Effective resistances for supercritical percolation clusters in boxes

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Abstract

Let $C^n$ be the largest open cluster for supercritical Bernoulli bond percolation in $[-n,n]^d \cap \mathbb{Z}^d$ with $d \geq 2$. We obtain a sharp estimate for the effective resistance on $C^n$. As an application we show that the cover time for the simple random walk on $C^n$ is comparable to $n^d (\log n)^2$. Noting that the cover time for the simple random walk on $[-n,n]^d \cap \mathbb{Z}^d$ is of order $n^d \log n$ for $d \geq 3$ (and of order $n^2 (\log n)^2$ for $d = 2$), this gives a quantitative difference between the two random walks for $d \geq 3$.

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1 Introduction

The effective resistance is a fundamental measurement of the conductivity for the electrical network. It has close connections with many subjects of reversible Markov chains such as transience/recurrence, heat kernels, mixing times and cover times. We refer to [17], [25], [26], [29] for the introduction of the theory of reversible Markov chains.

Effective resistances for percolation clusters in $\mathbb{Z}^d$ have been studied for a long time. Grimmett, Kesten and Zhang [19] showed almost-sure finiteness of the effective resistance from a fixed point to infinity on the infinite supercritical percolation cluster in $\mathbb{Z}^d$ for $d \geq 3$. (This is equivalent to almost-sure transience of the simple random walk on the cluster.) The result was extended to a series of works on more general energy of flows on percolation clusters [2, 9, 20, 21, 22, 28]. The study of effective resistances for boxes on $\mathbb{Z}^d$ under percolation goes back to 1980’s [11, 18, 23]. These results showed that critical phenomena occur at the critical percolation probability for effective resistances between opposing faces of boxes (see Remark 1.3 below). In [10], Boivin and Rau estimated the effective resistances from a fixed point to boundaries of boxes in the infinite

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supercritical percolation cluster on $\mathbb{Z}^2$ (see Remark 1.4 below). In [7], general upper bounds for effective resistances on general graphs are given by using isoperimetric inequalities. The estimates are used to obtain upper bounds of effective point-to-point resistances on supercritical percolation clusters in boxes on $\mathbb{Z}^2$; however, there seems to be a gap in this part (see Remark 1.2 below).

Many properties of simple random walks on supercritical percolation clusters in $\mathbb{Z}^d$ have been investigated such as transience [2, 9, 19, 20], mixing times [8, 13], heat kernel decays [4, 31], invariance principles [6, 30, 35], collisions of two independent simple random walks [5, 12], and the existence of the harmonic measure [10]. These properties are very similar to those of simple random walks on $\mathbb{Z}^d$. A famous folk conjecture is that most important properties of simple random walks have been investigated such as transience [2, 9, 19, 20], mixing times [8, 13], heat kernel decays [4, 31], invariance principles [6, 30, 35], collisions of two independent simple random walks [5, 12], and the existence of the harmonic measure [10].

In this paper, we consider the largest supercritical percolation cluster in $[-n,n]^d \cap \mathbb{Z}^d$. We obtain the correct order of the maximum of the effective resistances between vertices in the giant cluster. Applying the result, we obtain a sharp estimate of the cover time for the simple random walk on the largest cluster; the result shows that it is much larger than the cover time for the simple random walk on $[-n,n]^d \cap \mathbb{Z}^d$ when $d \geq 3$. This gives a negative answer to the folk conjecture.

In order to describe our results more precisely, we begin with some notation. Let $| \cdot |_1$ be the $\ell^1$-distance on $\mathbb{Z}^d$. We define the set of edges between all nearest-neighbour pairs on $\mathbb{Z}^d$ by $E(\mathbb{Z}^d) := \{\{x,y\} : x, y \in \mathbb{Z}^d, |x - y|_1 = 1\}$. For $p \in [0,1]$, let $\mathbb{P}_p$ be the product Bernoulli measure on $\{0,1\}^{E(\mathbb{Z}^d)}$ with $\mathbb{P}_p(e^0) = 1 - \mathbb{P}_p(e^1) = p$ for each $e \in E(\mathbb{Z}^d)$. We say that an edge $e$ is open for $\omega \in \{0,1\}^{E(\mathbb{Z}^d)}$ if $\omega(e) = 1$. For $A \subseteq \mathbb{Z}^d$, we define a random set of edges by

$$O_A = O_A(\omega) := \{\{x,y\} \in E(\mathbb{Z}^d) : x, y \in A, \omega(\{x,y\}) = 1\}. \quad (1.1)$$

An open cluster in $A$ is a connected component of the graph $(A, O_A)$. We define the critical probability $p_c(\mathbb{Z}^d)$ by

$$\inf\{p \in [0,1] : \mathbb{P}_p(\text{the cluster in } \mathbb{Z}^d \text{ containing the origin is infinite}) > 0\}. \quad (1.2)$$

We focus our attention to clusters in the box

$$B(n) := [-n,n]^d \cap \mathbb{Z}^d. \quad (1.3)$$

It is known that for $d \geq 2$ and $p > p_c(\mathbb{Z}^d)$, we have the unique largest open cluster in $B(n)$ whose size is proportional to $n^d$, $\mathbb{P}_p$-a.s., for large $n \in \mathbb{N}$ (see [14, Theorem 1] for $d = 2$ and [34, Theorem 1.2] for $d \geq 3$). We write $C^n$ to denote the largest open cluster in $B(n)$.

Let $G = (V(G), E(G))$ be a finite connected graph; the set $V(G)$ is the vertex set and $E(G)$ is the edge set. We define the Dirichlet energy by

$$\mathcal{E}(f) := \frac{1}{2} \sum_{u,v \in V(G)} (f(u) - f(v))^2, \quad f \in \mathbb{R}^{V(G)}.$$
For $A, B \subset V(G)$ with $A \cap B = \emptyset$, the effective resistance between $A$ and $B$ for $G$ is defined by

$$R_{\text{eff}}^G(A, B)^{-1} := \inf \{ \mathcal{E}(f) : f \in \mathbb{R}^{V(G)}, \mathcal{E}(f) < \infty, f|_A = 1, f|_B = 0 \}. $$

We write $R_{\text{eff}}^G(x, y)$ to denote $R_{\text{eff}}^G(\{x\}, \{y\})$. We now state our result on the effective resistance for $C^n$.

