Four notions of conjugacy for abstract semigroups

João Araújo
Universidade Aberta, R. Escola Politécnica, 147, 1269-001 Lisboa, Portugal (jaraujo@uab.pt)
and
CEMAT-Ciências, Departamento de Matemática, Faculdade de Ciências, Universidade de Lisboa, 1749-016, Lisboa, Portugal (jjarauerdo@fc.ul.pt)

Michael Kinyon
Department of Mathematics, University of Denver, 2360 S Gaylord St, Denver, CO 80208, USA (mkinyon@du.edu)
and
CEMAT-Ciências, Departamento de Matemática, Faculdade de Ciências, Universidade de Lisboa, 1749-016, Lisboa, Portugal

Janusz Konieczny
Department of Mathematics, University of Mary Washington, Fredericksburg, VA 22401, USA (jkoniecz@umw.edu)

António Malheiro
Departamento de Matemática and Centro de Matemática e Aplicações (CMA), Faculdade de Ciências e Tecnologia (FCT), Universidade Nova de Lisboa (UNL), 2829-516 Caparica, Portugal (ajm@fct.unl.pt)

(MS received 16 October 2015; accepted 25 April 2016)

The action of any group on itself by conjugation and the corresponding conjugacy relation play an important role in group theory. There have been many attempts to find notions of conjugacy in semigroups that would be useful in special classes of semigroups occurring in various areas of mathematics, such as semigroups of matrices, operator and topological semigroups, free semigroups, transition monoids for automata, semigroups given by presentations with prescribed properties, monoids of graph endomorphisms, etc. In this paper we study four notions of conjugacy for semigroups, their interconnections, similarities and dissimilarities. They appeared originally in various different settings (automata, representation theory, presentations, and transformation semigroups). Here we study them in full generality. The paper ends with a large list of open problems.

Keywords: conjugacy; symmetric inverse semigroups; epigroups

2010 Mathematics subject classification: Primary 20M07; 20M20; 20M15
1. Introduction and preliminaries

By a notion of conjugacy for a class of semigroups we mean an equivalence relation defined in the language of that class of semigroups and coinciding with the group theory notion of conjugacy whenever the semigroup is a group. We study three notions of conjugacy in the most general setting (that is, in the class of all semigroups) and, in view of its importance for representation theory, we also study one notion that was originally only defined for finite semigroups.

When generalizing a concept, it is sometimes tempting to think that there should be one correct, or even preferred, generalization. The view we take in this paper is that since semigroup theory is a vast subject, intersecting many areas of pure and applied mathematics, it is probably not reasonable to expect a one-size-fits-all notion of conjugacy suitable for all purposes. Searching for the ‘best’ notion of conjugacy is, from our point of view, akin to searching for, say, the ‘best’ topology. Instead, we think that the goal of studying conjugacy in semigroups is to determine what different notions of conjugacy look like in various classes of semigroups, and how they interact with each other and with other mathematical concepts. It is thus incumbent upon individual mathematicians to decide, given their needs, which particular notion fits best with the class of semigroups under consideration and within the particular context.

In this paper, we consider primarily four notions of conjugacy (and some variations) that we see as especially interesting given their properties and generality. However, as happens throughout mathematics, stronger notions can be obtained by requiring additional properties. Adding to the general requirements in the first paragraph above, one might require that the notion of conjugacy must be non-trivial, or first-order definable, or that a given set of results about conjugacy in groups carries over to some class of semigroups, etc. Therefore, the years to come will certainly see the rise of many more systems of equivalence relations for semigroups based on notions of conjugacy.

Before introducing the notions of conjugacy that will occupy us in this paper, we recall some standard definitions and notation (we generally follow [40]). Other needed definitions will be given in context.

For a semigroup \( S \), we denote by \( E(S) \) the set of idempotents of \( S \); \( S^1 \) is the semigroup \( S \) if \( S \) is a monoid, or otherwise denotes the monoid obtained from \( S \) by adjoining an identity element \( 1 \). The relation \( \leq \) on \( E(S) \) defined by \( e \leq f \) if \( ef = fe = e \) is a partial order on \( E(S) \) [40, p. 69]. A commutative semigroup of idempotents is said to be a semilattice.

An element \( a \) of a semigroup \( S \) is said to be regular if there exists \( b \in S \) such that \( aba = a \). Setting \( c = bab \), we get \( aca = a \) and \( cac = c \), so \( c \) is an inverse of \( a \). Since \( a \) is also an inverse of \( c \), we often say that \( a \) and \( c \) are mutually inverse. A semigroup \( S \) is regular if all elements of \( S \) are regular, and it is an inverse semigroup if every element of \( S \) has a unique inverse.

If \( S \) is a semigroup and \( a, b \in S \), we say that \( aLb \) if \( S^1a = S^1b \), \( aRb \) if \( aS^1 = bS^1 \), and \( aJb \) if \( S^1aS^1 = S^1bS^1 \). We define \( H = L \cap R \), and \( D = L \vee R \), that is, \( D \) is the smallest equivalence relation on \( S \) containing both \( L \) and \( R \). These five equivalence relations are known as Green’s relations [40, p. 45], and are among the most important tools in studying semigroups.
Four notions of conjugacy for abstract semigroups 1171

We now introduce the four notions of conjugacy that we will consider in this paper. As noted, we expect any reasonable notion of semigroup conjugacy to coincide in groups with the usual notion. For elements $a, b, g$ of a group $G$, if $a = g^{-1}bg$, then we say that $a$ and $b$ are conjugate and $g$ (or $g^{-1}$) is a conjugator of $a$ and $b$. Conjugacy in groups has several equivalent formulations that avoid inverses, and hence generalize syntactically to any semigroup. For example, if $G$ is a group, then $a, b \in G$ satisfy $a = g^{-1}bg$ (for some $g \in G$) if and only if $a = wg$ and $b = vu$ for some $u, v \in G$ (namely, $u = g^{-1}b$ and $v = g$). This last formulation has been used to define the following relation on a free semigroup $S$ (see [48]):

$$a \sim_p b \iff \exists u, v \in S \colon a = uv \text{ and } b = vu.$$  \hfill (1.1)

If $S$ is a free semigroup, then $\sim_p$ is an equivalence relation on $S$ [48, corollary 5.2], and so it can be considered as a notion of conjugacy in $S$. In a general semigroup $S$, the relation $\sim_p$ is reflexive and symmetric, but not transitive. If $a \sim_p b$ in a semigroup, we say that $a$ and $b$ are primarily related [47] (hence the subscript in $\sim_p$). The transitive closure $\sim_p^*$ of $\sim_p$ has been defined as a conjugacy relation in a general semigroup [39, 46, 47]. Lallement credited the idea of the relation $\sim_p$ to Lyndon and Schützenberger [51].

Again looking to group conjugacy as a model, for $a, b$ in a group $G$, $a = g^{-1}bg$ for some $g \in G$ if and only if $ag = gb$ for some $g \in G$ if and only if $bh = ha$, for some $h \in G$ (namely, $h = g^{-1}$). A corresponding semigroup conjugacy is defined as follows:

$$a \sim_o b \iff \exists g, h \in S \colon ag = gb \text{ and } bh = ha.$$  \hfill (1.2)

This relation was defined by Otto for monoids presented by finite Thue systems [54], and, unlike $\sim_p$, it is an equivalence relation in any semigroup. However, $\sim_o$ is the universal relation in any semigroup $S$ with zero. Since it is generally believed [25, 35, 56] that $\lim_{n \to \infty} z_n/s_n = 1$, where $s_n$ (respectively, $z_n$) is the number of semigroups (respectively, the number of semigroups with zero) of order $n$, it would follow that ‘almost all’ finite semigroups have a zero, and hence this notion of conjugacy might be of interest only in particular classes of semigroups.

In [19] a new notion of conjugacy was introduced. This notion coincides with Otto’s concept for semigroups without zero, but does not reduce to the universal relation when $S$ has a zero. The key idea was to restrict the set from which conjugators can be chosen. For a semigroup $S$ with zero and $a \in S \setminus \{0\}$, let $P(a)$ be the set of all elements $g \in S$ such that $(ma)g \neq 0$ for all $ma \in S^1a \setminus \{0\}$. We also define $P(0) = \{0\}$. If $S$ has no zero, we set $P(a) = S$ for every $a \in S$. Let $P^1(a) = P(a) \cup \{1\}$, where $1 \in S^1$. Define a relation $\sim_c$ on any semigroup $S$ by

$$a \sim_c b \iff \exists g \in P^1(a) \exists h \in P^1(b) \colon ag = gb \text{ and } bh = ha.$$  \hfill (1.3)

(See [19, §2] for the motivation of using the sets $P^1(a)$.) Restricting the choice of conjugators, as happens in the definition of $\sim_c$, is not unprecedented for semigroups. For example, if $S$ is a monoid and $G$ is the group of units of $S$, we say that $a$ and $b$ in $S$ are $G$-conjugated and write $a \sim_G b$ if there exists $g \in G$ such that $b = g^{-1}ag$ [46]. The restrictions proposed in the definition of $\sim_c$ are much less stringent. Their choice was motivated by considerations in the context of semigroups of transformations. The translation of these considerations into abstract semigroups
resulted in the sets $P^1(a)$. Roughly speaking, conjugators selected from $P^1(a)$ satisfy the minimal requirements needed to avoid the pitfalls of $\sim_0$.

The relation $\sim_c$, turns out to be an equivalence relation on an arbitrary semigroup $S$. Moreover, if $S$ is a semigroup without zero, then $\sim_c = \sim_o$. If $S$ is a free semigroup, then $\sim_c = \sim_o = \sim_p$. In the case in which $S$ has a zero, the conjugacy class of 0 with respect to $\sim_c$ is $\{0\}$.

The last notion of conjugacy that we will consider has been inspired by considerations in the representation theory of finite semigroups (for details we refer the reader to Steinberg’s book [59]). Let $M$ be a finite monoid and let $a, b \in M$. We say that $a \sim_{tr} b$ if there exist $g, h \in M$ such that $ghg = g$, $hgh = h$, $hg = a^\omega$, $gh = b^\omega$ and $ga^{\omega + 1}h = b^{\omega + 1}$, where, for $a \in M$, $a^\omega$ denotes the unique idempotent in the monogenic semigroup generated by $a$ (see [40, §1.2]) and $a^{\omega + 1} = aa^\omega$. The relation $\sim_{tr}$ is an equivalence relation in any finite monoid.

The same notion can be alternatively introduced (see, for example, [47]) via characters of finite-dimensional representations. Given a finite-dimensional complex representation $\varphi : S \to \text{End}_\mathbb{C}(V)$ of a semigroup $S$, the character of $\varphi$ is the function $\chi_{\varphi} : S \to \mathbb{C}$ defined by $\chi_{\varphi}(s) = \text{trace}(\varphi(s))$ for all $s \in S$. In a finite monoid $S$, $a \sim_{tr} b$ if and only if $\chi_{\varphi}(a) = \chi_{\varphi}(b)$ (see [52, theorem 2.2] or [59, proposition 8.9, 8.3 and theorem 8.10]). This explains the subscript notation $\sim_{tr}$.

The relation $\sim_{tr}$, in its equational definition, can be naturally extended from the class of finite monoids to the class of epigroups. We need some definitions first. Let $S$ be a semigroup. An element $a \in S$ is an epigroup element (or, more classically, a group-bound element) if there exists a positive integer $n$ such that $a^n$ belongs to a subgroup of $S$, that is, the $H$-class $H_{a^n}$ of $a^n$ is a group. If this positive integer is 1, then $a$ is said to be completely regular. If we denote by $e$ the identity element of $H_{a^n}$, then $ae$ is in $H_{a^n}$ and we define the pseudo-inverse $a'$ of $a$ by $a' = (ae)^{-1}$, where $(ae)^{-1}$ denotes the inverse of $ae$ in the group $H_{a^n}$ [58, (2.1)]. An epigroup is a semigroup consisting entirely of epigroup elements, and a completely regular semigroup is a semigroup consisting entirely of completely regular elements. Finite semigroups and completely regular semigroups are examples of epigroups. Following Petrich and Reilly [55] for completely regular semigroups and Shevrin [58] for epigroups, it is now customary to view an epigroup $(S, \cdot)$ as a unary semigroup $(S, \cdot', \omega)$ where $x \mapsto x'$ is the map sending each element to its pseudo-inverse. In addition, the superscript $\omega$ notation introduced above for finite semigroups can be extended to an epigroup $S$ [58, §2], where, for $a \in S$, $a^\omega$ denotes the idempotent of the group to which some power of $a$ belongs. (In the finite case, $a^\omega$ itself is a power of $a$.) We can therefore extend the definition of $\sim_{tr}$ from finite monoids to epigroups: for all $a, b \in S$,

$$a \sim_{tr} b \iff \exists g, h \in S \mid ghg = g, \ hgh = h, \ ga^{\omega + 1}h = b^{\omega + 1}, \ hg = a^\omega \text{ and } gh = b^\omega. \tag{1.4}$$

In any epigroup, we have $a^\omega = aa'$ (see [58, §2.2]), and therefore $a^{\omega + 1} = aa'a = a^n$. Thus, in epigroups, as is sometimes convenient, we can express the conjugacy relation $\sim_{tr}$ entirely in terms of pseudo-inverses: for all $a, b \in S$,

$$a \sim_{tr} b \iff \exists g, h \in S \mid ghg = g, \ hgh = h, \ ga''h = b'', \ hg = aa' \text{ and } gh = bb'. \tag{1.5}$$

We will refer to $\sim_p, \sim_p', \sim_o, \sim_c$, and $\sim_{tr}$ as $p$-conjugacy, $p'$-conjugacy, $o$-conjugacy, $c$-conjugacy, and trace conjugacy, respectively. Of course, $\sim_p$ is a valid notion...
Four notions of conjugacy for abstract semigroups

Figure 1. Inclusions between the four conjugacies.

of conjugacy only in the class of semigroups in which it is transitive, and trace conjugacy is only defined for epigroups.

For epigroups (and, in particular, for finite semigroups), we have the inclusions depicted in figure 1 (which will be justified later). The corresponding picture for arbitrary semigroups can be extracted from figure 1 by removing $\sim_{\text{tr}}$. The following semigroup $S$, which is $\text{SmallSemigroup}(7,542155)$ of [26], shows that all inclusions in figure 1 are strict:

$\begin{array}{ccccccc}
\cdot & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
0 & 0 & 0 & 0 & 0 & 4 & 4 & 0 \\
1 & 0 & 0 & 0 & 0 & 4 & 4 & 0 \\
2 & 0 & 0 & 0 & 0 & 4 & 4 & 0 \\
3 & 0 & 0 & 0 & 0 & 4 & 4 & 0 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
5 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
6 & 0 & 0 & 2 & 3 & 4 & 5 & 6 \\
\end{array}$

Since $S$ has a zero (the element 4) it follows that $\sim_o = S \times S$; in addition, it is obvious from the table that $\sim_p$ (viewed as a directed graph) consists of all loops together with the edges 0–2, 0–3, and 4–5. Therefore, the partition induced by $\sim_p$ is $\{\{0,2,3\}, \{4,5\}, \{1\}, \{6\}\}$. On the other hand, $\sim_{\text{tr}}$ induces the partition $\{\{0,1,2,3\}, \{5,6\}, \{4\}\}$. Finally, we have $P(0) = P(1) = P(2) = P(3) = P(6) = \{0,1,2,3,6\}; P(4) = \{4\}$, and $P(5) = \emptyset$. From that we infer that $\sim_c$ induces the partition $\{\{0,1,2,3,6\}, \{4\}, \{5\}\}$. Now, $\sim_c \cap \sim_p$ consists of all loops and the edges 0–2 and 0–3; $\sim_c \cap \sim^*_p$ induces the partition $\{\{0,2,3\}, \{4\}, \{5\}, \{1\}, \{6\}\}$; finally, $\sim_c \cap \sim_{\text{tr}}$ induces the partition $\{\{0,1,2,3\}, \{4\}, \{5\}, \{6\}\}$.

In §2 we study c-conjugacy, trace conjugacy, and p-conjugacy in one of the most important classes of inverse semigroups with proper divisors of zero, namely, symmetric inverse semigroups (see [40, theorem 5.1.5]). We give a complete description of the c-conjugacy classes, answering a question posed by the referee of [19]. In the symmetric inverse semigroup $\mathcal{I}(X)$ on a set $X$, we find that $\sim_c \subset \sim_p$ when $X$ is finite, and $\sim_p$ and $\sim_c$ are not comparable when $X$ is countably infinite. Note that
∼_p \subseteq \sim_o in every semigroup \( S \) [19, theorem 2.2]. However, as \( \mathcal{I}(X) \) shows, the relation between \( \sim_c \) and \( \sim_p \) is more complex.

In §3 we study the relationship between conjugacies and Green’s relations. We find that, in general, Green’s relations and the conjugacies under consideration are not comparable with respect to inclusion, but there are some comparison results for some transformation semigroups. Our general perception, however, is that conjugacies and Green’s relations form two ‘orthogonal’ systems of equivalence relations.

The bulk of our results is contained in §4 and §5. Roughly speaking, in the first we deal with conditions under which the conjugacies tend to be equal; in the second we deal with the opposite situation. Given the definition of \( \sim_{tr} \), epigroups form the largest class of semigroups in which all the notions are defined, and hence is the largest class in which all the relations could be equal; therefore §4 only deals with epigroups. In particular, to have \( \sim_p \) equal to one of the other notions of conjugacy, a necessary condition is the transitivity of \( \sim_p \). A complete classification of the semigroups in which \( \sim_p \) is transitive is still an open problem. Besides groups and free semigroups [48, corollary 5.2], a recent result of Kudryavtseva [44, corollary 4] shows that \( p \)-conjugacy is transitive in completely regular semigroups. We generalize this result by introducing a wider class of epigroups that contains completely regular semigroups and their variants.

In §5 we prove a number of properties and separation results of the four notions of conjugacy. We conclude the section by extending various results about conjugacy in groups to conjugacy in semigroups. For example, if \( \sim \) is any of \( \sim_p, \sim_c, \sim_o \) or \( \sim_{tr} \), then \( a \sim b \) implies \( a^k \sim b^k \), just like in groups.

Finally, §6 lists open problems regarding the notions of conjugacy under discussion, showing how wide open this topic is.

2. Conjugacy in symmetric inverse semigroups

The symmetric inverse semigroup on a non-empty set \( X \) is the semigroup \( \mathcal{I}(X) \) of partial injective transformations on \( X \) under composition [40, p. 148]. The aim of this section is to answer a question of the referee of [19] regarding \( c \)-conjugacy in \( \mathcal{I}(X) \) for a countable \( X \), and also compare these results with the existing ones on the other notions of conjugacy. For \( \mathcal{I}(X) \), with countable \( X \), \( p \)-conjugacy was described in [33] (for \( X \) finite) and [46] (for \( X \) countably infinite). It will follow from these descriptions and our result that in \( \mathcal{I}(X) \), \( \sim_c \subset \sim_p \) if \( X \) is finite, and \( \sim_c \) and \( \sim_p \) are not comparable (with respect to inclusion) if \( X \) is countably infinite. We note that since the semigroup \( \mathcal{I}(X) \) has a zero, \( o \)-conjugacy in \( \mathcal{I}(X) \) is universal for every \( X \). Also, if \( X \) is infinite, then \( \mathcal{I}(X) \) is not an epigroup, so trace conjugacy is only defined for \( \mathcal{I}(X) \) if \( X \) is finite. We will get back to this later.

The importance of symmetric inverse semigroups comes from the fact that every inverse semigroup can be embedded in \( \mathcal{I}(X) \) for some \( X \) [40, theorem 5.1.7]. The role of \( \mathcal{I}(X) \) in the theory of inverse semigroups is analogous to that of the symmetric group \( \text{Sym}(X) \) of permutations on \( X \) in group theory.

To describe \( \sim_c \) in \( \mathcal{I}(X) \), we will use the cycle-chain-ray decomposition of a partial injective transformation [43], which is an extension of the cycle decomposition of a permutation.

We will write functions on the right and compose from left to right; that is, for \( f: A \to B \) and \( g: B \to C \), we will write \( xf \) rather than \( f(x) \), and \( x(fg) \)
rather than $g(f(x))$. Let $\alpha \in \mathcal{I}(X)$. We denote the domain of $\alpha$ by $\text{dom}(\alpha)$ and the image of $\alpha$ by $\text{im}(\alpha)$. The union $\text{dom}(\alpha) \cup \text{im}(\alpha)$ will be called the span of $\alpha$ and be denoted $\text{span}(\alpha)$. We say that $\alpha$ and $\beta$ in $\mathcal{I}(X)$ are completely disjoint if $\text{span}(\alpha) \cap \text{span}(\beta) = \emptyset$.

