Milling Stability Analysis Based on Chebyshev Segmentation

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Abstract. Chebyshev segmentation method was used to discretize the time period contained in delay differential equation, then the Newton second-order difference quotient method was used to calculate the cutter motion vector at each time endpoint, and the Floquet theory was used to determine the stability of the milling system after getting the transfer matrix of milling system. Using the above methods, a two degree of freedom milling system stability issues were investigated, and system stability lobe diagrams were got. The results showed that the proposed methods have the following advantages. Firstly, with the same calculation accuracy, the points needed to represent the time period are less by the Chebyshev Segmentation than those of the average segmentation, and the computational efficiency of the Chebyshev Segmentation is higher. Secondly, if the time period is divided into the same parts, the stability lobe diagrams got by Chebyshev segmentation method are more accurate than those of the average segmentation.

1. Introduction

It is often required to mill a large number of materials from semifinished product with high efficiency and high precision in fields like tooling, electronics, aviation and aerospace. Practice indicates that chatter is an important factor for milling efficiently and precisely⁴, because it changes the relative position of the tool and the workpiece, reduces the efficiency of milling and the precision of the machined surface, exacerbates the wear of tool or even damages it shortening the service life of the machine and tools. Therefore, stability analysis of milling process and research on controlling the chatter in milling have become an important means of increasing milling efficiency, which is of great research value.

Stability analysis of milling process is an important way to reasonably select machining parameters such as spindle speed and the cutting depth to optimize the process parameters, avoid chatter and improve production efficiency. The periodic change of cutting force and the interruption of cutting process caused by the rotation of multi-tooth tool during milling make the stability analysis of milling process much harder than the stability analysis of continuous orthogonal cutting.

Scholars at home and abroad, aiming at the question of milling stability, have put forward many approximate calculation methods for the prediction of borderline curves of stability in the milling process involving critical process parameters, it is the so called stability lobe diagram (SLD)⁵. Insperger [3] has proposed a Semi-discretization Method (SDM) to judge the stability of a time delay system. Bayly [4] has
proposed a Temporal Finite Element Analysis (TFEA) for Single degree of freedom interrupted cutting. Ding [5], based on the dynamic response of a direct integration scheme in milling process, has put forward a Full-discretization Method (FDM), which is suitable for several semi-analytical operating modes like large (small) radial depth cuts and large (small) axial depth cuts. Insperger [6] has proved that Full-discretization Method and zero-order Semi-discretization Method have a local truncation error of the same order.

Aforementioned prediction methods of milling stability, such as Semi-discretization Method, Temporal Finite Element Analysis and Full-discretization Method, evenly split the time period of delay differential equations in milling. Chebyshev segmentation method presented in this paper unevenly segments time period T into m time units. State items and time period items of a tool at any time linearly represent the state items of the time unit endpoint and delayed items based on the Newton second-order difference quotient method represent by interpolation the state items of the time unit endpoint. One should firstly obtain the transfer matrix of milling system in every time unit, and then in a time period, finally using Floquet theory [7] to judge the stability of the milling system. From the results of time domain simulation we can get that the computational efficiency of Chebyshev unequally time-period-segmentation method is higher than that of equally time-period-segmentation method.

2. Chebyshev segmentation method
An approximate method to approximate function analytical expressions is to take n+1 different nodes in the function domain, where the function values are given. Then, the Lagrange and Newton interpolation method [8] constructed out of a single n degree interpolation polynomial to approximate the original function. By taking uniformly spaced points to construct an n+1 node interpolation integral formula, it has in general n degree algebraic accuracy. If nodes to construct the quadrature formula are properly selected, one can get a higher algebraic precision quadrature formula. The Chebyshev segmentation method [9] is such a method, its segmentation process is as follows.

Figure 1 shows the semicircle uniform angular segmentation according to angle \( \theta \) and the corresponding segmentation points of the arc projection on the x axis in the range \([-1,1]\).

\[
\theta = \frac{2N_0 + 1 - 2k}{2(N_0 + 1)} \pi (k = 0, 1, \ldots, N_0)
\]

\[
x_k' = \cos \theta = \cos \left( \frac{2N_0 + 1 - 2k}{2(N_0 + 1)} \pi \right)
\]

In the interval \([a, b]\), the x axis position expression is obtained as

\[
x_i = \frac{b - a}{2} x_i' + \frac{b + a}{2}
\]

\[
x_i = \frac{b - a}{2} \cos \left( \frac{2N_0 + 1 - 2k}{2(N_0 + 1)} \pi \right) + \frac{b + a}{2}
\]

\(k = 0, 1, \ldots, N_0\)

![Figure 1. Image of Chebyshev segmentation](image)
3. Full-discretization method

Consider regenerative chatter in milling whose delay differential equation system is

\[ X(t) = AX(t) + B(t)[X(t) - X(t - T)] \tag{4} \]

where \( A \) indicates the constant matrix of milling system features; \( B \) is the cycle matrix which depends on the regeneration effect of dynamic milling force, \( B(t) = B(t + T) \); \( T \) is the time period equal to the amount of time delay.

