The application of $\sigma$-LFSR in Key-Dependent Feedback Configuration for Word-Oriented Stream Ciphers

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Abstract. In this paper, we propose and evaluate a method for using $\sigma$-LFSRs with key-dependent feedback configurations (KDFC) in stream ciphers. This method can be applied to any stream cipher that uses a word-based LFSR. Here, a configuration generation algorithm uses the secret key (K) and the initialization vector (IV) to generate a feedback configuration. We have mathematically analysed the feedback configurations generated by this method. As a test case, we have applied this method on SNOW 2.0 and have analysed its impact on resistance to algebraic attacks. Further, we have also tested the generated keystream for randomness.

Keywords: Stream Cipher · $\sigma$-LFSR · Key-Dependent Feedback Configuration · Primitive Polynomial · Algebraic Attack.

1 Introduction

Stream ciphers are used in a variety of applications [ANA16, PBC19]. LFSRs (Linear Feedback Shift Register) are widely used as building blocks in stream ciphers because of their simple construction and easy implementation. Stream ciphers are vulnerable against algebraic attacks. For stream ciphers built on LFSRs, the knowledge of the feedback equation of the LFSR is critical for any algebraic attack. The generating polynomials (and hence the feedback configuration) of the LFSRs are publicly known in most ciphers, and confidentiality is provided only via secret initial values fed into LFSRs.

The literature in this area comes with a lot of research on algebraic attacks MPC04, Cou03, Dal06, KAAR17 as well as different approaches for improving the algebraic resistance of stream ciphers (resistance against algebraic attacks).
In this paper, we propose a solution for schemes like SNOW which consist of a word based LFSR connected with a Finite State Machine (FSM). We approach this problem making the feedback configuration of the $\sigma$-LFSR dependent on the secret key. During initialization, the feedback configuration of the $\sigma$-LFSR is assumed to be publicly known. The initial state of the $\sigma$-LFSR is dependent on the secret key. With this configuration the scheme generates a set of random numbers. These random numbers are then used to generate a feedback configuration. The scheme is operated using this feedback configuration. It is assumed that the random numbers generated during initialization are not available to the attacker. In the rest of this paper, we refer to this method as $\sigma$-KDFC (Key-Dependent Feedback Configuration).

The first novel aspect of the proposed approach is that the feedback configuration of the $\sigma$-LFSR is key dependent. This increases the non-linearity of the stream cipher and improves its resistance against algebraic attacks. The second novelty of a $\sigma$-KDFC is its flexibility in managing the trade-off between algebraic resistance and computational complexity. The randomness of the chosen configuration and the computational complexity of the configuration generation algorithm both increase with the number of iterations in the algorithm. $\sigma$-KDFC provides the flexibility required to manage this trade-off via allowing the designer to control the number of iterations.

We theoretically analyse the impact of $\sigma$-KDFC on algebraic resistance by calculating the degree of the polynomials that relate the configuration of the keystream generation module to the initial state of the $\sigma$-LFSR. Then, we study the integration of $\sigma$-KDFC with SNOW 2.0 as a case study. We refer to the resulting cipher stream as KDFC-SNOW. We use empirical tests to verify the randomness of the keystream generated by KDFC-SNOW.

In this paper, we have considered $\mathbb{F}_{p^n}$ as a finite field of $p^n$ elements over base field $\mathbb{F}_p$ with characteristic $p$, where $p$ is a prime number. $\mathbb{F}_2^n$ is $n$-dimensional vector space over base field $\mathbb{F}_2$. The $i^{th}$ row and $j^{th}$ column of a matrix $M \in \mathbb{F}_2^{n \times n}$ is denoted by $M[i,:]$ and $M[:,j]$ respectively. $M[i,j]$ is considered as a coordinate of matrix $M$. Minor of an entry in the $i^{th}$ row and $j^{th}$ column of matrix $M$ is expressed by $\mu(M[i,j])$. Besides, $\oplus$ and $+$ are used interchangeably to represent addition over $\mathbb{F}_2$.

The rest of this paper is organized as follows. Section 2 introduces LFSR, $\sigma$-LFSR and some related concepts. Section 3 examines $\sigma$-KDFC, its components and its time complexity. Section 4 presents the mathematical analysis on the algebraic resistance of $\sigma$-KDFC. Section 5 discusses the case study as well as the related evaluations. Section 6 concludes the paper.

## 2 Basic Concepts

A traditional LFSR of length $b$ is a linear bit sequence generator usually implemented using a shift register with $b$ flip-flops and feedback loop containing a few XOR gates. The XOR gates implement the GF(2) addition operation, and serve to establishing a linear transformation characterized by a fixed gen-
erating polynomial which represents the feedback configuration of the LFSR. An LFSR generates a maximum-period sequence if its generating polynomial is primitive [AG98]. The companion matrix of a primitive polynomial \( f(x) = x^b + c_{b-1}x^{b-1} + c_{b-2}x^{b-2} + \cdots + c_1x + c_0 \) wherein \( c_0, c_1, \ldots, c_{b-1} \in \mathbb{F}_2 \) is given in equation 1.

\[
P_f = \begin{bmatrix}
0 & 0 & \cdots & c_0 \\
1 & 0 & \cdots & c_1 \\
0 & 1 & \cdots & c_2 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & c_{b-2} \\
0 & 0 & \cdots & c_{b-1}
\end{bmatrix} \in \mathbb{F}_2^{b \times b} \tag{1}
\]

In addition to hardware implementation using XOR and XNOR gates [KGK19], LFSRs can be implemented in software [KKT10,Lau07]. Moreover, multi-bit input (parallel) LFSRs [?], multi-bit output (MBO) LFSRs [KKT10] have been proposed in order to improve speed via increased parallelism.

Multi-bit input multi-bit output (MIMO) LFSRs are more evolved variants of LFSRs wherein the ideas behind parallel and MBO LFSRs are combined [?]. MIMO LFSRs replace the scalar coefficients of the generating polynomial in traditional LFSRs by vectors over the base field (GF(2) in this paper).

\( \sigma \)-LFSRs are special MIMO LFSRs in which the coefficients of the generating polynomial are matrices over the base field. A \( \sigma \)-LFSR \( S \) is formally defined as follows.

**Definition 1.** Consider \( B_0, B_1, \ldots, B_{b-1} \in \mathbb{F}_2^{m \times m} \) where \( b \) is a positive integer. A \( \sigma \)-LFSR is a system with \( b \), \( m \)-input \( m \)-output delay blocks \( D_0, D_1, \ldots, D_{b-1} \) which satisfies the feedback equation \( D_{b-1}^{t+1} = \sum_{j=0}^{b-1} B_j \times D_j^t \) where \( D_r^t \) represents the content of \( D_r \) at time \( t \). The polynomial \( f_S(x) = x^n + B_{b-1}x^{n-1} + B_{b-2}x^{n-2} + \cdots + B_0 \) is referred to as the generating \( \sigma \)-polynomial of \( S \) [ZHH07].

