On the total edge irregularity strength of some copies of ladder graphs

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Abstract. Let $G = (V(G), E(G))$ be a graph and $k$ be a positive integer. A total $k$-labeling of $G$ is a map $f : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$. The edge weight $uv$ under the labeling $f$ is defined by $w_f(uv) = f(u) + f(uv) + f(v)$ and denoted by $w_f(uv)$ and. A total $k$-labeling of $G$ is called edge irregular if every two distinct edges have distinct weight. The total edge irregularity strength of $G$ is denoted by $tes(G)$ and defined by the minimum $k$ such that $G$ has an edge irregular total $k$-labeling. The labeling was introduced by Bača et al. in 2007. In this paper, we determine the total edge irregularity strength of some copies of ladder graphs.

1. Introduction

A total $k$-labeling of a graph $G$ is a map $f : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ for a positive integer $k$. A kind of total $k$-labeling of graph was introduced in 2007 by Bača et al. Name of the labeling is edge irregular total $k$-labeling. [1] The definition of the labeling is given in Definition 1.1. Definition 1.1. [1] Let $G = (V,E)$ be a graph. A total $k$-labeling $f : V \cup E \to \{1, 2, \ldots, k\}$, for an integer $k$, is called an edge irregular total $k$-labelling of $G$ if every two distinct edges $e = uv$ and $f = wx$ in $E$ satisfy $w_f(e) \neq w_f(f)$, where $w_f(e)$ is defined by $f(u) + f(e) + f(v)$.

The notation $w_f(e)$ is called by the weight of $e$ under the labeling $f$. Definition 1.2. [1] The total edge irregularity strength of $G$, denoted by $tes(G)$, is the minimum $k$ for which a graph $G$ has an edge irregular total $k$-labeling. A lower bound and an upper bound on $tes(G)$ for arbitrary graph $G$ is given by Bača et al [1]. The bounds are written in Theorem 1.1. Theorem 1.1 [1] Let $G = (V, E)$ be a graph, $V$ be the vertex set of $G$ and $E$ be the non empty edge set of $G$. Then

$$\left\lceil \frac{|E| + 2}{3} \right\rceil \leq tes(G) \leq |E|.$$ 

In the paper, Bača et al [1] also gave $tes(G)$ for $G$ are paths, cycles, and friendships [1]. Path with order $n$ is a connected graph with 2 vertices with degree 1 and $n - 2$ vertices with degree 2. Cycle with order $n$ is a connected graph such that every vertices have degree 2. The exact values of $tes$ of path and
cycle are given in Theorem 1.2 and Theorem 1.3, respectively. Theorem 1.2 [1] Let \( P_n \) be a path with \( n \) vertices. Then,

\[
tes(P_n) = \left\lfloor \frac{n+1}{3} \right\rfloor.
\]

Theorem 1.3 [1] Let \( C_n \) be a cycle with \( n \) vertices. Then,

\[
tes(C_n) = \left\lfloor \frac{n+2}{3} \right\rfloor.
\]

The *friendship graph* \( F_n \) is visualised by \( n \) triangles stick to a common vertex. Note that the number of edges and vertices of \( F_n \), respectively are \( 3n \) and \( 2n + 1 \). [1] The exact value of \( tes(F_n) \) is given in Theorem 1.4. Theorem 1.4 [1] \( tes(F_n) = \left\lfloor \frac{3n+2}{3} \right\rfloor \). A conjecture about tes of arbitrary graph \( G \neq K_5 \) was posed by Ivančo and S. Jendrol’ [2] as follows.

\[
tes(G) = \max \left\{ \left\lfloor \frac{|E(G)|+2}{3} \right\rfloor, \left\lfloor \frac{\Delta(G)+1}{2} \right\rfloor \right\}.
\]

Ramdani, et al [3] gave us an upper bound on tes of disjoint union of graphs. The result can be seen in Theorem 1.5. Theorem 1.5 [3] Let \( G_1, G_2, \ldots, G_m, m \geq 2 \), be graphs. The tes of disjoint union of the graphs is

\[
tes \left( \bigcup_{i=1}^{m} G_i \right) \leq \sum_{i=1}^{m} tes(G_i) - \left\lfloor \frac{m-1}{2} \right\rfloor.
\]

In Ref. [4], Nurdin et al gave the tes of the corona product of paths and some graphs. The *corona product* of a graph \( G \) with a graph \( H \), denoted by \( G \odot H \), is a graph obtained by taking one copy of a graph \( G \) with \( n \) vertices and \( n \) copies \( H_1, H_2, \ldots, H_n \) of \( H \), and joining the \( i \)-th vertex of \( G \) to every vertices in \( H_i \). The value of \( tes(P_m \odot P_n) \) and \( tes(P_m \odot P_n) \) are given in Theorem 1.6 and Theorem 1.7, respectively. Theorem 1.6 [4]. For any integer \( m, n \geq 2 \),

\[
tes(P_m \odot P_n) = \left\lfloor \frac{2mn+1}{3} \right\rfloor.
\]

