Open Wilson Lines and Generalized Star Product
in Noncommutative Scalar Field Theories

Youngjai Kiem $^a$, Soo-Jong Rey $^{b,c}$, Haru-Tada Sato $^a$, Jung-Tay Yee $^{b,c}$

BK21 Physics Research Division & Institute of Basic Science
Sungkyunkwan University, Suwon 440-746 KOREA $^a$

School of Physics & Center for Theoretical Physics
Seoul National University, Seoul 151-747 KOREA $^b$

Centre Emil Borel, Institut Henri Poincaré
11, rue Pierre et Marie Curie, Paris F-75231 FRANCE $^c$

ykiem, haru@newton.skku.ac.kr sjrey@gravity.snu.ac.kr jungtay@phya.snu.ac.kr

abstract

Open Wilson line operators and generalized star product have been studied extensively in noncommutative gauge theories. We show that they also show up in noncommutative scalar field theories as universal structures. We first point out that dipole picture of noncommutative geometry provides an intuitive argument for robustness of the open Wilson lines and generalized star products therein. We calculate one-loop effective action of noncommutative scalar field theory with cubic self-interaction and show explicitly that the generalized star products arise in the nonplanar part. It is shown that, at low-energy, large noncommutativity limit, the nonplanar part is expressible solely in terms of the scalar open Wilson line operator and descendants.

---

$^1$Work supported in part for SJR and JTY by BK-21 Initiative in Physics through (SNU-Project 2), KOSEF Interdisciplinary Research Grant 98-07-02-07-01-5, and KOSEF Leading Scientist Program 2000-1-11200-001-1, for HTS by KOSEF Brain-Pool Program, and for YK by KRF Grant 2001-015-DP0082.
1 Introduction and Summary

One of the most salient features of noncommutative field theories is that physical excitations are described by ‘dipoles’ — weakly interacting, nonlocal objects. Denote their center-of-mass momentum and dipole moment as \( k \) and \( \Delta x \), respectively. According to the ‘dipole’ picture, originally developed in [1] and more recently reiterated in [2], the two are related each other:

\[
\Delta x^a = \theta^{ab} k_b. \tag{1}
\]

Here, \( \theta^{ab} \) denotes the noncommutativity parameter:

\[
\{ x^a, x^b \}_\star = i \theta^{ab}, \tag{2}
\]

in which \( \{ \cdot \}_\star \) refers to the Moyal commutator, defined in terms of the \( \star \)-product:

\[
\{ A(x_1)B(x_2) \}_\star := \exp \left( \frac{i}{2} \partial_1 \land \partial_2 \right) A(x_1)B(x_2) \quad \text{where} \quad \partial_1 \land \partial_2 := \theta^{ab} \partial_a \partial_b. \tag{3}
\]

Evidently, in the commutative limit, the dipoles shrink in size and represent pointlike excitations.

In noncommutative gauge theories, (part of) the gauge orbit is the same as the translation along the noncommutative directions [3, 4, 5]. For example, in noncommutative U(1) gauge theory, the gauge potential \( A_\mu(x) \) and the neutral scalar field \( \Phi(x) \), both of which give rise to ‘dipoles’, transform in ‘adjoint’ representation:

\[
\delta_\epsilon A_\mu(x) = i \int \frac{d^2 k}{(2\pi)^2} \tilde{\epsilon}(k) \left[ (A_\mu(x + \theta \cdot k) - A_\mu(x - \theta \cdot k)) + ik_\mu \right] e^{ik \cdot x},
\]

\[
\delta_\epsilon \Phi(x) = i \int \frac{d^2 k}{(2\pi)^2} \tilde{\epsilon}(k) \left[ \Phi(x + \theta \cdot k) - \Phi(x - \theta \cdot k) \right] e^{ik \cdot x}, \tag{4}
\]

where the infinitesimal gauge transformation parameter is denoted as

\[
\epsilon(x) = \int \frac{d^2 k}{(2\pi)^2} e^{ik \cdot x} \tilde{\epsilon}(k).
\]

For fields transforming in ‘adjoint’ representations under the noncommutative gauge group, it has been shown that the only physical observables are the ‘open Wilson lines’ [3, 4, 5] along an open contour \( C \):

\[
W_k[C] = \mathcal{P} \int d^2 x \exp \left[ i \int_0^1 dt \dot{y}(t) \cdot A(x + y(t)) \right] e^{ik \cdot x}, \tag{5}
\]

and their descendant operators. The \( \star \)-product refers to the base point \( x \) of the open contour \( C \). Despite being defined over an open contour, the operator is gauge-invariant provided the momentum \( k \) is related to the geodesic distance \( y(1) - y(0) := \Delta x \) precisely by the ‘dipole
relation’, Eq.(1)). In other words, in noncommutative gauge theory, the open Wilson lines (physical observables) ought to obey the dipole relation Eq.(1) as an immediate consequence of the gauge invariance! For a straight Wilson line, expanding Eq.(5) in successive powers of the gauge field $A_\mu$, it was observed \[7, 8\] that generalized $\star$-product, $\star_n$, a structure discovered first in \[9, 10\] emerge:

$$W_k[C] = \int \! d^2x \left[ 1 - (\partial \wedge A) + \frac{1}{2!} (\partial \wedge A)^2 + \cdots \right] \star e^{ik \cdot x}.$$  \hspace{1cm} (6)

