Central spin dynamics and relaxation of antiferromagnetic order in a central-spin–XXZ-chain system

Jiaxiu Li,1,2 Ye Cao,2 and Ning Wu1,2

1Center for Quantum Technology Research, School of Physics, Beijing Institute of Technology, Beijing 100081, China
2Key Laboratory of Advanced Optoelectronic Quantum Architecture and Measurements (MOE), School of Physics, Beijing Institute of Technology, Beijing 100081, China

Using an equations-of-motion method based on analytical representations of spin-operator matrix elements in the XX chain, we obtain exact long-time dynamics of a composite system consisting of a spin-S central spin and an XXZ chain, with the two interacting via inhomogeneous XXZ-type hyperfine coupling. Three types of initial bath states, namely, the Néel state, the ground state, and the spin coherent state are considered. We study the reduced dynamics of both the central spin and the XXZ bath. For the Néel state, we find that strong hyperfine couplings slow down the initial decay but facilitate the long-time relaxation of the antiferromagnetic order. Moreover, for fixed hyperfine coupling a larger S leads to a faster initial decay of the antiferromagnetic order. We then study the purity dynamics of an S = 1 central spin coupled to an XXZ chain prepared in the ground state. The time-dependent purity is found to reach the highest values at the critical point. We finally study the polarization dynamics of the central spin homogeneously coupled to a bath prepared in the spin coherent state. Under the resonant condition, the polarization dynamics for S > 1 exhibits collapse-revival behaviors with fine structures. However, the collapse-revival phenomena is found to be fragile with respect to the anisotropic intrabath coupling.

I. INTRODUCTION

The study of real-time dynamics of a composite system made up of a central spin and a coupled quantum spin bath is important for understanding many physical phenomena in condensed matter and statistical physics [1]. In particular, the Gaudin model [2] and its variants, which describe a central spin coupled to spin baths without intrabath coupling, play an important role in quantum decoherence [3–10], quantum information [11, 12], quantum metrology [13], and even mathematical physics [14–19]. The dynamics of these noninteracting central spin models has been widely studied by various theoretical methods, including techniques based on the Bethe ansatz solutions [20, 21], quantum master equations [22–24], density matrix renormalization group method [25], and so on. In the special case of homogeneous hyperfine coupling, the polarization dynamics of the S = 1/2 central spin even admits analytical solutions [26–28]. Because of the existence of integrability or an extensive set of conserved quantities, the above-mentioned approaches can often deal with noninteracting baths having a large number of spins.

However, including the intrabath coupling among bath spins generally makes the evaluation of the dynamics difficult even for intermediate-size baths, mainly due to the induced breakdown of the integrability. In spite of the technical difficulties, it is however interesting and important to take into account the effect of environmental self-interaction on the central spin dynamics, as demonstrated in an early work using a fully connected spin–spin-bath model [21]. Recently, there appeared several theoretical works in which the decoherence of a qubit coupled to interacting quantum spin chains are investigated [22–24]. Among these, Wu et al. studied the decoherence of a qubit coupled to an XX chain via XX- [22] and XXZ-type [23] hyperfine couplings using an equations-of-motion method and a Chebyshev expansion technique. Based on the Bethe ansatz solution of the XXX chain, Lu et al. obtained the exact decoherence and polarization dynamics of a qubit coupled homogeneously to an XXX bath [24]. Nevertheless, the quantum dynamics of a spin-S central spin interacting inhomogeneously with an XXZ chain remains unexplored. In passing we mention that the reduced dynamics of a qubit locally coupled to a free-end XXZ chain was studied in Ref. [25] by using time-dependent density matrix renormalization group method.

As a paradigmatic spin model exhibiting strong correlations, the spin-1/2 XXZ chain has served as an ideal testbed for studying nonequilibrium quantum dynamics. Barmettler et al. studied the relaxation of antiferromagnetic order in an XXZ chain prepared in the Néel state [26]. It was found that the antiferromagnetic order experiences oscillatory or nonoscillatory relaxations depending on the anisotropy parameter and the relaxation time reaches its minimum at the critical point. The same dynamical protocol was later used to study dynamical quantum phase transitions in the XXZ chain [27].

The influence of a small quantum system and the induced frustrations on strongly correlated systems is another long-studied topic [28–29]. Richter and Voigt studied the static properties of a composite spin system named “frustrated Heisenberg star” [29], which is made up of a central spin and a homogeneously coupled XXX ring. The competition between the intrabath coupling
and the hyperfine coupling is found to result in interesting behaviors of ground-state spin correlations. It is therefore desirable and interesting to study the effect of the interplay between the two types of interactions on the internal dynamics of the interacting spin bath.

In this work, we obtain exact quantum dynamics of a composite system consisting of a spin-$S$ central spin and a coupled periodic XXZ chain. The two parts interact with each other through the usual XXZ-type homogeneous hyperfine coupling $\tilde{B}$. Due to the presence of the intrabath interaction, theoretical methods such as those based on the Bethe ansatz solution are no longer applicable. Here, we employ an equations-of-motion approach to treat the time evolution of the whole system. The usefulness of the method lies in the fact that each bath spin interacts locally to the central spin, while the matrix elements of local bath operators in the diagonal basis of the XX chain admit analytical expressions. Using the conservation of the total magnetization, we explicitly write out the equations of motion for the time-dependent amplitudes in the XX-chain basis, where the coefficients are associated with the spin-operator matrix elements.

By numerically solving the equations of motion in each magnetization sector, we are able to calculate the dynamics of the composite system prepared in a generic initial state. We consider three types of initial states for the XXZ bath, i.e., the Néel state, the ground state of the XXZ ring, and the spin coherent state. The Néel state is one of the two degenerate ground states of the antiferromagnetic XXZ chain in the large anisotropy limit and has been realized with high fidelity in cold atom systems. It has also been used to investigate the relaxation of antiferromagnetic order in the XXZ chain and to probe the decoherence dynamics of a qubit coupled to spin baths. In our setup, we assume that the composite system is prepared in a separable pure state, so that the dynamical protocol can be regarded as a simultaneous quench of both the anisotropy parameter (from infinity to a finite value) and the hyperfine couplings (from zero to finite values). We study both the central-spin decoherence and the relaxation of the staggered magnetization in the XXZ bath after such a quench. It is found that the intrabath coupling has little effect on the short-time dynamics of the decoherence factor, but can change the long-time coherence significantly. The central spin also has a great influence on the relaxation of the antiferromagnetic order within the bath. We observe that strong hyperfine coupling can slow down the short-time decay but facilitate the long-time relaxation of the staggered magnetization. In addition, increasing the quantum number $S$ of the central spin at fixed hyperfine coupling strength will accelerate the initial decay of the staggered magnetization.

