Spin-spin effects in radiating compact binaries

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The dynamics of a binary system with two spinning components on an eccentric orbit is studied, with the inclusion of the spin-spin interaction terms appearing at the second post-Newtonian order. A generalized true anomaly parametrization properly describes the radial component of the motion. The average over one radial period of the magnitude of the orbital angular momentum \( \bar{L} \) is found to have no nonradiative secular change. All spin-spin terms in the secular radiative loss of the energy and magnitude of orbital angular momentum are given in terms of \( \bar{L} \) and other constants of the motion. Among them, self-interaction spin effects are found, representing the second post-Newtonian correction to the 3/2 post-Newtonian order Lense-Thirring approximation.

I. INTRODUCTION

Neutron-star–black-hole and black-hole–black-hole binaries are among the most promising sources for the earth-based gravitational wave observatories, such as the Laser Interferometric Gravitational Wave Observatory LIGO [1], VIRGO [2], GEO [3] and TAMA [4]. They are also important sources for the forthcoming Laser Interferometer Space Antenna (LISA) [5,6]. It is generally agreed that in a post-Newtonian description of the motion and of the gravitational radiation of these binary systems a precision of 7/2 post-Newtonian orders has to be reached. At such high orders, due to the nonlinear character of the Einstein equations, various multipolar contributions to the gravitational radiation are present.

Computations based on the general expectation that the orbit of a binary will circularize have reached high post-Newtonian orders. However in various physical situations, as described in Refs. [7]-[10], the eccentricity of the orbits can play a significant role. An equally important feature of the binary system, often neglected, is the effect of the individual spins on the motion and on the radiation of the binary. The dynamics under the influence of the spins was investigated in Refs. [11] - [14]. Spin contributions already appear at the 3/2 post-Newtonian order, and exceptionally at the first post-Newtonian order when angular evolutions are monitored. A complete description should allow for both spin effects and eccentric motions.

In a recent series of papers Gergely, Perjés, and Vasúth have investigated the influence of the spins in the secular evolution and radiation back reaction of binary systems on eccentric orbit up to 3/2 post-Newtonian order. In the last paper [15] of these series, the secular radiative evolution of the energy \( E \) and magnitude of the orbital angular momentum \( \bar{L} \) were given. Both \( E \) and \( \bar{L} \) are constants of the non-radiative motion up to 3/2 post-Newtonian order. At this order, among the spin effects, only the spin-orbit interaction is involved in the radiative change of \( E \) and \( \bar{L} \). The results were in perfect agreement with computations of Rieth and Schäfer [16] in a different, noncovariant spin supplementary condition, with the energy and orbital momentum derived from a different action.

Also in [17], both the nonradiative and radiative evolution of three angles \( \kappa_i = \cos^{-1}(\hat{S}_i \cdot \hat{L}) \), \( (i = 1, 2) \) and \( \gamma = \cos^{-1}(\hat{S}_1 \cdot \hat{S}_2) \), characterizing the relative orientation of the spin vectors \( \hat{S}_1 \) and orbital angular momentum \( \hat{L} \), were given. By considering both the spin-orbit and the spin-spin interactions, the nonradiative change of the angles receives both first and 3/2 post-Newtonian order contributions, while the radiative changes are of 3/2 post-Newtonian order. Remarkably, the radiative change of the spins themselves gave no secular contribution to the radiative evolution of the angles.

Earlier results, pertinent for the physical situations of spinning binaries on eccentric orbit with comparable masses, but one of the spins dominating over the second [17], and a nonspinning particle orbiting about a spinning object (the Lense-Thirring approximation) [18] (see also Ref. [19]), can be obtained as limiting cases of Ref. [15].

The computations in Refs. [14], [17], and [18] were facilitated by the solution of the radial motion of the binary in terms of a generalized true anomaly parameter. Then all relevant integrals presented the remarkable property of having the only pole in the origin. Later a systematic analysis [20] has yielded the same type of parametrization for a wide class of perturbations of the Keplerian motion, including the generic perturbing force of Brumberg [21,22].

The main impediment in generalizing the description of Ref. [15] to include all spin-spin effects was that at the second post-Newtonian order the magnitude \( L \) of the orbital angular momentum, even in the absence of the radiation,

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1 An overhat denotes a unit vector.
is not a conserved quantity. This feature of the spin-spin type perturbation of the Keplerian motion renders the problem even outside the general framework settled in Ref. [20].

The purpose of this paper is to overcome the above difficulty and to give a solution to the radial motion in terms of the generalized true anomaly parametrization, which is valid at the second post-Newtonian order in the presence of the spin-spin interaction terms. This task will be completed by a careful analysis of the specific form of the evolution of the orbital angular momentum. In Sec. II we study the radial motion. We compute the instantaneous expression of $L$ and then we find that it has no secular change. Therefore the angular average value $\bar{L}$ has a deep significance: it has no other change during the whole adiabatic regime of the evolution of the system, then the radiative one. The average magnitude $\bar{L}$ of the orbital angular momentum provides the missing constant of the motion at the second post-Newtonian order, and in fact it coincides with the previously employed $\bar{L}$ up to $3/2$ post-Newtonian orders. As a by-product, the radial equation of motion is derived.

Based on the analysis of the radial motion, in Sec. III we introduce the generalized true and eccentric anomaly parametrizations. The latter is employed in the computation of the radial period, which is found to have the Keplerian expression in terms of the energy of the perturbed motion.

Section IV contains the main results of the paper: the spin-spin contributions to the instantaneous and secular losses of the energy and magnitude of orbital angular momentum. The computations are considerably facilitated by the advantageous properties of the true anomaly parametrization. Among these losses self-interaction type contributions appear.

In a second post-Newtonian order description of the compact binary there are no other spin terms in the radiative losses of energy and magnitude of orbital angular momentum then the spin-spin terms computed in this paper, together with the corresponding spin-orbit expressions from Ref. [15]. We note here that it is equally possible to give a description in terms of the time average $\langle L \rangle$ of the magnitude of orbital angular momentum. However, as will be deduced in the Appendix, the relation between $\langle L \rangle$ and $\bar{L}$ is cumbersome. The description in terms of $\bar{L}$ turns out to be simpler. We keep the velocity of light $c$ and the gravitational constant $G$ in all expressions.

