Indentation of an elastic half-space by the cylindrical flat punch with a rounded edge under non-monotonic loading

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Abstract. The paper presents a novel approach to solving the contact problem for interaction with friction between a rigid cylindrical flat punch with rounded corners and a linearly elastic half-space. Increasing and decreasing normal loads act on the punch sequentially. Coulomb's law is used to account for friction. The contact problem is reduced to nonlinear operator equations that must be solved at each step of the discrete loading process. The numerical method is applied. This method includes regularization of the equation, discretization of the regularized equations and the usage of an iterative process to obtain the approximate solution. The calculations were performed for different values of the relative radius of the flat area of the punch base. Distributions of contact traction and configuration of the stick and slip zones are studied at each monotonic stage of the loading process.

1. Introduction
The study of the contact of elastic bodies is an important problem of mechanics and is widely used in mechanical engineering, transport, construction mechanics and other industries. When determining contact stresses in interacting elements of various mechanical systems, it is often necessary to take into account the friction between the contacting surfaces of these elements. To account for friction in contact problems of elasticity theory, as a rule, Coulomb law is used [1].

The complexity of such problems is due to the fact that the contact surface, areas of adhesion and slippage are unknown in advance and can be complex and unpredictable. For the first time, approximate analytical solutions of the partial slip contact problems were independently obtained in papers [2] and [3]. The modern researches of the stick-slip contact problem are considered in articles [4–10] and books [11–15].

The numerical methods based on the variation statement of the problem are considered in researches [16–18]. The authors of papers [19–21] use methods based on reducing the problem to integral equations. Difficulties caused by the numerical realization of the variation methods for solving such complex contact problems are absent when we use the methods [22–24] based on nonlinear operator equations to simulate the contact interaction.

In this paper, nonlinear boundary integral equations method [24] was applied to obtain a numerical solution to the quasi-static contact problem of the interaction with Coulomb friction between the rigid cylindrical flat punch with rounded edges and a linearly elastic half-space when a non-monotonic normal load is acting on the punch.
2. Operator equation of the contact problem

Consider the three-dimensional quasi-static contact problem on interaction of two linearly elastic bodies in the presence of Coulomb friction [1]. This problem is reduced to the solution of the nonlinear operator equations [24]:

\[ \mathbf{p}_i = G_{\mu p_{i-1}} \left( \mathbf{p}_i - E \cdot (A(\mathbf{p}_i) - \mathbf{f}_i) \right), \quad i = 1, \]

(1)

where \( i \) is denotes step number of the discrete loading process; \( l \) is a total number of loading steps; \( \mathbf{p}_i = (p_{i1}(s), p_{i2}(s), p_{i3}(s)) \in L_2^3(\Omega) \) [22] are unknown vector-functions from Hilbert space \( L_2^3(\Omega) \) [22], where \( p_{i1}(s) \in L_2(\Omega) \) [25] are normal contact tractions, \( p_{i2}(s), p_{i3}(s) \in L_2(\Omega) \) are tangential contact tractions at a point \( s \in \Omega \) at the \( i \)th step of loading, where \( \Omega \) is bounded flat region, containing an unknown contact area at all loading steps; \( \mathbf{f}_i \) is known element of \( L_2^3(\Omega) \), that determines the configuration of bodies and the conditions for their loading; \( \mu \) is a friction coefficient; \( E \) is a positive const.

