Lower Bounds for Asymptotic Consensus in Dynamic Networks

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Abstract

In this work we study the performance of asymptotic and approximate consensus algorithms in dynamic networks. The asymptotic consensus problem requires a set of agents to repeatedly set their outputs such that the outputs converge to a common value within the convex hull of initial values. This problem, and the related approximate consensus problem, are fundamental building blocks in distributed systems where exact consensus among agents is not required, e.g., man-made distributed control systems, and have applications in the analysis of natural distributed systems, such as flocking and opinion dynamics. We prove new nontrivial lower bounds on the contraction rates of asymptotic consensus algorithms, from which we deduce lower bounds on the time complexity of approximate consensus algorithms. In particular, the obtained bounds show optimality of asymptotic and approximate consensus algorithms presented in [Charro-Bost et al., ICALP’16] for certain classes of networks that include classical failure assumptions, and confine the search for optimal bounds in the general case.

Central to our lower bound proofs is an extended notion of valency, the set of reachable limits of an asymptotic consensus algorithm starting from a given configuration. We further relate topological properties of valencies to the solvability of exact consensus, shedding some light on the relation of these three fundamental problems in dynamic networks.

1 Introduction

In the asymptotic consensus problem a set of agents, each starting from an initial value in \(\mathbb{R}^d\), update their values such that all agents’ values converge to a common value within the convex hull of initial values. The problem is closely related to the approximate consensus problem, in which agents have to irrevocably decide on values that lie within a predefined distance \(\epsilon > 0\) of each other. The latter is weaker than the exact consensus problem in which agents need to decide on the same value. Both the asymptotic and the approximate consensus problems have not only a variety of applications in the design of man-made control systems like sensor fusion [3], clock synchronization [19], formation control [15], rendezvous in space [20], or load balancing [13], but also for analyzing natural systems like flocking [29], firefly synchronization [24], or opinion dynamics [18]. These problems often have to be solved under harsh-environmental restrictions: with limited computational power and local storage, under restricted communication abilities, and in presence of communication uncertainty.

In this work we study asymptotic consensus in round-based computational models with a dynamic communication topology whose directed communication graphs are chosen each round from a predefined set of communication graphs, the so-called network model. In previous work [7], Charron-Bost et al. showed that asymptotic consensus is solvable precisely within rooted network models in which all communication graphs contain rooted spanning trees. These rooted spanning trees need not have any edges in common and can change from one round to the next.

An interesting special case of rooted network models are network models whose graphs are non-split, that is, any two agents have a common incoming neighbor. Their prominent role is motivated by two properties: (i) They occur as communication graphs in benign classical distributed failure models. For example, in synchronous systems with crashes, in asynchronous systems with a minority of crashes, and
synchronous systems with send omissions. (ii) In [7], Charron-Bost et al. showed that non-split graphs also play a central role in arbitrary rooted network models: they showed that any product of \( n \) - 1 rooted graphs with \( n \) nodes is non-split, allowing to transform asymptotic consensus algorithms for non-split network models into their amortized variants for rooted models.

Interestingly, solvability in any rooted network model is already provided by deceptively simple algorithms [7]: so-called averaging or convex combination algorithms, in which agents repeatedly broadcast their current value, and update it to some weighted average of the values they received in this round. One instance, proposed by Charron-Bost et al. [8] is the midpoint algorithm, in which agents update their value to the midpoint of the set of received values, i.e., the average of the smallest and the largest of the received values.

Regarding time complexity, for dimension \( d = 1 \), the amortized midpoint algorithm was shown to have a contraction rate (cf. Section 3 for a formal definition) of \( \sim \sqrt{\frac{1}{d}} \) in arbitrary rooted network models with \( n \) agents, and the midpoint algorithm of \( \frac{1}{d} \) in non-split network models [8]. The latter is optimal for “memoryless” averaging algorithms, which only depend on the values received in the current round [8].

The question arises whether non-averaging or non-memoryless algorithms, i.e., algorithms that (i) do not necessarily set their output values to within the convex hull of previously received values or (ii) whose output is a function not only of the previously received values, allow faster contraction rates. Indeed, algorithms that violate (i) and (ii) are studied in literature. As an example for (i), consider the algorithm where each agent sends an equal fraction of its current output value to all out-neighbors and combination algorithm as its output may lie outside the convex hull of the values of its in-neighbors. However, it solves asymptotic consensus algorithm for a fixed directed communication graph. Other examples of algorithms that violate (i) and (ii) are from control theory, where the usage of overshooting fast second-order controllers is common.

**Contribution.** In this work, we prove lower bounds on the contraction rate of any asymptotic consensus algorithm. In particular, the following lower bounds hold for a rooted network model \( \mathcal{N} \) with \( n \) agents: If exact consensus is solvable in \( \mathcal{N} \), an optimal contraction rate of 0 can be achieved. Otherwise,

- In a system with \( n = 2 \) agents, the contraction rate is lower bounded by 1/3 (Theorem 9).
- For an arbitrary communication graph \( G \), let \( \text{deaf}(G) = \{ F_1, \ldots, F_n \} \), where \( F_i \) is derived from \( G \) by making agent \( i \) deaf in \( F_i \), i.e., removing the incoming links of \( i \) in \( G \). In a system with \( n \geq 3 \) agents, if \( \mathcal{N} \) contains \( \text{deaf}(G) \), then the contraction rate is lower bounded by 1/2 (Theorem 10).
- In general, in a system with \( n \geq 3 \) agents, any asymptotic consensus algorithm must have a contraction rate of at least \( 1/(D+1) \), where \( D \) is the so-called \( \alpha \)-diameter of \( \mathcal{N} \), introduced in Section 6 (Theorem 20). This generalizes the previous two lower bounds.

All lower bounds hold regardless of the structure of the algorithm. In particular, algorithms can be full-information and agents can set their outputs outside the convex hull of received values. This, e.g., includes using higher-order filters in contrast to the 0-order filters of averaging algorithms.

Central to our proofs is the concept of the valency of a configuration of an asymptotic consensus algorithm, defined as the set of limits reachable from this configuration. By studying changes in valency along executions, we infer bounds on the contraction rate.

In Sections 4 and 5, we show tight lower bounds in network-models with \( n = 2 \) agents, and in case of non-split network models and \( d = 1 \) dimensional values. Notably, the lower bounds are valid for arbitrary dimensions. In recent work [9], an asymptotic consensus algorithm for non-split network models and dimension \( d = 2 \) with contraction rate \( 1/2 \), and an algorithm for arbitrary dimensions with contraction rate \( \frac{1}{d+1} \) in non-split models were presented. This also shows tightness of our lower bounds for dimension two in non-split network models. Table 1 summarizes lower and upper bounds.

In Section 6 we study the topological structure of the valencies with respect to the network model the asymptotic consensus algorithm is executed in (Theorem 18), and generalize our lower bounds to \( 1/(D+1) \) where \( D \) is the \( \alpha \)-diameter of the network model (Theorem 20).

In Section 7 we then extend our results on contraction rates to derive new lower bounds on the decision time of any approximate consensus algorithms in non-split network models: \( \log_3 \frac{D}{\delta} \) for \( n = 2 \) (Theorem 22), and in case \( n \geq 3 \), \( \log_2 \frac{D}{\delta} \) for models with \( \text{deaf}(G) \) for some communication graph \( G \).
Table 1: Summary of lower and upper bounds on contraction rates if consensus is not solvable. New lower bounds proved in this work are marked with a ∗. The three right columns distinguish between the case the network model is (i) non-split and contains deaf(G) for some communication graph G, (ii) is non-split, and (iii) is rooted.

| agents | dimension | network model in which exact consensus is unsolvable |  \( n = 2 \) \( d \geq 1 \) |  \( n \geq 3 \) \( d \in \{1,2\} \) |  \( n \geq 3 \) \( d \geq 3 \) |
|--------|-----------|-----------------------------------------------|-----------------|-----------------|-----------------|
|        |           | non-split with deaf graphs \( \leq \) non-split \( \leq \) rooted | \( \frac{1}{3} \) | \( \frac{d}{d+1} \) | \( \frac{d}{d+1} \) |

(Proposition 3, and \( \log_{D+1} \frac{\Delta}{\log n} \) for arbitrary models in which exact consensus is not solvable (Theorem 23). Again, deciding versions of the two asymptotic consensus algorithms for \( n = 2 \) and \( n \geq 3 \) from [8], have matching time complexity in non-split network models that include some deaf(G); showing optimality of these algorithms also for solving approximate consensus.

For proofs not part of the main body we would like to refer to the appendix.

Related work. The problem of asymptotic consensus in dynamic networks has been extensively studied, see e.g. [25, 11, 2, 5, 4]. The question of guaranteed convergence rates and decision times of the corresponding approximate consensus problems, naturally arise in this context. Algorithms with convergence times exponential in the number of agents have been proposed, e.g., in [5].

