Abstract. Kusuoka’s measure on fractals is a Gibbs measure of a very special kind, since its potential is discontinuous while the standard theory of Gibbs measures requires continuous (in its simplest version, Hölder) potentials. In this paper we shall see that for many fractals it is possible to build a class of matrix-valued Gibbs measures completely within the scope of the standard theory; there are naturally some minor modifications, but they are only due to the fact that we are dealing with matrix-valued functions and measures. We shall use these matrix-valued Gibbs measures to build self-similar bilinear forms on fractals. Moreover, we shall see that Kusuoka’s measure and bilinear form can be recovered in a simple way from the matrix-valued Gibbs measure.

Introduction. First of all, let us briefly explain what we mean by a fractal $G$ on $\mathbb{R}^d$: we mean the attractor of an iterated function system, an object less general than the ones considered in [16]. More precisely, we consider $n$ contractions \[ \psi_1, \ldots, \psi_n \in C^{1,\nu_0}(\mathbb{R}^d, \mathbb{R}^d) \] (0.1) with $\nu_0 \in (0, 1]$; it is standard ([13]) that there is a unique compact set $G \subset \mathbb{R}^d$ such that \[ G = \bigcup_{i=1}^n \psi_i(G). \]

We shall also suppose that the maps $\psi_i$ are the “branches of the inverse” of a Borel map $F: G \to G$. Since the maps $\{\psi_i\}_{i=1}^n$ are contractions, their inverse $F$ is expanding; in Dynamical Systems, expanding maps have been studied extensively (see for instance [17] or [28]); as we shall see, many results on expanding maps carry over to Dirichlet forms on fractals (see [13] or [23] for an introduction to this theory).

Applying ergodic theory to the study of Dirichlet forms on fractals is not new: in section 1.4 of [11] the idea of applying the Ruelle operator to the study of Kusuoka’s measure is attributed to Strichartz. In this paper we try to understand the results of [11] and [16] by looking at them from a slightly different perspective.

In order to explain the connection between expanding maps and Dirichlet forms, we begin recalling the scalar Gibbs measure; though it will play no rôle in our paper, it will guide us in the construction of the matrix-valued one.

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For $v_0 \in (0,1]$ we take $V \in C^{v_0}(G, \mathbb{R})$ and define the scalar Ruelle operator as

$$L_{sc}: C(G, \mathbb{R}) \to C(G, \mathbb{R}), \quad (L_{sc}v)(x) = \sum_{i=1}^{n} e^{V \circ \psi_i(x)} v \circ \psi_i(x). \quad (0.2)$$

Using the Perron-Frobenius theorem one can prove ([24]) that there is $\beta > 0$ and a continuous function $h > 0$ such that $L_{sc}h = \beta h$. Since $L_{sc}$ is a continuous operator from the space of continuous functions into itself, its adjoint $L_{sc}^*$ brings the space of Borel measures into itself; it can be shown that there is a measure $\mu$, called the Gibbs measure, such that $h\mu$ is probability and $L_{sc}\mu = \beta \mu$. One of the properties of $\mu$ is the following: for all $u, v \in C(G, \mathbb{R})$ we have that

$$\frac{1}{\beta} \int_{G} u(L_{sc}v) d\mu = \int_{G} (u \circ F) v d\mu. \quad (0.3)$$

We point out a few consequences of (0.3). First of all, we can write the adjoint $L_{sc}^*$ explicitly, at least for measures absolutely continuous with respect to $\mu$.

$$\frac{1}{\beta} L_{sc}^*(\nu u) = (u \circ F) \nu u,$$

Moreover, (0.3) tells us that $\frac{1}{\beta} L_{sc}v$ is the density of $F_{\nu}v$, where $F_{\nu}$ denotes the push-forward of the measure $\nu$ by $F$; the push forward is defined by

$$\int_{G} f d(F_{\nu}v) = \int_{G} f \circ F d\nu$$

for all $f \in C(G, \mathbb{R})$. Since $L_{sc}h = \beta h$, this implies that $F_{\nu}(h \nu) = h \mu$, i.e. that $h \mu$ is an invariant measure.

Moreover, we know how $\mu[\psi_{i_0} \circ \ldots \psi_{i_l}(G)]$ scales as $l \to +\infty$; namely, there is a constant $D_1 > 0$, independent of the sequence $i_0 i_1 \ldots$, such that

$$\frac{1}{D_1} \leq \beta^{-l} \cdot \exp(V(x) + V(F(x)) + \ldots + V(F^{l-1}(x))) \leq D_1$$

for all $x \in \psi_{i_0} \circ \ldots \psi_{i_l}(G)$.

Kusuoka’s measure $\kappa$ is defined by a similar scaling property; when the maps $\psi_i$ of (0.1) are affine, i.e. $D\psi_i$ is a constant matrix, we have

$$\kappa[\psi_{x_0} \circ \ldots \psi_{x_l}(G)] = \frac{1}{\beta^l} \cdot \text{tr}[\hat{Q}(D\psi_{x_0} \ldots D\psi_{x_l})Q^t(D\psi_{x_0} \ldots D\psi_{x_l})]$$

where $Q, \hat{Q}$ are two suitable symmetric $d \times d$ matrices, $\beta > 0$, $^tA$ denotes the transpose of the matrix $A$ and $\text{tr}$ is the trace. We recalled above that Gibbs measures are associated to a potential $V$ which in the standard theory is Hölder; however, in the case of the harmonic Sierpinski gasket, Bell, Ho and Strichartz proved in [3] that the potential of Kusuoka’s measure is not even continuous. However, Johansson, Öberg and Pollicott develop in [11] a theory of Ruelle’s operator which works in a space large enough to contain the potential of [3].

Our aim in this paper is to show that, if instead of a scalar Gibbs measure one considers a matrix-valued one, then the standard approach of [24] and [28] in $C^{v_0}$ works with only minor modifications.

More precisely, we denote by $M^d$ the space of symmetric $d \times d$ matrices and we define a Ruelle operator

$$L_G: C(G, M^d) \to C(G, M^d)$$
by
\[
(L_G A)(x) = \sum_{i=1}^{n} D\psi_i(x) A(\psi_i(x)) D\psi_i(x).
\]  
(0.4)

The dual space of \(C(G, M^d)\) is the space \(M(G, M^d)\) of \(M^d\)-valued measures on \(G\), and the adjoint \(L_G^*\) of \(L_G\) brings \(M(G, M^d)\) into itself. As we shall see, \(L_G\) and \(L_G^*\) have each a positive-definite eigenvector, which we shall call \(Q_G\) and \(\tau_G\) respectively; they share the same eigenvalue \(\beta > 0\). In lemma 4.8 below we shall prove the following version of (0.3): if \(g \in C(G, \mathbb{R})\) and \(A \in C(G, M^d)\), defining the integral as in section 2 below we have
\[
\int_G \left( g \cdot \left( \frac{1}{\beta} L_G A \right) \right) d\tau_G = \int_G (g \circ F \cdot A, d\tau_G)_{HS}.
\]

The formula above implies that the scalar measure \((Q_G, \tau_G)_{HS}\) (see section 2 for the definition, but HS stays for the Hilbert-Schmidt product of matrices) is invariant.

**Theorem 1.** Let the maps \(\psi_1, \ldots, \psi_n\) and \(F\) satisfy hypotheses (F1)-(F4) of section 1 below and (ND) with constant \(b > 0\) at the beginning of section 4. Let \(G\) be the fractal associated with \(\psi_1, \ldots, \psi_n\) and let \(M^d\) denote the space of \(d \times d\) symmetric matrices. Let the operator \(L_G\) be defined as in (0.4) and let the seminorm \(||D\psi_i||_{\nu_0}\) be smaller than a positive constant that depends on the parameter \(b > 0\) of (ND) (when the maps \(\psi_i\) are affine, this comes for free); then the following holds.

1) There are \(Q_G \in C(G, M^d)\) and \(\beta > 0\) such that
\[
(L_G Q_G)(x) = \beta Q_G(x) \quad \forall x \in G.
\]

The map \(Q_G\) belongs to \(C^{\nu_0}(G, M^d)\) and is unique up to multiplication by a constant; again up to multiplication by a constant, \(Q_G(x)\) is positive-definite for all \(x \in G\).

2) Let \(L_G^*\) denote the adjoint of \(L_G\); then, there is a Borel measure \(\tau_G\) on \(G\) which takes values in \(M^d\) and such that
\[
L_G^* \tau_G = \beta \tau_G.
\]

The measure \(\tau_G\) is unique up to multiplication by a constant; again up to multiplication by a constant, \(\tau_G\) takes values in semi-positive definite matrices.

3) The scalar measures \(||\tau_G||\) and \(\kappa_G = (Q_G, \tau_G)_{HS}\) are mutually absolutely continuous. Moreover, \(\kappa_G\) is ergodic for the map \(F\).

4) We define the form \(\mathcal{E} : C^1(\mathbb{R}^d, \mathbb{R}) \times C^1(\mathbb{R}^d, \mathbb{R}) \to \mathbb{R}\) in the following way (the notation for the integral is in section 2 below):
\[
\mathcal{E}(f, g) = \int_G \langle \nabla f(x), d\tau_G(x) \nabla g(x) \rangle.
\]

Then, \(\mathcal{E}\) is self-similar, i.e.
\[
\mathcal{E}(f, g) = \frac{1}{\beta^3} \sum_{i=1}^{n} \mathcal{E}(f \circ \psi_i, g \circ \psi_i)
\]
for all \(f, g \in C^1(\mathbb{R}^d)\).

5) Let the maps \(\psi_i\) be affine. Then, the measure \(\tau_G\) has the Gibbs property, i.e., for all \(x_0, \ldots, x_l \in \{1, \ldots, n\}\) we have that
\[
\tau_G(\psi_{x_0} \circ \cdots \circ \psi_{x_{l-1}}(G)) = \frac{1}{\beta^l} \cdot \left( D(\psi_{x_0} \circ \cdots \circ \psi_{x_{l-1}}) \right) \cdot \tau_G(G) \cdot t(\psi_{x_0} \circ \cdots \circ \psi_{x_{l-1}}).
\]

We haven't written explicitly the point where we calculate \(D(\psi_{x_0} \circ \cdots \circ \psi_{x_l})\) since these maps are affine and their derivative is constant.
Three remarks are in order, the first of which is the comparison with previous literature. We begin from [11]: the remark at the very end of our paper shows that, on the Harmonic Sierpinski gasket, we find the same measure. However, [11] looks for a scalar Gibbs measure and applies the Ruelle operator to a space of functions strictly larger than $C^{(0,\nu)}(G,\mathbb{R})$; as a bonus, it obtains the exponential decay of correlations (point 3) of proposition 4.5 below) in a space which is strictly larger than ours. Two other papers which use the Ruelle operators are [25] (which uses a Ruelle operator different from ours) and [18]; the Ruelle operator of the appendix of [18] is the same as ours (and of [15], though Kusuoka doesn’t use this name) but is applied to matrices and not to matrix fields. Our approach allows us to treat the case when the maps $\psi_i$ are not affine and recovers the classical fact that the Gibbs measure is an eigenvector of the adjoint of the Ruelle operator. As a last remark, both [18] and [25] are also concerned with the pressure, a concept which has been adapted to matrix products in [6].

