Renormalization group study of damping in nonequilibrium field theory

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In this paper we shall study whether dissipation in a $\lambda \phi^4$ theory may be described, in the long wavelength, low frequency limit, with a simple Ohmic term $\kappa \dot{\phi}$, as it is usually done, for example, in studies of defect formation in nonequilibrium phase transitions. We shall obtain an effective theory for the long wavelength modes through the coarse graining of shorter wavelengths. We shall implement this coarse graining by iterating a Wilsonian renormalization group transformation, where infinitesimal momentum shells are coarse-grained one at a time, on the influence action describing the dissipative dynamics of the long wavelength modes. To the best of our knowledge, this is the first application of the nonequilibrium renormalization group to the calculation of a damping coefficient in quantum field theory.

PACS numbers: 11.10.-z, 11.10.Hi, 05.40.Ca

I. INTRODUCTION

In this paper we shall study whether dissipation in a $\lambda \phi^4$ theory may be described, in the long wavelength, small frequency limit, with the addition of a simple Ohmic term $\kappa \dot{\phi}$, as it is usually done, for example, in studies of defect formation in nonequilibrium phase transitions. We shall obtain an effective theory for the long wavelength modes through the coarse graining of shorter wavelengths. We shall implement this coarse graining by iterating a Wilsonian renormalization group (RG) transformation [1, 2, 3], where infinitesimal momentum shells are coarse-grained one at a time, of the influence action describing the dissipative dynamics of the long wavelength modes. To the best of our knowledge, this is the first application of the nonequilibrium RG to the calculation of a transport coefficient in quantum field theory.

Understanding damping is one of the main goals of nonequilibrium quantum field theory, particularly with respect to (probably) its main applications, namely the generation of primordial fluctuations during Inflation [4], the modelling of reheating after Inflation [5], and the

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formation of defects in nonequilibrium phase transitions [6]. There is a tension between the needs of model builders, who, understandably, seek simple and model-robust solutions, and first principles calculations, whose generic result is that damping in field theory is a complicated, non Markovian phenomenon [7, 8, 9, 10]. A clear depiction of this tension can be seen in the contrast between the early models of reheating, based on a linear damping of the inflaton field, and the now accepted picture of preheating, where the main decay mechanism for the inflaton is the parametric amplification of matter fields. This phenomenon is exponentially suppressed in perturbative calculations [11].

The physical mechanism for damping in the long wavelength sector is the interaction with shorter wavelength modes. Damping is a feature of the effective theory where the shorter modes have been coarse-grained away [12]. Since this operation will leave the long wavelength modes in a mixed state, the natural description of the relevant sector is in terms of a density matrix [13], and the natural action functional encoding the effective dynamics is the Feynman-Vernon Influence action [14]. Now suppose we are given the influence action when all modes $k > \Lambda$ have been coarse-grained away, and we wish to further coarse grain the modes in the range $\Lambda \geq k > k_0$. A straightforward perturbative functional integration (for example, to two-loops accuracy) is likely to miss the most important effects. However, if we split the desired range into shells of infinitesimal thickness $\delta s$, then each shell can be integrated out exactly within one-loop accuracy, because each loop integration brings a factor of $\delta s$. Adding a change of units after each integration, we transform the shell coarse-graining into a RG flow in the space of influence actions. Because we shall not assume equilibrium conditions, this may be called the nonequilibrium renormalization group. Our goal is to study the flow of the damping parameter $\kappa$ in the infrared limit.

The nonequilibrium renormalization group should not be confused with the so-called dynamical RG, which is concerned with the long term behavior of solutions to evolution equations [15, 16, 17, 18, 19]. The nonequilibrium RG, on the other hand, is closely related to the Martin, Siggia and Rose formalism [20]. The equivalence between these two latter approaches has been discussed elsewhere [21, 22].

It is also important to stress the difference between our approach and the RG applied to dynamic phenomena, as in [23, 24, 25, 26, 27, 28]. In these applications, a dissipative dynamics is assumed from scratch, as well as a phenomenological noise term to induce the fluctuations to be coarse grained away. In our application, instead, we shall assume that the initial point of the RG trajectory corresponds to a nondissipative theory. The source of noise will be just the quantum fluctuations in the short wavelength modes, as well as any statistical noise coming from the initial density matrix (see below). The issue is whether the renormalization group flow alone brings in dissipation and noise.

It is important to stress two basic differences between the nonequilibrium and equilibrium renormalization groups [29, 30]. The influence action may be regarded as an action for a theory defined on a “closed time path” (CTP) composed of two branches [31]. The first branch goes from the initial time $t = 0$ to a later time $t = T$ when the relevant observations will be performed; that is why we need the density matrix at $T$. The second branch returns from $T$ to 0. Thus
each physical degree of freedom on the first branch acquires a twin on the second branch—we say the number of degrees of freedom is doubled. The influence action is not just a combination of the usual actions for each branch, but also admits direct couplings across the branches. The damping constant $\kappa$ is associated to one of these “mixed” terms. Therefore, the structure of the influence action (from now on, CTP action, to emphasize this feature) is much more complex than the usual Euclidean or “IN-OUT” action.

The second fundamental difference is the presence of the parameter $T$ itself. In nonequilibrium evolution, it is important to specify the time scale over which we shall observe the system [32]. The CTP action contains this physical time scale $T$. From the point of view of the RG, this adds one more dimensional parameter to the theory, much as an external field in the Ising model. Physically, because time integrations are restricted to the interval $[0,T]$, energy conservation does not hold at each vertex. This is of paramount importance regarding damping.

The Wilsonian RG for equilibrium quantum fields in the imaginary time formalism was studied in Ref. [33], and in the real time formalism in Ref. [29]. The RG for the CTP effective action (obtained by taking the limit $T \to \infty$) was studied by Dalvit and Mazzitelli [34]; see also [12] and [13]. Unlike those works, we focus on the dissipation and noise features of the effective dynamics, rather than in the running of the effective potential. For an application of similar ideas in a different field see [35].

We shall now describe the basic assumptions underlying our work.

In formulating a nonequilibrium RG, we must deal with the fact that the CTP action may have an arbitrary functional dependence on the fields and be nonlocal both in time and space. In principle, one can define an exact RG transformation [2, 34, 36], where all three functional dependencies are left open. However, the resulting formalism is too complex to be of practical use [34, 37]. At the same time, one must beware of restricting the form of the action to the point of leaving out an important process. An example is, if we were to assume that the CTP action is the difference of an action functional for each branch of the CTP, we shall miss the damping and noise terms. This is allowed, of course, if one interest is, for example, the running of the effective potential, but it would be disastrous to our present concern.

In our case, we will assume that the initial condition for the RG trajectory corresponds to a massless $\lambda \varphi^4$ theory, with no noise or dissipation. This introduces a small parameter $\lambda$ in the model, and we will consider only the class of action functionals which may be generated from this initial condition to order $\lambda^2$. This means we shall include quadratic, quartic and six-field interactions. We shall allow the interaction terms to be nonlocal in time and in space, to the extent demanded by closure of the RG transformation. There are further constraints coming from the CTP boundary conditions; these will be described below.

We shall work in three spatial dimensions. The structure of the nonequilibrium RG becomes much richer below this critical dimension, since there several fixed points (over and above the nontrivial fixed point of the equilibrium RG). These will be described in a separate publication. However, for the applications listed at the beginning, three spatial dimensions is the most relevant choice. We shall assume that the initial density matrix decomposes into an independent matrix for each mode. The formalism may be developed with great generality, but to obtain
definite results we will focus in the case were the initial density matrix corresponds to a *free* field at finite temperature. This is a nonequilibrium initial condition for the *interacting* field.

We are mostly interested in the case of small $T$ (that is, $\Lambda T \approx 1$), since for large $T$ we expect the nonequilibrium RG to converge to the usual one (we shall return to this point below). This means that in computing Feynman graphs, it is allowed to use propagators pertaining to the initial density matrix and couplings, as any drift term will bring additional powers of $\lambda$. To extend the nonequilibrium RG to a larger $T$ range (for example, to study thermalization within this approach) a fully self-consistent approach is necessary [39] in the spirit of the so-called “environmentally friendly” RG [40].

The paper is organized as follows. In the next two sections we provide an introduction to the CTP formalism, thus fixing our notation, and introduce the parametrization of the CTP action and of the initial density matrix. As we have remarked, the form of the CTP action is chosen to enforce closure of the RG transformation, within the constraints imposed by CTP boundary conditions. In Sec. [IV] we define the RG transformation. This is composed of two stages, first the elimination of one shell in momentum space, thus lowering the cutoff to $(1 - \delta s)\Lambda$, and then the rescaling of momenta, times and fields to restore the value of the cutoff and the coefficients of the kinetic terms in the CTP action. We will show that, in spite of rescaling, the parameter $T$ may be kept as a RG invariant. We then translate the change in the CTP action into a dynamic equation for the RG flow in parameter space. In Sec. [V] we write down these RG equations to order $\lambda^2$ and discuss their structure. In Sec. [VI] we discuss the flow of the damping parameter $\kappa$ and of the noise kernel $\nu$ in the regime outlined above.

We conclude the paper with some brief final remarks. We have concentrated most technical details into appendixes.

## II. OPEN SYSTEMS, CTP AND RG

The core of the RG transformation is the elimination of selected degrees of freedom. When we are only concerned with the lower wave number sector of a field, we can carry out explicitly the integration over the higher wave number modes in the density matrix. As a result, these modes are eliminated from the description. This partial integration will return a description for the lower wave number modes only. The influence of the higher modes is incorporated in the parameters which define the effective action for the surviving modes. The procedure may be seen as a straightforward application of the Feynman-Vernon influence functional techniques for open systems. In the problem at hand, the long modes are regarded as the system and the short ones as the environment.

We first review briefly the ideas behind the Feynman-Vernon formalism. We will deal with a system composed of two parts, which we call the relevant system, or simply the system, and the environment.
A. Open systems and Feynman-Vernon formalism

Let $\varphi$ be the field variables of the whole system, and $S$ its action. Then, its density matrix $\rho$ admits the following representation

$$\rho[\varphi^+, \varphi^-, T] = \langle \varphi^+ \big| U(0, T) \rho U(T, 0) \big| \varphi^- \rangle$$

$$= \int \mathcal{D}\varphi^+ \mathcal{D}\varphi^- \exp i \left( S[\varphi^+] - S[\varphi^-] \right) \rho[\varphi^+(0), \varphi^-(0), 0].$$

This expression says how to obtain the matrix elements of the density matrix operator in the Schrödinger representation at time $T$. The generating functional for the expectation values is obtained by adding current terms, and by closing the path of integration to take the trace

$$Z[J^+, J^-] = \int \mathcal{D}\varphi^+ \mathcal{D}\varphi^- \exp i \left\{ S[\varphi^+] - S[\varphi^-] + \int (J^+\varphi^+ + J^-\varphi^-) \right\} \rho[\varphi^+(0), \varphi^-(0), 0].$$

Derivatives of $Z$ with respect to the currents give expectation values of products of the fields (with a time ordering that depends on which derivatives have been taken). Equation (2) can be thought as an integral over single histories defined on a closed time path. This path has a first branch from 0 to $T$, where the history takes the values $\varphi^+(t)$, and a second branch from $T$ to 0, where the history takes the values $\varphi^-(t)$. The CTP condition

$$\varphi^+(T) = \varphi^-(T)$$

implies that each history is a continuous function of the time along the path. The object

$$S_{\text{CTP}}[\varphi^+, \varphi^-] = S[\varphi^+] - S[\varphi^-]$$

is called the CTP action. In general it can be written

$$\rho[\varphi^+, \varphi^-, T] = \int \mathcal{D}\varphi^+ \mathcal{D}\varphi^- \exp (iS_{\text{CTP}}[\varphi^+, \varphi^-]) \rho[\varphi^+(0), \varphi^-(0), 0].$$

Now, we make the division between relevant system and environment. Suppose that the whole system consists of two kinds of variables $\varphi_<$ and $\varphi_>$, being $\varphi_<$ the system variables, and $\varphi_>$ the environment variables. Depending on the context, this could be a division between system and bath fields in the thermodynamical sense, or between different particle fields, or, as will be in our case, two different sectors of the same field: the lower and the higher wave number sectors.

The whole system is described using the generating functional associated with the full density matrix, which depends on both system and environment variables. However, if we only wish to compute expectation values for the system observables, we can use the generating functional associated with the reduced density matrix that is obtained after taking the trace with respect to
the environment variables. To do this, suppose that the CTP action of the system + environment is $S_{CTP}[\varphi^+, \varphi^-]$, then write it as a functional of the two pairs of fields, $(\varphi^<_+, \varphi^-_+)$ and $(\varphi^+_+, \varphi^-_+)$, for the system and the environment, respectively,

$$S_{CTP}[\varphi^+, \varphi^-] = S_{CTP}[\varphi^+_+, \varphi^-_+] + \Delta S[\varphi^+_+, \varphi^-_+, \varphi^+_+, \varphi^-_+].$$  \hfill (6)

For future convenience, the same functional $S_{CTP}$ appears in both members; this is just the definition of $\Delta S$. The functional $\Delta S$ contains the information about the environment itself and about the interaction with the system. Let us further assume that at $t = 0$ system and environment are uncorrelated,

$$\rho(0) = \rho_s(0) \otimes \rho_e(0),$$  \hfill (7)

where $\rho_s(0)$ ($\rho_e(0)$) refers to the initial density matrix of the proper system (environment). After, replacing (6) in the Eq. (5) and integrating out the environment variables $\varphi^>_\pm$, we get an evolution law for the so called reduced density matrix

$$\rho_r[\varphi^-_+, \varphi^-_+, T] = \int D\varphi^+_+ D\varphi^-_+ \exp i \left( S_{CTP}[\varphi^-_+, \varphi^-_+] + S_{IF}[\varphi^-_+, \varphi^-_+] \right) \rho_s[\varphi^-_+(0), \varphi^-_+(0), 0],$$  \hfill (8)

where the influence action $S_{IF}$ is given by

$$e^{iS_{IF}[\varphi^-_+, \varphi^-_+]} = \int D\varphi^+_+ D\varphi^-_+ \exp i \left( \Delta S[\varphi^-_+, \varphi^-_+, \varphi^+_+, \varphi^-_+] \right) \rho_e[\varphi^+_+(0), \varphi^-_+(0), 0].$$  \hfill (9)

All the influence of the environment on the system is encoded into $S_{IF}$. The elimination of the fields $\varphi^>_+$ takes the initial CTP action, $S_{CTP}$, and returns $S_{CTP<}$, the coarse-grained CTP action for the proper system:

$$S_{CTP<}[\varphi^-_+, \varphi^-_+] = S_{CTP}[\varphi^-_+, \varphi^-_+] + S_{IF}[\varphi^-_+, \varphi^-_+],$$  \hfill (10)

which is not necessarily the difference of some functional evaluated in each branch, $\tilde{S}[\varphi^+_+] - \tilde{S}[\varphi^-_+]$ for some $\tilde{S}$ as in Eq. (4), but usually a more complex functional. It can entangle the two branches. CTP actions of this general type satisfy certain properties, namely

$$S_{CTP}[\varphi, \varphi] = 0,$$  \hfill (11)

$$S_{CTP}[\varphi^-, \varphi^+] = -S_{CTP}[\varphi^+, \varphi^-]^*. $$  \hfill (12)

Thus, if we introduce new variables

$$\phi = \varphi^+ - \varphi^-,$$  \hfill (13)

$$\varphi = \varphi^+ + \varphi^-,$$  \hfill (14)
the expansion of $S_{CTP}$ in powers of $\phi$, with functional coefficients $\sigma_n[\varphi; t_1, \ldots, t_n]$ depending on $\varphi$ and $n$ time variables

$$S_{CTP}[\phi, \varphi] = \sum_{n \geq 1} \int dt_1 \ldots \int dt_n \sigma_n[\varphi; t_1, \ldots, t_n] \phi(t_1) \ldots \phi(t_n),$$

should start from $n = 1$, and all the odd (even) terms should be real (imaginary). In what follows the fields $\phi$ and $\varphi$ will be used instead of $\varphi^+$ and $\varphi^-$; the CTP condition (3) is now expressed as

$$\phi(T) = 0.$$  \hspace{1cm} (16)

The application of this formalism to the case where the system and the environment are two sectors of the same scalar field is presented in Appendix A.

