Digital Quantum Estimation

Lorenzo Maccone\textsuperscript{1,*}, Majid Hassani\textsuperscript{2}, Chiara Macchiavello\textsuperscript{1}
\textsuperscript{1}Dip. Fisica and INFN Sez. Pavia, University of Pavia, via Bassi 6, I-27100 Pavia, Italy
\textsuperscript{2}Department of Physics, Sharif University of Technology, Tehran 14588, Iran
*Corresponding Author: maccone@unipv.it

Quantum Metrology calculates the ultimate precision of all estimation strategies, measuring what is their root mean-square error (RMSE) and their Fisher information. Here, instead, we ask how many bits of the parameter we can recover, namely we derive an information-theoretic quantum metrology. In this setting we redefine “Heisenberg bound” and “standard quantum limit” (the usual benchmarks in quantum estimation theory), and show that the former can be attained only by sequential strategies or parallel strategies that employ entanglement among probes, whereas parallel-separable strategies are limited by the latter. We highlight the differences between this setting and the RMSE-based one.

The theory of quantum metrology\textsuperscript{11} \textsuperscript{12} determines the ultimate precision in any estimation. The estimation of an unknown parameter generally requires a probe that interacts with the system to be sampled: the interaction encodes the parameter onto the probe(s) and is then measured. Clearly, if one uses \( N \) independent measurements, the root mean square error (RMSE) in the estimation scales as \( 1/\sqrt{N} \) (the standard quantum limit) as dictated by the central limit theorem. If one uses \( N \) parallel entangled probes or one probe sequentially \( N \) times, the error can be reduced to \( 1/N \) (the Heisenberg bound) \textsuperscript{1} \textsuperscript{13}. This precision can be attained without the use of entanglement at the measurement stage \textsuperscript{4}.

The RMSE is, however, ill suited for digital sensors, digital data processing, or even for the digital archival of parameters, where the number of significant digits (bits) is a more useful figure of merit. Moreover, the techniques used in the conventional theory (e.g. the use of NOON states \textsuperscript{14}) suffer from ambiguities in the typical case in which a phase is estimated \textsuperscript{15} \textsuperscript{16}, so that the reported RMSE does not typically refer to the true error in the estimation \textsuperscript{17} \textsuperscript{18} \textsuperscript{19}.

In this paper we overcome these problems by replacing RMSE (and Fisher information) with mutual information, which directly measures the number of bits of information that the quantum estimation strategy provides. Namely, we derive an information-theoretic quantum metrology, obtaining a number of results: (1) we redefine in a natural way the concepts of Heisenberg bound (using the Holevo theorem) and of standard quantum limit; (2) for parallel estimation strategies the Heisenberg bound can be attained, but only in the presence of entanglement, as in the RMSE case; (3) as expected, for parallel strategies without entanglement at the preparation, at most the standard quantum limit is achievable (and entanglement at the measurement stage is useless); (4) instead, for sequential strategies (where one of the probes performs most of the samplings) the Heisenberg bound is attainable without using entanglement, as in the RMSE case; (5) increasing the Hilbert space dimension of the probe is helpful, in contrast to the RMSE case where a two-dimensional subspace is sufficient; (6) the Heisenberg bound is achieved by the quantum phase estimation algorithm (QPEA) \textsuperscript{20} \textsuperscript{21} and by the Pegg-Barnett phase states \textsuperscript{22}, in contrast to the RMSE case \textsuperscript{17} \textsuperscript{18} \textsuperscript{23}.

Heisenberg bound and standard quantum limit:— In quantum metrology we estimate a parameter \( \varphi \) by first preparing one or more probes into an initial state \( \rho_0 \), then evolving them by applying \( N \) times the interaction \( U_\varphi \) that encodes the parameter onto the probe(s) and transforms the state into \( \rho_\varphi \), and finally measuring \( \rho_\varphi \). The aim is to find the ultimate precision attainable for the estimation strategy as a function of \( N \). If the probe is finite-dimensional, no estimation strategy can beat the Heisenberg bound \( \propto 1/N \) for the RMSE.

A natural way to extend the Heisenberg bound to an information-theoretic setting is to use the Holevo theorem \textsuperscript{24}, which gives the maximum number of bits \( I \) attainable on a parameter \( \varphi \) encoded into a state \( \rho_\varphi \), given the measurement results \( \vec{m} \):

\[
I(\vec{m}:\varphi) \leq S\left(\sum_\varphi \rho_\varphi\rho_\varphi\right) - \sum_\varphi \rho_\varphi S(\rho_\varphi),
\]

where \( S(\rho) = -\text{Tr}[\rho \log_2 \rho] \) is the von Neumann entropy, and \( p_\varphi \) the prior probability of the parameter \( \varphi \). Clearly the accessible information is largest when \( \rho_\varphi \) are all pure states, and in this case the last sum in (1) is null and the Holevo bound is attainable. We then define the info-theoretic Heisenberg bound as \( S(\sum_\varphi p_\varphi \rho_\varphi) \). This quantity scales as \( \log_2 N \) since we are applying \( N \) times the same transformation \( U_\varphi \) that encodes the unknown parameter \( \varphi \) (supplementary material). So the Heisenberg bound is \( I \propto \log_2 N \), at least asymptotically for large \( N \). In the RMSE case the best precision attainable for unentangled parallel strategies scales as the square root of the Heisenberg bound, so an intuitive definition of information-theoretic standard quantum limit is \( I \propto \log_2 \sqrt{N} = \frac{1}{2} \log_2 N \). As shown below, this is the correct definition since unentangled parallel strategies are indeed bounded by this quantity. These definitions are consistent with the RMSE based ones: an
error $\Delta \varphi \simeq 1/N$ leads to the expectation that roughly \( \log_2 N \) binary digits of the results are reliable, and similarly an error $\Delta \varphi \simeq 1/\sqrt{N}$ leads to the expectation that $\frac{1}{2} \log_2 N$ digits are reliable. Nonetheless, the RMSE and the mutual information capture different aspects of the estimation’s quality, as shown below.

Below we show which kinds of estimation strategies achieve these bounds. An example (the QPEA) shows that sequential and entangled-parallel strategies can achieve the info-theoretic Heisenberg bound. We then show that the optimal parallel-separable strategies can only attain the standard quantum limit. We finally discuss the role of the probe’s dimensionality.

