FOUR-DIMENSIONAL LIE ALGEBRAS WITH A
PARA-HYPERCOMPLEX STRUCTURE

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Abstract. The main goal is to classify 4-dimensional real Lie algebras \( g \) which admit a para-hypercomplex structure. This is a step toward the classification of Lie groups admitting the corresponding left-invariant structure and therefore possessing a neutral, left-invariant, anti-self-dual metric. Our study is related to the work of Barberis who classified real, 4-dimensional simply-connected Lie groups which admit an invariant hypercomplex structure.

1. Introduction

Our work is motivated by the work of Barberis [2] where invariant hypercomplex structures on 4-dimensional real Lie groups are classified (see Section 2 for definitions). In that case the corresponding hermitian metric is positive definite and unique up to a positive constant. Our main goal is to classify 4-dimensional real Lie algebras \( g \) which admit para-hypercomplex structures. This is a step toward the classification of the corresponding left invariant structures on Lie groups. In this case the corresponding hermitian pseudo-Riemannian metric determined by the para-hypercomplex structure is also unique up to a constant, but has to be of signature \((2, 2)\). This metric is anti-self-dual (see [4]).

In the paper [1] Andrada and Salamon have shown that any para-hypercomplex structure on a real Lie algebra \( g \) rise to a hypercomplex structure on its complexification \( g^\mathbb{C} \) (considered as a real Lie algebra). They referred to para-hypercomplex structure as complex product structure.

Let us remark that Snow [5] and Ovando [3] classified the invariant complex structures on 4-dimensional, solvable, simply-connected real Lie groups where the dimension of commutators is less than three and equal three, respectively. Since every para-hypercomplex manifold is also complex, the Lie algebras from our classification also appear in their lists.

Let us state the main theorem (proved in Subsection 3.4).

**Theorem 1.1.** Up to an isomorphism the only 4-dimensional Lie algebras \( g \) admitting an integrable para-hypercomplex structure are listed below.

1. \( [X, Y] = Z, [X, W] = W, [Y, W] = -X, [W, X] = Y \),
2. \( [X, Y] = W, [Y, W] = -X, [W, X] = Y \),
3. \( [X, Y] = Y, [X, W] = W \),
4. \( [X, Y] = Z \),

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(PHC5) \[ [X,Y] = X, \]
(PHC6) \[ [X,Y] = Z, \ [X,W] = X + aY + bZ, \ [W,Y] = Y, \]
(PHC7) \[ [X,Z] = X, \ [X,W] = Y, \ [Y,Z] = Y, \ [Y,W] = aX + bY, \ a,b \in \mathbb{R}, \]
(PHC8) \[ [X,Z] = X, \ [Y,W] = Y, \]
(PHC9) \[ [Z,W] = Z, \ [Y,W] = Y, \ [X,W] = cX + aY + bZ, \ c \neq 0, \ a,b \in \mathbb{R}, \]
(PHC10) \[ [Y,X] = Z, \ [W,Z] = cZ, \ [W,X] = \frac{1}{2}X + ay + bZ, [W,Y] = (c - \frac{1}{2})Y, c \neq 0. \]

In the previous list the additive basis of algebra \( \mathfrak{g} \) is \( (X,Y,Z,W) \), and only the non-zero commutators are given.

In the proof we study separately the cassis defined in the terms of metrics defined on the derived algebra \( \mathfrak{g}' \) by means of the para-hypercomplex structure.

Here is a brief outline of the paper. In Section 2 we first give necessary definitions and prove some basic properties of para-hypercomplex structures and a number of lemmas which we use in the sequel. In Section 3 we step-by-step prove Theorem 1.1. First, in Subsection 3.1 we classify 4-dimensional Lie algebras with a non-trivial center and admitting a para-hypercomplex structure. Further on we suppose that algebra \( \mathfrak{g} \) has a trivial center. In Subsection 3.2 and 3.3 we classify solvable 4-dimensional Lie algebras \( \mathfrak{g} \) admitting a para-hypercomplex structure (Theorems 3.2 and 3.3), and prove some basic properties of para-hypercomplex structures and a number of particular examples of para-hypercomplex structures on algebras PHC1-PHC10. Finally, in Section 4 we compare our results with the results of Barberi [2].

2. Preliminaries

Let \( V \) be a real vector space. A complex structure on \( V \) is an endomorphism \( J_1 \) of \( V \) satisfying the condition

\[ J_1^2 = -1. \]

Existence of a complex structure implies that \( V \) has to be of an even dimension. A product structure on \( V \) is an endomorphism \( J_2 \) of \( V \) satisfying the conditions

\[ J_2^2 = 1, \quad J_2 \neq \pm 1. \]

A para-hypercomplex structure on \( V \) is a pair \( (J_1,J_2) \) of anti-commuting complex structure \( J_1 \) and product structure \( J_2 \), i.e. satisfying the relations

\[ J_1^2 = -1, \quad J_2^2 = 1, \quad J_1J_2 = -J_2J_1. \]

If both structures \( J_1 \) and \( J_2 \) are complex then the pair \( (J_1,J_2) \) is called a hypercomplex structure on \( V \). In the sequel we concentrate on the case of para-hypercomplex structure.

It is customary to denote \( J_3 = J_1J_2 \). Note that the structure \( J_3 \) is a product structure. The Lie subalgebra of \( \text{End}(V) \) spanned by \( J_1, J_2 \) and \( J_3 \) is isomorphic to \( \mathfrak{sl}_2(\mathbb{R}) \). Any \( x = (x_1,x_2,x_3) \in \mathbb{R}^3 \) defines a structure by the formula

\[ J_x := x_1J_1 + x_2J_2 + x_3J_3. \]

Denote by

\[ \langle x,y \rangle = x_1y_1 - x_2y_2 - x_3y_3, \]
\[ x = (x_1,x_2,x_3), \ y = (y_1,y_2,y_3), \] the inner product in \( \mathbb{R}^3 = \mathbb{R}^{1,2} \) and by

\[ x \times y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1) \]
the usual cross product. The structure \( J_x \) is a complex structure provided that
\[
\langle x, x \rangle = x_1^2 - x_2^2 - x_3^2 = 1
\]
and a product structure provided that
\[
\langle x, x \rangle = x_1^2 - x_2^2 - x_3^2 = -1.
\]
Hence, a para-hypercomplex structure \((J_1, J_2)\) defines a 2-sheeted hyperboloid \( \mathbb{B}^- \) of complex structures and a 1-sheeted hyperboloid \( \mathbb{B}^+ \) of product structures.

**Proposition 2.1.** If \((J_1, J_2)\) is a para-hypercomplex structure on a vector space \(V\), then:

i) \(J_x J_y = -\langle x, y \rangle 1 + J_x \times y\).

ii) The pair \((J_x, J_y)\) \(\in \mathbb{B}^- \times \mathbb{B}^+\) is a para-hypercomplex structure if and only if \(x \perp y\).