### III. PARAMETRIZATION OF THE CTP ACTION AND OF THE INITIAL DENSITY MATRIX.

In this section we introduce the parametrization of the action and of the initial density matrix. The parametrization of the action will be quite general. In Sec. [IV] we will define the RG transformation for this general class of actions, without any further assumption regarding the initial condition at cutoff $\Lambda$. In Sec. [V] the initial condition will be fixed and an approximation scheme introduced. This will allow us to reduce the parameters to be considered to a finite number of couplings with simpler dependencies on the momenta and time variables.

We will use natural units in which $\Lambda = 1$. To compute the RG transformation we will assume that the action is

$$S[\phi, \varphi] = S_0[\phi, \varphi] + S_{\text{int}}[\phi, \varphi],$$

where the free action is

$$S_0[\phi, \varphi] = \int_0^T dt \int_{\Lambda} d^d k \left[ -\frac{1}{2} \phi(k, t) \phi(k, t) - \frac{1}{2} \phi(k, t) \left( k^2 + m^2 \right) \phi(-k, t) - \kappa \phi(k, t) \phi(-k, t) + i \nu \phi(k, t) \phi(-k, t) \right],$$

and the interaction part

$$S_{\text{int}}[\phi, \varphi] = \sum_{\text{even } n \geq 2} S_n[\phi, \varphi].$$

For $n = 2$

$$S_2[\phi, \varphi] = \int_0^T dt_1 \int_0^T dt_2 \int_{\Lambda} d^d k \left[ v_{21}(k; t_1, t_2) \phi(k, t_1) \phi(-k, t_2) + i v_{22}(k; t_1, t_2) \phi(k, t_1) \phi(-k, t_2) \right],$$

(19)
and for \( n > 2 \)

\[
S_n[\phi, \varphi] = \Omega_d^{1-n/2} \left( \prod_{j=1}^{n} \int_{0}^{T} dt_j \int_{\Lambda} d^d k_{j} \right) \delta^{d} \left( \sum_{l=1}^{n} k_l \right) \\
\times \sum_{m=1}^{n} i^{1+(-1)^{m}/2} v_{nm}(\{k\}; \{t\}) \phi^m \varphi^{n-m}.
\]

(20)

The factor \( \Omega_d^{1-n/2} \), where \( \Omega_d \) is the area of the unit sphere in \( d \) dimensions, is introduced for further convenience. The subscript \( \Lambda \) in the integral symbols means that the integrations are restricted to \( |k_i| \leq \Lambda = 1 \). Moreover, we have defined

\[
v_{nm}(\{k\}; \{t\}) = v_{nm}(k_1, \ldots, k_m, \ldots, k_n; t_1, \ldots, t_m, \ldots, t_n) \quad \text{(with} \quad 1 \leq m \leq n),
\]

\[
\phi^m \varphi^{n-m} = \phi(k_1, t_1) \cdots \phi(k_m, t_m) \varphi(k_{m+1}, t_{m+1}) \cdots \varphi(k_n, t_n).
\]

(21)

With these definitions, both the fields and the couplings are dimensionless. Note that the CTP action associated with the usual classical action for a massless \( \lambda \phi^4 \) is

\[
S[\phi, \varphi] = \int_{0}^{T} dt \int_{\Lambda} d^d k \left[ \frac{1}{2} \left( \dot{\phi}(k, t) \varphi(-k, t) - k^2 \phi(k, t) \varphi(-k, t) \right) \right] \\
- \frac{\lambda}{48} \int_{0}^{T} dt \int_{\Lambda} \frac{d^d k_1 \cdots d^d k_4}{(2\pi)^d} \delta^d \left( \sum_{l=1}^{4} k_{l} \right) \left[ \phi(k_1, t) \varphi(k_2, t) \varphi(k_3, t) \varphi(k_4, t) \right] \\
+ \phi(k_1, t) \phi(k_2, t) \phi(k_3, t) \varphi(k_4, t) \varphi(k_4, t).
\]

(22)

Hence, in the notation just introduced, only \( v_{41} \) and \( v_{43} \) are different from zero,

\[
v_{41} = v_{43} = -\frac{\Omega_d \Lambda^{d-3}}{48(2\pi)^d} \lambda \delta(t_1 - t_2) \delta(t_1 - t_3) \delta(t_1 - t_4).
\]

(23)

(The factor \( \Lambda^{d-3} \), actually equal to 1, has been included for completeness.)

According to the definitions (19) and (20), \( v_{nm} \) couples \( n \) fields altogether, \( m \) of type \( \phi \) and \( n - m \) of type \( \varphi \), always with \( m \geq 1 \). In addition, when in a given term the number \( m \) of fields of the type \( \phi \) is even, the coupling function appears multiplied by \( i \). Therefore all the couplings functions \( v_{nm} \) will be real [see Eq. (12)]. We will take the couplings symmetrical with respect to the permutations of the variables of each type of field; e.g., \( v_{42} \) is unchanged when \((k_1, t_1) \leftrightarrow (k_2, t_2) \) or \((k_3, t_3) \leftrightarrow (k_4, t_4) \). We will also assume that the \( v_{nm} \) couplings do not contain time derivatives of Dirac deltas.

To make sure that the RG transformation relates actions on the same class and preserves the CTP properties (11) and (12), the couplings \( v_{nm} \) must be such that at least one of the \( \phi \) fields in each interaction term is evaluated at a time \( t \) equal or later than the fields \( \varphi \). This condition is directly related to the fact that the expectation value \( \langle \varphi(t) \phi(t') \rangle \) is causal.
Finally, we give a prescription to separate out the free action from the quadratic interaction terms. It must be

$$
\int_0^T dt \int_0^T dt' v_{21}(0; t, t') = 0,
$$

$$
\frac{\partial^2}{\partial k^2} \left( \int_0^T dt \int_0^T dt' v_{21}(k; t, t') \right) \bigr|_{k=0} = 0,
$$

$$
\int_0^T dt \int_0^T dt' (t - t') v_{21}(k; t, t') = 0,
$$

$$
\int_0^T dt \int_0^T dt' v_{22}(k; t, t') = 0.
$$

(24)

In this way, the terms $m^2 \phi \varphi$, $k^2 \phi \varphi$, $\kappa \phi \dot{\varphi}$ and $\nu \phi^2$ are isolated within $S_0$. We will return to this point in Appendix C.

We have to chose the initial conditions $\rho[\phi(k, 0), \varphi(k, 0)]$. We will use

$$
\rho[\phi(k, 0), \varphi(k, 0)] = \exp \left\{ - \int d^d k \frac{a(k)}{4} \left[ \tanh \left( \frac{a(k) \beta(k)}{2} \right) \phi(k, 0) \phi(-k, 0) \right.ight.
$$

$$
\left. + \coth \left( \frac{a(k) \beta(k)}{2} \right) \varphi(k, 0) \varphi(-k, 0) \right] \right\}.
$$

(25)

This corresponds to uncoupled, free fields with frequency $a(k)$ and with a temperature $\beta(k)^{-1}$ which depends on the wave number. Observe, however, that this is a nonequilibrium initial condition for the interacting fields.

The propagators corresponding to the free action (18) and the density matrix (25) are given in Appendix B.

IV. THE RG TRANSFORMATION

In this section, we define the RG transformation for the general class of actions (17). The infinitesimal RG transformation, from which the finite transformation is obtained, is composed by two operations: i) elimination of the modes with wave numbers in an infinitesimal momentum shell, and ii) rescaling. We analyze each of these operations in the following two subsections. The joint effect of i) and ii) is written as a set of differential equations in Sec. IV C.

A. Mode elimination

The first step to obtain the RG equations is discussed in Appendix A. Here we will apply the results of that section to the action (17). Modes with momenta in the shell between $b$ and
1 are eliminated \((b = 1 - \delta s, 0 < \delta s \ll 1)\). After eliminating the modes in the shell, one obtains an effective action for the modes with momenta below \(b\). The effective action is given by Eqs. \((A9)\) and \((A10)\). It can be written as

\[
S_\ast = S'_0 + S'_\text{int}
\]  

where

\[
S'_0[\phi, \varphi] = \int_0^T dt \int_{b\Lambda} d^dk \left\{ \frac{1}{2} \dot{\phi} \dot{\varphi} - \frac{1}{2} \left[ b^{-2\eta} k^2 + (m^2 + \delta m^2) \right] \phi \varphi 
- (\kappa + \delta \kappa) \dot{\phi} \dot{\varphi} + \frac{i}{2} (\nu + \delta \nu) \phi \varphi \right\},
\]

and where \(S'_\text{int}\) has the same form that \(S_\text{int}\) in Eq. \((18)\), but with perturbed couplings

\[
v'_{nm} = v_{nm} + \delta v_{nm}.
\]

Note that the momenta in the integrals now go up to \(b\Lambda\). We discuss in the Appendix C the actual calculations involved.

At this stage, the initial density matrix remains unchanged.

**B. Rescaling**

Now we introduce the second step in the RG transformation. We redefine the fields and change integration variables in the action, writing

\[
b^{\alpha_\phi} \phi \left( b^{-1} k, b^{\alpha_t} t \right)
\]

and

\[
b^{\alpha_\varphi} \varphi \left( b^{-1} k, b^{\alpha_t} t \right)
\]

instead of \(\phi(k, t)\) and \(\varphi(k, t)\), and

\[
bk
\]

and

\[
b^{-\alpha_t} t
\]

instead of \(k\) and \(t\). The rescaling redefines fields, times and momenta and restores the cutoff to its original value \(\Lambda = 1\). The two exponents,

\[
\alpha_\phi = \frac{1}{2}(\eta - d - 1),
\]

\[
\alpha_t = 1 - \eta,
\]
are chosen to reduce to 1 the coefficients of the terms $\dot{\phi}\dot{\varphi}$ and $k^2\dot{\phi}\dot{\varphi}$ in the free action. Here $\eta$ is the quantity introduced in (27). Rescaling affects both the couplings in the action and the parameters which define the density matrix.

It is important to notice that, in spite of rescaling, the time $T$ may be kept as a RG invariant. This is a consequence of the CTP condition (16) and causality. This fact and other details concerning rescaling are shown in Appendix D.

C. RG equations and formal solutions

The effect of mode elimination and rescaling over the parameters which define the action and the density matrix can be summarized in the following differential equations

\[
\left( \frac{\partial}{\partial s} - 2 + 2\eta \right) m^2 = \frac{\delta m^2}{\delta s},
\]

\[
\left( \frac{\partial}{\partial s} - 1 + \eta + k \frac{\partial}{\partial k} \right) \kappa = \frac{\delta \kappa}{\delta s},
\]

\[
\left( \frac{\partial}{\partial s} - 2 + 2\eta + k \frac{\partial}{\partial k} \right) \nu = \frac{\delta \nu}{\delta s},
\]

\[
\left[ \frac{\partial}{\partial s} - \left( \frac{n}{2} - 1 \right) (3 - d) + \frac{3n}{2} \eta - 3 + \sum_{i=1}^{n} \left( k_i \frac{\partial}{\partial k_i} - \alpha_i t_i \frac{\partial}{\partial t_i} \right) \right] v_{nm} = \frac{\delta v_{nm}}{\delta s},
\]

\[
\left( \frac{\partial}{\partial s} - 1 + \eta + k \frac{\partial}{\partial k} \right) a = 0,
\]

\[
\left( \frac{\partial}{\partial s} + 1 - \eta + k \frac{\partial}{\partial k} \right) \beta = 0.
\]

[See Eqs. (D4)-(D10). Equations. (33) and (34) have been used for the exponents.]

Equations (35)-(40), together with the equation for $\eta$, are the RG equations. Integrating these differential equations is equivalent to iterate the infinitesimal transformation. They have the general form

\[
\left[ \frac{\partial}{\partial s} + \alpha_F + \sum_{i=1}^{n} \left( k_i \frac{\partial}{\partial k_i} - \alpha_i t_i \frac{\partial}{\partial t_i} \right) \right] F = g,
\]

where $\alpha_F$ and $\alpha_i$ are functions of $s$ alone. The formal solution $F(\{k\}; \{t\}; s)$, with the initial condition

\[
F(\{k\}; \{t\}; 0) = F_0(\{k\}; \{t\}),
\]
is

\[ F(\{k\}; \{t\}, s) = \int_0^s ds' e^{\beta_F(s')-\beta_F(s)} g \left( e^{s'-s} \{k\}; e^{\beta_t(s)-\beta_t(s')} \{t\}, s' \right) + e^{-\beta_F(s)} F_0 \left( e^{-s} \{k\}; e^{\beta_t(s)} \{t\} \right) , \]

where \( \beta_F(s) = \int_0^s ds' \alpha_F(s') \), and \( \beta_t(s) = \int_0^s ds' \alpha_t(s') \).

\[ \text{V. THE RG EQUATIONS TO ORDER } \lambda^2 \]

\[ \text{A. Reduced set of parameters} \]

So far, the RG transformation defined for the class of actions (17) is rather general. The transformation is closed with respect to this class, but infinitely many couplings have to be taken into account. To reduce the number of couplings that have to be considered, we will choose as the initial condition at the cutoff \( \Lambda \) the \( \lambda \phi^4 \) action (22), and compute the RG equations to order \( \lambda^2 \). The central question is to find the minimum set of parameters such that the transformation is closed to order \( \lambda^2 \) for this initial condition. This set, which has 18 elements, is found in Appendix [E] together with a number of constraints upon the dependencies on the momenta and time variables.

These results can be summarized by saying that, starting at \( s = 0 \) from the action (22), to order \( \lambda^2 \), the action for \( s > 0 \) can be written as \( S = S_0 + S_{\text{int}} \), where \( S_0 \) is the free action of Eq. (18), and where \( S_{\text{int}} \) is given by

\[ S_{\text{int}} = \int_0^T dt \int_\Lambda d^d k \, V_{21}(t, s) \, \phi(k, t) \varphi(-k, t) \]

\[ + \int_0^T dt \int_0^T dt' \int_\Lambda d^d k \, \left[ W_{21}(k; t, t'); \phi(k, t) \varphi(-k, t') + iW_{22}(k; t, t', s) \phi(k, t) \psi(-k, t') \right] \]

\[ + \Omega_4^{-1} \int_0^T dt \int_0^T dt' \int_\Lambda \prod_{i=1}^4 d^d k_i \, \delta^d (k_1 + \cdots + k_4) \left[ V_{41}(s) \, \phi_1 \varphi_2 \varphi_3 \varphi_4 + V_{43}(s) \, \phi_1 \phi_2 \phi_3 \varphi_4 \right] \]

\[ + \Omega_4^{-1} \int_0^T dt \int_0^T dt' \int_\Lambda \prod_{i=1}^4 d^d k_i \, \delta^d (k_1 + \cdots + k_4) \left[ W_{41}(t; t', s) \, \phi_1 \varphi_2 \varphi_3 \varphi_4' \right. \]

\[ + iW_{42}(s; t, t', s) \, \phi_1 \varphi_2 \varphi_3' \varphi_4 \]

\[ + \Omega_4^{-2} \int_0^T dt \int_0^T dt' \int_\Lambda \prod_{i=1}^6 d^d k_i \, \delta^d (k_1 + \cdots + k_6) \left[ v_{61}(s; t, t', s) \, \phi_1 \varphi_2 \varphi_3 \varphi_4 \varphi_5 \varphi_6' \right. \]

\[ + i v_{62}(s; t, t', s) \, \phi_1 \varphi_2 \varphi_3 \varphi_4 \varphi_5 \varphi_6' + v_{63}(s; t, t', s) \, \phi_1 \phi_2 \phi_3 \varphi_4 \varphi_5 \varphi_6' \]

\[ + v_{64}(s; t, t', s) \, \phi_1 \varphi_2 \varphi_3 \varphi_4 \varphi_5 \varphi_6' + i v_{65}(s; t, t', s) \, \phi_1 \varphi_2 \varphi_3 \varphi_4 \varphi_5 \varphi_6' \]

\[ + v_{65}(s; t, t', s) \, \phi_1 \varphi_2 \varphi_3 \varphi_4 \varphi_5 \varphi_6' + i v_{66}(s; t, t', s) \, \phi_1 \phi_2 \phi_3 \varphi_4 \varphi_5 \varphi_6' \right]. \]
Here $\phi_i = \phi(k_i, t)$, $\phi'_i = \phi(k_i, t')$ (same for $\varphi$), and
\[
Q_{12\ldots} = |k_1 + k_2 + \ldots|.
\] (45)

Moreover, $m^2$, $V_{21}$, $V_{41}$ and $V_{43}$ are of order $\lambda$, while $\kappa$, $\nu$, the rest of the couplings in $S_{int}$, and $\eta$ are of order $\lambda^2$.

