A micropolar isotropic plasticity formulation for non-associated flow rule and softening featuring multiple classical yield criteria
Part I - Theory
Andrea Panteghini, Rocco Lagioia
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1 Abstract

The Cosserat continuum is used in this paper to regularize the ill-posed governing equations of the Cauchy/Maxwell continuum. Most available constitutive models adopt yield and plastic potential surfaces with a circular deviatoric section. This is a too crude an approximation which hinders the application of the Cosserat continuum into practice, particularly in the geotechnical domain. An elasto-plastic constitutive model for the linear formulation of the Cosserat continuum is here presented, which features non-associated flow and hardening/softening behaviour, whilst linear hyper-elasticity is adopted to reproduce the recoverable response. For the formulation of the yield and plastic potential functions, a definition of the equivalent von Mises stress is used which is based on Hencky’s interpretation of the von Mises criterion and also on the theory of representations. The dependency on the Lode’s angle of both the yield and plastic potential functions is introduced through the adoption of a recently proposed Generalized classical criterion, which rigorously defines most of the classical yield and failure criteria.

2 Introduction

Elasto-plastic constitutive models of the classical Cauchy/Maxwell continuum are widely used to analyse structural and geotechnical problems. The need of accounting for important features of the material behaviour usually requires, particularly for soils, the introduction of non-associativeness and/or softening in the constitutive formulation, which unfortunately results in serious consequences.
It has long been known that numerical analyses of boundary value problems exhibit non-uniqueness of the solution which results in mesh sensitivity when the material is characterized by softening behaviour. This manifests itself as mesh dependent results, which do not converge to the correct solution when the domain discretisation is refined. Moreover the width of the band with localized deformations depends on the size of the elements (see e.g. De Borst [4], De Borst and Sluys [5], Stefanou et al. [20]). The adoption of a non-associated flow rule too results in mesh dependency and also in structural softening even if the material exhibits perfect plastic response (e.g. Sabet and De Borst [18]).

However one of the most severe practical consequences for the analyst is that numerical analyses of boundary value problems are very much prone to crushing and indeed there are a number of problems for which the analyses do not perform even a single step when non-associativity and/or softening is introduced. It is now well established that mesh sensitivity is the manifestation in numerical methods of the loss of well-posedness of the governing equations which for static problems lose ellipticity when localisations occur (e.g. Sabet and de Borst [18]).

The introduction of a characteristic length is a valid approach for regularizing the problem, and this is indeed what a Cosserat continuum does via the introduction of additional rotational degrees of freedom.

Most available constitutive models for the Cosserat continuum adopt yield and plastic potential surfaces defined by classical yield/failure criteria with a circular deviatoric section, such as the von Mises and the Drucker-Prager (e.g. Altenbach and Eremeyev [1], Russo et al. [17], Sabet and De Borst [18]). For many applications, such as most of those in the geotechnical domain, that is too much of a crude assumption, which, depending on how parameters are calibrated, results in excessively unconservative or overconservative predictions. This considerably hinders the application of the Cosserat continuum into practical engineering problems.

An elasto-plastic constitutive model for the Cosserat continuum is here presented featuring linear hyper-elastic behaviour, non-associated flow and hardening/softening response. The yield and the plastic potential surfaces are those of the main classical yield and failure criteria for metals and soils which are selected by choosing the appropriate set of parameters. This is achieved by adopting the mathematical formulation of those criteria recently presented by Lagioia and Panteghini [8]. The dependence of the yield and plastic potential functions on the Lode’s angle is hence introduced in the model. Moreover given the generality of the proposed model, other classical constitutive relationships such as the Modified Cam-Clay (e.g. Roscoe and Burland [16], Panteghini and Lagioia [14]) can be easily introduced.
3 Notation

In this manuscript we will follow the classical continuum mechanics sign convention, whereby tensile stresses and strains are positive.

Tensors of the first, second, third and fourth orders will be referred to in the compact representation using bold letters of any case, but for the purpose of clarity the order will be distinguished, similarly to Russo et al. [17], using a combination of bars and tildes below the letter. The equivalence between compact and indicial notation is then

\[
\bar{a} \rightarrow a_i \\
\tilde{a} \rightarrow a_{ij} \\
\bar{\tilde{a}} \rightarrow a_{ijh} \\
\tilde{\tilde{a}} \rightarrow a_{ijhk}
\]

The tensor product between two first and two second order tensors is defined as

\[
\bar{c} = \bar{a} \otimes \bar{b} \rightarrow c_{ij} = a_i b_j \\
\tilde{c} = \tilde{a} \otimes \tilde{b} \rightarrow c_{ij} = a_{ij} b_{hk}
\]

The single contraction (or scalar product) between a second order and first order tensors and between two second order tensors is

\[
\bar{c} = \bar{a} \cdot \bar{b} \rightarrow c_i = a_{ij} b_j \\
\tilde{c} = \tilde{a} \cdot \tilde{b} \rightarrow c_{ij} = a_{ik} b_{kj}
\]

whilst two double contractions (or double scalar products) will be used

\[
c = \bar{a} \cdot \tilde{b} = a_{ij} b_{ij} \\
c = \tilde{a} \cdot \bar{b} = a_{ij} b_{ji}
\]

The gradient of a tensor is

\[
\bar{c} = \bar{a} \otimes \nabla \rightarrow c_{ij} = \frac{\partial a_i}{\partial x_j} = a_{ij}
\]

