Hyers-Ulam stability for differential equations and partial differential equations via Gronwall Lemma

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Abstract

In this paper we will study Hyers-Ulam stability for Bernoulli differential equations, Riccati differential equations and quasilinear partial differential equations of first order, using Gronwall Lemma, following a method given by Rus.

Keywords: Hyers-Ulam stability, Hyers-Ulam -Rassias stability, Gronwall lemma

MSC: 26D10; 34A40; 39B82; 35B20

1. Introduction

In [24], [26], [27] Rus has obtained some results regarding Ulam stability of differential and integral equations, using Gronwall inequalities method and weak Picard operators technique. In [25] Rus and Lungu have studied the stability of a partial differential equation of order two of hyperbolic type using the same method. In [14] Craciun and Lungu have studied, using this method, a partial differential equation of order two having a general form. In this paper we use the same method in order to study the stability of Bernoulli and Riccati equations and also of quasilinear partial differential equations of first order. We mention that some results regarding Ulam stability of Bernoulli and Riccati differential equations was established by Jung and Rassias

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using the integrating factor method. The first result proved on the Hyers-Ulam stability of partial differential equations is due to A. Prastaro and Th.M. Rassias. Also Lungu and Popa and Marian and Lungu have obtained stability results from some partial differential linear and quasilinear equations. The Gronwall inequality is used in Quarawani in order to study Hyers-Ulam-Rassias stability for Bernoulli differential equations and it is also used in.

For a broader study of Hyers-Ulam stability for functional equations the reader is also referred to the following books and papers: 1, 2, 4, 5, 6, 7, 8, 9, 10, 12, 13, 18, 19, 20, 28.

In the following we will use Definition 2.1, 2.2, 2.3 from p.126 and Remark 2.1, 2.2.

2. Main results

2.1. Stability of Bernoulli differential equation

Let \((B, |·|)\) be a (real or complex) Banach space, \(a, b \in \mathbb{R}, a < b, p, q \in C ([a, b], B)\) and \(n \in \mathbb{R} \setminus \{0, 1\}\).

We consider the Bernoulli differential equation

\[
z' (x) = p (x) z (x) + q (x) z^n (x), x \in [a, b],
\]

and the inequation

\[
|y' (x) - p (x) y (x) - q (x) y^n (x)| \leq \varepsilon, x \in [a, b].
\]

From Remark 2.1 from p.127 follows that \(y \in C^1 ([a, b], B)\) is a solution of the inequation (2) if and only if there exists a function \(g \in C^1 ([a, b], B)\) (which depend on \(y\)) such that

(i) \(|g (x)| \leq \varepsilon, \forall x \in [a, b];\)

(ii) \(y' (x) = p (x) y (x) + q (x) y^n (x) + g (x), \forall x \in [a, b].\)

From Remark 2.2 from p.127 follows that if \(y \in C^1 ([a, b], B)\) is a solution of the inequation (2), then \(y\) is a solution of the following integral inequation

\[
\left|y (x) - y (a) - \int_a^x [p (t) y (t) + q (t) y^n (t)] dt\right| \leq (x - a) \varepsilon, \forall x \in [a, b].
\]
Theorem 2.1. If

(i) \( a < \infty, b < \infty \);

(ii) \( p, q \in C ([a, b], \mathbb{R}) \);

(iii) there exists \( L > 0 \) such that

\[
|q (x) y''(x) - q (x) z''(x)| \leq L |y(x) - z(x)| ,
\]

for all \( x \in [a, b] \) and \( y, z \in C^1 ([a, b], \mathbb{R}) \),

then the equation (1) is Hyers-Ulam stable.

Proof. Let \( y \in C^1 ([a, b], \mathbb{R}) \) be a solution of the inequation (2) and \( z \) the unique solution of the Cauchy problem

\[
\begin{align*}
\begin{cases}
  z'(x) = p (x) z(x) + q (x) z''(x), & x \in [a, b], \\
  z(a) = y(a)
\end{cases}
\end{align*}
\]

(3)

We have that

\[
z(x) = y(a) + \int_a^x [p (t) z (t) + q (t) z''(t)] \, dt, \quad x \in [a, b].
\]

Let

\[
M = \max_{x \in [a, b]} |p (x)|.
\]

