GAUSSIAN AND PRÜFER CONDITIONS IN BI-AMALGAMATED ALGEBRAS

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ABSTRACT. Let \( f : A \to B \) and \( g : A \to C \) be two ring homomorphisms and let \( J \) (resp., \( J' \)) be an ideal of \( B \) (resp., \( C \)) such that \( f^{-1}(J) = g^{-1}(J') \). In this paper, we investigate the transfer of the notions of Gaussian and Prüfer properties to the bi-amalgamation of \( A \) with \((B, C)\) along \((J, J')\) with respect to \((f, g)\) (denoted by \( A \bowtie \sqcup (J, J') \)), introduced and studied by Kabbaj, Louartiti and Tamkekante in 2013. Our results recover well known results on amalgamations in [15] and generate new original examples of rings satisfying these properties.

1. Introduction

All rings considered in this paper are assumed to be commutative, and have identity element and all modules are unitary.

In 1932, Prüfer introduced and studied in [32] integral domains in which every finitely generated ideal is invertible. In 1936, Krull [28] named these rings after H. Prüfer and stated equivalent conditions that make a domain Prüfer. Through the years, Prüfer domains acquired a great many equivalent characterizations, each of which was extended to rings with zero-divisors in different ways. In their recent paper devoted to Gaussian properties, Bazzoni and Glaz have proved that a Prüfer ring satisfies any of the other four Prüfer conditions if and only if its total ring of quotients satisfies that same condition [5, Theorems 3.3 & 3.6 & 3.7 & 3.12].

In 1970, Koehler [27] studied associative rings for which every cyclic module is quasi-projective. She noticed that any commutative ring satisfies this property. In [3], the authors examined the transfer of the Prüfer conditions and obtained further evidence for the validity of Bazzoni-Glaz conjectures sustaining that ’the weak global dimension of a Gaussian ring is 0, 1, or \( \infty \)’ [5]. Notice that both conjectures share the common context of rings. Abuihlail, Jarrar and Kabbaj studied in [1] the multiplicative ideal structure of commutative rings in which every finitely generated ideal is quasi-projective. They provide some preliminaries quasi-projective modules over commutative rings and they investigate the correlation with well-known Prüfer conditions; namely, they proved that this class of rings stands strictly between the two classes of arithmetical rings and Gaussian rings. Thereby, they generalized Osrofskys theorem on the weak global dimension of arithmetical rings and partially resolve Bazzoni-Glazs related conjecture on Gaussian rings. They

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also established an analogue of Bazzoni-Glaz results on the transfer of Prüfer conditions between a ring and its total ring of quotients. In [9], the authors studied the transfer of the notions of local Prüfer ring and total ring of quotients. They examined the arithmetical, Gaussian, fqp conditions to amalgamated duplication along an ideal. At this point, we make the following definition:

**Definition 1.1.** Let $R$ be a commutative ring.

1. $R$ is called an **arithmetical ring** if the lattice formed by its ideals is distributive (see [16]).
2. $R$ is called a **Gaussian ring** if for every $f, g \in R[X]$, one has the content ideal equation $c(fg) = c(f)c(g)$ (see [33]).
3. $R$ is called a **Prüfer ring** if every finitely generated regular ideal of $R$ is invertible (equivalently, every two-generated regular ideal is invertible), (See [7, 22]).

In the domain context, all these forms coincide with the definition of a Prüfer domain. Glaz [20] provides examples which show that all these notions are distinct in the context of arbitrary rings. The following diagram of implications summarizes the relations between them [4, 5, 19, 20, 29, 30, 33]:

- Arithmetical $\Rightarrow$ Gaussian $\Rightarrow$ Prüfer

and examples are given in [20] to show that, in general, the implications cannot be reversed.

