The Kurzweil integral and hysteresis

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Abstract. A hysteresis operator, called the play, with variable (possibly degenerate) characteristics, is considered in the space of right-continuous regulated functions. The Lipschitz continuity of the input-output mapping is proved by means of a new technique based on the Kurzweil integral.

Introduction
This paper is devoted to the Kurzweil integral description of a scalar rate-independent evolution variational inequality with a moving constraint in the space $G_{R}(0,T)$ of right-continuous regulated functions. In the hysteresis literature, this variational inequality defines the so-called "play operator with variable characteristic", cf. [3]. We prove here that the input-output operator is Lipschitz continuous with respect to the sup-norm in $G_{R}(0,T)$. This fact in principle can be derived from the general Lipschitz continuity result for polyhedral sweeping processes in [7]. Our objective here is to show that the Kurzweil formalism, going back to [8], see also [9, 10], provides simple and straightforward proofs. As our main tool, we derive a Kurzweil integral characterization of monotone functions (Proposition 1.9), which we believe to be of independent interest in real analysis.

The paper is divided into two sections. In Section 1 we recall some basic facts about Kurzweil integration and give a detailed proof of Proposition 1.9; the hysteresis problem itself is treated in Section 2.

1. The Kurzweil integral
In this section, we recall the definition and some basic properties of the Kurzweil integral introduced in [8] as a framework for solving ODEs with singular right-hand sides. We cite most of the results without proof, and an interested reader can find more information in [5, 9, 10]. Proposition 1.9, however, which plays an important role in the theory of Kurzweil integral variational inequalities, seems to be new in this setting and its detailed proof is given at the end of this section.

The original definition in [8] is not suitable for an integral formulation of discontinuous evolution variational inequalities, and this is why the Young integral was used instead in [2, 6]. The extension of the Kurzweil integral in [5] contains, however, the Young integral as a special case, preserving the advantage of the easy Kurzweil formalism. Here, we deal only with right-continuous evolution processes, and Definition 1.1 below, which goes back to [9], turns out to be sufficient for our purposes.
The basic concept in the Kurzweil integration theory is that of a \( \delta \)-fine partition. Consider a nondegenerate closed interval \([a, b] \subset \mathbb{R}\), and denote by \( \mathcal{D}_{a,b} \) the set of all divisions of the form
\[
d = \{t_0, \ldots, t_m\}, \quad a = t_0 < t_1 < \ldots < t_m = b.
\] (1.1)
With a division \( d = \{t_0, \ldots, t_m\} \in \mathcal{D}_{a,b} \) we associate partitions \( \mathcal{D} \) defined as
\[
\mathcal{D} = \{ (\tau_j, [t_{j-1}, t_j]) ; j = 1, \ldots, m \} ; \quad \tau_j \in [t_{j-1}, t_j] \quad \forall j = 1, \ldots, m.
\] (1.2)
We define the set \( \Gamma(a, b) := \{ \delta : [a, b] \to \mathbb{R} ; \delta(t) > 0 \quad \text{for every} \quad t \in [a, b] \} \).
(1.3)
An element \( \delta \in \Gamma(a, b) \) is called a gauge. For \( t \in [a, b] \) and \( \delta \in \Gamma(a, b) \) we denote
\[
I_\delta(t) := [t - \delta(t), t + \delta(t)].
\] (1.4)
**Definition 1.1** Let \( \delta \in \Gamma(a, b) \) be a given gauge. A partition \( D \) of the form (1.2) is said to be \( \delta \)-fine if for every \( j = 1, \ldots, m \) we have
\[
\tau_j \in [t_{j-1}, t_j] \subset I_\delta(\tau_j),
\]
and the following implications hold:
\[
\tau_j = t_{j-1} \Rightarrow j = 1, \quad \tau_j = t_j \Rightarrow j = m.
\]
The set of all \( \delta \)-fine partitions is denoted by \( \mathcal{F}_\delta(a, b) \).

It is easy to see that \( \mathcal{F}_\delta(a, b) \) is nonempty for every \( \delta \in \Gamma(a, b) \); this follows e.g. from [4, Lemma 1.2].

For given functions \( f, g : [a, b] \to \mathbb{R} \) and a partition \( D \) of the form (1.2) we define the Kurzweil integral sum \( K_D(f, g) \) by the formula
\[
K_D(f, g) = \sum_{j=1}^{m} f(\tau_j) (g(t_j) - g(t_{j-1})).
\] (1.5)
**Definition 1.2** Let \( f, g : [a, b] \to \mathbb{R} \) be given. We say that \( J \in \mathbb{R} \) is the Kurzweil integral over \([a, b]\) of \( f \) with respect to \( g \) and denote
\[
J = \int_{a}^{b} f(t) \, dg(t),
\] (1.6)
if for every \( \varepsilon > 0 \) there exists \( \delta \in \Gamma(a, b) \) such that for every \( D \in \mathcal{F}_\delta(a, b) \) we have
\[
|J - K_D(f, g)| \leq \varepsilon.
\] (1.7)
Using the fact that the implication
\[
\delta \leq \min\{\delta_1, \delta_2\} \quad \Rightarrow \quad \mathcal{F}_\delta(a, b) \subset \mathcal{F}_{\delta_1}(a, b) \cap \mathcal{F}_{\delta_2}(a, b)
\] (1.8)
holds for every \( \delta, \delta_1, \delta_2 \in \Gamma(a, b) \), we easily check that the value of \( J \) in Definition 1.2 is uniquely determined.

