SHIFTS OF A MEASURABLE FUNCTION AND CRITERION OF 
\(p\)-INTEGRABILITY

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Abstract. It is shown that two conditions \(f(a + \cdot) - f(\cdot) \in L^p(\mathbb{R})\), and 
\((\sin b)f(\cdot) \in L^p(\mathbb{R})\) guarantee \(f \in L^p(\mathbb{R})\), \(1 \leq p < \infty\), if and only if \(ab\) is not in \((\pi \mathbb{Z})\).

Когда б вы знали, из какого сора
Растут стихи, не ведая стыда,
Как желтый одуванчик у забора,
Как лопухи и лебеда.
А. Ахматова, 'Тайны ремесла'

1. Let a measurable function \(f\) on \(\mathbb{R} = (-\infty, \infty)\) have properties

\((1a)\) \(\forall t \in \mathbb{R}, \quad f(t + \cdot) - f(\cdot) \in L^2(\mathbb{R})\),

and

\((1b)\) \(\forall s \in \mathbb{R}, \quad \sin(s \cdot)f(\cdot) \in L^2(\mathbb{R})\).

If a Fourier transform \(\tilde{f}\) is reasonably defined then \((1b)\) is equivalent to \((1a)\) for \(\tilde{f}\).

Claim 1. Under conditions \((1a), (1b)\) we have \(f \in L^2(\mathbb{R})\).

Recently, A. M. Vershik brought attention of the 25th St. Petersburg Summer Meeting in Mathematical Analysis, June 25 – 30, 2016, to Claim 1. He recalled that the known proof “was done in terms of representation theory (of Heisenberg group) many years ago” but noted that “the simple proof still does not exist” and after many years it is important “to give a simple and direct proof.” A stronger form of Claim 1 and its elementary proof was given just during the Meeting’s session of A. Vershik’s talk on June 30. It is presented in Section 2. If the reader wants a proof only of Claim 1 there is no need to go beyond Section 2.

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\(^1\)English translation is given in Section 10
\(^2\)The presentation gives a more extended motivation and links to the uncertainty principle although no reference to a published source is given.
Claim 2. Let \( f \) be a measurable function on \( \mathbb{R} \), and the following two conditions hold:

\[
\Delta(x) = f \left( x + \frac{\pi}{2} \right) - f(x) \in L^2(\mathbb{R}).
\]

Then \( f \in L^2(\mathbb{R}) \).

Proof. Put \( E = \{ x : |x - k\pi| \leq 10^{-6} \text{ for some } k \in \mathbb{Z} \} \). Then \( \frac{1}{\sin x} \leq \frac{1}{\sin \delta} \leq 10^7 \) on \( E^c = \mathbb{R} \setminus E \), \( \delta = 10^{-6} \), so

\[
|f| E^c = (\sin x \cdot f(x)) \cdot \frac{1}{\sin x} \in L^2(E^c).
\]

With \( 2\delta < \frac{\pi}{2} \) we have \( E + \frac{\pi}{2} \subset E^c \) and \( f(x) = f \left( x + \frac{\pi}{2} \right) - \Delta(x) \) for \( x \in E \); therefore \( ||f|| \leq ||f|E^c|| + ||\Delta|| < \infty \), and together with (3) and (2a) we have \( f \in L^2(\mathbb{R}) \). \( \square \)

3.

Section 2 is an almost stenographic recording of what I have said at the Meeting’s June 30 session. Now we will talk about a more general setting (sorry, some repetition is unavoidable) and get negative results (Proposition 5 and Example 8) as well. Of course, \( L^2 \)-norm is not special in our analysis in Section 2. Instead of \( L^2 \) we can talk about any Banach space \( X \) of measurable functions on \( \mathbb{R} \) with two properties:

\[
\begin{align*}
(4a) & \quad g \in X \Rightarrow g(\cdot + t) \in X, \quad t \in \mathbb{R} \\
(4b) & \quad g \in X \Rightarrow g \cdot h \in X, \quad \forall h \in L^\infty(\mathbb{R}).
\end{align*}
\]

Moreover, we do not need global conditions (1a), (1b); just a pair \( (t; s) = \left( \frac{\pi}{2}; 1 \right) \) with (2a), (2b) holding was good enough for the proof in Section 2. More general than Claim 2 is true:

Proposition 3. Let \( X \) be a Banach space of measurable functions on \( \mathbb{R} \) with properties (4a), (4b) and

\[
(4c) \quad g \in X \Rightarrow g(a \cdot) \in X, \quad \forall a \neq 0.
\]

Let \( (t, s) \) be two real non-zero numbers such that

\[
(5) \quad st \neq k\pi, k \in \mathbb{Z}.
\]

If a measurable function \( f \) on \( \mathbb{R} \) satisfies conditions

\[
\begin{align*}
(6a) & \quad \Delta_t(x) = f(t + x) - f(x) \in X, \quad \text{and} \\
(6b) & \quad (\sin sx) \cdot f(x) \in X,
\end{align*}
\]

then

\[
(7) \quad f \in X.
\]
Proof of Proposition 3. The assumption (4c) permits us to rescale a variable $x$ and go to $F(\xi) = f\left(\frac{\xi}{s}\right)$. It brings us to the pair $(T; 1) = (ts; 1)$ instead of $(t, s)$, so we need to prove that $F \in X$ under the assumptions

\[(8a) \quad F(\xi + T) - F(\xi) \in X\]

and

\[(8b) \quad \sin \xi \cdot F(\xi) \in X,\]

with

\[(9) \quad T \neq k\pi, \quad k \in \mathbb{Z}.\]

We can choose $\tau$ and $m$ such that

\[(10) \quad T = m\pi + \tau, \quad m \in \mathbb{Z}, \quad 0 < \tau < \pi,\]

and $\delta > 0$,

\[(11) \quad 2\delta \leq \tau \leq \pi - 2\delta.\]

Put $E = \bigcup_{k \in \mathbb{Z}} [k\pi - \delta; k\pi + \delta]$ and

\[(12) \quad h(\xi) = \begin{cases} 1 \sin(\xi), & \xi \in E^c = \mathbb{R} \setminus E, \\ 0, & \xi \in E \end{cases}\]

so $h \in L^\infty$ with $\|h\|_{L^\infty} \leq \frac{1}{\sin \delta}$. Then

\[(13) \quad F(\xi) \cdot \chi_{E^c} = \left[\sin \xi \cdot F(\xi)\right] \cdot h(\xi)\]

is in $X$ by (8b) and (11). (11) guarantees that $E + T = E + \tau \subset E^c$. Now we use (8a) and the identity $F(\xi) = F(\xi + T) - \Delta(\xi)$, with $\Delta(\xi) = F(\xi + T) - F(\xi)$ in $X$ by (8a) to conclude that both $F(\xi + T) \sin \xi = F(\xi) \sin \xi + \Delta(\xi) \sin \xi$ and its shift

\[(14) \quad F(\xi) \sin(\xi - T)\]

are in $X$. By (10), (11) $\min_{\xi \in E} |\sin(\xi - \tau)| = \min_{\xi \in E} |\sin(\xi - \tau)| \geq \sin \delta$. Put — compare (12) —

\[(12) \quad H(\xi) = \begin{cases} \frac{1}{\sin(\xi - T)}, & \xi \in E \\ 0, & \xi \in E^c \end{cases}\]

so $H \in L^\infty(\mathbb{R})$, $\|H\|_{L^\infty} \leq \frac{1}{\sin \delta}$. Therefore,

\[(15) \quad F(\xi) \cdot \chi_{E}(\xi) = \left[\sin(\xi - T)F(\xi)\right] \cdot H(\xi)\]

and by (14) and (8b) the function (15) is in $X$. Together with (13) this observation completes the proof. \qed
5.

This is worth to notice that the quantization condition (5) or (9) is crucial. Indeed, for a pair \((T, 1), T = \pi\), now we’ll construct a function \(f(x)\) such that

\[
\int |f(x)|^2 \, dx = \infty.
\]

\[
f(x) \cdot \sin x = g(x) \in L^2
\]

\[
f(x) - f(x + \pi) \in L^2
\]

It will be bad for any \(T = m\pi, m \in \mathbb{Z}\), of course. Put

\[
a_0 = \frac{1}{4}, \quad a_k = \frac{1}{5|k|}, \quad k \neq 0
\]

and

\[
f(x) = \begin{cases} 1, & x \in I_k, \quad k \in \mathbb{Z} \\ 0, & \text{otherwise.} \end{cases}
\]

