MODULI SPACES OF $J$-HOLOMORPHIC CURVES WITH
GENERAL JET CONSTRAINTS

KE ZHU

Abstract. In this paper, we prove that the tangent map of the holomorphic $k$-jet evaluation $j^k_{hol}$ from the mapping space to holomorphic $k$-jet bundle, when restricted on the universal moduli space $\mathcal{M}_1^j(\Sigma, M, \beta)$ of simple $J$-holomorphic curves with one marked point, is surjective. From this we derive that for generic $J$, the moduli space of simple $J$-holomorphic curves with general jet constraints at marked points is a smooth manifold of expected dimension.

1. Introduction

Let $(M, \omega)$ be a symplectic manifold of dimension $2n$. Denote by $\mathcal{J}$ the set of almost complex structures $J$ on $M$ compatible with $\omega$. Let $\Sigma$ be a compact oriented surface without boundary, and $(j, u)$ a pair of complex structure $j$ on $\Sigma$ and a map $u : \Sigma \to M$. We say $(j, u)$ is a $J$-holomorphic curve if $\bar{\partial}^J_{j,u} := \frac{1}{2} (du + J \circ du \circ j) = 0$. We let $\mathcal{M}_1(\Sigma, M, J; \beta)$ be the standard moduli space of $J$-holomorphic curves in class $\beta \in H_2(M, \mathbb{Z})$ with one marked point, and $\mathcal{M}_1^j(\Sigma, M, J; \beta)$ be the set of simple (i.e. somewhere injective) $J$-holomorphic curves in $\mathcal{M}_1(\Sigma, M; \beta)$.

Since the birth of the theory of $J$-holomorphic curves, moduli spaces of $J$-holomorphic curves with constraints at marked points have lead to finer symplectic invariants like Gromov-Witten invariants and quantum cohomology. $J$-holomorphic curves with embedding property also plays important role in low dimensional symplectic geometry, like the works of [HT] and [Wen]. These constraints all can be viewed as partial differential relations in the 0-jet and 1-jet bundles. In relative Gromov-Witten theory, contact order of $J$-holomorphic curves with given symplectic hypersurfaces (divisors) was used to define the relevant moduli spaces. In the work of Cieliebak-Mohnke [CM] and Oh [Oh], the authors studied the moduli space of $J$-holomorphic curves with prescribed vanishing orders of derivatives at marked points. All these are vanishing conditions in $k$-jets bundles. It is then natural to ask what properties we can expect for moduli spaces of $J$-holomorphic curves with general constraints in jet bundles (while all constraints in previous examples are zero sections in various jet bundles).

The main purpose of this paper is to confirm that for a wide class of closed partial differential relations in holomorphic jet bundles (Definition 4 originally defined in [Oh]), the moduli spaces of $J$-holomorphic curves from $\Sigma$ to $M$ with given constraints at marked points behave well for generic $J$ (Theorem 8). Namely, they are smooth manifolds of dimension predicted by index theorem, and all elements in
the moduli spaces are Fredholm regular. During the proof it appears that holomorphic jet bundles are the natural framework to put jet constraints for J-holomorphic curves in order to obtain regularity of their moduli spaces. The regularity of J-holomorphic curve moduli spaces fails for general constraints in usual jet bundle (Remark 2), but still holds in a special case when the moduli space consists of immersed J-holomorphic curves (Theorem 2).

The key of the proof is to establish the surjective property of the linearization of k-jet evaluations on the universal moduli spaces of J-holomorphic curves at marked points inside the mapping space, including the parameter $J \in J_\omega$(Theorem 1). It is important to take the evaluations in holomorphic jet bundles in order to get the surjectivity of the linearization of the k-jet evaluation map.

Since $J_\omega$ is a huge parameter space to deform J-holomorphic curves, the surjective property here is a reminiscence of the classic Thom transversality theorem, which says that the k-jet evaluation on smooth mapping space to the k-jet bundle is transversal to any section there.

The framework of the paper is similar to [Oh], which in turn is a higher jet generalization of [OZ] for 1-jet transversality of J-holomorphic curves. The main steps of the paper are in order:

1. We set up the Banach bundle including the finite dimensional holomorphic k-jet subbundle $J^k_{hol}(\Sigma, M)$ over the mapping space $\mathcal{F}_1(\Sigma, M) \times J_\omega$ and define the section $\Upsilon_k = (\partial, j^k_{hol}(u(z_0)))$.