Olshesky and Tsitsiklis [28], proposed an algorithm with polynomial convergence time in bidirectional networks with certain stability assumptions on the occurring communication graphs. The bounds on convergence times were later on refined in [20]. Chazelle [11] proposed an averaging algorithm with polynomial convergence time, which works in any bidirectional connected network model.

To speed up decision times, algorithms where agents set their output based on values also received in previous than the current round, have also been considered in literature: Olshesky [27] proposed a linear convergence time algorithm that uses messages from two rounds, however, being restricted to fixed bidirectional communication graphs. In [30], a linear decision time algorithm for a possibly non-bidirectional fixed topology was proposed. It requires storing all received values. In previous work [8], Charro-Bost et al. proposed the midpoint algorithm that has constant decision time in non-split network models the amortized midpoint algorithm with linear decision time in rooted network models.

To the best of our knowledge, the only lower bound on convergence rate in dynamic networks has been shown in [3]: the authors proved that the convergence rate of a specific averaging algorithm in a non-split network model with \( n \) agents is at least \( 1 - \frac{1}{n} \).

In the context of classical distributed computing failure scenarios, Dolev et al. [14] studied the related approximate agreement problem: they considered fully-connected synchronous distributed systems with up to \( f \) Byzantine agents, and its asynchronous variant. The two presented algorithms require \( n \geq 3f + 1 \) for the synchronous and \( n \geq 5f + 1 \) for the asynchronous distributed system, the first of which is optimal in terms of resilience [17]. The latter result was later on improved to \( n \geq 3f + 1 \) in [11]. Both papers also address the question of optimal contraction rate in such systems. Since, however, in synchronous systems with \( n \geq 3f + 1 \) exact consensus is solvable, leading to a contraction rate of 0, the authors consider bounds for round-by-round contraction rates. In [14] they show that the achieved round-by-round contraction rate of \( \frac{1}{2} \) is actually tight for a certain class of algorithms that repeatedly set their output to the image of a so-called cautious function applied to the multiset of received values. A lower bounds for arbitrary algorithms, however, remains an open problem. In higher dimensions, Mendes et al. [23] proposed algorithms with convergence time of \( d \cdot \left[ \log_2 \sqrt{\Delta} \right] \) under the optimal resiliency condition \( n \geq f \cdot \max \{3, d + 1\} + 1 \).

Fekete [16] also studied round-by-round contraction rates for several failure scenarios, again, all in which exact consensus is solvable. He proved asymptotically tight lower bounds for synchronous distributed systems in presence of crashes, omission, and Byzantine agents. The bounds hold for approximate agreement algorithms that potentially take into account information from all previous rounds.
2 Model

We consider a set \([n] = \{1, \ldots, n\}\) of \(n\) agents (also classically called processes). We assume a distributed, round-based computational model in the spirit of the Heard-Of model [10]. Computation proceeds in rounds: In every round, each agent sends its state to its outgoing neighbors, receives messages from its incoming neighbors, and finally updates its state according to a deterministic local algorithm, i.e., a transition function that maps the collection of incoming messages to a new state. Rounds are communication closed in the sense that no agent receives messages in round \(t\) that are sent in a round different from \(t\).

Communications that occur in a round are modeled by a directed graph with a node for each agent. Since an agent can obviously communicate with itself instantaneously, every communication graph contains a self-loop at each node. In the following, we use the product of two communication graphs \(G\) and \(H\), denoted \(G \circ H\), which is the directed graph with an edge from \(i\) to \(j\) if there exists \(k\) such that \((i, k)\) and \((k, j)\) are two edges in \(G\) and \(H\), respectively.

We fix a non-empty set of communication graphs \(\mathcal{N}\) that determines the network model. To fully model dynamic networks in which topology may change continually and unpredictably, the communication graph at each round is chosen arbitrarily among \(\mathcal{N}\). Thus we form the infinite sequences of graphs in \(\mathcal{N}\) which we call communication patterns in \(\mathcal{N}\). In each communication pattern, the communication graph at round \(t\) is denoted by \(G_t\), and \(\text{In}_t(i) = \text{In}_t(G_t)\) and \(\text{Out}_t(i) = \text{Out}_t(G_t)\) are the sets of incoming and outgoing neighbors (in-neighbors and out-neighbors for short) of agent \(i\) in \(G_t\).

Let us fix an algorithm \(A\); a configuration is a collection of \(n\) agent states, one per agent. We assume that all agents have the same sets of initial states. Since agents are deterministic, given some configuration \(C\) and some communication graph \(G\), the algorithm \(A\) uniquely determines a new configuration which we simply denote \(G.C\) if no confusion can arise. Then the execution of \(A\) from the initial configuration \(C_0\) and with the communication pattern \((G_t)_{t \geq 1}\) is the sequence \(C_0, G_1, \ldots, C_{t-1}, G_t, C_t, \ldots\) of alternating configurations and communication graphs such that for each round \(t\), \(C_t = G_t.C_{t-1}\). The set of executions with communication patterns in \(\mathcal{N}\), denoted \(E^\mathcal{N}_A\), with the distance \(\text{dist}(E, E') = 1/2^\theta\), where \(\theta\) is the first index at which \(E\) and \(E'\) differ, is a compact metric space (e.g., see [21]).

Finally, any configuration that occurs in some execution with a communication pattern in \(\mathcal{N}\) is said to be reachable by \(A\) in \(\mathcal{N}\).

In the sequel, the algorithm and the network model are omitted if no confusion can arise.

2.1 Asymptotic Consensus

We assume that the local state of agent \(i\) includes a variable \(y^i\) in an Euclidean \(d\)-space, and we let \(y^i_{\perp}(t) \in \mathbb{R}^d\) denote the value of \(y^i\) at the end of round \(t\) in some execution \(E\). Then we let \(y^i_{\parallel}(t) = (y^i_{1\perp}(t), \ldots, y^i_{n\perp}(t))\). We write \(\text{diam}(A) = \sup_{x,y \in A} \|x - y\|\) for the diameter of a set \(A \subseteq \mathbb{R}^d\) and \(\Delta(y(t)) = \text{diam}(\{y^1(t), \ldots, y^n(t)\})\) for the diameter of the set of values in round \(t\).

We say an algorithm solves the asymptotic consensus problem in a network model \(\mathcal{N}\) if the following holds for every execution \(E\) with a communication pattern in \(\mathcal{N}\):

- **Convergence.** Each sequence \((y^i_{\parallel}(t))_{t \geq 0}\) converges.
- **Agreement.** If \(y^i_{\parallel}(t)\) and \(y^j_{\parallel}(t)\) converge, then they have a common limit.
- **Validity.** If \(y^i_{\parallel}(t)\) converges, then its limit is in the convex hull of the initial values \(y^i_{\perp}(0), \ldots, y^i_{\perp}(0)\).

Observe that the consensus function defined by \(y^* : E \in (\mathcal{E}, \text{dist}) \mapsto y^*_{\parallel} \in (\mathbb{R}^d, \|\cdot\|)\), where \(y^*_{\parallel}\) denotes the common limit of the \(n\) sequences \((y^i_{\parallel}(t))_{t \geq 0}\) is a priori not continuous. And indeed, there exist asymptotic consensus algorithms whose consensus functions are not continuous.

Alternatively, one may also consider the approximate consensus problem, in which convergence is replaced by a decision in a finite number of rounds and where agreement should be achieved with an arbitrarily small error tolerance (see, e.g., [22]). Formally, the local state of \(i\) is augmented with a variable \(d^i\) initialized to \(\bot\). Agent \(i\) is allowed to set \(d^i\) to some value \(v \neq \bot\) only once, in which case we say that \(i\) decides \(v\). In addition to the initial values \(y^i(0)\), agents initially receive the error tolerance \(\varepsilon\) and an upper bound \(\Delta\) on the maximum distance of initial values. An algorithm solves approximate
consensus in \( \mathcal{N} \) if for all \( \varepsilon > 0 \) and all \( \Delta \), each execution \( E \) with a communication pattern in \( \mathcal{N} \) with initial diameter at most \( \Delta \) satisfies:

- **Termination.** Each agent eventually decides.
- **\( \varepsilon \)-Agreement.** If agents \( i \) and \( j \) decide \( v \) and \( v' \), then \( \| v - v' \| \leq \varepsilon \).
- **Validity.** If agent \( i \) decides \( v \), then \( v \) is in the convex hull of initial values \( y^1_k(0), \ldots, y^n_k(0) \).

The above two problems are clearly closely related. However, the \( \varepsilon \)-agreement condition does not preclude the decisions of a given agent, as a function of the error tolerance parameter \( \varepsilon \), to diverge, i.e., a priori may lead to unstable decisions with respect to this parameter. This is the reason why we prefer to focus on asymptotic consensus rather than on approximate consensus.