It is straightforward to recast the general analysis of Sec. [IV] for this particular action. The RG group equations are given in the next section.

**B. The RG equations**

In the previous subsection we gave the reduced set of parameters that has to be considered in working out the equations to order $\lambda^2$. Now the RG equations will be grouped in the same order that will be solved later on. The groups form a hierarchy. The equations at the top are themselves a closed system. Once their solutions are known, the equations at the following level can be considered as effectively closed, and so on.

At the top of the hierarchy are the equations that determine the propagators $G$ and $G$. It is important to notice that in writing the RG equations to order $\lambda^2$, for most cases it is enough to know the propagators $G$ and $G$ to order zero in $\lambda$. At that order they are independent of the couplings and are determined by the functions $a$ and $\beta$ in Eq. (25) alone. The only exceptions are the equations for $V_{21}$ and $m^2$, where $G$ must be known to order $\lambda$, in which case it will depend on $m^2$. But these equations are at the end of the hierarchy, so they do not affect the equations that precede them and still will form a closed system.

Thus, for the moment, we only need the propagators to order zero in $\lambda$ evaluated at $k = \Lambda = 1$. From their general expressions (B.1) and (B.2) we find
\[
G(t, t') = 2 \sin(t - t') \Theta(t - t'),
\] (46)
and
\[
G(t, t', s) = \frac{2}{a(1, s)} \left[ 1 + 2 f \left( a \beta(1, s) \right) \right] \left\{ a(1, s)^2 \cos(t - t') + \left[ 1 - a(1, s)^2 \right] \cos(t + t') \right\}. \quad (47)
\]

The functions $a(1, s)$ and $\beta(1, s)$ to order zero in $\lambda$ are given by Eqs. [39] and [40] setting $\eta = 0$.

At the second level are the equations for $V_{41}$ and $V_{43}$
\[
\left( \frac{\partial}{\partial s} + d - 3 \right) V_{41}(s) = 18 V_{41}(s)^2 \{GG\}(s, T), \quad (48)
\]
\[
\left( \frac{\partial}{\partial s} + d - 3 \right) V_{43}(s) = 18 V_{41} V_{43}(s) \{GG\}(s, T), \quad (49)
\]
where
\[
\{GG\} = P(GG).
\] (50)
It has been used that \( v_{61}(1; t, t', s) = 0 \) for the chosen initial conditions, and that \( \eta \) can be set equal to zero in the left-hand side member of the RG equations to order \( \lambda^2 \) [see Eqs. (35)-(38)]. Note that if \( V_{41}(s) = V_{43}(s) \) for \( s = 0 \), then this relation holds for \( s > 0 \) as well. The solutions \( V_{41} \) and \( V_{43} \) will depend on \( T \) through the function \( \{GG\} \).

At the next level we find the equations for the \( v_{6i} \) couplings, Appendix E. They have the same general form; we write only the first three equations, that are all that is needed for present discussion

\[
(D + 2d - 5) v_{61}(k; t, t', s) = 3V_{41}(s)^2 G(t, t') \delta(k - 1^+),
\]

\[
(D + 2d - 5) v_{62}(k; t, t', s) = \frac{9}{2} V_{41}(s)^2 G(t, t') \delta(k - 1^+),
\]

\[
(D + 2d - 5) v_{63}^{(1)}(k; t, t', s) = 9V_{41}(s)V_{43}(s) G(t, t') \delta(k - 1^+).
\]

[See Eq. (D14) for a definition of \( \delta(k - 1^+) \).] Here

\[
D = \frac{\partial}{\partial s} + k \frac{\partial}{\partial k} - \left( t \frac{\partial}{\partial t} + t' \frac{\partial}{\partial t'} \right).
\]

According to the results of Appendix E, the factor \( \alpha_t \), which would have to appear in front of the time derivatives, has been set equal to one.

Next come the equations for \( W_{41}, W_{42} \) and \( W_{43} \), which are

\[
(D + d - 4) W_{41}(k; t, t', s) = 18\delta_{k,0} V_{41}(s)^2 G G(t, t', s) - 18V_{41}^2 \{GG\} \delta(t - t')
 + \int \frac{d\Omega_d}{\Omega_d} \left[ 6v_{61}(Qk\Omega; t, t', s) G(t, t', s) + 2v_{62}(Qk\Omega; t, t', s) G(t, t') \right],
\]

\[
(D + d - 4) W_{42}(k; t, t', s) = 9\delta_{k,0} V_{41}(s) \left[ V_{41}(s) G G(t, t', s) - 2V_{43}(s) [GG(t, t')]_{tt'} \right]
 - \int \frac{d\Omega_d}{\Omega_d} \left\{ 4 \left[ v_{63}^{(1)}(Qk\Omega; t, t', s) G(t, t') \right]_{tt'} + 4v_{62}(Qk\Omega; t, t', s) G(t, t', s) \right\},
\]

\[
(D + d - 4) W_{43}(k; t, t', s) = 18\delta_{k,0} V_{41} V_{43}(s) G G(t, t', s) - 18V_{41} V_{43}(s) \{GG\}(s) \delta(t - t')
 + \int \frac{d\Omega_d}{\Omega_d} \left[ 2v_{63}^{(1)}(Qk\Omega; t, t', s) G(t, t', s) + 6v_{64}(Qk\Omega; t, t', s) G(t, t') \right].
\]

We defined

\[
Q_{k\Omega} = |k + \hat{\Omega}|.
\]

The subscript \( tt' \) means symmetrization respect to \( t \) and \( t' \) (we used that \( v_{62} \) and \( G \) are already symmetric in these variables). The terms in Eqs. (55) and (57) proportional to \( \delta(t - t') \) are the necessary subtractions after isolating the contributions to \( \delta V_{41} \) and \( \delta V_{43} \).
Finally, we write the equations for the the 2 field couplings and $\eta$, which can be solved in closed form if the solutions of the previous equations are known. We define three auxiliary functions

$$f_1(t, s) = \int_0^t dt' \left[ W_{41}(0; t, t', s) G(t', t', s) + V_{41}(t', s) G(t', t', s) \right] + 3V_{41}(s) G(t, t, s), \quad (59)$$

$$f_2(k; t, t', s) = 2 \int \frac{d\Omega_d}{\Omega_d} \left[ W_{41}(Qk\Omega; t, t', s) G(t, t', s) + W_{42}(Qk\Omega; t, t', s) G(t, t') \right], \quad (60)$$

$$f_3(k; t, t', s) = \int \frac{d\Omega_d}{\Omega_d} \left[ W_{42}(Qk\Omega; t, t', s) G(t, t', s) + 2W_{43}(Qk\Omega; t, t', s) G(t, t') \right]. \quad (61)$$

Each function is the sum of several diagrams appearing in Appendix E.

In terms of these functions, it results

$$(D - 2) V_{21}(t, s) = f_1(t, s) - \mathcal{P} f_1(s), \quad (62)$$

$$(D - 3) W_{21}(k; t, t', s) = f_2(k; t, t', s) - \left[ \mathcal{P} f_2(0, s) + \frac{1}{2} k^2 (\mathcal{P} f_2)'(0, s) \right] \delta(t - t')$$

$$- \mathcal{Q} f_2(k, s) \left\{ 2 \left[ \frac{\partial}{\partial t'} + \delta(t') - \delta(0) \right] \delta(t - t') \right\}, \quad (63)$$

$$(D - 3) W_{22}(k; t, t', s) = f_3(k; t, t', s) - \mathcal{P} f_3(k, s) \delta(t - t'), \quad (64)$$

$$(D - 2) m^2(s) = -2 \left[ \mathcal{P} f_1(s) + \mathcal{P} f_2(0, s) \right], \quad (65)$$

$$(D - 1) \kappa(k, s) = -\mathcal{Q} f_2(k, s), \quad (66)$$

$$(D - 2) \nu(k, s) = 2 \mathcal{P} f_3(k, s), \quad (67)$$

$$\eta = -\frac{1}{2} \frac{\partial^2 \mathcal{P} f_2}{\partial k^2}(0, s). \quad (68)$$

The operators $\mathcal{P}$ and $\mathcal{Q}$ are defined in Appendix C, Eqs. C9 and C10. Note also that, as was mentioned at the beginning of this subsection, since $V_{41}$ is of order $\lambda$, in the last term of Eq. (59), $G$ must be written to order $\lambda$. In this way, the equations for $V_{21}$ and $m^2$ will include all the terms to order $\lambda^2$. As we will not attempt to find the solutions for $V_{21}$ and $m^2$ to order $\lambda^2$ we will not need to write $G$ to order $\lambda$ explicitly.

The way to integrate the RG equations starts by writing down the propagators $G$ and $G$ as functions of $s$. Next the equations for $V_{41}$ and $V_{43}$ can be integrated immediately. Once these
functions are known, the equations for the \( v_{6i} \) couplings are solved using the formal solution (43). With these solutions, the equations for the \( W_{4i} \) can be integrated, and then the equations for \( V_{21}, W_{2i}, m^2, \kappa, \nu \) and \( \eta \). This is immediate, except for \( V_{21} \) and \( m^2 \), because they are given by integro-differential equations. (They become ordinary differential equations to order \( \lambda \).)

We will work in \( d = 3 \). First we find the propagators to order zero in \( \lambda \). We need initial conditions for \( a \) and \( \beta \) in the density matrix. We choose

\[
a(k, 0) = \omega_0(k, 0) = k, \tag{69}
\]

and

\[
\beta(k, 0) = \beta_0(k). \tag{70}
\]

For the moment \( \beta_0 \) can remain unspecified. From Eq. (39), with \( \eta = 0 \)

\[
a(k, s) = k.\tag{71}
\]

Note that the evolution of the product \( a\beta \), that appears in the argument of the function \( f \) in Eq. (A17) for \( G \), is given by

\[
\left( \frac{\partial}{\partial s} + k \frac{\partial}{\partial k} \right) a\beta = 0 \tag{72}
\]

which has the immediate solution

\[
a\beta(k, s) = a\beta(ke^{-s}, 0).\tag{73}
\]

It is convenient to use the variable \( z = e^s \) instead of \( s \) in the propagators. Using (69) and (70) in (73), and then replacing in (47), setting \( k = 1 \)

\[
G(t, t', z) = 2 \left[ 1 + 2f\left( z^{-1}\beta_0(z^{-1}) \right) \right] \cos(t - t'), \tag{74}
\]

from which Eq. (51) gives

\[
\{GG\}(z, T) = \left[ 1 - \frac{1}{2T} \sin(2T) \right] \left[ 1 + 2f\left( z^{-1}\beta_0(z^{-1}) \right) \right]. \tag{75}
\]

The dependence of \( \{GG\} \) on \( T \) comes through the first term in Eq. (75). In Fig. 1 we plot \( 1 - (2T)^{-1} \sin(2T) \) as a function of \( x = 1/T \). Observe that for \( T \to \infty \) the function tends to 1.

Then, from Eqs. (48) and (49), to order \( \lambda^2 \), we get

\[
V_{41}(s) = V_{43}(s) = V_0 + 18V_0^2 \int_0^s ds' \{GG\}(e^{s'}, T), \tag{76}
\]

where

\[
V_0 = \left. -\frac{\Omega_d}{(2\pi)^d} \frac{\lambda}{48} \right|_{d=3} = -\frac{1}{2\pi^2} \frac{\lambda}{48}. \tag{77}
\]
FIG. 1: The graph of $1 - (2T)^{-1} \sin(2T)$ as a function of $x = T^{-1}$. The term $\{GG\}$ depends on $T$ through this function and has a definite limit when $T \to \infty$.

To solve the equations for the $v_{6i}$ couplings to order $\lambda^2$, in the second members of Eqs. (51)-(53) we can use the initial values of $V_{41}$ and $V_{43}$. Using (43)

$$v_{61}(k; t, t', s) = \frac{3}{k} V_0^2 \Theta(1^+ e^s - k) \Theta(k - 1^+),$$

(78)

$$v_{62}(k; t, t', s) = \frac{9}{2k} V_0^2 \Theta\left(k(t, t'), \frac{e^s}{k}\right) \Theta(1^+ e^s - k) \Theta(k - 1^+),$$

(79)

$$v_{63}^{(1)}(k; t, t', s) = \frac{9}{k} V_0^2 \Theta(1^+ e^s - k) \Theta(k - 1^+).$$

(80)

[See Eq. (E5) for a definition of $\Theta(k - 1^+)$.] The notation $\left(k(t, t')\right)$ means $(kt, kt')$, and will be used frequently.

Next the equations for the $W_{4i}$ are integrated:

$$W_{41}(k; t, t', s) = 18 \delta_{k,0} V_0^2 I_1[GG](t, t', e^s)$$

$$+ 9 V_0^2 \left(2 I_2[GG](t, t', e^s) + I_2[GG](k, t, t', e^s)\right) - \left[V_{41}(s) - V_0\right] \delta(t - t'),$$

(81)

$$W_{42}(k; t, t', s) = 9 \delta_{k,0} V_0^2 I_1[GG](t, t', e^s) - 2 I_2[GG](t, t', e^s)_{\Pi}$$

$$+ 18 V_0^2 \left[I_2[GG](k, t, t', e^s) - 2 I_2[GG](k, t, t', e^s)_{\Pi}\right],$$

(82)
\[ W_{43}(k; t, t', s) = 18 \delta_{k,0} V_0^2 \mathcal{I}_1[G \mathcal{G}](t, t', e^s) + 18 V_0^2 \left[ \mathcal{I}_2[\mathcal{G}, \mathcal{G}](k; t, t', e^s) + \mathcal{I}_2[\mathcal{G}, \mathcal{G}](k; t, t', e^s) \right] - \left[ V_{43}(s) - V_0 \right] \delta(t - t'), \] (83)  

where 

\[ \mathcal{I}_1[G_1G_2](t, t', z) = z \int_0^s ds' e^{-s'} G_1 G_2 \left( z e^{-s'}(t, t'), e^{s'} \right), \] (84)  

and  

\[ \mathcal{I}_2[G_1, G_2](k; t, t', z) = z \int_0^s ds' e^{-s'} G_1 \left( z e^{-s'}(t, t'), e^{s'} \right) \times \frac{\int_{S} d\Omega_3}{\Omega_3} \frac{G_2 \left( |z^{-1} e^{s'} k + \hat{\Omega}| z e^{-s'}(t, t'), |z^{-1} e^{s'} k + \hat{\Omega}|^{-1} e^{s'} \right)}{|z^{-1} e^{s'} k + \hat{\Omega}|}. \] (85)  

Here, the angular integration is over the region \( S \) defined by  

\[ 1 < |z^{-1} e^{s'} k + \hat{\Omega}| < e^{s'}, \] (86)  

so \( \mathcal{I}_2 = 0 \) for \( k = 0 \) (though its limit when \( k \to 0^+ \) is not necessarily 0). As before, \( z \) is identified with \( e^s \).