   We interprete the universal J-holomorphic curve moduli space as
   
   $\mathcal{M}(\Sigma, M) = \partial^{-1}(0) = \Upsilon_k^{-1}(0, J^k_{hol}(\Sigma, M)).$

2. We compute the linearization $D\Upsilon_k$ of the section $\Upsilon_k$. We express the submersion property of $\Upsilon_k$ as the solvability of a system of equations $D\Upsilon_k(\xi, B) = (\gamma, \alpha)$ for any $(\gamma, \alpha)$, where $(\xi, B) \in T_u\mathcal{F}_1(\Sigma, M) \times T_jJ_\omega$, or equivalently, the vanishing of the cokernel element $(\eta, \zeta)$ in the Fredholm alternative system: $F((\xi, B), (\eta, \zeta)) = 0$ for all $(\xi, B)$. This is called the cokernel equation.

3. Using the abundance of $B \in T_jJ_\omega$ we get $\text{supp}\eta \subset \{z_0\}$. Then we use a structure theorem in distribution to write $\eta$ as a linear combination of $\delta$ function and its derivatives at $z_0$, up to $(k-1)$-th order derivatives.

4. Since $\text{supp}\eta \subset \{z_0\}$ the cokernel equation is supported at $z_0$. We replace the $\xi$ in the cokernel equation by $\xi + h$ where $h = h(z, \tau)$ is a suitable polynomial in local coordinates nearby $z_0$, and set $B = 0$, so that the cokernel equation is reduced to $\langle D_u\partial_j, j^k_{hol}(u(z_0)) = 0$ for all $\xi$. The crucial observation is that to get $\langle D_u\partial_j, j^k_{hol}(u(z_0)) = 0$ for all $\xi$ we do not need so strong conditions of vanishing of $1 \sim k$-derivatives of $u$ at $z_0$ as in [Oh] and [CM]. This is by exploring the flexibility of $h$ to get rid of redundant terms from the original cokernel equation.

5. Then we apply elliptic regularity to conclude $\eta = 0$ and consequently $\zeta = 0$. Therefore we get the surjectivity of $D\Upsilon_k$ and $Dj^k_{hol}$.

6. Finally, there is an obstruction in step 4 to get $h$ when $\zeta_k = 0$, where $\zeta_k$ is the k-th component of $\zeta$. But when $\zeta_k = 0$ the cokernel equation is
reduced to the $(k - 1)$-jet evaluation setting, so we still get $(\eta, \zeta) = (0,0)$
by induction on $k$.

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2. Holomorphic jet bundle

We recall the holomorphic jet bundle from [Oh]. Given $\Sigma, M$, and $(z, x) \in \Sigma \times M$,
the $k$-jet with source $z$ and target $x$ is defined as (see [Hir])

$$J^k_{z, x}(\Sigma, M) = \prod_{l=0}^{k} \text{Sym}^l (T_z \Sigma, T_x M),$$

where $\text{Sym}^l (T_z \Sigma, T_x M)$ is the set of $l$-multilinear maps from $T_z \Sigma$ to $T_x M$ for $l \geq 1$.
Here for convenience we have set $\text{Sym}^0 (T_z \Sigma, T_x M) = M$. Let

$$J^k(\Sigma, M) = \bigcup_{(z, x) \in \Sigma \times M} J^k_{z, x}(\Sigma, M)$$

be the $k$-jet bundle over $\Sigma \times M$. For the mapping space

$$\mathcal{F}_1(\Sigma, M; \beta) = \{((\Sigma, j), u) \mid j \in \mathcal{M}(\Sigma), z \in \Sigma, u : \Sigma \to M, [u] = \beta\},$$

we consider the map

$$\mathcal{F}_1(\Sigma, M; \beta) \to \Sigma \times M, \quad (u, j, z) \mapsto (z, u(z)).$$

By this map we can pull back the bundle $J^k(\Sigma, M) \to \Sigma \times M$ to the base $\mathcal{F}_1(\Sigma, M; \beta)$.
By abusing notation, we still call the resulted bundle by $J^k(\Sigma, M)$. Then $J^k(\Sigma, M) \to \mathcal{F}_1(\Sigma, M; \beta)$
is a finite dimensional vector bundle over the Banach manifold $\mathcal{F}_1(\Sigma, M; \beta)$.
We define the $k$-jet evaluation

$$j^k : \mathcal{F}_1(\Sigma, M; \beta) \to J^k(\Sigma, M), \quad j^k((u, j), z) = j^k_z u \in J^k_{z, u(z)}(\Sigma, M).$$

Then $j^k$ is a smooth section. Classic Thom transversality theorem says that $j^k$ is transversal to any section in $J^k(\Sigma, M)$.

