GLOBAL CLASSICAL LARGE SOLUTION TO COMPRESSIBLE VISCOUS MICROPOLAR AND HEAT-CONDUCTING FLUIDS WITH VACUUM

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Abstract. In this paper we consider the non-stationary 1-D flow of a compressible viscous and heat-conducting micropolar fluid, assuming that it is in the thermodynamically sense perfect and polytropic. Since the strong nonlinearity and degeneracies of the equations due to the temperature equation and vanishing of density, there are a few results about global existence of classical solution to this model. In the paper, we obtain a global classical solution to the equations with large initial data and vacuum. Moreover, we get the uniqueness of the solution to this system without vacuum. The analysis is based on the assumption $\kappa(\theta) = O(1 + \theta^q)$ where $q \geq 0$ and delicate energy estimates.

1. Introduction. In this paper, we consider non-stationary 1-D flow of a compressible viscous and heat-conducting micropolar fluid with vacuum, being in a thermodynamical sense perfect and polytropic. This model can describe many phenomena appeared in a large number of complex fluids such as the suspensions, animal blood, liquid crystals which can not be characterized appropriately by the Navier-Stokes system. Mathematically, the motion of 1-D compressible viscous micropolar and heat-conducting fluid, which is thermodynamically perfect and polytropic, is described by the following system of four equations in Eulerian coordinates:

$$\begin{align*}
\rho_t + (\rho u)_x &= 0, \quad \rho \geq 0, \\
(\rho u)_t + (\rho u^2)_x + P_x &= u_{xx}, \\
(\rho \omega)_t + (\rho u \omega)_x + A \omega &= A \omega_{xx}, \\
(\rho \theta)_t + (\rho u \theta)_x + P u_x &= (u_x)^2 + (\kappa(\theta) \theta_x)_x + (\omega_x)^2 + \omega^2.
\end{align*}$$

Here, the unknown functions $\rho(x, t), u(x, t), \omega(x, t), \theta(x, t), P$ and $\kappa$ denote the density, velocity, microrotation velocity, absolute temperature, pressure and coefficient of heat conduction, respectively. $A$ is the constant microviscosity coefficient. The model of micropolar fluids is the local form of the conservation law for the mass, momentum, momentum moment and energy equations. The model of micropolar fluids, introduced by Eringen [12], has received considerable attention in the last two decades. For more background, we refer to [24] and references therein. There has been much research on the existence, uniqueness, regularity, and asymptotic

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behavior of the solutions. Mujaković first described the one-dimensional case of the model for compressible, viscous and heat-conducting micropolar fluid. She solved various problems for this model. In detail, the local, global in time existence theorem and regularity of solutions for the model with homogeneous boundary condition for velocity and microrotation were given in articles [25, 26] respectively. Similar results were proved in [29, 28] for the corresponding non-homogeneous boundary value problems. Other authors, such as Chen [1] proved the global existence of strong solutions to the 1-D isentropic model with initial vacuum. For the three-dimensional model, Chen and his collaborators in [2, 3] established different blowup criteria of strong solutions to the isentropic micropolar fluid system. In detail, the local existence, global existence, large time behavior of the 3-dimensional flow of a compressible viscous micropolar fluid. N. Mujaković also analyzed the large time behavior of the solution for the compressible micropolar fluid model in one dimension for the initial-boundary value problem in [27]. The regularity of the solution for 1-D compressible viscous micropolar fluid model with weighted small initial data was analyzed by Qin, Wang and Hu in [31]. Recently, Liu and Zhang [23] have obtained the optimal time decay of the 3-dimensional compressible micropolar fluid model. Refer to [5, 17, 30] etc., and the references therein for the well-posedness of incompressible micropolar fluid system.

In the presence of vacuum (i.e., $\rho$ may vanish), to our best knowledge, there are no results about the global well-posedness of the one-dimensional compressible viscous and heat-conducting micropolar fluid. However, we notice that the compressible viscous and heat-conducting micropolar fluid model has close relations with Navier-Stokes equations. If the microstructure of the fluid is not taken into account, i.e., $\omega = 0$, then system (1) reduces to the classical Navier-Stokes system. As it is well known that there have been a great number of mathematical nice papers for Navier-Stokes system, see the papers [6, 15, 16, 8, 14, 19, 20, 22] and reference therein. Wen and Zhu [33, 34] consider the 1-D full Navier-Stokes equations with thermally insulated boundary condition. These two papers [33, 34] are the first two results on globally classical solutions to the 1-D full Navier-Stokes equations with large initial data and vacuum. The coefficient of heat conductivity $\kappa$ excluded the constant in these two papers. Recently, Liang and Wu [21] applied Calderón-Zygmund decomposition technique to establish the global existence and uniqueness of classical solutions for the 1-D full Navier-Stokes equations with a constant heat conductivity. Motivated by their ideas in these papers, we consider to show the existence of global classical solutions to (1)-(3) with $\kappa = (1 + \theta^q)$ $(q \geq 0)$.

The main novelty and difficulty of the paper is determining how to control the strong non-linearity and the possible singularities caused by the presence of vacuum. Moreover, compared with [33, 21] which describes the classical solutions for non-isentropic Navier-Stokes equations in 1D, the main trouble term is that we must deal with the external term, i.e., microrotation velocity $\omega$. To overcome these difficulty, based on the observation of structure of system (1), we used the Calderón-Zygmund decomposition technique for the case of $\kappa = (1 + \theta^q)$ $(q = 0)$ and used the weighted density-dependent test function method for the case of $\kappa = (1 + \theta^q)$ $(q > 0)$.

In the present paper, this system is considered in the domain $Q_T = [0,1] \times [0,T]$, where $T > 0$ is arbitrary; and the initial and boundary conditions are given,
respectively,
\[(\rho, u, \omega, \theta)(x, 0) = (\rho_0, u_0, \omega_0, \theta_0)(x) \text{ in } [0, 1] \tag{2}\]
and
\[(u, \omega, \theta_x)|_{x=0,1} = 0, \quad t \geq 0. \tag{3}\]

We focus on the polytropic perfect and polytropic fluids and assume that
\[P = R\rho\theta, \tag{4}\]
where constant \(R > 0\) are given. For convenience, we let \(A, R\) equal to 1 in this paper.

We would like to give some notations which will be used throughout the paper.

**Notations.**
1. \(I = [0, 1], \partial I = \{0, 1\}, Q_T = I \times [0, T]\) for \(T > 0\).
2. For \(p \in [1, \infty]\), \(L^p = L^p(I)\) denotes the \(L^p\) space with the norm \(\|\cdot\|_{L^p}\). For \(k \geq 1\) and \(p \in [1, \infty]\), \(W^{k,p} = W^{k,p}(I)\) denotes the Sobolev space, whose norm is denoted as \(\|\cdot\|_{W^{k,p}, H^k = W^{k,2}(I)}\).
3. For an integer \(k \geq 0\) and \(0 < \alpha < 1\), let \(C^{k+\alpha}\) denote the Schauder space of functions on \(I\), whose \(k\)th-order derivative is Hölder continuous with exponents \(\alpha\), with the norm \(\|\cdot\|_{C^{k+\alpha}}\).

In this paper, our assumptions are the following:
1. \((A_1)\) \(\rho_0 \geq 0; \int_I \rho_0 > 0\).
2. \((A_2)\) \(P = R\rho\theta\).
3. \((A_3)\) \(\kappa \in C^2[0, \infty)\) satisfies
   \[C_1(1 + \theta^q) \leq \kappa(\theta) \leq C_2(1 + \theta^q)\]
for \(q \geq 0\) and some constants \(C_i > 0\) \((i = 1, 2)\).

**Remark 1.** Based on the good structure of this system, we use the Calderón-Zygmund decomposition technique to overcome the difficulty caused by possible vacuum for the case where the heat conductivity is constant. However, this technique does not apply to situation in which the conduction coefficient is not constant. For the case that \(q > 0\), we use the ideas in [33, 34] which used the weighted density-dependent test function to handle the possible singularities due to vacuum.

The theorem below is our main result.

**Theorem 1.1** (Classical solutions). In addition to \((A1)-(A3)\), we assume \(\rho_0 \geq 0\), \(\rho_0 \in H^2\), \((\sqrt{\rho_0})_x \in L^\infty\), \(u_0 \in H^3 \cap H^1_0\), \(\omega_0 \in H^3 \cap H^1_0\), \(\theta_0 \in H^3\), \(\partial_x \theta_0 |_{x=0,1} = 0\), and that the following compatibility conditions are valid:
\[
\begin{align*}
  u_{0xx} - |P(\rho_0, \theta_0)|_x &= \sqrt{\rho_0} g_1, \\
  \omega_{0xx} - \omega_0 &= \sqrt{\rho_0} g_2, \\
  \left[\kappa(\theta_0) \theta_{0x}\right]_x + |u_0|^2 + |\omega_0|^2 &= \sqrt{\rho_0} g_3,
\end{align*}
\tag{5}
\]
for some \(g_i \in L^2\), \(i = 1, 2, 3\); and \(\sqrt{\rho_0} g_1, \sqrt{\rho_0} g_2 \in H^1_0\), \(\sqrt{\rho_0} g_3 \in H^1\). Then there exists a global solution \((\rho, u, \omega, \theta)\) to \((1)-(5)\) such that for any \(T > 0\),
\[
\begin{align*}
  \rho &\in C([0,T]; H^2), \quad \rho_t \in C([0,T]; H^1), \quad \sqrt{\rho} \in W^{1,\infty}(Q_T), \\
  u, \omega, \theta &\in L^\infty([0,T]; H^3 \cap H^1_0), \quad \sqrt{\rho} u_t, \sqrt{\rho} \omega_t \in L^\infty([0,T]; L^2), \\
  \rho u_t, \rho \omega_t &\in L^\infty([0,T]; H^1_0), \quad u_t, \omega_t \in L^2([0,T]; H^3_0), \quad \sqrt{\rho} \theta_t \in L^\infty([0,T]; L^2), \\
  \rho \theta_t &\in L^\infty([0,T]; H^1), \quad \theta \in L^\infty([0,T]; H^3), \quad \theta_t \in L^2([0,T]; H^1).
\end{align*}
\]
If there exists a positive constant \(\delta_0\) such that \(\rho_0 > \delta_0\), we obtain the uniqueness of the classical solution to system \((1)-(3)\).
The paper is organized as follows. In Section 2 we introduce the mathematical formulation of our problem. In Section 3 we prove Theorem 1.1 by giving the initial density and the initial temperature a lower bound $\delta > 0$, getting a sequence of approximate solutions to (1)-(5), and taking $\delta \to 0^+$ after making some estimates uniformly for $\delta$. More precisely, based on Lemma 2.1 and the one-dimensional properties of the equations, we obtain $H^2$ estimates of the solutions. As in [34], by using weighted density-dependent test function to handle the possible singularities due to vacuum, we get $H^3$ estimates of $u$, $\omega$, $\theta$.

2. Examples.

2.1. Preliminaries. At first, we give some necessary lemmas and corollary, they are the same as in [21, 33], we omit the proof for brevity.

**Lemma 2.1.** Let $\Omega = [\bar{a}, \bar{b}] (\bar{a} < \bar{b})$ be a bounded domain in $\mathbb{R}$, and $\rho$ be a nonnegative function satisfying $\int_{\Omega} \rho > 0$. Then

$$\max_{\Omega} v \leq \|v_x\|_{L^1(\Omega)} + \frac{1}{\int_{\Omega} \rho dx} \left| \int_{\Omega} \rho v dx \right|$$

for any absolutely continuous function $v(x) \in \Omega$.

**Corollary 1.** Consider the same conditions as in Lemma 2.1, and in addition assume $\Omega = I$ and

$$\|\rho v\|_{L^1(I)} \leq \bar{C}.$$ 

Then for any $l > 0$, there exists a positive constant $C = C(M, K, l, \bar{C})$ such that

$$\|v^l\|_{L^\infty(I)} \leq C\|(v^l)_x\|_{L^2(I)} + C$$

for any $v^l \in H^1(I)$.

**Lemma 2.2.** For any $v \in H^1_0(I)$, we have

$$\|v\|_{L^\infty(I)} \leq \|v_x\|_{L^1(I)}.$$ 

The next lemma (see [21, 18]), which we will be used to overcome the difficult caused by possible vacuum in the case $q = 0$.

**Lemma 2.3** (Calderón-Zygmund). Let $\Omega = [a, b]$ be bounded. Suppose $0 \leq f \in L^1(\Omega)$ satisfies

$$\frac{1}{|\Omega|} \int_{\Omega} f(x) dx \leq \alpha_0.$$ 

Then for any $\alpha > \alpha_0$, there exists a sequence (non-overlapping) $\Omega_j$ included in $\Omega$ such that

$$f(x) \leq \alpha, \ a.e. \ x \in \Omega \setminus \bigcup_j \Omega_j, \ and \ \alpha \leq \frac{1}{|\Omega_j|} \int_{\Omega_j} f(x) dx \leq 2\alpha.$$ 

Moreover,

$$|\Omega_j| \leq |\bigcup_j \Omega_j| \leq \frac{\alpha_0 |\Omega|}{\alpha},$$

where $|\Omega|$ denotes the Lebesgue measure of $\Omega$.