**Proof:** From the relations
\[
J_1 J_2 = J_3 = -J_2 J_1, \quad J_1 J_3 = -J_2 = -J_3 J_1, \quad J_2 J_3 = -J_1 = -J_3 J_2
\]
the statement i) follows by a direct calculation.

Since \(J_x\) is a complex structure and \(J_y\) is a product structure, the pair \((J_x, J_y)\) is a para-hypercomplex structure if and only if \(J_x\) and \(J_y\) anti-commute. Using the relation i) and the anti-commutativity of the cross product we have
\[
0 = J_x J_y + J_y J_x = -2\langle x, y \rangle 1.
\]
Hence, the statement ii) is proved. \(\square\)

The para-hypercomplex structures \((J_1, J_2)\) and \((J_x, J_y)\) are called compatible. An almost para-hypercomplex structure on a manifold \(M\) is a pair \((J_1, J_2)\) of sections of \(\text{End}(TM)\) satisfying the relations \([1]\). It is a para-hypercomplex structure if both structures are integrable, that is, if the corresponding Nijenhuis tensors
\[
N_\alpha(X, Y) = [J_\alpha X, J_\alpha Y] - J_\alpha [X, J_\alpha Y] - J_\alpha [J_\alpha X, Y] \pm [X, Y],
\]
\(\alpha = 1, 2\), vanish on all vector fields \(X, Y\). In this formula sign \(-\) occurs in the case of a complex structure and sign \(+\) occurs in the case of a product structure.

If \(M = G\) is a Lie group we additionally assume that the para-hypercomplex structure is left invariant. This allows us to also describe a para-hypercomplex structure on its Lie algebra \(g\). Hence, a para-hypercomplex structure \((J_1, J_2)\) on \(g\) satisfies both relations \([1]\) and \([2]\).

**Proposition 2.2.** Let \((J_1, J_2)\) be an integrable para-hypercomplex structure on a Lie algebra \(g\).

i) The product structure \(J_3 = J_1 J_2\) is integrable.

ii) Any compatible para-hypercomplex structure \((J_x, J_y)\) is integrable.

**Proof:** The statement i) follows from the relation
\[
2N_3(X, Y) = N_1(J_2 X, J_2 Y) + N_2(J_1 X, J_1 Y) - J_1 N_2(J_1 X, Y) - J_2 N_1(J_2 X, Y) +
+ N_2(X, Y) - J_2 N_1(J_2 X, Y) - J_2 N_1(X, J_2 Y) - N_1(X, Y)
\]
where \(N_\alpha\) is the Nijenhuis tensor of the product structure \(J_3\).

To prove ii) denote by \(N_x\) the Nijenhuis tensor corresponding to the structure \(J_x, x = (x_1, x_2, x_3)\). One can check that
\[
N_x = x_1^2 N_1 + x_2^2 N_2 + x_3^2 N_3 + x_1 x_2 (J_3 N_1 + J_3 N_2 + J_3 N_3) +
+ x_2 x_3 (J_1 N_2 - J_1 N_3 - J_1 N_1 J_2) + x_1 x_3 (-J_2 N_1 - J_2 N_3 + J_2 N_2 J_3)
\]
holds, where we have used the notation, for instance
\[ J_2N_2J_3(X,Y) = J_2N_2(J_3X, J_3Y). \]
Now, statement ii) follows using statement i). \( \square \)

Let \( g \) be an inner product on the vector space \( V \). A para-hypercomplex structure \((J_1, J_2)\) on \( V \) is called hermitian with respect to \( g \) if
\[
(3) \quad g(J_3X, J_3Y) = -g(X, J_3Y), \quad X, Y \in V
\]
holds, i.e. if both structures \( J_1 \) and \( J_2 \) are hermitian. It is easy to prove that a hermitian complex structure is an isometry and a hermitian product structure is an anti-isometry, i.e.
\[
g(J_1X, J_1Y) = g(X, Y), \quad g(J_2X, J_2Y) = -g(X, Y).
\]
Existence of an anti-isometry implies that the inner product \( g \) must be of neutral, \((n,n)\) signature.

**Proposition 2.3.** Let \((J_1, J_2)\) be a para-hypercomplex structure hermitian with respect to the scalar product \( g \) on the vector space \( V \).

i) The product structure \( J_3 = J_1J_2 \) is hermitian.

ii) Any compatible para-hypercomplex structure \((J_x, J_y)\) is hermitian.

**Proof:**

i) If \( J_1 \) and \( J_2 \) are hermitian then \( J_3 \) is hermitian since we have
\[
\langle J_3X, Y \rangle = \langle J_1J_2X, Y \rangle = -\langle J_2X, J_1Y \rangle = \langle X, J_2J_1Y \rangle = -\langle X, J_3Y \rangle.
\]
i) Since the condition of any \( J_x \) to be hermitian is linear with respect to \( x \), the statement ii) follows from the statement i). \( \square \)

Now, we prove some lemmas which will be useful in the sequel.

**Lemma 2.1.** If \((J_1, J_2)\) is a para-hypercomplex structure on a real 4-dimensional vector space \( V \) then:

i) There is an inner product \( g \) on \( V \), unique up to a non-zero constant, such that the structure \((J_1, J_2)\) is hermitian with respect to \( g \).

ii) Any compatible para-hypercomplex structure \((J_x, J_y)\) determines the same inner product \( g \) on \( V \).

**Proof:** First, we prove the existence of such an inner product. If \((\cdot, \cdot)\) is an arbitrary inner product on \( V \), then the inner product
\[
(4) \quad g(X, Y) := (X, Y) + (J_1X, J_1Y) - (J_2X, J_2Y) - (J_3X, J_3Y)
\]
satisfies the properties (3).

To see the uniqueness let \( g'(\cdot, \cdot) \) be another inner product on \( V \) satisfying (3). As remarked before both products are of signature \((2,2)\). There exists a vector \( X \) which is not null with respect to the both inner products, for instance
\[
g(X, X) = 1, \quad g'(X, X) = \lambda \neq 0.
\]

The relations (1) and (3) imply that the vectors \( X, J_1X, J_2X, J_3X \) are mutually orthogonal with respect to both inner products. Moreover,
\[
\begin{align*}
g(X, X) &= g(J_1X, J_1X) = 1 = -g(J_2X, J_2X) = -g(J_3X, J_3X) \\
g'(X, X) &= g'(J_1X, J_1X) = \lambda = -g'(J_2X, J_2X) = -g'(J_3X, J_3X).
\end{align*}
\]
Hence, \( g(\cdot, \cdot) = \lambda g'(\cdot, \cdot), \ \lambda \neq 0. \)
ii) According to Proposition 2.3 the structure $(J_x, J_y)$ is hermitian with respect to $g$. The statement follows from the uniqueness of $g$ (up to a non-zero scalar). □

**Remark 2.1.** In the light of Lemma 2.1 we see that the notion of null vector $N$ (such that $g(N, N) = 0$) depends only on the hermitian structure $(J_1, J_2)$ and not on a particular inner product.

