S_k$. For 2-element subsets $P_1, \ldots, P_{k+1} \subset [n]$ such a subcomplex

$$P = P(P_1, \ldots, P_{k+1}) = P_1 \ast \ldots \ast P_{k+1}$$

is defined by the set $1 \times P_1 \sqcup \ldots \sqcup (k+1) \times P_{k+1}$ of its vertices. Its $k$-faces $a_1 \ast \ldots \ast a_{k+1}$, $a_i \in P_i$, are spanned by vertices $(i, a_i)$.

We consider only matrices with entries in $\mathbb{Z}_2$. The matrices are square matrices, unless otherwise specified.

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3The condition $P = X \oplus Y$ in additivity means that there is $i \in [2]$ such that $P_i = X_i \oplus Y_i$ and $P_{3-i} = X_{3-i} \oplus Y_{3-i}$. Additivity holds, for example, for the cycles $X = [2] \ast \{u, v\}$, $Y = [2] \ast \{u, w\}$ and $P = [2] \ast \{v, w\}$.

4A $k$-octahedron is uniquely defined by a parallelepiped $P_1 \times \ldots \times P_{k+1} \subset [n]^{k+1}$. So below one may work with parallelepipeds instead of $k$-octahedra. This is more convenient for formal statements (because parallelepipeds are simpler than $k$-octahedra), but less convenient for topological motivations. Everything that is said on the language of $k$-octahedra can be said in the dual language of parallelepipeds.
For a 2k-manifold $M$, a map $f: [n]^{*k+1} \to M$, and $k$-octahedra $P, Q$ denote

$$A(f)_{P,Q} := fP \cap_M fQ \in \mathbb{Z}_2.$$  

The obtained matrix $A(f)$ is symmetric. If $f$ is an embedding then the matrix $A(f)$ has the following three properties (which define an $(n,k)$-matrix.

Let $A$ be a symmetric matrix whose rows and whose columns correspond to all $k$-octahedra.\footnote{Such matrix is a block matrix of size $\binom{n}{2}$, where each block is a block matrix of size $\binom{n}{2}$, etc.}

The matrix $A$ is said to be independent if for any $k$-octahedra $P, Q$

$$A_{P,Q} = 0$$

if $P$ and $Q$ are vertex-disjoint.

It is obvious that $A(f)$ is independent.

The matrix $A$ is said to be additive if for any $k$-octahedra $P, Q$

$$A_{P,Q} = A_{X,Q} + A_{Y,Q}$$

if $P = X \oplus Y$ for some $k$-octahedra $X, Y$.

The additivity\footnote{For additivity it is not even required that $f$ is an embedding. The condition $P = X \oplus Y$ in additivity holds, for example, for $k$-octahedra $X, Y$ such that $X_j = Y_j$ for $j \neq i$, and $|X_i \cap Y_i| = 1$, for some $i \in [k+1]$. Then $P = X_1 \ast \ldots \ast X_i \ast (X_i \oplus Y_i) \ast X_{i+1} \ast \ldots \ast X_{k+1}$. Presumably there are no other octahedra such that $P = X \oplus Y$.} of $A(f)$ holds since the mod 2 intersection $\cap_M$ distributes over the mod 2 summation of $k$-cycles on $M$.

We shorten $\{1\}^{*k+1}$ to $1^{*k+1}$.

The matrix $A$ is said to be non-trivial if $SA = 1$, where $SA$ is the sum of $A_{P,Q}$ over all unordered pairs $\{P, Q\}$ of $k$-octahedra from $[3]^{*k+1}$ such that $P \cap Q = 1^{*k+1}$.

As an example we give explicit formulas for $SA$ (which are not used later). Denote $\mathcal{X} := \{1, x\}$ for $x \in \{2, 3\}$. Then

$$SA = A_{\{1,2\}, \{1,3\}} = A_{\mathcal{X}, \mathcal{X}},$$

$$SA = A_{\mathcal{X}, \mathcal{X}, \mathcal{X}} + A_{\mathcal{X}, \mathcal{X}, \mathcal{X}},$$

$$SA = A_{\mathcal{X}, \mathcal{X}, \mathcal{X}} + A_{\mathcal{X}, \mathcal{X}, \mathcal{X}} + A_{\mathcal{X}, \mathcal{X}, \mathcal{X}} + A_{\mathcal{X}, \mathcal{X}, \mathcal{X}},$$

$$SA = A_{\mathcal{X}, \mathcal{X}, \mathcal{X}} + A_{\mathcal{X}, \mathcal{X}, \mathcal{X}},$$

$$SA = A_{\mathcal{X}, \mathcal{X}, \mathcal{X}}.$$  

$k = 0$;

$k = 1$;

$k = 2$.

**Lemma 1.7** (Non-triviality; proved in §2). For any embedding $f: [n]^{*k+1} \to M$ to a 2k-manifold $M$ the matrix $A(f)$ is non-trivial.

A symmetric, independent, additive, non-trivial matrix is called an $(n,k)$-matrix.

Now the reader can read the proof of Theorem 1.5 (Low Rank) at the beginning of §2.

**Proof of Theorem 1.4.** Take any embedding $f: [n]^{*k+1} \to M$. The matrix $A(f)$ is the Gram matrix (with respect to $\cap_M$) of some homology classes in $H_k(M; \mathbb{Z}_2)$. Hence $\text{rk} A(f) \leq \beta_k(M)$ by the following well-known result.

Let $v_1, v_2, \ldots, v_r$ be vectors in a linear space $V$ over $\mathbb{Z}_2$ with a bilinear symmetric product. Then the rank of the Gram matrix of $v_1, v_2, \ldots, v_r$ does not exceed $\text{dim} V$.

The additivity and the independence are already proved after their definitions. Then the matrix $A(f)$ is an $(n,k)$-matrix by Lemma 1.7 (Non-triviality).

**Almost embeddability and $\mathbb{Z}_2$-embeddability.** Our topological results (Theorems 1.1, 1.2 and 1.4) are correct under the weaker assumption of almost embeddability defined in Remark 1.3.c (and even of $\mathbb{Z}_2$-embeddability defined below in Remark 1.9). This holds by the following stronger version of Theorem 1.4 (Embeddability).
Theorem 1.8 (Almost embeddability). Let $M$ be a closed $2k$-manifold. Let $\Omega_M$ be
• the identity matrix of size $\beta_k(M)$ if there is $x \in H_k(M; \mathbb{Z}_2)$ such that $x \cap_M x = 1$, and
• the direct sum of $\beta_k(M)/2$ hyperbolic matrices $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, otherwise (it is known that $\beta_k(M)$ is even in the ‘otherwise’ case for closed manifolds).

If $[n]^{k+1}$ has an almost embedding to $M$, then there is a $\beta_k(M) \times \begin{pmatrix} n \end{pmatrix}^{k+1}$-matrix $Y$ such that $Y^T \Omega_M Y$ is an $(n, k)$-matrix.

Proof. Let $f : [n]^{k+1} \to M$ be an almost embedding. By [IF, Theorem 6.1] there is a basis in $H_k(M; \mathbb{Z}_2)$ in which the matrix of $\cap_M$ is $\Omega_M$. Let $Y$ be the $\beta_k(M) \times \begin{pmatrix} n \end{pmatrix}^{k+1}$-matrix whose columns are coordinates of $k$-octahedra in this basis. Then $Y^T \Omega_M Y = A(f)$. The matrix $A(f)$ is an $(n, k)$-matrix because
• the additivity is already proved after the definition;
• the independence is still obvious;
• the proof of Lemma 1.7 (Non-triviality) presented in §2 works under the weaker assumption that $f$ is an almost embedding.

Denote by $h|_X$ the restriction of a map $h$ to a set $X$.
Denote by $|X|_2 \in \mathbb{Z}_2$ the parity of the number of elements in a finite set $X$.

Remark 1.9 (On $\mathbb{Z}_2$-embeddings). (a) Let $M$ be a $2k$-manifold, and $K$ be a $k$-complex. An accurate definition of general position maps $f : K \to M$ is given in [KS21e, the end of §1.1]. A general position map $f : K \to M$ is called a $\mathbb{Z}_2$-embedding if $|f \sigma \cap f \tau|$ is even for any vertex-disjoint faces $\sigma, \tau$.

Clearly, any almost embedding (defined in Remark 1.3.c) is a $\mathbb{Z}_2$-embedding. Observe that $\mathbb{Z}_2$-embeddability of $k$-complexes in $\mathbb{R}^{2k}$ does not imply almost embeddability, even for $k \geq 3$ [Me06, Example 3.6].

