Communication via Holomorphic Green Functions

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Abstract

Let $G(x_r - x_e)$ be the causal Green function for the wave equation in four spacetime dimensions, representing the signal received at the spacetime point $x_r$ due to an impulse emitted at the spacetime point $x_e$. Such emission and reception processes are highly idealized, since no signal can be emitted or received at a single (mathematical) point in space and time. We present a simple model for extended emitters and receivers by continuing $G$ analytically to a function $\tilde{G}(z_r - z_e)$, where $z_e = x_e + iy_e$ is a complex spacetime point representing a circular pulsed-beam emitting antenna dish centered at $x_e$ and emitting in the direction of $y_e$, and $z_r = x_r - iy_r$ represents a circular pulsed-beam receiving antenna dish centered at $x_r$ and receiving from the direction of $y_r$. The holomorphic Green function $\tilde{G}(z_r - z_e)$ represents the coupling between the emission from $z_e$ and the reception at $z_r$. To preserve causality and give nonsingular coupling, the orientation vectors $y_e$ and $y_r$ must belong to the future cone $V_+$ in spacetime. Equivalently, $z_e$ and $z_r$ belong to the future and past tubes in complex spacetime, respectively. The space coordinates of $y_e$ and $y_r$ give the spatial orientations and radii of the dishes, while their time coordinates determine the duration and focus of the emission and reception processes. The directivity $D(y)$ of the communication process is a convex function on $V_+$, i.e., $D(y_r + y_e) \leq D(y_r) + D(y_e)$. This shows that the efficiency of the communication can be no better than the sum of its emission and reception components.

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1 Introduction

I begin by summarizing earlier work. Physical wavelets were defined in [K94] as wavelet-like bases for spaces of solutions of the homogeneous wave equation (acoustic wavelets) or Maxwell’s equations (electromagnetic wavelets). This was motivated by the observation that information is often communicated by acoustic or electromagnetic waves, and this fact should be taken into account when “processing” the resulting signals. All such wavelets can be obtained from a single “mother” wavelet by translations, scaling, rotations and Lorentz transformations.

The construction of physical wavelets was based on a holomorphic extension $\tilde{F}(x + iy)$ of solutions $F(x)$ to complex spacetime, with the imaginary spacetime variables $y$ interpreted as singling out approximate directions and frequencies of propagation. Thus $\tilde{F}(x + iy)$ is a description of the wave intermediate between the spacetime domain (where exact positions and times are known but no directional or frequency information is given) and the Fourier domain (where exact directional and frequency information is known but no local spacetime information is given). This is an extension to spacetime of continuous wavelet analysis of one-dimensional time signals, whose wavelet transform is intermediate between the time domain and the frequency domain representations.

The physical wavelets of the homogeneous wave- and Maxwell equations were then shown to split into a sum of causal and anticausal wavelets. Essentially, the causal wavelets are holomorphic extensions, in the sense of positive-frequency analytic signals, of the causal (retarded) Green function, and the anticausal ones are similar extensions of the anticausal (advanced) Green function for the appropriate equation. The causal wavelets are pulsed-beam solutions emitted by disk-like sources. That is, that they represent well-directed acoustic or electromagnetic beams that are pulsed in time rather than going on forever. The direction, pulse width, and duration of these beams are determined by the imaginary spacetime variables $y$. Such objects have appeared previously in the engineering literature under the name complex-source pulsed beams (see Heyman and Felsen, 1989, and the references therein).