**Theorem 1.1** For $d \geq 2$ and $p \in (p_c(\mathbb{Z}^d), 1)$, there exist $c_1, c_2 > 0$ such that $\mathbb{P}_p$-a.s., for large $n \in \mathbb{N}$,

$$c_1 \cdot \log n \leq \max_{x, y \in C^n} R_{\text{eff}}^n(x, y) \leq c_2 \cdot \log n.$$

**Remark 1.2** Corollary 3.1 of [7] says that for $d = 2$ and $p > p_c(\mathbb{Z}^2)$, there exists $c_1 > 0$ such that $\lim_{n \to \infty} \mathbb{P}_{p}(\max_{x, y \in C^n} R_{\text{eff}}^n(x, y) \leq c_1 \cdot \log n) = 1$. The proof is based on general upper bounds for effective resistances given in Theorem 2.1 of [7] and an isoperimetric profile for $C^n$ studied in [8]. However, according to the latest arXiv version of their paper [7] and [24], the proof only implies the following: there exists $c_1 > 0$ such that $\lim_{n \to \infty} \mathbb{P}_{p}(\max_{x, y \in C^n} R_{\text{eff}}^n(x, y) \leq c_1 \cdot (\log n)^2) = 1$. This is due to bad isoperimetry of some small subsets of $C^n$.

**Remark 1.3** Fix $d \geq 2$. Let us consider a random graph $G_n$ which has the vertex set $[0, n]^d \cap \mathbb{Z}^d$ and the edge set $\mathcal{O}_{[0,n]^d \cap \mathbb{Z}^d}$. We write $R_n$ to denote the effective resistance

$$R_{\text{eff}}^G([0, n]^{d-1} \times \{0\}, [0, n]^{d-1} \times \{n\}).$$

In [7], it was shown that the following holds: there exist $c_1, c_2 > 0$ such that if $p < p_c(\mathbb{Z}^d)$,

$$R_n = \infty \text{ for large } n \in \mathbb{N}, \mathbb{P}_p - \text{a.s.},$$

and if $p_c(\mathbb{Z}^d) < p \leq 1$,

$$c_1 \leq \lim inf_{n \to \infty} n^{d-2}R_n \leq \lim sup_{n \to \infty} n^{d-2}R_n \leq c_2, \mathbb{P}_p - \text{a.s.}$$

**Remark 1.4** Let $C_{\infty}$ be the infinite Bernoulli bond percolation cluster in $\mathbb{Z}^2$. Define spheres in $C_{\infty}$ by

$$\partial B_{C_{\infty}}(x, r) = \{ y \in C_{\infty} : d_{C_{\infty}}(x, y) = r \}, \ x \in C_{\infty}, r \in \mathbb{N},$$

where $d_{C_{\infty}}$ is the graph distance of $C_{\infty}$. In Proposition 4.3 of the published version of [11], Bowin and Rau proved the following: for $p \in (p_c(\mathbb{Z}^2), 1]$, there exist $c_1, c_2 > 0$ such that $\mathbb{P}_p$-a.s., for all $x_0 \in C_{\infty}$, for large $n \in \mathbb{N}$,

$$c_1 \log n \leq R_{\text{eff}}^{C_{\infty}}(x_0, \partial B_{C_{\infty}}(x_0, n)) \leq c_2 \log n.$$
For a finite connected graph $G = (V(G), E(G))$ and $x \in V(G)$, let $\tau_x(G)$ be the hitting time of $x$ by the simple random walk on $G$. We define the cover time by

$$t_{cov}(G) := \max_{x \in V(G)} E_x \left( \max_{y \in V(G)} \tau_y(G) \right).$$

Applying Theorem 1.1, we can obtain an improvement of Proposition 3.3 of the arXiv version of [1].

**Theorem 1.5** For $d \geq 2$ and $p \in (p_c(Z^d), 1)$, there exist $c_1, c_2 > 0$ such that

$$c_1 \cdot n^d (\log n)^2 \leq t_{cov}(C^n) \leq c_2 \cdot n^d (\log n)^2.$$

**Remark 1.6** It is known that $t_{cov}(B(n))$ is comparable to $n^2 (\log n)^2$ when $d = 2$ and to $n^d \log n$ when $d \geq 3$ (see, for example, Section 11.3.2 of [26]). Therefore, Theorem 1.5 implies that the simple random walk on $C^n$ exhibits anomalous behavior when $d \geq 3$. This is due to irregularity of $C^n$; the cluster $C^n$ contains a lot of one-dimensional objects (see Lemma 2.6 below).

Let $G = (V(G), E(G))$ be any finite connected graph. Let us define the Gaussian free field on $G$. This is a centered Gaussian process $\{\eta^G_x\}_{x \in V(G)}$ with $\eta^G_{x_0} = 0$ for some fixed vertex $x_0 \in V(G)$ and the covariances are given by

$$E(\eta^G_x \eta^G_y) = \frac{1}{2} \left( R^G_{\text{eff}}(x, x_0) + R^G_{\text{eff}}(y, x_0) - R^G_{\text{eff}}(x, y) \right), \quad x, y \in V(G).$$

We define the expected maximum of the Gaussian free field by

$$M_G := E \left( \max_{x \in V(G)} \eta^G_x \right).$$

Note that $M_G$ does not depend on the choice of $x_0$. For a set $S$, we will write $|S|$ to denote the cardinality of $S$. In [26], Ding, Lee and Peres proved the following: there exist universal constants $c_1, c_2 > 0$ such that

$$c_1 \cdot |E(G)| \cdot (M_G)^2 \leq t_{cov}(G) \leq c_2 \cdot |E(G)| \cdot (M_G)^2. \quad (1.4)$$

By (1.4) and Theorem 1.5 we obtain the following estimate of $M_{C^n}$ immediately.

**Corollary 1.7** For $d \geq 2$ and $p \in (p_c(Z^d), 1)$, there exist $c_1, c_2 > 0$ such that

$$c_1 \cdot \log n \leq M_{C^n} \leq c_2 \cdot \log n.$$

**Remark 1.8** By Remark 1.6 and (1.4), $M_{B(n)}$ is comparable to $\log n$ when $d = 2$ and to $\sqrt{\log n}$ when $d \geq 3$. Thus there is a marked quantitative difference between $M_{C^n}$ and $M_{B(n)}$ when $d \geq 3$. 
Let us describe the outline of the paper. Section 2 gives preliminary lemmas about percolation estimates. In Subsection 3.1 we give a proof of Theorem 1.5 via Theorem 1.1. The upper bound is due to Theorem 1.1 and the Matthews bound [32]. The lower bound is based on the general bound on the cover time via the Gaussian free field given in [16]. The key of the proof of the lower bound is counting the number of one-way open paths of length of order \( \log n \). In Subsection 3.2, we construct the so-called “Kesten grid” (the terminology comes from Mathieu and Remy [31]). This is an analogue of the square lattice and consists of “white” sites on a renormalized lattice; the concept of “white” sites is from the renormalization argument of Antal and Pisztora [3]. In Subsection 3.3, we prove Theorem 1.1. The lower bound is immediately followed by the fact that \( C_n \) has a one-way open path of length of order \( \log n \). In the proof of the upper bound, we construct unit flows between vertices of \( C_n \) with energy of order \( \log n \). Thanks to Kesten grids, we can let the flows run almost on some “sheets” and adapt an argument of estimating upper bounds of effective resistances for \( \mathbb{Z}^2 \). We use a result on the chemical distance for \( C_n \) based on [13]; this guarantees that every vertex of \( C_n \) can be connected with a Kesten grid decorated with small boxes by an open path in \( C_n \) of length of order \( \log n \).

