Duality, Quantum Vortices and Anyons in Maxwell-Chern-Simons-Higgs Theories

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Abstract

The order-disorder duality structure is exploited in order to obtain a quantum description of anyons and vortices in: a) the Maxwell theory; b) the Abelian Higgs Model; c) the Maxwell-Chern-Simons theory; d) the Maxwell-Chern-Simons-Higgs theory. A careful construction of a charge bearing order operator ($\sigma$) and a magnetic flux bearing disorder operator (vortex operator) ($\mu$) is performed, paying attention to the necessary requirements for locality. An anyon operator is obtained as the product $\varphi = \sigma \mu$. A detailed and comprehensive study of the euclidean correlation functions of $\sigma$, $\mu$ and $\varphi$ is carried on in the four theories above. The exact correlation functions are obtained in cases a and c. The large distance behavior of them is obtained in cases b and d. The study of these correlation functions allows one to draw conclusions about the condensation of charge and magnetic flux, establishing thereby an analogy with the Ising model. The mass of vortex and anyon excitations is explicitly obtained wherever these excitations are present in the spectrum. The independence between the mechanisms of mass generation for the vortices and for the vector field is clearly exposed.
1) Introduction

The physics of 2 + 1 dimensional world has been the object of intense theoretical investigation in the last few years. One of the features which has aroused a lot of interest is the possibility of existence of states with generalized, continuous statistics. Chern-Simons field theories [1] have been playing a central role in this framework. An important reason for this is that the presence of a Chern-Simons term in the lagrangian of a vector field induces a change in the statistics of the charged particles eventually coupled to it [2]. This statistical transmutation effectively occurs because the coupling of a charged particle to a Chern-Simons vector field imparts to this particle a point magnetic flux [2] and, as was demonstrated in full generality [3], the bound state of a point charge and a point magnetic flux carries arbitrary statistics proportional to the product: charge $\times$ magnetic flux. A particular case of this fact was first observed in a specific model in [4].

The mechanism of statistical transmutation via the Chern-Simons lagrangian is a key ingredient of field theories proposed to describe very interesting condensed matter systems such as the two dimensional electron gas undergoing the Quantum Hall effect [5]. It was also invoked in the theory of superconductivity [6] involving particles with generalized statistics or anyons as they become known.

In view of their vast potential physical interest, a large amount of study of anyon properties was undergone recently [7]. In order to obtain a full quantum field theoretic description of anyons one should be able to evaluate their correlation functions, mass spectrum and scattering amplitudes. The purpose of this work is to accomplish some of these goals in the framework of four different quantum field theories: a) the Maxwell theory; b) the Abelian Higgs Model (Maxwell-Higgs theory); c) the Maxwell-Chern-Simons theory; d) the Maxwell-Chern-Simons-Higgs theory.

A lot of work has been devoted to the study of generalized statistics in 1 + 1 D [8,4,9]. Actually the investigation performed in 1 + 1D provided a fundamental clue for understanding the quantum description of anyons, either in 1 + 1D and 2 + 1D. This consists in the realization that there is a basic algebraic structure underlying
the existence of generalized statistics excitations, namely, the order-disorder duality [10]. The idea is to introduce order ($\sigma$) and disorder ($\mu$) operators satisfying a certain algebra (dual algebra) in such a way that the composite operator $\varphi = \sigma\mu$ made out of them is shown to be in general an anyon operator [4]. One then generalizes the method of Kadanoff and Ceva [10] for continuum field theory [4,11] in order to obtain the euclidean correlation functions of $\sigma$, $\mu$ and $\varphi$. A review of this method is in ref. [12]. Very interesting and important related works are found in [13].

A few years ago, the order-disorder duality structure was generalized for 2 + 1-dimensional continuum field theory [14]. A magnetic flux bearing disorder operator $\mu$ was introduced and its correlation functions obtained as euclidean functional integrals. It was also shown that the order operator $\sigma$ dual to $\mu$ should now be a charge bearing operator. As a consequence, the product $\sigma\mu$ is naturally seen to be an anyon operator. This description of anyons goes beyond the Chern-Simons mechanism because one can study generalized statistics excitations even in theories without a Chern-Simons term. Very interesting related works were done previously on the the lattice [15] and subsequently in the continuum [16].

An extremely interesting and appealing feature of this approach emerges when one realizes that the magnetic flux is the topological charge in 2 + 1D. Hence the disorder operator $\mu$ or the anyon itself, must be closely related to topological charge bearing excitations in 2 + 1D, namely, topological solitons or vortices. This relationship was investigated and demonstrated in [14,16] where the disorder variable $\mu$ was shown to be a quantum vortex creation operator.

The above ideas were recently applied in the complete bosonization of the Dirac fermion field in 2 + 1D [18]. In this case the $\sigma$ and $\mu$ dual operators were expressed in terms of a bosonic vector field $W_\mu$ with dynamics governed by a nonlocal version of the Maxwell action plus a Chern-Simons term.

It follows from the work done in [14] that some stringent requirements must be met in order for the vortex operator to be local. Also, a lot of care must be exercised in order to obtain the charge bearing local order operator $\sigma$ dual to it. We will carefully analyze the construction of local order, disorder and anyon operators in the various
theories under consideration, according to the requirements set forth in [14].

After expressing the fields $\sigma$, $\mu$ and $\varphi$ in terms of a vector field $W_\mu$, we make a detailed study of the euclidean correlation functions of these operators and evaluate the commutation rules among themselves as well as between them and the charge and magnetic flux operators in the four theories mentioned above. Exact results for all correlation functions are obtained in the case of the Maxwell and Maxwell-Chern-Simons theories. The long distance behavior of the correlation functions is obtained in the case of the Abelian Higgs Model and Maxwell-Chern-Simons-Higgs theories, both in the symmetric and broken phases.

We investigate the conditions which allow the presence of genuine vortex on anyon excitations in the spectrum and in these cases, evaluate their masses, for all of the above mentioned theories.

As in $1+1$D [4], we observe that a nontrivial commutation relation between operators, manifests itself in the multivaluedness of the corresponding euclidean correlation functions. Analytic continuation back to Minkowski space, from each sheet, will lead to the various orderings of operators. We are thereby able to retrieve the commutation rules directly from the study of euclidean correlation functions. We show that the general expression for the statistics of a field bearing a charge $Q$ and a magnetic flux $\Phi$ is $S = \tilde{Q}\Phi/4\pi$, where $\tilde{Q} = Q - \theta\Phi$ with $\theta$ being the coefficient of the Chern-Simons term eventually present in the theory.

We can envisage some interesting applications of our formalism in field theories related to condensed matter systems. Among them we could mention the Chern-Simons-Landau-Ginsburg theory of the Quantum Hall Effect [19] and the theories describing the Hall-Superconducting phase transition in anyon systems [20]. We are presently investigating such applications.

The material is organized in the next four chapters in such a way that each chapter corresponds to one of the theories mentioned above. Concluding remarks and the main conclusions are presented in chapter 6. Six Appendixes are included to demonstrate useful results.
2) The Maxwell Theory

2.1) Order and Disorder (Vortex) Operators

2.1.1) The Disorder (Vortex) Operator

Let us start by considering the Maxwell theory, described by

\[ \mathcal{L} = -\frac{1}{4} W_{\mu\nu} W^{\mu\nu} \]  

which leads to Maxwell equation

\[ \partial_\nu W^{\nu\mu} = 0 \]  

The identically conserved topological current is given by

\[ J^\mu = \epsilon^{\mu\alpha\beta} \partial_\alpha W_\beta \]  

We can also introduce the two indexes dual topological current

\[ \tilde{J}^{\mu\nu} \equiv \epsilon^{\mu\nu\alpha} J_\alpha \quad \text{or} \quad \tilde{J}^{\mu\nu} = W^{\mu\nu} \]  

\( \tilde{J}^{\mu\nu} \) is not identically conserved; rather, its conservation is implied by the field equation (2.2).

A topological charge (magnetic flux) bearing operator (vortex operator) was first introduced as a disorder variable in continuum 2 + 1 D field theory in [14]. A fundamental piece in the construction of the vortex operator in [14] and which was also used in [18] is the singular external field

\[ A_\mu(z; x; C; T_x) = \int_{T_x(C)} d^2 \xi \arg(\vec{\xi} - \vec{x}) \delta^3(z - \xi) \]  

In this expression, \( \arg(\vec{z}) \) is the angle the vector \( \vec{z} \) makes with the z\(^1\) axis (\( z = (z^0, \vec{z}) \)). \( T_x(C) \) is the portion of the \( R^2 \) plane at \( \xi^0 = x^0 \), external to the curve C. C is the curve depicted in Fig. 1 which contains the arc of a circumference of radius \( \rho \), centered on \( (x^0, \vec{x}) \) and two straight lines along the cut of \( \arg(\vec{z} - \vec{x}) \) separated an angle 2\( \delta \) apart.
The vortex operator (disorder operator) \( \mu \) is one of the two basic dual operators whose properties we are going to investigate in this work. It is obtained by coupling the \( A_\mu \) external field to the dual current \( \tilde{j}^{\mu\nu} = W^{\mu\nu} \) in the following way [14]

\[
\mu(x; C) = \exp\left\{-\frac{ib}{2} \int d^3z W^{\mu\nu} A_{\mu\nu}\right\}
\]

(2.6)

In this expression, \( b \) is a free real parameter with dimension of \((mass)^{-\frac{3}{2}}\) and \( A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). In [14] it is shown that the correlation functions of \( \mu \) are completely independent of the surface \( T_x \) appearing in the definition of \( A_\mu \), provided we introduce a surface renormalization counterterm depending on \( A_\mu \) and having the same form as the kinetic term of the \( W_\mu \) field, namely

\[
\mathcal{L}_R = -\frac{b^2}{4} A_{\mu\nu} A^{\mu\nu}
\]

(2.7)

In the presence of this counterterm, it is clear that the integrand in the functional integral defining \( \mu \) correlation functions depends on the external field \( A_\mu \) through the combination \( W_\mu + A_\mu \). (See Eq.(2.26)) This is a fundamental requirement for surface invariance because making a change of variable of the type \( W_\mu \rightarrow W'_\mu = W_\mu + \partial_\mu \omega \), which of course leaves the measure invariant we have \( W_\mu + A_\mu \rightarrow W'_\mu + A_\mu - \partial_\mu \omega \). One can demonstrate [14] that for an appropriate choice of \( \omega \), we get \( A_\mu(z; x; C; \tilde{T}_x) - \partial_\mu \omega = A_\mu(z; x; C; T_x) \), where

\[
A_\mu(z; x; C; \tilde{T}_x) = \int_{\tilde{T}_x} d^2\xi \arg(\vec{\xi} - \vec{x}) \delta^3(z - \xi) - \Theta(V(\tilde{T}_x)) \partial_\mu \arg(\vec{z} - \vec{x})
\]

(2.8)

Here \( \tilde{T}_x \) is an arbitrary surface bounded by \( C \) and \( V(\tilde{T}_x) \) is the volume enclosed by \( \tilde{T}_x(C) \) and \( T_x(C) \). \( \Theta(V(\tilde{T}_x)) \) is the three-dimensional heaviside function with support on \( V(\tilde{T}_x) \). (See Fig. 2 for \( \tilde{T}_x(C) \) and \( V(\tilde{T}_x) \)). It is not difficult to realize [14] that the \( \omega \) which leads from (2.5) to (2.8) is

\[
\omega = \Theta(V(\tilde{T}_x)) \arg(\vec{z} - \vec{x})
\]

(2.9)

\( A_\mu(z; x; C; \tilde{T}_x) \), eq. (2.8), is the most general form of the external field [14]. Observe that \( A_\mu(z; x; C; T_x) \), eq. (2.5), is a special case of \( A_\mu(z; x; C; \tilde{T}_x) \), eq. (2.8); when \( \tilde{T}_x \) reduces to \( T_x \), since, in this case \( V(\tilde{T}_x) \) goes to zero along with the second term in
(2.8). As one would expect, the external field intensity tensor is not changed by the transformation which takes (2.5) into (2.8), namely \( A_{\mu\nu}[A_{\mu}(T_x)] = A_{\mu\nu}[A_{\mu}(\tilde{T}_x)] \). We emphasize that in the presence of the counterterm (2.7), as we will explicitly see, the correlation functions of \( \mu \) are completely independent of \( T_x \) or \( \tilde{T}_x \), only depending on \( x \) and \( C \). A local limit for \( \mu \) can then be obtained by using \( \rho \), the radius of the circumference part of \( C \) and \( \delta \), the angular width of the region along the cut (See Fig. 1), as regulators and at the end of all calculations, taking the limit in which \( \rho \) and \( \delta \) vanish. As we will see, the \( \mu \) correlation functions will only depend on \( x \) in this limit. This is the procedure we are going to adopt in order to obtain local vortex (disorder) correlation functions and operators.

2.1.2) The Order Operator

Let us introduce here the order operator which is going to be dual to \( \mu \). The natural way of defining the order operator, which is suggested by the study of a wide class of models in \( 1+1D \) [12] and also in \( 2+1D \) [18,24], is to couple the external field \( A_{\mu} \) to the topological current \( J_{\mu} \) in the following way

\[
\Sigma(x; C) = \exp\left\{ia \int d^3z J_{\mu} A_{\mu} \right\} = \exp\left\{ia \int d^3z \varepsilon^{\mu\alpha\beta} A_{\mu} \partial_\alpha W_\beta \right\}
\]

(2.10a)

or

\[
\Sigma'(x; C) = \exp\left\{ia \int d^3z W_{\mu} J_{\mu}[A_{\mu}] \right\} = \exp\left\{ia \int d^3z \varepsilon^{\mu\alpha\beta} W_{\mu} \partial_\alpha A_\beta \right\}
\]

(2.10b)

which differs from \( \Sigma \) by a boundary term. In (2.10), \( a \) is a free real parameter with dimension of \((\text{mass})^{1/2}\).

It turns out that we cannot obtain a local operator out of \( \Sigma \) or \( \Sigma' \) even in the limit where \( \rho, \delta \to 0 \). No surface renormalization counterterm will render the \( \Sigma \) correlation functions surface invariant in theories containing a Maxwell term in the action. This is so, because in contraposition to the case of \( \mu \), no renormalization counterterm will ever be able to make the integrand in the functional integral describing \( \Sigma \) or \( \Sigma' \) correlation functions to depend on the external field \( A_{\mu} \) in the combination \( W_{\mu} + A_{\mu} \), such that the procedure employed in [14] to show the surface invariance of \( \mu \) could be applied. The only case in which the \( \Sigma \) or \( \Sigma' \) operators could be made local would
be a theory containing just the Chern-Simons term in its kinetic lagrangian. In this case the renormalization counterterm would be a Chern-Simons term involving the external field. In this case however, the \( \mu \) operator would no longer be local! Actually, as we will see, in theories containing both the Maxwell and Chern-Simons terms, we will have to redefine \( \mu \) by adding a \( \Sigma' \) piece, in order to obtain a local disorder (vortex) operator.

In order to construct a local operator \( \sigma(x) \) dual to \( \mu(x; C) \), we are going to take advantage of the Cauchy-Riemann equations relating the real and imaginary parts of the analytic function \( \ln z \)(see Appendix A). Let us define

\[
\sigma(x^0, \vec{x}) = \lim_{\rho, \delta \to 0} \exp \left\{ ia \int_{R_x(C)} d^2 \xi \left[ \epsilon^{ij} \partial_i \mathcal{G}_j \arg(\xi - \vec{x}) + \partial_i \ln |\xi - \vec{x}| W_j(x^0, \xi) \right] \right\} \tag{2.11}
\]

In the above expression \( a \) is a free real parameter with dimension of \((mass)^{1/2}\) and the integral is performed over the region \( R_x(C) \), depicted in Fig. 3. \( R_x(C) \) is the part of the \( R^2 \) plane external to the curve \( C \). This consists of the arc of a circumference of radius \( \rho \) centered on \( \vec{x} \) and crossing the cut of the function \( \arg(\xi - \vec{x}) \) and two straight lines, along this cut, separated an angle \( 2\delta \) apart.

Using the Cauchy-Riemann equation, eq.(A.2) one immediately realizes that the exponent in (2.11) is different from zero only on the singularities of the \( \arg(\xi - \vec{x}) \) and \( \ln |\xi - \vec{x}| \) functions, that is, the point \( x = (x^0, \vec{x}) \) and the cut of \( \arg(\xi - \vec{x}) \). It is therefore clear that when we take out the regulators: \( \rho, \delta \to 0, \sigma \) will only depend on \( x \). This will be made at the end of all calculations, as in the case of \( \mu \).

As we shall see, \( \sigma \) as given by (2.11) is the correct local operator dual to \( \mu \).

2.2) Commutation Rules. Anyon Operators

Let us study here the various commutation rules involving the \( \sigma \) and \( \mu \) operators. These will enable us to construct an anyon operator out of \( \sigma \) and \( \mu \).

The basic equal-time commutators of the Maxwell theory are

\[
[W^i, W^j] = [E^i, E^j] = [\Pi^i, \Pi^j] = 0 \quad i, j = 1, 2
\]

\[
[W^i(x), E^j(y)] = -[W^i(x), \Pi^j(y)] = -i\delta^{ij} \delta^2(\vec{x} - \vec{y}) \tag{2.12}
\]
Here $E^i = W^{i\omega}$ is the electric field and $\Pi^i = -E^i$ is the momentum canonically conjugate to $W^i$.

