Consistency of the Nonsymmetric Gravitational Theory

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Abstract

The NGT field equations with sources are expanded first about a flat Minkowski background and then about a GR background to first-order in the antisymmetric part of the fundamental tensor, $g_{\mu\nu}$. From the general, static spherically symmetric solution of the field equation in empty space, we establish that there are two conserved charges $m$ and $\ell^2$ corresponding to the two basic gauge invariances of NGT. There is no direct contribution to the flux of gravitational waves from the antisymmetric, $g_{[\mu\nu]}$, sector in the linearized, lowest order of approximation, nor in the non-linear theory. It is demonstrated that the flux of gravitational waves is finite in magnitude and positive definite for solutions of the field equations which satisfy the boundary condition of asymptotic flatness.
1. Introduction

An analysis of the properties of gravitational radiation in the nonsymmetric gravitational theory (NGT) is given, based on the field equations with sources. In previous work\(^1\), an analysis of the radiation problem in NGT, showed that the flux of gravitational radiation was the same as the leading order contribution in general relativity (GR). This result was shown to follow from the axisymmetric, time-dependent vacuum field equations in a first-order expansion about a curved GR background\(^2\). From an expansion of the exact, axisymmetric time-dependent field equations in inverse powers of \(r\), the same result was obtained\(^3\).

In the following, we shall analyze the problem using the NGT field equations with sources. In Sect. 2, we review the Lagrangian density and the field equations of NGT and in Sect. 3, we consider the results that follow from an expansion of the theory about Minkowski spacetime. Then in Sect. 4, we study the consequences that follow from the general, static spherically symmetric solution of the empty space field equations. We find that only two conserved charges occur in NGT, the mass \(m\) and \(\ell^2\). These charges are conserved by virtue of the two basic gauge invariances of NGT, namely, diffeomorphism invariance and a \(U(1)\) invariance of the Lagrangian density. The other source tensor \(T_{\mu\nu}\) contains only geometrical mass coupling. Indeed, when \(\ell^2 = 0\), NGT reduces to a purely geometrical theory of gravity involving only the gravitational charge \(m\). It is found that in the flat space linearized theory, there is no direct contribution to gravitational radiation in the wave-zone. Only the GR quadrupole radiation manifests itself in this approximation.

In Sect 5, it is shown that by using a spin-projection analysis, there are no propagating ghost modes. In Sect. 6, a calculation of the flux of gravitational radiation for plane waves is shown to be positive definite.
In Sect. 7, we expand the antisymmetric field equations about a classical curved GR background. We find that to first-order in the skew part of $g_{\mu\nu}$, the flux of energy is finite and positive for solutions of the field equations which satisfy the boundary condition of asymptotic flatness. As in the linear approximation, only the GR quadrupole radiation makes a contribution to the flux of gravitational waves.

An important aspect of the results confirming the physical consistency of NGT, is that at no time is there any need for a gauge invariance in the skew $g_{\mu\nu}$ sector of the theory. Indeed, such a general gauge invariance does not exist in the theory; the only invariances are general coordinate invariance and an Abelian gauge invariance, associated with the Lagrange multiplier field $W_\mu$. This work demonstrates that NGT is a consistent theory of gravity, avoiding the criticisms of Damour, Deser and McCarthy$^5$–$^7$.

2. NGT Field Equations with Sources

The Lagrangian density with sources, in NGT, is given by$^8$–$^{11}$:

$$\mathcal{L} = g^{\mu\nu} R_{\mu\nu}(W) + \mathcal{L}_M, \quad (2.1)$$

where $g^{\mu\nu} = \sqrt{-gg^{\mu\nu}}$ and $R_{\mu\nu}(W)$ is the NGT contracted curvature tensor:

$$R_{\mu\nu}(W) = W^{\beta}_{\mu,\nu,\beta} - \frac{1}{2}(W^{\beta}_{\mu,\nu} + W^{\beta}_{\nu,\mu}) - W^{\beta}_{\alpha\nu}W^{\alpha}_{\mu\beta} + W^{\beta}_{\alpha\beta}W^{\alpha}_{\mu\nu}, \quad (2.2)$$

defined in terms of the unconstrained nonsymmetric connection:

$$W^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \frac{2}{3}\delta^\lambda_\mu W_\nu, \quad (2.3)$$

where $W_\mu \equiv W^\lambda_{[\mu\lambda]} = \frac{1}{2}(W^\lambda_{\mu\lambda} - W^\lambda_{\lambda\mu})$. This equation leads to:

$$\Gamma_\mu = \Gamma^\lambda_{[\mu\lambda]} = 0. \quad (2.4)$$
The contravariant tensor $g^{\mu \nu}$ is defined in terms of the equation:

$$g^{\mu \nu}g_{\sigma \nu} = g^{\nu \mu}g_{\nu \sigma} = \delta_{\sigma}^{\mu}.$$ \hfill (2.5)

The NGT contracted curvature tensor can be written as

$$R_{\mu \nu}(W) = R_{\mu \nu}(\Gamma) + \frac{2}{3}W_{[\mu, \nu]},$$ \hfill (2.6)

where $R_{\mu \nu}(\Gamma)$ is defined by

$$R_{\mu \nu}(\Gamma) = \Gamma_{\mu \nu, \beta}^{\beta} - \frac{1}{2}\left(\Gamma_{(\mu \beta), \nu}^{\beta} + \Gamma_{(\nu \beta), \mu}^{\beta}\right) - \Gamma_{\alpha \nu}^{\beta}\Gamma_{\mu \beta}^{\alpha} + \Gamma_{(\alpha \beta)}^{\beta}\Gamma_{\mu \nu}^{\alpha}.$$ \hfill (2.7)

The Lagrangian density for the matter sources is given by ($G=c=1$):

$$\mathcal{L}_M = -8\pi g^{\mu \nu}T_{\mu \nu} + \frac{8\pi}{3}W_{\mu}S^{\mu}.$$ \hfill (2.8)

