FORMULA OF ENTROPY ALONG UNSTABLE FOLIATIONS FOR $C^1$ DIFFEOMORPHISMS WITH DOMINATED SPLITTING

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Abstract. Metric entropies along a hierarchy of unstable foliations are investigated for $C^1$ diffeomorphisms with dominated splitting. The analogues of Ruelle’s inequality and Pesin’s formula, which relate the metric entropy and Lyapunov exponents in each hierarchy, are given.

1. Introduction. It is well known that among the major concepts of smooth ergodic theory are the notions of invariant measures, entropy and Lyapunov exponents. Entropies, including measure-theoretic entropy and topological entropy, play important roles in the study of the complexity of a dynamical system. Intuitively, topological entropy measures the exponential growth rate in $n$ of the number of orbits of length $n$ up to a small error, measure-theoretic entropy gives the maximum average information with respect to some invariant measure one can get from a system. While Lyapunov exponents reflect the rate at which two nearby orbits separate from each other. What interests one is the relation between entropy and Lyapunov exponents. Let $f$ be a $C^1$ diffeomorphism on a compact Riemannian manifold $M$ without boundary. For any regular point in the sense of Oseledec [9] $x \in M$, let $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_r(x)$ denote its distinct Lyapunov exponents,
and $E_1(x) \oplus \cdots \oplus E_r(x)$ be the corresponding decomposition of its tangent space $T_x M$. In 1970s, Ruelle [14] gave the following inequality

$$h_\mu(f) \leq \int_M \sum_{\lambda_i(x) > 0} \lambda_i(x)m_i(x)d\mu(x)$$

for any $f$-invariant measure $\mu$, where $m_i(x) = \text{dim}E_i(x)$. Moreover, if $f$ is $C^2$ and $\mu$ is equivalent to the Riemannian measure on $M$, Pesin [10] proved that the equality (which is called Pesin’s entropy formula) holds in the above inequality.

In 1981 Mañe [8] gave an ingenious approach to prove Pesin’s entropy formula under the assumption that $f$ is $C^{1+\alpha}(\alpha > 0)$ and $\mu$ is absolutely continuous with respect to the Lebesgue measure. In 1985, Ledrappier and Young [5] proved that Pesin’s entropy formula holds for $C^2$ diffeomorphisms if and only if $\mu$ is an SRB measure. Furthermore, in [6] they gave a more general formula which is called the dimension formula for any $f$-invariant measure $\mu$ as follows

$$h_\mu(f) = \int_M \sum_{\lambda_i(x) > 0} \lambda_i(x)\gamma_i(x)d\mu(x),$$

where $\gamma_i(x)$ denotes the dimension of $\mu$ in the direction of the subspace $E_i(x)$. In their argument, they used the notion of “entropy along unstable foliations”, which reflects the complexity of the system at different levels.

Except for Ruelle’s inequality, all other results above require that $f$ is $C^{1+\alpha}$ or $C^2$, so it is interesting to investigate Pesin’s formula under $C^1$ differentiability hypothesis plus some additional conditions, for example, dominated splitting. Recently, Sun and Tian [16] applied Mañe’s method to prove that Pesin’s entropy formula holds if $f$ is a $C^1$ diffeomorphism with dominated splitting. In [3], Catsigeras, Cerminara, and Enrich considered a nonempty set of invariant measures which describe the asymptotic statistics of Lebesgue almost all orbits, and they proved that the measure-theoretic entropy of each of these measures is bounded from below by the sum of the Lyapunov exponents on the dominating subbundle. For more details about the dynamics of a system with dominated splitting, one can refer to [12] and [15]. Instead of the condition in [16] that $f$ admits a dominated splitting, Tian [17] gave the concept of nonuniformly-Hölder-continuity for an $f$-invariant measure, and proved that Pesin’s entropy formula holds under that assumption.

An interesting question is: can we get the formula of entropy along unstable foliations for a $C^1$ diffeomorphism with dominated splitting? In this paper, we give a positive answer of this question. The analogues of Ruelle inequality and Pesin’s entropy formula are given. In the proofs, we borrow some ideas from Hu, Hua and Wu’s paper [4] in which a variational principle relating the topological entropy and measure-theoretic entropy on the unstable foliation of a partially hyperbolic diffeomorphism is obtained.

This paper is organized as follows. In section 2, we give some preliminaries and the statement of our main results, and the proofs of the main results are given in the next two sections.

2. Preliminaries and statement of results. Throughout this paper, Let $M$ be a compact Riemannian manifold without boundary, $f$ a $C^1$ diffeomorphism on $M$, and $\mu$ an $f$-invariant Borel measure.
Let $\Gamma$ be the set of points which are regular in the sense of Oseledec [9]. For $x \in \Gamma$, let
\[ \lambda_1(x) > \lambda_2(x) > \cdots > \lambda_{r(x)}(x) \]
denote its distinct Lyapunov exponents and let
\[ T_x M = E_1(x) \oplus \cdots \oplus E_{r(x)}(x) \]
be the corresponding decomposition of its tangent space.

Now we give the definition of dominated splitting. Denote the minimal norm of an invertible linear map $A$ by $m(A) = \|A^{-1}\|^{-1}$.

**Definition 2.1.** (1) (Dominated splitting at one point) Let $x \in M$ and $T_x M = E(x) \oplus F(x)$ be a $Df$-invariant splitting on $\text{orb}(x)$. $T_x M = E(x) \oplus F(x)$ is called to be $(N(x), i(x))$-dominated splitting at $x$, if the dimension of $F$ is $i(x)(1 \leq i(x) \leq \dim M - 1)$ and there exists a constant $N(x) \in \mathbb{Z}^+$ such that
\[ \frac{\|Df^N(x)|_{F(f^n(x))}\|}{m(Df^N(x)|_{F(f^n(x))})} \leq \frac{1}{2}, \forall j \in \mathbb{Z}. \]

(2) (Dominated splitting on an invariant set) Let $\Delta$ be an $f$-invariant set and $T_\Delta M = E \oplus F$ be a $Df$-invariant splitting on $\Delta$. We call $T_\Delta M = E \oplus F$ to be $(N, i(y))$-dominated splitting, if the dimension of $F$ at $y$ is $i(y)(1 \leq i(y) \leq \dim M - 1)$ and there exists a constant $N \in \mathbb{Z}^+$ such that
\[ \frac{\|Df^N|_{F(y)}\|}{m(Df^N|_{F(y)})} \leq \frac{1}{2}, \forall y \in \Delta. \]

In the following, we consider two cases of the invariant measure $\mu$.

**Case 1.** $\mu$ is ergodic. In this case, the functions $x \mapsto r(x), \lambda_i(x)$ and $\dim E_i(x)$ are constant $\mu$-a.e., denote them by $r$, $\lambda_i$ and $m_i$ respectively. Let $u = \max\{i : \lambda_i > 0\}$, $u(i) = u - i + 1$, and $\delta_* = \min\{\lambda_{i+1} - \lambda_i, 1 \leq i \leq r - 1\}$. For $1 \leq i \leq u$, $x \in \Gamma$ and $0 < \varepsilon < \delta_*$, we define
\[ W^i(x) = \{y \in M : \limsup_{n \to \infty} \frac{1}{n} \log d(f^{-n}x, f^{-n}y) \leq -\lambda_{u(i)} + \varepsilon\}, \quad (1) \]
where $d$ is the Riemannian metric on $M$. The following result ensures that $W^i(x)$ is an immersed $C^1$-manifold under the assumption of dominated splitting.

