Noncritical Holographic QCD in External Electric Field

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Abstract

We investigate behavior of a noncritical model in external electric field and explore its phase structure in the quenched approximation $N_f \ll N_c$. We compute the conductivity of QCD plasma in this model and compare it with the predictions of Sakai-Sugimoto model, D3-D7 system and the lattice simulation. We find that, while the behavior of conductivity in noncritical model as a function of temperature and baryon density is similar to those of D3-D7 system, the phase diagram of noncritical model resembles the phase diagram of Sakai-Sugimoto model.
1 introduction

The AdS/CFT correspondence is a duality between the strongly coupled conformal field theories and string theory in a higher-dimensional Anti-de-Sitter space-time [1]. The generalization of AdS/CFT correspondence to more realistic gauge theories like QCD, provides new insights to understanding the dynamical non-perturbative effects in QCD, such as confinement, chiral symmetry breaking, color superconductivity and so on.

Dual gauge theories arising from brane constructions in ten-dimensional critical string theory are supersymmetric. In order to break supersymmetry, one may compactify the supersymmetric gauge theory on a circle of radius $R$ and impose anti-periodic boundary conditions for fermions around the circle. The resulting effective theory at low energy compared to the Kaluza-Klein mass scale, $M_{KK} \sim \frac{1}{R}$, is pure QCD without fundamental matter. Fundamental matter may be incorporated by adding flavor branes [2]. In principle, it is possible to extract the physics of strongly correlated QCD by using these holographic models. The main obstacle of this approach, however, is that resulting holographic QCD contains undesired Kaluza-Klein modes with the mass as the same order of hadrons and glueballs. One way to overcome this problem is to consider brane backgrounds in noncritical string theory [13]. Since in this case holographic backgrounds live in lower dimensions, the problem of extra KK modes is more tractable.

A noncritical holographic model of QCD has been introduced in [3], where a stack of $N_c$ D4 branes in six-dimensional noncritical string theory play the role of color branes. Fundamental matters may be added by inserting $N_f$ D4 branes and D4 branes in this background [4]. Further investigations [4,5] have shown that this model captures correctly many properties of low energy QCD, like mass spectrum of mesons, area low behavior for the Wilson loop, glueball mass spectra, etc. In particular, a holographic calculation of Wilson line has been performed recently in [14] for near extremal D3 branes, D4 branes, non-critical near extremal AdS$_6$ model and Klebanov-Strassler model. It has been shown that the noncritical background admits a reasonable fit to lattice results (Actually the Klebanov-Strassler background exhibits the best fit to lattice results, but noncritical model has a asymptotic AdS$_6$ metric, see for details [13]). The thermal phases of the model have been investigated in [6], where it was shown that there is a first order confined/deconfined phase transition at $T_c = \frac{1}{2\pi R}$. In similar to Sakai-Sugimoto model, chiral symmetry restores above a critical temperature $T_{\chi SB} = \frac{0.169}{L}$, where $L$ is the separation of flavors branes at infinity.

It is also interesting to investigate the behavior of QCD matter in presence of other fundamental interactions. Indeed the physics of some astrophysical phenomena like quark stars and notron stars is related to the behavior of thermal QCD in external electromagnetic field. For example conductivity of QCD matter encodes the timescale for expulsion of magnetic flux lines from the core of quark star (see for example [15]). Since QCD is strongly coupled at these circumstance, it is natural
to use AdS/CFT and extracting the properties of QCD at these high densities and temperatures. Using gauge/gravity duality, response of Sakai-Sugimoto model to external electric and magnetic field has been studied in [7–10], behavior of massive $\mathcal{N} = 2$ hypermultiplets in an $\mathcal{N} = 4$ SYM plasma (D3-D7 system) in presence of background electric and magnetic field was explored in [11,12,16] and the effect of magnetic field on dynamics of noncritical holographic model has been considered in [17].

Motivated by these discussions, in this paper, we consider a noncritical holographic model of QCD, and examine its behavior in the presence of external electric field. We compare the results of this model with the predictions of Sakai-Sugimoto model [7], D3-D7 system [16], and a lattice simulation [19]. We find that the behavior of conductivity in noncritical model as a function of temperature and baryon density is similar to the results of D3-D7 system. In weak field regime, where lattice simulation is accessible, noncritical holographic model predicts a finite conductivity for deconfined QCD matter which is linear in temperature, in good agreement with lattice QCD result. Also, the phase diagram of noncritical model resembles the phase diagram of Sakai-Sugimoto model. In particular we find that electric field reduces chiral-symmetry restoration temperature.