(b) Theorem 1.8 (Almost embeddability) holds under the weaker assumption of $\mathbb{Z}_2$-embeddability. Indeed, the argument changes only in the proof that $A(f)$ is an $(n, k)$-matrix, where
• the independence holds since for any vertex-disjoint $k$-octahedra $P, Q$

\[
A(f)_{P,Q} = f P \cap_M f Q = \sum_{(\sigma, \tau) \in P \times Q} |f \sigma \cap f \tau|_2 = 0;
\]

• the proof of Lemma 1.7 (Non-triviality) presented in §2 works under the weaker assumption that $f$ is a $\mathbb{Z}_2$-embedding.

A converse to the version of Theorem 1.8 for $\mathbb{Z}_2$-embeddings is proved in [SS23]. It allows to reduce the Kühnel conjecture for $\mathbb{Z}_2$-embeddings [SS23, Conjecture 1.6.a] to a purely algebraic problem.

(c) For a proof of non-$\mathbb{Z}_2$-embeddability of graphs to 2-manifolds the Euler inequality does not work, as opposed to non-embeddability; methods of [FK19] do work.

Idea of proof and corollaries.

Remark 1.10 (Idea of proof and its relation to known proofs). (a) Our proof of Theorem 1.2 (Joinpower) is a higher-dimensional generalization of the case $k = 1$ proved in [FK19] (under the weaker assumption of $\mathbb{Z}_2$-embeddability; see definition in Remark 1.9).

In particular, Theorems 1.4 and 1.8 (and thus the analogue of Lemma 1.7 (Non-triviality) for $\mathbb{Z}_2$-embeddings) for $k = 1$ are implicitly proved in [FK19]. Theorem 1.5
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(Low Rank) for \( k = 1 \) is also implicitly proved in \cite{FK19, §4}. Neither of these results is explicitly stated in \cite{FK19}, so we present in §3 a detailed and well-structured proof of Theorem 1.5 (Low Rank) for \( k = 1 \).

Our explicit statements for \( k = 1 \) illustrate our new ideas which work for higher dimensions. We did not succeed in generalizing to higher dimensions Lemma 3.4 (Independence), which implicitly appeared in \cite{FK19, §5}. So we observe that the additivity and the independence

- for \( k = 1 \) imply the property obtained in Lemma 3.4 (Independence), and
- are kept throughout the induction on \( k \) (see Lemma 2.1).

(b) Lemma 1.7 and Theorems 1.4 and 1.8 could be deduced using Lemma 2.3 (Combinatorial) from \cite[Lemma 2.3.1]{KS21e}, see the deduction in Remark 4.3. The independence and the additivity of Theorems 1.4 and 1.8 are trivial (and so are easier than the deductions). We present in §2 a direct proof of Lemma 1.7, because such a proof is not very much longer than Remark 4.3, and because the paper \cite{KS21e} is unpublished.

Lemma 1.7 for \( k = 1 \) is known \cite[Lemma 17]{FK19} (cf. the last paragraphs of Remark 1.6). The case \( k = 1 \) is easily reduced to a result on linking of points on the circle. For higher dimensions the corresponding linking results are cumbersome, see Remark 4.6, so we use a different approach.

Remark 1.11 (Corollaries). (a) Corollaries of Theorem 1.1 (or rather of its stronger form for almost embeddings; see the definition in Remark 1.3.c) are improved Radon type and (the following) Helly type results \cite[Theorem 2 and Corollary 3]{PT19} for set systems in \( 2k \)-manifolds.

Let \( M \) be a \( 2k \)-manifold and \( r \geq (k + 1)2^{k-1}\sqrt{\beta_k(M)} + 4k + 4 \). Take a finite family of subsets of \( M \) such that

- the intersection of any proper subfamily is either empty or \( k \)-connected;
- the intersection of any \( r \)-element subfamily is nonempty.

Then the intersection of all members of the family is nonempty.

Deduction of this result from Theorem 1.1 (Skeleton) is analogous to deduction of \cite[Theorem 2 and Corollary 3]{PT19} from the linear estimate \cite[Theorem 1]{PT19}.

(b) Theorem 1.1 (Skeleton) in a standard way gives lower estimation of crossing number of \( \Delta_n^k \). Given a general position map \( \Delta_n^k \to \mathbb{R}^{2k} \) with minimal number of crossings, one eliminates any crossing by adding handle \( S^k \times S^k \). So the crossing number of \( \Delta_n^k \) is equal to the number of added handles, which is at least \( \frac{n^2}{2^{(k+1)^2}} \) by Theorem 1.1 (Skeleton).

2. PROOFS OF THEOREM 1.5 AND LEMMA 1.7

In order to grasp the main idea the reader may first check the following proofs for \( k = 2 \).

Proof of Theorem 1.5 (Low Rank). Proof of Theorem 1.5 is by induction. The base \( k = 1 \) is proved implicitly in \cite[§4] {FK19} and explicitly in §3 (for idea of the proof see Remark 1.10.a).

Denote by

\[ \mathbb{Z}_2^{(n)^{k+1}} \times (n)^{k+1} \]

the set of matrices whose rows and columns are numerated by \( k \)-octahedra in \([n]^{k+1}\).
Take any $A \in \mathbb{Z}_2^{(n)(k+1)\times(k+1)}$. For each 2-element subsets $U, V \subset [n]$ define the $k$-coordinate block
\[
A_{U,V} \in \mathbb{Z}_2^{(n)(k)\times(k)} \quad \text{by} \quad (A_{U,V})_{P,Q} := A_{U\cap P, V\cap Q} \quad \text{for} \quad (k-1)\text{-octahedra} \ P, Q.
\]
Denote $\mathcal{F} := \{1, x\}$ for $x > 1$.

**Lemma 2.1** (Heredity; proved below). Suppose that $n \geq 4$, $k \geq 1$ and $A$ is an $(n, k)$-matrix. Then $A_{2,2} + A_{3,2}$ is an $(n, k-1)$-matrix.

**Inductive step** $k - 1 \rightarrow k$ in the proof of Theorem 1.5. Take an $(n, k)$-matrix $A$, and set $Z := A_{2,2} + A_{3,2}$. Then
\[
\text{rk} \ A \geq \frac{1}{2} \left( \text{rk} \ A_{2,2} + \text{rk} \ A_{3,2} \right) \geq \frac{1}{2} \text{rk} \ Z \geq \frac{(n-3)^2}{2k},
\]
where
- the first inequality holds since $\text{rk} \ A \geq \text{rk} \ A_{U,V}$ for any $U, V$,
- the second inequality holds by subadditivity of rank,
- the third inequality holds by Lemma 2.1 and the induction hypothesis applied to $Z$.

**Proposition 2.2** (One-coordinate swap). Suppose that $A \in \mathbb{Z}_2^{(n)(k+1)\times(k+1)}$ is independent and additive. Suppose that two $k$-octahedra $P = P_1 \ast \ldots \ast P_{k+1}$ and $Q = Q_1 \ast \ldots \ast Q_{k+1}$ ‘have only one common vertex’, i.e. for some $i \in [k+1]$ we have $|P_i \cap Q_i| = 1$, and for any $j \neq i$ we have $P_j \cap Q_j = \emptyset$. Then $A_{P,Q} = A_{P',Q}$ for any $k$-octahedron $P' = P_1 \ast \ldots \ast P_{i-1} \ast P_i' \ast P_{i+1} \ast \ldots \ast P_{k+1}$ such that $P_i' \cap Q_i = P_i \cap Q_i$.

**Proof.** For $P = P'$ this is a tautology. Otherwise the proposition follows since
\[
A_{P,Q} = A_{P',Q} + A_{P \oplus P',Q} = A_{P',Q},
\]
where
- the first equality holds by the additivity of $A$,
- the second equality holds by the independence of $A$, since the $k$-octahedra $Q$ and $P \oplus P' = P_1 \ast \ldots \ast P_{i-1} \ast (P_i \oplus P_i') \ast P_{i+1} \ast \ldots \ast P_{k+1}$ are vertex-disjoint.

**Proof of Lemma 2.1.** The additivity holds for $Z := A_{2,2} + A_{3,2}$ since it holds for $A_{2,2}$ and $A_{3,2}$.

The independence holds for $Z$ and $n \geq 4$, since for vertex-disjoint $(k-1)$-octahedra $P, Q$
\[
A_{2,2,P,3,2,Q} = A_{2,2,P,3,2,Q} = A_{2,2,P,3,2,Q} = A_{2,2,P,3,2,Q},
\]
where each equality holds by Proposition 2.2.

Since $A$ is symmetric, we have $A_{3,2,P,3,2,Q} = A_{3,2,Q,3,2,P}$, i.e. $A_{3,2} = A_{2,2}^T$. Hence $Z = A_{2,2} + A_{2,2}^T$ is symmetric.

