The minimum number of clique-saturating edges

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Abstract

Let $G$ be a $K_p$-free graph. We say $e$ is a $K_p$-saturating edge of $G$ if $e \not\in E(G)$ and $G + e$ contains a copy of $K_p$. Denote by $f_p(n,e)$ the minimum number of $K_p$-saturating edges that an $n$-vertex $K_p$-free graph with $e$ edges can have. Erdős and Tuza conjectured that $f_4(n, \lfloor n^2/4 \rfloor + 1) = (1 + o(1)) \frac{n^2}{16}$. Balogh and Liu disproved this by showing $f_4(n, \lfloor n^2/4 \rfloor + 1) = (1 + o(1)) \frac{n^2}{33}$. They believed that a natural generalization of their construction for $K_p$-free graph should also be optimal and made a conjecture that $f_{p+1}(n, \text{ex}(n, K_p) + 1) = \left( \frac{2(p-2)^2}{(p+1)^2 - 12p+8} + o(1) \right) n^2$ for all integers $p \geq 3$. The main result of this paper is to confirm the above conjecture of Balogh and Liu.

1 Introduction

Given a graph $H$, we say a graph $G$ is $H$-free if $G$ does not contain $H$ as a subgraph. Let the Turán number $\text{ex}(n, H)$ of $H$ denote the maximum number of edges in an $n$-vertex $H$-free graph. The study of Turán numbers can date back to the work of Mantel\textsuperscript{9} and is the central subject in extremal graph theory (see\textsuperscript{7} for a recent survey). The classical theorem of Turán\textsuperscript{10} states that for any integer $p \geq 2$, the unique $n$-vertex $K_{p+1}$-free graph attaining the maximum number $\text{ex}(n, K_{p+1})$ of edges is the $p$-partite Turán graph $T_p(n)$, i.e., the $n$-vertex complete balanced graph.

For $p \geq 3$, let $G$ be a $K_p$-free graph and $e$ be a non-edge of $G$ (i.e., an edge in the complement of $G$). We say $e$ is a $K_p$-saturating edge of $G$, if $G + e$ contains a copy of $K_p$. This notion is closely related to Turán numbers. Indeed, a $K_p$-free graph $G$ is maximal if and only if every non-edge of $G$ is a $K_p$-saturating edge (let us call this property $\star$). So in other words, Turán’s Theorem determines the maximum number of edges $e(G)$ over all $K_p$-free graphs $G$ satisfying the property $\star$. On the other hand, Zykov\textsuperscript{11} and independently Erdős, Hajnal and Moon\textsuperscript{6} determined the minimum number $e(G)$ over all $n$-vertex $K_p$-free graphs $G$ satisfying the property $\star$, which is uniquely attained by the $n$-vertex complement graph of a clique of size $n - p + 2$. For more references on this minimization problem, we refer interested readers to the recent surveys\textsuperscript{4,8} and to\textsuperscript{1,2} for related problems in the language of graph bootstrap percolation.

In this paper, we consider another type of extremal problems on the clique-saturating edges. For a $K_p$-free graph $G$, let $f_p(G)$ denote the number of $K_p$-saturating edges of $G$. Let $f_p(n,e)$ be the minimum number of $K_p$-saturating edges of an $n$-vertex $K_p$-free graph with $e$ edges. For all integers

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arXiv:2201.03983v2 [math.CO] 13 Jan 2022.
\(p \geq 3\), the example of the Turán graph \(T_{p-1}(n)\) shows that
\[
f_{p+1}(n, e) = 0 \quad \text{for all } 0 \leq e \leq \text{ex}(n, K_p).
\]

Erdős and Tuza (see [5]) proved that \(f_4(n, \lceil n^2/4 \rceil + 1) \geq cn^2\) for some constant \(c > 0\); that is, for the case \(p = 3\), if adding one more edge to the above extreme, then the function will suddenly jump from \(0\) to \(\Omega(n^2)\). Erdős and Tuza also made a conjecture that \(f_4\left(n, \left\lceil \frac{n^2}{4} \right\rceil + 1 \right) = (1 + o(1)) \left\lceil \frac{n^2}{16} \right\rceil\). This however was disproved by Balogh and Liu in [3], where they constructed an \(n\)-vertex \(K_4\)-free graph with \(\left\lceil \frac{n^2}{4} \right\rceil + 1\) edges and with only \((1 + o(1))2\left\lceil \frac{n^2}{33} \right\rceil K_4\)-saturating edges (see Figure 1 in the case \(p = 3\) for the construction). Furthermore, Balogh and Liu [3] showed that this construction is best possible.

**Theorem 1.1** (Balogh-Liu [3]). \(f_4(n, \lceil n^2/4 \rceil + 1) = (1 + o(1)) \left\lceil \frac{n^2}{33} \right\rceil\).

In fact, they proved a stronger statement that \(f_4(n, \lceil n^2/4 \rceil + t) = \frac{2}{33}n^2 + \Theta(n)\) for every \(1 \leq t \leq \frac{n}{66}\). Balogh and Liu [3] commented that a similar phenomenon like Theorem 1.1 should also hold for general \(p\) and thus made an explicit conjecture (see Remark (iii) in [3]) suggested by a natural generalization of their \(K_4\)-free construction that for all integers \(p \geq 3\),
\[
f_{p+1}(n, \text{ex}(n, K_p) + 1) = \left(\frac{2(p-2)^2}{p(4p^2-11p+8)} + o(1)\right)n^2.
\]

The main result of the present paper is to prove the above conjecture of Balogh and Liu [3].

**Theorem 1.2.** For all integers \(p \geq 3\), \(f_{p+1}(n, \text{ex}(n, K_p) + 1) = \left(\frac{2(p-2)^2}{p(4p^2-11p+8)} + o(1)\right)n^2\).

Most of this paper will be devoted to the lower bound of the following theorem, which implies the lower bound of Theorem 1.2. Note that for any integer \(p \geq 3\), \(f_{p+1}(G) = 0\) holds for \(G = T_{p-1}(n)\).

**Theorem 1.3.** Let \(p \geq 3\) and \(n \geq 8p^5\) be integers. Let \(G\) be the family consisting of all \(n\)-vertex \(K_{p+1}\)-free graphs with exactly \(\text{ex}(n, K_p)\) edges. Then
\[
\min_{G \in \mathcal{G}\setminus\{T_{p-1}(n)\}} f_{p+1}(G) = \frac{2(p-2)^2}{p(4p^2-11p+8)}n^2 - \frac{(p-2)(2p-3)}{4p^2-11p+8}n + O_p(1).
\]

In addition, if \(n\) is divisible by \(p(p-1)(4p^2-11p+8)\), then
\[
\min_{G \in \mathcal{G}\setminus\{T_{p-1}(n)\}} f_{p+1}(G) = \frac{2(p-2)^2}{p(4p^2-11p+8)}n^2 - \frac{(p-2)(2p-3)}{4p^2-11p+8}n.
\]

We refer readers to the beginning of Section 4 for a proof sketch of this theorem.