The \( \sigma \)-LFSR defined above is shown in figure 1.

![Fig. 1. Block diagram of \( \sigma \)-LFSR](image)
The matrices $B_0, B_1, \ldots, B_{b-1}$ in definition 1 are referred to as the gain matrices of $S$. The configuration matrix of $S$ is defined accordingly as given by equation 2.

\[
C_S = \begin{bmatrix}
0 & I & 0 & \cdots & 0 \\
0 & 0 & I & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & I \\
B_0 & B_1 & B_2 & \cdots & B_{b-1}
\end{bmatrix} \in \mathbb{F}_2^{mb \times mb}
\] (2)

In equation 2, $0, I \in \mathbb{F}_2^{m \times m}$ are all-zero and identity matrices respectively. If $f_S(x)$ is the generating $\sigma$-polynomial, then $C_S$ is referred to as its $M$-companion matrix. The characteristic polynomial of this $M$-companion matrix is known as the characteristic polynomial of the $\sigma$-LFSR.

Let $D^t_i$ represent the output of the $i^{th}$ delay block in $S$ at timestamp $t$ where $i \in \{0, 1, \ldots, b-1\}$. The state vector of the $\sigma$-LFSR $S$ at timestamp $t$ (after the $t^{th}$ clock cycle) is defined as follows.

\[
V^t_S = \begin{bmatrix}
D^t_0 \\
D^t_1 \\
\vdots \\
D^t_{b-1}
\end{bmatrix} \in \mathbb{F}_2^{mb}
\]

Further, the state transition equation of $\sigma$-LFSR is defined as follows.

\[
V^{t+1}_S = C_S \times V^t_S
\] (3)

Let $D^n$ represent the vector sequence (multisequence) generated by $S$. If the period of $D^n$ is equal to $2^{mb} - 1$, the $\sigma$-LFSR $S$ is called a primitive $\sigma$-LFSR, and its $\sigma$-polynomial $f_S(x)$ is referred to as a primitive $\sigma$-polynomial. The characteristic polynomial of such a $\sigma$-LFSR is a primitive element of $\mathbb{F}_2[x]$.

We refer to this kind of companion matrices as $M$-companion matrices.

A conjecture on the total number of existing primitive $\sigma$-LFSRs over $\mathbb{F}_2^m$ has been presented in [ZHH07] as given by equation 4.

\[
N_P = \frac{|GL(m, \mathbb{F}_2)|}{2^{mb} - 1} \times \frac{\phi(2^{mb} - 1)}{mb} \times 2^{m(m-1)(b-1)}
\] (4)

In equation 4, $GL(m, \mathbb{F}_2)$ is the general linear group of non-singular matrices $\in \mathbb{F}_2^{m \times m}$, and $\phi$ represents Euler’s totient.

This conjecture has been proved in [KP14]. Moreover, [KP14] gives an algorithm for the enumeration of all $\frac{|GL(m, \mathbb{F}_2)|}{2^{mb} - 1} \times 2^{m(m-1)(b-1)}$ possible configuration matrices for a given primitive polynomial.

3 \textbf{\textit{\sigma-KDFC}}

Stream ciphers, like the SNOW series of ciphers, use word based LFSRs along with an FSM module. The feedback configuration of the LFSR in such schemes...
is publicly known. This feedback relation is an integral part of most algebraic attacks on such schemes \cite{BG05}. The resistance of such schemes to algebraic attacks could potentially increase if the feedback configuration is made key dependent. The large number of primitive feedback configurations for $\sigma$-LFSRs makes this possible. $\sigma$-KDFC is a key dependent mechanism of generating a primitive feedback configuration for a $\sigma$-LFSR. Each time the secret key is changed the feedback configuration of the $\sigma$-LFSR is updated. The first thing the method needs to achieve this goal, is a key-dependent random number generation module. Then a systematic mechanism is required to generate feedback configurations for the $\sigma$-LFSR using these random numbers. Finally, a configurable keystream generation module is required into which this $\sigma$-LFSR configuration can be deployed. Figure \ref{fig:fig2} shows the schematic of a $\sigma$-KDFC and its interaction with the host stream cipher.

![Fig. 2. The Schematic of $\sigma$-KDFC](image)

As shown in figure \ref{fig:fig2} a control line fed into a set of demultiplexers selects the running phase of the system. For example, the initialization and operation phases can be selected by setting $\text{PhaseSelect} = 0$ and $\text{PhaseSelect} = 1$ respectively. During the initialization phase, the demultiplexers forward their inputs to the configuration generation algorithm, which uses them as input random numbers. In the operation phase, the demultiplexers provide the output keystream of the modified cipher. The phases of $\sigma$-KDFC are explained in the following.

### 3.1 The Initialization Phase

The initialization phase comprises of three steps as explained below.
Step 1: Random Number Generation In this step, the $\sigma$-LFSR has a publicly known feedback configuration and is initialized using a secret key and an Initialization Vector (IV) (as is normally done in word based stream ciphers like SNOW). It collaborates with the FSM part of the host stream cipher to provide a set of random numbers. For a $\sigma$-LFSR with $b$, $m$-input $m$-output delay blocks, $mb - m$ random vectors are generated. These random vectors are used to generate the feedback configuration of the $\sigma$-LFSR.

Step 2: Key-Based Configuration Generation This phase is performed by an algorithm that uses the random vectors generated in the previous steps along with the same number of primitive polynomials in order to create a random configuration matrix in each invocation. These polynomials can be arbitrarily chosen from existing lists of primitive polynomials [Ziv94].

At any iteration, the algorithm uses a higher-degree primitive polynomial, and generates a matrix with an additional column until the number of columns reaches $n = m \times b$. The primitive polynomial with the highest degree is the characteristic polynomial of the final configuration matrix. The number of the iterations in this algorithm is controlled by the number of input random vectors. As the number of input random vectors increases, the randomness of the output configuration matrix increases at the cost of a nonlinearly-increasing time complexity. This trade-off can be resolved by controlling the number of random vectors (runs of $S$).

The initialization algorithm generates a matrix $Q$ which is in turn used to generate an $M$-companion matrix by means of a similarity transformation. Let $P$ be the companion matrix of a primitive polynomial. Consider the function $S_P$ defined in the following equation

$$S_P : GL(n, \mathbb{F}_2) \rightarrow GL(n, \mathbb{F}_2) \quad S_P(Q) = Q \times P \times Q^{-1} \quad (5)$$

The function $S_P(Q)$ transforms the matrix $Q$ into an $M$-companion matrix with the same characteristic polynomial as $P$.

The following result gives the structure of the matrix $Q$.