The function \( \mathcal{I}_2 \) can be expanded as  

\[ \mathcal{I}_2[G_1, G_2] = \Theta(1 + k - z) \mathcal{I}_{2<}[G_1, G_2] + \Theta(z - 1 - k) \mathcal{I}_{2>}[G_1, G_2], \] (87)  

where  

\[ \mathcal{I}_{2<}[G_1, G_2] = \frac{1}{2k} \int_1^z dy \int_y^z dx \ G_1 \left( y(t, t'), \frac{z}{y} \right) G_2 \left( x(t, t'), \frac{z}{x} \right) \] (88)  

and  

\[ \mathcal{I}_{2>}[G_1, G_2] = \frac{1}{2k} \left( \int_1^{z-k} dy \int_y^{y+k} dx + \int_{z-k}^z dy \int_y^z dx \right) G_1 \left( y(t, t'), \frac{z}{y} \right) G_2 \left( x(t, t'), \frac{z}{x} \right). \] (89)  

\( \mathcal{I}_2 \) is continuous and have continuous first partial derivatives along the line \( z = k + 1 \).

We end this section taking the limit of \( T \to \infty \), and zero initial temperature, i.e. \( \beta_0^{-1} = 0 \). We focus in the equations for \( \lambda \) and \( m^2 \). In this limit, Eq. (75) gives  

\[ \{G \mathcal{G}\} = 1 \quad (T \to \infty, \beta_0^{-1} = 0). \] (90)  

With initial conditions  

\[ V_{41}(0) = V_{43}(0) = -\frac{\Omega_d}{(2\pi)^d} \frac{\lambda}{48}; \] (91)
the equations for $V_{41}$ and $V_{43}$ are

$$
\left( \frac{\partial}{\partial s} + d - 3 \right) V_{41}(s) = 18V_{41}(s)^2, \quad (92)
$$

$$
V_{43}(s) = V_{41}(s). \quad (93)
$$

For $s > 0$ we can define

$$
\lambda(s) = -\frac{48(2\pi)^d}{\Omega_d}V_{41}(s), \quad (94)
$$

which will satisfy the following differential equation

$$
\left( \frac{\partial}{\partial s} + d - 3 \right) \lambda = -\frac{3\Omega_d}{8(2\pi)^d} \lambda^2. \quad (95)
$$

On the other hand, to order $\lambda$, the equation (65) for $m^2$ is

$$
\left( \frac{\partial}{\partial s} - 2 \right) m^2 = \frac{\Omega_d}{4(2\pi)^d} \lambda. \quad (96)
$$

In computing $\mathcal{P}f_1$ to order $\lambda$ in Eq. (65), it has been used that

$$
\mathcal{P}\mathcal{G} = 2 \quad (T \to \infty, \quad \beta_0^{-1} = 0). \quad (97)
$$

Equations (95) and (97) are the CTP equivalents of the textbook equations derived using the RG group defined for an Euclidean action [41]. Observe that we are integrating out momentum shells of infinitesimal width in the spatial directions but leaving the time direction unrestricted. Thus the numerical coefficients in Eqs. (95) and (97) may be different from those obtained when a spherical shell in Euclidean momentum space is integrated out [33].

**VI. THE DAMPING CONSTANT $\kappa$ AND THE NOISE KERNEL $\nu$**

**A. The damping constant**

To compute $\kappa$ we need the function $f_2$, Eq. (61), and then $\mathcal{Q}f_2$. Not all the terms in $W_{41}$ and $W_{42}$, given in Eqs. (81) and (82), contribute to $\kappa$. The terms proportional to $\delta(t - t')$, for which the application of $\mathcal{Q}$ gives zero, can be discarded, as well as the terms proportional to the Kronecker deltas, because they do not contribute to the angular integrals. If Eq. (66) is rewritten retaining only the relevant terms it yields

$$
(D - 1)\kappa = -18V_0^2 \mathcal{Q}f_2^*, \quad (98)
$$
where

\[
    f^*_2(k; t, t', z) = \int \frac{d\Omega_3}{\Omega_3} \left[ 2 \mathcal{I}_2[G, G](Q_{k\Omega}; t, t', z) + \mathcal{I}_2[G, G](Q_{k\Omega}; t, t', z) \right] G(t, t', z) \\
    + \int \frac{d\Omega_3}{\Omega_3} 2 \left[ \mathcal{I}_2[G, G](Q_{k\Omega}; t, t', z) - \mathcal{I}_2[G, G](Q_{k\Omega}; t, t') \right] G(t, t').
\]

(99)

In the last line, it has been used that for the function \( G \) given in the Eq. (46) is

\[
    I_2[G, G](Q_{k\Omega}; t, t', e^s) \frac{G(t, t')}{tt'} = 1
\]

(100)

Using the formal solution in Eq. (43)

\[
    \kappa(k, s) = -18V_0^2 \int_0^s ds' e^{s-s'} Q f^*_2(e^{s-s} k, e^{s'}). 
\]

(101)

Due to the general structure of the functions \( \mathcal{I}_2 \), Eq. (87), \( \kappa \) will be given by an expression of the form

\[
    \kappa(k, s) = 18V_0^2 \int_0^s ds' e^{s-s'} \int \frac{d\Omega_3}{\Omega_3} \left[ \Theta \left( |k e^{s-s} + \Omega| + 1 - e^{s'} \right) A_<(|k e^{s-s} + \Omega|, e^{s'}) \\
    + \Theta \left( e^{s'} - |k e^{s-s} + \Omega| - 1 \right) A_>(|k e^{s-s} + \Omega|, e^{s'}) \right], 
\]

(102)

with

\[
    A_<= -Q \left[ 2\mathcal{I}_2[G, G] + \mathcal{I}_2[G, G] \right] G + 2 \left( \mathcal{I}_2[G, G] - \mathcal{I}_2[G, G] \right) G, 
\]

(103)

and with a similar expression for \( A_> \). Observe that both \( A_< \) and \( A_> \) depend on \( T \) through the application of \( Q \), and that this dependence will be inherited by \( \kappa \).

Changing variables, Eq. (102) can be rewritten as

\[
    \kappa(k, z) = 9V_0^2 \frac{z}{k} \int_1^z dy \int_{\frac{y-k}{y}}^{\frac{y+k}{y}} du \frac{u}{y^3} \left[ \Theta (u+1-y) A_<(u, y) \\
    + \Theta (y-u-1) A_>(u, y) \right]. 
\]

(104)

In doing this integral, there are three different regions of the plane \( kz \), restricted to \( 0 \leq k \leq 1 \) and \( 1 \leq z \), that must be analyzed separately: i) \( z \leq 2 - k \), ii) \( 2 - k < z \leq 2 + k \), and iii) \( 2 + k < z \). In each region \( \kappa \) is given by the following expressions:

Region i:

\[
    \kappa(k, z) = 9V_0^2 \frac{z}{k} \int_1^z dy \int_{\frac{y-k}{y}}^{\frac{y+k}{y}} du \frac{u}{y^3} A_<(u, y). 
\]

(105)
Region ii:

\[
\kappa(k, z) = 9V_0^2 \frac{z^2}{k} \left( \int_1^{\frac{2z}{z+k}} dy \int_1^{1+\frac{yk}{z}} du + \int_1^{\frac{2z}{z+k}} dy \int_1^{1+\frac{yk}{z}} du \right) \frac{u}{y^3} A_<(u, y)
\]

\[+ 9V_0^2 \frac{z^2}{k} \left( \int_1^{\frac{2z}{z+k}} dy \int_1^{\frac{y-1}{1-\frac{yk}{z}}} du \frac{u}{y^3} A_<(u, y) \right). \tag{106}\]

Region iii:

\[
\kappa(k, z) = 9V_0^2 \frac{z^2}{k} \left( \int_1^{\frac{2z}{z+k}} dy \int_1^{1+\frac{yk}{z}} du + \int_1^{\frac{2z}{z+k}} dy \int_1^{1+\frac{yk}{z}} du \right) \frac{u}{y^3} A_<(u, y)
\]

\[+ 9V_0^2 \frac{z^2}{k} \left( \int_1^{\frac{2z}{z+k}} dy \int_1^{\frac{y-1}{1-\frac{yk}{z}}} du \frac{u}{y^3} A_>(u, y) \right). \tag{107}\]

Note that for \( k = 0 \) the angular integral in Eq. (102) is trivial

\[
\kappa(0, z) = 18V_0^2 z \int_1^{z} \frac{dy}{y^2} A_<(1, y) \Theta(2 - z)
\]

\[+ 18V_0^2 z \left[ \int_1^{\frac{2}{2+k}} dy \frac{A_<(1, y)}{y^2} + \int_2^{\frac{2}{2+k}} dy \frac{A_>(1, y)}{y^2} \right] \Theta(z - 2), \tag{108}\]

which is also the limit when \( k \to 0 \) of the expressions (105) and (107) for \( z \leq 2 \) and \( z > 2 \), respectively.

For \( z \gg 1 \) \( \kappa \) becomes independent of \( k \). If \( A_>(1, y) \) grows slower than \( y \), for \( z \gg 1 \) it results

\[
\kappa(z) \sim 18V_0^2 z \left[ \int_1^{\frac{2}{2+k}} dy \frac{A_<(1, y)}{y^2} + \int_2^{\frac{2}{2+k}} dy \frac{A_>(1, y)}{y^2} \right] = \kappa_0 \ z. \tag{109}\]

This expression corresponds to what can be expected from dimensional arguments. The constant \( \kappa_0 \) can be thought as the \( \kappa \) accumulated during the first stages of evolution. The factor \( z \) comes solely from the rescaling. If \( A_>(1, y) \) grows as \( y \) or faster for \( y \to \infty \), to leading order, \( \kappa \) will be given by the last term in Eq. (108)

\[
\kappa(z) \sim 18V_0^2 z \int_2^{\frac{2}{2+k}} dy \frac{A_>(1, y)}{y^2}. \tag{110}\]

Thus, in this case \( \kappa \) will acquire an anomalous dimension.

B. The noise kernel \( \nu \)

The noise kernel \( \nu \) is obtained following the same steps used in the calculation of \( \kappa \). First, Eq. (67) is rewritten keeping in \( f_3 \) only the relevant terms. Term proportional to \( \delta(t - t') \) in the solution (83) for \( W_{43} \) drops out because \( W_{43} \) appears multiplied by \( G(t, t') \), which is zero
in \( t = t' \). The terms proportional to \( \delta_{k,0} \) also drop out because they do not contribute to the angular average. The equation for \( \nu \) reduces to

\[
(D - 2) \nu = 36V_0^2 \mathcal{P} f_3^*,
\]

where

\[
f_3^*(k; t, t', z) = 2 \int \frac{d\Omega_3}{\Omega_3} \left[ \mathcal{I}_2[G, G](Q_{k\Omega}; t, t', z) + \mathcal{I}_2[G, G](Q_{k\Omega}; t, t', z) \right] G(t, t', z) + \int \frac{d\Omega_3}{\Omega_3} \left[ \mathcal{I}_2[G, G](Q_{k\Omega}; t, t', z) - \mathcal{I}_2[G, G](Q_{k\Omega}; t, t') \right] G(t, t').
\]

Defining

\[
B_\leq = \mathcal{P} \left[ 2(\mathcal{I}_{2\leq} [G, G] + \mathcal{I}_{2\leq} [G, G]) G + (\mathcal{I}_{2\leq} [G, G] - \mathcal{I}_{2\leq} [G, G]) G \right],
\]

and analogously \( B_\geq \), it results

\[
\nu(k, z) = 18V_0^2 \frac{z^3}{k} \int_1^z dy \int_{1-y/k}^{1+y/k} du \frac{u}{y^4} \left[ \Theta(u + 1 - y) B_\leq(u, y) + \Theta(y - u - 1) B_\geq(u, y) \right].
\]

Again, note that the functions \( B \) will depend on \( T \) through the application of \( \mathcal{P} \), an so will \( \nu \). Expressions analogous to Eqs. (105)-(107) hold for \( \nu \). For \( k = 0 \)

\[
\nu(0, z) = 36V_0^2 z^2 \int_1^z \frac{dy}{y^3} B_\leq(1, y) \Theta(2 - z) + 36V_0^2 z^2 \left[ \int_1^2 \frac{dy}{y^3} B_\leq(1, y) + \int_2^\infty \frac{dy}{y^3} B_\geq(1, y) \right] \Theta(z - 2).
\]

For \( z \gg 1 \), \( \nu \) becomes independent of \( k \). If \( B_\geq(1, y) \) grows slower than \( y^2 \), the behavior of \( \nu \) is controlled by the rescaling

\[
\nu(z) \sim 36V_0^2 z^2 \left[ \int_1^2 \frac{dy}{y^3} B_\leq(1, y) + \int_2^\infty \frac{dy}{y^3} B_\geq(1, y) \right].
\]

Otherwise \( \nu \) acquires an anomalous dimension

\[
\nu(z) \sim 36V_0^2 z^2 \int_2^z \frac{dy}{y^3} B_\geq(1, y).
\]
C. $\kappa$ and $\nu$ for a specific choice of $\beta_0$

We compute $\kappa$ and $\nu$ choosing $\beta_0^{-1}(k) = 0$, namely the case of zero initial temperature. The function $1 + 2f$ in Eq. (71), with $f$ defined in Eq. (B5), is replaced by 1. We compute $\kappa$ and $\nu$ only for $k = 0$, since the dependence on $k$ for $z \gg 1$ is weak. The detailed expressions are given in Appendix F.

The actual calculation shows that $A(x, y)$ and $B(x, y)$ both grow as $y \to \infty$. This implies that, for $z \gg 1$, $\kappa$ will develop an anomalous dimension, while $\nu$ will go simply as $z^2$. Making explicit the dependence on $T$, for $z \gg 1$

$$\kappa(z, T) \sim \frac{9V_0^2}{4T} \log z \left[ 7 - 2T^2 - 8 \cos T + \cos(2T) \right]$$

and

$$\nu(k, T) \sim \frac{9V_0^2}{8T} \left\{ 135 + 4 \left[ \gamma_E - 34 \ln 2 - 7 \ln 3 + 7 \text{Ci}(3T) \right] - 3 \cos(4T) 
+ 4 \ln T - 16 \left[ 8\pi + 3T - 16 \text{Si}(2T) \right] \sin T + 4 \left[ 7\pi + 6T - 14 \text{Si}(4T) \right] \sin(2T) 
- 12 \left[ 19 - 4T^2 \right] \text{Ci}(T) + 8 \left[ 7 - 6T^2 \right] \text{Ci}(2T) + 4 \left[ 1 - 4T^2 \right] \text{Ci}(4T) - 8 \cos(3T) 
- 24 \left[ 7 - 8 \text{Ci}(2T) \right] \cos T + 4 \left[ 11 + 2 \text{Ci}(T) - 2 \text{Ci}(2T) - 14 \text{Ci}(4T) \right] \cos(2T) 
+ 4T \left[ 18\pi - 54 \text{Si}(T) + 28 \text{Si}(2T) - 6 \text{Si}(3T) - 2 \text{Si}(4T) + \sin(4T) \right] \right\},$$

(118)

where $\text{Si}$ and $\text{Ci}$ are the sin and the cos integral functions. Observe that the damping constant $\kappa$ is not definite positive. This suggests that the underlying mechanism could be similar to Landau damping [42].

