USING BOOLEAN CUMULANTS TO STUDY MULTIPLICATION AND ANTICOMMUTATORS OF FREE RANDOM VARIABLES

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Abstract. We study how Boolean cumulants can be used in order to address operations with freely independent random variables, particularly in connection to the $\ast$-distribution of the product of two selfadjoint freely independent random variables, and in connection to the distribution of the anticommutator of such random variables.

1. Introduction

1.1. Multiplication of free random variables, in terms of free cumulants.
Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space. It is known since the 90’s (cf. [8]) how to handle the multiplication of two freely independent elements of $\mathcal{A}$ in terms of free cumulants. More precisely, let $(\kappa_n : \mathcal{A}^n \to \mathbb{C})_{n=1}^\infty$ be the family of free cumulant functionals of $(\mathcal{A}, \varphi)$. If $a, b \in \mathcal{A}$ are freely independent, then the free cumulants of the product $ab$ are described by the formula

$$\kappa_n(ab, \ldots, ab) = \sum_{\pi \in NC(n)} \prod_{U \in \pi} \kappa_{|U|}(a, \ldots, a) \cdot \prod_{V \in Kr(\pi)} \kappa_{|V|}(b, \ldots, b),$$

where $NC(n)$ is the lattice of non-crossing partitions of $\{1, \ldots, n\}$, and $Kr : NC(n) \to NC(n)$ is an important anti-automorphism of this lattice, called Kreweras complementation map. The formula (1.1) is very useful because it allows one to take advantage of many pleasant properties the lattices $NC(n)$ are known to have. In particular, upon re-writing (1.1) in terms of formal power series and upon doing suitable manipulations, one can use it (cf. [9]) to derive the multiplicativity of the well-known $S$-transform of Voiculescu [15].

In view of how we will make our presentation of results below, it is worth mentioning here that the clearest proof of the formula (1.1) is made in 3 steps, as follows:

\begin{align}
\begin{aligned}
\text{Step 1.} & \quad \text{On the left-hand side of (1.1), use the formula} \\
& \quad \text{(with summation over $NC(2n)$) which describes} \\
& \quad \text{free cumulants with products as arguments.} \\
\end{aligned}
\end{align}

\begin{align}
\begin{aligned}
\text{Step 2.} & \quad \text{Use the fact that, due to the freeness of $a$ from $b$,} \\
& \quad \text{all their mixed free cumulants vanish.} \\
\end{aligned}
\end{align}

\begin{align}
\begin{aligned}
\text{Step 3.} & \quad \text{Perform a direct combinatorial analysis of the non-crossing} \\
& \quad \text{partitions in $NC(2n)$ which were not pruned in Step 2.} \\
\end{aligned}
\end{align}

Let us now upgrade to the framework where $(\mathcal{A}, \varphi)$ is a $\ast$-probability space, and where $a, b$ are two freely independent selfadjoint elements of $\mathcal{A}$. Since $ab$ isn’t generally selfadjoint, we now need to keep track of the joint moments, or equivalently of the joint free cumulants of $ab$ and $(ab)^\ast = ba$. That is, we now need to look at free cumulants of the form

$$\kappa_n((ab)^{\varepsilon(1)}, \ldots, (ab)^{\varepsilon(n)}), \quad \text{with } n \in \mathbb{N} \text{ and } \varepsilon = (\varepsilon(1), \ldots, \varepsilon(n)) \in \{1, \ast\}^n.$$

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The tools invoked in Steps 1 and 2 of (1.2) can still be used in connection to the cumulants from (1.3). But the combinatorial analysis in Step 3 (where some version of the Kreweras complementation map would be hoped to appear) becomes ad-hoc and does not seem to reveal a pattern – this is seen on very simple examples, e.g. when doing the calculation which expresses $\kappa_3(ab, ab, (ab)^\ast)$ in terms of the $\ast$-free cumulants of $a$ and those of $b$.

1.2. Use Boolean cumulants instead of free cumulants?

For a noncommutative probability space $(\mathcal{A}, \varphi)$, one can also consider the family $(\beta_n : \mathcal{A}^n \to \mathbb{C})_{n=1}^{\infty}$ of Boolean cumulant functionals of $(\mathcal{A}, \varphi)$. Boolean cumulants are the analogue of free cumulants in the parallel (and simpler) world of Boolean probability. They are known to have some direct connections with free probability, particularly in connection to a development called “Boolean Bercovici-Pata bijection”. An intriguing fact around this topic is that “the Boolean Bercovici-Pata bijection preserves the structure behind multiplication of free random variables”.

A possible way to pitch this fact is as follows: when one describes the multiplication of two freely independent elements $a, b \in \mathcal{A}$ in terms of Boolean cumulants, the resulting formula has exactly the same structure as in (1.1):  

\begin{equation}
\beta_n(ab, \ldots, ab) = \sum_{\pi \in NC(n)} \prod_{U \in \pi} \beta_{|U|}(a, \ldots, a) \cdot \prod_{V \in Kr(\pi)} \beta_{|V|}(b, \ldots, b), \quad \forall n \geq 1.
\end{equation}

The formula (1.4) was first found in [1, Theorem 2']. It can be proved via a strategy with 3 steps parallel to the one described in (1.2). (See also [12, Lemma 3.2] for a similar result stated in terms of the so-called “c-free cumulants”, which relate at the same time to free and to Boolean cumulants.)

The main point of the present paper is that, for Boolean cumulants, the strategy with 3 steps can be pushed to the framework where $(\mathcal{A}, \varphi)$ is a $\ast$-probability space and where we look at Boolean cumulants of the form  

\begin{equation}
\beta_n((ab)^{\varepsilon(1)}, \ldots, (ab)^{\varepsilon(n)}), \quad \text{with } n \in \mathbb{N} \text{ and } \varepsilon = (\varepsilon(1), \ldots, \varepsilon(n)) \in \{1, \ast\}^n.
\end{equation}

What makes this possible is the use of an alternative facet of Kreweras complementation, suggested by the work in [1]. This facet of Kreweras complementation is discussed in the next subsection. (We reiterate here that the possibility of using it is specific to Boolean cumulants, and – as seen on very simple examples – fails to work for free cumulants.)

1.3. Kreweras complementation and VNRP property.

The definition of the Kreweras complement for a partition $\pi \in NC(n)$, as originally given by G. Kreweras [7], goes via a maximality argument. One considers partitions (not necessarily non-crossing) of $\{1, \ldots, 2n\}$ of the form $\pi^{\text{odd}} \sqcup \rho^{\text{even}}$, which have a copy of $\pi$ placed on $\{1, 3, \ldots, 2n - 1\}$ and a copy of some other partition $\rho \in NC(n)$ placed on $\{2, 4, \ldots, 2n\}$. The Kreweras complement $Kr_n(\pi)$ is the maximal element, with respect to the reverse refinement order “$\leq$” on $NC(n)$, for the set $\{\rho \in NC(n) \mid \pi^{\text{odd}} \sqcup \rho^{\text{even}} \in NC(2n)\}$.

The paper [1] considered another partial order “$\lessim$” on $NC(n)$, coarser than the reverse refinement order $\leq$, which is useful for studying connections between free cumulants and Boolean cumulants. In Proposition 6.10 of [1], another maximality property related to Kreweras complements was noticed to hold: the partitions of the form $\pi^{\text{odd}} \sqcup (Kr_n(\pi))^{\text{even}}$
are the maximal elements with respect to $\ll$ for the set

\[(1.6) \quad \{ \sigma \in NC(2n) \mid \text{every block of } \sigma \text{ is contained either in } \{1,3,\ldots,2n-1\} \text{ or in } \{2,4,\ldots,2n\}, \} \; \text{and } \sigma \text{ has exactly two outer blocks} \]

moreover, for every $\sigma$ in the above set, there exists a unique $\pi \in NC(n)$ such that $\sigma \ll \pi^{(\text{odd})} \sqcup (\text{Kr}_n(\pi))^{(\text{even})}$.

[The concept of outer block, and the related concept of depth for a block of a non-crossing partition are reviewed in Section 2 below. The requirement “$\sigma$ has exactly two outer blocks” in (1.6) is a minimality condition, since the block of $\sigma$ which contains the number 1 and the block of $\sigma$ which contains the number $2n$ are always sure to be outer blocks.]

In the present paper we extend the result described above to a framework where considering parities is a special case of considering a colouring $c : \{1,\ldots,2n\} \to \{1,2\}$ (given by $c(i) = i \text{ (mod 2)}$ for $1 \leq i \leq 2n$). Upon examining the point of view of colourings, one finds that the property which isolates partitions of the form $\tau = \pi^{(\text{odd})} \sqcup (\text{Kr}_n(\pi))^{(\text{even})}$ within the set $\{\sigma \in NC(2n) \mid \text{every block of } \sigma \text{ is contained either in } \{1,3,\ldots,2n-1\} \text{ or in } \{2,4,\ldots,2n\}, \}$ is a certain vertical-no-repeat property, or VNRP for short. It is straightforward how to define VNRP for a general colouring (Definition 3.8 below). Given a partition $\sigma \in NC(m)$ and a colouring in $s$ colours $c : \{1,\ldots,m\} \to \{1,\ldots,s\}$ which is constant along the blocks of $\sigma$, the fact that $\sigma$ and $c$ have VNRP amounts to the requirement that

\[c(\text{Parent}_\sigma(V)) \neq c(V), \quad \text{for every inner block } V \text{ of } \sigma.\]

[Here we refer to the fairly intuitive fact that every inner block $V$ of a non-crossing partition $\sigma$ must have a “parent-block” $\text{Parent}_\sigma(V)$ into which it is nested. The precise definition of how this goes is reviewed in Section 2 below.]

The key-property of VNRP, extending the considerations on Kreweras complements from the preceding paragraph, is then stated as follows.

**Theorem 1.1.** (Key-property of VNRP.)

Let $m$ be in $\mathbb{N}$, let $c : \{1,\ldots,m\} \to \{1,\ldots,s\}$ be a colouring, and consider the set of partitions

\[NC(m;c) := \{ \sigma \in NC(m) \mid c \text{ is constant on every block of } \sigma \}.\]

For every $\sigma \in NC(m;c)$ there exists a $\tau \in NC(m;c)$, uniquely determined, such that $\sigma \ll \tau$ and such that $\tau$ has the VNRP property with respect to $c$.

### 1.4. Free independence in terms of Boolean cumulants.

An easy calculation based on Theorem 1.1 leads to a description of free independence in terms of Boolean cumulants, as follows.

**Theorem 1.2.** Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space and let $(\beta_n : \mathcal{A}^n \to \mathbb{C})_{n=1}^\infty$ be the family of Boolean cumulant functionals associated to it. Let $\mathcal{A}_1, \ldots, \mathcal{A}_s \subseteq \mathcal{A}$ be unital subalgebras. The following two statements are equivalent.

1. $\mathcal{A}_1, \ldots, \mathcal{A}_s$ are free with respect to $\varphi$.

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1 The correct formulation here would be to say that $\tau$ and $c$ (together) have VNRP. We will occasionally replace this with saying that “$\tau$ has VNRP with respect to $c$”, or that “$c$ has VNRP with respect to $\tau$.”
For every $n \in \mathbb{N}$, every colouring $c : \{1, \ldots, n\} \to \{1, \ldots, s\}$, and every $a_1 \in A_c(1), \ldots, a_n \in A_c(n)$, one has
\[ \beta_n(a_1, \ldots, a_n) = \sum_{\pi \in NC(n; c)} \prod_{V \in \pi} \left( \beta_{|V|}(a_1, \ldots, a_n)|V| \right). \]

Remark 1.3. (1) The essential part of Theorem 1.2 is the implication (1) $\Rightarrow$ (2), which gives an explicit formula for Boolean cumulants with free arguments. Once this is established, the converse (2) $\Rightarrow$ (1) follows by combining (1) $\Rightarrow$ (2) with a standard “replica trick”.

(2) Coming from the study of a very general notion of noncommutative independence, Proposition 4.30 of the recent paper [5] gives a description of free independence in terms of Boolean cumulants which (when considered with $\mathbb{C}$ as field of scalars) is equivalent to the above Theorem 1.2. To be precise, condition 2 in Proposition 4.30 of [5] is the moment formula which comes out when one performs an additional summation over interval partitions on both sides of (1.7) – see Corollary 4.4 below. Conversely, the latter moment formula can be used to retrieve Equation (1.7), via an easy application of Möbius inversion.

(3) Equation (1.7) implies an amusing formula for the Boolean cumulants of the sum of two freely independent random variables. The structure of this formula cannot be as simple as what one gets by using free cumulants, but we present it nevertheless in Proposition 4.6 below, in anticipation of the similarly looking formula concerning free anticommutators (where the use of free cumulants does not provide a simpler alternative).

1.5. Joint Boolean cumulants for $ab$ and $(ab)^*$, and free anticommutators.

We now continue the thread from Section 1.2, concerning the joint Boolean cumulants of $ab$ and $(ab)^*$, where $a$ and $b$ are freely independent selfadjoint elements in a $*$-probability space. We will put into evidence some special sets of non-crossing partitions which appear in the explicit formula for a joint cumulant $\beta_n((ab)^{\varepsilon(1)}, \ldots, (ab)^{\varepsilon(n)})$ as in (1.5) and then (upon doing a summation over $\varepsilon \in \{1, *\}^n$) in the explicit formula for the Boolean cumulants of a free anticommutator $ab + ba$. We will refer to these special non-crossing partitions by using the ad-hoc term of “anticommutator-friendly”, due to how they appear in the formula for the Boolean cumulants of a free anticommutator, in Theorem 1.8 below. Their actual definition doesn’t, however, require any knowledge of a free probabilistic framework, and is stated as follows.

Definition 1.4. Let $n$ be a positive integer, let $\sigma$ be a partition in $NC(2n)$, and let us consider the set $\text{OuterMax}(\sigma) := \{\max(W) \mid W$ is an outer block of $\sigma\}$. We will say that $\sigma$ is anticommutator-friendly when it satisfies the following two conditions.

(AC-Friendly1) $\text{OuterMax}(\sigma) \subseteq \{1, 3, \ldots, 2n - 1\} \cup \{2n\}$.

(AC-Friendly2) For every $j \in \{1, 3, \ldots, 2n - 1\} \setminus \text{OuterMax}(\sigma)$, one has $\text{depth}_\sigma(j) \neq \text{depth}_\sigma(j + 1)$, where “$\text{depth}_\sigma(j)$” stands for the depth of the block of $\sigma$ which contains the number $j$.

Notation and Remark 1.5. For every $n \in \mathbb{N}$ we will denote
\[ NC_{ac-friendly}(2n) := \{\sigma \in NC(2n) \mid \sigma \text{ is anticommutator-friendly}\}. \]

For instance $NC_{ac-friendly}(2)$ consists of only one partition, $\{\{1\}, \{2\}\} \in NC(2)$. (The partition $\sigma = \{\{1, 2\}\}$ is not anticommutator-friendly because it has $\text{depth}_\sigma(1) = \text{depth}_\sigma(2) =$...
The next such set, $NC_{ac\text{-friendly}}(4)$, consists of the 5 partitions which are depicted in Figure 1 below. The cardinalities of the sets of partitions $NC_{ac\text{-friendly}}(2n)$ are tractable, in the respect that their generating series satisfies an algebraic equation of order 4, which can be solved explicitly:

$$
\sum_{n=1}^{\infty} |NC_{ac\text{-friendly}}(2n)| z^n = \frac{1}{2} - \sqrt{1 - 8z - \sqrt{1 - 8z}} \frac{1 - 2z}{8z}.
$$

![Figure 1. The 5 partitions in $NC_{ac\text{-friendly}}(4)$](image_url)

The formulas to be stated in Theorems 1.7 and 1.8 below will also refer to a “canonical alternating colouring” of the blocks of a non-crossing partition, which is described next.

Definition 1.6. Let $m \in \mathbb{N}$ and $\sigma \in NC(m)$ be given. We will use the name canonical alternating colouring of $\sigma$ for the colouring $\text{calt}_\sigma : \{1, \ldots, m\} \to \{1, 2\}$ whose values on the blocks of $\sigma$ are determined by the following conditions:

(C-Alt1) Denoting by $W_1$ the block of $\sigma$ which contains the number 1, one has $\text{calt}_\sigma(W_1) = 1$.

(C-Alt2) If $W$ and $W'$ are “consecutive” outer blocks of $\sigma$, with $\min(W') = 1 + \max(W)$, then $\text{calt}_\sigma(W') \neq \text{calt}_\sigma(W)$.

(C-Alt3) If $V$ is an inner block of $\sigma$, then $\text{calt}_\sigma(V) \neq \text{calt}_\sigma(\text{Parent}_\sigma(V))$.

Note that if $\sigma \in NC(m)$ has a unique outer block, then $\text{calt}_\sigma$ simply follows the parities of the depths of blocks of $\sigma$. For a general $\sigma \in NC(m)$, $\text{calt}_\sigma$ first does an alternating colouring of the outer blocks of $\sigma$, going from left to right; then for every outer block $W$ of $\sigma$ one follows the vertical alternance idea in order to colour the blocks of $\sigma$ which are nested inside $W$.