We show in Appendix B that the $\mu$ operator, eq. (2.6) can be cast in the form

$$
\mu(x; C) = \exp\{ -ib \int_{T_x(c)} d^2\xi E^i(x^0, \xi) \partial_i \text{arg}(\xi - \bar{x}) \} 
$$

(2.13)

Using (2.12) we immediately see that $[\mu, \mu] = [\sigma \sigma] = 0$. The $\mu - \sigma$ commutation rule can be obtained by writing $\mu_{\rho\delta}(x; C) \equiv e^{A(x; C)}$, $\sigma_{\rho\delta}(y; C) \equiv e^{B(y; C)}$ and using the formula $e^A e^B = e^B e^A e^{[A, B]}$, which is valid when $[A, B]$ is a c-number. Making use of (2.13), (2.11) and (2.12), we find

$$
[A(x; C), B(y; C)] = ab \int_{T_y(x)} d^2\eta \int_{R_y(x)} d^2\eta \partial_i \text{arg}(\xi - \bar{x}) [\varepsilon^{kl}\partial_k \text{arg}(\eta - \bar{y}) \\
+ \partial_l \ln |\eta - \bar{y}| ]( -i ) \delta^{ij} \delta^2(\xi - \eta) 
$$

(2.14a)

or

$$
[A(x; C), B(y; C)] = -iab \int_{R_y(x) \cap T_y(x)} d^2\eta [\varepsilon^{kl}\partial_k \text{arg}(\eta - \bar{y}) + \partial_l \ln |\eta - \bar{y}| ]\partial_l \text{arg}(\eta - \bar{x}) 
$$

(2.14b)

In Appendix C it is shown that after taking the limit $\rho, \delta \to 0$ in $R_y(x)$, we obtain the following result for (2.14): $(B(y) \equiv \lim_{\rho, \delta \to 0} B(y; C))$

$$
[A(x; C), B(y)] = \begin{cases} 
-i2\pi ab \arg(\bar{y} - \bar{x}) & \bar{y} \in T_x(C) \\
0 & \bar{y} \notin T_x(C) 
\end{cases} 
$$

(2.15)

We therefore conclude that the equal-time commutation rule between $\sigma$ and $\mu$ is

$$
(\sigma(y) \equiv \lim_{\rho, \delta \to 0} \sigma_{\rho\delta}(y; C))
$$

$$
\mu_{\rho\delta}(x; C) \sigma(y) = \begin{cases} 
\sigma(y) \mu_{\rho\delta}(x; C) \exp\{ -i2\pi ab \arg(\bar{y} - \bar{x}) \} & \bar{y} \in T_x(C) \\
\sigma(y) \mu_{\rho\delta}(x; C) & \bar{y} \notin T_x(C) 
\end{cases} 
$$

(2.16)

This is precisely the U(1) dual algebra appropriate for a vortex creation operator introduced in [14]. Taking the local limit for $\mu_{\rho\delta}$, we obtain the local dual algebra (equal times) $(\mu(x) \equiv \lim_{\rho, \delta \to 0} \mu_{\rho\delta}(x; C))$

$$
\mu(x) \sigma(y) = \sigma(y) \mu(x) \exp\{ -i2\pi ab \arg(\bar{y} - \bar{x}) \} 
$$

(2.17)
Let us investigate now the commutation rules of $\sigma$ and $\mu$ with the charge and magnetic flux operators, given respectively by

$$Q = \int d^2x \rho(x) \quad \text{and} \quad \Phi = \int d^2x B(x)$$

where

$$\rho(x) = \partial_i E^i(x) \quad \text{and} \quad B(x) = -\epsilon^{ij} \partial_i W_j(x) \quad (2.18)$$

Using (2.12) we readily conclude that $[\sigma, B] = [\mu, \rho] = 0$, indicating that $\sigma$ does not bear magnetic flux and $\mu$ does not bear charge. The commutation rules $[\sigma, \rho]$ and $[\mu, B]$ may be obtained by the use of the formula $[e^\alpha, \beta] = e^\alpha [\alpha, \beta]$, valid when $[\alpha, \beta]$ is a c-number.

For $e^{\alpha(x)} \equiv \sigma(x)$ and $\beta(y) \equiv \rho(y)$, using (2.11), (2.18) and (2.12), we immediately get

$$[\alpha(x), \beta(y)] = \lim_{\rho, \delta \to 0} (-a) \int_{R_+(C)} d^2\xi \left[ e^{ij} \partial_i \arg (\xi - x) + \partial_j \lim |\xi - x| \partial_j^{(y)} \delta^2(\xi - y) \right] (2.19a)$$

Using the result (C.1) of Appendix C, and the fact that $\partial_j^{(y)} \delta(\xi - y) = -\partial^j(\xi) \delta(\xi - y)$, we obtain

$$[\alpha(x), \beta(y)] = 2\pi a \delta^2(\vec{x} - \vec{y}) \quad (2.19b)$$

We therefore have

$$[\sigma(x), \rho(y)] = 2\pi a \sigma(x) \delta^2(\vec{x} - \vec{y})$$

or

$$[[\sigma(x), Q] = 2\pi a \sigma(x) Q$$

and

$$[\sigma(x), \Phi] = 0 \quad (2.20)$$

This result indicates that $\sigma(x)$ creates states bearing $2\pi a$ units of electric charge.

Let us choose now $e^{\alpha(x)} \equiv \mu(x)$ and $\beta(y) \equiv B(y)$. Using (2.13), (2.18) and (2.12), we get

$$[\alpha(x), \beta(y)] = \lim_{\rho, \delta \to 0} (-b) \int_{T_+(C)} d^2\xi \left[ e^{ij} \partial_i \arg (\xi - x) + \partial_j \delta^2(\xi - y) \right] (2.21a)$$

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Using the result (C.2) of Appendix C, we obtain

\[ [\alpha(x), \beta(y)] = 2\pi b \delta^2(x - y) \]    \hspace{1cm} (2.21b)

We therefore have, in the case of \( \mu \):

\[ [\mu(x), B(y)] = 2\pi b \mu(x) \delta^2(x - y) \]

or

\[ [\mu(x), \Phi] = 2\pi b \mu(x) \]

and

\[ [\mu(x), Q] = 0 \]    \hspace{1cm} (2.22)

This result indicates that \( \mu(x) \) creates, states bearing \( 2\pi b \) units of magnetic flux (or \((-)\) topological charge).

Since \( \sigma \) bears charge and \( \mu \) bears magnetic flux one should expect that the composite operator

\[ \varphi(x) = \lim_{x \to y} \sigma(x) \mu(y) Z(x - y) \]    \hspace{1cm} (2.23)

–where \( Z(x - y) \) is some renormalization factor used to absorb the short distance singularities in the above operator product– is an operator with generalized statistics or an anyon operator. Indeed, using (2.23) and (2.17) one immediately finds the equal-times commutation rule for \( \varphi \):

\[ \varphi(x)\varphi(y) = \varphi(y)\varphi(x) \exp\{i2\pi ab[\arg(x - y) - \arg(y - x)]\} \]

or

\[ \varphi(x)\varphi(y) = \varphi(y)\varphi(x) \exp\{i2\pi ab\pi \epsilon[\arg(x - y) - \pi]\} \]    \hspace{1cm} (2.24)

where \( \epsilon(x) \equiv \text{sign}(x) \). Eq. (2.24) is the appropriate equal time commutation rule for anyon fields which generalizes the one-dimensional one [8,4]. Eq. (2.24) indicates that the spin of the anyon field \( \varphi \) is \( S_{\varphi} = \pi ab \), that is, \( S_{\varphi} = \frac{Q\Phi}{4\pi} \), in agreement with the observations made in [3,4].
Let us remark at this point that the charge bearing operator $\sigma(x)$, dual to $\mu(x)$ is, of course, gauge noninvariant. $\mu(x)$ is essentially the generator of a gauge transformation with parameter $arg(\vec{y} - \vec{x})$, as can be clearly seen from (2.17) and (2.16). A gauge invariant operator would commute with $\mu(x)$ and therefore could never possibly be its dual! Indeed, starting from (2.11) and using (C.1) we see that under the gauge transformation $W_\mu \rightarrow W_\mu + \Lambda$, we have $\sigma(x) \rightarrow \exp\{i2\pi a\Lambda(x)\}\sigma(x)$. With respect to gauge transformations and charge bearing, the order operator $\sigma(x)$ is very much like the scalar field $\phi$ of the abelian Higgs model or the electron field of Quantum Electrodynamics.

**2.3) Disorder (Vortex) Correlation Function**

Let us evaluate here the two-point correlation function of $\mu$ within the euclidean functional integral framework. Going to the euclidean space $(ix^0 \rightarrow x^3, iW^0 \rightarrow W^3, iA^0 \rightarrow A^3)$, we will have

$$iS = i \int d^3z\{-\frac{1}{4}W_{\mu\nu}W^{\mu\nu}\} \rightarrow -\int d^3z_E\frac{1}{4}W_{\mu\nu}W^{\mu\nu} =$$

$$-\int d^3z_E\frac{1}{2}W_\mu P^{\mu\nu}W_\nu = -S_E \quad (2.25a)$$

where $P^{\mu\nu} = -\nabla_E \delta^{\mu\nu} + \partial^\mu \partial^\nu$. The exponent of $\mu$ in (2.6) will transform as

$$-\frac{ib}{2} \int d^3z W^{\mu\nu} A_{\mu\nu} \rightarrow -\frac{b}{2} \int d^3z_E W^{\mu\nu} A_{\mu\nu} = -b \int d^3z_E W_\mu P^{\mu\nu} A_\nu \quad (2.25b)$$

We will also use a gauge fixing term of the form

$$iS_{GF} = i \int d^3z\{-\frac{\xi}{2} (\partial_\mu W_\mu)^2\} \rightarrow -\frac{\xi}{2} \int d^3z_E (\partial_\mu W_\mu)^2 =$$

$$-\frac{1}{2} \int d^3z_E W_\mu G^{\mu\nu} W_\nu = -S_{E,GF} \quad (2.25c)$$

where $G^{\mu\nu} = -\xi \partial^\mu \partial^\nu$ and $\xi$ is an arbitrary real parameter. The renormalization counterterm (2.7) transforms as

$$iS_R = i \int d^3z\{-\frac{b^2}{4} A_{\mu\nu} A^{\mu\nu}\} \rightarrow -\int d^3z_E\frac{b^2}{4} A_{\mu\nu} A^{\mu\nu} =$$

$$-\int d^3z_E\frac{b^2}{2} A_\mu P^{\mu\nu} A_\nu = -S_{E,R} \quad (2.25d)$$
Using (2.25) we can write the following expression for the euclidean two-point correlation function \( < \mu(x) \mu^*(y) > \) (we henceforth will drop the subscript E):
\[
< \mu(x) \mu^*(y) > = \lim_{\rho, \delta \to 0} Z^{-1} \int DW_\mu \exp \{- \int d^3z \left[ \frac{1}{4} W_{\mu\nu}^2 + \frac{1}{2} W_\mu G_{\mu\nu} W_\nu + W_{\mu\nu} A_{\mu\nu}(z; x, y) + \frac{1}{4} A_{\mu\nu}^2(z; x, y) \right] \}
\]
(2.26a)

\[
< \mu(x) \mu^*(y) > = \lim_{\rho, \delta \to 0} Z^{-1} \int DW_\mu \exp \{- \int d^3z \left[ \frac{1}{2} W_\mu [P_{\mu\nu} + G_{\mu\nu}] W_\nu + W_{\mu\nu} A_\nu(z; x, y) + \frac{1}{4} A_{\mu\nu}^2(z; x, y) \right] \}
\]
(2.26b)

In these expressions, Z is the vacuum functional and \( A_\mu(z; x, y) \equiv b[A_\mu(z; x; C; T_x) - A_\mu(z; y; C; T_y)] \) (the minus sign corresponds to the fact that we have the adjoint operator \( \mu^*(y) \)). The mixed \( W_\mu - A_\mu \) terms in the exponent in (2.26) come from the \( \mu(x) \mu^*(y) \) operators and the last term is the surface renormalization counterterm.

Observe that, as we put forward above, the integrand in (2.26) depends on the external field in the combination \( W_\mu + A_\mu \). Under the change of variable \( W_\mu \rightarrow W_\mu + \partial_\mu \omega \), with \( \omega \) given by (2.9), we can arbitrarily change the surfaces in \( A_\mu \) [14]. The gauge fixing term would be invariant under this transformation, provided we add the ghost term to (2.26) [21]. For simplicity reasons, however, we will neglect the ghost term in the abelian theories considered here. The functional integral in (2.26) can be evaluated with the help of the euclidean correlation function \( < W_\mu W_\nu > = [P_{\mu\nu} + G_{\mu\nu}]^{-1} \). This is given, in momentum space, by
\[
< W^\mu(k) W^\nu(-k) > = \frac{P_{\mu\nu}(k)}{k^4} + \frac{1}{\xi} \frac{k^\mu k^n}{k^4} ; \quad P_{\mu\nu}(k) = \delta_{\mu\nu} k^2 - k^\mu k^n \quad (2.27a)
\]
and in coordinate space by
\[
< W^\mu(x) W^\nu(y) > = \lim_{m \to 0} \frac{1}{m^4} \left[ P_{\mu\nu} - \frac{1}{\xi} \partial_\mu \partial^n \right] \left[ \frac{1}{m} - \frac{|x - y|}{8\pi} \right] ; \quad P_{\mu\nu} = -\delta_{\mu\nu} + \partial_\mu \partial^n \quad (2.27b)
\]

Here m is an infrared regulator used to control the singularities of the inverse Fourier transform of \( \frac{1}{k^n} \) (see(4.15)).

Using the results of Appendix B, we conclude that each piece of the second term in the exponent in (2.26b) can be put in the form
\[
-b \int d^3z W_\mu P_{\mu\nu} A_\nu = -b \frac{1}{2} \int d^3z W^\mu W^\nu \partial_\mu A_\nu = \int d^3z B_\gamma W^\gamma, \quad (2.28a)
\]
where
\[ B_\gamma(z; x) = b \int_{T_z(C)} d^2 \xi_\alpha \partial_\beta^{(\xi)} \arg(\bar{\xi} - \bar{x}) F^{\alpha\beta} \delta^\gamma(z - \xi), \] (2.28b)
in which
\[ F^{\alpha\beta} \gamma = \partial^\alpha \delta^\beta \gamma - \partial^\beta \delta^\alpha \gamma. \]

Performing the functional integral in (2.26), we get
\[ < \mu(x) \mu^*(y) > = \lim_{\rho, \delta, m \to 0} \exp\left\{ - \frac{1}{2} \int d^3 z d^3 z' B_\mu(z; x, y) B_\nu(z'; x, y) \right\} \times [P^{\mu\nu} + G^{\mu\nu}]^{-1}(z - z') - S_R[A_{\mu\nu}]. \] (2.29)

where \( B_\mu(z; x, y) = B_\mu(z; x) - B_\mu(z; y) \) and \( S_R[A_{\mu\nu}] \) is the last term in the exponent in (2.26). Using (2.28b), (2.27b) and integrating over \( z \) and \( z' \), we obtain
\[ < \mu(x) \mu^*(y) > = \lim_{\rho, \delta, m \to 0} \exp\left\{ \frac{b^2}{2} \sum_{i,j=1}^{2} \epsilon_i \epsilon_j \int_{T_{\xi_i}(C)} d^2 \xi_\mu \int_{T_{\xi_j}(C)} d^2 \eta_\nu \partial_\sigma^{(\xi)} \arg(\bar{\xi} - \bar{x}_i) \partial_\delta^{(\eta)} \arg(\bar{\eta} - \bar{x}_j) \right\} \times F^{\mu\nu}_{(\xi)} F^{\alpha\beta}_{(\eta)} \lambda [P^{\sigma\lambda}_{(\xi)} - \frac{1}{\xi} \partial_\sigma^{(\xi)} \partial_\delta^{(\xi)}] \left[ \frac{1}{m} - \frac{|\xi - \eta|}{8\pi} \right] - S_R[A_{\mu\nu}] \] (2.30)

In the above expression, \( x_1 \equiv x, \epsilon_1 \equiv 1 \) and \( x_2 \equiv y, \epsilon_2 \equiv -1 \).

We immediately see that, since \( F^{\alpha\beta} \lambda \partial^\lambda \equiv 0 \), the gauge dependent part of \( < W_\mu W_\nu > \) gives no contribution to \( < \mu \mu^* > \), as one should expect for a gauge invariant operator as \( \mu \). The only contribution comes from the gauge independent part of \( < W_\mu W_\nu > \), which is proportional to \(-\Box \delta^{\mu\nu}\).

Let us now make use of the very useful identity
\[ -\Box F^{\mu\nu}_{(\xi)} F^{\alpha\beta}_{(\eta)} \lambda \delta^{\sigma\lambda} = -\Box \epsilon^{\mu\nu\lambda} \epsilon^{\alpha\beta\gamma} (\partial_\sigma^{(\xi)} \partial_\delta^{(\eta)}) \delta^{\lambda\gamma} - \partial_\delta^{(\xi)} \partial_\delta^{(\eta)} \] (2.31)

which can be readily demonstrated by expanding the \( \epsilon \)'s in terms of Kronecker \( \delta \)'s. Inserting (2.31) in (2.30) and observing that
\[ -\Box \left[ \frac{1}{m} - \frac{|x|}{8\pi} \right] \equiv -\Box F^{-1}[\frac{1}{k^2}] = F^{-1}[\frac{1}{k^2}] = \lim_{\epsilon \to 0} \frac{1}{4\pi ||x||^2 + ||\epsilon||^2} \] (2.32)
(here \( \epsilon \) is an ultraviolet regulator used to control the short distance singularities of the inverse Fourier transform of \( \frac{1}{k^2} \)) we get
\[ < \mu(x) \mu^*(y) > = \lim_{\rho, \delta, \epsilon \to 0} \exp\left\{ -\frac{b^2}{2} \sum_{i,j=1}^{2} \epsilon_i \epsilon_j \right\} \]
\[
\times \int_{T_{x_i(C)}} d^2\xi \mu \int_{T_{x_j(C)}} d^2\eta \partial^{(\xi)}_\alpha \arg(\vec{\xi} - \vec{x}_i)\partial^{(\eta)}_\beta \arg(\vec{\eta} - \vec{x}_j) \\
\times \epsilon^{\mu\nu\lambda} \epsilon^{\alpha\beta\gamma} (\Box(\xi)) \delta^{\lambda\gamma} + \partial^{(\xi)}_\alpha \partial^{(\eta)}_\beta \right] \frac{1}{4\pi[|\xi - \eta|^2 + |\epsilon|^2]^{1/2}} - S_R[A_\mu] \} (2.33)
\]

Here we used the fact that \( \partial^{(\eta)} F(\xi - \eta) = -\partial^{(\xi)} F(\xi - \eta) \).

Observing that \( \lim_{\epsilon \to 0} -\Box [4\pi[|x|^2 + |\epsilon|^2]^{-\frac{1}{2}} = \delta^3(x) \) and using the results of Appendix B, we see that the first term in the exponent in (2.33) can be written as

\[
\frac{b^2}{2} \sum_{i,j=1}^2 \int_{T_{x_i(C)}} d^2\xi \mu \int_{T_{x_j(C)}} d^2\eta \partial^{(\xi)}_\alpha \arg(\vec{\xi} - \vec{x}_i)\partial^{(\eta)}_\beta \arg(\vec{\eta} - \vec{x}_j) \epsilon^{\mu\nu\lambda} \epsilon^{\alpha\beta\gamma} \delta^3(\xi - \eta) \\
= \frac{1}{4} \int d^3 z d^3 z' [\partial_\mu A_\nu(z; x, y) - (\mu \leftrightarrow \nu)] \delta^3(z - z') [\partial^\mu A^\nu(z'; x, y) - (\mu \leftrightarrow \nu)] \\
= \frac{1}{4} \int d^3 z A_\mu^2(z; x, y) = S_R[A_\mu] (2.34)
\]

We immediately conclude that this term is canceled by the surface renormalization counterterm in (2.33).

The second term in the exponent in (2.33) can be evaluated by making use of the results of Appendix C. It is surface independent and, in the limit \( \rho \to 0 \) only depends on the points \( x \) and \( y \). Indeed, according to (C.2) and already using (2.34)

\[
< \mu(x)\mu^*(y) > = \lim_{\epsilon \to 0} \exp \left\{ -\frac{b^2}{2} \sum_{i,j=1}^2 \epsilon_i \epsilon_j \left[ \frac{\pi}{|x_i - x_j|^2 + |\epsilon|^2]^{1/2}} \right] \right\} (2.35a)
\]
or
\[
< \mu(x)\mu^*(y) > = \lim_{\epsilon \to 0} \exp \left\{ \pi b^2 \left[ \frac{1}{|x - y|} - \frac{1}{\epsilon} \right] \right\} (2.35b)
\]

We now see explicitly that the renormalization term (2.7) indeed, renders the \( \mu \)-correlation function surface independent.

