A JUDGEMENT ON SINORS

A CHAMBLIN

&

G W GIBBONS

D.A.M.T.P.

University of Cambridge

Silver Street

Cambridge CB3 9EW

U.K.

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ABSTRACT

This note contains some comments on a recent paper by Friedman [1] on two-component spinors in spacetimes which do not admit a time-orientation, and is intended to clarify the relation of the work reported in that paper to previous literature.
Notation

In [1] Friedman introduces two two-fold covers of the group $L_+$ of space-orientation preserving Lorentz transformations, the so-called ‘ortho-chirous Lorentz group ’. In fact there are eight covers of the full Lorentz group, called by Dabrowski [2] $\text{Pin}^{a,b,c}$. If $\mathcal{P}$ and $\mathcal{T}$ cover $P$ and $T$ respectively then $a, b, c$, may be determined (in four spacetime dimensions and using his conventions) from the relations:

$$\mathcal{P}^2 = -a$$  \hfill (1)  \\
$$\mathcal{T}^2 = b$$  \hfill (2)  \\
and

$$\mathcal{PT} = abc\mathcal{T}\mathcal{P}.$$  \hfill (3)

The Cliffordian groups

$$\text{Pin}^{+-+-} \subset \text{Cliff}(3, 1; \mathbb{R}) \equiv \mathbb{R}(4)$$  \hfill (4)

and

$$\text{Pin}^{-++} \subset \text{Cliff}(1, 3; \mathbb{R}) \equiv \mathbb{H}(2)$$  \hfill (5)

associated with signature $+-+-$ and $-+++$ respectively are obtained by representing reflections by Clifford multiplication, i.e. one has $\mathcal{P} = \gamma_1\gamma_2\gamma_3$ and $\mathcal{T} = \gamma_0$. The Cliffordian groups act on spinors by Clifford multiplication. We shall call this the Cliffordian action of $\text{Pin}^{+-+-}$ or $\text{Pin}^{-++}$ on spinors respectively. In the Cliffordian case, the discrete groups generated by $\mathcal{P}$ and $\mathcal{T}$ are subgroups of the discrete groups Dirac(3, 1) or Dirac(1, 3) which double cover the groups generated by all possible reflections in four orthogonal directions.

We note en passant that the reason for the minus sign in (1) is that $a$ was originally defined (in all spacetime dimensions) by the sign of the cover of a single spatial reflection. A consequence of (3) is

$$(\mathcal{PT})^2 = -c.$$  \hfill (6)
Thus Friedman’s Sinor groups, since he is not concerned with the action of space-reflections, are the images of four 2-1 forgetful homomorphisms:

\[ \text{Pin}^{\pm\pm} \rightarrow \text{Sin}^+ \]  \hspace{1cm} (7)

and

\[ \text{Pin}^{\pm-\pm} \rightarrow \text{Sin}^- . \]  \hspace{1cm} (8)

We propose calling quantities transforming under the action of Sin\(^\pm\) ‘sinors’.

**Weyl Sinors**

It is well-known that one cannot represent time-reversal in a complex-linear fashion on two-component Weyl spinors. Although one may retain the complex notation one is in effect working with a real four dimensional vector bundle whose structural group is Sin\(^\pm\). Anti-linear actions of both \(T\) and \(P\) on two component spinors has been considered previously by Staruskiewicz [3]. From the point of view of the Penrose sphere construction, i.e. thinking projectively, The actions of both of \(T\) and \(P\) corresponds to the anti-podal map on the the two-sphere \(\equiv \mathbb{C}\mathbb{P}^1\). The possible spinors differ in how the action of \(P\) and \(T\) is lifted to \(\mathbb{C}^2\).

Staruskiewicz considers what he calls spinors of two ‘kinds’. From his table III it follows that for both kinds \(T^2 = -1\). Thus as far as time-reversal is concerned it is Friedman’s Sin\(^-\) which is involved. Staruskiewicz takes for the action of \(P\) either \(iT\) for the first kind or \(T\) for the third kind. It follows that \((PT)^2 = +1\) for the first kind while \((PT)^2 = -1\) for the third kind. In Dabrowski’s notation the first kind corresponds to Pin\(^{---}\) and the third kind to Pin\(^{+-+}\). Thus the group involved for spinors of the third kind is Cliffordian (with respect to signature \(++++\)). Note that what he calls space-reflections, i.e. \(P\) might well, from a physical point of view, be called \(\mathcal{C}P\), where \(\mathcal{C}\) is charge conjugation if one were thinking of its action on the solutions of the two-component Weyl neutrino equations, rather than just a spinor at one point in spacetime.
Majorana Spinors

