1 Introduction

Let $M$ be an $n$-dimensional complex projective manifold, $\tilde{G}$ a $g$-dimensional reductive connected complex Lie group, and $\nu : \tilde{G} \times M \to M$ a holomorphic action. A $\tilde{G}$-line bundle $(L, \tilde{\nu})$ on $M$ will mean the assignment of a holomorphic line bundle $L$ on $M$ together with a lifting (a linearization) $\tilde{\nu} : \tilde{G} \times L \to L$ of the action of $\tilde{G}$ (to simplify notation, we shall generally leave $\tilde{\nu}$ understood, and denote a $\tilde{G}$-line bundle by $A, B, L, \ldots$).

If $L$ is a $\tilde{G}$-line bundle, there is for every integer $k \geq 0$ an induced representation of $\tilde{G}$ on the complex vector space of holomorphic sections $H^0(M, L^{\otimes k})$. This implies a $\tilde{G}$-equivariant direct sum decomposition

$$H^0(M, L^{\otimes k}) \cong \bigoplus_{\mu \in \Lambda_+} H^0_\mu(M, L^{\otimes k}),$$

where $\Lambda_+$ is the set of dominant weights for a given choice of a maximal torus $\tilde{T} \subseteq \tilde{G}$ and of a fundamental Weyl chamber. For every dominant weight $\mu$, let us denote the associated irreducible finite dimensional representation of $\tilde{G}$ by $V_\mu$. For every $\mu \in \Lambda_+$, the summand $H^0_\mu(M, L^{\otimes k})$ is $\tilde{G}$-equivariantly isomorphic to a direct sum of copies of $V_\mu$.

Suppose that the line bundle $L$ is ample. We shall address in this paper the asymptotic growth of the dimensions

$$h^0_\mu(M, L^{\otimes k}) = \dim (H^0_\mu(M, L^{\otimes k}))$$

of the equivariant spaces of sections $H^0_\mu(M, L^{\otimes k})$, for $\mu$ fixed and $k \to +\infty$. This problem has been the object of much attention over the years, and has been approached both algebraically \cite{BE, BD} and symplectically - in the
latter sense it is part of the broad and general picture revolving around the quantization commutes with reduction principle \([\text{GS1}]\), \([\text{GGK}]\), \([\text{MS}]\), \([\text{S}]\).

In this paper, we shall study this problem under the general assumption that the stable locus of \(L\) is non-empty, \(M^s(L) \neq \emptyset\). Thus the most general result in our setting is now the Riemann-Roch type formula proved in \([\text{MS}]\). This is a deep and fundamental Theorem - with a rather complex symplectic proof. However, we adopt a different, more algebro-geometric approach, taking as point of departure the Guillemin-Sternberg conjecture for regular actions (in the sense of \(\S\)2 below). Our motivation was partly to understand the leading asymptotics for singular actions by fairly elementary algebro-geometric arguments. Besides the hypothesis that \(M^s(L) \neq \emptyset\), our arguments need an additional technical assumption, namely that the pair \((M, L)\) admits a Kirwan resolution with certain mildness properties. Roughly, every divisorial component of the unstable locus upstairs should map to the unstable locus downstairs (Definition 1); such resolutions will be called mild.

One can produce examples showing that \(h^0_\mu(M, L \otimes k)\) may not be described, in general, by an asymptotic expansion, even if the GIT quotient \(M/\tilde{G}\) is nonsingular (but see \(\S\)2 below and the discussion in \([P2]\)). Inspired by the notion of volume of a big line bundle \([\text{DEL}]\), we shall then introduce and study the \(\mu\)-equivariant volume of a \(G\)-line bundle \(L\), defined as

\[
v_\mu(L) =: \limsup_{k \to +\infty} \frac{(n-g)!}{k^{n-g}} h^0_\mu(X, L \otimes k).
\]  

Because it leads to a concise and simple statement, we shall focus on the special case where the stabilizer \(K \subseteq \tilde{G}\) of a general \(p \in M\) is a (necessarily finite) central subgroup. Our methods can however be applied with no conceptual difficulty to the case of an arbitrary principal type (the conjugacy class of the generic stabilizer). This will involve singling out for each \(\mu\) the kernel \(K_\mu \subseteq K\) of the action of \(K\) on the coadjoint orbit of \(\mu\), and considering the contribution coming from each conjugacy class of \(K_\mu\).

Let us then assume that \(K\) is a central subgroup. By restricting the linearization to \(K\), we obtain an induced character \(\chi_{K,L} : K \to \mathbb{C}^*\).