In order to state the explicit formula for a joint Boolean cumulant of the kind indicated in (1.5) of Section 1.2, there is one last observation we need to make, namely that: in the canonical alternating colouring of a partition $\sigma \in NC_{ac\text{-friendly}}(2n)$ one can naturally read (encoded in the colouring) a tuple $\varepsilon \in \{1, *\}^n$, which will be denoted as “oddtuple($\sigma$)” – see Notation 5.4 below for the precise definition. We then have the following theorem.

Theorem 1.7. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space and let $(\beta_n : \mathcal{A}^n \to \mathbb{C})_{n=1}^{\infty}$ be the family of Boolean cumulant functionals associated to it. Consider two selfadjoint elements $a, b \in \mathcal{A}$ such that $a$ is freely independent from $b$, and consider the sequences of Boolean cumulants $(\beta_n(a))_{n=1}^{\infty}$ and $(\beta_n(b))_{n=1}^{\infty}$ of $a$ and of $b$ (where we use natural abbreviations such as $\beta_n(a) := \beta_n(a, \ldots, a)$, $n \in \mathbb{N}$).

1. For $n \in \mathbb{N}$ and $\varepsilon = (\varepsilon(1), \ldots, \varepsilon(n)) \in \{1, *\}^n$ such that $\varepsilon(1) = 1$, one has

$$
\beta_n((ab)^{\varepsilon(1)}, \ldots, (ab)^{\varepsilon(n)}) =
$$
of joint free cumulants that an \( R \) same distribution (due to radial symmetry displayed by \( a \) and \( b \) with is that this problem is vastly simplified when we make the additional hypothesis that the common distribution of \( ab \) and \( ba \) Lecture 15 of [11]). From here it follows in particular that In this case, the nonselfadjoint element \( ab \) Remark 1.10. In order to put things into perspective, we give here some background on Remark 1.9. (1) In Theorem 1.7 it should be noted that if we make \( \beta \) (1.12) In Theorem 1.7, we arrive to a formula for the Boolean cumulants of a free anticommutator.

**Theorem 1.8.** Consider the same framework and notation as in Theorem 1.7. Then, for every \( n \in \mathbb{N} \), the \( n \)-th Boolean cumulant of \( ab + ba \) is

\[
\beta_n(ab + ba) = \sum_{\sigma \in NC_{ac-friendly}(2n), \text{ such that oddtuple}(\sigma) = \varepsilon} \left( \prod_{U \in \sigma, \text{ with } \text{calt}_\sigma(U) = 1} \beta_{|U|}(a) \right) \cdot \left( \prod_{V \in \sigma, \text{ with } \text{calt}_\sigma(V) = 2} \beta_{|V|}(b) \right)
\]

(1.11)

\[
+ \sum_{\sigma \in NC_{ac-friendly}(2n), \text{ such that oddtuple}(\sigma) = \varepsilon'} \left( \prod_{U \in \sigma, \text{ with } \text{calt}_\sigma(U) = 1} \beta_{|U|}(b) \right) \cdot \left( \prod_{V \in \sigma, \text{ with } \text{calt}_\sigma(V) = 2} \beta_{|V|}(a) \right).
\]

**Remark 1.9.** (1) In Theorem 1.7 it should be noted that if we make \( \varepsilon = (1, 1, \ldots, 1) \), then what comes out is precisely the formula from [11] which was reviewed in Equation (1.4). A discussion of why this is the case appears in Remark 5.8 below.

(2) In Theorem 1.8 we note that formula (1.11) simplifies a lot when \( a \) and \( b \) have the same distribution. In this case, denoting by \( (\lambda_n)_{n=1}^\infty \) the common sequence of Boolean cumulants of \( a \) and of \( b \), we find that the \( n \)-th Boolean cumulant of \( ab + ba \) is

\[
\beta_n(ab + ba) = 2 \cdot \sum_{\sigma \in NC_{ac-friendly}(2n)} \prod_{V \in \sigma} \lambda_{|V|}.
\]

**Remark 1.10.** In order to put things into perspective, we give here some background on the past work around the problem of the free anticommutator. A noteworthy fact to begin with is that this problem is vastly simplified when we make the additional hypothesis that \( a \) and \( b \) have symmetric distributions (that is, \( \varphi(a^{2n-1}) = 0 = \varphi(b^{2n-1}) \) for all \( n \in \mathbb{N} \)). In this case, the nonselfadjoint element \( ab \in A \) has a certain “\( R \)-diagonal” property (cf. Lecture 15 of [11]). From here it follows in particular that \( ab + ba \) and \( i(ab - ba) \) have the same distribution (due to radial symmetry displayed by \( R \)-diagonal elements); moreover, the common distribution of \( ab + ba \) and \( i(ab - ba) \) is tractable due to the very special form of joint free cumulants that an \( R \)-diagonal element and its adjoint are known to have.
It is interesting that the combinatorial study of $i(ab - ba)$ remains tractable even when we drop the assumption of $a$ and $b$ having symmetric distributions, because one can follow (cf. [10]) how terms cancel in the expansions of the free cumulants $\kappa_n(i(ab - ba))$. The situation is not at all the same concerning $ab + ba$, where there are no cancellations to be followed. Here the expansions for moments or cumulants just create some large summations, and the combinatorial line of attack goes via precise identification of the combinatorial structures which appear as index sets for these large summations.

Another noteworthy possibility to be mentioned is via approaches that are plainly analytic in nature, and produce systems of equations which can in principle be used to calculate the Cauchy transform of $ab + ba$. Such a system of equations is proposed in [14]. Another possibility of proceeding on these lines is suggested by the linearization method championed in [4].

1.6. Equations with $\eta$-series.

The possibility of approaching the distribution of $ab + ba$ via a system of equations in (not necessarily convergent) power series can also be pursued in the framework of the present paper. Here we use the generating power series for Boolean cumulants, which are also known as $\eta$-series: for $a \in \mathcal{A}$, we put

\[
\eta_a(z) := \sum_{n=1}^{\infty} \beta_n(a) z^n \in \mathbb{C}[[z]].
\]

In Section 6 of the paper we make a detailed analysis of the $\eta$-series $\eta_{ab+ba}$ of a free anticommutator, and we come up with a system of equations which, when solved, leads to the explicit determination of this $\eta$-series. Our derivation of this system of equations is combinatorial in nature, and is intimately related to the study of recursions satisfied by ac-friendly non-crossing partitions.

The best way to describe our system of equations leading to $\eta_{ab+ba}$ is in a $2 \times 2$ matrix form, where we make use of some auxiliary power series $f_{a,a}, f_{a,a^*}, f_{a^*,a}, f_{a^*,a^*}$ grouped\(^2\) in a matrix

\[
F_a = \begin{bmatrix} f_{a,a} & f_{a,a^*} \\ f_{a^*,a} & f_{a^*,a^*} \end{bmatrix},
\]

and of some power series $f_{b,b}, \ldots, f_{b^*,b^*}$ likewise grouped in a $2 \times 2$ matrix $F_b$. For illustration, in this Introduction we present the special case when $a$ and $b$ have the same distribution. In this case we only need to refer to the matrix $F_a$, and we have the theorem stated next. (In the case when $a$ and $b$ are not required to have the same distribution, we get a more involved system of equations, where we use both matrices $F_a$ and $F_b$. This is described in Theorem [6.1 below].)

**Theorem 1.11.** Let $(\mathcal{A}, \varphi)$ be a $*$-probability space, and let $a, b$ be selfadjoint elements of $\mathcal{A}$ such that $a$ is free from $b$ and such that $a, b$ have the same distribution.

1. The matrix $F_a$ from Equation (1.14) is obtained by solving the matrix equation

\[
F_a H_a = \eta_a(z H_a),
\]

where $\eta_a$ is the $\eta$-series of $a$ (as in (1.13), and

\[
H_a := \begin{bmatrix} (1 - f_{a,a^*})^{-1} f_{a^*,a} & f_{a^*,a^*}^{-1} f_{a,a} \\ (1 - f_{a,a^*})^{-1} f_{a^*,a} & (1 - f_{a,a^*})^{-1} f_{a,a} \end{bmatrix}.
\]

\(^2\)It is convenient to have the entries of $F_a$ indexed by symbols $a$ and $a^*$, even though the intended use of $F_a$ is when $a$ is selfadjoint. The rationale for this notation is given at the beginning of Section 6.1.
(2) The $\eta$-series of $ab + ba$ can be obtained from the entries of $F_a$ via the equation

$$\eta_{ab+ba}(z^2) = 2\left(f_{a,a}(z) + \frac{f_{a,a}(z)f_{a^*,a^*}(z)}{1 - f_{a^*,a}(z)}\right).$$

**Remark 1.12.** Very much in agreement with the discussion at the beginning of Remark 1.10, the study of free anticommutators via equations in $\eta$-series also simplifies substantially in the case when $a$ and $b$ have symmetric distributions. In this case the matrices $F_a$ and $F_b$ mentioned above are sure to have some vanishing entries ($f_{a,a} = f_{a^*,a^*} = 0$ and $f_{b,b} = f_{b^*,b^*} = 0$), and the systems of equations that have to be solved become simpler, as shown in Proposition 6.4 and Corollary 6.5.

While the examples of free anticommutators of symmetric distributions are covered by the methods from [10], it is nevertheless interesting to work out some examples of this kind and combine them with a use "in reverse" of Theorem 1.8 in order to obtain corollaries about the enumeration of ac-friendly non-crossing partitions. For instance, in order to count the non-crossing partitions $\sigma \in NC_{ac-friendly}(2n)$ with the property that all blocks $V \in \sigma$ have even cardinality, one uses elements $a, b \in \mathcal{A}$ which are freely independent and have distribution $\frac{1}{2}(\delta_0 + \delta_2)$. The reason for choosing the latter distribution is that the common sequence $(\lambda_n)_{n=1}^\infty$ of Boolean cumulants for $a$ and $b$ simply has $\lambda_n = 1$ for $n$ even and $\lambda_n = 0$ for $n$ odd. In view of Remark 1.9(2), the Boolean cumulant $\beta_n(ab + ba)$ is then equal to twice the cardinality we are interested to determine. Upon combining this with the explicit formula obtained for $\eta_{ab+ba}$, we can determine precisely what is the required cardinality, as explained in Example 6.9 and Corollary 6.10 below.

**Example 1.13.** In the framework of Theorem 1.11 it is instructive to consider the simplest possible non-symmetric example, where both $a$ and $b$ have distribution $\frac{1}{2}(\delta_0 + \delta_2)$.

![Figure 2. Plot of the density of distribution of $ab + ba$ for $a, b$ free and having distribution $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_2$, together with a histogram of eigenvalues of random matrix approximation.](image-url)
The explicit formula for the density of this distribution and the calculations leading to it are presented in Proposition 6.11 below.

Figure 2 shows the graph of the density $f(x)$ found for the law of $ab + ba$. For a check, Figure 2 also shows a histogram of empirical eigenvalues distribution for $AB + BA$ where $A$ is a diagonal $6000 \times 6000$ matrix with half of diagonal entries equal 0 and half equal 2, and $B = UAU^*$ where $U$ is a random unitary matrix.

This example offers a very good illustration of how one gets to have different distributions for the free commutator and anticommutator – indeed, the law of the commutator $i(ab - ba)$ is easily found to be the arcsine distribution on $[-2, 2]$ (cf. Example 6.6, and the discussion in the paragraph preceding Proposition 6.11). We point out that this example has a combinatorial significance as well, and can be used (cf. Corollary 6.12) to infer the formula indicated in Equation (1.8) for the generating series of cardinalities of sets $NC_{ac\text{-friendly}}(2n)$.

1.7. Organization of the paper.
Besides the present Introduction, we have five other sections. After a review of background in Section 2, we discuss VNRP and prove Theorem 1.1 in Section 3. In Section 4 we discuss the applications of VNRP to free independence via Boolean cumulants. In Section 5 we prove the results about the joint Boolean cumulants of $ab$ and $(ab)^*$ and about the Boolean cumulants of the free anticommutator which were advertised in Section 1.5 above. Finally, in Section 6 we consider the conversion from Boolean cumulants to $\eta$-series, and prove the results that were advertised in Section 1.6 above.

2. Background and Notation

In this section we review some background on set-partitions, and the two types of cumulants we want to work with.

2.1. Nestings and depths for blocks of a non-crossing partition.
We start by reviewing, for the sake of setting notation, the definition of the two basic types of set-partitions used in this paper, the non-crossing partitions and the interval partitions.

Definition 2.1. (1) Let $n$ be a positive integer and let $\pi = \{V_1, \ldots, V_k\}$ be a partition of $\{1, \ldots, n\}$; that is, $V_1, \ldots, V_k$ are non-empty pairwise disjoint sets (called the blocks of $\pi$) with $V_1 \cup \cdots \cup V_k = \{1, \ldots, n\}$. The number $k$ of blocks of $\pi$ will be denoted as $|\pi|$, and we will occasionally use the notation “$V \in \pi$” to mean that $V$ is one of $V_1, \ldots, V_k$.

We say that $\pi \in NC(n)$ is an interval partition to mean that every block $V$ of $\pi$ is of the form $V = [i, j] \cap \mathbb{N}$ for some $1 \leq i \leq j \leq n$.

We say that $\pi$ is a non-crossing partition to mean that for every $1 \leq i_1 < i_2 < i_3 < i_4 \leq n$ such that $i_1$ is in the same block with $i_2$ and $i_3$ is in the same block with $i_4$, it necessarily follows that all of $i_1, \ldots, i_4$ are in the same block of $\pi$.

(2) For every $n \in \mathbb{N}$, we denote by $\text{Int}(n)$ the set of all interval partitions of $\{1, \ldots, n\}$, and we denote by $NC(n)$ the set of all non-crossing partitions of $\{1, \ldots, n\}$.

Remark 2.2. Clearly, one has $\text{Int}(n) \subseteq NC(n)$ for all $n \in \mathbb{N}$. It is not hard to see that $|\text{Int}(n)| = 2^{n-1}$ and that $NC(n)$ is counted by the $n$-th Catalan number:

$$|NC(n)| = \text{Cat}_n := \frac{(2n)!}{n!(n+1)!}, \quad \forall n \in \mathbb{N}.$$
For a more detailed introduction to the $NC(n)$’s, one can for instance consult Lectures 9 and 10 of [11].

Given a non-crossing partition $\pi \in NC(n)$, it is convenient to formalize the notion of “relative nesting” for blocks of $\pi$, as follows.

**Notation and Remark 2.3.** Let $n$ be in $\mathbb{N}$ and let $\pi$ be a partition in $NC(n)$.

1. Let $V, W$ be blocks of $\pi$. We will write “$V \nest\leq W$” to mean that we have the inequalities
   $\min(W) \leq \min(V)$ and $\max(W) \geq \max(V)$.
   We will write “$V \nest< W$” to mean that $V \nest\leq W$ and $V \neq W$. We will occasionally also use the notations $W \nest\geq V$ instead of $V \nest\leq W$ and $W \nest> V$ instead of $V \nest< W$.

2. It is immediate that “$\nest\leq$” is a partial order relation on the set of blocks of $\pi$. A block $W \in \pi$ which is maximal with respect to $\nest\leq$ will be said to be an outer block. A block $V \in \pi$ which is not outer will be said to be an inner block.

**Remark and Definition 2.4.** Let $n$ be in $\mathbb{N}$, let $\pi$ be a partition in $NC(n)$, and let $V$ be a block of $\pi$. It is easy to check that the set $\{W \in \pi \mid W \nest\geq V\}$ is totally ordered by $\nest\leq$. That is, we can write

\begin{equation}
\{W \in \pi \mid W \nest\geq V\} = \{V_1, \ldots, V_k\}
\end{equation}

where $k \geq 1$ and $V_1 \nest< V_2 \nest< \cdots \nest< V_k$. In (2.1) we note, in particular, that $V_1 = V$ and that $V_k$ is an outer block. The depth of $V$ in $\pi$ is defined as

$\text{depth}_{\pi}(V) := k - 1,$

with $k$ picked from Equation (2.1). If $k \geq 2$ (which is equivalent to saying that $\text{depth}_{\pi}(V) \neq 0$, or that $V$ is an inner block), then the block $V_2$ appearing in (2.1) is called the parent-block for $V$, and will be denoted as Parent$_{\pi}(V)$. The parent-block could be equivalently introduced via the requirement that

\begin{equation}
\begin{cases}
(i) & V \nest< \text{Parent}_{\pi}(V), \\
(ii) & \text{There is no block } V' \in \pi \text{ such that } V \nest< V' \nest< \text{Parent}_{\pi}(V).
\end{cases}
\end{equation}

**Remark 2.5.** Let $n$ be in $\mathbb{N}$ and let $\pi$ be a partition in $NC(n)$.

1. The notion of depth for the blocks of $\pi$ could also be defined recursively, by postulating that outer blocks have depth 0 and by making the requirement that

$\text{depth}_{\pi}(V) = 1 + \text{depth}_{\pi}(\text{Parent}_{\pi}(V))$ for every inner block $V \in \pi$.