### 2.2 Solvability of Asymptotic Consensus with Convex Combination Algorithms

In a previous paper [7], Charron-Bost et al. proved the following characterization of network models in which asymptotic consensus is solvable.

**Theorem 1** ([7]). In any dimension \( d \), the asymptotic consensus problem is solvable in a network model \( \mathcal{N} \) if and only if each graph in \( \mathcal{N} \) has a rooted spanning tree.

For the proof of the sufficient condition, Charron-Bost et al. focused on convex combination algorithms where each agent \( i \) updates its variable \( y^i \) to a value within the convex hull of values \( y^j(t-1) \) it has just received. In particular, they showed in [7] that convex combination algorithms where agents update their \( y^i \) via a weighted average of the received values, where weights only depend on the currently received values, solve asymptotic consensus in rooted network models. Such algorithms are memoryless, require little computational overhead and, more importantly, have the benefit of working in anonymous networks. Interestingly, their consensus function \( y^* \) is continuous.

**Theorem 2.** The consensus function of every convex combination algorithm that solves asymptotic consensus is continuous on the set of its executions.

**Proof.** Let \( (E_s)_{s \geq 0} \) be a sequence of executions that converges to \( E \). By definition of the distance on the execution space, this in particular means that

\[
\forall t \geq 0 \exists s_t \forall s \geq s_t: \quad y_s(0) = y(0), \quad y_s(1) = y(1), \ldots, \quad y_s(t) = y(t)
\]

where \( y_s(t) \) and \( y(t) \) denote \( y_{E_s}(t) \) and \( y_{E}(t) \), respectively.

Let \( \varepsilon > 0 \). By definition of the limit \( y^* \) of execution \( E \), there exists some \( t \) such that

\[
\forall i \in [n]: \quad \| y^i(t) - y^* \| \leq \varepsilon/3 \ .
\]

By (1), there is an \( s_t \) such that

\[
\forall s \geq s_t \forall i \in [n]: \quad \| y^i_s(t) - y^* \| \leq \varepsilon/3 \ .
\]

By the triangle inequality, this means

\[
\forall s \geq s_t \forall i, j \in [n]: \quad \| y^i_s(t) - y^j_s(t) \| \leq 2\varepsilon/3 \ .
\]

Because the algorithm is a convex combination algorithm, the limit \( y^*_s \) lies in the convex hull of the points \( y^1_s(t), \ldots, y^n_s(t) \). That is,

\[
\forall s \geq s_t \forall i \in [n]: \quad \| y^i_s(t) - y^*_s \| \leq 2\varepsilon/3 \ .
\]

Combining these inequalities gives

\[
\forall s \geq s_t: \quad \| y^*_s - y^* \| \leq \| y^*_s - y^i_s(t) \| + \| y^i_s(t) - y^* \| \leq \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\]

where \( i \) is any agent. This proves \( \lim_{s \to \infty} y^*_s = y^* \) as required. \( \square \)
3 Valency and Contraction Rate

We now extend the notion of valency for a consensus algorithm to asymptotic consensus algorithms. We fix an asymptotic consensus algorithm \( \mathcal{A} \) that solves \( d \)-dimensional asymptotic consensus in a certain network model \( \mathcal{N} \) with \( n \geq 2 \) agents. Let \( C \) be a configuration reachable by \( \mathcal{A} \) in \( \mathcal{N} \). Then we define the valency of \( C \) by

\[
Y^*_N(C) = \{ y^*_E \in \mathbb{R}^d : C \text{ occurs in } E \in \mathcal{E}^N_A \}.
\]

In case the algorithm \( \mathcal{A} \) is clear from the context, we skip the index. Observe that in the case \( \mathcal{A} \) is a convex combination algorithm, the valency of a configuration \( C \) is a compact set of \( \mathbb{R}^d \) since the consensus function is continuous and the set of executions in which \( C \) occurs is a compact set. Set \( \delta_N(C) = \text{diam}(Y^*_N(C)) \) the diameter of the set of reachable limits starting from configuration \( C \).

It is \( \delta_N(C_t) \to 0 \) in any execution \( E = G_0, C_1, G_1, C_2, \ldots \) by Convergence and Agreement. To study the speed of convergence, we introduce the contraction rate of algorithm \( \mathcal{A} \) in network model \( N \) as

\[
\sup_{E \in \mathcal{E}^N_A} \limsup_{t \to \infty} \sqrt{\delta_N(C_t)}
\]

where \( E = C_0, G_1, C_1, G_2, \ldots \). In particular, any algorithm that guarantees \( \delta_N(C_t) \leq \alpha^t \delta_N(C_0) \) for all \( t \geq 0 \) has contraction rate of at most \( \alpha \).

We obtain the following properties for subsets of network models:

**Lemma 3.** Let \( \mathcal{N}, \mathcal{N}'' \) be two network models with \( \mathcal{N}'' \subseteq \mathcal{N} \). If \( \mathcal{A} \) is an algorithm that solves asymptotic consensus in \( \mathcal{N} \), then (i) it also solves asymptotic consensus in \( \mathcal{N}'' \), (ii) for every configuration \( C \) reachable by \( \mathcal{A} \) in \( \mathcal{N}'' \), we have \( Y^*_N(C) \subseteq Y^*_N(C) \), (iii) \( \delta_N(C) \leq \delta_N(C) \), and (iv) the contraction rate in \( \mathcal{N}'' \) is less or equal to the contraction rate in \( \mathcal{N} \).

**Proof.** Statements (i), (ii), and (iii) immediately follow from the definition of valency. It remains to show statement (iv). From \( \mathcal{E}^N_A \subseteq \mathcal{E}^N_A \) and (iii), we deduce

\[
\sup_{E \in \mathcal{E}^N_A} \limsup_{t \to \infty} \sqrt{\delta_N(C_t)} \leq \sup_{E \in \mathcal{E}^N_A} \limsup_{t \to \infty} \sqrt{\delta_N(C_t)},
\]

which concludes the proof.

We establish two branching properties of valency of configurations in execution trees.

**Lemma 4.** Let \( C \) be a configuration reachable by algorithm \( \mathcal{A} \) in network model \( \mathcal{N} \). Then

\[
Y^*_N(C) = \bigcup_{G \in \mathcal{N}} Y^*_N(G.C).
\]

**Proof.** First let \( y^* \in Y^*_N(C) \). By definition of \( Y^*_N(C) \), there exists an execution \( E = C_0, G_1, C_1, G_2, \ldots \) in \( \mathcal{E}^N_A \) and a \( t \geq 0 \) such that \( y^* = y^*_E \) and \( C = C_t \). Set \( G = G_{t+1} \). Hence we have \( C_{t+1} = G.C \). But this shows that \( y^* \in Y^*_N(G.C) \) since \( G.C \) occurs in execution \( E \) whose limit is \( y^* \). This shows inclusion of the left-hand side in the right-hand side.

Now let \( G \in \mathcal{N} \) and \( y^* \in Y^*_N(G.C) \). Then there is an execution \( E = C_0, G_1, C_1, G_2, \ldots \) in \( \mathcal{E}^N_A \) and a \( t \geq 0 \) such that \( y^* = y^*_E \) and \( G.C = C_t \). Since \( C \) is a reachable configuration, there exists an execution \( E' = C_0, G_1, C_1, G_2, \ldots \) in \( \mathcal{E}^N_A \) and an \( s \geq 0 \) such that \( C_s = C \). Then the sequence

\[
E'' = C_0', G_1', C_1', G_2', \ldots \in \mathcal{E}^N_A \text{ with } y^*_{E''} = y^* \text{. Hence } y^* \in Y^*_N(C) \text{ because } C \text{ occurs in } E''. \text{ This shows inclusion of the right-hand side in the left-hand side and concludes the proof.}
\]

**Lemma 5.** Let \( C \) be a configuration reachable by algorithm \( \mathcal{A} \) in network model \( \mathcal{N} \). Then there exist \( G, H \in \mathcal{N} \) such that

\[
\text{diam} \left( Y^*_N(C) \right) = \text{diam} \left( Y^*_N(G.C) \cup Y^*_N(H.C) \right).
\]
Proof. Set $Y = Y_N^*(C)$, and $Y_G = Y_N^*(G,C)$ for $G \in \mathcal{N}$. By Lemma 4 it is $Y = \bigcup_{G \in \mathcal{N}} Y_G$, which means that every sequence of pairs of points in $Y$ whose distances converge to $\text{diam}(Y)$ includes an infinite subsequence in some product $Y_G \times Y_H$ because there are only finitely many. Thus $\text{diam}(Y) \leq \text{diam}(Y_G \cup Y_H)$. The other inequality follows from $Y_G \cup Y_H \subseteq Y$.