### Methods

For the sake of simplicity we will first restrict to two-dimensional probes (qubits), for which $U_\varphi = |0⟩⟨0| + e^{i2\pi \varphi} |1⟩⟨1|$ (with $|0⟩$ and $|1⟩$ the eigenstates of the generator of $U_\varphi$), and then separately analyze what happens in the (finite) $d$-dimension case. We use finite-dimensional probes and unitaries, so the parameter $\varphi$ is periodic and we restrict to $\varphi \in [0,1]$. As is customary in quantum metrology, we request no prior knowledge on the parameter to be estimated (uniform prior).

![FIG. 1: Sequential and parallel-entangled strategies. (a) Sequential strategy, where a single probe (large triangle) samples $N$ unitaries $U_\varphi$ (black boxes) sequentially. Ancillary systems (small triangles) may interact through arbitrary intermediate unitaries (gray squares). (b) Quantum phase estimation algorithm (QPEA). To see that it is equivalent to a sequential strategy [21], where the last unitary is the inverse quantum Fourier transform (QFT$^\dagger$), use intermediate unitaries that swap the state of the ancillas with the state of the probe. The output (measured in the computational basis) is a $t$-bit digital estimate of the parameter $\varphi$ with $t = \log_2(N+1)$. (c) Parallel QPEA, which uses entangled N00N states (dashed boxes) composed of $1, 2, 4, \ldots, 2^{t-1}$ qubits. The circles represent C-NOT gates that remove the entanglement, and the cups represent the discarding of qubits in the state $|0⟩$.

**Sequential strategies:**— In sequential strategies [4, 21, 23] the transformations $U_\varphi$ act on a single probe sequentially and ancillas may interact with the probe at any intermediate stage. Fig. 1a. We consider the QPEA [20, 21] as an example of sequential strategy, Fig. 1b: it needs $t = \log_2(N+1)$ qubits initialized in $|+⟩ \propto |0⟩ + |1⟩$ states, where the zero-th qubit is subject to $U_\varphi$ once, and the $j$-th qubit is subject to $U_\varphi$ $2^j$ times. The $t$ qubits then undergo a quantum Fourier transform (QFT) and are measured in the computational basis, yielding a $t$-bit number $m$, from which $\varphi$ can be estimated as $m/2^t$. One can see that the QPEA is a sequential strategy by considering one of the qubits as the probe and the others as ancillas, and inserting appropriate swap-unitaries to swap the ancilla states and the probe state (the zero-th swap after a single $U_\varphi$ action, the $j$-th after $2^j$ actions) [21].

To evaluate how many of the bits of $m$ are reliable, one needs to calculate the mutual information $I(m : \varphi)$, using the QPEA conditional probability

$$p(m|\varphi) = \frac{\sin^2(\pi(N+1)\varphi)}{(N+1)^2 \sin^2(\pi(\varphi - m/(N+1)))}.$$ (2)

The mutual information obtained from it has an asymptotic scaling in $N$ given by (supplementary material)

$$I(m : \varphi) \rightarrow \log_2 N - 2 + 2^{1+\ln(2)/\ln(2)} - 1 \simeq \log_2 N - 1.2199,$$ (3)

where $\gamma$ is the Euler-Mascheroni constant. Namely, it (quickly) asymptotically achieves the info-theoretic Heisenberg bound, apart from a small additive constant, see Fig.[2].

The QPEA is known to also achieve the best estimation in terms of a window function cost [21], but it cannot achieve the RMSE-based Heisenberg bound unless one repeats it a few times [17, 14].

![FIG. 2: Heisenberg bound of the QPEA. (a) Plot of the mutual information $I(m : \varphi)$ as a function of $N$ (blue) and of the function $\log_2 N$ (dashed red). Note that $I$ quickly acquires the same linear dependence in a log scale as the Heisenberg bound. The inset shows the same behavior for large $N$. (b) Ratio between the mutual information and $\log_2 N = 1.2199$, showing the rapid onset of the asymptotic behavior to this quantity.

**Parallel entangled strategies:**— The proof that parallel entangled strategies can achieve the mutual-info Heisenberg bound is simple, since one can easily transform the sequential strategy detailed above into a parallel one by entangling the probes: see Fig. 1c. This means that one
uses $N$ probes grouped in $N00N$ states of increasing number of bits: \( |0⟩ + |1⟩, |00⟩ + |11⟩, \ldots, |0^k⟩ + |1^k⟩, \ldots \). When these $\log_2(N+1)$ groups interact in parallel with the $N$ transformations $U_ϕ$, the $j$th group acquires a phase of $2\pi 2^jϕ$, the same as the corresponding probe in the QPEA strategy of Fig. 11. A simple network of controlled-not gates can transfer this phase to one of the probes in each group and the other probes in the group are discarded. So the input to the final quantum Fourier transform is identical to the one of the conventional QPEA. Thus both the output probability and the mutual information are the same as the ones calculated above: also the parallel entangled strategy can achieve the Heisenberg bound (apart from a small additive constant).

Note that the use of controlled-not gates after the action of the transformations $U_ϕ$ imply that this procedure requires an entangled detection strategy (in contrast, the QFT does not require entanglement among probes). It is still an open question whether a parallel entangled strategy can achieve the info-theoretic Heisenberg bound with a separable detection, as is the case for the RMSE bound. The Heisenberg bound is not achieved (supplementary material) if one uses the same detection strategy as in the RMSE case (namely projecting each probe onto the $|±⟩ \propto |0 \pm 1⟩$ states) or if one employs the single-qubit optimal strategy according to Davies theorem (see below).

Parallel separable strategies:— To prove that without entanglement the parallel strategies cannot achieve the Heisenberg bound, one needs to analyze the optimal strategy and show that it can only achieve the standard quantum limit. (Whereas to prove that the sequential and entangled strategies can achieve the Heisenberg bound, we merely had to exhibit an example, the QPEA above.)

In the separable case, the optimal input state for each qubit probe is an equatorial state, such as $(|0⟩ + |1⟩)/\sqrt{2}$, which is evolved by $U_ϕ$ into $|ϕ⟩ = (|0⟩ + e^{i2ϕ}|1⟩)/\sqrt{2}$. Indeed equatorial states maximize the distinguishability between input and output. The $N$ parallel probes after the $U_ϕ$ evolutions emerge in a joint state

$$|ϕ⟩^N = \sum_{j=0}^{N} \frac{1}{\sqrt{N}} \binom{N}{j} e^{i2ϕj} |S_j⟩,$$

where $|S_j⟩$ is the normalized symmetric state obtained by summing over all possible permutations with $j$ ones, e.g. for $N = 4$, $|S_0⟩ \propto |0001⟩ + |0010⟩ + |0100⟩ + |1000⟩$, $|S_2⟩ \propto |0011⟩ + |0101⟩ + |0110⟩ + |1010⟩ + |1100⟩$.

