Asymptotics of flat, radiation universes in quadratic gravity

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Abstract

We consider the asymptotics of flat, radiation-dominated isotropic universes in four-dimensional theories with quadratic curvature corrections which may arise when contributions related to the string parameter $\alpha'$ are switched on. We show that all such universes are singular initially, and calculate all early time asymptotics using the method of asymptotic splittings. We also examine the late time asymptotic behaviour of these models and show that there are no solutions which diverge as $e^{t^2}$, the known radiation solution of general relativity is essentially the only late time asymptotic possibility in these models.

1 Introduction

A marked feature of theoretical cosmology during the last few decades has been the proliferation of different dynamical theories of gravity that are used to tackle the basic cosmological issues. With the advent of string theory and its subsequent generalizations, these theories acquired a status similar to the shafts and knobs of J. L. Synge’s allegorical box, an unopened (perhaps unopenable) box in which a vast number of shafts are connected to one another by many complicated laws, making it objectively impossible to
explain what really is inside. There is, it is true, an appealing commonsense in this new picture and, although the promise of hard rationality is not yet fulfilled, one hopes that in time some new forms will develop and the whole picture will come into better focus.

Another feature common to many string theory effective actions is the presence of higher order curvature corrections, the importance of which cannot be neglected unless the basic string parameter $\alpha'$ is small. The generic form of the higher order curvature terms present in these effective string actions is that of a typical Gauss-Bonnet combination, cf. [1, 2]. Such theories admit constant curvature vacua even in the absence of a cosmological constant, many of which are obviously stable [3].

The pertinent issue of the structure and nature of cosmological singularities becomes of interest in this new context for at least two reasons. First, it is important to know how different asymptotic behaviours can be compared to the known situation in general relativity, for instance whether or not simple isotropic solutions act as past attractors to more general homogeneous universes in quadratic theory, cf. [4, 5, 6]. Secondly, in the context of string theory the basic problem becomes that of understanding the structure of cosmological singularities and how the latter could be resolved using the notion of string dualities, cf. [7].

In this note we face the more humble task of analyzing the asymptotic form of certain simple cosmological solutions to the general quadratic theory in four dimensions. Our analysis includes both late and early time asymptotics. We shall show that contrary to claims usually made in the literature of such models (such claims were first made as early as 1969, cf. [8]), one cannot conclude that solutions (which are regular initially) fail to approach the general relativistic solution at late times. In fact, we show that there can be no regular solutions at early times and by applying the method of asymptotic splittings of [9], we further prove that there is an open set of initial conditions for which the general solution blows up at the big bang (collapse) singularity.

The plan of this short paper is as follows. In the next Section we write down the basic equations of the quadratic theory which we consider further in later Sections. We then
show that their late time asymptotics cannot be of the divergent form $e^{t^2}$, contrary to known claims found in the literature. In Section 3, we apply the method of asymptotic splittings to find the form of the general solution of these models near the collapse singularity in the past. We may therefore conclude that simple bouncing, flat, radiation universes with quadratic curvature corrections do not exist.

2 Field equations and late time asymptotics

We start with the action

$$S = \int_{M^4} \mathcal{L}_{\text{total}} d\mu_g, \quad d\mu_g = \sqrt{-g} d\Omega,$$

where $\mathcal{L}_{\text{total}}$ is the lagrangian density of the general quadratic gravity theory given in the form $\mathcal{L}_{\text{total}} = \mathcal{L}(R) + \mathcal{L}_{\text{matter}}$, with

$$\mathcal{L}(R) = R + BR^2 + CRic^2 + DRiem^2,$$

where $B, C, D$ are constants. Since in four dimensions we have the Gauss-Bonnet identity,

$$\delta \int_{M^4} R^2 GB d\mu_g = 0, \quad R^2 GB = R^2 - 4Ric^2 + Riem^2,$$

in the derivation of the field equations through variation of the action associated with (2), only terms up to $\text{Ric}^2$ will matter. Therefore the variational derivative of the action leads to the following field equations:

$$8\pi G c^4 T_{ij} = R^{ij} - \frac{1}{2} g^{ij} R + B \left[ 2RR^{ij} - \frac{1}{2} R^2 g^{ij} - 2(g^{ik}g^{jm} - g^{ij}g^{km})\nabla_k \nabla_m R \right] + C \left[ 2R^{ik} R^j_k - \frac{1}{2} g^{ij} R^{km} R_{km} + \nabla_k \nabla^k R^{ij} + g^{ij} \nabla_k \nabla_m R^{mk} - 2\nabla_k \nabla^j R^{k|i} \right].$$

