INTERPOLATION AND \( \Phi \)-MOMENT INEQUALITIES OF NONCOMMUTATIVE MARTINGALES

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Abstract. This paper is devoted to the study of \( \Phi \)-moment inequalities for noncommutative martingales. In particular, we prove the noncommutative \( \Phi \)-moment analogues of martingale transformations, Stein’s inequalities, Khintchine’s inequalities for Rademacher’s random variables, and Burkholder-Gundy’s inequalities. The key ingredient is a noncommutative version of Marcinkiewicz type interpolation theorem for Orlicz spaces which we establish in this paper.

0. Introduction

Given a probability space \( (\Omega, \mathcal{F}, P) \). Let \( \{\mathcal{F}_n\}_{n \geq 1} \) be a nondecreasing sequence of \( \sigma \)-subfields of \( \mathcal{F} \) such that \( \mathcal{F} = \vee \mathcal{F}_n \), and let \( \Phi \) be an Orlicz function with \( 1 < p_\Phi \leq q_\Phi < \infty \). If \( f = (f_n)_{n \geq 1} \) is a \( L_{\Phi} \)-bounded martingale, then

\[
\int \Omega \Phi \left[ \left( \sum_{n=1}^{\infty} |df_n|^2 \right)^{\frac{1}{2}} \right] dP \approx \sup_{n \geq 1} \int \Omega \Phi(|f_n|) dP.
\]

where \( df = (df_n)_{n \geq 1} \) is the martingale difference of \( f \) and “ \( \approx \) ” depends only on \( \Phi \). This result is the well-known Burkholder-Gundy inequality for convex powers \( \Phi(t) = t^p \) (see [9]) and proved in the general setting of Orlicz functions by Burkholder-Davis-Gundy [8]. In their remarkable paper [31], Pisier and Xu proved the noncommutative analogue of the Burkholder-Gundy inequality, which triggered a systematic research of noncommutative martingale inequalities (we refer to a recent book by Xu [37] for an up-to-date exposition of theory of noncommutative martingales). In this paper, we will extend their work to \( \Phi \)-moment versions, i.e., we will prove the noncommutative analogue of (0.1).

Let us briefly describe our \( \Phi \)-moment inequality. Let \( \mathcal{M} \) be a finite von Neumann algebra with a normalized normal faithful trace \( \tau \), and \( \{\mathcal{M}_n\}_{n \geq 0} \) be an increasing filtration of von Neumann subalgebras of \( \mathcal{M} \). Let \( \Phi \) be an Orlicz function and \( x = \{x_n\}_{n \geq 0} \) a noncommutative \( L_{\Phi} \)-martingale with

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respect to \( \{M_n\}_{n \geq 0} \). Then our result reads as follows. If \( 1 < p_\Phi \leq q_\Phi < 2 \), then

\[
\tau(\Phi[|x|]) \approx \inf \left\{ \tau\left(\Phi\left(\sum_{k=0}^{\infty} |dy_k|^2 \right)^{\frac{1}{2}}\right) + \tau\left(\Phi\left(\sum_{k=0}^{\infty} |dz_k|^2 \right)^{\frac{1}{2}}\right) \right\}
\]

where the infimum runs over all decomposition \( x_k = y_k + z_k \) with \( y_k \) in \( \mathcal{H}^p_C(M) \) and \( z_k \) in \( \mathcal{H}^p_R(M) \), and if \( 2 < p_\Phi \leq q_\Phi < \infty \), then

\[
\tau(\Phi[|x|]) \approx \max \left\{ \tau\left(\Phi\left(\sum_{k=0}^{\infty} |dx_k|^2 \right)^{\frac{1}{2}}\right), \tau\left(\Phi\left(\sum_{k=0}^{\infty} |dx_\ast_k|^2 \right)^{\frac{1}{2}}\right) \right\}.
\]

Here “\( \approx \)” depends only on \( \Phi \). Note that the Orlicz norm version of noncommutative analogue of (0.1) has been proved by the first named author [2]. Evidently, the \( \Phi \)-moment inequalities imply the norm version.

One interesting feature of our result, similar to that of Pisier-Xu [31], is that the square function is defined differently (and it must be changed!) according to \( q_\Phi < 2 \) or \( p_\Phi > 2 \). This surprising phenomenon was already discovered by F.Lust-Piquard in [21, 22] (see also [23]) while establishing noncommutative versions of Khintchine’s inequalities (see §5 also).

Stopping times and good-\( \lambda \) techniques developed by Burkholder et al [7] are two key ingredients in the proof of (0.1). Unfortunately, the concept of stopping times is, up to now, not well defined in the generic noncommutative setting (there are some works on this topic, see [1] and references therein). On the other hand, the noncommutative analogue of good-\( \lambda \) inequalities seems open. Then, in order to prove the noncommutative \( \Phi \)-moment inequalities (0.2) and (0.3) we need new ideas.

The style of proof of (0.2) and (0.3) is via interpolation. Our key ingredient is a noncommutative analogue of Marcinkiewicz type interpolation theorem for Orlicz spaces, which we will prove in this paper. Recall that the first interpolation theorem concerning Orlicz spaces as intermediate spaces is due to Orlicz [27]. Subsequently, the classical Marcinkiewicz interpolation theorem was extended to include Orlicz spaces as interpolation classes by A.Zygmund, A.P.Calderón, et al., for references see [25] and therein.

Now, let us briefly explain our strategy. Firstly, we prove \( \Phi \)-moment versions of noncommutative martingale transforms and Stein’s inequalities via interpolation. Then by interpolation again we prove \( \Phi \)-moment versions of noncommutative Khintchine’s inequalities (this is the key point of the proof). Finally, by these \( \Phi \)-moment inequalities we deduce (0.2) and (0.3). This argument seems new and that even in the classical case, it is simpler than all existing methods to the \( \Phi \)-moment inequalities of martingales.

The remainder of this paper is divided into six sections. In Section 1, we present some preliminaries and notations on the noncommutative Orlicz spaces and Orlicz-Hardy spaces of noncommutative martingales. Then, a noncommutative analogue of Marcinkiewicz type interpolation theorem for Orlicz spaces is proved in Section 2, which is the key ingredient for the proof
of the main result in this paper. Φ-moment versions of noncommutative martingale transforms and Stein’s inequalities are proved in Section 3. As an immediate application of Φ-moment inequalities of noncommutative martingale transforms, we will prove the UMD property of noncommutative Orlicz spaces. In Section 4, the noncommutative Φ-moment Khintchine inequalities for Rademacher’s random variables are proved via interpolation again. By the Φ-moment inequalities proved previously, we deduce the Φ-moment version of noncommutative Burkholder-Gundy’s martingale inequalities in Section 5. Finally, in Section 6, we make some remarks on our results and possible further researches.

In what follows, C always denotes a constant, which may be different in different places. For two nonnegative (possibly infinite) quantities X and Y by $X \approx Y$ we mean that there exists a constant $C > 1$ such that $C^{-1}X \leq Y \leq CX$.

1. Preliminaries

1.1. Noncommutative Orlicz spaces. We use standard notation and notions from theory of noncommutative $L_p$-spaces. Our main references are [32] and [37] (see also [32] for more historical references). Let $\mathcal{N}$ be a semifinite von Neumann algebra acting on a Hilbert space $\mathbb{H}$ with a normal semifinite faithful trace $\nu$. Let $L_0(\mathcal{N})$ denote the topological $*$-algebra of measurable operators with respect to $(\mathcal{N}, \nu)$. The topology of $L_0(\mathcal{N})$ is determined by the convergence of measure. The trace $\nu$ can be extended to the positive cone $L_0^+(\mathcal{N})$ of $L_0(\mathcal{N})$:

$$\nu(x) = \int_0^{\infty} \lambda d\nu(E_{\lambda}(x)),$$

where $x = \int_0^{\infty} \lambda dE_{\lambda}(x)$ is the spectral decomposition of $x$. Given $0 < p < \infty$, let

$$L_p(\mathcal{N}) = \{x \in L_0(\mathcal{N}) : \nu(|x|^p)^{\frac{1}{p}} < \infty\}.$$

We define

$$\|x\|_p = \nu(|x|^p)^{\frac{1}{p}}, \quad x \in L_p(\mathcal{N}).$$

Then, $(L_p(\mathcal{N}), \|\cdot\|_p)$ is a Banach (or quasi-Banach for $p < 1$) space. They are the noncommutative $L_p$-space associated with $(\mathcal{N}, \nu)$, denoted by $L_p(\mathcal{N}, \nu)$ or simply by $L_p(\mathcal{N})$. As usual, we set $L_\infty(\mathcal{N}, \nu) = \mathcal{N}$ equipped with the operator norm.

**Definition 1.1.** Let $\mathcal{N}$ be a semifinite von Neumann algebra on a Hilbert space $\mathbb{H}$ with a normal semifinite faithful trace $\nu$. Let $x \in L_0(\mathcal{N})$. Define

$$\lambda_s(x) = \nu(E_{(s, \infty)}(|x|)), \quad s > 0,$$

where $E_{(s, \infty)}(|x|)$ is the spectral projection of $x$ associated with the interval $(s, \infty))$. The function $s \mapsto \lambda_s(x)$ is called the distribution function of $x$. 
For $0 < p < \infty$, we have the following Kolmogorov inequality:
\begin{equation}
\lambda_s(x) \leq \frac{\|x\|^p_s}{s^p}, \quad \forall x \in L_p(N).
\end{equation}

**Definition 1.2.** Let $x$ be a $\tau$-measure operator and $t > 0$. The “$t$-th singular number of $x$” $\mu_t(x)$ is defined by
$$
\mu_t(x) = \inf \left\{ \|xe\| : e \text{ is any projection in } N \text{ with } \tau(e^\perp) \leq t \right\}.
$$

The $\mu_t(x)$ is finite valued and decreasing function on $(0, \infty)$. For further information on the generalised singular value we refer the reader to [12].