We consider the difference

\[
|y(x) - z(x)| \leq \left| y(x) - y(a) - \int_a^x [p (t) y(t) + q (t) y''(t)] \, dt \right| +
\]

\[
\left| \int_a^x [p (t) y(t) + q (t) y''(t) - p (t) z(t) - q (t) z''(t)] \, dt \right| \leq
\]

\[
\leq \varepsilon (x - a) + \int_a^x \left[ |p (t) y(t) - p (t) z(t)| + |q (t) y''(t) - q (t) z''(t)| \right] \, dt \leq
\]

\[
\leq \varepsilon (x - a) + \int_a^x \left[ |p (t)| |y(t) - z(t)| + L |y(t) - z(t)| \right] \, dt =
\]

\[
= \varepsilon (x - a) + \int_a^x [ |p (t)| + L ] |y(t) - z(t)| \, dt
\]
From Gronwall lemma (see [11], p. 6) we have that

\[
|y(x) - z(x)| \leq \varepsilon (x - a) e^{\int_a^x [p(t) + M] \, dt} \leq \varepsilon (b - a) e^{\int_a^b (M+L) \, dt} = \\
\varepsilon (b - a) e^{(M+L)(b-a)} = c \cdot \varepsilon,
\]

where \( c = (b - a) e^{(M+L)(b-a)} \).

**Example 2.2.** We consider the Bernoulli differential equation

\[
z' = xz + \frac{x}{1 + x^2} \sqrt{z},
\]

where \( x \in [a, b] \) and \( z \geq 1 \). We have \( p(x) = x \) and \( q(x) = \frac{x}{1+x} \). Let \( D = \{(x, z) \mid x \in [a, b], z \geq 1\} \) and \( f(x, z) = \frac{x}{1+x^2} \sqrt{z} \). We have

\[
\left| \frac{\partial f}{\partial z} \right| = \left| \frac{x}{1+x^2} \cdot \frac{1}{2 \sqrt{z}} \right| \leq \frac{1}{2} \left| \frac{x}{1+x^2} \right| \leq \frac{1}{4}, \forall (x, z) \in D,
\]

hence the function \( f \) satisfies a Lipschitz condition in the variable \( z \), on \( D \), with Lipschitz constant \( 1/4 \). Hence

\[
|f(x, y) - f(x, z)| \leq L |y - z| = \frac{1}{4} |y - z|,
\]

that is

\[
\left| \frac{x}{1+x^2} \sqrt{y} - \frac{x}{1+x^2} \sqrt{z} \right| \leq \frac{1}{4} |y - z|, x \in [a, b], y, z \geq 1.
\]

We apply Theorem 2.1 so the equation (4) is Hyers-Ulam stable. Let \( y \in C^1([a, b], \mathbb{B}) \) be a solution of the inequation

\[
\left| z' - xz - \frac{x}{1+x^2} \sqrt{z} \right| \leq \varepsilon,
\]

and \( z \) the unique solution of the Cauchy problem

\[
\begin{cases}
z' = xz + \frac{x}{1+x^2} \sqrt{z}, \\
z(a) = y(a)
\end{cases}
\]

We have

\[
z(x) = y(a) - \int_a^x \left[ tz + \frac{t}{1+t^2} \sqrt{z} \right] \, dt, \, x \in [a, b].
\]

Let

\[
M = \max_{x \in [a, b]} |p(x)| = |b|.
\]
We have
\[ |y(x) - z(x)| \leq \varepsilon (b - a) e^{(|b|+\frac{1}{4})(b-a)}. \]

2.2. Stability of Riccati differential equation

Let \((\mathbb{B}, |\cdot|)\) be a (real or complex) Banach space, \(a, b \in \mathbb{R}, a < b\) and \(p, q, r \in C ([a, b], \mathbb{B})\).

We consider the Riccati differential equation
\[ z'(x) = p(x) z^2(x) + q(x) z(x) + r(x), \quad x \in [a, b], \quad (7) \]
and the inequation
\[ |y'(x) - p(x) y^2(x) - q(x) y(x) - r(x)| \leq \varepsilon, \quad x \in [a, b]. \quad (8) \]

From Remark 2.1 from [24], p.127 follows that \(y \in C^1 ([a, b], \mathbb{B})\) is a solution of the inequation (8) if and only if there exists a function \(g \in C^1 ([a, b], \mathbb{B})\) (which depend on \(y\)) such that

(i) \(|g(x)| \leq \varepsilon, \forall x \in [a, b];\)

(ii) \(y'(x) = p(x) y^2(x) + q(x) y(x) + r(x) + g(x), \forall x \in [a, b].\)