**Definition 2.1.** Let $M$ be a set of pairwise completely disjoint elements of $\mathcal{I}(X)$. The join of the elements of $M$, denoted $\bigsqcup_{\gamma \in M} \gamma$, is the element of $\mathcal{I}(X)$ whose domain is $\bigcup_{\gamma \in M} \text{dom}(\gamma)$ and whose values are defined by

$$x \left( \bigsqcup_{\gamma \in M} \gamma \right) = x\gamma_0,$$

where $\gamma_0$ is the (unique) element of $M$ such that $x \in \text{dom}(\gamma_0)$. If $M = \emptyset$, we define $\bigsqcup_{\gamma \in M} \gamma$ to be 0 (the zero in $\mathcal{I}(X)$). If $M = \{ \gamma_1, \gamma_2, \ldots, \gamma_k \}$ is finite, we may write the join as $\gamma_1 \sqcup \gamma_2 \sqcup \cdots \sqcup \gamma_k$.

**Definition 2.2.** Let $\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots$ be pairwise distinct elements of $X$. The following elements of $\mathcal{I}(X)$ will be called basic partial injective transformations on $X$.

- A cycle of length $k$ ($k \geq 1$), denoted by $(x_0 x_1 \cdots x_{k-1})$, is an element $\delta \in \mathcal{I}(X)$ with $\text{dom}(\delta) = \{x_0, x_1, \ldots, x_{k-1}\}$, $x_i \delta = x_{i+1}$ for all $0 \leq i < k-1$, and $x_{k-1} \delta = x_0$.

- A chain of length $k$ ($k \geq 1$), denoted by $[x_0 x_1 \cdots x_k]$, is an element $\theta \in \mathcal{I}(X)$ with $\text{dom}(\theta) = \{x_0, \ldots, x_{k-1}\}$ and $x_i \theta = x_{i+1}$ for all $0 \leq i \leq k-1$.

- A double ray, denoted by $(\cdots x_{-1} x_0 x_1 \cdots)$, is an element $\omega \in \mathcal{I}(X)$ with $\text{dom}(\omega) = \{\ldots, x_{-1}, x_0, x_1, \ldots\}$ and $x_i \omega = x_{i+1}$ for all $i$.

- A right ray, denoted by $[x_0 x_1 x_2 \cdots]$, is an element $v \in \mathcal{I}(X)$ with $\text{dom}(v) = \{x_0, x_1, x_2, \ldots\}$ and $x_i v = x_{i+1}$ for all $i \geq 0$.

- A left ray, denoted by $(\cdots x_2 x_1 x_0]$, is an element $\lambda \in \mathcal{I}(X)$ with $\text{dom}(\lambda) = \{x_1, x_2, x_3, \ldots\}$ and $x_i \lambda = x_{i-1}$ for all $i > 0$.

By a ray we will mean a double, right, or left ray.

We note the following.

- The span of a basic partial injective transformation is exhibited by the notation. For example, the span of the right ray $[1 2 3 \cdots]$ is $\{1, 2, 3, \ldots\}$.

- The left bracket in $'\eta = [x \cdots]$' indicates that $x \notin \text{im}(\eta)$, while the right bracket in $'\eta = \cdots x]'$ indicates that $x \notin \text{dom}(\eta)$. For example, for the chain $\theta = [1 2 3 4]$, $\text{dom}(\theta) = \{1, 2, 3\}$ and $\text{im}(\theta) = \{2, 3, 4\}$.

- A cycle $(x_0 x_1 \cdots x_{k-1})$ differs from the corresponding cycle in the symmetric group of permutations on $X$ in that the former is undefined for every $x \in X \setminus \{x_0, x_1, \ldots, x_{k-1}\}$, while the latter fixes every such $x$.

The following decomposition result was proved in [43, proposition 2.4].

Four notions of conjugacy for abstract semigroups 1175
Proposition 2.3. Let $\alpha \in \mathcal{I}(X)$ with $\alpha \neq 0$. Then there exist unique sets $\Delta_\alpha$ of cycles, $\Theta_\alpha$ of chains, $\Omega_\alpha$ of double rays, $\Upsilon_\alpha$ of right rays, and $\Lambda_\alpha$ of left rays such that the transformations in $\Delta_\alpha \cup \Theta_\alpha \cup \Omega_\alpha \cup \Upsilon_\alpha \cup \Lambda_\alpha$ are pairwise completely disjoint and

$$\alpha = \bigsqcup_{\delta \in \Delta_\alpha} \delta \sqcup \bigsqcup_{\theta \in \Theta_\alpha} \theta \sqcup \bigsqcup_{\omega \in \Omega_\alpha} \omega \sqcup \bigsqcup_{\upsilon \in \Upsilon_\alpha} \upsilon \sqcup \bigsqcup_{\lambda \in \Lambda_\alpha} \lambda. \quad (2.1)$$

We will call the join (2.1) the cycle-chain-ray decomposition of $\alpha$. If $\eta \in \Delta_\alpha \cup \Theta_\alpha \cup \Omega_\alpha \cup \Upsilon_\alpha \cup \Lambda_\alpha$, we will say that $\eta$ is contained in $\alpha$ (or that $\alpha$ contains $\eta$). We note the following.

- If $\alpha \in \text{Sym}(X)$, then
  $$\alpha = \bigsqcup_{\delta \in \Delta_\alpha} \delta \sqcup \bigsqcup_{\omega \in \Omega_\alpha} \omega$$

  (since $\Theta_\alpha = \Upsilon_\alpha = \Lambda_\alpha = \emptyset$), which corresponds to the usual cycle decomposition of a permutation [57, §1.3.4].

- If $\text{dom}(\alpha) = X$, then
  $$\alpha = \bigsqcup_{\delta \in \Delta_\alpha} \delta \sqcup \bigsqcup_{\omega \in \Omega_\alpha} \omega \sqcup \bigsqcup_{\upsilon \in \Upsilon_\alpha} \upsilon$$

  (since $\Theta_\alpha = A_\alpha = \emptyset$), which corresponds to the decomposition given in [49].

- If $X$ is finite, then
  $$\alpha = \bigsqcup_{\delta \in \Delta_\alpha} \delta \sqcup \bigsqcup_{\theta \in \Theta_\alpha} \theta$$

  (since $\Omega_\alpha = \Upsilon_\alpha = \Lambda_\alpha = \emptyset$), which is the decomposition given in [50, theorem 3.2].

For example, if $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, then

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 6 & 5 & 9 & 8 & 2 \end{pmatrix} \in \mathcal{I}(X)$$

written in cycle-chain decomposition (no rays since $X$ is finite) is $\alpha = (26 8) \sqcup [13] \sqcup [45 9]$. The following $\beta$ is an example of an element of $\mathcal{I}(\mathbb{Z})$ written in cycle-chain-ray decomposition:

$$\beta = (24) \sqcup [68 10] \sqcup \langle \cdots -6 -4 -2 -1 -3-5\cdots \rangle \sqcup [15 9 13\cdots] \sqcup \langle \cdots 15 11 73 \rangle.$$
Four notions of conjugacy for abstract semigroups

Definition 2.6. Let $\alpha \in \mathcal{I}(X)$. The sequence of cardinalities

\[
(\langle |\Delta^1_{\alpha}|, |\Delta^2_{\alpha}|, |\Delta^3_{\alpha}|, \ldots; |\Theta^1_{\alpha}|, |\Theta^2_{\alpha}|, |\Theta^3_{\alpha}|, \ldots; |\Omega_{\alpha}|, |\tau_{\alpha}|, |A_{\alpha}|) \]

(indexed by the elements of the ordinal $2\omega + 3$) will be called the cycle-chain-ray type of $\alpha$. This notion generalizes the cycle type of a permutation [27, p. 126]. Suppose that $\text{dom}(\alpha)$ is finite. Then $\alpha$ does not have any rays and its cycle-chain-ray type reduces to the cycle-chain type

\[
(\langle |\Delta^1_{\alpha}|, |\Delta^2_{\alpha}|, |\Delta^3_{\alpha}|, \ldots; |\Theta^1_{\alpha}|, |\Theta^2_{\alpha}|, |\Theta^3_{\alpha}|, \ldots). \]

The cycle-chain-ray type of $\alpha$ is completely determined by the form of the cycle-chain-ray decomposition of $\alpha$. The form is obtained from the decomposition by omitting each occurrence of the symbol \ '-' and replacing each element of $X$ by some generic symbol, say \ '*'. For example, $\alpha = (268) \cup [13] \cup [459]$ has the form $(\ast \ast \ast) [\ast \ast \ast] [\ast \ast \ast]$, and

\[
\beta = (24) \cup [6810] \cup \{\cdots - 6 - 4 - 2 - 1 - 3 - 5 \cdots\} \cup [15913 \cdots] \cup \{\cdots 151173\}. \]

has the form $(\ast \ast \ast) [\ast \ast \ast] [\ast \ast \ast] [\ast \ast \ast] [\ast \ast \ast] [\ast \ast \ast].$

A directed graph (or a digraph) is a pair $\Gamma = (A, R)$, where $A$ is a set (not necessarily finite and possibly empty) and $R$ is a binary relation on $A$. Any element $x \in A$ is called a vertex of $\Gamma$, and any pair $(x, y) \in R$ is called an arc of $\Gamma$. We will call a vertex $y$ terminal if there is no $x \in A$ such that $(x, y) \in R$.

Let $\Gamma_1 = (A_1, R_1)$ and $\Gamma_2 = (A_2, R_2)$ be digraphs. A mapping $\phi: A_1 \to A_2$ is called a homomorphism from $\Gamma_1$ to $\Gamma_2$ if for all $x, y \in A_1$, if $(x, y) \in R_1$, then $(x\phi, y\phi) \in R_2$.

Definition 2.7. Let $\Gamma_1 = (A_1, R_1)$ and $\Gamma_2 = (A_2, R_2)$ be digraphs. A homomorphism $\phi: A_1 \to A_2$ is called a restrictive homomorphism (or an $r$-homomorphism) from $\Gamma_1$ to $\Gamma_2$ if for every terminal vertex $x$ of $\Gamma_1$, $x\phi$ is a terminal vertex of $\Gamma_2$.

Any partial transformation $\alpha$ on a set $X$ (injective or not) can be represented by the digraph $\Gamma(\alpha) = (A_\alpha, R_\alpha)$, where $A_\alpha = \text{span}(\alpha)$ and for all $x, y \in A_\alpha$, $(x, y) \in R_\alpha$ if and only if $x \in \text{dom}(\alpha)$ and $x\alpha = y$.

The following proposition is a special case of [19, theorem 3.8].

Proposition 2.8. For all $\alpha, \beta \in \mathcal{I}(X)$, $\alpha \sim_\epsilon \beta$ if and only if there are $\phi, \psi \in \mathcal{I}(X)$ such that $\phi$ is an $r$-homomorphism from $\Gamma(\alpha)$ to $\Gamma(\beta)$ and $\psi$ is an $r$-homomorphism from $\Gamma(\beta)$ to $\Gamma(\alpha)$.

Definition 2.9. Let $\ldots, x_{-1}, x_0, x_1, \ldots$ be pairwise distinct elements of $X$. Let

\[
\delta = (x_0 \cdots x_{k-1}), \quad \theta = [x_0 x_1 \cdots x_k], \quad \omega = \langle \cdots x_{-1} x_0 x_1 \cdots \rangle, \quad \upsilon = [x_0 x_1 x_2 \cdots], \quad \lambda = \langle \cdots x_2 x_1 x_0 \rangle. \]

For any $\eta \in \{\delta, \theta, \omega, \upsilon, \lambda\}$ and any $\phi \in \mathcal{I}(X)$ such that $\text{span}(\eta) \subseteq \text{dom}(\phi)$, we define $\eta\phi^*$ to be $\eta$ in which each $x_i$ has been replaced with $x_i\phi$. For example,

\[
\delta\phi^* = (x_0\phi x_1\phi \cdots x_{k-1}\phi) \quad \text{and} \quad \lambda\phi^* = \langle \cdots x_2\phi x_1\phi x_0\phi \rangle. \]

Consider $\theta = [x_0 x_1 \cdots x_k], \omega = \langle \cdots x_{-1} x_0 x_1 \cdots \rangle, \upsilon = [x_0 x_1 x_2 \cdots], \lambda = \langle \cdots x_2 x_1 x_0 \rangle$. Then any $[x_i x_{i+1} \cdots x_k] \ (0 \leq i < k)$ is a terminal segment of $\theta$; any $[x_i x_{i+1} x_{i+2} \cdots]$ is a terminal segment of $\omega$; any $[x_i x_{i+1} x_{i+2} \cdots] \ (i \geq 0)$ is a terminal segment of $\upsilon$; and any $[x_i x_{i-1} \cdots x_0] \ (i \geq 1)$ is a terminal segment of $\lambda$.\n
The following proposition follows easily from more general results proved in [19] (see [19, propositions 4.18 and 7.3]).

**Proposition 2.10.** Let $\alpha, \beta, \phi \in I(X)$. Then $\phi$ is an $r$-homo-morphism from $\Gamma(\alpha)$ to $\Gamma(\beta)$ if and only if for all $k \geq 1$, $\delta \in \Delta^k_\alpha$, $\omega \in \Omega_\alpha$, $\nu \in \Upsilon_\alpha$ and $\lambda \in \Lambda_\alpha$:

1. $\delta \phi^* \in \Delta^k_\beta$, $\omega \phi^* \in \Omega_\beta$ and $\lambda \phi^* \in \Lambda_\beta$;

2. either there is a unique $\theta_1 \in \Theta^m_\beta$ with $m \geq k$ such that $\theta \phi^*$ is a terminal segment of $\theta_1$ or there is a unique $\lambda_1 \in \Lambda_\beta$ such that $\theta \phi^*$ is a terminal segment of $\lambda_1$;

3. either there is a unique $\nu_1 \in \Upsilon_\beta$ such that $\nu \phi^*$ is a terminal segment of $\nu_1$ or there is a unique $\omega_1 \in \Omega_\beta$ such that $\nu \phi^*$ is a terminal segment of $\omega_1$.

**Definition 2.11.** Let $\alpha, \beta, \phi \in I(X)$ such that $\phi$ is an $r$-homo-morphism from $\Gamma(\alpha)$ to $\Gamma(\beta)$. We define a mapping $h_\phi$: $\Delta_\alpha \cup \Theta_\alpha \cup \Omega_\alpha \cup \Upsilon_\alpha \cup \Lambda_\alpha \rightarrow \Delta_\beta \cup \Theta_\beta \cup \Omega_\beta \cup \Upsilon_\beta \cup \Lambda_\beta$ by

$$
\eta h_\phi = \begin{cases} 
\eta \phi^* & \text{if } \eta \in \Delta_\alpha \cup \Omega_\alpha \cup \Lambda_\alpha, \\
\theta_1 & \text{if } \eta \in \Theta_\alpha \text{ and } \eta \phi^* \text{ is a terminal segment of } \theta_1 \in \Theta_\beta, \\
\lambda_1 & \text{if } \eta \in \Theta_\alpha \text{ and } \eta \phi^* \text{ is a terminal segment of } \lambda_1 \in \Lambda_\beta, \\
v_1 & \text{if } \eta \in \Upsilon_\alpha \text{ and } \eta \phi^* \text{ is a terminal segment of } \nu_1 \in \Upsilon_\beta, \\
\omega_1 & \text{if } \eta \in \Omega_\alpha \text{ and } \eta \phi^* \text{ is a terminal segment of } \omega_1 \in \Omega_\beta.
\end{cases}
$$

Note that $h_\phi$ is well defined (by proposition 2.10) and injective (since $\phi$ is injective).

For a countable set $X$, we define two cardinal numbers that will be crucial in our characterization of $c$-conjugacy in the semigroup $I(X)$. We denote by $\mathbb{Z}_+$ the set of positive integers and by $\mathbb{N}$ the set $\mathbb{Z}_+ \cup \{0\}$.

**Definition 2.12.** Let $X$ be countable and suppose that $\alpha \in I(X)$. We define $k_\alpha \in \mathbb{N} \cup \{0\}$ by

$$
k_\alpha = \sup \{k \in \mathbb{Z}_+: \Theta^k_\alpha \neq \emptyset\}.
$$

If $\Theta^k_\alpha = \emptyset$ for every $k \in \mathbb{Z}_+$, we define $k_\alpha$ to be 0.

Suppose that $k_\alpha \in \mathbb{Z}_+$, that is, $k_\alpha$ is the largest positive integer $k$ such that $\Theta^k_\alpha \neq \emptyset$. We define $m_\alpha \in \mathbb{N}$ by

$$
m_\alpha = \max \{m \in \{1, 2, \ldots, k_\alpha\}: |\Theta^m_\alpha| = \aleph_0\}.
$$

If $\Theta^m_\alpha$ is finite for every $m \in \{1, 2, \ldots, k_\alpha\}$, we define $m_\alpha$ to be 0.

For any chain $\theta$ in $I(X)$, we denote the length of $\theta$ by $l(\theta)$. For example, if $\theta = [123]$ then $l(\theta) = 2$.

**Lemma 2.13.** Let $X$ be countably infinite and let $\alpha, \beta \in I(X)$. Suppose that $k_\alpha = k_\beta = \aleph_0$. Then there exists an injective mapping $p: \Theta_\alpha \rightarrow \Theta_\beta$ such that for every $\theta \in \Theta_\alpha$, if $\theta \in \Theta^k_\alpha$ and $\theta p \in \Theta^m_\beta$, then $m \geq k$. 

Theorem 2.14. Suppose that $X$ is countable. Let $\alpha, \beta \in \mathcal{I}(X)$. Then $\alpha \sim_c \beta$ if and only if the following conditions are satisfied:

1. $|\Delta^k_\alpha| = |\Delta^k_\beta|$ for every $k \in \mathbb{Z}_+$, $|\Omega_\alpha| = |\Omega_\beta|$ and $|A_\alpha| = |A_\beta|$;

2. If $\Omega_\alpha$ is finite, then $|\Gamma_\alpha| = |\Gamma_\beta|$; and

3. If $A_\alpha$ is finite, then
   
   (i) $k_\alpha = k_\beta$; and
   
   (ii) $k_\alpha \in \mathbb{Z}_+$, then $m_\alpha = m_\beta$ and for every $k \in \{m_\alpha + 1, \ldots, k_\alpha\}$, $|\Theta^k_\alpha| = |\Theta^k_\beta|$.

Proof. Suppose that $\sim_c$. By proposition 2.8, there exists $\phi \in \mathcal{I}(X)$ such that $\phi$ is an $r$-homomorphism from $\Gamma(\alpha)$ to $\Gamma(\beta)$. Let $k \in \mathbb{Z}_+$. Define $f_k: \Delta^k_\alpha \to \Delta^k_\beta$ by $\delta f_k = \delta h$, $g: \Omega_\alpha \to \Omega_\beta$ by $\omega g = \omega h$, and $d: A_\alpha \to A_\beta$ by $\lambda d = \lambda h$. Each of the mappings $f_k$, $g$ and $d$ is injective since $h$ is injective. Thus, $|\Delta^k_\alpha| \leq |\Delta^k_\beta|$, $|\Omega_\alpha| \leq |\Omega_\beta|$ and $|A_\alpha| \leq |A_\beta|$. Hence, (1) holds.

Suppose that $\Omega_\alpha$ is finite. Then $g: \Omega_\alpha \to \Omega_\beta$ defined above is a bijection (since $g$ is injective and $|\Omega_\alpha| = |\Omega_\beta|$). Thus, for every $\omega \in \Omega_\beta$, there is $\omega \in \Omega_\alpha$ such that $\omega g = \omega h$. Since $h$ is injective, it follows that for every $v \in \Theta^k_\beta$, $\omega h \in \Theta^k_\alpha$ (since $vh$ can not belong to $\Omega_\beta$), which implies that $|\Gamma_\alpha| \leq |\Gamma_\beta|$. By symmetry, $|\Gamma_\beta| \leq |\Gamma_\alpha|$. Hence, (2) holds.

Suppose that $\Lambda_\alpha$ is finite. Then, by the foregoing argument for $\Omega_\alpha$ and $\Gamma_\alpha$ applied to $\Lambda_\alpha$ and $\Theta_\alpha$, we conclude that $|\Theta_\alpha| = |\Theta_\beta|$ and that for every $\theta \in \Theta_\alpha$, $\theta h \subseteq \Theta_\beta$. Suppose to the contrary that $k_\alpha \neq k_\beta$. We may assume that $k_\alpha > k_\beta$, then there exists $k \in \mathbb{Z}_+$ such that $k_\alpha > k \leq k_\beta$ and $\Theta^k_\alpha \neq \emptyset$. Select some $\theta \in \Theta^k_\alpha$. Then $\theta h \subseteq \Theta^k_\beta$. But this is a contradiction since $k > k_\beta$ and $\Theta^k_\beta = \emptyset$ for every $m > k_\beta$. Thus, $k_\alpha = k_\beta$.