According to Chebyshev segmentation method time period \( T \) was divided into \( m \) uniform time units.

\[ T = \sum_{k=0}^{m-1} \Delta t_k \quad (k = 0, 1, \ldots, m - 1) \tag{5} \]

\[ \Delta t_k = t_{k+1} - t_k = \frac{T}{2} (t_{k+1}' - t_k') \quad (k = 0, 1, \ldots, m - 1) \tag{6} \]

\[ t_k' = \cos \left( \frac{2m + 1 - 2k}{2(m + 1)} \pi \right) \tag{7} \]

\[ t_{k+1}' = \cos \left( \frac{2m + 1 - 2(k + 1)}{2(m + 1)} \pi \right) \tag{8} \]

In the first time unit \( t_k \leq t \leq t_{k+1} \), the response of the formula (4) can be written as the following direct integral:

\[ X(t) = e^{A(t-t_k)} X(t_k) + \int_0^{t_k} [e^{A(t-t)} B(\xi) X(\xi)] d\xi - \int_0^{t_k} [e^{A(t-t)} B(\xi) X(\xi - T)] d\xi \tag{9} \]

where \( X(t_k) \) represents the cutting tool state vector at \( t_k \). At the moment \( t_{k+1} \), cutting tool state vector \( X(t_{k+1}) \) can be represented as

\[ X(t_{k+1}) = e^{A(t_{k+1} - t_k)} X(t_k) + \int_0^{t_k} [e^{A(t_{k+1} - \xi)} B(t_{k+1} - \xi) X(t_{k+1} - \xi)] d\xi - \int_0^{t_k} [e^{A(t_{k+1} - \xi)} B(t_{k+1} - \xi) X(t_{k+1} - \xi - T)] d\xi \tag{10} \]

The integral time interval for the above formula is \([0, \Delta t_k] \). The periodic coefficient matrix \( B(t_{k+1} - \xi) \) are linear approximation on the time unit \([0, \Delta t_k] \). The delay item \( X(t_{k+1} - \xi - T) \) can be expressed in terms of the state vectors \( X(t_{k+1} - T) \) and \( X(t_k - T) \), which are the two endpoints of the milling cutter, linear representation by the Newton second-order difference quotient method on the time unit \([t_k - T, t_{k+1} - T] \), as in Eqs.(11); substituting Eqs.(11) into Eqs. (10), one gets Eqs.(12).

\[
X(t_{k+1} - \xi) = \frac{\xi^2 - \xi \Delta t_k + \Delta t_k \Delta t_{k+1}}{\Delta t_k (\Delta t_k + \Delta t_{k+1})} X(t_{k+1}) + \frac{\xi (\Delta t_k + \Delta t_{k+1} - \xi)}{\Delta t_k (\Delta t_k + \Delta t_{k+1})} X(t_k) + \frac{(\Delta t_{k+1} + \Delta t_k - \xi) (\Delta t_k - \xi)}{\Delta t_k (\Delta t_k + \Delta t_{k+1})} X(t_{k+1}) - \int_0^{t_k} [e^{A(t_{k+1} - \xi)} B(t_{k+1} - \xi) X(t_{k+1} - \xi - T)] d\xi \]

\[ X(t_{k+1}) = F_{k+1} X_{k+1} + (F_0 + F_{k+1}) X_k + F_{k+1} X_{k+1} - F_{k+1} X_{k+1} - F_{k+1} X_{k+1} - F_{k+1} X_{k+1} \tag{11} \]

where

\[
F_{k+1} = \frac{\Phi_3}{\Delta t_k} + \frac{\Phi_4}{\Delta t_k + \Delta t_{k+1}} + \frac{\Phi_5}{\Delta t_k (\Delta t_k + \Delta t_{k+1})} B_{k+1} \quad \Phi_2 = \frac{\Phi_3}{\Delta t_k} + \frac{\Phi_4}{\Delta t_k + \Delta t_{k+1}} + \frac{\Phi_5}{\Delta t_k (\Delta t_k + \Delta t_{k+1})} B_{k+1} \quad \Phi_1 = \frac{\Phi_3}{\Delta t_k} + \frac{\Phi_4}{\Delta t_k + \Delta t_{k+1}} + \frac{\Phi_5}{\Delta t_k (\Delta t_k + \Delta t_{k+1})} B_{k+1} \tag{12} \]