Apparently, the generalized $\star_n$ products give rise to different algebraic structures from Moyal’s $\star$-product. For instance, the first two, $\star_2, \star_3$ defined as:

\begin{align*}
[A(x_1)B(x_2)]_{\star_2} &:= \sin\left(\frac{1}{2} \partial_1 \wedge \partial_2\right) A(x_1)B(x_2) \\
[A(x_1)B(x_2)C(x_3)]_{\star_3} &:= \frac{\sin\left(\frac{1}{2} \partial_2 \wedge \partial_3\right) \sin\left(\frac{1}{2} \partial_1 \wedge (\partial_2 + \partial_3)\right)}{\frac{1}{2} (\partial_1 + \partial_2) \wedge \partial_3} + (1 \leftrightarrow 2) A(x_1)B(x_2)C(x_3)
\end{align*}

show that the $\star_n$’s are commutative but non-associative. Even though the definition of the open Wilson line is given in terms of path-ordered $\star$-product, its expansion in powers of the gauge potential involves the generalized $\star_n$-product at each $n$-th order. The complicated $\star_n$ products have arisen upon expansion in powers of the gauge potential, and are attributable again to dipole nature of the Wilson line and the gauge invariance thereof – each term in Eq.(6) is not gauge invariant, as the gauge transformation Eq.(4) mixes terms involving different $\star_n$’s. Indeed, the generalized $\star_n$ products are not arbitrary but obey recursive identities:

\begin{align*}
i [\partial_x A(x) \wedge \partial_x B(x)]_{\star_2} &= \{A(x), B(x)\}_{\star} \\
i \partial_x \wedge [A(x)B(x)\partial_x C(x)]_{\star_3} &= A(x) \star_2 \{B(x), C(x)\}_{\star} + B(x) \star_2 \{A(x), C(x)\}_{\star}. \hspace{1cm}
\end{align*}

These recursive identities are crucial for ensuring gauge invariance of the power-series expanded open Wilson line operator, Eq.(6).

The open Wilson lines and the generalized $\star$-products therein have been studied extensively in literatures, largely in the context of noncommutative gauge theories. On the other hand, the dipole relation Eq.(1) ought to be a universal relation, applicable for any theories defined over noncommutative spacetime. The aforementioned remark that noncommutative gauge invariance plays a prominent role in both deriving the relation Eq.(1) and the recursive relation among the generalized $\star$-products then raises an immediate question. Are these structures present also in noncommutative field theories, in which neither gauge invariance nor gauge field exists?

In this paper, we show that the answer to the above question is affirmatively yes by studying $d$-dimensional massive $\lambda [\Phi^3]_{\star}$ theory. We compute the one-loop effective action and find that
nonplanar contributions are expressible in terms of the generalized $\ast$-product. Most signifi-

cantly, at low-energy and large noncommutativity limit, we show that the nonplanar part of
the one-loop effective action can be resummed into a remarkably simple expression

$$
\Gamma_{np}[\Phi] = \frac{\hbar}{2} \int \frac{d^d k}{(2\pi)^d} W_k[\Phi] \tilde{K}_{-\frac{d}{2}}(k \circ k) W_{-k}[\Phi],
$$

where $W_k[\Phi]$ denotes the first descent of the scalar Wilson line operators:

$$
W_k[\Phi] := \mathcal{P}\int d^2 x \exp \left( -g \int_0^1 dt |\dot{y}(t)| \Phi(x + y(t)) \right) * e^{i k \cdot x}
$$

$$
(\Phi W)_k[\Phi] := \mathcal{P}\int d^2 x \left( \int_0^1 dt |\dot{y}(t)| \Phi(x + y(t)) \right) \exp \left( -g \int_0^1 dt |\dot{y}(t)| \Phi(x + y(t)) \right) * e^{i k \cdot x}
$$

$$
= \left( -\frac{\partial}{\partial g} \right) W_k[\Phi]
$$

$$
\ldots
$$

$$
(\Phi^n W)_k[\Phi] := \left( -\frac{\partial}{\partial g} \right)^n W_k[\Phi], \quad (g := \frac{\lambda}{4m})
$$

Here, $\tilde{K}_{-\frac{d}{2}}$ denotes a (Fourier-transformed) kernel and $k \circ k := (\theta \cdot k)^2$, etc. We trust that
the above result bears considerable implications to the noncommutative solitons studied in [11, 12]
and to the UV/IR mixing discovered in [13], and will report the details elsewhere.

