COMBINATORIAL IDENTITIES INVOLVING THE
CENTRAL COEFFICIENTS OF A SHEFFER MATRIX

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Given $m \in \mathbb{N}$, $m \geq 1$, and a Sheffer matrix $S = [s_{n,k}]_{n,k \geq 0}$, we obtain the
exponential generating series for the coefficients $(a + (m+1)n)^{-1}s_{a+(m+1)n,a+mn}$. Then, by using this series, we obtain two general combinatorial identities,
and their specialization to $r$-Stirling, $r$-Lah and $r$-idempotent numbers. In particular, using this approach, we recover two well known binomial identities, namely Gould’s identity and Hagen-Rothe’s identity. Moreover, we generalize these results obtaining an exchange identity for a cross sequence (or for two Sheffer sequences) and an Abel-like identity for a cross sequence (or for an $s$-Appell sequence). We also obtain some new Sheffer matrices.

1. INTRODUCTION

A Sheffer matrix [2, 26] is an infinite lower triangular matrix $S = [s_{n,k}]_{n,k \geq 0}$ whose columns have exponential generating series

$$s_k(t) = \sum_{n \geq k} s_{n,k} \frac{t^n}{n!} = g(t)\frac{f(t)^k}{k!} \quad (k \in \mathbb{N})$$

for two exponential series $g(t) = \sum_{n \geq 0} g_n \frac{t^n}{n!}$ and $f(t) = \sum_{n \geq 0} f_n \frac{t^n}{n!}$, with $g_0 = 1$ and $f_0 = 0$, $f_1 \neq 0$. In this case, we also write $S = (g(t), f(t))$ and we say that the pair $(g(t), f(t))$ is the spectral representation of $S$, or simply the spectrum of $S$.

Given $m \in \mathbb{N}$, $m \geq 1$, and $a \in \mathbb{N}$, the $m$-central coefficients of $S$ are the entries $c_{n}^{(m)} = s_{(m+1)n,an}$, while the shifted $m$-central coefficients are the entries $c_{n}^{(a,m)} = s_{a+(m+1)n,a+mn}$.

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A Sheffer sequence with spectrum \( (g(t), f(t)) \), \[2, 4, 8, 29, 32, 33, 34, 36, 37\], is the polynomials sequence \( \{s_n(x)\}_{n \in \mathbb{N}} \) of the row polynomials of the Sheffer matrix \( S = (g(t), f(t)) \), with generating exponential series

\[
\sum_{n \geq 0} s_n(x) \frac{t^n}{n!} = g(t) e^{xf(t)}.
\]

An \( s \)-Appell sequence \([26]\), with \( s \neq 0 \), is a Sheffer sequence with spectrum \((g(t), st)\). For \( s = 1 \), we have the ordinary Appell polynomials \([32\text{, p. 86}]\) \([34]\).

Several classical combinatorial sequences, such as the binomial coefficients, the Stirling numbers of the first and the second kind and the Lah numbers, form a Sheffer matrix. Similarly, several classical polynomial sequences, such as the falling and rising factorial powers, the generalized Hermite polynomials, the generalized Laguerre polynomials, the generalized Bernoulli and Euler polynomials, the exponential polynomials, the actuarial polynomials, the Cayley continuants, the Abel polynomials, form a Sheffer sequence.

The theory of Sheffer matrices (or sequences) provides a powerful tool for studying and deriving combinatorial identities \([25, 26, 39]\). In this paper, we start by deriving the exponential generating series for the coefficients

\[
\sum_{n \geq 0} c_n^{(a,m)} \frac{t^n}{n!} = \frac{t^{a+m}}{\varphi(t)^a \varphi(t)^m} g(\varphi(t)).
\]

(2) Then, by using this result, we obtain the spectral representation of the Sheffer matrix generated by these coefficients (Theorem 2) and two general combinatorial identities (Theorems 3 and 5). In particular, we specialize these identities to some combinatorial families of numbers, such as the \( r \)-Stirling numbers of the first and second kind, the \( r \)-Lah numbers, the \( r \)-idempotent numbers. In particular, using this approach, we recover two well known binomial identities, namely Gould’s identity \([16, 17]\) and Hagen-Rothe’s identity \([35, 21]\). Moreover, we generalize the results obtained for the \( r \)-idempotent numbers by determining an exchange identity for an arbitrary cross sequence (Theorem 18) or for two Sheffer sequences (Theorem 19), and an Abel-like identity for an arbitrary cross sequence (Theorem 20) or for an arbitrary \( s \)-Appell sequence (Theorem 21). These Abel-like identities generalize the classical Abel’s binomial identity \([1]\) (see formula (28)). Finally, we also obtain some new Sheffer matrices.

2. MAIN RESULTS

We start by determining the generating series for the coefficients (2).

**Theorem 1.** Let \( S = [s_{n,k}]_{n,k \in \mathbb{N}} = (g(t), f(t)) \) be a Sheffer matrix, and let \( c_n^{(a,m)} = s_{a+(m+1)n,a+mn} \) be the shifted \( m \)-central coefficients. Let \( F = [f_{n,k}]_{n,k \in \mathbb{N}} = (1, f(t)) \). Then, we have the exponential generating series

\[
\sum_{n \geq 0} c_n^{(a,m)} \frac{t^n}{n!} = \frac{t^{a+m}}{\varphi(t)^a \varphi(t)^m} g(\varphi(t)).
\]
where
\[
\varphi(t) = \sum_{n \geq 1} \frac{f_{(m+1)n-1, mn}}{(m+1)n-1} \frac{t^n}{n!}
\]
is the unique invertible exponential series satisfying the equation
\[
\varphi(t)^{m+1} = tf(\varphi(t))^m.
\]
Equivalently, \(\varphi(t)\) is the exponential series whose compositional inverse is
\[
\hat{\varphi}(t) = \frac{t^{m+1}}{f(t)^m} = t \left( \frac{t}{f(t)} \right)^m.
\]

Proof. Consider the bivariate series
\[
H(t, u) = \sum_{n, k \geq 0} H_{n,k} t^n u^k
\]
whose diagonal series is
\[
h(t) = \sum_{n \geq 0} H_{n,n} t^n = \sum_{n \geq 0} \frac{(a + mn)!}{n!(a + (m+1)n)!} s_{a+(m+1)n,a+mn} t^n.
\]
By Cauchy’s integral formula \cite{11} \cite{22}, we have
\[
h(t) = \frac{1}{2\pi i} \oint H(z, t/z) \frac{dz}{z} = \frac{1}{2\pi i} \oint g(z) \left( \frac{f(z)}{z} \right)^{a+m} e^{tf(z)} \frac{dz}{z}.
\]
Let \(z = \varphi(w)\), where \(\varphi\) is the unique invertible exponential series\footnote{Notice that equation (5) implies \(\varphi(0) = 0\) and \(\varphi'(0) = f_1^m \neq 0\).} defined by equation (5). Then \(w = f(z)^{m+1} = \hat{\varphi}(z)\), \(\left( \frac{f(z)}{z} \right)^m = \frac{\varphi(w)}{w} \), \(f(z) = \left( \frac{\varphi(w)}{w} \right)^{1/m}\),
\[ dz = \varphi'(w) \, dw \] and

\[ h(t) = \frac{1}{2\pi i} \oint e^{\frac{t}{z}} \frac{w \varphi'(w)}{\varphi(w)} \left( \frac{\varphi(w)}{w} \right)^{a/m} g(\varphi(w)) \frac{dw}{w} \, . \]