The continuous nonlinear operator \( G_{\mu g}: L_2^3(\Omega) \to L_2^3(\Omega) \) is given by the equalities

\[
\begin{align*}
\mathbf{x} &= (x_1, x_2, x_3), \quad \mathbf{y} = (y_1, y_2, y_3) \in L_2^3(\Omega); \\
\mathbf{y}_1(s) &= h(x_1(s)), \\
\mathbf{y}_2(s) &= q(x_2(s), x_3(s), \mu g(s)), \\
\mathbf{y}_3(s) &= q(x_3(s), x_2(s), \mu g(s)), \quad s \in \Omega,
\end{align*}
\]

(2)

where \( g(s) \in L_2(\Omega) \) is given positive function, the functions \( h(x) \) and \( q(x, y, z) \) have the form:

\[
\begin{align*}
h(x) &= \begin{cases} x, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0; \end{cases} \\
q(x, y, z) &= \begin{cases} x, & \text{if } x^2 + y^2 \leq z; \\ \frac{z}{\sqrt{x^2+y^2}}, & \text{if } x^2 + y^2 > z. \end{cases}
\end{align*}
\]

(3)

(4)

The linear bounded operator of influence \( A:L_2^3(\Omega) \to L_2^3(\Omega) \) is defined by the relations

\[
\begin{align*}
\mathbf{x} &= (x_1, x_2, x_3), \quad \mathbf{y} = (y_1, y_2, y_3) \in L_2^3(\Omega); \\
\mathbf{y} &= A(\mathbf{x}), \quad y_k = \sum_{j=1}^3 A_{kj}(x_j) \quad \forall k = 1, 3.
\end{align*}
\]

(5)

where \( A_{kj}: L_2(\Omega) \to L_2(\Omega) \) have the form:

\[ A_{kj}(x) = \int_{\Omega} K_{kj}(s, s') \cdot x(s')ds', \quad (k, j = 1, 3). \]

(6)

According to the Boussinesq-Cerruti solution [13] the functions \( K_{kj}(s, s') \) have the form:

\[
\begin{align*}
K_{11}(s, s') &= \frac{c_1}{r}; & K_{12}(s, s') &= -K_{21}(s, s') = \frac{c_2(x-x')}{r^2}; \\
K_{13}(s, s') &= -K_{31}(s, s') = \frac{c_2(y-y')}{r^2}; \\
K_{22}(s, s') &= \frac{c_1}{r} + \frac{c_3(x-x')^2}{r^3}; \\
K_{23}(s, s') &= -K_{32}(s, s') = \frac{c_3(x-x')(y-y')}{r^3}; \\
K_{33}(s, s') &= \frac{c_1}{r} + \frac{c_3(y-y')^2}{r^3};
\end{align*}
\]

(7)
\[
\begin{align*}
    c_1 &= \frac{1-v_1^2}{\pi E_1} + \frac{1-v_2^2}{2\pi E_2}, \\
    c_2 &= \frac{(1+v_2)(1-2v_2)}{2\pi E_2} - \frac{(1+v_1)(1-2v_1)}{2\pi E_1}, \\
    c_3 &= \frac{v_1(1+v_2)}{\pi E_1} + \frac{v_2(1+v_2)}{2\pi E_2}, \\
    r &= |s - s'| = \sqrt{(x - x')^2 - (y - y')^2}.
\end{align*}
\]

where \(E_1, E_2\) and \(v_1, v_2\) are Young’s modulus’s and Poisson’s ratios for first and second bodies; \(x, x'\) and \(y, y'\) are coordinates of points \(s, s'\) in the region \(\Omega\). If the first body is rigid then Young’s modulus \(E_1\) is equal infinity.

The method of the numerical solution of the operator equation (1) is considered in paper [24]. It consists of three nonlinear integra

\[\text{Numerical parameters } p_{i1}(s), p_{i2}(s), p_{i3}(s) \in L_2(\Omega).\]

