Infinite stable looptrees

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Abstract

In this note, we give a construction of an infinite stable looptree, which we denote by \( L^\infty \), and prove that it arises both as a local limit of the compact stable looptrees of [CK14], and as a scaling limit of the infinite discrete looptrees of [Ric17b] and [BS15]. As a consequence, we are able to prove various convergence results for volumes of small balls in compact stable looptrees, explored more deeply in the article [Arc19]. We also establish the spectral dimension of \( L^\infty \), and show that it agrees with that of its discrete counterpart. Moreover, we show that Brownian motion on \( L^\infty \) arises as a scaling limit of random walks on discrete looptrees, and as a local limit of Brownian motion on compact stable looptrees, which has similar consequences for the limit of the heat kernel of the Brownian motion studied in [Arc19].

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1 Introduction

Stable looptrees are a class of random fractal objects indexed by a parameter \( \alpha \in (1, 2) \) and can informally be thought of as the dual graphs of stable trees. Motivated by [LGM11], they were originally introduced by Curien and Kortchemski in [CK14], and along with their discrete counterparts have been shown to be of increasing significance in the study of critical percolation models on random planar maps. For example, the same authors showed in [CK15] that the rescaled boundary of a critical percolation cluster on the UIPT converges to a random stable looptree as its size is taken to infinity, and Richier showed in [Ric17b] that the incipient infinite cluster of the UIHPT has the form of an infinite discrete looptree. Further results along these lines can be found in [CK15], [CDKM15], [Ric17a], [Ric17b], [ÖS17], [BR18], [CR18] and [KR18], though this is a very non-exhaustive list. Moreover, they are also emerging as an important tool in the programme to reconcile the theories of random planar maps and Liouville quantum gravity, demonstrated for example in [MS15], [CPT18] and [BHS18].

Given a discrete tree \( T \), the corresponding discrete looptree \( \text{Loop}(T) \) as defined in [CK14] is constructed by replacing each vertex \( u \in T \) with a cycle of length equal to the degree of \( u \) in \( T \), and then gluing these cycles along the tree structure of \( T \). This is illustrated in Figure 1. This operation can also be applied in the case where \( T \) is an infinite tree.

In [CK14 Theorem 4.1], it is shown that if \( T_n \) is a Galton Watson tree conditioned to have \( n \) vertices with offspring distribution \( \xi \) such that \( \xi([k, \infty)) \sim ck^{-\alpha} \) as \( k \to \infty \) for some \( c \in (0, \infty) \), then we can define the \( \alpha \)-stable looptree (which we denote by \( L_\alpha \)) to be the random compact metric space such that

\[
n \xrightarrow{d} \text{Loop}(T_n) \xrightarrow{(d)} (c|\Gamma(-\alpha)|)^{-\frac{1}{\alpha}} L_\alpha
\]

in the Gromov-Hausdorff topology as \( n \to \infty \). A simulation is shown in Figure 2.

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The main purpose of this paper is to give a construction of infinite stable looptrees. The construction is similar in spirit to Duquesne’s construction of stable sin-trees in [Duq09], which is the continuum analogue of Kesten’s discrete construction of an infinite critical tree. Additionally, infinite discrete looptrees have been defined by Björnberg and Stefánsson in [BS15] by applying a related loop operation to Kesten’s infinite critical tree $T_\infty$, and similarly by Richier in [Ric17b] by applying a similar operation to a two-type version of Kesten’s tree.

As is done for stable sin-trees in [Duq09], we define the infinite stable looptree $L_\infty^\alpha$ from two independent stable Lévy processes, each of which is used to code the looptree on one-side of its singly infinite loopspine. This is the construction suggested in [Ric17b] Section 6] and is the natural extension of the coding mechanism used to define stable looptrees from stable Lévy excursions.

The construction is given in Section 4. The remainder of the article is devoted to proving various limit theorems to justify the definition, and then using these to make deductions about Brownian motion on compact stable looptrees, which is explored more deeply in the article [Arc19]. In particular, we prove a local limit theorem showing that $L_\infty^\alpha$ can be characterised as the limit of compact stable looptrees as their mass goes to infinity, and also as the scaling limit of infinite discrete looptrees. When combined with earlier results of Curien and Kortchemski, Björnberg and Stefánsson, and Richier, this shows that the diagram of Figure 3 commutes as indicated.
We start by giving the local limit result. In what follows, we let $L^\ell_\alpha$ be a compact stable looptree conditioned to have mass $\ell$, and let $L^\infty_\alpha$ be as above. We recall from [CK14] that $L^\ell_\alpha$ is endowed with a measure $\nu_\ell$ which is the natural analogue of uniform measure on $L^\alpha_\ell$. We will define a similar measure on $L^\infty_\alpha$ in Section 4, and denote it by $\nu^\infty$. We also recall from [CK14] (respectively [Arc19]) that there is a natural way to define shortest-distance metric (respectively a resistance metric) on $L^\ell_\alpha$, and we will define analogous metrics for $L^\infty_\alpha$ in Section 4.

**Theorem 1.1.** Let $L^\ell_\alpha$ be a compact stable looptree conditioned to have mass $\ell$, and let $L^\infty_\alpha$ be as above. Then,

$$(L^\ell_\alpha, d_\ell, \nu_\ell, \rho_\ell) \overset{(d)}{\to} (L^\infty_\alpha, \tilde{d}_\infty, \nu^\infty, \rho^\infty)$$

as $\ell \to \infty$, with respect to the Gromov-Hausdorff vague topology. Here $d_\ell$ and $\tilde{d}_\infty$ can denote either the geodesic metrics, or the effective resistance metrics on the respective spaces.

Similarly, we prove the following scaling result.

**Theorem 1.2.** Let $T^\infty_\alpha$ denote Kesten’s tree with critical offspring distribution in the domain of attraction of $\alpha$-stable law, say $\xi$ such that $\xi((k, \infty)) \sim c k^{-\alpha}$. Also let $\nu^{disc}$ denote the measure that gives mass 1 to every vertex of $\text{Loop}(T^\infty_\alpha)$. Then

$$(\text{Loop}(T^\infty_\alpha), C_\alpha^{-1} n^{-\frac{1}{\alpha}} \tilde{d}_n, \rho, n^{-1} \nu^{disc}) \overset{(d)}{\to} (L^\infty_\alpha, \tilde{d}_\infty, \rho^\infty, \nu^\infty)$$

with respect to the Gromov-Hausdorff vague topology as $n \to \infty$, where $C_\alpha = (c|\Gamma(-\alpha)|)^{-\frac{1}{\alpha}}$. Here $\tilde{d}_n$ and $\tilde{d}_\infty$ can denote either the geodesic metrics, or the effective resistance metrics on the respective spaces.

We will see in Section 3.3 that similar results hold for the infinite discrete looptrees defined in [BS15] and [Ric17b], constructed constructed similarly, though we have to take a slightly different measure in this case.

Given these two theorems, we are also in the right setting to apply results of [Cro18] about similar limits for stochastic processes on these spaces. In particular, we obtain the following results. Note that we formally define Brownian motion on stable looptrees in the article [Arc19] by defining it to be the stochastic process naturally associated with the effective resistance metric on them. In Section 4, we similarly define an effective resistance metric on $L^\infty_\alpha$ and in Section 4 we define Brownian motion on $L^\infty_\alpha$ to be the associated stochastic process. We denote it by $B^\infty$.

**Theorem 1.3.** Let $(B^\ell_t)_{t \geq 0}$ be Brownian motion on $L^\ell_\alpha$, and let $(B^\infty_t)_{t \geq 0}$ be Brownian motion on $L^\infty_\alpha$. Then there exists a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ on which we can almost surely define a common metric space $(M, R_M)$ in which the spaces $(L^\ell_\alpha, R_t, \nu_t, \rho_t)$ and $(L^\infty_\alpha, R^\infty, \nu^\infty, \rho^\infty)$ can all be embedded and such that

$$(L^\ell_\alpha, R_t, \nu_t, \rho_t) \to (L^\infty_\alpha, R^\infty, \nu^\infty, \rho^\infty)$$

Figure 3: Commuting Diagram.
with respect to the Gromov-Hausdorff-vague topology, and the required Hausdorff convergence specifically holds in the metric space \((M, R_M)\). Letting \((B^t)_{t \geq 1}\) and \(B^\infty\) be as above, we have that
\[
(B^t)_{t \geq 0} \overset{(d)}{\to} (B^\infty)_{t \geq 0}
\]
as \(t \to \infty\), considered on the space \(C(\mathbb{R}^+, M)\) endowed with the topology of uniform convergence on compact time intervals.

**Theorem 1.4.** Let \((\text{Loop}(T_{\infty}^\alpha), C^{-1}\alpha n^{-\frac{d}{2}} \tilde{d}, \rho, n^{-1} \nu_n)\) be as in Theorem 1.2. Then there exists a probability space \((\Omega, \mathcal{F}, P)\) on which we can almost surely define a common metric space \((M, R_M)\) in which the spaces \((\text{Loop}(T_{\infty}^\alpha), C^{-1}\alpha n^{-\frac{d}{2}} \tilde{d}, \rho, n^{-1} \nu)\) and \((C^{-1}\alpha n^{-\frac{d}{2}} \tilde{d}, \rho, n^{-1} \nu)\) can all be embedded and such that
\[
(\text{Loop}(T_{\infty}^\alpha), C^{-1}\alpha n^{-\frac{d}{2}} \tilde{d}, \rho, n^{-1} \nu_n) \overset{(d)}{\to} (C^{-1}\alpha n^{-\frac{d}{2}} \tilde{d}, \rho, n^{-1} \nu_n)
\]
with respect to the Gromov-Hausdorff-vague topology, and the required Hausdorff convergence specifically holds in the metric space \((M, R_M)\). Letting \(Y^{(n)}\) be a simple random walk on \((\text{Loop}(T_{\infty}^\alpha), C^{-1}\alpha n^{-\frac{d}{2}} \tilde{d}, \rho, n^{-1} \nu_n)\), and \(B^\infty\) be as above, we have that
\[
(C^{-1}\alpha n^{-\frac{d}{2}} Y^{(n)}_{\lfloor (4m^2 + 1)t \rfloor})_{t \geq 0} \overset{(d)}{\to} (B^\infty)_{t \geq 0}
\]
on the space \(D(\mathbb{R}^+, M)\) endowed with the Skorohod-J_1 topology as \(n \to \infty\).

Again, we will prove a similar result for random walks on the other infinite discrete looptrees in Section 7, but the one above is easiest to state as all vertices have degree 4 in \(\text{Loop}(T_{\infty}^\alpha)\). The process \(B^\infty\) is considered further in Section 7, where we prove the following results about the spectral dimension of \(C^{-1}\alpha n^{-\frac{d}{2}} \tilde{d}, \rho, n^{-1} \nu\). Recall that the spectral dimension of \(C^{-1}\alpha n^{-\frac{d}{2}} \tilde{d}, \rho, n^{-1} \nu\) is defined as
\[
d_S(C^{-1}\alpha n^{-\frac{d}{2}} \tilde{d}, \rho, n^{-1} \nu) = \lim_{t \to \infty} \frac{-2 \log(p^\infty_t(\rho, \rho))}{\log t},
\]
where \(p^\infty_t(\cdot, \cdot)\) is the transition density of the Brownian motion \(B^\infty\) defined above. We assume that \(C^{-1}\alpha n^{-\frac{d}{2}} \tilde{d}, \rho, n^{-1} \nu\) is defined on the probability space \((\Omega, \mathcal{F}, P)\), and let \(E\) denote the corresponding expectation.

**Theorem 1.5.** \(P\)-almost surely, \(d_S(C^{-1}\alpha n^{-\frac{d}{2}} \tilde{d}, \rho, n^{-1} \nu) = \frac{2\alpha}{\alpha + 1}\).

In light of Theorem 1.5, we call \(d_S(C^{-1}\alpha n^{-\frac{d}{2}} \tilde{d}, \rho, n^{-1} \nu)\) the quenched spectral dimension. We also define the annealed spectral dimension as
\[
d^a_S(C^{-1}\alpha n^{-\frac{d}{2}} \tilde{d}, \rho, n^{-1} \nu) = \lim_{t \to \infty} \frac{-2 \log(E[p^\infty_t(\rho, \rho)])}{\log t}.
\]
For a general space, the annealed heat kernel is trickier to bound than the quenched one defined above, since the expected transition density may not be finite. This is the case, for example, for the trees with heavy-tailed offspring distributions considered in [CK08]. In the case of stable looptrees however we are able to utilise scaling invariance of \(C^{-1}\alpha n^{-\frac{d}{2}} \tilde{d}, \rho, n^{-1} \nu\) to prove following (more precise) result.

**Theorem 1.6.** We have that
\[
d^a_S(C^{-1}\alpha n^{-\frac{d}{2}} \tilde{d}, \rho, n^{-1} \nu) = \frac{2\alpha}{\alpha + 1}.
\]
Moreover, there exists a constant \(c_1 \in (0, \infty)\) such that \(E[p^\infty_t(\rho, \rho)] = c_1 t^{-\frac{\alpha}{\alpha + 1}}\).

Both the quenched and annealed spectral dimensions the same as for the infinite discrete looptrees defined from offspring distributions in the domain of attraction of an \(\alpha\)-stable law in [BS15]. In our case, the annealed result follows straightforwardly from scaling invariance of \(C^{-1}\alpha n^{-\frac{d}{2}} \tilde{d}, \rho, n^{-1} \nu\), but in the discrete case the authors have to do some extra work to show that the annealed quantity is actually finite.

The results of this paper and in particular Theorem 1.1 are used in the paper [Arc19] to prove various limit results for volumes of small balls in compact stable looptrees, and also to obtain limiting heat kernel estimates.
in the regime $t \downarrow 0$. We refer the reader directly to [Arc19] for more details. Moreover, Richier showed in [Ric17b] that the incipient infinite cluster (IIC) of the Uniform Infinite Half-Planar Triangulation has the structure of an infinite discrete looptree, but where each of the loops are filled with independent critically percolated Boltzmann triangulations. The size of the loops of this looptree are given by a distribution in the domain of attraction of a $\frac{3}{2}$-stable law and an appropriate modification of Theorem 1.2 implies that the boundary of this cluster converges after rescaling to the infinite stable looptree $L^\infty_{3/2}$. The question of the scaling limit of the whole cluster is more subtle and is discussed in [Ric17b Section 6], but we hope the methods used in this article will be a good starting point for studying random walks on the IIC. In particular, we anticipate that such a random walk might fall into a framework similar to the discussions of [ARFK18], in that the looptree forming the boundary of the IIC may play a role analogous to that of the classical Sierpinski gasket in [ARFK18]. If this is the case, then understanding Brownian motion on $L^\infty$ and $L^\infty_\alpha$ is an important preliminary step to understanding the scaling limit of a random walk on the IIC.

This paper is organised as follows. In Section 2 we go over some preliminaries on Lévy processes and stochastic processes associated with resistance forms. In Section 3 we give some background on random trees and looptrees and explain how the stable versions can be coded by Lévy excursions. In Section 4 we give our construction of $L^\infty_\alpha$, which essentially involves replacing the Lévy excursion used to code a compact looptree by two independent Lévy processes. In Section 5 we prove some precise volume and resistance bounds for $L^\infty_\alpha$ by making comparisons with arguments of [Arc19]. We then proceed to prove Theorems 1.1 and 1.2 in Section 6 and explain how these are applied to study compact stable looptrees in [Arc19]. Finally, we conclude with a study of stochastic processes in Section 7 where we use Theorems 1.1 and 1.2 to prove Theorems 1.3 and 1.4 and also prove Theorems 1.5 and 1.6.

Throughout this paper, $C, C', c$ and $c'$ will denote constants, bounded above and below, that may change on each appearance. We will use the notation $B^\infty(x, r)$ to denote the open ball of radius $r$ around $x$ in $L^\infty_\alpha$, and $\overline{B}^\infty(x, r)$ its closure. We will instead use the superscript $\ell$ to denote the corresponding quantities on a compact looptree conditioned to have mass $\ell$.

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2 Preliminaries

2.1 Gromov-Hausdorff-Prohorov Topologies

In order to prove convergence results for measured metric spaces such as looptrees we will work in the pointed Gromov-Hausdorff-Prohorov topology. To define this, first let $F$ denote the set of quadruples $(F, R, \mu, \rho)$ such that $(F, R)$ is a boundedly finite Heine-Borel metric space, $\mu$ is a locally finite Borel measure of full support on $F$, and $\rho$ is a distinguished point of $F$, which we call the root. Let $F^c \subset F$ denote the subset of spaces where $(F, R)$ is compact.