We list below in Propositions 1.3, 1.4 some standard properties common to most integral concepts.
Proposition 1.3 Let \( f, f_1, f_2, g, g_1, g_2 : [a, b] \to \mathbb{R} \) be any functions. Then the following implications hold.

(i) If \( \int_a^b f_1(t) \, dg(t), \int_a^b f_2(t) \, dg(t) \) exist, then \( \int_a^b (f_1 + f_2)(t) \, dg(t) \) exists and
\[
\int_a^b (f_1 + f_2)(t) \, dg(t) = \int_a^b f_1(t) \, dg(t) + \int_a^b f_2(t) \, dg(t). \tag{1.9}
\]

(ii) If \( \int_a^b f(t) \, dg_1(t), \int_a^b f(t) \, dg_2(t) \) exist, then \( \int_a^b f(t) \, d(g_1 + g_2)(t) \) exists and
\[
\int_a^b f(t) \, d(g_1 + g_2)(t) = \int_a^b f(t) \, dg_1(t) + \int_a^b f(t) \, dg_2(t). \tag{1.10}
\]

(iii) If \( \int_a^b f(t) \, d(g)(t) \) exist, then \( \int_a^b f(t) \, d(\lambda g)(t) \) exist for every constant \( \lambda \in \mathbb{R} \), and
\[
\int_a^b \lambda f(t) \, d(g)(t) = \lambda \int_a^b f(t) \, d(g)(t). \tag{1.11}
\]

Proposition 1.4 Let \( f, g : [a, b] \to \mathbb{R} \) be given functions and let \( s \in ]a, b[ \) be given.

(i) Assume that \( \int_a^s f(t) \, dg(t) \) exists. Then \( \int_a^s f(t) \, dg(t), \int_a^b f(t) \, dg(t) \) exist.

(ii) Assume that \( \int_a^s f(t) \, dg(t), \int_a^b f(t) \, dg(t) \) exist. Then \( \int_a^b f(t) \, dg(t) \) exists and
\[
\int_a^b f(t) \, dg(t) = \int_a^s f(t) \, dg(t) + \int_s^b f(t) \, dg(t). \tag{1.12}
\]

In order to preserve the consistency of (1.12) also in the limit cases \( s = a \) and \( s = b \), we set
\[
\int_a^a f(t) \, dg(t) = 0 \quad \forall s \in [a, b], \forall f, g : [a, b] \to \mathbb{R}. \tag{1.13}
\]

Recall that a function \( f : [a, b] \to \mathbb{R} \) is said to be regulated if for every \( t \in [a, b] \) there exist both one-sided limits \( f(t^+), f(t^-) \in \mathbb{R} \) with the convention \( f(a^-) = f(a), f(b^+) = f(b) \), see [1]. Obviously, the set of discontinuity points of every regulated function is at most countable.

In agreement with [10], we denote by \( G(a, b) \) the set of all regulated functions \( f : [a, b] \to \mathbb{R} \). Let us introduce in \( G(a, b) \) a system of seminorms
\[
\|f\|_{[s,t]} := \sup \{|f(\tau)| : \tau \in [s,t]\} \tag{1.14}
\]
for any subinterval \([s, t] \subset [a, b]\). Indeed, \( \|\|_{[a,b]} \) is a norm. With this norm, \( G(a, b) \) becomes a Banach space. Let us note that the space \( C[a, b] \) of continuous functions \( f : [a, b] \to \mathbb{R} \) is a closed subspace of \( G(a, b) \) with respect to the norm \( \|\|_{[a,b]} \). Moreover, every regulated function can be uniformly approximated by step functions of the form

\[
w(t) = \sum_{k=0}^{m} c_k \chi_{[t_{k-1}, t_k]}(t) + \sum_{k=1}^{m} c_k \chi_{[t_k, t_{k+1}]}(t), \quad t \in [a, b], \tag{1.15}
\]
where \( d = \{t_0, \ldots, t_m\} \in \mathcal{D}_{a,b} \) is a given division, \( \chi_A \) for \( A \subset [a, b] \) is the characteristic function of the set \( A \), and \( c_0, \ldots, c_m, c_1, \ldots, c_m \) are given real numbers. We see in particular that the space \( BV(a, b) \) of all functions of bounded variation on \([a, b]\) is contained as a dense subset in \( G(a, b) \). In the next section, we will restrict ourselves to the spaces \( G_R(a, b), BV_R(a, b) \) of right-continuous functions from \( G(a, b), BV(a, b) \), respectively.
Remark 1.5 Proposition 1.4 needs some comment. Whenever we integrate functions \( f, g \) defined in \([a, b]\) over an interval \([r, s] \subset [a, b]\), we implicitly consider their restrictions \( f|_{[r, s]}, g|_{[r, s]} \). In particular, in case of regulated functions, we have e.g. \( f|_{[r, s]}(s) = f(s) \), \( f|_{[r, s]}(s-) = f(r) \).

Note that we deal here with functions that are defined for all \( t \in [a, b] \). The concept of “almost everywhere” is meaningless here.