From the proof of Lemma 2.1 we also obtain the following.

**Lemma 2.2.** If $(J_1, J_2)$ is a is a para-hypercomplex structure on a real 4-dimensional vector space $V$ then

$$(X, J_1 X, J_2 X, J_3 X)$$

is a basis of $V$ if $X$ is not null.

**Lemma 2.3.** If $N_\alpha$ is an endomorphism of a 4-dimensional Lie algebra $g$ such that $J_\alpha^2 = \pm 1$ and $(X, J_\alpha X, Y, J_\alpha Y)$ is a basis of $g$ then the corresponding Nijenhuis tensor $N_\alpha$ vanishes if and only if $N_\alpha(X, Y) = 0$.

**Proof:** One can easily show that $N_\alpha(J_\alpha X, Y) = -J_\alpha N_\alpha(X, Y)$. The lemma follows from the fact that $N_\alpha$ is antisymmetric and bilinear. □

**Lemma 2.4.** Let $(J_1, J_2)$ be a para-hypercomplex structure on a real 4-dimensional vector space $V$ and let $W \subset V$ be a 2-dimensional subspace. Then there exists a compatible para-hypercomplex structure $(J'_1, J'_2)$ such that:

1. If $W$ is definite (contains no null directions) then $J'_1 W = W$.
2. If $W$ is Lorentz (contains exactly two null directions) then $J'_2 W = W$.
3. If $W$ is totally null (every vector in $W$ is a null vector) then either
   a) $J'_1|_W = 1$, $V = W \oplus J'_1 W$, or
   b) there exists a non-null vector $X$ such that $W = R(J'_1 X + J'_2 X, X - J'_3 X)$, $J(W) = W$ for all $J \in g^\pm$.

4. If the induced metric on $W$ is of rank 1 (W contains exactly one null direction $N$) then $N = J'_1 X - J'_2 X$ for any given vector $X \in W$, $|X| /= 0$.

**Proof of i)** and **ii):** Let $(X, Y)$ be a pseudo-orthonormal basis of $W$ ($|X|^2 = -|Y|^2 = 1$ and $(X, Y) = 0$ with respect to the induced inner product on $W$). Then, according to Lemma 2.2 vectors $X$, $J_1 X$, $J_2 X$ and $J_3 X$ form a pseudo-orthonormal basis of $V$ and we have $Y = x_1 J_1 X + x_2 J_2 X + x_3 J_3 X$ with $x_1^2 - x_2^2 - x_3^2 = \pm 1$, where $-$ occurs if $W$ is Lorentz and $+$ if $W$ is positive or negative definite. The structure

$$J_x = x_1 J_1 + x_2 J_2 + x_3 J_3$$

preserves $W$. It is a product structure if $W$ is Lorentz (and we set $J'_2 = J_x$) and a complex structure if $W$ is definite (and we set $J'_1 = J_x$). The second structure can be chosen such that $(J'_1, J'_2)$ forms a compatible para-hypercomplex structure. Note that there cannot exists a product structure preserving a definite $W$ since a product structure is an anti-isometry. Similarly, a complex structure preserving a Lorentz $W$ cannot exist.

**Proof of iii)** Let $N_1 \in W$ be a null vector. There exists a non-null vector $X \in V$ perpendicular to $N_1$. Hence

$$N_1 = \alpha J_1 X + \beta J_2 X + \gamma J_3 X$$

and

$$\alpha^2 - \beta^2 - \gamma^2 = 0$$
so \( \alpha \neq 0 \) and we may assume that \( \alpha = 1 \). Then \( J'_2 = \beta J_2 + \gamma J_3 \) is a product structure, the structure \( (J'_1, J'_2) \), \( J'_1 = J_1 \) is a compatible para-hypercomplex structure and we have

\[
N_1 = J'_1X + J'_2X.
\]

Any null vector \( aX + bJ'_1X + cJ'_2X + dJ'_3X \) which is orthogonal to the vector \( N_1 \) is of the form

\[
N^\pm = aX + bJ'_1X + bJ'_2X \pm aJ'_3X.
\]

Notice that the vector \( N_1 \) is also of the form \( N^\pm \) and that there exist exactly two null planes \( W^\pm \) containing the vector \( N_1 \). They can be written in the form

\[
W^\pm = \mathbb{R}\langle N_1, N_2^\pm = X \pm J'_3X \rangle.
\]

The plane \( W^- \) is the \( +1 \)-eigenspace of the product structure \( J'_3 \) and the vectors \( N_1, N_2^- \), \( J'_1N_1, J'_2N_2^- \) are independent, so \( V = W^- \oplus J'_1W^- \) and iii)a holds.

In the case of the plane \( W^+ \) one easily checks that \( J'_1W^+ = W^+ = J'_2W^+ \) and hence statement iii)b follows.

**Proof of iv)** The proof is similar to the first part of the previous proof (with \( N_1 = N \)). \( \square \)

**Lemma 2.5.** Let \( (J_1, J_2) \) be a para-hypercomplex structure on a real 4-dimensional vector space \( V \) and let \( W \subset V \) be a 3-dimensional subspace such that the induced metric is degenerate. For \( N \in W^\perp \) and \( X \in W, \langle X \rangle \) \( \neq 0 \), there exists a compatible para-hypercomplex structure \( (J'_1, J'_2) \) on \( V \) such that \( N = J'_1X - J'_2X \) and the arbitrary null vector in \( W \) belongs to the union of two-dimensional planes \( \pi_1 = \mathbb{R}\langle N, J'_1N \rangle \) and \( \pi_- = \{ V \mid J'_3V = -V \} \), i.e.

\[
\text{null}(W) = \{ U \in W \mid |U|^2 = 0 \} = \pi_1 \cup \pi_- = \mathbb{R}\langle N, J'_1N \rangle \cup \{ V \mid J'_3V = -V \}.
\]

**Proof:** Since we have \( |N|^2 = 0, \langle X \rangle \neq 0, \langle N, X \rangle = 0 \) the existence of a compatible structure \( (J'_1, J'_2) \) such that \( N = J'_1X - J'_2X \) follows from the Lemma \( \text{2.4 iv).} \)

Moreover, \( \{ N, J'_1N, X, J'_3X \} \) is a basis of \( W \) and \( \{ N, J'_1N, X, J'_2X \} \) is a basis of \( V \). Thus, for \( U \in \text{null}(W) \) of the form \( U = \alpha N + \beta J'_1N + \gamma X \) we get

\[
0 = |U|^2 = \gamma (\gamma - 2\beta)|X|^2.
\]

The case \( \gamma = 0 \) gives the plane \( \pi_1 = \mathbb{R}\langle N, J_1N \rangle \). For \( \gamma = 2\beta \) one can check that \( J'_3(U) = -U \), so \( U \) belongs to the \(-1\) eigenspace of \( J'_3 \). \( \square \)

3. Lie algebras admitting a para-hypercomplex structure

### 3.1. Case when \( g \) has a non-trivial center.

In the following theorem the additive basis of the Lie algebra \( g \) is either \( \langle X, Y, Z, W \rangle \) or \( \langle X, Y, N_1, N_2 \rangle \). The vectors \( N_\alpha \) are null vectors.