Our field equations are given by

$$G_{\mu \nu}(W) = 8\pi T_{\mu \nu},$$ \hfill (2.9)

$$g^{[\mu \nu]}_{\ , \nu} = 4\pi S^{\mu},$$ \hfill (2.10)

where

$$G_{\mu \nu} = R_{\mu \nu} - \frac{1}{2}g_{\mu \nu}R.$$ \hfill (2.11)

The variation of the $W$ connection gives

$$g^{\mu \nu, \sigma} + g^{\rho \nu}W_{\rho \sigma}^{\mu} + g^{\mu \rho}W_{\sigma \rho}^{\nu} - g^{\mu \nu}W_{\sigma \rho}^{\rho} + \frac{2}{3}\delta_{\sigma}^{\nu}g^{\mu \rho}W_{\rho}^{\nu} + \frac{4\pi}{3}(S^{\mu} \delta_{\nu}^{\sigma} - S^{\nu} \delta_{\sigma}^{\mu}) = 0.$$ \hfill (2.12)

These equations can be written in the form:

$$g_{\mu \nu, \sigma} - g_{\rho \nu}\Lambda_{\mu \sigma}^{\rho} - g_{\mu \rho}\Lambda_{\sigma \nu}^{\rho} = 0,$$ \hfill (2.13)
where

\[ \Lambda_{\mu\nu}^\rho = \Gamma_{\mu\nu}^\rho + D_{\mu\nu}^\rho. \]  

(2.14)

Here, \( D_{\mu\nu}^\rho \) depends only on \( S^\mu \) and \( g_{\mu\nu} \) and is defined by

\[ g_{\rho\nu} D_{\mu\sigma}^\rho + g_{\mu\rho} D_{\sigma\nu}^\rho = -\frac{4\pi}{3} S^\rho (g_{\mu\sigma} g_{\rho\nu} - g_{\mu\rho} g_{\sigma\nu} + g_{\mu\nu} g_{[\sigma\rho]}). \]  

(2.15)

The variational principle yields for invariance under coordinate transformations the four Bianchi identities:

\[ [g^{\alpha\nu} G_{\rho\nu}(\Gamma) + g^{\nu\alpha} G_{\nu\rho}(\Gamma)]_{,\alpha} + g^{\mu\nu}_{,\rho} G_{\mu\nu}(\Gamma) = 0. \]  

(2.16)

The matter response equations are

\[ \frac{1}{2} (g_{\sigma\rho} T_{,\sigma}^{\rho\alpha} + g_{\rho\sigma} T_{,\rho}^{\alpha\sigma})_{,\alpha} - \frac{1}{2} g_{\alpha\beta,\rho} T^{\alpha\beta} + \frac{1}{3} W_{[\rho,\nu]} S_{\nu} = 0. \]  

(2.17)

From the invariance of \( \mathcal{L} \) under the \( U(1) \) gauge transformation:

\[ W'_\mu = W_\mu + \lambda_{,\mu}, \]  

(2.18)

we obtain from Noether’s theorem the identity:

\[ g^{[\mu\nu]}_{,\mu,\nu} \equiv 4\pi S_{[\mu,\nu]]} = 0. \]  

(2.19)

An important consideration, in determining the fundamental properties of NGT, is the possible invariances allowed in the theory. The rigorous NGT Lagrangian density contains the diffeomorphism invariance, associated with the invariance under general coordinate transformations, which leads to the existence of the four Bianchi identities (2.16). The only other invariance of the Lagrangian density is the Abelian invariance under the gauge transformation (2.18), from which follows the one identity (2.19). There is no rigorous, general invariance associated with the \( g_{[\mu\nu]} \), and there are no further identities
associated with such an invariance. However, as we shall demonstrate, there is no need
of any gauge invariance associated with $g_{\mu\nu}$. This will prove to be true in the linear
approximation to the theory, as well as in the higher-order nonlinear theory.

3. Field Equations in the Linear Approximation

Let us consider weak fields in NGT, whereby we expand $g_{\mu\nu}$ about flat Minkowski
space:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad (3.1)$$

where $|h_{\mu\nu}| \ll 1$ and $\eta_{\mu\nu}$ is the Minkowski metric: $\eta_{\mu\nu} = (-1, -1, -1, +1)$. We shall solve
the field equations to lowest order in $h_{\mu\nu}$. Raising and lowering is done using $\eta_{\mu\nu}$, so that from (2.5), we obtain

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + O(h^2), \quad (3.2)$$

$$h^{\mu\nu} = \eta^{\mu\lambda} \eta^{\sigma\nu} h_{\sigma\lambda}. \quad (3.3)$$

Solving for $\Lambda_{\mu\nu}^\lambda$ and $D_{\mu\nu}^\lambda$ using (2.13) and (2.15), we get

$$\Gamma_{\mu\nu}^\lambda = \frac{1}{2} \eta^{\lambda\sigma}(h_{\sigma\nu,\mu} + h_{\mu\sigma,\nu} - h_{\nu\mu,\sigma}) - \frac{1}{3} \left(\delta_{\lambda}^\nu h_{[\mu\beta]}^\alpha - \delta_{\nu}^\lambda h_{[\nu\beta]}^\alpha\right). \quad (3.4)$$

The full Lagrangian of the theory in the linear approximation is given by

$$\mathcal{L} = \mathcal{L}_{GR} + \mathcal{L}_S, \quad (3.5)$$

where

$$\mathcal{L}_{GR} = -\frac{1}{4} h^{(\mu\nu)} \Box h_{(\mu\nu)} - \frac{1}{2} h^{(\mu\sigma)} h_{(\mu\nu)} \eta^\nu - \frac{1}{2} h^{(\mu\alpha)} \eta_{(\mu\nu)} h_{(\mu\nu)^{\alpha}} - \frac{1}{2} h^{(\mu\nu)} h_{(\mu\nu)} + 8\pi h_{(\mu\nu)} T^{(\mu\nu)}, \quad (3.6)$$

and

$$\mathcal{L}_S = \frac{1}{2} \left(\frac{1}{2} h^{[\mu\nu],\lambda} h_{[\mu\nu],\lambda} + h^{[\mu\sigma]} \eta_{[\mu\nu],\sigma} \eta_{[\mu\nu],\nu}\right) - \frac{2}{3} h^{[\mu\nu]} \eta^\nu W_{\mu} - \frac{8\pi}{3} h^{[\mu\nu]} S_{[\mu,\nu]} - \frac{8\pi}{3} W_{\mu} S^\mu$$
\[ + \frac{4\pi}{3} S^\mu S_\mu - S[^{\mu\nu}] S_{[\mu\nu]} + 8\pi h_{[\mu\nu]} T^{[\mu\nu]} . \]  