The Levi-Civita pseudo or permutation tensor is indicated as

\[
\varepsilon \rightarrow e_{ijk} = \begin{cases} 
+1 & \text{if } (i,j,k) \in \{(1,2,3), (2,3,1), (3,1,2)\} \\
-1 & \text{if } (i,j,k) \in \{(1,3,2), (3,2,1), (2,1,3)\} \\
0 & \text{if } i = j \quad \text{or} \quad j = k \quad \text{or} \quad i = k
\end{cases}
\]
and the following second and forth order unit tensors will be used

\[ I \rightarrow \delta_{ij} \]
\[ I \rightarrow I_{ijhk} = \delta_{ih}\delta_{jk} \]
\[ I \rightarrow I_{ijhk} = \delta_{ik}\delta_{jh} \]
\[ I \rightarrow \bar{I}_{ijhk} = \delta_{ij}\delta_{hk} \]

\[ \bar{I}_{\text{sym}} = \frac{1}{2} (I + \bar{I}) \rightarrow I_{ijhk}^{\text{sym}} = \frac{1}{2} (\delta_{ih}\delta_{jk} + \delta_{ik}\delta_{jh}) \]
\[ \bar{I}_{\text{skw}} = \frac{1}{2} (I - \bar{I}) \rightarrow I_{ijhk}^{\text{skw}} = \frac{1}{2} (\delta_{ih}\delta_{jk} - \delta_{ik}\delta_{jh}) \]

where the first and second are the so called identity tensors such that
\[ I \cdot a = a \]
\[ I : a = a \]

Whilst the remaining are such that
\[ \bar{I} : a = a^T \]
\[ \bar{I} : a = \text{tr}(a)I \]
\[ I_{\text{sym}}^{\text{sym}}: a = \text{sym} a \]
\[ I_{\text{skw}}^{\text{skw}}: a = \text{skew} a \]

Finally the unit deviatoric forth order tensor is defined as
\[ \bar{T}^{d} \rightarrow I_{ijhk}^{d} = \delta_{ih}\delta_{jk} - \frac{1}{3}\delta_{hk}\delta_{ij} \]

and is such that
\[ \bar{T}^{d}: \varepsilon = \varepsilon \rightarrow I_{ijhk}^{d}\varepsilon_{hk} = \left(\delta_{ih}\delta_{jk} - \frac{1}{3}\delta_{hk}\delta_{ij}\right)\varepsilon_{hk} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{rr}\delta_{ij} = \varepsilon_{ij} \]

For the derivations presented in what follows we will use the direct and inverse relationships between the vector which defines a rigid body rotation \( \bar{\omega} = \omega_i \hat{e}_i \)

and its associated unit displacement gradient tensor \( \omega \)

\[ \omega = -\varepsilon \cdot \omega \quad \rightarrow \quad \omega_{ij} = -e_{ijk}\omega_k \]

and
\[ \omega = -\frac{1}{2} \varepsilon : \omega \quad \rightarrow \quad \omega_k = -\frac{1}{2}e_{ijk}\omega_{ij} \]

where \( \hat{e}_i \) with \( i = 1, 2, 3 \) is an orthonormal basis.
4 Cosserat continuum

The so-called linear formulation of the general Cosserat continuum (Cosserat & Cosserat [2]) is adopted in this study. In such a simplified version of the higher-order continuum developed by the two Cosserat brothers the micro-volume embedded in each point of the underlying material matrix is assumed to be rigid (Mindlin [11]) and solidly connected to that point so that it moves together with it, but it can also rigidly rotate relatively to the material matrix. Each point identified by the position vector

\[ \bar{x} = x_i \hat{e}_i \]

is thus characterized by six degrees of freedom (DoF) in the 3D space

\[ \{u_i, \theta_i\} \]

where \( u_i \) and \( \theta_i \) are the components of the displacement and rotational vector fields defined over the continuum

\[ u = u_i \hat{e}_i \]
\[ \theta = \theta_i \hat{e}_i \]

Materials with a particulate nature, such as granular media and soils, when deformed not only experience displacements but also rotation of the individual particles. The linear Cosserat continuum is hence inherently better suited than the classical Cauchy medium to reproduce their behaviour.

4.1 Kinematics

The unit variation of the displacement vector in the neighbourhood of a material point is

\[ \frac{du}{dS} = u \otimes \nabla \cdot \frac{dx}{dS} = u \otimes \nabla \cdot n \quad \text{or} \quad \frac{du_i}{dS} = u_{i,j} \frac{dx_j}{dS} = u_{i,j}n_j \]

where \( n \) is the unit directional vector pointing from \( x \) to \( x + dx \) and \( dS \) is the distance between the two points

\[ dS = \sqrt{dx \cdot dx} \quad \text{or} \quad dS = \sqrt{dx_i dx_i} \]

The gradient of the displacement vector field

\[ u \otimes \nabla \quad \text{or} \quad u_{i,j} = \frac{\partial u_i}{\partial x_j} \]

can be split into its symmetric

\[ \varepsilon = \frac{1}{2} (u \otimes \nabla + \nabla \otimes u) \quad \text{or} \quad \varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \]
and skew-symmetric parts
\[\omega^m = \frac{1}{2} (u \otimes \nabla - \nabla \otimes u) \quad \text{or} \quad \omega^m_{ij} = \frac{1}{2} (u_{i,j} - u_{j,i})\]
which represent the strain and the rigid body rotation tensors of the underlying material matrix, respectively. It should be noted that in the classical Cauchy continuum the rotation associated to the latter tensor occurs freely at constant energy, hence it is not associated to a stress variation. However, in the case of the Cosserat medium such a rotation does not necessarily occur freely, without providing any work, as it interacts with the rotation \(\bar{\theta}\) prescribed to the \textit{micro-volumes} which are themselves embedded in the matrix at each point \(\mathbf{x}\). If the rotation of the embedded \textit{micro-volumes} left free to follow that of the underlying matrix, then no work needs to be provided and no stress variation occurs. However if the matrix rotation \(\omega^m\) is larger than the rotations \(\bar{\theta}\) prescribed to the \textit{micro-volumes}, then work needs to be provided to rotate the matrix and additional stress will develop.