Since the Hadamard product \[11\] of two ordinary series \( a(t) = \sum_{n \geq 0} a_n t^n \) and \( b(t) = \sum_{n \geq 0} b_n t^n \) is given by

\[ a(t) \odot b(t) = \sum_{n \geq 0} a_n b_n t^n = \frac{1}{2\pi i} \oint a(t/z) b(z) \frac{dz}{z} , \]

then we have

\[ h(t) = \exp t \odot \left[ t \varphi'(t) \left( \frac{\varphi(t)}{t} \right)^{a/m} g(\varphi(t)) \right] . \]

Hence, if we set

\[ c^{(a,m)}(t) = \frac{t \varphi'(t)}{\varphi(t)} \left( \frac{f(\varphi(t))}{\varphi(t)} \right)^{a/m} g(\varphi(t)) = \sum_{n \geq 0} C_n \frac{t^n}{n!} , \]

then we have

\[ h(t) = \exp t \odot c^{(a,m)}(t) = \sum_{n \geq 0} \frac{t^n}{n!} \odot \sum_{n \geq 0} C_n \frac{t^n}{n!} = \sum_{n \geq 0} C_n \frac{t^n}{(n!)^2} \]

and consequently

\[ \frac{(a + mn)!}{n!(a + (m + 1)n)!} c^{(a,m)}_n = C_n \frac{C_n}{(n!)^2} \]

from which we obtain

\[ C_n = n!(a + mn)! \frac{C_n}{(a + (m + 1)n)!} c^{(a,m)}_n = \frac{c_n^{(m)}}{(a + (m + 1)n)!} . \]

This proves identity (3). Finally, by the Lagrange Inversion Formula \[40, p. 38\], we have

\[ [t^n] \varphi(t) = \frac{1}{n} \left[ t^{n-1} \left( \frac{t}{\varphi(t)} \right)^n \right] = \frac{1}{n} \left[ t^{n-1} \left( \frac{f(t)}{t} \right)^{mn} \right] = \left( \frac{mn)!}{n} \right) f^{(m+1)n-1,mn} (mn)! = \left( \frac{(m+1)n-1, mn}{n!} \right) f_{(m+1)n-1, mn} . \]

This proves identity (4).

As an immediate consequence of Theorem 1, we have the following result.
Theorem 2. Let \( m \in \mathbb{N}, \ m \geq 1 \). Let \( S = [s_{n,k}]_{n,k \in \mathbb{N}} = (g(t), f(t)) \) be a Sheffer matrix, and let \( \varphi(t) \) be series (4). Then, we have the Sheffer matrix

\[
\left[ \binom{n}{k} \frac{s_{(m+1)n-k, mn}}{(m+1)n-k} \right]_{n,k \geq 0} = \left( \frac{t \varphi'(t)}{\varphi(t)} g(\varphi(t)), \varphi(t) \right)
\]

Proof. Let \( k \in \mathbb{N} \) and \( a = mk \). Then, by series (3), we have the identity

\[
\frac{t \varphi'(t)}{\varphi(t)} g(\varphi(t)) = \sum_{n \geq 0} \left( \frac{n}{k} \right) c_n \left( \frac{mk}{m(k+1)n-k} \right) \frac{t^n}{n!} = \sum_{n \geq 0} \left( \frac{n}{k} \right) \frac{c_n}{(m+1)n-k} \frac{t^n}{n!}.
\]

This means that we have the Sheffer matrix (7). \( \square \)

Another consequence of Theorem 1 is the next property, giving our first main identity.

Theorem 3. Let \( a, b, m \in \mathbb{N}, \ m \geq 1 \). Given two Sheffer matrices

\[
S_1 = [s'_{n,k}]_{n,k \in \mathbb{N}} = (g_1(t), f(t)) \\
S_2 = [s''_{n,k}]_{n,k \in \mathbb{N}} = (g_2(t), f(t)),
\]

let \( c_1^{(a,m)}(t) \) and \( c_2^{(b,m)}(t) \) be the respective exponential generating series defined by (3). Then, we have the relation

\[
(8) \quad c_1^{(a,m)}(t) c_2^{(b,m)}(t) = c_1^{(0,m)}(t) c_2^{(a+b,m)}(t)
\]
or, equivalently, the identity

\[
(9) \quad \sum_{k=0}^{n} \binom{n}{k} \frac{s'_{a+(m+1)k, a+m k}}{(a+(m+1)k)} \frac{s''_{b+(m+1)(n-k), b+m(n-k)}}{(b+(m+1)(n-k)} = \sum_{k=0}^{n} \binom{n}{k} \frac{s'_{(m+1)k, mk}}{(m+1)k} \frac{s''_{a+b+(m+1)(n-k), a+b+m(n-k)}}{(a+b+(m+1)(n-k)}. \]

Proof. By Theorem 1, we have

\[
c_1^{(a,m)}(t) c_2^{(b,m)}(t) = \left( \frac{t \varphi'(t)}{\varphi(t)} \right)^2 \left( \frac{\varphi(t)}{t} \right)^{(a+b)/m} g_1(\varphi(t)) g_2(\varphi(t)) = \varphi(t) g_1(\varphi(t)) \cdot \varphi(t) g_2(\varphi(t)) = c_1^{(0,m)}(t) c_2^{(a+b,m)}(t).
\]

This yields identity (8).
To prove Theorem 5, we need the following result.

Lemma 4. Let $S = [s_{n,k}]_{n,k \in \mathbb{N}} = (g(t), f(t))$ be a Sheffer matrix and let $F = [f_{n,k}]_{n,k \in \mathbb{N}} = (1, f(t))$. Let $\varphi(t)$ be the exponential series defined by equation (5). For every $a \in \mathbb{N}$, $a \neq 0$, we have the exponential series

$$
(10) \quad \left( \frac{\varphi(t)}{t} \right)^a = \sum_{n \geq 0} \frac{f_{ma+(m+1)n, ma+mn} a^n}{(ma+mn)} \frac{a}{a+n} n!.
$$

Proof. By applying the Lagrange Inversion Formula [40, p. 38], we have

$$
[t^n] \left( \frac{\varphi(t)}{t} \right)^a = [t^{a+n}] \varphi(t)^a = \frac{a}{a+n} [t^n] \left( \frac{t}{\varphi(t)} \right)^{a+n} = \frac{a}{a+n} [t^n] \left( \frac{f(t)}{t} \right)^{m(a+n)}
$$

$$
= \frac{a}{a+n} (m(a+n))! \frac{f_{ma+(m+1)n, ma+(m+1)n}}{(ma+(m+1)n)!} \frac{a}{a+n} \frac{n!(m(a+n))!}{(ma+(m+1)n)!} f_{ma+(m+1)n, ma+(m+1)n} \frac{1}{n!}
$$

$$
= \frac{f_{ma+(m+1)n, ma+(m+1)n}}{(ma+(m+1)n)} \frac{a}{a+n} \frac{1}{n!}.
$$

This proves the claim.

Now, we can prove our second main identity.