In order to relate the Staruskiewicz’ and Friedman’s Weyl spinor formalism to
the Clifford algebra approach, we consider, as an alternative to using complex
two component spinors, the use of four real component Majorana spinors. Since
Cliff(3, 1; \mathbb{R}) \equiv \mathbb{R}(4), the algebra of four by four real matrices, this is most con-
veniently done using signature \(+ + + -\). In other words we adopt \(\gamma_0^2 = -1\) and
\(\gamma_i^2 = 1\). The relationship to the two-component formalism is that the Cliffordian
cover of total reflection \(PT\), i.e. \(\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_0\) acting on \(\mathbb{R}^4_{\text{Majorana}} \equiv \mathbb{C}^2_{\text{Weyl}}\) serves
as a complex structure [4,5]. Projectively speaking Majorana spinors correspond to
points in \(\mathbb{R}P^3\) and the Dirac group \(\text{Dirac}(3, 1)/\pm 1\) acts as the symmetry group of
the Kummer configuration [5].

The Cliffordian choice \(\text{Pin}^{+-+}\) is:

\[ P = \gamma_1\gamma_2\gamma_3 \] (9)

and

\[ T = \gamma_0 \] (10)

Since \(\gamma_5\) anti-commutes with \(\gamma_1\gamma_2\gamma_3\) and \(\gamma_0\) both \(P\) and \(T\) are anti-linear considered
as acting on \(\mathbb{C}^2_{\text{Weyl}}\) and thus we see the Cliffordian action corresponds precisely to
Staruskiewicz’s spinors of the third kind. His spinors of the first kind correspond
to the choice

\[ P = \gamma_0 \] (11)

and

\[ T = \gamma_0 \] (12)

which gives an action of the non-Cliffordian group \(\text{Pin}^{+-+}\).

This relationship between real Majorana spinors and complex Weyl spinors has
an analogue for Maxwell’s equations. One may think of the Hodge star operator,
which acting on 2-forms on an orientable spacetime of signature \(+ + + -\) or \(- - - +\)
satisfies $\star^2 = -1$, as a complex structure on the space of Maxwell fields. This is behind the well-known complex notation in which one writes Maxwell’s equations in the first-order form as:

$$\nabla \times (E + iB) = i \frac{\partial}{\partial t} (E + iB)$$

or in a more covariant notation

$$(d + \delta)F = 0.$$  

where $\delta$ is the adjoint of $d$.

Physically speaking the space of complex self-dual Maxwell fields describes photons of a particular helicity, for example circularly polarized photons with a right-handed polarization. On a non-orientable spacetime this reformulation of Maxwell’s equations is not possible. In other words on a non-orientable spacetime there is an obstruction to introducing this particular complex structure on the space of Maxwell fields. The physical interpretation is that on such spacetime it is not possible to speak of a right or left-handed photon. For some other speculations concerning the non-orientable case see [8].
Cliffordian versus non-Cliffordian

Evidently there are many possibilities for representing $P$ and $T$. The fact that some of the groups are non-Cliffordian or if they are Cliffordian they have non-Cliffordian actions is not in itself of great physical significance. Consider the classical Dirac equation in flat space with a mass:

$$\gamma^\mu \partial_\mu \psi + m\psi = 0. \quad (13)$$

This is invariant under time reversal in the sense that:

$$\psi(t, x) \rightarrow \gamma_1 \gamma_2 \gamma_3 \psi(-t, x) \quad (14)$$

takes solutions to solutions, but $T$ is not represented at even one point of spacetime by $\gamma_0$, rather since

$$\psi(t, x) \rightarrow \gamma_0 \psi(t, -x), \quad (15)$$

also takes solutions to solutions the group is the Cliffordian group $\text{Pin}^{+-+}$ but its action is not Cliffordian.

On the other hand, the standard actions of $P$ and $T$ on the full second quantized Hilbert space $\mathcal{H}$ in flat spacetime quantum field theory is of a non-Cliffordian group. Conventionally one seeks actions which take positive energy states to positive energy states. To that end one follows Wigner and chooses $P$ to act unitarily and scales it such that $P^2 = \pm 1$. By contrast, again following Wigner, one chooses $T$ to act anti-unitarily (on fermion states) and finds that $T^2 = -1$. Since in addition $P$ and $T$ anti-commute the relevant groups are $\text{Pin}^{\pm 1-1-1}$. These include the group associated to Staruskieiwicz’s spinors of the first kind, though of course it acts on a different space. In fact in flat space quantum field theory the Hilbert space $\mathcal{H}$ is constructed from the space of solutions of the Dirac equation and these carry other possible actions of $P$, $T$ and $PT$, both linear and anti-linear, in addition to the action of ‘charge conjugation’ $C$. Thus charge conjugation acts by ‘complex conjugation’ on the classical solutions but acting on $\mathcal{H}$ it acts unitarily, taking particle
to anti-particle states. The resolution of this apparent paradox is that the complex structure on $H$ should be distinguished from that used on the space of classical solutions [5].

