Discrete self-adjoint Dirac systems: asymptotic relations, Weyl functions and Toeplitz matrices

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Abstract

We consider discrete Dirac systems as an alternative (to the famous Szegő recurrences and matrix orthogonal polynomials) approach to the study of the corresponding block Toeplitz matrices. We prove an analog of the Christoffel–Darboux formula and derive the asymptotic relations for the analog of reproducing kernel (using Weyl–Titchmarsh functions of discrete Dirac systems). We study also the case of rational Weyl–Titchmarsh functions (and GBDT version of the Bäcklund-Darboux transformation of the trivial discrete Dirac system). We show that block diagonal plus semi-separable Toeplitz matrices appear in this case.

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1 Introduction

Self-adjoint discrete Dirac systems have been introduced in [10] and studied further in [11,29,33,34] following the case of skew-self-adjoint discrete Dirac systems in [17]. In particular, the paper [33] is dedicated to the interrelations between self-adjoint discrete Dirac systems and block Toeplitz matrices, which (in the scalar case) are in many respects similar to the interrelations
between the famous Szegő recurrences and Toeplitz matrices. The corresponding Verblunsky-type theorems are proved in [33]. Our present paper may be considered as an important development of the work started in [33]. (See also [34, Chapter 5] on discrete Dirac systems.) Some related recent research and references on continuous systems and convolution and other structured operators one can find in [22, 27, 34, 38].

Self-adjoint discrete Dirac system on the semi-axis $0 \leq k < \infty$ is the system of the form

$$y_{k+1}(\lambda) = \left( I_{2p} - \frac{i}{\lambda} jC_k \right) y_k(\lambda), \quad j := \begin{bmatrix} I_p & 0 \\ 0 & -I_p \end{bmatrix},$$

(1.1)

where $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{N}$ is the set of positive integers, $\lambda$ is the so-called spectral parameter, $y_{k+1}(\lambda)$ and $y_k(\lambda)$ are $2p \times 1$ vector functions, $I_p$ is the $p \times p$ identity matrix, $i$ stands for the imaginary unit ($i^2 = -1$), and the $2p \times 2p$ matrices $C_k$ have the following properties:

$$C_k > 0, \quad C_k jC_k = j \quad (0 \leq k < \infty).$$

(1.2)

System (1.1), (1.2) may be also considered as a special case of the discrete canonical system.

Here, we introduce analogs of Christoffel-Darboux formula and of reproducing kernels for system (1.1) such that (1.2) holds. The asymptotics of reproducing kernels is of essential interest in theory and applications to random processes (see, e.g., [4, 21] and references therein). In our paper, we study the asymptotics of the analogs of reproducing kernels using Weyl–Titchmarsh (Weyl) functions of system (1.1). Some of the possible applications are connected with the physical Gaussian models [32, 41].

We note that interesting and related papers on analogs of Weyl functions in the theory of orthogonal polynomials were written, for instance, by L. Golinskii and P. Nevai [14] and by B. Simon [39]. The case of matrix orthogonal polynomials is of interest as well (see, e.g., [1, 7, 8, 19]) but the corresponding analogs of Szegő recurrences are rather complicated whereas discrete Dirac systems, which we consider here, have the same form as in the scalar case.
Section 2 is dedicated to some basic preliminary results in order to make the paper self-contained. In Section 3, we consider Christoffel-Darboux formula and asymptotics of the analogs of reproducing kernels. Finally, in Section 4 we study the case of rational Weyl functions and our GBDT version of the Bäcklund-Darboux transformation of the trivial discrete Dirac system. (See [6, 12, 15, 20, 23, 34, 43] on Bäcklund-Darboux transformations and related commutation methods.) We show that block diagonal plus block semiseparable Toeplitz matrices appear in this case.

As usual, \( \mathbb{N} \) stands for the set of positive integers, \( \mathbb{R} \) stands for the real axis, \( \mathbb{D} \) stands for the unit disk \( \{ z : |z| < 1 \} \), \( \mathbb{C} \) stands for the complex plane and \( L^1(\mathbb{R}) \) denotes the class of absolutely integrable functions on \( \mathbb{R} \). The open upper (lower) half-plane is denoted by \( \mathbb{C}_+ (\mathbb{C}_-) \), and \( \overline{\mathbb{C}_+ (\mathbb{C}_-)} \) stands for the closed upper (lower) half-plane. The notation \( \overline{\lambda} \) means complex conjugate of \( \lambda \) and \( A^* \) means complex conjugate transpose of the matrix \( A \). The inequality \( S > 0 \) for some matrix \( S \) means that \( S \) is positive definite. The notation \( \Im(\alpha) \) stands for the imaginary part of matrix \( \alpha \) (i.e., \( \Im(\alpha) = \frac{1}{2i}(\alpha - \alpha^*) \)). Below in the text, we write sometimes Dirac system or discrete Dirac system meaning self-adjoint discrete Dirac system (1.1), where (1.2) holds.

## 2 Preliminaries

The fundamental solution of the Dirac system (1.1) (where (1.2) holds) is denoted by \( \{ W_k(\lambda) \} \), that is, \( W_{k+1}(\lambda) = \left( I_{2p} - \frac{1}{i} j C_k \right) W_k(\lambda) \). This solution is normalised by

\[
W_0(\lambda) \equiv I_{2p}.
\]

Let us recall the definition of the Weyl function of system (1.1) (see, e.g., [33]).

**Definition 2.1** A \( p \times p \) matrix function \( \varphi(\lambda) \) holomorphic in the lower complex half-plane \( \mathbb{C}_- \) is called a Weyl function for Dirac system (1.1), (1.2) if the inequality

\[
\sum_{k=0}^{\infty} [i\varphi(\lambda)^* I_p] q(\lambda)^k KW_k(\lambda)^* C_k W_k(\lambda) K^* \left[ \begin{array}{c} -i\varphi(\lambda) \\ I_p \end{array} \right] < \infty,
\]

holds.
holds for
\[ q(\lambda) := |\lambda^2||\lambda^2| + 1)^{-1}, \quad K := \frac{1}{\sqrt{2}} \begin{bmatrix} I_p & -I_p \\ I_p & I_p \end{bmatrix}. \]  

Weyl functions defined by the inequality (2.2) belong to Herglotz class, that is, \( \Im(\varphi(\lambda)) \leq 0 \) for \( \lambda \in \mathbb{C}_- \) (see [10, Corollary 5.5]). Formula (6.13) and Theorems 5.2 and 6.5 in [10] imply that the Weyl function \( \varphi \) of Dirac system (1.1), (1.2) is unique and admits Taylor representation

\[ i\varphi\left(\frac{iz + 1}{z - 1}\right) = \alpha_0 + \sum_{k=1}^{\infty} s_{-k}z^k, \]  

where (putting \( s_0 = \alpha_0 + \alpha_0^* \) and \( s_k = s_{-k}^* \)) we have

\[ S(N) = \{s_{k-i}\}_{i,k=1}^{N} > 0 \quad (1 \leq N < \infty). \]  

