Hook Interpolations

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Abstract

The hook components of $V^\otimes n$ interpolate between the symmetric power $\text{Sym}^n(V)$ and the exterior power $\wedge^n(V)$. When $V$ is the vector space of $k \times m$ matrices over $\mathbb{C}$, we decompose the hook components into irreducible $GL_k(\mathbb{C}) \times GL_m(\mathbb{C})$-modules. In particular, classical theorems are proved as boundary cases. For the algebra of square matrices over $\mathbb{C}$, a bivariate interpolation is presented and studied.

1 Introduction

The vector space $M_{k,m}$ of $k \times m$ matrices over $\mathbb{C}$ carries a (left) $GL_k(\mathbb{C})$-action and a (right) $GL_m(\mathbb{C})$-action. A classical Theorem of Ehresmann [2] describes the decomposition of an exterior power of $M_{k,m}$ into irreducible bimodules. The symmetric analogue was given later (cf. [6]). See Subsection 2.3 below.

In this paper we present a natural interpolation between these theorems, in terms of hook components of the $n$-th tensor power of $M_{k,m}$. Duality and asymptotics of the decomposition of hook components follow.

Similar methods are then applied to the diagonal two-sided $GL_k(\mathbb{C})$-action on the vector space of $k \times k$ matrices. Classical theorems of Thrall

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[18] and James [7] (for the symmetric powers of symmetric matrices), and of Helgason [4], Shimura [14] and Howe [5] (for the symmetric powers of anti-symmetric matrices) are extended, and a bivariate interpolation is presented.

Proofs are obtained using the representation theory of the symmetric and hyperoctahedral groups, together with Schur-Weyl duality; no use is made of highest weight theory.

1.1 Main Results

Let $M_{k,m}$ be the vector space of $k \times m$ matrices over $C$. The tensor power $M_{k,m} \otimes^n$ carries a natural $S_n$-action by permuting the factors. This action decomposes the tensor power into irreducible $S_n$-modules. Let $M_{k,m} \otimes^n(t)$ be the isotypic component of $M_{k,m} \otimes^n$ corresponding to the irreducible $S_n$-representation indexed by the hook $(n-t,1^t)$, where $0 \leq t \leq n-1$. This component still carries a $GL_k(C) \times GL_m(C)$-action.

**Theorem 1.1** Let $\lambda$ and $\mu$ be partitions of $n$, of lengths at most $k$ and $m$, respectively. For every $0 \leq t \leq n - 1$, the multiplicity of the irreducible $GL_k(C) \times GL_m(C)$-module $V^\lambda_k \otimes V^\mu_m$ in $M_{k,m} \otimes^n(t)$ is

$$\left(\begin{array}{c} n-1 \\ t \end{array}\right) \sum_{i=0}^{t} (-1)^{t-i} \sigma_{\lambda,\mu}(i) = \left(\begin{array}{c} n-1 \\ t \end{array}\right) \sum_{i=t+1}^{n} (-1)^{i-t} \sigma_{\lambda,\mu}(i)$$

where

$$\sigma_{\lambda,\mu}(i) := \sum_{\alpha+n-i, \beta+i} c^\lambda_{\alpha,\beta} c^\mu_{\alpha,\beta'},$$

$c^\lambda_{\alpha,\beta}$ are Littlewood-Richardson coefficients, and $\beta'$ is the partition conjugate to $\beta$.

See Theorem 3.3 below; for definitions and notation see Section 2 below. Theorem 1.1 interpolates between two well-known classical theorems (Theorems 2.4 and 2.5 below; see the remark following Theorem 3.3).

The following corollary generalizes the duality between Theorem 2.4 and Theorem 2.5.

**Corollary 1.2** Let $\mu \subseteq (m^m)$ and $\lambda$ be partitions of $n$. For every $0 \leq t \leq n - 1$ the multiplicity of $V^\lambda_k \otimes V^\mu_m$ in $M_{k,m} \otimes^n(t)$ is equal to the multiplicity of $V^\lambda_k \otimes V^\mu'_m$ in $M_{k,m} \otimes^n(n - 1 - t)$.  

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See Corollary 3.4 below.

Let $\lambda$ and $\mu$ be partitions of $n$. Define the distance
\[
d(\lambda, \mu) := \frac{1}{2} \sum_i |\lambda_i - \mu_i|.
\]

**Theorem 1.3** If $V^\lambda_k \otimes V^\mu_m$ appears as a factor in $M^{\otimes n}_{k,m}(t)$ (for some $0 \leq t \leq n - 1$) then
\[
d(\lambda, \mu) < km.
\]

See Theorem 4.3 below. This shows that, for $V^\lambda_k \otimes V^\mu_m$ to appear in a hook component, $\lambda$ and $\mu$ must be very “close” to each other (for $k$ and $m$ fixed, $n$ tending to infinity).

Consider now the vector space $M_{k,k}$ of $k \times k$ square matrices over $C$. Let $M^{\otimes n}_{k,k}(t,j)$ be the component of $M^{\otimes n}_{k,k}(t)$ consisting of tensors with $j$ skew symmetric and $n-j$ symmetric factors. $M^{\otimes n}_{k,k}(t,j)$ carries a $GL_k(C)$ two-sided diagonal action. The following theorem describes its decomposition as a $GL_k(C)$-module.

**Theorem 1.4** Let $\lambda$ be a partition of $2n$ of length at most $k$. For every $0 \leq t \leq n - 1$ and $0 \leq j \leq n$, the multiplicity of $V^\lambda_k$ in $M^{\otimes n}_{k,k}(t,j)$ is
\[
\binom{n-1}{t} \sum_{i=0}^{t} (-1)^{t-i} \sigma(\lambda, i, j) = \binom{n-1}{t} \sum_{i=t+1}^{n} (-1)^{t-1} \sigma(\lambda, i, j)
\]
where
\[
\sigma(\lambda, i, j) := \sum_{|\alpha|+|\beta|+|\gamma|+|\delta|=n, \ |\gamma|+|\delta|=i, \ |\beta|+|\delta|=j} c_{2-\alpha,(2,\beta)'-2\gamma,(2\delta)'-\lambda},
\]
and the sum is over all partitions $\alpha, \beta, \gamma, \delta$ with total size $n$ such that $\gamma$ and $\delta$ have distinct parts and total size $i$, and $\beta$ and $\delta$ have total size $j$. The operations $*$ and $\cdot$ are defined in Subsection 2.1. Definition of the (extended) Littlewood-Richardson coefficients is given in Subsection 2.2.

See Theorem 5.7 below. Theorem 1.4, for $t = 0$, interpolates between classical results, regarding symmetric powers of the spaces of symmetric and skew symmetric matrices (Theorems 2.6 and 2.7 below). Another boundary case, $t = n$, gives an interpolation between exterior powers of the same matrix spaces.
Corollary 1.5 Let $\lambda \subseteq (k^k)$ be a partition of $2n$. For every $0 \leq t \leq n - 1$ and $0 \leq j \leq n$, the multiplicity of $V^\lambda_k$ in $M_{k,k}^\otimes (t,j)$ is equal to the multiplicity of $V^\lambda_k$ in $M_{k,k}^\otimes (t,n-j)$.

See Corollary 5.8 below.

2 Background and Notation

2.1 Partitions

Let $n$ be a positive integer. A partition of $n$ is a vector of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ and $\lambda_1 + \ldots + \lambda_k = n$. We denote this by $\lambda \vdash n$. The size of a partition $\lambda \vdash n$, denoted $|\lambda|$, is $n$, and its length, $\ell(\lambda)$, is the number of parts. The empty partition $\emptyset$ has size and length zero: $|\emptyset| = \ell(\emptyset) = 0$. The set of all partitions of $n$ with at most $k$ parts is denoted by $\text{Par}_k(n)$.

