Marcinkiewicz–Zygmund Laws of Large Numbers under Sublinear Expectation

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In this paper, we derive the weak and strong results of Marcinkiewicz–Zygmund laws of large numbers under sublinear expectation. The results are extensions of the Kolmogorov laws of large numbers under sublinear expectation and the classical Marcinkiewicz–Zygmund laws of large numbers.

1. Introduction

The theory of sublinear expectation was initiated by Peng [1, 2] to describe the probability uncertainties in statistics, economics, finance, and other fields which are difficult to be handled by the classical probability theory. The classical laws of large numbers (LLNs for short) which reveal the almost sure laws of stabilized partial sum are of great significance in the probability theory. Recently, the LLNs under sublinear expectation got a lot of development, see for example, Marinacci [3]; Maccheroni and Marinacci [4]; Chen et al. [5]; Chen [6]; Zhang [7]; Hu [8]; Chen et al. [9]; and Hu [10].

Peng [11]; Chen et al. [9]; and Hu [12] gave three forms of weak LLNs under sublinear expectation under the first moment condition. That is, for any \( \varphi \in C_b(\mathbb{R}) \),

\[
\lim_{n \to \infty} \mathbb{E} \left[ \varphi \left( \frac{S_n}{n} \right) \right] = \sup_{\mu \leq x \leq \overline{\mu}} \varphi(x),
\]

for any \( \varepsilon > 0 \),

\[
\lim_{n \to \infty} \nu \left( \mu - \varepsilon < \frac{S_n}{n} < \mu + \varepsilon \right) = 1,
\]

and for any \( h \in [\mu, \overline{\mu}] \)

\[
\lim_{n \to \infty} \nu \left( h - \varepsilon < \frac{S_n}{n} < h + \varepsilon \right) = 1,
\]

where \((\mathbb{E}, \mathcal{E})\) denotes the pair of the sublinear expectation and the conjugate expectation and \((\vee, \nu)\) represent the induced capacities, \(\overline{\mu} = \mathbb{E}[X_1]\) and \(\mu = \mathbb{E}[X_1]\).

Chen et al. [5] obtained the Kolmogorov strong LLNs under sublinear expectation under the condition of finite \((1 + \alpha)\)th moments \((\alpha > 0)\) for sublinear expectation:

\[
\nu \left( \lim \inf_{n \to \infty} \frac{S_n}{n} < \mu \right) \cup \nu \left( \lim \sup_{n \to \infty} \frac{S_n}{n} > \overline{\mu} \right) = 0.
\]

Zhang [7] got the above strong LLNs under the condition of finite first moment for Choquet expectation. Hu [8] improved the above results under a general moment condition for sublinear expectation which is the weakest one for sublinear expectation.

The classical Marcinkiewicz–Zygmund strong LLNs generalized the Kolmogorov strong LLNs by extending the convergence rate of partial sum and give the relation between moment conditions and convergence rate. The norming constants become \(n^{1/p} (0 < p \leq 2)\) instead of \(n\) and the moment conditions depend on \(p\) accordingly. In sublinear situation, Feng and Lan [13] obtained the Marcinkiewicz–Zygmund strong LLNs for arrays of row wise independent random variables. Zhang and Lin [14]; Xu and Zhang [15] got the Marcinkiewicz–Zygmund strong LLNs under the condition of finite \(p\)th moments for Choquet expectation by different methods. We know that Choquet expectation is larger than sublinear expectation. The purpose of this paper is to generalize the weak and strong LLNs to the Marcinkiewicz–Zygmund LLNs under some moment conditions for sublinear expectation. We discuss the weak results under the condition of \(p\)-order uniform integrability for sublinear expectation and study...
the strong results under the condition a bit stronger than finite \( p \)th moments for sublinear expectation.

The plan of this paper is as follows. In Section 2, we introduce the basic concepts and lemmas under sublinear expectation. In Section 3, we prove some forms of Marcinkiewicz–Zygmund weak LLNs under sublinear expectation and discuss the equivalence relation among them. In Section 4, the Marcinkiewicz–Zygmund strong LLNs under sublinear expectation is given.

2. Preliminaries

Let \((\Omega, \mathcal{F})\) be a measurable space and \(\mathcal{H}\) be a linear space of real functions defined on \((\Omega, \mathcal{F})\) such that if \(X_1, X_2, \ldots, X_n \in \mathcal{H}\) then \(\varphi(X_1, X_2, \ldots, X_n) \in \mathcal{H}\) for each \(\varphi \in C_{\text{lip}}(\mathbb{R}^n)\) where \(C_{\text{lip}}(\mathbb{R}^n)\) denotes the linear space of local Lipschitz continuous functions \(\varphi\) satisfying

\[ |\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y|, \quad \forall x, y \in \mathbb{R}^n, \]

for some \(C > 0, m \in \mathbb{N}\) depending on \(\varphi\). \(\mathcal{H}\) contains all \(I_A\) where \(A \in \mathcal{F}\). We also denote \(C_{\text{lip}}(\mathbb{R}^n)\) as the linear space of bounded Lipschitz continuous functions \(\varphi\) satisfying

\[ |\varphi(x) - \varphi(y)| \leq C|x - y|, \quad \forall x, y \in \mathbb{R}^n, \]

for some \(C > 0\).

Definition 1. A function \(E: \mathcal{H} \rightarrow \mathbb{R}\) is said to be a sublinear expectation if it satisfies for \(\forall X, Y \in \mathcal{H}\),

a) Monotonicity: \(X \geq Y\) implies \(E[X] \geq E[Y]\)

b) Constant preserving: \(E[c] = c, \forall c \in \mathbb{R}\)

c) Positive homogeneity: \(E[\lambda X] = \lambda E[X], \forall \lambda \geq 0\)

d) Subadditivity: \(E[X + Y] \leq E[X] + E[Y]\) whenever \(E[X] + E[Y]\) is not of the form \(+\infty - \infty\) or \(-\infty + \infty\)

The triple \((\Omega, \mathcal{F}, E)\) is called the sublinear expectation space.

Remark 1. By Definition 1, we can obtain two properties of sublinear expectation \(E\):

(1) \(E[X + c] = E[X] + c, \forall c \in \mathbb{R}\)

(2) \(E[X - Y] \geq E[X] - E[Y]\)

Remark 2. Let \(\mathcal{P}\) be a family of probability measures defined on \((\Omega, \mathcal{F})\). For any random variable \(X \in \mathcal{F}\), the upper expectation defined by \(E[X]: = \sup_{\mathcal{P} \in \mathcal{P}} E_{\mathcal{P}}[X]\) is a sublinear expectation. So, the results in this paper can also be applied to upper expectation.