In this paper, we investigate the transfer of Gaussian and Prüfer properties in bi-amalgamation of rings, introduced and studied by Kabbaj, Louartiti and Tamekkante in [25] and defined as follow: Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and let $J$ and $J'$ be two ideals of $B$ and $C$, respectively, such that $I_0 := f^{-1}(J) = g^{-1}(J')$. The **bi-amalgamation** (or **bi-amalgamated algebra**) of $A$ with $(B, C)$ along $(J, J')$ with respect to $(f, g)$ is the subring of $B \times C$ given by

$$A \triangleright \triangleleft^{f,g}(J, J') := \{(f(a) + j, g(a) + j') \mid a \in A, (j, j') \in J \times J'\}.$$

This construction was introduced in [25] as a natural generalization of duplications [11, 14] and amalgamations [12, 13]. In [25], the authors provide original examples of bi-amalgamations and, in particular, show that Boisen-Sheldon’s CPI-extensions [6] can be viewed as bi-amalgamations (notice that [12, Example 2.7] shows that CPI-extensions can be viewed as quotient rings of amalgamated algebras). They also show how every bi-amalgamation can arise as a natural pullback (or even as a conductor square) and then characterize pullbacks that can arise as bi-amalgamations. Then, the last two sections of [25] deal, respectively, with the transfer of some basic ring theoretic properties to bi-amalgamations and the study of their prime ideal structures. All their results recover known results on duplications and amalgamations. Recently in [26], the authors established necessary and sufficient conditions for a bi-amalgamation to inherit the arithmetical property, with applications on the weak global dimension and transfer of the semihereditary
Moreover, by [25, Lemma 5.1], $f_p : A_p \rightarrow B_{S_p}$ and $g_p : A_p \rightarrow C_{S_p}$ be the canonical ring homomorphisms induced by $f$ and $g$. One can easily check that $f_p^{-1}(J_{S_p}) = g_p^{-1}(J'_{S_p}) = (I_0)_p$.

Moreover, by [25] Lemma 5.1, $P := p_{\Rightarrow f_0, g_0} (J, J')$ is a prime (resp., maximal) ideal of $A \Rightarrow f_0, g_0 (J, J')$ and, by [25] Proposition 5.7, we have

$$(A \Rightarrow f_0, g_0 (J, J'))_p \cong A_p \Rightarrow f_p, g_p (J_{S_p}, J'_{S_p}).$$

For a ring $R$, we denote by $Jac(R)$, the Jacobson radical of $R$.

2. Results

Let $f : A \rightarrow B$ and $g : A \rightarrow C$ be two ring homomorphisms and let $J$ and $J'$ be two ideals of $B$ and $C$, respectively, such that $I_0 := f^{-1}(J) = g^{-1}(J')$. All along this section, $A \Rightarrow f_0, g_0 (J, J')$ will denote the bi-amalgamation of $A$ with $(B, C)$ along $(J, J')$ with respect to $(f, g)$.

Our first result investigates the transfer of Gaussian and Prüfer properties in bi-amalgamated algebras in case $J \times J'$ contains a regular element.

**Theorem 2.1.** Assume $J \times J'$ is a regular ideal of $(f(A) + J) \times (g(A) + J')$. Then $A \Rightarrow f_0, g_0 (J, J')$ is Gaussian (resp., Prüfer) if and only if $J = B$, $J' = C$ and $B$ and $C$ are Gaussian (resp., Prüfer).

**Proof.** Assume that $A \Rightarrow f_0, g_0 (J, J')$ is Gaussian (resp., Prüfer). We claim that $I_0 = f^{-1}(J) = g^{-1}(J') = A$. Deny, suppose that there exists a maximal ideal $m$ of $A$ such that $I_0 \subseteq m$. From [25] Lemma 5.1, $M := m_{\Rightarrow f_0, g_0} (J, J')$ is a maximal ideal of $A_{\Rightarrow f_0, g_0} (J, J')$ and we have

$$(A_{\Rightarrow f_0, g_0} (J, J'))_M \cong A_m \Rightarrow f_m, g_m (J_{S_m}, J'_{S_m}) := D.$$ 

