WEIGHTED ESTIMATES FOR BILINEAR FRACTIONAL INTEGRAL OPERATOR ON THE HEISENBERG GROUP

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ABSTRACT. In this article, we introduce an analogue of Kenig and Stein’s bilinear fractional integral operator on the Heisenberg group $\mathbb{H}^n$. We completely characterize exponents $\alpha, \beta$ and $\gamma$ such that the operator is bounded from $L^p(\mathbb{H}^n, |x|^\alpha) \times L^q(\mathbb{H}^n, |x|^\beta)$ to $L^r(\mathbb{H}^n, |x|^{-\gamma})$.

1. Introduction and preliminaries

Fractional integral operators are classical objects in analysis pertaining to the study of smoothness of functions, potential theory and embedding theorems. Recall that for $0 < \lambda < n$, the fractional integral operator is defined as follows

$$ I_\lambda f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\lambda}} dy, \quad x \in \mathbb{R}^n. $$

The operators $I_\lambda$ are bounded off-diagonally and the characterization of weights for which $I_\lambda : L^p(w^p) \rightarrow L^q(w^q)$, with $1/q = 1/p - \lambda/n$, $1 < p < n/\lambda$, was obtained by Muckenhoupt and Wheeden in [MW74]. The appropriate class of weights are denoted as $A_{p,q}$ weights. The operator $I_\lambda$ and its analogues are also investigated beyond the Euclidean setting.

In this article we are interested in bilinear analogue of $I_\lambda$ on the Heisenberg group $\mathbb{H}^n$. Let us begin with the bilinear fractional integral operator $BI_\lambda$ on $\mathbb{R}^n$ defined as

$$ BI_\lambda(f,g)(x) = \int_{\mathbb{R}^n} \frac{f(x-y)g(x+y)}{|y|^{n-\lambda}} dy, \quad 0 < \lambda < n. $$

These operators are well studied, for example we refer the works [Gra92, KS99, Moe14]. They are also of interest due to their connections with the bilinear Hilbert transform of Lacey and Thiele (see [LT97]). It was proved in [KS99] that $BI_\lambda$ bounded from $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$ provided $1/r = 1/p + 1/q - \lambda/n > 0$ and $1 < p, q \leq \infty$, and also the expected weak type inequality holds if either $p$ or $q$ is 1. It is not difficult to see that using Hölder’s inequality and weighted boundedness of $I_\lambda$, we can obtain that $BI_\lambda : L^p(w^p_i) \times L^q(w^q_2) \rightarrow L^r(w^r_1w^r_2)$ provided $1/r = 1/p + 1/q - \lambda/n$, $1 < r, s < \infty$ and $w^{p/s}_i \in A_{p,q}$, where $1/s = 1/p + 1/q$. However, the above approach is not useful when $r < 1$ and it was also pointed out in the influential work of Lerner et al that linear Muckenhoupt classes are not the appropriate weights while studying bilinear operators. In [LOP+09], multilinear $A_\beta$ weights are introduced in connection with the multilinear Hardy–Littlewood maximal operator and multilinear Calderón–Zygmund operators. Subsequently, Kab Moen has initiated the study of fractional multilinear weights and proved the following: $BI_\lambda$ maps $L^p(w^p_i) \times L^q(w^q_2)$ to $L^r(w^r_1w^r_2)$ boundedly, when $1 < p, q < \infty$, $1/r = 1/p + 1/q - \lambda/n > 1$, and $\bar{w} \in A_{p,q,r}$, where the the class $A_{p,q,r}$ is defined as follows. We
say \( \vec{w} = (w_1, w_2) \in \mathcal{A}_{p, q, r} \) if

\[
\sup_Q \left( \frac{1}{|Q|} \int_Q (w_1 w_2)^{r/(1-r)} \right)^{(1-r)/r} \left( \frac{1}{|Q|} \int_Q w_1^{-p'} \right)^{1/p'} \left( \frac{1}{|Q|} \int_Q w_2^{-q'} \right)^{1/q'} \lesssim C < \infty,
\]

where the supremum is over all cubes with sides parallel to the coordinate axes. Though it is not yet known whether the condition \( \mathcal{A}_{p, q, r} \) is also necessary for the boundedness of \( BI_\alpha \).

Interestingly, if we only consider power weights then it was shown in [KF20] that it is possible to obtain both necessary and sufficient conditions on \( \alpha, \beta, \) and \( \gamma \) such that \( BI_\alpha \) is bounded from \( L^p(|x|^\alpha) \times L^q(|x|^\beta) \) to \( L^r(|x|^{-\gamma}) \), in the particular case when \( \gamma = -\alpha - \beta \).

Our primary goal in this article is to obtain a complete characterization of \( \alpha, \beta, \) and \( \gamma \) in full generality such that the bilinear fractional operator \( B_\alpha \) maps \( L^p(|x|^\alpha) \times L^q(|x|^\beta) \) to \( L^r(|x|^{-\gamma}) \) on the Heisenberg group \( \mathbb{H}^n \). We write our results on the Heisenberg group but it is not difficult to see that the same ideas also work for the Euclidean case, which in turn improves all the existing results in the Euclidean setting. We will explain it in detail in Remark 1.1.

To illustrate our results, let us recall the following preliminaries. Let \( \mathbb{H}^n := \mathbb{C}^n \times \mathbb{R} \) denotes the \((2n + 1)\)-dimensional Heisenberg group with the group law

\[
(z, t) \cdot (w, s) = \left( z + w, t + s + \frac{1}{2} \Im(z \cdot \vec{w}) \right), \quad \text{for all } (z, t), (w, s) \in \mathbb{H}^n. \tag{1.1}
\]

We have a family of non-isotropic dilations defined by \( \delta_r(z, t) := (rz, r^2 t) \), for all \( (z, t) \in \mathbb{H}^n \), for every \( r > 0 \). The Koranyi norm on \( \mathbb{H}^n \) is defined by

\[
|\langle z, t \rangle| := \left( |z|^4 + r^2 \right)^{1/4}, \quad (z, t) \in \mathbb{H}^n,
\]

which is homogeneous of degree 1, that is \( |\delta_r(z, t)| = r |\langle z, t \rangle| \). The Haar measure on \( \mathbb{H}^n \) coincides with the Lebesgue measure \( dx dt \). Let \( B(0, r) = \{ (z, t) \in \mathbb{H}^n : |\langle z, t \rangle| < r \} \) be the ball of radius \( r \) with respect to Koranyi norm. One has its measure \( |B(0, r)| = C \cdot r^Q \), where \( Q = (2n + 2) \) is known as the homogeneous dimension of \( \mathbb{H}^n \). The convolution of \( f \) with \( g \) on \( \mathbb{H}^n \) is defined by

\[
f \ast g (x) = \int_{\mathbb{H}^n} f(xy^{-1}) g(y) dy, \quad x \in \mathbb{H}^n.
\]

Recall that the fractional integral operator, \( I_\lambda \), on the Heisenberg group \( \mathbb{H}^n \) is defined as follows

\[
I_\lambda f(x) = \int_{\mathbb{H}^n} f(xy^{-1}) \frac{dy}{|y|^Q - \lambda}, \quad 0 < \lambda < Q.
\]

Fractional integral operators on the Heisenberg group has a long history, starting with the foundational work of Folland and Stein in [FS74], where it was shown that \( I_\lambda : L^p(\mathbb{H}^n) \to L^q(\mathbb{H}^n) \) with \( 1/q = 1/p - \lambda/Q, 1 < p < Q/\lambda \), also the natural end-point boundedness \( I_\lambda : L^1(\mathbb{H}^n) \to L^{Q/(Q-\lambda), \infty}(\mathbb{H}^n) \). We would like to mention the work [Kai14] where fractional integral operators are extended in the more general context of spaces of homogeneous type, and also the article [CF11] where the authors have treated an analogue of Kenig and Stein’s bilinear fractional integral operator on compact Lie groups. Motivated by the above discussion, let us define bilinear fractional integral operator on \( \mathbb{H}^n \).