From Remark 2.2 from [24], p.127 follows that if \(y \in C^1 ([a, b], \mathbb{B})\) is a solution of the inequation (8), then \(y\) is a solution of the following integral inequation
\[ \left| y(x) - y(a) - \int_a^x \left[ p(t) y^2(t) + q(t) y(t) + r(t) \right] dt \right| \leq (x - a) \varepsilon, \forall x \in [a, b]. \]

**Theorem 2.3.** If

(i) \(a < \infty, b < \infty;\)

(ii) \(p, q, r \in C ([a, b], \mathbb{B});\)

(iii) there exists \(L > 0\) such that
\[ |p(t) y^2(x) - p(t) z^2(x)| \leq L |y(x) - z(x)|, \]
for all \(x \in [a, b]\) and \(y, z \in C^1 ([a, b], \mathbb{B}),\)
then the equation (7) is Hyers-Ulam stable.

**Proof.** Let \( y \in C^1 ([a, b], \mathbb{B}) \) be a solution of the inequation (8) and \( z \) the unique solution of the Cauchy problem

\[
\begin{cases}
z'(x) = p(x) z^2(x) + q(x) z(x) + r(x), x \in [a, b], \\
z(a) = y(a)
\end{cases}
\tag{9}
\]

We have that

\[
z(x) = y(a) + \int_a^x \left[ p(t) y^2(t) + q(t) z(t) + r(x) \right] dt, \forall x \in [a, b].
\]

Let

\[
M = \max_{x \in [a, b]} |q(x)|.
\]

We consider the difference

\[
\begin{align*}
|y(x) - z(x)| & \leq |y(x) - y(a) - \int_a^x \left[ p(t) y^2(t) + q(t) y(t) + r(x) \right] dt| + \\
& \left| \int_a^x \left[ p(t) y^2(t) + q(t) y(t) - p(t) z^2(t) - q(t) z(t) \right] dt \right| \\
& \leq \varepsilon (x - a) + \int_a^x \left[ |p(t) y^2(t) - p(t) z^2(t)| + |q(t) y(t) - q(t) z(t)| \right] dt \\
& \leq \varepsilon (x - a) + \int_a^x \left( L |y(t) - z(t)| + |q(t)||y(t) - z(t)|| \right) dt = \\
& \leq \varepsilon (x - a) + \int_a^x [L + |q(t)|] |y(t) - z(t)| dt.
\end{align*}
\]

From Gronwall lemma (see [11], p. 6) we have that

\[
|y(x) - z(x)| \leq \varepsilon (x - a) e^{\int_a^x (L + |q(t)|) dt} \leq \varepsilon (b - a) e^{\int_a^b (L + M) dt} = \\
= \varepsilon (b - a) e^{(M + L)(b-a)} = c \cdot \varepsilon,
\]

where \( c = (b - a) e^{(M + L)(b-a)} \).

2.3. **Hyers-Ulam stability of quasilinear partial differential equation**

2.3.1. **Hyers-Ulam stability**

Let \((\mathbb{B}, |\cdot|)\) be a (real or complex) Banach space, \( a, b \in (0, \infty) \), \( \varepsilon \) a positive real number, \( \varphi \in C ([0, a) \times [0, b], \mathbb{R}_+) \) and \( p, q, r \in C ((0, a) \times [0, b] \times \mathbb{B}, \mathbb{R}) \) and \( p(x, y, u) \neq 0 \).
We consider the following quasilinear partial differential equation of first order

\[
\frac{\partial u(x, y)}{\partial x} = -\frac{q(x, y, u)}{p(x, y, u)} \frac{\partial u}{\partial y} + \frac{r(x, y, u)}{p(x, y, u)},
\]

and the following partial differential inequation

\[
\left| \frac{\partial v(x, y)}{\partial x} + \frac{q(x, y, v)}{p(x, y, v) \partial y} - \frac{r(x, y, v)}{p(x, y, v)} \right| \leq \varepsilon,
\]

Remark 2.4. A function \( v \in C ((0, a) \times (0, b), \mathbb{R}) \) is a solution of the inequation (11) if and only if there exists a function \( g \in C ((0, a) \times (0, b), \mathbb{R}) \) such that

(i) \( |g(x, y)| \leq \varepsilon, \forall (x, y) \in [0, a) \times [0, b); \)

(ii) \( \frac{\partial v(x, y)}{\partial x} = -\frac{q(x, y, v(x, y))}{p(x, y, v(x, y))} v_y(x, y) + \frac{r(x, y, v(x, y))}{p(x, y, v(x, y))} + g(x, y), \) where \( v_y = \frac{\partial v}{\partial y}. \)