Suppose $(F, R, \mu, \rho)$ and $(F', R', \mu', \rho')$ are elements of $F^c$. The pointed Gromov-Hausdorff-Prohorov distance between them is given by

$$d_{GHP}((F, R, \mu, \rho), (F', R', \mu', \rho')) = \inf_{\varphi, \varphi', M} \left\{ d^H_M(\varphi(F), \varphi'(F')) + d^P_M(\mu \circ \varphi^{-1}, \mu' \circ \varphi'^{-1}) + d_M(\varphi(\rho), \varphi'(\rho')) \right\},$$

where the infimum is taken over all isometric embeddings $\varphi, \varphi'$ of $(F, R)$ and $(F', R')$ respectively into a common metric space $(M, d_M)$. Here $d^H_M$ denotes the Hausdorff distance between two sets in $M$, and $d^P_M$ the Prohorov distance between two measures, as defined in [Bil68 Chapter 1]. It is well-known (for example, see [ADH13 Theorem 2.3]) that this defines a metric on the space of equivalence classes of $F^c$, where we say that two spaces $(F, R, \mu, \rho)$ and $(F', R', \mu', \rho')$ are equivalent if there is a measure and root preserving isometry between them.
The pointed Gromov-Hausdorff distance \(d_{GH}(\cdot, \cdot)\), which is defined by removing the Prohorov term from \([1]\) above, can be helpfully defined in terms of correspondences. A correspondence \(\mathcal{R}\) between \((F, R, \mu, \rho)\) and \((F', R', \mu', \rho')\) is a subset of \(F \times F'\) such that for every \(x \in F\), there exists \(y \in F'\) with \((x, y) \in \mathcal{R}\), and similarly for every \(y \in F'\), there exists \(x \in F\) with \((x, y) \in \mathcal{R}\). We define the distortion of a correspondence by
\[
\text{dis}(\mathcal{R}) = \sup_{(x,y),(x',y') \in \mathcal{R}} |R(x,y) - R(x',y')|.
\]
It is then straightforward to show that
\[
d_{GH}((F, R, \mu, \rho), (F', R', \mu', \rho')) = \frac{1}{2} \inf_{\mathcal{R}} \text{dis}(\mathcal{R}),
\]
where the infimum is taken over all correspondences \(\mathcal{R}\) between \((F, R, \mu, \rho)\) and \((F', R', \mu', \rho')\) that contain the point \((\rho, \rho')\).

In this article, we will prove pointed Gromov-Hausdorff-Prohorov convergence by first proving pointed Gromov-Hausdorff convergence using correspondences, and then show Prohorov convergence of the measures on the appropriate metric space.

For non-compact elements of \(F\), we will need a generalised notion of Gromov-Hausdorff-Prohorov convergence. This is provided by the Gromov-Hausdorff-vague topology of \([ALW16]\). To define it, suppose that \((F, R, \mu, \rho)\) and \((F_n, R_n, \mu_n, \rho_n)_{n \geq 0}\) are elements of \(F \setminus F^c\). For \(r > 0\), we let \(B_r(F)\) denote the quadruple \((B_T(\rho, r), R_{B_T(\rho, r)}, \mu|_{B_T(\rho, r)}, \rho)\), and similarly for \(B_r(F_n)\). Recall that we are restricting to Heine-Borel metric spaces of full support, so that weak convergence is metrized by the Prohorov metric. Following \([ALW16]\) Definition 5.8], we say that \((F_n, R_n, \mu_n, \rho_n)\) converges to \((F, R, \mu, \rho)\) in the Gromov-Hausdorff-vague topology if
\[
d_{GHP}(B_r(F_n), B_r(F)) \to 0
\]
for Lebesgue almost every \(r > 0\). The following proposition will be useful.

**Proposition 2.1.** \([ALW16]\ Proposition 5.12\]. The space of Heine-Borel boundedly finite measure spaces equipped with the Gromov-Hausdorff-vague topology is a Polish space.

### 2.2 Stochastic Processes Associated with Resistance Metrics

To study Brownian motion and random walks on metric spaces we will be using the theory of resistance forms and resistance metrics, developed by Kigami in \([Kig01]\) and \([Kig12]\).

Let \(G = (V, E)\) be a discrete graph equipped with edge conductances \(c(x, y)_{(x, y) \in E}\) and a measure \((\mu(x))_{x \in V}\). Effective resistance on \(G\) is a function \(R\) on \(V \times V\) defined by
\[
R(x,y)^{-1} = \inf\{E(f,f)| f : V \to \mathbb{R}, f(x) = 1, f(y) = 0\},
\]
where \(E(f,f)\) is an energy functional given by
\[
E(f,g) = \frac{1}{2} \sum_{x,y \in V} c(x,y)(f(y) - f(x))(g(y) - g(x)).
\]

\(R(x,y)\) corresponds to the usual physical notion of electrical resistance between \(x\) and \(y\) in \(G\). It can be shown (e.g. see \([Tet91]\)) that \(R\) is a metric on \(G\), and that \(E\) is a Dirichlet form on \(L^2(V, \mu)\).

This definition can be extended to the continuum as follows.

**Definition 2.2.** \([Kig01]\ Definition 2.3.2\]. Let \(F\) be a set. A function \(R : F \times F\) is known as a resistance metric on \(F\) if and only if for every finite subset \(E \subset F\), there exists a weighted graph with vertex set \(V\) such that \(R|_{E \times E}\) is the effective resistance on \(V\).
A resistance metric on a set $F$ can be naturally associated with a stochastic process on $F$ via the theory of resistance forms. In particular, by [Kig01, Theorems 2.3.4 and 2.3.6], there is a one-to-one correspondence between resistance metrics and resistance forms on $F$ (see [Kig12] for more on resistance forms). Moreover, if the corresponding resistance form is regular (see [Kig12, Definition 6.2] for the definition of regular), then it can be used to define a regular Dirichlet form on the space $L^2(F,\mu)$ in a natural way, which in turn is naturally associated with a Hunt process on $F$ as a consequence of [FOT01] Theorem 7.2.1. This is automatically the case when $(F,R)$ is a compact resistance metric space endowed with a finite Borel measure $\mu$ of full support, for example, but in the case of infinite looptrees we will have to put some extra work into proving that the resistance form associated with $L^\infty$ is regular. This is done in Proposition 7.1.

We have tried to keep background on resistance forms and Dirichlet forms to a minimum in this article, but see [Kig12] for more on this.

This correspondence allows us to use results about scaling limits of measured resistance metric spaces to prove results about scaling limits of stochastic processes as detailed in the following result of [Cro18].

**Theorem 2.3.** ([Cro18, Theorem 1.2]). Suppose that $(F_n,R_n,\mu_n,\rho_n)_{n\geq 0}$ is a sequence in $\mathbb{F}$ such that

$$(F_n,R_n,\mu_n,\rho_n) \to (F,R,\mu,\rho)$$

Gromov-Hausdorff-vaguely for some $(F,R,\mu,\rho)$ and $R,(R_n)_{n\geq 1}$ are resistance metrics on the respective spaces. Assume further that

$$\lim_{n\to\infty} \liminf_{r\to\infty} R_n(\rho_n,B_n(\rho_n,r)^c) = \infty. \quad (3)$$

Let $(Y^n_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ be the stochastic processes respectively associated with $(F_n,R_n,\mu_n,\rho_n)$ and $(F,R,\mu,\rho)$ as described above. Then it is possible to isometrically embed $(F_n,R_n)_{n\geq 1}$ and $(F,R)$ into a common metric space $(M,d_M)$ so that

$$\mathbb{P}_{\rho_n}((Y^n_t)_{t\geq 0} \in \cdot) \to \mathbb{P}_{\rho}((Y_t)_{t\geq 0} \in \cdot)$$

weakly as probability measures as $n \to \infty$ on the space $D(\mathbb{R}_+,M)$ equipped with the Skorohod $J_1$-topology.

For more on the Skorohod-$J_1$ topology, see [Bill99, Chapter 3]. The intuition behind the result above is that the convergence of metrics and measures respectively give the appropriate spatial and temporal convergences of the stochastic processes. We will apply it several times in this paper to take limits of stochastic processes on looptrees.

### 2.3 Stable Lévy Excursions

Following the presentations of [Duq03] and [CK14], we now introduce stable Lévy excursions, which will be used to code stable trees and looptrees in Section 3.

Given an interval $I \subset \mathbb{R}$, we first recall that $D(I,\mathbb{R})$ represents the space of càdlàg functions from $I$ to $\mathbb{R}$. For an interval $[0,\ell] \subset \mathbb{R}$, we also define the càdlàg excursion space $D^{exc}([0,\ell],\mathbb{R})$ to be the space

$$D^{exc}([0,\ell],\mathbb{R}) = \{ e \in D([0,\ell],\mathbb{R}_{\geq 0}) : e(0) = e(\ell) = 0, e(t) > 0 \text{ for all } t \in (0,\ell) \}.$$ 

Throughout this article, we take $\alpha \in (1,2)$, and $X$ will be an $\alpha$-stable spectrally positive Lévy process as in [Ber96, Section 8], normalised so that

$$\mathbb{E}[e^{-\lambda X_t}] = e^{\lambda^\alpha t}$$

for all $\lambda > 0$. $X$ takes values in the space $D([0,\infty),\mathbb{R})$ of càdlàg functions, endowed with the Skorohod-$J_1$ topology, and satisfies the scaling property that for any constant $c > 0$, $(e^{-\frac{t}{c}}X_{ct})_{t\geq 0}$ has the same law as $(X_t)_{t\geq 0}$. $X$ has Lévy measure

$$\Pi(dx) = \frac{\alpha(\alpha - 1)}{\Gamma(2-\alpha)} x^{-\alpha-1} 1_{(0,\infty)}(x)dx.$$

To define a normalised excursion of $X$, we follow [Chin97] and let $X_s = \inf_{t\in[0,\ell]} X_t$ denote its running infimum process, and set

$$g_1 = \sup\{ s \leq 1 : X_s = X_s \}, \quad d_1 = \inf\{ s > 1 : X_s = X_s \}.$$
Note that $X_{g_1} = X_{d_1}$ almost surely. Following [Cha97, Proposition 1], we define the normalised excursion $X^\text{exc}$ of $X$ above its infimum at time 1 by

$$X^\text{exc}_s = (d_1 - g_1)^{-\frac{1}{\alpha}}(X_{g_1 + s(d_1 - g_1)} - X_{g_1})$$

for every $s \in [0, 1]$. $X^\text{exc}$ is almost surely an $\alpha$-stable càdlàg function on $[0, 1]$ with $X^\text{exc}(s) > 0$ for all $s \in (0, 1)$, and $X^\text{exc}_0 = X^\text{exc}_1 = 0$.

### 2.3.1 Itô excursion measure

We can alternatively define $X^\text{exc}$ using the Itô excursion measure. For full details, see [Ber96, Chapter IV], but the measure is defined by applying excursion theory to the process $X - \overline{X}$, which is strongly Markov for which the point 0 is regular for itself. We normalise local time so that $-\overline{X}$ denotes the local time of $X - \overline{X}$ at its infimum, and let $(g_j, d_j)_{j \in I}$ denote the excursion intervals of $X - \overline{X}$ away from zero. For each $i \in I$, the process $(e^i_0 \leq s \leq d_i - g_i)$ defined by

$$e^i(s) = X_{g_i} + s - X_{g_i}$$

is an element of the excursion space $E = \bigcup_{\ell > 0} D^\text{exc}([0, \ell], \mathbb{R})$.

$\zeta(e) = \sup\{s > 0 : e(s) > 0\}$ is called the lifetime of the excursion $e$. It was shown in [Ito72] that the measure

$$N(dt, de) = \sum_{i \in I} \delta(-X_{g_i}, e^i)$$

is a Poisson point measure of intensity $dtN(de)$, where $N$ is a $\sigma$-finite measure on the set $E$ known as the Itô excursion measure.

The measure $N(\cdot)$ also inherits a scaling property from the $\alpha$-stability of $X$. Indeed, for any $\lambda > 0$ we define a mapping $\Phi_\lambda : E \to E$ by $\Phi_\lambda(e)(t) = \lambda e(\frac{t}{\lambda})$. It is shown in [Wat10] (and should be clear from the scaling property of $X$) that $N \circ \Phi_\lambda^{-1} = \lambda^{\frac{1}{\alpha}}N$.

More concretely, it follows from the results in [Ber96, Section IV.4] that we can uniquely define a set of conditional measures $(N(s), s > 0)$ on $E$ such that:

- (i) For every $s > 0$, $N(s)(\zeta = s) = 1$.
- (ii) For every $\lambda > 0$ and every $s > 0$, $\Phi_\lambda(N(s)) = N(\lambda s)$.
- (iii) For every measurable $A \subset E$

$$N(A) = \int_0^\infty \frac{N(s)(A)}{\alpha \Gamma(1 - \frac{1}{\alpha}) s^{\frac{1}{\alpha} + 1}} ds.$$  

$N(s)$ is therefore used to denote the law $N(\cdot | \zeta = s)$. The probability distribution $N(1)$ coincides with the law of $X^\text{exc}$ as constructed above.

### 2.3.2 Relation between $X$ and $X^\text{exc}$

It is easier to analyse an unconditioned Lévy process rather than an excursion, so throughout this paper we will use the following two tools to compare the probability of an event defined in terms of $X^\text{exc}$ to that of the same event defined in terms of $X$. The first tool is the Vervaat transform of the following proposition, which allows us to compare to a stable bridge $X^\text{br}$ as an intermediate step. This is particularly useful as we will at times consider our looptrees to be rooted at a uniform point.

**Theorem 2.4.** [Cha97, Théorème 4]. Vervaat Transform.
1. Let $X^{\text{exc}}$ be as above, and take $U \sim \text{Uniform}([0,1])$. Then the process $(X_t^{br})_{0 \leq t \leq 1}$ defined by

$$X_t^{br} = \begin{cases} X_{U+t}^{exc} & \text{if } U + t \leq 1, \\ X_{U+t-1}^{exc} & \text{if } U + t > 1. \end{cases}$$

has the law of a spectrally positive stable Lévy bridge on $[0,1]$. 

2. Now let $X^{br}$ be a spectrally positive stable Lévy bridge on $[0,1]$, and let $m$ be the (almost surely unique) time at which it attains its minimum. Define an excursion $X^{\text{exc}}$ by

$$X_t^{\text{exc}} = \begin{cases} X_{m+t}^{br} & \text{if } m + t \leq 1, \\ X_{m+t-1}^{br} & \text{if } m + t > 1. \end{cases}$$

Then $X^{\text{exc}}$ has the law of a spectrally positive stable Lévy excursion.

An event defined for the stable bridge on the interval $[0,T]$ can then be transferred to the unconditioned process using the fact that the law of the bridge is absolutely continuous with respect to the law of the process, with Radon-Nikodym derivative

$$p_{1-T}(-X_T) / p_{1}(0)$$

for $T \in (0,1)$ (see [Ber96] Section VIII.3, Equation (8)).

2.3.3 Other useful results

Next, we introduce the notion of a descent of a Lévy process, following the presentation of [CK14] Section 3.1.3. Let $X^1$ and $X^2$ be two independent spectrally positive $\alpha$-stable Lévy processes as defined above, and define a two-sided process $X$ by setting

$$X_t = \begin{cases} X_t^1 & \text{if } t \geq 0, \\ -X_{t-}^2 & \text{if } t < 0. \end{cases}$$

For every $s, t \in \mathbb{R}$, we write $s \leq t$ if and only if $s \leq t$ and $X_{s-} \leq \inf_{s \leq t} X$, and in this case we set

$$x_s^t(X) = \inf_{s \leq t} X - X_{s-}, \text{ and } u_s^t(X) = \frac{x_s^t(X)}{\Delta X_s}.$$ 

We write $s < t$ if $s \leq t$ and $s \neq t$. As in [CK14], for any $t \in \mathbb{R}$, we will call the collection $\{x_s^t(X), u_s^t(X) : s \leq t\}$ the descent of $t$ in $X$.