The ultraviolet singularity in (2.35) may be eliminated by a renormalization of the disorder(vortex) operator:

\[
\mu_R(x) \equiv \lim_{\epsilon \to 0} \mu(x) \exp \left\{ \frac{\pi b^2}{2|\epsilon|} \right\} (2.36)
\]

We therefore finally arrive at the result

\[
< \mu_R(x)\mu^*_R(y) > = \exp \left\{ \frac{\pi b^2}{|x - y|} \right\} (2.37)
\]

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for the disorder (vortex) euclidean correlation function in the Maxwell theory. Observe that
\[
\lim_{|x-y| \to \infty} \langle \mu_R(x) \mu_R^*(y) \rangle = 1 \quad (2.38)
\]
This indicates that \( \mu_R \) does not create genuine vortex excitations, in the true Maxwell theory. One should expect this result since already at the classical level the theory does not possess topological excitations. Another way of describing this fact is to say that there is a vortex condensate in the Maxwell theory. In section 3 we will examine how this can be changed in the Maxwell-Higgs theory.

2.4) Order Correlation Function

Let us evaluate now the two-point euclidean correlation function of \( \sigma \). From (2.11) we see that there is no change in \( \sigma \) as we go to euclidean space. Using the results of the previous subsection for the euclidean action \( S \) and gauge fixing term \( S_{GF} \), as well as the expression of \( \sigma(x) \) eq.(2.11), we may write
\[
\langle \sigma(x) \sigma^*(y) \rangle = \lim_{\rho, \delta \to 0} Z^{-1} \int DW_\mu \exp\{- \int d^3z \left[ \frac{1}{2} W_\mu [P^{\mu\nu} + G^{\mu\nu}] W_\nu + C_\mu(z; x, y) W_\mu \right] \}
\] (2.39)

In this expression, \( C_\mu(z; x, y) = [C_\mu(z; x; C; T_x) - C_\mu(z; y; C; T_y)] \) (as in the case of \( \mu \), the minus sign corresponds to the fact that we have the adjoint operator \( \sigma^*(y) \)) where
\[
C^\mu(z; x; C; T_\xi) = \left\{ \begin{array}{ll}
\int_R e^{i[\xi]_{\nu}} \frac{1}{2} \partial^{(\xi)}_k \partial^k \arg (\xi - x) + \partial^k \ln |\xi - x| \delta^3(z - \xi) & \text{for } \mu = i \\
0 & \text{for } \mu = 0
\end{array} \right. \quad (2.40)
\]

Performing the functional integral in (2.39), we get
\[
\langle \sigma(x) \sigma^*(y) \rangle = \lim_{\rho, \delta \to 0} \exp\left\{ \frac{1}{2} \int d^3z' d^3z' C_\mu(z; x, y) C_\nu(z'; x, y) [P^{\mu\nu} + G^{\mu\nu}]^{-1}(z - z') \right\}
\] (2.41)

The \( \sigma \) correlation function, of course is not gauge invariant since \( \sigma \) itself is not. In order to obtain the gauge independent part of \( \langle \sigma \sigma^* \rangle \), let us use in (2.41) just the gauge independent part of \( \langle W_\mu W_\nu \rangle \), namely, the \( \delta^{\mu\nu} \)-proportional part of
< W_μ W_ν > in (2.27b), which is explicitly given by (2.32) (times δ^μν). Inserting this and (2.40) in (2.41), and integrating on z and z', we get

\[ < σ(x)σ^*(y) > = \lim_{ρ,δ,ε→0} \exp\left\{ -\frac{a^2}{2} \sum_{i,j=1}^{2} ε_iε_j \int_{R_ρ(C)} d^2ξ \int_{R_δ(C)} d^2η \times \right\} \]

\[ \left[ e^{ki} δ^{(ξ)} \arg(ξ-ξ_i) + δ^{(η)} \ln |ξ-ξ_i| \right] \left[ e^{ji} δ^{(η)} \arg(η-η_j) + δ^{(η)} \ln |η-η_j| \right] \frac{δ^{ij}}{4π||ξ-η||^2 + |ε|^2} \]  

(2.42)

Let us now use the identity

\[ \delta^{μν} \frac{ε^{-m||x^2 + |ε|^2|^{1/2}}}{||x||^2 + |ε|^2} = \partial^μ \partial^ν \left[ \frac{ε^{-m||x^2 + |ε|^2|^{1/2}}}{m} \right] + \]

\[ + \frac{x^μ x^ν}{||x||^2 + |ε|^2} \left( \frac{1}{||x||^2 + |ε|^2} + m \right) e^{-m||x^2 + |ε|^2|^{1/2}} \]  

(2.43a)

which in the limit m → 0 reduces to

\[ \frac{δ^{μν}}{||x||^2 + |ε|^2} = \partial^μ \partial^ν \left[ \frac{1}{m} + ||x||^2 + |ε|^2 \right] + \frac{x^μ x^ν}{||x||^2 + |ε|^2} \]  

(2.43b)

Insertion of (2.43b) in (2.42) produces two pieces. The one coming from the second term in (2.43b) vanishes in the limit ρ, δ → 0 as is shown in Appendix C. The one coming from the first term in (2.43b) can be computed with the help of (C.1), after using the fact that \( \partial^ν(ξ) F(ξ-η) = -\partial^ν(η) F(ξ-η) \). We therefore conclude, according to (C.1) that

\[ < σ(x)σ^*(y) > = \lim_{m,ε→0} \exp\left\{ \frac{a^2}{8π} \sum_{i,j=1}^{2} ε_iε_j (2\pi)^2 \left[ -\frac{1}{m} + ||x_i - x_j||^2 + |ε|^2 \right] \right\} \]

(2.44a)

or

\[ < σ(x)σ^*(y) > = \exp\{ -πa^2 |x - y| \} \]  

(2.44b)

Observe that the infrared singularities at m → 0 completely cancel. If we were calculating a charge nonconserving correlation function (as < σσ >, for instance) we would have ε_i = ε_j and the \( \frac{1}{m} \) factors would no longer cancel, implying < σσ > = 0 as m → 0. We see that, as in 1 + 1D [4], an infrared singularity is responsible for enforcing the selection rule for the dual operators. Observe that

\[ \lim_{|x-y|→0} < σ(x)σ^*(y) > = 0 \]  

(2.45)
indicating that the charged states created by $\sigma$ are orthogonal to the vacuum and charge is a conserved quantity as one would expect in the pure Maxwell Theory.

2.5) The Mixed and Anyon Correlation Functions

Let us examine now the mixed order-disorder euclidean correlation function from which we will be able to obtain the anyon correlation function.

Combining (2.26) with (2.39), we can write

$$< \sigma(x_1) \mu(x_2) \sigma^*(y_1) \mu^*(y_2) > = \lim_{\rho,\delta \to 0} Z^{-1} \int DW_\mu \exp \left\{ - \int d^3z \left[ \frac{1}{2} W_\mu (F^\mu_{\nu} + G^\mu_{\nu}) W_\nu + C_\mu (z; x_1, y_1) W_\mu + W_\mu P^\mu_{\nu} A_\nu (z; x_2, y_2) + \frac{1}{4} A^2_{\mu\nu} (z; x_2, y_2) \right] \right\}$$

The functional integral can be evaluated as before, yielding

$$< \sigma(x_1) \mu(x_2) \sigma(y_1) \mu(y_2) > = \lim_{\rho,\delta \to 0} \exp \left\{ \frac{1}{2} \int d^3z d^3z' [B_\mu (z; x_2, y_2) + C_\mu (z; x_1, y_1)] \times [B_\nu (z'; x_2, y_2) + C_\nu (z'; x_1, y_1)] P^{\mu \nu} + G^{\mu \nu} \right\}^{-1} (z - z') - S_R [A_\mu (x_2, y_2)]$$

The BPB and CPC terms were evaluated before. The novel terms which appear here are the BPC terms (already considering the two of them):

$$BPC = \int d^3z d^3z' B_\mu (z; x_2, y_2) C_\mu (z'; x_1, y_1) P^{\mu \nu} + G^{\mu \nu} \right\}^{-1} (z - z') = \lim_{\rho,\delta,m \to 0} \sum_{i,j=1}^{2} \epsilon_i \epsilon_j$$

$$\times \int_{T_{i,j}(C)} d^2 \xi_{\mu} \int_{R_{s_j}(C)} d^2 \eta \partial^\xi_\nu \arg (\tilde{\xi} - \tilde{\eta}_i) [\epsilon_k \partial^\nu_\eta \arg (\tilde{\eta} - \tilde{s}_j) + \partial^\nu \ln |\tilde{\eta} - \tilde{s}_j|]$$

$$\times F^{\mu \nu}_{\xi} \sigma [P^{\sigma i}_{\xi} - \frac{1}{\xi} \partial^\xi_\sigma \partial^\sigma_\nu] \frac{1}{m} \frac{|\xi - \eta|}{8\pi}$$

(2.48)

Here $r_1 \equiv x_2$, $r_2 \equiv y_2$, $s_1 \equiv x_1$ and $s_2 \equiv y_1$. As before $\epsilon_1 \equiv 1$ and $\epsilon_2 \equiv -1$. As in the case of $< \mu \mu^* >$, the gauge dependent part of the propagator in (2.48) gives no contribution, because $F^{\mu \nu}_{\sigma} \partial^\sigma \equiv 0$. Only the $-\partial \delta^i_\sigma$ part contributes:

$$F^{\mu \nu}_{\sigma} P^{\sigma i} = -\square (\partial^\mu \delta^\nu_i - \partial^\nu \delta^\mu_i) F^{-1} \left[ \frac{1}{k^2} \right] = (\partial^\mu \delta^\nu_i - \partial^\nu \delta^\mu_i) F^{-1} \left[ \frac{1}{k^2} \right]$$

(2.49)

The $\delta^\mu_i$-part of (2.49) vanishes because $d^2 \xi^\mu$ is orthogonal to $R_{s_j}(C)$. The remaining part gives

$$BPC = \lim_{\rho,\delta \to 0} \sum_{i,j=1}^{2} \epsilon_i \epsilon_j \int_{R_{s_j}(C)} [\epsilon_k \partial^\nu_\eta \arg (\tilde{\eta} - \tilde{s}_j) + \partial^\nu_\eta \ln |\tilde{\eta} - \tilde{s}_j|]$$

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\[
\times \int_{T_c(C)} d^2 \xi_\mu \partial_\mu^{(\xi)} \arg(\xi - \vec{r}_i) \partial_\mu^{(\eta)} \left[ \frac{1}{4\pi |\xi - \eta|} \right]
\] (2.50)

The above integrals are evaluated in Appendix D. Using (D.1), we get

\[\text{BPC} = i\pi ab \sum_{i,j} \epsilon_i \epsilon_j arg(\vec{s}_j - \vec{r}_i)\]

or

\[\text{BPC} = i\pi ab \left[ arg(\vec{x}_1 - \vec{x}_2) + arg(\vec{y}_1 - \vec{y}_2) - arg(\vec{x}_1 - \vec{y}_2) - arg(\vec{y}_1 - \vec{x}_2) \right] \] (2.51)

Combining (2.51) with the previous results for the BPB and CPC terms, we obtain

\[< \sigma(x_1) \mu_R(x_2) \sigma^*(y_1) \mu_R^*(y_2) > = \exp\{-\pi a^2 |x_1 - y_1| + \frac{\pi b^2}{|x_2 - y_2|} + i\pi ab [arg(\vec{x}_1 - \vec{x}_2) + arg(\vec{y}_1 - \vec{y}_2) - arg(\vec{x}_1 - \vec{y}_2) - arg(\vec{y}_1 - \vec{x}_2)]\} \] (2.52)

The anyon correlation function can now be obtained in a straightforward manner. Introducing the anyon field

\[
\varphi(x) = \lim_{x_1 \to x_2 \equiv x} \sigma(x_1) \mu_R(x_2) \exp[-i\pi ab \ arg(\vec{x}_1 - \vec{x}_2)]
\] (2.53)

we immediately get

\[< \varphi(x) \varphi^*(y) > = \exp\{-\pi a^2 |x - y| + \frac{\pi b^2}{|x - y|} - i\pi ab [arg(\vec{x} - \vec{y}) + arg(\vec{y} - \vec{x})]\} \] (2.54)

Observe that \(< \varphi \varphi^* > \) is multivalued, the ambiguous phase being \(\exp[i2\pi(\pi ab)]\). As in the case of 1 + 1D [4], we interpret this fact as being the manifestation in the framework of euclidean correlation functions of the nontrivial commutation rule of \(\varphi\). Of course, the same functional integral describes the two possible orderings of operators in the left hand side of (2.54). Making the analytic continuation from euclidean to Minkowski space from each sheet of (2.54) would reproduce each possible ordering of \(\varphi[4]\). This, of course implies that the spin of \(\varphi\) is \(s_\varphi = \pi ab\), confirming the result we found above by direct computation of the \(\varphi\) commutator.

The \(\varphi\)-correlation function (2.54) decays, at large distances as \(\exp[-\pi a^2 |x - y|]\). This implies the anyon states created by \(\varphi\) possess a mass \(M = \pi a^2\).
3) The Maxwell-Higgs Theory
(Abelian Higgs Model)

3.1) Introduction

Let us study in this section the properties of the dual operators $\sigma$ and $\mu$ introduced before, in the framework of the Abelian Higgs Model (AHM), described by

\[ \mathcal{L} = -\frac{1}{4} W_{\mu\nu}^2 + |D_\mu \phi|^2 - g^2 \phi^* \phi - \frac{\lambda}{4} (\phi^* \phi)^2 \]  

(3.1)

where $D_\mu = \partial_\mu + ieW_\mu$.

The theory exists in two phases: i) a symmetric phase, for $g^2 > 0$, where $\langle \phi \rangle = 0$ and $W_\mu$ is massless and ii) a broken phase, for $g^2 < 0$, where $\langle \phi \rangle = \kappa \neq 0$ and $W_\mu$ acquires a mass $M = e\kappa$ through the Higgs mechanism.

Let us introduce in this theory the $\sigma$ and $\mu$ operators defined in section 2. Since the equal time commutators of the AHM are exactly the same as in (2.12), we conclude that all commutation rules (equal times) evaluated in section 2.2 for the pure Maxwell theory remain valid here.

As the surface independence of $\mu$ is concerned, let us note that the same counterterm (2.7) will be sufficient to make the $\mu$-correlation functions surface independent here, because also in the AHM it will make the integrand in the functional integral defining $\langle \mu \mu^* \rangle$ to depend on the external field through the combination $W_\mu + A_\mu$.

The remaining three last terms in (3.1) will be invariant under the change of variable (involving $\omega$, eq. (2.9)) needed to show surface independence. As was shown in [14], $\mu$ is the operator which creates the quantum states associated to the classical soliton (vortex) solution of Nielsen and Olesen [22].

In this interacting theory, of course, we will no longer be able to compute exact correlation functions. Instead, we will evaluate the long distance behavior of them. As we will see, this will be enough to obtain interesting physical consequences.

3.2) The Unbroken Phase

Let us study here the symmetric phase ($g^2 > 0$), in which $\langle \phi \rangle = 0$. We start
with the $\mu$ correlation function which is given by an expression similar to (2.26):

$$
< \mu(x)\mu^*(y) > = \lim_{\rho,\delta \to 0} Z^{-1} \int DW_\mu \exp \left\{ - \int d^3 z \left[ \frac{1}{2} W_\mu [P^{\mu\nu} + G^{\mu\nu}] W_\nu + |D_\mu \phi|^2 + V(\phi) + W_\mu P^{\mu\nu} A_\nu(z; x, y) + \frac{1}{4} A_\mu^2(z; x, y) \right] \right\}
$$

(3.2)

Using (2.28), we see that $< \mu\mu^* >$ is obtained by coupling the external field $B_\mu(z; x, y)$ to the AHM lagrangian, in the way given by (2.28a). We therefore conclude that

$$
< \mu(x)\mu^*(y) > = \exp \{ \Lambda(x, y) - S_R[A_\mu(x, y)] \}
$$

(3.3)

where, using a diagrammatic language, $\Lambda(x, y)$ is given by the sum of all Feynman diagrams with the field $B_\mu(z; x, y)$ in the external legs.

We are only interested here in the long distance behavior of $< \mu\mu^* >$. As was shown in [21], only two-leg graphs contribute to (3.3) in this limit. We are going to evaluate $\Lambda(x, y)$ by making a loop expansion (in powers of $\hbar$). The only vertex involving the external field $B_\mu$ is given in Fig. 4a. The lowest order (two-leg) graph contributing to $\Lambda(x, y)$ is given in Fig. 4b. Inserting the euclidean propagator for $W_\mu$, eq. (2.27) we immediately see that the graph of Fig. 4b. is identical to the first term in the exponent in (2.29). It follows that in this order of approximation ($0(\frac{1}{\hbar})$, the large distance behavior of $< \mu\mu^* >$ is given exactly by the same expression as in the Maxwell theory, namely

$$
< \mu_R(x)\mu_R^*(y) > \xrightarrow{|x-y| \to \infty} \exp\{ \frac{\pi b^2}{|x-y|} \} \xrightarrow{|x-y| \to \infty} 1
$$

(3.4)

We have renormalized $\mu$ in the same way as in (2.36).

From (3.4) we conclude that in the symmetric phase of the AHM, in analogous way as in the true Maxwell theory, the disorder operator $\mu$ does not create states orthogonal to the vacuum, that is, genuine excitations, as one should expect. Also the symmetric phase of the AHM can be viewed as a vortex condensate.

Let us turn now to the order correlation function. This is now given by

$$
< \sigma(x)\sigma^*(y) > = \lim_{\rho,\delta \to 0} Z^{-1} \int DW_\mu \exp \left\{ - \int d^3 z \left[ \frac{1}{2} W_\mu [P^{\mu\nu} + G^{\mu\nu}] W_\nu + \right] \right\}
$$

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\[ + |D_\mu \phi|^2 + V(\phi) + C_\mu(z; x, y)W^\nu] \]  
(3.5)

where \( C_\mu(z; x, y) \) is defined in (2.39) and (2.40). Now,

\[ < \sigma(x)\sigma^*(y) > = \exp \{ \Gamma(x, y) \} \]
(3.6)

where \( \Gamma(x, y) \) is the sum of all Feynman graphs containing the field \( C_\mu(z; x, y) \) in the external legs. Again, only two-leg graphs contribute to the long distance behavior of \( < \sigma\sigma^* > [21] \). The only vertex involving \( C_\mu \) is the same as for \( B_\mu \) and is shown in Fig. 4a. Again, making an expansion in loops, the lowest order graph will be the one of Fig. 4b. Inserting the \( W_\mu \)-propagator, eq.(2.27), we conclude that in this order \((O(\frac{1}{\lambda}))\) the large distance behavior of the gauge invariant part of \( < \sigma\sigma^* > \) is given by the same expression as in the Maxwell theory, namely

\[ < \sigma(x)\sigma^*(y) > \xrightarrow{|x-y| \to \infty} \exp \{ -\pi a^2|x - y| \} \xrightarrow{|x-y| \to \infty} 0 \]  
(3.7)

The long distance behavior of the anyon field could be obtained as well, by exchanging \( B_\mu(z; x, y) \) for \( B_\mu(z; x, y) + C_\mu(z; x, y) \) in (3.2). In the lowest order of approximation, we would get \( < \varphi(x)\varphi^*(y) > \) behaving asymptotically as (2.54).

