Linear block and convolutional MDS codes to required rate, distance and type

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Abstract

Algebraic methods for the design of series of maximum distance separable (MDS) linear block and convolutional codes to required specifications and types are presented. Algorithms are given to design codes to required rate and required error-correcting capability and required types. Infinite series of block codes with rate approaching a given rational $R$ with $0 < R < 1$ and relative distance over length approaching $(1 - R)$ are designed. These can be designed over fields of given characteristic $p$ or over fields of prime order and can be specified to be of a particular type such as (i) dual-containing under Euclidean inner product, (ii) dual-containing under Hermitian inner product, (iii) quantum error-correcting, (iv) linear complementary dual (LCD). Convolutional codes to required rate and distance and infinite series of convolutional codes with rate approaching a given rational $R$ and distance over length approaching $2(1 - R)$ are designed. The designs are algebraic and properties, including distances, are shown algebraically. Algebraic explicit efficient decoding methods are referenced.

1 Introduction

1.1 Motivation, summary

Linear block and convolutional codes are error-correcting codes which are used extensively in many applications including digital video, radio, mobile communication, and satellite/space communications. Maximum distance separable (MDS) codes are of particular interest and dual-containing codes, which lead to the design of quantum error-correcting codes, QECCs, and linear complementary dual, LCD, codes are also of great interest with many applications. Codes for which there exist efficient decoding methods are required and necessary for applications.

This paper gives design methods for both linear block codes and convolutional codes of the highest distance possible for a particular length and rate. The design methods are then extended to particular types of codes. Types here include DC (dual-containing), QECC (quantum error-correcting codes), LCD (linear complementary dual). MDS convolutional codes are designed where MDS here means the codes attain the GSB (generalised Singleton bound, see section 1.2.1 below for definition) for convolutional codes. The methods allow the design of codes to given specifications and to design infinite series of such codes. The block linear MDS DC codes are designed under both

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Euclidean and Hermitian inner products and lead to MDS QECCs under Euclidean or Hermitian inner products.

For given (allowable) rate $R$ and given distance $(2t + 1)$ (that is, for specified error-correcting capability), design methods for MDS block linear codes with efficient decoding algorithms are given. Design methods for types of such are then derived, where ‘types’ can be DC or LCD; QECCs are obtainable from DC codes. These are further specified to be over fields of characteristic a fixed prime $p$ or over a field of prime order. In fields of prime order the arithmetic is modular arithmetic which is particularly nice and efficient. Infinite series of codes, which can be required to be DC or LCD, in which the rate approaches a given $R$ and the relative distance (ratio of distance over length) approaches $(1 - R)$ are designed. These infinite series can be specified to be codes with characteristic a given prime $p$ or to be codes over a field of prime order. Again note that QECCs are obtainable from DC codes, and if the DC code is MDS then the QECC obtained is maximum distance attainable for a QECC with those parameters.

In general the convolutional codes designed offer better distances than the equivalent block linear code of the same length and rate. Note that the convolutional codes are designed algebraically.

1.2 Background, notation

Background on coding theory may be found in [1], [2], [3], [4], [5] and many others.

The notation for linear block codes is fairly standard. Here $[n, r, d]$ denotes a linear block code of length $n$, dimension $r$ and distance $d$. The maximum distance attainable by an $[n, r]$ linear block code is $(n - r + 1)$ and this is known as the Singleton bound, see [1] or [4]. A linear code $[n, r]$ attaining the maximum distance possible is called an MDS (maximum distance separable) code. The ratio of distance over length features here and we refer to this as the relative distance, $r_{\text{dist}}$ for short, of the code. The MDS linear block codes are those with maximum error correcting capability for a given length and dimension. MacWilliams and Sloane refer to MDS codes in their book [5] as “one of the most fascinating chapters in all of coding theory”; MDS codes are equivalent to geometric objects called $n$-arcs and combinatorial objects called orthogonal arrays, [5], and are, quote, “at the heart of combinatorics and finite geometries”.

A dual-containing, DC, code $C$ is a code which contains its dual $C^\perp$; thus a DC code is a code $C$ such that $C \cap C^\perp = C^\perp$.

A linear complementary dual, LCD, code is one such that its intersection with its dual is zero, that is, it’s a code $C$ such that $C \cap C^\perp = 0$

LCD codes and DC codes are ‘supplemental’ to one another in the sense that $C$ is DC if $C \cap C^\perp = C^\perp$ and $C$ is LCD if $C \cap C^\perp = 0$. We shall see this further in action when MDS DC block linear codes are extended to LCD MDS convolutional codes and LCD MDS codes are extended to MDS DC convolutional codes.

Why DC? DC codes have been studied extensively in particular since they lead by the CSS construction to the design of quantum error-correcting codes, QECCs, see [23] and also [25]. The CSS constructions are specified as follows:

- Let $C$ be a linear block code $[n, k, d]$ over $GF(q)$ containing its dual $C^\perp$. The CSS construction derives a quantum (stabilizer) $[[n, 2k - n, \geq d]]$ code over $GF(q)$.

\[\text{There exist very few algebraic constructions for designing convolutional codes and search methods limit their size and availability, see McEliece [1] for discussion and also [10, 11, 12, 13].}\]
Let $\mathcal{D}$ be a linear block code over $GF(q^2)$ containing its Hermitian dual $\mathcal{D}^\perp$. The CSS construction derives a quantum (stabilizer) code $[[n, 2k-n, \geq d]]$ code over $GF(q^2)$.

For more details on CSS constructions of QECCs see [47, 48]; proofs of the above may also be found therein. The work of [47] follows from Rains’ work on non-binary codes [46]. As noted in for example [47] if the DC code used for the CSS construction is an MDS linear code then the quantum code obtained is a quantum MDS code which means it has the best possible distance attainable for such a quantum code.

Why LCD? LCD codes have been studied extensively in the literature. For background, history and general theory on LCD codes, consult the nice articles [14, 15, 16, 22] by Carlet, Mesnager, Tang, Qi and Pelikaan. LCD codes were originally introduced by Massey in [20, 21]. These codes have been studied amongst other things for improving the security of information on sensitive devices against side-channel attacks (SCA) and fault non-invasive attacks, see [17], and have found use in data storage and communications’ systems.

### 1.2.1 Notation for convolutional codes

Notation(s) for convolutional codes can be confusing. Different equivalent definitions are given in the literature and these are analysed nicely in [37]. The following definition is followed here. A rate $\frac{k}{n}$ convolutional code with parameters $(n, k, \delta)$ over a field $F$ is a submodule of $F[z]^n$ generated by a reduced basic matrix $G[z] = (g_{ij}) \in F[z]^{r \times n}$ of rank $r$ where $n$ is the length, $\delta = \sum_{i=1}^{k} \delta_i$ is the degree with $\delta_i = \max_{1 \leq j \leq k} \deg g_{ij}$. Also $\mu = \max_{1 \leq i \leq r} \delta_i$ is known as the memory of the code and then the code may be given with parameters $(n, k, \delta; \mu)$. The parameters $(n, r, \delta; \mu, d_f)$ are used for such a code with free (minimum) distance $d_f$. Suppose $C$ is a convolutional code in $F[z]^n$ of rank $k$. A generating matrix $G[z] \in F[z]_{k \times n}$ of $C$ having rank $k$ is called a generator or encoder matrix of $C$. A matrix $H \in F[z]_{n \times (n-k)}$ satisfying $C = \ker H = \{ v \in F[z]^n : vH = 0 \}$ is said to be a control matrix or check matrix of the code $C$.

Convolutional codes can be catastrophic or non-catastrophic; see for example [3] for the basic definitions. A catastrophic convolutional code is prone to catastrophic error propagation and is not much use. A convolutional code described by a generator matrix with right polynomial inverse is a non-catastrophic code; this is sufficient for our purposes. The designs given here for the generator matrices allow for specifying directly the control matrices and right polynomial inverses of the generator matrices.

By Rosenthal and Smarandache, [35], the maximum free distance attainable by an $(n, r, \delta)$ convolutional code is $(n-r)(\lceil \frac{\delta}{r} \rceil + 1) + \delta + 1$. The case $\delta = 0$, which is the case of zero memory, corresponds to the linear Singleton bound $(n-r+1)$. The bound $(n-r)(\lceil \frac{\delta}{r} \rceil + 1) + \delta + 1$ is then called the generalised Singleton bound, [35], GSB, and a convolutional code attaining this bound is known as an MDS convolutional code. The papers [35] and [51] are major contributions to the area of convolutional codes.

In convolutional coding theory, the idea of dual code has two meanings. The one used here is what is referred to as the convolutional dual code, see [30] and [7], and is known also as the module-theoretic dual code. The other dual is called the sequence space dual code. The two generator matrices for these ‘duals’ are related by a specific formula. If $G[z]H^T[z] = 0$ for a generator matrix $G[z]$, then $H[z^{-1}]z^m$ for memory $m$ generates the convolutional/(module theoretic) dual code. The code is then dual-containing provided the code generated by $H[z^{-1}]z^m$ is contained in the code.
generated by $G[z]$.

The papers [28], [38] introduce certain algebraic decoding techniques for convolutional codes. Viterbi or sequential decoding are available for convolutional codes, see [1], [2] or [3] and references therein. The form of the control matrix derived here leads to algebraic implementable error-correcting algorithms.

The MDS block linear codes to requirements rate $R$ are extended to MDS convolutional codes with rate $R$ and with the order of twice the distance of the linear block MDS codes of the same length. These may also be specified to be (i) of characteristic a fixed prime $p$, (ii) of prime order, (iii) DC, or (iv) LCD. Noteworthy here is how MDS DC block linear lead to convolutional MDS with memory 1 LCD codes, and in characteristic 2, LCD MDS block linear codes lead to the design of DC memory 1 convolutional MDS codes. The DC codes may be designed with Euclidean inner product or with Hermitian inner product.

1.2.2 Previously

In [43] a general method for deriving MDS codes to specified rate and specified error-correcting capability is established; [9] gives a general method for designing DC MDS codes of arbitrary rate and error-correcting capability, from which MDS QECCs can be specified, [8] specifies a general method for designing LCD MDS codes to arbitrary requirements and [42] gives general methods for designing convolutional codes. The unit-derived methods devised in [31, 33, 39, 40, 41] are often in the background.

1.3 Abbreviations

DC: dual-containing
LCD: linear complementary dual
QECC: quantum error-correcting code
MDS: maximum distance separable
GSB: Generalised Singleton bound
rdist: relative distance, which is the ratio of distance over length.

2 Summary of design methods

Section 5 describes explicitly the design algorithms in detail. Here’s a summary of the main design methods.

2.1 Linear block MDS

Design methods are given initially for:
(i) MDS linear block codes to rate $\geq R$ and distance $\geq (2t + 1)$ with efficient decoding algorithms.
(ii) MDS linear block codes to rate $\geq R$ and distance $\geq (2t + 1)$ with efficient decoding algorithms over fields of (fixed) characteristic $p$.

3This has different parameter requirements for linear block codes, convolutional codes and QECCs.
(iii) MDS linear block codes to rate $\geq R$ and distance $\geq (2t+1)$ with efficient decoding algorithms over prime order fields.

Then types of codes are required. Thus design methods are obtained, as in the above (i)-(iii), where “MDS linear block” is replaced by “MDS linear block of type X” where of type X is (a) DC or (b) LCD. These may be designed with Hermitian inner product when working over fields of type $GF(q^2)$. For Hermitian inner products, in item (iii), the ‘prime order fields’ needs to be replaced by ‘fields of order $GF(p^2)$, where $p$ is prime’. The DC codes designed are then used to design MDS QECCs.