Another character \(\tilde{\mu}_K : K \to \mathbb{C}^*\) is associated to the choice of a dominant weight \(\mu \in \lambda_+ \subseteq t^*\). Namely, \(K\) lies in the chosen maximal torus, and \(\tilde{\mu}_K\) is the restriction to \(K\) of the character \(\tilde{\mu} : \tilde{T} \to \mathbb{C}^*\) induced by \(\mu\) by exponentiation, \(\exp_{\tilde{G}}(\xi) \mapsto e^{2\pi i \langle \mu, \xi \rangle}\). We may define \(\tilde{\mu}_K =: \tilde{\mu}|_K : K \to \mathbb{C}^*\) (the restriction of \(\tilde{\mu}\) to \(K\)). An alternative description of \(\tilde{\mu}_K\) is as follows: Let \(G \subseteq \tilde{G}\) be a maximal compact subgroup, \(T \subseteq G\) a maximal torus, and suppose \(\mu \in t^*\), where \(t^*\) is the Lie algebra of \(T\). If \(g\) is the Lie algebra of \(G\), let \(O_\mu \subseteq g^*\) be the coadjoint orbit of \(\mu\). Since \(\mu\) is an integral weight, the
natural Kähler structure on $O_\mu$ is in fact a Hodge form, that is, it represents an integral cohomology class. The associated ample holomorphic line bundle $A_\mu \to O_\mu$ is a $G$-line bundle in a natural manner. Since the action of $K$ is trivial on $O_\mu$, the linearization induces the character $\tilde{\mu}_K$ on $K$.

We then have:

**Theorem 1.** Let $M$ be a complex projective manifold, $\tilde{G}$ a reductive complex Lie group, $\nu : \tilde{G} \times M \to M$ a holomorphic action. Suppose for simplicity that the stabilizer of a general $p \in M$ is a central subgroup $K \subseteq \tilde{G}$. Let $L$ be an ample $\tilde{G}$-line bundle on $M$ such that $M^s(L) \neq \emptyset$ and admitting a mild Kirwan resolution. Let $M_0 = : M/\!/\tilde{G}$ be the GIT quotient with respect to the linearization $L$. Let $\mu \in \Lambda_+$ be a dominant weight. Let $\chi_{K,L}, \tilde{\mu}_K : K \to \mathbb{C}^*$ be the characters introduced above. Then:

i): If for every $r = 1, \ldots, |K|$ we have $\chi_r \cdot \tilde{\mu}_K \neq 1$ (the constant character equal to 1), then $H^0_\mu(M, L^\otimes k) = 0$ for every $k = 1, 2, \ldots$;

ii): Assume that for some $r \in \{1, \ldots, |K|\}$ we have $\chi_r \cdot \tilde{\mu}_K = 1$. Then

$$v_\mu(L) = \dim(V_\mu)^2 \cdot \text{vol}(\tilde{M}_0, \tilde{\Omega}_0) > 0.$$ 

Here $\tilde{M}_0$ is an orbifold, $\varphi : \tilde{M}_0 \to M_0$ is a partial resolution of singularities, $L$ induces on $\tilde{M}_0$ a natural nef and big line-orbibundle $\tilde{L}_0$, with first Chern class $c_1(\tilde{L}_0) = \left[\tilde{\Omega}_0\right]$, and

$$\text{vol}(\tilde{M}_0, \tilde{\Omega}_0) = : \int_{\tilde{M}_0} \tilde{\Omega}_0^{n-g}.$$ 

2 The case $M^s(L) = M^{ss}(L) \neq \emptyset$.

Let us begin by considering the case where the stable and semistable loci for the $\tilde{G}$-line bundle are equal and nonempty: $M^s(L) = M^{ss}(L) \neq \emptyset$. First, as we shall make use of the Riemann-Roch formulae for multiplicities for regular actions conjectured by Guillemin and Sternberg, and first proved by Meinrenken [M1], [M2], it is in order to recall how these algebro-geometric hypothesis translate in symplectic terms. Let us choose a maximal compact subgroup $G$ of $\tilde{G}$. Thus $G$ is a $g$-dimensional real Lie group, and $\tilde{G}$ is the complexification of $G$. Let $\mathfrak{g}$ denote the Lie algebra of $G$. We may without loss choose a $G$-invariant Hermitian metric $h_L$ on $L$, such that the unique covariant derivative on $L$ compatible with $h_L$ and the holomorphic structure has curvature $-2\pi i\Omega$, where $\Omega$ is a $G$-invariant Hodge form on $M$. The
given structure of $G$-line bundle of $L$, furthermore, determines (and, up to topological obstructions, is equivalent to) a moment map $\Phi = \Phi_L : M \to \mathfrak{g}^*$ for the action of $G$ on the symplectic manifold $(M, \Omega)$ \cite{GS1}. The hypothesis that $M^s(L) = M^s(L) \neq \emptyset$ may be restated symplectically as follows: $0 \in \mathfrak{g}^*$ is a regular value of $\Phi$, and $\Phi^{-1}(0) \neq \emptyset$ \cite{Ki1}. In this case, $P =: \Phi^{-1}(0)$ is a connected $G$-invariant codimension $g$ submanifold of $M$.