We use standard notations on graphs throughout the paper. Let \(G\) be a graph. For a subset \(U \subseteq V(G)\), the subgraph of \(G\) induced by the vertex set \(U\) is denoted by \(G[U]\), while the subgraph obtained from \(G\) by deleting all vertices in \(U\) is expressed by \(G\setminus U\). Let \(N_G(U) = \bigcap_{v \in U} N_G(v)\) be the common neighborhood of all vertices of \(U\) in \(G\). Suppose that \(U, W\) are two disjoint vertex subsets in \(G\). We denote \(E_G(U, W)\) to be the set of edges of \(G\) between \(U\) and \(W\) and let \(e_G(U, W) = |E_G(U, W)|\). We often drop the above subscripts when they are clear from context. For positive integers \(k\), we write \([k]\) for the set \(\{1, 2, \ldots, k\}\) and the notation \(\left\lceil x \right\rceil\) means the function \(x(x-1)/2\) for all reals \(x\). We often omit floors and ceilings whenever they are not critical.
The rest of the paper is organized as follows. In Section 2, we provide constructions which match with the upper bounds of Theorems 1.2 and 1.3. In Section 3, we prove Theorem 1.2 by using Theorem 1.3. In Section 4, we give a complete proof of Theorem 1.3. Finally, we conclude the paper with some remarks.

2 The constructions for the upper bounds

In this section, we establish the upper bounds of Theorems 1.2 and 1.3 by defining some explicit $K_{p+1}$-free graphs. These graphs are suggested by Balogh and Liu in [3], each of which is an appropriate blow-up of the following graph: take a complete $(p-1)$-partite graph $K = K_{2,\ldots,2}$ and add a new vertex by making it adjacent to exactly one vertex in each partite set of $K$.

In the rest of this section, we write $n = p(p-1)(4p^2 - 11p + 8)x + y$, where $x, y$ are integers such that $x \geq 0$ and $0 \leq y < p(p-1)(4p^2 - 11p + 8)$. By Turán’s Theorem, we have $\text{ex}(n, K_p) = \frac{p^2(p-1)^2(4p^2 - 11p + 8)^2}{2(p-1)} + 2(p-2)(4p^2 - 11p + 8)xy + t_{p-1}(y)$, where $t_{p-1}(y) = e(T_{p-1}(y))$.

First, we prove the desired upper bound of Theorem 1.3.

The upper bound of Theorem 1.3. In this case, we will construct an $n$-vertex $K_{p+1}$-free graph $H_1$ with exactly $\text{ex}(n, K_p)$ edges and $f_{p+1}(H_1) = \frac{2(p-2)^2}{p(4p^2 - 11p + 8)}n^2 - \frac{(p-2)(2p-3)}{p(4p^2 - 11p + 8)}n + O_p(1)$.

To do so, we first construct a graph $H_0$ as follows (see Figure 1). First, take a $(p-1)$-partite complete graph $K_{2,\ldots,2}$ with vertex set $\{v_i, u_i : i \in [p-1]\}$, where $v_i$ and $u_i$ are in the same part for $i \in [p-1]$. Next, take a new vertex $v_0$ and make it adjacent to each $v_i$ for $i \in [p-1]$. Finally, let $H_0$ be obtained by blowing-up $v_0$ into an independent set $V_0$ of size $2(p-1)(p-2)^2x$, blowing-up each $v_i$ into an independent set $V_i$ of size $4(p-1)^2(p-2)x$ for $i \in [p-1]$, blowing-up each $u_i$ into an independent set $U_i$ of size $p(3p-4)x$ for $i \in [p-1]$. We can check that $H_0$ is $K_{p+1}$-free on $p(p-1)(4p^2 - 11p + 8)x$ vertices with $\frac{p^2(p-1)^2}{2(p-1)} + 2(p-2)(4p^2 - 11p + 8)^2x^2$ edges.

Next we construct the desired graph $H_1$ from $H_0$ by enlarging the size of $V_0$ with $2y$ more new vertices and deleting $y$ vertices which form a $T_{p-1}(y)$ in $H_0 \left[ \bigcup_{i=1}^{p-1} U_i \right]$. Indeed, since $n \geq 8p^5$ and by our definitions of $x$ and $y$, we can check that $p(p-1)(3p-4)x > y$. Thus, the above deletion process succeed. By a careful calculation, we can derive that $H_1$ is $K_{p+1}$-free on $n$ vertices with $\text{ex}(n, K_p)$ edges. The only $K_{p+1}$-saturating edges are the pairs in $V_i$ for $0 \leq i \leq p-1$. This (see Appendix A) leads to

$f_{p+1}(H_1) = \frac{2(p-2)^2}{p(4p^2 - 11p + 8)}n^2 - \frac{(p-2)(2p-3)}{p(4p^2 - 11p + 8)}n + \frac{8(p-1)^3}{p(4p^2 - 11p + 8)}y^2 - \frac{2(p-1)^2}{4p^2 - 11p + 8}y

= \frac{2(p-2)^2}{p(4p^2 - 11p + 8)}n^2 - \frac{(p-2)(2p-3)}{4p^2 - 11p + 8}n + O_p(1),$

completing the proof for the upper bound.

\footnote{All the detailed calculations in this proof can be found in Appendix A.}
The construction for the upper bound of Theorem 1.2 is quite similar to the one above. The only differences are the sizes of the parts in the blow-up.

**The upper bound of Theorem 1.2.** In this case, we will construct an $n$-vertex $K_{p+1}$-free graph $G$ with exactly $\text{ex}(n, K_p) + 1$ edges and $f_{p+1}(G) = \frac{2(p-2)^2}{p(4p^2 - 11p + 8)} n^2 - \frac{(p-2)(2p^2 - 5p + 4)}{p(4p^2 - 11p + 8)} n + O_p(1)$.

