Continuity of family of Calderón projections

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Abstract We consider a continuous family of linear elliptic differential operators of arbitrary order over a smooth compact manifold with boundary. Assuming constant dimension of the spaces of inner solutions, we prove that the orthogonalized Calderón projections of the underlying family of elliptic operators form a continuous family of projections. Hence, its images (the Cauchy data spaces) form a continuous family of closed subspaces in the relevant Sobolev spaces. We use only elementary tools and classical results: basic manipulations of operator graphs and other closed subspaces in Banach spaces; elliptic regularity; Green’s formula and trace theorems for Sobolev spaces; well-posed boundary conditions; duality of spaces and operators in Hilbert space; and the interpolation theorem for operators in Sobolev spaces.

Keywords Calderón projection · Cauchy data spaces · Elliptic differential operators · Green’s formula · Interpolation theorem · Manifolds with boundary · Parameter dependence · Trace theorem · Variational properties

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1 Introduction

This paper provides a new approach to the investigation of Cauchy data spaces (made of the normal traces at the boundary up to the order $d - 1$ of the kernel of elliptic differential operators of order $d \geq 1$, that can be obtained as images of certain pseudo-differential projections over the boundary, called Calderón projections) under continuous or smooth variation of the underlying operators over a fixed manifold with boundary. The concept of the Calderón projection originated from Calderón’s observation in [12].

Previous approaches to the variational problem were based either on purely functional-analytic, symplectic and topological arguments or on geometric and holomorphic analysis. For the first type of approach we refer to [4] that dealt with symmetric operators admitting self-adjoint Fredholm extensions and a certain unique continuation property (UCP). The variation was restricted to compact perturbations. By those assumptions, the authors achieved the continuous variation of the Cauchy data spaces in symplectic quotient Hilbert spaces, namely as Lagrangian subspaces.

The second type of approach is based on investigating spectral projections and exploiting the pseudo-differential calculus, via canonical and explicit constructions of Poisson operators and the Calderón projection. See, e.g., the classical [25, 26] for Dirac type operators, based on the invertible double via gluing of [7], or our [5, 6, 3] for arbitrary elliptic differential operators of first order with UCP, based on the ideas of general invertible doubles via a system of boundary value problems in [20].

Our present approach is a hybrid, changing repeatedly between the calculus of closed subspaces of the graph-theoretical approach and the geometric analysis of the Calderón projections of the pseudo-differential approach. In that way we obtain the wanted generalization to linear elliptic differential operators of order $d \geq 1$ with weakened UCP requirements and, as a bonus, a much shorter path to the quoted results.

1.1 Structure of the paper

This paper consists of three sections. In this Section 1 we explain the structure of the paper and state our main result.

In Section 2 we fix the notations. The main topics are Sobolev spaces and domains of elliptic differential operators on manifolds with boundary; Green’s forms; Cauchy data spaces; the homogenized Cauchy trace operator; the classical properties of the Calderón projection; and Neubauer’s classical $\cap$ and $+$ arithmetic of pairs of families of closed subspaces in Banach space [24].

The proof of Theorem 1.2 is in Section 3. In Section 3.1 assuming $s \geq \frac{d}{2}$, we obtain first the continuous variation of the solution spaces in the Sobolev space of order $s + \frac{d}{2}$. By the continuity and surjectivity of the adjusted trace operator, that yields a continuous variation of the Cauchy data spaces in the Sobolev
space of order \( s \) over the boundary, and, furthermore, the continuous variation of the family of \( L^2 \)-orthogonalized Calderón projections in the operator norm of these Sobolev spaces. This part of our results has been announced in \([9, \text{Proof of Proposition 4.5.2, first part}]\). In Section 3.2, we use the results of Section 3.1 to prove our theorem for \( s < \frac{d}{2} \) by duality and interpolation property of spaces and operators in Sobolev scales. In the following Appendix, we show that the assumption about the constant dimension of the spaces of inner solutions in Theorem 1.2 can be weakened a little by finer analysis above.

1.2 Our main result

**Notation 1.1** Let \( B \) be a topological space and \( \mathcal{M} \) a compact smooth Riemannian manifold with boundary \( \Sigma \). Let \((A_b)_{b \in B}\) be a family of linear elliptic differential operators of order \( d \geq 1 \), acting between sections of complex finite-dimensional Hermitian vector bundles \( E, F \) over \( \mathcal{M} \).

Let \( \rho^d \) denote the Cauchy trace operator, mapping sections of \( E \) over \( \mathcal{M} \) to \( d \)-tuples of jets over \( \Sigma \) in normal direction (these jets can be adjusted, i.e., homogenized to sections of the bundle \( E^{nd} := (E|\Sigma)^d \), for details see Proposition 2.3). Let

\[
Z_{+,0}(A_b) := \{ u \in H^d(\mathcal{M}; E) \mid A_b u = 0 \text{ and } \rho^d u = 0 \}
\]

denote the space of all inner solutions. It is the finite-dimensional kernel of the closed minimal realization associated with \( A_b \). Correspondingly, \( Z_{-,0}(A_b) := Z_{+,0}(A^*_b) \) denotes the kernel of the closed minimal realization associated with the formal adjoint \( A^*_b \).

For the interesting case of Cauchy data spaces and \( L^2 \)-orthogonalized (and so uniquely determined) Calderón projections (see Section 2.4), we shall prove

**Theorem 1.2 (Main result)** Assume that

(i) for \( s \geq \frac{d}{2} \), the two families of bounded extensions

\[
(A_{b,s+d/4} : H^{s+d/4}(\mathcal{M}; E) \rightarrow H^{s-\frac{d}{4}}(\mathcal{M}; F))_{b \in B}
\]

and

\[
(A^*_b,s+d/4) : H^{s+d/4}(\mathcal{M}; F) \rightarrow H^{s-\frac{d}{4}}(\mathcal{M}; E))_{b \in B}
\]

are continuous in the respective operator norms \( \| \cdot \|_{s+d/4,s-\frac{d}{4}} \), and that the family of adjusted Green’s forms (of Equation 2.7) \( (\tilde{J}_{b,s} : H^s(\Sigma; F^{nd}) \rightarrow H^*(\Sigma; E^{nd}))_{b \in B} \) is continuous in the operator norm \( \| \cdot \|_{s,s} \);

(ii) \( \dim Z_{+,0}(A_b) \) and \( \dim Z_{-,0}(A_b) \) do not depend on \( b \in B \).

Then for any \( s \in \mathbb{R} \), the family of \( L^2 \)-orthogonalized Calderón projections \( (C^s(A_b))_{b \in B} \) is continuous in the operator norm of the corresponding Sobolev space \( H^s(\Sigma; E^{nd}) \).
Remark 1.1 (a) Assumption (i) can be weakened by demanding continuous variation only for \( s \geq \frac{d}{2} \) and \( s + \frac{d}{2} \in \mathbb{N} \).

(b) Let \( M \) be covered by coordinate charts \( (U, \varphi) \in \mathcal{A} \) for an atlas \( \mathcal{A} \) with local trivializations \( E|_U, F|_U \). If for any \( (U, \varphi) \in \mathcal{A} \), all multiple-order partial derivatives of the coefficients of the operators \( A_b \) are continuous and uniformly bounded on \( U \times B \) (cf. [11] Section 1), then Assumption (i) follows.

(c) In the literature on families of elliptic operators over manifolds with boundary, strict weak inner unique continuation property is commonly assumed (that is, \( Z_{+,0}(A_b) = \{0\} \)). Relaxing that assumption to Assumption (ii), i.e., the constant dimensions of the spaces of inner solutions, was suggested in [20]. To prove the continuity of \( (\ker A_{b,s+\frac{d}{2}})_{b \in B} \) for \( s \geq \frac{d}{2} \) (see Proposition 3.1), we assume that the spaces \( Z_{-,0}(A_b) \) are of finite constant dimension. Actually, the two statements are equivalent by Lemma 2.3(a). Once we have obtained the continuity of \( (\ker A_{b,s+\frac{d}{2}})_{b \in B} \), the assumption that the spaces \( Z_{+,0}(A_b) \) are of finite constant dimension is equivalent to our conclusion that the family of the images of the corresponding Calde rón projections is continuous (see Proposition 3.2).

In some special example, Assumption (ii) can be weakened. Please see the Appendix.

(d) In [20] we define the Cauchy data space \( A_{-,\frac{d}{2}}(A_b) \subset H^{-\frac{d}{2}}(\Sigma; E^d) \) as the space of the homogenized Cauchy traces of the weak solutions \( u \) of \( A_b u = 0 \) and also in [2,10] the Cauchy data spaces \( A_{s}(A_b) \) for \( s \geq \frac{d}{2} \). According to Theorem 2.2 and Corollary 2.2 these spaces are precisely the images of the corresponding Calderón projections.

Clearly, the continuity of a family of projections of a Banach space in operator norm implies the continuity of their images in the gap topology, see also [22, Section I.4.6]. Hence one can read Theorem 1.2 as the claim of a continuous variation of the Cauchy data spaces depending on the parameter \( b \) for each of these Sobolev orders \( s \) — under the assumption of constant dimensions of the spaces of inner solutions.

(e) One of the most fundamental examples is the continuous variation of the Riemannian metric on a fixed smooth manifold, i.e., that in local coordinates all derivatives of the component functions of the metric vary continuously. Then the induced Laplace operators vary continuously in the sense of (b). The weak inner unique continuation property holds for Laplace operators. So both Assumptions (i) and (ii) are satisfied, and as a consequence the corresponding Cauchy data spaces vary continuously.

(f) For applications of our results we refer to the spectral flow formulae for operator families with varying maximal domains as in [9]; and, consequently, to the possibility of determining the precise number of negative eigenvalues in stability analysis of an essentially positive differential operator \( A \) (appearing in the descriptions of, e.g., reaction-diffusion, wave propagation and scattering systems) by calculating the spectral flow of \( (1 - b)A + bA_+ \) \( b \in [0,1] \), where \( A_+ \) is a suitably chosen strictly positive differential operator, or more advanced expressions, cf. [2][8][38] in the tradition of Bott’s Sturm type theorems [10].
2 Main tools and notations of elliptic operators on manifolds with boundary

Before proving the theorem, we fix the notations and recall the most basic concepts and tools. We begin with a single operator.

2.1 Our data

1. $\mathcal{M}$ is a smooth compact Riemannian manifold of dimension $n$ with boundary $\partial \mathcal{M} =: \Sigma$.
2. $E, F \to \mathcal{M}$ are Hermitian vector bundles of fiber dimension $m$ with metric connections $\nabla^E, \nabla^F$. As in Notation 1.1, we set $E'|\Sigma := E|\Sigma$ and $F'|\Sigma := F|\Sigma$.
3. $\mathcal{C}^\infty(\mathcal{M}; E)$ denotes the space of smooth sections of $E$; $\mathcal{C}^\infty_c(\mathcal{M}; E)$ denotes the space of smooth sections of $E$ with compact support in $\mathcal{M}$.
4. $A: \mathcal{C}^\infty(\mathcal{M}; E) \to \mathcal{C}^\infty(\mathcal{M}; F)$ is an elliptic differential operator of order $d$.
5. $A_0: \mathcal{C}^\infty_c(\mathcal{M}; E) \to \mathcal{C}^\infty_c(\mathcal{M}; F)$, where $A_0 = A|_{\mathcal{C}^\infty_c(\mathcal{M}; E)}$.
6. $A'_0: \mathcal{C}^\infty_c(\mathcal{M}; F) \to \mathcal{C}^\infty_c(\mathcal{M}; E)$, where $A'_0$ denotes the formal adjoint of $A$.
7. $\mathcal{A}_{\text{min}} := \overline{A_0}$, $\mathcal{A}'_{\text{min}} := \overline{A'_0}$, where we consider $A_0: D(A_0) \to L^2(\mathcal{M}; F)$ as an unbounded densely defined operator from $L^2(\mathcal{M}; E)$ to $L^2(\mathcal{M}; F)$, and denote its closure by $\overline{A_0}$, see Section 2.2 in particular Proposition 2.1 below. We write $\mathcal{A}_{\text{max}} := (A'_0)^*$, i.e.,

$$D(\mathcal{A}_{\text{max}}) = \{ u \in L^2(\mathcal{M}; E) \mid Au \in L^2(\mathcal{M}; F) \text{ in the distribution sense} \},$$

where $D(\cdot)$ denotes the domain of an operator.

Note that $\mathcal{A}_{\text{min}}, \mathcal{A}_{\text{max}}$ are the closed minimal and maximal extensions of $A_0$. For a section $u \in D(\mathcal{A}_{\text{max}})$, the intermediate derivatives $D^\alpha u$ (with $|\alpha| \leq d$) need not exist as sections on $\mathcal{M}$, even though $Au$ does in the distribution sense, see [19, Section 4.1, p. 61].

2.2 The Sobolev scale and special relations for elliptic operators

For real $s$, we recall the definition of the Sobolev scale $H^s(\mathcal{M}; E)$ for a complete smooth Riemannian manifold $\mathcal{M}$ without boundary. Then, for a compact manifold $\mathcal{M}$ with smooth boundary, the Sobolev scale is induced for non-negative $s$ by embedding and restriction.

We follow mostly CALDERÓN [13, Section 3.1], as reproduced and elaborated in FREY [17, Chapters 0 and 1], supplemented by LIONS and MAGENES [23, Sections 1.7 and 1.9] and TRÉVES [32, Section III.2]. We replace the regular subsets of $\mathbb{R}^n$ in the classical literature by a smooth compact manifold with boundary embedded in a complete manifold without boundary. So, without restricting the general validity of our results, we assume, as we may, that
our compact Riemannian manifold \((\mathcal{M}, g)\) with boundary is embedded in a (metrically) complete smooth Riemannian manifold \((\mathcal{M}, g)\) of the same dimension \(n\) without boundary,

- our bundles \(E, F\) are extended to smooth Hermitian vector bundles \(E, F\) over \(\mathcal{M}\),

- the elliptic differential operator \(A\) is defined on \(\mathcal{M} \cup N\) where \(N\) denotes a collar neighbourhood of \(\Sigma\) in \(\mathcal{M} \setminus \mathcal{M}^0\).

### Sobolev scale on complete manifolds without boundary.

First we recall the concept of the Sobolev scale for functions. The immediate generalization for sections of Hermitian bundles follows then.