For a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ define the conjugate partition $\lambda' = (\lambda'_1, \ldots, \lambda'_t)$ by letting $\lambda'_i$ be the number of parts of $\lambda$ that have size at least $i$.

A partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ may be viewed as the subset

$$\{(i,j) \mid 1 \leq i \leq k, 1 \leq j \leq \lambda_i\} \subseteq \mathbb{Z}^2,$$

the corresponding Young diagram. Using this interpretation, we may speak of the intersection $\lambda \cap \mu$ and the set difference $\lambda \setminus \mu$ of any two partitions. The set difference is called a skew shape; when $\mu \subseteq \lambda$ it is usually denoted $\lambda/\mu$.

Let $(k^m) := (k, \ldots, k)$ ($m$ equal parts). Thus, for example, $\lambda \subseteq (k^m)$ means $\lambda_1 \leq k$ and $\lambda'_1 \leq m$.

We shall also use the Frobenius notation for partitions, defined as follows: Let $\lambda$ be a partition of $n$ and set $d := \max\{i \mid \lambda_i - i \geq 0\}$ (i.e., the length of the main diagonal in the Young diagram of $\lambda$). Then the Frobenius notation for $\lambda$ is $(\lambda_1 - 1, \ldots, \lambda_d - d \mid \lambda'_1 - 1, \ldots, \lambda'_d - d)$.

For any partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of $n$ define the following doubling operation

$$2 \cdot \lambda := (2\lambda_1, \ldots, 2\lambda_k) \vdash 2n.$$

If all the parts of $\lambda$ are distinct, define also

$$2 \ast \lambda := (\lambda_1, \ldots, \lambda_k \mid \lambda_1 - 1, \ldots, \lambda_k - 1) \vdash 2n$$

in the Frobenius notation.
2.2 Representations

For any group $G$ denote the trivial representation by $1_G$. In this paper we shall denote the irreducible $S_n$-modules (Specht modules) by $S^\lambda$, and the irreducible $GL_k(C)$-modules (Weyl modules) by $V^\lambda_k$.

The Littlewood-Richardson coefficients describe the decomposition of tensor products of Weyl modules. Let $\mu \vdash t$ and $\nu \vdash n-t$. Then

$$V^\mu_k \otimes V^\nu_k \cong \bigoplus_{\lambda \vdash n} c^\lambda_{\mu,\nu} V^\lambda_k,$$

for $k \geq \max\{\ell(\lambda), \ell(\mu), \ell(\nu)\}$ (and the coefficients $c^\lambda_{\mu,\nu}$ are then independent of $k$).

By Schur-Weyl duality they are also the coefficients of the outer product of Specht modules. Namely,

$$(S^\mu \otimes S^\nu) \uparrow_{S_{t^1} \times S_{n-t^1}}^{S_n} \cong \bigoplus_{\lambda \vdash n} c^\lambda_{\mu,\nu} S^\lambda.$$

The following identity is well-known: For all triples of partitions $\lambda, \mu, \nu$

$$(2.a) \quad c^\lambda_{\mu,\nu} = c^\lambda_{\mu',\nu'}.$$

We shall also use the following notation for extended Littlewood-Richardson coefficients:

$$c^\lambda_{\alpha,\beta,\gamma,\delta} := \sum_{\mu,\nu} c^\lambda_{\alpha,\mu} c^\mu_{\beta,\nu} c^\nu_{\gamma,\delta};$$

so that

$$V^\alpha_k \otimes V^\beta_k \otimes V^\gamma_k \otimes V^\delta_k = \bigoplus_{\lambda} c^\lambda_{\alpha,\beta,\gamma,\delta} V^\lambda_k.$$

Let $B_n$ be the Weyl group of type $B$ and rank $n$, also known as the hyperoctahedral group or the group of signed permutations. A bipartition of $n$ is an ordered pair $(\mu, \nu)$ of partitions of total size $|\mu| + |\nu| = n$. The irreducible characters of $B_n$ are indexed by bipartitions of $n$; denote by $\chi^{\mu,\nu}$ the character indexed by $(\mu, \nu)$.

Consider the following natural embeddings of $S_n$ into $B_n$ and of $B_n$ into $S_{2n}$: $S_{2n}$ is the group of permutations on $\{-n, \ldots, -1, 1, \ldots, n\}$. $B_n$ is embedded as the subgroup of all $\pi \in S_{2n}$ satisfying $\pi(-i) = -\pi(i)$ ($1 \leq i \leq n$).
$S_n$ is embedded as the subgroup of all $\pi \in B_n$ satisfying also $\pi(i) > 0 \ (1 \leq i \leq n)$.

The following lemmas, used in Section 5, describe certain induced characters via the above embeddings. Lemma 2.1 is an immediate consequence of [12, Ch. I §7 Ex 4, Ch. I §8 Ex 5-6, and Ch. VII (2.4)]. See also [16].

**Lemma 2.1**

(a) $1_{B_n} \uparrow_{B_n}^{S_{2n}} = \chi^{(n)}, \emptyset \uparrow_{B_n}^{S_{2n}} = \sum_{\lambda \vdash n} \chi^{2\cdot\lambda}$;

(b) $\chi^{\emptyset, (n)} \uparrow_{B_n}^{S_{2n}} = \sum_{\lambda \vdash n} \chi^{(2\cdot\lambda)'},$

(c) $\chi^{(1^n), \emptyset} \uparrow_{B_n}^{S_{2n}} = \sum_{\lambda \vdash n} \chi^{2\cdot\lambda}$;

(d) $\chi^{\emptyset, (1^n)} \uparrow_{B_n}^{S_{2n}} = \sum_{\lambda \vdash n} \chi^{(2\cdot\lambda)'},$

where the last two sums are over partitions with distinct parts.

**Lemma 2.2**

(a) $\chi^{(n)} \uparrow_{S_n}^{B_n} = \sum_{i=0}^{n} \chi^{(i), (n-i)}$.

(b) $\chi^{(1^n)} \uparrow_{S_n}^{B_n} = \sum_{i=0}^{n} \chi^{(1^i), (1^{n-i})}$.

For a proof, see Section 6.1.

The following lemma is a special case of the Littlewood-Richardson rule for $B_n$, cf. [17, Lemma 7.1].

**Lemma 2.3**

$$\chi^{(i), (n-i)} = (\chi^{(i)}, \emptyset \otimes \chi^{\emptyset, (n-i)}) \uparrow_{B_i \times B_{n-i}}^{B_{n-i}}.$$
2.3 Symmetric and Exterior Powers of Matrix Spaces

In this subsection we cite well-known classical theorems, concerning the decomposition into irreducibles of symmetric and exterior powers of matrix spaces, which are to be generalized in this paper.

Let $M_{k,m}$ be the vector space of $k \times m$ matrices over $\mathbb{C}$. Then $M_{k,m}$ carries a (left) $GL_k(\mathbb{C})$-action and a (right) $GL_m(\mathbb{C})$-action. A classical Theorem of Ehresmann [2] (see also [10]) describes the decomposition of an exterior power of $M_{k,m}$ into irreducible $GL_k(\mathbb{C}) \times GL_m(\mathbb{C})$-modules.

**Theorem 2.4** The $n$-th exterior power of $M_{k,m}$ is isomorphic, as a $GL_k(\mathbb{C}) \times GL_m(\mathbb{C})$-module, to

$$\bigwedge^n(M_{k,m}) \cong \bigoplus_{\lambda \vdash n \text{ and } \lambda \subseteq (m^k)} V_{k}^{\lambda} \otimes V_{m}^{\lambda'},$$

where $\lambda'$ is the partition conjugate to $\lambda$.