2 Scalar Open Wilson Lines as Dipoles

2.1 Dipole Relation for Scalar Fields

We begin with a proof that the dipole relation Eq.(1) holds for scalar fields as well, wherein no
gauge invariance is present. Recall that, in noncommutative spacetime, energy-momentum is a
good quantum number, as the noncommutative geometry is invariant under translation along
the noncommutative directions. As such, consider the following set of operators, so-called Parisi
operators [14]:

$$
\mathcal{O}_n(x_1, \cdots, x_n; k) = \int d^2 z \Phi_1(x_1 + z) * \Phi_2(x_2 + z) * \cdots * \Phi_n(x_n + z) * e^{i k \cdot x},
$$

viz. Fourier-transform of a string of elementary fields, $\Phi_k(x)$ ($k = 1, 2, \cdots$). Consider the
one-point function:

$$
G_1(x, k) = \mathcal{O}_2(x, k) = \langle \int d^2 z \Phi(z) * \Phi(x + z) * e^{i k \cdot x} \rangle.
$$

In terms of Fourier decomposition of the scalar field:

$$
\Phi(x) = \int \frac{d^2 k}{(2\pi)^2} e^{i k \cdot x} \Phi(k),
$$

3
one obtains that
\[ G_1(x, k) = \int \frac{d^2l}{(2\pi)^2} \Phi(l) \Phi(-l + k) \exp \left[ il \cdot \left(x + \frac{1}{2} \theta \cdot k\right)\right]. \]

Consider ‘wave-packet’ of the scalar particle, \( \Phi(z) = \Phi_0 \delta^{(2)}(z) \) and \( \Phi(x + z) = \Phi_0 \delta^{(2)}(x + z) \) so that \( \tilde{\Phi}(l) = \Phi_0 \exp(il \cdot x) \). From the above equation, one then finds that
\[ G_1(x, k) = \Phi_0^2 \delta^{(2)} \left(x + \frac{1}{2} \theta \cdot k\right). \]
Thus, one finds that the stationary point of the correlator is given by
\[ \Delta x^a \sim \theta^{ab} k_b \]
and hence precisely by the ‘dipole relation’, Eq. (10).

### 2.2 Scalar Open Wilson Lines and Generalized Star Products

First, as in the case of the gauge open Wilson lines, Eq.(5), we will show that power-series expansion of the scalar open Wilson line operators Eq.(9) gives rise to generalized \( \ast_n \) products, and take this as the definition of the products for ‘adjoint’ scalar fields.

Begin with a remark concerning symmetry of the open Wilson line operators. The open Wilson line operator Eq.(5) possesses reparametrization invariance:
\[ t \rightarrow \tau(t) \quad \text{such that} \quad \tau(0) = 0, \tau(1) = 1 \quad \text{and} \quad \dot{\tau}(t) > 0, \]
as the gauge-element \( dy(t) \cdot A(x + y) \) is invariant under the reparametrization, Eq.(11):
\[ dt \dot{y}(t) \cdot A(x + y(t)) = \left(d\tau |\dot{\tau}(t)|^{-1}\right) (\dot{\tau}(t) \dot{y}(\tau)) \cdot A(x + y(\tau)) = d\tau \dot{y}(\tau) \cdot A(x + y(\tau)). \]
Likewise, the scalar open Wilson line operator Eq.(9) possesses reparametrization invariance, as the scalar-element \( |dy(t)|\Phi(x + y) \) is invariant under the reparametrization, Eq.(11):
\[ dt |\dot{y}(t)|\Phi(x + y(t)) = \left(d\tau |\dot{\tau}(t)|^{-1}\right) ( |\dot{\tau}(t) \dot{y}(\tau)|) \Phi(x + y(\tau)) = d\tau |\dot{y}(\tau)|\Phi(x + y(\tau)). \]
In fact, the reparametrization invariance, along with rotational invariance on the noncommutative subspace and reduced Lorentz invariance on the complement spacetime, puts a powerful constraint enough to determine uniquely structure of the scalar open Wilson line operator, Eq.(3), much as for that of the gauge open Wilson line operator, Eq.(3).

For illustration, we will examine a simplified form of the scalar open Wilson line with an insertion of a local operator \( O \) at a location \( R \) on the Wilson line:
\[ (O_R W)_k[\Phi] := \mathcal{P}_t \int d^2x O(x + R) \ast \exp \left(-g \int_0^1 dt |\dot{y}(t)|\Phi(x + y(t))\right) \ast e^{ik \cdot x}. \]
Take a straight Wilson line:
\[ y(t) = Lt \] where \[ L^a = \theta^{ab}k_b := (\theta \cdot k)^a, \quad L := |L|, \]
corresponding to a uniform distribution of the momentum \( k \) along the Wilson line. As the path-ordering progresses to the right with increasing \( t \), power-series expansion in \( gL\Phi \) yields:
\[
(\mathcal{O}_R W)_k[\Phi] = \int d^2x \mathcal{O}(x + R) * e^{ik \cdot x} \\
+ (-gL) \int d^2x \int_0^1 dt \mathcal{O}(x + R) * \Phi(x + Lt) * e^{ik \cdot x} \\
+ (-gL)^2 \int d^2x \int_0^1 dt_1 \int_0^1 dt_2 \mathcal{O}(x + R) * \Phi(x + Lt_1) * \Phi(x + Lt_2) * e^{ik \cdot x} \\
+ \cdots.
\]
Each term can be evaluated, for instance, by Fourier-transforming \( \mathcal{O} \) and \( \Phi \)'s:
\[
\mathcal{O}(x) = \int \frac{d^2k}{(2\pi)^2} \tilde{\mathcal{O}}(k) T_k, \quad \Phi(x) = \int \frac{d^2k}{(2\pi)^2} \tilde{\Phi}(k) T_k \quad \text{where} \quad T_k = e^{ik \cdot x},
\]
taking the \(*\)-products of the translation generators \( T_k \)'s
\[
T_k * T_1 = e^{ik \cdot 1} T_{k+1},
\]
and then evaluating the parametric \( t_1, t_2, \cdots \) integrals. Fourier-transforming back to the configuration space, after straightforward calculations, one obtains
\[
(\mathcal{O}_R W)_k[\Phi] = \int d^2x e^{ik \cdot x} \mathcal{O}(x + R) \\
+ (-gL) \int d^2x e^{ik \cdot x} [\mathcal{O}(x + R)\Phi(x)]_{*2} \\
+ \frac{1}{2!}(-gL)^2 \int d^2x e^{ik \cdot x} [\mathcal{O}(x + R)\Phi(x)\Phi(x)]_{*3} \\
+ \cdots,
\]
where the products involved, \( *_2, *_3, \cdots \), are precisely the same generalized star products as those appearing in the gauge open Wilson line operators.