The following explicit formulas can easily be derived from the definition.

**Proposition 1.6** For every \( f : [a, b] \to \mathbb{R}, g \in G(a, b), a \leq r \leq b \), we have

(i) \( \int_{a}^{b} \chi_{[r]} (t) \, dg(t) = g(r+) - g(r-) \),

(ii) \( \int_{a}^{b} f(t) \, d \left( \chi_{\{r\}} \right) (t) = \begin{cases} 0 & \text{if } r \in ]a, b[, \\ -f(a) & \text{if } r = a, \\ f(b) & \text{if } r = b, \end{cases} \)

(iii) \( \int_{a}^{b} \chi_{[r,s]} (t) \, dg(t) = g(s-) - g(r+) \quad \forall s \in ]r, b[ \),

(iv) \( \int_{a}^{b} f(t) \, d \left( \chi_{[r,s]} \right) (t) = f(r) - f(s) \quad \forall s \in ]r, b[ \).

We see in particular that the integral \( \int_{a}^{b} f(t) \, dg(t) \) exists whenever one of the functions \( f, g \) is a step function and the other one is regulated. By a density argument, we obtain the following result.

**Theorem 1.7** (Properties of the Kurzweil integral)

(i) If \( f \in G(a, b) \) and \( g \in BV(a, b) \), then \( \int_{a}^{b} f(t) \, dg(t) \) exists and satisfies the estimate

\[ \left| \int_{a}^{b} f(t) \, dg(t) \right| \leq \|f\|_{[a,b]} \text{Var}_{[a,b]} g. \quad (1.16) \]

(ii) If \( f \in BV(a, b) \) and \( g \in G(a, b) \), then \( \int_{a}^{b} f(t) \, dg(t) \) exists and satisfies the estimate

\[ \left| f(a) g(a) + \int_{a}^{b} f(t) \, dg(t) \right| \leq \left( |f(b)| + \text{Var}_{[a,b]} f \right) \|g\|_{[a,b]} \quad (1.17) \]

(iii) For every \( f \in G(a, b), g \in BV(a, b) \) we have the integration-by-parts formula

\[ \int_{a}^{b} f(t) \, dg(t) + \int_{a}^{b} g(t) \, df(t) = f(b) g(b) - f(a) g(a) + \sum_{t \in [a, b]} \left( \left( f(t) - f(t-) \right) (g(t) - g(t-)) - \left( f(t+) - f(t) \right) (g(t+) - g(t)) \right). \quad (1.18) \]

(iv) If \( f_{n} \in G(a, b) \) and \( g_{n} \in BV(a, b) \) are such that \( \|f_{n} - f\|_{[a,b]} \to 0, \|g_{n} - g\|_{[a,b]} \to 0 \) as \( n \to \infty \), and \( \text{Var}_{[a,b]} g_{n} \leq C \) independently of \( n \), then

\[ \lim_{n \to \infty} \int_{a}^{b} f_{n}(t) \, dg_{n}(t) = \int_{a}^{b} f(t) \, dg(t). \quad (1.19) \]
Corollary 1.8 For every $g \in BV(a, b)$ we have
\[
\int_a^b g(t+) \, dg(t) = \frac{1}{2} \left( |g(b)|^2 - |g(a)|^2 \right) + \frac{1}{2} \sum_{t \in [a, b]} |g(t+) - g(t-)|^2. \tag{1.20}
\]

We conclude this section by a Kurzweil integral characterization of monotone functions, which will be referred to several times in the next section in the context of Kurzweil integral variational inequalities.

Proposition 1.9 Let $f \in G(a, b)$ and $g \in BV(a, b)$ be such that

(i) $f(t) > 0$ for every $t \in [a, b]$;

(ii) $\int_s^t f(\tau) \, dg(\tau) \geq 0$ for every $a \leq s < t \leq b$.

Then $g$ is nondecreasing in $[a, b]$.

The proof of this statement will be divided into several steps. Let us start with an easy convergence result.

Lemma 1.10 For every $f \in G(a, b)$ and $g \in BV(a, b)$ we have
\[
\lim_{h \to 0+} \int_{t-h}^{t+h} f(\tau) \, dg(\tau) = f(t) \, (g(t+) - g(t^)) \quad \forall t \in [a, b]. \tag{1.21}
\]

Proof. Let $t \in [a, b]$ be fixed. For $\tau \in [a, b]$ put $f_1(\tau) = f(t+), \; f_2(\tau) = (f(t)-f(t+)) \, \chi_{(t)}(\tau), \; f_3(\tau) = f(\tau) - f_1(\tau) - f_2(\tau)$. Let $\varepsilon > 0$ be arbitrary, and let $h \in ]0, b-t[$ be such that
\[
|f(\tau) - f(t+)| \leq \varepsilon, \quad |g(\tau) - g(t+)| \leq \varepsilon \quad \text{for } \tau \in [t, t+h] .
\]

Then $|f_3(\tau)| \leq \varepsilon$ for all $\tau \in [t, t+h]$, hence, by (1.16),
\[
\int_{t-h}^{t+h} f_3(\tau) \, dg(\tau) \leq \varepsilon \, \text{Var}_{[a,b]} g. \tag{1.23}
\]