The central spin dynamics depends not only on the hyperfine coupling strength but also on the internal phase of the XXZ bath. Our second choice for the initial bath state is the ground state of the XXZ chain. In this case, we focus on the purity dynamics of an $S = 1$ central spin. We find that in the strong hyperfine coupling regime the time-dependent purity acquires the highest values when the bath is prepared in the ground state at the critical point. Finally we study the polarization dynamics of a central spin homogeneously coupled to an XXZ chain in the spin coherent state. For an XXX bath and $S = 1/2$, we recover the results in Ref. [13]. For $S > 1/2$, we find that the polarization dynamics still exhibits collapse-revival behaviors under the resonant condition. However, the collapse-revival phenomena are destroyed once the anisotropic intrabath coupling is introduced.

The rest of the paper is organized as follows. In Sec. II we introduce the central-spin-XXZ-chain model and provide details of the equations-of-motion approach. In Sec. III we present the numerical results for the three types of bath initial states. Conclusions are drawn in Sec. IV.

### II. MODEL AND METHODOLOGY

#### A. Hamiltonian

We consider an interacting central spin model described by the Hamiltonian (see Fig. I)

$$ H = H_S + H_B + H_{SB}. \tag{1} $$

The system part

$$ H_S = \omega S_z + \lambda S_z^2, \tag{2} $$

describes a central spin $\vec{S} = (S_x, S_y, S_z)$ of size $S \geq 1/2$, where $\omega$ is the Larmor frequency due to the applied magnetic field. We also include the single-ion anisotropy of the central spin with strength $\lambda$. The spin bath takes the form of a spin-1/2 XXZ chain

$$ H_B = H_{XY} + H_Z, $$

$$ H_{XY} = J \sum_{j=1}^{N} (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y), $$

$$ H_Z = J' \sum_{j=1}^{N} S_j^z S_{j+1}^z, \tag{3} $$

where $\vec{S}_j = (S_j^x, S_j^y, S_j^z)$ is the spin-1/2 operator for the $j$th bath spin. We have separated the bath Hamiltonian into the in-plane component $H_{XY}$ and the Ising component $H_Z$. For simplicity, we assume that $N$ is even and impose periodic boundary conditions. We set $J > 0$ and the sign of $J'$ essentially determines the quantum phase of $H_B$ [32]. The XXZ-type hyperfine coupling between the
central spin and the spin bath reads \[ H_{SB} = 2 \sum_{j=1}^{N} [g_j (S_x S_x^j + S_y S_y^j) + g'_j S_z S_z^j] \]

\[ = \sum_{j=1}^{N} [g_j (S_x + S_y - S_z) + 2 g'_j S_z S_z^j], \]

where \{g_j\} and \{g'_j\} are, respectively, the in-plane and Ising parts of the (inhomogeneous) exchange interaction constants. It is usually the case that \( g'_j/g_j = \Lambda, \forall j \), where \( \Lambda \) measures the anisotropy of the system-bath coupling. In the case of \( J = J' = 0 \), the bath becomes noninteracting and we recover the usual Gaudin model that admits Bethe ansatz solutions under certain conditions \[2,16,18,19\].

Let \( L = \sum_{j=1}^{N} \vec{S}_j \) be the collective angular momentum operator of the spin bath, it can be easily checked that the total magnetization \( \vec{M} = S_z + L_z \) is conserved. The angular momentum of the central spin \( \vec{S}_c \) is also conserved. However, the total angular momentum of the spin bath, \( \vec{L}^2 \), is not conserved unless \( J = J' \) and \{g_j\} and \{g'_j\} are both homogeneous \[22\]. Below the eigenvalue of \( \vec{M}, S_z \), and \( L_z \) will be denoted as \( M, s_z, \) and \( l_z \), respectively.

The total magnetization \( M \) can take the following \( 2S + N + 1 \) possible values: \( M = -S - \frac{N}{2}, -S - \frac{N}{2} + 1, \ldots, S + \frac{N}{2} \). The structure of the states in an individual \( M \)-subspace depends on whether \( S < \frac{N}{2} \) or \( S \geq \frac{N}{2} \). In this paper, we focus on the case of \( S < \frac{N}{2} \) (see Appendix [A] for details). To get a universal short-time dynamics for different numbers of bath spins, we introduce the energy scale

\[ \omega_{\text{Hub}} = 2 \left( \sum_{j=1}^{N} g_j^2 \right)^{\frac{1}{2}}, \]

which is associated with the fluctuation of the Overhauser field \[33\].

B. Method: spin-operator matrix elements

To numerically simulate the real-time dynamics of the composite system, we use the representation in which the noninteracting Hamiltonian \( H_0 = H_S + H_{XY} \) is diagonal. This is motivated by the fact that the matrix elements of each term in the remaining part of the Hamiltonian, \( H_1 = H - H_0 \), can be expressed in this representation in terms of the so-called spin-operator matrix elements for the XX chain \[30\]. The eigenbasis of \( H_0 \) is spanned by the following \((2S + 1)^N\) states

\[ \{ |s_z \rangle | \vec{\eta}_n \rangle \}, \quad s_z = S, S - 1, \ldots, -S; \quad n = 0, 1, \ldots, N, \]

where

\[ S_z | s_z \rangle = s_z | s_z \rangle, \]

\[ H_{XY} | \vec{\eta}_n \rangle = E_{\vec{\eta}_n} | \vec{\eta}_n \rangle, \]

with \( E_{\vec{\eta}_n} = \sum_{n=1}^{N} J \cos K^{(\sigma_n)}_{\vec{\eta}_n} \). Here, \(| \vec{\eta}_n \rangle \) is an eigenstate of \( H_{XY} \) having \( n \) fermionic excitations labeled by the tuple \( \vec{\eta}_n = (\eta_1, \ldots, \eta_n) \) (with the convention \( 1 \leq \eta_1 < \cdots < \eta_n \leq N \)) with respect to the vacuum state \(|0\rangle = | \downarrow \cdots \downarrow \rangle [30]\). The corresponding eigenenergy \( E_{\vec{\eta}_n} \) depends on the parity of \( n \) through wave numbers

\[ K^{(\sigma_n)}_{\vec{\eta}_n} = -\pi + \frac{2 \eta_1 + 1}{2} (\sigma_n - 3) \frac{\pi}{N}, \]

with \( \sigma_n = 1 \) (even \( n \)) or \( \sigma_n = -1 \) (odd \( n \)). For later convenience, we also introduce \( \alpha = s_z + n = M + \frac{N}{2} \), which is also conserved and can take values from \( \alpha = -S \) to \( \alpha = S + N \).