On \(\mathcal{M}\) with Riemannian metric \(g\), let \(|d\mathbf{vol}|\) denote the volume density derived from the metric. Recall the Hodge–Laplace operator
\[
\Delta^\mathcal{M}_0 := d^t d : C_\infty^c(\mathcal{M}) \rightarrow C_\infty^c(\mathcal{M}),
\]
acting on functions, where \(d^t\) denotes the formal adjoint of the exterior differential \(d : C_\infty(\mathcal{M}) \rightarrow C_\infty(\mathcal{M}; \Lambda^1(\mathcal{M}))\). The operator \(-\Delta^\mathcal{M}_0\) is equal to the Laplace–Beltrami operator on the Riemannian manifold \((\mathcal{M}, g)\). Let \(L^2(\mathcal{M})\) denote the completion of \(C_\infty^c(\mathcal{M})\) with respect to the norm induced by the \(L^2\)-inner product
\[
\langle u, v \rangle_{L^2(\mathcal{M})} := \int_\mathcal{M} \bar{u} v |d\mathbf{vol}|,
\]
where \(\bar{v}\) denotes the complex conjugate of \(v\). Since \(\mathcal{M}\) is complete with respect to \(g\), \(\Delta^\mathcal{M}_0\) is essentially self-adjoint (see [16, Theorem 3] or [15, Section 3(A)]). So the closure of \(\Delta^\mathcal{M}_0\), \(\Delta^\mathcal{M}\), is a non-negative self-adjoint operator. It gives rise to the Sobolev spaces
\[
H^s(\mathcal{M}) := D((\Delta^\mathcal{M})^{s/2}), \quad s \geq 0,
\]
equipped with the graph norm. By [23 Theorem 1.1.2], we regain, for \(\mathcal{M} = \mathbb{R}^n\) and \(s \in \mathbb{N} \cup \{0\}\) the usual Hilbert space
\[
H^s(\mathcal{M}) = \{ u \in L^2(\mathcal{M}) \mid D^\alpha u \in L^2(\mathcal{M}) \text{ for } |\alpha| \leq s \},
\]
where the partial differentiation \(D^\alpha\) with multi-index \(\alpha\) is applied in the distribution sense and the scalar product and norm are defined by
\[
\langle u, v \rangle_s := \sum_{|\alpha| \leq s} \langle D^\alpha u, D^\alpha v \rangle_{L^2(\mathcal{M})} \quad \text{and} \quad \|u\|_s := \sqrt{\langle u, u \rangle_s}.
\]

For \(s > 0\), we define the space \(H^{-s}(\mathcal{M})\) of distributions to be the so-called \(L^2\)-dual of \(H^s(\mathcal{M})\), i.e.,
\[
H^{-s}(\mathcal{M}) := \{ u \in \mathcal{D}'(\mathcal{M}) \mid \exists c \forall v \in H^s(\mathcal{M}) |u(v)| = |\langle v, u \rangle_{s,-s}| \leq c \|v\|_{H^s(\mathcal{M})} \},
\]
here \(\langle v, u \rangle_{s,-s} := u(\bar{v})\) with a distribution \(u \in \mathcal{D}'(\mathcal{M})\) acting on a test function \(v\). Hence for \(u \in L^2(\mathcal{M})\) we have \(\langle v, u \rangle_{s,-s} = \langle v, u \rangle_{L^2(\mathcal{M})}\), as nicely explained in [19 Section 8.2] and [18 Section 1.1].
The above constructions can be generalized for sections of any bundle \( E \rightarrow M \) carrying an Hermitian structure \( M \ni p \mapsto \langle \cdot, \cdot \rangle_{E_p} \) and an Hermitian connection. Let

\[
\nabla^E: C^\infty(M; E) \longrightarrow C^\infty(M; T^*M \otimes E) \quad \text{and} \\
\nabla^F: C^\infty(M; F) \longrightarrow C^\infty(M; T^*M \otimes F)
\]

be Hermitian connections, i.e., connections that are compatible with the Hermitian metrics on \( E \) and \( F \) respectively. To define Sobolev spaces of sections in vector bundles, one replaces the Laplacian \( d^t d \) in the previous definition by the Bochner–Laplacian \( (\nabla^E)^t \nabla^E \) and \( (\nabla^F)^t \nabla^F \).

**Sobolev scale on compact smooth manifolds with boundary.** For functions, the corresponding Sobolev space on the compact submanifold \( \mathcal{M} \) with boundary \( \Sigma \) is defined as the quotient

\[
H^s(\mathcal{M}) := H^s(M) / \{ u \in H^s(M) \mid u|_{\mathcal{M}} = 0 \}, \quad s \in \mathbb{R} \text{ and } s \geq 0.
\]

In other words, \( H^s(\mathcal{M}) \) coincides algebraically with the space of restrictions to \( \mathcal{M}^\circ \) of the elements of \( H^s(M) \). The norm of \( H^s(\mathcal{M}) \) is given by the quotient norm, that is,

\[
\| u \|_{H^s(\mathcal{M})} = \inf \{ U \in H^s(M) \mid u = U \text{ a.e. on } \mathcal{M} \} \quad \text{for all } U \in H^s(M) \text{ with } U = u \text{ a.e. on } \mathcal{M}^\circ.
\]

In our smooth case, the definition coincides with the interpolation \( H^s(\mathcal{M}) = [H^m(\mathcal{M}), H^0(\mathcal{M})]^\theta \), \( (1 - \theta)m = s \), \( m \) integer, \( 0 \leq \theta \leq 1 \). See [23, Theorems 1.9.1 and 1.9.2]. For \( s \geq 0 \), an important subspace is the function space

\[
H^s_0(\mathcal{M}; E) := C^\infty_c(\mathcal{M}^\circ; E|_{\mathcal{M}^\circ}) \| H^s(\mathcal{M}; E) \quad \text{for } s \in \mathbb{R}, \ s \geq 0.
\]  

(2.3)

**Sobolev scale on closed manifolds.** For any Hermitian vector bundle \( G \) over the closed manifold \( \Sigma \), we can define the Sobolev spaces \( H^s(\Sigma; G) \) for all \( s \in \mathbb{R} \) as in [19, Section 8.2] or [18, Section 1.3]. Note that \( C^\infty(\Sigma; G) \) is dense in \( H^s(\Sigma; G) \) for all \( s \in \mathbb{R} \). Then the \( L^2 \)-scalar product for smooth sections can be extended to a perfect pairing between \( H^s(\Sigma; G) \) and \( H^{-s}(\Sigma; G) \) for all \( s \in \mathbb{R} \). That is, from [13, Lemma 1.3.5(e)], the pairing \( (f, h)_{L^2(\Sigma; G)} \) extends continuously to a perfect pairing

\[
H^s(\Sigma; G) \times H^{-s}(\Sigma; G) \longrightarrow \mathbb{C},
\]

which we denoted by \( \langle \cdot, \cdot \rangle_{s, -s} \) in (2.2).
Remark 2.1 Let $H, K$ be Hilbert spaces. A bounded sesquilinear form $\Phi: H \times K \to \mathbb{C}$ is called a perfect pairing if it induces on each of $H, K$ an isomorphism to the dual of the other. More precisely, we obtain the induced conjugate linear map from $K$ to the space of bounded linear functionals on $H$ by

$$v \mapsto \Phi(\cdot, v) \quad \text{for} \quad v \in K,$$

the induced linear map from $H$ to the space of bounded conjugate linear functionals on $K$ by

$$u \mapsto \Phi(u, \cdot) \quad \text{for} \quad u \in H.$$

Both are isomorphisms; moreover, the functionals are bounded by

$$\|v\|_K = \sup_{0 \neq u \in H} \left| \frac{\Phi(u, v)}{\|u\|_H} \right| \quad \text{and} \quad \|u\|_H = \sup_{0 \neq v \in K} \left| \frac{\Phi(u, v)}{\|v\|_K} \right|.$$

Notation 2.1 Let $X, Y$ be normed spaces, we denote the normed algebra of bounded linear operators from $X$ to $Y$ by $\mathcal{B}(X, Y)$; for $X = Y$, $\mathcal{B}(X, X) = \mathcal{B}(X)$. We use shorthand $\|\cdot\|_s$ for the norm in $H^s(\cdot; \cdot)$, $s \in \mathbb{R}$; and $\|\cdot\|_{r,s}$ for the operator norm in $\mathcal{B}(H^r(\cdot; \cdot), H^s(\cdot; \cdot))$, $r, s \in \mathbb{R}$.

Special relations for elliptic operators. We fix the notation, in particular the sign conventions. Let $d_g(\cdot, \cdot)$ be the distance function; (locally) it is the arc length of the minimizing geodesic. In a collar neighbourhood of $\Sigma$ in $\mathcal{M}$, say $V$, the function

$$V \ni p \mapsto x_1(p) := d_g(p, \Sigma), \quad p \in \mathcal{M}$$

is smooth and defines the inward unit normal field $\nu := \text{grad} x_1$ and inward unit co-normal field $\nu^\flat := d x_1$.

Let $T^*\mathcal{M}$ denote the cotangent vector bundle of $\mathcal{M}$, $S(\mathcal{M})$ the unit sphere bundle in $T^*\mathcal{M}$ (relative to the Riemannian metric $g$), and $\pi: S(\mathcal{M}) \to \mathcal{M}$ the projection. Then associated with any linear differential operator $A$ of order $d$ there is a vector bundle homomorphism

$$\sigma_d(A): \pi^*E \to \pi^*F,$$

which is called the principal symbol of $A$. In terms of local coordinates, $\sigma_d(A)$ is obtained from $A$ by replacing $\partial/\partial x_j$ by $i\xi_j$ in the highest order terms of $A$ (here $\xi_j$ is the $j$th coordinate in the cotangent bundle). $A$ elliptic means that $\sigma_d(A)$ is an isomorphism.

For elliptic operators there is an important relation (the Gårding inequality) between the graph norm, originating from the basic $L^2$ Hilbert space, and the corresponding Sobolev norm. More precisely, we recall from [17, Proposition 1.1.1]

Proposition 2.1 Assume that $A$ is an elliptic operator of order $d$. Then

(a) The graph norm of $A$ restricted to $C^\infty_c(\mathcal{M}^\circ; E)$ is equivalent to the Sobolev norm $\|\cdot\|_{H^d(\mathcal{M}; E)}$.

(b) In particular, $\mathcal{D}(A_{\min}) = H^d_0(\mathcal{M}; E)$ and $\mathcal{D}(A_{\max}^t) = H^d_0(\mathcal{M}; F)$.

(c) $H^d(\mathcal{M}; E) \subset \mathcal{D}(A_{\max})$ is dense.
2.3 Green’s formula, traces of Sobolev spaces over the boundary, and weak traces for elliptic operators

Let \( j \in \mathbb{N} \cup \{0\} \). Let \( \gamma^j : C^\infty(\mathcal{M}; E) \to C^\infty(\Sigma; E') \) denote the trace map \( \gamma^j u := (\nabla E^j u)|_{\Sigma} \) yielding the \( j \)-th jet in normal direction. Set

\[
\rho^d := (\gamma^0, \ldots, \gamma^{d-1}) : C^\infty(\mathcal{M}; E) \to C^\infty(\Sigma; E'^{\underline{d}}).
\]

(2.4)

Analogously, \( \nabla F \) gives rise to trace maps \( \gamma^j : C^\infty(\mathcal{M}; F) \to C^\infty(\Sigma; F') \). The corresponding maps for \( F \) will also be denoted by \( \gamma^j \) and \( \rho^d \).

We recall Green’s Formula, e.g., from Seeley [28, Equation 7], Trèves [32, Equation III.5.41], Grubb [19, Proposition 11.3], or Frey [17, Proposition 1.1.2], with a description of the operator \( J \) in the error term:

**Proposition 2.2 (Green’s Formula for differential operators of order \( d \geq 1 \))** Let \( A : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F) \) be a linear differential operator of order \( d \). Then there exists a (uniquely determined) differential operator

\[
J : C^\infty(\Sigma; E'^{\underline{d}}) \to C^\infty(\Sigma; F'^{\underline{d}}),
\]

such that for all \( u \in C^\infty(\mathcal{M}; E), v \in C^\infty(\mathcal{M}; F) \) we have

\[
(Au, v)_{L^2(\mathcal{M}; E)} - (u, A^t v)_{L^2(\mathcal{M}; F)} = (J \rho^d u, \rho^d v)_{L^2(\Sigma; F'^{\underline{d}})}.
\]

(2.5)

\( J \) is a matrix of differential operators \( J_{kj} \) of order \( d - 1 - k - j \), \( 0 \leq k, j \leq d - 1 \), and \( J_{kj} = 0 \) if \( k + j > d - 1 \) (\( J \) is upper skew-triangular). Moreover, for \( j = d - 1 - k \) we have explicitly given homomorphisms

\[
J_{k,d-1-k} = i^d(-1)^{d-1-k} \sigma_d(A)(\nu^j).
\]

(2.6)

**Remark 2.2** (a) For \( d = 1, 2, 3 \), we visualize the structure of the matrix \( J \),

\[
\begin{pmatrix}
J_{00}^0, & J_{01}^1, & J_{10}^1, & J_{02}^2, & J_{11}^1, & 0 \\
0, & 0, & 0, & 0, & 0, & 0
\end{pmatrix}, \quad \text{etc.,}
\]

where the orders of the differential operators of the entries were marked by a superscript \([\text{order}]\).

(b) From the explicit form of the skew diagonal elements of \( J \) in (2.3), we get that \( J \) is invertible for any elliptic operator \( A \).

(c) If \( J \) is invertible, Green’s Formula (2.5) extends to \( (u, v) \in \mathcal{D}(A_{\text{max}}) \times H^d(\mathcal{M}; F) \), where the right-hand side is interpreted as the \( L^2 \)-dual pairing

\[
\oplus_{j=0}^{d-1} H^{-d+j+\frac{1}{2}}(\Sigma; E') \times \oplus_{j=0}^{d-1} H^{-d-j-\frac{1}{2}}(\Sigma; F') \to \mathbb{C}.
\]

With [171, Theorem 1.1.4], we obtain a slight reformulation, sharpening, and generalization of the classical Sobolev Trace Theorem (see also [19, Section 9.1] and [30, Lemma 16.1]):

**Proposition 2.3 (Sobolev Trace Theorem)**
We have continuous trace maps $\rho^d$ (obtained by continuous extension):

(a) $\rho^d: H^{d+s}(M; E) \longrightarrow \bigoplus_{j=0}^{d-1} H^{d+s-j-\frac{1}{2}}(\Sigma; E')$ for $s > \frac{1}{2}$,

(b) $\rho^d: \mathcal{D}(A_{\text{max}}) \longrightarrow \bigoplus_{j=0}^{d-1} H^{-j-\frac{1}{2}}(\Sigma; E')$.

Moreover, the map (a) is surjective and has a continuous right-inverse $\eta^d$.

