THE PENTAGON EQUATION AND THE CONFLUENCE RELATIONS

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Abstract. We show an equivalence of Drinfeld’s pentagon equation and Hirose-Sato’s confluence relations. As a corollary, we obtain a ‘pentagon-free’ presentation of the Grothendieck-Teichmüller group $\text{GRT}_1$ and associators.

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1. Introduction

This paper discusses an equivalence of two types of relations (Theorem 1):

\begin{itemize}
  \item Drinfeld’s pentagon equation ([3]) which is the main defining equation of associators and the Grothendieck-Teichmüller group.
  \item Hirose-Sato’s confluence relations ([8]) which are conjectured to exhaust all the relations among multiple zeta values (cf. §2).
\end{itemize}

As a corollary, a new formulation of associators and the Grothendieck-Teichmüller group is obtained (Corollary 2).

Let $\hat{f}_2$ be the free Lie algebra over $\mathbb{Q}$ with two variables $f_0$ and $f_1$ and $U_2 := \mathbb{Q}\langle f_0, f_1 \rangle$ be its universal enveloping algebra. We denote $\hat{f}_2$ and $\hat{U}_2 := \mathbb{Q}\langle \langle f_0, f_1 \rangle \rangle$ to be their completions by degrees. An associator ([3] [4]) is a series $\varphi = \varphi(f_0, f_1)$ in $\hat{U}_2$ with non-zero quadratic terms which satisfies the following:

\begin{itemize}
  \item the commutator group-like condition: $\varphi \in \exp[\hat{f}_2, \hat{f}_2]$.
\end{itemize}
Here exp[\hat{f}_2, \hat{f}_2] is the image of the topological commutator \([\hat{f}_2, \hat{f}_2]\) of \(\hat{f}_2\) under the exponential map and \(\mathfrak{P}_5\) is the Lie algebra generated by \(t_{ij}\) (\(i, j \in \mathbb{Z}/5\)) with the relations

- \(t_{ij} = t_{ji}, \quad t_{ii} = 0,\)
- \(\sum_{j \in \mathbb{Z}/5} t_{ij} = 0 \quad (\forall i \in \mathbb{Z}/5),\)
- \([t_{ij}, t_{kl}] = 0 \quad \text{for} \quad \{i,j\} \cap \{k,l\} = \emptyset.\)

For \(i, j, k \in \mathbb{Z}/5, \varphi_{ijk}\) means the image of \(\varphi\) under the embedding \(\widehat{U}_{f_2} \to \widehat{\mathfrak{P}}_5\) sending \(f_0 \mapsto t_{ij}\) and \(f_1 \mapsto t_{jk}\).

**Theorem 1.** Let \(\varphi\) be a commutator group-like series in \(\widehat{U}_{f_2}\), i.e. \(\varphi \in \exp[\hat{f}_2, \hat{f}_2]\). Then it satisfies the pentagon equation if and only if it satisfies the confluence relations (cf. \(\S 2\)).

As a corollary, we obtain a new formulation of the set \(M(\mathbb{Q})\) of associators and also that of the graded Grothendieck-Teichmüller group \(GRT_1(\mathbb{Q})\), the set of ‘associators without quadratic terms’ (\(\S 3\)).

**Corollary 2.** There are equalities:

\[
M(\mathbb{Q}) = \{\varphi \in \exp[\hat{f}_2, \hat{f}_2] \mid <e_0 e_1 | \varphi > \neq 0\ and \ <l | \varphi > = 0 \ for \ l \in \mathcal{I}_{CF}\},
\]

\[
GRT_1(\mathbb{Q}) = \{\varphi \in \exp[\hat{f}_2, \hat{f}_2] \mid <e_0 e_1 | \varphi > = 0\ and \ <l | \varphi > = 0 \ for \ l \in \mathcal{I}_{CF}\}.
\]

For the set \(\mathcal{I}_{CF}\) of confluence relations, consult (22) and for the pairing \(<\cdot | \cdot>\), see (23). It is remarkable that the right hand side of the second equation turns to be a group under the operation

\[
\varphi_1 \circ \varphi_2 = \varphi_1(\varphi_2(f_0, f_1)f_0\varphi_2(f_0, f_1)^{-1}, f_1)\varphi_2(f_0, f_1)
\]

because \(GRT_1(\mathbb{Q})\) forms a group (\(\S 3\)). Its associated Lie algebra \(\mathfrak{grt}_1(\mathbb{Q})\) is described as

\[
\mathfrak{grt}_1(\mathbb{Q}) = \{\psi \in [\hat{f}_2, \hat{f}_2] \mid <e_0 e_1 | \psi > = 0\ and \ <l | \psi > = 0 \ for \ l \in \mathcal{I}_{CF}\}.
\]

It might be worthy to pursue a pentagon-free presentation of the filtered version \(GT_1(\mathbb{Q})\) and the profinite version \(\widehat{GT}_1\) of the Grothendieck-Teichmüller group which were both introduced in (3).

In (8) they showed that the confluence relations imply the regularized (generalized) double shuffle relations (cf. Theorem (3)). By combining it with Theorem (1) we recover the result in (5) that the associator relation implies the generalized double shuffle relations. The author expects that our new formulation would give us a further understanding on the implication.

Here is the plan of the proof of Theorem (1). Under the commutator group-like condition, we show that the pentagon equation implies the confluence relations (Theorem (1)) and vice versa (Theorem (18)).

2. Confluence relations

We recall the definition of the confluence relations by employing their original symbols in (8).
Let \( z \neq 0, 1 \in \mathbb{C} \). Put \( \mathcal{A}_z = \mathbb{Q}(e_0, e_z, e_1) \). We consider the following sequence of linear subspaces \( \mathcal{A}_z \supset \mathcal{A}_z^1 \supset \mathcal{A}_z^0 \supset \mathcal{A}_z^{-1} \supset \mathcal{A}_z^{-2} \), where

\[
\begin{align*}
\mathcal{A}_z^1 &= \mathbb{Q} \oplus \mathcal{A}_z e_1 \oplus \mathcal{A}_z e_z, \\
\mathcal{A}_z^0 &= \mathbb{Q} \oplus \mathbb{Q} e_z \oplus \mathbb{Q} e_0 \oplus \mathcal{A}_z e_0 e_b, \\
\mathcal{A}_z^{-1} &= \mathbb{Q}(e_z) \cdot \mathcal{A}_z^{-2}, \\
\mathcal{A}_z^{-2} &= \mathbb{Q} \oplus \mathbb{Q} e_0 \mathcal{A}_z e_1 \oplus \mathbb{Q} e_0 \mathcal{A}_z e_z.
\end{align*}
\]

(For our purpose, we reverse the orders in their definitions of the subspaces in \([8]\), that is, we read them backwards.) For \( \mathcal{A} = \mathbb{Q}(e_0, e_1) \), set \( \mathcal{A}^t := \mathcal{A} \cap \mathcal{A}_z^t \). We have \( \mathcal{A}^{-2} = \mathcal{A} = \mathcal{A}^0 \). All the spaces form algebras under the shuffle product \( \otimes \).

They considered the algebra homomorphism \( \text{Const} : \mathcal{A}_z \to \mathcal{A} \) defined by \( \text{Const}(e_0, e_z) = e_0 e_z \) if \( e_0 \neq e_z \) and \( 0 \) otherwise. We have \( \lim_{z \to 0} L(l) = L(\text{Const}(l)) \).

The image is a holomorphic function on \( z \) (Its relation with multiple polylogarithm will be discussed in Remark \([8]\)]. Particularly the image of \( \mathcal{A}^0 \) is given by a multiple zeta value (MZV in short):

\[
L(e_{n_1} \cdots e_{n_{m-1}} e_1) := \int_{0 < t_1 < \cdots < t_n < 1} \frac{dt_n}{t_n - a_n} \cdots \frac{dt_1}{t_1 - a_1}
\]

and \( L(1) = 1 \). It yields a linear map \( L : \mathcal{A}_z^0 \to \text{Hol}_z(\mathbb{C} \setminus [0, 1]) \) to the space of holomorphic functions on \( \mathbb{C} \setminus [0, 1] \).

The shuffle algebra homomorphism \( \text{Const} : \mathcal{A}_z \to \mathcal{A} \) is defined by \( \text{Const}(e_{a_1} \cdots e_{a_n}) = e_{a_1} \cdots e_{a_n} \) if \( a_i \neq z \) for all \( i \), and \( 0 \) otherwise. We have \( \lim_{z \to \infty} L(l) = L(\text{Const}(l)) \).

The linear operator \( \partial_{z, \alpha} : \mathcal{A}_z \to \mathcal{A}_z \) (\( \alpha = 0, 1 \)) is defined by

\[
\partial_{z, \alpha}(e_{a_1} \cdots e_{a_n}) = \sum_{i=1}^n \left( \delta_{\{a_i, a_{i+1}\}, \{z, \alpha\}} - \delta_{\{a_{i-1}, a_i\}, \{z, \alpha\}} \right) e_{a_1} \cdots \hat{e}_{a_i} \cdots e_{a_n},
\]

and \( \partial_{z, 0}(1) = 0 \) with \( a_0 = 0 \) and \( a_{n+1} = 1 \). Here \( \delta_{\{a_1, a_2\}, \{\beta_1, \beta_2\}} \) is the Kronecker delta function, i.e. 1 if \( \{a_1, a_2\} = \{\beta_1, \beta_2\} \) as sets and 0 otherwise. It is shown in \([7]\) that

\[
\frac{d}{dz} L(l) = \frac{1}{z} L(\partial_{z, 0}(l)) + \frac{1}{z - 1} L(\partial_{z, 1}(l)).
\]

The set of standard relations is defined to be the subspace of \( \mathcal{A}_z^0 \),

\[
\mathcal{I}_{ST} := \{ l \in \mathcal{A}_z^0 \mid \text{Const}(\partial_{z, \alpha_1} \cdots \partial_{z, \alpha_r} l) = 0 \text{ for } r \geq 0, \alpha_1, \ldots, \alpha_r \in \{0, 1\} \},
\]

which actually forms an ideal of \( (\mathcal{A}_z^0, \otimes) \). An element of \( \mathcal{I}_{ST} \) is called a standard relation. They showed that \( L(\mathcal{I}_{ST}) = 0 \).

