A SELF-CONTAINED PROOF TO MARTIO’S CONJECTURE IN THE CLASS OF BLD-MAPS

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Abstract. We provide a self-contained proof to so-called Martio’s conjecture in the class of mappings of bounded length distortion. Unlike the earlier proofs, our proof is not based on the modulus of continuity estimate of Martio from 1970.

1. Introduction

In this article we study the sufficient conditions for the local invertibility of mappings. Our work is motivated by the well-known inverse function theorem which states that every continuously differentiable mapping

\[ f \in C^1(U, \mathbb{R}^n) \quad (U \subset \mathbb{R}^n \text{ open and connected set}) \]

is a local \( C^1 \)-diffeomorphism outside the zero set of its Jacobian determinant. We are interested in the conditions under which one can recover the local invertibility when the usual assumptions of the inverse function theorem are not satisfied. This leads us to the following two questions:

(Q1) How to recover local invertibility when the mappings are less than \( C^1 \)-regular?
(Q2) How to recover local invertibility near the singular set of the Jacobian determinant of mappings?

We study these questions in the class of quasiregular mappings, that is, in the class of Sobolev mappings

\[ f \in W_\text{loc}^{1,n}(U, \mathbb{R}^n) \quad (U \subset \mathbb{R}^n \text{ open and connected set with } n \geq 2) \]

for which the operator norm of the weak differential matrix satisfies the following distortion inequality

\[
(1.1) \quad \|Df(x)\|^n \leq K \det Df(x) := KJ(x; f) := KJ_f(x) \quad \text{a.e.}
\]

for some constant \( K \geq 1 \). Homeomorphic quasiregular maps form the well-studied class of quasiconformal mappings. For the basic properties and the background of quasiregular and quasiconformal mappings we refer to monographs [AIM09, IM01, Res89, Ric93, Yuo88, Vä71].

Next we point out that even if the definition of quasiregularity is purely analytical it still harbors a great deal of topological information per se. Especially, the distortion inequality can be applied to provide surprisingly vast amount of information on the invertibility properties of quasiregular
mappings. This was first observed by Reshetnyak who originally introduced quasiregular mappings by the name of \textit{mappings of bounded distortion} and discovered their basic properties in a series of papers in 1966–1969. One of the deepest discoveries of these works was that non-constant quasiregular mappings are discrete and open, see e.g. \cite{Res89}. This observation connected the study of quasiregular mappings to the earlier studies on \textit{branched coverings} in geometric topology, see e.g \cite{Ce64, Ce65, CH60, CH61, CH63}.

In the critical step of the proof of Reshetnyak’s celebrated discreteness and openness theorem one applies non-linear potential theory and non-linear PDEs to transfer analytical data into topological information. This step can be carried out by studying the geometric size of the polar sets of the solutions to the quasilinear elliptic partial differential equation

\begin{equation}
- \text{div}\left( (G_{f}^{-1} \nabla u, \nabla u)^{(n-2)/2} G_{f}^{-1} \nabla u \right) = 0,
\end{equation}

where

\[
G_{f}^{-1}(x) = \begin{cases}
\text{cof} \frac{Df(x)}{\det Df(x)} & \text{if } J_f(x) > 0 \\
1 & \text{otherwise},
\end{cases}
\]

stands for the inverse dilatation tensor and \text{cof} \(Df(x)\) denotes the cofactor matrix of the differential matrix. This way one eventually obtains that

\[H^1(f^{-1}(y)) = 0 \text{ for every } y \in \mathbb{R}^n,\]

which implies that every quasiregular mapping is \textit{light} in the sense that the preimage of every point is totally disconnected. After this Reshetnyak’s result follows by showing that quasiregular mappings are sense-preserving and by obtaining that sense-preserving and light maps between connected oriented manifolds are discrete and open, see e.g. \cite{BI83, Hei02, IM01, MV98} for further details.

Reshetnyak’s theorem and its techniques have been studied further by several authors, see e.g. \cite{GoVy76, HK93, HM02, IS93, MV98, MV95, MRS08, Raj11}. In this process the development of \textit{mappings of finite distortion} \cite{HeKo14, IM01, MRSY09} and the study of their connection to the non-linear elasticity theory of Ball, Antman, and Ciarlet \cite{Ball81, Ant76, Cia88} have been main driving forces. In this context, the generalizations of Reshetnyak’s theorem have been applied to investigate \textit{impenetrability of matter} of deformations in non-linear elasticity theory. However, usually these techniques have only been used to obtain discreteness and openness of deformations instead of recovering the actual local invertibility.

