Abstract

Conservation of the total momentum of isolated multiparticle systems implies that there are constraints on the internal dynamics of the system. Meanwhile, constraints on the dynamics lead to constraints on the kinematics in spacetime. These kinematical constraints force the particles into orbits that span a 3+1-dimensional pseudo-Riemannian manifold. This manifold is described by the field equations of conformal gravity. A quantisation of this manifold leads to a consistent quantum gravity with conformal gravity as its quasi-classical limit of large quantum numbers. The close connection between the conservation of momentum and gravity provides a new, in-depth understanding of both classical and quantum gravity.

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1 Introduction

Symmetries have long played an important role in theoretical physics because they are closely related to conserved quantities (constants of motion), whose identification can drastically simplify the mathematical formulation of physical problems. Noether’s theorem \cite{1} plays a central role in this context. Generally, Noether’s theorem assigns a conservation law to every continuous symmetry. Thus, the invariance of an isolated (non-dissipative) multiparticle system under translation leads to the conservation of the total momentum, without requiring knowledge of the internal dynamics or kinematics of the system.

In contrast, a similar conclusion cannot be drawn for the momenta of the individual particles because there is (in general) no translational symmetry of the individual particles with respect to each other. Without translational symmetry, a simple inversion of Noether’s theorem suggests that the particle momenta are not conserved but are time dependent. However, these time dependencies are constrained by the fact that the total momentum is conserved. Constraints on the dynamics of particle momenta imply that there are constraints on the internal kinematics of the system. Similar to the conservation law of total momentum, these kinematic constraints are independent of the details of the internal dynamics (i.e., the causes of the time dependence of the particle momenta). In other words, Noether’s theorem and its inverse predict an extension of the conservation law of momentum that constrains the internal kinematics of isolated multiparticle systems. This article will examine the mathematical structure and physical meaning of these constraints.

2 Conservation of momentum in isolated multiparticle systems

The total momentum $\mathbf{p}$ of an isolated system of massive particles, as described by Newtonian mechanics, is expressed as the sum over the particle momenta

$$\sum_{i=1}^{N} \mathbf{p}_i(t) = \mathbf{p}. \quad (1)$$

Because the system is assumed to be isolated, it is invariant under synchronous translation of all particles; therefore, by Noether’s theorem, $\mathbf{p}$ is a constant over time. However, this does not apply to the individual particle momenta, which are therefore treated as time-dependent.

It is obvious that equation (1) is invariant under translations and covariant under rotations, and boosts operations applied to the individual particle momenta. Moreover, this equation is covariant under conformal scaling of the individual particle momenta because the conservation law of momentum is not bound to certain values of the particle masses. This is a self-evident yet fundamental property of this equation.

By Newton’s second law, a time-dependent particle momentum $\mathbf{p}_i(t)$ defines a force that is equal to the time derivative of that momentum, whatever the reason for that time dependence. Given that the total momentum is conserved, this force must be compensated by forces from other particles. Thus, we can describe the time dependence of the particle momenta by an exchange of forces or, equivalently, by a virtual exchange of momentum between the particles.

Note that the concept of momentum exchange does not involve a model assumption. Similar to the concept of forces, it is a generally valid descriptive element of Newtonian mechanics. In
the following, this concept is used as a convenient description of curved trajectories under the constraint of conserved total momentum.

If Eq. (1) is rewritten in the form

\[ p_1(t) = p - \sum_{i=2}^{N} p_i(t), \]  

(2)

it then becomes evident that the trajectory of particle 1 is completely determined by the time-dependent momenta of particles 2 to \( N \), together with the position and momentum of particle 1 at \( t = 0 \). If we had not known that we are dealing with the conservation of total momentum, then we might have suspected that the mere presence of neighbouring matter bends the trajectory of particle 1. This clearly evokes associations with the phenomenon of gravitation and suggests that particles 2 to \( N \) generate a virtual gravitational field that bends the trajectory of particle 1 such that the total momentum of the system is conserved at any time.

Despite the virtual character of this field, its structure is principally measurable by observing the trajectories of particle 1. A structure that is measurable should also be theoretically describable. This description must be compatible with Einstein’s principle of relativity and should be transferable into the realm of quantum mechanics. Both requirements should be possible to satisfy because the law of conservation of momentum holds in relativistic classical mechanics, as well as in quantum mechanics.