The conjugate expectation of \(E\) is defined by

\[ E^*[X] := -E[-X], \quad \forall X \in \mathcal{H}. \]

Obviously, for all \(X \in \mathcal{H}\), \(E^*[X] \leq E[X]\).

Definition 2 (see [16]). A set function \(V: \mathcal{F} \rightarrow [0, 1]\) is called a capacity if it satisfies

a) \(V(\emptyset) = 0, V(\Omega) = 1\)

b) \(V(A) \leq V(B), A \subseteq B, A, B \in \mathcal{F}\)

In this paper, we consider the capacities induced by sublinear expectation and the conjugate expectation: \(\forall (A): = E[I_A], v(A): = E^*[I_A] = 1 - V(A), \forall A \in \mathcal{F}\).

The continuity from above and continuity from below of sublinear expectation and capacity can also be defined similar to the classical probability theory (see [7]).

Definition 3 (independence). \(Y = (Y_1, \ldots, Y_n) (Y_i \in \mathcal{H})\) is said to be independent of \(X = (X_1, \ldots, X_m) (X_i \in \mathcal{H})\), if, for each test function \(\varphi \in C_{\text{lip}}(\mathbb{R}^m \times \mathbb{R}^n)\)

\[ E[\varphi(X, Y)] = E[E[\varphi(x, Y)]|_{x=X}], \]

whenever the sublinear expectations on both sides are finite.

\[ \{X_n\}_{n=1}^\infty \] is said to be a sequence of independent random variables, if \(X_{n+1}\) is independent of \((X_1, \ldots, X_n)\) for each \(n \geq 1\).

Remark 3. Peng [2] also gave the definition of identical distribution under the sublinear expectation space. The results in this paper do not need the random variables to be identically distributed.

To prove our main results, we need the following lemmas. The proofs of Lemma 1, Lemma 2, and Lemma 3 can be found in [5] and [8].

Lemma 1 (Borel–Cantelli lemma). If sublinear expectation \(E\) is continuous from below and \(\sum_{n=1}^\infty V(A_n) < \infty\), then

\[ \bigcap_{n=1}^\infty \bigcup_{m=n}^\infty A_m = 0. \]

(9)

Lemma 2 (Chebyshev’s inequality). Let \(f(x) > 0\) be a nondecreasing function on \(\mathbb{R}\). Then, for any real \(x\),

\[ \forall (X \geq x) \leq \frac{E[f(X)]}{f(x)}, \]

\[ \forall (X \geq x) \leq \frac{E^*[f(X)]}{f(x)}. \]

(10)

Lemma 3. If \(E[|X|] < \infty\), then \(v(|X| < \infty) = 1\).

In the following sections, we consider the sequence \(\{X_n\}_{n=1}^\infty\) of independent random variables defined on a sublinear expectation space \((\Omega, \mathcal{F}, E)\) with \(E[X_n] = \mu\), \(E^*[X_n] = \mu\) for each \(n \geq 1\). Denote \(S_n = \sum_{i=1}^n X_i, S_0 = 0\).

3. The Marcinkiewicz–Zygmund Weak LLNs

Theorem 1

(1) If \(\lim_{n \rightarrow \infty} \sup_{m \geq n} E[|X_m|^p I(|X_m| > n)] = 0\) for some \(1 \leq p < 2\), then for any \(\varepsilon > 0\),

\[ \lim_{n \rightarrow \infty} \bigcup_{m=n}^{\infty} \left\{ \left\{ S_n - \frac{n \mu}{n^{1/p}} \leq -\varepsilon \right\} \bigcup \left\{ S_n - \frac{n \mu}{n^{1/p}} \geq \varepsilon \right\} \right\} = 0, \]

(11)
and for any $h \in [\mu, \overline{\mu}]$,
\[
\lim_{n \to \infty} \mathbb{P}\left( -\varepsilon < \frac{S_n - nh}{n^{1/p}} < \varepsilon \right) = 1. \tag{12}
\]

(2) If $\lim_{n \to \infty} \sup_{m \geq 1} \mathbb{E}[|X_m|^p I(|X_m| > n)] = 0$ for some $0 < p < 1$, then
\[
\lim_{n \to \infty} \mathbb{P}\left( \frac{|S_n|}{n^{1/p}} \geq \varepsilon \right) = 0. \tag{13}
\]

**Proof**

(1) For the case $1 \leq p < 2$, we only give the proof of (11). The proof of (12) is similar to Theorem 3.1 of Hu [10], so we omit it.

For any fixed $\varepsilon > 0$, define a nondecreasing function $\phi(x) \in C^1_k(\mathbb{R})$ such that $\phi(x) = 0$ when $x \leq (\varepsilon/2)$, $0 \leq \phi(x) \leq 1$ when $(\varepsilon/2) < x \leq \varepsilon$, and $\phi(x) = 1$ when $x > \varepsilon$. It is obvious that $I(x \geq \varepsilon) \leq \phi(x)$ and $\phi(0) = 0$. It follows that
\[
\mathbb{P}\left( \frac{S_n - \bar{m}}{n^{1/p}} \geq \varepsilon \right) 
\leq \mathbb{E}\left[ \phi\left( \frac{S_n - \bar{m}}{n^{1/p}} \right) \right] - \mathbb{E}\left[ \phi\left( \frac{S_{m-1} - (m-1)\bar{m}}{n^{1/p}} \right) \right]
= \sum_{m=1}^{n} \left[ \mathbb{E}\left[ \phi\left( \frac{S_m - m\bar{m}}{n^{1/p}} \right) \right] - \mathbb{E}\left[ \phi\left( \frac{S_{m-1} - (m-1)\bar{m}}{n^{1/p}} \right) \right] \right]. \tag{14}
\]

Let $f(x) = \mathbb{E}[\phi(x + (X_m/n^{1/p}))]$. By the independence of $\{X_m\}_{m=1}^{n}$, we have
\[
\mathbb{E}\left[ \phi\left( \frac{S_m - m\bar{m}}{n^{1/p}} \right) \right] = \mathbb{E}\left[ \mathbb{E}\left[ \phi\left( \frac{x + X_m}{n^{1/p}} \right) \right] \right]_{x = (S_{m-1} - m\bar{m})/n^{1/p}}
= \mathbb{E}\left[ f\left( \frac{S_{m-1} - m\bar{m}}{n^{1/p}} \right) \right]. \tag{15}
\]