Let $(j, j')$ be a regular element of $J \times J'$. It is easy to see that $j/1$ (resp., $j'/1$) is also a regular element of $B_{S_m}$ (resp., $C_{S_m}$). Using the fact $A_{\Rightarrow f_0, g_0} (J, J')$ is Gaussian (resp., Prüfer), then by [22] Theorem 13, the ideals $(j/1, 0)D$ and $(j/1, 1/1)D$ are comparable. Since $0 \neq j'/1$, then necessarily $(j/1, 0)D \subseteq (j/1, 1/1)D$. Thus, there exist $\alpha \in A_m, \beta \in J_{S_m}$ and $\gamma \in J'_{S_m}$ such that $(j/1, 0) = (j/1, 1/1)(f_m(\alpha) + \beta, g_m(\alpha) + \gamma)$. Hence, it follows that $f_m(\alpha) + \beta = 1$ and $g_m(\alpha) + \gamma = 0$. Thus, $\alpha \in (I_0)_m$ and so $f_m(\alpha) \in J_{S_m}$ and $1 = f_m(\alpha) + \beta \in J_{S_m}$. Therefore, $J_{S_m} = B_{S_m}$. Then $(I_0)_m = A_m$, which is a contradiction since $I_0 \subseteq m$. Hence, $I_0 = f^{-1}(J) = A$ and so $J = B$ and $J' = C$ and $A \Rightarrow f_0, g_0 (J, J') = B \times C$ which is Gaussian (resp., Prüfer). It is known that Gaussian (resp., Prüfer) notion is stable under finite products. It follows that $B$ and $C$ are Gaussian (resp., Prüfer). The converse is straightforward. \(\Box\)
Recall that the amalgamation of $A$ with $B$ along $J$ with respect to $f$ is given by

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}.$$ 

Clearly, every amalgamation can be viewed as a special bi-amalgamation, since $A \bowtie^f J = A \bowtie^{d_{A,J}}(f^{-1}(J), J)$.

The following result is an immediate consequence of Theorem 2.1 and recovers [15] Theorem 3.1.

**Corollary 2.2.** Under the above notation, assume that $f^{-1}(J) \times J$ is a regular ideal of $A \times f(A) + J$. Then $A \bowtie^f J$ is Gaussian (resp., Pr"ufer) if and only if $f^{-1}(J) = A$ and $J = B$ and both $A$ and $B$ are Gaussian (resp., Pr"ufer).

Let $I$ be a proper ideal of $A$. The (amalgamated) duplication of $A$ along $I$ is a special amalgamation given by

$$A \bowtie I := A \bowtie^{d_{I \times I}} I = \{(a, a + i) \mid a \in A, i \in I\}.$$ 

The next corollary is an immediate consequence of Corollary 2.2 on the transfer of Gaussian and Pr"ufer properties to duplications and capitalizes [15] Corollary 3.3.

**Corollary 2.3.** Let $A$ be a ring and $I$ be a regular ideal of $A$. Then $A \bowtie I$ is Gaussian (resp., Pr"ufer) if and only if $A$ is Gaussian (resp., Pr"ufer) and $I = A$.

The next result investigates when the bi-amalgamation is local Gaussian in case $J \times J'$ is not a regular ideal. We recall an important characterization of a local Gaussian ring $A$. Namely, for any two elements $a$ and $b$ in the ring $A$, we have $(a, b)^2 = (a^2)$ or $(b^2)$; moreover if $ab = 0$ and $(a, b)^2 = (a^2)$, then $b^2 = 0$ (see [5, Theorem 2.2]).

**Proposition 2.4.** Assume that $(A, m)$ be a local ring and $J$ (resp., $J'$) be a nonzero proper ideal of $B$ (resp., $C$) such that $J \times J' \subseteq \text{Jac}(B \times C)$. Then the following statements hold:

1. If $A \bowtie^{f,g} (J, J')$ is Gaussian, then so are $f(A) + J$ and $g(A) + J'$.
2. If $A$, $f(A) + J$ and $g(A) + J'$ are Gaussian, $J^2 = 0$, $J'^2 = 0$, $\forall a \in m$, $f(a)J = f(a)^2J$ and $g(a)J' = g(a)^2J'$, then $A \bowtie^{f,g} (J, J')$ is Gaussian.
3. Assume that $A$ is Gaussian, $J^2 = 0$, $J'^2 = 0$ and $I_0$ is a prime ideal of $A$. Then $A \bowtie^{f,g} (J, J')$ is Gaussian if and only if $f(A) + J$, $g(A) + J'$ are Gaussian, $\forall a \in m$, $f(a)J = f(a)^2J$ and $g(a)J' = g(a)^2J'$.