3.3) The Broken Phase

3.3.1) The Order Correlation Function

Let us consider now the case in which \( g^2 < 0 \) and \( < \phi > = \kappa = \left( \frac{|g^2|}{\lambda} \right)^{\frac{1}{4}} \). We are going to write \( \phi \) in terms of its real components: \( \phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \) and choose \( < \phi_1 > = \kappa \) and \( < \phi_2 > = 0 \).

Let us compute here the order correlation function\(< \sigma\sigma^* >\). This is given by (3.5), and (3.6). Again, in lowest order \( < \sigma\sigma^* > \) will be given by the graph of Fig. 4a. There is, however, an important difference. We must shift the Higgs field around the vacuum value. This will generate the following quadratic part for the euclidean lagrangian \( \mathcal{L}[W_\mu, \phi_1, \phi_2] \)

\[ \mathcal{L}^{(2)}[W_\mu, \phi_1, \phi_2] = \frac{1}{4} W^\mu_\nu W^\nu_\mu + \frac{1}{2} M^2 W_\mu W_\mu + \frac{1}{2} (\partial_\mu \phi_1)^2 + \frac{1}{2} m^2_1 \phi_1^2 + \frac{1}{2} (\partial_\mu \phi_2)^2 + MW_\mu \partial^\mu \phi_2 \]
(3.8)
where $M = e\kappa$ and $m_1 = 2|g^2|$. Using the Lorentz gauge ($\xi \to \infty$), we see that $W_{\mu}$ and $\phi_2$ decouple in the quadratic part (since $\partial_{\mu} W_{\mu} = 0$). The euclidean propagators for $W_{\mu}$, $\phi_1$ and $\phi_2$ are now given respectively by

$$D_{\mu\nu}(k) = \lim_{\xi \to \infty} \left[ \frac{P_{\mu\nu}}{k^2(k^2 + M^2)} + \frac{k_\mu k_\nu}{k^2(\xi k^2 + M^2)} \right]$$ (3.9a)

$$\Delta_1(k) = \frac{1}{k^2 + m_1^2}; \quad \Delta_2(k) = \frac{1}{k^2}$$ (3.9b)

Inserting the $W_{\mu}$-propagator in the graph of Fig. 4b, we see that, the lowest order $(0(\frac{1}{\hbar}))$ contribution to large distance behavior of the gauge invariant part of $<\sigma \sigma^*>$ is given by an expression identical to (2.41) and (2.42), except for the fact that the last term between square brackets in (2.42), namely $F^{-1}[\frac{1}{k^2 + M^2}]$, is exchanged by $F^{-1}[\frac{1}{k^2 + M^2}]$, that is

$$\delta^{\mu\nu} F^{-1}[\frac{1}{k^2 + M^2}] = \lim_{\epsilon \to 0} \delta^{\mu\nu} \frac{e^{-||x||^2 + |\epsilon|^2}^{1/2}}{||x||^2 + |\epsilon|^2}$$ (3.10)

Using the identity (2.43a) and following the same steps which led us to (2.44), we immediately conclude that in lowest order $(0(\frac{1}{\hbar}))$,

$$<\sigma(x)\sigma^*(y)>_{|x-y|\to \infty} \exp\left\{ \frac{\pi a^2}{M} e^{-M|x-y|} - 1 \right\}$$ (3.11a)

or

$$<\sigma_R(x)\sigma_R^*(y)>_{|x-y|\to \infty} \exp\left\{ \frac{\pi a^2}{M} e^{-M|x-y|} \right\}$$ (3.11b)

where $\sigma_R \equiv \sigma \exp[\frac{\pi a^2}{2M}]$.

Observe that in the broken phase, the charge bearing operator $\sigma$ no longer creates states orthogonal to the vacuum. This corresponds to the screening of charge associated with the mass generation to the field $W_{\mu}$ through the Higgs mechanism. Notice that in the limit $M \to 0$, $<\sigma \sigma^*>$ reduces to the expression found in (3.7) for the symmetric phase and the zero mass singularities cancel in charge selection rule respecting correlation function.

3.3.2) The Disorder (Vortex) Correlation Function
Let us evaluate here the \( \mu \)-correlation function in the broken phase of the AHM. As we infer from (3.2), we can write \(< \mu \mu^* > \) as

\[
< \mu(x)\mu^*(y) > := \lim_{\rho, \delta \to 0} Z^{-1} \int DW_\mu \exp\{-S[W_{\mu\nu} + A_{\mu\nu}, \phi, D_\mu\phi] - S_{GF}[W_\mu]\} \tag{3.12}
\]

where \( S[W_{\mu\nu}, \phi, D_\mu\phi] \) is the action associated with (3.1).

Performing the change of variable \( W_\mu \to W_\mu + A_\mu(z; x, y) \), we get

\[
< \mu(x)\mu^*(y) > := \lim_{\rho, \delta \to 0} \int D_\mu \exp\{-S[W_{\mu\nu}, \phi, \tilde{D}_\mu\phi] - S_{GF}[W_\mu - A_\mu]\} \tag{3.13}
\]

where \( \tilde{D}_\mu = \partial_\mu + ie[W_\mu - A_\mu(x, y)] \). It will be more convenient to use (3.13) to compute \(< \mu \mu^* > \) in the broken phase. Now, we can write

\[
< \mu(x)\mu^*(y) > = \exp\{\tilde{\Lambda}(x, y)\} \tag{3.14}
\]

where \( \tilde{\Lambda}(x, y) \) is the sum of all Feynman graphs with the \( A_\mu(z; x, y) \) field in the external legs and computed with the Feynman rules coming from (3.13). We are again going to make an expansion in loops and furthermore, in powers of \( M \), the gauge field mass.

Before evaluating \( \tilde{\Lambda}(x, y) \), we shift the Higgs field around the vacuum value in (3.13). This operation commutes with the change of variable \( W_\mu \to W_\mu + A_\mu \). The vertices relevant for the lowest order computation of \( \tilde{\Lambda}(x, y) \) are depicted in Fig. 5. Their contribution, in this order, is given by the graphs of Fig. 6. The gauge dependent terms, which appear in Fig. 6a, vanish for all values of \( \xi \) as can be easily seen from (3.9a). The first nonzero contribution for \( \tilde{\Lambda}(x, y) \), therefore is given by the graphs of Fig. 6b, which are of the order \( 0(M^2) \). Using (3.9), in the graphs of Fig. 6b, we find

\[
\tilde{\Lambda}(x, y) \bigg|_{|x-y| \to \infty} = -\frac{M^2}{2} \int d^3z d^3z' A_\mu(z; x, y) A_\nu(z'; x, y) F^{\mu\nu}(z-z') \tag{3.15}
\]

where \( F(z-z') = F^{-1}\left[\begin{array}{c} 1 \\ i \end{array}\right]\right| = \Delta_2(z-z') \) and is given by (2.32). Eq. (3.15) can be written as

\[
\tilde{\Lambda}(x, y) \bigg|_{|x-y| \to \infty} = -\frac{M^2}{2} \int d^3z d^3z' \epsilon^{\mu\lambda\nu\delta} A_\mu(z; x, y) F(z-z') \epsilon^{\nu\beta\lambda\delta} A_\nu(z'; x, y) \tag{3.16}
\]

Using the result of Appendix B, we get, after integrating over \( z \) and \( z' \) and using (2.32):

\[
\tilde{\Lambda} \bigg|_{|x-y| \to \infty} \lim_{\rho, \delta, \epsilon \to 0} -\frac{M^2 b^2}{2} \sum_{i,j=1}^2 \epsilon_i \epsilon_j \int_{T_{x_i}(C)} d^2\xi \int_{T_{x_j}(C)} d^2\eta \partial^{(c)}(\arg(\tilde{\xi} - \tilde{x}_i) \partial^{(a)}(\arg(\tilde{\eta} - \tilde{x}_j)) \tag{3.17}
\]
Making use of the identity (2.43b) we see that (3.16) contains two terms, corresponding to the two pieces in the right hand side of (2.43b). The second one is surface dependent but, as was shown in [17], for \( i \neq j \), it vanishes in the limit \( |x - y| \to \infty \). This therefore does not contribute to (3.16). The \( i = j \) term is just a self-energy that will renormalize \( \mu \) [17]. The first term of (3.16) can be evaluated immediately with the help of result (C.2) of Appendix C:

\[
\tilde{\Lambda} \left. \frac{|x - y| \to \infty}{\epsilon \to 0} \lim \frac{\pi M^2 b^2}{2} \sum_{i,j=1}^{2} \epsilon_i \epsilon_j \left[ -\frac{1}{m} + \frac{|x_i - x_j|^2 + |\epsilon|^2}{2} \right] \right] (3.17a)
\]

or

\[
\tilde{\Lambda}(x, y) \left. \frac{|x - y| \to \infty}{\epsilon \to 0} \right. -\pi M^2 b^2 |x - y| (3.17b)
\]

We therefore conclude that

\[
< \mu_R(x) \mu_R^*(y) > \left. \frac{|x - y| \to \infty}{\epsilon \to 0} \lim \exp \left\{ -\pi M^2 b^2 |x - y| \right\} \right. \left. \frac{|x - y| \to \infty}{\epsilon \to 0} \right. 0 (3.18)
\]

We see that in the broken phase the vortex operator \( \mu \) indeed creates states orthogonal to the vacuum, i.e., genuine excitations in the spectrum. According to (3.18), the mass of these excitations is \( M_v = \pi M^2 b^2 \), up to the order of approximation we are working (0(\( M^2 \bar{\hbar} \))). It is interesting to see that choosing \( b^2 = e^{-2} [21] \), the mass of the quantum vortices created by \( \mu \) coincides with the classical vortex energy found in [23]. In ref. [21] one-loop corrections to the above result were also evaluated.

Our study of the AHM and Maxwell theory clearly exposes the reason why we call \( \sigma \) and \( \mu \) “order” and “disorder” operators. \( \sigma \) has a nonzero vacuum expectation value in the broken (ordered) phase, while the vacuum expectation value of \( \mu \) is different from zero in the symmetric (disordered) phases and vice-versa. In the broken (ordered) phase \( \mu \) creates genuine quantum soliton states. The transition to the symmetric (disordered) phases can be viewed as a condensation of topological charge in the same way as the transition to a broken (symmetric) phase can be viewed as a condensation of charge as occurs in superconductivity. We see that in a certain sense, the AHM
is quite similar to the Ising model, the phase transition which generates mass to the
gauge field being analogous to the phase transition which takes place in the latter.

A final word is in order about the procedure adopted for the computation of
$<\mu\mu^*>$ in the broken phase of the AHM. Observe that we shifted $W_\mu$ around $A_\mu$
before evaluating this correlation function, contrary to what we did in other cases. We
proceeded this way because without shifting, the lowest order in our approximation
scheme would produce a trivial result as the computations would be effectively done
in the Proca theory. We would have to go to higher orders in order to get a nontrivial
result, while with the shifting trick we already get it in lowest order. Of course the
two ways of calculating should produce the same result if one was able to sum the
whole perturbation series.

4) The Maxwell-Chern-Simons Theory

4.1) Order and Disorder (Vortex) Operators

Let us study now the Maxwell-Chern-Simons (MCS) theory, described by
\[ L = -\frac{1}{4} W_{\mu\nu} W^{\mu\nu} + \frac{\theta}{2} \epsilon^{\mu\nu\alpha\beta} W_\mu \partial_\alpha W_\beta \] (4.1)
which leads to the field equation
\[ \partial_\nu W^{\mu\nu} = \theta \epsilon^{\mu\alpha\beta} \partial_\alpha W_\beta \] (4.2)
The parameter $\theta$ has dimension of mass.

In addition to the identically conserved current (2.3), let us introduce the two
indexes current $\tilde{J}_\theta^{\mu\nu}$ which generalizes (2.4):
\[ \tilde{J}_\theta^{\mu\nu} = W^{\mu\nu} - \theta \epsilon^{\mu\nu\alpha} W_\alpha \] (4.3)
$\tilde{J}_\theta^{\mu\nu}$ is conserved as a consequence of the field equation (4.2) in analogy with (2.4).

Let us introduce also the disorder operator $\mu_\theta$ which generalizes (2.6) for $\theta \neq 0$, by coupling $\tilde{J}_\theta^{\mu\nu}$ to $A_{\mu\nu}$:
\[ \mu_\theta(x; C) = \exp\left\{-\frac{ib}{2} \int d^3 z \tilde{J}_\theta^{\mu\nu} A_{\mu\nu}\right\} \]
or

\[ \mu_{\theta}(x; C) = \exp\left\{ \frac{ib}{2} \int d^3z W^{\mu} A_{\mu} + ib\theta \int d^3z e^{\mu\alpha\beta} W_{\mu} \partial_\alpha A_\beta \right\} \quad (4.4) \]

We see that \( \mu_{\theta} \equiv \mu(b)\Sigma'(\theta b) \), where \( \Sigma' \) is given by (2.10b).

Making use of the surface renormalization counterterm (2.7), we immediately see that the integrand in the functional integral defining correlation functions of \( \mu_{\theta} \) will depend on \( A_\mu \) through the combination \( W_{\mu} + A_\mu \) and therefore, we expect these correlation functions to be surface independent as before. We are going to see explicitly that this is indeed the case.

The \( \sigma \) operator needs no modification in the MCS theory. It is given, as before, by eq. (2.11).

### 4.2) Commutation Rules

Let us obtain here the relevant commutators involving the \( \sigma \) and \( \mu_{\theta} \) operators in the MCS theory.

The momentum canonically conjugate to \( W^i \) is now given by \( \Pi^i = -E^i + \frac{\theta}{2} \epsilon^{ij} W^j \), with \( E^i = W^{i\alpha} \) [1]. The basic commutators of the MCS theory are [1] (equal times)

\[
[W^i, W^j] = [\Pi^i, \Pi^j] = 0
\]

\[
[W^i(x), E^j(y)] = -[W^i(x), \Pi^j(y)] = -i\delta^{ij}(\vec{x} - \vec{y})
\]

\[
[E^i(x), E^j(y)] = -i\theta\epsilon^{ij} \delta^2(\vec{x} - \vec{y})
\]

Observe that contrary to the Maxwell case, the electric field \( E^i \) no longer commutes with itself.

Using the results of Appendix B, we see that \( \mu_{\theta} \) can be written as

\[ \mu_{\theta}(x; C) = \exp\{-ib \int_{T_x(C)} d^2\xi [E^i(x^0, \vec{\xi}) + \theta \epsilon^{ij} W_j(x^0, \vec{\xi})] \partial_i \text{arg}(\vec{\xi} - \vec{x}) \} \quad (4.6) \]

Using (4.5) we immediately see that the commutation rule between \( \sigma \) and \( \mu_{\theta} \) in the MCS theory is identical to that in the Maxwell theory involving \( \sigma \) and \( \mu \), namely (2.16) or (2.17). Also in the MCS theory, we have \([\sigma, \sigma] = 0\). \( \mu_{\theta} \), however, no longer commutes with itself. Writing \( \mu_{\theta}(x; C) \equiv e^{A(x; C)} \), we have according to (4.5),

\[ [A(x; C), A(y; C)] = \lim_{\rho, \delta \to 0} -i\theta b^2 \int_{T_x(C)} d^2\xi \int_{T_y(C)} d^2\eta \partial_\xi^{(i)} \text{arg}(\vec{\xi} - \vec{x}) \partial_\eta^{(j)} \text{arg}(\vec{\eta} - \vec{y}) \]
\[ \times \epsilon^{ij} \delta^2 (\xi - \eta) \]  

(4.7a)

or

\[
[A(x; C), A(y; C)] = \lim_{\rho,\delta \to 0} -i \theta b^2 \int_{T_x(C) \cap T_y(C)} d^2 \xi \epsilon^{ij} \partial_i \arg(\xi - \bar{x}) \partial_j \arg(\xi - \bar{y})
\]  

(4.7b)

It is shown in Appendix C, that, after taking the limit \( \rho, \delta \to 0 \), in \( T_x(C) \) and \( T_y(C) \), eq. (4.7b) is given by

\[
[A(x; C), A(y; C)] \lim_{\rho,\delta \to 0} -i \theta b^2 \left[ \epsilon \left[ \arg(x - \bar{y}) - \arg(y - \bar{x}) \right] \right] = -i \theta b^2 \left[ \epsilon \left[ \arg(x - \bar{y}) - \arg(\bar{x} - y) \right] \right]
\]  

(4.8)

where \( \epsilon(x) = \text{sign}(x) \). It follows that, at equal times

\[
\mu_\theta(x) \mu_\theta(y) = \mu_\theta(y) \mu_\theta(x) \exp\left\{ -i \theta b^2 \epsilon \left[ \arg(x - \bar{y}) - \pi \right] \right\}
\]  

(4.9)

indicating that in the MCS theory the disorder (vortex) operator is itself anyonic, carrying statistics \( S_{\mu_\theta} = \pi \theta b^2 \).

Let us investigate now whether the commutation rules of \( \sigma \) and \( \mu_\theta \) with the charge and magnetic flux operators, (2.18), are changed in the MCS theory.

According to (4.5), we immediately see that the commutation rules involving \( \sigma \) are the same as in Maxwell theory, namely, eq. (2.20) remains valid, indicating that \( \sigma \) bears charge but not magnetic flux. The commutator between \( \mu_\theta \) and the magnetic flux (topological charge) operator is also identical to the one in Maxwell theory. Let us evaluate the commutator between \( \mu_\theta \) and the charge operator. Writing \( \mu_\theta(x) \equiv e^{\alpha(x)} \) and \( \rho(y) = \partial_j E_j(y) \equiv \beta(y) \), we have, according to (4.5)

\[
[\alpha(x), \beta(y)] = \lim_{\rho,\delta \to 0} -ib \int_{T_x(C)} d^2 \xi \partial_i^{(x)} \arg(\xi - \bar{x}) \partial_j^{(y)} \left[ -i \theta \epsilon^{ij} + i \theta \epsilon^{jk} \delta^{ik} \right] \delta^2 (\xi - y) = 0
\]  

(4.10)

We see that also in MCS theory we have \([\mu_\theta, Q] = 0\) and eq. (2.22) remains fully valid.

A very interesting phenomenon occurs in the MCS theory, involving the statistics of \( \mu_\theta \). In spite of bearing only magnetic flux (and no net charge) \( \mu_\theta \) has generalized statistics \( S_{\mu_\theta} = \pi \theta b^2 \). This can be understood as follows. In the pure Maxwell
theory, the two quantities determining the statistics were the magnetic flux and the 
charge, whose densities were given respectively by the zeroth components of \(-J^\mu\) and 
\(\tilde{J}_\mu \equiv \partial_\nu \tilde{J}^{\nu \mu}\), where \(\tilde{J}^{\nu \mu}\) is given by (2.4). In the MCS theory, however, \(\tilde{J}^{\mu \nu}\) must be 
exchanged by \(\tilde{J}^{0 \nu \mu\theta}\), eq. (4.3). It follows that in addition to the magnetic flux, the 
other quantity relevant for determining the statistics in the MCS theory will have its 
density given by the zeroth component of \(\tilde{J}^{0 \mu\theta\equiv \partial_\nu \tilde{J}^{0 \nu \mu\theta}}\), or 
\(\tilde{J}^{0\theta} = \rho - \theta B\). In other 
words, the statistics in the MCS theory will be determined by the product of \(\Phi\) and 
the effective charge \(\tilde{Q} = Q - \theta \Phi\). Indeed, we see that according to (2.22) (which is 
also valid for \(\mu_\theta\)) \(\mu_\theta\) carries \(2\pi b\) units of \(\Phi\) and \(-2\pi \theta b\) units of \(\tilde{Q}\), implying that it will 
have statistics \(S_{\mu_\theta} = \frac{|\tilde{Q}|}{4\pi} = \pi \theta b^2\)  (Observe also the sign in (4.9).