**Theorem 3.1.** A 4-dimensional Lie algebra \( g \) admitting a para-hypercomplex structure and with a non-trivial center \( Z(g) \) is one of algebras \( \text{PHC}1-\text{PHC}6 \).

As a consequence of Levi decomposition theorem and the classification of real semisimple Lie algebras the only non-solvable Lie algebras which are 4-dimensional are \( \mathbb{R} \oplus \text{so}(3) \) and \( \mathbb{R} \oplus \text{sl}_2(\mathbb{R}) \). Since they both have a non-trivial center, as a consequence of Theorem \( 3.1 \) we have the following corollary.

**Corollary 3.1.** The only non-solvable, real 4-dimensional Lie algebra admitting a para-hypercomplex structure is \( \mathbb{R} \oplus \text{sl}_2(\mathbb{R}) \).
Proof of Theorem 3.1: In order to prove that these are the only Lie algebras with non-trivial center which admit a para-hypercomplex structure we consider two cases.

Case 1: there exists a non-null central element $Z$. Let $(J_1, J_2)$ be a para-hypercomplex structure on $g$ and denote

$X = J_1 Z, \ Y = J_2 Z, \ W = J_3 Z.$

Then

$$[X, Y] = aZ + bX + cY + dW.$$  \hspace{1cm} (5)

According to Lemma 2.3 integrability of $J_1$ is equivalent to

$$0 = N_1(Z, Y) = [X, W] - J_1[X, Y].$$  \hspace{1cm} (6)

Similarly, the integrability of $J_2$ is equivalent to

$$0 = N_2(X, Z) = [Y, W] - J_2[X, Y].$$  \hspace{1cm} (7)

From the relations (5), (6) and (7) we get

$$[X, W] = -bZ + aX - dY + cW, \ [Y, W] = cZ - dX + aY - bW.$$  \hspace{1cm} (8)

The Jacobi identity is equivalent to

$$0 = [\ [X, Y], W] + [\ [Y, W], X] + [\ [W, X], Y] = 2(-a^2 - b^2 + c^2)Z - 2cdX - 2dbY - 2adW.$$  \hspace{1cm} (9)

If $a = b = c = d = 0$ then the algebra $g$ is abelian, i.e. PHC1. If $a = b = c = 0$ and $d \neq 0$ then after scaling $g \cong R \oplus sl_2(R)$, i.e. PHC2.

If $d = 0$ and $0 \neq c^2 = a^2 + b^2$ then the derived algebra $g' = [g, g]$ of $g$ is 2-dimensional since

$$c[Y, W] = a[X, Y] + b[W, X]$$

It is generated by the vectors $W_1 = [X, Y], \ Y_1 = [W, X]$. The vectors $Z, X_1 = \frac{1}{c} X, \ Y_1$ and $W_1$ are linearly independent and we get algebra PHC3.

Case 2: all central vectors are null vectors. Denote one of them by $N$. According to Lemma 2.4 iv), we can assume that $N = J_1 X - J_2 X$ for a non-null vector $X \in g'$. Then the vectors $N, J_1 N, X$ and $J_1 X$ form a basis of $g$ and the structure $J_2$ expressed in the terms of that basis reads

$$J_2 X = J_1 X - N, \ J_2 J_1 N = N, \ J_2 J_1 X = J_1 N + X, \ J_2 N = J_1 N.$$  \hspace{1cm} (10)

The integrability of the structure $J_1$ gives the following conditions

$$0 = N_1(X, N) = [J_1 X, J_1 N] - J_1[X, J_1 N].$$  \hspace{1cm} (11)

Since the vectors $N, J_2 N, X$ and $J_2 X$ form a basis of $g$, the integrability of the product structure $J_2$ is equivalent to

$$0 = N_2(X, N) = [J_1 X, J_1 N] - J_2[X, J_1 N].$$  \hspace{1cm} (12)

The vector $[X, J_1 N]$ is of the form $[X, J_1 N] = aN + bJ_1 N + cX + dJ_1 X$. Using the relations (11) and (12) we get that

$$X, J_1 N = aN + bJ_1 N + 2bX, \ [J_1 X, J_1 N] = -bN + aJ_1 N + 2bJ_1 X.$$
If we write \([X, J_1 X] = \alpha N + \beta J_1 N + \gamma X + \delta J_1 X\) and impose the Jacobi identity on the vectors \(J_1 N, X\) and \(J_1 X\) we get the following system of equations:

\[-4ab - b^2 - \delta b + \gamma a - a^2 = 0,\]
\[-4b\beta + a\delta + b\gamma = 0,\]
\[b(a + \gamma) = 0,\]
\[b(b - \delta) = 0.\]

The system has three classes of solutions.

1) \(a = 0 = b\). In this case the only non-zero commutator is

\([X, J_1 X] = \alpha N + \beta J_1 N + \gamma X + \delta J_1 X.\)

If \(\gamma = 0 = \delta\), the change of the basis \(Y = J_1 X, N_1 = \alpha N + \beta J_1 N, N_2 \in \mathbb{R}\langle N, J_1 N \rangle\) gives the relations PHC4. If \(\delta \neq 0\) then the change \(Y = \frac{1}{\delta}[X, J_1 X], N_1 = N, N_2 = J_1 N\) gives the relations PHC5. The case \(\delta = 0, \gamma \neq 0\) similarly reduces to the relations PHC5.

2) \(b = \delta \neq 0, a = -\gamma\). This case reduces to the relations PHC3.

3) \(a = \gamma \neq 0\). This immediately gives the commutator relations PHC6. \(\square\)

3.2. Case of solvable Lie algebra \(g\) and \(\dim g' \leq 2\).

**Theorem 3.2.** Let \(g\) be a 4-dimensional real Lie algebra admitting a para-hypercomplex structure and \(\dim g' = 1\). Then \(g\) is one of the algebras PHC1, PHC2 from Theorem 3.1.