As we will see, the equations of motion of the system in the basis \{\(|s_z\rangle | \vec{\eta}_n \rangle \}\} involve the following matrix elements

\[ F_{\vec{\eta}_{n+1}, \vec{\eta}_n}(\{g_j\}) = \langle \vec{\eta}_n | \sum_{j=1}^{N} g_j S_j^x | \vec{\eta}_{n+1} \rangle, \]

\[ G_{\vec{\eta}_n, \vec{\eta}_{n+1}'}(\{g'_j\}) = \langle \vec{\eta}_n | \sum_{j=1}^{N} g'_j S_j^y | \vec{\eta}_{n+1}' \rangle, \]

\[ \tilde{G}_{\vec{\eta}_n, \vec{\eta}_{n+1}'} = \langle \vec{\eta}_n | \sum_{j=1}^{N} S_j^z S_{j+1}^z | \vec{\eta}_{n+1}' \rangle, \]

For the homogeneous XX ring described by \( H_{XY} \), it is shown in Ref. \[31\] that \( F_{j; \vec{\eta}_{n+1}, \vec{\eta}_n} \equiv \langle \vec{\eta}_n | S_j^x | \vec{\eta}_{n+1} \rangle \) admits a simple factorized form,

\[ F_{j; \vec{\eta}_{n+1}, \vec{\eta}_n} = \frac{1}{\sqrt{N}} \left( \frac{2}{N} \right)^n e^{i(j-n)\Delta_{\vec{\eta}_{n+1}, \vec{\eta}_n}} h_{\vec{\eta}_{n+1}, \vec{\eta}_n}, \]

where

\[ \Delta_{\vec{\eta}_{n+1}, \vec{\eta}_n} = \sum_{j=1}^{n+1} K^{(\sigma_{n+1})}_{\eta_j} - \sum_{i=1}^{n} K^{(\sigma_n)}_{\chi_i} \]

FIG. 1: A spin-S central spin interacts with an interacting spin bath modeled by an XXZ ring via inhomogeneous XXZ-type hyperfine coupling.
is the momentum transfer between $|\tilde{\eta}_{n+1}\rangle$ and $|\tilde{\chi}_n\rangle$ and
\[
h_{\tilde{\eta}_{n+1},\tilde{\chi}_n} = \prod_{i>j}(e^{-ikx_i} - e^{-ikx_j}) \prod_{i>j'}(e^{ikx_i} - e^{-ikx_{j'}})
\prod_{i=1}^{n+1} \prod_{j=1}^{n+1} (1 - e^{-i(K_{ij}^{(n+1)} - K_{ij}^{(n)})})
\]
(12)
is a function of the momenta [34]. From Eq. (10), we immediately get
\[
F_{\tilde{\eta}_{n+1},\tilde{\chi}_n}(\{g_j\})
= \left(\frac{2}{N}\right)^n \frac{\tilde{g}_{\Delta\tilde{\eta}_{n+1},\tilde{\chi}_n} e^{-in\Delta\tilde{\eta}_{n+1},\tilde{\chi}_n}}{\sqrt{N}} h_{\tilde{\eta}_{n+1},\tilde{\chi}_n},
\]
(13)
where
\[
\tilde{g}_q = \sum_{j=1}^{N} e^{iqj} g_j
\]
is the Fourier transform of $\{g_j\}$. Using $S_j^z = \frac{1}{2} - S_j^\sigma$, we similarly obtain
\[
G_{\tilde{\eta}_{n+1},\tilde{\chi}_n}(\{g_j\}) = \frac{1}{2} \delta_{\tilde{\eta}_{n+1},\tilde{\chi}_n} \sum_j g_j^\prime
- \left(\frac{2}{N}\right)^{2n} \frac{\tilde{g}_{\Delta\tilde{\eta}_{n+1},\tilde{\chi}_n} e^{in\Delta\tilde{\eta}_{n+1},\tilde{\chi}_n}}{N} \tilde{h}_{\tilde{\eta}_{n+1},\tilde{\chi}_n},
\]
(15)
where
\[
\tilde{h}_{\tilde{\eta}_{n+1},\tilde{\chi}_n} = \sum_{\tilde{\eta}_{n+1}} h_{\tilde{\eta}_{n+1},\tilde{\chi}_n} \tilde{h}_{\tilde{\eta}_{n+1},\tilde{\chi}_n}.
\]
As a byproduct, the matrix elements of the staggered magnetization,
\[
m_s = \frac{1}{N} \sum_{j=1}^{N} (-1)^j S_j^z,
\]
which measures the antiferromagnetic order in the XXZ chain with $J'/J > 0$, can be obtained by setting $g_j^\prime = \frac{1}{N} e^{i\pi j}$ in Eq. (15):
\[
m_{s;\tilde{\chi}_n,\tilde{\chi}_n} = (-1)^{n-1} \left(\frac{2}{N}\right)^{2n} \frac{\delta(\Delta\tilde{\chi}_n,\tilde{\chi}_n,\pi)}{N} \tilde{h}_{\tilde{\eta}_{n+1},\tilde{\chi}_n},
\]
(18)
where
\[
\delta(x, y) = \begin{cases} 1, & x - y = 2\pi m, \quad m \in Z, \\ 0, & \text{otherwise.} \end{cases}
\]
Finally, the matrix elements $G_{\tilde{\chi}_n,\tilde{\chi}_n}^\prime$ can also be calculated from Eq. (10) and have the form
\[
G_{\tilde{\eta}_{n+1},\tilde{\chi}_n} = \left(n - \frac{3N}{4}\right) \delta_{\tilde{\eta}_{n+1},\tilde{\chi}_n}
+ \left(\frac{2}{N}\right)^{4n} \frac{\delta(\Delta\tilde{\chi}_n,\tilde{\chi}_n,0)}{N} \sum_{\tilde{\eta}_{n+1}} e^{i\Delta\tilde{\eta}_{n+1}} \tilde{h}_{\tilde{\eta}_{n+1},\tilde{\chi}_n} \tilde{h}_{\tilde{\eta}_{n+1},\tilde{\chi}_n}.
\]
(20)
The advantage of using the eigenbasis of the XX chain now becomes clear: the system-bath coupling constants simply enter the matrix elements $F_{\tilde{\eta}_{n+1},\tilde{\chi}_n}(\{g_j\})$ and $G_{\tilde{\chi}_n,\tilde{\chi}_n}(\{g_j^\prime\})$ through the Fourier transforms $\tilde{g}_{\Delta\tilde{\eta}_{n+1},\tilde{\chi}_n}$ and $\tilde{g}_{\Delta\tilde{\chi}_n,\tilde{\chi}_n}$. The main task is to calculate the function $h_{\tilde{\eta}_{n+1},\tilde{\chi}_n}$ given by Eq. (12). Moreover, the matrix elements $G_{\tilde{\chi}_n,\tilde{\chi}_n}$ given by Eq. (20) also provide an alternative way to diagonalize the XXZ chain in a basis where $H_{\text{XX}}$ is diagonal (in contrast, the Ising term $H_Z$ is diagonal in the real basis formed by the Ising configurations).