Defining

\[
\Phi_0 = e^{A t_0} \quad \Phi_1 = \int_0^{t_0} e^{A t} d\xi \\
\Phi_2 = \int_0^{t_0} e^{A t} \cdot \xi d\xi \\
\Phi_3 = \int_0^{t_0} e^{A t} \cdot \xi^2 d\xi \\
\Phi_4 = \int_0^{t_0} e^{A t} \cdot \xi^3 d\xi \tag{13} \]

the variables in formula (12) can be expressed as

\[
F_{k+1} = (\Phi_1 - \frac{\Phi_2}{\Delta t_k} - \frac{\Phi_3}{\Delta t_k + \Delta t_{k+1}} + \frac{\Phi_4}{\Delta t_k (\Delta t_k + \Delta t_{k+1})}) B_{k+1} + (\Phi_2 - \frac{\Phi_3}{\Delta t_k} - \frac{\Phi_4}{\Delta t_k + \Delta t_{k+1}} + \frac{\Phi_5}{\Delta t_k (\Delta t_k + \Delta t_{k+1})}) B_{k+1} \tag{14} \]
\[ F_{k+1} = (\Delta t_{k} (\Delta t_{k} + \Delta t_{k-1}) \Delta t_{k} + \Delta t_{k-1}) B_{0}^{(i)} + (\Delta t_{k} (\Delta t_{k} + \Delta t_{k-1}) \Delta t_{k} + \Delta t_{k-1}) B_{1}^{(i)} \]  
\[ F_{k,i} = (\Delta t_{k-1} + \Delta t_{k-2} \Delta t_{k} - \Delta t_{k-1} \Delta t_{k} - \Delta t_{k-2} \Delta t_{k}) B_{0}^{(i)} + (\Delta t_{k-1} + \Delta t_{k-2} \Delta t_{k} - \Delta t_{k-1} \Delta t_{k} - \Delta t_{k-2} \Delta t_{k}) B_{1}^{(i)} \]  

Assuming \([I - F_{k,i}]^{-1}\) exists, the transfer matrix of each time element \(\Psi_{i}\) in the milling system can be expressed as:

\[
\Psi_{i} = \begin{bmatrix}
[I - F_{k,i}]^{-1}(F_{0} + F_{k,i}) & [I - F_{k,i}]^{-1} F_{k-1} & 0 & \cdots & 0 & -[I - F_{k,i}]^{-1} F_{k-1} & -[I - F_{k,i}]^{-1} F_{n} \\
I & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & I & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & I & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & I \\
\end{bmatrix}
\]  

The transfer matrix \(\Psi\) of milling system in a time period \(T\) can be expressed by the sequence \(\Psi_{i}\) (\(k=0, 1, \cdots, m-1\)) of the transfer matrix of the time element in this period, as

\[
\Psi = \Psi_{m-1} \Psi_{m-2} \cdots \Psi_{1} \Psi_{0}
\]  

Finally, according to Floquet theory, the stability of the milling system can be determined by the characteristic values of the system transfer matrix \(\Psi\): if all the moduli of the characteristic values of the transfer matrix are less than 1, the system is stable, otherwise, the system is unstable.

4. Stability analysis with Chebyshev segmentation method

4.1. Parameters of milling system

Two degree of freedom milling system is as follows\([12]\): Groove type straight milling cutter, number of cutter teeth \(N=2\), mass \(m_{tx}=0.03993\) kg, \(m_{ty}=0.03993\) kg; damping ratio \(\zeta=0.011\), natural frequency of the system \(f_{n}=\omega_{n}/(2\pi)\)=922 Hz, The range of milling speed \(\Omega\) is \(0.5 \times 10^{4} \rightarrow 2.5 \times 10^{4}\) r/min; the depth of cutting \(\omega\) varies in the range of \(0 \sim 0.01\)m; the linear tangential cutting force coefficient \(K_{t}=6 \times 10^{8}\)N/m²; the linear normal cutting force coefficient \(K_{n}=2 \times 10^{8}\)N/m².

The constant matrix \(A\) of milling system characteristics is as follows:

\[
A = \begin{bmatrix}
-63.72 & 0 & 25.04 & 0 \\
0 & -63.72 & 0 & 25.04 \\
-1.34 \times 10^{6} & 0 & -63.72 & 0 \\
0 & -1.34 \times 10^{6} & 0 & -63.72 \\
\end{bmatrix}
\]