The following three results on symmetric powers were proved several times independently; these results may be found in [6] and [3].

The symmetric analogue of Theorem 2.4 was studied, for example, in [6, (11.1.1)] and [3, Theorem 5.2.7].

**Theorem 2.5** The $n$-th symmetric power of $M_{k,m}$ is isomorphic, as a $GL_k(\mathbb{C}) \times GL_m(\mathbb{C})$-module, to

$$\text{Sym}^n(M_{k,m}) \cong \bigoplus_{\lambda \vdash n \text{ and } t(\lambda) \leq \min(k,m)} V_{k}^{\lambda} \otimes V_{m}^{\lambda},$$

Let $M_{k,k}^+$ be the vector space of symmetric $k \times k$ matrices over $\mathbb{C}$. This space carries a natural two sided $GL_k(\mathbb{C})$-action. The following theorem describes the decomposition of its symmetric powers into irreducible $GL_k(\mathbb{C})$-modules.

**Theorem 2.6** The $n$-th symmetric power of $M_{k,k}^+$ is isomorphic, as a $GL_k(\mathbb{C})$-module, to

$$\text{Sym}^n(M_{k,k}^+) \cong \bigoplus_{\lambda \in \text{Par}_k(n)} V_{k}^{2\lambda}.$$
This theorem was proved by A.T. James [7], but had already appeared in an early work of Thrall [18]. See also [5], [14], [6, (11.2.2)] and [3, Theorem 5.2.9] for further proofs and references.

Let $M_{k,k}^-$ be the vector space of skew symmetric $k \times k$ matrices over $C$. Then

**Theorem 2.7** The $n$-th symmetric power of $M_{k,k}^-$ is isomorphic, as a $GL_k(C)$-module, to

$$\text{Sym}^n(M_{k,k}^-) \cong \bigoplus_{(2i,i) \in \text{Par}_k(2n)} V_{k}^{(2i,i)}.$$ 

This theorem is proved in [4], [5], [14]. See also [6, (11.3.2)] and [3, Theorem 5.2.11].

### 3 Hook Components of $M_{k,m}^\otimes n$

Consider $M = M_{k,m} = C^{k \times m}$, the vector space of $k \times m$ matrices over $C$. Then $M \cong V \otimes W$, where $V \cong C^k$ and $W \cong C^m$. Thus $M$ carries a (left) $GL(V)$-action and a (right) $GL(W)$-action, which commute. Its tensor power $M^\otimes n \cong V^\otimes n \otimes W^\otimes n$ thus carries a $GL(V) \times S_n \times S_n \times GL(W)$ linear representation; one copy of the symmetric group $S_n$ permutes the factors in $V^\otimes n$, and the other copy of $S_n$ permutes the factors in $W^\otimes n$. The actions of all four groups clearly commute. We are interested in the $GL(V) \times S_n \times GL(W)$-action on $M^\otimes n$ obtained through the diagonal embedding $S_n \hookrightarrow S_n \times S_n$, $\pi \mapsto (\pi, \pi)$.

**Lemma 3.1**

$$M^\otimes n \cong \bigoplus_{\lambda \in \text{Par}_k(n), \mu \in \text{Par}_m(n)} \alpha_{\lambda \mu \nu} V_k^{\lambda} \otimes S^\mu \otimes V_m^\nu,$$

where

$$\alpha_{\lambda \mu \nu} := \langle \chi^{\lambda}, \chi^{\mu}, \chi^{\nu}, 1_S \rangle.$$

**Proof.** By Schur-Weyl duality (the double commutant theorem) [3, Theorem 9.1.2]

$$V^\otimes n \cong \bigoplus_{\lambda \in \text{Par}_k(n)} V_k^\lambda \otimes S^\lambda.$$
as $GL(V) \times S_n$-modules. Similarly,

$$W^\otimes n \cong \bigoplus_{\lambda \in \text{Par}_m(n)} V^\lambda_k \otimes S^\lambda_m$$

as $GL(W) \times S_n$-modules. Therefore

$$M^\otimes n \cong V^\otimes n \otimes W^\otimes n \cong \bigoplus_{\lambda \in \text{Par}_n(n)} \bigoplus_{\mu \in \text{Par}_m(n)} V^\lambda_k \otimes (S^\lambda \otimes S^\mu) \downarrow_{S_n \times S_n} \otimes V^\mu_m$$

as $GL(V) \times S_n \times S_n \times GL(W)$-modules.

Using the diagonal embedding $S_n \hookrightarrow S_n \times S_n$,

$$M^\otimes n \cong \bigoplus_{\lambda \in \text{Par}_n(n)} \bigoplus_{\mu \in \text{Par}_m(n)} V^\lambda_k \otimes (S^\lambda \otimes S^\mu) \downarrow_{S_n \times S_n} \otimes V^\mu_m$$

as $GL(V) \times S_n \times GL(W)$-modules.

Note that the $S_n$-character of $(S^\lambda \otimes S^\mu) \downarrow_{S_n \times S_n}$ is the standard inner tensor product (sometimes called Kronecker product) of the $S_n$-characters $\chi^\lambda$ and $\chi^\mu$. Hence, by elementary representation theory

$$(S^\lambda \otimes S^\mu) \downarrow_{S_n \times S_n} \cong \bigoplus_{\nu \vdash n} \alpha_{\lambda\mu\nu} S^\nu,$$

where

$$\alpha_{\lambda\mu\nu} = \langle \chi^\lambda \chi^\mu, \chi^\nu \rangle = \frac{1}{n!} \sum_{\pi \in S_n} \chi^\lambda(\pi) \chi^\mu(\pi) \chi^\nu(\pi) = \langle \chi^\lambda \chi^\mu \chi^\nu, 1_{S_n} \rangle.$$

\[\square\]

In particular, Lemma 3.1 gives Theorem 2.4 and Theorem 2.5.

**Corollary 3.2**

(1) $\text{Sym}^n(M) \cong \bigoplus_{\lambda \vdash n \text{ and } \ell(\lambda) \leq \min(k,m)} V^\lambda_k \otimes V^\lambda_m.$

(2) $\wedge^n(M) \cong \bigoplus_{\lambda \vdash n \text{ and } \lambda \subseteq (m^k)} V^\lambda_k \otimes V^\lambda_m.$
**Proof.** \( \text{Sym}^n(M) \) is the isotypic component of \( M^\otimes n \) corresponding to the trivial character \( \chi^{(n)} \) of the symmetric group. Thus, by Lemma 3.1

\[
\text{Sym}^n(M) \cong \bigoplus_{\lambda \in \text{Par}_k(n) \text{ and } \mu \in \text{Par}_m(n)} \alpha_{\lambda,\mu,(n)} V_k^\lambda \otimes S^{(n)} \otimes V_m^\mu.
\]

But by the orthonormality of irreducible characters

\[
\alpha_{\lambda,\mu,(n)} = \langle \chi^\lambda \chi^\mu, \chi^{(n)} \rangle = \langle \chi^\lambda, \chi^\mu \rangle = \delta_{\lambda\mu}.
\]

This proves (1), namely Theorem 2.5.