Let $G$ and $\Phi$ be as above. The action of $G$ on $\Phi^{-1}(0)$ is locally free, and the GIT quotient $M/\tilde{G} = M^s(L)/\tilde{G}$ may be identified in a natural manner with the symplectic reduction $M_0 =: \Phi^{-1}(0)/G$, and is therefore a Kähler orbifold. The quantizing line bundle $L$ descends to a line orbibundle $L_0$ on $M_0$.

Similar considerations apply to symplectic reductions at coadjoint orbits sufficiently close to the origin. If the $G$-line bundle $L$ is replaced by its tensor power $L^\otimes k$, the Hodge form and the moment map get replaced by their multiples $k\Omega$ and $\Phi_k := k\Phi$. Given any $\mu \in \mathfrak{g}^*$, there exists $k_0$ such that $\mu$ is a regular value of $\Phi_k$ if $k \geq k_0$. The relevant asymptotic information about the multiplicity of $V_\mu$ in $H^0(M, L^\otimes k)$ may then be determined by computing appropriate Riemann-Roch numbers on these orbifolds \cite{Ka}, \cite{M2}.

Let $O_\mu \subseteq \mathfrak{g}^*$ be the coadjoint orbit through $\mu$; since $\mu$ is integral, the Kirillov symplectic form $\sigma_\mu$ is a Hodge form on the complex projective manifold $O_\mu$. By the Konstant version of the Borel-Bott theorem, there is an ample line bundle $A_\mu$ on $O_\mu$ such that $H^0(O_\mu, A_\mu)$ is the irreducible representation of $G$ with highest weight $\mu$.

Let then $M^{(k)}_\mu$ be the Weinstein symplectic reduction of $M$ at $\mu$ with respect to the moment map $\Phi_k = k\Phi_L$ ($k \gg 0$). Using the normal form description of the symplectic and Hamiltonian structure of $(M, \Omega)$ in the neighbourhood of the coisotropic submanifold $P = \Phi^{-1}(0)$ \cite{M2}, \cite{G}, \cite{GS3}, one can verify that $M^{(k)}_\mu$ is, up to diffeomorphism, the quotient of $P \times O_\mu$ by the product action of $G$. In other words, $M^{(k)}_\mu$ is the fibre orbibundle on $M_0 = P/G$ associated to the principal $G$-orbibundle $q : P \to M_0$ and the $G$-space $O_\mu$ (endowed with the opposite Kähler structure); in particular, its diffeotype is independent of $k$ for $k \gg 0$. Let $p_\mu : M^{(k)}_\mu \to M_0$ be the projection.

Let $\theta$ be a connection 1-form for $q$ \cite{GKK}, Appendix B). By the shifting trick, the symplectic structure $\Omega^{(k)}_\mu$ of the orbifold $M^{(k)}_\mu$ is obtained by descending the closed 2-form $k! \iota^* (\Omega^+) < \mu, F(\theta) > - \sigma_\mu$ on $P \times O_\mu$ down to the quotient (the symbols of projections are omitted for notational simplicity). The minimal coupling term $< \mu, F(\theta) > - \sigma_\mu$ is the curvature of the line orbibundle $R_\mu = (P \times A_\mu)/G$ on $M^{(k)}_\mu$. Thus, $\Omega^{(k)}_\mu$ is the curvature form of the line orbibundle $p_\mu^*(L_0^\otimes k) \otimes R_\mu$.

4
Let
\[ \tilde{P}_\mu =: \{(p, \mu', g) \in P \times O_\mu \times G : g \cdot (p, \mu') = (p, \mu')\}, \]
\[ \tilde{P}_{\mu,K} =: P \times O_\mu \times K. \]

There is a natural inclusion \( \tilde{P}_{\mu,K} \subseteq P_\mu \). Now let \( \Sigma_\mu =: \tilde{P}_\mu / G \), \( \Sigma_{\mu,K} =: \tilde{P}_{\mu,K} / G = M^{(k)}_\mu \times K \). There is a natural orbifold complex immersion \( \Sigma_\mu \rightarrow M^{(k)}_\mu \), with complex normal orbi-bundle \( N_{\Sigma_\mu} \), and \( \Sigma_{\mu,K} \subseteq \Sigma_\mu \) is the union of the \( |K| \) connected components mapping dominantly (and isomorphically) onto \( M^{(k)}_\mu \). The orbifold multiplicity of \( \Sigma_{\mu,K} \) is constant and equal to \( |K| \).