**Proof.** Notice that from [25] Proposition 5.4 (2)), $(A \bowtie^{f,g} (J, J'), m \bowtie^{f,g} (J, J'))$ is local since $(A, m)$ is local and $J \times J' \subseteq \text{Jac}(B \times C)$.

1. Since the Gaussian property is stable under factor rings (here, $f(A) + J = \frac{A \bowtie^{f,g}(J, J')}{fJ}$ and $g(A) + J' = \frac{A \bowtie^{f,g}(J, J')}{Jx0}$ by [25] Proposition 4.1 (2)), then result is straightforward.

2. Assume that $A$, $f(A) + J$ and $g(A) + J'$ are Gaussian, $J^2 = 0$, $J'^2 = 0$ and $\forall a \in m$, $f(a)J = f(a)^2J$ and $g(a)J' = g(a)^2J'$. Our aim is to show that $A \bowtie^{f,g} (J, J')$ is Gaussian. Let $(f(a) + i, g(a) + i')$ and $(f(b) + j, g(b) + j') \in A \bowtie^{f,g} (J, J')$. Two cases are possible:
Case 1: \( a \) or \( b \) \( \notin \) \( m \). Assume without loss of generality that \( a \notin m \). Then \((f(a)+i, g(a)+i') \notin m \Rightarrow (J, J')\). So \((f(a)+i, g(a)+i')\) is invertible in \( A \Rightarrow (J, J')\). Therefore, \(((f(a)+i, g(a)+i'), (f(b)+j, g(b)+j'))^2 = ((f(a)+i, g(a)+i')^2) = A \Rightarrow (J, J')\). Moreover, if \(((f(a)+i, g(a)+i'), (f(b)+j, g(b)+j'))^2 = ((f(a)+i, g(a)+i')^2) = A \Rightarrow (J, J')\) and \((f(a)+i, g(a)+i')(f(b)+j, g(b)+j') = (0, 0)\), then it follows that \((f(b)+j, g(b)+j') = (0, 0)\), making \((f(b)+j, g(b)+j')^2 = (0, 0)\), as desired.

Case 2: \( a \) and \( b \) \( \in \) \( m \). Using the fact that \( A \) is local Gaussian, then \((a, b)^2 = (a^2)\) or \((b^2)\). We may assume that \((a, b)^2 = (a^2)\). So we have, \(b^2 = a^2x\) and \(ab = a^2y\) for some \(x, y \in A\). Moreover \(ab = 0\) implies that \(b^2 = 0\). So \(b^2 = f(a)^2f(x)\), \(b^2 = g(a)^2g(x)\) and \(f(a)f(b) = f(a)^2f(y)\), \(g(a)g(b) = g(a)^2g(y)\). By assumption, \(2f(b)i, f(b)i \in f(b)^2J\) and \(2f(a)f(x), f(a)i, f(a)f(y) \in f(a)^2J\). Therefore, there exist \(j_1, j_2, j_3 \in J\) such that \(2f(b)i = f(a)^2j_1, f(b)i = f(a)^2f(x)i_2, 2f(a)f(y) = f(a)^2i_3\) and similarly, there exist \(j_1', j_2', j_3' \in J\) such that \(2g(b)f(x) = g(a)^2g(x)f_1', 2g(a)^2g(x) = g(a)^2f_2'\), \(g(a)f_2' = g(a)^2g(x)f_2'\) and \(2g(a)^2g(y) = g(a)^2f_3'\). In view of the fact that \(J^2 = 0\) and \(J^2 = 0\), one can easily check that \((f(b)+j, g(b)+j')^2 = (f(a)+i, g(a)+i')^2f(x)+f(x)i_1, g(a)+i')\) and \((f(b)+j, g(b)+j') = (0, 0)\). Hence, \((f(a)+i, f(b)+j) = 0\) and \((g(a)+i', g(b)+j') = 0\). Since \((f(a)+i, f(b)+j) = 0\) and \((g(a)+i', g(b)+j') = 0\), \((f(a)+i, (f(b)+j) = ((f(a)+i, f(b)+j) = 0\). Consequently, \((f(a)+i, (f(b)+j) = 0\). And so there exists \((f(b)+j, g(b)+j') \in A \Rightarrow (J, J')\) such that \((f(b)+j, g(b)+j') = 0\). Therefore, \((f(b)+j, g(b)+j')^2 = 0\). Finally, \(A \Rightarrow (J, J')\) is Gaussian, as desired.