**Proof.** The proof can refer to [18, Lemma 3.7, Chap. 3]. We omit the detail here.
Lemma 2.4 (see [32]). Assume $X \subset E \subset Y$ are Banach spaces and $X \hookrightarrow E$. Then the following embeddings are compact:

(i) \((\varphi : \varphi \in L^q(0, T; X), \frac{\partial \varphi}{\partial t} \in L^1(0, T; Y)) \hookrightarrow L^q(0, T; E)\) if \(1 \leq q \leq \infty\);

(ii) \((\varphi : \varphi \in L^\infty(0, T; X), \frac{\partial \varphi}{\partial t} \in L^r(0, T; Y)) \hookrightarrow C([0, T]; E)\) if \(1 \leq r \leq \infty\).

3. Proof of Theorem 1.1. In this section, we get a global solution to (1.1)-(1.3) with initial density and initial temperature having \(a\), respectively, lower bound \(\delta > 0\) by using some \textit{a priori} estimates of the solutions based on the local existence. Theorem 1.1 would be gotten after making some \textit{a priori} estimates uniformly for \(\delta\) and taking limit of \(\delta \to 0^+\).

Denote \(\rho_0^\delta = \rho_0 + \delta\) and \(\theta_0^\delta = \theta_0 + \delta\) for \(\delta \in (0, 1)\). Throughout this section, we denote \(C\) to be a generic constant depending on \(\rho_0, u_0, \omega_0, \theta_0\), and some other known constants but independent of \(\delta\) for any \(\delta \in (0, 1)\).

For any given \(\delta \in (0, 1)\), let \(u_0^\delta\) be the solution to the following elliptic equation:

\[
\begin{cases}
 u_{0xx}^\delta - (P_0^\delta)_{x} = \sqrt{\rho_0} g_1, \\
 u_{0x}^{\delta}|_{x=0,1} = 0,
\end{cases}
\]  

(6)

and let \(\omega_0^\delta\) be the solution to the elliptic equation:

\[
\begin{cases}
 \omega_{0xx}^\delta - \omega_{0}^\delta = \sqrt{\rho_0} g_2, \\
 \omega_{0x}^{\delta}|_{x=0,1} = 0.
\end{cases}
\]  

(7)

Since \(\rho_0^\delta = \rho_0 + \delta \in H^2(I), \theta_0^\delta = \theta_0 + \delta \in H^3(I)\), and \(\sqrt{\rho_0} g_1, \sqrt{\rho_0} g_2 \in H^1_0(I)\), by the elliptic theory (see for instance [13]), (5), (6) and (7), we have \(u_0^\delta, \omega_0^\delta \in H^3(I) \cap H^2_0(I)\)

\[
\begin{cases}
 u_0^\delta \to u_0 \text{ in } H^3, \omega_0^\delta \to \omega_0 \text{ in } H^3 \text{ as } \delta \to 0, \\
 \|u_0^\delta - u_0\|_{H^2} \leq C\delta, \|\omega_0^\delta - \omega_0\|_{H^2} \leq C\delta.
\end{cases}
\]  

(8)

Before proving Theorem 1.1, we need the following auxiliary theorem.

Theorem 3.1. Consider the same assumption as in Theorem 1.1. Then for any given \(\delta \in (0, 1)\), there exists a unique global solution \((\rho, u, \omega, \theta)\) to (1.1)-(1.3) with initial data replaced by \((\rho_0^\delta, u_0^\delta, \omega_0^\delta, \theta_0^\delta)\), such that for any \(T > 0\),

\[\rho \in C([0, T]; H^2), \rho_t \in C([0, T]; H^1), \rho_{tt} \in L^2([0, T]; L^2), \rho \geq \delta > 0,\]

\[u \in C([0, T]; H^3 \cap H^2_0), u_t \in C([0, T]; H^1_0) \cap L^2([0, T]; H^2), u_{tt} \in L^2([0, T]; L^2),\]

\[\omega \in C([0, T]; H^3 \cap H^2_0), \omega_t \in C([0, T]; H^1_0) \cap L^2([0, T]; H^2), \omega_{tt} \in L^2([0, T]; L^2),\]

\[\theta \geq C_\delta > 0, \theta_t \in C([0, T]; H^3), \theta_t \in C([0, T]; H^1_0) \cap L^2([0, T]; H^2), \theta_{tt} \in L^2([0, T]; L^2),\]

where \(C_\delta\) is a constant depending on \(\delta\), but independent of the solutions.

Proof of Theorem 3.1. We can get the local solutions of Theorem 3.1 by the successive approximations as in [6]. One can refer Appendix 2. The regularities guarantee the uniqueness (the details please refer for instance to Appendix 1). Based on it, Theorem 3.1 can be proved by some \textit{a priori} estimates globally in time.

For any given \(T \in (0, \infty)\), let \((\rho, u, \omega, \theta)\) be the solution to (1.1)-(1.3) as in Theorem 3.1.

Then we have the following basic energy estimate.
Lemma 3.2. Under the conditions of Theorem 3.1, we have for any \( 0 \leq t \leq T \),
\[
\int_t^1 \rho(1 + u^2 + \omega^2 + \theta)(t) \leq C.
\]

Proof. Multiplying (1) by \( u \), it gives
\[
\frac{1}{2}(\rho u^2)_t + \frac{1}{2}(\rho u^2)_{xx} + (\rho \theta)_x u = (uu_x)_x - u_x^2,
\]
and multiplying (1) by \( \omega \), we obtain
\[
\frac{1}{2}(\rho \omega^2)_t + \frac{1}{2}(\rho \omega^2)_{xx} + \omega^2 = (\omega \omega_x)_x - \omega_x^2.
\]
Adding above equations (9), (10) to (1) and integrating the resulting equation and (1) over \( I \times [0, T] \), it gives
\[
\int_I \rho \left( 1 + \frac{1}{2}u^2 + \frac{1}{2}\omega^2 + \theta \right) \leq \varepsilon_0 = \int_I \rho_0 \left( 1 + \frac{1}{2}u_0^2 + \frac{1}{2}\omega_0^2 + \theta_0 \right).
\]

Lemma 3.3 (see [33]). Under the conditions of Theorem 3.1, it holds that for any \((x, t) \in Q_T\),
\[
\begin{align*}
0 &< \rho(x, t) \leq C, \\
\theta(x, t) &> 0.
\end{align*}
\]

Lemma 3.4 (see [21]). Given \( \alpha > R\varepsilon_0 \), there exists a sequence of (non-overlapping) intervals \( \{ \Omega_j \} \) in \( I \) that for every \( t \in [0, T] \)
\[
P(x, t) \leq \alpha, \ a.e. \ x \in I \setminus \bigcup_j \Omega_j, \ \text{and} \ \alpha \leq \frac{1}{|\Omega_j|} \int_{\Omega_j} P(x, t) dx \leq 2\alpha.
\]
Moreover,
\[
|\bigcup_j \Omega_j| \leq \frac{R\varepsilon_0}{\alpha}.
\]

Lemma 3.5. Under the conditions of Theorem 6, it holds that for \( q = 0 \),
\[
\sup_{0 \leq t \leq T} \int_t^1 \rho(u^4 + \omega^4 + \theta^2) + \int_0^T \int_I (u^2 |u_x|^2 + \omega^2 |\omega_x|^2 + |\theta_x|^2) \leq C.
\]

Proof. Adding (9) and (10) into (1), we obtain
\[
\left( \rho \theta + \frac{1}{2} u^2 + \frac{1}{2} \omega^2 \right)_t + \left( \rho u \theta + \frac{1}{2} u^2 + \frac{1}{2} \omega^2 \right)_x = (uu_x)_x + (\omega \omega_x)_x + \kappa \theta_{xx}.
\]
Let \( \Psi = \theta + \frac{1}{2} u^2 + \frac{1}{2} \omega^2 \), the equation (12) is equivalent to
\[
(\rho \Psi)_t + (\rho u \Psi)_x + (Pu)_x = \kappa \Psi_{xx} + (1 - \kappa)(uu_x + \omega \omega_x)_x.
\]
Multiplying the above equation by \( \Psi \) and integrating the resulting equation over \( Q_T \), we have
\[
\int_I \rho \Psi^2(x, t) + \int_0^T \int_I |\Psi|^2 \leq C + C \int_0^T \int_I (\rho^2 \theta^2 u^2 + u^2 |u_x|^2 + \omega^2 |\omega_x|^2).
\]
In addition, multiplying (1) by \( u^3, \omega^3 \) respectively, we have
\[
\int_I \rho (u^4(x, t) + \omega^4(x, t)) + \int_0^T \int_I (u^2 |u_x|^2 + \omega^2 |\omega_x|^2) \leq C + C \int_0^T \int_I \rho^2 \theta^2 u^2.
\]
Combing the last two inequalities we get
\[
\int_l \rho (u^4 + \omega^4 + \theta^2) + \int_0^T \int_l \left( u^2 |u_x|^2 + \omega^2 |\omega_x|^2 + |\theta_x|^2 \right) \leq C + C \int_0^T \int_l \rho^2 \theta^2 u^2.
\]

We only need to estimate the second term of the right hand, the proof can refer to [21], it mainly applies the Lemma 3.4 and exquisite analysis. We omit the detail for brevity.

The next estimate is a corollary of Lemma 3.5, which argument can be seen in [21]. For completeness, we present the proof.

**Corollary 2.** Under the conditions of Theorem 6, it holds that for \( q = 0 \),

\[
\int_0^T \left( \|\theta\|_{L^\infty}^2 + \|u_x\|_{L^2}^2 \right) \leq C. \tag{14}
\]

**Proof.**

\[
\int_0^T \|\theta\|_{L^\infty}^2 \leq C \int_0^T \int_l \theta_x^2 + C \leq C,
\]

where we have used the Lemmas 2.1, 3.2 and 3.5. We multiply (1)_2 by \( u \) and integrate the resulting equation to deduce

\[
\sup_{0 \leq t \leq T} \int_l \rho u^2 + \int_0^T \int_l |u_x|^2 \leq C + C \int_0^T \int_l \rho^2 \theta^2 \leq C + C \int_0^T \|\theta\|_{L^\infty}^2 \leq C.
\]

\[\square\]

**Lemma 3.6.** Under the conditions of Theorem 3.1, it holds that for \( q > 0 \) and for any given \( 0 < \alpha < \min\{1, q\} \),

\[
\int_{Q_T} \left( \frac{u_x^2 + \omega_x^2 + \omega^2}{\theta^\alpha} + \frac{(1 + \theta^q)\theta^2}{\theta^{1+\alpha}} \right) \leq C,
\]

where \( C \) may depend on \( \alpha \).

**Proof.** Similar to the proof in [34], multiplying (1)_4 by \( \theta^{-\alpha} \), integrating the resulting equation on \( Q_T \) and using integration by parts, we have

\[
\int_{Q_T} \left( \frac{u_x^2 + \omega_x^2 + \omega^2}{\theta^\alpha} + \frac{\alpha \kappa(\theta)\theta^2}{\theta^{1+\alpha}} \right) = \int_l \rho \int_0^T \frac{1}{\xi^\alpha} - \int_l \rho_0 \int_0^T \frac{1}{\xi^\alpha} + \int_{Q_T} \rho \theta^{1-\alpha} u_x \leq C + C \int_l \rho \theta^{1-\alpha} + \frac{1}{2} \int_{Q_T} \frac{u_x^2}{\theta^\alpha} + C \int \rho^2 \theta^{2-\alpha} \leq C + \frac{1}{2} \int_{Q_T} \frac{u_x^2}{\theta^\alpha} + C_0 \int_0^T \max_{x \in I} \theta^{1-\alpha}, \tag{15}
\]

where we have used the Cauchy inequality, and Lemmas 3.2 and 3.3. Now we estimate the last term of (15) as follows:
Case 1. \(0 < q < 1 - \alpha\)

\[
C_0 \int_0^T \max_{x \in I} \theta^{1-\alpha} \leq C + \int_0^T \|\theta^{-\alpha} \theta_x\|_{L^2}
\]

\[
\leq C + C \int_0^T \left( \int_I \frac{\theta^2 \theta_x^2 \theta^{1-\alpha-q}}{\theta^{1+\alpha}} \right)^{\frac{1}{2}}
\]

\[
\leq C + \frac{1}{2} \alpha \int_{Q_T} \frac{\kappa \theta^2_x}{\theta^{1+\alpha}} + C \int_0^T \|\theta\|_{L^{1-\alpha-q}}
\]

\[
\leq C + \frac{1}{2} \alpha \int_{Q_T} \frac{\alpha K(\theta) \theta^2_x}{\theta^{1+\alpha}} + \frac{1}{2} C_0 \int_0^T \|\theta\|_{L^{1-\alpha}}
\]  

(16)

where we have used Corollary 1, Lemma 3.2, the Cauchy inequality and Young Inequality.