The second remark concerns the bilinear form $E$ defined in point 4) of theorem 1 above: its domain is a very familiar space, $C^1(\mathbb{R}^d,\mathbb{R})$. Suppose now that we are given a probability measure $m$ on the fractal $G$; for instance, $m$ could be Kusuoka’s measure $\kappa_G$, though many other choices are possible. Then we don’t know whether $E$ is closable in $L^2(G,\mu)$. In the case of the Harmonic Sierpinski gasket, what we build in point 4) of theorem 1 is Kusuoka’s Dirichlet form; this is already known to be closable for all measures $m$ positive on open sets. However, in general we don’t have ready any yes/no criterion. As a negative example, if $G$ is totally disconnected and $m = \kappa_G$, it is easy to see that the form we find is not closable.

The question whether a bilinear form on a fractal is closable is delicate; we refer the reader to [21] for a counterexample and to [22] (see also [9]) for a theorem on this problem and a presentation of its history; the paper [23] contains an introduction to Dirichlet forms on fractals in general. Our paper does not deal with closability: our aim is to show that the so-called “energy measures” are intimately connected with Gibbs measures, which are the natural measures from the point of view of the dynamics.

If however $E$ can be extended to a closed form, then it is a Dirichlet form (the textbook on the theory is [7], an elementary introduction is in [10]) and we can associate to it a stochastic process on the fractal. Actually, by theorem 3.1.2 of [7], $E$ extends to a local Dirichlet form, and the stochastic process is a diffusion (always [7]) or, very loosely speaking, a Brownian motion; though somewhat imprecise, the term was used at the beginning of the theory ([1], [2], [8], [19], [20]).

Our last remark is the connection of theorem 1 with the harmonic coordinates. Very roughly (the details are, for instance, in [23] and [27]) the usual way to build a fractal $\hat{G} \subset \mathbb{R}^s$ is to start from the pre-fractals, which are finite sets of points $V_0 \subset V_1 \subset \cdots \subset \mathbb{R}^s$. One starts from a quadratic form (or symmetric matrix $D_0$) on $\mathbb{R}^{V_0}$ and defines a symmetric matrix $D_l$ on $V_l$ by

$$D_l = \sum_{i_0,\ldots,i_{l-1} \in V_0} t(M_{i_0\ldots i_{l-1}})D_0(M_{i_0\ldots i_{l-1}}).$$

We don’t dwell on the procedure of harmonic extension which yields the matrices $M_{i_0\ldots i_{l-1}}$; the details are in lemma 3.6 of [27]. On many fractals, self-similarity implies that there are matrices $M_1, \ldots, M_n$ such that

$$M_{i_0i_1\ldots i_{l-1}} = M_{i_0}M_{i_1} \cdots M_{i_{l-1}}.$$
which is another avatar of the Gibbs formula; indeed, both above and in the Gibbs formula we are doing the push-forward of a Riemannian structure on smaller and smaller cells of the fractal. It turns out that the maps $M_i$ are the derivatives of our maps $\psi_i$; the fractal $G$ generated by the maps $\psi_i$ is homeomorphic to $\hat{G}$; the homeomorphism is called a system of harmonic coordinates on $\hat{G}$. In a sense, our paper branches off from this construction and studies the Gibbs measure induced by arbitrary contractions $\psi_i$.

The paper is organised as follows. In section 1 we recall the notation and the basic facts about the Perron-Frobenius theorem, fractal sets and Dirichlet forms. In section 2 we define the convex cones to which we are going to apply the Perron-Frobenius theorem. In section 3 we define the Ruelle operator $L_G$ on matrices and show that the fixed points of its adjoint $L_G^*$ induce a self-similar quadratic form $E$ on $C^1(R^d)$. In section 4, we apply the Perron-Frobenius theorem to find the maximal eigenvector of $L_G$ and the matrix-valued Gibbs measure $\tau_G$. In section 5, we show that $\tau_G$ has the Gibbs property.

1. Preliminaries and notation. The Perron-Frobenius theorem. We follow [28] (see also [4] for the original treatment).

Let $X$ be a real vector space; we say that $C \subset X \setminus \{0\}$ is a cone if

$$v \in C \quad \text{and} \quad t > 0 \quad \text{implies that} \quad tv \in C.$$ 

Let $C \subset X$ be a convex cone; we say that $w \in \bar{C}$ if there are $v \in C$ and $t_n \searrow 0$ such that $w + t_nv \in C$ for all $n \geq 1$. In what follows, we shall suppose that $C$ is a convex cone such that $
abla C \cap (-\bar{C}) = \{0\}$. (1.1)

If $v_1, v_2 \in C$, we define

$$\alpha(v_1, v_2) = \sup \{t > 0 : v_2 - tv_1 \in C\} \quad (1.2)$$

$$\frac{1}{\beta(v_1, v_2)} = \sup \{t > 0 : v_1 - tv_2 \in C\} \quad (1.3)$$

and

$$\theta(v_1, v_2) = \log \frac{\beta(v_1, v_2)}{\alpha(v_1, v_2)}. \quad (1.4)$$

Since $\theta(v, \lambda v) = 0$ for all $\lambda > 0$, we identify the points of a ray; namely, we say that $v_1 \sim v_2$ if $v_2 = tv_1$ for some $t > 0$; we shall denote by $\frac{C}{\sim}$ the set of equivalence classes.

We have that $\theta(v_1, v_2) \in [0, +\infty]$ for all $v_1, v_2 \in C$; if $\theta$ never assumes the value $+\infty$, then $\theta$ is a distance on $\frac{C}{\sim}$.

The following proposition from [28] allows us to use the contraction principle.

**Proposition 1.1.** 1) Let $L \colon X \to X$ be a linear operator such that $L(C) \subset C$ and let us define

$$D = \sup \{\theta(Lv_1, Lv_2) : v_1, v_2 \in C\}.$$ 

Then, if $D < +\infty$, $L$ is a contraction on $(\frac{C}{\sim}, \theta)$, namely

$$\theta(Lv_1, Lv_2) \leq (1 - e^{-D})\theta(v_1, v_2) \quad \forall v_1, v_2 \in C.$$ 

2) As a consequence of 1), if $D < +\infty$ and $(\frac{C}{\sim}, \theta)$ is a complete metric space, there is $(\lambda, v) \in (0, +\infty) \times \frac{C}{\sim}$, unique in $(0, +\infty) \times \frac{C}{\sim}$, such that

$$Lv = \lambda v.$$
Moreover, if \( w \in C \), then
\[
\theta(L^n w, v) \leq \theta(w, v) \frac{(1 - e^{-D})^n}{e^{-D}}. \tag{1.5}
\]

**Fractal sets.** We make the following hypotheses on the fractal set. 

**(F1)** There is \( \nu_0 \in (0, 1] \) and diffeomorphisms
\[
\psi_1, \ldots, \psi_n \in C^{1,\nu_0}(\mathbb{R}^d, \mathbb{R}^d) \tag{1.6}
\]
satisfying
\[
\eta := \sup_{i \in \{1, \ldots, n\}} \text{Lip}(\psi_i) < 1. \tag{1.7}
\]

By theorem 1.1.7 of [13], this implies that there is a unique non empty compact set \( G \subset \mathbb{R}^d \) such that
\[
G = \bigcup_{i=1}^n \psi_i(G). \tag{1.8}
\]

In the following, we shall always rescale the norm of \( \mathbb{R}^d \) in such a way that
\[
\text{diam}(G) \leq 1. \tag{1.9}
\]

If (F1) holds, then the dynamics of \( F \) on \( G \) can be coded. Indeed, we define \( \Sigma \) as the space of sequences
\[
\Sigma = \{1, \ldots, n\}^\mathbb{N} = \{\{x_i\}_{i \geq 0} : x_i \in \{1, \ldots, n\}, \forall i \geq 0\}
\]
with the product topology. This is a metric space; for instance, if \( \gamma \in (0, 1) \), we can define the metric
\[
d_\gamma(\{x_i\}_{i \geq 0}, \{y_i\}_{i \geq 0}) = \gamma^k
\]
where
\[
k = \inf\{i \geq 0 : x_i \neq y_i\},
\]
with the convention that the inf of the empty set is \(+\infty\) and \( \gamma^{+\infty} = 0 \).

We define the shift \( \sigma \) as
\[
\sigma : \Sigma \to \Sigma, \quad \sigma : \{x_0, x_1, x_2, \ldots\} \to \{x_1, x_2, x_3, \ldots\}.
\]

If \( x_0, \ldots, x_l \in \{1, \ldots, n\} \), we define the cylinder
\[
[x_0 \ldots x_l] = \{\{y_i\}_{i \geq 0} : y_i = x_i \text{ for } i \in \{1, \ldots, l\}\}.
\]

We also set
\[
\psi_{x_0 \ldots x_l} = \psi_{x_0} \circ \cdots \circ \psi_{x_l}
\]
and
\[
[x_0 \ldots x_l]_G = \psi_{x_0} \circ \psi_{x_1} \circ \cdots \circ \psi_{x_l}(G). \tag{1.10}
\]
If \( x = (x_0 x_1 \ldots) \) we set \( (ix) = (ix_0 x_1 \ldots) \). Now (1.10) implies that
\[
\psi_i([x_1 \ldots x_l]_G) = [ix_1 \ldots x_l]_G. \tag{1.11}
\]

Since the maps \( \psi_i \) are continuous and \( G \) is compact, the sets \( [x_0 \ldots x_l]_G \subset G \) are compact. By (1.8) we have that \( \psi_i(G) \subset G \) for \( i \in \{1, \ldots, n\} \); this implies that, for all \( \{x_i\}_{i \geq 0} \in \Sigma \)
\[
[x_0 \ldots x_l-1]_G \subset [x_0 \ldots x_{l-1}]_G.
\]

From (1.7), (1.9) and (1.10) we get that
\[
\text{diam}([x_0 \ldots x_l]_G) \leq \eta^l. \tag{1.12}
\]
Let \( \{x_i\}_{i \geq 0} \subset \Sigma \); by the last two formulas and the finite intersection property we have that
\[
\bigcap_{l \geq 1} [x_0 \ldots x_l]_G
\]
is a single point, which we call \( \Phi(\{x_i\}_{i \geq 0}) \); formula (1.12) implies in a standard way that the map \( \Phi: \Sigma \to G \) is continuous. It is not hard to prove, using (1.8), that \( \Phi \) is surjective. We shall call \( d \) the distance on \( G \) induced by the Euclidean distance on \( \mathbb{R}^d \) and, from now on, in our choice of the metric on \( \Sigma \) we take \( \gamma \in (\eta, 1) \); this implies by the definition of \( d, \gamma \) and (1.12) that \( \Phi \) is 1-Lipschitz.

(F2) If \( i \neq j \), \( \psi_i(G) \cap \psi_j(G) \) is a finite set. We set
\[
\mathcal{F} = \bigcup_{i \neq j} \psi_i(G) \cap \psi_j(G).
\]
We ask that there are disjoint open sets \( \mathcal{O}_1, \ldots, \mathcal{O}_n \subset \mathbb{R}^d \) such that \( G \in \mathcal{O}_i = \psi_i(G) \setminus \mathcal{F} \) for \( i \in (1, \ldots, n) \).

We briefly prove that this implies that the coding \( \Phi \) is finite-to-one and that the set \( N \subset \Sigma \) where \( \Phi \) is not injective is countable. More precisely, we assert that the set with only one preimage by \( \Phi \) contains
\[
G \setminus \bigcup_{l \geq 0} \bigcup_{x_0 \ldots x_l} \psi_{x_0 \ldots x_l}(\mathcal{F})
\]
and that the points on the union on the right have at most \( n \) preimages.

