Remarks on E11 approach

H. Mkrtchyan and R. Mkrtchyan

Abstract

We consider a few topics in E11 approach to superstring/M-theory: even subgroups (Z2 orbifolds) of En, n=11,10,9 and their connection to Kac-Moody algebras; EE11 subgroup of E11 and coincidence of one of its weights with the l1 weight of E11, known to contain brane charges; possible form of supersymmetry relation in E11; decomposition of l1 w.r.t. the SO(10,10) and its square root at first few levels; particle orbit of l1 ⋉ E11. Possible relevance of coadjoint orbits method is noticed, based on a self-duality form of equations of motion in E11.

1E-mail:hike@r.am
2E-mail: mrl@r.am
1 Introduction

One of the recent ideas on a hidden structures in M-theory is that of a hidden $E_{11}$ and/or $E_{10}$ Kac-Moody Lie algebra symmetry [1, 2, 3], generalizing an U-duality of compactified superstrings/M-theory. Appearance of E series of Lie algebras as a symmetry of supergravity theories started from the discovery of $E_7$ as a symmetry (of equations of motion) of maximal four-dimensional supergravity [4]. Afterwards, $E_n$ type algebras, with Dynkin diagrams given below were discovered to appear in compactifications of 11d supergravity and superstrings to $11 - n$ dimensions, including $E_9$ affine algebra in 2d [5] and $E_{10}$ in a 1-dimensional reduction in a form of a particle motion in an $E_{10}$ Weyl chamber (see [6]).

Dynkin diagrams of $E_{12-n}$ are given by nodes $n$, $n+1$, ..., 11

It was the main idea of [1] to consider $E_{11}$ as a symmetry of M-theory. Formally, its Dynkin diagram appears in U-duality considerations of M-theory compactified to 0 dimensions [7]. The point is that $E_{11}$ was suggested in [1] as a symmetry of opposite extreme - completely uncompactified theory. There is a number of arguments in favor of this idea: the field content of model of [1] recovers (the first levels of) the M-theory, the T-duality between IIA and IIB theories appears to be a simple property of $E_{11}$ Dynkin diagram, the brane charges seem to fill in one of the fundamental representations of $E_{11}$, and others. The accompanied difficulties can be seen from this last observation: since in the usual approach the space-time is associated with point-like charge, which now is the part of $E_{11}$ irrep, the $E_{11}$ covariance requires substitution of space-time with duals of all (infinite number of) brane charges, which apparently is not a standard situation in field or string theories.

The similar $E_{10}$ suggestion [3] is much more compact and more precise. It deals with 1d sigma model, instead of infinite-dimensional one, namely that
based on a coset of $E_{10}$ group, with fields depending on one parameter in a
reparametrization invariant way. The price is the loss of (at least explicit)
Lorentz invariance, since the space-time is introduced in this approach by
assumption that coefficients of expansion of fields over coordinates appear
on the higher levels of algebra (the number of which is infinite, due to the
Kac-Moody nature of $E_{10}$).

The $E_{11}$ model will be our main object of study in this paper, we will
present a few results on different aspects of the topic.

According to [1, 2], the hypothesis is that sigma model over coset space
$E_{11}/K_{11}$ gives some description of M-theory. To define $K_{11}$ we first introduce
notations for Kac-Moody Lie algebra with Cartan matrix $A_{ij}$. It’s the Lie
algebra with generators $h_i, e_i, f_i$ and relations:

\[
\begin{align*}
[h_i, e_j] &= A_{ij}e_j \\
[h_i, f_j] &= -A_{ij}f_j \\
[e_i, f_j] &= \delta_{ij}h_j \\
ad(e_i)^{(1-A_{ij})}e_j &= 0, \\
ad(f_i)^{(1-A_{ij})}f_j &= 0
\end{align*}
\]

All other generators should be obtained from these prime generators by
all possible multiple commutators, factorized over relations (2)-(6). Matrix
$A_{ij}$ has the following properties: $A_{ii} = 2$, $A_{ij}$ are non-positive integers such
that from $A_{ij} = 0$ follows $A_{ji} = 0$. $A_{ij}$ alternatively and equivalently can be
presented by Dynkin diagram, with simple rules of equivalence. For example,
for symmetric $A_{ij}$ with non-diagonal entries 0 or -1 Dynkin diagram is given
by nodes equal to dimensionality of $A_{ij}$, nodes $i$ and $j$ are connected by
simple line if and only if $A_{ij} = -1$.