This implies

\[
C_0 \int_0^T \|\theta\|_{L^{1-\alpha}} \leq \frac{1}{2} \alpha \int_{Q_T} \frac{\kappa \theta^2_x}{\theta^{1+\alpha}} + C.
\]  

(17)

Case 2. \(q \geq 1 - \alpha\)

Using Corollary 1, Lemma 3.3 and Young inequality, we have

\[
C_0 \int_0^T \max_{x \in I} \theta^{1-\alpha} \leq C + \int_0^T \|\theta^{-\alpha} \theta_x\|_{L^2}
\]

\[
\leq C + C \int_0^T \left( \int_I \frac{\theta^2 \theta_x^2 \theta^{1-\alpha}}{\theta^{1+\alpha}} \right)^{\frac{1}{2}}
\]

\[
\leq C + \frac{1}{2} \alpha \int_{Q_T} \frac{\kappa \theta^2_x}{\theta^{1+\alpha}}
\]  

(18)

By (15), (16), (17) and (18), the proof of Lemma 3.6 is completed. \(\square\)

Lemma 3.6 implies the following corollary, which proof can be found in [33, 34].

For completeness, we present the proof.

Corollary 3. Under the conditions of Theorem 3.1, it holds that

\[
\int_0^T \|\theta\|_{L^{1-\alpha}} \leq C.
\]

Proof. By Corollary 1 and Lemma 3.2, we have

\[
\int_0^T \|\theta\|_{L^{1-\alpha}} = \int_0^T \int_I \left| \frac{\theta^2 \theta_x^2 \theta^{1-\alpha}}{\theta^{1+\alpha}} \right| dx
\]

\[
\leq C \int_0^T \int_I \frac{\theta^{1-\alpha+1} \theta_x^2}{\theta^{1+\alpha}}
\]

\[
= C \int_0^T \int_I \frac{\theta^2 \theta_x^2}{\theta^{1+\alpha+1}}
\]

\[
\leq C.
\]

Remark 2. Next, we will first complete the proof of Theorem 3.1 under the case \(q > 0\). For the case \(q = 0\), the following proofs are valid due to Corollary 2.
Lemma 3.7. Under the conditions of Theorem 3.1, it holds that
\[ \int_{Q_T} u_t^2 \leq C. \]

Proof. Multiplying (1) by \( u_t \), and integrating by parts over \( I \), we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_I \rho u_t^2 + \int_I u_t^2 = \int_I P u_x \\
\leq \frac{1}{2} \int_I u_t^2 + C \int I \rho^2 \theta^2 \\
\leq \frac{1}{2} \int_I u_t^2 + C \sup_{x \in I} \int_I \rho \theta \\
\leq \frac{1}{2} \int_I u_t^2 + C \sup_{x \in I} \theta,
\]
where we have used the Cauchy inequality and Lemmas 3.2, 3.3. We have \( q - \alpha + 1 > 1 \) due to \( q > \alpha > 0 \). By Young inequality we have
\[
\frac{d}{dt} \int_I \rho u_t^2 + \int_I u_t^2 \leq C \sup_{x \in I} \theta^{q-\alpha+1} + C.
\]
Integrating over \((0, T)\) and using Corollary 3, we complete the proof of Lemma 3.7. \( \square \)

Lemma 3.8. Under the conditions of Theorem 3.1, we have for any \( 0 \leq t \leq T \),
\[ \int (u_x^2 + \omega_x^2 + \omega^2 + P^2 + \rho \theta q^2) + \int_{Q_T} (\rho u_t^2 + \rho \omega_t^2 + (1 + \theta^q)^2 \theta_t^2) \leq C. \]

Proof. We multiply the second equation of (1) by \( u_t \), use integration by parts, Lemmas 2.2 and 3.3, and the Cauchy inequality, to discover
\[
\int_I \rho u_t^2 + \frac{1}{2} \frac{d}{dt} \int_I u_t^2 = \frac{d}{dt} \int_I P u_x - \int_I \rho uu_x u_t - \int_I P_t u_x \\
\leq \frac{1}{2} \int_I \rho u_t^2 + \frac{1}{2} \frac{d}{dt} \int_I u_t^2 + \frac{d}{dt} \int_I P u_x - \int_I P_t u_x \\
\leq \frac{1}{2} \int_I \rho u_t^2 + C \left( \int_I u_t^2 \right)^2 + \frac{d}{dt} \int_I P u_x \\
- \int_I P_t (u_x - P) - \frac{1}{2} \frac{d}{dt} \int_I P^2,
\]
which implies
\[
\int_I \rho u_t^2 + \frac{d}{dt} \int_I u_t^2 \leq C \left( \int_I u_t^2 \right)^2 + \frac{d}{dt} \int_I P u_x - \frac{d}{dt} \int_I P^2 - 2 \int_I P_t (u_x - P). \quad (19)
\]
Since \( W^{1,1} \to L^\infty \), it follows from (1) and Lemma 3.2 that
\[
\|u_x - P\|_{L^\infty} \leq \|(u_x - P)\|_{L^1} + \|(u_{xx} - P_x)\|_{L^1} \leq C + C \int_I (|u_x|^2 + \rho |u|^2). \quad (20)
\]
Owing to (20), we use (1) and integration by parts to conclude
\[-2 \int P_t(u_x - P)\]
\[= -2 \int (\rho \theta_t)(u_x - P)\]
\[= -2 \int [(\kappa(\theta)\theta_x)_x + u_x^2 + \omega_x^2 + \omega^2 - (\rho u \theta)_x - (\rho \theta)_{xx}](u_x - P)\]
\[= 2 \int \kappa(\theta)\theta_x(u_{xx} - P_x) - 2 \int (u_x^2 + \omega_x^2 + \omega^2)(u_x - P)\]
\[- 2 \int \rho \theta(u_{xx} - P_x) + 2 \int \rho \theta u(x_u - P)\]
\[\leq 2 \int \kappa(\theta)(\rho u_t + \rho uu_x) + 2 \|u_x - P\|_{L^\infty} \int (u_x^2 + \omega_x^2 + \omega^2)\]
\[- 2 \int \rho \theta u(\rho u_t + \rho uu_x) + \rho \theta^2 \int \sup_{x \in I} \theta \int \sup_{x \in I} \rho \theta^2\]
\[\leq \frac{1}{4} \int \rho u_t^2 + C \int \kappa^2 \theta_x^2 + C \left( \int u_x^2 \right)^2 + \sup_{x \in I} \rho \theta^2 \int \frac{1}{4} \int \rho uu_t^2 + C \int \kappa^2 \theta_x^2 + C \left( \int u_x^2 \right)^2 + C \left( \int \omega^2 \right)^2 + C \sup_{x \in I} \rho \theta^2 \int \frac{1}{4} \int \rho uu_t^2 + C \int \kappa^2 \theta_x^2 + C \left( \int u_x^2 \right)^2 + C \left( \int \omega^2 \right)^2 + C \sup_{x \in I} \rho \theta^2 \int \frac{1}{4} \int \rho uu_t^2 + C \int \kappa^2 \theta_x^2 + C \left( \int u_x^2 \right)^2 + C \left( \int \omega^2 \right)^2 + C \sup_{x \in I} \rho \theta^2 \int \frac{1}{4} \int \rho uu_t^2 + C \int \kappa^2 \theta_x^2 + C \left( \int u_x^2 \right)^2 + C \left( \int \omega^2 \right)^2 + C \sup_{x \in I} \rho \theta^2 \int \frac{1}{4} \int \rho uu_t^2 + C \int \kappa^2 \theta_x^2 + C \left( \int u_x^2 \right)^2 + C \left( \int \omega^2 \right)^2 + C \sup_{x \in I} \rho \theta^2 \int .
\](21)

Substituting (21) into (19), we have
\[\frac{1}{2} \int \rho u_t^2 + \frac{d}{dt} \int u_x^2 \leq C \left( \int (u_x^2 + \omega_x^2 + \omega^2) \right)^2 + 2 \frac{d}{dt} \int Pu_x - \frac{d}{dt} \int P^2\]
\[+ C \int \kappa^2 \theta_x^2 + C \int u_x^2 \int \rho \theta^2\]
\[+ C \sup_{x \in I} \rho \theta^2 \int (u_x^2 + \rho \theta^2) + C.
\](22)

Multiplying the third equation of (1) by the \(\omega_t\), and integrating the resulting equation, it gives
\[\int \rho \omega_t^2 + \frac{1}{2} \frac{d}{dt} \int \omega^2 + \frac{1}{2} \frac{d}{dt} \int \omega_x^2 = - \int \rho u \omega_x \omega_t\]
\[\leq \frac{1}{2} \int \rho \omega_t^2 + C \left( \int u_x^2 \right)^2 + C \left( \int \omega_x^2 \right)^2 + C,
\](23)
which implies
\[\int \rho \omega_t^2 + \frac{d}{dt} \int \omega^2 + \frac{d}{dt} \int \omega_x^2 \leq C \left( \int u_x^2 \right)^2 + C \left( \int \omega_x^2 \right)^2.
\](24)
Adding (24) to (22), we obtain that
\[
\int_{I} \rho u_{t}^{2} \, dt + \int_{I} \rho u_{x}^{2} \, dt + \frac{d}{dt} \int_{I} (\omega^{2} + \omega_{x}^{2} + u_{x}^{2}) \, dx \\
\leq C \left( \int_{I} u_{x}^{2} \right)^{2} + C \left( \int_{I} \omega_{x}^{2} \right)^{2} + C \left( \int_{I} \omega^{2} \right)^{2} \\
+ 2 \frac{d}{dt} \int_{I} P u_{x} - \frac{d}{dt} \int_{I} P^{2} + C \int_{I} \kappa^{2} \theta_{x}^{2} + C \int_{I} u_{x}^{2} \, dx \int_{I} \rho \theta^{2} \\
+ C \sup_{x \in I} \theta^{q-\alpha+1} \int_{I} (u_{x}^{2} + \rho \theta^{2}) + C.
\] (25)

Integrating the above result on \((0, T)\) and using the Cauchy inequality, we receive that
\[
\int_{0}^{T} \int_{I} \rho u_{t}^{2} + \int_{0}^{T} \int_{I} \rho u_{x}^{2} + \int_{I} (u_{x}^{2} + \omega_{x}^{2} + \omega^{2} + P^{2}) \\
\leq C \int_{0}^{T} \left\{ \left( \int_{I} u_{x}^{2} \right)^{2} + \left( \int_{I} \omega_{x}^{2} \right)^{2} + \left( \int_{I} \omega^{2} \right)^{2} \right\} + 2 \int_{I} \rho \theta u_{x} + C \int_{0}^{T} \int_{I} \kappa^{2} \theta_{x}^{2} \\
+ C \int_{0}^{T} \int_{I} u_{x}^{2} \int_{I} \rho \theta^{2} + C \int_{0}^{T} \sup_{x \in I} \theta \int_{I} u_{x}^{2} + C \int_{0}^{T} \sup_{x \in I} \theta \int_{I} \rho \theta^{2} + C \\
\leq C \int_{0}^{T} \left( \int_{I} u_{x}^{2} + \int_{I} \omega_{x}^{2} + \int_{I} \omega^{2} + \int_{I} P^{2} \right)^{2} + C \int_{I} \rho \theta^{q} + \frac{1}{2} \int_{I} u_{x}^{2} + C \int_{0}^{T} \int_{I} \kappa^{2} \theta_{x}^{2} \\
+ C \int_{0}^{T} \int_{I} u_{x}^{2} \int_{I} \rho \theta^{2} + C \int_{0}^{T} \sup_{x \in I} \theta^{q-\alpha+1} \int_{I} u_{x}^{2} + C \int_{0}^{T} \sup_{x \in I} \theta^{q-\alpha+1} \int_{I} \rho \theta^{2} + C.
\] (26)

After the third term of the right side is absorbed by the left, we have
\[
\int_{0}^{T} \int_{I} \rho u_{t}^{2} + \int_{0}^{T} \int_{I} \rho u_{x}^{2} + \int_{I} (u_{x}^{2} + \omega_{x}^{2} + \omega^{2} + P^{2}) \\
\leq C \int_{0}^{T} \left( \int_{I} u_{x}^{2} + \int_{I} \omega_{x}^{2} + \int_{I} \omega^{2} + \int_{I} P^{2} \right)^{2} + C \int_{I} \rho \theta^{q+2} + C \int_{0}^{T} \int_{I} \kappa^{2} \theta_{x}^{2} \\
+ C \int_{0}^{T} \int_{I} u_{x}^{2} \int_{I} \rho \theta^{2} + C \int_{0}^{T} \sup_{x \in I} \theta^{q-\alpha+1} \int_{I} u_{x}^{2} \\
+ C \int_{0}^{T} \sup_{x \in I} \theta^{q-\alpha+1} \int_{I} \rho \theta^{2} + C.
\] (27)

Here we have used Lemma 3.2 and Young's inequality on the second term of the right side. Obviously, we only need to handle the terms about \(\theta\) in (27). To do this, we have to use the third equation of (1).