Recall that Toeplitz matrices satisfy matrix identities (see a detailed discussion and references in [33]):

\[ AS(N) - S(N)A^* = i\Pi J \Pi^*; \quad \Pi = \begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix}, \]  

where \( \Pi = \Pi(N), \quad \Phi_1 = \Phi_1(N), \quad \Phi_2 = \Phi_2(N), \)

\[ A(N) = \{a_{k-i}\}_{i,k=1}^{N}, \quad a_k = \begin{cases} \frac{1}{2}I_p & \text{for } k = 0 \\ \frac{1}{i}I_p & \text{for } k < 0 \end{cases}, \quad J = \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix}; \]  

\[ \Phi_1(N) = \begin{bmatrix} I_p \\ I_p \\ \cdots \\ I_p \end{bmatrix}, \quad \Phi_2(N) = \begin{bmatrix} s_0/2 \\ s_0/2 + s_{-1} \\ \cdots \\ s_0/2 + s_{-1} + \cdots + s_{1-N} \end{bmatrix} + i\Phi_1(N)\nu, \]  

and \( \nu = \nu^* \). Moreover, in view of Verblunsky-type results [33, Theorem 2.6] (see also the proof of [33, Theorem 2.4]) there is a one to one correspondence
between the sets \{\nu\} \cup \{s_{-k}\} (0 \leq k < \infty) such that (2.5) holds and Dirac systems (1.1), (1.2). This correspondence is given (in one direction) by (2.4) and by the equalities

\[ \nu = \Im(\alpha_0), \quad s_0 = \alpha_0 + \alpha_0^*. \]  

(2.9)

The transfer matrix function in Lev Sakhnovich form [35, 37] is given by the formula

\[ w_A(N, \lambda) = I_{2p} - i \Pi(N)^* S(N)^{-1} (A(N) - \lambda I_{np})^{-1} \Pi(N), \]  

(2.10)

and the recovery of Dirac system from the set \{\nu\} \cup \{s_{-k}\} (the Verblunsky-type correspondence in another direction) is equivalent to the factorisation of the transfer matrix functions \(w_A(N, \lambda)\) [33]. More precisely, by virtue of [33, (2.11)] we have

\[ W_N(\lambda) = \lambda^{-N}(\lambda + i)^N K^* w_A(N, -\lambda/2) K \quad (0 < N < \infty). \]  

(2.11)

**Remark 2.2** If we substitute \(\mathbb{C}_+\) for \(\mathbb{C}_-\), Definition 2.1 defines Weyl functions in \(\mathbb{C}_+\). According to [10, Theorem 6.8], the Weyl functions \(\varphi(z)\) in \(\mathbb{C}_+\) and \(\mathbb{C}_-\) are unique and are connected by the relation

\[ \varphi(\lambda) = \varphi(\bar{\lambda})^*. \]  

(2.12)

Relations (2.4), (2.5) provide Verblunsky-type one to one correspondence between Dirac systems and Toeplitz matrices (with additional matrix \(\nu = \nu^*\) given by (2.9)) in one direction and relations (2.10), (2.11) provide Verblunsky-type mapping in the opposite direction (as well as constitute the main part of the solution of the inverse problem to recover Dirac system from the Weyl function), see [33].

3 Asymptotics of the analog of reproducing kernel

1. Christoffel functions, reproducing kernels and Christoffel–Darboux formula are important components of the theory of orthogonal polynomials
(see, e.g., [5, 21, 25, 26] and references therein). In this section, we consider an analog of the Christoffel–Darboux formula for the discrete Dirac system and study the asymptotics of the analog of reproducing kernel. First, we prove the following Christoffel–Darboux-type formula.

**Theorem 3.1** Let $W_k(\lambda)$ be the fundamental solution of the discrete Dirac system (1.1), (1.2) normalized by (2.1). Then, we have

$$
\sum_{k=0}^{N} c(\lambda, \mu)^k W_k(\mu)^* C_k W_k(\lambda) = i \frac{1 + \lambda \mu}{\mu - \lambda} \left( c(\lambda, \mu)^{N+1} W_{N+1}(\mu)^* j W_{N+1}(\lambda) - j \right),
$$

where

$$
c(\lambda, \mu) = \frac{\lambda \mu}{1 + \lambda \mu}.
$$

We note that $c(\lambda, \lambda) = q(\lambda)$ for $q$ introduced in (2.3).

**Proof of Theorem 3.1.** Taking into account (1.1) and (1.2), we obtain

$$
W_{k+1}(\mu)^* j W_{k+1}(\lambda) = W_{k+1}(\mu)^* \left( I_{2p} + \frac{i}{\mu} C_k j \right) j \left( I_{2p} - \frac{i}{\lambda} j C_k \right) W_{k+1}(\lambda)
$$

$$
= W_k(\mu)^* \left( \frac{1}{c(\lambda, \mu)} j + \frac{i(\lambda - \mu)}{\lambda \mu} C_k \right) W_k(\lambda).
$$

Formula (3.3) yields

$$
W_k(\mu)^* C_k W_k(\lambda) = \frac{i(1 + \lambda \mu)}{\mu - \lambda} \left( c(\lambda, \mu) W_{k+1}(\mu)^* j W_{k+1}(\lambda) - W_k(\mu)^* j W_k(\lambda) \right).
$$

Substitute (3.4) into the left-hand side of (3.1) in order to derive (3.1). $\blacksquare$

2. Let us fix the set $\{\nu\} \cup \{s_{-k}\}$ $(0 \leq k < \infty)$ such that (2.5) holds. For asymptotical results in this section, we use [32, Example 1] and [10, Theorem 6.5]. The notations in [32] differ from the notations here and we give some explanations. We recall that the Weyl function $\varphi(\lambda)$ is constructed in [10, Theorem 6.5] as the unique function belonging to the intersection of
the Weyl circles generated by some linear fractional transformations. The corresponding linear fractional transformations are given by [10, (5.7)] and closely related transformations are given by the formulas [32, (11)]. The matrices of coefficients of these transformations are denoted by \( W \) in [10] and by \( A \) in [32], where 

\[
W(\lambda) = W_N(\lambda) = kW_N(\lambda)^* \quad (N \in \mathbb{N}),
\]

\( A \) (in the notations of this paper) takes the form

\[
A(\zeta) = A_N(\zeta) = j(I_{2p} + i\zeta J\Pi(N)^* (I_p + \zeta A(N)^*)^{-1} S(N)^{-1}\Pi(N))j,
\]

and \( K \) and \( J \) are introduced in (2.3) and (2.7), respectively. Simple calculations show that these matrices (matrix functions) satisfy the equality

\[
W(\lambda) = \left( \frac{\lambda - i}{\lambda} \right)^N jJ\mathcal{A} \left( \frac{2}{\lambda} \right) JjK.
\]

Moreover, it is easy to see that

\[
K^*JK = j, \quad jJK = K^*.
\]

We note that linear-fractional transformations considered in [32] have the form

\[
\omega_N(\zeta) = i(a(\zeta)R(\zeta) + b(\zeta)Q(\zeta))(c(\zeta)R(\zeta) + d(\zeta)Q(\zeta))^{-1}, \quad A_N =: \begin{bmatrix} a & b \\ c & d \end{bmatrix},
\]

where \( \zeta \in \mathbb{C}_+; \) \( a, b, c, d \) are \( p \times p \) blocks of \( \mathcal{A} \) and \( \{R(\zeta), Q(\zeta)\} \) are pairs of \( p \times p \) matrix-valued functions, which are meromorphic in \( \mathbb{C}_+ \) and satisfy inequalities:

\[
R(\zeta)^*R(\zeta) + Q(\zeta)^*Q(\zeta) > 0, \quad \begin{bmatrix} R(\zeta)^* & Q(\zeta)^* \end{bmatrix} J \begin{bmatrix} R(\zeta) \\ Q(\zeta) \end{bmatrix} \geq 0
\]

