Power Systems Dynamic State Estimation With the Two-Step Fault Tolerant Extended Kalman Filtering

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ABSTRACT Bad data may lead to performance degradation or even instability of a power system, which can be caused by various factors: unintentional PMU abnormalities, topology error, malicious cyber-attacks, electromagnetic interference, temporary loss of communication links, external disturbances, extraneous noise biases, etc. In order to develop a more resilient and reliable state estimation technique, this manuscript presents a novel two-step fault tolerant extended Kalman filter framework for discrete-time stochastic power systems, under bad data, PMU failures, external disturbances, extraneous noise, and bounded observer-gain perturbation conditions. The failure mechanisms of multiple phasor measurement units are assumed to be independent of each other with various bad data or malfunction rates. The benchmark IEEE standard test systems are utilized as a demonstrative example to carry out computer simulation studies and to examine different estimation algorithms. Experimental results demonstrates that the proposed second-order fault tolerant extended Kalman filter provides more accurate estimation results, in comparison with traditional first- and second-order extended Kalman filter, and the unscented Kalman filter. The proposed two-step fault-tolerant extended Kalman filter can serve as a powerful alternative to the existing dynamic power system state estimation techniques.

INDEX TERMS Dynamic state estimation, phasor measurement unit, extended Kalman filtering, bad data, sensor failures.

I. INTRODUCTION
Modern smart grid has been envisioned to improve the robustness, efficiency of the traditional power grid with the advancement of power electronics, computing, control and communication technologies. It is enabled by the latest advancements in real-time measurement, sensing and control devices, capable of two-way communications among independent system operator (ISO), power generation, transmission, distribution, and loads parts of power grid by interchanging information about the grid states to consumers, operators, automated devices. Hence, dynamic state estimation (DSE) is essential technology in establishing modern real-time models of power grid in energy management centers (EMC). DSE is also the enabling technology for developing real-time control schemes for power grid, such as through actions on FACTS devices, wide-area power system stabilizers, though which the overall power grid performance and stability can be enhanced.

Typically, the following types of data may be utilized for power system state estimation:

- The active power and reactive power of various devices including the load, generator, transformers, etc.
- The voltage phasors at power system buses from PMUs
- The current phasors from PMUs
- The frequency measurements
- The on/off status of power switching devices, such as circuit breakers, reclosers, power converters, and transformers taps, which governs the topology of the entire power system
- Zero injections

Due to the dynamic nature of modern power grid, and thanks to the technology maturity and widespread deployment of wide-area synchrophasor measurement units, latest advancements in power system estimation are mostly focused on dynamic state estimation approaches. Over the past 50 years, the celebrated Kalman filtering have been...
applied to various industrial applications involving dynamic state estimation. The original work of celebrated Kalman filter can be found in [1], [2]. The first-order extended Kalman filter, also known as the quasilinear Kalman filter, have been reported in [3]–[6]. Among them, the stochastic stability of extended Kalman filter has been investigated by Reif, Gunther, Yaz, Unbehauen in [5] and [6]. The second-order extended Kalman filter was proposed in the paper written by Athens, Wishner, and Bertolini in [7].

Recent development of the Kalman filtering-based dynamic power system state estimation can be found in literature [8]–[26]. In [8], Huang, Schneider and Nieplocha concluded that there is a promising path to apply Kalman filtering-based dynamic state estimation together with emerging PMU measurement technologies. Motivated by this feasibility study, Fan and Wehbe in [9] proposed an extended Kalman filter (EKF)-based real-time dynamic state and parameter estimation utilizing PMU measurement data. To further improve the performance of EKF-based dynamic state estimation, Ghahremani and Kamwa considered a modified extended Kalman filter to cope with the situations, where the field voltage \( E_{fd} \) is not directly accessible to metering due to brushless excitation systems in [10]. This work was later on been improved by considering the unscented Kalman filter (UKF) in [11], and been even further extended with the development of a decentralized scheme for synchronous generator states estimation to enable the wide-area power system stabilizer (WA-PSS) and system integrity protection scheme (SIPS) in [12].

Among various improved version of EKF, the iterated EKF (IEKF) and unscented Kalman filter (UKF) have gain much popularity. IEKF linearizes the nonlinear power system equation iteratively, to compensating for higher-order terms of Taylor series expansion, to achieve better estimation performance. And UKF leverages the unscented transformation to characterize the probability density function through a finite-set of sigma points, without truncations of Taylor series terms, in order to achieve a better performance. The claimed advantages of UKF are that it propagates the first two moments of the posterior distribution, and it does not require gradient computation of the system nonlinear dynamics. A more detailed relations between EKF and UKF has been summarized in [13]. [11] considered applying UKF-based dynamic state estimator for a single-machine infinite bus power system. A centralized UKF estimator was developed for a multi-machine power system in [14], while a decentralized UKF scheme was investigated in [15] in order to reduce the computational complexity and easier to be implemented. A comparative study of EKF and UKF leveraged for power system dynamic state estimation and load modeling were studied in [16].

However, it has also been demonstrated that the performance EKF, IEKF and UKF is significantly degraded or even become unstable in the presence of bad data injection to the power systems [17]. Due to the importance of dynamic state estimation, the negative effect of injecting bad data to power systems have also been studied in literature [18]–[26]. It is not uncommon that PMU measurements do not contain accurate signals, but corrupted signals, commonly referred to as bad data, due to unintentional PMU abnormalities, topology error, malicious cyber-attacks, electromagnetic interference, temporary loss of communication links, external disturbances, extraneous noise biases, delays, attenuation, distortions, multi-paths, etc. Bad data may severely degrade the overall power system performance and stability, robust state estimation is of great importance in modern dynamic and data-intensive smart grid applications.

To mitigate the effect of bad data, Hounkpevi and Yaz investigated a state estimator for nonlinear stochastic system in the presence of multiple measurement unit failures (the sensing measurement devices may malfunction independently) in [18]. Hu, Wang, Gao, and Stergioulas applied the extended Kalman filter to estimate the state variables of stochastic nonlinear systems with multiple missing measurement in [19]. To further improve the performance, in [20], Rouhani and Abur developed a linear phasor estimator assisted dynamic state estimator, which is robust against PMU measurement errors. The linear phasor estimator which provides the required inputs to the distributed dynamic state estimators for the generators uses least-absolute-value (LAV) method in minimizing the \( L_1 \) norm of the measurement residuals. The developed dynamic state estimator (DSE), which is implemented in [20], estimates the dynamic state variables through UKF. Aiming at developing a robustly weighted Schewpe-type generalized maximum likelihood (GM) estimator based on projection statistics, in [21] and [22], Mili et. al. proposed a robust Kalman filter for linear dynamical system. Extending this linear robust estimator to nonlinear power system applications, Zhao and Mili in [23] and [24] proposed PMU-based robust nonlinear state estimation method for real-time power systems dynamic state estimation (DSE), based on the generalized maximum likelihood approach.

Different from the other power system DSE approaches, by modeling bad data as binary Bernoulli distributed random variables, Wang and Yaz proposed a first-order fault tolerant extended Kalman filter in [25], [26], which has been successfully implemented in CompactRIO and dSPACE hardware platform.

The purpose of this manuscript is to present a second-order fault tolerant estimator for dynamic power system state estimation, from corrupted observations of its outputs with bad data, made in discrete instants of time. Leveraging our previous effort in [25]–[27], we propose a novel two-step second-order, locally unbiased, minimum estimation error covariance based fault-tolerant nonlinear observer, which is customized for power system state estimation under bad data, PMU failures, bounded observer gain perturbation, external disturbances, and extraneous noise. In comparison with the traditional nonlinear estimation such as the extended Kalman filter (EKF) and the unscented Kalman filter (UKF), the proposed second-order fault-tolerant extended Kalman filter.
filter (SOFETKF) can provide significant improvements in estimation accuracy, without an increased level of computational effort.