Let $\Phi$ be an Orlicz function on $[0, \infty)$, i.e., a continuous increasing and convex function satisfying $\Phi(0) = 0$ and $\lim_{t \to \infty} \Phi(t) = \infty$. Recall that $\Phi$ is said to satisfy the $\Delta_2$-condition if there is a constant $C$ such that $\Phi(2t) \leq C\Phi(t)$ for all $t > 0$. In this case, we denote by $\Phi \in \Delta_2$. It is easy to check that $\Phi \in \Delta_2$ if and only if for any $a > 0$ there is a constant $C_a > 0$ such that $\Phi(at) \leq C_a \Phi(t)$ for all $t > 0$.

We will work with some standard indices associated to an Orlicz function. Given an Orlicz function $\Phi$. Let
$$
M(t, \Phi) = \sup_{s > 0} \frac{\Phi(ts)}{\Phi(s)}, \quad t > 0.
$$

Define
$$
p_\Phi = \lim_{t \to 0} \frac{\log M(t, \Phi)}{\log t}, \quad q_\Phi = \lim_{t \to \infty} \frac{\log M(t, \Phi)}{\log t}.
$$

All the following properties we will use in the sequel are classical and can be found in [24]:

(1) $1 \leq p_\Phi \leq q_\Phi \leq \infty$.

(2) We have the following characterizations of $p_\Phi$ and $q_\Phi$:
$$
p_\Phi = \sup \left\{ p > 0 : \int_0^t s^{-p} \Phi(s) \frac{ds}{s} = O(t^{-p}\Phi(t)), \forall t > 0 \right\};
$$
$$
q_\Phi = \inf \left\{ q > 0 : \int_t^\infty s^{-q} \Phi(s) \frac{ds}{s} = O(t^{-q}\Phi(t)), \forall t > 0 \right\}.
$$

(3) $\Phi \in \Delta_2$ if and only if $q_\Phi < \infty$ if and only if $\sup_{t > 0} t\Phi'(t)/\Phi(t) < \infty$.

($\Phi'(t)$ is defined for each $t > 0$ except for a countable set of points in which we take $\Phi'(t)$ as the derivative from the right.)

See [24, 25] for more information on Orlicz functions and Orlicz spaces.

For an Orlicz function $\Phi$, the noncommutative Orlicz space $L_\Phi(N)$ is defined as the space of all measurable operators with respect to $(N, \nu)$ such that
$$
\nu\left(\Phi\left(\frac{|x|}{c}\right)\right) < \infty
$$
for some $c > 0$. The space $L_\Phi(N)$, equipped with the norm
$$
\|x\|_\Phi = \inf \left\{ c > 0 : \nu\left(\Phi\left(\frac{|x|}{c}\right)\right) < 1 \right\},
$$
is a Banach space. If $\Phi(t) = t^p$ with $1 \leq p < \infty$ then $L_\Phi(N) = L_p(N)$. Noncommutative Orlicz spaces are symmetric spaces of measurable operators as defined in [36].

Let $a = (a_n)$ be a finite sequence in $L_\Phi(N)$, we define

$$
\|a\|_{L_\Phi(N, \ell^2_C)} = \left\| \left( \sum_n |a_n|^2 \right)^{1/2} \right\|_\Phi \quad \text{and} \quad \|a\|_{L_\Phi(N, \ell^2_R)} = \left\| \left( \sum_{n \geq 0} |a_n|^2 \right)^{1/2} \right\|_\Phi,
$$

respectively. This gives two norms on the family of all finite sequences in $L_\Phi(N)$. To see this, let us consider the von Neumann algebra tensor product $N \otimes B(\ell^2)$ with the product trace $\nu \otimes \text{tr}$, where $B(\ell^2)$ is the algebra of all bounded operators on $\ell^2$ with the usual trace $\text{tr}$. $\nu \otimes \text{tr}$ is a semifinite normal faithful trace. The associated noncommutative Orlicz space is denoted by $L_\Phi(N \otimes B(\ell^2))$. Now, any finite sequence $a = (a_n)_{n \geq 0}$ in $L_\Phi(N)$ can be regarded as an element in $L_\Phi(N \otimes B(\ell^2))$ via the following map

$$
a \mapsto T(a) = \begin{pmatrix}
a_0 & 0 & \ldots \\
a_1 & 0 & \ldots \\
\vdots & \vdots & \ddots
\end{pmatrix},
$$

that is, the matrix of $T(a)$ has all vanishing entries except those in the first column which are the $a_n$’s. Such a matrix is called a column matrix, and the closure in $L_\Phi(N \otimes B(\ell^2))$ of all column matrices is called the column subspace of $L_\Phi(N \otimes B(\ell^2))$. Since

$$
\|a\|_{L_\Phi(N, \ell^2_C)} = \|T(a)\|_{L_\Phi(N \otimes B(\ell^2))} = \|T(a)\|_{L_\Phi(N \otimes B(\ell^2))},
$$

then $\|\cdot\|_{L_\Phi(N, \ell^2_C)}$ defines a norm on the family of all finite sequences of $L_\Phi(N)$. The corresponding completion $L_\Phi(N, \ell^2_C)$ is a Banach space. It is clear that a sequence $a = (a_n)_{n \geq 0}$ in $L_\Phi(N)$ belongs to $L_\Phi(N, \ell^2_C)$ if and only if

$$
\sup_{n \geq 0} \left\| \left( \sum_{k=0}^n |a_k|^2 \right)^{1/2} \right\|_\Phi < \infty.
$$

If this is the case, $(\sum_{k=0}^\infty |a_k|^2)^{1/2}$ can be appropriately defined as an element of $L_\Phi(N)$. Similarly, $\|\cdot\|_{L_\Phi(N, \ell^2_R)}$ is also a norm on the family of all finite sequence in $L_\Phi(N)$, and the corresponding completion $L_\Phi(N, \ell^2_R)$ is a Banach space, which is isometric to the row subspace of $L_\Phi(N \otimes B(\ell^2))$ consisting of matrices whose nonzero entries lie only in the first row. Observe that the column and row subspaces of $L_\Phi(N \otimes B(\ell^2))$ are 1-complemented by Theorem 3.4 in [11].

**Definition 1.3.** Let $\Phi$ be an Orlicz function. The space $CR_\Phi[L_\Phi(N)]$ is defined as follows:

1. If $q_\Phi < 2$,

$$
CR_\Phi[L_\Phi(N)] = L_\Phi(N, \ell^2_C) + L_\Phi(N, \ell^2_R)
$$
equipped with the sum norm:
\[ \| (x_n) \|_{CR\Phi[L\Phi(N)]} = \inf \{ \| (y_n) \|_{L\Phi(N, \ell^2_1)}, \| (y_n) \|_{L\Phi(N, \ell^2_2)} \}, \]
where the infimum runs over all decomposition \( x_n = y_n + z_n \) with \( y_n \) and \( z_n \) in \( L\Phi(N) \).

(2) If \( 2 \leq p_\Phi \),
\[ CR\Phi[L\Phi(N)] = L\Phi(N, \ell^2_1) \cap L\Phi(N, \ell^2_2) \]
equipped with the intersection norm:
\[ \| (x_n) \|_{CR\Phi[L\Phi(N)]} = \max \{ \| (x_n) \|_{L\Phi(N, \ell^2_1)}, \| (x_n) \|_{L\Phi(N, \ell^2_2)} \}. \]

In the sequel, unless otherwise specified, we always denote by \( \Phi \) an Orlicz function.

1.2. Noncommutative martingales. Let \( \mathcal{M} \) be a finite von Neumann algebra with a normalized normal faithful trace \( \tau \). Let \( (\mathcal{M}_n)_{n \geq 0} \) be an increasing sequence of von Neumann subalgebras of \( \mathcal{M} \) such that \( \bigcup_{n \geq 0} \mathcal{M}_n \) generates \( \mathcal{M} \) (in the \( w^* \)-topology). \( (\mathcal{M}_n)_{n \geq 0} \) is called a filtration of \( \mathcal{M} \). The restriction of \( \tau \) to \( \mathcal{M}_n \) is still denoted by \( \tau \). Let \( \mathcal{E}_n = \mathcal{E}(\cdot | \mathcal{M}_n) \) be the conditional expectation of \( \mathcal{M} \) with respect to \( \mathcal{M}_n \). Then \( \mathcal{E}_n \) is a norm 1 projection of \( L\Phi(\mathcal{M}) \) onto \( L\Phi(\mathcal{M}_n) \) (see Theorem 3.4 in [11]) and \( \mathcal{E}_n(x) \geq 0 \) whenever \( x \geq 0 \).

A non-commutative \( L\Phi \)-martingale with respect to \( (\mathcal{M}_n)_{n \geq 0} \) is a sequence \( x = (x_n)_{n \geq 0} \) such that \( x_n \in L\Phi(\mathcal{M}_n) \) and
\[ \mathcal{E}_n(x_{n+1}) = x_n \]
for any \( n \geq 0 \).

Let \( \| x \|_\Phi = \sup_{n \geq 0} \| x_n \|_\Phi \). If \( \| x \|_\Phi < \infty \), then \( x \) is said to be a bounded \( L\Phi \)-martingale.

Remark 1.1. (1) Let \( x_\infty \in L\Phi(\mathcal{M}) \). Set \( x_n = \mathcal{E}_n(x_\infty) \) for all \( n \geq 0 \). Then \( x = (x_n) \) is a bounded \( L\Phi \)-martingale and \( \| x \|_{L\Phi(\mathcal{M})} = \| x_\infty \|_{L\Phi(\mathcal{M})} \).

(2) Suppose \( \Phi \) is an Orlicz function with \( 1 < p_\Phi \leq q_\Phi < \infty \). Then \( L\Phi(\mathcal{M}) \) is reflexive. By the standard argument we conclude that any bounded noncommutative martingale \( x = (x_n) \) in \( L\Phi(\mathcal{M}) \) converges to some \( x_\infty \) in \( L\Phi(\mathcal{M}) \) and \( x_n = \mathcal{E}_n(x_\infty) \) for all \( n \geq 0 \).