(excluding, possibly, some isolated points). Such pairs \( \{R(\zeta), Q(\zeta)\} \) are called nonsingular and \( J \)-nonnegative. The first equality in (3.8) shows that the matrix \( JjK \) maps \(-j\)-nonegative pairs \( \{\tilde{R}, \tilde{Q}\} \) (more precisely,
the columns \( \text{col} \begin{bmatrix} \tilde{R} \\ \tilde{Q} \end{bmatrix} \) which are used in the linear fractional transformations \([10, (5.7)]\) onto the set of \( J \)-nonnegative pairs \( \{ R, Q \} \) which are used in the linear fractional transformations \([32, (11)]\). In view of \([10, (5.7)], [32, (11)]\) and the equality \((3.7)\) above, it follows that our Weyl function \( \varphi \) is connected with the function \( \omega \) corresponding to \( \{ \nu \} \cup \{ s_k \} \) \((0 \leq k < \infty)\) in \([32]\) by the relation

\[
\omega(\zeta) = -\varphi(2/\zeta). \tag{3.11}
\]

Since \( \varphi(\lambda) \) is Herglotz function in \( \mathbb{C}_- \), \( \omega(\zeta) \) belongs to Herglotz class in \( \mathbb{C}_+ \) (i.e., \( \Im(\omega(\zeta)) \geq 0 \)).

3. Now, consider Herglotz representation

\[
\omega(\zeta) = \beta \zeta + \gamma + \int_{-\infty}^{\infty} \frac{1 + t \zeta}{(t - \zeta)(1 + t^2)} d\tau(t); \tag{3.12}
\]

\[
\beta \geq 0, \quad \gamma = \gamma^*, \quad \int_{-\infty}^{\infty} (1 + t^2)^{-1} d\tau(t) < \infty, \tag{3.13}
\]

and assume that \( \tau' \) (the positive semi-definite derivative of the absolutely continuous part of \( \tau \)) satisfies the Szegö condition

\[
\int_{-\infty}^{\infty} (1 + t^2)^{-1} \ln \left( \det \tau'(t) \right) > -\infty. \tag{3.14}
\]

The factors of \( \tau' \) will play a crucial role in our further considerations. The factorisation of positive semi-definite integrable matrix functions is one of the classical domains connected with the names of A. Beurling, N. Wiener, P.R. Masani, H. Helson, D. Lowdenslager, M.G. Krein, Yu.A. Rozanov, D. Sarason and many others (see, e.g., \([16, 24, 30, 31, 42]\) and the bibliography in the interesting paper \([18]\)). We will use the results formulated in \([16, \text{Theorem 9}]\) and \([18, \text{Theorem 5.2}]\).

Notice that the functions \( (\zeta_0 z - \zeta_0)(z - 1)^{-1} \), where \( \zeta_0 \in \mathbb{C}_+ \), map unit disk \( \mathbb{D} \) \((|z| < 1)\) onto \( \mathbb{C}_+ \). Let us introduce the class of functions \( \tilde{H} \) on \( \mathbb{C}_+ \). We say that the \( p \times p \) matrix function \( G(\zeta) \) belongs to \( \tilde{H} \) if the entries of \( G \left( \frac{\zeta_0 z - \zeta_0}{z - 1} \right) \) belong to the Hardy class \( H^2(\mathbb{D}) \) and \( \det \left( G \left( \frac{\zeta_0 z - \zeta_0}{z - 1} \right) \right) \) is an outer function. Setting

\[
t = (\zeta_0 z - \zeta_0)(z - 1)^{-1} = \zeta_0 + (\zeta_0 - \zeta_0)(z - 1)^{-1}, \quad z = e^{i\theta} \quad (0 \leq \theta < 2\pi),
\]
and taking into account that \(z(z−1) = z−1\) and \(z(\zeta_0z−\zeta_0) = z−1\)
we obtain
\[
\frac{dt}{1+t^2} = -\frac{i(\zeta_0−\zeta_0)zd\theta}{(z−1)^2 + (\zeta_0z−\zeta_0)^2} = \frac{i(\zeta_0−\zeta_0)d\theta}{|z−1|^2 + |\zeta_0z−\zeta_0|^2}.
\]
(3.15)

From (3.13) and (3.15), it follows that
\[
\int_0^{2\pi} \tau'(z(\zeta_0e^{i\theta}−\zeta_0))d\theta < \infty.
\]
In other words, \(\tau'(z(\zeta_0e^{i\theta}−\zeta_0))\) is integrable. From (3.14) and (3.15) we derive that
\[
\int_0^{2\pi} \ln \det \left(\tau' \left(\frac{z(\zeta_0e^{i\theta}−\zeta_0)}{e^{i\theta}−1}\right)\right) d\theta > −\infty,
\]
and (in view of the equality \(\ln(\det(\tau')) = p\text{tr}(\ln(\tau'))\)) the condition (74) of [16, Theorem 9] is fulfilled. Thus, there is a factorisation
\[
\tau'(t) = G_\tau(t)^*G_\tau(t),
\]
(3.16)
where \(G_\tau(\zeta) \in \tilde{H}\) and \(G_\tau(t)\) is the boundary function of \(G_\tau(\zeta)\). Here, we reversed the order of factors in [16, Theorem 9], which does not matter (see, e.g., [16, p. 195]). One can use [18, Theorem 5.2] on inner-outer factorisation to show that an outer matrix function \(G \left(\frac{\zeta_0z−\zeta_0}{z−1}\right)\) (such that \(\det(G)\) is a scalar outer function) may be chosen uniquely up to a constant unitary factor. The definition of \(\tilde{H}\) does not depend on the choice of \(\zeta_0 \in \mathbb{C}_+\) because the Hardy classes \(H^p\) are invariant under conformal one-to-one transformations of \(\mathbb{D}\).

The matrix functions \(R_k(\lambda, \mu)\) (i.e., \(\rho_k(\lambda, \mu)\) in [32]) are introduced by the equality
\[
R_k(\lambda, \mu) = \Phi_1(k)^*(I_{kp} + \lambda A(k)^*)^{-1}S(k)^{-1}(I_{kp} + \mu A(k))^{-1}\Phi_1(k).
\]
(3.17)

**Remark 3.2** Clearly, the matrices \(R_k(\zeta, \overline{\zeta})\) are well-defined and positive-definite for \(\zeta \neq -2i\). According to [32, Theorem 2], the sequence \(R_k(\zeta, \overline{\zeta})\) \((\zeta \in \mathbb{C}, \zeta \neq -2i)\) is nondecreasing.

From [32, Example 1], [32, Theorem 4] and [32, Remark 1], we obtain the following theorem.
**Theorem 3.3** Let us fix the set of $p \times p$ matrices $\{\nu\} \cup \{s_{-k}\}$ $(0 \leq k < \infty)$, where $\nu = \nu^*$, $s_0 = s_0^*$ and we set $s_k = s_{-k}^*$ for $k > 0$. Assume that the inequalities (2.5) hold for all $1 \leq N < \infty$ and that Szegő condition (3.14) is fulfilled. Then,

$$\lim_{k \to \infty} R_k(\zeta, \zeta)^{-1} = 2\pi i(\overline{\zeta} - \zeta)G_\tau(\zeta)^*G_\tau(\zeta) \quad (\zeta \in \mathbb{C}_+).$$

(3.18)

**Remark 3.4** In view of Remark 3.2, the limit in (3.18) is uniform for $\zeta$ belonging to the compact subsets of $\mathbb{C}_+$.