They considered the algebra homomorphism \( N : (\mathcal{A}_z^0, \otimes) \to (\mathcal{A}_z^{-1}, \otimes) \) which is defined by the composition of the following algebra homomorphisms with respect to \( \otimes \)

\[
N : \mathcal{A}_z^{-2} \otimes (\mathcal{A}_z^0 \cap \mathbb{Q}(e_1, e_z)) \xrightarrow{id \otimes \tau_z} \mathcal{A}_z^{-2} \otimes (\mathcal{A}_z^0 \cap \mathbb{Q}(e_0, e_z)) \to \mathcal{A}_z^{-1}.
\]

Here \( \text{reg}_{z, 1} \) in the first map is caused by the isomorphism \( \mathcal{A}_z^{-2} \otimes (\mathcal{A}_z^0 \cap \mathbb{Q}(e_1, e_z)) \approx \mathcal{A}_z^0; u \otimes v \mapsto u \cup v \), \( \tau_z \) appearing in the second map is the anti-automorphism...
$\tau_z : (A_z, \cdot) \to (A_z, \cdot)$ such that
\[\begin{align*}
\tau_z & : (A_z, \cdot) \to (A_z, \cdot) \\
e_0 & \mapsto e_z - e_1, \\
e_1 & \mapsto e_z - e_0, \text{ and the third map is the surjection} \\
e_z & \mapsto e_z,
\end{align*}\]
simply induced by the shuffle algebra homomorphism $A_z \otimes A_z \to A_z$; $u \otimes v \mapsto u \cup_1 v$.

Then they introduced the algebra homomorphism $\lambda : (A^0, \cup) \to (A^0, \cup)$ which is defined by the composition of the following algebra homomorphisms
\[
\lambda : A^0 \overset{N}{\to} A^{-1} \overset{\text{reg}}{\to} Q\langle e_z \rangle \otimes A^{-2} \overset{\text{const} \otimes \text{id}}{\to} A^{-2} \overset{\tau_1 \mapsto 1}{\to} A^0.
\]

Here $\text{reg}_z$ is caused by the isomorphism $Q\langle e_z \rangle \otimes A^{-2}_z \simeq A^{-1}_z$; $u \otimes v \mapsto u \cup_1 v$, $\text{const} = \text{Const}_{Q\langle e_z \rangle}$ and $\tau_{z \mapsto 1} : A^{-2}_z \to A^0$; $l \mapsto \tau_{z \mapsto 1}$ is the algebra homomorphism sending $e_z \mapsto e_1$ and $e_i \mapsto e_i$ for $i = 0, 1$.

The set of confluence relations is defined to be the image of $\mathcal{I}_{ST}$ in $A^0$ under the map $\lambda$,
\[
\mathcal{I}_{CF} := \lambda(\mathcal{I}_{ST}).
\]

An element of $\mathcal{I}_{CF}$ is called a confluence relation.

**Theorem 3 ([8]).** (i). We have $L(\mathcal{I}_{CF}) = 0$, that is, the confluence relations give linear relations among MZV’s.

(ii). The confluence relations imply the regularized (generalized) double shuffle relations and also the duality relation, namely we have $\mathcal{I}_{CF} \supset \mathcal{I}_{RDS}$ and $\mathcal{I}_{CF} \supset \mathcal{I}_{\Delta}$.

Here $\mathcal{I}_{RDS}$ (resp. $\mathcal{I}_{\Delta}$) is the ideal of $(A^0, \cup)$ generated by \{reg$_{\cup}(u \cup_1 v - u * v) \mid u \in A^1, v \in A^0\}$ (resp. \{w - $\tau_{\infty}(w) \mid w \in A^0\}$ where $\tau_{\infty}$ is the anti-automorphism of $A$ sending $\tau_{\infty}(e_0) = -e_1$, $\tau_{\infty}(e_1) = -e_0$) (for definitions of reg$_{\cup}$ and $*$, consult [8]).

In [8] Conjecture 24, it is conjectured that the confluence relations exhaust all the relations among MZV’s, i.e. $\mathcal{I}_{CF} = \ker\{L : A^0 \to \mathbb{R}\}$.

3. The Pentagon Equation Implies the Confluence Relations

We prove that the pentagon equation implies the confluence relation under the commutator group-like condition (Theorem 4).

We regard $Uf_2 = Q\langle f_0, f_1 \rangle$ as the dual of $A = Q\langle e_0, e_1 \rangle$ by the pairing
\[
< \cdot, \cdot > : A \otimes Uf_2 \to Q
\]
such that $< W_1(e_0, e_1) \mid W_2(f_0, f_1) > = \delta_{W_1, W_2}$ for any words $W_1, W_2$. Here a word means a monic monomial element of the free monoid generated by two elements, say $A$ and $B$. For a word $W$, we denote $W(e_0, e_1)$ (resp. $W(f_0, f_1)$) to be the corresponding element in $A$ (resp. $Uf_2$) obtained by the substitution of $e_0, e_1$ (resp. $f_0, f_1$) to $A, B$.

**Theorem 4.** Let $\varphi \in \exp[\hat{f}_2, \hat{f}_2]$. Assume that $\varphi$ satisfies the pentagon equation. Then for any $l \in \mathcal{I}_{CF}$ we have $< l \mid \varphi > = 0$. In other word, for any $l \in \mathcal{I}_{ST}$ we have $< \lambda(l) \mid \varphi > = 0$.

3.1. Bar Construction and Oi-Ueno Decomposition. We prepare basic tools of bar-constructions of varieties, $\mathcal{M}_{0,5}$, $\mathcal{M}_{0,4}$, $\mathcal{X}(z)$, $\mathcal{Y}(w)$ and recall two decompositions of the reduced bar algebra $B$ of $\mathcal{M}_{0,5}$ by Oi-Ueno [9].

We consider three varieties. Let $\mathcal{M}_{0,4} := \{(p_1, \ldots, p_4) \in \mathbb{P}^1(Q)^4 \mid p_i \neq p_j\}/\text{PGL}(2)$ and $\mathcal{M}_{0,5} := \{(p_1, \ldots, p_5) \in \mathbb{P}^1(Q)^5 \mid p_i \neq p_j\}/\text{PGL}(2)$ be the moduli spaces of...
the projective line \( \mathbb{P}^1(\mathbb{Q}) \) with 4 and 5 marked points respectively. For \( i = 1, \ldots, 5 \), we have the projections \( \pi_i : \mathcal{M}_{0,5} \to \mathcal{M}_{0,4} \) induced by the omission of the \( i \)-th parametrized point respectively, which are encoded with the action of the symmetric groups \( \mathfrak{S}_4 \) and \( \mathfrak{S}_5 \). By taking the normalized coordinate \( (0, x, 1, \infty) \), we identify \( \mathcal{M}_{0,4} \) with the space \( \{ x \in \mathbb{Q} \mid x \neq 0, 1 \} \) and by taking the normalized coordinate \( (0, w, z, 1, \infty) \), we identify \( \mathcal{M}_{0,5} \) with \( \{(w, z) \in \mathbb{Q}^2 \mid z, w \neq 0, 1, z \neq w\} \). Put \( \mathcal{X}(z) := \{ w \in \mathbb{Q} \mid w \neq 0, 1, z \} \) for a fixed \( z \). We often regard \( \mathcal{X}(z) \) be a subspace of \( \mathcal{M}_{0,5} \) under \( jX : w \mapsto (w, z) \). Similarly we consider \( \mathcal{Y}(w) := \{ z \in \mathbb{Q} \mid z \neq 0, 1, w \} \) for a fixed \( w \) and regard it as a subspace of \( \mathcal{M}_{0,5} \) under \( jy : z \mapsto (\frac{1}{z}, 1) \).

**Remark 5.** There is a geometric picture which might help intuitively our later arguments: Our \( \mathcal{M}_{0,5} \) is regarded to be \( \overline{\mathcal{M}_{0,5}} \setminus \cup_{i \neq j} L_{ij} \). Here \( \overline{\mathcal{M}_{0,5}} \) is the stable compactification of \( \mathcal{M}_{0,5} \), which is a blowing-up of \( \mathbb{P}^1 \times \mathbb{P}^1 \) at \( (z, w) = (0, 0), (1, 1), (\infty, \infty) \). Its real structure is depicted in Figure 3.1. Here \( L_{ij} \) means the boundary component given by \( p_i = p_j \), which is isomorphic to \( \mathbb{P}^1(\mathbb{Q}) \). Particularly \( L_{14}, L_{15}, L_{14} \) are the exceptional divisors. They are depicted as if they were hexagons but they represent \( \mathbb{P}^1(\mathbb{Q}) \) which identify interfacing vertices and edges. We note that our parameter \( t_{ij} \in U_{P5} \) corresponds to the local monodromy around \( L_{ij} \). For some 5-tuples \( \{i, j, k, l, m\} = \{1, 2, 3, 4, 5\} \), the symbol \( \varphi_{ijklm} \) is depicted aside the oriented edge on \( L_{ij} \) connecting the two vertices \( v_{ijkl} = L_{ij} \cap L_{kl} \) and \( v_{ijlm} = L_{ij} \cap L_{lm} \). It represents the image of \( \varphi \in U_{P2} \) under the embedding \( U_{P2} \to U_{P5} : f_0 \mapsto t_{kl} \) and \( f_1 \mapsto t_{im} \), which is caused by the identification \( \mathbb{P}^1(\mathbb{Q}) \) with \( L_{ij} \) such that \( 0 \mapsto v_{ijkl} \) and \( 1 \mapsto v_{ijlm} \) and the inclusion \( L_{ij} \hookrightarrow \overline{\mathcal{M}_{0,5}} \).