In this paper we further develop the above analysis by showing that the holomorphic extension of the causal Green function describes not only the emission but also the reception of a pulsed beam, and so represents a communication between the emitting and receiving antenna dishes.
2 Holomorphic Green Functions

For simplicity, we concentrate on the wave equation in four-dimensional spacetime $\mathbb{R}^4$. The causal Green function is a fundamental solution of the wave equation,

$$(\partial_t^2 - \Delta)G(x,t) = \delta(x,t), \quad \Delta \equiv \Delta_x,$$

given by

$$G(x,t) = \frac{\delta(t - |x|)}{4\pi|x|}.$$  \hspace{1cm} (2)

Its analytic extension to complex spacetime is obtained as follows. First we extend the delta function to the lower-half time plane by taking its positive-frequency (analytic signal) part. This gives the Cauchy kernel:

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} d\omega \to \tilde{\delta}(\tau) = \frac{1}{2\pi} \int_{0}^{\infty} e^{-i\omega \tau} d\omega = \frac{1}{2i\pi \tau},$$  \hspace{1cm} (3)

where

$$\tau = t - is \quad \text{with} \quad s > 0$$  \hspace{1cm} (4)

is necessary for convergence. Next, we extend the Euclidean distance $r \equiv |x|$ to complex space:

$$r = \sqrt{x \cdot x} \quad \rightarrow \quad \tilde{r} \equiv \sqrt{z \cdot z}, \quad z = x - iy \in \mathbb{C}^3.$$  \hspace{1cm} (5)

Writing

$$|x| = r \quad \text{and} \quad |y| = a,$$  \hspace{1cm} (6)

we see that

$$\tilde{r} = \sqrt{r^2 - a^2 - 2iar \cos \theta},$$  \hspace{1cm} (7)

where $\theta$ is the angle between $x$ and $y$. The complex root has branch points when $r = a$ and $\theta = 0$. For fixed $y$, these form a circle of radius $a$ in the plane orthogonal to $y$. In order to make $\tilde{r}$ a single-valued function, we choose the branch defined by

$$\Re \tilde{r} \geq 0, \quad \text{so that} \quad y \rightarrow 0 \Rightarrow \tilde{r} \rightarrow r.$$  \hspace{1cm} (8)
The branch cut (again, for fixed $y$) is then the disk
\[ S(y) = \{ x : r = 0 \} = \{ x : r \leq a, \ \theta = 0 \}. \tag{9} \]

As with ordinary branch cuts in the complex plane, the disk $S(y)$ can be deformed continuously to a membrane, as long as its boundary ($r = a, \ \theta = 0$) remains invariant. The extended Coulomb potential
\[ \tilde{\phi}(z) \equiv -\frac{1}{4\pi r(z)} \tag{10} \]
is a holomorphic extension of the fundamental solution for the Laplacian in $\mathbb{R}^3$. The distribution defined by
\[ \tilde{\delta}(z) \equiv \Delta \tilde{\phi}(z), \tag{11} \]
where $\Delta$ is the distributional Laplacian with respect to $x$, is an extended source distribution which contracts to the delta function as $y \to 0$ \cite{K00}:
\[ y \to 0 \Rightarrow \tilde{\delta}(x - iy) \to \delta(x). \tag{12} \]

Since the Coulomb potential $\phi(x)$ is harmonic outside the origin, it follows that $\tilde{\phi}(z)$ is harmonic outside the branch disk $S(y)$, and so the distribution $\tilde{\delta}$ is supported on $S(y)$. Thus $S(y)$ acts as an extended source generalizing the usual point source of the Coulomb potential, and this source has been constructed simply by analytic continuation.

We now have all the ingredients for extending the causal Green function $G(x, t)$ in (2) to complex spacetime. To simplify the notation, denote real spacetime points by
\[ x = (x, t) \in \mathbb{R}^4, \quad y = (y, s) \in \mathbb{R}^4 \tag{13} \]
and complex spacetime points by
\[ z = (z, \tau) \in \mathbb{C}^4, \quad z = x - iy \in \mathbb{C}^3, \quad \tau = t - is, \ s > 0, \tag{14} \]
so that
\[ z = x - iy. \tag{15} \]
The holomorphic Green function for the wave equation is now defined by

\[
\tilde{G}(z) = \tilde{G}(\mathbf{z}, \tau) = \frac{\delta(\tau - \tilde{r}(\mathbf{z}))}{4\pi \tilde{r}(\mathbf{z})} \approx \frac{1}{8i\pi^2 \tilde{r}(\tau - \mathbf{r})}.
\]  