Throughout the paper, we will write \( c, c', c'' \) to denote positive constants depending only on the dimension of the lattice and the percolation parameter. Values of \( c, c' \) and \( c'' \) will change from line to line. We use \( c_1, c_2, \cdots \) to denote constants whose values are fixed within each argument. If we cite elsewhere the constant \( c_1 \) in Lemma 2.4, we write it as \( c_1(2.4) \), for example.

2 Percolation estimates

In this Section, we collect some useful results on percolation estimates. We need the following facts to prove Theorem 1.1 and Theorem 1.5.

(1) Size and connectivity pattern of \( C^n \)
A path is a sequence \((x_0, x_1, \cdots, x_\ell)\) satisfying that \( |x_{i-1} - x_i|_1 = 1 \) for all \( 1 \leq i \leq \ell \). An open path is a path all of whose edges are open. We write \( |\cdot|_\infty \) to denote the \( \ell_\infty \)-distance on \( \mathbb{Z}^d \). The diameter of an open path \((x_0, x_1, \cdots, x_\ell)\) is defined by \( \max_{0 \leq i,j \leq \ell} |x_i - x_j|_\infty \). For \( \kappa > 0 \), let \( A^n_\kappa \) be the event that every open path in \( B(n) \) with diameter larger than \( \kappa \log n \) is contained in \( C_n \). The following fact is easily followed by [13] Theorem 1 and Theorem 9] for \( d = 2 \) and [34] Theorem 1.2 and Theorem 3.1] for \( d \geq 3 \).

Lemma 2.1 Fix \( d \geq 2 \) and \( p > p_c(\mathbb{Z}^d) \). There exist \( c_1, c_2 > 0 \) and \( \kappa_0, n_0 \in \mathbb{N} \) such that for all \( \kappa \geq \kappa_0 \) and \( n \geq n_0 \),

\[
\mathbb{P}_p(A^n_\kappa \cap \{|C^n| \geq c_1 n^d\}) \geq 1 - 2\exp(-c_2 \kappa \log n).
\]

Remark 2.2 By Lemma 2.1 and the Borel-Cantelli lemma, there exist \( c_1, c_2 > 0 \) such that \( \mathbb{P}_p \)-a.s., for large \( n \in \mathbb{N} \), the event \( A^n_1 \cap \{|C^n| \geq c_2 n^d\} \) holds.
Let $d_{C^n}$ be the graph distance of the graph $(C^n, \mathcal{O}_{C^n})$. The following result is immediately followed by Lemma 3.2 and (4.1) in [13].

**Lemma 2.3** Fix $d \geq 2$ and $p > p_c(\mathbb{Z}^d)$. There exist $c_1 > 0$ and $\kappa_0 \in \mathbb{N}$ such that the following holds for all $\kappa \geq \kappa_0$, $\mathbb{P}_p$-a.s., for large $n \in \mathbb{N}$: for all $x, y \in C^n$ with $|x - y|_1 \leq \kappa \log n$,

$$d_{C^n}(x, y) \leq c_1 \kappa \log n.$$

(3) Crossing probabilities

In (3), we restrict our attention to Bernoulli site percolation on the square lattice. Let $Q_q$ be the product Bernoulli measure on $\{0, 1\}^{\mathbb{Z}^2}$ with $Q_q(\omega(x) = 1) = 1 - Q_q(\omega(x) = 0) = q$ for each $x \in \mathbb{Z}^2$. For $\omega \in \{0, 1\}^{\mathbb{Z}^2}$ and $x \in \mathbb{Z}^2$, we will say that $x$ is occupied if $\omega(x) = 1$. The critical probability $q_c(\mathbb{Z}^2)$ is defined similarly to (1.2). We write $x \cdot y$ to denote the inner product of $x$ and $y$. A self-avoiding path is a path all of whose vertices are distinct. Let $e_1, e_2$ be the standard basis for $\mathbb{Z}^2$. Fix $m, n \in \mathbb{N}$. A crossing in $([0, m] \times [0, n]) \cap \mathbb{Z}^2$ is a self-avoiding path $(x_0, x_1, \cdots, x_\ell)$ with $x_0 \cdot e_1 = 0$ and $x_\ell \cdot e_1 = m$.  

**Lemma 2.4** [23, Theorem 11.1] For $d = 2$ and $q > q_c(\mathbb{Z}^2)$, there exist $c_1, c_2, c_3 > 0$ such that

$$Q_q \left( \text{there exist at least } c_1 n \text{ disjoint crossings consisting of occupied sites in } ([0, m] \times [0, n]) \cap \mathbb{Z}^2 \right) \geq 1 - c_2 \cdot m \exp(-c_3 n).$$

(4) Renormalization argument

We recall the renormalization argument of [3]. We will write $a, b, \cdots$ rather than $x, y, \cdots$ to denote vertices of the renormalized lattice. Fix a positive integer $K$. To $a \in \mathbb{Z}^d$, we associate boxes

$$B_a(K) := (2K + 1)a + [-K, K]^d \cap \mathbb{Z}^d,$$

$$B_a'(K) := (2K + 1)a + \left[-\frac{5}{4}K, \frac{5}{4}K\right]^d \cap \mathbb{Z}^d.$$  

We will say that an open cluster crosses a box if the cluster intersects all the faces of the box. For $a \in \mathbb{Z}^d$, we define an event $R^K_a$ satisfying the following:

- There exists a unique open cluster $C$ in $B_a'(K)$ crossing $B_a'(K)$.
- The open cluster $C$ crosses all the subboxes of $B_a'(K)$ of side length larger than $\frac{K}{10}$.
- Any open paths in $B_a'(K)$ of diameter larger than $\frac{K}{10}$ are contained in $C$. 

