Finite difference method based on polynomial interpolation for solving Helmholtz equation

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ABSTRACT. The generalized finite difference formula plays an important role in the meshless method for solving differential equations. The main purpose of this paper is to find the numerical solution of Helmholtz equation, an elliptic partial differential equation describing electromagnetic waves in physics, by using the generalized finite difference formula. Firstly, this paper introduces a simple and practical nodal distribution, which not only guarantees the uniqueness of multivariate polynomial interpolation, but also makes the matrix triangular, so that the constructed basic polynomials can be transformed into Lagrange basis polynomials. Secondly, the generalized finite difference formula is created by polynomial interpolation. Finally, a numerical example of Helmholtz equation under general boundary conditions is given.

1. Introduction

In solving the problem of partial differential equation in physical mechanics, if the equation has no analytical solution, we often hope to get the numerical solution of the equation. Finite element method (FEA) is a popular method to solve partial differential equations in mechanics. It is a modern calculation method developed rapidly with the development of computer. However, because the finite element method is based on the numerical algorithm with mesh, when solving problems involving large deformation such as metal forming and stamping process, as well as problems such as dynamic crack propagation, explosion impact, high-speed collision, the method is faced with the problems of mesh redivision complexity and mesh distortion, which seriously affect the efficiency and accuracy of calculation. If the mesh remodeling is not good enough in the process of calculation, such as when the mesh is discontinuous or does not meet the requirements of pre-processing, it is difficult to carry on the whole process of mesh based solution and iteration. On this basis, in order to make up for the problems of traditional finite element calculation, the meshless method[1-3] is proposed in the academic circle.

Mesh less method is a kind of solution technology which does not consider the mesh division and only uses the node distribution to realize the modeling and Simulation of physical problems. This method not only gets rid of the constraints of mesh initialization and mesh reconstruction, but also ensures the accuracy of the solution [4]. At present, there are more than ten mature meshless methods, including smoothed particle hydrodynamics (SPH), meshless Galerkin method (EFGM), meshless local Galerkin method (mlpgm), diffusion element method (DEM), HP clouds meshless method. Because this method does not need to mesh and is flexible, it is widely used in computational mechanics, large deformation problems and emerging nanoscale multi-scale problems, cell infiltration, blood flow, biological microelectronic systems and other fields.

Helmholtz equation is an elliptic partial differential equation describing electromagnetic waves, named after Helmholtz, a German physicist. Because of its relationship with wave equation, Helmholtz
equation appears in the fields of electromagnetic radiation, seismology and acoustics in physics. In this paper, the generalized finite difference method, a meshless method proposed by Huang Xiaowei and Wu Chuansheng [5], is used to solve the numerical solution of Helmholtz equation under the general boundary condition, that is, the Dirichlet problem. This method is based on local polynomial interpolation method, and requires local nodes to have a special distribution proposed by Gasca and maetzu, so as to maintain the existence and uniqueness of Multivariable Interpolation [6-8]. On this basis, the Newton interpolation basis function is constructed directly, and the generalized finite difference formula with simple and stable algorithm is established. This method overcomes the singularity problem that may exist in the generalized finite difference method, and the determination of the finite difference formula only needs to solve the linear equations whose coefficient matrix is a triangular matrix, which reduces the amount of calculation to produce the generalized finite difference formula.

In this paper, we first introduce a simple method to determine the node distribution and interpolation node set, then derive the approximation function \( f(x) \) at the point \( p \) \((p \in \Omega, \Omega \) is a given bounded region of the problem under study), and finally give a numerical example of solving Helmholtz equation with the generalized finite difference method under general boundary conditions.

2. Interpolation node and interpolation node set

The distributed generation of nodes is very important to the numerical solution of partial differential equations by meshless method. We hope that the node distribution algorithm is simple enough and can satisfy the uniqueness and Solvability of multivariate polynomial interpolation. Then, according to the selected nodes, the generalized finite difference formula is constructed by polynomial interpolation. Finally, the finite difference formula is introduced into the partial differential equation and the numerical solution of the problem is obtained.