This paper is organized as follows: In section 2 we briefly review the noncritical model. In section 3 we analyze the dynamics of flavor branes in presence of electric field and extract the phase diagram of the model. We check our result by making use of Kubo formula in section 4 and section 5 is devoted to brief summary and conclusions.

2 Review of the model

In this section we briefly review the noncritical model of [3]. The model is based on D4/D4-D4 brane system, where $N_c$ D4-branes compactified on $S^1$ with radius $R$ and $N_f$ D4-D4 flavor branes are transverse to the $S^1$. By imposing periodic boundary condition on bosonic field and antiperiodic boundary condition on fermionic field along the circle $S^1$, supersymmetry is broken and one obtains QCD spectrum at the energies below the Kaluza-Klein scale. The brane configuration of the system is

|   | $t$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
|---|-----|------|------|------|------|------|
| D4 | ×   |      |      |      |      |      |
| D4-D4 | × | ×   | ×   | ×   | ×   |      |

where $x_4$ is the coordinate of $S^1$. At zero temperature the 6-dimensional background metric is given by
\[ ds_6^2 = \left( \frac{u}{R_{\text{AdS}}} \right)^2 (-dt^2 + dx^i dx^i + f(u)\, dx_4^2) + \left( \frac{R_{\text{AdS}}}{u} \right)^2 \frac{du^2}{f(u)} \]  

(2.1)

\[ f(u) = 1 - \left( \frac{u_{\Lambda}}{u} \right)^5, \quad R_{\text{AdS}} = \sqrt{\frac{15}{2}} \ell_s \]

There is also a constant dilaton and a 6-form field strength

\[ F_{(6)} = -Q_c \left( \frac{u}{R_{\text{AdS}}} \right)^4 dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge du \]

\[ e^\phi = \frac{2\sqrt{2}}{\sqrt{3} Q_c} \]

The space in \((x_4, u)\) plane looks like a cigar where tip located at \(u = u_{\Lambda}\), and to avoid a conical singularity the periodicity of \(x_4\) should satisfies

\[ x_4 \sim x_4 + 2\pi R = x_4 + \frac{4\pi R_{\text{AdS}}^2}{5u_{\Lambda}} \]  

(2.2)

Kaluza-Klein energy scale is \(M_{KK} = \frac{1}{R} = \frac{5u_{\Lambda}}{2R_{\text{AdS}}^2}\); below this scale the effective theory is QCD. At nonzero temperature there are two solutions with the same boundary condition [6]. The one which dominates at low temperature is given by (2.1) where the periodicity of euclidian time is arbitrary, \(t_E \sim t_E + \beta\), and periodicity of \(x_4\) is given by (2.2). The flavor branes form a U embedding configuration, so this background corresponds to the confined phase with broken chiral symmetry.

By increasing temperature a confinement/deconfinement phase transition occurs at \(T_c = \frac{1}{2\pi R}\), and the background for \(T_c < T\) is represented by

\[ ds^2 = \left( \frac{u}{R_{\text{AdS}}} \right)^2 (f(u) dt_E^2 + dx_0 dx_1 + dx_2 dx_3) + \left( \frac{R_{\text{AdS}}}{u} \right)^2 \frac{1}{f(u)} du^2 \]  

(2.3)

\[ f(u) = 1 - \left( \frac{u_T}{u} \right)^5, \quad t_E \sim t_E + \frac{4\pi R_{\text{AdS}}^2}{5u_T} \]

In this case the periodicity of \(x_4\) circle is arbitrary and temperature is given by \(T = \frac{5u_T}{4\pi R_{\text{AdS}}}\). This background allows two embeddings for D4-D4 flavor branes: U embedding which is preferred configuration for \(T_c < T < T_{\chi_{SB}}\) and parallel embedding which dominates for \(T > T_{\chi_{SB}}\), where \(T_{\chi_{SB}} = 0.169\). Thus chiral symmetry restores at temperatures above \(T_{\chi_{SB}}\).

3 Adding U(1) gauge field

In order to accommodate an external electric field, following [16], we turn on a U(1) gauge field on flavor branes
where time dependence part of gauge field describes a static electric field on the boundary and \( u \) dependence part encodes the response current. In this section we analyze various phases of noncritical holographic QCD in external electric field by making use the method of [16] and [7].