Now we turn to the case when $\Sigma$ and $M$ are equipped with (almost) complex structures $j$ and $J$ respectively. The corresponding concept is the holomorphic jet bundle defined in [Oh]. With respect to $(j_z, j_x)$, $\text{Sym}^l_{z, x}(\Sigma, M)$ splits into summands indexed by the bigrading $(p, q)$ for $p + q = k$:

$$\text{Sym}^l_{z, x}(\Sigma, M) = \text{Sym}^{(l, 0)}(T_z \Sigma, T_x M) \oplus \text{Sym}^{(0, l)}(T_z \Sigma, T_x M) \oplus \text{“mixed parts”}$$

Let

$$H^{(l, 0)}_{j_z, j_x}(\Sigma, M) = \text{Sym}^{(l, 0)}(T_z \Sigma, T_x M),$$

$$H^{(l, 0)}_{j_z, j_x}(\Sigma, M) = \bigcup_{(z, x) \in \Sigma \times M} H^{(l, 0)}_{j_z, j_x}(\Sigma, M).$$

Given $(j, J)$, the $(j, J)$-*holomorphic jet bundle* $J^k_{(j, J)\text{hol}}(\Sigma, M)$ is defined as

$$J^k_{(j, J)\text{hol}}(\Sigma, M) = \prod_{l=0}^{k} H^{(l, 0)}_{j, J}(\Sigma, M),$$

(2.1)

$$J^k_{(j, J)\text{hol}}(\Sigma, M) = \prod_{l=0}^{k} H^{(l, 0)}_{j, J}(\Sigma, M),$$
which is a finite dimensional vector bundle over $\Sigma \times M$.

We define the bundle

$$J^k_{hol}(\Sigma, M) = \bigcup_{(j,J) \in M(\Sigma) \times J_\omega} J^k_{hol}(\Sigma, M).$$

$J^k_{hol}(\Sigma, M) \to \Sigma \times M \times M(\Sigma) \times J_\omega$ is a finite dimensional vector bundle over the base Banach manifold. Using the pull back of the map

$$ev : F_1(\Sigma, M; \beta) \times J_\omega \to \Sigma \times M \times M(\Sigma) \times J_\omega, \quad ((u, j), z, J, \beta) \to (z, u(z), j, J),$$

$ev^* (J^k_{hol}(\Sigma, M))$ is a finite dimensional vector bundle over the Banach manifold $F_1(\Sigma, M; \beta) \times J_\omega$. By abusing of notation, we still call $ev^* (J^k_{hol}(\Sigma, M))$ by $J^k_{hol}(\Sigma, M)$.

**Definition 1.** $J^k_{hol}(\Sigma, M) \to F_1(\Sigma, M; \beta) \times J_\omega$ is called the holomorphic $k$-jet bundle.

Let $\pi_{hol} : J^k(\Sigma, M) \to J^k_{hol}(\Sigma, M)$ be the bundle projection. We define the holomorphic $k$-jet evaluation

$$j^k_{hol} = \pi_{hol} \circ j^k.$$

It is not hard to see $j^k_{hol}$ is a smooth section of the Banach bundle $F_1(\Sigma, M; \beta) \times J_\omega \to J^k_{hol}(\Sigma, M)$. According to the summand $(2.1)$, we write $j^k_{hol}$ in components

$$j^k_{hol} = \prod_{l=0}^{k-1} \sigma^l,$$

where the $l$-th component is

$$\sigma^l : F_1(\Sigma, M; \beta) \times J_\omega \to H^{l,0}_{j,J}(\Sigma, M), \quad ((u, j), J, \beta) \to \pi_{hol}^{l,0}(d^l u(z)).$$

We remark that if $J$ is integrable, $\sigma^l$ corresponds to the $l$-th holomorphic derivative $\frac{\partial}{\partial z^l} u$ of $u$ at $z$.

The important point is that the holomorphic $k$-jet bundle and the section $j^k_{hol}$ are canonically associated to the pair $(\Sigma, j)$ and $(M, J)$ in the “off-shell level”, i.e. on the space of all smooth maps, not only $J$-holomorphic maps. This enables us to formulate the jet constraints for $J$-holomorphic maps as some submanifold in the bundle $J^k_{hol}(\Sigma, M) \to F_1(\Sigma, M; \beta) \times J_\omega$.