Two configurations $C$ and $C'$ are called indistinguishable for agent $i$, denoted $C \sim_i C'$, if $i$ is in the same state in $C$ as in $C'$.

As an immediate consequence of the above definition, we obtain:

**Lemma 6.** Let $C$ and $C'$ be two reachable configurations, and let $G$ and $G'$ be communication graphs from $\mathcal{N}$. If some agent $i$ has the same in-neighbors in $G$ and $G'$ and if $C \sim_j C'$ for each of $i$’s in-neighbors $j$, then $G.C \sim_i G'.C'$.

An agent $i$ is said to be **deaf in a communication graph** $G$ if $i$ has a unique in-neighbor in $G$, namely $i$. We are now in position to relate valencies of successor configuration $s$.

**Lemma 7.** If the agent $i$ has the same in-neighbors in two communication graphs $G$ and $G'$ in $\mathcal{N}$, and if there exists a communication graph in $\mathcal{N}$ in which $i$ is deaf, then $Y_N^*(G.C) \cap Y_N^*(G'.C') \neq \emptyset$.

**Proof.** From Lemma 6 we have $G.C \sim_i G'.C$.

Let $D_i$ be a communication graph in $\mathcal{N}$ in which the agent $i$ is deaf. Then we consider an execution $E$ in which $C$ occurs at some round $t_0 - 1$, $G$ is the communication graph at round $t_0$, and from there on all communication graphs are equal to $D_i$. Analogously, let $E'$ be an execution identical to $E$ except that the communication graph at round $t_0$ is $G'$ instead of $G$. By inductive application of Lemma 6 we show that for all $t \geq t_0$, we have $C_t \sim_i C'_t$. In particular, we obtain $y^i_{e}(t) = y^i_{e'}(t)$. Thus $y^e_i = y'^e_i$ which shows that $Y_N^*(G.C)$ and $Y_N^*(G'.C)$ intersect.

From Lemma 7 we determine the valency of any initial configuration when the network model contains certain communication graphs. If every agent is deaf in some communication graph of the network model $\mathcal{N}$, then the next lemma shows that the diameter of the valency of any initial configuration is equal to the diameter of the set of its initial values.

**Lemma 8.** If for every agent $i$, there is a communication graph in $\mathcal{N}$ in which $i$ is deaf, then each initial configuration $C_0$ satisfies $\delta_N(C_0) = \Delta(y(0))$. In particular, there is an initial configuration for which $\delta_N(C_0) > 0$.

**Proof.** Since $Y_N^*(C_0)$ is a subset of the convex hull of $\{y^1(0), \ldots, y^n(0)\}$ by the Validity property of asymptotic consensus and since the diameter of the convex hull of $\{y^1(0), \ldots, y^n(0)\}$ is equal to $\Delta(y(0))$, we have the inequality $\delta_N(C_0) \leq \Delta(y(0))$.

To show the converse inequality, let $i$ and $j$ be two agents such that $\|y^i(0) - y^j(0)\| = \Delta(y(0))$. Let $E$ be the execution with initial configuration $C_0$ and a constant communication graph in which agent $i$ is deaf. Now consider $C^{(i)}_0$, an initial configuration such that all initial values are set to $y^i(0)$, and the execution $E^{(i)}$ from $C^{(i)}_0$ with the same communication pattern as in $E$.

By a repeated application of Lemma 7 we see that at each round $t$, we have $C_t \sim_i C^{(i)}_t$. Hence, $y^e_i = y^e_{E^{(i)}}$.

From the Validity condition, we deduce that $y^e(E^{(i)}) = y^i(0)$. It then follows that $y^i(0) \in Y_N^*(C_0)$. By a similar argument, we see $y^j(0) \in Y_N^*(C_0)$. Hence

$$\delta_N(C_0) \geq \|y^i(0) - y^j(0)\| = \Delta(y(0)),$$

which concludes the proof.

## 4 Lower Bound for $n = 2$

In this section, we prove a lower bound of $1/3$ on the contraction rate of algorithms that solve asymptotic consensus in the network model of all rooted (and here also non-split) communication graphs with two agents. Combined with Algorithm 1 which achieves this lower bound, we have indeed identified a
Theorem H
Lemma 7

We show the stronger statement that for every initial configuration $C_0$, there is an execution $E = C_0, G_1, G_2, \ldots$ starting from $C_0$ such that

$$\delta_N(C_t) \geq \frac{1}{3^t} \cdot \delta_N(C_0)$$

for all $t \geq 0$. This, applied to an initial configuration with $\delta_N(C_0) > 0$, which exists by Lemma 8, then shows the theorem.

Note that it suffices to show (5) for the specific network model $N = \{H_0, H_1, H_2\}$ because $\delta_N(C_t) \geq \delta_{N'}(C_t)$ by Lemma 8 and $\delta_N(C_0) = \delta_{N'}(C_0)$ by Lemma 8 whenever $N \supseteq N'$. We hence suppose $N = N'$ in the rest of the proof.

The proof is by inductive construction of an execution $E = C_0, G_1, G_2, \ldots$ whose configurations $C_t$ satisfy (5). Equation (5) is trivial for $t = 0$.

Now assume $t \geq 0$ and that Equation (5) holds for $t$. There are three possible successor configurations of $C_t$, one for each of the communication graphs $H_0$, $H_1$, and $H_2$ in $N'$. Set $C_{t+1}^k = H_k.C_t$. Further let $Y = Y_{N'}(C_t)$, and $Y_k = Y_{N'}(C_{t+1}^k)$.

We will show that there is some $k \in \{0, 1, 2\}$ with $\text{diam}(Y_k) \geq \text{diam}(Y)/3$. We then define $G_{t+1} = H_k$ and $C_{t+1} = C_{t+1}^k$. By the induction hypothesis, we then have

$$\delta_{N'}(C_{t+1}) \geq \delta_{N'}(C_t)/3 \geq \delta_{N'}(C_0)/3^{t+1},$$

i.e., Equation (5) holds for $t + 1$.

Assume by contradiction that $\text{diam}(Y_k) < \text{diam}(Y)/3$ for all $k \in \{0, 1, 2\}$. From Lemma 8, we have $Y = Y_0 \cup Y_1 \cup Y_2$. Noting that agent 1 is deaf in $H_1$ and that agent 2 is deaf in $H_2$ and agent 1 has the same incoming edges as in $H_0$, and that agent 2 is deaf in $H_2$ and agent 1 has the same incoming edges as in $H_0$, we obtain from Lemma 4 that

$$Y_0 \cap Y_1 \neq \emptyset \quad \text{and} \quad Y_0 \cap Y_2 \neq \emptyset.$$ 

The sets $Y_0$ and $Y_1$ intersecting means

$$\text{diam}(Y_0 \cup Y_1) \leq \text{diam}(Y_0) + \text{diam}(Y_1) < \frac{2}{3} \text{diam}(Y).$$

Algorithm 1

**Algorithm with contraction rate 1/3 for $n = 2$**

**Initialization:**

1. $y' \in R$

**In round $t \geq 1$ do:**

2. send $y'$ to other agent and receive $y'$ if $j \in \text{In}_i(t)$
3. if $y'$ was received then
4. $y' \leftarrow y'/3 + 2y'/3$
5. end if

**Figure 1:** The rooted communication graphs $H_0$, $H_1$, and $H_2$ for $n = 2$
Further, the sets $Y_0 \cup Y_1$ and $Y_2$ intersecting means
\[
\text{diam}(Y) = \text{diam}(Y_0 \cup Y_1 \cup Y_2) \leq \text{diam}(Y_0 \cup Y_1) + \text{diam}(Y_2) < \text{diam}(Y), \tag{9}
\]
a contradiction. This concludes the proof. \qed

5 Lower Bound for $n \geq 3$

In this section, we prove a lower bound of $1/2$ on the contraction rate of asymptotic consensus algorithms for $n \geq 3$ agents, in a network model that includes graphs derived from a communication graph $G$, where agents are made deaf in the derived graphs. As a special case this includes the network model of all non-split communication graphs. Charron-Bost et al. \cite{8} presented the midpoint algorithm (given in Algorithm 2) for dimension $d = 1$ with contraction rate $1/2$ for non-split communication graphs. Together this shows tightness of our lower bound in dimension one.