Theorem 5. Let $S = [s_{n,k}]_{n,k \in \mathbb{N}} = (g(t), f(t))$ be a Sheffer matrix and let $F = [f_{n,k}]_{n,k \in \mathbb{N}} = (1, f(t))$. Then, for every $a, b \in \mathbb{N}$, $b \neq 0$, we have the identity

$$
(11) \quad \sum_{k=0}^{n} \binom{n}{k} \frac{s_{a+(m+1)n, a+mk}}{(a+(m+1)n)} \frac{f_{mb+(m+1)(n-k), mb+m(n-k)}}{(mb+(m+1)(n-k)} \frac{b}{b+n-k} = \frac{s_{a+mb+(m+1)n, a+mb+mn}}{(a+mb+mn)}.
$$

In particular, if $g(t) = 1$, then

$$
(12) \quad \sum_{k=0}^{n} \binom{n}{k} \frac{f_{a+(m+1)n, a+mk}}{(a+(m+1)n)} \frac{f_{mb+(m+1)(n-k), mb+m(n-k)}}{(mb+(m+1)(n-k)} \frac{b}{b+n-k} = \frac{f_{a+mb+(m+1)n, a+mb+mn}}{(a+mb+(m+1)n)}.
$$
Moreover, if \( g(t) = f'(t) \), then
\[
\sum_{k=0}^{n} \binom{n}{k} \frac{f_{a+(m+1)k+1,a+mk+1}}{(a+(m+1)k)} \frac{f_{mb+(m+1)(n-k),mb+m(n-k)}}{(mb+(m+1)(n-k))} \frac{b}{b+n-k} =
\]
\[
= \frac{f_{a+mb+(m+1)n+1,a+mb+mn+1}}{(a+mb+(m+1)n)}.
\]

**Proof.** By series (3), we have
\[
e^{(a,m)}(t) \left( \frac{\varphi(t)}{t} \right)^{b} = \left( \frac{\varphi(t)}{t} \right)^{(a+mb)/m} g(\varphi(t)) = e^{(a+mb,m)}(t).
\]

By Theorem 1 and Lemma 4, this equation is equivalent to identity (11). In particular, if \( g(t) = 1 \), then \( s_{n,k} = f_{n,k} \) and we have identity (12). If \( g(t) = f'(t) \), then \( s_{n,k} = f_{n+1,k+1} \) and we have identity (13).

\[
3. \text{ EXAMPLES}
\]

In this section, we exemplify the identities obtained in Theorems 3 and 5 by considering some specific Sheffer matrices of combinatorial interest. In some cases, we also give explicitly the spectral representation of the Sheffer matrix obtained in Theorem 2.

**3.1 \( r \)-Stirling numbers of the second kind**

The \( r \)-Stirling numbers of the second kind \([5, 13]\) are the entries of the Sheffer matrix
\[
S^{(r)} = \begin{bmatrix} \{n\} \\ \{k\} \end{bmatrix}_{r,n,k \geq 0} = (e^{rt}, e^{t} - 1).
\]

Equivalently, they have exponential generating series
\[
\sum_{n \geq k} \binom{n}{k} \frac{t^n}{n!} = e^{rt} \left( e^{t} - 1 \right)^k.
\]

In particular, for \( r = 0 \) and \( r = 1 \), we have the ordinary Stirling numbers of the second kind \([11, \text{ p. 310}] [30, \text{ p. 48}] [38, \text{ A008277}] \}; \{n\}_0 = \{k\} \text{ and } \{n\}_1 = \{k+1\} \text{.}

The row polynomials are related to the actuarial polynomials \([32, \text{ p. 123}] [41] \).

**Theorem 6.** For every \( a, b, r, s, m, n \in \mathbb{N}, m \geq 1 \), we have the identities
\[
\sum_{k=0}^{n} \binom{n}{k} \frac{\{a+(m+1)k\}}{a+mk} = \frac{\{a+b+(m+1)(n-k)\}}{a+b+m(n-k)} = \sum_{k=0}^{n} \binom{n}{k} \frac{\{a+(m+1)k\}}{a+mk} \frac{\{a+b+(m+1)(n-k)\}}{a+b+m(n-k)}
\]
\[
\sum_{k=0}^{n} \binom{n}{k} \frac{\{a+(m+1)k\}}{a+mk} = \frac{\{a+b+(m+1)(n-k)\}}{a+b+m(n-k)} = \sum_{k=0}^{n} \binom{n}{k} \frac{\{a+(m+1)k\}}{a+mk} \frac{\{a+b+(m+1)(n-k)\}}{a+b+m(n-k)} \frac{b}{b+n-k} = \frac{\{a+b+(m+1)n\}}{a+b+mn} \frac{b}{a+b+mn} \quad (b \neq 0).
\]
Proof. If \( S_1 = S^{(r)} \) and \( S_2 = S^{(s)} \), then identity (9) yields the first identity. Similarly, if \( S = S^{(r)} \), then \( F = S^{(0)} \) and (11) yields the second identity.

\[ \text{Remark 7.} \text{ In particular, for } r = s = 0, \text{ we have the identities} \]
\[
\sum_{k=0}^{n} \binom{n}{k} \frac{a+(m+1)k}{a+mk} \left\{ \begin{array}{c} \sum_{k=0}^{n} \binom{n}{k} b+(m+1)(n-k) \\ b+m(n-k) \end{array} \right\} = \sum_{k=0}^{n} \binom{n}{k} \frac{a+b+(m+1)(n-k)}{a+b+m(n-k)} \\
\sum_{k=0}^{n} \frac{a+(m+1)k}{a+mk} \left\{ \begin{array}{c} \sum_{k=0}^{n} \binom{n}{k} mb+(m+1)(n-k) \\ mb+m(n-k) \end{array} \right\} \frac{b}{b+n-k} = \frac{a+mb+(m+1)n}{a+mb+mn} (b \neq 0).
\]

and, for \( r = s = 1 \), we have the identities
\[
\sum_{k=0}^{n} \binom{n}{k} \frac{a+(m+1)k+1}{a+mk+1} \left\{ \begin{array}{c} \sum_{k=0}^{n} \binom{n}{k} b+(m+1)(n-k)+1 \\ b+1+m(n-k) \end{array} \right\} = \sum_{k=0}^{n} \binom{n}{k} \frac{a+b+(m+1)(n-k)+1}{a+b+m(n-k)} \\
\sum_{k=0}^{n} \frac{a+(m+1)k+1}{a+mk+1} \left\{ \begin{array}{c} \sum_{k=0}^{n} \binom{n}{k} mb+(m+1)(n-k) \\ mb+m(n-k) \end{array} \right\} \frac{b}{b+n-k} = \frac{a+mb+(m+1)n+1}{a+mb+mn+1} (b \neq 0).
\]

### 3.2 \( r \)-Stirling numbers of the first kind

The \( r \)-Stirling numbers of the first kind \([5, 23]\) are the entries of the Sheffer matrix
\[
\mathcal{S}^{(r)} = \left[ \begin{array}{c} n \\ k \end{array} \right]_{r, n,k \geq 0} = \left( \frac{1}{(1-t)^r}, \log \frac{1}{1-t} \right).
\]

Equivalently, they have exponential generating series
\[
\sum_{n \geq k} \left[ \begin{array}{c} n \\ k \end{array} \right]_{r} \frac{t^n}{n!} = \frac{1}{(1-t)^r} \frac{1}{k!} \left( \log \frac{1}{1-t} \right)^k.
\]

In particular, for \( r = 0 \) and \( r = 1 \), we have the ordinary Stirling numbers of the first kind \([11, \text{p. } 310]\) \([30, \text{p. } 48]\) \([38, A132393, A008275]\): \( \left[ \begin{array}{c} n \\ k \end{array} \right]_{0} = \left[ \begin{array}{c} n \\ k \end{array} \right] \) and \( \left[ \begin{array}{c} n \\ k \end{array} \right]_{1} = \left[ \begin{array}{c} n+1 \\ k+1 \end{array} \right] \).