4. The analogs of the reproducing kernels are matrix functions $W_k(\overline{\mu})^*jW_k(\lambda)$ (see, e.g., (3.4)). Let us rewrite this expression in terms of $\mathfrak{A}_k$. From the definitions (2.3), (2.10) and (3.6) we derive

$$w_A(k, -\overline{\mu}/2)^* = jJA_k(2/\mu)Jj, \quad w_A(k, -\lambda/2) = jJA_k(2/\lambda)^*Jj. \quad (3.19)$$

Using (2.11), (3.8) and (3.19) we obtain

$$W_k(\overline{\mu})^*jW_k(\lambda) = \frac{(\mu-i)^k(\lambda+i)^k}{(\mu \lambda)^k}K^*jJA_k(2/\mu)JjKjK^*jJA_k(2/\lambda)^*JjK$$

$$= -\frac{(\mu-i)^k(\lambda+i)^k}{(\mu \lambda)^k}K^*jJA_k(2/\mu)JjA_k(2/\lambda)^*K^*.$$ \quad (3.20)

Thus, we can study the asymptotics of $\mathfrak{A}_k(\zeta)J\mathfrak{A}_k(\overline{\zeta})^*$ instead of the asymptotics of $W_k(\overline{\lambda})^*jW_k(\mu)$. We set

$$\mathcal{M}(k, \zeta, \xi) := \mathfrak{A}_k(\zeta)J\mathfrak{A}_k(\overline{\zeta})^*. \quad (3.21)$$

Relations (3.19) and (3.21) yield

$$\mathcal{M}(k, \zeta, \xi) = -Jw_A(k, -1/\overline{\zeta})^*Jw_A(k, -1/\xi)jJ. \quad (3.22)$$

Hence, using either the properties of the transfer function $w_A$ (see, e.g., [34, Corollary 1.15]) or direct calculation (which takes into account (2.6)), we have

$$\mathcal{M}(k, \zeta, \xi) = J + i(\xi - \zeta)J\Pi(k)^*$$

$$\times (I_{kp} + \zeta A(k)^*S(k)^{-1}(I_{kp} + \xi A(k))^{-1}\Pi(k))^*J.$$

(3.23)
Partition $\mathcal{M}$ into $p \times p$ blocks $\mathcal{M} = \{\mathcal{M}_{ik}\}_{i,k=1}^2$. From (3.17) and (3.23) it follows that

$$\mathcal{M}_{22}(k, \zeta, \xi) = i(\xi - \zeta)\mathcal{R}_k(\zeta, \xi).$$  \hspace{1cm} (3.24)

Thus, Theorem 3.3 describes the asymptotics of $\mathcal{M}_{22}(k, \zeta, \xi)$:

$$\lim_{k \to \infty} \mathcal{M}_{22}(k, \zeta, \xi) = (1/2\pi)(G_r(\zeta)^*G_r(\zeta))^{-1} \quad (\zeta \in \mathbb{C}_+).$$  \hspace{1cm} (3.25)

Another way to obtain this asymptotics is to use the note [3]. According to Remark 3.4, the limit (3.25) is uniform.

5. Next, using (for the asymptotics of $\mathcal{M}_{22}(k, \zeta, \xi)$) an approach from the theory of orthogonal polynomials (see, e.g., [2]) we prove the following theorem.

**Theorem 3.5** Let us fix the set of $p \times p$ matrices $\{\nu\} \cup \{s_{-k}\}$ ($0 \leq k < \infty$), where $\nu = \nu^*$, $s_0 = s_{0}^*$ and we set $s_k = s_{-k}^*$ for $k > 0$. Assume that the inequalities (2.5) hold for all $1 \leq N < \infty$ and that Szegő condition (3.14) is fulfilled. Let complex values $\zeta$ and $\xi$ belong to some compact subset in $\mathbb{C}_+ \backslash \{2i\}$. Then, uniformly with respect to $\zeta$ and $\xi$ in this compact subset, we have

$$\lim_{k \to \infty} \mathcal{M}(k, \zeta, \xi) = \frac{1}{2\pi} \begin{bmatrix} -i\omega(\zeta) \\ I_p \end{bmatrix} G_r(\zeta)^{-1} (G_r(\xi)^*)^{-1} \begin{bmatrix} i\omega(\xi)^* \\ I_p \end{bmatrix},$$  \hspace{1cm} (3.26)

where $\mathcal{M}(k, \zeta, \xi) = \mathfrak{A}_k(\zeta)\mathfrak{A}_k(\xi)^*$, $\mathfrak{A}_k$ is given by (3.6), $\omega$ is given by (3.11), and $G$ is the factoring multiplier from (3.16) ($G(\zeta) \in \tilde{H}$).

**Proof.** Step 1. Denote by $\Gamma_r$ the curve $| (\zeta - \xi)(\zeta - \xi)^{-1} | = r \leq 1$, where $\xi$ is some fixed point in $\mathbb{C}_+$, choose anticlockwise orientation for this $\Gamma_r$ and
put
\[
\psi(k, r, \xi) := \frac{1}{2\pi i} \int_{\Gamma_r} \left( 2\pi G_\tau(\xi)M_{22}(k, \zeta, \bar{\xi})^* G_\tau(\zeta)^* - I_p \right) d\xi \]
\times \left( 2\pi G_\tau(\zeta)M_{22}(k, \zeta, \bar{\xi})^* G_\tau(\xi)^* - I_p \right) \frac{(\xi - \bar{\xi})}{(\xi - \bar{\xi})(\zeta - \bar{\xi})}; \quad (3.27)
\]
\[
\tilde{\psi}(k, r, \xi) := 2\pi i \int_{\Gamma_r} \frac{G_\tau(\xi)M_{22}(k, \zeta, \bar{\xi})^* G_\tau(\xi)^* G_\tau(\zeta)M_{22}(k, \zeta, \bar{\xi})}{(\zeta - \bar{\xi})(\zeta - \xi)} \times \frac{G_\tau(\xi)^* (\bar{\xi} - \xi)}{(\zeta - \xi)(\zeta - \xi)}. \quad (3.28)
\]
Clearly, for \( r < 1 \) we have
\[
\frac{1}{2\pi i} \int_{\Gamma_r} G_\tau(\zeta)M_{22}(k, \zeta, \bar{\xi})^* G_\tau(\zeta)^* \frac{(\xi - \bar{\xi})}{(\zeta - \xi)(\zeta - \bar{\xi})} = G_\tau(\xi)M_{22}(k, \xi, \bar{\xi}) G_\tau(\xi)^*. \quad (3.29)
\]
Moreover, the curve \( \Gamma_r \) may be rewritten in the form \( (\zeta - \zeta) (\zeta - \bar{\xi})^{-1} = re^{i\theta} \),
which yields:
\[
\zeta - \bar{\xi} = \frac{\xi - \bar{\xi}}{1 - re^{i\theta}}, \quad \zeta - \xi = (\xi - \bar{\xi}) \frac{re^{i\theta}}{1 - re^{i\theta}}.
\]
Hence, the equality
\[
\frac{(\xi - \bar{\xi})}{(\zeta - \xi)(\zeta - \bar{\xi})} = i d\theta \quad (3.30)
\]
follows, and (in view of (3.29)) we obtain
\[
\frac{1}{2\pi i} \int_{\Gamma_r} G_\tau(\zeta)M_{22}(k, \zeta, \bar{\xi})^* G_\tau(\xi)^* \frac{(\xi - \bar{\xi})}{(\zeta - \xi)(\zeta - \bar{\xi})} = \frac{1}{2\pi} \int_0^{2\pi} G_\tau(\zeta(\theta))M_{22}(k, \zeta(\theta), \bar{\xi})^* G_\tau(\zeta(\theta))^* d\theta
\]
\[
= \left( \frac{1}{2\pi} \int_0^{2\pi} G_\tau(\zeta(\theta))M_{22}(k, \zeta(\theta), \bar{\xi})G_\tau(\zeta)^* d\theta \right)^*
\]
\[
= (G_\tau(\xi)M_{22}(k, \xi, \bar{\xi}) G_\tau(\xi)^*)^* = G_\tau(\xi)M_{22}(k, \xi, \bar{\xi}) G_\tau(\xi)^*. \quad (3.31)
\]
Here, we used the equality $M_{22}(k, \xi, \bar{\xi}) = M_{22}(k, \xi, \bar{\xi})^*$, which is immediate from (3.22). Equalities (3.27)–(3.29) and (3.31) imply that