(2) If $u \in \mathcal{D}(A_{\text{max}})$, then $u \in H^d(M; E)$ if and only if

$$\rho^d u \in H^{d-\frac{1}{2}}(\Sigma; E') \oplus \cdots \oplus H^\frac{1}{2}(\Sigma; E').$$

(3) For $\rho^d$ on $\mathcal{D}(A_{\text{max}})$ we have $\ker \rho^d = H^0_d(M; E)$.

**Remark 2.3**

(a) Following GRUBB [19 Section 9.1], we call the preceding scale of operators $\rho^d$ with domains in different Sobolev spaces by one name: the 

**Cauchy trace operator** associated with the order $d$.

(b) It is well known that the trace operators do not extend to the whole $L^2(M; E)$. For the special case of the half-space in $\mathbb{R}^n$, it is shown in [19 Remark 9.4] that the 0-trace map $\gamma^0$ makes sense on $H^s(\mathbb{R}^n_+)$ if and only if $s > \frac{n}{2}$. The Cauchy trace operator $\rho^d$ extends, however, to $\mathcal{D}(A_{\text{max}})$, though in a way that depends on the choice of the elliptic operator $A$ but with $\ker (\rho^d|_{\mathcal{D}(A_{\text{max}})})$ is obvious. There remains to be shown that the converse holds. Via local maps and claim 2, this reduces to the Euclidean case (see [18 Theorem 9.6]).

**Homogenization of Sobolev orders.**

Following CALDERÓN [14 Section 4.1, p. 76] and using the notation of [17 p. 26], we introduce a homogenized (adjusted) Cauchy trace operator $\tilde{\rho}^d$. We set $\Delta^E := (\nabla^E)^t \nabla^E$, where $\nabla^E$ denotes the restriction of $\nabla$ to $\Sigma$. Since $\Delta^E + 1$ is a positive symmetric elliptic differential operator of second order on the closed manifold $\Sigma$, it possesses a discrete spectral resolution (e.g., [15 Section 1.6]). Then $\Phi := (\Delta^E + 1)^{1/2}$ is a pseudo-differential operator of order 1 which induces an isomorphism of Hilbert spaces

$$\Phi_{(s)}: H^s(\Sigma; E') \longrightarrow H^{s-1}(\Sigma; E')$$

for all $s \in \mathbb{R}$

and, in fact, generates the Sobolev scale $H^s(\Sigma; E')$, see the Sobolev scale of an unbounded operator in [11 Section 2.4].

In order to achieve that all boundary data are of the same Sobolev order, we introduce the matrix

$$\Phi_d := \begin{pmatrix} \Phi_{d-1} & 0 & \cdots & 0 \\ 0 & \Phi_{d-2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Phi_0 \end{pmatrix}.$$
We set $\bar{\rho}^d := \Phi_d \circ \rho^d$, $\bar{\eta}^d := \eta^d \circ \Phi_d^{-1}$. So, we obtain a condensed and adjusted Trace Theorem as a corollary to Proposition 2.3.

**Corollary 2.1 (Homogenized trace map)** We have continuous trace maps $\bar{\rho}^d$ (obtained by continuous extension):

(a) $\bar{\rho}^d : H^{s+\frac{d}{2}}(\mathcal{M}; E) \rightarrow H^s(\Sigma; E^{\text{nd}})$ for $s > \frac{d}{2} - \frac{1}{2}$,

(b) $\bar{\rho}^d : D(A_{\text{max}}) \rightarrow H^{-\frac{d}{2}}(\Sigma; E^{\text{nd}})$.

Furthermore, the map (a) is surjective and has a continuous right-inverse, $\bar{\eta}^d$.

**Remark 2.4** (a) We have $D(A_{\text{min}}) = \ker(\bar{\rho}^d : D(A_{\text{max}}) \rightarrow H^{-\frac{d}{2}}(\Sigma; E^{\text{nd}}))$.

(b) Analogous constructions for the bundle $F$ lead to $\Delta F' : C^\infty(\Sigma; F') \rightarrow C^\infty(\Sigma; F')$. Whenever this causes no ambiguity we will denote the corresponding matrices, $\Phi F'$ and $\Phi_{d'}$, again by $\Phi$ and $\Phi_d$, respectively.

We can replace the boundary operator $J$ of Green’s Formula (Proposition 2.2) by its adjusted version

$$\tilde{J} := (\Phi F_d')^{-1} \circ J \circ (\Phi_{d'} F)^{-1}.$$  (2.7)

It follows that all components of $\tilde{J}$,

$$\tilde{J}_{ij} = \Phi^{\frac{i+j+2d-2}{2}} J_{ij} \Phi^{\frac{i+j-2d}{2}}$$

are pseudo-differential operators of order $i + j + (1 - d) + (d - 1) - i - j = 0$. So for any $s \in \mathbb{R}$,

$$\tilde{J} : H^s(\Sigma, E^{\text{nd}}) \rightarrow H^s(\Sigma, F^{\text{nd}}).$$

$\tilde{J}$ is upper skew-triangular, with invertible elements on the skew diagonal for elliptic operator $A$.

For later use we give the adjusted version of Green’s Formula. It is valid for arbitrary linear differential operators of order $d \geq 1$ with smooth coefficients acting between sections of smooth Hermitian vector bundles $E$ and $F$ over a smooth compact Riemannian manifold $\mathcal{M}$ with boundary $\Sigma$.

With the preceding notations of $\tilde{J}$ for the adjusted Green boundary operator of (2.7) and $\bar{\rho}^d$ for the adjusted Cauchy trace operators of Corollary 2.1, we have

$$(Au, v)_{L^2(\mathcal{M}; E)} - (u, A^t v)_{L^2(\mathcal{M}; E)} = (\tilde{J} \bar{\rho}^d u, \bar{\rho}^d v)_{L^2(\Sigma, F^{\text{nd}})}$$  (2.8)

for $s \geq \frac{d}{2}$ and all $u \in H^{s+\frac{d}{2}}(\mathcal{M}; E)$, $v \in H^{s+\frac{d}{2}}(\mathcal{M}; F')$. For elliptic differential operators, remains valid for $u \in D(A_{\text{max}}), v \in H^d(\mathcal{M}; F)$ in the following extended version

$$(Au, v)_{L^2(\mathcal{M}; F)} - (u, A^t v)_{L^2(\mathcal{M}; E)} = (\tilde{J} \bar{\rho}^d u, \bar{\rho}^d v)_{L^2(\Sigma, F^{\text{nd}})},$$

where $\bar{\rho}^d u \in H^{-\frac{d}{2}}(\Sigma; E^{\text{nd}})$ and $\bar{\rho}^d v \in H^\frac{d}{2}(\Sigma; F^{\text{nd}})$. 


2.4 Properties of the Calderón projection

We give our version of perhaps not widely known classical results concerning the Calderón projection. In his famous note [12] of 1963, A. Calderón introduced the concept of a pseudo-differential projection onto the Cauchy data of solutions of a system of homogeneous elliptic differential equations over smooth compact manifolds with boundary, later called the Calderón projection, see below Theorem 2.2 and Corollary 2.2. While the Cauchy trace operator was introduced for sections in Equation (2.4) and Corollary 2.1, we define the Cauchy data spaces for elliptic differential operators. We recall

**Definition 2.1 (Cauchy Data Spaces)** Let $A, d, \mathcal{M}, \Sigma, E, F$ be given as in Section 2.1. Based on the homogenized Cauchy trace operators in Corollary 2.1, we define the Cauchy data spaces as follows:

(a) \[
\Lambda_{-d/2}(A) := \{ h \in H^{-d/2}(\Sigma; E^{\text{ad}}) \mid \exists u \in \ker \max A \text{ with } \tilde{\rho}^d u = h \},
\]

i.e., the space of boundary values of weak solutions to $A$ (= sections belonging to the maximal domain of $A$ that vanish under the operation of $A$ in the distribution sense).

(b) \[
A_s(A) := \{ h \in H^s(\Sigma; E^{\text{ad}}) \mid \exists u \in H^{s+d/2}(\mathcal{M}; E), Au = 0 \text{ with } \tilde{\rho}^d u = h \}.
\]

**Remark 2.5** (a) For later use, we rewrite the Cauchy data spaces:

\[
A_{-d/2}(A) = \tilde{\rho}^d (\ker \max A), \quad \text{and}
\]

\[
A_s(A) = \tilde{\rho}^d (\ker A_{s+d/2}) = \tilde{\rho}^d (\ker \max A \cap H^{s+d/2}(\mathcal{M}; E)) \quad \text{for } s \geq d/2,
\]

where $A_{s+d/2} : H^{s+d/2}(\mathcal{M}; E) \to H^{s-d/2}(\mathcal{M}; F)$ is the extension of $A$. In fact, for $u \in H^{s+d/2}(\mathcal{M}; E)$, $v \in C_c(\mathcal{M}; F)$, we have

\[
(A_{s+d/2} u, v)_{L^2(\mathcal{M}; F)} = (u, A_v^t v)_{L^2(\mathcal{M}; E)},
\]

so $\max |_{H^{s+d/2}(\mathcal{M}; E)} = A_{s+d/2}$, thus (2.11) follows.

(b) For a subspace $V$ of $L^2(\Sigma; F^{\text{ad}})$, we denote by

$V_{L^2} := \{ g \in L^2(\Sigma; F^{\text{ad}}) \mid (f, g)_{L^2(\Sigma; F^{\text{ad}})} = 0 \text{ for all } f \in V \}$.

Let $\tilde{J}$ be the adjusted Green’s form of (2.7). Now we recall an important relationship of the Cauchy data spaces between $A$ and $A'$:

\[
A_{\tilde{J}}(A') = \left( \tilde{J} A_{\tilde{J}}(A) \right)^{L^2} \cap H^{d/2}(\Sigma; F^{\text{ad}}).
\]

It was proved in [17, Proposition 2.1.1] and will be used in the proof of our Corollary 2.3.
There is a vast literature on pseudo-differential operators and the symbolic calculus. We shall only draw on the general knowledge regarding pseudo-differential operators over closed manifolds. Let \( k \in \mathbb{R} \). Let \( \Psi_k(\Sigma; G_1, G_2) \) denote the space of all \( k \)th order pseudo-differential operators mapping sections of a smooth vector bundle \( G_1 \) to sections of a smooth vector bundle \( G_2 \) over the same closed manifold \( \Sigma \); for \( P \in \Psi_k(\Sigma; G_1, G_2) \), let \( \sigma_k(P) \) denote the principle symbol of \( P \).

Proposition 2.4 (cf. [18] Lemmas 1.34 and 1.35)

(a) For \( P \in \Psi_k(\Sigma; G_1, G_2) \), there is a unique formal adjoint of \( P \), denoted by \( P^* \), such that
\[
(Pf,h)_{L^2(\Sigma;G_1)} = (f,P^*h)_{L^2(\Sigma;G_2)} \quad \text{for all } f \in C^\infty(\Sigma;G_1), h \in C^\infty(\Sigma;G_2),
\]
and \( P^* \in \Psi_k(\Sigma; G_2, G_1) \), \( \sigma_k(P^*) = \sigma_k(P)^* \).

(b) If \( Q \in \Psi_k(\Sigma; G_1, G_2) \), \( P \in \Psi_e(\Sigma; G_2, G_3) \), then \( PQ \in \Psi_{k+e}(\Sigma; G_1, G_3) \) and \( \sigma_{k+e}(PQ) = \sigma_k(P)\sigma_e(Q) \).

(c) Continuity property with respect to Sobolev spaces: Each \( P \in \Psi_k(\Sigma; G_1, G_2) \) extends to a continuous linear map form \( H^{s+k}(\Sigma; G_1) \) to \( H^s(\Sigma; G_2) \) for all real \( s \).

With the preceding notations we recall the classical knowledge about Calderón projections.

Theorem 2.2 (A. Calderón 1963; R. T. Seeley 1966, 1969) Let \( A \) be an elliptic differential operator of order \( d \) over a smooth compact Riemannian manifold \( \mathcal{M} \) with boundary \( \Sigma \), acting between sections of Hermitian vector bundles \( E, F \) over \( \mathcal{M} \). Then there exists a (in general, not uniquely determined) zeroth order classical pseudo-differential operator, called the Calderón projection of \( A \), \( C(A) = C_\infty(A): C^\infty(\Sigma; E^{\text{ad}}) \to C^\infty(\Sigma; E^{\text{ad}}) \), such that \( C(A) \) is idempotent, i.e., \( C(A)^2 = C(A) \) and
\[
\text{im} \ C_{-\Psi}^{-}(A) = \tilde{\rho}^d(\ker A_{\text{max}}).
\]
(2.13)

Here, for every \( s \in \mathbb{R} \), we denote the extended projection by \( G_s(A): H^s(\Sigma; E^{\text{ad}}) \to H^s(\Sigma; E^{\text{ad}}) \).

Remark 2.6 A detailed construction of the Calderón projection and a careful proof of [2.13] can be found in [17] Section 2.3, originally from Seeley’s [27][28][29] and Calderón’s [13] and inspired by Hörmander’s [21]; for the properties see also [19] Section 11.1 and, in the special case \( d = 1 \), [7] Chapters 12-13 and [8] Section 5.

Remark 2.7 The Calderón projections are very useful in the treatment of boundary value problems for elliptic differential operators. We recall some relevant definitions and properties directly from [17].

(a) Assume \( P \in \Phi_0(\Sigma; E^{\text{ad}}, E^{\text{ad}}) \) idempotent. To consider \( P \) as a boundary condition we associate with it the realisation \( A_P: D(A_P) \to L^2(\mathcal{M}; F) \) with
\[
D(A_P) := \{ u \in H^d(\mathcal{M}; E) \mid P\partial^d u = 0 \}.
\]
We define the weak domain by
\[ \mathcal{D}(A_{\max}, p) := \{ u \in \mathcal{D}(A_{\max}) \mid P \bar{p}^d u = 0 \}. \]

(b) Following [17] Definition 1.2.5, we call \( P \) a regular boundary condition if
\[ \mathcal{D}(A_P) = \mathcal{D}(A_{\max}, p). \]
P is called well-posed if it is regular and \( \text{im} A_P \) has finite codimension.

From [17] Proposition 2.1.2 we recall equivalent conditions:
- \( P \) is a regular boundary condition if and only if \( A_P : \mathcal{D}(A_P) \to L^2(\mathcal{M}; F) \) is left-Fredholm, i.e., \( \dim \ker A_P < \infty \) and \( \text{im} A_P \) closed.
- The boundary condition \( P \) is well-posed if and only if \( A_P : \mathcal{D}(A_P) \to L^2(\mathcal{M}; F) \) is Fredholm.