3.1 Deconfined Phase

Let us start with investigating the effect of an external electric field on high temperature \((T > T_c)\) phase of QCD. By using \( \xi = (t_E, \vec{x}, u) \) for parametrization of D4-D4 flavor branes, the induced metric on flavor branes is given by

\[
ds^2 = \left( \frac{u}{R_{AdS}} \right)^2 (f(u)dt_E^2 + dx_i dx_i) + \left( \frac{u}{R_{AdS}} \right)^2 x_4^2 + \left( \frac{R_{AdS}}{u} \right)^2 \frac{1}{f(u)} du^2 \tag{3.2}
\]

DBI action in presence of background gauge field \((3.1)\) takes the following form

\[
S = \frac{\mathcal{N}}{R_{AdS}^3} \int du u^5 \sqrt{\left( f(u)x_4^2 + \frac{R_{AdS}^4}{u^4} \right)(1 - \frac{e^2 R_{AdS}^4}{u^4 f(u)}) + \frac{f(u)R_{AdS}^4}{u^4} a_x'^2} \tag{3.3}
\]

where \( \mathcal{N} = 2N_fT_4e^{-\phi}, \ e = 2\pi\alpha' \), and \( a_x' = 2\pi\alpha'A_x' \). By making use of the first integrals of motion one can find the asymptotic forms of gauge field and \( x_4' \) at large \( u \)

\[
a_x' \sim \frac{j}{u^3}, \quad x_4' \sim \frac{c}{u^2} \tag{3.5}
\]

where \( j \) and \( c \) are identified with the current and chiral condensate in dual gauge theory. By doing holographic renormalization one can show that the physical current on the boundary is

\[
J^x = \frac{2\pi\alpha'\mathcal{N}}{R_{AdS}^3} j \tag{3.7}
\]

By writing the action in terms of current, \( j \), we have

\[
S = \frac{\mathcal{N}}{R_{AdS}^3} \int du u^5 \sqrt{(f(u)x_4^2 + \frac{R_{AdS}^4}{u^4})(f(u) - \frac{e^2 R_{AdS}^4}{u^4})(f(u) - \frac{j^2}{u^6})^{-1}} \tag{3.8}
\]

\[\text{Throughout this paper we work with action density} \]
As we have mentioned in the previous section, deconfined background possesses two embeddings, U embedding and parallel embedding. Our aim is to determine current $j$ for a given external electric field in these embeddings and then extracting conductivity of these configurations.

First consider the U embedding. In the case of vanishing response current, $j = 0$, the solution takes the following form

$$x_4'(u) = \frac{R_{AdS}^2}{u^2 \sqrt{f}} \left[ \frac{u^{10}(f(u) - \frac{e^2 R_{AdS}^4}{u_0^4})}{u_0^{10}(f(u_0) - \frac{e^2 R_{AdS}^4}{u_0^4})} - 1 \right]^{\frac{1}{2}} \tag{3.9}$$

where $u_0$ is the turning point, $x_4'(u_0) = 0$. By inserting the above solution into the action we arrive at

$$S_U = \frac{N}{R_{AdS}^3} \int_{u_0}^{\infty} du \, u^3 \sqrt{1 - e^2 R_{AdS}^4 \frac{u_{10}}{u^4}} \left[ 1 - \frac{u_0^{10}(f(u_0) - \frac{e^2 R_{AdS}^4}{u_0^4})}{u^{10}(f(u) - \frac{e^2 R_{AdS}^4}{u^4})} \right]^{\frac{1}{2}} \tag{3.10}$$

As long as $e_0^2 \leq \frac{1}{R_{AdS}^4} u_{0}^4 f(u_0)$, the action is real and embedding is acceptable. Since by turning on the current, action increased, the fevered configuration is a U embedding with vanishing current, $j = 0$ \[7\]. In dual QCD this means that the deconfined chiral-symmetry breaking phase is an insulator.

A natural question is that what happens for $e > e_0$?. Figure 1 depicts $L$ as a function of $c$ for different values of electric field. As it is evident from this figure, $L$ is a decreasing function of $e$, so there is a maximal value of $e$ at fixed values of $L$ and $T$, such that above which there are no U embedding solution. Thus we expect that the favorite solution in this regime becomes parallel embedding and there should be a phase transition from chiral-broken phase to chiral symmetric phase by increasing electric field \[7\]. We come back to this issue after discussing the behavior of parallel embedding in electric field.