For convenience, we denote \(E = (u_{x}^{2} + \omega_{x}^{2} + \omega^{2})\) in the following context. Multiplying (1)_3 by \(\int_{0}^{\theta} \kappa(\xi) d\xi\) and using integration by parts over \(I\), we have
\[
\frac{d}{dt} \int_{I} \rho \int_{0}^{\theta} \int_{0}^{\theta} \kappa(\xi) d\xi d\eta + \int_{I} \kappa^{2} \theta_{x}^{2} \\
= \int_{I} E \int_{0}^{\theta} \kappa(\xi) d\xi - \int_{I} \rho \theta u_{x} \int_{0}^{\theta} \kappa(\xi) d\xi.
\]
\[
\leq C\|(1 + \theta^q)\theta\|_{L^\infty} \int_I E + C\|(1 + \theta^q)\theta\|_{L^\infty} \int_I \rho \theta |u_x|.
\] (28)

Using Corollary 3 and (A3), we have
\[
\|(1 + \theta^q)\theta\|_{L^\infty} \leq C\|\kappa \theta_x\|_{L^2} + C.
\] (29)

Substituting (29) into (28) and using the Hölder inequality, the Cauchy inequality, and Lemma 3.3, we have
\[
\frac{d}{dt} \int_I \rho \int_0^\eta \kappa(\xi) d\xi d\eta + \int_I \kappa^2 \theta_x^2 \\
\leq C\|\kappa \theta_x\|_{L^2} \int_I E + C \int_I E + C\|\kappa \theta_x\|_{L^2} \int_I \rho \theta |u_x| + C \int_I \rho \theta |u_x| \\
\leq C\|\kappa \theta_x\|_{L^2} \left( \int_I E + \|\rho \theta\|_{L^2} \|u_x\|_{L^2} \right) + C \int_I E + C \int_I \rho \theta^2 \\
\leq \frac{1}{2} \int_I \kappa^2 \theta_x^2 + C \left( \int_I E \right)^2 + C \int_I \rho \theta^2 \int_I u_x^2 + C \int_I \rho \theta^2 + C,
\]
which implies
\[
\frac{d}{dt} \int_I \rho \int_0^\eta \kappa(\xi) d\xi d\eta + \frac{1}{2} \int_I \kappa^2 \theta_x^2 \leq C \left( \int_I E \right)^2 + C \int_I \rho \theta^2 \int_I u_x^2 + C.
\]

Integrating over \((0, T)\), and using (A3), Lemma 3.2, and Corollary 3, we get
\[
\int_I \rho \theta^q + \int_0^T \int_I \kappa^2 \theta_x^2 \leq C \int_0^T \left( \int_I E \right)^2 + C \int_0^T \left( \int_I \rho \theta^2 \int_I u_x^2 \right) + C. \quad (30)
\]

Using (27), (30), Corollary 3, Lemma 3.7, and the Gronwall inequality, we complete the proof.

**Corollary 4** (see [34]). Under the conditions of Theorem 3.1, we have
\[
\int_0^T \|\theta\|_{L^\infty}^{2q+2} \leq C.
\]

**Proof.** By Corollary 1 and Cauchy inequality, we obtain
\[
\int_0^T \sup_{x \in I} \theta^{2q+2} \leq \int_0^T \int_I \theta^{2q+1} |\theta_x| + C \leq \frac{1}{2} \int_0^T \sup_{x \in I} \theta^{2q+2} + C \int_{Q_T} \theta^{2q} \theta_x^2,
\]
which together with Lemma 3.8, we complete the proof of Corollary 4.

**Lemma 3.9** (see [33]). Under the conditions of Theorem 3.1, we have for any \(0 \leq t \leq T\),
\[
\int_I (\rho_x^2 + \rho_t^2) + \int_{Q_T} u_x^2 \leq C.
\]

**Lemma 3.10.** Under the conditions of Theorem 3.1, it holds that for any \(0 \leq t \leq T\),
\[
\int_{Q_T} \omega_{xx}^2 \leq C.
\]
Proof. Squaring both right and left of the third equation of (1) and integrating over $Q_T$, we have
\[
\int_{Q_T} \omega^2_{xx} \leq C \int_{Q_T} \rho \omega_i^2 + C \int_{Q_T} \rho^2 u^2 \omega_i^2 + C \int_{Q_T} \omega^2
\]
\[\leq C \int_{Q_T} \rho \omega_i^2 + C\]
\[\leq C,
\]
where we have used Lemma 3.8, the Sobolev inequality, the Cauchy inequality.

Lemma 3.11. Under the conditions of Theorem 3.1, it holds that for any $0 \leq t \leq T$,
\[
\int (\rho u_i^2 + \rho \omega_i^2 + \theta_i^2) + \int_{Q_T} (u_{xxt}^2 + \omega_{xxt}^2 + \omega_t^2 + \rho \theta_i^2) \leq C.
\]

Proof. Differentiating the second equation of (1) with respect to $t$, we obtain
\[
\rho_{utt} + \rho_t u_t + \rho_t uu_x + \rho u_{xt} + P_{xt} = u_{xxt}.
\]
(31)
Multiplying (31) by $u_t$ and integrating the resulting equation over $I$, similar to the proof in the [33] we have
\[
\int_{t} \rho u_i^2 + \int_0^T \int_I u_{xxt}^2 \leq C + C \int_0^T \int_I u_{xxt}^2 \int_t^T \rho u_i^2 + C \int_0^T \int_I \rho(1 + \theta^q) \theta_i^2.
\]
(32)
Multiplying the fourth equation of (1) by $(\int_0^\theta \kappa(\xi) d\xi)_t$, integrating the resulting equation over $I$, and using integration by parts, Lemmas 2.2, 3.3, 3.8 and the Cauchy inequality, we have for any $\varepsilon > 0$
\[
\int_I \rho \kappa \theta_i^2 + \frac{1}{2} \frac{d}{dt} \int_I \kappa^2 \theta_i^2
\]
\[\leq \frac{1}{2} \int_I \rho \kappa \theta_i^2 + \int_I \rho \kappa u_i \theta_i + \int_I u_i^2 \left( \int_0^\theta \kappa(\xi) d\xi \right)_t
\]
\[\leq \frac{1}{2} \int_I \rho \kappa \theta_i^2 + C \int_I \rho \kappa u_i \theta_i + \int_I \rho \kappa^2 u_i^2
\]
\[+ \frac{d}{dt} \left( \int_I E \int_0^\theta \kappa(\xi) d\xi \right) - 2 \int_I (u_x u_{xt} + \omega_x \omega_{xt} + \omega \omega_t) \int_0^\theta \kappa(\xi) d\xi
\]
\[\leq \frac{1}{2} \int_I \rho \kappa \theta_i^2 + C \int_I (1 + \theta^q)^2 \theta_x^2 + C \left( 1 + \int_I u_{xx}^2 \right) \int_I \rho(1 + \theta^q) \theta^2
\]
\[+ \frac{d}{dt} \left( \int_I E \int_0^\theta \kappa(\xi) d\xi \right) + \varepsilon \int_I (u_{xxt}^2 + \omega_{xxt}^2 + \omega_t^2) + C \sup_{x \in I} (1 + \theta^q)^2 \theta^2.
\]
The first term of the right side can be absorbed by the left. After that, combining Lemmas 2.2, 3.3, 3.8 implies
\[
\int_I \rho \kappa \theta_i^2 + \frac{d}{dt} \int_I \kappa^2 \theta_i^2
\]
\[\leq C \int_I (1 + \theta^q)^2 \theta_x^2 + C \int_I u_{xx}^2 + C + \frac{d}{dt} \left( \int_I u_x^2 \int_0^\theta \kappa(\xi) d\xi \right)
\]
Under the conditions of Theorem 3.1, it holds that for any $0 \leq t \leq T$,

$$\|u\|_{W^{1,\infty}(Q_T)} + \|\omega\|_{W^{1,\infty}(Q_T)} + \int_t^T (u_{xx}^2 + \omega_{xx}^2) + \int_{Q_T} \theta_{xx}^2 \leq C.$$

**Corollary 5.** Under the conditions of Theorem 3.1, it holds that

$$\|\theta\|_{L^\infty(Q_T)} \leq C.$$
Proof. Squaring both side of the equation \((1)_2\) and integrating over \(I\), we have
\[
\int_I u_{xx}^2 \leq C \int_I \rho u_t^2 + C \left( \int_I u_x^2 \right)^2 + C \int_I \rho_x^2 \theta^2 + C \int_I \theta_x^2 \rho^2
\]
\[
\leq C \int_I \rho u_t^2 + C \sup_{x \in I} \theta^2 \int_I \rho_x^2 + C \int_I (1 + \theta^q)^2 \theta_x^2 + C
\]
\[
\leq C \int_I \rho u_t^2 + C \sup_{x \in I} (1 + \theta^q)^2 \int_I \rho_x^2 + C \int_I (1 + \theta^q)^2 \theta_x^2 + C,
\]
which, combining Lemmas 3.9, 3.11, Corollary 4 and Corollary 5, gives
\[
\int_I u_{xx}^2 \leq C.
\]
This, together with Lemmas 2.2 and 3.8 and the Sobolev inequality, ensures
\[
\|u\|_{W^{1,\infty}(I)} \leq C.
\]
After that, similarly, we have
\[
\int_I \omega_{xx}^2 \leq C \int_I \rho \omega_t^2 + C \int_I \rho^2 \omega_x^2 + C \int_I \omega^2
\]
\[
\leq C \int_I \rho \omega_t^2 + C
\]
\[
\leq C,
\]
where we have used Lemmas 3.8, 3.11, 3.3 and (40). Similarly, we have
\[
\|\omega\|_{W^{1,\infty}(I)} \leq C.
\]
After that, we have to estimate \(\int_{Q_T} \theta_{xx}^2\):
\[
\int_I \theta_{xx}^2 \leq \int_I \theta_t^2 + \int_I u_x^2 + \int_I \omega_x^2 + \int_I \omega^4 + \int_I \rho \theta_t^2 + C \int_I u^2 \theta_x^2 + \int_I \theta_x^2 \rho_t^2
\]
\[
\leq C \|\theta_x \theta_{xx}\|_{L^1} \int_I \theta_x^2 + \int_I \rho \theta_t^2 + C
\]
\[
\leq C \|\theta_{xx}\|_{L^2} \int_I \rho \theta_t^2 + C
\]
\[
\leq \frac{1}{2} \int_I \theta_{xx}^2 + \int_I \rho \theta_t^2 + C,
\]
which implies
\[
\int_I \theta_{xx}^2 \leq \int_I \rho \theta_t^2 + C.
\]
Integrating (44) over \([0, T]\), and using Lemma 3.11, we get
\[
\int_{Q_T} \theta_{xx}^2 \leq C,
\]
which, combining (40), (42), (39), we have proved Corollary 6. \qed

Lemma 3.12. Under the conditions of Theorem 3.1, it holds that for any \(0 \leq t \leq T\),
\[
\|\rho\|_{W^{1,\infty}(Q_T)} + \|\rho_t\|_{L^\infty(Q_T)} + \int_I (\rho_{xx}^2 + \rho_{xt}^2) + \int_{Q_T} (\rho_{tt}^2 + u_{xxx}^2 + \omega_{xxx}^2) \leq C.
\]
Proof. Since the estimate of $u$ and $\rho$ is similar to the proof in [33], for brevity, we omit the detail. We only need to get the estimate of $\omega$. Differentiating the third equation of (1) with respect to $x$, we obtain

$$\omega_{xxx} = \rho_x \omega_t + \rho \omega_x + \rho_x u \omega_x + \rho u_x \omega_x + \rho u \omega_x + \omega_x.$$  \hspace{1cm} (45)

By Lemmas 2.2, 3.3 and 3.10, Corollaries 5 and 6 and the Sobolev inequality and Cauchy inequality, we have

$$\int_I \omega_{xxx}^2 \leq C \int_I \rho^2 \omega_{xt}^2 + C \int_I \rho_x \omega_t^2 + C \int_I \rho u \omega_x^2 + \omega_x^2.$$ \hspace{1cm} (46)

Integrating (46) over $[0,T]$ and using Lemma 3.11, we obtain

$$\int_{Q_T} \omega_{xxx}^2 \leq C.$$  \hspace{1cm} This proves Lemma 3.12. \hfill \Box

**Lemma 3.13.** Under the conditions of Theorem 3.1, we have for any $0 \leq t \leq T$,

$$\int_I \rho \theta_t^2 + \int_{Q_T} |(\kappa \theta_x)_t|^2 \leq C.$$  

Proof. Differentiating the fourth equation of (1) with respect to $t$, we have

$$\rho \theta_t + \rho_t \theta_t + (\rho \theta \theta_x)_t + (\rho \theta u_x)_t = 2(u_x u_{xt} + \omega_x \omega_{xt} + \omega \omega_t) + (\kappa \theta_x)_{xt}.$$ \hspace{1cm} (47)

Multiplying (47) by $\int_0^\theta \kappa(\xi) \, d\xi$, integrating over $I$, and using integration by parts, (1)$_1$, Corollary 5, Lemma 3.3, Corollary 6, and the Hölder inequality, we infer

$$\frac{1}{2} \frac{d}{dt} \int_I \rho \kappa \theta_t^2 + \int_I |(\kappa \theta_x)_t|^2$$

$$= -\frac{1}{2} \int_I \rho \kappa \theta_t^2 + \frac{1}{2} \int_I \rho \kappa \theta_t^2 \theta_t - \int_I (\rho \theta \theta_x)_t \kappa \theta_t - \int_I (\rho \theta u_x)_t \kappa \theta_t$$

$$+ 2 \int_I (u_x u_{xt} + \omega_x \omega_{xt} + \omega \omega_t) \kappa \theta_t$$

$$\leq \frac{1}{2} \int_I (\rho u)_x \kappa \theta_t^2 + C \|\kappa \theta_t\|_{L^\infty} \int_I \rho \theta_t^2 - \int_I (\rho u \kappa \theta_x)_t \theta_t + \int_I \rho u k \theta_t^2 \theta_x$$

$$- \int_I (\rho u \kappa \theta_x)_t \theta_t + C \int_I (u_x^2 + \omega_x^2 + \omega_t^2) + C \int_I \rho \theta_t^2$$

$$- \int_I \rho_t u_x \kappa \theta_t + C \|\kappa \theta_t\|_{L^\infty} \left( \|u_x\|_{L^2} \|u_x\|_{L^2} + \|\omega_x\|_{L^2} \|\omega_x\|_{L^2} \right).$$

