COMPOSITION OPERATORS IN THE LIPSCHITZ SPACE OF THE POLYDISCS

ZHONGSHAN FANG AND ZEHUA ZHOU

Abstract. In 1987, Shapiro showed that composition operator induced by symbol \( \varphi \) is compact on the Lipschitz space if and only if the infinity norm of \( \varphi \) is less than 1 by a spectral-theoretic argument, where \( \varphi \) is a holomorphic self-map of the unit disk. In this paper, we shall generalize Shapiro’s result to the \( n \)-dimensional case.

1. Introduction

Let \( U^n \) be the unit polydiscs of \( n \)-dimensional complex spaces \( C^n \) with boundary \( \partial U^n \), the class of all holomorphic functions on domain \( U^n \) will be denoted by \( H(U^n) \). Let \( \varphi(z) = (\varphi_1(z), \ldots, \varphi_n(z)) \) be a holomorphic self-map of \( U^n \), composition operator is defined by

\[ C_\varphi(f)(z) = f(\varphi(z)) \]

for any \( f \in H(U^n) \) and \( z \in U^n \).

In the past few years, boundedness and compactness of composition operators between several spaces of holomorphic functions have been studied by many authors: by Jarchow and Ried [4] between generalized Bloch-type spaces and Hardy spaces, between Bloch spaces and Besov spaces and BMOA and VMOA in Tian’s thesis [10].

More recently, there have been many papers focused on studying the same problems for \( n \)-dimensional case: by Zhou and Shi [15] [16] [17] on the Bloch space in polydisk or classical symmetric domains, Gorkin and MacCluer [3] between hardy spaces in the unit ball.

For the Lipschitz case, the compactness of \( C_\varphi \) is characterized by ”little-oh” version of Madigan’s [6] the boundedness condition, the same results in polydisc were obtained by Zhou [11] and by Zhou and Liu

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*Zehua Zhou, corresponding author. Supported in part by the National Natural Science Foundation of China (Grand Nos. 10671141, 10371091).
In all these works the main goal is to relate function theoretic properties of $\phi$ to boundedness and compactness of $C_{\phi}$.

To our surprise, by a spectral-theoretic argument, Shapiro [9] obtained the following fact: $C_{\phi}$ is compact on the Lipschitz space $L_\alpha(D)$ if and only if $||\varphi||_\infty < 1$. In this paper, we shall generalize Shapiro’s result to the unit polydisc.

2. Notation and background

Throughout the paper, $D$ is the unit disk in one dimensional complex plane, and $|||z||| = \max_{1 \leq j \leq n} |z_j|$ stands for the sup norm on the unit polydisc. Define $Rf(z) = \langle \nabla f(z), \bar{z} \rangle$ where $z = (z_1, \cdots, z_n) \in U^n$, and $H(U^n, D)$ for the class of the holomorphic mappings from $U^n$ to $D$. For $0 < \alpha < 1$, it is well known that the Lipschitz space $L_{1-\alpha}(U^n)$ is equivalent to $\alpha - \text{Bloch}$ space, which is defined to be the space of holomorphic functions $f \in U^n$ such that

$$|||f|||_{1-\alpha} = \sup_{z \in U^n} \sum_{j=1}^n (1 - |z_j|^2)^\alpha |\partial f/\partial z_j(z)| < \infty.$$  

Here, Lipschitz space $L_{1-\alpha}(U^n)$ is a Banach space with the equivalent norm

$$||f|| = |f(0)| + |||f|||_{1-\alpha}.$$  

The Kobayashi distance $k_{U^n}$ of $U^n$ is given by

$$k_{U^n}(z, w) = \frac{1}{2} \log \frac{1 + |||\phi_z(w)|||}{1 - |||\phi_z(w)|||},$$  

where $\phi_z : U^n \to U^n$ is the automorphism of $U^n$ given by

$$\phi_z(w) = \left( \frac{w_1 - z_1}{1 - \bar{z}_1 w_1}, \cdots, \frac{w_n - z_n}{1 - \bar{z}_n w_n} \right).$$  

Since the map $t \to \log \frac{1+t}{1-t}$ is strictly increasing on $[0, 1)$, it follows that

$$k_{U^n}(z, w) = \max_{1 \leq j \leq n} \left\{ \frac{1}{2} \log \frac{1 + |w_j - z_j|}{1 - |w_j - z_j|} \right\} = \max_{1 \leq j \leq n} \{ \rho(z_j, w_j) \},$$  

where $\rho$ is the Poincaré distance on the unit disk $D \subset C$.