2. As mentioned in the Introduction, for an $i \in \{1, \ldots, n\}$ we will sometimes write “$\text{depth}_{\pi}(i)$” in order to refer to the depth of the block of $\pi$ which contains the number $i$. Thus $\text{depth}_{\pi}$ can be viewed as a special example of colouring of $\pi$ (a function from $\{1, \ldots, n\}$ to $\mathbb{Z}$ which is constant along the blocks of $\pi$).

**Remark 2.6.** Let $n$ be in $\mathbb{N}$ and let $\pi$ be a partition in $NC(n)$. It is easy to see that one can always list the set of outer blocks of $\pi$ as $\{W_1, \ldots, W_\ell\}$ in such a way that

\begin{equation}
\begin{cases}
\min(W_1) = 1, \max(W_\ell) = n, \text{ and } \\
\min(W_{i+1}) = 1 + \max(W_i) \text{ for every } 1 \leq i < \ell.
\end{cases}
\end{equation}
In the case when the number $\ell$ of outer blocks of $\pi$ is $\ell = 1$, the second condition in (2.2) is vacuous ($\pi$ has a unique outer block $W$, with $1, n \in W$).

In the notation from (2.2): the interval partition $\pi := \{J_1, \ldots, J_\ell\}$ with $J_i := [\min(W_i), \max(W_i)] \cap \mathbb{N}$, for $1 \leq i \leq \ell$ is sometimes called the *closure* of $\pi$ in the set of interval-partitions. Note that knowing what is $\pi$ provides exactly the same information as knowing the set $\text{OuterMax}(\pi)$ which was introduced in Definition 1.4 of the Introduction.

### 2.2. Review of free and of Boolean cumulant functionals.

Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space (in purely algebraic sense) – that is, $\mathcal{A}$ is a unital algebra over $\mathbb{C}$ and $\varphi : \mathcal{A} \to \mathbb{C}$ a linear functional with $\varphi(1_\mathcal{A}) = 1$. In this subsection we briefly review the definition of the free and the Boolean cumulants of $(\mathcal{A}, \varphi)$.

Before starting, we record a customary notation which will appear in the formulas for both types of cumulants: given an $n \in \mathbb{N}$, a tuple $(a_1, \ldots, a_n) \in \mathcal{A}^n$, and a non-empty subset $S = \{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}$ with $i_1 < \cdots < i_m$, we denote

\[(a_1, \ldots, a_n) | S := (a_{i_1}, \ldots, a_{i_m}) \in \mathcal{A}^m.\]

**Definition 2.7.** Notations as above.

(1) The *free cumulants* associated to $(\mathcal{A}, \varphi)$ are the family of multilinear functionals $(\kappa_n : \mathcal{A}^n \to \mathbb{C})_{n=1}^\infty$ which is uniquely determined by the requirement that

\[\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} \prod_{V \in \pi} \kappa_{|V|}( (a_1, \ldots, a_n) | V ),\]

holding for all $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathcal{A}$.

(2) The *Boolean cumulants* associated to $(\mathcal{A}, \varphi)$ are the family of multilinear functionals $(\beta_n : \mathcal{A}^n \to \mathbb{C})_{n=1}^\infty$ which is uniquely determined by the requirement that

\[\varphi(a_1 \cdots a_n) = \sum_{\pi \in \text{Int}(n)} \prod_{V \in \pi} \beta_{|V|}( (a_1, \ldots, a_n) | V ),\]

holding for all $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathcal{A}$.

**Remark 2.8.** It is easy to see that the families of equations indicated in either (2.4) or (2.5) have unique solutions. Indeed, all that actually matters is that the index sets $\text{Int}(n)$ and $NC(n)$ for the summations on the right-hand sides of these equations contain the partition, usually denoted as “$1_n$”, of the set $\{1, \ldots, n\}$ into only one block. For instance in connection to (2.4): by separating the term indexed by $1_n$ on the right-hand side, this equation can be written as

\[\kappa_n(a_1, \ldots, a_n) = \varphi(a_1, \cdots, a_n) - \sum_{\substack{\pi \in NC(n), \ \pi \neq 1_n}} \prod_{V \in \pi} \kappa_{|V|}( (a_1, \ldots, a_n) | V ).\]

Then (2.6) can be used as an explicit definition of the functional $\kappa_n$, under the assumption that explicit formulas for $\kappa_1, \ldots, \kappa_{n-1}$ have already been determined. A similar recursive argument holds in connection to solving the system of equations indicated in (2.5)
It is in fact not difficult to write in a really explicit way some formulas giving the $\kappa_n$’s and the $\beta_n$’s in terms of $\varphi$. This is not needed in the present paper, so we only mention that the way to do it goes by using some standard elements of “Möbius inversion theory in a partially ordered set” (as presented e.g. in Chapter 3 of the monograph [13]).

There also is a nice direct formula which expresses Boolean cumulants in terms of free cumulants, as follows.

**Proposition 2.9.** Let $(A, \varphi)$ be a noncommutative probability space, and let $(\kappa_n : A^n \to \mathbb{C})_{n=1}^\infty$ and $(\beta_n : A^n \to \mathbb{C})_{n=1}^\infty$ be the free and respectively the Boolean cumulants associated to $(A, \varphi)$. For every $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in A$ one has

$$\beta_n(a_1, \ldots, a_n) = \sum_{\pi \in NC(n) \text{ with unique outer block}} \prod_{V \in \pi} \kappa_{|V|}(a_1, \ldots, a_n | V).$$

The proof of Proposition 2.9 can e.g. be obtained by an immediate re-phrasing of the argument proving Proposition 3.9 in [1].

**Notation and Remark 2.10.** Let $(A, \varphi)$ be a noncommutative probability space, and let $(\kappa_n : A^n \to \mathbb{C})_{n=1}^\infty$ and $(\beta_n : A^n \to \mathbb{C})_{n=1}^\infty$ be the free and respectively the Boolean cumulants associated to $(A, \varphi)$. Let $a \in A$ be given. We will use the abbreviations $\beta_n(a) := \beta_n(a, \ldots, a)$ and $\kappa_n(a) := \kappa_n(a, \ldots, a)$, $n \in \mathbb{N}$.

The power series

$$M_a(z) = \sum_{n=1}^\infty \varphi(a^n)z^n, \ R_a(z) = \sum_{n=1}^\infty \kappa_n(a)z^n \text{ and } \eta_a(z) = \sum_{n=1}^\infty \beta_n(a)z^n$$

are called the moment series, the $R$-transform and respectively the $\eta$-series associated to $a$.

In this paper, an important role is played by $\eta$-series. We note that, as an easy consequence of the formula (2.5) connecting moments to Boolean cumulants, one has a very simple relation between $\eta_a$ and $M_a$:

$$M_a(z) = \eta_a(z)/(1 - \eta_a(z)), \text{ or equivalently, } \eta_a(z) = M_a(z)/(1 + M_a(z)).$$

### 2.3. Cumulants with products as arguments.

When working with cumulants of any kind, it is good to have an efficient formula for what happens when every argument of the cumulant is a product of elements of the underlying algebra. For free cumulants, this formula was put into evidence in [3]. We will need here the analogous fact for Boolean cumulants. In order to state this fact and to explain the analogy with [6], we need to use the lattice structure (with respect to the partial order by reverse refinement) on $NC(n)$ and on $\text{Int}(n)$, so we first do a brief review of this structure.

**Definition 2.11.** Let $n$ be a positive integer.

(1) On $NC(n)$ we consider the partial order by reverse refinement, where for $\pi, \rho \in NC(n)$ we put

$$\pi \leq \rho \quad \text{def} \quad \left( \text{every block of } \rho \text{ is a union of blocks of } \pi \right).$$

The partially ordered set $(NC(n), \leq)$ turns out to be a lattice. That is, every $\pi_1, \pi_2 \in NC(n)$ have a least common upper bound, denoted as $\pi_1 \vee \pi_2$, and have a greatest common
lower bound, denoted as $\pi_1 \land \pi_2$. One refers to $\pi_1 \lor \pi_2$ and to $\pi_1 \land \pi_2$ as the *join* and respectively as the *meet* of $\pi_1$ and $\pi_2$ in $NC(n)$.

We will use the notation $0_n$ for the partition of $\{1, \ldots, n\}$ into $n$ singleton blocks and the notation $1_n$ for the partition of $\{1, \ldots, n\}$ into one block. It is immediate that $0_n, 1_n \in NC(n)$ and that $0_n \leq \pi \leq 1_n$ for all $\pi \in NC(n)$.

(2) Consider the restriction of the partial order by reverse refinement from $NC(n)$ to $\text{Int}(n)$. For $\pi_1, \pi_2 \in \text{Int}(n)$, the partitions $\pi_1 \lor \pi_2, \pi_1 \land \pi_2 \in NC(n)$ which were defined in (1) above turn out to still belong to $\text{Int}(n)$. As a consequence, $(\text{Int}(n), \leq)$ is a lattice as well, and for $\pi_1, \pi_2 \in \text{Int}(n)$ there is no ambiguity in the meaning of what are $\pi_1 \lor \pi_2$ and $\pi_1 \land \pi_2$ (considering the join and meet of $\pi_1$ and $\pi_2$ in $\text{Int}(n)$ gives the same result as when considering them in $NC(n)$).

The special partitions $0_n$ and $1_n$ considered in (1) belong to $\text{Int}(n)$, hence they also serve as minimum and maximum elements for the poset $(\text{Int}(n), \leq)$.

The formula for Boolean cumulants with products as entries is then stated as follows.

**Proposition 2.12.** Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space, and let $(\beta^n : \mathcal{A}^n \to \mathbb{C})_{n=1}^{\infty}$ be the family of Boolean cumulant functionals of $(\mathcal{A}, \varphi)$. Consider a Boolean cumulant of the form $\beta_m(x_1, \ldots, x_m)$ where each of the elements $x_1, \ldots, x_m \in \mathcal{A}$ is written as a product:

$$x_i = a_1 \cdots a_{i(1)}, \quad x_2 = a_{i(1)+1} \cdots a_{i(2)}, \ldots, \quad x_m = a_{i(m-1)+1} \cdots a_{i(m)},$$

where $1 \leq i(1) < i(2) < \cdots < i(m) =: n$ are some positive integers, and where $a_1, \ldots, a_n \in \mathcal{A}$. Then one has

$$\beta_m(x_1, \ldots, x_m) = \sum_{\pi \in \text{Int}(n) \text{ such that } \pi \lor \sigma = 1_n} \prod_{V \in \pi} \beta_{|V|}(\{a_1, \ldots, a_n\} \setminus V),$$

with

$$\sigma := \{\{1, \ldots, i(1)\}, \{i(1)+1, \ldots, i(2)\}, \ldots, \{i(m-1)+1, \ldots, i(m)\}\} \in \text{Int}(n).$$

The proof of Proposition 2.12 is left as an exercise to the reader. A way to do it is by going over the development presented on pages 178-180 of [11] about free cumulants with products as arguments, and by replacing everywhere on those pages the occurrences of lattices of non-crossing partitions by occurrences of lattices of interval partitions. The statement of Proposition 2.12 should then emerge as the Boolean analogue of Theorem 11.12(2) of [11].

**Remark 2.13.** The lattice $\text{Int}(n)$ is in fact a Boolean lattice. Indeed, one has a natural bijection which identifies $\text{Int}(n)$ to the lattice of subsets of $\{1, \ldots, n-1\}$, by sending a partition $\pi = \{J_1, \ldots, J_k\} \in \text{Int}(n)$ to the set $\{\max(J_1), \ldots, \max(J_k)\} \setminus \{n\} \subseteq \{1, \ldots, n-1\}$. By using this fact it is easy to see that, with $\sigma$ as defined in Equation (2.10), a partition $\pi \in \text{Int}(n)$ has

$$(\pi \lor \sigma = 1_n) \iff (i(p) \text{ and } i(p) + 1 \text{ belong to the same block of } \pi, \text{ for all } 1 \leq p \leq m - 1).$$

This allows for a somewhat more convenient re-phrasing of the join condition invoked on the right-hand side of Equation (2.9).
3. The partial order $\ll$, VNRP, and the proof of Theorem 1.1

3.1. The partial order $\ll$ on $NC(n)$, and its upper ideals.

In this paper we also make use of another partial order relation on $NC(n)$, coarser than reverse refinement, which is denoted as “$\ll$”. The partial order $\ll$ has been used for some time in free probability (starting with [1]), in the description of relations between free and Boolean cumulants.

Definition and Remark 3.1. (The partial order “$\ll$”.)

(1) For $\pi, \rho \in NC(n)$, we will write $\pi \ll \rho$ to mean that $\pi \leq \rho$ and that, in addition, for every block $W$ of $\rho$ there exists a block $V$ of $\pi$ such that $\min(W), \max(W) \in V$.

(2) Since in this paper we have a lot of occurrences of the special case “$\pi \ll 1_n$”, let us record the obvious fact that this simply amounts to requiring $\pi$ to have a unique outer block $W$, with $1, n \in W$.

(3) It is immediate that $\text{Int}(n)$ is precisely equal to the set of maximal elements of the poset $(NC(n), \ll)$.

(4) A significant point about the partial order $\ll$ on $NC(n)$ is that we have a nice structure for its upper ideals, that is, for the sets of non-crossing partitions of the form

\begin{equation}
\{ \rho \in NC(n) \mid \rho \gg \pi \}, \text{ for a fixed } \pi \in NC(n).
\end{equation}

This was noticed in Section 2 of [1], but the discussion around the set (3.1) was mostly done in a proof (cf. proof of Proposition 2.13 in [1]), and it will be useful for our present purposes to spell that out in more detail. It turns out to be convenient to use a notion of “projection map” for blocks of a fixed $\pi \in NC(n)$, as introduced in the next definition.

Definition 3.2. Let $n$ be in $\mathbb{N}$ and let $\pi$ be a partition in $NC(n)$. A block-projection for $\pi$ is a map $\Phi : \pi \to \pi$ which has the following properties.

(i) $\Phi$ is a projection map; that is, $\Phi \circ \Phi = \Phi$.

(ii) If $A, B \in \pi$ and if $A \leq B$, then it follows that $\Phi(A) \leq \Phi(B)$.

(iii) $A \leq \Phi(A)$ for all $A \in \pi$.

For such a $\Phi$ we will denote $\text{Ran}(\Phi) := \{ B \in \pi \mid \exists A \in \pi \text{ such that } \Phi(A) = B \}$. Note that, due to the property (i) satisfied by $\Phi$, we can also write $\text{Ran}(\Phi) = \{ B \in \pi \mid \Phi(B) = B \}$.

Remark 3.3. Let $n$ be in $\mathbb{N}$ and let $\pi$ be a partition in $NC(n)$.

(1) Let $\Phi : \pi \to \pi$ be a block-projection for $\pi$. Property (iii) satisfied by $\Phi$ implies that $\text{Ran}(\Phi)$ contains all the outer blocks of $\pi$.

(2) Let $\Phi, \Psi : \pi \to \pi$ be block-projections for $\pi$, and suppose that $\text{Ran}(\Phi) = \text{Ran}(\Psi)$. Then $\Phi = \Psi$. Indeed, for every block $A \in \pi$ we can apply $\Psi$ to both sides of the relation $A \leq \Phi(A)$ to get that $\Psi(A) \leq \Psi(\Phi(A)) = \Phi(A)$, (where the latter equality holds because $\Phi(A) \in \text{Ran}(\Phi) = \text{Ran}(\Psi)$, hence $\Phi(A)$ is fixed by $\Psi$). A symmetric argument gives that $\Phi(A) \leq \Psi(A)$, and it follows that $\Phi(A) = \Psi(A)$, as required.

Lemma 3.4. Let $n$ be in $\mathbb{N}$ and let $\pi$ be a partition in $NC(n)$. Let $\mathcal{M}$ be a subset of $\pi$ such that $\mathcal{M}$ contains all the outer blocks of $\pi$. Then there exists a block-projection $\Phi : \pi \to \pi$, uniquely determined, such that $\text{Ran}(\Phi) = \mathcal{M}$.
Proof. Uniqueness of $\Phi$ follows from the preceding remark. In order to prove existence, we use the following prescription to define $\Phi$:

$$
(3.2) \quad \begin{cases} 
\text{if } A \in \mathcal{M}, \text{ then } \Phi(A) = A; \\
\text{if } A \in \pi \setminus \mathcal{M}, \text{ then } \Phi(A) = \Phi(\text{Parent}_\pi(A)).
\end{cases}
$$

The definition proposed via Equations (3.2) is consistent because if we start with any $A \in \pi$ and do iterations of the Parent$_\pi$ map on it, we will eventually have to find a block that belongs to $\mathcal{M}$. Or more precisely: if we start with $A \in \pi$ and we write explicitly

$$
\{B \in \pi \mid B \geq A\} = \{B_1, \ldots, B_k\}
$$

in the way indicated in Remark 2.34, then Equations (3.2) define $\Phi(A) = B_j$ with $j := \min\{i \in \{1, \ldots, k\} \mid B_i \in \mathcal{M}\}$. □

Remark 3.5. One has a natural construction of block-projection map $\Phi : \pi \rightarrow \pi$ which arises whenever we are given two partitions $\pi, \rho \in NC(n)$ such that $\pi \ll \rho$. Recall that, in this situation, for every block $X \in \rho$ there exists a block $B \in \pi$ such that

$$
(3.3) \quad B \subseteq X \text{ and } \min(B) = \min(X), \max(B) = \max(X).
$$

We then define $\Phi : \pi \rightarrow \pi$ as follows: for every $A \in \pi$ we consider the (unique) block $X \supseteq A$, and then we define $\Phi(A) := B$, where $B$ is as in (3.3). It is easy to check that the map $\Phi : \pi \rightarrow \pi$ defined in this way fulfills the conditions (i), (ii) and (iii) from Definition 3.2, hence is indeed a block-projection map for $\pi$.