Then infinite series of such block block codes are designed so that the rate approaches $R$ and $\text{rdist}$ approaches $(1-R)$ for given $R$, $0 < R < 1$; for DC codes it is required $\frac{1}{2} < R < 1$. The infinite series of MDS QECCs designed from the DC codes have $\text{rdist}$ approaching $(2R-1)$ and rate approaching $R$ for given $R$, $1 > R > \frac{1}{2}$.

Specifically:

- Design of infinite series of linear block codes $[n_i, r_i, d_i]$, such that $\lim_{i \to \infty} \frac{r_i}{n_i} = R$, and $\lim_{i \to \infty} \frac{d_i}{n_i} = 1 - R$

- Design of infinite series of MDS block linear codes $[n_i, r_i, d_i]$, such that $\lim_{i \to \infty} \frac{r_i}{n_i} = R$ and $\lim_{i \to \infty} \frac{d_i}{n_i} = 1 - R$ in fields of characteristic $p$.

- Design of infinite series of MDS block linear codes $[n_i, r_i, d_i]$, such that $\lim_{i \to \infty} \frac{r_i}{n_i} = R$ and $\lim_{i \to \infty} \frac{d_i}{n_i} = 1 - R$ in prime order fields or in fields $GF(p^2)$ for Hermitian inner product.

Further in the above a ‘type’ may be included in the infinite series designed where ‘type’ could be ‘DC’ or ‘LDC’. For DC, it is necessary that $R > \frac{1}{2}$, and then infinite series of QECCs $[[n_i, 2r_i - n_i, d_i]]$ are designed where $\lim_{i \to \infty} \frac{2r_i - n_i}{n_i} = 2R - 1$ (limit of rates) but still $\lim_{i \to \infty} \frac{d_i}{n_i} = 1 - R$, for given $R$, $R > \frac{1}{2}$.

### 2.2 Convolutional MDS

Convolutional MDS codes and infinite series of convolutional MDS codes are designed. These are designed to specified rate and distance and in general have better relative distances. In memory 1 the distances obtained are of the order of twice that of the corresponding MDS linear block codes with the same length and rate. Higher memory convolutional codes are briefly discussed.

Thus memory 1 MDS convolutional codes are designed as follows.

- Design of MDS convolutional codes of memory 1 with the same length and rate as the corresponding MDS linear block code but with twice the distance less 1.

- Design of MDS convolutional in characteristic $p$ of the same length and rate as the corresponding MDS block linear codes but twice the distance less 1.

- Design of MDS convolutional over a prime field of the same length and rate as the corresponding MDS block linear codes but with twice the distance, less 1.

These may be extended to higher memory MDS convolutional codes.
These are then specified for particular types such as DC or LCD. The LCD linear block codes when ‘extended’ to convolutional codes give rise to DC codes, and the DC linear block codes when ‘extended’ to convolutional codes give rise to LCD codes in characteristic 2.

Infinite series of convolutional codes are designed as follows.

- Design of infinite series of MDS convolutional codes \((n_i, r_i, n_i - r_i; 1, 2(n_i - r_i))\) such that 
  \[
  \lim_{i \to \infty} \frac{r_i}{n_i} = R, \quad (2R - 1), \quad \lim_{i \to \infty} \frac{d_i}{n_i} = (2R - 1).
  \]

- Design of such infinite series over fields of (fixed) characteristic \(p\).

- Design of such infinite series over fields of prime order.

Then such infinite series designs of convolutional codes are described for ‘types’ of codes such as DC or LCD or QECCs, which may be designed from DC codes.

2.3 Characteristic 2 and prime fields

The designs over the fields \(GF(2^i)\) and over prime fields \(GF(p) = \mathbb{Z}_p\) have a particular interest and have nice properties. The designs to specific requirements and specific type both linear block and convolutional can be constructed over such fields. \(GF(p), p \text{ a prime, has an element of order } (p-1)\) which is easily found and arithmetic within \(GF(p)\) is modular arithmetic.

2.4 Examples

Examples are given throughout. Although there is no restriction on length in general, examples explicitly written out here are limited in their size. Example 3.1 below is a prototype example of small order which has some hallmarks of the general designs; it is instructional and may be read now with little preparation. Example 4.3 is an instructional example on the design methods for linear block (MDS) codes to meet specific rates and distances. The general designs are more powerful and include designs for convolutional codes.

3 Constructions

The following constructions follow from [43]; see also [31], [33].

**Construction 3.1** Design MDS linear block codes.

- Let \(F_n\) be a Fourier \(n \times n\) matrix over a finite field.

- Taking \(r\) rows of \(F_n\) generates an \([n, r]\) code. A check matrix for the code is obtained by eliminating the corresponding columns of the inverse of \(F_n\).

- Let \(r\) rows of \(F_n\) be chosen in arithmetic sequence such that the arithmetic difference \(k\) satisfies \(\gcd(k, n) = 1\). The code generated by these rows is an MDS \([n, r, n - r + 1]\) code. There exists an explicit efficient decoding algorithm of \(O(\max\{n \log n, t^2\})\), 
  
  \[
  t = \left\lfloor \frac{n - r}{2} \right\rfloor, \quad t \text{ is the error-correcting capability of the code.}
  \]

  In particular this is true when \(k = 1\), that is when the rows are taken in sequence.
General Vandermonde matrices may be used instead of Fourier matrices but are not necessary. For a given Fourier $n \times n$ matrix $F_n$ under consideration the rows of $F_n$ in order are denoted by $\{e_0, e_1, \ldots, e_{n-1}\}$ and $n$ times the columns in order are denoted by $\{f_0, f_1, \ldots, f_{n-1}\}$. $F_n$ is generated by a primitive $n^{th}$ root of unity $\omega$; thus $\omega^n = 1$, $\omega^r \neq 1, 0 < r < n$. Hence $e_i = (1, \omega^i, \omega^{2i}, \ldots, \omega^{(n-1)i})$. Indices are taken modulo $n$ so that $e_i + n = e_i$. The arithmetic sequences in Construction 3.1 may wrap around; for example when $n = 10$, such arithmetic sequences include $\{e_8, e_9, e_0, e_1\}$, $\{e_3, e_6, e_9, e_2, e_5\}$.

Note also that if $B$ is a check matrix then so also is $nB$ for any $n \neq 0$. Thus in the above Construction 3.1 the check matrix may be obtained from $n$ times the columns of the inverse.

**Construction 3.2** Design DC MDS linear block codes.

This construction follows from [9].

- Let $F_n$ be a Fourier $n \times n$ matrix with rows $\{e_0, \ldots, e_{n-1}\}$ in order and $n$ times the columns of the inverse in order are denoted by $\{f_0, \ldots, f_{n-1}\}$.
- Let $r > \lceil \frac{n}{2} \rceil$ and define $A = \begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_{r-1} \end{pmatrix}$.
- A check code for the code generated by $A$ is $B = (f_r, f_{r+1}, \ldots, f_{n-1})$.
- Then $B^T = \begin{pmatrix} f_r^T \\ f_{r+1}^T \\ \vdots \\ f_{n-1}^T \end{pmatrix} = \begin{pmatrix} e_{n-r} \\ e_{n-r-1} \\ \vdots \\ e_1 \end{pmatrix}$. Hence the code generated by $A$ is a DC $[n, r, n-r+1]$ MDS code.
- This works for any $r$ such that $n > r > \lceil \frac{n}{2} \rceil$.

**Construction 3.3** Design LCD MDS linear block codes.

The design technique is taken from [8].

- Construct a Fourier $n \times n$ matrix. Denote its rows in order by $\{e_0, \ldots, e_{n-1}\}$ and $n$ times the columns of the inverse in order are denoted by $\{f_0, \ldots, f_{n-1}\}$.
- Design $A$ as follows. $A$ consists of first row $e_0$ and rows $\{e_1, e_{n-1}, e_2, e_{n-2}, \ldots, e_r, e_{n-r}\}$ for $r \leq \lceil \frac{n}{2} \rceil$. ($A$ consists of $e_0$ and pairs $\{e_i, e_{n-i}\}$ starting with $\{e_1, e_{n-1}\}$.)
- Set $B^T = (f_{r+1}, f_{n-r-1}, \ldots, f_{\frac{n-1}{2}}, f_{\frac{n+1}{2}})$ when $n$ is odd and $B^T = (f_{r+1}, f_{n-r-1}, \ldots, f_{\frac{n-1}{2}}, f_{\frac{n+1}{2}}, f_{\frac{n+3}{2}}, f_{\frac{n+5}{2}})$ when $n$ is even.
- Then $AB^T = 0$ and $B$ generates the dual code $\mathcal{C}^\perp$ of the code $\mathcal{C}$ generated by $A$.
- Using $f_i^T = e_{n-i}$ it is easy to check that $\mathcal{C} \cap \mathcal{C}^\perp = 0$.
- The rows of $A$ are in sequence $\{n-r, n-r+1, \ldots, n-1, 0, 1, \ldots, r-1\}$ and so $A$ generates an MDS LCD linear block $[n, 2r+1, n-2r]$ code.
The general method of constructing MDS codes by choosing rows from Fourier matrices does not take into account the power of the other non-chosen rows. This can be remedied by going over to convolutional codes. The convolutional codes obtained carefully in this way are MDS convolutional codes with free distance of the order of twice the distance of the corresponding MDS linear block code with the same length and rate.

**Lemma 3.1** Let $F$ be a Fourier $n \times n$ matrix with rows $\{e_0, e_1, \ldots, e_{n-1}\}$. Define $A = \begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_{n-1} \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ e_r \\ \vdots \\ e_{n-1} \end{pmatrix}$ where $r > \left\lfloor \frac{n}{2} \right\rfloor$ and the first $(2r-n)$ rows of $B$ consist of zeros. Let $P$ be a non-zero vector of length $n$. Then $\text{wt}(P + Bz) \geq 2(n-r) + 1$.

**Proof:** $PA$ has wt $\geq (n-r+1)$ as $A$ generates an $[n,r,n-r+1]$ code. $PB$ has weight $\geq r + 1$ except when $P$ has the last $(n-r)$ entries consisting of zeros, as the non-zero rows of $B$ generate an $[n,n-r,r+1]$ code. Now $(n-r+1) + r + 1 = n + 2 > 2(n-r) + 1$. When $P$ has last $(n-r)$ entries consisting of zeros then $PA$ contains a non-zero sum of $\{e_0, e_1, \ldots, e_{2r-n}\}$ which is part of an $[n,2r-n,2r-n-n+1] = [n,2r-n,2(n-r)+1]$ code and so has weight $\geq 2(n-r) + 1$ as required. \hfill \square

In fact if $P$ is a polynomial of degree $t$ then $\text{wt}(P + Bz) \geq 2(n-r) + t + 1$ so weight increases with the degree of the multiplying polynomial vector.

**Lemma 3.2** Let $F$ be a Fourier $n \times n$ matrix with rows $\{e_0, e_1, \ldots, e_{n-1}\}$. Let $A$ be chosen by taking $r$ rows, $r > \left\lfloor \frac{n}{2} \right\rfloor$, of the Fourier $n \times n$ matrix in arithmetic sequence with arithmetic difference $k$ satisfying $\gcd(k,n) = 1$. Let $B$ be the matrix with first $(2r-n)$ rows consisting of zeros and the last $(n-r)$ rows consisting of the rest of the rows of $F$ not in $A$; these last rows of $B$ are also in sequence with arithmetic difference $k$ satisfying $\gcd(k,n) = 1$. Let $P$ be any non-zero vector of length $n$. Then $\text{wt}(P + Bz) \geq 2(n-r) + 1$.

Before describing the general design methods, it is instructional to consider the following example. See also Example 4.3 below for a larger example demonstrating the design techniques for constructing a code to given rate and distance.