Let $k_\alpha \in \mathbb{Z}_+$. Suppose to the contrary that $m_\alpha \neq m_\beta$. We may assume that $m_\alpha > m_\beta$. By definition, $|\Theta^m_\alpha| = \aleph_0$. For every $\theta \in \Theta^m_\alpha$, $\theta h \subseteq \Theta_\beta^l$ for some $l$ with $k_\beta \geq l \geq m_\alpha > m_\beta$. But this is a contradiction since $h$ is injective, the set $\{\theta h: \theta \in \Theta^m_\alpha\}$ is infinite, and the set $\Theta^m_\beta \cup \cdots \cup \Theta^k_\beta$ is finite. Thus, $m_\alpha = m_\beta$.

Finally, suppose to the contrary that there exists $k \in \{m_\alpha + 1, \ldots, k_\alpha\}$ such that $|\Theta^k_\alpha| \neq |\Theta^k_\beta|$. Select the largest such $k$. We may assume that $|\Theta^k_\alpha| > |\Theta^k_\beta|$. Then $|\Theta^k_\alpha \cup \cdots \cup \Theta^m_\alpha| > |\Theta^k_\beta \cup \cdots \cup \Theta^m_\beta|$ and $h$ maps $\Theta^k_\alpha \cup \cdots \cup \Theta^m_\alpha$ to $\Theta^k_\beta \cup \cdots \cup \Theta^m_\beta$, which is a contradiction since $h$ is injective. Hence, $|\Theta^k_\alpha| = |\Theta^k_\beta|$ for every $k \in \{m_\alpha + 1, \ldots, k_\beta\}$. We have proved (3), which concludes the direct part of the proof.

Conversely, suppose that conditions (1), (2) and (3) are satisfied. We will define an injective homomorphism $\phi$ from $\Gamma(\alpha)$ to $\Gamma(\beta)$. By (1), for every $k \in \mathbb{Z}_+$, there is an injective mapping $f_k: \Delta^k_\alpha \to \Delta^k_\beta$.
Suppose that both $\Omega_\alpha$ and $A_\alpha$ are infinite. Then $|\Omega_\alpha| = |\Omega_\beta|$ and $|A_\alpha \cup \Theta_\alpha| = |A_\beta|$, and so there are injective mappings $g: \Omega_\alpha \cup \Theta_\alpha \to \Omega_\beta$ and $d: A_\alpha \cup \Theta_\alpha \to A_\beta$. For all $k \geq 0$, $\delta \in \Delta^k$, $\omega \in \Omega_\alpha$, $\lambda \in A_\alpha$, $v \in \Theta_\alpha$ and $\theta \in \Theta^k_\alpha$, we define $\phi$ on $\text{span}(\delta) \cup \text{span}(\omega) \cup \text{span}(\lambda) \cup \text{span}(v) \cup \text{span}(\theta)$ in such a way that $\delta \phi^* = \delta f_k$, $\omega \phi^* = \omega g$, $\lambda \phi^* = \lambda d$, $v \phi^*$ is a terminal segment of $vg$, and $\theta \phi^*$ is a terminal segment of $\theta d$. Note that this defines $\phi$ for every vertex $x$ in $\Gamma(\alpha)$. By the definition of $\phi$ and proposition 2.10, $\phi \in \mathcal{I}(X)$ and $\phi$ is an $r$-homomorphism from $\Gamma(\alpha)$ to $\Gamma(\beta)$.

Suppose that $\Omega_\alpha$ is finite and $A_\alpha$ is infinite. Then $|\Omega_\alpha| = |\Omega_\beta|$ by (2), and so there exists an injective mapping $j: \Theta_\alpha \to \Theta_\beta$. Let $f_k: \Delta^k \to \Delta^k_\beta$ ($k \in \mathbb{Z}_+$) and $d: A_\alpha \cup \Theta_\alpha \to A_\beta$ be the injective mappings defined in the previous paragraph. Since $|\Omega_\alpha| = |\Omega_\beta|$, there exists an injective mapping $g: \Omega_\alpha \to \Omega_\beta$. We define $\phi$ as in the previous paragraph, except that $v \phi^* = vj$ for every $v \in \Theta_\alpha$. Again, $\phi \in \mathcal{I}(X)$ and $\phi$ is an $r$-homomorphism from $\Gamma(\alpha)$ to $\Gamma(\beta)$.

Suppose that $\Omega_\alpha$ is infinite and $A_\alpha$ is finite. Then $k_\alpha = k_\beta$ by (3)(i). Let $f_k: \Delta^k \to \Delta^k_\beta$ ($k \in \mathbb{Z}_+$) and $g: \Omega_\alpha \cup \Theta_\alpha \to \Omega_\beta$ be the injective mappings defined in the case in which both $\Omega_\alpha$ and $A_\alpha$ are infinite. Since $|\Omega_\alpha| = |\Omega_\beta|$, there exists an injective mapping $d: A_\alpha \to A_\beta$.

Suppose that $k_\alpha = \aleph_0$. Then by lemma 2.13, there is an injective mapping $p: \Theta_\alpha \to \Theta_\beta$ such that for every $\theta \in \Theta_\alpha$, if $\theta \in \Theta^k_\alpha$ and $\theta p \in \Theta^m_\beta$, then $m \geq k$. We define $\phi$ as in the case in which both $\Omega_\alpha$ and $A_\alpha$ are infinite, except that $\theta \phi^*$ is a terminal segment of $\theta p$ for every $\theta \in \Theta_\alpha$. Again, $\phi \in \mathcal{I}(X)$ and $\phi$ is an $r$-homomorphism from $\Gamma(\alpha)$ to $\Gamma(\beta)$.

Suppose that $k_\alpha < \aleph_0$. If $k_\alpha = 0$, then $\Theta_\alpha = \Theta_\beta = \emptyset$. Suppose that $k_\alpha \in \mathbb{Z}_+$. Then by (3)(ii), $m_\alpha = m_\beta$ and for every $k \in \{m_\alpha + 1, \ldots, k_\alpha\}$, $|\Theta^k_\alpha| = |\Theta^k_\beta|$. Let $m = m_\alpha$. We have $|\Theta^1_\alpha \cup \cdots \cup \Theta^m_\alpha| = |\Theta^m_\beta| = \aleph_0$ and $|\Theta^k_\alpha| = |\Theta^k_\beta|$ for every $k > m$. Thus, there are injective mappings $s: \Theta^1_\alpha \cup \cdots \cup \Theta^m_\alpha \to \Theta^m_\beta$ and $t_k: \Theta^k_\alpha \to \Theta^k_\beta$ for every $k > m$. We define $\phi$ (whether $k_\alpha$ is 0 or not) as in the case when both $\Omega_\alpha$ and $A_\alpha$ are infinite, except that for every $\theta \in \Theta_\alpha$, $\theta \phi^*$ is a terminal segment of $\theta s$ if $\theta \in \Theta^k_\alpha$ with $1 \leq k \leq m$, and $\theta \phi^*$ is a terminal segment of $\theta t_k$ if $\theta \in \Theta^k_\alpha$ with $k > m$. As in the previous cases, $\phi \in \mathcal{I}(X)$ and $\phi$ is an $r$-homomorphism from $\Gamma(\alpha)$ to $\Gamma(\beta)$.

Finally, if both $\Omega_\alpha$ and $A_\alpha$ are finite, we define an injective $r$-homomorphism $\phi$ from $\Gamma(\alpha)$ to $\Gamma(\beta)$ as in the case in which $\Omega_\alpha$ is infinite and $A_\alpha$ is finite, except that $v \phi^* = vj$ for every $v \in \Theta_\alpha$, where $j: \Theta_\alpha \to \Theta_\beta$ is an injective mapping from the case in which $\Omega_\alpha$ is finite and $A_\alpha$ is infinite.

We have proved that there exists an injective $r$-homomorphism $\phi$ from $\Gamma(\alpha)$ to $\Gamma(\beta)$. By symmetry, there exists an injective $r$-homomorphism $\psi$ from $\Gamma(\beta)$ to $\Gamma(\alpha)$. Hence, $\alpha \sim_c \beta$ by proposition 2.8.

Suppose that $X$ is finite. Then for every $\alpha \in \mathcal{I}(X)$, $\Omega_\alpha = \Theta_\alpha = A_\alpha = \emptyset$, $k_\alpha \neq \aleph_0$, and $m_\alpha = 0$ if $k_\alpha \in \mathbb{Z}_+$. Thus, theorem 2.14 implies the following corollary, which generalizes the result for the symmetric group Sym(X) [27, proposition 11, p. 126].

**Corollary 2.15.** Suppose that $X$ is finite. Then for all $\alpha, \beta \in \mathcal{I}(X)$, $\alpha \sim_c \beta$ if and only if $\alpha$ and $\beta$ have the same cycle-chain type.
Remark 2.16. By corollary 2.15, for a finite set $X$, the relation $\sim_c$ on $I(X)$ can also be characterized by $\alpha \sim_c \beta$ if and only if there exists a permutation $\sigma$ on the set $X$ such that $\alpha = \sigma^{-1}\beta\sigma$.

Corollary 2.15 implies that if $X$ is finite, then in $I(X)$, $\sim_c$ is strictly included in $\sim_p$.

**Proposition 2.17.** Suppose that $X$ is finite with $|X| \geq 2$. Then $\sim_c \subset \sim_p$ in $I(X)$.

**Proof.** Let $\alpha, \beta \in I(X)$ and suppose that $\alpha \sim_c \beta$. By remark 2.16, there exists $\sigma \in \text{Sym}(X)$ such that $\sigma^{-1}\alpha\sigma = \beta$. For $\mu = \alpha\sigma$ and $\nu = \sigma^{-1}$ in $I(X)$, we have $\mu\nu = \alpha$ and $\nu\mu = \beta$, and so $\alpha \sim_p \beta$.

We have proved that $\sim_c \subseteq \sim_p$. The inclusion is strict. Select $x, y \in X$ with $x \neq y$. Then for $\alpha = [x y]$ and $\beta = 0$ in $I(X)$, $\alpha \sim_p \beta$ (since $\alpha = \alpha(y)$ and $\beta = (y)\alpha$) but $(\alpha, \beta) \notin \sim_c$ by corollary 2.15. 

Since $\sim_p \subseteq \sim_p^*$ in any semigroup, we also have $\sim_c \subset \sim_p$ in $I(X)$ when $X$ is finite. The relation $\sim_p^*$ in a finite $I(X)$ was characterized by Ganyushkin and Kormysheva [33] (see also [46, theorem 1]): for all $\alpha, \beta \in I(X)$, $\alpha \sim_p^* \beta$ if and only if $\alpha$ and $\beta$ have the same cycle type (while there are no restrictions on the chain type of $\alpha$ and $\beta$).

Regarding $\sim_{tr}$ in $I(X)$, for a finite $X$, we have $\alpha \sim_{tr} \beta$ if and only if $\alpha$ and $\beta$ have the same cycle type [59, example 8.4]. Therefore, in these semigroups, $\sim_{tr} = \sim_p^*$.

Thus, in $I(X)$ and for finite $X$, we have the following chain:

\[
\sim_o = I(X)^2 \quad \sim_p^* = \sim_{tr} \quad \sim_p \quad \sim_c
\]

Proposition 2.17 does not extend to the infinite case. Suppose that $X$ is countably infinite. Consider the following transformations in $I(X)$:

\[
\alpha = [y_0 y_1 y_2] \cup \{ \cdots x_2^1 x_1^1 \} \cup \{ \cdots x_2^2 x_1^2 \} \cup \{ \cdots x_2^3 x_1^3 \} \cup \cdots,
\]

\[
\beta = \{ \cdots z_1^1 z_0^1 \} \cup \{ \cdots z_1^2 z_0^2 \} \cup \{ \cdots z_1^3 z_0^3 \} \cup \cdots.
\]

Then $\Delta_o = \Delta_o = \Omega_o = \Omega_o = \Omega_o = \Omega_o = \emptyset$ and $A_o = A_o = \mathbb{N}_0$. Thus, $\alpha \sim_c \beta$ by theorem 2.14. By [46, lemma 4], if $\alpha$ and $\beta$ were p-conjugate, then there would exist an injective mapping $j : \Theta^2_o \to \Theta^2_o \cup \Theta^3_o \cup \Theta^3_o$. Since $\Theta^2_o = \{ [y_0 y_1 y_2] \}$ and $\Theta^3_o \cup \Theta^3_o \cup \Theta^3_o = \emptyset$, such a mapping does not exist, and so $(\alpha, \beta) \notin \sim_p$.

Now consider $\alpha = [y_0 y_1 y_2]$ and $\beta = [z_0 z_1]$ in $I(X)$. Then $\alpha \sim_p \beta$ by [46, lemma 4], but $\alpha$ and $\beta$ are not c-conjugate by theorem 2.14 (since $A_o = \emptyset$, $k_o = 2$, and $k_o = 1$). Thus $(\alpha, \beta) \notin \sim_c$.

The foregoing examples prove the following proposition.
Proposition 2.18. Suppose that $X$ is countably infinite. Then, with respect to inclusion, $\sim_p$ and $\sim_c$ are not comparable in $\mathcal{I}(X)$.

Since $\sim_p^*$ is the transitive closure of $\sim_p$ and $\sim_c$ is an equivalence relation, it follows from proposition 2.18 that if $X$ is infinitely countable, then $\sim_p^*$ and $\sim_c$ are not comparable in $\mathcal{I}(X)$ either. For a countably infinite set $X$, the relation $\sim_p^*$ in $\mathcal{I}(X)$ was characterized by Kudryavtseva and Mazorchuk [46, theorem 2].

Therefore, in $\mathcal{I}(X)$, for a countably infinite $X$, we have the following diamond:

If $X$ is infinite, the semigroup $\mathcal{I}(X)$ is not an epigroup, and hence $\sim_{tr}$ is not defined in $\mathcal{I}(X)$. However, in §4, we show that $\sim_{tr}$ can be defined, and is an equivalence relation, on the set of epigroup elements of an arbitrary semigroup. We then characterize $\sim_{tr}$ as the relation on the set of epigroup elements of $\mathcal{I}(X)$ for a countably infinite $X$ (theorem 4.12).

3. Conjugacy and Green’s relations

Green’s relations play an important role in studying semigroups. In a group, any two elements are Green’s relations play an important role in studying semigroups. In a group, any two elements are comparable in general, but there are some inclusion results for the symmetric inverse semigroup $\mathcal{I}(X)$ and its subsemigroup consisting of full injective transformations on $X$.

Fixing some terminology, for a set $X$ and $\alpha: X \rightarrow X$, the kernel of $\alpha$ is the equivalence relation on $X$ defined by $\ker(\alpha) = \{ (x,y) \in X \times X : x\alpha = y\alpha \}$.

Theorem 3.1. Let $\mathcal{G}$ be any Green relation and let $\sim \in \{ \sim_p, \sim_p^*, \sim_{tr}, \sim_c, \sim_o \}$. Then there exists a semigroup $S$ such that $\mathcal{G} \nsubseteq \sim$ and $\sim \nsubseteq \mathcal{G}$ in $S$.

Proof. Suppose that $\sim \in \{ \sim_p, \sim_p^*, \sim_{tr} \}$ and consider $S = \mathcal{I}(X)$, where $X = \{1, 2\}$. In any $\mathcal{I}(X)$, we have $\alpha \mathcal{J} \beta \iff |\dom(\alpha)| = |\dom(\beta)|$ and $\alpha \mathcal{H} \beta \iff (\dom(\alpha) = \dom(\beta))$ and $\im(\alpha) = \im(\beta))$. In any semigroup, $\mathcal{J}$ is the largest and $\mathcal{H}$ is the smallest Green relation with respect to inclusion. Let $\alpha = [12]$ and $\beta = 0$ in $\mathcal{I}(X)$. Then $\alpha \sim_p \beta$ since $\alpha = \alpha(2)$ and $\beta = (2)\alpha$, but $\alpha, \beta \notin \mathcal{J}$ since $|\dom(\alpha)| = 1$ and $|\dom(\beta)| = 0$. Hence, $\sim_p \nsubseteq \mathcal{J}$, and so $\sim_p \nsubseteq \mathcal{G}$. It follows that $\sim_p^*, \sim_{tr} \nsubseteq \mathcal{G}$ since $\sim_p \subseteq \sim_p^* \subseteq \sim_{tr}$ in any finite semigroup (see figure 1). Now let $\gamma = (1) \cup (2) = \text{id}_X$ and $\delta = (12)$ in $\mathcal{I}(X)$. Then $\gamma \mathcal{H} \delta$, but $\gamma, \delta \notin \sim_{tr}$ since, by [59, example 8.4], for $X$ finite, $\gamma \sim_{tr} \delta$ in $\mathcal{I}(X)$ if and only if $\gamma$ and $\delta$ have the same cycle type. Hence, $\mathcal{H} \nsubseteq \sim_{tr}$, and so $\mathcal{G} \nsubseteq \sim_{tr}$. It follows that $\mathcal{G} \nsubseteq \sim_p, \sim_p^*$ since $\sim_p \subseteq \sim_p^* \subseteq \sim_{tr}$.
Suppose that $\sim = \sim_c$ and consider $S = T(X)$, where $X = \{1, 2, 3\}$ and $T(X)$ is the semigroup of all full transformations on $X$. In any $T(X)$, we have $\alpha J \beta \iff |\text{im}(\alpha)| = |\text{im}(\beta)|$ and $\alpha H \beta \iff (\ker(\alpha) = \ker(\beta)$ and $\text{im}(\alpha) = \text{im}(\beta))$. Let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}$$

in $T(X)$. Then $\alpha \sim_c \beta$ by [19, corollary 6.3], but $(\alpha, \beta) \notin J$ since $|\text{im}(\alpha)| = 1$ and $|\text{im}(\beta)| = 2$. Hence, $\sim_c \not\subseteq J$, and so $\sim_c \not\subseteq G$. Now let $\gamma = (1) \cup (2) \cup (3) = \text{id}_X$ and $\delta = (1 2 3)$ in $T(X)$. Then $\gamma H\delta$, but $(\gamma, \delta) \notin \sim_c$ by [19, corollary 6.3]. Hence, $H \not\subseteq \sim_c$, and so $G \not\subseteq \sim_c$. Since $T(X)$ does not have a zero, we have $\sim_c = \sim_o$ in $T(X)$. Thus, the foregoing argument can be applied to $\sim_c$, which concludes the proof.

Although c-conjugacy is not comparable with Green’s relations in general, it is strictly included in Green’s relation $J$ in the symmetric inverse semigroup on a countable set.

**Proposition 3.2.** Suppose that $X$ is countable with $|X| \geq 2$. Then $\sim_c \subset J$ in $I(X)$.

**Proof.** Let $\alpha, \beta \in I(X)$ with $\alpha \sim_c \beta$. Suppose that $\text{dom}(\alpha)$ is infinite. Then $\text{dom}(\beta)$ is also infinite by theorem 2.14. Thus, $|\text{dom}(\alpha)| = |\text{dom}(\beta)| = \aleph_0$, which implies that $\alpha J \beta$. Suppose that $\text{dom}(\alpha)$ is finite. Then, by theorem 2.14, $\alpha$ and $\beta$ have the same cycle-chain decomposition, which implies that $|\text{dom}(\alpha)| = |\text{dom}(\beta)|$. Thus, $\alpha J \beta$ in this case also. We have proved that $\sim_c \subseteq J$. The inclusion is strict since for $x, y \in X$ with $x \neq y$, $\alpha = (x) \cup (y)$ and $\beta = (x y)$ in $I(X)$ are $J$-related but not c-conjugate.

By the proof of theorem 3.1, $\sim_p \not\subseteq J$ in $I(X)$ when $|X| \geq 2$. However, $\sim_p$ is strictly included in $J$ in the semigroup of full injective transformations on a countably infinite set $X$.

Denote by $I^*(X)$ the subsemigroup of $I(X)$ consisting of all transformations $\alpha \in I(X)$ with $\text{dom}(\alpha) = X$. If $X$ is finite, then $I^*(X) = \text{Sym}(X)$ but this is not the case for an infinite $X$. The semigroup $I^*(X)$ is universal for right cancellative semigroups with no idempotents (except possibly the identity), that is, any such semigroup can be embedded in $I^*(X)$ for some $X$ [23, lemma 1.0].

If $\alpha \in I^*(X)$, then there are no chains or left rays in the cycle-chain-ray decomposition of $\alpha$, that is, $\Theta^*_\alpha = \Lambda^*_\alpha = \emptyset$. By [42, theorem 2.3], for all $\alpha, \beta \in I^*(X)$, $\alpha J \beta$ if and only if $|X \setminus \text{im}(\alpha)| = |X \setminus \text{im}(\beta)|$. For every $\alpha \in I^*(X)$, the set $X \setminus \text{im}(\alpha)$ consists of the initial points of the right rays on $\alpha$, so $|X \setminus \text{im}(\alpha)| = |Y_\alpha|$. Thus, for all $\alpha, \beta \in I^*(X)$,

$$\alpha J \beta \iff |Y_\alpha| = |Y_\beta|.$$  \hfill (3.1)

**Lemma 3.3.** For all $\alpha, \beta \in I^*(X)$, $\alpha \sim_p \beta$ in $I^*(X)$ if and only if $\alpha \sim_p \beta$ in $I(X)$.