VII. FINAL REMARKS

In this paper we have investigated the nonequilibrium dynamics of the low frequency, long wavelength modes of a self-interacting real scalar field. We have computed the influence functional encoding the back-reaction on these modes of the higher frequency, shorter wavelength sector. We have obtained the coefficients of the influence functional by solving the RG equations which describe the change in the influence functional induced by the progressive averaging over momentum shells.

The main finding of this paper is that the influence functional for the low frequency modes contains terms associated with damping and noise. It is most important to stress that we have not put these terms by hand; contrariwise, we have assumed that they are absent at high energies. Damping and noise are forced upon us by the RG flow itself.

We have been able to retrieve these terms because we have gone beyond the adiabatic approximation for the environment (high frequency) modes. A crucial step in this direction is the recognition of the role of the parameter $T$ in nonequilibrium evolution. Because time-integration
is restricted to the lapse from 0 to $T$, energy fluctuations, and thus particle creation from the vacuum in the environment sector, are allowed even in simple approximations to the dynamics. Although the finite temperature RG in the real time formulation is well known, and also the running of the effective potential in a CTP framework, to the best of our knowledge this is the first instance were the RG is used to compute an intrinsically nonequilibrium feature in relativistic field theory.

The most limiting approximation we have made is the use of fixed propagators in internal lines, disregarding the changes in the propagators caused (mainly, but not only) by the generation of mass, dissipation and noise terms through the RG flow. This approximation forces us to restrict our analysis to very short times (a few inverse cutoffs). A fully self-consistent RG should overcome this shortcoming. Also we have only considered a very simple -though nonequilibrium-initial condition for the field. A more flexible approach regarding initial conditions is required for most interesting applications.

The methods advanced in this paper are relevant to essentially all the applications of nonequilibrium quantum field theory, from high energy ones like thermalization in relativistic heavy ion collisions and reheating after inflation to low energy applications such as the dynamics of glasses and Bose-Einstein condensates. It is clear that we have the bare outlines of a framework, but we look forward to see this framework develop and fructify in manifold ways.

Acknowledgments

We acknowledge Joan Sola and Enric Verdaguer for their kind hospitality at Barcelona, and Enric Verdaguer for discussions. This work was supported by Universidad de Buenos Aires, CONICET and ANPCYT.

APPENDIX A: INTEGRATING OUT MODES IN A SHELL

In the problem at hand, the relevant system and the environment are sectors of the same scalar field, being $\phi>$ and $\varphi>$ the variables associated with the short scales. The elimination of these modes is the fundamental step in the RG transformation, and follows the lines sketched in Sec. II A.

The CTP generating functional is

$$Z[J,j] = \int\mathcal{D}\varphi\mathcal{D}\phi \exp i \left\{ S_{\text{CTP}}[\phi, \varphi] + \int_0^T dt \int_\Lambda dk \left[ J(k,t)\varphi(k,t) + j(k,t)\phi(k,t) \right] \right\} \rho[\phi_0, \varphi_0]. \quad (A1)$$

Here the integral is over histories with $\phi(k,T) = 0$, and $\phi_0$ and $\varphi_0$ stand for the fields at $t = 0$. We retain explicitly the dimensional parameter $T$ at which both histories on the CTP integral coincide. The field configurations contain modes up to momentum $\Lambda$: $\phi(k,T)$ and $\varphi(k,t)$ are defined in the region $|k| = k \leq \Lambda$, and so are the currents $J$ and $j$. If necessary, fields and currents can be defined to be zero outside this domain. The subscript in the momentum integral indicates that the integration domain is bounded by $k = \Lambda$. 
The density matrix is known at \( t = 0 \). For simplicity we assume that
\[
\rho[\phi, \varphi] = \rho[\phi_>, \varphi_<] \otimes \rho[\phi_>, \varphi_>],
\]
with the same functional \( \rho \) everywhere [cf. Eq. (25)].

Only modes with wave numbers within an infinitesimal shell will be integrated out. Let \( 0 < \delta s \ll 1 \) the infinitesimal parameter of the transformation, and define
\[
b = 1 - \delta s.
\]
The momentum domain \( k \leq \Lambda \) is divided in two regions: the shell
\[
b\Lambda < k \leq \Lambda,
\]
and its interior:
\[
k \leq b\Lambda.
\]
Accordingly, the fields are split in two parts, one which contains the modes within the shell, indicated by \( \phi_> \), and another part containing the modes with \( k \leq b\Lambda \), indicated by \( \phi_< \), i.e.
\[
\phi = \phi_< + \phi_>,
\]
where
\[
\phi_<(k,t) = \phi(k,t) \Theta(b\Lambda - k),
\]
\[
\phi_>(k,t) = \phi(k,t) \left[ \Theta(k - b\Lambda) - \Theta(k - \Lambda) \right],
\]
and so for \( \varphi \). Here \( \Theta \) is the unit step function. Equation (6) reads
\[
S_{\text{CTP}}[\phi, \varphi] = S_{\text{CTP}}[\phi_<, \varphi_<] + \Delta S[\phi_<, \varphi_<, \phi_>, \varphi_>].
\]

After integrating out the modes within the shell, the effective action for the surviving modes is
\[
S_{\text{CTP}}<[\phi_<, \varphi_<] = S_{\text{CTP}}[\phi_<, \varphi_<] + \delta S[\phi_<, \varphi_<],
\]
where
\[
e^{i\delta S[\phi_<, \varphi_<]} = \int \mathcal{D}\phi_> \mathcal{D}\phi_> \exp \left( i\Delta S[\phi_<, \varphi_<, \phi_>, \varphi_> \right) \rho[\phi_>(k,0), \varphi_>(k,0)].
\]

This is essentially Eq. (10). The task here is to compute \( \delta S \) to order \( \delta s \).
1. Computing \( \delta S \) to order \( \delta s \), perturbative approach and Feynman rules

We will compute \( \delta S \) perturbatively. To do this, we will separate in \( \Delta S \), Eq. (A8), a free CTP action \( S_0 \), which will define the propagators to be used in the Feynman diagrams,

\[
\Delta S[\phi_-, \varphi_-, \phi_+, \varphi_+] = S_0[\phi_+, \varphi_+] + S_I[\phi_-, \varphi_-, \phi_+, \varphi_+].
\] (A11)

Momentarily and for sake of brevity, we will write \( \phi_+ \) meaning the pair \( \{ \phi_+, \varphi_+ \} \), and \( \phi_0 \) instead of \( \phi_+(k, 0) \) (analogous meaning for \( \phi_- \)). Equation (A10) reads

\[
e^{i\delta S[\phi_-]} = \int D\phi_+ \exp i \left( S_0[\phi_+] + S_I[\phi_-, \phi_+] \right) \rho[\phi_0].
\] (A12)

This is identical to the usual expression for a vacuum-vacuum amplitude. Hence,

\[
\delta S[\phi_-] = -i \log \left( \int D\phi_+ e^{iS_0[\phi_+]} \rho[\phi_0] \right) - i \sum_{n=1}^{\infty} \frac{i^n}{n!} \langle S^n_I \rangle_c[\phi_-]
\] (A13)

where

\[
\langle S^n_I \rangle_c[\phi_-] = \left. \frac{\int D\phi_+ e^{iS_0[\phi_+]} \rho[\phi_0] S_I[\phi_-, \phi_+]^n}{\int D\phi_+ e^{iS_0[\phi_+]} \rho[\phi_0]} \right|_{\text{connected}}
\] (A14)

is the sum of the connected diagrams drawn from \( S^n_I \), with the fields \( \phi_- \) acting as external currents. The first, constant term in (A13) can be omitted. Thus, essentially, we can write (restoring the full notation for \( \phi \) and \( \varphi \))

\[
\delta S[\phi_-, \varphi_-] = \langle S_I \rangle_c[\phi_-, \varphi_-] + \frac{i}{2!} \langle S^2_I \rangle_c[\phi_-, \varphi_-] + \ldots
\] (A15)

Internal lines in the diagrams will correspond to the propagators deduced from \( S_0[\phi, \varphi] \). There are two propagators

\[
\langle \varphi(k, t) \phi(k', t') \rangle_0 = -iG(k, t, t') \delta^d(k + k'),
\] (A16)

\[
\langle \varphi(k, t) \varphi(k', t') \rangle_0 = G(k, t, t') \delta^d(k + k').
\] (A17)

The subscript 0 means that the expectation values are taken with respect to the free action \( S_0 \). In general, the function \( G \) is zero if \( t \leq t' \). The third propagator \( \langle \phi \phi' \rangle \) is always zero. These are two general properties of the CTP formalism.

It is clear that diagrams with more than one loop can be ignored to order \( \delta s \). This is because for each loop there would be one independent momentum integration over a region of volume of order \( \delta s \). Thus, the resulting term would be (at least) proportional to \( (\delta s)^L \), where \( L \) is the number of loops in the diagram. Hence, it has to be \( L \leq 1 \). In fact, diagrams with \( L = 0 \) and \( L = 1 \) are both at least of order \( \delta s \). Owing to the fact that internal lines carry momenta in the infinitesimal momentum shell, when computing \( \delta S \) to order \( \delta s \), it is enough to consider two types of diagrams [2], shown in Fig. 2 and Fig. 3.
FIG. 2: Type 1 diagrams have one loop with momentum $k = \Lambda$. The total external momentum at each vertex is zero.

FIG. 3: Type 2 diagrams are tree diagrams with no external momentum entering at the intermediate vertices.
Diagrams of the first type are shown in Fig. 2. They are one-loop diagrams. External lines attached to each vertex must have total momentum equal to zero: for each vertex with external lines carrying momenta $q_1, q_2, \ldots$, there will be a Kronecker delta

$$\delta_{Q;0}.$$  \hspace{1cm} (A18)

where

$$Q = q_1 + q_2 + \ldots$$ \hspace{1cm} (A19)

Thus, the same momentum $k = \Lambda \hat{\Omega}$, with $|\hat{\Omega}| = 1$, traverses the loop at every point. The modulus of the momentum in the loop is fixed, but not its direction $\hat{\Omega}$. Hence, instead of an volume integration over $k$, there will be just angular average over $\hat{\Omega}$ and a multiplicative factor accounting for the volume of the shell, that is

$$\Omega_d \Lambda^d \delta s \int \frac{d\Omega_d}{\Omega_d}.$$ \hspace{1cm} (A20)

The fact that external lines at each vertex must carry total momentum equal to zero can be understood graphically. Consider for example the one-loop diagram in Fig. 4.

The analytic expression for this diagram will include an integration of the form

$$\int d^d k \ G_1(k) G_1(|k + Q|) f(q_1, q_2, q_3, q_4, k, -k - Q),$$ \hspace{1cm} (A21)

where $Q = q_1 + q_2$ is the external momentum entering at the left vertex, and $G_1$ and $G_2$ are propagators for modes of the field in the momentum shell $b\Lambda < k \leq \Lambda$. Then, two conditions will determine the integration volume, namely

$$b\Lambda < k \leq \Lambda$$ \hspace{1cm} (A22)

and

$$b\Lambda < |k + Q| \leq \Lambda.$$ \hspace{1cm} (A23)

These conditions are depicted in Fig. 5. For $\delta s \ll 1$, the volume in which the two conditions
FIG. 5: In $d = 2$, the integration region for the diagram in Fig. 4 is the intersection of the two shells. Its volume is of order $\delta s^2$ unless $Q \sim \delta s$.

are satisfied is of order $\delta s^2$, except when $Q$ is itself of order $\delta s$ or superior. In this case the two shells coincide with a precision at least of order $\delta s$, and the integration volume is of order $\delta s$. In the limit in which $\delta s \to 0$, the diagram will be null unless $Q = 0$. Then, if $k = k \hat{\Omega}$, we can rewrite Eq. (A21)

$$\delta_{Q;0} \Lambda^d \Omega_d \delta_s$$

Diagrams with more than two vertices can be analyzed in similar terms.

There are two other rules for the diagrams of the first type. For each internal line connecting fields at times $t$ and $t'$, there will be a factor $\mathcal{G}(\Lambda, t, t')$ for contractions of the type $\langle \varphi \varphi \rangle$, or $-i \mathcal{G}(\Lambda, t, t')$ in the case of $\langle \varphi \phi \rangle$. For each internal line there will be two time integrals between 0 and $T$, associated with the times of the two fields of type $>$ joined by the line.

Diagrams of the second type, shown in Fig. 3, consist of a single chain of propagators, all carrying the same momentum $k$, with $|k| = \Lambda$. For each internal line connecting fields at times $t$ and $t'$, there will be a factor $\mathcal{G}(\Lambda, t, t')$ or $\mathcal{G}(\Lambda, t, t')$. For each intermediate vertex (i.e., not at the extremes of the chain) with external lines carrying total momentum equal to $q$, there will be a Kronecker delta $\delta_{q;0}$ (the argument is essentially the same that was given above for diagrams of the first type). Finally, if the total external momentum entering to the diagram at one of its extremes is $k$, there will be a factor $\Lambda \delta(k - \Lambda)$. As before, there will be two time integrals for each internal line.

Note that in both types of diagrams, two internal lines at most are attached to each vertex.
APPENDIX B: PROPAGATORS

The propagators obtained for the free action $S_0$ of Eq. (18), with initial conditions given by Eq. (25), are defined as in Eqs. (A16) and (A17), with

$$G(k, t, t') = \frac{2}{\omega} \sin[\omega(t - t')] e^{-\kappa(t-t')} \Theta(t - t'), \quad (B1)$$

and

$$G(k, t, t') =$$

$$\frac{2}{a} e^{-\kappa(t+t')} \left\{ \left[ 1 + 2 f(a \beta) \right] - \frac{a\nu}{2\kappa\omega_0^2} \right\} \left\{ \left[ \frac{\omega_0^2}{\omega^2} \cos[\omega(t - t')] - \frac{\kappa^2}{\omega^2} \cos[\omega(t + t')] + \frac{\kappa}{\omega} \sin[\omega(t + t')] \right] \right\}$$

$$+ \frac{\nu}{\kappa\omega_0^2} \left( \left[ \cos[\omega(t - t')] + \frac{\kappa}{\omega} \sin[\omega(t - t')] \right] e^{-\kappa(t-t')} \Theta(t - t') + (t \leftrightarrow t') \right)$$

$$+ D_2^2 \left( \frac{a^2 - \omega_0^2}{\omega^2} \right) \left[ 1 + 2 f(a \beta) \right] \left\{ \cos[\omega(t - t')] - \cos[\omega(t + t')] \right\} e^{-\kappa(t+t')} \right\}$$

$$+ \frac{\nu}{\kappa\omega_0^2} \left( \left[ \cos[\omega(t - t')] + \frac{\kappa}{\omega} \sin[\omega(t - t')] \right] e^{-\kappa(t-t')} \Theta(t - t') + (t \leftrightarrow t') \right)$$

$$+ D_2^2 \left( \frac{a^2 - \omega_0^2}{\omega^2} \right) \left[ 1 + 2 f(a \beta) \right] \left\{ \cos[\omega(t - t')] - \cos[\omega(t + t')] \right\} e^{-\kappa(t+t')} \right\}$$

$$+ \frac{\nu}{\kappa\omega_0^2} \left( \left[ \cos[\omega(t - t')] + \frac{\kappa}{\omega} \sin[\omega(t - t')] \right] e^{-\kappa(t-t')} \Theta(t - t') + (t \leftrightarrow t') \right)$$

$$+ D_2^2 \left( \frac{a^2 - \omega_0^2}{\omega^2} \right) \left[ 1 + 2 f(a \beta) \right] \left\{ \cos[\omega(t - t')] - \cos[\omega(t + t')] \right\} e^{-\kappa(t+t')} \right\}$$

$$+ \frac{\nu}{\kappa\omega_0^2} \left( \left[ \cos[\omega(t - t')] + \frac{\kappa}{\omega} \sin[\omega(t - t')] \right] e^{-\kappa(t-t')} \Theta(t - t') + (t \leftrightarrow t') \right)$$

$$+ D_2^2 \left( \frac{a^2 - \omega_0^2}{\omega^2} \right) \left[ 1 + 2 f(a \beta) \right] \left\{ \cos[\omega(t - t')] - \cos[\omega(t + t')] \right\} e^{-\kappa(t+t')} \right\}$$

with

$$\omega_0^2 = k^2 + m^2, \quad (B3)$$

$$\omega^2 = \omega_0^2 - \kappa^2, \quad (B4)$$

and

$$f(x) = (e^x - 1)^{-1}. \quad (B5)$$

APPENDIX C: OBTAINING $\delta S$

Here we will sketch how to compute the actual values of $\eta$, $\delta \kappa$, etc. from Eq. (A15). We will follow the procedure presented in Appendix A. To illustrate the method, we will give some examples of diagram calculations, and show how to extract from $\delta S$ the corrections to the parameters in the free action and the value of $\eta$.