(3) Let \( \mathcal{M} \) be a semifinite von Neumann algebra with a semifinite normalized trace \( \tau \). Let \( (\mathcal{M}_n)_{n \geq 0} \) be a filtration of \( \mathcal{M} \) such that the restriction of \( \tau \) to each \( \mathcal{M}_n \) is still semifinite. Then we can define noncommutative martingales with respect to \( (\mathcal{M}_n)_{n \geq 0} \). All results on noncommutative martingales that will be presented below in this paper can be extended to this semifinite setting.

Let \( x \) be a noncommutative martingale. The martingale difference sequence of \( x \), denoted by \( dx = (dx_n)_{n \geq 0} \), is defined as
\[ dx_0 = x_0, \quad dx_n = x_n - x_{n-1}, \quad n \geq 1. \]
Set

\[ S_{C,n}(x) = \left( \sum_{k=0}^{n} |dx_k|^2 \right)^{1/2} \quad \text{and} \quad S_{R,n}(x) = \left( \sum_{k=0}^{n} |dx_k^*|^2 \right)^{1/2}. \]

By the preceding discussion, \( dx \) belongs to \( L_{\Phi}(\mathcal{M}, \ell^2) \) (resp. \( L_{\Phi}(\mathcal{M}, \ell^2_R) \)) if and only if \( (S_{C,n}(x))_{n \geq 0} \) (resp. \( (S_{R,n}(x))_{n \geq 0} \)) is a bounded sequence in \( L_{\Phi}(\mathcal{M}) \); in this case, \( S_{C}(x) = \left( \sum_{k=0}^{\infty} |dx_k|^2 \right)^{1/2} \) and \( S_{R}(x) = \left( \sum_{k=0}^{\infty} |dx_k^*|^2 \right)^{1/2} \) are elements in \( L_{\Phi}(\mathcal{M}) \). These are noncommutative analogues of the usual square functions in the commutative martingale theory. It should be pointed out that the two sequences \( S_{C,n}(x) \) and \( S_{R,n}(x) \) may not be bounded in \( L_{\Phi}(\mathcal{M}) \) at the same time.

We define \( H_{\Phi C}(\mathcal{M}) \) (resp. \( H_{\Phi R}(\mathcal{M}) \)) to be the space of all \( L_{\Phi}-\)martingales with respect to \( (M_n)_{n \geq 0} \) such that \( dx \in L_{\Phi}(\mathcal{M}, \ell^2) \) (resp. \( dx \in L_{\Phi}(\mathcal{M}, \ell^2_R) \), equipped with the norm

\[ \|x\|_{H_{\Phi C}(\mathcal{M})} = \|dx\|_{L_{\Phi}(\mathcal{M}, \ell^2)} \quad \text{(resp. \( \|x\|_{H_{\Phi R}(\mathcal{M})} = \|dx\|_{L_{\Phi}(\mathcal{M}, \ell^2_R)} \))}. \]

\( H_{\Phi C}(\mathcal{M}) \) and \( H_{\Phi R}(\mathcal{M}) \) are Banach spaces. Note that if \( x \in H_{\Phi C}(\mathcal{M}) \),

\[ \|x\|_{H_{\Phi C}(\mathcal{M})} = \sup_{n \geq 0} \|S_{C,n}(x)\|_{L_{\Phi}(\mathcal{M})} = \|S_{C}(x)\|_{L_{\Phi}(\mathcal{M})}. \]

The similar equalities hold for \( H_{\Phi R}(\mathcal{M}) \).

Now, we define the Orlicz-Hardy spaces of noncommutative martingales as follows: If \( q_{\Phi} < 2 \),

\[ H_{\Phi}(\mathcal{M}) = H_{\Phi C}(\mathcal{M}) + H_{\Phi R}(\mathcal{M}), \]

equipped with the norm

\[ \|x\| = \inf \{ \|y\|_{H_{\Phi C}(\mathcal{M})} + \|z\|_{H_{\Phi R}(\mathcal{M})} : x = y + z, \ y \in H_{\Phi C}(\mathcal{M}), \ z \in H_{\Phi R}(\mathcal{M}) \}. \]

If \( 2 \leq p_{\Phi} \),

\[ H_{\Phi}(\mathcal{M}) = H_{\Phi C}(\mathcal{M}) \cap H_{\Phi R}(\mathcal{M}), \]

equipped with the norm

\[ \|x\| = \max \{ \|x\|_{H_{\Phi C}(\mathcal{M})}, \ \|x\|_{H_{\Phi R}(\mathcal{M})} \}. \]

The reason that we have defined \( H_{\Phi}(\mathcal{M}) \) differently according to \( q_{\Phi} < 2 \) or \( 2 \leq p_{\Phi} \) will become clear in the next section. This has been used first in [31, 32] and also in [23].
2. An interpolation theorem

The main result of this section is a Marcinkiewicz type interpolation theorem for noncommutative Orlicz spaces. It is the key to our proof of \( \Phi \)-moment inequalities of the noncommutative martingales. We first introduce the following definition.

**Definition 2.1.** Let \( \mathcal{N}_1 \) (resp. \( \mathcal{N}_2 \)) be a semifinite von Neumann algebra on a Hilbert space \( \mathbb{H}_1 \) (resp. \( \mathbb{H}_2 \)) with a normal semifinite faithful trace \( \nu_1 \) (resp. \( \nu_2 \)). A map \( T : L_0(\mathcal{N}_1) \to L_0(\mathcal{N}_2) \) is said to be sublinear if for any operators \( x, y \in L_0(\mathcal{N}_1) \), there exist isometries \( u, v \in \mathcal{N}_2 \) such that

\[
|T(x + y)| \leq u^*|Tx|u + v^*|Ty|v, \quad |T(\alpha x)| \leq |\alpha||Tx|, \quad \forall \alpha \in \mathbb{C}.
\]

This definition of sublinear operators in the noncommutative setting belongs to Q.Xu, which first appeared in Ying Hu’s thesis \cite{14} (see also \cite{15}). We recall the definition that a sublinear operator \( T : L_0(\mathcal{N}_1) \to L_0(\mathcal{N}_2) \) is of weak type \((p, q)\) with \( 1 \leq p \leq q \leq \infty \). This means that there is a constant \( C > 0 \), so that for every \( x \in L_p(\mathcal{N}_1) \)

\begin{equation}
\lambda_\alpha(|Tx|) \leq \left( \frac{C\|x\|_p}{\alpha} \right)^q, \quad \forall \alpha > 0.
\end{equation}

If \( q = \infty \), it means that \( \|Tx\|_q \leq C\|x\|_p \).

The classical Marcinkiewicz interpolation theorem has been extended to include Orlicz spaces as interpolation classes by A.Zygmund, A.P.Calderón, S.Koizumi, I.B.Simonenko, W.Riordan, H.P.Heinig and A.Torchinsky (for references see \cite{25} and therein). The following result is a noncommutative analogue of the Marcinkiewicz type interpolation theorem for Orlicz spaces.

**Theorem 2.1.** Let \( \mathcal{N}_1 \) (resp. \( \mathcal{N}_2 \)) be a semifinite von Neumann algebra on a Hilbert space \( \mathbb{H}_1 \) (resp. \( \mathbb{H}_2 \)) with a normal semifinite faithful trace \( \nu_1 \) (resp. \( \nu_2 \)). Suppose \( 1 \leq p_0 < p_1 \leq \infty \). Let \( T : L_0(\mathcal{N}_1) \to L_0(\mathcal{N}_2) \) be a sublinear operator and simultaneously of weak types \((p_i, p_i)\) for \( i = 0 \) and \( i = 1 \). If \( \Phi \) is an Orlicz function with \( p_0 < p_\Phi \leq q_\Phi < p_1 \), then there exists a constant \( C \) depending only on \( p_0, p_1 \) and \( \Phi \), such that

\begin{equation}
\nu_2(\Phi(|Tx|)) \leq C\nu_1(\Phi(|x|)),
\end{equation}

for all \( x \in L_\Phi(\mathcal{N}_1) \).

**Proof.** At first, we take \( p_1 < \infty \). For \( \alpha > 0 \), let \( x = x_0^\alpha + x_1^\alpha \), where \( x_0^\alpha = xE_{(\alpha, \infty)}(|x|) \). From the sublinearity of \( T \), it follows that

\begin{equation}
\lambda_{2\alpha}(|Tx|) \leq \lambda_\alpha(|Tx_0^\alpha|) + \lambda_\alpha(|Tx_1^\alpha|).
\end{equation}