(16)

But recall from (4) that the imaginary part of the argument of the numerator had to be negative. We must therefore require that

\[-\Re(\tau - \tilde{r}) = s + \Re \tilde{r} > 0.\]

(17)

It can be shown [K00] that this is equivalent to requiring the imaginary spacetime coordinates \(y = (y, s)\) to satisfy

\[|y| < s, \quad \text{or} \quad y \in V_+,\]

(18)

where \(V_+\) is the future cone in spacetime. This means that the argument \(z\) of \(\tilde{G}\) belongs to the past tube \(T_\tau\) in complex spacetime [K94].

3 Pulsed-Beam Wavelets

We now show that \(\tilde{G}(x - iy)\) describes the emission of a pulsed beam by an elementary “antenna dish” that can be identified with the imaginary spacetime variable \(y \in V_+\), as observed at the spacetime point \(x\). Note that when \(y \to 0\), this reduces to the usual interpretation of \(G(x)\) as the signal observed at \(x\) due to an idealized impulse emitted at the origin.

To simplify the analysis, we suppose that the observer is far from the source disk \(S(y)\) of (9). By (7),

\[r \gg a \Rightarrow \tilde{r} \approx r - ia \cos \theta,\]

(19)

where the choice of branch \(\Re \tilde{r} \geq 0\) has been enforced. Substituting this into (16) gives the far-zone approximation

\[
\tilde{G}(x - iy, t - is) \approx \frac{1}{8i\pi^2 r} \cdot \frac{1}{t - r - iT(\theta)},
\]

(20)

where

\[T(\theta) = s - a \cos \theta > 0 \quad \text{since} \quad (y, s) \in V_+ .\]

(21)
At a fixed position \( \mathbf{x} \), (20) is easily seen to be a pulse passing the observer at time \( t = r \), in accordance with causality and Huygens’ principle. The duration of this pulse is given by \( T(\theta) \). The pulse is shortest and strongest when the observer is on the front axis of the disk \( S(\mathbf{y}) \) (\( \mathbf{x} \) parallel to \( \mathbf{y} \)), and longest and weakest on the rear axis (\( \mathbf{x} \) antiparallel to \( \mathbf{y} \)). By making \( s - a \) small, we obtain a well-focused pulsed beam concentrated around the front axis. The smaller \( s - a \), the better the focus.

Thus \( y = (\mathbf{y}, s) \in V_+ \) controls the shape of the pulsed beam \( \tilde{G}(x - iy) \) observed at \( x \). Namely, \( \mathbf{y} \) determines the radius \( a = |\mathbf{y}| \) and orientation \( \mathbf{y}/a \) of the source disk \( S(\mathbf{y}) \), while \( s - a \) controls the focus of the emitted pulsed beam and its duration along the beam axis. We will label these emission parameters by a subscript \( e \):

\[
y \rightarrow y_e \equiv (\mathbf{y}_e, s_e) \in V_+.
\] (22)

The above pulsed beam is emitted near the origin \( \mathbf{x} = 0 \) around the time \( t = 0 \). To emit a pulsed beam from any point \( \mathbf{x}_e \) at any time \( t_e \), we need only perform a spacetime translation:

\[
\tilde{G}(x - y_e) \rightarrow \tilde{G}(x - x_e - iy_e) = \tilde{G}(x - z_e),
\] (23)

where

\[
z_e = (\mathbf{x}_e, t_e) + i(\mathbf{y}_e, s_e) = x_e + iy_e
\] (24)

belongs to the future tube \( \mathcal{T}_+ \) in complex spacetime since \( y_e \in V_+ \).