Another result of tes of a graph was given by Siddiqui et al. in the Ref. [5]. In the paper, they considered the tes of disjoint union of suns. A *sun* \( M_n \) is the graph obtained by adding an edge incident with pendant vertex to every vertices of cycle \( C_n \). [5] Theorem 1.8. [5] Let \( n \geq 3 \) and \( p \) be two integers. Then tes of disjoint union of \( p \) isomorphic sun graphs is

\[
\left\lfloor \frac{2(pn+1)}{3} \right\rfloor.
\]

In the Ref. [6], Ahmad et al determined tes of disjoint union of friendship graphs. The results are given in theorem and corollary as follows. Theorem 1.10 [6] For \( n_j \geq 3 \) and \( 1 \leq j \leq m \), and \( m \geq 2 \), let \( F_{n_j} \) be a friendship graph with \( n_j \) triangles. Let \( G \cong \bigcup_{j=1}^{m} F_{n_j} \) be disjoint union of \( F_{n_j} \). Then

\[
tes \left( \bigcup_{j=1}^{m} F_{n_j} \right) = 1 + \sum_{j=1}^{m} n_j.
\]

Corollary 1.1. [6] For \( n \geq 3 \), let \( F_n \) be a friendship graph with \( n \) triangles, and \( mF_n \) be \( m \) copies of \( F_n \). Then,
\[
\text{tes}(mR_n) = mn + 1.
\]

In the Ref. [7], Indriati et al determined tes of generalized web graph. For \( n \geq 3 \) and \( m \geq 2 \),
generalized web graph \( W_{n,m} \) is obtained by joining all vertices \( v_{i,m} \) of the generalized prism \( P^n_m \), for \( 1 \leq i \leq n \), to a center vertex \( w \). The \( \text{tes}(W_{n,m}) \) is given in Theorem 1.11. Theorem 1.11 [7] Let \( W_{n,m} \), \( n \geq 3, m \geq 2 \), be a generalized web graph. Then,
\[
\text{tes}(W_{n,m}) = \left\lfloor \frac{2mn + 2}{3} \right\rfloor.
\]

In Ref. [8], Jeyanthi and Sudha proved that tes of flower graph \( F\ell_n \) is \( \left\lceil \frac{4n+2}{3} \right\rceil \). The tes of disjoint union graphs is also given by Siddiqui et al in the Ref. [9]. The result is as follows. Theorem 1.12 [9] Let \( m, n \geq 3 \) be integers. Then, the tes of a \( m \) copies of a helm graph \( mH_n \) is \( mn + 1 \). From the results before, we interest to get tes of another graph, which is a graph with two operations, that is disjoint union and Cartesian product. The graph is disjoint union of isomorf ladders, denoted by \( m(P_2 \square P_n) \).

2. Methods
To get tes of \( m \) copies of ladder, we consider a lower bound also an upper bound on tes of the graph. We use Theorem 1 to get a lower bound on \( \text{tes}(m(P_2 \square P_n)) \). On the other hand, to get an upper bound on \( \text{tes}(m(P_2 \square P_n)) \), we construct an edge irregular total labeling by minimizing the maximum label.

3. Results and Discussion
In this paper, we determine tes of disjoint union the Cartesian product of \( P_2 \) and \( P_n \), namely ladder graph. The Cartesian product \( G \square H \) of graphs \( G \) and \( H \) is a graph with that the vertex set \( V(G) \times V(H) \) and any two vertices \( (u, u_0) \) and \( (v, v_0) \) are adjacent in \( G \square H \) if and only if either \( u_0 \) is adjacent with \( v_0 \) in \( H \) and \( u = v \), or \( u \) is adjacent with \( v \) in \( G \) and \( u_0 = v_0 \). The tes of ladder graphs is given in Theorem 2.1. Theorem 2.1. Let \( P_2 \square P_n \) be the ladder graph, which is the Cartesian product of paths \( P_2 \) and \( P_n \). Let \( m(P_2 \square P_n) \) be the \( m \) copies of \( P_2 \square P_n \). Then for \( n \geq 2 \) and \( m \geq 1 \),
\[
\text{tes}(m(P_2 \square P_n)) = (n - 1)m + \left\lceil \frac{m+2}{3} \right\rceil.
\]

Proof. Let the vertex set of the \( j \)-th copy of \( m(P_2 \square P_n) \) is
\[
\{u_{i,j} \mid i = 1,2,\ldots,n\} \cup \{v_{i,j} \mid i = 1,2,\ldots,n\}
\]
(2)

And the edge set is
\[
\{u_{i,j}v_{i,j} \mid i = 1,2,\ldots,n\} \cup \{u_{i,j}u_{(i+1),j} \mid i = 1,2,\ldots,n - 1\} \cup \{v_{i,j}v_{(i+1),j} \mid i = 1,2,\ldots,n - 1\}
\]
(3)

For every \( j = 1,2,\ldots,m \).