To obtain the POVM that maximizes the mutual information on this state, we use Davies’ theorem [27]: If the input is covariant with respect to a group that admits an irreducible unitary representation $U_ϕ$, then there exists a unit vector $|r⟩$ such that the mutual information is maximized by the positive operator-valued measure (POVM)

$$\Pi_ϕ = \frac{1}{2π} U_ϕ |r⟩⟨r| U_ϕ^†,$$ (5)

where $d$ is the dimension of the system Hilbert space and $|G|$ is the number of elements in the group [27]. Davies’ theorem can be extended to continuous parameters $ϕ$ by requiring the compactness of the group [28] and to unitary representations that are irreducible only on equatorial states [28].

Since the state $|ϕ⟩^N$ spans only the $N+1$-dimensional symmetric subspace of the $N$-qubits space, we can limit ourselves to it. So the optimal POVM is given by $\Pi_ϕ = \Pi$ with $d = N + 1$, $|G| = 1$ and $|r⟩$ a state in the symmetric subspace: $|r⟩ = \sum_j α_j |S_j⟩$. Apart from an irrelevant phase factor, this state is uniquely determined by the POVM’s normalization condition $\int dϕ \Pi_ϕ = 1$ (see [30]). Indeed, this condition is satisfied only if $|α_j| = 1/\sqrt{N + 1}$ for all $j$. Hence an optimal POVM is

$$\Pi_ϕ = (N + 1)|ϕ⟩⟨ϕ|, \text{ with } |ϕ⟩ = \frac{1}{\sqrt{N + 1}} \sum_{n=0}^N e^{i2πϕn} |S_n⟩.$$

Then, the conditional probability of finding the result $ϕ$ (which is our estimate of the unknown parameter) when the true value is $ϕ$ is

$$p(ϕ|ϕ) = (⟨ϕ|^N)\Pi_ϕ(⟨ϕ|)^N$$ (7)

$$= \sum_{n,n'} \frac{1}{N} \binom{N}{n} \binom{N}{n'} e^{i2π(ϕ−ϕ)(n−n')}$$ (8)

whence one can calculate the mutual information $I(ϕ : ϕ)$. Its asymptotic scaling (supplementary material) is

$$I(ϕ : ϕ) \rightarrow \frac{1}{2} \log_2 N + \frac{1}{2} \log_2 \frac{2π}{e} \approx \frac{1}{2} \log_2 N + 0.6,$$

namely the standard quantum limit for the mutual information (apart from a small additive constant). The explicit evaluation of $I(m : ϕ)$ shows that it quickly attains the asymptotic expression, Fig. 14. This proves that separable probes can achieve at most the standard quantum limit.

The above strategy uses separable input states, but an entangled POVM $\Pi_ϕ$ (the states $|ϕ⟩$ are entangled). We now show that the standard quantum limit can be achieved also by a strategy separable both at the input and at the measurement. Indeed, consider the strategy in which we measure the separable state $|ϕ⟩^N$ with a projective POVM which projects onto the states $|±⟩ \propto |0 \pm 1⟩$ each of the $N$ qubits separately. The outcome will be a string $m$ of $N$ zero-one results corresponding to outcome “+” or “−” at each qubit respectively. The probability of each outcome is $p(+|ϕ) = \cos^2(ϕ)$ and $p(−|ϕ) = \sin^2(ϕ)$, so the probability of obtaining the whole string $m$ is

$$p(m|ϕ) = \sin^2κ(ϕ) \cos^{2(N−κ)}(ϕ),$$ (10)
where $\kappa$ is the number of ones in the string $\vec{m}$ (its Hamming weight). The unknown parameter is easily estimated from the vector $\vec{m}$ as $\kappa/N$. The marginal probability of the string $\vec{m}$ is then

$$p(\vec{m}) = \int_0^1 d\varphi p(\vec{m}|\varphi) = \frac{(2(N-\kappa))!(2\kappa)!}{2^{2N}(N-\kappa)!\kappa!N!}.$$ (11)

Whence mutual information is (supplementary material)

$$I(\vec{m} : \varphi) = N/\ln 2 + \sum_{\kappa=0}^N \frac{2(N-\kappa))!(2\kappa)!}{4^N((N-\kappa)!^2(\kappa)!^2}\log_2 \frac{(N-\kappa)!\kappa!N!}{(2(N-\kappa))!(2\kappa)!}.$$ (12)

The asymptotic scaling of $I(\vec{m} : \varphi)$ of Eq. (12) for large $N$ was numerically checked (Fig. 3a) and goes as $\sim \log(N)/2 - 0.395$ (the constant was evaluated numerically), as expected from the standard quantum limit.

**Beyond qubits:** We now drop the assumption of two-dimensional probes (qubits) and consider the effect of a $d$-dimensional Hilbert space of the probes. In this case, we must consider the transformation $U_{\varphi} = \sum_{n=1}^{d-1} e^{i2\pi n\varphi} |n\rangle \langle n|$, where $|n\rangle$ are eigenstates with eigenvalue $n$ of the generator $H$ of $U_{\varphi}$. Intuitively, one expects that a two-dimensional probe will give outcomes in bits (base-2 numbers) and that a $d$-dimensional probe will give outcomes in base-$d$ numbers. We will see that this intuition is correct: one will asymptotically gain the factor $\log_2 d$ of a change of basis in the logarithms in the mutual information definition. We can prove this result using a $d$-dimensional extension of the QPEA for the sequential and entangled protocols, and using the Pegg-Barnett states for the separable protocol (also shown in [23]).

The QPEA for $d$-dimensional systems is a straightforward extension of the QPEA. Its output is a number $m$ composed of $t$ base-$d$ digits, whence the parameter $\varphi$ can be estimated as $m/d^t$. The conditional probability of obtaining $m$ given $\varphi$ is

$$p(m|\varphi) = \frac{\sin^2(\pi \varphi d^t)}{d^{2t} \sin^2[\pi(\varphi - m/d^t)]},$$ (13)

analogous to (2). The mutual information is then

$$I(m : \varphi) = t \log_2 d + \int_0^1 d\varphi \sum_m p(m|\varphi) \log_2 p(m|\varphi)$$

$$\rightarrow t \log_2 d - 1.2199,$$ (14)

where the asymptotic scaling is derived in the same way as for (3). The $t \sim \log_2 N$ factor in (14) accounts for the Heisenberg scaling of the QPEA, while the $\log_2 d$ term accounts for the increase in the dimensionality of the probes. The form of $I(m : \varphi)$ as a function of $d$ is the same as the one shown in Fig. 2 if one replaces $N+1$ with $d^t$; compare (12) with (4). Hence as in the previous case, the asymptotic scaling (14) kicks in very rapidly.