In this article we study the local invertibility of quasiregular mappings by utilizing the earlier studies of Onninen and Zhong \cite{OZ08} on Reshetnyak’s theorem in order to study Martio’s conjecture which states that every non-constant quasiregular mapping

\[f : U \to f(U) \subset \mathbb{R}^n \quad (U \subset \mathbb{R}^n \text{ open set with } n \geq 3)\]

with an \textit{inner dilatation}

\[K_I(f) := \text{ess sup}_{x \in U} \frac{\|\text{cof} Df(x)\|^n}{\det Df(x)^{n-1}} \]

less than two is a local homeomorphism. This long-standing unconfirmed conjecture was originally stated by Martio, Rickman, and Väisälä in \cite{MRV71}.
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and it was motivated by the preliminary work of Martio [Mar70], where the conjecture was confirmed when the branch set

\[ B_f := \{ x \in U : f \text{ is not a local homeomorphism at } x \} \]

of a quasiregular map contains a rectifiable curve. From a technical point of view it is usually more natural to study the following generalization of the conjecture from [Ten]:

**Conjecture 1.1** (Strong Martio’s conjecture). The inner dilatation of a non-constant quasiregular mapping

\[ f : U \to f(U) \subset \mathbb{R}^n \ (U \subset \mathbb{R}^n \text{ open set with } n \geq 3) \]

satisfies

\[ \inf_{x \in B_f} i(x, f) \leq K_1(f), \]

where and in what follows \( i(x, f) \) stands for the local topological index of a point \( x \in U \) under the mapping \( f \), see [Ric93, Chapter I].

We point out for the reader that the standard quasiregular \( m \)-to-1 winding mapping

\[ (r, \theta, z) \mapsto (r, m\theta, z) \quad (z \in \mathbb{R}^{n-2}), \]

written here in the cylindrical coordinates, is an external for the conjecture. In addition, the holomorphic function

\[ f : \mathbb{C} \to \mathbb{C}, \quad f(z) = z^m \]

shows the conjecture to fail in the planar case. In [KLT21] Kauranen, Luisto, and Tengvall verified the conjecture for mappings of bounded length distortion, also known as BLD-mappings. This class consists of those quasiregular Lipschitz mappings

\[ f : U \to f(U) \subset \mathbb{R}^n \ (U \subset \mathbb{R}^n \text{ open set with } n \geq 2) \]

for which we have

\[ \det Df(x) > c \quad \text{a.e.} \]

for some constant \( c > 0 \), see [MV88]. In [Ten] Tengvall relaxed the boundedness condition (1.3) even further by proving the conjecture under certain integrability condition on the reciprocal of the Jacobian determinant. He also offered several alternative proofs for the conjecture in the BLD-class. However, none of the above-mentioned proofs from [KLT21, Ten] is self-contained as each one of them heavily relies on the following well-known local modulus of continuity estimate

\[ |f(x) - f(y)| \leq C|x - y| \left( \frac{1}{|x'|} \right)^{\frac{1}{m-1}} \quad \text{for all } y \in B(x, r) \]

by Martio [Mar70] which can be also found from [Ric93, Theorem III.4.7]. The proof of this estimate requires several layers of preliminary results which makes it technical and rather lengthy. In this article we prove the strong Martio’s conjecture for BLD-mappings without any use of the estimate (1.4) by providing a rather short and self-contained proof for the following result from [KLT21] which is valid also in the planar case:
Theorem 1.2 (Kauranen, Luisto, and Tengvall, 2021). Every non-constant BLD-mapping

\[ f : U \to f(U) \subset \mathbb{R}^n \quad (U \subset \mathbb{R}^n \text{ open set with } n \geq 2) \]
satisfies \( i(x, f) \leq K_I(f) \) for every \( x \in U \).