**Obstructions to Sinors**

In flat spacetime the distinctions we have drawn above between the various Pin groups and their actions might seem to have little physical consequence. However spacetime is not flat and may well not be space and time orientable. The distinctions then become vital because the obstructions to the global existence of a given Pinor bundle depends upon precisely what Pin group we are considering.

Karoubi has shown [9] that the obstruction to a *Cliffordan* $\text{Sin}^\pm$ structure is

$$w_2(M) + w_1^-(M, g_L) \sim w_1^-(M, g_L).$$

where $w_2(M)$ is the second Stiefel-Whitney class of $TM$ the tangent bundle of $M$ and $w_1^-(M, g_L)$ is the first Stiefel-Whitney class of the bundle associated to the negative sign in the metric. Note that $w_2(M)$ is a topological invariant independent of the existence of any metric on $M$.

Thus for signature $+++-$, i.e. for Pin$^{+-+}$ $\sim$ Sin$^-$ we have

$$w_1^-(M, g_L) = w_1^T(M, g_L)$$

where $w_1^T(M, g_L)$ is the element of $H^1(M; \mathbb{Z}_2)$ giving the obstruction to time-orientability. On the other hand for signature $--++$, i.e for Pin$^{++-}$ $\sim$ Sin$^+$

$$w_1^-(M, g_L) = w_1^S(M, g_L)$$

where $w_1^S(M, g_L)$ is the obstruction to space-orientability. Note that both $w_1^T(M, g_L)$ and $w_1^S(M, g_L)$ depend on the existence of a Lorentzian metric $g_L$ but their sum is a topological invariant:

$$w_1^S(M, g_L) + w_1^T(M, g_L) = w_1(M)$$
where \( w_1(M) \) is the first Stiefel-Whitney class of the tangent bundle \( TM \) and is the obstruction to time orientability. If \( w_2^\pm \) are the second Stiefel-Whitney classes of the indicated bundles then

\[
 w_2(M) = w_2^+(M, g_L) + w_2^-(M, g_L) + w_1^+(M, g_L) \sim w_1^-(M, g_L).
\]

However the second Stiefel-Whitney class of a one-dimensional bundle vanishes and thus

\[
 w_2(M) = w_2^S(M, g_L) + w_1^T(M, g_L) \sim w_1^S(M, g_L).
\]

Thus the \( \text{Pin}^{++-} \sim \text{Sin}^- \) the obstruction is

\[
 w_2(M) + w_1^T(M, g_L) \sim w_1^T(M, g_L),
\]

while for \( \text{Pin}^{--} \sim \text{Sin}^+ \) the obstruction is

\[
 w_2(M) + w_1^S(M, g_L) \sim w_1^S(M, g_L).
\]

Consider, as does Friedman, a space-orientable but time non-orientable spacetime. Then \( w_1^S(M, g_L) = 0 \). The obstruction to \( \text{Pin}^{+-} \sim \text{Sin}^- \) is therefore

\[
 w_2(M) + w_1^T(M, g_L) \sim w_1^T(M, g_L)
\]

while to \( \text{Pin}^{--} \sim \text{Sin}^+ \) it is just

\[
 w_2(M).
\]

A particularly interesting case is \( \{M, g_L\} \equiv \text{antipodally identified De-Sitter spacetime} \). Topologically we may think of \( M \) as \( \mathbb{R}P^4 - \{\text{pt.}\} \). This has the same first and second Stiefel-Whitney classes as \( \mathbb{R}P^4 \), thus \( w_2(M) = 0 \). Moreover we clearly have for this choice of metric that \( w_1^T(M, g_L) \) is given by the unique element of \( H^1(M) \). It follows immediately that antipodally identified De-Sitter spacetime admits a \( \text{Pin}^{--} \sim \text{Sin}^+ \) structure but not a \( \text{Pin}^{++} \sim \text{Sin}^- \) structure. This is in
agreement with a direct construction of the relevant bundles carried out by one of us with L Dabrowski some time ago and reported in [10] It also agrees with Friedman.

In fact the argument will generalize to manifolds of the form \( M = \tilde{\Sigma} \times \mathbb{R}/A \) where \( A = T \circ I \) where \( T \) is time-reversal on the \( \mathbb{R} \) factor, which corresponds to time, and \( I \) is an space-orientation preserving involution which acts freely on \( \tilde{\Sigma} \). We claim that \( w_2(M) = 0 \). To see this we use proposition 4 of [11] which states that if the tangent bundle of a 4-manifold admits three linearly independent non-vanishing cross-sections then \( w_k(M) = 0 \) for \( k > 1 \). To construct the requisite cross-sections we parallelize \( \Sigma/I \) and lift the parallelization to \( \Sigma \). The three cross-sections are invariant under the action of \( T \) and thus descend to \( M \). Thus, in accord with Friedman’s result, all manifolds of this type admit \( \text{Spin}^+ \), and admit \( \text{Spin}^- \) if and only if

\[
    w_1^T(M) \sim w_1^T(M) = 0.
\]

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