Following [1], the horosphere $E(x, R)$ of center $x \in \partial U^n$ and radius $R$ and the Korányi region $H(x, M)$ of vertex $x$ and amplitude $M$ are defined by

$$E(x, R) = \{ z \in U^n : \limsup_{w \to x} [k_{U^n}(z, w) - k_{U^n}(0, w)] < \frac{1}{2} \log R \}$$  

$$H(x, M) = \{ z \in U^n : k_{U^n}(z, x) < M \}$$.
and

\[ H(x, M) = \{ z \in U^n : \limsup_{w \to x}[k_{U^n}(z, w) - k_{U^n}(0, w)] + k_{U^n}(0, z) < \log M \}. \]

We say that \( f \) has \( K \)-limit \( L \in C \) at \( x \) if \( f(z) \to L \) as \( z \to x \) inside any Korányi region \( H(x, M) \), we shall write \( \tilde{K} \)-\( \lim f(z) = L \).

Let \( f \in H(U^n, D) \) and \( x \in \partial U^n \). If there is \( \delta \) such that

\[ \liminf_{w \to x} \frac{1 - |f(w)|}{1 - ||w||} = \delta < \infty, \]

we call \( f \) is \( \delta \)-Julia at \( x \). If there exists \( \tau \in \partial U^n \) such that

\[ f(E(x, R)) \subseteq E(\tau, \delta R) \]

for all \( R \), we call this \( \tau \) is the restricted \( E \)-limit of \( f \) at \( x \).

It should be noticed that \( \delta > 0 \). In fact,

\[ \rho(0, f(w)) \leq \rho(0, f(0)) + \rho(f(0), f(w)) \leq \rho(0, f(0)) + k_{U^n}(0, w); \]

therefore

\[ 1 - |f(w)| > 0. \]

3. SOME LEMMAS

**Lemma 1.** (Julia-Wolff-Carathéodory Theorem, Theorem 4.1 in [1])

Let \( f \in H(U^n, D) \) be \( \delta \)-Julia at \( x \in \partial U^n \), and \( \tau \in \partial U \) be the restricted \( E \)-limit of \( f \) at \( x \), then

\[ \tilde{K} \lim_{z \to x} \frac{\partial f}{\partial x}(z) = \delta \tau. \]

**Lemma 2.** (Theorem 1 in [11] or Corollary 4.1 in [14]) Composition operator \( C_\varphi \) is bounded on the Lipschitz space \( L^{1-\alpha}(U^n) \) if and only if there is a constant \( M > 0 \) such that

\[ \sum_{k,j=1}^n \left| \frac{\partial \varphi_j}{\partial z_k}(z) \right| \left( \frac{1 - |z_k|^2}{1 - |\varphi_j(z)|^2} \right)^\alpha \leq M \]

for \( z \in U^n \).

**Lemma 3.** (Theorem 2 in [11] or Corollary 4.2 in [14]) Composition operator \( C_\varphi \) is compact on the Lipschitz space \( L^{1-\alpha}(U^n) \) if and only if

\[ \lim_{\delta \to 0} \sup_{\text{dist}(\varphi(z), \partial U^n) < \delta} \sum_{k,j=1}^n \left| \frac{\partial \varphi_j}{\partial z_k}(z) \right| \frac{(1 - |z_k|^2)^\alpha}{(1 - |\varphi_j(z)|^2)^\alpha} = 0. \]

**Lemma 4.** (Lemma 3.2 in [1]) Let \( f \in H(U^n, D) \) and \( x \in \partial U^n \). Then

\[ \liminf_{w \to x} \frac{1 - |f(w)|}{1 - ||w||} = \liminf_{t \to 1^-} \frac{1 - |f(\varphi_x(t))|}{1 - t}, \]

where \( \varphi_x(z) = zx \) for any \( z \in D \).
4. Main theorem

Theorem 1. Suppose $C_\varphi$ is bounded on $L_{1-\alpha}(U^n)$, then for every $1 \leq l \leq n$ and $\xi \in \partial U^n$ with $|\varphi_l(\xi)| = 1$, $\varphi_l$ is $\delta$–Julia at $\xi$.