Below we focus exclusively in spatially flat universes of the form

$$ds^2 = dt^2 - b(t)^2 (dx^2 + dy^2 + dz^2),$$

\footnote{the conventions for the metric and the Riemann tensor are those of [10].}
which are radiation dominated \( P = \rho/3 \). For such spacetimes we have a second useful identity,
\[
\delta \int_{\mathcal{M}^4} (R^2 - 3\text{Ric}^2) d\mu_g = 0,
\]
which further enables us to include the contribution of the \( \text{Ric}^2 \) term into the coefficient of \( R^2 \), altering only the arbitrary constants. In this case the field equations (4) simplify as follows:
\[
R^{ij} - \frac{1}{2} g^{ij} R + \frac{\kappa}{6} \left[ 2R R^{ij} - \frac{1}{2} R^2 g^{ij} - 2(g^{ik} g^{jm} - g^{ij} g^{km}) \nabla_k \nabla_m R \right] = \frac{8\pi G}{c^4} T^{ij},
\]
where \( \kappa = 6B + 2C \). This naturally splits into 00— and \( ii \)—components, but only the 00—component of (7) will be used below. This reads:
\[
\frac{\dot{b}^2}{b^2} - \kappa \left[ 2 \frac{\dddot{b}}{b^2} + 2 \frac{\dot{b}^2}{b^2} - \frac{\dddot{b}^2}{b^2} - 3 \frac{\dot{b}^4}{b^4} \right] - \frac{b_1^2}{b^4} = 0,
\]
where \( b_1 \) is a constant defined by
\[
\frac{8\pi G \rho}{3c^4} = \frac{b_1^2}{b^4}, \quad \text{(from } \nabla_i T^{0i} = 0).\]
Note that the Friedmann solution \( \sqrt{2} b_1 t \) of general relativity satisfies the above equation.

In the rest of this Section we shall consider the problem of the late time asymptotics of solutions to Eq. (8). We say that a solution \( b(t) \) is asymptotic to another solution \( a(t) \) provided that the following two conditions hold (the first is subdivided):

(i) Either (1) \( a(t) \) is an exact solution of the system, or (2) \( a(t) \) is a solution of the system (substitution gives \( 0 = 0 \)) as \( t \to \infty \),

(ii) \( b(t) = a(t)[1 + g(t)], \quad g(t) \to 0, \) as \( t \) tends to infinity.

If either of these two conditions is not satisfied, then \( b(t) \) cannot be asymptotic to \( a(t) \).

Let us assume, following [8], that Eq. (8) has a solution with a regular minimum at the arbitrary time \( t_0 \), \( \dot{b}_0 \equiv \dot{b}(t = t_0) = 0 \) and \( b_0 \equiv b(t = t_0) \neq 0 \). We can then expand this solution as a Taylor series
\[
b(t) = b_0 + \frac{\dot{b}_0}{2} (t - t_0)^2 + \frac{\ddot{b}_0}{6} (t - t_0)^3 + \cdots.
\]
and substitute this form back to Eq. (8), to see that this restricts the value of the constant $\kappa$ to $\kappa = (b_1/(b_0\ddot{b}_0))^2 > 0$.

Reduction of the order of Eq. (8) can be achieved if we set $f = (b\dot{b})^{3/2}$ and $\xi = 12^{-3/4}b^3$, so as to obtain the following second order differential equation:

$$f'' - \frac{1}{\kappa \xi^{2/3}}(f^{-1/3} - b_0^2f^{-5/3}) = 0,$$

(11)

where the derivative is with respect to $\xi$. For large values of $\xi$, an exact asymptotic solution of (11), in the sense of condition (i2) above, is given by

$$f \sim \left(\frac{4}{3\kappa}\right)^{3/4} \xi (\ln \xi)^{3/4}, \quad \text{for large } \xi,$$

(12)

which in terms of the variables $b$ and $t$ has the form

$$b(t) \sim 12^{1/4}e^{(t-t_0)^2/12\kappa}.$$

(13)

However, we stress that this is no more an asymptotic solution of the original equation (8) (as it is, for instance, claimed in [8]). In fact, a simple substitution of (13) into the left hand side of Eq. (8) gives the result:

$$-b_1^2e^{-(t-t_0)^2/3\kappa} + \frac{1}{36\kappa},$$

which never goes over to zero as $t$ tends to infinity, so that condition (i2) can never be satisfied. In other words, an asymptotic solution of Eq. (11) does not necessarily translate into an asymptotic solution of Eq. (8), and this is the case for (12). These results lead us to conclude that there exist no solutions of the field equation (11) which have late time asymptotics of the form (13).