(c) With these notations we obtain from Seeley’s achievements in [29], reproduced and worked out in [17] Theorem 2.1.4(ii) the following operational conditions on the principle symbols of boundary pseudo-differential operators: Regularity and moreover well-posedness hold respectively if and only if for all \( q \in \Sigma, \xi \in T^*_q \Sigma, \xi \neq 0 \)
\[ \sigma_0(P)(q, \xi) : \text{im} \sigma_0(C(A))(q, \xi) \to E^d|_q \text{ injective}, \]
respectively,
\[ \sigma_0(P)(q, \xi) : \text{im} \sigma_0(C(A))(q, \xi) \to \text{im} \sigma_0(P)(q, \xi) \text{ invertible.} \]

In particular, since \( \sigma_0(\text{Id}) = \text{Id} \) and
\[ \sigma_0(C(A))(q, \xi) : \text{im} \sigma_0(C(A))(q, \xi) \to \text{im} \sigma_0(C(A))(q, \xi) \]
is just the identity, \( \text{Id} \) is a regular boundary condition and \( C(A) \) is a well-posed boundary condition.

(d) For regular \( P \) we have a lifting jack for regularity, also called higher regularity (see [17] Theorems 2.2.1 and 2.2.3): Let \( s \in \mathbb{N} \cup \{0\} \). Assume that
\[ u \in L^2(\mathcal{M}; E) \]
then \( u \in H^{s+\frac{d}{2}}(\Sigma; E^d) \). When \( P \) is also well-posed, then this regularity argument holds for all real \( s \geq 0 \).

(e) For later use we give the following simple description of \( \text{im} C_s(A) \) for all \( s \in \mathbb{R} \):
\[ \text{im} C_s(A) = \text{im} C_t(A) \cap H^s(\Sigma; E^d) \text{ for any real } t \leq s. \] (2.14)
Since \( s \geq t, H^s(\Sigma; E^d) \subset H^t(\Sigma; E^d) \), then \( \text{im} C_s(A) = \text{im} (C_t(A)|_{H^s(\Sigma; E^d)}) \).
Hence, the inclusion \( \subset \) of (2.14) is trivial. For the opposite inclusion we exploit that \( C(A) \) is a zeroth order pseudo-differential idempotent: so, for any \( C_t(A)f \in H^s(\Sigma; E^d) \), we have
\[ C_t(A)f = C_t^2(A)f = C_s(A)C_t(A)f. \]
In particular, for all \( s \geq -\frac{d}{2} \):

\[
\text{im} C_s(A) = \text{im} C_{-\frac{d}{2}}(A) \cap H^s(\Sigma; E^{\text{ed}}) = \Lambda_{-\frac{d}{2}}(A) \cap H^s(\Sigma; E^{\text{ed}}). \tag{2.15}
\]

As a side result, we obtain for all \( s \in \mathbb{R} \)

\[
\text{im} C_s(A) = \text{im} C_{\infty}(A)^t_{\|H^s(\Sigma; E^{\text{ed}})} = \text{im} C_t(A)^t_{\|H^s(\Sigma; E^{\text{ed}})} \text{ for any } t \geq s. \tag{2.16}
\]

According to the preceding Remark 2.7, the image of \( C_s(A) \), \( s \in \mathbb{R} \), does not depend on the choices of Calderón projection \( C(A) \) in the Calderón–Seeley Theorem 2.2. Moreover we can prove the following generalization of Equation (2.13).

\textbf{Corollary 2.2} The validity of the claim (2.13) in Theorem 2.2 holds also for Sobolev orders \( s \geq \frac{d}{2} \), yielding

\[
\text{im} (C_s(A)) = \rho^d(\ker A_{\max} \cap H^{s+\frac{d}{2}}(\mathcal{M}; E)) \text{ for } s \geq \frac{d}{2}. \tag{2.17}
\]

\textbf{Proof} By the Sobolev Trace Theorem (cf. Corollary 2.1) and Equations (2.13) and (2.15), we obtain

\[
\text{im} C_s(A) \supset \rho^d(\ker A_{\max} \cap H^{s+\frac{d}{2}}(\mathcal{M}; E)) \text{ for } s > \frac{d}{2} - \frac{1}{2},
\]

i.e., the inclusion \( \subset \) of (2.17), actually for a wider range of \( s \) than claimed.

Now we turn to the proof of the inclusion \( \supset \) for \( s \geq \frac{d}{2} \). First by the preceding Remark 2.7, \( C(A) \) is a well-posed boundary condition. Let \( s \geq \frac{d}{2} \). If \( f \in \text{im} C_s(A) = \Lambda_{-\frac{d}{2}}(A) \cap H^s(\Sigma; E^{\text{ed}}) \), then there is a \( u \in \ker A_{\max} \), such that \( f = \rho^d u \), so \( C(A)\rho^d u = \rho^d u \in H^s(\Sigma; E^{\text{ed}}) \). By the higher regularity for well-posed boundary conditions of the preceding Remark 2.7, we have \( u \in H^{s+\frac{d}{2}}(\mathcal{M}; E) \), so \( f \in \rho^d(\ker A_{\max} \cap H^{s+\frac{d}{2}}(\mathcal{M}; E)) \). Thus we get (2.17). \( \square \)

\textbf{Remark 2.8} In Definition 2.1, we defined the Cauchy data spaces for \( s = -\frac{d}{2} \) and \( s \geq \frac{d}{2} \). By Theorem 2.2 and Corollary 2.2, these spaces coincide with the images of the extensions \( C_s(A) \) of the Calderón projection for those \( s \). Moreover by (2.14) and Corollary 2.2, the images of \( C_s(A) \) for all \( s \in \mathbb{R} \) are unique determined by the Cauchy data spaces of the elliptic operator. That motivates us to define the Cauchy data spaces \( \Lambda_s(A) \) for all \( s \in \mathbb{R} \) as the images of \( C_s(A) \), yielding a \textit{chain of Cauchy data spaces}

\[
\Lambda_s(A) := \text{im} C_s(A) \text{ for all } s \in \mathbb{R}. \tag{2.18}
\]

By (2.12) and Corollary 2.2, we immediately have

\[
\text{im} C_{\frac{d}{2}}(A^t) = \left( J(\text{im} C_{\frac{d}{2}}(A)) \right)^t_{\perp L^2} \cap H^\frac{d}{2}(\Sigma; E^{\text{ed}}). \tag{2.19}
\]

Now we can prove the following generalization of Equation (2.19). The preceding corollary and the following one will be used in Section 4 in the proof of our Main Theorem (Theorem 1.2).
Corollary 2.3 There is an $L^2$-orthogonal decompositions of complementary closed subspaces

$$\text{im} \ C_s(A) \oplus_{L^2} \tilde{J} \left(\text{im} \ C_s(A') \right) = H^s(\Sigma; E^{id}) \text{ for } s \geq 0, \tag{2.20}$$

where $\tilde{J}$ is defined as in (2.24).

Our proof of Corollary 2.3 below needs the (uniquely determined) $L^2$-orthogonalized Calderón projection which we are going to introduce now – and use extensively later in our Section 3.

Lemma 2.1 Let $C := C(A)$ be a Calderón projection as introduced in Theorem 2.2, i.e., a zeroth order classical pseudo-differential operator,

$$C(A) : C^\infty(\Sigma; E^{id}) \rightarrow C^\infty(\Sigma; E^{id})$$

with $C^2 = C$ and $\text{im} \ C_{\frac{1}{2}}(A) = \tilde{\rho}(\ker A_{\text{max}})$. Then there exists a unique zeroth classical pseudo-differential operator $C_{\text{ort}} = C_{\text{ort}}(A) : C^\infty(\Sigma; E^{id}) \rightarrow C^\infty(\Sigma; E^{id})$ with

$$(C_{\text{ort}})^2 = C_{\text{ort}}, \quad CC_{\text{ort}} = C_{\text{ort}}, \quad C_{\text{ort}} C = C, \tag{2.21}$$

with self-adjoint $L^2$-extension $C_{\text{ort}}(A)$ on $H^0(\Sigma; E^{id})$ and with

$$\text{im} \ C_{\text{ort}}(A) = \text{im} \ C_s(A) \text{ for all } s \in \mathbb{R}. \tag{2.22}$$

Proof (of the lemma) Since $CC^d + (\text{Id} - C^d)(\text{Id} - C)$ is a formally self-adjoint elliptic pseudo-differential operator with trivial kernel, it is invertible. As in [7, Lemma 12.8], we define $C_{\text{ort}}$ by

$$C_{\text{ort}} := CC^d (CC^d + (\text{Id} - C^d)(\text{Id} - C))^{-1},$$

and infer that it is still a classical pseudo-differential operator of order 0. Moreover, it is symmetric

$$(C_{\text{ort}} f, h)_{L^2(\Sigma; E^{id})} = (f, C_{\text{ort}} h)_{L^2(\Sigma; E^{id})}, \text{ for all } f, h \in C^\infty(\Sigma; E^{id}) \tag{2.23}$$

which implies (over the closed manifold $\Sigma$) that its $L^2$-extension is self-adjoint. The symmetry property (2.23) and the algebraic equalities of (2.21) follow as in loc. cit. by calculation, then the invariance of the range in (2.22) and the uniqueness of $C_{\text{ort}}$ follow.

Proof (of Corollary 2.3) For $s \geq \frac{d}{2}$, $\text{im} \ C_s(A) \ni f = \tilde{\rho}^d u$ and $\text{im} C_s(A') \ni g = \tilde{\rho}^d v$ with $u \in \ker A_{\text{max}} \cap H^{s+\frac{d}{2}}(\mathcal{M}; E)$, $v \in \ker A'_{\text{max}} \cap H^{s+\frac{d}{2}}(\mathcal{M}; E)$, we obtain

$$(f, \tilde{J} g)_{L^2(\Sigma; E^{id})} = (\tilde{J} \tilde{\rho}^d u, \tilde{\rho}^d v)_{L^2(\Sigma; E^{id})}$$

$$= (Au, v)_{L^2(\mathcal{M}; E)} - (u, A' v)_{L^2(\mathcal{M}; E)} = 0 - 0 = 0.$$ 

That proves the $L^2$-orthogonality in (2.20).
Now let \( f \in H^s(\Sigma; E^d) \) for \( s = \frac{d}{2} \). To prove
\[
f \in \text{im} \, C^\sharp_4(A) + \tilde{J}^t(\text{im} \, C^\sharp_4(A^t)),
\]
i.e., the claimed decomposition in (2.20) for \( s = \frac{d}{2} \), we rewrite
\[
f = C^\text{ort}_4(A)f + f - C^\text{ort}_4(A)f.
\]
We shall show that \( f - C^\text{ort}_4(A)f \in \tilde{J}^t(\text{im} \, C^\sharp_4(A^t)) \). First we observe that
\[
f - C^\text{ort}_4(A)f \in (\text{im} \, C^\sharp_4(A))^{1-L^2} \cap H^{\sharp}(\Sigma; E^d).
\]
In fact, \( h' \in \text{im} \, C^\sharp_4(A) = \text{im} \, C^\text{ort}_4(A) \) implies that \( h' = C^\text{ort}_4(A)h \) for some \( h \in H^{\sharp}(\Sigma; E^d) \). Then we have
\[
(f - C^\text{ort}_4(A)f, h')_{L^2(\Sigma; E^d)} = (f - C^\text{ort}_4(A)f, C^\text{ort}_4(A)h)_{L^2(\Sigma; E^d)} = (C^\text{ort}_4(A)(f - C^\text{ort}_4(A)f), h)_{L^2(\Sigma; E^d)} = (0, h')_{L^2(\Sigma; E^d)} = 0.
\]
Next from the fact that \( \tilde{J} \) is an invertible zeroth pseudo-differential operator, we obtain for any \( h \in H^{\sharp}(\Sigma; E^d) \),
\[
0 \overset{\text{(2.20)}}{=} (f - C^\text{ort}_4(A)f, C^\text{ort}_4(A)h)_{L^2(\Sigma; E^d)} = (f - C^\text{ort}_4(A)f, \tilde{J}^{-1} C^\text{ort}_4(A)h)_{L^2(\Sigma; E^d)} = ((\tilde{J}^t)^{-1}(f - C^\text{ort}_4(A)f), \tilde{J} C^\text{ort}_4(A)h)_{L^2(\Sigma; E^d)},
\]
so
\[
((\tilde{J}^t)^{-1}(f - C^\text{ort}_4(A)f) \in (\tilde{J}(\text{im} \, C^\sharp_4(A)))^{1-L^2} \cap H^{\sharp}(\Sigma; F^d) \overset{\text{(2.10)}}{=} \text{im} \, C^\sharp_4(A^t),
\]
and we are done for \( s = \frac{d}{2} \).

For \( s > \frac{d}{2} \), the \( L^2 \)-complement in \( H^s(\Sigma; E^d) \) of (2.20) follows from the preceding result for \( s = \frac{d}{2} \) and the facts that \( C(A), C(A^t) \) and \( C^\text{ort} \) are pseudo-differential projections of order zero and \( \tilde{J} \) is an invertible pseudo-differential operator of order zero. Finally, (2.20) holds for \( 0 \leq s < \frac{d}{2} \), since \( H^{\sharp}(\Sigma; E^d) \) is dense in \( H^s(\Sigma; E^d) \) and \( H^s(\Sigma; E^d) \subset L^2(\Sigma; E^d) \).

\[\square\]

**Remark 2.9** (a) In the proof of Corollary 2.7, we get that
\[
\ker C^\text{ort}_s(A) = \text{im} (\text{Id} - C^\text{ort}_s(A)) = \tilde{J}^t(\text{im} \, C_s(A^t)) = \tilde{J}^t(\Lambda_s(A^t)) \quad \text{for} \quad s \geq 0,
\]
which will be used in Section 3.1.

(b) For a symmetric elliptic differential operator \( A \) and \( s \geq 0 \), \( \tilde{J}^t \) defines a (strong) symplectic form on \( (L^2(\Sigma; E^d), H^s(\Sigma; E^d)) \) with the Cauchy data space as a Lagrangian subspace according to (2.20).
2.5 Neubauer’s arithmetic of families of closed linear subspaces in Banach space

We refer to a functional analysis fact from our [9] Appendix A.3 regarding the continuity of families of closed subspaces in Banach space. We restate it in the following lemma that is based on Neubauer’s elementary, but deeply original [24]. We impose the gap topology on the space of closed linear subspaces of a given Banach space. We recall the concept of the gap between subspaces and the quantity $\gamma$ ("angular distance") that is useful in our estimates.

**Definition 2.2** (cf. [22] Sections IV.2.1 and IV.4.1) Let $X$ be a Banach space.
(a) Denote by $S_X$ the unit sphere of $X$ for any closed linear subspace $M$ of $X$. For any two closed linear subspaces $M, N$ of $X$, we set

$$
\delta(M, N) := \begin{cases} 
\sup_{u \in S_M} \text{dist}(u, N), & \text{if } M \neq \{0\}, \\
0, & \text{if } M = \{0\}.
\end{cases}
$$

$$
\hat{\delta}(M, N) := \max\{\delta(M, N), \delta(N, M)\}.
$$

$\hat{\delta}(M, N)$ is called the gap between $M$ and $N$.