II. THE RADIAL MOTION

The Keplerian motion of a reduced mass particle $\mu$ in the gravitational potential generated by the total mass $m = m_1 + m_2$ is governed by the Lagrangian

$$L_N = \frac{\mu v^2}{2} + \frac{Gm\mu}{r}, \quad (2.1)$$

or equivalently, by the acceleration

$$a_N = -\frac{Gm}{r^3} \quad (2.2)$$

Among other dynamical quantities, the Newtonian energy $E_N$ and magnitude $L_N$ of the Newtonian orbital angular momentum are conserved. In terms of the spherical polar coordinates $r, \theta$ and $\varphi$ they are expressed as

$$E_N = \frac{\mu v^2}{2} - \frac{Gm\mu}{r} = \frac{\mu}{2}[r^2 + r^2(\dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2)] - \frac{Gm\mu}{r}, \quad (2.3)$$

$$L_N^2 = \mu^2 r^4(\dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2). \quad (2.4)$$

From the first expression of $E_N$ we find the velocity $v$. By suitably combining the second expression for $E_N$ with Eq. (2.4) we obtain the equation for the radial motion given by $\dot{r}$:

$$v^2 = \frac{2E_N}{\mu} + \frac{2Gm}{r}, \quad (2.5)$$

$$\dot{r}^2 = \frac{2E_N}{\mu} + \frac{2Gm}{r} - \frac{L_N^2}{\mu^2 r^2}. \quad (2.6)$$

When the Keplerian motion is perturbed, Eqs. (2.5) and (2.6) still hold as identities. However the Newtonian expressions $E_N$ and $L_N$ are not constants of motion any more. The procedure used in Refs. [13], [14], and [15] was to express them in terms of the constants of motion $E, L, \kappa_i$, and $\gamma$ of the perturbed motion and of the radius $r$ alone.
There is no simple way to fulfill a similar task when the perturbation is due to spin-spin interaction, as will be seen in what follows. In fact a milder requirement will be enforced: \( v^2 \) and \( \dot{r}^2 \) will be expressed in terms of constants of motion and a suitably introduced true anomaly parameter \( \chi \).

The motion of the binary under the influence of the spin-spin interactions is governed by the Lagrangian
\[
\mathcal{L} = \mathcal{L}_N + \mathcal{L}_{SS},
\]
\[
\mathcal{L}_{SS} = \frac{G}{c^2 r^3} \left[ (\mathbf{S}_1 \cdot \mathbf{S}_2) - \frac{3}{r^2} (\mathbf{r} \cdot \mathbf{S}_1) (\mathbf{r} \cdot \mathbf{S}_2) \right].
\] (2.7)

We have deliberately omitted those first and second post-Newtonian order terms (PN, 2PN) which do not contain the spins and the spin-orbit (SO) terms. Their contributions were discussed previously in Refs. [7] and [15], and they add linearly to the spin-spin perturbative terms computed in this paper.

An equivalent characterization of the motion is provided by the acceleration
\[
\mathbf{a} = \mathbf{a}_N + \mathbf{a}_{SS},
\]
\[
\mathbf{a}_{SS} = -\frac{3G}{c^2 r^5} \{ r^2 [\mathbf{r} (\mathbf{S}_1 \cdot \mathbf{S}_2) + \mathbf{S}_1 (\mathbf{r} \cdot \mathbf{S}_2) + \mathbf{S}_2 (\mathbf{r} \cdot \mathbf{S}_1)]
-5\mathbf{r} (\mathbf{r} \cdot \mathbf{S}_1) (\mathbf{r} \cdot \mathbf{S}_2) \}.
\] (2.8)

The perturbed Keplerian motion (2.7) is characterized [13], [14] by the conservation of the total energy
\[
E = E_N + E_{SS},
\]
\[
E_{SS} = -\mathcal{L}_{SS}
\] (2.10)

and of the total angular momentum vector
\[
\mathbf{J} = \mathbf{L} + \mathbf{S},
\] (2.11)
where \( \mathbf{L} = \mathbf{L}_N = \mu (\mathbf{r} \times \mathbf{v}) \) is the total orbital angular momentum (there is no spin-spin contribution to the orbital angular momentum and the PN, 2PN, SO contributions are not listed here). \( \mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 \) is the total spin. Both spin vectors undergo a precessional motion due to the spin-spin interaction
\[
\dot{\mathbf{S}}_i = \frac{G}{c^2 r^3} \left[ \frac{3}{r^2} (\mathbf{r} \cdot \mathbf{S}_j) \mathbf{r} - \mathbf{S}_j \right] \times \mathbf{S}_i,
\] (2.12)
where \( i \neq j \).

From the conservation of the total angular momentum \( \mathbf{J} \), Eq. (2.11) and the precession equations (2.12) we find that the orbital angular momentum evolves as
\[
\dot{\mathbf{L}} = \frac{3G}{c^2 r^5} [ (\mathbf{r} \cdot \mathbf{S}_2) \mathbf{S}_1 + (\mathbf{r} \cdot \mathbf{S}_1) \mathbf{S}_2 ] \times \mathbf{r}.
\] (2.13)

The magnitude of the orbital angular momentum changes according to \( \dot{L} = \dot{\mathbf{L}} \cdot \mathbf{L} \), thus
\[
\dot{L} = -\frac{3G}{c^2 r^5} \mathbf{S}_1 \mathbf{S}_2 \left\{ (\mathbf{r} \cdot \dot{\mathbf{S}}_2) \left[ \dot{\mathbf{L}} \cdot (\mathbf{r} \times \dot{\mathbf{S}}_1) \right] + (\mathbf{r} \cdot \dot{\mathbf{S}}_1) \left[ \dot{\mathbf{L}} \cdot (\mathbf{r} \times \dot{\mathbf{S}}_2) \right] \right\}.
\] (2.14)