Furthermore, we have by Remark 1.5 and Proposition 1.6 that
\[
\int_t^{t+h} f_1(\tau) \, dg(\tau) = f(t+) \, (g(t+h) - g(t)), \tag{1.24}
\]
\[
\int_t^{t+h} f_2(\tau) \, dg(\tau) = (f(t) - f(t+)) \, (g(t) - g(t+)), \tag{1.25}
\]

hence, as a consequence of (1.24)–(1.25),
\[
\left| \int_t^{t+h} (f_1(\tau) + f_2(\tau)) \, dg(\tau) - f(t) \, (g(t) - g(t+)) \right| = |f(t+) \, (g(t+h) - g(t+))| \leq \varepsilon \|f\|_{[a,b]}, \tag{1.26}
\]

and (1.21) follows from (1.23) and (1.26). The proof of (1.22) is similar.

Under the hypotheses of Proposition 1.9, we thus have in particular
\[
g(t-) \leq g(t) \leq g(t+) \quad \forall t \in [a, b], \tag{1.27}
\]

As a next step, we prove the following lemma.
Lemma 1.11 Let the hypotheses of Proposition 1.9 hold. Then for every non-negative function $w \in G(a,b)$ we have

$$
\int_a^b w(\tau) f(\tau) \, dg(\tau) \geq 0. \tag{1.28}
$$

Proof. It suffices to assume that $w$ is a step function of the form (1.15). Indeed, for an arbitrary function $w \in G(a,b)$ we find a sequence $\{w_n\}$ of step functions such that $\|w-w_n\|_{[a,b]} \to 0$ as $n \to \infty$, and use Proposition 1.7(iv).

Hence, let $w$ be as in (1.15), and let us consider any $h > 0$ such that $t_k - h < t_k - h$ for all $k = 1, \ldots, m$. We then have

$$
\int_a^b w(\tau) f(\tau) \, dg(\tau) = \sum_{k=1}^m \left( \int_{t_k-1+h}^{t_k} + \int_{t_k-1+h}^{t_k-h} + \int_{t_k} \right) w(\tau) f(\tau) \, dg(\tau),
$$

where

$$
\int_{t_k-1+h}^{t_k} w(\tau) f(\tau) \, dg(\tau) = c_k \int_{t_k-1+h}^{t_k-h} f(\tau) \, dg(\tau) \geq 0
$$

for all $h$ and $k$ by hypothesis. Letting $h$ tend to $0+$ we obtain from Lemma 1.10 and formula (1.27) that

$$
\int_a^b w(\tau) f(\tau) \, dg(\tau) \geq \sum_{k=1}^m (c_k-1 f(t_k-1)-(g(t_k-1)+g(t_k-1)) + c_k f(t_k)(g(t_k)-g(t_k))) \geq 0. \tag{1.29}
$$

Lemma 1.12 Let the hypotheses of Proposition 1.9 hold, and let in addition $f(t+) > 0$, $f(t-) > 0$ for all $t \in [a,b]$. Then $g$ is nondecreasing in $[a,b]$.

Proof. There exists some $r > 0$ such that $f(t) \geq r$ for all $t \in [a,b]$. Hence, we may use Lemma 1.11 with the function

$$
w(\tau) = \frac{1}{f(\tau)} \chi_{[s,t]}(\tau)
$$

for any choice of $]s,t[ \subset [a,b]$, and the desired inequality

$$
0 \leq \int_a^b w(\tau) f(\tau) \, dg(\tau) = \int_b^t 1 \, dg(\tau) - \int_s^t 1 \, dg(\tau) - \int_s^t \chi_{[s,t]} \, dg(\tau) = g(t) - g(s) - (g(s) - g(t)) - (g(t) - g(t)) \leq g(t) - g(s),
$$

follows easily.

We now pass to the proof of Proposition 1.9.

Proof of Proposition 1.9. We define the sets

$$
\mathcal{N}_+ = \{ t \in [a,b] : f(t+) = 0 \}, \quad \mathcal{N}_- = \{ t \in [a,b] : f(t-) = 0 \}, \quad \mathcal{N} = \mathcal{N}_+ \cup \mathcal{N}_-.
$$

For every $t \in \mathcal{N}$, we have either $f(t) > f(t-)$ or $f(t) > f(t+)$, hence $\mathcal{N}$ is at most countable. Furthermore, $\mathcal{N}$ is closed. Indeed, if for instance $\{t_i\}$ is a sequence in $\mathcal{N}$, $t_i \not\to t$, then in a neighborhood of each $t_i$ there exists $\hat{t}_i$ such that $f(\hat{t}_i) < 1/i$, $\hat{t}_i \not\to t$, hence $f(t-) = 0$. A similar argument works for $t_i \searrow t$. 