In the parallel embedding the DBI action becomes

$$S_{||} = \frac{N}{R_{AdS}^3} \int_{u_T}^{\infty} du \, u^3 \sqrt{\frac{f(u) - \frac{e^2 R_{AdS}^4}{u^4}}{f(u) - \frac{e^2 R_{AdS}^4}{u^4}}} \tag{3.11}$$

From this expression, it is clear that action becomes complex somewhere unless a nonzero current being turned on. The magnitude of current is given by

$$j = R_{AdS}^2 \, e \, u_c \tag{3.12}$$

where $u_c$ is the root of numerator

$$f(u_c) - \frac{e^2 R_{AdS}^4}{u_c^4} = 0 \tag{3.13}$$
Therefore the chiral-symmetry restoration phase behaves like a conductor by following conductivity

\[ \sigma_{\text{non-critical}} = \frac{J^x}{E} = \frac{(2\pi\alpha')^2 N j}{R_{AdS}^3} e = \frac{(2\pi\alpha')^2 N}{R_{AdS}} u_c(e, T) \]  

(3.14)

There is no algebraic solution for (3.13), however, we can study the weak filed and strong field behavior of conductivity

\[ \sigma_{\text{non-critical}} = \begin{cases} \sqrt{\frac{12\pi}{5}} (2\pi\alpha')^{\frac{2}{5}} N T & E \ll \frac{12\pi}{5} T^2 \\ (2\pi\alpha')^{\frac{2}{5}} N E^{\frac{2}{3}} & E \gg \frac{12\pi}{5} T^2 \end{cases} \]  

(3.15)

It is interesting to compare this results with the predictions of Sakai-Sugimoto model, \( \sigma_{S-S} \), and D3-D7 system, \( \sigma_{D3-D7} \). The weak and high field behavior of conductivity in these models are given by [7,16]

\[ \sigma_{S-S} = \begin{cases} \frac{N_f N_c}{27\pi} \lambda_5^2 T^2 & E \ll \frac{8\pi^2}{27}\lambda_5 T^3 \\ \frac{N_f N_c}{12\pi^{\frac{2}{3}}} \lambda_5^{\frac{1}{3}} E^{\frac{2}{3}} & E \gg \frac{8\pi^2}{27}\lambda_5 T^3 \end{cases} \]

\[ \sigma_{D3-D7} = \begin{cases} \frac{N_f N_c}{4\pi} T & E \ll \frac{\pi}{2}\sqrt{\lambda_4 T^2} \\ \frac{N_f N_c}{(2\pi)^{\frac{2}{3}}} \lambda_4^{\frac{1}{3}} E^{\frac{2}{3}} & E \gg \frac{\pi}{2}\sqrt{\lambda_4 T^2} \end{cases} \]

where \( \lambda_5 \) and \( \lambda_4 \) stands for five and four dimensional t’ Hooft coupling. As it is clear from above expressions, the behaviour of conductivity as a function of temperature and electric field in noncritical model is similar to the D3-D7 system.

Now, let us consider to the phase structure of deconfined background. Since external electric field bring the system out of equilibrium, as argued in [7], to determine which phase is favorite, one can use a Maxwell Like construction for order
parameter \( c = \frac{\partial F}{\partial L} |_{e,T} \). According to Maxwell construction, for a fixed value of \( L \) and \( T \), the transition occurs at the value of \( e \) such that two areas A and B (see figure 2(a)) become equal. By changing electric field we find the phase diagram as figure 2(b), which is, remarkably, similar to those of Sakai-Sugimoto model \[7\]. According to this phase diagram, we observe that the critical temperature decreases with increasing external electric field, as we expected from the polarization effect of electric field. Also at zero electric field we get the result of \[6\] for chiral symmetry restoration temperature, \( T_{\chi B} = \frac{0.169}{L} \).

![Figure 2: (a) Maxwell construction (b) phase diagram for \( L = 1 \) in presence of external electric field in deconfined background](image)

### 3.1.1 finite density

In this subsection, we generalize our analysis in the parallel embedding for finite baryon density by turning on a nontrivial zero-component of the gauge field \( a_0(u) \). The resulting DBI action has the form

\[
S = \frac{N}{R_{AdS}^5} \int du u^5 \sqrt{(f(u)x'^2 + R_{AdS}^4(1 - \frac{e^2 R_{AdS}^4}{u^4 f(u)}) + \frac{f(u)R_{AdS}^4}{u^4} a'_x^2 - \frac{R_{AdS}^4}{u^4} a_0^2)}
\]

Solving the equation of motion one finds the asymptotic behavior of gauge fields as

\[
a_x \simeq \text{constant} - \frac{j}{2u^2} \quad a_0 \simeq \text{constant} + \frac{d}{2u^2} \quad (3.16)
\]

where \( j \) and \( d \) are related to the physical current and charge density via

\[
J = \frac{(2\pi\alpha')N}{R_{AdS}^3} j \quad D = \frac{(2\pi\alpha')N}{R_{AdS}^3} d \quad (3.17)
\]