For \( \mathcal{M} = \mathcal{M}_{0,5}, \mathcal{M}_{0,4}, \mathcal{X}(z) \) and \( \mathcal{Y}(w) \) we consider its Chen’s \([1]\) reduced bar algebra \( H^0 \bar{B}(\Omega^*_{DR}(\mathcal{M})) \). It is calculated to be the graded Hopf algebra \( B(\mathcal{M}) = \oplus_{m=0}^{\infty} B(\mathcal{M})_m (\subset \mathcal{T}B(\mathcal{M})) = \oplus_{m=0}^{\infty} B(\mathcal{M})_{\leq m} \) over \( \mathbb{Q} \), where \( B(\mathcal{M})_0 = \mathbb{Q}, B(\mathcal{M})_1 = H^1_{DR}(\mathcal{M}) \) and \( B(\mathcal{M})_m \) is the totality of linear combinations (finite sums)

\[
\sum_{I=(i_m, \ldots, i_1)} c_I [\omega_{i_m}| \cdots |\omega_{i_1}] \in B(\mathcal{M})_{\leq m}^m
\]

\((c_I \in \mathbb{Q}, \omega_{i_j} \in B(\mathcal{M})_1, [\omega_{i_m}| \cdots |\omega_{i_1}] := \omega_{i_m} \otimes \cdots \otimes \omega_{i_1})\) satisfying the integrability condition

\[
\sum_{I=(i_m, \ldots, i_1)} c_I [\omega_{i_m}| \cdots |\omega_{i_{j+1}} \wedge \omega_{i_j}| \cdots |\omega_{i_1}] = 0
\]

in \( B(\mathcal{M})_{\leq m}^m \otimes H^2_{DR}(\mathcal{M}) \otimes B(\mathcal{M})_{\leq j-1} \) for all \( j \) \((1 \leq j < m)\). Its product is given by the shuffle product \( \shuffle \) and its coproduct \( \delta \) is given by deconcatenation.

**Lemma 6.** We have the following isomorphisms of Hopf algebras:

\[
H^0 \bar{B}(\Omega^*_{DR}(\mathcal{M}_{0,4})) \simeq A, \quad H^0 \bar{B}(\Omega^*_{DR}(\mathcal{X}(z))) \simeq A_z, \quad H^0 \bar{B}(\Omega^*_{DR}(\mathcal{Y}(w))) \simeq A_w,
\]

\[
H^0 \bar{B}(\Omega^*_{DR}(\mathcal{M}_{0,5})) \simeq U_{P5}^*.
\]

Here \( U_{P5}^* \) is the graded linear dual of \( U_{P5} \).

The first isomorphism is given by the correspondence \( d\log x \mapsto e_0, d\log(x-1) \mapsto e_1 \). The second one is given by the correspondence \( d\log w \mapsto e_0, d\log(w-1) \mapsto e_2 \), \( d\log(w-2) \mapsto e_1 \). The third one is given by \( d\log z \mapsto e_0, d\log(z-w) \mapsto e_w, d\log(z-1) \mapsto e_1 \). Though the last one is explained in \([5]\), we further investigate it below.
Put $B := H^0\tilde{\mathcal{B}}(\Omega^1_{\text{DR}}(\mathcal{M}_{0,5}))$ and fix the $\mathbb{Q}$-linear basis of $B(\mathcal{M}_{0,5})_1$ as
\[ e_{21} = d \log w, \quad e_{23} = d \log (w - z), \quad e_{24} = d \log (w - 1), \quad e_{31} = d \log z, \quad e_{34} = d \log (z - 1). \]

We note that $t_{21}, t_{23}, t_{24}, t_{31}, t_{34}$ forms a basis of the degree 1 part of $\hat{U}\mathfrak{P}_5$ and $e_{21}, e_{23}, e_{24}, e_{31}, e_{34}$ gives its dual basis.

Following the setting in \[9\], we employ the symbols below:
\[ \eta_2 = d \log w = e_{21}, \quad \eta_{22} = d \log (1 - w) = e_{24}, \]
\[ \eta_3 = d \log \frac{1}{z} = -e_{31}, \quad \eta_{33} = d \log (1 - \frac{1}{z}) = -e_{31} + e_{34}, \]
\[ \eta_{23} = d \log (1 - \frac{w}{z}) = -e_{31} + e_{23}, \quad \text{and} \quad \eta_{23}^{(2)} = \frac{-dw}{z - w}, \quad \eta_{23}^{(3)} = \frac{dz}{z - w} - \frac{dz}{z}. \]

We introduce the non-commutative polynomial algebras $A_z(\eta_2, \eta_{23}^{(2)}, \eta_{22}, \eta_{33}) := \mathbb{Q}\langle \eta_2, \eta_{23}^{(2)}, \eta_{22}, \eta_{33} \rangle$ (resp. $A(\eta_3, \eta_{33}) := \mathbb{Q}\langle \eta_3, \eta_{33} \rangle$). By the map $e_0 \mapsto \eta_2, \quad e_z \mapsto \eta_{23}^{(2)}, \quad e_1 \mapsto \eta_{22}$ (resp. $e_0 \mapsto \eta_3, \quad e_1 \mapsto \eta_{33}$), it is identified with $A_z$ (resp. $A$). We set $A^1_z(\eta_2, \eta_{23}^{(2)}, \eta_{22})$, $A^0_z(\eta_2, \eta_{23}^{(2)}, \eta_{22})$ (resp. $A^1(\eta_3, \eta_{33})$) to be the subspaces corresponding to $A^1_z$ and $A^0_z$ (resp. $A^1$) under the identification. Similarly we define corresponding subspaces.
for $A_w(\eta_3, \eta_2^{(3)}, \eta_{33}) := \mathbb{Q}(\eta_3, \eta_2^{(3)}, \eta_{33})$ (resp. $A(\eta_2, \eta_{22}) := \mathbb{Q}(\eta_2, \eta_{22})$) by the identification with $A_z$ (resp. $A$) given by $e_0 \mapsto \eta_3$, $e_z \mapsto e_z^{(3)}$, $e_1 \mapsto \eta_{33}$ (resp. $e_0 \mapsto \eta_2$, $e_1 \mapsto \eta_{22}$).

**Lemma 7.** There are the following isomorphisms of shuffle $\mathbb{Q}$-algebras

$$
\text{dec}_2 := \left(\text{pr}^{(2)}_{\otimes 3} \otimes \text{pr}^{(3)}_{\otimes 2}\right) \circ \delta : B \simeq A_z(\eta_2, \eta_2^{(2)}, \eta_{22}) \otimes \mathbb{Q} A(\eta_3, \eta_{33}),
$$

$$
\text{dec}_3 := \left(\text{pr}^{(3)}_{\otimes 2} \otimes \text{pr}^{(2)}_{\otimes 3}\right) \circ \delta : B \simeq A_w(\eta_3, \eta_2^{(3)}, \eta_{33}) \otimes \mathbb{Q} A(\eta_2, \eta_{22}).
$$

Here $\text{pr}_{\otimes 3}^{(2)}$ is the algebra homomorphism sending $\eta_3, \eta_{23} \mapsto 0$, $\eta_{23} \mapsto \eta_2^{(2)}$ and $\text{pr}_{\otimes 2}^{(3)}$ is the algebra homomorphism sending $\eta_2, \eta_{23}, \eta_{22} \mapsto 0$. Similarly the algebra homomorphisms $\text{pr}_{\otimes 2}^{(3)}$ (resp. $\text{pr}_{\otimes 3}^{(2)}$) is defined by $\eta_2, \eta_{22} \mapsto 0$, $\eta_{23} \mapsto \eta_3^{(3)}$ (resp. $\eta_3, \eta_{23}, \eta_{33} \mapsto 0$).

**Proof.** This is nothing but a reformulation of [9] Proposition 9.3, where they employ the different coordinate $(z_1, z_2)$ corresponding to our coordinate $(w, \frac{1}{z})$. Actually their terminologies are translated to ours as follows:

$$
\xi_1 = \eta_2, \xi_{11} = -\eta_{22}, \xi_2 = \eta_3, \xi_{22} = -\eta_{33}, \xi_{12} = -\eta_{23}
$$

and $t_{1,2} = \text{dec}_2, t_{2,1} = \text{dec}_3$.