For any $l \in \{k-1, k\}$ denote by $G_l$ the set of unordered pairs of $l$-octahedra from $[3]^{l+1}$ whose intersection is $1^{l+1}$. The non-triviality holds for $Z$ since
\[
S_Z = S_{A_{2,2}} + S_{A_{3,2}} = \sum_{\{P,Q\} \in G_{k-1}} (A_{2,2,P,3,2,Q} + A_{3,2,P,2,2,Q}) = \sum_{\{P',Q'\} \in G_k} A_{P',Q'} = 1,
\]
where the last equality is the non-triviality of $A$.\)

---

\[\text{One may say, } P' \text{ shares with } Q \text{ the same common vertex } P_i \cap Q_i, \text{ and intersects } P \text{ by the cone over common } (k-1)\text{-octahedron.}\]

\[\text{The independence does not hold for blocks } A_{2,2} \text{ and } A_{3,2} \text{ alone, only for their sum.}\]
Proof of Lemma 1.7 (Non-triviality).

Proof of Lemma 1.7. Recall that Int $M$ is the interior of $M$. Denote by $\partial x$ the boundary of $x$, where $x$ is either a $k$-face or a $2k$-ball. We may assume that $n = 3$.

The join [Ma03, §4.2] of $s$ non-empty complexes is $(s - 2)$-connected [Ma03, Proposition 4.4.3]. So $[3]^{*k+1}$ is $(k - 1)$-connected. Let $L := ([3]^{*k+1})^{(k-1)}$ be the union of all those faces of $[3]^{*k+1}$ whose dimension is less than $k$. Then $f|_L$ is null-homotopic. We shall use the following Borsuk Homotopy Extension Theorem [FF89, §5.5]:

if $(K, L)$ is a polyhedral pair, $Z \subset \mathbb{R}^m$, $F: L \times I \to Z$ is a homotopy, and $g: K \to Z$ is a map such that $g|_L = F|_{L \times 0}$, then $F$ extends to a homotopy $G: K \times I \to Z$ such that $g = G|_{K \times 0}$.

Hence $f$ is homotopic to a map $f'' : [3]^{*k+1} \to M$ such that $f'' L$ is a point. Take a $2k$-ball $B \subset $ Int $M$ such that $B \cap f''([3]^{*k+1}) = f'' L \cap \partial B$. Then $f''$ (and so $f$) is homotopic to a general position map $f': [3]^{*k+1} \to M - $ Int $B$ such that $f' L \subset \partial B$. (An accurate definition of a general position map is given in [KS21e, the end of §1.1].) By general position we may assume that $f' L$ is an embedding.$^9$

Define the map $g: [3]^{*k+1} \to B$ to be $f'$ on $L$, and to be the cone map over $f'|_{\partial B}$ with a vertex in Int $B$ on every $k$-face $\sigma$ of $[3]^{*k+1}$. By proper choosing these vertices we may assume that $g$ is a general position map. Then

$$SA(f) = SA(f') = \sum_{\{P, Q\} \in G_k} f' P \cap_M f' Q = \sum_{\{P, Q\} \in G_k} f'_g P \cap_M f'_g Q = \sum_{\{P, Q\} \in G_k ; \{\alpha, \beta\} \in T(P, Q)} f'_g \alpha \cap_M f'_g \beta = \sum_{\{\alpha, \beta\} \in H} |f'_g \alpha \cap f'_g \beta|_2 = \sum_{\{\alpha, \beta\} \in H} (|f'_g \alpha \cap f'_g \beta| + |g \alpha \cap g \beta|)_2 = 1.$$

Here

- equality (1) holds since $SA(f)$ is independent of homotopy of $f$;
- $G_k$ is the set of unordered pairs of $k$-octahedra from $[3]^{*k+1}$ whose intersection is $1^{*k+1}$;
- equality (2) is the definition of $SA(f')$;
- $f'_g \xi := f' \xi \cup g \xi$, where $\xi$ is either a $k$-octahedron or a $k$-face;
- equality (3) holds since $g P$ and $g Q$ are null-homologous;
- $T(P, Q)$ is the set$^{10}$ of pairs $\{\alpha, \beta\}$ formed by $k$-faces $\alpha, \beta$ of $[3]^{*k+1}$ such that either $\alpha \in P$ and $\beta \in Q$, or vice versa (note that $\alpha = \beta = 1^{*k+1}$ is possible);
- equality (4) holds since for any $\{P, Q\} \in G_k$

$$f'_g P \cap_M f'_g Q = \sum_{\{\alpha, \beta\} \in P \times Q} f'_g \alpha \cap_M f'_g \beta = \sum_{\{\alpha, \beta\} \in T(P, Q)} f'_g \alpha \cap_M f'_g \beta,$$

where

- the second (and the first) term is meaningful for any unordered pair $\{P, Q\}$ since the term is symmetric in $P, Q$;
- the first equality holds since $f'_g P = \bigoplus_{\alpha \in P} f'_g \alpha$ for $k$-cycles $f'_g \alpha$, and since we have the analogous equality for $f'_g Q$ and $f'_g \beta$.

$^9$Observe that $f'$ is not necessarily an embedding, almost embedding or a $\mathbb{Z}_2$-embedding. This paragraph is analogous to [PT19, Lemma 12], [KS21e, §2.3, beginning of proof of Lemma 2.3.1].

$^{10}$Note that $T(P, Q)$ is the image of the torus $P \times Q$ under the projection to the quotient of $[3]^{*k+1} \times [3]^{*k+1}$ under the symmetry exchanging factors.
we prove the second equality as follows: since $P \cap Q = 1^{*k+1}$, for $\{\alpha, \beta\} \in T\{P, Q\}$ exactly one of the pairs $(\alpha, \beta)$ and $(\beta, \alpha)$ lies in $P \times Q$; hence the formula $(\alpha, \beta) \mapsto \{\alpha, \beta\}$ gives a bijection $P \times Q \to T\{P, Q\}$; this implies the second equality;
• $H$ is the set of all unordered pairs of vertex-disjoint $k$-faces of $[3]^{*k+1}$;
• equality (5) holds since $H = \bigoplus_{\{P, Q\} \in G_k} T\{P, Q\}$, which is a reformulation of Lemma 2.3

(Combinatorial) below;
• equality (6) holds by definition of $\cap_M$ since $f', g$ are general position maps;
• equality (7) holds since $f'\alpha \cap g\beta = \emptyset$ for vertex-disjoint $\alpha, \beta$; this holds since
$$f'[3]^{*k+1} \subset M - \text{Int } B, \quad g[3]^{*k+1} \subset B, \quad f'|_L = g|_L,$$
and $g$ is a general position map;
• equality (9) is the result of van Kampen [vK32, Satz 5] (see Lemma 4.2).

It remains to prove equality (8). For a general position map $h: [3]^{*k+1} \to M$ the van Kampen number
$$v(h) := \sum_{\{\alpha, \beta\} \in H} |h\alpha \cap h\beta|_2$$
is the parity of the number of all pairs $\{\alpha, \beta\}$ of vertex-disjoint $k$-faces of $[3]^{*k+1}$ such that $|h\alpha \cap h\beta|_2 = 1$. Then (8) holds since
$$v(f') = v(f) = 0.$$

Here the second equality holds since $f$ is an embedding. Let us present a fairly standard argument for the first equality.

For a general position map $h: [3]^{*k+1} \to M$ the intersection cocycle $\nu(h) \subset H$ is the set of all pairs $\{\sigma, \tau\}$ such that $|h\sigma \cap h\tau|_2 = 1$; so $v(h) = |\nu(h)|_2$. For vertex-disjoint $(k-1)$-face $e$ and $k$-face $\alpha$ the elementary coboundary of $(\alpha, e)$ is the set of all unordered pairs $\{\alpha, \beta\}$ of vertex-disjoint $k$-faces such that $e \subset \beta$. Since $f'$ is homotopic to $f$, by [KS21e, Lemma 2.3.2] of van Kampen-Shapiro-Wu-Johnson, $\nu(f)$ and $\nu(f')$ are cohomologous, i.e. $\nu(f) \oplus \nu(f')$ is the mod 2 sum of some elementary coboundaries. For any $(k-1)$-face $e = e_1 \ldots e_{k+1}$ there is the unique $t(e) \in [k+1]$ such that $e_{t(e)} = \emptyset$. For a $k$-face $\alpha$ and a $(k-1)$-face $e$ the elementary coboundary of $(\alpha, e)$ consists of pairs $\{\alpha, \beta = \beta_1 \ldots \beta_{k+1}\}$ such that $\beta_{t(e)} \neq \alpha_{t(e)}$ and $\beta_s = e_s$ for every $s \neq t(e)$. Hence any elementary coboundary consists of two elements. Since the size of any elementary coboundary is even, $|\nu(f) \oplus \nu(f')|_2 = 0$. Hence $|\nu(f)|_2 = |\nu(f')|_2$.