In fact, for the straight open Wilson lines, one can take each term of the Taylor expansion in Eq.(12) as the definition of the generalized \( *_n \) products. Denote the exponent of the scalar open Wilson line operator as:
\[
\Phi_\ell (x, k) := \int_0^1 dt_\ell |\theta \cdot k|\Phi(x + \theta \cdot kt_\ell).
\]
We have taken the open Wilson lines as straight ones. Then, analogous to [10], the generalized \( *_n \) products are given by:
\[
[\Phi *_n \Phi]_k := \mathcal{P}_l \int d^d x_1 \Phi_1(x, k) * \Phi_2(x, k) * \cdots * \Phi_n(x, k) * e^{ik \cdot x} \\
= \int d^d x \Phi *_n(x) * e^{ik \cdot x}.
\]
3 One-Loop Effective Action in $\lambda[\Phi^3]_*$ Theory

Given the above results, we assert that the scalar open Wilson line operators play a central role in noncommutative field theories. To support our assertion, we now show that, for the case of $\lambda[\Phi^3]_*$ scalar field theory, the effective action is expressible entirely in terms of the open Wilson line operators and nothing else. Begin with the classical action

$$S_{NC} = \int d^dx \left[ -\frac{1}{2} \left( \partial_\mu \Phi \right)^2 - \frac{1}{2} m^2 \Phi^2 - \frac{\lambda}{3!} \Phi \Phi \Phi \right].$$

At one-loop order, the effective action can be obtained via the background field method—expand the scalar field around a classical configuration, $\Phi = \Phi_0 + \varphi$,

$$S_{NC} = \int d^dx \frac{1}{2} \varphi(x) \left[ -\partial_\mu^2 - m^2 - \lambda \Phi_0(x) \right] \加重 \varphi(x)$$

and then integrate out the quantum fluctuation, $\varphi$. Fourier transform yields the interaction vertex of the form:

$$iS_{int} = -i \frac{\lambda}{2} \int \frac{d^dk_a}{(2\pi)^d} \tilde{\varphi}(k_1) \tilde{\Phi}_0(k_2) \tilde{\varphi}(k_3) \left[ e^{-\frac{2}{\lambda} k_1 \wedge k_2} + e^{\frac{2}{\lambda} k_1 \wedge k_2} \right] (2\pi)^d \delta^{(d)}(k_1 + k_2 + k_3) = (U + T),$$

under the rule that $\varphi$’s are Wick-contracted in the order they appear in the Taylor expansion of $\exp(iS_{int})$. The $U$ and $T$ parts refer to the untwisted and the twisted vertex interactions, respectively. In Eq.(16), relative sign between $T$ and $U$ is $+$, opposite to those for gauge fields. This can be traced to the fact that, under ‘time-reversal’ $t \rightarrow (1 - t)$, the exponents in gauge and scalar open Wilson lines, Eq.(13) and Eq.(9), transform as even and odd, respectively. The N-point correlation function is given by $(iS_{int})^N/N!$ term in the Taylor expansion and, as shown by Filk [16], consists of planar and non-planar contributions. In the binomial expansion of $(iS_{int})^N = (U + T)^N$, the two terms, $[U]^N$ and $[T]^N$, yield the planar contribution, while the rest comprises the non-planar contribution. Evaluation of the N-point correlation function is straightforward. Summing over $N$, the one-loop effective action is given as:

$$\Gamma[\Phi_0] := -\frac{\hbar}{2} \ln \det \left[ -\partial_\mu^2 - m^2 - \lambda \Phi_0(x) \right] + n$$

$$= \sum_{N=1}^\infty \int d^dq \prod_{\ell=1}^N \frac{d^dq}{(2\pi)^d} \tilde{\Phi}_0(q_1) \cdots \tilde{\Phi}_0(q_N) \Gamma^{(N)} [q_1, \cdots, q_N].$$

Here, the N-point correlation function takes schematically the following form:

$$\Gamma^{(N)} [q_1, \cdots, q_N] = (2\pi)^d \delta^{(d)}(q_1 + \cdots + q_N) S_N \cdot A(q_1, \cdots, q_N) \cdot B(q_1, \cdots, q_N),$$

where $S_N$ refers to the combinatorics factor, $A$ is the Feynman loop integral, and $B$ consists of the noncommutative phase factors.