First of all, if \( x \in G \setminus \mathcal{F} \), then \( x = \Phi(\{x_j\}_{j \geq 0}) \) can belong to at most one \( \psi_i(G) \), and thus there is only one choice for \( x_0 \). Using the fact that \( \psi_x \) is a diffeo, we see that there is at most one choice for \( x_1 \) if moreover \( x \notin \psi_{x_0}(\mathcal{F}) \); iterating, we see that the points in the set above have at most one preimage.

If \( x \in \mathcal{F} \), then \( x \) can belong to at most \( n \) sets \( \psi_i(G) \), and there are at most \( n \) choices for \( x_0 \). As for the choice for \( x_1 \), once we have fixed \( x_0 \) then we have that \( x \in \psi_{x_0} \circ \psi_{x_1}(G) \), i.e. that \( \psi_{x_0}^{-1}(x) \in \psi_{x_1}(G) \). Since \( x \in \mathcal{F} \), (F3) implies that \( \psi_{x_0}^{-1}(x) \notin \mathcal{F} \) and thus we have at most one choice for \( x_1 \). Iterating, we see that there is at most one choice for \( x_2, x_3, \text{etc...} \) if \( x \notin \mathcal{F} \) but \( \psi_{x_0}^{-1}(x) \in \mathcal{F} \), then we have one choice for \( x_0 \) but \( n \) choices for \( x_1 \); arguing as above we see that there is at most one choice for \( x_2, x_3, \text{etc...} \) Iterating, we get the assertion.

(F4) We ask that there are disjoint open sets \( \mathcal{O}_1, \ldots, \mathcal{O}_n \subset \mathbb{R}^d \) such that
\[
G \cap \mathcal{O}_i = \psi_i(G) \setminus \mathcal{F} \quad \text{for} \quad i \in (1, \ldots, n).
\]
We define a map \( F: \bigcup_{i=1}^n \mathcal{O}_i \to \mathbb{R}^d \) by
\[
F(x) = \psi_i^{-1}(x) \quad \text{if} \quad x \in \mathcal{O}_i.
\]
If moreover we ask that \( \mathcal{O}_i \subset \psi_i^{-1}(\mathcal{O}_i) \) (or, equivalently, that \( \psi_i(\mathcal{O}_i) \subset \mathcal{O}_i \), since the maps \( \psi_i \) are diffeos), this implies the first equality below.

\[
F \circ \psi_i(x) = x \quad \forall x \in \mathcal{O}_i \subset \psi_i^{-1}(\mathcal{O}_i) \quad \text{and} \quad \psi_i \circ F(x) = x \quad \forall x \in \mathcal{O}_i. \tag{1.13}
\]
We call \( a_i \) the unique fixed point of \( \psi_i \); note that, by (1.8), \( a_i \in G \). If \( x \in \mathcal{F} \), we define \( F(x) = a_j \) for some arbitrary \( a_j \). This defines \( F \) as a Borel map on all of \( G \), which satisfies (1.13).

By (1.10) and the definition of \( \Phi \) we easily get that \( [x_0 \ldots x_l] \subset \Phi^{-1}([x_0 \ldots x_l]_G) \); since \( \Phi \) is finite to one and the set where it is not injective is countable, we have that
\[
\sharp(\Phi^{-1}([x_0 \ldots x_l]_G) \setminus [x_0 \ldots x_l]) \leq \sharp \mathbb{N}. \tag{1.14}
\]
Note that, if \( x = (x_0 x_1 \ldots) \), then by the definition of \( \Phi \)
\[
\Phi \circ \sigma (x) = \bigcap_{l \geq 1} [x_1 \ldots x_l]_G.
\]
The definition of \( \Phi \) implies the first equality below; if we suppose that \( \Phi(x) \in O_{x_0} \)
and recall that \( F = \psi_{x_0}^{-1} \) on \( O_{x_0} \) we get the middle one while the last equality comes
from the formula above.
\[
F \circ \Phi(x) = F \left( \bigcap_{l \geq 1} [x_0 \ldots x_l]_G \right) = \bigcap_{l \geq 1} [x_1 \ldots x_l]_G = \Phi \circ \sigma (x).
\]
In other words, the first equality below holds save when \( \Phi(x) \in F \). The second
equality below follows for all \( x \in G \) from (1.11).
\[
\begin{cases}
\Phi \circ \sigma (x) = F \circ \Phi(x) & \text{save possibly when } \Phi(x) \in F, \\
\Phi(i, x) = \psi_i(\Phi(x)) & \forall x \in \Sigma,
\end{cases}
\quad (1.15)
\]
In other words, up to a change of coordinates, shifting the coding one place to the
left is the same as applying \( F \). Iterating the first one of (1.15) we get that, for all \( l \geq 1, \)
\[
\Phi \circ \sigma^l (x) = F^l \circ \Phi(x) \quad \text{save possibly for } x \in \bigcup_{j \geq 0} \sigma^{-j}(\Phi^{-1}(F)).
\quad (1.16)
\]
Note that the union on the right is a countable set, since \( F \) is finite by (F2) and \( \Phi \)
and \( \sigma \) are finite to one.

A particular case we have in mind is the harmonic Sierpinski gasket on \( \mathbb{R}^2 \) ([12],
[14], see also [5] for the geometry of the \( n \)-dimensional harmonic gasket). We set
\[
T_1 = \left( \frac{3}{5}, 0 \right), \quad T_2 = \left( \frac{3}{10}, \frac{\sqrt{3}}{10} \right), \quad T_3 = \left( \frac{3}{10}, \frac{\sqrt{3}}{10}, \frac{1}{2} \right),
\]
\[
A = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}, \quad C = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}
\]
and
\[
\psi_1(x) = T_1(x), \quad \psi_2(x) = B + T_2 (x - B), \quad \psi_3(x) = C + T_3 (x - C).
\]
Referring to the figure below, \( \psi_1 \) brings the triangle \( ABC \) into \( Abc \); \( \psi_2 \) brings \( ABC \)
into \( Bac \) and \( \psi_3 \) brings \( ABC \) into \( Cba \). We take \( O_1, O_2, O_3 \) as three disjoint open
sets which contain, respectively, the triangle \( Abc \) minus \( b, c \), \( Bca \) minus \( c, a \) and \( Cba \)
minus \( a, b \).

We define the map \( F \) as
\[
F(x) = \psi_i^{-1}(x) \quad \text{if } x \in O_i
\]
and we extend it as in (F4) on \( \{a, b, c\} \).
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Fig. 1. The first stage of the harmonic gasket.

It is easy to check that the fractal $G$ generated by $\psi_1, \psi_2, \psi_3$ satisfies hypotheses (F1)-(F4) above; it is easy to check that it also satisfies (ND) of section 4 below.

The invariant Dirichlet form. Let $(X, \mathring{d}, \nu)$ be a metric measure space; we suppose for simplicity that $(X, \mathring{d})$ is compact and $\nu$ is probability.

A Dirichlet form is a symmetric bilinear form

$$\mathcal{E}: \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \to \mathbb{R}$$

defined on a dense set $\mathcal{D}(\mathcal{E}) \subseteq L^2(X, \nu)$ such that the two conditions below hold.

(D1) $\mathcal{D}(\mathcal{E})$ is closed under the graph norm; in other words, $\mathcal{D}(\mathcal{E})$ is a Hilbert space for the norm

$$||u||^2_{\mathcal{D}(\mathcal{E})} = ||u||^2_{L^2(S, \nu)} + \mathcal{E}(u, u).$$

(D2) $\mathcal{E}$ is Markovian, i. e.

$$\mathcal{E}(\eta \circ f, \eta \circ f) \leq \mathcal{E}(f, f)$$

for all $f \in \mathcal{D}(\mathcal{E})$ and all 1-Lipschitz maps $\eta: \mathbb{R} \to \mathbb{R}$ with $\eta(0) = 0$.

We list some additional properties a Dirichlet form can have.

(D3) $\mathcal{E}$ is regular; this means that $\mathcal{D}(\mathcal{E}) \cap C(X, \mathbb{R})$ is dense in $C(X, \mathbb{R})$ for the topological and in $\mathcal{D}(\mathcal{E})$ for the graph norm (1.17).

(D4) $\mathcal{E}$ is strongly local, i. e.

$$\mathcal{E}(f, g) = 0$$

whenever $f, g \in \mathcal{D}(\mathcal{E})$ and $f$ is constant in a neighbourhood of the support of $g$.

It can be proven ([7]) that, if $\mathcal{E}$ satisfies (D1)-(D4), then it is the Dirichlet form of a Hunt process with continuous trajectories. Historically, the diffusion process was built first ([1], [2], [8], [15]); then it was realised ([19], [20]) that it was more convenient to build the Dirichlet form and deduce the existence of the Brownian motion from the theorem of [7] we just quoted.

(D5) $\mathcal{E}$ is self similar, i. e. there is $\beta > 0$ such that

$$\mathcal{E}(u, v) = \beta \sum_{i=1}^{n} \mathcal{E}(u \circ \psi_i, v \circ \psi_i)$$

for all $u, v \in \mathcal{D}(\mathcal{E})$. 

As we stated in the introduction, we shall be able to build a local and self-similar form on \( C^1(\mathbb{R}^d, \mathbb{R}) \), but not to prove its closability.

2. Function spaces and cones. We begin listing three equivalent ways to define the norm of a matrix \( A \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d) \).

The first one is the sup norm

\[
\|A\| = \sup\{\|Av\| : \|v\| \leq 1\}.
\]

We denote by \( ^tA \) the adjoint of the matrix \( A \) and by \( \text{tr}(A) \) its trace; on the space of all matrices we can define the inner product

\[
(A, B)_{HS} = \text{tr}(^tAB).
\]

This inner product induces the Hilbert-Schmidt norm

\[
\|A\|^2_{HS} = \text{tr}(^tAA).
\]

If \( A \) is symmetric we can set

\[
\|A\| = \sup\{(Av, v) : \|v\| \leq 1\}
\]

where we have denoted by \((\cdot, \cdot)\) the inner product of \( \mathbb{R}^d \).

Clearly, if \( A \) is symmetric, \( \|A\| = \|A\|_1 \) is the modulus of the largest eigenvalue of \( A \), while \( \|A\|_{HS} \) is the quadratic mean of the eigenvalues; it is standard that there is \( D_1 > 0 \) such that

\[
\frac{1}{D_1}\|A\|_{HS} \leq \|A\| \leq D_1\|A\|_{HS} \tag{2.1}
\]

for all \( A \) symmetric.

We define \( M^d \) as the space of \( d \times d \) symmetric matrices; we recall that \( A \in M^d \) is positive semidefinite if

\[
(v, Av) \geq 0 \quad \forall v \in \mathbb{R}^d. \tag{2.2}
\]

It is standard that \( B \in M^d \) is positive semidefinite if and only if

\[
(A, B)_{HS} \geq 0 \tag{2.3}
\]

for every \( A \in M^d \) satisfying (2.2). We briefly prove this fact. Let \( B \) satisfy (2.3); if we let let \( A \) vary among the one-dimensional projections we easily see that all the eigenvalues of \( B \) are positive, and (2.2) follows. For the converse, we note that, since \( A \) and \( B \) are symmetric and semi positive definite, their square roots are symmetric and real, which implies the first and third equalities below; for the second one, we use the fact that the trace of a product is invariant under cyclic permutations.

\[
\text{tr}(AB) = \text{tr}(\sqrt{A}\sqrt{A}\sqrt{B}\sqrt{B}) = \text{tr}((\sqrt{A}\sqrt{B})(\sqrt{A}\sqrt{B})) \geq 0.
\]

Essentially, (2.3) implies that the angle at the vertex of the cone of positive semidefinite matrices is smaller than \( \frac{\pi}{2} \).