The plan of this manuscript is organized as follows: In Section II, multi-machine power system dynamical model is formulated. In Section III, we describe the dynamic power system state estimation problem, and review the statistical and mathematical background needed for our derivations. After that, we formulate it as a quadratic optimization problem, and derived the two-step second-order fault-tolerant extended Kalman filter in Section IV. And in Section V, we expand our approach to consider extreme cases, and we obtained the second-order fault tolerant extended Kalman filter without estimator gain uncertainties, and the traditional two-step EKF as special cases. In Section VI, the proposed two-step fault tolerant extended Kalman filter is applied to IEEE standard-test power systems for dynamic state estimation accuracy, without an increased level of computational effort.

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II. MULTI-MACHINE POWER SYSTEM DYNAMICAL MODEL FORMULATION

A. LOAD ADMITTANCE CALCULATION

For a certain load with voltage $V_Li$ and complex power $S_{Li} = V_LiP_{Li} + jQ_{Li}$, the equivalent shunt load admittance $y_Li = g_{Li} + jb_{Li}$ can be computed as

$$y_Li = \frac{I_Li}{V_Li} = \frac{S_{Li}^*}{|V_Li|^2} = \frac{P_{Li} - jQ_{Li}}{|V_Li|^2}$$

B. INTERNAL INDUCED VOLTAGE OF SYNCHRONOUS GENERATORS

Denote the internal induced voltage of synchronous generators as $E_i = |E_i|\angle \delta_i$. Denote the per-phase terminal voltage as $V_i = |V_i|\angle \beta_i$, which is also bus $i$ voltage. By taking the terminal voltage $V_i$ as the reference, we have

$$|E_i|\angle \delta'_i = |V_i| + jx_{di}I_i$$

$$= |V_i| + jx_{di}\frac{S_{di}^*}{|V_i|}$$

$$= |V_i| + jx_{di}\frac{P_{ei} - jQ_{ei}}{|V_i|}$$

$$= (|V_i| + \frac{Q_{ei}x_{di}}{|V_i|}) + j(\frac{P_{ei}x_{di}}{|V_i|})$$

where $\delta'_i$ is the relative angle difference between the internal and the terminal voltage, i.e.,

$$\delta_i = \delta'_i + \beta_i$$

C. REDUCED BUS ADMITTANCE MATRIX $Y_{bus}$

Consider an s-bus power system with n synchronous generators. Denote the system admittance bus as $Y_{bus}$, which satisfies:

$$I = Y_{bus}V$$

where $I \in \mathbb{R}^{(n+s) \times 1}$ represents the injected current at each node, i.e.

$$I = [I_n \quad 0_{1 \times s}]^T$$

where $I_n \in \mathbb{R}^{1 \times n}$ represents the injected current from n-generators, where the current injection to the remaining s nodes is $0_{1 \times s}$.

We may partition $Y_{bus}$ and $V$ as

$$I = \begin{pmatrix} I_n \\ 0 \end{pmatrix} = \begin{pmatrix} Y_{nn} & Y_{ns} \\ Y_{sn} & Y_{ss} \end{pmatrix} \begin{pmatrix} E_n \\ V_s \end{pmatrix}$$

where $E_n$ denotes the internal induced voltage of n-synchronous generators. And $V_s$ denotes voltage vector of the remaining s nodes.

$Y_{nn}$ is a diagonal matrix consisting the reciprocals of generator direct-axis impedance

$$Y_{nn} = diag\left(\frac{1}{jx_{d1}}, \frac{1}{jx_{d2}}, \ldots, \frac{1}{jx_{dn}}\right)$$

And, the $ij$ element of $Y_{ns}$ matrix is defined as follows

$$(Y_{ns})_{ij} = \begin{cases} -\frac{1}{jx_{dk}}, & \text{if } i = k \text{ and } j = k, \text{ for } k = 1, \ldots, n. \\ 0, & \text{otherwise}. \end{cases}$$

Notice that the voltage at the internal generator nodes are given by the internal emf’s. Hence, we have

$$I_n = Y_{nn}E_n + Y_{ns}V_s$$

$$0 = Y_{sn}E_n + Y_{ss}V_s$$

Applying Kron reduction to eliminate $V_s$, we have

$$I_n = (Y_{nn} - Y_{ns}Y_{ss}^{-1}Y_{sn})E_n = Y_{bus}E_n$$

Hence, the reduced admittance bus $Y_{bus}$ is reached.

D. SWING EQUATION FOR EACH GENERATOR

For the $i^{th}$ synchronous generator, a departure from the steady state due to an external disturbance results [28]

$$\frac{d^2\theta_{mi}}{dt^2} = \tau_{mi} - \tau_{ei} - \tau_{di}$$

where $J_i$ is the moment of inertia of the rotor for the $i^{th}$ generator. $\theta_{mi}$ is the rotor angle is mechanical radians. $\tau_{mi}$ is the applied torque input from prime mover. And $\tau_{ei}$ is the developed electromechanical torque of generator $i$. An external disturbance $\tau_{di}$ represents the viscous damping torque.

Considering the $i^{th}$ synchronous generator is operated in steady-state, i.e., initially rotor is rotating at the constant synchronous speed. Hence

$$\theta_{mi} = \omega_{sm}t + \delta_{mi}$$

where $\omega_{sm}$ is the constant synchronous speed in mechanical radians per second; and $\delta_{mi}$ is the initial rotor position in mechanical radians.
Therefore, by taking the second-order derivative of (12), we have
\[
\frac{d^2 \theta_{mi}}{dt^2} = \frac{d^2 \delta_{mi}}{dt^2}
\]
and (11) can be rewritten as
\[
J \frac{d^2 \delta_{mi}}{dt^2} = \tau_{mi} \cdot \omega_{mi} - \tau_{di} \cdot \omega_{di} - \tau_e \cdot \omega_{mi} = P_{mi} - P_{ei} - P_{di}
\]
where \( P_{mi} \) is the applied mechanical power from the prime mover, \( P_{ei} \) is the developed electromechanical power of the \( i^{th} \) synchronous generator, and \( P_{di} \) is the viscous damping power specified by
\[
P_{di} = D_i \frac{d \delta_{mi}}{dt} = D_i \omega_{mi}
\]
(14)

Denote the inertia constant \( M_i \) as
\[
J \omega_{mi} = M_i
\]
(15)

It should be noted that the inertia constant \( M_i \) is closely related to the kinetic energy of the rotating mass \( W_k \) by
\[
W_{ki} = \frac{1}{2} J \omega_{mi} = \frac{1}{2} M_i \omega_{mi}
\]
(16)

Hence, we have
\[
M_i \frac{d \omega_{mi}}{dt} = P_{mi} - P_{ei} - P_{di}
\]
(18)

Denote electrical angle \( \delta_i = \frac{\sigma}{2} \delta_{mi} \) and electrical angular speed \( \omega_i = \frac{\sigma}{2} \omega_{mi} \), we can convert (18) into p.u. system by dividing both sides of the equation by three-phase power base \( S_B \) as
\[
\frac{2}{P_B} \frac{M_i}{S_B} \frac{d \omega_i}{dt} = \frac{P_{mi}}{S_B} - \frac{P_{ei}}{S_B} - \frac{P_{di}}{S_B}
\]
(19)

From (17) and (19), we have
\[
\frac{2}{P_B} \frac{2 W_{ki}}{\omega_{mi} \omega_{syn} S_B} \frac{d \omega_i}{dt} = \frac{P_{mi}}{S_B} - \frac{P_{ei}}{S_B} - \frac{P_{di}}{S_B}
\]
(20)