Next, Chevalley involution of an arbitrary Kac-Moody algebra is the Lie
algebra automorphism

\[
\begin{align*}
h_i &\rightarrow -h_i \\
e_i &\rightarrow -f_i \\
f_i &\rightarrow -e_i
\end{align*}
\]

Subalgebra $K_{11}$ consists of generators invariant under Chevalley involution,
it is generated by elements $e_i - f_i$. $K_n$ is a key object for the discussion
of supersymmetry, also, see below. The study of $K_9$ see in [3].
In the body of paper, in Section 2 we consider the even \((Z_2)\) invariant, see (11)-(13) subalgebras of \(E_n\) algebras. From the M-theory viewpoint they are relevant, particularly, for orbifold considerations [9]. Moreover, for finite dimensional algebras \(E_7, E_8\) the even subalgebras actually coincide with Chevalley-invariant subalgebras \(K_7, K_8\). This statement actually extends to all finite-dimensional algebras where corresponding \(K_n\) has rank \(n\). In [10] it was suggested that this coincidence extends to Kac-Moody algebras \(E_n, n = 9, 10, 11\), however, it seems that this assumption is not correct. Which concerns a description of even subalgebras, our claim is that the even roots of \(E_{11}, E_{10}\) and \(E_9\) coincide with all roots of \(EE_{11}, DE_{10}\) and \(D_{8}^{(1)}\), respectively. For \(E_{11}\) that is supported by numerical calculations, for \(E_{10}\) and \(E_9\) that is proved below, for \(E_9\) that gives the complete coincidence of algebras, since the multiplicities of imaginary roots coincide, also.

Next Section 3 considers the weights of fundamental representations of \(EE_{11}\), introduced in previous Section 2. Observation is that one of its fundamental weights coincides with weight of \(l_1\) - first fundamental weight of \(E_{11}\), known to contain the brane charges [11]. This kind of considerations are aimed to discussion of possible supersymmetry relation in \(E_{11}\) theory:

\[
\{Q, Q\} = Z
\]  

(10)

In usual supersymmetric theories the supercharges \(Q\) should be a representations of both compact subgroup \(K_n\) and Lorentz group. E.g., 3d compactification of 11d supergravity gives a \(E_8/\text{SO}(16)\) supersymmetric sigma-model [12], with supercharges in spinor representation of Lorentz group and vector of \(\text{SO}(16)\). In \(E_{11}\) approach these two groups are joined into \(K_{11}\) group. From the other side, the anticommutator of supercharges gives the brane charges \(Z\), which, as argued in [11], is \(l_1\), the irreducible representation of \(E_{11}\). So, roughly speaking, the symmetric square of \(Q\) representation of \(K_n\) gives a fundamental irrep of \(E_{11}\). More precisely, one can imagine that some Klebsh-Gordon coefficients can enter in r.h.s. of (10), so \(l_1\) is one of irreps, appearing in the r.h.s after decomposition to irreducible representations. The symmetric square of highest-weight representation gives, particularly, the irrep with doubled highest weight. So, we see that the coefficient 2 is missing in abovementioned statement of coincidence of one of weights of \(EE_{11}\) with weight of \(l_1\), saying least. Nevertheless, the coincidence of highest weights can signal on some relevance of \(EE_{11}\). The attempt to introduce a supersymmetry in the \(E_{11}\) approach was done in [13], where the Killing spinor
equations are constructed, and fermionic generators are introduced into a
part of $G_{11}$ algebra, which was an intermediate step in construction of $E_{11}$
in [I]. It seems, however, that these results are not relevant for supersym-
metrization of $E_{11}$ itself, (10), due to the few reasons, one of which is that
group, considered in [13], includes momenta which is not the part of $E_{11}$.
Further discussion of problems of [13] see in Section 4

The discussion of supposed susy relation (10) is continued in the next
Section 4, where we study the expansion of first fundamental weight $l_1$ of
$E_{11}$ w.r.t. the levels of root $e_{11}$ of $E_{11}$, (rightmost root of diagram (11)) and
show that the subset of representations at first three levels can be obtained
as a symmetric square of representations of corresponding compact subgroup
$SO(10) \times SO(10)$. This result is another face of similar phenomena found in
[14].