We associate the above two decompositions with the following inclusions and projections of shuffle algebras

$$
i_2 : A \hookrightarrow B, \quad j_2 : A_z \hookrightarrow B,
$$

$$
i_3 : A \hookrightarrow B, \quad j_3 : A_w \hookrightarrow B,
$$

$$
r_2 : B \twoheadrightarrow A_z(\eta_2, \eta_2^{(2)}, \eta_{22}) \otimes \mathbb{Q} A(\eta_3, \eta_{33}),
$$

$$
r_3 : B \twoheadrightarrow A_w(\eta_3, \eta_2^{(3)}, \eta_{33}) \otimes \mathbb{Q} A(\eta_2, \eta_{22}).
$$

In precise, they are defined by

$$
i_2 : A \simeq A(\eta_3, \eta_{33}) \overset{1 \otimes \text{id}}{\hookrightarrow} A_z(\eta_2, \eta_2^{(2)}, \eta_{22}) \otimes \mathbb{Q} A(\eta_3, \eta_{33}) \overset{\text{dec}_2^{-1}}{\twoheadrightarrow} B,
$$

$$
j_2 : A_z \simeq A_z(\eta_2, \eta_2^{(2)}, \eta_{22}) \overset{\text{id} \otimes 1}{\hookrightarrow} A_z(\eta_2, \eta_2^{(2)}, \eta_{22}) \otimes \mathbb{Q} A(\eta_3, \eta_{33}) \overset{\text{dec}_2^{-1}}{\twoheadrightarrow} B,
$$

$$
r_2 : B \overset{\text{dec}_2}{\twoheadrightarrow} A_z(\eta_2, \eta_2^{(2)}, \eta_{22}) \otimes \mathbb{Q} A(\eta_3, \eta_{33}) \overset{\text{id} \otimes \epsilon_A}{\hookrightarrow} A_z(\eta_2, \eta_2^{(2)}, \eta_{22}) \simeq A_z,
$$

where $\epsilon_A$ is the augmentation map. The maps $i_3, j_3, r_3$ are defined similarly. We have $r_2 \circ j_2 = \text{id}$ and $r_3 \circ j_3 = \text{id}$. We note that $i_2$ (resp. $i_3$) is the Hopf algebra homomorphism induced by $\text{pr}_2 \circ (3,4)$ (resp. $\text{pr}_3$) and $r_2$ (resp. $r_3$) is the Hopf algebra homomorphism induced by the embedding $j_X : \mathcal{X}(z) \hookrightarrow \mathcal{M}(\mathcal{M}(0,5))$. By our construction, any element $b \in B$ is decomposed as $b = \sum_k i_2(a_k) \cup j_2(a'_k)$ with $a_k \in A$ and $a'_k \in A_z$.

We denote $U_{f_z} = \mathbb{Q}\langle f_0, f_z, f_1 \rangle$ to be the universal enveloping algebra of the free Lie algebra $f_z$ of three variables $f_0, f_z, f_1$ and regard it as the linear dual of $A_z$ by the pairing $\langle \cdot, \cdot \rangle$. Since $r_2$ is induced by the embedding $\mathcal{X}(z) \hookrightarrow \mathcal{M}(\mathcal{M}(0,5))$, we have

$$\langle r_2(l'), \varphi' \rangle = \langle l' \mid r_2(\varphi') \rangle
$$

for any $l' \in B$ and $\varphi' \in U_{f_z}$. Here $r_2$ on the right hand side means the induced map $U_{f_z} \hookrightarrow U_{f_z}$ sending $f_0 \mapsto t_{21}$, $f_z \mapsto t_{23}$, $f_1 \mapsto t_{24}$. Similarly we have

$$\langle r_3(l'), \varphi' \rangle = \langle l' \mid r_3(\varphi') \rangle
$$

for any $l' \in B$ and $\varphi' \in U_{f_z}$. Here $r_3$ on the right hand side means the induced map $U_{f_z} \hookrightarrow U_{f_z}$ sending $f_0 \mapsto -t_{31}$, $f_w \mapsto -t_{31} + t_{33}$, $f_1 \mapsto -t_{31} + t_{34}$.  

For $k = 1, \ldots, 5$, we denote $\text{pr}_k : \hat{\mathcal{Q}}_5 \to \hat{U}_2$ to be the projection induced from $\text{pr}_k : \mathcal{M}_{0,5} \to \mathcal{M}_{0,4}$, which actually sends $s_{ij}$ to 0 when $i = k$ or $j = k$. Since $i_3$ and $i_4$ are induced from $i_3 \circ (3,4)$ and $i_4$ respectively, we have

\[(3.4) \quad \langle i_4(l) | \varphi' \rangle = \langle l | \text{pr}_2 \circ (3,4)(\varphi') \rangle, \quad \langle i_3(l) | \varphi' \rangle = \langle l | \text{pr}_3(\varphi') \rangle \]

for any $l \in \mathcal{A}$ and $\varphi' \in \hat{\mathcal{Q}}_5$. For our further use, we also consider the inclusion $i_4 : \mathcal{A} \to \mathcal{B}$ of Hopf algebras sending $e_0 \mapsto e_21 - e_31$ and $e_1 \mapsto e_23 - e_31$, which is induced from the projection $i_4 : \mathcal{M}_{0,5} \to \mathcal{M}_{0,4}$. It induces an identification of pairings

\[(3.5) \quad \langle i_4(l) | \varphi' \rangle = \langle l | \text{pr}_4(\varphi') \rangle \]

We put $\mathcal{B}^1$ (N.B. it was denoted by $\mathcal{B}^0$ in \cite{9}) to be the subalgebra of $\mathcal{B}$ generated by elements which have no terms ending with $\eta_2$ and $\eta_3$.

**Lemma 8.** We have the following decompositions:

\[(3.6) \quad \text{dec}_2(\mathcal{B}^1) = A^1_2(\eta_2, \eta_{23}^{(2)}, \eta_{22}), \quad \text{dec}_3(\mathcal{B}^1) = A^1_3(\eta_3, \eta_{23}^{(3)}, \eta_{33}).\]

**Proof.** It is stated in \cite{9} Proposition 9.4. We get the claim by restriction of the isomorphisms of Lemma \cite{7} to $\mathcal{B}^1$. \qed

It follows

\[(3.8) \quad \mathcal{B}^1 \cap j_2(A_z) = j_2(A^1_z) \supset j_2(A^0_z)\]

and also $\mathcal{B}^1 \cap j_3(A_w) = j_3(A^1_w) \supset j_3(A^0_w)$.

### 3.2. Differentials and multiple polylogarithms

We introduce the linear operator $\partial_\alpha$ ($\alpha = 0, 1$) on $j_3(A_w)$ which extends to the $i_3(A)$-linear operator $\partial_\alpha$ on $\mathcal{B}$ and show that it restricts to Hirose-Sato’s differential operator $\partial_{z, \alpha}$ on $j_2(A_z)$ by showing how they are connected to multiple polylogarithms.

Let $\mathcal{P}((z, w))$ to the set of piece-wise smooth paths from $(\infty, 0)$ to $(z, w)$ on $\mathcal{M}_{0,5}$. We consider the $\mathbb{Q}$-linear map

$\bar{\rho}_5 : T\mathcal{B}(\mathcal{M}_{0,5}) \to \text{Map}(\mathcal{P}((z, w)), \mathbb{C})$.

defined by regularized iterated integral with tangential basepoints (for its treatment see also \cite{2}§15). Particularly it is given by

$\bar{\rho}_5([\omega_{i_1} \cdots \omega_{i_1}]) \gamma = \int_{0 < t_1 < \cdots < t_m < 1} \omega_{i_m}(\gamma(t_m)) \cdot \omega_{i_{m-1}}(\gamma(t_{m-1})) \cdots \omega_{i_1}(\gamma(t_1))$

when the integral is convergent. We define $I_{(z, w)}$ to be the homotopy invariant part of $\text{Im} \bar{\rho}_5$. We often regard it to be a subspace of $\text{Map}(\pi_1(\mathcal{M}_{0,5}; (\infty, 0), (z, w)), \mathbb{C})$.

By Chen’s theory \cite{11}, we have a shuffle (actually Hopf) algebra isomorphism

$\rho_5 : H^0\bar{\mathcal{B}}(\Omega_{dr}^*(\mathcal{M}_{0,5})) \simeq I_{(z, w)}(\mathcal{M}_{0,5})$.

with $\rho_5 = \bar{\rho}_5|_{H^0\bar{\mathcal{B}}(\Omega_{dr}^*(\mathcal{M}_{0,5}))}$. Similarly we have isomorphisms

$\rho_z : H^0\bar{\mathcal{B}}(\Omega_{dr}^*(\mathcal{X}(z))) \simeq I^z_0(\mathcal{X}(z))$,

$\rho_w : H^0\bar{\mathcal{B}}(\Omega_{dr}^*(\mathcal{Y}(w))) \simeq I^w_\infty(\mathcal{Y}(w))$. 


By the isomorphisms in Lemma 3 we identify $\rho_5$, $\rho_z$, $\rho_w$ with the isomorphisms $B \simeq I_{(\infty,0)}^z(M_{0,5})$, $A_z \simeq I_w^z(\mathcal{X}(z))$, $A_w \simeq I_{(\infty,0)}^w(\mathcal{Y}(w))$, and use the same symbols.

**Remark 9.** (i). We note that for each $l \in \mathcal{A}_0^z$

$$\rho_z(l)(\text{dch}_{0,1}) = L(l)$$

where $\text{dch}_{0,1}$ is the straight line path from 0 to 1 when $w = 1$ and $L(l)$ is in (27).