The next proposition describes the law of descents from a typical point of $X$. We let $X_s = \sup \{X_s : 0 \leq s \leq t\}$ denote the running supremum process of $X$. The process $\overline{X} - X$ is strong Markov and 0 is regular for itself, allowing the use of the excursion theory. Let $(L_t)_{t \geq 0}$ denote the local time of $\overline{X} - X$ at 0. Note that, by [Ber96] Chapter VIII, Lemma 1, $L^{-1}$ is a $(1 - \frac{1}{\alpha})$-stable subordinator, and $(X_{L^{-1}(t)})_{t \geq 0}$ is an $(\alpha - 1)$-stable subordinator, so we can normalise so that $E[\exp(-\lambda X_{L^{-1}(t)})] = \exp(-t\lambda^{\alpha-1})$ for all $\lambda > 0$. Finally, if $X_s > \overline{X}_{s-}$, set

$$x_s = \overline{X}_s - \overline{X}_{s-}, \quad u_s = \frac{\overline{X}_s - \overline{X}_{s-}}{\overline{X}_s - \overline{X}_{s-}}.$$ 

**Proposition 2.5.** ([CK14] Proposition 3.1, [Ber92] Corollary 1). Let $X$ be a two-sided spectrally positive $\alpha$-stable process as above. Then

(i)

$$\{(-s, x_s^0(X), u_s^0(X)) : s \leq 0\} \overset{(d)}{=} \{s, x_s, u_s : s \geq 0 \text{ such that } \overline{X}_s > \overline{X}_{s-}\}.$$
(ii) The point measure

\[ \sum_{X_s > X_s -} \delta(L_s, u_s) \]

is a Poisson point measure with intensity \( dl \cdot z \Pi(dx) \cdot 1_{[0,1]}(u) du \).

We also give a technical lemma which will be used at various points in the paper. This appeared previously in [CK14, Section 3.3.1] and uses an argument from [Ber96]. The final claim follows by bounded convergence.

First recall that for a function \( f : [0, \infty) \to \mathbb{R} \) and \([a, b] \subset [0, \infty)\), we define

\[ \text{Osc}_{[a,b]} f := \sup_{s,t \in [a,b]} |f(t) - f(s)|. \]

Lemma 2.6. Let \( E \) be an exponential random variable with parameter 1, and let \( X \) be a spectrally positive \( \alpha \)-stable Lévy process conditioned to have no jumps of size greater than 1 on \([0, E]\). Let \( \tilde{\text{Osc}} = \text{Osc}_{[0,E]} X \).

Then there exists \( \theta > 0 \) such that \( \mathbb{E} \left[ e^{\theta \tilde{\text{Osc}}} \right] < \infty \). Moreover, \( \mathbb{E} \left[ e^{\theta \tilde{\text{Osc}}} \right] \downarrow 1 \) as \( \theta \downarrow 0 \).

Remark 2.7. The same results holds if \( E \) is set to be deterministically equal to 1 rather than an exponential random variable. The proof is almost identical to the one above, with one minor modification.

3 Background on Stable Trees and Looptrees

3.1 Discrete Trees

Before defining stable trees and looptrees, we briefly recap some notation for discrete trees, following the formalism of [Nev86]. Firstly, let \( U = \bigcup_{n=0}^{\infty} \mathbb{N}^n \) be the Ulam-Harris tree. By convention, \( \mathbb{N}^0 = \{\emptyset\} \).

If \( u = (u_1, \ldots, u_n) \) and \( v = (v_1, \ldots, v_m) \in U \), we let \( uv = (u_1, \ldots, u_n, v_1, \ldots, v_m) \) be the concatenation of \( u \) and \( v \).

Definition 3.1. A plane tree \( T \) is a finite subset of \( U \) such that

(i) \( \emptyset \in T \),
(ii) If \( v \in T \) and \( v = uv \) for some \( j \in \mathbb{N} \), then \( u \in T \),
(iii) For every \( u \in T \), there exists a number \( k_u(T) \geq 0 \) such that \( uj \in T \) if and only if \( 1 \leq j \leq k_u(T) \).

We let \( T \) denote the set of all plane trees.

A plane tree \( T \in T \) with \( n+1 \) vertices labelled according to the lexicographical order as \( u_0, u_1, \ldots, u_n \) can be coded by its height function, contour function, or Lukasiewicz path, defined as follows.

- The height function \( (H^T_m)_{0 \leq m \leq n} \) is defined by considering the vertices \( u_0, u_1, \ldots, u_n \) in lexicographical order, and then setting \( H^T_1 \) to be the generation of vertex \( u_i \).
- The contour function \( (C^T_t)_{0 \leq t \leq 2n} \) is defined by considering a particle that starts at the root \( \emptyset \) at time zero, and then continuously traverses the boundary of \( T \) at speed one, respecting the lexicographical order where possible, until returning to the root. \( C^T(t) \) is equal to the height of the particle at time \( t \).
- The Lukasiewicz path \( (W^T_m)_{0 \leq m \leq n} \) is defined by setting \( W^T_0 = 0 \), then by considering the vertices \( u_0, u_1, \ldots, u_n \) in lexicographical order and setting \( W^T_{m+1} = W^T_m + k_{u_m}(T) - 1 \).
These are illustrated in Figure 4, together with points corresponding to specific vertices in the tree, and the part of each excursion coding the subtree rooted at the red vertex, which we denote by \( \tau_1(T) \). For further details, see [DLG02, Section 0.1].

These functions all uniquely define the tree \( T \). This can be written particularly conveniently in the case of the contour function, since for any \( s, t \in \{0, \ldots, 2(n-1)\} \), we can define a distance function on \( \{0, \ldots, 2(n-1)\} \times \{0, \ldots, 2(n-1)\} \) by

\[
d^T(s,t) = C^T(s) + C^T(t) - 2 \inf_{s \leq r \leq t} C^T(r),
\]

and this corresponds to the distance between the points corresponding to \( s \) and \( t \) in the tree \( T \).

We will work mainly with the Lukasiewicz path \( (W^T_m)_{0 \leq m \leq n} \) in this paper. It is not too hard to see that \( W^T_m \geq 0 \) for all \( 0 \leq m \leq n-1 \), and \( W^T_n = -1 \). Moreover, the height function can be defined as a function of the Lukasiewicz path (see [DLG02, Equation (1)]) by setting

\[
H^T(m) = \left| \left\{ k \in \{0, 1, \ldots, m-1\} : W^T_k = \inf_{k \leq l \leq m} W^T_l \right\} \right|.
\]

### 3.2 Stable Trees

We now introduce stable trees. These are closely related to stable looptrees, and were introduced by Le Gall and Le Jan in [GL98] and Duquesne and Le Gall in [DLG02, DLG05]. For \( \alpha \in (1, 2) \) we define the stable tree \( T_\alpha \) from a spectrally positive \( \alpha \)-stable Lévy excursion, which plays the role of the Lukasiewicz path introduced above. Given such an excursion \( X^\text{exc} \), we define the height function \( H^\text{exc} \) to be the continuous modification of the process satisfying

\[
H^\text{exc}(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t 1\{X^\text{exc}_s < I^*_s + \varepsilon\} ds,
\]

where the limit exists in probability (e.g. see [DLG02, Lemma 1.1.3]). We define a distance function on \([0, 1]\) by

\[
d(s,t) = H^\text{exc}(s) + H^\text{exc}(t) - 2 \inf_{s \leq r \leq t} H^\text{exc}(r),
\]

and an equivalence relation on \([0, 1]\) by setting \( s \sim t \) if and only if \( d(s,t) = 0 \). \( T_\alpha \) is the quotient space \( ([0, 1]/\sim, d) \), and we let \( \pi \) denote the canonical projection from \([0, 1]\) to \( T_\alpha \). If \( u, v \in T_\alpha \), we let \([u, v]\) denote the unique geodesic between \( u \) and \( v \) in \( T_\alpha \).

This construction also provides a natural way to define a measure \( \mu \) on \( T_\alpha \) as the image of Lebesgue measure on \([0, 1]\) under the quotient operation.
Stable trees arise naturally as scaling limits of discrete plane trees with appropriate offspring distributions. More specifically, let $T_n$ be a discrete tree conditioned to have $n$ vertices and with critical offspring distribution $\xi$ in the domain of attraction of an $\alpha$-stable law, i.e. a law $\xi$ such that $\xi([k,\infty)) \sim c k^{-\alpha}$ as $k \to \infty$ for some $c \in (0,\infty)$. We then have that

$$n^{-(1-\frac{1}{\alpha})} T_n \xrightarrow{d} C_\alpha \cdot T_\alpha$$

in the Gromov-Hausdorff topology as $n \to \infty$, where $C_\alpha = (c \Gamma(-\alpha))^{-\frac{1}{\alpha}}$, as in Theorem 1.2.

### 3.2.1 Re-rooting Invariance for Stable Trees and Loop Trees

In [DLG05], Duquesne and Le Gall prove that stable Lévy trees are invariant under uniform rerooting. More formally, if $U$ is a uniform point in $[0,1]$, and we define a new height function $H^U : [0,1] \to \mathbb{R}$ from the original height function $H^{\text{exc}}$ by

$$H^U(x) = \begin{cases} H^{\text{exc}}(U) + H^{\text{exc}}(U + x) - 2 \min_{0 \leq s \leq U + x} H^{\text{exc}}(s) & \text{if } U + x \leq 1, \\ H^{\text{exc}}(U) + H^{\text{exc}}(U + x - 1) - 2 \min_{U + x - 1 \leq s \leq U} H^{\text{exc}}(s) & \text{if } U + x > 1, \end{cases}$$

then $H^U \xrightarrow{d} H^{\text{exc}}$. This property is just saying that if we pick a uniform point $U \in [0,1]$, and reroot the tree $T_\alpha$ at $\pi(U)$, then the resulting tree has the same distribution as the original one.

The problem of uniform rerooting invariance of continuum fragmentation trees was also considered in the paper [HPW09], where the authors additionally show that stable trees are the only fragmentation trees for which this property holds. Duquesne and Le Gall also prove a similar result for rerooting at a deterministic point $u \in [0,1]$ in the paper [DLG09], and [CK14] Remark 4.6 addresses the question of uniform rerooting invariance for stable looptrees.

### 3.3 Random Loop Trees

Discrete looptrees are best described by Figure 1 in the introduction. As outlined there, stable loop trees can be defined as scaling limits of their discrete counterparts. That is, if $T_n$ is a Galton Watson tree conditioned to have $n$ vertices with offspring distribution $\xi$ such that $\xi([k,\infty)) \sim c k^{-\alpha}$ as $k \to \infty$ for some $c \in (0,\infty)$, then

$$n \xrightarrow{d} \text{Loop}(T_n) \xrightarrow{d} (c \Gamma(-\alpha))^{-\frac{1}{\alpha}} L_\alpha$$

with respect to the Gromov-Hausdorff topology as $n \to \infty$ (see [CK14] Theorem 4.1).

By comparison with (4), $L_\alpha$ can therefore be thought of as the looptree version of the Lévy tree $T_\alpha$. We now explain how this intuition can be used to code $L_\alpha$ from a stable Lévy excursion, in such a way that $L_\alpha$ can be heuristically obtained from the corresponding stable tree $T_\alpha$ by replacing each branch point by a loop with length proportional to the size of the branch point, and gluing these loops together along the tree structure of $T_\alpha$.

This construction was introduced in [CK14] Section 2.3. The Lévy excursion itself plays the role of a continuum Łukasiewicz path. It was shown in [Mie05, Proposition 2] that if we define the width of a branch point in $T_\alpha$, coded by a jump at $t \in [0,1]$, by

$$\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \mu(\{v \in T_\alpha, d(\pi(t), v) \leq \varepsilon\}),$$

then the limit almost surely exists and is equal to $\Delta_t$. It is therefore natural that a jump of size $\Delta$ in $X^{\text{exc}}$ should code a loop of length $\Delta$ in $L_\alpha$.

Accordingly, using the notation of Section 2.3.3 for every $t \in [0,1]$ with $\Delta_t > 0$, the authors in [CK14] Section 2.3 equip the segment $[0,\Delta_t]$ with the pseudodistance

$$\delta_t(a,b) = \min\{|a-b|, (\Delta_t - |a-b|)\}, \quad \text{for } a,b \in [0,\Delta_t],$$

(5)
and define a distance function on \([0, 1]\) by first setting
\[
d_0(s, t) = \sum_{s < u \leq t} \delta_u(0, x_u^t)
\]
whenever \(s \leq t\), and
\[
d(s, t) = \delta_{s \wedge t}(x_{s \wedge t}^s, x_{s \wedge t}^t) + d_0(s \wedge t, s) + d_0(s \wedge t, t)
\]
for arbitrary \(s, t \in [0, 1]\).

They show that \(d\) as defined above is almost surely a continuous pseudodistance on \([0, 1]\), and then define an equivalence relation \(\sim\) on \([0, 1]\) by setting \(s \sim t\) if \(d(s, t) = 0\), and define the stable looptree \(L^\alpha\) as the quotient space
\[
L^\alpha = ([0, 1]/ \sim, d)
\]
in [CK14] Definition 2.3. We let \(p : [0, 1] \to L^\alpha\) denote the canonical projection under the quotient operation, and let \(\nu\) denote the image of Lebesgue measure on \([0, 1]\) under \(p\). \(\nu\) therefore denotes the natural analogue of uniform measure on \(L^\alpha\).

In [Arc19], we also define a resistance metric \(R\) on stable looptrees. By analogy with the construction above, this is done by first replacing \(\delta_u\) with the quantity \(r_\Delta\) defined by
\[
\Delta \alpha = \left( \sum_{s < u \leq t} |a - b|(\Delta_\tau - |a - b|) \right) \Delta_\Delta, \quad \text{for } a, b \in [0, \Delta_\tau].
\]
Note that this corresponds to the effective resistance across two parallel edges of lengths \(|a - b|\) and \(|\Delta_\tau - |a - b||\).

For \(s, t \in [0, 1]\) with \(s \leq t\), we then set
\[
R_\tau(s, t) = \sum_{s < u \leq t} r_\tau(0, x_u^t).
\]
For arbitrary \(s, t \in [0, 1]\), we set
\[
R(s, t) = \frac{1}{\Delta_\tau} \left( \frac{1}{\Delta_\tau} + \frac{1}{\Delta_\tau} \right) \Delta_\Delta, \quad \text{for } a, b \in [0, \Delta_\tau].
\]

We show in [Arc19] that \(R\) defined in this way is a resistance metric on \(L^\alpha\) in the sense of Definition 2.2. Moreover, in [Arc19] Lemma 4.1 we show that for any \(s, t \in [0, 1]\), we have that \(\frac{1}{2}d(s, t) \leq R(s, t) \leq d(s, t)\), and define the resistance looptree \(L^R\) (which we sometimes denote \((L^\alpha, R)\)) as
\[
L^\alpha = ([0, 1]/ \sim, R).
\]
As a consequence, we also show in [Arc19] Corollary 4.2 that the looptrees \((L^\alpha, d)\) and \((L^\alpha, R)\) are homeomorphic.

A jump of size \(\Delta\) therefore corresponds naturally to a cycle of length \(\Delta\) in \(L^\alpha\), which we will call a “loop”.

A key result of the [CK14] is a Gromov-Hausdorff invariance principle. We extended the result to include convergence of measures in [Arc19] Proposition 4.6. Moreover, the Gromov-Hausdorff convergence of [CK14] Theorem 4.1] was originally stated with the geodesic metric \(d\) in place of the resistance metric \(R\), but equally holds for \(R\). This results in the following proposition.

**Proposition 3.2.** Let \((\tau_n)_{n=1}^\infty\) be a sequence of trees with \(|\tau_n| \to \infty\) and corresponding Lukasiewicz paths \((W^n)_{n=1}^\infty\), and let \(R_n\) denote the effective resistance metric on \(\text{Loop}(\tau_n)\) obtained by letting an edge between any two adjacent vertices have length 1. For a given realisation of \(\tau_n\), this can be computed explicitly using the series and parallel laws for effective resistance. Additionally let \(\nu_n\) be the uniform measure that gives mass 1 to each vertex of \(\text{Loop}(\tau_n)\), and let \(\rho_n\) be the root of \(\text{Loop}(\tau_n)\), defined to be the vertex representing the edge joining the root of \(\tau_n\) to its first child. Suppose that \((C_n)_{n=1}^\infty\) is a sequence of positive real numbers such that
\[
(i) \quad \left( \frac{1}{C_n} W^n_{|\tau_n| t}(\tau_n) \right)_{0 \leq t \leq 1} \xrightarrow{d} X_{exc} \text{ as } n \to \infty,
\]

\begin{enumerate}
\item 
\end{enumerate}
(ii) $\frac{1}{C_n} \text{Height}(\tau_n) \xrightarrow{p} 0$ as $n \to \infty$.