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We say $a \in \mathbb{Z}^d$ is white for $\omega \in \{0, 1\}^{E(\mathbb{Z}^d)}$ if $1_{R_a^K}(\omega) = 1$. By [14, Theorem 9] for $d = 2$ and [34, Theorem 3.1] for $d \geq 3$, if $p > p_c(\mathbb{Z}^d)$, we have for all $a \in \mathbb{Z}^d$

$$
\lim_{K \to \infty} \mathbb{P}_p(R_a^K) = 1.
$$

By [27, Theorem 0.0 (ii)], the following holds:

**Lemma 2.5** For $d \geq 2$ and $p > p_c(\mathbb{Z}^d)$, there exists a function $q : \mathbb{N} \to [0, 1]$ with

$$
\lim_{K \to \infty} q(K) = 1
$$

such that

the process $(1_{R_a^K})_{a \in \mathbb{Z}^d}$ stochastically dominates

a Bernoulli site percolation process in $\mathbb{Z}^d$ with parameter $q(K)$.

(5) Special open paths

Let $e_1, \ldots, e_d$ be the standard basis for $\mathbb{Z}^d$. We define a one-sided boundary of $B(n)$ by

$$
\partial_1 B(n) := \{ x \in B(n) : x \cdot e_1 = n \}.
$$

We will say that $x \in \partial_1 B(n)$ is $m$-special if it is a base point of a special open path; that is, the edge $\{x + ie_1, x + (i + 1)e_1\}$ is open for each $0 \leq i \leq m - 1$ and the vertex $x + ie_1$ does not have any other open edges for each $1 \leq i \leq m$.

See Figure 1. The following is a key to prove the lower bounds of Theorems 1.1 and 1.5.

**Lemma 2.6** For $d \geq 2$ and $p \in (p_c(\mathbb{Z}^d), 1)$, there exist $c_1, c_2 > 0$ such that $\mathbb{P}_p$-a.s., for large $n \in \mathbb{N}$,

$$
|\{ x \in \partial_1 B(n) \cap C^n : x \text{ is } c_1 \log n \text{-special} \}| \geq n^{c_2}.
$$

**Proof.** Fix a large constant $K > 0$. We define a subset of $\partial_1 B(n)$ by

$$
\partial_{1,2}^K B(n) := \{ x \in \partial_1 B(n) : |x \cdot e_i| \leq \frac{5}{4} K \text{ for all } 3 \leq i \leq d \}.
$$
Let $S^K_n$ be the “square” defined by
\[
\left\{ a \in \mathbb{Z}^d : |a \cdot e_1| \leq \left\lfloor \frac{n}{2K+1} \right\rfloor, |a \cdot e_2| \leq \left\lfloor \frac{n-5K/4}{2K+1} \right\rfloor, a \cdot e_i = 0 \text{ for } 3 \leq i \leq d \right\}.
\]
Note that $S^K_n$ is isomorphic to $[0, 2\left\lfloor \frac{n}{2K+1} \right\rfloor] \times [0, 2\left\lfloor \frac{n-5K/4}{2K+1} \right\rfloor] \cap \mathbb{Z}^2$. Let $\varepsilon > 0$ be a small constant. By Lemma 2.4 and Lemma 2.5, the following holds: there exist $c, c' > 0$ such that with $P_p$-probability at least $1 - cn \exp(-c' n)$, we have more than $\varepsilon n$ disjoint crossings of white sites in $S^K_n$. By this fact and the definition of white sites together with Lemma 2.1, the event $G^K_n := \{ |\partial_1 B(n) \cap C^n| < \varepsilon n \}$ satisfies the following: there exist $c, c' > 0$ and $\kappa_0 \in \mathbb{N}$ such that for all $\kappa \geq \kappa_0$ and sufficiently large $n \in \mathbb{N}$,
\[
P_p(G^K_n) \leq c \exp(-c' \kappa \log n). \tag{2.3}
\]
Take $c_1 > 0$ small enough to make $r := 1 - c_1 \left( \log \frac{1}{p} + 2(d-1) \log \frac{1}{1-p} \right)$ positive. Fix $0 < c_2 < r$. We write $\text{Bin}(m, q)$ to denote a binomial random variable with parameter $m$ and $q$. Let $q_n$ be the probability of the event that a fixed vertex in $\partial_1 B(n)$ is $|c_1 \log n|$-special. Consider the $\sigma$-field generated by finite-dimensional cylinders associated with configurations restricted to $B(n)$. Note that conditioned on the $\sigma$-field, events $\{ x \text{ is } |c_1 \log n|\text{-special} \}$ and $\{ y \text{ is } |c_1 \log n|\text{-special} \}$ are independent for $x, y \in \partial_1 B(n)$ with $|x - y|_{\infty} \geq 2$. By conditioning on the $\sigma$-field, we have for some $c, c' > 0$ and some $0 < c'' < \varepsilon$,
\[
P_p \left( \left\{ |\{ x \in \partial_1 B(n) \cap C^n : x \text{ is } |c_1 \log n|\text{-special} \}| \leq n^{c_2} \right\} \cap (G^K_n)^c \right)
\leq P_p \left( \text{Bin} \left( \lfloor \varepsilon' n \rfloor, q_n \right) \leq n^{c_2} \right)
\leq c \exp(-c'' n^r). \tag{2.4}
\]
In the last inequality, we have used the Chebyshev inequality and the fact that $n q_n \geq c n^r$ for some $c > 0$. By (2.3) and (2.4) together with the Borel-Cantelli lemma, we obtain the conclusion. \[\square\]

3 Proof of Theorem 1.1 and Theorem 1.5

3.1 Cover time estimate

In this Subsection we prove Theorem 1.5 via Theorem 1.1. We begin with general bounds on cover times based on [16, 32]. The following is immediately followed by [24, Proposition 10.6 and Theorem 11.2] (see also [15, 32] and [10, Theorem 1.1 and Lemma 1.11].
Lemma 3.1 Let $G = (V(G), E(G))$ be any finite connected graph.