**Theorem 8.** For every \( a, b, r, s, m, n \in \mathbb{N}, m \geq 1 \), we have the identities
\[
\sum_{k=0}^{n} \binom{n}{k} \frac{a+(m+1)k}{a+mk} \left\{ \begin{array}{c} \sum_{k=0}^{n} \binom{n}{k} b+(m+1)(n-k) \\ b+m(n-k) \end{array} \right\} = \sum_{k=0}^{n} \binom{n}{k} \frac{a+b+(m+1)(n-k)}{a+b+m(n-k)} \\
\sum_{k=0}^{n} \frac{a+(m+1)k}{a+mk} \left\{ \begin{array}{c} \sum_{k=0}^{n} \binom{n}{k} mb+(m+1)(n-k) \\ mb+m(n-k) \end{array} \right\} \frac{b}{b+n-k} = \frac{a+mb+(m+1)n}{a+mb+mn} (b \neq 0).
\]

Proof. If \( S_1 = \mathcal{S}^{(r)} \) and \( S_2 = \mathcal{S}^{(s)} \), then identity (9) yields the first identity. Similarly, if \( S = \mathcal{S}^{(r)} \), then \( F = \mathcal{S}^{(0)} \) and (11) yields the second identity. \( \square \)
Remark 9. In particular, for \( r = s = 0 \), we have the identities

\[
\sum_{k=0}^{n} \binom{n}{k} \frac{(a+(m+1)k)}{a+mk} \frac{(b+(m+1)(n-k))}{b+m(n-k)} = \sum_{k=0}^{n} \binom{n}{k} \frac{(m+1)k}{mk} \frac{(a+b+(m+1)(n-k))}{a+b+m(n-k)}
\]

\[
\sum_{k=0}^{n} \binom{n}{k} \frac{(a+(m+1)k)}{a+mk} \frac{(mb+(m+1)(n-k))}{mb+m(n-k)} \frac{b}{b+n-k} = \frac{a+mb+(m+1)n}{a+mb+mn} \quad (b \neq 0)
\]

and, for \( r = s = 1 \), we have the identities

\[
\sum_{k=0}^{n} \binom{n}{k} \frac{(a+(m+1)k+1)}{a+mk+1} \frac{(b+(m+1)(n-k)+1)}{b+m(n-k)+1} = \sum_{k=0}^{n} \binom{n}{k} \frac{(m+1)k+1}{mk+1} \frac{(a+b+(m+1)(n-k)+1)}{a+b+m(n-k)+1}
\]

\[
\sum_{k=0}^{n} \binom{n}{k} \frac{(a+(m+1)k+1)}{a+mk+1} \frac{(mb+(m+1)(n-k)+1)}{mb+m(n-k)+1} \frac{b}{b+n-k} = \frac{a+mb+(m+1)n+1}{a+mb+mn+1} \quad (b \neq 0).
\]

3.3 \( r \)-Lah numbers

The \( r \)-Lah numbers \([28]\), defined by \( \binom{n}{k}_r = \binom{n+2r-1}{k+2r-1}_r \), are the entries of the Sheffer matrix

\[
L(r) = \begin{bmatrix} \frac{n!}{k! \prod_{i=k+1}^{n} (1-t^i)} \end{bmatrix}_{n,k \geq 0} = \left( \frac{1}{(1-t)^{2r}}, \frac{t}{1-t} \right).
\]

Equivalently, they have exponential generating series

\[
\sum_{n \geq k} \binom{n}{k}_r \frac{t^n}{n!} = \frac{1}{(1-t)^{2r}} \frac{1}{k!} \left( \frac{t}{1-t} \right)^k = \frac{t^{k+t}}{k! (1-t)^{2r+k}}.
\]

In particular, for \( r = 0 \) and \( r = 1 \), we have the ordinary Lah numbers \([11\), p. 156]\ [30, p. 44] [38, A008297, A271703]: \( \binom{n}{k}_0 = \binom{n}{k} \) and \( \binom{n}{k}_1 = \binom{n+1}{k+1} \).

Theorem 10. For every \( a, b, r, s, m, n \in \mathbb{N}, m \geq 1 \), we have the identities

\[
\sum_{k=0}^{n} \binom{n}{k} \frac{(a+(m+1)k)}{a+mk} \frac{(b+(m+1)(n-k))}{b+m(n-k)} = \sum_{k=0}^{n} \binom{n}{k} \frac{(m+1)k}{mk} \frac{(a+b+(m+1)(n-k))}{a+b+m(n-k)}
\]

\[
\sum_{k=0}^{n} \binom{n}{k} \frac{(a+(m+1)k)}{a+mk} \frac{(mb+(m+1)(n-k))}{mb+m(n-k)} \frac{b}{b+n-k} = \frac{a+mb+(m+1)n}{a+mb+mn} \quad (b \neq 0).
\]

Proof. If \( S_1 = L^{(r)} \) and \( S_2 = L^{(s)} \), then identity (9) yields the first identity. Similarly, if \( S = L^{(r)} \), then \( F = L^{(0)} \) and (11) yields the second identity. \( \square \)
Theorem 11. We have the Sheffer matrix
\[
\left( \begin{array}{c}
\binom{n}{k} \\
\binom{2n-k}{n}
\end{array} \right)_{n,k \geq 0} = \left( \begin{array}{c} 1 + \sqrt{1 - 4t} \\
\frac{1 - \sqrt{1 - 4t}}{2\sqrt{1 - 4t}} \end{array} \right)^{2r} \frac{1 - \sqrt{1 - 4t}}{2}.
\]

Proof. Let \( m = 1 \). By identity (5), the series \( \varphi(t) \) associated with the Sheffer matrix (14) is defined by the equation
\[
\varphi(t)^2 = \frac{t\varphi(t)}{1 - \varphi(t)} \quad \text{or} \quad \varphi(t)^2 - \varphi(t) + t = 0.
\]
Hence, we have \( \varphi(t) = \frac{1 - \sqrt{1 - 4t}}{2} \) and \( \varphi'(t) = \frac{1}{\sqrt{1 - 4t}} \). By Theorem 2, we obtain the Sheffer matrix (15).

3.4 Binomial coefficients
Consider the Sheffer matrix
\[
B^{(\alpha)} = \left[ B^{(\alpha)}_{n,k} \right]_{n,k \geq 0} = \left( \begin{array}{c}
\binom{n}{k} \\
\binom{2n-k}{n}
\end{array} \right)_{n,k \geq 0} = \left( \begin{array}{c} 1 + \sqrt{1 - 4t} \\
\frac{1 - \sqrt{1 - 4t}}{2\sqrt{1 - 4t}} \end{array} \right)^{2r} \frac{1 - \sqrt{1 - 4t}}{2}.
\]
where
\[
B^{(\alpha)}_{n,k} = 4^{n-k} \binom{n + \alpha - 1/2}{n-k} n! = \frac{(2n+2\alpha)}{(n-k)} \frac{(2n-2k)}{n-k} n!.
\]

For \( \alpha = 0 \), we have sequence A048854 in [38] (see also [32, p. 25]). For \( \alpha = 1 \), we have sequence A286724 in [38]. For \( \alpha = 2r - 1/2 \), we have \( B^{(\alpha)}_{n,k} = \binom{n}{k} 4^{n-k} \), where the coefficients \( \binom{n}{k} \) are the \( r \)-Lah numbers.

In this case, we have the following result.