**Proposition 1** ([1, Proposition 8.9]). Let $\mu$ be an ergodic measure whose support admits a dominated splitting $E \oplus F$, let $\lambda^+_E < \lambda^-_F$ be the maximal Lyapunov exponent in $E$ and the minimal Lyapunov exponent in $F$ of the measure $\mu$.

If $\lambda^+_E$ is strictly negative, then at $\mu$-a.e. point $x \in M$, there exists an injectively immersed $C^1$-manifold $W_E(x)$ with $\dim W_E(x) = \dim E$, tangent to $E_x$, which is a stable manifold, and for any $\lambda \leq 0$ contained in $(\lambda^+_E, \lambda^-_F)$ and $\mu$-a.e. point $x$ we have
\[ W_E(x) = \{y \in M : d(f^n x, f^n y) e^{-\lambda n} \to 0, \text{ as } n \to \infty\}. \]

For $x \in \Gamma$ and $1 \leq i \leq u$, let
\[ E(x) = \bigoplus_{u(i)+1 \leq j \leq r} E_j(x) \quad \text{and} \quad F(x) = \bigoplus_{1 \leq j \leq u(i)} E_j(x). \]

**Assumption 1** (ergodic case). For each $1 \leq i \leq u$, $T_{\Gamma \cdot} M = E \oplus F$ is $(N, I(i))$-dominated splitting for some $N \in \mathbb{Z}^+$, where $I(i) = \sum_{1 \leq j \leq u(i)} m_j$.

**Remark 1.** It is obvious that any hyperbolic automorphism on two dimensional tori satisfies Assumption 1.
Under Assumption 1, we know that, by Proposition 1, $W^i(x)$ is a $C^1$ $I(i)$-dimensional immersed submanifold of $M$ tangent at $x$ to $F(x)$ by replacing $f$ with $f^{-1}$. It is called the $i$th unstable manifold of $f$ at $x$. $\{W^i(x): x \in \Gamma\}$ is called the $W^i$-foliation.

A measurable partition $\xi_i$ of $M$ is said to be subordinate to $W^i$ if for $\mu$ -a.e. $x \in \Gamma$, $\xi_i(x) \subset W^i(x)$ and contains an open neighborhood of $x$ in $W^i(x)$. An important property with respect to such a partition is that there is a canonical system of conditional measures $\{\mu^i_x\}$. The following lemma ensures the existence of such partitions.

**Lemma 2.2.** Let $\mu$ be an ergodic measure, then there exists a measurable partition $\xi_i$ of $M$ satisfying the following properties:

(1) $\xi_i$ is a partition subordinate to $W^i$;
(2) $\xi_i$ is increasing, i.e., $f^{-1}\xi_i \supseteq \xi_i$;
(3) $\sum_{i=1}^{\infty} e^{-n}\xi_i = \epsilon$, where $\epsilon$ is the partition of $M$ into points;
(4) $\bigvee_{n=1}^{\infty} f^n\xi_i = \mathcal{H}(\Pi^+)$, where $\Pi^+$ is the partition of $M$ into global $i$th unstable manifolds, and $\mathcal{H}(\Pi^+)$ is the measurable hull of $\Pi^+$.

**Proof.** For the proof, the reader can refer to [11].

For more details about measurable partitions and conditional measures the reader can refer to Section 0.1 – 0.3 in [7] and Section 3 and 4 in [13].

Let $\xi_i$ be a measurable partition subordinate to $W^i$ with conditional measures $\{\mu^i_x\}$. Define $h^i_\mu(f, x): \Gamma \rightarrow \mathbb{R}$ by

$$
h^i_\mu(f, x) = h^i_\mu(f, x, \xi_i) = h_i(f, x, \xi_i, \{\mu^i_x\}) = \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \inf_n - \frac{1}{n} \log \mu^i_x V^i(f, x, n, \varepsilon) = \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} - \frac{1}{n} \log \mu^i_x V^i(f, x, n, \varepsilon),
$$

where $V^i(f, x, n, \varepsilon) = \{y \in W^i(x): d^k(f^k(y), f^k(x)) < \varepsilon, 0 \leq k \leq n - 1\}$, and $d^k$ is the metric on $W^i(x)$ given by the Riemannian structure inherited from $M$.

$h^i_\mu(f, x)$ is well defined and is independent of the choice of $\xi_i$ or $\mu^i_x$, and it is easy to verify that $h^i_\mu(f, x) = h^i_\mu(f, x, \mu)$ $\mu$-a.e.(cf.[6]). Since $\mu$ is ergodic, $h^i_\mu(f, x)$ is constant $\mu$-a.e.

**Definition 2.3.** We define the entropy of $f$ along ith unstable foliation by

$$
h^i_\mu(f) = \int_M h^i_\mu(f, x) d\mu(x).
$$

Since $\mu$ is ergodic, we know that $h^i_\mu(f, x) = h^i_\mu(f)$, for $\mu$-a.e.$x \in M$.

**Case 2.** $\mu$ is arbitrary. In this case, the functions $x \mapsto r(x)$, $\lambda_i(x)$ and $\dim E_i(x)$ are now measurable. Let $u(x) = \max\{i: \lambda_i(x) > 0\}$, $u(i, x) = u(x) - i + 1$, and $\Gamma_i = \{x \in \Gamma: u(i, x) > 0\}$. Then we can define $W^{u(i, x)}(x)$ as in (1) except that $u(i)$ and $\lambda_{u(i)}$ should be replaced by $u(i, x)$ and $\lambda_{u(i, x)}(x)$ respectively, and the choice of $\varepsilon$ depends on $x$ such that $\varepsilon < \lambda_{u(i, x)}(x) - \lambda_{u(i, x) + 1}(x)$.
For $x \in \Gamma$ and $1 \leq i \leq u(x)$, let

$$E(x) = \oplus_{u(i,x)+1 \leq j \leq r(x)} E_j(x) \quad \text{and} \quad F(x) = \oplus_{1 \leq j \leq u(i,x)} E_j(x).$$

**Assumption 2 (general case).** For each $x \in \Gamma$ and $1 \leq i \leq u(x)$, $T_{\text{orb}(x)} M = E \oplus F$ is $(N(i,x), I(i,x))$-dominated splitting, where $N(i, \cdot): \Gamma_i \rightarrow Z^+$ is a measurable function and $I(i,x) = \sum_{1 \leq j \leq u(i,x)} m_j(x)$.

Similar to that in Proposition 1, under Assumption 2, $W^i(x)$ is a $C^1 I(i,x)$-dominated immersed submanifold of $M$ tangent to $F(x)$ at $x$. It is also called the $i$th unstable manifold of $f$ at $x$. $\{W^i(x) : x \in \Gamma_i\}$ is called the $W^i$-foliation.

A measurable partition $\xi^i$ of $M$ is said to be subordinate to $W^i$ on $\Gamma_i$, if for $\mu$-a.e. $x \in \Gamma_i$, $\xi^i(x) \subset W^i(x)$ and contains an open neighborhood of $x$ in $W^i(x)$. For the existence of such $\xi^i$, one can simply disintegrate $\mu$ into its ergodic components and note that the entire leaf $W^i(x)$ is contained in the ergodic component of $x$ (cf. [6]). There is a canonical system of conditional measures $\{\mu^j_i\}$ as $\mu$ is ergodic. Then for $x \in \Gamma_i$, we can define $h^i_{\mu}(f,x): \Gamma_i \rightarrow \mathbb{R}$ as $\mu$ is ergodic. And we define the entropy along $W^i$ on $\Gamma_i$ which is still denoted by $h^i_{\mu}(f)$ as in Definition 2.3 except that $M$ now should be replaced by $\Gamma_i$.