(3.7)

Here, \( h = \eta^\mu{}^\nu h_{\mu\nu} \) and \( \Box = \partial^\mu \partial_\mu \). The field equations to lowest order are:

\[ h_{[\mu\beta]} \beta = 4\pi S_\mu, \quad (3.8) \]

\[-\frac{1}{2} (\Box h_{\nu\mu} - h_{(\nu\sigma),\mu} \sigma + h_{,\mu\nu} - \frac{1}{3} h_{[\mu\sigma],\nu} \sigma + \frac{1}{3} h_{[\nu\sigma],\mu} \sigma) = -\frac{2}{3} W_{[\mu,\nu]} + 8\pi (T_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} T), \quad (3.9)\]

where \( T = \eta^{\mu\nu} T_{\mu\nu} \).

By choosing the coordinate conditions:

\[ h_{(\sigma\mu)} \sigma = \frac{1}{2} h_{,\sigma}, \]

(3.10)

the field equations become (3.8) and

\[ \Box h_{\mu\nu} = -16\pi \tilde{T}_{(\mu\nu)}, \]

(3.11)

\[ \Box h_{[\mu\nu]} = -\frac{4}{3} W_{[\mu,\nu]} - \frac{8\pi}{3} S_{[\mu,\nu]} + 16\pi T_{[\mu\nu]}, \]

(3.12)

where

\[ \tilde{T}_{(\mu\nu)} = T_{(\mu\nu)} - \frac{1}{2} \eta_{\mu\nu} T. \]

(3.13)

In the linear approximation, the skew Lagrangian \( \mathcal{L}_S \) is not invariant under the gauge transformation:

\[ \delta h_{[\mu\nu]} = \epsilon_{\mu,\nu} - \epsilon_{\nu,\mu}. \]

(3.14)

However, the presence of the Lagrange multiplier \( W_\mu \), in \( \mathcal{L}_S \), is of crucial importance, for it guarantees that unphysical ghost pole contributions will not occur in the theory. If the Lagrange multiplier \( W_\mu \) were absent, then it would be vital to have invariance under the transformation (3.14) to avoid any unphysical propagating modes.
It is important to distinguish between those degrees of freedom that propagate as physical dynamical modes, and those that do not couple dynamically to the matter sources. We shall therefore define longitudinal and transverse projection operators:

\[ P^L_{\mu\nu} = \frac{\partial_{\mu} \partial_{\nu}}{\Box}, \quad P^T_{\mu\nu} = \eta_{\mu\nu} - \frac{\partial_{\mu} \partial_{\nu}}{\Box}, \]  

(3.15)

and make the general decomposition:

\[ h_{[\mu\nu]} = \alpha_{\mu,\nu} - \alpha_{\nu,\mu} + \epsilon_{\mu\nu\kappa\lambda} \beta^{[\kappa,\lambda]}, \]  

(3.16)

and

\[ T_{[\mu\nu]} = K_{[\mu,\nu]} + \epsilon_{\mu\nu\kappa\lambda} J^{[\kappa,\lambda]}, \]  

(3.17)

where \( K_{\mu} \) and \( J^{\mu} \) are a vector and a pseudovector, respectively.

We choose the gauge condition:

\[ \alpha_{\mu,}^{\cdot\mu} = 0, \]  

(3.18)

and find that

\[ h^{TT}_{[\mu\nu]} = P^T_{[\mu} P^T_{\nu]} h_{[\alpha\beta]} = \epsilon_{\mu\nu\kappa\lambda} \beta^{[\kappa,\lambda]}, \]  

(3.19)

\[ h^{LL}_{[\mu\nu]} = P^L_{[\mu} P^L_{\nu]} h_{[\alpha\beta]} = 0, \]  

(3.20)

\[ h^{LT}_{[\mu\nu]} = P^L_{[\mu} P^T_{\nu]} h_{[\alpha\beta]} = \alpha_{\mu,\nu} - \alpha_{\nu,\mu}. \]  

(3.21)

Using the gauge condition \( W_{\alpha,}^{\cdot\alpha} = 0 \), we have

\[ W^{TT}_{[\mu,\nu]} = W^{LL}_{[\mu,\nu]} = 0, \quad W^{LT}_{[\mu,\nu]} = W_{[\mu,\nu]}. \]  

(3.22)

Moreover, we find that

\[ T^{TT}_{[\mu\nu]} = \epsilon_{\mu\nu\kappa\lambda} J^{[\kappa,\lambda]}, \]  

(3.23)
\[ T_{[\mu\nu]}^{LL} = 0, \quad (3.24) \]
\[ T_{[\mu\nu]}^{LT} = K_{[\mu,\nu]}. \quad (3.25) \]

It follows that
\[ T_{[\mu\nu]}^{TT,\nu} = 0. \quad (3.26) \]

The field equations with sources now become:
\[ h_{[\mu\beta]}^{LT,\beta} = 4\pi S_\mu, \quad (3.27) \]
\[ \Box h_{[\mu\nu]}^{TT} = 16\pi \epsilon_{\mu\nu\kappa\lambda} J^{[\kappa,\lambda]}, \quad (3.28) \]
\[ \Box h_{[\mu\nu]}^{LT} = -\frac{4}{3} W_{[\mu,\nu]} - \frac{8\pi}{3} S_{[\mu,\nu]} + 16\pi K_{[\mu,\nu]}. \quad (3.29) \]

By using (3.18), (3.21) and (3.27), we have
\[ \Box \alpha_\mu = 4\pi S_\mu. \quad (3.30) \]

Then, (3.21) and (3.30) give
\[ \Box h_{[\mu\nu]}^{LT} = 8\pi S_{[\mu,\nu]}, \quad (3.31) \]

and from (3.29), it follows that
\[ W_{[\mu,\nu]} = -8\pi S_{[\mu,\nu]} + 12\pi K_{[\mu,\nu]}. \quad (3.32) \]

Thus, from (3.28) and (3.31), we see that the auxiliary (Lagrange multiplier) contribution \( W_{[\mu,\nu]} \) can be eliminated from the field equations in the presence of sources.