**Lemma 1.** (Lemma 4.5 proven in [KP14]) Let $P$ be the companion matrix of a given irreducible polynomial $f(x)$. Given an $m$-tuple $N = (n_1, n_2, \ldots, n_m) \in \mathbb{Z}^m_+$ such that $n_1 + n_2 + \cdots + n_m = n$, the matrix $P^* = Q \times P \times Q^{-1}$ is in the $M$-companion form if and only if $Q$ is an invertible matrix of the form

$$Q = [e^n_1; v_1; v_2; \cdots; v_{m-1}; e^n_1P; v_1P; \cdots; v_{m-1}P; \cdots; e^n_1P^{n_1-1}; v_1P^{n_2-1}; \cdots; v_{m-1}P^{n_m-1}]$$

where $v_2, \cdots, v_m \in \mathbb{F}_q^n$ are row vectors and $N = (n_1, n_2, \cdots, n_m)$ are such

$$\sum_{i=1}^b n_i = n, \quad e^n_1 = \begin{pmatrix} 0 \vdots 1 \\ n-1 \text{ times} \end{pmatrix} \in \mathbb{F}_2^n \quad \text{and} \quad v_i \in \mathbb{F}_2^n \quad \text{for} \quad i = 2, 3, \ldots, n.$$
generated in the first step of the initialization process along with a list of arbitrary primitive polynomials. These are used to generate the matrix $Q$ which is then acted upon by the function $S_P$. The algorithm guarantees to create a unique $Q$-matrix from each set of random vectors for a given set of primitive polynomials [KP14].

Algorithm 1 Full Rank Matrix Y Generation Algorithm

Input:
1. A full rank matrix stored in $M \in \mathbb{F}_2^{m \times m}$.
2. A set of $n - m$ primitive polynomials of degrees \{m, m + 1, \ldots, n\} stored in an array $L$. The corresponding companion matrices are represented by $P_{L[i]}$ for $m \leq i \leq n$. These polynomials can be arbitrarily selected from the lists of primitive polynomials which are available in literature [Ziv94].
3. A set of random binary vectors $R_i \in \mathbb{F}_2^n$, for $i \in \{0, \ldots, n - m\}$. (These are the \((n - m)\) random vectors generated in the first step of the initialization process.)

Output: A full rank matrix $Y \in \mathbb{F}_2^{m \times n}$.

1: procedure Pillai($M, L, R$)
2: $Y \leftarrow M$
3: $d \leftarrow \text{Dimension}(Y)$
4: $t \leftarrow 0$
5: while $t \leq (n - m)$ do
6: \hspace{1em} $c \leftarrow Y[t \, (\text{mod } m); ]$
7: \hspace{1em} $\lambda \leftarrow \text{Lin - solver}(c, P_{L[i]})$
8: \hspace{1em} $Y \leftarrow Y \times \lambda$
9: \hspace{1em} $Y[:, d + t - 1] \leftarrow R_t$
10: \hspace{1em} $Y[c, d + t - 2] \leftarrow 0$
11: \hspace{1em} $Y[c, d + t - 1] \leftarrow 1$
12: \hspace{1em} $t \leftarrow t + 1$
13: end while
14: Return $Y$
15: end procedure

Note that at every step, the size of $c$ vector increases. Therefore the time taken for each iteration of Algorithm 1 increases with each iteration. To circumvent this problem, a few of these iterations could be run offline in a server and the resulting $Y$ matrix could be made public. Let this matrix be denoted by $Y_{\text{init}}$. The remaining iterations can be done during the initialization phase of the keystream generator. When this is done, the variable $Y$ in Algorithm 1 will be initialized as $Y_{\text{init}}$, and the variable $t$ will be initialized as $k \, (\text{mod } m)$, where $k(m < k < n)$ is the number of iteration done offline. Further, the number of random vectors generated in the first step will now be $n - m - k$. The array $L$ will contain primitive polynomials with degrees \((m + k, \ldots, n)\). The 'for' loop will run $n - m - k$ times.
Algorithm 2 Find the power of $A$, $\lambda$, such that $c \times \lambda = e_{|c|}^1$

**Input:**

1. $c \in \mathbb{F}_2^1 \times \mathbb{F}_2^n$.
2. A companion matrix $A \in \mathbb{F}_2^{n \times n}$.

**Output:** The power of $A$, $\lambda$, such that $c \times \lambda = e_{|c|}^1$.

1: **procedure** Lin−solver($c, A$)
2: $X \leftarrow \begin{bmatrix} c \\ c \times A \\ c \times A^2 \\ \vdots \\ c \times A^{n-1} \end{bmatrix} \in \mathbb{F}_2^{n \times n}$
3: Solve the linear equation $y \times X = e_1^n$ for $y$.
4: $\lambda \leftarrow y[0] \times I + y[1] \times X + \cdots + y[n-1] \times X^{n-1}$
5: Return $\lambda$
6: **end procedure**

In algorithm 1, $\lambda = Lin − solver(c, P_L[t])$ is a matrix such that $c \times \lambda = e_{|c|}^1$. Algorithm 2 calculates $\lambda$ by solving a linear equation.

Algorithm 3 Configuration matrix Generation

**Input:**

1. Companion matrix of a primitive polynomial of degree $n$, $P_z$.
2. Full rank matrix $Y \in \mathbb{F}_2^{m \times n}$ generated from Algorithm 1.

**Output:** A random configuration matrix for $\mathcal{S}$ is stored in $C_S$.

1: **procedure** FIND-CONFIG($P_z, Y$)
2: $P_z \leftarrow P_{L[n-m-1]}$

8
Step 3: Configuration Deployment

Algorithm 3 generates a configuration matrix in the form of equation 2. The last $m$ rows in the generated configuration matrix contain the gain matrices $B_0$ through $B_{b-1}$. The feedback gains of the $\sigma$-LFSR are set as $B_0, B_1, \ldots, B_{b-1}$.

3.2 An Analysis on the Initialization Phase

In this subsection, we analyse the time complexity of the initialization phase. Then we discuss the flexibility of $\sigma$-KDFC, which makes it possible to manage the tradeoff between the complexity and the algebraic resistance.

Configuration Generation Time Complexity

Algorithm 2 is of time complexity $O(n^3)$ as it uses Gaussian elimination of type LU(Lower-Upper Triangular) decomposition to solve the system of linear equations. On the other hand, algorithm 1 calls algorithm 2 once in each of its $n - m$ iterations. Thus, the time complexity of algorithm 1 is of order $O(n^4)$.

Flexibility

The structure of algorithm 1 allows for a trade-off between time complexity on one hand and randomness and algebraic resistance on the other. The total time taken to calculate the feedback configuration can be reduced by increasing the number of iterations of Algorithm 1 done in the server. This however reduces the number of possible configurations and hence compromises on the randomness.
3.3 The Operation Phase

In this phase, the \(\sigma\)-LFSR with the updated feedback configuration is used along with the FSM to generated the keystream. Its configuration is updated by the mentioned algorithm each time the secret key is changed. \(\sigma\)-KDFC does not change the FSM part of the host stream cipher.