Theorem 12. Let \( \alpha, \beta, \gamma \) and \( \delta \) arbitrary symbols. We have the identities
\[
\sum_{k=0}^{n} \binom{\alpha + k\beta}{k} \binom{\gamma + (n-k)\beta}{n-k} = \sum_{k=0}^{n} \binom{\alpha + \delta + k\beta}{k} \binom{\gamma - \delta + (n-k)\beta}{n-k}
\]
\[
\sum_{k=0}^{n} \binom{\alpha + k\beta}{k} \binom{\gamma + (n-k)\beta}{n-k} \frac{\gamma}{\gamma + (n-k)\beta} = \binom{\alpha + \gamma + n\beta}{n}
\]

Proof. Let \( a, b, m, n \in \mathbb{N}, m \geq 1 \). If \( S_1 = B^{(\alpha)} \) and \( S_2 = B^{(\beta)} \), then identity (9) becomes
\[
\sum_{k=0}^{n} \binom{n}{k} B^{(\alpha)}_{a+(m+1)k,a+mk} B^{(\beta)}_{mb+(m+1)(n-k),mb+m(n-k)}
\]
\[
= \sum_{k=0}^{n} \binom{n}{k} B^{(\alpha)}_{(m+1)k,mk} B^{(\beta)}_{a+mb+(m+1)(n-k),a+mb+m(n-k)}
\]

This identity can be simplified noticing that

\[
(19) \quad \frac{B_{n,k}^{(\alpha)}}{\binom{n}{k}} = 4^{n-k}(n-k)! \binom{n+\alpha-1/2}{n-k}.
\]

Indeed, after some simplifications, we have

\[
\sum_{k=0}^{n} \binom{a + (m+1)k + \alpha - 1/2}{k} \binom{mb + (m+1)(n-k) + \beta - 1/2}{n-k} = \sum_{k=0}^{n} \binom{(m+1)k + \alpha - 1/2}{k} \binom{a + mb + (m+1)(n-k) + \beta - 1/2}{n-k}.
\]

Setting \(A = a + \alpha - 1/2, B = m+1, C = mb + \beta - 1/2\) and \(D = -a\), we obtain the identity

\[
\sum_{k=0}^{n} \binom{A + kB}{k} \binom{C + (n-k)B}{n-k} = \sum_{k=0}^{n} \binom{A + D + kB}{k} \binom{C - D + (n-k)B}{n-k}.
\]

For the arbitrariness of the parameters \(A, B, C\) and \(D\), this identity is equivalent to identity (17).

Furthermore, if \(S = L^{(\alpha)}\), then \(F = (1, \frac{1}{1-4t}) = \binom{n!}{k^{4n-k}}\) \(n,k \geq 0\) and identity (11) becomes

\[
\sum_{k=0}^{n} \binom{n}{k} B_{a+(m+1)k,a+mk}^{(\alpha)} \binom{mb+(m+1)(n-k)}{mb+m(n-k)} \frac{4^{n-k}b}{b+n-k} = \frac{B_{a+mb+(m+1)n,a+mb+mn}^{(\alpha)}}{a+mb+(m+1)n}
\]

provided that \(b \neq 0\). Also this identity can be simplified using formula (19) and noticing that

\[
\frac{\binom{n}{k}}{\binom{n}{k}} = (n-k)! \binom{n-1}{k-1} = (n-k)! \binom{n}{k} \frac{k}{n} \quad (n \neq 0).
\]

In this way, after some simplifications, the above identity becomes

\[
\sum_{k=0}^{n} \binom{a + (m+1)k + \alpha - 1/2}{k} \binom{mb + (m+1)(n-k)}{n-k} \frac{mb}{mb + (m+1)(n-k)} = \frac{(a + mb + (m+1)n + \alpha) - 1/2}{n}.
\]

Setting \(A = a + \alpha - 1/2, B = m+1\) and \(C = mb\), we have the identity

\[
\sum_{k=0}^{n} \binom{A + kB}{k} \binom{C + (n-k)B}{n-k} \frac{C}{C + (n-k)B} = \frac{(A + nB)}{n}.
\]

For the arbitrariness of the parameters \(A, B\) and \(C\), this identity is equivalent to identity (18).
Notice that identities (17) and (18) are well known. More precisely, identity (17) is Gould’s identity \[16, 17, 20, 39\] \[18\], p. 41, Formula (3.143)] [24, Formula (1.3)], while identity (18) is Hagen-Rothe’s identity \[35, 21, 16, 17, 9, 10, 20\] [31, Section 4.5] [19, p. 202, Formula (5.62)].

A similar result can also be obtained from Theorem 10. More precisely, from the first identity we can recover identity (17), while from the second identity we can derive the identity

\[
\sum_{k=0}^{n} \binom{\alpha + k\beta}{k} \binom{\gamma + (n-k)\beta}{n-k} \frac{\gamma + 1}{\gamma + 1 + (n-k)(\beta - 1)} = \binom{\alpha + \gamma + n\beta + 1}{n}.
\]

Theorem 13. We have the Sheffer matrix

\[
\begin{bmatrix}
\binom{4n-2k+2\alpha}{2n-2k} \\
\binom{(2n-k+\alpha)}{n-k}
\end{bmatrix} \frac{2n-2k}{n-k} n! \bigg|_{n,k \in \mathbb{N}} =
\begin{bmatrix}
1 + \sqrt{1 - 16t} \left(\frac{1 - \sqrt{1 - 16t}}{8t}\right)^{\alpha + \frac{1}{2}} , 1 - \sqrt{1 - 16t} \\
2\sqrt{1 - 16t} \left(\frac{1 - \sqrt{1 - 16t}}{8t}\right)^{\alpha + \frac{1}{2}} , 1 - \sqrt{1 - 16t}
\end{bmatrix}.
\]

Proof. Let \( m = 1 \). By identity (5), the series \( \varphi(t) \) associated with the Sheffer matrix (16) is defined by the equation

\[
\varphi(t)^2 = \frac{t\varphi(t)}{1 - 4\varphi(t)} \quad \text{or} \quad 4\varphi(t)^2 - \varphi(t) + t = 0.
\]

Hence, we have \( \varphi(t) = \frac{1 - \sqrt{1 - 16t}}{8} \) and \( \varphi'(t) = \frac{1}{\sqrt{1 - 16t}} \). By Theorem 2, we obtain the Sheffer matrix (20).

3.5 \( r \)-idempotent numbers

We define the \( r \)-idempotent numbers as the entries of the Sheffer matrix

\[
J^{(r)} = \begin{bmatrix}
\binom{n}{k} (r+k)^{n-k}
\end{bmatrix} \bigg|_{n,k \geq 0} = (e^{rt}, t^{r}).
\]

For \( r = 0 \), we have the idempotent numbers [38, A059297], and for \( r = 1 \) we have sequence A154372 in [38].

Remark 14. For \( r \in \mathbb{N} \), the \( r \)-idempotent numbers admit the following simple combinatorial interpretation. An idempotent map \( f : X \to X \) on a set \( X \) is a map such that \( f^2 = f \), i.e. a map where each element of \( X \) is a fixed point or is mapped into a fixed point. If \( X = \{1, 2, \ldots, r + 1, \ldots, n + r\} \) and \( R = \{1, 2, \ldots, r\} \), then the numbers \( \binom{n}{k} (r+k)^{n-k} \) count the idempotent maps \( f : X \to X \) with \( r+k \) fixed points, where the first elements 1, 2, \ldots, \( r \) are all fixed points. Indeed, an idempotent map \( f : X \to X \) with this property is equivalent to a pair \((K, \varphi)\), where \( K \) is a \( k \)-subset of \( X \setminus R \), so that \( R \cup K \) is the set of all fixed points, and \( \varphi \) is a function from \( X \setminus (R \cup K) \) to \( R \cup K \). The subset \( K \) can be chosen in \( \binom{n}{k} \) different ways, and the function \( \varphi \) can be chosen in \( (r+k)^{n-k} \) different ways.
Theorem 15. For every \(a, b, r, m, n \in \mathbb{N}\), we have the identities
\[
\begin{align*}
\sum_{k=0}^{n} \binom{n}{k} (a + r + mk)^k (b - mk)^{n-k} &= \sum_{k=0}^{n} \binom{n}{k} (r + mk)^k (a + b - mk)^{n-k} \\
x \sum_{k=0}^{n} \binom{n}{k} (x + mk)^{k-1} (y - mk)^{n-k} &= (x + y)^n.
\end{align*}
\]

Proof. Let \(S_1 = J^{(r)}\) and \(S_2 = J^{(s)}\), and apply Theorem 3. Replacing \(s + b + mn\) by \(b\), we have the first identity. If \(S = J^{(r)}\), then \(F = J^{(0)}\) and identity (11) becomes
\[
mb \sum_{k=0}^{n} \binom{n}{k} (r + a + mk)^k (m(b + n) - mk)^{n-k-1} = (r + a + m(b + n))^n.
\]
Setting \(x = r + a\) and \(y = m(b + n)\), we have
\[
(y - mn) \sum_{k=0}^{n} \binom{n}{k} (x + mk)^k (y - mk)^{n-k-1} = (x + y)^n
\]
or, equivalently,
\[
(y - mn) \sum_{k=0}^{n} \binom{n}{k} (y - mn + mk)^{k-1} (x + mn - mk)^{n-k} = (x + y)^n.
\]
Now, replacing \(y - mn\) by \(x\) and \(x + mn\) by \(y\), we derive the second identity. \(\square\)

Notice that the second identity is a particular case of Abel’s identity (see Remark 22).