**Proof:** If \(g\) has a non-trivial center \(\xi\) then from Theorem 3.1 we get the algebras PHC1 and PHC2. Now, as in [2], Proposition 3.2, let \(\xi = \{0\}\) and let \(X\) be a non-zero element of \(g'\). There exists \(Y\) such that \([Y, X] = X\). Then \(g\) decomposes as

\[g = \ker(\text{ad}_X) \cap \ker(\text{ad}_Y) \oplus \mathbb{R}X \oplus \mathbb{R}Y.\]

From the Jacobi identity we get that \(\xi = \ker(\text{ad}_X) \cap \ker(\text{ad}_Y),\) a contradiction. Hence solvable \(g\) without center and with \(\dim g' = 1\) does not exist (this does not depend on the existence of para-hypercomplex structure). \(\square\)

**Theorem 3.3.** Let \(g\) be a 4-dimensional solvable Lie algebra admitting a para-hypercomplex structure and with \(\dim g' = 2\). If \(g\) has a non-trivial center then \(g\) is algebra PHC2. If \(g\) has a trivial center then \(g\) is one of algebras PHC7-PHC9.

**Remark 3.1.** Using the notation introduced by Snow [5], these Lie algebras are S11, S8 and S10 respectively. The class S11 contains as a special case the Lie algebra \(\text{aff}(\mathbb{C})\) which is the unique solvable Lie algebra with 2-dimensional derived algebra which admits hypercomplex structure [2].

**Proof:** Suppose that the center of \(g\) is trivial and that \((J_1, J_2)\) is a para-hypercomplex structure on \(g\). According to Lemma 2.4 and Remark 2.1 the structure \((J_1, J_2)\) determines the inner product on \(g = V\) and the notion of a null vector. As in Lemma 2.4 we have to consider the cases concerning the rank and the signature of the induced inner product on \(g' = W\).

**Case i):** Induced metric on \(g'\) is definite. Because of Lemma 2.4 i) we may assume that \(g'\) is invariant with respect to the complex structure \(J_1, J_2g' = g',\) and \(g = g' \oplus J_2g'.\) Let \(\{X, J_1 X = Y\}\) be a basis of \(g'\) and \(\{X, Y, J_2X, J_2Y\}\) be a basis
Since dim $g$ and we get algebra PHC7 for $a$ and hence
\begin{equation}
[J_2 X, J_2 Y] = 0, \quad [J_2 X, Y] = [J_2 Y, X].
\end{equation}
Because of the integrability of the complex structure $J_1$, $N_1(X, J_2 X) = 0$ and
\begin{equation}
[X, J_2 X] = -[Y, J_2 Y].
\end{equation}
For arbitrary vectors $V$ and $W$ in $g$,
\begin{equation}
[V, W] = \alpha(V, W)X + \beta(V, W)Y,
\end{equation}
where $\alpha$ and $\beta$ are skew-symmetric bilinear forms on $g$. From the Jacobi identity we have
\begin{equation}
\alpha(X, J_2 X) = \beta(X, J_2 Y), \quad \alpha(J_2 Y, X) = \beta(X, J_2 X)
\end{equation}
and the bracket in $g$ is determined by $c = \alpha(X, J_2 X)$ and $d = \beta(X, J_2 X)$ as follows:
\begin{equation}
[X, J_2 X] = -[Y, J_2 Y] = cX + dY, \quad [X, J_2 Y] = [Y, J_2 X] = -dX + cY.
\end{equation}
Since dim $g' = 2$, $c^2 + d^2 \neq 0$ and we may choose
\begin{align*}
\tilde{X} &= (c^2 + d^2)^{-1}(cX + dY), \\
\tilde{Y} &= (c^2 + d^2)^{-1}(-dX + cY), \\
\tilde{Z} &= (c^2 + d^2)^{-1}(cJ_2 X - dJ_2 Y), \\
\tilde{W} &= (c^2 + d^2)^{-1}(dJ_2 X + cJ_2 Y),
\end{align*}
and hence
\begin{equation}
[\tilde{X}, \tilde{Z}] = \tilde{X}, \quad [\tilde{Y}, \tilde{Z}] = \tilde{Y}, \\
[\tilde{Y}, \tilde{W}] = -\tilde{X}, \quad [\tilde{Y}, \tilde{W}] = -\tilde{X},
\end{equation}
so we get the algebra PHC7 for $a = -1, b = 0$. Note that $g = \text{aff}(\mathbb{C})$.

**Case ii):** Induced metric on $g'$ is indefinite, of Lorentz type $(-+)$. Because of Lemma 2(ii) we may assume that $g'$ is invariant with respect to the product structure $J_2$, $J_2 g' = g'$, and $g = g' \oplus J_1 g'$. Let $\{X, J_2 X = Y\}$ be a basis of $g'$ and $\{X, Y, J_1 X, J_1 Y\}$ be a basis of $g$. By the integrability of the complex structure $J_1$, $N_1(X, Y) = 0$ and
\begin{equation}
[J_1 X, J_1 Y] = 0, \quad [J_1 X, Y] = [J_1 Y, X].
\end{equation}
Because of the integrability of the product structure $J_2$, $N_2(X, J_1 X) = 0$ and
\begin{equation}
[X, J_1 X] = [Y, J_1 Y].
\end{equation}
From the Jacobi identity we have
\begin{equation}
\alpha(X, J_1 X) = \beta(X, J_1 Y), \quad \alpha(J_1 Y, X) = -\beta(X, J_1 X),
\end{equation}
and the bracket in $g$ is determined by $c = \alpha(X, J_1 X)$ and $d = \beta(X, J_1 X)$ as follows:
\begin{equation}
[X, J_1 X] = [Y, J_1 Y] = cX + dY, \quad [X, J_1 Y] = [Y, J_1 X] = dX + cY.
\end{equation}
Since dim $g' = 2$, $c^2 - d^2 \neq 0$ and we may choose
\begin{align*}
\tilde{X} &= (c^2 - d^2)^{-1}(cX + dY), \\
\tilde{Y} &= (c^2 - d^2)^{-1}(-dX + cY), \\
\tilde{Z} &= (c^2 - d^2)^{-1}(cJ_1 X - dJ_1 Y), \\
\tilde{W} &= (c^2 - d^2)^{-1}(dJ_1 X + cJ_1 Y),
\end{align*}
and hence
\begin{equation}
[\tilde{X}, \tilde{Z}] = \tilde{X}, \quad [\tilde{Y}, \tilde{Z}] = \tilde{Y}, \\
[\tilde{Y}, \tilde{W}] = -\tilde{X}, \quad [\tilde{Y}, \tilde{W}] = -\tilde{X},
\end{equation}
and we get algebra PHC7 for $a = 1, b = 0$. 

4. FOUR-DIMENSIONAL LIE ALGEBRAS WITH A PARA-HYPERCOMPLEX STRUCTURE
Case iii): $g'$ is a totally null plane. According to Lemma 2.4 iii) we have to consider two geometrically different cases.