Algorithm 2 Midpoint algorithm

\begin{algorithm}
\caption{Midpoint algorithm}
\begin{algorithmic}
\State \textbf{Initialization:} \stepcounter{algorithm}
\State $y' \in \mathbb{R}$
\State \textbf{In round} $t \geq 1$ \textbf{do:}
\State 2: send $y'$ to all agents in $\text{Out}_i(t)$ and receive $y'$ from all agents $j$ in $\text{In}_i(t)$
\State 3: $m' \leftarrow \min \{y' \mid j \in \text{In}_i(t)\}$
\State 4: $M' \leftarrow \max \{y' \mid j \in \text{In}_i(t)\}$
\State 5: $y' \leftarrow (m' + M')/2$
\end{algorithmic}
\end{algorithm}

One can apply the algorithm componentwise in dimension $d = 2$ to show tightness of our lower bound also there. Unfortunately, componentwise application in dimension $d \geq 3$ does not yield an asymptotic consensus algorithm \cite{9}.

We start with a lower bound proof for the network models that include certain deaf graphs. Let $G$ be an arbitrary communication graph. Consider a system with $n \geq 3$ agents, and the $n$ communication graphs $F_1, \ldots, F_n$ where $F_i$ is obtained by making $i$ deaf in $G$, i.e., by removing all the edges towards $i$ except the self-loop $(i, i)$: let $\text{deaf}(G) = \{F_1, \ldots, F_n\}$ with $F_i = G \setminus \{(j, i) : j \in [n] \setminus \{i\}\}$.

**THEOREM 10.** The contraction rate of any asymptotic consensus algorithm for $n \geq 3$ agents in a network model that includes deaf(G) is greater or equal to $1/2$.

*Proof.* We show the stronger statement that for every initial configuration $C_0$ there is an execution $E = C_0, C_1, C_2, \ldots$ starting at $C_0$ such that
\[
\delta_N(C_t) \geq \frac{1}{2^t} \delta_N(C_0), \tag{10}
\]
for all $t \geq 0$. It suffices to show \cite{11} for the specific network model $N' = \text{deaf}(G)$ because $\delta_N(C_t) \geq \delta_{N'}(C_t)$ by Lemma \cite{8} and $\delta_{N'}(C_0) = \delta_N(C_0)$ by Lemma \cite{8} whenever $N' \supseteq N'$. We hence suppose $N = N'$ in the rest of the proof. The proof is by inductive construction of an execution $E = C_0, C_1, C_2, \ldots$ whose configurations $C_t$ satisfy \cite{10}. This, applied to an initial configuration with $\delta_N(C_0) > 0$, which exists by Lemma \cite{8}, then shows the theorem.

For $t = 0$ the inequality holds trivially.

Now let $t$ be any positive integer and assume that Equation \cite{11} holds for $t$. There are $n$ possible successor configurations based on the applicable communication graphs $F_1, \ldots, F_n$. We denote $C^k_{t+1} = F_k.C_t$, for any agent $k$. Further let $Y = Y_{N'}(C_t)$, and $Y_k = Y_{N'}(C^k_{t+1})$.

We will show that there exists some agent $k \in [n]$ such that
\[
\text{diam}(Y_k) \geq \text{diam}(Y)/2. \tag{11}
\]
We then define $G_{t+1} = F_k$ and $C_{t+1} = C^k_{t+1}$. By \cite{11} and the induction hypothesis, we have
\[
\delta_{N'}(C_{t+1}) \geq \frac{\delta_{N'}(C_t)}{2} \geq \frac{1}{2^{t+1}} \delta_{N'}(C_0), \tag{12}.
\]
i.e., Equation (13) holds for $t + 1$.

Assume by contradiction that $\text{diam}(Y_k) < \text{diam}(Y)/2$ for all $k \in [n]$. Recall that agent $i$ is deaf in $F_i$ and has the same in-neighbors in all the communication graphs $F_j$ with $j \neq i$. Since $n \geq 3$, for any pair of agents $i, j$ we may select an agent $\ell$ different from $i$ and $j$ such that $\ell$ has the same in-neighbors in $F_i$ as in $F_j$. Lemma 7 with the assumption that $F_i$ is in $\mathcal{N}$ shows that for any pair of agents $i, j$, we have

$$Y_i \cap Y_j \neq \emptyset .$$  

(13)

By Lemma 5, there exist $k, k' \in [n]$ such that $\text{diam}(Y_k \cup Y_{k'}) = \text{diam}(Y)$. In particular, we can choose $i = k$ and $j = k'$, which implies that

$$\text{diam}(Y) = \text{diam}(Y_k \cup Y_{k'}) \leq \text{diam}(Y_k) + \text{diam}(Y_{k'}) < \text{diam}(Y)$$  

(14)

which is a contradiction and concludes the proof. $\Box$

Note that the network model deaf$(K_n)$ is a subset of the network model that contains all non-split communication graphs. Hence the lower bound holds and since Algorithm 2 is applicable the claim of a solvable (Theorem 18) and nontrivial lower bounds on the contraction rates whenever exact consensus is not a characterization of the topological structure of valencies with respect to solvability of exact consensus relation between these two problems by studying valencies and convergence rates. Our main results are Charron-Bost et al. showed that asymptotic consensus is solvable in a significantly broader class: it is (Theorem 20 and Corollary 21).

We next show properties of subsets of network model $\mathcal{N}$ that are $\beta_N$-classes.

6 Relation to Exact Consensus and Generalized Bounds

An algorithm solves exact consensus in network model $\mathcal{N}$ if in all its executions with communication graphs in $\mathcal{N}$, the following properties hold:

- **Termination.** Each agent eventually decides.
- **Agreement.** If agents $i$ and $j$ decide $v$ and $v'$, then $v = v'$.
- **Validity.** If agent $i$ decides $v$, then $v$ is among the initial values $y_1^1(0), \ldots, y_n^1(0)$.

In [12], Coulouma et al. characterized the network models in which exact consensus is solvable. In [7], Charron-Bost et al. showed that asymptotic consensus is solvable in a significantly broader class: it is solvable if and only if a network model is rooted. In this section we aim to shed light on the deeper relation between these two problems by studying valencies and convergence rates. Our main results are a characterization of the topological structure of valencies with respect to solvability of exact consensus (Theorem 15) and nontrivial lower bounds on the contraction rates whenever exact consensus is not solvable (Theorems 20 and Corollary 21).

We start with recalling some definitions from Coulouma et al. [12]. In the following, we denote by $R(G)$ the set of roots of a communication graph $G$, i.e., the set of agents that have a directed path to all other agents in $G$. For a set $S \subseteq [n]$, let $\text{In}_S(G) = \bigcup_{j \in S} \text{In}_j(G)$. The set $\text{Out}_S(G)$ is defined analogously.

**Definition 11** (Definition 4.7 in [12]). Let $\mathcal{N}$ be a network model. Given $G, H, K \in \mathcal{N}$, we define $G_{\alpha_N,K}H$ if $\text{In}_{R(K)}(G) = \text{In}_{R(K)}(H)$. The relation $\alpha^*_N$ is the transitive closure of the union of relation $\alpha_{N,K}$ where $K$ varies in $\mathcal{N}$.

**Definition 12** (Definition 4.8 in [12]). Let $\mathcal{N}$ be a network model. We define $\beta_N$ to be the coarsest equivalence relation included in $\alpha^*_N$ such that for all $G, H$ holds:

**Closure Property** If $G_{\beta_N}H_q$ then there exists a nonnegative integer $q$ and communication graphs $H_0, \ldots, H_q \in \mathcal{N}$ and $K_1, \ldots, K_q \in \mathcal{N}$ such that

- (i) $G = H_0$ and $H = H_q$
- (ii) $\forall r \in [q]: H_r_{\beta_N}G$ and $K_r_{\beta_N}G$
- (iii) $\forall r \in [q]: H_{r-1_{\alpha_N,K_r}}H_r$

We next show properties of subsets of network model $\mathcal{N}$ that are $\beta_N$-classes.
Lemma 13. Let $\mathcal{N}$ be a network model and let $\mathcal{N}' \subseteq \mathcal{N}$ be a $\beta_{\mathcal{N}}$-class. Then $\alpha_{\mathcal{N}'} H$ and $\beta_{\mathcal{N}'} H$ for all $G, H \in \mathcal{N}'$.

Proof. Let $G, H \in \mathcal{N}'$. Since $G \beta_{\mathcal{N}} H$, there is a $q$ and $H_0, \ldots, H_q \in \mathcal{N}$ and $K_1, \ldots, K_q \in \mathcal{N}$ such that (i) $G \sim H_0$ and $H \sim H_q$ (ii) $H_r \beta_{\mathcal{N}} G$ and $K_r \beta_{\mathcal{N}} G$ for all $r \in [q]$, and (iii) $H_r \alpha_{\mathcal{N}'} K_r$ for all $r \in [q]$. Condition (ii) implies $H_0, \ldots, H_q \in \mathcal{N}'$ and $K_1, \ldots, K_q \in \mathcal{N}'$ since they belong to the same $\beta_{\mathcal{N}}$-class as $G$, i.e., $\mathcal{N}'$. Since all $H_r$ are in $\mathcal{N}'$, condition (iii) can be strengthened to $H_r \alpha_{\mathcal{N}'} K_r \beta_{\mathcal{N}'} H_r$ for all $r \in [q]$.