We set

$$
\gamma(M, N) := \begin{cases} 
\inf_{u \in M \setminus N} \frac{\text{dist}(u, N)}{\text{dist}(u, M \setminus N)}, & \text{if } M \not\subset N, \\
1, & \text{if } M \subset N.
\end{cases}
$$

(b) We say, a sequence $(M_n)_{n=1,2, \ldots}$ of closed linear subspaces converges to $M$ if $\hat{\delta}(M_n, M) \to 0$ for $n \to \infty$. We write $M_n \to M$. Correspondingly we declare when a mapping $M$ from a topological space $B$ to the space of closed subspaces shall be called continuous at $b_0 \in B$.

**Remark 2.10** Denote the set of all closed operators from $X$ to $Y$ by $\mathcal{C}(X, Y)$. If $A_1, A_2 \in \mathcal{C}(X, Y)$, their graphs $\mathcal{G}(A_1) := \{(x, A_1x) \in X \times Y \mid x \in D(A_1)\}$, $\mathcal{G}(A_2)$ are closed linear subspaces in the product Banach space $X \times Y$. We use the gap $\hat{\delta}(\mathcal{G}(A_1), \mathcal{G}(A_2))$ to measure the "distance" between $A_1$ and $A_2$. Obviously, for $A', A \in \mathcal{B}(X, Y)$, we have (cf. [22] Theorem IV.2.14)

$$
\hat{\delta}(\mathcal{G}(A'), \mathcal{G}(A)) \leq \|A' - A\|.
$$

**Lemma 2.2** (cf. [22] Theorem IV.4.2]) Let $X$ be a Banach space and let $M, N$ be closed subspaces of $X$. In order that $M + N$ be closed, it is necessary and sufficient that $\gamma(M, N) > 0$.

**Lemma 2.3** (cf. [9] Proposition A.3.13 and Corollary A.3.14) Let $X$ be a Banach space and let $(M_b)_{b \in B}, (N_b)_{b \in B}$ be two families of closed subspaces of $X$, where $B$ is a parameter space. Assume that $M_{b_0} + N_{b_0}$ is closed for some $b_0 \in B$, and $(M_b)_{b \in B}, (N_b)_{b \in B}$ are both continuous at $b_0$.

(a) Then $(M_b \cap N_b)_{b \in B}$ is continuous at $b_0$ if and only if $(M_b + N_b)_{b \in B}$ is continuous at $b_0$.

(b) Assume furthermore that for $b \in B$, $\dim(M_b \cap N_b) \equiv \text{constant} < +\infty$ or $\dim(X/(M_b + N_b)) \equiv \text{constant} < +\infty$. Then the families $(M_b \cap N_b)_{b \in B}$ and $(M_b + N_b)_{b \in B}$ are both continuous at $b_0$. 


Remark 2.11 For better understanding the proof of the preceding lemma in [9], note that: according to [9, Corollary A.3.12b] and [22, Theorem IV.4.2]), if $M_{b_0} + N_{b_0}$ is closed and $(M_b)_{b \in B}$, $(N_b)_{b \in B}$ and $(M_b \cap N_b)_{b \in B}$ are all continuous at $b_0$, then we get that $M_b + N_b$ is closed in a whole neighbourhood of $b_0$ in $B$.

3 Proof of our main theorem

We shall divide the proof of Theorem 1.2 into two cases, both under the assumption of constant dimensions of the spaces of inner solutions $Z_{+,0}(A_b) = \ker A_{b,\min}$ and $Z_{-,0}(A_b) = \ker A_{b,\min}^t$.

(1) In Section 3.1 we deal with the case $s \geq \frac{d}{2}$ in three steps. (i) In Proposition 3.1 we obtain that $(\ker A_{b,s+\frac{d}{2}})_{b \in B}$ is continuous in $H^{s+\frac{d}{2}}(\mathcal{M}; E)$. (ii) Since the Cauchy trace map is bounded and surjective for $s \geq \frac{d}{2}$, we can deduce the main achievement of this Subsection, Proposition 3.2, and obtain that the family $(\tilde{\rho}^d(\ker A_{b,s+\frac{d}{2}}))_{b \in B}$ is continuous in $H^{s}(\Sigma; E^{adj})$. That means $(\text{im} C_{s}^{\text{ort}}(A_b))_{b \in B}$ is continuous in $H^{s}(\Sigma; E^{adj})$. (iii) So, according to Corollary 3.1 we can conclude that the corresponding family $(C_{s}^{\text{ort}}(A_b): H^{s}(\Sigma; E^{adj}) \leftarrow \rightarrow)_{b \in B}$ of Calderón projections is continuous in the operator norm for all $s \geq \frac{d}{2}$. That proves our Main Theorem for such $s$.

(2) We use the results of case (1) (i.e., $s \geq \frac{d}{2}$) to show that for $s < \frac{d}{2}$ the family $(C_{s}^{\text{ort}}(A_b): H^{s}(\Sigma; E^{adj}) \leftarrow \rightarrow)_{b \in B}$ of Calderón projections is continuous in the operator norm by duality and interpolation property of spaces and operators in Sobolev scales. That is the content of Section 3.2.

Remark 3.1 We emphasize that all the Calderón projections in this section are assumed to be $L^2$-orthogonalized, that is, with Lemma 2.1

$$C = C^{\text{ort}}, \quad C_s(A) = C_{s}^{\text{ort}}(A) \text{ for } s \in \mathbb{R}.$$
Notation 3.2 Based on Remark 2.7a, for \( s \geq \frac{d}{2} \), we denote by \( A_{s+\frac{d}{2},P} \) the operators

\[
A_{s+\frac{d}{2},P} : \{ u \in H^{s+\frac{d}{2}}(\mathcal{M}; E) \mid P\tilde{\rho}^d u = 0 \} \longrightarrow H^{s-\frac{d}{2}}(\mathcal{M}; F),
\]

for any boundary condition \( P : C^\infty(\Sigma; E^{\prime}d) \rightarrow C^\infty(\Sigma; E^{\prime}d). \) We write shorthand \( A_P := A_{d,P} \).

For any elliptic operator \( A \) over a smooth compact manifold with boundary, recall \( A_{\min} : H^d_0(\mathcal{M}; E) \rightarrow L^2(\mathcal{M}; F) \). It is well known and was emphasised above in Notation 1.1 that \( \ker A_{\min} \) consists only of smooth sections and is finite-dimensional. That follows from the interior regularity for elliptic operators (e.g., [31, Theorem 5.11.1]) and one can use the interior elliptic estimate to prove that \( A_{\min} \) is left-Fredholm, i.e., \( \dim \ker A_{\min} < +\infty \) and \( \text{im} A_{\min} \) is closed (e.g., [17, Propositions 1.1.1 and A.1.4]). Later we shall use the following slight generalization:

Lemma 3.1 For \( s \geq \frac{d}{2} \), \( \ker A_{s+\frac{d}{2},1d} = \ker A_{\min} \) is finite-dimensional and consists of smooth sections.

Proof We only need to prove the equality. By Proposition 2.3(3), \( \mathcal{D}(A_{1d}) = H^d_0(\mathcal{M}; E) \), so \( A_{1d} = A_{\min} \). As just emphasized, we have \( \ker A_{\min} \subset \{ u \in C^\infty(\mathcal{M}; E) \mid Au = 0 \text{ and } \tilde{\rho}^d u = 0 \} \). Obviously we have for \( s \geq \frac{d}{2} \)

\[
\{ u \in C^\infty(\mathcal{M}; E) \mid Au = 0 \text{ and } \tilde{\rho}^d u = 0 \} \subset \ker A_{s+\frac{d}{2},1d} \subset \ker A_{\min}.
\]

So we get the equality. \( \square \)

In the following lemma we prove that \( \text{im} A_{s+\frac{d}{2}} \) is closed in \( H^{s-\frac{d}{2}}(\mathcal{M}; F) \) and get information about the quotient space \( H^{s-\frac{d}{2}}(\mathcal{M}; F)/\text{im} A_{s+\frac{d}{2}} \).

Lemma 3.2 For \( s \geq \frac{d}{2} \), there is an \( L^2 \)-orthogonal decomposition of complementary closed subspaces

\[
H^{s-\frac{d}{2}}(\mathcal{M}; F) = \text{im} A_{s+\frac{d}{2}} \oplus L^2 \ker A_{\min}.
\]

Proof The \( L^2 \)-orthogonality follows directly from Green’s Formula (Proposition 2.8) and in adjusted form (2.8). In fact, for \( s \geq \frac{d}{2} \), \( u \in H^{s+\frac{d}{2}}(\mathcal{M}; E) \) and \( v \in \ker A_{\min} \), we have

\[
(Au, v)_{L^2(\mathcal{M}; F)} = (u, A^*v)_{L^2(\mathcal{M}; E)} + (\tilde{J}\tilde{\rho}^d u, \tilde{\rho}^d v)_{L^2(\Sigma; E^{\prime}d)} = 0.
\]

Next we prove

\[
L^2(\mathcal{M}; F) = \text{im} A_C \oplus \ker A_{\min},
\]

where \( C := C^{\text{ort}}(A) \) denotes the \( L^2 \)-orthogonalized Calderón projection defined in Lemma 2.1.
By Remarks 2.7, b, the Calderón projection is a well-posed boundary condition; hence \( A_C : D(A_C) \to L^2(\mathcal{M}; F) \) is Fredholm, where \( D(A_C) = \{ u \in H^d(\mathcal{M}; E) \mid C\tilde{\rho}^d u = 0 \} \). Thus \( \text{im } A_C \) is closed, then we have the decomposition \( L^2(\mathcal{M}; F) = \text{im } A_C \oplus \ker (A_C)^* \), where we consider \( A_C \) as an unbounded densely defined operator from \( L^2(\mathcal{M}; E) \) to \( L^2(\mathcal{M}; F) \) and denote its adjoint by \((A_C)^*\). So (3.2) will follow from

\[
\ker (A_C)^* = \ker A^t_{\text{min}}. \tag{3.3}
\]

Now we shall prove (3.3). In fact, according to [17, Proposition 1.2.6],

\[
(A_C)^* = A^t_{\text{max}, C^{\text{adm}}} \quad \text{with } C^{\text{adm}} := (\tilde{J})^{-1}(\text{Id} - C^t)\tilde{J}.
\]

Note that \( C^{\text{adm}} \in \mathcal{V}_0(\Sigma; F^{2d}, F^d) \) is idempotent and defines a well-posed boundary condition for \( A^t \). According to our assumption \( C = C^{\text{ort}} \), we have \( C = C^t \).

By Corollary 2.3, we have

\[
\tilde{J}^t(\text{im } C_\frac{1}{2}(A^t)) = \text{im } (\text{Id} - C^{\text{ort}}(A)).
\]

Then we get \( \text{im } C_\frac{1}{2}(A^t) = C^{\text{adm}} \), where \( C^{\text{adm}} : H^\frac{1}{2}(\Sigma; F^{2d}) \to H^\frac{1}{2}(\Sigma; F^d) \).

Thus

\[
C^{\text{adm}} : \text{im } C_\frac{1}{2}(A^t) \to \text{im } C^{\text{adm}} \tag{3.4}
\]

is just the identity. So by [17, Proposition 2.1.2], \( C^{\text{adm}} \) is a well-posed boundary condition for \( A^t \). Then by Remark 2.7, we have \( A^t_{\text{max}, C^{\text{adm}}} = A^t_{C^{\text{adm}}} \), thus

\[
\ker A^t_{\text{max}, C^{\text{adm}}} = \ker A^t_{C^{\text{adm}}}
\]

\[
= \{ u \in H^d(\mathcal{M}; F) \mid A^t u = 0, C^{\text{adm}} \tilde{\rho}^d u = 0 \}
\]

\[
= \{ u \in H^d(\mathcal{M}; F) \mid A^t u = 0, \tilde{\rho}^d u = 0 \} = \ker A^t_{\text{min}},
\]

where in the last line we used

\[
\text{im } C_\frac{1}{2}(A^t) = \{ \tilde{\rho}^d u \mid u \in H^d(\mathcal{M}; F), A^t u = 0 \} \text{ and (3.4).}
\]

Now (3.3) is done.

Note that \( \ker A^t_{\text{min}} \) consists of smooth sections and is finite-dimensional. Thus we can use (3.2), i.e., the decomposition in \( L^2(\mathcal{M}; F) \) to get our results for \( s \geq \frac{d}{2} \):

\[
H^{s-\frac{d}{2}}(\mathcal{M}; F) = L^2(\mathcal{M}; F) \cap H^{s-\frac{d}{2}}(\mathcal{M}; F)
\]

\[
= (\text{im } A_C \oplus \ker A^t_{\text{min}}) \cap H^{s-\frac{d}{2}}(\mathcal{M}; F)
\]

\[
= (\text{im } A_C \cap H^{s-\frac{d}{2}}(\mathcal{M}; F)) \oplus \ker A^t_{\text{min}}
\]

\[
= \text{im } A^{s-\frac{d}{2}, C} \oplus \ker A^t_{\text{min}},
\]
where \( \mathcal{D}(A_{s+\frac{d}{2}}, C) = \{ u \in H^{s+\frac{d}{2}}(\mathcal{M}; E) \mid C\phi^2 u = 0 \} \) and we have used higher regularity for well-posed boundary conditions of Remark 2.7. So im \( A_{s+\frac{d}{2}} \) is finite-codimensional in \( H^{s+\frac{d}{2}}(\mathcal{M}; F) \). Since

\[
\text{im } A_{s+\frac{d}{2}} \subset \text{im } A_{s+\frac{d}{2}} \subset H^{s+\frac{d}{2}}(\mathcal{M}; F),
\]

the space \( \text{im } A_{s+\frac{d}{2}} \) is also finite-codimensional and thus closed in \( H^{s+\frac{d}{2}}(\mathcal{M}; F) \). So we get (3.1).

\[ \square \]

**Proposition 3.1** Let \( s \geq \frac{d}{2} \). If \( \dim A_{b,\min}^t = \kappa_- \) constant for all \( b \in B \), then Assumption 3.7, i.e., that the family \( \{ A_{b,t+s} \}_{b \in B} \) is continuous in the operator norm, implies that the family \( \{ \ker A_{b,t+s} \}_{b \in B} \) of closed linear subspaces is continuous in \( H^{s+\frac{d}{2}}(\mathcal{M}; E) \).