In the first case we can assume that $J_2|g'| = 1$ and $g = g' + J_1g'$ holds. If $(X, Y)$ is a basis of $g'$ we have

$$J_2 X = X, \quad J_2 Y = Y, \quad J_2 J_1 X = -J_1 X, \quad J_2 J_1 Y = -J_1 Y.$$ 

One easily checks that the integrability of the complex structure $J_1$ is equivalent to the relations

$$[J_1 X, J_1 Y] = 0, \quad [X, J_1 Y] = [Y, J_1 X].$$ 

It is interesting that the product structure $J$ with respect to $J_1$ or equivalently

$$(16) \quad (e - d)X' + fY' - cT' = 0, \quad (a - f)X' + bY' - cT' = 0,$$

or equivalently

$$e(e - d) + c(f - a) = 0, \quad ef = bc, \quad af - f^2 + bd - be = 0.$$ 

If $X'$ is a zero vector then we get the algebra PHC8. Suppose that $X'$ is a non-zero vector. If $Y'$ or $T'$ is a zero vector then we get an algebra PHC7 for $a = 0 = b$.

Suppose that none of the vectors $X', Y', Z'$ is the zero vector. We can suppose that one of the pairs $X', Y'$ and $X', T'$ is independent, say $X', T'$. If the vectors $X'$ and $Y'$ are collinear then we get the algebra PHC7 for $a = 0, b = 1$. Finally, if the both pairs $X', T'$ and $X', Y'$ are independent then introduce a new basis $(X', Y', Z', W')$ satisfying

$$Z' = \frac{1}{D}(fJ_1 X - bJ_1 Y), \quad W' = \frac{1}{D}(-eJ_1 X + aJ_1 Y),$$

where $D = af - be \neq 0$. In the new basis the commutator relations take the very simple form

$$[X', Z'] = X', \quad [X', W'] = Y', \quad [Y', Z'] = Y', \quad [Y', W'] = \frac{fc - de}{D}X' + \frac{ad - be}{D}Y'.$$

Since $X'$ and $Y'$ are independent then $cf - de \neq 0$, that is, $a \neq 0$ in the algebra PHC7.

In the second case we can assume that $(N_1, N_2)$ is a basis of $g'$ and $g'$ is invariant with respect to $J_1$, $J_2$, $J_3$. Then a possible basis of $g$ is

$$N_1 = J_1 X + J_2 X, \quad N_2 = X - J_3 X, \quad N_3 = J_1 X - J_2 X, \quad N_2 = X + J_3 X.$$ 

We calculate the structures in terms of that basis:

$$J_1 N_1 = -N_2, \quad J_1 N_3 = -N_4, \quad J_2 N_1 = N_2, \quad J_2 N_3 = -N_4, \quad J_3 N_1 = N_1, \quad J_3 N_2 = -N_2, \quad J_3 N_3 = -N_3, \quad J_3 N_4 = N_4.$$ 

By the integrability of $J_3$,

$$J_3[N_1, N_4] = [N_1, N_4], \quad J_3[N_2, N_3] = -[N_2, N_3].$$
Thus,
\[ [N_1, N_4] = \mu N_1, \quad [N_2, N_3] = \lambda N_2. \]
The integrability of \( J_1 \) and \( J_2 \) is equivalent to
\[ 0 = -[N_2, N_4] - \lambda N_1 + \mu N_2 + [N_1, N_3] \]
After imposing the Jacobi identity this reduces to the algebra PHC3.

**Case iv):** the induced metric on \( g' \) is of rank 1. Denote by \( N \) the null vector belonging to \( g' \) (which is unique up to a scaling constant).

According to Lemma (2.3) iv) we can choose a product structure \( J_2 \) such that for the basis \((X, N)\) of \( g' \) one has
\[ N = J_1 X - J_2 X, \quad N \text{ is null.} \]
Then \((X, N, J_1 X, J_1 N)\) is a basis of \( g \). One easily calculates the following relations
\[ J_2 X = J_1 X - N, \quad J_2 N = J_1 N. \]
The integrability of \( J_1 \) is equivalent to \( J_1 [X, N] = 0 \), i.e. to the relations
\[ [J_1 X, J_1 N] = 0, \quad [X, J_1 N] = [N, J_1 X]. \]
Since \((X, N, J_2 X, J_2 N)\) is a basis of \( g \) the integrability of the product structure \( J_2 \) is equivalent to \( J_2 [X, N] = 0 \) which gives the condition
\[ [N, J_1 N] = 0. \]
The commutator relations now read
\[ [X, J_1 X] = a X + b N, \quad [X, J_1 N] = c X + d N, \]
where \( a, b, c, d \) are unknown coefficients. The Jacobi identity is now equivalent to the following relations
\[ c = 0, \quad d(a - d) = 0. \]
The case \( d = 0 \) gives the algebra with \( \dim g' = 1 \) which we have already discussed. The remaining case \( a = d \neq 0 \), after the change
\[ \tilde{Y} = N, \quad \tilde{Z} = J_1 N, \quad \tilde{X} = \frac{1}{a} X, \quad \tilde{W} = \frac{1}{a} J_1 X - \frac{b}{a^2} J_1 N, \]
takes the form
\[ [\tilde{Y}, \tilde{Z}] = 0, \quad [\tilde{Y}, \tilde{W}] = \tilde{Y}, \quad [\tilde{X}, \tilde{Z}] = \tilde{Y}, \quad [\tilde{X}, \tilde{W}] = \tilde{X} \]
of the algebra PHC7 for \( a = 0 = b \).

### 3.3. Case of solvable Lie algebra \( g \) with \( \dim g' = 3 \).

**Theorem 3.4.** Let \( g \) be a 4-dimensional solvable Lie algebra admitting a para-hypercomplex structure and with \( \dim g' = 3 \). If \( g \) has a nontrivial center it is algebra PHC6, otherwise it is algebra PHC9 or PHC10.

**Proof:** If the algebra \( g \) is solvable then its derived algebra \( g' \) is nilpotent. Up to isomorphism the only 3-dimensional nilpotent Lie algebras are Abelian algebra and the Heizenberg algebra generated by \( X, Y \) and \( Z \) with nonzero commutator
\[ [X, Y] = Z. \]
Let \( g \) be with trivial center, admitting a para-hypercomplex structure \((J_1, J_2)\) and let \((\cdot, \cdot)\) be a compatible inner product on \( g \). First, we discuss the case of \( g' \) being abelian.
Suppose that $\mathfrak{g}'$ is nondegenerate subspace and $X$ is normal vector of $\mathfrak{g}'$. Then $|X|^2 \neq 0$ and $\mathfrak{g}' = \mathbb{R}\langle J_1 X, J_2 X, J_3 X \rangle$. From the integrability of $J_1$ and $J_2$ we have

$$[X, J_\alpha J_\beta X] = J_\alpha [X, J_\beta X],$$

for $\alpha, \beta \in 1, 2, 3$, $\alpha \neq \beta$. Hence, $[X, J_\alpha X] = \lambda J_\alpha$, and we get the algebra PHC9 for $a = 0 = b$. (the Lie algebra corresponding to the real hyperbolic spaces).