**Definition 1.1.** For \( 0 < \lambda < Q \), the bilinear fractional integral operator \( B_\lambda \) on \( \mathbb{H}^n \) is defined as follows

\[
B_\lambda(f, g)(x) = \int_{\mathbb{H}^n} f(xy^{-1}) g(xy) \frac{dy}{|y|^Q - \lambda}.
\]

Our first result addresses the unweighted boundedness of the operator \( B_\lambda \) on \( \mathbb{H}^n \), thus extending the result of Kenig and Stein to the Heisenberg group.

**Theorem 1.1** (Unweighted boundedness). Let \( 0 < \lambda < Q, \) \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{\lambda}{Q} > 0, \) and that \( f \in L^p(\mathbb{H}^n), g \in L^q(\mathbb{H}^n), 1 \leq p, q \leq \infty \). Then,
(a) If \( 1 < p, q \leq \infty \),

\[ \| B_\lambda(f,g) \|_{L^r(H^n)} \leq K \| f \|_{L^p(H^n)} \| g \|_{L^q(H^n)}, \tag{1.2} \]

(b) If \( 1 \leq p, q \leq \infty \), with either \( p = 1 \) or \( q = 1 \),

\[ \| B_\lambda(f,g) \|_{L^r,\infty(H^n)} \leq K \| f \|_{L^p(H^n)} \| g \|_{L^q(H^n)}. \tag{1.3} \]

Now, we present the main result of this article concerning the characterization of power weights for the boundedness of \( B_\lambda \). Precisely, we obtain the following:

**Theorem 1.2** (Characterization of power weights). Let \( 1 < p, q < \infty \), \( 0 < r < \infty \) and \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q'} \). Let \( 0 < \lambda < Q \) and

\[ \alpha < \frac{Q}{p'}, \quad \beta < \frac{Q}{q'} \quad \text{and} \quad \gamma < \frac{Q}{r}. \tag{1.4} \]

Further, assume \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{\lambda - \alpha - \beta - \gamma}{Q} > 0 \).

Then, the following are equivalent:

(a) There exists a constant \( K > 0 \), such that

\[ \| |x|^{-\gamma} B_\lambda(f,g) \|_{L^r(H^n)} \leq K \| |x|^{\alpha} f \|_{L^p(H^n)} \| |x|^{\beta} g \|_{L^q(H^n)}, \tag{1.5} \]

for all \( f \in L^p(H^n) \) and \( g \in L^q(H^n) \);

(b) The exponents \( \alpha, \beta \) and \( \gamma \) satisfy

\[ (I) -Q + \lambda \leq \beta + \gamma, \quad (II) -Q + \lambda \leq \gamma + \alpha, \quad (III) -Q + \lambda \leq \alpha + \beta, \quad \text{and} \quad (IV) \alpha + \beta + \gamma \geq 0. \tag{1.6} \]

The proof of the above theorem is quite involved. Since the operator \( B_\lambda \) involves the product of the form \( f(xy^{-1})g(xy) \), in order to obtain the necessary conditions we need to construct functions \( f \) and \( g \) with very delicate precision. In the following remark we would like to mention some key features of Theorem 1.2.

**Remark 1.1.** We point out that our characterization, that is, Theorem 1.2 is sharp. Moreover, if restricted to the Euclidean setting, it improves all the previously known results. We mention some of them here.

- Recall that the condition obtained in Theorem 2 in [KF20] were \( \alpha \leq n - \lambda, \beta \leq n - \lambda, \) and \( -n + \lambda \leq \alpha + \beta \) which certainly implies our conditions. Also it is easy to see that by suitably choosing \( \epsilon > 0 \) we can construct triplets \( \alpha = n - \lambda + \epsilon, \beta = 0, \gamma = -n + \lambda \) which satisfy conditions of Theorem 1.2 but it is not covered by Theorem 2 in [KF20].

- A simple computation shows that our result also recovers the well-known result of Hoang and Moen, see Theorem 10.1 in [HM18].

- In [BLO21], as an application of Brascamp-Lieb forms, the authors have obtained boundedness \( (1.5) \) when one has \( (1.6) \) with strict inequalities in the first three conditions and the \( r \) is additionally restricted by the requirement \( 1 < r < \infty \). The limitation \( 1 < r < \infty \) is due to use of duality arguments in their proof.

Another interesting aspect related to the study of fractional integral operator is the Stein-Weiss inequality obtained in 1958 by Stein and Weiss in [SW58]. We recall it here. The inequality

\[ \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x|^\alpha |x-y|^\beta |y|^\beta} dxdy \right| \leq \| f \|_{L^p(\mathbb{R}^n)} \| g \|_{L^q(\mathbb{R}^n)}, \tag{1.7} \]

holds, where \( 1 < p, q < \infty, 0 < \lambda < n, \alpha + \beta \geq 0, 1/p + 1/q + (\alpha + \beta + \lambda)/n = 2 \) with \( \alpha < n/p', \beta < n/q' \). This was extended to the Heisenberg group in [HLZ12]. In this article, we will also prove an analogue of \( (1.7) \) for the bilinear operator \( B_\lambda \) on the Heisenberg group \( H^n \), see
Theorem 3.1. We end this section with the following bilinear interpolation theorem for Lorentz spaces which will be very useful for our purpose.

**Theorem 1.3 ([Jan88], Theorem 3 [KS99]).** Suppose that a bilinear operator $T : L^{p_1,1} \times L^{q_1,1} \to L^{r_1,\infty}$, where $0 < p_i, q_i \leq \infty$, $0 < r_i \leq \infty$, for three points $\left(\frac{1}{p_i}, \frac{1}{q_i}\right)$, $i = 1, 2, 3$ in $\mathbb{R}^2$, that are non-collinear. Suppose, further, that there are $\theta_0, \theta_1, \theta_2 \in \mathbb{R}$ with $\theta_1, \theta_2 > 0$ so that $\frac{1}{r_i} = \frac{\theta_0}{p_i} + \frac{\theta_1}{q_i}$, $i = 1, 2, 3$. Then,

$$T : L^p \times L^q \to L^r,$$

provided $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} \geq \frac{1}{r}$ and $\left(\frac{1}{p}, \frac{1}{q}, \frac{1}{r}\right)$ lies in the open convex hull of $\left(\frac{1}{p_i}, \frac{1}{q_i}, \frac{1}{r_i}\right)$.

The article is organized as follows. In the next section we prove Theorem 1.1. Section 3 is dedicated to the proof of our main result Theorem 1.2. Finally, as a consequence of Theorem 1.2, we conclude this article by Stein-Weiss inequality for the bilinear fractional integral operator $B_\lambda$ on $\mathbb{H}^n$. Throughout this article, we write $A \lesssim B$ and $B \gtrsim A$ to abbreviate $A \leq CB$ for some constant $C$ is independent of $A$ and $B$, and $A \simeq B$ means both $A \lesssim B$ and $A \gtrsim B$. We write the Euclidean convolution of $f$ and $g$ on $\mathbb{R}^{2n+1}$ by $f *_e g(x) := \int_{\mathbb{R}^{2n+1}} f(y)g(x-y)\,dy$.

## 2. Proof of Theorem 1.1

We first prove part (b) in Theorem 1.1 when $p = q = 1$. This is the key estimate for proving Theorem 1.1. Let us introduce the following operators which are pieces of the operator $B_\lambda$.

$$B(f, g)(x) = \int_{|y| = 1} f(xy^{-1})g(xy)\,dy,$$

and

$$B_k(f, g)(x) = \int_{|y| \approx 2^{-k}} f(xy^{-1})g(xy)\,dy.$$

Our main goal is to establish the end-point weak type boundedness $L^1(\mathbb{H}^n) \times L^1(\mathbb{H}^n) \to L^{1/2}(\mathbb{H}^n)$ for the pieces $B_k$. We address this as the following lemma.

**Lemma 2.1.** The following statements hold:

(i) $\|B(f, g)\|_{L^{1/2}(\mathbb{H}^n)} \lesssim \|f\|_{L^1(\mathbb{H}^n)}\|g\|_{L^1(\mathbb{H}^n)}$.