**Remark 2.** It is easy to check that when $\mu$ is ergodic, $\Gamma_i = \Gamma$ for $1 \leq i \leq u$. So when $\mu$ is ergodic, the entropy along $i$th unstable foliation on $\Gamma_i$ coincides with the entropy along $i$th unstable foliation. So we call the entropy defined as above the entropy of $f$ along $i$th unstable foliation.

**Standing hypotheses for the remaining of this paper:** When $\mu$ is ergodic, we set Assumption 1, and when $\mu$ is arbitrary, we set Assumption 2.

Now we are ready to state our main results of this paper:

**Theorem A.** Let $\mu$ be an invariant measure. Then we have the following inequality

$$h^i_{\mu}(f) \leq \int_{\Gamma_i} \sum_{j \leq u(i,x)} m_j(x) \lambda_j(x) d\mu(x).$$

In particular, if $\mu$ is ergodic, then $h^i_{\mu}(f, x), \lambda_i(x), m_i(x)$ and $u(i,x)$ are constant, then we have

$$h^i_{\mu}(f) \leq \sum_{j \leq u(i)} m_j \lambda_j.$$

Moreover, if $\mu$ satisfies some additional conditions, we have the following theorem.

**Theorem B.** Let $\mu$ be an invariant measure satisfying that for $\mu$-a.e. $x \in M$ and every measurable partition $\xi^i$ subordinate to $W^i$, $\mu^i \ll \lambda^i$, where $\lambda^i$ is the corresponding Riemannian measure on $W^i(x)$. Then we have the following entropy formula

$$h^i_{\mu}(f) = \int_{\Gamma_i} \sum_{j \leq u(i,x)} m_j(x) \lambda_j(x) d\mu(x).$$

In particular, if $\mu$ is ergodic, then we have

$$h^i_{\mu}(f) = \sum_{j \leq u(i)} m_j \lambda_j.$$
Remark 3. We only need to prove the ergodic versions of Theorem A and Theorem B respectively, and the nonergodic versions of them follow immediately from the ergodic versions by decomposing $\mu$ into ergodic components (just as that has been done in [6]). So in the following two sections, we always assume that $\mu$ is ergodic.

In the following, we relate the entropy $h^i$ along the unstable foliation $W^i$ with the supremum of certain conditional entropy of finite partitions with respect to a measurable partition subordinate to $W^i$. This idea derives from [4].

Definition 2.4. Let $\mu$ be an ergodic $f$-invariant measure and $\mathcal{P}_M$ denote the set of all finite Borel partitions of $M$. The conditional entropy of $\alpha \in \mathcal{P}_M$ with respect to $\xi$ is defined as

$$H^i(\alpha|\xi) = \int_{\Gamma_i} -\log \mu_\xi^i(\alpha(x) \cap \xi(x))d\mu(x).$$

The following proposition gives an equivalent definition of $h^i$.

Proposition 2. Let $\mu$ be an ergodic measure, then we have

$$h^i(\mu) = \sup_{\alpha \in \mathcal{P}_M} \limsup_{n \to \infty} \frac{1}{n} H^i(\bigvee_{i=0}^{n-1} f^{-i}\alpha|\xi).$$

Proof. Similar to the proof of $h^u(\mu) = \sup_{\alpha \in \mathcal{P}_M} \limsup_{n \to \infty} \frac{1}{n} H^u(\bigvee_{i=0}^{n-1} f^{-i}\alpha|\xi)$ in [4], where $\xi$ is a partition subordinate to the unstable foliation $W^u$. We omit the details. \(\Box\)

Remark 4. In fact, the partitions used in Definition 2.4 and Proposition 2 can be replaced by some more natural partitions. Roughly speaking, such partition is constructed via the intersection of a finite partition and the local unstable manifolds. For more details, the reader can refer to [4].

As the classical measure-theoretic entropy and the topological entropy, the entropy along $i$th unstable foliation also has the so-called power rule.

Proposition 3 (Power rule). For $m \geq 1$, we have

$$h^i_m(f^m) = mh^i(f).$$

Proof. Let $\xi$ be a measurable partition of $M$ subordinate to $W^i$. Fix $\varepsilon > 0$. It is clear that

$$V^i(f^m, x, n, \varepsilon) \supset V^i(f, x, mn, \varepsilon),$$

so, we have

$$-\frac{1}{n} \log \mu^i_\xi f^m, x, n, \varepsilon) \leq -m \frac{1}{mn} \log \mu^i_\xi V^i(f, x, mn, \varepsilon).$$

Let $n \to \infty$, and then $\varepsilon \to 0$, we obtain

$$h^i_m(f^m, x) \leq mh^i(f, x).$$

On the other hand, pick $\delta_0 > 0$, define $W^i(x, \delta_0) = \{y : y \in W^i(x), d^i(y, x) < \delta_0\}$. Because of the compactness of $W^i(x, \delta_0)$, we can pick $0 \leq \delta \leq \varepsilon < \delta_0$ such that if $d^i(x, y) \leq \delta$, we have

$$d^i(f^j(x), f^j(y)) \leq \varepsilon, \forall 0 \leq j \leq m - 1.$$ 

It follows that

$$V^i(f^m, x, n, \delta) \subset V^i(f, x, mn, \varepsilon),$$
and hence,
\[ -\frac{1}{n} \log \mu_x^i V^i(f^n, x, n, \delta) \geq -m \frac{1}{mn} \log \mu_x^i V^i(f, x, mn, \varepsilon). \]
Let \( n \to \infty \), and then \( \varepsilon \to 0 \) (hence \( \delta \to 0 \)), we obtain
\[ h^i_n(f^m, x) \geq mh^i_n(f, x). \] (4)

(2) follows from (3) and (4) immediately. \( \Box \)

3. **Proof of Theorem A.** Now we complete the proof of Theorem A. Firstly, we need the following definition from [4].

**Definition 3.1.** Pick \( 0 < \delta < \tau \), where \( \tau \) is as in the proof of Lemma 2.2, and \( \varepsilon > 0 \) small enough. Let \( S \subseteq \overline{W^s}(x, \delta) \) satisfying
\[ d(f^jy, f^jz) \geq \varepsilon, \exists 0 \leq j \leq n - 1, \forall y, z \in S, \]
we call \( S \) an \((n, \varepsilon)\) i-separated set of \( \overline{W^s}(x, \delta) \). Let \( N^i(f, \varepsilon, n, x, \delta) \) denote the largest cardinality of any \((n, \varepsilon)\) i-separated set in \( \overline{W^s}(x, \delta) \).

Let \( R \subseteq \overline{W^u}(x, \delta) \) satisfying
\[ d(f^jy, f^jz) < \varepsilon, 0 \leq j \leq n - 1, \forall y, z \in S, \]
we call \( R \) an \((n, \varepsilon)\) i-spanning set of \( \overline{W^u}(x, \delta) \). Let \( S^i(f, \varepsilon, n, x, \delta) \) denote the smallest cardinality of any \((n, \varepsilon)\) i-spanning set in \( \overline{W^u}(x, \delta) \).

The following lemma gives us a relation between \( N^i(f, \varepsilon, n, x, \delta) \) and \( S^i(f, \varepsilon, n, x, \delta) \).

**Lemma 3.2.**
\[ \lim_{n \to \infty} \limsup \frac{1}{n} \log N^i(f, \varepsilon, n, x, \delta) = \lim_{n \to \infty} \limsup \frac{1}{n} \log S^i(f, \varepsilon, n, x, \delta). \]

**Proof.** cf. the proof of Lemma 3.5 in [4]. \( \Box \)

The estimation of \( h^i_\mu(f) \) from above is based on the following lemma.