Finally we highlight the connection of the generalized Liouville’s theorem to our studies on local invertibility of quasiregular mappings. This rigidity result states that non-planar, non-constant quasiregular mappings with

\[ K_I(f) = 1 \]

are restrictions of Möbius transformations. The result generalizes the well-known Liouville’s theorem [Cap86, Lio50, Har47, Har68] on rigidity of non-planar conformal diffeomorphisms. The original proofs of Gehring [Geh62] and Reshetnyak [Res67] for the result are based on the study of regularity properties of the solutions to the non-linear \( n \)-harmonic equation

\[ -\text{div}(|\nabla u(x)|^{n-2}\nabla u(x)) = 0. \]

This equation can be obtained from (1.2) when

\[ G_f^{-1}(x) = \text{id} \quad \text{a.e.} \]

and in the planar case it reduces to the usual Laplace equation. In the earlier-mentioned work [MRV71] of Martio, Rickman, and Väisälä (see also [Gol71]) generalized Liouville’s theorem was applied with a compactness argument to obtain that non-planar, non-constant quasiregular mappings with an inner dilatation close to one are local homeomorphisms. Later a quantitative version of this result was obtained by Rajala [Raj05].

As by the generalized Liouville’s theorem all non-planar, non-constant quasiregular mappings with the property (1.5) coincide with the identity map up to a conjugation by restrictions of Möbius transformations it is natural to ask whether similar kind of phenomenon occurs also for quasiregular mappings with

\[ K_I(f) = \inf_{x \in B_f} i(x, f). \]

In the light of current knowledge it seems that up to a conjugation by Möbius transformations the only non-planar quasiregular mapping with the property (1.7) is the standard \( m \)-to-1 winding map. Therefore, we conjecture:

Conjecture 1.3 (Rigidity conjecture). Every non-planar quasiregular mapping with

\[ K_I(f) = m_f, \quad \text{where } m_f := \begin{cases} 1, & \text{if } B_f = \emptyset \\ \inf_{x \in B_f} i(x, f), & \text{if } B_f \neq \emptyset, \end{cases} \]
equals to the standard \( m_f \)-to-1 winding mapping up to a conjugation by restrictions of Möbius transformations.
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2. Preliminaries

In this section we recall some basic facts on mappings of bounded length distortion from [MV88] and on discrete and open maps from [Ric93]. If the reader is well-aware of the basic properties and the notation related to these mapping classes, reading this section is not necessary.

2.1. Preliminary properties for BLD-maps. We recall that mappings of bounded length distortion form a subclass of quasiregular maps. Thus, it follows from Reshetnyak’s theorem that these mappings are continuous, sense-preserving, discrete, and open. In addition, by the characterization [MV88, Theorem 2.16] of these mappings every BLD-map

\[ f : U \rightarrow f(U) \subset \mathbb{R}^n \quad (U \subset \mathbb{R}^n \text{ open set with } n \geq 2) \]

satisfies the following length distortion bounds

\[ \ell(\gamma)/L \leq \ell(f \circ \gamma) \leq L\ell(\gamma) \]

for every path \( \gamma \) in \( U \) with some length distortion constant \( L \geq 1 \), where \( \ell(\gamma) \) stands for the length of a path \( \gamma \).

2.2. Preliminary properties for discrete and open maps. As mappings of bounded length distortion form a subclass of discrete and open maps we may apply all the basic results on these maps in order to study BLD-maps further. In this section we recall the definitions and results on discrete and open maps from [Ric93, Chapter I] that are later needed for the proof of Theorem 1.2. We start by recalling that an open, connected set \( D \subset U \) is called a normal domain of a continuous, discrete, and open mapping

\[ f : U \rightarrow f(U) \subset \mathbb{R}^n \quad (U \subset \mathbb{R}^n \text{ open set with } n \geq 2) \]

if it satisfies

\[ f(\partial D) = \partial f(D). \]

If a normal domain \( D \) satisfies

\[ D \cap f^{-1}(f(x)) = \{x\}, \]

then it is called a normal neighborhood of a point \( x \in U \). In addition, for a given point \( x \in U \) we denote

\[ U(x, f, r) := \text{“the } x\text{-component of the preimage } f^{-1}(B(f(x), r))\text{”}. \]
With the notation introduced above we may recall the following standard lemma from [Ric93, Lemma I.4.9] which is applied frequently throughout the article:

**Lemma 2.1.** Let

\[ f : U \to f(U) \subset \mathbb{R}^n \quad (U \subset \mathbb{R}^n \text{ open set with } n \geq 2) \]

be a continuous, discrete, and open mapping. Then for every \( x \in U \) there exists a radius \( r_x > 0 \) for which \( U(x, f, r) \) is a normal neighborhood of \( x \) such that

\[ f(U(x, f, r)) = B(f(x), r) \quad \text{for every } 0 < r \leq r_x. \]