According to the definitions (A8) and (17), and due to the quadratic nature of $S_0$, it results

$$\Delta S[\phi_-, \varphi_-, \phi_+, \varphi_+] = S_0[\phi_+, \varphi_+] + S_{int}[\phi_+ + \phi_-, \varphi_+ + \varphi_-] - S_{int}[\phi_-, \varphi_-]. \quad (C1)$$

Then, from definition (A11), it is

$$S_{I}[\phi_-, \varphi_-, \phi_+, \varphi_+] = S_{int}[\phi_+ + \phi_-, \varphi_+ + \varphi_-] - S_{int}[\phi_-, \varphi_-]. \quad (C2)$$

We will take as an example the term $v_{41} \phi \varphi^3$. Writing $\phi = \phi_+ + \phi_-$ and $\varphi = \varphi_+ + \varphi_-$, it yields

$$\phi_1 \varphi_2 \varphi_3 \varphi_4 = \phi_1 \varphi_2 \varphi_3 \varphi_4 + 3 \phi_1 \varphi_2 \varphi_3 \varphi_4 + 3 \phi_1 \varphi_2 \varphi_3 \varphi_4 + 3 \phi_1 \varphi_2 \varphi_3 \varphi_4 + 3 \phi_1 \varphi_2 \varphi_3 \varphi_4 + 3 \phi_1 \varphi_2 \varphi_3 \varphi_4 + 3 \phi_1 \varphi_2 \varphi_3 \varphi_4 + 3 \phi_1 \varphi_2 \varphi_3 \varphi_4 + 3 \phi_1 \varphi_2 \varphi_3 \varphi_4 + 3 \phi_1 \varphi_2 \varphi_3 \varphi_4 + 3 \phi_1 \varphi_2 \varphi_3 \varphi_4. \quad (C3)$$
Here $\phi_1<$ stands for $\phi_<(k_1,t_1)$, and so on. The fact that the function $v_{41}$ is symmetric with respect to the variables corresponding to the $\varphi$ fields accounts for the factors 3. To write $S_I$, the last term in Eq. \((C3)\), which corresponds to the term $S_{\text{int}}[\phi_>,\varphi_<]$ in Eq. \((C2)\), must be taken apart. Moreover, since, as it was said before, at most two internal lines can be attached to any vertex, we can discard the terms in Eq. \((C3)\) with more than two fields of the type $>$. Hence, corresponding to $\phi\varphi^3$ in $S_{\text{int}}$, $S_I$ will display the following terms

\[
\Omega_d^{-1} \left( \prod_{i=1}^{4} \int_{0}^{T} dt_i \int_{\Lambda} d^d k_i \right) \delta^d (k_1 + \cdots + k_4) \left( 3 \phi_1>\varphi_2>\varphi_3<\varphi_4< + 3 \phi_1<\varphi_2<\varphi_3>\varphi_4> + \phi_1>\varphi_2<\varphi_3<\varphi_4< + 3 \phi_1<\varphi_2<\varphi_3>\varphi_4> \right) v_{41}(\{k\}; \{t\}).
\]

\((C4)\)

Using solely these terms, we can construct two diagrams with one vertex (shown in Fig. 6) and which correspond to the term $\langle S_I \rangle$ in Eq. \((A15)\), and four diagrams with two vertices (shown in Fig. 7) corresponding to the term $\langle S^2_I \rangle$ in Eq. \((A15)\).

Since all the propagators will be evaluated at $\Lambda$, we will write $G(t,t')$ for $G(\Lambda,t,t')$, and so for $\mathcal{G}$.

![Diagram 1.I](diag1i.png)

![Diagram 1.II](diag1ii.png)

**Fig. 6.** One-vertex diagrams constructed using the first two terms in Eq. \((C4)\). Continuous (dashed) lines represent $\varphi$ ($\varphi$) fields.
FIG. 7. Two-vertex diagrams constructed using the last two terms in Eq. (C4). Continuous (dashed) lines represent $\varphi$ ($\phi$) fields.

We use continuous lines for $\varphi$ fields and dashed for $\phi$. In this way, internal lines corresponding to propagators $G$ are represented by continuous lines, and corresponding to propagators $-iG$ by half dashed, half continuous lines. We use $\phi_{<i}$ for $\phi(k_i, t_i)$ and so on. Now we analyze each diagram in separated subsections.

1. Diagram 1.I

The first one-vertex diagram in Fig. 6 comes from the term $3\phi_{<\varphi_{<\varphi_{>\varphi_{>}}}}$ in Eq. (C4), after contracting the last two $\varphi_{>}$ fields. There just one way in which this contraction can be made, so there is no additional combinatorial factors. According to the rules enunciated at the end of Sec. A1 there would be a Kronecker delta $\delta_{|k_1+k_2|=0}$, but the condition $|k_1+k_2|=0$ is already granted by the Dirac delta of momentum conservation. Thus, in this case the Kronecker delta can be omitted. The contribution of this diagram to $\delta S$ reads

$$\int_0^T dt_1 \int_0^T dt_2 \int_{k\Lambda} d^4k \phi(k, t_1) \varphi(-k, t_2) v(k; t_1, t_2) \delta s,$$  (C5)
where
\[ v(k; t_1, t_2) = 3 \int_0^T dt_3 \int_0^T dt_4 \, v_{41}(k, -k, \Omega, -\dot{\Omega}; t_1, t_2, t_3, t_4) G(t_3, t_4). \] (C6)

We have written \( v \delta s \) and not directly something like \( \delta v^{(1,1)} \), because this is not yet the actual contribution of Diagram 1.I to \( \delta v_{21} \). We have to remove from \( v \) those terms associated with \( \eta \), \( \delta m^2 \) and \( \delta \kappa \). We must separate in \( v(k; t_1, t_2) \) something proportional to \( \delta(t_1 - t_2) \) and something proportional to \( \partial \delta(t_1 - t_2) \). Two projectors are introduced. Given a function of two times \( v(k; t_1, t_2) \), we define
\[ P_v(k; t_1, t_2) = P_v(k) \delta(t_1 - t_2), \] (C7)
and, if \( v(k; t_1, t_2) = 0 \) for \( t_2 > t_1 \),
\[ Q_v(k; t_1, t_2) = Q_v(k) \left[ 2 \left( \frac{\partial}{\partial t_2} + \delta(t_2) - \delta(0) \right) \delta(t_1 - t_2) \right], \] (C8)
where
\[ P_v(k) = \frac{1}{T} \int_0^T dt_1 \int_0^T dt_2 \, v(k; t_1, t_2), \] (C9)
and
\[ Q_v(k) = \frac{1}{T} \int_0^T dt_1 \int_0^T dt_2 \, v(k; t_1, t_2) (t_2 - t_1). \] (C10)

It is easy to verify that \( P^2 = P \), \( Q^2 = Q \), and that \( QP = PQ = 0 \). This proves that the decomposition
\[ v(k; t_1, t_2) = P_v(k; t_1, t_2) + Q_v(k; t_1, t_2) + \Delta v(k; t_1, t_2) \] (C11)
is unique. Defining
\[ v_0 = P_v, \] (C12)
and
\[ v_1 = Q_v, \] (C13)
and using (C11) it yields
\[ \int_0^T dt_1 \int_0^T dt_2 \int_{b^A} d^4k \, \phi(k, t_1) \varphi(-k, t_2) \, v(k; t_1, t_2) = \]
\[ \int_0^T dt_1 \int_{b^A} d^4k \, \phi(k, t_1) \varphi(-k, t_1) \, v_0(k) + \int_0^T dt_1 \int_0^T dt_2 \int_{b^A} d^4k \, \phi(k, t_1) \varphi(-k, t_1) \, v_1(k) \]
\[ + \int_0^T dt_1 \int_{b^A} d^4k \, \phi(k, t_1) \varphi(-k, t_2) \, \Delta v(k; t_1, t_2). \] (C14)
[Observe that the extra delta factors in (C8) get rid of the boundary terms, and that the factor 2 appears because the deltas are evaluated at the extreme of integration.] In this way, we have extracted from \( v(k; t_1, t_2) \) two quantities: \( v_0(k) \) which acts as a momentum dependent mass squared term, and \( v_1(k) \) which is equivalent to a damping constant. Indeed, comparing with (27), we get

\[
\delta \kappa^{(1,1)} = -v_1(k) \delta s. \tag{C15}
\]

From \( v_0(k) \) we can extract contributions to \( \delta m^2 \) and \( \eta \). Writing

\[
v_0(k) = v_0(0) + k \frac{\partial v_0(0)}{\partial k} + \frac{k^2}{2!} \frac{\partial^2 v_0(0)}{\partial k^2} + \Delta v_0(k), \tag{C16}
\]

matching terms with Eq. (27), we identify

\[
\delta m^{2(1,1)} = -2v_0(0) \delta s \tag{C17}
\]

and

\[
\eta^{(1,1)} = -\frac{1}{2} \frac{\partial^2 v_0(0)}{\partial k^2}. \tag{C18}
\]

After subtracting from \( v(k; t_1, t_2) \) all the terms belonging to the free action, the net contribution from Diagram (1.I) to \( \delta v_{21}(k; t_1, t_2) \) comes from Eqs. (C11) and (C16), and is

\[
\delta v_{21}^{(1,1)}(k; t_1, t_2) = \Delta v(k; t_1, t_2) + \Delta v_0(k) \delta(t_1 - t_2). \tag{C19}
\]

If \( v(k; t_1, t_2) \) were local in time but not constant, that is, if it were \( v(k; t_1) \delta(t_1 - t_2) \), it could be interpreted as a time dependent mass. Our definition in this case would give for \( \delta m^{2(1,1)} \) the time average of \( v(0; t_1) \delta s \). The definition of \( \mathcal{P} \) can be generalized for functions \( g(k; t) \) depending on just one time, writing

\[
\mathcal{P} g(k; t) = \mathcal{P} g(k), \tag{C20}
\]

with

\[
\mathcal{P} g(k) = \frac{1}{T} \int_0^T dt \, g(k; t). \tag{C21}
\]

Something analogous can be said about corrections of the form

\[
i \int_0^T dt_1 \int_0^T dt_2 \int d^4k \, \phi(k, t_1) \phi(-k, t_2) \, w(k; t_1, t_2) \delta s. \tag{C22}
\]

We have to subtract from \( w \delta s \) the part that corresponds to the term

\[
\delta \nu \, \phi(k, t) \phi(-k, t) \tag{C23}
\]

in (27). In this case is

\[
\delta \nu = 2w_0(k) \delta s \tag{C24}
\]

where \( w_0 = \mathcal{P} w \).
2. Diagram 1.II

Diagram 1.II in Fig. 6 comes from the term \(3\phi_\varphi \varphi \varphi_\varphi\) in Eq. (C4) after contracting the first two fields. As could be suspected, it is zero. Terms with no \(\phi\) fields are not included in the definition of the action [Eq. (17) and below], and owing to the causal nature of the propagator \(\varphi\), Eq. (A17), they cannot be generated. Remember that we have defined the action in Sec. III in such a way that there is always at least one field \(\phi\) (let us call it \(\phi^*\)) evaluated at a time \(t^*\) greater or equal than the times of all the \(\varphi\) fields. When any of the \(\varphi(k,t)\) fields is paired with \(\phi^*\), there will be a \(\Theta(t-t^*)\), that means that \(t\) must be greater than \(t^*\). The two conditions are mutually exclusive and the resulting diagram is effectively zero. In the case of the vertex \(v_{41}\) there is just one \(\phi\), which must be \(\phi^*\) necessarily.

This is essentially the mechanism which made all the diagrams with no external \(\varphi\) fields equal to zero.

3. Diagram 2.I

The first diagram in Fig. 7 contributes to \(\delta v_{41}\). It has a combinatorial factor equal to 3 (ways of choosing which one of the \(\varphi\) in the left vertex is \(\varphi_\varphi\)) \(\times 3\) (ways of choosing the \(\varphi_\varphi\) in the right vertex) \(\times 2\) (ways of pairing the fields \((\varphi_\varphi,\varphi_\varphi)\) on the left with the fields \((\varphi_\varphi,\varphi_\varphi)\) on the right) \(\times 2\) (combinatorial factor from the expansion of \(S_\varphi^2\)). There is also a factor \(-i^2/2\), which comes from the \(i/2\) in Eq. (A15) and the \(-i\) in \(-iG\). The result is

\[
\delta v_{41}^{(2)}(\{k\}; \{t\}) = 18 \delta s \left[ \delta_{|k_1+k_2|=0} \int_0^T dt_1' \ldots \int_0^T dt_4' \int d\Omega_d \frac{d\Omega_d}{\Omega_d} v_{41}(k_1, k_2, \hat{\Omega}, -\hat{\Omega}, t_1, t_2, t_3, t_4') \times G(t_3', t_1') G(t_4', t_2') v_{41}(\hat{\Omega}, -\hat{\Omega}, k_3, k_4; t_1', t_2, t_3, t_4) \right], \tag{C25}
\]

where the subscript \(ij\ldots\) means symmetrization with respect to the given variables (remember that according to our definitions, \(v_{nm}\) is symmetrical with respect to the permutations of the \(m\) fields of type \(\phi\), and of the \(n-m\) of type \(\varphi\)).

4. Diagram 2.II

The second diagram in Fig. 7 contributes to \(\delta v_{42}\), has a combinatorial factor equal to \(3 \times 3 \times 2\) and it gives

\[
\delta v_{42}^{(2)}(\{k\}; \{t\}) = 9 \delta s \left[ \delta_{|k_1+k_3|=0} \int_0^T dt_1' \ldots \int_0^T dt_4' \int d\Omega_d \frac{d\Omega_d}{\Omega_d} v_{41}(k_1, k_3, \hat{\Omega}, -\hat{\Omega}, t_1, t_3, t_1', t_2') \times G(t_1', t_3') G(t_2', t_4') v_{41}(k_2, k_4, \hat{\Omega}, -\hat{\Omega}, t_2, t_4, t_3', t_4') \right]. \tag{C26}
\]
5. Diagram 2.III

The third diagram in Fig. 7 is zero due to causality, which prevents diagrams with no external \( \phi \) fields.