Section 5 is devoted to the study of hypothesis that finally symmetry
group should be extended to semidirect product of $l_1$ and $E_{11}$ [15]. We
calculate the little group for particle orbit, i.e. for a given point in the space
$l_1$ of brane charges, when all charges are zero, except the particle one, we
calculate its stabilizer in $E_{11}$. It appears to have an explicit description in
terms of basic generators of $E_{11}$.

Conclusion contains the resume of results and ways of their development,
particularly, possible relevance of coadjoint orbits of $E_{11}$ is discussed.

2 Even subalgebras of $E_n$

We consider involution of $E_n$ algebras, given by

$$
\begin{align*}
    h_i &\rightarrow h_i \\
    e_i &\rightarrow -e_i \\
    f_i &\rightarrow -f_i
\end{align*}
$$

(11) (12) (13)

The corresponding invariant subalgebra is given by generators $h_i$ and
those of even power of $e_i$ or $f_i$. Let’s denote that by $Z_2(E_n)$. Study of this
subalgebras is relevant for $Z_2$ orbifolds [I]. For finite dimensional Lie algebras
g with the property that rank of CSA of $K_n$ (Chevalley-invariant subalgebra)
is maximal, i.e. equal to n, $K_n$ coincide with $Z_2(g)$,

$$
K_n(g) = Z_2(g)
$$

(14)
It is suggested in [10] that (14) extends to Kac-Moody algebras. However, the problem is in Cartan subalgebra. Although one can find [10] a lot of commuting generators, even with hermiticity properties, their diagonalizability is questionable, since one can show that they have to be diagonalized in the infinite subspaces 1.

As proposed in [14] and [16], the study of bilinear invariant forms can shed light on a problem of connection of compact subalgebra with Kac-Moody algebras. Particularly, in [16] is shown, that special contravariant Hermitian bilinear form is positively defined on $K_n$, and this leads to a conclusion that $K_n$ is not a semisimple Kac-Moody algebra.

Now we shall try to construct even roots of $E_{11}$ from some basic even roots. We would like to introduce the subalgebra of $E_{11}$ generated by CSA and following generators (Lie algebra commutators are implied) and their opposite roots partners:

$$
a_1 = e_7e_8e_9e_{10}, a_2 = e_1e_2, a_3 = e_3e_4, a_4 = e_5e_6, \quad (15)$$
$$a_5 = e_7e_8, a_6 = e_{10}e_{11}, a_7 = e_8e_9, a_8 = e_6e_7,$n$$a_9 = e_8e_{10}, a_{10} = e_4e_5, a_{11} = e_2e_3,

Definition of $a_1$ is actually unique, up to overall sign, since although Lie brackets can be arranged in different ways, results coincide. Roots of (15) are real. One can find the corresponding Cartan matrix and Dynkin diagram:

$$EE_{11} = \begin{pmatrix} 2 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \quad (16)$$

