Scaling and local limits of Baxter permutations and bipolar orientations through coalescent-walk processes

Jacopo Borgia  
UZH ZÜRICH

(joint work with M. Maazoun)
Permuton limits

We look at permutations from a geometric perspective:
Consider the permutation \( \sigma = (1, 2, 3, 4, 5, 6, 7) \) 
\( \overset{\text{\( \Gamma = \)} \quad \text{\( \rightarrow \) \quad \( \mathcal{M}_\sigma = \)} \quad \text{Probability measure on the unit-square with uniform marginals}}{\begin{array}{c}
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\end{array}
\end{array}} \)
**Def.** A **permution** is a probability measure on the square $[0,1]^2$ with uniform marginals.

**Remark.** We have a natural notion of convergence of such objects: the **weak convergence**. This defines a nice compact Polish space. Limits of permutations are permutations, i.e., potential limits of sequences of permutations also have uniform marginals.
**Def:** Baxter permutations are permutations avoiding the patterns 2413 & 3142.

Dokos & Pak (2014) explored the expected shape of doubly alternating Baxter permutations, i.e. Baxter perm. $\sigma$ s.t. $\sigma$ and $\sigma^{-}$ are alternating and they claimed that "it would be nice to compute the limit shape of Baxter permutations."
Bipolar orientations and walks in cones

Bonichon, Bousquet-Mélou & Fusy (2011) showed that Baxter permutations are in bijection with plane bipolar orientations.

Def: A **plane bipolar orientation** is a planar map (connected graphs properly embedded in the plane up to continuous deformations) equipped with an **acyclic orientation** of the edges with exactly one **source** (a vertex with only outgoing edges) and one **sink** (a vertex with only incoming edges) both on the outer face.
Kenyon, Miller, Sheffield & Wilson (2019) constructed a bijection $\text{OW}$ from bipolar orientations to a specific family of two-dimensional walks in the non-negative quadrant, called tandem walks.

**Theorem:** (Gwynne, Holden, Sun 2016)
The pairs of height functions for an infinite-volume random bipolar triangulation and its dual converge jointly in law to the two Brownian motions which encode the same $\sqrt{13}$-LQG surface decorated by both an SLE$_{12}$ and the “dual” SLE$_{12}$ which travels in a perpendicular direction.
So far...

(0,2), (0,3), (0,3),
(1,2), (2,1), (0,3),
(1,2), (2,1), (3,0), (2,0).

OP

OW

OP o OW⁻¹

We want "to read" the patterns of a permutation in the corresponding walk.
Let $W_t = (X_t, Y_t)$ be a tandem walk & $\tau = OP\circ OW^{-1}(W)$ be the corresponding Baxter permutation.

**IDEA:** Given $i < j$, we want to find a way in order to "read" in $W_t$ if $\tau(i) < \tau(j)$ or $\tau(j) < \tau(i)$.

**SOLUTION:** COALESCENT-WALK PROCESSES

i.e. a collection of walks $(Z_t)_{i \geq t}$ that "follow" $Y_t$ when they are positive and $-X_t$ when they are negative.
Def: Let \((W_t)_{t \in \mathbb{N}} = (X_t, Y_t)_{t \in \mathbb{N}}\) be a tandem walk of length \(n \in \mathbb{N}\). The **coalescent walk process** associated to \((W_t)_{t \in \mathbb{N}}\) is a collection of \(n\) one-dimensional walks \((Z^{(\ell)}_t)_{t \in \mathbb{N}} = WC(W)\) defined for every \(t \in \mathbb{N}\) by:

\[
Z^{(\ell)}_t = 0, \quad Z^{(\ell)}_K = \begin{cases} 
Z^{(\ell)}_{K-1} + (Y_K - Y_{K-1}) & \text{if } Z^{(\ell)}_{K-1} \geq 0 \\
Z^{(\ell)}_{K-1} - (X_K - X_{K-1}) & \text{if } Z^{(\ell)}_{K-1} < 0 \text{ and } Z^{(\ell)}_{K-1} - (X_K - X_{K-1}) < 0 \\
Y_K - Y_{K-1} & \text{if } Z^{(\ell)}_{K-1} < 0 \text{ and } Z^{(\ell)}_{K-1} - (X_K - X_{K-1}) \geq 0
\end{cases}
\]

\[
(0,2),(0,3),(0,3), (1,2),(2,1),(0,3), (1,2),(2,1),(3,0),(2,0).
\]

\[
\text{OW(m)} \quad \text{WC}
\]

\[
Y_t + 1
\]

\[
-X_t - 1
\]
**Theorem**: Let \( W = (W_t)_{t \in [n]} \) be a tandem walk and \( \sigma = \text{OP} \circ \text{OW}^{-1}(W) \) the corresponding Baxter permutation. Then \( \text{CP} \circ \text{WC}(W) = \sigma \).