(3) If \(A \), \(f(A)+J\), \(g(A)+J'\) are Gaussian, \(J^2 = 0\), \(\forall a \in m\), \(f(a)J = f(a)^2J\), \(J^2 = 0\) and \(g(a)J' = g(a)^2J'\), then by statement (2) above, \(A \Rightarrow (J, J')\) is Gaussian. Conversely, assume that \(A \Rightarrow (J, J')\) is Gaussian. Then by statement (1) above, \(f(A)+J\) and \(g(A)+J'\) are Gaussian. Next, we show that \(\forall a \in m\), \(f(a)J = f(a)^2J\). It is clear that \((f(a)^2J \subseteq f(a)J\). On the other hand, let \(a \in m\) and \(0 \neq x \in J\). If \(f(a) = 0\), then \(f(a)J = f(a)^2J\), as desired. We may assume that \(f(a) \neq 0\). Then obviously, \((0, 0) \neq (f(a), g(a))\) and \((0, 0) \neq (x, 0)\) are elements of \(A \Rightarrow (J, J')\). Using the fact \(A \Rightarrow (J, J')\) is (local) Gaussian, then \(((f(a), g(a)), (x, 0))^2 = ((f(a), g(a))^2\) or \(((x, 0))^2\). Since \(J^2 = 0\), say \(((f(a), g(a)), (x, 0))^2 = ((f(a), g(a))^2\). If \((f(a), g(a))^2 = (0, 0)\), then it follows that \(xf(a) = 0\) and so \(f(a)J \subseteq f(a)^2J\), as desired. We may assume that \((f(a), g(a))^2 \neq (0, 0)\). And so there exists \((f(b)+j, g(b)+j') \in A \Rightarrow (J, J')\) such that \((xf(a), 0) = (f(a)^2, g(a)^2)(f(b)+j, g(b)+j')\). Therefore,

\[
\begin{align*}
    xf(a) &= (f(a)^2)(f(b)+j) & \text{(i)} \\
    0 &= (g(a)^2b + g(a)^2j') & \text{(ii)}
\end{align*}
\]

From equation (ii), it follows that \(a^2b \in I_0\) which is prime ideal of \(A\). So \(a^2 \in I_0\) or \(b \in I_0\). Two cases are possible:

Case 1: \(a^2 \in I_0\). Then \(a \in I_0\) and \(f(a) \in J\). Therefore, \(f(a)J = f(a)^2J = 0\) (as \(J^2 = 0\)).
Case 2: $b \in I_0$. Then $f(b) \in J$ and $f(b) + j \in J$. Consequently, $xf(a) = (f(a^2)(f(b) + j) \in f(a)^2J$. Hence, $f(a)J \subseteq f(a)^2J$, as desired. Next, it remains to show that $orall a \in m$, $g(a)J' = g(a)^2J'$. Clearly, $g(a)^2J' \subseteq g(a)J'$. On the other hand, let $a \in m$ and $0 \neq x' \in J'$. With similar argument as previously, it follows that $g(a)J' \subseteq g(a)^2J'$, as desired.

□

Proposition 2.6 enriches the literature with new original examples of non-arithmetical Gaussian rings. Recall that for a ring $A$ and an $A$–module $E$, the trivial ring extension of $A$ by $E$ (also called idealization of $E$ over $A$) is the ring $R := A \bowtie E$ whose underlying group is $A \times E$ with multiplication given by $(a, e)(a', e') = (aa', ae' + ea')$.