C. Initial states, time-evolved states, and equations of motion

We assume a separable initial state for the whole system,
\[
|\psi(0)\rangle = |\phi^{(S)}\rangle \otimes |\phi^{(B)}\rangle,
\]
(21)
where $|\phi^{(S)}\rangle$ is a general pure state of the central spin,
\[
|\phi^{(S)}\rangle = \sum_{s_z = -S}^{S} a_{s_z} |s_z\rangle,
\]
(22)
with $\sum_{s_z = -S}^{S} |a_{s_z}|^2 = 1$. Similarly, $|\phi^{(B)}\rangle$ is a pure state of the spin bath and can generally be written as a linear combination of the component states having fixed number of fermionic excitations:
\[
|\phi^{(B)}\rangle = \sum_{n=0}^{N} \sum_{\tilde{\eta}_{n}} b_{\tilde{\eta}_{n}} |\tilde{\eta}_{n}\rangle,
\]
(23)
where $\sum_{n=0}^{N} \sum_{\tilde{\eta}_{n}} |b_{\tilde{\eta}_{n}}|^2 = 1$. Since the time evolution occurs in each sector with fixed $\alpha$, the most general form of the time-evolved state is
\[
|\psi(t)\rangle = |\psi^{(I)}(t)\rangle + |\psi^{(II)}(t)\rangle + |\psi^{(III)}(t)\rangle.
\]
(24)
According to the classification of different structures of the magnetization sectors listed in Appendix A, the three parts of the time-evolved state read
\[
|\psi^{(I)}(t)\rangle = \sum_{\alpha = -S}^{S} \sum_{n=0}^{N+\alpha} A_{\alpha-n,\tilde{\eta}_{n}}^{(I,\alpha)} |\alpha - n\rangle |\tilde{\eta}_{n}\rangle,
\]
\[
|\psi^{(II)}(t)\rangle = \sum_{\alpha = S+1}^{N+\alpha} \sum_{n=0}^{N+\alpha} A_{\alpha-n,\tilde{\eta}_{n}}^{(II,\alpha)} |\alpha - n\rangle |\tilde{\eta}_{n}\rangle,
\]
\[
|\psi^{(III)}(t)\rangle = \sum_{\alpha = S}^{\alpha - S} \sum_{n=0}^{N+\alpha} A_{\alpha-n,\tilde{\eta}_{n}}^{(III,\alpha)} |\alpha - n\rangle |\tilde{\eta}_{n}\rangle,
\]
(25)
with initial conditions
\[
A_{\alpha-n,\tilde{\eta}_{n}}^{(I,\alpha)} = a_{\alpha-n} b_{\tilde{\eta}_{n}}, \quad i = I, II, III.
\]
(26)
the spin coherent state states for the spin bath, i.e., the Néel state $\alpha$ subspace with fixed of each amplitude vector $H(\alpha)$. Similar analysis can be made for categories II and III. To obtain the time-evolved $H$ is the binomial coefficient). The structure of the block Hamiltonian $H^{L,\alpha}$ is shown in Fig. 2. Similar analysis can be made for categories II and III. To obtain the time-evolved state $\hat{A}^{l,\alpha}$ then read

$$i\frac{d}{dt}\hat{A}^{l,\alpha} = H^{L,\alpha}\hat{A}^{l,\alpha},$$

where $H^{L,\alpha}$ is a $D^{l,\alpha} \times D^{l,\alpha}$ matrix with $D^{l,\alpha} = \sum_{n=0}^{\alpha+\alpha} n! C_n^{\alpha}$ ($C_n^{\alpha} = \binom{N}{n(N-n)}$ is the binomial coefficient). The structure of $H^{L,\alpha}$ is shown in Fig. 2. Similar analysis can be made for categories II and III. To obtain the time-evolved state $\hat{A}^{l,\alpha}$, we need only to simulate the time evolution of each amplitude vector $\vec{a}$ governed by $H^{i,\alpha}$ in each subspace with fixed $\alpha$.

III. RESULTS

In this work, we mainly consider three types of initial states for the spin bath, i.e., the Néel state $|AF\rangle = |↓↑\cdots ↓↑\rangle$, the ground state $|G_{XXZ}\rangle$ of the XXZ chain, and the spin coherent state $|\Omega\rangle$.

A. The Néel state $|AF\rangle$

The Néel state $|AF\rangle = |↓↑\cdots ↓↑\rangle$ is one of the two degenerate ground states of the XXZ chain in the Ising limit $J'/J \to \infty$. It has been used to detect the relaxation of antiferromagnetic order in the XXZ chain after a quantum quench [20], to probe the decoherence dynamics of a qubit coupled to both noninteracting [2] and interacting [23] spin baths, and so on.

We use the equations-of-motion approach described in the last section to calculate the decoherence dynamics of a qubit coupled to an XXZ bath with $N = 16$ sites. The initial state of the central spin (spin bath) is chosen as $|\phi(S)\rangle = \frac{1}{2}(|↑\rangle + |↓\rangle)$ ($|\phi(B)\rangle = |AF\rangle$). We use the following inhomogeneous hyperfine coupling [11]

$$g_j = \frac{g_j'}{\Lambda} = \frac{g_0}{N} e^{-\frac{3n+2}{2}}$$

which corresponds to a Gaussian wave function in a two-dimensional quantum dot [33]. Unless otherwise specified, we always use the inhomogeneous coupling given by Eq. (25), where the parameter $\Lambda$ defines the overall energy scale through the relation [23]

$$\omega_{\text{tune}} = \frac{2g}{N} \sqrt{\frac{e^2 - 1}{N \frac{e^2 - 1}{e^{2n+1}}}}$$

Note that the Fourier transform of $g_j$ has a simple form,

$$\tilde{g}_q = \frac{1 - e^{-i\pi N - 1}}{N \left(e^{-i\pi q} - e^{-\frac{1}{\pi}}\right)}$$

It is obvious that $|AF\rangle$ lives in the manifold with excitation number $n = N/2$, so that $\alpha$ takes two possible values, $\alpha = (N \pm 1)/2$, and the time-evolved state is thus of type II (we assume $2S < \frac{N}{2}$), as can be seen from Eq. (25). The dimension of the relevant Hilbert space is $2 \times 24310 = 48620$ and the simulation can be performed on a personal workstation.