3 Relativistic field equations

In establishing the field equations of general relativity, Einstein required that "The general laws of nature have to be expressed by equations that are valid for all coordinate systems, that is, that are covariant (generally covariant) with respect to arbitrary substitutions" ([2], p. 776, translation by the author). We adopt this requirement and also the coordinate-independent description of curved trajectories in four-dimensional spacetime with the tools of differential geometry.

The concept of momentum exchange is easily extended to relativistic mechanics by combining momentum \( p_i(t) \) and energy \( p_0^i(t) \) into a four-momentum \( p_i(t) \). Following Einstein’s requirement, the four-momenta \( p_i(t) \) will become part of a covariant energy-momentum tensor \( T^{\mu\nu}(x) \) (of massive point-like particles; see, e.g., [3]).

Within the concept of exchanged momenta, the variations of momentum along a trajectory ‘generate’ exchange momenta. Because these exchange momenta are derived from the time-dependent particle momenta alone, they are independent of the trace \( T^{\mu\nu} \) corresponding to the (time-constant) particle masses. Simultaneously, other particles ‘annihilate’ the exchange momenta, which causes a change of their trajectories. This change can again be described by time-dependent momenta, which can be directly observed as a curvature of the trajectories. Again, the trace of \( T^{\mu\nu} \) is not involved. Therefore, the entire exchange process is completely described by the traceless part of \( T^{\mu\nu} \), namely

\[ T^{\mu\nu} = T^{\mu\nu} - \frac{1}{4} g^{\mu\nu} T. \]  

(3)

This is in line with the fact that the particle masses do not enter into equation (1). Therefore, the exchange process is invariant under conformal scaling, just like the conservation law of momentum on which it is based.

Alternatively to the description by the traceless energy-momentum tensor (3), the trajectories can be described by their curvatures in spacetime. Obviously, both descriptions are physically
equivalent. This (apparently trivial) equivalence will now be formulated as a law of nature (in the sense of Einstein’s requirement); that is, as a covariant relation between the traceless energy-momentum tensor, considered as the source of exchange momenta, and a geometric tensor $W^{\mu\nu}$, derived from a curvature tensor and considered as a sink. Together, these tensors are expected to form the covariant equation

$$\kappa W^{\mu\nu} = T^{\mu\nu} - \frac{1}{4} g^{\mu\nu} T,$$

where $\kappa$ is a constant required for dimensional reasons. (If appropriate units are used, $\kappa$ can take the numerical value 1.) As a counterpart to the traceless energy-momentum tensor, $W^{\mu\nu}$ must also be traceless.

To determine the explicit form of $W^{\mu\nu}$, we follow Mannheim [4] and use the variation principle of least action. This requires a suitable chosen action $I = I_W + I_M$, the sum of the geometric action $I_W$ and the material action $I_M$, which is invariant under all relevant coordinate transformations, including local conformal scaling. The functional variation of $I$ with respect to the metric tensor $g^{\mu\nu}$ should then yield the expected covariant relationship between the curvature tensor and the energy-momentum tensor.

The only geometric action that is invariant under local conformal scaling is the Weyl action (see [4])

$$I_W = -\frac{1}{4} \kappa \int d^4x (-g)^{1/2} C_{\lambda\mu\nu\xi} C_{\lambda\mu\nu\xi},$$

where $C_{\lambda\mu\nu\xi}$ is the Weyl tensor, which is the traceless part of the Riemann curvature tensor $R_{\lambda\mu\nu\xi}$. The Weyl tensor has the suitable property that it is not only covariant but can even be invariant under local conformal scaling.

For the material action,

$$I_M = \int d^4x (-g)^{1/2} g_{\mu\nu} (T^{\mu\nu} - \frac{1}{4} g^{\mu\nu} T)$$

can be chosen. The term under the integral is the trace of the traceless energy-momentum tensor, which is zero in any coordinate system: $g_{\mu\nu} (T^{\mu\nu} - \frac{1}{4} g^{\mu\nu} T) = g_{\mu\nu} T^{\mu\nu} - T = T - T = 0$. Thus, $I_M$ is trivially invariant under local conformal scaling.

The factor $(-g)^{1/2}$ in the integrals $I_W$ and $I_M$ can be dropped if $x$ refers to a fixed Minkowskian coordinate system (see [2], p. 801).