One candidate solution (leading to a well-posed asymptotic problem for late or early times) that satisfies condition (i) of the definition and so qualifies to perturb is the radiation solution proportional to $\sqrt{t}$ as we already noted after Eq. (9). The problem of the late (as well as early) time asymptotics of solutions to the field equations of the quadratic theories was taken up in Refs. [4, 5] using a detailed perturbation analysis.
of the FRW radiation solutions (flat and non-flat) of the form \( a(t) = (t - \sigma t^2)^{1/2} \) (in our present work we focus in the flat case, \( \sigma = 0 \)). The conclusion is that, although the non-flat radiation solutions are generically unstable, in the case of a flat universe there is a parameter region in which all late time solutions are stable with respect to perturbations, the latter decaying as \( g(t) \sim t^{-1} + t^{-3/4} \), (cf., e.g., [3], Eq. (21)). Therefore in such models all relevant solutions of our higher order gravity theory asymptotically approach the flat, radiation solution of general relativity. In view of the non-existence of an \( e^{t^2} \) late time asymptotic shown currently, these perturbation conclusions suggest that the flat, radiation form \( \sqrt{t} \) is a unique late time asymptotic solution in the category considered here.

3 Early asymptotic splittings

We now move on to perform a local asymptotic analysis in order to find the general behaviour of the solutions of Eq. (8) near the initial singularity. This analysis is based on the use of the method of asymptotic splittings expounded in Ref. [9], we follow their notation closely. As a first step, after setting \( b = x, \dot{b} = y \) and \( \ddot{b} = z, \) Eq. (8) can be written as a dynamical system of the form

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= z, \\
\dot{z} &= \frac{y}{2\kappa} - \frac{b^2}{2\kappa y x^2} - \frac{yz}{x} + \frac{z^2}{2y} + \frac{3y^3}{2x^2}.
\end{align*}
\]

(14)

If \( a = (\alpha, \beta, \gamma) \), and \( p = (p, q, r) \), we denote by \( x(\tau) \) the solution

\[
x(\tau) = a \tau^p = (\alpha \tau^p, \beta \tau^q, \gamma \tau^r),
\]

(15)

and by direct substitution in our system (14), we look for the possible scale invariant solutions of this form\(^2\). There are two possible combinations. The first, which is the most interesting one, has dominant part

\[
f^{(0)} = \left(y, z, \frac{z^2}{2y} - \frac{zy}{x} + \frac{3y^3}{2x^2}\right),
\]

(16)

\(^2\)The vector field \( f \) is called scale invariant if \( f(a \tau^p) = \tau^{-p-1} f(a) \)
while the subdominant part reads
\[ f^{\text{sub}} = \left(0, 0, -\frac{b_1^2}{2\kappa y x^2} + \frac{y}{2\kappa}\right), \quad (17) \]

with \( f = f^{(0)} + f^{\text{sub}} \). The dominant balance (of order 3) turns out to be
\[ (a, p) = \left(\left(\alpha, \frac{\alpha}{2}, -\frac{\alpha}{4}\right), \left(\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right)\right), \quad (18) \]

where \( \alpha \) is an arbitrary constant. We recognize this as being what we want, but we are not done yet. We now calculate the eigenvalues of an important matrix, called the K-matrix, which signify the places in a series expansion of the solution around the finite time singularity where arbitrary constants appear, offering thus a clue as to how general the found solution is. The Kowalevskaya exponents for this particular decomposition, eigenvalues of the matrix \( K = Df(a) - \text{diag}(p) \), are \( \{-1, 0, 3/2\} \) with corresponding eigenvectors \( \{(4, -2, 3), (4, 2, -1), (1, 2, 2)\} \). The arbitrariness coming from the coefficient \( \alpha \) in the dominant balance reflects the fact that one of the dominant exponents is zero with multiplicity one.

Keeping with the method of asymptotic splittings [9], we proceed to construct series expansions which are local solutions around movable singularities. In our particular problem, the expansion around the singularity turns out to be a Puiseux series of the form
\[ x(t) = \sum_{i=0}^{\infty} c_{1i}(t - t_0)^{\frac{i}{2} + \frac{1}{2}}, \quad y(t) = \sum_{i=0}^{\infty} c_{2i}(t - t_0)^{\frac{i}{2} - \frac{1}{2}}, \quad z(t) = \sum_{i=0}^{\infty} c_{3i}(t - t_0)^{\frac{i}{2} - \frac{3}{2}}, \quad (19) \]

where \( t_0 \) is arbitrary and \( c_{10} = \alpha, c_{20} = \alpha/2, c_{30} = -\alpha/4 \). For these series expansions to be valid the compatibility condition
\[ (1, 2, 2) \cdot \begin{pmatrix} -2c_{13} + c_{23} \\ -c_{23} + c_{33} \\ -\frac{1}{2}c_{13} + \frac{5}{3}c_{23} - c_{33} \end{pmatrix} = 0, \quad (20) \]

must be satisfied. Substitution of Eq. (19) into Eq. (14) leads to recursion relations that determine the unknowns \( c_{1i}, c_{2i}, c_{3i} \) term by term. After verifying that Eq. (20) is
indeed true, we may write the final series expansion corresponding to the balance (18). It is:

\[ x(t) = \alpha (t - t_0)^{\frac{1}{2}} + c_{13} (t - t_0)^2 + \frac{\alpha^4 - 4b_1^2}{24\kappa\alpha^3} (t - t_0)^{\frac{7}{2}} + \cdots. \]  