1. There exists a universal constant $c_1 > 0$ such that
   \[ t_{\text{cov}}(G) \leq c_1 \cdot |E(G)| \cdot \left( \max_{x,y \in V(G)} R_{\text{eff}}^{G}(x,y) \right) \cdot \log |V(G)|. \]

2. There exists a universal constant $c_1 > 0$ such that for all subset $\tilde{V} \subset V(G)$,
   \[ t_{\text{cov}}(G) \geq c_1 \cdot |E(G)| \cdot \left( \min_{x,y \in \tilde{V}} R_{\text{eff}}^{G}(x,y) \right) \cdot \log |\tilde{V}|. \]

Proof of Theorem 1.5 via Theorem 1.1. The upper bound is immediately followed by Lemma 3.1 (1) and Theorem 1.1. From now, we prove the lower bound. By Remark 2.2, we have
   \[ \mathbb{P}_{p}\text{-a.s., for large } n \in \mathbb{N}, \quad C_{n} - \lceil \frac{c_2}{c_0} \log n \rceil \subseteq C_{n}. \] (3.1)

Set $m_n := n - \lfloor \frac{c_0}{c_2} \log n \rfloor$. We define a set $V_n$ of tips of special open paths by
   \[ \{ x + \lfloor \frac{c_0}{c_2} \log m_n \rfloor e_1 : x \in \partial_1 B(m_n) \cap C^{m_n} \text{ and } x \text{ is } \lfloor \frac{c_0}{c_2} \log m_n \rfloor \text{-special} \}. \]

By (3.1), Lemma 2.6 and the Nash-Williams inequality (see, for example, [26, Proposition 9.15]), we have the following $\mathbb{P}_{p}\text{-a.s., for large } n \in \mathbb{N}:$ for all $x, y \in V_n$ with $x \neq y$,
   \[ V_n \subseteq C^{n}, |V_n| \geq (m_n) \lfloor \frac{c_0}{c_2} \log m_n \rfloor \text{ and } R_{\text{eff}}^{C^{n}}(x, y) \geq \lfloor \frac{c_0}{c_2} \log m_n \rfloor. \]

Therefore, by Remark 2.2 and Lemma 3.1 (2) with $\tilde{V} = V_n$, we obtain the lower bound. □

3.2 Kesten grid

In this Subsection, we construct the so-called “Kesten grids” on the renormalized lattice (recall Section 2 (4)) in order to obtain the upper bound of Theorem 1.1. The terminology is due to Mathieu and Remy [31]. We note that the construction of the Kesten grid is based on Theorem 11.1 of [23] (see Lemma 2.4).

Let us define some notations. Recall that $e_1, \cdots, e_d$ is the standard basis for $\mathbb{Z}^d$. Fix a sufficiently large positive integer $K$. Recall the notation (2.2). For a subset $S$ of the renormalized lattice, we define a fattened version of $S$ by
   \[ W(S) := \bigcup_{a \in S} B_{\ell_n}(K). \] (3.2)

Let $\alpha$ be a positive constant. We will choose $\alpha$ sufficiently large later. Let $\ell_n$ be the largest integer $\ell$ satisfying that
   \[ (2K + 1)((2 \lceil \alpha \log n \rceil + 1)\ell + \lceil \alpha \log n \rceil) + \frac{5}{4} K \leq n, \]
and set

\[ \tilde{\ell}_n := (2\lceil \alpha \log n \rceil + 1)\ell_n + \lceil \alpha \log n \rceil. \]  

(3.3)

Note that we have \( W(B(\tilde{\ell}_n)) \subset B(n) \) (recall the notation (1.4)). Thus we can regard \( B(\tilde{\ell}_n) \) as a box in the renormalized lattice corresponding to the original box \( B(n) \). The number of two-dimensional sections in \( B(\tilde{\ell}_n) \) is \( \frac{d(d-1)}{2}(2\tilde{\ell}_n + 1)^{d-2} \); each of them is isomorphic to the square \([-\tilde{\ell}_n, \tilde{\ell}_n]^2 \cap \mathbb{Z}^2 \). In Subsection 3.3 below, we will focus our attention on one of them, namely

\[ F_1 := \{ k_1e_1 + k_2e_2 : \tilde{\ell}_n \leq k_1, k_2 \leq \tilde{\ell}_n \}. \]  

(3.4)

We write

\[ F_i, \ 2 \leq i \leq \frac{d(d-1)}{2}(2\tilde{\ell}_n + 1)^{d-2} \]  

(3.5)

to denote the other two-dimensional sections of \( B(\tilde{\ell}_n) \). For \(-\ell_n \leq m \leq \ell_n\), we define the \( m \)-th horizontal strip of \( F_1 \)

\[ R_1^m := \{ k_1e_1 + k_2e_2 : \tilde{\ell}_n \leq k_1 \leq \tilde{\ell}_n, |k_2 - (2\lceil \alpha \log n \rceil + 1)m| \leq \lceil \alpha \log n \rceil \} \]  

and the \( m \)-th vertical strip of \( F_1 \)

\[ R_2^m := \{ k_1e_1 + k_2e_2 : \tilde{\ell}_n \leq k_2 \leq \tilde{\ell}_n, |k_1 - (2\lceil \alpha \log n \rceil + 1)m| \leq \lceil \alpha \log n \rceil \}. \]  

(3.6)

(3.7)

We define a horizontal (respectively, vertical) crossing of \( F_1 \) as a self-avoiding path with endvertices \( a, b \) satisfying \( a \cdot e_1 = -\ell_n \) and \( b \cdot e_1 = \ell_n \) (respectively, \( a \cdot e_2 = -\ell_n \) and \( b \cdot e_2 = \ell_n \)). We define strips and crossings for the other two-dimensional sections of \( B(\tilde{\ell}_n) \) in a similar fashion. We note that each strip is isomorphic to \(([0, 2\tilde{\ell}_n] \times [0, 2\lceil \alpha \log n \rceil]) \cap \mathbb{Z}^2 \). Taking \( \alpha \) large enough, the following holds immediately by Lemma 2.4 and Lemma 2.5.

**Corollary 3.2** Fix \( d \geq 2 \) and \( p > p_c(\mathbb{Z}^d) \). The following holds \( \mathbb{P}_p \)-a.s., for large \( n \in \mathbb{N} \) and for all \( 1 \leq i \leq \frac{d(d-1)}{2}(2\tilde{\ell}_n + 1)^{d-2} \) and \(-\ell_n \leq m \leq \ell_n\), there exist at least \( 2\lceil \alpha \log n \rceil \) disjoint horizontal (respectively, vertical) crossings consisting of white sites in the \( m \)-th horizontal (respectively, vertical) strip of \( F_i \).

Set \( L_n := \left[ 2\lceil \alpha \log n \rceil \right] \). Fix \( 1 \leq i \leq \frac{d(d-1)}{2}(2\tilde{\ell}_n + 1)^{d-2} \) and a configuration satisfying the event of Corollary 3.2. We can choose \( L_n \) disjoint crossings of white sites in each strip of \( F_i \). Since \( F_i \) is isomorphic to \([-\tilde{\ell}_n, \tilde{\ell}_n]^2 \cap \mathbb{Z}^2 \), the horizontal and the vertical crossings intersect (see Figure 2) and form a grid on \( F_i \). We will call it a Kesten grid. Note that Kesten grids on \( F_i \) and \( F_j \) do not intersect if \( i \neq j \) in general.