This change in \( L \) is of second post-Newtonian order. Therefore Eq. (2.14) can be simplified by inserting the previously derived Newtonian expressions, Eqs. (2.19) and (2.22) of Ref. [15]:
\[
\mathbf{r} \cdot \dot{\mathbf{S}}_i = r \sin \kappa_i \cos (\psi - \psi_i),
\] (2.15)
\[
\dot{\mathbf{L}} \cdot (\mathbf{r} \times \dot{\mathbf{S}}_i) = -r \sin \kappa_i \sin (\psi - \psi_i),
\] (2.16)
where the angles \( \psi, \psi_i \) are subtended by the node line with the position \( \mathbf{r} \) and the projections of the spins in the plane of the orbit, respectively, see Fig. 1 in Ref. [15]. In order to derive Eq. (2.16) the auxiliary relation
\[
\mathbf{v} \cdot \dot{\mathbf{S}}_i = \dot{r} \sin \kappa_i \cos (\psi - \psi_i) - \frac{L}{\mu} \sin \kappa_i \sin (\psi - \psi_i)
\] (2.17)
was employed. The angles \( \psi_i \) are given implicitly in terms of \( \kappa_i \) and \( \gamma \) by Eq. (2.20) of Ref. [15]

\[
S_1 \sin \kappa_1 \cos \psi_1 + S_2 \sin \kappa_2 \cos \psi_2 = 0
\]  

(2.18)

and by the spherical cosine identity

\[
\cos \gamma = \cos \kappa_1 \cos \kappa_2 + \sin \kappa_1 \sin \kappa_2 \cos \Delta \psi,
\]

(2.19)

where \( \Delta \psi = \psi_2 - \psi_1 \). We introduce the additional notation \( \bar{\psi} = (\psi_1 + \psi_2)/2 \).

In terms of the above angles the variation of the magnitude of orbital angular momentum takes the simple form

\[
\dot{L} = \frac{3Gc^2}{r^3} S_1 S_2 \sin \kappa_1 \sin \kappa_2 \sin 2(\psi - \bar{\psi})
\]

(2.20)

We remark that none of the perturbations not treated here (PN, 2PN, SO) would give contributions to the above expression for \( \dot{L} \). This is to be contrasted with \( \dot{L} \), which receives also the SO-type contribution [Eq. (2.11) in Ref. [15]].

The change in the magnitude of the orbital angular momentum in the second post-Newtonian order is a feature not encountered in the discussion of the PN, 2PN, SO effects. This was the major inconvenience in generalizing the description of Ref. [15] to include the spin-spin contributions. We will see in what follows how to handle this difficulty by means of an alternative description based on a properly defined average value \( \bar{L} \) of \( L \).

The Newtonian limit of a generalized true anomaly parameter \( \chi \) (to be defined later in this paper) has to coincide with the Keplerian true anomaly. This feature is enough to introduce it in second post-Newtonian expressions already at this stage. As implied by Eq. (2.20), the magnitude \( L(\chi) \) of the orbital angular momentum at some value of the true anomaly \( \chi \in [0, 2\pi] \) differs in the second post-Newtonian order from \( L_0 = L(0) \):

\[
L(\chi) = L_0 + \int_0^\chi \frac{\dot{L}}{d\chi'} d\chi' .
\]

(2.21)

We compute the integral by inserting Eq. (2.20) and the following Newtonian expressions:

\[
\frac{dt}{d\chi} = \frac{\mu r^2}{L},
\]

(2.22)

\[
r = \frac{\bar{L}^2}{\mu(Gm\mu + \bar{A}\cos \chi)},
\]

(2.23)

\[
\psi = \chi + \psi_0 .
\]

(2.24)

The angle \( \psi_0 \) is subtended by the direction of the periastron and the node line, see Fig. 1 in Ref. [15] and we have denoted by \( \bar{L} \) the average value of \( L(\chi) \) over the range \( (0, 2\pi) \). The quantity \( \bar{A} \) defined by

\[
\bar{A} = \left( G^2 m^2 \mu^2 + \frac{2E\bar{L}^2}{\mu} \right)^{1/2}
\]

(2.25)

is the magnitude of the Laplace-Runge-Lenz vector for a Keplerian motion characterized by \( E \) and \( \bar{L} \). The integration yields

\[
L(\chi) = L_0 + \frac{G\mu^2}{2c^2 L^3} S_1 S_2 \sin \kappa_1 \sin \kappa_2 \Theta ,
\]

\[
\Theta = 3Gm\mu \left[ \cos 2(\psi_0 - \bar{\psi}) - \cos 2(\chi + \psi_0 - \bar{\psi}) \right] + 4\bar{A} \left[ \cos 2(\psi_0 - \bar{\psi}) - \cos \chi \cos 2(\chi + \psi_0 - \bar{\psi}) \right] - 2\bar{A} \sin \chi \sin 2(\chi + \psi_0 - \bar{\psi}).
\]

(2.26)

It is straightforward to verify that \( L(0) = L(2\pi) = L_0 \), because all three terms of \( \Theta \) independently vanish at \( \chi = 0, 2\pi \).

Therefore there is no change due to the spin-spin interaction in the magnitude of the orbital angular momentum over one radial period.

A second integration over the parameter \( \chi \) from 0 to \( 2\pi \) gives the angular average of the magnitude of orbital angular momentum:
\[ \dot{L} = \frac{1}{2\pi} \int_0^{2\pi} L(\chi) d\chi = L_0 + \frac{G\mu^2(3G\mu + 4\bar{A})S_1S_2\beta_0}{2c^2L^3}, \]  

(2.27)

where we have introduced the notation

\[ \beta_0 = \sin \kappa_1 \sin \kappa_2 \cos 2(\psi_0 - \tilde{\psi}). \]  

(2.28)

In terms of \( \dot{L} \) the expression of \( L(\chi) \) reduces to

\[ L(\chi) = \dot{L} - \frac{G\mu^2}{2c^2L^3}S_1S_2 \sin \kappa_1 \sin \kappa_2 \{2\bar{A} \cos(\chi + 2(\psi_0 - \tilde{\psi})) + (3G\mu + 2\bar{A} \cos \chi) \cos 2(\chi + \psi_0 - \tilde{\psi}) \}. \]  

(2.29)

A considerably simpler task is to find the spin-spin part \( E_{SS} \) of the energy. This can be expressed from Eq. (2.10) by use of Eqs. (2.15) and (2.19) (the azimuthal angle \( \psi \) being eliminated by its expression (2.24)):

\[ E_{SS}(r, \chi) = -\frac{G}{2c^2r^4}S_1S_2 \alpha(\chi), \]  