**Lemma 3.3.** Let \( \Sigma \subseteq M \) with \( \mu(\Sigma) = 1 \), and assume that \( \lambda_i(x), m_i(x) \) are constant when \( x \in \Sigma \), then for any \( \rho > 0 \), there exists \( x \in \Sigma \) such that
\[ h^i_\mu(f) - \rho \leq \lim_{n \to \infty} \frac{1}{n} \log S^i(f, \varepsilon, n, x, \delta). \]

**Proof.** Let \( \xi_i = \xi \) be any measurable partition subordinate to ith unstable foliation as in Lemma 2.2. Since \( \mu \) is ergodic, then we can pick \( x \in \Sigma \) with the following property: there exists a set \( B \subseteq \xi(x) \) with \( \mu^i_\mu(B) = 1 \), such that
\[ h^i_\mu(f, y, \xi) = h^i_\mu(f, \xi), \forall y \in B. \]

In fact, \( h^i_\mu(f, x, \xi) \) is \( \mu \)-a.e. constant, let \( \Sigma_1 \) be the set of \( x \in M \) where \( h^i_\mu(f, x, \xi) \) is constant. Let \( \Sigma_2 \) be the set of \( x \in M \) such that \( \mu^i_\mu(\xi(x)) = 1 \), it is clear that \( \mu(\Sigma_2) = 1 \). We can pick \( x \in \Sigma \cap \Sigma_1 \cap \Sigma_2 \), and let \( B = \xi(x) \cap \Sigma \cap \Sigma_1 \cap \Sigma_2 \). It is clear that \( \mu^i_\mu(B) = 1 \) and \( B \) satisfies the property above.

The property above implies that for any \( \rho > 0 \) and \( y \in B \), there exists \( \varepsilon_0(y) \), such that if \( 0 < \varepsilon < \varepsilon_0(y) \), then
\[ \liminf_{n \to \infty} -\frac{1}{n} \log \mu^i_\mu(V^i(f, y, n, \varepsilon)) \geq h^i_\mu(f, \xi) - \rho. \] (5)
Denote \( B_\varepsilon := \{ y \in B | \|y\|_0 \geq \varepsilon \} \), then \( B = \bigcup_{\varepsilon > 0} B_\varepsilon \). So there exists \( \varepsilon_1 > 0 \) such that \( \mu^\varepsilon(B_\varepsilon) > 1 - \rho \) for any \( \varepsilon < \varepsilon_1 \). Fix such an \( \varepsilon \). (5) implies that for any \( y \in B_\varepsilon \) there exists \( N = N(y) > 0 \) such that if \( n \geq N \), then
\[
\mu^\varepsilon(B(y, n, \varepsilon)) \leq e^{-(h(n,f,\xi) - \rho)}.
\]

Denote \( B^n_\varepsilon := \{ y \in B_\varepsilon | N(y) \leq n \} \). Then \( B = \bigcup_{n=1}^\infty B^n_\varepsilon \). So there exists \( N \) large enough such that \( \mu^\varepsilon(B^n_\varepsilon) > \mu^\varepsilon(B_\varepsilon) - \rho > 1 - 2\rho \). Since \( y \in \xi(x) \), \( \mu^\varepsilon(B^n_\varepsilon) = \mu^\varepsilon \), for any \( n \geq N \), one has
\[
\mu^\varepsilon(B(y, n, \varepsilon)) \leq e^{-(h(n,f,\xi) - \rho)}.
\]

Now we take \( \delta > 0 \) with \( W^i(x, \delta) \supset \xi(x) \). Then there exists a set \( R_n \) with cardinality no more than \( S^i(f, \frac{\varepsilon}{2}, n, x, \delta) \), such that
\[
W^i(x, \delta) \cap B \subseteq \bigcup_{z \in R_n} V^i(f, z, n, \frac{\varepsilon}{2}),
\]
and \( V^i(f, z, n, \frac{\varepsilon}{2}) \cap B \neq \emptyset \). Choose an arbitrary point in \( V^i(f, z, n, \frac{\varepsilon}{2}) \cap B \) and denote it by \( y(z) \). Then we have
\[
1 - 2\rho \leq \mu^\varepsilon(B(y(z), n, \frac{\varepsilon}{2})) \leq \mu^\varepsilon(B(y(z), n, \frac{\varepsilon}{2} \cap B)) \leq \sum_{z \in R_n} \mu^\varepsilon(V^i(f, z, n, \frac{\varepsilon}{2})) \leq \sum_{z \in R_n} \mu^\varepsilon(V^i(f, y(z), n, \frac{\varepsilon}{2})) \leq S^i(f, \frac{\varepsilon}{2}, n, x, \delta) e^{-(h(n,f,\xi) - \rho)}.
\]

And hence \( S^i(f, \frac{\varepsilon}{2}, n, x, \delta) \geq e^{n(h(n,f,\xi) - \rho)} \). Thus we have
\[
h^i(f) - \rho = h^i(f, \xi) - \rho \leq \lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n} \log S^i(f, \varepsilon, n, x, \delta).
\]

Now we begin the proof of Theorem A. Let \( S_\varepsilon \subseteq W^i(x, \delta) \) be an \( (n, \varepsilon) \) i-separated set with the largest cardinality. When \( n \) is large enough, we can pick \( y_n \in S_\varepsilon \) such that
\[
N^i(f, \varepsilon, n, x, \delta) \leq \frac{\tilde{V}_{x,\delta}}{\text{Vol}(\exp^{-1} W^i(f, y_n, n, \frac{\varepsilon}{2}))}, \quad (6)
\]
where \( \exp_x \) is the exponential map at \( x \), \( \tilde{V}_{x,\delta} = \text{Vol}(\exp^{-1} W^i(x, \delta)) \), and \( \text{Vol}(\cdot) \) denotes the volume function.

Because of the compactness of \( M \), we can choose \( \delta > 0 \) small enough such that \( \exp_x \) is a diffeomorphism and \( d(\exp_v, x) = \|v\| \) for \( v \in T_x M, \|v\| < \delta \). In order to avoid a cumbersome computation, for every \( x \in M \), we treat the tangent space \( T_x M \) as it were \( \mathbb{R}^n \). We denote the Jacobian determinant of \( \exp_x^{-1} f^{-1} \exp_y f(x) | f(x) \) at \( y \in F(x) \) by \( J_x(y) \). For any \( \varepsilon > 0 \), we can choose \( 0 < \delta(\varepsilon) < \frac{\varepsilon}{2} \) such that \( |J_x(y_1)| - |J_x(y_2)| < \varepsilon \), for any \( x \in M \) and \( y_1, y_2 \in \pi_x B(0_x, \delta(\varepsilon)) \), where \( \pi_x : T_x M \to F(x) \).
is the projection and 0_e is the null vector in T_x M. Let \( \varepsilon_0 : = \frac{1}{2} \inf \{|J_x(y)| : x \in M, y \in \pi_x B(0_x, \frac{\varepsilon}{2})\} \). Then for any \( x \in M \), we have
\[
\frac{1}{2} < \frac{|J_x(y_1)|}{|J_x(y_2)|} < 2, \tag{7}
\]
for any \( y_1, y_2 \in \pi_x B(0_x, \delta_0) \), where \( \delta_0 = \delta(\varepsilon) \).