**Proof** Assumption 3.7 implies that the graphs \( \{ \mathfrak{G}(A_{b,t+s}) \}_{b \in B} \) make a continuous family of closed linear subspaces of \( H^{s+\frac{d}{2}}(\mathcal{M}; E) \times H^{s-\frac{d}{2}}(\mathcal{M}; F) \). Here we impose the gap topology of Definition 2.2 on the space of closed linear subspaces of the product space. Actually, the two claims are equivalent by [22, Theorem IV.2.23 a)].

For Banach spaces \( X, Y \) and any bounded linear map \( Q : X \to Y \), we recall the elementary formulae

\[
\mathfrak{G}(Q) + X \times \{ 0 \} = X \times \text{im } Q \quad \text{and} \quad \mathfrak{G}(Q) \cap (X \times \{ 0 \}) = \ker Q \times \{ 0 \}.
\]

Together with Lemma 3.2 for \( X := H^{s+\frac{d}{2}}(\mathcal{M}; E), Y := H^{s-\frac{d}{2}}(\mathcal{M}; F) \) and \( Q \) right-Fredholm, i.e., im \( Q \) is closed and finite-codimensional, we have

\[
\dim \frac{X \times Y}{\mathfrak{G}(Q)+X \times \{ 0 \}} = \dim \frac{Y}{\text{im } Q} \quad \text{for } Q = A_{b,t+s}^t \quad \text{dim ker } A_{b,\min}^t = \kappa_-.
\]

Now we consider the two following continuous families of closed subspaces of \( X \times Y \):

\( M_b := \mathfrak{G}(A_{b,t+s}) \) with \( b \) running in \( B \) and the constant family \( N_b := H^{s+\frac{d}{2}}(\mathcal{M}; E) \times \{ 0 \} \). By Lemma 3.2

\[
M_b + N_b = \mathfrak{G}(A_{b,t+s}) + H^{s+\frac{d}{2}}(\mathcal{M}; E) \times \{ 0 \} = H^{s+\frac{d}{2}}(\mathcal{M}; E) \times \text{im } A_{b,t+s}^t
\]

are closed. Then by Lemma 2.3, the constance of \( \kappa_- \) implies that the family \( M_b \cap N_b = \mathfrak{G}(A_{b,t+s}) \cap (H^{s+\frac{d}{2}}(\mathcal{M}; E) \times \{ 0 \}) = \ker A_{b,t+s}^t \times \{ 0 \} \)

is continuous on \( B \) and so \( \{ \ker A_{b,t+s}^t \}_{b \in B} \), and the proposition is proved. \[ \square \]

Now we turn to the Cauchy traces \( \tilde{\rho}^t(\ker A_{b,t+s}^t) \). Note that for \( s \geq \frac{d}{2} \)
the Cauchy trace operator \( \tilde{\rho}^t : H^{s+\frac{d}{2}}(\mathcal{M}; E) \to H^s(\Sigma; E^d) \) is surjective and bounded.
**Proposition 3.2** Additionally to Assumption 3.1, we assume for all \( b \in B \), \( \dim \ker A_{b, \text{min}} = \kappa_+ \) constant and \( \dim \ker A_{b, \text{min}}^1 = \kappa_- \) constant. Then the family

\[
\left( \tilde{p}^d(\ker A_{b, s + \frac{d}{2}}) \right)_{b \in B} = (\im C_s(A_b))_{b \in B}
\]

makes a continuous family of closed subspaces in \( H^s(\Sigma; E^d) \) for all \( s \geq \frac{d}{2} \).

Our proof of Proposition 3.2 will use the following functional-analytic estimate.

**Lemma 3.3** Let \( X, Y \) be Banach spaces, \( p: X \to Y \) be surjective and bounded linear. Then there exist positive constants \( c \) and \( \bar{c} \) such that for any closed linear subspaces \( M, N \) of \( X \) with \( M, N \supset \ker p \), we have

\[
\bar{c}\delta(M, N) \leq \delta(p(M), p(N)) \leq c\delta(M, N)
\]

where \( \delta(\cdot, \cdot) \) is defined in Definition 2.2b.

**Proof** Note that \( \ker p \) is a closed linear subspace of \( X \). For the quotient map \( q: X \to X/\ker p \), we have \( \delta(M, N) = \delta(q(M), q(N)) \) (cf. [5] Lemma A.3.1(d))]. We define the induced map \( \tilde{p}: X/\ker p \to Y \) by \( \tilde{p}(x + \ker p) := p(x) \), then \( p = \tilde{p} \circ q \). Since the bounded linear transformation \( \tilde{p}: X/\ker p \to Y \) is bijective, the Inverse Mapping Theorem implies that \( \tilde{p} \) is a homeomorphism. So the lemma holds. \( \square \)

**Proof (of Proposition 3.2)** By Proposition 3.1, the family \( \left( \ker A_{b, s + \frac{d}{2}} \right)_{b \in B} \) of closed linear subspaces is continuous in \( H^{s + \frac{d}{2}}(\mathcal{M}; E) \). For closed subspaces \( M_b := \ker A_{b, s + \frac{d}{2}}, N_b := \ker (\tilde{p}^d|_{H^{s + \frac{d}{2}}(\mathcal{M}; E)}) \) of \( H^{s + \frac{d}{2}}(\mathcal{M}; E) \),

\[
\im C_s(A_b) = \tilde{p}^d(\ker A_{b, s + \frac{d}{2}}) = \tilde{p}^d(M_b + N_b).
\]

Moreover by Lemma 3.1 and this proposition’s assumption, the spaces

\[
M_b \cap N_b = \ker A_{b, s + \frac{d}{2}} \cap \ker \tilde{p}^d = \ker A_{b, \text{min}}
\]

are of finite constant dimension \( \kappa_+ \) for all \( b \in B \). Since \( \im C_s(A_b) \) is closed in \( H^s(\Sigma; E^d) \), the subspace

\[
M_b + N_b = (\tilde{p}^d)^{-1}(\im C_s(A_b))
\]

is closed in \( H^{s + \frac{d}{2}}(\Sigma; E^d) \). So by Lemma 2.3b, the continuous variation of \( M_b = \ker A_{b, s + \frac{d}{2}} \) and the constancy of the family \( N_b = \ker (\tilde{p}^d|_{H^{s + \frac{d}{2}}(\mathcal{M}; E)}) \) imply that the family \( (M_b + N_b)_{b \in B} \) is continuous. From Lemma 3.3 we get the continuous variation of \( \tilde{p}^d(\ker A_{b, s + \frac{d}{2}}) = \tilde{p}^d(M_b + N_b). \) \( \square \)

Next we will provide a non-trivial jump from the continuity of the Cauchy data spaces to the continuity of the Calderón projections. Our arguments are based on the following observation: Given a family of bounded projections in a Banach space, if their images and kernels are continuous in the gap topology, then this family of bounded projections is continuous in the operator norm. More precisely we have
Lemma 3.4 Let $X$ be a Banach space and $B$ be a topological space. Let $(P_b \in B(X))_{b \in B}$ be a family of projections, that is, $P_b^2 = P_b$ for every $b \in B$. If either

(1) \[ \lim_{b \to b_0} \delta(\text{im} P_b, \text{im} P_{b_0}) \to 0 \quad \text{and} \quad \lim_{b \to b_0} \delta(\ker P_b, \ker P_{b_0}) \to 0, \]  

(2.26) or

(2) \[ \lim_{b \to b_0} \delta(\text{im} P_b, \text{im} P_{b_0}) \to 0 \quad \text{and} \quad \lim_{b \to b_0} \delta(\ker P_b, \ker P_{b_0}) \to 0; \]  

then

(3.7) \[ \lim_{b \to b_0} \| P_b - P_{b_0} \| \to 0. \]

Proof We will use the quantity $\gamma(\cdot, \cdot)$ in (2.26) to get the estimate of the operator norm $\| P_b - P_{b_0} \| := \sup_{z \in X, \| z \| = 1} \| (P_b - P_{b_0})z \|$. 

(1) We begin to prove (3.5) $\Rightarrow$ (3.7). First we recall the definition and properties of $\gamma(\cdot, \cdot)$. Since $X = \text{im} P_b \oplus \ker P_b$, by the definition of $\gamma(\cdot, \cdot)$ in (2.26), for any $x' \in \text{im} P_b, y' \in \ker P_b$, we have

(3.8) \[ \| x' + y' \| \geq \| x' \| \gamma(\text{im} P_b, \ker P_b) \quad \text{and} \quad \| x' + y' \| \geq \| y' \| \gamma(\ker P_b, \text{im} P_b); \]

and $\gamma(\text{im} P_b, \ker P_b) > 0, \gamma(\ker P_b, \text{im} P_b) > 0$ (cf. [22, Theorem IV.4.2]).

Then we use $\delta(\cdot, \cdot)$ and $\gamma(\cdot, \cdot)$ to give the estimate of the norm $\| P_b - P_{b_0} \|$. Take $\delta_1 := \delta(\text{im} P_b, \text{im} P_{b_0})$, $\delta_2 := \delta(\ker P_b, \ker P_{b_0})$. By the definition of $\delta(\cdot, \cdot)$ (see also [22, IV (2.3)]), for any $\varepsilon > 0$ and any $z' = x' + y'$ with $x' \in \text{im} P_b, y' \in \ker P_b$, we can correspondingly choose $x \in \text{im} P_{b_0}, y \in \ker P_{b_0}$ such that

(3.9) \[ \| x' - x \| \leq (\delta_1 + \varepsilon)\| x' \|, \quad \| y' - y \| \leq (\delta_2 + \varepsilon)\| y' \|. \]

So we have

\[ \| (P_b - P_{b_0})z' \| = \| (P_b - P_{b_0})(x' + y') \| \]
\[ = \| x' - P_{b_0}(x' + y') + P_{b_0}(x + y) - x \| \]
\[ = \| x' - x + P_{b_0}(x' + y') - P_{b_0}(x + y) \| \]
\[ \leq \| x' - x \| + \| P_{b_0} \| (\| x' \| + \| y' \|) \]
\[ \leq (\| P_{b_0} \| + 1)(\delta_1 + \delta_2 + \varepsilon)(\| x' \| + \| y' \|) \] by (3.8)
\[ \leq (\| P_{b_0} \| + 1)(\delta_1 + \delta_2 + \varepsilon)(\frac{\| x' + y' \|}{\gamma(\text{im} P_b, \ker P_b)} + \frac{\| x' + y' \|}{\gamma(\ker P_b, \text{im} P_b)}) \] by (3.8).

Since $\varepsilon > 0$ and $z' \in X$ are both arbitrary, we have

(3.10) \[ \| P_b - P_{b_0} \| \leq (\| P_{b_0} \| + 1)(\delta_1 + \delta_2) \left( \frac{1}{\gamma(\text{im} P_b, \ker P_b)} + \frac{1}{\gamma(\ker P_b, \text{im} P_b)} \right). \]
Finally, we give the positive lower bound estimate of \( \gamma(\text{im } P_b, \ker P_b) \). By [24, Lemma 1.4], if \( \gamma(\text{im } P_{b_0}, \ker P_{b_0}) - \delta_1 \cdot \gamma(\text{im } P_{b_0}, \ker P_{b_0}) - \delta_1 - \delta_2 > 0 \), then

\[
\gamma(\text{im } P_b, \ker P_b) \geq \frac{\gamma(\text{im } P_{b_0}, \ker P_{b_0}) - \delta_1 \cdot \gamma(\text{im } P_{b_0}, \ker P_{b_0}) - \delta_1 - \delta_2}{1 + \delta_2}.
\]

Together with (3.3), we have

\[
\lim \inf_{b \to b_0} \gamma(\text{im } P_b, \ker P_b) \geq \gamma(\text{im } P_{b_0}, \ker P_{b_0}) > 0. \tag{3.11}
\]

Similarly,

\[
\lim \inf_{b \to b_0} \gamma(\ker P_b, \text{im } P_b) \geq \gamma(\ker P_{b_0}, \text{im } P_{b_0}) > 0. \tag{3.12}
\]

Combining (3.10), (3.11) and (3.12), we get (3.7).

(2) Now we are going to prove (3.6) \( \Rightarrow \) (3.7). Take \( \delta_3 := \delta(\text{im } P_{b_0}, \text{im } P_{b_0}) \), \( \delta_4 := \delta(\ker P_{b_0}, \text{im } P_{b_0}) \). Similar to (3.10), we have

\[
\|P_b - P_{b_0}\| \leq (\|P_b\| + 1)(\delta_3 + \delta_4) \left( \frac{1}{\gamma(\text{im } P_{b_0}, \ker P_{b_0})} + \frac{1}{\gamma(\ker P_{b_0}, \text{im } P_{b_0})} \right).
\]

Take \( \alpha := \frac{1}{\gamma(\text{im } P_{b_0}, \ker P_{b_0})} + \frac{1}{\gamma(\ker P_{b_0}, \text{im } P_{b_0})} \). Since \( \|P_b\| \leq \|P_b - P_{b_0}\| + \|P_{b_0}\| \), we have

\[
\|P_b\|(1 - \alpha(\delta_3 + \delta_4)) \leq \|P_{b_0}\| + \alpha(\delta_3 + \delta_4).
\]

Together with (3.3) and (3.13), we get (3.7). \qed

By the preceding lemma and the definition of the gap, we can conclude

**Corollary 3.1** A sufficient and necessary condition for the continuity of a family of bounded projections in a fixed Banach space, parameterized by a topological space, is that their kernels and images are both continuous in the gap topology.

**Proof** (of Theorem 1.2 for \( s \geq \frac{d}{2} \)) According to Corollary 2.3, we have, for \( s \geq \frac{d}{2} \)

\[
\ker C_s^\text{ort}(A) = \tilde{J}_b^t \left( \text{im } C_s(A) \right).
\]

So under the assumptions of Theorem 1.2 by Proposition 3.2 we get that, for \( s \geq \frac{d}{2} \), \( \left( \text{im } C_s^\text{ort}(A_b) \right)_{b \in B} \) and \( \left( \ker C_s^\text{ort}(A_b) \right)_{b \in B} \) are both continuous in \( H^s(S; E^{ad}) \). Then by Corollary 3.1 we get that the family \( \left( C_s^\text{ort}(A_b) \right)_{b \in B} \) is continuous in the operator norm \( \| \cdot \|_{s,s} \) for all \( s \geq \frac{d}{2} \). \qed
3.2 Proof of our main theorem for $s < \frac{d}{2}$

Interpolation theory can be applied easily for intermediate Sobolev spaces between two given Sobolev spaces to establish an estimate for the operator norm of an intermediate operator, see Calderón’s [13] or [23] by J.-L. Lions and Magenes.

We give a slimmed-down version of interpolation theory for intermediate spaces.