Writing the action in terms of current and charge density we arrive at
\[ S = \frac{N}{R_{\text{AdS}}^5} \int du u^3 \sqrt{\frac{1 - \frac{e^2 R_{\text{AdS}}^4}{u^4 f(u)} a}{1 - \frac{j^2 - d^2 f(u)}{u^4 f(u)}}} \] (3.18)

Reality of action implies a nonzero current

\[ j = R_{\text{AdS}}^2 \sqrt{\frac{u^2}{u_c^2} + \frac{d^2}{u_c^2}} e \] (3.19)

from which conductivity at finite charge density can be read as

\[ \sigma = \frac{J}{E} = \frac{(2\pi\alpha')^2 N}{R_{\text{AdS}}} \sqrt{\frac{u^2}{u_c^2} + \frac{d^2}{u_c^2}} \] (3.20)

At high density approximation it becomes

\[ \sigma_{\text{noncritical}} = \frac{25\alpha'^2 N}{4R_{\text{AdS}}^5} \frac{d}{T^2} = \frac{5}{12\pi} \frac{D}{T^2} \] (3.21)

Let us compare this result with that one obtained in Sakai-Sugimoto model and D3-D7 system

\[ \sigma_{\text{Sakai–Sugimoto}} = \frac{27}{8\pi^2 \lambda_5} \frac{D}{T^3} \quad \sigma_{\text{D3–D7}} = \frac{2}{\pi \sqrt{\lambda_4}} \frac{D}{T^2} \] (3.22)

Again we observe that charge density dependence of conductivity in noncritical model and D3-D7 system is the same.

### 3.2 confined phase

In this section we will study the response of confined phase to external electric field. This background dominates for \( T < T_c \) and in the case of absence external field, the only allowed embedding for flavor branes is U embedding. To proceed we start with the DBI action for flavor branes

\[ S = \frac{N}{R_{\text{AdS}}^5} \int du u^5 \sqrt{\frac{(f(u)x_4'^2 + R_{\text{AdS}}^4)}{u^4} \left(1 - \frac{e^2 R_{\text{AdS}}^4}{u^4 f(u)}\right) + \frac{R_{\text{AdS}}^4 a x_4'^2}{u^4}} \] (3.23)

First consider the U embedding with \( j = 0 \). The equations of motions gives

\[ x_4'(u) = \frac{R_{\text{AdS}}^2}{u^2 f(u)} \left[ \frac{u^{10} f(u)\left(1 - \frac{e^2 R_{\text{AdS}}^4}{u^4 f(u)}\right)}{u_0^{10} f(u_0)\left(1 - \frac{e^2 R_{\text{AdS}}^4}{u_0^4}\right)} - 1 \right]^{-\frac{1}{2}} \] (3.24)
with the following asymptotic behavior near the boundary

\[ x'_4 \simeq \frac{c}{u^4} \]  

where

\[ c = R_{AdS}^2 u_0^5 \sqrt{f(u_0)(1 - \frac{e^2 R_{AdS}^4}{u_0^4})} \]  

(3.26)

In order to have a physical solution, it is required \( u_0 > e^{\frac{1}{2}} R_{AdS} \). Since the range of coordinate \( u \) is \((u_{KK}, \infty)\), if \( e < \frac{u_{KK}^2}{R_{AdS}} \), \( u_0 \) can interpolates between \( u_{KK} \) and \( \infty \). Figure 3(a) shows the behavior of \( L \) in terms of \( c \), in this case. On the other hand, for \( e > \frac{u_{KK}^2}{R_{AdS}} \), the range of \( u_0 \) would be \((e^{\frac{1}{2}} R_{AdS}, \infty)\) and the plot of \( L(c) \) is like deconfind background (figure 3(b)). Then for \( e > \frac{u_{KK}^2}{R_{AdS}} \), there is a maximal \( L \) for a given \( e \). Since by increasing \( e \) the maximal of \( L \) decreases, there is a maximal \( e \) at any fixed value of \( L \), such that for \( e > e_{max} \), \( U \) embedding is not a valid solution. As discussed in [7] the solution for \( e > e_{max} \) is a V shape embedding, where flavor branes follow parallel radial geodesics. Thus this solution satisfies \( x'_4(u) = 0 \) except at the tip. DBI action for D4-D4 branes in this background is as follows

\[ S_V = \frac{N}{R_{AdS}^3} \int_{u_T}^{\infty} du \frac{u^3}{\sqrt{f(u)}} \sqrt{1 - \frac{e^2 R_{AdS}^4}{u_0^4} \left( 1 - \frac{u^4}{R_{AdS}^4} - \frac{u^6}{u_0^4} \right)} \]  

(3.27)

From which one can read the conductivity of \( V \) embedding

\[ \sigma = (2\pi \alpha')^{\frac{1}{2}} N \sqrt{E} \]  

(3.28)

Therefore the confined phase exhibits two phases in external electric field. \( U \) embedding which describes an insulating phase and \( V \) embedding with a finite electric conductivity. In order to explore the phase diagram, we apply the same method
used for deconfined phase. The resulting phase diagrams for fixed value of $u_{KK}$ and $L$ is shown in Figure 4, which are similar to those of Sakai-Sugimoto model [7]. From this diagram we observe that U embedding is dominate background for $e < \frac{u_{KK}}{R_{AdS}}$, disregard how much $L$ is, as we expected.