This, combining integration by parts and the Cauchy inequality, we conclude from Lemmas 2.1, 2.2, Corollaries 5, 6, Lemmas 3.11, 3.12 that

$$\frac{1}{2} \frac{d}{dt} \int_I \rho \kappa \theta_t^2 + \int_I |(\kappa \theta_x)_t|^2$$

$$\leq -\frac{1}{2} \int_I \rho u k \theta_x^2 - \int_I \rho u k \theta_x \theta_t + C \|\kappa \theta_t\|_{L^\infty} \int_I \rho \theta_t^2$$
Corollary 7. Under the conditions of Theorem 3.1, it holds that
\[ C \| \sqrt{\rho} u_x \|_{L^2} \| (\kappa \theta_{xx})_t \|_{L^2} + C \| \theta_x \|_{L^\infty} \int_I \rho \theta_t^2 + C \| \kappa \theta_t \|_{L^\infty} \]
\[ + C \int_I (u_{xx}^2 + \omega_{xx}^2 + \omega_t^2) + C \int_I \rho \theta_t^2 + C \| \kappa \theta_t \|_{L^\infty} (\| u_{xt} \|_{L^2} + \| \omega_{xt} \|_{L^2} + \| \omega_t \|_{L^2}) \]
\[ \leq C \| \theta_{xx} \|_{L^2} \int_I \rho \theta_t^2 + C \| \kappa \theta_t \|_{L^\infty} \int_I \rho \theta_t^2 + C \| \sqrt{\rho} (\kappa \theta_{xx})_t \|_{L^2} + C \| \kappa \theta_t \|_{L^\infty} \]
\[ + C \int_I (u_{xx}^2 + \omega_{xx}^2 + \omega_t^2) + C \int_I \rho \theta_t^2 + C \| \kappa \theta_t \|_{L^\infty} \]
\[ + C \int_I \rho \kappa \| \theta_t \| + C \int_I (u_{xx}^2 + \omega_{xx}^2 + \omega_t^2) + C \int_I \rho \theta_t^2 + \]
\[ + C \left( \| (\kappa \theta_t)_x \|_{L^2} + \int_I \rho \kappa \| \theta_t \| \right) (\| u_{xt} \|_{L^2} + \| \omega_{xt} \|_{L^2} + \| \omega_t \|_{L^2}), \]
which together with Cauchy inequality, Lemma 3.3, and Corollary 5 gives
\[ \frac{1}{2} \frac{d}{dt} \int_I \rho \kappa \theta_t^2 + \int_I |(\kappa \theta_t)_x|^2 \leq C \int_I \theta_{xx}^2 + C \left( \int_I \rho \theta_t^2 \right)^2 + C \int_I (u_{xx}^2 + \omega_{xx}^2 + \omega_t^2) + C. \quad (48) \]

Integrating it over \((0,T)\) and using Lemma 3.11, Corollary 6, compatibility conditions (5) and Gronwall inequality, we complete the proof.

The next two corollaries are needed in our analysis, whose proof is available in [33].

Corollary 7. Under the conditions of Theorem 3.1, it holds that
\[ \int_0^T \| \theta_t \|_{L^\infty}^2 \leq C. \]

Corollary 8. Under the conditions of Theorem 3.1, it holds that
\[ \int_{Q_T} \theta_{xx}^2 \leq C. \]

Corollary 9. Under the conditions of Theorem 3.1, we have for any \(0 \leq t \leq T\),
\[ \| \theta \|_{W^{1,\infty}(I)} + \int_I \theta_{xx}^2 + \int_{Q_T} \theta_{xxx}^2 \leq C. \]

Proof. From (44) and Lemma 3.13, we have
\[ \int_I \theta_{xx}^2 \leq C, \quad (49) \]
which, combining Corollary 5, Lemma 3.11, and the Sobolev inequality, we obtain
\[ \| \theta \|_{W^{1,\infty}(I)} \leq C. \quad (50) \]

Differentiating (1) with respect to \(x\), we have
\[ \kappa \theta_{xxx} = -3 \kappa x' \theta_x \theta_{xx} - \kappa' \theta_x^3 - 2 (u_x u_{xx} + \omega_x \omega_{xx} + \omega \omega_x) + \rho \theta_{xx} + \rho_x \theta_t \]
\[ + (\rho u \theta_x)_x + (\rho u \theta_x)_{x}. \quad (51) \]
Lemma 3.14. Following. Lemma 3.15. Under the conditions of Theorem 3.1, we have for any $t$

By (49), (50), (51), Lemmas 3.12, 3.13, and Corollary 6, we have

$$\int I \rho^2 \theta_{xx}^2 \leq C \int I \theta_x^2 + C \int I \theta_{xt}^2 + C \int I (u_x^2 u_{xx}^2 + \omega_x^2 \omega_{xx}^2 + \omega_x^2) + \int I \rho^2 \theta_{xt}^2$$

$$+ \int I |(\rho u \theta_x)_{xx}|^2 + C \int I |(\rho \theta u_x)_x|^2 + C$$

$$\leq C \int I \theta_{xt}^2 + C \int I \rho^2 \theta_t^2 + C$$

$$\leq C \int I \theta_{xt}^2 + C \sup_{x \in I} \theta_t^2 + C.$$  \hfill (52)

Integrating it over $(0, T)$ and using Corollaries 7 and 8, we have

$$\int_{Q_T} \theta_{xx}^2 \leq C.$$

The next lemma, which plays a key role in getting $H^3$ estimates of $\theta$ in the following.

Lemma 3.14 (see [33]). Under the conditions of Theorem 3.1, it holds that

$$\|\sqrt{\theta}\|_{L^\infty(Q_T)} + \|\sqrt{\rho}\|_{L^\infty(Q_T)} \leq C.$$

To get $H^3$ estimates of $\theta$, we also need to use the next lemma.

Lemma 3.15. Under the conditions of Theorem 3.1, we have for any $0 \leq t \leq T$

$$\int I \rho^2 |(\kappa \theta_x)_x|^2 + \int_{Q_T} \rho^3 \theta_{tt}^2 \leq C.$$

Proof. Multiply (47) by $\rho^2 (\kappa \theta_t)_t$, and using integration by parts, we have

$$\int I \rho^3 \kappa \theta_{tt}^2 + \frac{1}{2} \frac{d}{dt} \int I \rho^2 |(\kappa \theta_x)_x|^2$$

$$= \int I \rho \rho_t (|(\kappa \theta_x)_x|^2 - 2 \int I \rho u \kappa \theta_{tt}(\kappa \theta_x)_t - 2 \int I \rho \kappa \theta_{tt}^2 (\kappa \theta_x)_t$$

$$+ 2 \int I (u_x u_{xt} + \omega_x \omega_{xt} + \omega_x \omega_t)(\rho^2 \kappa \theta_{tt} + \rho^2 \kappa \theta_t^2) - \int I \rho^3 \kappa \theta_t^2 \theta_{tt}$$

$$- \int I (\rho_t \theta_t + (\rho u \theta_x)_t + (\rho \theta u_x)_t)(\rho^2 \kappa \theta_{tt} + \rho^2 \kappa \theta_t^2)$$

$$\leq C \int I |(\kappa \theta_x)_x|^2 - 4 \int I \rho^2 (\sqrt{\rho}) \kappa \theta_{tt}(\kappa \theta_x)_t$$

$$+ C \int I \rho^2 + C \int I u_{xt}^2 + C \int I \theta_{xt}^2 + C$$

$$+ C \left( \int I \rho^3 \kappa \theta_{tt}^2 \right) \frac{1}{4} \left\{ 1 + \left( \int I \rho \theta_t^2 \right)^{\frac{1}{2}} + \|\theta_t\|_{L^2} + \|\sqrt{\theta u_t}\|_{L^2} + \|u_t\|_{L^2} + \|\sqrt{\theta t}\|_{L^2} \right\}$$

$$\leq \frac{1}{4} \int I \rho^3 \kappa \theta_{tt}^2 + C \int I |(\kappa \theta_x)_x|^2 + C \int I \theta_{tt}^2 + C \int I u_{xt}^2 + C.$$ \hfill (53)

This implies

$$\int I \rho^3 \kappa \theta_{tt}^2 + \frac{d}{dt} \int I \rho^2 |(\kappa \theta_x)_x|^2 \leq C \int I |(\kappa \theta_x)_x|^2 + C \|\theta_t\|_{L^\infty}^2 + C \int I \theta_{tt}^2 + C \int I u_{xt}^2 + C.$$  \hfill (54)

This, along with Lemmas 3.11, 3.13, Corollaries 7, 8 and compatibility conditions (5), we have

$$\int I \rho^2 |(\kappa \theta_x)_x|^2 + \int_0^t \int I \rho^3 \theta_{tt}^2 \leq C.$$
This completes the proof. □

**Corollary 10.** Under the conditions of Theorem 3.1, we have for any \(0 \leq t \leq T\),
\[
\int_I (\theta_{xxx}^2 + \rho^2 \theta_{xt}^2) \leq C.
\]

**Proof.** The proof of this lemma see [33], we omit the detail here. □

The next two lemmas will be used to get \(H^3\) estimates of \(u\) and \(\omega\).

**Lemma 3.16 (see [33]).** Under the conditions of Theorem 3.1, it holds that for any \(0 \leq t \leq T\),
\[
\int_I \rho^2 u_{xt}^2 + \int_{Q_T} \rho^3 u_{tt}^2 \leq C.
\]

**Lemma 3.17.** Under the conditions of Theorem 3.1, it holds that for any \(0 \leq t \leq T\),
\[
\int_I \rho^2 \omega_{xt}^2 + \int_{Q_T} \rho^3 \omega_{tt}^2 \leq C.
\]

**Proof.** Similarly to Lemma 3.16, multiplying (35) by \(\rho^2 \omega_{tt}\), and integrating over \(I\), we have
\[
\int_I \rho^3 \omega_{tt}^2 + \frac{1}{2} \frac{d}{dt} \int_I \rho^2 \omega_{xt}^2
\]
\[
= \int_I \rho \rho_t \omega_{tt}^2 - 2 \int_I \rho \rho_x \omega_{xt} \omega_{tt} - \int_I \rho^2 \omega_{tt} (\rho \omega_t + \rho_t u \omega_x + \rho u_t \omega_x + \rho u \omega_{xt} + \omega_t) 
\]
\[
\leq C \int_I \omega_{xt}^2 - 4 \int_I \rho^3 (\sqrt{\rho}) \omega_{xt} \omega_{tt} + \frac{1}{4} \int_I \rho^3 \omega_{tt}^2 + C \int_I \rho \omega_t^2 + C
\]
\[
\leq \frac{1}{2} \int_I \rho^3 \omega_{tt}^2 + C \int_I \omega_{xt}^2 + C.
\]

After the first term of the right side is absorbed by the left, we obtain
\[
\int_I \rho^3 \omega_{tt}^2 + \frac{1}{2} \frac{d}{dt} \int_I \rho^2 \omega_{xt}^2 \leq C \int_I \omega_{xt}^2 + C.
\]

Integrating this inequality on both sides over \((0, T)\), and using Lemma 3.11, 3.12, 3.14 and Corollaries 6, 9, we have completed the proof. □

**Corollary 11.** Under the conditions of Theorem 3.1, it holds that for any \(0 \leq t \leq T\),
\[
\int_I (u_{xxx}^2 + \omega_{xxx}^2) \leq C.
\]

**Proof.** Differentiating the second equation of (1) with respect to \(x\), we obtain
\[
u_{xxx} = \rho_x u_t + \rho u_{xt} + \rho_x uu_x + \rho u_x^2 + \rho uu_{xx} + (\rho \theta)_{xx}.
\]
By Lemmas 2.2, 3.3, 3.9-3.12 and 3.14, and Corollaries 5, 6 and 9, and the Sobolev inequality, we have
\[
\int_I u_{xxx}^2 \leq C \int_I \rho^2 u_{zt}^2 + C \int_I \rho^2 u_t^2 + C \int_I \rho_{xx}^2 + C \int_I \theta_{xx}^2 + C
\]
\[
\leq C \int_I \rho^2 u_{zt}^2 + C \int_I \rho_{xx}^2 + C \int_I \theta_{xx}^2 + C
\]
\[
\leq C.
\]
Similarly, differentiating the third equation of (1) in $x$, we have
\[
\omega_{xxx} = \rho_x \omega_t + \rho \omega_{xx} + \rho_x u \omega + \rho u_x \omega_x + \rho u \omega_{xx} + \omega_x,
\] (57)
\[
\int \omega_{xxx}^2 = \int \rho_x^2 \omega_t^2 + \int \rho^2 \omega_{xx}^2 + \int \rho_x^2 u_x^2 \omega_x^2 + \int \rho^2 u_x^2 \omega_{xx}^2 + \int \rho^2 u_x \omega_{xx}^2 + \int \omega_x^2
\leq C.
\] (58)

Therefore, using the previous estimates, we immediately have the following lemma.