(2.30)

\[ \alpha(\chi) = 3 \cos \kappa_1 \cos \kappa_2 - \cos \gamma \]  

(2.31)

\[ -3 \sin \kappa_1 \sin \kappa_2 \cos 2(\chi + \psi_0 - \tilde{\psi}). \]

\( E_{SS} \) is given solely in terms of constants of the motion, radius \( r \) and true anomaly parameter \( \chi \). Now we are in position to write the expressions for \( v^2 \), Eq. (2.3) and for \( \dot{r}^2 \), Eq. (2.6) in suitable form:

\[ v^2 = \frac{2[E - E_{SS}(r, \chi)]}{\mu r} + \frac{2Gm}{r}, \]  

(2.32)

\[ \dot{r}^2 = \frac{2[E - E_{SS}(r, \chi)]}{\mu r} + \frac{2Gm}{r} - \frac{L(\chi)^2}{\mu^2r^2}, \]  

(2.33)

with \( L(\chi) \) and \( E_{SS}(r, \chi) \) given by Eqs. (2.28) and (2.30).

As the angles \( \kappa_1 \) are constant up to the first post-Newtonian order and \( \gamma \) up to one-half post-Newtonian order [see Eqs. (2.17) in Ref. [13] ], they do not vary in the above expressions. In principle it is possible to rewrite Eqs. (2.32) and (2.33) in terms of constants of motion and \( r \) alone. This is because the true anomaly parameter \( \chi \) enters only at Newtonian order, in which order the relation \( \chi = \chi(r) \) is given by Eq. (2.23). We avoid to write such cumbersome expressions, as our task will be exactly the converse: to parametrize all expressions in terms of \( \chi \).

### III. THE TRUE AND ECCENTRIC ANOMALY PARAMETRIZATIONS

In this section first we introduce the generalized true anomaly parametrization \( r = r(\chi) \) for the radial motion (2.33). For this purpose we have to find the turning points \( r_{\text{min}} \) and \( r_{\text{max}} \) defined by

\[ r_{\text{min}} = r(0), \quad \dot{r}^2(0) = 0, \]  

\[ r_{\text{max}} = r(\pi), \quad \dot{r}^2(\pi) = 0. \]  

(3.1)

As from Eq. (2.23) the magnitude of the orbital angular momentum in the turning points is found in the form \( L(0) = \dot{L} + 2L_\pm \) and \( L(\pi) = \dot{L} + 2L_\pm \) with

\[ \delta L_\pm = -\frac{G\mu^2}{2c^2L^3}(3G\mu + 4\bar{A})S_1S_2\beta_0, \]  

(3.2)

the radial equations (3.1) evaluated at the turning points have the explicit form:

\[ 0 = \frac{2E}{\mu} + \frac{2Gm}{r_{\text{max}} \text{min}} \frac{\dot{L}(\dot{L} + 2\delta \dot{L}_\pm)}{\mu^2r_{\text{max}} \text{min}} + \frac{GS\alpha_0}{\mu^2r_{\text{max}} \text{min}}. \]  

(3.3)

Here we have denoted \( \alpha(0) = \alpha(\pi) = \alpha_0 \). We seek for solutions in the form \( r_{\text{max}} = r_\pm + \epsilon_\pm \), where \( \epsilon_\pm \) are small corrections to the expressions of the turning points of a Keplerian motion characterized by \( \dot{L} \) and \( \bar{A} \).
\[ r_{\pm} = \frac{G\mu \pm \bar{A}}{-2E} = \frac{\bar{L}^2}{\mu(G\mu \mp \bar{A})}. \quad (3.4) \]

The turning points are
\[
\begin{align*}
  r_{\text{max}}_{\text{min}} &= \frac{G\mu \pm \bar{A}}{-2E} - \frac{G\mu}{2c^2AL^2}S_1S_2\rho_\mp, \\
  \rho_\mp &= \alpha_0(\bar{A} + G\mu) + \beta_0(4\bar{A} + 3G\mu). \quad (3.5)
\end{align*}
\]

Then we define the generalized true anomaly parameter \( \chi \) by the relation
\[
\frac{2}{r} = \left( \frac{1}{r_{\text{min}}} + \frac{1}{r_{\text{max}}} \right) + \left( \frac{1}{r_{\text{min}}} - \frac{1}{r_{\text{max}}} \right) \cos \chi, \quad (3.6)
\]
which yields the expression
\[
r = \frac{\bar{L}^2}{\mu(G\mu + A\cos \chi)} - \frac{G\mu S_1 S_2 \Lambda}{2c^2 A L^2 (G\mu + A\cos \chi)^2}, \quad (3.7)
\]
with
\[
\Lambda = \bar{A}[\bar{A}^2(\alpha_0 + 4\beta_0) + (G\mu)^2(3\alpha_0 + 10\beta_0)] + G\mu \bar{A}^2(3\alpha_0 + 11\beta_0) + (G\mu)^2(\alpha_0 + 3\beta_0) \cos \chi. \quad (3.8)
\]

Applications of the true anomaly parametrization (3.7) will emerge in the next section when the instantaneous expressions of radiative losses, represented here by a generic function \( f \) will be averaged as
\[
\langle f \rangle = \frac{1}{T} \int_0^{2\pi} f(\chi) \, dt \, d\chi. \quad (3.9)
\]

In \( dt/d\chi = (1/\dot{r})(dr/d\chi) \) the first factor comes from a Taylor series expansion of the radial equation (2.33) and the second is obtained from the derivative of Eq. (3.6):
\[
\frac{dr}{d\chi} = \frac{1}{2} \left( \frac{1}{r_{\text{min}}} - \frac{1}{r_{\text{max}}} \right) r^2 \sin \chi. \quad (3.10)
\]

The parametrization \( z = \exp(i\chi) \) has the remarkable property that the integrals (3.9) are given by the residues in the origin [20]. (The only exception under this rule is the time average \( \langle L \rangle \), which is treated in the Appendix.)