Then by (17), (18)

\[
\int |f(x)|^2 \, dx = \sum_{k \in \mathbb{Z}} \int_{0}^{a_k} 1 \, dt = \infty,
\]

but with

\[
\sin x \leq x, \quad 0 \leq x \leq \frac{1}{4}, \quad \text{and } |\sin(x + j\pi)| = |\sin x| \forall j \in \mathbb{Z},
\]

\[
\int |g(x)|^2 \, dx = \sum_{k \in \mathbb{Z}} \int_{0}^{a_k} (\sin t)^2 \, dt
\]

\[
\leq \sum_{k \in \mathbb{Z}} \int_{0}^{a_k} t^2 \, dt = \frac{1}{3} \sum_{k \in \mathbb{Z}} a_k^3 < \infty.
\]

We still need to check (16c). First, we analyze the case \(x \geq 0, k \geq 0\).

On \(I_k \cap (I_{k+1} - \pi) = I_{k+1} - \pi\)

\[
f(x) - f(x + \pi) = 0
\]

but on \(I_k \setminus (I_{k+1} - \pi) = k\pi + [a_{k+1}, a_k]\)

\[
f(x) - f(x + \pi) = f(x) = 1,
\]

and equals 0 otherwise. If \(-x < 0, k < 0, (I_{k+1} - \pi) \cap I_k\) so

\[
f(x) - f(x + \pi) = \begin{cases} 0 & \text{on } I_k, \\ -1 & \text{on } (I_{k+1} - \pi) \setminus I_k = k\pi + [a_k, a_{k+1}], \\ 0 & \text{otherwise.} \end{cases}
\]

Therefore, by (17)

\[
\int |f(x) - f(x + \pi)|^2 \, dx \leq 2 \sum_{k \geq 0} \int_{a_{k+1}}^{a_k} 1 \, dt = 2a_0 = \frac{1}{2}.
\]

Remark 4. The same example, i.e., a function (18), is good to show that (16b), (16c), (16a) holds if \(L^2\) is changed to \(L^p(\mathbb{R}), 1 \leq p < \infty\).
The function \(18\) gives a counterexample for a pair \((\pi, 1)\), so it is good for any \((t, s) \in (\pi \mathbb{Z}) \times \mathbb{Z}\) if \(t \neq 0, s \neq 0\). Together with Proposition 3 this observation implies the following.

**Proposition 5.** Let \(X = \mathbb{L}^p(\mathbb{R}), 1 \leq p < \infty\). Then conditions (6a), (6b) imply (7) if and only if \(ts \notin \pi \mathbb{Z}\).

**Proof.** To complete the proof we need to give counterexamples when \(ts = 0\). If \(t = 0\) the condition (6a) trivially holds for any measurable \(f\), and (6b) is satisfied if we put

\[
f(x) = \begin{cases} 
  \frac{1}{x}, & 0 < x < 1 \\
  0, & \text{otherwise.}
\end{cases}
\]

But \(f\) is not in \(X = \mathbb{L}^p\). If \(s = 0\) put \(g(x) = (1 + |x|)^{-a}, \frac{1}{p} - 1 < a < \frac{1}{p}\), or just \(g(x) = 1\); then \((6a), (6b)\) hold for \(g\) but \(g\) is not in \(X = \mathbb{L}^p\). \(\square\)

6.

With some adjustments, we can give the multidimensional analogs of Propositions 3 and 5.

Let \(Y\) be a Banach space of measurable functions on \(\mathbb{R}^n\) with properties

(21a) \(g \in Y \Rightarrow g(x + t) \in Y, \forall t \in \mathbb{R}^n\)

(21b) \(g \in Y \Rightarrow g \cdot h \in Y, \forall h \in \mathbb{L}^\infty(\mathbb{R}^n)\)

(21c) \(g \in Y \Rightarrow g(Cx) \in Y\) for any \((n \times n)\)-matrix \(C, \det C \neq 0\).

**Definition 6.** We say that a pair \(A, B \subset \mathbb{R}^n\) of subsets is assertive in \(Y\) if two conditions on a measurable function \(F\) on \(\mathbb{R}^n\)

(22a) \(F(\cdot + t) - F(\cdot) \in Y, \forall t \in A,\)

(22b) \(F(x) \cdot (\sin(x, b)) \in Y, \forall b \in B\)

imply

(22c) \(F(x) \in Y\).