Lemma 2.3 (Combinatorial). The following two sets\textsuperscript{11} are equal:
• the set of all ordered pairs $(\sigma, \tau)$ of vertex-disjoint $k$-faces of $[3]^{*k+1}$;
• the mod 2 sum of products $P \times Q$ over all ordered pairs $(P, Q)$ of $k$-octahedra from $[3]^{*k+1}$ whose intersection is $1^{*k+1}$.

\textsuperscript{11}In the dual language of parallelepipeds (see footnote 4) Lemma 2.3 states that the following two sets are equal:
• the set of all ordered pairs of vectors in $[3]^{[*k+1]}$ such that the vectors have no equal components;
• the mod 2 sum of products $P \times Q$ over all ordered pairs $(P, Q)$ of parallelepipeds whose intersection is $1^{*k+1}$.
idea of [PT19, Lemma 20], and is implicit in [PT19, proof of Lemma 20]; cf. [KS21, Proposition 2.2].

Proof of Lemma 2.3. It suffices to prove that

(A) for any vertex-disjoint $k$-faces $\alpha, \beta \in [3]^{k+1}$ there is exactly one ordered pair $(P, Q)$ of $k$-octahedra from $[3]^{k+1}$ such that $P \cap Q = 1^{k+1}$ and $(\alpha, \beta) \in P \times Q$; and

(B) for any $k$-faces $\alpha, \beta \in [3]^{k+1}$ sharing a common vertex there is an even number of ordered pairs $(P, Q)$ of $k$-octahedra from $[3]^{k+1}$ such that $P \cap Q = 1^{k+1}$ and $(\alpha, \beta) \in P \times Q$.

For a $k$-octahedron $P$ from $[3]^{k+1}$ containing the $k$-face $1^{k+1}$ denote by $\sigma^P$ the $k$-face of $P$ that is opposite to $1^{k+1}$.

Proof of (A). Take any $k$-octahedra $P,Q$ from $[3]^{k+1}$ such that $\alpha \in P$, $\beta \in Q$ and $P \cap Q = 1^{k+1}$. Take any $i \in [k+1]$.

Suppose that $\alpha_i \neq 1$. Since $\alpha \in P$, it follows that $\sigma^P_i = \alpha_i$. Since $P \cap Q = 1^{k+1}$, we have $\sigma^Q_i = 5 - \sigma^P_i = 5 - \alpha_i$.

Suppose that $\alpha_i = 1$. Then $\beta_i \neq 1$. Hence $\sigma_i^Q = \beta_i$ and $\sigma_i^P = 5 - \beta_i$ analogously to the previous paragraph.

Hence the $i$-th coordinates $\sigma_i^P$ and $\sigma_i^Q$ are uniquely defined for each $i \in [k+1]$. Thus there is exactly one pair $(P, Q)$ from the statement of (A).

Proof of (B). Since $\alpha$ and $\beta$ share a common vertex, we may assume that $\alpha_1 = \beta_1$ (the other cases are analogous).

Suppose that $\alpha_1 \neq 1$. Take any $k$-octahedra $P, Q$ from $[3]^{k+1}$ such that $P \cap Q = 1^{k+1}$. Since $\sigma^P, \sigma^Q$ are vertex-disjoint, we have $\sigma^P_1 \neq \sigma^Q_1$. Hence either $\alpha_1^P \neq \alpha_1$ or $\alpha_1^Q \neq \beta_1$. Then either $\alpha \not\in P$ or $\beta \not\in Q$. Thus $(\alpha, \beta) \not\in P \times Q$.

Suppose that $\alpha_1 = 1$. For every $k$-octahedron $R = \overline{r_1 * r_2 * \ldots * r_{k+1}} \subset [3]^{k+1}$ denote $R' := \overline{5 - r_1 * r_2 * \ldots * r_{k+1}}$. Clearly, $(R')' = R$, and if $P \cap Q = 1^{k+1}$, so $P' \cap Q' = 1^{k+1}$. Thus the pairs $(P, Q)$ from (B) split into couples corresponding to ‘opposite’ pairs $(P, Q)$ and $(P', Q')$. Since $\alpha_1 = 1$, the $k$-face $\alpha$ is contained either in both $P$ and $P'$ or in none. Analogously for $\beta$. Then for every couple $\{(P, Q), (P', Q')\}$ the pair $(\alpha, \beta)$ is contained either in both $P \times Q$ and $P' \times Q'$ or in none. This implies (B). \hfill \Box

3. APPENDIX: PROOF OF THEOREM 1.5 (LOW RANK) FOR $k = 1$

Here and below rows and columns of matrices are not necessarily numerated by octahedra (as opposed to §1.2).

For a (block) matrix $X$ whose rows are numerated by $[\ell] \times [m]$, and any $i,j \in [\ell]$ define the $m \times m$-block $X_{i,j}$ by $(X_{i,j})_{a,b} := X_{(i,a)(j,b)}$.

Denote by $0_m$ the zero $m \times m$-matrix, and by $J_m$ the $m \times m$-matrix consisting of units.

Lemma 3.1. Suppose $N$ is a matrix whose rows are numerated by $[\ell] \times [m]$, such that for every $i,j \in [\ell]$

\[
\begin{cases}
N_{i,j} = 0_m, & i \leq j, \\
N_{i,j} \in \{0_m, J_m\}, & i > j.
\end{cases}
\]

Then $\text{rk} N \leq \ell - 1$.

Proof. Define the matrix $F$ of size $\ell$ so that $F_{i,j} = 0$ if and only if $N_{i,j} = 0_m$. Clearly, $F_{i,j} = 0$ if $i \leq j$. Then the first row of $F$ consists of zeros. Thus $\text{rk} N = \text{rk} F \leq \ell - 1$. \hfill \Box
A matrix $Y$ with $\mathbb{Z}_2$-entries is said to be **tournament** if $Y_{a,b} + Y_{b,a} = 1$ for all $a \neq b$. In other words, $Y$ is a tournament matrix if $Y + Y^T$ is the inverted identity matrix, i.e. the sum of the identity matrix and $J_m$, where $m$ is the number of rows of $Y$.

**Lemma 3.2** (Tournament; [Ca91, Theorem 1]). The rank of a tournament $m \times m$-matrix is at least $\frac{m-1}{2}$.

**Proof.** For a tournament $m \times m$-matrix $Y$

$$
\text{rk}\ Y = \frac{\text{rk}\ Y + \text{rk}\ Y^T}{2} = \frac{\text{rk}(Y + Y^T)}{2} = \frac{\text{rk}(I_m + J_m)}{2} = \frac{\text{rk}\ I_m - \text{rk}\ J_m}{2} = \frac{m-1}{2},
$$

where

- $I_m$ is the identity matrix,
- the inequalities hold by subadditivity of rank,
- the middle equality holds since $Y$ is a tournament matrix, so $Y + Y^T = I_m + J_m$.

\[ \square \]

Recall that

$$[m_1] \cup [m_2] \cup \ldots \cup [m_\ell] = 1 \times [m_1] \cup 2 \times [m_2] \cup \ldots \cup \ell \times [m_\ell],$$

so that the disjoint union of $\ell$ copies of $[m]$ is $[\ell] \times [m]$.

Take a (block) matrix $X$ whose rows are numerated by $[m_1] \cup \ldots \cup [m_\ell]$.

For $i, j \in [\ell]$ define the $m_i \times m_j$-block $X_{i,j}$ by $(X_{i,j})_{a,b} := X_{(i,a)(j,b)}$.

The matrix $X$ is said to be **tournament-like** if for every $i \in [\ell]$ the diagonal block $X_{i,i}$ is a tournament matrix.

The matrix $X$ is said to be **diagonal-like** if it is obtained by removing some rows and columns symmetric to the rows, from a block matrix $\bar{X}$ with the following properties:

- its rows are numerated by $[\ell] \times [m]$,
- $m \geq m_i$ for every $i \in [\ell]$,
- the under-diagonal block $\bar{X}_{i,j}$ is a diagonal matrix for every $i, j \in [\ell]$, $i > j$.

**Lemma 3.3** (Diagonal-Tournament). The rank of any tournament-like diagonal-like matrix, whose rows are numerated by $[m_1] \cup \ldots \cup [m_\ell]$, is at least $\frac{\ell}{2} \sum_{i=1}^{\ell} m_i - \frac{1}{2}$.