Figure 3 (a) shows the time-evolution of the decoherence factor $|\psi(t)\rangle = (|S_+(t)\rangle/\langle S_+|)$ [34] for both a noninteracting bath ($J/\omega_{\text{tune}} = J'/\omega_{\text{tune}} = 0$) and an XXX chain ($J'/J = 1$). In the former case, the result is fully consistent with that obtained by the Chebyshev expansion technique for $N = 16$ [23]. The latter case can be viewed as a simultaneous quench of both the anisotropy parameter and hyperfine coupling: $J'/J = \infty \to 1$ and $g_j = g_j' = 0 \to \frac{g_j}{\Lambda} e^{-\frac{3n+2}{2}}$. The short-time dynamics of $|\psi(t)\rangle$ seems to be independent of the value of $J/\omega_{\text{tune}}$. However, $|\psi(t)\rangle$ starts to exhibit oscillations at long times and acquires a lower value when $J/\omega_{\text{tune}}$ becomes finite, demonstrating the role played by intrabath interactions on the central-spin decoherence.

Figure 3 (b) shows the corresponding dynamics of the staggered magnetization $\langle m_s(t)\rangle$. The relaxation of antiferromagnetic order after a sudden quench of the anisotropy parameter $J'/J$ in a pure XXZ chain has been thoroughly studied in Ref. [20] for large systems.
relaxes to a nearly zero value around $Jt$ (Fig. 3). Actually, the largest hyperfine interaction out system-bath coupling (comparing to the green dash-dotted curve). The inset shows the short-time dynamics up to $Jt = 15$. Parameters: $N = 16$, $\Lambda = 1$, and $\omega = \lambda = 0$.

using the infinite-size time-evolving block decimation algorithm. It was found that the relaxation time is minimal for a quench to the isotropic point $J'/J = 1$, which is the critical point separating the Luttinger liquid phase ($J'/J < 1$) and the gapped antiferromagnetic phase ($J'/J > 1$). The green dash-dotted curve in Fig. 3(b) shows the result for $N = 16$ when the central spin and the XXZ chain are decoupled. It can be seen that $\langle m_s(t) \rangle$ relaxes to a nearly zero value around $Jt \approx 7.5$ [inset of Fig. 3(b)], indicating the occurrence of the relaxation. The revivals appearing at long times are due to the finite-size effect.

The blue dashed curve in Fig. 3(b) shows $\langle m_s(t) \rangle$ for $J'/J = 1$. It can be seen that at short times $\langle m_s(t) \rangle$ behaves similarly to the result without system-bath coupling (comparing to the green dash-dotted curve). Actually, the largest hyperfine interaction is $2g/N \approx 0.37\omega_{\text{fluc}}$ for $N = 16$, indicating that the system lies in the weak system-bath coupling regime. The red solid curve in Fig. 3(b) shows the result for a strong hyperfine coupling ($J'/\omega_{\text{fluc}} = 1$). Interestingly, we found that $\langle m_s(t) \rangle$ drops more smoothly in the initial stage of the evolution. Moreover, there is a long-period collapse of $\langle m_s(t) \rangle$ in the time window $\omega_{\text{fluc}}t \in (47, 63)$. For a noninteracting bath with $J = J' = 0$, the staggered magnetization dynamics is solely driven by the hyperfine coupling and $\langle m_s(t) \rangle$ reaches a minimum at $\omega_{\text{fluc}}t \approx 24$ (black dotted curve). These observations show that strong coupling to the central spin can suppress both the short-time decay and long-time oscillations of the antiferromagnetic order within the bath, even for systems of intermediate size.

To see the effect of the value of $S$ on the staggered magnetization dynamics, we plot in Fig. 4 $\langle m_s(t) \rangle$ for $S = 1/2, 1, 3/2, 2$ at a strong hyperfine coupling ($J'/\omega_{\text{fluc}} = 1$). It can be seen that an increase in $S$ tends to accelerate the initial decay of $\langle m_s(t) \rangle$, due to the increase of the number of bath states involved in the composite dynamics.

**B. The ground state $|G_{\text{XXZ}}\rangle$**

It is known that for $J'/J > -1$ the ground state of $H_0$ is nondegenerate and possesses magnetization $l_z = 0$; while for $J'/J < -1$ the bath is ferromagnetic and has two degenerate ground states, i.e., the two fully polarized states $| \uparrow \cdots \uparrow \rangle$ and $| \downarrow \cdots \downarrow \rangle$ [57]. Below we focus on the case of $J'/J > 0$, so that the initial bath state $|\phi^{(B)}\rangle = \sum_{\vec{n}} b_{\vec{n}} |\vec{n}\rangle \overline{|\vec{n}\rangle}$ lives in the subspace with $n = N/2$. The coefficients $\{b_{\vec{n}}\}$ can be determined nu-
merically by solving an eigenvalue problem of \( H_B \) in the \( l_z = 0 \) sector.

Recently, the purity dynamics of a qubit coupled to a bosonic bath has been studied and the long-time recovery of purity under low-temperature and weak system-bath coupling conditions is observed \[35\]. It is therefore interesting to study the purity dynamics of a central spin coupled to an interacting spin bath at zero temperature. This protocol can be considered as a sudden quench in the hyperfine coupling strength: at \( t = 0^- \) the system lies in a separable eigenstate associated with \( g_j = g_j' = 0, \forall j \), and then one suddenly changes the coupling constants to the finite values given by Eq. \[28\].