All elements of \( \mathcal{N} \) can be ordered into a sequence \( \{s_j : j \in \mathbb{N}\} \). Let \( \varepsilon > 0 \) be given. For every \( j \in \mathbb{N} \) we find \( h_j > 0 \) such that
\[
\begin{align*}
|g(\tau) - g(s_j +)| & \leq \varepsilon 2^{-j - 1} & \text{for } \tau \in [s_j, s_j + h_j], \\
|g(\tau) - g(s_j -)| & \leq \varepsilon 2^{-j - 1} & \text{for } \tau \in [s_j - h_j, s_j].
\end{align*}
\]
(1.29)

The set \( \mathcal{N} \) is compact, we can therefore find a finite covering
\[
\mathcal{N} \subset \bigcup_{k=0}^m [s_{jk} - h_{jk}, s_{jk} + h_{jk}],
\]
(1.30)
with \( s_{j0} < s_{j1} < \ldots < s_{jm} \). We may assume that
\[
s_{jk-1} - h_{jk-1} < s_j - h_{jk}, \quad s_{jk-1} + h_{jk-1} < s_j + h_{jk}
\]
for all \( k = 1, \ldots, m \); otherwise we remove the redundant intervals from the covering (1.30). We now set \( a_0 = a \), \( b_m = b \), and for \( k = 1, \ldots, m \) choose
\[
\begin{align*}
b_{k-1} & = s_{jk-1} + h_{jk-1}, \quad a_k = s_j - h_{jk}, \\
b_k & = a_k \in [s_{jk} - h_{jk}, s_{jk-1} + h_{jk-1} \cap s_{jk}, s_{jk}] \mathcal{N}
\end{align*}
\]
if \( s_{jk-1} + h_{jk-1} \leq s_j - h_{jk} \), \( k = 1, \ldots, m \) choose
\[
\begin{align*}
b_{k-1} & = s_{jk-1} + h_{jk-1}, \quad a_k = s_j - h_{jk}, \\
b_k & = a_k \in [s_{jk} - h_{jk}, s_{jk-1} + h_{jk-1} \cap s_{jk}, s_{jk}] \mathcal{N}
\end{align*}
\]
if \( s_{jk-1} + h_{jk-1} > s_j - h_{jk} \).
(1.31)

Then
\[
\mathcal{N} \subset [a_0, b_0] \cup \bigcup_{k=1}^m [a_k, b_k] \cup [a_m, b_m].
\]

For \( a_k \leq s < t \leq b_k \), \( k = 0, \ldots, m \), we have by (1.27) and (1.29) that \( g(s) \leq g(t) + \varepsilon 2^{-jk} \). On the other hand, for \( b_{k-1} \leq s < t \leq a_k \), \( k = 1, \ldots, m \), it follows from Lemma 1.12 that \( g(s) \leq g(t) \). Consequently, for all \( a \leq s < t \leq b \) we have
\[
g(s) \leq g(t) + \varepsilon \sum_{k=0}^m 2^{-jk} \leq g(t) + \varepsilon.
\]

Since \( \varepsilon > 0 \) has been chosen arbitrarily, we obtain the assertion. \( \blacksquare \)

2. A variational inequality

We now consider a fixed interval \([0, T]\) and two (input) functions \( u, r \in G_R(0, T) \) such that \( r(t) \geq 0 \) for all \( t \in [0, T] \), and an initial condition \( x_0 \in [-r(0), r(0)] \). The problem consists in finding the output \( \xi \in G_R(0, T) \) such that
\[
\begin{align*}
|u(t) - \xi(t)| & \leq r(t) & \forall t \in [0, T], \\
\int_0^T (u(t) - \xi(t) - y(t)) \, d\xi(t) & \geq 0 & \forall y \in G(0, T), \quad |y(t)| \leq r(t) & \forall t \in [0, T], \\
u(0) - \xi(0) & = x_0.
\end{align*}
\]
(2.1)

This extends the model considered in \([3]\) in two respects: we admit discontinuous inputs, and \( r(t) \) is allowed to vanish at any point. The setting of \([3]\) however contains additional nonlinearities that we neglect here for simplicity.

We first construct a solution to Problem (2.1) if the inputs are step functions of the form
\[
u(t) = \sum_{k=1}^m u_{k-1} \chi_{[t_{k-1}, t_k]}(t) + u_m \chi_{[T]}(t), \quad r(t) = \sum_{k=1}^m r_{k-1} \chi_{[t_{k-1}, t_k]}(t) + r_m \chi_{[T]}(t),
\]
(2.2)
where $0 = t_0 < t_1 < \ldots < t_m = T$ is a given division, and $u_k, r_k$ for $k = 0, \ldots, m$ are given real numbers. To this end, we use the projection $Q_c(x) = \max\{-c, \min\{x, c\}\}$ for $c \geq 0$ and its complement $P_c(x) = x - Q_c(x)$ for $x \in \mathbb{R}$, and claim that the solution $\xi$ is given by the formula

$$
\xi(t) = \sum_{k=1}^{m} \xi_{k-1} \chi_{[u_{k-1}, u_k]}(t) + \xi_m \chi_{[T]}(t)
$$

with $\xi_0 = u_0 - x_0$ and

$$
\xi_k = \xi_{k-1} + P_{r_k}(u_k - \xi_{k-1}) \quad \text{for } k = 1, \ldots, m.
$$

Indeed, we have $u_k - \xi_k = Q_{r_k}(u_k - \xi_{k-1}) \in [-r_k, r_k]$. Using Proposition 1.6, we can evaluate the integral in (2.1) explicitly and obtain

$$
\int_0^T (u(t) - \xi(t) - y(t)) \, d\xi(t) = \sum_{k=1}^{m} (\xi_k - \xi_{k-1})(u_k - \xi_k - y(t_k))
$$

for every regulated function $y$ such that $|y(t)| \leq r(t)$ for all $t$. For all $x \in \mathbb{R}$ and $|y| \leq c$ we have $P_c(x)(Q_c(x) - y) \geq 0$. Putting $z_k = u_k - \xi_{k-1}$, we thus can rewrite (2.5) as

$$
\int_0^T (u(t) - \xi(t) - y(t)) \, d\xi(t) = \sum_{k=1}^{m} P_{r_k}(z_k)(Q_{r_k}(z_k) - y(t_k)) \geq 0,
$$

hence (2.1) is satisfied.