**Proof.** Let $\alpha, \beta \in I^*(X)$. If $\alpha \sim_p \beta$ in $I^*(X)$, then $\alpha \sim_p \beta$ in $I(X)$ since $I^*(X) \subseteq I(X)$. Conversely, suppose that $\alpha \sim_p \beta$ in $I(X)$. Then $\alpha = \mu \nu$ and $\beta = \nu \mu$ for some $\mu, \nu \in I(X)$. Since $\text{dom}(\alpha) = X$ and $\alpha = \mu \nu$, we have $\text{dom}(\mu) = X$. Similarly, $\text{dom}(\nu) = X$. Thus, $\mu, \nu \in I^*(X)$, and so $\alpha \sim_p \beta$ in $I^*(X)$. \hfill $\Box$
Proposition 3.4. Suppose that $X$ is countably infinite. Then $\sim_p = \sim_c \cap \mathcal{J}$ in $\mathcal{I}^*(X)$. Moreover, $\sim_p \subset \sim_c$ and $\sim_p \subset \mathcal{J}$.

Proof. The equality $\sim_p = \sim_c \cap \mathcal{J}$ follows immediately from (3.1), (3.2) and (3.3). Thus, $\sim_p \subseteq \sim_c$ and $\sim_p \subseteq \mathcal{J}$. Let $X = \{x^i : i, j \in \mathbb{Z}_+ \text{ with } i \geq 1\} \cup \{y_j : j \in \mathbb{Z}_+\}$. Consider

$$\alpha = [y_0 \ y_1 \ y_2 \ y_3 \cdots] \cup \{\cdots x_{-3}^3 x_0^2 x_1^2 \cdots \} \cup \{\cdots x_{-1}^2 x_0^1 x_1^1 \cdots \}$$

and

$$\beta = \{\cdots y_{-3}^3 y_0^2 y_1^2 \cdots \} \cup \{\cdots x_{-3}^3 x_0^2 x_1^2 \cdots \} \cup \{\cdots x_{-1}^2 x_0^1 x_1^1 \cdots \}$$

in $\mathcal{I}^*(X)$. Then $\Delta_\alpha = \Delta_\beta = \emptyset$, $|\Omega_\alpha| = |\Omega_\beta| = \aleph_0$, $|\mathcal{T}_\alpha| = 1$ and $|\mathcal{T}_\beta| = 0$. Thus, $\sim_p \subseteq \sim_c$ by (3.3), but $(\alpha, \beta) \notin \sim_p$ by (3.2). Hence $\sim_p \subset \sim_c$. Now, let $X = \{x, y\} \cup \{z_1, z_2, z_3, \ldots\}$ and consider

$$\gamma = (x \ y) \cup \{z_1 \ z_2 \ z_3 \cdots\} \text{ and } \delta = (x) \cup (y) \cup \{z_1 \ z_2 \ z_3 \cdots\}$$

in $\mathcal{I}^*(X)$. Then $\gamma \mathcal{J} \delta$ by (3.1), but $(\gamma, \delta) \notin \sim_p$ by (3.2). Hence, $\sim_p \subset \mathcal{J}$. 

Transformations $\alpha$ and $\beta$ from the proof of theorem 3.4 are $c$-conjugate but not $\mathcal{J}$-related. Thus, in $\mathcal{I}^*(X)$, where $|X| = \aleph_0$, $\sim_c$ is not included in $\mathcal{J}$. However, the following result holds for an arbitrary infinite set $X$.

Proposition 3.5. Suppose that $X$ is infinite. Let $\alpha, \beta \in \mathcal{I}^*(X)$ be transformations such that $\alpha$ has finitely many double chains. If $\alpha \sim_c \beta$, then $\alpha \mathcal{J} \beta$.

Proof. Suppose that $\alpha \sim_c \beta$. Then $|\Omega_\alpha| = |\Omega_\beta|$ and $|\mathcal{T}_\alpha| + |\Omega_\alpha| = |\mathcal{T}_\beta| + |\Omega_\beta|$ by (3.2). Since $|\Omega_\alpha|$ is finite, it follows that $|\mathcal{T}_\alpha| = |\mathcal{T}_\beta|$, and so $\alpha \mathcal{J} \beta$ by (3.1).

Since the semigroup $\mathcal{I}^*(X)$ does not have a zero, $\sim_c = \sim_o$ in $\mathcal{I}^*(X)$, so theorem 3.4 and proposition 3.5 also hold for $o$-conjugacy. The symmetric inverse semigroup $\mathcal{I}(X)$ does have a zero, so $o$-conjugacy is the universal relation in any $\mathcal{I}(X)$. Since $\sim_c$ and $\mathcal{J}$ are equivalence relations in any semigroup, it follows from theorem 3.4 that $\sim_p$ is transitive in $\mathcal{I}^*(X)$ for a countably infinite $X$. Thus, theorem 3.4 also holds for $\sim_p$. Trace conjugacy is not defined in $\mathcal{I}(X)$ or $\mathcal{I}^*(X)$ when $X$ is infinite.
4. Conjugacy in epigroups and epigroup elements

The principal aim of this section is to explore the relations between the four conjugacies in epigroups, the largest class for which all four notions can be defined. We will prove that in any epigroup, \( \sim_p \subseteq \sim_n \subseteq \sim_{tr} \subseteq \sim_o \) (see figure 1). We will also investigate when and which conjugacies coincide in a variety of epigroups that contains all variants of completely regular semigroups. For background information on epigroups, we refer the reader to the survey paper of Shevrin [58].

Let \( S \) be a semigroup. As noted in the introduction, an element \( a \in S \) is an epigroup element (or a group-bound element) if there exists a positive integer \( n \) such that \( a^n \) is contained in a subgroup of \( S \). The smallest \( n \) for which this is satisfied is the index of \( a \), and, for all \( k \geq n \), \( a^k \) is contained in the group \( H \)-class of \( a^n \). Let \( \text{Epi}(S) \) denote the set of all epigroup elements of \( S \) and let \( \text{Epi}_n(S) \) denote the subset of \( \text{Epi}(S) \) consisting of elements of index no more than \( n \). Thus, \( \text{Epi}_m(S) \subseteq \text{Epi}_n(S) \) for \( m \leq n \) and \( \text{Epi}(S) = \bigcup_{n \geq 1} \text{Epi}_n(S) \). The elements of \( \text{Epi}_1(S) \) are more commonly called completely regular (or group elements).

For \( a \in \text{Epi}_n(S) \), the maximum subgroup of \( S \) containing \( a^n \) is its \( H \)-class \( H \). Let \( e \) denote the identity element of \( H \). Then \( ae = ea = e \) in \( H \) and we define the pseudo-inverse \( a' \) of \( a \) by \( a' = (ae)^{-1} \), the inverse of \( ae \) in the group \( H \) [58, (2.1)]. This leads to a characterization: \( a \in \text{Epi}(S) \) if and only if there exists a positive integer \( n \) and a (necessarily unique) \( a' \in S \) such that the following hold (see [58, §2]):

\[
a'aa' = a', \quad aa' = a'a, \quad a^{n+1}a' = a^n. \tag{4.1}
\]

If \( a \) is an epigroup element, then so is \( a' \) with \( a'' = aa'a \). The element \( a'' \) is always completely regular and \( a''' = a' \). Borrowing finite semigroups standard notation (see [56,59]), for an epigroup element \( a \), we set \( a^{-} = aa'a \). We also have \( a_{-} = a'aa' \), \( (a')_{-} = (a')aa' = a^{-} \), and more generally \( a_{-} = (aa')^m = (a')^m aa' = a'^m (a')^m \) for all \( m > 0 \).

A semigroup \( S \) is said to be an epigroup if \( \text{Epi}(S) = S \). If \( \text{Epi}_1(S) = S \) (that is, if \( S \) is a union of groups), then \( S \) is called a completely regular semigroup. For \( n > 0 \), the class \( \mathcal{E}_n \) consists of all epigroups \( S \) such that \( S = \text{Epi}_n(S) \); thus, \( \mathcal{E}_1 \) is the class of completely regular semigroups.

The conclusion of the following lemma is an identity in epigroups, but here we need a version for epigroup elements. The lemma seems to be a folk result, but we include a brief proof for completeness.

**Lemma 4.1.** Let \( S \) be a semigroup and suppose that \( xy, yx \in \text{Epi}(S) \) for some \( x, y \in S \). Then \( (xy)'x = x(yx)' \).

**Proof.** Let \( n \) denote the larger of the indices of \( xy \) and \( yx \). Then

\[
(xy)^{n}x = ((xy)'y)^n (xy)^n x = ((xy)'^n x(yx)^n + 1
= ((xy)'^n x(yx)^n (yx)' = ((xy)'^n x(yx)^n x(yx)' = (xy)'x(yx)'.
\]

By a dual calculation, we also have \( x(yx)^{n} = (xy)'x(yx)' \), and thus

\[
(xy)^{n}x = x(yx)^{n}. \tag{4.2}
\]
Now we compute
\[
(\mathfrak{xy})'x = (\mathfrak{xy})'(\mathfrak{xy})^\omega x \overset{(4.2)}{=} (\mathfrak{xy})'x(\mathfrak{yx})^\omega = (\mathfrak{xy})'x(\mathfrak{yx})'
\]
\[
= (\mathfrak{yx})^\omega x(\mathfrak{yx})' \overset{(4.2)}{=} x(\mathfrak{yx})^\omega(\mathfrak{yx})' = x(\mathfrak{yx})',
\]
as claimed. \(\square\)

Throughout the rest of the section, the conditions \(gh = a^\omega\), \(hg = b^\omega\) for some \(a, b \in \text{Epi}(S)\), some \(g, h \in S^1\), will recur frequently (as, for example, in the definition of \(\sim_{tr}\)). We record two obvious consequences of this for later use:
\[
a^\omega g = gb^\omega \quad \text{and} \quad b^\omega h = ha^\omega. \quad \quad (4.3)
\]
Indeed, both sides of the first equation are equal to \(ghg\) and both sides of the second are equal to \(hgh\).

The relation \(\sim_{tr}\) is not, in general, well defined for an arbitrary semigroup \(S\), but it is a well-defined relation on \(\text{Epi}(S)\): for \(a, b \in \text{Epi}(S)\), we set
\[
a \sim_{tr} b \iff \exists g, h \in S^1 \text{ such that } ghg = g, \quad hgh = h, \quad ha^\omega g = b^\omega, \quad gb^\omega = ha^\omega. \quad \quad (4.4)
\]
In fact many of the results on \(\sim_{tr}\) do not require the whole semigroup to be an epigroup, rather only the involved elements must be epigroup elements; as an illustration, the next eight results will be proved on \(\sim_{tr}\) restricted to epigroup elements.

We start by observing that the asymmetry in our definition of \(\sim_{tr}\), which follows [59], is only for the sake of brevity.

**Lemma 4.2.** Let \(S\) be a semigroup, let \(a, b \in \text{Epi}(S)\), and suppose that there exist \(g, h \in S^1\) such that \(gh = a^\omega\) and \(hg = b^\omega\). The following are equivalent:

1. \(ha^\omega g = b^\omega\),
2. \(gb^\omega h = a^\omega\),
3. \(a^\omega g = gb^\omega\),
4. \(b^\omega h = ha^\omega\),
5. \(ha'g = b'\),
6. \(gb'h = a'\),
7. \(a'g = gb'\),
8. \(b'h = ha'\).

**Proof.**

(1) \(\Rightarrow\) (2) \(\Rightarrow\) (3) \(\Rightarrow\) (4) \(\Rightarrow\) (5) \(\Rightarrow\) (6) \(\Rightarrow\) (7) \(\Rightarrow\) (8) \(\Rightarrow\) (6)

To get (5) \(\Rightarrow\) (6) \(\Rightarrow\) (7) \(\Rightarrow\) (5) and (5) \(\Rightarrow\) (8) \(\Rightarrow\) (6), we just repeat the same calculations with \(a'\) in place of \(a\) and \(b'\) in place of \(b\). Here we use \(a'' = a'\), \(b'' = b'\), \((a')^\omega = a^\omega\) and \((b')^\omega = b^\omega\).

Showing (3) \(\iff\) (7) will conclude the proof. Assume (3). Then

\[
a'g = a'a^\omega g \overset{(4.3)}{=} a'gb^\omega = a'gb'b' = a'a''gb' = a''gb' = gb'b = gb'.
\]
This establishes (7). Conversely, if (7) holds, then since \(a'' = a'\), \(b'' = b'\), we may repeat the same calculation, replacing \(a\) with \(a'\) and \(b\) with \(b'\) to get (3). \(\square\)
Proposition 4.3. Let $S$ be a semigroup and let $a, b \in \text{Epi}(S)$. Then $a \sim_{tr} b$ if and only if $a' \sim_{tr} b'$.

Proof. This follows from lemma 4.2 together with $a''' = a'$, $b''' = b'$, $a^\omega = (a')^\omega$ and $b^\omega = (b')^\omega$.

One theme of this section is to discuss when various notions of conjugacy coincide. The following lemma will be useful later when we discuss epigroups in which all notions on the right-hand side of figure 1 coincide. Although we will not use it right away, we state it here because it is a lemma about epigroup elements (in fact, idempotents) in arbitrary semigroups.

Lemma 4.4. Let $S$ be a semigroup. Suppose that $e, f \in E(S)$ satisfy $e \leq f$ and $e \sim_{tr} f$. Then $e = f$.

Proof. Since $e \sim_{tr} f$, there exist $g, h \in S^1$ such that $ghg = g$, $hgh = h$, $gh = e$, $hg = f$ and $hec = f$ (using $e'' = e^\omega = e$ and $f'' = f^\omega = f$). We have $he = h(gh) = (hg)h = fh$, and so $e = fe = f(he) = (fh)g = heg = f$.

We now provide alternative definitions of $\sim_{tr}$ and compare trace conjugacy to $p$-conjugacy. In particular, we show that the requirement that $g$ and $h$ be mutually inverse can be omitted from the definition of $\sim_{tr}$ (see (4.4)).

Theorem 4.5. Let $S$ be a semigroup. For $a, b \in \text{Epi}(S)$, the following are equivalent:

1. $a \sim_{tr} b$;
2. $\exists g, h \in S^1 : ha''g = b''$, $gh = a^\omega$, $hg = b^\omega$;
3. $\exists g, h \in S^1 : a''g = gb''$, $gh = a^\omega$, $hg = b^\omega$;
4. $\exists g, h \in S^1 : ag = gb$, $bh = ha$, $gh = a^\omega$, $hg = b^\omega$;
5. $\exists g, h \in S^1 : hgh = h$, $ha''g = b''$, $gb''h = a''$;
6. $a'' \sim_p b''$.

The asymmetries in the statements of the theorem are explained by lemma 4.2.

Proof. The implication (1) $\implies$ (2) is trivial. Assume (2) and set $\bar{g} = a^\omega g$ and $\bar{h} = b^\omega h$. Then

\[
\bar{h}a''\bar{g} = b^\omega ha''a^\omega g = b^\omega ha''g = b^\omega b'' = b'', \quad \bar{g}h = a^\omega gb''h \overset{(4.3)}{=} a^\omega gh a^\omega = a^\omega a^\omega a^\omega = a^\omega, \\
\bar{h}g = b^\omega ha''g \overset{(4.3)}{=} b^\omega hgb'' = b^\omega b^\omega b^\omega = b^\omega, \\
\bar{g}h\bar{g} = a^\omega a^\omega g = a^\omega g = \bar{g}, \\
\bar{h}\bar{g}h = b^\omega b^\omega h = b^\omega h = \bar{h}.
\]

This proves (1). The equivalence (2) $\iff$ (3) follows from lemma 4.2.
Assume (3). Since we have already proved that (3) implies (1), we can conclude by lemma 4.2 that there are \( g, h \in S^1 \) such that \( ghg = g, hgh = h, a''g = gb'', \)
\( gh = a^\omega \) and \( hg = b^\omega \). Thus,
\[
ag = aghg = a^\omega g = a''g = gb'' = gb^\omega b = ghgb = gb.
\]
This proves half of (4) and the proof of the other part is similar. Assume (4). Then \( a''g = a^\omega ag = a^\omega gb = ghgb = gb^\omega b = gb'' \), which proves (3).

So far, we have proved \( (1) \iff (2) \iff (3) \iff (4) \). In view of lemma 4.2, (1) clearly implies (5).

Assume (5). Set \( u = gb'' \), \( v = h \). Then \( uv = gb''h = a'' \) and \( vu = hgb'' = hgha''g = hha''g = b'' \). Thus, \( a'' \sim_p b'' \), which proves (6).

Finally, assume (6). Then \( a'' = uv, b'' = vu \) for some \( u, v \in S^1 \), which implies that
\[
\begin{aligned}
a''u &= ub'' \quad \text{and} \quad b''v = va''. \\
\end{aligned}
\]
Since \( a' = a'' = (uv)' \) and \( b' = b'' = (vu)' \), lemma 4.1 implies that
\[
\begin{aligned}
a'u &= ub' \\
b'v &= va'.
\end{aligned}
\]

Now set \( g = a'u \) and \( h = b''v \). Then
\[
\begin{aligned}
gh &= a'ub''v = a'ub'b''v = a'ub'va'' = a'uvd'a'' = a''uva'' \\
&= a'uvd'a'' = a'a''a^\omega = a^\omega = a^\omega, \\
hg &= b''v'a'u = b''vub' = b'b'b' = b^\omega, \\
a''g &= a''a'u = a''ub' = ub'b' = ub'b' = a'ub'' = gb''.
\end{aligned}
\]
This proves (3) and completes the proof of the theorem. \( \square \)

The equivalence of (5) and (6) in theorem 4.5 was proved for regular semigroups by Kudryavtseva [44, corollary 6 and theorem 2]. The equivalence of (1) and (6) for finite semigroups can also be extracted from the literature since each is equivalent to the notion of conjugacy defined by having all characters coincide; see [52, theorem 2.2], [59]. A direct proof of the equivalence in the finite case is also straightforward (B. Steinberg 2014, personal communication).

If we adapt the implications \( (1) \iff (3) \iff (5) \iff (6) \) of theorem 4.5 to completely regular elements, we obtain the following.

**Corollary 4.6.** Let \( S \) be a semigroup and let \( a, b \in \text{Epi}_1(S) \). The following are equivalent:

1. \( a \sim_tr b; \)
2. \( \exists_{g, h \in S^1} ag = gb, gh = a^\omega, hg = b^\omega; \)
3. \( \exists_{g, h \in S^1} ghg = g, hgh = h, hag = b, gbh = a; \)
4. \( a \sim_p b. \)

The equivalence of (3) and (4) in corollary 4.6 was proved by Kudryavtseva [44, theorem 1(1)].
Theorem 4.7. Let \( S \) be a semigroup. Then

1. \( \sim_{tr} \) is an equivalence relation on \( \text{Epi}(S) \);
2. for all \( x \in \text{Epi}(S) \), \( x \sim_{tr} x'' \);
3. for all \( x, y \in S \) such that \( xy, yx \in \text{Epi}(S) \), \( xy \sim_{tr} yx \);
4. \( \sim_{tr} \) is the smallest equivalence relation on \( \text{Epi}(S) \) such that (2) and (3) hold.

Proof. (1) The proof of [59, proposition 8.2] can be repeated verbatim in this setting.
(2) Setting \( g = x'', h = x' \), we have \( ghg = g \), \( hgh = h \), \( hx''g = x'x''x' = x'' = (x'')'' \) and \( gh = hg = (x'')'' = x'' \).
(3) Since \( (xy)''' = xy(xy)'xy = x \cdot y(xy)'xy \) and \( (yx)''' = yx(yx)'yx = y(xy)'xy \cdot x \),
using lemma 4.1, we have \( (xy)''' \sim_p (yx)''' \), and so \( xy \sim_{tr} yx \) by theorem 4.5.
(4) Suppose that \( \theta \) is an equivalence relation on \( \text{Epi}(S) \) such that \( x\theta x'' \) for all \( x \in \text{Epi}(S) \) and \( xy\theta yx \) for all \( x, y \in S \) such that \( xy, yx \in \text{Epi}(S) \). If \( a \sim_{tr} b \) for some \( a, b \in \text{Epi}(S) \), then by theorem 4.5 there exist \( u, v \in S^1 \) such that \( a'' = uv \), \( b'' = vu \). Thus, \( a\theta a'' = uv\theta vu = b''\theta b \). Therefore, \( \sim_{tr} \subseteq \theta \), as claimed.

Now we have reached one of our goals of this section, which is to verify the inclusions on the right-hand side of figure 1.

Theorem 4.8. Let \( S \) be a semigroup. As relations on \( \text{Epi}(S) \), the following inclusions hold:

\[ \sim_p \subseteq \sim_{tr} \subseteq \sim_o . \]

Proof. The second inclusion follows from theorem 4.7. The third inclusion follows from theorem 4.5.

The transitivity of \( \sim_p \) on completely regular elements, a result first obtained by Kudryavtseva [44, corollary 4], now follows easily. We interpret it here as the equality of certain notions of conjugacy.