In this subsection, we mainly focus on the case of \( S = 1 \), for which the reduced density matrix of the central spin read \[39\]

\[
\begin{align*}
\rho_{11} &= 1 + \frac{1}{2}(a_j^z - q_j^{xx} - q_j^{yy}), \\
\rho_{22} &= -1 + q_j^{xx} + q_j^{yy}, \\
\rho_{33} &= 1 - \rho_{11} - \rho_{22}, \\
\rho_{12} &= \frac{1}{2\sqrt{2}}(a_j^x + q_j^{yy} - i(a_j^y + q_j^{yz})), \\
\rho_{13} &= \frac{1}{2}(q_j^{xx} - q_j^{yy} - iq_j^{xy}), \\
\rho_{23} &= \frac{1}{2\sqrt{2}}(a_j^y - q_j^{zz} - i(a_j^x - q_j^{xz})),
\end{align*}
\]

where

\[
\begin{align*}
a_j^\alpha &= \langle S_j^\alpha \rangle, \quad \alpha = x, y, z, \\
q_j^{\alpha\alpha} &= \langle (S_j^\alpha)^2 \rangle, \quad \alpha = x, y, \\
q_j^{\alpha\beta} &= \langle S_j^\alpha S_j^\beta + S_j^\beta S_j^\alpha \rangle, \quad \alpha\beta = xy, yz, zx.
\end{align*}
\]

The purity of the central spin is defined as

\[
P(t) = \text{Tr}[\rho(t)]^2.
\]

Figure 5 shows \( P(t) \) after a sudden quench to the strong hyperfine coupling regime with \( J/\omega_{\text{fluc}} = 0.1 \) for an XXZ chain with \( N = 14 \) sites. The results for various values of \( J'/J \) are shown to see the influence of different quantum phases of the XXZ chain on the purity dynamics of the central spin. In the limit of \( J' = 0 \), the XXZ chain is reduced to the XX chain whose ground state is simply the fermionic Fock state \( |\bar{\eta}_j\rangle = |1, 2, 3, 4, 12, 13, 14\rangle \) for \( J > 0 \). In this case, the purity drops rapidly at short times and gradually approaches its minimal value \( \sim 1/3 \) at long times after experiencing some oscillation in the intermediate-time regime (solid black curve). The overall profile of \( P(t) \) is found to be lifted up as \( J'/J \) increases from 0 to 1 within the gapless phase. Remarkably, we observe that \( P(t) \) acquires the highest values after a quench from the ground state at the critical point \( J'/J = 1 \) (red curve), beyond which its magnitude drops as \( J'/J \) increases further in the gapped phase. Specially, in the large \( J'/J \) limit \( P(t) \) drops more abruptly at the beginning and approaches a steady value close to the minimal value 1/3, indicating that the central spin is approximated in a maximally mixed state at long times. These dynamical behaviors of the purity indicate that not only the system-bath coupling but also the internal phases of the XXZ bath have a significant influence on the central spin dynamics.

Figure 6 shows the purity dynamics after a sudden quench to the weak hyperfine coupling regime with \( J/\omega_{\text{fluc}} = 1 \), for which \( P(t) \) exhibits a less regular dependence on the parameter \( J'/J \). Nevertheless, the initial
The polarization dynamics of a qubit coupled to a spin bath prepared in the spin coherent state has been studied in several prior works \cite{11,12,17,22}. In this subsection, we choose the initial state as
\[ |\psi(0)\rangle = |S\rangle \otimes |\Omega\rangle, \tag{34} \]
where the spin coherent state of the spin bath is defined as \cite{40}
\[ |\Omega\rangle = e^{-iL_z \phi} e^{-iL_y \theta} |N/2, N/2\rangle = \sum_{n=0}^{N} Q_n |n/2, n/2\rangle, \tag{35} \]
with \( Q_n = \frac{z^n}{(1+z^n)^{1/2}} \sqrt{C_n} \) and \( z = \frac{1}{2} e^{-i\phi} \). Here, \( |N/2, n/2\rangle \) is the Dicke state belonging to \( l = N/2 \) and has magnetization \( I_z = n - N/2 \). The parameter \( \alpha \) therefore takes \( N+1 \) values, \( \alpha = S, S+1, \ldots, S+N \). As a result, all the three types of states in Eq. (26) are involved. The Dicke case can be expanded in terms of the fermionic states as \cite{12,22}
\[ \frac{N}{2}, n - N/2 = \frac{1}{\sqrt{C_n}} \sum_{j_n} \left( \sum_{j_n} S^*_{\gamma_n,j_n} \right) |\bar{\eta}_n\rangle, \tag{36} \]
where \( S_{\gamma_n,j_n} = \text{det}(O) \) is the determinant of an \( n \times n \) matrix \( O_{a,b} = e^{iK_{j_n} \sigma_n} / \sqrt{N} \). This gives the following nonvanishing initial values for the amplitudes
\[ A^1_{S,\bar{\eta}_0}(0) = Q_0, \]
\[ A^{II,III+n}_{S,\eta_0}(0) = Q_n \sum_{j_n} S^*_{\eta_n,j_n}, \quad n = 1, \ldots, N - 2S - 1, \]
\[ A^{III+n}_{S,\bar{\eta}_0}(0) = Q_n \sum_{j_n} S^*_{\bar{\eta}_n,j_n}, \quad n = N - 2S, \ldots, N. \tag{37} \]

Below we focus on the polarization dynamics \( \langle S_z(t) \rangle = \langle \psi(0)| e^{iHt} S_z e^{-iHt} |\psi(0)\rangle \) of the central spin.

Let us first look at the special case of homogeneous hyperfine couplings, i.e., \( g_j = g, g'_{j'} = g', \forall j \). In this case, the Hamiltonian \( H \) becomes
\[ H_{\text{hom}} = \omega S_z + \lambda S_y^2 + J \sum_{j=1}^{N} S_j \cdot S_{j+1} \]
\[ + (J' - J) \sum_{j=1}^{N} S_j^z S_{j+1}^z \]
\[ + 2g(S_z L_x + S_y L_y) + g'S_z L_z. \tag{38} \]

\[ \text{FIG. 7: Dynamics of the central-spin polarization } \langle S_z(t) \rangle / S \]

\[ \text{for } J = J' \text{ and homogeneous system-bath coupling } (g_j = g, g'_{j'} = g', \forall j). \]

\[ \text{Parameters: } N = 14, \theta = \frac{\pi}{2}, \phi = 0, \omega = g = g', \text{and } \lambda = 0. \]

For \( J = J' \) and \( S = 1/2 \), \( H_{\text{hom}} \) is reduced to the model studied in Ref. \[24\], which conserves the total angular momentum \( \vec{L}^2 \) of the bath. If one further sets \( J = 0 \), then \( H_{\text{hom}} \) is reduced to the qubit—big-spin model studied in Ref. \[13\]. The dynamics of such a model can be analytically solved by using either a recurrence method \[13\] or an interaction-picture method \[12\].