Therefore,
\[
\mathbb{E}\left[ \phi\left( \frac{S_m - m\bar{m}}{n^{1/p}} \right) \right] - \mathbb{E}\left[ \phi\left( \frac{S_{m-1} - (m-1)\bar{m}}{n^{1/p}} \right) \right]
\leq \mathbb{E}\left[ f\left( \frac{S_{m-1} - m\bar{m}}{n^{1/p}} \right) \right] - \mathbb{E}\left[ \phi\left( \frac{S_{m-1} - (m-1)\bar{m}}{n^{1/p}} \right) \right]
\leq \sup_{x \in \mathbb{R}} \left\{ f(x) - \phi(x + \frac{\bar{m}}{n^{1/p}}) \right\}
= \sup_{x \in \mathbb{R}} \left\{ \mathbb{E}\left[ \phi\left( \frac{x + X_m}{n^{1/p}} \right) \right] - \phi(x + \frac{\bar{m}}{n^{1/p}}) \right\}
= \sup_{x \in \mathbb{R}} \left\{ \mathbb{E}\left[ \phi\left( \frac{x + X_m - \bar{m}}{n^{1/p}} \right) - \phi(x) \right] \right\}. \tag{16}
\]

For any $1 \leq m \leq n$, there exist two random variables $\zeta_m, \bar{\zeta}_m \in [0, 1]$ satisfying
\[
\phi\left( x + \frac{X_m - \bar{m}}{n^{1/p}} \right) - \phi(x) = \phi'(x) \frac{X_m - \bar{m}}{n^{1/p}} + \left( \phi'\left( x + \frac{X_m - \bar{m}}{n^{1/p}} \right) - \phi'(x) \right) \frac{X_m - \bar{m}}{n^{1/p}},
\]
\[
\phi'(x + \theta_m \frac{X_m - \bar{m}}{n^{1/p}}) - \phi'(x) = \phi''(x + \theta_m \frac{X_m - \bar{m}}{n^{1/p}}) \cdot \theta_m \frac{X_m - \bar{m}}{n^{1/p}}. \tag{17}
\]

Thus, by $\phi'(x) \geq 0$, for any $\delta > 0$, we have
\[
\mathbb{P}\left( \frac{S_n - n\bar{m}}{n^{1/p}} \geq \varepsilon \right) \leq \sum_{m=1}^{n} \sup_{x \in \mathbb{R}} \left\{ \mathbb{E}\left[ \phi\left( x + \frac{X_m - \bar{m}}{n^{1/p}} \right) - \phi(x) \right] \right\}
\leq \sum_{m=1}^{n} \sup_{x \in \mathbb{R}} \left\{ \mathbb{E}\left[ \phi'(x) \frac{X_m - \bar{m}}{n^{1/p}} \right] + \mathbb{E}\left[ \left| \phi'(x + \theta_m \frac{X_m - \bar{m}}{n^{1/p}}) \right| \cdot \theta_m \frac{X_m - \bar{m}}{n^{1/p}} \right] \right\}
\leq \sum_{m=1}^{n} \sup_{x \in \mathbb{R}} \left\{ \phi'(x) \frac{ \frac{X_m - \bar{m}}{n^{1/p}}}{n^{1/p}} + \mathbb{E}\left[ \left| \phi'(x + \theta_m \frac{X_m - \bar{m}}{n^{1/p}}) \right| \cdot \theta_m \frac{X_m - \bar{m}}{n^{1/p}} \right] \right\}
- \phi'(x) \frac{X_m - \bar{m}}{n^{1/p}} I\left( |X_m| > n^{1/p} \delta \right) + \phi''(x + \theta_m \frac{X_m - \bar{m}}{n^{1/p}}) \cdot \theta_m \frac{X_m - \bar{m}}{n^{1/p}} \cdot \theta_m \frac{X_m - \bar{m}}{n^{1/p}} \cdot I\left( |X_m| \leq n^{1/p} \delta \right) \right\}. \tag{18}
\]
Let \( M = \max \{ \sup_{x \in \mathbb{R}} |\phi(x)|, \sup_{x \in \mathbb{R}} |\phi'(x)|, \sup_{x \in \mathbb{R}} |\phi''(x)| \} \). It follows that

\[
\begin{align*}
\mathbb{P} \left( \frac{S_n - n\mu}{n^{1/p}} \geq \varepsilon \right) &\leq \frac{2M}{n^{1/p}} \sum_{m=1}^{n} \mathbb{E} \left[ \left| X_m - \mu \right| I \left( \left| X_m \right| > n^{1/p} \delta \right) \right] + \frac{M}{n^{1/p}} \sum_{m=1}^{n} \mathbb{E} \left[ \left| X_m - \mu \right|^2 I \left( \left| X_m \right| \leq n^{1/p} \delta \right) \right] \\
&\leq \frac{2M}{n^{1/p}} \sum_{m=1}^{n} \mathbb{E} \left[ \left| X_m \right| I \left( \left| X_m \right| > n^{1/p} \delta \right) \right] + \frac{2Mn^{1/p}}{n^{2/p}} \sum_{m=1}^{n} \mathbb{P} \left( \left| X_m \right| > n^{1/p} \delta \right) \]
\end{align*}
\]

(19)

\[
\begin{align*}
&\leq \frac{2M}{n^{1/p}n^{1/p} - n^{1/p} \delta} \sum_{m=1}^{n} \mathbb{E} \left[ \left| X_m \right|^p I \left( \left| X_m \right| > n^{1/p} \delta \right) \right] + \frac{2Mn^{1/p}}{n^{2/p} \delta} \sum_{m=1}^{n} \mathbb{E} \left[ \left| X_m \right|^p I \left( \left| X_m \right| > n^{1/p} \delta \right) \right] \\
&\quad + \frac{2Mn^{1/p}n^{1/p} \delta}{n^{2/p}} \sum_{m=1}^{n} \mathbb{P} \left( \left| X_m \right|^p \right) + \frac{2Mn^{1/p}}{n^{2/p}} \sup_{m \geq n} \mathbb{E} \left[ \left| X_m \right|^p \right] \\
&\longrightarrow 2M \delta^{2-p} \sup_{m \geq n} \mathbb{E} \left[ \left| X_m \right|^p \right], \text{ as } n \to \infty.
\end{align*}
\]

Then, by the arbitrariness of \( \delta \), we obtain

\[
\lim_{n \to \infty} \mathbb{P} \left( \frac{S_n - n\mu}{n^{1/p}} \geq \varepsilon \right) = 0. \tag{20}
\]

Considering \( \{-X_i\}_{i=1}^{\infty} \), applying the above consequence, we have

\[
\lim_{n \to \infty} \mathbb{P} \left( \frac{-S_n - n\mathbb{E}[-X_n]}{n^{1/p}} \geq \varepsilon \right) = 0. \tag{21}
\]

Equivalently,

\[
\lim_{n \to \infty} \mathbb{P} \left( \frac{S_n - n\mu}{n^{1/p}} \leq -\varepsilon \right) = 0. \tag{22}
\]

Noting that \( \mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B) \), we obtain (11).