(ii). For a multi-index $k = (k_1, \ldots, k_m) \in \mathbb{N}^m$, set $e_k(1) = e_0^{k_1-1}e_1 \cdots e_0^{k_m-1}e_1$ and $e_k(z) = e_0^{k_1-1}e_z \cdots e_0^{k_m-1}e_z$. Then the set

$$\{e_{k_1}(1)e_{k_2}(z)e_{k_3}(1) \cdots e_{k_N}(*) \mid N > 0, m_1, \ldots, m_N > 0, k_i \in \mathbb{N}^m\}$$

(* is 1 or $z$ according to the parity of $N$) forms a basis of $\mathcal{A}_1^z$. For such $l = e_{k_1}(1)e_{k_2}(z)e_{k_3}(1) \cdots e_{k_N}(*)$, its image under $\rho_z$ is calculated to be

$$\rho_z(l) = (-1)^{\sum_{i=1}^N m_i} \ell_{k_1, k_2, \ldots, k_N}.$$ 

Here $\ell_{k_1, k_2, \ldots, k_N}$ is the element in $I_w^z(\mathcal{X}(z))$ which is associated with the series

$$\text{Li}_{k_1, k_2, \ldots, k_N}(w, 1^{m_1-1}, z^{1-1}, 1^{m_2-1}, z, 1^{m_3-1}, z^{1-1}, \ldots, 1^{m_N-1})$$

when $m_1 = 0$, it stands for $\text{Li}_{k_2, \ldots, k_N}(wz^{-1}, 1^{m_2-1}, z, 1^{m_3-1}, z^{1-1}, \ldots, 1^{m_N-1})$, that is, the restriction of the *multiple polylogarithm*

$$\text{Li}_{k_1, \ldots, k_m}(s_1, \ldots, s_m) := \sum_{n_1 > \cdots > n_m > 0} \frac{s_1^{n_1} \cdots s_m^{n_m}}{n_1! \cdots n_m!}$$

to $(s_1, \ldots, s_m) = (w, 1^{m_1-1}, z^{1-1}, 1^{m_2-1}, z, 1^{m_3-1}, z^{1-1}, \ldots, 1^{m_N-1})$ where $(k_1, \ldots, k_m)$ is the juxtaposition of $k_1, k_2, \ldots, k_N$. Strictly speaking (3.9) is merely a series converging when $|w| < |z|$, however it determines an element of $I_w^z(\mathcal{X}(z))$ by its iterated integral presentation even if $|w| \geq |z|$.

(iii). By abuse of notation, we denote $j_2 : I_w^z(\mathcal{X}(z)) \hookrightarrow I_{(\infty,0)}^z(M_{0,5})$ the embedding induced from $j_2$ under the two isomorphisms $\rho_z$ and $\rho_5$:

$$\begin{array}{ccc}
     & B & I_{(\infty,0)}^z(M_{0,5}) \\
\rho_5 & & \downarrow j_2 \\
B & \uparrow j_2 & I_w^z(\mathcal{X}(z)) \\
\rho_z & & \end{array}$$

By our construction, the image $j_2(\ell_{k_1, k_2, \ldots, k_N})$ is nothing but the element $\rho_5^{w, z^{-1}}_{k_1, k_2, \ldots, k_N}$ in $I_{(\infty,0)}^z(M_{0,5})$ which is associated with the series regarding both $w$ and $z$ as variables in (3.9). Recursive differentiations of (3.9) with respect to $w$ and $z$ reveal the expression of $\rho_5^{-1}(j_2(\ell_{k_1, k_2, \ldots, k_N}))$ in $B$. The differentiations assure

$$\text{decr}(\rho_5^{-1}(\ell_{k_1, k_2, \ldots, k_N})) = \rho_z^{-1}(\ell_{k_1, k_2, \ldots, k_N}) \otimes 1$$

because each term of $\rho_5^{-1}(\ell_{k_1, \ldots, k_N})$ never ends in $\eta_3$ or $\eta_{33}$.

(iv). Similarly we have an element $\ell_{k_1, k_2, \ldots, k_N}^{z^{-1}, w}$ in $I_{(\infty,0)}^z(M_{0,5})$ which is defined by

$$\text{Li}_{k_1, k_2, \ldots, k_N}(z^{-1}, 1^{m_1-1}, w, 1^{m_2-1}, w^{-1}, 1^{m_3-1}, w, \ldots, 1^{m_N-1})$$

and is also equal to $j_3 \circ \rho_w((-1)^{\sum_{i=1}^N m_i} e_{k_1}(1)e_{k_2}(w)e_{k_3}(1) \cdots e_{k_N}(*)$).

We consider new operators $\partial_{\alpha}$ on $A_w$ and $\tilde{\partial}_{\alpha}$ on $B$ ($\alpha = 0, 1$) below:
Definition 10. (1). For each \( l \in A_w \) with the presentation \( l = [e_0]_{l_1} + [e_w]_{l_2} + [e_1]_{l_3} \) with \( l_1, l_2, l_3 \in A_w \), we define

\[
\partial_0(l) = -l_1 - l_2 - l_3 \in A_w, \quad \partial_1(l) = l_2 + l_3 \in A_w,
\]

and \( \partial_0(1) = \partial_1(1) = 0 \in A_w \), which yield linear operators \( \partial_0 \) and \( \partial_1 \) on \( A_w \).

(2). By using dec, we extend \( \partial_\alpha \) to \( i_3(A) \)-linear (under the shuffle product) operator \( \hat{\partial}_\alpha \) on \( B \) for \( \alpha = 0, 1 \):

\[
\hat{\partial}_\alpha(j_3(l) \shuffle i_3(m)) = j_3(\partial_\alpha(l)) \shuffle i_3(m)
\]

for \( l \in A_w \) and \( m \in A \).

By Remark 9 (iv), the image of \( l \in B \) with the presentation \( l = [e_31]_{l_1} + [e_34]_{l_2} + [e_{23}]_{l_3} + [e_{21}]_{l_4} + [e_{24}]_{l_5} \) \((l_1, \ldots, l_5 \in B)\) is calculated to be

\[
(3.11) \quad \hat{\partial}_0(l) = l_1 \in B, \quad \hat{\partial}_1(l) = l_2 + l_3 \in B.
\]

The following is a key proposition which says that our differential operator \( \hat{\partial}_\alpha \) extends the one \( \partial_{z,\alpha} \) of Sato-Hirose.

Proposition 11. The operators \( \hat{\partial}_0 \) and \( \hat{\partial}_1 \) on \( B \) restrict to the linear operators \( \partial_{z,0} \) and \( \partial_{z,1} \) on \( A_z \), that is, we have the following commutative diagram for \( \alpha = 0, 1 \):

\[
\begin{array}{ccc}
B & \xrightarrow{\partial_\alpha} & B \\
\downarrow{j_2} & & \downarrow{j_2} \\
A_z & \xrightarrow{\partial_{z,\alpha}} & A_z
\end{array}
\]

Proof. By the same arguments of the proof of [6] Theorem 2.1 (for admissible indices), we have

\[
dl(a_0; a_1, \ldots, a_m; a_{m+1})
\]

\[
= \sum_{i=1}^{m} I(a_0; a_1, \ldots, \tilde{a}_i, \ldots; a_{m+1}) \cdot \{d \log(a_i - a_{i+1}) - d \log(a_i - a_{i-1})\}
\]

for \( I(a_0; a_1, \ldots, a_m; a_{m+1}) = \int_{a_0}^{a_{m+1}} \frac{dt_m}{t_m - a_m} \wedge \cdots \wedge \frac{dt_1}{t_1 - a_1} \). Then we have

\[
(3.12) \quad \frac{d}{dz} \rho_z(l') (\gamma') = \frac{1}{z} \rho_z(\partial'_{z,0}(l')) (\gamma') + \frac{1}{z - \rho_z(\partial'_{z,1}(l')) (\gamma')} + \frac{1}{z - w} \rho_z(\partial'_{z,w}(l')) (\gamma')
\]

for any \( l' \in A_z \) and \( \gamma' \in \pi_1(X(z), 0, w) \), where \( \partial'_{z,\alpha} \) \((\alpha = 0, 1, z, w)\) is the linear operator defined by

\[
\partial'_{z,\alpha}(e_{a_0} \cdots e_{a_1}) = \sum_{i=1}^{n} (\delta_{a_0, a_{i+1}})_{(z, w)} - \delta_{a_{i-1}, a_1})_{(z, w)} e_{a_0} \cdots \tilde{e}_{a_i} \cdots e_{a_1}
\]

with \( a_0 = 0 \) and \( a_{n+1} = w \). By comparing this with (2.2), we get

\[
(3.13) \quad \partial_{z,0} = \partial'_{z,0} \quad \text{and} \quad \partial_{z,1} = \partial'_{z,1} + \partial'_{z,w}.
\]

While, by definition, for \( l \in B \) with \( l = [e_31]_{l_1} + [e_34]_{l_2} + [e_{23}]_{l_3} + [e_{21}]_{l_4} + [e_{24}]_{l_5} \) \((l_1, \ldots, l_5 \in B)\), we have

\[
d\rho_z(l) (\gamma) = \frac{dz}{z} \rho_z(l_1) (\gamma) + \frac{dz}{z - 1} \rho_z(l_2) (\gamma) + \frac{dw - dz}{w - z} \rho_z(l_3) (\gamma)
\]

\[
+ \frac{dw}{w} \rho_z(l_4) (\gamma) + \frac{dw}{w - 1} \rho_z(l_5) (\gamma)
\]
for any \( \gamma \in \pi_1(\mathcal{M}_{0,5}; (\infty, 0), (z, w)) \). Particularly we have
\[
\frac{d}{dz} \rho_5(l)(\gamma) = \frac{1}{z} \rho_5(l_1)(\gamma) + \frac{1}{z-1} \rho_5(l_2)(\gamma) + \frac{1}{z-w} \rho_5(l_3)(\gamma).
\]
Hence for \( L \in \mathcal{B} \) such that
\[
\frac{d}{dz} \rho_5(L)(\gamma) = \frac{1}{z} \rho_5(L_1)(\gamma) + \frac{1}{z-1} \rho_5(L_2)(\gamma) + \frac{1}{z-w} \rho_5(L_3)(\gamma)
\]
with \( L_i \in \mathcal{B} \) (\( i = 1, 2, 3 \)), we have
\[
\partial_0(L) = L_1 \quad \text{and} \quad \partial_1(L) = L_2 + L_3
\]
by (3.11).

By Remark 9, (iii), the restrictions of \( \partial_0 \) and \( \partial_1 \) to \( \mathcal{I}_0^w(\mathcal{X}(z)) \) are also calculated by (3.14) and (3.15). Hence by (3.12) we have
\[
\tilde{\partial}_0(j_2(l')) = j_2(\partial_{x,0}(l')), \quad \tilde{\partial}_1(j_2(l')) = j_2(\partial_{x,1}(l') + \partial'_{x,w}(l'))
\]
for \( l' \in A_z \).