From \[ J \sum_{j=1}^{N} S_j \cdot S_{j+1}, L_α \rangle = 0 \ (α = x, y, z), \] we have
\[ J \sum_{j=1}^{N} S_j \cdot S_{j+1} |Ω\rangle \]
\[ = e^{-iL_z \phi} e^{-iL_y \theta} J \sum_{j=1}^{N} S_j \cdot S_{j+1} |N/2, N/2\rangle \]
\[ = \frac{N}{4} |Ω\rangle. \]

In other words, the spin coherent state \( |Ω\rangle \) is an eigenstate of \( J \sum_{j=1}^{N} S_j \cdot S_{j+1} \) with eigenvalue \( \frac{N}{4} \). Therefore, the dynamics generated by \( H_{\text{hom}} \) is independent of the value of \( J \) at the isotropic point \( J = J' \), where \( J \sum_{j=1}^{N} S_j \cdot S_{j+1} \) commutes with the Hamiltonian \( H_{\text{hom}} \).

The top panel of Fig. 7 shows the polarization dynamics \( \langle S_z(t) \rangle / S \) of an \( S = 1/2 \) central spin for \( J = J' \) and under the resonant condition \( ω = g = g' \). It can be seen that the polarization exhibits the so-called collapse-revival behavior and the revival peaks occur at \( gt \approx mN\pi \) (\( m \in \mathbb{Z} \)), recovering the analytical results presented in Ref. \[13\]. The middle and bottom panels of Fig. 7 show \( \langle S_z(t) \rangle / S \) for \( S = 1 \) and \( S = 3/2 \), respectively. The polarization still shows collapse and revivals during the evolution, but with rich fine structures. For example, the initial revival region seems show 2S discrete
sub-peaks before the first collapse occurs. These structures reappear after the regular revival region consisting of $2S + 1$ packets.

Our formalism allows us to calculate the polarization dynamics in the presence of the intrabath interaction. Figure 8 shows $(S_j(t))/S$ for various pairs of $(J/g, J'/g)$. It can be seen that the collapse-revival behaviors are generally destroyed by the intrabath coupling, although for $(J, J')/g = (1, 0.8)$ and $(1, 1.2)$ there is some evidence of collapse (middle column of Fig. 8) at short time since they are close to the isotropic point $J'/J = 1$. Note that $(J' - J) \sum_{j=1}^{N} S_j^z S_{j+1}^z$ does not commute with the remaining part of $H_{\text{hom}}$, the dynamics thus depends not only on $J' - J$ but also on $J$ for $J \neq J'$ (right column of Fig. 8). Actually, since the term $(J' - J) \sum_{j=1}^{N} S_j^z S_{j+1}^z$ breaks the conservation of $L^2$, the time-evolved state will run out of the $l = N/2$ subspace, making the collapse-revival phenomena fragile with respect to anisotropic intrabath coupling.

IV. CONCLUSIONS AND DISCUSSIONS

In this work, we obtain exact dynamics of a composite system made up of a spin-$S$ central spin and a coupled XXZ ring. The two parts interact with each other through inhomogeneous XXZ-type hyperfine coupling. We use the analytical representations of local spin-operator matrix elements in the XX chain to write out the equations of motion of the time-dependent amplitudes in each sector with fixed total magnetization. By solving these equations of motion under three types of initial bath states, i.e., the Néel states, the ground state of the XXZ chain, and the spin coherent state, we investigate the reduced dynamics of both the central spin and the spin bath.

Under the Néel bath initial condition, we first simulate the decoherence dynamics of a spin-1/2 central spin inhomogeneously coupled to a noninteracting bath with $N = 16$ sites and get consistent result with that obtained by the Chebyshev expansion technique [23]. This demonstrates the validity of the equations-of-motion method. By turning on the nearest-neighbor coupling within the bath, we find that the intrabath coupling has significant effect on the central spin decoherence. On the other hand, the central spin also alters the dynamical behavior of the antiferromagnetic order measured by the staggered magnetization. We find that in the strong hyperfine coupling regime the short-time decay of the staggered magnetization is slowed down, while the long-time oscillations get suppressed, which facilitates the relaxation of the antiferromagnetic order. For fixed hyperfine couplings, we also find that an increase in $S$ tends to accelerate the initial decay of the staggered magnetization.

We then turn to study the purity dynamics of an $S = 1$ central spin coupled an XXZ chain prepared in its ground state. It is found that in the strong hyperfine coupling regime the purity reaches the highest values at the critical point of the XXZ chain. Finally, we study the polarization dynamics of a spin-$S$ central spin homogeneously coupled to an XXZ chain in the spin coherent state. Under the resonant condition [13] and for $S > 1/2$, we observe collapse-revival behaviors having fine structures. Including the anisotropic intrabath coupling generally destroys the collapse-revival phenomena due to the breakdown of angular momentum conservation for the bath.

Our work implies that not only the intrabath coupling can have significant influence on the central spin dynamics but also the central spin can affect the internal dynamics of the interacting spin bath. The results and theoretical method presented in this work may simulate further investigations on interacting central spin models.

Acknowledgements: N.W. thanks X.-W. Guan, H. Katsura, and S.-W. Li for useful discussions. This work was supported by the Natural Science Foundation of China (NSFC) under Grant No. 11705007, and partially by the Beijing Institute of Technology Research Fund Program for Young Scholars.