\( T \) is the radial period, to be conveniently computed by means of the generalized eccentric anomaly parametrization, defined as
\[
2r = (r_{\text{max}} + r_{\text{min}}) - (r_{\text{max}} - r_{\text{min}}) \cos \xi, \quad (3.11)
\]
which yields the explicit form
\[
r = \frac{G\mu - \bar{A}\cos \xi}{-2E} - \frac{G\mu S_1 S_2 \Xi}{2c^2 A L^2}, \quad (3.12)
\]
with
\[
\Xi = \bar{A}(\alpha_0 + 4\beta_0) + G\mu(\alpha_0 + 3\beta_0) \cos \xi. \quad (3.13)
\]

For the evaluation of the period
\[
T = \int_0^{2\pi} \frac{dt}{d\xi} d\xi = \int_0^{2\pi} \frac{1}{\dot{r}} \frac{dr}{d\xi} d\xi, \quad (3.14)
\]
the factor \( 1/\dot{r} \) is expressed again from Eq. (2.33) by a Taylor series expansion. However this time we have to employ also the relations between the Newtonian true and eccentric anomaly parameters.
\[
\begin{align*}
\cos \chi &= \frac{Gm\mu \cos \xi - \bar{A}}{Gm\mu - A \cos \xi}, \\
\sin \chi &= \sqrt{-\frac{2E/\mu \bar{L}}{Gm\mu - A \cos \xi}}. 
\end{align*}
\]

(3.15)

Either the immediate integration or the residue theorem yields

\[
T = 2\pi Gm \left( \frac{\mu}{-2E} \right)^{3/2}. 
\]

(3.16)

Similarly as in the case of the spin-orbit perturbations, here the period is given by its Keplerian expression (with \(E\) characterizing the perturbed dynamics).

IV. SPIN-SPIN CONTRIBUTIONS TO THE INSTANTANEOUS AND SECULAR LOSSES

The instantaneous radiative losses of the energy \(E\) and angular momentum \(\mathbf{J}\) were given in terms of symmetric trace-free (STF) multipole moments by Thorne [23]. To the required order they are

\[
\begin{align*}
\frac{dE}{dt} &= -\frac{G}{5c^5} \left[ I^{(3)jl}_N f^{(3)j|l|} + \frac{5}{189c^2} f^{(4)jk} f^{(4)jk} + \frac{16}{9c^2} J^{(3)jl} f^{(3)jl} \right], \\
\frac{dJ}{dt} &= -\frac{2G}{5c^5} \epsilon^{ijk} \left[ f^{(2)jl} f^{(3)kl} + \frac{5}{126c^2} f^{(3)jm} f^{(4)jm} + \frac{16}{9c^2} J^{(2)jl} f^{(3)kl} \right],
\end{align*}
\]

(4.1)

(4.2)

where \(\epsilon^{ijk}\) is the completely antisymmetric Levy-Civita symbol, any number in brackets denotes a corresponding order derivative and the expressions of the symmetric trace-free (STF) multipole moments can be derived by use of the Blanchet-Damour-Iyer formalism [24]. The Newtonian part of the mass quadrupole moment \(I^{jl}_N\) and the spin-orbit part of the velocity quadrupole moment \(J^{jl}_{SO}\) are given by

\[
\begin{align*}
I^{jl}_N &= \mu (x^j x^l)^{STF} , \\
J^{jl}_{SO} &= \frac{3\mu}{2} \left[ x^j \left( \frac{S_1}{m_1} - \frac{S_2}{m_2} \right) \right]^{STF}.
\end{align*}
\]

respectively. As estimates of post-Newtonian orders show [14], these are the only moments needed for evaluation of the spin-spin type contribution to radiation losses.

A. Energy loss

The Newtonian and spin-spin contribution to the instantaneous radiative loss in the energy is found from Eq. (4.1):

\[
\frac{dE}{dt} = -\frac{G}{5c^5} I^{(3)jl}_N (a_N) f^{(3)j|l|} (a_N) - \frac{2G}{5c^5} \left[ I^{(3)jl}_N (a_{SS}) I^{(3)j|l|}_N (a_N) + \frac{8}{9c^2} J^{(3)jl}_{SO} (a_N) J^{(3)j|l|}_{SO} (a_N) \right]. 
\]

(4.5)

The arguments of time derivatives of the momenta contain the contribution to the acceleration to be inserted in the respective terms. The leading order contribution to Eq. (4.5) was given by Peters [25] and the \(S_1 S_2\) contribution by Kidder [14], in terms of \(r, v, S_i\), and \(\dot{r}\). Remarkably the last term of Eq. (4.5) generates also self-interaction contributions (containing either \(S_1^2\) or \(S_2^2\)) [26]. Putting together all these terms, the radiated energy has the expression

\[\text{The poles of the integrand inside the unit circle of the complex parameter plane } w = \exp(i\xi) \text{ are in the origin and at } w_1 = \left( \frac{Gm^2 - \sqrt{-2\mu EL^2}}{Gm^2 + \sqrt{-2\mu EL^2}} \right)^{1/2}. \text{ The residue at } w_1 \text{ vanishes, thus the period (3.16) is given by the residue at the origin as } T = 2\pi iR(w = 0).\]
\[ \frac{dE}{dt} = -\frac{8G^3m^2\mu^2}{15c^2r^6}(12v^2 - 11r^2) \]
\[ - \frac{4G^3m\mu}{15c^2r^8}(-171r^6[(v \cdot S_1)(r \cdot S_2) + (r \cdot S_1)(v \cdot S_2)] \]
\[ + r^2[3(47v^2 - 55r^2)(S_1 \cdot S_2) - 3(168v^2 - 269r^2)(r \cdot S_1)(r \cdot S_2) + 71(v \cdot S_1)(v \cdot S_2)] \]
\[ - \frac{2G^3m^2\mu^2}{15c^2r^8} \sum_i \frac{1}{m_i^2} \left[ 3r^2(v^2 + 3r^2)S_i^2 + 9r^2(r \cdot S_1)(r \cdot S_1) - 6r^4(r \cdot S_1)(v \cdot S_1) + r^2(v \cdot S_1)^2 \right]. \] (4.6)

The summation in the last term is understood over the components of the binary.