An immediate consequence of (2.3) is the following: if \( A, B, C \in M^d \), if \( A \geq 0 \) and \( B \leq C \), then

\[
(A, B)_{HS} \leq (A, C)_{HS} \tag{2.4}
\]
Let now \((E, \tilde{d})\) be a compact metric space; in the following, \((E, \tilde{d})\) will be either one of \((G, \tilde{d})\) or \((\Sigma, d_\Sigma)\). We define \(C(E, M^d)\) as the space of continuous functions from \(E\) to \(M^d\); for \(A \in C(E, M^d)\) we define
\[
||A||_\infty = \sup_{x \in E} ||A(x)||_{HS}.
\]
Let us call \(\mathcal{M}(E, M^d)\) the space of the Borel measures on \(E\) valued in \(M^d\). Putting on \(M^d\) the Hilbert-Schmidt norm, we can define in the usual way the total variation \(||\tau||\) of a measure \(\tau \in \mathcal{M}(E, M^d)\): by \([26]\), \(||\tau||\) is a scalar-valued, non-negative, finite measure on the Borel sets of \(E\).

If \(\tau \in \mathcal{M}(E, M^d)\) and \(B: E \to M^d\) belongs to \(L^1(E, ||\tau||)\), we define the real number
\[
\int_E (B_x, d\tau(x))_{HS} = \int_E (B_x, T_x)_{HS} d||\tau||(x)
\]
where \(\tau = T_x||\tau||\) is the polar decomposition of \(\tau\); we recall from \([26]\) that \(||T_x||_{HS} = 1\) for \(||\tau||\)-a.e. \(x \in S\).

Several other products are possible; for instance, if \(u, v: E \to \mathbb{R}^d\) are Borel vector fields such that
\[
||u(x)|| : ||v(x)|| \in L^1(E, ||\tau||),
\]
we can define the real number
\[
\int_E (u(x), d\tau(x)v(x)) = \int_E (u(x), T_x v(x)) d||\tau||(x)
\]
where, again, \(\tau = T_x||\tau||\) is the polar decomposition of \(\tau\). Analogously, if \(A: E \to M^d\) is a Borel field of matrices such that \(||A_x||_{HS} \in L^1(E, ||\tau||)\), we can define the two matrices
\[
\int_E A_x d\tau(x) = \int_E A_x T_x d||\tau||(x)
\]
and
\[
\int_E d\tau(x) A_x = \int_E T_x A_x d||\tau||(x).
\]
If \(Q \in C(E, M^d)\) and \(\tau \in \mathcal{M}(E, M^d)\), we define the scalar measure \((Q, \tau)_{HS}\) in the following way: if \(B \subset E\) is Borel, then
\[
(Q, \tau)_{HS}(B) = \int_B (Q, d\tau)_{HS}.
\]
In other words, \((Q, \tau)_{HS} = (Q_x, T_x)_{HS}||\tau||\).

By Riesz’s representation theorem, \(\mathcal{M}(E, M^d)\) is the dual space of \(C(E, M^d)\); the duality coupling
\[
\langle \cdot, \cdot \rangle: C(E, M^d) \times \mathcal{M}(E, M^d) \to \mathbb{R}
\]
is given by
\[
\langle B, \tau \rangle = \int_E (B_x, d\tau(x))_{HS}.
\]
By Lusin’s theorem we get in the usual way that, if \(B \subset E\) is a Borel set, then
\[
||\tau||(B) = \sup_B \int (A, d\tau)_{HS}
\]
where the sup is over all \(A \in C(E, M^d)\) such that \(||A||_{\infty} \leq 1\).

**Remark.** In the discussion above, we should have distinguished between \(M^d\) and its dual \((M^d)^*\); strictly speaking, the dual of \(C(E, M^d)\) is \(\mathcal{M}(E, (M^d)^*)\). In order
to have a simpler notation, we identify $M^d$ and $(M^d)^*$ thanks to the Riemannian structure on $\mathbb{R}^d$. For the same reason, if $f \in C^1(\mathbb{R}^d, \mathbb{R})$, we shall deal with its gradient $\nabla f$ and not with its differential $df$.

We say that $\tau \in \mathcal{M}^+(E, M^d)$ if $\tau \in \mathcal{M}(E, M^d)$ and $\tau(B)$ is a non-negative definite matrix for all Borel sets $B \subset E$. By Lusin’s theorem, this is equivalent to

$$\int_E \langle v_x, d\tau(x)v_x \rangle \geq 0 \quad \forall v \in C(E, \mathbb{R}^d).$$

In turn, by (2.3) this is equivalent to

$$\int_E \langle A_x, d\tau(x) \rangle_{HS} \geq 0$$

for all $A \in C(E, M^d)$ such that $A_x$ is positive semidefinite for all $x \in E$.

Let now $Q \in C(E, M^d)$ such that $Q_x$ is positive-definite for all $x \in E$; since $E$ is compact there is $D_2 > 0$ such that

$$\frac{1}{D_2} \text{Id} \leq Q_x \leq D_2 \text{Id} \quad \forall x \in E. \quad (2.7)$$

For $Q$ satisfying (2.7) we define $\mathcal{P}_Q(E, M^d)$ as the set of all $\tau \in \mathcal{M}^+(E, M^d)$ such that

$$\int_E \langle Q, d\tau \rangle_{HS} = 1.$$

**Lemma 2.1.** Let $Q \in C(E, M^d)$ satisfy (2.7).

1) There is $D_3 > 0$ (depending on the constant $D_2$ of (2.7)) such that for all $\tau \in \mathcal{M}^+(E, M^d)$ and all Borel sets $B \subset E$ we have

$$||\tau||(B) \leq D_3 \cdot \langle Q, \tau \rangle_{HS}(B) \quad (2.8)$$

where $||\cdot||$ denotes total variation.

2) $\mathcal{P}_Q(E, M^d)$ is a convex set of $\mathcal{M}(E, M^d)$, compact for the weak* topology.

**Proof.** We begin with point 1). By the definition of total variation, we must find $D_3 > 0$ with the following property: for all Borel sets $B \subset E$ and all countable Borel partitions $\{B_i\}_{i \geq 1}$ of $B$ we have that

$$\sum_{i \geq 1} ||\tau(B_i)||_{HS} \leq D_3 \cdot \langle Q, \tau \rangle_{HS}(B).$$

By (2.1), this follows if we show that, for the constant $D_2$ of (2.7),

$$\sum_{i \geq 1} ||\tau(B_i)|| \leq D_2 \cdot \langle Q, \tau \rangle_{HS}(B). \quad (2.9)$$

By the definition of $||\tau(B_i)||$ before (2.1), we can find unit vectors $v_i$ such that

$$||\tau(B_i)|| = \langle v_i, \tau(B_i)v_i \rangle \quad \forall i \geq 1.$$

Let now $v \in \mathbb{R}^d$; the inequality below follows in a standard way from the fact that $\tau(B_i)$ is symmetric and positive-semidefinite; the equality comes from the definition of the Hilbert-Schmidt product.

$$\langle v, \tau(B_i)v \rangle \leq \text{tr}(\tau(B_i)||v||^2 = \langle \tau(B_i), \text{Id} \rangle_{HS}||v||^2.$$

Since $v_i$ has unit length, the last two formulas imply the first inequality below; the first equality follows since $\tau$ is a measure and $\{B_i\}_{i \geq 1}$ is a partition of $B$. Since $\tau \in \mathcal{M}^+(G, M^d)$, $\tau(B)$ is positive semidefinite; in particular, (2.4) holds and
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... implies the second inequality below. The last equality follows from the definition of the measure \((Q, \tau)_{HS}\).

\[
\sum_{i \geq 1} ||| \tau(B_i) ||| \leq \sum_{i \geq 1} (\tau(B_i), Id)_{HS} = (\tau(B), Id)_{HS} \leq D_2 \cdot \int_B (Q, d\tau)_{HS} = D_2 \cdot (Q, \tau)_{HS}(B).
\]

This is (2.9) and we are done.

In order to prove point 2), we note that, by (2.6), \(\mathcal{M}^+(E, M^d)\) is a convex set of \(\mathcal{M}(E, M^d)\) closed for the weak* topology; as a consequence, also \(\mathcal{P}_Q(E, M^d)\) is a closed convex set, while (2.8) implies that it is relatively compact for the weak* topology. □

**Definitions.** Let \((E, \hat{d})\) be a compact metric space with \(\text{diam}(E) \leq 1\). We define \(C_+\) as the set of all the \(A \in C(E, M^d)\) such that \(A_x\) is positive-definite for all \(x \in E\); since \(E\) is compact, if \(A \in C_+\), there is \(\epsilon > 0\) (depending on \(A\)) such that \(A_x \geq \epsilon \text{Id} \quad \forall x \in E\). (2.10)

For \(a > 0\) and \(\nu \in (0, 1]\) we define \(C_+(E, a, \nu)\) as the set of all the \(A \in C_+\) such that \(A_x e^{-ad(x,y)^\nu} \leq A_y \leq A_y e^{ad(x,y)^\nu} \quad \forall x, y \in E\). (2.11)

We also define \(C_\nu(E, M^d)\) as the set of all \(\nu\)-Hölder maps from \(E\) to \(M^d\), with the seminorm \(||| A |||_\nu = \sup_{x \neq y \in E} \frac{||| A_x - A_y |||}{\hat{d}(x, y)^\nu}\).

As lemma 2.3 below shows, the last two formulas are two different ways to look at the same seminorm, but we shall need both.

**Lemma 2.2.** Let \(\epsilon > 0\) and let \(A, B \in M^d\) such that

\[ A, B \geq \epsilon \text{Id}. \]

Then, there is \(D_3 = D_3(\epsilon, B) > 0\) such that

\[ Be^{-D_3|||B-A|||_{HS}} \leq A \leq Be^{D_3|||B-A|||_{HS}}. \] (2.13)

For fixed \(\epsilon\), the function \(D_3(\epsilon, B)\) is bounded when \(B\) is bounded.