Then

$$\left(\text{Loop}(\tau_n), \frac{1}{C_n} R_n, \frac{1}{n} \nu_n, \rho_n\right) \xrightarrow{(d)} \left(\mathcal{L}_\alpha, R, \nu, \rho\right)$$

as $n \to \infty$ with respect to the Gromov-Hausdorff-Prohorov topology.

More generally, if $f$ is a function in $D^{exc}([0, \ell])$ for some $\ell \in (0, \infty)$, with only positive jumps, we can replace $X^{exc}$ with $f$ in the construction above to define the associated continuum looptree $\mathcal{L}_f$. Moreover, if $f_n$ is a sequence in $D^{exc}([0, \ell])$ converging to $f$, also all with only positive jumps, then we can prove a similar invariance principle for the sequence of corresponding continuum looptrees.

There are minor differences in the assumptions required for the continuum convergence. In particular, note that the second condition of Proposition 3.2 that $\frac{1}{C_n} \text{Height}(\tau_n) \to 0$ as $n \to \infty$ is important there because it ensures that in the limit, distances in the rescaled discrete looptrees come from the loop structure and not from the height of the corresponding tree. More formally, in the proof of [CK14, Theorem 4.1] it is used to make a comparison between the expressions $\frac{1}{C_n} \sum_{u_n \leq v_n} x^{v_n}_{u_n}$ and $\sum_{u \leq v} x^v_u$ for the discrete and continuum trees respectively, where $x^{v_n}_{u_n}$ is the discrete analogue of $x^v_u$. For a sequence of trees $\tau_n$ with $W^n \to f$ in the setting of Proposition 3.2, we have for any $v_n \in \text{Loop}(\tau_n)$ and $v \in \mathcal{L}_f$ that

$$\sum_{u_n \leq v_n} x^{v_n}_{u_n} = \text{Height}(v_n) + W^n(v_n), \quad \sum_{u \leq v} x^v_u = f(v). \quad (10)$$

If $v$ and $v_n$ are in correspondence with each other, after being careful with left and right limits we can essentially apply the result that $\frac{1}{C_n} W^n(v_n) \to f(v)$ to deduce that the $\frac{1}{C_n} \sum_{u_n \leq v_n} x^{v_n}_{u_n}$ also converges to $\sum_{u \leq v} x^v_u$ in the limit to prove the invariance. To obtain this result, it is therefore crucial that the contribution from the height function goes to zero.

If, however, we replace the sequence of rescaled discrete looptrees with a sequence of continuum looptrees, say coded by the functions $\left(f_n\right)_{n \geq 1}$ each with support $[0, 1]$ and such that $f_n \to f$ in the Skorohod-$J_1$ topology as $n \to \infty$, then the height function won’t appear in any of the new terms in (10) and so the continuum analogue of condition (ii) of Proposition 3.2 is not required for convergence of the corresponding looptrees.

In this sense, condition (ii) reflects the fact the looptree $\text{Loop}(\tau_n)$ isn’t quite the same as the looptree $\mathcal{L}_{W^n}$. Condition (ii) is precisely what is required to say that the difference between $\text{Loop}(\tau_n)$ and $\mathcal{L}_{W^n}$ becomes negligible in the limit.

Hence, in the continuum, the same proof gives the following result.

**Proposition 3.3.** Let $\left(f_n\right)_{n \geq 1}$ be a sequence in $D([0, 1], \mathbb{R})$, and $f \in D^{exc}([0, 1], \mathbb{R})$ be such that $f_n \to f$ as $n \to \infty$ with respect to the Skorohod-$J_1$ topology. Additionally let $\nu$ and $\nu_n$ be the projections of Lebesgue measure via $p_f$ and $p_{f_n}$ onto the spaces $\mathcal{L}_f$ and $\mathcal{L}_{f_n}$ respectively. Then

$$d_{GHP}\left(\left(\mathcal{L}_{f_n}, \hat{d}_n, \nu_n, \rho_n\right), \left(\mathcal{L}_f, \hat{d}_f, \nu_f, \rho_f\right)\right) \to 0$$

as $n \to \infty$.

Here $\hat{d}$ can denote either the shortest-distance metric of [CK14], or the resistance metric of [9], but defined using the function $f$ in place of $X^{exc}$.

The result follows exactly as in the proof of [CK14, Theorem 4.1] by defining a correspondence between $\mathcal{L}_f$ and $\mathcal{L}_{f_n}$ to consist of all pairs $(t, \lambda_n(t))$, where $\lambda_n$ is the Skorohod homeomorphism that minimises the Skorohod distance between $f_n$ and $f$. The extension to include convergence of measures can be obtained exactly as in [Arc19, Proposition 4.6].

Clearly the result of the proposition will hold for functions defined on any compact time interval, not just $[0, 1]$. We will use this in Section 6 to prove Theorem 1.1.
At some points in this paper, we will refer to the “corresponding” or “underlying” stable tree of $\mathcal{L}_\alpha$, by which we mean the stable tree $\mathcal{T}_\alpha$ coded by the same excursion that codes $\mathcal{L}_\alpha$. We let $\mathcal{L}_\alpha$ denote a compact stable looptree conditioned on $\nu(\mathcal{L}_\alpha) = 1$, but at various points we will let $\hat{\mathcal{L}}_\alpha$ denote a generic stable looptree coded by an excursion under the Itô measure but without any conditioning on its total mass. We will also let $\mathcal{L}_\alpha$ denote a stable looptree but conditioned so that its underlying tree has height 1. However, we will make this explicit at the time of writing.

The height of a stable tree $\mathcal{T}_\alpha$ is defined as $H_{\max} = \sup_{u \in \mathcal{T}_\alpha} d_{\mathcal{T}_\alpha}(\rho, u)$. As the height process is almost surely continuous, this maximum is almost surely realised by at least one $u \in \mathcal{T}_\alpha$. Moreover, we see from [DW17, Equation (23)] (and references therein) that there is almost surely a unique $\mathcal{L}$-Height to $\mathcal{T}_\alpha$, and such that a proportion $u$ of the loop is on the “left” of the $\mathcal{L}$-Height.

(i) We define its $L^W$-Height to be the looptree distance from $\rho$ to $u_H$,

(ii) We define its $L^m$-Height to be $\sup_{u \in \mathcal{L}_\alpha} d_{\mathcal{L}_\alpha}(\rho, u)$.

In general, these are not the same. At times, we will also use the notation $T^W$-Height and $T^m$-Height to denote the length of the corresponding spine in the underlying tree.

3.3.1 Williams’ Decomposition of Stable Looptrees

The Williams’ Decomposition for stable trees was given in [AD09]. There, the authors show that if we define the W-spine of a stable Lévy tree $\mathcal{T}_\alpha$ to be the unique path from its root to $u_H$, then $\mathcal{T}_\alpha$ can be broken along this W-spine and that the resulting fragments form a collection of smaller Lévy trees. As a consequence, we immediately have a similar decomposition result for looptrees.

The Williams’ Decomposition for stable trees given in [AD09] encodes this decomposition of $\mathcal{T}_\alpha$ along its W-spine in a Poisson process. In the Brownian case of $\alpha = 2$, this corresponds to Williams’ decomposition of Brownian motion. Letting $H_{\max}$ and $u_H$ be as above, we define the Williams’ spine (or W-spine) of $\mathcal{T}_\alpha$ to be the segment $[\rho, u_H]$, and take the Williams’ loops (or W-loops) in the corresponding looptree $\mathcal{L}_\alpha$ to be the closure of the set of loops coded by points in $[\rho, u_H]$. One of the main results of [AD09] is a theorem which firstly gives the distribution of the loop lengths along the W-loopspine, and additionally the distribution of the fragments obtained by decomposing along it.

Given the spine from $\rho$ to $u_H$, and conditional on $H_{\max} = H$, the loops along the W-loopspine can be represented by a Poisson point measure $\sum_{j \in J} \delta(l_j, t_j, u_j)$ on $\mathbb{R}^+ \times [0, H] \times [0, 1]$ with a certain intensity. A point $(l, t, u)$ corresponds to a loop of length $l$ in the W-loopspine, occurring on the W-spine at distance $t$ from the root in the corresponding tree $\mathcal{T}_\alpha$, and such that a proportion $u$ of the loop is on the “left” of the W-loopspine, and a proportion $1 - u$ is on the “right”. In [AD09], this is written in terms of the exploration process on $\mathcal{T}_\alpha$, but we interpret their result below in the context of looptrees.

We note that when stating this result, we are not conditioning on the total mass of $\mathcal{T}_\alpha$: only the maximal height. In particular, the mass of $\mathcal{T}_\alpha$ will depend on its height via the joint laws for these under the Itô excursion measure.

**Theorem 3.4.** (Follows directly from [AD09, Lemma 3.1 and Theorem 3.2]).

(i) Conditionally on $H_{\max} = H$, the set of loops in the W-loopspine forms a Poisson point process $\mu_{\text{W-loopspine}} = \sum_{j \in J} \delta(l_j, t_j, u_j)$ on the W-spine in the corresponding tree with intensity

$$1_{[0,1]}(u)1_{[0,H]}(t)l!\exp\{-l(H-t)^{\frac{\alpha}{\alpha-1}}\}du dt \Pi(dl),$$

where $\Pi$ is the underlying Lévy measure, with $\Pi(dl) = \frac{1}{\Gamma(\alpha)}l^{\alpha-1}1_{(0,\infty)}(l)dl$ in the stable case. We will denote the atom $\delta(l_j, t_j, u_j)$ by Loop$_j$. 

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(ii) Let $\delta(l, t, u)$ be an atom of the Poisson process described above. The set of sublooptrees grafted to the $W$-loopspine at a point in the corresponding loop can be described by a random measure $M(l) = \sum_{i \in I} \delta(l)(E_i, D_i)$, where $E_i$ is a Lévy excursion that codes a looptree in the usual way, and $D_i$ represents the distance going clockwise around the loop from the point at which this sublooptree is grafted to the loop, to the point in the loop that is closest to $\rho$. This measure has intensity

$$N(\cdot, H_{\text{max}} \leq H - t) \times 1_{([0, l])} dD.$$

In particular, since the sublooptrees are coded by the Itô excursion measure, they are just rescaled copies of our usual normalised compact stable looptrees, and each of these is grafted to the loop on the $W$-loopspine at a uniform point around the loop lengths.

**Remark 3.5.** Point (ii) is a slight extension of the results of [AD09] since the authors of that paper are only concerned with stable trees, and consequently are not interested in how the sublooptrees are distributed around the each loop in the $W$-loopspine. Instead they write that the intensity of subtrees incident to the $W$-spine at the corresponding node has intensity $lN(\cdot, H_{\text{max}} \leq m - t)$. In fact, in our proofs we will only be counting sublooptrees grafted to entire loops so the distribution of these around each individual loop will not matter. However, it should be clear from equation (11) and the paragraph following it in [DLG05] that they are actually distributed uniformly around each loop.

Recalling from Section 3.3 that loops correspond to jumps in the coding Lévy excursion, an atom $\text{Loop}_j$ therefore corresponds to a loop in the $W$-loopspine.

If $p(t)$ is a point on the $W$-loopspine, this definition of the $W$-loopspine also gives a natural way to define the “clockwise” or “anti-clockwise” distance from the root to that point. Formally, we define the clockwise distance from $\rho$ to $p(t)$ to be $\sum_{0 < u \leq t} x_u$, and the anticlockwise distance to be $\sum_{0 < u \leq t} (\Delta u - x_u) + \sum_{t < u \leq h} \Delta u$.

In Proposition 5.2, we will have to decompose along the loopspine from the root to a point attaining the distance of the $L^m$-Height from the root. By analogy with the notation above, we will call this the $m$-loopspine, and the corresponding spine in the underlying tree the $m$-spine. Although we do not prove a specific distribution for the decomposition along this $m$-loopspine, in the proof of Proposition 5.2 we will make a comparison with the Williams’ decomposition that will allow us to apply certain results obtained for the Williams’ decomposition in [Arc19] in the case of the $m$-loopspine.

### 3.4 Infinite Critical Trees and Looptrees

In this section we introduce Kesten’s tree $T_\infty$ for a given critical offspring distribution $\xi$. In light of Theorem 3.7, it is the natural way to construct such an infinite tree.

**Definition 3.6.** ([AD13, Definition 2.9], adapted from [Kes86]). Let $\xi$ be a critical offspring distribution, and define its size biased version $\xi^*$ by

$$\xi^*(n) = n\xi(n).$$

The **Kesten’s tree** $T_\infty$ associated to the probability distribution $\xi$ is a two-type Galton-Watson tree distributed as follows:

- Individuals are either normal or special.
- The root of $T_\infty$ is special.
- A normal individual produces only normal individuals according to $\xi$.
- A special individual produces individuals according to the size-biased distribution $\xi^*$. Of these, one of them is chosen uniformly at random to be special, and the rest are normal.

Almost surely, the special vertices form a unique infinite backbone of $T_\infty$. Note that this is one-ended. Aldous in [Ald91] coined the term sin-trees for such trees, since they have a single infinite spine.

The following local limit theorem was originally proved by Kesten in [Kes80] under a second moment condition, but was proved as stated in [Jan12, Theorem 7.1], and demonstrates that this construction is the right one to take.
Theorem 3.7. ([Kes86, AD15, Theorem 2.1.1], [Jan12, Theorem 7.1]). Let \( \xi \) be a critical offspring distribution with \( \xi(0) + \xi(1) < 1 \) and define \( T_\infty \) as in Definition 3.6. Let \( T_n \) be a Galton-Watson tree with offspring distribution \( \xi \) conditioned on having height at least \( n \). Then

\[
T_n \xrightarrow{d} T_\infty
\]

with respect to the Gromov-Hausdorff-vague topology as \( n \to \infty \).

Note that the convergence is actually stated in a stronger topology in the original literature, but we are mainly interested in Gromov-Hausdorff-vague convergence in this paper.

Kesten’s construction has been imitated in the continuum by Duquesne in [Duq09], who constructs continuum sin-trees and shows that these arise as the appropriate local limit of compact continuum trees conditioned on being large. By analogy with the compact continuum case, Duquesne’s construction involves defining two height functions from two independent Lévy processes in the same way as done with the excursion in (3.2). These respectively code the tree structure on the left and right sides of the spine in the usual way.

The construction was further extended to infinite discrete looptrees in [BS15], where the authors define the infinite looptree associated with a critical offspring distribution \( \xi \) to simply be \( \text{Loop}'(T_\infty) \), where \( T_\infty \) is constructed as in Definition 3.6 and \( \text{Loop}' \) is an operation very close to \( \text{Loop} \), as defined in [CK14, Section 4]. This infinite looptree thus inherits the structure of having a loopspine with loop sizes determined by a size-biased version of \( \xi \), to which usual compact discrete looptrees are grafted. The local limit theorem of Theorem 3.7 thus passes directly to the looptree case by continuity of the \( \text{Loop} \) operation (see [BS15, Corollary 2.3], the proof of which can easily be adapted to \( \text{Loop} \) rather than \( \text{Loop}' \)).

Finally, Kesten’s construction of Definition 3.6 was extended to critical multi-type Galton Watson trees in [Ste18, Theorem 3.1] satisfying an analogous local limit theorem. Richier in [Ric17b] then used this to define an infinite two-type looptree and showed in [Ric17a] this also arises as a similar local limit under appropriate conditions.

The concept of an infinite stable looptree has thus left a gap in the literature and the purpose of this paper is to fill that gap. The construction is the one suggested in [Ric17b, Section 6] and extends the construction of their discrete counterparts. The resulting local limit theorem allows us to prove various volume and heat kernel convergence results for compact stable looptrees in [Arc19].

4 Construction of Infinite Stable Looptrees

Our construction uses two stable Lévy processes to code each side of the loopspine, in place of the excursion. This is the approach suggested in [Ric17b, Section 6] and our construction is merely the continuum version of the discrete construction of [Ric17b, Section 3], except that we have essentially turned this construction “upside down” to match the original coding mechanism for compact looptrees.

The proof of Theorem 1.1 exploits the uniform rerooting invariance of stable looptrees. By taking a stable looptree coded by an excursion \( X^{\text{exc},\ell} \) of length \( \ell \), and taking the root to be a uniform point in \( U \in [0, \ell] \), it follows from the Vervaat transformation that the processes \( (X_{t}^{\text{exc},\ell})_{0 \leq t \leq U} \) and \( (X_{t}^{\text{exc},\ell})_{U \leq t \leq \ell} \) are distributed respectively as the post- and pre-minimum parts of a stable Lévy bridge. Standard convergence results then imply that on any compact interval, these converge in distribution to stable Lévy processes as \( \ell \to \infty \).