This implies that the Cosserat strain tensor must be defined as
\[\gamma = u \otimes \nabla - \theta \quad \text{or} \quad \gamma_{ij} = u_{i,j} - \theta_{ij}\]
or equivalently
\[\gamma = \varepsilon + \omega \quad \text{or} \quad \gamma_{ij} = \varepsilon_{ij} + \omega_{ij}\] (8)
where \(\omega\) is usually referred to as the relative rotation and defined as
\[\omega = \omega^m - \theta \quad \text{or} \quad \omega_{ij} = \omega^m_{ij} - \theta_{ij}\]
or
\[\omega = -\varepsilon \cdot (\omega^m - \theta) \quad \text{or} \quad \omega_{ij} = -e_{ijk}(\omega^m_k - \theta_k)\]
and represents the relative rotation of the material matrix with respect to that prescribed to the \textit{micro-volumes} embedded at each location \(\mathbf{x}\).

Fig. 1 shows a plane strain exemplification of the role played by the relative rotation into the Cosserat strain definition. The number of DoF reduces to three
\[u = u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2 \quad \theta = \theta_3 \mathbf{e}_3\]
In Fig. 1a the strain tensor of the material matrix \(\varepsilon\) is nil, whilst its rotation and the rotation of the embedded \textit{micro-volumes} have the same value, so that the relative rotation \(\omega\) vanishes. No stress state acts on such a Cosserat infinitesimal element. In Fig. 1b the strain tensor of the material matrix is still nil and its rotation is \(\omega^m > 0\), whilst the rotation of the embedded
Figure 1: Role played by the relative rotation tensor into the Cosserat strain and stress definitions in a plane strain exemplification of the Cosserat continuum.

micro-volume is set to zero, so that a positive relative rotation $\omega$ applies. Whilst the stress state associated to the deformation $\varepsilon$ of the material matrix is nil, the anti-clockwise and positive rotation of the matrix $\omega^m$ requires an anti-clockwise tangential stress state on the material element. It should be noted that whilst the stress tensor associated to the symmetric part of the strain tensor $\gamma$ is also symmetric, that of the skew-symmetric part of $\gamma$ is also skew-symmetric.

In addition to the strain tensor a curvature or wryness tensor characterizes the Cosserat continuum

$$\chi = \theta \otimes \nabla \quad \text{or} \quad \chi_{ij} = \theta_{i,j}$$

which accounts for the different prescribed rotations of infinitesimally close micro-volumes. It should be noted that the principal diagonal $\chi_{i,i}$ and the off-diagonal $\chi_{i,j}$ terms of this tensor, i.e. its spherical and deviatoric components, represent torsional and bending curvatures, respectively.

### 4.2 Statics

Let $\sigma$ and $\mu$ indicate the stress and couple-stress tensors conjugated in the meaning of Hill [6] to the strain $\gamma$ and wryness $\chi$ tensors introduced in Eqs. (8) and (9). The rate of the internal work of deformation over the volume $\Omega$ is

$$\dot{W}_{int} = \int_{\Omega} (\sigma : \dot{\gamma} + \mu : \dot{\chi}) \, d\Omega$$

However this can be reformulated by exploiting the decomposition of a general second order tensor into its symmetric and skew-symmetric parts, which
for the Cosserat stress and strain tensors gives

\[
\sigma = \text{sym} \sigma + \text{skew} \sigma = \sigma_{\text{sym}} + \sigma_{\text{skw}}
\]
\[
\gamma = \text{sym} \gamma + \text{skew} \gamma = \varepsilon + \omega
\] (11)

Since the double contraction of a symmetric and a skew-symmetric tensor is nil, the term \(\sigma : \dot{\gamma}\) can be rewritten as

\[
\sigma : \dot{\gamma} = (\text{sym}\sigma + \text{skew}\sigma) : (\text{sym}\dot{\gamma} + \text{skew}\dot{\gamma}) = \sigma_{\text{sym}} : \dot{\varepsilon} + \sigma_{\text{skw}} : \dot{\omega}
\] (12)

and Eq. (10) then becomes

\[
\dot{W}_{\text{int}} = \int_{\Omega} (\sigma_{\text{sym}} : \dot{\varepsilon} + \sigma_{\text{skw}} : \dot{\omega} + \mu : \dot{\chi}) \, d\Omega
\] (13)

which indicates that the symmetric part of the stress tensor performs work only for the classical Cauchy strain tensor whilst the skew-symmetric part of the same tensor performs work for the relative-rotation tensor only.