3. Method of the numerical solution

The method of the numerical solution of the operator equation (1) is considered in paper [24]. It consists of the regularization of the equation (1), the discretization of the regularized equation and the use of an iterative process for solving the discrete analogue of the regularized equation. To obtain such a discrete analogue, we define the region \(\Omega\) in the form of an open square. We split \(\Omega\) into \(n^2\) square regions \(\omega_1, \omega_2, \ldots, \omega_{n^2}\). Let unknown functions \(p_{i1}(s), p_{i2}(s), p_{i3}(s)\) is constants \(x_{3k-2i}, x_{3k-1i}, x_{3ki}\) on each boundary element \(\omega_k\). Then, obtaining an approximate solution of the equation (1) at the \(i\)th step of loading is reduced to obtaining unknown numerical parameters \(x_{1i}, x_{2i}, \ldots, x_{3n^2i}\) from the following system of \(3n^2\) scalar equations:

\[
\begin{align*}
    x_{3k-2i} &= h(x_{3k-2i} - E \cdot (\sum_{j=1}^{3n^2} a_{3k-2j} \cdot x_j - b_{3k-2i})); \\
    x_{3k-1i} &= q(x_{3k-1i} - E \cdot (\sum_{j=1}^{3n^2} a_{3k-1j} \cdot x_j - b_{3k-1i}), x_{3ki} - E \cdot (\sum_{j=1}^{3n^2} a_{3k-j} \cdot x_j - b_{3ki}), \mu \cdot x_{3k-2i}); \\
    x_{3ki} &= q(x_{3ki} - E \cdot (\sum_{j=1}^{3n^2} a_{3k-j} \cdot x_j - b_{3ki}), x_{3ki} - E \cdot (\sum_{j=1}^{3n^2} a_{3k-j} \cdot x_j - b_{3ki}), \mu \cdot x_{3k-2i});
\end{align*}
\]

where \(k = \overline{1, n^2}, i = \overline{1, l}\).

Numerical parameters \(a_{kj}, b_{ki}\) in system (10) are determined by relations

\[
\begin{align*}
    a_{3k-d 3k-d} &= \varepsilon + \int_{\omega_k} K_{3-j 3-j}(s_k, s)ds; \\
    a_{3k-2k-1} &= a_{3k-2k-3} = a_{3k-1k-2} = a_{3k-1k-3} = a_{3k 3k-2} = a_{3k 3k-1} = 0; \\
    a_{3k-3d-3} &= mes(\omega_j) \cdot K_{3-j 3-j}(s_k, s_j); b_{3k-2k-1} = -\delta_0(s_k) - \Delta_{1i}, \\
    b_{3k-1i} &= -\Delta_{2i}(s_k), b_{3ki} = -\Delta_{3i}(s_k); \forall k, j = \overline{1, n^2} (k \neq j); \forall d, e = 0, 2,
\end{align*}
\]

where \(\varepsilon > 0\) is regularization parameter; \(s_j\) is the center of the square \(\omega_j; mes(\omega_j)\) is the area of the square \(\omega_j\).
The approximate solution of the system (10) can be found by the iterative process:

\[
\begin{align*}
&\begin{cases}
(x^{(0)}_1, x^{(0)}_2, \ldots, x^{(0)}_{3n^2}) \in R^{3n^2}; \\
(x^{(m+1)}_{3k-2 i} = h(x^{(m)}_{3k-2 i} - E \cdot (\sum_{j=1}^{3n^2} a_{3k-2 j} \cdot x^{(m)}_{j i} - b_{3k-2 i}))), \\
(x^{(m+1)}_{3k-1 i} = q(x^{(m)}_{3k-1 i} - E \cdot (\sum_{j=1}^{3n^2} a_{3k-1 j} \cdot x^{(m)}_{j i} - b_{3k-1 i})), x^{(m)}_{3k i} - E \cdot (\sum_{j=1}^{3n^2} a_{3k j} \cdot x^{(m)}_{j i} - b_{3k i})), \mu \cdot x_{3k-1 i} \\
(x^{(m+1)}_{3k i} = q(x^{(m)}_{3k i} - E \cdot (\sum_{j=1}^{3n^2} a_{3k j} \cdot x^{(m)}_{j i} - b_{3k i})), x^{(m)}_{3k-1 i} - E \cdot (\sum_{j=1}^{3n^2} a_{3k-1 j} \cdot x^{(m)}_{j i} - b_{3k-1 i})), \mu \cdot x_{3k-2 i} - 1); \\
\end{cases}
\end{align*}
\]

where \(i = 1, l; k = 1, n^2; m = 0, 1, 2, \ldots \) is number of iteration step.