Moreover, if we think of the loopspine as the sequence of loops coded by jump points at times \( 0 \leq t \leq U \), then \( (X_{t}^{\text{exc},\ell})_{0 \leq t \leq U} \) codes for the loopspine along with everything grafted to the left hand side of it, and \( (X_{t}^{\text{exc},\ell})_{U \leq t \leq \ell} \) codes for everything grafted to the right hand side of it. It is therefore natural to replace each of these by unconditioned Lévy process in the infinite volume limit.

We start by writing this below as an equivalent construction of compact stable looptrees. We give the construction for a looptree of mass \( \ell \).
Two-sided Construction of Compact Stable Looptrees

1. Let $X^{\text{br},\ell}$ be a spectrally positive, $\alpha$-stable Lévy bridge of lifetime $\ell$. Let $m = m_\ell$ be the (almost surely unique) time at which $X^{\text{br},\ell}$ attains its infimum.

2. Let $(X^{(2,\ell)}_t)_{t \geq 0}$ be the pre-infimum process, and $(X^{(1,\ell)}_t)_{t \geq 0}$ be the time-reversed post-infimum process, extended to stay constant after times $m$ and $1 - m$ respectively. That is,

$$X^{(2,\ell)}_t = \begin{cases} X^{\text{br}}_t & \text{for } t \in [0, m], \\ X^\text{br}_m & \text{for } t > m; \end{cases} \quad X^{(1,\ell)}_t = \begin{cases} X^\text{br}_{\ell - t} & \text{for } t \in [0, 1 - m], \\ X^\text{br}_m & \text{for } t > \ell - m. \end{cases}$$

3. Define a function $X^\ell : \mathbb{R} \to \mathbb{R}$ by

$$X^\ell_t = \begin{cases} X^{(2,\ell)}_t & \text{if } t \geq 0, \\ X^{(1,\ell)}_{-t} & \text{if } t < 0. \end{cases}$$

It should be clear from the Vervaat transform that $X^\ell$ is just a shifted Lévy excursion.

4. For $s, t \in \mathbb{R}$, we define resistances $r^\ell$, $R^\ell_s$ and $R^\ell$ from $X^\ell$ exactly as in (7), (8) and (9). We can similarly define distances $\delta^\ell$, $d^\ell_s$ and $d^\ell$ exactly as in [CK14, Section 2.3]. We then set $\mathcal{L}_\alpha^\ell = (\mathbb{R} / \sim, d^\ell)$, and $\mathcal{L}^\ell_R = (\mathbb{R} / \sim, R^\ell)$.

Due to the Vervaat transformation, this construction is entirely equivalent to the original construction of looptrees using the Lévy excursion, but we have now split the coding into two functions which define each side of the loopspine. To code the infinite looptree, we will take limits of each of these functions and use these to code each side of the infinite loopspine.

We first give the construction, and then prove Theorem 1.1.
Construction of Infinite Stable Looptrees

1. Let $X$ be an $\alpha$-stable, spectrally positive Lévy process, and let $X'$ be an $\alpha$-stable, spectrally negative Lévy process.

2. Define a function $X^\infty : \mathbb{R} \to \mathbb{R}$ by

$$X^\infty_t = \begin{cases} X_t & \text{if } t \geq 0, \\ X'_{-t} & \text{if } t < 0. \end{cases}$$

3. As in the discrete construction of [CK14, Section 2.3] and Section 3.3 if $t$ is a jump point of $X^\infty$ with jump size $\Delta_t$ and $a, b \in [0, \Delta_t]$, set

$$\delta_t^\infty(a, b) = \min\{|a - b|, \Delta_t - |a - b|\},$$

$$r_t^\infty(a, b) = \left( \frac{1}{|a - b|} + \frac{1}{\Delta_t - |a - b|} \right)^{-1} = \frac{|a - b|}{\Delta_t}.$$  

Additionally, as before, for $s, t \in \mathbb{R}$ with $s \leq t$ set $I^\infty_{s, t} = \inf_{r \in [s, t]} X^\infty_r$, and $X^\infty_{s, t} = I^\infty_{s, t} - X^\infty_s$.

For $s, t \in \mathbb{R}$ we again write $s < t$ if $s \leq t$ and $s \neq t$. Then, if $s \leq t$ set

$$d^\infty_0(s, t) = \sum_{s < u \leq t} \delta^\infty_u(0, x^t_u),$$

$$R^\infty_0(s, t) = \sum_{s < u \leq t} r^\infty_u(0, x^t_u).$$

Then, for general $s, t \in \mathbb{R}$, set

$$d^\infty(s, t) = \delta^\infty_{s,t}(x^\infty_{s\wedge t, s}, x^\infty_{s\wedge t, t}) + d^\infty_0(s \wedge t, s) + d^\infty_0(s \wedge t, t),$$

$$R^\infty(s, t) = r^\infty_{s,t}(x^\infty_{s\wedge t, s}, x^\infty_{s\wedge t, t}) + R^\infty_0(s \wedge t, s) + R^\infty_0(s \wedge t, t).$$  

Finally, define an equivalence relation $\sim$ on $\mathbb{R}$ by $s \sim t$ if and only if $d^\infty(s, t) = 0$. We define the infinite looptrees $\mathcal{L}^\infty_\alpha$ and $\mathcal{L}^\infty_\alpha R$ by

$$\mathcal{L}^\infty_\alpha = (\mathbb{R} / \sim, d^\infty),$$

$$\mathcal{L}^\infty_\alpha R = (\mathbb{R} / \sim, R^\infty).$$

For ease of notation and intuition, we will focus on $\mathcal{L}^\infty_\alpha$ rather than $\mathcal{L}^\infty_\alpha R$ in the next section, but given that we still have the relation $\frac{1}{2} \delta^\infty_t(a, b) \leq r^\infty_t(a, b) \leq \delta^\infty_t(a, b)$ for all $t \in \mathbb{R}$, and $a, b \in [0, \Delta_t]$, the results will also hold in the resistance setting.

As in the compact case, we can define the projection $p^\infty : \mathbb{R} \to \mathcal{L}^\infty_\alpha$, which is almost surely continuous, and endow $\mathcal{L}^\infty_\alpha$ with the measure $\nu^\infty$ which is defined to be the pushforward of Lebesgue measure on the real line to $\mathcal{L}^\infty_\alpha$ via $p^\infty$.

We also have the following proposition, as a direct consequence of the scale invariance of the stable Lévy process.

**Proposition 4.1** (Scale invariance of $\mathcal{L}^\infty_\alpha$). For any $c > 0$,

$$(\mathcal{L}^\infty_\alpha, cd, \rho^\infty, c^\alpha \nu^\infty) \overset{(d)}{=} (\mathcal{L}^\infty_\alpha, \hat{d}, \rho^\infty, \nu^\infty),$$

where $\hat{d}$ here can be equal to either $d^\infty$ or $R^\infty$.

We also record the following result, which arises as a consequence of Theorem 1.1 [CK14 Corollary 4.4], and [BB10] Theorem 7.5.1.

**Corollary 4.2.** Almost surely, $\mathcal{L}^\infty_\alpha$ is a length space.
5 Volume Bounds and Resistance Estimates for Infinite Stable Looptrees

In this section, we prove precise estimates on the resistance growth properties of infinite stable looptrees, which we later use in Section 7 to prove Theorems 1.5 and 1.6. We consider similar quantities for compact stable looptrees in the article [Arc19], and the arguments there can also be applied to the case of infinite stable looptrees to give bounds on $\nu^\infty(B^\infty(\rho^\infty, r))$. The only modifications are as follows.

(i) Rather than applying the Vervaat transform and absolute continuity relation at various points in Sections 5.1 and 5.4 of [Arc19], we can work directly with the Lévy process.

(ii) In the first iteration of the iterative procedure described in Section 5.2, we need to remove the exponential term from (17), and replace $\mathbb{1}\{t \in [0, H]\}$ with $\mathbb{1}\{t \in [0, \infty)\}$, but this does not affect the rest of the proof of Lemma 5.5 of that paper.

(iii) Again in the first iteration of the iterative procedure described in Section 5.2, we no longer need to thin the Itô excursion measure in Lemma 5.6. However, since we stochastically dominate by the unthinned version in the proof, this does not affect the result.

By applying the same arguments, we therefore get the following results for infinite stable looptrees.

**Proposition 5.1.** There exist constants $c, c', C, C' \in (0, \infty)$ such that

$$C \exp\{-c\lambda^{\alpha-1}\} \leq \mathbb{P}(\nu^\infty(B^\infty(\rho^\infty, r)) < r^\alpha \lambda^{-1}) \leq C' \exp\{-c'\lambda^{\frac{\alpha-1}{2}}\} \quad C e^{-c\lambda} \leq \mathbb{P}(\nu^\infty(B^\infty(\rho^\infty, r)) \geq r^\alpha \lambda) \leq C'\lambda^{\frac{\alpha-1}{2}} e^{-c'\lambda^{\frac{\alpha-1}{2}}}.$$ 

By applying Borel-Cantelli arguments along the sequence of integral radii we therefore deduce that $\nu^\infty(B^\infty(\rho^\infty, r))$ also has at most log-logarithmic fluctuations about the term $r^\alpha$ as $r \to \infty$.

We also get similar bounds on the resistance fluctuations.

**Proposition 5.2.** There exist constants $C, c \in (0, \infty)$ such that

$$\mathbb{P}(R^\infty(\rho^\infty, B^\infty(\rho^\infty, r)^c) \leq r\lambda^{-1}) \leq C\lambda^{\frac{\alpha-1}{2}} e^{-c\lambda^{\frac{\alpha-1}{2}}}$$

for all $r > 0, \lambda > 1$.

**Proof.** By scaling invariance of $L^\infty_\alpha$, it is sufficient to prove the result for $r = 1$. The proof technique is similar to the decomposition argument used in [Arc19] Section 5.2 to prove upper volume bounds for compact stable looptrees. We define an iterative algorithm as we did in Section 5.2.2 there, except that we perform subsequent iterations around sublooptrees of large $L^m$-Height rather than those of large volume. We do not present all the fine technical details here, since this would be rather lengthy, and instead refer the reader to [Arc19] Section 5] for these.

In summary, we define a branching process, which we will index by a tree $T_{\text{res}}$, as follows. Firstly, we let the root $\emptyset$ of $T_{\text{res}}$ represent the whole looptree $L^\infty_\alpha$. We start by performing a decomposition of $L^\infty_\alpha$ along its infinite loospine, and select all sublooptrees that are grafted to this loospine within distance $\frac{1}{4}\lambda^{-\alpha}$ of the root, and that such that their $L^m$-Height is at least $\frac{1}{2}$. These sublooptrees form the offspring of $\emptyset$ in $T_{\text{res}}$. We proceed inductively: for each sublooptree that is an element of $T_{\text{res}}$, we perform a decomposition along its m-loospine and define its offspring to be those sublooptrees that are grafted to the m-loospine at a point with distance $\lambda^{-\alpha}$ of its root, and that also have $L^m$-Height at least $\frac{1}{2}$.

The aim will be to define a set of points $A \subset L^\infty_\alpha$ such that any path from $B^\infty(\rho^\infty, \lambda^{-\alpha})$ to $B^\infty(\rho^\infty, 1)^c$ must pass through at least one point in $A$. We will show that, with high probability, the total progeny of $T_{\text{res}}$ is at most $\lambda^{\alpha-1}$, and that we can then select a set $A$ of cardinality at most $2\lambda^{2(\alpha-1)}$. On this event, it
follows that \( R^\infty(\rho^\infty, B^\infty(\rho^\infty, r)^c)^{-1} \leq 2\lambda^{3\alpha-2} \), and consequently, that \( R^\infty(\rho^\infty, B^\infty(\rho^\infty, r)^c) \geq \frac{1}{2}\lambda^{-(3\alpha-2)} \). The result will then follow as stated.

We adopt some of the notation of \[Arc19\] Section 5.2. In particular, for any \( R > 0 \), and any compact looptree \( \tilde{\mathcal{L}}_\alpha \) (or infinite looptree \( \mathcal{L}_\alpha^\infty \)), we let \( I^m_R \) be the closure in \( \mathcal{L}_\alpha \) (or \( \mathcal{L}_\alpha^\infty \)) of the union of all the loops in the m-loopspine (or infinite loopspine) that intersect \( B'(\rho', R) \) (or \( B^\infty(\rho^\infty, R) \)). Additionally, we let \( |I^m_R| \) be the sum of the lengths of these loops. We use the superscript ‘W’ to denote the corresponding quantities along the W-loopspine. By arguments similar to \[Arc19\] Lemma 5.5, we claim that

\[
P\left( |I_{\frac{1}{4}\lambda^{-\alpha}}^m| \geq \frac{3}{4} \lambda^{-1} \left| L^m\text{-Height}(\tilde{\mathcal{L}}_\alpha) \geq \frac{1}{2} \right) \leq C e^{-c\lambda^{\alpha-1}}. \tag{12} \]

Indeed, take the setup as in the proof of \[Arc19\] Lemma 5.5, but with \( \beta_4 = 1 \). In the case that the \( T^m\text{-Height} \) of \( \tilde{\mathcal{L}}_\alpha \) is at least \( (\frac{1}{4}\lambda^{1-\alpha})^{-1} \), we can immediately apply the argument used there, noting that although the exponential penalty in the Williams’ decomposition in Theorem 3.4(i) will be slightly different along the \( m \)-spine, it will still be monotone in \( t \), allowing a similar argument. Otherwise, we have to modify the first step of the proof. In this case, let \( M' \) be the total number of goodish loops on the m-loopspine (i.e. the total number of loops of length at least \( \lambda^{-\alpha} \)). It then follows that

\[
P\left( M' \leq \lambda^{\alpha-1} \left| L^m\text{-Height}(\tilde{\mathcal{L}}_\alpha) \geq \frac{1}{2} \right) T^m\text{-Height}(\tilde{\mathcal{L}}_\alpha) \leq \left(\frac{1}{4}\lambda^{1-\alpha}\right)^{-1} \right)
\leq P\left( M' \leq \lambda^{\alpha-1}, L^m\text{-Height}(\tilde{\mathcal{L}}_\alpha) \geq \frac{1}{2} \left| T^m\text{-Height}(\tilde{\mathcal{L}}_\alpha) \leq \left(\frac{1}{4}\lambda^{1-\alpha}\right)^{-1} \right) \right)
\leq P\left( \text{Sub}_{\lambda^{-\alpha}} \geq \frac{1}{2} - \lambda^{-1} \left| \text{no jumps at least } \lambda^{-\alpha} \right) \right),
\]

where \( \text{Sub} \) is a subordinator with (time-dependent) jump measure

\[C_\alpha \mathbb{I}_{\{[0,1]\}}(u) \mathbb{I}_{\{[0,H^\infty]\}}(t) \lambda^{-\alpha} \text{pen}(l, H^m, t) du dt dl,
\]

where \( C_\alpha = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} \), as before, \( H^m = T^m\text{-Height}(\tilde{\mathcal{L}}_\alpha) \), and \( \text{pen} \) is a penalty term that is increasing in \( t \), decreasing in \( H^m - t \), and always less than 1. Note that \( \text{Sub} \) is almost an \( (\alpha - 1) \)-stable subordinator, but with the extra penalty against larger jumps. We therefore let \( \text{Sub}^{\alpha-1} \) denote an \( (\alpha - 1) \)-stable subordinator. It follows that for any \( k > 1 \), and any \( t, x, y > 0 \):

\[
P(\text{Sub}_y \geq x \mid \text{no jumps at least } y) \leq P\left( \text{Sub}^{\alpha-1}_x \geq x \left| \text{no jumps at least } y \right) \right)
\leq P\left( \text{Sub}^{\alpha-1}_{kx} \geq kx \mid \text{no jumps at least } ky \right).
\]

Taking \( k = \frac{1}{4}\lambda^{-1} \), we therefore see that

\[
P\left( M' \leq \lambda^{\alpha-1} \left| L^m\text{-Height}(\tilde{\mathcal{L}}_\alpha) \geq \frac{1}{2} \right) T^m\text{-Height}(\tilde{\mathcal{L}}_\alpha) \leq \left(\frac{1}{4}\lambda^{1-\alpha}\right)^{-1} \right)
\leq P\left( \text{Sub}_{\lambda^{-\alpha}} \geq \frac{1}{8} \lambda^{\alpha-1} - \lambda^{\alpha-2} \left| \text{no jumps at least } \frac{1}{4}\lambda^{-1} \right) \right)
\leq E\left[ e^{\theta \text{Sub}^{\alpha-1}_{\lambda^{-\alpha}}} \right] e^{-\theta \lambda^{\alpha-1}}
\]

for sufficiently small \( \theta > 0 \), where the existence of the exponential moment in the last line follows from Remark 2.7. Conditional on \( M' > \lambda^{\alpha-1} \), it follows that the probability that there exists a good loop in the first half of the m-loopspine is at least \( 1 - P(\text{Geo}(\frac{1}{4}) \geq \lambda^{\alpha-1}) \), where we recall that a good loop is a goodish loop that also satisfies the condition that the associated uniform random variable (that dictates the ratio of the two segments it splits into on either side of the loopspine) is in the interval \([\frac{1}{4}, \frac{1}{2}]\). This in turn is at least \( 1 - C e^{-c\lambda^{\alpha-1}} \) by a Chernoff bound. Conditional on this, we can then proceed exactly as in the proof of \[Arc19\] Lemma 5.5 to prove (12).