Corollary 4.9. Let \( S \) be a semigroup. As relations on \( \text{Epi}_1(S) \), we have \( \sim_p = \sim_{tr} = \sim_o \). In particular, \( \sim_p \) is transitive on completely regular semigroups.

Proof. This follows from corollary 4.6 and theorem 4.7(1).
Lemma 4.10. Let $\alpha \in \mathcal{I}(X)$. Then $\alpha$ is an epigroup element if and only if $\Omega_\alpha = \Upsilon_\alpha = \Lambda_\alpha = \emptyset$ and there is a positive integer $n$ such that $\Theta_\alpha^k = \emptyset$ for all $k > n$.

Recall that an idempotent $\varepsilon \in \mathcal{I}(X)$ is completely determined by its domain: for every $x \in \text{dom}(\varepsilon)$, $x\varepsilon = x$. For $A \subseteq X$, we will denote the idempotent in $\mathcal{I}(X)$ with domain $A$ by $\varepsilon_A$.

Lemma 4.11. Let $\alpha \in \text{Epi}(\mathcal{I}(X))$. Then $\alpha$ and $\alpha''$ have the same cycle type.

Proof. By lemma 4.10, $\alpha$ does not contain any rays and there is a positive integer $n$ such that $\Theta_\alpha^k = \emptyset$ for all $k > n$. Thus, $\alpha = \bigsqcup_{\delta \in \Delta_\alpha} \delta \sqcup \bigsqcup_{\theta \in \Theta_\alpha} \theta$ and its cycle-chain type is

$$\langle |\Delta_\alpha^1|, |\Delta_\alpha^2|, \ldots; |\Theta_\alpha^1|, |\Theta_\alpha^2|, \ldots; |\Theta_\alpha^n| \rangle.$$  

Then $\alpha^n$ is in a group $\mathcal{H}$-class of $\mathcal{I}(X)$ whose identity is the idempotent $\alpha^\omega = \varepsilon_A$, where $A = \bigcup\{\text{dom}(\delta) : \delta \in \Delta_\alpha\}$. Thus,

$$\alpha'' = (\alpha')^{-1} = ((\alpha^\omega)^{-1})^{-1} = \alpha \varepsilon_A = \bigsqcup_{\delta \in \Delta_\alpha} \delta,$$

and the result follows.

Theorem 4.12. Let $X$ be a countably infinite set. Then for all $\alpha, \beta \in \text{Epi}(\mathcal{I}(X))$,

$\alpha \sim_{\text{tr}} \beta$ if and only if $\alpha$ and $\beta$ have the same cycle type.

Proof. Let $\alpha, \beta \in \text{Epi}(\mathcal{I}(X))$. The following statements are true:

(a) $\alpha \sim_{\text{tr}} \beta$ if and only if $\alpha'' \sim_p \beta''$ (by theorem 4.5);

(b) $\alpha'' \sim_p \beta''$ if and only if $\alpha''$ and $\beta''$ have the same cycle type (by [46, lemma 4]);

(c) $\alpha$ and $\alpha''$ have the same cycle type, and the same is true for $\beta$ and $\beta''$ (by lemma 4.11).

The result clearly follows from (a)–(c).

Now we would like to exhibit a larger class of semigroups in which $\sim_p = \sim_p^\ast \subset \sim_{\text{tr}}$, where the last inclusion is proper. At this point we will no longer work with epigroup elements in arbitrary semigroups, but rather with epigroups. In particular, this means we will change our point of view about the role of pseudo-inverses.

Following Petrich and Reilly [55] for completely regular semigroups and Shevrin [58] for epigroups, it is now customary to view an epigroup $(S, \cdot)$ as a unary semigroup $(S, \cdot')$, where $x \mapsto x'$ is the map sending each element to its pseudo-inverse. By a variety of epigroups, we will mean a class of epigroups viewed as a variety of unary semigroups in the usual sense: closed under unary subsemigroups, homomorphic images and direct products. The class of all epigroups is not a variety because it is not closed under arbitrary direct products, but the following identities, all of
Four notions of conjugacy for abstract semigroups

which we have already seen, hold in epigroups:

\[ x'xx' = x', \quad (4.7) \]
\[ xx' = x'x, \quad (4.8) \]
\[ xx'x = x'', \quad (4.9) \]
\[ (xy)'x = x(yx)', \quad (4.10) \]
\[ x''' = x'. \quad (4.11) \]

We note that the class \( E_n \) (that is, the epigroups \( S \) such that \( S \in \text{Epi}_n(S) \)) is a variety of epigroups axiomatized [58, proposition 2.10] by associativity, (4.7), (4.8) and

\[ x^{n+1}x' = x^n. \quad (4.12) \]

Let \( W \) be the class of semigroups \( S \) such that the subsemigroup \( S^2 := \{ab \mid a, b \in S\} \) is completely regular. This class contains all completely regular semigroups, all null semigroups (semigroups satisfying the identity \( xy = uv \)) and, more generally, all variants of completely regular semigroups. (We will recall the definition of a variant of a semigroup later in the section.) We first prove that \( W \) is a variety of epigroups.

**Proposition 4.13.** Any semigroup in \( W \) is an epigroup. The following proper inclusions of epigroup varieties hold: \( E_1 \subset W \subset E_2 \).

**Proof.** For \( S \in W \), every \( a \in S \) satisfies \( a^2 \in \text{Epi}_1(S) \), that is, \( a^2 \) lies in a subgroup of \( S \). Thus, \( S \in \mathcal{E}_2 \), which both verifies the first assertion and the second inclusion. The second inclusion is also proper, as can be seen by considering a three-element monoid \( S = \{e, a, b\} \), where \( e \) is the identity element and \( \{a, b\} \) is a null subsemigroup with \( xy = a \) for all \( x, y \in \{a, b\} \). Then \( S \) is clearly in \( \mathcal{E}_2 \), but \( ea = a \) is not completely regular, so \( S \) is not in \( W \).

Finally, the first inclusion is obvious from the definition of \( W \), and since every null semigroup is in \( W \), the inclusion is also proper.

The following result characterizes \( W \) in terms of pseudo-inverses.

**Proposition 4.14.** Let \( S \) be a semigroup. Then \( S \) is in \( W \) if and only if \( S \) is an epigroup in \( E_2 \) satisfying the additional identity

\[ (xy)'' = xy. \quad (4.13) \]

**Proof.** If \( S \) is in \( W \), then \( S \) is in \( \mathcal{E}_2 \) by proposition 4.13. We have already noted that the completely regular elements \( a \) in an epigroup are characterized by the equation \( a'' = a \), so (4.13) holds by the definition of \( W \).

Conversely, if \( S \) is an epigroup in \( \mathcal{E}_2 \) satisfying (4.13), then combining (4.9) and (4.13) shows that each \( xy \) lies in a subgroup of \( S \).

**Theorem 4.15.** Let \( S \) be an epigroup in \( W \). Then \( \sim_p = \sim_p^* \subset \sim_{tr} \).

**Proof.** Suppose that \( a \sim_p b \) and \( b \sim_p c \), that is, \( a = uv, b = vu = xy \) and \( c = yx \) for some \( u, v, x, y \in S^1 \). If \( a = b \) or \( b = c \), then clearly \( a \sim_p c \). Otherwise,
$a, b, c \in S^2 \subseteq \text{Epi}_1(S)$, so $a \sim_p c$ by corollary 4.9. Thus, $\sim_p$ is transitive, and so $\sim_p = \sim_p^*$.

To see that the inclusion $\sim_p^* \subset \sim_{tr}$ is proper, consider a two-element null semigroup $S = \{a, b\}$ with $xy = a$ for all $x, y \in S$. Then $a' = b' = a$. As already noted, null semigroups are in $\mathcal{W}$. Since $a'' = b''$, we have $a \sim_{tr} b$ (by theorem 4.5), but $a$ and $b$ are evidently not $p$-conjugate.

To show that the variety $\mathcal{W}$ is of more than just formal interest, we will now show that it contains all variants of completely regular semigroups. First, we recall the notion of variant.

Let $S$ be a semigroup and let $a \in S$. Then the pair $(S, \circ)$, where $\circ$ is a binary operation on $S$ defined by $x \circ y = xay$, is called the variant of $S$ at $a$. Variants of semigroups are semigroups. Besides giving a construction of new semigroups from old ones, variants also provide an interesting interpretation of Nambooripad’s natural partial order on regular semigroups [53]. (See [37, 38] and also [41, 45].)

Since $\mathcal{W}$ can be viewed as a variety of unary semigroups, we will also find it helpful to introduce unary variants. Let $(S, ., ')$ be a unary semigroup, and fix $a \in S$. Then the unary semigroup $(S, \circ, .^*)$, where $(S, \circ)$ is the variant of $S$ at $a$ and $x^* = (xa)'x(ax)'$, is called the unary variant of $S$ at $a$. Since it will always be clear from the context when we mean a unary variant, we will usually drop the word ‘unary’ when referring to variants.

Variants of completely regular semigroups are not, in general, completely regular.

**Example 4.16.** Let $S = \{0, 1\}$ be the 2-chain. Since $S$ is a semilattice, it is certainly completely regular. However, its variant at 0 is the null semigroup, which is not even regular.

**Theorem 4.17.** Let $(S, ., ')$ be a completely regular semigroup, and fix $a \in S$. Let $(S, \circ, .^*)$ be the variant of $S$ at $a$, that is,

$$x \circ y = xay \quad \text{and} \quad x^* = (xa)'x(ax)'$$

for all $x, y \in S$. Then $(S, \circ, .^*)$ is in $\mathcal{W}$.

**Proof.** All we need to show is that $S \circ S$ is a subsemigroup of $(S, \circ, .^*)$ that is completely regular. We will first prove that $(S, \circ, .^*)$ is an epigroup in $\mathcal{E}_2$, which implies that $S \circ S$ is also an epigroup, and then show that $S \circ S$ satisfies the identity (4.12).

We begin by proving that $x^* \circ x \circ x^* = x^*$. Indeed, we have

$$x^* \circ x \circ x^* = (xa)'x(ax)'ax^a x \underbrace{ax}_{ax} x(ax)'$$

$$\stackrel{(4.10)}{=} (xa)'x(ax)'ax^a x(ax)'$$

$$\stackrel{(4.7)}{=} (xa)' x ax (ax)'$$

$$\stackrel{(4.7)}{=} (xa)' x(ax)'$$

$$= x^*.$$
Then we also have $x \circ x^* = x^* \circ x$ since

\[
x \circ x^* = x(ax)^' = (xa)^' x(ax)^' = (xa)^' x(ax)^' x = x^* \circ x.
\]

Finally, $x^3 \circ x^k = x^2$ since

\[
x \circ x \circ x^* = xaxaxa(xa)^' x(ax)^' = xaxaxa(xa)^' x(ax)^' (S, \cdot, ') \text{ is completely regular}
\]

\[
= xax (S, \cdot, ') \text{ is completely regular}
\]

\[
= x \circ x,
\]

and so $(S, \circ, \cdot^*)$ is an epigroup of $E_2$.

Given an element $x \circ y$ of $S \circ S$ we will show that $(x \circ y)^2 \circ (x \circ y)^* = x \circ y$. Indeed,

\[
(x^*)^* = (x \circ y) \circ (x \circ y) \circ (x \circ y)^*
\]

\[
= xayaxay(axay)^' xay(axay)^' y\]

\[
= xay(axay)^' xay(axay)^' y (S, \cdot, ') \text{ is completely regular),}
\]

\[
= xay(axay)^' xay(axay)^' y (S, \cdot, ') \text{ is completely regular),}
\]

\[
= xay (S, \cdot, ') \text{ is completely regular)
\]

\[
= x \circ y.
\]

**Corollary 4.18.** The relation $\sim_p$ is transitive in every variant of a completely regular semigroup.

From the preceding result, it is natural to conjecture that if $p$-conjugacy is transitive in some epigroup, then perhaps the relation is transitive in all of the epigroup’s variants. The following example shows this is not true even for regular epigroups from $E_2$. 
Example 4.19. Let $S$ be the following semigroup, which is both regular and in $\mathcal{E}_2$:

\[
\begin{array}{cccc}
\cdot & 0 & 1 & 2 & 3 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 2 \\
2 & 0 & 1 & 2 & 1 & 2 \\
3 & 0 & 0 & 0 & 3 & 4 \\
4 & 0 & 3 & 4 & 3 & 4 \\
\end{array}
\]

Let $T$ be the variant of $S$ at 1:

\[
\begin{array}{cccc}
\circ & 0 & 1 & 2 & 3 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 1 & 2 \\
3 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 3 & 4 \\
\end{array}
\]

In $S$, $p$-conjugacy is an equivalence relation that induces the partition $\{\{0, 1\}, \{2, 3, 4\}\}$. However, in $T$, $p$-conjugacy is not transitive because $2 \sim_p 0$ and $0 \sim_p 1$, but $(2, 1) \notin \sim_p$.

Next we will consider epigroups in which all notions of conjugacy on the right-hand side of figure 1 coincide. An obvious necessary condition is that $\sim^*_p = \sim_p$, that is, that $\sim_p$ must be transitive. Another necessary condition follows from just the assumed equality of $\sim^*_r$ and $\sim_o$.

Proposition 4.20. Let $S$ be an epigroup in which $\sim^*_r = \sim_o$. Then $E(S)$ is an antichain.

Proof. Suppose that $e, f \in E(S)$ satisfy $e \preceq f$. Setting $g = h = e$, we have $eg = ee = e = ef = gf$ and $fh = fe = e = ee = he$. Thus, $e \sim_o f$. Since $\sim^*_r = \sim_o$, we have $e \sim^*_r f$, and so $e = f$ by lemma 4.4. It follows that $E(S)$ is an antichain. \qed

A natural class of semigroups in which $\sim_p$ is transitive and idempotents form an antichain is the class of completely simple semigroups. A semigroup $S$ is simple if it has no proper ideals [40, p. 66]. A simple semigroup $S$ is called completely simple if it has a primitive idempotent (that is, an idempotent that is minimal with respect to the partial order $\preceq$) [40, p. 77]. This turns out to be equivalent to every idempotent in $S$ being primitive, that is, the idempotents in $S$ forming an antichain.

A completely simple semigroup can be identified with its Rees matrix representation $\mathcal{M}(G; I, J; P)$, with elements from $I \times G \times \Lambda$, where $I$ and $\Lambda$ are non-empty sets, $G$ is a group, and multiplication is defined by

\[
(i, a, \lambda)(j, b, \mu) = (i, ap_{\lambda j}b, \mu),
\]

where $P = (p_{\lambda j})$ is a $\Lambda \times I$ matrix with entries in $G$ [40, theorem 3.3.1]. From this characterization, it is clear that every element of a completely simple semigroup is contained in a subgroup, that is, completely simple semigroups are completely regular.
Theorem 4.21. In completely simple semigroups, we have $\sim_p = \sim^*_p = \sim_{tr} = \sim_o$.

Proof. By theorem 4.8, it suffices to prove that $\sim_o \subseteq \sim_p$. To do this, we identify $S$ with its Rees matrix representation $S = M(G; I, J; P)$. Let $(i, a, \lambda), (j, b, \mu) \in S$ and suppose that $(i, a, \lambda) \sim_o (j, b, \mu)$. Then, by (4.14), there exist $(i, c, \mu), (j, d, \lambda) \in S$ such that

$$(i, a, \lambda)(i, c, \mu) = (i, c, \mu)(j, b, \mu) \quad \text{and} \quad (j, b, \mu)(j, d, \lambda) = (j, d, \lambda)(i, a, \lambda),$$

which implies that

$$ap_{\lambda,c} = cp_{\mu,j}b \quad \text{and} \quad bp_{\mu,j}d = dp_{\lambda,i}a. \quad (4.15)$$

Consider $x = (dp_{\lambda,i})^{-1}b$ and $y = d$. Then, by (4.15),

$$(i, x, \mu)(j, y, \lambda) = (i, xp_{\mu,j}y, \lambda) = (i, (dp_{\lambda,i})^{-1}bp_{\mu,j}d, \lambda) = (i, (dp_{\lambda,i})^{-1}dp_{\lambda,i}a, \lambda) = (i, a, \lambda),$$

$$(j, y, \lambda)(i, x, \mu) = (j, yp_{\lambda,i}x, \mu) = (j, dp_{\lambda,i}(dp_{\lambda,i})^{-1}b, \mu) = (j, b, \mu),$$

which implies that $(i, a, \lambda) \sim_p (j, b, \mu)$. \hfill $\square$

Theorem 4.22. Let $S$ be a regular epigroup without zero. The following are equivalent:

1. $\sim_p = \sim_o$ in $S$,
2. $S$ is completely simple.

Proof. Suppose that $\sim_o = \sim_p$ in $S$. Since $S$ is an epigroup, we also have $\sim_{tr} = \sim_o$, and thus $E(S)$ is an antichain by proposition 4.20, that is, every idempotent in $S$ is primitive. Since $S$ is also regular, we conclude that $S$ is completely simple [40, theorem 3.3.3]. The converse follows from theorem 4.21. \hfill $\square$

Theorem 4.23. Let $S$ be an epigroup in $W$ without zero. The following are equivalent:

1. $\sim_p = \sim_o$ in $S$,
2. $S$ is completely simple.

Proof. Suppose that $\sim_o = \sim_p$ in $S$. Arguing as in the preceding proof, we have that every idempotent in $S$ is primitive. Next, since $\sim_o = \sim_p$, we have $x \sim_p x''$, by theorem 4.7 and theorem 4.8. Hence, there exist $u, v \in S^1$ such that $x'' = uv$ and $x = vu$. But then $x'' = (vu)v = vu = x$, using (4.13) (since $S$ is in $W$). Therefore, $x$ is completely regular. It follows that $S$ is completely regular. Finally, since $S$ is completely regular and every idempotent is primitive in $S$, it follows that $S$ is completely simple [40, theorem 3.3.3]. The converse follows from theorem 4.21. \hfill $\square$

We now give two examples of inverse epigroups (epigroups that are also inverse semigroups) to illustrate some possible relations between the conjugacies in the variety $E_2$. 
Example 4.24. In a semigroup from the epigroup variety $\mathcal{E}_2$, we can have $\sim_p \subset \sim_p^\ast = \sim_{tr} \subset \sim_c = \sim_o$, where the inclusions are strict. (In particular, $\sim_p$ need not be transitive in a semigroup from $\mathcal{E}_2$.) Consider, for example, the inverse semigroup $S$ given by the following multiplication table:

| . | 0   | 1   | 2 | 3 | 4 | 5 |
|---|-----|-----|---|---|---|---|
| 0 | 0   | 0   | 3 | 3 | 3 | 3 |
| 1 | 0   | 1   | 0 | 3 | 4 | 3 |
| 2 | 0   | 0   | 2 | 3 | 3 | 5 |
| 3 | 3   | 3   | 3 | 0 | 0 | 0 |
| 4 | 3   | 3   | 4 | 0 | 0 | 1 |
| 5 | 3   | 5   | 3 | 0 | 2 | 0 |

This is an $E$-unitary inverse semigroup. (An inverse semigroup $S$ is $E$-unitary if for all $e, a \in S$, if $e$ and $ea$ are idempotents, then $a$ is an idempotent.) This semigroup is in $\mathcal{E}_2$ since every entry on the main diagonal of the table is an idempotent, but it is not Clifford (that is, both completely regular and inverse), not even in $W$, which can be checked directly, but also follows because p-conjugacy in $S$ is not transitive. Indeed, we have $4 \sim_p 3$ (since $4 = 1 \cdot 4$ and $3 = 4 \cdot 1$) and $3 \sim_p 5$ (since $3 = 1 \cdot 5$ and $5 = 5 \cdot 1$), but there are no $x, y$ such that $4 = xy$ and $5 = yx$. It is straightforward to check that $\sim_p$ is the symmetric and reflexive closure of $\{(1, 2), (3, 4), (3, 5)\}$, that $\sim_p^\ast = \sim_{tr}$, and that $\sim_c = \sim_o$ has equivalence classes $\{0, 1, 2\}$ and $\{3, 4, 5\}$. Thus, we have the claimed strict inclusions.

Example 4.25. There are epigroups in $\mathcal{E}_2$ but not $W$ in which p-conjugacy is transitive. Consider, for example, the following inverse semigroup $S$, which is an ideal extension of the group $\{1, a\}$ by the Brandt semigroup $\{0, b, c, e, f\}$ [40, p. 152]:

| . | 1   | a   | 0   | b   | c   | e   | f   |
|---|-----|-----|-----|-----|-----|-----|-----|
| 1 | 1   | a   | 0   | b   | c   | e   | f   |
| a | a   | 1   | 0   | e   | f   | b   | c   |
| 0 | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| b | b   | f   | 0   | 0   | f   | b   | 0   |
| c | c   | e   | 0   | e   | 0   | c   | 0   |
| e | e   | c   | 0   | c   | e   | 0   | 0   |
| f | f   | b   | 0   | b   | 0   | 0   | f   |

The semigroup $S$ is an $E^\ast$-unitary inverse monoid. (An inverse semigroup $S$ with zero is $E^\ast$-unitary if for all $e, a \in S$, if $e$ and $ea$ are non-zero idempotents, then $a$ is an idempotent.) Again, $S$ is in $\mathcal{E}_2$ since every entry on the main diagonal of the table is an idempotent, but it is not Clifford because neither $b$ nor $c$ are completely regular, not even in $W$ because, for instance, $a \cdot e = b$.