A composite anyon operator can also be constructed in the MCS theory through 
an expression like (2.23) or (2.53). Using the \(\sigma - \mu_\theta\) and \(\mu_\theta - \mu_\theta\) commutation rules, 
we immediately find the equal-time commutator 

\[
[\varphi(x), \varphi(y)] = \varphi(y)\varphi(x) \exp\{i2\pi[\pi ab - \pi \theta b^2] \epsilon[\arg(\vec{x} - \vec{y}) - \pi]\}
\]

This indicates that the composite anyon field has now statistics \(S_\varphi = \pi b|a - \theta b|\). The 
field \(\varphi\) carries magnetic flux \(\Phi = 2\pi b\) and effective charge \(\tilde{Q} = 2\pi(a - \theta b)\). We see 
that here also the formula \(S_\varphi = \frac{|\tilde{Q}|}{4\pi}\) holds true. For the special case \(a = \theta b\), \(\varphi\) will 
be a boson, corresponding to the fact that \(\tilde{Q} = 0\) for this value of \(a\).

4.3) Order Correlation Functions

Let us compute now the euclidean order correlation function in the MCS theory. 
In order to do this we will need the analytic continuation of the Chern-Simons action 
to euclidean space:

\[
iS_{CS} = i\frac{\theta}{2} \int d^3 z \epsilon^{\mu \alpha \nu} W_\mu \partial_\alpha W_\nu \rightarrow i\frac{\theta}{2} \int d^3 z E \epsilon^{\mu \alpha \nu} W_\mu \partial_\alpha W_\nu \\
\equiv -\frac{\theta}{2} \int d^3 z E W_\mu C^{\mu \nu} W_\nu \equiv -S_{CS, E} ; \quad C^{\mu \nu} \equiv -i \epsilon^{\mu \alpha \nu} \partial_\alpha
\]

Following the same steps as in the case of Maxwell theory and using (4.12), we arrive 
at an expression similar to (2.39) for \(<\sigma\sigma^*>:\)

\[
<\sigma(x)\sigma^*(y)> = \lim_{\rho,\delta \to 0} Z^{-1} \int DW_\mu \exp\{-\int d^3 z \left[\frac{1}{2} W_\mu \left[P^{\mu \nu} + \theta C^{\mu \nu} + G^{\mu \nu}\right] W_\nu + \right.
\]

29
where \( C_\mu(z; x, y) \) was defined in (2.39-40).

Integrating over \( W_\mu \) we readily obtain

\[
< \sigma(x)\sigma^*(y) > = \lim_{\rho, \delta \to 0} \exp \left\{ \frac{1}{2} \int d^3z d^3z' C_\mu(z; x, y) C_\nu(z'; x, y) [P^{\mu\nu} + \theta C^{\mu\nu} + G^{\mu\nu}]^{-1}(z-z') \right\}
\]  

(4.14)

where \([P^{\mu\nu} + \phi C^{\mu\nu} + G^{\mu\nu}]^{-1} \equiv < W^{\mu} W^{\nu} >_{MCS}\) is the euclidean propagator of the \( W_\mu \) field in the MCS theory. This is given in momentum space by

\[
< W^{\mu}(k) W^{\nu}(-k) >_{MCS} = [P^{\mu\nu}(k) - \theta \epsilon^{\mu\alpha\nu} k_\alpha] \frac{1}{k^2(k^2 + \theta^2)} + \frac{k^\mu k^\nu}{\xi k^4} \quad (4.15a)
\]

and in coordinate space by

\[
< W^{\mu}(x) W^{\nu}(y) >_{MCS} = [P^{\mu\nu} + i \theta \epsilon^{\mu\alpha\nu} \partial_\alpha] \left[ \frac{1 - e^{-\theta|x-y|}}{4\pi\theta^2|x-y|} \right] - \lim_{m \to 0} \frac{1}{\xi} \partial^\mu \partial^\nu \left[ \frac{1}{m} - \frac{|x-y|}{8\pi} \right] \quad (4.15b)
\]

Here, the second expression between brackets is \( F^{-1}\left[ \frac{1}{k^2(k^2 + \theta^2)} \right] \) and the third is \( F^{-1}\left[ \frac{1}{k^4} \right] \)

(as before, \( m \) is an infrared regulator). Observe that (4.15) reduces to (2.27) in the limit \( \theta \to 0 \) (\( m \equiv \frac{1}{4\pi\theta} \) in this limit).

As before, we want to extract the gauge independent part of \(< \sigma \sigma^* >\). This will be achieved by inserting the gauge independent part of \(< W^{\mu} W^{\nu} >_{MCS}\), namely the \( \delta^{\mu\nu} \) and \( \epsilon^{\mu\alpha\nu} \)-proportional terms of (4.15), in (4.14). The \( \delta^{\mu\nu} \)-proportional part of (4.15) is

\[
-\Box \left[ \frac{1 - e^{-\theta|x-y|}}{4\pi\theta^2|x-y|} \right] = -\Box F^{-1}\left[ \frac{1}{k^2(k^2 + \theta^2)} \right] = F^{-1}\left[ \frac{1}{k^2 + \theta^2} \right] = \frac{e^{-\theta|x-y|}}{4\pi|x-y|} \quad (4.16)
\]

We can now use the identity (2.43a) and go through the same steps which led to (2.44). As before, the second term in (2.43a) will give a null contribution in the limit \( \rho, \delta \to 0 \), as is shown in Appendix C. The \( \epsilon^{\mu\alpha\nu} \)-proportional term in (4.15) also gives a vanishing contribution in this limit, for the same reason (see Appendix C).

The contribution from the first term in (2.43a) can be evaluated in the same way as we did in (2.44), leading to

\[
< \sigma(x)\sigma^*(y) >= \lim_{\epsilon \to 0} \exp \left\{ \frac{a^2}{8\pi} \sum_{i,j=1}^{2} \epsilon_i \epsilon_j (2\pi)^2 \left[ \frac{\exp\{-\theta|x_i - x_j|^2 + |\epsilon|^2\}}{\theta} \right] \right\} \quad (4.17a)
\]
\[< \sigma(x)\sigma^*(y) > = \exp\left\{ \frac{\pi a^2}{\theta} [e^{-\theta |x-y|} - 1]\right\} \tag{4.17b}\]

or

\[< \sigma_R(x)\sigma^*_R(y) > = \exp\left\{ \frac{\pi a^2}{\theta} e^{-\theta |x-y|}\right\} \tag{4.17c}\]

where \(\sigma_R \equiv \sigma \exp\left[\frac{\pi a^2}{2\theta}\right]\). Observe that (4.17) reduces to (2.44) in the limit \(\theta \to 0\).

Now,

\[
\lim_{|x-y| \to \infty} < \sigma_R(x)\sigma^*_R(y) > = 1 \tag{4.18}
\]

This result reflects the screening of charge associated with the mass generation to the gauge field induced by the Chern-Simons term.

4.4) Disorder (Vortex) Correlation Function

Let us evaluate here the \(\mu_\theta\)-correlation function in the MCS theory. Using (4.4), (4.12), (2.25) and (2.7), we can write

\[
< \mu_\theta(x)\mu^*_\theta(y) > = \lim_{\rho,\delta \to 0} Z^{-1} \int DW_\mu \exp\{-\int d^3z [\frac{1}{2} W_\mu [P^{\mu\nu} + \theta C^{\mu\nu} + G^{\mu\nu}] W_\nu + W_\mu [P^{\mu\nu} + \theta C^{\mu\nu}] A_\nu(z; x, y) + \frac{1}{4} A^{\alpha\beta}_\mu(z; x, y)]\} \tag{4.19}
\]

where \(A_\mu(z; x, y)\) was defined in (2.26).

Notice that, since \(S_{CS}[A_\mu] \equiv 0\) because \(A_\mu\) only has the \(\mu = 3\) component different from zero, the integrand in (4.9) depends on \(A_\mu\) through the combination \(W_\mu + A_\mu\). The method employed in [14], therefore, can also be used to prove the surface invariance of \(< \mu_\theta\mu^*_\theta >\) in the MCS theory.

Integrating over \(W_\mu\) with the help of (4.15) and using the results of Appendix B, we get

\[
< \mu_\theta(x)\mu^*_\theta(y) > = \lim_{\rho,\delta \to 0} \exp\left\{ \frac{1}{2} \int d^3z d^3z' [B_\mu(z; x, y) + D_\mu(z; x, y)] 
\times [B_\nu(z'; x, y) + D_\nu(z'; x, y)] [P^{\mu\nu} + \theta C^{\mu\nu} + G^{\mu\nu}]^{-1}(z - z') - S_R[A_\mu]\right\} \tag{4.20}
\]

In the above expression, \(B_\mu(z; x, y)\) was defined in (2.28) and (2.29) and according to (B.5), \(D_\mu(z; x, y) = D_\mu(z; x) - D_\mu(z; y)\), where

\[
D^\mu(z; x) = -i\theta b \int_{T_{x}(C)} d^2\xi_{\alpha} \epsilon^{\alpha\beta\mu} \partial_\beta arg(\xi - \bar{\xi}) \tag{4.21}
\]
Inserting $B_\mu, D_\mu$ and (4.15) in (4.20), we will get six terms: BPB, BPD and DPD, corresponding to the $[P^{\mu\nu} + \partial^\mu \partial^\nu]$ part of (4.15) and BCB, BCD and DCD, corresponding to the $\epsilon^{\mu\alpha\nu}$ part of (4.15). One of these, namely DPD will be gauge dependent, as a consequence of the fact that the $\theta$-dependent part of $\mu_\theta$ is not gauge invariant. As before, we will only consider the gauge independent part of this term. Let us evaluate now each one of the six terms above.

The BPB term can be computed exactly as in (2.30), except for the fact that now the $P^{\sigma\lambda}$-proportional term between square brackets in (2.30) is replaced by $F^{-1}[\frac{1}{k^2(k^2 + \theta^2)}]$ as one can infer from (4.15). We will arrive at the expression in the exponent of (2.33) but with (2.32) replaced by (4.16) in the square brackets in (2.33). Following the same steps that we took after (2.33) we arrive at

$$BPB = \lim_{\epsilon \to 0} \frac{\pi b^2}{e^{-\theta|x-y|}} - \frac{1}{\epsilon} + \frac{1}{4} \int d^3z d^3z' A_{\mu\nu}(z;x,y) F(z-z') A_{\mu\nu}(z';x,y)$$

where

$$F(z-z') = F^{-1}[\frac{k^2}{k^2 + \theta^2}] = -\frac{e^{-\theta|x-z'|}}{4\pi |z-z'|}$$

Observe that the last term in (4.22) is surface dependent but no longer canceled by the renormalization counterterm $S_R[A_\mu]$.

Let us obtain now the BCD term. This is given by (notice the factor two because there are actually two crossed terms)

$$BCD = \lim_{\rho,\delta \to 0} \frac{\theta^2 b^2}{2} \sum_{i,j=1} e_i e_j \int_{T_{x_i}(C)} d^2\xi \int_{T_{x_j}(C)} d^2\eta \partial^\xi \partial^\eta \arg(\xi - \eta_i) \partial^\eta \arg(\eta - \eta_j)$$

$$\times F^{\mu\nu}_{(\xi)} \epsilon^{\alpha\beta\gamma} \epsilon^{\sigma\rho\lambda} \partial_\rho \frac{1 - e^{-\theta|x-y|}}{4\pi \theta^2 |\xi - \eta|}$$

Using an identity similar to (2.31), namely,

$$F^{\mu\nu}_{\sigma} \epsilon^{\alpha\beta\gamma} \epsilon^{\sigma\rho\lambda} \partial_\rho = \epsilon^{\mu\nu\lambda} \epsilon^{\alpha\beta\gamma} (\Box \delta^\lambda \gamma + \partial^\lambda \partial^\gamma)$$

we arrive at an expression identical to the exponent in (2.33) except for the prefactor and for the expression between square brackets which is replaced by $F^{-1}[\frac{1}{k^2(k^2 + \theta^2)}]$. 

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Following the same steps we took after (2.33) we immediately obtain

\[
BCD = 2\pi\theta^2 b^2 \left[ \frac{1 - e^{-\theta|z - y|}}{\theta|x - y|} - 1 \right] + \frac{2\theta^2}{4} \int d^3zd^3z' A_{\mu\nu}(z;x,y)G(z - z')A_{\mu\nu}(z';x,y)
\]

(4.26)

where \(G(z - z') = \mathcal{F}^{-1}[\frac{1}{k^2 + \theta^2}]\) is given by the last equality in (4.16). Again, the last term in (4.26) is surface dependent.

Let us now consider the DPD term. As we remarked before, it is gauge dependent. In order to get the gauge invariant part, let us take the \(\delta_{\mu\nu}\)-proportional part of (4.15).

Then, using (4.21), we get an expression identical to (2.34), except for the prefactor and for the \(\delta_3(\xi - \eta)\) function which is replaced by \(\mathcal{F}^{-1}[\frac{1}{k^2 + \theta^2}] = G(z - z')\), namely

\[
DPD = -\frac{\theta^2}{4} \int d^3zd^3z' A_{\mu\nu}(z;x,y)G(z - z')A_{\mu\nu}(z';x,y)
\]

(4.27)

The gauge invariant part of DPD only contains the surface dependent term above.

Let us evaluate now the crossed BPD term. This is given by

\[
BPD = \lim_{\rho,\delta,m \to 0} -i\theta^2 \sum_{i,j=1}^2 \epsilon_i\epsilon_j \int_{T_{x_i}(C)} d^2\xi_\mu \int_{T_{x_j}(C)} d^2\eta_\sigma \partial_\nu^{(\xi)} \text{arg}(\vec{\xi} - \vec{x}_i) \partial^{(\eta)}_\beta \text{arg}(\vec{\eta} - \vec{x}_j)
\]

\[
\times \epsilon^{\nu\alpha\beta} \partial^{\mu}_{(\xi)} \mathcal{F}^{-1}[\frac{1}{k^2 + \theta^2}](\xi - \eta)\]

(4.28)

All the gauge dependent (derivative) terms vanish because \(F^{\mu\nu}_{\sigma\tau} \equiv 0\). Let us observe now that

\[
F^{\mu\nu}_{(\xi)} P^\sigma_{(\xi)} = -\Box (\partial^{\mu}_{(\xi)} \epsilon^{\nu\alpha\beta} - \partial^{\nu}_{(\xi)} \epsilon^{\mu\alpha\beta})
\]

(4.29)

Inserting (4.29) in (4.28), we see that the second term vanishes because \(d^2\xi^\mu / d^2\eta^\alpha\). The first term gives

\[
BPD = \lim_{\rho,\delta,m \to 0} -i\theta^2 \sum_{i,j=1}^2 \epsilon_i\epsilon_j \int_{T_{x_i}(C)} d^2\xi_\mu \int_{T_{x_j}(C)} d^2\eta_\sigma \partial_\nu^{(\xi)} \text{arg}(\vec{\xi} - \vec{x}_i) \partial^{(\eta)}_\beta \text{arg}(\vec{\eta} - \vec{x}_j)
\]

\[
\times \epsilon^{\nu\alpha\beta} \partial^{\mu}_{(\xi)} \mathcal{F}^{-1}[\frac{1}{k^2 + \theta^2}](\xi - \eta)
\]

(4.30)

where we used (4.16).

Writing

\[
\frac{1}{k^2 + \theta^2} = \frac{1}{k^2} - \frac{\theta^2}{k^2(k^2 + \theta^2)}
\]

(4.31)

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and using the last equality in (2.32), we see that the first term of (4.31) when inserted in (4.30) leads to an integral identical to the one evaluated in Appendix D. Hence, using (D.2) and (4.31), we obtain

\[ BPD = -\pi \theta b^2 \sum_{i,j=1}^{2} \epsilon_i \epsilon_j [\arg(\vec{x}_i - \vec{x}_j)] + K(T_x, T_y) \]  

(4.32a)

or

\[ BPD = \lim_{\epsilon \to 0} i2\pi \theta b^2 \left[ \arg(\vec{x} - \vec{y}) + \arg(\vec{y} - \vec{x}) - 2\arg(\vec{z}) \right] + K(T_x, T_y) \]  

(4.32b)

where

\[ K(T_x, T_y) = \lim_{\rho, \delta \to 0} i2\theta b^2 \sum_{i,j=1}^{2} \epsilon_i \epsilon_j \int_{T_{x_i}(C)} d^2 \xi \mu \int_{T_{x_j}(C)} d^2 \eta \nu \partial_\xi(\xi - \vec{x}_i) \partial_\eta(\eta - \vec{x}_j) \]

\[ \times \partial_\mu F(\xi - \eta) \]  

(4.33)

where \( F(\xi - \eta) = F^{-1}[\frac{1}{k^2(k^2 + \theta^2)}] \).

Observe that as the previous terms, also BPD contains a surface dependent term, namely \( K(T_x, T_y) \), given by (4.33).

Let us consider now the DCD term. Using (4.21) and the \( \theta \)-dependent part of (4.15), we find immediately

\[ DCD = \lim_{\rho, \delta \to 0} \frac{-i\theta b^2}{2} \sum_{i,j=1}^{2} \epsilon_i \epsilon_j \int_{T_{x_i}(C)} d^2 \xi \mu \int_{T_{x_j}(C)} d^2 \eta \nu \partial_\xi(\xi - \vec{x}_i) \partial_\eta(\eta - \vec{x}_j) \]

\[ \times \epsilon^{\mu \nu \lambda} \epsilon^{\alpha \beta \lambda} \partial_\rho F(\xi - \eta) \]  

(4.34)

where \( F(\xi - \eta) = F^{-1}[\frac{1}{k^2(k^2 + \theta^2)}] \). Observing that

\[ \epsilon^{\mu \nu \lambda} \epsilon^{\alpha \beta \lambda} \partial_\rho = \epsilon^{\alpha \beta \nu} \partial_\rho(\xi) - \epsilon^{\alpha \beta \mu} \partial_\nu(\xi) \]  

(4.35)

we see that the second term gives a vanishing contribution because \( d^2 \xi \mu / d^2 \xi \alpha \). The first term is identical to (4.33) and we get

\[ DCD = -\frac{1}{2} K(T_x, T_y) \]  

(4.36)
Let us obtain, finally, the BCB term. This is given by

\[ BCB = \lim_{\rho, \delta \to 0} \frac{i\theta b^2}{2} \sum_{i,j=1}^{2} \epsilon_i \epsilon_j \int_{T_{x_i}(C)} d^2 \xi \int_{T_{x_j}(C)} d^2 \eta \eta_{\mu} \partial_{\nu}^{(\xi)} arg(\xi - \bar{x}_i) \partial_{\beta}^{(n)} arg(\eta - \bar{x}_j) \]

\[ \times F^{\mu \nu}_{(\xi)} \sigma F^{\alpha \beta}_{(\eta)} \lambda \epsilon^{\rho \lambda} \partial_{\rho}^{(\xi)} F(\xi - \eta) \]

(4.37)

where again \( F(\xi - \eta) = \mathcal{F}^{-1} \left[ \frac{1}{k^2 + \theta^2} \right] \).