Let us record that, in the terminology introduced in [1], a block $B$ as in (3.3) is said to be a $\rho$-special block of $\pi$. The block-projection $\Phi$ constructed above is characterized by the fact that $\text{Ran}(\Phi)$ is precisely the set of all $\rho$-special blocks of $\pi$.

We now come to the main point concerning the set of partitions indicated in (3.1), namely that it is actually “parametrized ” by the set of block-projection maps for $\pi$, where the parametrization is just the inverse of the natural construction indicated in Remark 3.5. The formal statement of how this works is recorded in the next proposition.

Proposition 3.6. Let $n$ be in $\mathbb{N}$, let $\pi$ be a partition in $NC(n)$, and let $\Phi : \pi \rightarrow \pi$ be a block-projection map. Then there exists $\rho \in NC(n)$, uniquely determined, such that $\rho \gg \pi$ and such that $\Phi$ is obtained from $\pi$ and $\rho$ by using the recipe described in Remark 3.5. If we list the range of $\Phi$ as $\text{Ran}(\Phi) = \{B_1, \ldots, B_p\}$, then the partition $\rho$ can be described explicitly as $\rho = \{X_1, \ldots, X_p\}$, where

$$
(3.4) \quad X_j = \cup_{A \in \Phi^{-1}(B_j)} A, \quad 1 \leq j \leq p.
$$

The proof of Proposition 3.6 amounts essentially to reproducing the proof of Proposition 2.13 from [1], but where we work with $\Phi$ itself rather than writing all the arguments in terms of the set of blocks $\text{Ran}(\Phi)$. We note that in view of Lemma 3.4 the parametrization of $\{\rho \in NC(n) \mid \rho \gg \pi\}$ in terms of block-projections for $\pi$ can also be viewed as a parametrization in terms of subsets of $\pi$ which contain all the outer blocks – this is, actually, what was observed in Proposition 2.13 of [1] and in the proof of that proposition.

We conclude the discussion about $\ll$ with an observation that will be needed in the next subsection. This observation does not depend on Proposition 3.6; it is just a direct consequence of how the notion of “$\rho$-special block of $\pi$”, is defined. It goes as follows.

Lemma 3.7. Let $n$ be in $\mathbb{N}$, and let $\pi, \rho \in NC(n)$ be such that $\pi \ll \rho$. Let $A$ be a $\rho$-special block of $\pi$ which is not outer, and let $B = \text{Parent}_\pi(A) \in \pi$. Let $X,Y$ be the blocks of $\rho$ determined by the requirements that $X \supseteq A$ and $Y \supseteq B$. Then $Y = \text{Parent}_\rho(X)$. □
The proof of Lemma 3.7 is done by an elementary argument, directly from the definitions of the notions involved. (One must keep in mind, of course, that the hypothesis “\(A\) is \(\rho\)-special” means, by definition, that \(\min(A) = \min(X)\) and \(\max(A) = \max(X)\).)

### 3.2. VNRP, and the proof of Theorem 1.1

Throughout this whole subsection we fix the data used in the statement of Theorem 1.1. That is, we fix two positive integers \(m\) and \(s\), and a function \(c : \{1, \ldots, m\} \to \{1, \ldots, s\}\). (We think of \(c\) as of a “colouring of \(\{1, \ldots, m\}\) in \(s\) colours”.) We will denote by \(NC(m; c)\) the subset of \(NC(m)\) defined by

\[
NC(m; c) := \{\sigma \in NC(m) \mid c \text{ is constant on every block of } \sigma\}.
\]

For \(\sigma \in NC(m; c)\) and \(A \in \sigma\), we will use the notation \(c(A)\) for the common value \(c(a) \in \{1, \ldots, s\}\) taken by \(c\) on all \(a \in A\). Note that on \(NC(m; c)\) we have two partial order relations “\(<\)” (reverse refinement) and “\(<\)”, induced from \(NC(n)\).

The definition of VNRP goes as follows.

**Definition 3.8.** A partition \(\sigma \in NC(m; c)\) will be said to have the **vertical no-repeat property** with respect to \(c\) when the following happens: for every inner block \(A \in \sigma\), one has

\[
c(\text{Parent}_\sigma(A)) \neq c(A).
\]

As already done in the Introduction, we will refer to the vertical no-repeat property by using the acronym “VNRP”. (Note that if the number of colours \(s\) would happen to be \(s = 2\), VNRP could also go under the name of “vertical alternance property”.)

Our goal for the section is to prove Theorem 1.1 stated in the Introduction. In order to do so, we start with an adjustment of Proposition 3.6 to the present framework which uses coloured partitions from \(NC(m; c)\).

**Proposition 3.9.** Let \(\sigma\) be a partition in \(NC(m; c)\). Let \(\Phi : \sigma \to \sigma\) be a block-projection map, and let \(\rho \in NC(m)\) be the partition with \(\rho \gg \sigma\) which is parametrized by \(\Phi\) in Proposition 3.6. We have that:

\[
\rho \in NC(m; c) \iff (c(\Phi(A)) = c(A), \ \forall A \in \sigma).
\]

**Proof.** “\(\Rightarrow\)” From the concrete description of \(\rho\) provided by Equation (3.4) of Proposition 3.6, it is clear that for every \(A \in \sigma\), the blocks \(A\) and \(\Phi(A)\) are contained in the same block \(X\) of \(\rho\). The latter fact implies in particular that \(c(A) = c(X) = c(\Phi(A))\). The condition on \(\Phi\) listed on the right-hand side of (3.5) thus follows.

“\(\Leftarrow\)” Let \(X\) be a block of \(\rho\), and consider the block \(B\) of \(\sigma\) such that \(\min(X), \max(X) \in B\). The explicit description of \(\rho\) provided by Equation (3.4) in Proposition 3.6 tells us that

\[
X = \cup_{A \in \Phi^{-1}(B)} A.
\]

Denoting \(c(B) = s_o \in \{1, \ldots, s\}\), we see that \(c(A) = s_o\) for every \(A\) appearing in the union from (3.6) – indeed, one has \(\Phi(A) = B\), hence the hypothesis \(c(A) = c(\Phi(A))\) gives \(c(A) = c(B) = s_o\). This makes it clear that \(c\) is constantly equal to \(s_o\) on \(X\), and completes the verification that \(\rho \in NC(n; c)\).

**Corollary 3.10.** Let \(\sigma, \rho\) be partitions in \(NC(m; c)\) such that \(\sigma \ll \rho\). Let \(\Phi : \sigma \to \sigma\) be the block-projection map which corresponds to \(\rho\) in Remark 3.5. One has

\[
\text{Ran}(\Phi) \supseteq \{A \in \sigma \mid A \text{ is inner and } c(\text{Parent}_\sigma(A)) \neq c(A)\}.
\]
Proof. We prove the reverse inclusion for the complements of the sets indicated in in (3.7). That is: we pick \( A_o \in \sigma \setminus \text{Ran}(\Phi) \), and we prove that \( A_o \) belongs to the complement of the set on the right-hand side of (3.7). The condition \( A_o \not\in \text{Ran}(\Phi) \) implies in particular that \( A_o \) is inner, so what we have to prove is the equality \( c(\text{Parent}_o(A_o)) = c(A_o) \).

We denote \( \text{Parent}_o(A_o) = A_1 \). It is easy to see (directly from the properties of \( \Phi \) listed in Definition 3.2) that the assumption \( A_o \not\in \text{Ran}(\Phi) \) (which is equivalent to \( \Phi(A_o) \neq A_o \)), hence to \( A_o < \Phi(A_o) \) entails the equality \( \Phi(A_o) = \Phi(A_1) \). On the other hand, the assumption that \( \rho \in NC(m; c) \) entails, via Proposition 3.9, the equalities \( c(A_o) = c(\Phi(A_o)) \) and \( c(A_1) = c(\Phi(A_1)) \). By putting all these things together we find that \( c(A_o) = c(\Phi(A_o)) = c(\Phi(A_1)) = c(A_1) \). Hence \( c(\text{Parent}_o(A_o)) = c(A_o) \), as required.

The next proposition addresses the uniqueness part in the statement of Theorem 1.1 by giving an explicit description of the set of \( \tau \)-special blocks of \( \sigma \), for the partition \( \tau \gg \sigma \) which is needed in the conclusion of the theorem.

Proposition 3.11. Let \( \sigma \) be a partition in \( NC(m; c) \). Suppose that \( \tau \in NC(m; c) \) has VNRP, and is such that \( \tau \gg \sigma \). Then the set of \( \tau \)-special blocks of \( \sigma \) is equal to

\[
\{ A \in \sigma \mid A \text{ is outer} \} \cup \{ A \in \sigma \mid A \text{ is inner and } c(\text{Parent}_\tau(A)) \neq c(A) \}.
\]

Proof. Let \( \Phi : \sigma \to \tau \) be the block-projection map which parametrizes \( \tau \) in the way described in Proposition 3.6. The set of \( \tau \)-special blocks of \( \sigma \) is thus the same as \( \text{Ran}(\Phi) \). Corollary 3.10 then assures us that the set of \( \tau \)-special blocks of \( \sigma \) contains all blocks \( A \in \sigma \) such that \( A \) is inner and \( c(\text{Parent}_\tau(A)) \neq c(A) \). Since \( \text{Ran}(\Phi) \) is also sure to contain all the outer blocks of \( \sigma \) (Remark 3.3(1)), it follows that \( \text{Ran}(\Phi) \) must contain the set of blocks indicated in formula (3.8).

Let us assume, for contradiction, that \( \text{Ran}(\Phi) \) is strictly larger than the set from (3.8), i.e. that it contains a block \( V \in \sigma \) which is inner and has \( c(\text{Parent}_\tau(V)) = c(V) \). This \( V \) is a \( \tau \)-special block of \( \sigma \) (since it is in \( \text{Ran}(\Phi) \)), hence we can use Lemma 3.7 with \( A := V \) and \( B := \text{Parent}_\sigma(V) \). Denoting by \( X, Y \) the blocks of \( \tau \) which contain \( A \) and \( B \), respectively, we get from Lemma 3.7 that \( \text{Parent}_\tau(X) = Y \). Now, we must have \( c(A) = c(X) \) and \( c(B) = c(Y) \) (since \( A \subseteq X \) and \( B \subseteq Y \)), so from \( c(A) = c(B) \) it follows that \( c(X) = c(Y) \). This contradicts the VNRP of \( \tau \), and concludes the proof.

For the existence part in Theorem 1.1 one could go by showing directly that the “candidate for \( \tau^\prime \)” suggested by Proposition 3.11 does indeed the required job. We leave this approach as an exercise to the interested reader, and we just invoke here a simple maximality argument.

Lemma 3.12. Consider the partial order given by \( \ll \) on \( NC(m; c) \). Every maximal element of \( (NC(m; c), \ll) \) has the VNRP property.

Proof. We prove the contrapositive: if \( \pi \in NC(m; c) \) does not have VNRP, then it cannot be a maximal element with respect to \( \ll \). Indeed, let us pick a \( \pi \in NC(m; c) \) without VNRP, and let \( V, V' \) be blocks of \( \pi \) such that \( V' = \text{Parent}_\pi(V) \) but \( V, V' \) have the same colour. Let \( \rho \) be the partition of \( \{1, \ldots, n\} \) which is obtained out of \( \pi \) by joining together the blocks \( V \) and \( V' \). Then \( \rho \in NC(m; c) \) and \( \pi \ll \rho \) (this is a slight modification of Lemma 6.4.3 from [1]), showing in particular that \( \pi \) is not maximal with respect to \( \ll \).
Remark 3.13. We leave it as an exercise to the reader to check that the converse of Lemma 3.12 is true as well: the maximal elements of \((NC(m;c), \ll)\) are precisely the partitions which have VNRP with respect to the colouring \(c\).

3.14. Proof of Theorem 1.1. We fix a partition \(\sigma \in NC(m;c)\) and we have to prove that there exists a \(\tau \in NC(m;c)\), uniquely determined, such that \(\sigma \ll \tau\) and such that \(\tau\) has VNRP with respect to \(c\). And indeed: the uniqueness of \(\tau\) with the required properties follows from Proposition 3.11. On the other hand, the existence of \(\tau\) follows from Lemma 3.12 and the fact that \(\sigma\) must have a majorant which is maximal with respect to \(\ll\). □

4. Free independence in terms of Boolean cumulants, via VNRP

Based on Theorem 1.1, one gets the characterization of free independence in terms of Boolean cumulants which was announced in Theorem 1.2 from the Introduction.

4.1. Proof of Theorem 1.2. Recall that in this theorem we are given a noncommutative probability space \((A, \varphi)\) and some unital subalgebras \(A_1, \ldots, A_s \subseteq A\), and we have to prove the equivalence of two statements (1) and (2) concerning \(A_1, \ldots, A_s\). Throughout the proof we use the notation \((\beta_n : A^n \to \mathbb{C})_{n=1}^{\infty}\) and respectively \((\kappa_n : A^n \to \mathbb{C})_{n=1}^{\infty}\) for the families of Boolean and respectively free cumulant functionals of \((A, \varphi)\).

Proof that (1) \(\Rightarrow\) (2). Here we know that \(A_1, \ldots, A_s\) are freely independent with respect to \(\varphi\). We consider an \(n \in \mathbb{N}\), a colouring \(c : \{1, \ldots, n\} \to \{1, \ldots, s\}\), and some elements \(a_1 \in A_{c(1)}, \ldots, a_n \in A_{c(n)}\), and we have to prove that

\[
\beta_n(a_1, \ldots, a_n) = \sum_{\pi \in NC(n;c), \pi \ll 1_n} \prod_{V \in \pi} \beta_{|V|}((a_1, \ldots, a_n) | V).
\]

To that end, we write

\[
\beta_n(a_1, \ldots, a_n) = \sum_{\pi \in NC(n), \pi \ll 1_n} \prod_{V \in \pi} \kappa_{|V|}((a_1, \ldots, a_n) | V) = \sum_{\pi \in NC(n,c), \pi \ll 1_n} \prod_{V \in \pi} \kappa_{|V|}((a_1, \ldots, a_n) | V),
\]

where at the first equality sign we used the formula expressing Boolean cumulants in terms of free cumulants, and at the second equality sign we used the vanishing of mixed free cumulants with entries from the free subalgebras \(A_1, \ldots, A_s\).

We now invoke Theorem 1.1 and group the partitions \(\pi \in NC(n,c), \pi \ll 1_n\) which index the latter sum according to the unique \(\rho \in NC(n;c)\) such that \(\pi \ll \rho\) and \(\rho\) has VNRP with respect to \(c\). When doing so, we continue the above equalities with

\[
\sum_{\rho \in NC(n,c), \rho \ll 1_n} \left( \sum_{\pi \ll \rho \text{ V has VNRP}} \prod_{V \in \pi} \kappa_{|V|}((a_1, \ldots, a_n) | V) \right).
\]

Finally, in the latter double sum we note that the inside summation comes to
\[ \prod_{V \in \rho} \beta_V((a_1, \ldots, a_n) \mid V) \] (again due to how Boolean cumulants are expressed in terms of free cumulants). This leads precisely to the formula for \( \beta_n(a_1, \ldots, a_n) \) that was stated in Equation (4.1).

**Proof that (2) \( \Rightarrow \) (1).** In order to prove that \( A_1, \ldots, A_s \) are free, we consider the free product \( \tilde{A}, \tilde{\varphi} = \ast (A_i, \varphi_i)_{i=1,\ldots,s} \). Then for any \( a_1, \ldots, a_n \) such that \( a_k \in A_{i(k)} \), formula (4.1) holds for Boolean cumulants related with \( \varphi_i \), denoted by \( \beta_{\varphi_i} \) by assumption. On the other hand (4.1) holds also for \( \beta_{\tilde{\varphi}} \), i.e. Boolean cumulants related with \( \tilde{\varphi} \), since \( a_1, \ldots, a_n \) are free wrt \( \tilde{\varphi} \) hence the implication (1) \( \Rightarrow \) (2), proved above, can be applied. Since for any \( a_1, \ldots, a_n \) such that \( a_k \in A_{i(k)} \) we have \( \beta_{\varphi_i}(a_1, \ldots, a_n) = \beta_{\tilde{\varphi}}(a_1, \ldots, a_n) \), then also all joint moments with respect to \( \varphi \) and \( \tilde{\varphi} \) coincide. Since \( A_1, \ldots, A_s \) are free with respect to \( \tilde{\varphi} \), we get that \( A_1, \ldots, A_s \) are free with respect to \( \varphi \). \( \Box \)

**Example 4.2.** For the sake of clarity, we give a concrete example of how the formula (4.1) works. Let \( A_1, A_2 \) be freely independent subalgebras of \( A \). Suppose we pick elements \( a, a', a'' \in A_1 \) and \( b, b', b'' \in A_2 \), and we are interested in the Boolean cumulant \( \beta_6(a, b, a', b', b'', a'') \).