Moreover, we have

\[ \text{diam}(U(x, f, r)) \to 0 \quad \text{as } r \to 0. \]

**2.3. Path lifting.** In order to study the compression of BLD-maps later in section 3 we need the following path lifting lemma that follows directly from [Ric93, Proposition I.4.10] and [Ric93, Corollary II.3.4]:

**Lemma 2.2.** Let

\[ f : U \to f(U) \subset \mathbb{R}^n \quad (U \subset \mathbb{R}^n \text{ open set with } n \geq 2) \]

be a continuous, discrete, and open mapping and let \( D \subset \subset U \) be a normal neighborhood of a point \( x_0 \in U \) under the mapping \( f \). In addition, denote

\[ m := i(x_0, f). \]

Then for every path

\[ \beta : [a, b) \to f(D) \]

starting at \( f(x_0) \) there exist paths

\[ \alpha_j : [a, b) \to D \quad (j = 1, \ldots, m) \]

each of which starts at \( x_0 \) and such that the following conditions are satisfied:

(i) \( f \circ \alpha_j = \beta \) for each \( j = 1, \ldots, m. \)

(ii) \( \text{card}\{j : \alpha_j(t) = x\} = i(x, f) \) for \( x \in D \cap f^{-1}(\beta(t)) \).

(iii) For the traces of the above-mentioned paths we have the following relation

\[ |\alpha_1| \cup \cdots \cup |\alpha_m| = D \cap f^{-1}(|\beta|). \]

**3. Compression of BLD-mappings**

In this section we shortly discuss the local compression of BLD-maps and show that these maps cannot compress material too much together in the following sense:

**Lemma 3.1.** Every non-constant BLD-mapping

\[ f : U \to f(U) \subset \mathbb{R}^n \quad (U \subset \mathbb{R}^n \text{ open set with } n \geq 2) \]

satisfies

\[ B(f(x), r/L) \subset f(B(x, r)) \quad \text{for some } L \geq 1, \]

whenever \( r > 0 \) is sufficiently small.
Proof. Fix a point $x \in U$. By Lemma 2.1 we may find a radius $r_x > 0$ such that the set $U(x, f, r)$ is a normal neighborhood of $x$ whenever $0 < r \leq r_x$.

Fix any radius $r > 0$ so small that

$$B(x, r) \subset U(x, r_x, f),$$

and denote by $\tilde{r} > 0$ the largest radius such that

$$B(f(x), \tilde{r}) \subset f(B(x, r)).$$

Then to conclude the proof it suffices to show that

$$\tilde{r} \geq r/L \quad \text{for some constant } L \geq 1. \quad (3.1)$$

For this purpose, we obtain that by the openness of $f$ we have

$$\partial f(B(x, r)) \subset f(\partial B(x, r)),$$

and therefore we may find a point $z \in \partial B(x, r)$ such that

$$f(z) \in \partial B(f(x), \tilde{r}).$$

Let us next consider the line-segment

$$I := [f(x), f(z)].$$

By applying Lemma 2.2 we may then find a path $\gamma$ in $U(x, f, \tilde{r})$ from $x$ to $z$ such that

$$f \circ \gamma = I.$$ 

However, then by (2.1) we get

$$\tilde{r} = \ell(I) = \ell(f \circ \gamma) \geq \ell(\gamma)/L \geq r/L$$

for some constant $L \geq 1$. Thus, we have verified (3.1) and the claim follows. □

We point out that Lemma 3.1 can be actually obtained directly from [MV88, Lemma 4.6]. However, for the convenience of the reader and for the self-containedness of this article we have sketched a proof above as well. One should also notice that Lemma 3.1 is not valid for quasiregular maps in general. This can be demonstrated by the radially symmetric $K$-quasiconformal map

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad f(x) = \begin{cases} \frac{x}{|x|^{K-1}}, & \text{if } x \neq 0 \\
0, & \text{if } x = 0, \end{cases}$$

with a constant $K > 1$. In this context we note for the reader that obtaining Lemma 3.1 is the only step of our proof for Theorem 1.2 where the BLD-assumption is needed as all the other steps are valid for general quasiregular mappings.
4. IMPROVING THE LEMMA OF ONNINEN AND ZHONG