According to the theory of polynomial interpolation, if the generated nodes are on a series of parallel planes and the points on the plane are on a series of parallel lines, the interpolation nodes of n-ary polynomials can be easily selected. Based on this principle, we propose a two-dimensional linear node distribution model.

2.1. Distribution of linear nodes in two-dimensional region

For the two-dimensional bounded connected region \( \Omega \subset R^2 \), we select the linear nodes parallel to the y-axis. In order to obtain a group of parallel linear nodes, we use \( \Delta \) to represent the discretization level of the planar region. The generation process of node distribution is as follows:

First, the projection region \([a_1, a_2]\) of region \( \Omega \) on the x-axis is defined as follows:

\[
a_1 = \inf \{x| (x, y) \in \Omega\} \quad a_2 = \sup \{x| (x, y) \in \Omega\}
\]

Then, the number of columns \( m \) of the linear node is determined by the discretization level \( \Delta \) of the region.

\[
M = \begin{cases} 
1, & a_1 = a_2, \\
2, & 0 < a_2 - a_1 < \Delta, \\
\left\lfloor \frac{a_2 - a_1}{\Delta} \right\rfloor + 1, & a_2 - a_1 \geq \Delta;
\end{cases}
\]

Then a group of equidistant dividing points and a series of lines parallel to the y-axis are generated on the projection area \([a_1, a_2]\) of the x-axis.

\[
x_i = a_1 + \frac{a_2 - a_1}{M-1}(i - 1), \quad i = 1, \ldots, M
\]

We subdivide the intersection \( l_i \cap \Omega \) of line \( l_i \) and region \( \Omega \) into several straight line segments, and we make the following formula true.

\[
b_{i1} = \inf \{y| (x_i, y) \in \Omega\}, b_{i2} = \sup \{y| (x_i, y) \in \Omega\}
\]
The column number $N_i$ of linear node is determined by discretization level $\Delta$ of region.

$$
N_i = \begin{cases} 
1, & b_{i1} = b_{i2}, \\
2, & 0 < b_{i2} - b_{i1} < \Delta, \\
\left\lfloor \frac{b_{i2} - b_{i1}}{\Delta} \right\rfloor + 1, & b_{i2} - b_{i1} \geq \Delta,
\end{cases} \tag{5}
$$

Similarly, equidistant nodes are generated on these straight line segments.

$$
y_{ij} = \begin{cases} 
b_{i1}, & N_i = 1, \\
b_{i1} + \frac{b_{i2} - b_{i1}}{N_i - 1} (j - 1), & N_i > 1, j = 1, \ldots, N_i
\end{cases} \tag{6}
$$

Finally, the nodes on the region $\Omega \subset \mathbb{R}^2$ are obtained as follows:

$$
T = \bigcup_{i=1}^{l} \{(x_i, y_{ij}) | i = 1, \ldots, M; j = 1, \ldots, N_i\}. \tag{7}
$$

In MATLAB, we choose parameter $\Delta = \frac{1}{10}$ to get the node distribution of circular area as shown in Figure 1.

![Figure 1. Node distribution](image_url)

As can be seen from the above figure, 281 nodes are generated, and the execution time is about 0.0035s.

The rotation of x-axis and y-axis in the algorithm can produce linear nodes parallel to y-axis as shown in Figure 2. In this paper, the two-dimensional discrete nodes are composed of linear nodes parallel to x-axis.
2.2. Interpolation node set

After obtaining the linear node set $T$ in the region $\Omega$ parallel to the y-axis by the above method, this section will use the radial basis function interpolation method to select the interpolation node set $\lambda_n(p)$ near the node $p$, so that the polynomial interpolation problem on $\prod_n^d$ has a unique feasible solution with respect to the interpolation node set $\lambda_n(p)$.