For the case \( 0 < p < 1 \), applying the function \( \phi(x) \in C_{\text{c}}^2(\mathbb{R}) \) in (1) (actually, we only need \( \phi(x) \in C_{\text{b}}(\mathbb{R}) \) here), we have

\[
\lim_{n \to \infty} \mathbb{P} \left( \frac{S_n - n\mu}{n^{1/p}} \geq \varepsilon \right) \leq \mathbb{E} \left[ \phi \left( \frac{S_n}{n^{1/p}} \right) \right] \tag{23}
\]

Also, by \( f(x) = \mathbb{E}[\phi(x + (X_m/n^{1/p}))] \), we get

\[
\begin{align*}
\mathbb{E} \left[ \phi \left( \frac{S_m}{n^{1/p}} \right) \right] &= \mathbb{E} \left[ \mathbb{E} \left[ \phi \left( x + \frac{X_m}{n^{1/p}} \right) \right] \right] = \mathbb{E} \left[ \mathbb{E} \left[ \phi \left( x + \frac{X_m}{n^{1/p}} \right) \right] \right] \\
&\leq \mathbb{E} \left[ \phi \left( x + \frac{X_m}{n^{1/p}} \right) \right].
\end{align*}
\]

(24)

For any \( 1 \leq m \leq n \), there exists some random variables \( \lambda_m \in [0, 1] \) satisfying

\[
\phi \left( x + \frac{X_m}{n^{1/p}} \right) - \phi(x) = \phi \left( x + \lambda_m \frac{X_m}{n^{1/p}} \right) \tag{25}
\]

So, for any \( \delta > 0 \),
For any \(0 < p < 2\), the following two statements are equivalent:

1. For any \(\varepsilon > 0\),
   \[
   \lim_{n \to \infty} \mathbb{V} \left( \frac{S_n}{n^{1/p}} \geq \varepsilon \right) = 0.
   \]

2. For any \(\varphi \in C_b(\mathbb{R})\),
   \[
   \lim_{n \to \infty} \mathbb{E} \left[ \varphi \left( \frac{S_n}{n^{1/p}} \right) \right] = \varphi(0).
   \]

Proof. (1) \(\implies\) (2): Let \(L = \sup_n |\varphi(x)|\), then for any \(\varepsilon > 0\),
   \[
   \mathbb{E} \left[ \varphi \left( \frac{S_n}{n^{1/p}} \right) \right] = \mathbb{E} \left[ \varphi \left( \frac{|S_n|}{n^{1/p}} \right) \right] \leq L \mathbb{E} \left[ \frac{|S_n|}{n^{1/p}} \right] = L \sup_{-\infty < x < \infty} \varphi(x),
   \]
   as \(n \to \infty\),
   \[
   \sup_{-\infty < x < \infty} \varphi(x), \quad \text{as} \quad n \to \infty.
   \]

(2) \(\implies\) (1): For any \(\varepsilon > 0\), define a function \(\phi(x) \in C_b(\mathbb{R})\) as follows:
   \[
   \phi(x) = \begin{cases}
   1, & x < -\varepsilon \text{ or } x > \varepsilon, \\
   -\frac{|x|}{\varepsilon} & -\varepsilon \leq x \leq \varepsilon.
   \end{cases}
   \]

We can easily obtain that \(I(|x| \geq \varepsilon) \leq \phi(x)\) and \(\phi(0) = 0\). Then,
   \[
   \mathbb{V} \left( \frac{|S_n|}{n^{1/p}} \geq \varepsilon \right) \leq \mathbb{E} \left[ \phi \left( \frac{|S_n|}{n^{1/p}} \right) \right] \to \phi(0) = 0, \quad \text{as} \quad n \to \infty.
   \]

Theorem 1 together with Theorem 2 gives birth to the following result.

Corollary 1. If \(\lim_{n \to \infty} \sup_{n \geq 1} \mathbb{E} |X_n|^{p} I(|X_n| > n) = 0\),
for some \(0 < p < 2\), \(\mathbb{E} [X_n] = \mathbb{E} \mathbb{E} [X_n] = 0\), for each \(n \geq 1\) when \(1 \leq p < 2\). Then, for any \(\varphi \in C_b(\mathbb{R})\),
   \[
   \lim_{n \to \infty} \mathbb{E} \left[ \varphi \left( \frac{S_n}{n^{1/p}} \right) \right] = \varphi(0).
   \]

Remark 4. If sublinear expectation \(\mathbb{E}\) coincides with the classical expectation \(\mathbb{E}\), where \(\mathbb{P}\) is classical probability, (11)–(13) and (34) all can be reduced to the classical forms.

4. The Marcinkiewicz–Zygmund Strong LLNs

To study the moment conditions for strong LLNs, we need the following concept of function class which was introduced by Petrov [17]. Let \(\Phi_c\) (or, respectively, \(\Phi_d\)) denote the set of nonnegative functions \(\psi(x)\) defined on \([0, \infty)\) satisfying

1. \(\psi(x)\) is positive and nondecreasing on \((0, \infty)\). The series \(\sum_{n=1}^{\infty} (1/\psi(n))\) converges (respectively, diverges).

2. For any fixed \(a > 0\), there exists \(C > 0\) such that \(\psi(x + a) \leq C \psi(x)\) for any \(x > 0\).

Functions \(x^a\) and \((\ln(1 + x))^{1+a} (a > 0)\) are examples of \(\Phi_c\). The functions \(\ln(1 + x)\) and \(\ln(\ln(e^x))\) belong to \(\Phi_d\).

Before we give the main results, we need the following two lemmas which are used to cope with the truncated parts.

Lemma 4. Suppose that \(\psi(x) \in \Phi_c\). Then, for any \(0 < p < 2\),
   \[
   \sum_{n=1}^{\infty} \frac{1}{n^p \psi(n^{1/p} / (\ln(1 + n)))} < \infty.
   \]

Proof. Since \(\psi(x) \in \Phi_c\), we have \(\psi(n^{1/p} / (\ln(1 + n))) \geq \psi(n^{1/p})\) for each \(n \geq 1\). The rest of the proof is similar to Lemma 2.5 of Hu [8].