### 3. Fredholm set up

The Fredholm set up is the same as in [Oh], with the simplification that we only need one marked point on $\Sigma$. The case with more marked points has no essential difference. We introduce the standard bundle

$$\mathcal{H}'' = \bigcup_{((u,j),J)} \mathcal{H}''((u,j),J), \quad \mathcal{H}''((u,j),J) = \Omega^{0,1}_{j,J}(u^*TM)$$

and define the section

$$\Upsilon_k : F_1(\Sigma, M; \beta) \times J_\omega \to \mathcal{H}'' \times J^k(\Sigma, M)$$

as

$$\Upsilon_k ((u, j), z, J) = (\mathcal{J}(u, j, J); j^k_{hol}(u, j, J, z)), $$

where

$$\mathcal{J}(u, j, J) = \mathcal{J}_{j,J}(u) = \frac{du + J \circ du \circ j}{2}.$$
Given $\beta \in H_2(M, \mathbb{Z})$, let
\[ \mathcal{M}_1 (\Sigma, M; \beta) = \bigcup_{J \in \mathcal{J}_\omega} \mathcal{M}_1 (\Sigma, M, J; \beta) \]
be the universal moduli space of $J$-holomorphic curves in class $\beta$ with one marked point. Its open subset consisting of somewhere injective $J$-holomorphic curves is denoted by $\mathcal{M}_1^*(\Sigma, M; \beta)$. It is a standard fact in symplectic geometry that $\mathcal{M}_1^*(\Sigma, M; \beta)$ is a Banach manifold.

Now we make precise the necessary regularity requirement for the Banach manifold set-up:

1. To make sense of the evaluation of $j^ku$ at a point $z$ on $\Sigma$, we need to take at least $W^{k+1,p}$-completion with $p > 2$ of $\mathcal{F}_1(\Sigma, M; \beta)$ so $j^ku \in W^{1,p} \rightarrow C^0$. To make the section $\Upsilon_k$ differentiable we need to take $W^{k+2,p}$ completion, since in (4.2) $(k + 1)$-th derivatives of $u$ are involved. To apply Sard-Smale theorem, we actually need to take $W^{N,p}$ completion with sufficiently large $N = N(\beta, k)$.

2. We provide $\mathcal{H}''$ with topology of a $W^{N,p}$ Banach bundle.

3. We also need to provide the Banach manifold structure of $\mathcal{J}_\omega$. We can borrow Floer’s scheme \cite{Floer} for this whose details we refer readers thereto.

4. Transversality

**Theorem 1.** At every $J$-holomorphic curve $((u, j), z, J) \in \mathcal{M}_1^*(\Sigma, M; \beta) \subset \mathcal{F}_1(\Sigma, M; \beta) \times \mathcal{J}_\omega$, the linearization $DT_k$ of the map
\[ \Upsilon_k = (\overline{\partial}, j^k_{\text{hol}}) : \mathcal{F}_1(\Sigma, M; \beta) \times \mathcal{J}_\omega \rightarrow \mathcal{H}'' \times J^k_{\text{hol}} (\Sigma, M) \]
is surjective. Especially the linearization $Dj^k_{\text{hol}}$ of the holomorphic $k$-jet evaluation
\[ j^k_{\text{hol}} : \mathcal{F}_1(\Sigma, M; \beta) \times \mathcal{J}_\omega \rightarrow J^k_{\text{hol}} (\Sigma, M) \]
on $\mathcal{M}_1^*(\Sigma, M; \beta)$ is surjective.

To prove theorem we need to verify that at each $((u, j), z, J) \in \mathcal{M}_1^*(\Sigma, M; \beta)$, the system of equations
\[ D_{\tau, (j, u)} \overline{\partial} (B, (b, \xi)) = \gamma \]
\[ D_{\tau, (j, u)} j^k_{\text{hol}} (B, (b, \xi)) (z) + \nabla_v (j^k_{\text{hol}} (u)) (z) = \alpha \]
has a solution $(B, (b, \xi) , v) \in T_j \mathcal{J}_\omega \times T_j \mathcal{M}(\Sigma) \times T_u \mathcal{F}_1(\Sigma, M; \beta) \times T_z \Sigma$ for each given data
\[ \gamma \in \Omega^{(0,1)}_{N-1,p} (u^*TM), \quad \zeta = (\zeta_0, \zeta_1, \ldots, \zeta_k) \in J^k_{\text{hol}} (T_z \Sigma, T_u(z)M) \]
It will be enough to consider the triple with $b = 0$ and $v = 0$ which we will assume from now on.