Fix \( 0 < \varepsilon < \delta_0 \), pick \( \tilde{y}_n \in W^s(x, \delta) \) such that \( f^n(\tilde{y}_n) \in B(f^n(y_n), \frac{\varepsilon}{2}) \cap W^s(f^n(x)) \) and let
\[
B^i_n = \exp^{-1}_{f^n(\tilde{y}_n)} B(f^n(y_n), \frac{\varepsilon}{2}) \cap W^s(f^n(x)).
\]
Then we have
\[
\text{Vol}(\exp^{-1}_{\tilde{y}_n} f^{-n} \exp f^n(\tilde{y}_n) B^i_n) = \text{Vol}(\exp^{-1}_{\tilde{y}_n} f^{-1} \exp f(\tilde{y}_n) \exp^{-1}_{\tilde{y}_n} f^{-1} \exp f(\tilde{y}_n) \cdots \exp^{-1}_{\tilde{y}_n} f^{-1} \exp f(\tilde{y}_n) B^i_n)
\]
\[
= \int_{B^i_n} |J_{\tilde{y}_n}(\tilde{f}^{-n-1}(y))||J_{\tilde{y}_n}(\tilde{f}^{-n-2}(y))| \cdots |J_{\tilde{y}_n}(\tilde{f}^{-n+1}(y))|d\lambda,
\]
where \( \tilde{f}_j(y) = \exp^{-1}_{f^{j-1}(\tilde{y}_n)} f^{-j} \exp f^{j-1}(\tilde{y}_n), j = 1, 2, \cdots, n - 1 \) and \( \lambda \) is the Lebesgue measure on \( B^i_n \).

Notice the definition of \( B^i_n \), so we have
\[
\tilde{f}^{-n-j}(y) \in \pi_x B(0_{f^{-1-j}(\tilde{y}_n)}, \frac{\varepsilon}{2}),
\]
for \( j = 1, 2, \cdots, n - 1 \) and \( y \in B^i_n \). Hence by (7) we have
\[
|J_{f^{-1}(\tilde{y}_n)}(\tilde{f}^{-n-j}(y))| > \frac{1}{2} |J_{f^{-1}(\tilde{y}_n)}(0_{f^{-1}(\tilde{y}_n)})|.
\]
So we have
\[
\text{Vol}(\exp^{-1}_{\tilde{y}_n} f^{-n} \exp f^n(\tilde{y}_n) B^i_n) \geq \frac{1}{2^n} \prod_{j=1}^{n-1} |J_{f^{-1}(\tilde{y}_n)}(0_{f^{-1}(\tilde{y}_n)})||J_{f^{-n-j}(\tilde{y}_n)}(0_{f^{-1}(\tilde{y}_n)})|d\lambda
\]
\[
= \frac{1}{2^n} \text{Vol}(D_f^n(\tilde{y}_n) f^{-n} B^i_n).
\]
The last equality follows that \( D f^n |_{\tilde{y}_n} \) is an identity.

Let \( R_{m,\varepsilon} \) be the set of \( x \in M \) such that for \( \varepsilon' > 0 \), any \( n \geq m \) and \( v \in E_i(x) \), we have \( \|D_x f^{-n} v\| \geq e^\varepsilon (-\lambda_i - \varepsilon') \). By Oseledec’s Theorem, we know that
\[
\lim_{m \to \infty} \mu(R_{m,\varepsilon'}) = 1.
\]
Without loss of generality, we assume that for any \( y \in M \) and \( a > 0 \), \( \mu(B(y, a)) > 0 \). In fact, let
\[
A = \{ y \in M : \exists a > 0 \text{ such that } \mu(B(y, a)) = 0 \}.
\]
It is easy to check that \( A \) is an \( f \)-invariant set and \( \mu(A) = 0 \). So for \( \varepsilon \), there exists \( N > 0 \) such that for any \( n \geq N \),
\[
B(y_n, \frac{\varepsilon}{2}) \cap R_{n,\varepsilon'} \neq \emptyset.
\]
And for every \( n \geq N \) we can choose an appropriate \( x_n \in M \) such that
\[
B(y_n, \frac{\varepsilon}{2}) \cap W^s(f^n(x_n)) \cap R_{n,\varepsilon'} \neq \emptyset.
\]
So when \( n \) is large enough such that
\[
\frac{1}{n} \log \frac{1}{C} + \frac{1}{n} \dim M \log \varepsilon < \varepsilon',
\]
we can pick \( \tilde{y}_n \) from the set \( B(y_n, \frac{\varepsilon}{2}) \cap W^i(f^n(x_n)) \cap R_{n, \varepsilon'} \). Hence when \( n \) is large enough, we have
\[
- \frac{1}{n} \log \Vol(\exp_{\tilde{y}_n}^{-1} f^{-n} \exp_{\tilde{y}_n} B^i_{n}) \\
\leq - \frac{1}{n} \log \frac{1}{2^n} \Vol(D_{f^n(\tilde{y}_n)} f^{-n} B^i_{n}) \\
\leq - \frac{1}{n} \log \frac{1}{2^n} C \prod_{j=1}^{u_i} \varepsilon e^{nm_j} (-\lambda_j - \varepsilon') \\
\leq \sum_{j=1}^{u_i} \lambda_j m_j + \log 2 + \dim M \varepsilon' + \frac{1}{n} \log \frac{1}{C} + \frac{1}{n} \dim M \log \varepsilon \\
\leq \sum_{j=1}^{u_i} \lambda_j m_j + \log 2 + (\dim M + 1) \varepsilon'.
\]
where the constant \( C \) only related to \( f \).

Since
\[
\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \text{N}^i(f, \varepsilon, n, x, \delta) \\
\leq \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} (\log \text{V}_{x, \delta} - \log \Vol(\exp_{x}^{-1} V^i(f, y_n, n, \frac{\varepsilon}{2}))),
\]
and notice that
\[
\limsup_{n \to \infty} \frac{1}{n} \log \text{V}_{x, \delta} = 0,
\]
so let \( \varepsilon' \to 0 \), using Lemma 3.2 and Lemma 3.3, we obtain
\[
h^i_{\mu}(f) - \rho \leq \sum_{j \leq u(i)} \lambda_j m_j + \log 2.
\]
Let \( \rho \to 0 \), we obtain
\[
h^i_{\mu}(f) \leq \sum_{j \leq u(i)} \lambda_j m_j + \log 2.
\]
For \( N > 0 \), let \( g = f^N \), then we have
\[
h^i_{\mu}(f) = \frac{1}{N} h^i_{\mu}(g) \leq \sum_{j \leq u(i)} \lambda_j m_j + \frac{1}{N} \log 2.
\]
Let \( N \to \infty \), we obtain
\[
h^i_{\mu}(f) \leq \sum_{j \leq u(i)} \lambda_j m_j.
\]
Now we have completed the proof of Theorem A.
4. Proof of Theorem B. Now we start to prove Theorem B. By Theorem A, we only need to complete the estimation of $h^1_\mu(f)$ from below. Firstly, we give the following lemma.

**Lemma 4.1.** Let $\lambda^i_\mu$ be the corresponding Riemannian measure on $W^i(x)$ and $\xi^i = \xi$ be a measurable partition as in Lemma 2.2. If for $\mu$-a.e. $x \in M$, $\mu_x^\xi \ll \lambda^i_\mu$, we have

$$h^1_\mu(f) \geq \liminf_{n \to \infty} \frac{1}{n} \int_{\Gamma_\varepsilon} \log \lambda^i_\mu(V^i(f, x, n, \varepsilon)) d\mu(x).$$

**(Proof.** Let $\alpha$ be a finite Borel partition of $M$ with $diam(\alpha) \leq \varepsilon$ and let $\alpha_n := \alpha \lor \cdot^{-1} \lor \cdot^{-n} \lor \cdot^{-n} \alpha$.