Lemma 5. Suppose that \(|X_n - \mu| \leq 2n^{1/p} / \ln(1 + n)\) for some \(0 < p < 2\), and
Since \( w \), then, for any \( \frac{1}{2} \leq \alpha \leq 1 \), for some \( \psi \in \Phi \), when \( 1 \leq p < 2 \),
\[
\sup_{n \geq 1} \mathbb{E}[|X_n|^p \ln^{\alpha - 1}(1 + |X_n|)\psi(|X_n|)] < \infty, \quad \text{for some } \psi \in \Phi, \text{ when } 0 < p < 1.
\] (36)

Then, for any \( m > 1 \), we have
\[
\sup_{n \geq 1} \mathbb{E} \left[ \exp \left( \frac{m \ln (1 + n)}{n^{1/p}} \sum_{i=1}^{n} (X_i - \mu) \right) \right] < \infty. \quad (37)
\]

Proof. Since \( e^x \leq 1 + x + (x^2/2)e^{2x} \) for all real \( x \), and for \( i \leq n \),
\[
\frac{\ln(1 + n)}{n^{1/p}} |X_i - \mu| \leq \frac{\ln(1 + n)}{n^{1/p}} \cdot \frac{2^{1/p}}{\ln(1 + i)} \leq 2m, \quad (38)
\]
we have for any \( m > 1 \),
\[
\exp \left( \frac{m \ln (1 + n)}{n^{1/p}} (X_i - \mu) \right) \leq 1 + \frac{m \ln (1 + n)}{n^{1/p}} (X_i - \mu) + \frac{m^2 \ln^2 (1 + n)}{2n^{2/p}} |X_i - \mu|^2 \exp \left( \frac{m \ln (1 + n)}{n^{1/p}} |X_i - \mu| \right). \quad (39)
\]

Taking expectations, for \( i \leq n \),
\[
\mathbb{E} \left[ \exp \left( \frac{m \ln (1 + n)}{n^{1/p}} (X_i - \mu) \right) \right] \leq 1 + \frac{m^2 \ln^2 (1 + n)}{2n^{2/p}} \mathbb{E}[|X_i - \mu|^2]. \quad (40)
\]

Since \( \psi \in \Phi \), the function \( f(x) = \psi(x)/\ln(1 + x) \) converges to \( \infty \) as \( x \to \infty \). Then, for any \( 0 < a < 1 \),
\[
\ln^2 (1 + n) \mathbb{E}[|X_i - \mu|^2] \leq \ln^2 (1 + n) \mathbb{E}[|X_i - \mu|^2 I(|X_i - \mu| \leq n^a)] + \ln^2 (1 + n) \sup_{i \leq n} \mathbb{E}[|X_i - \mu|^2 I(n^a \leq |X_i - \mu| \leq \frac{2^{1/p}}{\ln(1 + i)})]. \quad (41)
\]

For \( 1 \leq p < 2 \), by noting that
\[
\sup_{n \geq 1} \mathbb{E}[|X_n - \mu|^p \ln^{\alpha - 1}(1 + |X_n - \mu|)\psi(|X_n - \mu|)] \\
\leq \sup_{n \geq 1} \mathbb{E} \left[ |X_n| + |\mu| \right]^p \ln^{\alpha - 1}(1 + |X_n| + |\mu|)\psi(|X_n| + |\mu|)] \\
\leq C \sup_{n \geq 1} \mathbb{E}[|X_n|^p + |\mu|^p] \ln^{\alpha - 1}(1 + |X_n|)\psi(|X_n|)] \leq \infty
\] (42)

we have
\[
\ln^2 (1 + n) \mathbb{E}[|X_i - \mu|^2] \leq \ln^2 (1 + n) \sup_{i \leq n} \mathbb{E}[|X_i - \mu|^2 I(|X_i - \mu| \leq n^a)] + \ln^2 (1 + n) \sup_{i \leq n} \mathbb{E}[|X_i - \mu|^2 I(n^a \leq |X_i - \mu| \leq \frac{2^{1/p}}{\ln(1 + i)})]. \quad (43)
\]

\[
= \frac{2^{-p} \psi(2^{-p}/p)}{\ln^{1-p}(1 + n) \psi(n^a)} \cdot \sup_{i \leq n} \mathbb{E}[|X_i - \mu|^p \ln^{\alpha - 1}(1 + |X_i - \mu|)\psi(|X_i - \mu|)] \\
\leq \frac{\ln^2 (1 + n)}{n^{2/p-1-2a}} + \frac{\ln^2 (1 + n)}{n^{2/p-1}} \cdot 1 \cdot \frac{1}{f(n^a)} \cdot \sup_{i \leq n} \mathbb{E}[|X_i - \mu|^p \ln^{\alpha - 1}(1 + |X_i - \mu|)\psi(|X_i - \mu|)] \\
= \ln^2 (1 + n) \mathbb{E}[|X_i - \mu|^2] + O \left( \frac{1}{f(n^a)} \right). \quad (44)
\]

\[
\frac{\ln^2 (1 + n) \mathbb{E}[|X_i - \mu|^2]}{n^{2/p-1-2a}} + 0 \left( \frac{1}{f(n^a)} \right).
\]
For $0 < p < 1$, by noting that

$$\sup_{n \leq 1} E \left[ |X_n - \overline{p}|^p \psi(|X_n - \overline{p}|) \right] \leq \sup_{n \leq 1} E \left[ \left( |X_n| + |\overline{p}| \right)^p \psi(|X_n| + |\overline{p}|) \right]$$

$$\leq C \sup_{n \leq 1} E \left[ (|X_n|^p + |\overline{p}|^p) \psi(|X_n|) \right] < \infty,$$  \hspace{1cm} (44)

we have

$$\frac{\ln^2 (1 + n)}{n^{2/p - 1}} \sup_{i \leq n} E \left[ |X_i - \overline{p}|^2 \right]$$

$$\leq \frac{\ln^2 (1 + n)}{n^{2/p - 1}} \cdot n^{- \alpha^2} + \frac{\ln^2 (1 + n)}{n^{2/p - 1}} \cdot \frac{2^{2 - p} n^{(2 - p)/p}}{\ln^2 (1 + n)^i} \cdot \frac{1}{\psi(n^i)}$$

$$\cdot \sup_{i \leq n} E \left[ |X_i - \overline{p}|^p \psi(|X_i - \overline{p}|) \right]$$

$$\leq \frac{\ln^2 (1 + n)}{n^{2/p - 1 - 2a}} \cdot \frac{\ln^2 (1 + n)}{n^{2/p - 1}} \cdot \frac{2^{2 - p} n^{(2 - p)/p}}{\ln^2 (1 + n)^i} \cdot \frac{1}{\ln (1 + n^i) f (n^i)}$$

$$\cdot \sup_{i \leq n} E \left[ |X_i - \overline{p}|^p \psi(|X_i - \overline{p}|) \right]$$

$$= \frac{\ln^2 (1 + n)}{n^{2/p - 1 - 2a}} + O \left( \frac{1}{\ln (1 + n) f (n^i)} \right).$$  \hspace{1cm} (45)