(ii) $\|B(f, g)\|_{L^1(\mathbb{H}^n)} \lesssim \|f\|_{L^1(\mathbb{H}^n)}\|g\|_{L^1(\mathbb{H}^n)}$.

(iii) $\|B_k(f, g)\|_{L^{1/2}(\mathbb{H}^n)} \lesssim 2^{-Qk}\|f\|_{L^1(\mathbb{H}^n)}\|g\|_{L^1(\mathbb{H}^n)}$.

(iv) $\|B_k(f, g)\|_{L^1(\mathbb{H}^n)} \lesssim \|f\|_{L^1(\mathbb{H}^n)}\|g\|_{L^1(\mathbb{H}^n)}$.

**Proof of Lemma 2.1.** The statements (iii) and (iv) follow from (i) and (ii), respectively, by scaling: Let $r = 2^{-k}$ and $B_r(f, g)(x) = \int_{|y| = r} f(xy^{-1})g(xy)\,dy$ then

$$\delta_r [B(\delta_r f, \delta_r g)] = r^{-Q} B_r(f, g), \quad Q = 2n + 2.$$
We assume, without loss of generality, that \( f \geq 0, g \geq 0 \). We begin with proving (i). For \( a \in \mathbb{Z}^{2n+1} \), let \( Q_a = a \cdot Q_0 \), where \( Q_0 = [0,1)^{2n+1} \). Then,

\[
\|B(f,g)\chi_{Q_a}\|_{L^{1/2}(\mathbb{H}^n)} \leq \frac{\int_{Q_a} B(f,g)(x)dx \leq \int_{x \in Q_a} \int_{y \in B(0,1)} f(xy^{-1}) g(xy) dy \, dx}{\int_{y \in B(0,1)} f(y) g(xy^{-1}) \, dy} \\
\leq \int_{y \in B(0,1)} f(y) g(xy^{-1}) \, dy \\
= \int_{x \in Q_a} f(y) \int_{x \in Q_a} g(xy^{-1}) \, dx \, dy \\
\leq 2^{-2n-1} \int_{y \in Q_a} g(y) \, dx \\
\]

where \( Q_a := a \cdot Q_0 \cdot B(0,1) \cdot Q_0^{-1}, Q_0 \subset a \cdot ( [-4,4]^{2n} \times [-16,16] ) \).

Observe that \( Q_a \) and \( Q_a^* \) have bounded overlapping and covers whole of \( \mathbb{H}^n \). Indeed, let \( (z,t) = (x + iy,t) \in \mathbb{H}^n \). Choose an \( a' \in \mathbb{Z}^{2n} \) such that \( a' \leq (x,y) < a' + (1,\ldots,1) \), component wise. Having chosen \( a' \), choose integer, say \( a_{2n+1} \) such that \( t - \frac{1}{2} \Im(a' \cdot \bar{z}) \in [a_{2n+1},a_{2n+1} + 1] \). Then we have an \( a = (a',a_{2n+1}) \in \mathbb{Z}_n \) such \( a^{-1} \cdot (z,t) \in Q_0 \). So \( \mathbb{H}^n = \bigcup_{a \in \mathbb{Z}^{2n+1}} Q_a \). For bounded overlapping of \( Q_a \), let us fix an \( a \in \mathbb{Z}^{2n+1} \) and consider \( \bar{a} \in \mathbb{Z}^{2n+1} \) such that

\[
a \cdot Q_0 \cap \bar{a} \cdot Q_0 \neq \emptyset.
\]

Equivalently, \( Q_0 \cap a^{-1} \bar{a} \cdot Q_0 \neq \emptyset \). Let \( (z,t) = a^{-1} \bar{a} \) and \( (w,s) \in Q_0 \). Let \( \|z\| \) means the Euclidean norm of \( z \in \mathbb{C}^{2n} \). Then, if \( \|z\| > 2\sqrt{n} \), then \( (z,t) \cdot (w,s) = (z+w, t+s + \frac{1}{2} \Im(z \cdot \bar{w})) \notin Q_0 \). If \( \|z\| \leq 2\sqrt{n} \) but \( |t| > 2n + 2 \), then \( |t+s + \frac{1}{2} \Im(z \cdot \bar{w})| \geq 2(n+1) - (n+1) = n+1 \). So, again \( (z,t) \cdot (w,s) \notin Q_0 \). If \( \|z\| \leq 2\sqrt{n} \) and \( |t| \leq 2n + 2 \), then for fixed \( a \), we are, at most, counting the number of lattice points \( \bar{a} \in \mathbb{Z}^{2n+1} \) such that \( \bar{a} \in a \cdot B(0,4\sqrt{n}) \) which is, clearly, \( \approx n^{Q/2} \).

Similarly, we can argue for the sets \( Q_a^* \).

So,

\[
\|B(f,g)\|_{L^{1/2}(\mathbb{H}^n)} \approx \sum_{a \in \mathbb{Z}^{2n+1}} \|B(f,g)\chi_{Q_a}\|_{L^{1/2}(\mathbb{H}^n)}^{1/2} \\
\approx \sum_{a \in \mathbb{Z}^{2n+1}} \|f \chi_{Q_a^*}\|_{L^1(\mathbb{H}^n)}^{1/2} \|g \chi_{Q_a^*}\|_{L^1(\mathbb{H}^n)}^{1/2} \\
\leq \left( \sum_{a \in \mathbb{Z}^{2n+1}} \|f \chi_{Q_a^*}\|_{L^1(\mathbb{H}^n)} \right)^{1/2} \left( \sum_{a \in \mathbb{Z}^{2n+1}} \|g \chi_{Q_a^*}\|_{L^1(\mathbb{H}^n)} \right)^{1/2} \\
\approx \|f\|_{L^1(\mathbb{H}^n)} \|g\|_{L^1(\mathbb{H}^n)},
\]

establishing (i).

Next, (ii) follows from using the same set of change of variables and Fubini's theorem as in (2.1). Thus, completing the proof of Lemma 2.1. \( \square \)
Returning to the proof of part (b) in Theorem 1.1 when \( p = q = 1 \), \( \frac{1}{r} = 2 - \frac{\lambda}{Q} \). Let \( \|f\|_{L^1(\mathbb{H}^n)} = \|g\|_{L^1(\mathbb{H}^n)} = 1 \). Let us decompose the operator \( B_\lambda \) as

\[
B_\lambda(f, g)(x) \simeq \sum_{k \in \mathbb{Z}} 2^{k(Q-\lambda)} B_k(f, g)(x)
\]

\[
= \sum_{k \leq k_0} + \sum_{k \geq k_0} =: F_1 + F_2,
\]

and for \( F_1 \) and \( F_2 \) we have, using (iii) and (iv) in Lemma 2.1,

\[
\|F_1\|_{L^1(\mathbb{H}^n)} \leq \sum_{k \leq k_0} 2^{k(Q-\lambda)} \|B_k(f, g)\|_{L^1(\mathbb{H}^n)} \lesssim \sum_{k \leq k_0} 2^{k(Q-\lambda)} \simeq 2^{k_0(Q-\lambda)}
\]

and

\[
\|F_2\|_{L^{1/2}(\mathbb{H}^n)}^{1/2} \leq \sum_{k \geq k_0} 2^{k(Q-\lambda)/2} \|B_k(f, g)\|_{L^{1/2}(\mathbb{H}^n)}^{1/2} \lesssim \sum_{k \geq k_0} 2^{k(Q-\lambda)/2} 2^{-k_0Q} \simeq 2^{-\frac{\lambda}{2}k_0}.
\]

Then, for all \( t > 0 \),

\[
|\{B_\lambda(f, g) > t\}| \lesssim \left| \left\{ F_1 > \frac{Ct}{2} \right\} \right| + \left| \left\{ F_2 > \frac{Ct}{2} \right\} \right| \lesssim \frac{|F_1|_{L^1(\mathbb{H}^n)}}{t} + \frac{|F_2|_{L^{1/2}(\mathbb{H}^n)}}{t^{1/2}}.
\]