Technical details of the following results will be reported in a separate paper [13].
3.1 Effective Action: Non-planar Part

We are primarily interested in the nonplanar part. After a straightforward algebra, we have obtained the nonplanar part as a double sum involving the generalized star products:

\[ \Gamma_{np} = \hbar \sum_{N=2}^{\infty} \left( -\frac{\lambda}{4} \right)^N \frac{1}{N!} \int d^d x \left( \frac{1}{2} \sum_{n=1}^{N-1} \binom{N}{n} \left[ \Phi_0 \ast_n \Phi_0 \right](x) \cdot \mathcal{K}_{N-n} \cdot \left[ \Phi_0 \ast_{N-n} \Phi_0 \right](x) \right). \]

Here, \( \left[ \Phi_0 \ast_n \Phi_0 \right](x) \) refers to the generalized \( \ast_n \) products of \( \Phi_0 \)'s and the combinatoric factor \( \frac{1}{2} \binom{N}{n} \) originates from the binomial expansion of \( (U + T)^N \) modulo an inversion symmetry, corresponding to Hermitian conjugation. The kernel \( \mathcal{K}_n \) is defined as

\[ \mathcal{K}_n := \mathcal{K}_n(-\partial_x \circ \partial_x) \quad \text{where} \quad \mathcal{K}_n(|z|^2) = 2 \left( \frac{1}{2\pi} \right)^{\frac{d}{4}} \left( \frac{|z|}{m} \right)^n K_n(m|z|) \quad (18) \]

in terms of the modified Bessel functions, \( K_n \).

To proceed further, we will be taking the low-energy, large noncommutativity limit:

\[ q_\ell \sim \epsilon, \quad \text{Pf} \theta \sim \frac{1}{\epsilon^2} \quad \text{as} \quad \epsilon \to 0 \quad (19) \]

so that

\[ q_\ell \cdot q_m \sim O(\epsilon^2) \to 0, \quad q_\ell \wedge q_m \to O(1), \quad q_\ell \circ q_m \sim O(\epsilon^{-2}) \to \infty. \quad (20) \]

In this limit, the modified Bessel function \( K_n \) exhibits the following asymptotic behavior:

\[ K_n(mz) \to \sqrt{\frac{\pi}{2mz}} e^{-m|z|} \left[ 1 + O \left( \frac{1}{m|z|} \right) \right]. \quad (21) \]

Most importantly, the asymptotic behavior is independent of the index \( n \). Hence, in the low-energy limit, the Fourier-transformed kernels, \( \tilde{\mathcal{K}}_n \)'s obeys obey the following recursive relation:

\[ \tilde{\mathcal{K}}_{n+1}(k \circ k) = \left( \frac{\theta \cdot k}{m} \right) \tilde{\mathcal{K}}_n(k \circ k), \quad (22) \]

viz.

\[ \tilde{\mathcal{K}}_n(k \circ k) = \left( \frac{\theta \cdot k}{m} \right)^n \tilde{Q}(k \circ k). \quad (23) \]

Here, the kernel \( \tilde{Q} \) is given by:

\[ \tilde{Q}(k \circ k) = (2\pi)^{(1-d)/2} \left( \frac{1}{m \theta \cdot k} \right)^{1/2} \exp \left( -m|\theta \cdot k| \right). \quad (24) \]

Note that, in power series expansion of the effective action, natural expansion parameter is \( |\theta \cdot k| \).

\[ ^3\text{As will be shown momentarily, this implies that manifestly reparametrization invariant open Wilson line operator Eq. (6) is more fundamental than those defined in [17] and utilized in [18, 19, 20].} \]
Thus, the nonplanar one-loop effective action in momentum space is expressible as:

\[
\Gamma_{np}[\Phi] = \frac{\hbar}{2} \int \frac{d^dk}{(2\pi)^d} \tilde{K}_\Phi (k \circ k) \sum_{N=2}^{\infty} \sum_{n=1}^{N-1} \left( -\frac{\lambda}{4m} \right)^N \\
\times \left( \frac{1}{n!} |\theta \cdot k|^n \left[ \bar{\Phi} \ast_n \Phi \right]_k \right) \left( \frac{1}{(N-n)!} |\theta \cdot k|^{N-n} \left[ \bar{\Phi} \ast_{N-n} \Phi \right]_{-k} \right).
\] (25)

Utilizing the relation between the generalized \(*_n\) products and the scalar open Wilson line operators, as elaborated in section 2, the nonplanar one-loop effective action can be summed up into a remarkably simple closed form. Denote the rescaled coupling parameter as \(g := \lambda / 4m\) (see Eq.(9)). Exploiting the exchange symmetry \(n \leftrightarrow (N-n)\), one can rearrange the summations into decoupled ones over \(n\) and \((N-n)\). Moreover, because of \([\bar{\Phi} \ast_0 \Phi]_k = (2\pi)^d \delta^{(d)}(k)\), the summations can be extended to \(n = 0, (N-n) = 0\) terms, as they yield identically vanishing contribution after \(k\)-integration. One finally obtains:

\[
\Gamma_{np}[\Phi] = \frac{\hbar}{2} \int \frac{d^dk}{(2\pi)^d} W_k[\Phi] \cdot \tilde{K}_\Phi (k \circ k) \cdot W_{-k}[\Phi],
\]

yielding precisely the aforementioned result, Eq.(7).