We now have the identity

\[ F^{\mu \nu}_{(\xi)} \sigma F^{\alpha \beta}_{(\eta)} \lambda \epsilon_{\rho \lambda} \partial_{\rho}^{(\xi)} = \epsilon_{\nu \gamma \beta} \partial_{\gamma}^{(\xi)} \partial_{\alpha}^{(\eta)} + \epsilon_{\mu \gamma \alpha} \partial_{\gamma}^{(\xi)} \partial_{\beta}^{(\eta)} - \epsilon_{\nu \gamma \alpha} \partial_{\gamma}^{(\xi)} \partial_{\beta}^{(\eta)} - \epsilon_{\mu \gamma \beta} \partial_{\gamma}^{(\xi)} \partial_{\alpha}^{(\eta)} \]

(4.38)

Again, the contribution of the 2\textsuperscript{nd} term vanishes because \( d^2 \xi / d^2 \eta \). In Appendix E we show that the contribution of the 3\textsuperscript{rd} and 4\textsuperscript{th} terms also vanish. We also show in Appendix E that the 1\textsuperscript{st} term gives

\[ BCB = -\frac{1}{2} BPD \]

(4.39a)

or

\[ BCB = \lim_{\epsilon \to 0} -i\pi \theta b^2 [arg(\bar{x} - \bar{y}) + arg(\bar{y} - \bar{x}) - 2arg(\bar{\epsilon})] - \frac{1}{2} K(T_x, T_y) \]

(4.39b)

Collecting the six terms given by (4.22), (4.26), (4.27), (4.32), (4.36) and (4.39), we see that the surface dependent part of the last three terms, namely \( K(T_x, T_y) \) exactly cancel! On the other hand, we can see, by using the fact that

\[ \mathcal{F}^{-1} \left[ \frac{k^2}{k^2 + \theta^2} \right] + \theta^2 \mathcal{F}^{-1} \left[ \frac{1}{k^2 + \theta^2} \right] = \mathcal{F}^{-1} [1] = \delta^3(z - z') \]

(4.40)

that the sum of the surface dependent pieces of the first three terms precisely add up to \( S_{R}[A_{\mu}] \) and therefore are exactly canceled by the renormalization counterterm in (4.20)! Just the surface independent part of the above six terms contributes to the correlation function and we get:

\[ < \mu_{\theta, R}(x) \mu_{\theta, R}^{*}(y) > = \exp \left\{ \pi b^2 \left[ \frac{e^{-\theta|x-y|}}{|x-y|} \right] + 2\pi \theta^2 b^2 \left[ \frac{1 - e^{-\theta|x-y|}}{|x-y|} \right] \right\} \]

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where we defined the renormalized $\mu_\theta$ as

\[
\mu_{\theta,R} = \lim_{\epsilon \to 0} \mu_\theta \exp\left\{ \frac{\pi b^2}{2|\epsilon|} \right\} + \pi b^2 \theta^2 - i \pi \theta b^2 \text{arg}(\epsilon) \right\} (4.42)
\]

Observe that for $\theta \to 0$ (4.41) reduces to (2.37). Eq. (4.41) is multivalued up to a factor $\exp[i2\pi(\pi\theta b^2)]$, indicating the $\mu_\theta$ has statistics $S_{\mu_\theta} = \pi \theta b^2$. This is in agreement with the previous result (4.9).

We see that $\mu_\theta$ is indeed a local field. Notice that the correlation function of $\mu$ would be given by $\exp[B\mu B + B\mu B]$, indicating therefore that $\mu$ would be a nonlocal (surface dependent) field in the MCS theory. The same would be true for $\Sigma'$ (even in Maxwell theory) whose correlation function would be given by $\exp[D\Sigma D + D\Sigma D]$. We see that the operators studied in [24] are nonlocal in the MCS theory and therefore cannot be used as bona fide dual operators.

The above study reveals how delicate and stringent are the conditions necessary for the obtainment of local dual operators.

Observe that

\[
< \mu_{\theta,R}(x)\mu_{\theta,R}^*(y) > \to \frac{|x-y|}{\to \infty} \text{constant} \neq 0
\]

indicating that $\mu_\theta$ does not create genuine vortex excitations in the MCS theory. In section 5 we will see how this result is changed when we add a Higgs potential to the MCS theory.

### 4.5) Mixed and Composite Anyon Correlation Functions

Let us study here the mixed $\sigma - \mu_\phi$ correlation function, from which we will obtain the composite anyon $\varphi$ correlation function.

Combining (4.13) with (4.19), we can write

\[
< \sigma(x_1)\mu_\theta(x_2)\sigma^*(y_1)\mu_\theta^*(y_2) > = \lim_{\rho,\delta \to 0} Z^{-1} \int DW_{\mu} \exp\left\{ - \int d^3z \left[ \frac{1}{2} W_{\mu}[P^{\mu \nu} + \theta C^{\mu \nu} + G^{\mu \nu}] W_{\nu} \\
+ C_{\mu}(z; x_1, y_1)W^\mu + W_{\mu}[P^{\mu \nu} + \theta C^{\mu \nu}] A_\nu(z; x_2, y_2) + \frac{1}{4} A_{\mu \nu}^2(x_2, y_2) \right] \right\} (4.43)
\]
Integrating over $W_\mu$, we get

$$<\sigma(x_1)\mu_\theta(x_2)\sigma^*(y_1)\mu_\theta^*(y_2)> - \lim_{\rho,\beta,\epsilon \to 0} \exp\{\frac{1}{2} \int d^3z d^3z' [C_\mu(z; x_1, y_1) + B_\mu(z; x_2, y_2)$$

$$+ D_\mu(z; x_2, y_2)][C_\nu(z'; x_1, y_1) + B_\nu(z'; x_2, y_2) + D_\nu(z'; x_2, y_2)]$$

$$\times [P^{\mu\nu} + \theta C^{\mu\nu} + G^{\mu\nu}]^{-1}(z - z') - S_R[A_\mu(x_2, y_2)]\} \tag{4.44}$$

where $C_\mu$ was defined in (2.39-40), $B_\mu$ in (2.28-29) and $D_\mu$ in (4.21). Inserting expression (4.15) for $[P^{\mu\nu} + \theta C^{\mu\nu} + G^{\mu\nu}]^{-1}$, we will get, in addition to the six terms computed in (4.4) six more terms involving $C_\mu$, namely CPC, CPB, and CPD, corresponding to the $[P^{\mu\nu} + \partial^\mu \partial^\nu]$ part of (4.15) and CCC, CCB and CCD, corresponding to the $\epsilon^{\mu\nu\alpha\beta}$ part of (4.15). As before, we only consider the gauge invariant part of eventually gauge dependent terms.

The CPC and CCC terms were computed in section 4.3. Their sum is given by the exponent in (4.17b). The CPB term was computed in section 2.5. It is given by (2.51). The gauge invariant part of the CPD term is given by

$$CPD = \lim_{\rho,\beta,\epsilon \to 0} ab \theta \sum_{i,j=1}^2 \epsilon_i \epsilon_j \int_{R_{i}(C)} d^2 \xi \int_{T_{j}(C)} d^2 \eta_{\alpha}[\epsilon^{ik} \partial_{k}^{(\epsilon)} \arg(\xi - \tilde{r}_i) + \partial_{\beta}^{(\epsilon)} \ln|\xi - \tilde{r}_i|]$$

$$\times \partial_{\beta}^{(\epsilon)} \arg(\eta - \tilde{s}_j) e^{\alpha \beta \lambda}[\delta^{\alpha \lambda} \frac{e^{-\theta |\xi - \eta|^2 + |\epsilon|^2}/2}{4\pi |\xi - \eta|^2 + |\epsilon|^2}]}\tag{4.45}$$

where we used (4.16), and introduced the ultraviolet regulator $\epsilon$. (We use the same convention for $r_i$ and $s_i$ as in section 2.5). Using now (2.43a) and the results of Appendix C, we find

$$CPD = \lim_{\epsilon \to 0} \frac{-ab \theta}{4\pi} \sum_{i,j=1}^2 \epsilon_i \epsilon_j (2\pi)^2 \left[-\exp\left[-\frac{\theta |r_i - s_j|^2 + |\epsilon|^2}{\theta}\right]\right]\tag{4.46a}$$

or

$$CPD = -\pi ab [e^{-\theta |x_1 - y_2|} + e^{-\theta |x_2 - y_1|} - e^{-\theta |x_1 - x_2|} - e^{-\theta |y_1 - y_2|}]\tag{4.46b}$$

Let us consider now the CCD term. This is given by

$$CCD = \lim_{\rho,\beta,\epsilon \to 0} iab \theta \sum_{i,j=1}^2 \epsilon_i \epsilon_j \int_{R_{i}(C)} d^2 \xi \int_{T_{j}(C)} d^2 \eta_{\alpha}[\epsilon^{ik} \partial_{k}^{(\epsilon)} \arg(\xi - \tilde{r}_i) + \partial_{\beta}^{(\epsilon)} \ln|\xi - \tilde{r}_i|]$$
As is shown in Appendix C, the only possibility for the \( \xi \) contraction with the immediately see that the derivative part of the identity would vanish, because of the such as (2.43). Such identity, however, would also be contracted with (4.49) and we nonzero is the expression between brackets (in (4.48)) being contracted with a derivative \( \partial^i \). We could try to force the appearance of such a derivative by using an identity such as (2.43). Such identity, however, would also be contracted with (4.49) and we immediately see that the derivative part of the identity would vanish, because of the contraction with the \( \epsilon \)'s and \( \partial^\rho \). We conclude, therefore, that \( CCB = 0 \).

Collecting all terms contributing to the mixed correlation function, we get

\[
< \sigma(x_1)\mu_{\theta,R}(x_2)\sigma^*(y_1)\mu_{\theta,R}(y_2) >= \exp\{\frac{\pi a^2}{\theta}[e^{-\theta|x_1-y_1|} - 1] +
\pi b^2\left[e^{-\theta|x_2-y_2|} - 2\pi b^2\left[1 - e^{-\theta|x_2-y_2|}\right]\right] - \pi b\left[e^{-\theta|x_2-y_1|} - e^{-\theta|x_1-x_2|} - e^{-\theta|x_1-y_2|} - e^{-\theta|y_1-y_2|}\right]
+ i\pi ab[\arg(x_1 - \bar{x}_2) + \arg(y_1 - \bar{y}_2) - \arg(x_1 - \bar{y}_2) - \arg(y_1 - \bar{x}_2)]
+ i\pi \theta b^2[\arg(x_2 - \bar{y}_2) + \arg(y_2 - \bar{x}_2)]\}
\]  

(4.50)

Observe that this expression reduces to (2.52) when \( \theta \to 0 \). The correlation function for the composite anyon field (2.53) can now be easily obtained by taking the limit \( x_1 \to x_2 \) and \( y_1 \to y_2 \) in (4.50). Introducing the same renormalization factor as in (2.53), we get

\[
< \varphi(x)\varphi^*(y) >= \exp\{\frac{\pi a^2}{\theta} - 2\pi ab[e^{-\theta|x-y|} - 1] + \pi b^2\left[\frac{e^{-\theta|x-y|}}{|x-y|}\right] \}
\]

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Notice that the multivaluedness of \(< \varphi \varphi^* >\) corresponds to a phase \(\exp[i2\pi(\pi ab - \pi \theta b^2)]\). This is in agreement with (4.11) and indicates that \(\varphi\) has statistics \(S_\varphi = \pi b|a - \theta b|\). Observe that for \(\theta \to 0\), (4.51) reduces to (2.54), which we obtained in Maxwell theory.

5) The Maxwell-Chern-Simons-Higgs Theory

5.1) The Unbroken Phase

Let us study now the Maxwell-Chern-Simons-Higgs (MCSH) theory which is essentially the Abelian Higgs Model plus a Chern-Simons term:

\[
\mathcal{L} = -\frac{1}{4} W_{\mu\nu}^2 + \frac{\theta}{2} \epsilon^{\mu\alpha\beta} W_\mu \partial_\alpha W_\beta + |D_\mu \phi|^2 - g^2 \phi^* \phi - \frac{\lambda}{4} (\phi^* \phi)^2
\]  

(5.1)

As in the AHM, we have two phases, according to whether: i) \(g^2 > 0\), \(< \phi > = 0\) or ii) \(g^2 < 0\), \(< \phi > = \kappa \neq 0\). Shifting the Higgs field around \(\kappa\) (for \(g^2 < 0\)), as before, will generate a mass term \((M = e\kappa)\) for \(W_\mu\).

Let us introduce here the operators \(\sigma\) and \(\mu_\theta\) considered before. The equal time commutation rules of the MCSH theory are the same as in (4.5) and therefore all the commutators evaluated in section 4.2 remain valid here.

The same remarks concerning the surface independence of \(\mu\) in the AHM may be applied to \(\mu_\theta\) here and therefore we conclude that (2.7) is the appropriate surface renormalization counterterm for \(\mu_\theta\) in the MCSH theory too.

Let us consider firstly the symmetric phase \((g^2 > 0\), \(< \phi > = 0\)). The \(\mu_\theta\)-correlation function is given by

\[
< \mu_\theta(x) \mu_\theta^*(y) > \lim_{\rho, \delta \to 0} Z^{-1} \int DW_\mu \exp \{- \int d^3 z \left[ \frac{1}{2} W_\mu [P^{\mu\nu} + \theta C^{\mu\nu} + G^{\mu\nu}] W_\nu + |D_\mu \phi|^2 + V(\phi) + W_\mu [P^{\mu\nu} + \theta C^{\mu\nu}] A_\mu(z; x, y) + \frac{1}{4} A_{\mu\nu}(z; x, y) \right] \}
\]  

(5.2)

As in the AHM, \(< \mu_\theta \mu_\theta^* >\) will be given by (3.3). Again, only two-leg graphs will contribute to the large distance limit in (3.3) [17]. Also here, we are going to make a
loop expansion for the computation of $\Lambda(x, y)$ in (3.3). As in the AHM, the leading contribution will be given by the graph of Fig. 4b. The only difference now is that the $W_\mu$-propagator is given by (4.15) and the external field is $B_\mu(z; x, y) + D_\mu(z; x, y)$, where $B_\mu$ is given by (2.28) and $D_\mu$ by (4.21). We immediately see that the graph of Fig. 4b, for these external fields and propagator is identical to the first term in the exponent in (4.20). It follows that

$$\lim_{|x-y|\to\infty} <\mu_{\theta, R}(x)\mu^*_{\theta, R}(y)>_{MCSH} = <\mu_{\theta, R}(x)\mu^*_{\theta, R}(y)>_{MCS} \to \text{constant} \neq 0$$

(5.3)

where $<\mu_{\theta, R}\mu^*_{\theta, R}>_{MCS}$ is given by (4.41) and we renormalized $\mu_\theta$ as in the MCS theory.

We conclude that also in the ordered phase of the MCSH theory the $\mu_\theta$ operator does not create genuine vortex excitations. We will see that this is no longer true in the broken phase of the theory.

Let us evaluate now the order correlation function $<\sigma\sigma^*>$. This is given now by

$$<\sigma(x)\sigma^*(y)> \lim_{\rho, \delta \to 0} Z^{-1} \int DW_\mu \exp\{-\int d^3z [\frac{1}{2}W_\mu [P_{\mu\nu} + \theta C_{\mu\nu} + G_{\mu\nu}]]W_\nu + |D_\mu \phi|^2$$

$$+ V(\phi) + C_\mu(z; x, y)W_\mu]\}$$

(5.4)

where $C_\mu(z; x, y)$ was defined in eqs. (2.39) and (2.40). As in the AHM, $<\sigma\sigma^*>$ is given by (3.6). Making exactly the same approximation we did in section 3.2 we see that $\Gamma(x, y)$ will be given by the graph in Fig. 4b but with $C_\mu$ as the external field and (4.15) as the $W_\mu$-propagator. It is easy to see that this is given by the exponent in (4.14). We readily conclude that in lowest order, the long distance behavior of the gauge invariant part of $<\sigma\sigma^*>$ in the MCSH theory is given by (4.17), namely,

$$\lim_{|x-y|\to\infty} <\sigma_R(x)\sigma^*_R(y)>_{MCSH} = <\sigma_R(x)\sigma^*_R(y)>_{MCS} \to 1$$

(5.5)

where $<\sigma_R\sigma^*_R>_{MCS}$ is given by (4.17c) and we renormalized $\sigma$ as in the MCS theory.

We now see that the behavior of $<\sigma_R\sigma^*_R>$ indicates charge screening in spite of the fact that we are in the symmetric phase. This naturally happens, because as in the MCS theory, the $W_\mu$-field acquired a mass through the Chern-Simons mechanism.
As before, the long distance behavior of the composite anyon field could be obtained by just exchanging $B_\mu(z; x, y) + D_\mu(z; x, y)$ by $B_\mu(z; x, y) + C_\mu(z; x, y) + D_\mu(z; x, y)$ in (5.2). We would obtain (4.51) for the long distance behavior of $< \varphi \varphi^\ast >$.

**5.2) The Broken Phase**

Let us investigate now the broken phase, where $< \phi > \neq 0$. Shifting the Higgs field around the vacuum value, as before, we can see that a mass term will be generated for $W_\mu$. The quadratic lagrangian, however, contains a Chern-Simons term in addition to (3.8). The euclidean $W_\mu$- propagator will be now given by

$$D_{\mu\nu}(k) = \frac{P^{\mu\nu}(1 + \frac{M^2}{k^2}) - \theta \epsilon^{\mu\nu\kappa} k_\kappa}{(k^2 + M^2)^2 + k^2 \theta^2} + \frac{k^\mu k^\nu}{k^2 (k^2 + M^2)}$$  

(5.6)

Observe that this reduces to (3.9a) for $\theta \to 0$ and to (4.15a) for $M \to 0$.

Let us evaluate the order correlation function $< \sigma \sigma^\ast >$. This is given by (5.4) and (3.6). Making the same approximation as before, the leading contribution to the long distance behavior of $< \sigma \sigma^\ast >$ will be given by the graph of Fig. 4b with $C_\mu$ as the external field which, as we pointed out before, leads to an expression similar to (4.14). The only difference is that now we must use the $M \neq 0$ and $\theta \neq 0$ propagator (5.6) instead of (4.15). As in the previous cases we are going to compute the gauge independent part of $< \sigma \sigma^\ast >$, namely the $\delta^{\mu\nu}$ and $\epsilon^{\mu\nu\alpha}$ proportional terms.

