RIGIDITY OF LOW INDEX SOLUTIONS ON $S^3$
VIA A FRANKEL THEOREM
FOR THE ALLEN–CAHN EQUATION

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Abstract. We prove a rigidity theorem in the style of Urbano for the
Allen–Cahn equation on the three-sphere: the critical points with Morse
index five are symmetric functions that vanish on a Clifford torus. More-
over they realised the fifth width of the min-max spectrum for the Allen–
Cahn functional. We approach this problem by analysing the nullity and
symmetries of these critical points. We then prove a suitable Frankel-
type theorem for their nodal sets, generally valid in manifolds with posi-
tive Ricci curvature. This plays a key role in establishing the conclusion,
and further allows us to derive ancillary rigidity results in spheres with
larger dimension.

1. Introduction

F. Urbano gave a description of the minimal surfaces in the round $S^3$ with
low Morse index [Urb90]. The only embedded surfaces with index at most
five are the equatorial spheres $S \subset S^3$, with index one, and the Clifford tori
$T \subset S^3$, with index five. Our aim here is to prove an analogous result in
the setting of the Allen–Cahn equation.

Consequences of Urbano’s theorem have found applications in numerous
contexts, most notably the proof of the Willmore conjecture by Marques–
Neves [MN14]. Their proof relied on the development of the so-called Almgren–
Pitts min-max methods. In recent years an alternative min-max theory
has grown, whose underlying idea is to first find critical points of a cer-
tain semilinear elliptic functional. (For the sake of this exposition we work
in $S^3$, although much of the discussion could remain unchanged in higher-
dimensional spheres or general closed manifolds.) This is the Allen–Cahn
functional $E_\epsilon(u) = \int_{S^3} \frac{1}{2} |\nabla u|^2 + \epsilon^{-1} W(u)$, which depends on a small param-
eter $\epsilon > 0$. (For the sake of this exposition we may use the potential function
$W(t) = \frac{1}{4} (1 - t^2)^2$.) This models the phase transition in a two-phase liquid.
Its critical points solve the elliptic Allen–Cahn equation

$$\epsilon \Delta u - \epsilon^{-1} W'(u) = 0;$$

we write $Z_\epsilon$ for the set of (classical) solutions of this equation. As $\epsilon \to 0$
their values tend to congregate near $\pm 1$, separated by a thin transition layer
around a minimal surface.

A number of authors have made contributions to this alternative min-max
theory; we give a condensed summary of only those results most pertinent
here. Hutchinson–Tonegawa [HT00] proved that given a sequence of positive
$\epsilon_j \to 0$ and critical points $u_j \in Z_{\epsilon_j}$, one can form a sequence of varifolds $V_j$
which converge weakly to a stationary integral varifold $V$. When the Morse
index of the $u_j$ is bounded along the sequence, then the combined work of Tonegawa–Wickramasekera [TW12] and Guaraco [Gua18] establishes that

$$V_j \to Q|\Sigma| \text{ as } j \to \infty,$$

where $\Sigma \subset S^3$ is a smooth embedded minimal surface and $Q \in \mathbb{Z}_{>0}$ is its so-called multiplicity. (Both of these rely fundamentally on the deep results of Wickramasekera [Wic14] on the regularity of stable codimension one stationary varifolds.) Following this, Chodosh–Mantoulidis [CM20] proved that the convergence is actually with multiplicity one, that is $Q = 1$. (This is the main element not available in higher dimensions, although it does hold in three-manifolds with positive Ricci curvature.)

It was Guaraco [Gua18] and Gaspar–Guaraco [GG18] who respectively implemented one-and multi-parameter min-max methods in the Allen–Cahn setting. When working with $p \in \mathbb{Z}_{>0}$ parameters, these produce a critical point which realises the $p$-th width $c_\epsilon(p)$ of the min-max spectrum of the Allen–Cahn functional. With fixed $\epsilon > 0$ and varying $p$, these form a monotone increasing sequence,

$$0 < c_\epsilon(1) \leq \cdots \leq c_\epsilon(p) \leq c_\epsilon(p + 1) \leq \cdots .$$

One is naturally led to draw comparisons with the sequence of widths attained by Almgren–Pitts methods: $(\omega(p) \mid p \in \mathbb{Z}_{>0})$. Nurser [Nur16] computed the first terms in this sequence in the round $S^3$. The first four have values $\omega(1) = \cdots = \omega(4) = 4\pi$, and are realised by equatorial spheres, and the next three widths are $\omega(5) = \omega(6) = \omega(7) = 2\pi^2$, and are realised by Clifford tori. Recently Caju–Gaspar–Guaraco–Matthiesen [CGGM20] proved an analogue for the first four min-max widths for the Allen–Cahn functional, namely they are equal and are realised by symmetric functions vanishing on equatorial spheres. Moreover, they point out that for topological reasons the next width $c_\epsilon(5)$ must be distinct from the first four. (The lower bounds for the phase transition spectrum of [GG18] give an alternative justification for this.)

We prove that in fact $c_\epsilon(5)$ is realised by a function vanishing on a Clifford torus $T$, with the same symmetries as $T$. This result, stated in Corollary 2 below, is a direct consequence of the following rigidity theorem.

**Theorem 1.** Given any $C > 1$, there is $\epsilon_0 > 0$ so that for all $0 < \epsilon < \epsilon_0$ the following holds. If a solution $u \in Z_\epsilon$ on $S^3$ has index $u \leq 5$ and $E_\epsilon(u) \leq C$, then its nodal set is either an equatorial sphere or a Clifford torus, and $u$ is a symmetric solution around it.

Its proof is given in the last section. We proceed by studying the nullity and comparing it to the Killing nullity of $u$; this allows conclusions about the symmetry of $u$.

**Corollary 2.** There is $\epsilon_0 > 0$ so that if $0 < \epsilon < \epsilon_0$ then any $u \in Z_\epsilon$ on $S^3$ with $E_\epsilon(u) = c_\epsilon(5)$ is a symmetric solution vanishing on a Clifford torus.