Assume now that $\mathfrak{g}'$ is degenerate subspace and $N$ is normal vector of $\mathfrak{g}'$. Then $|N|^2 = 0$ and $N \in \mathfrak{g}'$. According to Lemma 2.4 iv) we can choose a compatible structure $(J_1, J_2)$ such that $N = J_1 X - J_2 X$ for any $X \in \mathfrak{g}'$, $|X|^2 \neq 0$. Since $J_1 N$ is orthogonal to $N$ we also have $J_1 N \in \mathfrak{g}'$. Hence we may suppose that $\mathfrak{g}' = \mathbb{R}\langle N, J_1 N, X \rangle$. Moreover the $(N, J_1 N, X, J_1 X)$ is a basis of $\mathfrak{g}$. The integrability of $J_1$ and $J_2$ implies

$$[J_1 N, J_1 X] = J_1 [N, J_1 X] = J_2 [N, J_1 X],$$

i.e. $[N, J_1 X] = dN$ and $[J_1 N, J_1 X] = dJ_1 N$, $d \neq 0$ what after scaling reduces to algebra PHC9.

Now we turn to the case when $\mathfrak{g}'$ is Heizenberg algebra. Let $\mathfrak{g}' = \mathbb{R}\langle X, Y, Z \rangle$ and $\mathfrak{g} = \mathbb{R}\langle X, Y, Z, W \rangle$. One can easily check that the center $\mathbb{R}\langle Z \rangle$ is an ideal of $\mathfrak{g}$ and hence

$$[W, Z] = \lambda Z, \; \lambda \neq 0,$$

no matter how the vector $W$ that does not belong to $\mathfrak{g}'$ is chosen. At the other side, independently of the choice of non-central vectors $X, Y \in \mathfrak{g}'$ their commutator is always in the center, i.e.

$$[X, Y] = \mu Z, \; \mu \neq 0.$$

Here, $\mu \neq 0$ since $\mathfrak{g}'$ is not abelian and $\lambda \neq 0$ since otherwise $Z$ would be a non-zero central element of $\mathfrak{g}$. Hence, it remains to calculate the commutators $[W, X]$ and $[W, Y]$. This approach we use to prove the remaining part of the theorem.

We consider the cases depending on degeneracy of $\mathfrak{g}'$ with respect to the induced compatible metric. Also there are different subcases depending on the norm of a central element of $\mathfrak{g}'$.

i) Suppose that $\mathfrak{g}'$ is not degenerated, and let $W$ be its normal vector. Denote by $Z = \xi(\mathfrak{g}')$ a non-zero central element of $\mathfrak{g}'$. As an element of $\mathfrak{g}'$, $Z$ is orthogonal to $W$. Now we have the following cases.

**W and Z have the same sign:** Using the Lemma 2.4 i) we may choose a compatible structure $(J_1, J_2)$ such that $Z = J_1 W$. Then the $(J_1 W, J_2 W, J_3 W)$ is a basis of $\mathfrak{g}'$. After a simple calculation (and scaling) we get the commutator relation:

$$[W, J_1 W] = 2J_1 W, \; [W, J_2 W] = J_2 W, \; [W, J_3 W] = J_3 W, \; [J_2 W, J_3 W] = J_1 W.$$

That is a special form of algebra PHC10.

**W and Z have the opposite sign:** Using Lemma 2.4 ii) we may choose a compatible structure $(J_1, J_2)$ such that $Z = J_2 W$. Then the $(J_1 W, J_2 W, J_3 W)$ is a basis of $\mathfrak{g}'$. After a simple calculation (and scaling) we get the commutator relation:

$$[W, J_1 W] = J_1 W, \; [W, J_2 W] = 2J_2 W, \; [W, J_3 W] = J_3 W, \; [J_1 W, J_3 W] = J_2 W.$$

That is again a special form of algebra PHC10.
The center $Z$ of $g'$ is a null vector: We have: $|W|^2 \neq 0$, $|Z|^2 = 0$, $Z \perp X$, so using the Lemma 2.4 iv) we may choose a structure $(J_1, J_2)$ such that
$$N = Z = J_1W - J_2W.$$ Moreover there is a decomposition
$$g = g' \oplus RW = \mathbb{R} \langle N, J_1W, J_3W \rangle \oplus RW.$$ Now we have
$$[J_1W, J_3W] = \lambda N, \ [W, N] = \mu N, \ \lambda, \mu \neq 0.$$ After imposing the integrability condition for the structure $(J_1, J_2)$ we get $\mu = 0$ what is a contradiction. Hence, this case does not give a solution.

ii) Suppose that $g'$ is degenerated, and let $N \in g'$ be its normal vector and $Z \in g'$, a non-zero central element of $g'$. We now discuss cases depending on the type of vector $Z$.

$Z$ is a non null vector, $|Z|^2 \neq 0$: Let $X = Z$. Consider the basis:
$$g = \mathbb{R} \langle N, J_1N, X, J_1X \rangle, \ g' = \mathbb{R} \langle N, J_1N, X \rangle.$$ Let $[N, J_1N] = \mu X$ and $[J_1X, X] = \lambda X$. Then
$$(J_1 - J_2)[N, J_1X] = -\mu X.$$ Thus, $\mu = 0$ and $g'$ is Abelian, what is again a contradiction. \ \Box

$Z$ is a null vector, $|Z|^2 = 0$: According to Lemma 2.3 all null vectors of $g'$ are contained in two 2-dimensional planes:
$$\text{null}(g') = \pi_1 \cup \pi_2 = \mathbb{R} \langle N, J_1N \rangle \cup \{V|J_3V = -V\}.$$ We now study three possible cases $Z = N, Z \in \pi_2, Z \in \pi_1$.

$Z = N$ (the normal to $g'$ is a center of $g'$): Then we have a decomposition:
$$g = \mathbb{R} \langle N, J_1N, X, J_1X \rangle, \ g' = \mathbb{R} \langle N, J_1N, X \rangle.$$ Because of the integrability of para-hypercomplex structure $(J_1, J_2)$ we have
$$[J_1N, X] = \lambda N, \ [J_1X, N] = \mu N, \ [J_1X, X] = aN + bJ_1N + cX, \ \lambda, \mu \neq 0.$$ The Jacobi identity is equivalent to $c = \lambda$. After some scaling we get the algebras PHC10.