As a converse, there is \(D_5 > 0\) such that the following holds. Let \(A, B \in M^d\) be semi-positive definite and let us suppose that there is \(D_4 > 0\) such that

\[ e^{-D_4} B \leq A \leq e^{D_4} B. \] (2.14)

Then,

\[ ||| B - A |||_{HS} \leq D_5(e^{D_4} - 1)||| A |||_{HS}. \] (2.15)

**Proof.** We begin with the direct part. Let \(C \in M^d\); it is easy to see (for instance, choosing a base in which \(C\) is diagonal) that

\[ C \leq |||C|||_{HS} \text{Id}. \]

Let \(A, B \in M^d\); if we apply the formula above to \(C = A - B\) we get that

\[ A \leq B + ||| B - A |||_{HS} \text{Id}. \] (2.16)
Since $B$ satisfies (2.12), this implies the first inequality below.

$$A \leq B \left(1 + \frac{1}{\epsilon}||B - A||_{HS}\right) \leq Be^{\frac{1}{2}||B - A||_{HS}}.$$  

This yields the inequality on the right of (2.13). We prove the inequality on the left; the first inequality below is (2.16) with the names changed, the second one follows from (2.12).

$$A \geq B - ||B - A||_{HS}Id \geq B \left(1 - \frac{1}{\epsilon}||B - A||_{HS}\right).$$

Again by (2.12) this implies that

$$A \geq \begin{cases} 
B \left(1 - \frac{1}{\epsilon}||B - A||_{HS}\right) & \text{if } ||B - A||_{HS} \leq \frac{\epsilon}{2}, \\
\epsilon Id & \text{if } ||B - A|| > \frac{\epsilon}{2}.
\end{cases}$$

The left hand side of (2.13) now follows from two facts: the first one is that, for $D_3$ large enough,

$$1 - \frac{t}{\epsilon} \geq e^{-D_3t} \quad \text{if} \quad 0 \leq t \leq \frac{\epsilon}{2}.$$ 

The second one is the formula below. The first inequality comes taking $||B - A||_{HS} \geq \frac{\epsilon}{2}$, the second one taking $\gamma > 0$ so small that $\gamma B \leq Id$; the third one taking $D_3$ so large that $\frac{\epsilon}{2} e^{-D_3t} \leq \epsilon$.

$$Be^{-D_3||B - A||_{HS}} \leq Be^{-\frac{\epsilon}{2}} \leq \frac{1}{\gamma} Id \cdot e^{-D_3t} \leq \epsilon Id.$$ 

We prove the converse. By (2.14) we have that

$$-(1 - e^{-D_4})(Bx, x) \leq ((A - B)x, x) \leq (e^{D_4} - 1)(Bx, x).$$

By the definition of $||\cdot||$ and the fact that $e^{D_4} - 1 \geq 1 - e^{-D_4}$ this implies that

$$||A - B|| \leq (e^{D_4} - 1)||B||.$$  

Now (2.15) follows from (2.1). \qed

**Lemma 2.3.** Let $a > 0$ and let $\nu \in (0, 1]$; then, the following holds.

1) The sets $C_\pi(E, a, \nu)$ are convex cones in $C(E, M^d)$ which satisfy (1.1).

2) There is $D_5 > 0$ such that for all $A \in C_\pi(E, a, \nu)$ we have that

$$||A_x - A_y|| \leq D_5||A||_{\infty} \cdot \hat{d}(x, y)^\nu \quad \forall x, y \in G.$$  

(2.17)

Conversely, if $A \in C_\pi \cap C_\nu(E, M^d)$, then $A \in C_\pi(E, a, \nu)$ for some $a > 0$ (which depends on $A$).

**Proof.** We don’t dwell on the proof of point 1), since it follows immediately from the definitions of $C_\pi$ and $C_\pi(E, a, \nu)$.

We prove point 2). Let $A \in C_\pi(E, a, \nu)$; by (2.11) we have that, if $x, y \in E$,

$$e^{-ad(x, y)^\nu} A_y \leq A_x \leq e^{ad(x, y)^\nu} A_y.$$ 

This is (2.14) for $D_4 = \hat{d}(x, y)^\nu$; by lemma 2.2, (2.15) holds, i. e.

$$||A_x - A_y||_{HS} \leq D_5 \left(e^{ad(x, y)^\nu} - 1\right)||A_x||_{HS} \quad \forall x, y \in E.$$ 

Since we are supposing that the diameter of $E$ is 1 (for $E = G$ this is (1.9)), we get (2.17).
Conversely, let $A \in C_+ \cap C^\nu(E, M^d)$; since $E$ is compact, we easily see that there is $\epsilon > 0$ such that $A_x \geq \epsilon d$ for all $x \in E$. Thus, setting $A = A_x$ and $B = A_y$, we have that $A$ and $B$ satisfy (2.12); by lemma 2.2 also (2.13) holds, i.e.

$$A_y e^{-D_3||A_x - A_y||_{HS}} \leq A_x \leq A_y e^{D_3||A_x - A_y||_{HS}}$$

for some $D_3 = D_3(\epsilon, A_y) > 0$. Since $A \in C^\nu(E, M^d)$, we have that

$$||A_x - A_y||_{HS} \leq D_7 \hat{d}(x - y)\nu.$$  

The last two formulas imply that

$$A_y e^{-D_7 D_3(\epsilon, A_y) \hat{d}(x, y)\nu} \leq A_x \leq A_y e^{D_7 D_3(\epsilon, A_y) \hat{d}(x, y)\nu}. $$

Thus, $A \in C_+(E, a, \nu)$ if

$$a \geq D_7 \sup \{D_3(\epsilon, A_y) : y \in E\}.$$  

Note that the term on the right is finite since, by lemma 2.2, $D_3(\epsilon, \cdot)$ is bounded on bounded sets and $||A||_\infty$ is finite. \hfill \Box

3. **The Ruelle operator.** From now on, we suppose that (F1)-(F4) hold; we let $(G, \hat{d})$ be the fractal defined by (1.8) with the distance $\hat{d}$ induced by the immersion in $\mathbb{R}^d$.

We define the Ruelle operator $L_G$ on $G$ as

$$L_G : C(G, M^d) \to C(G, M^d)$$

$$(L_G A)(x) = \sum_{i=1}^n \tau D\psi_i(x) A_{\psi_i(x)} D\psi_i(x). \quad (3.1)$$

We also define a Ruelle operator $L_\Sigma$ on $C(\Sigma, M^k)$: first, if $x = (x_0, x_1, \ldots) \in \Sigma$, we set $(ix) = (ix_0, x_1, \ldots)$. Then, we define

$$L_\Sigma : C(\Sigma, M^k) \to C(\Sigma, M^k)$$

$$(L_\Sigma A)(x) = \sum_{i=1}^n \Phi D\psi_i|_{\Phi(x)} A_{(ix)} D\psi_i|_{\Phi(x)}. \quad (3.2)$$

Note also that: $x \to D\psi_i|_{\Phi(x)}$ is $\nu_0$-Hölder, since $D\psi_i$ is $\nu_0$-Hölder by (F1) and we saw in section 1 that $\Phi$ is Lipschitz; if $A = \hat{A} \circ \Phi$ for some $\hat{A} \in C(G, M^d)$, we get by the second formula of (1.15) that

$$L_\Sigma A(x) = \sum_{i=1}^n \Phi(x) \hat{A}_{\psi_i(x)} \psi_i(x) D\psi_i|_{\Phi(x)}.$$

The next lemma shows that the fixed points of the adjoint of $L_G$, which we call $L^*_G$, induce a self-similar form on $C^1(\mathbb{R}^d)$.

**Lemma 3.1.** Let $G$ be a fractal satisfying (F1)-(F4). Let $(\beta, \tau) \in (0, +\infty) \times M^+(G, M^d)$ be such that

$$L^*_G \tau = \beta \tau.$$  

Let $f, g \in C^1(\mathbb{R}^d, \mathbb{R})$ and let us define, with the notation of section 2 for the integral,

$$\mathcal{E}_\tau(f, g) = \int_G (\nabla f(x), d\tau \nabla g(x)).$$

Then,

$$\sum_{i=1}^n \mathcal{E}_\tau(f \circ \psi_i, g \circ \psi_i) = \beta \mathcal{E}_\tau(f, g). \quad (3.3)$$
Proof. By the polarisation identity, it suffices to show (3.3) when \( f = g \); we shall take advantage of the fact that \( \nabla f \otimes \nabla f \in C(G, M^d) \), i.e. is in the domain of \( L_G \). We recall that, if \( a \in \mathbb{R}^d \) and \( A \in M^d \), then
\[
(a, Aa) = (a \otimes a, A)_{HS}
\]
where by \( a \otimes a \) we denote the tensor product of the column vector \( a \) with itself:
\[
a \otimes a = a \cdot t^a.
\]
As we shall see in the formula below, this explains the position of the transpose sign in (3.1).

The definition of \( E_\tau \) and the formula above imply the first equality below; the second one comes from the chain rule (recall that \( \nabla (f \circ \psi_i) = t^\psi_i \cdot \nabla f \)) and the definition of \( L_G \); this third one follows since \( L_G \beta = \beta \tau \) and the last one is again the definition of \( E_\tau \).
\[
\sum_{i=1}^n E_\tau(f \circ \psi_i, f \circ \psi_i) = \sum_{i=1}^n \int_G (\nabla (f \circ \psi_i)(x) \otimes \nabla (f \circ \psi_i)(x), d\tau(x))_{HS} = \\
\int_G (L_G (\nabla f \otimes \nabla f)(x), d\tau(x))_{HS} = \beta \int_G (\nabla f \otimes \nabla f, d\tau)_{HS} = \beta E_\tau(f, f).
\]

Remark. We can read lemma 3.1 as a statement about the push-forward of the measure \( \tau \), the positive eigenvector of \( L_G^* \). Indeed, let \( f, g \in C^1(\mathbb{R}^d) \); by (F3), \( f \circ F \) and \( g \circ F \) are not defined at the points of \( F \), which are a finite set. As we shall see in lemma 5.3 below, the measure \( \tau \) of point 2) of theorem 1 is non-atomic; in particular, the points where \( f \circ F \) and \( g \circ F \) are not defined have measure zero. Together with (1.13), this implies the first equality below; the second one follows from lemma 3.1; the last one follows from the chain rule.
\[
\int_G (\nabla f, d\tau \nabla g) = \frac{1}{n} \sum_{i=1}^n \int_G (\nabla (f \circ F \circ \psi_i), d\tau \nabla (g \circ F \circ \psi_i)) = \\
\frac{\beta}{n} \int_G (\nabla (f \circ F), d\tau \nabla (g \circ F)) = \frac{\beta}{n} \int_G (t^F \nabla f \big|_{F(x)} \otimes t^F \nabla g \big|_{F(x)}, d\tau(x))_{HS}.
\]

In other words, denoting by \( F_* \omega \) the pull-back by \( F \) of the two-tensor \( \omega \), we have that
\[
\tau = \frac{\beta}{n}(F_*)^* \tau
\]
where the “push-forward” \( (F_*)^* \tau \) is defined as in the last term on the right in the formula above. Though this is not the standard push-forward operator, it is natural if we regard \( \tau \) not as a measure, but as a linear operator on 2-tensors.

4. Fixed points of the Ruelle operator. We shall suppose that the maps \( \{\psi_i\}_{i=1}^n \) satisfy the following nondegeneracy condition: it is stronger than the one in [16], but it allows us to use the Perron-Frobenius theorem without modifications.

(ND) We suppose that, for all \( v \in \mathbb{R}^d \setminus \{0\} \) and all \( x \in G \), the set \( \{t^\psi_i(x)v\}_{i=1}^n \) generates \( \mathbb{R}^d \). Actually, we ask for a quantitative version of this, i.e. that there is
\( b > 0 \) such that the following holds. Let \( v, v_0 \in \mathbb{R}^d \) and let \( x \in G \); then, there is \( i \in (1, \ldots, n) \), depending on \( x, v \) and \( v_0 \), such that
\[
(D\psi_i(x)v, v_0) \geq b\|v\| \cdot \|v_0\|. \tag{4.1}
\]
If we denote by \( P \) the orthogonal projection on \( v_0 \), the formula above implies the inequality below.
\[
\|P \cdot D\psi_i(x)v\| \geq b\|v\|. \tag{4.2}
\]
It is easy to verify that the harmonic Sierpinski gasket of section 1 satisfies (ND); as we shall see in the next lemma, (ND) implies a bound from below on \( L_A \).