By applying the Green’s theorem to \(\sigma_{\text{sym}} : \dot{\varepsilon}\), \(\sigma_{\text{skw}} : \dot{\omega}\) and to \(\mu : \dot{\chi}\) one obtains

\[
\int_{\Omega} \sigma_{\text{sym}} : \dot{\varepsilon} \, d\Omega = \int_{\partial\Omega} \dot{u} \cdot \sigma_{\text{sym}} \cdot n \, dS - \int_{\Omega} \dot{u} \cdot \sigma_{\text{sym}} \cdot \nabla \, d\Omega
\]
\[
\int_{\Omega} \sigma_{\text{skw}} : \dot{\omega} \, d\Omega = \int_{\partial\Omega} \dot{u} \cdot \sigma_{\text{skw}} \cdot n \, dS - \int_{\Omega} \dot{u} \cdot \sigma_{\text{skw}} \cdot \nabla \, d\Omega - \int_{\Omega} 2\sigma_{\text{skw}} \cdot \dot{\theta} \, d\Omega
\]
\[
\int_{\Omega} \mu : \dot{\chi} \, d\Omega = \int_{\partial\Omega} \dot{\theta} \cdot \mu \cdot n \, dS - \int_{\Omega} \dot{\theta} \cdot \mu \cdot \nabla \, d\Omega
\]

where \(n\) is the outer normal to the continuum, \(\partial\Omega\) is the surface enclosing the continuum and Eq. (10) has been used to transform the skew stress and micro-volumes rotation tensors into their vector counterparts.

Finally the substitution of these results into Eq. (13) yields

\[
\dot{W}_{\text{int}} = \int_{\partial\Omega} \dot{u} \cdot \sigma \cdot n \, dS - \int_{\Omega} \dot{u} \cdot \sigma \cdot \nabla \, d\Omega - \int_{\Omega} 2\sigma_{\text{skw}} \cdot \dot{\theta} \, d\Omega + \int_{\partial\Omega} \dot{\theta} \cdot \mu \cdot n \, dS - \int_{\Omega} \dot{\theta} \cdot \mu \cdot \nabla \, d\Omega
\] (14)

The rate of work of deformation of the external forces over the volume \(\Omega\) and bounding surface \(\partial\Omega\) of the continuum is

\[
\dot{W}_{\text{ext}} = \int_{\Omega} (f \cdot \dot{u} + c \cdot \dot{\theta}) \, d\Omega + \int_{\partial\Omega} (\dot{t} \cdot \dot{u} + \dot{m} \cdot \dot{\theta}) \, dS
\] (15)

where \(f\) and \(c\) are the body forces and couples per unit volume and \(\dot{t}\) and \(\dot{m}\) are the surface tractions and couples per unit surface.
The balance equation of the internal and external work of deformation in static applications can then be written as

$$W_{ext} - W_{int} = \int_{\partial \Omega} \dot{\mathbf{u}} \cdot (\dot{\mathbf{t}} - \dot{\mathbf{\sigma}} \cdot \mathbf{n}) \, dS + \int_{\partial \Omega} \dot{\theta} \cdot (\dot{\mathbf{m}} - \dot{\mathbf{\mu}} \cdot \mathbf{n}) \, dS + \int_{\Omega} \dot{\mathbf{u}} \cdot (\mathbf{\sigma} \cdot \nabla + \mathbf{f}) \, d\Omega + \int_{\Omega} \dot{\theta} \cdot (2\mathbf{\sigma}_{skw} + \mathbf{c} + \dot{\mathbf{\mu}} \cdot \nabla) = 0$$

(16)

from which the strong form of the balance equations is retrieved

$$\mathbf{\sigma} \cdot \nabla + \mathbf{f} = 0$$

$$\mathbf{\mu} \cdot \nabla + 2\mathbf{\sigma}_{skw} + \mathbf{c} = 0$$

(17)

and on the surface of the continuum

$$\mathbf{\sigma} \cdot \mathbf{n} = \dot{\mathbf{t}}$$

$$\mathbf{\mu} \cdot \mathbf{n} = \dot{\mathbf{m}}$$

(18)

The second of Eqs (17) shows that due to the couple-stress tensor the stress tensor is not symmetric and that its skew component is

$$\mathbf{\sigma}_{skw} = \mathbf{c} \cdot \frac{1}{2} (\dot{\mathbf{\mu}} \cdot \nabla + \mathbf{c})$$

(19)

The sign convention for both stress and couple-stress tensors is shown in Fig 2. It should be noted that the second subscript indicates the normal to the plane.

5 Elastic behaviour

It is assumed that the reversible behaviour of the Cosserat continuum is described by means of a linear hyper-elastic model. The work of deformation
is then stored in each material point in the form of an elastic potential energy \( \Psi \) defined as

\[
\Psi = \int d\Psi = \int (\sigma : \dot{\gamma}^e + \mu : \dot{\chi}^e)
\]  

(20)

where \( \gamma^e \) and \( \chi^e \) indicate the elastic components of the respective tensors. However, the decomposition of all stress and strain tensors into their symmetric and skew-symmetric parts allows a simplification based on the observation that the double contraction of a symmetric and skew-symmetric tensor is nil. Eq. (20) can hence be rewritten as

\[
\Psi = \int d\Psi = \int (\sigma_{\text{sym}} : \dot{\varepsilon}^e + \sigma_{\text{skw}} : \dot{\omega}^e + \mu_{\text{sym}} : \dot{\chi}_{\text{sym}}^e + \mu_{\text{skw}} : \dot{\chi}_{\text{skw}}^e)
\]  