![Figure 4: phase diagram in confined phase, (left) $u_{KK} = .5$, (right) L=1](image)

3.3 electric susceptibility

The response of an insulating phase to an external electric field can be measured by electric susceptibility, which defined by

$$\chi_e = -\frac{\partial^2 F_e}{\partial e^2}$$  \hspace{1cm} (3.29)

Since there is an insulating phase in both deconfined and confined background, it is interesting to use holographic description to calculate the susceptibility in the dual QCD. Indeed holographic free energy is divergent and one has to add a counterterm to find finite results. We apply regularization of [7]

$$\chi_e = -\frac{\partial^2 F_e}{\partial e^2} + \frac{\partial^2 F_e}{\partial e^2} \big|_{e=0}$$  \hspace{1cm} (3.30)

Using this expression, we find electric susceptibility in deconfined and confined phase according to figure 5.

4 Kubo formula

The response of a thermodynamic system to an applied external field is described by transport coefficients of the system. For small deviation from equilibrium, Kubo formula relates transport coefficients to the equilibrium retarded green’s functions of the system. In particular real-time correlator of two electromagnetism current determines electric conductivity of the medium via
\[ \sigma = \frac{1}{2} \lim_{\omega \to 0} \frac{1}{\omega} \text{Im} G_{\alpha\beta} R(\omega, k) \]  

(4.1)

where \( G_{\alpha\beta} \) is retarded green's function of two transverse electromagnetic current. In this section we calculate retarded green’s function of two EM currents in the dual QCD by using of the Lorenzian AdS/CFT prescription \[18\]. To do so, we turn off external electric field, \( e = 0 \), thus the system is in equilibrium and turn on a nontrivial time-component gauge field \( A_0 \) as a background field, corresponding to a finite charge density in the dual gauge theory. Therefore the background is given by

\[ ds^2|_{D4} = \left( \frac{u}{R_{\text{AdS}}} \right)^2 (-f(u)dt^2 + dx^i dx^j) + \left( \frac{R_{\text{AdS}}}{u} \right)^2 \frac{du^2}{f(u)} \]  

(4.2)

\[ F_{tu} = -A'_t(t, u) \]

Where \( ds^2|_{D4} \) is the induced metric on D4–\overline{D4} flavor branes. According to holographic dictionary we focus on the linearized fluctuations of U(1) gauge field on the gravitational background. By expanding the DBI action up to second order in the field strength around the background (4.2), we arrive at

\[ S = \frac{(2\pi\alpha')^2 N}{2R_{\text{AdS}}^3} \int \frac{du}{\sqrt{u^6 + C^2}} \left[ -\frac{(u^6 + C^2)}{f(u)} (R_{\text{AdS}})^4 F_{ti}^2 - \frac{(u^6 + C^2)^2}{u^6} F_{0u}^2 \right. \]

\[ + R_{\text{AdS}}^4 u^2 \sum_{i<j} F_{ij}^2 + f(u)(u^6 + C^2) \sum F_{iu}^2 \]  

(4.3)

where, \( C = \frac{-R_{\text{AdS}}^2 D}{(2\pi\alpha')^N} \), it is convenient to change variable to

\[ y = \frac{R_{\text{AdS}}^2}{u} \]  

(4.4)
\[
S = \frac{(2\pi \alpha')^2 N \mathcal{R}_{\text{AdS}}}{2} \int \frac{dyy^{-1}}{\sqrt{1 + C'^2 y^6}} \left[ -\frac{(1 + C'^2 y^6)}{f(y)} F_{ii}^2 - (1 + C'^2 y^6) F_{iy}^2 + \sum_{i<j} F_{ij}^2 \right] + f(y)(1 + C'^2 y^6) \sum_{i<j} F_{iy}^2
\]

where \( y_T = \frac{R_{\text{AdS}}}{a_T} = \frac{5}{4\pi T} \) and

\[
f(y) = 1 - \frac{y^5}{y_T^5} 
C' = \frac{C}{F_6} \quad (4.5)
\]