Proof. For every $1 \leq l \leq n$ and $\xi \in \partial U^n$ with $\varphi_l(\xi)| = \eta$ and $\eta = e^{\theta_0}$, we will show that $\varphi_l$ is $\delta$–Julia at $\xi$ according to the following cases.

Case 1: $\xi = (\xi_1, \xi')$, $\xi_1 = e^{\theta_1}$ and $\|\xi'\| < 1$.

First we consider the special case for $\xi = e_1 = (1, 0, \cdots, 0)$ and $\eta = 1$.

For $r \in (1/2, 1)$, define $\sigma(r) = (r, 0, \cdots, 0) = re_1$ such that

$$\lim_{r \to 1^-} \varphi_l(\sigma(r)) = 1.$$ 

Setting $g(r) = \varphi_l(re_1)$, then $g'(r) = \frac{\partial g}{\partial z}(re_1)$. It follows from Lemma 2 that the boundedness of $C_\varphi$ implies that

$$h(r) = R\varphi_l(re_1)(\frac{1-r}{1-\varphi_l(re_1)})^\alpha = rg'(r)(\frac{1-r}{1-g(r)})^\alpha$$

is bounded.

Putting $u(r) = \frac{1-g(r)}{1-r}$, it is easy to see that $g'(r) = -(1-r)u'(r) + u(r)$ and

$$h(r) = ru(r)^{-\alpha}[-(1-r)u'(r) + u(r)].$$

If we write $v(r) = u(r)^{1-\alpha}$, then

$$-\frac{1}{1-\alpha}(1-r)v'(r) + v(r) = \frac{h(r)}{r}$$

the general solution of this differential equation is

$$v(r) = -\frac{1-\alpha}{(1-r)^{1-\alpha}} \int_1^r \frac{h(s)}{s(1-s)^\alpha} ds + \frac{C}{(1-r)^{1-\alpha}}.$$ 

Since $h$ is bounded, the first term in the right above is a bounded function of $r$, and moreover $v(r)$ is of the order $o(\frac{1}{(1-r)^{1-\alpha}})$ as $r \to 1^-$, so we have $C = 0$. Hence $v$, and moreover $u$ is also bounded, according to Lemma 4, for some $\delta$, $\varphi_l$ is $\delta$–Julia at $e_1$.

Now we return to the proof in case 1. Considering the mapping $\tilde{\varphi}_l : U^n \to U^n$, where

$$\tilde{\varphi}_l(z_1, z') = e^{-i\theta_0} \cdot \varphi_l(e^{i\theta_1}z_1, \phi_\varphi(z'))$$

for $z = (z_1, z') \in U^n$. It is easy to check that $C_{\tilde{\varphi}_l}$ is bounded on $L_{1-\alpha}(U^n)$ and $\tilde{\varphi}_l(e_1) = 1$. 
By the above argument, we get
\[
\liminf_{t \to 1^-} \frac{1 - |\varphi_l(t\xi_1, \xi')|}{1 - t} = \delta < +\infty,
\]
that is
\[
\liminf_{t \to 1^-} \frac{1 - |\varphi_l(t\xi_1, \xi')|}{1 - t} = \liminf_{r \to 1^-} \frac{1 - |\varphi_l(t\xi_1, r\xi')|}{1 - r} \geq \liminf_{t \to 1^-} \frac{1 - |\varphi_l(t\xi_1, t\xi')|}{1 - t}.
\]
It follows from Lemma 4 that
\[
\liminf_{w \to \xi_1} \frac{1 - |\varphi_l(\xi)|}{1 - ||\xi||} = \delta < +\infty.
\]