The \( n \)-th exterior power is the isotypic component of \( M^\otimes n \) corresponding to the sign character \( \chi^{(1^n)} \) of the symmetric group. Recall that for any partition \( \mu \vdash n \), \( \chi^\mu \chi^{(1^n)} = \chi^{\mu'} \). Thus

\[
\alpha_{\lambda,\mu,(1^n)} = \langle \chi^\lambda \chi^\mu, \chi^{(1^n)} \rangle = \langle \chi^\lambda, \chi^{\mu'} \rangle = \delta_{\lambda\mu'}.
\]

This proves (2), namely Theorem 2.4.

\[\square\]

Let \( M \) be the vector space of \( k \times m \) matrices as before. The tensor power \( M^\otimes n \) carries a natural \( S_n \)-action by permuting the factors. This action decomposes into irreducible \( S_n \)-representations. Let \( M^\otimes n(t) \) be the component of \( M^\otimes n \), corresponding to the irreducible hook representation \( (n-t,1^t) \), \( 0 \leq t \leq n-1 \). This component carries a \( GL_k(C) \times GL_m(C) \)-action.

**Theorem 3.3** Let \( \lambda \in \text{Par}_k(n) \) and \( \mu \in \text{Par}_m(n) \). For every \( 0 \leq t \leq n-1 \), the multiplicity of the irreducible \( GL_k(C) \times GL_m(C) \)-module \( V_k^\lambda \otimes V_m^\mu \) in \( M^\otimes n(t) \) is

\[
\binom{n-1}{t} \sum_{i=0}^t (-1)^{t-i} \sigma_{\lambda,\mu}(i) = \binom{n-1}{t} \sum_{i=t+1}^n (-1)^{i-t-1} \sigma_{\lambda,\mu}(i)
\]

where

\[
\sigma_{\lambda,\mu}(i) := \sum_{\alpha \vdash n-i, \beta \vdash i} c_{\alpha\beta}^\lambda c_{\alpha\beta'}^\mu,
\]

\( c_{\alpha\beta}^\lambda \) are Littlewood-Richardson coefficients, and \( \beta' \) is the partition conjugate to \( \beta \).
Remark. Theorem 3.3 may be considered as an interpolation between Theorem 2.4 and Theorem 2.5. Taking $t = 0$ gives $M^{\otimes n}(0) \cong \text{Sym}^n(M)$, and $\beta \vdash 0$ means $\beta = \emptyset$. Hence $\lambda = \alpha = \mu$, yielding multiplicity $\delta_{\lambda\mu}$. This gives Theorem 2.5.

Similarly, taking $t = n - 1$ gives $M^{\otimes n}(n - 1) \cong \wedge^n(M)$, and $\alpha \vdash 0$ means $\alpha = \emptyset$. Hence $\lambda = \beta = \mu'$, yielding multiplicity $\delta_{\lambda\mu'}$. This gives Theorem 2.4.

Proof. By Lemma 3.1

$$M^{\otimes n}(t) \cong \bigoplus_{\lambda \in \text{Par}_k(n) \text{ and } \mu \in \text{Par}_m(n)} \alpha_{\lambda,\mu,(n-t,1^t)} V_k^\lambda \otimes S^{(n-t,1^t)} \otimes V_m^\mu$$

is the decomposition of this component into irreducibles.

Denote by $1_t$ and $\varepsilon_t$ the trivial and sign characters, respectively, of $S_t$.

By the combinatorial interpretation of the Littlewood-Richardson rule (cf. [8, Theorem 2.8.13]), for every $0 \leq t \leq n$

$$(3.a) \quad (1_{n-t} \otimes \varepsilon_t)^{S_n}_{S_{n-t} \times S_t} = \chi^{(n-t,1^t)} + \chi^{(n-t+1,1^{t-1})}.$$ 

Hence, by Frobenius reciprocity

$$\alpha_{\lambda,\mu,(n-t,1^t)} + \alpha_{\lambda,\mu,(n-t+1,1^{t-1})} = \langle \chi^\lambda \chi^\mu, \chi^{(n-t,1^t)} + \chi^{(n-t+1,1^{t-1})} \rangle =$$

$$= \langle \chi^\lambda \chi^\mu, (1_{n-t} \otimes \varepsilon_t)^{S_n}_{S_{n-t} \times S_t} \rangle = \langle (\chi^\lambda \chi^\mu)^{S_n}_{S_{n-t} \times S_t}, 1_{n-t} \otimes \varepsilon_t \rangle =$$

$$= \langle \chi^\lambda \chi^{S_n}_{S_{n-t} \times S_t}, \chi^\mu \chi^{S_n}_{S_{n-t} \times S_t} \cdot (1_{n-t} \otimes \varepsilon_t) \rangle.$$ 

By the Littlewood-Richardson rule the last expression is equal to

$$\langle \sum_{a \vdash n-t, \beta \vdash t} c_{a\beta}^\lambda \chi^a \otimes \chi^\beta, \sum_{a \vdash n-t, \beta \vdash t} c_{a\beta}^\mu \chi^a \otimes \chi^\beta \cdot (1_{n-t} \otimes \varepsilon_t) \rangle =$$

$$= \langle \sum_{a \vdash n-t, \beta \vdash t} c_{a\beta}^\lambda \chi^a \otimes \chi^\beta, \sum_{a \vdash n-t, \beta' \vdash t} c_{a\beta'}^\mu \chi^a \otimes \chi^\beta' \rangle = \sum_{a \vdash n-t, \beta \vdash t} c_{a\beta}^\lambda c_{a\beta'}^\mu,$$

which was denoted $\sigma_{\lambda,\mu}(t)$ in the statement of the theorem. Alternating summation and the well-known fact

$$\dim S^{(n-t,1^t)} = \binom{n-1}{t}$$

now complete the proof. $\square$

The following corollary generalizes the “duality” of Theorem 2.4 and Theorem 2.5.
Corollary 3.4 Let $\lambda \in \operatorname{Par}_k(n)$, and let $\mu, \mu' \in \operatorname{Par}_m(n)$ be conjugate partitions. Then, for every $0 \leq t \leq n - 1$, the multiplicity of $V_\lambda^k \otimes V_\mu^m$ in $M^\otimes n(t)$ is equal to the multiplicity of $V_\lambda^k \otimes V_{\mu'}^m$ in $M^\otimes n(n - 1 - t)$.

Proof. By Theorem 3.3, we need to show that

$$\binom{n - 1}{t} \sum_{i=0}^{t} (-1)^{t-i} \sigma_{\lambda,\mu}(i) = \binom{n - 1}{n - 1 - t} \sum_{j=n-t}^{n} (-1)^{j-n+t} \sigma_{\lambda,\mu'}(j).$$

This follows from

$$\sigma_{\lambda,\mu}(i) = \sigma_{\lambda,\mu'}(n-i),$$

which in turn follows from (2.a).

Examples. Let $\lambda \in \operatorname{Par}_k(n)$, $\mu, \mu' \in \operatorname{Par}_m(n)$. The multiplicities of $V_\lambda^k \otimes V_\mu^m$ in $M^\otimes n(t)$ for $t = 0$ and $t = n - 1$ are given by Theorems 2.5 and 2.4. Consider two other pairs of $t$-values.

- $t = 1$: For $\lambda = \mu$ the multiplicity is $n - 1$ times the number of (inner) corners in $\lambda$, minus 1. For $\lambda \neq \mu$ it is $n - 1$ if $|\lambda \setminus \mu| = 1$, and zero otherwise.

- $t = n - 2$: For $\lambda = \mu'$ the multiplicity is $n - 1$ times the number of (inner) corners in $\lambda$, minus 1. For $\lambda \neq \mu'$ it is $n - 1$ if $|\lambda \setminus \mu'| = 1$, and zero otherwise.