Given (12), it follows that the sequence of sublooptrees incident to the m-loopspine at a point in \( I_{\frac{1}{4}\lambda^{-\alpha}}^m \) can still be stochastically dominated by those coded by the classical (unthinned) Itô excursion measure along
this segment. It therefore follows exactly as in [Arc19, Lemma 5.7] that
\[ \mathbb{P}(|T_{\text{res}}| \geq \lambda \alpha^{-1}) \leq C \lambda^{-1} e^{-c \lambda^{-1}} + \mathbb{P}(|\hat{T}| \geq \lambda \alpha^{-1}) \leq C \lambda^{-1} e^{-c \lambda^{-1}}, \]
where \( \hat{T} \) is a Galton-Watson tree with \( \text{Poisson}(\tilde{C} \lambda^{-1}) \) offspring distribution, for some constant \( \tilde{C} < \infty \).

Assuming now that \( |T_{\text{res}}| < \lambda \alpha^{-1} \), we claim that we can pick a set \( A \) of cardinality at most \( 2 \lambda^{2(\alpha-1)} \). The argument is similar to that in [Arc19, Proposition 6.3] and illustrated in Figure 7 of that paper. In particular, given a subloop tree that is an element of \( T_{\text{res}} \), and looking at the proof of (12), it follows from the event constructed there that we can assume that there are at most \( \lambda \alpha^{-1} \) goodish loops on its m-loopspine until we reach the first good one. It also follows that the sum of the lengths of the smaller loops between each goodish loop is at most \( \lambda \alpha^{-1} \). By arguments similar to those illustrated in [Arc19, Figure 7], it then follows that for each of these subloop trees we can pick two points on each of the goodish loops, and two points on the first good loop, to be in \( A \), and that all of these points lie outside \( B^\infty(\rho^\infty, \lambda^{-\alpha}) \), but within \( B^\infty(\rho^\infty, \lambda^{-1} + \frac{1}{2}) \). Moreover, any path from \( B^\infty(\rho^\infty, \lambda^{-\alpha}) \) to \( B^\infty(\rho^\infty, \lambda^{-1} + \frac{1}{2}) \) must pass through at least one point in \( A \).

Since the cardinality of \( A \) is at most \( 2 \lambda^{2(\alpha-1)} \), it thus follows from the parallel law for resistance that
\[ R^\infty(\rho^\infty, B^\infty(\rho^\infty, 1)^c)^{-1} \leq R^\infty(\rho^\infty, B^\infty(\rho^\infty, \frac{1}{2} + \lambda^{-1})^c)^{-1} \leq 2 \lambda^{3\alpha-2}, \]
and consequently
\[ R^\infty(\rho^\infty, B^\infty(\rho^\infty, 1)^c) \geq \frac{1}{2} \lambda^{-(3\alpha-2)}. \]

Combining all the results above, we see that
\[ \mathbb{P} \left( R^\infty(\rho^\infty, B^\infty(\rho^\infty, 1)) \leq \frac{1}{2} \lambda^{-(3\alpha-2)} \right) \leq C \lambda^{-1} e^{-c \lambda^{-1}}, \]
which is equivalent to the stated result.

It therefore follows from the first Borel-Cantelli Lemma that there almost surely exist constants \( c, c', C' \in (0, \infty) \) such that
\[ cr(\log \log r^{-1})^{-(3\alpha-2)} \leq R^\infty(\rho^\infty, B^\infty(\rho^\infty, 1)^c) \leq r, \]
\[ c' r^\alpha (\log r^{-1})^{-\alpha} \leq \nu(B^\infty(u, r)) \leq C' r^\alpha (\log r^{-1})^{\frac{3\alpha-3}{\alpha}} \]
for all \( r > 0 \).

6 Limit Theorems

In this Section we prove Theorems 1.1 and 1.2 and explore their consequences.

6.1 Proof of Theorem 1.1

Theorem 1.1 is proved by applying Proposition 3.3 to the following convergence result.

Proposition 6.1. Let \( X^{br, \ell} \) be a spectrally positive, \( \alpha \)-stable Lévy bridge of lifetime \( \ell \), let \( X \) be an \( \alpha \)-stable, spectrally positive Lévy process, and let \( X' \) be an independent \( \alpha \)-stable, spectrally negative Lévy process. Also let \( m_t \) be the (almost surely unique) time at which \( X^{br, \ell} \) attains its minimum. Then, for any \( T_1, T_2 > 0 \), letting \( f \) and \( g \) be any bounded continuous functions \( D([0, T], \mathbb{R}) \rightarrow \mathbb{R} \), we have that
\[ \mathbb{E} \left[ f \left( (X^{br, \ell}_{t \wedge m_t})_{t \in [0, T_1]} \right) g \left( (X^{br, \ell}_{(t-\ell) \vee m_t})_{t \in [0, T_2]} \right) \right] \rightarrow \mathbb{E} \left[ f \left( (X_t)_{t \in [0, T_1]} \right) \right] \mathbb{E} \left[ g \left( (X'_t)_{t \in [0, T_2]} \right) \right] \]
as \( \ell \rightarrow \infty \).
Given this result, the proof of Theorem 1.1 proceeds as follows by applying Proposition 3.3 to the functions $X$ and $X'$ on compact time intervals.

**Proof of Theorem 1.1 assuming Proposition 6.1.** We need to show that for Lebesgue almost every $r > 0$,

$$
\mathcal{B}_r(\mathcal{L}_\alpha^\ell) \overset{(d)}{\rightarrow} \mathcal{B}_r(\mathcal{L}_\alpha^\infty).
$$

To this end, take some $r > 0$. We define two times $t_g(r)$ and $t_d(r)$ by

$$
t_g(r) = \inf\{s \geq 0 : \Delta_s \geq 4r, \delta^\infty_s(x_{-s,0}) \geq r\}, \quad t_d(r) = \inf\{s \geq 0 : X_s^\infty \leq X_{-t_g(r) -}^\infty\}.
$$

Note that $t_g(r)$ is almost surely finite, since letting $L_s$ denote the local time spent by $(X^\infty_{t})_{t \geq 0}$ at its infimum by time $s$, normalised so that $E[e^\lambda X_{-1}^\infty] = e^{-\lambda - 1}$, we have from Proposition 2.5 that the measure

$$
\sum_{s \in J} \delta_{(L_s, \Delta_s)}
$$

is a Poisson point measure of intensity $dl \cdot x1\{x^{-\alpha} \geq 4r\}dx$, where $J$ is the set $\{s \geq 0 : \Delta_s \geq 4r, \delta^\infty_s(x_{-s,0}) \geq r\}$. Moreover, by [Ber96, Chapter VIII, Lemma 1] we know that $L^{-1}$ is a stable subordinator of parameter $1 - \frac{1}{\alpha}$, and hence $L_s \to \infty$ almost surely as $t \to \infty$. It follows that $t_g(r)$ is almost surely finite for all $r > 0$. Similarly, since $\lim \inf_{t \to \infty} X_t^\infty = -\infty$ almost surely, $t_d(r)$ is also almost surely finite for all $r > 0$.

For notational convenience, we write $t_g = t_g(r)$ and $t_d = t_d(r)$ from now on.

Since the interval $[-t_d - 1, t_d + 1]$ is almost surely compact, and the space of càdlàg functions with compact support endowed with the Skorohod-$J_1$ topology is separable, it follows by the Skorohod Representation Theorem and Proposition 6.1 that there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which $(X^{\text{br}, \ell})_{t \in [-t_d - 1, t_d + 1]} \to (X^\infty)_t$ almost surely. We henceforth work in this space.

For each $\ell > 0$, let $\lambda_\ell$ be the Skorohod homeomorphism (defined pointwise on $\Omega$) from $[-t_d - 1, t_d + 1] \to [-t_g - 1, t_d + 1]$ that minimises the Skorohod distance between these $X^{\text{br}, \ell}$ and $X^\infty$ on this interval. Then set $t_d' = \lambda_\ell(t_d)$, and similarly $t_g' = \lambda_\ell(t_g)$.

The correspondence consisting of all pairs $[t, \lambda_\ell(t)]$ for $t \in [-t_g, t_d]$ is a subset of the correspondence used to minimise the Gromov-Hausdorff distance in the proof of Theorem 3.3, so letting $\mathcal{L}_\alpha^{\infty,r} = p^r((X^{\text{br}, \ell})_{t \in [-t_g', t_d']})$ for each $\ell > 0$ and $\mathcal{L}_\alpha^{-\infty,r} = p^\infty((X_t)_{t \in [-t_g', t_d]})$, it follows from Theorem 3.3 that $d_{\text{GHP}}(\mathcal{L}_\alpha^{\infty,r}, \mathcal{L}_\alpha^{-\infty,r}) \to 0$ as $\ell \to \infty$. Since $\mathcal{B}_r(\mathcal{L}_\alpha^\infty) \subset \mathcal{L}_\alpha^{\infty,r}$ and $\mathcal{B}_r(\mathcal{L}_\alpha^{-\infty}) \subset \mathcal{L}_\alpha^{-\infty,r}$, it thus follows that $\mathcal{B}_r(\mathcal{L}_\alpha^\ell) \overset{(d)}{\rightarrow} \mathcal{B}_r(\mathcal{L}_\alpha^\infty)$ for Lebesgue almost every $r' < r$. By taking a countable sequence $r_n \to \infty$ we therefore deduce the result for Lebesgue almost-every $r > 0$, and the theorem follows.

We now conclude the proof of Theorem 1.1 by proving Proposition 6.1.

**Proof of Proposition 6.1.** The key point is that the two sides of the bridge have a density with respect to the laws of $X$ and $X'$, in that for any $f, g$ as in the statement of the proposition, and any $\ell > T_1 + T_2$, it follows from [Ber96] Chapter VIII.3, Equation (8) that

$$
E\left[f\left((X^{\text{br}, \ell})_{t \in [0, T_1]}\right)g\left((X^{\text{br}, \ell})_{(t-L)}_{t \in [0, T_2]}\right)\right] = E\left[f\left((X_t)_{t \in [0, T_1]}\right)g\left((X_t)_{t \in [0, T_2]}\right)\frac{p_{t-T_1-T_2}(X_{T_2} - X_{T_1})}{p(0)}\right]
$$

where $p$ here denotes the transition density of $X$. The proof then essentially just uses the fact that $m_\ell$ and $\ell - m_\ell$ tend to infinity in probability as $\ell \to \infty$, and then the fact that with high probability, $X_{T_1}$ and $X'_{T_2}$ will also not be too large. There are two main steps. We first note that the quantity

$$
E\left[f\left((X^{\text{br}, \ell})_{t \in [0, T_1]}\right)g\left((X^{\text{br}, \ell})_{(t-L)}_{t \in [0, T_2]}\right)\right] - E\left[f\left((X^{\text{br}, \ell})_{t \in [0, T_1]}\right)g\left((X^{\text{br}, \ell})_{(t-L)}_{t \in [0, T_2]}\right)\right]
$$

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is upper bounded by
\[ \leq 2\|f\|_\infty \|g\|_\infty \left( \mathbb{P}\left( m_1 < \frac{T_1}{\ell} \right) + \mathbb{P}\left( m_1 > 1 - \frac{T_2}{\ell} \right) \right), \]
which converges to 0 as \( \ell \to \infty \). This allows us to apply [15] as follows. First, note that it follows from scaling invariance that
\[ \frac{p_{\ell-T_1-T_2}(X_{T_2}^\ell - X_{T_1})}{p_{\ell}(0)} = \left( \frac{\ell}{\ell - T_1 - T_2} \right)^\frac{1}{2} \frac{p_1((\ell - T_1 - T_2)\frac{T_1}{\ell}(X_{T_2}^\ell - X_{T_1}))}{p_1(0)}. \]
We denote this latter quantity by \( p(\ell, X, X', T_1, T_2) \), so that
\[ \mathbb{E} \left[ f\left((X_t^\text{br,}e)_{t \in [0,T_1]} g\left((X_t^\text{br,}e)_{t \in [0,T_2]} \right) \right) - \mathbb{E} \left[ f\left((X_t')_{t \in [0,T_1]} g\left((X_t')_{t \in [0,T_2]} \right) \right) \right] \]
\[ = \mathbb{E} \left[ f\left((X_t)_{t \in [0,T_1]} g\left((X_t')_{t \in [0,T_2]} \right) \right) \left( p(\ell, X, X', T_1, T_2) - 1 \right) \right]. \]
We then take some \( 0 < \varepsilon \ll \frac{1}{\ell} \), and decompose further on the event \( \{|X_{T_1}| \lor |X_{T_2}^\ell| \leq (\ell - T_1 - T_2)^{\frac{1}{2} - \varepsilon} \} \) and its complement by writing this as the sum
\[ \mathbb{E} \left[ f\left((X_t)_{t \in [0,T_1]} g\left((X_t')_{t \in [0,T_2]} \right) \right) \left( p(\ell, X, X', T_1, T_2) - 1 \right) \mathbb{1}\{|X_{T_1}| \lor |X_{T_2}^\ell| \leq (\ell - T_1 - T_2)^{\frac{1}{2} - \varepsilon} \} \]
\[ + \mathbb{E} \left[ f\left((X_t)_{t \in [0,T_1]} g\left((X_t')_{t \in [0,T_2]} \right) \right) \left( p(\ell, X, X', T_1, T_2) - 1 \right) \mathbb{1}\{|X_{T_1}| \lor |X_{T_2}^\ell| > (\ell - T_1 - T_2)^{\frac{1}{2} - \varepsilon} \} \right]. \]
We deal with each of these two terms separately. For the first term, note that by continuity of the transition density,
\[ \sup_{|x| \leq 2(\ell - T_1 - T_2)^{\frac{1}{2} - \varepsilon}} \left\{ p_1\left(x(\ell - T_1 - T_2)^{\frac{1}{\ell}}\right) \right\} \to p_1(0) \]
as \( \ell \to \infty \). It therefore follows that
\[ \left| \left( p(\ell, X, X', T_1, T_2) - 1 \right) \mathbb{1}\{|X_{T_1}| \lor |X_{T_2}^\ell| \leq (\ell - T_1 - T_2)^{\frac{1}{2} - \varepsilon} \} \right|_\infty \]
\[ \leq \frac{1}{p_1(0)} \left( \left( \frac{\ell}{\ell - T_1 - T_2} \right)^{\frac{1}{2}} - 1 \right) \sup_{|x| \leq 2(\ell - T_1 - T_2)^{\frac{1}{2} - \varepsilon}} \left\{ p_1\left(x(\ell - T_1 - T_2)^{\frac{1}{\ell}}\right) \right\} \]
\[ + \sup_{|x| \leq 2(\ell - T_1 - T_2)^{\frac{1}{2} - \varepsilon}} \left\{ p_1\left(x(\ell - T_1 - T_2)^{\frac{1}{\ell}}\right) - p_1(0) \right\}, \]
and we deduce that the first term in (16) converges to zero as \( \ell \to \infty \), since \( f \) and \( g \) are also bounded. To deal with the second term, we write
\[ \mathbb{E} \left[ f\left((X_t)_{t \in [0,T_1]} g\left((X_t')_{t \in [0,T_2]} \right) \right) \left( p(\ell, X, X', T_1, T_2) - 1 \right) \mathbb{1}\{|X_{T_1}| \lor |X_{T_2}^\ell| > (\ell - T_1 - T_2)^{\frac{1}{2} - \varepsilon} \} \]
\[ \leq \|f\|_\infty \|g\|_\infty \frac{1}{p_1(0)} \|p_1\|_\infty \mathbb{P}\left( |X_{T_1}| \lor |X_{T_2}^\ell| > (\ell - T_1 - T_2)^{\frac{1}{2} - \varepsilon} \right), \]
which also vanishes as \( \ell \to \infty \).
It therefore follows from (15) that
\[ \mathbb{E} \left[ f\left((X_t^\text{br,}e)_{t \in [0,T_1]} g\left((X_t^\text{br,}e)_{t \in [0,T_2]} \right) \right) \right] - \mathbb{E} \left[ f\left((X_t)_{t \in [0,T_1]} g\left((X_t')_{t \in [0,T_2]} \right) \right) \right] \to 0 \]
as \( \ell \to \infty \), as claimed. We can then factorise the final term by independence of \( X \) and \( X' \).
6.2 Applications to Volume Results for Compact Stable Looptrees

As a result of Theorem 1.1, we are able to prove various volume convergence results that are exploited in [Arc19] to study Brownian motion on compact stable looptrees. The main applicable result is the following theorem. Here we let \( \nu \) denote the intrinsic measure on a compact stable looptree \( \mathcal{L}_\alpha \) as defined in Section 3.3 and \( B(\rho, r) \) denote the open ball of radius \( r \) around the root in \( \mathcal{L}_\alpha \), and \( \overline{B}(\rho, r) \) its closure.