But this means that the pair $(G, H)$ is in the transitive closure of the union of the relations $\alpha_{\mathcal{N}'} K_1, \ldots, \alpha_{\mathcal{N}'} K_q$, and thus in $\alpha_{\mathcal{N}'}$. Hence $\alpha_{\mathcal{N}'} = \mathcal{N}' \times \mathcal{N}'$, i.e., the first part of the lemma.

To show the second part, define relation $\tilde{\beta} = \mathcal{N}' \times \mathcal{N}'$, which, as we just proved, is included in $\alpha_{\mathcal{N}'}$. But it also satisfies the closure property in $\mathcal{N}'$. Since $\tilde{\beta}$ is the coarsest equivalence relation on $\mathcal{N}'$, we thus have $\beta_{\mathcal{N}'} = \tilde{\beta} = \mathcal{N}' \times \mathcal{N}'$, i.e., the second part of the lemma. \hfill \qed

Definition 14 (Definition 4.5 in [12]). A network model $\mathcal{N}$ is called source-incompatible if

$$
\bigcap_{G \in \mathcal{N}} R(G) = \emptyset.
$$

The proof of Coulouma et al. [12] actually shows a stronger version of their theorem (they focus on binary consensus), stated below:

Theorem 15 (Generalization of Theorem 4.10 in [12]). Let $\mathcal{N}$ be a network model. Exact consensus is solvable in $\mathcal{N}$ if and only if each $\beta_{\mathcal{N}}$-class is not source-incompatible.

We start with showing a generalization of Lemma 7 that allows us to induce non-empty intersection of valencies.

Lemma 16. Let $C$ be a configuration of an asymptotic consensus algorithm $A$ for $\mathcal{N}$. For all configurations $C$ in an execution of $A$ in $\mathcal{N}$, and for all $G, H, K \in \mathcal{N}$, if $G \alpha_{\mathcal{N}'} K$ then $Y_{\mathcal{N}'}(G, C) \cap Y_{\mathcal{N}'}(H, C) \neq \emptyset$.

Proof. By the definition of $G \alpha_{\mathcal{N}'} K$ it is $\text{In}_{R(K)}(G) = \text{In}_{R(K)}(H)$. Hence, together with Lemma 6 it follows that $G.C \sim H.C \forall$ nodes $i$ in $R(K)$. We consider an execution $E$ in which $C$ occurs at some $t_0 - 1$, $G$ is the communication graph at $t_0$ and all following graphs are equal to $K$. Analogously, let $E'$ be an execution identical to $E$ except that the communication graph at round $t_0$ is $H$ instead of $G$. By inductive application of Lemma 6 we show that for all $t \geq t_0$, we have $C_t \sim C'_t$. In particular, we obtain $y_{\mathcal{N}'}^g(t) = y_{\mathcal{N}'}^{g'}(t)$. Thus $y_{\mathcal{N}'}^g = y_{\mathcal{N}'}^{g'}$, which shows that $Y_{\mathcal{N}'}(G, C)$ and $Y_{\mathcal{N}'}(H, C)$ intersect. \hfill \qed

We next establish that for network models in which exact consensus is not solvable, asymptotic consensus algorithms must have initial configurations that can be extended to executions with different limit outputs.

Lemma 17. Let $\mathcal{N}$ be a network model in which exact consensus is not solvable. Then for all asymptotic consensus algorithms $A$, there exists an initial configuration $C_0$ such that $Y_{\mathcal{N}'}(C_0)$ is not a singleton.

More precisely, for every $\Delta > 0$, there exists an initial configuration $C_0$ such that $\Delta(y(0)) \leq \Delta$ and $\delta_{\mathcal{N}'}(C_0) \geq \Delta/n$.

Proof. We assume without loss of generality that $d = 1$. If not, we embed the initial values in any 1-dimensional affine subspace.

Let $\mathcal{N}' \subseteq \mathcal{N}$ be any source-incompatible $\beta_{\mathcal{N}}$-class, which exists by Theorem 15. Consider the $n + 1$ initial configurations $C^{(k)}_0$ where $0 \leq k \leq n$ with initial values

$$
y_{i}^{(k)}(0) = \begin{cases} 
\Delta & \text{if } i \leq k \\
0 & \text{if } i > k.
\end{cases}
$$

For all these initial configurations, we have $\Delta(y^{(k)}(0)) \leq \Delta$. Define $a(k) = \inf Y_{\mathcal{N}'}(C^{(k)}_0)$ and $b(k) = \sup Y_{\mathcal{N}'}(C^{(k)}_0)$. By Validity, $Y_{\mathcal{N}'}(C^{(0)}_0) = \{0\}$ and $Y_{\mathcal{N}'}(C^{(n)}_0) = \{\Delta\}$, which means $a(0) = b(0) = 0$ and $a(n) = b(n) = \Delta$. There exists some $k$ with $1 \leq k \leq n$ such that $b(k-1) < b(k) = \Delta/n$ since otherwise $0 = b(0) > b(n) = \Delta = 0$. Because $\mathcal{N}'$ is source-incompatible, for every agent $k$, there exists a communication
graph $G^{(k)} \in \mathcal{N}$ such that $k \not\in S(G^{(k)})$. Since $C_0^{(k-1)} \sim C_0^{(k)}$ for all $i \in S(G^{(k)})$, choosing two executions with all communication graphs equal to $G^{(k)}$ shows that $Y_{\mathcal{N}'}(C_0^{(k-1)}) \cap Y_{\mathcal{N}'}(C_0^{(k)}) \neq \emptyset$, which implies $a(k) \leq b(k - 1)$. Combining both inequalities gives $a(k) \leq b(k) - \Delta/n$ and shows that $\delta_{\mathcal{N}'}(C_0^{(k)}) = b(k) - a(k) \geq \Delta/n$. We hence choose the initial configuration $C_0 = C_0^{(k)}$.

This shows $\delta_{\mathcal{N}'}(C_0) \geq \delta_{\mathcal{N}'}(C_0) \geq \Delta/n$ by Lemma 3 and concludes the proof.

This finally allows us to derive one of our main results of this section: a characterization of network models in which exact consensus is solvable by the topological structure of valencies of asymptotic consensus algorithms.

**Theorem 18.** Let $\mathcal{N}$ be a network model. Exact consensus is solvable in $\mathcal{N}$ if and only if there exists an asymptotic consensus algorithm $\mathcal{A}$ for $\mathcal{N}$ such that $Y_{\mathcal{N}''-\mathcal{A}}(C_0)$ is either a singleton or disconnected for all network models $\mathcal{N}'' \subseteq \mathcal{N}$ and all initial configurations $C_0$ of $\mathcal{A}$.

**Proof.** ($\Rightarrow$): Assume that exact consensus is solvable in $\mathcal{N}$, an let $\mathcal{A}'$ be an algorithm that solves exact consensus in $\mathcal{N}$. Let $\mathcal{A}$ be the algorithm derived from $\mathcal{A}'$ in that deciding is replaced by setting its output variable to the decision value of $\mathcal{A}'$ and not changing it anymore. Before the decision of algorithm $\mathcal{A}'$, algorithm $\mathcal{A}$ outputs its initial value. Then $\mathcal{A}$ is an asymptotic consensus algorithm in $\mathcal{N}$. Further, from Validity of exact consensus, for any initial configuration $C_0$, the valency $Y_{\mathcal{N}''-\mathcal{A}}(C_0)$ is a subset of the set of initial values in $C_0$. As the set of initial values of $C_0$ is finite, so is $Y_{\mathcal{N}''-\mathcal{A}}(C_0)$ and, by Lemma 3 also $Y_{\mathcal{N}''-\mathcal{A}}(C_0)$ for all $\mathcal{N}'' \subseteq \mathcal{N}$. Since any finite set is either a singleton or disconnected, the claim follows.

($\Leftarrow$): We assume without loss of generality that $d = 1$. If not, we embed the initial values in any 1-dimensional affine subspace.

We proceed by means of contradiction. Assume that exact consensus is unsolvable in $\mathcal{N}$. We will show that for all asymptotic consensus algorithms $\mathcal{A}$ for $\mathcal{N}$, there exists an initial configurations $C_0$ and a network model $\mathcal{N}'' \subseteq \mathcal{N}$ such that $Y_{\mathcal{N}''-\mathcal{A}}(C_0)$ is a nontrivial interval.

By Theorem 15 there is a source-incompatible $\beta_{\mathcal{N}''}$-class. Choose $\mathcal{N}''$ to be equal to such a class. We choose $C_0$ via Lemma 4 such that $Y_{\mathcal{N}''}(C_0)$ is not a singleton.