- $t = 2$ ($n > 2$): For $\lambda = \mu$ the multiplicity is nonzero iff $\lambda$ has at least 3 inner corners. For $\lambda \neq \mu$ it is nonzero iff there is a partition $\alpha$ of $n - 2$ such that $\lambda / \alpha$ is a horizontal strip and $\mu / \alpha$ is a vertical strip, or vice versa.

- $t = n - 3$ ($n > 2$): Analogous to the previous case, with $\mu$ replaced by $\mu'$.

4 Asymptotics

Let $\lambda$ and $\mu$ be partitions of the same number $n$. Recalling the definition of the set difference $\lambda \setminus \mu$ from Subsection 2.1, define the distance

$$d(\lambda, \mu) := |\lambda \setminus \mu| = \frac{1}{2} \sum_{i} |\lambda_i - \mu_i|.$$
Lemma 4.1 If $V_k^\lambda \otimes V_m^\mu$ appears as a factor in $M^\otimes n(t)$ (for some $0 \leq t \leq n - 1$) then $d(\lambda, \mu) \leq t$ and $d(\lambda, \mu') \leq n - 1 - t$.

Proof. By Theorem 3.3, if $V_k^\lambda \otimes V_m^\mu$ appears as a factor in $M^\otimes n(t)$ then there exists a pair of partitions, $\alpha \vdash n - i$ and $\beta \vdash i$, with $0 \leq i \leq t$, such that $c_{\alpha\beta}^\lambda c_{\alpha\beta'}^\mu \neq 0$. $c_{\alpha\beta}^\lambda \neq 0 \Rightarrow \alpha \subseteq \lambda$, and $c_{\alpha\beta'}^\mu \neq 0 \Rightarrow \alpha \subseteq \mu$. Hence

$$|\lambda \setminus \mu| \leq |\lambda \setminus \alpha| = i \leq t.$$  

Using the second expression in Theorem 3.3, if $V_k^\lambda \otimes V_m^\mu$ appears as a factor in $M^\otimes n(t)$ then there exists a pair of partitions, $\alpha \vdash n - j$ and $\beta \vdash j$, with $t + 1 \leq j \leq n$, such that $c_{\alpha\beta}^\lambda c_{\alpha\beta'}^\mu \neq 0$. $c_{\alpha\beta}^\lambda \neq 0 \Rightarrow \beta \subseteq \lambda$, and $c_{\alpha\beta'}^\mu \neq 0 \Rightarrow \beta \subseteq \mu'$. Hence

$$|\lambda \setminus \mu'| \leq |\lambda \setminus \beta| = n - j \leq n - t - 1.$$  

Altogether, we get the desired claim. $\square$

Let $\psi$ be an $S_n$-character (not necessarily irreducible). Define the height of $\psi$ by

$$\text{height}(\psi) := \max \{ \ell(\nu) \mid \nu \vdash n, \langle \psi, \chi^\nu \rangle \neq 0 \}.$$  

The following result was proved by Regev.

Lemma 4.2 [13, Theorem 12] For any $\lambda, \mu \vdash n$,

$$\text{height}(\chi^\lambda \chi^\mu) \leq \ell(\lambda) \cdot \ell(\mu).$$

Theorem 4.3 If $V_k^\lambda \otimes V_m^\mu$ appears as a factor in $M^\otimes n(t)$ (for some $0 \leq t \leq n - 1$) then

$$d(\lambda, \mu) < km.$$  

Proof.  

$$d(\lambda, \mu) \leq t \leq \text{height}(\chi^\lambda \chi^\mu) - 1 \leq \ell(\lambda) \cdot \ell(\mu) - 1 \leq km - 1.$$  

Inequalities (1), (2) and (3) follow from Lemmas 4.1, 3.1 (for $\nu = (n - t, 1^t)$) and 4.2, respectively. $\square$

Let $\psi$ be an $S_n$-character (not necessarily irreducible). Define the width of $\psi$ by

$$\text{width}(\psi) := \max \{ \nu \mid \nu \vdash n, \langle \psi, \chi^\nu \rangle \neq 0 \}.$$  

The following result of Dvir strengthens Lemma 4.2.
Lemma 4.4 [1, Theorem 1.6] For any $\lambda, \mu \vdash n$,

(1) \hspace{1cm} \text{width}(\chi^\lambda \chi^\mu) = |\lambda \cap \mu|$

and

(2) \hspace{1cm} \text{height}(\chi^\lambda \chi^\mu) = |\lambda \cap \mu'|.

This result gives another way of proving Theorem 4.3.

Second Proof of Theorem 4.3.

\[ d(\lambda, \mu) = |\lambda \setminus \mu| = n - |\lambda \cap \mu| \leq t \leq \text{height}(\chi^\lambda \chi^\mu) - 1 \leq |\lambda \cap \mu'| - 1 \leq km - 1. \]

Inequality (1) follows from Lemma 4.4(1), since $n - t \leq \text{width}(\chi^\lambda \chi^\mu)$. Inequality (2) follows from Lemma 3.1. Equality (3) is Lemma 4.4(2).

Note: For any two partitions $\lambda, \mu$ of $n$ with $\ell(\lambda) \leq k$ and $\ell(\mu) \leq m$, $V_k^\lambda \otimes V_m^\mu$ appears as a factor in $M^{\otimes n}$. Theorem 4.3 shows that, in order to appear in a hook component, $\lambda$ and $\mu$ must be very “close” to each other (for $k$ and $m$ fixed, $n$ tending to infinity).

5 Square Matrices

Consider now the vector space $M_k = M_{k,k}$ of $k \times k$ matrices over $\mathbb{C}$. This space carries a diagonal (left and right) $GL_k(\mathbb{C})$-action, defined by

\[ g(m) := g \cdot m \cdot g^t \quad (\forall g \in GL_k(\mathbb{C}), \forall m \in M_k). \]

5.1 Symmetric Powers

Recall from Section 2.1 the definition of $2 \cdot \lambda$, for a partition $\lambda$.

Theorem 5.1 For $\lambda \in Par_k(2n)$, the multiplicity of $V_k^\lambda$ in $\text{Sym}^n(M_k)$ is

\[ \sum_{|\mu|+|\nu|=n} c_{2\mu,(2\cdot\nu)'}^\lambda. \]

Corollary 5.2 Let $\lambda \in Par(2n)$, $\lambda \subseteq (k^k)$ (i.e., $\lambda, \lambda' \in Par_k(2n)$). Then the multiplicities of $V_k^\lambda$ and of $V_k^{\lambda'}$ in $\text{Sym}^n(M_k)$ are equal.
Proof. This is an immediate consequence of Theorem 5.1, applying identity (2.a).

Proof of Theorem 5.1. Let $V \cong \mathbb{C}^k$. Then $V \otimes V$ carries a diagonal (left) $GL_k$-action, and

$$M_k \cong V \otimes V$$

as $GL_k$-modules. Thus

$$M_k^\otimes n \cong V^\otimes 2n$$

as $GL_k$-modules. Moreover, $V^\otimes 2n$ carries an $S_{2n} \times GL_k$-action: $S_{2n}$ permutes the $2n$ factors in the tensor product, and $GL_k$ acts on all of them simultaneously (on the left). The $S_{2n}$- and $GL_k$-actions satisfy Schur-Weyl duality (the double commutant theorem), so that

$$V^\otimes 2n \cong \bigoplus_{\lambda \in \text{Par}_k(2n)} V^\lambda_k \otimes S^\lambda,$$

as $GL_k \times S_{2n}$-modules.