**Definition 3.1 (Interpolation property)** We follow [30, Definitions 21.4 and 21.5]. Let $E_0$ and $E_1$ be normed spaces with $E_1 \hookrightarrow E_0$ continuously embedded and dense.

(a) An **intermediate space** between $E_1$ and $E_0$ is any normed space $E$ such that $E_1 \subset E \subset E_0$ (with continuous embeddings).

(b) An **interpolation space** between $E_1$ and $E_0$ is any intermediate space $E$ such that every linear mapping from $E_0$ into itself which is continuous from $E_0$ into itself and from $E_1$ into itself is automatically continuous from $E$ into itself. It is said to be of exponent $\theta$ (with $0 < \theta < 1$), if there exists a constant $c_0$ such that

\[
\|A\|_{B(E,E)} \leq c_0 \|A\|_{B(E_1,E_0)}^{1-\theta} \|A\|_{B(E_0,E_0)}^\theta \quad \text{for all } A \in B(E_1,E_1) \cap B(E_0,E_0).
\]

(c) Moreover, if $E_0$ and $E_1$ are Banach spaces, for $0 < \theta < 1$, we can define the complex interpolation space $[E_1, E_0]_\theta$ in loc. cit.

**Remark 3.2** The construction of the complex interpolation space uses analytic functions with values in the Banach space $E_0$. Using the classical Three Lines Theorem (mainly about the maximum modulus principle), one can show that $[E_1, E_0]_\theta$ with a kind of quotient norm is also a Banach space (cf. [23, Section 1.14.1] or [31, Section 4.2]). By [30, Lemma 21.6], the interpolation property holds for $[E_1, E_0]_\theta$ with $c_0 = 1$ in (3.14).

**Definition 3.2** (cf. [11, Definition 2.5]) Slightly more generally, we call a family $(H^s)_{s \in \mathbb{R}}$ a **scale of Hilbert spaces** if

1. $H^s$ is a Hilbert space for each $s \in \mathbb{R}$,
2. $H^{s'} \hookrightarrow H^s$ embeds continuously for $s \leq s'$,
3. if $s < t$, $0 < \theta < 1$, then the complex interpolation space belongs to the scale with $[H^t, H^s]_\theta = H^{(1-\theta)t+\theta s}$,
4. $H^\infty := \cap_{s \in \mathbb{R}+} H^s$ is dense in $H^t$ for each $t \in \mathbb{R}$,
5. the $H^0$-scalar product, denoted by $(\cdot, \cdot)$, restricted to $H^\infty$ extends to a perfect pairing between $H^s$ and $H^{-s}$, denoted by $(\cdot, \cdot)_{s,-s}$, for all $s \in \mathbb{R}$.

Let $(H^s)_{s \in \mathbb{R}}$ be a scale of Hilbert spaces.
Definition 3.3 A linear map $T: H^\infty \to H^\infty$ is called an operator of order 0, if it extends to a continuous linear map $T_s: H^s \to H^s$ for all $s \in \mathbb{R}$. We denote the vector space of all operators of order 0 by $\text{Op}^0((H^s)_{s \in \mathbb{R}})$. For $T_s \in \mathcal{B}(H^s)$, we denote its operator norm by $\|T_s\|_{s,s}$.

Lemma 3.5 Let $(H^s)_{s \in \mathbb{R}}$ be a scale of Hilbert spaces and $T \in \text{Op}^0((H^s)_{s \in \mathbb{R}})$. We assume that the continuous extension $T_0: H^0 \to H^0$ is self-adjoint. Then

1. for $t > 0$, $\|T_{-t}\|_{-t,-t} = \|T_t\|_{t,t}$;
2. for $s_0 < s < s_1$, $\|T_s\|_{s,s} \leq \left( \|T_{s_0}\|_{s_0,s_0} \right)^{\frac{s-s_0}{s_1-s_0}} \left( \|T_{s_1}\|_{s_1,s_0} \right)^{\frac{s_1-s_0}{s_1-s_0}}$.

Proof (1) Fix any $s \in \mathbb{R}$. Let $(H^s)^*$ denote the the space of bounded linear functionals on $H^s$. The norm of $\phi \in (H^s)^*$ is given by

$$\|\phi\| := \sup_{f \in H^s, \|f\|_s \leq 1} |\phi(f)|.$$  

By Definition 5.2(5), $H^{-s}$ can be identified with $(H^s)^*$ by the isometric isomorphism,

$$H^{-s} \to (H^s)^*,$$

$$h \mapsto \langle \cdot, h \rangle_{s,-s},$$

where isometric means that: if $\phi(f) := \langle f, h \rangle_{s,-s}$ for every $f \in H^s$, then $\|\phi\| = \|h\|_{-s}$. According to the above identification, we can define the adjoint operator of $T_s$

$$(T_s)^*: H^{-s} \to H^{-s} \text{ by setting for } h \in H^{-s}$$

$$(f, (T_s)^* h)_{s,-s} := \langle T_s f, h \rangle_{s,-s} \text{ for all } f \in H^s,$$  

(3.15)

then $(T_s)^*$ is also a bounded linear operator and

$$\|(T_s)^*\|_{-s,-s} = \|T_s\|_{s,s}.$$  

(3.16)

For $t > 0$, we claim that $(T_t)^* = T_{-t}$. In fact, since $t > 0$, $T_0|_{H^t} = T_t$, $T_{-t}|_{H^0} = T_0$, and for any $f \in H^t \subset H^0, h \in H^0 \subset H^{-t}$, we have

$$\langle T_t f, h \rangle_{t,-t} = (T_t f, h) = (T_0 f, h) = (f, T_0 h) = \langle f, T_0 h \rangle_{t,-t} = \langle f, T_{-t} h \rangle_{t,-t},$$

(3.17)

where we have used the assumption that $T_0: H^0 \to H^0$ is self-adjoint. So by (3.15) and (3.17), for any $f \in H^t, h \in H^0$, we have

$$\langle f, (T_t)^* h \rangle_{t,-t} = \langle f, T_{-t} h \rangle_{t,-t}.$$  

This implies

$$(T_t)^* h = T_{-t} h \text{ for any } h \in H^0.$$  

(3.18)

Since $(T_t)^*, T_{-t}$ are bounded linear operators on $H^{-t}$ and since $H^0$ is dense in $H^{-t}$, we get

$$(T_t)^* = T_{-t} \text{ on } H^{-t}.$$
Finally by (3.10) and (3.18), we get $\|T_{t,-t}\|_{t,-t} = \|(T_t)^\ast\|_{t,-t} = \|T_0\|_{t,t}$.

(2) Since $[H^{s_1}, H^{s_0}]_\theta = H^{(1-\theta)s_1+\theta s_0}$, for $0 < \theta < 1$, by the interpolation property for $[H^{s_1}, H^{s_0}]_\theta$ (cf. (3.14)), we obtain, for $s_0 < s < s_1$

$$\|T_s\|_{s,s} \leq \left( \|T_{s_1}\|_{s_1,s_1} \right)^{s-s_0/s_1-s_0} \left( \|T_{s_0}\|_{s_0,s_0} \right)^{s_1-s}.$$

$\square$

**Theorem 3.3** Let $B$ be a topological space and $T_b \in \text{Op}^0((H^s)_{s \in \mathbb{R}})$ for all $b \in B$. Assume that the extended bounded linear maps $T_{b,0}: H^0 \to H^0$ are self-adjoint for all $b \in B$. If $(T_{b,t})_{b \in B}$ is continuous on $B$ in the operator norm for some $t \in \mathbb{R}_+$, then $(T_{b,s})_{b \in B}$ is continuous on $B$ in the operator norm for all $s \in [-t,t]$.

**Proof** For any $b_1, b_2 \in B$, the linear map $T_{b_1} - T_{b_2} \in \text{Op}^0((H^s)_{s \in \mathbb{R}})$. According to Lemma 3.5 we have

$$\|T_{b_1} - T_{b_2}\|_{t,-t} = \|T_{b_1,t} - T_{b_2,t}\|_{t,t},$$

and

$$\|T_{b_1,s} - T_{b_2,s}\|_{s,s} \leq \left( \|T_{b_1,s_1} - T_{b_2,s_1}\|_{s_1,s_1} \right)^{s-s_0/s_1-s_0} \left( \|T_{b_1,s_0} - T_{b_2,s_0}\|_{s_0,s_0} \right)^{s_1-s},$$

where $s_0 \leq s \leq s_1$ and we use the situation $s_0 := -t, s_1 := t$. So for any $s \in [-t,t]$, the continuity of $(T_{b,s})_{b \in B}$ on $B$ in operator norm follows.

$\square$

For the chain of Sobolev spaces over our closed manifold $\Sigma$ and $s_0 < s_1$ we set $E_0 := H^{s_0}(\Sigma; E^{td})$ and $E_1 := H^{s_1}(\Sigma; E^{td})$. We exploit that the Sobolev spaces are Hilbert (or Hilbertable) spaces and admit a densely defined self-adjoint positive operator $\Lambda$ in $E_0$ with domain $\mathcal{D}(\Lambda) = E_1$.

**Proposition 3.3 (Interpolation between Sobolev spaces)** For each $s \in ]s_0,s_1[$ the Sobolev space $H^s(\Sigma; E^{td})$ is an interpolation space between $E_1 := H^{s_1}(\Sigma; E^{td})$ and $E_0 := H^{s_0}(\Sigma; E^{td})$ of exponent

$$\theta(s) = \frac{s_1 - s}{s_1 - s_0}.$$

More precisely, we have for all $\theta \in ]0,1[$ and corresponding $s = (1-\theta)s_1 + \theta s_0$:

1. Identifying Sobolev spaces with interpolation spaces, [23] Definition 1.2.1 and Section 1.7.1: $H^s(\Sigma; E^{td}) = \mathcal{D}(\Lambda^{1-\theta}) = [E_1, E_0]_\theta$ with equivalent norms. The norm on $[E_1, E_0]_\theta$ is equivalent to the graph norm of $\Lambda^{1-\theta}$, i.e., $\left( \|u\|^2_{E_0} + \|\Lambda^{1-\theta} u\|^2_{E_0} \right)^{1/2}$.

2. Interpolation property of (Sobolev) norms, [23] Proposition 1.2.3: There exists a constant $c$ such that $\|u\|_{[E_1, E_0]_\theta} \leq c \|u\|^{1-\theta}_{E_1} \|u\|^\theta_{E_0}$ for all $u \in E_1$. 


Remark 3.3 For our Hilbert spaces we have $[E_1, E_0]_\theta = \mathcal{D}(A^{1-\theta})$. The proof can be found in [23, Theorem 1.14.1] or [31, Section 4.2]. Then statements (1), (2) are immediate from the definition of the Sobolev spaces; for (2) see also [19, Theorem 7.22] with Grubb’s four-line proof in the Euclidean case based on the Hölder Inequality.

According to Proposition 3.3 and the facts about the chain of Sobolev spaces over a closed manifold (cf. Section 2.2), the family $(H^s(\Sigma; E^{2d}))_{s \in \mathbb{R}}$ satisfies Definition 3.2.

Proof (of Theorem 1.2 for $s < \frac{d}{2}$) We set $H^s = H^s(\Sigma; E^{2d})$, $s \in \mathbb{R}$, and $T_b = C^{\infty}(A_0)$, $b \in B$ in Theorem 3.3. By the continuity results for $s \geq \frac{d}{2}$ in Section 3.1, we obtain our Main Theorem.

Appendix: Weaker conditions than Assumption (ii)

In this Appendix, we will prove that Assumption (ii) in Theorem 1.2 can be weakened a little by finer analysis above. First, we give a kind of example about special perturbations of formally self-adjoint elliptic operator:

**Theorem 3.4** Let $A$: $C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; E)$ be a formally self-adjoint elliptic operator of order $d$, i.e., $A = A^t$. Denote by $I$: $E \to E$ the identity bundle map. Then for any $s \in \mathbb{R}$, the family of $L^2$-orthogonalized Calderón projections $(\mathcal{C}_{\nu}(A - bI))_{b \in \mathbb{R}}$ is continuous at $b = 0$ in the operator norm of the corresponding Sobolev space $H^s(\Sigma; E^{2d})$.

**Proof** According to Theorem 3.3 and Proposition 3.3, we only need to consider the case $s \geq \frac{d}{2}$. Then by Corollary 2.3, Lemmas 3.3 and 3.4, we only need to prove for $s \geq \frac{d}{2}$

$$\lim_{b \to 0} \delta(b) \ker(A_{s+\frac{d}{2}} - bI) + \ker(\rho^d), \ker(A_{s+\frac{d}{2}} + \ker(\rho^d)) = 0,$$  \hspace{1cm} (3.19)

where $\ker(A_{s+\frac{d}{2}} - bI) + \ker(\rho^d) = (\rho^d)^{-1}(\text{im} C_s(A - bI))$, $\ker(A_{s+\frac{d}{2}} - bI)$ and $\ker(\rho^d)$ are all closed subspaces of $H^{s+\frac{d}{2}}(\mathcal{M}; E)$.

Let $s \geq \frac{d}{2}$. Since $\ker(A_{s+\frac{d}{2}} - bI) = \mathcal{G}(A_{s+\frac{d}{2}}) \cap (H^{s+\frac{d}{2}}(\mathcal{M}; E) \times \{0\})$ and $\mathcal{G}(A_{s+\frac{d}{2}}) + H^{s+\frac{d}{2}}(\mathcal{M}; E) \times \{0\} = H^{s+\frac{d}{2}}(\mathcal{M}; E) \times \text{im} A_{s+\frac{d}{2}}$ is closed, by [9] Proposition A.3.5a, we have

$$\delta(\ker(A_{s+\frac{d}{2}} - bI), \ker(A_{s+\frac{d}{2}})) \leq \frac{2\delta(\mathcal{G}(A_{s+\frac{d}{2}} - b), \mathcal{G}(A_{s+\frac{d}{2}}))}{\gamma(\mathcal{G}(A_{s+\frac{d}{2}}), H^{s+\frac{d}{2}}(\mathcal{M}; E) \times \{0\})}.$$  \hspace{1cm} (3.20)

So we get

$$\lim_{b \to 0} \delta(b) \ker(A_{s+\frac{d}{2}} - bI), \ker(A_{s+\frac{d}{2}}) = 0.$$  \hspace{1cm} (3.21)

Again by [9] Proposition A.3.5a, we have

$$\lim_{b \to 0} \delta(b) \cap \text{im} A_{s+\frac{d}{2}} - bI, \ker(A_{s+\frac{d}{2}} \cap \text{im} A_{s+\frac{d}{2}}) = 0.$$  \hspace{1cm} (3.22)
Since $C := C^{\alpha t}(A)$ is a well-posed boundary condition, by Lemma 3.2, we have $\ker(A_{s+\frac{d}{2}} - bI) \subseteq \ker(A_{s+\frac{d}{2}}) \cap H^{s+\frac{d}{2}}(\mathcal{M}; E) = \ker(A_{s+\frac{d}{2}}) \cap H^{s+\frac{d}{2}}(\mathcal{M}; E) = \ker(A_{s+\frac{d}{2}}) = \ker(A_{s+\frac{d}{2}})$. 