Denote the p.u. inertia constant of generator \( i \) as [28]
\[
H_i = \frac{W_{ki}}{S_B} = \frac{\text{kinetic energy in MJ at rated speed}}{\text{machine rating in MVA}}
\]
(21)

Considering the relations between electrical and mechanical synchronous speed, (22) can be reduced to
\[
\frac{2 H_i}{\omega_{syn}} \frac{d \omega_i}{dt} = \frac{P_{mi}}{S_B} - \frac{P_{ei}}{S_B} - \frac{P_{di}}{S_B}
\]
(22)

Lastly, denote the normalized inertia constant of the \( i^{th} \) generator, \( M_i \) as
\[
M_i = \frac{2 H_i}{\omega_{syn}}
\]
(23)

By dropping the per unit notation, the classical generator model for the \( i^{th} \) power generation unit, for \( i = 1, \ldots, n \), can be expressed as follows:
\[
\frac{d \delta_i}{dt} = \omega_i
\]
\[
M_i \frac{d \omega_i}{dt} = P_{mi} - P_{ei} - P_{di}
\]
(24)

where (24) is known as the swing equation in p.u. form, and \( \delta_i, \omega_i \) are electrical angle and angular speed of rotation for the \( i^{th} \) synchronous generator.

**E. DEVELOPED POWER FROM EACH SYNCHRONOUS GENERATOR**

In the swing equation (24), the developed electromechanical power from the \( i^{th} \) synchronous generator is denoted as \( P_{ei} \), which is also the electrical power injected to the grid by generator \( i \). The power flow equation of \( P_{ei} \) can be expressed as follows, for \( i = 1, 2, \ldots, n \) [28]:
\[
P_{ei} = \text{Re}\{E_i I_i^*\} = \text{Re}\{E_i \sum_{j=1}^{n} |E_j||Y_{ji}|\angle(\delta_i - \delta_j - \theta_j)\}
\]

\[
= \text{Re}\{\sum_{j=1}^{n} |E_i||E_j||Y_{ji}|\angle(\delta_i - \delta_j - \theta_j)\}
\]

(25)

where \( E_i = |E_i|\angle \delta_i \) is the phasor voltage of \( i^{th} \) generator’s emf; \( E_j = |E_j|\angle \delta_j \) is the phasor voltage of \( j^{th} \) generator’s emf. The diagonal elements of \( Y_{bas} \) is denoted as
\[
Y_{ii} = |Y_{ii}|\angle \theta_{ii} = |Y_{ii}|\cos \theta_{ii} + j|Y_{ii}|\sin \theta_{ii} = G_{ii} + jB_{ii}
\]
(26)

The off-diagonal elements of \( Y_{bas} \) is denoted as follows, for \( i \neq j \):
\[
Y_{ij} = |Y_{ij}|\angle \theta_{ij} = |Y_{ij}|\cos \theta_{ij} + j|Y_{ij}|\sin \theta_{ij} = G_{ij} + jB_{ij}
\]
(27)

Equivalently, (25) can be written as
\[
P_{ei} = \sum_{j=1}^{n} |E_i||E_j||Y_{ji}|\cos(\delta_i - \delta_j - \theta_j)
\]
\[
= |E_i|^2 G_{ii} + \sum_{j=1,j\neq i}^{n} |E_i||E_j||Y_{ji}|\cos(\delta_i - \delta_j - \theta_j)
\]
\[
= |E_i|^2 G_{ii} + \sum_{j=1,j\neq i}^{n} |E_i||E_j||Y_{ji}|\cos(\delta_i - \delta_j)\cos \theta_{ij}
\]
\[
+ \sum_{j=1,j\neq i}^{n} |E_i||E_j||Y_{ji}|\sin(\delta_i - \delta_j)\sin \theta_{ij}
\]
\[
= |E_i|^2 G_{ii} + \sum_{j=1,j\neq i}^{n} |E_i||E_j|G_{ij}\cos(\delta_i - \delta_j)
\]
\[
+ \sum_{j=1,j\neq i}^{n} |E_i||E_j|B_{ij}\sin(\delta_i - \delta_j)
\]
(28)
Applying (28), the overall multi-machine power system model can be derived from (24) as
\[
\frac{d\delta_i}{dt} = \omega_i \tag{29}
\]
\[
M_i \frac{d\omega_i}{dt} = P_{ni} - P_{di} - |E_i|^2 G_{ii} - \sum_{j=1, j \neq i}^{n} |E_i||E_j|[G_{ij}\cos(\delta_i - \delta_j) + B_{ij}\sin(\delta_i - \delta_j)] \tag{30}
\]
for \(i = 1, 2, \ldots, n\).

III. PROPOSED STRUCTURE OF THE PLANT

Consider the following discrete-time nonlinear stochastic system
\[
x_k = f(x_{k-1}, u_k) + v_k \\
y_k = \begin{bmatrix}
\gamma_1^k h^1(x_k, u_k) + w^1_k \\
\vdots \\
\gamma_p^k h^p(x_k, u_k) + w^p_k
\end{bmatrix} \tag{31}
\]
where
\[
x_k \in \mathbb{R}^n \quad \text{state space variable} \\
u_k \in \mathbb{R}^m \quad \text{control input} \\
v_k \in \mathbb{R}^n \quad \text{process disturbance and perturbation} \\
y_k \in \mathbb{R}^p \quad \text{measurement output} \\
w_k \in \mathbb{R} \quad \text{measurement disturbance in each sensor} \\
f, h \quad \text{nonlinear } C^k \text{ with } k \geq 2 \text{ vector functions}
\]

The mean of initial state \(x_0 \) is \(E[x_0] = \bar{x}_0 \) and covariance \(X_0 = E[(x_0 - \bar{x}_0)(x_0 - \bar{x}_0)^T] \). The noise processes, \(v_k \) and \(w_k \), are white, zero mean, uncorrelated with each other and with \(x_0 \), and have covariance \(V_k \) and \(W_k \) respectively:
\[
v_k \sim (0, V_k), w_k \sim (0, W_k), \tag{32}
\]
\[
E[v_k w_k^T] = V_k \delta_{k-j}, E[w_k w_k^T] = W_k \delta_{k-j}, \\
E[v_k w_k^T] = 0, E[v_k x_0^T] = 0, E[w_k x_0^T] = 0 \tag{33}
\]

The scalar binary Bernoulli distributed random variables \(\gamma_i \) are with mean \(\pi_i \) and variance \(\pi_i(1 - \pi_i) \) whose possible outcomes 0, 1 are defined as \(P(\gamma_i^k = 1) = \pi_i \) and \(P(\gamma_i^k = 0) = 1 - \pi_i \). The formulation involves hard sensor failures, i.e., the sensor either works normally or fails to provide measurement. The detailed process to identify and detect bad data can be found in Algorithm 1 of our previous work [26].

Let’s denote the sensing condition matrix as
\[
\Gamma_k = diag[\gamma_1^k, \gamma_2^k, \ldots, \gamma_p^k] \tag{34}
\]

the measurement dynamics matrix as
\[
h(x_k, u_k) = diag[h^1(x_k, u_k), h^2(x_k, u_k), \ldots, h^p(x_k, u_k)] \tag{35}
\]
and the extraneous measurement noise vector as
\[
w_k = [w^1_k, w^2_k, \ldots, w^p_k]^T \tag{36}
\]
Hence, the measurement equation can be written as
\[
y_k = \Gamma_k h(x_k, u_k) + w_k \tag{37}
\]

For dynamic power system state estimation, we consider the state variable \(x \) and measurement \(y \) as
\[
x = [\delta_1, \delta_2, \ldots, \delta_n, \omega_1, \omega_2, \ldots, \omega_n] \\
y = [P_{e1}, P_{e2}, \ldots, P_{en}, Q_{e1}, Q_{e2}, \ldots, Q_{en}, \\
|V_1|, |V_2|, \ldots, |V_n|, \beta_1, \beta_2, \ldots, \beta_n] \tag{38}
\]
where the state variables are the angle and angular velocity of \(n \) synchronous generators. The measurements are the active and reactive power at \(n \) generators, and the \(s \) buses voltage magnitudes and angles.