By (2.1), there are two constants \( A_0, A_1 > 0 \) such that for any \( \alpha > 0 \)

\begin{equation}
\lambda_\alpha(|Tx|) \leq A_0^p\alpha^{-p_0}\|x\|_{p_0}^{p_0}, \quad \forall x \in L_{p_0}(\mathcal{N}_1),
\end{equation}

\begin{equation}
\lambda_\alpha(|Tx|) \leq A_1^p\alpha^{-p_1}\|x\|_{p_1}^{p_1}, \quad \forall x \in L_{p_1}(\mathcal{N}_1).
\end{equation}
Using (2.3), (2.4) and (2.5), we have

\[ \nu_2(\Phi(| Tx |)) = \int_0^\infty \lambda_{2\alpha}(| Tx |) d\Phi(2\alpha) \]

\[ \leq \int_0^\infty \lambda_\alpha(| Tx_0^\alpha |) d\Phi(2\alpha) + \int_0^\infty \lambda_\alpha(| Tx_1^\alpha |) d\Phi(2\alpha) \]

\[ \leq A_0^{p_0} \int_0^\infty \alpha^{-p_0} \| T x_0^\alpha \|_{p_0} d\Phi(2\alpha) + A_1^{p_1} \int_0^\infty \alpha^{-p_1} \| T x_1^\alpha \|_{p_1} d\Phi(2\alpha) \]

\[ \leq A_0^{p_0} \int_0^\infty \alpha^{-p_0} \nu_1( \| T x_0^\alpha \|_{E(\alpha, \infty)}(| x |) ) d\Phi(2\alpha) \]

\[ + A_1^{p_1} \int_0^\infty \alpha^{-p_1} \nu_1( \| T x_1^\alpha \|_{E(\alpha, \infty)}(| x |) ) d\Phi(2\alpha) \]

\[ \leq A_0^{p_0} \int_0^\infty \alpha^{-p_0} \left( \int_0^\infty t^{p_0} d\nu_1(E_t(| x |)) \right) d\Phi(2\alpha) \]

\[ + A_1^{p_1} \int_0^\infty \alpha^{-p_1} \left( \int_0^\infty t^{p_1} d\nu_1(E_t(| x |)) \right) d\Phi(2\alpha) \]

\[ = A_0^{p_0} \int_0^t t^{p_0} \left( \int_0^t \alpha^{-p_0} d\Phi(2\alpha) \right) d\nu_1(E_t(| x |)) \]

\[ + A_1^{p_1} \int_0^t t^{p_1} \left( \int_0^t \alpha^{-p_1} d\Phi(2\alpha) \right) d\nu_1(E_t(| x |)). \]

By the assumption, we know that \( \Phi \) satisfies the \( \Delta_2 \)-condition. This implies that

\[ \sup_{t > 0} \frac{t \Phi'(t)}{\Phi(t)} < \infty. \]

Then, we have

\[ \nu_2(\Phi(| Tx |)) \leq C_\Phi \left[ A_0^{p_0} \int_0^t t^{p_0} \left( \int_0^t \alpha^{-p_0-1} \Phi(\alpha) d\alpha \right) d\nu_1(E_t(| x |)) \right] \]

\[ + A_1^{p_1} \int_0^t t^{p_1} \left( \int_0^t \alpha^{-p_1-1} \Phi(\alpha) d\alpha \right) d\nu_1(E_t(| x |)). \]

On the other hand, by the assumption we have

\[ \int_0^t s^{-p_0}(s) \frac{ds}{s} = O(t^{-p_0}(t)) \] and \( \int_t^\infty s^{-p_1}(s) \frac{ds}{s} = O(t^{-p_1}(t)) \)

for all \( t > 0 \), respectively. Hence,

\[ \nu_2(\Phi(| Tx |)) \leq C_\Phi \left[ A_0^{p_0} + A_1^{p_1} \right] \int_0^\infty \Phi(t) d\nu_1(E_t(| x |)) = C \nu_1(\Phi(| x |)), \]

where \( C \) depends only on \( p_0, p_1 \) and \( \Phi \), i.e., (2.2) holds.

Let \( p_1 = \infty \) and let \( x = x_0^\alpha + x_1^\alpha \) as above. Then

\[ \| T x_1^\alpha \|_{L_\infty} \leq A_1 \| x_1^\alpha \|_{L_\infty} \leq A_1 \alpha \]
and \( \lambda_{A}(|Tx_1^n|) = 0 \). According to the above estimate, one obtains
\[
\lambda_{A}(|Tx|) \leq \lambda_{A}(|Tx_0^\alpha|) \leq \frac{\|Tx_0^\alpha\|_{p_0}^{p_0}}{(A_1 \alpha)^{p_0}} \leq \left( \frac{A_0}{A_1} \right)^{p_0} \alpha^{-p_0} \|x_0^\alpha\|_{p_0}.
\]
Then, by the same argument as above we have
\[
\nu_2(\Phi(|Tx|)) = \int_0^\infty \lambda_{A_{1}}(|Tx|) d\Phi(A_1 \alpha)
\leq \left( \frac{A_0}{A_1} \right)^{p_0} \int_0^\infty \alpha^{-p_0} \|x_0^\alpha\|_{p_0} d\Phi(A_1 \alpha)
= \left( \frac{A_0}{A_1} \right)^{p_0} \int_0^\infty \int_0^\infty \alpha^{-p_0} \nu_1(\|x\|_{p_0} E_{(\alpha, \infty)}(\|x\|)) d\Phi(A_1 \alpha)
= \left( \frac{A_0}{A_1} \right)^{p_0} \int_0^\infty \int_0^\infty \int_0^t \alpha^{-p_0} d\Phi(A_1 \alpha) d\nu_1(E_t(\|x\|))
\leq C_\Phi \left( \frac{A_0}{A_1} \right)^{p_0} \int_0^\infty \int_0^t \alpha^{-p_0-1} \Phi(\alpha) d\alpha d\nu_1(E_t(\|x\|))
\leq C_\Phi \left( \frac{A_0}{A_1} \right)^{p_0} \int_0^\infty \Phi(t) d\nu_1(E_t(\|x\|))
= C\nu_1(\Phi(|x|)),
\]
where \( C \) depends only on \( p_0, p_1 \) and \( \Phi \). This completes the proof. \( \square \)

**Remark 2.1.** (1) If \( T \) is of strong type \((p, p)\), i.e., there exists a constant \( C > 0 \) such that \( \|Tx\|_p \leq C\|x\|_p \) for any \( x \in L_p(N) \), then by the Kolmogorov inequality (1.1) we have
\[
\lambda_{\alpha}(|Tx|) \leq \alpha^{-p}\|Tx\|_p^p \leq C^p \alpha^{-p}\|x\|_p^p,
\]
that is, \( T \) is of weak type \((p, p)\). Consequently, if \( T \) is simultaneously of strong types \((p_i, p_i)\) for \( i = 0 \) and \( i = 1 \), then the above Theorem still holds.

(2) If we only consider the spaces of Hermitian operators, that is,
\[
L_p(N)_{\mathrm{Her}} = \{x \in L_p(N) : x^* = x\},
\]
the corresponding result of Theorem 2.1 also holds. The proof is the same as above and omitted.

### 3. \( \Phi \)-Moment Inequalities of Martingale Transforms

In the sequel, \((\mathcal{M}, \tau)\) always denotes a finite von Neumann algebra with a normalized normal faithful trace \( \tau \) and \((\mathcal{M}_n)_{n \geq 0}\) an increasing filtration of subalgebras of \( \mathcal{M} \) which generate \( \mathcal{M} \). We keep all notations introduced in the previous sections.
**Definition 3.1.** Let $\alpha = (\alpha_n) \subset \mathbb{C}$ be a sequence. Define a map $T_\alpha$ on the family of martingale difference sequences by $T_\alpha(dx) = (\alpha_n dx_n)$. $T_\alpha$ is called the martingale transform of symbol $\alpha$.

It is clear that $(\alpha_n dx_n)$ is indeed a martingale difference sequence. The corresponding martingale is

$$T_\alpha(x) = \sum_n \alpha_n dx_n.$$ 

The first application of Theorem 2.1 is to obtain $\Phi$-moment inequalities of martingale transforms as follows.

**Theorem 3.1.** Let $\alpha = (\alpha_n) \subset \mathbb{C}$ be a bounded sequence and $T_\alpha$ the associated martingale transform. Let $\Phi$ be an Orlicz function with $1 < p_\Phi \leq q_\Phi < \infty$. Then, there is a positive constant $C_{\Phi,\alpha}$ such that for all bounded $L_{\Phi}$-martingales $x = (x_n)$,

$$\tau(\Phi(|T_\alpha(x)|)) \leq C_{\Phi,\alpha} \tau(\Phi(|x|)),$$

where $C_{\Phi,\alpha}$ depends only on $\Phi$ and $\sup_n |\alpha_n|$.

**Proof.** Let $1 < p < p_\Phi \leq q_\Phi < q < \infty$. As the consequence of the noncommutative Burkholder-Gundy inequality as proved in Pisier-Xu [31] (see Remark 2.4 there) we have

$$T_\alpha : L_p(M) + L_q(M) \to L_p(M) + L_q(M)$$

with $\|T_\alpha\|_p \leq C_{p,\alpha}$ and $\|T_\alpha\|_q \leq C_{q,\alpha}$, where $C_{p,\alpha}, C_{q,\alpha}$ are both positive constants depending only on $p, q$ and $\sup_n |\alpha_n|$. Then, it follows from Theorem 2.1 that there is a constant $C_{\Phi,\alpha}$ such that

$$\tau(\Phi(|T_\alpha(x)|)) \leq C_{\Phi,\alpha} \tau(\Phi(|x|)),$$

as required. \qed

**Remark 3.1.** It is proved by Randrianantoanina [33] that $T_\alpha$ is of weak type $(1,1)$, from which we also conclude Theorem 3.1.

**Corollary 3.1.** Let $\Phi$ be an Orlicz function with $1 < p_\Phi \leq q_\Phi < \infty$. Then,

$$\tau(\Phi\left(\left\{ \sum \varepsilon_n dx_n \right\}\right)) \approx \tau(\Phi\left(\left\{ \sum dx_n \right\}\right)), \quad \forall \varepsilon_n = \pm 1$$

for all bounded $L_{\Phi}$-martingales $x = (x_n)$, where “$\approx$” depends only on $\Phi$.