In this section we provide the key lemma for our proof of Theorem \[1.2\]. For this purpose we first recall the result of Onninen and Zhong from \[OZ08\] according to which for every

\[ f \in C^\infty(U, \mathbb{R}^n) \quad \text{and} \quad \Psi \in C^1([0, \infty), [0, \infty)) , \]

and for every compactly supported test-function \( \eta \in C_c^\infty(U, [0, \infty)) \) we have

\[
\left| \int_U \eta^n [\eta (|f|^2) + 2|f|^2 \Psi'(|f|^2) J_f] \right| \leq C \int_U \eta^{n-1} |\nabla \eta| |\Psi(|f|^2)| \|Df\|^{n-1} ,
\]

where the constant \( C > 0 \) depends on the dimension of the underlying open set \( U \subset \mathbb{R}^n \). In what follows, we provide a sharp version of this estimate in Lemma \[4.2\] below. The proof of this refinement is based on the following identity:

**Lemma 4.1.** Let \( U \subset \mathbb{R}^n \) be an open set with \( n \geq 2 \) and suppose that

\[ f : U \to f(U) \subset \mathbb{R}^n \quad \text{and} \quad \eta : U \to \mathbb{R} \]

are differentiable at a point \( x \in U \). Then

\[
\sum_{i=1}^n f_i J(x; f_1, \ldots, f_{i-1}, \eta, f_{i+1}, \ldots, f_n) = \nabla \eta(x)^T \cof Df(x) f(x) .
\]

**Proof.** Let us denote

\[ A_{i,j} := [\cof Df(x)]_{i,j} . \]

Then by direct computation we obtain that

\[
\nabla \eta(x)^T \cof Df(x) f(x) = \begin{bmatrix} \partial_1 \eta & \cdots & \partial_n \eta \end{bmatrix} \begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,n} \end{bmatrix} \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \\

(4.1) = \left( \sum_{i=1}^n f_i \partial_i \eta A_{i,1} \right) + \cdots + \left( \sum_{i=1}^n f_n \partial_i \eta A_{i,n} \right) \\

= \sum_{i=1}^n \sum_{j=1}^n \partial_i \eta A_{i,j} f_j .
\]

On the other hand, for each \( i = 1, \ldots, n \) we have

\[
f_i J(x; f_1, \ldots, f_{i-1}, \eta, f_{i+1}, \ldots, f_n) = \sum_{j=1}^n \partial_j \eta A_{i,j} f_i ,
\]

and by summing over \( i \)'s we get

\[
(4.2) \sum_{i=1}^n f_i J(x; f_1, \ldots, f_{i-1}, \eta, f_{i+1}, \ldots, f_n) = \sum_{i=1}^n \sum_{j=1}^n \partial_j \eta A_{i,j} f_i .
\]

By combining \[4.1-4.2\] we have

\[
\sum_{i=1}^n f_i J(x; f_1, \ldots, f_{i-1}, \eta, f_{i+1}, \ldots, f_n) = \nabla \eta(x)^T \cof Df(x) f(x) ,
\]

which completes the proof. \( \square \)
Lemma 4.2. Let $U \subset \mathbb{R}^n$ be an open set with $n \geq 2$ and suppose that
\[ f \in C^\infty(U, \mathbb{R}^n) \quad \text{and} \quad \Psi \in C^1([0, \infty), [0, \infty)). \]
Then for every test-function $\eta \in C_0^\infty(U, [0, \infty))$ we have
\[ \left| \int_U \eta \left[ n\Psi(|f|^2) + 2|f|^2\Psi'(|f|^2) J_f \right] \right| \leq \int_U |\nabla \eta||f|\Psi(|f|^2) \|\cof Df\| . \]