The iterative process (12) converges to the solution of the system (10) regardless of the choice of the initial approximation \((x^{(0)}_1, x^{(0)}_2, \ldots, x^{(0)}_{3n^2}) \in R^{3n^2}\) assuming a positive constant \(E\) satisfies inequality \(E \leq \left(\frac{\max_{1 \leq i \leq 3n^2} |a_{ij}|}{\sum_{j=1}^{3n^2} a_{ij}}\right)^{-1}\) [26].

4. Numerical results and discussion

The proposed approach was applied to obtain a numerical solution of the contact problem of indentation of an elastic half-space by a rigid cylindrical flat punch with rounded edges (Figure 1).

![Figure 1. Diagram of contact.](image)

The indenter profile has the shape

\[
\delta_0(r) = \begin{cases} 
0, & r \leq R_1, \\
\frac{(r-R_1)^2}{2R}, & r > R_1,
\end{cases}
\]

where \(r = \sqrt{x^2 + y^2}; R\) is the radius of curvature of the rounded corner; \(R_1\) is the radius of the flat punch surface.

The process of loading consisted of 3\(l\) steps in accordance with the following law of changing of the normal displacement of the punch:

\[
\Delta_{1l} = \begin{cases} 
-\Delta, & i \leq l, \\
-\Delta + \frac{(1-W)_l}{l} \cdot (i - l), & l+1 \leq i \leq 2l, \\
-\Delta \cdot W - \frac{(1-W)_l}{l} \cdot (i - 2l), & 2l+1 \leq i \leq 3l,
\end{cases}
\]

4
where $\Delta$ is the maximum indentation depth; $0 < W < 1$ is constant. The tangential displacements $\Delta_{2i}$ and $\Delta_{3i}$ are equal zero for all steps of loading.

Formula (14) defines non-monotonic normal loading, which consists of three stages. At the first stage ($1 \leq i \leq l$) the punch is pressed into the half-space to the maximum indentation depth $\Delta$, in the second stage ($l+1 \leq i \leq 2l$) the punch is moved in the opposite direction to the indentation depth $\Delta W$, in the third stage ($2l+1 \leq i \leq 3l$) the punch is again pressed into the half-space to the maximum depth $\Delta$. The solution of the problem for the case of monotonic normal load was obtained in our earlier paper [27].

In the calculations, we used the following parameters: Young’s modulus $E = 3.5 \cdot 10^4$ MPa; Poisson’s ratio $\nu = 0.44$; friction coefficient $\mu \approx 0.07$; $\mu/\gamma = 0.665$, where $\gamma = (1 - 2\nu)/(2 - 2\nu)$ [16]. In the formulas (14) we used the following parameters: $l = 20$, $\Delta = 10^{-4}$ m, $W = 0.547$.

Numerical solutions were obtained for different values of parameter $\alpha = R_1/a$, where $a$ is a maximum contact radius. The radius of curvature $\tilde{R}$ in each case was selected from the condition of the constancy of the ratio $\Delta/a = 0.002$.

The calculation results are illustrated by the graphs (Figure 2-10). Here $\bar{q} = p_2(x,0)/p_1(0,0)$ is normalized tangential traction acting on the punch, $\bar{p} = p_1(x,0)/p_1(0,0)$ is normalized contact pressure (normal traction), $\bar{x} = \tilde{x}/a$ is dimensionless distance to the center of the punch base.

In these figures, dotted lines with triangles correspond to the state after the first stage of loading ($i = l$ in formula (14) ), with squares - after the second stage ($i = 2l$), with circles - after the third stage ($i = 3l$). Solid and dashed lines in Figure 8 correspond to the known solution [16] for a flat punch after the first and after the second stages of loading, respectively.