4. General Static Spherically Symmetric Solution

In order to establish unequivocally the number of conserved “charges” in NGT, we shall study the consequences of the general, static spherically symmetric solution of the NGT field equations (2.9) and (2.10) in the absence of sources: \( S^\mu = T_{\mu\nu} = 0. \)
In the case of a static spherically symmetric field\textsuperscript{13}, Papapetrou has derived the canonical form of $g_{\mu\nu}$:

$$
\begin{pmatrix}
-\alpha(r) & 0 & 0 & w(r) \\
0 & -\beta(r) & f(r)\sin\theta & 0 \\
0 & -f(r)\sin\theta & -\beta(r)\sin^2\theta & 0 \\
-w(r) & 0 & 0 & \gamma(r)
\end{pmatrix}.
$$

(4.1)

A general solution of the NGT field equations in vacuum was obtained by Vanstone\textsuperscript{13}, and is given by

$$
f + i\beta = \frac{\lambda b_1(i - b)}{4(1 + b^2)y} \sinh^{-2}\left[\frac{\sqrt{b_1}}{2}(\ln y - a)\right], \\
\alpha = \frac{(f^2 + \beta^2)(y')^2}{\lambda y}, \quad \gamma = \frac{\ell^4 + f^2 + \beta^2}{f^2 + \beta^2} y, \quad w = \frac{\ell^2 y'}{\sqrt{\lambda}},
$$

(4.2)

where $y$ is an arbitrary function of $r$, $a$ is a complex constant, $b_1 = 1 + is$, and $b, s, \lambda$ and $\ell^2$ are real constants.

Consider now the asymptotically flat boundary conditions as $r \to \infty$:

$$
\alpha \to 1, \quad \beta \to r^2, \quad \gamma \to 1, \quad w \to 0.
$$

(4.3)

Vanstone proved that the only possibility to satisfy these conditions is for $f$ to vanish identically. If we adopt the weaker condition that $f/r^2 \to 0$ as $r \to \infty$, and that the solution (4.2) must reduce to the Schwarzschild solution when $g_{\mu\nu} = 0$, then we get

$$
y = 1 - \frac{2m}{r}, \quad \lambda = 4m^2, \quad b = 0, \quad a = 0.
$$

(4.4)

where the radial variable is only equivalent to the $r$ variable in the GR Schwarzschild solution in the limit $\beta \to r^2$ as $r \to \infty$.

We now put $\sqrt{b_1} = \mu + iv$, and obtain $\mu = \pm[1/2 + 1/2\sqrt{(1 + s^2)}]^{1/2}, \nu = \pm s[2 + 2\sqrt{(1 + s^2)}]^{-1/2}$. Then we find that

$$
f = \frac{2m^2[\sinh\xi\sin\eta - s(\cosh\xi\cos\eta - 1)]}{(1 - \frac{2m}{r})(\cosh\xi - \cos\eta)^2},
$$

(4.5)
\[ \beta = \frac{2m^2[\sinh \xi \sin \eta + \cosh \xi \cos \eta - 1]}{(1 - \frac{2m}{r})(\cosh \xi - \cos \eta)^2}, \quad (4.6) \]
\[ \alpha = \left( \frac{f^2 + \beta^2}{r^4} \right)^{-1}, \quad (4.7) \]
\[ \gamma = \left( 1 + \frac{\ell^2}{f^2 + \beta^2} \right) \left( 1 - \frac{2m}{r} \right), \quad (4.8) \]
\[ w = \pm \frac{\ell^2}{r^2}, \quad (4.9) \]

where \( \xi = \mu \ln y, \eta = \nu \ln y \). In the limit that \( s = \ell^2 = 0 \), we obtain the Schwarzschild solution:
\[ \alpha = \left( 1 - \frac{2m}{r} \right)^{-1}, \quad \beta = r^2, \quad \gamma = 1 - \frac{2m}{r}. \quad (4.10) \]

Expanding (4.5)-(4.8) for small values of \( s \), we get
\[ f = \frac{m^2 s}{3} \left( 1 - \frac{2m}{r} + \ldots \right), \quad (4.11) \]
\[ \beta = r^2 \left( 1 - \frac{2s^2 m^4}{15r^4} + \ldots \right), \quad (4.12) \]
\[ \gamma = \left( 1 + \frac{\ell^4}{r^4} + \ldots \right) \left( 1 - \frac{2m}{r} \right), \quad (4.13) \]
\[ \alpha = \left( 1 - \frac{s^2 m^4}{45r^4} + \ldots \right) \left( 1 - \frac{2m}{r} \right). \quad (4.14) \]

The first important consequence of the general, static spherically symmetric solution is that there are only two conserved charges in NGT, the mass \( m \) and \( \ell^2 \). These are both conserved by virtue of the Bianchi identities (2.16) or (2.17), as in GR, and the NGT conservation law (2.19) with the identification:
\[ \ell^2 = \int d^3 x S^0. \quad (4.15) \]

Thus, only two gauge invariances play a role in NGT, the diffeomorphism invariance and the invariance associated with the NGT Lagrangian density under the \( U(1) \) transformation.
(2.18). There is no need for any extra gauge invariance associated with the \( g_{\mu\nu} \) sector, for there does not exist a conservation law of a new charge in the theory besides \( m \) and \( \ell^2 \).