Theorem 16. For any \(m \in \mathbb{N}, m \neq 0\), we have the Sheffer matrix
\[
\left[ \binom{n}{k} (r + mn)^{n-k} \right] = \left( \frac{1}{1 - c(mt)} \right)^{r} \binom{c(mt)}{mt}^{m}
\]

where
\[
c(t) = \sum_{n \geq 1} n^{n-1} \frac{t^n}{n!}.
\]

Proof. By identity (5), the series \(\varphi(t)\) associated with the Sheffer matrix \(J^{(r)}\) is defined by the equation \(\varphi(t)^{m+1} = t \varphi(t)^{m} e^{m \varphi(t)}\), that is \(\varphi(t) = te^{m \varphi(t)}\). So \(\varphi(t) = c(mt)/m = -W(-mt)/m\), where \(c(t) = -W(-t)\) is the Cayley series [30, p. 128] [40, p. 25], giving the exponential generating series for the labeled rooted trees, and \(W(t)\) is the Lambert series [12]. Since
\[
\frac{t \varphi'(t)}{\varphi(t)} = \frac{1}{1 - m \varphi(t)} = \frac{1}{1 - c(mt)},
\]
by applying Theorem 2, we obtain the Sheffer matrix (21).
4. SHEFFER POLYNOMIALS

The results obtained in Subsection 3.5, for the \( r \)-idempotent numbers, can be generalized as follows. Given a Sheffer sequence \( \{s_n(x)\}_{n \in \mathbb{N}} \), with spectrum \( (g(t), f(t)) \), we can always consider the associated cross sequence \( \{s_n^{(\lambda)}(x)\}_{n \in \mathbb{N}} \) of index \( \lambda \) [32, p.140] defined as the Sheffer sequence with spectrum \( (g(t)^\lambda, f(t)) \).

**Lemma 17.** Given a Sheffer sequence \( \{s_n(x)\}_{n \in \mathbb{N}} \), with spectrum \( (g(t), f(t)) \), we have the Sheffer matrix

\[
\begin{pmatrix}
(n_k) 
\end{pmatrix}^{(\lambda+k\alpha)}(x+kz) \bigg|_{n,k \geq 0}.
\]

**Proof.** The exponential generating series for the \( k \)th column is

\[
g(t)^\lambda \ e^{xf(t)} \left( \frac{t^k}{k!} \right) = \left( \frac{t^k}{k!} \right) \sum_{n \geq 0} \left( \frac{n+k}{k} \right) s_n^{(\lambda+k\alpha)}(x+kz) \frac{t^n}{n!}.
\]

This implies identity (23).

Then, we have the next exchange identity for a cross sequence.

**Theorem 18.** Let \( \{s_n(x)\}_{n \in \mathbb{N}} \) be a Sheffer sequence with spectrum \( (g(t), f(t)) \). Then, we have the identity

\[
\sum_{n=0}^{\infty} \binom{n}{k} s_k^{(\lambda+\mu+k\alpha)}(w+x+kz) s_{n-k}^{(\nu-k\alpha)}(y-kz) =
\]

\[
= \sum_{n=0}^{\infty} \binom{n}{k} s_k^{(\mu+k\alpha)}(x+kz) s_{n-k}^{(\lambda+\nu-k\alpha)}(w+y-kz).
\]

**Proof.** If \( S_1 = (g(t)^\mu e^{xf(t)}, tg(t)^\nu e^{zf(t)}) \) and \( S_2 = (g(t)^\nu e^{zf(t)}, tg(t)^\alpha e^{zf(t)}) \), then identity (9), for \( m = 1 \) and \( b = 0 \), becomes

\[
\sum_{n=0}^{\infty} \binom{n}{k} s_k^{(\alpha+\mu+k\alpha)}(az+x+kz) s_{n-k}^{(\nu+n\alpha-k\alpha)}(y+nz-kz) =
\]

\[
= \sum_{n=0}^{\infty} \binom{n}{k} s_k^{(\mu+k\alpha)}(x+kz) s_{n-k}^{(\alpha+\nu+n\alpha-k\alpha)}(az+y+nz-kz).
\]

Setting \( \lambda = a\alpha \) and \( w = az \), and replacing \( \nu + n\alpha \) by \( \nu \) and \( y + nz \) by \( y \), we obtain identity (24).

We also have the following exchange identity for two Sheffer sequences.
Theorem 19. Let \( \{s_n(x)\}_{n \in \mathbb{N}} \) and \( \{t_n(x)\}_{n \in \mathbb{N}} \) be two Sheffer sequences, with spectrum \((g_1(t), f(t))\) and \((g_2(t), f(t))\), respectively. Then, we have the identity

\[
(25) \quad \sum_{k=0}^{n} \binom{n}{k} s_k(w + x + kz)t_{n-k}(y - kz) = \sum_{k=0}^{n} \binom{n}{k} s_k(x + kz)t_{n-k}(w + y - kz).
\]

Proof. If \( S_1 = (g_1(t) e^{zf(t)}, te^{zf(t)}) \) and \( S_2 = (g_2(t) e^{zf(t)}, te^{zf(t)}) \), then identity (9), for \( m = 1 \), becomes

\[
\sum_{k=0}^{n} \binom{n}{k} s_k(az + x + kz)t_{n-k}(y + bz + nz - kz) = \sum_{k=0}^{n} \binom{n}{k} s_k(x + kz)t_{n-k}(az + y + bz + nz - kz).
\]

Setting \( w = az \) and replacing \( y + bz + nz \) by \( y \), we obtain identity (25). \( \square \)

Moreover, we have the following Abel-like identity for a cross sequence.

Theorem 20. Let \( \{s_n(x)\}_{n \in \mathbb{N}} \) be a Sheffer sequence with spectrum \((g(t), f(t))\). Then, for the associated cross sequence, we have the identity

\[
(26) \quad x \sum_{k=0}^{n} \binom{n}{k} \frac{s_n^{(\lambda x-kz)}(x-kz)}{x-kz} s_n^{(\nu+k\lambda z)}(y+kz) = s_n^{(\lambda x+\nu)}(x+y).
\]

Proof. Let \( S \) be the matrix (23). For \( \lambda = 0 \) and \( x = 0 \), we have

\[
F = \left( 1, tg(t)^{\alpha} e^{zf(t)} \right) = \left[ \binom{n}{k} s^{(k\alpha)}_{n-k}(kz) \right]_{n,k \geq 0}.
\]

Hence, by identity (11) with \( m = 1 \) and \( a = 0 \) (but without loss of generality), we obtain

\[
\sum_{k=0}^{n} \binom{n}{k} s_n^{(\nu+k\alpha)}(x+kz)s_n^{(b+n)\alpha-k\alpha}((b+n)z-kz)\frac{b}{b+n-k} = s_n^{(\nu+(b+n)\alpha)}(x+(b+n)z).
\]