However, this time, $\sim_p$ is an equivalence relation, with the equivalence classes $\{1\}, \{a\}, \{0, b, c\}$ and $\{e, f\}$. Also $\sim_p = \sim_{tr}$. This semigroup, incidentally, is the smallest example of an inverse semigroup that is not completely regular but in which p-conjugacy is transitive. Note that $\sim_c$ has equivalence classes $\{1\}, \{a\}, \{0\}, \{b, c\}$ and $\{e, f\}$, and therefore $\sim_c \subset \sim_p$. 
Let us now turn our attention to semigroups with zero. A semigroup $S$ with zero is 0-simple if $S^2 \neq \{0\}$ and $\{0\}$ and $S$ are the only ideals of $S$ [40, p. 66]. A 0-simple semigroup $S$ is called completely 0-simple if it contains a primitive idempotent [40, p. 70]. A completely 0-simple semigroup $S$ can be identified with its Rees matrix representation $M^0(G; I, \Lambda; P)$, with elements from $(I \times G \times \Lambda) \cup \{0\}$, where $I$ and $\Lambda$ are non-empty sets, $G$ is a group, and multiplication is defined by $(i, a, \lambda)(j, b, \mu) = (i, ap_{ij}b, \mu)$ if $p_{ij} \neq 0$, $(i, a, \lambda)(j, b, \mu) = 0$ if $p_{ij} = 0$, and $(i, a, \lambda)0 = 0(i, a, \lambda) = 0$, where $P = (p_{ij})$ is a $\Lambda \times I$ matrix with entries in $G \cup \{0\}$ such that no row or column of $P$ consists entirely of zeros [40, theorem 3.2.3].

Theorem 4.23 does not remain true if $\sim_o$ is replaced with $\sim_c$ and ‘completely simple’ with ‘completely 0-simple.’ Indeed, suppose that in the matrix $P$, we have $p_{ij} \neq 0$ and $p_{ji} = 0$. Let $a, b \in G$. Then $(i, a, \lambda)(j, b, \mu) = (i, ap_{ij}b, \mu) \neq 0$ and $(j, b, \mu)(i, a, \lambda) = 0$. Thus, $(i, ap_{ij}b, \mu) \sim_p 0$, while $(i, ap_{ij}b, \mu)$ and 0 are not $\sim_c$-related since in every semigroup with zero the c-conjugacy class of 0 is $\{0\}$ [19, lemma 2.3]. Hence, $\sim_c \neq \sim_p$ in completely 0-simple semigroups.

We have, however, the following results.

**Proposition 4.26.** For a completely 0-simple semigroup $M^0(G; I, \Lambda; P)$, we have $\sim_c \subseteq \sim_p$. Moreover, $\sim_c = \sim_p$ if and only if the sandwich matrix $P$ has only non-zero elements.

**Proof.** Let $(i, a, \lambda), (j, b, \mu)$ be non-zero elements of $S = M^0(G; I, \Lambda; P)$ such that $(i, a, \lambda) \sim_c (j, b, \mu)$. By (1.3), there exist non-zero elements $(i, c, \mu),(j, d, \lambda)$ with $p_{ci} \neq 0$, $p_{dj} \neq 0$ such that

$$
(i, a, \lambda)(i, c, \mu) = (i, c, \mu)(j, b, \mu) \quad \text{and} \quad (j, b, \mu)(j, d, \lambda) = (j, d, \lambda)(i, a, \lambda).
$$

Using the same arguments as in the proof of theorem 4.21, we obtain $(i, a, \lambda) \sim_p (j, b, \mu)$.

Now, suppose first that $\sim_c = \sim_p$. By the argument showing that $\sim_c \neq \sim_p$ in completely 0-simple semigroups (see the paragraph above this proposition), we can conclude that whenever $p_{ji} = 0$, for some $i \in I$ and $\mu \in \Lambda$, then $p_{ij} = 0$, for all $j \in I$ and $\lambda \in \Lambda$.

Conversely, suppose that the sandwich matrix $P$ has only non-zero elements. Then we have that $S$ is isomorphic to $T^o$, where $T$ is the completely simple semigroup $M(G; I, \Lambda; P)$. But then, by theorem 4.21, $\sim_p^T = \sim_o^T$ in $T$. Since $S$ has no zero divisors, we have $\sim_c^S = \{(0,0)\} \cup \sim_o^T$ and $\{0\}$ is one of the p-conjugacy classes of $S$. Therefore, $\sim_c^S = \{(0,0)\} \cup \sim_p^T = \sim_p^S$. □

**Lemma 4.27.** Let $S$ be an epigroup with zero and suppose that $\sim_c \subseteq \sim_{tr}$. Then $E(S) \setminus \{0\}$ is an antichain.

**Proof.** Suppose that $e, f \in E(S)$ with $0 \neq e \leq f$. Since $e$ is in both $P^1(e)$ and $P^1(f)$, $ee = e = ef$ and $fe = e = ee$, we have $e \sim_c f$. Since $\sim_c \subseteq \sim_{tr}$, we have $e \sim_{tr} f$, and so $e = f$ by lemma 4.4. It follows that $E(S) \setminus \{0\}$ is an antichain. □

A semigroup $S$ with zero is called a 0-direct union of completely 0-simple semigroups if $S = \bigcup_{i \in I} S_i$, where each $S_i$ is a completely 0-simple semigroup and $S_i \cap S_j = S_iS_j = \{0\}$ if $i \neq j$ [40, pp. 79–80].
THEOREM 4.28. Let $S$ be a regular epigroup with zero. The following are equivalent:

(1) $\sim_c \subseteq \sim_p$,

(2) $\sim_c \subseteq \sim_{tr}$,

(3) $S$ is a 0-direct union of completely 0-simple semigroups.

Proof. (1) $\implies$ (2) is true because $\sim_p \subseteq \sim_{tr}$ in any epigroup.

Assume (2). By lemma 4.27, every non-zero idempotent is primitive. Since $S$ is also regular, then by [40, theorem 3.3.4], we obtain (3).

Now assume (3), that is, $S = \bigcup_{i \in I} S_i$, where each $S_i$ is a completely 0-simple semigroup and $S_i \cap S_j = S_i S_j = \{0\}$ if $i \neq j$.

We will show that if $a \sim_c b$ in $S$, then both $a$ and $b$ belong to the same subsemigroup $S_i$, for some $i \in I$, and that $a \sim_{S_i}^c b$ in $S_i$. Since the $c$-conjugacy class of 0 is $\{0\}$, we may assume that $a, b \neq 0$. Suppose that $a \sim_c b$ in $S$. By (1.3) there exist non-zero elements $g, h \in S$, with $ag \neq 0$ and $bh \neq 0$, satisfying $ag = gb$ and $bh = ha$. Thus, since $S_i S_j = \{0\}$, for all $i, j \in I$, we conclude that $a, b, g, h \in S_i$ for some $i \in I$. But then $a \sim_{S_i}^c b$ in $S_i$.

So any two $c$-conjugate elements in $S$ are $c$-conjugate elements in a completely 0-simple semigroup $S_i$. Hence, by proposition 4.26, any two $c$-conjugate elements are also $p$-conjugate in $S_i$, for some $i \in I$. Since $p$-conjugate elements in $S_i$ are also $p$-conjugate in $S$, we have $\sim_c \subseteq \sim_p$ in $S$. \hfill $\square$

In the last part of this section, we will examine $o$-conjugacy in all epigroups and $c$-conjugacy in the variety $W$.

If two elements $a, b$ with $a \neq b$ of a semigroup are $o$-conjugate, say, $ag = gb$ and $bh = ha$, then in general there is no apparent connection between $g$ and $h$ beyond these two equations. In a group, of course, one may assume without loss of generality that $h = g^{-1}$. The next result shows that in epigroups we may similarly restrict the choice of conjugators for $\sim_o$ without loss of generality.

THEOREM 4.29. Let $S$ be an epigroup and suppose that $a \sim_o b$ for some $a, b \in S$. Then there exist mutually inverse $g, h \in S^1$ such that $ag = gb$ and $bh = ha$.

Proof. Since $a \sim_o b$, there exist $c, d \in S^1$ such that $ac = cb$ and $bd = da$. These imply that $acd = cda$, a fact we will use without comment in what follows. Set

$$h = da(cda)^t \quad \text{and} \quad g = chc.$$ \hfill (4.16)

Then $hch = da(cda)^t cda(cda)^t (\overset{4.7}{=} da(cda)^t = h$. Thus, $h$ is regular and so an inverse of $h$ is given by $chc = g$, that is, $g$ and $h$ are mutually inverse as claimed.

Now we check that $g$ and $h$ are conjugators of $a$ and $b$. First, we have

$$bh = bd a(cda)^t = da a(cda)^t (\overset{4.10}{=} da(cda)^t a = ha.$$ \hfill (4.10)

Then we use this in the third step of the following calculation:

$$ag = ac \quad hc = c \quad bh \quad c = ch \quad ac = cheb = gb.$$

This completes the proof. \hfill $\square$
Example 4.30. In the completely regular case, it is not possible, in general, to choose the mutually inverse $g$ and $h$ of theorem 4.29 to be $g$ and $g' = g^{-1}$, the commuting inverse of $g$. To see this, consider a two-element left zero semigroup \{a, b\}. Since $aba = a$, $bab = b$, $a$ and $b$ are mutually inverse. We also have $aa = ab$ and $bb = ba$, so $a \sim b$. However, $a' = a$ and $b' = b$, so we cannot have both $ax = xb$ and $bx' = x'a$ for either $x = a$ or $x = b$.

Now we consider $c$-conjugacy. We do not know if there is a full analogue of theorem 4.29 for all epigroups, but there is one for our variety $W$. First we need the following result.

Lemma 4.31. Let $S$ be an epigroup with zero in $W$. If $st = 0$ for some $s, t \in S^1$, then $sxt = 0$ for all $x \in S^1$.

Proof. First,

$$ts \overset{(4.13)}{=} (ts)'' \overset{(4.9)}{=} ts(ts)'ts \overset{(4.8)}{=} ts(t_s)'s = 0.$$ 

Then

$$sxt \overset{(4.13)}{=} (sxt)'' \overset{(4.9)}{=} sxt(sxt)'' \overset{(4.8)}{=} sxxt_xt(sxt)' = 0. \quad \square$$

Theorem 4.32. Let $S$ be an epigroup with zero in $W$ and suppose that $a \sim c b$ for some $a, b \in S$. Then there exist mutually inverse $g \in P^1(a)$, $h \in P^1(b)$ such that $ag = gb$ and $bh = ha$.

Proof. We may assume that $a, b \neq 0$. Since $a \sim c b$, there exist $c \in P^1(a)$, $d \in P^1(b)$ such that $ac = cb$ and $bd = da$. As before, we will use $acd = cda$ without comment.

Define $h$ and $g$ by (4.16). By the proof of theorem 4.29, we have that $g, h$ are mutually inverse and satisfy $ag = gb$, $bh = ha$. What remains is to show that $h \in P^1(b)$ and $g \in P^1(a)$.

Suppose that $(mb)h = 0$ for some $m \in S^1$. We wish to prove $mb = 0$. By lemma 4.31, $mxbh = 0$ for all $x \in S^1$, and so, in particular, we have $mcbh = 0$. Thus,

$$0 = mc \overset{b}{\sim} mcha \overset{c}{\sim} mcd(a)c'a.$$ 

Multiply both sides on the right by $cd$ to get

$$0 = mcd(a)c'a \overset{ac}{\sim} mcd(a)d'a \overset{c}{\sim} mdc'da \overset{mc}{\sim} mcd.$$

Now, since $d \in P^1(b)$, the result of this last calculation implies that $mcb = 0$. Thus, $0 = mcb = mac$. Since $c \in P^1(a)$, we conclude that $ma = 0$. Using lemma 4.31 once again, $mxa = 0$ for all $x \in S^1$. In particular, we have $0 = mda = mbd$. Since $d \in P^1(b)$, we obtain $mb = 0$ as claimed.

Finally, suppose that $(ma)g = 0$ for some $m \in S^1$. We wish to prove $ma = 0$. Thus,

$$0 = mag \overset{m}{\sim} h = mc \overset{b}{\sim} c = mchac.$$ 

Since $c \in P^1(a)$, we have $mchac = 0$, that is, $mcbh = 0$. Since $h \in P^1(b)$, $0 = mcb = mac$. Using $c \in P^1(a)$ one last time, we get $ma = 0$ as claimed. \qed
The proof of theorem 4.32 depends heavily on the epigroup $S$ being in the variety $W$, and indeed the method of proof does not work for all epigroups in general. For example, consider the commutative monoid $S$ with zero defined by the following multiplication table:

|   | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 |
| 2 | 0 | 2 | 0 | 0 | 2 | 2 |
| 3 | 0 | 3 | 0 | 0 | 2 | 2 |
| 4 | 0 | 4 | 2 | 2 | 5 | 5 |
| 5 | 0 | 5 | 2 | 2 | 5 | 5 |

This is an epigroup with pseudo-inverse given by $0' = 2' = 3' = 0$, $1' = 1$, $4' = 5' = 5$. It is easy to see that $S$ is in $E_2$ since every element on the diagonal is an idempotent. $S$ is not in $W$ because, for instance, $(2 \cdot 4)'' = 2'' = 0 \neq 2 \cdot 4$. If $a = 2$, $b = 3$, $c = 4$, then $ac = cb$, $bd = da$, $c \in P^1(a)$ and $d \in P^1(b)$. Thus, $a \sim_c 3$.

Note that $c$ is not regular, but if we try to define $g$, $h$ by (4.16), we get $g = h = 0$. Thus, the proof of theorem 4.32 does not apply here. However, note that by setting $g = h = 5$, we do obtain mutually inverse $g$, $h$, which will suffice. Therefore, in this example, the conclusion of theorem 4.32 is still correct.

5. Comparison results

In this section, we compare the four notions of conjugacy under discussion in various settings. In every semigroup, $\sim_p \subseteq \sim^*_p \subseteq \sim_{tr} \subseteq \sim_o$ and $\sim_p \subseteq \sim_o$.

Regarding $\sim_p$ and $\sim_c$, we have the following result.

**Theorem 5.1.** For each of the conditions

(a) $\sim_c \subseteq \sim_p$,

(b) $\sim_p \subseteq \sim_c$,

(c) $\sim_p$ and $\sim_c$ are not comparable with respect to inclusion,

there exists a semigroup with zero in which the condition holds.

**Proof.** Proposition 2.17 shows that $\sim_c \subseteq \sim_p$ in any symmetric inverse semigroup $\mathcal{I}(X)$, where $2 \leq |X| < \infty$.

Example 4.24 provides an example of a semigroup $S$ without zero in which $\sim_p \subseteq \sim_p^* \subseteq \sim_c = \sim_o$. Let $S^0$ denote the semigroup obtained from $S$ by adding an extra element $0$ acting as a zero. Then $\sim_p^* = \sim_p^* \cup \{(0, 0)\}$ and $\sim_p^{S^0} = \sim_c^{S^0} \cup \{(0, 0)\}$. Thus, $S^0$ is a semigroup with zero in which $\sim_p \subseteq \sim_c$ as claimed.

Finally, by proposition 2.18, relations $\sim_p$ and $\sim_c$ are not comparable with respect to inclusion in the symmetric inverse semigroup $\mathcal{I}(X)$ on a countably infinite set $X$. There are also finite semigroups in which $\sim_p$ and $\sim_c$ are not comparable. Indeed,
let $S = \{0, 1, 2, 3, 4\}$ be the monoid given by the following multiplication table:

\[
\begin{array}{c|cccc}
\cdot & 0 & 1 & 2 & 3 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 2 & 3 & 4 \\
2 & 0 & 2 & 0 & 2 & 0 \\
3 & 0 & 3 & 4 & 3 & 4 \\
4 & 0 & 4 & 0 & 4 & 0 \\
\end{array}
\]

It is straightforward to check that $1 \sim_c 3$ and all other $\sim_c$-classes are trivial, while $2 \sim_p 4$ and all other $\sim_p$-classes are trivial.

Regarding $\sim_{tr}$ and $\sim_c$, we have the following result.

**Theorem 5.2.** For each of the conditions

(a) $\sim_c \subset \sim_{tr}$,

(b) $\sim_{tr} \subset \sim_c$,

(c) $\sim_{tr}$ and $\sim_c$ are not comparable with respect to inclusion,

there exists a semigroup with zero in which the condition holds.

**Proof.** The following semigroup [26, SmallSemigroup(4,22)] satisfies condition (a):

\[
\begin{array}{c|cccc}
\cdot & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 \\
2 & 0 & 0 & 0 & 1 \\
3 & 0 & 1 & 1 & 3 \\
\end{array}
\]

where $\sim_{tr} = \{\{0, 1, 2\}, \{3\}\}$ and $\sim_c = \{\{0\}, \{1, 2\}, \{3\}\}$.

The following semigroup [26, SmallSemigroup(4,113)] satisfies condition (b):

\[
\begin{array}{c|cccc}
\cdot & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 \\
2 & 0 & 1 & 2 & 1 \\
3 & 0 & 3 & 3 & 3 \\
\end{array}
\]

where $\sim_{tr} = \{\{0\}, \{2\}, \{1, 3\}\}$ and $\sim_c = \{\{0\}, \{1, 2, 3\}\}$.

Finally, the following semigroup [26, SmallSemigroup(4,56)] satisfies condition (c):

\[
\begin{array}{c|cccc}
\cdot & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 2 & 2 \\
3 & 0 & 0 & 2 & 3 \\
\end{array}
\]

where $\sim_{tr} = \{\{0, 1\}, \{2\}, \{3\}\}$ and $\sim_c = \{\{0\}, \{1\}, \{2, 3\}\}$. 

\qed
Our next result separates $c$-conjugacy and $o$-conjugacy. As already mentioned, $\sim_o$ is the universal relation in any semigroup with zero and $\sim_c = \sim_o$ in any semigroup without zero. Therefore, a trivial way of separating $\sim_c$ and $\sim_o$ is to consider any semigroup without zero and then adjoin a zero to that semigroup.

Less trivially, we can separate $\sim_c$ and $\sim_o$ in semigroups with proper zero divisors. The next theorem shows that the two notions might be different in such a semigroup in the most extreme way – see figure 2.

\textbf{Theorem 5.3.} In a semilattice $S$ that is an anti-chain with 0 and 1, $\sim_o$ is universal, while $\sim_c$ is the identity.

\textit{Proof.} Observe that $\mathbb{P}^1(1) = \{1\}$, $\mathbb{P}^1(0) = \{0\}$ and $\mathbb{P}^1(x) = \{x, 1\}$ for all $x \in S \setminus \{0, 1\}$. Therefore, in this semigroup $\sim_c$ is the identity, while $\sim_o$ is the universal relation. \hfill \Box

The same result holds for every null semigroup. Table 1 was produced using the SMALLSEMI package for GAP [26]. It contains data illustrating how common the extreme behaviour of $\sim_c$ is in monoids with zero divisors. In table 1, $|S|$ is the order of the semigroup; the column labelled by ‘# of monoids with 0-divisors’ contains the number of monoids of order $|S|$ that have a zero and zero divisors; the column ‘$\sim_c$ is the identity’ contains the number of such monoids in which $\sim_c$ is the identity relation; the column ‘$\sim_c$ is ‘universal’ contains the number of such monoids in which all non-zero elements form a single conjugacy class.

For a large proportion of the monoids from table 1, $c$-conjugacy is the identity. Observe that in groups, conjugacy is the identity relation if and only if the group is abelian. This is not the case for $c$-conjugacy in monoids, as the following monoid
with proper divisors of zero shows:

|   | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 1 |
| 2 | 0 | 0 | 0 | 0 | 2 |
| 3 | 0 | 0 | 1 | 0 | 3 |
| 4 | 0 | 1 | 2 | 3 | 4 |

Every element in this monoid is only c-conjugate to itself, and the monoid is not commutative. This monoid is \texttt{SmallSemigroup(5,110)} in the \texttt{SMALLEMI} package for GAP [26].

However, the result analogous to group conjugacy holds for p-conjugacy.

**Theorem 5.4.** Let $S$ be a semigroup. Then $\sim_p$ is the identity relation in $S$ if and only if $S$ is commutative.

**Proof.** If $S$ is commutative and if $x = uv$ and $y = vu$, then obviously $x = uv = vu = y$, and so $\sim_p$ is the identity relation. Conversely, suppose that each element of $S$ is p-conjugate only to itself. For all $a, b \in S$, $ab \sim_p ba$, and so $ab = ba$ by the assumption. \hfill $\Box$

**Theorem 5.5.** Let $S$ be an epigroup. Then $\sim_{tr}$ is the identity relation in $S$ if and only if $S$ is a commutative completely regular epigroup.