### 3.2 Explicit Calculations

To convince the readers that the expression Eq.(7) is indeed correct, we will evaluate below the simplest yet nontrivial correlation functions: \(N = 3, 4\). Utilizing the factorization property \([21]\), begin with nonplanar \(N\)-point correlation functions, in which one of the external legs is twisted. Denoting the twisted vertex as the \(N\)-th, the relevant Feynman diagram is given in Fig.(1): Evaluating Fig.(1) explicitly, one obtains the nonplanar one-loop correlation functions of the form Eq.(17). The Feynman loop integral \(\mathcal{A}\) is independent of the noncommutativity and hence has the same form as in commutative counterpart. In the parametrization of the internal and the external momenta as in Fig.(1), after Wick rotation to Euclidean spacetime, the loop integral is given by

\[
\mathcal{A}(q_1, \cdots, q_N) = \int \frac{d^dk}{(2\pi)^d} \prod_{\ell=0}^{N-1} \frac{1}{(k + q_1 + \cdots + q_\ell)^2 + m^2}.
\] (26)

Introduce the Schwinger-Feynman parametrization for each propagator. This leads to

\[
\mathcal{A} = \int \frac{d^dk}{(2\pi)^d} \int_0^\infty \prod_{\ell=1}^{N} d\alpha_\ell \exp \left[ - \alpha(m^2 + k^2) - 2 \left( \sum_{m=1}^{N-1} \alpha_{m+1}(q_1 + \cdots + q_m) \right) \cdot k + \mathcal{O}(q^2) \right],
\]

where \(\alpha := \sum_{\ell=1}^{N} \alpha_\ell\) serves as the moduli measuring perimeter of the one-loop graph. As we are mainly concerned with the low-energy, large noncommutativity limit, Eqs.(19, 20), we will drop terms of order \(\mathcal{O}(q^2)\) in what follows.

\(^4\) We assume that the noncommutativity is turned only on two-dimensional subspace.
The Moyal phase factor, $\mathcal{B}$, is extractible from the $\mathbf{U}$ and $\mathbf{T}$ factors for untwisted and twisted interaction vertices, respectively. One finds, in counter-clockwise convention for Wick contraction,

$$
\mathcal{B}(q_1, \cdots, q_N) = \exp \left[ ik \wedge \left( \sum_{\ell=1}^{N-1} q_\ell \right) \right] \exp \left[ \frac{i}{2} \sum_{\ell < m} q_\ell \wedge q_m \right].
$$

The integrand in $\mathcal{A}$ is simplified once the loop momentum variable is shifted as:

$$
k^\mu \longrightarrow k^\mu - \frac{1}{\alpha} \left[ \sum_{m=1}^{N-1} \alpha_{m+1} (q_1 + \cdots + q_m) \right].
$$

Change the the Schwinger-Feynman moduli parameters into those for the ordered interaction vertices around the circumference of the one-loop diagram:

$$
\alpha_\ell \longrightarrow \alpha (x_{\ell-1} - x_\ell) \quad \text{where} \quad (1 > x_1 > x_2 > \cdots > x_{N-1} > 0).
$$

Accordingly, $\mathcal{B}$ is multiplied by a moduli-dependent phase factor:

$$
\mathcal{B}(q_1, \cdots, q_N) \longrightarrow \mathcal{B}(q_1, \cdots, q_N) \times \exp \left( -i \sum_{\ell < m} (x_\ell - x_m) q_\ell \wedge q_m \right),
$$

and hence mixes with $\mathcal{A}$ through the $(N-1)$ moduli-parameter integrations.

The loop momentum and overall moduli integrals can be calculated explicitly. The loop momentum integral yields:

$$
\int \frac{d^d k}{(2\pi)^d} \exp \left( -\alpha k^2 + iq_N \wedge k \right) = \left( \frac{1}{4\pi\alpha} \right)^\frac{d}{2} \exp \left( -\frac{q_N \circ q_N}{4\alpha} \right), \quad \text{(27)}
$$
while the $\alpha$-moduli integral yields:

$$
\int_0^\infty d\alpha \alpha^{N-1-\frac{d}{2}} \exp \left[ -m^2 \alpha - \frac{q_N \circ q_N}{4\alpha} \right] = 2^{4-N+1} \left( \frac{q_N \circ q_N}{m^2} \right)^{\frac{N}{2} - 4} K_{4-N} \left( m | q_N \circ q_N | \right). \quad (28)
$$

The remaining integrals over $(N-1)$ moduli parameters are given by:

$$
\tilde{B}_{\text{total}}^{(N)} = \exp \left( \frac{i}{2} \sum_{\ell<m}^{N-1} q_{\ell} \wedge q_m \right) \int_0^1 dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{N-2}} dx_{N-1} \exp \left( -i \sum_{\ell<m}^{N-1} (x_\ell - x_m) q_{\ell} \wedge q_m \right) + \text{(permutations)}.
$$

The (permutations) part refers to $(N-1)!$ diagrams of the same sort, differing from one another by permutation of the $(N-1)$ untwisted interaction vertices. Half of these diagrams are Hermitian conjugates of the other, corresponding to topological reversal between the inner and the outer sides. The complete nonplanar $N$-point correlation function with a single twisted vertex is given by all possible cyclic permutation of the twisted interaction vertex with the $(N-1)$ untwisted ones.