Define an edge irregular total \( \left( (n - 1)m + \left\lceil \frac{m+2}{3} \right\rceil \right) \)-labeling \( f \) of \( m(P_2 \square P_n) \) as follows.
\[
f(u_{i,j}) = (n - 1)(j - 1) + \left\lceil \frac{j + 1}{3} \right\rceil \text{ for } 1 \leq i \leq n \text{ and } 1 \leq j \leq m;
\]
\[
f(v_{i,j}) = (n - 1)j + \left\lceil \frac{j + 2}{3} \right\rceil \text{ for } 1 \leq i \leq n \text{ and } 1 \leq j \leq m;
\]
\[
f(u_{i,j}u_{(i+1),j}) = n(j-1) - \frac{j-2}{3} + i \text{ for } 1 \leq i \leq n - 1 \text{ and } 1 \leq j \leq m;
\]
\[
f(v_{i,j}v_{(i+1),j}) = n(j-1) - \frac{j-1}{3} + i + 1 \text{ for } 1 \leq i \leq n - 1 \text{ and } 1 \leq j \leq m;
\]
\[
f(u_{i,j}v_{i,j}) = n(j-1) - \frac{2j-2}{3} + i \text{ for } 1 \leq i \leq n \text{ and } 1 \leq j \leq m.
\]

Form the labeling \( f \), we have the weight of edges of \( m(P_2 \square P_n) \) as follows.
\[
w_f(u_{i,j}u_{(i+1),j}) = (3n-2)(j-1) + i + 2 \text{ for } 1 \leq i \leq n - 1 \text{ and } 1 \leq j \leq m;
\]
\[
w_f(u_{i,j}v_{i,j}) = (3n-2)(j-1) + n + i + 1 \text{ for } 1 \leq i \leq n \text{ and } 1 \leq j \leq m;
\]
\[
w_f(v_{i,j}v_{(i+1),j}) = (3n-2)(j-1) + 2n + i + 1 \text{ for } 1 \leq i \leq n \text{ and } 1 \leq j \leq m.
\]

Form the weight formula above, there are no two edges with the same weight. So, \( f \) is a total edge irregular \( (n-1)m + \left\lceil \frac{m+2}{3} \right\rceil \) – labeling of \( m(P_2 \square P_n) \). So that, we have an inequality as follows.
\[
es(m(P_2 \square P_n)) \leq (n-1)m + \left\lceil \frac{m+2}{3} \right\rceil.
\]

On the other hand, the number of edges of \( m(P_2 \square P_n) \) is \( m(3n-2) \). So, by using Theorem 1.1, we have
\[
tes(m(P_2 \square P_n)) \geq \frac{m(3n-2)+2}{3}
\]
\[
= \frac{3mn-2m+2}{3}
\]
\[
= \frac{3mn-3m+m+2}{3}
\]
\[
= \frac{3(mn-m)+m+2}{3}
\]
\[
= mn - m + \left\lceil \frac{m+2}{3} \right\rceil
\]
\[
= (n-1)m + \left\lceil \frac{m+2}{3} \right\rceil.
\]

So that, we have an Inequality (8).
\[
tes(m(P_2 \square P_n)) \geq (n-1)m + \left\lceil \frac{m+2}{3} \right\rceil.
\]

From Inequality (6) and (8), we have an equality as follows.
\[
tes(m(P_2 \square P_n)) = (n-1)m + \left\lceil \frac{m+2}{3} \right\rceil.
\]

Figure 1 gives an illustration of the notating of vertices in \( m(P_2 \square P_n) \) for \( m = 3 \) and \( n = 5 \).
In the next figure, can be seen the edge irregular total \( (5 - 1)3 + \lceil \frac{3+2}{3} \rceil = 14 \) – labeling \( f \) of 3\((P_2 \Box P_5)\).

The weight of each edge of 3\((P_2 \Box P_5)\), under the labeling in Figure 2, can be seen in the Figure 3.
It can be seen that under the labeling $f$, there are no two edges in $3(P_2 \square P_5)$ with the same weight.

4. Conclusion

By using Theorem 1.1 and construct an edge irregular total labeling of $m(P_2 \square P_n)$, we can conclude that exact value of the tes of $m$ copies of ladder, donated by $m(P_2 \square P_n)$, is $(n - 1)m + \left\lceil \frac{m+2}{3} \right\rceil$.

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