Neglect the higher-order terms, the second-order Taylor series expansions of \(f(x_{k-1}, u_k) \) and \(h(x_k, u_k) \) around the estimated state \(\hat{x}_k \) can be expressed as
\[
f(x_{k-1}, u_k) = f(\hat{x}_{k-1}, u_k) + \frac{\partial f}{\partial x_{k-1}}|_{\hat{x}_{k-1}} (x_{k-1} - \hat{x}_{k-1}) + \frac{1}{2} \sum_{i=1}^{n} \phi_i^f(x_{k-1} - \hat{x}_{k-1})^T \frac{\partial^2 f_i}{\partial x_{k-1}^2}|_{\hat{x}_{k-1}} (x_{k-1} - \hat{x}_{k-1}) \tag{39}
\]
and
\[
h(x_k, u_k) = h(\hat{x}_k, u_k) + \frac{\partial h}{\partial x_k}|_{\hat{x}_k} (x_k - \hat{x}_k) + \frac{1}{2} \sum_{i=1}^{p} \phi_i^h(x_k - \hat{x}_k)^T \frac{\partial^2 h_i}{\partial x_k^2}|_{\hat{x}_k} (x_k - \hat{x}_k) \tag{40}
\]
where \(f_i \) and \(h_i \) are the \(i^{th} \) element of \(f(x_{k-1}, u_k) \) and \(h(x_k, u_k) \), respectively. \(\phi_i^f \) and \(\phi_i^h \) are \(n \times 1 \) and \(p \times 1 \) column vectors respectively. \(\phi_i \) denotes a column vector with all zeros except for a one in the \(i^{th} \) element, i.e.,
\[
\phi_i = [0 \ldots 0 \quad 1 \quad 0 \ldots 0]^T \tag{41}
\]
The quadratic terms in (37) and (38) can be written as
\[
(x_{k-1} - \hat{x}_{k-1})^T \frac{\partial^2 f_i}{\partial x_{k-1}^2}|_{\hat{x}_{k-1}} (x_{k-1} - \hat{x}_{k-1}) = tr\{ \frac{\partial^2 f_i}{\partial x_{k-1}^2}|_{\hat{x}_{k-1}} (x_{k-1} - \hat{x}_{k-1})(x_{k-1} - \hat{x}_{k-1})^T \} \tag{42}
\]
and
\[
(x_k - \hat{x}_k)^T \frac{\partial^2 h_i}{\partial x_k^2}|_{\hat{x}_k} (x_k - \hat{x}_k) = tr\{ \frac{\partial^2 h_i}{\partial x_k^2}|_{\hat{x}_k} (x_k - \hat{x}_k)(x_k - \hat{x}_k)^T \} \tag{43}
\]
where \(tr\{\cdot\} \) denotes the trace of a matrix.

By denoting the estimation error covariance matrix \(P_k \) as
\[
P_k = E[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T] \tag{44}
\]
then (40) and (41) can be approximated as
\[
(x_{k-1} - \hat{x}_{k-1})^T \frac{\partial^2 f_l}{\partial x_{k-1}^2} (x_{k-1} - \hat{x}_{k-1}) \approx tr\left( \frac{\partial^2 f_l}{\partial x_{k-1}^2} \| P_{k-1} \right)
\] (43)
and
\[
(x_k - \hat{x}_k)^T \frac{\partial^2 h_l}{\partial x_k^2} (x_k - \hat{x}_k) \approx tr\left( \frac{\partial^2 h_l}{\partial x_k^2} \| P_k \right)
\] (44)

Hence, if we evaluate (37) at \(x_{k-1} = \hat{x}_{k-1}\) and substitute (43) in the summation, we have
\[
f(x_{k-1}, u_k) \approx f(\hat{x}_{k-1}, u_k) + \frac{\partial f}{\partial x_{k-1}} \| \hat{x}_{k-1} - x_{k-1} \] + \frac{1}{2} \sum_{i=1}^{n} \phi_i^T tr\left( \frac{\partial^2 f_l}{\partial x_{k-1}^2} \| P_{k-1} \right)
\] (45)

Likewise, if we evaluate (38) at \(x_k = \hat{x}_k\) and substitute (44) in the summation, we have
\[
h(x_k, u_k) \approx h(\hat{x}_k, u_k) + \frac{\partial h}{\partial x_k^2} \| \hat{x}_k - x_k \] + \frac{1}{2} \sum_{i=1}^{p} \phi_i^T tr\left( \frac{\partial^2 h_l}{\partial x_k^2} \| P_k \right)
\] (46)

IV. THE STRUCTURE OF THE TWO-STEP SECOND-ORDER FAULT TOLERANT EXTENDED KALMAN FILTER

In this section, we propose a novel two-step fault tolerant extended Kalman filter, which is robust against external disturbances, extraneous noise, bad data, sensor failures, and bounded observer-gain perturbations.

First, let’s consider the auxiliary vector
\[
z_k = \begin{bmatrix} x_0 \\ \vdots \\ x_k \end{bmatrix} \in \mathbb{R}^{(k+1)n}
\]

By generalizing the cost function as
\[
J_k(z_k) = E\left\{ \frac{1}{2} \sum_{i=0}^{k} ||x_i - f(x_{i-1}, u_i)||^2_{V_i} \right\} + \frac{1}{2} \sum_{i=1}^{k} ||y_i - \Gamma_i h(x_i, u_i)||^2_{W_i}
\] (47)

We can rewrite (47) as two recursive equations, which represent the process update and measurement update respectively.
\[
J_{k|k-1}(z_{k}) = J_{k-1}(z_{k-1}) + \frac{1}{2} ||x_i - f(x_{i-1}, u_i)||^2_{V_i}
\] (48)
and
\[
J_k(z_k) = J_{k|k-1}(z_k) + \frac{1}{2} ||y_i - \Gamma_i h(x_i, u_i)||^2_{W_i}
\] (49)

To develop a resilient nonlinear estimator against observer gain perturbation, Kalman gain \(K_k + \Delta_k\) is utilized in (64).

Though the Kalman filter gain should be \(K_k\), due to computational or tuning uncertainties \(\Delta_k\), it is erroneously implemented as \(K_k + \Delta_k\).

The term \(\Gamma_k\), the reliability expectation matrix of \(p\) independent sensors, can be defined as
\[
\Gamma_k = E[\Gamma_k] = \text{diag}[\pi_1, \pi_2, \ldots, \pi_p]
\] (50)
with \(\pi_i\) is the probability of the \(i^{th}\) sensor to work perfectly, i.e., being an accurate sensor to provide reliable measurements.