Optimising the right hand side of the above with respect to \( k_0 \), that is, choosing \( k_0 \) such that \( 2^{k_0(Q-\lambda)}/t = 2^{-\frac{\lambda}{2}k_0}/t^{1/2} \), gives the desired estimate

\[
|\{B_\lambda(f, g) > t\}| \lesssim \frac{1}{t^{1/2}}, \quad \frac{1}{r} = 2 - \frac{\lambda}{Q},
\]

which settles the proof of (b), in Theorem 1.1 when \( p = q = 1 \). To finish part (b), observe that, if \( g \in L^\infty(\mathbb{H}^n) \), we have

\[
B_\lambda(f, g)(x) \leq \|g\|_{L^\infty(\mathbb{H}^n)} \left( f * \frac{1}{|y|^{Q-\lambda}} \right)(x), \quad x \in \mathbb{H}^n.
\]

So, from linear fractional integration on \( \mathbb{H}^n \),

\[
\|B_\lambda(f, g)\|_{L^r(\mathbb{H}^n)} \leq \|g\|_{L^\infty(\mathbb{H}^n)} \left\| f * \frac{1}{|y|^{Q-\lambda}} \right\|_{L^r,\infty(\mathbb{H}^n)} \lesssim \|g\|_{L^\infty(\mathbb{H}^n)} \|f\|_{L^1(\mathbb{H}^n)},
\]

if \( \frac{1}{r} = 1 - \frac{\lambda}{Q} \) which is, indeed, the situation when \( p = 1, q = \infty \). If \( g \in L^q(\mathbb{H}^n), 1 < q < \infty \), then (b) follows from linear interpolation, by fixing \( f \in L^1(\mathbb{H}^n) \), and using bounds for \( B_\lambda \) in \( L^1(\mathbb{H}^n) \times L^1(\mathbb{H}^n) \to L^{r,\infty}(\mathbb{H}^n) \) (the key estimate) and \( L^1(\mathbb{H}^n) \times L^\infty(\mathbb{H}^n) \to L^{r,\infty}(\mathbb{H}^n) \) (the sub case discussed above).

The part (a) is obtained from (b), by applying bilinear interpolation Theorem 1.3. For the sake of completeness, we briefly explain this point from [KS99]. Consider the open convex set in \( \mathbb{R}^2 \),

\[
C := \left\{ (x, y) \in \mathbb{R}^2 : x + y > \frac{\lambda}{Q}, 0 < x, y < 1 \right\}, \quad 0 < \lambda < Q.
\]
Observe that the interior of $C$ is precisely the union of interior of triangles whose vertices lie on different sides of the square $[0,1]^2$ intersected with $C$. Thus, by symmetry and part (b), it suffices to establish the “weak-type” inequality for $p = \infty$ and $1 < q < \infty$. But for $f \in L^\infty(\mathbb{H}^n)$,

$$B_\lambda(f, g)(x) \leq \|f\|_{L^\infty(\mathbb{H}^n)} \left( g * \left[ \frac{1}{|y|^{Q-\lambda}} \right] \right)(x), \quad x \in \mathbb{H}^n,$$

which implies

$$\|B_\lambda(f, g)\|_{L^r(\mathbb{H}^n)} \leq \|f\|_{L^\infty(\mathbb{H}^n)} \left\| g * \left[ \frac{1}{|y|^{Q-\lambda}} \right] \right\|_{L^r(\mathbb{H}^n)} \lesssim \|f\|_{L^\infty(\mathbb{H}^n)} \|g\|_{L^q(\mathbb{H}^n)},$$

which is guaranteed by the strong type boundedness of linear fractional operator on $\mathbb{H}^n$ provided,

$$\frac{1}{r} = \frac{1}{q} - \frac{\lambda}{Q} > 0, \ 1 < q < \infty,$

which is indeed, true in this case.

3. Characterization of power weights

In this section we provide the proof of Theorem 1.2. Let us start with proving the sufficient part. Our proof of the sufficient part involves delicate analysis of singularities of the operator $B_\alpha$. Subsequently, we decompose it appropriately to estimate each piece individually. In contrast with the proof of [KF20], we provide a unified approach to handle the operator $B_\lambda$ irrespective of the sign of $\alpha$ and $\beta$.

3.1. Proof of the sufficient part.

Proof of (1.6) $\implies$ (1.5): Since $|B_\lambda(f, g)| \leq B_\lambda(|f|, |g|)$, throughout this proof we will assume that $f, g$ are non-negative functions. First we will prove the following weak type estimate

$$\|S(f, g)\|_{L^{r,\infty}(\mathbb{H}^n)} \leq K \|f\|_{L^r(\mathbb{H}^n)} \|g\|_{L^q(\mathbb{H}^n)},$$

(3.1)

where,

$$S(f, g) = S_{\alpha, \beta, \gamma}(f, g)(x) := |x|^{-\gamma} \int_{\mathbb{H}^n} \frac{f(xy^{-1}) \ g(xy)}{|xy|^{-\alpha} \ |xy|^\beta \ |y|^{Q-\lambda}} \ dy.$$

Once the proof of (3.1) is complete, as an application of bilinear interpolation, we can conclude the required strong type estimates.

We analyse the operator $S$ into three parts. Namely the following: $S(f, g)(x) = \sum_{i=1}^{3} J_i(x)$, where

$$J_1(x) := |x|^{-\gamma} \int_{y \in B(0, \frac{|x|}{2}) \ x} \frac{f(xy^{-1}) \ g(xy)}{|xy|^{-\alpha} |xy|^\beta |y|^{Q-\lambda}},$$

$$J_2(x) := |x|^{-\gamma} \int_{y \in x^{-1} \ B(0, \frac{|x|}{2})} \frac{f(xy^{-1}) \ g(xy)}{|xy|^{-\alpha} |xy|^\beta |y|^{Q-\lambda}},$$

$$J_3(x) := |x|^{-\gamma} \int_{\mathbb{H}^n \ \ B(0, \frac{|x|}{2}) \ x \ \ x^{-1} \ B(0, \frac{|x|}{2})} \frac{f(xy^{-1}) \ g(xy)}{|xy|^{-\alpha} |xy|^\beta |y|^{Q-\lambda}} \ dy.$$

We first estimate $J_3$.

Estimate for $J_3$: Let us denote the set $\mathbb{H}^n \ \ B(0, \frac{|x|}{2}) \ x \ \ x^{-1} \ B(0, \frac{|x|}{2})$ by $G_x$. One can decompose $J_3$ as follows:

$$J_3(x) \leq J_{31}(x) + J_{32}(x),$$

where,

$$J_{31}(x) := |x|^{-\gamma} \int_{\{y: |y| \geq |x|\}} \frac{f(xy^{-1}) \ g(xy)}{|xy|^{-\alpha} |xy|^\beta |y|^{Q-\lambda}},$$

$$J_{32}(x) := |x|^{-\gamma} \int_{\{y: |y| \leq |x|\}} \frac{f(xy^{-1}) \ g(xy)}{|xy|^{-\alpha} |xy|^\beta |y|^{Q-\lambda}}.$$
\[ J_{32}(x) := |x|^{-\gamma} \int_{\{y \in G_x : |y| < 2|x|\}} \frac{f(xy^{-1}) g(xy)}{|xy^{-1}|^\alpha |y| Q^{-\lambda}} \, dy. \]

Observe that for \( y \) such that \(|y| \geq 2|x|\) we have \(|xy^{-1}| \simeq |y|\) and \(|xy| \simeq |y|\). According to our hypothesis \(-Q + \lambda \leq \alpha + \beta\). First, let us consider the case when \(-Q + \lambda = \alpha + \beta\). Therefore,

\[ J_{31}(x) \lesssim |x|^{-\gamma} (f * e g)(2x). \]

We also have \(\gamma \geq -\alpha - \beta - Q = -\lambda > 0\), and \(\frac{1}{r} = \frac{\gamma}{Q} + \frac{1}{s}\), where, \(\frac{1}{s} = \frac{1}{p} + \frac{1}{q} - 1\). Using Hölder’s inequality for weak-type spaces and Young’s convolution inequality subsequently, we obtain \(\|J_{31}\|_{L^{r,\infty}(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}\), which is the required estimate.