$Z \in \pi_2, Z \neq N, (Z$ is $\lambda$ eigenvector of $J_3$). Then $Z = aN + b(J_1N + 2X)$ and we have the decomposition:
$$g = \mathbb{R} \langle N, J_1N, Z, J_1Z \rangle, \ g' = \mathbb{R} \langle N, J_1N, Z \rangle.$$ Due to the Heisenberg algebra structure of $g'$ we may assume
$$[Z, J_1Z] = \mu Z, \ [N, J_1N] = \lambda Z, \ \mu, \lambda \neq 0.$$ Because of the integibility of $J_1$ and $J_2$ we have
$$[J_1N, J_1Z] = J_1[N, J_1Z] = J_2[N, J_1Z],$$ and then
$$[N, J_1Z] = aN, \ \text{and} \ [J_1N, J_1Z] = \alpha J_1N, \alpha \neq 0.$$ Now, by the Jacobi identity,
$$[N, J_1Z] = \alpha N, \ \ [Z, J_1Z] = 2\alpha Z,$$
$$[J_1N, J_1Z] = \alpha J_1N, \ \ [Y, X] = \lambda Z,$$
$\alpha, \lambda \neq 0$. After scaling it is a special case of relations PHC10.
\( Z \in \pi_1, Z = aN + J_1N, a \in \mathbb{R} \). Consider the decomposition
\[
g = \mathbb{R}\langle N, Z, X, J_1 \rangle, \quad g' = \mathbb{R}\langle N, Z \rangle.
\]
Let \([N, X] = \mu Z\) and \([J_1X, Z] = \lambda Z\). By the integrability,
\[
(J_1 - J_2)[N, J_1X] = 2\lambda Z - 2\mu aN,
\]
what implies \(\lambda = 0\), i.e. \(Z\) is in the center of \(g\). That is a contradiction.

3.4. The proof of Theorem 1.1 According to the Levi decomposition theorem every Lie algebra \(g\) decomposes into direct sum
\[
g = \mathfrak{r} \oplus \mathfrak{s},
\]
where \(\mathfrak{r}\) is maximal solvable ideal (radical) and \(\mathfrak{s}\) is semisimple part. Since \(\mathfrak{so}(3)\) and \(\mathfrak{sl}_2(\mathbb{R})\) are the only semisimple Lie algebras of dimension less or equal to 4, the only non-solvable Lie algebras of dimension four are
\[
\mathbb{R} \oplus \mathfrak{so}(3) \quad \text{and} \quad \mathbb{R} \oplus \mathfrak{sl}_2(\mathbb{R}).
\]
They both have a non-trivial center \(\mathbb{R}\), so from Theorem 3.1 we conclude that the unique non-solvable Lie algebra admitting a para-hypercomplex structure is \(\mathbb{R} \oplus \mathfrak{so}(3)\), i.e. PHC2. Solvable 4-dimensional Lie algebras with nontrivial center and admitting a para-hypercomplex structure are PHC1 and PHC3-PHC6 (Theorem 3.4). Solvable 4-dimensional Lie algebras with trivial center and admitting a para-hypercomplex structure are PHC7-PHC10 (theorems 3.2, 3.3 and 3.4).

It remains to prove that algebras PHC1-PHC10 possess an integrable para-hypercomplex structure. We construct the structures below and leave the reader to check the integrability conditions for \(J_1\) and \(J_2\) and the relations (1) by direct calculation.

PHC1 and PHC2:
\[
J_1Z = X, \quad J_1Y = W, \quad J_2Z = Y, \quad J_2X = -W.
\]

PHC3:
\[
J_1Z = X, \quad J_1Y = W, \\
J_2Z = W - Z, \quad J_2X = X + Y, \quad J_2Y = -Y, \quad J_2W = W.
\]

PHC4 and PHC5:
\[
J_1Z = W, \quad J_1X = Y, \\
J_2Z = W, \quad J_2X = Y - Z, \quad J_2Y = X + W.
\]

PHC7
\[
J_1X = Z, \quad J_1Y = W, \\
J_2X = X, \quad J_2Y = Y, \quad J_2Z = -Z, \quad J_2W = -W
\]

PHC8:
\[
J_1X = -Y, \quad J_1Z = -W, \quad J_2X = Y, \quad J_2Z = -W
\]

PHC6, PHC9 and PHC10:
\[
J_1Z = Y, \quad J_1X = W, \\
J_2Z = Y, \quad J_2X = W - Z, \quad J_2W = X + Y.
\]
\[\square\]
In this section we compare our results with the classification of hypercomplex structures in the paper of Barberis [2]. We see that there are many more 4-dimensional Lie algebras with para-hypercomplex structure than Lie algebras with hypercomplex structure.

Namely, we have the following.

**Theorem 4.1.** (2) The only 4-dimensional Lie algebras admitting an integrable hypercomplex structure are:

- (HC1) \( \mathfrak{g} \) is abelian,
- (HC2) \([X,Y] = W, [Y,W] = X, [W,X] = Y\),
- (HC3) \([X,Z] = X, [X,W] = Y, [Y,Z] = Y, [Y,W] = -Y, [W,X] = Y\),
- (HC4) \([W,X] = X, [W,Y] = Y, [W,Z] = Z\),
- (HC5) \([W,X] = X, [W,Y] = \frac{1}{2}Y, [W,Z] = \frac{1}{2}Z, [Z,Y] = X\).

The Lie algebra HC2 is isomorphic to \( \mathbb{R} \oplus \mathfrak{so}(3) \) and it does not admit a para-hypercomplex structure. Its counterpart admitting a para-hypercomplex (but not hypercomplex) structure is algebra \( \mathbb{R} \oplus \mathfrak{sl}(2) \) given by the relations PHC2.

No algebra \( \mathfrak{g} \) with \( \dim \mathfrak{g}' = 1 \) admits a hypercomplex structure, while algebras PHC4 and PHC5 admit a para-hypercomplex structure and satisfy \( \dim \mathfrak{g}' = 1 \).

The Lie algebra HC3 is isomorphic to \( \mathfrak{aff}(\mathbb{C}) \) and it is the only Lie algebra with \( \dim \mathfrak{g}' = 2 \) admitting a hyper-complex structure. It also admits a para-hypercomplex structure (PHC7 for \( a = 1, b = -1 \)).

The Lie algebra HC4 corresponds to real hyperbolic space \( \mathbb{R}H^4 \). It admits both hypercomplex and para-hypercomplex structure (PHC9 for \( a = 0, b = 0 \)).

Finally, the Lie algebra HC5 corresponds to complex hyperbolic space \( \mathbb{C}H^2 \). It admits both hypercomplex and para-hypercomplex structure (PHC10 for \( c = 1, a = b = 0 \)).

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