1We are indebted to H.Nicolai for e-mail correspondence stressing the importance of diagonalizability of Cartan generators.
where simple roots in (17) are enumerated in agreement with (16). One can construct the corresponding abstract Kac-Moody algebra, we denote it by $EE_{11}$ since it contains two E type tails, and this notation is similar to that of hyperbolic algebras - AE, BE, CE, DE. Since roots (15) are real, according to [18], algebra (17) is isomorphic to above defined subalgebra in $E_{11}$. Our hypothesis is that this algebra has the same roots as $Z_2(E_{11})$, i.e. all even roots of $E_{11}$. Statement seems to be simple, nevertheless, we were not able to prove it algebraically, due to unknown structure of roots system. Instead we checked that up to level 146 by the help of computer program (available upon request), which generates the roots for an arbitrary input Dynkin diagram. The number of roots up to the level 146 ( inclusively) is 19661788 (without counting multiplicity), so coincidence is considerable. Since multiplicity of real roots is one, this statement means that these two algebras coincide at least in the sector of real roots. However, as shown in [9] for similar considerations for $E_{10}$ (see below), for imaginary roots there is difference in multiplicities in the case of $Z_2(E_{10})$ and $DE_{10}$, so we are confident in the same statement for $E_{11}$.

Similar statements (on the coincidence of even roots and roots of subalgebras of simple even roots) for $E_{10}$ and $E_9$ can be proved. Corresponding composite roots (generators) and Dynkin diagram for $Z_2(E_{10})$ are:

\[
\begin{align*}
  a_1 &= e_6e_7e_8e_9, a_2 = e_2e_3, a_3 = e_4e_5, a_4 = e_6e_7, a_5 = e_9e_{10}, \\
  a_6 &= e_7e_8, a_7 = e_5e_6, a_8 = e_7e_9, a_9 = e_3e_4, a_{10} = e_1e_2
\end{align*}
\]
Dynkin diagram of $DE_{10}$ algebra \hfill (19)

For $E_9$:

$$a_1 = e_1 e_2, a_2 = e_5 e_6 e_7 e_8, a_3 = e_3 e_4, a_4 = e_5 e_6,$$

$$a_5 = e_8 e_9, a_6 = e_6 e_7, a_7 = e_4 e_5, a_8 = e_6 e_8, a_9 = e_2 e_3$$ \hfill (20)

Dynkin diagram of $D_{8}^{(1)}$ algebra \hfill (21)

Both for $E_{10}$ and $E_9$ coincidence of roots follows from two facts. First, root lattices coincide \cite{9}, namely, even root sublattice of $E_{10}$ ($E_9$) coincide with lattice of $DE_{10}$ \hfill (19) ($D_{8}^{(1)}$ \hfill (21)). Second, description of all roots is given in \hfill (19), p.67), in terms of lattices - all real roots are all those elements of lattice with square equal two, and all other roots (i.e. imaginary ones) are all those elements of lattice with square less or equal to zero. For $E_9$ that means complete coincidence of algebras:

$$Z_2(E_9) = D_{8}^{(1)}$$ \hfill (22)
since multiplicities of imaginary roots coincide (multiplicities of $Z_2(E_9)$ are that of $E_9$, which is 8, since that is an affine algebra $E_8^{(1)}$, and multiplicity of roots of $D_8^{(1)}$ is 8, also).

For $E_{10}$ the problem of multiplicities is more complicated, and in [9] it is shown that actually multiplicities of $Z_2(E_{10})$ and $DE_{10}$ are different. It would be interesting to describe that difference explicitly.

For $E_{11}$ the coincidence of its even lattice of and that of $EE_{11}$ can be proved, also, but it is not enough for coincidence of all roots, since for these non-hyperbolic algebras there is no similar description of roots.

Coincidence of even roots of $E_{10}$ and all roots of $DE_{10}$ also follows from statements on a level decompositions of these two algebras, proved in Section 4.2 of [17].

3 On a representations of $EE_{11}$

The weights of the fundamental representations of $E_{11}$ can be obtained by the rows of an inverse Cartan matrix $(EE_{11})^{(-1)}$ of [17]. We would like to compare these weights with those of $E_{11}$:

$$E_{11}^{-1} = \frac{1}{2}$$

$$\begin{bmatrix}
-1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 3 & 4 & 2 \\
0 & 0 & 2 & 4 & 6 & 8 & 10 & 12 & 6 & 8 & 4 \\
1 & 2 & 3 & 6 & 9 & 12 & 15 & 18 & 9 & 12 & 6 \\
2 & 4 & 6 & 8 & 12 & 16 & 20 & 24 & 12 & 16 & 8 \\
3 & 6 & 9 & 12 & 15 & 20 & 25 & 30 & 15 & 20 & 10 \\
4 & 8 & 12 & 16 & 20 & 24 & 30 & 36 & 18 & 24 & 12 \\
5 & 10 & 15 & 20 & 25 & 30 & 35 & 42 & 21 & 28 & 14 \\
6 & 12 & 18 & 24 & 30 & 36 & 42 & 48 & 24 & 32 & 16 \\
3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & 11 & 16 & 8 \\
4 & 8 & 12 & 16 & 20 & 24 & 28 & 32 & 16 & 20 & 10 \\
2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 8 & 10 & 4
\end{bmatrix}$$

(23)

The only subtlety is that rows of inverse Cartan matrix express weights in the basis of simple roots of a given algebra, so for comparison we should express both in the same basis, e.g. in a basis of simple roots of $E_{11}$. The expression of simple roots of $EE_{11}$ through the simple roots of $E_{11}$ is given in [15]. So we should multiply the matrix $(EE_{11})^{(-1)}$ from the right by the transformation matrix.

9
and obtain

\[
(EE_{11})^{-1}T = \frac{1}{8} \begin{bmatrix}
2 & 8 & 10 & 16 & 18 & 24 & 26 & 32 & 14 & 20 & 12 \\
-4 & 0 & 4 & 8 & 12 & 16 & 20 & 24 & 12 & 16 & 8 \\
0 & 8 & 8 & 16 & 24 & 32 & 40 & 48 & 24 & 32 & 16 \\
4 & 16 & 20 & 32 & 36 & 48 & 60 & 72 & 36 & 48 & 24 \\
6 & 16 & 22 & 32 & 38 & 48 & 54 & 64 & 34 & 44 & 20 \\
8 & 16 & 24 & 32 & 40 & 48 & 56 & 64 & 32 & 40 & 16 \\
10 & 16 & 26 & 32 & 42 & 48 & 58 & 64 & 30 & 44 & 20 \\
12 & 16 & 28 & 32 & 44 & 48 & 60 & 72 & 36 & 48 & 24 \\
6 & 8 & 14 & 16 & 22 & 24 & 30 & 32 & 18 & 20 & 12 \\
8 & 8 & 16 & 16 & 24 & 32 & 40 & 48 & 24 & 32 & 16 \\
4 & 0 & 4 & 8 & 12 & 16 & 20 & 24 & 12 & 16 & 8 \\
\end{bmatrix}
\]

The second row of (24) and first row of (25) coincide. As mentioned in the introduction, this statement has some resemblance with the one that is implied by (supposed) supersymmetry relation (10), although it differs in few important points - coefficient 2 is missing, and, more important, $EE_{11}$ is not a $K_{11}$, although can have similar properties. It is also worth recalling the discussion of [20], where the traces of needed phenomena were noticed, namely at certain dimensions certain brane charges are not only in the representation of corresponding susy algebra’s automorphism group, but also combine into representations of U-duality group. For example, consider maximal susy algebra in 4 dimensions, obtained from reduction to 4d of 11d susy algebra.
The corresponding 4d algebra has SU(8) as an automorphism group. Scalar central charges appear from a few sources: 7 from vector, 21 from membrane charge, and 28 from five-brane charge, altogether they combine into 56 of 4d U-duality group $E_7$, which contains SU(8) as its maximal compact subgroup. This field worths further study, the key element should be the theory of $K_{11}$ group’s representations.

4 $l_1$ expansion over $SO(10, 10)$

Let’s consider the expansion of both sides of supposed susy relation (10) over $SO(20)$ ($SO(10, 10)$ if reality properties taken into account) subgroup of $E_{11}$, obtained by removing most right root (number 11 on diagram (11). The corresponding compact group is $SO(10) \times SO(10)$. This expansion can shed some light on whether such a relation can exist, at least on a first few terms of expansion. The level expansion of $l_1$ with respect to the generator $e_9$ was suggested in [14], where it was shown, that usual brane charges($P_\mu, Z_{\mu\nu}, Z_{\mu_1\mu_2\mu_3\mu_4\mu_5}$) appear on first three levels of that expansion. As is well-known, one can fulfill relation (10) with Q as a spinors of corresponding compact group $SO(11)$. We would like to consider the same problem with expansion over $e_{11}$, with compact group $SO(10) \times SO(10)$, which was not supported by existence of any supersymmetric theories, but, from the other side, should exist, provided $E_{11}$ hypothesis and relation (10) are correct.