$$\psi(k, r, \xi) = I_p + \tilde{\psi}(k, r, \xi) - 4\pi G_\tau(\xi) M_{22}(k, \xi, \bar{\xi}) G_\tau(\xi)^*. \quad (3.32)$$

We note that $\Gamma_1 = \mathbb{R}$ (where the curves $\Gamma_r$ were introduced at the beginning of the proof), and so (in view of (3.16), (3.17), (3.24) and (3.28)) we have

$$\tilde{\psi}(k, 1, \xi) = 2\pi i (\xi - \xi) G_\tau(\xi) \Phi_1(k)^* (I_{kp} + \zeta A(k)^*)^{-1} S(k)^{-1} \times \int_{-\infty}^{\infty} (I_{kp} + \zeta A(k))^{-1} \Phi_1(k)^* (I_{kp} + \zeta A(k)^*)^{-1} d\zeta \times S(k)^{-1} (I_{kp} + \bar{\xi} A(k))^{-1} \Phi_1(k) G_\tau(\xi)^*. \quad (3.33)$$

From [32, (7) and (9)], we obtain the following representation of $S(k)$ (in the present notations):

$$S(k) = A^{-1} \Phi_1(k) \beta \Phi_1(k)^* (A^{-1})^* + \int_{-\infty}^{\infty} (I_{kp} + t A(k))^{-1} \Phi_1(k)^* (I_{kp} + t A(k)^*)^{-1}, \quad (3.34)$$

where $\Phi_1$ is introduced in (2.8) and $\beta$ and $\tau$ are given by the Herglotz representation (3.12). Thus, it is easy to see that

$$\int_{-\infty}^{\infty} (I_{kp} + \zeta A(k))^{-1} \Phi_1(k)^* (I_{kp} + \zeta A(k)^*)^{-1} d\zeta \leq S(k). \quad (3.35)$$

Finally, relations (3.24), (3.33) and (3.35) yield

$$\tilde{\psi}(k, 1, \xi) \leq 2\pi G_\tau(\xi) M_{22}(k, \xi, \bar{\xi}) G_\tau(\xi)^*. \quad (3.36)$$

This implies that the entries of $G_\tau(\zeta) M_{22}(\zeta, \bar{\zeta})$ belong (after substitution $\zeta = (\bar{\zeta}_0 z - \zeta_0) (z - 1)^{-1}$, $\zeta_0 \in \mathbb{C}_+$) to the space $H^2$ of the analytic functions of $z$ in the unit disk. Then, using theorem of F. Riesz (see, e.g., formula (4.1.1) in [28]) and properties of subharmonic functions we have

$$\tilde{\psi}(k, r, \xi) \leq \tilde{\psi}(k, 1, \xi) \quad (r \leq 1). \quad (3.37)$$
According to (3.27) and (3.30) the inequality \( \psi(k, r, \xi) \geq 0 \) is valid. Since \( \psi(k, r, \xi) \geq 0 \), we take into account (3.25), (3.36) and (3.37) and derive from (3.32) that

\[
\lim_{k \to \infty} \psi(k, r, \xi) = 0,
\]

uniformly in \( \xi \) and \( r \). Using (3.38) and expanding

\[
G_r \left( (\bar{\zeta}_0 z - \zeta_0) (z - 1)^{-1} \right) \mathcal{M}_{22} \left( (\bar{\zeta}_0 z - \zeta_0) (z - 1)^{-1}, \bar{\xi} \right)
\]

in series in \( z \), we obtain the assertion:

\[
\lim_{k \to \infty} \mathcal{M}_{22}(k, \zeta, \bar{\xi}) = \left( \frac{1}{2\pi} \right)(G_r(\xi)^*G_r(\zeta))^{-1} \quad (\zeta, \xi \in \mathbb{C}_+)
\]

uniformly on the compacts in \( \mathbb{C}_+ \). Formula (3.39) coincides with the restriction of (3.26) to \( \mathcal{M}_{22}(k, \zeta, \bar{\xi}) \).

Step 2. The set of the matrix-valued functions given by the linear-fractional (Möbius) transformations (3.9), where \( \{R(\zeta), Q(\zeta)\} \) are nonsingular, \( J \)-nonnegative pairs, is denoted by \( \mathcal{N}(\mathfrak{A}_N) \) (\( N \in \mathbb{N} \)), and the set of values which these matrix functions take at \( \zeta \in \mathbb{C}_+ \) is denoted by \( \mathcal{N}(\mathfrak{A}_N)(\zeta) \). The sets \( \mathcal{N}(\mathfrak{A}_N) \) are embedded (i.e., \( \mathcal{N}(\mathfrak{A}_N) \subseteq \mathcal{N}(\mathfrak{A}_{\hat{N}}) \) for \( N > \hat{N} \)), and their intersection consists of one function \( \omega(\zeta) \); see, for instance, Proposition 5.7 and Theorem 6.4 in [10]. (One easily deletes the requirement from [10, Definition 5.3] that the pairs \( \{R, Q\} \) are well-defined at some fixed point). Moreover, according to (3.11) and [10, p. 227] we have

\[
\bigcap_{N \geq 1} \mathcal{N}(\mathfrak{A}_N)(\zeta) = \{\omega(\zeta)\}.
\]

Next, we show that \( \mathcal{N}(\mathfrak{A}_N)(\zeta) \) form Weyl-type disks and consider these disks. By virtue of (3.21) and (3.23) (see also [32, (17)]), we have

\[
\mathfrak{A}_N(\zeta)J \mathfrak{A}_N(\bar{\zeta})^* = J = \mathfrak{A}_N(\bar{\zeta})^*J \mathfrak{A}_N(\zeta).
\]

Since \( \mathfrak{A}_N(\bar{\zeta})^*J \mathfrak{A}_N(\zeta) = J \), it is immediate that equality (3.9) (where (3.10) holds) is equivalent to

\[
\left[ i\omega_N(\zeta) I_p \right] J \mathfrak{A}_N(\bar{\zeta})J \mathfrak{A}_N(\zeta)^* J \left[ -i\omega_N(\zeta) \begin{bmatrix} I_p \end{bmatrix} \right] \geq 0.
\]

\[\text{14}\]
Setting
\[ \mathcal{F}(\zeta) := J \mathcal{A}_N(\zeta) J^* \mathcal{A}_N(\zeta) = \{ \mathcal{F}_{ik}(\zeta) \}_{i,k=1}^2, \]  
(3.43)
where \( \mathcal{F}_{ik} \) are \( p \times p \) blocks of \( \mathcal{F} \), we rewrite (3.42) in the form

\[ \omega_N(\zeta)^* (- \mathcal{F}_{11}(\zeta)) \omega_N(\zeta) + i (\mathcal{F}_{21}(\zeta) \omega_N(\zeta) - \omega_N(\zeta)^* \mathcal{F}_{12}(\zeta)) \leq \mathcal{F}_{22}(\zeta). \]