Analysis of the results shows that after the first stage of loading the contact pressure and tangential traction reach their maximum values at the points belonging to the rounded edges of the punch. On the flat punch surface, the contact traction increase monotonically with increasing distance from the center of the punch base. With a decrease in the radius of the flat section $R_1$ from 0.8$a$ to 0.2$a$, the maxima of contact traction shift closer to the center of the contact area (Figure 2-7).

After the second stage of loading (monotonic unloading to the indentation depth 0.547$\Delta$), the contact pressures decrease (Figure 2, 4, 6) and their distribution (dotted line with squares) is similar to the first stage of loading (dotted line with triangles). The distribution of tangential traction (dotted line with squares in Figure 3, 5, 7) differs from their distribution after the first stage. At points near the boundary of the contact area, these stresses change sign to the opposite. The radii of the contact area and the central stick zone decrease; a second stick zone is formed at the boundary, while slipping occurs at the points on the contact area boundary (Figure 8-10). With a decrease in the radius of the flat region ($\alpha = 0.8$ 0.5 0.2) we see a decrease in the radius of the central adhesion zone, while the annular stick zone at the boundary has approximately the same width. Comparison of the results with the known solution [16] of this problem for a flat punch (Figure 8) shows that after the first stage of loading, the radius of the stick zone for all considered values of the parameter $\alpha$ is 0.68$a$, which is approximately equal to its value for a flat punch. After the second stage of loading, the radius of the central stick zone for a flat punch is close to the value of 0.56$\alpha$, obtained at $\alpha = 0.8$. For the cases of $\alpha$ equal to 0.5 and 0.2 (Figure 9, 10) this radius is equal to 0.48$\alpha$ and 0.44$\alpha$, respectively.

After the third stage of loading, when the punch is again pressed into the half-space to the maximum indentation depth $\Delta$, the contact area is again divided into a circular adhesion zone in the center and a slip zone at the boundary. As seen from Figure 2, 4, 6 the distribution of contact pressure (dotted line with circles) in this case coincides with the distribution obtained after the first stage (dotted line with triangles). In the graphs Figure 3, 5, 7 decrease in the values of tangential tractions in the slip zone formed after the second stage of loading is observed. This leads to an increase in the radius of the adhesion zone in comparison with its value achieved after the first stage of loading (by 24% for the case of $\alpha = 0.8$ and by 12% for the cases of $\alpha = 0.5$ and 0.2) (Figure 8-10).
Figure 2. Contact pressure distribution, $\alpha = 0.8$.

Figure 3. Tangential traction distribution, $\alpha = 0.8$.

Figure 4. Contact pressure distribution, $\alpha = 0.5$.

Figure 5. Tangential traction distribution, $\alpha = 0.5$.

Figure 6. Contact pressure distribution, $\alpha = 0.2$.

Figure 7. Tangential traction distribution, $\alpha = 0.2$. 
Figure 8. Solution to the problem, $\alpha = 0.8$.

Figure 9. Solution to the problem, $\alpha = 0.5$.

Figure 10. Solution to the problem, $\alpha = 0.2$. 
5. Conclusions

The new approach to solving three-dimensional quasi-static contact problems of the interaction of elastic bodies with Coulomb friction is proposed. The contact problems are formulated using nonlinear operator equations for the unknown components of the contact traction. Each equation corresponds to a step in the discrete loading process. For the numerical solution of these equations, they were discretized and an iterative process was used to obtain an approximate solution to their discrete analogs.

The proposed approach is applied to solve the contact problem of interaction of a rigid cylindrical flat punch with rounded edge and an elastic half-space under non-monotonic normal loading, including the stages of loading, incomplete unloading and reloading. The distributions of normal and tangential contact tractions corresponding to each stage of loading were obtained for different values of the radius of the flat region of the punch base. Analysis of the obtained numerical results showed that the non-monotonic loads can lead to a change in the distribution of contact tractions and the configurations of the stick and slip zones formed at the initial stage of loading.

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