\(K_k\) is the observer gain with additive uncertainty \(\Delta_k\), while \(\Delta_k\) is considered to have zero mean, bounded second moment satisfying
\[
E[\Delta_k\Delta_k^T] \leq \delta I, \ E[\Delta_k^T x_0] = 0, \ E[\Delta_k^T W_k] = 0
\] (51)

**Theorem 1:** The Two-Step Second-Order Fault Tolerant Extended Kalman Filter

The two-step second-order fault tolerant extended Kalman filter is defined as follows:

1) Initialization
\[
\hat{x}_0 = E[x_0], \ P_0 = E[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^T]
\] (52)

2) Computation of Jacobian matrices
\[
A_k = \frac{\partial f(x_{k-1}, u_k)}{\partial x_{k-1}} \| \hat{x}_{k-1} \]
\[
C_k = \frac{\partial h(x_k, u_k)}{\partial x_k} \| \hat{x}_k \]
(53)
(54)

3) Process update equations are given as follows:
\[
\dot{\hat{x}}_{k|k-1} = f(\hat{x}_{k-1}, u_k) + \frac{1}{2} \sum_{i=1}^{n} \phi_i^T tr\left( \frac{\partial^2 f_i}{\partial x_{k-1}^2} \| P_{k-1} \right)
\]
\[
P_{k|k-1} = A_k P_{k-1} A_k^T + V_k
\] (55)

4) Measurement update equations are summarized below:
\[
\dot{\hat{x}}_k = \dot{\hat{x}}_{k|k-1} + (K_k \Delta_k + \Lambda_k)(y_k - \Gamma_k h(\hat{x}_k))
\]
\[
- \Gamma_k \frac{1}{2} \sum_{i=1}^{p} \phi_i^T tr\left( \frac{\partial^2 h_i}{\partial x_k^2} \| P_k \right)
\] (56)
\[
K_k^\delta = P_{k|k-1} C_k^T \Gamma_k \left\{ \Gamma_k C_k P_{k|k-1} C_k^T \Gamma_k - \delta I \right\}
\]
\[
+ \beta \otimes [G(\hat{x}_{k|k-1}) \Gamma_k (\hat{x}_{k|k-1}) + C_k P_{k|k-1} C_k^T] + W_k
\] (57)
\[
P_k = P_{k|k-1} + \lambda_{\max} \left\{ \Gamma_k C_k P_{k|k-1} C_k^T \Gamma_k^T + W_k \right\}
\]
\[
+ \beta \otimes \left\{ [G(\hat{x}_{k|k-1}) \Gamma_k (\hat{x}_{k|k-1}) + C_k P_{k|k-1} C_k^T] \right\} \delta I
\]
\[
- P_{k|k-1} C_k^T \Gamma_k \left\{ \Gamma_k C_k P_{k|k-1} C_k^T \Gamma_k^T + W_k \right\}
\]
\[
+ \beta \otimes \left\{ [G(\hat{x}_{k|k-1}) \Gamma_k (\hat{x}_{k|k-1}) + C_k P_{k|k-1} C_k^T] \right\}^{-1}
\] (58)
The a priori estimation error covariance matrix

\[ P_{k|k-1} = E\{e_k|k-1\} = E\{(x_k - \hat{x}_k)|k-1\} = A_k P_{k-1} A_k^T + V_k \]

By applying Taylor series expansion results in (37), we have

\[ \hat{x}_{k|k-1} = E\{x_k|k-1\} = E\{f(x_{k-1}, u_k) + v_k\} \]

\[ = E\{f(\hat{x}_{k-1}, u_k) + \frac{\partial f}{\partial x_{k-1}}|_{\hat{x}_{k-1}} (x_{k-1} - \hat{x}_{k-1}) \} + \sum_{i=1}^{n} \phi_i^f \cdot (x_{k-1} - \hat{x}_{k-1})^T \cdot \frac{\partial^2 f}{\partial x_{k-1}^2}|_{\hat{x}_{k-1}} \]
Given the measurement $y_k$, we derive the measurement update using (38) as follows:

Suppose we have

$$\hat{x}_k = \hat{x}_{k|k-1} + (K_k + \Delta K)(y_k - \bar{\Gamma}_k h(\hat{x}_k)) - \varrho_k \quad (64)$$

Denote the estimation error as

$$e_k = x_k - \hat{x}_k$$

$$e_{k|k-1} = x_k - \hat{x}_{k|k-1} \quad (65)$$

Hence, we have

$$e_k = e_{k|k-1} + (K_k + \Delta K)(y_k - \bar{\Gamma}_k h(\hat{x}_k)) + \varrho_k \quad (66)$$

Equivalently, we have

$$e_k = e_{k|k-1} - (K_k + \Delta K)[\Gamma_k h(\hat{x}_{k|k-1}, u_k) + C_k e_k]$$

$$+ \Gamma_k \frac{1}{2} P_k \frac{1}{2} \sum_{i=1}^{p} \phi^h_i \cdot (x_k - \hat{x}_{k|k-1})^T \left( \frac{\partial^2 h_i}{\partial x_k^2} \right) \mid_{\hat{x}_{k|k-1}}$$

$$+ w_k - \bar{\Gamma}_k h(\hat{x}_{k|k-1}, u_k) + \varrho_k + O(e_k) \quad (67)$$

For unbiased estimation, we demand that $E[e_k] = 0$. The choice of $\varrho_k$ makes $\hat{x}_{k+1}$ an unbiased estimate, therefore, we have

$$\varrho = (K_k + \Delta K)\bar{\Gamma}_k \frac{1}{2} P_k \frac{1}{2} \sum_{i=1}^{p} \phi^h_i \cdot (x_k - \hat{x}_{k|k-1})^T \left( \frac{\partial^2 h_i}{\partial x_k^2} \right) \mid_{\hat{x}_{k|k-1}} \quad (68)$$

By neglecting the higher-order error term $O(e_k^2)$, and applying (66), we have

$$e_k = [I - (K_k + \Delta K)\Gamma_k C_k] e_{k|k-1} - (K_k + \Delta K)w_k$$

$$- (K_k + \Delta K)\bar{\Gamma}_k \left[ h(\hat{x}_{k|k-1}, u_k) \right.$$

$$+ \frac{1}{2} P_k \frac{1}{2} \sum_{i=1}^{p} \phi^h_i \cdot \left( \frac{\partial^2 h_i}{\partial x_k^2} \right) \mid_{\hat{x}_{k|k-1}} \left( \right) \quad (69)$$

where

$$\bar{\Gamma}_k = \Gamma_k - \bar{\Gamma}_k \quad (70)$$

To simplify (69) in the derivation process, let us denote

$$\zeta(\hat{x}_{k|k-1}) = h(\hat{x}_{k|k-1}, u_k) + \frac{1}{2} P_k \frac{1}{2} \sum_{i=1}^{p} \phi^h_i \cdot \left( \frac{\partial^2 h_i}{\partial x_k^2} \right) \mid_{\hat{x}_{k|k-1}} \quad (71)$$

To derive the optimal estimator gain $K_k$, we need to consider an upper bound on the estimation error covariance matrix.

$$P_k = E[e_k e_k^T] \quad (72)$$

Applying (69), the error covariance matrix evolves as

$$P_k = E[e_k e_k^T]$$

$$= E[(I - (K_k + \Delta K)\Gamma_k C_k) e_{k|k-1} - (K_k + \Delta K)w_k$$

$$- (K_k + \Delta K)\bar{\Gamma}_k \zeta(\hat{x}_{k|k-1}) \times [\cdots]^T]$$

$$= E[(I - (K_k + \Delta K)\Gamma_k C_k) e_{k|k-1}] + \varrho_k \quad (73)$$

Each individual term of (73) can be reduced leveraging the Rayleigh’s inequality, i.e., for matrices $X = X^T \in \mathbb{R}^{n \times n}$ and $Y \in \mathbb{R}^{m \times n}$, the matrix inequality $\lambda_{\min}(X)YY^T \leq YXY^T \leq \lambda_{\max}(X)YY^T$ holds.