**Lemma 4.1.** Let the maps \( \{\psi_i\}_{i=1}^n \) satisfy (F1)-(F4) and (ND) for some \( b > 0 \). Let \((E, \mathcal{L})\) denote either one of \((G, \mathcal{L}_G)\) or \((\Sigma, \mathcal{L}_\Sigma)\).

Then, for all \( a > 0 \) there is \( D_1 = D_1(a, b) > 0 \) such that the following happens. Let \( \nu \in (0, v_0) \) and let \( A \in C_+(E, a, v) \); then, for all \( x \in E \),
\[
\frac{1}{D_1(a, b)} \|A\|_{\infty} \cdot \|\nu\| \leq L_A x \leq D_1(a, b)\|A\|_{\infty} \cdot \|\nu\|. \tag{4.3}
\]

**Proof.** We prove the left hand side of (4.3); for the right hand side it suffices to note that \( L \) is continuous from the \( \| \cdot \|_{\infty} \) topology to itself.

By compactness, there is \( x_{\text{max}} \in E \) such that
\[
\|A_{x_{\text{max}}}\|_{HS} = \|A\|_{\infty}. \tag{4.4}
\]
By the definition of \( \|A_{x_{\text{max}}}\| \) we can find \( v_{\text{max}} \in \mathbb{R}^d \) with \( \|v_{\text{max}}\| = 1 \) such that
\[
\|A_{x_{\text{max}}}\| = (A_{x_{\text{max}}} v_{\text{max}}, v_{\text{max}}). \tag{4.5}
\]
By (2.11) and the fact that \( \text{diam}(E) = 1 \) we have that
\[
e^{-a}(A_{x_{\text{max}}} v, v) \leq (A_x v, v) \leq e^a(A_{x_{\text{max}}} v, v) \quad \forall x \in E, \quad \forall v \in \mathbb{R}^d.
\]
Let \( x \in G, v \in \mathbb{R}^d \) and let \( i \in (1, \ldots, n) \); the formula above implies the first inequality below. Since \( v_{\text{max}} \) is an eigenvector of the symmetric matrix \( A_{x_{\text{max}}} \), we have that \( A_{x_{\text{max}}} \) preserves the space generated by \( v_{\text{max}} \) and its orthogonal complement. Thus, if we denote by \( P \) the orthogonal projection on \( v_{\text{max}} \), we get the second inequality below. Next, we choose \( i \) in such a way that (4.1) holds with \( v_0 = v_{\text{max}} \); the choice of \( i \) depends on \( x \) and \( v \). By (4.2) we get the third inequality below; the last equality comes from (4.5) and the last inequality from (2.1) and (4.4).
\[
(A_{\psi_i(x)} D\psi_i(x)v, D\psi_i(x)v) \geq e^{-a} (A_{x_{\text{max}}} D\psi_i(x)v, D\psi_i(x)v) \geq e^{-a} (A_{x_{\text{max}}} P D\psi_i(x)v, P D\psi_i(x)v) \geq e^{-a} b^2 (A_{x_{\text{max}}} v_{\text{max}}, v_{\text{max}}) \cdot \|v\|^2 = e^{-a} b^2 \|A_{x_{\text{max}}}\| \cdot \|v\|^2 \geq D_4 \cdot e^{-a} b^2 \cdot \|A\|_{\infty} \cdot \|v\|^2.
\]
We choose \( i \) as above; the definition of \( L \) implies the first inequality below; the second one comes from the formula above.
\[
((L_A x) v, v) \geq (A_{\psi_i(x)} D\psi_i(x)v, D\psi_i(x)v) \geq D_4 \cdot e^{-a} b^2 \cdot \|A\|_{\infty} \cdot \|v\|^2 \quad \forall x \in E.
\]
then for all \( x, y \in G \) we have that
\[
e^{-||x-y||^\omega} \sum_{i=1}^{n} t D\psi_i(x) A_{\psi_i(x)} D\psi_i(x) \leq \sum_{i=1}^{n} t D\psi_i(y) A_{\psi_i(x)} D\psi_i(y) \leq e^{||x-y||^\omega} \sum_{i=1}^{n} t D\psi_i(x) A_{\psi_i(x)} D\psi_i(x).
\] (4.7)_G

Note that, if the maps \( \psi_i \) are affine and satisfy (ND), (4.6) is always verified.

Analogously, possibly reducing \( \omega(a,b) \) in (4.6), for all \( A \in C_+ (\Sigma, a, \nu_0) \) and all \( x, y \in \Sigma \) we have that
\[
e^{-d_\omega(x,y)^\nu_0} \sum_{i=1}^{n} t D\psi_i|_{\Phi(x)} A_{(ix)} D\psi_i|_{\Phi(x)} \leq \sum_{i=1}^{n} t D\psi_i|_{\Phi(y)} A_{(ix)} D\psi_i|_{\Phi(y)} \leq e^{d_\omega(x,y)^\nu_0} \sum_{i=1}^{n} t D\psi_i|_{\Phi(x)} A_{(ix)} D\psi_i|_{\Phi(x)}.
\] (4.7)_\Sigma

Proof. We shall prove (4.7)_G, since (4.7)_\Sigma is analogous. We begin recalling an inequality on matrices. Let \( B \in M^d \) be positive semidefinite and let \( C, C' \) two invertible matrices; we suppose that
\[
||C||_{HS}, ||C'||_{HS} \leq D_1
\] (4.8)
for some \( D_1 > 0. \)

It is easy to see that, since \( B \) is symmetric,
\[
(\tr C' B C x, x) = (\tr C B C' x, x).
\]

Together with a simple calculation, this implies that
\[
(\tr C' B C' x, x) = (BC x, C x) + 2(B(C' - C) x, C x) + (B(C' - C) x, (C' - C) x).
\]

Since \( ||C - C'||_{HS} \leq 2D_1 \) by (4.8), this implies that, for some \( D_2 > 0 \) depending only on \( D_1 \), but not on \( C, C' \) and \( B, \)
\[
\tr C BC - D_2 ||B||_{HS} ||C - C'||_{HS} I_d \leq \tr C' B C' \leq \tr C BC + D_2 ||B||_{HS} ||C - C'||_{HS} I_d.
\]

We set
\[
B = A_{\psi_i(x)}, \quad C' = D\psi_i(y) \quad \text{and} \quad C = D\psi_i(x)
\]
which by (F1) implies that
\[
||C - C'||_{HS} \leq ||D\psi_i||_{\nu_0} ||x - y||^\nu_0
\]
where the Hölder norm has been defined before lemma 2.2. Recalling that by (F1) (4.8) holds with \( D_1 = \eta \), we get from the last three formulas that
\[
\tr D\psi_i(x) A_{\psi_i(x)} D\psi_i(x) - D_5 ||A_{\psi_i(x)}||_{HS} ||D\psi_i||_{\nu_0} ||x - y||^\nu_0 I_d \leq \tr D\psi_i(y) A_{\psi_i(x)} D\psi_i(y) \leq \tr D\psi_i(x) A_{\psi_i(x)} D\psi_i(x) + D_5 ||A_{\psi_i(x)}||_{HS} ||D\psi_i||_{\nu_0} ||x - y||^\nu_0 I_d.
\]
Summing over \( i \in (1, \ldots, n) \) and setting

\[
\|D\psi\|_{\nu_0} : = \sup_{i \in (1, \ldots, n)} \|D\psi_i\|_{\nu_0}
\]

we get

\[
\sum_{i=1}^{n} t D\psi_i(x) A_{\psi_i(x)} D\psi_i(x) - D_5 \sum_{i=1}^{n} \|A_{\psi_i(x)}\|_{HS} \cdot \|x - y\|_{\nu_0} \cdot Id \leq \sum_{i=1}^{n} t D\psi_i(y) A_{\psi_i(x)} D\psi_i(y) \leq \sum_{i=1}^{n} t D\psi_i(x) A_{\psi_i(x)} D\psi_i(y) + \|D\psi\|_{\nu_0} \cdot D_5 \sum_{i=1}^{n} \|A_{\psi_i(x)}\|_{HS} \cdot \|x - y\|_{\nu_0} \cdot Id.
\]

Since \( A \in C_+(a, \nu) \) we can apply lemma 4.1 and get that there is \( D_6 = D_6(a, b) \) such that

\[
[1 - D_6(a, b) \|D\psi\|_{\nu_0} \cdot \|x - y\|_{\nu_0}] \sum_{i=1}^{n} t D\psi_i(x) A_{\psi_i(x)} D\psi_i(x) \leq \sum_{i=1}^{n} t D\psi_i(y) A_{\psi_i(x)} D\psi_i(y) \leq [1 + D_6(a, b) \|D\psi\|_{\nu_0} \cdot \|x - y\|_{\nu_0}] \sum_{i=1}^{n} t D\psi_i(x) A_{\psi_i(x)} D\psi_i(x).
\]

We take

\[
\omega(a, b) = \frac{1}{4D_6(a, b)}
\]

and we recall that, if \( x \in [0, 1] \),

\[
1 - \frac{1}{4} x \geq e^{-x} \quad \text{and} \quad 1 + \frac{1}{4} x \leq e^x.
\]

From the last three formulas, (4.6) and the fact that \( \text{diam}(E) = 1 \) we get that

\[
e^{-\|x - y\|_{\nu_0}} \sum_{i=1}^{n} t D\psi_i(x) A_{\psi_i(x)} D\psi_i(x) \leq \sum_{i=1}^{n} t D\psi_i(y) A_{\psi_i(x)} D\psi_i(y) \leq e^{\|x - y\|_{\nu_0}} \sum_{i=1}^{n} t D\psi_i(x) A_{\psi_i(x)} D\psi_i(x)
\]

which is (4.7)\(_G\).

\[\square\]

**Lemma 4.3.** Let \((E, \mathcal{L})\) be either one of \((G, \mathcal{L}_G)\) or \((\Sigma, \mathcal{L}_\Sigma)\); let \((F1)-(F4)\) and \((ND)\) hold. Then, there is \(a_0 > 0\) such that, for \(a > a_0\) and \(\sup_{i \in (1, \ldots, n)} \|D\psi_i\|_{\nu_0} \leq \omega(a, b)\),

\[
\mathcal{L}(C_+(E, a, \nu_0)) \subset C_+(E, a - 1, \nu_0).
\]

**Proof.** We follow [28]. It is immediate from the definition of \( \mathcal{L} \) that \( \mathcal{L}(C_+) \subset C_+ \).