**Case 2:** \(\xi = (\xi_1, \xi_2, \xi')\), \(\xi_1 = e^{\theta_1}, \xi_2 = e^{\theta_2}\) and \(||\xi'|| < 1\).

Now assume \(\varphi_l(1, 1, 0, \cdots, 0) = 1\), and set \(g(r) = \varphi_l(r, r, 0, \cdots, 0)\) for \(r \in (1/2, 1)\). Then \(g'(r) = \frac{\partial \varphi_l}{\partial z_1}(r, r, 0, \cdots, 0) + \frac{\partial \varphi_l}{\partial z_2}(r, r, 0, \cdots, 0)\), and so \(R\varphi_l(r, r, 0, \cdots, 0) = rg'(r)\), we can deal with it as in the case 1, and we can get \(u\) is bounded, furthermore
\[
\liminf_{w \to \xi_1} \frac{1 - |\varphi_l(\xi)|}{1 - ||\xi||} = \delta < +\infty.
\]

**Case 3:** For the case \(\varphi_l(\xi) = 1\) with \(\xi = \sum_{k=1}^{n} \beta_k e_k\), where \(\beta_k = 0\) or 1, and \(e_k = (0, 0, \cdots, 1, 0, \cdots, 0)\) with the \(k-\)th component is 1, otherwise 0; and even more general case, in a similar argument with the cases 1 and 2, we can also show
\[
\liminf_{w \to \xi_1} \frac{1 - |\varphi_l(\xi)|}{1 - ||\xi||} = \delta < +\infty.
\]
This completes the proof of this theorem. \(\square\)

**Theorem 2.** \(C_\varphi\) is compact on \(L_{1-\alpha}(U^n)\) if and only if \(\varphi_j \in L_{1-\alpha}(U^n)\) and \(||\varphi_j||_\infty < 1\) for each \(j = 1, 2, \cdots, n\).

**Proof.** Sufficiency is obvious. Now we just turn to the necessity. Suppose to the contrary that there exists \(l (1 \leq l \leq n)\) satisfying \(|\varphi_l(\xi)| = 1\) for some \(\xi \in \partial U^n\). It follows from Theorem 1 that \(\varphi_l\) is \(\delta - \text{Julia}\) at \(\xi\), therefore by Lemma 1, we have \(R\varphi_l(z)\) has \(K-\text{limit}\) at \(\xi\). Hence
\[
\sum_{k,l=1}^{n} |\frac{\partial \varphi_l}{\partial z_k}(z)| \frac{(1 - |z_k|^2)^\alpha}{(1 - |\varphi_l(z)|^2)^\alpha}
\]
\[
\geq \sum_{k,l=1}^{n} \left| \frac{\partial \varphi_l(z)}{\partial z_k} \right| \frac{(1 - ||z||^2)^{\alpha}}{(1 - |\varphi_l(z)|^2)^{\alpha}}
\]

\[
\geq \sum_{k,l=1}^{n} \left| z_k \cdot \frac{\partial \varphi_l(z)}{\partial z_k} \right| \frac{(1 - ||z||^2)^{\alpha}}{(1 - |\varphi_l(z)|^2)^{\alpha}}
\]

\[
\geq C \sum_{l=1}^{n} |R\varphi_l(z)| \frac{(1 - ||z||^2)^{\alpha}}{(1 - |\varphi_l(z)|^2)^{\alpha}}
\]

\[
\geq C \delta^{1-\alpha}
\]

as \( z \to \xi \) inside any Korányi region, where we can take \( C = \frac{1}{2^\alpha} \). It is a contradiction to the compactness of \( C_\varphi \) by Lemma 3. Now the proof of Theorem 2 is completed. \( \square \)

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**DEPARTMENT OF MATHEMATICS**  
**TIANJIN POLYTECHNIC UNIVERSITY**  
TIANJIN 300160  
P.R. CHINA.  
*E-mail address: fangzhongshan@yahoo.com.cn*

**DEPARTMENT OF MATHEMATICS**  
**TIANJIN UNIVERSITY**  
TIANJIN 300072  
P.R. CHINA.  
*E-mail address: zhuahou2003@yahoo.com.cn*