When $\gcd(p,n) = 1$, OrderMod$(p,n)$ denotes the least positive power $s$ such that $p^s \equiv 1 \mod n$.

**Example 3.1** Consider $n = 7$. Now OrderMod$(2,7) = 3$ so Fourier $7 \times 7$ matrix may be constructed over $GF(2^3)$. The Fourier $7 \times 7$ matrix may also be constructed over $GF(3^6)$ as OrderMod$(3,7) = 6$ and over many other fields whose characteristic does not divide 7. It may be formed over the prime field $GF(29)$ as OrderMod$(29,7) = 1$; arithmetic in $GF(29)$ is then modular arithmetic.

We’ll stick to $GF(2^3)$ for the moment; when Hermitian inner product is required we’ll move to $GF(2^6)$. 

The rows in order of a Fourier $7 \times 7$ matrix under consideration are denoted by $\{e_0, e_1, e_2, \ldots, e_6\}$ and $(7$ times$)$ the columns of the inverse in order are denoted by $\{f_0, \ldots, f_6\}$; note $7 = 1$ in characteristic 2. Then $e_if_j = \delta_{ij}, e_i^T = f_{7-i}, f_i^T = e_{7-i}$.

1. Construct $A = \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix}$. Then $A * (f_4, f_5, f_6) = 0$. The Euclidean dual matrix is $(f_4, f_5, f_6)^T = \begin{pmatrix} f_4^T \\ f_5^T \\ f_6^T \end{pmatrix} = \begin{pmatrix} e_3 \\ e_4 \\ e_5 \end{pmatrix}$. Thus the code generated by $A$ is a DC code $[7, 4, 4]$ code.

2. To obtain a DC code relative to the Hermitian inner product in $GF(2^6)$. Again the rows of the Fourier $7 \times 7$ matrix over $GF(2^6)$ are denoted by $\{e_0, \ldots, e_6\}$. Here $e_l^T = e_{il}$ as explained below where $l = 2^3$ and thus since $2^3 \equiv 1 \mod 7$ it follows that $e_l^T = e_{il} = e_i$. Thus the code generated by $A = \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix}$ is an Hermitian DC MDS $[7, 4, 4]$ code.

3. A DC MDS convolutional code over $GF(2^3)$ and a DC Hermitian MDS convolutional code over $GF(2^6)$ are obtained as follows. The distance obtained is of the order of twice the distance of the corresponding MDS linear block code.

(In characteristic 2, $-1 = +1$.)

4. Now design $G[z] = \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix} + \begin{pmatrix} 0 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix} z$. Then $G[z] * ((f_4, f_5, f_6) + (f_1, f_2, f_3))z = 0$. $H^T[z] = (f_1, f_2, f_3)z$. Then $H[z^{-1}] = \begin{pmatrix} e_3 \\ e_4 \end{pmatrix} z^{-1}$. Thus a control matrix is $K[z] = \begin{pmatrix} e_3 \\ e_4 \end{pmatrix} z + \begin{pmatrix} e_4 \\ e_5 \end{pmatrix}$. It is easy to show that the convolutional code generated by $K[z]$ has trivial intersection with the convolutional code generated by $G[z]$. Thus the convolutional code generated by $G[z]$ is a LCD $(7, 4, 3; 1, d_f)$ code. The GSB for a code of this form is $3([\frac{3}{4} + 1] + 3 + 1 = 7$. The free distance of the one constructed may be shown to be 7 directly or from the general Lemma [3, 1] below.

5. Starting with the MDS DC $[7, 4, 4]$ code, a corresponding convolutional code $(7, 4, 3; 1, 7)$ is designed of memory 1 which is LCD and has almost twice the distance of the DC linear block code.

6. Is it possible to go the other way? Methods for designing MDS LCD linear block codes are established in [8]. Following the method of [8], let $A = \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix}$ and hence $A * (f_3, f_4) = 0$. Then $(f_3, f_4)^T = (e_4)$ giving that the code generated by $A$ is an MDS $[7, 5, 3]$ LCD code.

Then $G[z] = \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} z = A + Bz$, say, gives a convolutional $(7, 5, 2; 1, 5)$ MDS code. A control matrix is $H^T[z] = (f_3, f_4) - (f_2, f_3)z$. $H[z^{-1}] = (e_4) + (e_5) z^{-1}$. Thus the dual code has generating matrix $(e_3) z + (e_5) e_2$. Now it is necessary to show that the code generated by $G[z]$ is DC.

Note $(0, 0, 0, 0, 0, 0, 0, 0) \ast \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix} = (e_5) z + (e_4) z$. Hence the code generated by $G[z]$ is DC over $GF(2^3)$ and is Hermitian DC over $GF(2^6)$. 

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7. Construct \( G[z] = \left( e_0 \over e_1 \right) + \left( 0 \over e_2 \right) z + \left( e_3 \over 0 \right) z^2 \)

Then \( G[z] \) is a convolutional code of type \((7, 3, 5; 2)\); the degree is 5. The GSB for such a code is \((7 - 3)(\left\lfloor \frac{5}{3} \right\rfloor + 1) + 5 + 1 = 4 \times 2 + 5 + 1 = 14\). In fact the free distance of this code is actually 14. This may be shown in an analogous way to the proof of Lemma 3.1. \( G[z] \ast ((f_0, f_1, f_2)) = 7I_3 \).

The result is that the code generated by \( G[z] \) is a non-catastrophic convolutional \( MDS \) code. Note the free distance attained is \( 5 \times 3 - 1 \) where 5 is the free distance of a \([7, 3, 5] \) MDS code; the distance is tripled less 1. This is a general principle for the more general cases – the free distance is of order three times the distance of the same length and dimension \( MDS \) code.

It’s not a dual-containing code nor a \( LCD \) code. To get such codes requires a compromise on the distance. \( G[z] = \left( e_0 \over e_1 \right) + \left( 0 \over e_2 \right) z + \left( e_3 \over 0 \right) z^2 \). This give \((7, 3, 4; 2)\) convolutional code which turns out to be an \( LCD \) code but the free distance is only 7. The GSB for such a code is 13.

8. Now for memory 3 define \( G[z] = \left( e_0 \over e_1 \right) + \left( e_2 \over e_3 \right) z + \left( e_4 \over e_5 \right) z^2 + \left( 0 \over e_6 \right) z^3 \). The GSB for such a code is \((7 - 2)(\left\lfloor \frac{5}{2} \right\rfloor + 1) + 5 + 1 = 21\). The free distance of the code is actually 21 so the code is a \((7, 2, 5; 3, 21)\) convolutional \( MDS \) code. This is \( 6 \times 4 - 3 \) where 6 is the free distance of the corresponding block linear \( MDS \) code \([7, 2, 6]\).

9. Ultimately get a convolutional code \( e_0 + e_1 z + e_2 z^2 + e_3 z^3 + e_4 z^4 + e_5 z^5 + e_6 z^6 \) which is the convolutional \( MDS \) code \((7, 1, 6; 6, 47)\) code which is repetition convolutional code.

4 Specify the codes

4.1 Matrices to work and control

Many of the designs hold using general Vandermonde matrices but the Fourier matrix case is considered for clarity.

If the Fourier \( n \times n \) matrix \( F_n \) exists over a finite field then the characteristic \( p \) of the field does not divide \( n \) which happens if and only if \( \gcd(p, n) = 1 \).

Let \( F_n \) denote a Fourier matrix of size \( n \). Over which finite fields precisely may this matrix be constructed? Suppose \( \gcd(p, n) = 1 \) mod \( n \), where \( \phi \) is the Euler \( \phi \)-function. Hence there exists a least positive power \( s \) that \( p^s \equiv 1 \) mod \( n \); this \( s \) is called the order of \( p \) modulo \( n \). Use \( \text{OrderMod}(p, n) \) to denote the order of \( p \) mod \( n \).

**Lemma 4.1** Let \( p \) be any prime such that \( p \nmid n \) and \( s = \text{OrderMod}(p, n) \).

(i) There exists an element of order \( n \) in \( GF(p^s) \) from which the Fourier \( n \times n \) matrix may be constructed over \( GF(p^s) \).

(ii) The Fourier \( n \times n \) matrix cannot exist over a finite field of characteristic \( p \) of order smaller than \( GF(p^s) \).

(iii) There exists a Fourier \( n \times n \) matrix over any \( GF(p^{rs}), r \geq 1 \) and in particular over \( GF(p^{2s}) \).

**Proof:** (i) There exists an element of order \( p^s - 1 \) in \( GF(p^s) \), that is for some \( \omega \in GF(p^s) \), \( \omega^{p^s - 1} = 1 \). Now \( (p^s - 1) = qn \) for some \( q \) and so \( (\omega^q)^n = 1 \) in \( GF(p^s) \), giving an element of order \( n \) in \( GF(p^s) \). This element may then be used to construct the Fourier \( n \times n \) matrix over \( GF(p^s) \).
Proofs of (ii) and (iii) are omitted. □

For a vector $v = (v_1, v_2, \ldots, v_r)$ define $v^t = (v_1^t, v_2^t, \ldots, v_r^t)$.

The following lists some properties of a Fourier matrix of size $n$ over a finite field. These are used throughout.

1. Let $F_n$ be a Fourier $n \times n$ matrix over a field generated by $\omega$, where $\omega^n = 1$ and $\omega^r \neq 1$ for $0 < r < n$.

2. Denote the rows of $F_n$ by $\{e_0, e_1, \ldots, e_{n-1}\}$ in order and $n$ times the columns of the inverse of $F_n$ in order by $\{f_0, f_1, \ldots, f_{n-1}\}$.

3. Then $e_if_j = \delta_{ij}, e_i^T = f_{n-i}, f_i^T = e_{n-i}$.

4. The rows of the $F_n$ are given by $e_i = (1, \omega^i, \omega^{2i}, \ldots, \omega^{(n-1)i})$. Indices are to be taken modulo $n$.

5. $e_i^t = (1, \omega^i, \omega^{2i}, \ldots, \omega^{(n-1)i}) = e_{il}$.

6. Note that if $l \equiv 1 \mod n$ then $e_i^t = e_{il} = e_i$.

Within $GF(p^{2s})$ the Hermitian inner product is defined by $\langle u, v \rangle_H = \langle u, v^* \rangle_E$ where $l = p^s$. In this setup $\langle e_i, e_j \rangle_H = \langle e_i, e_j^t \rangle_E = \langle e_i, e_{il} \rangle_E$. This facilitates the construction of Hermitian inner product codes over $GF(2^{2s})$.

**Example 4.1** Consider length $n = 10$. Now OrderMod$(3, 10) = 4$. Construct the Fourier $10 \times 10$ matrix $F_{10}$ over $GF(3^4)$. Denote the rows in order of $F_{10}$ by $\{e_0, e_1, \ldots, e_9\}$ and the $10$ times the columns of the inverse of $F_{10}$ in order by $\{f_0, f_1, \ldots, f_9\}$. Then $e_if_j = \delta_{ij}$. Also $e_i^T = f_{10-i}$, $f_i^T = e_{10-i}$.

1. As in [33] construct $A = \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \\ e_8 \\ e_9 \end{pmatrix}$. Then $A \ast (f_6, f_7, f_8, f_9) = 0$. Now by Construction 3.1 [33] the code generated by $A$ is an MDS [10, 6, 5] code.

2. The Euclidean dual of $A$ is $(f_6, f_7, f_8, f_9)^T = \begin{pmatrix} f_6^T \\ f_7^T \\ f_8^T \\ f_9^T \end{pmatrix} = \begin{pmatrix} e_4 \\ e_3 \\ e_2 \\ e_1 \end{pmatrix}$. Thus $A$ is a DC code.

3. Now $e_i^2 = e_{9+1} = e_{-1} = e_{10-i}$. Thus $A$ is not a DC code under the Hermitian inner product induced in $GF(3^4)$.