Applying (51) and Rayleigh’s matrix inequality, the first term can be simplified to

$$E[(I - (K_k + \Delta K)\Gamma_k C_k) e_{k|k-1}] = 0$$

$$E[(I - (K_k + \Delta K)\Gamma_k C_k) e_{k|k-1}] = 0$$

$$E[(I - (K_k + \Delta K)\Gamma_k C_k) e_{k|k-1}] = 0$$

Since $e_{k|k-1}$ and $w_k$ are uncorrelated, we have

$$E[(I - (K_k + \Delta K)\Gamma_k C_k) e_{k|k-1}] = 0$$

$$E[(I - (K_k + \Delta K)\Gamma_k C_k) e_{k|k-1}] = 0$$

Since $e_{k|k-1}, \Delta K, \bar{\Gamma}_k$ are mutually uncorrelated and $E[e_k] = 0, E[\bar{\Gamma}_k] = 0$, the following term is zero as shown below

$$E[(I - (K_k + \Delta K)\Gamma_k C_k) e_{k|k-1}] = 0$$

$$E[(I - (K_k + \Delta K)\Gamma_k C_k) e_{k|k-1}] = 0$$

Applying (51), the term yields

$$E[(K_k + \Delta K)w_k (K_k + \Delta K)^T]$$

$$E[(K_k + \Delta K)w_k (K_k + \Delta K)^T]$$

Since $w_k, \Delta K, \bar{\Gamma}_k$ are mutually uncorrelated, and $E[w_k] = 0, E[\bar{\Gamma}_k] = 0$, the term is zero as follows

$$E[(K_k + \Delta K)w_k (K_k + \Delta K)^T] = 0$$

$$E[(K_k + \Delta K)w_k (K_k + \Delta K)^T] = 0$$

$$E[(K_k + \Delta K)w_k (K_k + \Delta K)^T] = 0$$

(74)
Applying (51), the term has an upper-bound as shown below
\[
\begin{align*}
E\{&(K_k + \Delta_k) \hat{\Gamma}_k \zeta (\hat{x}_k|k-1) T (\hat{x}_k|k-1) \hat{\Gamma}_k^T (K_k + \Delta_k)^T \} \\
= & \ K_k E[\hat{\Gamma}_k \zeta (\hat{x}_k|k-1) T (\hat{x}_k|k-1) \hat{\Gamma}_k^T] K_k^T \\
+ & \ E[\Delta_k \hat{\Gamma}_k \zeta (\hat{x}_k|k-1) T (\hat{x}_k|k-1) \hat{\Gamma}_k^T \Delta_k^T] \\
\leq & \ K_k E[\hat{\Gamma}_k \zeta (\hat{x}_k|k-1) T (\hat{x}_k|k-1) \hat{\Gamma}_k^T] K_k^T \\
+ & \ \lambda_{\max}(E[\hat{\Gamma}_k \zeta (\hat{x}_k|k-1) T (\hat{x}_k|k-1) \hat{\Gamma}_k^T]) \delta I \\
\end{align*}
\]
(78)

Hence, based on the aforementioned relations in (74)-(78), (73) yields
\[
P_k \leq [I - K_k \hat{\Gamma}_k C_k] P_{k|k-1} [I - K_k \hat{\Gamma}_k C_k]^T \\
+ K_k E[\hat{\Gamma}_k C_k P_{k|k-1} C_k^T \hat{\Gamma}_k^T] K_k^T \\
+ \lambda_{\max}(E[\hat{\Gamma}_k C_k P_{k|k-1} C_k^T \hat{\Gamma}_k^T]) \delta I \\
+ K_k W_k K_k^T \\
+ K_k E[\hat{\Gamma}_k \zeta (\hat{x}_k|k-1) T (\hat{x}_k|k-1) \hat{\Gamma}_k^T] \delta I \\
+ \lambda_{\max}(E[\hat{\Gamma}_k \zeta (\hat{x}_k|k-1) T (\hat{x}_k|k-1) \hat{\Gamma}_k^T]) \delta I \\
= [I - K_k \hat{\Gamma}_k C_k] P_{k|k-1} [I - K_k \hat{\Gamma}_k C_k]^T \\
+ K_k E[\hat{\Gamma}_k C_k P_{k|k-1} C_k^T \hat{\Gamma}_k^T] K_k^T \\
+ \hat{\Gamma}_k \zeta (\hat{x}_k|k-1) T (\hat{x}_k|k-1) \hat{\Gamma}_k^T + W_k K_k^T \\
+ \lambda_{\max}(E[\hat{\Gamma}_k C_k P_{k|k-1} C_k^T \hat{\Gamma}_k^T]) \delta I \\
+ E[\hat{\Gamma}_k C_k P_{k|k-1} C_k^T \hat{\Gamma}_k^T] K_k^T \\
+ \hat{\Gamma}_k \zeta (\hat{x}_k|k-1) T (\hat{x}_k|k-1) \hat{\Gamma}_k^T + W_k K_k^T \\
+ \lambda_{\max}(E[\hat{\Gamma}_k C_k P_{k|k-1} C_k^T \hat{\Gamma}_k^T]) \delta I \\
\] 
(79)

Denote the Hadamard product of $n \times m$ matrices $A$ and $B$ as $A \odot B$. Equivalent, $[A \odot B]_{i,j} = [A]_{i,j} [B]_{i,j}$ for $1 \leq i \leq n$, $1 \leq j \leq m$. Then, we have
\[
E\{[\hat{\Gamma}_k \zeta (\hat{x}_k|k-1)] [\hat{\Gamma}_k \zeta (\hat{x}_k|k-1)]^T + [\hat{\Gamma}_k C_k] P_{k|k-1} [\hat{\Gamma}_k C_k]^T \} \\
= \ \Upsilon \odot (\zeta (\hat{x}_k|k-1) T (\hat{x}_k|k-1) + C_k P_{k|k-1} C_k^T) \\
\] 
(80)

where
\[
\Upsilon = \text{diag}(\pi_1(1 - \pi_1), \pi_2(1 - \pi_2), \ldots, \pi_p(1 - \pi_p)) \\
\]
(81)

The upper bound on the error covariance equation can be obtained as
\[
P_k = [I - K_k \hat{\Gamma}_k C_k] P_{k|k-1} [I - K_k \hat{\Gamma}_k C_k]^T \\
+ K_k [W_k + \Upsilon \odot (\zeta (\hat{x}_k|k-1) T (\hat{x}_k|k-1) + C_k P_{k|k-1} C_k^T)] K_k^T \\
+ \lambda_{\max}(E[\hat{\Gamma}_k C_k P_{k|k-1} C_k^T \hat{\Gamma}_k^T]) \delta I \\
+ \hat{\Gamma}_k \zeta (\hat{x}_k|k-1) T (\hat{x}_k|k-1) + C_k P_{k|k-1} C_k^T \] 
(82)

Equivalently, it can be organized as
\[
P_k = \Omega_k + K_k \Lambda_k + \Lambda_k^T K_k^T + K_k \Phi_k K_k^T \\
\] 
(83)

where
\[
\Phi_k = \hat{\Gamma}_k C_k P_{k|k-1} C_k^T + W_k \\
+ \Upsilon \odot (\zeta (\hat{x}_k|k-1) T (\hat{x}_k|k-1) + C_k P_{k|k-1} C_k^T) \\
\]

\[
\Lambda_k = -\hat{\Gamma}_k C_k P_{k|k-1} \\
\Omega_k = P_{k|k-1} + \lambda_{\max}(E[\hat{\Gamma}_k C_k P_{k|k-1} C_k^T \hat{\Gamma}_k^T]) \delta I \\
+ \Upsilon \odot (\zeta (\hat{x}_k|k-1) T (\hat{x}_k|k-1) + C_k P_{k|k-1} C_k^T) \] 
(84)

Applying completing the square in observer gain $K_k$, equivalently, the following equation holds
\[
P_k = \Omega_k + (K_k - K_k^o) \Phi_k (K_k - K_k^o)^T + K_k^o \Phi_k K_k^o \] 
(85)

For (83) to be equal to (85), the following condition must hold
\[
K_k \Lambda_k = -K_k \Phi_k K_k^o \] 
(86)