To do so, we first let $H_0$ be the same as above (see Figure 1). We then construct a graph $H_2$ from $H_0$ by enlarging the size of $V_0$ with $2y + 1$ more new vertices and deleting $y + 1$ vertices which forms a $T_{p-1}(y + 1)$ in $H_0 \left[ \bigcup_{i=1}^{p-1} U_i \right]$. By a similar calculation as the previous case, one can derive that $H_2$ is $K_{p+1}$-free on $n$ vertices with

$$\text{ex}(n, K_p) + \frac{(p-2)^3}{p(p-1)(4p^2 - 11p + 8)} n + O_p(1)$$

edges. Again, the only $K_{p+1}$-saturating edges are the pairs in $V_i$ for $0 \leq i \leq p - 1$. By some careful calculation, one can derive that

$$f_{p+1}(H_2) = \frac{2(p-2)^2}{p(4p^2 - 11p + 8)} n^2 - \frac{(p-2)(2p^2 - 5p + 4)}{p(4p^2 - 11p + 8)} n + O_p(1).$$

Now one can easily remove some edges from $H_2$ (i.e., edges incident with vertices in $U_i$’s) without changing the number of $K_{p+1}$-saturating edges until the remaining graph $G$ has exactly $\text{ex}(n, K_p) + 1$ edges. In this way, we obtain the desired graph $G$ with $f_{p+1}(G) = f_{p+1}(H_2)$ and thus prove that

$$f_{p+1}(n, \text{ex}(n, K_p) + 1) \leq \frac{2(p-2)^2}{p(4p^2 - 11p + 8)} n^2 - \frac{(p-2)(2p^2 - 5p + 4)}{p(4p^2 - 11p + 8)} n + O_p(1)$$

holds for all integers $n$. \[\blacksquare\]
3 Proof of Theorem [1.2]

In this section, assuming Theorem 1.3 we complete the proof of Theorem 1.2. The upper bound of Theorem 1.2 is given by the last section, so it suffices to prove the lower bound. Let $G$ be a $K_{p+1}$-free graph with $\text{ex}(n, K_p) + 1$ edges. By Turán’s Theorem, $G$ contains a copy of $K_p$. Let $G'$ be obtained from $G$ by removing a single edge such that $G'$ still contains a $K_p$. Then $G'$ is $K_{p+1}$-free with $\text{ex}(n, K_p)$ edges. As $G'$ contains a $K_p$, it cannot be the Turán graph $T_{p+1}(n)$. By Theorem 1.3 we have

$$f_{p+1}(G) \geq f_{p+1}(G') \geq \frac{2(p - 2)^2}{p(4p^2 - 11p + 8)}n^2 - \frac{(p - 2)(2p - 3)}{4p^2 - 11p + 8}n + O_p(1),$$

finishing the proof of Theorem 1.2. 

We remark that the above proofs actually show a sharper bound than the statement of Theorem 1.2. Namely, for all integers $p \geq 3$ and $n$, if we write

$$g_p(n) = f_{p+1}(n, \text{ex}(n, K_p) + 1) - \frac{2(p - 2)^2}{p(4p^2 - 11p + 8)}n^2,$$

then we have

$$-\frac{(p - 2)(2p - 3)}{4p^2 - 11p + 8}n + O_p(1) \leq g_p(n) \leq -\frac{(p - 2)(2p^2 - 5p + 4)}{p(4p^2 - 11p + 8)}n + O_p(1)$$

such that $g_p(n) \to -\frac{n}{2} + o(n)$ as $p \to \infty$.

4 Proof of Theorem 1.3

We begin with a sketch of the proof of Theorem 1.3. Let $G$ be an $n$-vertex $K_{p+1}$-free graph with $\text{ex}(n, K_p)$ edges and containing at least one copy of $K_p$. Following the approach of [3], we partition the vertex set of $G$ into two parts $V(\mathcal{R})$ and its complement $V(G) \setminus V(\mathcal{R})$, where $\mathcal{R}$ is a maximum family of vertex-disjoint $K_p$'s in $G$ and $V(\mathcal{R})$ denotes the set of all vertices contained in $\mathcal{R}$. Then all $K_p$-saturating edges of $G$ can be divided into two types, the first type of which are those $K_p$-saturating edges incident to $V(\mathcal{R})$ and the other type are those contained in $V(G) \setminus V(\mathcal{R})$. Estimations on the number of saturating edges of these two types have been established in [3], respectively, which work quite well when $p$ is small. The problem is that when $p$ is getting bigger, the complexity of computations based on these estimations will be difficult to handle. So some novel ideas will be needed. A key motivation for us comes after Lemma 1.4 which roughly says that for any $p$-clique $R$ in $\mathcal{R}$, as long as there are enough edges between $R$ and $V(G) \setminus V(\mathcal{R})$, any $p - 1$ vertices of $R$ have some common neighbors in $V(G) \setminus V(\mathcal{R})$ (it can even be set up as $\Omega(1)$ many if required). Therefore one may hope to use similar proof ideas as in Hajnal-Szemerédi Theorem to find a larger collection of vertex-disjoint $p$-cliques than $\mathcal{R}$ and thus obtain a contradiction to the maximality of $\mathcal{R}$. This indeed would work. And it turns out that if we choose $\mathcal{R}$ with an additional requirement on the number of edges contained in $V(G) \setminus V(\mathcal{R})$, then the proof can be shortened and a contradiction can already be reached using some appropriate vertex-switching techniques (in fact this would provide a shortcut for the proof in [3] for the case $p = 3$ as well).

Throughout the rest of this section, we present the proof of Theorem 1.3. The upper bounds
of Theorem 1.3 are given by the aforementioned constructions. Consider any integers \( p \geq 3 \) and \( n \geq 120p^2 \). (As \( n \geq 8p^5 \geq 120p^2 \), here we remark that \( n \geq 120p^2 \) is enough to show the lower bounds of Theorem 1.3.) Let \( G \) be any \( n \)-vertex \( K_{p+1} \)-free graph with \( \text{ex}(n, K_p) \) edges, but not the \((p - 1)\)-partite Turán graph \( T_{p-1}(n) \). It suffices to show that \( f_{p+1}(G) \) is bounded from below by the desired formula. By Turán’s Theorem, \( G \) contains at least one copy of \( K_p \) with 

\[
e(G) = \text{ex}(n, K_p) = \frac{p - 2}{2(p - 1)} n^2 - \delta, \tag{1}
\]

where \( \delta = \frac{t(p - 1 - t)}{2(p - 1)} \) for \( t \in \{0, 1, \ldots, p - 2\} \) with \( t \equiv n \mod (p - 1) \). We note that \( 0 \leq \delta \leq \frac{p - 1}{8} \), and \( \delta = 0 \) if and only if \( n \) is divisible by \( p - 1 \).

We now partition \( V(G) \) into two parts \( V(R) \) and \( V(G) \setminus V(R) \) satisfying the following conditions

(i). \( R \) is a maximum family of vertex-disjoint \( K_p \)'s in \( G \), and

(ii). subject to (i), the remaining graph \( G \setminus V(R) \) has the maximum number of edges.