The second significant consequence of the static solution is that in the linear approximation, neglecting contributions of order \( h^2_{(\mu\nu)} \), we see from (4.11) that \( h_{[23]} \) is zero to lowest order. From (3.28), it follows that to lowest order of approximation (neglecting terms quadratic and higher order in \( m \)) in the static limit: \( h_{TT}^{[23]} = 0 \) and therefore in general: \( h_{[m\mu\nu]}^{TT} = 0 \). Then, to lowest order we have

\[
J^\mu = 0, \quad T_{[\mu\nu]} = K_{[\mu,\nu]}.
\] (4.16)

It now follows from (3.32) that to lowest order of approximation:

\[
W_{[\mu,\nu]} = 0,
\] (4.17)

in the wave-zone. Because \( T_{[\mu\nu]} \) is a compact, localized source tensor, then from (3.17) and (4.16), it follows that \( K_{[\mu,\nu]} \) has compact support and vanishes in the wave-zone together with \( S^\mu \) and (4.16) follows immediately.

In the wave-zone at infinity, the skew vacuum field equations are given by:

\[
h^{LT,\beta}_{[\mu}\beta = 0,
\] (4.18)

\[
\Box h^{LT}_{[\mu\nu]} = 0.
\] (4.19)

The solution for \( h_{(\mu\nu)} \) coincides to lowest order with the GR solution\(^{14}\):

\[
h_{(\mu\nu)}(x, t) = -4 \int d^3x' \frac{T_{(\mu\nu)}(x', t - |x - x'|)}{|x - x'|}.
\] (4.20)

To this order \( h_{[\mu\nu]}^{TT} = 0 \) and the solution of equation (3.31) is given by

\[
h^{LT}_{[\mu\nu]}(x, t) = 2 \int d^3x' \frac{S_{[\mu,\nu]}(x', t - |x - x'|)}{|x - x'|}.
\] (4.21)
5. Spin Projection Analysis of the Linearized Theory

Let us write the skew part of the Lagrangian $L_S$ in the form\(^\text{15}\):

$$ L_S = \frac{1}{2} \sum_{A,B} \phi_A O_{AB} \phi_B, \quad (5.1) $$

where $\phi_A = (h_{[\mu\nu]}, W_{\mu})$ and $O_{AB}$ is the wave operator. Using the methods of ref. (16), we can decompose the fields into subspaces with spin-parity $J^P$ and invert the operator $O_{AB}$ to obtain the saturated propagator:

$$ \Pi = - \sum_{\psi_A, \phi_B} S_A O_{AB}^{-1} S_B, \quad (5.2) $$

where $S_A = (L_{[\mu\nu]}, S_\mu)$ and

$$ L_{[\mu\nu]} = T_{[\mu\nu]} + S_{[\mu,\nu]}, \quad (5.3) $$

Expanding the operator $O_{AB}$ yields

$$ L_S = \sum_{\psi_A, \phi_B, i,j,J^P} a_{ij}^{\psi\phi} (J^P) \psi_A P_{ij}^{\psi\phi} (J^P) \phi_B, \quad (5.4) $$

and

$$ \Pi = - \sum_{\psi_A, \phi_B, i,j,J^P} a_{ij}^{-1} \psi \phi (J^P) S_A P_{ij}^{\psi\phi} (J^P) AB S_B, \quad (5.5) $$

where $a_{ij}^{\psi\phi} (J^P)$ are matrix coefficients. A calculation of the spin-projection operators yields

$$ P_{ij} (1^+) = \frac{1}{2} (\theta_{\mu\alpha} \theta_{\nu\beta} - \theta_{\nu\alpha} \theta_{\mu\beta}), \quad (5.6) $$

$$ P_{ij} (1^-) = \left( \frac{1}{2} (\theta_{\mu\alpha} \omega_{\nu\beta \text{ beta}} - \theta_{\mu\beta} \omega_{\nu\alpha} - \theta_{\nu\alpha} \omega_{\mu\beta} + \theta_{\nu\beta} \omega_{\mu\alpha}) - \frac{1}{\sqrt{2}} (\hat{k}_\nu \theta_{\mu\alpha} - \hat{k}_\mu \theta_{\nu\alpha}) \right), \quad (5.7) $$

$$ P_{ij} (0^+) = \omega_{\alpha\beta}, \quad (5.8) $$

where

$$ \theta_{\mu\alpha} = \eta_{\mu\alpha} - k_\mu k_\alpha k^{-2}, \quad \omega_{\mu\alpha} = k_\mu k_\alpha k^{-2}, \quad \hat{k}_\mu = k_\mu (k^2)^{-1/2}. \quad (5.9) $$
These operators are orthonormal and complete within \((h_{[\mu\nu]}, W_\mu)\). The Lagrangian \(\mathcal{L}_S\) now yields the matrix coefficients:

\[
a(1^+) = \frac{1}{2} k^2, \tag{5.10}
\]

\[
a_{ij}(1^-) = \begin{pmatrix}
  k^2 & -\frac{\sqrt{2}i}{3} (k^2)^{1/2} \\
  \frac{\sqrt{2}i}{3} (k^2)^{1/2} & 0
\end{pmatrix}, \tag{5.11}
\]

\[
a(0^+) = 0. \tag{5.12}
\]

The result (4.12) follows from the gauge invariance under the transformation (2.18) and the conservation equation (2.19). Inverting (4.10) and (4.11) yields

\[
a^{-1}(1^+) = \frac{2}{k^2}, \tag{5.13}
\]

\[
a^{-1}_{ij}(1^-) = \begin{pmatrix}
  0 & -\frac{3i}{\sqrt{2}} (k^2)^{1/2} \\
  \frac{3i}{\sqrt{2}} (k^2)^{1/2} & -\frac{9}{2} k^2
\end{pmatrix} \frac{1}{k^2}. \tag{5.14}
\]

A calculation determines the saturated propagator to be of the form:

\[
\Pi = -\frac{2}{k^2} T^{[\mu\nu]} P^{(1^+)\mu\nu\alpha\beta} T^{[\alpha\beta]} + \frac{3i}{2k^2} S^\alpha (k_\nu \theta_{\mu\alpha} - k_\mu \theta_{\nu\alpha}) L^{[\mu\nu]}
- \frac{3i}{2k^2} L^{[\mu\nu]} (k_\mu \theta_{\nu\alpha} - k_\nu \theta_{\mu\alpha}) S^\alpha + \frac{9}{2} (S^\alpha \theta_{[\alpha\beta]} S^{\beta}). \tag{5.15}
\]

Here, we have used the condition \(k^\alpha \theta_{[\alpha\beta]} = 0\) to remove terms of the form \(P(1^+)_{\mu\nu\alpha\beta} S^{[\alpha\beta]}\).