Expansion over $e_{11}$ goes with non-negative powers. This is clear when recalling an approach of [13] - in that paper $l_1$ is identified with the subspace of $E_{12}$, linear over $e_0$. This subspace evidently is a representation of $E_{11}$, and actually the first fundamental representation, with $e_0$ as a highest weight vector. The same is true for the zeroth order over $e_{11}$ - it is a representation with highest vector $e_0$, since all $f_i, i > 0$ commute with $e_0$ and action of $h_i, i > 0$ on $e_0$ gives the only non-zero Dynkin label $p_1 = 1$, i.e. that of vector representation of $SO(20)$, 20. Next is linear over $e_{11}$ representation of $SO(20)$. It is easy to understand, that the highest vector for that representation is unique, namely that given by unique nonzero commutator of $e_0, e_1, ..., e_{11}$ (except $e_9$). The only nonzero Dynkin label is $p_9 = 1$, i.e. that is one of two Weyl spinor representations of $SO(20)$, of dimensionality 512. Next is the second order over $e_{11}$ representation, which is the last one we are interested in. It can be shown to be fifth-rank antisymmetric tensor $Z_5$, with highest vector...
\[ \sum n_i e_i, \ n_0 = \ldots = n_5 = 1, n_6 = 2, n_7 = 3, n_8 = 4, n_9 = 2, n_{10} = 3, n_{11} = 2. \]

The corresponding Dynkin label is \( p_5 = 1 \) (other \( p_i = 0 \)). We are interested in decomposition of these few first level representations over compact subgroup of \( \text{SO}(10,10) \). The \( 20 \) is of course \( 10 + 10 \), \( 512 \) is \((16,16) + (16,16)\) (chirality of \( \text{SO}(20) \) representation choose chirality of \( \text{SO}(10) \)). \( Z_5 \) is decomposed in more or less clear way, according to all possible choices of belonging of indexes to two product \( \text{SO}(10) \) groups: 
\[
Z_5 = Z_{5,0} + Z_{4,1} + Z_{3,2} + Z_{2,3} + Z_{1,4} + Z_{0,5} = (252,1) + (210,10) + (120,45) + (45,120) + (10,210) + (1,252)
\]
Moreover, fifth index tensors should be decomposed into their chiral parts, i.e. into self-dual and anti-self-dual tensors of dimensionality 136. Here we imagine complex field of coefficients, neglecting reality properties of groups involved, their will tune themselves according to the choice of Chevalley subgroup.

After this preparatory work we can look for an answer on a possibility of taking a square root of \( l_1 \), i.e. finding a representation of \( K_{11} \), symmetric square of which contains \( l_1 \). It appears that one can take the following combination of representation of \( \text{SO}(10) \times \text{SO}(10) \). This combination should be considered as a decomposition of irrep of \( K \) we are seeking for, w.r.t. the \( \text{SO}(10) \times \text{SO}(10) \):
\[
(16,1) + (1,16)
\]
Symmetric square of this representation is
\[
(10,1) + (1,10) + (16,16) + (136,1) + (1,136)
\]
This includes representations of first level, second level, and part of representations of third level, obtained above in decomposition of \( l_1 \) w.r.t. the powers of \( e_{11} \). I.e. we find the square root of (part of the) first few levels of \( l_1 \). Finally, the susy relation, in this approximation, can be represented in a well-known form
\[
\{Q_\alpha, Q_\beta\} = Z_{\alpha\beta}
\]
This is a standard form of 11d susy relation, where supersymmetry charge is 32 dimensional, giving in the r.h.s all possible 528 central charges. That means that we are dealing with the same \( \text{SL}(32) \) invariant susy algebra, decomposed with respect to different subalgebras. The natural question is whether there exist \( \text{SO}(10) \times \text{SO}(10) \) covariant supergravity theories, with
susy algebra \((28)\). The \(SL(32)\) invariance of \((28)\) doesn’t persist on the higher levels, since \(E_{11}\) does not have such a subgroup (see discussion in \((21)\), where it was shown that antisymmetric tensor representations, precisely corresponding to those of \(SL(32)\) decomposed w.r.t. the \(SL(11)\) can be identified on a first 4 levels of \(K_{11}\)).