To extend the set admissible inputs, we state and prove the main result of this section on Lipschitz continuity of the input-output mapping defined by (2.1).

**Theorem 2.1** Let $u_i, r_i \in G_R(0, T)$ and initial conditions $x_i^0 \in [-r_i(0), r_i(0)]$ be given, $i = 1, 2$. Let $\xi_1, \xi_2$ be corresponding solutions to (2.1), and assume that $\xi_i \in BV_R(0, T)$ for $i = 1, 2$. Then for every $t \in [0, T]$ we have

$$
|\xi_1(t) - \xi_2(t)| \leq \max\{|\xi_1(0) - \xi_2(0)|, \|u_1 - u_2\|_{[0, t]} + \|r_1 - r_2\|_{[0, t]}\}.
$$

Before passing to the proof of Theorem 2.1, we derive an easy auxiliary result.

**Lemma 2.2** Let $u, r \in G_R(0, T)$ and an initial condition $x_0 \in [-r(0), r(0)]$ be given. Let $\xi$ be a solution to (2.1), and assume that $\xi \in BV_R(0, T)$. Then for every $0 \leq s < t \leq T$ and every regulated function $y$ such that $|y(\tau)| \leq r(\tau)$ for all $\tau$, we have

$$
\int_s^T (u(\tau) - \xi(\tau) - y(\tau)) \, d\xi(\tau) \geq 0.
$$

Furthermore, if both $u$ and $r$ are constant in an interval $[t_0, t_1] \subset [0, T]$, then $\xi$ is constant in $[t_0, t_1]$.

**Proof.** Set $y^*(\tau) = y(\tau) \chi_{[s, t]}(\tau) + (u(\tau) - \xi(\tau)) \left( \chi_{[0, s]} + \chi_{[t, T]} \right)(\tau)$ for $\tau \in [0, T]$. Using Propositions 1.4, 1.6, and Remark 1.5 (note that $\xi$ is right-continuous!), we obtain that

$$
0 \leq \int_0^T (u(\tau) - \xi(\tau) - y^*(\tau)) \, d\xi(\tau)
= \int_s^T (u(\tau) - \xi(\tau) - y(\tau)) \, d\xi(\tau)
+ \int_t^T (u(\tau) - \xi(\tau) - y(\tau)) \chi_{[t, T]}(\tau) \, d\xi(\tau)
- \int_s^t (u(\tau) - \xi(\tau) - y(\tau)) \chi_{[s, t]}(\tau) \, d\xi(\tau)
= \int_s^T (u(\tau) - \xi(\tau) - y(\tau)) \, d\xi(\tau).
$$
Assume now that $u$ and $r$ are constant in $[t_0, t_1]$. For all $t \in [t_0, t_1]$ we then have
\[
\int_{t_0}^{t}(u(t_0) - \xi(t)) d\xi(t) \geq 0.
\]
We may choose $y(t) = u(t_0) - \xi(t_0)$ and use (1.20) for $g(t) = u(t_0) - \xi(t)$ to obtain
\[
0 \leq \int_{t_0}^{t}(\xi(t_0) - \xi(t)) d\xi(t) = -\frac{1}{2}|\xi(t_0) - \xi(t)|^2 - \frac{1}{2} \sum_{\tau \in [t_0, t]} |\xi(\tau) - \xi(\tau - \tau)|^2,
\]
hence $\xi(t) = \xi(t_0)$ for all $t \in [t_0, t_1]$.

The above proof shows why it was convenient to choose the test functions $y$ in Problem (2.1) in $G(0, T)$ and not in $G_R(0, T)$. More refined results in [2] however show that even the choice $y \in BV_R(0, T)$ is sufficient.