In the separable case, we can find a similar $\log_2 d$ factor by preparing each $d$-dimensional probe in the Pegg-Barnett state $\sum_n |n\rangle/\sqrt{d}$ [22], which is evolved by $U_{\varphi}$ into the state $|\varphi, d\rangle = \sum_n e^{2\pi i n \varphi} |n\rangle/\sqrt{d}$. A measurement that extracts information from the probe asymptotically approaching $\log_2 d$ bits is a projective POVM onto the states $|\varphi_j, d\rangle$ with $\varphi_j = j/d$ (with $j = 0, \cdots, d-1$) [24]. This is equivalent to the above $d$-dimensional QPEA for a single probe $t = 1$, so the mutual information of this Pegg-Barnett procedure is given by Eq. (14) with $t = 1$, where again we find a $\log_2 d$ factor. Hence, also in the separable case, an increase in the probes dimension leads to a $\log_2 d$ increase in the estimation precision.

Note that, also in the RMSE case, an increase in the probe dimension increases the precision, because we can access larger eigenvalues of the generator of $U_{\varphi}$. However, in that case, one can always restrict the probes to a two-dimensional subspace, spanned by the eigenvectors $|0\rangle$ and $|d-1\rangle$ relative to the minimum and maximum eigenvalues of the generator $H$ [3]. In the mutual-info case, this is not true anymore: the above $\log_2 d$ increase in precision is absent if we limit the probe states to the subspace spanned by these two states (supplementary material). Interestingly, the Pegg-Barnett states are known to be useless in achieving the RMSE base Heisenberg scaling in the dimension $d$ [22], in contrast to the above $\log_2 d$ scaling result. These two facts emphasize that, although RMSE and mutual-info give consistent indications on the measurement precision, they really capture different aspects of it.

**Conclusions:** In conclusion, we have given an information-theoretic version of quantum metrology, leading to the main results of ordinary RMSE-based...
quantum metrology, but highlighting some peculiar differences from it. We did not consider the effect of noise and experimental imperfections here, leaving it to future work, since this substantially complicates the theory, as happens in the RMSE-case, e.g. \[ \text{[10]} \text{[33]} \text{[36]} \].

Acknowledgments

LM acknowledges the FQXi foundation for funding. MH thanks A.T. Rezakhani for support and P. Perinotti for useful discussions, the MSRT of Iran and Iran Science Elites Federation for funding, and the University of Pavia for hospitality.

[1] M. Zwierz, C. A. Pérez-Delgado, P. Kok, Ultimate limits to quantum metrology and the meaning of the Heisenberg limit, Phys. Rev. A 85, 042112 (2012).
[2] S. L. Braunstein, C. M. Caves, G. J. Milburn, Generalized uncertainty relations: Theory, examples, and Lorentz invariance, Ann. Phys. 247, 135-173 (1996); S. L. Braunstein, C. M. Caves, Statistical distance and the geometry of quantum states. Phys. Rev. Lett. 72, 3439 (1994).
[3] V. Giovannetti, S. Lloyd, L. Maccone, Advances in quantum metrology, Nature Phot. 5, 222 (2011).
[4] V. Giovannetti, S. Lloyd, L. Maccone, Quantum metrology, Phys. Rev. Lett. 96, 010401 (2006).
[5] M. G. A. Paris, Quantum estimation for quantum technology, Int. J. Quantum Inf. 7, 125 (2009).
[6] R. Demkowicz-Dobrzański, M. Jarzyna, J. Kolodynski, Quantum limits in optical interferometry, Progress in Optics 60, 345 (2015).
[7] G. Toth, I. Apellaniz, Quantum metrology from a quantum information science perspective, J. Phys. A: Math Th. 47, 424006 (2014).
[8] R. Demkowicz-Dobrzański, J. Kołodynski, M. Guta, The elusive Heisenberg limit in quantum-enhanced metrology, Nature Comm. 3, 1063 (2012).
[9] M. Kacprzowicz, R. Demkowicz-Dobrzański, W. Wasilewski, K. Banaszek, I. A. Walmsley, Experimental quantum-enhanced estimation of a lossy phase shift, Nature Photonics 4, 357 (2015).
[10] B. M. Escher, R. L. de Matos Filho, L. Davidovich, General framework for estimating the ultimate precision limit in noisy quantum-enhanced metrology, Nature Phys. 7, 406 (2011).
[11] D.S. Simon, G. Jaeger, A.V. Sergienko, Quantum Metrology, Imaging, and Communication (Springer, 2017).
[12] J. C. F. Matthews, et al. Towards practical quantum metrology with photon counting, NPJ Quantum Inf. 2, 16023 (2016).
[13] V. Giovannetti, S. Lloyd, L. Maccone, Quantum-enhanced measurements: Beating the standard quantum limit, Science 306, 1330 (2004).
[14] H. Lee, P. Kok, J. P. Dowling, A quantum Rosetta stone for interferometry, J. Mod. Opt. 49, 2325 (2002).
[15] M. de Burgh, S. D. Bartlett, Quantum methods for clock synchronization: Beating the standard quantum limit without entanglement, Phys. Rev. A 72, 042301 (2005).
[16] B. L. Higgins, et al. Demonstrating Heisenberg-limited unambiguous phase estimation without adaptive measurements, New J. Phys. 11, 073023 (2009).
[17] D.W. Berry, et al. How to perform the most accurate possible phase measurements, Phys. Rev. A 80, 052114 (2009).
[18] M. W. Mitchell, Metrology with entangled states, Proc. SPIE 5893, 589310 (2005).
[19] J. Combes, H. M. Wiseman, States for phase estimation in quantum interferometry, J. Opt. B, Quant. Semiclass. Opt. 7, 14 (2005).
[20] R. Cleve, A. Ekert, M. Macchiavello, M. Mosca, Quantum algorithms revisited, Proc. R. Soc. London A 454, 339 (1998).
[21] W. van Dam, G. M. D’Ariano, A. Ekert, M. Macchiavello, M. Mosca, Optimal quantum circuits for general phase estimation, Phys. Rev. Lett. 98, 090501 (2007).
[22] D. T. Pegg, S. M. Barnett, Phase properties of the quantized single-mode electromagnetic field, Phys. Rev. A 39, 1665 (1989); S. M. Barnett, D. T. Pegg, Limiting procedures for the optical phase operator, J. Mod. Opt. 39, 2121 (1992).
[23] V. Buzek, R. Derka, S. Massar, Optimal quantum clocks, Phys. Rev. Lett. 82, 2207 (1999).
[24] A. S. Holevo, Probabilistic and Statistical Aspect of Quantum Theory (North-Holland, Amsterdam, 1982).
[25] B. L. Higgins, D. W. Berry, S. D. Bartlett, H. M. Wiseman, G. J. Pryde, Entanglement-free Heisenberg-limited phase estimation, Nature 450, 393 (2007).
[26] R. B. Griffiths, C.-S. Niu, Semi-classical fourier transform for quantum computation, Phys. Rev. Lett. 76, 3228 (1996).
[27] E. B. Davies, Information and quantum measurement, IEEE Trans. Inf. Theory 24, 596 (1978).
[28] G. Chiribella, Optimal estimation of quantum signals in the presence of symmetry, PhD thesis (2006).
[29] M. Sasaki, S. M. Barnett, R. Jozsa, M. Osaki, O. Hirota, Accessible information and optimal strategies for real symmetrical quantum sources, Phys. Rev. A 59, 3325 (1999).
[30] G. M. D’Ariano, C. Macchiavello, M. F. Sacchi, On the general problem of quantum phase estimation, Phys. Lett. A 248, 103 (1998).
[31] C. Ye, P. Shi-Guo, Z. Chao, L. Gui-Lu, Quantum fourier transform and phase estimation in qudit system, Commun. Theor. Phys. 55, (2011)
[32] This was proven in [24], where they use a sin^2 window function which (for small phases) is equivalent to the variance.
[33] S. F. Huelga, et al. Improvement of Frequency Standards with Quantum Entanglement, Phys. Rev. Lett. 79, 3865 (1997).
[34] R. Demkowicz-Dobrzański, L. Maccone, Using entanglement against noise in quantum metrology, Phys. Rev. Lett. 113, 250801 (2014).
[35] K. Micadei, et al. Coherent measurements in quantum metrology. New J. Phys. 17, 023057 (2015).
[36] S. Alipour, M. Mehboudi, A. T. Rezakhani, Quantum metrology in open systems: Dissipative Cramér-Rao bound, Phys. Rev. Lett. 112, 120405 (2014).
Supplementary Material