We see that the massless spin \(1^-\) ghost particles do not propagate, and only contact terms occur in this sector, whereas the spin \(1^+\) sector is neither a ghost particle nor a tachyon. For the real version of NGT, we see that the theory is completely ghost free, even with a nonvanishing \(T_{[\mu\nu]}\).

The nonpropagating spin \(1^-\) sector corresponds to the \(h^{LT}_{[\mu\nu]}\) components, while the propagating spin \(1^+\) sector corresponds to the transverse \(h^{TT}_{[\mu\nu]}\) components of \(h_{[\mu\nu]}\). In
Sect. 6, we will prove that the \( h^{LT}_{[\mu\nu]} \) do not propagate in the wave-zone. Although the skew field equations are not invariant under the transformation (3.14), there does exist a restricted gauge invariance under the transformations:

\[
\delta h^{LT}_{[\mu\nu]} = \epsilon_{\mu,\nu} - \epsilon_{\nu,\mu}, \quad \delta h^{TT}_{[\mu\nu]} = \epsilon_{\mu\nu\kappa\lambda} \zeta^{[\kappa,\lambda]},
\]

provided that \( \epsilon_\mu \) and \( \zeta_\mu \) satisfy:

\[
\Box \epsilon_\mu = 0, \quad \epsilon_\mu^{,\mu} = 0, \quad \Box \zeta_\mu = 0, \quad \zeta_\mu^{,\mu} = 0.
\]

This restricted gauge invariance reduces the three degrees of freedom of the propagating spin \( 1^+ \) sector to one spin \( 0^+ \) degree of freedom. It should be stressed at this point that we do not need to use this restricted gauge invariance to remove potential ghost modes in the linear approximation, for such ghost modes simply do not exist.

A similar result occurs for a spin \( 1^- \) field \( A_\mu \) with a scalar Lagrange multiplier \( \phi \), if we replace \( h_{[\mu\nu]} \) with \( A_\mu \) and \( W_\mu \) with \( \phi \). The spin \( 0^+ \) ghost particle becomes nonpropagating because of the Lagrange multiplier \( \phi \).

If we had included a mass term in \( \mathcal{L}_S \) of the form \( d g^{[\mu\nu]} g_{[\mu\nu]} \), then we would be forced to have an unphysical particle spectrum, which contains either ghosts \((d < 0)\) or tachyons \((d > 0)\). Only for the massless theory are we guaranteed the absence of ghosts, due to the presence of the Lagrange multiplier \( W_\mu \). The suggestion by Damour, Deser and McCarthy \(^{5-7}\) that a mass term be introduced into NGT to avoid bad asymptotic behavior would necessarily lead to an unphysical theory.
6. Plane Wave Solutions in the Linear Approximation

The plane wave solutions for \( h_{(\mu\nu)} \) are identical to those in GR. The solution of Eqs. (4.19) is

\[
h^{LT}_{[\mu\nu]} = e^{LT}_{[\mu\nu]} \exp(ik\lambda x^\lambda) + e^{*LT}_{[\mu\nu]} \exp(-ik\lambda x^\lambda),
\]

where \( k^\mu k_\mu = 0 \),

\[
k^\mu e_{[\mu\nu]} = 0,
\]

and to this order \( h^{TT}_{[\mu\nu]} = 0 \).

Let us now calculate the energy-momentum tensor of the plane waves. This is done using the energy-momentum pseudotensor \( t_{\mu\nu} \) of NGT. We have to lowest order\(^\text{12}\):

\[
t_{\mu\nu} = \frac{1}{8\pi} (R^{(2)}_{\mu\nu}(\Gamma) - \frac{1}{2} \eta_{\mu\nu} R^{(2)}(\Gamma)),
\]

where \( R^{(2)}_{\mu\nu}(\Gamma) \) is \( R_{\mu\nu}(\Gamma) \) to \( O(h^2) \). We only need the average expression, \( \langle t_{(\mu\nu)} \rangle \), since this is what is measured. Averaging over space and time in a region much larger than \( |k|^{-1} \), we have

\[
\langle t_{(\mu\nu)} \rangle = \frac{k_\mu k_\nu}{16\pi} (e^{(\beta\gamma)} e^{*}_{(\beta\gamma)} - \frac{1}{2} |\eta^{\beta\gamma} e_{(\beta\gamma)}|^2 + e^{[\beta\gamma]} e^{*}_{[\beta\gamma]}).
\]

This expression is positive definite for the real (or hyperbolic complex\(^\text{11}\)) version of NGT. Here, we have taken into account that \( W_{[\mu,\nu]} \) is zero to lowest order of approximation in the wave-zone.

7. Generation of Gravitational Waves in the Linear Approximation

The analysis of the gravitational waves generated by the GR source, \( T_{(\mu\nu)} \), in the linear approximation is well understood\(^\text{14}\). Let us consider the possible existence of
radiation generated by the NGT sources, \( S^\mu \) and \( T_{[\mu \nu]} \), in terms of their Fourier components \( S^\mu (x, t) \) and \( T_{[\mu \nu]} (x, t) \) in the linear approximation:

\[
S^\mu (x, t) = \int_0^\infty d\omega S^\mu (x, \omega) \exp(i\omega t) + CC, \tag{7.1}
\]

\[
T_{[\mu \nu]} (x, t) = \int_0^\infty d\omega K_{[\mu, \nu]} (x, \omega) \exp(i\omega t) + CC, \tag{7.2}
\]

where \( CC \) means complex conjugate.