The functional derivative of $I_W + I_M$ with respect to $g^{\mu\nu}$ yields the fourth-order partial differential equations

$$\kappa W^{\mu\nu} = T^{\mu\nu} - \frac{1}{4} g^{\mu\nu} T,$$

where [4]

$$W^{\mu\nu} = \frac{1}{2} g^{\mu\nu}(R^a_a)_{;\beta} + R^{\mu\nu;\beta} - R^{\mu\beta;\gamma} ; \gamma - R^{\gamma\beta;\mu} ; \beta - 2R^{\mu\beta} R^{\nu}_{\beta} + \frac{1}{2} g^{\mu\nu} R_{ab} R^{ab} - 2 \frac{2}{3} g^{\mu\nu} (R^a_a)_{;\beta} + \frac{2}{3} (R^a_a)_{;\mu} ; \nu + \frac{2}{3} R^a_a R^{\mu\nu} - \frac{1}{6} g^{\mu\nu} (R^a_a)^2$$

$$= 2 C_{\mu\lambda\nu\xi}^{;\beta} - C_{\mu\lambda\nu\xi}^{;\beta} R_{\lambda\xi}.$$

Surprisingly, these field equations have the same structure as Mannheim’s equations of conformal gravity, although the latter are based on completely different physical premises. While
Mannheim postulated a gauge invariance of Nature under local conformal scaling, conformal covariance here refers to the aforementioned property of the conservation law of momentum to be covariant under particle-individual conformal scaling. In contrast to Mannheim’s approach, the particles involved here need not be massless.

As in Einstein’s general theory of relativity, the metric tensor $g^{\mu\nu}(x)$ defines a pseudo-Riemannian manifold. Observers moving freely along a trajectory in this manifold will experience a curved spacetime, whereas they will infer the existence of a gravitational field from the Minkowskian coordinate system.

Mathematically, the field equations express nothing more than the (obvious) equivalence of the description of trajectories by time-dependent momenta—that is, by $T^{\mu\nu} - \frac{1}{4} g^{\mu\nu} T$—and their description by curvatures—that is, by $W^{\mu\nu}$. (Remember that this was the starting point of the derivation of the field equations.) However, when applied to a real multiparticle system (subject to the conservation law of momentum), they translate the dynamical constraints on the particle momenta (i.e. conservation of total momentum) into equivalent kinematic constraints on the particle trajectories in spacetime. Therefore, the field equations describe the structure of spacetime of isolated multiparticle systems, if the energy-momentum tensor is given, and vice versa. As Wheeler [5] put it (slightly modified by the author): Matter tells spacetime how to curve, while spacetime tells matter how to move.

It is noteworthy that the form of the field equations is completely determined by the local conformal invariance of the conservation law of momentum.

4 Quantum gravity

4.1 Irreducible multiparticle representations

In (relativistic) quantum mechanics, isolated systems of massive particles with well-defined total linear and angular momentum are described by irreducible unitary multiparticle representations of the Poincaré group $P(3,1)$. Similar to the coupling of two or more angular momenta to a resulting angular momentum, irreducible multiparticle representations of the Poincaré group can be obtained from the reduction of the tensor products of unitary single-particle representations (see, e.g., [6] and references therein). Irreducible representations of $P(3,1)$ are characterised by fixed eigenvalues of two Casimir operators corresponding to a mass and an angular momentum, which is analogous to the mass and spin of irreducible single-particle representations (see textbooks on quantum field theory, e.g. [7], pp. 36–53).

By fixing the eigenvalue of the first Casimir operator, the individual particle momenta are correlated such that the mass defined by the total momentum keeps a fixed value. This correlates the particle momenta in the same manner as the classical Eq. (1).

Fixing the eigenvalue of the second Casimir operator forces the individual particle states into a rotational-symmetric momentum-entangled structure, such as in the two-particle eigenstate of the total linear and angular momenta [8]

$$|\mathbf{p},w\rangle = \int_{\Omega} d\omega_{\mathbf{k}} |\mathbf{p}_1+\mathbf{k}\rangle c_{\mathbf{p},w}(\mathbf{k}) |\mathbf{p}_2-\mathbf{k}\rangle.$$  \hspace{1cm} (9)

The integration over $\mathbf{k}$ is over a circular path $\Omega$. This structure describes two particles that are correlated by the virtual exchange of momenta $\mathbf{k}$. These momenta are the counterpart to the Newtonian exchanged momenta, which was addressed in Section 2, and to the momenta of exchanged
gauge bosons in the Standard Model. This structure clearly shows that the particle momenta cannot (not even approximately) be eigenstates simultaneously with the total linear momentum and angular momentum. In fact, the commutation relations of the infinitesimal generators of $P(3,1)$ (see [7], p. 45) allow the simultaneous diagonalisation of a linear momentum (the total momentum) and a component of angular momentum with an axis of rotation parallel to the linear momentum; however, they do not allow the diagonalisation of another momentum that is not parallel to the axis of rotation. Thus, unlike product representations (on which the Standard Model is based), irreducible multiparticle representations do not contain pure (i.e. non-entangled) products of momentum eigenstates of individual particles.