For $N=3$ correlation function, two diagrams of ordering $(1-2-3)$ and $(2-1-3)$, in which $3$ refers to the twisted vertex, contribute. The result is:

$$
\tilde{B}_{\text{total}}^{(3)} = e^{\frac{i}{2} q_1 \wedge q_2} \int_0^1 dx \int_0^x dy \exp \left( -i (x - y) q_1 \wedge q_2 \right) + (1 \leftrightarrow 2) = \sin \left( \frac{q_1 \wedge q_2}{2} \right) / \left( \frac{q_1 \wedge q_2}{2} \right), \quad (29)
$$

yielding precisely the generalized star product $\star_2$. Note that, in deriving the result, higher-powers of the momenta cancel out each other \[^{[21]}\], leaving only the $\star_2$ part.

For $N=4$ correlation function, there are $3!$ diagrams. Three diagrams with ordering $(1-2-3-4)$, $(1-2-[4]-3)$, and $(1-[4]-2-3)$ contribute, while the other three diagrams $(1-[4]-3-2)$, $(1-3-[4]-2)$, and $(1-3-2-[4])$ are Hermitian conjugates, respectively. Summing them up, one finds:

$$
\tilde{B}_{\text{total}}^{(4)} = e^{\frac{i}{2} (q_1 \wedge q_2 + q_1 \wedge q_3 + q_2 \wedge q_3)} \int \cdots \int e^{-i[(x_1-x_2)q_1 \wedge q_2 + (x_1-x_3)q_1 \wedge q_3 + (x_2-x_3)q_2 \wedge q_3]} + e^{\frac{i}{2} (q_3 \wedge q_1 + q_3 \wedge q_1 + q_1 \wedge q_2)} \int \cdots \int e^{-i[(x_1-x_2)q_3 \wedge q_1 + (x_1-x_3)q_3 \wedge q_2 + (x_2-x_3)q_1 \wedge q_2]} + e^{\frac{i}{2} (q_3 \wedge q_1 + q_3 \wedge q_1 + q_1 \wedge q_2)} \int \cdots \int e^{-i[(x_1-x_2)q_3 \wedge q_3 + (x_1-x_3)q_3 \wedge q_2 + (x_2-x_3)q_1 \wedge q_1]} + \text{(h.c.)} = \left( \sin \frac{q_2 \wedge q_3}{2} \right) / \left( \frac{q_1 + (q_2 + q_3)}{2} \right) + \left( \sin \frac{q_1 \wedge (q_2 + q_3)}{2} \right) / \left( q_1 \wedge (q_2 + q_3) \right) + (1 \leftrightarrow 2),
$$

yielding precisely the generalized $\star_3$ product.

The $N=4$ correlation function contains another type of nonplanar diagram, in which two interaction vertices are twisted. It consists of six diagrams of distinct permutations, among
which half are Hermitian conjugates of the others. Adding them up, one easily finds
\[ \Gamma^{(4)} = \left( \frac{\lambda}{2} \right)^4 (2\pi)^d \tilde{K}_{4-d/2} \left( (q_3 + q_4) \circ (q_3 + q_4) \right) \left( \sin \frac{q_1 \cdot \phi}{2} \right) \left( \sin \frac{q_1 \cdot \phi}{2} \right), \]

viz. product of two \( \star_2 \)'s.

Hence, up to \( \mathcal{O}(\lambda^4) \), after Wick-rotation back to Minkowski spacetime, the non-planar part of the one-loop effective action is given by:

\[ \Gamma_{np}[\Phi] = \frac{1}{3!} \left( -\frac{\lambda}{4} \right)^3 \int d^d x \left( 3 \Phi(x) \Phi(x) \right)_{\star_2} K_{3-d/4} (-\partial_x \circ \partial_x) \Phi(x) \]
\[ + \frac{1}{4!} \left( -\frac{\lambda}{4} \right)^4 \int d^d x \left( 3 \Phi(x) \Phi(x) \right)_{\star_2} K_{4-d/4} \left( -(\partial_{x_1} + \partial_{x_2}) \circ (\partial_{x_1} + \partial_{x_2}) \right) \Phi(x_1) \Phi(x_2) \]  
\[ + 4 \left( \Phi(x) \Phi(x) \Phi(x) \right)_{\star_3} K_{4-d/4} (-\partial_x \circ \partial_x) \Phi(x) \]  
\[ + \cdots, \tag{30} \]

where, for the second line, \( x_1, x_2 \rightarrow x \) is assumed in the end. The final expression is in complete agreement with power-series expansion of Eq.(7), as given in Eq.(25).

### 3.3 Further Remarks

We close this section with two remarks concerning our result. First, contribution of planar part to the one-loop effective action can be deduced straightforwardly by replacing, in Eqs.(7, 25), \( |\theta \cdot k| \) into \( 2/\Lambda_{UV} \), where \( \Lambda_{UV} \) is the ultraviolet cutoff, and all the generalized \( \star_n \) products into Moyal’s \( \star \)-product. In the limit \( \Lambda_{UV} \rightarrow \infty \), utilizing Taylor expansion of the modified Bessel function \( K_n \), one easily recognizes that the planar part of the effective action reduces to ‘Coleman-Weinberg’-type potential and appears not to be expressible in terms of the open Wilson line operators, even including those of zero momentum.