Set $\mathcal{A}$ be the $\sigma$-algebra generated by partitions $\alpha_n$, $n \geq 0$. Let $\tilde{\mu}_x^\xi$, $\tilde{\lambda}_x^i$ be two measures on $M$ satisfying

$$\tilde{\mu}_x^\xi(M \setminus \xi(x)) = 0, \tilde{\mu}_x^\xi|_{\xi(x)} = \mu_x^\xi,$$

and

$$\tilde{\lambda}_x^i(M \setminus \xi(x)) = 0, \tilde{\lambda}_x^i|_{\xi(x)} = \mu_x^\xi.$$

It is easy to verify that $\tilde{\mu}_x^\xi \ll \tilde{\lambda}_x^i$. Let $k: M \to \mathbb{R}$ be a $\tilde{\lambda}_x^i$-integrable function with respect to $\mathcal{A}$ such that

$$\int_{\mathcal{A}} kd\tilde{\lambda}_x^i = \tilde{\mu}_x^\xi(A), \forall A \in \mathcal{A}.$$  

Such a function exists because that $\tilde{\mu}_x^\xi \ll \tilde{\lambda}_x^i$. It follows from (9) that

$$\lim_{n \to \infty} \frac{\tilde{\mu}_x^\xi(\alpha_n(x))}{\tilde{\lambda}_x^i(\alpha_n(x))} = k(x), \text{ for } \tilde{\lambda}_x^i\text{-a.e. } x \in M. \quad (10)$$

And hence we have

$$-\frac{1}{n} \log \tilde{\mu}_x^\xi(\alpha_n(x)) = -\frac{1}{n} \log \tilde{\lambda}_x^i(\alpha_n(x)) - \frac{1}{n} \log \frac{\tilde{\mu}_x^\xi(\alpha_n(x))}{\tilde{\lambda}_x^i(\alpha_n(x))}.$$  

Using (10), we obtain

$$\liminf_{n \to \infty} -\frac{1}{n} \log \tilde{\mu}_x^\xi(\alpha_n(x)) = \liminf_{n \to \infty} -\frac{1}{n} \log \tilde{\lambda}_x^i(\alpha_n(x)).$$

Note that $\{y \in M : d(f^k(y), f^k(x)) < \varepsilon, 0 \leq k \leq n - 1\} =: V(f, x, n, \varepsilon) \supset \alpha_n(x)$, so we have

$$\liminf_{n \to \infty} -\frac{1}{n} \log \mu_x^\xi(\alpha_n(x)) = \liminf_{n \to \infty} -\frac{1}{n} \log \lambda_x^i(V(f, x, n, \varepsilon)).$$

Observe that for $\varepsilon_0 > 0$ small enough, we can take $C > 0$, such that for any $x \in M$,

$$\frac{1}{C} d(y, z) \leq d^i(y, z) \leq C d(y, z), \forall y, z \in W^i(x, \varepsilon_0).$$

Notice the relation between $\tilde{\mu}_x^\xi$ and $\mu_x^\xi$, $\tilde{\lambda}_x^i$ and $\lambda_x^i$ respectively, if $\varepsilon \leq \varepsilon_0$ is small enough, we have

$$\liminf_{n \to \infty} -\frac{1}{n} \log \mu_x^\xi(\alpha_n(x) \cap \xi(x)) \geq \liminf_{n \to \infty} -\frac{1}{n} \log \lambda_x^i(V^i(f, x, n, C\varepsilon))). \quad (11)$$

Integrating both side of (11), we obtain

$$\int_{\Gamma_\varepsilon} \liminf_{n \to \infty} -\frac{1}{n} \log \mu_x^\xi(\alpha_n(x) \cap \xi(x)) d\mu(x)$$

$$\geq \int_{\Gamma_\varepsilon} \liminf_{n \to \infty} -\frac{1}{n} \log \lambda_x^i(V^i(f, x, n, C\varepsilon))) d\mu(x). \quad (12)$$
By Fatou’s Lemma we obtain
\[ \limsup_{n \to \infty} \frac{1}{n} H^i(\alpha_n | \xi) = \limsup_{n \to \infty} \frac{1}{n} \int_{\Gamma_1} -\log \mu_\xi^E(\alpha_n(x) \cap \xi(x))d\mu(x) \]
\[ \geq \liminf_{n \to \infty} \frac{1}{n} \int_{\Gamma_1} -\log \mu_\xi^E(\alpha_n(x) \cap \xi(x))d\mu(x) \]
\[ \geq \int_{\Gamma_1} \liminf_{n \to \infty} -\frac{1}{n} \log \mu_\xi^E(\alpha_n(x) \cap \xi(x))d\mu(x). \]  
(13)
Hence by Proposition 2, we have
\[ h_\mu^i(f) \geq \limsup_{n \to \infty} \frac{1}{n} H^i(\alpha_n | \xi). \]  
(14)
Combining (12), (13), and (14), we obtain
\[ h_\mu^i(f) \geq \int_{\Gamma_1} \liminf_{n \to \infty} \frac{1}{n} \log \lambda_x^i(V^i(f, x, n, C\varepsilon))d\mu(x). \]  
(15)

Let \( \{\varepsilon_k\}_{k \geq 1} \) be a sequence such that \( \varepsilon_k > 0 \) and \( \varepsilon \to 0 \) as \( k \to \infty \). Then by the monotone convergence theorem, we have
\[ \lim_{\varepsilon \to 0} \int_{\Gamma_1} \liminf_{n \to \infty} \frac{1}{n} \log \lambda_x^i(V^i(f, x, n, C\varepsilon))d\mu(x) \]
\[ = \lim_{k \to \infty} \int_{\Gamma_1} \liminf_{n \to \infty} \frac{1}{n} \log \lambda_x^i(V^i(f, x, n, C\varepsilon_k))d\mu(x) \]
\[ = \int_{\Gamma_1} \lim_{k \to \infty} \liminf_{n \to \infty} \frac{1}{n} \log \lambda_x^i(V^i(f, x, n, C\varepsilon_k))d\mu(x) \]
\[ = \int_{\Gamma_1} \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log \lambda_x^i(V^i(f, x, n, C\varepsilon))d\mu(x). \]  
(16)
Therefore (8) follows from (15) and (16). \( \square \)

Before going into the proof of Theorem B, we need a technical lemma from [16]. In the statement of the lemma, we will use the following definition from [8].

**Definition 4.2.** Let \( E \) be a normed space and \( E = E_1 \oplus E_2 \) be a splitting. Define \( \gamma(E_1, E_2) \) as the supremum of norms of the projections \( \pi: E \to E_i, i = 1, 2 \), associated with the splitting. Moreover, we say that a subset \( G \subseteq E \) is a \((E_1, E_2)\)-graph if there exists an open set \( U \subseteq E_2 \) and a \( C^1 \) map \( \psi: U \to E_1 \) satisfying
\[ G = \{ x + \psi(x) | x \in U \}. \]
The number \( \sup \{ \frac{\| \psi(x) - \psi(y) \|}{\| x - y \|} | x \neq y \in U \} \) is called the dispersion of \( G \).

The following lemma about graph transform on dominated bundles is a generalization of Lemma 3 in [8] by Sun and Tian [16].