Proof. The proof follows the same steps as the proofs of [OZ08] Lemma 2.1 and [HeKo14] Lemma 3.19. Indeed, first by fixing $i \in \{1, \ldots, n\}$ we obtain by applying Stokes’ theorem that
\[ (4.3) \quad \int_U J(x; f_1, \ldots, f_{i-1}, \eta \Psi(|f|^2) f_i, f_{i+1}, \ldots, f_n) \, dx = 0. \]
On the other hand, the chain rule gives us
\[ J(x; f_1, \ldots, f_{i-1}, \Psi(|f|^2), f_i, f_{i+1}, \ldots, f_n) \]
\[ = \sum_{j=1}^n 2\Psi'(|f|^2) f_j J(x; f_1, \ldots, f_{i-1}, f_j, f_{i+1}, \ldots, f_n) \]
\[ = 2\Psi'(|f|^2) f_i J(x; f_1, \ldots, f_n). \]
Thus, by the product rule we obtain
\[ J(x; f_1, \ldots, f_{i-1}, \eta \Psi(|f|^2) f_i, f_{i+1}, \ldots, f_n) \]
\[ = \Psi(|f|^2) f_i J(x; f_1, \ldots, f_{i-1}, f_i, f_{i+1}, \ldots, f_n) + 2\eta f_i^2 \Psi'(|f|^2) J_f + \eta \Psi(|f|^2) J_f . \]
By combining this with (4.3) we get
\[ \int_U \eta \left[ \Psi(|f|^2) + 2\Psi'(|f|^2) f_i^2 \right] J_f \]
\[ = - \int_U \Psi(|f|^2) f_i J(x; f_1, \ldots, f_{i-1}, f_i, f_{i+1}, \ldots, f_n) , \]
and by summing over $i$’s and applying Lemma 4.1 we have
\[ \left| \int_U \eta \left[ n\Psi(|f|^2) + 2\Psi'(|f|^2) |f|^2 \right] J_f \right| \]
\[ = \sum_{i=1}^n \int_U \Psi(|f|^2) f_i J(x; f_1, \ldots, f_{i-1}, f_i, f_{i+1}, \ldots, f_n) \]
\[ = \int_U \Psi(|f(x)|^2) \nabla \eta(x)^T \cof Df(x) f(x) \]
\[ = \int_{U \setminus \{x:|f(x)|=0\}} \Psi(|f(x)|^2) \nabla \eta(x)^T \cof Df(x) f(x) \]
\[ \leq \int_{U \setminus \{x:|f(x)|=0\}} \Psi(|f(x)|^2) |\nabla \eta(x)||f(x)| \|\cof Df(x)||f(x)|| dx \]
\[ \leq \int_U \Psi(|f(x)|^2) |\nabla \eta(x)||f(x)||\cof Df(x)|| dx , \]
which ends the proof. \qed
5. Proof of Theorem 1.2

To prove Theorem 1.2 we first assume without loss of generality that
\[ f(0) = 0 \quad \text{and} \quad i(0, f) = m. \]
By applying Lemma 2.1 we find a radius \( r_0 > 0 \) such that \( U(0, f, t) \) defines a normal neighborhood of the origin whenever \( 0 < t \leq r_0 \). Next, we fix a radius
\[ 0 < R < \min\{1, r_0\} \quad \text{so small that} \quad f(B(0, R)) \subset B(0, r_0), \]
and assume first that \( f \in C^\infty(U, \mathbb{R}^n) \).
Suppose now that \( t \in (0, R) \) and let \( \varepsilon > 0 \). By applying Lemma 4.2 with the standard cut-off function
\[ \eta_\varepsilon(s) = \begin{cases} 1, & \text{if } 0 \leq s \leq t - \varepsilon \\ \frac{s}{t - s}, & \text{if } t - \varepsilon < s < t \\ 0, & \text{if } s \geq t, \end{cases} \]
we get for any function \( \Psi \in C^1([0, \infty), [0, \infty)) \) the following estimate
\[
\left| \int_{B_t} \eta_\varepsilon [n \Psi(|f|^2) + 2|f|^2 \Psi'(|f|^2)] J_f \right| \leq \int_{B_t} |\nabla \eta_\varepsilon||f|\Psi(|f|^2)||\text{cof} \, Df| \leq \frac{1}{\varepsilon} \int_{B_t \setminus B_{t-\varepsilon}} |f|\Psi(|f|^2)||\text{cof} \, Df|,
\]
where and in what follows we denote \( B_t := B(0, t) \) for a given radius \( t > 0 \).
By letting \( \varepsilon \to 0 \) we conclude from (5.1) that
\[
\left| \int_{B_t} n \Psi(|f|^2) + 2|f|^2 \Psi'(|f|^2) \right| J_f \leq \int_{\partial B_t} |f|\Psi(|f|^2)||\text{cof} \, Df| dH^{n-1},
\]
see [EG92, 3.4.4]. Next we observe that for the function
\[ \Psi(s) = \frac{s^{-\frac{2}{n}}}{\log^{n-1}(1/s)} \]
we have
\[ \Psi'(s) = -\frac{n}{2\log^{n-1}(1/s)s^{\frac{2}{n}+1}} + \frac{n - 1}{\log^n(1/s)s^{\frac{2}{n}+1}}. \]
By approximating \( \Psi \) with \( C^1 \)-functions we conclude from (5.2) the following estimate
\[
2(n - 1) \int_{B_t} \frac{J_f}{|f|^{n+1}\log^n(1/|f|^2)} \leq \int_{\partial B_t} \frac{||\text{cof} \, Df||}{|f|^{n+1}\log^{n-1}(1/|f|^2)} dH^{n-1}.
\]
Now, the standard approximation argument shows that for almost every radius \( t \in (0, R) \) the estimate (5.3) is valid also for continuous Sobolev mappings
\[ f \in W^{1,n}_{\text{loc}}(U, \mathbb{R}^n) \cap C(U, \mathbb{R}^n). \]
Therefore, under the assumptions of the theorem we obtain by applying the estimate (5.3) together with the inner dilatation inequality