### 3.3 Effective resistance estimate

In this Subsection we prove Theorem 1.1 To prove it, we only need to estimate the upper bound of effective resistances for pairs of vertices on \( W(F_i) \cap C^n \), \( 1 \leq i \leq \frac{d(d-1)}{2}(2\tilde{\ell}_n + 1)^{d-2} \) (recall the notations 3.2 - 3.5).
Figure 2: An illustration of disjoint crossings of white sites (thin solid lines) in strips (rectangles with dotted boundaries) of $F_1$ (square with thick solid boundary). These horizontal and vertical crossings intersect since $F_1$ is isomorphic to $[-l_n, l_n]^2 \cap \mathbb{Z}^2$.

**Proposition 3.3** Fix $d \geq 2$ and $p > p_c(\mathbb{Z}^d)$. There exists $c_1 > 0$ such that $\mathbb{P}_p$-a.s., for large $n \in \mathbb{N}$, the following holds: for all $1 \leq i \leq \frac{d(d-1)}{2} (2\ell_n + 1)^d - 2$ and $x, y \in W(F_1) \cap C^n$, 

$$R^c_{x,y}(x, y) \leq c_1 \cdot \log n.$$

**Proof of Theorem 1.1 via Proposition 3.3** Since the triangle inequality holds for the effective resistance (see, for instance, [26, Corollary 10.8]), the upper bound is immediately followed by using Lemma 2.3 and Proposition 3.3 repeatedly. The lower bound holds by Lemma 2.6 and the Nash-Williams inequality. □

In the proof of Proposition 3.3 below, we will focus our attention to $W(F_1) \cap C^n$. Recall the constant $\alpha$ in Subsection 3.3. To $k_1, k_2 \in \mathbb{Z}$, we associate the “square” defined by

$$S^n_{k_1, k_2} := \{ s_1e_1 + s_2e_2 : |s_i - (2[\alpha \log n] + 1)k_i| \leq \lceil \alpha \log n \rceil, i = 1, 2 \}.$$

Fix some integers $r, r'$ with $r < r'$. A left-to-right (respectively, right-to-left) path in $[r, r'] \times \mathbb{Z}^{d-1}$ is a path in $[r, r'] \times \mathbb{Z}^{d-1}$ whose initial and final vertices $a, b$ satisfy $a \cdot e_1 = r$ and $b \cdot e_1 = r'$ (respectively, $a \cdot e_1 = r'$ and $b \cdot e_1 = r$). A bottom-to-top path in $\mathbb{Z} \times [r, r'] \times \mathbb{Z}^{d-2}$ is a path in $\mathbb{Z} \times [r, r'] \times \mathbb{Z}^{d-2}$ whose initial and final vertices $a, b$ satisfy $a \cdot e_2 = r$ and $b \cdot e_2 = r'$. For a path $\Gamma$ and $a, a' \in \Gamma$ (we assume that $a$ appears before $a'$ in $\Gamma$), we write $\Gamma[a, a']$ to denote a part of $\Gamma$ from $a$ to $a'$. In particular, when $a'$ is the final vertex of $\Gamma$, we write
Figure 3: Illustrations of paths $\Gamma_i$, $\Gamma(i,j)$ and $\gamma(i,j)$.  

$\Gamma[a]$ in the place of $\Gamma[a,a']$. 

**Proof of Proposition 3.3.** We fix a configuration satisfying the events of Remark 2.2, Corollary 3.2, and Lemma 2.3 with $\kappa$ large enough. Let $m_1, m_2, \tilde{m}_2$ be integers satisfying the following: $m_2 < \tilde{m}_2$, $\tilde{m}_2 - m_2$ is even, and $m_1 + \frac{\tilde{m}_2 - m_2}{2} \leq \ell_n$ (recall the definition of $\ell_n$ below (3.2)). For simplicity, we only estimate the effective resistance between $x \in W(S_{m_1, m_2}^n) \cap C^n$ and $y \in W(S_{m_1, \tilde{m}_2}^n) \cap C^n$. For more general cases, we can apply a similar argument and use Lemma 2.3. We omit the details. We will construct a unit flow between $x$ and $y$. Our argument is based on [29, Proposition 2.15] and Section 4.1 of the arXiv version of [10].

We first construct a random self-avoiding open path from $x$ to $y$, most of whose parts lie on $W(F_1) \cap C^n$. Recall the notations (3.6) and (3.7). A Kesten grid guarantees the existence of the following self-avoiding paths:

- disjoint left-to-right (respectively, right-to-left) self-avoiding paths of white sites $H_{m_1}^n, \cdots, H_{\tilde{m}_2}^n$ contained in $R_1^n(m_2 + m)$ for $1 \leq m \leq \frac{\tilde{m}_2 - m_2}{2}$ (respectively, $\frac{\tilde{m}_2 - m_2}{2} < m < \tilde{m}_2 - m_2$),
- disjoint bottom-to-top self-avoiding paths of white sites $V_{m_1}^n, \cdots, V_{\tilde{m}_2}^n$ contained in $R_2^n(m_1 + m)$ for $0 \leq m \leq \frac{\tilde{m}_2 - m_2}{2}$,
- left-to-right self-avoiding paths of white sites $H_x$ and $H_y$ of length at most $c \log n$ for some $c > 0$, contained in $S_{m_1, m_2}^n$ and in $S_{m_1, \tilde{m}_2}^n$ respectively.

From now, we construct an open path from $x$ to $y$ corresponding to a fixed label $(i,j)$ with $i \in \{0, \cdots, \frac{\tilde{m}_2 - m_2}{2}\}$ and $j \in \{1, \cdots, L_n\}$ by the following three steps.

Step 1: For convenience, we will associate a vertex $(k_1, k_2) \in \mathbb{Z}^2$ to $S_{k_1, k_2}^n$. Let $f : \{0, \cdots, 2i\} \to \{0, \cdots, i\}$ be a function defined by $f(r) = r$ for $0 \leq r \leq i$ and $f(r) = 2i - r$ for $i < r \leq 2i$. Take a self-avoiding path $\Gamma_i$ as follows:
Let \((s_r)_{0 \leq r \leq 2i+1}\) be a sequence with \(0 = s_0 < s_1 < \cdots < s_{2i+1} = \tilde{m}_2 - m_2\).
For \(0 \leq r \leq 2i\), set \(v_{2r} = (m_1 + f(r), m_2 + s_r)\) and \(v_{2r+1} = (m_1 + f(r), m_2 + s_{r+1})\). The self-avoiding path \(\Gamma_i\) is obtained from the sequence \((v_r)_{0 \leq r \leq 2i+1}\) by linear interpolation.