**Proof.** Suppose first that $\sim_{tr}$ is the identity relation. Since $\sim_p \subseteq \sim_{tr}$, it follows that $\sim_p$ is also the identity relation, and hence, by theorem 5.4, $S$ is commutative. In every epigroup, we have $a \sim_{tr} a''$ by theorem 4.7. Since $\sim_{tr}$ coincides with equality, we have $a = a''$ for all $a \in S$. Thus, $S$ is a commutative completely regular epigroup (or, equivalently, a commutative inverse epigroup).

Conversely, if $S$ is a commutative completely regular epigroup, then $\sim_{tr} = \sim_p$ by corollary 4.6, and so $\sim_{tr}$ is the identity relation by theorem 5.4. \hfill $\Box$

The corresponding result for o-conjugacy is as follows.

**Theorem 5.6.** Let $S$ be a semigroup. Then

1. if $S$ is commutative, then $\sim_o$ is the minimum cancellative congruence on $S$;
2. $\sim_o$ is the identity relation in $S$ if and only if $S$ is commutative and cancellative.

**Proof.** (1) Suppose that $S$ is commutative. Then for all $a, b \in S$, $a \sim_o b$ if and only if $ag = bg$ for some $g \in S^1$. Thus, whenever $a \sim_o b$ we have $ca \sim_o cb$ and $ac \sim_o bc$, for all $c \in S$, which implies that $\sim_o$ is a congruence. Denote the congruence class of $x \in S$ by $\bar{x}$. Let $\bar{a}, \bar{b}, \bar{c} \in S/\sim_o$ and suppose that $\bar{a} \bar{b} = \bar{a} \bar{c}$. Then $ab \sim_o ac$, and so $(ab)g = (ac)g$ for some $g \in S^1$. Since $S$ is commutative, we have $b(ag) = c(ag)$, and so $b \sim_o c$. Hence, $\bar{b} = \bar{c}$, which implies that $S/\sim_o$ is cancellative. Now let $\theta$ be any cancellative congruence on $S$ and suppose that $a \sim_o b$, where $a, b \in S$. Then $ag = bg$ for some $g \in S^1$, so $g\theta gb$. Since $\theta$ is cancellative, it follows that $a\theta b$. Therefore, $\sim_o \subseteq \theta$, which proves that $\sim_o$ is the minimum cancellative congruence on $S$. 
(2) If $S$ is commutative and cancellative, then (1) implies that $\sim_o$ must be the identity relation. For the converse, note that $xy \sim_o yx$ in any semigroup (since $(xy)x = x(yx)$ and $(yx)y = y(xy)$). Thus, if $\sim_o$ is the identity relation, then $xy = yx$ for all $x, y \in S$, that is, $S$ is commutative. By (1), $S \cong S/\sim_o$ is cancellative.

Observe that in left zero semigroups (those satisfying the identity $xy = x$), $\sim_o$ is the universal relation, and thus a congruence, but the semigroup is not commutative.

In commutative semigroups, p-conjugacy is the identity, and non-trivial cancellative semigroups cannot have a zero. Thus, the following result holds.

**Corollary 5.7.** Let $S$ be a commutative and cancellative semigroup. Then $\sim_p$, $\sim_o$ and $\sim_c$ all coincide and are equal to the identity relation.

By the definition of the notion of conjugacy, all semigroup conjugacies coincide in a group. The following result is a sort of converse.

**Corollary 5.8.** Let $S$ be an epigroup. Then $\sim_p$, $\sim_o$, $\sim_{tr}$ and $\sim_c$ all coincide and are equal to the identity relation if and only if $S$ is a commutative group.

**Proof.** Regarding the direct implication, observe that if $\sim_{tr}$ is the identity, then the semigroup is completely regular and commutative; in addition, $\sim_o$ trivial implies that $S$ is cancellative. It is well known that a regular cancellative semigroup is a group.

The converse is obvious.

Now we discuss conditions under which our various notions of conjugacy on a semigroup $S$ coincide with the universal relation $S \times S$. Regarding o-conjugacy, no characterization seems likely, because of what we have noted multiple times already: $\sim_o$ is universal in any semigroup with a zero.

Thus, we pass immediately to trace conjugacy in epigroups. One would guess that in epigroups with universal trace conjugacy, each subgroup is trivial, and this does indeed turn out to be the case; see part (2) of the following result. More interestingly, the theorem shows that the class of epigroups in which trace conjugacy is universal forms a variety.

**Theorem 5.9.** Let $S$ be an epigroup. The following are equivalent:

1. $\sim_{tr}$ is the universal relation;
2. $E(S)$ is an antichain and for all $x \in S$, $x'' = x^\omega$;
3. for all $x, y \in S$, $x'yx' = x'$;
4. for all $x, y \in S$, $x^\omega yx^\omega = x^\omega$;
5. for all $x \in S$, $e \in E(S)$, $exe = e$.

**Proof.** We prove (1) $\implies$ (2) $\implies$ (3) $\iff$ (1) and (3) $\iff$ (4) $\iff$ (5).

Assume that (1) holds, that is, $\sim_{tr} = S \times S$. Since $\sim_{tr} \subseteq \sim_o$, it follows that $\sim_{tr} = \sim_o$. By proposition 4.20, $E(S)$ is an antichain. Now fix an idempotent $e$. 

Since $\sim_U$ is universal, $e \sim_U x'$ for all $x \in S$. Thus, by theorem 4.5(6), there exist $u, v \in S^1$ such that $e = e'' = uv$ and $x' = (x')'' = vu$. Now,

$$x^\omega = (x')^\omega = x' x'' = vu(vu) = (vu)u = ve'u = veu.$$

Thus,

$$x'' = x' x^\omega = vuveu = veeu = veu = x^\omega. \tag{5.1}$$

This establishes (2).

Assume that (2) holds. Note that for all $x \in S$, $x' = x'' = (x^\omega)' = x^\omega$, so $x'$ is idempotent. We show that for all $x, y \in S$, $x'(yx')'$ is idempotent, freely using (4.10) to rewrite this as $(x'y')'x'$:

$$x'(yx')' = (x'y')'x'(yx')' = (x'y')'x'(yx') = x'(yx')'(yx') = x'(yx').$$

Next we show that $x' \leq x'(yx')': x'x'(yx')' = x'(yx')'$ and $x'(yx')'x' = (x'y')'x'x' = (x'y')'x' = x'(yx')'$. Now, since $E(S)$ is an antichain and $x', x'(yx')' \in E(S)$, we conclude that $x'(yx')' = x'$. Finally, we have

$$(yx')' = (yx')^\omega = y x'(yx') = yx'.$$

Therefore, $x'yx' = x'(yx')' = x'$, which establishes (3).

Assume that (3) holds. For $x, y \in S$, set $u = x''y''$ and $v = y''x''$. Then $x'' = x''y''x'' = uv$ and $y'' = y''x''y'' = vu$. Thus, $x'' \sim_p y''$, and so $x \sim_U y$ by theorem 4.5(6). Thus, $\sim_U$ is the universal relation, that is, (1) holds.

Next, once again assume that (3) holds. Taking $y = x''$, we have $x^\omega yx^\omega = x'(yx)x' = xx' = x^\omega$, so that (4) holds.

Assume that (4) holds. Taking $y = x$, we obtain $x^\omega = x^\omega xx^\omega = x''xx^\omega = x''$. Thus, $xx' = x''$, and so $x' = x'xx' = x''x' = (x')^\omega = x^\omega$. Therefore, $x'yx' = x^\omega yx^\omega = x^\omega = x'$. This establishes (3).

Finally, the equivalence of (4) and (5) is obvious. \qed

Next we discuss semigroups in which $p$-conjugacy is the universal relation. Our description is complete for semigroups with idempotents, and partial for semigroups without idempotents.

First we need a definition. A rectangular band is an idempotent semigroup satisfying the identity $xyx = x$ for all $x, y$. Every rectangular band is completely simple, and, in fact, is isomorphic to the Rees matrix semigroup $I \times G \times A$ with $G = \{1\}$ [40, theorem 1.1.3].

**Theorem 5.10.** Let $S$ be a semigroup.

(1) If $S$ is a rectangular band, then $\sim_p$ is the universal relation.

(2) If $\sim_p$ is the universal relation in $S$, then $S$ is simple. If, in addition, $S$ contains an idempotent, then $S$ is a rectangular band.

**Proof.** (1) Let $S$ be a rectangular band. For $x, y \in S$, set $u = xy, v = yx$. Then $x = xyx = xyyx = uu$ and $y = yxy = yxyy = vv$. Therefore, $x \sim_p y$ for all $x, y \in S$, that is, $\sim_p$ is universal.
Let $S$ be a semigroup in which $\sim_p$ is the universal relation. We first show that $S$ is simple. For $a \in S$, $S^3aS^1$ is the principal ideal of $S$ generated by $a$. We want to show that $S^3aS^1 = S$. Let $b \in S$. If $b = a$, then clearly $b \in S^3aS^1$. Suppose that $b \neq a$. Since $a \sim_p b$ and $a \neq b$, there exist $u_1, v_1 \in S$ such that $a = u_1v_1, b = v_1u_1$. Note that $u_1 \neq v_1$ (since otherwise $a = b$), so there exist $u_2, v_2 \in S$ such that $u_1 = u_2v_2, v_1 = v_2u_2$. Now, if $u_2 = a$, then $b = v_1u_1 = v_2au_1 \in SaS$, so we may assume that $u_2 \neq a$. Then there exist $u_3, v_3 \in S$ such that $a = u_3v_3, u_2 = v_3u_3$, and so

$$b = v_1u_1 = v_2u_2v_2u_2 = v_2v_3u_3v_3v_2 = v_2v_3u_3v_2 \in SaS.$$ 

Hence $S^3aS^1 = S$, and so $S$ is simple.

Now suppose that $S$ has an idempotent $e$. We will show that $S$ satisfies the identity $x^3 = x^2$. Since $x \sim_p e$, there exist $u, v \in S^1$ with $x = uv, e = vu$. Then

$$xxx = uvuv = uev = evu = uuv = xx.$$ 

The identity $x^3 = x^2$ implies that $S$ is an epigroup in $E_2$ with $x' = x^2$, that is, $x'xx' = x^5 = x^2 = x'$, $xx' = x'x$ and $x^3x' = x^2$. Since $\sim_p \subseteq \sim_{tr}$, we have that $\sim_{tr}$ is the universal relation. By theorem 5.9(2), $E(S)$ is a chain, that is, every idempotent is primitive.

We have now shown that $S$ is completely simple. In particular, $S$ is completely regular and the epigroup pseudo-inverse $x' = x^2$ is actually an inverse. Thus, $x = xx'x = x^4$. But this together with $x^3 = x^2$ imply $x^2 = x$ for all $x \in S$, that is, $S$ is an idempotent semigroup. Now using theorem 5.9(3), we conclude that $xyx = x'yx' = x$ for all $x, y \in S$, that is, $S$ is a rectangular band.

**Example 5.11.** By theorem 5.10, if $\sim_p$ is universal in a semigroup $S$, then $S$ is simple. If $S$ does not have an idempotent, then the converse is not necessarily true. Let $X$ be a countably infinite set. Denote by $\Gamma(X)$ the semigroup of all injective mappings from $X$ to $X$. For $\alpha \in \Gamma(X)$, let $\text{im}(\alpha)$ denote the image of $\alpha$. The set $S$ consisting of all $\alpha \in \Gamma(X)$ such that $X \setminus \text{im}(\alpha)$ is infinite is a subsemigroup of $\Gamma(X)$ called the Baer–Levi semigroup [24, §8.1]. The Baer–Levi semigroup is simple without idempotents [24, theorem 8.2]. Partition the set $X$ as follows:

$$X = \{x, y\} \cup \{z_1, z_2, \ldots\} \cup \{z^1_1, z^2_2, \ldots\} \cup \{z^3_3, z^4_4, \ldots\} \cup \cdots.$$ 

Define $\alpha, \beta \in S$ by

$$\alpha(x) = x, \quad \alpha(y) = y, \quad \alpha(z^i_j) = z^i_{j+1},$$

$$\beta(x) = y, \quad \beta(y) = x, \quad \beta(z^i_j) = z^i_{j+1}.$$ 

Then $(\alpha, \beta) \notin \sim_p$ by [46, proposition 4], so $\sim_p$ is not the universal relation.

Since, for example, every finite semigroup has an idempotent, theorem 5.10 implies an immediate corollary.

**Corollary 5.12.** In a finite semigroup (or, more generally, an epigroup) $S$, $\sim_p$ is the universal relation if and only if $S$ is a rectangular band.

We conclude this section with some results that extend to semigroups familiar results on conjugacy in groups.
For elements $a_1, a_2, b_1, b_2$ in a group, if $a_1a_2$ is conjugate to $b_1b_2$, then $a_2a_1$ is conjugate to $b_2b_1$. This result carries over to semigroups as follows. A semigroup $S$ with zero is categorical at zero if for all $a, b, c \in S$, $ab \neq 0$ and $bc \neq 0$ imply that $abc \neq 0$ [24, p. 73].

**Theorem 5.13.** Let $S$ be a semigroup.

1. For all $a_1, a_2, b_1, b_2 \in S$, $a_1a_2 \sim_0 b_1b_2$ implies that $a_2a_1 \sim_0 b_2b_1$.

2. If $S$ is categorical at zero and $a_1a_2, a_2a_1, b_1b_2, b_2b_1 \neq 0$, then $a_1a_2 \sim_c b_1b_2$ implies that $a_2a_1 \sim_c b_2b_1$.

3. The following statements are equivalent:
   
   (a) $\sim_p$ is transitive in $S$;
   
   (b) For all $a_1, a_2, b_1, b_2 \in S$, $a_1a_2 \sim_p b_1b_2$ implies that $a_2a_1 \sim_p b_2b_1$.

4. For all $a_1, a_2, b_1, b_2 \in S$ such that $a_1a_2, b_1b_2, a_2a_1, b_2b_1 \in \text{Epi}(S)$, $a_1a_2 \sim_{tr} b_1b_2$ implies that $a_2a_1 \sim_{tr} b_2b_1$.

**Proof.** Let $a_1a_2 \sim_0 b_1b_2$. This implies that, for some $c, d \in S$, $a_1a_2c = cb_1b_2$ and $b_1b_2d = da_1a_2$. Then

$$a_2a_1(a_2cb_1) = a_2(a_1a_2c)b_1 = a_2(cb_1b_2)b_1 \quad \text{and} \quad b_2b_1(b_2da_1) = (b_2da_1)a_2a_1. \quad (5.2)$$

Thus, $a_2a_1 \sim_0 b_2b_1$. We have proved (1).

Regarding $\sim_c$, suppose that $S$ is categorical at zero and let $a_1a_2, a_2a_1, b_1b_2, b_2b_1 \neq 0$. Suppose that $a_1a_2 \sim_c b_1b_2$. This implies that $a_1a_2c = cb_1b_2$ and $b_1b_2d = da_1a_2$ for some $c \in \mathbb{P}^1(a_1a_2)$ and $d \in \mathbb{P}^1(b_1b_2)$. As in the proof of (1), we obtain (5.2). It remains to prove that $a_2cb_1 \in \mathbb{P}^1(a_2a_1)$ and $b_2da_1 \in \mathbb{P}^1(b_2b_1)$. First we observe that in any semigroup categorical at zero, $x \in \mathbb{P}^1(y)$ if and only if $yx \neq 0$. Since $c \in \mathbb{P}^1(a_1a_2)$, it follows that $cb_1b_2 = a_1a_2c \neq 0$, and hence $a_2c \neq 0 \neq cb_1$. Thus, $a_2cb_1 \neq 0$ since $S$ is categorical at zero. Similarly, since $a_2a_1 \neq 0$ and $a_1a_2 \neq 0$, we have $a_2a_1a_2 \neq 0$. Now $a_2a_1a_2 \neq 0$ and $a_2cb_1 \neq 0$ imply that $a_2a_1a_2cb_1 \neq 0$, which implies that $a_2cb_1 \in \mathbb{P}^1(a_2a_1)$. Similarly, $b_2da_1 \in \mathbb{P}^1(b_2b_1)$, which concludes the proof of (2).

Regarding $\sim_p$, we start by proving (a) $\implies$ (b). Suppose that $\sim_p$ is transitive and let $a_1a_2 \sim_p b_1b_2$. By the definition of $\sim_p$, we have $xy \sim_p yx$ for all $x, y \in S$. Thus

$$a_2a_1 \sim_p a_1a_2 \sim_p b_1b_2 \sim_p b_2b_1,$$

which implies that $a_2a_1 \sim_p b_2b_1$ since $\sim_p$ is transitive.

For (b) $\implies$ (a), assume that $a_1a_2 \sim_p b_1b_2$ implies that $a_2a_1 \sim_p b_2b_1$ for all $a_1, a_2, b_1, b_2 \in S$. Let $a, b, c \in S$ and suppose that $a \sim_p b$ and $b \sim_p c$. Then $a = xy$, $b = yx = uv$ and $c = vu$ for some $x, y, u, v \in S^1$. Thus $yx \sim_p uv$ (since $xy = uv = b$), and hence $xy \sim_p vu$ (by the hypothesis), that is, $a \sim_p c$. Therefore, $\sim_p$ is transitive.

Finally, the result for $\sim_{tr}$ follows from theorem 4.7(3).

In a group, if $a$ and $b$ are conjugate, then $a^k$ and $b^k$ are also conjugate for all positive integers $k$. This fact generalizes to the conjugacies $\sim_p$, $\sim_c$ and $\sim_o$ in semigroups.
**Theorem 5.14.** Let $S$ be a semigroup and let $\sim \in \{\sim_o, \sim_c, \sim_p\}$. Then for all $a, b \in S$ and integers $k \geq 1$, $a \sim b$ implies $a^k \sim b^k$.

**Proof.** Let $a, b \in S$ and $c \in S^1$ be such that $ac = cb$. We claim that $a^k c = cb^k$ for all integers $k \geq 1$. We proceed by induction on $k$. The claim is certainly true for $k = 1$. Let $k \geq 1$ and suppose that $a^k c = cb^k$. Then $a^{k+1} c = a(a^k c) = a(cb^k) = (ac)b^k = cb^{k+1}$. The claim has been proved. The result follows immediately for $\sim_o$ and $\sim_c$.

For $\sim_p$, the desired result is [44, lemma 2]: if, say, $a = cd$ and $b = dc$, then $a^k = ((cd)^{k-1}c)d$ while $b^k = d((cd)^{k-1}c)$. □

The same result is true for trace conjugacy and epigroup elements.

**Theorem 5.15.** Let $S$ be a semigroup. Then for all $a, b \in \text{Epi}(S)$ and integers $k \geq 1$, $a \sim_{tr} b$ implies $a^k \sim_{tr} b^k$.

**Proof.** Suppose that $a \sim_{tr} b$. Then $a'' \sim_p b''$ by theorem 4.5, and so $(a'')^k \sim_p (b'')^k$ by theorem 5.14. Since $(a'')^k = (a^k)'$ and $(b'')^k = (b^k)'$, we have $(a^k)' \sim_p (b^k)'$, and so $a^k \sim_{tr} b^k$ by theorem 4.5. □

In a group, if $a$ and $b$ are conjugate, then $a^{-1}$ and $b^{-1}$ are also conjugate. This fact generalizes to $\alpha$-conjugacy and $\rho$-conjugacy in epigroups. (See proposition 4.3 for a stronger result for trace conjugacy.)

**Theorem 5.16.** Let $S$ be an epigroup and let $\sim \in \{\sim_o, \sim_p\}$. Then for all $a, b \in S$, $a \sim b$ implies $a' \sim b'$.

**Proof.** Suppose that $a \sim_o b$, so $ac = cb$ and $da = bd$ for some $c, d \in S^1$. Set $g = aa'cb'$ and $h = bb'da'$. Then

$$a'g = a'aa'cb' \overset{(4.7)}{=} a'c \overset{(4.7)}{=} a'c b' b' \overset{(4.8)}{=} a' cb' b'$$

and an almost identical calculation shows that $b'h = ha'$. Thus, $a' \sim_o b'$.

Now suppose that $a \sim_p b$. Then $a = cd$ and $b = dc$ for some $c, d \in S^1$. Set $u = c, v = d(cd)'(cd)'$. Then $uv = cd(cd)'(cd)' = (cd)'cd(cd)' = (cd)' = a'$, using (4.8) and (4.7), and $vu = d(cd)'(cd)'e = (dc)'(dc)'e = (dc)' = b'$, using (4.10) twice followed by (4.7). Thus, $a' \sim_{tr} b'$. □

In a group, if $a$ and $b$ are conjugate and $a^m = a^k$ for some integers $m, k \geq 1$, then $b^m = b^k$. This result does not hold in general for semigroups, but we have the following for $\sim_p$.