These are not the first rigidity results for the Allen–Cahn equation. Besides the aforementioned [CGGM20] concerning the ground states, there is also recent work of Guaraco–Marques–Neves [GNM19] near non-degenerate minimal surfaces. In [GNM19] the authors adapt the curvature estimates
Low index solutions of the Allen–Cahn equation on $S^3$

We sidestep the technical difficulties tied to such an approach, and instead of [CM20] to produce a non-trivial Jacobi field on the limit minimal surface. Frankel’s theorem states that any two minimal hypersurfaces in $M$ must intersect. This is not quite true for the nodal sets $Z(\nu^1) = \{u^1 = 0\}$ of two solutions $u^1 \in \mathcal{Z}$ on $M$; see for example [GNM19, Expl. 1]. However, we prove that under mild topological hypotheses on the nodal sets a Frankel-type result actually does hold. The following theorem combines the statements of Corollary 9 and Proposition 10.

**Theorem 3.** Let $(M, g)$ be a closed manifold with positive Ricci curvature. Frankel’s theorem states that any two minimal hypersurfaces in $M$ must intersect. Moreover it allows us to obtain ancillary rigidity results in spheres with dimension larger than three, the caveat being that in absence of [CM20] the convergence must be assumed to be with multiplicity one. We illustrate this with the Clifford-type minimal hypersurfaces $$T_{p,q} = S^p(\sqrt{p/n}) \times S^q(\sqrt{q/n}) \subset S^{n+1},$$ where $p, q \in \mathbb{Z}_{>0}$ and $n = p + q \geq 2$. By [AA81, HL71] these are known to have $\nu(T_{p,q}) = \text{nullity } T_{p,q} = (p+1)(q+1)$. Moreover they are homogeneous, and are stabilised by $SO(p) \times SO(q) \subset SO(n+2)$. The rigidity result valid near these surfaces is similar to Theorem 1, except that here no claim is made about the precise location of the nodal set. (Here and in the remainder we write $(T_{p,q})_\delta \subset S^{n+1}$ for the tubular neighbourhood of $T_{p,q}$ with size $\delta > 0$.)

**Theorem 4.** Let $p, q \in \mathbb{Z}_{>0}$ and $n = p + q \geq 2$. There exist $0 < \epsilon_0, \delta < 1$ so that for all $0 < \epsilon < \epsilon_0$ there is a unique solution $u \in \mathcal{Z}$ on $S^{n+1}$ with $Z(u) \subset (T_{p,q})_\delta$ and $(1 - \delta)\mathcal{H}^n(T_{p,q}) \leq E_\epsilon(u) \leq (1 + \delta)\mathcal{H}^n(T_{p,q})$, up to rotation and change of sign. Moreover $u$ is $SO(p) \times SO(q)$-symmetric, up to conjugation.

In the more symmetric case where $p = q$ and $n = 2p$, the information about the nodal set can be recovered. We state this corollary in a slightly different way.

**Corollary 5.** Let $p \in \mathbb{Z}_{>0}$ and $n = 2p$. Let $\epsilon_j \to 0$, and $u_j \in \mathcal{Z}$ be so that $V(\epsilon_j, u_j) \to |T_{p,p}|$. For large $j$, the nodal set of $u_j$ is a rigid copy of $T_{p,p}$ and $u_j$ is a symmetric solution around it.

Theorem 4 and Corollary 5 are proved in essentially the same way as the main result, Theorem 1. We give the necessary modifications to the higher-dimensional setting in Appendix A but do not repeat the overlapping arguments. In general, beyond the restrictive assumptions on the limit minimal surface—homogeneity and integrability through rotations—our arguments are quite flexible, and apply in closed manifolds with positive Ricci curvature, of dimension three or larger.
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2. Preliminary results

Consider $S^3$ equipped with the standard round metric. For every $\epsilon > 0$ the $\epsilon$-Allen–Cahn equation is the semilinear partial differential equation

$$\epsilon \Delta u - \epsilon^{-1} W'(u) = 0,$$

where $W : \mathbb{R} \to \mathbb{R}$ is a regular non-negative even potential with a positive, non-degenerate maximum at the origin and two minima at $\pm 1$, where $W(\pm 1) = 0$. (For example one may take $W(t) = (1 - t^2)^2/4$.)

This is the Euler–Lagrange equation of the Allen–Cahn functional $E_\epsilon(u) = \int_{S^3} \frac{1}{2} |\nabla u|^2 + \epsilon^{-1} W(u)$. We write $Z_\epsilon$ for the space of (classical) solutions of this equation. There is a standard way to associate to each solution $u = u_\epsilon$ a varifold $V_\epsilon = V(\epsilon, u)$, meaning a Radon measure on the Grassmann bundle $Gr_2(S^3)$. Given any $\varphi \in C(Gr_2(S^3))$, this is defined by

$$V_\epsilon(\varphi) = \frac{1}{2\sigma} \int_{S^3} \epsilon |\nabla u(X)|^2 \varphi(X, T_X\{u = u(X)\}) \, d\mathcal{H}^3(X)$$

where $\{u = u(X)\}$ is the level set of $u$ through $X$ and $\sigma = \int_{-1}^1 \sqrt{W(t)/2} \, dt$ is a normalising constant. (We use very little in the way of varifold theory in the remainder, though one may consult [Sim84] for unfamiliar vocabulary.)