As in the MCS theory, the $\epsilon^{\mu\alpha\nu}$-proportional term vanishes (see Appendix C). The $\delta^{\mu\nu}$-proportional contribution, namely

$$\delta^{\mu\nu} \mathcal{F}^{-1} \left[ \frac{k^2 + M^2}{(k^2 + M^2)^2 + k^2 \theta^2} \right] = \frac{\delta^{\mu\nu}}{2\pi^2} \int_0^\infty dk \frac{k^2}{|x|} \frac{k^2 + M^2}{(k^2 + M^2)^2 + k^2 \theta^2}$$  

(5.7)

can be calculated with the help of the identity

$$\delta^{\mu\nu} \frac{\sin k|x|}{|x|} = \partial^\mu \partial^\nu \left[ - \frac{\cos k|x|}{k} + \frac{x^\mu x^\nu}{|x|^2} \left( \frac{\sin k|x|}{|x|} - k \cos k|x| \right) \right]$$  

(5.8)

which generalizes (2.43) when inserted in (5.7). Proceeding exactly as we did to go from (2.42) to (2.44), we see that the contribution from the last two terms in (5.8) vanishes. The contribution from the first term gives

$$< \sigma(x)\sigma^\ast(y) > \xrightarrow{|x-y| \to \infty} \exp \{ 4\pi^2 a^2 [I(x - y) - I(0)] \}$$  

(5.9a)
\( < \sigma_R(x)\sigma_R^*(y) > \frac{|x-y| \to \infty}{\to} \exp\{4\pi^2 a^2 I(x - y)\} \)

(5.9b)

where

\[
I(x - y) = \frac{1}{2\pi^2} \int_0^\infty dk \cos k|x - y| \left[ \frac{k^2 + M^2}{(k^2 + M^2)^2 + k^2\theta^2} \right]
\]

(5.10)

and \( \sigma_R \equiv \sigma \exp[2\pi^2 a^2 I(0)] \). Observe that

\[
\lim_{\theta \to 0} I(x - y) = \frac{e^{-M|x-y|}}{4\pi M}
\]

(5.11)

and (5.9) reduces to (3.11) in the AHM.

It follows from the Riemann-Lebesgue lemma that \( \lim_{|x-y| \to \infty} I(x - y) = 0 \). Hence,

\[
< \sigma_R(x)\sigma_R^*(y) > \frac{|x-y| \to \infty}{\to} 1
\]

This result indicates that also in the broken phase of the MCSH theory the charge is screened. Actually it is doubly screened, both by \( M \) and by \( \theta \), namely by the Higgs mechanism and by the Chern-Simons term. Notice that for \( M \to 0 \) and \( \theta \to 0 \), we have

\[
\lim_{M,\theta \to 0} I(x - y) = \frac{1}{4\pi} \left[ \frac{1}{M} - |x - y| \right]
\]

(5.12)

and (5.9) reduces to the expression (3.7) found in the symmetric phase of the AHM, where charge screening no longer occurs.

Let us consider now the disorder correlation function \( < \mu_\theta \mu^*_\theta > \). According to (5.2), we can write \( < \mu_\theta \mu^*_\theta > \) exactly as in (3.12), with the only difference that now \( S[W_{\mu\nu}, \phi, D_{\mu}\phi] \) is the action corresponding to (5.1). Making, as before, the change of variable \( W_\mu \to W_\mu + A_\mu(z; x; y) \), we get an expression identical to (3.13), where again, \( S \) is associated with (5.1). The correlation function \( < \mu_\theta \mu^*_\theta > \) therefore, will still be given by (3.14), the only difference being that the \( W_\mu^* \) propagator used in the computation of Feynman graphs must be (5.6) instead of (3.9a). As before, making an expansion in loops and in powers of \( M^2 \), it follows that the leading contribution to \( \tilde{\Lambda}(x, y) \) is given by the graphs of Fig. 6. Using (5.6), it is easy to see that the sum of the gauge dependent graphs of Fig. 6a identically vanishes for all values of the gauge parameter \( \xi \). The first contribution (of order \( 0(M^2) \)) is given, as in the AHM, by the graphs of Fig. 6b. These are independent of \( \theta \) and we see that in lowest
order the $\mu_{\theta,R}$-correlation function behaves asymptotically at large distances exactly as (3.18). The $\theta$-dependence will only be introduced by 1-loop corrections, through the $W_\mu$-propagator.

The result $\langle \mu_{\theta,R}(x)\mu_{\theta,R}^*(y) \rangle \sim |x-y|^{\infty} \rightarrow 0$ in the broken phase of the MCSH theory shows that $\mu_{\theta,R}$ creates true topological charge bearing excitations in this case. The mass of these quantum solitons is $M_\nu = \pi M^2 b^2$, within our approximation.

Let us remark finally that the (charge and magnetic flux bearing) composite anyon operator $\varphi$ would be the creation operator of the quantum excitations corresponding to the classical electrically charged vortices of ref. [25].

6) Conclusions and Remarks

The order-disorder duality structure was exploited in order to introduce dual operators corresponding to a local U(1) symmetry and carrying respectively, charge and magnetic flux, in some $2 + 1$ dimensional field theories involving a vector field. The conditions for the obtainment of local order($\sigma$), disorder (vortex) ($\mu$) or anyon ($\varphi$) operators must be carefully analyzed in each case. Local, surface independent correlation functions of these operators can be obtained only after some very stringent requirements are met.

In the absence of a Chern-Simons term, namely, in the Maxwell and in the Abelian-Higgs theories, the behavior of $\langle \sigma \rangle$ and $\langle \mu \rangle$ which can be inferred from the long distance behavior of the $\sigma$ and $\mu$ correlation functions is analogous to the behavior of the corresponding operators in the Ising model. The Maxwell and symmetric Higgs phases correspond to the disordered phase of the latter and the broken Higgs phase corresponds to the ordered phase of it. The ordered phase contains vortex excitations whose mass is explicitly evaluated. The disordered phase may be viewed a vortex condensate. Massive anyon excitations occur in all cases. The screening of charge in the broken phase is clearly exposed by the behavior of the $\sigma$-correlation function.

In the presence of a Chern-Simons term, namely, in the Maxwell-Chern-Simons and Maxwell-Chern-Simons-Higgs theories, we have two concurrent mechanisms of charge screening (mass generation for the vector field). The behavior of $\sigma$-correlation
functions expresses the Chern-Simons induced charge screening in the MCS and symmetric MCSH phases. In the broken MCSH phase we can infer from the long distance behavior of $<\sigma\sigma^*>$ the action of both mechanisms of charge screening, namely, the Chern-Simons and Higgs mechanisms.

Massive vortex excitations are still only seen to occur in the broken phase of the MCSH theory. We therefore can clearly distinguish the process which generates a mass for the vector field from that which generates a mass for the vortices. The first one can occur either via the Chern-Simons or Higgs mechanisms, while the second one is only induced by the latter.

In the presence of a Chern-Simons term we show that the statistics is no longer determined by the product $Q\Phi$ (charge $\times$ flux) but rather, by $\tilde{Q}\Phi$, where $\tilde{Q} = Q - \theta\Phi$. As a consequence, the exclusively flux bearing disorder (vortex) operator $\mu_\theta$ is seen to be anyonic.

The charge and magnetic flux carrying quantum vortex excitations corresponding to the classic solutions of the MCSH theory found in [25] will be described by the composite anyon operator $\varphi = \sigma\mu_\phi$.

Let us remark that even in the presence of a Chern-Simons term, an Ising-like behavior for the order and disorder variables could be obtained, by introducing a new order operator $\sigma' = \sigma\phi$, where $\phi$ is the Higgs field. This could be done naturally in the Abelian Higgs and MCSH theories but could also be easily achieved in the Maxwell and MCS theories by considering the weak coupling limit of them with a complex scalar field (without self-interaction, for instance). The behavior of $<\sigma'>$ in the MCS and MCSH symmetric and broken phases would be the same as that of $<\sigma>$ in the Maxwell and Abelian Higgs (symmetric and broken) phases. Furthermore, the short distance behavior of $<\sigma'\sigma'^*>$ in the Maxwell and MCS theories would have the singularity one would expect usually. The main advantage of the operator $\sigma$ is that it provides a reliable order parameter which works well whether or not a Higgs field is present in the theory.
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Appendix A

Let us show here the Cauchy-Riemann equations involving the real and imaginary parts of the analytic function \( \ln(z - x) \) that we use in this work.

The first one is (we will be in euclidean space throughout this Appendix)

\[
\partial_i \arg(z - x) = -\epsilon^{ij} \partial_j \ln |z - x| + 2\pi \int_{x,L}^\infty d\xi_j \epsilon^{ij} \delta^2(z - \xi), \ i = 1, 2 \tag{A.1}
\]

The line integral corresponds to the singularity of the derivative of \( \arg(z - x) \) along its cut \( L \), chosen to be the straight line going from \( x \) to \( \infty \) along the \( z^1 \) axis.

From (A.1) we get a second equation by contracting with \( \epsilon^{ij} \):

\[
\epsilon^{ij} \partial_i \partial_j \arg(z - x) = -\partial^j \ln |z - x| + 2\pi \int_{x,L}^\infty d\xi_j \delta^2(z - \xi), \ i = 1, 2 \tag{A.2}
\]

From (A.2) we obtain

\[
\epsilon^{ij} \partial_i \partial_j \arg(z - x) = 2\pi \delta^2(z - x) + 2\pi \int_{x,L}^\infty d\xi_j \partial^j \delta^2(z - \xi) \tag{A.3}
\]

where we used the fact that \( \partial_i \partial^i \ln |z - x| = 2\pi \delta^2(z - x) \).

Appendix B

Let us derive here eqs. (2.13) and (4.6). We start with (2.6) and observe that

\[
A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \int_{T_{x}(C)} d^2\xi \arg(\xi - x) \partial_\mu \delta^3(z - \xi) - (\mu \leftrightarrow \nu)
\]

\[
+ \oint_{C(x)} \epsilon_{\mu\nu\alpha} \arg(\xi - x) \delta^3(z - \xi) \tag{B.1}
\]
In this equation, the second term comes from the discontinuity of $A_{\mu}$ at $C(x)$, the boundary of $T_x(C)$ (see Fig. 1). Inserting (B.1) in (2.6), we get

$$
\mu(x; C) = \exp \{ib \int_{T_x(C)} d^2 \xi^\nu \partial_\mu W^{\mu \nu} \arg(\vec{\xi} - \vec{x}) - \frac{ib}{2} \oint_{C(x)} d\xi^\alpha \epsilon_{\mu \nu \alpha} W^{\mu \nu} \arg(\vec{\xi} - \vec{x}) \} \quad (B.2)
$$

Observing that $\partial_\mu W^{\mu \nu} = \epsilon_{\mu \sigma \rho \nu} \partial^\sigma \partial_\lambda W^\mu$ and integrating by parts the first term in (B.2), with the help of Stokes theorem we see that the boundary term exactly cancels the last term in (B.2). The remaining term gives

$$
\mu(x; C) = \exp \{ -ib \int_{T_x(C)} d^2 \xi^\nu W^{\mu \nu} \partial_\mu \arg(\vec{\xi} - \vec{x}) \} \quad (B.3)
$$

Noting that $d^2 \xi^\nu$ only has the zeroth component and remembering that $E^i = W^{io}$, eq. (2.13) immediately follows.

In order to obtain eq. (4.6) for $\mu_\theta$, let us consider $\Sigma'$, eq. (2.10b). Inserting (B.1) in (2.10b) (we can write this equation in terms of $A_{\mu \nu}$ divided by two) we get

$$
\Sigma'(x; C) = \exp \{ ia \int_{T_x(C)} d^2 \xi^\nu \epsilon^{\mu \alpha \beta} \partial_\alpha W_\beta \arg(\vec{\xi} - \vec{x}) + ia \oint_{C(x)} d\xi^\mu W^\mu \arg(\vec{\xi} - \vec{x}) \} \quad (B.4)
$$

Integrating by parts the first term with the help of Stokes theorem we again find that the boundary term exactly cancels the last term. The remaining term yields the following expression:

$$
\Sigma'(x; C) = \exp \{ -ia \int_{T_x(C)} d^2 \xi^\nu \epsilon^{\mu \alpha \beta} W_\beta \partial_\alpha \arg(\vec{\xi} - \vec{x}) \} \quad (B.5)
$$

This equation, for $a = \theta b$ together with (B.3) leads to expression (4.6) for $\mu_\theta$.

**Appendix C**

Let us demonstrate here the following two useful results:

$$
R_1 = \lim_{\rho, \delta \to 0} \int_{R_x(C)} d^2 \xi [\epsilon^{ij} \partial_j \arg(\vec{\xi} - \vec{x}) + \partial_j \ln |\vec{\xi} - \vec{x}|] \partial_j \Lambda(x^0, \vec{\xi}) = 2\pi \Lambda(x^0, \vec{x}) \quad (C.1)
$$

and

$$
R_2 = \lim_{\rho, \delta \to 0} \int_{T_x(C)} d^2 \xi \epsilon^{ij} \partial_j \arg(\vec{\xi} - \vec{x}) \partial_j \Lambda(x^0, \vec{x}) = 2\pi \Lambda(x^0, \vec{x}) \quad (C.2)
$$

for an arbitrary function $\Lambda(x)$. 

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Let us start with (C.1). We may write $R_1$ as (we omitted the argument $x^0$)

$$R_1 = \lim_{\rho,\delta \to 0} \int_{R_x(C)} d^2\xi [\epsilon^{ij} \partial_j [\partial_i \arg(\vec{\xi} - \vec{x}) \Lambda(\vec{\xi})] - \epsilon^{ij} \partial_j \partial_i \arg(\vec{\xi} - \vec{x}) \Lambda(\vec{\xi})]$$

$$+ \partial_i [\partial_i \ln |\vec{\xi} - \vec{x}| \Lambda(\vec{\xi})] - \partial_i \partial_i \ln |\vec{\xi} - \vec{x}| \Lambda(\vec{\xi})]$$

(C.3)

Using (A.2) or (A.3), we see that the sum of the $2^{nd}$ and $4^{th}$ terms is:

$$2^{nd} + 4^{th} = \lim_{\rho,\delta \to 0} 2\pi \int_{R_x(C)} d^2\xi \int_\infty^{\infty} d\eta \partial_j^{(\eta)} \delta^2(\xi - \eta) \Lambda(\vec{\xi})$$

(C.4)

$$= \lim_{\rho,\delta \to 0} 2\pi \int_{x,L \cap R_x(C)} d\eta \partial_j^{(\eta)} \Lambda(\vec{\eta}) = \lim_{\rho \to 0} 2\pi [\Lambda(\vec{x} + \rho \hat{\xi}_1) - \Lambda(\vec{x})] = 0$$

(C.5)

where the last integral is performed over the section of $L$ contained in $R_x(C)$. Observe that (A.3) is not zero in a cut plane like $R_x(C)$. It would also be different from zero in a punctured plane, for instance (this is the case considered in [11]).

The $1^{st}$ and $3^{rd}$ terms in (C.3) may be evaluated with the help of the Stokes and (two dimensional) Gauss theorems, respectively. The result is

$$R_1 = \lim_{\rho,\delta \to 0} \{- \oint_{C_\infty + C_\delta + C_\rho} d\xi [\partial_i \arg(\vec{\xi} - \vec{x}) + \epsilon^{ij} \partial_j \ln |\vec{\xi} - \vec{x}|] \Lambda(\vec{\xi})\}$$

(C.6)

In this expression, $C_\rho$ is the arc of circumference of radius $\rho$ and $C_\delta$, the two straight lines, in Fig. 3. $C_\infty$ is the arc of circumference with infinite radius closing $C$ at infinity. We immediately see that the contribution from $C_\delta$ is zero because the integrand is orthogonal to the integration element along $C_\delta$. The contribution of $C_\infty$ also vanishes because of (A.1). The only contribution to (C.6) comes from $C_\rho$. Inserting (A.1) in (C.6) we get

$$R_1 = \lim_{\rho \to 0} 2\pi \Lambda(x^0, \vec{x} + \rho \hat{\xi}_1) = 2\pi \Lambda(x^0, \vec{x})$$

(C.7)

thus establishing (C.1).

Let us observe that replacing $\partial_j \Lambda$ by a vector $\Lambda_j$ (not a derivative!) in (C.1) would yield a zero result. Indeed, inserting (A.2) in (C.2) we would obtain an integral identical to (C.5), with $\partial_j \Lambda$ replaced by $\Lambda_j$. This would vanish at $\rho \to 0$ for any regular $\Lambda_j$. The boundary terms, eq. (C.6) which are only present when $\Lambda_j = \partial_j \Lambda$
are responsible for making $R_1 \neq 0$. This is the reason why expressions of the type we found in the last terms of (2.43) and (5.8) do not contribute to correlation functions.

Let us turn now to (C.2). Observe that $\epsilon^{ij} \partial_j \partial_i \arg(\vec{\xi} - \vec{x}) = 0$ over the surface $T_x(C)$, Fig. 1. Using this fact, we can make the $\partial_j$ derivative total in (C.2) and then use Stokes theorem to get

$$R_2 = \lim_{\rho,\delta \to 0} \oint_{C_{\infty} + C_{\delta} + C_{\rho}} d\xi^i \partial_i \arg(\vec{\xi} - \vec{x})$$

(C.8)

Here $C_{\rho}$ is the arc of circumference of radius $\rho$ and $C_{\delta}$, the straight lines in Fig. 1. $C_{\infty}$ is the infinite radius arc of circumference closing $C$ at infinity. The contribution from $C_{\delta}$ again vanishes because the integrand is orthogonal to $d\xi^i$ along it. The contribution from $C_{\infty}$ vanishes for $\Lambda(\vec{\xi})$-functions going to zero at infinity. This is true for all cases considered in this work. Even for the function between square brackets in (2.43b) this is seen to be valid by adopting the following procedure before taking the limit $x \to \infty$: we put the system inside a circular box of radius $R \equiv \frac{1}{m}$. We immediately see that the above mentioned function will vanish on the boundary of this box. Then we make $R \to \infty$ as $x \to \infty$.

The only nonvanishing contribution to (C.8) therefore comes from $C_{\rho}$. Making a Taylor expansion in $\Lambda(\vec{\xi})$ around $\vec{\xi} = \vec{x}$, we see that in the limit $\rho \to 0$ only the zeroth order term is nonvanishing and we obtain:

$$R_2 = \lim_{\rho,\delta \to 0} \Lambda(x^0, \vec{x}) \int_{C_{\rho}} d\xi^i \partial_i \arg(\vec{\xi} - \vec{x}) = 2\pi \Lambda(x^0, \vec{x})$$

(C.9)

This establishes (C.2).

In eq. (4.7b), we have an expression similar to (C.2), the difference being that the boundary of the integration region now contains two curves: $C_x$ and $C_y$. Using the Stokes theorem with $\partial_j$ as the total derivative in $C_x$ and with $\partial_i$ in $C_y$, eq. (4.8) immediately follows from (C.2).

**Appendix D**

Let us demonstrate here the following two useful results:

$$R_3 \equiv \lim_{\rho_x, \delta_x, \rho_y, \delta_y \to 0} \int_{R_y(C)} d^2\eta [\epsilon^{ki} \partial_k^{(n)} \arg(\vec{\eta} - \vec{y}) + \partial_i^{(n)} \ln |\vec{\eta} - \vec{y}|]$$
\[
\times \int_{T_x(C)} d^2 \xi \partial_{i}^{(\xi)} \text{arg} (\xi - \vec{x}) \partial_{\xi}^{\mu} \left[ \frac{1}{4\pi |\xi - \eta|} \right] = \pi \text{arg}(\vec{y} - \vec{x}) \quad (D.1)
\]

and

\[
R_4 \equiv \lim_{\rho_x, \delta_x \to 0} \int_{T_x(C)} d^2 \eta \delta^{(n)}_{\beta} \text{arg}(\eta - \vec{y}) \int_{T_x(C)} d^2 \xi \partial_{\xi}^{(\xi)} \text{arg} (\xi - \vec{x}) \partial_{\xi}^{\mu} \left[ \frac{1}{4\pi |\xi - \eta|} \right] = \pi \text{arg}(\vec{y} - \vec{x}) + \text{arg}(\vec{x} - \vec{y}) \quad (D.2)
\]

Let us first demonstrate the following result that we will need later

\[
I_1 \equiv \lim_{\rho_x, \delta_x \to 0} \int_{T_x(C)} d^2 \xi \text{arg} (\xi - \vec{x}) \partial_{\xi}^{\mu} \left[ \frac{1}{4\pi |\xi - \eta|} \right] = \frac{1}{2} \text{arg}(\vec{y} - \vec{x}) + \pi \quad (D.3)
\]

It may be written as

\[
I_1 = \lim_{\rho_x, \delta_x \to 0} \int_{\delta_x}^{2\pi - \delta_x} d\varphi \varphi \int_{\rho_x}^{\infty} dr rH[L^2 + r^2 - 2Dr \cos(\varphi - \varphi_0)]^{-3/2} \quad (D.4)
\]

where \( L = |\eta - x|, \varphi_0 = \text{arg}(\vec{y} - \vec{x}), H = L \cos \theta \) and \( D = L \sin \theta \), where \( \theta \) is the angle the 3-vector \( \eta - x \) makes with the \( \xi_3 \)-axis in 3D-euclidean space.