**Proposition**: Let \( \sigma \) be a Baxter permutation of size \( n \in \mathbb{N} \) corresponding to a coalescent walk process \( (Z^{(t)})_{t \in [n]} \). Then for \( i < j \)

\[
\sigma(i) < \sigma(j) \iff Z^{(i)}_{j} < 0
\]
Theorem:
(B.-Mazzoun 2020*)

This diagram commutes.
This diagram commutes.
Scaling limits of coalescent-walk processes

The continuous coalescent-walk process

Consider a two dimensional process $W(t) = (X(t), Y(t))_{t \in I}$ and the following family of stochastic differential equations (SDEs) indexed by $\mu \in I$

\[
\begin{aligned}
    dZ^{(\mu)}(t) &= \mathbb{1}_{\{Z^{(\mu)}(t) > 0\}} \, dY(t) - \mathbb{1}_{\{Z^{(\mu)}(t) < 0\}} \, dX(t), \quad t \in (\mu, \infty) \cap I, \\
    Z^{(\mu)}(t) &= 0, \quad t \in (-\infty, \mu] \cap I.
\end{aligned}
\]

**THEOREM** (Prokaj 2013, Çaglar-Hajri-Karakus 2018)

Let $(W(t))_{t \in I}$ be a two-dimensional Brownian motion with covariance matrix $(1 \ p)$ for some $p \in (-1, 1)$. Fix $\mu \in I$. We have path-wise uniqueness and existence of a strong solution for the SDE $(\ast)$ driven by $W(t)$. 
\[
\begin{align*}
\{ dZ^{(u)}(t) &= \begin{cases} 
-1 & \{Z^{(u)}(t) > 0\}, \\
1 & \{Z^{(u)}(t) \leq 0\}
\end{cases} \, dY(t) - \begin{cases} 
-1 & \{Z^{(u)}(t) > 0\}, \\
1 & \{Z^{(u)}(t) \leq 0\}
\end{cases} \, dX(t) \quad t > u, \\
Z^{(u)}(t) &= 0, \quad t \leq u
\end{align*}
\]

\text{Def:} \text{ We call \textbf{CONTINUOUS COALESCENT-WALK PROCESS} (driven by } W \text{) the collection of solutions } \{Z^{(u)}\}_{u \in \mathbb{R}} \text{ where properly defined.}

The previous theorem + Fubini-Tonelli imply that:
For almost every } w, Z^{(u)} is a solution for almost every } u.
Let $\overline{W} = (\overline{X}, \overline{Y}) = (\overline{X}_k, \overline{Y}_k)_{k \geq 0}$ be a two-dimensional random walk having value $(0, 0)$ at time 0 and step distribution

$$\nu = \frac{1}{2} \delta_{(1,1)} + \sum_{i,j \geq 0} 2^{-i-j-3} \delta_{(-i,j)}$$

**Proposition:** The following is a uniform tandem walk of length $n$:

$$W_n := \left( (\overline{W}_t)_{1 \leq t \leq n} \mid \overline{W}_0 = (0,0), \overline{W}_{n+1} = (0,0), (\overline{W}_t)_{0 \leq t \leq n+1} \in \mathbb{Z}_{\geq 0}^2 \right)$$

Let $W_n$ be the associated rescaled continuous process that interpolates the steps of $W_n$.

**Theorem (Barlow-Muzon 2020):** Let $\mu \in (0,1)$. We have the following joint convergence in $C([0,1], \mathbb{R}^3)$

$$\left( W_n, \mathcal{X}_n^{(\mu)} \right) \xrightarrow{d} \left( W, \mathcal{X}_\mu \right)$$

- Continuous interpolation of the walk starting at time $\mu$ in the discrete coalescent process associated to $W_n$.
- 2D Brownian excursion in the quadrant with covariance $(\frac{4-\mu}{1-\mu}, \frac{2}{1-\mu})$.
- Associated continuous coalescent process.
Theorem: Let $\sigma_n$ be a uniform Baxter permutation of size $n$. We have the following convergence in the space of permutations

\[ \mu_{\sigma_n} \xrightarrow{d} \mu_{\mathcal{X}} : = \phi(\{X^{(\mu)}_{\mathcal{X}}\}_{\mu \in [0,1]}) \]

Proof based on:

- Proposition: Let $\sigma$ be a Baxter permutation of size $n \in \mathbb{N}$ corresponding to a coalescent-walk process $(Z^{(\sigma)}_t)_{t \in [1]}$. Then for $i < j$

\[ \sigma(i) < \sigma(j) \iff Z^{(\sigma)}_j < 0 \]

- Theorem: Let $(\mu_i)_{i \geq 1}$ be a sequence of iid uniform random variables on $[0,1]$ independent of all other variables. Then

\[ (\mathcal{X}_n, (Z^{(\mu)}_n)_{i \geq 1}) \xrightarrow{d} (\mathcal{W}, (X^{(\mu)}_{\mathcal{X}})_{i \geq 1}) \]
Final comments

- Our results imply convergence of finite-volume bip-orientations to a $\sqrt{4/3}$-LQG. What is the connection between our approach and the LQG approach?
- The convergence of all the objects (walks, permutations, map, card-walk proc.) holds jointly.
- We also proved joint Benjamini-Schramm local limits (both in the annealed & quenched sense) for all the objects involved in the commutative diagram.
- We believe that our techniques are rather general: we would like to consider other families of permutations (and maps?) encoded by two-dimensional walks.
- We would also like to investigate better the Baxter permutation. For instance, what is $E [xyz] = ?$. 

Thank you!

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