**Lemma 3.18.** There exists a uniform constant $C$ such that the following estimate holds for the solution on $[0, T] \times \mathbb{R}$, for any $T > 0$,
\[
\begin{align*}
&\|\sqrt{\rho} \|_{L^\infty} + \|\sqrt{\rho_t} \|_{L^\infty} + \|\rho \|_{H^2} + \|\rho_t \|_{H^1} + \|\theta \|_{H^3} \\
&+ \|\omega \|_{H^3} + \|\rho u_t \|_{H^1} + \|\rho \omega_t \|_{H^1} + \|\sqrt{\rho u_t} \|_{L^2} + \|\sqrt{\rho \omega_t} \|_{L^2} + \|\theta \|_{H^3} + \|\sqrt{\theta_t} \|_{H^1} \\
&+ \int_{Q_T} (u_x^2 + \omega^2_x + \rho_t^2 + \theta_t^2 + \theta^2 + \rho^3 u_t^2 + \rho^3 \omega_t^2 + \rho^3 \theta_t^2)
\leq C.
\end{align*}
\] (59)

**Corollary 12.** Under the conditions of Theorem 3.1, there is a positive constant $C_\delta$ depending on $\delta$ such that for any $(x, t) \in Q_T$, it holds that
\[
\begin{align*}
\rho(x, t) &\geq \frac{\delta}{C} > 0, \\
\theta(x, t) &\geq C_\delta.
\end{align*}
\] (60)

From (59), (60), (31), (35) and (47) we have
\[
\begin{align*}
&\|\rho \|_{H^2} + \|\rho_t \|_{H^1} \\
&+ \|\omega \|_{H^3} + \|u_t \|_{H^1} + \|\omega_t \|_{H^1} + \|\theta \|_{H^3} + \|\theta_t \|_{H^1} \\
&+ \int_{Q_T} (u_x^2 + u_x^2 + \omega_x^2 + \omega_{xx} + \rho_t^2 + \theta_t^2 + \theta^2 + \theta_x^2 + u_t^2 + \omega_t^2 + \theta_t^2)
\leq C_\delta.
\end{align*}
\] (61)

The proof of Theorem 3.1 is completed.

**Proof of Theorem 1.1.** Consider (1)-(3) with initial data exchanged $(\rho_0^\delta, u_0^\delta, \omega_0^\delta, \theta_0^\delta)$. We get a global solution $(\rho^\delta, \omega^\delta, u^\delta, \theta^\delta)$ to (1)-(3) for each $\delta > 0$. These *a priori* estimates (59) and (60) are true for $(\rho^\delta, \omega^\delta, u^\delta, \theta^\delta)$. Since constant $C$ in the estimate of (59) is independent of $\delta$, we can take $\delta \to 0^+$ (take subsequence if necessary) to get a solution to (1)-(3) denoted by $(\rho, u, \omega, \theta)$ which satisfies (59) by the lower semicontinuity of the norms. This proves the existence of Theorem 1.1. We can prove the uniqueness of the solution to system (1) without vacuum by the standard methods, see for the appendix. We complete the proof of Theorem 1.1. \qed

**Remark 3.** We can't get the uniqueness of the solution to this system with vacuum due to our boundary condition (3) and the term $(\kappa(\theta_x) \theta_x)_x$. In particular, for the temperature equation, we can't make all terms $\|\theta_t \|_{L^2}$ become $\int \rho_t \theta^2 dx$ on the right hand of inequality, which leads us to not use Gronwall's inequality. However, we obtain the uniqueness of this system without vacuum. The detailed proof of uniqueness is given in Appendix.
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Appendix.

Appendix 1. In this appendix, we will give the detailed proofs of uniqueness of the solution to the compressible viscous micropolar fluid without vacuum.

Proof. If \( \rho_0 \) has a positive lower bound \( \delta_0 \), recalling the proof of Lemma 3.1 in [33], we get \( \rho \) has a positive lower bound \( \delta \), which depends on \( \delta_0 \). Let \((\rho_1, u_1, \theta_1, \omega_1)\) and \((\rho_2, u_2, \theta_2, \omega_2)\) be two classical solutions to problem satisfying the regularity, and we denote

\[
\bar{\rho} = \rho_1 - \rho_2, \quad \bar{u} = u_1 - u_2, \quad \bar{\theta} = \theta_1 - \theta_2 \text{ and } \bar{\omega} = \omega_1 - \omega_2.
\]

Then from (1), we derive the equations for the differences

\[
\bar{\rho}_t + \bar{\rho}_x u_2 + \rho_1 u_1 \bar{u} + \bar{\rho} u_{2x} + \rho_1 \bar{u}_x = 0. \tag{62}
\]

Multiplying (62) by \( \bar{\rho} \) and integrating over \( \Omega \), we obtain that

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_I \bar{\rho}^2 dx &\leq \left( \frac{1}{2} \|u_{2x}\|_{L^\infty} + C \|\rho_1 x\|_{L^\infty}^2 + \|\rho_1\|_{L^\infty}^2 \right) \int_I \bar{\rho}^2 dx + \frac{1}{8} \int_I |\bar{u}_x|^2 dx. \tag{63}
\end{align*}
\]

For \( u \), we have

\[
\begin{align*}
\rho_1 \bar{u}_t + \rho_1 u_1 \bar{u}_x - \bar{u}_{xx} + R((\rho_1 \theta_1)_x - (\rho_2 \theta_2)_x) \\
= \bar{\rho}(-u_{2t} - u_{2x}) - \rho_1 \bar{u}_x.
\end{align*} \tag{64}
\]

Multiplying the equation (64) with \( \bar{u} \) and integrating the result equation over \( \Omega \), we have

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_I \rho_1 \bar{u}^2 dx + \int_I |\bar{u}_x|^2 dx &\leq \int_I |\bar{u}_x|^2 dx + C \|u_{2t} - u_{2x}\|_{L^\infty}^2 \int_I \bar{\rho}^2 dx + \|\rho_1\|_{L^\infty} \int_I \rho_1 \bar{u}^2 \\
&\quad + \|\rho_1\|_{L^\infty}^2 \int_I \bar{\rho}^2 dx + ||\bar{\theta}_2||_{L^\infty}^2 \int_I \bar{\rho}^2 dx. \tag{65}
\end{align*}
\]

Similarly, we have

\[
\begin{align*}
\rho_1 \bar{\omega}_t + \rho_1 u_1 \bar{\omega}_x - A \bar{\omega}_{xx} + A \bar{\omega} = \bar{\rho}(-\omega_{2t} - \omega_{2x}) - \rho_1 \bar{u}\omega_{2x}. \tag{66}
\end{align*}
\]

Multiplying the result equation (66) with \( \bar{\omega} \), we obtain

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_I \rho_1 \bar{\omega}^2 dx + A \int_I \bar{\omega}^2 dx &\leq A \int_I \bar{\omega}^2 dx + C(\|\rho_1\|_{L^\infty} \|\omega_{2x}\|_{L^\infty}^2) \int_I \rho_1 \bar{\omega}^2 dx + \frac{1}{8} \int_I |\bar{u}_{2x}|^2 dx \\
&\quad + (\|\omega_{2t}\|_{L^\infty}^2 + \|\omega_{2x}\|_{L^\infty}^2) \int_I \bar{\rho}^2 dx. \tag{67}
\end{align*}
\]

For \( \theta \), we have

\[
\begin{align*}
\rho_1 \bar{\theta}_t + \rho_1 u_1 \bar{\theta}_x - [\kappa(\theta_1) \bar{\theta}_x + (\kappa(\theta_1) - \kappa(\theta_2)) \bar{\theta}_{2x}]_x \\
= \bar{\rho}(-\theta_{2t} - \theta_{2x}) - \rho_1 \bar{u}\theta_{2x} - R \rho_1 u_{1x} \bar{\theta} - R \bar{u}\omega_{2x} \theta_2 - R \rho_1 \bar{u}_x \theta_2 \\
+ (u_1 + u_2)_x \bar{u}_x + (\omega_1 + \omega_2)_x \bar{\omega}_x + (\omega_1 + \omega_2) \bar{\omega}.
\end{align*} \tag{68}
\]
Multiplying the equation (68) with \( \tilde{\theta} \), we obtain that
\[
\frac{d}{dt} \left( \int \rho \tilde{\theta}^2 + C_1 \int |\tilde{x}|^2 \right) \\
\leq C \left( |\theta_{2t}|_{L^2}^2 + |\theta_{2x}|_{L^\infty}^2 + |\rho_{1x}|_{L^\infty}^2 |\theta_{2x}|_{L^\infty}^2 + |\rho_{1x}|_{L^\infty} |u_{1x}|_{L^\infty} + |\theta_{2x}|_{L^\infty}^2 \right) \\
+ C \left( \frac{1}{2} |\rho_{1}|_{L^\infty}^2 + \int_1^t \rho_{1} \tilde{\theta}^2 dx \right) + C_1 \int \tilde{x}^2 dx,
\]
where \( \xi = \alpha_1 \theta + (1 - \alpha_1) \theta_2 \), and \( \alpha_1 \in [0, 1] \). Then, combing (63), (65), (67) and (69) we deduce that
\[
\frac{d}{dt} \left( \int \tilde{\rho}^2 + \int \sqrt{\rho_{1}} \tilde{u}^2 + \int \sqrt{\rho_{1}} \bar{\omega}^2 + \int \sqrt{\rho_{1}} \bar{\theta}^2 \right) \\
+ C \left( |\theta_{2t}|_{L^2}^2 + |\theta_{2x}|_{L^\infty}^2 + |\rho_{1x}|_{L^\infty}^2 |\theta_{2x}|_{L^\infty}^2 + |\rho_{1x}|_{L^\infty} |u_{1x}|_{L^\infty} + |\theta_{2x}|_{L^\infty}^2 \right) \\
+ \frac{1}{2} \left( |\bar{\rho}^2 + |\sqrt{\rho_{1}} \bar{\omega}^2 + |\sqrt{\rho_{1}} \bar{\theta}^2 \right) \\
\leq C \left( 1 + |\theta_{2t}|_{L^2}^2 + |\rho_{1x}|_{L^\infty}^2 + |u_{1x}|_{L^\infty} + |u_{2x}|_{L^\infty} + |\rho_{1x}|_{L^\infty}^2 + |\rho_{1x}|_{L^\infty}^2 \right) \\
+ \int_1^t \rho_{1} \tilde{\theta}^2 dx + \int_1^t \rho_{1} \tilde{\theta}^2 dx + C \left( |\theta_{2t}|_{L^2}^2 + |\theta_{2x}|_{L^\infty}^2 + |\rho_{1x}|_{L^\infty}^2 |\theta_{2x}|_{L^\infty}^2 + |\rho_{1x}|_{L^\infty} |u_{1x}|_{L^\infty} + |\theta_{2x}|_{L^\infty}^2 \right) \\
+ \frac{1}{2} \left( |\tilde{\rho}^2 + |\sqrt{\rho_{1}} \tilde{u}^2 + |\sqrt{\rho_{1}} \bar{\omega}^2 + |\sqrt{\rho_{1}} \bar{\theta}^2 \right).\]
Integrating the inequality (70) over \( [0, T] \) and noting that \( \rho_{1} > \delta \), we have
\[
\int_1^T \frac{d}{dt} \left( \int \tilde{\rho}^2 + \int \sqrt{\rho_{1}} \tilde{u}^2 + \int \sqrt{\rho_{1}} \bar{\omega}^2 + \int \sqrt{\rho_{1}} \bar{\theta}^2 \right) dx \\
\leq G(t) \left( \int \tilde{\rho}^2 + \int \sqrt{\rho_{1}} \tilde{u}^2 + \int \sqrt{\rho_{1}} \bar{\omega}^2 + \int \sqrt{\rho_{1}} \bar{\theta}^2 \right),
\]
where \( G(t) = 1 + |\theta_{2t}|_{L^2}^2 + |\rho_{1x}|_{L^\infty}^2 + |u_{1x}|_{L^\infty} + |u_{2x}|_{L^\infty} + |\rho_{1x}|_{L^\infty}^2 + |\theta_{2x}|_{L^\infty}^2 \).

Notice from estimates of Theorem 3.1 that \( G(t) \in L^1(0, T) \).

Therefore, in view of Gronwall’s inequality, we conclude that \( \tilde{\rho} = \tilde{u} = \tilde{\omega} = 0 \) and \( \tilde{\theta} = 0 \) in \( (0, T) \times \Omega \), which implies the uniqueness of classical solutions. \( \square \)

**Appendix 2.** In this appendix, we will give a sketch proof of existence of the local solution to the compressible viscous micropolar fluid.