**Example 7.** Take two singletons \(A = \{a\}, B = \{b\}, a, b \in \mathbb{R}^n\). They are assertive if a scalar product \(t \equiv \langle a, b \rangle \neq \pi m, m \in \mathbb{Z}\). Indeed, with \(b \neq 0\) we can rescale by (21c) a variable \(x\) and assume

(23) \(b = e_1, a = te_1 + a', \langle a', e_1 \rangle = 0,\)

With \(\delta = \frac{1}{4} \tau, \tau = \min\{|t - k\pi| : k \in \mathbb{Z}\}\) define

\[E = \{x \in \mathbb{R}^n : |(\sin x_1)| < \delta\}\]

and notice that \(E + a \subseteq E^\delta\). We can repeat (with proper adjustment) the proof of Proposition 3.

If, however, \(t \equiv \langle a, b \rangle = m\pi, m \in \mathbb{Z}, A, B\) are not assertive in \(\mathbb{L}^p(\mathbb{R}^n), 1 \leq p < \infty\) – see Claim 3 below.
Example 8. Now take $A = \alpha \mathbb{Z}^n, B = \beta \mathbb{Z}^n$.

This pair is assertive in $L^p(\mathbb{R}^n), 1 \leq p < \infty$, if and only if $t = \alpha \cdot \beta \notin \pi \mathbb{Z}$.

Just two points $\alpha e_1 \in A, \beta e_1 \in B$ guarantee (after Example 7) that $A, B$ are assertive if $t \neq \pi m, m \in \mathbb{Z}$.

Now we will construct a bad function $F$ for $\alpha = \pi, \beta = 1$, i.e., such $F$ on $\mathbb{R}^n$ that

\begin{align}
(24a) & \int_{\mathbb{R}^n} |F(x)|^p \, dx = \infty, \\
(24b) & F(x) \cdot (\sin x_j) \in L^p(\mathbb{R}^n), 1 \leq j \leq n \\
(24c) & \int_{\mathbb{R}^n} |F(x) - F(x + \pi e_j)|^p \, dx < \infty, 1 \leq j \leq n
\end{align}

It will be bad for $A$ and $B$ if we will make two observations.

(24b) Conditions (24b) guarantee that $F(x) \cdot (\sin \langle b, x \rangle) \in L^p(\mathbb{R}^n)$ for any $b \in \mathbb{Z}^n$.

Indeed, by induction, one can explain that

$$\sin \langle b, x \rangle = \sum_{j=1}^n Q_j^b(x) \sin(x_j),$$

where $Q_j^b$ are trigonometric polynomials on $\mathbb{R}^n$ of period $2\pi$, i.e., algebraic polynomials of $\cos x_j, \sin x_j, 1 \leq j \leq n$. They are bounded, i.e., $Q_j^b(x) \in L^\infty(\mathbb{R}^n)$, so

$$(24b) \implies (24b').$$

Any $a \in A$ is a finite sum of vectors $\pm \pi e_j, 1 \leq j \leq n$, so

(24c) the conditions (24c) guarantee that $F(x) - F(x + a) \in L^p(\mathbb{R}^n) \forall a \in A = \pi \mathbb{Z}^n$.

Now we focus on (24a) – (24c).

We now present some preliminary facts about the construction blocks. Define

(25) $$\sigma(n; a) = \sigma(a) = \left\{ \xi \in \mathbb{R}^n : 0 \leq \xi_j, 1 \leq j \leq n; \sum_{j=1}^n \xi_j \leq a \right\}, a > 0.$$  

Then

(26) $$\int_{\sigma(a)} 1 d^n \xi = \frac{1}{n!} a^n,$$
and
\[
\int \xi_j^p d^n\xi = \int_0^a t^p dt \int_{\sigma(n-1,a-t)} d^{n-1}t'
\]
(27)
\[
\leq \frac{1}{(n-1)!} \int_0^a t(a-t)^{n-1} dt
\]
\[
= \frac{1}{(n-1)!} a^{n+1} \int_0^1 t(1-t)^{n-1} dt = \frac{1}{(n+1)!} a^{n+1}.
\]

With notations
\[
\kappa = (k_1, \ldots, k_n) \in \mathbb{Z}^n, k(\kappa) \equiv k = \sum_{j=1}^n |k_j|,
\]
let us observe the following:

(28)
\[
\mathcal{D}_1(n)(k+1)^{n-1} \leq \# \left\{ \kappa \in \mathbb{Z}_+^n : \sum_{j=1}^n k_j = k \right\} \leq \mathcal{D}_2(n)(k+1)^{n-1} \quad \text{for all } k \geq 1,
\]
where constants $0 < \mathcal{D}_1(n) < \mathcal{D}_2(n)$ do not depend on $k$. If $n = 2$ the number $\# = k + 1$. Thus we can by induction explain (28) for any $n$.