**Theorem 6.2.** There exists a random variable \( (V_t)_{t \geq 0} : \Omega \to D([0, \infty), [0, \infty)) \) such that the finite dimensional distributions of the process

\[
(r^{-\alpha} \nu(\overline{B}(\rho, rt)))_{t \geq 0}
\]

converge to those of \( (V_t)_{t \geq 0} \) as \( r \to 0 \), and \( V_t \) denotes the volume of a closed ball of radius \( t \) around the root in \( \mathcal{L}_\alpha^\infty \). Moreover, for any \( p \in [1, \infty) \), we have that

\[
r^{-\alpha p} \mathbb{E}[\nu(\overline{B}(\rho, r))^p] \to \mathbb{E}[V_t^p]
\]

as \( r \to 0 \), and \( V_1 \) is a \((0, \infty)\)-valued random variable with all moments finite.

**Remark 6.3.** We have taken closed balls rather than open ones simply so that \( V \) is càdlàg. We conjecture that the volume processes are in fact continuous, and that the convergence of the theorem can be extended to hold uniformly on compacts. However, due to the complex nature of looptrees, this is not straightforward to prove. In particular it is difficult to replicate the argument used to prove a similar result for stable trees, since looptrees do not have such a straightforward regeneration structure around the boundary of a ball of radius \( r \).

**Proof.** By the separability of Theorem 2.1, we can work on a probability space on which \( \mathcal{L}_\alpha^\ell \to \mathcal{L}_\alpha^\infty \) almost surely as \( \ell \to \infty \). By standard results on metric space convergence, it follows that almost surely, \( \nu'(B^\ell(\rho^\ell, t)) \to \nu^\infty(B^\infty(\rho^\infty, t)) \) for all \( t \) such that \( \nu^\infty(\partial B^\infty(\rho^\infty, t)) = 0 \). However, since this must hold for Lebesgue almost every \( t \), it therefore follows that almost surely, \( \nu'(B^\ell(\rho^\ell, t)) \to \nu^\infty(B^\infty(\rho^\infty, t)) \) for Lebesgue almost every \( t \). Moreover, by scaling invariance of \( \mathcal{L}_\alpha^\infty \), there are no “special” values of \( t \), so we deduce that for any fixed sequence \( 0 < t_0 < t_1 < \ldots < t_n < \infty \), the convergence almost surely holds simultaneously for all of the points \( t_i, 0 \leq i \leq n \). The first result then follows.

Since \( (\nu'(B^\ell(\rho^\ell, t)))_{t \geq 0} \overset{(d)}{=} (\nu B(\rho, \ell^{-\alpha} t))_{t \geq 0} \), by writing \( \ell = r^{-\alpha} \) we therefore deduce the result as stated.

In particular, it follows that \( \nu(t B(\rho t, 1)) \overset{(d)}{=} \nu^\infty(B^\infty(\rho^\infty, 1)) \) as \( \ell \to \infty \).

We now set \( V := V_1 = \nu^\infty(B^\infty(\rho^\infty, 1)) \).

We claim that \( V \in (0, \infty) \) almost surely, with all moments finite. This follows immediately from the exponential upper tails of Proposition 5.1, namely that

\[
\mathbb{P}(V \geq \lambda) \leq C \lambda^{\frac{\alpha-1}{\alpha+1}} e^{-c \lambda^{\frac{\alpha-1}{\alpha+1}}}.
\]

We also claim that the moments of \( r^{-\alpha} \nu_1(B(\rho_1, r)) \) converge to those of \( V \). To see this, note that by scaling invariance the results of [Arc19] Section 5.2 can be applied uniformly along the sequence \( \mathcal{L}_\alpha^\ell \) to give constants \( c, C \in (0, \infty) \) such that

\[
\mathbb{P}^\ell(\nu'(B^\ell(\rho, r)) \geq r^\alpha \lambda) \leq C \lambda^{\frac{\alpha-1}{\alpha+1}} e^{-c \lambda^{\frac{\alpha-1}{\alpha+1}}}
\]

for all \( \ell \geq 1 \). It follows that the sequence \( (r^{-\alpha p}(\nu'(B^\ell(\rho, r)))^p)_{\ell \geq 1} \) is uniformly integrable for all \( p \geq 1 \) and so setting \( C_p = \mathbb{E}[V^p] \) we have that

\[
r^{-\alpha p} \mathbb{E}[(\nu_1(B(\rho_1, r)))^p] \to C_p
\]

for all \( p \geq 1 \).
6.3 Scaling Limits of Infinite Discrete Looptrees

In this section, we prove that infinite stable looptrees are scaling limits of infinite discrete looptrees commutes. We start by proving Theorem 1.2.

Just as in the compact discrete case, we can endow Loop(T∞) with either a geodesic metric d or a resistance R metric, and a uniform measure ν which gives mass 1 to every vertex.

Proof of Theorem 1.2. We will prove the result with ̃d = d and note that due to the equivalence of metrics, the corresponding result for ̃d = R follows by the same arguments. The proof is again a consequence of Theorem 3.2. The point is that the looptree Loop(T∞) is coded by a two sided infinite Lukasiewicz path in exactly the same way that discrete looptrees are coded by Lukasiewicz paths as described in Section 3.3. In this sense, we define one discrete time path (W′ n)N≥0 with step distribution ν(k) = ξ(k + 1) to code the right side of the tree, and a second independent path (W−n)N≥0 to code the left side, with step distribution ν(−k) = ξ(k + 1) and (W′ n) = (W0) = 0. It then follows from standard results in [BGT89, Chapter 8] that both (C−1αn−1W′ n)N≥0 and (C−1αn−1W−n)N≥0 converge in distribution (once interpolated to non-integer values) to independent copies of (Xt)N≥0 and (X′ t)N≥0 as n → ∞, where X and X′ are as they were in the construction of L∞ κ given in Section 3.

The proof is almost identical to the proof of Theorem 1.1, so we omit the details. As we did there, we define two times t̃g(R) and t̃d(R) by

\[
\begin{align*}
  t̃g(R) &= \inf\{ s \geq 0 : \Delta_s, \delta_s, (s^{d^{0}}_{\alpha}) \geq R \} \\
  t̃d(R) &= \inf\{ s \geq 0 : \tilde{X}_s \leq \tilde{X}_{-t̃g(R)} \}.
\end{align*}
\]

It then follows by the Skorohod Representation Theorem that there exists a probability space (Ω, F, P) upon which \((C_{n}^{-1}n^{-1/2}W_{nt})_{0 \leq t \leq t_{0}} \sim (X_{t})_{0 \leq t \leq t_{0}}\) and \((C_{n}^{-1}n^{-1/2}W_{nt})_{0 \leq t \leq t_{0}} \sim (X'_{t})_{0 \leq t \leq t_{0}}\) almost surely with respect to the Skorohod-J₁ topology. As in the proof of Theorem 1.1 for each n ∈ N let λₙ be the Skorohod homeomorphism \([-t̃g - 1, -t̃d + 1] \rightarrow [-t̃g - 1, t̃d + 1] \) that minimises the Skorohod-J₁ distance between the concatenations of these two functions, and set \(t^n_{g} = \lambda_{n}(t_{d})\), and similarly \(t^n_{d} = \lambda_{n}(t_{g})\).

By repeating the arguments of the proof of Theorem 1.1 and noting that condition (ii) of Proposition 3.2 is satisfied just as in the proof of [CK14, Theorem 4.1], we deduce that the looptrees coded by \((C_{n}^{-1}n^{-1/2}W_{nt})_{0 \leq t \leq t_{n}}\) and \((C_{n}^{-1}n^{-1/2}W_{nt})_{0 \leq t \leq t_{n}}\) converge to the looptree coded by \((X_{t})_{t \geq 0}\) on the interval \([-t̃g, t̃d] \) as \(n \rightarrow \infty\). The result then follows as in the proof of Theorem 1.1.

The infinite discrete looptrees defined by Björnberg and Stéfansson in [BS15] are formed by first taking a critical offspring distribution ξ in the domain of attraction of an α-stable law, say with ξ(k, ∞) ∼ κx^{−α}, and forming Kesten’s tree T∞ κ as outlined in Section 3.4. Recall that this has a unique infinite spine of vertices with a size-biased version of the offspring distribution. The authors define their looptree as Loop'(T∞ κ). Here Loop' is an operation very similar to Loop, and \(d_{GH}(\text{Loop}(T∞ κ), \text{Loop}'(T∞ κ)) \leq 2\) (see [CK14, Proof of Theorem 4.1]). We let \(L^∞ κ, = \text{Loop}'(T∞ κ)\).

We then have the following scaling result, analogous to Theorem 1.2. The only difference is that we take a slightly different measure on \(L^∞ κ,\) to account for the fact that different vertices have different degrees. We let ν’ denote the measure on the vertices of \(L^∞ κ,\) given by ν'(x) = deg(x). The Gromov-Hausdorff convergence arises from Theorem 1.2, and the fact that \(d_{GH}(\text{Loop}(T∞ κ), \text{Loop}'(T∞ κ)) \leq 2\), and the Prokhorov convergence of metrics can be proved by combining the appropriate parts of the proofs of 1.1 and Proposition 3.2.

Theorem 6.4. Let \(C_{α} = (c\Gamma(-α))^{-1/α},\) and let \(L^∞ κ,\) be as above, with ν’ the measure on \(L^∞ κ,\) such that ν'(x) = deg(x) for all x ∈ \(L^∞ κ,\). Then

\[
(L^∞ κ, , C_{α}^{-1}n^{-1/2}, \rho, n^{-1/2}ν') \overset{(d)}{\rightarrow} (L^∞ κ, , d^∞ κ, , ρ^∞ κ, , ν^∞ κ, )
\]

with respect to the Gromov-Hausdorff vague topology as n → ∞. Here \(d^∞ κ,\) (respectively \(d^∞ κ,\)) can denote either the geodesic metric d (respectively \(d^∞ κ,\)), or the effective resistance metric R (respectively \(R^∞ κ,\)).
Richier also gave a similar definition of infinite discrete looptrees in [Ric17b] Section 3]. The construction is slightly more complicated and involves two-type Galton-Watson trees so we do not repeat it here. However, modulo turning various coding mechanisms upside down and back to front, they are coded in exactly the same way as our infinite stable looptrees, but instead the functions $(W^l_n)_{n \geq 0}$ and $(W^l_{-n})_{n \geq 0}$ mentioned in the proof above are used to code the left and right sides of the tree as in the continuum case. Letting $(W_t)_{t \geq 0}$ denote the natural concatenation of these functions, the difference with the construction of Björnberg and Stefánsson is that Richier’s looptree is exactly the looptree $L_W$, whereas $W$ plays the role of the Lukasiewicz path in [BS14]. The difference is mainly technical but one consequence is that several different sublooptrees can be grafted to the same point on the loopspine in the latter case.

For now, when $W$ is as above, we denote the infinite looptrees of Richier by $L^\infty_{\alpha}$. We then have the following convergence result, analogous to Theorem 6.4.

**Theorem 6.5.** Let $L^\infty_{\alpha}$ be as above, and let $\nu'$ be the measure on $L^\infty_{\alpha}$ such that $\nu'(x) = \deg(x)$ for all $x \in L^\infty_{\alpha}$. Then

$$
(L^\infty_{\alpha}, C^{-1}_\alpha n^{-\frac{1}{2}} \hat{d}, \rho, n^{-1}\nu') \xrightarrow{d} (L^\infty, \hat{d}, \rho^\infty, \nu^\infty)
$$

with respect to the Gromov-Hausdorff vague topology as $n \to \infty$. Again, here $\hat{d}$ (respectively $\hat{d}^\infty$) can denote either the geodesic metric $d$ (respectively $d^\infty$), or the effective resistance metric $R$ (respectively $R^\infty$).

**Proof.** The proof is identical to the proof of Theorem 6.4 above, except that we invoke Proposition 3.3 rather than Proposition 5.2 to prove the convergence on the compact interval. \qed

### 7 Consequences for random walk limits

#### 7.1 Brownian motion and spectral dimension of $L^\infty_{\alpha}$

As in the case of compact looptrees, the looptree convergence results can be used to give a collection of limit results for random walks and Brownian motion on sequences of looptrees. Before we do this, we have to show that the resistance form associated with the metric space $(L^\infty_{\alpha}, R^\infty)$ is regular, so that we can deduce that there is an associated stochastic process. This is done in the following proposition.

**Proposition 7.1.** Almost surely, the resistance form associated with the metric space $(L^\infty_{\alpha}, R^\infty)$ is regular.

**Proof.** We let $(\mathcal{E}_{\infty}, \mathcal{F}_{\infty})$ denote the resistance form on $L^\infty_{\alpha}$ associated with the resistance metric $R^\infty$. By [Kig12] Definition 6.2, we need to show that for any $f \in C_0(L^\infty_{\alpha})$ and any $\varepsilon > 0$, we can find $g' \in \mathcal{F}_{\infty} \cap C_0(L^\infty_{\alpha})$ such that $\|f - g'\|_{\infty} \leq \varepsilon$. The key point is that by cutting of the infinite loopspine of $L^\infty_{\alpha}$ at an appropriate cutpoint, any such $f$ is also a compactly supported function on a compact stable looptree, and therefore approximable on this compact looptree, since all resistance forms on compact spaces are regular. Formallly, we proceed as follows.

First, note that since $f$ is compactly supported, then it must be contained in $B(\rho_{\infty}, r)$ for some $r > 0$. Taking such an $r$, we then define $t_g(r)$ and $t_d(r)$ exactly as we did in the proof of Theorem 1.1 that is, we set

$$
t_g(r) = \inf\{s \geq 0 : \Delta_{-s} \geq 4r, \delta^\infty_{\alpha}(x_{-s,0}, r) \geq r\}, \quad t_d(r) = \inf\{s \geq 0 : X^\infty_{s} \leq X^\infty_{-t_d(r)-}\}.
$$

As in the proof there, it then follows that $B(\rho_{\infty}, r) \subset P^\infty([-t_g(r), t_d(r)])$, and $P^\infty(-t_g(r)) = P^\infty(t_d(r))$. We denote this projected point by $v_r$. Moreover, removal of $v_r$ divides $L^\infty_{\alpha}$ into two connected components, of which the compact one $P^\infty([-t_g(r), t_d(r)])$ codes a compact stable looptree. We denote this looptree by $L_{\alpha}(r)$. We endow it with a metric and a measure by restricting $R^\infty$ and $\nu^\infty$ to $L_{\alpha}(r)$. It follows from [Kig12] Theorem 8.4 that the associated resistance form is obtained as the trace of $(\mathcal{E}_{\infty}, \mathcal{F}_{\infty})$ on $L_{\alpha}(r)$, and is such that for any $f \in \mathcal{F}_{\alpha}(r), \mathcal{E}_{\alpha}(r)(f, f) = \mathcal{E}_{\infty}(h(f), h(f))$, where $h(f)$ is the unique harmonic extension of $f$ to $L^\infty_{\alpha}$.