Remark 2.5. If \( v \in C ((0, a) \times (0, b), B) \) is a solution of the inequation (11), then \( v \) is a solution of the following integral inequation

\[
\left| v(x, y) - v(0, y) - \int_{0}^{x} \left[ -\frac{q(s, y, v(s, y))}{p(s, y, v(s, y))} v_y(s, y) + \frac{r(s, y, v(s, y))}{p(s, y, v(s, y))} \right] ds \right| \leq \varepsilon x,
\]

\( \forall x \in [0, a), y \in [0, b). \)

Indeed, by Remark 2.4 we have that

\[
\frac{\partial v(x, y)}{\partial x} = -\frac{q(x, y, v(x, y))}{p(x, y, v(x, y))} v_y(x, y) + \frac{r(x, y, v(x, y))}{p(x, y, v(x, y))} + g(x, y),
\]

\( \forall x \in [0, a), y \in [0, b). \) This implies that

\[
v(x, y) = v(0, y) + \int_{a}^{x} \left[ -\frac{q(s, y, v(s, y))}{p(s, y, v(s, y))} v_y(s, y) + \frac{r(s, y, v(s, y))}{p(s, y, v(s, y))} + g(s, y) \right] ds.
\]

From this it follows that

\[
\left| v(x, y) - v(0, y) - \int_{0}^{x} \left[ -\frac{q(s, y, v(s, y))}{p(s, y, v(s, y))} v_y(s, y) + \frac{r(s, y, v(s, y))}{p(s, y, v(s, y))} \right] ds \right| \leq \int_{0}^{x} |g(s, y)| ds \leq \varepsilon x.
\]
Theorem 2.6. We suppose that

(i) $a < \infty, b < \infty$;

(ii) $p, q, r \in C ([0, a] \times [0, b] \times \mathbb{R}, \mathbb{R}), p \neq 0$;

(iii) there exists $l_1, l_2 > 0$ such that

$$\left| \frac{q(x, y, v_1)}{p(x, y, v_1)} v_1, (x, y) - \frac{q(x, y, v_2)}{p(x, y, v_2)} v_2, (x, y) \right| \leq l_1 |v_1 - v_2|,$$

$$\left| \frac{r(x, y, v_1)}{p(x, y, v_1)} - \frac{r(x, y, v_2)}{p(x, y, v_2)} \right| \leq l_2 |v_1 - v_2|,$$

$\forall v_1, v_2 \in \mathbb{R}, \forall (x, y) \in [0, a] \times [0, b]$.

Then:

(a) for $\psi \in C ([0, a], \mathbb{R})$ the equation (10) has a unique solution with

$$u(0, y) = \psi(y), \forall y \in [0, b];$$

(b) the equation (10) is Hyers-Ulam stable.

Proof.

(a) This is a known result (see [23]).

(b) Let $v$ be a solution of the inequation (11). Denote by $u$ the unique solution of the equation (10) which satisfies the condition

$$u(0, y) = v(0, y), \forall y \in [0, b].$$

From Remark 2.5 and condition (iii) we have that

$$|v(x, y) - u(x, y)| \leq$$

$$\leq \left| v(x, y) - v(0, y) - \int_0^x \left[ - \frac{q(s, y, v(s, y))}{p(s, y, v(s, y))} v_1(s, y) + \frac{r(s, y, v(s, y))}{p(s, y, v(s, y))} \right] ds \right| +$$

$$+ \int_0^x \left| \frac{q(s, y, v(s, y))}{p(s, y, v(s, y))} v_1(s, y) + \frac{r(s, y, v(s, y))}{p(s, y, v(s, y))} \frac{q(s, y, u(s, y))}{p(s, y, u(s, y))} u_0(s, y) - \frac{r(s, y, u(s, y))}{p(s, y, u(s, y))} \right| ds$$

$$\leq \varepsilon x + \int_0^x \left[ l_1 |v(s, y) - u(s, y)| + l_2 |v(s, y) - u(s, y)| \right] ds.$$
Or,
\[|v(x, y) - u(x, y)| \leq \varepsilon x + \int_0^x [l_1 + l_2] |v(s, y) - u(s, y)| ds.\]

From Gronwall lemma (see [11], p. 6) we have
\[|v(x, y) - u(x, y)| \leq ae^{(l_1 + l_2)} \cdot \varepsilon = c \cdot \varepsilon,
\]
where \(c = ae^{(l_1 + l_2)}\).