Let \( L_0 \) be the line orbi-bundle on \( M_0 \) determined by descending \( L \), and let \( \tilde{L}_0 \) be its pull-back to \( \Sigma_0 \). Let \( r \) be the complex dimension of \( O_\mu \), so that \( \dim M^{(k)}_\mu = n - g + r \). After \([\text{M1}]\) and \([\text{M2}]\), the multiplicity \( N^{(k)}(\mu) \) of the irreducible representation \( V_\mu \) in \( H^0(M, L^{(k)}) \) is then given by:

\[
N^{(k)}(\mu) = \int_{\Sigma_0} \frac{1}{d_{\Sigma_\mu}} \frac{Td(\Sigma_\mu) Ch^{\Sigma_\mu}(p_\mu^*(L_0^{(k)}) \otimes R_{\mu})}{D^{\Sigma_\mu}(N_{\Sigma_\mu})} \]
\[
= k^{n-g} \sum_{h \in K} \chi_{K,L}(h)^k \overline{\mu_K(h)} \int_{M^{(k)}_\mu} \frac{(k c_1(L_0) + c_1(R_{\mu}))^{n-g+r}}{(n-g+r)!} \]
\[
+ O(k^{n-g-1}).
\]

Now suppose that \( \chi_{K,L} \cdot \bar{\mu}_K \neq 1 \). Then the action of \( K \) on \( j^*(L^{(k)}) \otimes \overline{A}_\mu \) is not trivial, where \( j : P \hookrightarrow M \) is the inclusion. Therefore, the fiber of \( p_\mu^*(L_0^{(k)}) \otimes R_{\mu} \) on the smooth locus of \( M_0 \) is a nontrivial quotient of \( \mathbb{C} \), and \( N^{(k)}(\mu) = 0 \) in this case. If there exists \( k \) such that \( \chi_{K,L} \cdot \bar{\mu}_K \equiv 1 \), on the other hand, the same condition holds with \( k \) replaced by \( k + \ell e \), where \( e \) is the period of \( \chi_{K,L} \) and \( \ell \in \mathbb{Z} \) is arbitrary. Thus \( k \) may be assumed arbitrarily large. Passing to the original Kähler structure of \( O_\mu \) in the computation, and recalling that \( \dim(V_\mu) = (r!)^{-1} \int_{O_\mu} \sigma^r \), we easily obtain:

\[
N^{(k)}(\mu) = \dim(V_\mu) \frac{k^{n-g}}{(n-g)!} \int_{M_0} c_1(L_0)^{(n-g)} + O(k^{n-g-1}).
\]

3 The asymptotics of equivariant volumes.

Let \( f : \tilde{M} \rightarrow M \) be a Kirwan desingularization of the action \([\text{K1}]\). This means that \( f \) is a \( G \)-equivariant birational morphism, obtained as a sequence of blow-ups along \( G \)-invariant smooth centers, such that for all \( a \gg 0 \) the ample \( G \)-line bundle \( B =: f^*(L^{(a)})(-E) \) satisfies \( M^a(B) = M^{as}(B) \supseteq f^{-1}(M^a(L)) \). Here \( E \subseteq \tilde{M} \) is an effective exceptional divisor for \( f \). Clearly \( v_\mu(F) = v_\mu(f^*(F)) \) for every \( G \)-line bundle \( F \) on \( M \).
Definition 1. Let $M_u(B)_{\text{div}} \subseteq M_u(B)$ be the divisorial part of the unstable locus of $B$; in other words, $M_u(B)_{\text{div}}$ is the union of the irreducible components of $M_u(B)$ having codimension one in $	ilde{M}$. We shall say that the Kirwan resolution $f$ is mild if $f(M_u(B)_{\text{div}}) \subseteq M_u(L)$.

We have:

Theorem 2. Let $L$ be an ample $G$-line bundle on $M$ such that $M^s(L) \neq \emptyset$. Suppose that $f : \tilde{M} \to M$ is a mild Kirwan resolution of $(M, L)$. Let $H$ be a $G$-line bundle on $\tilde{M}$ such that $\chi_{K,H} = 1$. Let $\mu \in \Lambda_+$ be a dominant weight. Then for any $\epsilon > 0$ there exist arbitrarily large positive integers $m$ (how large depending on $\epsilon$ and $\mu$) such that

$$v_\mu(f^*(L)^{\otimes m} \otimes H^{-1}) \geq m^n (v_\mu(f^*(L)) - \epsilon).$$

More precisely, this will hold whenever $m = 1 + pe$, where $e$ is the period of $\chi_{K,L}$ and $p \in \mathbb{N}$, $p \gg 0$.