The path \(\Gamma_i\) lies within 1 in \(\ell^\infty\)-distance from the piecewise line segments connecting \((m_1, m_2)\), \((m_1 + i, m_2 + \tilde{m}_2)\) and \((m_1, \tilde{m}_2)\). See the left-hand side of Figure 3.

Step 2: Recall the notation before the proof of Proposition 3.3. We define vertices \(a_1, \ldots, a_{4i+2}\) as follows:

- For \(0 \leq r \leq 2i\), \(a_{2r+1}\) is the last-visited vertex of \(H^s_j[a_{2r}]\) by \(V^f_j\), where \(a_0 := h_x\) and \(H^{s_0}_j := H_x\).
- For \(0 \leq r \leq 2i\), \(a_{2r+2}\) is the last-visited vertex of \(V^f_j[a_{2r+1}]\) by \(H^{s_{r+1}}_j\), where \(H^{s_{2i+1}}_j := H_y\).

See Figure 4. We can obtain a self-avoiding path \(\Gamma(i, j)\) of white sites from \(h_x\) to \(h_y\) in \(C_n\) via \(\gamma_x\), \(\tilde{\gamma}(i, j)\) and \(\gamma_y\). See the middle of Figure 3.

Step 3: Recall the notation. Fix vertices \(\hat{x} \in B_{h_x}(K) \cap C^n\) and \(\hat{y} \in B_{h_y}(K) \cap C^n\). Indeed, we can take such \(\hat{x}\) and \(\hat{y}\) by Remark 2.2. By the definition of white sites, we can take a self-avoiding open path \(\hat{\gamma}(i, j)\) from \(\hat{x}\) to \(\hat{y}\) contained in \(W(\Gamma(i, j))\). By Lemma 2.3, we have self-avoiding open paths \(\gamma_x\) and \(\gamma_y\) in \(C^n\) from \(x\) to \(\hat{x}\) and from \(\hat{y}\) to \(y\) respectively of length at most \(c \log n\) for some \(c > 0\). We can get a self-avoiding open path \(\gamma(i, j)\) from \(x\) to \(y\) in \(C^n\) via \(\gamma_x\), \(\hat{\gamma}(i, j)\) and \(\gamma_y\). See the right-hand side of Figure 3.
Let $X$ and $Y$ be random variables distributed uniformly on $\{0, \cdots, \frac{\tilde{m}_2-m_2}{2}\}$ and on $\{1, \cdots, L_n\}$ respectively; they are defined on a probability space with probability measure $P$. Recall the notation (1.1). From now, we construct a unit flow from $x$ to $y$ via the random open path $\gamma(X,Y)$. We define a random function $\psi$ on $C^n \times C^n$ by

$$\psi(u,v) := \begin{cases} 
1 & \text{if } \{u, v\} \in O_{C^n} \text{ and } \gamma(X,Y) \text{ passes } u, v \text{ in this order,} \\
-1 & \text{if } \{u, v\} \in O_{C^n} \text{ and } \gamma(X,Y) \text{ passes } v, u \text{ in this order,} \\
0 & \text{otherwise.}
\end{cases}$$

We define a function $\theta$ on $C^n \times C^n$ by

$$\theta(u,v) := E(\psi(u,v)) \text{ for } u, v \in C^n.$$

Since $\psi$ is a unit flow from $x$ to $y$, $P$-a.s., $\theta$ is a unit flow from $x$ to $y$. In order to bound $\theta$, let us define the following function on $C^n \times C^n$:

$$p(u,v) := \begin{cases} 
P(\gamma(X,Y) \text{ passes the edge } \{u, v\}) & \text{if } \{u, v\} \in O_{C^n}, \\
0 & \text{otherwise.}
\end{cases}$$

For $0 \leq r \leq \tilde{m}_2 - m_2$, we define a set of labels of the $r$-th level by

$$D_r := \begin{cases} 
\{(m_1 + s, m_2 + r) \in \mathbb{Z}^2 : 0 \leq s \leq r + 1\} & \text{if } r \leq \frac{\tilde{m}_2-m_2}{2}, \\
\{(m_1 + s, m_2 + r) \in \mathbb{Z}^2 : 0 \leq s \leq \tilde{m}_2 - m_2 - r + 1\} & \text{otherwise.}
\end{cases}$$

Let $U$ be the union of $\gamma_x, \gamma_y, W(H_x)$ and $W(H_y)$. By Thomson’s principle (see, for example, [26, Theorem 9.10]) and the construction of $\gamma(X,Y)$, we have

$$P_{\text{eff}}^C(x,y) \leq \frac{1}{2} \sum_{u,v \in C^n} \theta(u,v)^2$$

$$\leq \sum_{u,v \in U} p(u,v)^2 + \frac{\tilde{m}_2-m_2}{2} \sum_{r=0}^{(\tilde{m}_2-m_2)/2} \sum_{(k_1, k_2) \in D_r} \sum_{u,v \in W(S_{k_1, k_2}^n)} p(u,v)^2$$

$$+ \sum_{r=(\tilde{m}_2-m_2)/2+1} \frac{\tilde{m}_2-m_2}{2} \sum_{(k_1, k_2) \in D_r} \sum_{u,v \in W(S_{k_1, k_2}^n)} p(u,v)^2.$$  \hspace{1cm} (3.8)

Since $\gamma_x, \gamma_y, H_x$ and $H_y$ are self-avoiding paths of length of order $\log n$, the first term on the right-hand side of (3.8) is bounded by $c \log n$ for some $c > 0$. Fix $1 \leq r \leq \frac{\tilde{m}_2-m_2}{2}$, $(k_1, k_2) \in D_r$ and $u, v \in W(S_{k_1, k_2}^n)$ with $v \notin U$ and $\{u,v\} \in O_{C^n}$. By the construction of the random open path $\gamma(X,Y)$, we have for some $c > 0$,

$$p(u,v) \leq \frac{c}{r \cdot L_n}.$$
So we have for some $c, c' > 0$,

$$
\sum_{r=0}^{(\tilde{m}_2-m_2)/2} \sum_{(k_1,k_2) \in D_r} \sum_{u,v \in W(S_{k_1,k_2})} p(u,v)^2 \leq c \sum_{r=1}^{(\tilde{m}_2-m_2)/2} \frac{1}{r} \leq c' \log n.
$$

By a similar argument, the last term on the right-hand side of (3.8) is bounded by $c \log n$ for some $c > 0$. Therefore by (3.8), we obtain the conclusion. \(\square\)

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**References**

[1] Y. Abe. Cover times for sequences of reversible Markov chains on random graphs. To appear in Kyoto J. Math.; [arXiv:1206.0398](http://arxiv.org/abs/1206.0398).