THE HEISENBERG BOUND IS \( \simeq \log_2 N \)

We prove that the Heisenberg bound is asymptotically equal to \( \log_2 N \) for large \( N \). This follows from the proof that \( N \) applications of the unitary \( U_\varphi \) for unknown \( \varphi \) cannot increase the entropy beyond \( \log_2(N+1) \), whenever the initial probe state \( \rho_0 \) is pure. Consider the projector \( P_k \) that projects onto an \( N \)-qubit state with \( k \) ones, e.g. for \( N = 3 \)
\[
P_0 = |000\rangle\langle 000|, \quad P_1 = |001\rangle\langle 001| + |010\rangle\langle 010| + |100\rangle\langle 100|, \quad P_2 = |110\rangle\langle 110| + |101\rangle\langle 101| + |011\rangle\langle 011|, \quad P_3 = |111\rangle\langle 111|. 
\]
The set \( \{P_k\} \) is a projective POVM, namely \( P_k^2 = P_k \) and \( \sum_{k=0}^N P_k = 1 \). Use this POVM to build the following chain of inequalities
\[
S(\sum_\varphi p_\varphi \rho_\varphi) = S(\sum_\varphi p_\varphi U_\varphi \otimes \rho_0 U_\varphi^\dagger \otimes \rho_0 N^\dagger) \leq S(\sum_{k=0}^N P_k \rho_\varphi U_\varphi \otimes \rho_0 U_\varphi^\dagger P_k) = S(\sum_{k=0}^N P_k \rho_0 P_k) \leq \log_2(N+1), \quad (S1)
\]
where the first inequality follows from the fact that a projective measurement increases entropy, the following equality follows from the fact that the projection removes any \( \varphi \) dependence, and the last inequality follows from the fact that there are \( N+1 \) terms in the sum over \( k \). A similar proof holds also if we apply sequentially the \( N \) unitaries \( U_\varphi \) to a single probe or for hybrid sequential/parallel strategies.

We are interested in the asymptotic scaling for large \( N \), so \( \log_2(N+1) \simeq \log_2 N \) and we consider this last as the Heisenberg bound scaling, since joining Eq. (S1) with the Holevo bound, we find that
\[
I(\vec{m} : \varphi) \leq \log_2(N+1) \simeq \log_2 N. \quad (S2)
\]

THE MUTUAL INFORMATION OF THE QPEA IS \( \simeq \log_2(N+1) - 1.2199 \)

We prove that the mutual information of the QPEA is asymptotically equal to \( \log_2 N - 1.2199 \) for large \( N \). By using Eq.(2) of the main text, we have
\[
I(m : \varphi) = \sum_{m=0}^N \int_0^1 d\varphi \ p(m|\varphi)p(\varphi) \log_2 \left( \frac{p(m|\varphi)}{p(m)} \right) \\
= \log_2(N+1) + \sum_{m=0}^N \int_0^1 d\varphi \frac{1}{(N+1)^2} \frac{\sin^2((N+1)\pi\varphi)}{\sin^2(\pi(\varphi - m/(N+1)))} \log_2 \left( \frac{1}{(N+1)^2} \frac{\sin^2((N+1)\pi\varphi)}{\sin^2(\pi(\varphi - m/(N+1)))} \right) \\
= \log_2(N+1) + \int_0^1 d\varphi \ \log_2 \left[ \sin^2((N+1)\pi\varphi) \right] \\
- \sum_{m=0}^N \int_0^1 d\varphi \frac{1}{(N+1)^2} \frac{\sin^2((N+1)\pi\varphi)}{\sin^2(\pi(\varphi - m/(N+1)))} \log_2 \left[ (N+1)^2 \sin^2(\pi(\varphi - m/(N+1))) \right]. \quad (S3)
\]
The second term of Eq. (S3) is equal to \(-2\) for all \( N \). We focus on the third term of Eq. (S3). First we can get rid of the sum over \( m \) by noticing that \( \sin^2((N+1)\pi\varphi) = \sin^2((N+1)\pi(\varphi - \frac{m}{N+1})) \) for integer \( m \). This implies that, changing the integration variable \( \varphi \to \varphi - \frac{m}{N+1} \) and using the periodicity of the sine to restore the integration extremes, that integral can be also written as
\[
- \sum_{m=0}^N \frac{1}{(N+1)^2} \int_0^1 d\varphi \frac{\sin^2((N+1)\pi\varphi)}{\sin^2(\pi\varphi)} \log_2 \left[ (N+1)^2 \sin^2(\pi\varphi) \right] \quad (S4)
\]
\[
= -\frac{1}{(N+1)} \int_0^1 d\varphi \frac{\sin^2((N+1)\pi\varphi)}{\sin^2(\pi\varphi)} \log_2 \left[ (N+1)^2 \sin^2(\pi\varphi) \right]. \quad (S5)
\]
Now we can change the integration variable to \( x = (N+1)\pi\varphi \) to write Eq. (S5) as
\[
- \frac{1}{(N+1)} \int_0^{(N+1)} \frac{dx}{\pi(N+1)\sin^2(\frac{x}{N+1})} \log_2 \left[ (N+1)^2 \sin^2(\frac{x}{N+1}) \right]. \quad (S6)
\]
IF. 4: Explicit evaluation of the quantity in Eq. \((S8)\) as a function of \(N\). (a) Plot up to \(N = 127(t = 7)\). (b) Plot up to \(N = 100000\). It shows that Eq. \((S8)\) is equal to \(-1.299\) for large \(N\).