As a corollary, we obtain the following equivariant version of Lemma 3.5 of [DEL] (for the case of finite group actions, see Lemma 3 of [PI]).

Corollary 1. Under the same hypothesis, let $H$ be a $G$-line bundle on $M$. Then for any $\epsilon > 0$ there exist arbitrarily large integers $m > 0$ (how large depending on $\epsilon$ and $\mu$) such that $v_\mu(L^{\otimes m} \otimes H^{-1}) \geq m^n (v_\mu(L) - \epsilon)$.

By definition, $v_\mu(L) \geq v_\mu(L^{\otimes m})/m^{n-g}$ for every $m > 0$. Thus Corollary 1 with $H = \mathcal{O}_M$ implies:

Corollary 2. Let $L$ be a $G$-line bundle with $M^s(L) \neq \emptyset$. Then

$$v_\mu(L) = \limsup_{m \to +\infty} \frac{v_\mu(L^{\otimes m})}{m^{n-g}}.$$

Similarly,

Corollary 3. Under the same hypothesis,

$$v_\mu(L) = \limsup_{m \to +\infty} \frac{v_\mu(f^*(L^{\otimes m})(-E))}{m^{n-g}}.$$
Lemma 1. Fix $k$ with $s_r \geq 0$ and perhaps after passing to a subsequence, we may assume without loss of generality that if $\nu \equiv r \pmod{e}$, then $\nu \equiv r' \pmod{m}$, for a fixed $0 \leq r' \leq m - 1$. Thus, $s_\nu = e_k m + r'$. Let now $x > 0$ be an integer of the form $x = sm + r' - e$, $s \in \mathbb{N}$. Then $xpe = x(m-1)$ and $s_\nu + xpe = (\ell \nu + x)m + r' - x = (\ell \nu + x - s)m + e$.

**Proof of Theorem 2.** The proof is inspired by arguments in [DEL]. If $v_\mu(L) = 0$, there is nothing to prove; we shall assume from now on that $v_\mu(L) > 0$, and for simplicity write $L$ for $f^*(L)$. Thus there exists $0 \leq r < e$ such that $\chi_{K,L} \cdot \tilde{\mu}_K = 1$, where $e$ is the period of $\chi_{K,L}$. If $\ell \gg 0$, by the above $H^0(\tilde{M}, B^{\otimes \ell}e)^g \neq 0$, so that $v_\mu(L^{\otimes m} \otimes H^{-1}) \geq v_\mu(L^{\otimes m} \otimes H^{-1} \otimes B^{\otimes \ell e})$. Thus there is no loss of generality in replacing $H$ by $H \otimes B^{\otimes \ell e}$ for some $\ell \gg 0$. In view of the hypothesis on $H$, we may thus assume without loss of generality that $0 \neq \sigma \in H^0(\tilde{M}, H)^G \neq \{0\}$, with invariant divisor $D \in \left| H^0(\tilde{M}, H)^G \right|$.

Since furthermore the class of $B$ in the $G$-ample cone introduced in [DH] lies in the interior of some chamber, the class of $H \otimes B^{\otimes \ell e}$ lies in the same chamber for $\ell \gg 0$. Hence we may as well assume that $H$ is a very ample $G$-line bundle satisfying $M^s(H) = M^s(B) = M^s(H) \neq \emptyset$.

By definition of $v_\mu$, there exists a sequence $s_\nu \uparrow +\infty$ such that

$$h^0_\mu(M, L^{\otimes s_\nu}) \geq \frac{s_\nu^{n-g}}{(n-g)!} \left( v_\mu(L) - \frac{\epsilon}{3} \right). \quad (5)$$

Necessarily $s_\nu \equiv r \pmod{e}, \forall \nu \gg 0$. We shall show that the stated inequality holds if $p \gg 0$ and $m =: 1 + pe$.

**Lemma 1.** Fix $p \gg 0$ and let $m =: 1 + pe$. There is a sequence $k_\nu \uparrow +\infty$ such that

$$h^0_\mu(M, L^{\otimes k_\nu}) \geq \frac{k_\nu^{n-g}}{(n-g)!} \left( v_\mu(L) - \frac{\epsilon}{2} \right), \quad (6)$$

with $k_\nu \equiv r \pmod{e}$ and furthermore $k_\nu \equiv e \pmod{m}$ for every $\nu$.