We set $a$ such that $2/p - 1 - 2a > 0$, i.e., $a < (1/p) - (1/2)$. So, for $0 < p < 2$, we have

$$\lim_{n \to \infty} \frac{\ln^2 (1 + n)}{n^{2/p - 1}} \sup_{i \leq n} E \left[ |X_i - \overline{p}|^2 \right] = 0.$$  \hspace{1cm} (46)

Then, there exists some $M > 0$ such that for any $n \geq 1$,

$$\frac{\ln^2 (1 + n)}{n^{2/p}} \sup_{i \leq n} E \left[ |X_i - \overline{p}|^2 \right] \leq M/n.$$  \hspace{1cm} (47)

By the independence of $\{X_n\}_{n=1}^\infty$, we have

$$E \left[ \exp \left( \frac{m \ln (1 + n)}{n^{1/p}} \sum_{i=1}^n (X_i - \overline{p}) \right) \right] = \prod_{i=1}^n E \left[ \exp \left( \frac{m \ln (1 + n)}{n^{1/p}} (X_i - \overline{p}) \right) \right]$$

$$\leq \left( \exp \left( \frac{M n^2 e^{2m}}{2n} \right) \right)^n$$

$$= e^{(M n^2 e^{2m})/2} < \infty.$$  \hspace{1cm} (48)

\[ \square \]

**Theorem 3.** Suppose that $E$ is continuous from below.

(1) If $\sup_{n \leq 1} E \left[ |X_n|^p \ln^{p-1} (1 + |X_n|) \psi(|X_n|) \right] < \infty$ for some $\psi \in \Phi_{\psi}$ and $1 \leq p < 2$, then

$$\forall \left( \liminf_{n \to \infty} \frac{S_n - n \overline{\mu}}{n^{1/p}} < 0 \right) \cup \left( \limsup_{n \to \infty} \frac{S_n - n \overline{\mu}}{n^{1/p}} > 0 \right) = 0.$$  \hspace{1cm} (49)

(2) If $\sup_{n \leq 1} E \left[ |X_n|^p \ln^p (1 + |X_n|) \psi(|X_n|) \right] < \infty$ for some $\psi \in \Phi_{\psi}$ and $0 < p < 1$, then

$$\forall \left( \lim_{n \to \infty} \frac{S_n}{n^{1/p}} > 0 \right) = 0.$$  \hspace{1cm} (50)

**Proof.**

(1) For $1 \leq p < 2$, define $f_n(x) = (-n^{1/p} \ln (1 + n)) X_n (X_n - \overline{p}) (X_n - x)$. Then, $f_n(\cdot), f_n(\cdot)$ is $C_{1, 1/p}$.

Let $Y_n = f_n(X_n - \overline{p}) - E[f_n(X_n - \overline{p}) + \overline{p}]$, and denote $\overline{S}_n = \sum_{i=1}^n Y_i$. Then,

$$X_n - \overline{p} = Y_n - \overline{p} + f_n(X_n - \overline{p}) + E[f_n(X_n - \overline{p})],$$

$$\frac{1}{n^{1/p}} (S_n - n \overline{\mu}) = \frac{1}{n^{1/p}} (S_n - n \overline{\mu}) + \frac{1}{n^{1/p}} \sum_{i=1}^n f_i(X_i - \overline{p})$$

$$+ \frac{1}{n^{1/p}} \sum_{i=1}^n E[f_i(X_i - \overline{p})].$$  \hspace{1cm} (52)

Note that

$$E[f_i(X_i - \overline{p})] = E[X_i - \overline{p} - f_i(X_i - \overline{p})]$$

$$\leq E[X_i - \overline{p}] + E[-f_i(X_i - \overline{p})]$$

$$\leq E[f_i(X_i - \overline{p})].$$  \hspace{1cm} (53)

Therefore,

$$\frac{1}{n^{1/p}} (S_n - n \overline{\mu}) \leq \frac{1}{n^{1/p}} (S_n - n \overline{\mu}) + \frac{1}{n^{1/p}} \sum_{i=1}^n f_i(X_i - \overline{p})$$

$$+ \frac{1}{n^{1/p}} \sum_{i=1}^n E[f_i(X_i - \overline{p})].$$  \hspace{1cm} (54)
We can easily obtain that \( \{Y_n\}_{n=1}^\infty \) are independent, \( |Y_n| \leq (2n^{1/p}/\ln(1+n)) \), \( \mathbb{E}[Y_n] = \overline{\mu} \) and \( \sup_{n \geq 1} \mathbb{E}[|Y_n|^p \ln^{p-1}(1+|Y_n|)\psi(|Y_n|)] < \infty. \)

For any \( \varepsilon > 0 \), we set \( m > (1/\varepsilon) \). Then, by Chebyshev’s inequality,

\[
\mathcal{V}\left( \frac{S_n - n\overline{\mu}}{n^{1/p}} \geq \varepsilon \right) = \mathcal{V}\left( \frac{m \ln(1+n)}{n^{1/p}} - \sum_{i=1}^n (Y_i - \overline{\mu}) \geq \varepsilon m \ln(1+n) \right) \\
\leq \exp\left( \frac{m \ln(1+n)}{n^{1/p}} \sum_{i=1}^n (Y_i - \overline{\mu}) \right) \\
\leq \frac{1}{(1+n)^{\epsilon m}} \sup_{n \geq 1} \mathbb{E}\left( \exp\left( \frac{m \ln(1+n)}{n^{1/p}} \sum_{i=1}^n (Y_i - \overline{\mu}) \right) \right).
\]

By Lemma 5 and \( \sum_{n=1}^\infty (1/(1+n)^{\epsilon m}) < \infty \), we have

\[
\sum_{n=1}^\infty \mathcal{V}\left( \frac{S_n - n\overline{\mu}}{n^{1/p}} \geq \varepsilon \right) < \infty. \quad (56)
\]