4 Reception and Communication of Pulsed Beams

The holomorphic Green function \( \tilde{G}(x - z_e) \), defined in the past tube \( \mathcal{T}_- \), represents a wave emitted by an extended source described by \( z_e = x_e + iy_e \), with \( x_e \) giving the spacetime coordinates of the center of the source and \( y_e \) giving the spacetime extension about this center (the radius and orientation of the emitting disk, as well as the duration of the emitted pulse). By contrast, the original Green function \( G(x - x_e) \) describes an idealized spherical impulse emitted from the single spacetime point \( x_e \). By making the coordinates \( x_e \) complex, we have thus obtained a more realistic and physically interesting model for emission.
However, our model for reception is still highly idealized since the observer is supposed to measure the pulsed beam at the single spacetime point $x$. We now remedy this by making the observation point complex as well:

$$x \rightarrow z_r \equiv (x_r, t_r) - i(y_r, s_r) = x_r - i y_r,$$  \hspace{1cm} (25)

where we have labeled the complex reception point $z_r$ with a subscript $r$. The change in sign as compared with (24) will be explained below.

With the formal substitution (25), we have

$$\tilde{G}(x - z_e) \rightarrow \tilde{G}(z_r - z_e) = \tilde{G}(x_r - x_e - i(y_r + y_e)).$$  \hspace{1cm} (26)

Since the argument of $\tilde{G}$ must belong to the past tube $\mathcal{T}_-$, we have to require that

$$y_r + y_e \in V_+ \text{ for all } y_e \in V_+.$$  \hspace{1cm} (27)

This implies that $y_e \in V_+$, which explains our choice of sign in (25).

The emission point $z_e$ must belong to the future tube $\mathcal{T}_+$, and the reception point $z_r$ must belong to the past tube $\mathcal{T}_-$. $\tilde{G}(z_r - z_e)$ represents the coupling between $z_e$ and $z_r$, giving the strength of the overall communication process.

These requirements also make intuitive sense, since emission creates a signal in the future while reception measures a signal from the past. From now on we identify $z_e \in \mathcal{T}_+$ with the emitting dish and $z_r \in \mathcal{T}_-$ with the receiving dish. Note that this includes the durations of the emission and the reception processes. (In reception, “duration” is interpreted as the integration time.) Our use of the term “dish” therefore stretches the usual meaning, being a spacetime concept rather than merely spatial.

The condition $z_r \in \mathcal{T}_-$ was derived from the mathematical requirement that $z_r - z_e \in \mathcal{T}_-$ for all $z_e \in \mathcal{T}_+$. We now confirm that our model also makes physical sense by studying the communication $\tilde{G}(z_r - z_e)$ in the far-zone approximation. Writing

$$r = |x_r - x_e|, \quad t = t_r - t_e, \quad a = |y_r + y_e|, \quad s = s_e + s_e,$$  \hspace{1cm} (28)

(20) gives

$$r \gg a \Rightarrow \tilde{G}(z_r - z_e) \approx \frac{1}{8i\pi^2 r} \cdot \frac{1}{t - r - iT(\theta)},$$  \hspace{1cm} (29)

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where \( \theta \) is the angle between \( \mathbf{x}_r - \mathbf{x}_e \) and \( \mathbf{y}_r + \mathbf{y}_e \) and

\[
T(\theta) = s - a \cos \theta
\]  

(30)

now denotes the duration of the overall communication process. Let us fix the distance \( r \) between the centers of the emitting and receiving disks \( S(y_e) \) and \( S(y_r) \), as well as their radii \( a_e = |y_e| \) and \( a_r = |y_r| \) and the duration parameters \( s_e \) and \( s_r \).

To maximize the communication (29), we need to minimize the duration function \( T(\theta) \). By Schwarz’s inequality,

\[
a \leq a_r + a_e, \quad \text{with} \quad a = a_r + a_e \quad \text{iff} \quad \mathbf{y}_r \text{ is parallel to } \mathbf{y}_e.
\]  

(31)

Thus (30) shows that the communication is maximal when

1. the emitting and receiving dishes are synchronized for causal communication, so that \( t = r \);

2. the spatial direction vectors \( \mathbf{y}_r \) and \( \mathbf{y}_e \) are parallel to one another (to maximize \( a \)) and also parallel to \( \mathbf{x}_r - \mathbf{x}_e \) (to make \( \theta = 0 \)).