Equivalently, we have

\[ \omega_N(\zeta)^* (- \mathcal{F}_{11}(\zeta)) \omega_N(\zeta) + i (\mathcal{F}_{21}(\zeta) \omega_N(\zeta) - \omega_N(\zeta)^* \mathcal{F}_{12}(\zeta)) + \mathcal{F}_{21}(\zeta) (- \mathcal{F}_{11}(\zeta))^{-1} \mathcal{F}_{12}(\zeta) \leq \mathcal{F}_{22}(\zeta) - \mathcal{F}_{21}(\zeta) \mathcal{F}_{11}(\zeta)^{-1} \mathcal{F}_{12}(\zeta), \]

(3.44)
where

\[ - \mathcal{F}_{11}(\zeta) = i (\zeta - \bar{\zeta}) R_N(\zeta, \zeta) > 0. \]

(3.45)

From (3.43) (using again (3.41)), we obtain

\[ \mathcal{F}(\zeta)^{-1} = \{ (\mathcal{F}(\zeta)^{-1})_{ik} \}_{i,k=1}^2 = \mathcal{A}_N(\zeta) J \mathcal{A}_N(\zeta)^*, \]
(3.46)
where \( (\mathcal{F}(\zeta)^{-1})_{ik} \) are \( p \times p \) blocks of \( \mathcal{F}(\zeta)^{-1} \). Taking into account (3.21), (3.24) and (3.46) we derive

\[ (\mathcal{F}(\zeta)^{-1})_{22} = i (\zeta - \bar{\zeta}) R_N(\zeta, \zeta) > 0. \]

(3.47)

Since \( (\mathcal{F}(\zeta)^{-1})_{22} \) is invertible, we express \( ((\mathcal{F}(\zeta)^{-1})_{22})^{-1} \) in terms of the blocks \( \mathcal{F}_{ik} \) (see, e.g., [37, Section 1.2]):

\[ ((\mathcal{F}(\zeta)^{-1})_{22})^{-1} = \mathcal{F}_{22}(\zeta) - \mathcal{F}_{21}(\zeta) \mathcal{F}_{11}(\zeta)^{-1} \mathcal{F}_{12}(\zeta). \]

(3.48)

In view of (3.45) and (3.47), the matrix functions \( \Lambda_l(N, \zeta) \) and \( \Lambda_r(N, \zeta) \) (which we will show to be the left and right radii) are well-defined by the relations

\[ \Lambda_l(N, \zeta)^2 = (- \mathcal{F}_{11}(\zeta))^{-1}, \quad \Lambda_l(N, \zeta) > 0; \]
\[ \Lambda_r(N, \zeta)^2 = ((\mathcal{F}(\zeta)^{-1})_{22})^{-1}, \quad \Lambda_r(N, \zeta) > 0. \]

(3.49)  
(3.50)
Using (3.48)–(3.50), we rewrite (3.44) in the form
\[(\Lambda_l^{-1}\omega_N - i\Lambda_l\tilde{\mathcal{F}}_{12})^*(\Lambda_l^{-1}\omega_N - i\Lambda_l\tilde{\mathcal{F}}_{12}) \leq \Lambda_r^2.\]
That is, we parametrize \(\mathcal{N}(\mathfrak{A}_N)(\zeta)\) via contractive \(p \times p\) matrices \(u\) as a matrix circle
\[
\omega_N(\zeta) = \Lambda_l(N, \zeta)u\Lambda_r(N, \zeta) + i\Lambda_l(N, \zeta)^2\tilde{\mathcal{F}}_{12}(\zeta) \quad (u^*u \leq I_p),
\] (3.51)
where \(\Lambda_l(N, \zeta)\) and \(\Lambda_r(N, \zeta)\) are, indeed, the left and right radii, and \(N\) is omitted in the notations connected with \(\mathfrak{F}\). According to (3.45), (3.49) and (3.47), (3.50), we obtain
\[
\Lambda_l(N, \zeta)^2 = i(\zeta - \overline{\zeta})^{-1}R_N(\overline{\zeta}, \zeta)^{-1}, \quad \Lambda_r(N, \zeta)^2 = i(\zeta - \overline{\zeta})^{-1}R_N(\zeta, \overline{\zeta})^{-1}.
\] (3.52)
Relations (3.18), (3.52) and Remark 3.4 imply that uniformly on each compact in \(\mathbb{C}\) we have
\[
\lim_{k \to \infty} \Lambda_r(k, \zeta) = \sqrt{2\pi G_\tau(\zeta)^*G_\tau(\zeta)} > 0.
\] (3.53)
Hence, equalities (3.40) and (3.51) yield
\[
\lim_{k \to \infty} \Lambda_l(k, \zeta) = 0 \quad (\zeta \in \mathbb{C}_+, \ \zeta \neq 2i).
\] (3.54)
Taking into account (3.45), (3.49) and (3.54) we obtain
\[
\lim_{k \to \infty} \mathcal{R}_k(\overline{\zeta}, \zeta)^{-1} = 0 \quad (\zeta \in \mathbb{C}_+, \ \zeta \neq 2i).
\] (3.55)
Moreover, according to [32, Theorem 2] the sequences of matrices \(\mathcal{R}_k(\overline{\zeta}, \zeta)\) (matrices \(\rho_k\) in the notations of [32]) are nondecreasing. Thus, (3.55) (and so (3.53) as well) holds uniformly for the values \(\zeta\) on any compact \(\mathcal{C} \subset \mathbb{C}_+ \setminus \{2i\}\).

Step 3. Using (3.21), (3.43) and (3.46) (and substituting \(\zeta\) instead of \(\zeta\)), we rewrite (3.48) in the form
\[
\mathcal{M}_{22}(N, \zeta, \zeta)^{-1} = \tilde{\mathcal{F}}_{22}(\zeta) - \tilde{\mathcal{F}}_{21}(\zeta)\mathcal{M}_{22}(N, \zeta, \zeta)^{-1}\tilde{\mathcal{F}}_{12}(\zeta) \quad (\zeta \neq \zeta, \ \zeta \neq \pm 2i).
\] (3.56)
In order to right down the necessary expressions for $\mathfrak{F}_{22}$, $\mathfrak{F}_{21}$ and $\mathfrak{F}_{12}$, we need the representation

$$
\begin{bmatrix}
a(N, \zeta) & b(N, \zeta)
\end{bmatrix} = a(N, \zeta)c(N, \zeta)^{-1} \begin{bmatrix}
c(N, \zeta) & d(N, \zeta)
\end{bmatrix}
- q(N, \zeta) \begin{bmatrix}
0 & d(N, \zeta)
\end{bmatrix},
$$

(3.57)

$$
q(N, \zeta) := a(N, \zeta)c(N, \zeta)^{-1} - b(N, \zeta)d(N, \zeta)^{-1}.
$$

(3.58)