Let \( H_R := G \setminus V(R) \) and \(|R| := rn \). Since \( G \) contains a \( K_p \), we have 

\[
1/n \leq r \leq 1/p. \tag{2}
\]

By the choice of (i), we know that \( H_R \) is \( K_p \)-free with \((1 - pr)n \) vertices, thus by Turán’s Theorem,

\[
e(H_R) \leq \frac{(p - 2)}{2(p - 1)} (1 - pr)^2 n^2. \tag{3}
\]

For any \( p \)-clique \( R \in \mathcal{R} \) and \( 0 \leq j \leq p \), we let

\[
Z_j(R) = \{ \text{all vertices in } H_R \text{ that has exactly } j \text{ neighbors in } V(R) \} \quad \text{and} \quad z_j(R) := |Z_j(R)|/n.
\]

By the assumption that \( G \) is \( K_{p+1} \)-free, it is clear that \( Z_p(R) = \emptyset \). So for any \( p \)-clique \( R \in \mathcal{R} \),

\[
\sum_{j=0}^{p-1} z_j(R) = 1 - pr. \tag{4}
\]

We will also need to consider a refined partition of \( Z_{p-1}(R) \) as follows. Let \( \{v_1, v_2, \ldots, v_p\} \) represent the vertex set of a given \( p \)-clique \( R \in \mathcal{R} \). For any \( i \in [p] \), define 

\[
A_i(R) := N_{H_R}(R \setminus \{v_i\})
\]

to be the common neighborhood of \( V(R) \setminus \{v_i\} \) in \( V(H_R) \). Let us observe that \( A_i(R) \)'s are pairwise vertex-disjoint independent sets in \( Z_{p-1}(R) \) (for otherwise \( \bigcup A_i(R) \cup R \) would contain a copy of \( K_{p+1} \), a contradiction to \( G \) is \( K_{p+1} \)-free). In particular, we have

\[
\sum_{i=1}^{p} |A_i(R)|/n = z_{p-1}(R). \tag{5}
\]

\footnote{Throughout we will write \( V(\mathcal{R}) \) for the union of the vertex sets of all \( K_p \)'s in \( \mathcal{R} \).}
It is crucial to see that every non-edge inside each $A_i(R)$ is a $K_{p+1}$-saturating edge in $G$.

![Figure 2. The proof of Lemma 4.1](image)

The following lemma is key in our proof. It shows that by the choice of $\mathcal{R}$ and $H_R$, there are enough many edges incident to new $p$-cliques obtained from some $R \in \mathcal{R}$ by switching some vertices in $R$ with vertices in $H_R$ of equal size.

**Lemma 4.1.** Let $R \in \mathcal{R}$ be a $p$-clique and $C$ be a subclique of $R$. If there exists a clique $C'$ in $H_R$ of equal size as $C$ such that $R' := (R \setminus C) \cup C'$ remains a clique in $G$, then $\mathcal{R}' := (\mathcal{R} \setminus \{R\}) \cup \{R'\}$ is also a maximum family of vertex-disjoint $K_p$’s in $G$ with $e(R', H_{R'}) \geq e(R, H_R)$, where $H_{R'} = G \setminus V(\mathcal{R}')$

**Proof.** First observe that $\mathcal{R}'$ is also a maximum family of $rn$ vertex-disjoint $K_p$’s. Let $H_{R'} = G \setminus V(\mathcal{R}')$. So $H_{R'} = (H_R \setminus C') \cup C$ (see Figure 2). By (ii), we have $e(H_R) \geq e(H_{R'})$. Since $e(C') = e(C)$,

$$e(H_R) = e(C') + e(C', H_R \setminus C') + e(H_R \setminus C')$$

and $e(H_{R'}) = e(C) + e(C, H_R \setminus C') + e(H_R \setminus C')$, it follows that

$$e(C', H_R \setminus C') \geq e(C, H_R \setminus C').$$

Therefore, as $e(R \setminus C, C') = e(R \setminus C, C)$, one can derive that

$$e(R', H_{R'}) - e(R, H_R) = e(C', H_R \setminus C') - e(C, H_R \setminus C') \geq 0.$$

This completes the proof of Lemma 4.1.

Next we proceed to prove three technical lemmas and we should emphasize in advance that these lemmas hold for any family $\mathcal{R}$ solely satisfying the condition (i). The first one says that for any family $\mathcal{R}$ satisfying the condition (i), there is a $R^* \in \mathcal{R}$ such that $e(R^*, H_R)$ is large.

**Lemma 4.2.** Suppose that $\mathcal{R}$ is under the condition (i) and $H_R = G \setminus V(\mathcal{R})$. Then there exists a $p$-clique $R^* \in \mathcal{R}$ such that

$$e(R^*, H_R) \geq \left( \frac{p(p-2)}{p-1} - \frac{p(2p^2 - 4p + 1)}{2(p-1)} \right) n - \frac{\delta}{rn}. \quad (6)$$

Moreover, for any $R^* \in \mathcal{R}$ satisfying (i), it holds that

$$z_{p-1}(R^*) \geq \frac{p-2}{p-1} - \frac{p(2p-3)}{2(p-1)} r - \frac{\delta}{rn^2}. \quad (7)$$
Proof. Note that the edge set of $G$ can be partitioned into $E(H_{R}), E(V(R), H_{R})$ and $E(G[V(R)])$. Since $G$ is $K_{p+1}$-free, by Turán’s Theorem $e(G[V(R)]) \leq \text{ex}(prn, K_{p+1}) = (\ell_2^p) r^2 n^2$. Together with (1) and (4), we have that

$$e(V(R), H_{R}) = e(G) - e(H_{R}) - e(G[V(R)])$$

$$\geq \left( \frac{p - 2}{2(p - 1)} - \frac{(p - 2)}{2(p - 1)}(1 - pr)^2 - \frac{p(p - 1)}{2} \right) n^2 - \delta$$

$$= \left( \frac{p(p - 2)}{p - 1} - \frac{p(2p^2 - 4p + 1)}{2(p - 1)} \right) n^2 - \delta.$$

By averaging, there exists a clique $R^* \in R$ with

$$e(R^*, H_{R}) \geq \frac{e(V(R), H_{R})}{rn} \geq \left( \frac{p(p - 2)}{p - 1} - \frac{p(2p^2 - 4p + 1)}{2(p - 1)} \right) n - \frac{\delta}{rn}.$$

As $G$ is $K_{p+1}$-free, every vertex in $H_{R}$ has at most $r - 1$ neighbors in $V(R^*)$. So we have $e(R^*, H_{R}) \leq |Z_{p-1}(R^*)| + (p - 2) \sum_{j=0}^{p-1} |Z_j(R^*)|$, which by (1) implies that

$$z_{p-1}(R^*) \geq \frac{e(R^*, H_{R})}{n} - (p - 2)(1 - pr) \geq \frac{p - 2}{p - 1} - \frac{p(2p - 3)}{2(p - 1)} r - \frac{\delta}{rn^2}.$$

This completes the proof of Lemma 4.2.