Consider $GF(3^8)$. This has an element of order $10$, as $3^8 - 1 = (3^4 - 1)(3^4 + 1)$, and so the Fourier $10 \times 10$ matrix may be constructed over $GF(3^8)$. Here then the Hermitian inner product is $\langle u, v \rangle_H = \langle u, v^{3^4} \rangle_E$, where the suffix $E$ denotes the Euclidean inner product.

Now $e_i^t = e_{il} = e_{i+1} = e_i$. Thus $\langle e_i, e_j \rangle_H = \langle e_i, e_j \rangle_E$. Hence here then the code generated by $A$, constructed in $GF(3^4)$, is a DC code under the Hermitian inner product.

4. Also OrderMod$(7, 10) = 4$ so the above works over $GF(7^4)$ and over $GF(7^8)$ when seeking Hermitian DC codes.
5. Better though is the following. OrderMod(11, 10) = 1 and so the prime field \( GF(11) \) may be considered. \( A \) as above is then a DC code over the prime field \( GF(11) \) and a DC Hermitian code when considered over \( GF(11^2) \).

6. What is an element of order 10 in \( GF(11) \)? In fact \( \omega = (2 \mod 11) \) works. In \( GF(11) \) the arithmetic is modular arithmetic. An element of order 10 is required in \( GF(11^2) \). Now \( GF(11^2) \) is constructed by finding an irreducible polynomial of degree 2 over \( Z_{11} \). A primitive element is easily found.

7. Let \( A \) be as above and \( B = \begin{pmatrix} 0 & 0 \\ e_2 & e_8 \\ e_6 & e_9 \end{pmatrix} \). Define \( G[z] = A + Bz \). The code \( C \) generated by \( G[z] \) is a \( (10, 6, 4; 1, d_f) \) convolutional code.

8. Then \( G[z] * \{f_0, f_7, f_8, f_9\} - \{f_2, f_3, f_4, f_5\} = 0 \) and \( G[z] * \{f_0, f_1, f_2, f_3, f_4, f_5\} = I_5 \). Thus \( G[z] \) is a non-catastrophic generator for the code.

9. The GSB of such a code is \((10 - 6)(\lfloor \frac{4}{17} \rfloor) + 4 + 1 = 4 + 4 + 1 = 9 \). The free distance of \( C \) may be shown, using Lemma 3.1 essentially, to be 9 and so is thus an MDS convolutional \((10, 6, 4; 1, 9)\) code.

Since OrderMod(11, 10) = 1 the calculations may be done over the prime field \( GF(11) \), and over \( GF(11^2) \) when Hermitian codes are required.

**Example 4.2** Consider \( n = 2^5 - 1 = 31 \). Then OrderMod(2, 31) = 5 and so the Fourier \( 31 \times 31 \) matrix may be constructed over \( GF(2^5) \) but also over \( GF(2^{10}) \). Let \( r = 17 \), and \( A \) consist of rows \( \langle e_0, e_1, \ldots, e_{16} \rangle \) and \( B = \langle f_{17}, f_{18}, \ldots, f_{30} \rangle \). Then \( A \) generates an \([31, 17, 15] \) MDS code. Now \( B^T \) consists of rows \( \{e_{14}, e_{13}, \ldots, e_1\} \), using \( f_i^T = e_{31-i} \). Thus The code generated by \( A \) is a DC MDS \([31, 17, 15] \) code Euclidean over \( GF(2^5) \) and Hermitian over \( GF(2^{10}) \).

**Example 4.3** Example of a general technique.

It is required to construct a rate \( \geq \frac{7}{8} \) codes which can correct 25 errors; thus a distance \( \geq 51 \) is required. LINEAR BLOCK:

1. An \([n, r, n - r + 1] \) type code with \( \frac{r}{n} \geq \frac{7}{8} = R \) and \( n - r + 1 \geq 51 \) is required. Thus \( n - r \geq 50 \) giving \( n(1 - R) \geq 50 \) and so it is required that \( n \geq 400 \).

2. Construct Fourier matrix \( F_{400} \) of size \( 400 \times 400 \) over some suitable field, to be determined. The rows are denoted by \( \{e_0, \ldots, e_{399}\} \) and the columns of 400 times the inverse by \( \{f_0, \ldots, f_{399}\} \).

3. Define \( A \) to be the matrix with rows \( \{e_0, \ldots, e_{399}\} \). Then by [3] \( A \) is a DC \([400, 350, 51] \) MDS code. By the CSS construction a \([400, 300, 51] \) MDS QECC is designed.

- Over which fields can \( F_{400} \) be defined? The characteristic must not divide 400 but otherwise the fields can be determined by finding OrderMod\((p, n) \) where \( \gcd(p, n) = 1 \).

- Now OrderMod(3, 400) = 20, OrderMod(7, 400) = 4, OrderMod(401, 400) = 1 so it may be constructed over \( GF(3^{20}), GF(7^4), GF(401) \) and others.
• Now $GF(401)$ is a prime field and arithmetic therein is modular arithmetic; it is in fact the smallest field over which the Fourier $400 \times 400$ can be constructed.

• An Hermitian dual-containing code may be obtained by working over $GF(p^{2l})$ when there exists an element of the required order in $GF(p^l)$. Just define $A$ as above to be a Fourier $400 \times 400$ matrix over say $GF(401^2)$ using a $400^{th}$ root of unity in $GF(401^2)$.

• The $e_i$ in this case satisfy $e_i^{401} = e_{i*401} = e_i$ as 401 $\equiv 1 \bmod 400$. Thus the code obtained is an Hermitian dual-containing $[400,350,51]$ code from which a QECC $[[400,300,51]$ MDS code is designed.

By taking ‘only’ $\{e_0, \ldots, e_{349}\}$ the full power of all the rows of the Fourier matrix is not utilised.

1. Define $A$ as before and $B$ to be the matrix whose last 50 rows are $\{e_{350}, \ldots, e_{399}\}$ and whose first 350 rows are zero vectors.

2. Define $G[z] = A + Bz$. The convolutional code generated by $G[z]$ is a $(400,350,50;1)$ code. It may be shown to be non-catastrophic by writing down the right inverse of $G[z]$.

3. The GSB for such a code is $(450 - 350)((\frac{50}{350}) + 1) + 50 + 1 = 101$. Using Lemma 3.1 it may be shown that the free distance of the code generated by $G[z]$ is 101 so it’s an MDS convolutional $(400,350,50;1,101)$ code. The distance is twice less 1 of the distance of an MDS $[400,350,51]$ linear block code.

To get an LCD linear block code of rate $\geq \frac{7}{8}$ and $d \geq 51$ it is necessary to go to length 401 or higher. Use the methods of [8].

• For length 401, let $A$ be the matrix generated $\{e_0,e_1,e_{400},e_2,e_{399}, \ldots, e_{175},e_{226}\}$. (The selection includes pairs $e_i,e_{401-i}$.)

• Then the code generated by $A$ is an LCD $[401,351,51]$ MDS code.

• The fields required for $n = 401$ are fairly large. Go to $511 = 2^9 - 1$ as here $\text{OrderMod}(2,511) = 9$ so the field $GF(2^9)$ works and has characteristic 2.

• Require $r \geq 511 * \frac{7}{8}$ for a rate $\geq \frac{7}{8}$. Thus require $r \geq 448$. Take $r = 449$ for reasons which will appear later.

• Let $F_{511}$ be the Fourier $511 \times 511$ matrix over $GF(2^9)$ or for the Hermitian case over $GF(2^{18})$.

• Let $A$ be the matrix with rows $\{e_0,e_1,\ldots,e_{448}\}$.

• Then $A$ is an $[511,449,63]$ MDS DC code – and DC Hermitian code over $GF(2^{18})$.

• From this QECC MDS codes $[[511,387,63]]$ are designed and are Hermitian over $GF(2^{18})$.

Convolutional

1. Let $B$ be the matrix of size $449 \times 511$ with last 62 rows consisting of $\{e_{449}, \ldots, e_{510}\}$ and other rows consisting of zero vectors.
2. Define $G[z] = A + Bz$. Then the code generated by $G[z]$ is a non-catastrophic MDS $(511, 449, 62 : 1, 125)$ convolutional code. The proof of the distance follows the lines of Lemma 3.1. It has twice the distance less 1 of the corresponding MDS block linear $[511, 449, 63]$ code.

Now design LCD codes in $GF(2^9)$.

1. Let $A$ be the matrix with rows $\{e_0, e_1, e_{510}, e_2, e_{509}, \ldots, e_{509}, e_{224}, e_{287}\}$. Notice the rows are in sequence and so $A$ generates an $[511, 449, 63]$ linear block code.

2. The check matrix is $H^T = (f_{286}, f_{285}, f_{226}, \ldots, f_{256}, f_{255})$.

3. Then $H$ consists of rows $\{e_{225}, e_{286}, \ldots, e_{255}, e_{256}\}$. Thus the code generated by $A$ has trivial intersection with the code generated by $H$ and so the code is an MDS LCD block linear $[511, 449, 63]$ code. This is an Hermitian LCD MDS code over $GF(2^{18})$.

4. Let $B$ be the matrix whose last 62 rows are $\{e_{225}, e_{286}, \ldots, e_{255}, e_{256}\}$ and whose first 449 rows consists of zero vectors.

5. Define $G[z] = A + Bz$. A check matrix for the code generated by $G[z]$ is $(f_{286}, f_{285}, f_{226}, \ldots, f_{256}, f_{255}) - (0, 0, \ldots, 0, f_{254}, f_{257}, \ldots, f_{224}, f_{287}) = C + Dz$, say.

6. Recall we are in characteristic 2. Now $C^T + D^T z^{-1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ e_{225} \\ e_{286} \\ e_{285} \\ e_{226} \\ e_{286} \\ e_{285} \\ \vdots \\ \vdots \\ \vdots \\ e_{257} \\ e_{254} \\ e_{286} \\ e_{285} \\ e_{254} \\ e_{257} \\ e_{224} \\ e_{286} \\ e_{285} \\ e_{256} \\ e_{255} \end{pmatrix} + \begin{pmatrix} e_{225} \\ e_{286} \\ e_{285} \\ e_{226} \\ e_{286} \\ e_{285} \\ \vdots \\ \vdots \\ \vdots \\ e_{257} \\ e_{254} \\ e_{286} \\ e_{285} \\ e_{254} \\ e_{257} \\ e_{224} \\ e_{286} \\ e_{285} \end{pmatrix} z^{-1}$.

7. Thus the dual matrix is $\begin{pmatrix} 0 \\ 0 \\ 0 \\ e_{225} \\ e_{286} \\ e_{285} \\ e_{226} \\ e_{286} \\ e_{285} \\ \vdots \\ \vdots \\ \vdots \\ e_{257} \\ e_{254} \\ e_{286} \\ e_{285} \end{pmatrix} + \begin{pmatrix} e_{225} \\ e_{286} \\ e_{285} \\ e_{226} \\ e_{286} \\ e_{285} \\ \vdots \\ \vdots \\ \vdots \\ e_{257} \\ e_{254} \\ e_{286} \\ e_{285} \end{pmatrix} z = E + Fz$, say.

8. It is relatively easy to show that there is a matrix $K$ such that $K(A + Bz) = E + Fz$ and so the code generated by $A + Bz$ is a DC convolutional $(511, 449, 62; 1)$ code.

9. The GSB for such a code is 125 and this is the distance attained, so the code is a DC MDS convolutional $(511, 449, 62; 1, 125)$ code. From this a quantum convolutional code may be designed. To obtain Hermitian DC, work in $GF(2^{18})$.

This can be extended to higher degrees.