Therefore, the robust optimal feedback gain
\[
K_k^o = -\frac{\Lambda_k^T K_k^o}{K_k^o \Phi_k K_k^o} = P_{k|k-1} C_k^T \hat{\Gamma}_k C_k P_{k|k-1} C_k^T \hat{\Gamma}_k^T + W_k \\
+ \Upsilon \odot (\zeta (\hat{x}_k|k-1) T (\hat{x}_k|k-1) + C_k P_{k|k-1} C_k^T) + W_k \\
+ \Upsilon \odot (\zeta (\hat{x}_k|k-1) T (\hat{x}_k|k-1) + C_k P_{k|k-1} C_k^T) - 1 \hat{\Gamma}_k C_k P_{k|k-1} \\
\] 
(87)

By setting $K_k = K_k^o$, the resulting matrix difference equation for the minimum of the upper bound on the estimation error covariance is given as
\[
P_k = \Omega_k + K_k \Lambda_k + \Lambda_k^T K_k^T + K_k \Phi_k K_k^T \\
+ K_k [W_k + \Upsilon \odot (\zeta (\hat{x}_k|k-1) T (\hat{x}_k|k-1) + C_k P_{k|k-1} C_k^T)] K_k^T \\
+ \lambda_{\max}(E[\hat{\Gamma}_k C_k P_{k|k-1} C_k^T \hat{\Gamma}_k^T]) \delta I \\
+ \hat{\Gamma}_k \zeta (\hat{x}_k|k-1) T (\hat{x}_k|k-1) + C_k P_{k|k-1} C_k^T \] 
(88)

This concludes the proof of Theorem 1.

\section*{V. SPECIAL CASES}

\subsection*{A. THE ONE-STEP SECOND-ORDER FAULT TOLERANT EXTENDED KALMAN FILTER}

We refer to Theorem 1 as the two-step second-order fault-tolerant extended Kalman filter. However, it is possible to combine the process update and measurement update steps into a single step, which yield the one-step second-order fault tolerant extended Kalman filter in [27].

\subsection*{B. THE TWO-STEP SECOND-ORDER FAULT TOLERANT EXTENDED KALMAN FILTER WITHOUT ESTIMATOR GAIN UNCERTAINTIES}

As a limiting case, by setting $\delta = 0$, we can derive the two-step second-order fault tolerant extended Kalman filter without perturbations on estimator gain, as follows.

Process update equations are given as follows:
\[
\hat{x}_{k|k-1} = f(\hat{x}_{k-1}, u_k) + \frac{1}{2} \sum_{i=1}^{n} \phi_i \rho_i |\frac{\partial^2 f_i}{\partial x_i^2}|_{\hat{x}_{k-1}} P_{k-1} \\
P_{k|k-1} = A_k P_{k-1} A_k^T + V_k \\
\]
(89)
Measurement update equations are summarized below:
\[
\dot{x}_k = \hat{x}_{k|k-1} + K_k^o[y_k - \hat{h}(\hat{x}_k)]
\]
\[
-\dot{\hat{y}}_k + \sum_{i=1}^{P} \frac{\partial^2 f_i}{\partial x_k^2} \bigg|_{\hat{x}_{k|k-1}} \begin{pmatrix} P_k \end{pmatrix}
\]
\[
K_k^o = P_k|k-1|C_k^T \hat{h}_k(T_{k|k-1}C_k^T \hat{h}_k(T_{k|k-1} + C_kP_k|k-1C_k^T) + W_k)^{-1}
\]
\[
P_k = P_k|k-1| - P_k|k-1|C_k^T \hat{h}_k(T_{k|k-1} + C_kP_k|k-1C_k^T) + W_k)
\]
\[
+ \hat{\gamma} \otimes [\xi(\hat{x}_{k|k-1})\xi^T(\hat{x}_{k|k-1}) + C_kP_k|k-1C_k^T]^{-1} \hat{h}_k(T_{k|k-1}C_k^T \hat{h}_k(T_{k|k-1} + C_kP_k|k-1C_k^T) + W_k)
\]

C. THE TWO-STEP SECOND-ORDER EXTENDED KALMAN FILTER

By further setting \( y_i^j = 1 \) for all sensors, then \( \hat{y}_k = I \) and \( \hat{\gamma} = 0 \). In this case, all measurements are reliable and accurate at all time with no sensor failures, then traditional second-order extended Kalman filter can be derived as a special case of the proposed two-step fault-tolerant extended Kalman filter as follows:

Process update equations are given as follows:
\[
\dot{x}_{k|k-1} = f(\hat{x}_{k-1}, u_k) + \frac{1}{2} \sum_{i=1}^{P} \frac{\partial^2 f_i}{\partial x_k^2} \bigg|_{\hat{x}_{k|k-1}} \begin{pmatrix} P_{k-1} \end{pmatrix}
\]
\[
P_{k|k-1} = A_kP_{k-1}A_k^T + V_k
\]

Measurement update equations are summarized below:
\[
\dot{x}_k = \hat{x}_{k|k-1} + K_k^o[y_k - \hat{h}(\hat{x}_k)]
\]
\[
-\frac{1}{2} \sum_{i=1}^{P} \frac{\partial^2 h_i}{\partial x_k^2} \bigg|_{\hat{x}_{k|k-1}} \begin{pmatrix} P_k \end{pmatrix}
\]
\[
K_k^o = P_k|k-1|C_k^T \left( C_kP_k|k-1C_k^T + W_k \right)^{-1}
\]
\[
P_k = P_k|k-1| - P_k|k-1|C_k^T \left( C_kP_k|k-1C_k^T + W_k \right)^{-1} C_kP_k|k-1|C_k^T + W_k
\]

VI. APPLICATIONS TO IEEE 14-BUS POWER SYSTEM BENCHMARK PROBLEM

In this section, we consider the IEEE 14-bus power system model as our testing system, which is shown in Fig.3. In this standard grid model, there are 5 synchronous generators, and 14 buses interconnected via branches, i.e., \( n = 5 \) and \( s = 14 \).

The standard first order extended Kalman filter (EKF), the second order extended Kalman filter (SOEKF), and the unscented Kalman filter (UKF) are also implemented for performance comparison. In the simulation studies, the active and reactive power injections from generators

\[ P_{ei}, Q_{ei}, \text{ for } i = 1, \ldots, n \text{ and bus-voltage phasors } V_j, \text{ for } j = 1, \ldots, s \text{ are assumed to be directly measured in real-time through the phasor measurement units (PMU).} \]

A random Gaussian noise with zero mean and standard deviation of \( 10^{-2} \) is assumed for either system and measurement noise. The diagonal elements of the initial estimation error covariance matrix of UKF are set to be \( 10^{-4} \). The initial values of the state vector are arbitrarily chosen for all four nonlinear estimators. The damping coefficient \( D \) is set to \( 0.005 \) for all generators, and the PMU measurements are assumed to be received at a rate of 48 samples per second. At time \( t = 2 \text{ sec.}, \) due to symmetrical fault, the transmission line 2-5 is switched off by the protective relays and circuit breakers, which causes a large disturbance to the entire power grid. We considered the bad data effect in our simulation with PMU reliability rate \( \pi = 0.95 \) for all phasor measurement units.

The simulation results are summarized as follows: Fig. 3 provides the measurement of active power generated by the generator 1 at PMU bad-rate of 5%. At the time during
Fig. 4. State $\omega_1$ estimation comparison with SOEKF, UKF and SOFTEKF at PMU bad-data rate of $(1 - \pi) = 5\%$.

Fig. 5. Zoomed-in figure for state $\omega_1$ estimation comparison.

Fig. 6. State $\delta_2$ estimation comparison with SOEKF, UKF and SOFTEKF at PMU bad-data rate of $(1 - \pi) = 5\%$.