6. Diagram 2.IV

The last diagram in Fig. 7 is an example of a tree diagram. It contributes to \( \delta v_{61} \), is proportional to \( \delta(|k_1 + k_2 + k_3| - 1) \), and has a combinatorial factor equal to 3, which is the number of ways of choosing \( \varphi \) among the 3 \( \varphi \) fields [see Eq. (C4)]. It results

\[
\delta v_{61}^{(2.IV)}(\{k\}; \{t\}) = 3 \left[ \delta(|k_1 + k_2 + k_3| - 1) \int_0^T dt_1 \int_0^T dt_4' v_{41}(k_1, k_2, k_3, -k_1 - k_2 - k_3; t_1, t_2, t_3, t_4') \times G(t_4', t_1') v_{41}(k_1 + k_2 + k_3, k_4, k_5, k_6; t_1', t_2, t_3, t_4) \right].
\]

(C27)

APPENDIX D: RESCALING

To see in detail the effect of the rescaling, take a generic term \( v_{nm} \) in the interaction part (terms in the free action can be analyzed in the same way). After step i) has been performed the corresponding term in the action is (constant factors omitted)

\[
\int_0^T dt_1 \cdots \int_0^T dt_n \int_{b\Lambda} d^d k_1 \cdots \int_{b\Lambda} d^d k_n \delta^d(k_1 + \cdots + k_n) \times v_{nm}'(k_1, \ldots, k_n; t_1, \ldots, t_n) \left[ \phi(k_1, t_1) \cdots \phi(k_m, t_m) \right] \left[ \varphi(k_{m+1}, t_{m+1}) \cdots \varphi(k_n, t_n) \right],
\]

(D1)

where \( v_{nm}' \) is given in Eq. (28). Then, redefine the fields and change integration variables according to Eqs. (29)-(32). The momentum integrals are again restricted to \( k \leq \Lambda = 1 \), but the time interval goes up to \( b^{\alpha} T \):

\[
\int_0^{b^{\alpha} T} dt_1 \cdots \int_0^{b^{\alpha} T} dt_n \int_{\Lambda} d^d k_1 \cdots \int_{\Lambda} d^d k_n \delta^d(k_1 + \cdots + k_n) b^{d(n-1)-n\alpha + n\alpha_\phi} \times v_{nm}'(b k_1, \ldots, b^{-\alpha} t_1, \ldots, b^{-\alpha} t_n) \left[ \phi(k_1, t_1) \cdots \right] \left[ \varphi(k_{m+1}, t_{m+1}) \right].
\]

(D2)

We expand \( v_{nm}'(b k_1, \ldots, b^{-\alpha} t_1, \ldots, b^{-\alpha} t_n) \) around \( k_i \) and \( t_i \), and the time integrals as functions of their superior extreme around \( T \). To order \( \delta s \) we get

\[
\int_0^T dt_1 \cdots \int_0^T dt_n \int_{\Lambda} d^d k_1 \cdots \int_{\Lambda} d^d k_n \delta^d(k_1 + \cdots + k_n) \times \left[ 1 + \left\{ -d(n-1) + n\alpha_t - n\alpha_\phi - k_i \frac{\partial}{\partial k_i} + \alpha_t t_i \frac{\partial}{\partial t_i} \right\} \delta s \right] v_{nm}'(k_1, \ldots; t_1, \ldots) \times \left[ \phi(k_1, t_1) \right] \left[ \varphi(k_{m+1}, t_{m+1}) \right].
\]
\[-\alpha_t T \delta s \sum_{j=1}^{n} \int_0^T dt_1 \cdots \int_0^T dt_n \int_\Lambda d^4k_1 \cdots \int_\Lambda d^4k_n \delta^d(k_1 + \cdots + k_n) \]
\[\times 2 \delta(t_j - T) v'_{nm}(k_1, \ldots; t_1, \ldots) \left[ \phi(k_1, t_1) \right] \ldots \left[ \phi(k_{m+1}, t_{m+1}) \right]. \quad \text{(D3)}\]

Because of the \(\delta(t_j - T)\), the last two lines are a sum of terms with \(n-1\) time integrals. One field (the one with index \(j\)) in each term of the sum is evaluated at time \(T\). Thus, in principle boundary terms could appear in the action. But this is not the case: if \(j \leq m\), the field evaluated at \(t_j = T\) is of type \(\phi\), and the corresponding term is zero, because \(\phi(k, T) = 0\) by the CTP condition (16). If \(j \geq m\) then the field evaluated at the boundary if of type \(\varphi\). But we have imposed the condition that in each term of the action there is always a field \(\phi\) evaluated at a time \(t^*\) which is equal or greater than the times at which are evaluated all the \(\varphi\) fields. Because \(t_j = T, t^*\) must be equal to \(T\). But, again, the CTP condition implies that the term will be zero.

Thus, we could rewrite Eq. (D2) keeping \(T\) as the superior limit of the time integrals, which means that \(T\) can be considered as an invariant quantity with respect to the RG transformation.

In conclusion, rescaling takes the resulting parameters after mode elimination, Eqs. (27) and (28), and returns (to order \(\delta s\))

\[m^2 + \delta m^2 + (-2\alpha_\phi + \alpha_t - d) m^2 \delta s, \quad \text{(D4)}\]
\[\kappa + \delta \kappa + \left( -2\alpha_\phi - d - k \frac{\partial}{\partial k} \right) \kappa \delta s, \quad \text{(D5)}\]
\[\nu + \delta \nu + \left( -2\alpha_\phi + \alpha_t - d - k \frac{\partial}{\partial k} \right) \nu \delta s, \quad \text{(D6)}\]
\[v_{nm} + \delta v_{nm} + \left[ -n\alpha_\phi + n\alpha_t - d(n-1) - k_i \frac{\partial}{\partial k_i} + \alpha_t t_i \frac{\partial}{\partial t_i} \right] v_{nm} \delta s. \quad \text{(D7)}\]

Rescaling also affects the density matrix. After rescaling it results

\[\rho'[\phi(k, 0), \varphi(k, 0)] = \exp \left\{ - \int_\Lambda d^4k \frac{b^{d+2\alpha_\phi}}{4} \left[ \tanh \left( \frac{a(bk)\beta(bk)}{2} \right) \phi(k, 0)\phi(-k, 0) \right] \right. \]
\[\left. + \coth \left( \frac{a(bk)\beta(bk)}{2} \right) \varphi(k, 0)\varphi(-k, 0) \right\} = \]
\[\exp \left\{ - \int_\Lambda d^4k \left[ \frac{a(bk)b^{d+2\alpha_\phi}}{4} \left[ \tanh \left( \frac{a(bk)b^{d+2\alpha_\phi}}{2} \right) \phi(k, 0)\phi(-k, 0) + \ldots \right] \right. \right. \]
\[\left. \left. + \coth \left( \frac{a(bk)b^{d+2\alpha_\phi}}{2} \right) \varphi(k, 0)\varphi(-k, 0) \right\} \right\}. \quad \text{(D8)}\]

Then, rescaling takes \(a\) and \(\beta\), and returns

\[a + \left( -d - 2\alpha_\phi - k \frac{\partial}{\partial k} \right) a \delta s, \quad \text{(D9)}\]
\[\beta + \left( d + 2\alpha_\phi - k \frac{\partial}{\partial k} \right) \beta \delta s. \quad \text{(D10)}\]
1. Final remark

Before leaving this section, we note a special case of rescaling. In principle, one does not have to rescale the arguments of δκ, δν and δv nm, because they are already of order δs. Note that there are no k-derivatives nor t-derivatives of these quantities in the differential equations of Sec. IV C. But it may happen (for example in the case of tree diagrams) that

\[ \delta v_{nm}(\{k\}; \{t\}) \propto \delta(|K| - \Lambda) \delta s, \]  
(D11)

where K is a linear combination of the k_i. Rescaling k_i gives

\[ \delta v_{nm}(b\{k\}; \{t\}) \propto b^{-1} \delta(|K| - b^{-1}) \delta s, \]  
(D12)

or

\[ \delta v_{nm}(b\{k\}, \{t\}) \propto b^{-1} \delta(|K| - b^{-1}) \delta s. \]  
(D13)

The first \( b^{-1} \) can be replaced by 1, but \( b^{-1} \approx 1 + \delta s \) inside the delta must be conserved because it can define how to take limit values of the functions. We define \( \delta(k - 1^+) \) such that

\[ \int_0^q dk \delta(k - \Lambda^+) = \begin{cases} 0, & \text{if } q \leq \Lambda, \\ 1, & \text{if } q > \Lambda \end{cases} \]  
(D14)

and then rewrite Eq. (D13) as

\[ \delta v_{nm}(b\{k\}, \{t\}) \propto \delta(|K| - \Lambda^+) \delta s. \]  
(D15)

APPENDIX E: CLOSED SET OF COUPLINGS TO ORDER \( \lambda^2 \)

Here we find the set of couplings for which the RG transformation is closed to order \( \lambda^2 \). Starting from the usual \( \lambda \phi^4 \) theory, Eq. (22), new terms generated by the RG, and \( v_{41} \) and \( v_{43} \) themselves, will not longer be local in time. After one infinitesimal step, there will be terms with fields evaluated at two times; for example the one generated by the diagram in Fig. 8.

![Diagram](image-url)

**FIG. 8.** One-loop diagram drawn using two identical vertices local in time, with coupling constant λ. The resulting term in the effective action will be no longer local in time, but will couple fields at t (\( \phi \phi' \)) and \( t' (\phi' \phi') \).
We used $\phi$ and $\varphi$ for fields evaluated at time $t$, and $\phi'$ and $\varphi'$ at $t'$. If we iterate the transformation once more, we will get terms with fields evaluated at 3 and 4 times. We may suppose that, basically, $v_{41}$ and $v_{43}$ remain local and time independent, and that the non local and time dependent terms are corrections.

We assume that $v_{41}$ and $v_{43}$ are of order $\lambda$ and compute the RG equations to order $\lambda^2$. We have to find what kind of couplings can be generated when proceeding up to this order. We assume that for $s \geq 0$ in the interaction part of the action there will be two terms local in time,

$$\int_0^T dt \int_{b\Lambda} d^4k_1 \ldots d^4k_4 \delta^4 (k_1 + \ldots + k_4) \ V_{41}(s) \phi(k_1, t)\varphi(k_2, t)\varphi(k_3, t)\varphi(k_4, t) \quad (E1)$$

and

$$\int_0^T dt \int_{b\Lambda} d^4k_1 \ldots d^4k_4 \delta^4 (k_1 + \ldots + k_4) \ V_{43}(s) \phi(k_1, t)\varphi(k_2, t)\varphi(k_3, t)\varphi(k_4, t), \quad (E2)$$

where $V_{41}$ and $V_{43}$ are order $\lambda$ and do not depend on $\{k\}$, and satisfy

$$V_{41}(0) = V_{43}(0) = -\frac{\Omega_d \lambda}{48(2\pi)^d}. \quad (E3)$$

Starting from these vertices we perform successive infinitesimal RG transformations until no new terms are generated. The finite RG transformation will be closed with respect to the resulting set of couplings, a desired property which was stressed before. We do not need to keep track of the precise value of each coupling from step to step. At this point we just want to enumerate the couplings that have to be considered to be consistent to order $\lambda^2$, and find their general functional form. Rescaling have to be taken into account only in the deltas appearing in tree diagrams.

Then, for the first step, we start with $V_{41}$ and $V_{43}$ and construct all the possible diagrams of order $\lambda$ and $\lambda^2$, shown in Fig. 9. They all will give new terms. Below each diagram we have written schematically the corresponding term, indicating the name of the coupling and the variables on which depends. We have defined

$$Q_{12\ldots} = |k_1 + k_2 + \ldots| \quad (E4)$$

The couplings have, by definition, the same symmetry that the set of fields that they couple. Most diagrams give a result that has already the required symmetry.

Next, for the second step, we combine the original terms with the new ones and construct all possible diagrams of order $\lambda$ and $\lambda^2$, shown in Fig. 10. Only two new terms are generated at this step, with couplings $W_{21}$ and $W_{22}$, all of order $\lambda^2$. They cannot be used to produce new terms, because combined with any other vertex the resulting diagram would be at least of order $\lambda^3$. Hence, a third step fails to create new terms. In Fig. 10, we have drawn the diagrams for the two field couplings at the end, because, in some sense, they are at the deepest level and depend on all the terms which precede them. Note that diagram (8) in Fig. 10 will not depend on $k$, owing to the fact that $W_{41}$ is evaluated at $Q = 0$, and hence the dependence on $k$ cancels out.
FIG. 9. Set of diagrams obtained after performing one infinitesimal step of RG transformation starting from the initial condition given by Eq. (22). Below each diagram we indicate the coupling generated, with its corresponding dependence on the momentum and time variables, with $Q_{ij\ldots} = |k_1 + k_2 + \ldots|$. 
FIG. 10. Set of diagrams obtained after performing two infinitesimal steps of RG transformation starting from the initial condition given by Eq. (22). A third step does not create new couplings.
Some conclusions can be drawn. If $m^2$, $\kappa$ and $\nu$ are initially zero, $m^2$ will be of order $\lambda$ through $V_{41}$ in diagram 1 of Fig. 9, and $\kappa$ and $\nu$ will be at least of order $\lambda^2$, through the one vertex diagrams of Fig. 10. Note, as well, that $\eta$ will be of order $\lambda^2$ owing to the fact that $W_{21}$ is order $\lambda^2$ and is the only term of the type $\phi \varphi$ which depends on $k$. This means that, at order $\lambda^2$, $\eta$ can be omitted and $\alpha_t$ can be set equal to 1 in the left-hand side members of Eq. (38), and Eqs. (35)-(37). We will assume that this is the case.

The vertices which depend on two times, have half of the fields evaluated at $t$ and half at $t'$. Diagrams which depend on two times with an uneven number of fields at $t$ and $t'$ can be drawn using a $v_{6i}$ vertex, but they are zero. This fact can be understood noting that, after $N$ iterations of the RG transformation, $v_{6i}(k; t, t')$ will be given by a sum of $N$ terms, with the $n$th term proportional to $\delta(k - b^{-n})$ (remember that $\Lambda = 1$). So, if $k$ is not in the interval $[b^{-1}, b^{-N}]$, $v_{6i}(k; t, t')$ is zero, or in other words (and it will shown below)

$$v_{6i}(k; t, t') \propto \Theta(k - 1^+)\Theta(\epsilon^1 - k), \quad (E5)$$

where we use $1^+$ meaning that $\Theta(k - 1^+)$ is strictly zero if $k = 1$. Then, consider for example the case of

$$v_{61}(|k_1 + k_2 + k_3|; t, t') \phi \varphi \varphi' \varphi' \varphi'. \quad (E6)$$

If we connect the two $\varphi$ fields to generate a term of the form

$$\phi \varphi' \varphi' \varphi', \quad (E7)$$

we must set $k_2 = -k_3 = 1$ and integrate over $\hat{\Omega}$. But because of $\Theta$, the unpaired $\phi$ field should have momentum

$$|k_4| > 1, \quad (E8)$$

which is impossible, because $k_4 \leq 1$. The same occurs if we connect two of the $\varphi'$. The conservation delta, $\delta^d(k_1 + k_2 \ldots)$, which is always present, allows us to write the first condition on Eq. (E5) as

$$\Theta(|k_4 + k_5 + k_6| - 1^+). \quad (E9)$$

Setting, for example, $k_5 = -k_6 = 1$, it should be $|k_4| > 1$, and the conclusion follows.

There are other possible couplings depending on $t$ and $t'$, with an even number of fields at each time, that due to causality are not generated to order $\lambda^2$. For example, from

$$v_{63}^{(2)} \phi \phi \varphi' \varphi' \varphi' \varphi'. \quad (E10)$$

joining one of the $\phi$ with one of the $\varphi'$ we cannot generate a new term of the form

$$\phi \phi \varphi' \varphi', \quad (E11)$$
because, on one hand, necessarily is \( t' \leq t \) (see Sec. III), and on the other hand, the propagator \( G(t', t) \) connecting \( \varphi' \) with \( \phi \) entails the opposite condition \( t' \geq t \). Also, \( v_{65} \phi \phi \phi' \phi' \varphi' \) cannot create a term of the form \( \phi' \phi' \phi' \), because by construction (but not necessarily, as in the previous example) is \( t' \leq t \).