To show that $Y_{\mathcal{N}''}(C_0)$ is connected, we assume to the contrary that it is not and derive a contradiction. The set $Y_{\mathcal{N}''}(C_0)$ not being connected means the existence of some $z \not\in Y_{\mathcal{N}''}(C_0)$ such that

$$\exists z_1, z_2 \in Y_{\mathcal{N}''}(C_0): z_1 < z < z_2.$$ (15)

We will inductively construct an execution $E = C_0, G_1, C_1, G_2, \ldots$ such that

$$\exists z_1, z_2 \in Y_{\mathcal{N}''}(C_t): z_1 < z < z_2$$ (16)

for all $t \geq 0$. Setting $m(t) = \inf Y_{\mathcal{N}''}(C_t)$ and $M(t) = \sup Y_{\mathcal{N}''}(C_t)$, we then have $m(t) \leq z \leq M(t)$ by (16) and $M(t) - m(t) = \delta_{\mathcal{N}''}(C_t) \to 0$ by Convergence and Agreement. Hence $\lim_{t \to \infty} m(t) = \lim_{t \to \infty} M(t) = z$, which means

$$\lim_{t \to \infty} Y_{\mathcal{N}''}(C_t) = \bigcap_{t \geq 0} Y_{\mathcal{N}''}(C_t) = \{z\},$$

where the first equality follows from Lemma 4. In particular $z \in Y_{\mathcal{N}''}(C_0)$, which gives the desired contradiction.

It thus suffices to construct execution $E$ satisfying (16). Assume that (16) holds for a given $t \geq 0$ and let $z_1^{(t)}, z_2^{(t)} \in Y_{\mathcal{N}''}(C_t)$ with $z_1^{(t)} < z < z_2^{(t)}$. By Lemma 4 it follows that there are communication graphs $G, H \in \mathcal{N}$ with $z_1^{(t)} \in Y_{\mathcal{N}'}(G,C)$ and $z_2^{(t)} \in Y_{\mathcal{N}'}(H,C)$. By Lemma 3 we have $G \alpha_{\mathcal{N}'} H$. Thus there exists a chain $G = H_0, H_1, \ldots, H_q = H \in \mathcal{N}$ and communication graphs $K_1, \ldots, K_q \in \mathcal{N}''$ such that $H_{r-1} \alpha_{\mathcal{N}''} K_r H_r$ for all $r \in [q]$. From Lemma 10 we thus know that

$$Y_{\mathcal{N}'}(H_{r-1}, C) \cap Y_{\mathcal{N}'}(H_r, C) \neq \emptyset$$ (17)

for all $r \in [q]$. Set $f(r) = \inf Y_{\mathcal{N}'}(H_r, C)$ and $g(r) = \sup Y_{\mathcal{N}'}(H_r, C)$ for $r \in \{0, \ldots, q\}$, and

$$\hat{r} = \min \{ r \in \{0, \ldots, q\} \mid g(r) > z \}.$$ (18)

Then $f(0) \leq z_1^{(t)} \leq g(0)$ and $f(q) \leq z_2^{(t)} \leq g(q)$. The quantity $\hat{r}$ is finite since $g(q) \geq z_2^{(t)} > z$. We show $f(\hat{r}) < z$ by distinguishing two cases:
1. \( \hat{r} = 0 \): Then \( f(\hat{r}) = f(0) \leq z_i^{(1)} < z \).

2. \( \hat{r} \geq 1 \): Then, by (17) and the definition of \( \hat{r} \), we have \( f(\hat{r}) \leq g(\hat{r} - 1) < z \).

In both cases, we showed \( f(\hat{r}) < z < g(\hat{r}) \). Choosing \( G_{t+1} = H_k \) and \( C_{t+1} = G_{t+1,C_t} \), we hence proved (16) for \( t + 1 \). This concludes the proof.

We next introduce the \( \alpha \)-diameter of a network model \( \mathcal{N} \), which we will then (cf. Theorem 20 and Corollary 21) show to be directly linked to a nontrivial lower bound on the contraction rate in \( \mathcal{N} \) if exact consensus is not solvable in \( \mathcal{N} \). Note, that in case exact consensus is solvable in \( \mathcal{N} \), the optimal contraction rate always is 0, obtained by a reduction argument to exact consensus.

**Definition 19.** Let \( \mathcal{N} \) be a network model. The \( \alpha \)-diameter of \( \mathcal{N} \) is the smallest \( D \geq 1 \) such that for all \( G, H \in \mathcal{N} \) there exist communication graphs \( H_0, \ldots, H_q \in \mathcal{N} \) and \( K_1, \ldots, K_q \in \mathcal{N} \) with \( q \leq D \) such that \( G = H_0 \), \( H = H_q \), and \( H_r \to_{\alpha K_r} H_{r+1} \) for all \( r \in [q] \). In case it does not exists we set \( D = \infty \).

Observe, that for the network model \( \{ H_0, H_1, H_q \} \) from Theorem 9 it is \( D = 2 \). Further, for network model deaf(\( G \)), where \( G \) is an arbitrary communication graph \( G \), we have \( D = 1 \). The following theorem and corollary thus generalize Theorems 9 and 10 to arbitrary network models in which exact consensus is not solvable.

**Theorem 20.** Let \( \mathcal{N} \) be a network model in which exact consensus is not solvable. The contraction rate of any asymptotic consensus algorithm in \( \mathcal{N} \) is greater or equal to \( 1/(D+1) \) where \( D \) is the \( \alpha \)-diameter of \( \mathcal{N} \).

**Proof.** We show the stronger statement that for every initial configuration \( C_0 \) there is an execution \( E = C_0, G_1, C_1, G_2, \ldots \) starting at \( C_0 \) such that

\[
\delta_\mathcal{N}(C_t) \geq \frac{1}{(D+1)^t} \delta_\mathcal{N}(C_0)
\]

for all \( t \geq 0 \). This, applied to an initial configuration with \( \delta_\mathcal{N}(C_0) > 0 \), which exists by Lemma 14, then shows the theorem.

For the case \( D = \infty \), the above statement follows trivially. We hence suppose \( D < \infty \). The proof is by inductive construction of an execution \( E = C_0, G_1, C_1, G_2, \ldots \) whose configurations \( C_t \) satisfy (18).

For \( t = 0 \) the inequality trivially holds.

Now let \( t \) be any nonnegative integer and assume that Equation (18) holds for \( t \). By Lemma 5, there exist \( G, H \in \mathcal{N} \) such that \( \delta_\mathcal{N}(C_t) = \delta_\mathcal{N}(Y_t(C_t)) = \delta_\mathcal{N}(Y_t^\alpha(G.C_t) \cup Y_t^\alpha(H.C_t)) \). Because the \( \alpha \)-diameter of \( \mathcal{N} \) is equal to \( D < \infty \), there exist communication graphs \( H_0, \ldots, H_q \in \mathcal{N} \) and \( K_1, \ldots, K_q \in \mathcal{N} \) with \( q \leq D \) such that \( G = H_0 \), \( H = H_q \), and \( H_r \to_{\alpha K_r} H_{r+1} \) for all \( r \in [q] \).

Define \( Y_t = Y_t^\alpha(C_t) \) and \( Y_r = Y_t^\alpha(H_r,C_t) \). We have \( \delta_\mathcal{N}(Y_t) = \delta_\mathcal{N}(Y_0 \cup Y_q) \) by choice of \( G = H_0 \) and \( H = H_q \). We show that there exists some \( r \in \{0, \ldots, q\} \) such that \( \delta_\mathcal{N}(Y_r) \geq \delta_\mathcal{N}(Y_t)/(q+1) \) and then set \( G_{t+1} = H_r \) and \( C_{t+1} = H_r.C_t \). Then, by the induction hypothesis, we have

\[
\delta_\mathcal{N}(C_{t+1}) \geq \frac{\delta_\mathcal{N}(C_t)}{q+1} \geq \frac{\delta_\mathcal{N}(C_t)}{D+1} \geq \frac{1}{(D+1)^{t+1}} \delta_\mathcal{N}(C_0),
\]

i.e., Equation (18) holds for \( t+1 \).

Assume by contradiction that \( \delta_\mathcal{N}(Y_r) < \delta_\mathcal{N}(Y_t)/(q+1) \) for all \( r \in \{0, \ldots, q\} \). By Lemma 16 we have \( Y_{r-1} \cap Y_r \neq \emptyset \) for all \( r \in [q] \). Inductively, we prove

\[
\delta_\mathcal{N}(\bigcup_{s=0}^{r} Y_s) < \frac{r}{q+1} \cdot \delta_\mathcal{N}(Y_t)
\]

for all \( r \in \{0, \ldots, q\} \). In particular for \( r = q \), which leads to \( \delta_\mathcal{N}(Y_t) < \delta_\mathcal{N}(Y_0 \cup Y_q) < \delta_\mathcal{N}(Y_0) \), which is a contradiction and concludes the proof.