\[ \|\text{cof} \ Df(x)\|^{n} \leq K_{I}J_{f}(x)^{n-1} := K_{I}J_{f}(x)^{n-1} \text{ a.e.}, \]

and Hölder’s inequality that for almost every \( t \in (0, R) \) one has

\[
2(n-1) \int_{B_{t}} \frac{J_{f}}{|f|^{n} \log^{n}(1/|f|^{2})} \leq \int_{\partial B_{t}} \frac{\|\text{cof} \ Df\|}{|f|^{n-1} \log^{n-1}(1/|f|^{2})} \\
\leq K_{I}^{\frac{1}{n}} \int_{\partial B_{t}} \frac{J_{f}^{\frac{n-1}{n}}}{|f|^{n-1} \log^{n-1}(1/|f|^{2})} \\
\leq K_{I}^{\frac{1}{n}} |\partial B_{t}|^{\frac{1}{n}} \left( \int_{\partial B_{t}} \frac{J_{f}}{|f|^{n} \log^{n}(1/|f|^{2})} \right)^{\frac{n-1}{n}} \\
= K_{I}^{\frac{1}{n}} |\partial B_{t}|^{\frac{1}{n}} \left( \int_{\partial B_{t}} \frac{J_{f}}{|f|^{n} \log^{n}(1/|f|^{2})} \right)^{\frac{n-1}{n}},
\]

where \( \omega_{n-1} \) denotes the surface measure of the unit sphere and \( |\partial B_{t}| \) denotes the surface measure of the sphere \( \partial B_{t} \). By considering the increasing, non-negative function

\[ \varphi(t) = \int_{B_{t}} \frac{J_{f}}{|f|^{n} \log^{n}(1/|f|^{2})} \]

we may rewrite the estimate (5.4) as follows

\[ \varphi(t) \leq C_{n} t^{\frac{n-1}{n}} \left[ \varphi'(t) \right]^{\frac{n-1}{n}} \text{, where } C_{n} := \frac{K_{I}^{\frac{1}{n}} \omega_{n-1}^{\frac{1}{n}}}{2(n-1)}. \]

Therefore, we have

\[ \frac{1}{C_{n}^{\frac{1}{n-1}}} t \leq \frac{\varphi'(t)}{\varphi(t)\frac{n}{n-1}} = -(n-1) \frac{d}{dt} \left[ \varphi(t)^{-\frac{1}{n-1}} \right], \]

and by integrating both sides over an interval \( (r, R) \) we get

\[ \frac{1}{C_{n}^{\frac{1}{n}}} \log(R/r) \leq -(n-1) \left[ \varphi(R)^{-\frac{1}{n-1}} - \varphi(r)^{-\frac{1}{n-1}} \right] \leq (n-1) \varphi(r)^{-\frac{1}{n-1}}. \]

This gives us

\[ \varphi(r) \leq \frac{C_{n}^{n}(n-1)^{n-1}}{\log^{n}(1/R/r)}. \]

Next, we recall that by Lemma 3.1 we have

\[ B_{r/L} \subset f(B_{r}) \text{ for all sufficiently small radii } r > 0. \]

In addition, for a given set \( A \subset \mathbb{R}^{n} \) we recall the standard notation

\[ N(y, f, A) := \text{card } f^{-1}(y) \cap A. \]
Then by applying the change of variables formula [HeKo14, Theorem A.35] and the inclusion (5.6) we obtain the following lower estimate

\[ \psi(r) = \int_{B_r} \frac{J_f}{|f|^n \log^n(1/|f|^2)} = \int_{f(B_r)} \frac{N(y, f, B_r)}{|y|^n \log^n(1/|y|^2)} \, dy \]