Recall that a Banach space $X$ is called a UMD space if for some $q \in (1, \infty)$ (or equivalently, for every $q \in (1, \infty)$) there exists a constant $C$ such that for any finite $L_q$-martingales $f$ with values in $X$ one has

$$\left\| \sum \varepsilon_n df_n \right\|_{L_q(\Omega; X)} \leq C \sup_{n \geq 1} \|f_n\|_{L_q(\Omega; X)}, \quad \forall \varepsilon_n = \pm 1.$$ 

Then, a Banach space $X$ is a UMD space if and only if for any $L_\infty$-bounded Walsh-Paley martingale $f$ with values in $X$, the series $\sum \varepsilon_n df_n$ converges in probability (cf., see [20]).
Let \((\Omega, \mathcal{F}, P)\) be a probability space equipped with \((\mathcal{F}_n)\) a filtration of \(\sigma\)-subalgebras of \(\mathcal{F}\) such that \(\bigcup_n \mathcal{F}_n\) generates \(\mathcal{F}\). Let \((\mathcal{N}, \nu)\) be a noncommutative probability space. Put \(\mathcal{M} = L_\infty(\Omega, \mathcal{F}, P) \otimes \mathcal{N}\) equipped with the tensor product trace, and \(\mathcal{M}_n = L_\infty(\Omega, \mathcal{F}_n, P) \otimes \mathcal{N}\) for every \(n\). Then \((\mathcal{M}_n)\) is a filtration of von Neumann subalgebras of \(\mathcal{M}\). Recall that \(L_p(\mathcal{M}) = L_p(\Omega; L_p(\mathcal{N}))\) for all \(0 < p < \infty\). In this case, the noncommutative \(L_p\)-martingales with respect to \((\mathcal{M}_n)\) coincide with the usual \(L_p\)-martingales with respect to \((\mathcal{F}_n)\) but with values in \(L_p(\mathcal{N})\). Hence, by (3.2), for all bounded \(L_\Phi\)-martingales \(f = (f_n)\) with values in \(L_\Phi(\mathcal{N})\), we have

\[
\int_{\Omega} \nu \left( \Phi \left( \left| \sum \varepsilon_n df_n \right| \right) \right) \, dP \approx \int_{\Omega} \nu \left( \Phi \left( |f| \right) \right) \, dP, \quad \forall \varepsilon_n = \pm 1,
\]

where “\(\approx\)” depends only on \(\Phi\).

**Corollary 3.2.** Let \((\mathcal{N}, \nu)\) be a noncommutative probability space and \(\Phi\) an Orlicz function with \(1 < p_\Phi \leq q_\Phi < \infty\). Then \(L_\Phi(\mathcal{N})\) is a UMD space.

**Proof.** Let \(f = (f_n)\) be a \(L_\infty\)-bounded Walsh-Paley martingale with values in \(L_\Phi(\mathcal{N})\). By (3.3), we have

\[
\int_{\Omega} \nu \left( \Phi \left( \left| \sum \varepsilon_n df_n \right| \right) \right) \, dP \leq C \int_{\Omega} \nu \left( \Phi \left( |f| \right) \right) \, dP, \quad \forall \varepsilon_n = \pm 1,
\]

from which it follows that \(\Phi \left( \left| \sum \varepsilon_n df_n \right| \right) < \infty\) a.e., or \(\| \sum \varepsilon_n df_n \|_\Phi < \infty\) a.e.. Therefore, by Remark 1.1 (2), the series \(\sum \varepsilon_n df_n\) converges almost everywhere. This yields that \(L_\Phi(\mathcal{N})\) is a UMD space.

**Remark 3.2.** The above result on the UMD property of \(L_\Phi(\mathcal{N})\) remains true when \(\nu\) is a normal semifinite faithful trace and \(1 < p_\Phi \leq q_\Phi < \infty\). Indeed, there exists an increasing family \((e_j)_{j \in J}\) of projection of \(\mathcal{N}\) such that \(\nu(e_j) < \infty\) for every \(j \in J\) and such that \(e_j\) converges to the unit element of \(\mathcal{N}\) in the strong operator topology. Hence, \(\nu(e_j \Phi(|x|)) \to \nu(\Phi(|x|))\) for any \(x \in L_\Phi(\mathcal{N})\), since \(\Phi(|x|) \in L_1(\mathcal{N})\). Therefore, by approximation, one can easily reduce the semifinite case to the finite one. Alternately, the preceding argument continues to work for normal semifinite trace \(\nu\) on \(\mathcal{N}\) because the subalgebras \(\mathcal{M}_n = L_\infty(\Omega, \mathcal{F}_n, P) \otimes \mathcal{N}\) of \(\mathcal{M} = L_\infty(\Omega, \mathcal{F}, P) \otimes \mathcal{N}\) satisfy the condition in Remark 1.1 (3).

At the end of this section, by our interpolation result Theorem 2.1 we easily obtain the following noncommutative analogue of the Stein inequality for Orlicz spaces.

**Theorem 3.2.** Let \(\Phi\) be an Orlicz function with \(1 < p_\Phi \leq q_\Phi < \infty\) and \(\ell = (a_n)_{n \geq 0}\) a finite sequence in \(L_\Phi(\mathcal{M})\). Then, there exists a constant \(C_\Phi\) such that

\[
\tau \left( \Phi \left( \left( \sum_{n} \left| \mathcal{E}_n(a_n) \right|^2 \right)^{\frac{1}{2}} \right) \right) \leq C_\Phi \tau \left( \Phi \left( \left( \sum_{n} \left| a_n \right|^2 \right)^{\frac{1}{2}} \right) \right).
\]

Similar assertion holds for the row subspace \(L_\Phi(\mathcal{M}; \ell_1^2)\).
Proof. Let us consider the von Neumann algebra tensor product \( \mathcal{M} \otimes \mathcal{B}(l^2) \) with the product trace \( \tau \otimes \text{tr} \). Evidently, \( \tau \otimes \text{tr} \) is a semi-finite normal faithful trace. Let \( L_\Phi(\mathcal{M} \otimes \mathcal{B}(l^2)) \) be the associated non-commutative \( L_\Phi \) space. Then, \( L_\Phi(\mathcal{M} \otimes \mathcal{B}(l^2)) \) is an interpolation space for the couple \( (L_p(\mathcal{M} \otimes \mathcal{B}(l^2)), L_q(\mathcal{M} \otimes \mathcal{B}(l^2))) \), where \( 1 < p \leq q < q < \infty \). We define

\[
T : L_p(\mathcal{M} \otimes \mathcal{B}(l^2)) + L_q(\mathcal{M} \otimes \mathcal{B}(l^2)) \rightarrow L_p(\mathcal{M} \otimes \mathcal{B}(l^2)) + L_q(\mathcal{M} \otimes \mathcal{B}(l^2)),
\]

by

\[
T \left( \begin{array}{ccc}
 a_{11} & \ldots & a_{1n} \\
 a_{21} & \ldots & a_{2n} \\
 \vdots & \ddots & \vdots \\
 a_{n1} & \ldots & a_{nn}
\end{array} \right) = \left( \begin{array}{ccc}
 \xi_n(a_{11}) & 0 & 0 \\
 \xi_n(a_{21}) & 0 & 0 \\
 \vdots & \vdots & \vdots \\
 \xi_n(a_{n1}) & 0 & 0
\end{array} \right).
\]

Theorem 2.3 in [31] gives that \( T \) is a bounded linear operator on both \( L_p(\mathcal{M} \otimes \mathcal{B}(l^2)) \) and \( L_q(\mathcal{M} \otimes \mathcal{B}(l^2)) \). Thus, by Theorem 2.1 we obtain the desired result. \( \square \)

Remark 3.3. The noncommutative analogue of the classical Stein inequality in \( L_\nu \)-spaces is first presented in [31], which is one of key ingredients in their proof of the noncommutative Burkholder-Gundy inequality.

4. \( \Phi \)-moment Khintchine’s inequalities

In this section, we will prove a noncommutative analogue of \( \Phi \)-moment Khintchine’s inequality for Rademacher’s random variables.

Let \( \mathbb{T} \) be the unit circle of the complex plane equipped with the normalized Haar measure denoted by \( dm \). Let \( \mathcal{M} \) be a finite von Neumann algebra with a normalized normal faithful trace \( \tau \). Put \( \mathcal{N} = L_\infty(\mathbb{T})\mathcal{M} \) equipped with the tensor product trace \( \nu = \int \otimes \tau \) and \( \mathcal{A} = \mathcal{H}_\infty(\mathbb{T})\mathcal{M} \). Then, \( \mathcal{A} \) is a finite maximal subdiagonal algebras of \( \mathcal{N} \) with respect to \( \mathcal{E} = \int \otimes I_\mathcal{M} : \mathcal{N} \rightarrow \mathcal{M} \) (e.g., see [5]).

Lemma 4.1. Let \( \Phi \) be an Orlicz function with \( 1 < p_\Phi \leq q_\Phi < \infty \). Let \( \Phi^{(2)}(t) = \Phi(t^2) \). Then, for any \( f \in \mathcal{H}_\Phi(\mathcal{N}) \) and \( \varepsilon > 0 \), there exist two functions \( g, h \in \mathcal{H}\Phi(\mathcal{N}) \) such that \( f = gh \) with

\[
\max \left\{ \int_{\mathbb{T}} \tau[\Phi(|g|^2)] dm, \int_{\mathbb{T}} \tau[\Phi(|h|^2)] dm \right\} \leq \int_{\mathbb{T}} \tau(\Phi(|f|)) dm + \varepsilon.
\]

Proof. Using Theorem 6.2 of [26], we obtain

\[
(4.1) \quad \mathcal{H}_{p_\Phi}(\mathcal{N}) \cap L_{q_\Phi}(\mathcal{N}) = \mathcal{H}_\Phi(\mathcal{N}), \quad \mathcal{H}_\Phi(\mathcal{N}) \cap L_{q_\Phi(2)}(\mathcal{N}) = \mathcal{H}_{\Phi(2)}(\mathcal{N}).
\]

Let \( w = (f^*f + \varepsilon)^{1/2} \). Then \( w \in L_{q_\Phi}(\mathcal{N}) \) and \( w^{-1} \in \mathcal{N} \). Let \( v \in \mathcal{N} \) be a contraction such that \( f = vw \). Applying Theorem 4.8 of [5] to \( w^{1/2} \), we have \( w^{1/2} = uh \), where \( u \) is a unitary in \( \mathcal{N} \) and \( h \in \mathcal{H}^{2p_\Phi}(\mathcal{N}) \) such that \( h^{-1} \in \mathcal{A} \).