The generalized finite difference formula approximates the derivative of the calculation point by the linear combination of the function values of the adjacent nodes near the calculation point, so it is necessary to select the set of adjacent nodes near the calculation point to establish the generalized finite difference formula. The radial basis function interpolation method used in this section first finds the node closest to the calculation point $p$, and then selects the interpolation node set $\lambda_n(p)$ near the node $p$ according to the given number of nodes $n$.

Firstly, in the generated node set $T = \bigcup_{i=1}^M \{ (x_i, y_{ij}) | i = 1, \ldots, M; j = 1, \ldots, N_i \}$, the $n + 1$-points nearest to node $p$ are determined to form the point set $J$, and then the node set $G$ is transformed into a node set $S_1, S_2, \ldots, S_{k_1}$, and we make the following formula true.

$$ (T^{(k)})_{k=1}^K = \{ T_{ij} | i = 1, \ldots, M; j = 1, \ldots, N_i \}, \quad K = \sum_{i=1}^M N_i $$

(8)

Then, we apply the greedy strategy. Firstly, we define index set $S = \{ 1, \ldots, K \}$, and the point set closest to point $p$ is selected as $\rho_0(p)$ in $S_1, S_2, \ldots, S_{k_1}$. the point set $S_1, S_2, \ldots, S_{k_2}$ is composed of $n$ points which are different from $\rho_0(p)$ and closest to point $p$. And so on, until all $C_{n+k_2}^2$ nodes are selected on $n + 1$ different grid lines, the interpolation node set $\lambda_n(p) = \bigcup_{r=0}^n \rho_r(p)$ is constructed, that is, through the following rule iteration:

$$ V = \{ k \in S | |T^{(k)}| \geq n - r + 1 \} $$

(9)

For any $k \in V$, select $n - r + 1$ points in $T^{(k)}$ which are closest to point $p$ to form point set $U_r^{(k)}$. 

Figure 2. Node distribution
\[ k_r = \arg \min_{k \in V} \min_{q \in U^{(k)}_r} || p - q ||, \quad \rho_r(p) = U^{(k_r)}_r \]

Finally, the interpolation node set is obtained as follows:

\[ S = \mathcal{S} \setminus \{ k_r \}. \]  

We implement it in MATLAB. Figure 2 shows the interpolation nodes of a given point in the circle.

Figure 3. Interpolation node set, denotes the given point \( p = (0.086, 0.095) \)

Given node \( p = (0.086, 0.095) \), \( n = 6 \), 28 interpolation nodes are generated, and the execution time is about 0.012s.

3. Generalized finite difference formula

Based on the plane line type node distribution and interpolation node set selection method introduced in the above section, the Newton polynomial interpolation basis function and Lagrange polynomial interpolation basis function are derived, and the generalized finite difference formula is finally derived. In this paper, we only consider the two-dimensional case, i.e. given region \( \Omega \in \mathbb{R}^2 \).

First, according to the method of the upper section, the plane linear node distribution \( P \in \Omega \) is constructed. For any given point \( p \in \Omega \), the set of two-dimensional interpolation nodes is obtained:

\[ \chi_n(p) = \{ q_\beta = (u_r, v_s) || \beta = (r, s) \in I_6^2 \}, \]  

According to the conclusion of Gasca et al. [4, 5], for the given set of interpolation nodes, the Newton polynomial interpolation basis functions of two-dimensional polynomial space are constructed as follows:

\[ \phi_{\alpha}(x, y) = \mu_i(x) \lambda_{ij}(y), \quad \alpha = (i, j) \in I_n^2. \]  

In this formula:
\[ \mu_i(x) = \begin{cases} 1, & i = 1 \\ \Pi_{i=1}^{r-1} \frac{x - u_r}{u_i - u_r}, & i > 1, \end{cases} \quad (15) \]

\[ \lambda_{ij}(y) = \begin{cases} 1, & i = 1 \\ \Pi_{i=1}^{s-1} \frac{y - v_{is}}{v_{ij} - v_{is}}, & i > 1, \end{cases} \quad (16) \]