Thus in a sense:

| DC block linear | $\rightarrow$ LCD convolutional degree 1 at twice the distance. |
| LCD block linear | $\rightarrow$ DC convolutional degree 1 at twice the distance. |

The LCD block linear to give DC convolutional requires characteristic 2.
5 Algorithms

Algorithm 5.1 Construct block linear codes of rate $\geq R$ and distance $\geq (2t + 1)$ for $0 < R < 1$, and positive integer $t$, and with efficient decoding algorithm.

A $[n, r, n - r + 1]$ linear block code will be designed. Thus $\frac{r}{n} \geq R$, $(n - r + 1) \geq 2t + 1$. This requires $n - r \geq 2t$, $n(1 - R) \geq 2t$ and so require $n \geq \frac{2t}{1 - R}$.

1. Choose $n \geq \frac{2t}{1 - R}$ and construct the Fourier $n \times n$ matrix $F_n$ over a suitable field. Now choose $r \geq nR$.
2. Select any $r$ rows of $F_n$ in arithmetic sequence with arithmetic difference $k$ satisfying $\gcd(k, n) = 1$ and form the matrix $A$ consisting of these rows.
3. The block linear code with generator matrix $A$ is an MDS $[n, r, n - r + 1]$ code.
4. The rate is $\frac{r}{n} \geq R$, and the distance $d = n - r + 1 = n(1 - R) + 1 \geq 2t + 1$ as required.

Algorithm 5.2 Construct block linear codes of rate $\geq R$ and distance $\geq (2t + 1)$ for given rational $R$, $0 < R < 1$, and positive integer $t$, with efficient decoding algorithms, over fields of characteristic $p$.

A $[n, r, n - r + 1]$ code is designed in characteristic $p$. As in Algorithm 5.1 it is required that $n \geq \frac{2t}{1 - R}$. Require in addition that $\gcd(p, n) = 1$.

1. Require $n \geq \frac{2t}{1 - R}$ and $\gcd(p, n) = 1$.
2. Construct the Fourier $n \times n$ matrix over a field of characteristic $p$.
3. Proceed as in items 1-4 of Algorithm 5.1.

Algorithm 5.3 Construct block linear DC codes of rate $\geq R$ with $\frac{1}{2} < R < 1$ and distance $\geq (2t + 1)$ for positive integer $t$, with efficient decoding algorithm.

A $[n, r, n - r + 1]$ block dual-containing code is required. As before require $n \geq \frac{2t}{1 - R}$.

1. Choose $n \geq \frac{2t}{1 - R}$ and construct the Fourier $n \times n$ matrix $F_n$ over a suitable field.
2. Choose $r \geq nR$. As $R > \frac{1}{2}$ then $r > \lfloor \frac{n}{2} \rfloor + 1$.
3. Select $A$ to be the first $r$ rows of $F_n$, that is $A$ consists of rows $\{e_0, e_1, \ldots, e_{r-1}\}$. Then $B^T = (f_r, \ldots, f_{n-1})$ satisfies $AB^T = 0$. Now $B = \begin{pmatrix} e_{n-r} \\ e_{n-r+1} \\ \vdots \\ e_1 \end{pmatrix}$. Thus the code generated by $A$ is a dual-containing MDS $[n, r, n - r + 1]$ code.
4. The rate is $\frac{r}{n} \geq R$, and the distance $d = n - r + 1 = n(1 - R) + 1 \geq (2t + 1)$ as required.

Algorithm 5.4 Design block linear DC codes of rate $\geq R > \frac{1}{2}$ and distance $\geq (2t + 1)$ for given $R, t$, with efficient decoding algorithm, over fields of characteristic $p$.

A $[n, r, n - r + 1]$ block dual containing code is required. As before require $n \geq \frac{2t}{1 - R}$.
1. Choose \( n \geq \frac{2t}{1-R} \) and also such that \( \gcd(p, n) = 1 \).

2. Construct the Fourier \( n \times n \) matrix \( F_n \) over a field of characteristic \( p \).

3. Now proceed as in items 1-4 of Algorithm 5.3.

**Algorithm 5.5**

(i) Design MDS QECCs of form \([n, 2r - n, n - r + 1]\).

(ii) Design MDS QECCs of form \([[n, 2r - n, n - r + 1]]\) over a field of characteristic \( p \).

Method:

(i) By Algorithm 5.3 construct MDS DC codes \([n, r, n - r + 1]\). Then by CSS construction, construct the MDS \([n, 2r - n, n - r + 1]\) QECC.

(ii) By Algorithm 5.4 construct MDS DC codes \([n, r, n - r + 1]\) over a field of characteristic \( p \). Then by CSS construction, construct the MDS \([[n, 2r - n, n - r + 1]]\) QECC in a field of characteristic \( p \).

**Algorithm 5.6**

Design LCD MDS codes of rate \( \geq R \) and distance \( \geq 2t + 1 \).

This design follows from [8].

1. Choose \( n \geq \frac{2t}{1-R} \) and \( r \geq nR \).

2. For such \( n \) let \( F_n \) be a Fourier \( n \times n \) matrix with rows \( \{e_0, \ldots, e_{n-1}\} \) in order and \( n \) times the columns of the inverse in order are denoted by \( \{f_0, \ldots, f_{n-1}\} \).

3. For \( 2r + 1 \geq nR \) and \( r \leq \lfloor \frac{n}{2} \rfloor \) define \( A \) as follows. \( A \) consists of row \( e_0 \) and rows \( \{e_1, e_{n-1}, e_2, e_{n-2}, \ldots, e_r, e_{n-r}\} \) for . (\( A \) consists of \( e_0 \) and pairs \( \{e_i, e_{n-i}\} \) starting with \( e_1, e_{n-1} \).)

4. Set \( B^T = (f_{r+1}, f_{n-r-1}, \ldots, f_{n-1}, f_{n+1}) \) when \( n \) is odd and \( B^T = (f_{r+1}, f_{n-r-1}, \ldots, f_{n-1}, f_{n+1}, f_{n+2}, f_{n+3}) \) when \( n \) is even.

5. Then \( AB^T = 0 \) and \( B \) generates the dual code \( C^\perp \) of the code \( C \) generated by \( A \).

6. Using \( f_i^T = e_{n-i} \) it is easy to check that \( C \cap C^\perp = 0 \).

7. Now the rows of \( A \) are in sequence \( \{n-r, n-r+1, \ldots, n-1, 0, 1, \ldots, r-1\} \) and so \( A \) generates an MDS, LCD, \([n, 2r + 1, n - 2r]\) code.

**Algorithm 5.7**

Construct MDS and MDS DC codes and MDS QECCs, all over prime fields. Construct Hermitian such codes over \( GF(p^2) \) for a prime \( p \).

- Let \( GF(p) = \mathbb{Z}_p \) be a prime field. This has a primitive element of order \( (p - 1) = n \), say.
- Construct the Fourier \( n \times n \) matrix over \( GF(p) \).
- Choose \( r > \frac{p-1}{2} \).
- Construct \( A = \begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_{r-1} \end{pmatrix} \).
- \( H^T = (f_r, f_{r+1}, \ldots, f_{n-1}) \) is a check matrix.
• \(H\) consists of rows \(\{e_{n-r}, e_{n-r-1}, \ldots, e_1\}\) and so the code generated by \(A\) is a DC \([n, r, n-r+1]\) MDS code in \(GF(p)\).

• Over \(GF(p^2)\) with \((u, v)_H = (u, v^p)_E\) (and different \(e_i\)) the code generated by \(A\) is a Hermitian DC \([n, r, n-r+1]\) code.

• Construct \(\left(\begin{array}{c} e_0 \\ e_1 \\ \vdots \\ e_r \\ e_{r-1} \end{array}\right) + \left(\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ e_{r+1} \end{array}\right)\) where the second matrix has \((2r-n)\) initial zero rows. This gives a convolutional \((n, r, n-r; 1, 2(n-r) + 1)\) code over \(GF(p)\). This code is not dual-containing even over \(GF(p^2)\).

Algorithm 5.8 Design infinite series of DC codes \([n_i, r_i, d_i]\), (i) with rates and rdists satisfying \(\lim_{i \to \infty} \frac{r_i}{n_i} = \frac{1}{2}\) and \(\lim_{i \to \infty} \frac{d_i}{n_i} = \frac{1}{2}\), (ii) with rates and rdists satisfying \(\lim_{i \to \infty} \frac{r_i}{n_i} = \frac{2}{q}\) and \(\lim_{i \to \infty} \frac{d_i}{n_i} = 1 - \frac{2}{q}\).

1: Consider a series of \(n_1 < n_2 < n_3 < \ldots\) and let \(p_i\) be a prime such that \(p_i \not| n_i\). Then \(\text{OrderMod}(p_i, n_i) = s_i\), for some \(s_i\).

2: Construct the Fourier \(n_i \times n_i\) matrix over \(GF(p_i^{s_i})\). Let \(r_i = \lfloor \frac{n_i}{2} \rfloor + 1\). Then \(A = \left(\begin{array}{c} e_0 \\ \vdots \\ e_{r_i-1} \end{array}\right)\) generates a DC \([n_i, r_i, n_i - r_i + 1]\) MDS code over \(GF(p_i^{s_i})\).

3: Construct the Fourier \(n_i \times n_i\) matrix over \(GF(p_i^{2s_i})\). Let \(r_i = \lfloor \frac{n_i}{2} \rfloor + 1\). Then \(A = \left(\begin{array}{c} e_0 \\ \vdots \\ e_{r_i-1} \end{array}\right)\) generates an Hermitian DC \([n_i, r_i, n_i - r_i + 1]\) MDS code over \(GF(p_i^{2s_i})\).

4: \(\lim_{i \to \infty} \frac{r_i}{n_i} = \frac{1}{2}\), \(\lim_{i \to \infty} \frac{n_i - r_i + 1}{n_i} = \frac{1}{2}\).

Algorithm 5.9 Design infinite series of DC codes \([n_i, r_i, d_i]\) over fields of given characteristic \(p\) with (i) rates and rdists satisfying \(\lim_{i \to \infty} \frac{r_i}{n_i} = \frac{1}{2}\), \(\lim_{i \to \infty} \frac{d_i}{n_i} = \frac{1}{2}\), (ii) rates and rdists satisfying \(\lim_{i \to \infty} \frac{r_i}{n_i} = \frac{2}{q} > \frac{1}{2}\), \(\lim_{i \to \infty} \frac{d_i}{n_i} = 1 - \frac{2}{q}\).

1: Consider a series of \(n_1 < n_2 < n_3 < \ldots\) where \(\gcd(p, n_i) = 1\). Then \(\text{OrderMod}(p, n_i) = s_i\) for some \(s_i\).

2: Construct the Fourier \(n_i \times n_i\) matrix over \(GF(p^{s_i})\). Let \(r_i = \lfloor \frac{n_i}{2} \rfloor + 1\). Then \(A = \left(\begin{array}{c} e_0 \\ \vdots \\ e_{r_i-1} \end{array}\right)\) is a DC \([n_i, r_i, n_i - r_i + 1]\) code over \(GF(p^{s_i})\).

3: Construct the Fourier \(n_i \times n_i\) matrix over \(GF(p^{2s_i})\). Let \(r_i = \lfloor \frac{n_i}{2} \rfloor + 1\). Then \(A = \left(\begin{array}{c} e_0 \\ \vdots \\ e_{r_i-1} \end{array}\right)\) is an Hermitian DC \([n_i, r_i, n_i - r_i]\) code over \(GF(p^{2s_i})\).

4: \(\lim_{i \to \infty} \frac{r_i}{n_i} = \frac{1}{2}, \lim_{i \to \infty} \frac{n_i - r_i + 1}{n_i} = \frac{1}{2}\).