Evaluating the \( r \)-integral [26], we obtain

\[
I_1 = \lim_{\delta_x \to 0} \int_{\delta_x}^{2\pi - \delta_x} d\varphi \varphi \frac{H}{L - D \cos(\varphi - \varphi_0)} \quad (D.5)
\]

Performing now the \( \varphi \)-integral [27], we get the last equality in (D.3).

Let us consider now the last integral in (D.1) and (D.2) (observe that \( \nu \equiv i = 1, 2 \) in (D.2), because \( d^2 \eta^\alpha \) has only the 3-component different from zero. Also \( \mu \equiv 3 \) in (D.1) and (D.2) for similar reason):

\[
I_2 \equiv \lim_{\rho_x, \delta_x \to 0} \int_{T_x(C)} d^2 \xi \partial_{i}^{(\xi)} \text{arg} (\xi - \vec{x}) \partial_{\xi}^{3} \left[ \frac{1}{4\pi |\xi - \eta|} \right] \quad (D.6)
\]

Integrating by parts and using the 2D-Gauss theorem, we may write (D.6) as

\[
I_2 = \lim_{\rho_x, \delta_x \to 0} \{ \partial_{i}^{(n)} \int_{T_x(C)} d^2 \xi \text{arg}(\xi - \vec{x}) \partial_{\xi}^{3} \left[ \frac{1}{4\pi |\xi - \eta|} \right] + \int_{C(\xi)} d\xi e^{ji} \text{arg} (\xi - \vec{x}) \partial_{\xi}^{3} \left[ \frac{1}{4\pi |\xi - \eta|} \right] \} \quad (D.7)
\]

Using (D.3) in the first term of (D.7), inserting the result in (D.1) and (D.2) and using (C.1) and (C.2), respectively, we see that the first term of (D.7) contributes a term \( \pi \text{arg}(\vec{y} - \vec{x}) \) to (D.1) and (D.2).
We will now show that the second term (boundary term) in (D.7) only contributes to (D.2). Indeed, using (A.2) we see that in the limit \( \delta_x \to 0 \), the first expression between brackets in (D.1) is orthogonal to \( \epsilon^{ji} d\xi^j \) along the two straight lines in \( C(x) \). The contribution of the arc of circumference with radius \( \rho \) vanishes for \( \rho \to 0 \). Hence, only the first term of (D.7) contributes to (D.1) and the result of the previous paragraph establishes (D.1).

Let us compute now the contribution from the second term in (D.7) to (D.2). Inserting this term in (D.2), we obtain the boundary contribution to \( R_4 \):

\[
R_4^{(b)} = \lim_{\rho_x, \delta_x \to 0} \oint d\xi^j \arg(\xi - \vec{x}) \int_{T_y(C)} d^2\eta \partial_j^{(\eta)} \arg(\eta - \vec{x})(-) \partial^3_{(\eta)} \left[ \frac{1}{4\pi} |\xi - \eta| \right] \quad (D.8)
\]

Observe that the second integral again has an expression like (D.6). Inserting (D.7) in (D.8) we see that the second term vanishes because \( d\xi^j \) is orthogonal to \( \epsilon^{kj} d\eta^k \) along the straight lines of \( C_x \) and \( C_y \). As before, the contribution from the arcs of circumference vanish for \( \rho_x, \rho_y \to 0 \). Using (D.3) we see that the first term of (D.7) gives the following contribution to (D.8):

\[
R_4^{(b)} = \frac{1}{2} \lim_{\rho_x, \delta_x \to 0} \oint_{C_x} d\xi^j \arg(\xi - \vec{x}) \partial_j^{(\xi)} \arg(\xi - \vec{y})
\]

\[
= \frac{1}{2} \lim_{\rho_x, \delta_x \to 0} \oint_{C_x} d\xi^j [\partial_j^{(\xi)} \arg(\xi - \vec{x})] \arg(\xi - \vec{y}) = \pi \arg(\vec{x} - \vec{y}) \quad (D.9)
\]

where we used (C.9). Combining this term with the contribution from the first term of (D.7) to \( R_4 \), we immediately establish the last equality in (D.2).

Let us finally mention that actually a more general result could be established for (D.1) and (D.2). Before taking the limit \( \rho_x, \delta_x \to 0 \) in (D.1) one would obtain \( R_3 = \frac{1}{2} \arg(\vec{y} - \vec{x}) \Omega(\vec{y}, T_x(C)) \), where \( \Omega(\vec{y}, T_x(C)) \) is the solid angle determined by the point \( \vec{y} \) and the surface \( T_x(C) \). For \( \rho_x, \delta_x \to 0 \), \( \Omega(\vec{y}, T_x(C)) \to 2\pi \) and we recover (D.1). Also, without taking the limits in (D.2) one would obtain \( R_4 = \frac{1}{2} [\arg(\vec{y} - \vec{x}) \Omega(\vec{y}, T_x(C)) + \arg(\vec{x} - \vec{y}) \Omega(\vec{x}, T_y(C))] \), which reduces to (D.2) for \( \rho_x, \delta_x \to 0 \) and \( \rho_y, \delta_y \to 0 \). These results would be relevant for the computation of correlation functions of the loop dependent operators \( \mu(x; C) \) (finite \( \rho, \delta \)).
Appendix E

Let us demonstrate here the result (4.39) for the BCB term which is given by eq. (4.37).

We start showing that the contribution from the last two terms in (4.38) vanishes. Insertion of the 3rd term of (4.38) in (4.37) yields an integral

\[
H(x_i, x_j) = \lim_{\rho, \delta \to 0} \int_{T_{x_i}(C)} d^2 \xi \int_{T_{x_j}(C)} d^2 \eta \partial^\mu(\xi) \arg(\vec{\xi} - \vec{x}_i) \partial^\nu(\eta) \arg(\vec{\eta} - \vec{x}_j)
\]

\[
\times e^{\nu\gamma\alpha} \partial^\mu(\xi) \partial^\nu(\eta) \partial^{\alpha}(\xi) \partial^{\beta}(\eta) F(\xi - \eta)
\]

(E.1)

Making the change of variable \( \xi \leftrightarrow \eta \) and exchanging \( \alpha \leftrightarrow \mu, \beta \leftrightarrow \nu \), and using the fact that \( \partial^{(\xi)} F(\xi - \eta) = -\partial^{(\eta)} F(\xi - \eta) \), we immediately conclude that the 4th term in (4.38) would yield an integral \( H(x_j, x_i) \). Hence, summing over \( i \) and \( j \) in (4.37) we see that the contributions from the last two terms in (4.38) are identical. Let us now show that these contributions actually vanish. Using the fact that \( d^2 \xi^\mu \) and \( d^2 \eta^\alpha \) only have the 3-component different from zero and noting that \( \arg(\vec{\eta} - \vec{x}_j) = 0 \) along the surface \( T_{x_j}(C) \), we may write

\[
H(x_i, x_j) = \lim_{\rho, \delta \to 0} \int_{C(x_i)} d^2 \xi e^{kj} \partial^l(\xi) \arg(\vec{\xi} - \vec{x}_i)
\]

\[
\times \partial^j(\xi) \int_{T_{x_j}(C)} d^2 \eta \partial_i^{(\eta)} \arg(\vec{\eta} - \vec{x}_j) F(\xi - \eta) \] 

(E.2)

Using now the Stokes and 2D-Gauss theorems, we get

\[
H(x_i, x_j) = \lim_{\rho, \delta \to 0} \int_{C(x_i)} d^2 \xi e^{kj} \partial^l(\xi) \arg(\vec{\xi} - \vec{x}_i) \int_{C(x_j)} d^2 \eta \partial_i^{(\eta)} \arg(\vec{\eta} - \vec{x}_j) \partial^3(\xi) F(\xi - \eta)
\]

(E.3a)

\[
= \lim_{\rho, \delta \to 0} 2\pi \int_{C(x_j)} d^2 \eta \partial_i^{(\eta)} \ln |\vec{\eta} - \vec{x}_j| \partial^3(\xi) F(x_i - \eta)
\]

(E.3b)

where we used (C.9) and (A.2).

The contribution from the arc of circumference of radius \( \rho \) (Fig. 1) to (E.3b) vanishes because \( d\eta^j \) is orthogonal to the integrand along this curve. The contributions from each one of the straight lines in \( C(x_j) \) (Fig.1) cancel each other in the limit \( \delta_{x_j} \to 0 \) because the integrand is regular along the cut. Hence we conclude that \( H(x_i, x_j) = 0 \).
The only nonvanishing contribution to (4.37) comes from the first term in (4.38). Insertion of this term in (4.37) yields the following integral

\[
J(x_i, x_j) = \lim_{\rho, \delta \to 0} \int_{T_{x_i}(C)} d^2 \xi \int_{T_{x_j}(C)} d^2 \eta \partial_{\alpha} \arg(\vec{\xi} - \vec{x}_i) \partial_{\beta} \arg(\vec{\eta} - \vec{x}_j) \\
\times \epsilon^{\nu\gamma\beta} \partial_{(\xi)} \partial_{(\eta)} \frac{F(\xi - \eta)}{F(0)}
\]

where \(F(\xi - \eta) = \mathcal{F}^{-1}[\frac{1}{k^2(k^2 + m^2)}]\).

Before evaluating (E.4), let us demonstrate the following results that we will need later:

\[
\int_{T_x(C)} d^2 \xi \partial_{\mu} \arg(\vec{\xi} - \vec{x}) \frac{(\xi - y)^\mu}{|\xi - y|} f(|\xi - y|) = 0 \quad (E.5a)
\]

and

\[
\int_{T_x(C)} d^2 \xi \partial_{\nu} \partial_{\mu} \arg(\vec{\xi} - \vec{x}) \frac{(\xi - y)^\mu}{|\xi - y|} f(|\xi - y|) = 0 \quad (E.5b)
\]

where \(f\) is an arbitrary function.

We may write (E.5a) as

\[
\int_0^\infty r \, dr \int_0^{2\pi-\delta} d\varphi \frac{1}{r} \frac{D \sin(\varphi - \varphi_0)}{[D^2 + r^2 - 2Dr \cos(\varphi - \varphi_0)]^{1/2}} f((L^2 + r^2 - 2Dr \cos(\varphi - \varphi_0))^{1/2})
\]

where \(L, D\) and \(\varphi_0\) were defined in Appendix D. The angular integration in (E.6) is easily seen to vanish, because the contribution from the first two quadrants exactly cancels the one coming from the last two. This establishes (E.5a). Eq. (E.5b) immediately follows if we note that it will be given by an expression like (E.6) but with \(\frac{1}{2}\) replaced by \(-\frac{\varphi_0}{r^2}\).

Let us now return to (E.4). Observe that \(\partial_{(\eta)}\) only has the 3-component different from zero (because it is contracted with \(d^2 \eta^3\)). Since \(\arg(\vec{\eta} - \vec{x}_j)\) does not depend on \(\eta^3\), we may make \(\partial_{(\eta)}\) a total derivative and use Gauss theorem on the closed surface \(S\), consisting of \(T_{x_j}(C) \cup T_{x_j}(C) \cup \tilde{T}_{x_j}(C)\). Here \(T_{x_j}(C)\) is a copy of \(T_{x_j}(C)\) but at \(\eta^3 = \infty\) and \(\tilde{T}_{x_j}(C)\) is the surface parallel to the \(\eta^3\)-axis, connecting \(T_{x_j}(C)\) and \(T_{x_j}(C)\), which is obtained by translating the curve \(C(x_j)\) from \(\eta^3 = x_j^3\) to \(\eta^3 = \infty\) along the \(\eta^3\)-axis. For \(\rho, \delta \to 0\) of course, \(\tilde{T}_{x_j}(C)\) collapses to zero. After using the Gauss theorem we may write (E.4) as,

\[
J(x_i, x_j) = \lim_{\rho, \delta \to 0} \int_{T_{x_i}(C)} d^2 \xi \int_{V(S)} d^3 \eta \epsilon^{\nu\gamma\beta} \partial_{(\xi)} \arg(\vec{\xi} - \vec{x}_i) \partial_{(\eta)} \arg(\vec{\eta} - \vec{x}_j)
\]

\[
\times \epsilon^{\nu\gamma\beta} \partial_{(\xi)} \partial_{(\eta)} \frac{F(\xi - \eta)}{F(0)}
\]

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\[ \times \partial^{\mu}_{(\xi)} \partial^{\gamma}_{(\xi)} \left[ -\Box \eta F(\xi - \eta) \right] \] (E.7)

where \( V(S) \) is the volume bounded by \( S \) (note that \( d^2 \eta^\alpha \) points inwards \( V(S) \)). To arrive at (E.7), we used (E.5b) and the fact that \( \Box \text{arg}(\vec{\eta} - \vec{x}_j) = 0 \) on the volume \( V(S) \). Observe that the contribution from \( T_{\infty}(C) \) vanishes because of the boundary conditions at infinity we impose on the function \( F \) (see Appendix C). One may also show that the contribution from \( \tilde{T}_{x_j}(C) \) vanishes for \( \rho, \delta \to 0 \) as should be expected.

Observe now that \( \partial^{\gamma}_{(\xi)} \) also only has the 3-component different from zero in (E.7). This is so because it is contracted with \( \epsilon^{\nu\gamma\beta} \) and \( \nu, \beta = 1, 2 \) since these indexes are contracted with derivatives of \( \text{arg} \) functions. We therefore can pull \( \partial^{\gamma}_{(\xi)} \) as a total derivative in (E.7) (after using the fact that \( \partial^{\gamma}_{(\xi)} = -\partial^{\gamma}_{(\eta)} \)) in the same way we did with \( \partial^{\alpha}_{(\eta)} \) in (E.4) and use Gauss theorem backwards to obtain

\[ J(x_i, x_j) = \lim_{\rho, \delta \to 0} \int_{R_{x_i}(C)} d^2 \xi \int_{T_{x_j}(C)} d^2 \eta \partial^{(\xi)}_{\nu} \text{arg}(\vec{\xi} - \vec{x}_i) \partial^{(\eta)}_{\beta} \text{arg}(\vec{\eta} - \vec{x}_j) \times \epsilon^{\nu\alpha\beta} \partial^{\mu}_{(\xi)} \left[ -\Box \eta F(\xi - \eta) \right] \] (E.8)

Again we used the fact that the contribution from \( T_{\infty}(C) \) is zero for the same reason above. The contribution from \( \tilde{T}_{x_j}(C) \) also vanishes because the integrand in (E.8) is orthogonal to \( d^2 \eta^\alpha \) along \( \tilde{T}_{x_j}(C) \).

The integral \( J(x_i, x_j) \) in (E.8) is identical to the one in (4.30) \((-\Box F = \mathcal{F}^{-1}[(k^2 + \theta^2)^{-1}])\), therefore, insertion of (E.8) in (4.37) immediately demonstrates (4.39).

**Appendix F**

Let us demonstrate here that (4.47) is equal to zero. Using the identity \( \epsilon^{\alpha\beta\lambda} \epsilon^{i\rho\lambda} = \delta^{\alpha i} \delta^{\beta \rho} - \delta^{\alpha \rho} \delta^{\beta i} \), we see that the first term vanishes because \( d^2 \eta^\alpha \) is orthogonal to the expression between square brackets in (4.47). The second term, when inserted in (4.47) yields the integral

\[ L(x_i, x_j) = \lim_{\rho, \delta \to 0} (-) \int_{R_{x_i}(C)} d^2 \xi \int_{T_{x_j}(C)} d^2 \eta \left[ \epsilon^{k \alpha} \partial^{(\xi)}_{k} \text{arg}(\vec{\xi} - \vec{r}_i) + \partial^{i}_{(\xi)} \ln |\vec{\xi} - \vec{r}_i| \right] \times \partial^{(\eta)}_{i} \text{arg}(\vec{\eta} - \vec{s}_j) \partial^{\alpha}_{(\xi)} F(\xi - \eta) \] (F.1)

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Observing that $\partial(\xi) = -\partial(\eta)$ only has the 3-component different from zero and using the Gauss theorem exactly in the same way we did in Appendix E, we may write (F.1) as

$$L(x_i, x_j) = \lim_{\rho, \delta \to 0} \int_{R_{\rho}(C)} d^2 \xi \int_{V(S)} d^3 \eta [\epsilon^{ik} \partial(\xi_k) \text{arg}(\vec{\xi} - \vec{r}_i) + \partial(\xi_k) \ln |\vec{\xi} - \vec{r}_i|]$$

$$\times \partial(\eta_k) \text{arg}(\vec{\eta} - \vec{s}_j) [-\Box \eta F(\xi - \eta)]$$ (F.2)

where $V(S)$ is the volume introduced in Appendix E.

As we have shown in Appendix C, the only possibility for the $\xi$-integral to give a nonzero result is the $\eta$-integral in (F.2) producing a total derivative. Using the identity given by (2.43a), for $[-\Box F]$ in (F.2), we see that the $\eta$-integral of the derivative term vanishes according to (E.5a). Hence we conclude that $\lim_{\rho, \delta \to 0} L(x_i, x_j) = 0$ and therefore $CCD = 0$, in eq. (A.47).

It is worth mentioning that for the special case $F = F^{-1}[\frac{1}{\Box}]$, $\Box F = \delta^3(\xi - \eta)$ in (F.2) and then, we would obtain $L(x_i, x_j) = 2\pi \text{arg}(\vec{r}_i - \vec{s}_j)$. This, however is not the case of the theories studied here.

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Figure Captions

Fig. 1: Surface used in the definition of $\mu(x;C)$.  
Fig. 2: Surface used in the definition of $\sigma(x)$.  
Fig. 3: Surfaces used in the definition of the external field $A_\mu(x;C)$. 
Fig. 4a: Vertex involving the external fields $B_\mu, C_\mu$ or $D_\mu$.  
Fig. 4b: Leading graph contributing to the long distance behavior of correlation functions in the AHM and MCSH theory. 
Fig. 5: Vertices relevant for the evaluation of $<\mu\mu^*>$ in the broken phase of the AHM and MCSH theory: a) gauge independent; b) gauge dependent.  
Fig. 6: Leading graphs contributing to the long distance behavior of $<\mu\mu^*>$ in the broken phase of the AHM and MCSH theory: a) gauge dependent; b) gauge independent.