2.1. Index and nullity. Let $(\epsilon_j \mid j \in \mathbb{N})$ be a positive sequence with $\epsilon_j \to 0$ as $j \to \infty$, and $(u_j \mid j \in \mathbb{N})$ be a corresponding sequence with $u_j \in Z_{\epsilon_j}$ for all $j$. We assume uniform bounds for the energy and that the Morse index of these solutions is fixed, namely there are $C > 0$ and $k \in \mathbb{Z}_{\geq 0}$ so that for all $j \in \mathbb{N}$,

$$E_{\epsilon_j}(u_j) \leq C$$

and

$$\text{index } u_j = k.$$

A subsequence of the $V_j = V(\epsilon_j, u_j)$ (which we extract without relabelling) converges to a stationary integral varifold, as was shown by Hutchinson–Tonegawa [HT00]. Relying on the regularity theory developed by Wickramasekera [Wic14], the combined work of Tonegawa–Wickramasekera [TW12] and Guaraco [Gua18] established that this limit is supported in a smooth embedded minimal surface $\Sigma \subset S^3$, and

$$V_j \to Q|\Sigma|$$

with integer multiplicity $Q \in \mathbb{Z}_{\geq 0}$. (Here and throughout we write $|\Sigma|$ for the unit density varifold associated to $\Sigma$.) Using different methods, Hiesmayr [Hie18] and Gaspar [Gas20] both proved that

$$\text{index } \Sigma \leq k,$$

the former building on previous work of Tonegawa [Ton05] in the stable case, the latter by adapting computations of Le [Le11, Le15].
As the three-sphere $S^3$ has positive Ricci curvature, the results of Chodosh–Mantoulidis [CM20] give that the convergence is with multiplicity one, meaning $Q = 1$ and $V_j \to |\Sigma|$ as $j \to \infty$. They also show that
\[
\limsup_{j \to \infty} (\text{index } u_j + \text{nullity } u_j) \leq \text{index } \Sigma + \text{nullity } \Sigma.
\]
Combining this with the lower semicontinuity of the Morse index, we find that for large enough $j$,
\[
\text{nullity } u_j \leq \text{nullity } \Sigma.
\]
We are especially interested in the situation where the Morse index of the $u_j$ is low, specifically where it is five at the most. In this range the classification of minimal surfaces of Urbano [Urb90] shows that for the limit minimal surface $\Sigma$,
- either index $\Sigma = 1$ and $\Sigma = S$ is an equatorial sphere,
- or index $\Sigma = 5$ and $\Sigma = T$ is a Clifford torus.
By [HL71] they respectively have nullity $S = 3$ and nullity $T = 4$.

2.2. **Action by rotations.** The orthogonal group $SO(4)$ acts on the space of minimal surfaces in $S^3$, and preserves their index and nullity. Given a surface $\Sigma \subset S^3$, let as usual $\text{Stab } \Sigma = \{P \in SO(4) \mid P(\Sigma) = \Sigma\}$ and $\text{Orb } \Sigma = \{P(\Sigma) \subset S^3 \mid P \in SO(4)\}$. The former is a closed Lie subgroup of $SO(4)$ denoted $G_\Sigma$, and $\dim \text{Stab } \Sigma + \dim \text{Orb } \Sigma = 6$, the dimension of $SO(4)$.

Let $\Sigma$ be minimal and $L_\Sigma$ be the *Jacobi operator* on $\Sigma$. We write index $\Sigma \in \mathbb{Z}_{\geq 0}$ for the number of strictly negative eigenvalues of $L_\Sigma$, counted with multiplicity. The kernel of $L_\Sigma$ is spanned by the so-called *Jacobi fields*. Those that stem from the action of $SO(4)$ are also called *Killing Jacobi fields*. They span a linear subspace of $L^2(\Sigma)$ of dimension $\nu(\Sigma) = \dim \text{Orb } \Sigma$; this is the *Killing nullity*. (Here we follow the conventions of Hsiang–Lawson [HL71].)

For any minimal surface $\Sigma \subset S^3$,
\[
\nu(\Sigma) \leq \text{nullity } \Sigma.
\]
We are especially interested in surfaces for which $\nu$ equals the full nullity. Surfaces with this property are sometimes said to be *integrable through rotations*. The equatorial sphere and the Clifford torus both have this property, and quoting [HL71] they respectively have
\[
\text{nullity } S = \nu(S) = 3 \quad \text{and} \quad \text{nullity } T = \nu(T) = 4.
\]

Let $\Sigma \subset S^3$, with isotropy subgroup $G_\Sigma \subset SO(4)$. The surface is called *homogeneous* if $G_\Sigma$ acts transitively on it. This is equivalently expressed by saying that $\Sigma$ is an orbit of the action of $G_\Sigma$ on $S^3$. The equatorial sphere and the Clifford torus both satisfy this property; in fact they are the only homogeneous minimal surfaces in $S^3$ [HL71].

These notions have natural analogues in the setting of the Allen–Cahn equation. The group $SO(4)$ gives a right action on functions defined on $S^3$ by pre-composition, where $P \in SO(4)$ sends a function $u$ to $u \circ P$. Given a function $u$ on $S^3$ we write $\text{Stab } u = \{P \in SO(4) \mid u \circ P = u\}$ and $\text{Orb } u = \{u \circ P \mid P \in SO(4)\}$. Again we have $\dim \text{Stab } u + \dim \text{Orb } u = 6$. (To see this fix any $\alpha \in (0,1)$; by elliptic regularity any critical point has
$u \in C^{2,\alpha}(S^3)$. Consider the orbit map $P \in SO(4) \mapsto u \circ P \in C^{2,\alpha}(S^3)$. Upon quotienting by the stabiliser $G_u = \text{Stab } u$ this defines a homeomorphism onto its image. In particular $SO(4)/G_u$ and $\text{Orb } u$ have the same dimension, namely $6 - \dim G_u$. Note that a rigorous redervation of the analogous orbit-stabiliser identity we quoted above for minimal surfaces $\Sigma \subset S^3$ would go along the same lines, working in a suitable Banach space of embeddings into $S^3$.