From (3.21), (3.43), (3.56) and (3.57) (omitting variables $\zeta$ and $N$ in the matrix functions $a, b, c, d$ and $q$), we obtain

$$
\mathcal{M}_{22}(N, \zeta, \zeta)^{-1} = ab^* + ba^* - (ac^{-1}\mathcal{M}_{22}(N, \zeta, \zeta) - qdc^*)\mathcal{M}_{22}(N, \zeta, \zeta)^{-1} 
\times (ac^{-1}\mathcal{M}_{22}(N, \zeta, \zeta) - qdc^*)^*,
$$

(3.59)

where $\mathcal{M}_{22}(N, \zeta, \zeta) = cd^* + dc^*$. Thus, it is easily checked that

$$
ab^* + ba^* - ac^{-1}(cd^* + dc^*)(c^{-1})^*a^* + ac^{-1}cd^*q^* + qdc^*(c^{-1})^*a^* = 0,
$$

and (3.59) takes the form

$$
\mathcal{M}_{22}(N, \zeta, \zeta)^{-1} = - q(N, \zeta)d(N, \zeta)c(N, \zeta)^*\mathcal{M}_{22}(N, \zeta, \zeta)^{-1} 
\times c(N, \zeta)d(N, \zeta)^*q(N, \zeta)^* \quad (\zeta \neq \overline{\zeta}, \zeta \neq \pm 2i).
$$

(3.60)

In view of the uniform limits (3.25) and (3.55) and equality (3.60), we derive (uniformly for the values $\zeta$ on any compact $C \subset \mathbb{C} + \{2i\}$) the relation

$$
\lim_{k \to \infty} \|q(k, \zeta)d(k, \zeta)c(k, \zeta)^*\| = 0.
$$

(3.61)

According to (3.24) and Remark 3.2, the inequality

$$\Re\left( c(N, \xi)^{-1}d(N, \xi) \right) > 0$$

is valid for $\xi \in \mathbb{C}_+$. Hence, we have

$$2\Re\left( c(N, \xi)c(N, \xi)^{-1}\mathcal{M}_{22}(N, \xi, \overline{\xi}) \right) > \mathcal{M}_{22}(N, \xi, \overline{\xi}) \quad (\xi \in \mathbb{C}_+).$$

(3.62)

Using (3.62) (and (3.39)) one easily proves (by negation) that for any compact $C \subset \mathbb{C}_+$ there are such $\hat{N}$ and $M > 0$ that for each pair $\xi, \zeta \in C$ and $N > \hat{N}$ the inequality

$$\|c(N, \xi)c(N, \zeta)^{-1}\| < M
$$

(3.63)
holds. It is immediate from (3.61) and (3.63) that we have a uniform limit

\[
\lim_{k \to \infty} \|q(k, \zeta)d(k, \zeta)c(k, \xi)^*\| = 0 \quad (\zeta, \xi \in \mathbb{C} \setminus \{2i\}).
\]  

(3.64)

Taking into account (3.9), (3.51), and uniform limits (3.53) and (3.54), we see that

\[
\lim_{k \to \infty} a(k, \zeta)c(k, \xi)^{-1} = \lim_{k \to \infty} b(k, \zeta)d(k, \zeta)^{-1} = -i\omega(\zeta)
\]  

(3.65)

uniformly for the values \(\zeta\) on any compact \(\mathcal{C} \subset \mathbb{C} \setminus \{2i\}\).

Relations (3.57), (3.64) and (3.65) imply the equality

\[
\lim_{k \to \infty} \mathcal{M}_{12}(k, \zeta, \bar{\xi}) = a(N, \zeta)d(N, \xi)^* + b(N, \zeta)c(N, \xi)^* \\
= (-i/2\pi)\omega(\zeta)(G_{\tau}(\xi)^*G_{\tau}(\zeta))^{-1} \quad (\zeta, \xi \in \mathbb{C} \setminus \{2i\}),
\]  

(3.66)

which holds uniformly on any compact \(\mathcal{C} \subset \mathbb{C} \setminus \{2i\}\). In this way, the reductions of (3.26) for the blocks \(\mathcal{M}_{12}(k, \zeta, \bar{\xi})\) and \(\mathcal{M}_{21}(k, \zeta, \bar{\xi}) = \mathcal{M}_{12}(k, \bar{\xi}, \zeta)^*\) of \(\mathcal{M}(k, \zeta, \bar{\xi})\) are proved.

Finally, using again (3.57), (3.64) and (3.65), we see that the asymptotic equality

\[
\lim_{k \to \infty} \mathcal{M}_{11}(k, \zeta, \bar{\xi}) = a(N, \zeta)b(N, \xi)^* + b(N, \zeta)a(N, \xi)^* \\
= (1/2\pi)\omega(\zeta)(G_{\tau}(\xi)^*G_{\tau}(\zeta))^{-1}\omega(\xi)^*
\]  

(3.67)

holds uniformly for \(\zeta, \xi\) on any compact \(\mathcal{C} \subset \mathbb{C} \setminus \{2i\}\). Formulas (3.39), (3.66) and (3.67) prove the theorem. \(\blacksquare\)

**Remark 3.6** We note that the asymptotic formula (3.39) for \(\mathcal{M}_{22}(k, \zeta, \bar{\xi})\) holds uniformly for \(\zeta, \xi\) belonging to the compacts \(\mathcal{C}\) from \(\mathbb{C}_+\) (and not to \(\mathcal{C} \subset \mathbb{C}_+ \setminus \{2i\}\)). In the same way, uniform limits for \(\mathcal{M}_{11}(k, \zeta, \bar{\xi})\) and \(\mathcal{M}_{12}(k, \zeta, \bar{\xi})\) can be proved as well. Moreover, the mentioned above proofs of (3.26) admit generalizations for a wide class of interpolation problems (see the abstract interpolation in [36] and the proof of (3.18) for interpolation problems in [32]).

**Remark 3.7** We note also that formula (3.55) in the proof of Theorem 3.5 is of independent interest.
4 Explicit formulas

Let us consider the case, where the matrices $C_k$ are obtained explicitly, namely, the case of the GBDT transformations of the trivial discrete Dirac system, where $C_k \equiv I_{2p}$. Each GBDT is determined by some $n \in \mathbb{N}$ and an admissible triple consisting of $n \times n$ matrices $\mathcal{A}$ and $\mathcal{S}_0$ and of $n \times 2p$ matrix $\Pi_0$. The admissible triples are defined here as follows.

**Definition 4.1** The triple $\{\mathcal{A}, \mathcal{S}_0, \Pi_0\}$ is called admissible if

$$\mathcal{A}\mathcal{S}_0 - \mathcal{S}_0\mathcal{A}^* = i\Pi_0 j \Pi_0^*, \quad \text{det} \mathcal{A} \neq 0, \quad \mathcal{S}_0 > 0. \quad (4.1)$$

According to [10, Propositions 2.4, 3.1], each admissible triple determines the potential $\{C_k\}$ ($k \geq 0$) satisfying (1.2) in the following way:

$$\Pi_{k+1} = \Pi_k + i\mathcal{A}^{-1}\Pi_k j \quad (k \geq 0), \quad (4.2)$$

$$\mathcal{S}_{k+1} = \mathcal{S}_k + \mathcal{A}^{-1}\mathcal{S}_k(\mathcal{A}^*)^{-1} + \mathcal{A}^{-1}\Pi_k \Pi_k^*(\mathcal{A}^*)^{-1} \quad (k \geq 0), \quad (4.3)$$

$$C_k := I_{2p} + \Pi_k^* \mathcal{S}_k^{-1} \Pi_k - \Pi_k^* \mathcal{S}_{k+1}^{-1} \Pi_{k+1}. \quad (4.4)$$

We note that the matrices $\Pi_k$ in (4.1)-(4.4) have nothing to do with $\Pi(k)$ in Section 3. The Weyl function of Dirac system determined by the admissible triple $\{\mathcal{A}, \mathcal{S}_0, \Pi_0\}$ is given by the formulas [10, (4.7), (4.28)], which (in our notations) take the form

$$\varphi(\lambda) = -i(I_p - \phi(\lambda))(I_p + \phi(\lambda))^{-1}, \quad (4.5)$$

$$\phi(\lambda) = -i\vartheta_1^* \mathcal{S}_0^{-1}(\mathcal{A}^* - \lambda I_n)^{-1} \vartheta_2, \quad \mathcal{A}^* = \mathcal{A} + i\vartheta_2 \vartheta_2^* \mathcal{S}_0^{-1}, \quad (4.6)$$

where $\vartheta_1$ are $p \times p$ blocks of $\Pi_0$, that is, $\Pi_0 := \begin{bmatrix} \vartheta_1 & \vartheta_2 \end{bmatrix}$.