(21)

and since \( d\Psi \) is a perfect differential

\[
\begin{align*}
\sigma_{\text{sym}} &= \frac{\partial \Psi}{\partial \varepsilon^e} \\
\sigma_{\text{skw}} &= \frac{\partial \Psi}{\partial \omega^e} \\
\mu_{\text{sym}} &= \frac{\partial \Psi}{\partial \chi_{\text{sym}}^e} \\
\mu_{\text{skw}} &= \frac{\partial \Psi}{\partial \chi_{\text{skw}}^e}
\end{align*}
\]  

(22)

In the case of linear elastic behaviour the energy potential is a quadratic function given by

\[
\Psi (\varepsilon^e, \omega^e, \chi_{\text{sym}}^e, \chi_{\text{skw}}^e) =
\begin{align*}
\frac{1}{2} \varepsilon^e : \frac{\partial^2 \Psi}{\partial \varepsilon^e \otimes \partial \varepsilon^e} : \varepsilon^e + \frac{1}{2} \omega^e : \frac{\partial^2 \Psi}{\partial \omega^e \otimes \partial \omega^e} : \omega^e + \\
\frac{1}{2} \chi_{\text{sym}}^e : \frac{\partial^2 \Psi}{\partial \chi_{\text{sym}}^e \otimes \partial \chi_{\text{sym}}^e} : \chi_{\text{sym}}^e + \frac{1}{2} \chi_{\text{skw}}^e : \frac{\partial^2 \Psi}{\partial \chi_{\text{skw}}^e \otimes \partial \chi_{\text{skw}}^e} : \chi_{\text{skw}}^e
\end{align*}
\]  

(23)

and the constant forth order isotropic symmetric and skew symmetric stiffness tensors are

\[
\begin{align*}
D_s &= \frac{\partial^2 \Psi}{\partial \varepsilon^e \otimes \partial \varepsilon^e} = \left( K - \frac{2}{3} G \right) I_{\text{sym}} + 2 G I_{\text{skw}} \\
D_w &= \frac{\partial^2 \Psi}{\partial \omega^e \otimes \partial \omega^e} = 2 G c I_{\text{skw}} \\
D_{\chi_{\text{sym}}} &= \frac{\partial^2 \Psi}{\partial \chi_{\text{sym}}^e \otimes \partial \chi_{\text{sym}}^e} = T I_{\text{sym}} + 2 B c I_{\text{sym}} \\
D_{\chi_{\text{skw}}} &= \frac{\partial^2 \Psi}{\partial \chi_{\text{skw}}^e \otimes \partial \chi_{\text{skw}}^e} = 2 B c I_{\text{skw}}
\end{align*}
\]  

(24)

In these equations \( K \) is the bulk modulus, hence \( K - \frac{2}{3} G \) and \( G \) are the usual Lamé’s constants (clearly associated to the symmetric of the tensors only) whilst \( G_c \) is an additional constitutive parameter also with dimension of a force per unit area. \( T, B \) and \( B_c \) are additional parameters which account...
for the torsional and bending stiffness with dimension of a moment per unit length.

The linear elastic stress-strain relationships can then be formulated as

\[
\sigma = \left( K - \frac{2}{3} G \right) \text{tr} \varepsilon^e I + 2G \varepsilon^e + 2G_c \omega^e
\]
\[
\mu = T \text{tr} \chi^e I + 2B \chi^e \text{sym} + 2B_c \chi^e \text{skw}
\]

The latter can be further rewritten considering the spherical and deviatoric components of the stress and couple-stress tensors and of the strain and wryness tensors

\[
\begin{align*}
\dot{s}_{\text{sym}} &= \sigma_{\text{sym}} - \frac{1}{3} \text{tr} \sigma I \\
\dot{m}_{\text{sym}} &= \mu_{\text{sym}} - \frac{1}{3} \text{tr} \mu I \\
\dot{e}_{\text{sym}} &= \varepsilon - \frac{1}{3} \text{tr} \varepsilon I \\
\dot{g}_{\text{sym}} &= \chi - \frac{1}{3} \text{tr} \chi I
\end{align*}
\]

whilst the skew-symmetric part of all four tensors is clearly deviatoric only. The elastic constitutive equations can then be reformulated as

\[
\begin{align*}
\sigma &= K \text{tr} \varepsilon^e I + 2G \varepsilon^e + 2G_c \omega^e \\
\mu &= K_c \text{tr} \chi^e I + 2B \dot{g}_{\text{sym}} + 2B_c \dot{g}_{\text{skw}}
\end{align*}
\]

where to simplify the notation

\[
K_c = (T + \frac{2}{3} B)
\]

6 Equivalent von Mises stress

For the formulation of the yield function a definition of the equivalent von Mises stress is required for the Cosserat continuum. The von Mises [22] criterion assumes that yield in a Cauchy continuum occurs when the second invariant \( J_2 = \frac{1}{2} s_{\text{sym}} : s_{\text{sym}} \) of the deviatoric stress tensor attains a critical value. According to the physical interpretation of that criterion given by Hencky [5], yield in a specific material occurs when the energy of the linear elastic distortion reaches a critical value. This can be obtained by subtracting from Eq. (21) the energy associated to the volumetric deformation

\[
\psi_D = \frac{1}{2} \left[ \dot{s}_{\text{sym}} : \varepsilon^e + \dot{g}_{\text{skw}} : \omega^e + \dot{m}_{\text{sym}} : \dot{g}_{\text{sym}} + \dot{m}_{\text{skw}} : \dot{g}_{\text{skw}} + \frac{1}{3} \text{tr} \mu \text{tr} \chi^e \right]
\]

where the decomposition given in Eqs [26] of the stress, strain, couple stress and curvature tensors into their respective deviatoric and spherical parts has
been used. It should be noted that whilst the spherical component of the
strain tensor is indeed not associated to distortion, the principal diagonal
terms of the curvature tensors account for the rate of variation along one
axis of the component along the same axis of the rotation vector of the
micro-volumes, i.e. they define a torsional curvature (e.g. Russo et al. \[17\]).
Since those terms are clearly related to a distortion, the energy associated
to the spherical component of the curvature and couple-stress tensors needs
to be included in the definition of the energy of distortion.