Let $M_\epsilon$ be the semilinear operator corresponding to (1). The linearisation of $M_\epsilon$ around a function $u$ is (up to multiplication by $\epsilon$) the linear operator $L_{\epsilon, u} = \Delta - \epsilon^{-2}W''(u)$. (This describes the second variation of $-E_\epsilon$ at $u$ via integration by parts.) This operator has finite-dimensional kernel, whose dimension is denoted $\text{nullity } u \in Z_{\geq 0}$. We also write $\text{index } u \in Z_{\geq 0}$ for the number of strictly negative eigenvalues of $L_{\epsilon, u}$ counted with multiplicity. The action of $SO(4)$ preserves the Allen–Cahn functional, $E_\epsilon(u \circ P) = E_\epsilon(u)$ for all $P \in SO(4)$. It also maps $Z_{\epsilon}$ to itself, and for all $P \in SO(4)$, $\text{index } u \circ P = \text{index } u$ and nullity $u \circ P = \text{nullity } u$. The invariance of $E_\epsilon$ under $SO(4)$ also means that the action generates functions in the kernel of $L_{\epsilon, u}$, which span a space of dimension $\nu(u) \in Z_{\geq 0}$, the Killing nullity of $u$. As above we have

$$\nu(u) = \dim \text{Orb } u$$

(6)

(The fact that $\nu(u) = \dim \text{Orb } u = 6 - \dim G_u$ is not hard to see. To derive this rigorously requires working again with the orbit map $P \in SO(4) \mapsto u \circ P \in C^{2,\alpha}(S^3)$ we used above for the orbit-stabiliser identity. This differentiates to the map $A \in so(4) \mapsto -\langle \nabla u, \xi_A \rangle$, where $\xi_A$ is the Killing vector field corresponding to $A$. Writing $g_u \subset so(4)$ for the Lie algebra corresponding to $G_u$, we obtain a linear isomorphism between $so(4)/g_u$ and the tangent space to the orbit of $u$. In particular there are precisely $\nu = 6 - \dim g_u = 6 - \dim G_u$ Killing vector fields so that $\langle \nabla u, \xi_1 \rangle, \ldots, \langle \nabla u, \xi_\nu \rangle$ forms a linearly independent family.)

2.3. Consequences of multiplicity one convergence. Let $\epsilon_j \to 0$ and $(u_j \mid j \in \mathbb{N})$ be a sequence of critical points in $S^3$ with $V(\epsilon_j, u_j) \to |\Sigma|$, where $\Sigma \subset S^3$ is an embedded minimal surface. Using either a simple calculation as in [CG19] or Lemma 13 as justification, one finds that for large $j$,

$$\nu(\Sigma) \leq \nu(u_j).$$

(7)

Assume additionally that $\Sigma$ is integrable through rotations, that is $\nu(\Sigma) = \text{nullity } \Sigma$. Stringing together (5), (6) and (7), we find that eventually

$$\text{nullity } u_j = \nu(u_j).$$

We now specialise this to the setting of low index in $S^3$, and show that if index $u_j \leq 5$ then either index $u_j = 1$ or 5, provided $j$ is large enough. To this end assume that $E_{\epsilon_j}(u_j) \leq C$ and index $u_j \leq 4$ along the sequence. By (3) and Urbano’s classification, $V(\epsilon_j, u_j)$ converges to an equatorial sphere $S \subset S^3$; moreover by [CM20] the convergence is with multiplicity one, that is $V(\epsilon_j, u_j) \to |S|$ as $j \to \infty$. From the above we have that for large $j$, $\text{nullity } u_j = \nu(u_j) = 3$. At the same time $4 \geq \text{index } u_j + \text{nullity } u_j \geq \text{index } u_j + 3$ by (4). As $u_j$ is unstable, we find $\text{index } u_j = 1$. 

We apply this observation to the context of Corollary 2, where no assumption is made about the index of the solutions. Instead there only imposes that the \( u_j \in \mathcal{Z}_c \) have \( E_{\epsilon_j}(u_j) = c_{\epsilon_j}(5) \). This notwithstanding, critical points obtained through five-parameter min-max arguments as in [GG18] have index at most five. Taking the conclusions of Theorem 1 for granted, and as \( c_{\epsilon_j}(4) < c_{\epsilon_j}(5) \), the functions obtained from min-max methods must be symmetric solutions vanishing on a Clifford torus. Moreover \( c_{\epsilon_j}(5) \to 2\pi^2 \) as \( j \to \infty \). By [MN14], the Clifford tori are the only minimal surfaces in \( S^3 \) whose area has this value (or a fraction thereof), so that \( V(\epsilon_j, u_j) \to [T] \), after extracting a subsequence if necessary. Arguing as above one obtains index bounds for the \( u_j \), which prove that they too must be of the rigid form described in Theorem 1.

3. A Frankel-type result for the Allen–Cahn equation

Let \((M, g)\) be a closed manifold with dimension \( n + 1 \) and positive Ricci curvature. We formulate two types of results for the Allen–Cahn equation in the style of Frankel’s theorem [Fra61]: the first concerning the solutions themselves, and the second their nodal sets.

**Proposition 6.** Let \( \epsilon > 0 \) and \( u^1_\epsilon \neq u^2_\epsilon \) be two solutions of (1) on \( M \). If \( u^1_\epsilon \leq u^2_\epsilon \) then one of the two is constant.

We postpone the proof of this for now, and move on to the Frankel-type result for nodal sets. Let us remark first that the nodal sets of two solutions \( u^1_\epsilon, u^2_\epsilon \in \mathcal{Z}_\epsilon \) do not intersect in general. Indeed, Guaraco–Marques–Neves [GMN19, Expl. 1] give a short construction of solutions near equatorial spheres in \( S^{n+1} \) for which this fails. (In their example one critical point vanishes on the equatorial sphere, and the other vanishes along two hypersurfaces lying on either side of it, a small distance away.) One must therefore impose natural hypotheses on the nodal sets to ensure that they meet. Given a function \( u \) on \( M \), we write \( Z(u) = \{ u = 0 \} \) for its nodal set, and say that it is separating if \( M \setminus Z(u) \) has exactly two connected components.