Denote by $\ell_1^R$ the number of $K_{p+1}$-saturating edges incident to $V(R)$, and by $\ell_2^R$ the number of $K_{p+1}$-saturating edges in $H_{R}$. Obviously $f_{p+1}(G) = \ell_1^R + \ell_2^R$. The lemma below gives a lower bound on $\ell_1^R$, which in particular shows that Theorem 1.3 holds in case $r$ is close to $1/p$.

Lemma 4.3. Suppose that $R$ is under the condition (i). Then

$$\ell_1^R \geq \left( \frac{p - 2}{p - 1} - \frac{p(p - 2)}{2(p - 1)} \right) n^2 - \frac{p}{2} n - \delta.$$

Moreover, if $r > \frac{2(p - 2)(2p - 3)}{p(4p^2 - 11p + 8)}$, then Theorem 1.3 holds.

Proof. Let $\mathcal{R} = \{R_1, R_2, ..., R_{rn}\}$, $H_{R} = G \backslash V(R)$ and $r_i = e(R_i, G \backslash \bigcup_{j=1}^{i} R_j)$ for $i \in [rn]$ such that

$$\sum_{i=1}^{rn} r_i = e(G) - e(H_{R}) - \left( \frac{p}{2} \right) rn.$$

Since $G$ is $K_{p+1}$-free, every vertex has at most $p - 1$ neighbors on each $R_i$. So there exist at least $r_i - (p - 2)(n - pi)$ vertices in $G \backslash \bigcup_{j=1}^{i} R_j$ with exactly $p - 1$ neighbors in $V(R_i)$, each of which
contributes a $K_{p+1}$-saturating edges to $\ell^R_i$. Therefore, we have

\[
\ell^R_i \geq \sum_{i=1}^{rn} (r_i - (p - 2)(n - pi)) = e(G) - e(H_R) - \binom{p}{2}rn - \frac{p(p-2)}{2}(rn+1)rn
\]

\[
\geq \left(\frac{p-2}{2(p-1)} - \frac{p(p-2)}{2(p-1)}(1-pr)^2 - (p-2)r + \frac{p(p-2)}{2}r^2\right)n^2 - \frac{pr}{2}n - \delta
\]

\[
= \left(\frac{p-2}{p-1}r - \frac{p(p-2)}{2(p-1)}r^2\right)n^2 - \frac{pr}{2}n - \delta,
\]

where the last inequality follows from (1) and (3).

For the second statement of this lemma, by (2) and the assumption therein, we have $\frac{2(p-2)(2p-3)}{p(4p^2-11p+8)} \leq r \leq \frac{1}{p}$ Then the first statement implies that

\[
f_{p+1}(G) \geq \ell^R_i \geq \left(\frac{p-2}{p-1}r - \frac{p(p-2)}{2(p-1)}r^2\right)n^2 - \frac{pr}{2}n - \delta
\]

\[
\geq \frac{p-2}{p-1}r \left(1 - \frac{pr}{2}\right)n^2 - \frac{n}{2} - \delta
\]

\[
> \left(\frac{2(p-2)^2(2p-3)}{p(p-1)(4p^2-11p+8)} - \frac{2(p-2)^3(2p-3)^2}{p(p-1)(4p^2-11p+8)^2}\right)n^2 - \frac{n}{2} - \delta
\]

\[
= \frac{4(p-1)(p-2)^2(2p-3)}{p(4p^2-11p+8)^2}n^2 - \frac{n}{2} - \delta
\]

\[
\geq \frac{2(p-2)^2}{p(4p^2-11p+8)}n^2 - \frac{(p-2)(2p-3)}{4p^2-11p+8}n - \delta,
\]

where the second last inequality holds because $g(r) = r(1 - \frac{pr}{2})$ is increasing for $r \leq \frac{1}{p}$ and the last inequality holds whenever $n \geq 120p^2$. This matches the lower bounds of Theorem 1.3 (also for the case when $n$ is divisible by $p(p-1)(4p^2-11p+8)$, as for which $\delta = 0$). Now Lemma 4.3 is completed. \( \square \)

The next lemma says that for any $R^* \in \mathcal{R}$ satisfying the conclusion of Lemma 4.2 one may assume that the set $A_i(R^*)$ for every $i \in [p]$ is non-empty.

**Lemma 4.4.** Suppose that $\mathcal{R}$ is under the condition (i). Let $R^* \in \mathcal{R}$ be any clique satisfying (6). If there exists some $i \in [p]$ such that $A_i(R^*) = \emptyset$, then Theorem 1.3 holds.

**Proof.** Let $A_i = A_i(R^*)$ and $z_{p-1} = z_{p-1}(R^*)$. Without loss of generality, we assume that $A_p = \emptyset$. Recall that each pair of vertices in $A_i$ is a $K_{p+1}$-saturating edge in $H_R$ and by (6), $\sum_{i=1}^{p-1} |A_i|/n = \sum_{i=1}^{p} |A_i|/n = z_{p-1}$. Using Jensen’s inequality, we get that

\[
\ell^R_2 \geq \sum_{i=1}^{p-1} \frac{|A_i|}{2} \geq (p-1) \frac{z_{p-1}}{2} = \frac{z_{p-1}}{2}n - \frac{z_{p-1}}{2}n.
\]

Since $\mathcal{R}$ satisfies the condition (i), by Lemma 4.3, we may assume that $r \leq \frac{2(p-2)(2p-3)}{p(4p^2-11p+8)}$. By (2), we
have $r \geq 1/n$. Since $\delta \leq \frac{n-1}{8}$, we can derive from (8) that

$$z_{p-1} \geq \frac{p-2 - p(2p-3)\frac{r}{2(p-1)} - \frac{\delta}{r n^2}}{p-1}$$

$$\geq \frac{p-2 - p(2p-3)\frac{2(p-2)(2p-3)}{p(4p^2 - 11p + 8)} - \frac{p-1}{8n}}{4p^2 - 11p + 8} - \frac{p-1}{8n} \geq \frac{p-1}{2n},$$

where the last inequality holds as $n \geq 120p^2$. Note that $h(z_{p-1}) = \frac{z_{p-1}^2}{2(p-1)} n^2 - \frac{z_{p-1}}{2}$ is increasing in the range of $z_{p-1} > \frac{1}{2n}$ and takes its minimum at the smallest value that $z_{p-1}$ can take. Thus

$$\ell_2^R \geq h(z_{p-1}) = \frac{(2(p-2) - p(2p-3)r)^2 n^2}{8(p-1)^3} - \frac{p-2}{2(p-1)^2} + \frac{p(2p-3)}{2(p-1)^2} + \frac{1}{2rn} \geq - \frac{p-2}{(p-1)^2 r}.$$