It is easy to verify that \( \phi_{\alpha}(q_\beta) \) satisfies the following equation:

\[ \phi_{\alpha}(q_\beta) = \begin{cases} 1, & \alpha = \beta \\ 0, & \alpha \neq \beta \end{cases} \quad (17) \]

In this formula, \( \alpha, \beta \in \mathbb{I}_n^2 \). Moreover, the two-dimensional Newton polynomial interpolation basis function derived has the same properties as the one-dimensional Newton polynomial interpolation basis function, and the two-dimensional polynomial interpolated on \( \mathbb{I}_n^2 \) is unique for \( \chi_n(p) \).

Similarly, for the interpolation node set \( \chi_n(p) \), the Lagrange polynomial interpolation basis function is constructed so that the function is 1 at a given interpolation node and 0 at other interpolation nodes. The expression is as follows:

\[ l_{\alpha}(q_\beta) = \begin{cases} 1, & \alpha = \beta \\ 0, & \alpha \neq \beta \end{cases} \quad (18) \]

\( \{\phi_{\alpha}\}_{\alpha \in \mathbb{I}_n^2} \) and \( \{l_{\alpha}\}_{\alpha \in \mathbb{I}_n^2} \) form column vectors \( \Phi \) and \( L \) respectively according to the dictionary order of subscripts. The values of column vector \( \Phi \) at each interpolation node are arranged into a square matrix \( A = [\Phi(q_\beta)]_{\beta \in \mathbb{I}_n^2} \) according to the dictionary order of node labels \([12,13]\). From (2-5) formula, it is easy to prove that the matrix \( A \) is a triangular matrix of unit, and the determinant value of the matrix is 1, that is, the matrix is nonsingular. According to the uniqueness of interpolation polynomial, Newton polynomial interpolation basis function can be expressed by Lagrange polynomial interpolation basis function in the following matrix form:

\[ \Phi = AL \quad (19) \]

Then the Lagrange polynomial interpolation basis function can be obtained

\[ L = A^{-1} \Phi \quad (20) \]

According to the uniqueness of the polynomial interpolation problem, we can prove that the Lagrange polynomial interpolation basis function \( \{l_{\alpha}\}_{\alpha \in \mathbb{I}_n^2} \) of our given interpolation node set \( \chi_n(p) \) satisfies the following conditions for \( \forall \gamma, \gamma' \in \mathbb{N}_0^2, |\gamma'| \leq n \).

\[ \sum_{\alpha \in \mathbb{I}_n^2} (q_\alpha - p)^{\gamma'} D^{\gamma} q_\alpha(p) = \begin{cases} \gamma! & \gamma = \gamma' \\ 0 & \gamma \neq \gamma' \end{cases} \quad (21) \]

In the formula \( D^{\gamma} l_{\alpha}(p) \) is the \( \gamma \) -th derivative of \( l_{\alpha}(q) \).

Finally, the generalized finite difference formula is derived, that is, given \( f: \mathbb{R}^2 \to \mathbb{R} \), the Lagrange interpolation polynomial of given interpolation node set \( \chi_n(p) \) is:

\[ \sum_{\alpha \in \mathbb{I}_n^2} f(q_\alpha) l_{\alpha}(p), \quad q \in \mathbb{R}^2. \quad (22) \]
According to equation (2-9), the polynomial in (2-10) is differentiated and substituted into \( q = p \), and the approximate derivative of \( f \) at the point is obtained as follows:

\[
f^{(y)}(p) \approx \sum_{a \in \mathbb{A}} f(q_a)D^y l_a(p),
\]
(23)

4. Numerical experiment
In this paper, we consider the general problem of Helmholtz equation[14] in \( D \in \mathbb{R}^2 \) (Dirichlet and Neumann problem) in the case of sound soft and sound hard.