**Lemma 4.3.** Given \( \alpha > 0, \beta > 0 \) and \( c > 0 \), there exists \( \tau > 0 \) with the following property. If \( E \) is a finite-dimensional normed space and \( E = E_1 \oplus E_2 \) a splitting with \( \gamma(E_1, E_2) \leq \alpha \), and \( F \) is a \( C^1 \) embedding of a ball \( B_\delta(0) \subseteq E \) into another Banach space \( E' \) satisfying
\[ (1) \ D_0F \text{ is an isomorphism and } \gamma((D_0F)E_1, (D_0F)E_2) \leq \alpha; \]
\[ (2) \ \| D_0F - D_2F \| \leq \tau \text{ for all } x \in B_\delta(0); \]
\[ (3) \ \frac{\| D_0F |_{E_2} \|}{\| D_0F |_{E_2} \|} \leq \frac{1}{2}; \]
\[ (4) \ m(D_0F |_{E_2}) \geq \beta; \]
then for every \((E_1, E_2)\)-graph \(G\) with dispersion \(\leq c\) contained in the ball \(B_\delta(0)\), its image \(\mathcal{F}(G)\) is a \(((D_0,\mathcal{F})E_1, (D_0,\mathcal{F})E_2)\)-graph with dispersion \(\leq c\).

The following lemma is also useful for the proof of Theorem B.

**Lemma 4.4.** Let \(g \in \text{Diff}^1(M)\) and \(\Lambda\) be a \(g\)-invariant subset of \(M\). If there is a \((1, i(x))\)-dominated splitting on \(\Lambda\): \(T_\Lambda M = E \oplus F\), then for any \(c > 0\), there exists \(\delta > 0\) such that for every \(x \in \Lambda\) and any \((E_x, F_x)\)-graph \(G\) with dispersion \(\leq c\) contained in the Bowen ball \(V(g,x,n,\delta)(n \geq 0)\), its image \(g^n(G)\) is a \((D_xg^nE_x, D_xg^nF_x)\)-graph with dispersion \(\leq c\).

**Proof.** cf. the proof of Lemma 3.4 in [16].

Now we are ready to prove Theorem B. Fix any \(\varepsilon > 0\). Take \(N_0\) so large that the set \(\Gamma_{i,\varepsilon} = \{x \in \Gamma : D_i(x) \leq N_0\}\) has \(\mu\)-measure larger than \(\mu(\Gamma_i) - \varepsilon\). Let \(N = N_0!\) and \(g = f^N\), then the splitting \(T_{\Gamma_{i,\varepsilon}} M = E \oplus F\) satisfies \((1, i(x))\)-dominated with respect to \(g\):

\[
\frac{\|Dg|_{E(x)}\|}{m(Dg)_{F(x)}} \leq \prod_{j=0}^{N-1} \frac{\|Df^{D_i(x)}|_{E(f^{D_i(x)}(x))}\|}{m(Df^{D_i(x)}|_{F(f^{D_i(x)}(x))})} \leq \left(\frac{1}{2}\right)^{\frac{N}{m(0)}} \leq \frac{1}{2} \forall x \in \Gamma_{i,\varepsilon}.
\]

Note that \(\Gamma_{i,\varepsilon}\) is \(f\)-invariant and thus is \(g\)-invariant. In what follows, in order to avoid a cumbersome and conceptually unnecessary use of coordinate charts, we shall treat \(M\) as if it were a Euclidean space, and let \(\lambda\) be the Lebesgue measure on \(M\). The reader will observe that all our arguments can be easily formalized by a completely straightforward use of local coordinates.

Since dominated splitting can be extended on the closure of \(\Gamma_{i,\varepsilon}\), and dominated splitting is always continuous(see [2]), we can fix two constants \(c > 0\) and \(a > 0\) so small that if \(x \in \Gamma_{i,\varepsilon}\), \(y \in M\) and \(d(x,y) < a\), then for every linear subspace \(E \subseteq T_y M\) which is a \((E(x), F(x))\)-graph with dispersion \(\leq c\) we have

\[
|\log |\det(D_y g)|_E| - \log |\det(D_x g)|_{F(x)}| \leq \varepsilon.
\]

Thus

\[
|\det(D_y g)|_E \geq \det |(D_x g)|_{F(x)}| \cdot e^{-\varepsilon}.
\]

By Lemma 4.4, there exists \(\delta \in (0, a)\) such that for every \(x \in \Gamma_{i,\varepsilon}\) and any \((E_x, F_x)\)-graph \(G\) with dispersion \(\leq c\) contained in the ball \(V(g,x,n,\delta)(n \geq 0)\). Its image \(g^n(G)\) is a \(((D_xg^nE_x, D_xg^nF_x)\)-graph with dispersion \(\leq c\).

The estimation of \(h^i_\mu(f)\) from below is based on the following fact.

**Fact.** For every \(x \in \Gamma_{i,\varepsilon}\), we have

\[
\liminf_{n \to \infty} \frac{1}{n} \left(- \log \lambda^i_x(V^i(g,x,n,\delta))\right) \geq N \sum_{j \leq u(i)} m_j(x) \lambda_j(x) - \varepsilon.
\]

**Proof.** Fix any \(x \in \Gamma_{i,\varepsilon}\), there exists \(B_x > 0\) satisfying

\[
\lambda^i_x(V^i(g,x,n,\delta)) = B_x \lambda(V(g,x,n,\delta)).
\]

It is clear that we can choose a positive constant \(B_1\) such that \(B_x \leq B_1\), for all \(x \in M\). Fix \(x \in \Gamma_{i,\varepsilon}\), now we consider the measure of \(V(g,x,n,\delta)\), which we have that there is a constant \(B > 0\) satisfying

\[
\lambda(V(g,x,n,\delta)) = B \int_{E(x)} \lambda((y + F(x)) \cap V(g,x,n,\delta)) \, d\lambda(y).
\]
It follows that
\[ \liminf_{n \to \infty} \inf_{y \in E(x)} \frac{1}{n} \left[ -\log \lambda(\Lambda_n(y)) \right] \geq N \sum_{j \leq u(i)} m_j(x) \lambda_j(x) - \varepsilon, \] (19)
where
\[ \Lambda_n(y) = (y + F(x)) \cap V(g, x, n, \delta). \]
If \( \Lambda_n(y) \) is not empty, by Lemma 4.4, we have \( g^n(\Lambda_n(y)) \) is a \( (E(g^n(x)), F(g^n(x))) \)-graph with dispersion \( \leq c \).