The periodic matrix \(B\) determined by the regeneration effect of the dynamic milling force is as follows:

\[
B(t) = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-w_{h_{x}}(t) & -w_{h_{y}}(t) & 0 & 0 \\
-w_{h_{y}}(t) & -w_{h_{x}}(t) & 0 & 0 \\
\end{bmatrix}
\]

\[
h_{xx}(t) = \sum_{j=1}^{N} g(\phi_{j}(t)) \sin(\phi_{j}(t)) \left[ K_{t} \cos(\phi_{j}(t)) + k_{r} \sin(\phi_{j}(t)) \right]
\]

\[
h_{xy}(t) = \sum_{j=1}^{N} g(\phi_{j}(t)) \cos(\phi_{j}(t)) \left[ K_{t} \cos(\phi_{j}(t)) + k_{r} \sin(\phi_{j}(t)) \right]
\]

\[
h_{yx}(t) = \sum_{j=1}^{N} g(\phi_{j}(t)) \sin(\phi_{j}(t)) \left[ -K_{t} \sin(\phi_{j}(t)) + k_{r} \sin(\phi_{j}(t)) \right]
\]
\[ h_{xy}(t) = \sum_{j=1}^{N} g(\phi_j(t)) \cos(\phi_j(t)) \left[-K_1 \sin(\phi_j(t)) + k_r \sin(\phi_j(t))\right] \]

where \( w \) is the axial cutting depth; \( \phi_j(t) \) is the angular position of the \( j \)th tooth; \( g(\phi_j(t)) \) is the window function, which is utilized to demonstrate whether the \( j \)th tooth in or out of the cut.

### Table 1. Comparison of two degrees of freedom model of milling algorithm

| \( a/D \) | Full-discretization of Chebyshev segmentation (\( m=30 \)) | Full-discretization of average segmentation (\( m=40 \)) |
|-----------|-------------------------------------------------|-------------------------------------------------|
| \( 0.1 \) | ![Graph](image1.png) | ![Graph](image2.png) |
| computation time (s) | 242.819 | 501.195 |
| \( 0.2 \) | ![Graph](image3.png) | ![Graph](image4.png) |
| computation time (s) | 251.978 | 509.060 |

#### 4.2. Results of stability analysis

The rotation speed-axial cutting depth plane is equally divided into \( 200 \times 100 \) grid point. It means the rotational speed step is 100r/min, and the axial cutting depth step is 0.1mm. The limit of the milling stability of the radial immersion ratio \( a/D \) is 0.1 and 0.2 are calculated respectively. According to the theoretical basis and algorithm principle, we can write a calculation program and draw the stability lobes diagram on a personal computer [Intel Core 3, 4GHz, 2GB]

![Stability diagram](image5.png)

**Figure 2. Comparison of two different segmentation stability lobes diagram.**

(down-milling \( N=2, \ a/D=0.1 \))

From Table 1, one can see that using a lobe diagram to predict the stability of milling system, with the condition that the number of points of Chebyshev segmentation is lower than that of the average
segmentation method, the lobe diagram can still achieve the same accuracy. From comparison of the two methods’ computation time, it also can be seen the Chebyshev segmentation method has the better computational efficiency than the average segmentation method when these two methods obtain the same accuracy of lobes diagram.

Figure 2 shows the comparison chart of two different segmentation stability lobes diagram with a given parameter $a/D=0.1$, and $m=40$. This chart suggests that the stability ranges of the milling system obtained by these two methods are different. In Figure 2, four working points were taken. The working point’s milling parameters are shown in table 2. The results of time domain simulation for the working point listed in table 2 are shown in Figure 3 (a) (b) (c) (d). In figure 3 (a) (b) (c), the milling process corresponding to working point 1, 2, 3 is stable, in figure(d) the milling process corresponding to working point 4 is unstable. The stability of the four points of time domain simulation results shown in figure 3 are fully consistent with the results predicted by Chebyshev segmentation method shown in table 2, which is much different from the stability of the four working points predicted by the average segmentation method. So it can be concluded that the milling system stability prediction results of Chebyshev segmentation method have better accuracy compared to the average segmentation method, when the number of segmentation points are same.

| Working point | Spindle speed $\Omega$ (r/min) | Axial cutting depth $w$ (mm) | Chebyshev segmentation | Average segmentation |
|---------------|-------------------------------|-----------------------------|------------------------|----------------------|
| point 1       | 15000                         | 1.5                         | stable                 | unstable             |
| point 2       | 20000                         | 1                           | stable                 | stable               |
| point 3       | 20000                         | 2                           | stable                 | unstable             |
| point 4       | 20000                         | 3                           | unstable               | unstable             |
5. Conclusions

This paper describes the Chebyshev segmentation method and the Newton second-order difference quotient method for interpolation integral method of discrete items. Analysis of the two milling systems DOF example, by the drawing system stability lobe diagram available: (1) To achieve the same prediction accuracy condition, Chebyshev unequal segmentation method needs fewer points than average points needed for segmentation, and the calculating efficiency of chebyshev unequal segmentation method is higher than that of average segmentation methods; (2) Under the condition of the same number of segmentation points, Chebyshev unequal segmentation method to predict the stability of the system is more accurate than the average segmentation method to predict the stability of the system.

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