We have seen that \( \mathbf{y}_e \) gives the direction of propagation of the emitted pulsed-beam wavelet. If we similarly assume that \( \mathbf{y}_r \) gives the direction in which the receiving disk is pointed, the above result clearly runs against common sense since it states that reception is greatest when the receiver points directly away from the transmitter. Rather, we must interpret \( \mathbf{y}_r \) as a vector pointing into the receiver, so that the dish \( z_r = x_r - iy_r \) is configured to receive signals coming from the direction of \( \mathbf{y}_r \). With this interpretation, the above results are in complete harmony with intuition.

The communication between an emitting dish \( \mathbf{z}_e = x_e - iy_e \) and a receiving dish \( \mathbf{z}_r = x_r - iy_r \) is greatest when the two dishes are synchronized for causal communication and each is pointed towards the center of the other.

5 The Convex Directivity Function

According to the above, the peak value of the pulsed beam emitted by \( \mathbf{z}_e \) and received by \( \mathbf{z}_r \) is obtained when \( \mathbf{y}_r, \mathbf{y}_e \) and \( \mathbf{x}_r - \mathbf{x}_e \), are all parallel, so that

\[
r = t, \quad a = a_e + a_r, \quad \theta = 0
\]  

(32)
and

\[ \tilde{G}(z_r - z_e) \approx \frac{1}{8\pi^2 r} \cdot \frac{1}{s - a}. \]  \hfill (33)

A dimensionless measure of the directivity of the communication, independent of \( r \), may be given as

\[ D \equiv \frac{a}{s - a} = \frac{a_r + a_e}{s_r + s_e - a_r - a_e}. \]  \hfill (34)

Since this expression depends only on \( y_r + y_e \in V_+ \), it defines a function \( D(y) \) on \( V_+ \). Note that

\[ 0 \leq D(y) < \infty, \text{ with } D(y) = 0 \text{ iff } a = 0. \]  \hfill (35)

But under the above assumptions, \( a = 0 \) implies \( y_r = y_e = 0 \). The directivity \( D(y_r + y_e) \) therefore vanishes if and only if the emitting and receiving disks both shrink to points, making the communication process entirely direction-free. (But note that we still have \( s_r > 0 \) and \( s_e > 0 \), so that the communicated signal remains a pulse rather than an impulse.) This helps justify the term “directivity.”

But \( D \) has another attractive property that goes deeper than the above. For all \( y_r, y_e \in V_+ \) we have \( s_r - a_r > 0 \) and \( s_e - a_e > 0 \), hence

\[ D(y_r + y_e) = \frac{a_r + a_e}{(s_r - a_r) + (s_e - a_e)} \leq \frac{a_r}{s_r - a_r} + \frac{a_e}{s_e - a_e}, \]  \hfill (36)

thus

\[ D(y_r + y_e) \leq D(y_r) + D(y_e). \]  \hfill (37)

Now \( V_+ \) is a convex cone in \( \mathbb{R}^4 \), and (37) shows that \( D \) is a convex function on \( V_+ \). This is an important property with an immediate physical interpretation. \( D(y_e) \) measures the directivity of the communication when the receiver is a spacetime point \((y_r = 0)\), so that \( \tilde{G}(x_r - z_e) \) represents a pure emission. Similarly, \( D(y_r) \) measures the directivity of the communication when the emitter is a spacetime point, so that \( \tilde{G}(z_r - x_e) \) represents a pure reception. Then (37) states that the efficiency of the overall communication can be no better than the sum of its separate emission and reception components. Further developments of these ideas will appear in [K01].
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