Now, $\text{Sym}^n(M_k)$ is the part of $M_k^\otimes n$ which is invariant under the action of $S_n \hookrightarrow S_{2n}$, where the embedding $S_n \hookrightarrow S_n \times S_n \subseteq S_{2n}$ is diagonal: $\pi \mapsto (\pi, \pi)$. It follows that the multiplicity of $V^\lambda_k$ in $\text{Sym}^n(M_k)$ is equal to the multiplicity of the trivial character $1_{S_n}$ in the restriction $\chi^\lambda \uparrow_{S_{2n}}^{S_n}$, where $S_n$ is diagonally embedded.

By Frobenius reciprocity,

$$\langle 1_{S_n}, \chi^\lambda \uparrow_{S_{2n}}^{S_n} \rangle = \langle 1_{S_n} \uparrow_{S_n}^{B_n}, \chi^\lambda \rangle.$$

We conclude that, for $\lambda \in \text{Par}_k(2n)$, the multiplicity of $V^\lambda_k$ in $\text{Sym}^n(M_k)$ is

$$\langle 1_{S_n} \uparrow_{S_n}^{B_n}, \chi^\lambda \rangle.$$

We shall compute these multiplicities in several steps.

First, we induce in two steps:

$$1_{S_n} \uparrow_{S_{2n}}^{S_n} = (1_{S_n} \uparrow_{S_n}^{B_n}) \uparrow_{B_n}^{S_{2n}}.$$

By Lemmas 2.2(a) and 2.3,

$$(\chi^{(n)} \uparrow_{S_n}^{B_n}) \uparrow_{B_n}^{S_{2n}} = \sum_{i=0}^{n} \chi^{(i), (n-i)} \uparrow_{B_n}^{S_{2n}} =$$
\[
\sum_{i=0}^{n} (\chi(i), \emptyset \otimes \chi^{(n-i)}) \uparrow_{B_i \times B_{n-i}}^{B_n} = \\
\sum_{i=0}^{n} (\chi(i), \emptyset \otimes \chi^{(n-i)}) \uparrow_{B_i \times B_{n-i}}^{S_{2n}}.
\]

Again, let us induce in two steps:
\[
(\chi(i), \emptyset \otimes \chi^{(n-i)}) \uparrow_{B_i \times B_{n-i}}^{S_{2n}} = (\chi(i), \emptyset \otimes \chi^{(n-i)}) \uparrow_{B_i \times B_{n-i}}^{S_{2i} \times S_{2n-2i}}.
\]

By Lemma 2.1 (a)-(b), the right hand side is equal to
\[
(\sum_{\mu \vdash i} \chi^{2\mu} \otimes \chi^{(2\nu)'}) \uparrow_{S_{2i} \times S_{2n-2i}}^{S_{2n} \times S_{2n-2i}}.
\]

We conclude that
\[
1_{S_n} \uparrow_{S_{2n}} = \sum_{i=0}^{n} \sum_{\mu \vdash i, \nu \vdash n-i} (\chi^{2\mu} \otimes \chi^{(2\nu)'}) \uparrow_{S_{2i} \times S_{2n-2i}}^{S_{2n} \times S_{2n-2i}}.
\]

Applying the Littlewood-Richardson rule completes the proof.

\[\boxdot\]

5.2 A Graded Refinement of Symmetric Powers

The space \(M_k^{\otimes n}\) carries not only an \(S_n\)-action but also a \(B_n\)-action, where the signed permutation \((i, -i)\) \((1 \leq i \leq n)\) acts by transposing the \(i\)-th factor in the tensor product of \(n\) square matrices. \(M_k = M_k^+ \oplus M_k^-\), where \(M_k^+ (M_k^-)\) is the vector space of symmetric (skew symmetric) matrices of order \(k \times k\). Consequently, \(M_k^{\otimes n}\) is graded by the number of skew symmetric factors. The component of \(M_k^{\otimes n}\) with \(i\) skew symmetric factors, denoted \(M_k^{\otimes n}(i)\), is invariant under the \(B_n\)-action, as well as under the diagonal two-sided \(GL_k\)-action.

**Lemma 5.3** If the irreducible \(B_n\)-character \(\chi^{\mu, \nu}\) appears in the decomposition of the \(B_n\)-action on \(M_k^{\otimes n}(i)\), then \(|\nu| = i\).

For a proof see Section 6.2.

Since the components \(M_k^{\otimes n}(i)\) are invariant under the \(S_n\)-action, the \(S_n\)-invariant subspace \(\text{Sym}^n(M_k)\) inherits the grading by the number of skew symmetric factors. Let \(\text{Sym}^n(M_k)\) denote the component with \(i\) skew symmetric factors. The following theorem refines Theorem 5.1.
Theorem 5.4  For \( \lambda \in \text{Par}_k(2n) \), the multiplicity of \( V^\lambda_k \) in \( \text{Sym}^n_{i}(M_k) \) is
\[
\sum_{\mu \vdash n-i, \nu \vdash i} c^\lambda_{2 \mu,(2 \nu)'}.
\]

Note: Theorem 5.4 interpolates between two classical results, Theorem 2.6 and Theorem 2.7. Indeed, \( \text{Sym}^n_0(M_k) = \text{Sym}^n(M_k^+) \) is the \( n \)-th symmetric power of the vector space of symmetric matrices. In this case \( i = 0 \), so \( \nu = \emptyset \). Hence
\[
\sum_{\mu \vdash n} c^\lambda_{2 \mu,\emptyset} = \begin{cases} 1, & \text{if } \lambda = 2 \cdot \mu \text{ for some } \mu \vdash n; \\ 0, & \text{otherwise.} \end{cases}
\]
This gives Theorem 2.6. Similarly, \( \text{Sym}^n_n(M_k) = \text{Sym}^n(M_k^-) \). In this case \( i = n \), \( \mu = \emptyset \), and a similar computation gives Theorem 2.7.

An analogue of Corollary 3.4 follows.

Corollary 5.5  Let \( \lambda, \lambda' \in \text{Par}_k(2n) \) be conjugate partitions. Then, for every \( 0 \leq i \leq n \), the multiplicity of \( V^\lambda_k \) in \( \text{Sym}^n_{i}(M_k) \) is equal to the multiplicity of \( V^{\lambda'}_k \) in \( \text{Sym}^n_{n-i}(M_k) \).

Proof. Combine Theorem 5.4 with identity (2.a).

Proof of Theorem 5.4. This is a refinement of the proof of Theorem 5.1. In this refinement the group \( B_n \) appears in an essential way, whereas in the proof of Theorem 5.1 it was used only as a technical tool.

\( M^\otimes_n \) is a \( B_n \)-module, and \( \text{Sym}^n(M_k) \) is its submodule, for which the \( B_n \)-action, when restricted to \( S_n \), is trivial. Hence, if the irreducible \( B_n \)-character \( \chi^{\mu,\nu} \) appears in \( \text{Sym}^n(M_k) \), then
\[
\langle \chi^{\mu,\nu} \uparrow_{S_n}^{B_n}, 1_{S_n} \rangle \neq 0.
\]

By Lemma 2.2(a),
\[
\langle \chi^{\mu,\nu} \uparrow_{S_n}^{B_n}, 1_{S_n} \rangle = \langle \chi^{\mu,\nu}, 1_{S_n} \uparrow_{S_n}^{B_n} \rangle = \langle \chi^{\mu,\nu}, \sum_{j=0}^{n} \chi^{(n-j),(j)} \rangle,
\]
and this is nonzero (and equal to 1) if and only if \( \mu = (n-j) \) and \( \nu = (j) \) for some \( 0 \leq j \leq n \).