4 Algebraic Resistance Analysis

In this subsection, we analyse the complexity and algebraic resistance of \(\sigma\)-KDFC.

4.1 Definitions

An \(n\)-variable Boolean function \(f\) is a map, \(f : \mathbb{F}_2^n \rightarrow \mathbb{F}_2\). It can be written in the Algebraic Normal Form (ANF) representation as follows [Car].

\[
f = \bigoplus_{I \in P(l)} a_I \left( \prod_{i \in I} x_i \right) = \bigoplus_{I \in P(n)} a_I x^I
\]  

(6)

where, \(P(n)\) denotes the power set of \(N = \{1, \ldots, n\}\) and \(f\) belongs to the ring \(\mathbb{F}_2[x_0, x_1, \ldots, x_{n-1}] / < x_0^2 + x_0, x_2^2 + x_2, \ldots, x_{n-1}^2 + x_{n-1}>\). The algebraic degree of the Boolean function \(f\), denoted by \(|f|\) is simply defined as \(\{\max |T| | a_T \neq 0\}\), where \(|T|\) represents the cardinality of the set \(T\).

Observe that the entries of the last \(m\)-rows of the configuration matrix \(C_S\), i.e. the entries of the \(B_i\)s in Equation 2 are polynomials in the entries of the matrix \(Y\) which is calculated in Algorithm 1. These entries are in turn related to the initial state of the \(\sigma\)-LFSR which is unknown to the adversary. Therefore,

\[
B_{k(i,j)} = f_{k(i,j)}(U)
\]  

(7)

where \(U\) represents the entries of the last \(m-1\) rows of the matrix \(Y\) (The first row of \(Y\) is \(e_1^n\)). The algebraic degree of \(C_S\) is defined by equation 8.

\[
\Theta(C_S) = \max_{k,i,j} (|f_{k(i,j)}(U)|)
\]  

(8)

In the rest of this paper, we simply use \(\Theta\) instead of \(\Theta(C_S)\) to represent the algebraic degree of \(C_S\). \(\Theta\) can be considered as a measure of the algebraic resistance of \(\sigma\)-KDFC. In the succeeding subsection we calculate a lower bound for \(\Theta\).

4.2 Analysis

In this section we analyse the entries of the \(B_i\)s, generated in the proposed method, as boolean functions and give a lower bound on the maximum degree of these functions. We begin our analyses making a few observations.
Algorithm 1 generates a matrix $Y \in \mathbb{F}_{2}^{m \times n}$ with first row $e_{1}^{n}$. Let the remaining rows of $Y$ be $v_{1}, v_{2}, \ldots, v_{m-1}$. For $1 \leq i \leq m - 1$, let the entries of $v_{i}$ be $v_{i,1}, v_{i,2}, \ldots, v_{i,n}$. These $v_{i,j}$s are functions of the initial state of the stream cipher. In Algorithm 3, $Y$ is in turn used to generate the following matrix.

\[
Q = \begin{bmatrix}
    e_{1}^{n} \\
    v_{1} \\
    v_{2} \\
    \vdots \\
    v_{m-1} \\
    e_{1}^{n}P_{z} \\
    v_{1}P_{z} \\
    \vdots \\
    v_{m-1}P_{z} \\
    \vdots \\
    e_{1}^{n}P_{z}^{b-1} \\
    v_{1}P_{z}^{b-1} \\
    \vdots \\
    v_{m-1}P_{z}^{b-1}
\end{bmatrix}
\]  \hspace{1cm} (9)

where $P_{z}$ is the companion matrix of a publicly known primitive characteristic polynomial of the $\sigma$-LFSR. Finally, the configuration matrix $C_{S}$ is generated by the formula $C_{S} = Q \times P_{z} \times Q^{-1}$. Since $Q$ is an inevitable boolean matrix, the determinant of $Q$ is always 1. This results in $Q^{-1} = Q^{(a)}$ where $Q^{(a)}$ is the adjugate of $Q$. Moreover, since the elements of $Q$ belong to $\mathbb{F}_{2}$, the co-factors are equal to minors of $Q$. The rows of $Q$ can be permuted to get the following matrix $Q_{P}$.
\[ Q^P = \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 1 & * \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & * \\
0 & 0 & \cdots & 0 & 1 & * & * \\
v_{1,1} & v_{1,2} & \cdots & v_{1,n-b} & v_{1,n-b+1} & \cdots & v_{1,n-1} & v_{1,n} \\
v_{1,2} & v_{1,3} & \cdots & v_{1,n-b+1} & v_{1,n-b+2} & \cdots & v_{1,n} & * \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & * \\
v_{1,b} & v_{1,b+1} & \cdots & v_{1,n} & v_{1,n+1} & \cdots & * & * \\
v_{2,1} & v_{2,2} & \cdots & v_{2,n-b} & v_{2,n-b+1} & \cdots & v_{2,n-1} & v_{2,n} \\
v_{2,2} & v_{2,3} & \cdots & v_{2,n-b+1} & v_{2,n-b+2} & \cdots & v_{2,n} & * \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & * \\
v_{2,b} & v_{2,b+1} & \cdots & v_{2,n} & v_{2,n+1} & \cdots & * & * \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & * \\
v_{m-1,1} & v_{m-1,2} & \cdots & v_{m-1,n-b} & v_{m-1,n-b+1} & \cdots & v_{m-1,n-1} & v_{m-1,n} \\
v_{m-1,2} & v_{m-1,3} & \cdots & v_{m-1,n-b+1} & v_{m-1,n-b+2} & \cdots & v_{m-1,n} & * \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & * \\
v_{m-1,b} & v_{m-1,b+1} & \cdots & v_{m-1,n} & v_{m-1,n+1} & \cdots & * & * \\
\end{pmatrix} \in \mathbb{F}_2^{n \times n}
\]

(10)

where \(*\) denotes the linear combination of \((v_{i,1}, v_{i,2}, \cdots, v_{i,n})\). Note that \(Q^{-1}\)
can be got by permuting the rows of \(Q_P^{-1}\). The matrix \(Q_P\) can be decomposed into four sub-matrices:

- \(Q_1 = [Q_P[i,j]] 1 \leq i \leq b, 1 \leq j \leq n - b\).
- \(Q_2 = [Q_P[i,j]] 1 \leq i \leq b, n - b + 1 \leq j \leq n\).
- \(Q_3 = [Q_P[i,j]] b + 1 \leq i \leq n, 1 \leq j \leq n - b\).
- \(Q_4 = [Q_P[i,j]] b + 1 \leq i \leq n, n - b + 1 \leq j \leq n\).
These sub-matrices appear as follows:

\[
\begin{align*}
Q_1 &= \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}_{b \times n-b} \\
Q_2 &= \begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
1 & \cdots & * \end{pmatrix}_{b \times b}
\end{align*}
\]

Let us consider \( Q_P \) is invertible, then \( \det(Q_P) = \det(Q_3) = 1 \). We now proceed to analyse some of the minors of \( Q_P \) in the following lemma.