After the substitution of the linear elastic constitutive relationship of
Eqs. (27) the energy becomes

$$\Psi_D = \frac{1}{4G} \left[ 3 \text{sym} : 3 \text{sym} + \frac{G}{G_c} 3 \text{skw} : 3 \text{skw} + \frac{G}{B} m_{\text{sym}} : m_{\text{sym}} + \frac{G}{B_c} m_{\text{skw}} : m_{\text{skw}} + \frac{2G \text{tr}^2 \mu}{K_c} \right]$$

(30)

If the energy associated to the linear elastic distortion is the quantity re-
sponsible for yielding, then by imposing the equivalence with that of a bar
of a Cauchy material subjected to a uni-axial stress state $q$

$$\Psi_D = \frac{1}{2} \frac{q^2}{3G}$$

(31)

the equivalent von Mises stress is obtained

$$q = \left\{ \frac{3}{2} \left[ 3 \text{sym} : 3 \text{sym} + \frac{G}{G_c} 3 \text{skw} : 3 \text{skw} + \frac{G}{B} m_{\text{sym}} : m_{\text{sym}} + \frac{G}{B_c} m_{\text{skw}} : m_{\text{skw}} \right] + \frac{2G \text{tr}^2 \mu}{K_c} \right\}^{\frac{1}{2}}$$

(32)

For dimensional correctness it is clear that three quantities

$$\ell_1 = \sqrt{\frac{B}{G}}$$

$$\ell_2 = \sqrt{\frac{B_c}{G}}$$

$$\ell_3 = \sqrt{\frac{K_c}{2G}}$$

(33)

have the dimension of a length and can be considered as three characteristic
lengths of the Cosserat material. It should be noted that in plane strain
problems the third characteristic length does not play any role, since the
spherical component of the curvature tensor vanishes.
This result is coherent with the theory of representations which states that an isotropic scalar valued function of a finite number of symmetric $A_i$ and skew-symmetric $W_i$ second order tensors has a complete and irreducible representation $f$ in terms of a number of invariants expressed as the trace of the scalar product of those tensors. A list of those invariants for different situations can be found in e.g. Smith and Smith, [19], Zheng [23]. In the two simplest cases where the arguments of the function are either a single symmetric tensor or a symmetric and a skew-symmetric tensors, the complete and irreducible representations are

$$f = f(\text{tr}A, \text{tr}A^2, \text{tr}A^3)$$ (34)

and

$$f = f(\text{tr}A, \text{tr}A^2, \text{tr}A^3, \text{tr}W^2, \text{tr}A \cdot W^2, \text{tr}A^2 \cdot W^2, \text{tr}A^2 \cdot W^2 \cdot A \cdot W)$$ (35)

respectively. The former expression is often used in Cauchy continua to formulate the isotropic scalar-valued functions which define yield and failure criteria (e.g. von Mises and Drucker-Prager) and strain energy potentials (e.g. Houlsby [7], Lagioia and Panteghini [9]), often dropping one or two of the three invariants, resulting in incomplete formulations. For example the third invariant is dropped in the Drucker-Prager criterion, thus loosing the dependency on the relative magnitude of the intermediate principal stress $\sigma_{II}$ with respect to $\sigma_I$ and $\sigma_{III}$, where $\sigma_I \geq \sigma_{II} \geq \sigma_{III}$. Similarly if in Eq. (35) some of the invariants are dropped, retaining only $\text{tr}A^2, \text{tr}W^2$, then a general expression for the isotropic scalar-valued function of the equivalent von Mises stress of Eq. (32) is obtained (to be precise two skew-symmetric tensors are used in that case).

Finally it should be noted that since

$$a_3\text{tr}\sigma_{sym}^2 + a_4\text{tr}\sigma_{skw}^2$$

$$a_1\sigma : \sigma + a_2\sigma : \sigma^T$$ (36)

are equivalent, provided that

$$a_1 = \frac{a_3 - a_4}{2}$$

$$a_2 = \frac{a_3 + a_4}{2}$$ (37)

for plane strain conditions, Eq. (32) can also be formulated as

$$q = \left\{ \frac{3}{2} \left[ \frac{G_c - G}{2G_c} s : s + \frac{G_c + G}{2G_c} s^T : s + \frac{G B_c - B}{B} \frac{m \cdot m}{2B_c} \right] \right\}^{\frac{1}{2}}$$ (38)
which coincides with that proposed by Muehlhaus and Vardulakis [12]. It is interesting to observe that those Authors derived their definition of $q$ using only micromechanical and averaging considerations, and yet it is equivalent to that obtained using either the theory of representations or Hencky’s energy approach.

7 Constitutive model

We will confine our work to isotropic materials. Constitutive models, even with such a frequent and simplifying assumption, are usually formulated and numerically integrated in terms of components of the stress and strain tensors. However it has been recently shown by Panteghini and Lagioia [13] [15] that if stress and strain invariants are adopted, an extremely efficient integration algorithm can be written which requires the solution of one equation in one unknown only, rather than a system of seven by seven equations and unknowns as in the standard integration. This not only results in considerable reduction of the machine run-time but also in a more robust scheme.