**Proof.** The simple case when there are no units in the under-diagonal blocks follows by Lemma 3.2 because all under-diagonal blocks are zero matrices.

The proof is by induction on $m_1 + \ldots + m_\ell$. The base follows by the above simple case. Let us prove the inductive step in the case when there is a unit in the union of under-diagonal blocks. Denote by $D$ the given matrix. Arrange the rows and the columns of $D$ lexicographically. Let $(i, a)$ be the lexicographically maximal (i.e. the lowest) row whose intersection with the union of under-diagonal blocks of $D$ is non-zero. Let $(j, b)$ be the lexicographically minimal (i.e. the leftmost) column whose intersection with the row $(i, a)$ is non-zero. Formally, $(i, a)$ is the lexicographically maximal row such that there is

$$(j, b) \quad \text{for which } \quad i > j, \quad D_{(i,a)(j,b)} = 1, \quad \text{and } \quad D_{(i,a)(j',b')} = 0 \quad \text{for all } \quad (j', b') < (j, b).$$

(Note that by the choice of the row, $D_{(i',a')(j,b)} = 0$ for all $(i', a') > (i, a)$.)

Let $D'$ be the matrix obtained from $D$ by adding the row $(i, a)$ to all other rows whose intersection with the column $(j, b)$ is non-zero. Let $D''$ be the matrix obtained from $D'$

\[ \text{lexicographically smaller than } (y, b) \text{ if either } x = y \text{ and } a < b, \text{ or } x < y. \]
by adding the column \((j, b)\) to all other columns whose intersection with the row \((i, a)\) is non-zero. In \(D'\) the union of the row \((i, a)\) and the column \((j, b)\) contains only one unit, located at the intersection of these row and column.

Let \(D''\) be the matrix obtained from \(D'\) by removing the rows \((i, a)\), \((j, b)\), and the columns \((i, a)\), \((j, b)\). Then \(D'' \in \mathbb{Z}_2^{[n_1-1] \times [n_2-1]}\) for \(n_i = m_i - 1\), \(n_j = m_j - 1\), and \(n_s = m_s\) for \(s \not\in \{i, j\}\), and \(D''\) is a tournament-like diagonal-like matrix. Then by induction hypothesis
\[
\text{rk } D \geq \text{rk } D'' + 1 \geq \sum_{s=1}^{\ell} \frac{n_s - 1}{2} + 1 = \sum_{s=1}^{\ell} \frac{m_s - 1}{2}.
\]

Recall that
\[
\overline{a} = \{1, a\} \quad \text{for } a > 1.
\]
For \(A \in \mathbb{Z}_2^{(\overline{a})^2 \times (\overline{a})^2}\) define the matrix
\[
B = B(A) \in \mathbb{Z}_2^{[n-1]^2 \times [n-1]^2} \quad \text{by} \quad B_{(i, a)(j, b)} := A_{i+1 \ast a+1, j+1 \ast b+1}
\]
(this notation helps to make a transition from all cycles of length 4 in \(K_{n,n}\) to cycles containing the edge \((1, 1')\), and from matrices whose rows and columns are numerated by such cycles to block matrices).

Recall that an \textit{inversed diagonal matrix} is the sum of some diagonal matrix and \(J_m\).

**Lemma 3.4 (Independence).** Suppose that \(n \geq 4\) and \(A \in \mathbb{Z}_2^{(\overline{a})^2 \times (\overline{a})^2}\) is independent and additive. Take \(B = B(A)\). Then for any pairwise distinct \(i, j, s \in [n-1]\) the residue \(B_{(i, a)(j, b)} + B_{(s, a)(j, b)}\) does not depend on distinct \(a, b \in [n-1]\). In other words, the sum \(B_{i,j} + B_{s,j}\) of blocks is either a diagonal matrix or an inversed diagonal matrix.

**Proof.** For fixed \(i, j, s\) denote
\[
P(a) := i + 1 \ast a + 1 \oplus s + 1 \ast a + 1 = \{i + 1, s + 1\} \ast a + 1
\]
and
\[
Q(b) := j + 1 \ast b + 1.
\]
By the additivity,
\[
B_{(i, a)(j, b)} + B_{(s, a)(j, b)} = A_{P(a), Q(b)}.
\]
The residue \(B_{(i, a)(j, b)} + B_{(s, a)(j, b)}\) does not depend on \(a\) since for any \(a' \in [n-1]\) distinct from \(a\) and \(b\)
\[
A_{P(a), Q(b)} = A_{P(a'), Q(b)}
\]
by Proposition 2.2 (One-coordinate swap). Analogously the residue does not depend on \(b\).

Now, the residue does not depend on both \(a\) and \(b\) since \(n - 1 \geq 3\). \(\square\)

**Lemma 3.5 (Pre-tournament-like).** Suppose that \(n \geq 4\) and \(A\) is an \((n, 1)\)-matrix. Then for any \(j > 1\) the off-diagonal block \(B(A)_{1,j}\) is a tournament matrix.

**Proof.** Denote \(B := B(A)\). First, let us show that \(B_{1,2}\) is a tournament matrix. Take the matrix \(Z := A_{\overline{1}\overline{1}} + A_{\overline{x}\overline{y}}\), which is an \((n, 0)\)-matrix by Lemma 2.1 (Heredity). By the symmetry of \(A\) and \(Z\) it suffices to check that \(Z_{\overline{x,y}} = 1\) for any numbers \(1 < x < y\). This follows since
\[
Z_{\overline{x,y}} = Z_{\overline{2,y}} = Z_{\overline{2,3}} = 1,
\]
where
each of the first and the second equalities is either a tautology or holds by Proposition 2.2 (One-coordinate swap),

• the last equality is the non-triviality of $Z$.

Now for any $j > 2$ the matrix $B_{1,2} + B_{1,j}$ is either a diagonal matrix or an inversed diagonal matrix, by the symmetry of $B$ and Lemma 3.4. Then $B_{1,j}$ is a tournament matrix. □

Proof of Theorem 1.5 for $k = 1$. Take $B = B(A)$. Define the matrix $C$ whose rows are numerated by $[n - 2] \times [n - 1]$, by $C_{i,j} := B_{i+1,j+1} + B_{1,j+1}$, i.e. $C$ is obtained from $B$ by row addition and taking submatrix. By Lemma 3.4 (Independence), for every $i \neq j$ the block $C_{i,j}$ is either a diagonal matrix or an inversed diagonal matrix. Thus the following formula defines the matrix $D$ of the same block structure as $C$: for $i, j \in [n - 2]$

$$D_{i,j} := \begin{cases} C_{i,j}, & \text{if either } i \leq j \text{ or } C_{i,j} \text{ is a diagonal matrix;} \\ C_{i,j} + J_{n-1}, & \text{if both } i > j \text{ and } C_{i,j} \text{ is an inversed diagonal matrix.} \end{cases}$$

Now the theorem follows since

$$\text{rk } A \geq \text{rk } B \geq \text{rk } C \geq \text{rk } D - \text{rk } (C + D) \geq \frac{(n - 2)^2}{2} - (n - 3) \geq \frac{(n - 3)^2}{2}.$$ 

Here

• the first and second inequalities follow by definition of $B$ and $C$, respectively;

• the third inequality holds by subadditivity of rank; 

• the last inequality is obvious.

The fourth inequality holds by Lemma 3.3 (Diagonal-Tournament) applied to $D$, $\ell = n - 2$ and $m_1 = \ldots = m_\ell = n - 1$, and Lemma 3.1 applied to $D + C$, $\ell = n - 2$ and $m = n - 1$. It is obvious that the hypotheses of Lemma 3.1 are fulfilled for $D + C$. It remains to prove that the hypotheses of Lemma 3.3 (Diagonal-Tournament) are fulfilled for $D$.

Clearly, $D$ is diagonal-like. We prove that $D$ is tournament-like as follows. Take any $s \in [n - 2]$. By Lemma 3.5 for $j = s + 1$, the block $B_{1,s+1}$ is a tournament matrix. Since $A$ is symmetric, $B_{s+1,s+1}$ is symmetric. Then $D_{s,s} = C_{s,s} = B_{s+1,s+1} + B_{1,s+1}$ is a tournament matrix. □

Remark 3.6 (On generalization of linear algebraic properties). During the work with similar linear algebraic properties of $(n,k)$-matrices (e.g. Proposition 2.2, Lemma 3.4) we got the impression that probably there may be another simple property, which can help with the estimate $c_k n^{k+1}$. However, we did not succeed in obtaining it.