When \(-Q + \lambda < \alpha + \beta\), together with the condition \(\gamma > Q/r\), we can ensure that \(\lambda - \alpha - \beta < Q(\frac{1}{p} + \frac{1}{q})\). Now choosing \(\mu > 0\) such that \(\mu \in (\alpha - \beta, \beta - Q, Q(\frac{1}{p} + \frac{1}{q}) \cap (\lambda - \alpha - \beta, Q))\), we conclude the following

\[ J_{31}(x) \lesssim |x|^{-\gamma} \int_{\{y \in G_x : |y| < 2|x|\}} \frac{|y|^{-\alpha - \beta + \lambda - \mu} f(xy^{-1}) g(xy) \, dy}{|y| |y|^{Q - \lambda}}, \]

\[ = |x|^{-\gamma} \int_{\{y \in G_x : |y| < 2|x|\}} |y|^{-\alpha - \beta + \lambda - \mu} f(xy^{-1}) g(xy) \, dy \]

\[ \lesssim |x|^{-\gamma - \alpha - \beta + \lambda - \mu} B_{\mu}(f,g)(x). \] (3.2)

Define \(\frac{1}{s} := \frac{1}{p} + \frac{1}{q} - \frac{\mu}{Q}\), then it is trivial to see that \(\frac{1}{r} = \frac{1}{s} + \frac{\alpha + \beta + \gamma + \mu - \lambda}{Q}\). Denote \(h(x) = |x|^{-\gamma - \alpha - \beta + \mu}\). Using Hölder’s inequality for weak-type spaces we obtain

\[ \|J_{31}\|_{L^{r,\infty}(\mathbb{R}^n)} \lesssim \|h\|_{L^{\frac{Q}{\alpha + \beta + \gamma + \mu - \lambda} \infty}^Q} \|B_{\mu}(f,g)\|_{L^{r,\infty}} \lesssim K_{\alpha,\beta,\gamma,Q,\lambda} \|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}, \]

where we have used Theorem 1.1 in the last inequality. This completes the estimates for \(J_{31}\).

To estimate \(J_{32}\), observe that for points \(y \in G_x\) with \(|y| < 2|x|\), we have \(|xy^{-1}| = |yx^{-1}| \simeq |x|\) and \(|xy| \simeq |x|\). Assuming that \(\alpha + \beta + \gamma > 0\), one can choose \(\mu_1 \in (0, Q(\frac{1}{p} + \frac{1}{q}))\) such that \(-\alpha - \beta - \gamma + \lambda < \mu_1 < \lambda\). Now

\[ J_{32}(x) \lesssim |x|^{-\gamma - \alpha - \beta} \int_{\{y \in G_x : |y| < 2|x|\}} |y|^{-\mu_1} f(xy^{-1}) g(xy) \, dy \]

\[ \lesssim |x|^{-\gamma - \alpha - \beta + \lambda - \mu_1} B_{\mu_1}(f,g)(x). \]

At this point we follow the argument provided after equation (3.2) to conclude that \(\|J_{32}\|_{L^{r,\infty}(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}\). Similarly, if \(\alpha + \beta + \gamma = 0\), we have \(J_{32}(x) \lesssim B_{\lambda}(f,g)(x)\), then also we have the required estimate invoking Theorem 1.1.

**Estimate for \(J_1\):** Let \(y \in B(0, \frac{|x|}{2}) \cdot x\) then \(y = \xi \cdot x\) for some \(\xi \in B(0, \frac{|x|}{2})\). Observe that \(\frac{xy}{x} = \xi \cdot x = 2x + \xi\) and \(|xy| = |2x + \xi| \geq |2x| - |\xi| \geq c|x|\), for some fixed constant \(1 < c < \infty\). Moreover, \(|xy| \lesssim |x|\). Again observe that \(|y| = |\xi \cdot x| \leq |\xi| + |x| \leq \frac{3}{2}|x|\), \(|y| = |\xi \cdot x| \geq |x| - |\xi| \geq \frac{1}{2}|x|\). Incorporating these estimates we obtain

\[ J_1(x) \lesssim |x|^{-\gamma - \beta - Q + \lambda} \int_{B\left(0, \frac{|x|}{2}\right)} \frac{f(xy^{-1}) g(xy)}{|xy^{-1}|^\alpha} \, dy. \] (3.3)

Our hypothesis \(\alpha < Q/p\) and \(-Q + \lambda < \beta + \gamma\) allow us to choose \(\mu > 0\) such that \(Q(1 - \frac{1}{p} - \frac{1}{q}) < \mu < Q(1 - \frac{1}{p})\) and \(\alpha < \mu < \alpha + \beta + \gamma + Q - \lambda\). Now

\[ J_1(x) \lesssim |x|^{-\gamma - \beta - Q + \lambda} \int_{B\left(0, \frac{|x|}{2}\right)} |xy^{-1}|^{\mu - \alpha} f(xy^{-1}) g(xy) \, dy \]
\[ |x|^{-\gamma - \beta - Q + \lambda + \mu - \alpha} \int_{y \in B(0, \frac{1}{p} \cdot x)} \frac{f(xy^{-1})g(xy)}{|xy^{-1}|^\alpha} dy \text{ (since } |xy^{-1}| \leq c|x| \]

\[ \lesssim |x|^{-\gamma - \beta - Q + \lambda + \mu - \alpha} \int_{\mathbb{R}^{2n+1}} \frac{f(z)g(2x + z^{-1})}{|z|} dz \]

\[ \lesssim |x|^{-\gamma - \beta - Q + \lambda + \mu - \alpha} \left( \frac{f}{| \cdot |^\alpha} * e \right)(2x). \]

Write \( \frac{1}{s} = \frac{1}{p} + \frac{1}{q} - 1 + \frac{\alpha}{Q} \), then \( \frac{1}{r} = \frac{1}{s} + \frac{\alpha + \beta + \gamma - Q - \lambda - \mu}{Q} \). This implies

\[ \| J_1 \|_{L^r, \infty(\mathbb{H}^n)} \lesssim \| x^{-\gamma - \beta - Q + \lambda + \mu - \alpha} \|_{L^{\alpha + \beta + \gamma + Q - \lambda - \mu}(\mathbb{R}^{2n+1})} \left\| \left( \frac{f}{| \cdot |^\alpha} * e \right) \right\|_{L^{s, \infty}(\mathbb{C}^n \times \mathbb{R})}. \]

Define \( \frac{1}{s} = \frac{1}{p} + \frac{1}{q} - 1 \). By Young’s inequality, we obtain

\[ \left\| \left( \frac{f}{| \cdot |^\alpha} * e \right) \right\|_{L^{s, \infty}(\mathbb{R}^{2n+1})} \lesssim \left\| \frac{f}{| \cdot |^\alpha} \right\|_{L^{s, \infty}(\mathbb{R}^{2n+1})} \left\| g \right\|_{L^q(\mathbb{R}^{2n+1})}. \]

The required estimate i.e., \( \| J_1 \|_{L^r, \infty(\mathbb{H}^n)} \lesssim \| f \|_{L^p(\mathbb{H}^n)} \| g \|_{L^q(\mathbb{H}^n)} \), follows once we use the inequality \( \left\| \frac{f}{| \cdot |^\alpha} \right\|_{L^{s, \infty}(\mathbb{R}^{2n+1})} \lesssim \| f \|_{L^p(\mathbb{H}^n)} \). This completes this case.