(ii) The general fraction \(R = \frac{p}{q}\) may be obtained by choosing \(r_i = \lfloor \frac{p}{q} n_i \rfloor\) in item 2.
By taking a series of odd integers \(2n_1 + 1 < 2n_2 + 1 < \ldots\) infinite such series are obtained over fields of characteristic 2. The odd series \((2^2 - 1) < (2^3 - 1) < (2^4 - 1) < \ldots\) is particularly noteworthy. Note OrderMod\((2, n) = s\) for \(2^s - 1 = n\) and \(GF(2^s)\) contains an element of order \(2^s - 1\) describing all the non-zero elements of \(GF(2^s)\). The Fourier matrix of size \(n \times n\) may then be constructed over \(GF(2^s)\). This has many nice consequences for designing codes of different types in characteristic 2.

This is illustrated as follows for the case \(GF(2^5)\).

**Example 5.1** Construct codes and particular types of codes over \(GF(2^5)\) and Hermitian such codes over \(GF(2^{10})\).

Let \(F_{15} = F\) be the Fourier matrix of size \(15 \times 15\) over \(GF(2^5)\) or as relevant over \(GF(2^{10})\).

1: For \(r > 7\), \(A = \left( \begin{array}{c} e_0 \\ \vdots \\ e_{r-1} \end{array} \right)\) generates a DC MDS code \([15, r, 15 - r + 1]\).

2: For \(r > 7\), \(A = \left( \begin{array}{c} e_0 \\ \vdots \\ e_{r-1} \end{array} \right)\) considered as a matrix over \(GF(2^{10})\) generates an Hermitian DC MDS code \([15, r, 15 - r + 1]\).

3: \(G[z] = \left( \begin{array}{c} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \\ e_8 \\ e_9 \\ e_{10} \\ e_{11} \\ e_{12} \\ e_{13} \\ e_{14} \end{array} \right) + \left( \begin{array}{c} 0 \\ e_8 \\ e_9 \\ e_{10} \\ e_{11} \\ e_{12} \\ e_{13} \\ e_{14} \end{array} \right) z\) generates an MDS convolutional code \((15, 8, 7; 1, 15)\). This has distance twice the distance less 1 of the linear MDS code \([15, 8, 8]\). But also this code is an LCD code.

4: \(A = \left( \begin{array}{c} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \\ e_8 \end{array} \right)\) generates a LCD, MDS code \([15, 9, 7];\) see [S] for details. \(A * (f_5, f_{10}, f_6, f_9, f_7, f_8) = 0\).

5: Consider \(A\) as above constructed in \(GF(2^{10})\). Then \(A\) is a Hermitian LCD code over \(GF(2^{10})\).

6: \(A = \left( \begin{array}{c} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \\ e_8 \end{array} \right), B = \left( \begin{array}{c} 0 \\ 0 \\ e_{10} \\ e_9 \\ e_6 \\ e_7 \\ e_8 \end{array} \right)\) designs the MDS convolutional code \((15, 9, 6; 1, 13)\), generated by \(G[z] = A + Bz\). This code in addition is a dual-containing MDS convolutional code.

7: By taking \(r = \lfloor \frac{3}{2} 15 \rfloor + 1 = 12\) or \(r = \lfloor \frac{3}{2} 15 \rfloor = 11\) codes of rate ‘near’ \(\frac{3}{2}\) are obtained; in this case codes of rate \(\frac{12}{15} = \frac{4}{5}\) or \(\frac{11}{15}\) attaining the MDS are obtained.

**Algorithm 5.10** Construct infinite series MDS codes of various types of linear block and convolutional codes over fields of the form \(GF(2^i)\) and Hermitian such codes over fields of the form \(GF(2^{2i})\).

1. First note OrderMod\((2, 2^i - 1) = i\) for given \(i\).

2. Construct the \(n_i \times n_i\) Fourier matrix \(F_{n_i}\) over \(GF(2^i)\) with \(n_i = 2^i - 1\).
3. For \( r_i > \frac{3n}{4} \) let \( C_{r_i} \) be the code generated by rows \( \langle e_0, e_1, \ldots, e_{r_i-1} \rangle \). Then \( r_i \) is an \([n_i, r_i, n_i - r_i + 1]\) dual-containing MDS linear code over \( GF(2^i) \).

This gives an infinite series of codes \( C_{r_i} \) of type \([n_i, r_i, n_i - r_i + 1]\).

4. For \( r_i = \frac{3n}{2} \) this gives an infinite series \( C_{r_i} \) with \( \lim_{i \to \infty} \frac{r_i}{n_i} = \frac{1}{2} \) and \( \lim_{i \to \infty} \frac{d_i}{n_i} = \frac{1}{2} \) where \( d_i = n_i - r_i + 1 \) is the distance.

5. For \( r_i = \lfloor \frac{3n}{4} \rfloor \) this gives an infinite series \( C_{r_i} \) with \( \lim_{i \to \infty} \frac{r_i}{n_i} = \frac{3}{4} \) and \( \lim_{i \to \infty} \frac{d_i}{n_i} = \frac{1}{4} \) where \( d_i = n_i - r_i + 1 \) is the distance.

6. Other such infinite series can be obtained with different fractions by letting \( r_i = \lfloor Rn_i \rfloor \) (\( R \) rational) and then series \( C_{r_i} \) with \( \lim_{i \to \infty} \frac{r_i}{n_i} = R \) and \( \lim_{i \to \infty} \frac{d_i}{n_i} = 1 - R \) are obtained where \( d_i = n_i - r_i + 1 \) is the distance.

Construct the \( n_i \times n_i \) Fourier matrix \( F_{n_i} \) over \( GF(2^{2i}) \) with \( n_i = 2^i - 1 \). For \( r_i > \frac{n_i}{2} \) let \( C_{r_i} \) be the code generated by rows \( \langle e_0, e_1, \ldots, e_{r_i-1} \rangle \). Then \( C_{r_i} \) is an \([n_i, r_i, n_i - r_i + 1]\) DC MDS Hermitian linear code over \( GF(2^{2i}) \).

This gives an infinite series of \( C_{r_i} \) of type \([n_i, r_i, n_i - r_i + 1]\) which are Hermitian DC.

For \( r_i = \frac{3n}{2} \) this gives an infinite series of Hermitian DC \( C_{r_i} \) codes with \( \lim_{i \to \infty} \frac{r_i}{n_i} = \frac{1}{2} \) and \( \lim_{i \to \infty} \frac{d_i}{n_i} = \frac{1}{2} \) where \( d_i = n_i - r_i + 1 \) is the distance.

\[ r_i = \lfloor \frac{3n}{4} \rfloor \] designs an infinite series of Hermitian DC codes \( C_{r_i} \) with \( \lim_{i \to \infty} \frac{r_i}{n_i} = \frac{3}{4} \) and \( \lim_{i \to \infty} \frac{d_i}{n_i} = \frac{1}{4} \) where \( d_i = n_i - r_i + 1 \) is the distance.

Other such infinite series of Hermitian codes can be obtained with different fractions by choosing \( r_i = \frac{3n}{4} \) for a fraction \( \frac{3}{4} < 1 \).

**Algorithm 5.11** Construct infinite series of codes of various ‘types’ over prime fields \( GF(p^i) \) and with Hermitian inner product over fields \( GF(p^2) \).

Let \( p_1 < p_2 < \ldots \) be an infinite series of primes.

For \( p_i \) construct the Fourier \( n_i \times n_i \) matrix over \( GF(p_i) \) where \( n_i = p_i - 1 \).

For \( r_i > \frac{n_i}{2} \) let \( C_{r_i} \) be the code generated by rows \( \langle e_0, e_1, \ldots, e_{r_i-1} \rangle \). Then \( C_{r_i} \) is an \([n_i, r_i, n_i - r_i + 1]\) DC MDS linear code over \( GF(p_i) \). The arithmetic is modular arithmetic.

This gives an infinite series of codes \( C_{r_i} \) of type \([n_i, r_i, n_i - r_i + 1]\).

For \( r_i = \frac{3n}{2} + 1 \) this gives an infinite series \( C_{r_i} \) of type \([n_i, r_i, n_i - r_i + 1]\) DC codes over prime fields with \( \lim_{i \to \infty} \frac{r_i}{n_i} = \frac{1}{2} \) and \( \lim_{i \to \infty} \frac{d_i}{n_i} = \frac{1}{2} \) where \( d_i = n_i - r_i + 1 \) is the distance.

For \( r_i = \lfloor \frac{3n}{4} \rfloor \) this gives an infinite series of such \( C_{r_i} \) with \( \lim_{i \to \infty} \frac{r_i}{n_i} = \frac{3}{4} \) and \( \lim_{i \to \infty} \frac{d_i}{n_i} = \frac{1}{4} \) where \( d_i = n_i - r_i + 1 \) is the distance.

For \( r_i = \lfloor \frac{3n}{8} \rfloor \) infinite series are obtained with \( \lim_{i \to \infty} \frac{r_i}{n_i} = \frac{3}{8} \) and \( \lim_{i \to \infty} \frac{d_i}{n_i} = 1 - \frac{3}{8} \) where \( d_i = n_i - r_i + 1 \) is the distance.

Construct the \( n_i \times n_i \) Fourier matrix \( F_{n_i} \) over \( GF(p_i^2) \). For \( r_i > \frac{n_i}{2} \) let \( C_{r_i} \) be the code generated by rows \( \langle e_0, e_1, \ldots, e_{r_i-1} \rangle \). Then \( C_{r_i} \) is an \([n_i, r_i, n_i - r_i + 1]\) Hermitian DC MDS linear code over \( GF(p_i^2) \).

This gives an infinite series of \( C_{r_i} \) of type \([n_i, r_i, n_i - r_i + 1]\) which are Hermitian DC over \( GF(p_i^2) \).
For \( r_i = \left\lfloor \frac{n_q}{4} \right\rfloor + 1 \) this gives an infinite series of Hermitian DC \( \mathcal{C}_{r_i} \) codes with \( \lim_{i \to \infty} \frac{r_i}{n_i} = \frac{1}{2} \) and \( \lim_{i \to \infty} \frac{d_i}{n_i} = \frac{1}{2} \) where \( d_i = n_i - r_i + 1 \) is the distance.

For \( r_i = \left\lfloor \frac{3n_q}{4} \right\rfloor \) this gives an infinite series of Hermitian DC \( \mathcal{C}_{r_i} \) codes with \( \lim_{i \to \infty} \frac{r_i}{n_i} = \frac{3}{4} \) and \( \lim_{i \to \infty} \frac{d_i}{n_i} = \frac{1}{4} \) where \( d_i = n_i - r_i + 1 \) is the distance.

For \( r_i = \left\lceil \frac{p}{q} n_i \right\rceil \) an infinite series are obtained with \( \lim_{i \to \infty} \frac{r_i}{n_i} = \frac{p}{q} \) and \( \lim_{i \to \infty} \frac{d_i}{n_i} = 1 - \frac{p}{q} \) where \( d_i \) is the distance.

The following algorithm explains how to design convolutional codes with order twice the distance of the corresponding MDS code of the same length and rate.

**Algorithm 5.12** Design convolutional MDS memory 1 codes to the order of twice the distance of the linear block MDS codes of the same length and rate.

1. Let \( F_n \) be a Fourier \( n \times n \) matrix. Denote the rows in order of \( F_n \) by \( \{e_0, e_1, \ldots, e_{n-1}\} \) and \( n \) times the columns of the inverse of \( F_n \) by \( \{f_0, f_1, \ldots, f_{n-1}\} \). Then \( e_i f_j = \delta_{ij} \). \( f_i^T = e_{n-i} \) with indices taken mod \( n \).