We compute the $D_{\tau, (j, u)} j^k_{\text{hol}} (B, (b, \xi)) (z)$. It is enough to compute $D_{\tau, (j, u)} \sigma^l (B, (0, \xi)) (z)$ for $l = 0, 1, \cdots k$. We have
\[ D_{\tau, (j, u)} \sigma^l (B, (0, \xi)) (z) = \pi_{\text{hol}} \left( (\nabla_{da})^l \xi (z) + \sum_{0 \leq s, t \leq l} B(z) \cdot F_{st} (z) ((\nabla_{da})^s \xi (z), \nabla^t u (z)) \right) \]
where $F_{st} (\cdot, \cdot)$ is some vector-valued monomial, and $B(z)$ is a matrix valued function, both smoothly depending on $z$. There is no derivative of $B$ in the above formula, because for any $l$, $\sigma^l$ is the projection of the tensor $d^l u \in$
the above equation into
\[ (4.4) \quad \pi_{\text{hol}} \left( (\nabla_{du})^l \xi (z) \right) = \left( \nabla'_{du} \right)^l \xi (z), \]
where \( \nabla'_{du} = \pi_{\text{hol}} \nabla_{du} = D_u \partial_{j,J}. \) There is a formula for \( D_u \partial_{j,J} \) and \( D_u \partial'_{j,J} \) nearby \( z_0 \) (see [Si]):
\[ (4.5) \quad D_u \partial'_{j,J} \xi = \overline{\partial} \xi + A(z) \partial \xi + C(z) \xi \]
\[ D_u \partial_{j,J} \xi = \partial \xi + G(z) \overline{\partial} \xi + H(z) \xi \]
where \( A(z), C(z), G(z), H(z) \) are matrix-valued smooth functions, all vanishing at \( z_0. \)

Now we study the solvability of (4.1) and (4.2) by Fredholm alternative. We regard
\[ \Omega_{N-1,p}^{(0,1)} (u^*TM) \times J^k_{\text{hol}} (T_z \Sigma, T_{u(z)} M) \]
as a Banach space with the norm
\[ \| \cdot \|_{N-1,p} + \Sigma_{i=1}^k |\cdot|_i \]
where \(|\cdot|_i\) is any norm induced by an inner product on the 2n-dimensional vector space \( \text{Sym}^{(0,1)}_{j,J} (T_z \Sigma, T_{u(z)} M) \cong \mathbb{C}^n. \)

We denote the natural pairing
\[ \Omega_{N-1,p}^{(0,1)} (u^*TM) \times \left( \Omega_{N-1,p}^{(0,1)} (u^*TM) \right)^* \rightarrow \mathbb{R} \]
by \( \langle \cdot, \cdot \rangle \) and the inner product on \( \text{Sym}^{(0,1)}_{j,J} (T_z \Sigma, T_{u(z)} M) \) by \( \langle \cdot, \cdot \rangle_{z}. \)

Let \( (\eta, \zeta) \in \left( \Omega_{N-1,p}^{(0,1)} (u^*TM) \right)^* \times J^k_{\text{hol}} (T_z \Sigma, T_{u(z)} M) \) for \( \zeta = (\zeta_1, \ldots, \zeta_k) \) such that
\[ (4.6) \quad \langle D_{J,(j,u)} \overline{\partial} (B, (0, \xi)) , (\eta) \rangle + \Sigma_{i=1}^k \langle D_{J,(j,u)} J^l (B, (0, \xi)) (\xi) , (\zeta)_i \rangle_z = 0 \]
for all \( \xi \in \Omega_{N-1,p}^{(0,1)} (u^*TM) \) and \( B \in T_J \mathcal{J}_\omega. \) We want to show \( (\eta, \zeta) = (0, 0). \) The idea is to change the above equation into
\[ \langle D_{J,(j,u)} \overline{\partial} (B, (0, \xi)) , (\eta) \rangle = 0 \]
for all \( \xi \) and \( B \) by judiciously modifying \( \xi \) by a Taylor polynomial nearby \( z \), and then use standard techniques in \( J \)-holomorphic curve theory to show \( \eta = 0 \), and after that use Cauchy integral to show \( \zeta = 0. \) We first deal with \( N = k \) case, and later raise the regularity by ellipticity of Cauchy-Riemann equation.

Let \( \xi = 0 \), then (4.6) becomes
\[ \left\langle \frac{1}{2} B \circ du \circ j, (\eta) \right\rangle = 0. \]

Using the abundance of \( B \in T_J \mathcal{J}_\omega, \) and that \( u \) is a simple \( J \)-holomorphic curve, by standard technique (for example [MS]) we get \( \eta = 0 \) on \( \Sigma \setminus \{ z_0 \} \), namely \( \text{supp} \eta \subset \{ z_0 \}. \) Since \( \eta \in (W^{k,p})^* \), by the structure theorem of distribution with point support (see [GS]), we have
\[ (4.7) \quad \eta = P \left( \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial \overline{z}} \right) \delta_{z_0} \]
where $\delta_{z_0}$ is the delta function supported at $z_0$, and $P$ is a polynomial in two variables with degree $\leq k - 1$: this is because the evaluation at a point of the $k$-th derivative of $W^{k,p}$ maps does not define a continuous functional on $W^{k,p}$.