We are left with the case when \( \beta + \gamma = -Q + \lambda \). As a consequence of the condition \( \alpha + \beta + \gamma \geq 0 \) we obtain \( \alpha \geq Q - \lambda > 0 \). Now (3.3) implies

\[ J_1(x) \lesssim \int_{y \in B(0, \frac{1}{p} \cdot x)} f(xy^{-1})g(xy) xy^{-1}|\alpha| dy \lesssim \left( \frac{f}{| \cdot |^\alpha} * e \right)(2x). \]

Observe that in this case \( \frac{1}{s} = \frac{1}{p} + \frac{1}{q} - 1 + \frac{\alpha}{Q} \). Define \( \frac{1}{r} = \frac{1}{p} + \frac{\alpha}{Q} \), therefore \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \). Therefore, by Young’s inequality, we obtain

\[ \| J_1 \|_{L^r, \infty(\mathbb{R}^{2n+1})} \lesssim \left\| \frac{f}{| \cdot |^\alpha} * e \right\|_{L^{s, \infty}(\mathbb{R}^{2n+1})} \lesssim \left\| \frac{f}{| \cdot |^\alpha} \right\|_{L^{s, \infty}(\mathbb{R}^{2n+1})} \left\| g \right\|_{L^q(\mathbb{R}^{2n+1})} \lesssim \| f \|_{L^p(\mathbb{H}^n)} \left\| g \right\|_{L^q(\mathbb{H}^n)}. \]

**Estimate for \( J_2 \):** This case is similar to \( J_1 \), so we skip it.

This completes the proof of inequality 3.1. Once we have the weak type inequalities, achieving the strong type inequality just uses the multilinear interpolation Theorem 1.3. We explain it here. For fixed \( \alpha, \beta \) and \( \gamma \) in Theorem 1.2, we have

\[ 0 < \frac{1}{p} < \min \left[ 1, 1 - \frac{\alpha}{Q} \right], \quad \text{and} \quad 0 < \frac{1}{q} < \min \left[ 1, 1 - \frac{\beta}{Q} \right]. \]  

(3.4)

The condition \( \frac{1}{r} \leq \frac{1}{p} + \frac{1}{q} \) is equivalent to \( \alpha + \beta + \gamma \leq \lambda \), which combined with \( 0 \leq \alpha + \beta + \gamma \), gives \( 0 \leq \alpha + \beta + \gamma \leq \lambda \). Further, the conditions \( r < \infty \) and \( \gamma < \frac{Q}{p} \) being, respectively, equivalent to \( \frac{\lambda - (\alpha + \beta + \gamma)}{Q} < \frac{1}{p} + \frac{1}{q} \) and \( \frac{-\alpha - \beta + \lambda}{Q} < \frac{1}{p} + \frac{1}{q} \), lead to

\[ \frac{-\alpha - \beta + \lambda}{Q} + \max \left[ -\frac{\gamma}{Q} , 0 \right] < \frac{1}{p} + \frac{1}{q}. \]

Therefore, consider the open convex set in \( \mathbb{R}^2 \),
\[ C_{\alpha,\beta,\gamma} := \left\{ (x, y) \in (0, 1)^2 : x + y > \frac{-\alpha - \beta + \lambda}{Q} + \max \left[ -\frac{\gamma}{Q}, 0 \right], \right. \\
0 < x < \min \left[ 1, 1 - \frac{\alpha}{Q} \right], \left. \left. 0 < y < \min \left[ 1, 1 - \frac{\beta}{Q} \right] \right\} \right. . \]

Depending on the sign of \( \alpha, \beta \) and \( \gamma \), the set \( C_{\alpha,\beta,\gamma} \subseteq (0, 1)^2 \) changes. But, in all cases, for each point \((\frac{1}{p}, \frac{1}{q})\) in \( C_{\alpha,\beta,\gamma} \) one can always choose three non-collinear points inside \( C_{\alpha,\beta,\gamma} \) such that \((\frac{1}{p}, \frac{1}{q})\) is contained in the interior of the solid triangle inside \( C_{\alpha,\beta,\gamma} \), determined by these three points. Therefore, in view of Theorem 1.3, it suffices to show the “weak-type” inequality for \((\frac{1}{p}, \frac{1}{q})\) in \( C_{\alpha,\beta,\gamma} \).

\[ \square \]

### 3.2. The necessary conditions.

In [KF20], some counter examples were constructed to conclude necessary conditions for the boundedness of \( BI_\lambda \) on the real line. Here, we construct them on the Heisenberg group \( H^n \) of any dimension.

Recall that the inequality (1.5) is equivalent to the following unweighted boundedness

\[ \| S_{\alpha,\beta,\gamma}(f, g) \|_{L^r(H^n)} \leq K \| f \|_{L^p(H^n)} \| g \|_{L^q(H^n)}, \]

where the operator \( S_{\alpha,\beta,\gamma}(f, g) \) is defined as follows

\[ S_{\alpha,\beta,\gamma}(f, g)(x) := |x|^{-\gamma} \int_{\mathbb{H}^n} \frac{f(xy^{-1}) g(xy)}{|xy|^{\alpha} |y|^{\beta} |y|^{Q - \lambda}} dy. \]  

**Necessity of \(-Q + \lambda \leq \alpha + \beta\) in Theorem 1.2**: Suppose \( \alpha + \beta < -Q + \lambda \). Since \( \gamma < \frac{Q}{r} \), whence

\[ (-\alpha - \beta - Q + \lambda), \quad \frac{Q}{r} - \gamma > 0. \]

Also, recall that

\[ \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{\lambda}{Q} + \frac{\alpha + \beta + \gamma}{Q}, \]

which implies that \( \frac{1}{p} + \frac{1}{q} > 1 \).

Let \( N, M \gg 1 \) to be specified later. For \( a \in \mathbb{Z}^{2n+1} \), consider sets

\[ E_a = a \cdot \delta_{r_a} Q(0, 1) \cdot a, \]

where \( r_a = |a|^{-N-1} \) and \( Q(0, r) := [0, r]^{2n} \times [0, r^2], 0 < r < \infty \). Observe that

\[ E_a = 2a + Q(0, r_a). \]

Here, “ + ” denotes usual addition in \( \mathbb{R}^{2n+1} \). Take functions

\[ f(\xi) := \sum_{a \in \mathbb{Z}^{2n+1} \setminus \{0\}} |a|^{M/p} \chi_{E_a}(\xi), \quad \xi \in \mathbb{H}^n, \]

and

\[ g(\xi) := \sum_{a \in \mathbb{Z}^{2n+1} \setminus \{0\}} |a|^{M/q} \chi_{E_a}(\xi), \quad \xi \in \mathbb{H}^n, \]

The functions \( f \in L^p(\mathbb{H}^n) \) and \( g \in L^q(\mathbb{H}^n) \)

\[ M < QN. \]

We will show that for these choice of functions, \( S_{\alpha,\beta,\gamma}(f, g)(x) \chi_{|x|<1} \notin L^r(\mathbb{H}^n) \).
Fix $x \in \mathbb{R}_+^{2n+1} := (0, \infty)^{2n+1}$ and choose $K_0 \gg 1$ such that $(K_0 + 1)^{-N - 1} \leq |x| < K_0^{-N - 1}$. Fix $a$ such that $|a| < K_0/2$. Consider sets
\[ \widetilde{E}_{a,x} := E_a \cdot x \cap x^{-1} \cdot E_a. \]
By definition, whenever $y \in \widetilde{E}_{a,x}$ then $yx^{-1}, xy \in E_a.$

For $|a| < K_0/2$, we see that $|\widetilde{E}_{a,x}| \gtrsim r_a^Q = |a|^{-QN - Q}$. Indeed, we observe that
\[ \widetilde{E}_{a,x} = x^{-1} \cdot (x \cdot E_a \cdot x \cap E_a) = x^{-1} \cdot \left( [2x + 2a + Q(0,r)] \cap [2a + Q(0,r)] \right). \]
Thus, $|\widetilde{E}_{a,x}| = \left[ \prod_{j=1}^{2n}(r_a - 2x_{2n+1}) \right] (r_a^2 - 2x_{2n+1}).$ From our choice of $K_0$, we have $x_{2n+1} < K_0^{-2(N+1)}$, whence $r_a^2 - 2x_{2n+1} > |a|^{-2(N+1)} - 2K_0^{-2(N+1)} \gtrsim |a|^{-2(N+1)} = r_a^2$, since $|a| < K_0/2$. Similarly, $r_a - 2x_j \gtrsim r_a, j = 1, \ldots, 2n$. Combining the above estimates, we have $|\widetilde{E}_{a,x}| \gtrsim r_a^Q$.