Thus, it suffices to show that, if \( A \in C_+ \) satisfies (2.11) for \(a\) and \(\nu_0\), then \( \mathcal{L}A \) satisfies (2.11) for \(a - 1\) and \(\nu_0\), provided \(a\) is large enough.
We shall prove the lemma on $G$, since the proof on $\Sigma$ is analogous. Let $x, y \in G$; the first equality below is the definition of $\mathcal{L}_G$ in (3.1); the first inequality is the right hand side of (4.7) and holds if $||D\psi_i||_{\nu_0} \leq \omega(a, b)$. The second one follows from two facts: the map $: A \to tBAB$ is order-preserving and $A \in C_+(G, a, \nu_0)$, i. e. it satisfies (2.11). The third inequality comes from the fact that $\eta$ is the common Lipschitz constant of the maps $\psi_i$, i. e. formula (1.7).

\[
(\mathcal{L}_G A)(y) = \sum_{i=1}^{n} tD\psi_i(y)A_{\psi_i(y)}D\psi_i(y) \leq \\
e^{||x-y||\nu_0} \sum_{i=1}^{n} tD\psi_i(x)A_{\psi_i(x)}D\psi_i(x) \leq \\
e^{||x-y||\nu_0 + a||\psi_i(x) - \psi_i(y)||\nu_0} \sum_{i=1}^{n} tD\psi_i(x)A_{\psi_i(x)}D\psi_i(x) \leq \\
e^{(1+a\eta\nu_0)||x-y||\nu_0} \sum_{i=1}^{n} tD\psi_i(x)A_{\psi_i(x)}D\psi_i(x).
\]

Since $\eta \in (0, 1)$ we can choose $a_0$ so large that, for $a \geq a_0$,

\[1 + a\eta\nu_0 \leq a - 1.\]

From the last two formulas we get that

\[
(\mathcal{L}_G A)(y) \leq e^{(a-1)||x-y||\nu_0} (\mathcal{L}_G A)(x).
\]

The opposite inequality follows similarly, implying (4.9).

**Definitions.** Let $\lambda_1, \epsilon > 0$; we denote by $C'_+(E, \lambda_1 a, \nu_0)$ the subset of the $A \in C_+(E, \lambda_1 a, \nu_0)$ such that, for all $x \in E$,

\[
A_x \geq \epsilon ||A||_{\infty} Id. \quad (4.10)
\]

Moreover, we shall call $\theta_{(a, \nu_0)}$ the hyperbolic distance on $C_+(E, a, \nu_0)$ and $\theta_+$ the hyperbolic distance on $C_+$; we recall that the hyperbolic distance on a cone has been defined at the beginning of section 1.

**Lemma 4.4.** Let $E$ be either one of $G$ or $\Sigma$; let $a > 0$. Then, the following holds.

1) $(C_+(E, \epsilon a, \nu_0), \theta_{(a, \nu_0)})$ is a complete metric space.

2) Let $C'_+(E, \lambda_1 a, \nu_0)$ be defined as in (4.10). If $\lambda_1 \in (0, 1)$, then

\[
\text{diam}_{\theta_{(a, \nu_0)}} C'_+(E, \lambda_1 a, \nu_0) < +\infty.
\]

**Proof.** Again we follow closely [28]; we begin with point 1). We consider a Cauchy sequence $\{A_n\}_{n \geq 1}$ in $(C_+(E, \lambda_1 a, \nu_0), \theta_{(a, \nu_0)})$; we choose the representatives which satisfy

\[
||A_n||_{\infty} = 1 \quad \forall n \geq 1. \quad (4.11)
\]

**Step 1.** We begin to show that $\{A_n\}_{n \geq 1}$ converges uniformly to $A \in C(E, M^d)$.

Since $C_+(E, a, \nu_0) \subset C_+$, the definition of the hyperbolic distance implies that $\theta_+ \leq \theta_{(a, \nu_0)}$; thus, $\{A_n\}_{n \geq 1}$ is Cauchy also for the $\theta^+$ distance; the definition of $\theta^+$ in (1.4) implies that

\[
\frac{\beta^+(A_m, A_n)}{\alpha^+(A_m, A_n)} \to 1 \quad \text{as} \quad n, m \to +\infty. \quad (4.12)
\]
By the definition of $\alpha^+$ and $\beta^+$ in (1.2) and (1.3) respectively, we get that, for all $x \in E$,
$$\alpha^+(A_m, A_n)A_m(x) \leq A_n(x) \leq \beta^+(A_m, A_n)A_m(x).$$
Let $\delta > 0$; by (4.12) we have that, for $n$ and $m$ large enough and all $x \in E$,
$$(1 - \delta)\beta^+(A_m, A_n)A_m(x) \leq A_n(x) \leq \beta^+(A_m, A_n)A_m(x).$$
By the converse part of lemma 2.2 this implies that, for $n$ and $m$ large,
$$||A_n - \beta^+(A_m, A_n)A_m||_\infty \leq D_5 \cdot \delta||A_n||_\infty.$$
By (4.11) and the triangle inequality this implies that $\beta^+(A_m, A_n) \to 1$; again by the formula above, $\beta^+(A_m, A_n) \to 1$ implies that $\{A_n\}_{n \geq 1}$ is Cauchy for $|| \cdot ||_\infty$; thus, there is $A \in C(G, M^d)$ such that $A_n \to A$ uniformly.

**Step 2.** We show that $g^{(a, \nu_0)}(A_n, A) \to 0$ as $n \to +\infty$. It is easy to see that $g^{(a, \nu_0)}$ is lower semicontinuous under uniform convergence; together with step 1, this yields the first inequality below. Since $\{A_n\}_{n \geq 1}$ is Cauchy for $\theta^{(a, \nu_0)}$, there is $\delta_n \to 0$ such that also the second inequality below holds.
$$g^{(a, \nu_0)}(A, A_n) \leq \liminf_{m \to +\infty} \theta^{(a, \nu_0)}(A_m, A_n) \leq \delta_n.$$

**End of the proof of point 1.** It only remains to prove that $A \in C_+(E, a, \nu_0)$. First of all, $A$ satisfies (2.11), since this condition is closed under uniform convergence. We have to show that $A_x$ is positive-definite for all $x \in E$. We recall that $A_{n,x}$ is positive-definite for all $x \in E$, since $A_n \in C_+(E, a, \nu_0)$; since $\theta^+(A_n, A) < +\infty$, we have that $\alpha^+(A_n, A) > 0$; since by (1.2) $A \geq \alpha^+(A_n, A)A_n$ and $A_n$ satisfies (2.10) for some $\epsilon > 0$ (which depends on $A_n$), we get that $A_x$ is positive-definite for all $x \in E$.

**Proof of point 2.** The proof is in two steps: first, we show that
$$\text{diam}_{g^{(a, \nu_0)}} C^*_+(E, \lambda_1 a, \nu_0) < +\infty$$
(4.13)
and then that
$$\text{diam}_{g^{(a, \nu_0)}} C^*_+(E, \lambda_1 a, \nu) \leq \text{diam}_{g^{(a, \nu_0)}} C^*_+(E, \lambda_1 a, \nu_0) + D_5 (\lambda_1).$$
(4.14)

**Step 3.** We prove (4.13); this follows if we show that $C^*_+(E, \lambda_1 a, \nu_0)$ is compact in $(C_+, \theta^+)$. Thus, let $\{A_n\}_{n \geq 1} \subset C^*_+(E, \lambda_1 a, \nu_0)$; we can suppose that $\{A_n\}_{n \geq 1}$ is normalised, i.e. that (4.11) holds. Now, point 2) of lemma 2.3 implies that the Hölder seminorm of $A_n$ is bounded. Thus, by Ascoli-Arzelà there is a subsequence $\{A_{n_k}\}_{k \geq 1}$ which converges uniformly to $A \in C(E, M^d)$; we see as in point 1) that $A$ satisfies (2.11); it also satisfies (4.10) because this formula is stable under uniform convergence. Thus, $A \in C^*_+(E, \lambda_1 a, \nu_0)$.

**Step 4.** We prove (4.14); recall that we are working on $E$ with the distance $\hat{d}$. For starters, let us see how $\alpha^+$ and $\alpha^{(a, \nu_0)}$ are related. Let $A_1, A_2 \in C_+(E, a, \nu_0)$; by the definition of $\alpha^{(a, \nu)}(A_1, A_2)$ we have that
$$A_2 - \alpha^{(a, \nu_0)}(A_1, A_2)A_1 \in C_+(E, a, \nu_0)$$
which by (2.11) implies that, for all $x, y \in E$,
$$e^{-\hat{d}(x,y)^\nu} |A_2(x) - \alpha^{(a, \nu_0)}(A_1, A_2)A_1(x)| \leq A_2(y) - \alpha^{(a, \nu_0)}(A_1, A_2)A_1(y) \leq e^{\hat{d}(x,y)^\nu} |A_2(x) - \alpha^{(a, \nu_0)}(A_1, A_2)A_1(x)|.$$
Rearranging the terms of the inequality on the left, we get that
\[ e^{-ad(x,y)v_0} A_2(x) - A_2(y) \leq [e^{-ad(x,y)v_0} A_1(x) - A_1(y)] \alpha(a,v_0)(A_1, A_2) \]
for all \( x, y \in E \). Since \( A_1, A_2 \) satisfy (2.11) for \( \lambda_1 a \), the formula above implies that
\[ A_2(x) \leq \frac{e^{-ad(x,y)v_0} - e^{-\lambda_1 ad(x,y)v_0}}{e^{-ad(x,y)v_0} - e^{-\lambda_1 ad(x,y)v_0}} \alpha(a,v_0)(A_1, A_2)A_1(x). \quad (4.15) \]
We set
\[ D_6 = \sup \left\{ \frac{z - z^{\lambda_1}}{z - z^{-\lambda_1}} : z \in (0, 1) \right\} \]
and by a function study we see that \( D_6 \in (0, 1) \). By (4.15) we get that
\[ A_2(x) \leq D_6 \alpha(a,v_0)(A_1, A_2)A_1(x) \]
which by the definition of \( \alpha^+ \) implies that
\[ D_6 \alpha(a,v_0)(A_1, A_2) \geq \alpha^+(A_1, A_2). \quad (4.16) \]
Analogously, we can set
\[ D_7 = \inf \left\{ \frac{z - z^{-\lambda_1}}{z - z^{\lambda_1}} : z > 1 \right\}. \]
A function study shows that \( D_7 > 1 \) and the same argument that yielded (4.16) yields
\[ \beta(a,v)(A_1, A_2) \leq D_7 \beta^+(A_1, A_2). \]
Using this, (4.16) and the definition of \( \theta^+ \) in (1.4) we get that
\[ \theta(a,v)(A_1, A_2) \leq \theta^+(A_1, A_2) + \log D_7 - \log D_6 \]
which ends the proof of (2.18). \qed

**Proposition 4.5.** Let \((E, \mathcal{L})\) be either one of \((G, \mathcal{L}_G)\) or \((\Sigma, \mathcal{L}_\Sigma)\). Let (F1)-(F4) and (ND) with constant \( b > 0 \) hold. Then, if \( \sup_{i \in \{1, \ldots, n\}} ||D\psi_i||_{v_0} \) is small enough, the following holds.

1) There is a simple, positive eigenvalue \( \beta \) of
\[ \mathcal{L} : C(E, M^d) \to C(E, M^d). \]
Denoting by \( Q \) the eigenfunction of \( \beta \), we have that \( Q \in C_+(E, a, v_0) \). In particular, \( Q_x \) is positive-definite for all \( x \in E \). If the maps \( \psi_i \) are affine, then there is \( \bar{Q} \in M^d \) such that \( Q_x = \bar{Q} \) for all \( x \in E \).

2) Recall that after formula (2.7) we defined \( P_Q(E, M^d) \); we assert that there is \( \tau \in P_Q(E, M^d) \) such that \( \mathcal{L}^* \tau = \beta \tau \).

3) If \( B \in C(E, M^d) \), we have
\[ \frac{1}{\beta^l} \mathcal{L}^l B \to Q \int_E (B, d\tau)_{HS} \quad (4.17) \]
uniformly on \( E \). Note that this implies that the measure \( \tau \) of the previous point is unique and that \( \beta \) is simple eigenvalue of \( \mathcal{L} \). Moreover, if \( B \in C_+(E, a, v) \) with \( v \in (0, v_0) \), the convergence above is exponentially fast.