**Lemma 2.** If \( i = b \) and \( 1 \leq j \leq (n - b) \), then \( \mu(Q_P[i,j]) \in \Gamma_{n-b} \).

**Proof.** For two matrices \( A \) and \( B \) with the same number of rows, let \([AB]_{p,q}\) represent the matrix which results from removing the \( p^{th} \) column from \( A \) and appending the \( q^{th} \) column of \( B \) to \( A \). For \( i = b \) and \( 1 \leq j \leq (n - b) \), \( \mu(Q_P[i,j]) \) is given by:

\[
\mu(Q_P[i,j]) = \det([Q_3Q_4]_{j,1})
\]

Recall that for a binary matrix \( M \in \mathbb{F}_2^{n \times n} \) its determinant is given by the following formula

\[
\det(M) = \sum_{f \in S_n} \prod_{1 \leq i \leq n} M(i, f(i))
\]

where \( S_n \) is the set of permutations on \( (1,2,\ldots,n) \). Observe that the diagonal elements of \( ([Q_3Q_4])_{j,1} \) are distinct \( v_{i,k} \& s \). Their product corresponds to the identity permutation in the determinant expansion formula for \( [Q_3Q_4]_{j,1} \). The resultant monomial has degree \( n - b \). Further, this monomial will not occur as a result of any other permutation. Hence \( \det([Q_3Q_4]_{j,1}) \) is always a polynomial of degree \( n - b \).
Lemma 3. If \( 1 \leq i \leq b \) then

\[
\mu(Q_P[i,j]) = \begin{cases} 
\det(Q_3) & i + j = n + 1 \\
0 & i + j \geq n + 1
\end{cases} \tag{14}
\]

Proof. Observe that, for \( 1 \leq i \leq b \) and \( i + j = n + 1 \), the \( Q_P[i,j] \)'s are the anti-diagonal elements of \( Q_2 \). Clearly, the minors of these elements are all equal to the determinant of \( Q_3 \). As we have already seen the invertibility of \( Q_P \) implies that this determinant is always 1. Therefore, \( \mu(Q_P[i,j]) = 1 \) when \( i + j = n + 1 \).

Note that, for \( 1 \leq i \leq b \) and \( i + j > n + 1 \), the \( Q_P[i,j] \)'s are the elements of \( Q_2 \) that are below the anti-diagonal. Observe that if the row and column corresponding to such an element are removed from \( Q_P \) then the first \( b - 1 \) rows of the resulting matrix are always rank deficient. Therefore, the determinant of this matrix is always 0. Therefore, \( \mu(Q_P[i,j]) = 0 \).

Lemma 4. If \( b + 1 \leq i \leq n \) and \( 1 \leq j \leq n - b \), then \( \mu(Q_P[i,j]) \in \Gamma_{n-b-1} \).

Proof. Observe that the elements of \( Q_P \) considered in this lemma are elements of the sub-matrix \( Q_3 \). Therefore, \( \mu(Q_P[i,j]) \), for the range of \( i \) and \( j \) considered, is nothing but the determinant of the sub-matrix of \( Q_3 \) got by deleting the \( i^{th} \) row and \( j^{th} \) column of \( Q_3 \). The diagonal elements of such a sub-matrix are distinct \( v_{i,j} \)'s. Their product will result in a monomial of degree \( n - b - 1 \). This corresponds to the identity permutation in the determinant expansion formula given by Equation \( \text{12} \). Observe that no other permutation generates this monomial. Hence, the minor will always have a monomial of degree \( n - b - 1 \). Therefore, \( \mu(Q_P[i,j]) \in \Gamma_{n-b-1} \).

Lemma 5. If \( b + 1 \leq i \leq n \) and \( n - b + 1 \leq j \leq n \), then \( \mu(Q_P[i,j]) = 0 \).

Proof. The elements of \( Q_P \) considered in this lemma are elements of the sub-matrix \( Q_4 \). Whenever the row and column corresponding to such an element is removed from \( Q_P \), the rows of the submatrix \( Q_2 \) become linearly dependent. Therefore, the first \( b \) rows of the resultant matrix are always rank deficient. Consequently, \( \mu(Q_P[i,j]) = 0 \).

For a given matrix \( A \) with polynomial entries, let \( \Theta(A) \) be the maximum degree among all the entries of \( A \). As there are \( n - b \) rows in \( Q_P \) with variable entries, \( \Theta(Q_P^{-1}) \leq n - b \). Therefore, we get the following as a consequence of Lemma \( \text{2} \)

\[
\Theta(Q^{-1}) = \Theta(Q_P^{-1}) = n - b \tag{15}
\]

Recall that the configuration matrix \( C_S \) is given by \( QP_ZQ^{-1} \) where \( P_Z \) is the companion matrix of the characteristic polynomial of the \( \sigma \)-LFSR. We now use the above developed machinery to calculate \( \Theta(C_S) \).

Theorem 1. \( \Theta(C_S) \geq n - b \).
Proof. Observe that the gain matrices $B_0, B_1, \ldots, B_{b-1}$ appear in the last $m$ rows of $C_S$. These rows are generated by multiplying the last $m$ rows $QP_Z$ with $Q^{-1}$. The last $m$ rows of $QP_Z$ are as follows

$$
\begin{pmatrix}
0 & 0 & \cdots & 1 & \cdots & * & * & * \\
v_{1,b+1} & v_{1,b+2} & \cdots & v_{1,n} & \cdots & * & * & * \\
v_{2,b+1} & v_{2,b+2} & \cdots & v_{2,n} & \cdots & * & * & * \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\
v_{m-1,b+1} & v_{m-1,b+2} & \cdots & v_{m-1,n} & \cdots & * & * & * \\
\end{pmatrix} \in \mathbb{F}_2^{b \times n} \tag{16}
$$

The element $C_S[n-m+1, n-m+1]$ is got by multiplying the $(n-m+1)$-th row of $QP_Z$ with the $(n-m+1)$-th column of $Q^{-1}$. Note that the $(n-m+1)$-th column of $Q^{-1}$ is equal to the $b$-th column of $Q_P^{-1}$. As a consequence of Lemmas 2 and 3 this column has the following form.

$$
Q^{-1}[; n-m+1] = (P_1, P_2, \ldots, P_{n-b}, 1, 0, \ldots, 0)^T \tag{17}
$$

where $P_1, P_2, \ldots, P_{n-b} \in \Gamma(n-b)$. Therefore,

$$
C_S[n-b+1, n-b+1] = (0, 0, \cdots, 1, *, \cdots, *, *) \times (P_1, P_2, \ldots, P_{n-b}, 1, 0, \cdots, 0)^T \\
\text{with } (n-b) \text{ entries} \\
= P_{n-b}
$$

Hence, it is proved that $\Theta(C_S) \geq n-b$.