Let $B = 0$. By (4.3) and (4.7), (4.6) becomes

$$ (D_u \bar{\partial}_j, \xi, \eta) + ( (D_{J,(j,u)} j^{k}_{hol}) \xi (z_0), \zeta )_{z_0} = 0. $$

Since $\xi$ is arbitrary, we can replace $\xi$ by $\xi + \chi(z)h(z, \overline{z})$ in the above identity, where $h(z, \overline{z})$ is a vector-valued polynomial in $z$ and $\overline{z}$, and $\chi(z)$ is a smooth cut-off function equal to 1 in a coordinate neighborhood of $z_0$ and 0 outside a slightly larger neighborhood, so that $\chi(z)h(z, \overline{z})$ is a well defined and smooth on whole $\Sigma$. We want (4.8) becomes $\langle D_u \bar{\partial}_j, \xi, \eta \rangle = 0$ after that replacement. For this purpose the $h(z, \overline{z})$ should satisfy

$$ (D_u \bar{\partial}_j, h, P \left( \frac{\partial}{\partial z} \right)^{\delta_{z_0}} + \left( (D_{J,(j,u)} j^{k}_{hol}) h(z_0), \zeta \right)_{z_0} = - \left( (D_{J,(j,u)} j^{k}_{hol}) \xi (z_0, \overline{z}_0), \zeta \right)_{z_0} $$

After simplification, the above is a differential equation about $h$:

$$ Q \left( \frac{\partial}{\partial z} \right) h(z_0, \overline{z}_0) = w $$

where $Q(s, t)$ is a vector-valued polynomial in two variables $s, t$, $Q \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial \overline{z}} \right)$ acts on $h(z, \overline{z})$ with vector coefficients paired with those of $h$ by inner product, and $w := - \left( (D_{J,(j,u)} j^{k}_{hol}) \xi (z_0), \zeta \right)_{z_0}$ is a constant.

Here comes the crucial observation: when $\zeta_k \neq 0$, the highest degree of $s$ in $Q(s, t)$ is in the term $\zeta_k s^k$. This is because $P(s, t)$ has degree $\leq k - 1$ and after integration by parts, $\frac{\partial}{\partial z}$ can fall at $D_u \bar{\partial}_j, h$ of most $(k - 1)$ times, and in (4.10),

$$ D_u \bar{\partial}_j, h = \overline{h} + A(z) \partial h + C(z) h, $$

where $A(z_0) = 0$. On the other hand, in $\left( (D_{J,(j,u)} j^{k}_{hol}) h \right)$ by (4.3), (4.4), (4.5), the highest derivative for $\frac{\partial}{\partial z}$ is $\left( \frac{\partial}{\partial z} \right)^k$, and is paired with the coefficient $\zeta_k$ in (4.10).

When $\zeta_k \neq 0$, we take $h(z, \overline{z}) = \frac{-\zeta_k}{|\zeta_k|^{k+1}} k \left( z - z_0 \right)^k w$, then $h$ solves (4.10). This is because of the following: $h$ is holomorphic nearby $z_0$, so we can ignore all terms in $Q \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial \overline{z}} \right)$ involving $\frac{\partial}{\partial \overline{z}}$. For the remaining terms in $Q \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial \overline{z}} \right)$, they must be of the form $\left( \frac{\partial}{\partial z} \right)^l$ with $0 \leq l \leq k$, and only $\left( \frac{\partial}{\partial z} \right)^k h(z_0) \neq 0$.

With this $h$, we reduce the cokernel equation to $\langle D_u \bar{\partial}_j, \xi, \eta \rangle = 0$. Since $\eta$ is a weak solution of $\left( D_u \bar{\partial}_j \right)^* \eta = 0$ on $\Sigma$, by ellipticity of the $\left( D_u \bar{\partial}_j \right)^*$ operator, the distribution solution $\eta$ is smooth on $\Sigma$ (See [15]). Since $\eta = 0$ on $\Sigma \setminus \left\{ z_0 \right\}$, $\eta = 0$ on $\Sigma$. Then it is not hard to conclude $\zeta = 0 = 0$ by Cauchy integral formula as in [17] and [18]. Therefore the system of equations (4.1) and (4.2) is solvable for any $\eta \in W^{k,p}$ and $\alpha \in j^k_{hol}(T_{z_0} \Sigma, T_{u(z_0)} M)$.