Combining this with Lemma 5.3 we conclude that \( \chi^{(n-i),(i)} \) is the unique irreducible \( B_n \)-character in \( \text{Sym}^n_i(M_k) \).
Now, as in the proof of Theorem 5.1, the multiplicity of \( V^\lambda_k \) in \( \text{Sym}_i^n(M_k) \) is
\[
\langle \chi^\lambda \uparrow_{S^2_n} B_n, \chi^{(n-i),(i)} \rangle = \langle \chi^\lambda, \chi^{(n-i),(i)} \uparrow_{S^2_n} B_n \rangle.
\]
By Lemmas 2.3 and 2.1(a)-(b),
\[
\chi^{(n-i),(i)} \uparrow_{S^2_n} B_n = (\chi^{(n-i),0} \otimes \chi^0,(i)) \uparrow_{S^2_n} B_{n-i} \times B_i = \left( \sum_{\mu \vdash n-i} \chi^{2\mu} \otimes \sum_{\nu \vdash i} \chi^{(2\nu)'}, \uparrow_{S^2_n} S^2_n - 2i \times S^2_i \right).
\]
The Littlewood-Richardson rule completes the proof of Theorem 5.4.

\[\square\]

### 5.3 Hook Components of Tensor Powers

In this subsection we generalize the results of the previous sections to obtain a bivariate interpolation between symmetric and exterior powers of symmetric and skew symmetric matrices.

As before, the \( n \)-th tensor power \( M_k \otimes^n \) carries an \( S_n \)-action. The symmetric power \( \text{Sym}^n(M_k) \) is the \( S_n \)-invariant part, i.e., corresponds to the trivial character \( \chi^{(n)} \). The exterior power corresponds to the sign character \( \chi^{(1^n)} \). We shall denote the factor corresponding to the hook character \( \chi^{(n-t,1^t)} \) (\( 0 \leq t \leq n-1 \)) by \( M_k \otimes^n(t) \). Then

**Theorem 5.6** For every \( 0 \leq t \leq n-1 \) and \( \lambda \in \text{Par}_k(2n) \), the multiplicity of \( V^\lambda_k \) in \( M_k \otimes^n(t) \) is
\[
\binom{n-1}{t} \sum_{i=0}^t (-1)^{t-i} \sigma_\lambda(i) = \binom{n-1}{t} \sum_{i=t+1}^n (-1)^{i-t-1} \sigma_\lambda(i)
\]
where
\[
\sigma_\lambda(i) := \sum_{|\alpha|+|\beta|=n-i, |\gamma|+|\delta|=i} c^\lambda_{2\alpha,(2\beta)' \ast 2\gamma,(2\delta)'}
\]
and the sum runs over all partitions \( \alpha \) and \( \beta \) with total size \( n - i \), and all partitions \( \gamma \) and \( \delta \) with distinct parts and total size \( i \). The operations \( \ast \), \( \ast \) are as defined in Subsection 2.1, and the extended Littlewood-Richardson coefficients are as defined in Subsection 2.2.
Proof. Similar arguments to those in the proof of Theorem 5.1 show that the multiplicity of $V_k^\lambda$ in the hook component $M_k^{\otimes n}(t)$ is

$$\binom{n-1}{t} \langle \chi^{(n-t,1^t)} \uparrow_{S_n}^{S_{2n}} \chi^\lambda \rangle.$$

By (3.a),

$$\langle (\chi^{(n-t,1^t)} + \chi^{(n-t+1,1^{t-1})}) \uparrow_{S_n}^{S_{2n}} \chi^\lambda \rangle =$$

$$= \langle (\chi^{(n-t)} \otimes \chi^{(1^t)}) \uparrow_{S_{n-t} \times S_t}^{S_{2n}} \uparrow_{S_n}^{S_{2n}} \chi^\lambda \rangle =$$

$$= \langle (\chi^{(n-t)} \otimes \chi^{(1^t)}) \uparrow_{S_{n-t} \times S_t}^{B_{n-t} \times B_t} \uparrow_{S_n}^{S_{2n}} \chi^\lambda \rangle.$$

By Lemmas 2.2 and 2.3, for every $t$

$$\langle \chi^{(n-t)} \otimes \chi^{(1^t)} \rangle \uparrow_{S_{n-t} \times S_t}^{B_{n-t} \times B_t} = \langle \chi^{(n-t)} \rangle \uparrow_{S_{n-t} \times S_t}^{B_{n-t} \times B_t} \langle \chi^{(1^t)} \rangle \uparrow_{S_{n-t} \times S_t}^{B_{n-t} \times B_t} =$$

$$= \sum_{i=0}^{n-t} \chi^{(i),(n-t-i)} \otimes \sum_{j=0}^{t} \chi^{(1^t)} \uparrow_{B_{n-t} \times B_t}^{B_{n-t} \times B_t}.$$

Hence

$$\langle \chi^{(n-t)} \otimes \chi^{(1^t)} \rangle \uparrow_{S_{n-t} \times S_t}^{S_{2n}} =$$

$$\sum_{i=0}^{n-t} \chi^{(i),(n-t-i)} \otimes \chi^{(1^t)} \uparrow_{B_{n-t} \times S_{n-i} \times S_t}^{B_{n-t} \times B_{n-t-i} \times B_t \times S_{n-i} \times S_t} \otimes \sum_{j=0}^{t} \chi^{(1^t)} \uparrow_{B_{n-t} \times B_t}^{B_{n-t} \times B_t}.$$

Let $M_k^{\otimes n}(t, j)$ be the component of $M_k^{\otimes n}(t)$ with $j$ skew symmetric factors. The following result is a common refinement of Theorems 5.4 and 5.6.

**Theorem 5.7** For every $0 \leq t \leq n - 1$, $0 \leq j \leq n$ and $\lambda \in Par_k(2n)$, the multiplicity of $V_k^\lambda$ in $M_k^{\otimes n}(t, j)$ is

$$\binom{n-1}{t} \sum_{i=0}^{t} (-1)^{t-i} \sigma(i, j) = \binom{n-1}{t} \sum_{i=t+1}^{n} (-1)^{t-1} \sigma(i, j).$$
where
\[ \sigma(\lambda(i,j)) := \sum_{|\alpha|+|\beta|+|\gamma|+|\delta|=n, |\gamma|+|\delta|=i, |\beta|+|\delta|=j} c_{\alpha,\beta,\gamma,\delta}^2 \]
and the sum is over all partitions \( \alpha, \beta, \gamma, \delta \) with total size \( n \) such that \( \gamma \) and \( \delta \) have distinct parts and total size \( i \), and \( \beta \) and \( \delta \) have total size \( j \).

**Proof.** Lemma 5.3, used as in the proof of Theorem 5.4, shows that the factors of \( M^\otimes(n,k) \), in the decomposition given in Theorem 5.6, are those for which \( |\beta| + |\delta| = j \).

\[ \blacksquare \]

**Corollary 5.8** Let \( \lambda \subseteq (k^k) \) be a partition of \( 2n \). For every \( 0 \leq t \leq n-1 \) and \( 0 \leq j \leq n \), the multiplicity of \( V^\lambda_k \) in \( M^\otimes(t,j) \) is equal to the multiplicity of \( V^{\lambda'}_k \) in \( M^\otimes(t,n-j) \).

**Proof.** By Theorem 5.7, it suffices to show that
\[ \left( \frac{n-1}{t} \right) \sum_{i=0}^{t} (-1)^{t-i} \sigma(\lambda(i,j)) = \left( \frac{n-1}{t} \right) \sum_{i=0}^{t} (-1)^{t-i} \sigma(\lambda'(i,n-j)). \]

This follows from
\[ \sigma(\lambda(i,j)) = \sigma(\lambda'(i,n-j)), \]
which in turn follows from (2.a).

\[ \blacksquare \]

6 Appendices.

6.1 Proof of Lemma 2.2

Lemma 2.2 follows from a more general result.