Let $h(\kappa)$ be a positive function on $\mathbb{Z}^n$ such that

(29)
\[
h(\kappa) = H(1 + k(\kappa)), \quad \text{where } H \text{ is a function on } [1, \infty).
\]

Then

(30)
\[
\sum_{\kappa \in \mathbb{Z}^n} h(\kappa) \geq \sum_{\kappa \in \mathbb{Z}_+^n} h(\kappa) \geq \mathcal{D}_1(n) \sum_{k=0}^{\infty} (1 + k)^{n-1} H(1 + k),
\]
and

(31)
\[
\sum_{\kappa \in \mathbb{Z}^n} h(\kappa) \leq 2^n \sum_{\kappa \in \mathbb{Z}_+^n} h(\kappa) \leq 2^n \mathcal{D}_2(n) \sum_{k=0}^{\infty} (1 + k)^{n-1} H(1 + k).
\]

Now choose

(32)
\[
r(\kappa) = R(1 + k), \quad R(x) = \left( \frac{1}{x} \right)^\gamma, \quad 1 \geq \gamma > 1 - \frac{1}{n+1}.
\]

and define

(33)
\[
F(x) = \begin{cases} 1, & x \in \bigcup_{\kappa \in \mathbb{Z}^n} I(\kappa), \quad I(\kappa) = \{ \pi \kappa + \sigma(r(\kappa)) \}, \\ 0, & \text{otherwise}. \end{cases}
\]

We claim that (24a) – (24c) hold. Indeed, by (26) and (30), (32)

(34)
\[
\int_{\mathbb{R}^n} F(x)^p dx = \sum_{\kappa \in \mathbb{Z}^n} \int_{\sigma(r(\kappa))} 1 dx = \sum_{\kappa \in \mathbb{Z}^n} r^n(\kappa) \frac{1}{n!} \geq \frac{\mathcal{D}_1(n)}{n!} \sum_{k=0}^{\infty} (1+k)^{n-1} \left( \frac{1}{1+k} \right)^{\gamma n} = \infty.
\]

To check (24b) let us notice that
\[
\frac{\sin t}{t} \leq 1, \quad 0 \leq t \leq \pi, \quad \text{and } |\sin(t + \pi \ell)| = |\sin t|, \quad \forall \ell \in \mathbb{Z}.
\]
Therefore, by (27) and (31)

\[
\int_{\mathbb{R}^n} |(\sin x_j) F(x)|^p \, dx \leq \sum_{\kappa \in \mathbb{Z}^n} \int |\sin(x_j)|^p \, d^n x
\]

(35)

\[
= \frac{1}{(n+1)!} \sum_{\kappa \in \mathbb{Z}^n} r(\kappa)^{n+1}
\]

\[
\leq \frac{2^n D_2(n)}{(n+1)!} \sum_{k=0}^{\infty} (1+k)^n \cdot \left( \frac{1}{k+1} \right)^{\gamma(n+1)} \quad < \infty
\]

if \( n - 1 - \gamma(n+1) < -1 \), i.e., \( \gamma > 1 - \frac{1}{n+1} \). This is the condition (32) so (24b) holds.