**Proposition 7.** Let \( \epsilon > 0 \) and \( u^1_\epsilon, u^2_\epsilon \neq \pm 1 \) be two solutions of (1) on \( M \). If \( Z(u^2_\epsilon) \) is separating, then either \( Z(u^1_\epsilon) \cap Z(u^2_\epsilon) \neq \emptyset \) or

\[
Z(u^2_\epsilon) \cap \{ u^1_\epsilon > 0 \} \neq \emptyset \text{ and } Z(u^2_\epsilon) \cap \{ u^1_\epsilon < 0 \} \neq \emptyset.
\]

**Proof.** We argue by contradiction, and assume that \( Z(u^2_\epsilon) \subset \{ u^1_\epsilon > 0 \} \). Divide the complement of \( Z(u^2_\epsilon) \) into its connected components, \( M \setminus Z(u^2_\epsilon) = \bigcup_{j=0}^N U_j \) where \( N \in \mathbb{Z}_{\geq 0} \cup \{ \infty \} \). (There are at most countably many of these, and the proof is the same whether \( N \) is finite or infinite.) We decompose \( \{ u^1_\epsilon < 0 \} \) by conditioning on the \( U_j \), and write \( \{ u^1_\epsilon < 0 \} = \bigcup_{j=0}^N \{ u^1_\epsilon < 0 \} \cap U_j \).

As \( Z(u^1_\epsilon) \) is separating, only of these is non-empty, and \( \{ u^1_\epsilon < 0 \} \subset U_0 \) say. Now on the one hand we may assume that \( u^2_\epsilon < 0 \) on \( U_0 \), flipping its sign if necessary. On the other hand, \( \partial U_0 \subset \{ u^1_\epsilon > 0 \} \) and thus \( \partial(U_0 \cap \{ u^1_\epsilon < 0 \}) \subset \partial \{ u^1_\epsilon < 0 \} \). Therefore \( u^1_\epsilon = 0 \) on \( \partial(U_0 \cap \{ u^1_\epsilon < 0 \}) \). A simple consequence of the maximum principle—see [GMN19, Cor. 7.4]—means that \( u^2_\epsilon < u^1_\epsilon \) on \( U_0 \cap \{ u^1_\epsilon \leq 0 \} \). This actually holds on the whole \( U_0 \), as \( u^2_\epsilon < 0 \leq u^2_\epsilon \) on \( U_0 \setminus \{ u^1_\epsilon \leq 0 \} \). Let \( U_j \) be one of the remaining components. If \( u^2_\epsilon \leq 0 \) in \( U_j \) then \( u^2_\epsilon \leq u^1_\epsilon \), so we may assume \( u^2_\epsilon \geq 0 \). Then \( u^2_\epsilon = 0 \) on \( \partial U_j \) while
Corollary 9. If an inspection of the proof reveals this hypothesis to not be necessary.

Remark. Technically [GNM19, Cor. 7.4] is stated on regular domains, but an inspection of the proof reveals this hypothesis to be not necessary.

The following is an immediate consequence.

Corollary 9. If $Z(u^1_f)$ is separating and $Z(u^2_f) \neq \emptyset$ is connected then $Z(u^1_f) \cap Z(u^2_f) \neq \emptyset$.

The next proposition can be derived independently, using a similar argument. Either Corollary 9 or Proposition 10 would be sufficient for our purposes; we include both for the sake of completeness.

Proposition 10. If $Z(u^1_f), Z(u^2_f) \neq \emptyset$ are connected then $Z(u^1_f) \cap Z(u^2_f) \neq \emptyset$.

Proof. Again we argue by contradiction, assuming that $Z(u^1_f) \cap Z(u^2_f) = \emptyset$. Split $Z = Z(u^1_f) = Z_\pm \cup Z_\pm$, according to the sign of $u^2_f$. As $Z$ is connected, only one of these can be non-empty, and $Z = Z_\pm$ say. After perhaps flipping the sign of $u^1_f$, we find that $u^2_f < u^1_f$ in $\{u^1_f \leq 0\}$ by [GNM19, Cor. 7.4]. The mirror argument shows that also $u^2_f < u^1_f$ in $\{u^2_f \geq 0\}$. On the remaining region $u^2_f < 0 < u^1_f$, which means $u^2_f < u^1_f$ is established on the entirety of $M$; the conclusion follows from Proposition 6.

We now turn to the proof of Proposition 6. For this we use the parabolic Allen–Cahn equation with initial data a bounded function $u_0 \in C^1(M)$, which given $\epsilon > 0$ and $T \in (0, \infty]$ is defined to be

$$
\begin{align*}
\left\{ \begin{array}{ll}
\epsilon u_t - \epsilon^{-1}W''(u) & \text{in } M \times [0, T), \\
\epsilon u(0, \cdot) = u_0 & \text{on } M.
\end{array} \right.
\end{align*}
$$

Lemma 11. Let $u_0 \in C^1(M)$ be a weak subsolution of (1) with $|u_0| \leq 1$. Then the solution to (8) exists for all time, and $-1 \leq u(s, \cdot) \leq u(t, \cdot) \leq 1$ for all $0 \leq s \leq t$. As $t \to \infty$, $u(t, \cdot)$ converges to a smooth function $u_+$. Moreover $u_+$ is a stable solution of (1), and can be characterised as the least element of $\{v \in Z | v \geq u_0\}$.

Proof. Using the classical theory of parabolic PDE, we find that the solution $u$ of (8) is unique, smooth and exists for all time. To obtain the monotonicity of the flow, note that $u_t$ satisfies the following parabolic equation: $\partial_t u_t = \epsilon \Delta u_t - \epsilon^{-1}W''(u)u_t$. As initially $u_t(0, \cdot) \geq 0$, the parabolic maximum principle forces $u_t \geq 0$ for all $t \geq 0$. Moreover, this is strict unless $u_t = 0$, which happens precisely if $u_0$ is a solution of (1). Together with classical parabolic Schauder estimates, this monotonicity guarantees the convergence of $u(t, \cdot)$ as $t \to \infty$, with limit the smooth function $u_+$. Moreover $u_+$ is a classical solution of (1). As the set $\{v \in Z | v \geq u_0\}$ works as a barrier for the flow, the function $u_+$ must be its least element.