So for $b \neq 0$, 

$$\ker(A_{s+\frac{d}{2}} - bI) \subseteq \ker(A_{s+\frac{d}{2}}) \cap \ker \rho^{d} = \{0\}. \tag{3.22}$$

By Lemma 3.2, we also have 

$$\ker(A_{s+\frac{d}{2}}) \cap \ker \rho^{d} = \{0\}. \tag{3.23}$$

By Lemma 1.4, (3.22), (3.21) and (3.23), we have 

$$\delta(\ker(A_{s+\frac{d}{2}} - bI) \cap \ker \rho^{d}) \leq \frac{\delta(\ker(A_{s+\frac{d}{2}} - bI), \ker(A_{s+\frac{d}{2}}))}{\gamma(\ker(A_{s+\frac{d}{2}} - bI), \ker \rho^{d})},$$

and 

$$\liminf_{\delta \neq b \to 0} \gamma(\ker(A_{s+\frac{d}{2}} - bI) \cap \ker \rho^{d}) \geq \gamma(\ker(A_{s+\frac{d}{2}}) \cap \ker \rho^{d}) > 0. \tag{3.24}$$

So together with (3.20), we get (3.19). \hfill \Box

In general, we will show that Assumption (ii) in Theorem 1.2 can be weakened to Assumption (ii') in the following Theorem 3.6. Let $B, \mathcal{M}, \Sigma, E, F, d, (A_b)_{b \in B}$ be given as in Notation 1.1. For $s \geq \frac{d}{2}$, fix $b_0 \in B$ and let 

$$A_{b_0, \min}^{-1} (\ker A_{b_0, \min}^{-1}) = \{ u \in H^{s+\frac{d}{2}}(\mathcal{M}; E) \mid A_{b_0, \min}^{-1} u \in \ker A_{b_0, \min}^{-1} \}, \tag{3.25}$$

$$A_{b_0, \min}^{-1} (\ker A_{b_0, \min}^{-1}) = \{ u \in H^{s+\frac{d}{2}}(\mathcal{M}; F) \mid A_{b_0, \min}^{-1} u \in \ker A_{b_0, \min}^{-1} \}. \tag{3.26}$$

Clearly, 

$$\ker A_{b, \min}^{-1} \subseteq (A_{b_0, \min}^{-1})^{-1} (\ker A_{b_0, \min}^{-1}), \quad \ker A_{b, \min}^{-1} \subseteq (A_{b_0, \min}^{-1})^{-1} (\ker A_{b_0, \min}^{-1}). \tag{3.27}$$

Without Assumption (ii), we still have 

**Lemma 3.6** Let $s \geq \frac{d}{2}$. Assumption (3.25), i.e., that the family $(A_{b, \min}^{-1})_{b \in B}$ is continuous in the operator norm, implies 

$$\delta(\ker A_{b_0, \min}^{-1} (\ker A_{b_0, \min}^{-1})) \to 0, \quad \text{when} \ b \to b_0. \tag{3.28}$$
\textbf{Proof} Assumption 3.1 implies that the graphs \((\mathcal{G}(A_{b,s+\frac{d}{2}}))_{b \in B}\) make a continuous family of closed linear subspaces of \(H^{s+\frac{d}{2}}(\mathcal{M}; E) \times H^{s+\frac{d}{2}}(\mathcal{M}; F)\). For \(s \geq \frac{d}{2}\) and \(b \in B\),

\[
\{(u, A_{b,s+\frac{d}{2}}u) \in H^{s+\frac{d}{2}}(\mathcal{M}; E) \times \ker A_{b_0,\text{min}}^t\} = \mathcal{G}(A_{b,s+\frac{d}{2}}) \cap (H^{s+\frac{d}{2}}(\mathcal{M}; E) \times \ker A_{b_0,\text{min}}^t).
\]

By Lemma 3.2 \(\mathcal{G}(A_{b,s+\frac{d}{2}}) \cap (H^{s+\frac{d}{2}}(\mathcal{M}; E) \times \ker A_{b_0,\text{min}}^t) = \ker A_{b_0,s+\frac{d}{2}} \times \{0\}\).

Since the following proof holds for any \(s \geq \frac{d}{2}\), we fix an \(s \geq \frac{d}{2}\) and write shorthand \(A_b := A_{b,s+\frac{d}{2}}\), \(X := H^{s+\frac{d}{2}}(\mathcal{M}; E)\), \(Y := H^{s+\frac{d}{2}}(\mathcal{M}; F)\).

First, we prove that Assumption 3.1 implies

\[
\lim_{b \to b_0} \delta(\mathcal{G}(A_b) \cap (X \times \ker A_{b_0,\text{min}}^t), \mathcal{G}(A_{b_0}) \cap (X \times \ker A_{b_0,\text{min}}^t)) \to 0. \tag{3.25}
\]

By Lemma 2.8b, we just need prove that Assumption 3.1 implies

\[
\lim_{b \to b_0} \delta(\mathcal{G}(A_b) + (X \times \ker A_{b_0,\text{min}}^t), \mathcal{G}(A_{b_0}) + (X \times \ker A_{b_0,\text{min}}^t)) \to 0. \tag{3.26}
\]

In fact, for any \(b \in B\), \(\mathcal{G}(A_b) + X \times \ker A_{b_0,\text{min}}^t = X \times (\text{im} A_b + \ker A_{b_0,\text{min}}^t)\). So by Lemma 3.2 \(\mathcal{G}(A_{b_0}) + X \times \ker A_{b_0,\text{min}}^t = X \times Y\). On one hand, the closed subspace \(\mathcal{G}(A_b) + X \times \ker A_{b_0,\text{min}}^t \subset X \times Y\). On the other hand, by [24, Lemma 1.4],

\[
\delta(\mathcal{G}(A_b) + X \times \ker A_{b_0,\text{min}}^t, \mathcal{G}(A_{b_0}) + X \times \ker A_{b_0,\text{min}}^t) \leq \frac{\delta(\mathcal{G}(A_{b_0}), \mathcal{G}(A_b))}{\gamma(\mathcal{G}(A_{b_0}), X \times \ker A_{b_0,\text{min}}^t)}.
\]

So we get (3.26). Thus (3.25) holds. Then, by the definition of the gap, Assumption 3.1 and (3.24) imply (3.24).

In fact, one one hand,

\[
\delta(\ker A_{b_0}, A_b^{-1}(\ker A_{b_0,\text{min}}^t)) \leq \delta(\mathcal{G}(A_{b_0}) \cap (X \times \{0\}), \mathcal{G}(A_{b_0}) \cap (X \times \ker A_{b_0,\text{min}}^t));
\]

on the other hand,

\[
\delta(A_b^{-1}(\ker A_{b_0,\text{min}}^t), \ker A_{b_0}) \leq \sqrt{\|A_b\|^2 + 1} \cdot \delta(\mathcal{G}(A_b) \cap (X \times \ker A_{b_0,\text{min}}^t), \mathcal{G}(A_{b_0}) \cap (X \times \ker A_{b_0,\text{min}}^t)).
\]

We also have the following lemma analogous to Lemma 3.1

\textbf{Lemma 3.7} For \(s \geq \frac{d}{2}\),

\[
A_{b,s+\frac{d}{2}}(\ker A_{b_0,\text{min}}^t) \cap \ker \tilde{\rho}^d = A_{b_0,s+\frac{d}{2}}^{-1}(\ker A_{b_0,\text{min}}^t) \cap \ker \tilde{\rho}^d
\]

is finite-dimensional and consists of smooth sections.
Proof Since $\ker A_{b,\min} = \ker A_{b,\min}^t \cap \ker \rho^d$ and $\ker A_{b,\min}^t$ are both finite-dimensional, $A_{b,s+\frac{d}{2}}^{-1}(\ker A_{b,\min}^t) \cap \ker \rho^d$ is finite-dimensional for $s \geq \frac{d}{2}$. Since $\ker A_{b,\min}^t \subset C^\infty(A; F)$, by the interior regularity for elliptic operators, we have

$$A_{b,d}^{-1}(\ker A_{b,\min}^t) \cap \ker \rho^d \subset \{ u \in C^\infty(A; E) \mid A_b u \in \ker A_{b,\min}^t \text{ and } \rho^d u = 0 \}.$$ 

Obviously, we have for $s \geq \frac{d}{2}$,

$$\{ u \in C^\infty(A; E) \mid A_b u \in \ker A_{b,\min}^t \text{ and } \rho^d u = 0 \} \subset A_{b,s+\frac{d}{2}}^{-1}(\ker A_{b,\min}^t) \cap \ker \rho^d \subset A_{b,d}^{-1}(\ker A_{b,\min}^t) \cap \ker \rho^d$$

So we get the equality. \qed

Moreover, we have

**Lemma 3.8** Let $s \geq \frac{d}{2}$, (1) $\ker A_{b,\min} \subset A_{b,s+\frac{d}{2}}^{-1}(\ker A_{b,\min}^t) \cap \ker \rho^d$; (2) Assumption [9] implies that $\dim(A_{b,s+\frac{d}{2}}^{-1}(\ker A_{b,\min}^t) \cap \ker \rho^d) \leq \dim Z_{+0}(A_{b,0})$, when $b$ in a sufficient small neighbourhood of $b_0$ in $B$.

Proof (1) is obvious. (2) follows from Lemma [4.6] [9] Proposition A.3.5a and [22] Corollary IV.2.6. In fact, $Z_{+0}(A_{b,0}) = \ker A_{b,0,\min} = \ker A_{b_0,s+\frac{d}{2}} \cap \ker \rho^d$ and $\lim_{b \to b_0} \delta(A_{b,s+\frac{d}{2}}^{-1}(\ker A_{b,\min}^t) \cap \ker \rho^d, \ker A_{b_0,s+\frac{d}{2}} \cap \ker \rho^d) \to 0$. \qed

Now, we can prove

**Theorem 3.5** Assume that

(i) for $s \geq \frac{d}{2}$, the two families of bounded extensions

$$(A_{b,s+\frac{d}{2}} : H^{s+\frac{d}{2}}(\mathcal{M}; E) \to H^{s-\frac{d}{2}}(\mathcal{M}; F))_{b \in B}$$

and

$$(A_{b,s+\frac{d}{2}}^t : H^{s+\frac{d}{2}}(\mathcal{M}; F) \to H^{s-\frac{d}{2}}(\mathcal{M}; E))_{b \in B}$$

are continuous in the respective operator norms $\| \cdot \|_{s+\frac{d}{2},s-\frac{d}{2}}$, and that the family of adjusted Green’s forms (of Equation [2.7]) $(J_{b,s}^s : H^s(\Sigma; F^{\text{mod}}) \to H^s(\Sigma; E^{\text{mod}}))_{b \in B}$ is continuous in the operator norm $\| \cdot \|_{s,s}$;

(ii) $\dim(A_{b,d}^{-1}(\ker A_{b,\min}^t) \cap \ker \rho^d) = \dim Z_{+0}(A_{b,0})$ and $\dim(A_{b,d}^{-1}(\ker A_{b,\min}^t) \cap \ker \rho^d) = \dim Z_{-0}(A_{b,0})$ hold for $b$ in a neighbourhood of $b_0$ in $B$.

Then for any $s \in \mathbb{R}$, the family of $L^2$-orthogonalized Calderón projections $(C_s^{\text{orth}}(A_b))_{b \in B}$ is continuous at $b_0$ in the operator norm of the corresponding Sobolev space $H^s(\Sigma; E^{\text{mod}})$. 
Proof According to Theorem 3.3 and Proposition 3.3 we only need to prove the case \( s \geq \frac{d}{2} \). Let \( s \geq \frac{d}{2} \) in the following. By Lemmas 3.6, 3.7 and (3.23), Assumption (3.1) and \( \dim(A_{b,s}^{-1}(\ker A_{b_0, \text{min}}) \cap \ker \rho^d) = \dim Z_{+0}(A_{b_0}) \) imply

\[
\lim_{b \to b_0} \delta(A_{b,s}^{-1}(\ker A_{b_0, \text{min}}) + \ker \rho^d, \ker A_{b_0,s} + \ker \rho^d) = 0.
\]

Since \( \ker A_{b,s} + \frac{d}{2} \subset A_{b,s}^{-1}(\ker A_{b_0, \text{min}}) \), (3.27) implies

\[
\lim_{b \to b_0} \delta(\ker A_{b,s} + \frac{d}{2} + \ker \rho^d, \ker A_{b_0,s} + \ker \rho^d) \to 0.
\]

Similarly, (i) and \( \dim((A^t)_{b,s}^{-1}(\ker A_{b_0, \text{min}}) \cap \ker \rho^d) = \dim Z_{-0}(A_{b_0}) \) imply

\[
\lim_{b \to b_0} \delta(\ker A_{b_0,s} + \frac{d}{2} + \ker \rho^d, \ker A_{b_0,s} + \ker \rho^d) \to 0.
\]

According to Corollary 2.3 we have

\[
\im C^\sigma_s(A_b) = \tilde{\rho}^d(\ker A_{b,s} + \frac{d}{2}) \quad \text{and} \quad \ker C^\sigma_s(A_b) = \tilde{J}^d \rho^d(\ker A_{b,s} + \frac{d}{2}).
\]

Then applying Lemmas 3.3 and 3.4 we get the continuity of \( L^2 \)-orthogonalized Calderón projections in the operator norm of the corresponding Sobolev space \( H^s(\Sigma; E^{d/2}) \) for \( s \geq \frac{d}{2} \), then by the discussion above we get the same conclusion for all \( s \in \mathbb{R} \).

\( \square \)

Remark 3.4 (a) Theorem 3.3 can be seen as a direct corollary of Theorem 3.5. For \( A = A^t \) and \( b \in C, (A - bI)^{-1}(\ker A_{\text{min}}) = \ker (A - bI) + \ker A_{\text{min}} \). So when \( b \to 0 \) and \( b \neq 0 \), we have \( \dim((A - bI)^{-1}(\ker A_{\text{min}}) \cap \ker \rho^d) = \dim \ker A_{\text{min}} \) and \( Z_{+0}(A - bI) = \{0\} \). (b) By Lemma 3.8, Assumption (ii) in Theorem 1.2 implies Assumption (ii') in Theorem 3.5.

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