Set \( g = vh \). Then \( f = gh \), so \( g = fh^{-1} \). Since \( g \in \mathcal{H}_\Phi(\mathcal{N}) \) and \( h^{-1} \in \mathcal{A} \), \( f \in \mathcal{H}_\Phi(\mathcal{N}) \). By (4.1), \( g, h \in \mathcal{H}_{\Phi(2)}(\mathcal{N}) \). The integral estimate is clear. \( \square \)
Lemma 4.2. Let $\Phi$ be an Orlicz function with $1 < p_\Phi \leq q_\Phi < \infty$. Let $\{I_n = (\frac{2^n}{5}, 3^n] : n \in \mathbb{N}\}$ and $\Delta_n$ the Fourier multiplier by the indicator function $\chi_{I_n}$, i.e.

$$\Delta_n(f)(z) = \sum_{k \in I_n} \hat{f}(k)z^k$$

for any trigonometric polynomial $f$ with coefficients in $L_\Phi(M)$. Then, there exists a constant $C_\Phi > 0$ such that

$$\int_{\mathbb{T}} \tau \left( \Phi \left( \left( \sum_n \Delta_n(f)^* \Delta_n(f) \right)^{\frac{1}{2}} \right) \right) \nu dm \leq C_\Phi \int_{\mathbb{T}} \tau (\Phi(|f|)) dm,$$

for any $f \in L_\Phi(M)$.

Proof. Let $N = L_\infty(\mathbb{T}) \hat{\otimes} M$ equipped with the tensor product trace $\nu = \int \otimes \tau$, $1 < p < \infty$, then $L_p(\mathbb{T}, L_p(M)) = L_p(N)$. By Theorem 4 of [6] (see also the proof of Theorem III.1.1 of [23]) there exists a constant $C_p > 0$ such that for all $f \in L_p(\mathbb{T}, L_p(M))$, we have that

$$\left\| \left( \sum_n \Delta_n(f)^* \Delta_n(f) \right)^{\frac{1}{2}} \right\|_{L_p(\mathbb{T}, L_p(M))} \leq C_p \| f \|_{L_p(\mathbb{T}, L_p(M))},$$

that is,

$$\left\| \left( \sum_n \Delta_n(f)^* \Delta_n(f) \right)^{\frac{1}{2}} \right\|_p \leq C_p \| f \|_p, \quad \forall f \in L_p(N).$$

Since the mapping $T : N \mapsto N \hat{\otimes} B(\ell^2)$ is sublinear, where

$$Tf = \left( \sum_n \Delta_n(f)^* \Delta_n(f) \right)^{\frac{1}{2}}, \quad \forall f \in N,$$

by Theorem 2.1 we obtain the required result. \hfill \Box

Let $(\varepsilon_n)$ be a Rademacher sequence on a probability space $(\Omega, P)$. In the sequel, without specified, $N$ denotes a semifinite von Neumann algebra on a Hilbert space $\mathbb{H}$ with a normal semifinite faithful trace $\nu$.

Lemma 4.3. Let $\Phi$ be an Orlicz function. Suppose $x = (x_0, x_1, \ldots, x_n)$ is a finite sequence in $L_\Phi(N)$.

1. If $1 < p_\Phi \leq q_\Phi < 2$, then

$$\int_{\Omega} \nu \left( \Phi \left( \left[ \sum_{k=0}^n x_k \varepsilon_k \right] \right) \right) dP \leq \min C_\Phi \left\{ \nu \left( \Phi \left( \sum_{k=0}^n |x_k|^2 \right)^{\frac{1}{2}} \right), \nu \left( \Phi \left( \sum_{k=0}^n |x_k^*|^2 \right)^{\frac{1}{2}} \right) \right\}.$$
Consequently,

\[
\int_\Omega \nu \left( \Phi \left[ \left| \sum_{k=0}^n x_k \varepsilon_k \right| \right] \right) \, dP \\
\leq C_{\Phi} \inf \left\{ \nu \left( \Phi \left[ \left( \sum_{k=0}^n |y_k|^2 \right)^{\frac{1}{2}} \right] \right) \right. \\
+ \left. \nu \left( \Phi \left[ \left( \sum_{k=0}^n |z_k|^2 \right)^{\frac{1}{2}} \right] \right) \right\},
\]

where the infimum runs over all decomposition \( x_k = y_k + z_k \) with \( y_k \) and \( z_k \) in \( \mathcal{L}_\Phi(N) \).

(2) If \( 2 < p_{\Phi} \leq q_{\Phi} < \infty \), then

\[
\max \left\{ \nu \left( \Phi \left[ \left( \sum_{k=0}^n |x_k|^2 \right)^{\frac{1}{2}} \right] \right), \nu \left( \Phi \left[ \left( \sum_{k=0}^n |x^*_k|^2 \right)^{\frac{1}{2}} \right] \right) \right\} \\
\leq \int_\Omega \nu \left( \Phi \left[ \left| \sum_{k=0}^n x_k \varepsilon_k \right| \right] \right) \, dP.
\]

(4.3)

Proof. (1) We define \( T : \mathcal{L}_p(N \otimes \mathcal{B}(\ell^2)) \to \mathcal{L}_p(N \otimes \mathcal{L}_\infty(\Omega, P)) \) by

\[
T(a_{ij}) = \sum_i \varepsilon_i a_{i1}, \quad \forall (a_{ij}) \in \mathcal{L}_p(N \otimes \mathcal{B}(\ell^2)).
\]

Since \( \mathcal{L}_p(N, \ell^2_C) \) is 1-complemented in \( \mathcal{L}_p(N \otimes \mathcal{B}(\ell^2)) \), by the noncommutative Khintchine inequalities [21, 23] we conclude that \( T \) is bounded from \( \mathcal{L}_1(N \otimes \ell^2_C) \) into \( \mathcal{L}_1(N \otimes \mathcal{L}_\infty(\Omega, P)) \) and \( \mathcal{L}_2(N \otimes \ell^2_C) \) into \( \mathcal{L}_2(N \otimes \mathcal{L}_\infty(\Omega, P)) \) simultaneously. Then, by Theorem 2.1 we have

\[
\int_\Omega \nu \left( \Phi \left[ \left| \sum_{k=0}^n x_k \varepsilon_k \right| \right] \right) \, dP \leq C_{\Phi} \nu \left( \Phi \left[ \left( \sum_{k=0}^n |x_k|^2 \right)^{\frac{1}{2}} \right] \right).
\]

Similarly, if we let

\[
T(a_{ij}) = \sum_j \varepsilon_j a_{1j}, \quad \forall (a_{ij}) \in \mathcal{L}_p(N \otimes \mathcal{B}(\ell^2)),
\]

then we obtain

\[
\int_\Omega \nu \left( \Phi \left[ \left| \sum_{k=0}^n x_k \varepsilon_k \right| \right] \right) \, dP \leq C_{\Phi} \nu \left( \Phi \left[ \left( \sum_{k=0}^n |x^*_k|^2 \right)^{\frac{1}{2}} \right] \right).
\]

Hence, (4.2) holds.

To prove the second inequality take a decomposition \( x_k = y_k + z_k \). Then there exist two isometries \( U, V \in N \) such that

\[
\left| \sum_{k=0}^n x_k \varepsilon_k \right| \leq U^* \left| \sum_{k=0}^n y_k \varepsilon_k \right| U + V^* \left| \sum_{k=0}^n z_k \varepsilon_k \right| V.
\]
Consequently, by Proposition 4.6 (ii) in [12] and (4.2) we have
\[
\int_{\Omega} \nu\left( \Phi\left[ \left| \sum_{k=0}^{n} x_k \varepsilon_k \right| \right] \right) dP \\
\leq C \Phi \left\{ \int_{\Omega} \nu\left( \Phi\left[ \left| \sum_{k=0}^{n} y_k \varepsilon_k \right| \right] \right) dP + \int_{\Omega} \nu\left( \Phi\left[ \left| \sum_{k=0}^{n} z_k \varepsilon_k \right| \right] \right) dP \right\} \\
\leq C \Phi \inf \left\{ \nu\left( \Phi\left[ \left( \sum_{k=0}^{n} |y_k|^2 \right)^{\frac{1}{2}} \right] \right) + \nu\left( \Phi\left[ \left( \sum_{k=0}^{n} |z_k|^2 \right)^{\frac{1}{2}} \right] \right) \right\},
\]
where we have used the fact that \( \Phi \in \Delta_2 \) in the first inequality.

(2) Without loss of generality, we assume that \( \{\varepsilon_i\} \) is a (unconditional) basis in \( L_p(\Omega, P) \) \((1 < p < \infty)\), i.e., \( \text{span}\{\varepsilon_i\} \) is dense in \( L_p(\Omega, P) \). We let
\[
S\left( \sum_{k} x_k \varepsilon_k \right) = \begin{pmatrix} x_1 & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots \\
x_k & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \ddots \end{pmatrix}
\]
for any finite sequence \( \{x_k\} \) in \( L_p(\mathcal{N}) \). By the noncommutative Khintchine inequalities [21, 23] we conclude that \( S \) is well defined and extends to a bounded operator from \( L_p(\mathcal{N} \otimes L_{\infty}(\Omega, P)) \) into \( L_p(\mathcal{N} \otimes \mathcal{B}(\ell^2)) \) for every \( 2 \leq p < \infty \). Hence, by Theorem 2.1 we have
\[
\nu\left( \Phi\left[ \left( \sum_{k=0}^{n} |x_k|^2 \right)^{\frac{1}{2}} \right] \right) \leq \int_{\Omega} \nu\left( \Phi\left[ \left| \sum_{k=0}^{n} x_k \varepsilon_k \right| \right] \right) dP.
\]
Similarly, if we set
\[
S\left( \sum_{k} x_k \varepsilon_k \right) = \begin{pmatrix} x_1 & \ldots & x_k & \ldots \\
0 & \ldots & 0 & \ldots \\
\vdots & \vdots & \vdots & \ddots \end{pmatrix}
\]
for any finite sequence \( \{x_k\} \) in \( L_p(\mathcal{N}) \), then we have
\[
\nu\left( \Phi\left[ \left( \sum_{k=0}^{n} |x_k|^2 \right)^{\frac{1}{2}} \right] \right) \leq \int_{\Omega} \nu\left( \Phi\left[ \left| \sum_{k=0}^{n} x_k \varepsilon_k \right| \right] \right) dP.
\]
Hence, (4.3) holds. \( \Box \)

As following is the noncommutative analogue of \( \Phi \)-moment version of Khintchine’s inequalities for Rademacher’s sequences.