There is one case left: that is when $\zeta_k = 0$. We still need to show $(\eta, \zeta) = (0, 0)$. If $k = 1$, then $\zeta_1 = 0 \Leftrightarrow \zeta = 0$ so it has been done as above. If $k > 1$, we notice that the cokernel equation (4.4) now is the cokernel equation for the section $D \Upsilon_{k-1}$, since the $k$-th jet is paired with $\zeta_k$ there,

$$ \left( (D_{J,(j,u)} j^k_{hol}) \xi (z_0), \zeta \right)_{z_0} = \left( (D_{J,(j,u)} j^{k-1}_{hol}) \xi (z_0), \zeta \right)_{z_0}. $$

By induction assumption on $k$, $D \Upsilon_{k-1}$ has trivial cokernel hence $(\eta, \zeta) = (0, 0)$.
Last we raise the regularity from $W^{k+1,p}$ to $W^{N,p}$, for any $N > k$. For $\eta \in W^{N-1,p} \subset W^k,p$, by the above argument we can find a solution $\xi \in W^{k+1,p}$ in (4.1). By elliptic regularity, the solution $\xi \in W^{N,p}$. Therefore (4.1) and (4.2) is solvable in $W^{N,p}$ setting. This finishes induction hence the proof of Theorem.

Remark 1. In the above proof, the induction starts from $k = 1$. In [OZ], $k = 1$ case was treated in the framework of 1-jet transversality at $(u, z_0)$ where $du(z_0) = 0$. The above proof includes the $k = 1$ case as well, but the way of choosing $h$ does not rely on $du(z_0) = 0$ and applies to any $z_0$ on $\Sigma$.

Remark 2. It is crucial that we use the holomorphic $k$-jet bundle instead of the usual $k$-jet bundle to get the surjective property of $D\Gamma_k$. Otherwise, as the usual jet evaluation involves mixed derivatives, given $\zeta_{(k,0)} = 0$ we cannot reduce the cokernel equation to the $(k-1)$ case by induction, and when $k = 1$, $\zeta_{(1,0)} = 0$ does not imply $\xi = 0$. In the case $k = 1$, we can explicitly see why this submersion property fails in the usual 1-jet bundle: for a $J$-holomorphic curve $u$ with $du(z_0) = 0$, and $\Gamma_1 = (\overline{\partial}, j^1) : \mathcal{F}_1(\Sigma, M; \beta) \times \mathcal{J}_\omega \to \mathcal{H}'' \times J^1(\Sigma, M)$, calculations in [OZ] yield

$$D\Gamma_1(\xi, B) = (D_u\overline{\partial}_{j,J} \xi, D_u\overline{\partial}_{j,J} \xi(z_0), D_u\partial_{j,J} \xi(z_0))$$

therefore there is no solution for $\eta, \alpha(0,1), \alpha(1,0)$ if $\eta(z_0) \neq \alpha(0,1)$.

However, if $du(z_0) \neq 0$ then the surjective property still holds in the usual jet bundles. More precisely we have the following

Theorem 2. At any $J$-holomorphic curve $((u, j), z_0, J) \in \mathcal{M}_1^J(\Sigma, M; \beta) \subset \mathcal{F}_1(\Sigma, M; \beta) \times \mathcal{J}_\omega$ with $du(z_0) \neq 0$, the linearization $D\Gamma_k$ of the section

$$\Gamma_k = (\overline{\partial}, j^k) : \mathcal{F}_1(\Sigma, M; \beta) \times \mathcal{J}_\omega \to \mathcal{H}'' \times J^k(\Sigma, M)$$

is a surjective. Especially the linearization $Dj^k$ of $k$-jet evaluation

$$j^k : \mathcal{F}_1(\Sigma, M; \beta) \times \mathcal{J}_\omega \to J^k(\Sigma, M)$$

at $((u, j), z_0, J)$ is surjective.