Let’s mention a difference of previous considerations with approach of \((13)\). First, as mentioned in the Introduction, the groups considered are different - \(G_{11}\) of \((13)\) includes momenta \(P\mu\), which is not the part of \(E_{11}\). Moreover, it is argued in \((13)\) that two spinor generators are needed in supersymmetrization of \(G_{11}\), on the basis that 11d conformal group \(SO(2,11)\) should be, finally, part of the symmetries considered. That statement stress that \(SO(2,11)\) group is not part of \(E_{11}\), from the same fact of absence of momenta \(P\mu\) in \(E_{11}\) (as well as its conformal counterpart \(K\mu\)).

This discussion stress the problem of finding the generalization of \(E_{11}\), which will include momenta and possibly the whole conformal group \(SO(2,11)\). As mentioned in Introduction, the generalization including momenta, actually the whole multiplet \(l_1\), is suggested in \((13)\) as semidirect product of \(l_1\) on \(E_{11}\). Inclusion of conformal \(SO(2,11)\) will require further extension of this group.

Next remark concerns the possibility of continuation of above analysis to higher levels of \(e_{11}\). The problem is in that there is no corresponding grading of desired representation of \(K_{11}\). So it is not clear how to continue finding next terms of decomposition of that representation w.r.t. the \(SO(10) \times SO(10)\).

It is worth mentioning here the connection between \(SO(10) \times SO(10)\) subgroups of \(SO(20) \subset E_{11}\), with \(EE_{11}\). From the diagram of \(EE_{11}\) one can easily read off two \(SO(10)\) subgroups, constructed from two \(D_5\) subdiagrams, that appear after removing the middle root (number six). One can easily understand, that they are the same \(SO(10)\) subgroups of \(SO(20)\) and \(E_{11}\). It follows from the fact that eleventh root \(e_{11}\) of \(E_{11}\) enters in the sixth root of \(EE_{11}\), only, and from the fact mentioned in Section 1, that compact subgroup of \(D_{10}\) can be represented as its even subgroup.

## 5 Particle orbit in \(l_1 \ltimes E_{11}\)

In the search of a space-time in the \(E_{11}\) approach, West \((14)\) suggested to extend the symmetry group to semidirect product \(l_1 \ltimes E_{11}\), which is similar to Poincare group. Then one has to consider a unitary representation of this
group, which can be constructed by Wigner’s little group method, which is recently applied to construction of irreps of semidirect product of Lorentz and tensorial translations group \[22\]. The method requires choosing of the orbit of action of \(E_{11}\) on \(l_1\), and then construction of unitary irreps of little group - stabilizer of a given point of the orbit in the \(E_{11}\). We will apply this method to particle orbit.

According to suggestion \[14\] representation \(l_1\) contains all brane charges. Particularly, the decomposition of \(l_1\) w.r.t. the \(SL(11)\) subgroup of \(E_{11}\) starts from vector representation \(P_\mu\). Particle can be naturally defined as a configuration of brane charges when all of them are zero, except \(P_\mu\). Note that we are dealing with general linear \(GL_{11}\) group, which gives usual Lorentz \(SO(11)\) after taking a Chevalley-invariant (=compact) subgroup of \(GL_{11}\). Correspondingly, an arbitrary vector \(P_\mu\) can be transformed into any other vector, so we can choose

\[
P_\mu = (1, 0, 0, ...)
\]  

\((29)\)