**Theorem 4.1.** Let \( \Phi \) be an Orlicz function and \( \{\varepsilon_i\} \) a Rademacher’s sequence.
(1) If $1 < p_\Phi \leq q_\Phi < 2$, then for any finite sequence $\{x_k\}$ in $L_\Phi(N)$,
\[
\int_\Omega \nu\left(\Phi\left[\sum_{k=0}^n x_k \epsilon_k\right]\right) dP \approx \inf \left\{ \nu\left(\Phi\left[\left(\sum_{k=0}^n \left|y_k\right|^2\right)^{\frac{q_\Phi}{2}}\right]\right) + \nu\left(\Phi\left[\left(\sum_{k=0}^n \left|z_k^*\right|^2\right)^{\frac{q_\Phi}{2}}\right]\right) \right\},
\]
where the infimum runs over all decomposition $x_k = y_k + z_k$ with $y_k$ and $z_k$ in $L_\Phi(N)$ and "$\approx$" depends only on $\Phi$.

(2) If $2 < p_\Phi \leq q_\Phi < \infty$, then for any finite sequence $\{x_k\}$ in $L_\Phi(N)$,
\[
\int_\Omega \nu\left(\Phi\left[\sum_{k=0}^n x_k \epsilon_k\right]\right) dP \approx \max \left\{ \nu\left(\Phi\left[\left(\sum_{k=0}^n \left|x_k\right|^2\right)^{\frac{q_\Phi}{2}}\right]\right), \nu\left(\Phi\left[\left(\sum_{k=0}^n \left|x_k^*\right|^2\right)^{\frac{q_\Phi}{2}}\right]\right) \right\},
\]
where "$\approx$" depends only on $\Phi$.

**Proof.** (1) By Lemma 4.3 (1), we need only to prove the lower estimate of (4.4). By the Khintchine-Kahane inequality [30] and Theorem 2.1, we are reduced to show for any finite sequence $\{x_k\}$ in $L_\Phi(N)$,
\[
\inf \left\{ \nu\left(\Phi\left[\left(\sum_{k=0}^n \left|y_k\right|^2\right)^{\frac{q_\Phi}{2}}\right]\right) + \nu\left(\Phi\left[\left(\sum_{k=0}^n \left|z_k^*\right|^2\right)^{\frac{q_\Phi}{2}}\right]\right) \right\}
\leq A' \int_\mathbb{T} \nu\left(\Phi\left[\sum_{k=0}^n x_k \epsilon_k \zeta_k\right]\right) dm(z),
\]
where the infimum runs over all decomposition $x_k = y_k + z_k$ with $y_k$ and $z_k$ in $L_\Phi(N)$. To prove (4.6), we can clearly assume, by approximation, that $\nu$ is finite, and even $\nu(1) = 1$. Thus, let $\{x_k\} \subset L_\Phi(M)$ be a fixed finite sequence, and set
\[
f(z) = \sum_{k=0}^n x_k z^k,
\]
then $f \in \mathcal{H}_\Phi(N)$. Given $\varepsilon > 0$ let $g$ and $h$ be the two functions in $\mathcal{H}_{\Phi(\varepsilon)}(N)$ associated to $f$ and $\varepsilon$ as in Lemma 4.1. Then for any $k$
\[
\hat{f}(3^k) = \sum_{0 \leq m \leq 3^k} \hat{g}(3^k - m) \hat{h}(m).
\]
Let
\[
a_k = \sum_{0 \leq m \leq \frac{3^k}{2}} \hat{g}(3^k - m) \hat{h}(m) \quad \text{and} \quad b_k = \sum_{\frac{3^k}{2} < m \leq 3^k} \hat{g}(3^k - m) \hat{h}(m).
Thus we have a decomposition $x_k = a_k + b_k$. It remains to estimate

$$\nu\left( \Phi\left[ \left( \sum_{k=0}^{n} |a_k|^2 \right)^{1/2} \right] \right) \quad \text{and} \quad \nu\left( \Phi\left[ \left( \sum_{k=0}^{n} |b_k|^2 \right)^{1/2} \right] \right),$$

respectively. To this end, let

$$g_k(z) = \sum_{\frac{3k}{T} < m \leq 3k} \hat{g}(m) z^m.$$

Observe that

$$a_k = \hat{g}h(3^k) = \int_T g_k(z) h(z) z^{-3^k} dm(z).$$

Then, by the Jensen and Hölder inequalities we have

$$\nu\left( \Phi\left[ \left( \sum_{k=0}^{n} |a_k|^2 \right)^{1/2} \right] \right)$$

$$= \nu\left( \Phi\left[ \left( \sum_{k=0}^{n} \left| \int_T g_k(z) h(z) z^{-3^k} dm(z) \right|^2 \right)^{1/2} \right] \right)$$

$$\leq \int_T \nu\left( \Phi\left[ \left( \sum_{k=0}^{n} |g_k(z)| h(z) z^{-3^k} \right)^2 \right] \right) dm(z)$$

$$= \int_T \nu\left( \Phi\left[ \left( h(z)^* \sum_{k=0}^{n} g_k(z)^* g_k(z) h(z) \right)^{1/2} \right] \right) dm(z)$$

$$= \int_T \int_0^\infty \Phi \left[ \mu_T \left\{ h(z) \left( \mu_T \left\{ \sum_{k=0}^{n} g_k(z)^* g_k(z) \right\} \right)^{1/2} \right\} \right] dt dm(z)$$

$$\leq \int_T \int_0^\infty \Phi \left[ \mu_T \left\{ h(z) \right\} \left( \mu_T \left\{ \sum_{k=0}^{n} g_k(z)^* g_k(z) \right\} \right)^{1/2} \right] dt dm(z)$$

$$\leq C \Phi \int_T \int_0^\infty \Phi \left[ \mu_T \left( |h(z)|^2 \right) \right] dt dm(z)$$

$$+ \int_T \int_0^\infty \Phi \left[ \mu_T \left( \sum_{k=0}^{n} g_k(z)^* g_k(z) \right) \right] dt dm(z)$$

$$\leq C \Phi \left\{ \int_T \nu\left( \Phi(|h(z)|^2) \right) dm + \int_T \nu\left( \Phi \left[ \sum_{k=0}^{n} g_k(z)^* g_k(z) \right] \right) dm \right\}.$$
Now let $I_k = \{ m \in \mathbb{Z} : \frac{3^k-1}{3^k} < m \leq 3^k \}$. Then by Lemma 4.2 we have

$$
\int_T \nu \left( \Phi \left[ \left\{ g_k(z)^* g_k(z) \right\} \right] \right) dm = \int_T \nu \left( \Phi^{(2)} \left[ \left\{ \left( \sum_{k=0}^n g_k(z)^* g_k(z) \right)^{1/2} \right\} \right] \right) dm \\
\leq C_\Phi \int_T \nu \left( \Phi^{(2)} \left[ \left| g(z) \right| \right] \right) dm \\
= C_\Phi \int_T \nu \left( \Phi \left[ \left| g(z) \right|^2 \right] \right) dm,
$$

since $1 < 2p_\Phi \leq p_{\Phi^{(2)}} \leq q_{\Phi^{(2)}} \leq 2q_\Phi < \infty$. Thus, we deduce that

$$
\nu \left( \Phi \left[ \left( \sum_{k=0}^n \left| a_k \right|^2 \right)^{1/2} \right] \right) \leq C_\Phi \left( \int_T \tau \left( \Phi \left[ \left| f \right| \right] \right) dm + \varepsilon \right).
$$

Similarly, we have

$$
\nu \left( \Phi \left[ \left( \sum_{k=0}^n \left| b_k^* \right|^2 \right)^{1/2} \right] \right) \leq C_\Phi \left( \int_T \tau \left( \Phi \left[ \left| f \right| \right] \right) dm + \varepsilon \right).
$$

Hence, we obtain (4.6) and complete the proof of (1).

(2) The upper estimate of (4.5) immediately follows from Lemma 4.3 (2). To prove the upper estimate of (4.5), we consider first the case that $x_0, x_1, \ldots, x_n$ are Hermitian operators in $L_\Phi(N)$. Define $T$ as in the proof of Lemma 4.3 (1). By the noncommutative Khintchine’s inequality [23] (also see [21]) and the fact that $L_p(N \otimes B(\ell_2))_\text{Her}$ is also 1-complemented in $L_p(N \otimes L_p(\Omega, P))_\text{Her}$ (i.e., $(a_{ij}) \in L_p(N \otimes B(\ell_2))_\text{Her}$ if $a_{ij}$‘s are Hermitian operators and $(a_{ij}) \in L_p(N \otimes B(\ell_2))$), we obtain that $T$ is bounded from $L_p(N \otimes L_p(\Omega, P))_\text{Her}$ for $2 \leq p < \infty$. Consequently, by Theorem 2.1 (see Remark 2.1 (2) there) there exists a constant $C_\Phi$ such that

$$
\int_\Omega \nu \left( \Phi \left[ \left| \sum_{k=0}^n x_k \varepsilon_k \right| \right] \right) dP \leq C_\Phi \nu \left( \Phi \left[ \left( \sum_{k=0}^n x_k^2 \right)^{1/2} \right] \right).
$$