It is clear that the major contributions to this integral come from the regions where the \(\sin^2(\frac{y}{(N+1)})\) in the integrand is null, namely for \(x \to 0^+\) and \(x \to \pi(N+1)^-\). Indeed the integrand has a logarithmic divergence there. To look at the asymptotic behavior for \((N+1) \to \infty\), it is better to move this region entirely in the vicinity of \(x \to 0\) by using the periodicity of the integrand to change the integration extremes from \(\int_0^{\pi(N+1)} dx\) to \(\int_0^{\pi(N+1)/2} dy\). Expanding to first order in \(\frac{y}{(N+1)}\) the integrand (which is a good expansion in the region of interest \(y \to 0\)), we get

\[
-\int_{-\pi(N+1)/2}^{\pi(N+1)/2} dy \frac{\sin^2 y}{y^2} \log_2 y^2 \to -\int_{-\infty}^{\infty} dy \frac{\sin^2 y}{y^2} \log_2 y^2 = 2 \left(\frac{\gamma + \ln \frac{2}{\ln 2}}{2}\right) \simeq 0.7801,
\]

where \(\gamma\) is the Euler-Mascheroni constant. Hence, we proved

\[
\sum_{m=0}^{N} \int_0^1 d\varphi \frac{1}{(N+1)^2} \frac{\sin^2((N+1)\varphi)}{(\pi(\varphi - m/(N+1)))^2} \log_2 \left[ \frac{1}{(N+1)^2} \frac{\sin^2((N+1)\varphi)}{(\pi(\varphi - m/(N+1)))^2} \right] \to -2 + 2 \left(\frac{\gamma + \ln \frac{2}{\ln 2}}{2}\right) \simeq -1.2199.
\]

Also the numerical calculations shows that Eq. \((S8)\) converges to \(-1.299\) for large \(N\), see Fig. 4. For the QPEA \(I(m, \varphi) \to \log_2 N - 1.299\).

**THE MUTUAL INFORMATION OF THE QPEA WITH A SEPARABLE DETECTION**

We show that, if one uses the single-qubit measurement which projects each probe onto \(|\pm\rangle\) for the QPEA, the Heisenberg bound is not achieved. Consider the parallel QPEA (Fig. 1c). The state of \(j\)-th probe after the transformations \(U_\varphi = \frac{1}{\sqrt{2}} (|0\rangle + e^{i\pi \varphi} |1\rangle)\) (with \(j = 0, 1, ..., t - 1\)). One can measure each qubit of the probe separately with the projective operators which project onto \(|\pm\rangle \propto |0\rangle \pm |1\rangle\). Therefore the outcome will be string \(\vec{m}\) of \(2^j\) bits corresponding to outcome “+” or “−” at each qubit respectively. If the number of ones in the string (its Hamming weight) is even (odd), the probability \(p(\vec{m} | \varphi)\) is \(\frac{1}{2(2^j-1)} \cos^2(\pi \varphi 2^j) \left(\frac{1}{2(2^j-1)} \sin^2(\pi \varphi 2^j)\right)\). Hence the mutual information of \(j\)-th probe is

\[
I^{(j)}(\vec{m} : \varphi) = \log_2 2^{(2^j)} + \int_0^1 d\varphi \cos^2(\pi \varphi 2^j) \log_2 \left[ \frac{1}{2(2^j-1)} \cos^2(\pi \varphi 2^j) \right] + \sin^2(\pi \varphi 2^j) \log_2 \left[ \frac{1}{2(2^j-1)} \sin^2(\pi \varphi 2^j) \right]
\]

\[
= 1 + \int_0^1 d\varphi \cos^2(\pi \varphi 2^j) \log_2 \left[ \cos^2(\pi \varphi 2^j) \right] + \sin^2(\pi \varphi 2^j) \log_2 \left[ \sin^2(\pi \varphi 2^j) \right]
\]

\[
= 1 + \log_2 \frac{e}{4} \simeq 0.44 .
\]

The total mutual information of \(t\) probes is \(0.44 t = 0.44 \log_2 (N + 1)\) and the Heisenberg bound is not achieved. We cannot even attain the standard quantum limit \(\frac{1}{2} \log_2 N\). Since this is the same result that is obtained from the optimal POVM of the Davies theorem, we can conclude that this POVM is also an optimal one. Then, using Davies’ POVM separately on each qubit would not give any advantage over the above calculation.
THE MUTUAL INFORMATION OF THE SEPARABLE PARALLEL STRATEGY WITH AN ENTANGLED MEASUREMENT

We show that the mutual information of the parallel strategy with a separable initial state and entangled POVM is asymptotically equal to $\frac{1}{2} \log_2 N + 0.6$. The conditional probability for the measurement described by the POVM $\Pi_\phi$ of Eqs. (5,6) of the main text is

$$p(\phi|\varphi) = \langle \varphi^{\otimes N} | \Pi_\phi | \varphi^{\otimes N} \rangle = \sum_{n',n=0}^{N} \sqrt{\frac{1}{2N}} \left( \begin{array}{c} N \\ n' \end{array} \right) \sqrt{\frac{1}{2N}} \left( \begin{array}{c} N \\ n \end{array} \right) \exp [i2\pi(\phi - \varphi)(n' - n)]. \tag{S10}$$