Example 1. Consider a primitive $\sigma$–LFSR with 4, 2-input 2-output delay blocks i.e. $m = 2$ and $b = 4$. Therefore $n = mb = 8$. The primitive polynomial for the companion matrix $P_z$ is $f(x) = x^8 + x^4 + x^3 + x^2 + 1$. The corresponding matrices $Q$ and $Q_P$ have the following structure:

$$
Q = 
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_1 + x_3 + x_4 + x_5 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_1 + x_3 + x_4 + x_5 & x_1 + x_3 + x_4 + x_5 + x_6 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
x_4 & x_5 & x_6 & x_7 & x_8 & x_1 + x_3 + x_4 + x_5 & x_2 + x_4 + x_5 + x_6 & x_3 + x_5 + x_6 + x_7 \\
\end{pmatrix} \tag{18}
$$
SNOW 2.0 (Adopted by ISO/IEC standard IS 18033-4) was introduced later in [EJ02] as a modified version of SNOW 1.0. This version was shown to be vulnerable to algebraic attacks [BG05]. We consider SNOW 2.0 as a test case and demonstrate how replacing the LFSR in this scheme with a $\sigma$-LFSR increases its resistance to algebraic attacks.

The block diagram of SNOW 2.0 is shown in figure 3.
In figure 3, + and ⊕ represent the $GF(2)$ addition operation (implemented in hardware by XOR gates) and integer addition modulo $2^{32}$ respectively. As shown in figure 3, the keystream generator in SNOW 2.0 consists of an LFSR and an FSM (Feedback State Machine). The generating polynomial of the word based LFSR is given by:

$$F_S(x) = (\alpha x^{16} + x^{14} + \alpha^{-1}x^5 + 1) \in \mathbb{F}_{2^{32}}[X]$$

(20)

where, $\alpha$ is the root of the following primitive polynomial

$$G_S(x) = (x^4 + \beta^{23}x^3 + \beta^{245}x^2 + \beta^{248}x + \beta^{239}) \in \mathbb{F}_{2^8}[X]$$

where $\beta$ is the root of the following primitive polynomial.

$$H_S(x) = x^8 + x^7 + x^5 + x^3 + 1 \in \mathbb{F}_2[X]$$

The state transition equation of the LFSR in figure 3 is as follows:

$$D^t_{16} = \alpha^{-1}D^t_{11} + D^t_2 + \alpha D^t_0.$$ 

where $D^t_i \in \mathbb{F}_{2^{32}}$ is the value stored in the $i^{th}$ delay block at time $t$ (after the $t^{th}$ clock cycle).

The FSM contains two 32-bit registers $R1$ and $R2$. These registers are connected by means of an S-Box which is made using four AES S-boxes. This S-box serves as the source of nonlinearity.

5.2 KDFC-SNOW:

In the proposed modification, we replace the LFSR part of SNOW 2.0 by a $\sigma$-LFSR having 16, 32-input 32 output delay blocks. The configuration matrix of
the $\sigma$-LFSR is generated using Algorithms 1-3. We shall refer to the modified scheme, shown in Figure 4 as KDFC-SNOW.

During initialization the configuration matrix of the $\sigma$-LFSR is publicly known. This could correspond to the feedback configuration of the LFSR in SNOW-2 where multiplication by finite field elements are expressed using suitable matrices. As in SNOW 2.0, the $\sigma$-LFSR is initialized using a 128-bit IV and a 128/256-bit secret key $K$. KDFC-SNOW is run with this configuration for 32 clock cycles without producing any symbols at the output. The random numbers thus generated are used in Algorithms 1-3 to generate a new feedback configuration. This configuration replaces the original one and the resulting set-up is used to generate the keystream. As we have already mentioned, some of the iterations of Algorithm 1 could be precalculated and the remaining could be done as a part of the initialization process. In the particular case that we have considered, it is assumed that 468 iterations of Algorithm 1 are precalculated and the result, $Y_{\text{init}} \in F_{2}^{32 \times 500}$, is publicly known. The last 12 iterations of Algorithm 1 are carried out as part of the initialization process.

### 5.3 Initialization of KDFC-SNOW

- The delay blocks $D_0, \cdots, D_{15}$ are initialized using the 128/256 bit secret key $K$ and a 128 bit IV in exactly the same manner as SNOW 2.0. The registers $R_1$ and $R_2$ are set to zero.
- The initial feedback configuration of the $\sigma$-LFSR is identical to SNOW 2.0. This is done by setting $B_{11}$ and $B_0$ as matrices that represent multiplication by $\alpha^{-1}$ and $\alpha$ respectively. Further, $B_2$ is set to identity. The other gain matrices are set to zero.
KDFC-SNOW is run in this configuration for 32 clock cycles without making the output externally available. During this phase the feedback equation is given as follows,

\[ D_{t+1}^{15} = B_0 D_t^0 + B_1 D_t^1 + \cdots + B_{15} D_t^{15} + F^t \]  

(21)

where \( F^t \) is the output of the FSM at the \( t \)-th instant. The last 12 values of \( F^t \) are used as the random numbers in Algorithm 1.

A new configuration matrix is calculated using Algorithms 1-3 and the corresponding feedback configuration replaces the original one.

### 5.4 Generation of Nonlinear Equation of KDFC-SNOW

We now discuss the equations that govern the working of KDFC-SNOW. The outputs of the delay blocks of the \( \sigma \)-LFSR are related as per the following equation:

\[ D_{t+1}^{15} = B_0 D_t^0 + B_1 D_t^1 + \cdots + B_{15} D_t^{15} \]  

(22)

where \( D_k^t \) is the value of the \( k \)-th delay block at the \( t \)-th time instant. Note that

\[
D_k^t = \begin{cases} 
D_{k+t}^0 & 0 \leq k + t \leq 15 \\
B_0 D_t^0 + B_1 D_t^1 + \cdots + B_{15} D_t^{15} & k + t > 15 
\end{cases}
\]  

(23)

The value of the keystream at the \( t \)-th time instant is given by the following equation. Let \( F_t \) be the output of the FSM at time \( t \),

\[ F_t = (D_t^{15} \oplus R_1^t) + R_2^t \]  

(24)

The registers are updated as follows:

\[ R_1^{t+1} = D_5^t \oplus R_2^t \]  

(25)

\[ R_2^{t+1} = S(R_1^t) \]  

(26)

\[ z^t = R_1^t \oplus D_t^{15} + R_0^t + D_0^t = (R_2^{t-1} \oplus D_1^t) \oplus D_t^{15} + R_2^t + D_0^t \]  

(27)

where \( R_1^t \) and \( R_2^t \) represent the values of registers \( R_1 \) and \( R_2 \) at time instant \( t \). The operation \( \oplus \) is defined as follows:

\[ x \oplus y = (x + y) \mod 2^{32} \]  

(28)

The challenge for an adversary in this scheme is to find the gain matrices \( \{B_0, B_1, \cdots, B_{15}\} \) in addition to the initial state \( \{D_0^0, \cdots, D_{15}^0\} \).