It remains to see the stability of $u_+$; there are various ways to obtain this. For example, assume that $u_+$ is unstable and let $\varphi_1 > 0$ be its first eigenfunction, with eigenvalue $\lambda_1 < 0$. A quick computation reveals that for a suitably small $\theta \in (0, 1)$, $u_+ - \theta \varphi_1 < u_+$ is a supersolution of (1). Indeed
If we were to apply the arguments above, we would find that solving the parabolic Allen–Cahn equation with initial datum $u_+ - \theta \varphi_1$ yields a strictly decreasing solution. Perhaps after adjusting $\theta$ to a smaller value so that $u_+ - \theta \varphi_1 > u_0$, this acts as an upper barrier and makes $u(t, \cdot) \to u_+$ absurd. □

We use this to prove Proposition 6.

Proof. In a manifold with positive Ricci curvature, the only stable solutions of (1) are the constant functions with values one of $\{-1, 0, 1\}$. This is because, as a quick computation shows, $\delta^2 E_\epsilon(u)(|\nabla u|, |\nabla u|) = -\epsilon \int_M |\nabla^2 u|^2 - |\nabla|\nabla u||^2 + \text{Ric}(\nabla u, \nabla u)$. This is non-negative precisely when $|\nabla u| \equiv 0$ and $u$ is constant.

Let $u_1 \leq u_2$ be two distinct solutions of (1), and assume that $u_1$ is not constant equal $\pm 1$. By the maximum principle $u_1 < u_2$ on $M$. (If $u_1 = 0$ then this would imply that $u_2$ is constant equal one.) Let $\varphi_1 > 0$ be the first eigenfunction of the linearised operator $L_{\epsilon, u_1}$, with eigenvalue $\lambda_1 < 0$. The same computation as in the proof of Lemma 11 shows that for sufficiently small $\epsilon \in (0, 1)$, $u_1 + \theta \varphi_1$ is a subsolution of (1). Take $\theta > 0$ small enough that still $u_1 + \theta \varphi_1 < u_2$. Then solving (8) with this function as an initial datum we obtain a strictly increasing family $u(t, \cdot)$ of functions with $u_1 < u(t, \cdot) \leq u_2$. By Lemma 11 this converges to a stable solution $u_+$ when we let $t \to \infty$. The characterisation of $u_+$ given there shows that $u_+ \leq u_2$, while the fact that $\text{Ric} > 0$ means that $u_+$ is constant equal 1. □

4. Symmetry of solutions

4.1. Stabilisers and weak convergence. The Lie group $SO(4)$ can be endowed with a bi-invariant metric, which induces a distance function $d$. Write $\mathcal{G}$ for the set of closed subgroups of $SO(4)$. When endowed with the topology induced by the Hausdorff distance, the couple $(\mathcal{G}, d_H)$ forms a compact metric space. Moreover, we have the following useful result. (We state this for an arbitrary compact Lie group $G$, although for our purposes $G = SO(4)$ or $SO(n + 2)$ would be sufficient. Moreover given a closed subgroup $H \subset G$, we let $(H)_r = \{P \in G \mid \text{dist}(P, H) < r\}$ be its open tubular neighbourhood of size $r > 0$.)

Lemma 12 ([MZ42]). Let $G$ be a compact Lie group, and $H$ be a closed subgroup of $G$. There is $r > 0$ so that every subgroup $H' \in \mathcal{G}$ with $H' \subset (H)_r$ is conjugate to a subgroup of $H$.

Let $\epsilon_j \to 0$ be a sequence of positive scalars and $(u_j \mid j \in \mathbb{N})$ be a sequence of solutions of (1), with respectively $u_j \in \mathcal{G}_{\epsilon_j}$. We assume that they additionally have uniformly bounded Allen–Cahn energy and index $u_{\epsilon_j} = 5$ for all $j$, whence $V(\epsilon_j, u_j) \to |T|$ as $j \to \infty$. We abbreviate $G_j = \text{Stab} u_j$ and $G_T = \text{Stab} T$.

Moreover, we may assume throughout that $j$ is large enough that (5) holds, ensuring that nullity $u_j \leq 4$. We use Lemma 12 to show that eventually the stabilisers $G_j$ and $G_T$ are conjugate subgroups of $SO(4)$.
First we point out that $SO(4)$ defines a natural action on the space of two-varifolds $\mathbf{V}_2(\mathbb{S}^3)$. A rotation $P \in SO(4)$ maps $V \in \mathbf{V}_2(\mathbb{S}^3)$ to its pushforward $P_\# V$. (Here we implicitly identify $P \in SO(4)$ with the isometry it induces on $\mathbb{S}^3$, as we have been doing.) This action is continuous in the varifold topology. To see this, let $(P,V) \in SO(4) \times \mathbf{V}_2(\mathbb{S}^3)$ be arbitrary, and consider two sequences $P_k \to P$ and $V_k \to V$, the latter being in the varifold topology. Given $\varphi \in C(Gr_2(\mathbb{S}^3))$, one has $(P_k \# V_k - P_\# V)\varphi = P_k \# (V_k - V)\varphi + (P_k \# V - P_\# V)\varphi$, both of which tend to zero as $k \to \infty$.

We define the stabiliser and orbit of $V$ in the usual way. For an embedded surface $\Sigma \subset \mathbb{S}^3$ and all $P \in SO(4)$, it holds that $P_\# |\Sigma| = |P(\Sigma)|$ and we need not distinguish between the stabiliser and orbit of $\Sigma$ as an embedded surface or as a varifold.