By using the following approximation that is valid for $N \gg 1$:

$$\left( \begin{array}{c} N \\ n \end{array} \right) \approx 2^N \sqrt{\frac{2}{\pi N}} \exp \left[ -\frac{(n - N/2)^2}{N} \right], \tag{S11}$$

in Eq. (S10), we have

$$p(\phi|\varphi) \approx \sqrt{\frac{2}{\pi N}} \sum_{n'=0}^{N} \exp \left[ -\frac{(n' - N/2)^2}{N} \right] \exp \left[ -\frac{(n - N/2)^2}{N} \right] \exp [i2\pi(\phi - \varphi)(n' - n)]$$

$$= \sqrt{\frac{2}{\pi N}} \left( \sum_{n=0}^{N} \exp \left[ -\frac{(n - N/2)^2}{N} \right] \exp [i2\pi(\phi - \varphi)(n - N/2)] \right)$$

$$\times \left( \sum_{n'=0}^{N} \exp \left[ -\frac{(n' - N/2)^2}{N} \right] \exp [i2\pi(\phi - \varphi)(n' - N/2)] \right). \tag{S12}$$

Considering $n'$ and $n$ as continuous variables and replacing the sum with an integral we obtain

$$p(\phi|\varphi) \approx \sqrt{2\pi N} \exp [-2\pi^2(\phi - \varphi)^2 N]. \tag{S13}$$

Substitute Eq. (S13) in the mutual information relation (using uniform prior $p(\phi) = 1$)

$$I(\phi : \varphi) = \int_0^1 \int_0^1 d\varphi \ d\phi \ p(\phi|\varphi) \log_2 [p(\phi|\varphi)]$$

$$\approx \int_0^1 \int_0^1 d\varphi \ d\phi \ \sqrt{2\pi N} \exp [-2\pi^2(\phi - \varphi)^2 N] \log_2 \left[ \sqrt{2\pi N} \exp [-2\pi^2(\phi - \varphi)^2 N] \right]$$

$$= \frac{1}{2} \log_2 N + \log_2 \sqrt{2\pi - (2\pi^2 N)} (\log_2 e) \int_0^1 \int_0^1 d\varphi \ d\phi \ \sqrt{2\pi N} (\phi - \varphi)^2 \exp [-2\pi^2(\phi - \varphi)^2 N]. \tag{S14}$$

Consider the third term of Eq. (S13) and define $\omega := 2\pi^2 N$,

$$- \sqrt{2\pi N} \omega (\log_2 e) \int_0^1 \int_0^1 d\varphi \ d\phi \ (\phi - \varphi)^2 \exp [-\omega(\phi - \varphi)^2]$$

$$= - \sqrt{2\pi N} \omega (\log_2 e) \int_0^1 \int_0^1 d\varphi \ d\phi \ \left[ -\frac{d}{d\omega} (\exp [-\omega(\phi - \varphi)^2]) \right]$$

$$= \sqrt{2\pi N} \omega (\log_2 e) \left[ -\frac{1 + e^{-\omega} + \sqrt{\pi} \text{erf}(\sqrt{\omega})}{\omega} \right]$$

$$= \sqrt{2\pi N} (\log_2 e) \left[ -\frac{1}{2} \frac{\sqrt{\pi}}{\omega} \text{erf}(\sqrt{\omega}) + \frac{1}{\omega} (1 - e^{-\omega}) \right]$$

$$\approx -\frac{1}{2} \log_2 e \quad N \gg 1, \tag{S15}$$

where $\text{erf}(x) := \frac{1}{\sqrt{\pi}} \int_{-x}^{x} dt \ e^{-t^2}$ denotes the error function. This function is equal to 1 for large $x$. Replacing Eq. (S15) in Eq. (S14)

$$I(\phi : \varphi) \rightarrow \frac{1}{2} \log_2 N + \log_2 \sqrt{\frac{2\pi}{e}} \approx \frac{1}{2} \log_2 N + 0.6 \quad N \gg 1. \tag{S16}$$
We define \( t = x \int \) the main text, we have:

\[
I(m : \varphi) = \sum_{\kappa=0}^{N} \int_{0}^{1} d\varphi \left( \frac{N}{\kappa} \right) (\cos^2(\pi\varphi))^{N-\kappa} (\sin^2(\pi\varphi))^{\kappa} \log_2 \left[ (\cos^2(\pi\varphi))^{N-\kappa} (\sin^2(\pi\varphi))^{\kappa} \right] \\
+ \sum_{\kappa=0}^{N} \int_{0}^{1} d\varphi \left( \frac{N}{\kappa} \right) (\cos^2(\pi\varphi))^{N-\kappa} (\sin^2(\pi\varphi))^{\kappa} \log_2 \left[ \frac{4^N (N-\kappa)!\kappa!\Gamma(2\kappa)!}{(2(N-\kappa))!(\kappa)!} \right].
\]  

(S17)

The first part of Eq. (S17) is equal to

\[
\sum_{\kappa=0}^{N} \int_{0}^{1} \left( \frac{N}{\kappa} \right) (\cos^2(\pi\varphi))^{N-\kappa} (\sin^2(\pi\varphi))^{\kappa} (N-\kappa) \log_2 [\cos^2(\pi\varphi)] d\varphi \\
+ \sum_{\kappa=0}^{N} \int_{0}^{1} \left( \frac{N}{\kappa} \right) (\cos^2(\pi\varphi))^{N-\kappa} (\sin^2(\pi\varphi))^{\kappa} \kappa \log_2 [\sin^2(\pi\varphi)] d\varphi,
\]

\[
= \int_{0}^{1} \log_2 [\cos^2(\pi\varphi)] \sum_{\kappa=0}^{N} \left( \frac{N}{\kappa} \right) (\cos^2(\pi\varphi))^{N-\kappa} (\sin^2(\pi\varphi))^{\kappa} (N-\kappa) \, d\varphi \\
+ \int_{0}^{1} \log_2 [\sin^2(\pi\varphi)] \sum_{\kappa=0}^{N} \left( \frac{N}{\kappa} \right) (\cos^2(\pi\varphi))^{N-\kappa} (\sin^2(\pi\varphi))^{\kappa} \kappa \, d\varphi
\]

\[
= N \int_{0}^{1} \{ (\cos^2(\pi\varphi)) \log_2 [\cos^2(\pi\varphi)] + \sin^2(\pi\varphi) \log_2 [\sin^2(\pi\varphi)] \} \, d\varphi,
\]

\[
= N \log_2 \left( \frac{e}{4} \right),
\]

(S18)

where the second equality uses the moment-generating function method.