So, the equation (10) is Hyers-Ulam stable.

2.3.2. Hyers-Ulam-Rassias stability of equation (10)

Let us consider the equation (10) and the inequation (12) in the case \(a = \infty, b = \infty\).

Theorem 2.7. We suppose that

(i) \(p, q, r \in C ([0, a] \times [0, b] \times \mathbb{R}, \mathbb{B}), p \neq 0;\)

(ii) there exists \(l_1, l_2 \in C^1 ([0, a] \times [0, b], \mathbb{R}_+)\) such that
\[
\left| \frac{q(x, y, v_1)}{p(x, y, v_1)} v_{1y} (x, y) - \frac{q(x, y, v_2)}{p(x, y, v_2)} v_{2y} (x, y) \right| \leq l_1 (x, y) |v_1 - v_2|;
\]
\[
\left| \frac{r(x, y, v_1)}{p(x, y, v_1)} - \frac{r(x, y, v_2)}{p(x, y, v_2)} \right| \leq l_2 (x, y) |v_1 - v_2|;
\]
\(\forall v_1, v_2 \in \mathbb{B}, \forall (x, y) \in [0, a] \times [0, b] ;\)

(iii) \(e^{\int_0^x [l_1(x, y) + l_2(x, y)] ds}\) is convergent and there exists a real number \(M\) such that \(e^{\int_0^x [l_1(x, y) + l_2(x, y)] ds} \leq M, \forall y \in [0, b];\)

(iv) there exists \(\lambda_x > 0\) such that
\[
\int_0^x \varphi (s, y) ds \leq \lambda_x \cdot \varphi (x, y), \forall (x, y) \in [0, a] \times [0, b)
\]

and \(\varphi\) increasing.

Then the equation (10) \((a = \infty, b = \infty)\) is Hyers-Ulam-Rassias stable.
Proof. Let \( v \) be a solution of the inequation (12). Denote by \( u \) the unique solution of the problem

\[
\begin{align*}
\frac{\partial u(x,y)}{\partial x} &= - \frac{q(x,y,u(x,y))}{p(x,y,u(x,y))} u_y(x,y) + \frac{r(x,y,u(x,y))}{p(x,y,u(x,y))} \\
u(0,y) &= v(0,y).
\end{align*}
\]

We have

\[
u(x,y) = v(0,y) + \int_0^x \left[ - \frac{q(s,y,u(s,y))}{p(s,y,u(s,y))} u_y(s,y) + \frac{r(s,y,u(s,y))}{p(s,y,u(s,y))} \right] ds
\]

and

\[
\left| \nu(x,y) - v(0,y) - \int_0^x \left[ - \frac{q(s,y,v(s,y))}{p(s,y,v(s,y))} v_y(s,y) + \frac{r(s,y,v(s,y))}{p(s,y,v(s,y))} \right] ds \right| \leq \varepsilon \int_0^x \varphi(s,y) ds \leq \varepsilon \lambda \varphi \cdot \varphi(x,y).
\]

Then we have

\[
|\nu(x,y) - u(x,y)| \leq |\nu(x,y) - v(0,y) - \int_0^x \left[ - \frac{q(s,y,v(s,y))}{p(s,y,v(s,y))} v_y(s,y) + \frac{r(s,y,v(s,y))}{p(s,y,v(s,y))} \right] ds| + \int_0^x \left| - \frac{q(s,y,v(s,y))}{p(s,y,v(s,y))} v_y(s,y) + \frac{r(s,y,v(s,y))}{p(s,y,v(s,y))} \right| ds
\]

\[
\leq \varepsilon \lambda \varphi \cdot \varphi(x,y) + \int_0^x \left[ |l_1(s,y)| \varphi(s,y) - u(s,y)| + l_2(s,y) |v(s,y) - u(s,y)| \right] ds \leq \\
\leq \varepsilon \lambda \varphi \cdot \varphi(x,y) + \int_0^x \left[ l_1(s,y) + l_2(s,y) \right] |v(s,y) - u(s,y)| ds.
\]

From Gronwall lemma (see [11], p. 6) we have that

\[
|\nu(x,y) - u(x,y)| \leq \varepsilon \lambda \varphi \cdot \varphi(x,y) e^{\int_0^x [l_1(s,y) + l_2(s,y)] ds} \leq c \varphi \cdot \varphi \cdot \varphi(x,y),
\]

where \( c \varphi = \lambda \varphi \cdot M \).

So, the equation (10) is generalized Hyers-Ulam-Rassias stable.

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