The approach described by Panteghini and Lagioia is here further extended to formulate and integrate an elasto-plastic constitutive model for a Cosserat continuum. The recoverable behaviour is assumed to be hyperelastic and linear as described previously, whilst the irrecoverable one is defined within the theory of plasticity with isotropic hardening/softening.

The adopted framework is that typically used for elastic-perfect plastic materials, where the yield and the plastic potential surfaces are described by classical yield/failure criteria. However an hardening rule is also introduced which governs the evolution of the intercept of the yield surface with the equivalent von Mises stress axis.

The isotropic scalar-valued function which defines the yield and plastic potential surfaces needs to be defined entirely in terms of stress invariants

$$f = f(p, q, \theta_s)$$

where $p$, $q$ and $\theta_s$ for a Cauchy continuum are the mean pressure, equivalent von Mises stress (also known as deviatoric stress) and the Lode’s angle of the stress tensor.

This expression can also be employed for a Cosserat continuum, if the definition of $p$ is kept unchanged, i.e. $p = \frac{1}{3} \text{tr} \tilde{\sigma}$, whilst the deviatoric stress is substituted with that given by Eq. (32) and the Lode’s angle is that of the symmetric part only of the deviatoric stress tensor $\tilde{s}_{sym}$

$$\theta_s = \frac{1}{3} \text{arcsin} \left( \frac{27 \det \tilde{s}_{sym}}{2 q_s^2} \right)$$
where

\[ q_s = \sqrt{\frac{3}{2} s_{\text{sym}} : s_{\text{sym}}} \]  \hspace{1cm} (39)

Following Panteghini and Lagioia \[15\] considerable advantages are achieved if a slightly different structure is adopted

\[ f = f(p, q, \Gamma(\theta_s)) \]  \hspace{1cm} (40)

in which \( \Gamma(\theta_s) \) is the reciprocal of the function which defines the shape of the yield and plastic potential surfaces in the deviatoric plane, and the Lode’s angle \( \theta_s \) is restricted in the interval \( \theta_s \in [-\frac{\pi}{6}, \frac{\pi}{6}] \). It should be noted that in triaxial compression conditions, associated to \( \theta_s = -\frac{\pi}{6} \), \( \Gamma \) evaluates to unity, whilst in triaxial extension conditions, characterized by \( \theta_s = \frac{\pi}{6} \), the value of \( \Gamma \) is larger than unity and results \( \Gamma (\frac{\pi}{6}) = \frac{q_{\text{y}}(-\frac{\pi}{6})}{q_{\text{y}}(\frac{\pi}{6})} \), where the subscript \( y \) indicates yield in the case of metals and failure in the case of soils. The role of \( \Gamma(\theta_s) \) is that of modulating the value of the equivalent von Mises stress \( q \) so that the meridional sections of the yield and plastic potential surfaces are accordingly reduced or expanded.

The yield and the plastic potential surfaces are defined by a function recently proposed by Lagioia and Panteghini \[8\]. That function was mathematically demonstrated to define classical yield and failure criteria for metals and soils, i.e. von Mises, Tresca, Drucker-Prager, Matsuoka-Nakai, Lade-Duncan and Mohr-Coulomb. For the last failure criterion parameters were also retrieved which result in smooth versions circumscribed to and approximately inscribed in the original one. The set of published parameters is likely to be not conclusive and other classical criteria can possibly be included. As an example, recently Lester and Sloan \[10\] retrieved parameters for a rigorously inscribed rounded Mohr-Coulomb and named the Lagioia and Panteghini formulation generalized classical yield function (GC).

Even if Eq. \[40\] can be used to define yield surfaces with any meridional section, for for those criteria it is linear and Eq. \[40\] particularizes to the Generalised Classical yield function, which with the sign convention adopted in this paper (tensile stresses are positive) becomes

\[ f(p, q, \theta_s) = q \Gamma(\theta_s) + M_c p - \sigma_0(\lambda) \]  \hspace{1cm} (41)

where, as shown in Fig. \[4\], \( M_c \) is the slope of the criterion in the meridional section associated to a Lode’s angle \( \theta_s = -\frac{\pi}{6} \) and is defined as

\[ M_c = \frac{6 \sin \phi}{3 - \sin \phi} \]  \hspace{1cm} (42)

where \( \phi \) is the angle of shearing resistance.

The function \( \Gamma \) is

\[ \Gamma(\theta_s) = \alpha_f \cos \left[ \arccos \left( -\beta_f \sin 3\theta_s \right) \frac{3}{3} - \gamma_f \frac{\pi}{6} \right] \]  \hspace{1cm} (43)
(a) Mohr-Coulomb, Matsuoka-Nakai (b) Tresca, von Mises and Drucker-Prager

Figure 3: Exemplification of the shapes provided by the $\Gamma(\theta_s)$ function of the GC failure criterion (from Lagioia and Panteghini [8]). For the Tresca and Mohr-Coulomb criteria rounded versions are also shown.

Figure 4: Significance of the cohesion term in the failure criterion.

and $\alpha_f$, $\beta_f$ and $\gamma_f$ are the three parameters which rule the shape of the yield function in the deviatoric plane and are provided in Lagioia and Panteghini [8] without the $f$ subscript (Table 1). An additional set of parameters will be also used for the plastic potential surface, distinguished from the previous one using a $g$ subscript, so that the possibility is given to adopt surfaces with different deviatoric sections. It should also be noted that the minus sign before the $\beta_f$ parameter is introduced in Eq. 43 to account for the sign convention adopted in this paper. An instance of the shapes given by the $\Gamma(\theta_s)$ function are shown in Fig. 3.