Setting \( y = (b+n)z \) and \( \mu = (b+n)\alpha \), we have \( bz = y - nz \), \( ay - \mu z = 0 \) and

\[
\sum_{k=0}^{n} \binom{n}{k} s_n^{(\nu+k\alpha)}(x+kz)s_n^{(\mu-k\alpha)}(y-kz)\frac{y-nz}{y-kz} = s_n^{(\mu+\nu)}(x+y).
\]

Equivalently, we have

\[
(y-nz) \sum_{k=0}^{n} \binom{n}{k} s_{n-k}^{(\nu+n\alpha-k\alpha)}(x+nz-kz)s_k^{(\mu-n\alpha+k\alpha)}(y-nz+kz)\frac{y-nz+kz}{y-nz+kz} = s_n^{(\mu+\nu)}(x+y).
\]
Now, replacing $\nu + n\alpha$ by $\nu$, $\mu - n\alpha$ by $\mu$, $y - nx$ by $x$, $x + nz$ by $y$ and $z$ by $-z$, we obtain the identity

$$x \sum_{k=0}^{n} \binom{n}{k} s_k^{(\mu-\alpha)}(x-kz) s_{n-k}^{(\nu-k\alpha)}(y+kz) = s_n^{(\mu+\nu)}(x+y)$$

subject to the condition $\alpha x - \mu z = 0$. Finally, by setting $\alpha = \lambda z$ and $\mu = \lambda x$, we obtain identity (26).

Furthermore, we have the following Abel-like identity for $s$-Appell polynomials (not necessarily forming a cross sequence).

**Theorem 21.** Let $\{a_n(x)\}_{n \in \mathbb{N}}$ be an $s$-Appell sequence with spectrum $(g(t), st)$. Then, we have the identity

$$(27) \quad x \sum_{k=0}^{n} \binom{n}{k} \frac{s_k(x-kz)x}{s_{n-k}(y+kz)} a_{n-k}(y+kz) = a_n(x+y).$$

**Proof.** Let $\lambda = 0$ and $\nu = 1$ in identity (27). Then, we have

$$x \sum_{k=0}^{n} \binom{n}{k} a_k^{(0)}(x-kz) s_{n-k}(x+kz) = a_n(x+y).$$

Since $\sum_{n \geq 0} a_k^{(0)}(x) \frac{t^n}{n!} = e^{sx}$, we have $a_k^{(0)}(x) = (sx)^n$. Hence $a_k^{(0)}(x-kz) = s_k(x-kz)$, and we have identity (27).

**Remark 22.** For the ordinary powers $x^n$, which form an Appell sequence, identity (27) yields the original Abel’s binomial identity [1] [11, p. 128] [31, p. 18] [32, p. 73]

$$(28) \quad x \sum_{k=0}^{n} \binom{n}{k} (x-kz)^k (y+kz)^n = (x+y)^n.$$

In this way, we also reobtain the second identity stated in Theorem 15.

Furthermore, we have the following Sheffer matrices.

**Theorem 23.** Let $\{s_n(x)\}_{n \in \mathbb{N}}$ be a Sheffer sequence with spectrum $(g(t), f(t))$. Then, we have the Sheffer matrix

\[
\left[ \binom{n}{k} s_n^{(\lambda+n\alpha)}(x+nz) \right]_{n,k \geq 0} = \\
\left( \frac{g(\varphi(t))^\lambda}{1-z\varphi(t)f'(\varphi(t)) - \alpha \frac{\varphi(t)}{g(\varphi(t))}} \left( \frac{\varphi(t)}{fg(\varphi(t))} \right)^{x/z}, \varphi(t) \right)
\]
where $\varphi(t)$ is the unique invertible solution of the equation

$$
\varphi(t) = tg(\varphi(t))^{\alpha} e^{s f(\varphi(t))}.
$$

In particular, if $\{a_n(x)\}_{n \in \mathbb{N}}$ is an $s$-Appell sequence, then we have the Sheffer matrix

$$
\left[ \begin{array}{c}
\binom{n}{k} a_{n-k}(x+nz) \\
k 
\end{array} \right]_{n,k \geq 0} = \left( \frac{g(c(stz))}{1 - c(stz)} \right) \left( \frac{c(st)}{st} \right)^{x/z},
$$

where $c(t)$ is the Cayley series \(22\).

Proof. For $m = 1$, the series $\varphi(t)$ associated with the Sheffer matrix \(23\) is defined (by identity \(5\)) by the equation

$$
\varphi(t) = tg(\varphi(t))^{\alpha} e^{s f(\varphi(t))}.
$$

Since $t\varphi'(t) - z\varphi(t)f'(\varphi(t)) - \alpha \varphi(t)g(\varphi(t))$ and $e^{s f(t)} = \left( \frac{\varphi(t)}{tg(\varphi(t))} \right)^{x/z}$,

we obtain the first Sheffer matrix, by applying Theorem 2. For an $s$-Appell sequence, we have $\lambda = 1$, $\alpha = 0$, $f(t) = st$, $f'(t) = s$ and $\varphi(t) = te^{sz(t)}$. Hence, we have $\varphi(t) = \frac{c(st)}{sz}$ and, consequently, the obtain the second Sheffer matrix.

We conclude with some examples.

Examples

1. The falling factorials $x^n = x(x-1)(x-2)\cdots(x-n+1)$ form a Sheffer sequence \([32, p. 57]\), with exponential generating series

$$
\sum_{n \geq 0} x^n t^n = (1+t)^x.
$$

Then, after some simplifications, the exchange identity \(25\) becomes

$$
\sum_{k=0}^{n} \binom{n}{k} \frac{w+x+kz}{k} \frac{y+kz}{n-k} = \sum_{k=0}^{n} \binom{x+kz}{k} \frac{w+y-kz}{n-k}.
$$

Notice that, setting $\alpha = w+x$, $\beta = z$, $\gamma = y-nz$ and $\delta = -w$, this identity becomes Gould's identity \(17\).

A similar result can be obtained for the rising factorials, or Pochhammer symbol, $(x)_n = x(x+1)(x+2)\cdots(x+n-1)$ \([32, p. 5]\).

2. The generalized Hermite polynomials $H_n^{(\nu)}(x)$ form a $2$-Appell sequence and a cross sequence \([14, Vol. 2, p.192]\) \([26]\), with exponential generating series

$$
\sum_{n \geq 0} H_n^{(\nu)}(x) \frac{t^n}{n!} = e^{2xt-\nu t^2} = e^{-\nu t^2} e^{2xt}.
$$
Then, the exchange identity (24) becomes
\[
\sum_{k=0}^{n} \binom{n}{k} H_k^{(\lambda + \mu + k\alpha)}(w + x + k z) H_{n-k}^{(\nu - k\alpha)}(y - k z) = 
\sum_{k=0}^{n} \binom{n}{k} H_k^{(\mu + k\alpha)}(x + k z) H_{n-k}^{(\lambda + \nu - k\alpha)}(w + y - k z)
\]
and the Abel-like identity (26) becomes
\[
x \sum_{k=0}^{n} \binom{n}{k} \frac{H_k^{(\lambda(x-kz))}(x - k z)}{x - k z} H_{n-k}^{(\nu + \lambda k z)}(y + k z) = H_n^{(\lambda x + \nu)}(x + y).
\]

In particular, the Abel-like identity (27) becomes
\[
x \sum_{k=0}^{n} \binom{n}{k} 2^k (x - k z)^{k-1} H_{n-k}^{(\nu)}(y + k z) = H_n^{(\nu)}(x + y).
\]

3. The generalized Laguerre polynomials \( L_n^{(\nu)}(x) \) form a Sheffer sequence (but not a cross sequence) \[14\], Vol. 2, p. 189], with exponential generating series
\[
\sum_{n \geq 0} L_n^{(\nu)}(x) \frac{t^n}{n!} = e^{-\frac{xt}{1-t}}.
\]

The polynomials \( L_n^{(\nu-1)}(x) \) form a \((-1)\)-Appell sequence and a cross sequence \[14\], Vol. 2, p. 189, Formula (19)], with exponential generating series
\[
\sum_{n \geq 0} L_n^{(\nu-1)}(x) \frac{t^n}{n!} = (1 + t)^{\nu} e^{-xt}.
\]

Since the sequence is not \(s\)-Appell, identity (27) does not hold.