[2] O. Angel, I. Benjamini, N. Berger and Y. Peres. Transience of percolation clusters on wedges. *Electron. J. Probab.* 11 (2006), 655-669.

[3] P. Antal and A. Pisztora. On the chemical distance for supercritical Bernoulli percolation. *Ann. Probab.* 24 (1996), 1036-1048.

[4] M. T. Barlow. Random walks on supercritical percolation clusters. *Ann. Probab.* 32 (2004), 3024-3084.

[5] M. T. Barlow, Y. Peres and P. Sousi. Collisions of random walks. *Ann. Inst. H. Poincaré Probab. Statist.* 48 (2012), 922-946.

[6] N. Berger and M. Biskup. Quenched invariance principle for simple random walk on percolation clusters. *Probab. Theory Relat. Fields.* 137 (2007), 83-120.

[7] I. Benjamini and G. Kozma. A resistance bound via an isoperimetric inequality. *Combinatorica* 25 (2005), 645-650; [arXiv:math/0212322v2](http://arxiv.org/abs/math/0212322v2).

[8] I. Benjamini and E. Mossel. On the mixing time of a simple random walk on the super critical percolation cluster. *Probab. Theory Relat. Fields.* 125 (2003), 408-420.
[9] I. Benjamini, R. Pemantle and Y. Peres. Unpredictable paths and percolation. Ann. Probab. 26 (1998), 1198-1211.

[10] D. Boivin and C. Rau. Existence of the harmonic measure for random walks on graphs and in random environments. J. Stat. Phys. 150 (2013), 235-263; arXiv:1111.5326v2.

[11] J. T. Chayes and L. Chayes. Bulk transport properties and exponent inequalities for random resistor and flow networks. Commun. Math. Phys. 105 (1986), 133-152.

[12] X. Chen and D. Chen. Two random walks on the open cluster of $\mathbb{Z}^2$ meet infinitely often. Sci China Math. 53 (2010), 1971-1978.

[13] Z.-Q. Chen, D. A. Croydon and T. Kumagai. Quenched invariance principles for random walks and elliptic diffusions in random media with boundary. arXiv:1306.0076v1.

[14] O. Couronné and R. J. Messikh. Surface order large deviations for 2D FK-percolation and Potts models. Stochastic Processes Appl. 113 (2004), 81-99.

[15] A. K. Chandra, P. Raghavan, W. L. Ruzzo, R. Smolensky and P. Tiwari. The electrical resistance of a graph captures its commute and cover times. Comput. Complexity 6 (1996/1997), 312-340.

[16] J. Ding, J. R. Lee, and Y. Peres. Cover times, blanket times, and majorizing measures. Ann. of Math. 175 (2012), 1409-1471.

[17] P. G. Doyle and J. L. Snell. Random Walks and Electric Networks. Carus Math. Monogr. 22, Mathematical Association of America, Washington, DC, 1984.

[18] G. Grimmett and H. Kesten. First-passage percolation, network flows and electrical resistances. Z. Wahrsch. Verw. Gebiete. 66 (1984), 335-366.

[19] G.R. Grimmett, H. Kesten and Y. Zhang. Random walk on the infinite cluster of the percolation model. Probab. Theory Relat. Fields. 96 (1993), 33-44.

[20] O. Häggström and E. Mossel. Nearest-neighbor walks with low predictability profile and percolation in $2 + \varepsilon$ dimensions. Ann. Probab. 26 (1998), 1212-1231.

[21] C. Hoffman. Energy of flows on $\mathbb{Z}^2$ percolation clusters. Random Struct. Alg. 16 (2000), 143-155.

[22] C. Hoffman and E. Mossel. Energy of flows on percolation clusters. Potential Anal. 14 (2001), 375-385.
[23] H. Kesten. *Percolation Theory for Mathematicians*. Progress in Probability and Statistics, Birkhäuser, Boston, 1982.

[24] G. Kozma. Personal communication.

[25] T. Kumagai. *Random walks on disordered media and their scaling limits*. Book in preparation. Current version available at [http://www.kurims.kyoto-u.ac.jp/~kumagai/StFlour-Cornell.html](http://www.kurims.kyoto-u.ac.jp/~kumagai/StFlour-Cornell.html).

[26] D. A. Levin, Y. Peres, and E. L. Wilmer. *Markov Chains and Mixing Times*. American Mathematical Society, Providence, RI, 2009. With a chapter by James G. Propp and David B. Wilson.

[27] T. M. Liggett, R. H. Schonmann and A. M. Stacey. Domination by product measures. *Ann. Probab.* **25** (1997), 71-95.

[28] D. Levin and Y. Peres. Energy and cutsets in infinite percolation clusters. In: Random walks and discrete potential theory (Cortona, 1997), Sympos. Math., XXXIX. Cambridge: Cambridge Univ. Press, 1999.

[29] R. Lyons and Y. Peres. *Probability on trees and networks*. Book in preparation. Current version available at [http://mypage.iu.edu/~rdlyons/](http://mypage.iu.edu/~rdlyons/).

[30] P. Mathieu and A. L. Piatnitski. Quenched invariance principles for random walks on percolation clusters. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **463** (2007), 2287-2307.

[31] P. Mathieu and E. Remy. Isoperimetry and heat kernel decay on percolation clusters. *Ann. Probab.* **32** (2004), 100-128.

[32] P. Matthews. Covering problems for Brownian motion on spheres. *Ann. Probab.* **16** (1988), 189-199.

[33] G. Pete. A note on percolation on $\mathbb{Z}^d$: Isoperimetric profile via exponential cluster repulsion. *Elect. Comm. in Probab.* **13** (2008), 377-392.

[34] A. Pisztora. Surface order large deviations for Ising, Potts and percolation models. *Probab. Theory Relat. Fields.* **104** (1996), 427-466.

[35] V. Sidoravicius and A.-S. Sznitman. Quenched invariance principles for walks on clusters of percolation or among random conductances. *Probab. Theory Relat. Fields.* **129** (2004), 219-244.