**Proof.** Let $x > 0$ be an integer. We may assume that there is a non-zero section $0 \neq \tau \in H^0(M, L^{\otimes xpe})^G$. Thus, there are injections

$$H^0_\mu(M, L^{\otimes s_\nu}) \hookrightarrow H^0_\mu(M, L^{\otimes (s_\nu + xpe)}),$$

and for $\nu \gg 0$ we have

$$h^0_\mu(X, L^{\otimes (s_\nu + xpe)}) \geq \frac{s_\nu^{n-g}}{(n-g)!} \left( v_\mu(L) - \frac{\epsilon}{3} \right) \geq \frac{(s_\nu + xpe)^{n-g}}{(n-g)!} \left( v_\mu(L) - \frac{\epsilon}{2} \right). \quad (7)$$

Perhaps after passing to a subsequence, we may assume without loss of generality that $s_\nu \equiv r' \pmod{m}$, for a fixed $0 \leq r' \leq m - 1$. Thus, $s_\nu = e_k m + r'$. Let now $x > 0$ be an integer of the form $x = sm + r' - e$, $s \in \mathbb{N}$. Then $xpe = x(m-1)$ and $s_\nu + xpe = (\ell \nu + x)m + r' - x = (\ell \nu + x - s)m + e$.
Now we need only set $k_\nu =: s_\nu + xpe$.

Set $\ell_\nu = \left[ \frac{kw}{m} \right]$. Thus $k_\nu = \ell_\nu m + e$.

**Lemma 2.** There exists $a > 0$ such that $H^0(\tilde{M}, H^{\otimes ma} \otimes L^{\otimes -a})^{G} \neq \{0\}$ for every $m \geq 1$.

**Proof.** If $r \gg 0$, the equivalence class of the $G$-line bundles $H^{\otimes re} \otimes L^{-e}$ in the $G$-ample cone lie in the interior of the same chamber as the class of $H$. Thus, they share the same stable and semistable loci, and determine the same genuine line bundles.

We may decompose $D_{\tilde{M}}$ for $0 \leq j < s$, to conclude inductively that

$$h_{\mu}(\tilde{M}, L^{\otimes k_\nu} \otimes H^{\otimes (-s)}) \geq h_{\mu}(\tilde{M}, L^{\otimes k_\nu}) - \sum_{0 \leq j < s} h_{\mu}(D, L^{\otimes k_\nu} \otimes H^{\otimes (-s)}|_D)$$

We may decompose $D$ as $D = D_u + D_s$, where $D_u, D_s \geq 0$ are effective divisors on $\tilde{M}$. $D_u$ is supported on the unstable locus of $B$, $M^u(B) \subseteq M$, and no irreducible component of $D_s$ is supported on $M^u(B)$. Let $D_u = \sum_j D_{uj}$ and $D_s = \sum_i D_{si}$ be the decomposition in irreducible components.

**Lemma 3.** If $D \in H^0(\tilde{M}, H)^{G}$ is general, then $D_s$ is reduced, and it is nonsingular away from the unstable locus $\tilde{M}^u(H) = M^u(B)$.

**Proof.** Perhaps after replacing $H$ by some appropriate power we may assume that the linear series $H^0(\tilde{M}, H)^{G}$ is base point free away from $\tilde{M}^u(H)$. The claim then follows from Bertini’s Theorem.
We now make use of the mildness assumption on $f$. If $m \gg 0$ is fixed, $\nu \gg 0$ and $0 \leq s \leq \ell + am$, then the moment map of the (not necessarily ample) line bundle $L^{\otimes k\nu} \otimes H^{\otimes -s}$ is bounded away from 0 in the neighbourhood of $D_u$. An adaptation of the arguments in §5 of [GS1] (applied on some resolution of singularities of each $D_{u_j}$) then shows that $h^0_\mu(D_u, L^{\otimes k\nu} \otimes H^{\otimes (-s)}|_{D_u}) = 0$ (if $D_u$ is not reduced, we need only filter $O_{D_{u_j}}(k_u L - s H)$ by an appropriate chain of line bundles).

Since furthermore on each $D_{s_j}$ we may find a non-vanishing invariant section of $H$, we obtain:

$$h^0_\mu(M, L^{\otimes k\nu} \otimes H^{\otimes (-s)}) \geq h^0_\mu(M, L^{\otimes k\nu}) - \sum_{0 \leq j < s} h^0_\mu(D_s, L^{\otimes k\nu} \otimes H^{\otimes (-s)} \otimes O_{D_s})$$

$$\geq h^0_\mu(M, L^{\otimes k\nu}) - sh^0_\mu(D_s, L^{\otimes k\nu} \otimes O_{D_s})$$

$$\geq h^0_\mu(M, L^{\otimes k\nu}) - sh^0_\mu(D_s, (L \otimes B)^{\otimes k\nu} \otimes O_{D_s})$$

**Proposition 1.** There exists $C > 0$ constant such that if $D \in \left[H^0(\tilde{M}, H)^G\right]$ is general then

$$h^0_\mu(D_s, (L \otimes B)^{\otimes k} \otimes O_{D_s}) \leq Ck^{n-g-1}$$

for every $k \gg 0$.