(5.7)

\[ \geq \int_{B_{r/L}} \frac{N(y, f, U_{r/L})}{|y|^n \log^n(1/|y|^2)} \, dy \]

\[ = i(0, f) \int_{B_{r/L}} \frac{\log(1/|y|^2)}{|y|^n \log^n(1/|y|^2)} \, dy \]

for all sufficiently small \( r > 0 \), where the last equality follows from the fact that

\[ U_{r/L} := U(0, f, r/L) \]

is a normal neighborhood of the origin and from [Ric93, Proposition I.4.10]. By a direct calculation we get

\[ \int_{B_{r/L}} \frac{dz}{|y|^n \log^n(1/|y|^2)} = \int_0^{r/L} \frac{1}{\partial B_t} \frac{dt}{t \log^n(1/t^2)} \]

(5.8)

\[ = -\frac{\omega_{n-1}}{2^n} \int_{0}^{\log(L/r)} \frac{ds}{s^n} \]

\[ = -\frac{\omega_{n-1}}{2^n} \frac{\log(L/r)}{(n-1)2^n \log^{n-1}(\frac{L}{r})} \]

for all sufficiently small radii \( r > 0 \). Thus, by combining the estimates (5.5) and (5.7)–(5.8) we get

\[ i(0, f) \omega_{n-1} \frac{\log(L/r)}{(n-1)2^n \log^{n-1}(\frac{L}{r})} \leq \psi(r) \leq C_n(n-1)^{n-1} \frac{\log^{n-1}(R/r)}{2^n (n-1)^n \log^{n-1}(R/r)} \]

By simplifying both sides we finally obtain

\[ i(0, f) \leq K_I \left( \frac{\log(L/r)}{\log(R/r)} \right)^{n-1} \]

for all small \( r > 0 \).

Thus, by letting \( r \to 0 \) we observe that \( i(0, f) \leq K_I \) which concludes the proof.

6. Final remarks

We close the article with further observations on rigidity and Martio’s conjectures. We start by pointing out that in the literature several constructions on quasiregular mappings [BH04, GV73, KTW05], on mappings of bounded length distortion [MV88], and on mappings of finite distortion [GHT20] have the conjugated form

\[ h \circ w_m \circ g \]

(6.1)

where \( w_m \) stands for the standard \( m \)-to-1 winding map, and both \( h \) and \( g \) are some suitable homeomorphisms. In addition to this, by the results of Church and Hemmingsen [CH60, Theorem 4.1], Martio, Rickman, and
Väisälä [MRV71, Lemma 3.20], and Luisto and Prywes [LP21, Theorem 1.1] quasiregular maps with reasonably regular branch sets or branch set images are indeed topologically equivalent to the standard winding map. Therefore, studying this type of maps is not useful only for the sake of the rigidity and Martio’s conjectures, but also for its own right. On the other hand, from the point of view of these conjectures the following question stands out:

**Question 6.1.** Does the standard \( m \)-to-1 winding map minimize the inner dilatation for the non-planar quasiregular maps of the conjugated form \( (6.1) \) in the following sense

\[
K_I(w_m) = \min \left\{ K_I(h \circ w_m \circ g) : g \text{ and } h \text{ quasiconformal} \right\}?
\]

By answering the question above one would solve Martio’s conjecture for the conjugated quasiregular mappings. As one may notice, Theorem 1.2 already answers the question in the class of BLD-maps. Therefore, for BLD-maps a natural next step is to study the uniqueness of the winding map by investigating the accuracy of the rigidity conjecture.

We also point out that by the constructions in [BH04, KTW05] there exists even reasonably smooth quasiregular maps with branching. Furthermore, these examples can be even written as \( (6.1) \) for some quasiconformal maps \( h \) and \( g \). Therefore, by current knowledge studying Martio’s conjecture in the class of continuously differentiable quasiregular maps makes perfect sense as well as investigating the following question:

**Question 6.2.** Let us consider a continuously differentiable non-planar quasiregular map

\[
f : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad f = h \circ w_m \circ g,
\]

where \( g \) and \( h \) are quasiconformal maps. Does it follow that \( K_I(f) \geq m \)?

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