**Theorem 4.2** Let Dirac system (1.1), (1.2) be determined by some admissible triple $\{\mathcal{A}, \mathcal{S}_0, \Pi_0\}$. Then, the set $\nu \cup \{s_{-k}\}$ ($0 \leq k < \infty$) (or, equivalently, the $p \times p$ matrix $\nu = \nu^*$ and the semi-infinite Toeplitz matrix) corresponding to this Dirac system via (2.4) is given explicitly by the formulas:

$$\nu = \Im(I_p + 2i\vartheta_1^* \mathcal{S}_0^{-1}(\bar{\mathcal{A}} + iI_n)^{-1} \vartheta_2), \quad \bar{\mathcal{A}} := \mathcal{A} + i\vartheta_2(\vartheta_2 - \vartheta_1)^* \mathcal{S}_0^{-1}; \quad (4.7)$$

$$s_0 = 2\Re(I_p + 2i\vartheta_1^* \mathcal{S}_0^{-1}(\bar{\mathcal{A}} + iI_n)^{-1} \vartheta_2), \quad (4.8)$$

$$s_{-k} = 2i\vartheta_1^* \mathcal{S}_0^{-1}(\bar{\mathcal{A}} + iI_n)^{-1} \mathcal{U}^{-1}(\mathcal{U} - I_n) \vartheta_2 \quad (k \geq 1), \quad (4.9)$$
where $\Im$ and $\Re$ denote imaginary and real parts of the matrices, $\tilde{A} + iI_n$ is invertible and

$$U = (\tilde{A} - iI_n)(\tilde{A} + iI_n)^{-1}. \quad (4.10)$$

**Proof.** Using (4.5) and (4.6), we will write down $\varphi(\lambda)$ in a more simple way. Indeed, it is well-known from system theory (and is easily checked directly) that

$$\left(I_{2p} - i\vartheta_1^* S_0^{-1} (A^x - \lambda I_n)^{-1} \vartheta_2 \right)^{-1} = I_{2p} + i\vartheta_1^* S_0^{-1} \left(\tilde{A} - \lambda I_n\right)^{-1} \vartheta_2, \quad (4.11)$$

where $\tilde{A} = A^x - i\vartheta_2 \vartheta_1^* S_0^{-1}$. Clearly this definition of $\tilde{A}$ coincides with the one in formula (4.7). Moreover, the definition of $\tilde{A}$ yields

$$i^2 \vartheta_1^* S_0^{-1} (A^x - \lambda I_n)^{-1} \vartheta_2 \vartheta_1^* S_0^{-1} \left(\tilde{A} - \lambda I_n\right)^{-1} \vartheta_2$$

$$= i\vartheta_1^* S_0^{-1} (A^x - \lambda I_n)^{-1} (A^x - \tilde{A}) \left(\tilde{A} - \lambda I_n\right)^{-1} \vartheta_2$$

$$= i\vartheta_1^* S_0^{-1} \left(\tilde{A} - \lambda I_n\right)^{-1} \vartheta_2 - i\vartheta_1^* S_0^{-1} \left(A^x - \lambda I_n\right)^{-1} \vartheta_2. \quad (4.12)$$

From (4.5)–(4.12) we derive

$$\varphi(\lambda) = -i \left(I_{2p} + 2i\vartheta_1^* S_0^{-1} \left(\tilde{A} - \lambda I_n\right)^{-1} \vartheta_2\right), \quad (4.13)$$

and the representation

$$i\varphi \left(\frac{z + 1}{z - 1}\right) = I_{2p} + 2i\vartheta_1^* S_0^{-1} \left(\tilde{A} - i\frac{z + 1}{z - 1} I_n\right)^{-1} \vartheta_2 \quad (4.14)$$

follows. Since

$$\Pi_0 = \begin{bmatrix} \vartheta_1 & \vartheta_2 \end{bmatrix}, \quad A^x = A + i\vartheta_2 \vartheta_1^* S_0^{-1}, \quad \tilde{A} = A^x - i\vartheta_2 \vartheta_1^* S_0^{-1}, \quad (4.15)$$

the matrix identity in (4.1) can be rewritten in the form

$$\tilde{A} S_0 - S_0 \tilde{A}^* = i(\vartheta_1 - \vartheta_2)(\vartheta_1 - \vartheta_2)^*. \quad (4.16)$$

Taking into account that $S_0 > 0$ and (4.16) is valid, we see that $\sigma(\tilde{A}) \subset \mathbb{C}_+$, where $\sigma$ means spectrum. In particular, $\det(A + i I_n) \neq 0$, and the matrix $A + i I_n$ is, indeed, invertible.
From (4.10) and (4.14) we derive that
\[
\varphi\left(\frac{z+1}{z-1}\right) = I_p + 2i(z-1)\vartheta_1^* S_0^{-1}((z-1)\tilde{A} - i(z+1)I_n)^{-1}\vartheta_2
\]
\[
= I_p - 2i(z-1)\vartheta_1^* S_0^{-1}(\tilde{A} + iI_n)^{-1}(I_n - z\mathcal{U})^{-1}\vartheta_2.
\]  
(4.17)

Using equality \(-(z-1)(I_n - z\mathcal{U})^{-1} = I_n + z(I_n - z\mathcal{U})^{-1}(\mathcal{U} - I_n)\) and (4.17), we obtain a so called realisation of \(\varphi\left(\frac{z+1}{z-1}\right)\):
\[
\varphi\left(\frac{z+1}{z-1}\right) = I_p + 2i\vartheta_1^* S_0^{-1}(\tilde{A} + iI_n)^{-1}\vartheta_2 + 2iz\vartheta_1^* S_0^{-1}(\tilde{A} + iI_n)^{-1}(I_n - z\mathcal{U})^{-1}
\times (\mathcal{U} - I_n)\vartheta_2.
\]  
(4.18)

Finally, relations (2.4), (2.9), and (4.18) yield (4.7)–(4.9).

\[\blacksquare\]

**Remark 4.3** In view of (4.10), \(\mathcal{U}\) is invertible in the case \(i \not\in \sigma(\tilde{A})\). In this case, the Toeplitz matrices considered in Theorem 4.2 (and given by the relations (4.8), (4.9) and \(s_k = s_k^*\)) are block diagonal plus block semiseparable matrices.

We note that semiseparable matrices have been studied in various papers (see, e.g., [9, 13, 40] and the references therein).

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