![Diagram](image1.png)

**Figure 3(a).** Some coloured partitions in \( NC(6) \), with VNRP.

We get

\[
(4.2) \quad \beta_6(a, b, a', b', b'', a'') = \beta_3(a, a', a'')\beta_1(b)\beta_1(b')\beta_1(b'') + \beta_3(a, a', a'')\beta_1(b)\beta_2(b', b'') + \beta_2(a, a'')\beta_1(a')\beta_3(b, b', b'') + \beta_2(a, a'')\beta_1(a')\beta_2(b, b')\beta_1(b''),
\]

where the four terms on the right-hand side of the above equation correspond to the four partitions in \( NC(6) \) that are listed in Figure 3(a). It is instructive to note that one also has two partitions in \( NC(6) \), shown in Figure 3(b), which satisfy the colouring condition (i.e. they separate \( a \)'s from \( b \)'s in the tuple \( (a, b, a', b', b'', a'') \)) and also satisfy the requirement of having a unique outer block), but don’t contribute to the sum on the right-hand side of (4.2) because they don’t have VNRP.

![Diagram](image2.png)

**Figure 3(b).** Some coloured partitions in \( NC(6) \), without VNRP.

**Remark 4.3.** It is useful to note a special situation when we are sure to get “vanishing of mixed Boolean cumulants with free arguments”: consider the setting of Theorem 1.2.
and let \( c : \{1, \ldots, n\} \to \{1, \ldots, s\} \) be a colouring such that \( c(1) \neq c(n) \). Then for every \( a_1 \in \mathcal{A}_{c(1)}, \ldots, a_n \in \mathcal{A}_{c(n)} \), one has that \( \beta_n(a_1, \ldots, a_n) = 0 \). Indeed, in this special case the index set for the summation on the right-hand side of Equation (4.1) is the empty set.

In the remaining part of this section, we record some easy consequences of Theorem 1.2. First, we note that from Equation (1.7) one can derive a formula for moments – this is precisely the \( \mathbb{C} \)-valued case of the moment formula found in Proposition 4.30 of [5], and is stated as follows.

**Corollary 4.4.** For every \( n \in \mathbb{N} \), every colouring \( c : \{1, \ldots, n\} \to \{1, \ldots, s\} \), and every \( a_1 \in \mathcal{A}_{c(1)}, \ldots, a_n \in \mathcal{A}_{c(n)} \), one has

\[
\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC(n;c)} \prod_{V \in \pi} \beta_{|V|}((a_1, \ldots, a_n) | V).
\]

**Proof.** Perform an additional summation over interval partitions on both sides of Equation (1.7), and use the formula which expresses moments in terms of Boolean cumulants. \( \square \)

A basic fact concerning free cumulants is that \( \kappa_n(a_1, \ldots, a_n) = 0 \) whenever \( n \geq 2 \) and there exists an \( m \in \{1, \ldots, n\} \) such that \( a_m \) is a scalar multiple of the unit. The next corollary gives the analogue of this fact when one uses Boolean cumulants.

**Corollary 4.5.** Let \((\mathcal{A}, \varphi)\) be a noncommutative probability space and let \((\beta_n : \mathcal{A}^n \to \mathbb{C})_{n=1}^{\infty}\) be the family of Boolean cumulant functionals associated to it. Let \( a_1, \ldots, a_n \) be elements of \( \mathcal{A} \), where \( n \geq 2 \), and suppose we are given an index \( m \in \{1, \ldots, n\} \) for which it is known that \( a_m = 1_\mathcal{A} \). Then

\[
\beta_n(a_1, \ldots, a_n) = \begin{cases} 0, & \text{if } m = 1 \text{ or } m = n, \\ \beta_{n-1}(a_1, \ldots, a_{m-1}, a_{m+1}, \ldots, a_n), & \text{if } 1 < m < n. \end{cases}
\]

**Proof.** Consider the unital subalgebras \( \mathcal{A}_1, \mathcal{A}_2 \) of \( \mathcal{A} \) defined by \( \mathcal{A}_1 = \mathcal{A} \) and \( \mathcal{A}_2 = \mathbb{C}1_\mathcal{A} \), and consider the colouring \( c : \{1, \ldots, n\} \to \{1, 2\} \) defined by

\[
c(i) = \begin{cases} 1, & \text{if } i \in \{1, \ldots, n\} \setminus \{m\}, \\ 2, & \text{if } i = m. \end{cases}
\]

We then have \( a_i \in \mathcal{A}_{c(i)} \) for all \( 1 \leq i \leq n \), with \( \mathcal{A}_1 \) being freely independent of \( \mathcal{A}_2 \), hence Theorem 1.2 applies to this situation and gives us a formula for \( \beta_n(a_1, \ldots, a_n) \). If \( m = 1 \) or \( m = n \), then \( \beta_n(a_1, \ldots, a_n) = 0 \) by Remark 4.3. If \( 1 < m < n \), then an immediate inspection shows that the only partition \( \pi \in NC(n) \) with unique outer block and with VNRP is

\[
\pi = \{ \{1, \ldots, m-1, m+1, \ldots, n\}, \{m\} \}. 
\]

Thus the sum on the right-hand side of (4.1) has in this case only one term, which is as indicated in (4.4) above (since \( \beta_1(a_m) = \beta_1(1_\mathcal{A}) = 1 \)). \( \square \)

From Theorem 1.2 one can also get an explicit description of the Boolean cumulants of the sum of two free elements, as follows.

**Proposition 4.6.** Let \((\mathcal{A}, \varphi)\) be a noncommutative probability space and let \( a, b \in \mathcal{A} \) be such that \( a \) is freely independent from \( b \). Consider the sequences of Boolean cumulants...
On the other hand, one can write
\[
\beta_n(a + b) = \sum_{\pi \in NC(n), \atop \pi \ll 1_n} \left( \prod_{U \in \pi, \text{ with depth}_n(U) \text{ even}} \beta_{|U|}(a) \right) \cdot \left( \prod_{V \in \pi, \text{ with depth}_n(V) \text{ odd}} \beta_{|V|}(b) \right)
\]
\[
+ \sum_{\pi \in NC(n), \atop \pi \ll 1_n} \left( \prod_{U \in \pi, \text{ with depth}_n(U) \text{ even}} \beta_{|U|}(b) \right) \cdot \left( \prod_{V \in \pi, \text{ with depth}_n(V) \text{ odd}} \beta_{|V|}(b) \right).
\]

**Proof.** We expand $\beta_n(a + b, \ldots , a + b)$ as a sum of $2^n$ terms by multilinearity, and then for each of the $2^n$ terms we use the formula for Boolean cumulants with free entries. This takes us to an expression of the form stipulated on the right-hand side of Equation (4.5) – it is a large sum where every term of the sum is a product of Boolean cumulants of $a$ and of $b$.

We next make the following observation: for every $\pi \in NC(n)$ such that $\pi \ll 1_n$ there exist precisely two ways of colouring the blocks of $\pi$ in the colours “$a$” and “$b$” such that VNRP holds; indeed, once we decide what is the colour of the unique outer block of $\pi$, everything else is determined. By using this observation, we sort out the terms of the large sum indicated in the preceding paragraph, and we organize these terms into two separate sums, arriving precisely to the formula announced in (4.5).

**Remark 4.7.** The statement of Proposition 4.6 simplifies quite a bit when the two elements $a$ and $b$ have the same distribution. In this case we only have one sequence $(\lambda_n)_{n=1}^{\infty}$, giving the Boolean cumulants for both $a$ and $b$, and Equation (4.5) becomes
\[
\beta_n(a + b) = 2 \cdot \sum_{\pi \in NC(n), \atop \pi \ll 1_n} \left( \prod_{V \in \pi} \lambda_{|V|} \right).
\]

We note that Equation (4.6) can alternatively be obtained by using some known facts about the Boolean Bercovici-Pata bijection introduced in [3]. Indeed, let us apply this Bercovici-Pata bijection to the common distribution $\mu$ of $a$ and of $b$, and let us denote the resulting distribution by $\nu$. (Here we view both $\mu$ and $\nu$ as linear functionals on $\mathbb{C}[X]$, with $\nu$ being defined in terms of $\mu$ via the requirement that its free cumulants\(^3\) satisfy $\kappa_n(\nu) = \beta_n(\mu)$, $\forall n \in \mathbb{N}.$) It is known (cf. Theorem 1.2 in [2]) that the Boolean cumulants of $\nu$ satisfy the relation
\[
\beta_n(\nu) = \frac{1}{2} \beta_n(\mu \boxplus \mu), \quad n \in \mathbb{N},
\]
where $\mu \boxplus \mu$ (the “free additive convolution” of $\mu$ with itself) is the distribution of $a + b$.

On the other hand, one can write
\[
\beta_n(\nu) = \sum_{\pi \in NC(n), \atop \pi \ll 1_n} \prod_{V \in \pi} \kappa_{|V|}(\nu) = \sum_{\pi \in NC(n), \atop \pi \ll 1_n} \prod_{V \in \pi} \beta_{|V|}(\mu),
\]

\(^3\)The free cumulant $\kappa_n(\nu)$ is defined as the free cumulant $\kappa_n(X, \ldots , X)$ in the noncommutative probability space $(\mathbb{C}[X], \nu)$, while the Boolean cumulant $\beta_n(\mu)$ is defined as the Boolean cumulant $\beta_n(X, \ldots , X)$ in the noncommutative probability space $(\mathbb{C}[X], \mu)$. 
where the first equality in (4.8) is the identity expressing Boolean cumulants in terms of free cumulants. Putting together (4.7) and (4.8) leads to

\[(4.9) \quad \beta_n (\mu \boxplus \mu) = 2 \cdot \sum_{\pi \in NC(n), \pi \ll 1_n} \left( \prod_{V \in \pi} \beta_{|V|} (\mu) \right), \quad n \in \mathbb{N},\]

which is a re-phrasing of (4.6).

**Remark 4.8.** One can do a bit of further combinatorial analysis following to the statement of Proposition 4.6 in order to go to the level of power series, and thus come up with some equations in \(\eta\)-series which can be used to obtain \(\eta_{a+b}\).

More precisely, let us consider, same as in Proposition 4.6, a noncommutative probability space \((A, \varphi)\), and let \(a, b \in A\) be free. Remark 4.3 implies that the \(\eta\)-series \(\eta_{a+b}(z) := \sum_{n=1}^{\infty} \beta_n (a+b) z^n\) splits as \(\eta_{a+b}(z) = B_a(z) + B_b(z)\), where

\[B_a(z) = \sum_{n=1}^{\infty} \left( \sum_{c_2, \ldots, c_{n-1} \in \{a, b\}} \beta_n (a, c_2, \ldots, c_{n-1}, a) \right) z^n,\]

\[B_b(z) = \sum_{n=1}^{\infty} \left( \sum_{c_2, \ldots, c_{n-1} \in \{a, b\}} \beta_n (b, c_2, \ldots, c_{n-1}, b) \right) z^n.\]

From Theorem 1.2 it follows that \(\beta_n (a, c_2, \ldots, c_{n-1}, a)\) is expressed as a sum over coloured non-crossing partitions with one outer block. Upon sorting out terms according to what is this outer block, one obtains the formula

\[B_a(z) = \beta_1 (a) z + \beta_2 (a) z^2 \left( \sum_{m=0}^{\infty} \left( \beta_1 (b) z + \beta_2 (b, b) z^2 + (\beta_3 (b, a, b) + \beta_3 (b, b, b)) z^3 + \ldots \right)^m \right) + \eta_3 (a) z^2 \left( \sum_{m=0}^{\infty} \left( \eta_1 (b) z + \eta_2 (b, b) z^2 + (\eta_3 (b, a, b) + \eta_3 (b, b, b)) z^3 + \ldots \right)^m \right)^2 + \ldots\]

After summing the geometric series that have appeared, and after doing the similar calculation for \(B_b(z)\), one arrives to the system of equations

\[(4.10) \quad \begin{cases} B_a(z) = \eta_a \left( \frac{z}{1 - B_b(z)} \right) (1 - B_b(z)), \\ B_b(z) = \eta_b \left( \frac{z}{1 - B_a(z)} \right) (1 - B_a(z)). \end{cases}\]

Solving the system (4.10) may be used as a path towards the explicit calculation of the \(\eta\)-series \(\eta_{a+b} = B_a + B_b\). This is of course not as smooth as using free cumulants and \(R\)-transforms (where one has the simplest possible formula, \(R_{a+b} = R_a + R_b\)), but we indicated how this goes in anticipation of the analogous development for anticommutators which we present in Section 6 below, and where free cumulants do not provide a simpler alternative.
5. Boolean cumulants of products and of anticommutators

This section is concerned with studying $*$-Boolean cumulants of products of $*$-free random variables; as a byproduct of that, we obtain the formula for Boolean cumulants of a free anticommutator which was announced in Theorem 1.8 of the Introduction.

For clarity, we start by discussing a concrete low-order example.

Example 5.1. Assume that $a, b$ are $*$-free. We are interested to calculate

\begin{equation}
\beta_3(ab, ab, b^* a^*) = ?,
\end{equation}

where the expression sought on the right-hand side should be a polynomial expression in the Boolean $*$-cumulants of $a$ and those of $b$. We reach this goal in three steps.

Step 1. Use the formula for Boolean cumulants with products as entries.

This step expresses $\beta_3(ab, ab, b^* a^*)$ as a sum of 8 terms, indexed by the collection of partitions $\sigma \in \text{Int}(6)$ with the property that $\sigma \vee \{\{1, 2\}, \{3, 4\}, \{5, 6\}\} = 1_6$ (or, equivalently, that $\sigma \geq \{\{1\}, \{2, 3\}, \{4, 5\}, \{6\}\}$). By keeping in mind the Remark 4.3 we see that 5 of these terms are sure to be equal to 0, and thus the conclusion of this step is that

\begin{equation}
\beta_3(ab, ab, b^* a^*) = \beta_1(a) \beta_4(b, a, b^*) \beta_1(a^*) \\
+ \beta_3(a, b, a) \beta_2(b, b^*) \beta_1(a^*) + \beta_6(a, b, a, b^*, a^*).
\end{equation}

Step 2. Use the formula for Boolean cumulants with free entries.

This step takes on the Boolean cumulants which appear on the right-hand side of (5.2) and still mix together $a$’s and $b$’s – we use Theorem 1.2 to express these Boolean cumulants as polynomials which separate the $a$’s from the $b$’s. In order to process $\beta_6(a, b, a, b^*, a^*)$ we refer to Example 4.2 and we also do the immediate calculation that

\[ \beta_4(b, a, b^*) = \beta_1(a) \beta_3(b, b^*), \quad \beta_3(a, b, a) = \beta_2(a, a) \beta_1(b). \]

The overall result of the calculation is that

\begin{equation}
\beta_3(ab, ab, b^* a^*) = \beta_1(a) \beta_1(a^*) \cdot \beta_3(b, b, b^*) + \beta_2(a, a) \beta_1(a^*) \cdot \beta_1(b) \beta_2(b, b^*) \\
+ \beta_2(a, a^*) \beta_1(a) \cdot \beta_2(b, b) \beta_1(b^*) + \beta_2(a, a^*) \beta_1(a) \beta_3(b, b, b^*) \\
\beta_3(a, a, a^*) \cdot \beta_1(b) \beta_1(b) \beta_1(b^*) + \beta_3(a, a, a^*) \beta_1(b) \beta_2(b, b^*).
\end{equation}

The right-hand side of (5.3) is indeed a sum of products of $*$-cumulants of $a$ and of $b$, as we wanted.

Step 3. In order to understand what is going on, we record Equation (5.3) in the form:

\begin{equation}
\beta_3(ab, ab, b^* a^*) = \sum_{\pi \in \Pi} \prod_{U \in \pi, \text{ of ‘colour’ } a} \beta_{|U|}((a, b, a, b^*, a^*) \mid U) \cdot \prod_{V \in \pi, \text{ of ‘colour’ } b} \beta_{|V|}((a, b, a, b^*, a^*) \mid V),
\end{equation}

where $\Pi$ is the set of 6 non-crossing partitions depicted (in a way which includes colouring of blocks) in Figure 4. The question then becomes: can we put into evidence the structure which underlies this special set $\Pi \subseteq NC(6)$? Upon staring a bit at Figure 4, it becomes quite appealing to believe that VRNP must be involved here! This hunch is confirmed by Theorem 5.6 below, where the set $\Pi \subseteq NC(6)$ from Equation (5.4) becomes the correct indexing set corresponding to the tuple $\varepsilon = (1, 1, *) \in \{1, *\}^3$. 