By Borel–Cantelli lemma, we have

\[
\mathcal{V}\left( \limsup_{n \to \infty} \frac{S_n - n\overline{\mu}}{n^{1/p}} \geq \varepsilon \right) = 0. \quad (57)
\]

The continuity from below of \( \mathcal{V} \) can be deduced by the continuity from below of \( \mathbb{E} \). So, we have

\[
\mathcal{V}\left( \limsup_{n \to \infty} \frac{S_n - n\overline{\mu}}{n^{1/p}} > 0 \right) = 0. \quad (58)
\]

Moreover,

\[
\sum_{i=1}^\infty \mathbb{E}\left[ \left| f_i(X_i - \overline{\mu}) \right| \right] \leq \sum_{i=1}^\infty \mathbb{E}\left[ |X_i - \overline{\mu}| \left( |X_i - \overline{\mu}| > (i^{1/p}/\ln(1+i)) \right) \right] \\
\leq \sum_{i=1}^\infty \mathbb{E}\left[ |X_i - \overline{\mu}|^{p-1} \ln^{p-1}(1+|X_i - \overline{\mu}|) \psi(|X_i - \overline{\mu}|) \right] \\
\leq \sum_{i=1}^\infty \mathbb{E}\left[ |X_i - \overline{\mu}|^{p-1} (1+i^{1/p}/\ln(1+i))^{p-1} \ln^{p-1}(1+i^{1/p}/\ln(1+i)) \psi(i^{1/p}/\ln(1+i)) \right]. \quad (59)
\]

where

\[
\frac{i^{1/p}}{\ln(1+i)} \ln^{p-1}(1+i^{1/p}/\ln(1+i))^{p-1} \psi\left( \frac{i^{1/p}}{\ln(1+i)} \right) \\
\geq i^{1/p} - (1+i^{1/p}/\ln(1+i))^{p-1} \ln^{p-1}(1+i^{1/p}/\ln(1+i))^{p-1} \psi\left( \frac{i^{1/p}}{\ln(1+i)} \right) \\
\geq i^{1/p} - \left( \frac{1}{p} \ln i - \ln \ln(1+i) \right) \psi\left( \frac{i^{1/p}}{\ln(1+i)} \right) \\
\geq i^{1/p} \ln(1+i) \psi\left( \frac{i^{1/p}}{\ln(1+i)} \right). \quad (60)
\]

By Lemma 4, we have

\[
\sum_{i=1}^\infty \mathbb{E}\left[ \left| f_i(X_i - \overline{\mu}) \right| \right] < \infty. \quad (61)
\]

By Kronecker lemma, we have

\[
\lim_{n \to \infty} \frac{1}{n^{1/p}} \sum_{i=1}^n \mathbb{E}\left[ \left| f_i(X_i - \overline{\mu}) \right| \right] = 0. \quad (62)
\]

On the other hand, by the continuity from below of \( \mathbb{E} \),

\[
\mathbb{E}\left[ \sum_{i=1}^\infty \left| f_i(X_i - \overline{\mu}) \right| \right] \leq \sum_{i=1}^\infty \mathbb{E}\left[ \left| f_i(X_i - \overline{\mu}) \right| \right] < \infty. \quad (63)
\]
Then, by Lemma 3, we have
\[ \nu\left( \sum_{i=1}^{\infty} \frac{\bar{f}_i(X_i - \bar{p})}{i^{1/p}} \leq \infty \right) = 1, \]  
which implies
\[ \nu\left( \lim_{n \to \infty} \frac{1}{n^{1/p}} \sum_{i=1}^{n} \bar{f}_i(X_i - \bar{p}) = 0 \right) = 1. \]  
(65)

Taking \( \limsup_{n \to \infty} \) on both sides of (54), and by (58), (62), and (65), we have
\[ \nu\left( \limsup_{n \to \infty} \frac{S_n - n\bar{\mu}}{n^{1/p}} \leq 0 \right) = 1. \]  
Equivalently,
\[ \nu\left( \limsup_{n \to \infty} \frac{S_n - n\bar{\mu}}{n^{1/p}} > 0 \right) = 0. \]  
(67)
Considering \( \{-X_i\}_{i=1}^{\infty} \) with \( \mathbb{E}[-X_i] = -\mu \), we have
\[ \nu\left( \limsup_{n \to \infty} \frac{-S_n - n(-\mu)}{n^{1/p}} > 0 \right) = 0. \]  
(68)
Equivalently,
\[ \nu\left( \liminf_{n \to \infty} \frac{S_n - n\mu}{n^{1/p}} < 0 \right) = 0. \]  
(69)
Noting that \( \nu(A \cup B) \leq \nu(A) + \nu(B) \), we obtain (49).

(2) For \( 0 < p < 1 \), let \( Z_n = f_n(X_n) - \mathbb{E}[f_n(X_n)] \) and denote \( S_n^* = \sum_{i=1}^{n} Z_i \). Then,
\[ X_n = Z_n + \bar{f}_n(X_i) + \mathbb{E}[f_n(X_n)], \]
\[ \frac{1}{n^{1/p}} S_n = \frac{1}{n^{1/p}} S_n^* + \frac{1}{n^{1/p}} \sum_{i=1}^{n} \bar{f}_i(X_i) + \frac{1}{n^{1/p}} \sum_{i=1}^{n} \mathbb{E}[f_i(X_i)]. \]  
(70)
We can easily obtain that \( \{Z_i\}_{i=1}^{\infty} \) are independent, \( |Z_n| \leq (2n^{1/p}/\ln(1 + n)) \), \( E[Z_n] = 0 \), and \( \sup_{n \geq 1} \mathbb{E}[|Z_n|^{1/p}] \mathbb{P}(1 + |Z_n|) < \infty \).