The term $\sigma_0$, also shown in Fig. 4 accounts for the so called cohesion in the classical criterion and is expressed as

$$\sigma_0(\lambda) = c'(\lambda) \frac{6 \cos \phi}{3 - \sin \phi}$$ (44)
Figure 5: Adopted softening of the cohesion term of the failure criterion.

| Model                  | $\alpha$ | $\beta$ | $\gamma$ |
|------------------------|----------|----------|-----------|
| von Mises              | 1        | 0        | 1         |
| Drucker-Prager         | 1        | 0        | 1         |
| Tresca                 | $\text{sec} \left( \frac{\pi}{6} \right)$ | 1 | 1 |
| Mohr-Coulomb           | $\text{sec} \left[ (\bar{\gamma} + 1) \frac{\pi}{6} \right]$ | 1 | $(1 - \bar{\gamma})$ |
| Matsuoka-Nakai         | $\frac{2}{3} \sqrt{A_1 M_c}$ | $\frac{A_2}{A_1^3/2}$ | 0         |
| Lade-Duncan            | $\frac{2}{3} \sqrt{A_1 M_c}$ | $\frac{A_2}{A_1^3/2}$ | 0         |

Table 1: Parameters for the shape function of the generalized criterion of [8]. In addition $\bar{\gamma} = \frac{2}{3} \arctan \left( \frac{\sin \phi}{\sqrt{3}} \right)$, $\phi$ is the angle of shearing resistance. For Matsuoka-Nakai $A_1 = \frac{K_{MN}-3}{K_{MN}-9}$, $A_2 = \frac{K_{MN}}{K_{MN}-9}$ and $K_{MN} = \frac{9 - \sin^2 \phi}{1 - \sin^2 \phi}$, whilst for Lade-Duncan $A_1 = \frac{K_{LD}}{K_{LD}-2\pi}$, $A_2 = A_1$ and and $K_{LD} = \frac{(3 - \sin \phi)^3}{(1 - \sin \phi)(1 - \sin \phi)^2}$ (modified from [8]).

It defines the intercept of the yield/failure criterion in the triaxial compression section (i.e. $\theta_s = -\frac{\pi}{6}$) with the q axis. It should be noted that in Eq. (44) $\lambda$ is a generic scalar function of plastic strains, so that isotropic strain hardening/softening can be accounted for. In this paper a simple hardening rule will be adopted, defined as

$$c' (\lambda) = c'_f + (c'_i - c'_f) \exp(-a\lambda)$$  \hspace{1cm} (45)$$

where $c'_i$ and $c'_f$ are the initial and the final cohesions and $a$ is a parameter
which controls the rate of the reduction (Fig. 5). A similar softening rule is often adopted in numerical analyses of geotechnical problems both in drained and undrained conditions. As an example Summersgill et al. [21] use a linear reduction of the cohesion term. In this paper a very non-linear hardening law has been used in order to test the capability of the Cosserat continuum to regularize the boundary value problem even in extreme conditions. It should be noted, however that adopting a more suitable hardening/softening rule can be done without modifying the general structure of the model.

In what follows the plastic potential function will be indicated as

$$g(p, q, \theta_s) = q \hat{\Gamma}(\theta_s) + \hat{M}_c p$$  \hspace{1cm} (46)

where the $\hat{\cdot}$ symbol is introduced to differentiate from the yield function.

Plastic strain tensors are obtained through the plastic potential as

$$\dot{\gamma}^p = \frac{\partial g(p, q, \theta_s)}{\partial \sigma} \dot{\lambda},$$

$$\dot{\chi}^p = \frac{\partial g(p, q, \theta_s)}{\partial \mu} \dot{\lambda}$$  \hspace{1cm} (47)

where $\dot{\lambda}$ is the plastic multiplier and the standard assumption is made that the strain and curvature increments can be split into their elastic and plastic components

$$\dot{\gamma} = \dot{\gamma}^e + \dot{\gamma}^p,$$

$$\dot{\chi} = \dot{\chi}^e + \dot{\chi}^p$$  \hspace{1cm} (48)

8 Conclusions

An elasto-plastic constitutive model for the linear formulation of the Cosserat continuum has been presented. The model features non-associated flow and hardening/softening behaviour, whilst linear hyper-elasticity is adopted to reproduce the recoverable response.

For the formulation of the yield and plastic potential functions, the definition of the equivalent von Mises stress used in the Cauchy continuum has been extended to the Cosserat material exploiting Hencky’s [5] interpretation of the von Mises criterion. The resulting expression for the equivalent von Mises stress coincides with that of Muehlhaus and Vardoulakis [12] which was obtained on the basis of considerations of micro-mechanics and averaging. The same expression can also be obtained using the theory of representations.

The key feature of the model is the capability of providing different shapes of the yield and plastic potential surfaces in the deviatoric plane. Whilst most models in the literature adopt surfaces with a circular shape in the deviatoric plane, hence dropping the dependency on the Lode’s angle,
a general shape function has been used in the proposed model for the deviatoric section. The dependency of the those functions on the Lode’s angle has hence been introduced in the constitutive model, which makes it significant for practical applications. Both surfaces are defined using the function, proposed by Lagioia and Panteghini [8], which is an exact definition of most classical yield and failure criteria.

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