4. Appendix: Simplicial Embeddings and Non-Triviality

Remark 4.1. For a complex $K$ denote by $f_j = f_j(K)$ the number of $j$-faces. Denote $\gamma = \gamma(K) := f_k - (k + 2)f_{k-1}$.

(a) Conjecture.\footnote{In comments on the proof of this conjecture presented in [Ad18, Remark 4.9] it is not explained how to get rid of the condition that $M$ embeds into $\mathbb{R}^{2k+1}$. This condition is present in [Ad18, Corollary 4.8], so presumably is used in the proof, so presumably has to be used in the proof of this conjecture.} For any subcomplex $K$ of any triangulation of any $2k$-manifold $M$ we have

$$\binom{2k+1}{k+1} b_k(M) \geq \gamma(K).$$
(b) **Deduction of the asymptotic version of Conjecture 1.3.d from Conjecture (a).** The asymptotic version states that
\[
\binom{2k + 1}{k + 1} b_k(M) \gtrsim \frac{n^{k+1}}{(k+1)!} \quad \text{as} \quad n \to \infty \quad \text{for fixed} \quad k \geq 1.
\]

This follows because by (a)
\[
\binom{2k + 1}{k + 1} b_k(M) \geq \gamma(\Delta_n^k) = \binom{n+1}{k+1} - (k+2) \binom{n+1}{k} \sim \frac{n^{k+1}}{(k+1)!}.
\]

(c) **Conjecture 1.3.d** (even with a typo \(n \to n + 1\) from [Ad18, Remark 4.9]) does not follow from Conjecture (a) by the argument of (b). Indeed,
\[
\binom{n}{k+1} - (k+2) \binom{n}{k} < \binom{n+1}{k+1} - (k+2) \binom{n+1}{k} < \binom{n-k-1}{k}
\]
for \(n\) large comparative to \(k\). Here the first inequality is obvious; let us prove the second one. Let \(N := n + 1\). Both parts of the equivalent inequality
\[
N(N-1) \ldots (N-k+1)(N-k-(k+1)(k+2)) < (N-k-2)(N-k-2) \ldots (N-2k-2)
\]
are unitary polynomials in \(N\) of degree \(k + 1\). For the coefficients of \(N^k\) we have
\[
1 + \ldots + (k-1) + k + (k+1)(k+2) = \frac{k(k+1)}{2} + (k+1)(k+2) = (k+1)\frac{(k+2) + (2k+2)}{2} = (k+2) + (k+3) + \ldots + (2k+2).
\]

For the coefficients of \(N^{k-1}\) we have
\[
\sum_{i,j=1}^{k} i j + \sum_{i=1}^{k-1} i(k+1)(k+2) = \frac{k^2(k+1)^2}{4} + \frac{k(k-1)(k+1)(k+2)}{2} = \frac{k+1}{4} (3k^3 - k^2 - 2k) < \frac{k+1}{4} 9k^3 < \frac{(k+1)^2(k+2+2k+2)}{4} = \sum_{i,j=k+2}^{2k+2} i j. \quad \square
\]

(d) The asymptotic version (analogous to (b)) of the Kühnel conjecture on embeddings (Remark 1.3.a) follows (analogously to the argument of (b)) from Conjecture (a) and the inequality \(\gamma(K) \geq \gamma(\Delta_n^k)\) for any subdivision \(K\) of \(\Delta_n^k\). However, this inequality is not clear.\(^{14}\)

Now we discuss the non-triviality.

Recall that for a general position map \(g: [3]^{\ast k+1} \to \mathbb{R}^{2k}\) the van Kampen number \(v(g) \in \mathbb{Z}_2\) is the parity of the number of all unordered pairs \(\{\sigma, \tau\}\) of vertex-disjoint \(k\)-faces of \([3]^{\ast k+1}\) such that \(|g\sigma \cap g\tau|_2 = 1\).

**Lemma 4.2** (van Kampen; [vK32, Satz 5]). **For any general position map** \(g: [3]^{\ast k+1} \to \mathbb{R}^{2k}\) **we have** \(v(g) = 1\).

---

\(^{14}\)The \(k\)- and \((k-1)\)-skeleta of \(K\) are larger than those of \(\Delta_n^k\), so we only have \(f_k(K) \geq f_k(\Delta_n^k)\) and \(f_{k-1}(K) \geq f_{k-1}(\Delta_n^k)\), which does not imply the inequality from (d). The inequality from (d) is clear when \(K\) is obtained from \(\Delta_n^k\) by subdivision of an edge, but is not clear for subsequent subdivisions.
The following proof, except the last paragraph, is alternative to known proofs.\textsuperscript{15}

Proof of Lemma 4.2. Let \( \gamma(t) = (t, \ldots, t^2k) \) be the moment curve in \( \mathbb{R}^{2k} \). Let \( g: [3]^{k+1} \rightarrow \mathbb{R}^{2k} \) be the linear map such that
\[
g(\mathcal{O}^i * a * \mathcal{O}^{k-i}) = \gamma(a + 3i) \quad \text{for every} \quad i \in \{0, 1, \ldots, k\} \quad \text{and} \quad a \in [3].
\]
It is well known that every at most \( 2k + 1 \) points on \( \gamma \) are affine independent (for proof see e.g. [St24, Lemma 5]). Then \( g \) is a general position map. In the following paragraph we prove that \( v(g) = 1 \).

It is known that for vertex-disjoint \( k \)-faces \( \sigma = \sigma_1 * \ldots * \sigma_{k+1} \) and \( \tau = \tau_1 * \ldots * \tau_{k+1} \) their images \( g\sigma \) and \( g\tau \) intersect (at a single point) if and only if the vertices of the images alternate along the moment curve\textsuperscript{16}. The alternation means that either
\[
\sigma_1 < \tau_1 < 3 + \sigma_2 < 3 + \tau_2 < \ldots < 3k + \sigma_{k+1} < 3k + \tau_{k+1} \quad \text{or}
\]
\[
\tau_1 < \sigma_1 < 3 + \tau_2 < 3 + \sigma_2 < \ldots < 3k + \tau_{k+1} < 3k + \sigma_{k+1}.
\]
The alternation is equivalent to `either \( \sigma_i < \tau_i \) for every \( i \in [k+1] \) or \( \sigma_i > \tau_i \) for every \( i \in [k+1] \)`.

Then
\[
v(g) = \left| \left\{ (\sigma_1, \ldots, \sigma_{k+1}, \tau_1, \ldots, \tau_{k+1}) \in [3]^{2k+2} : \sigma_i < \tau_i \text{ for every } i \in [k+1] \right\} \right|_2 = 1.
\]
Here the last equality is proved as follows. For every \((2k+2)\)-tuples \((\sigma_1, \ldots, \sigma_{k+1}, \tau_1, \ldots, \tau_{k+1})\) every pair \((\sigma_i, \tau_i)\) is either \((1, 2)\), or \((1, 3)\), or \((2, 3)\). Then by the Cartesian product rule the number of such \((2k+2)\)-tuples is \(3^{k+1} \equiv 1 \pmod{2}\).

Proof that \( v(g) = v(g') \) for any general position maps \( g, g': [3]^{k+1} \rightarrow \mathbb{R}^{2k} \), repeats the fairly standard part in the last paragraph of the proof of Lemma 1.7 (Non-triviality). \(\square\)

Recall that
\begin{itemize}
  \item \( H \) is the set of all unordered pairs of vertex-disjoint \( k \)-faces of \( [3]^{k+1} \);
  \item \( G_k \) is the set of unordered pairs of \( k \)-octahedra from \( [3]^{k+1} \) whose intersection is \( 1^{k+1} \);
  \item for \( \{P, Q\} \in G_k \) we denote by \( T\{P, Q\} \) the set of pairs \( \{\alpha, \beta\} \) formed by (not necessary distinct) \( k \)-faces \( \alpha, \beta \) of \( [3]^{k+1} \) such that either \( \alpha \in P \) and \( \beta \in Q \), or vice versa.
\end{itemize}

Remark 4.3 (Deduction of Theorem 1.8 (Almost Embeddability) from [KS21e]).