2. Let \( r > \left\lfloor \frac{n}{2} \right\rfloor \). Let \( A \) be the matrix with first \( r \) rows \( \{e_0, e_1, \ldots, e_{r-1}\} \) of \( F_n \) and let \( B \) be the matrix whose last rows are \( \{e_r, \ldots, e_{n-1}\} \) in order and whose first \( (n-r) \) rows consists of zero vectors.

3. Define \( G[z] = A + Bz \). Then \( G[z] \) is a generating matrix for a non-catastrophic convolutional code \( (n, r, n-r; 1, 2(n-r)+1) \) of free distance \( 2(n-r)+1 \). A control matrix is easily written down, as is a right inverse for \( G[z] \).

Thus the MDS convolutional code produced of rate \( \frac{n}{r} \) has twice the distance, less 1, of the MDS linear code \( [n, r, r-n+1] \) with the same rate and length.

Note that \( r \) can be any integer > \( \left\lfloor \frac{n}{2} \right\rfloor \) so all rates \( \frac{n}{r} \) for \( \left\lfloor \frac{n}{2} \right\rfloor < r < n \) are obtainable. The dual code has rate \( (1-R) \) where \( R = \frac{n}{r} \) is the rate of \( \mathcal{C} \) so rates \( R \) with \( R < \frac{1}{2} \) are obtainable.

Alternatively item 2. of Algorithm 5.12 may be replaced by taking rows in arithmetic sequence as follows:

Let \( r > \left\lfloor \frac{n}{2} \right\rfloor \) and \( A \) be formed from \( F_n \) by taking \( r \) rows in geometric sequence with geometric difference \( k \) satisfying \( \text{gcd}(k, n) = 1 \). Define \( B \) to be the matrix whose last rows are the other rows of \( F_n \) not in \( A \) (which also are in geometric sequence satisfying the gcd condition) and whose first \( (n-r) \) rows consist of zero vectors.

The methods are illustrated in the following examples.

**Example 5.2** (i) Construct DC MDS codes of length 255 of various permissible rates over \( GF(2^8) \).

(ii) Construct Hermitian DC MDS codes of length 255 of various permissible rates over \( GF(2^{16}) \).

(iii) Construct QECC MDS codes of length 255 of various rates over \( GF(2^8) \).

(iv) Construct Hermitian QECC MDS codes of length 255 of various rates over \( GF(2^{16}) \).

(v) Construct LCD MDS codes of length 255 of various rates over \( GF(2^8) \).

(vi) Construct Hermitian LCD MDS codes of length 255 of various rates over \( GF(255^2) \).
1. Over $GF(2^8)$ construct the Fourier $255 \times 255$ matrix $F$.

2. For $r > \lfloor \frac{255}{2} \rfloor = 127$ let $A$ be the code generated by the first $r$ rows of $F$. Then $A$ is a DC $[255, r, 255 - r + 1]$ code.

3. For $r = 128$ the DC $[255, 128, 128]$ code of rate about $\frac{1}{2}$ and rdist of about $\frac{1}{7}$ is designed.

4. For $r = \lfloor \frac{255 \times 3}{4} \rfloor = 191$ the code $[255, 191, 65]$ of rate about $\frac{3}{4}$ and rdist of about $1/4$ is obtained.

5. For $r = \lfloor \frac{255 \times 7}{8} \rfloor = 223$ the code $[255, 223, 33]$ of rate about $\frac{7}{8}$ and rdist of about $\frac{1}{8}$ is obtained. This can correct 16 errors.

6. Over $GF(2^{16})$ construct the Fourier $255 \times 255$ matrix.

7. For $r > \lfloor \frac{255}{2} \rfloor = 127$, let $A$ be the code generated by the first $r$ rows of $F$. Then $A$ is an Hermitian DC $[255, r, 255 - r + 1]$ code.

8. For $r = 128$ the Hermitian DC $[255, 128, 128]$ code of rate about $\frac{1}{2}$ and rdist of about $\frac{1}{5}$ is designed.

9. For $r = \lfloor \frac{255 \times 3}{4} \rfloor = 191$ the Hermitian DC code $[255, 191, 65]$ of rate about $\frac{3}{4}$ and rdist of about $\frac{1}{4}$ is obtained.

10. For $r = \lfloor \frac{255 \times 7}{8} \rfloor = 223$ the Hermitian DC code $[255, 223, 33]$ of rate about $\frac{7}{8}$ and rdist of about $\frac{1}{8}$ is obtained. This can correct 16 errors.

To obtain the QECCs, apply the CSS construction to the DC codes formed.

LCD codes are designed as follows; see [8] where the method is devised.

1. Let $C$ be the code generated by the rows $\langle e_0, e_1, e_{254}, e_2, e_{253}, e_3, e_{252}, \ldots, e_r, e_{255-r} \rangle$ for $2r < 255$.

2. Then $C$ is a $[255, 2r + 1, 255 - 2r]$ code; notice that the rows of $A$ are in sequence $\{255 - r, 255 - r + 1, \ldots, 254, 0, 1, 2, \ldots, r\}$ so the code is MDS.

3. The dual code of $C$ is the code generated by the transpose of $(f_{255-r-1}, f_{r+1}, f_r+2, f_{255-r-2}, \ldots, f_r+2, f_{255-r-2})$ and this consists of rows $\{e_{r+1}, e_{255-r-1}, \ldots, e_{255-r-2}, e_r+2\}$. Thus $C \cap C^\perp = \{0\}$ and so the code is an LCD MDS code.

4. To get LCD MDS Hermitian codes of length 255 of various rates as above, work in $GF(2^{16})$.

**Lemma 5.1** Let $A, B, C, D$ be matrices of the same size $r \times n$. Suppose the code generated by $A$ intersects trivially the code generated by $C$ and the code generated by $B$ intersects trivially the code generated by $D$. Then the convolutional code generated by $A + Bz$ intersects trivially the convolutional code generated by $C \pm Dz$.

**Proof:** Compare coefficients of $P[z](A + Bz)$ with coefficients of $Q[z](C \pm Dz)$ for $1 \times r$ polynomial vectors $P[z], Q[z], P[z] = P_0 + P_1z + \ldots, Q[z] = Q_0 + Q_1z + \ldots$. In turn get $P_0 = 0 = Q_0$ then $P_1 = 0 = Q_1$ and so on. □ Using this, LCD MDS convolutional codes may be designed leading on from the DC codes designed in Algorithm 5.4.
Algorithm 5.14 Design MDS LCD convolutional codes of the order of twice the distance of the MDS DC block code with the same length and rate. First of all design the DC codes as in Algorithm 5.12.

1. Let $F_n$ be a Fourier $n \times n$ matrix.

2. Let $r > \lfloor \frac{n}{2} \rfloor$. Define $A$ to be the matrix of the first $r$ rows $(e_0, e_1, \ldots, e_{r-1})$ of $F_n$ and define $B$ to be the matrix whose last rows are $e_r, \ldots, e_{n-1}$ in order and whose first $n-r$ rows consist of zero vectors.

3. Define $G[z] = A + Bz$. Then $G[z]$ is a generating matrix for a non-catastrophic convolutional code $(n, r, n-r; 1, 2(n-r)+1)$ of free distance $2(n-r)+1$. A check matrix is easily written down, as is a right inverse for $G[z]$.

4. The code generated by $G[z]$ is an LCD MDS convolutional code. This is shown as follows:

A control matrix for the code is $H^T[z] = (f_r, \ldots, f_{n-1}) - (f_{n-r}, f_{n-r+1}, \ldots, f_{r-1})z$. Then

$$H[z^{-1}] = \begin{pmatrix} e_{n-r} & e_{n-r+1} & \cdots & e_1 \\ e_{n-r-1} & e_{n-r} & \cdots & e_0 \\ \vdots & \vdots & \ddots & \vdots \\ e_1 & e_2 & \cdots & e_{n-r+1} \\ e_0 & e_1 & \cdots & e_{n-r} \end{pmatrix} z^{-1}.$$

Thus a control matrix is $\begin{pmatrix} e_{n-r} & e_{n-r+1} & \cdots & e_1 \\ e_{n-r-1} & e_{n-r} & \cdots & e_0 \\ \vdots & \vdots & \ddots & \vdots \\ e_1 & e_2 & \cdots & e_{n-r+1} \end{pmatrix}$ equal to say $-C + Dz$. Now the code generated by $C$ has trivial intersection with the code generated by $A$ and the code generated by $D$ has trivial intersection with the code generated by $B$. Hence by Lemma 5.1 the convolutional code $C$ generated by $A + Bz$ has trivial intersection with the code generated by $C - Dz$. Hence $C$ is an LCD convolutional MDS $(n, r, n-r; 1, 2(n-r)+1)$ code.

5.1 QECC Hermitian

Of particular interest are QECCs with Hermitian inner product. These need to be designed over fields $GF(q^2)$ where the Hermitian inner product is defined by $\langle u, v \rangle_H = \langle u, v^q \rangle_E$. Hermitian QECCs can be designed by the CSS construction from DC Hermitian codes. A separate algorithm is given below although it follows by similar methods to those already designed.

Algorithm 5.14 Construct QECCs over $GF(q^2)$.

$GF(q^2)$ has an element of order $q^2 - 1$ and hence an element of order $q-1$ as $q^2 - 1 = (q-1)(q+1)$. Let $\omega$ be an element of order $(q-1)$ in $GF(q^2)$. Let $n = (p-1)$ and construct the Fourier $n \times n$ matrix $F_n$ with this $\omega$. The rows of $F_n$ are denoted by $\{e_0, e_1, \ldots, e_{n-1}\}$.

Let $r > \lfloor \frac{n}{2} \rfloor$ and define $A$ to be the matrix with rows $\{e_0, \ldots, e_{r-1}\}$. Then the code generated by $A$ is an Hermitian DC MDS $[n, r, n-r+1]$ code over $GF(q^2)$.

Use the CSS construction to form a QECC MDS Hermitian code $[[n, 2r - n, n-r + 1]]$ code.

For $r = \lfloor \frac{n}{4} \rfloor + 1$ a DC MDS code of rate about $\frac{1}{4}$ is obtained and an MDS QECC of rate 0 and rdist of about $\frac{1}{2}$. For $r = \lfloor \frac{3n}{4} \rfloor$ a DC MDS code is obtained of rate about $\frac{3}{4}$ and a MDS, QECC of rate about $\frac{1}{2}$ and rdist of about $\frac{3}{4}$.

Higher rates may be obtained.
Infinite series of such codes may also be obtained. The following Algorithm gives an infinite series of characteristic 2 such codes but other characteristics are obtained similarly; the characteristics may be mixed.

Algorithm 5.15 Construct infinite series of characteristic 2 Hermitian DC \([n_i, r_i, n_i - r_i + 1]\) codes \(C_i\) in which (i) \(\lim_{i \to \infty} \frac{r_i}{n_i} = R\), for \(1 > R \geq \frac{1}{2}\) and (ii) \(\lim_{i \to \infty} \frac{d_i}{n_i} = 1 - R\); \(R\) here is rational. From this derive infinite series of Hermitian MDS QECCs \(D_i\) of form \([n_i, 2r_i - n_i, n_i - r_i + 1]\) in which \(\lim_{i \to \infty} \frac{2r_i - n_i}{n_i} = (2R - 1)\), and (i) \(\lim_{i \to \infty} \frac{d_i}{n_i} = (1 - R)\).

Consider \(GF(2^8)\). This has an element of order \((2^i - 1) = n_i\) and use this to form the Fourier \(n_i \times n_i\) matrix over \(GF(2^8)\). Let \(r_{j,i} > \left\lfloor \frac{n_i}{2} \right\rfloor\) and \(A\) be the matrix with rows \(\{e_0, \ldots, e_{r_{j,i} - 1}\}\). Then the code \(C_{i,j}\) generated by \(A\) is a Hermitian dual-containing code \([n_i, r_{j,i}, n_i - r_{j,i} + 1]\) code. This gives an infinite series \(C_{i,j}\) of Hermitian dual-containing codes in characteristic 2. The \(r_i, j\) can vary for each \(GF(2^8)\).