For partitions \( \lambda = (\lambda_1, \ldots, \lambda_k) \) and \( \mu = (\mu_1, \ldots, \mu_m) \), let \( \lambda \oplus \mu \) be the skew shape defined by
\[ \lambda \oplus \mu := (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots, \lambda_k + \mu_k, \mu_1, \mu_2, \ldots, \mu_m) / (\mu_1^k). \]

**Theorem 6.1** If \( (\lambda, \mu) \) is a bipartition of \( n \) then the restriction
\[ \chi^{\lambda \mu} \downarrow_{B_n}^{S_n} = \chi^{\lambda \oplus \mu}. \]
Proof. The characters $\chi^{\lambda \oplus \mu}$ and $\chi^{\lambda \mu}$, evaluated at elements of $S_n$, have the same recursive formula (Murnaghan-Nakayama rule). For $\chi^{\lambda \mu}$ see [17, Theorem 4.3]. For $\chi^{\lambda \oplus \mu}$ see [9, Theorem 5.6.16].

Proof of Lemma 2.2.
(a) Let $(\lambda, \mu)$ be a bipartition of $n$. By Frobenius reciprocity and Theorem 6.1,
\[ \langle \chi^{(n)} \uparrow B_n, \chi^{\lambda \mu} \downarrow B_n \rangle = \langle \chi^{(n)} \downarrow B_n, \chi^{\lambda \mu} \uparrow B_n \rangle = \langle \chi^{(n)}, \chi^{\lambda \oplus \mu} \rangle = \begin{cases} 1, & \max \{ \ell(\lambda), \ell(\mu) \} \leq 1; \\ 0, & \text{otherwise.} \end{cases} \]
The last equality follows from the Littlewood-Richardson rule, reformulated for skew shapes [15, (7.64)]. By this rule, $\langle \chi^{(n)}, \chi^{\lambda \oplus \mu} \rangle$ is nonzero (and equal to 1) if and only if $\lambda \oplus \mu$ is a horizontal strip (i.e., each column contains at most one box).

(b) The proof for $\chi^{(1^k)}$ is similar.

6.2 Proof of Lemma 5.3

Proof. Let $\sigma_i := (i, -i) \in B_n$ (1 ≤ $i$ ≤ $n$), and let $\eta$ be the sum $\sum_{i=1}^{n} \sigma_i \in C[B_n]$. Consider the tensor product $w = w_1 \otimes w_2 \otimes \cdots \otimes w_n \in M^{\otimes n}_k$, where each $w_i$ is either a symmetric or a skew symmetric matrix. Then according to the $B_n$-action, defined in Section 5.2,
\[ \sigma_i(w) = \begin{cases} w, & \text{if } w_i \text{ is symmetric;} \\ -w, & \text{if } w_i \text{ is skew symmetric.} \end{cases} \]
Hence, for every vector $v \in M^{\otimes n}_k(i)$
\[(6.a) \quad \eta(v) = (n - 2i)v. \]

On the other hand, the set $\{ \sigma_i \mid 1 \leq i \leq n \}$ is a conjugacy class in $B_n$. Thus the element $\eta = \sum_{i=1}^{n} \sigma_i$ is in the center of $C[B_n]$. By Schur’s Lemma, for every vector $v$ in the irreducible $B_n$-module $S^{\mu, \nu}$
\[ \eta(v) = c^{\mu, \nu} \cdot v, \]
where
\[ c^{\mu, \nu} = \frac{\chi^{\mu, \nu}(\eta)}{\chi^{\mu, \nu}(id)} = \frac{n\chi^{\mu, \nu}(\sigma_1)}{\chi^{\mu, \nu}(id)}. \]
Let $f^\lambda, f^{\mu,\nu}$ be the number of standard Young tableaux (bitableaux) of shapes $\lambda, (\mu, \nu)$ respectively. Recall that
\[ \chi^{\mu,\nu}(id) = f^{\mu,\nu} = \binom{n}{|\nu|} f^\mu f^\nu, \]
and that $\chi^{\mu,\nu}(\sigma_1)$ is equal to the number of standard Young bitableaux of shape $(\mu, \nu)$, in which the digit 1 is in the first diagram $\mu$, minus the number of those in which 1 is in the second diagram $\nu$. Thus
\[ \chi^{\mu,\nu}(\sigma_1) = \left( \frac{n-1}{|\nu|} \right) f^\mu f^\nu - \left( \frac{n-1}{|\nu| - 1} \right) f^\mu f^\nu = \frac{n - 2|\nu|}{n} \left( \binom{n}{|\nu|} \right) f^\mu f^\nu. \]

It follows that
\[ c^{\mu,\nu} = \frac{n\chi^{\mu,\nu}(\sigma_1)}{\chi^{\mu,\nu}(id)} = n - 2|\nu|, \]
and therefore
\[ (6.b) \quad \eta(v) = (n - 2|\nu|)v \quad (\forall v \in S^{\mu,\nu}). \]

Combining (6.a) with (6.b) completes the proof.

\[ \square \]

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References

[1] Y. Dvir, On the Kronecker product of $S_n$ characters. J. Algebra 154 (1993), 125–140.

[2] C. Ehresmann, Sur la topologie de certains espaces homogènes. Ann. of Math. 35 (1934), 396–443.

[3] R. Goodman and N. R. Wallach, Representations and Invariants of the Classical Groups. Encyclopedia of Math. and its Appl. Vol. 68, Cambridge University Press, 1998.

[4] S. Helgason, A duality for symmetric spaces with applications. Adv. Math. 5 (1970), 1–54.
[5] R. Howe, Remarks on Classical Invariant Theory. Trans. Amer. Math. Soc. 313 (1989), 539–570.

[6] R. Howe and T. Umeda, The Capelli identity, the double commutant theorem, and multiplicity-free actions. Math. Ann. 290 (1991), 565–619.

[7] A. T. James, Zonal polynomials of the real positive definite matrices. Annals of Math. 74 (1961), 456–469.

[8] G. D. James and A. Kerber, The Representation Theory of the Symmetric Group. Encyclopedia of Math. and its Appl. Vol. 16, Addison-Wesley, 1981.

[9] A. Kerber, Algebraic combinatorics via finite group actions. Bibliographisches Institut, Mannheim, 1991.

[10] B. Kostant, Lie algebra cohomology and the generalized Borel Weil theorem. Ann. of Math. 74 (1961), 329–387.

[11] J. P. Serre, Linear Representations of Finite Groups. Springer-Verlag, 1977.

[12] I. G. Macdonald, Symmetric Functions and Hall Polynomials. second edition, Oxford Math. Monographs, Oxford Univ. Press, Oxford, 1995.

[13] A. Regev, The Kronecker product of \(S_n\) characters and an \(A \otimes B\) theorem for Capelli identities. J. Algebra 66 (1980), 505–510.

[14] G. Shimura, On differential operators attached to certain representations of classical groups. Invent. Math. 77 (1984), 463–488.

[15] R. P. Stanley, Enumerative Combinatorics, Volume II. Cambridge Univ. Press, Cambridge, 1999.

[16] J. R. Stembridge, On Schur Q-functions and the primitive idempotents of commutative Hecke algebra. J. Alg. Combin. 1 (1992), 71–95.

[17] J. Stembridge, On the eigenvalues of representations of reflection groups and wreath products. Pacific J. Math. 140 (1989), 359–396.

[18] R. Thrall, On symmetrized Kronecker powers and the structure of the free Lie ring. Amer. J. Math. 64 (1942), 371–388.