USE OF BOOLEAN CUMULANTS FOR FREE RANDOM VARIABLES 23
Remark 5.2. A similar calculation to the one shown in Example 5.1 could be done in the world of free cumulants, and would lead to a similarly looking formula which expresses the free cumulant $\kappa_3(ab,ab,b^*a^*)$ in terms of the $*$-free cumulants of $a$ and the $*$-free cumulants of $b$. The formula with free cumulants is “just a bit” more complicated than the one for Boolean cumulants obtained in (5.3) above: it has 7 terms instead of 6, where 6 of the 7 terms are having the same structure as in (5.3), while the 7th term is

$$\kappa_2(a,a^*)\kappa_1(a)\kappa_1(b)\kappa_2(b,b^*),$$

corresponding to the partition $\{\{1,6\},\{2\},\{3\},\{4,5\}\} \in NC(6)$. But, unlike the partitions corresponding to the other 6 terms, this latter partition does not satisfy VNRP! (It is one of the two partitions without VNRP that were featured in Figure 3(b) in Section 4.)

We now move to discuss Boolean cumulants for general words made with $ab$ and with $(ab)^*$. To this end, we recall from the Introduction the terminology about ac-friendly partitions and about the canonical alternating colouring of a non-crossing partition.

In the Introduction, the definition of what is an ac-friendly partition in $NC(2n)$ was made by only referring to depths of blocks. It will come in handy to note that this notion can also be approached via canonical alternating colourings, as explained in the next proposition.

**Proposition 5.3.** Let $n$ be a positive integer and let $\pi$ be a partition in $NC(2n)$. Then $\pi$ belongs to the set $NC_{ac-frienly}(2n)$ introduced in Definition 1.4 if and only if it satisfies the following two conditions:

1. **(AC-Friendly1)** $\text{OuterMax}(\pi) \subseteq \{1,3,\ldots,2n-1\} \cup \{2n\}$.
2. **(AC-Friendly2')** $\text{calt}_\pi(2i-1) \neq \text{calt}_\pi(2i)$, $\forall 1 \leq i \leq n$, where $\text{calt}_\pi : \{1,\ldots,2n\} \to \{1,2\}$ is the canonical alternating colouring associated to $\pi$.

**Proof.** “⇒” We assume that (AC-Friendly1) and (AC-Friendly2) hold. Suppose, toward a contradiction, that (AC-Friendly2') fails. Then there exists an $i$ such that $\text{calt}_\pi(2i-1) = \text{calt}_\pi(2i)$. This can only happen if $2i-1$ and $2i$ belong to distinct blocks $A$ and $B$ that are “siblings,” i.e., have same parent $C$. But then, for $j = 2i-1$, we have that $\text{depth}_\pi(j) = \text{depth}_\pi(C) + 1 = \text{depth}_\pi(j+1)$ contradicting (AC-Friendly2).

“⇐” We assume that (AC-Friendly1) and (AC-Friendly2') hold. Suppose, toward a contradiction, that (AC-Friendly2) fails. Let $j = 2i-1 \notin \text{OuterMax}(\pi)$ be such that...
depth_\pi(j) = depth_\pi(j + 1). Since j and j + 1 cannot belong to distinct outer blocks (as then we would have j ∈ OuterMax(\pi)) this can only happen if they either belong to the same block or they belong to blocks that have the same parent. In both of these cases we then have that calt_\pi(2i − 1) = calt_\pi(2i), contradicting (AC-Friendly2'). □

**Notation 5.4.** Let n be a positive integer, let \pi be a partition in \textsc{NC}_{ac-friendly}(2n), and consider the canonical alternating colouring calt_\pi : \{1, \ldots, 2n\} → \{1, 2\}. We use the values calt_\pi(1), calt_\pi(3), \ldots, calt_\pi(2n − 1) in order to create a tuple \varepsilon ∈ \{1, *\}^n, as follows:

\[(5.5)\]
\[
\varepsilon(i) = \begin{cases} 1, & \text{if } calt_\pi(2i − 1) = 1, \\ * , & \text{if } calt_\pi(2i − 1) = 2 \end{cases}, 1 ≤ i ≤ n.
\]

The tuple \varepsilon ∈ \{1, *\}^n so defined will be denoted as oddtuple(\pi).

**Remark 5.5.** Let n be a positive integer and let \pi be a partition in \textsc{NC}_{ac-friendly}(2n). Knowing what is the tuple oddtuple(\pi) ∈ \{1, *\}^n gives us precisely the same information as if we knew what is the colouring calt_\pi : \{1, \ldots, 2n\} → \{1, 2\}. Indeed, if we know oddtuple(\pi) then we can use Equation (5.5) to find the values calt_\pi(1), calt_\pi(3), \ldots, calt_\pi(2n − 1), after which we can also find out what are calt_\pi(2), calt_\pi(4), \ldots, calt_\pi(2n) based on the fact that calt_\pi(2i) ≠ calt_\pi(2i − 1), 1 ≤ i ≤ n.

We can now state the desired formula about the joint Boolean cumulants of ab and ba.

**Theorem 5.6.** Let (A, \varphi) be a *-probability space and let (\beta_n : A^n → \mathbb{C})_{n=1}^\infty be the family of Boolean cumulant functionals associated to it. We consider two selfadjoint elements a, b ∈ A such that a is freely independent from b, and we consider the sequences of Boolean cumulants (\beta_n(a))_{n=1}^\infty and (\beta_n(b))_{n=1}^\infty.

1. Let n be a positive integer and let \varepsilon = (\varepsilon(1), \ldots, \varepsilon(n)) ∈ \{1, *\}^n such that \varepsilon(1) = 1. One has

\[(5.6)\]
\[
\sum_{\substack{\pi ∈ \textsc{NC}_{ac-friendly}(2n), \\
\text{such that} \\
\text{oddtuple}(\pi) = \varepsilon}} \prod_{\substack{U ∈ \pi, \text{ with} \\
calt_\pi(U) = 1}} \beta_{|U|}(a) \cdot \prod_{\substack{V ∈ \pi, \text{ with} \\
calt_\pi(V) = 2}} \beta_{|V|}(b).
\]

2. Let n be a positive integer and let \varepsilon = (\varepsilon(1), \ldots, \varepsilon(n)) ∈ \{1, *\}^n such that \varepsilon(1) = *. Consider the complementary tuple \varepsilon′ ∈ \{1, *\}^n, uniquely determined by the requirement that \varepsilon′(i) ≠ \varepsilon(i), for all 1 ≤ i ≤ n. One has

\[(5.7)\]
\[
\sum_{\substack{\pi ∈ \textsc{NC}_{ac-friendly}(2n), \\
\text{such that} \\
\text{oddtuple}(\pi) = \varepsilon'}} \prod_{\substack{U ∈ \pi, \text{ with} \\
calt_\pi(U) = 1}} \beta_{|U|}(b) \cdot \prod_{\substack{V ∈ \pi, \text{ with} \\
calt_\pi(V) = 2}} \beta_{|V|}(a).
\]

**Proof.** We will assume the case (1), i.e., that \varepsilon(1) = 1 (the case \varepsilon(1) = * is the ‘mirror’ image of this case and is left to the reader). Note that \varepsilon induces a colouring c : \{1, \ldots, 2n\} → \{1, 2\} by c(2i − 1) = 1, c(2i) = 2 when \varepsilon(i) = 1 and c(2i − 1) = 2, c(2i) = 1 when \varepsilon(i) = *; in other words, c assigns 1 to positions where there is an a in (ab)^{\varepsilon(1)} \ldots (ab)^{\varepsilon(n)} and 2 to positions where there is a b.
We do again the steps presented in Example 5.1, where the discussion is now made to go in reference to an abstract tuple $\varepsilon$, rather than the special case of $(1,1,*) \in \{1,*\}^3$.

Step 1 takes us to a sum over interval partitions partitions $\sigma = \{J_1,\ldots, J_p\} \in \text{Int}(2n)$ such that $\sigma \vee \{\{1\}, \{2,3\}, \ldots, \{2n-2,2n-1\}, \{2n\}\} = 1_{2n}$. The condition $\sigma \vee \{\{1\}, \{2,3\}, \ldots, \{2n-2,2n-1\}, \{2n\}\} = 1_{2n}$ is equivalent to saying that $\max(J_k) \in \{1,3,\ldots, 2n-1\}$ for all $1 \leq k < p$. The latter can be expressed directly in terms of the cardinalities of the blocks of $\sigma$: either $\sigma = 1_{2n}$, or it has $|J_1|, |J_p|$ odd and $|J_k|$ even for all $1 < k < p$. This implies, among other things, that for all $k < p$ we have that $c(\min(J_k)) \neq c(\min(J_{k+1}))$.

The observation in Remark 4.3 that certain mixed Boolean cumulants with free entries have to vanish, then yields that we are left to only consider the cases where for each $k$ we have that $c(\min(J_k)) = c(\max(J_k))$.

Step 2: We now apply the separation formula from Theorem 1.2 to each block $J_k$ to get that $C_{\varepsilon}(ab)\varepsilon(1)\ldots, (ab)\varepsilon(n)$ is equal to

$$\sum_{\sigma = \{J_1,\ldots, J_p\} \in \text{Int}(2n)} \sum_{\pi \in \text{NC}(2n,c), \pi \leq \sigma} \sum_{\forall k,c(\min(J_k)) = c(\max(J_k))} \prod_{U \in \pi, \text{ with } c(U) = 1} \beta_{|U|}(a) \cdot \prod_{V \in \pi, \text{ with } c(V) = 2} \beta_{|V|}(b).$$

Step 3: We claim that partitions $\pi$ appearing in the summation above are precisely those in $\text{NC}_{\text{ac-friendly}}(2n)$ for which oddtuple($\pi$) = $\varepsilon$ and that we always have $\text{calt}_{\pi} = c$ from whence the formula (5.6) follows.

Since consecutive blocks of $\sigma$ start with distinct colours we have that the consecutive outer blocks of $\pi$ have distinct colours. When we combine this observation with the facts that the first block of $\pi$ has colour 1 and that $\pi$ has VNRP (with respect to $c$) we get that $c = \text{calt}_\pi$.

As already mentioned above we have that the condition $\pi \leq \sigma$, $\sigma \in \text{Int}(2n)$, $\sigma \vee \{\{1\}, \{2,3\}, \ldots, \{2n-2,2n-1\}, \{2n\}\} = 1_{2n}$ is equivalent to $\pi$ satisfying (AC-Friendly1).

It now remains to prove that every $\pi$ in the above summation formula satisfies (AC-Friendly2). Suppose the converse. Then there is an odd number $j \notin \text{OuterMax}(\pi)$ such that $\text{depth}_\pi(j) = \text{depth}_\pi(j+1)$. Since $j$ and $j+1$ have distinct $c$-colours we have that $j$ and $j+1$ belong to distinct blocks. They cannot both belong to outer blocks (in this case it would follow that $j \in \text{OuterMax}(\pi)$). The equality $\text{depth}_\pi(j) = \text{depth}_\pi(j+1)$ then implies that the blocks containing $j$ and $j+1$ must have the same parent, and considering the $c$-colour of the parent-block leads to an immediate contradiction with VNRP. \qed

Remark 5.7. We note that Theorem 5.6 could be stated in the framework where $(\mathcal{A}, \varphi)$ is a plain noncommutative probability space, and we look at joint Boolean cumulants of $ab$ and $ba$, with $a$ free from $b$.

The statement of Theorem 5.6 could also be extended to the case when we deal with joint Boolean cumulants of $ab$ and $(ab)^*$ in a $*$-probability space, without assuming that $a$ and $b$ are selfadjoint. The proof would be the same, only that it would result in stiffer formulas.

Remark 5.8. The special case $\varepsilon = (1,1,\ldots,1)$ of Theorem 5.6 gives a formula for the Boolean cumulant $C_n(ab,ab,\ldots,ab)$. We explain here that this is precisely the formula from (1.4) which was reviewed in Equation (1.4) of the Introduction.

To this end, let us first note that if a partition $\sigma \in \text{NC}_{\text{ac-friendly}}(2n)$ has oddtuple($\sigma$) = $(1,1,\ldots,1)$, then the canonical alternating colouring $\text{calt}_\sigma$ must have $\text{calt}_\sigma(2i-1) = 1$ and $\text{calt}_\sigma(2i) = 2$ for all $1 \leq i \leq n$. So $\text{calt}_\sigma$ is the colouring of $\{1,\ldots,2n\}$ by parity, which implies that every block of $\sigma$ is contained either in $\{1,3,\ldots,2n-1\}$ or in $\{2,4,\ldots,2n\}$. We note moreover that such $\sigma$ is sure to only have two outer blocks, the blocks $W$ and
\[ W^n \] which contain the numbers 1 and 2\(n\), respectively. Indeed, if \( \sigma \) had an outer block \( W \neq W' \), \( W'' \), then the condition (AC-Friendly1) satisfied by \( \sigma \) would imply that \( \min(W) \) is an even number and \( \max(W) \) is an odd number – not possible! Knowing these things about \( \sigma \), plus the fact that \( \sigma \) has VNRP, leads to the conclusion that \( \sigma \) must be of the form \( \sigma = \pi^{(\text{odd})} \cup (\text{Kr}_n(\pi))^{(\text{even})} \) for some \( \pi \in NC(n) \); this is precisely the content of Lemma 6.8 in [1].

Conversely, let \( \pi \) be in \( NC(n) \) and consider the partition \( \sigma = \pi^{(\text{odd})} \cup (\text{Kr}_n(\pi))^{(\text{even})} \in NC(2n) \). Then Lemma 6.6 of [1] assures us that the canonical alternating colouring of \( \sigma \) is the colouring of \( \{1, \ldots, 2n\} \) by parity. The same lemma of [1] also records the fact that \( \sigma \) has exactly two outer blocks, the ones containing the numbers 1 and 2\(n\), and this clearly entails the condition (AC-Friendly1) from Proposition 5.3 above. In view of Proposition 5.3 we then conclude that \( \sigma \in NC_{\text{ac-friendly}}(2n) \) and at the same time we see that the tuple oddtuple(\( \sigma \)) is equal to (1, 1, \ldots, 1).

The discussion from the preceding two paragraphs shows that when we make \( \varepsilon = (1, 1, \ldots, 1) \) in Theorem 5.6(1), the summation on the right-hand side of Equation (5.6) is made over partitions of the form \( \pi^{(\text{odd})} \cup (\text{Kr}_n(\pi))^{(\text{even})} \), with \( \pi \) running in \( NC(n) \). Moreover, when looking at the term of the summation which is indexed by a \( \pi \in NC(n) \), one sees that the two products appearing there are precisely \( \prod_{U \in \pi, \text{calt}_{\pi}(U) = 1} \beta_{[U]}(a) \) and \( \prod_{V \in \text{Kr}_n(\sigma) \beta_{[V]}(b) \cdots} \). Hence this special case of Equation (5.6) retrieves Equation (1.4), as claimed at the beginning of the remark.

By starting from Theorem 5.6 it is easy to prove the formula announced in Theorem 1.8 of the Introduction (and repeated below), concerning the Boolean cumulants of a free anticommutator.