Let $u \in \mathcal{Z}$, and the varifold $V(\epsilon, u)$ be defined as in (2). Via a simple change of variable we find

$$P_\# V(\epsilon, u) = V(\epsilon, u \circ P^{-1}) \text{ for all } P \in SO(4).$$

To see this, let $X \in \mathbb{S}^3$ be arbitrary and $Y = P(X)$. Assume without loss of generality that $|\nabla u|(X) = |\nabla (u \circ P^{-1})|(Y) > 0$, ensuring that the level sets $\{u = u(X)\}$ and $\{u \circ P^{-1} = (u \circ P^{-1})(Y)\}$ are regular near $X$ and $Y$ respectively. The map $P$ induces on $Gr_2(\mathbb{S}^3)$ sends $(X, TX\{u = u(X)\})$ to $(Y, TY\{u \circ P^{-1} = (u \circ P^{-1})(Y)\})$. Given any $\varphi \in C(Gr_2(\mathbb{S}^3))$, we may thus compute $[P_\# V(\epsilon, u)](\varphi) = V(\epsilon, u)(\varphi \circ P)$ to be equal $V(\epsilon, u \circ P^{-1})(\varphi)$:

$$\frac{1}{2\sigma} \int_{\mathbb{S}^3} \epsilon |\nabla u|^2(X)(\varphi \circ P)(X, TX\{u = u(X)\}) \, d\mathcal{H}^3(X)$$

$$= \frac{1}{2\sigma} \int_{\mathbb{S}^3} \epsilon |\nabla (u \circ P^{-1})|^2(Y)\varphi(Y, TY\{u \circ P^{-1} = (u \circ P^{-1})(Y)\}) \, d\mathcal{H}^3(Y).$$

Although the compared actions are on the right and left respectively, it follows from (9) that $\text{Stab} u \subset \text{Stab} V(\epsilon, u)$, and specialised to our sequence $G_j \subset \text{Stab} V(\epsilon_j, u_j)$ for all $j$.

**Lemma 13.** Let $\epsilon_j \to 0$ and $u_j \in \mathcal{Z}_{\epsilon_j}$ be as described above, with $V(\epsilon_j, u_j) \to |T|$ as $j \to \infty$. For large $j$, $G_j$ is conjugate to a subgroup of $G_T = SO(2) \times SO(2)$.

**Proof.** Consider a sequence $(P_j \mid j \in \mathbb{N})$ with $P_j \in \text{Stab} u_j$. Upon extracting a subsequence we may assume that it converges to some $P \in SO(4)$. As $V(\epsilon_j, u_j) \to |T|$ we get on the one hand $P_j \# V(\epsilon_j, u_j) \to P_\# |T|$. On the other hand $P_j \# V(\epsilon_j, u_j) = V(\epsilon_j, u_j \circ P_j^{-1}) = V(\epsilon_j, u_j)$ by construction, so $P_\# |T| = |T|$ and $P \in G_T = \text{Stab} T$. Via another extraction argument, we find that given any $\tau > 0$ there is $J(\tau) \in \mathbb{N}$ so that $G_j \subset (G_T)_\tau$ when $j \geq J(\tau)$. By Lemma 12, $G_j$ is conjugate to a subgroup of $G_T$. $\Box$

Therefore eventually $\dim \text{Stab} u_j \leq 2$ and $\nu(u_j) \geq 4$. Combining this with (5), we find that for large $j$

$$\nu(u_j) = \text{nullity } u_j = 4.$$

Thus $G_j$ is conjugate to $G_T$; further given any $\tau > 0$ there is $J(\tau) \in \mathbb{N}$ so that for $j \geq J(\tau)$, there is $P_j \in SO(4)$ with $d(P_j, I) < \tau$ and $P_j^{-1}G_j P_j = G_T$. 

4.2. Proof of Theorem 1. Applying the results of [BO86] in the present setting, one obtains the following; see also [CGGM20] for an alternative construction of the symmetric critical point.

**Lemma 14.** There is $\epsilon_0 = \epsilon_0(T) > 0$ so that for all $0 < \epsilon < \epsilon_0$ there is a unique function $u_{T, \epsilon} = u \in \mathcal{Z}_\epsilon$ with $\{u = 0\} = T$, up to change of sign. Moreover $u_{T, \epsilon}$ is invariant under $G_T$.

**Proof.** It is convenient to write $\mathbb{S}^3 = \{(X, Y) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid |X|^2 + |Y|^2 = 1\}$. The complement of $T$ has two connected components $U_\pm$, and we write $U_+ = \{(X, Y) \in \mathbb{S}^3 \mid |X| < |Y|\}$ and $U_- = \{(X, Y) \in \mathbb{S}^3 \mid |X| > |Y|\}$. These two regions are isometric via the involution $(X, Y) \in \mathbb{S}^3 \mapsto (Y, X)$.

Let $U = U_\pm$ and $\lambda_1(\Delta; U)$ be the first eigenvalue of the Laplacian on $U$ with Dirichlet eigenvalues. By [BO86], provided
\[ 0 < \epsilon < \left\{ W''(0)/\lambda_1(\Delta; U) \right\}^{1/2}, \]

there is a unique solution $u_\pm$ on $U_\pm$ respectively of the system
\[
\begin{cases}
\epsilon \Delta u - \epsilon^{-1}W'(u) = 0 & \text{in } U, \\
u > 0 & \text{in } U, \\
u = 0 & \text{on } \partial U.
\end{cases}
\]

As this system is invariant under the action of $G_T$, the solutions $u_\pm$ must further be $G_T$-invariant. Moreover they are symmetric under the involution above, that is $u_+(X, Y) = u_-(Y, X)$ for all $(X, Y) \in \mathbb{S}^3$ with $|X| < |Y|$. By elliptic regularity $u_\pm$ are respectively smooth up to and including the boundary of $U_\pm$. We define the function $u$ by setting
\[ u(X, Y) = \begin{cases} u_+(X, Y) & \text{if } |X| \leq |Y|, \\
u_-(X, Y) & \text{if } |X| > |Y|. \end{cases} \]

By the symmetry of $u_\pm$ under the involution, this function is smooth and solves (1) on $\mathbb{S}^3$. Thus there exists at least one solution of (1) which vanishes precisely on $T$. Conversely, if an arbitrary solution $v \in \mathcal{Z}_\epsilon$ had $Z(v_e) = T$ then its restrictions to $U_\pm$ would both have a sign, and thus $v$ is forced to coincide with $u$ up to change of sign. \hfill \Box

We conclude with a proof of the main result: Theorem 1.