To calculate the integral in the second line of Eq. (S17) consider the definition of gamma function as \( \Gamma(z) := \int_{0}^{\infty} t^{z-1} e^{-t} dt \), so we have

\[
\Gamma(z)\Gamma(s) := \int_{0}^{\infty} \int_{0}^{\infty} t^{z-1} u^{s-1} e^{-(t+s)} \, dt \, du.
\]

(S19)

We define \( t := x^2 \), \( u := y^2 \) and replace them in Eq. (S19)

\[
\Gamma(z)\Gamma(s) := 4 \int_{0}^{\infty} \int_{0}^{\infty} x^{2z-1} y^{2s-1} e^{-(x^2+y^2)} \, dx \, dy.
\]

(S20)

For simplicity, we use the polar coordinate so \( x = r \cos \theta, y = r \sin \theta \)

\[
\Gamma(z)\Gamma(s) := 4 \int_{0}^{\frac{\pi}{2}} d\theta \cos^{2z-1} \theta \sin^{2s-1} \theta \int_{0}^{\infty} r^{2(z+s)-2} \, e^{-r^2} \\
= 4 \left( \int_{0}^{\frac{\pi}{2}} d\theta \cos^{2z-1} \theta \sin^{2s-1} \theta \right) \frac{1}{2} \left( \int_{0}^{\infty} 2r dr \left( \frac{r^2}{\pi} \right)^{(z+s)-1} e^{-r^2} \right)
\]

\[
= 2 \left( \int_{0}^{\frac{\pi}{2}} d\theta \cos^{2z-1} \theta \sin^{2s-1} \theta \right) \left( \int_{0}^{\infty} \frac{dr}{r^{(z+s)-1}} e^{r} \right) = 2 \left( \int_{0}^{\frac{\pi}{2}} d\theta \cos^{2z-1} \theta \sin^{2s-1} \theta \right) \Gamma(z + s),
\]

(S21)

Therefore

\[
\int_{0}^{\frac{\pi}{2}} d\theta (\cos^2 \theta)^{z-\frac{1}{2}}(\sin^2 \theta)^{s-\frac{1}{2}} = \frac{1}{2} \frac{\Gamma(z)\Gamma(s)}{\Gamma(z + s)}.
\]

(S22)
We show that the mutual information of the probe which is spanned by the states $|\phi\rangle$ that maximizes the mutual information as $\Pi$, after the evolution is $|\pm\rangle \propto |\phi\rangle$, so the mutual information is given by

$$I(\phi: \psi) = \log_2 2 + 2 \int_0^1 \int_0^1 \cos^2(\pi(\phi - \varphi)(d-1)) \log_2 \left[ \cos^2(\pi(\phi - \varphi)(d-1)) \right] \, d\varphi \, d\phi,$$

$$= 1 + \log_2 \frac{e}{4} \approx 0.44 .$$

The numerical calculation of $f(N)$ shows that this term is asymptotically a constant and $f(N) \to -0.395$, Fig. 5.

**THE MUTUAL INFORMATION FOR THE PROBE IN A 2-DIMENSIONAL SUBSPACE**

We show that the mutual information of the probe which is spanned by the states $|0\rangle$ and $|d-1\rangle$ does not increase with the probe Hilbert space dimension $d$. Suppose the initial state is $|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |d-1\rangle)$ and the quantum state after the evolution is $|\psi_\varphi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi\varphi(d-1)}|d-1\rangle)$. One can use the Davies theorem to find the optimal POVM that maximizes the mutual information as $\Pi_\varphi = 2U_\varphi |r\rangle \langle r| U_\varphi^\dagger$, when restricting the Hilbert space to the subspace spanned by $|0\rangle$ and $|d-1\rangle$ the Davies theorem gives that $|r\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |d-1\rangle)$, so the mutual information is

$$I(\phi: \varphi) = \log_2 2 + 2 \int_0^1 \int_0^1 \cos^2(\pi(\phi - \varphi)(d-1)) \log_2 \left[ \cos^2(\pi(\phi - \varphi)(d-1)) \right] \, d\varphi \, d\phi,$$

$$= 1 + \log_2 \frac{e}{4} \approx 0.44 .$$

Also one can use the POVM $|\pm\rangle \propto |0\rangle \pm |d-1\rangle$ to obtain the same result. In this case the mutual information does not depend on $d$. 

For non-negative integer value of $a$, we have $\Gamma(a + \frac{1}{2}) = \frac{(2a)!}{4^a a!} \sqrt{\pi}$. By substituting Eq. (S22) in the second line of Eq. (S17), we get

$$\sum_{\kappa=0}^{N} \frac{(2(N - \kappa))!(2\kappa)!}{4^N((N - \kappa)!)^2(\kappa)!} \log_2 \left[ \frac{4^N(N - \kappa)!k!N!}{(2(N - \kappa))!(2\kappa)!} \right] \simeq 2N + \frac{1}{2} \log_2 N + \log_2 \sqrt{2\pi} + N \log_2 N - N \log_2 e$$

$$+ \sum_{\kappa=0}^{N} \frac{(2(N - \kappa))!(2\kappa)!}{4^N((N - \kappa)!)^2(\kappa)!} \log_2 \left[ \frac{(N - \kappa)!k!}{(2(N - \kappa))!(2\kappa)!} \right],$$

where we use the Stirling’s formula ($\nu! \sim \sqrt{2\pi\nu}(\frac{e}{\nu})^\nu$ for large $\nu$). By substituting Eq. (S18) and Eq. (S23) in Eq. (S17), we get

$$I \simeq \frac{1}{2} \log_2 N + \log_2 \sqrt{2\pi} + N \log_2 N + \sum_{\kappa=0}^{N} \frac{(2(N - \kappa))!(2\kappa)!}{4^N((N - \kappa)!)^2(\kappa)!} \log_2 \left[ \frac{(N - \kappa)!k!}{(2(N - \kappa))!(2\kappa)!} \right],$$

$$= \frac{1}{2} \log_2 N + f(N).$$

The numerical calculation of $f(N)$ shows that this term is asymptotically a constant and $f(N) \to -0.395$, Fig. 5.