Thus, using $r \geq 1/n$ and $\delta \leq \frac{n-1}{8}$, we have

$$\ell_2^R \geq \frac{(2(p-2) - p(2p-3)r)^2 n^2}{8(p-1)^3} - \frac{p-2}{2(p-1)^2} - \frac{p(2p-3)}{2(p-1)^2} + \frac{1}{2rn} \geq - \frac{p-2}{(p-1)^2 r}.$$

Next we claim that Theorem 1.3 holds in case $r \leq \frac{1}{40(p-2)(2p-3)}$. Indeed, by the above lower bound of $\ell_2^R$, we have

$$f_{p+1}(G) \geq \ell_2^R \geq \frac{(2(p-2) - p(2p-3)r)^2 n^2}{8(p-1)^3} - \frac{5}{8n} \geq \frac{2(p-2)^2}{8(p-1)^3 n^2} - \frac{(p-2)(2p-3)}{4p^2 - 11p + 8} - \frac{5}{8n}$$

where the last inequality holds whenever $n \geq 120p^2$ and $p \geq 3$ (see the verification in Appendix B). This matches the lower bounds of Theorem 1.3 and thus proves the above claim.

Therefore in the following of the proof, we may assume that $r \leq \frac{1}{40(p-2)(2p-3)}$. By (8) and (9), we see that $F(n, p, r, \delta) \geq \frac{p-2}{(p-1)^2 r} \geq \frac{40(p-2)^2(2p-3)}{p(4p^2-11p+8)} \geq -40(p-2)(2p-3)$, and thus

$$\ell_2^R \geq \frac{(2(p-2) - p(2p-3)r)^2 n^2}{8(p-1)^3} - \frac{2(p-2) - p(2p-3)r}{4(p-1)} n - 40\delta(p-2)(2p-3).$$
This together with the estimation on $\ell_1^R$ from Lemma 4.3 give that

$$f_{p+1}(G) = \ell_1^R + \ell_2^R$$

$$\geq \left( \frac{p - 2}{p - 1} r - \frac{p(p - 2)}{2(p - 1)} r^2 \right)n^2 - \frac{p}{2} r n - \delta + \frac{(2(p - 2) - p(2p - 3)r)^2}{8(p - 1)^3} n^2$$

$$- \frac{2(p - 2) - p(2p - 3)r}{4(p - 1)} n - 40\delta(p - 2)(2p - 3)$$

$$= \frac{p(4p^2 - 11p + 8)n^2}{8(p - 1)^3} r^2 - \frac{2(p - 2)n^2 + p(p - 1)^2 n}{4(p - 1)^3} r + \frac{(p - 2)^2}{2(p - 1)^3} n^2 - \frac{p - 2}{2(p - 1)} n$$

$$- \delta (40(p - 2)(2p - 3) + 1)$$

$$\geq \frac{2(p - 2)^2}{p(4p^2 - 11p + 8)n^2} - \frac{(p - 2)(2p - 3)}{4p^2 - 11p + 8} n - \frac{p(p - 1)}{8(4p^2 - 11p + 8)} - \delta (40(p - 2)(2p - 3) + 1)$$

$$= \frac{2(p - 2)^2}{p(4p^2 - 11p + 8)n^2} - \frac{(p - 2)(2p - 3)}{4p^2 - 11p + 8} n + O_p(1),$$

where the last inequality holds since the quadratic function on $r$ formed by the first two terms on one side is minimized at $r = \frac{2(p-2)^2}{p(4p^2-11p+8)} + \frac{(p-1)^2}{4p^2-11p+8}$ (for the detailed calculation, see Appendix C). This matches the lower bounds of Theorem 1.3. For the case when $n$ is divisible by $p(p-1)(4p^2-11p+8)$, we have $\delta = 0$ and in this case, the optimal $r$ for the last inequality should be chosen as $r = \frac{2(p-2)^2}{p(4p^2-11p+8)}$, so that $r_n$ is an integer. Repeating the above calculation, it would exactly imply that $f_{p+1}(G) \geq \frac{2(p-2)^2}{p(4p^2-11p+8)n^2} - \frac{(p-2)(2p-3)}{4p^2-11p+8}n$. Now Lemma 4.4 is completed. \hfill \square

Finally we are ready to finish the proof of Theorem 1.3. By Lemma 4.4, for any $\mathcal{R}$ satisfying the condition (i) and for any $R_0 \in \mathcal{R}$ satisfying (i), we may assume that $A_i(R_0) \neq \emptyset$ for each $i \in [p]$, i.e., any $p-1$ vertices in $V(R_0)$ have at least one common neighbor in $H_\mathcal{R} = G \setminus V(\mathcal{R})$.

Let $R^* \in \mathcal{R}$ be the $p$-clique obtained from Lemma 4.2. So $R^*$ satisfies (i). Let $C$ be a clique in $H_\mathcal{R}$ of maximum size such that $R^* \cup C$ contains a $p$-clique $R'$ in $G$ covering all the vertices of $C$. Since $A_i(R^*) \neq \emptyset$ for each $i \in [p]$, such a clique $C$ exists in $H_\mathcal{R}$ (for instance, one can just take one vertex in $A_1(R^*)$). Let $V(R') = \{v_1, \ldots, v_p\}$ and $V(C) = \{x_1, \ldots, x_c\}$ for some integer $c \geq 1$. Without loss of generality we may assume that

$$V(R') = \{x_1, \ldots, x_c, v_{c+1}, \ldots, v_p\}.$$

In what follows, we should complete the proof by deriving the final contradiction that $c \geq p$.

Suppose that $c \leq p-1$. In this case, we are always able to find a clique in $H_\mathcal{R}$ of larger size than $C$ and satisfying the above conditions required for $C$. To see this, let $\mathcal{R}' = (\mathcal{R} \setminus \{R^*\}) \cup \{R'\}$ and $H_{\mathcal{R}'} = G \setminus V(\mathcal{R}')$. So $\mathcal{R}'$ also satisfies the condition (i) and

$$V(H_{\mathcal{R}'}) = V(H_{\mathcal{R}}) \setminus \{x_1, \ldots, x_c\} \cup \{v_1, \ldots, v_c\}.$$

Applying Lemma 4.1 with the clique $R$ therein being $R^*$, we know that

$$e(R', H_{\mathcal{R}'}) \geq e(R^*, H_{\mathcal{R}}) \geq \left( \frac{p(p - 2)}{p - 1} - \frac{p(2p^2 - 4p + 1)}{2(p - 1)} \right) n - \frac{\delta}{r_n},$$