(a) [KS21e, Lemma 2.3.1 and footnote 4] If a \((k - 1)\)-connected \( k \)-complex \( K \) is almost embeddable to a \( 2k \)-manifold \( M \), then there are a collection of \( k \)-cycles \( y_\sigma \) in \( M \), parametrized by \( \sigma \)-faces of \( K \), and a general position map \( g: K \rightarrow \mathbb{R}^{2k} \) such that
\[
y_\sigma \cap_M y_\tau = \left| g\sigma \cap g\tau \right|_2 \quad \text{for any vertex-disjoint \( \sigma, \tau \)-faces of \( K \)).}
\]

(b) The deduction. Take \( y_\sigma \) and \( g \) from (a). For \( k \)-octahedra \( P, Q \) define
\[
A_{P,Q} := \sum_{(\sigma, \tau) \in P \times Q} y_\sigma \cap_M y_\tau.
\]
Take a basis in \( H_k(M; \mathbb{Z}_2) \) in which the matrix of \( \cap_M \) is \( \Omega_M \). Let \( Y \) be the \( \beta_k(M) \times \binom{n}{2}^{k+1} \)-matrix whose columns are coordinates of \( y_\sigma \) in this

\textsuperscript{15}The first part of the proof (namely, everything except the last paragraph) is construction of some map \( g: [3]^{k+1} \rightarrow \mathbb{R}^{2k} \) such that \( v(g) = 1 \). In the original proof van Kampen constructed another such \( g \), splitting vertices on groups of three and placing the groups in different hyperplanes. Another construction of such a map \( g \) is given in [Me06, the second paragraph of Example 3.5]: take \( n = k \) and \( n_i = 0 \) for \( i \in [k+1] \); take \( +\Delta n^i = 1^{k+1} \) and \( +\partial\Delta n_i^{k+1} = [2]^{k+1} \).

\textsuperscript{16}For clear exposition see [St24, Lemma 6]; for an earlier reference see [Br73, Theorem] (in [Br73] the statement of the Theorem has undefined \( A \) and \( B \), and uses the term ‘the primitive Radon partition’ defined elsewhere).
basis. Then $A = Y^T \Omega_M Y$. So it remains to prove that $A$ is an $(n, k)$-matrix. (Cf. [SS23, §2, proof of the ‘if’ part of Theorem 1.4].)

The additivity is obvious.

The independence holds since for vertex-disjoint $k$-octahedra $P, Q$ we have

$$A_{P,Q} = \sum_{(\sigma, \tau) \in P \times Q} y_\sigma \cap_M y_\tau = |g_P \cap g_Q|_2 = 0,$$

where the last equality holds by the parity lemma (see e.g. [Sk, Lemma 5.3.4]).

The non-triviality holds since

$$SA = \sum_{\{P, Q\} \in G_k} A_{P, Q} = \sum_{\{P, Q\} \in G_k} \sum_{\{\alpha, \beta\} \in T\{P, Q\}} y_\alpha \cap_M y_\beta = \sum_{\{\alpha, \beta\} \in H} y_\alpha \cap_M y_\beta = \sum_{\{\alpha, \beta\} \in H} |g_\alpha \cap g_\beta|_2 = 1,$$

where the third equality holds by Lemma 2.3 (Combinatorial), and the last equality is the result of van Kampen [vK32, Satz 5], see Lemma 4.2 (van Kampen).

(c) Lemma 1.7 (for almost embeddings) is deduced from (a) with the following addendum (which is essentially obtained in [KS21e, §2.3]): if $P$ and $Q$ are $k$-cycles in $K$ and $f: K \to M$ is an almost embedding, then $A(f)_{P,Q} = \sum_{(\sigma, \tau) \in P \times Q} y_\sigma \cap_M y_\tau$.

**Remark 4.4** (On alternative definition of non-triviality). Under the assumptions that $A$ is symmetric, independent and additive, the following is an equivalent definition of the non-triviality (the equivalence is clear from Lemma 3.5). The matrix $A$ is said to be non-trivial if for any complex $K \subset [n]^{*k+1}$ isomorphic to $[3]^{*k+1}$ and any $k$-face $\alpha \subset K$ the sum $S_{\alpha, K} A = 1$, where $S_{\alpha, K} A$ is the sum of $A_{P, Q}$ over all unordered pairs $\{P, Q\}$ of $k$-octahedra in $[3]^{*k+1}$ such that $P \cap Q = \alpha$.

The analogue of Lemma 1.7 (Non-triviality) for the new definition is correct because all the proof presented work for any subcomplex of $[n]^{*k+1}$ isomorphic to $[3]^{*k+1}$ and any its $k$-face.

Below we give an alternative proof of Lemma 1.7 (Non-triviality), which we did not succeed to generalize to $\mathbb{Z}_2$-embeddings.

**Proposition 4.5** (Intersection formula). For any embedding $f: [3]^{*k+1} \to M$ into a $2k$-manifold $M$ there is a general position map $g: [3]^{*k+1} \to \mathbb{R}^{2k}$ such that for any $k$-octahedra $P, Q$ whose intersection is $1^{*k+1}$

$$f_P \cap_M f_Q = \sum_{\{\alpha, \beta\} \in T\{P, Q\}} |g(\alpha - \beta) \cap g(\beta - \alpha)|_2.$$

**Proof.** Recall that $I = [0, 1] \subset \mathbb{R}$.

By ambient isotopy of $M$ we may assume that the image of $f$ is in the interior of $M$. Since $f$ is an embedding and since $1^{*k+1} \cong I^k$ is collapsible, by [RS72, Corollary 3.27] there is an embedding $i: I^{2k} \to M$ in general position to $f$, and such that

$$i I^{2k} \supset f(1^{*k+1}) \quad \text{and} \quad i I^{2k} \cap f(\{2, 3\}^{*k+1}) = \emptyset.$$

Take any general position map

$$g: [3]^{*k+1} \to \mathbb{R}^{2k} \quad \text{such that} \quad f^{-1}(i I^{2k}) = g^{-1}(I^{2k}) =: Z \quad \text{and} \quad f|_Z = ig|_Z.$$
Here the property \( f|_Z = ig|_Z \) means that \( f(x) = ig(x) \) when \( f(x) \in iI^{2k} \) (or, equivalently, when \( g(x) \in I^{2k} \)).

Take any general position map \( g' : [3]^{*k+1} \to \mathbb{R}^{2k} \) such that

\[
g|_{g^{-1}(\mathbb{R}^{2k} - I^{2k})} = g'|_{g'^{-1}(\mathbb{R}^{2k} - I^{2k})}
\]

(which means that \( g^{-1}(\mathbb{R}^{2k} - I^{2k}) = g'^{-1}(\mathbb{R}^{2k} - I^{2k}) \), and the restrictions of \( g \) and \( g' \) to \( g^{-1}(\mathbb{R}^{2k} - I^{2k}) \) coincide), and

\[\text{(CGP) } g \text{ and } g' \text{ are close, and } (g \sqcup g')|_{g^{-1}(I^{2k})} \text{ is a general position map.}\]

Then for any \( k \)-octahedra \( P, Q \) whose intersection is \( 1^{*k+1} \)

\[
fP \cap_M fQ \overset{(4.5.a)}{=} \left|(igP \cap iI^{2k}) \cap (ig'Q \cap iI^{2k})\right|_2 \overset{(4.5.b)}{=} \left|(gP \cap I^{2k}) \cap (g'Q \cap I^{2k})\right|_2 \overset{(4.5.c)}{=} \left|(gP - I^{2k}) \cap (g'Q - I^{2k})\right|_2 \overset{(4.5.d)}{=} \sum_{\{\alpha,\beta\} \in T\{P, Q\}} \left|g(\alpha - \beta) \cap g(\beta - \alpha)\right|_2.
\]

Let us prove the equalities.

- Equality (4.5.a) is proved as follows. Since \( P \cap Q = 1^{*k+1} \), we have \( fP \cap fQ \subset iI^{2k} \). This, \( f|_Z = ig|_Z \), and (CGP), imply (4.5.a).
- Equality (4.5.b) holds since \( i \) is an embedding.
- Equality (4.5.c) holds by the parity lemma (see e.g. [Sk, Lemma 5.3.4]) and since \( g|_P \sqcup g'|_Q \) is a general position map. On \( g^{-1}(I^{2k}) \) this map is in general position by (CGP). On \( g^{-1}(\mathbb{R}^{2k} - I^{2k}) \) this map is in general position since
  - \( g|_{g^{-1}(\mathbb{R}^{2k} - I^{2k})} = g'|_{g'^{-1}(\mathbb{R}^{2k} - I^{2k})}, \)
  - \( g \) is a general position map,
  - \( P \cap Q = 1^{*k+1}, \)
  - \( iI^{2k} \supset f\{1^{*k+1}\} \text{ and } iI^{2k} \cap f\{2, 3\}^{*k+1} = \emptyset \), and
  - \( f|_Z = ig|_Z.\)
- Equality (4.5.d) holds since \( g|_{g^{-1}(\mathbb{R}^{2k} - I^{2k})} = g'|_{g'^{-1}(\mathbb{R}^{2k} - I^{2k})}. \)
- Equality (4.5.e) is proved as follows. For \( \{\alpha, \beta\} \in T\{P, Q\} \) we have \( \alpha \cap \beta \subset 1^{*k+1}. \) This, \( iI^{2k} \supset f\{1^{*k+1}\}, iI^{2k} \cap f\{2, 3\}^{*k+1} = \emptyset \) and \( f|_Z = ig|_Z \) imply

\[
(gP - I^{2k}) \cap (gQ - I^{2k}) = \bigcup_{\{\alpha, \beta\} \in T\{P, Q\}} g(\alpha - \beta) \cap g(\beta - \alpha).
\]

\[\square\]