(21)

The series expansions for \( y(t) \) and \( z(t) \) are given by the first and second time derivatives of the above expressions respectively.

Our series (21) has three arbitrary constants, \( \alpha, c_{13}, t_0 \) (the last corresponding to the arbitrary position of the singularity) and is therefore a local expansion of the general solution around the movable singularity \( t_0 \). Since the leading order coefficients can be taken to be real, by a theorem of Goriely-Hyde [11], we conclude that there is an open set of initial conditions for which the general solution blows up at the finite time (initial) singularity at \( t_0 \). Finally, we observe that near the initial singularity, all flat, radiation solutions of the quadratic gravity theory considered here are Friedmann-like regardless of the sign of the \( R^2 \) coefficient, while away from the singularity they strongly diverge from such forms. This proves the stability of our solutions in the neighborhood of the singularity.

The second possibility of a scale invariant solution has dominant part

\[ f^{(0)} = \left( y, z, -\frac{b_1^2}{2\kappa y x^2} + \frac{z^2}{2y} - \frac{zy}{x} + \frac{3y^3}{2x^2} \right), \]  

(22)

and subdominant part

\[ f^{\text{sub}} = \left( 0, 0, \frac{y}{2\kappa} \right). \]  

(23)

The dominant balance (of order 2) at the singularity turns out to be

\[ \{(a, p)\} = \{((\alpha, \alpha, 0), (1, 0, -1)), ((-\alpha, -\alpha, 0), (1, 0, -1))\}, \]  

(24)

where \( \alpha \) a constant defined by \( \alpha = (b_1^2/3\kappa)^{1/4} \in \mathbb{R} \). The Kowalevskaya exponents are \( \{-1, \sqrt{6}, -\sqrt{6}\} \), therefore this decomposition leads to a particular solution (two arbitrary constants) whose dominant term near the initial singularity is \( b(t) \sim (t - t_0) \) and therefore does not provide any bounce solutions.
4 Conclusions

We have considered the behaviour of flat, radiation-dominated solutions to the general quadratic theory of gravity in four dimensions. This theory is supposed to arise as part of an effective string theory action truncated to first order in $\alpha'$. We found that in these models there are no bounce solutions, all have a collapse, initial, isotropic singularity and we have given explicit forms of the approach to the initial finite time singularity. However, our results suggest that it is not possible to inhibit the existence of late time behaviour similar to the observed, even though any such solution must necessarily be singular initially.

There are many ways to extend these results. An obvious one is to take into account the dilaton dependence present in the full string action (cf. [1], Eq. (4.3)), in other words to consider the present problem in the string frame (and not in the Einstein conformal frame of the present paper). We also wish to see how our results are altered (or not altered) when we pass on to models with curvature, especially near the singularity.

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References

[1] M. Gasperini, M. Maggiore and G. Veneziano, Nucl. Phys. B494 (1997) 315-328.

[2] K. A. Meissner, Phys. Lett. B392 (1997) 298-304.
[3] S. Deser and B. Tekim, Phys. Rev. D67 (2003) 084009.

[4] S. Cotsakis and G. P. Flessas, Phys. Rev. D48 (1993) 3577.

[5] S. Cotsakis and G. P. Flessas, Phys. Rev. D51 (1995) 4160.

[6] J. D. Barrow and S. Hervik, Phys. Rev. D74 (2006) 124017.

[7] E. Witten, *Singularities in String Theory*, arXiv:hep-th/0212349.

[8] T. V. Ruzmaikina and A. A. Ruzmaikin, Zh. Eksp. Teor. Fiz. 57, 680 (1969).

[9] S. Cotsakis, J. D. Barrow, *The Dominant Balance at Cosmological Singularities*, arXiv:gr-qc/0608137, to appear in the Proceedings of the Greek Relativity Meeting NEB12, June 29-July 2, 2006, Nauplia, Greece.

[10] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, (Pergamon Press, 1975).

[11] A. Goriely and C. Hyde, J. Diff. Eq., 161, 422 (2000).