Proof of Theorem 2.1. The process defined by (2.1) is rate-independent, hence it suffices to prove that
\[
|\xi_1(t) - \xi_2(t)| \leq \max\{|\xi_1(0) - \xi_2(0)|, \|u_1 - u_2\|_{[0, T]} + \|r_1 - r_2\|_{[0, T]}\}
\]
for all $t \in [0, T]$. Set $\bar{u} = u_1 - u_2$, $\bar{r} = r_1 - r_2$, $\bar{\xi} = \xi_1 - \xi_2$, and assume that there exists $t_1 \in ]0, T]$ such that $\bar{\xi}(t_1) > \|\bar{u}\|_{[0, T]} + \|\bar{r}\|_{[0, T]}$. Putting $y(t) = Q_{r_1(t)}(u_2(t) - \xi_2(t))$ and $y(t) = Q_{r_2(t)}(u_1(t) - \xi_1(t))$ in the variational inequalities for $\xi_1$ and $\xi_2$, respectively, we obtain for all $0 \leq s < t \leq t_1$ that
\[
\int_{s}^{t}(\bar{u}(\tau) - \bar{\xi}(\tau) + P_{r_1(\tau)}(u_2(\tau) - \xi_2(\tau))) d\xi_1(\tau) \geq 0, \quad (2.11)
\]
\[
\int_{s}^{t}(\bar{u}(\tau) + \bar{\xi}(\tau) + P_{r_2(\tau)}(u_1(\tau) - \xi_1(\tau))) d\xi_2(\tau) \geq 0. \quad (2.12)
\]
We have for all $c, c' > 0$ and $x \in \mathbb{R}$ the implication $|x| \leq c' \Rightarrow |P_c(x)| \leq |c - c'|$, hence,
\[
|P_{r_1(\tau)}(u_2(\tau) - \xi_2(\tau))| \leq \|\bar{r}\|_{[0, T]}, \quad |P_{r_2(\tau)}(u_1(\tau) - \xi_1(\tau))| \leq \|\bar{r}\|_{[0, T]}
\]
for all $\tau \in [0, T]$. Letting $t \neq t_1$ in (2.11)-(2.12), and using (1.22), we obtain $\xi_1(t_1 -) = \xi_0(t_1)$, $\xi_2(t_1 -) = \xi_0(t_1)$, hence $\bar{\xi}(t_1) = \xi_0(t_1) > \|\bar{u}\|_{[0, T]} + \|\bar{r}\|_{[0, T]}$. There exists therefore an interval $[t_0, t_1] \subset [0, T]$ such that $\bar{\xi}(t) > \|\bar{u}\|_{[0, T]} + \|\bar{r}\|_{[0, T]}$ for all $t \in [t_0, t_1]$. Applying Proposition 1.9 to (2.11)-(2.12), we conclude that $\xi_1$ is nonincreasing, $\xi_2$ is nondecreasing, hence $\bar{\xi}$ is nonincreasing in $[t_0, t_1]$. Let $t^*$ be the infimum of all $t_0 \in [0, t_1]$ such that $\bar{\xi}(t) > \|\bar{u}\|_{[0, T]} + \|\bar{r}\|_{[0, T]}$ for every $t \in [t_0, t_1]$. The above argument yields that $t^* = 0$ and that $\bar{\xi}$ is nonincreasing in $[0, t_1]$, which we wanted to prove. The case $\bar{\xi}(t_1) < -\|\bar{u}\|_{[0, T]} - \|\bar{r}\|_{[0, T]}$ is similar. The proof of Theorem 2.1 is complete.

Let now $u, r \in G_R(0, T)$ be given, $r(t) \geq 0$ for all $t$, and for some fixed interval $[t_0, t_1] \subset [0, T]$ set $u_1(t) = u_2(t) = u(t)$, $r_1(t) = r_2(t) = r(t)$ for $t \in [0, t_0]$, $u_1(t) = u(t)$, $r_1(t) = r(t)$ $u_2(t) = u(t_0)$, $r_2(t) = r(t_0)$ for $t \in [t_0, t_1]$. Then Theorem 2.1 and Lemma 2.2 yield
\[
|\xi(t_1) - \xi(t_0)| \leq \|u - u(t_0)||_{[t_0, t_1]} + \|r - r(t_0)||_{[t_0, t_1]}.
\]
In particular, if both $u$ and $r$ belong to $BV_R(0, T)$, then so does $\xi$ and we have
\[
\text{Var } \xi \leq \text{Var } u + \text{Var } r.
\]
We have seen that Problem (2.1) has a unique solution $\xi$ whenever $u$ and $r$ are step functions. Since every $BV$-function can be uniformly approximated by step functions with uniformly bounded variation, we obtain, as immediate consequence of Theorems 2.1 and 1.7 (iv), the following result.

**Corollary 2.3** Problem (2.1) has a unique solution $\xi \in BV_R(0,T)$ for every $u,r \in BV_R(0,T)$, $r(t) \geq 0$ for all $t \in [0,T]$, and for every $x_0 \in [-r(0),r(0)]$. Moreover, the solution operator $P : (u,r,x_0) \mapsto \xi$ admits a Lipschitz continuous extension to $P : D \to G_R(0,T)$, where $D = \{(u,r,x_0) \in G_R(0,T) \times G_R(0,T) \times \mathbb{R} : r(t) \geq 0 \text{ for } t \in [0,T], x_0 \in [-r(0),r(0)]\}$, and inequality (2.7) holds for all $(u_i,r_i,x_i^0) \in D$, $\xi_i = P(u_i,r_i,x_i^0)$, $i = 1,2$.

We do not know if the extended operator $P : D \to G_R(0,T)$ still admits the Kurzweil integral representation (2.1), because in general, $\int^b_a f(t) \, dg(t)$ is not well defined if both $f$ and $g$ are only regulated. There are some indications that (2.1) is meaningful in this situation, too, but no proof is yet available. Things are different if $r$ is bounded from below by a positive constant. Then, similarly as in [6], $\xi$ has bounded variation and the integral formula holds. To conclude this paper, we give an elementary proof of a slightly stronger version of this fact.