**Proof of Lemma 1.7 (Non-triviality).** Take a map \( g : [3]^{*k+1} \to \mathbb{R}^{2k} \) from Proposition 4.5. Then

\[
SA(f) \overset{(1.7.a)}{=} \sum_{\{P, Q\} \in G_k} fP \cap_M fQ \overset{(1.7.b)}{=} \sum_{\{P, Q\} \in G_k} \sum_{\{\alpha, \beta\} \in T\{P, Q\}} \left|g(\alpha - \beta) \cap g(\beta - \alpha)\right|_2 \overset{(1.7.c)}{=} \sum_{\{\alpha, \beta\} \in H} \left|g(\alpha - \beta) \cap g(\beta - \alpha)\right|_2 \overset{(1.7.d)}{=} \sum_{\{\alpha, \beta\} \in H} |g\alpha \cap g\beta|_2 \overset{(1.7.e)}{=} 1.
\]

Here

- equality (1.7.a) is the definition of \( SA(f) \);
- equality (1.7.b) holds by Proposition 4.5;
- equality (1.7.c) holds by Lemma 2.3 (Combinatorial);
• equality (1.7.d) holds since \( \alpha, \beta \) are vertex-disjoint;
• equality (1.7.e) is the result of van Kampen [vK32, Satz 5] (see Lemma 4.2).

\[ \square \]

**Remark 4.6** (Relation to intrinsic linking results). (a) Non-embeddability of \( \Delta_{2k+2}^k \) into \( \mathbb{R}^{2k} \) is related to a congruence analogous to \( SA(f) = 1 \) [PT19, Proposition 16.ii], [KS21, §1, non-triviality], and to an intrinsic linking result for \( (k - 1) \)-complexes in \( \mathbb{R}^{2k-1} \) [KS21, Theorem 1.6.odd] (see also [KS21, proof of Lemma 1.5]). Non-embeddability of \( K_5^k \) into \( \mathbb{R}^k \) is related to an intrinsic linking result for \( (k - 1) \)-complexes in \( \mathbb{R}^{2k-1} \) [Sk03]. Analogously, the proof of (well-known) non-embeddability of \( [3]^k+1 \) in \( \mathbb{R}^k \) given by Lemma 4.2 (van Kampen) is related to the congruence \( SA(f) = 1 \), and to certain intrinsic linking result for a \( (k - 1) \)-complex in \( \mathbb{R}^{2k-1} \). For \( k = 2 \) this is a result on a graph in \( \mathbb{R}^3 \); see (b,c,d) below.

(b) Let \( G \) be the graph with the vertex set \( \mathbb{Z}_4 \times \mathbb{Z}_3 \) and edges joining the following pairs of vertices:

\[ (i, j)(i + 1, j) \quad \text{and, for} \quad i = 0, 2, \quad (i + 1, 0)(i + 1, 1), \quad (i, 1)(i, 2), \quad (i + 1, 2)(i, 0). \]

Let \( S_j \) be the induced subgraph on vertices \( (i, j), i \in \mathbb{Z}_4 \). Clearly, \( S_j \) is a cycle of length 4.

An *octahedral cycle of length 6* in \( G \) is any of the following 8 cycles for \( i = 0, 2 \) and \( \varepsilon_1, \varepsilon_2 = \pm 1 \):

\[ (i, 0) (i + \varepsilon_1, 0) (i + \varepsilon_1, 1) (i + \varepsilon_1 + \varepsilon_2, 1) (i + \varepsilon_1 + \varepsilon_2, 2) (i + 1, 2). \]

The two cycles for fixed \( \varepsilon_1, \varepsilon_2 \) and different \( i \) are called **involutive**. Clearly, involutive cycles are disjoint.

Take the (octahedral) cycles

\[ T_1 := (0, 0)(1, 2)(2, 2)(3, 2)(2, 0)(3, 0), \]
\[ T_2 := (3, 1)(2, 1)(1, 1)(0, 0)(2, 0)(3, 0), \]
\[ T_3 := (2, 2)(3, 2)(0, 2)(0, 1)(3, 1)(2, 1). \]

Clearly, \( T_j \cap S_j = \emptyset \).

(c) **Assertion.** [Ni] Suppose that \( f : G \to \mathbb{R}^3 \) is an embedding. Then the sum of the three pairwise linking numbers of \( f(S_j) \) and \( f(T_j) \), \( j = 1, 2, 3 \), and the four linking numbers of involutive octahedral cycles, is odd.

This follows from [Sa81]. Indeed, in the graph \( G \), by contracting three edges \( (0, 0)(1, 2), \)
\( (0, 2)(0, 1) \) and \( (1, 1)(1, 0) \), we obtain a proper minor of \( G \) isomorphic to the graph \( G_9 \) in the Petersen family [Sa81]. (Further, certain \( \Delta Y \)-move yields the Petersen graph \( P_{10} \).) All disjoint cycle pairs of \( G_9 \) consist of six \( (4, 5) \)-cycle pairs and exactly one \( (3, 6) \)-cycle pair. It is known that for every embedding \( G_9 \to \mathbb{R}^3 \) the sum of the linking numbers over all of the constituent 2-component links is odd [Sa81]. Three of the six \( (4, 5) \)-cycle pairs and exactly one \( (3, 6) \)-cycle pair of \( G_9 \) correspond to the four involutive octahedral cycle pairs of \( G \). The three remaining \( (4, 5) \)-cycle pairs of \( G_9 \) correspond to pairs \( S_j, T_j \) of \( G \). Thus the assertion follows.

(d) The following holds both by (c) (see (e)) and by our proof of Lemma 1.7 (see (f)). Suppose that \( f : G \to \mathbb{R}^3 \) is an embedding such that the images \( f(S_1), f(S_2), f(S_3) \) lie in pairwise disjoint 3-balls, and the image of any edge outside \( S_1 \cup S_2 \cup S_3 \) is disjoint with one of the three balls. Then the sum of the four linking coefficients of involutive octahedral cycles is odd.

(e) [Ni] Denote by \( B_j \supseteq f(S_j) \) the mutually disjoint 3-balls. Then for \( f(S_1 \cup T_1) \), two edges \( f((3, 2)(2, 2)) \) and \( f((0, 0)(1, 2)) \) miss \( B_1 \), and the other edges of \( f(T_1) \) also miss \( B_1 \) because they are contained in \( B_2 \) and \( B_3 \). Thus the linking number of \( f(S_1) \) and \( f(T_1) \) is
zero. Analogously the linking number of \(f(S_j)\) and \(f(T_j)\) is zero for each \(j = 2, 3\). This implies (d).

(f) Take the following representation of \(G\). Vertices of \(G\) correspond to edges \(a \ast b \ast c\) of \([3]^3\), where among \(a, b, c\) there is exactly one ‘3’, and there is exactly one \(\emptyset\). We denote such a vertex by \(abc\). Edges of \(G\) correspond to faces \(3 \ast b \ast c\) or \(3 \ast 3 \ast c\) of \([3]^3\), and three times as many symmetric faces (i.e. faces obtained by changing the place of ‘3’s). So edges are \(3\emptyset, 3\emptyset c\) (short edges), \(3\emptyset c, 3\emptyset c\) (long edges), and symmetric edges.

There are three cycles \(S_1, S_2, S_3\) of length four obtained by changing the place of ‘3’ from the cycle \(32\emptyset, 3\emptyset 2, 31\emptyset, 3\emptyset 1\) of short edges. For each \(c \in \{2\}\) there are three edges obtained by changing the place of \(c\) in the long edge \(3\emptyset c, 3\emptyset c\).

An octahedral cycle of length 6 is the cycle \(3b\emptyset, 3\emptyset c, 3\emptyset 3, a\emptyset 3, a\emptyset 3, \emptyset b 3\), where \(a, b, c \in \{2\}\). This is \(*(a \ast b \ast c) \cap g^{-1}(\partial I^{2k})\). Take the involution on \(G\) defined by interchanging 1 and 2. Then the eight octahedral cycles split into pairs of involutive cycles, and involutive octahedral cycles are disjoint.

Now (d) follows by transforming the left-hand side of (4.5,e) analogously to [KS21, (2) in the proof of Lemma 1.5].

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