Second, one might consider the results in this work trifling as, at low-energy, large noncommutativity limit Eqs.(19, 20), the kernel \( \tilde{K}_{-d/2} \sim \exp(-m|\theta \cdot k|) \) is exponentially suppressed. Quite to the contrary, we actually believe that the exponential suppression indicates a sort of holography in terms of a gravity theory. Indeed, in the context of noncommutative D3-branes, as shown in [4], Green’s functions of supergravity fields exhibit precisely the same, exponentially suppressed propagation in the region where the noncommutativity is important, viz. the dipole-dominated, ‘UV-IR proportionality’ region. We view this as an evidence that, in noncommutative field theories, the dipoles are provided by open Wilson line operators and that their dynamics describes, via holography, gravitational interactions. The latter then defines an effective field theory of the dipoles.
Acknowledgement

SJR would like to thank H. Liu for stimulating correspondences, and L. Susskind for useful remarks.

References

[1] N. Read, Semicond. Sci. Technol. **9** (1994) 1859; Surf. Sci. **361** (1996) 7; V. Pasquier, (unpublished); R. Shankar and G. Murthy, Phys. Rev. Lett. **79** (1997) 4437; D.-H. Lee, Phys. Rev. Lett. **80** (1998) 4745; V. Pasquier and F.D.M. Haldane, Nucl. Phys. **B516**[FS] (1998) 719; A. Stern, B.I. Halperin, F. von Oppen and S. Simon, Phys. Rev. **B59** (1999) 12547.

[2] D. Bigatti and L. Susskind, Phys. Rev. **D66** (2000) 066004, [hep-th/9908056](http://arxiv.org/abs/hep-th/9908056).

[3] S.-J. Rey and R. von Unge, Phys. Lett. **B499** (2001) 215, [hep-th/0007083](http://arxiv.org/abs/hep-th/0007083).

[4] S.R. Das and S.-J. Rey, Nucl. Phys. **B590** (2000) 453, [hep-th/0008042](http://arxiv.org/abs/hep-th/0008042).

[5] D.J. Gross, A. Hashimoto and N. Itzhaki, [hep-th/0008075](http://arxiv.org/abs/hep-th/0008075).

[6] N. Ishibashi, S. Iso, H. Kawai and Y. Kitazawa, Nucl. Phys. B **573** (2000) 573, [hep-th/9910004](http://arxiv.org/abs/hep-th/9910004).

[7] T. Mehen and M.B. Wise, J. High-Energy Phys. **0012** (2000) 008, [hep-th/0010204](http://arxiv.org/abs/hep-th/0010204).

[8] H. Liu, [hep-th/0011125](http://arxiv.org/abs/hep-th/0011125).

[9] M. R. Garousi, Nucl. Phys. **B579** (2000) 209, [hep-th/9909214](http://arxiv.org/abs/hep-th/9909214).

[10] H. Liu and J. Michelson, [hep-th/0008205](http://arxiv.org/abs/hep-th/0008205).

[11] R. Gopakumar, S. Minwalla and A. Strominger, J. High-Energy Phys. **0005** (2000) 020, [hep-th/0003160](http://arxiv.org/abs/hep-th/0003160).

[12] J.A. Harvey, P. Kraus, F. Larsen and E.J. Martinec, J. High-Energy Phys. **0007** (2000) 042, [hep-th/0005031](http://arxiv.org/abs/hep-th/0005031).

[13] S. Minwalla, M. Van Raamsdonk and N. Seiberg, J. High-Energy Phys. **0002** (2000) 020, [hep-th/9912072](http://arxiv.org/abs/hep-th/9912072).

M. Van Raamsdonk and N. Seiberg, J. High-Energy Phys. **0003** (2000) 035, [hep-th/0002186](http://arxiv.org/abs/hep-th/0002186).

M. Hayakawa, Phys. Lett. **B478** (2000) 394, [hep-th/9912167](http://arxiv.org/abs/hep-th/9912167).
A. Matusis, L. Susskind and N. Toumbas, J. High-Energy Phys. **0012** (2000) 002, hep-th/0002075.

[14] G. Parisi, Phys. Lett. **B112** (1982) 463.

[15] Y. Kiem, S.-J. Rey, H. Sato, and J.-T. Yee, Anatomy of One-Loop Effective Action of Noncommutative Scalar Field Theory, to appear.

[16] T. Filk, Phys. Lett. **B376** (1996) 53.

[17] Y. Okawa and H. Ooguri, Nucl. Phys. **B599** (2001) 55, hep-th/0012218.

[18] S. R. Das and S. P. Trivedi, J. High-Energy Phys. **0102** (2001) 046, hep-th/0011131.

[19] S. Mukhi and N. V. Suryanarayana, J. High-Energy Phys. **0105** (2001) 023, hep-th/0104045.

[20] H. Liu and J. Michelson, hep-th/0101016.

[21] Y. Kiem, D. H. Park and S. Lee, Phys. Rev. **D63** (2001) 126006, hep-th/0011233.