This expression, however, is not suitable for averaging by the method described previously. Therefore we insert in the Newtonian terms of Eq. (4.4) the expressions (2.32) and (2.33) for \( v^2 \) and \( r^2 \), respectively, with \( L(\chi) \) and \( E(r, \chi) \) given by Eqs. (2.23) and (2.30). Then we insert in the spin-spin terms of Eq. (4.6) the Newtonian expressions for \( v^2 \), the Eqs. (2.13), (2.17) (both with \( \dot{L} \) in place of \( L \)), Eq. (2.24) and

\[ \dot{r} = \frac{\ddot{L}}{L} \sin \chi, \] (4.7)

obtaining the following expression for the instantaneous loss of the energy in terms of the true anomaly parameter \( \chi \) and radius \( r \) alone:

\[ \frac{dE}{dt} = \frac{dE}{dt}_N + \left( \frac{dE}{dt} \right)_{SS-self} + \left( \frac{dE}{dt} \right)_{S_1S_2}, \] (4.8)

\[ \left( \frac{dE}{dt} \right)_N = \frac{8G^3m^2}{15c^2r^6}(2\mu Er^2 + 2Gm\mu^2r + 11\dot{L}^2), \] (4.9)

\[ \left( \frac{dE}{dt} \right)_{SS-self} = \frac{G^3m^2}{15c^2r^6} \sum_i \frac{S_i}{m_i}^2 \left\{ (6 + \sin^2\kappa_i)(8\mu Er^2 + 8Gm\mu^2r - 3\dot{L}^2) \right. \]
\[ \left. + \sin^2\kappa_i[4\mu \dot{r} \sin \chi \sin 2(\chi + \psi_0 - \psi_i) + (8\mu Er^2 + 8Gm\mu^2r - 5\dot{L}^2) \cos 2(\chi + \psi_0 - \psi_i)] \right\}, \] (4.10)

\[ \left( \frac{dE}{dt} \right)_{S_1S_2} = \frac{2G^3mS_1S_2}{15c^2\mu \dot{L}^2 r^8} \left\{ \frac{3}{n=1} \sum a_n \cos[n\chi + 2(\psi_0 - \psi_i)] \sin \kappa_1 \sin \kappa_2 + a_4 \cos \kappa_1 \cos \kappa_2 + a_5 \cos \gamma \right\}. \] (4.11)

The coefficients \( a_k \) are

\[ a_1 = 4\mu \dot{r}(33Gm\mu^2 r - 25\dot{L}^2), \]
\[ a_2 = -64\mu \dot{r}E \dot{L}^2 r^2 + 132G^2m^2 \mu^4 r^2 - 52Gm\mu^2 \dot{L}^2 r + 607\dot{L}^4, \]
\[ a_3 = 4\mu \dot{r}(11Gm^2 r + 25\dot{L}^2), \]
\[ a_4 = \dot{L}^2(64\mu Er^2 + 52Gm\mu^2 r - 465\dot{L}^2), \]
\[ a_5 = \dot{L}^2(32\mu Er^2 + 36Gm\mu^2 r + 135\dot{L}^2). \] (4.12)

By use of the true anomaly parameterization \( r(\chi), \) Eq. (4.7) we find the energy loss in terms of \( \chi \) alone. When we pass to the complex parameter \( z = \exp(i\chi) \) the only pole of \( dE/dt \) is in the origin. Applying the method described in the previous section, the secular change of the energy emerges in terms of constants of motion and the average value \( \dot{L} \) of the magnitude of orbital momentum:

\[ \left\langle \frac{dE}{dt} \right\rangle = \left\langle \frac{dE}{dt} \right\rangle_N + \left\langle \frac{dE}{dt} \right\rangle_{SS-self} + \left\langle \frac{dE}{dt} \right\rangle_{S_1S_2}, \] (4.13)

\[ \left\langle \frac{dE}{dt} \right\rangle_N = -\frac{G^2m(-2E\mu)^{3/2}}{15c^2\dot{L}^2}(148E^2\dot{L}^4 + 732G^2m^2 \mu^3 E\dot{L} + 425G^4m^4 \mu^6), \] (4.14)

\[ \left\langle \frac{dE}{dt} \right\rangle_{SS-self} = \frac{G^2m(-2E\mu)^{3/2}}{960c^2\dot{L}^{11}} \sum_i \left( \frac{S_i}{m_i} \right)^2 \left[ C_1 \sin^2 \kappa_i \cos 2(\psi_0 - \psi_i) + C_2(6 + \sin^2 \kappa_i) \right], \] (4.15)

\[ \left\langle \frac{dE}{dt} \right\rangle_{S_1S_2} = -\frac{G^2(-2E\mu)^{3/2} S_1S_2}{480c^2\dot{L}^{11}} (C_3 \sin \kappa_1 \sin \kappa_2 \cos 2(\psi_0 - \psi_i) + C_4 \cos \kappa_1 \cos \kappa_2 + C_5 \cos \gamma), \] (4.16)

with the following coefficients \( C_k \):
\[ C_1 = -(72E^3\tilde{L}^6 + 428G^2m^2\mu^3E^2\tilde{L}^4 + 406G^4m^4\mu^6E\tilde{L}^2 + 105G^6m^6\mu^9), \]
\[ C_2 = -4(72E^3\tilde{L}^6 + 660G^2m^2\mu^3E^2\tilde{L}^4 + 910G^4m^4\mu^6E\tilde{L}^2 + 315G^6m^6\mu^9), \]
\[ C_3 = -(60744E^3\tilde{L}^6 + 602668G^2m^2\mu^3E^2\tilde{L}^4 + 189798G^4m^4\mu^6E\tilde{L}^2 + 260825G^6m^6\mu^9), \]
\[ C_4 = 4(17064E^3\tilde{L}^6 + 241140G^2m^2\mu^3E^2\tilde{L}^4 + 453670G^4m^4\mu^6E\tilde{L}^2 + 199899G^6m^6\mu^9), \]
\[ C_5 = -12(2056E^3\tilde{L}^6 + 28260G^2m^2\mu^3E^2\tilde{L}^4 + 52430G^4m^4\mu^6E\tilde{L}^2 + 22911G^6m^6\mu^9). \]  

(4.17)