Next, our aim is to define an orbit of this point under action of the whole group \(E_{11}\). Desired orbit is a factor of \(E_{11}\) over \(L\), the stabilizer of \(P_\mu\), i.e. subgroup of \(E_{11}\), which leaves \(P_\mu\) unchanged. So, our task is to find the subgroup \(L\). Note that \(P_\mu\) \((29)\) is represented by just \(e_0\). So, we are seeking a stabilizer of \(e_0\) in \(E_{11}\). Evidently, among generators of \(E_{11}\), commuting with \(e_0\), are \(E_{10}\) generators, constructed from elements \(e_i, h_i, f_i\) with indexes starting from 2. Next, among them are all generators of \(E_{11}\) with non-zero power of \(f_1\). It remains to consider generators of \(e_{11}\) with non-zero power of \(e_1\). They all have nonzero commutators with \(e_0\), which is evident from the rules of construction of roots - the scalar product of such generators with \(e_0\) are nonzero negative integers (equal to the power of \(e_1\)), so real root \(\alpha_0\) can be added to the given root of \(E_{11}\). So, the stabilizer of \(e_0\) within \(E_{11}\) is \((E_{10}, (f_1)^n ... (n > 0))\), where \((f_1)^n ... (n > 0))\) denotes all roots of \(E_{11}\) with nonzero power of \(f_1\). This is a semidirect product of \(E_{10}\) and \((f_1)^n ... (n > 0))\). So, particularly, each unitary representation of \(E_{10}\) gives rise to induced unitary irrep of \(l_1 \ltimes E_{11}\).
6 Conclusion

In the body of paper we discuss some features of $E_{11}$ approach, which can help in study of different aspects of theory - such as orbifolds, supersymmetry relation, induced representations, etc. We introduce an even subalgebras $Z_2(E_n)$ and find a description of corresponding roots through $EE_{11}$ (for $E_{11}$), $DE_{10}$ (for $E_{10}$), and $D_8^{(1)}$ (for $E_9$). It is proved that for last two cases even roots are completely given by algebras mentioned, for $E_{11}$ that is a hypothesis, supported by computer calculations up to level 142. The possible form of supersymmetry relation (10) is considered. It requires that compact subgroup $K_n$ has a representation the symmetric square of which contains $l_1$ representation of $E_{11}$. In view of that we consider the expansion of space of brane charges, i.e. $l_1$ w.r.t. the $SO(10,10)$ subgroup of $E_{11}$, and show that first few representations of $SO(10,10)$ in $l_1$ can be represented in required form. Corresponding relation (28) is a standard 11d supersymmetry relation, decomposed w.r.t. the $SO(10) \times SO(10)$ subgroup. Another result, which may have relation with supersymmetry (and not only) is the finding of Section 3, that second fundamental weight of $EE_{11}$ coincides with the weight of $l_1$ irrep of $E_{11}$. Other existing approaches to supersymmetrization of $E_{11}$ are discussed.

Finally, precise answer is obtained for a little group of particle orbit in a semidirect product group $l_1 \ltimes E_{11}$, assumed to be an extended symmetry group of $E_{11}$ theory.

In conclusion, we would like to discuss the possible connection of $E_{11}$ approach to well-known method of coadjoint orbits. According to the construction of [1] fields of $E_{11}/K_{11}$ manifold contains both fields and their duals, as is precisely shown for lower level, and corresponding equations of motion are those of generalized self-duality. If one neglects dependence of fields from (infinite number of?) space-time coordinates, that will mean that $E_{11}/K_{11}$ is not a configuration space but rather phase space, since it includes both fields and their conjugate momenta. Taking into account the existence of natural (Kirillov-Kostant) Poisson bracket on the coadjoint orbits of Lie algebras, one can ask whether $E_{11}/K_{11}$ is such an orbit. That would mean that $K_{11}$ is a stabilizer of some element of $E_{11}$ algebra. It is easy to show that it is not the case. Of course, according to the previous discussion it is not a necessary feature, one should take into account a space coordinates, to make a statement precise. We think that application of coadjoint orbit method to $E_{11}$ works further study.
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