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To do this, we first define a sequence \( (g_r) \). To bound the second term in (17), we therefore construct an appropriate function \( \hat{\alpha} \). That is, if \( \hat{\alpha} \) is a minimiser; that is, if \( \hat{\alpha} \) is coded by jump point of \( X^\infty \), that also separates \( \rho_\infty \) from \( \infty \), but such that \( R^\infty(\rho_\infty, v_r') > R^\infty(\rho_\infty, v_r) \). It follows that \( v_r' \) is coded by jump point of \( X^\infty \) at a time \(-t_{g,2}(r)\), where \( t_{g,2}(r) > t_g(r) \) and \(-t_{g,2}(r) \leq 0\). For any \( s \) with \(-t_{g,2}(r) \leq s < -t_g(r)\), set \( a_s = \delta_s(x^\infty_{g,0}) \), and \( b_s = \Delta_s - \delta_s(x^\infty_{g,0}) \), so that \( a_s \) gives the length of the “shorter” segment of the corresponding loop in the loopspine, and \( b_s \) gives the length of the “longer” segment. Set

\[
d_s = \sum_{-t_{g,2}(r) \leq s < -t_g(r)} a_s, \quad d_t = \sum_{-t_{g,2}(r) \leq s < -t_g(r)} b_s.
\]

These are defined so that \( d_s \) gives the looptree distance between \( v_r \) and \( v_r' \), and \( d_t \) gives the “longer distance” between them, which is the length of the path between them that traverses the longer side of all the loops in the loopspine that lie between \( v_r \) and \( v_r' \) (see Figure 5).

![Figure 5: Illustration of how we cut the infinite loopspine.](image)

Additionally, let \( t_{d,2}(r) = \inf\{ s \geq 0 : X^\infty_s \leq X^\infty_{t_{d,2}(r)} \} \). Then \( \rho^\infty([t_{d,2}(r), t_{d,2}(r) - 1]) \) codes another compact stable looptree which we denote by \( \mathcal{L}_\alpha(r)' \), satisfying \( \mathcal{L}_\alpha(r) \subseteq \mathcal{L}_\alpha(r)' \subseteq \mathcal{L}_\alpha^\infty \).

Since \( \mathcal{L}_\alpha(r) \) is compact, it follows that \( (\mathcal{E}\mathcal{L}_\alpha(r), \mathcal{F}\mathcal{L}_\alpha(r)) \) is regular, so there exists \( g \in \mathcal{F}\mathcal{L}_\alpha(r) \cap C_0(\mathcal{L}_\alpha(r)) \) with \( \|f|_{\mathcal{L}_\alpha(r)} - g\|_{\infty, \mathcal{L}_\alpha(r)} \leq \varepsilon \). We therefore define a function \( g' \in C_0(\mathcal{L}_\alpha^\infty) \) by setting \( g' = g \) on \( \mathcal{L}_\alpha(r) \), \( g' = 0 \) on \( \mathcal{L}_\alpha^\infty \setminus \mathcal{L}_\alpha(r)' \), and extending harmonically on \( \mathcal{L}_\alpha(r)' \setminus \mathcal{L}_\alpha(r) \).

Since \( g \) approximates \( f|_{\mathcal{L}_\alpha(r)} \), it follows that \( \|g(v_r)\| \leq \varepsilon \), and moreover it then follows by the maximum principle for harmonic functions that \( \|g'|_{\mathcal{L}_\alpha(r)' \setminus \mathcal{L}_\alpha(r)}\|_{\infty} \leq \varepsilon \). Consequently, \( \|f - g'|_{\infty} \leq \varepsilon \). It therefore just remains to show that \( \mathcal{E}_\infty(g', g') < \infty \).

Since the sets \( \mathcal{L}_\alpha(r), \mathcal{L}_\alpha(r)' \setminus \mathcal{L}_\alpha(r) \) and \( \mathcal{L}_\alpha^\infty \setminus \mathcal{L}_\alpha(r)' \) are disjoint, it follows by bilinearity and from consistency properties of resistance forms and their traces given in [Kig12 Section 8] that

\[
\mathcal{E}_\infty(g', g') = \mathcal{E}_{\mathcal{L}_\alpha(r)}(g, g) + \mathcal{E}_{\mathcal{L}_\alpha(r)'}(g'|_{\mathcal{L}_\alpha(r)' \setminus \mathcal{L}_\alpha(r)}, g'|_{\mathcal{L}_\alpha(r)' \setminus \mathcal{L}_\alpha(r)}).
\]

The first term is finite since \( \mathcal{L}_\alpha(r) \) is regular. Since \( g' \) is harmonic, it also follows that \( g' \) is an energy minimiser; that is, if \( \tilde{g} \) is any other continuous function defined on \( \mathcal{L}_\alpha(r)' \setminus \mathcal{L}_\alpha(r) \) with the same boundary conditions, then

\[
\mathcal{E}_{\mathcal{L}_\alpha(r)'}(g'|_{\mathcal{L}_\alpha(r)' \setminus \mathcal{L}_\alpha(r)}, g'|_{\mathcal{L}_\alpha(r)' \setminus \mathcal{L}_\alpha(r)}) \leq \mathcal{E}_{\mathcal{L}_\alpha(r)'}(\tilde{g}|_{\mathcal{L}_\alpha(r)' \setminus \mathcal{L}_\alpha(r)}, \tilde{g}|_{\mathcal{L}_\alpha(r)' \setminus \mathcal{L}_\alpha(r)}).
\]

To bound the second term in (17), we will therefore construct an appropriate function \( \tilde{g} \) with finite energy. To do this, we first define a sequence \( (\varepsilon_s)_{-t_{g,2}(r) \leq s < -t_g(r)} \) by setting

\[
\varepsilon_s = \frac{R^\infty_{g,2}(v_r, v_r')}{R^\infty(\rho_\infty, v_r')},
\]
where \( R_s^\infty = (a_s^{-1} + b_s^{-1})^{-1} \) is the resistance across the loop coded by the jump at time \( s \). We then define \( \hat{g} \) on the set \( \{ \rho^\infty(s) : -t_g,2(r) \leq s < -t_g(r) \} \) by setting
\[
\hat{g}(\rho^\infty(s)) = \sum_{-t_g,2(r) \leq s < -t_g(r)} \varepsilon_q.
\]
Note that \( \hat{g}(v_r^\epsilon) = 0 \), and \( \hat{g}(v_r) = \varepsilon \), so \( \hat{g} \) has the required boundary conditions. For each loop in the part of \( \mathcal{L}_\alpha(r) \setminus \mathcal{L}_\alpha(r) \) intersecting the infinite loop spine, we then extend \( \hat{g} \) harmonically to the rest of the loop, and on each sublooptree grafted to the loop spine, we take \( \hat{g} \) to be constant.

Summing the energy contributions across each loop, and using the equivalence between \( R^\infty \) and \( d^\infty \), it then follows that:
\[
E_{\mathcal{L}_\alpha(r)}(\hat{g}|_{\mathcal{L}_\alpha(r) \setminus \mathcal{L}_\alpha(r)}, \hat{g}|_{\mathcal{L}_\alpha(r) \setminus \mathcal{L}_\alpha(r)}) \leq 2 \sum_{-t_g,2(r) \leq s < -t_g(r)} \left( a_s \left( \frac{\varepsilon_s}{a_s} \right)^2 + b_s \left( \frac{\varepsilon_s}{b_s} \right)^2 \right) = 2 \sum_{-t_g,2(r) \leq s < -t_g(r)} \frac{a_s^{-1} + b_s^{-1}}{R^\infty(v_r, v_r^\epsilon)} \varepsilon^2.
\]
Since this is finite, it therefore follows from (17) that \( E^\infty(g', g') < \infty \), and hence \( g \in \mathcal{F}_\infty \cap \mathcal{C}_0(\mathcal{L}_\alpha^\infty) \). The result follows.

As a result, we deduce that the resistance metric space is naturally associated with a Hunt process on \( (\mathcal{L}_\alpha^\infty, R^\infty) \), which we call Brownian motion on \( \mathcal{L}_\alpha^\infty \) and denote by \( B^\infty \). We can therefore apply Theorem 2.3 to use Theorems 1.1 and 1.2 to deduce corresponding convergence results for stochastic processes on these spaces. The only additional detail in the proofs of these results is that we have to check that the non-explosion condition at (3) is satisfied, that is that
\[
\lim_{r \to \infty} \liminf_{\ell \to \infty} R_{\ell}(\rho_\ell, B^\ell(\rho_\ell, r)_c) = \infty
\]
amost surely, where \( R_{\ell} \) here denotes the resistance metric on \( \mathcal{L}_\alpha^\ell \).

However, this follows immediately as a consequence of Proposition 5.2. In particular, the arguments used to prove Proposition 5.2 are also valid for compact stable looptrees, so we deduce that the resistance bounds of (13) almost surely hold along the sequence \( (\mathcal{L}_\alpha^\ell)_{\ell \in \mathbb{N}} \).

### 7.2 Local Limits

The local limit theorem of Theorem 1.1 immediately allows us to apply Theorem 2.3 to deduce that Brownian motion on \( \mathcal{L}_\alpha^\ell \) converges in distribution to Brownian motion on \( \mathcal{L}_\alpha^\infty \) as \( \ell \to \infty \) on compact time intervals. Indeed, it follows from the Theorem 2.1 and the Skorohod Representation Theorem that there exists a probability space on which the convergence on Theorem 1.1 holds almost surely. Theorem 1.3 then follows by a direct application of Theorem 2.3.

### 7.3 Scaling Limits

We can also deduce similar results from Theorems 1.4, 6.4 and 6.5. In the first case, the non-explosion condition is almost surely satisfied as a result of similar arguments to those above and [BS15, Lemma 3.5]. The stochastic process associated with the set \( \text{Loop}(T_\infty^\alpha), C_\alpha^{-1} n \frac{\epsilon}{d^\alpha} \rho, n^{-1} \nu \) is a constant speed random walk on \( \text{Loop}(T_\infty^\alpha) \) (since all vertices of \( \text{Loop}(T_\infty^\alpha) \) have degree 4), and therefore (by applying Kolmogorov’s Maximal Inequality to the time index as in the proof of [Arc19, Theorem 1.1]) it will also hold for a simple random walk on \( \text{Loop}(T_\infty^\alpha) \) sped up by a factor of 4.
Theorem 1.4 therefore follows by an immediate application of [Cro18 Theorem 1.2].

In the case of $L^\alpha_{\text{an}}$ and $L^\alpha_{\text{var}}$, we have to take the scaling limit of a variable speed random walk since the vertices in $L^\alpha_{\text{an}}$ have varying degree. Denote this by $Y^\var$. The non-explosion condition is again satisfied by the same arguments as in Section 7.2 above. We then have the following analogues of Theorem 1.4.

**Theorem 7.2.** There exists a probability space $(\tilde{\Omega}', \mathcal{F}', \tilde{P}')$ on which we can almost surely define a common metric space $(M, R_M)$ in which the spaces $(L^\alpha_{\text{an}}, C^\alpha_{\text{an}} n^{\frac{\alpha}{2}} \hat{d}, \rho, n^{-1} \nu)$ and $L^\infty_{\alpha}, R^\infty, \nu^\infty, \rho^\infty)$ can all be embedded and such that

$$(L^\alpha_{\text{an}}, C^\alpha_{\text{an}} n^{\frac{\alpha}{2}} \hat{d}, \rho, n^{-1} \nu) \xrightarrow{\text{d}} (L^\infty_{\alpha}, \hat{d}^\infty, \rho^\infty, \nu^\infty)$$

with respect to the Gromov-Hausdorff-vague topology, and the convergence specifically holds on the metric space $(M, R_M)$. Letting $Y$ and $B^\infty$ be as above, we have that

$$(C^\alpha_{\text{an}} n^{\frac{\alpha}{2}} Y^\var_{\lfloor \frac{1}{4n^{1+\frac{1}{\gamma}} t} \rfloor})_{t \geq 0} \xrightarrow{\text{d}} (B^\infty_t)_{t \geq 0}$$

on the space $D(\mathbb{R}^+, M)$ as $n \to \infty$.

**Theorem 7.3.** There exists a probability space $(\Omega', \mathcal{F}', \mathcal{P}')$ on which we can define a common metric space $(M, R_M)$ in which the spaces $(L^\alpha_{\text{var}}, C^\alpha_{\text{var}} n^{\frac{\alpha}{2}} \hat{d}, \rho, n^{-1} \nu)$ and $L^\infty_{\alpha}, R^\infty, \nu^\infty, \rho^\infty)$ can all be embedded and such that

$$(L^\alpha_{\text{var}}, C^\alpha_{\text{var}} n^{\frac{\alpha}{2}} \hat{d}, \rho, n^{-1} \nu) \xrightarrow{\text{d}} (L^\infty_{\alpha}, \hat{d}^\infty, \rho^\infty, \nu^\infty)$$

almost surely with respect to the Gromov-Hausdorff-vague topology, and the convergence specifically holds on the metric space $(M, R_M)$. Letting $Y^\var$ and $B^\infty$ be as above, we have that

$$(C^\alpha_{\text{var}} n^{\frac{\alpha}{2}} Y^\var_{\lfloor \frac{1}{n^{1+\frac{1}{\gamma}} t} \rfloor})_{t \geq 0} \xrightarrow{\text{d}} (B^\infty_t)_{t \geq 0}$$

on the space $D(\mathbb{R}^+, M)$ as $n \to \infty$.

**Remark 7.4.** We could also prove other convergence results, for example by taking increasing sequences of increasingly rescaled discrete looptrees to approximate $L^\infty_{\alpha}$, in some sense combining Theorems 1.4 and 3.3, and deduce similar convergence results for random walks, exactly as we did in the cases above.

### 7.4 Heat Kernel Convergence and Spectral Dimension

We now show how Theorem 1.1 can be applied to give results on the heat kernel of Brownian motion on compact stable looptrees. First, note that it follows from the scaling invariance of Proposition 4.1 that the annealed heat kernel for $L^\infty_{\alpha}$ satisfies the scaling relation

$$E[p^\infty_k(\rho, \rho)] = k^{\frac{\alpha}{\gamma}} E[p^\infty_{k\ell}(\rho, \rho)]$$

for any $k > 0$. Similarly, if we let $p^\ell_1$ denote the transition density of Brownian motion on a looptree coded by an excursion of length $\ell$, we have that

$$E[p^\ell_1(\rho, \rho)] = k^{\frac{\alpha}{\gamma}} E[p^{k\ell_1}_{k\ell}(\rho, \rho)].$$

Setting $k = t^{-1}$ we see that

$$t^{\frac{\alpha}{\gamma}} E[p^\ell_t(\rho, \rho)] = E[p^{\frac{1}{t\ell}}_1(\rho, \rho)].$$

It then follows from [CH08 Theorem 2 and Proposition 14] that

$$t^{\frac{\alpha}{\gamma}} p^\ell_t(\rho, \rho) \xrightarrow{\text{d}} p^\ell_{\infty}(\rho, \rho)$$

as $t \downarrow 0$. To deduce that the corresponding expectations also converge, note that it follows from Propositions 5.1 and 5.2 that we also have upper exponential tails for the transition density, and from (a continuum
version of) [KM08, Proposition 1.4] that $E[p_1^\infty(\rho, \rho)]$ is finite, so we can apply similar arguments to those in the previous section to deduce that

$$t^{\frac{2\alpha}{\alpha+1}} E[p_1^t(\rho, \rho)] \to E[p_1^\infty(\rho, \rho)]$$

as $t \to \infty$. This is stated as [Arc19, Theorem 1.8], where Brownian motion on $L_\alpha$ is studied more closely.

Similarly, it also follows from [KM08, Theorem 1.5, Part II] (adapted to the continuum) that the heat kernel $p_1^\infty(\rho, \rho)$ almost surely experiences at most log-logarithmic fluctuations around a leading term of $t^{\frac{2\alpha}{\alpha+1}}$ as $t \uparrow \infty$ and as $t \downarrow 0$, and therefore that the quenched spectral dimension of $L_\alpha$ is almost surely equal to $\frac{2\alpha}{\alpha+1}$.

To establish the annealed spectral dimension, we observe that

$$E[p_1^\infty(\rho, \rho)] = t^{\frac{2\alpha}{\alpha+1}} E[p_1^\infty(\rho, \rho)].$$

Since $E[p_1^\infty(\rho, \rho)]$ is finite, we deduce that the annealed spectral dimension is also equal to $\frac{2\alpha}{\alpha+1}$. This concludes the proof of Theorem 1.6.

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