**Proof.** Given the equivariant injective morphism of structure sheaves $O_{D_s} \rightarrow \bigoplus_i O_{D_{s_i}}$, we may as well assume that $D_s$ is a reduced and irreducible $G$-invariant divisor, descending to a Cartier divisor $D_0$ on the quotient.

Let $R = L \otimes B$, with associated moment map $\Phi_R$. By the generality in its choice, we may assume that $D_s \in \left[H^0(\tilde{M}, H)^G\right]$ is non-singular in the neighbourhood of $\Phi^{-1}_R(0)$, and is transversal to it. In fact, the singular locus of $\left[H^0(\tilde{M}, H)^G\right]$ is the unstable locus of $H$. Furthermore, by the arguments of Lemma 3 in [P2] and compactness one can see the following: There exist a finite number of holomorphic embeddings $\varphi_i : B \rightarrow \tilde{M}$, where $B \subseteq \mathbb{C}^{n-g}$ is the unit ball, satisfying i): $\varphi_i(B) \subseteq \Phi^{-1}_R(0)$; ii): a submanifold of $\Phi^{-1}_R(0)$, $\varphi_i(B)$ is transversal to every $G$-orbit; iii): the union $\bigcup_i \varphi_i(B)$ maps surjectively onto $\tilde{M} / \tilde{G}$. In view the local analytic proof of Bertini’s theorem in [GH], we may assume that $D$ is transversal to each $\varphi_i(B)$. By $G$-invariance, it is then transversal to all of $\Phi^{-1}_R(0)$.

Let $g : \tilde{D}_s \rightarrow D_s$ be a $G$-equivariant resolution of singularities [EH], [EV]. For $s \gg 0$, $g^*(R^{\otimes s})(-F)$ is an ample $G$-line bundle on $\tilde{D}_s$, where $F$ is some effective exceptional divisor. Since $0 \in g^*$ is a regular value of $g \circ \Phi_R : \tilde{D}_s \rightarrow g^*$ and belongs to its image, the same holds for the moment.
map of $g^*(R^\otimes s)(-F)$, for $s \gg 0$. Having fixed $s \gg 0$, let us also choose $r \gg 0$ such that $H^0(\tilde{M}, g^*(R^\otimes sre)(-reF))^G \neq 0$. The choice of $0 \neq \sigma \in H^0(\tilde{D}_s, g^*(R^\otimes sre)(-reF))^G$ determines injections

$$H^0(\tilde{D}_s, R^\otimes k \otimes \mathcal{O}_{\tilde{D}_s}) \xrightarrow{\otimes \sigma \otimes k} H^0(\tilde{D}_s, R^\otimes k(1+sre)(-rekF) \otimes \mathcal{O}_{\tilde{D}_s}).$$

This implies the statement by the arguments in §2, since $\dim(\tilde{D}_s) = n-1$.

Given (5), (9) and Proposition 1, we get

$$h^0_\mu(\tilde{M}, L^\otimes k \otimes H^\otimes (-s)) \geq \frac{k^{n-g}_\nu}{(n-g)!} \left( v_\mu(L) - \frac{\ell}{2} \right) - s C k^{n-g-1}_\nu. \quad (11)$$

Now, in view of (8), we set $s = \ell_\mu + am$ to obtain:

$$h^0_\mu\left(\tilde{M}, \mathcal{O}_{\tilde{M}}(\ell_\mu(mL - H))\right) \geq \frac{k^{n-g}_\nu}{(n-g)!} \left( v_\mu(L) - \frac{\ell}{2} \right) - C(\ell_\nu + am)k^{n-g-1}_\nu$$

$$\geq \frac{k^{n-g}_\nu m^{n-g}}{(n-g)!} \left( v_\mu(L) - \frac{\ell}{2} \right) - C(\ell_\nu + am)(\ell_\nu + 1)^{n-g-1}m^{n-g-1}. \quad (12)$$

The proof of Theorem 2 follows by taking $\ell_\nu \gg m \gg 1$ (see [DEL], Lemma 3.5).

Remark 3.1. The arguments used in the proof of Theorem 2 may be applicable in other situations. For example, suppose that $L$ is a $G$-ample line bundle with $\text{vol}_0(L) > 0$; assume that the equivalence class of $L$ in the $G$-ample cone [DH] lies on a face of measure zero, and that - say - the $G$-ample line bundles in the interior of an adjacent chamber have unstable locus of codimension $\geq 2$. If $A$ is a $G$-ample line bundle in the interior of the chamber, one may apply the previous arguments to tensor powers of the form $L^\otimes k \otimes A$.