Fig. 7. Zoomed-in figure for state $\delta_2$ estimation.

The occurrence of bad data, the measurements are considered to be 0.

Fig. 4 shows the performance comparison among the second-order extended Kalman filter (SOEKF), the unscented Kalman filter (UKF) and the novel second-order fault-tolerant extended Kalman filter (SOFTEKF) for estimating the angular velocity of generator 1, $\omega_1$. Notice that due to faulty measurement of PMU, SOEKF failed to provide a reliable estimate. While UKF gives decent tracking performance, the proposed second-order fault tolerant extended Kalman filter (SOFTEKF) offers superior dynamic state estimation performance. Please also notice that this conclusion also holds for all other state variables estimations, under various IEEE standard power systems DSE applications. To better demonstrate the superior performance of SOFTEKF, a zoomed-in version of Fig. 4, is shown in Fig. 5. Notice that SOFTEKF shown in black-colored dashed line closely tracks the reference shown in red-colored solid line, while UKF is shown in green-colored dash-dotted line and SOEKF is shown in blue-colored dashed line.

Fig. 6 illustrates the performance comparison among the second-order extended Kalman filter (SOEKF), the unscented Kalman filter (UKF) and the novel second-order fault-tolerant extended Kalman filter (SOFTEKF) for estimating the angle position of generator 2, $\delta_2$. A zoomed-in version of Fig. 6 is shown in Fig. 7. Again, the proposed second-order fault tolerant extended Kalman filter, as shown in black-colored dashed line, shows superior performance compared with UKF and SOEKF. Second-order EKF is not included in Fig. 7, since it greatly deviates from the actual state trajectory.

To better evaluate the performance improvement of the proposed second-order fault-tolerant extended Kalman filter. The performance metric we used to evaluate second-order FTEKF and other nonlinear estimation methods is based on the $L_1$ norm of estimation error deviation at time step $k$, which is given by

$$
\varepsilon_k = \frac{1}{2n} \sum_{k=1}^{2n} |\hat{x}_k^i - x_k^i|
$$

(97)
where \( n \) is the total number of buses. Denote the total simulation time steps as \( N \), the overall simulation error is defined by
\[
\varepsilon = \frac{1}{N} \sum_{k=1}^{N} \varepsilon_k
\]
(98)

Hence, the estimation errors for angular speed \( \omega \) and angle position \( \delta \) are given as
\[
\varepsilon_\omega = \frac{1}{N} \sum_{k=1}^{N} \frac{1}{n} \sum_{i=1}^{n} |\hat{\omega}_k[i] - \omega_k[i]|
\]
\[
\varepsilon_\delta = \frac{1}{N} \sum_{k=1}^{N} \frac{1}{n} \sum_{i=1}^{n} |\hat{\delta}_k[i] - \delta_k[i]|
\]
(99)

The performance indices of the traditional first-order EKF (EKF), the second order EKF (SOEKF), the unscented Kalman filter (UKF) and the proposed second-order fault tolerant extended Kalman filter (SOFTEKF) for the IEEE 14-bus power system dynamic state estimation are summarized in Tab. 1., under PMU bad data rate 5%. Experimental data demonstrate that the proposed SOFTEKF is more robust and resilient against bad data, PMU failures, extraneous noise, and external disturbances.

It should also be noted that, in order to verify the performance of the proposed second-order fault-tolerant extended Kalman filter, various power systems have been examined, including the IEEE standard 5-bus, 30-bus, 118-bus power systems in Tab.2-Tab.4. Results are clearly encouraging based on the \( \mathcal{L}_1 \) norm of estimation error comparisons. We conclude that the proposed two-step fault tolerant extended Kalman filter shows superior performance in robustness, resiliency, and estimation accuracy.

### VII. CONCLUSION AND FUTURE WORK

Robust nonlinear estimation under bad data, faulty sensor measurements, extraneous noise, and external disturbances conditions is of great importance in modern smart grid applications. In this paper, a novel second-order fault tolerant extended Kalman filter (SOFTEKF) has been developed for tracking the dynamic state variables of power systems. Computer simulation studies carried out on the benchmark IEEE standard-test power systems provide promising results, which demonstrate the tolerance for PMU faults, robustness against extraneous noise and disturbances, computational efficiency, and stochastic resiliency of the proposed second-order FTEKF. Studies indicate that second-order fault tolerant extended Kalman filter provides superior accuracy than the traditional first- and second order extended Kalman filter, and the unscented Kalman filter. The impact on the accuracy improvement of the second-order FTEKF estimation error deviation over the first- and second-order EKF, and UKF has been shown to be major, in the view of similar computational complexity and running time. Hence, the proposed second-order fault tolerant extended Kalman filter is suitable for robust and resilient dynamic state estimation for smart grid applications. As a future work, the proposed method will be extended to a decentralized framework for real-time monitoring and control of a large-scale power network. Also, based on [29], we will try to leverage the extended set of sigma points to explore a more computationally efficient implementation of the second-order fault tolerance extended Kalman filter, without any analytical or numerical explicit expressions of the Jacobian or Hessian matrices required.

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**TABLE 1.** Performance comparison of EKF, SOEKF, UKF and SOFTEKF under PMU bad-data rate 5% for IEEE Standard 5-Generator 14-Bus Power System.

| Metrics | EKF | SOEKF | UKF | SOFTEKF |
|---------|-----|-------|-----|---------|
| \( \varepsilon \) | 9.2823 | 7.3321 | 0.0291 | 0.0120 |
| \( \varepsilon_\omega \) | 0.0087 | 0.0021 | 5.6822 \times 10^{-4} | 3.0945 \times 10^{-4} |
| \( \varepsilon_\delta \) | 13.3583 | 12.5796 | 0.0577 | 0.0236 |

**TABLE 2.** Performance comparison of EKF, SOEKF, UKF and SOFTEKF under PMU bad-data rate 5% for IEEE Standard 3-generator 5-bus power system.

| Metrics | EKF | SOEKF | UKF | SOFTEKF |
|---------|-----|-------|-----|---------|
| \( \varepsilon \) | 9.1235 | 5.8996 | 0.0297 | 0.0187 |
| \( \varepsilon_\omega \) | 0.0064 | 0.0031 | 7.3453 \times 10^{-4} | 3.0089 \times 10^{-4} |
| \( \varepsilon_\delta \) | 13.8763 | 12.4521 | 0.0913 | 0.0123 |

**TABLE 3.** Performance comparison of EKF, SOEKF, UKF and SOFTEKF under PMU bad-data rate 5% for IEEE Standard 6-generator 30-bus power system.

| Metrics | EKF | SOEKF | UKF | SOFTEKF |
|---------|-----|-------|-----|---------|
| \( \varepsilon \) | 9.4391 | 8.4211 | 0.0519 | 0.0377 |
| \( \varepsilon_\omega \) | 0.0094 | 0.0041 | 3.3114 \times 10^{-4} | 3.0471 \times 10^{-4} |
| \( \varepsilon_\delta \) | 13.1941 | 12.1741 | 0.0641 | 0.0237 |

**TABLE 4.** Performance comparison of EKF, SOEKF, UKF and SOFTEKF under PMU bad-data rate 5% for IEEE Standard 19-generator 118-bus power system.

| Metrics | EKF | SOEKF | UKF | SOFTEKF |
|---------|-----|-------|-----|---------|
| \( \varepsilon \) | 9.4341 | 6.8939 | 0.0415 | 0.0183 |
| \( \varepsilon_\omega \) | 0.0054 | 0.0031 | 9.3717 \times 10^{-4} | 3.3745 \times 10^{-4} |
| \( \varepsilon_\delta \) | 13.3951 | 12.9981 | 0.0947 | 0.2681 |
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