Firstly, we consider the following linearized problem:
\[
\begin{aligned}
&\rho_t + (\rho v)_x = 0, \quad \rho \geq 0, \\
&(\rho u)_t + (\rho u v)_x + P_x = u_{xx}, \\
&(\rho \omega)_t + (\rho u \omega)_x + A \omega = A \omega_{xx}, \\
&(\rho \theta)_t + (\rho u \theta)_x + P v = (u v)_x + \kappa \theta x + (\omega v)_x + \omega^2, \\
&\left( \begin{array}{l}
\rho, u, \theta, \omega
\end{array} \right)|_{t=0} = (\rho_0, u_0, \theta_0, \omega_0), \\
&(u, \omega) = (0, 0), \quad \theta = 0 \text{ on } (0, T) \times \partial I,
\end{aligned}
\]
where \( v \) is a known vector field on \( (0, T) \times I \), and \( \tilde{\theta} \) is a known scalar.
Lemma 3.19. Assume that $\rho_0$, $u_0$, $\omega_0$, $\theta_0$, $v$, and $\hat{\theta}$ satisfy the properties

$$
\rho_0 \geq \delta > 0, \rho_0 \in H^2, u_0, \omega_0 \in H^3 \cap H^1_0, \theta_0 \in H^3,
$$

$$
v \in C([0, T]; H^3 \cap H^1_0), v_t \in C([0, T]; H^1_0) \cap L^2([0, T]; H^2), v_{tt} \in L^2([0, T]; L^2),
$$

$$
\hat{\theta} \in C([0, T]; H^1), \hat{\theta}_t \in C([0, T]; H^1) \cap L^2([0, T]; H^2), \hat{\theta}_{tt} \in L^2([0, T]; L^2).
$$

Then there exists a classical solution $(\rho, u, \omega, \theta)$ to the initial boundary value problem (72) such that

$$
\rho \in C([0, T]; H^2), \rho_t \in C([0, T]; H^1), \rho_{tt} \in L^2([0, T]; L^2),
$$

$$
u, \omega \in C([0, T]; H^3 \cap H^1_0), u_t, \omega_t \in C([0, T]; H^1_0) \cap L^2([0, T]; H^2),
$$

$$
u_{tt}, \omega_{tt}, \theta_{tt} \in L^2([0, T]; L^2),
$$

$$
\theta \in C([0, T]; H^3), \theta_t \in C([0, T]; H^1) \cap L^2([0, T]; H^2),
$$

and

$$
\rho \geq \delta \text{ on } [0, T] \times \bar{I} \text{ for some constant } \delta \geq 0.
$$

Proof. The existence and regularity of a solution $\rho$ of the linear hyperbolic problem see [7, Lemma 1]. Moreover, it follows from the representation formula (2.2) in [7, Lemma 1] that

$$
\rho(t, x) \geq (\inf_{I} \rho_0) \exp(-C \int_0^t \| v(s) \|_{H^2} \, ds) \geq \delta > 0 \tag{73}
$$

The existence and regularity of solutions $\omega$, $\theta$ and then $u$ to the corresponding linear parabolic problems have been well known and we omit the details. For instance, we may apply a semi-discrete Galerkin method.

Then we derive local (in time) a priori estimates for classical solutions to the linearized problem (72), which are independent of a lower bound $\delta$ of the initial density $\rho_0$. Let $(\rho_0, u_0, \theta_0, \omega_0)$ be a given initial data satisfying the hypotheses of Lemma 3.19, and let us choose any fixed $c_0$ so that

$$
c_0 \geq 1 + \| \rho_0 \|_{H^2} + \| (\sqrt{\rho_0})_x \|_{L^\infty} + \| u_0 \|_{H^3 \cap H^1_0} + \| \omega_0 \|_{H^3 \cap H^1_0} + \| \theta_0 \|_{H^3} + \|(g_1, g_2, g_3)\|_{L^2}.\]

Moreover let $v$ and $\hat{\theta}$ satisfying the regularity stated in Lemma 3.19, and assume that $v(0) = u_0$, $\theta(0) = \theta_0$ and

$$
\sup_{0 \leq t \leq T^*} \left( \| v(t) \|_{H^3} + \| \hat{\theta} \|_{H^3} + \| v(t) \|_{H^1_0} + \| v_t(t) \|_{H^1_0} + \| \hat{\theta}_t \|_{H^1} \right)
$$

$$
+ \int_0^{T^*} \| v_t(t) \|_{H^2}^2 + \| v_{tt} \|_{L^2}^2 + \| \hat{\theta}_t \|_{H^2}^2 + \| \hat{\theta}_{tt} \|_{L^2}^2 \, dt \leq c_1. \tag{74}
$$

Now let we derive some a priori estimates for the solution $(\rho, u, \omega, \theta)$ which are independent of $\delta$.

The estimate the density $\rho$ please refer [7, lemma 5]

$$
\sup_{0 \leq t \leq T^*} \left( \| \rho \|_{H^2} + \| \rho_t \|_{H^1} \right) + \int_0^{T^*} \| \rho_{tt} \|_{L^2}^2 \, dt \leq C. \tag{75}
$$

Then we get the estimate of $\omega$ by using the parabolic and elliptic theory.

$$
\sup_{0 \leq t \leq T^*} \left( \| \omega \|_{H^3 \cap H^1_0} + \| \omega_t \|_{H^1_0} \right) + \int_0^{T^*} \left( \| \omega_t \|_{H^2}^2 + \| \omega_{tt} \|_{L^2}^2 \, dt \leq C. \tag{76}
$$
To derive estimate for $\theta$, we differentiate Eq (72) with respect to $t$ and obtain
\[
\rho \dot{\theta}_t + \rho v \dot{\theta}_x + \nu (\partial_1^2 \theta)_x + (R \theta_0 + v_x) = ((v_x)^2 + \omega_x^2) \dot{t} - \rho v \theta_x - \rho \partial_1 \theta_t.
\]
(77)

Then multiplying this by $\theta_t$, integrating over $I$ and using transport equation, noticing that $\kappa(\theta) > 1$ we have
\[
\frac{1}{2} \frac{d}{dt} \int \rho \theta_t^2 dx + \int |\theta_{xt}|^2 dx \\
\leq C \int (|\rho||v||\theta_x| + |\rho v_t||\theta_t| + |\rho v||\theta_{xt}| + |(\rho \theta)_t| + |v_x| + |\theta| + \rho |\theta| v_x)|\theta_t| \\
+ (|v_x||v_{xt}| + |\omega_x||\omega_{xt}| + |\omega||\omega_t|)|\theta_t| + |\nu_t| + |\theta_t|) dx \equiv \sum_{j=1}^7 I_j.
\]
(78)

Making use of (74), (75) and (76), we can estimate each term $I_j = I_j(t)$ for $0 \leq t \leq \min(T^*, T_1)$ as follows:

\[
I_1 \leq C ||\rho||_L^2 ||v||_L^2 ||\theta_x||_L^2 ||\theta_{xt}||_L^2 \leq C \|\theta_x\|_2^2 + C \|\theta_{xt}\|_2^2 + C \|\sqrt{\rho} \theta_t\|_2^2,
\]

\[
I_2, I_5 \leq ||\rho||_2^2 ||v_x||_L^2 ||\theta_x||_L^2 \sqrt{\rho} \theta_t \|_2 \|\theta_t\|_2 \leq C \sqrt{\rho} \theta_t \|_2 \leq C ||\theta_x\|_2^2 + C ||\theta_{xt}\|_2^2,
\]

\[
I_3 \leq C ||\rho||_2^2 ||v_{xt}||_L^2 \sqrt{\rho} \theta_t \|_2 \|\theta_t\|_2 \leq C \sqrt{\rho} \theta_t \|_2 \leq C ||\theta_x\|_2^2 + C ||\theta_{xt}\|_2^2,
\]

\[
I_4 \leq ||\rho||_2^2 \sqrt{\rho} \theta_t \|_2 \|\theta_x||_L^2 \leq C ||\theta_x\|_2^2 + C ||\theta_{xt}\|_2^2 + C \|\theta_t\|_2^2 + C \|\theta_{xt}\|_2^2 + C \|\theta_t\|_2^2,
\]

\[
I_6 \leq \|v_x||v_{xt}||_L^2 + \|\omega_x||\omega_{xt}||_L^2 \leq C \|v_x||v_{xt}||_L^2 + \|\omega_x||\omega_{xt}||_L^2 \leq C \|\theta_x\|_2^2 + C \|\theta_{xt}||_2^2 + C \|\theta_t\|_2^2,
\]

\[
I_7 \leq C \|\theta_x\|_2^2 + C \|\theta_{xt}||_2^2 + C \|\theta_t\|_2^2.
\]

Substituting these estimates into (78), we have
\[
\frac{d}{dt} \|\sqrt{\rho} \theta_t\|_2^2 + \|\theta_{xt}\|_2^2 \leq C \|\sqrt{\rho} \theta_t\|_2^2 + C \|\theta_x\|_2^2 + \|u_{xt}\|_2^2 + \|\omega_{xt}\|_2^2 + \|\omega_t\|_2^2,
\]

then using the method in [6] and noting the Corollary 1, we derive estimates for $\theta$ as follows
\[
\sup_{0 \leq t \leq T^*} (||\theta||_{H^2} + ||\theta_t||_{H^1}) + \int_0^{T^*} (||\theta_t||_{L^2}^2 + ||\theta_{xt}||_{L^2}^2) dt \leq C.
\]
(79)

Then we obtain the estimate of $\theta$ by using the similar method in [6], we omit the detail for brevity.
\[
\sup_{0 \leq t \leq T^*} (||u||_{H^\infty} + ||u_t||_{H^1}) + \int_0^{T^*} (||u_t||_{H^2}^2 + ||u_{xt}||_{L^2}^2) dt \leq C.
\]
(80)

Now we are devoted to providing a sketch proof of the local existence of the solution. Our proof will be based on the usual iteration argument and on the above results. Then there exists a classical solution $(\rho^*, u^*, \omega^*, \theta^*)$ to the linearized problem (72) with $v, \theta$ replaced by $(u_0, \theta_0)$, which satisfies the regularity estimate (75), (76), (79) and (80) with $T^*$ replaced by $T_1$. Similarly, we construct approximate solutions $(\rho^k, u^k, \omega^k, \theta^k)$, inductively, as follows: assuming that $(u^{k-1}, \theta^{k-1})$ was
defined for $k \geq 1$, let $(\rho^k, u^k, \omega^k, \theta^k)$ be the solution to (72) with $(v, \tilde{\theta})$ replaced by $(u^{k-1}, \theta^{k-1})$. Then it also follows from that there exists a constant $C > 1$ such that
\[
\sup_{0 \leq t \leq T_1} \left( \|\rho^k(t)\|_{H^1} + \|u^k(t)\|_{H^1} + \|\omega^k(t)\|_{H^1} + \|\theta^k\|_{H^1}\right) \\
+ \|\omega_{t}^k\|_{H^1} + \|u_{t}^k\|_{H^1} + \|\theta_{t}^k\|_{H^1}) \\
+ \int_{0}^{T_1} \left(\|\rho_{tt}^k\|_{L^2}^2 + \|u_{tt}^k\|_{L^2}^2 + \|\omega_{tt}^k\|_{L^2}^2 + \|\omega_{\xi}^k\|_{L^2}^2 + \|\theta_{tt}^k\|_{L^2}^2 + \|\theta_{\xi}^k\|_{L^2}^2\right) \\
\leq C.
\] (81)

From now on, we show that the full sequence $(\rho^k, u^k, \theta^k, \omega^k)$ converges to a solution to the original nonlinear problem (1) in a strong sense. Let us define
\[
\tilde{\rho}^{k+1} = \rho^{k+1} - \rho^k, \quad \tilde{u}^{k+1} = u^{k+1} - u^k, \quad \tilde{\theta}^{k+1} = \theta^{k+1} - \theta^k, \quad \tilde{\omega}^{k+1} = \omega^{k+1} - \omega^k.
\] Then from (1), we derive the equations for the differences
\[
\tilde{\rho}_t^{k+1} + (\tilde{\rho}^{k+1} u^k)_x + (\tilde{\rho}^k u^k)_x = 0, \quad (82)
\]
\[
\tilde{\rho}^{k+1} u_{t}^{k+1} + \rho^{k+1} u \tilde{u}_{x}^{k+1} - \tilde{u}_{xx}^{k+1} \\
= \tilde{\rho}^{k+1} (-u^{k+1} - u^k) - \rho^{k+1} \tilde{\omega}^k u^k - R(\rho^{k+1} \tilde{\theta}^k - \tilde{\rho}^{k+1} \theta^k)_x, \quad (83)
\]
\[
\tilde{\rho}^{k+1} \tilde{\theta}^k_t + \rho^{k+1} \tilde{\theta}^k_x + \rho^{k+1} (-u^{k+1} \tilde{\omega}^k - R \tilde{\theta}^k u^{k+1}) \\
- \tilde{\rho}^{k+1} (-\tilde{\omega}^k + R \tilde{\theta}^k u^k + R \tilde{\theta}^k \tilde{u}^k) + (\omega_{x}^{k+1})^2 - (\omega_{x}^k)^2 + (\omega^{k+1})^2 - (\omega^k)^2, \quad (84)
\]
\[
\begin{align*}
\tilde{\rho}^{k+1} \tilde{\omega}_{t}^{k+1} + \rho^{k+1} \tilde{\omega}_{x}^{k+1} + A \tilde{\omega}^{k+1} - A \tilde{\omega}_{x}^{k+1} \\
= \tilde{\rho}^{k+1} (-\tilde{\omega}^k - u^{k+1} \tilde{\omega}^k) - \rho^{k+1} \tilde{\omega}^k \omega^k. \quad (85)
\end{align*}
\]

Multiplying (82) by $\tilde{\rho}^{k+1}$ and integrating over $\Omega$, we obtain
\[
\frac{d}{dt} \int |\tilde{\rho}^{k+1}|^2 dx \leq C \int (|u_{x}^k||\tilde{\rho}^{k+1}|^2 + |\rho^k||\tilde{u}^k||\tilde{\rho}^{k+1}| + |\rho^k||\tilde{u}^k|^2)|\tilde{\rho}^{k+1}| dx \\
\leq C (||u^k||_{L^\infty} \|\tilde{\rho}^{k+1}\|_{L^2}^2 + (||\rho^k||_{L^\infty} + ||\rho^k||_{L^\infty}^2)) ||\tilde{u}^k||_{L^2} \|\tilde{\rho}^{k+1}\|_{L^2},
\] (86)

Hence, by virtue of Young’s inequality, we have
\[
\frac{d}{dt} \|\tilde{\rho}^{k+1}\|_{L^2}^2 \leq A_k^h(t) \|\tilde{\rho}^{k+1}\|_{L^2}^2 + \eta ||\tilde{u}^k||_{L^2}^2, \quad (87)
\]
where $A_k^h(t) = C ||u_{x}^k(t)||_{L^\infty} + \eta C (||\rho^k||_{L^\infty}^2 + ||\rho^k(t)||_{L^\infty}^2)$. Notice from estimate (81) that $A_k^h(t) \in L^1(0, T_1)$ and $\int_{0}^{T_1} A_k^h(s) dt \leq C + C_\eta t$ for all $k \geq 1$ and $t \in [0, T_1]$.