4 Proof of Theorem 1

Given the Kirwan resolution $f : \tilde{M} \to M$, for $m \gg 0$ the equivalence classes of the ample $G$-line bundles $f^*(L^\otimes m)(-E)$ in the $G$-ample cone of $\tilde{M}$ all lie to the interior of the same chamber. Therefore, they determine the same GIT quotient $\tilde{M}_0 = \tilde{M}///\tilde{G}$. The latter is an $(n-g)$-dimensional complex projective orbifold, which partially resolves the singularities of $M//\tilde{G}$ [K].

Being $G$-line bundles on $\tilde{M}$, $f^*(L)$ and $\mathcal{O}_{\tilde{M}}(-E)$ descend to line orb-bundles on $\tilde{M}_0$. Fixing $G$-invariant forms $\tilde{\Omega}$ and $\Omega_{-E}$ on $\tilde{M}$ representing the first Chern class of $f^*(L)$ and $\mathcal{O}_{\tilde{M}}(-E)$, we obtain forms $\tilde{\Omega}_0$ and $\Omega_{-E0}$ on $\tilde{M}_0$. 

10
By Corollary 3, we have
\[ v_\mu(L) = \limsup_{m \to +\infty} \frac{v_\mu(f^*(L^\otimes m)(-E))}{m^{n-g)}}. \]

By the results in §2, under the appropriate numerical hypothesis,
\[ v_\mu(f^*(L^\otimes m)(-E)) = \dim(V_\mu)^2 \cdot \text{vol}\left(\tilde{M}_0, m\tilde{\Omega}_0 + \Omega_{-E_0}\right). \]

The statement follows.

References

[B] M. Brion, *Sur les modules de covariants*, Ann. Sci. École Norm. Sup. 26 (1993), 1-21

[BD] M. Brion, J. Dixmier, *Comportement asymptotique des dimensions des covariants*, Bull. Soc. Math. Fr. 119 (1991), 217-230

[DEL] J.-P. Demailly, L. Ein, R. Lazarsfeld, *A subadditivity property of multiplier ideals*, Michigan Math. J. 48 (2000), 137-156

[DH] I. V. Dolgachev, Y. Hu, *Variation of geometric invariant theory quotients*. With an appendix by N. Ressayre. Inst. Hautes Études Sci. Publ. Math. 87 (1998), 5–56

[EH] S. Encinas, H. Hauser, *Strong resolution of singularities in characteristic zero*, Comment. Math. Helv. 77 (2002), no. 4, 821–845

[EV] S. Encinas, O. Villamayor, *A new proof of desingularization over fields of characteristic zero*, Proceedings of the International Conference on Algebraic Geometry and Singularities (Sevilla, 2001), Rev. Mat. Iberoamericana 19 (2003), no. 2, 339–353

[G] M. J. Gotay, *On coisotropic embeddings of presymplectic manifolds*, Proc. Am. Math. Soc. 84 (1982), 111-114

[GH] P. Griffiths, J. Harris, *Principles of algebraic geometry*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1994

[GGK] V. Guillemin, V. Ginzburg, Y. Karshon *Moment maps, cobordism, and Hamiltonian group actions*, Mathematical Surveys and Monographs 98, A.M.S. (2002)
[GS1] V. Guillemin, S. Sternberg, *Geometric quantization and multiplicities of group representations*, Inv. Math. **67** (1982), 515-538

[GS3] V. Guillemin, S. Sternberg, *Symplectic techniques in physics*, Cambridge University Press 1984

[Ka] T. Kawasaki, *The Riemann-Roch theorem for complex V-manifolds*, Osaka J. Mat. **16** (1979), 151-157

[Ki1] F. Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, Mathematical Notes **31**, Princeton University Press, Princeton, NJ, 1984

[Ki2] F. Kirwan, *Partial desingularizations of quotients of nonsingular varieties and their Betti numbers*, Ann. of Math. **122** (1985), 41-85

[M1] E. Meinrenken, *On Riemann-Roch formulas for multiplicities*, Journal of the A.M.S. **9** (1996), 373-389.

[M2] E. Meinrenken, *Symplectic surgery and the Spin$^c$-Dirac operator*, Adv. Math. **134** (1998), no. 2, 240-277.

[MS] E. Meinrenken, R. Sjamaar *Singular reduction and quantization*, Topology **38** (1999), no. 4, 699–762

[P1] R. Paoletti, *The asymptotic growth of equivariant sections of positive and big line bundles*, to appear in Rocky Mountain J. Math.

[P2] R. Paoletti, *Moment maps and equivariant Szegö kernels*, J. Symplectic Geom. **2** (2003), no. 1, 133 - 175,

[S] R. Sjamaar, *Symplectic reduction and Riemann-Roch formulas for multiplicities*, Bull. Amer. Math. Soc. (N.S.) **33** (1996), no. 3, 327–338