Now fix \(r_{j,i} = \left\lfloor \frac{n_i}{2} \right\rfloor + 1 = r_i\) for each \(C_{i,j}\) and let \(C_i\) be the codes obtained. This gives the infinite series \(C_i\) of \([n_i, r_i, n_i - r_i + 1]\) codes and \(\lim_{i \to \infty} \frac{r_i}{n_i} = \frac{1}{2}\), \(\lim_{i \to \infty} \frac{d_i}{n_i} = \frac{1}{2}\).

Fixing \(r_{j,i} = \left\lfloor \frac{n_i}{2} \right\rfloor = r_i\) gives an infinite series \(C_i\) of \([n_i, r_i, n_i - r_i + 1]\) codes with \(\lim_{i \to \infty} \frac{r_i}{n_i} = \frac{3}{4}\), \(\lim_{i \to \infty} \frac{d_i}{n_i} = \frac{1}{4}\).

Fixing \(r_{j,i} = \left\lfloor \frac{n_i}{2} \right\rfloor = r_i\), \(1/2 < p/q < 1\) gives an infinite series \(C_i\) of \([n_i, r_i, n_i - r_i + 1]\) codes and \(\lim_{i \to \infty} \frac{r_i}{n_i} = \frac{p}{q}\), \(\lim_{i \to \infty} \frac{d_i}{n_i} = 1 - \frac{p}{q}\).

The infinite series of Hermitian QECCs with limits as specified is immediate.

5.2 Higher memory

Higher memory MDS convolutional codes may be obtained by this general method of using all the rows of an invertible ‘good’ matrix. The principle is established in [42] where rows of an invertible matrix are used to construct convolutional codes. Here just one example is given and the general construction is left for later work; some extremely nice codes are obtainable by the method.

Example 5.3 Consider again, Example 3.1, the Fourier \(7 \times 7\) matrix over \(GF(3^2)\) with rows \(\{e_0, \ldots, e_6\}\) and 7 the columns of the inverse denoted by \(\{f_0, \ldots, f_6\}\).

Construct \(G[z] = \left( \begin{array}{c} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{array} \right) z + \left( \begin{array}{c} e_0 \\ e_4 \\ e_5 \end{array} \right) z^2\)

Then \(G[z]\) is a convolutional code of type \((7, 3, 5; 2)\); the degree is 5. The GSB for such a code is \(7 - 3)(\frac{5}{2} + 1) + 5 + 1 = 4 \ast 2 + 5 + 1 = 14\). In fact the free distance is actually 14. This may be shown in an analogous way to the proof of Lemma 3.1.

\(G[z] * (f_3, f_4, f_5, f_6) - (f_1, f_2, 0, 0) - (0, 0, f_0, f_2) z^2) = 0, G[z] * (f_0, f_1, f_2) = 7I_3\).

The result is that the code generated by \(G[z]\) is a non-catastrophic convolutional MDS \((7, 3, 5; 2, 14)\) code.

Note the free distance attained is \(5 \ast 3 - 1\) where \(5\) is the free distance of a \([7, 3, 5]\) MDS code; the distance is tripled less 1. This is a general principle – the free distance is of order three times the distance of the same length and dimension MDS code.

It’s not a dual-containing code nor a LCD code. To get such codes requires a compromise on the distance. \(G[z] = \left( \begin{array}{c} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{array} \right) z + \left( \begin{array}{c} 0 \\ e_4 \\ e_5 \end{array} \right) z^2\). This gives \((7, 3, 4; 2)\) convolutional code which turns out to be an LCD code but the free distance is only 7. The GSB for such a code is 13.
Convolutional codes \((n, r, \delta)\) have maximum free distance \(n - r + |\lfloor \frac{\delta}{r} \rfloor| + 1 + \delta + 1\). When \(r > \delta\) this maximum free distance is \(n - r + \delta + 1\). Here is a design method for maximum free distance memory 1 convolutional codes.

**Example 5.4** Construct a convolutional code of rate \(\frac{15}{16}\) which has free distance \(\geq 61\).

It is required to construct a \((n, r, n-r)\) convolutional code such that \(\frac{r}{n} \geq \frac{15}{16}\) and \(2(n-r)+1 \geq 61\). Thus require \((n-r) \geq 60\) and hence require \(n(1-R) \geq 30\). Thus \(n \geq 30/1-R \geq 30*16 = 480\).

Construct the Fourier \(480 \times 480\) matrix over a suitable field. Require \(r \geq \frac{15}{16}\) and \(r \geq \frac{15}{16}*480 = 450\).

Now by Algorithm 5.12 construct the \((480, 450, \delta)\) convolutional code with \(\delta = 30\). This code has free distance \(2(n-r) + 1 = 60 + 1\) as required. The rate is \(\frac{450}{480} = \frac{15}{16}\).

The Fourier \(480 \times 480\) may be constructed over a field of characteristic \(p\) where \(\gcd(p, 480) = 1\). Now \(7^4 \equiv 1 \mod 480\) so the field \(GF(7^4)\) can be used. This has an element of order 480 and the Fourier matrix of \(480 \times 480\) exists over \(GF(7^4)\).

Suppose now a field of characteristic 2 for example is required. Then replace “\(n \geq 480\)” by “\(n \geq 480\) and \(\gcd(2, n) = 1\)”. As we shall see, it is convenient to take \(n\) to be \(2^s - 1\) and in this case take \(n = 2^6 - 1 = 511\) in which case the arithmetic is done in \(GF(2^6)\).

The first prime greater than \(480\) is \(487\) so the construction can be done over the prime field \(GF(487)\).

These codes have twice the error-correcting capability as MDS codes and of the same rate so should be very useful as codes.

Series of block linear codes which are DC, QECCs, LDC are designed in the Algorithms 5.1 to 5.4. Now we work on the types of convolutional codes that can be formed from these types when extending according to Algorithm 5.12. Thus design methods for convolutional DC, QECCs and LCD codes are required.

### 5.2.1 Comment

From a recent article: “Far more efficient quantum error-correcting codes are needed to cope with the daunting error rates of real qubits. The effort to design better codes is “one of the major thrusts of the field,” Aaronson said, along with improving the hardware.

Ahmed Almheiri, Xi Dong and Daniel Harlow \[6\] did calculations suggesting that this holographic “emergence” of space-time works just like a quantum error-correcting code. They conjectured in the Journal of High Energy Physics that space-time itself is a code — in anti-de Sitter (AdS) universes, at least. This lead to a wave of activity in the quantum gravity community, leading to new new impulse to quantum error-correcting codes that could capture more properties of space-time.

What this is saying is that “Ahmed Almheiri, Xi Dong and Daniel Harlow originated a powerful new idea that the fabric of space-time is a quantum error-correcting code”.

### 5.3 DC LCD convolutional

Convolutional DC codes over fields of characteristic 2 may be designed as follows.
Algorithm 5.16  1. Let $F_{2m+1}$ be a Fourier matrix over a field of characteristic 2. Denote its rows in order by $\{e_0, e_1, \ldots, e_{n-1}\}$ and the columns of its inverse times $n$ is denoted by $\{f_0, f_1, \ldots, f_{n-1}\}$ in order, where $n = 2m + 1$. Then $e_if_j = ij, f_iT = e_{n-i}, e_iT = f_{n-i}$.

2. Choose the matrix $A$ as follows. Let $e_0$ be its first row and then choose $r$ pairs $\{e_i, e_{n-i}\}$ for the other rows and such that $2r \geq m$. Thus $A$ has $(2r + 1)$ rows and $A$ is an $(2r + 1) \times n$ matrix.

3. Choose $B$ with first $(4r - 2m + 1)$ rows consisting of the zero vector and the other $2(m - r)$ rows consisting of the rest of the pairs $e_i, e_{n-i}$ ($m - r$ pairs) not used in item $\underline{2}$. Then $B$ is a $(2r + 1) \times n$ matrix.

4. Construct $G[z] = A + Bz$.

5. $G[z]$ generates a convolutional dual-containing code from which quantum convolutional codes may be constructed. The control matrix of the code is easy to construct. There is a matrix $K$ such that $GK = I_{2r+1}$ thus ensuring the code is non-catastrophic. The degree, $\delta$, of the code is $2(m - r)$.

6. The GSB of such a $(n, 2r + 1, \delta)$ code is $(n - 2r - 1)(\lfloor \frac{\delta}{2r + 1} \rfloor + \delta + 1) = n - 2r + \delta = n - 2r + 2m - 2r = 4(m - r) + 1$.

7. It may be shown that the code generated by $G[z]$ is an MDS convolutional MDS $(n, 2r + 1, 2(m - r) + 1, 4(m - r) + 1)$ code.

Consider the field $GF(2^n)$. This has an element of order $2^n - 1 = q$ and it seems best to construct the Fourier $F_q \times F_q$ over $GF(2^n)$.

Example 5.5  Construct $31 \times 31$ DC convolutional codes.

The order of 2 mod 31 is 5 and thus work in the field $GF(2^5)$. Form the $F_{31} \times F_{31}$ matrix over $GF(2^5)$. Now proceed as in Algorithm 5.16. For example let $\{A, B\}$ have rows $\langle e_0, e_1, e_3, e_2, e_9, e_3, e_2, e_4, e_27, e_5, e_6, e_25, e_7, e_24, e_8, e_23 \rangle$, $\langle 0, 0, 0, e_9, e_22, e_{10}, e_{21}, e_{11}, e_{20}, e_{12}, e_{19}, e_{13}, e_{18}, e_{14}, e_{17}, e_{15}, e_{16} \rangle$ respectively.

Now form $G[z] = A + Bz$. The code generated by $G[z]$ is then a $(31, 17, 14)$ DC convolutional code. The GSB for such a code is $(n - r)(\lfloor \frac{\delta}{r} \rfloor + 1) + \delta + 1 = (14)(1) + 14 + 1 = 29$. The generators of $A$ may be arranged in arithmetic sequence with difference 1 and so these form a $[31, 17, 15]$ MDS linear code. Similarly the non-zero vectors in $B$ generate a $[31, 14, 18]$ MDS linear code. Using these it may be shown that this code is an MDS convolutional code.

A quantum convolutional code may be designed from this.

Larger rate DC convolutional MDS codes may also be derived.

For example let $A, B$ have rows $\langle e_0, e_1, e_3, e_2, e_9, e_3, e_2, e_4, e_27, e_5, e_6, e_25, e_7, e_24, e_8, e_23, e_9, e_{22}, e_{10}, e_{21} \rangle$, $\langle 0, 0, 0, 0, 0, 0, e_{11}, e_{20}, e_{12}, e_{19}, e_{13}, e_{18}, e_{14}, e_{17}, e_{15}, e_{16} \rangle$ respectively.

This gives a $(31, 21, 10)$ DC code. The GSB for such a code is $(n - r)(\lfloor \frac{\delta}{r} \rfloor + 1) + \delta + 1 = 10 + 11 = 21$. The free distance of this code is exactly 21. Similarly $(31, 23, 8)$ codes with free distance 17, $(31, 25, 6)$ with free distance 11 and so on may be obtained.
5.4 Addendum

It is shown in [12] how orthogonal matrices may be used to construct convolutional codes. Using orthogonal matrices does not allow the same control on the distances achieved as can for Vandermonde/Fourier matrices.

Low density parity check (LDPC) codes have important applications in communications. Linear block LDPC codes are constructed algebraically in [50] and the methods can be extended to obtain convolutional LDPC codes. This is dealt with separately.

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