By (3.13) and (3.10), we have
\[
\partial_0(j_2(l')) = j_2(\partial_{x,0}(l')) = j_2(\partial_{z,0}(l'))
\]
\[
\partial_1(j_2(l')) = j_2(\partial_{x,1}(l') + \partial'_{x,w}(l')) = j_2(\partial_{z,1}(l')).
\]
Therefore our claim follows. \( \square \)

3.3. Upgrading of the standard relations and the involution \( \tau_z \). We extend the set of standard relations to a subspace of \( \mathcal{B} \) and give its explicit presentation. We also extend the involution \( \tau_z \) in \( A_z \) to the one in \( \mathcal{B} \) and show its property.

We start by preparing the \( \mathbb{Q} \)-linear subspace of \( \mathcal{B} \)
\[
\mathcal{L}_{ST} := \{ j_3(l - l_1 + w) \mid l \in A_w^1, \ m \in A_1 \} \subset \mathcal{B}.
\]
Here \( l_1 + w \) means the image of \( l \) by the algebra homomorphism \( A_w \to A_w \) sending \( e_0 \mapsto e_0, e_w \mapsto e_w, e_1 \mapsto e_1 \).

Remark 12. By the isomorphism \( \rho_5 : \mathcal{B} \simeq \mathcal{I}_{(\infty, 0)}^{(z,w)}(\mathcal{M}_{0,5}) \) and the decompositions \( \text{dec}_2 \) and \( \text{dec}_3 \), the \( \mathbb{Q} \)-linear subspaces \( \mathcal{L}^1 \) and \( \mathcal{L}_{ST} \) are described in terms of the elements of Remark 9.

(i) By (3.6), we see that \( \rho_5(\mathcal{L}^1) \) is linearly spanned by the basis \( \ell_{k_1, k_2, \ldots, k_N}^{w, z^{-1}}, \ell_{w}^{z^{-1}} \) (where \( \ell_{h}^{z^{-1}} \) is the element in \( \mathcal{I}_{(\infty, 0)}^{(z,w)}(\mathcal{M}_{0,5}) \) associated with \( \text{LL}_h(z^{-1}, 1, 1, \ldots) \)), which is an element of \( \mathcal{I}_{(\infty, 0)}^{(z,w)}(\mathcal{M}_{0,5}) \) determined by
\[
\text{LL}_{k_1, k_2, \ldots, k_N}^{w, z^{-1}}(w, 1^{m_1-1}, z^{-1}, 1^{m_2-1}, z, \ldots) \cdot \text{LL}_h(z^{-1}, 1, 1, \ldots),
\]
for all multi-indices \( h, k_1, \ldots, k_N, N > 0 \) (\( h \) and \( k_1 \) can be empty).

While by (3.7), similarly we see that \( \rho_5(\mathcal{L}^1) \) is linearly spanned by the basis \( \ell_{k_1, k_2, \ldots, k_N}^{w, z^{-1}}, \ell_{w}^{z^{-1}} \), which is an element of \( \mathcal{I}_{(\infty, 0)}^{(z,w)}(\mathcal{M}_{0,5}) \) determined by
\[
\text{LL}_{k_1, k_2, \ldots, k_N}^{w, z^{-1}}(z^{-1}, 1^{m_1-1}, w, 1^{m_2-1}, w^{-1}, \ldots) \cdot \text{LL}_h(w, 1, 1, \ldots),
\]
for all multi-indices \( h, k_1, \ldots, k_N, N > 0 \) (\( h \) and \( k_1 \) can be empty).

(ii) Particularly \( \rho_5(\mathcal{L}_{ST}) \) is linearly spanned by the basis
\[
\{ \ell_{k_1, k_2, \ldots, k_N}^{w, z^{-1}}, \ell_{w, (k_1, k_2, \ldots, k_N)}^{z^{-1}, w} \} \cdot \ell_{w}^{z^{-1}},
\]
for all multi-indices \( h, k_1, \ldots, k_N, N > 0 \) (\( h \) and \( k_1 \) can be empty).
that is, the element of \( I_{1(\infty,0)}(M_{0,5}) \) determined by
\[
\left\{ \text{Li}_{k_1,\ldots,k_N}(z^{-1}, 1^{m_1-1}, w, 1^{m_2-1}, w^{-1}, \ldots) - \text{Li}_{k_1,\ldots,k_N}(wz^{-1}, 1^{m_1\cdot\cdots\cdot m_N-1}) \right\}
\cdot \text{Li}_h(w, 1, 1, \ldots),
\]
for all multi-indices \( h, k_1, \ldots, k_N, N > 0 \) (\( h \) and \( k_1 \) can be empty).

We extend the set of standard relations to a subspace of \( B \).

**Definition 13.** (i) Let \( \epsilon : A_w \to \mathbb{Q} \) be the augmentation map. By using \( \text{dec}_3 \), we extend \( \epsilon \) to \( i_3(A) \)-linear map \( \tilde{\epsilon} : B \to A \), that is,
\[
\tilde{\epsilon}(j_3(l) \cup i_3(m)) = j_3(\epsilon(l)) \cup i_3(m) = \epsilon(l)i_3(m)
\]
for \( l \in A_w \) and \( m \in A \).

(ii) We define the \( \mathbb{Q} \)-linear subspace of \( B \)
\[
\tilde{I}_{ST} := \{ l \in B^1 \mid \tilde{\epsilon} \tilde{\partial}_{\alpha_1} \cdots \tilde{\partial}_{\alpha_r}(l) = 0 \text{ for } r \geq 0, \alpha_1, \ldots, \alpha_r \in \{0, 1\} \}.
\]
The map \( \tilde{\epsilon} : B \to A \) is calculated to be the shuffle algebra homomorphism sending
\[
e_{31} \mapsto 0, \quad e_{23} \mapsto 0, \quad e_{34} \mapsto 0, \quad e_{21} \mapsto e_0, \quad e_{24} \mapsto e_1,
\]
which yields the following commutative diagram:

\[
\begin{array}{ccc}
B & \xrightarrow{\tilde{\epsilon}} & A \\
| & | & | \\
\tilde{I}_{ST} & \xrightarrow{\text{Const}} & A
\end{array}
\]

Then by (3.8), (3.8) and Proposition 11 we have
\[
(3.19) \quad \tilde{I}_{ST} \cap j_2(A_w^0) = j_2(I_{ST}).
\]
Actually \( \tilde{I}_{ST} \) coincides with our previous space \( \tilde{L}_{ST} \) in (3.17).

**Proposition 14.** We have
\[
\tilde{I}_{ST} = \tilde{L}_{ST}.
\]

**Proof.** Since \( i_3(A) \)-linear maps \( \tilde{\epsilon} \) and \( \tilde{\partial}_{\alpha} \) are constructed as scalar extensions of \( \mathbb{Q} \)-linear maps \( \epsilon \) and \( \partial_{\alpha} \) and we have \( B^1 = j_3(A_w) \cup i_3(A^1) \), we have
\[
\tilde{I}_{ST} = \{ \tilde{I}_{ST} \cap j_3(A_w) \} \cup i_3(A^1).
\]
While we have
\[
\tilde{L}_{ST} = \langle j_3(l - l_{1 \to w}) \mid l \in A_w^1 \rangle \cup i_3(A^1).
\]
It is enough to show
\[
\tilde{I}_{w} = L_{w}
\]
with \( \tilde{I}_{w} := \{ l \in A_w^1 \mid \epsilon \partial_{\alpha_1} \cdots \partial_{\alpha_r}(l) = 0 \text{ for } r \geq 0, \alpha_1, \ldots, \alpha_r \in \{0, 1\} \} \) and \( L_{w} := \langle l - l_{1 \to w} \mid l \in A_w^1 \rangle \). Put \( P = e_0, Q = e_1 + e_w, R = e_1 - e_w \) in \( A_w \) and express any element \( l \in A_w \) as \( f(P, Q, R) \) with a 3 variables non-commutative polynomial \( f \). By definition, \( l \in \tilde{I}_{w} \) if and only if \( \partial_{\alpha}(l) \in \tilde{I}_{w} \), \( \partial_l(l) \in \tilde{I}_{w} \) and \( \epsilon(l) = 0 \). It is reformulated to be
\[
f_1(P, Q, R), f_2(P, Q, R), f_3(P, Q, R) \in \tilde{I}_{w} \text{ and } f(0,0,0) = 0, \text{ when it is given by } f(P, Q, R) = P f_1(P, Q, R) + Q f_2(P, Q, R) + R f_3(P, Q, R) + f(0, 0, 0). \]
Actually the conditions are equivalent to \( f(P, Q, 0) = 0 \), from which we learn \( \tilde{I}_{w} = L_{w} \). \( \square \)
There is an action of the symmetric group $\mathfrak{S}_5$ on $\mathcal{M}_{0,5}$ by permutation of 5 marked points. Particularly we consider the involution $\tau = (1,4)(3,5) \in \mathfrak{S}_5$. It induces the involution $\tau$ on $U\mathfrak{P}_5$ by $t_{i,j} \mapsto t_{\tau(i),\tau(j)}$ and hence on its dual space $\mathcal{B}$ so that for any $l \in U\mathfrak{P}_5$ and $\phi \in \mathcal{B}$

$$<\tau(l)|\phi> = l|\tau(\phi)>.$$  

**Lemma 15.** Let $S$ be the antipode of $A_z$. The following diagram is commutative

```
B ----\tau----> B
        \downarrow r_1 \quad \downarrow r_2
A_z ----\tau \circ S----> A_z
```

**Proof.** In our coordinate $(z, w)$, the involution $\tau$ is described as $(z, w) \mapsto (z, \frac{z(w-1)}{w-z})$. Hence the induced involution $\tau$ on $B$ is presented as $e_{21} \mapsto e_{31} + e_{24} - e_{23}$, $e_{23} \mapsto e_{31} + e_{34} - e_{23}$, $e_{24} \mapsto e_{21} + e_{34} - e_{23}$, $e_{31} \mapsto e_{31}$. While the action $\tau$ on $\mathcal{M}_{0,5}$ stabilizes the fiber $p_2^{-1}(z) = \mathcal{A}(z)$ for a fixed $z$ and induces the Möbius transformation $t \mapsto \frac{z(t-1)}{t-z}$ there. It causes the involution on $A_z$ given by $e_0 \mapsto e_1 - e_z$, $e_z \mapsto -e_z$, $e_1 \mapsto e_0 - e_z$ (cf. [7]). It coincides with the automorphism $S \circ \tau_z$ of $A_z$. Since the map $r_2$ is given by $r_2(e_{21}) = e_0$, $r_2(e_{23}) = e_z$, $r_2(e_{24}) = e_1$, $r_2(e_{31}) = 0$, $r_2(e_{34}) = 0$, we see that the diagram commutes. \qed

### 3.4. Computations of the pairing.
We prove a key formula on the pairing in Proposition[16] and then prove Theorem[4].