Since transverse part of electromagnetic current couples to transverse part of bulk electric field, we consider the e.o.m for transverse part of electric field. It is convenient to work in fourier space

\[
A_M(x^\mu, y) = \int \frac{d^4k}{(2\pi)^4} e^{ik.x} A_M(k_\mu, y) \quad k_\mu = (-\omega, q, 0, 0) \quad (4.6)
\]

and equation of motion for transverse components, \( E_\alpha = \omega A_\alpha \, (\alpha = 2, 3) \), becomes

\[
E''_\alpha + \frac{y}{f(y) \sqrt{1 + C'^2 y^6}} \left( y^{-1} f(y) \sqrt{1 + C'^2 y^6} \right) E'_{\alpha} + \frac{\omega^2 - \frac{q^2 f(y)}{f(y)^2}}{f(y)} E_{\alpha} = 0 \quad (4.7)
\]

From this equation the near horizon behavior of \( E_\alpha \) can be read as

\[
E_\alpha = (y_T - y)^{\pm i\omega/2} \quad (4.8)
\]

where \( \omega = \frac{2\pi y_T}{\sqrt{y_T}} = \frac{\omega}{2\pi T} \) and \( \pm \) represent ingoing and outgoing wave into horizon, respectively. By imposing ingoing boundary condition at the horizon, the behavior of solution near \( y = 0 \) becomes

\[
E_{\alpha} = A + B y^2 \quad (4.9)
\]

The relevant part of boundary action for calculating retarded corelalator is

\[
S = \frac{(2\pi \alpha')^2 N \mathcal{R}_{\text{AdS}}}{2} \int d^{d+1}x \sqrt{-\mathcal{g}} g^\alpha g_{\alpha} A_\alpha \partial_\alpha A_\alpha \quad (4.10)
\]

\[
= \frac{(2\pi \alpha')^2 N \mathcal{R}_{\text{AdS}}}{2} \int dwdq \frac{2AB}{(2\pi)^2 \omega^2} \quad (4.11)
\]

By applying AdS/CFT reception for calculating real-time correlator, we have
$$G^{R}_{\alpha\alpha} = \frac{\delta^2 S}{\delta A_{\alpha}\delta A_{\alpha}} = \omega^2 \frac{\delta^2 S}{\delta E_{\alpha}\delta E_{\alpha}} = 2(2\pi\alpha')^2 N R_{AdS} Im[A/B] \quad (4.12)$$

In appendix we have calculated $A/B$ at low frequency, $w \rightarrow 0$, the result is

$$\frac{A}{B} = 1 + \frac{5}{4} i \sqrt{1 + C'y_T^6 w y_T^2} \quad (4.13)$$

substituting this result into (4.1) we arrive at

$$\sigma = \frac{(2\pi\alpha')^2 N}{R_{AdS} u_T} \sqrt{1 + \frac{d^2}{u_T^6}} \quad (4.14)$$

which is exactly (3.20) with $e = 0$.

5 Conclusion

We have studied a noncritical holographic model in external electric field and compare its results with the ones of Sakai-Sugimoto model and D3-D7 system. We found that, remarkably, the behavior of conductivity as a function of temperature, electric field and baryon density is similar to the result of D3-D7 system. In particular in weak-field regime the conductivity grows linear with temperature, $\sigma \propto T$.

Computation of transport coefficients by using lattice QCD requires an analytic continuation to real-time space, which leads to systematic errors in results. However, lattice results with small errors provide insights about strongly coupled regime of QCD. In [19] putting systematic errors under control, a lattice simulation of conductivity in QCD has been done, the reported result is

$$\frac{\sigma(T)}{T} = C_{EM} \left\{ \begin{array}{ll} 7.5 \pm 0.8, & T = 1.5T_c \\ 7.7 \pm 0.6, & T = 2T_c \\ 7.0 \pm 0.4, & T = 3T_c \end{array} \right. \quad (5.1)$$

where electromagnetic vertex factor is given by $C_{EM} = 4\pi\alpha \sum e_f^2$, in which $\alpha$ is fine structure constant and $e_f$ is electric charge of a quark with flavor $f$ ($C_{EM} \approx \frac{1}{20}$ for two flavors [19]). If we can trust this result, it shows that, with good accuracy, conductivity is linear in temperature, in accordance with the predictions of noncritical models and D3-D7 system (soft wall model also predicts a linear dependence $\sigma \propto T$ [21]).