The rotational symmetry of the eigenstates of the orbital angular momentum is preserved in the quasi-classical limit of large quantum numbers. The corresponding wave functions define probability distributions in the form of well-defined rotationally symmetric closed orbits (see [9], pp. 27–29). Without going into detail, we can expect that wave packets moving along such orbits will have time-dependent momenta. Therefore, irreducible multiparticle representation of $P(3,1)$ provide the necessary ingredients (i.e. time-dependent particle momenta with total momentum preserved) for the formulation of the field equations of conformal gravity, which then describe the system in the quasi-classical limit. The phenomenon of gravity can thus be traced back to the correlations, or more precisely to the entanglement, of the individual particle momenta within an eigenstate of total linear and orbital angular momentum.

4.2 Quantisation of spacetime

As described, irreducible multiparticle representations of the Poincaré group in the limiting case of large quantum numbers lead to curved trajectories in a pseudo-Riemannian spacetime. Therefore, the inversion of this limit consideration can be understood as ‘quantisation of spacetime’. This means that continuous spacetime is replaced by discrete trajectories, which are then replaced by wave functions of an irreducible multiparticle representation of $P(3,1)$, describing particles whose states are correlated by the fixed eigenvalues of two Casimir operators. This quantification rule is transparent and well founded, both physically and mathematically. Unlike other attempts to quantise spacetime, such as loop quantum gravity or string theory, this offers quantisation of spacetime at experimentally accessible scales.

5 Conclusions

Symmetry considerations in the spirit of Noether's theorem and its inverse have revealed universally valid constraints on the kinematics of particles in isolated multiparticle systems that are characterised by well-defined and conserved total linear and orbital angular momentum. These constraints imply that in these systems the particle trajectories form a pseudo-Riemannian manifold, which is described by the field equations of conformal gravity (CG). These are purely mathematical consequences of the conservation law of total momentum and its symmetries; that is, covariance under local (i.e. particle-individual) transformations of the Poincaré group and under local conformal scaling of the particle momenta. In brief, what in momentum space is the conservation of total momentum is in space-time the phenomenon of gravitation.

The field equations of CG are currently being discussed as an alternative to the field equations of general relativity (GR) (see e.g. [4]). In the form presented here, they apply to isolated multiparticle systems with Poincaré symmetry consisting of massive spinless particles or macroscopic
bodies. For the solar system, they not only reproduce the Schwarzschild solutions of GR but (in contrast to GR) they are also able to correctly describe galactic rotation curves without the aid of exotic dark matter (see [4] and the references given there). Thus, not only Einstein–Newton gravity but also the puzzling galactic rotation curves can be regarded as manifestations of the constrained kinematics of isolated multiparticle systems or (equivalently) as consequences of the conservation law of total momentum.

If we agree upon defining quantum gravity as the description of multiparticle systems that is in agreement with the axioms of quantum mechanics and that in the quasi-classical limit leads to the field equations of CG, then irreducible unitary multiparticle representations of the Poincaré group are the answer to the current search for a quantum theory of gravity. These representations are well-understood and provide the simplest description that matches this definition.

This quantum mechanical description requires neither a specific quantum theory nor an interaction with hypothetical ‘gravitons’. It is sufficient to recall the representation theory of the Poincaré group [10] within the framework of ordinary, well-understood relativistic (Poincaré-invariant) quantum mechanics of independent particles. The absence of explicit interaction terms means that there is a drastic reduction in mathematical complexity when compared to perturbation schemes that are based on quantum field theoretic methods as known from the Standard Model. A reformulation of the Standard Model based on irreducible multiparticle representations of the Poincaré group would not only allow a smooth integration of gravity into the model but would also eliminate any renormalisation problems.

Similar results have recently been obtained for irreducible two-particle representations of spin-1/2 particles, which (as a welcome addition) provide the correct value for the electromagnetic coupling constant [8]. The agreement of the calculated coupling constant with the experimental value not only identifies the corresponding kinematic constraints as ‘electromagnetic interaction’ but also confirms the use of irreducible unitary multiparticle representations of the Poincaré group as the mathematical basis of quantum mechanical descriptions of multiparticle systems.

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