Take \( D > 0 \) such that \( D > \text{vol}(G) \) for every \( (E(w), F(w)) \)-graph \( G \) with dispersion \( \leq c \) contained in \( B_\delta(w), w \in \Gamma_{i,\varepsilon} \). Observe that
\[ g^n(\Lambda_n(y)) \subseteq g^n V(g, x, n, \delta) \subseteq B_\delta(g^n(x)), g^n(x) \in \Gamma_{i,\varepsilon}. \]
We have
\[ D > \text{vol}(g^n(\Lambda_n(y))) = \int_{\Lambda_n(y)} |\det(Dzg^n)| |T_{z\Lambda_n(y)}| d\lambda(z). \]
Since
\[ g^j(\Lambda_n(y)) \subseteq g^j V(g, x, n, \delta) \subseteq B_\delta(g^j(x)) \subseteq B_a(g^j(x)), j = 0, 1, 2, \ldots, n, \]
we have for any \( z \in \Lambda_n(y), \)
\[ d(g^j(z), g^j(x)) < a, j = 0, 1, 2, \ldots, n. \]
By (17), we have
\[
\frac{1}{n} \log D \geq \frac{1}{n} \log \int_{\Lambda_n(y)} |\det(Dzg^n)| |T_{z\Lambda_n(y)}| d\lambda(z)
\]
\[ \geq \frac{1}{n} \log \int_{\Lambda_n(y)} |\det(Dzg^n)| \cdot e^{-\varepsilon} d\lambda(z)
\]
\[ = \frac{1}{n} \log \lambda(\Lambda_n(y)) \cdot \frac{1}{n} \log |\det(Dzg^n)| \cdot e^{-\varepsilon}. \]
Hence,
\[
\frac{1}{n} \log D \geq \frac{1}{n} \log \int_{\Lambda_n(y)} |\det(Dzg^n)| |T_{z\Lambda_n(y)}| d\lambda(z)
\]
\[ \geq \frac{1}{n} \log \int_{\Lambda_n(y)} |\det(Dzg^n)| \cdot e^{-\varepsilon} d\lambda(z)
\]
\[ = \frac{1}{n} \log \lambda(\Lambda_n(y)) \cdot |\det(Dzg^n)| \cdot e^{-\varepsilon} \]
\[ = \frac{1}{n} \log \lambda(\Lambda_n(y)) + \frac{1}{n} \log |\det(Dzg^n)| \cdot e^{-\varepsilon}. \]
It follows that
\[ \lim_{n \to \infty} -\frac{1}{n} \log \lambda(\Lambda_n(y)) \geq \frac{1}{n} \log |\det(Dzg^n)| \cdot e^{-\varepsilon}. \]
Combining this inequality and the fact from Oseledec Theorem [9], we obtain
\[ \lim_{n \to \infty} -\frac{1}{n} \log |\det(Dzg^n)| = N \sum_{j \leq u(i)} m_j(x) \lambda_j(x), \]
Now we have completed the proof of (19), then using (18), we obtain

\[
\liminf_{n \to \infty} \frac{1}{n} \left( - \log \lambda^i_x(V^i(g, x, n, \delta)) \right)
\]

\[
= \liminf_{n \to \infty} \frac{1}{n} \left( - \log B_x \lambda(V(g, x, n, \delta)) \right)
\]

\[
\geq \liminf_{n \to \infty} \frac{1}{n} \left( - \log B_1 \lambda(V(g, x, n, \delta)) \right)
\]

\[
= \liminf_{n \to \infty} \frac{1}{n} \left( - \log B_1 B \int_{E(x)} \lambda([y + F(x)] \cap V(g, x, n, \delta)] \right) \text{d}\lambda(y)
\]

\[
\geq N \sum_{j \leq u(i)} m_j(x) \lambda_j(x) - \varepsilon,
\]

which completes the proof of the fact.

Now we can complete the estimation of \( h^i_\mu(f) \) from below. By (8), we have

\[
h^i_\mu(g) \geq \int_{\Gamma_i} \liminf_{n \to \infty} \frac{1}{n} \left( - \log \lambda^i_x(V^i(g, x, n, \varepsilon)) \right) \text{d}\mu(x).
\]

Using the fact, we obtain

\[
h^i_\mu(g) \geq \int_{\Gamma_i} N \sum_{j \leq u(i)} m_j(x) \lambda_j(x) - \varepsilon \text{d}\mu(x)
\]

\[
= \int_{\Gamma_i} N \sum_{j \leq u(i)} m_j(x) \lambda_j(x) \text{d}\mu(x) - \int_{\Gamma_i \setminus \Gamma_i} N \sum_{j \leq u(i, x)} m_j(x) \lambda_j(x) \text{d}\mu(x)
\]

\[
- \varepsilon \mu(\Gamma_i \setminus \Gamma_i)
\]

\[
\geq \int_{\Gamma_i} N \sum_{j \leq u(i)} m_j(x) \lambda_j(x) \text{d}\mu(x) - NC \dim(M) \varepsilon - \varepsilon,
\]

where \( C = \max_{x \in M} \log \|D_x f\| \). Hence,

\[
h^i_\mu(f) = \frac{1}{N} h^i_\mu(g) \geq \int_{\Gamma_i} N \sum_{j \leq u(i)} m_j(x) \lambda_j(x) \text{d}\mu(x) - NC \dim(M) \varepsilon - \varepsilon.
\]

Since \( \varepsilon \) is arbitrary, this completes the estimation of \( h^i_\mu(f) \) from below.

REFERENCES

[1] F. Abdenur, C. Bonatti and S. Crovisier, Nonuniform hyperbolicity for \( C^1 \)-generic diffeomorphisms, Israel J. of Math., 183 (2011), 1–60.

[2] C. Bonatti, L. Díaz and M. Viana, Dynamics beyond Uniform Hyperbolicity: A Global Geometric and Probabilistic Perspective, vol. 102 of Encyclopedia Math. Sci., Springer-Verlag, Berlin, 2005.

[3] E. Catsigeras, M. Cerminara and H. Enrich, The Pesin entropy formula for \( C^1 \) diffeomorphisms with dominated splitting, Ergodic Theory Dynam. Systems, 35 (2015), 737–761.

[4] H.-Y. Hu, Y.-X. Hua and W.-S. Wu, Unstable entropies and variational principle for partially hyperbolic diffeomorphisms, Advances in Mathematics, 321 (2017), 31–68.

[5] F. Ledrappier and L.-S. Young, The metric entropy of diffeomorphisms: Part I: Characterization of measures satisfying Pesin’s entropy formula, Ann. of Math., 122 (1985), 509–539.

[6] F. Ledrappier and L.-S. Young, The metric entropy of diffeomorphisms: Part II: Relations between entropy, exponents and dimension, Ann. of Math., 122 (1985), 540–574.

[7] P.-D. Liu and M. Qian, Smooth Ergodic Theory of Random Dynamical Systems, vol. 1606 of Lecture Notes in Math., Springer-Verlag, Berlin, 1995.
[8] R. Mañé, A proof of Pesin’s formula, Ergodic Theory Dynam. Systems, 1 (1981), 95–102.
[9] V. I. Oseledec, A multiplicative ergodic theorem. Lyapunov characteristic numbers for dynamical systems, (Russian) Trudy Moskov. Mat. Obšč. , 19 (1968), 179–210.
[10] Y. B. Pesin, Characteristic Lyapunov exponents and smooth ergodic theory, Russian Math. Surveys, 32 (1977), 55–112.
[11] Y. B. Pesin and Y. G. Sinai, Gibbs measures for partially hyperbolic attractors, Ergodic Theory Dynam. Systems, 2 (1982), 417–438.
[12] E. R. Pujals, From hyperbolicity to dominated splitting, in Partially Hyperbolic Dynamics, Laminations, and Teichmüller Flow (eds. G. Forni, M. Lyubich, C. Pugh and M. Shub), vol. 51 of Fields Inst. Commun., Amer. Math. Soc., Providence, RI, (2007), 89–102.
[13] V. A. Rokhlin, On the fundamental ideas of measure theory, Amer. Math. Soc. Transl., 1952 (1952), 55pp.
[14] D. Ruelle, An inequality for the entropy of differentiable maps, Bull. Braz. Math. Soc., 9 (1978), 83–87.
[15] M. Sambarino, A (short) survey on dominated splitting, Mathematical Congress of the Americas, 149–183, Contemp. Math., 656, Amer. Math. Soc., Providence, RI, 2016.
[16] W.-X. Sun and X.-T. Tian, Dominated splitting and Pesin’s entropy formula, Discrete Contin. Dyn. Syst., 32 (2010), 1421–1434.
[17] X.-T. Tian, Pesin’s entropy formula for systems between $C^1$ and $C^{1+\alpha}$, J. Stat. Phys., 156 (2014), 1184–1198.

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