Since the triple $(B, \underline{w}, \delta)$ is the dual of the Hopf algebra $(U\mathfrak{P}_5, \Delta)$ with the standard coproduct given by $\Delta(t_{ij}) = t_{ij} \otimes 1 + 1 \otimes t_{ij}$, we have

$$<l_1 \sqcup l_2|\psi> = \sum_i <l_1|\psi^{(i)}_1> \cdot <l_2|\psi^{(i)}_2>$$

for $l_1, l_2 \in B$ and $\psi \in U\mathfrak{P}_5$ with $\Delta(\psi) = \sum_i \psi^{(i)}_1 \otimes \psi^{(i)}_2$, and

$$<l|\psi_1 \cdot \psi_2> = \sum_i <l^{(i)}_1|\psi_1> \cdot <l^{(i)}_2|\psi_2>$$

for $\psi_1, \psi_2 \in U\mathfrak{P}_5$ and $l \in B$ with $\delta(l) = \sum_i l^{(i)}_1 \otimes l^{(i)}_2$. The same equations hold for the pairing $<\cdot, \cdot>$ between $A$ and $U\mathfrak{P}_2$.

**Proposition 16.** For $l \in A_0$ and $\varphi \in \exp[s_2, f_2]$, we have

$$<\lambda(l)|\varphi> = <j_2(l)|\varphi_{243}^{-1}\varphi_{215}\varphi_{534} > .$$

**Proof.** Since $\lambda : A_0 \rightarrow A^0$ is a shuffle algebra homomorphism, $A_0$ is isomorphic to $A^{-2}_{\mathbb{Z}} \otimes (A_0 \cap \mathbb{Q} \langle e_1, e_z \rangle)$ as shuffle algebras and $\varphi$ is group-like, it is enough to check the equality for the case (i) $l \in A^{-2}_{\mathbb{Z}}$ or (ii) $l \in A_0 \cap \mathbb{Q} \langle e_1, e_z \rangle$ by (3.21).

(i). It is enough to check for $l \in A^{-2}_{\mathbb{Z}}$ determined by (3.9). By definition, we have $\lambda(l) = l|_{z \rightarrow 1}$. Hence

$$<j_2(l)|\varphi_{243}^{-1}\varphi_{215}\varphi_{534} > = <j_2(l)|\varphi_{215}\varphi_{534} >= <j_2(l)|\varphi_{215} > = <l|\varphi(f_0, f_1 + f_z) > = <l|_{z \rightarrow 1} | \varphi > = <\lambda(l)|\varphi> .$$

Here
The first equality is by (3.11), (3.22) and \( \varphi(f_0, 0) = \varphi(0, f_1) = 1 \) since \( j_2(l) \) is calculated by the prescription in Remark 9(iii); Recursive differentiations of (3.9) with respect to \( w \) and \( z \) tell us that there always appears \( e_{31} \) or \( e_{31} \) prior to \( e_{24} \) in each term of \( j_2(l) \). Hence \( \varphi_{243} \) never contribute to the pairing.

The second one is again by (3.9) and (3.22): \( j_2(l) \) is of the form \([l^2 e_{24} - c_{31}] (l', l'' \in B)\). So \( \varphi_{534} \) never contribute to the pairing because \( \varphi(t_{33}, t_{34}) = \varphi(-t_{33} - t_{32} - t_{34}, t_{34}) \).

The third one follows from \( r_2 \circ j_2 = \text{id} \), (3.22) and \( \varphi(t_{21}, t_{15}) = \varphi(t_{21}, t_{23} + t_{24} + t_{34}) = \varphi(t_{21}, t_{23} + t_{24}) \) where we use \([t_{34}, t_{21}] = [t_{34}, t_{23} + t_{24}] = 0 \) in the second equality.

The fourth and the last ones are immediate.

(ii). It is enough to check for \( l \in A^0 \cap \mathbb{Q}(e_1, e_2) \) determined by (3.9) with \((k_1, \ldots, k_N) = (1^{m_1}, \ldots, 1^{m_N})\). By definition, we have
\[
< \lambda(l) | \varphi > = < \text{Const} \otimes \text{id} \circ \text{reg}_z \otimes N(l) | \varphi(f_0, f_1 + f_2) > = < S \circ \tau_2(l) | \varphi(f_0, f_1 + f_2) > = < \tau_2(l) | \varphi(f_0, f_1 + f_2) > \leq < S \circ \tau_2(l) | \varphi(f_0, f_1 + f_2)^{-1} >
\]
\[
= < r_2 \circ \tau \circ j_2(l) | \varphi(f_0, f_1 + f_2)^{-1} > = < \tau \circ j_2(l) | \varphi_{215}^{-1} > = < j_2(l) | \varphi_{243}^{-1} > = < j_2(l) | \varphi_{243} | \varphi_{215} | \varphi_{534} >.
\]

Here

- The first equality is by definition.
- The second one follows from the fact that \( \varphi \) is commutator group-like and \( N(l) \in A_{-2}^0 \).
- The third one is due to \( \tau_2(l) \in \mathbb{Q}(e_0, e_2) \).
- The fourth one is since we have \( S_U(\varphi) = \varphi^{-1} \) for a group-like series \( \varphi \) where \( S_U \) is the antipode of \( U_{f_2} \).
- The fifth one is by \( r_2 \circ j_2 = \text{id} \) and Lemma 15.
- The sixth one is due to \( \varphi(t_{21}, t_{15}) = \varphi(t_{21}, t_{23} + t_{24}) \) and (3.22).
- The seventh one is by (3.22).
- The eighth one is by (3.22) and \( \varphi(t_{21}, t_{15}) = \varphi(t_{21}, t_{23} + t_{24}) \); Since \( e_{31} \) never appear in \( j_2(l) \), \( \varphi_{215} \) never contribute to the pairing.
- The last one is due to the same reason to the second one of our previous case (i).

Therefore we get the claim. \( \square \)

The following is also required to the proof of Theorem 4.

**Lemma 17.** For any \( \ell \in \mathcal{I}_{ST} \), we have
\[
< \ell | \varphi_{351} \varphi_{124} > = 0.
\]

**Proof.** For any \( l \in A_1 \), we have
\[
< j_3(l) | \varphi_{351} \varphi_{124} > = < j_3(l) | \varphi_{351} > = < l | \varphi(3f_0 - f_w - f_1, -2f_0 + f_w + f_1) >.
\]

Here

- The first equality is by the same reason to the second equality in the case (i) of the proof of Proposition 10. Since \( j_3(l) \) is of the form \([l' e_{31} - e_{34}] + [l'' e_{23} - e_{31}] (l', l'' \in B)\), \( \varphi_{124} \) never contribute to the pairing.
The second one follows from $r_3 \circ j_3 = \text{id}$, and $\varphi_{351} = \varphi(t_{35}, t_{51}) = \varphi(-t_{31} - t_{23} - t_{34}, t_{23} + t_{34})$ where we use $[t_{24}, t_{35}] = [t_{24}, t_{23} + t_{34}] = 0$ in the last equality.

It is immediate to see

$$< l | \varphi(3f_0 - f_w - f_1, -2f_0 + f_w + f_1) > = < l_{1 \rightarrow w} | \varphi(3f_0 - f_w - f_1, -2f_0 + f_w + f_1) >$$

for any $l \in A_w$. Therefore we have

$$< j_3(l) | \varphi_{351} > = < j_3(l_{1 \rightarrow w}) | \varphi_{351} >,$$

whence $< j_3(l) | \varphi_{351} \varphi_{124} > = < j_3(l_{1 \rightarrow w}) | \varphi_{351} \varphi_{124} >$ for any $l \in A_w^1$. Thus we have

$$< j_3(l - l_{1 \rightarrow w}) \cup i_3(m) | \varphi_{351} \varphi_{124} >$$

$$= < j_3(l - l_{1 \rightarrow w}) | \varphi_{351} \varphi_{124} > \cdots < i_3(m) | \varphi_{351} \varphi_{124} > = 0$$

for any $l \in A_w^1$ and $m \in A^1$, which proves our claim by Proposition [14].