In the wave-zone, we get

\[ |x - x'| = r - \hat{x} \cdot x' + ... \tag{7.3} \]

We have \( h^{TT}_{[\mu \nu]} = 0 \) and from (3.31), we obtain

\[
h^{LT}_{[\mu \nu]} = \partial_\nu \left( \int d^3 x' S^\mu (x', \omega) \exp[i\omega (t - |x - x'|)] \frac{|x - x'|}{|x - x'|} \right) \]

\[ - \partial_\mu \left( \int d^3 x' S^\nu (x', \omega) \exp[i\omega (t - |x - x'|)] \frac{|x - x'|}{|x - x'|} \right). \tag{7.4} \]

By using the expansion (7.3), we get

\[
h^{LT}_{[\mu \nu]} = e^{LT}_{[\mu \nu]} (x, \omega) \exp(i k_\lambda x^\lambda) + CC, \tag{7.5} \]

where

\[
e^{LT}_{[\mu \nu]} = \frac{1}{r} [k_\nu S^\mu (k, \omega) - k_\mu S^\nu (k, \omega)] + O(1/r^2). \tag{7.6} \]

When we substitute (7.6) into (6.4), we find that

\[
< t_{(\mu \nu)} > = \frac{k_\mu k_\nu}{16\pi} [ (e^{(\beta \gamma)} e^{(\beta \gamma)} - \frac{1}{2} |e|^2)] . \tag{7.7} \]

Thus, the \( e_{[\mu \nu]} \) do not contribute to the gravitational wave flux through the source \( S_{[\mu, \nu]} \), and this demonstrates that \( h^{LT}_{[\mu \nu]} \) does not propagate. This agrees with the conclusion, reached in Sect. 4, that the spin 1\(^{-}\) sector does not propagate. Since the NGT pseudovector
source \( J^\mu = 0 \) to linear order, the gravitational wave flux is the same to leading order as in GR. This coincides with the results obtained previously\(^{1-3} \). We observe that \( < t_{(\mu\nu)} > \) is positive definite, and the linear approximation is physically consistent.

8. NGT on a Curved GR Background

We shall expand the NGT field equations in powers of \( h_{[\mu\nu]} \), taking the symmetric part \( g_{(\mu\nu)} \) to be an exact GR background, \( g^{GR}_{\mu\nu} \). To lowest order in \( h_{[\mu\nu]} \)\(^{19,1,2,4-7} \):

\[
g_{\mu\nu} = g^{GR}_{\mu\nu} + h_{[\mu\nu]}.
\] (8.1)

The field equations with sources become in lowest order:

\[
R^{GR}_{\mu\nu} - \frac{1}{2} g^{GR}_{\mu\nu} R^{GR} = 8\pi T_{(\mu\nu)},
\] (8.2)

\[
D^\alpha F_{\mu\nu\alpha} + 4R^{GR}_{\mu\nu} \frac{\alpha}{\beta} h_{[\alpha\beta]} = -\frac{4}{3} W_{[\mu,\nu]} - \frac{16\pi}{3} S_{[\mu,\nu]} + 16\pi T_{[\mu\nu]},
\] (8.3)

\[
D^\nu h_{[\mu\nu]} = 4\pi S_{\mu},
\] (8.4)

where \( D^\alpha \) is the GR background covariant derivative and

\[
F_{\mu\nu\rho} = h_{[\mu\nu],\rho} + h_{[\nu\rho],\mu} + h_{[\rho\mu],\nu}.
\] (8.5)

We observe that these equations are not invariant under the gauge transformation (3.14), as is expected, for this invariance does not occur in the theory.

Because of (3.17) and (8.5), we can rewrite Eq. (8.3) in the form:

\[
D^\alpha D_\alpha h_{[\mu\nu]} + 4R^{GR}_{\mu\nu} \frac{\alpha}{\beta} h_{[\alpha\beta]} = \tilde{W}_{[\mu,\nu]} + 16\pi \epsilon_{\mu\nu\alpha\beta} J^{\alpha\beta},
\] (8.6)

where

\[
\tilde{W}_{[\mu,\nu]} = -\frac{4}{3} W_{[\mu,\nu]} - \frac{8\pi}{3} S_{[\mu,\nu]} + 16\pi K_{[\mu,\nu]}.
\] (8.7)
By taking the curl of (8.6), we obtain the field equation:

$$D^\alpha D_\alpha F_{\mu\nu\rho} + D_{\{\rho} Q_{[\mu\nu]}\} = 0,$$

(8.8)

where \{\ldots\} denotes the curl. Moreover, we have

$$Q_{[\mu\nu]} = 4R_{GR}^\alpha \mu^\beta h_{[\alpha\beta]} - 16\pi \epsilon_{\mu\nu\alpha\beta} J^{\alpha,\beta}.$$  

(8.9)

On the other hand, we could take the divergence of Eq. (8.6) and get:

$$D^\sigma D_\sigma \tilde{W}_\mu = -D^\nu (8R_{GR}^\alpha \mu^\beta h_{[\alpha\beta]}),$$

(8.10)

where we have used the gauge condition:

$$D^\mu \tilde{W}_\mu = 0.$$  

(8.11)

Eq. (8.10) represents four integrability conditions that must be satisfied, once the solutions for $g_{\mu\nu}^{GR}$ and $h_{[\mu\nu]}$ are obtained. The boundary conditions are imposed on $g_{\mu\nu}^{GR}$ and $h_{[\mu\nu]}$ by demanding that $g_{\mu\nu}^{GR} \to \eta_{\mu\nu}$ and $h_{[\mu\nu]} \to 0$.