By studying the phase structure of the model, we observed that the electric field reduces chiral-restoration temperature. Also the general structure of phase diagram closely resemble phase diagram of Sakai-Sugimoto model [7]. In addition we have checked our results by using Kubo formula in section 3.
Acknowledgment

The author would like to thank Mohsen Alishahiha for many helpful discussions and encouragement and for carefully reading and commenting on the manuscript. I would also like to thank H.R. Afshar, A. Akhavan, M. Ali-Akbari, D. Allahbakhshi, K. Bitaghsir Fadafan, R. Farezghbal, A.E. Mosaffa, A. Naseh, A.A. Varshovi, and A.V. Zayakin for discussions.

Appendix

In this appendix we explore the low-frequency behavior of retarded green’s function, by solving (4.7) perturbatively in $\omega$. Let us first rewrite (4.7) in terms of more convenient variable

$$E_{\perp} = (y_T - y)^{-i\omega/2} \phi(y)$$  \hspace{1cm} (5.2)

substituting this into (4.7) we find the following equation for $\phi(y)$

$$\phi''(y) + \mathfrak{P} \phi'(y) + \mathfrak{R} \phi(y) = 0$$  \hspace{1cm} (5.3)

where we have rescaled variables as $y \rightarrow y_T y$ and $C' \rightarrow \tilde{C} = y_T^6 C'$, for simplicity and

$$\mathfrak{P} = \frac{i\omega}{1 - y} + \frac{(1 + 4y^5 - 2\tilde{C}y^6 + 7\tilde{C}y^{11})}{(-1 + y)y(1 + y + y^2 + y^3 + y^4)(1 + \tilde{C}y^6)}$$

$$\mathfrak{R} = \frac{w(-2i + w)}{4(-1 + y)^2} + \frac{25w^2 y^5 (1 + \tilde{C}y)}{4(-1 + y)^2 (1 + \tilde{C}y^6)} - \frac{i\omega(1 + 4y^5 - 2\tilde{C}y^6 + 7\tilde{C}y^{11})}{2(-1 + y)^2 y(1 + y + y^2 + y^3 + y^4)(1 + \tilde{C}y^6)}$$

In order to solve this equation perturbatively in $w$, we consider following expansion for $\phi$

$$\phi = \phi_0 + \omega \phi_1$$  \hspace{1cm} (5.4)

plugging this into (5.3) and imposing regularity condition at the horizon, we find that $\phi_0$ is a constant and $\phi_1$ satisfies following equation

$$\phi_1''(y) + P(y) \phi_1'(y) = R(y)$$  \hspace{1cm} (5.5)

where

$$P(y) = \frac{1 + 4y^5 - 2y^6\tilde{C} + 7y^{11}\tilde{C}}{(-y + y^6)(1 + y^6\tilde{C})}$$  \hspace{1cm} (5.6)

$$R(y) = i\left[ -\frac{1}{2(-1 + y)^2} + \frac{1 + 4y^5 - 2y^6\tilde{C} + 7y^{11}\tilde{C}}{2(-1 + y)^2 y(1 + y + y^2 + y^3 + y^4)(1 + y^6\tilde{C})} \right]$$  \hspace{1cm} (5.7)
The general solution of above equation is
\[ \phi_1(y) = \lambda \tilde{\phi}(y) + \tilde{\phi}(y) \int_0^y dx \frac{R(x)}{\tilde{\phi}'(x)} - \int_0^y dx \frac{R(x)\tilde{\phi}(x)}{\tilde{\phi}'(x)} \] (5.8)

where \( \lambda \) is a constant and particular solution, \( \tilde{\phi}(y) \), is given by
\[ \tilde{\phi}(y) = \int_0^y dx e^{-\int_0^x ds P(s)} \] (5.9)

For our purpose, it is sufficient to consider the near horizon and near boundary form of the solution. First consider the near horizon behavior of solution
\[ \phi(y) \approx \frac{\lambda}{5\sqrt{1 + \tilde{C}}} \ln(y - 1) + \frac{iw}{2} \ln(y - 1) \] (5.10)

regularity at horizon implies that
\[ \lambda = -\frac{5iw}{2} \sqrt{1 + \tilde{C}} \] (5.11)

Take the limit \( y \to 0 \) in (5.8) and using (5.11), we find the asymptotic form of solution near the boundary as
\[ \phi(y) \approx 1 - iw(\sqrt{1 + \tilde{C}y^2} - \frac{y^2}{4} + \frac{y}{2}) \] (5.12)

and in original coordinate we have
\[ E_\alpha(y) \mid_{y \to 0} \approx 1 + \frac{5iw}{4} \sqrt{1 + C' y_T^2 \frac{y^2}{y_T^2}} \] (5.13)

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