The proof of Theorem 18 goes as follows: Assume that $\varphi$ is a commutator group-like series satisfying the pentagon equation. By [4], we have the 2-cycle relation $\varphi(f_0, f_1) \varphi(f_1, f_0) = 1$. Then we have $\varphi_{215} \varphi_{334} = \varphi_{243} \varphi_{351} \varphi_{124}$. By Proposition 16, we have for any $l \in A_0^0$

$$< \lambda(l) | \varphi > = < j_2(l) | \varphi_{215} \varphi_{334} > = < j_2(l) | \varphi_{351} \varphi_{124} > .$$

By (3.19) and (3.23), we have $< j_2(l) | \varphi_{351} \varphi_{124} > = 0$ for any $l \in I_{ST} \subset A_0^0$, which proves $< \lambda(l) | \varphi > = 0$ for any $l \in I_{ST}$. □

4. The Confluence Relations imply the Pentagon Equation

We prove that the confluence relations imply the pentagon equation under the commutator group-like condition (Theorem 18).

Theorem 18. Let $\varphi \in \exp[\hat{f}_2, \hat{f}_2]$. If it satisfies the confluence relation, i.e. $< l | \varphi > = 0$ for any $l \in I_{CP}$, then it satisfies the pentagon equation.

4.1. Construction of a standard relation. We show how to construct a standard relation with a given element $l \in A_0^0$.

Lemma 19. For each $l \in A_0^0$, decompose $j_2(l) = \sum_i j_3(l^{(i)}) \cup i_3(m^{(i)})$ with $l^{(i)} \in A_1^1$ and $m^{(i)} \in A^1$ by dec. Then $\sum_i j_3(l^{(i)}_{1 \rightarrow w}) \cup i_3(m^{(i)})$ belongs to $j_2(A_0^0)$.

Proof. We may assume that $l$ corresponds to $\ell = (k_1, k_2, \ldots, k_N) \in I_{ST}^N(X(z))$ under the isomorphism $\rho_2$. By the description of the isomorphism $\rho_5$ in Remark 10, $j_3(l^{(i)}_{1 \rightarrow w})$ corresponds to a linear combination of $\text{Lh}(wz^{-1}, 1, \ldots, 1)$ with multi-indices $k$ ($k$ can be empty). Hence it lies on $j_2(A_0^0)$.

While, by Remark 12, $i_3(m^{(i)})$ corresponds to a linear combination of $\text{Lh}(w, 1, \ldots, 1)$ with multi-indices $h$ ($h$ can be empty). By $l \in A_0^0$, $k_1$ is admissible (that means $e_{k_1} \in A^0$) or empty.

- When $k_1$ is admissible, recursive differentiation of (3.9) shows that all $h$ appearing are admissible. Whence $i_3(m^{(i)})$ is always in $j_2(A_0^0)$.
- When $k_1$ is empty, a non-admissible index $h$ might occur to contribute in $i_3(m^{(i)})$ for some $i$. However such contribution is cancelled out when we consider the summation $\sum_i j_3(l^{(i)}_{1 \rightarrow w}) \cup i_3(m^{(i)})$. It is because recursive differentiation of (3.9) implies $j_3(l^{(i)}) \in A_2^1(e_1 - e_w)$ for such $i$. So a nontrivial contribution in the summation occurs only in the case of $i_3(m^{(i)}) \in j_2(A_0^0)$. 
Therefore we have $\sum_i j_3(l(i)) \sqcup i_3(m(i)) \in j_2(A_2^0)$. □

Then we put
\[ \tilde{l} := j_2^{-1} \left( \sum_i j_3(l(i)_{1 \rightarrow w}) \sqcup i_3(m(i)) \right) \in A_2^0. \]

**Lemma 20.** For $l \in A_2^0$, we have $l - \tilde{l} \in I_{ST}$.

**Proof.** By (4.1) and the previous lemma, it is enough to prove $j_2(l - \tilde{l}) \in \tilde{I}_{ST}$. Since
\[ j_2(l - \tilde{l}) = \sum_i j_3(l(i) - l(i)_{1 \rightarrow w}) \sqcup i_3(m(i)), \]
we have $l - \tilde{l} \in I_{ST}$ by Proposition 13. □

4.2. **Computation of the pairing.** The following lemma is auxiliary to prove Theorem 15.

**Lemma 21.** Let $\varphi \in \exp[\tilde{f}_2, \tilde{f}_2]$ satisfies the confluence relation. Then it also satisfies the 2-cycle relation $\varphi(f_0, f_1)\varphi(f_1, f_0) = 1$.

**Proof.** Let $\varphi$ be as above. Since it is shown in [8] Theorem 28 that the confluence relations imply the duality relation (cf. Theorem 8), we have $S_U(\varphi(f_1, f_0)) = \varphi(f_0, f_1)$ with the antipode $S_U$ of $U \tilde{f}_2$. While since $\varphi$ is group-like, we have $S_U(\varphi(f_1, f_0)) = \varphi(f_1, f_0)^{-1}$. Thus we get the claim. □

The proof of Theorem 15 goes as follows: Assume that $\varphi \in \exp[\tilde{f}_2, \tilde{f}_2]$ satisfies the confluence relation, which is also equivalent to saying that $<\lambda(l) \mid \varphi > = 0$ for any $l \in I_{ST}$. Then for any $l \in I_{ST}$, we have
\[ 0 = <j_2(l)\varphi^{-1}>_{\tilde{f}_2\tilde{f}_2} \cdot \varphi_{215}\varphi_{534} > = <j_2(l)\varphi_{153}\varphi_{342}\varphi_{215}\varphi_{534} >. \]

Here
- The first equality follows from the key formula in Proposition 16.
- The second one is due to Lemma 21.
- The last one follows from the second equality of (3.21), (3.22), and the same arguments to the proof of (3.25) because $j_2(l) \in \tilde{I}_{ST}$ is of the form with $\sum_i j_3(l(i) - l(i)_{1 \rightarrow w}) \sqcup i_3(m(i))$ by Proposition 14.

Set $P'_{15342} = \varphi_{153}\varphi_{342}\varphi_{215}\varphi_{534} \in \exp[\tilde{f}_2]$. Let $l = \rho_5^{-1}(l_{k_1, k_2, \ldots, k_N}) \in A_o^0$ with $N \neq 1$. Then we have
\[ <j_2(l)P'_{15342} >= <j_2(l)\rho_5> |P'_{15342} > = \sum_i <j_3(l(i)_{1 \rightarrow w}) |P'_{15342} > \cdot <i_3(m(i))|P'_{15342} > = \sum_i <i_4(n^{(i)}) |P'_{15342} > \cdot <i_3(m(i))|P'_{15342} > = 0 \]
when the decomposition of $j_2(l)$ is given by $j_2(l) = \sum_i j_3(l(i)) \sqcup i_3(m(i))$. Here
- The first equality is by (4.1) as we have $l - \tilde{l} \in I_{ST}$ by Lemma 20.
- The second one is by (3.21).
- In the third one, $n^{(i)} \in A^1$ is chosen to be $i_4(n^{(i)}) = j_3(l(i)_{1 \rightarrow w})$: Such an element always exists because $j_3(l(i)_{1 \rightarrow w})$ corresponds to a linear combination of $Lwz^{-1}, 1, \ldots, 1$'s, which span $i_4(A^1)$, under $\rho_5.$
The last one follows from (3.5) because we have $\text{pr}_4(P_{15342}^\prime) = 1$ by Lemma 21 and $\deg n^{(i)} > 0$ by $N > 1$.

While by Lemma 21, we have $\text{pr}_2(P_{15342}^\prime) = 1$. Since the kernel of the restriction $\text{pr}_2: \hat{\mathfrak{g}}_5 \to \hat{\mathfrak{f}}_2$ is $\hat{\mathfrak{f}}_3(t_{21}, t_{23}, t_{24})$ and $P_{15342}^\prime$ is in $\exp \hat{\mathfrak{g}}_5$, we see that $P_{15342}^\prime$ is in $\exp \hat{\mathfrak{f}}_3(t_{21}, t_{23}, t_{24})$. Here $\hat{\mathfrak{f}}_3(t_{21}, t_{23}, t_{24})$ is the Lie subalgebra of $\hat{\mathfrak{g}}_5$ freely generated by the three elements. Therefore $P_{15342}^\prime$ is described as

$$P_{15342}^\prime = r_2(P')$$

for a $P' \in \exp \hat{\mathfrak{f}}_2$.

By (1.2), we have $< j_2(l) | P_{15342}^\prime > = 0$ for $l = \rho_2^{-1}(\ell_{k_1, k_2, \ldots, k_N}) \in A_0^0$ with $N \neq 1$. By $r_2 \circ j_2 = \text{id}$ and (3.2), we have

$$< l | P' > = < j_2(l) | P_{15342}^\prime > = 0$$

for such $l \in A_2$. By $\varphi \in \exp [\hat{\mathfrak{f}}_2, \hat{\mathfrak{f}}_2]$, there is no linear terms in $P_{15342}^\prime \in U\hat{\mathfrak{g}}_5$. Thus we have $P' \in \exp \hat{\mathfrak{f}}_2 \subset \exp \hat{\mathfrak{f}}_2$. Considering the case of $N = 1$, that is, $l = \rho_2^{-1}(\ell_{k_1})$, we have

$$< l | P' > = < j_2(l) | P_{15342}^\prime > = < i_3(l) | P_{15342}^\prime > = < l | \text{pr}_3(P_{15342}^\prime) > = < l | \varphi >$$

by $r_2 \circ j_2 = \text{id}$. (3.2) $j_2(l) = i_3(l)$ for such $l$ and the second equality of (3.3). Therefore we have $P' = \varphi$, which says $P_{15342}^\prime = \varphi_{124}$. So we have $\varphi_{1534324\varphi_{215}534\varphi_{421}} = 1$. By Lemma 21, By considering the action of (25) $\in S_5$, we get $\varphi_{1234534512434} = 1$, whence we obtain the pentagon equation for $\varphi$. □

By combining Theorem 4 and Theorem 18 we settle the proof of Theorem 11.

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