Other typical cancellation occurs in the diagram of Fig. 11

\[
\begin{array}{c}
\phi \\
\gamma_{43} \\
\phi
\end{array} \propto \Theta(t - t')
\]
\[
\begin{array}{c}
V_{43} \\
\gamma_{21} \\
V_{21}
\end{array} = 0
\]

\[
\begin{array}{c}
\phi \\
\gamma_{t'-t}
\end{array} \propto \Theta(t' - t)
\]

FIG. 11. Example of a diagram which is null due to a closed chain of steps functions of the form: \( \Theta(t_1 - t_2)\Theta(t_2 - t_3) \ldots \Theta(t_{n-1} - t_n)\Theta(t_n - t_1) \). This is typical of diagrams drawn using vertices which are local in time.

There are other cases that can be analyzed in similar terms.

1. Final remark

We have seen in Sec. III the method used to extract from diagrams with two external lines the terms which by definition belong to the free action, Eq. (18). Something analogous must be done with diagrams which give terms with 4 external fields of the form

\[
\phi \varphi' \varphi' \varphi'
\]

and

\[
\phi \varphi \varphi' \phi'.
\]

For example, the contributions of diagrams (2) and (5) of Fig. 10 can be written as

\[
\int_0^T dt \int_0^T dt' \int_{b \Lambda} d^4k_1 \ldots d^4k_4 \delta^d (k_1 + \ldots) \phi(k_1, t) \varphi(k_2, t) \varphi(k_3, t') \varphi(k_4, t') v(Q_{12}; t, t') \delta s, \tag{E14}
\]

\[
\int_0^T dt \int_0^T dt' \int_{b \Lambda} d^4k_1 \ldots d^4k_4 \delta^d (k_1 + \ldots) \phi(k_1, t) \varphi(k_2, t) \phi(k_3, t') \phi(k_4, t') w(Q_{12}; t, t') \delta s, \tag{E15}
\]

where

\[
v(k; t, t') = 18 \delta_{k;0} V_{41}^2 G(t, t'), \tag{E16}
\]

and

\[
w(k; t, t') = 18 \delta_{k;0} V_{41} V_{43} G(t, t'). \tag{E17}
\]
In principle, \( v \, \delta s \) and \( w \, \delta s \) are not directly identified with \( \delta W_{41} \) and \( \delta W_{43} \), but can contain contributions to \( \delta V_{41} \) and \( \delta V_{43} \). Using the definition (C9), we write

\[
v(k; t, t') \, \delta s = \delta(t - t') \, \delta V_{41} + \delta W_{41}(k; t, t'),
\]

(E18)

\[
w(k; t, t') \, \delta s = \delta(t - t') \, \delta V_{43} + \delta W_{43}(k; t, t')
\]

(E19)

with

\[
\delta V_{41} = \mathcal{P} v(0) \, \delta s,
\]

(E20)

and so for \( V_{43} \). In this way, the corrections to \( V_{41} \) and \( V_{43} \) are isolated, while \( \delta W_{41} \) and \( \delta W_{43} \) give the net contributions to the \( W_{41} \) and \( W_{43} \).

**APPENDIX F: DETAILED EXPRESSIONS FOR \( \kappa \) AND \( \nu \)**

Here, the formulas for \( \kappa \) and \( \nu \), with \( \beta_0^{-1} = 0 \) are given. The expressions are for \( k = 0 \), but are asymptotically valid for \( 0 \leq k \) when \( 1 \ll z \). They are composed by two parts, corresponding to \( z < 2 \) and \( z > 2 \), which join smoothly for \( z \to 2 \).

\[
\kappa(z, T) = \kappa_<(z, T) \, \Theta(2 - z) + \kappa_>(z, T) \, \Theta(z - 2),
\]

(F1)

\[
\nu(z, T) = \nu_<(z, T) \, \Theta(2 - z) + \nu_>(z, T) \, \Theta(z - 2).
\]

(F2)

**1. Expressions for \( \kappa \)**

We write

\[
\kappa_< = V_0^2 \sum_{i=1}^{7} \kappa_{<i},
\]

(F3)

and an analogous equation for \( \kappa_> \), where

\[
\kappa_{<1} = \frac{3}{2T} \left\{ -63z \log 3 - \frac{4}{z}(3 - 4z + z^2) + 6(-5 + 3z) \cos T - 14(-1 + z) \cos(3T) \right\},
\]

(F4)

\[
\kappa_{<2} = \frac{18}{2T} \left\{ - \cos \left[T(-2 + z)\right] + \frac{2}{z} \cos(Tz) - 7 \cos \left[T(2 + z)\right] + \cos \left[T(1 - 2z)\right] + 7 \cos \left[T(1 + 2z)\right] \right\},
\]

(F5)
\[ \kappa_{<3} = \frac{9}{2T} \left\{ z \log(-2 + z) - 2z(5 + 4 \cos T) \log z - 7z \log(2 + z) \right. \\
-4z \log(-1 + 2z) + 28z \log(1 + 2z) \right\}, \quad (F6) \]

\[ \kappa_{<4} = -18 \left[ z \sin T - \sin(Tz) \right], \quad (F7) \]

\[ \kappa_{<5} = 3 \left\{ -3(1 + 3z) \text{Si}(T) - 21(-1 + z) \text{Si}(3T) - 3(-2 + z) \text{Si}[T(-2 + z)] \\
+12 \text{Si}(Tz) - 21(2 + z) \text{Si}[T(2 + z)] + 3(-1 + 2z) \text{Si}[T(-1 + 2z)] \\
+21(1 + 2z) \text{Si}[T(1 + 2z)] \right\}, \quad (F8) \]

\[ \kappa_{<6} = 18zT \left[ \text{Ci}(T) - \text{Ci}(Tz) \right] + \frac{9z}{2T} \left\{ -3\text{Ci}(T) - \text{Ci}[T(-2 + z)] + 7\text{Ci}[T(2 + z)] \\
+21\text{Ci}(3T) + 4\text{Ci}[T(-1 + 2z)] - 28\text{Ci}[T(1 + 2z)] \right\}, \quad (F9) \]

\[ \kappa_{<7} = \frac{9z}{T} \left\{ -12 \left[ \text{Ci}(2T) - \text{Ci}(2Tz) \right] \cos T + 3 \left[ \text{Ci}(T) - \text{Ci}(Tz) \right] \cos(2T) \\
16 \left[ \text{Si}(2T) - \text{Si}(2Tz) \right] \sin T - 4 \left[ \text{Si}(T) - \text{Si}(Tz) \right] \sin(2T) \right\}, \quad (F10) \]

\[ \kappa_{>1} = -\frac{201}{8T} - 9T + \frac{63}{8Tz} + \frac{3z}{32T} \left\{ 97 \\
-24 \left[ \gamma_E + 21 \log 3 + \log 64 + \log(T - (2 + \log 4)T^2) \right] \right\}, \quad (F11) \]

\[ \kappa_{>2} = \frac{3}{32Tz} \left\{ z^2 \left[ 144 \cos T - 24(1 + \log 4) \cos(2T) - 112 \cos(3T) - 105 \cos(4T) \\
-28 \cos(2Tz) + z \left\{ -240 \cos T + 120 \cos(2T) + 112 \cos(3T) - 84 \cos(4T) \\
-24 \cos[2T(-1 + z)] + 48 \cos[T(1 - 2z)] + 336 \cos[T(1 + 2z)] \right\} \right\}, \quad (F12) \]

\[ \kappa_{>3} = -\frac{9}{8} \left\{ z \left[ 8 \sin T - 4 \sin(2T) - 7 \sin(4T) \right] + 14 \sin(2Tz) \right\}, \quad (F13) \]
\[
\kappa_{>4} = -\frac{9}{4T} \left[ 2(9 + T^2) + 8 \cos T - \cos(2T) \right] \log z \\
-\frac{9}{4T} \left[ - \log(-1 + z) + 4 \log(-1 + 2z) - 28 \log(1 + 2z) \right], \tag{F14}
\]

\[
\kappa_{>5} = \frac{9z}{4T} \left\{ (-3 + 4T^2) \text{Ci}(T) + \text{Ci}(2T) + 21 \text{Ci}(3T) + 7 \text{Ci}(4T) - \text{Ci}[2T(-1 + z)] \\
-2T^2 \left[ 2 \text{Ci}(2T) + 7 \text{Ci}(4T) - 7 \text{Ci}(2Tz) \right] + 4 \text{Ci}[T(-1 + 2z)] - 28 \text{Ci}[T(1 + 2z)] \right\}, \tag{F15}
\]

\[
\kappa_{>6} = \frac{9}{2} \left\{ -(1 + 3z) \text{Si}(T) + (3 + z) \text{Si}(2T) - 7(-1 + z) \text{Si}(3T) - 7(1 + z) \text{Si}(4T) \\
+(1 - z) \text{Si}[2T(-1 + z)] - 7 \text{Si}(2Tz) + (-1 + 2z) \text{Si}[T(-1 + 2z)] \\
+7(1 + 2z) \text{Si}[T(1 + 2z)] \right\}, \tag{F16}
\]

\[
\kappa_{>7} = \frac{9z}{8T} \left\{ \left[ 12 \text{Ci}(T) - 12 \text{Ci}(2T) - 2 \text{Ci}(4T) + 2 \text{Ci}(2Tz) \right] \cos(2T) \\
+64 \left[ \text{Si}(2T) - \text{Si}(2Tz) \right] \sin T - 2 \left[ 8 \text{Si}(T) - 8 \text{Si}(2T) + \text{Si}(4T) - \text{Si}(2Tz) \right] \sin(2T) \\
+48 \left[ - \text{Ci}(2T) + \text{Ci}(2Tz) \right] \cos T \right\}. \tag{F17}
\]

The functions Si and Ci are the sin and cos integral functions, and \( \gamma_E \) is the Euler’s constant.

2. **Expressions for \( \nu \)**

We write

\[
\nu_\prec = V_0^2 \sum_{i=1}^{7} \nu_{\prec i}, \tag{F18}
\]

and an analogous equation for \( \nu_\succ \), where

\[
\nu_{\prec 1} = \frac{9}{4T} \left[ -12 - \log 9 - 32z + (44 - \log 2187) z^2 \right], \tag{F19}
\]

\[
\nu_{\prec 2} = 27z \left[ -z \sin T + \sin(Tz) \right], \tag{F20}
\]

\[
\nu_{\prec 3} = \frac{9}{2T} \left\{ -3(1 + 7z^2) \cos T - (-1 + z^2) \cos(3T) + (-2 + z) \cos[T(-2 + z)] \\
+18 \cos(Tz) - (2 + z) \cos[T(2 + z)] + 7(-1 + 2z) \cos[T(1 - 2z)] \\
+(1 + 2z) \cos[T(1 + 2z)] \right\}, \tag{F21}
\]
\[ \nu_4 = \frac{9}{2} \left\{ -3(1 + 9z^2) \text{Si}(T) - 3(-1 + z^2) \text{Si}(3T) + (-2 + z)^2 \text{Si}[T(-2 + z)] \\
+ 24z \text{Si}(Tz) - (2 + z)^2 \text{Si}[T(2 + z)] - 7(1 - 2z)^2 \text{Si}[T(1 - 2z)] \\
+ (1 + 2z)^2 \text{Si}[T(1 + 2z)] \right\}, \quad (F22) \]

\[ \nu_5 = \frac{9}{4T} \left\{ 3 \left[ -2 + (-19 + 4T^2)z^2 \right] \text{Ci}(T) + (2 + 7z^2) \text{Ci}(3T) \\
+ (-4 + z^2) \text{Ci}[T(-2 + z)] + 12(2 - T^2z^2) \text{Ci}(Tz) + (-4 + z^2) \text{Ci}[T(2 + z)] \\
+ 14(-1 + 4z^2) \text{Ci}[T(-1 + 2z)] + 2(1 - 4z^2) \text{Ci}[T(1 + 2z)] \right\}, \quad (F23) \]

\[ \nu_6 = \frac{9}{4T} \left\{ (4 - z^2) \log(-4 + z^2) - 2(12 - 25z^2) \log z \\
- 2(-1 + 4z^2) \left[ 7 \log(-1 + 2z) - \log(1 + 2z) \right] \right\}, \quad (F24) \]

\[ \nu_7 = \frac{9z^2}{2T} \left\{ 24 [\text{Ci}(2T) - \text{Ci}(2Tz)] \cos T + [\text{Ci}(T) - \text{Ci}(Tz)] \cos(2T) \\
+ 32 [\text{Si}(2T) - \text{Si}(2Tz)] \sin T \right\}, \quad (F25) \]

\[ \nu_1 = \frac{9}{16T} \left\{ -4 (7 + 4\gamma_E + 2 \log 1536 + 4 \log T) - 56z \\
+ [135 + 4\gamma_E + 28 \log(4/3) + 4 \log T] z^2 \right\}, \quad (F26) \]

\[ \nu_2 = \frac{9}{4} \left\{ -2z \sin(2Tz) - [12 \sin T - 6 \sin(2T) - \sin(4T)] z^2 \right\}, \quad (F27) \]

\[ \nu_3 = \frac{9}{16T} \left\{ -24(1 + 7z^2) \cos T + 4(14 - 4z + 11z^2) \cos(2T) \\
- 8(-1 + z^2) \cos(3T) - (8 + 3z^2) \cos(4T) - 56(-1 + z) \cos[2T(-1 + z)] \\
- 12 \cos(2Tz) + 56(-1 + 2z) \cos[T(1 - 2z)] + 8(1 + 2z) \cos[T(1 + 2z)] \right\}, \quad (F28) \]

\[ \nu_4 = \frac{9}{4T} \left\{ 3 \left[ -2 - (19 - 4T^2)z^2 \right] \text{Ci}(T) + 2 \left[ 5 - (-7 + 6T^2)z^2 \right] \text{Ci}(2T) \\
+ (2 + 7z^2) \text{Ci}(3T) - (2 - z^2 + 4T^2z^2) \text{Ci}(4T) + 14(1 - z^2) \text{Ci}[2T(-1 + z)] \\
- (2 - 4T^2z^2) \text{Ci}[2Tz] - 14(1 - 4z^2) \text{Ci}[T(-1 + 2z)] + (2 - 8z^2) \text{Ci}[T(1 + 2z)] \right\}, \quad (F29) \]
\[\nu_{>5} = \frac{9}{2} \left\{ -3(1 + 9z^2)\text{Si}(T) + 2(7 - 2z + 7z^2)\text{Si}(2T) + 3(1 - z^2)\text{Si}(3T) \\
+ (-4 - z^2)\text{Si}(4T) - 14(1 - 2z + z^2)\text{Si}[2T(-1 + z)] - 4z\text{Si}(2Tz) \\
- 7(1 - 4z + 4z^2)\text{Si}[T(1 - 2z)] + (1 + 4z + 4z^2)\text{Si}[T(1 + 2z)] \right\}, \quad (F30)\]

\[\nu_{>6} = \frac{9}{2T} \left\{ 7(-1 + z^2)\log(-1 + z) + (1 + 17z^2)\log z \\
- (-1 + 4z^2)[7\log(-1 + 2z) - \log(1 + 2z)] \right\}, \quad (F31)\]

\[\nu_{>7} = \frac{9z^2}{2T} \left\{ 24[\text{Ci}(2T) - \text{Ci}(2Tz)]\cos T \\
+ [\text{Ci}(T) - \text{Ci}(2T) - 7\text{Ci}(4T) + 7\text{Ci}(2Tz)]\cos(2T) \\
+ 32[\text{Si}(2T) - \text{Si}(2Tz)]\sin T + 7[-\text{Si}(4T) + \text{Si}(2Tz)]\sin(2T) \right\}. \quad (F32)\]

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