4.1 Dirichlet problem
Firstly, the solution of Helmholtz equation with Dirichlet boundary condition is considered

\[
\begin{cases}
\Delta u + k^2 u = 0, & \text{in } D \\
u = f, & \text{on } \partial D
\end{cases}
\]
(24)

The non-zero constant is the transmission eigenvalue. Substituting (2-11) into (3-1), we can get

\[
\begin{pmatrix}
L \\
C
\end{pmatrix}
\begin{pmatrix}
u_0 \\
u_1
\end{pmatrix} +
\begin{pmatrix}
k^2 & 0 \\
0 & k^2
\end{pmatrix}
\begin{pmatrix}
u_0 \\
u_1
\end{pmatrix} =
\begin{pmatrix}
0 \\
f
\end{pmatrix}
\]
(25)

\( C = (0,E), \ u_0 = (u_{00},u_{11}), f_0 = (f_0,f_1) \). Therefore, (3-2) can also be written and integrated into the following matrix expression

\[
\begin{pmatrix}
L_0 + k^2 \\
0
\end{pmatrix}
\begin{pmatrix}
u_0 \\
u_1
\end{pmatrix} +
\begin{pmatrix}
L_1 + k^2 \\
E
\end{pmatrix}
\begin{pmatrix}
u_0 \\
u_1
\end{pmatrix} =
\begin{pmatrix}
0 \\
f
\end{pmatrix}
\]
(26)

the solution of Helmholtz equation is

\[
\begin{pmatrix}
u_0 \\
u_1
\end{pmatrix} =
\begin{pmatrix}
L_0 + k^2 & L_1 + k^2 \\
0 & E
\end{pmatrix}^{-1}
\begin{pmatrix}
0 \\
f
\end{pmatrix}
\]
(27)

4.2. Neumann problem
Then consider the solution of Helmholtz equation under Neumann boundary condition

\[
\begin{cases}
\Delta u + k^2 u = 0, & \text{in } D \\
\frac{\partial u}{\partial v} = g, & \text{on } \partial D
\end{cases}
\]
(28)

\( v \) is the unit normal vector on the boundary and the non-zero constant \( k \) is the transmission eigenvalue, Similarly, the solution of Helmholtz equation is obtained

\[
\begin{pmatrix}
u_0 \\
u_1
\end{pmatrix} =
\begin{pmatrix}
L_0 + k^2 & L_1 + k^2 \\
0 & B
\end{pmatrix}^{-1}
\begin{pmatrix}
0 \\
f
\end{pmatrix}
\]
(29)

\( B \) is differential operator, \( u_0 = (u_{00},u_{11}), f_0 = (f_0,f_1) \)

4.3. Graphic conclusion
Given \( \Delta = \frac{1}{10}, n = 6, D = \{(x,y) | x^2 + y^2 \leq 4\}, \partial D = \{(x,y) | x^2 + y^2 = 1\} \, , \, f = e^x \). Substituting program, in order to improve the degree of visualization, take each different node as the initial point of interpolation, and get the corresponding numerical solution value of the function, which are drawn as Figure 4 and Figure 5 respectively.
5. Conclusion

Helmholtz equation describes the harmonic motion (when the wave is harmonic, the wave equation and Maxwell equation will be transformed into Helmholtz equation; at the same time, the Helmholtz equation will be obtained when the wave equation and heat conduction equation are separated into variables). This paper proposes the node distribution of parallel lines, which is convenient to select the interpolation node set and ensures the uniqueness of multivariate Lagrange polynomial interpolation. One is solvability[15]. An example is given to solve the Helmholtz equation under two different boundary conditions. The results show that the interpolation function can replace the Laplace operator into the partial differential equation, and the result is reliable.
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Zihan Zhang and Silong Fang contributed equally to this work, and they are the co-first authors of this article.

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