Theorem 5.9. Let \((\mathcal{A}, \varphi)\) be a noncommutative probability space and let \(a, b \in \mathcal{A}\) be such that \(a\) is freely independent from \(b\). Consider the sequences of Boolean cumulants \((\beta_n(a))_{n=1}^{\infty}\) and \((\beta_n(b))_{n=1}^{\infty}\) for a and for \(b\), respectively. Then, for every \(n \geq 1\), the \(n\)-th Boolean cumulant of \(ab + ba\) is

\[
\beta_n(ab + ba) = \sum_{\pi \in \text{NC}_{\text{ac-friendly}}(2n)} \left( \prod_{U \in \pi, \text{calt}_{\pi}(U) = 1} \beta_{[U]}(a) \right) \cdot \left( \prod_{V \in \pi, \text{calt}_{\pi}(V) = 2} \beta_{[V]}(b) \right) \\
+ \sum_{\pi \in \text{NC}_{\text{ac-friendly}}(2n)} \left( \prod_{U \in \pi, \text{calt}_{\pi}(U) = 1} \beta_{[U]}(b) \right) \cdot \left( \prod_{V \in \pi, \text{calt}_{\pi}(V) = 2} \beta_{[V]}(a) \right).
\]

Proof. We write that

\[
(5.8) \quad \beta_n(ab + ba, \ldots, ab + ba) = \sum_{\varepsilon \in \{1, \ast\}^n} \beta_n\left( (ab)^{\varepsilon(1)}, \ldots, (ab)^{\varepsilon(n)} \right) = \\
\sum_{\varepsilon \in \{1, \ast\}^n, \varepsilon(1) = 1} \beta_n\left( (ab)^{\varepsilon(1)}, \ldots, (ab)^{\varepsilon(n)} \right) + \sum_{\varepsilon \in \{1, \ast\}^n, \varepsilon(1) = \ast} \beta_n\left( (ab)^{\varepsilon(1)}, \ldots, (ab)^{\varepsilon(n)} \right).
\]

Note that every \( \pi \in \text{NC}_{\text{ac-friendly}}(2n) \) determines a unique \( \varepsilon := \text{oddtuple}(\pi) \in \{1, \ast\}^n \) such that \( \varepsilon(1) = 1 \). Also recall that every \( \varepsilon \in \{1, \ast\}^n \) for which \( \varepsilon(1) = \ast \) we have \( \varepsilon' \in \{1, \ast\}^n \) determined by \( \varepsilon'(k) \neq \varepsilon(k) \) for all \( k \) (in particular \( \varepsilon'(1) = 1 \)). We invoke the formulas found
in Theorem 5.6 to get
\[\sum_{\varepsilon \in \{1, \ast\}^n, \varepsilon(1)=1} \beta_n((ab)^{(1)}, \ldots, (ab)^{(n)})\]
\[= \sum_{\varepsilon \in \{1, \ast\}^n, \varepsilon(1)=1} \sum_{\pi \in NC_{ac}-friendly(2n), such that} \prod_{U \in \pi, with \text{calt}_\pi(U)=1} \beta_{|U|}(a) \cdot \prod_{V \in \pi, with \text{calt}_\pi(V)=2} \beta_{|V|}(b),\]

and
\[\sum_{\varepsilon \in \{1, \ast\}^n, \varepsilon(1)=\ast} \beta_n((ab)^{(1)}, \ldots, (ab)^{(n)})\]
\[= \sum_{\varepsilon \in \{1, \ast\}^n, \varepsilon(1)=\ast} \sum_{\pi \in NC_{ac}-friendly(2n), such that} \prod_{U \in \pi, with \text{calt}_\pi(U)=1} \beta_{|U|}(b) \cdot \prod_{V \in \pi, with \text{calt}_\pi(V)=2} \beta_{|V|}(a),\]

and
\[\sum_{\varepsilon \in \{1, \ast\}^n, \varepsilon(1)=\ast} \beta_n((ab)^{(1)}, \ldots, (ab)^{(n)})\]
\[= \sum_{\varepsilon \in \{1, \ast\}^n, \varepsilon(1)=\ast} \sum_{\pi \in NC_{ac}-friendly(2n), such that} \prod_{U \in \pi, with \text{calt}_\pi(U)=1} \beta_{|U|}(b) \cdot \prod_{V \in \pi, with \text{calt}_\pi(V)=2} \beta_{|V|}(a).\]

\[\square\]

6. On the \(\eta\)-series of a free anticommutator

In this section we show how observations about Boolean cumulants of a free anticommutator from the previous section can be captured in the form of a system of equation at the level of \(\eta\)-series.

6.1. Equations in power series.
Throughout this subsection we fix a \(*\)-probability space \((\mathcal{A}, \varphi)\) and two elements \(a, b \in \mathcal{A}\). For clarity of arguments it is better if at first we do not assume that \(a\) and \(b\) are selfadjoint, and we discuss the formal power series \(\eta_{ab, b^* a^*} \in \mathbb{C}((z_a, z_a^*, z_b, z_b^*))\) defined as

\[(6.1) \quad \eta_{ab, b^* a^*} = \sum_{n=1}^{\infty} \sum_{\varepsilon(1), \ldots, \varepsilon(n) \in \{1, \ast\}^n} \beta_n((ab)^{(1)}, \ldots, (ab)^{(n)})(z_a z_b)^{(1)} \cdots (z_a z_b)^{(n)},\]

where on the right-hand side of (6.1) we make the convention to put \((z_a z_b)^* = z_b^* z_a^*\) (so, for instance, the term corresponding to \(\varepsilon = (1, 1, \ast) \in \{1, \ast\}^3\) is \(\beta_3(ab, ab, b^* a^*) z_a z_b z_a z_b z_b^* z_a^*\)). At some point down the line we will however switch to the special case when \(a = a^*, b = b^*\) and \(z_a = z_a^* = z_b = z_b^* =: z\); in this special case the series from Equation (6.1) becomes a series of one variable, which is nothing but \(\eta_{ab+ba}(z^2)\).

Returning to Equation (6.1) we observe that \(\eta_{ab, b^* a^*}\) splits naturally as a sum,

\[(6.2) \quad \eta_{ab, b^* a^*} = \eta^{1,1} + \eta^{1,\ast} + \eta^{1,\ast} + \eta^{\ast,\ast},\]
where $\eta^{1,1}$ contains those terms of $\eta_{ab,b^*a^*}$ which correspond to Boolean cumulants beginning and ending with $ab$, $\eta^{1,*}$ contains those terms of $\eta_{ab,b^*a^*}$ which correspond to Boolean cumulants that begin with $ab$ and end with $b^*a^*$, etc. Under the assumption that $\{a,a^*\}$ is free from $\{b,b^*\}$, one can then make a number of structural observations about the four power series introduced in Equation (6.2).

Let us start with $\eta^{1,1}$. Every term of this series is of the form
\[
(6.3) \quad \beta_n(ab,\ldots,ab)z_az_b\cdots z_a z_b.
\]
To the Boolean cumulant appearing in (6.3) we apply the formula for Boolean cumulants with products as entries from Proposition 2.12, together with the property that a Boolean cumulant which starts with $a$ or $a^*$ and ends with $b$ or $b^*$ (or vice versa) vanishes, as noticed in Remark 4.3. Then it follows from Theorem 5.9 that the cumulant from (6.3) is a sum of terms of the form
\[
(6.4) \quad \beta_{l_0}(a,b,\ldots,a)\left(\prod_{j=1}^{n-1}\beta_{k_j}(b,\ldots,b^*)\beta_j(a^*,\ldots,a)\right)\beta_{k_n}(b,\ldots,a,b),
\]
for $n \geq 1$, $l_0, k_n \geq 1$ and $l_j, k_j \geq 2$ for $j = 1, \ldots, n - 1$.

For $(a,a^*) \in \{a, a^*\}$ or $(b,b^*)$ we define power series $f_l, f_{l^*} \in \mathbb{C}\langle\langle z_a, z_{a^*}, z_b, z_{b^*}\rangle\rangle$ as power series which contain Boolean cumulants starting with $l$ and ending with $l'$ which can appear in the expression above. To be more precise, consider $\eta_{a,a^*,b,b^*} \in \mathbb{C}\langle\langle z_a, z_{a^*}, z_b, z_{b^*}\rangle\rangle$ the joint $\eta$–series of $a, a^*, b, b^*$ then $f_{l,l^*}$ is a restriction of $\eta_{a,a^*,b,b^*}$ to these terms which begin with $l$, end with $l'$, and corresponding word $z_l \cdots z_{l'}$ is a subword of some word of the type $(z_0 z_b)^{l_1}(z_{b^*} z_{a^*})^{k_1} \cdots (z_{a^*} z_a)^{k_n}$ with $n \geq 1$ and $l_1, k_n \geq 0$ and $k_1, \ldots, k_{n-1}, l_2, \ldots, l_n \geq 1$. We have for example
\[
(6.5) \quad \begin{cases}
\beta_{a,a} = \beta_1(a)z_a + \beta_3(a, b, a)z_azz_b + \beta_5(a, b, b^*, a^*, a)z_azz_bz_{b^*} + \ldots \\
\beta_{a^*, a} = \beta_2(a^*, a)z_{a^*}z_a + \beta_4(a^*, a, b, a)z_{a^*}z_azz_a + \ldots
\end{cases}
\]
With such notation, Equation (6.4) can be written as
\[
\eta^{1,1} = f_{a,a}(1 - f_{b,b^*}f_{a^*,a})^{-1}f_{b,b^*}.
\]
with the convention
\[
(1 - f_{b,b^*}f_{a^*,a})^{-1} = \sum_{n=0}^{\infty} (f_{b,b^*}f_{a^*,a})^n.
\]

Similar analysis to the one done above for $\eta^{1,1}$ can be done for the remaining three power series from the right-hand side of (6.2). We have for example
\[
\eta^{1,*} = f_{a,a^*} + f_{a,a}(1 - f_{b,b^*}f_{a^*,a})^{-1}f_{b,b^*}f_{a^*,a},
\]
where the additional term $f_{a,a^*}$ comes from the term of the expansion of $\beta_n(ab,\ldots,b^*a^*)$ with one outer block.

The system of equations which comes out of the preceding discussion can be nicely written in matrix form, as follows:
\[
(6.6) \quad \begin{bmatrix}
\eta^{1,1} & \eta^{1,*} \\
\eta^{*,1} & \eta^{*,*}
\end{bmatrix} = 
\begin{bmatrix}
f_{a,a}(1 - f_{b,b^*}f_{a^*,a})^{-1}f_{b,b^*}f_{a^*,a^*} + f_{a,a}(1 - f_{b,b^*}f_{a^*,a})^{-1}f_{b,b^*}f_{a^*,a^*} \\
(f_{b,b^*} + f_{b,b^*}(1 - f_{a^*,a}f_{b,b^*}^{-1})^{-1}f_{a^*,a}f_{b,b^*}^{-1})f_{b,b^*}^{-1}f_{a^*,a^*}
\end{bmatrix}.
\]

Once that Equation (6.6) is put into evidence, the problem of computing the $\eta$-series $\eta_{ab,b^*a^*}$ from (6.1) is reduced to the one of computing the power series $f_{l,l^*}$. We will take on this job in the special case when $a$ and $b$ are assumed to be (freely independent and) selfadjoint. In this special case, the determination of the series $f_{l,l^*}$ has to be made in terms
of the Boolean cumulants of $a$ and of $b$, or equivalently, in terms of the $\eta$-series of these elements. The mechanism for doing so is provided by the following theorem.

**Theorem 6.1.** Notation as above, where we assume that $a, b$ are selfadjoint and freely independent, and we put $z_a = z_{a^*} = z_b = z_{b^*} =: z$. Define

$$F_a = \begin{bmatrix} f_{a,a} & f_{a,a^*} \\ f_{a^*,a} & f_{a^*,a^*} \end{bmatrix}, \quad F_b = \begin{bmatrix} f_{b,b} & f_{b,b^*} \\ f_{b^*,b} & f_{b^*,b^*} \end{bmatrix}$$

and

$$H_a = \begin{bmatrix} f_{bb}(1 - f_{b^*b})^{-1} & f_{b,b^*} + f_{b,b}(1 - f_{b^*b})^{-1}f_{b^*b^*} \\ (1 - f_{b^*b})^{-1} & (1 - f_{b^*b})^{-1}f_{b^*b^*} \end{bmatrix},$$

$$H_b = \begin{bmatrix} (1 - f_{a,a^*})^{-1}f_{a,a} & (1 - f_{a,a^*})^{-1}f_{a,a^*} \\ f_{a^*,a} + f_{a^*,a^*}(1 - f_{a,a^*})^{-1}f_{a,a^*} & f_{a^*,a^*}(1 - f_{a,a^*})^{-1} \end{bmatrix}.$$  

Then one has

$$(6.7) \quad \begin{cases} F_a H_a = \eta_a(z H_a), \\ F_b H_b = \eta_b(z H_b). \end{cases}$$

where $\eta_a$ and $\eta_b$ are the $\eta$-series of $a$ and of $b$ (as reviewed in Notation 2.10).

**Proof.** The main tool we will use in order to establish the relations stated in (6.7) is the VNRP property from Theorem 1.2. We will only prove the first of the two relations, as the proof of the second one is analogous. Note that, even though our current hypotheses are such that $(ab)^* = b^*a^* = ba$, we will nevertheless continue to allow for occurrences of $a^*$ and $b^*$, which we will use to distinguish between $a$'s and $b$'s coming from a product $ab$ versus $a$'s and $b$'s coming from a product $ba$.

For $l, l' \in \{a, a^*\}$ consider $f_{l,l'}$. Each term of $f_{l,l'}$ is a joint Boolean cumulant of $a, a^*, b, b^*$ which starts with $l$ and ends with $l'$. According to Theorem 1.2 all partitions in the expansion have a unique outer block. We will sort the terms of $f_{l,l'}$ according to the structure of this outer block. So let us fix a possibility for what the outer block could be – this has to start with $l$, has to end with $l'$, and must contain only $a$'s and/or $a^*$'s. By $h_{k,k'}$ we will denote the power series which occurs between consecutive $k$ and $k'$ in the outer block. Then we have

$$f_{l,l'} = \sum_{n=1}^{\infty} \sum_{w(1), \ldots, w(n) \in \{a, a^*\}} \beta_n \left( w(1), \ldots, w(n) \right) z_{w(1)} h_{w(1), w(2)} z_{w(2)} \cdots z_{w(n)} h_{w(n-1), w(n)} z_{w(n)}.$$

For example we have

$$f_{a,a} = \beta_1(z) z_a + \beta_2(a, a) z_a h_{a,a} z_a + \beta_3(a, a^*, a) z_a h_{a,a^*} z_a^* h_{a^*,a} z_a + \ldots$$

Consider then the matrix $H_a$ defined by

$$\bar{H}_a := \begin{bmatrix} h_{a,a} & h_{a,a^*} \\ h_{a^*,a} & h_{a^*,a^*} \end{bmatrix}.$$  

Since we assume $a = a^*$ the relation between $f_{l,l'}$ and $h_{k,k'}$ can be written on the level of $2 \times 2$ matrices in the form

$$F_a = \sum_{n=1}^{\infty} \beta_n(a) \begin{bmatrix} z_a & 0 \\ 0 & z_{a^*} \end{bmatrix} \left( \bar{H}_a \begin{bmatrix} z_a & 0 \\ 0 & z_{a^*} \end{bmatrix} \right)^{n-1}.$$
We assumed that \( z_a = z_{a^*} = z \), thus multiplying both sides by \( \tilde{H}_a \) immediately gives
\[
F_a \tilde{H}_a = \eta_a(z \tilde{H}_a).
\]
We are left to prove that the matrix \( \tilde{H}_a \) is in fact the same as the \( H_a \) defined in the statement of the theorem, i.e. that the series \( h_{1,b} \) which have appeared in the discussion are such that
\[
\begin{bmatrix}
    h_{a,a} & h_{a,a^*} \\
    h_{a^*,a} & h_{a^*,a^*}
\end{bmatrix}
= \begin{bmatrix}
    f_{b,b}(1 - f_{b^*,b})^{-1} & f_{b,b^*} + f_{b,b}(1 - f_{b^*,b})^{-1} f_{b^*,b^*} \\
    (1 - f_{b^*,b})^{-1} & (1 - f_{b^*,b})^{-1} f_{b^*,b^*}
\end{bmatrix}.
\]
We will analyze separately each of the four entries of this matrix equality. The observation that is used in the discussion of each of the four entries is as follows: due to VNRP, it is immediate that between any two consecutive elements of the outer block we will have products of Boolean cumulants starting with \( b \) or \( b^* \) and ending with \( b \) or \( b^* \).

1. **Entry (1,1).** Assume that the two consecutive variables in the outer block are \( a \) and \( a^* \). Observe that the element coming right after the first \( a \) must be a \( b \). The element appearing immediately to the left of the second \( a \) from the outer block could be \( a^* \) or \( b \), but if it was \( a^* \), then by VNRP this \( a^* \) would be in the outer block and thus the consecutive elements considered in the outer block would be \( a \) and \( a^* \) (rather than \( a \) and \( a \)). We thus conclude that the element appearing immediately to the left of the second \( a \) from the outer block is a \( b \). In order to not violate VNRP between \( a \) and \( a \) we can get a cumulant \( \beta_n(b,\ldots,b) \) or for some \( k > 1 \) we can get \( \beta_n(b_1,\ldots,b_k) \) for \( n_i \geq 2 \). In \( \eta_n(b,\ldots,b) \) and \( \eta_n(a,\ldots,a^*) \) there are \( a \) and \( a^* \) as arguments but by Theorem 12 writing the term as a joint cumulant gives exactly all terms with VNRP property. We conclude that in each \( a,a \) pocket we can get any term of the power series \( f_{bb}(1 - f_{b^*,b})^{-1} \)

2. **Entry (1,2).** Assume that the two consecutive variables in the outer block are \( a \) and \( a^* \). In this case we find, by a similar argument as above that between \( a \) and \( a^* \) one can get one block of the form \( \beta_m(b,\ldots,b^*) \) for \( m \geq 2 \) or if there are more blocks they are of the form \( \beta_n(b_1,\ldots,b_k) \beta_n(b^{*1},\ldots,b^{*k}) \) for \( k \geq 0 \) and \( n_k \geq 2 \) and \( n_{k+1} \geq 1 \) for \( k \). Thus in each \( a,a \) pocket we get the power series \( f_{bb}(1 - f_{b^*,b})^{-1} f_{b^*,b^*} \).

3. **Entry (2,1).** Assume that the two consecutive variables in the outer block are \( b^* \) and \( b \). Then similar analysis shows that in each pocket we can get \( (1 - f_{b^*,b})^{-1} \).