Example 2.5. Let $(A, m) := (A_1 \bowtie E_1, m_1 \bowtie E_1)$ be the trivial ring extension of $A_1$ by $E_1$ which is a non-arithmetical Gaussian ring with $m_1^2 = 0$, (for instance $(A_1, m_1) := (\mathbb{Z}/4\mathbb{Z}, 2\mathbb{Z}/4\mathbb{Z})$, $E_1$ be a nonzero $\frac{A_1}{m_1}$–vector space. By [26] Theorem 2.1 (2) and (3)], $A$ is a non-arithmetical Gaussian ring, as $A_1$ is not a field). Let $B := A \bowtie E$ be the trivial ring extension of $A$ by a nonzero $A/m$–vector space $E$. Consider

$$f : A \rightarrow B$$

$$(a_1, e_1) \mapsto f((a_1, e_1)) = ((a_1, e_1), 0)$$

be an injective ring homomorphism and $J := m \bowtie E = (m_1 \bowtie E_1) \bowtie E$ be the maximal ideal of $B$. Let $C := A_1$ and let

$$g : A \rightarrow C$$

$$(a_1, e_1) \mapsto g((a_1, e_1)) = a_1$$

be a surjective ring homomorphism and $J' := m_1$ be the maximal ideal of $C$. Clearly, $f^{-1}(J) = g^{-1}(J') = m_1 = E_1$. Then:

(1) $A \bowtie f, g (J, J')$ is Gaussian.

(2) $A \bowtie f, g (J, J')$ is not arithmetical.

Proof. (1) One can verify that $J^2 = 0$, $J'^2 = 0$, $f(a)J = f(a)^2J = 0$, $g(a)J' = g(a)^2J' = 0$ for all $a \in m$. Hence by using statement (2) of Proposition 2.4 it follows that $A \bowtie f, g (J, J')$ is Gaussian.

(2) By [26] Theorem 2.1 (2)], $A \bowtie f, g (J, J')$ is not arithmetical since $f(A) + J = A \bowtie 0 + m \bowtie E = A \bowtie E$ which is not arithmetical (by [3] Theorem 3.1 (3)], as $A$ is not a field).

Total rings of quotients are important source of Prüfer rings. Next, we study the transfer of this notion to bi-amalgamated algebras, in case $J \times J'$ is not a regular ideal of $(f(A) + J) \times (g(A) + J')$. For any ring $R$ and $J$ an ideal of $R$, we denote by $Z(R)$ (resp., $Ann(J)$), the set of zero-divisor elements of $R$ (resp., the annihilator of $J$).

Proposition 2.6. Let $(A, m)$ be a local total ring of quotients, $f : A \rightarrow B, g : A \rightarrow C$ be two ring homomorphisms, and let $J$ (resp., $J'$) be a nonzero proper ideal of
ring extension of $A$ using [3, Theorem 3.1 (1) and (2)]. Let $(J, J')$ be a nonzero proper ideal of $A$ and let $J$ be injective in $B$ (resp., $C$) such that $f^{-1}(J) = g^{-1}(J')$, $J \times J' \subseteq \text{Jac}(B \times C)$. Assume that $f$ is injective, $J^2 = 0$ and $J'^2 = 0$. Then $A \bowtie_{f,g} (J, J')$ is a local total ring of quotients. In particular, $A \bowtie_{f,g} (J, J')$ is Prüfer.

**Proof.** Assume that $f$ is injective, $J^2 = 0$ and $J'^2 = 0$. By [25, Proposition 5.4 (2)], $(A \bowtie_{f,g} (J, J'), m \bowtie_{f,g} (J, J'))$ is local since $(A, m)$ is local and $J \times J' \subseteq \text{Jac}(B \times C)$. Our aim is to show that $A \bowtie_{f,g} (J, J')$ is a total ring of quotients, we have to prove that each element $(f(a) + i, g(a) + i')$ of $A \bowtie_{f,g} (J, J')$, is invertible or zero-divisor element.

Let $(f(a) + i, g(a) + i')$ be an element of $A \bowtie_{f,g} (J, J')$. If $a \neq m$, then $a$ is invertible. And so $(f(a) + i, g(a) + i') \notin m \bowtie_{f,g} (J, J')$. Consequently, $(f(a) + i, g(a) + i')$ is invertible in $A \bowtie_{f,g} (J, J')$, as desired.