Note that Equations 24 to 27 are got from the FSM. Since the FSM part of the keystream generator is identical for SNOW 2.0 and KDFC-SNOW these equations are identical for both schemes.
5.5 Security enhancement due to KDFC-SNOW

We first briefly the Algebraic attack on SNOW 2 described in [BG05] and demonstrate why this attack becomes difficult with KDFC-SNOW. This attack first attempts to break a modified version of the scheme where the $\boxplus$ operator is approximated by $\oplus$. The state of LFSR and the value of the registers at the end of the 32 initialization cycles are considered unknown variables. This accounts for a total of $512 + 32 = 544$ unknown variables. The algebraic degree of each of the S-box(S) equations (156 linearly independent quadratic equations in each clock cycle) is 2. Rearranging the terms in Equation 27 we get the following:

$$R_t^2 = (R_t^{t-1} \boxplus D_t^1) \boxplus D_{15}^t + D_0^t + z^t. \quad (29)$$

Note that $R_0^1 = R_2^0 + z_0^0 + D_0^0 + D_{15}^0$. Therefore, by approximating $\boxplus$ as $\oplus$, Equation 29 expands to the following:

$$R_2^t = R_2^0 + \sum_{i=0}^t z^i + \sum_{i=0}^t (D_i^1 + D_{15}^t + D_0^t) \quad (30)$$

Further Equation 26 can be expanded as follows:

$$R_{t+1}^1 = \mathcal{S}(R_t^1) = \mathcal{S}(R_2^0 + z^t + D_{15}^t + D_0^t) \quad (31)$$

In equation 31 the outputs of the delay blocks can be related to the initial state of the LFSR using equation 23. Because of the nature of the S-Box, Equation 31 gives rise to 156 quadratic equations per time instant ([BG05]). When these equations are linearized, the number of variables increases to $\sum_{i=0}^{544} i = 2^{17}$. Therefore with $2^{17}/156 \approx 951$ samples, we get a system of equations, which can be solved in $\mathcal{O}(2^{51})$ time, to obtain the initial state of the LFSR and the registers. This attack is then modified to consider the $\boxplus$ operator. This attack has a time complexity of approximately $\mathcal{O}(2^{294})$.

When the LFSR in SNOW 2.0 is replaced by a $\sigma$-LFSR, the feedback equation is no longer known. If the entries of the feedback gain matrices are considered as unknowns, then there are a total of $16 \times m^2 + mb + m = 16928$ unknown variables. The output of the delay blocks at a given instant are functions of these variables. The degree of these functions increase with each time instant till the degree reaches $16m^2 + n = 16896$. Therefore, the equations that are generated by Equation 31 are no longer quadratic in the set of all unknowns. These equations have a significantly higher degree and higher number of unknowns. Therefore, solving these equations using linearization is not feasible.

One could instead consider the rows of the matrix $Y$ generated by Algorithm 1 as unknowns. Assuming that the first row is $e_1^n$, the total number of unknowns will now be $31 \times 512 = 15872$. As we have already seen, the entries of the feedback matrices ($B_i$s) are polynomials in these variables. From Theorem 1 the maximum degree of these polynomials is atleast $n - b$. Therefore, the maximum degree of the equations generated by Equation 31 will be atleast $n - b + 1 = 497$. Therefore, linearizing this system of equations gives rise to a system of linear
equations in $N = \sum_{i=0}^{197} \binom{16416}{i} \approx \mathcal{O}(2^{3207})$ unknowns. Therefore, an algebraic attack on this scheme that uses linearization seems unfeasible.

Further, the feedback equation of the LFSR is central to known Guess and Determine attacks on SNOW 2.0 (\cite{AE09}, \cite{NP14}). Since this equation is not known apriori in KDFC-SNOW, launching such attacks is much more difficult.

### 5.6 Randomness Test

In this subsection, we evaluate the randomness of the keystream generated by KDFC-SNOW.

#### Test Methodology

We have used the NIST randomness test suite to evaluate the randomness of a keystream generated by KDFC-SNOW. There are are 16 randomness tests in the suite. Each test returns a level of significance i.e. $P - Value$. If this value is above 0.01 for a given test, then the keystream is considered to be random for that test.

KDFC-SNOW has been implemented using SageMath 8.0. The NIST randomness tests have been conducted on the generated keystream using Python.

The characteristic polynomial of the $\sigma$-LFSR has been taken as $f(x) = x^{512} + x^{510} + x^{504} + x^{502} + x^{501} + x^{494} + x^{493} + x^{490} + x^{486} + x^{485} + x^{483} + x^{481} + x^{480} + x^{478} + x^{477} + x^{471} + x^{470} + x^{469} + x^{466} + x^{462} + x^{461} + x^{459} + x^{458} + x^{452} + x^{449} + x^{446} + x^{445} + x^{444} + x^{441} + x^{438} + x^{437} + x^{434} + x^{433} + x^{432} + x^{431} + x^{429} + x^{427} + x^{424} + x^{423} + x^{420} + x^{419} + x^{414} + x^{412} + x^{411} + x^{409} + x^{405} + x^{402} + x^{400} + x^{399} + x^{398} + x^{396} + x^{395} + x^{392} + x^{390} + x^{388} + x^{387} + x^{385} + x^{375} + x^{374} + x^{372} + x^{371} + x^{366} + x^{365} + x^{362} + x^{359} + x^{357} + x^{356} + x^{355} + x^{354} + x^{353} + x^{352} + x^{351} + x^{350} + x^{347} + x^{345} + x^{344} + x^{343} + x^{341} + x^{339} + x^{338} + x^{337} + x^{336} + x^{333} + x^{330} + x^{329} + x^{326} + x^{324} + x^{322} + x^{319} + x^{310} + x^{306} + x^{305} + x^{304} + x^{303} + x^{301} + x^{299} + x^{298} + x^{297} + x^{296} + x^{295} + x^{294} + x^{293} + x^{292} + x^{291} + x^{289} + x^{286} + x^{285} + x^{283} + x^{282} + x^{281} + x^{278} + x^{276} + x^{274} + x^{271} + x^{269} + x^{264} + x^{262} + x^{259} + x^{258} + x^{257} + x^{255} + x^{253} + x^{251} + x^{249} + x^{248} + x^{243} + x^{240} + x^{239} + x^{238} + x^{236} + x^{235} + x^{233} + x^{232} + x^{230} + x^{229} + x^{228} + x^{227} + x^{226} + x^{222} + x^{217} + x^{216} + x^{214} + x^{213} + x^{211} + x^{208} + x^{206} + x^{203} + x^{201} + x^{199} + x^{193} + x^{190} + x^{181} + x^{179} + x^{177} + x^{175} + x^{174} + x^{173} + x^{172} + x^{171} + x^{169} + x^{167} + x^{164} + x^{163} + x^{158} + x^{156} + x^{155} + x^{153} + x^{152} + x^{151} + x^{149} + x^{147} + x^{146} + x^{143} + x^{141} + x^{138} + x^{136} + x^{132} + x^{131} + x^{129} + x^{128} + x^{126} + x^{125} + x^{123} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} + x^{116} + x^{115} + x^{113} + x^{112} + x^{111} + x^{109} + x^{105} + x^{104} + x^{103} + x^{102} + x^{98} + x^{97} + x^{94} + x^{93} + x^{89} + x^{88} + x^{87} + x^{81} + x^{78} + x^{76} + x^{75} + x^{73} + x^{72} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{63} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{53} + x^{51} + x^{50} + x^{49} + x^{47} + x^{46} + x^{45} + x^{44} + x^{41} + x^{39} + x^{37} + x^{36} + x^{33} + x^{30} + x^{26} + x^{25} + x^{21} + x^{20} + x^{19} + x^{16} + x^{5} + 1.$