4. **Entry (2,2).** Assume that the two consecutive variables in the outer block are \( a^* \) and \( a^* \). Then similar analysis shows that in each pocket we can get \( (1 - f_{b^*,b})^{-1} f_{b^*,b^*} \).

**Remark 6.2.** (1) Suppose that \( (\mathcal{A},\varphi) \) is a \( C^* \)-probability space. In this case the \( \eta \)-series of \( a \) and \( b \) are convergent power series around zero and the system of equations (6.7) can be solved for analytic functions in some neighbourhood of zero.

(2) By specializing to the case when \( a \) and \( b \) have the same distribution, one immediately obtains Theorem 1 of the Introduction. Stated in a bit more detail, this goes as follows.

**Corollary 6.3.** Consider the setting of Theorem 6.1, where we make the additional assumption that \( a \) and \( b \) have the same distribution. Then one has
\[
\begin{align*}
    f_{b,b} &= f_{a^*,a^*}, & f_{b,b^*} &= f_{a^*,a}, \\
    f_{b^*,b} &= f_{a,a^*}, & f_{b^*,b^*} &= f_{a,a}.
\end{align*}
\]

The system of equations (6.7) reduces to a single equation,
\[
F_a H_a = \eta_a(z H_a), \tag{6.8}
\]
where
\[ H_a = \begin{bmatrix} f_{a^*,a^*}(1 - f_{a,a^*})^{-1} & f_{a^*,a} + f_{a^*,a^*}(1 - f_{a,a^*})^{-1}f_{a,a} \\ (1 - f_{a,a^*})^{-1} & (1 - f_{a,a^*})^{-1}f_{a,a} \end{bmatrix}. \]

Moreover, in this case the formula for the \( \eta \)-series of \( ab + ba \) also simplifies, and from Equation (6.6) one gets that
\[ \eta_{ab+ba}(z^2) = 2 \left( f_{a,a^*}(z) + \frac{f_{a,a}(z)f_{a^*,a^*}(z)}{1 - f_{a^*,a}(z)} \right). \]

\[ \square \]

6.2. The special case of symmetric distributions.
As explained in Remark 1.10, the calculation of the distribution of a free anticommutator \( ab + ba \) is much more approachable in the case when \( a \) and \( b \) have symmetric distributions. This fact also manifests itself in the framework of Theorem 6.1, where the hypothesis that \( a \) and \( b \) have symmetric distributions leads to the simplified statement of the next proposition.

**Proposition 6.4.** Consider the framework and notation of Theorem 6.1, and let us also assume that \( a \) and \( b \) have symmetric distributions. Then the power series appearing in Theorem 6.1 are such that \( f_{a,a} = f_{a^*,a^*} = f_{b,b} = f_{b^*,b^*} = 0 \), and the matrices \( H_a \) and \( H_b \) from that theorem become:
\[ H_a = \begin{bmatrix} 0 & f_{b,b^*} \\ (1 - f_{b^*,b})^{-1} & 0 \end{bmatrix}, \quad H_b = \begin{bmatrix} 0 & (1 - f_{a,a^*})^{-1} \\ f_{a^*,a} & 0 \end{bmatrix}. \]

The system of equations (6.7) simplifies to
\[ \begin{bmatrix} f_{a,a^*}(1 - f_{b^*,b})^{-1} & 0 \\ 0 & f_{a^*,a}f_{b,b^*} \end{bmatrix} = \eta_a(zH_a), \]
\[ \begin{bmatrix} f_{b,b^*}f_{a^*,a} & 0 \\ 0 & (1 - f_{a,a^*})^{-1}f_{b^*,b} \end{bmatrix} = \eta_b(zH_b), \]

and one has \( \eta_{ab+ba}(z^2) = f_{a,a^*}(z) + f_{b^*,b}(z) \).

**Proof.** We first note that from the fact that \( \varphi(a^{2n-1}) = \varphi(b^{2n-1}) = 0 \) for all \( n \in \mathbb{N} \) and from the formulas connecting Boolean cumulants to moments we get \( \beta_{2n-1}(a) = \beta_{2n-1}(b) = 0 \) for all \( n \in \mathbb{N} \). Now, the coefficients of the power series \( f_{a,a}, f_{a^*,a^*}, f_{b,b}, f_{b^*,b^*} \) are odd length joint Boolean cumulants of \( a \) and \( b \), thus each of them contains either an odd number of \( a \)'s or an odd number of \( b \)'s. From Theorem 1.2 one gets that every such joint cumulant is zero (since upon writing it as a sum of products in the way indicated by Theorem 1.2 each term will contain an odd length Boolean cumulant of \( a \) or of \( b \)). Thus we get \( f_{a,a} = f_{a^*,a^*} = f_{b,b} = f_{b^*,b^*} = 0 \), and the claims of the proposition then follow from the general formulas obtained in Theorem 6.1. \( \square \)

**Corollary 6.5.** In the framework of Proposition 6.4, assume moreover that \( a \) and \( b \) have the same distribution. Then the Equations (6.10) of Proposition 6.4 further simplify to
\[ \begin{bmatrix} f_{a,a^*}(1 - f_{a,a^*})^{-1} & 0 \\ 0 & f_{a^*,a}^2 \end{bmatrix} = \eta_a(zH_a), \]
where
\[
H_n = \begin{bmatrix} 0 & f_{a,a^*} \\ (1 - f_{a,a^*})^{-1} & 0 \end{bmatrix}.
\]
Moreover, in this case one gets that \( \eta_{ab+ba}(z^2) = 2f_{a,a^*}(z) \).

While the examples of free anticommutators of symmetric distributions can be handled with the methods from \cite{10}, we nevertheless discuss two such examples, mostly in order to point out that Theorem \ref{thm1.8} can be used (in some sense) in reverse, for getting corollaries about the enumeration of \( ac \)-friendly non-crossing partitions.

**Example 6.6.** Suppose that \( p \) and \( q \) are two free projections in a \( * \)-probability space \((\mathcal{A}, \varphi)\), such that \( \varphi(p) = \varphi(q) = 1/2 \), and let \( a := 2p - 1, b = 2q - 1 \). Then \( a \) and \( b \) are as in Corollary \ref{cor6.5} where the common distribution of \( a \) and \( b \) is the symmetric Bernoulli distribution \( \frac{1}{2}(\delta_1 + \delta_1) \). The distribution of \( ab + ba \) can be computed by using Corollary \ref{cor6.5} but this special example is actually much easier to handle, since it is immediate that \( u := ab \) is a Haar unitary in \((\mathcal{A}, \varphi)\), which implies by direct calculation (cf. Example 1.14 in Lecture 1 of \cite{11}) that \( ab + ba = u + u^* \) has the arcsine distribution with density \( (\pi \sqrt{1 - t^2})^{-1} \) on the interval \([-2, 2] \).

On the other hand, one can also look at what Theorem \ref{thm1.8} has to say in connection to this example, and this leads to the corollary stated next. For this corollary, recall that a non-crossing partition \( \sigma \in NC(2n) \) is said to be a pairing when every block of \( \sigma \) has size 2. The set of all non-crossing pairings in \( NC(2n) \) is denoted by \( NC_2(2n) \). It is well-known that the cardinality of \( NC_2(2n) \) is equal to \( \text{Cat}_n \), the same Catalan number which counts all the non-crossing pairings in \( NC(n) \).

**Corollary 6.7.** For every \( m \in \mathbb{N} \), one has that \( |NC_{ac\text{-friendly}}(4m) \cap NC_2(4m)| = \text{Cat}_{m-1} \) and that \( NC_{ac\text{-friendly}}(4m-2) \cap NC_2(4m-2) = \emptyset \).

**Proof.** Take \( a \) and \( b \) as in the preceding example. It is an easy exercise to check that the common Boolean cumulants \((\lambda_n)_{n=1}^{\infty} \) for \( a \) and \( b \) come out as \( \lambda_2 = 1 \) and \( \lambda_n = 0 \) for all \( n \neq 2 \). Equation \((1.12)\) from Remark \ref{rem1.9} thus tells us that
\[
\beta_n(ab + ba) = 2 \cdot |NC_{ac\text{-friendly}}(2n) \cap NC_2(2n)|, \quad \forall n \in \mathbb{N}.
\]
On the other hand, the direct calculation of the \( \eta \)-series of the arcsine distribution gives us that \( \beta_n(ab + ba) = 2 \text{Cat}_{(n-2)/2} \) when \( n \) is even, and that \( \beta_n(ab + ba) = 0 \) when \( n \) is odd, which leads to the formulas stated in the corollary.

**Remark 6.8.** Let \( m \) be a positive integer. It is easy to easy that if \( \pi \in NC_2(2m) \) and if \( \pi \) has \( \{1, 2m\} \) as a pair, then the natural process of “doubling” the pairs of \( \pi \) leads to a pairing in \( NC_{ac\text{-friendly}}(4m) \). This construction produces \( \text{Cat}_{m-1} \) examples of pairings in \( NC_{ac\text{-friendly}}(4m) \), and the above corollary tells us that all the pairings in \( NC_{ac\text{-friendly}}(4m) \) are obtained in this way.

**Example 6.9.** Suppose we want to repeat the trick from Corollary \ref{cor6.7} in order to calculate the number of partitions \( \sigma \in NC_{ac\text{-friendly}}(2n) \) with the property that every block \( V \in \sigma \) has even cardinality. To this end we now start with two selfadjoint elements \( a, b \) in a \( * \)-probability space \((\mathcal{A}, \varphi)\) such that \( a \) is free from \( b \) and such that both \( a \) and \( b \) have distribution
\[
(6.12) \quad \frac{1}{4}\delta_{-\sqrt{2}} + \frac{1}{2}\delta_0 + \frac{1}{4}\delta_{\sqrt{2}}
\]
with respect to $\varphi$. The reason for choosing to use the distribution in (6.12) is that its $\eta$-series is $\frac{z^2}{(1 - z^2)}$, which makes the common sequence $(\lambda_n)_{n=1}^\infty$ of Boolean cumulants for $a$ and for $b$ to be given by
\begin{equation}
(6.13) \quad \lambda_n = \begin{cases} 
1, & \text{if } n \text{ is even}, \\
0, & \text{if } n \text{ is odd}.
\end{cases}
\end{equation}

When applying Corollary 6.5 to the $a$ and $b$ of the present example, the system of equations presented in (6.11) becomes
\[
\begin{cases}
f_{a,a^*}(1 - z^2 f_{a^*,a})(1 - f_{a,a^*})^{-1} = z^2 f_{a^*,a} \\
f_{a^*,a}(1 - z^2 f_{a^*,a})(1 - f_{a,a^*})^{-1} = z^2 (1 - f_{a,a^*})^{-1}.
\end{cases}
\]

Upon some further processing, we find that the series $f_{a,a^*}$ satisfies the equation
\[
f_{a,a^*}(z)(1 - f_{a,a^*}(z))^3 = z^4.
\]

Lagrange inversion formula gives
\[
f_{a,a^*}(z) = \sum_{n=1}^\infty \frac{3}{4n - 1} \left(\frac{4n - 1}{n - 1}\right) z^{4n},
\]
and (in view of the formula $\eta_{ab+ba}(z^2) = 2f_{a,a^*}(z)$ from Corollary 6.5) we come to the conclusion that the $\eta$-series of $ab+ba$ is
\begin{equation}
(6.14) \quad \eta_{ab+ba}(z) = 2 \sum_{n=1}^\infty \frac{3}{4n - 1} \left(\frac{4n - 1}{n - 1}\right) z^{2n}.
\end{equation}

When we look at what Theorem 1.8 has to say in this particular case, we obtain the corollary stated next. In the corollary we will use the notation
\[
NC^{(even)}(2n) := \{\sigma \in NC(2n) \mid \text{every block } V \in \sigma \text{ has even cardinality}\}.
\]

**Corollary 6.10.** For every $m \in \mathbb{N}$, one has that
\[
|NC_{ac-friendly}(4m) \cap NC^{(even)}(4m)| = \frac{3}{4m - 1} \left(\frac{4m - 1}{m - 1}\right)
\]
and that $NC_{ac-friendly}(4m - 2) \cap NC^{(even)}(4m - 2) = \emptyset$.

**Proof.** Take $a$ and $b$ as in the preceding example. Equation (1.12) from Remark 1.9(2) (used in conjunction to the formula for $\lambda_n$’s found in (6.13)) tells us that
\[
\beta_n(ab + ba) = 2 \cdot |NC_{ac-friendly}(2n) \cap NC^{(even)}(2n)|, \quad \forall n \in \mathbb{N}.
\]

On the other hand, $\beta_n(ab+ba)$ is obtained by extracting the coefficient of $z^n$ in the equality of power series which appeared in (6.14). This immediately leads to the formulas stated in the corollary. \qed

### 6.3. A non-symmetric example.

In this subsection we look at the example where $a$ and $b$ have distribution $\frac{1}{2}(\delta_0 + \delta_2)$.

This example offers a very good illustration of how one gets to have different distributions for the free commutator and anticommutator. The commutator $i(ab - ba)$ has exactly the same arcsine distribution as in Example 6.6; indeed, the $a,b$ of the current example are obtained by adding 1 to the $a,b$ of Example 6.6 and the translation by 1 does not affect the commutator. The anti-commutator $ab+ba$ turns out to have a different distribution, as stated in the next proposition. (Recall that the graph of the density $f(x)$ indicated in the
proposition was shown in Figure 2 at the end of the Introduction, together with a histogram of eigenvalues of random matrix approximation.)

**Proposition 6.11.** Notations as above, with \(a, b\) free and having distribution \(\frac{1}{2}(\delta_0 + \delta_2)\). Then the distribution of \(ab + ba\) is the absolutely continuous measure on the interval \([-1, 8]\) which has density \(f(x)\) described as follows:

\[
 f(x) = \begin{cases} 
 \frac{\sqrt{2}}{\pi} \frac{\sqrt{-1 - \sqrt{x^2 - 4}}}{8 - 3\sqrt{(x-8)x-x}} & \text{for } x \in (-1, 0), \\
 \frac{1}{\pi} \sqrt{z(4\sqrt{1+z}+x-4)+3\sqrt{(8-x)(4\sqrt{1+z}+x+4)-8\sqrt{4\sqrt{1+z}+x-4}x}} & \text{for } x \in (0, 8).
\end{cases}
\]

**Proof.** We have \(\eta_a(z) = z/(1-z)\), hence Equation (6.8) amounts here to

\[
(6.15) 
F_a = z(1-zH_a)^{-1},
\]

where \((1-zH_a)^{-1}\) can be written explicitly as

\[
\frac{1}{z^2f_{a,a} + z(f_{a,a} + f_{a,a^*}) + f_{a,a^*} - 1} \left[ zf_{a,a} + f_{a,a^*} - 1 - zf_{a,a^*} f_{a,a^*} - 1 \right].
\]

When solving Equation (6.15), one gets 6 solutions. However, when plugged into the formula for \(\eta_{ab+ba}(z^2)\) given in Corollary 6.3, only one of the 6 solutions satisfies the condition that \(\eta_{ab+ba}(z^2)\) has a double zero at \(z = 0\). After substituting \(z\) by \(\sqrt{z}\) in this solution, we come to the conclusion that

\[
(6.16) 
\eta_{ab+ba}(z) = 1 - \sqrt{(1-8z)(1-2z-\sqrt{1-8z})}. 
\]

From the latter formula for the \(\eta\)-series, a routine calculation takes us to the Cauchy transform of the distribution of \(ab + ba\), which is

\[
G_{ab+ba}(z) = \frac{\sqrt{2} \sqrt{1 + \sqrt{\frac{z-8}{z} + \frac{4}{z}}}}{8 + 3\sqrt{(z-8)z-z}}.
\]

Finally, by using the Stieltjes inversion formula on \(G_{ab+ba}(z)\), we find the form of the density \(f(x)\) which was stated in the proposition.

Same as the examples from the preceding subsection, the example considered here has a combinatorial significance, and can be used to infer the formula indicated in Equation (1.8) of the Introduction for the generating series of cardinalities of sets of ac-friendly partitions.

**Corollary 6.12.** The generating function for cardinalities of sets \(NC_{ac\text{-friendly}}(2n)\) is

\[
\sum_{n=1}^{\infty} |NC_{ac\text{-friendly}}(2n)|z^n = \frac{1}{2} - \sqrt{(1-8z)(1-2z-\sqrt{1-8z})}. 
\]

**Proof.** All the Boolean cumulants of \(a\) and of \(b\) in this example are equal to 1. As a consequence of this, Equation (1.12) from Remark 1.9(2) simply says that

\[
\beta_n(ab + ba) = 2 \cdot |NC_{ac\text{-friendly}}(2n)|, \quad \forall n \in \mathbb{N}.
\]

The generating function for the cardinalities of \(NC_{ac\text{-friendly}}(2n)\) is thus equal to \(\eta_{ab+ba}(z)/2\). We substitute this into the formula for \(\eta_{ab+ba}(z)\) which appeared in Equation (6.16) during the proof of the preceding proposition, and the corollary follows. \(\square\)
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