\footnote{Note that this value of $r$ corresponds to the exact construction in Section 2}
where the last inequality holds as $R^*$ satisfies (3). That says, $R' \in R'$ also satisfies (3). As discussed earlier, by Lemma 1.1 any $p - 1$ vertices in $V(R')$ have at least one common neighbor in $H_{R'}$. In particular, there exists a vertex $y \in V(H_{R'})$ such that it is not adjacent to $v_p$ but is adjacent to all other vertices of $V(R')$. Obviously, $y \notin \{v_1, \ldots, v_c\}$, since $v_pv_p \in E(G)$ for each $i \in [c]$. So it must be the case that $y \in V(H_R) \setminus \{x_1, \ldots, x_c\}$. Now let $C' = \{x_1, \ldots, x_c, y\} \subseteq V(H_R)$. Then $C'$ is a clique in $H_R$ of size larger than $C$ such that $C' \cup \{v_{c+1}, \ldots, v_{p-1}\}$ is a $p$-clique contained in $R' \cup C'$ and covering all vertices of $C'$. This is a contradiction to our choice of $C$. Therefore, we must have that $c \geq p$. However, it is also a contradiction to the fact that $H_R$ is $K_p$-free, proving Theorem 1.3.

5 Concluding remarks

In this paper, we determine the order of $f_{p+1}(n, ex(n, K_p) + 1)$, confirming a conjecture of Balogh and Liu [3]. Balogh and Liu proved a stronger result in [3] that $f_{4}(n, \lceil \frac{n^2}{4} \rceil + t) = \frac{22}{27}n^2 + \Theta(n)$ holds for every positive integer $t$ up to $\frac{n}{66}$. We remark that the upper bound construction of Theorem 1.2 as well as the proof of Section 3 also show that

$$f_{p+1}(n, ex(n, K_p) + t) = \frac{2(p - 2)^2}{p(4p^2 - 11p + 8)}n^2 + \Theta(n)$$

holds for any integer $1 \leq t \leq \frac{(p-2)^2}{p(4p^2 - 11p + 8)}n$. It is interesting to determine the function of $f_{p+1}(n, m)$ for every integer $m$ between $ex(n, K_p)$ and $ex(n, K_{p+1})$. We would like to ask if for all $m$, the extremal $K_{p+1}$-free graph attaining this minimum number is always obtained from an appropriate blow-up of the same graph suggested in [3] (i.e., the graph obtained by taking a complete $(p - 1)$-partite graph $K = K_2, \ldots, 2$ and adding a new vertex by making it adjacent to exactly one vertex in each partite set of $K$) by deleting $O(n)$ edges.

Let $H$ be a given graph. For an $H$-free graph $G$, a non-edge of $G$ is called an $H$-saturating edge, if $G + e$ contains a copy of $H$. Let $f_H(G)$ denote the number of $H$-saturating edges of $G$ and let $f_H(n, m)$ denote the minimum of $f_H(G)$ over all $H$-free $n$-vertex graphs $G$ with $m$ edges. It is natural to consider the same minimization problem $f_H(n, m)$ for general $H$. The following family of graphs seems to be of particular interest. A pair of two edges $e, f$ in $H$ is called critical if $\chi(H - \{e, f\}) = \chi(H) - 2$. It is clear that such two edges $e, f$ must be vertex-disjoint. We say a graph $H$ is double-edge-critical if it contains a critical pair of two edges $e, f$. We point out that there are many double-edge-critical graphs, for example, any join obtained from a $p$-clique for $p \geq 4$ and an arbitrary graph is double-edge-critical. From the definition, we see that such $H$ is also edge-critical, so is each of $H - e$ and $H - f$. Let $\chi(H) = p + 1$. Then it follows that $f_H(n, m) = 0$ for all integers $m \leq \chi(n, K_p)$. We believe that the same phenomenon as Theorem 1.2 holds for any double-edge-critical graph $H$, that is, $f_H(n, \chi(n, K_p) + 1)$ would suddenly jump to $\Omega(n^2)$. We wonder if $f_H(n, \chi(n, K_p) + 1)$ can be determined for every double-edge-critical graph $H$ with $\chi(H) \geq 4$.

Acknowledgements. The authors would like to thank Jozsef Balogh for suggesting the problem.

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4Here, we may allow that some vertices are blowing up to empty sets. For instance, a blow-up of $K_p$ counts.

5A graph $F$ is edge-critical if there exists an edge $e^*$ such that $\chi(F - e^*) = \chi(F) - 1$. A classical result of Erdős and Simonovits states that for sufficiently large $n$, if $F$ is an edge-critical graph, then the unique $n$-vertex $F$-free extremal graph for $ex(n, F)$ is $T_{\chi(F)-1}(n)$. 

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of [3] to the fourth author in 2020 Summer. The fourth author additionally thanks him for fruitful discussions.

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Appendix A  Calculations for the upper bound of Theorem [1.3]

We first calculate $|V(H_0)|$ and $e(H_0)$ (see Figure 1). It is easy to see that $|V(H_0)| = 2(p-1)(p-2)^2 x + 4(p-1)^3(p-2)x + p(p-1)(3p-4)x = p(p-1)(4p^2 - 11p + 8)x$, and

$$e(H_0) = 8(p-1)^4(p-2)^3 x^2 + \frac{p-2}{2(p-1)} \cdot (p-1)^2 \cdot (4(p-1)^2(p-2) + p(3p-4))^2 x^2$$

$$= \frac{p-2}{2(p-1)} \cdot p^2(p-1)^2 (4p^2 - 11p + 8)^2 x^2.$$
By our definition of $H_1$, we get that $|V(H_1)| = |V(H_0)| + y = p(p-1)(4p^2 - 11p + 8)x + y = n$, and

$$e(H_1) = e(H_0) + 2y \cdot 4(p-1)^3(p-2)x - y \cdot 4(p-1)^2(p-2)^2x - y \cdot p(p-2)(3p-4)x + t_{p-1}(y)$$

$$= e(H_0) + p(p-2)(4p^2 - 11p + 8)xy + t_{p-1}(y)$$

$$= \frac{p-2}{2(p-1)} \cdot p^2(p-1)^2(4p^2 - 11p + 8)^2x^2 + p(p-2)(4p^2 - 11p + 8)xy + t_{p-1}(y) = ex(n, K_p).$$

Since the only $K_{p+1}$-saturating edges are the pairs in $V_i$ for $0 \leq i \leq p-1$, we get that

$$f_{p+1}(H_1) = \left(\frac{2(p-1)(p-2)^2x + 2y}{2}\right) + (p-1)\left(\frac{4(p-1)^2(p-2)x}{2}\right)$$

$$= 2(p-1)^2(p-2)^2(4p^2 - 11p + 8)x^2 + 4(p-1)(p-2)^2xy$$

$$- p(p-1)(p-2)(2p-3)x + 2y^2 - y$$

$$= \frac{2(p-2)^2}{p(4p^2 - 11p + 8)}(p(p-1)(4p^2 - 11p + 8)x + y)^2$$

$$- \frac{(p-2)(2p-3)}{4p^2 - 11p + 8}(p(p-1)(4p^2 - 11p + 8)x + y)$$

$$+ 2y^2 - \frac{2(p-2)^2}{p(4p^2 - 11p + 8)}y^2 + \frac{(p-2)(2p-3)}{4p^2 - 11p + 8}y - y$$

$$= \frac{2(p-2)^2}{p(4p^2 - 11p + 8)}n^2 - \frac{(p-2)(2p-3)}{4p^2 - 11p + 8}n + \frac{8(p-1)}{p(4p^2 - 11p + 8)}y^2 - \frac{2(p-1)^2}{4p^2 - 11p + 8}y,$$

as desired.