(This polynomial is the characteristic polynomial of the LFSR in SNOW 2.0 when it is implemented as a $\sigma$-LFSR i.e. when multiplication by $\alpha$ and $\alpha^{-1}$ are represented by matrices).

The keystream has been generated using the following key (K) and initialization vector (IV).
\[ K = [681, 884, 35, 345, 203, 50, 912, 358], IV = [645, 473, 798, 506] \quad (32) \]

**Test Results**

The results obtained from 14 NIST tests are shown in table 1.

| Number | Test                                                   | P-Value                  | Random |
|--------|--------------------------------------------------------|--------------------------|--------|
| 01.    | Frequency Test (Monobit)                               | 0.35966689490586123      | ✓      |
| 02.    | Frequency Test within a Block                          | 0.24374184001729746      | ✓      |
| 03.    | Run Test                                               | 0.9038184342134019       | ✓      |
| 04.    | Longest Run of Ones in a Block                         | 0.5246846287441829       | ✓      |
| 05.    | Binary Matrix Rank Test                                | 0.1371167998339736       | ✓      |
| 06.    | Discrete Fourier Transform (Spectral) Test             | 0.1371167998339736       | ✓      |
| 07.    | Non-Overlapping Template Matching Test                 | 0.3189818228443801       | ✓      |
| 08.    | Overlapping Template Matching Test                     | 0.211350493609367        | ✓      |
| 09.    | Maurer’s Universal Statistical test                    | 0.452108320974134        | ✓      |
| 10.    | Linear Complexity Test                                 | 0.1647939201114819       | ✓      |
| 11.    | Serial Test                                            | 0.7821664366290292       | ✓      |
| 12.    | Approximate Entropy Test                               | 0.880218270580662        | ✓      |
| 13.    | Cumulative Sums (Forward) Test                         | 0.3463079954965923       | ✓      |
| 14.    | Cumulative Sums (Reverse) Test                         | 0.663368600204551        | ✓      |

**Table 1. NIST Randomness Test**

The results obtained for the *Random Excursions Test* are shown in table 2.

| State | CHI SQUARED  | P-Value                  | Random |
|-------|--------------|--------------------------|--------|
| -4    | 9.375081555789523 | 0.09500667227464867       | ✓      |
| -3    | .9069118454935624  | 0.969735280059932         | ✓      |
| -2    | 3.1963121920209367 | 0.6697497097941535       | ✓      |
| -1    | 5.343347639484978  | 0.375429196798428         | ✓      |
| 1     | 5.446351931330472  | 0.363864453873992         | ✓      |
| 2     | 6.937635775976262  | 0.22531988887331122       | ✓      |
| 3     | 13.843145064377687  | 0.016637085511558194      | ✓      |
| 4     | 6.790226890440857   | 0.579995946955087         | ✓      |

**Table 2. 15. Random Excursions Test**

Table 3 shows the results for the 16th test i.e. the *Random Excursions Variant Test*.
| State | Count | P-Value     | Random | State | Count | P-Value     | Random |
|-------|-------|-------------|--------|-------|-------|-------------|--------|
| -9.0  | 270   | 0.11944065987006025 | ✓  | +1.0  | 467   | 0.97386995237389 | ✓  |
| -8.0  | 307   | 0.178703957218327  | ✓  | +2.0  | 488   | 0.6773674079894312 | ✓  |
| -7.0  | 359   | 0.3310083710716354 | ✓  | +3.0  | 470   | 0.9532739974827851 | ✓  |
| -6.0  | 389   | 0.44696915370831947 | ✓  | +4.0  | 416   | 0.5358953455898371 | ✓  |
| -5.0  | 418   | 0.6002107789999439  | ✓  | +5.0  | 386   | 0.3823929438406025 | ✓  |
| -4.0  | 426   | 0.6204409395957975  | ✓  | +6.0  | 397   | 0.495576078534262  | ✓  |
| -3.0  | 439   | 0.6924575808023399  | ✓  | +7.0  | 430   | 0.7436251044167517 | ✓  |
| -2.0  | 486   | 0.705256223122887   | ✓  | +8.0  | 454   | 0.9191606776606087 | ✓  |
| -1.0  | 486   | 0.512389348919496   | ✓  | +9.0  | 481   | 0.9051424340008056 | ✓  |

Table 3. Randomness Excursions Variant Test

These results are comparable to that of SNOW 2.0. Note that the feedback configuration of SNOW 2.0 is one of the possible feedback configurations in the $\sigma$-KDFC scheme.

6 Conclusions

In this paper, we have described a method of using $\sigma$-LFSRs with key dependent feedback configurations in stream ciphers that use word based LFSRs. In this method, an iterative configuration generation algorithm (CGA) uses key-dependent random numbers to generate a random feedback configuration for the $\sigma$-LFSR. We have theoretically analysed the algebraic degree of the resulting feedback configuration. As a test case, we have demonstrated how this scheme can be applied to SNOW 2.0. We have shown that the resulting keystream generator is resistant to algebraic attacks that have been launched against SNOW 2.0. Further, the keystreams generated by the proposed method are comparable to SNOW 2.0 from a randomness point of view.

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