[1] N. V. Prokof’ev and P. C. E. Stamp, Rep. Prog. Phys. 63, 669 (2000).
[2] M. Gaudin, J. Phys. France 37, 1087 (1976).
[3] A. V. Khaetskii, D. Loss, and L. Glazman, Phys. Rev. Lett. 88, 186802 (2002).
[4] M. Bortz, S. Eggert, C. Schneider, R. Stübler, and J. Stolze, Phys. Rev. B 82, 161308(R) (2010).
[5] E. Barnes, L. Cywiński, and S. Das Sarma, Phys. Rev. B 84, 155315 (2011).
[6] E. Barnes, L. Cywiński, and S. Das Sarma, Phys. Rev. Lett. 109, 140403 (2012).
[7] A. Ricottone, Y. N. Fang, and W. A. Coish, Phys. Rev.
B 102, 085413 (2020).
[8] D. Stanek, C. Raas, and G. S. Uhrig, Phys. Rev. B 88, 155305 (2013).
[9] A. Faribault and D. Schuricht, Phys. Rev. Lett. 110, 040405 (2013).
[10] A. Faribault and D. Schuricht, Phys. Rev. B 88, 085323 (2013).
[11] S. Dooley, F. McCrossan, D. Harland, M. J. Everitt, and T. P. Spiller, Phys. Rev. A 87, 052323 (2013).
[12] Z. Li, P. Yang, W.-L. You, and N. Wu, Phys. Rev. A 102, 032409 (2020).
[13] W.-B. He, S. Chesi, H.-Q. Lin, and X.-W. Guan, Phys. Rev. B 99, 174308 (2019).
[14] D. Garajeu and A. Kiss, J. Math. Phys. 42, 3497 (2001).
[15] P. W.Claeys, S. De Baerdemacker, M. Van Raemdonck, and D. Van Neck, Phys. Rev. B 91, 155102 (2015).
[16] N. Wu, Physica A 501, 308 (2018).
[17] R. I. Nepomechie and X.-W. Guan, J. Stat. Mech. (2018) 103104.
[18] T. Villazón, A. Chandran, and P. W. Claeys, Phys. Rev. Research 2, 032052(R) (2020).
[19] N. Wu, X.-W. Guan, and J. Links, Phys. Rev. B 101, 155145 (2020).
[20] D. Fioretto, J.-S. Caux, and V. Gritsev, New J. Phys. 16, 043024 (2014).
[21] L. Tessier and J. Wilkje, J. Phys. A 36, 12305 (2002).
[22] N. Wu, A. Nanduri, and H. Rabitz, Phys. Rev. A 89, 062105 (2014).
[23] N. Wu, N. Fröhling, X. Xing, J. Hackmann, A. Nanduri, F.B. Anders, and H. Rabitz, Phys. Rev. B 93, 035430 (2016).
[24] P. Lu, H.-L. Shi, L. Cao, X.-H. Wang, T. Yang, J. Cao, and W.-L. Yang, Phys. Rev. B 101, 184307 (2020).
[25] C. Y. Lai, J. T. Hung, C. Y. Mou, and P. Chen, Phys. Rev. B 77, 205419 (2008).
[26] P. Barmettler, M. Punk, V. Gritsev, E. Demler, and E. Altman, Phys. Rev. Lett. 102, 130603 (2009).
[27] M. Heyl, Phys. Rev. Lett. 113, 205701 (2014).
[28] P.G.J. van Dongen, J.A. Vergés and D. Vollhardt, Z. Phys. 84, 383 (1991).
[29] J. Richter and A. Voigt, J. Phys. A: Math. Gen. 27, 1139 (1994).
[30] N. Wu, Phys. Rev. B 97, 014301 (2018).
[31] S. Trotzky, P. Cheinet, S. Fölling, M. Feld, U. Schnorrberger, A. M. Rey, A. Polkovnikov, E. A. Demler, M. D. Lukin, and I. Bloch, Science 319, 295 (2008).
[32] N. Nagaosa, Quantum field theory in strongly correlated electronic systems (Springer-Verlag, 1999).
[33] I. A. Merkulov, A. L. Efros, and M. Rosen, Phys. Rev. B 65, 205309 (2002).
[34] T. Tokihiro, Y. Manabe, and E. Hanamura, Phys. Rev. B 47, 2019 (1993).
[35] W. A. Coish and D. Loss, Phys. Rev. B 70, 195340 (2004).
[36] F. M. Cucchietti, J. P. Paz, and W. H. Zurek, Phys. Rev. A 72, 052113 (2005).
[37] I. Affleck and E. H. Lieb, Lett. Math. Phys. 12, 57 (1986).
[38] S. Chatterjee and N. Makri, J. Phys. Chem. Lett. 11, 8592 (2020).
[39] L. E. Ballentine, Quantum Mechanics: A Modern Development (World Scientific, Singapore, 1998).
[40] F. T. Arecchi, E. Courtens, R. Gilmore, and H. Thomas, Phys. Rev. A 6, 2211 (1972).

Appendix A: Structure of the $M$-sectors for $S < \frac{N}{2}$

Starting with the state with the lowest magnetization $M = -S - \frac{N}{2}$ and denote $(s_z, l_z)$ as the configuration with fixed magnetizations for the central spin and the spin bath, then each $M$ corresponds to the following configurations:

$M = -S - \frac{N}{2}$: $(S, -\frac{N}{2})$;
$M = -S - \frac{N}{2} + 1$: $(S, -1, -\frac{N}{2} + 1)$, $(-S + 1, -\frac{N}{2})$;
$M = S - \frac{N}{2}$: $(S, 2S - \frac{N}{2})$, \ldots, $(S, -\frac{N}{2})$;
$M = S - \frac{N}{2} + 1$: $(S, 2S - \frac{N}{2} + 1)$, \ldots, $(S, -\frac{N}{2} + 1)$;
$M = -S + \frac{N}{2} - 1$: $(S, \frac{N}{2} - 1)$, \ldots, $(S, \frac{N}{2} - 2S - 1)$;
$M = -S + \frac{N}{2}$: $(S, \frac{N}{2})$, \ldots, $(S, -2S)$;

In summary, the $M$-sectors can be classified into three categories:

I) For $-S - \frac{N}{2} \leq M \leq -S - \frac{N}{2} + 1$, there are $S + \frac{N}{2} + 1 + M$ configurations of $(s_z, l_z)$ in each $M$-sector, among which $s_z$ can take values from $M + \frac{N}{2}$ to $-S$, with the corresponding $l_z$ running from $-\frac{N}{2}$ to $M + S$. The dimension of this $M$-sector is $d_M = \sum_{j=M}^{M+S+\frac{N}{2}} C_N^j$, where $C_N^j = \frac{N!}{j!(N-j)!}$ is the binomial coefficient.

II) For $S - \frac{N}{2} + 1 \leq M \leq S + \frac{N}{2} - 1$, there are $2S + 1$ configurations of $(s_z, l_z)$ in each $M$-sector, among which $s_z$ can take all the values from $S$ to $-S$, with the corresponding $l_z$ running from $M - S$ to $M + S$. The dimension of this $M$-sector is $d_M = \sum_{j=0}^{N-M+S+\frac{N}{2}} C_N^j$. 

III) For $-S + \frac{N}{2} \leq M \leq S + \frac{N}{2}$, there are $1 + S + \frac{N}{2} - M$ configurations of $(s_z, l_z)$ in each $M$-sector, among which $s_z$ can take values from $S$ to $M - \frac{N}{2}$, with the corresponding $l_z$ running from $M - S + \frac{N}{2}$ to $N$. The dimension of this $M$-sector is $d_M = \sum_{j=0}^{S-M+S} C_N^j$.