Direct application of Theorem 20 to a network model \( \mathcal{N} \) in which exact consensus is not solvable may yield a trivial bound of 0 in case its \( \alpha \)-diameter is \( \infty \). Indeed, we can, however, use Lemma 3 to derive a strictly positive bound for any \( \mathcal{N} \) in which exact consensus is not solvable: By Theorem 15 and Lemma 13 network model \( \mathcal{N} \) contains a source-incompatible \( \beta_\mathcal{N} \)-class, which has a finite \( \alpha \)-diameter.
Corollary 21. Let $\mathcal{N}$ be a network model in which exact consensus is not solvable. The contraction rate of any asymptotic consensus algorithm in $\mathcal{N}$ is greater or equal to $1/(D+1)$ where $D$ is the smallest $\alpha$-diameter of $\mathcal{N}' \subset \mathcal{N}$ in which exact consensus is not solvable.

Proof. Set $\mathcal{N}' \subset \mathcal{N}$ equal to the network model with the smallest $\alpha$-diameter in which exact consensus is not solvable. Applying Theorem 20 to $\mathcal{N}'$, and Lemma 3 (iv) to $\mathcal{N}'$ and $\mathcal{N}$ yields the corollary. 

7 Approximate Consensus

In this section, we extend our lower bounds on the contraction rate of asymptotic consensus to lower bounds on the decision time of approximate consensus. In particular, we show optimality of the decision times of the algorithms of Charron-Bost et al. \[8\] of $[\log_3 \frac{\Delta}{\varepsilon}]$ for $n = 2$ and $[\log_2 \frac{\Delta}{\varepsilon}]$ for $n \geq 3$.

We start with the case of two agents in Theorem 22. The proof is by reducing asymptotic consensus to approximate consensus, arriving at a contradiction with Theorem 9 for too fast approximate consensus algorithms.

Theorem 22. Let $\Delta > 0$ and $\varepsilon > 0$. In a network model of $n = 2$ agents that includes the three communication graphs $H_0$, $H_1$, and $H_2$, all approximate consensus algorithms have an execution with initial diameter $\Delta(y(0)) \leq \Delta$ and decision time greater or equal to $\log_3 \frac{\Delta}{\varepsilon}$.

Proof. Assume to the contrary that algorithm $\mathcal{A}$ solves approximate consensus in some network model $\mathcal{N} \supseteq \{H_0, H_1, H_2\}$ that decides in $T < \log_3 \frac{\Delta}{\varepsilon}$ rounds for all vectors of initial values $y(0)$ with $\Delta(y(0)) \leq \Delta$ and some $\varepsilon > 0$.

Choose any $y(0)$ with $\Delta(y(0)) = \Delta$. Define algorithm $\tilde{\mathcal{A}}$ by running algorithm $\mathcal{A}$, updating $y$ to the agents’ decision values in round $T$, and then running Algorithm 1 with the initial values $y'(T) = d_i$ from round $T + 1$ on. Because Algorithm 1 is an asymptotic consensus algorithm and the decision values $y(T)$ of $\mathcal{A}$ satisfy the Validity condition of approximate consensus, algorithm $\tilde{\mathcal{A}}$ is an asymptotic consensus algorithm.

Let $C_0$ be an initial configuration of $\tilde{\mathcal{A}}$ with initial values $y(0)$. By the proof of Theorem 9, namely (5), there is an execution $E = C_0, G_1, C_1, G_2, \ldots$ starting from $C_0$ such that

$$\delta_{\mathcal{A}}(C_T) \geq \frac{1}{3^T} \cdot \delta_{\mathcal{A}}(C_0).$$

(21)

It is $\delta_{\mathcal{A}}(C_0) = \Delta(y(0)) = \Delta$ by Lemma 5 and $\delta_{\mathcal{A}}(C_T) \leq \Delta(y(T)) \leq \varepsilon$ by Validity of Algorithm 1 and $\varepsilon$-Agreement of algorithm $\mathcal{A}$. But this means $T \geq \log_3 \frac{\Delta}{\varepsilon}$, a contradiction. 

With a similar proof, but using (10) instead of (5), we also get the lower bound for approximate consensus with $n \geq 3$ agents:

Theorem 23. Let $\Delta > 0$ and $\varepsilon > 0$. In a network model of $n \geq 3$ agents that includes the communication graphs deaf($G$), all approximate consensus algorithms have an execution with initial diameter $\Delta(y(0)) \leq \Delta$ and decision time greater or equal to $\log_3 \frac{\Delta}{\varepsilon}$.

In case the network model does not include the graphs deaf($G$), we obtain the following general bound on the termination time:

Theorem 24. Let $\Delta > 0$ and $\varepsilon > 0$. In a network model in which exact consensus is not solvable, all approximate consensus algorithms have an execution with initial diameter $\Delta(y(0)) \leq \Delta$ and decision time greater or equal to $\log_{D+1} \frac{\Delta}{\varepsilon}$, where $D$ is the $\alpha$-diameter of the network model.

Proof. Assume to the contrary that algorithm $\mathcal{A}$ solves approximate consensus in some network model $\mathcal{N}$ in which exact consensus is not solvable and that decides in $T < \log_3 \frac{\Delta}{\varepsilon}$ rounds for all vectors of initial values $y(0)$ with $\Delta(y(0)) \leq \Delta$ and some $\varepsilon > 0$.

Define algorithm $\tilde{\mathcal{A}}$ by repeatedly running algorithm $\mathcal{A}$, updating $y$ to the agents’ decision values in round $kT$, and then restarting $\mathcal{A}$ in round $kT + 1$ with the decision values from the previous phase. Then $\tilde{\mathcal{A}}$ is an asymptotic consensus algorithm.
Let \( C_0 \) be an initial configuration of \( \tilde{A} \) with \( \Delta(y(0)) \leq \Delta \) and \( \delta_X(C_0) \geq \Delta/n \) By the proof of Theorem 20 namely \((18)\), there is an execution \( E = C_0, G_1, C_1, G_2, \ldots \) starting from \( C_0 \) such that

\[
\delta_X(C_T) \geq \frac{1}{(D+1)^T} \cdot \delta_X(C_0).
\]

(22)

It is \( \delta_X(C_0) \leq \Delta(y(0)) \leq \Delta/n \) and \( \delta_X(C_T) \leq \Delta(y(T)) \leq \varepsilon \) by \( \varepsilon \)-Agreement of algorithm \( \mathcal{A} \). But this means \( T \geq \log_{D+1} \frac{\Delta}{\varepsilon n} \), a contradiction.

From Theorem 24 and the fact that \( \mathcal{N}' \subseteq \mathcal{N} \) implies \( \mathcal{E}' \subseteq \mathcal{E} \) for the corresponding sets of executions of algorithm \( \mathcal{A} \), we get:

**Corollary 25.** Let \( \Delta > 0 \) and \( \varepsilon > 0 \). In a network model in which exact consensus is not solvable, all approximate consensus algorithms have an execution with initial diameter \( \Delta(y(0)) \leq \Delta \) and decision time greater or equal to \( \log_{D+1} \frac{\Delta}{\varepsilon n} \), where \( D \) is the smallest \( \alpha \)-diameter of a network model \( \mathcal{N}' \subseteq \mathcal{N} \) in which exact consensus is not solvable.

## 8 Conclusions

In this work we introduced the notion of valency for asymptotic consensus algorithms, generalizing the concept of valency from exact consensus algorithms. Based on the study of valency diameters along executions we proved lower bounds on the contraction rates of asymptotic consensus algorithm in arbitrary network models: In particular, together with previously published averaging algorithms in \([8]\), we showed tight bounds for one and two dimensions in the network model containing all non-split graphs. Furthermore we obtained a general lower bound of \( 1/(D+1) \) for any network model in which exact consensus is not solvable; here \( D \) denotes the newly introduced \( \alpha \)-diameter of the network model. We established a connection between the topological structure of valencies and the solvability of exact consensus, and finally, extended our lower bounds to lower bounds on termination times of approximate consensus algorithms in arbitrary network models.

We believe that the study of valencies is an interesting vehicle to approach several open questions: In future work we plan to close the remaining gaps illustrated in Table 1. A further interesting extension is the addition of liveness properties. Moreover we aim to find simple exact consensus algorithms for general network models, based on an in-depth understanding of how valencies change over executions.

## Acknowledgments

We would like to thank Bernadette Charron-Bost for the many fruitful discussions and her valuable input which greatly helped improve the paper. The research was partially funded by the Austrian Science Fund (FWF) project SIC (P26436).

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