It has been proved by explicitly solving the system of six equations:

$$D^\nu h_{[\mu\nu]} = 4\pi S_\mu,$$  

(8.12)  

$$D^\alpha D_\alpha F_{\mu\nu\rho} = -D_{\{\rho} Q_{[\mu\nu]}\},$$  

(8.13)

that consistent solutions can be obtained for the six $h_{[\mu\nu]}$, which satisfy the physical boundary condition of asymptotic flatness at infinity$^{1-4}$. It is important to observe that any $h_{[\mu\nu]}$ solutions of (8.12) and (8.13) must satisfy the primary equations (8.6). The same is true for any $\tilde{W}_\mu$ solution of (8.10)$^{4,20}$. 
The skew contribution to the stress-energy pseudotensor, \( t^{(\mu\nu)} \), can be expressed in the form\(^6\):
\[
t_s^{(\mu\nu)} = \frac{1}{2} (F^{\mu\alpha\beta} F^\nu_{\alpha\beta} - \frac{1}{12} g^{GR\mu\nu} F^2) + 2 \left(\frac{2}{3} h^{[\mu\alpha]} g^{GR\nu\beta} W_{[\beta,\alpha]} \right) + \frac{2}{3} h^{[\nu\alpha]} g^{GR\mu\beta} W_{[\beta,\alpha]} - \frac{1}{3} g^{GR\mu\nu} h^{[\alpha\beta]} W_{[\alpha,\beta]} - (3 R^{GR} (\mu \gamma \alpha \beta h^{[\nu\alpha]} h^{[\gamma\beta]}) - D_\beta D_\alpha (h^{[\gamma\beta]} h^{[\alpha\nu]}),
\]
(8.14)

The GR Riemann tensor must have dimensions of \([\text{length}]^{-2}\) and contains at most two time derivatives. The dimensionless quantity, \( \dot{M} \), (where \( M \) is the mass function) appears, together with the quantity \( \ddot{M} \), and therefore the Riemann tensor behaves in the wave-zone as \( \sim 1/r \). The rate of energy loss for an isolated body, in NGT, is given by
\[
\frac{dE}{dt} = -R^2 \int d\Omega t^{(0i)} \hat{n}_i,
\]
(8.15)

where the integration is over a sphere of radius \( R \) in the wave-zone at infinity and \( \hat{n}_i \) is an outward pointing unit vector. At infinity, the first term in (8.14) is positive, while the second term does not contribute, since in the flat spacetime limit \( W_{[\mu,\nu]} \) is zero to lowest order, as was shown in Sect. 4. The third term goes to zero at least as \( \sim 1/r^3 \) and does not contribute to the energy-momentum flux, while the fourth term has the form:
\[
\partial_\alpha \partial_\beta N^{[\mu\alpha][\nu\beta]},
\]
making no contribution. Therefore, the total energy-momentum flux in the wave zone is finite and positive definite.

In the case of an axisymmetric, time-dependent solution of the NGT field equations to first-order in \( h_{[\mu\nu]} \) on a curved GR background\(^2\), and for the generalized Bondi, van der Burg, Metzner and Sachs (BBMS) solution\(^3,21\), the calculations show that only the quadrupole GR radiation manifests itself in the energy-momentum flux, for solutions that obey the asymptotic flatness boundary condition at future and past null infinity. This agrees with the result obtained in the linear approximation in flat spacetime. However, as
in the case of GR, it is not possible in the generalized BBMS solution to interpret directly the nature of the near-zone NGT sources. The properties of these sources can only be inferred from multipole moment expansions and their effects on wave solutions at future and past null infinity. It should be stressed that the NGT source in the axisymmetric, reflexion symmetric solution represents a generic source in NGT, for it possesses all the GR and NGT moments of a general gravitational wave source.

9. Conclusions

The proof of the physical consistency of NGT does not depend on the existence of any form of gauge invariance in the $g_{[\mu\nu]}$ sector, for there exist only two conserved charges consistent with the two gauge invariances, namely, diffeomorphism invariance and the $U(1)$ gauge invariance that guarantees the conservation of the charge $\ell^2$. For an isolated body, the physical boundary condition demands that in orthonormal, Cartesian coordinates $g_{\mu\nu}^{GR} \to \eta_{\mu\nu}$ and $g_{[\mu\nu]} \to 0$ as $r \to \infty$, which in turn requires that for the static spherically symmetric NGT solution in vacuum, $g_{[23]} = 0$ to the lowest order of approximation. In the higher non-linear orders of approximation there is no direct contribution to the flux of gravitational waves from the skew sector. It then follows that NGT gives physically sensible answers for the radiation of gravitational waves.

Axisymmetric time-dependent solutions have been obtained, which show that the expected consistency of the field equations does hold. A solution to the axisymmetric case was derived to first-order in $h_{[\mu\nu]}$, in an expansion about a curved GR background. The cross-coupling term between the background GR Riemann curvature tensor and $h_{[\mu\nu]}$ does not cause any inconsistency, provided the field equations are correctly solved using the boundary conditions imposed on $g_{\mu\nu}$, namely, that the solutions are asymptotically flat at past and future null infinity. The exact axisymmetric NGT vacuum field equations
were expanded in inverse powers of $r$, and it was found that the gravitational wave flux of energy was positive definite\(^3\). This result extended the proof of the positivity of the gravitational wave flux in GR, obtained by Bondi \textit{et al.} and Sachs\(^{21,22}\).

Because there are only two conserved charges, $m$ and $\ell^2$, in NGT, it can be proved that there cannot exist any direct contribution from $g_{[\mu\nu]}$, which has the form of a retarded or advanced wave with the asymptotic behavior $g_{[\mu\nu]} \approx 1/r$. This follows from the fact that $\ell^2$ has the dimensions of a $[\text{length}]^2$, whereas $m$ (and the electric charge $e$ in electromagnetism) has the dimensions of a length and can generate a $1/r$ retarded wave solution and quadrupole radiation\(^{23}\).

When the NGT charge $\ell^2 = 0$, the static spherically symmetric solution only contains the mass $m$ and the theory becomes purely geometrical. Test particles will follow geodesic equations determined by the Christoffel symbols of NGT\(^{24}\).

From Eq. (8.2), we see that to first-order in $h_{[\mu\nu]}$, a positive energy theorem follows trivially using the methods of ref. 25. However, further work must be carried out to derive a general, rigorous positive energy theorem in NGT.

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