Proof. It is enough to show that the cokernel equation

$$\left< D_u\overline{\partial}_{j,J} \xi + \frac{1}{2} B \circ du \circ j, \eta \right> + (D_{j,J(u)} j^k) \xi(z_0), \zeta \right>_{z_0} = 0, \text{ for all } \xi, B$$

only has trivial solution $(\eta, \zeta) = (0, 0)$. To do this, using standard argument in [MS] we again get $\supp \eta \subset \{z_0\}$. Given $\zeta \in J^k(T_{z_0} \Sigma, T_{u(z_0)} M)$, by Taylor polynomial we can construct a smooth $\xi$ supported in arbitrarily small neighborhood of $z_0 \in \Sigma$, such that $(D_{J(u)} j^k) \xi(z_0) = \zeta$. When $du(z_0) \neq 0$, by linear algebra (namely the abundance of $T_j \mathcal{J}_\omega$) and perturbation method we can construct $B \in T_J \mathcal{J}_\omega$ such that $D_u \overline{\partial}_{j,J} \xi + \frac{1}{2} B \circ du \circ j = 0$ on $\Sigma$ (see [MS]). So we get from the cokernel equation that $0 + |\zeta|^2 = 0$, i.e. $\zeta = 0$. Let $B = 0$ in the cokernel equation, we get $\left< D_u\overline{\partial}_{j,J} \xi, \eta \right> = 0$ for all $\xi$. Then by elliptic regularity we conclude $\eta = 0$ on the whole $\Sigma$.\]

The following theorem is a direct consequence of Theorem 1 by applying Sard-Smale theorem.
Theorem 3. Let $S$ be any smooth section of the holomorphic $k$-jet bundle $J^k_{hol}(\Sigma, M) \rightarrow \mathcal{F}_1(\Sigma, M; \beta) \times \mathcal{J}_\omega$. Then the section $\Upsilon_k$ is transversal to the section $(0, S)$. The moduli space

$$\mathcal{M}_S := (J^k_{hol})^{-1}(S) \cap \mathcal{M}_1^* (\Sigma, M; \beta) = \Upsilon_k^{-1}(0, S)$$

is a Banach submanifold of codimension $2kn$ in $\mathcal{M}_1^* (\Sigma, M; \beta)$. Under the natural projection $\pi : \mathcal{F}_1(\Sigma, M; \beta) \times \mathcal{J}_\omega \rightarrow \mathcal{J}_\omega$, there exists $\mathcal{J}_{reg} \subset \mathcal{J}_\omega$ of second category, such that for any $J \in \mathcal{J}_{reg}$, the moduli space $\mathcal{M}_J^2 := \mathcal{M}_S \cap \pi^{-1}(J)$ is a smooth manifold in $\mathcal{M}_1^* (\Sigma, M; J; \beta)$, with dimension

$$\dim \mathcal{M}_J^2 = \dim \mathcal{M}_1^* (\Sigma, M; J; \beta) - 2kn,$$

and all the elements in $\mathcal{M}_J^2$ are Fredholm regular.

Remark 3. In [Oh], the $S$ is the zero section of the holomorphic $k$-jet bundle $J^k_{hol}(\Sigma, M)$, so $\mathcal{M}_J^2$ is the set of $J$-holomorphic curves with prescribed ramification degrees at the marked points. The $J$-holomorphic curves in our moduli space $\mathcal{M}_J^2$ can obey more general constraint $S$. Similar to [Oh], the theorem also has the version with more than one marked point. Also the constant $S$ need not to be a full section over the base, but only a closed submanifold in $J^k_{hol}(\Sigma, M)$ whose tangent space projects onto the horizontal distribution of the bundle $J^k_{hol}(\Sigma, M) \rightarrow \mathcal{F}_1(\Sigma, M; \beta) \times \mathcal{J}_\omega$, because the essential part in the proof the theorem is that $DY_k|_{(0, S)}$ is surjective.

The theorem appears to be a good start of studying moduli spaces of $J$-holomorphic curves satisfying general jet constraints in the holomorphic jet bundle; for example, moduli spaces of $J$-holomorphic curves with self tangency. Also in [CM], jet constraints from symplectic hypersurfaces were used to get rid of multicovering bubbling spheres. This enables them to define genus zero Gromov-Witten invariants without abstract perturbations.

The above theorem tells that the moduli spaces $\mathcal{M}_J^2$ are well-behaved, and the $\{J_t\}_{0 \leq t \leq 1}$ family version of the above theorem tells that they are cobordant to each other by moduli spaces $\{\mathcal{M}_J^2\}_{0 \leq t \leq 1}$ for generic path $J_t \subset \mathcal{J}_\omega$. It is interesting to see if the moduli spaces $\mathcal{M}_J^2$ can be used to construct new symplectic invariants.

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DEPARTMENT OF MATHEMATICS, THE CHINESE UNIVERSITY OF HONG KONG, SHATIN, HONG KONG

E-mail address: kzhu@math.cuhk.edu.hk