For any \( \epsilon > 0 \), we set \( m > (1/\epsilon) \). Then, by Chebyshev’s inequality and Lemma 5, we have
\[ \nu\left( \frac{S_n^*}{n^{1/p}} \geq \epsilon \right) = \nu\left( \frac{m \ln(1 + n)}{n^{1/p}} \sum_{i=1}^{n} Z_i \geq \epsilon \ln(1 + n) \right) \]
\[ \leq \frac{1}{(1 + n)^m} \sup_{n \geq 1} \mathbb{E}\left[ \exp\left( \frac{m \ln(1 + n)}{n^{1/p}} \sum_{i=1}^{n} Z_i \right) \right] < \infty. \]  
(71)
By Borel–Cantelli lemma, we have
\[ \nu\left( \limsup_{n \to \infty} \frac{S_n^*}{n^{1/p}} \geq \epsilon \right) = 0. \]  
(72)
Hence, by the continuity from below of \( \nu \), we have
\[ \nu\left( \limsup_{n \to \infty} \frac{S_n^*}{n^{1/p}} > 0 \right) = 0. \]  
(73)
Let \( B_i = \{|X_i| > (i^{1/p}/\ln(1 + i))\} \) for \( i \geq 1 \). Then,
\[ \sum_{i=1}^{\infty} \nu(B_i) \leq \sum_{i=1}^{\infty} \frac{\mathbb{E}[|X_i|^{1/p} \mathbb{P}(1 + |X_i|)] \mathbb{P}(1 + |X_i|)}{(i^{1/p}/\ln(1 + i))^{1/p} \mathbb{P}(1 + (i^{1/p}/\ln(1 + i))) \mathbb{P}(1 + (i^{1/p}/\ln(1 + i)))} \]
\[ \leq \frac{1}{1 - \epsilon} \sum_{i=1}^{\infty} \mathbb{E}[|X_i|^{1/p} \mathbb{P}(1 + |X_i|)] \sum_{i=1}^{\infty} \frac{1}{(i^{1/p}/\ln(1 + i))^{1/p} \mathbb{P}(1 + (i^{1/p}/\ln(1 + i)))} < \infty. \]  
(74)
By Borel–Cantelli lemma, we have
\[ \nu\left( \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} B_i \right) = 0. \]  
(75)
Then, for any \( \omega \in \bigcup_{i=1}^{\infty} \bigcap_{n=1}^{\infty} B_n^\omega \), there exists \( n(\omega) \in \mathbb{N}^* \) such that for any \( i > n(\omega) \), \( |X_i(\omega)| \leq (i^{1/p}/\ln(1 + i)) \). For these \( \omega \),
\[ \sum_{i=1}^{n(\omega)} \frac{|\bar{f}_i(X_i)|}{i^{1/p}} \leq \sum_{i=1}^{n(\omega)} \frac{|X_i| \mathbb{P}(1 + |X_i|)}{i^{1/p} \mathbb{P}(1 + (i^{1/p}/\ln(1 + i)))} = \left( \sum_{i=1}^{n(\omega)} + \sum_{i=n(\omega)+1}^{\infty} \right) \frac{|X_i| \mathbb{P}(1 + |X_i|)}{i^{1/p} \mathbb{P}(1 + (i^{1/p}/\ln(1 + i)))} < \infty. \]  
(76)
Hence,
\[ \nu\left( \sum_{i=1}^{\infty} \frac{|\bar{f}_i(X_i)|}{i^{1/p}} < \infty \right) = 1, \]  
(77)
which implies
\[ \nu\left( \lim_{n \to \infty} \frac{1}{n^{1/p}} \sum_{i=1}^{n} \bar{f}_i(X_i) = 0 \right) = 1. \]  
(78)
Furthermore,

\[
\sum_{i=1}^{\infty} \mathbb{E}\left[ f_i(X_i) \right] \leq \sum_{i=1}^{\infty} \mathbb{E}\left[ |X_i| \mathbb{I}\{ |X_i| \leq (i^{1/p} / \ln(1 + i)) \} \right] + \sum_{i=1}^{\infty} \mathbb{E}\left[ |X_i| \mathbb{I}\{ |X_i| > (i^{1/p} / \ln(1 + i)) \} \right] \\
\leq \sum_{i=1}^{\infty} \mathbb{E}\left[ ||X_i||^p \mathbb{I}\{ |X_i| \leq (i^{1/p} / \ln(1 + i)) \} \right] \frac{i^{(1-p)/p}}{\ln^p(1 + i)} + \sum_{i=1}^{\infty} \mathbb{E}\left[ |X_i| \mathbb{I}\{ |X_i| > (i^{1/p} / \ln(1 + i)) \} \right] \\
\leq \sup_{i \geq 1} \mathbb{E}\left[ |X_i|^p \mathbb{I}\{ |X_i| \leq (i^{1/p} / \ln(1 + i)) \} \right] \frac{1}{\ln^p(1 + i)} + \sup_{i \geq 1} \mathbb{E}\left( B_i \right) \\
< \infty,
\]

which implies

\[
\lim_{n \to \infty} \frac{1}{n^{1/p}} \sum_{i=1}^{n} \mathbb{E}\left[ f_i(X_i) \right] = 0.
\]

Taking \( \lim \sup_{n \to \infty} \) on both sides of (70) and by (73), (78), and (80), we have

\[
\psi\left( \lim \sup_{n \to \infty} \frac{S_n}{n^{1/p}} \right) \leq 0.
\]

Equivalently,

\[
\psi\left( \lim \sup_{n \to \infty} \frac{S_n}{n^{1/p}} > 0 \right) = 0.
\]

Considering \( \{ -X_i \}_{i=1}^{\infty} \), we have

\[
\psi\left( \lim \inf_{n \to \infty} \frac{S_n}{n^{1/p}} < 0 \right) = 0.
\]

\[\square\]

**Remark 5.** The two moment conditions for case 1 \( \leq p < 2 \) and case 0 < \( p \leq 1 \) in Theorem 3 are both stronger than \( \sup_{\alpha \geq 1} \mathbb{E}\left[ |X_{\alpha}|^p \right] < \infty \) but weaker than \( \sup_{\alpha \geq 1} \mathbb{E}\left[ |X_{\alpha}|^{p+a} \right] < \infty (\alpha > 0) \). Zhang and Lin [14] obtained that the \( \beta \)th moments for Choquet expectation is the necessary and sufficient conditions of the Marcinkiewicz–Zygmund strong LLNs. One can find a counterexample that the \( \beta \)th moments for sublinear expectation is finite but the \( \beta \)th moments for Choquet expectation is not finite (similar to Example 4.1 of Hu [12]). So the \( \beta \)th moments for sublinear expectation cannot maintain the Marcinkiewicz–Zygmund strong LLNs.

**Remark 6.** If sublinear expectation \( \mathbb{E} \) coincides with the classical expectation \( E_p \) where P is classical probability, Theorem 3 can be reduced to the classical Marcinkiewicz–Zygmund strong LLNs:

\[
P\left( \lim_{n \to \infty} \frac{S_n}{n^{1/p}} = 0 \right) = 1,
\]

where \( a = 0 \) if 0 < \( p < 1 \) and \( a = E[X_i] \) if 1 \( \leq p < 2 \).

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**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare no conflicts of interest.

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