**Theorem 5.17.** Let $S$ be a semigroup and let $a, b \in S$ such that $b$ is an epigroup element with $b^t$ ($t \geq 1$) lying in a subgroup of $S$. If $a \sim_{p} b$ and $a^m = a^k$ for some integers $m, k \geq t$, then $b^m = b^k$.

**Proof.** Since $a \sim_p b$, $a = cd$ and $b = dc$ for some $c, d \in S^1$. Since $b^t$ is in a subgroup of $S$, we have, by (4.1), $b^{n+1}b' = b^n$ for every integer $n \geq t$. Thus

$$b^m = b^{m+1}b' = d(cd)^m cb' = da^m cb' = da^k cb' = (dc)^{k+1}b' = b^{k+1}b' = b^k,$$

which completes the proof. □
Corollary 5.18. Let $S$ be an epigroup in $W$. If $a, b \in S$ satisfy $a \sim_p b$ and $a^m = a^k$ for some integers $m, k \geq 1$, then $b^m = b^k$.

Proof. Since $a \sim_p b$, we have $a = cd$ and $b = dc$ for some $c, d \in S^1$. Since $b'' = (dc)' = dc = b$ by (4.13), $b$ is completely regular, so theorem 5.17 applies with $t = 1$.

Theorem 5.17 fails for $\sim_o$. Indeed, if $S$ has a zero as its unique idempotent, then $\sim_o$ is the universal relation, but $0^2 = 0$ while $a^2 \neq a$ for every non-zero $a \in S$.

6. Open problems

We conclude this paper with some natural questions related to conjugacy.

In § 2, we characterized $c$-conjugacy in the symmetric inverse semigroup $I(X)$ for a countable set $X$. Descriptions of $\sim_p$ in this semigroup can be found in [33] and [46].

Problem 6.1. Characterize the relations $\sim_c$ and $\sim_p$ in $I(X)$ for an uncountable set $X$.

A characterization of $c$-conjugacy in the full transformation semigroup $T(X)$ on any set $X$ was obtained in [19]. For a finite set $X$, $p$-conjugacy in $T(X)$ was described in [46]. The partition semigroup $P_X$ on a set $X$ [28, 29] has both $T(X)$ and the symmetric inverse semigroup $I(X)$ as subsemigroups.

Problem 6.2. Characterize the relations $\sim_c$ and $\sim_p$ in $P_X$, and $\sim_{tr}$ restricted to the epigroup elements.

We proved in § 4 that $p$-conjugacy is transitive in completely regular semigroups and their variants, but noted that the epigroup variety $W$ does not include all epigroups in which $\sim_p$ is transitive.

Problem 6.3. Find other classes of semigroups in which $p$-conjugacy is transitive. Describe the $(E$-unitary) inverse semigroups in which $p$-conjugacy is transitive. Ultimately, classify the class of semigroups in which $\sim_p$ is transitive.

As already noted, $\sim_p$ is transitive in free semigroups. Free semigroups are both cancellative and embeddable in groups.

Problem 6.4. Is $\sim_p$ transitive in every cancellative semigroup? Is it transitive in every semigroup that is embeddable in a group?

In this paper we studied conjugacy in the symmetric inverse semigroup $I(X)$, but many other transformation semigroups, or endomorphism monoids of some relational algebras, may be considered.

Problem 6.5. For $\sim_c$, $\sim_p$ and $\sim_{tr}$, characterize the conjugacy classes and calculate their number for other transformation semigroups such as, for example, those appearing in the problem list of [9, § 6] or those appearing in the large list of transformation semigroups included in [30]. Especially interesting would be a characterization of the conjugacy classes in the centralizers of idempotents [6, 7], or in semigroups whose group of units has an especially rich structure [4, 12, 17, 20].
The classes described in the preceding problem have linear analogues and hence can be extended to the more general setting of independence algebras.

**Problem 6.6.** Characterize $\cong_c$, $\cong_p$ and $\cong_{tr}$ in the endomorphism monoid of an independence algebra. In [18] a problem on independence algebras was solved using their classification theorem; it is reasonable to guess that the same technique can be used to solve the problem proposed here. (For historical notes on how a problem on idempotent generated semigroups [10, 13] led to these algebras, see [5, 16]; for definitions and basic results, see [1–3,11,14,22,31,32,34].)

Similarly interesting would be the characterization of the conjugacy classes for the endomorphism monoids of free objects [15] or for the endomorphisms of algebras admitting some general notion of independence [14]. Regarding the latter, we propose the problem of calculating the conjugacy classes in the endomorphisms of $MC$-algebras, $MS$-algebras, $SC$-algebras, and $SC$-ranked algebras [14, ch. 8]. A first step would be to solve the conjugacy problem for the endomorphism monoid of an $SC$-ranked free $M$-act [14, ch. 9], and for an $SC$-ranked free module over an $\aleph_1$-Noetherian ring [14, ch. 10].

Since all varieties of bands are known, especially interesting would be the description of the conjugacy classes of the endomorphism monoid of the free objects of each variety of bands (for details and references, see [8]).

The study of the intersection of $\cong_c$ with other conjugacies was omitted from this paper. This suggests the following problem.

**Problem 6.7.** Let $\sim \in \{\cong_o, \cong_p, \cong_{tr}\}$. Study the notion of conjugacy $\cong_c \cap \sim$. In particular, describe it in the various types of transformation semigroups listed in the previous problems.

We have proved that if a semigroup $S$ has an idempotent, then $\cong_p$ is the universal relation in $S$ if and only if $S$ is a rectangular band. We have also proved that every semigroup in which $\cong_p$ is universal is simple, and noted that there are simple semigroups without idempotents in which $\cong_p$ is not universal.

**Problem 6.8.** Describe the simple semigroups without idempotents in which $p$-conjugacy is the universal relation.

We know that $o$-conjugacy is universal in the semigroups with zero.

**Problem 6.9.** Describe the semigroups without zero in which $o$-conjugacy (and thus $c$-conjugacy) is the universal relation.

We will say that a given conjugacy $\sim$ is *partition covering* if for every set $X$ and for every partition $\tau$ of $X$, there exists a semigroup $S$ with universe $X$ such that the $\sim$-conjugacy classes on $S$ form the same partition as $\tau$.

**Problem 6.10.** Is it true that $o$-conjugacy (p-conjugacy, $\cong_{tr}$-conjugacy) is a partition-covering relation?

We have used the GAP package SMALLSEMI [26] to check that this is true for all $X = \{1, \ldots, n\}$, where $1 \leq n \leq 6$, and $\cong_o$ or $\cong_p$. As SMALLSEMI contains all semigroups up to order 8, the following special case of the preceding problem might take a long time to compute, but it is certainly computationally feasible.
Problem 6.11. Is it true that o-conjugacy (p-conjugacy, trace conjugacy) is a partition-covering relation for all sets of size at most 8? What about 9?

In theorem 4.29 we showed that o-conjugacy in epigroups is equivalent to a stronger notion of conjugacy. Call elements \( a, b \) of a semigroup \( S \) strongly o-conjugate, denoted by \( a \sim_{so} b \), if there exist mutually inverse \( g, h \in S^1 \) such that \( ag = gb \) and \( bh = ha \). The relation \( \sim_{so} \) is evidently reflexive and symmetric, and \( \sim_{so} \subseteq \sim_{o} \). Theorem 4.29 can be restated as saying that in epigroups, \( \sim_{so} = \sim_{o} \). This result is not true in general. For example, the transformations \( \alpha \) and \( \beta \) defined in the proof of theorem 3.4 are o-conjugate but not strongly o-conjugate in the semigroup \( I^*(X) \).

Problem 6.12. Find natural classes of semigroups in which \( \sim_{so} = \sim_{o} \).

Since \( \sim_{o} \) is transitive in arbitrary semigroups, theorem 4.29 implies that \( \sim_{so} \) is transitive in epigroups. It is also easy to see that \( \sim_{so} \) is transitive in inverse semigroups. (If \( a \sim_{so} b \sim_{so} c \), then \( ag = gb, bg^{-1} = g^{-1}a, bk = kc, ck^{-1} = k^{-1}b \) for some \( g, k \). Thus, \( agk = gbk = gkc \) and \( c(gk)^{-1} = ck^{-1}g^{-1} = k^{-1}bg^{-1} = k^{-1}g^{-1}a = (gk)^{-1}a \).)

Problem 6.13. Is \( \sim_{so} \) transitive in arbitrary semigroups? Is it transitive in regular semigroups?

The analogue of strong o-conjugacy for \( \sim_{c} \) is as follows. Call elements \( a, b \) of a semigroup \( S \) strongly c-conjugate, denoted by \( a \sim_{sc} b \), if there exist \( g \in P^1(a), h \in P^1(b) \) such that \( g, h \) are mutually inverse and \( ag = gb, bh = ha \). Evidently, \( \sim_{sc} \subseteq \sim_{c} \). Theorem 4.32 can be rephrased as saying that for epigroups in \( W \), \( \sim_{c} = \sim_{sc} \).

Problem 6.14. Does theorem 4.32 generalize to all epigroups? Does there exist a semigroup with a pair of c-conjugate elements that are not strongly c-conjugate? A regular such semigroup? An inverse semigroup?

Problem 6.15. Is it possible to prove a result similar to theorem 4.28, replacing regular epigroups by epigroups in \( W \)? For semigroups without zero we have a similar result. Possibly, it is necessary to start by proving that \( x \sim_{c} x'' \) for all \( x \) such that \( x'' \neq 0 \). If such a result could be proved, then the result would follow as in the case without zero.

Problem 6.16. Is there an example of a semigroup \( S \) in which \( \sim_{o} \) is a congruence, but \( S/\sim_{o} \) is not cancellative?

The coordinatization theorem (see [56, definition A.4.18]) for rectangular bands is probably the most basic such result involving two of Green’s relations.

Problem 6.17. Find a class of semigroups admitting a coordinatization theorem in terms of \( \sim_{c} \) and \( \sim_{tr} \) (respectively, \( \sim_{c} \) and \( \sim_{p}^* \)). In particular, classify the semigroups in which \( \sim_{c} \cap \sim_{tr} \) (respectively, \( \sim_{c} \cap \sim_{p}^* \)) is the identity relation.

The class \( W \) seems a very interesting generalization of the class of completely regular semigroups. It is likely that many of the results for the latter carry over to the former.
Problem 6.18. Generalize for $W$ the main results on completely regular semigroups. In particular, is it true that $\sim_p$ is transitive in the variants of $W$?

Consider the variety $V$ of unary semigroups $(S, \cdot)$ defined by associativity, $x'xx' = x', xx' = x'x$ and
\begin{align}
  x''y &= xy, \\
  xy'' &= xy.
\end{align}

(6.1) (6.2)

This class also generalizes completely regular semigroups and appears to be as interesting as $W$.

Problem 6.19. Generalize for $V$ the main results on completely regular semigroups. In particular, is it true that $\sim_p$ is transitive in the variants of $V$?

In [21] there are two generalizations of the notion of variants of semigroups; one appears in proposition 2.1 and relies on translations, and the other is provided by the concept of interassociates (for definitions we refer the reader to [21]).

Problem 6.20. Do the results on variants in this paper carry over to the two generalizations introduced in [21]?

As seen in figure 1, $\sim_c$ is not related to $\sim_p$ or $\sim_{tr}$.

Problem 6.21. Is it possible to find an infinite set of first-order definable notions of conjugacy for semigroups such that these notions form an anti-chain (infinite chain)?

The final problem deals with the converse of example 4.19.

Problem 6.22. Is it true that if $\sim_p$ is transitive in all variants of a semigroup, then it is also transitive in the semigroup itself?

Acknowledgements

The authors thank the referee for the excellent suggestions that led to a much improved paper. J.A. and M.K. were partly supported by the Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) through Project no. CEMAT-CIENCIAIS UID/Multi/04621/2013, and through the project ‘Hilbert’s 24th problem’ (Project no. PTDC/MHC-FIL/2583/2014). J.K. was supported by the 2011–12 University of Mary Washington Faculty Research Grant. A.M. was partly supported by the Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) through Project no. UID/MAT/00297/2013 (Centro de Matemática e Aplicações), and by the FCT project ‘Hilbert’s 24th problem’ (Project no. PTDC/MHC-FIL/2583/2014).

References

1 J. Araújo. Generators for the semigroup of endomorphisms of an independence algebra. *Alg. Colloq.* 9 (2002), 375–382.
2 J. Araújo. Normal semigroups of endomorphisms of proper independence algebras are idempotent generated. *Proc. Edinb. Math. Soc.* 45 (2002), 205–217.
Four notions of conjugacy for abstract semigroups

3 J. Araújo. Idempotent generated endomorphisms of an independence algebra. *Semigroup Forum* **67** (2003), 464–467.

4 J. Araújo and P. J. Cameron. Two generalizations of homogeneity in groups with applications to regular semigroups. *Trans. Am. Math. Soc.* **368** (2016), 1159–1188.

5 J. Araújo and J. Fountain. The origins of independence algebras. In *Semigroups and languages*, pp. 54–67 (World Scientific, 2004).

6 J. Araújo and J. Konieczny. Automorphisms groups of centralizers of idempotents. *J. Alg.* **269** (2003), 227–239.

7 J. Araújo and J. Konieczny. Semigroups of transformations preserving an equivalence relation and a cross-section. *Commun. Alg.* **32** (2004), 1917–1935.

8 J. Araújo and J. Konieczny. Automorphisms of endomorphism monoids of relatively free bands. *Proc. Edinb. Math. Soc.* **50** (2007), 1–21.

9 J. Araújo and J. Konieczny. Centralizers in the full transformation semigroup. *Semigroup Forum* **86** (2013), 1–31.

10 J. Araújo and J. M. Mitchell. An elementary proof that every singular matrix is a product of idempotent matrices. *Am. Math. Mon.* **112** (2005), 641–645.

11 J. Araújo and J. M. Mitchell. Relative ranks in the monoid of endomorphisms of an independence algebra. *Monatsh. Math.* **151** (2007), 1–10.

12 J. Araújo and C. Schneider. The rank of the endomorphism monoid of a uniform partition. *Semigroup Forum* **78** (2009), 498–510.

13 J. Araújo and F. C. Silva. Semigroups of linear endomorphisms closed under conjugation. *Commun. Alg.* **28** (2000), 3679–3689.

14 J. Araújo and F. Wehrung. Embedding properties of endomorphism semigroups. *Fund. Math.* **202** (2009), 125–146.

15 J. Araújo, J. M. Mitchell and N. Silva. On generating countable sets of endomorphisms. *Alg. Univers.* **50** (2003), 61–67.

16 J. Araújo, M. Edmundo and S. Givant. $v^*$-algebras, independence algebras and logic. *Int. J. Alg. Comput.* **21** (2011), 1237–1257.

17 J. Araújo, P. J. Cameron, J. D. Mitchell and M. Neunhöffer. The classification of normalizing groups. *J. Alg.* **373** (2013), 481–490.

18 J. Araújo, W. Bentz and J. Konieczny. The largest subsemilattices of the semigroup of endomorphisms of an independence algebra. *Linear Alg. Applic.* **458** (2014), 50–79.

19 J. Araújo, J. Konieczny and A. Malheiro. Conjugation in semigroups. *J. Alg.* **403** (2014), 93–134.

20 J. Araújo, W. Bentz, J. D. Mitchell and C. Schneider. The rank of the semigroup of transformations stabilising a partition of a finite set. *Math. Proc. Camb. Phil. Soc.* **159** (2015), 339–353.

21 S. Boyd, M. Gould and A. Nelson. Interassociativity of semigroups. In *Proceedings of the Tennessee topology conference*, pp. 33–32 (World Scientific, 1997).

22 P. J. Cameron and C. Szabó. Independence algebras. *J. Lond. Math. Soc.* **61** (2000), 321–334.

23 A. H. Clifford and G. B. Preston. *The algebraic theory of semigroups, volume I*, Mathematical Surveys and Monographs, vol. 7 (Providence, RI: American Mathematical Society, 1961).

24 A. H. Clifford and G. B. Preston. *The algebraic theory of semigroups, volume II*, Mathematical Surveys and Monographs, vol. 7 (Providence, RI: American Mathematical Society, 1967).

25 A. Distler and J. D. Mitchell. The number of nilpotent semigroups of degree 3. *Electron. J. Combinator.* **19** (2012), paper 51.

26 A. Distler and J. D. Mitchell. GAP package – smallsemi, v. 0.6.8 (2014). (Available at http://www-groups.mcs.st-andrews.ac.uk/~jamesm/smallsemi.)

27 D. S. Dummit and R. M. Foote. *Abstract algebra*, 3rd edn (John Wiley and Sons, 2004).

28 J. East. Generators and relations for partition monoids and algebras. *J. Alg.* **339** (2011), 1–26.

29 J. East and D. G. FitzGerald. The semigroup generated by the idempotents of a partition monoid. *J. Alg.* **372** (2012), 108–133.
V. H. Fernandes. Presentations for some monoids of partial transformations on a finite chain: a survey. In *Semigroups, algorithms, automata and languages*, pp. 363–378 (World Scientific, 2002).

J. Fountain and A. Lewin. Products of idempotent endomorphisms of an independence algebra of finite rank. *Proc. Edinb. Math. Soc.* 35 (1992), 493–500.

J. Fountain and A. Lewin. Products of idempotent endomorphisms of an independence algebra of infinite rank. *Math. Proc. Camb. Phil. Soc.* 114 (1993), 303–319.

O. Ganyushkin and T. Kormysheva. The chain decomposition of partial permutations and classes of conjugate elements of the semigroup $\mathcal{IS}_n$. *Visnyk Kyiv Univ.* 2 (1993), 10–18.

V. Gould. Independence algebras. *Alg. Univers.* 33 (1995), 294–318.

P. A. Grillet. Counting semigroups. *Commun. Alg.* 43 (2015), 574–596.

P. Hell and J. Nešetřil. *Graphs and homomorphisms* (Oxford University Press, 2004).

J. Hickey. Semigroups under a sandwich operation. *Proc. Edinb. Math. Soc.* 26 (1983), 371–382.

J. Hickey. On variants of a semigroup. *Bull. Austral. Math. Soc.* 34 (1986), 447–459.

P. M. Higgins. The semigroup of conjugates of a word. *Int. J. Alg. Comput.* 16 (2006), 1015–1029.

J. M. Howie. *Fundamentals of semigroup theory* (Oxford University Press, 1995).

T. A. Khan and M. V. Lawson. Variants of regular semigroups. *Semigroup Forum* 62 (2001), 358–374.

J. Konieczny. Centralizers in the semigroup of injective transformations on an infinite set. *Bull. Austral. Math. Soc.* 82 (2010), 305–321.

J. Konieczny. Centralizers in the infinite symmetric inverse semigroup. *Bull. Austral. Math. Soc.* 87 (2013), 462–479.

G. Kudryavtseva. On conjugacy in regular epigroups. Preprint, 2006. (Available at https://arxiv.org/abs/math/0605698v1.)

G. Kudryavtseva and V. Maltcev. On representations of variants of semigroups. *Bull. Austral. Math. Soc.* 73 (2006), 273–283.

G. Kudryavtseva and V. Mazorchuk. On conjugation in some transformation and Brauer-type semigroups. *Publ. Math. Debrecen* 70 (2007), 19–43.

G. Kudryavtseva and V. Mazorchuk. On three approaches to conjugacy in semigroups. *Semigroup Forum* 78 (2009), 14–20.

G. Lallement. *Semigroups and combinatorial applications* (John Wiley and Sons, 1979).

I. Levi. Normal semigroups of one-to-one transformations. *Proc. Edinb. Math. Soc.* 34 (1991), 65–76.

S. Lipscomb. *Symmetric inverse semigroups*, Mathematical Surveys and Monographs, vol. 46 (Providence, RI: American Mathematical Society, 1996).

R. C. Lyndon and M. P. Schützenberger. The equation $a^m = b^n c^p$ in a free group. *Michigan Math. J.* 9 (1962), 289–298.

D. B. McAlister. Characters of finite semigroups. *J. Alg.* 22 (1972), 183–200.

K. S. S. Nambooripad. The natural partial order on a regular semigroup. *Proc. Edinb. Math. Soc.* 23 (1980), 249–260.

F. Otto. Conjugacy in monoids with a special Church–Rosser presentation is decidable. *Semigroup Forum* 29 (1984), 223–240.

M. Petrich and N. R. Reilly. *Completely regular semigroups* (John Wiley and Sons, 1999).

J. Rhodes and B. Steinberg. *The q-theory of finite semigroups*, Springer Monographs in Mathematics (Springer, 2009).

W. R. Scott. *Group theory* (Englewood Cliffs, NJ: Prentice Hall, 1964).

L. N. Shevrin. Epigroups. In *Structural theory of automata, semigroups, and universal algebra* (ed. V. B. Kudryavtsev and I. G. Rosenberg), NATO Science Series II: Mathematics, Physics and Chemistry, vol. 207, pp. 331–380 (Springer, 2005).

B. Steinberg. *The representation theory of finite monoids*, Springer Monographs in Mathematics (Springer, 2015).