For $y \in \widetilde{E}_{a,x}$, we have $|xy|, |yx^{-1}|, |y| \simeq |a|$. Indeed, $xy \in E_a$ so $xy = a \cdot \xi \cdot a = 2a + \xi$ for some $\xi \in Q(0,r)$. Thus, $|xy|^4 = \left[ \sum_{j=1}^{2n}(2a_j + \xi_j)^2 \right]^2 + (2a_{2n+1} + \xi_{2n+1})^2 \geq |a|^4$. So, we have $|a| \leq |xy| \leq |x| + |y|$ which gives $|y| \geq |a| - |x| \gtrsim |a|$. Since $y \in E_a \cdot x$, so $|y| \simeq |a|$ which in turn implies $|xy|, |y| \simeq |a|$. Similarly, $|yx^{-1}| \simeq |a|$.

For fixed $|x| \ll 1$, the collection $\{E_{a,x}, a \in \mathbb{Z}^{2n+1}\}$ is a disjoint family of sets. Indeed, $\widetilde{E}_{a,x}$’s are disjoint if and only if the sets $x \cdot E_a \cdot x \cap E_a = [2x + 2a + Q(0,r)] \cap [2a + Q(0,r)]$ are disjoint, which is true since $|x| \ll 1$.

Therefore, for $(K_0 + 1)^{-N - 1} \leq |x| < K_0^{-N - 1},$
\[
S_{a,\beta,\gamma}(f,g)(x) \gtrsim |x|^{-\gamma} \sum_{a \in \mathbb{Z}^{2n+1} \setminus \{0\}, |a| < K_0/2} \int_{y \in \widetilde{E}_{a,x}} \frac{f(yx^{-1})}{|yx^{-1}|^\alpha} |x|^{-\gamma} |y|^{-Q - \lambda} \ dy \gtrsim |x|^{-\gamma} \sum_{a \in \mathbb{Z}^{2n+1} \setminus \{0\}, |a| < K_0/2} |a|^{-Q - \lambda} + M \left( \frac{1}{p} + \frac{1}{q} \right) |a|^{-QN - Q},
\]
Assuming,
\[
M \left( \frac{1}{p} + \frac{1}{q} \right) = QN > 0,
\]
we are dealing with the sum of the form
\[ \sum_{a = (a', a_{2n+1}) \in \mathbb{Z}^{2n+1} \setminus \{0\}, |a'|^4 + a_{2n+1}^2} 1^{1/4} \leq K_0/2 \]
Hence,
\[ S_{a,\beta,\gamma}(f,g)(x) \gtrsim |x|^{-\gamma} \frac{(-\alpha - \beta - Q + \lambda) + M \left( \frac{1}{p} + \frac{1}{q} \right) - QN}{N + 1} \chi_{|x| \ll 1}, \]
which implies $\|S_{a,\beta,\gamma}(f,g)\|_{L^r(\mathbb{R}^n)}$ will diverge if
\[
\gamma + \frac{(-\alpha - \beta - Q + \lambda) + M \left( \frac{1}{p} + \frac{1}{q} \right) - QN}{N + 1} \geq \frac{Q}{r} \iff \gamma(N + 1) - Q(N + 1) + \lambda - (\alpha + \beta) + M \left( \frac{1}{p} + \frac{1}{q} \right) \geq \frac{Q}{r} + QN \left( \frac{1}{p} + \frac{1}{q} \right) - N\lambda + (\alpha + \beta + \gamma)N \iff (-\alpha - \beta - Q + \lambda)(N + 1) > \left( \frac{Q}{r} - \gamma \right) + (QN - M) \left( \frac{1}{p} + \frac{1}{q} \right). \]
Here, we have used (3.8). First pick out $N \gg 1$. Since $\frac{1}{p} + \frac{1}{q} > 1$, we can choose $M$ close to $QN$ such that (3.9) and (3.10) are satisfied. Subsequently, the last inequality holds true for large $N$ because of (3.7). Thus we arrive at a contradiction. Therefore, we must have $\alpha + \beta \geq -Q + \lambda$.

**Necessity of $-Q + \lambda \leq \beta + \gamma$ in Theorem 1.2**: Assume $-\beta - \gamma - Q + \lambda > 0$. For $x \in \mathbb{R}_{+}^{2n+1}$ such that $|x| \gg 1$, consider the following portion of $S_{\alpha, \beta, \gamma}(f, g)(x)$:

$$
\int_{y \in Q(0, |x|/2\sqrt{n}) \cdot x} \frac{|x|^{-\gamma}}{|xy|^\beta |y|^{Q - \lambda}} f(xy^{-1})g(xy) \frac{dy}{|xy^{-1}|^\alpha}. \tag{3.11}
$$

Arguing as in the previous example, we see that if $y \in Q(0, |x|/2\sqrt{n}) \cdot x$ then $|xy|, |y| \simeq |x|$. Therefore, (3.11) is bounded below by a constant times of the following

$$
|x|^{-\beta - \gamma + \lambda - Q} \int_{y \in Q(0, |x|/2\sqrt{n}) \cdot x} f(xy^{-1})g(xy) \frac{dy}{|xy^{-1}|^\alpha}. \tag{3.12}
$$

Take $f(y) = |y|^{-s} \chi_{Q(0,1)}(y)$ with $s < \frac{Q}{p}$ so that $f \in L^p(\mathbb{H}^n)$. Performing the change of variables $y \to y \cdot x$, (3.12) becomes

$$
|x|^{-\beta - \gamma + \lambda - Q} \int_{y \in Q(0, |x|/2\sqrt{n})} g(x \cdot y \cdot x) \frac{dy}{|y|^{s + \alpha}}. \tag{3.13}
$$

Next, we choose

$$
g(\xi) := \sum_{a \in \mathbb{Z}^{2n+1} : |a| > e} |a|^{Q(N-1)/q} (\log |a|)^{-2/q} \chi_{E_a}(\xi), \quad \xi \in \mathbb{H}^n,
$$

where $E_a := a \cdot Q(0, r_a) \cdot a$, $r_a = \frac{1}{|a|^{Q+1}}$. The function $g \in L^q(\mathbb{H}^n)$, which follows from disjointness of the sets $E_a = 2a + Q(0, r_a)$ and their measure $|E_a| = |Q(0, r_a)| = |a|^{-QN}$.

Setting $E_a := a^{1/2} \cdot Q(0, r_a/4) \cdot a^{1/2}$, where the notation $(z, t)^{1/2}$ means $(z/2, t/2)$ for $(z, t) \in \mathbb{H}^n$.

If $x \in E_a := a^{1/2} \cdot Q(0, r_a/4) \cdot a^{1/2}$ and $y \in Q(0, r_a/2)$, then $xyx \in E_a = a \cdot Q(0, r_a) \cdot a$ and $|x| \simeq |a|$. Indeed, for such $x$ and $y$, $xyx = 2x + y \in 2(a^{1/2} \cdot Q(0, r_a/4) \cdot a^{1/2}) + Q(0, r_a/2) = 2[a + Q(0, r_a/4)] + Q(0, r_a/2) \subset 2a + 2Q(0, r_a/4) + Q(0, r_a/2) \subset 2a + Q(0, r_a) = a \cdot Q(0, r_a) \cdot a = E_a$. Also, $x = a + \xi$, for some $\xi = (\xi', \xi_{2n+1}) \in [0, \frac{1}{2})^{2n} \times [0, \frac{1}{4})^2$. Writing $a = (a', a_{2n+1})$, we have $|x|^4 = ||a' + \xi||^4 + (a_{2n+1} + \xi_{2n+1})^2 \geq |a|^4$, where $||a'||$ is the Euclidean norm of $a' \in \mathbb{R}^{2n+1}$.

Further, since $E_a = a^{1/2} \cdot Q(0, r_a/4) \cdot a^{1/2} = a + Q(0, r_a/4)$, so clearly $|E_a| = |Q(0, r_a)| \simeq r_a^Q = |a|^{-QN}$ and $\{E_a\}_{a \in \mathbb{Z}^{2n+1}}$ is a disjoint collection.