B. Loss in the magnitude of orbital angular momentum

A somewhat more complicated computation yields the similar expressions for the instantaneous and averaged radiative losses in \( L \). To complete the task we pass through the following steps. We start from the instantaneous radiative loss of \( \mathbf{J} \) given in Ref. [14]:

\[
\frac{d\mathbf{J}}{dt} = -\frac{8G^2m\mu}{5c^5\rho^3} \mathbf{L}_N \left( 2\nu^2 - 3\dot{\nu}^2 + \frac{2Gm}{r} \right) - 2G \frac{\rho}{5c^5}\epsilon_{ijk} J^{(2)ij}_N (a_N) J^{(3)kl}_N (a_{SS}) + J^{(2)ij}_N (a_{SS}) J^{(3)kl}_N (a_N) + \frac{16}{9c^2} J^{(2)ij}_N (a_N) J^{(3)kl}_N (a_N)) .
\]  

(4.18)

We compute the instantaneous radiative loss in \( L \) as

\[
\frac{d\mathbf{L}}{dt} = \mathbf{\dot{L}} \cdot \frac{d(\mathbf{J} - \mathbf{S})}{dt} = \mathbf{\dot{L}} \cdot \frac{d\mathbf{J}}{dt} - \mathbf{\dot{L}} \cdot \frac{d\mathbf{S}_1}{dt} - \mathbf{\dot{L}} \cdot \frac{d\mathbf{S}_2}{dt} .
\]  

(4.19)

In Ref. [13], starting from the Burke-Thorne potential, we have already derived the radiative loss in the spins

\[
\frac{1}{S_i} \frac{d(S_i)}{dt} = 2G \frac{\rho}{5c^5}\epsilon_{ijk} J^{(5)ij}_N (\mathbf{S}_1) \epsilon_{i\rho\sigma} J^{(5)\rho\sigma}_N (\mathbf{S}_1) \cdot \epsilon_{\sigma\iota\tau} \frac{\Theta_1}{\Theta_i} \cdot \epsilon_{\iota\tau\nu} J^{(5)\nu\iota\tau}_N (\mathbf{\dot{S}}_1) .
\]  

(4.20)

\( \Theta_1 \) and \( \Theta_i \) being the principal moments of inertia and \( \Omega_i \) the angular velocity of the \( i \)-th spinning axisymmetric body, related to the spins by \( S_i = \Theta_i \Omega_i \). These losses appear at the second post-Newtonian order, thus they are not negligible in our approach. From Eq. (4.20) \( d\mathbf{S}_i/dt \) are found to be perpendicular to the respective spins \( \mathbf{S}_i \), therefore \( d\mathbf{S}_i/dt = S_i d\mathbf{S}_i/dt \). Also in Ref. [13] we have shown that

\[
\left< \mathbf{\dot{L}} \cdot \frac{d\mathbf{S}_i}{dt} \right> = 0 .
\]  

(4.21)

This result still holds here, in spite of the fact that the dynamics of the orbital motion and consequently the true anomaly parametrization employed for averaging are different. What matters in the derivation of the above result is only the Keplerian limit of the parametrizations, which by definition, coincide.

Therefore in the secular loss \( (d\mathbf{L}/dt) \) only the first term of Eq. (4.19) is relevant, which will be computed in what follows. As PN, SO, and 2PN contributions are not considered in our treatment, we insert \( \mathbf{L}_N = \mathbf{L} \) in the Newtonian term of (4.18), together with the previously derived expressions (2.32) and (2.33) for \( u^2 \) and \( \dot{u}^2 \), respectively. The spin-spin perturbation term of Eq. (4.18) can be easily dealt with by use of the Cartesian coordinates \( (x = r \cos \psi, y = r \sin \psi, z = 0) \). Then to the required, leading order \( \mathbf{\dot{L}} = (0, 0, 1) \) (see also Fig.1. of Ref. [13]). We employ in addition the Keplerian relations (2.22) and (2.24) which imply

\[
\psi = \frac{\tilde{L}}{\mu r^2} .
\]  

(4.22)

A lengthy but straightforward computation yields the following implicit function of the parameter \( \chi \) for \( \mathbf{\dot{L}} \cdot d\mathbf{J}/dt \):

\[
\mathbf{\dot{L}} \cdot d\mathbf{J}/dt = \left< \mathbf{\dot{L}} \cdot d\mathbf{J}/dt \right>_N + \left< \mathbf{\dot{L}} \cdot d\mathbf{J}/dt \right>_{SS-self} + \left< \mathbf{\dot{L}} \cdot d\mathbf{J}/dt \right>_{S_1S_2} ,
\]  

(4.23)

\[
\left< \mathbf{\dot{L}} \cdot d\mathbf{J}/dt \right>_N = \frac{8G^2m\tilde{L}}{5c^5\mu r^5} (2\mu Ey^2 - 3\tilde{L}^2) ,
\]  

(4.24)
spin-spin interaction as the perturbation, was derived in terms of the true anomaly parametrization. The secular losses of the system. In order to complete the task, a solution to the radiation back reaction on the energy and magnitude of orbital angular momentum of the coalescing binary from Eqs. (4.1), (4.2) of Ref. [15] (with the replacements order. Such a description is in the process of completion [28]. On the other hand, the situation is more sophisticated the evolution of the spinning binary then Eq. (2.7) could give new contributions even at the second post-Newtonian order. We would like to stress the appearance of the self-interaction terms (4.25) in the instantaneous loss of \( L \) either. They originate in the last term of Eq. (4.18).

A similar averaging procedure as in the case of the energy loss, gives here

\[
\left\langle \frac{dL}{dt} \right\rangle = \left\langle \frac{dL}{dt} \right\rangle_N + \left\langle \frac{dL}{dt} \right\rangle_{SS-\text{self}} + \left\langle \frac{dL}{dt} \right\rangle_{S_1S_2},
\]

\[
\left\langle \frac{dL}{dt} \right\rangle_N = -\frac{4G^2m(-2\mu E)^{3/2}}{5\epsilon^5 L^4} (14E\bar{L}^2 + 15G^2m^2\mu^3),
\]

\[
\left\langle \frac{dL}{dt} \right\rangle_{SS-\text{self}} = -\frac{G^2m(-2\mu E)^{3/2}}{20\epsilon^5 L^8} D_1 \sum_i \left( \frac{S_i}{m_i} \right)^2 (1 + \sin^2 \kappa_i),
\]

\[
\left\langle \frac{dL}{dt} \right\rangle_{S_1S_2} = \frac{G^2(-2\mu E)^{3/2}S_1S_2}{10\epsilon^5 L^8} [D_2 \sin \kappa_1 \sin \kappa_2 \cos 2(\psi_0 - \bar{\psi}) + D_3 \cos \kappa_1 \cos \kappa_2 + D_4 \cos \gamma],
\]

where the coefficients \( D_k \) are given below

\[
D_1 = 12E^2\bar{L}^4 + 60G^2m^2\mu^3E\bar{L}^2 + 35G^4m^4\mu^6,
\]

\[
D_2 = 12(62E^2\bar{L}^4 + 211G^2m^2\mu^3E\bar{L}^2 + 90G^4m^4\mu^6),
\]

\[
D_3 = -(1524E^2\bar{L}^4 + 7260G^2m^2\mu^3E\bar{L}^2 + 4865G^4m^4\mu^6),
\]

\[
D_4 = 24(22E^2\bar{L}^4 + 105G^2m^2\mu^3E\bar{L}^2 + 70G^4m^4\mu^6).
\]