Multiplying (85) by $\tilde{\omega}^{k+1}$, integrating over $I$ and recalling that
\[
(\rho^{k+1})_x + (\rho^{k+1} u^k)_x = 0.
\] (88)
we obtain
\[
\frac{1}{2} \frac{d}{dt} \int \rho^{k+1} |\tilde{\omega}^{k+1}|^2 dx + A \int |\tilde{\omega}^{k+1}|^2 dx + A \int |\tilde{\omega}_{x}^{k+1}|^2 dx \\
\leq C \int ([|\tilde{\rho}^{k+1}| + |u^{k-1}||\omega^k|] |\tilde{\omega}^{k+1}| + |\rho^{k+1}||\tilde{u}^k||\omega^k| |\tilde{\omega}^{k+1}|) dx. \quad (89)
\]
Then it follows from (81) that
\[ \frac{d}{dt} \int \rho^{k+1} \bar{\omega}^{k+1} |^2 dx + A \int |\bar{\omega}_x^{k+1} |^2 dx + A \int |\bar{\omega}_x^{k+1} |^2 dx \leq B_\rho^k(t) \left( \| \bar{\rho}^{k+1} \|_{L^2}^2 + \| \sqrt{\rho^{k+1} \bar{\omega}^{k+1} } \|_{L^2}^2 \right) + \eta \| u_x^k \|_{L^2}^2. \]  

(90)

Multiplying (84) by \( \bar{\theta}^{k+1} \) and integrating over I, we have
\[
\frac{1}{2} \frac{d}{dt} \int \rho^{k+1} |\bar{\theta}^{k+1} |^2 dx + C_1 \int |\bar{\theta}_x^{k+1} |^2 dx 
\leq C \int \left( |u_x^k| + |u_x^{k-1}| \right) |\bar{\theta}^{k+1} |^2 dx + |\theta^{k+1} | |\bar{\theta}_x^{k+1} |^2 dx + |\bar{\theta}^{k+1} | |u_x^k| |\bar{\theta}_x^{k+1} |^2 dx + |\bar{\theta}_x^{k+1} | |\bar{\theta}^{k+1} | |u_x^k| |\bar{\theta}_x^{k+1} |^2 dx 
\leq C \int |\bar{\theta}^{k+1} |^2 dx + C \int |\bar{\theta}_x^{k+1} |^2 dx + C \int |\bar{\theta}^{k+1} |^2 dx + C \int |\bar{\theta}^{k+1} |^2 dx + C \int |\bar{\theta}_x^{k+1} |^2 dx + C \int |\bar{\theta}^{k+1} |^2 dx + C \int |\bar{\theta}_x^{k+1} |^2 dx 
\leq C \int |\bar{\theta}^{k+1} |^2 dx + C \int |\bar{\theta}_x^{k+1} |^2 dx + C \int |\bar{\theta}^{k+1} |^2 dx + C \int |\bar{\theta}^{k+1} |^2 dx + C \int |\bar{\theta}_x^{k+1} |^2 dx + C \int |\bar{\theta}^{k+1} |^2 dx + C \int |\bar{\theta}_x^{k+1} |^2 dx. 
\]  

Hence it follows from (81) that
\[
\frac{d}{dt} \| \sqrt{\rho^{k+1} \bar{\theta}^{k+1} } \|_{L^2}^2 + C_1 \| \bar{\theta}_x^{k+1} |^2 dx 
\leq C \left( \| \bar{\rho}^{k+1} \|_{L^2}^2 + \| \sqrt{\rho^{k+1} \bar{\theta}^{k+1} } \|_{L^2}^2 + \| \sqrt{\rho^{k+1} \bar{\theta}^{k+1} } \|_{L^2}^2 \right) 
+ C \left( \| u_x^k \|_{L^2}^2 + \| \bar{\theta}_x \|_{L^2}^2 + \| \bar{\theta} \|_{L^2}^2 \right). 
\]  

(92)

Finally, multiplying (83) by \( u_t^{k+1} \) and integrating over I, we have
\[
\frac{1}{2} \frac{d}{dt} \int \rho^{k+1} |u_t^{k+1} |^2 dx + \int |u_x^{k+1} |^2 dx 
\leq C \int \left( |u_x^{k+1} | + |u_x^{k-1} u_x^k \right) |u_t^{k+1} | + \| \rho^{k+1} | u_x^k \| |u_x^{k+1} | 
+ \| (\rho^{k+1} | \bar{\theta}^{k+1} | + |\rho^{k+1} | \theta^{k+1} | |u_x^{k+1} | | \bar{\theta}_x^{k+1} | \right) dx. 
\]  

(93)

Then it also follows from (81) that
\[
\frac{d}{dt} \| \sqrt{\rho^{k+1} u^{k+1} } \|_{L^2}^2 + \| u_t^{k+1} \|_{L^2}^2 \leq D_\rho^k(t) \| \rho^{k+1} \|_{L^2}^2 + \| \sqrt{\rho^{k+1} \bar{u}^{k+1} } \|_{L^2}^2 
+ C \| \sqrt{\rho^{k+1} \bar{u}^{k+1} } \|_{L^2}^2 + \eta \| u_{xx}^k \|_{L^2}^2. 
\]  

(94)

for some \( D_\rho^k(t) \in L^1(0, T_1) \) such that \( \int_0^t D_\rho^k(s) ds \leq C + Ct \) for \( 0 \leq t \leq T_1 \) and \( k \geq 1 \).
Therefore, combing (87)-(94) and defining $\psi^{k+1}(t) = \|\rho^{k+1}\|_{L^2}^2 + \|\sqrt{\rho^{k+1}}\omega^{k+1}\|_{L^2}^2 + \frac{1}{C_1}\|\sqrt{\rho^{k+1}}\theta^{k+1}\|_{L^2}^2 + \|\sqrt{\rho^{k+1}}u^{k+1}\|_{L^2}^2$ We deduce that

$$\frac{d}{dt}\psi^{k+1} + A\|\omega^{k+1}\|_{L^2}^2 + \|\theta^{k+1}\|_{L^2}^2 + \|\tilde{u}^{k+1}\|_{L^2}^2 + A\|\tilde{\omega}^{k+1}\|_{L^2}^2$$

$$\leq E_\eta(t)(\psi^{k+1} + \frac{\eta}{C_1}\|\sqrt{\rho^2}\theta^{k+1}\|_{L^2}^2 + C(4\eta\|\tilde{u}^{k+1}\|_{L^2}^2 + \eta\|\tilde{\omega}^{k+1}\|_{L^2}^2 + \eta\|\tilde{\omega}^{k+1}\|_{L^2}^2)$$

(95)

for some $E_\eta(t) \in L^1(0, T_1)$ such that $\int_0^t E_\eta(s)ds \leq C + C_\eta t$ for $0 \leq t \leq T_1$ and $k \geq 1$.

Now considering for any $n > 1$, we have

$$\frac{d}{dt}\left(\sum_{k=1}^n \psi^{k+1}(t) + \sum_{k=1}^n \int_0^t (\eta\|\theta^{k+1}\|_{L^2}^2 + \|\tilde{u}^{k+1}\|_{L^2}^2 + A\|\omega^{k+1}\|_{L^2}^2 + A\|\tilde{\omega}^{k+1}\|_{L^2}^2)ds \right)$$

$$\leq C\sum_{k=1}^n \psi^{k+1}(t) + \sum_{k=1}^n C(4\eta\|\tilde{u}^{k+1}\|_{L^2}^2 + \eta\|\tilde{\omega}^{k+1}\|_{L^2}^2 + \eta\|\tilde{\omega}^{k+1}\|_{L^2}^2) + C\|\sqrt{\rho^2}\theta^{k+1}\|_{L^2}^2$$

(96)

Then recalling that $\psi^{k+1}(0) = 0$, (81) and using Gronwall’s inequality, we deduce from that

$$\sum_{k=1}^n \psi^{k+1}(t) + \sum_{k=1}^n \int_0^t (\eta\|\theta^{k+1}\|_{L^2}^2 + A\|\omega^{k+1}\|_{L^2}^2 + A\|\tilde{\omega}^{k+1}\|_{L^2}^2 + \|\tilde{u}^{k+1}\|_{L^2}^2)ds$$

$$\leq [C\eta\sum_{k=1}^n \int_0^t (4\|\tilde{u}^{k+1}\|_{L^2}^2 + \|\tilde{\omega}^{k+1}\|_{L^2}^2 + \|\tilde{\omega}^{k+1}\|_{L^2}^2) + C\|\sqrt{\rho^2}\theta^{k+1}\|_{L^2}^2)ds$$

$$\leq \eta\int_0^t (4\|\tilde{u}^{k+1}\|_{L^2}^2 + \|\tilde{\omega}^{k+1}\|_{L^2}^2 + \|\tilde{\omega}^{k+1}\|_{L^2}^2)ds \exp^{Ct}.$$  

Hence choosing small constant $\eta > 0$ and $T_3 > 0$ so that

$$4C\eta \exp(CT_3) = \frac{1}{2} \min\{1, A\}$$

we easily deduce that

$$\sum_{k=1}^n \sup_{0 \leq t \leq T_3} \psi^{k+1}(t) + \sum_{k=1}^n \int_0^{T_3} \|\omega^{k+1}\|_{L^2}^2 + \|\omega^{k+1}\|_{L^2}^2 + \eta\|\theta^{k+1}\|_{L^2}^2 + \|\tilde{u}^{k+1}\|_{L^2}^2 \leq \tilde{C} < \infty,$$

where $T^* = \min\{T_2, T_3\}$ and $\tilde{C}$ is independent on $n$. Then, we have

$$\sum_{k=1}^\infty \sup_{0 \leq t \leq T^*} \psi^{k+1}(t) + \sum_{k=1}^\infty \int_0^{T^*} \|\omega^{k+1}\|_{L^2}^2 + \|\omega^{k+1}\|_{L^2}^2 + \eta\|\theta^{k+1}\|_{L^2}^2 + \|\tilde{u}^{k+1}\|_{L^2}^2 \leq \tilde{C} < \infty,$$

Therefore, we conclude that the full sequence $(\rho^k, u^k, \omega^k, \theta^k)$ converges to a limit $(\rho, u, \omega, \theta)$ in the following strong sense:

$$\begin{cases}
\rho^k \to \rho \text{ in } L^\infty(0, T^*; L^2), \\
(\omega^k, \theta^k, u^k) \to (\omega, \theta, u) \text{ in } L^2(0, T^*; L^2).
\end{cases}$$

(97)
Furthermore, it follows from (81) that the limit \( (\rho, u, \omega, \theta) \) satisfies the following regularity estimate:

\[
\begin{align*}
\sup_{0 \leq t \leq T^*} \left( \|\rho\|_{H^2} + \|\rho_t\|_{H^1} + \|\omega\|_{H^3 \cap H^1_0} + \|\omega_t\|_{H^1_0} + \|\theta\|_{H^3} 
\right. \\
+ \|\theta_t\|_{H^1} + \|u\|_{H^3 \cap H^1_0} + \|u_t\|_{H^1_0} \\
+ \int_0^{T^*} \left( \|\rho_t\|_{L^2}^2 + \|\omega_t\|_{L^2}^2 + \|\omega_{tt}\|_{L^2}^2 + \|\theta_t\|_{L^2}^2 + \|\theta_{tt}\|_{L^2}^2 + \|u_t\|_{L^2}^2 + \|u_{tt}\|_{L^2}^2 \right) dt \\
\left. \leq C. \right)
\end{align*}
\]

(98)

It is now easy to show that \( (\rho, u, \omega, \theta) \) is a classical solution to the original nonlinear problem (1). This proves the local existence of a classical solution of Theorem 3.1.

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