**Appendix B  Verifying an inequality in the proof of Lemma 4.4**

We want to verify that the following inequality appeared as the last inequality in the second last paragraph of the proof of Lemma 4.4 (see page 10) holds:

$$\frac{4(p-2)^2 - 0.1}{8(p-1)^3}n^2 - \frac{5}{8}n \geq \frac{2(p-2)^2}{p(4p^2 - 11p + 8)}n^2 - \frac{(p-2)(2p-3)}{4p^2 - 11p + 8}n$$

for any $n \geq 120p^2$ and $p \geq 3$. First observe that the above inequality is equivalent to

$$\left(\frac{4(p-2)^2 - 0.1}{8(p-1)^3} - \frac{2(p-2)^2}{p(4p^2 - 11p + 8)}\right)n \geq \frac{5}{8} - \frac{(p-2)(2p-3)}{4p^2 - 11p + 8},$$

which can be further simplified to

$$\frac{p(4p^2 - 16p + 15.9)(4p^2 - 11p + 8) - 16(p-1)^3(p-2)^2}{8p(1)^3(4p^2 - 11p + 8)}n \geq \frac{4p^2 + p - 8}{8(p^2 - 11p + 8)}.$$

Let $f(p) = p(4p^2 - 16p + 15.9)(4p^2 - 11p + 8) - 16(p-1)^3(p-2)^2 = 4p^4 - 32.4p^3 + 97.1p^2 - 128.8p + 64$. Then we only need to show that $nf(p) \geq p(p-1)^3(4p^2 + p - 8)$. First from the calculation by python (see Figure A), we can see that $f(p) \geq 0$ for all $p \geq 3$.

Since $f(p) \geq 0$, as $n \geq 120p^2$, we get that $nf(p) \geq 120p^2f(p)$. Thus we only need to show that $120p^2f(p) - (p-1)^3(4p^2 + p - 8) \geq 0$. Let $g(p) = 120p^2f(p) - (p-1)^3(4p^2 + p - 8) = 476p^5 - 3877p^4 + 11651p^3 - 15479p^2 + 7705p - 8$. Then, from the calculation by python (see Figure B), we can see that

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indeed, $g(p) \geq 0$ for all $p \geq 3$. This proves the desired inequality.

\section*{Appendix C \hspace{1em} On an immediate step in the proof of Lemma \ref{lem:4.4}}

Here we want to show that the last inequality in the last paragraph of the proof of Lemma \ref{lem:4.4} (see page 11) holds. This is equivalent to show that

\[ h(n, p, r) \geq 2^4 \left( p - 2 \right)^2 \cdot \frac{2(p - 1)}{8(p - 1)^3} \cdot \left( p - 2 \right)^2 \cdot \frac{2(p - 1)}{(4p^2 - 11p + 8)n^2} \cdot r \]

where $h(n, p, r) = \frac{p(4p^2 - 11p + 8)n^2 - (p - 2)(2p - 3)n - p(p - 1)}{8(4p^2 - 11p + 8)}$. Reformulating the above inequality by using factorization, we let

\[ H(r) = 8p(p - 1)^3(4p^2 - 11p + 8) \cdot h(n, p, r) \]

\[ = p^2(4p^2 - 11p + 8)^2 \cdot r^2 - \left( 4p(p - 2)^2(4p^2 - 11p + 8)n^2 + 2p^2(p - 1)^2(4p^2 - 11p + 8)n \right) \cdot r \]

\[ + 4p(p - 2)^2(4p^2 - 11p + 8)n^2 - 4p(p - 1)^2(p - 2)(4p^2 - 11p + 8)n. \]

Then, it becomes to show that $H(r) \geq 16(p - 1)^3(4p - 2)^2n^2 - 8p(p - 1)^3(4p - 2)(2p - 3)n - p^2(p - 1)^4$. Since $H''(r) > 0$, $H(r)$ is a convex quadratic function on $r$ and minimized at $r = \frac{2(p - 2)^2}{p(4p^2 - 11p + 8)} + \frac{(p - 1)^2}{(4p^2 - 11p + 8)n}$. 

\[ \frac{[3,0.7]}{\text{Fig 1}} \]

\[ \frac{[3,4]}{\text{Fig 2}} \]
(i.e., the solution of the equation $H'(r) = 0$). Thus, we have

$$H(r) \geq H\left(\frac{2(p-2)^2}{p(4p^2-11p+8)} + \frac{(p-1)^2}{(4p^2-11p+8)n}\right)$$

$$= p^2(4p^2 - 11p + 8)^2n^2 \cdot \left(\frac{4(p-2)^4}{p^2(4p^2 - 11p + 8)^2} + \frac{4(p-1)^2(p-2)^2}{p(4p^2-11p+8)^2n} + \frac{(p-1)^4}{(4p^2-11p+8)^2n^2}\right)$$

$$- (4p(p-2)^2(4p^2-11p+8)n^2 + 2p^2(p-1)^2(4p^2 - 11p + 8)n) \cdot \left(\frac{2(p-2)^2}{p(4p^2 - 11p + 8)} + \frac{(p-1)^2}{(4p^2 - 11p + 8)n}\right)$$

$$+ 4p(p-2)^2(4p^2 - 11p + 8)n^2 - 4p(p-1)^2(p-2)(4p^2 - 11p + 8)n$$

$$= (4(p-2)^4 - 8(p-2)^4 + 4p(p-2)^2(4p^2 - 11p + 8)) \cdot n^2 + (4p(p-1)^2(p-2)^2 - 4p(p-1)^2(p-2)^2 -$$

$$- 4p(p-1)^2(p-2)^2 - 4p(p-1)^2(p-2)(4p^2 - 11p + 8)) \cdot n + (p^2(p-1)^4 - 2p^2(p-1)^4)$$

$$= 16(p-1)^3(p-2)^2n^2 - 8p(p-1)^3(2p-3)n - p^2(p-1)^4,$$

as we wanted. This verifies the inequality under consideration.

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