The polynomials \( L_n^{(\nu-n)}(x) \) form a \((-1)\)-Appell sequence and a cross sequence \[14\], Vol. 2, p. 189, Formula (19)], with exponential generating series
\[
\sum_{n \geq 0} L_n^{(\alpha-n)}(x) \frac{t^n}{n!} = (1 + t)^{\alpha} e^{-xt}.
\]
Then, the Abel-like identity (26) becomes
\[
x \sum_{k=0}^{n} \binom{n}{k} \frac{L_k^{(\lambda(x-kz)-k)}(x-kz)}{x-kz} \frac{L_{n-k}^{(\nu+\lambda kz-n+k)}(y+kz)}{L_{n-k}^{(\lambda x+\nu-n)}(x+y)} = L_{n-k}^{(\lambda x+\nu-n)}(x+y).
\]

In particular, the Abel-like identity (27) becomes
\[
x \sum_{k=0}^{n} \binom{n}{k} (-1)^k(x-kz)^{k-1} L_{n-k}^{(\nu-n+k)}(y+kz) = L_{n-k}^{(\nu-n)}(x+y).
\]

4. The generalized Bernoulli polynomials \(B_n^{(\nu)}(x)\) and the generalized Euler polynomials \(E_n^{(\nu)}(x)\) form two Appell sequences and two cross sequences, \([32, p. 93, p. 100]\) and \([14, Vol. 3, p. 252]\), with exponential generating series
\[
\sum_{n \geq 0} B_n^{(\nu)}(x) \frac{t^n}{n!} = \left( \frac{t}{e^t - 1} \right)^\nu e^{xt}
\]
\[
\sum_{n \geq 0} E_n^{(\nu)}(x) \frac{t^n}{n!} = \left( \frac{2}{e^t + 1} \right)^\nu e^{xt}.
\]

In this case, in addition to the exchange identity (24), we also have the exchange identity (25), that becomes
\[
\sum_{k=0}^{n} \binom{n}{k} B_k^{(\mu)}(w+x+kz) E_{n-k}^{(\nu)}(y-kz) = \sum_{k=0}^{n} \binom{n}{k} B_k^{(\mu)}(x+kz) E_{n-k}^{(\nu)}(w+y-kz).
\]

Moreover, the Abel-like identity (26) becomes
\[
x \sum_{k=0}^{n} \binom{n}{k} \frac{B_k^{(\lambda(x-kz))}(x-kz)}{x-kz} B_{n-k}^{(\nu+\lambda kz)}(y+kz) = B_{n-k}^{(\lambda x+\nu)}(x+y)
\]
\[
x \sum_{k=0}^{n} \binom{n}{k} \frac{E_k^{(\lambda(x-kz))}(x-kz)}{x-kz} E_{n-k}^{(\nu+\lambda kz)}(y+kz) = E_{n-k}^{(\lambda x+\nu)}(x+y)
\]

and the Abel-like identity (27) becomes
\[
x \sum_{k=0}^{n} \binom{n}{k} (x-kz)^{k-1} B_{n-k}^{(\nu)}(y+kz) = B_{n-k}^{(\nu)}(x+y)
\]
\[
x \sum_{k=0}^{n} \binom{n}{k} (x-kz)^{k-1} E_{n-k}^{(\nu)}(y+kz) = E_{n-k}^{(\nu)}(x+y).
\]
5. The actuarial polynomials $a_n^{(\nu)}(x)$ form a cross sequence [32, p. 123] [41], with exponential generating series

$$\sum_{n \geq 0} a_n^{(\nu)}(x) \frac{t^n}{n!} = e^{\nu t - x(e^t - 1)}.$$ 

In this case, the exchange identity (24) becomes

$$\sum_{k=0}^n \binom{n}{k} a_k^{(\lambda + \mu + k\alpha)}(w + x + k\alpha) a_{n-k}^{(\nu - k\alpha)}(y - k\alpha) =$$

$$= \sum_{k=0}^n \binom{n}{k} a_k^{(\mu + k\alpha)}(x + k\alpha) a_{n-k}^{(\lambda + \nu - k\alpha)}(w + y - k\alpha)$$

and the Abel-like identity (26) becomes

$$x \sum_{k=0}^n \binom{n}{k} a_k^{(\lambda(x - k\alpha))}(x - k\alpha) a_{n-k}^{(\nu + \lambda k\alpha)}(y + k\alpha) = a_n^{(\lambda x + \nu)}(x + y).$$

Also in this case, identity (27) does not hold.

6. The Cayley continuants $U_n^{(\nu)}(x)$ form a cross sequence [7, 27], with exponential generating series

$$\sum_{n \geq 0} U_n^{(\nu)}(x) \frac{t^n}{n!} = (1 - t^2)^{\nu/2} \left( \frac{1 + t}{1 - t} \right)^{x/2}.$$ 

In this case, the exchange identity (24) becomes

$$\sum_{k=0}^n \binom{n}{k} U_k^{(\lambda + \mu + k\alpha)}(w + x + k\alpha) U_{n-k}^{(\nu - k\alpha)}(y - k\alpha) =$$

$$= \sum_{k=0}^n \binom{n}{k} U_k^{(\mu + k\alpha)}(x + k\alpha) U_{n-k}^{(\lambda + \nu - k\alpha)}(w + y - k\alpha)$$

and the Abel-like identity (26) becomes

$$x \sum_{k=0}^n \binom{n}{k} U_k^{(\lambda(x - k\alpha))}(x - k\alpha) U_{n-k}^{(\nu + \lambda k\alpha)}(y + k\alpha) = U_n^{(\lambda x + \nu)}(x + y).$$

7. The generalized rencontres polynomials $D_n^{(\nu)}(x)$ form an Appell sequence (but not a cross sequence), with exponential generating series

$$\sum_{n \geq 0} D_n^{(\nu)}(x) \frac{t^n}{n!} = \frac{e^{(x-1)t}}{(1 - t)^{\nu + 1}}.$$ 

\[2\] See [6, 15] for a slightly different generalization of the rencontres polynomials.
So, the Abel-like identity (27) becomes
\[
x \sum_{k=0}^{n} \binom{n}{k} (x - kz)^{k-1} D_{n-k}^{(\nu)} (y + kz) = D_{n}^{(\nu)} (x + y).
\]

8. The Abel polynomials \( A_{n}^{(\nu)} (x) = x(x - \nu n)^{n-1} \) form a Sheffer sequence (but not a cross sequence) [32, p. 72], with exponential generating series
\[
\sum_{n \geq 0} A_{n}^{(\nu)} (x) \frac{t^n}{n!} = e^{x W(\nu t)}
\]
where \( W(t) \) is the Lambert series. In this case, we have only the exchange identity (25), that becomes
\[
\sum_{k=0}^{n} \binom{n}{k} A_{k}^{(\nu)} (w + x + kz) A_{n-k}^{(\nu)} (y - kz) =
\]
\[
= \sum_{k=0}^{n} \binom{n}{k} A_{k}^{(\nu)} (x + kz) A_{n-k}^{(\nu)} (w + y - kz).
\]

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