Incorporating the above, (3.13) implies

$$
\|S_{\alpha, \beta, \gamma}(f, g)\|_{L^r(\mathbb{H}^n)} \gtrsim \sum_{|a| > e} \left( \int_{x \in E_a} \left| x \right|^{-\beta - \gamma + \lambda - Q} \int_{y \in Q(0, r_a/2)} \frac{g(x \cdot y \cdot x)}{|y|^{s + \alpha}} \frac{dy}{|y|^{N}} \right)^r dx 
\simeq \sum_{|a| > e} |a|^{-\beta - \gamma + \lambda - Q} |a|^{\frac{Q(N-1)}{q}} (\log |a|)^{-\frac{2N}{q}} |a|^{(s + \alpha - Q)Nr} |a|^{-QN}.
$$

Therefore, $\|S_{\alpha, \beta, \gamma}(f, g)\|_{L^r(\mathbb{H}^n)}$ diverges provided
\[-\beta - \gamma + \lambda - Q r + \frac{rQ(N-1)}{q} + (s + \alpha - Q) N r - Q N \geq -Q\]
\[\iff -\beta - \gamma + \lambda - Q + \frac{Q(N-1)}{q} + (s + \alpha - Q) N \geq \frac{Q(N-1)}{r}\]
\[\iff -\beta - \gamma + \lambda - Q + \frac{Q(N-1)}{q} + (s + \alpha - Q) N \geq \frac{(Q N}{p} - \frac{1}{p}) + \frac{Q(N-1)}{q} + (N-1)(\alpha + \beta + \gamma - \lambda)\]
\[\iff N \left(\frac{1}{\alpha} - \gamma - \lambda + \frac{Q}{p} - s\right) \geq \frac{1}{p'} - \alpha.\]

Since \(-\beta - \gamma - Q + \lambda > 0\), we choose \(s < \frac{Q}{p}\) sufficiently close to \(\frac{Q}{p}\) so that \((-\beta - \gamma - Q + \lambda) - \frac{Q}{p} - s > 0\) and then, taking \(N\) large, we have that the last inequality holds true.

**Necessity of \(\alpha + \beta + \gamma \geq 0\) in Theorem 1.2:** Contrarily, suppose \(\gamma_0 := \alpha + \beta + \gamma < 0\). Then, the homogeneity condition takes the form of \(\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{\lambda}{2} > 0\).

Consider the portion of \(S_{\alpha,\beta,\gamma}(f,g)(x)\) in the set \(\{y \in \mathbb{H}^n : |y| \ll |x|\}\), wherein, one has \(|x| \gtrsim |xy|, |xy^{-1}| \gtrsim |x| - |y| \gtrsim |x|:\)
\[S_{\alpha,\beta,\gamma}(f,g)(x) \gtrsim |x|^{-\gamma_0} \int_{\{y \in \mathbb{H}^n : |y| \ll |x|\}} f(xy^{-1})g(xy) \frac{dy}{|y|^{Q-\lambda}}.\] (3.14)
Take \(N \gg 1\), to be specified later, and consider functions
\[f(y) = \sum_{a \in \mathbb{Z}^{2n+1}, |a| > e} |a|^{Q(N-1)/p} (\log |a|)^{-2/p} \chi_{E_a}(y),\]
and
\[g(y) = \sum_{a \in \mathbb{Z}^{2n+1}, |a| > e} |a|^{Q(N-1)/q} (\log |a|)^{-2/q} \chi_{E_a}(y), \quad y \in \mathbb{H}^n,\]
with \(E_a := a \cdot B(0, r_a)^2\), \(r_a = \frac{1}{|a|^n}\). Observe that \(|E_a| = |B(0, r_a)^2| \sim r_a^Q = \frac{1}{|a|^N}\) and that \(E_a, a \in \mathbb{Z}^{2n+1}\), are disjoint sets. Therefore, \(f \in L^p(\mathbb{H}^n)\) and \(g \in L^q(\mathbb{H}^n)\).

Define sets \(E_a := a \cdot B(0, r_a).\) For \(x \in E_a\) and \(y \in B(0, r_a)\), we have \(xy^{-1}, xy \in a \cdot B(0, r_a)^2 = E_a\). Therefore, from (3.14),
\[\|S_{\alpha,\beta,\gamma}(f,g)\|_{L^r(\mathbb{H}^n)} \gtrsim \sum_{|a| > e} \int_{E_a} |x|^{-\gamma_0} \int_{|y| < r_a} f(xy^{-1})g(xy) \frac{dy}{|y|^{Q-\lambda}} dx\]
\[\gtrsim \sum_{|a| > e} \int_{E_a} |x|^{-\gamma_0} |a|^{rQ(N-1)(\frac{1}{p} + \frac{1}{q})} (\log |a|)^{-2r(\frac{1}{p} + \frac{1}{q})} \frac{1}{|a|^{NQ}} dx\]
\[\gtrsim \sum_{|a| > e} |a|^{-\gamma_0} |a|^r Q(N-1)(\frac{1}{p} + \frac{1}{q}) - r N \lambda (\log |a|)^{-2r(\frac{1}{p} + \frac{1}{q})} \frac{1}{|a|^{NQ}}\]
\[= \sum_{|a| > e} |a|^R (\log |a|)^{-2r(\frac{1}{p} + \frac{1}{q})},\]
which diverges if \(R \geq -Q\), wherein we have used that the sets \(E_a, a \in \mathbb{Z}^{2n+1}\), are disjoint, and that for \(x \in E_a = a \cdot B(0, r_a), |x| \sim |a|\). Therefore, in view of the homogeneity condition, it
suffices to check whether

$$-\gamma_0 r + rQ(N - 1) \left( \frac{1}{p} + \frac{1}{q} \right) - rN\lambda - NQ \geq -Q$$

$$\iff -\gamma_0 + Q(N - 1) \left( \frac{1}{p} + \frac{1}{q} \right) - N\lambda \geq \frac{(N - 1)Q}{r}$$

$$\iff -\gamma_0 - Q \left( \frac{1}{p} + \frac{1}{q} \right) + QN \left( \frac{1}{p} + \frac{1}{q} - \frac{\lambda}{Q} - \frac{1}{r} \right) + \frac{Q}{r} \geq 0$$

$$\iff -\gamma_0 - Q \left( \frac{1}{p} + \frac{1}{q} \right) - \gamma_0 N + \frac{Q}{r} \geq 0,$$

which is true for $N$ sufficiently large, since $\gamma_0 < 0$.

**Necessity of $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{q}$ in Theorem 1.2:**

On the contrary, let us assume $\frac{1}{r} > \frac{1}{p} + \frac{1}{q}$. Take $f(x) := |x|^{-Q/p}(\log |x|)^{-\gamma_1} \chi_{\{|x| > 16\}}$ and $g(x) := |x|^{-Q/q}(\log |x|)^{-\gamma_2} \chi_{\{|x| > 16\}}$. It is not hard to see that $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ provided $\gamma_1 > 1/p$ and $\gamma_2 > 1/q$, respectively. For $|x| \gg 1$, we see that

$$S_{\alpha,\beta,\gamma}(f,g)(x) \gtrsim |x|^{-\gamma} \int_{B(0,\frac{|x|}{16})} \frac{f(xy^{-1})g(xy)}{|xy|^{1+\alpha} |xy|^{\beta} |y|^{Q-\lambda}} dy$$

$$\gtrsim |x|^{-\gamma}(\alpha+\beta+\gamma) \int_{B(0,\frac{|x|}{16})} f(xy^{-1})g(xy) \frac{dy}{|y|^{Q-\lambda}}$$

$$\gtrsim |x|^{-\gamma}(\alpha+\beta+\gamma)-Q(\frac{1}{p} + \frac{1}{q})+\lambda(\log |x|)^{\gamma_1+\gamma_2} = |x|^{-Q/r}(\log |x|)^{\gamma_1+\gamma_2}.$$
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