Now, we may assume that $a = m$. If $a = 0$, then $(f(a) + i, g(a) + i') = (i, i') \in Z(A \bowtie_{f,g} (J, J'))$, since $J^2 = J'^2 = 0$. We may assume $a \neq 0$. Since $A$ is local total ring of quotients, there exists $0 \neq b \in A$ such that $ab = 0$. So $f(a)(b) = 0$ and $g(a)(g(b)) = 0$. Two cases are then possible:

Case 1: $f(b) \in \text{Ann}(J)$ and $g(b) \in \text{Ann}(J')$. Using the fact that $f$ is injective, then there exists $(0, 0) \neq (f(b), g(b)) \in A \bowtie_{f,g} (J, J') / (f(a) + i, g(a) + i')(f(b), g(b)) = (0, 0)$. Consequently, $(f(a) + i, g(a) + i') \in Z(A \bowtie_{f,g} (J, J'))$.

Case 2: Assume that $f(b) \notin \text{Ann}(J)$ or $g(b) \notin \text{Ann}(J')$. Then there exists $0 \neq k \in J$ or $0 \neq k' \in J'$ such that $f(b)k = 0$ or $g(b)k' = 0$. So, $(f(a) + i, g(a) + i')(f(b)k, 0) = (0, 0)$ or $(f(a) + i, g(a) + i')(0, g(b)k') = (0, 0)$. Hence, $(f(a) + i, g(a) + i') \in Z(A \bowtie_{f,g} (J, J'))$. Thus, $A \bowtie_{f,g} (J, J')$ is a local total ring of quotients. In particular, $A \bowtie_{f,g} (J, J')$ is Prüfer.

Proposition 2.6 enriches the current literature with new original examples of Prüfer rings which are not Gaussian rings.

**Example 2.7.** Let $(A, m)$ be a non Gaussian local total ring of quotient (for instance $(A, m) := (A_1 \bowtie_{m_1} A_1, m_1 \bowtie_{m_1} A_1)$ with $(A_1, m_1)$ be a local ring that is not Gaussian, by using [3, Theorem 3.1 (1) and (2)]). Let $(B, N) := (A \bowtie E, m \bowtie E)$ be the trivial ring extension of $A$ by the nonzero $\frac{A}{m}$–vector space $E$ and $C := B \bowtie E$ be the trivial ring extension of $B$ by the nonzero $\frac{B}{N}$–vector space $E'$. Consider

\[
\begin{align*}
f & : A \\ (a_1, e_1) & \mapsto f((a_1, e_1)) = ((a_1, e_1), 0)
\end{align*}
\]

be an injective ring homomorphism and $J := 0 \bowtie E$ be a nonzero proper ideal of $B$ and let

\[
\begin{align*}
g & : A \\ (a_1, e_1) & \mapsto g((a_1, e_1)) = ((a_1, e_1), 0), 0)
\end{align*}
\]

be an injective ring homomorphism and let $J' := J \bowtie E'$ be a proper ideal of $C$. Obviously, $f^{-1}(J) = g^{-1}(J') = 0$. Then :

(1) $A \bowtie_{f,g} (J, J')$ is Prüfer.
(2) $A \bowtie_{f,g} (J, J')$ is not Gaussian.
Proof. (1) We claim that $A \triangleright \triangleleft_{f,g} (J, J')$ is a local total ring of quotients. Indeed, by [25, Proposition 5.3], $A \triangleright \triangleleft_{f,g} (J, J')$ is local since $A$ is local and $J \times J' \subseteq \text{Jac}(B \times C)$. One can easily check that $J^2 = 0$, $J'^2 = 0$. Hence, by using Proposition 2.6, it follows that $A \triangleright \triangleleft_{f,g} (J, J')$ is a total ring of quotients. Hence, $A \triangleright \triangleleft_{f,g} (J, J')$ is Prüfer.

(2) By (1) of Proposition 2.4, $A \triangleright \triangleleft_{f,g} (J, J')$ is not Gaussian since $f(A) + J = A \triangleleft 0 + 0 \triangleleft E = A \triangleleft E$ is not Gaussian (By [3, Theorem 3.1 (2)], since $A$ is not Gaussian, as $A_1$ is not Gaussian).

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