**Proposition 2.4** Let $r_0 > 0$ be given, and let $(u,r,x_0) \in D$ be such that $r(t) \geq r_0$ for all $t \in [0,T]$. Then $\xi = P(u,r,x_0)$ is piecewise monotone and, in particular, belongs to $BV_R(0,T)$.

**Proof.** Since both $u$ and $r$ are regulated, there exists an integer $n^*$ with the property that for every sequence $a_1 < b_1 \leq a_2 < b_2 \leq \ldots \leq a_n < b_n$ of points in $[0,T]$, the following implication holds:

$$
|u(b_k) - u(a_k)| + |r(b_k) - r(a_k)| \geq \frac{2}{3} r_0 \quad \forall k = 1,\ldots,n \quad \Rightarrow \quad n \leq n^*. \quad (2.15)
$$

We claim that $\xi$ has at most $(3n^* + 1)$ monotonicity intervals.

To prove this conjecture, we tacitly consider uniformly convergent $BV$-approximations of $u$ and $r$ preserving the property (2.15), so that the variational inequality (2.1) is meaningful. Since the number of monotonicity intervals (hence the total variation) of $\xi$ remains bounded independently of the approximations, we may pass to the limit using Theorem 1.7 and obtain the assertion.

Assume that there exists $m > n^*$ and points $0 \leq t_0 < t_1 < \ldots < t_{2m} \leq T$ such that

$$
\xi(t_0) < \xi(t_1) > \xi(t_2) < \ldots > \xi(t_{2m}).
$$

We make use of the following implications:

$$
\begin{align*}
&u(\tau) - \xi(\tau) < r_0 \quad \forall \tau \in [s,t] \quad \Rightarrow \quad \xi \text{ is nonincreasing in } [s,t], \\
&u(\tau) - \xi(\tau) > -r_0 \quad \forall \tau \in [s,t] \quad \Rightarrow \quad \xi \text{ is nondecreasing in } [s,t].
\end{align*}

(2.16)
$$

Indeed, (2.16) follows from Proposition 1.9 provided we put in (2.8) $y(\tau) = \pm r_0$. Hence, the sets

$$
\begin{align*}
A_{2j-1} &= \{\tau \in [t_{2j-2},t_{2j-1}] : u(\tau) - \xi(\tau) \geq r_0\}, \\
A_{2j} &= \{\tau \in [t_{2j-1},t_{2j}] : u(\tau) - \xi(\tau) \leq -r_0\}
\end{align*}
$$

are non-empty for all $j = 1,\ldots,m$, and we may set

$$
s_k = \sup A_k, \quad k = 1,\ldots,2m. \quad (2.17)
$$

By (2.16) and by the right-continuity of $\xi$, we have $\xi(s_{2j-1}) \geq \xi(t_{j-1})$, $\xi(s_{2j}) \leq \xi(t_{2j})$ for $j = 1,\ldots,m$, hence

$$
\xi(s_1) > \xi(s_2) < \ldots > \xi(s_{2m}). \quad (2.18)
$$
and, by definition of $s_k$, we have
\[ u(s_{2j-1}) - \xi(s_{2j-1}) \geq r_0, \quad u(s_{2j}) - \xi(s_{2j}) \leq -r_0 \quad \text{for } j = 1, \ldots, m. \]

Furthermore, by (2.13), we have
\[ |\xi(s_k) - \xi(s_k^-)| \leq |u(s_k) - u(s_k^-)| + |r(s_k) - r(s_k^-)| \quad \forall k = 1, \ldots, 2m. \]

Using (2.18), we thus obtain for $j = 1, \ldots, m$ that
\[
\begin{align*}
    u(s_{2j-1}) - u(s_{2j}) &\geq 2r_0 + \xi(s_{2j-1}) - \xi(s_{2j}) \\
    &\geq 2r_0 - |u(s_{2j-1}) - u(s_{2j-1^-})| - |r(s_{2j-1}) - r(s_{2j-1^-})| \\
    &\quad - |u(s_{2j}) - u(s_{2j^-})| - |r(s_{2j}) - r(s_{2j^-})|,
\end{align*}
\]
and similarly, for $j = 2, \ldots, m$,
\[
\begin{align*}
    u(s_{2j-1}) - u(s_{2j^-}) &\geq 2r_0 - |u(s_{2j}) - u(s_{2j^-})| - |r(s_{2j}) - r(s_{2j})| \\
    &\quad - |u(s_{2j-2}) - u(s_{2j-2^-})| - |r(s_{2j-2}) - r(s_{2j-2^-})|. (2.19)
\end{align*}
\]

The set $M := \{k = 1, \ldots, 2m ; |u(s_k) - u(s_k^-)| + |r(s_k) - r(s_k^-)| \geq (2/3)r_0\}$ contains, by (2.15), at most $n^*$ elements. For $2m - 1 - 2n^*$ indices $k \in \{1, \ldots, 2m\}$ we thus have by (2.19)–(2.20) that
\[ |u(s_k^-) - u(s_k^-)| \geq 2r_0 - \frac{1}{3}r_0 = \frac{2}{3}r_0. \]

From (2.15) we conclude that $2m - 1 - 2n^* \leq n^*$, that is, $2m \leq 3n^* + 1$ as conjectured. This completes the proof of Proposition 2.4. \qed

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