We emphasize that self-interaction terms (4.15) and (4.30) are present in both the secular losses of \( E \) and \( L \).

V. CONCLUDING REMARKS

The main result of this paper consists in the derivation of the second post-Newtonian order spin-spin contribution to the radiation back reaction on the energy and magnitude of orbital angular momentum of the coalescing binary system. In order to complete the task, a solution \( r(\chi) \) of the quasi-Keplerian radial motion of the binary, with the spin-spin interaction as the perturbation, was derived in terms of the true anomaly parametrization. The secular losses of \( E \) and \( L \), Eqs. (4.13)-(4.16) and of \( L \), Eqs. (4.28)-(4.31), in complement with the corresponding spin-orbit terms from Eqs. (4.1), (4.2) of Ref. [15] (with the replacements \( L \rightarrow \bar{L} \) and \( A_0 \rightarrow \bar{A} \), which are allowed up to the 3/2 post-Newtonian order), represent the total spin-induced back reaction of the radiation up to the second post-Newtonian order. Among the loss terms due to the spin-spin interaction, terms originating in the self-interaction of the spins were found. While all other spin-spin terms vanish in the one-spin limit \( S_2 = 0 \), (half of) these terms survive, representing the second post-Newtonian correction to the losses derived earlier in the Lense-Thirring approximation [18].

In the nonradiative description of the angles \( \kappa_i \) and \( \gamma \) in principle not all second post-Newtonian order spin terms can be found from the Lagrangian (2.7). Similarly as the leading spin-orbit and spin-spin terms in the spin precession equations (exceptionally) are of first and 3/2 post-Newtonian orders, respectively [15], any higher order description of the evolution of the spinning binary then Eq. (2.7) could give new contributions even at the second post-Newtonian order. Such a description is in the process of completion [28]. On the other hand, the situation is more sophisticated...
when it comes to the radiative evolution of the angles $\kappa_i$ and $\gamma$. New terms beyond those already given in Ref. [15] can arise in two ways. The first type of these terms would come from corrections to the Burke-Thorne potential [27]. It was already shown [29] that the first correction is one post-Newtonian order higher than the potential itself, therefore any correction from the radiative loss of the spins would contribute only at 5/2 post-Newtonian order (above the leading radiation term in the orbital angular momentum loss). However new terms in the radiative angular evolution appear at the second post-Newtonian order due to $(d\mathbf{j}/dt)_{SS}$. A computation in progress reveals a complicated structure of these terms. We defer the study of the angular evolutions at the second post-Newtonian order to a subsequent work.

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APPENDIX A: THE RELATION BETWEEN TIME AND ANGULAR AVERAGES OF $L$

We start from the expression (2.28) for $L(\chi)$. The time average is obtained as

$$\langle L \rangle = \frac{1}{T} \int_0^T L(\chi) d\chi = \frac{1}{T} \int_0^{2\pi} \frac{L(\chi) d\chi}{d\chi} .$$  \hspace{1cm} (A1)

Here $T$ is given by Eq. (3.16) and $d\chi/d\chi$ is derived from the radial equation as explained in Sec. III. When expressed in terms of the complex parameter $z = \exp(i\chi)$, the integrand in Eq. (A3) has two poles inside the unit circle: one in the origin $z_0 = 0$ and the other at

$$z_1 = -w_1 = -\left[ \frac{Gm\mu^2 - \sqrt{-2\mu EL^2}}{Gm\mu^2 + \sqrt{-2\mu EL^2}} \right]^{1/2} .$$  \hspace{1cm} (A2)

From the residue theorem, $\langle L \rangle$ is given as the sum of the residues at $z_0$ and $z_1$ multiplied with $2\pi i/T$.

Thus the desired relation between the angular average $\bar{L}$ and the time average $\langle L \rangle$ emerges as

$$\langle L \rangle = \bar{L} - \frac{G\mu E_1}{2c^2 A^2 L^3 E_2} S_1 S_2 \sin \kappa_1 \sin \kappa_2 \cos 2(\psi_0 - \bar{\psi})$$  \hspace{1cm} (A3)

with the coefficients $E_{1,2}$ given by

$$E_1 = 2(-2\mu E)^{1/2} \tilde{L}[\tilde{A}^6 - 15G^2 m^2 \mu^2 \tilde{A}^4 + 32G^4 m^4 \mu^4 \tilde{A}^2 - 16G^6 m^6 \mu^6]$$

$$+ Gm\mu^2[-11\tilde{A}^6 + 58G^2 m^2 \mu^2 \tilde{A}^4 - 80G^4 m^4 \mu^4 \tilde{A}^2 + 32G^6 m^6 \mu^6]$$

$$E_2 = 4Gm(-2\mu E)^{1/2} \tilde{L}[\tilde{A}^2 - 2G^2 m^2 \mu^2] + \tilde{A}^4 - 8G^2 m^2 \mu^2 \tilde{A}^2 + 8G^4 m^4 \mu^4 .$$  \hspace{1cm} (A4)

Whenever needed, all expressions can be rewritten in terms of $\langle L \rangle$ by use of Eq. (A3). However, by inserting $\bar{L}$ expressed from Eq. (A3) in Eq. (2.29) we obtain a considerably longer expression for $L(\chi)$ in terms of $\langle L \rangle$ than the original Eq. (2.29). This is an indication that the description in terms of the angular average $\bar{L}$ is better suited for the problem.

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