**Proof.** Let $u_\epsilon \in \mathcal{Z}_\epsilon$ be a solution with $E_\epsilon(u_\epsilon) \leq C$ and index $u_\epsilon = 5$. Let a small $\tau > 0$ be given, and $\epsilon > 0$ be small enough in terms of $\tau$, $D$ that (10) holds and there is $P_\epsilon \in SO(4)$ with $d(P_\epsilon, I) < \tau$ and $P_\epsilon^{-1}G_\epsilon P_\epsilon = G_T$, where $G_\epsilon = \text{Stab } u_\epsilon$. As the nodal set $Z(u_\epsilon) = \{u_\epsilon = 0\}$ converges to $T$ with respect to Hausdorff distance we may moreover assume that $Z(u_\epsilon) \subset (T)_\tau$. Let $v_\epsilon = u_\epsilon \circ P_\epsilon \in \mathcal{Z}_\epsilon$; the energy, index and nullities are unaffected by this operation and $v_\epsilon$ is stabilised by $G_T$. Therefore its nodal set is union of orbits of $G_T$. Appealing to [CM20] for example, it must be connected and so be of the form $Z(v_\epsilon) = \{\text{dist}(-, T) = \delta\}$ for some $\delta$ which tends to zero as $\tau, \epsilon \to 0$.

Let $u_{T, \epsilon}$ be the symmetric solution around the Clifford torus from Lemma 14. The Frankel-type property of Corollary 9 forces $\delta = 0$ and $Z(v_\epsilon) = Z(u_{T, \epsilon})$. (In fact here both nodal sets are connected and separating.) By Lemma 14 up to a change of sign $v_\epsilon = u_{T, \epsilon}$, which concludes the proof. \hfill \Box
Appendix A. Modifications in higher dimensions

The only result that needs to be slightly altered in higher dimensions, when working with the hypersurface $T_{p,q} \subset S^{n+1}$, is Lemma 14. Even then, in case $p = q$ the statement and its proof remain valid with no changes. However the less symmetric case where $p \neq q$ calls for a more complicated construction; Caú–Gaspar [CG19] prove the following. (Their result is valid more broadly; we give a modified statement specialised to the present context.)

Lemma A.1 ([CG19, Thm. 1.1]). Let $p, q > 0$ and $n + p + q \geq 3$. Given any $\delta > 0$ there is $\epsilon_0 > 0$ so that for all $0 < \epsilon < \epsilon_0$ there is a solution $u_{p,q,\epsilon} \in \mathcal{Z}_\epsilon$ on $S^{n+1}$ with $Z(u_{p,q,\epsilon}) \subset (T_{p,q})$ and $(1 - \delta)\mathcal{H}^n(T_{p,q}) \leq E_{\epsilon}(u_{p,q,\epsilon}) \leq (1 + \delta)\mathcal{H}^n(T_{p,q})$.

Applying [BO86] here gives that any other solution $v_\epsilon$ of (1) on $S^{n+1}$ with $Z(v_\epsilon) = Z(u_{p,q,\epsilon})$ must coincide with $u_{p,q,\epsilon}$ up to a possible change of sign. There are two ways to justify this here. The first is via the results of [CM20], which show that for $\epsilon > 0$ small enough the nodal set of the $u_{p,q,\epsilon}$ converge smoothly to the limit surface $T_{p,q}$. The uniqueness is then a direct consequence of [BO86], applied in the two regions making up $S^{n+1} \backslash Z(u_{p,q,\epsilon})$. For an argument that does not rely open the convergence of the nodal sets, one may combine [HL89] with [BO86] to obtain the following general lemma.

Lemma A.2. Let $(M, g)$ be closed. Let $\epsilon > 0$ and $u_1^\epsilon, u_2^\epsilon$ be two solutions of (1) on $M$. If $Z(u_1^\epsilon) = Z(u_2^\epsilon)$, then $u_1^\epsilon = \pm u_2^\epsilon$.

Proof. Write $Z = Z(u_1^\epsilon) = Z(u_2^\epsilon)$. The cases where $Z = \emptyset$ or $Z = M$ are trivial, and we leave them aside. Divide the complement of $Z$ into its connected components, say $M \backslash Z = \bigcup_{j=0}^N U_j$, where $N \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. The two functions $u_1^\epsilon, u_2^\epsilon$ have a sign in the interior of each $U_j$, and vanish on $\partial U_j$. By [BO86] they are equal up to a change of sign; however this needs to be chosen consistently across all regions of $M \backslash Z$. Call two regions $U_j, U_k$ adjacent if $\mathcal{H}^{n-1}(\partial U_j \cap \partial U_k) \neq 0$. By [HL89] the nodal set may be decomposed like $Z = R \cup S$, where $R$ is the set of regular points, that is those $X \in Z$ so that for some $\rho > 0$, $B_\rho(X) \cap Z$ is a $C^1$ $(n - 1)$-dimensional submanifold, and $S$ is a countably $(n - 2)$-rectifiable set. If $U_j, U_k$ are adjacent then $\partial U_j \cap \partial U_k$ contains a regular point $X$ say, and there is $\rho > 0$ so that $\partial U_j \cap \partial U_k \cap B_\rho(X) = R \cap B_\rho(X)$. The regularity of the boundary near $X$ allows the application of [BO86, Lem. 1] to deduce that $\frac{\partial u_1}{\partial \nu}, \frac{\partial u_2}{\partial \nu} \neq 0$ on $\partial U_j \cap \partial U_k \cap B_\rho(X)$. It follows that the respective signs of $u_1, u_2$ on $U_j$ determines their signs on $U_k$, and vice-versa. Now let $U_j, U_k$ be two connected components, which are not necessarily adjacent. There is a path $\gamma : [0, 1] \to M$ with endpoints $\gamma(0) \in U_j$ and $\gamma(1) \in U_k$. Using a perturbation analogous to that used in [SW16, Lem. A.1] one may arrange for $\gamma([0, 1]) \cap S = \emptyset$. The curve $\gamma$ runs through finitely many regions of $M \backslash Z$. List them as $U_{j_1} = U_j, U_{j_2}, \ldots, U_{j_d} = U_k$, which are pairwise adjacent in this order. Therefore the sign of $u_1, u_2$ on $U_j$ determines their sign on $U_k$ and vice-versa; this concludes the proof. \qed

For the remaining steps in the proof of Theorem 4 one may thus follow the arguments we used for Theorem 1.
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