Finally, by (26) and (33), (31)

\[
\int_{\mathbb{R}^n} |F(x) - F(x + \pi e_j)|^p \, dx = \sum_{\kappa \in \mathbb{Z}^n} \frac{1}{n!} |r(\kappa)^n - r^n(\kappa + \pi e_j)|
\]

\[
\leq \frac{2^n}{n!} \sum_{\kappa \in \mathbb{Z}^n_{+}} (r(\kappa) - r(\kappa + \pi e_1)^n)
\]

\[
= \frac{2^n}{n!} \sum_{\kappa \in \mathbb{Z}^n_{+}} \left[ R^n(1+k(\kappa)) - R^n(k(\kappa)+2) \right]
\]

(36)

\[
= \frac{2^n}{n!} \sum_{\kappa \in \mathbb{Z}^n_{+}} \left[ \left( \frac{1}{1+k(\kappa)} \right)^{\gamma^n} - \left( \frac{1}{2+k(\kappa)} \right)^{\gamma^n} \right]
\]

\[
\leq \frac{2^{n+1}}{(n-1)!} \sum_{\kappa \in \mathbb{Z}^n_{+}} \left( \frac{1}{1+k(\kappa)} \right)^{\gamma(n+1)}
\]

\[
\leq \frac{2^{n+1} D_2(n)}{(n-1)!} \sum_{k=0}^{\infty} (1+k)^{n-1} \left( \frac{1}{1+k} \right)^{\gamma(n+1)} \quad < \infty
\]

if \( n - 1 - (\gamma n + 1) = n(1-\gamma) - 2 < -1 \), i.e., \( \gamma > 1 - \frac{1}{n} \). This condition follows from (32) so (24c) holds.

8.

After analysis of Example 8 we can explain the following.

**Claim 9.** With notation of Example 7, if \( t = \langle a, b \rangle = m\pi, m \in \mathbb{Z} \), there exists a measurable function \( F(x) \) on \( \mathbb{R}^n \) such that

\[
(37a) \quad \int_{\mathbb{R}^n} |F(x)|^p \, dx = \infty,
\]

\[
(37b) \quad F(x) \cdot (\sin(x, b)) \in L^p(\mathbb{R}^n),
\]

\[
(37c) \quad \int_{\mathbb{R}^n} |F(x) - F(x + a)|^p \, dx < \infty.
\]
Proof. If vectors $a, b \in \mathbb{R}^n$ are linearly dependent we can assume (compare a rescaling in Proposition 3 use (21c) if necessary) that $a = m\pi e_1, b = e_1$. Then a function

$$F(x) = f(x_1) \cdot \varphi(x'), \quad x' = (x_2, \ldots, x_n),$$

where $f \in L^{\infty}$ and $\varphi \in L^p(\mathbb{R}^{n-1})$, say, $\varphi = \left(1 + \sum_{j=2}^{n} x_j^2\right)^{-n}$, satisfies the conditions (37a) – (37c). If $a, b$ are linearly independent, we can assume

$$b = e_1, \quad a = m\pi e_1 + \tau e_2, \quad \tau > 0,$$

and choose

$$F(x) = f(x_1, x_2) \varphi(x''), \quad x'' = (x_3, \ldots, x_n),$$

where $\varphi = \left(1 + \sum_{j=3}^{n} x_j^2\right)^{-n}$ and

$$f = \begin{cases} 1, & (x_1, x_2) \in I_\lambda, \quad \lambda = (\ell_1, \ell_2) \in \mathbb{Z}^2 \\ 0, & \text{otherwise,} \end{cases}$$

with

$$I_\lambda = \{\ell_1(\pi e_1) + \ell_2(\tau e_2) + \sigma(2; r_\lambda)\}, \quad r_\lambda = R(1 + |\ell_1| + |\ell_2|), \quad R(x) = \frac{1}{x}, \quad x \geq 1.$$ 

All technicalities to explain (37a) – (37c) are already done in Example 8. □

After Examples 7, 8 and Claim 9 it would be interesting to describe all assertive (for $L^p$, $1 \leq p < \infty$) pairs $A, B \subset \mathbb{R}^n$. It seems reasonable to conjecture that the pair $A, B \subset \mathbb{R}^n$ is assertive if and only if there are vectors $a \in A, b \in B$ such that $\langle a, b \rangle \not\in \pi \mathbb{Z}$. If $n = 1$ this is the statement of Proposition 5. For any $n > 1$ the “if” is explained in Example 7.

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10. Translation of the epigraph (as given in [1])

If you only knew what kind of trash
Poems shamelessly grow in:
Like weeds under the fence,
Like crabgrass, dandelions.
A. Akhmatova, “Secrets of the Trade.”

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[2] Anatolii M. Vershik. “More about uncertainty principle”. July 9, 2016. URL: http://gauss40.pdmi.ras.ru/ma25/presentations/Vershik.pdf

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