The general case follows from the above special case. Indeed, let $x_k = y_k + iz_k$ (1 \leq k \leq n), where $y_k, z_k$ are Hermitian operators. Since

$$
y_k^2 + z_k^2 = \frac{1}{2} [x_k^* x_k + x_k x_k^*],
$$

we have

$$
\sum_{k=0}^n y_k^2 \leq \frac{1}{2} \sum_{k=1}^n [x_k^* x_k + x_k x_k^*] \quad \text{and} \quad \sum_{k=0}^n z_k^2 \leq \frac{1}{2} \sum_{k=1}^n [x_k^* x_k + x_k x_k^*].
$$
Hence,

\[
\int_\Omega \nu \left( \Phi \left[ \sum_{k=0}^{n} x_k \varepsilon_k \right] \right) dP \\
= \int_\Omega \int_0^\infty \Phi \left[ \mu_t \left( \sum_{k=0}^{n} x_k \varepsilon_k \right) \right] dt dP \\
\leq \int_\Omega \int_0^\infty \Phi \left[ \mu_t \left( \sum_{k=0}^{n} y_k \varepsilon_k \right) + \mu_t \left( \sum_{k=0}^{n} z_k \varepsilon_k \right) \right] dt dP \\
\leq \frac{1}{2} \int_0^\infty \left\{ \Phi \left[ 2 \mu_t \left( \sum_{k=0}^{n} y_k \varepsilon_k \right) \right] + \Phi \left[ 2 \mu_t \left( \sum_{k=0}^{n} z_k \varepsilon_k \right) \right] \right\} dt dP \\
\leq C_\Phi \int_0^\infty \left\{ \nu \left( \Phi \left[ \left( \sum_{k=0}^{n} y_k^2 \right)^{\frac{1}{2}} \right] \right) + \nu \left( \Phi \left[ \left( \sum_{k=0}^{n} z_k^2 \right)^{\frac{1}{2}} \right] \right) \right\} dt \\
\leq C_\Phi \nu \left( \Phi \left[ \left( \frac{1}{2} \sum_{k=1}^{n} \left[ x_k^* x_k + x_k x_k^* \right] \right)^{\frac{1}{2}} \right] \right) \\
= C_\Phi \int_0^\infty \Phi \left[ \nu \left( \left( \sum_{k=0}^{n} \left[ x_k^2 \right]^2 \right)^{\frac{1}{2}} \right) \right] dt \\
\leq C_\Phi \max \left\{ \nu \left( \Phi \left[ \left( \sum_{k=0}^{n} \left[ x_k^2 \right]^2 \right)^{\frac{1}{2}} \right) \right), \nu \left( \Phi \left[ \left( \sum_{k=0}^{n} \left[ x_k^* \right]^2 \right)^{\frac{1}{2}} \right) \right) \right\},
\]

where \( C_\Phi \)'s may be different in different lines. This completes the proof. \( \square \)

**Remark 4.1.** Note that Khintchine's inequality is valid for \( L_1 \)-norm in both commutative and noncommutative settings (cf., [23]). We could conjecture that the right condition in Theorem 4.1 (1) should be \( q_\Phi < 2 \) without the additional restriction condition \( 1 < p_\Phi \). However, our argument seems to be inefficient in this case. We need new ideas to approach it.

5. \( \Phi \)-moment Burkholder-Gundy’s inequalities

Now, we are in a position to state and prove the \( \Phi \)-moment version of noncommutative Burkholder-Gundy martingale inequalities.

**Theorem 5.1.** Let \( \Phi \) be an Orlicz function and \( x = (x_n)_{n \geq 0} \) a noncommutative \( L_\Phi \)-martingale.
When \(0\) information yet. However, \(\Phi\) (c < 0
q
)
Example 5.2. Let (5.3)
Therefore, integrating on \(\Omega\) we have
(5.2).
Example 5.1. Let \(\xi\) noncommutative Burkholder-Gundy's inequalities obtained above.
Proof. (1) Let \(x\) be any finite martingale in \(L\Phi\) \((\xi)\) a Rademacher sequence on a probability space \((\Omega, P)\). Then, by (3.2) we have
\[
\tau\left(\Phi\left[\sum_{n=0}^{\infty} \xi_n d\xi_n\right]\right) \approx \tau\left(\Phi\left[\sum_{n=0}^{\infty} d\xi_n\right]\right).
\]
Therefore, integrating on \(\Omega\) we have
\[
\tau\left(\Phi\left[\sum_{n=0}^{\infty} d\xi_n\right]\right) \approx \int_{\Omega} \tau\left(\Phi\left[\sum_{n=0}^{\infty} \xi_n d\xi_n\right]\right) dP.
\]
It follows from Theorem 4.1 (1) that
\[
\tau\left(\Phi\left[\sum_{n=0}^{\infty} d\xi_n\right]\right) \approx \inf \left\{ \tau\left(\Phi\left[\sum_{n=0}^{\infty} |d\xi_n|^{1/2}\right]\right) \right\}
\]
where the infimum runs over all decomposition \(x_n = y_n + z_n\) with \(y_n\) and \(z_n\) in \(L\Phi\) \((\xi)\). Then, using Theorem 3.2 we get (5.1).
(2) Similarly, using (5.3) and Theorem 4.1 (2) we obtain the desired result (5.2).

As follows, we give two examples for illustrating the \(\Phi\)-moment version of noncommutative Burkholder-Gundy's inequalities obtained above.

**Example 5.1.** Let \(\Phi(t) = t^a \ln(1 + t^b)\) with \(a > 1\) and \(b > 0\). It is easy to check that \(\Phi\) is an Orlicz function and
\[
p_{\Phi} = a \quad \text{and} \quad q_{\Phi} = a + b.
\]
When \(1 < a < a + b < 2\), we have (5.1), while \(a > 2\) we have (5.2). However, when \(1 < a \leq 2 \leq a + b\) Theorem 5.1 gives no information.

**Example 5.2.** Let \(\Phi(t) = t^p(1 + c \sin(p \ln t))\) with \(p > 1/(1 - 2c)\) and 0 < \(c\) < 1/2. Then, \(\Phi\) is an Orlicz function and
\[
p_{\Phi} = q_{\Phi} = p.
\]
When \(0 < c < 1/4\), \(p_{\Phi} = q_{\Phi} = 2\) occurs. In this case, Theorem 5.1 gives no information yet. However, \(\Phi\) is equivalent to \(t^p\) and so the corresponding Burkholder-Gundy's inequality holds. On the other hand, in general \(p_{\Phi} = q_{\Phi} = p\) does not imply that \(\Phi\) is equivalent to \(t^p\) (see [24, 25] for details).
In this section, we make some remarks on our results and possible further researches.

(1) As indicated in Examples 5.1 and 5.2, Φ-moment Burkholder-Gundy’s inequalities of noncommutative martingales in the cases of $1 < p_\Phi \leq 2 \leq q_\Phi < \infty$ remain open. Our interpolation argument seems to be inefficient to approach them. (It is clear that our argument is efficient for all the case $1 < p_\Phi \leq q_\Phi < \infty$ in the commutative setting.) On the other hand, one encounters some substantial difficulties in trying to adapt the classical techniques, which used stopping times, to the noncommutative setting. As a good substitute for stopping times, Cuculescu’s projections [10] played an important role for establishing weak-type inequalities [34, 35] and a noncommutative analogue of the Gundy’s decomposition [28]. However, these projections do not seem to be powerful enough for noncommutative Φ-moment inequalities (see also [4] for the noncommutative atomic decomposition and [29] for the noncommutative Davis’ decomposition). We need new ideas beyond interpolation and Cuculescu’s projections.

(2) In [8], the authors proved the following Φ-moment martingale inequality: Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{\mathcal{F}_k\}$ a increasing sequence of $\sigma$-subfields of $\mathcal{F}$. If $\Phi$ is an Orlicz function satisfying $\Delta_2$-condition, then for any sequence $\{f_k\}$ of nonnegative $\mathcal{F}$-measurable functions

$$
E\Phi\left(\sum_k E[f_k | \mathcal{F}_k]\right) \leq C_\Phi E\Phi\left(\sum_k f_k\right).
$$

(See also [13] for an another proof.) Stopping times and good-λ techniques developed by Burkholder et al. [7] are two key ingredients in the proof of (6.1). In $L_p$-cases, (6.1) is the so-called dual version of Doob’s maximal inequality. The noncommutative analogue of (6.1) in the $L_p$-case plays a crucial role in Junge’s approach [16] to noncommutative Doob’s inequality. Unfortunately, our interpolation argument is unavailable in the approach to (6.1) in noncommutative setting for Orlicz functions. As expected, the good-λ techniques in the noncommutative setting should be developed and it might be efficient for this goal.

(3) We end the paper with a note on Φ-moment inequalities on the conditioned square function $\sigma(f) = \left( \sum_n E_{n-1}[|df_n|^2] \right)^{1/2}$ and maximal function $f^* = \sup_n |f_n|$ for a martingale $f = \{f_n\}$. Let us recall the Φ-moment version of the classical Burkholder-Davis-Gundy theorem for martingales (see [8]): Let $\Phi$ be an Orlicz function satisfying the $\Delta_2$-condition. Then

$$
E\Phi(f^*) \approx E\Phi[S(f)] \text{ with } S(f) = \left( \sum_n |df_n|^2 \right)^{1/2},
$$

for all martingales $f$, where “$\approx$” depends only on $\Phi$. The noncommutative case is surprisingly different as noted in [19]. Indeed, it was shown in [19, Corollary 14], that (6.2) does not hold for $\Phi(t) = t$ in general. Instead, a
noncommutative analogue of
\begin{equation}
E[S(f)] \approx \inf \left\{ E[\sigma(g)] + E\left( \sum_n |dh_n| \right) \right\},
\end{equation}
holds as shown in [29], where the infimum runs over all decompositions $f = g + h$ with $g, h$ being two martingales adapted to the same filtration. Motivated by this result and the commutative case, we would carry out a noncommutative analogue of the $\Phi$-moment version of (6.3) elsewhere [3]. Again, the interpolation argument will play a key role in this problem.

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