**Proof.** **Step 1.** Let \( \sup_{i \in \{1, \ldots, n\}} ||D\psi_i||_{v_0} \) be so small that lemma 4.3 hold. Since \( \mathcal{L} : C(E, M) \to C(E, M) \) is continuous, the left hand side of (4.3) implies that, possibly increasing the constant \( D_1(a, b) \),
\[ (\mathcal{L} A)_x \geq \frac{1}{D_1(a, b)} ||\mathcal{L} A||_\infty \cdot Id \quad \forall x \in E. \]
Together with lemma 4.3 this implies that
\[ \mathcal{L}(C_+(E, a, \nu_0)) \subset C'_+(E, \lambda_1 a, \nu_0) \]
for some \( \lambda_1 \in (0, 1) \) and \( \epsilon \) equal to the constant \( \frac{1}{m_{(a,b)}} \) of the formula above; by point 2) of lemma 4.4 this implies that
\[ \text{diam}_{\mathcal{L}(a,\nu)} \mathcal{L}(C_+(E, a, \nu_0)) < +\infty. \]
By point 1) of proposition 1.1, we get that \( \mathcal{L} \) is a contraction of \( C_+(E, a, \nu) \) into itself; since
\[ \left( \frac{C_+(E, a, \nu_0)}{\sim}, \theta(a,\nu_0) \right) \]
is complete by point 1) of lemma 4.4, we get that \( \mathcal{L} \) has a unique fixed point in \( C_+(E, a, \nu_0) \). In other words, there are
1) \( Q \in C_+(E, a, \nu_0) \), unique up to multiplication by a scalar, and
2) a unique \( \beta \in (0, +\infty) \) such that
\[ \mathcal{L}Q = \beta Q. \] (4.18)
Since \( Q \) is unique up to multiplication by a scalar, we can normalise it in such a way that \( ||Q||_{\infty} = 1 \).

**Step 2.** If the maps \( D\psi_\ell \) are constant, we see that, if \( A \) is a constant matrix, then \( \mathcal{L}A \) is constant too. Applying the Perron-Frobenius theorem to the positive cone of \( M^d \), we can find a constant, positive-definite matrix \( Q \) and \( \beta' > 0 \) such that \( \mathcal{L}Q = \beta' Q \). By the uniqueness of step 1, we have that \( Q \equiv \lambda Q \) for some \( \lambda > 0 \).

**Step 3.** We prove point 2). We saw in lemma 2.1 that \( \mathcal{P}_Q(E, M^d) \) is a convex, compact set of \( \mathcal{M}(E, M^d) \); thus, by Schauder’s fixed point theorem, it suffices to show that \( \frac{1}{\beta} \mathcal{L}^* \) brings \( \mathcal{P}_Q(E, M^d) \) into itself. Let \( \hat{\tau} \in \mathcal{P}_Q(E, M^d) \); we skip the proof that \( \frac{1}{\beta} \mathcal{L}^* \hat{\tau} \) is non-negative definite (it follows easily by (2.3) and the definition of the adjoint), but we show that its integral against \( Q \) is 1. The first equality below is the definition of the adjoint, the second one is point 1) and the last follows since \( \hat{\tau} \in \mathcal{P}_Q(E, M^d) \).
\[
\int_E \left( Q, d\left( \frac{1}{\beta} \mathcal{L}^* \hat{\tau} \right) \right)_{HS} = \int_E \left( \frac{1}{\beta} \mathcal{L}Q, d\hat{\tau} \right)_{HS} = \int_E (Q, d\hat{\tau})_{HS} = 1.
\]

**Step 4.** We prove point 3). Since \( \mathcal{L} \) is linear and \( B = B^+ - B^- \) with \( B^+, B^- \geq 0 \), it suffices to prove (4.17) when \( B \in C(E, M^d) \) and \( B \geq 0 \); in other words, when \( B \in C_+ \).
We begin to show that, if \( B \in C_+ \), then
\[ \theta_+(\mathcal{L}^*B, Q) \to 0. \] (4.19)
It is clear that (4.19) follows from the three points below.

a) For all \( \epsilon > 0 \) there is \( a_0 > 0 \) and \( \hat{B} \in C_+(E, a_0, \nu_0) \) such that \( \theta^+(B, \hat{B}) < \epsilon \). This follows, for instance, since Hölder functions are dense for the \( ||\cdot||_{\infty} \) topology. We can also require that \( \hat{B} \in C_+(E, a, \nu_0) \) for some \( a \geq a_0 \).

b) If \( a \) is large enough, \( \theta(a,\nu_0)(\mathcal{L}^*B, Q) \to 0 \). This follows since \( \hat{B} \in C_+(E, a, \nu_0) \) and \( \mathcal{L} \) is a contraction on \( C_+(E, a, \nu_0) \) by step 2. By (1.5), this convergence is exponentially fast. If we apply this argument to \( B \in C_+(E, a, \nu_0) \) we get the last assertion of the thesis.

c) Since \( C_+(E, a, \nu_0) \subset C_+ \), the definition of hyperbolic distance in section 1 immediately implies that \( \theta_+ \leq \theta(a,\nu_0) \); by the triangle inequality, this implies the first
inequality below; the second one comes from the fact that, since $\mathcal{L}(C_+) \subset C_+$, then $\text{Lip}_+(\mathcal{L}) \leq 1$; this is proven as point 1) of proposition 1.1.

$$\theta_+(\mathcal{L}'B, Q) \leq \theta_+(\mathcal{L}'B, \mathcal{L}'\tilde{B}) + \theta^{(a,\omega)}(\mathcal{L}'\tilde{B}, Q) \leq \theta_+(B, \tilde{B}) + \theta^{(a,\omega)}(\mathcal{L}'\tilde{B}, Q).$$

Now the first term on the right is arbitrarily small by point a) and the second one tends to zero by point b).

We show how (4.19) implies (4.17) when $B \in C_+$. We begin to note that, since $L^* \tau = \beta \tau$ by point 2) of the thesis, we have for all $l \geq 1$

$$\int_E \left( \left( \frac{1}{\beta} L \right)^l B, d\tau \right)_{HS} = \int_E (B, d\tau)_{HS}. \quad (4.20)$$

Since $B \in C_+$, we easily see that the integral on the right is positive; by (4.20) this implies a bound from below on $\| \left( \frac{1}{\beta} L \right)^n B \|_\infty$. Applying lemma 4.1 we get that the matrices $\left( \frac{1}{\beta} L \right)^n B$ are uniformly positive-definite. Together with (4.19) and lemma 2.2 this implies that there is a sequence $\{\alpha_l\}_{l \geq 1} \subset (0, +\infty)$ such that

$$\|\alpha_l \left( \frac{1}{\beta} L \right)^l B - Q\|_\infty \to 0. \quad (4.21)$$

In view of (4.20), this implies that

$$\alpha_l \int_E (B, d\tau)_{HS} \to \int_E (Q, d\tau)_{HS}.$$

The right hand side in the formula above is 1 since $\tau \in \mathcal{P}_Q(G, M^d)$; this implies that

$$\alpha_l \to \alpha : = \frac{\int_E (Q, d\tau)_{HS}}{\int_E (B, d\tau)_{HS}}.$$  

Note that the numerator is 1 since $\tau \in \mathcal{P}_Q(G, M^d)$; the denominator is different from zero since $B \in C_+$, (2.4) holds and $\tau \in \mathcal{P}_Q(E, M^d)$.

Recall that (4.21) and the last formula imply that

$$\left( \frac{1}{\beta} L \right)^l B \to \frac{1}{\alpha} Q$$

uniformly. Now (4.17) follows from the last two formulas.

The last case is when $B \in \bar{C}_+ \setminus \{0\}$. In this case, we consider $B + \delta Id$ for $\delta > 0$; since $B + \delta Id \in C_+$ we have just shown that

$$\left( \frac{1}{\beta} L \right)^l (B + \delta Id) \to Q \int_G (B + \delta Id, d\tau)_{HS} \quad (4.22)$$

uniformly as $n \to +\infty$. We saw above that, since $Id \in C_+$,

$$\left( \frac{1}{\beta} L \right)^l Id \to Q \int_E (Id, d\tau)_{HS}$$

uniformly. Now the thesis follows subtracting the last two formulas and using the fact that $\mathcal{L}$ is linear. $\square$

**Definition.** By point 1) of proposition 4.5, the operator $\mathcal{L}_G$ on $C(G, M^k)$ has a couple eigenvalue-eigenvector which we call $(\beta_G, Q_G)$; the operator $\mathcal{L}_\Sigma$ on $C(\Sigma, M^k)$
has a couple eigenvalue-eigenvector which we call \((\beta_G, Q_G)\). By point 2) of proposition 4.5 there is a Gibbs measure on \(G\), which we call \(\tau_G\), and one on \(\Sigma\), which we call \(\tau\). We shall say that \(\kappa_G := (Q_G, \tau_G)_{HS}\) is Kusuoka’s measure on \(G\) and that \(\kappa_\Sigma := (Q_\Sigma, \tau_\Sigma)_{HS}\) is Kusuoka’s measure on \(\Sigma\). Since \(\tau_G \in \mathcal{P}_{Q_G}\) and \(\tau_\Sigma \in \mathcal{P}_{Q_\Sigma}\), \(\kappa_G\) and \(\kappa_\Sigma\) are both probability measures.

The next lemma shows that there is a natural relationship between these objects.

**Lemma 4.6.** We have that \(\beta_G = \beta_\Sigma\); we shall call \(\beta\) their common value. Up to multiplying one of them by a positive constant, we have that \(Q_\Sigma = Q_G \circ \Phi\). With this choice for \(Q_\Sigma\) and \(Q_G\) we have that \(\tau_G = \Phi_\sharp \tau_\Sigma\) and \(\kappa_G = \Phi_\sharp \kappa_\Sigma\).

**Proof.** The first equality below comes from the formula after (3.2) and the second one from the definition of \(\mathcal{L}_G\) in (3.1).

\[
\mathcal{L}_G(A \circ \Phi)(x) = \sum_{i=1}^n tD\psi_i|_{\Phi(x)} A\psi_i \circ \Phi(x) D\psi_i|_{\Phi(x)} = \mathcal{L}_G(A) \circ \Phi(x) \quad \text{for all} \quad A \in C(G, M^k). \tag{4.23}
\]

Since \(Q_G\) is a fixed point of \(\mathcal{L}_G\) on \(\mathcal{C}_{\Sigma}(G, a, m_a)\), the formula above implies that \(Q_G \circ \Phi\) is a fixed point of \(\mathcal{L}_G\) on \(\mathcal{C}_{\Sigma}(\Sigma, a, m_a)\). By the uniqueness of proposition 4.5 we get that, up to multiplying one of them by a positive constant, \(Q_\Sigma = Q_G \circ \Phi\). Since \(\beta_G Q_\Sigma = \mathcal{L}_G(Q_\Sigma) = \mathcal{L}_G(Q_G \circ \Phi) = \mathcal{L}_G(Q_G) \circ \Phi = \beta_G Q_G \circ \Phi\) we get that \(\beta_G = \beta\).

We prove the relation between the Gibbs measures. Let \(A \in C(G, M^k)\); the first equality below is the definition of the adjoint, the second one follows from the definition of push-forward; the third one comes from (4.23) while the fourth one comes from the fact that \(\tau_\Sigma\) is an eigenvector of \(\mathcal{L}_G\); the last one comes again by the definition of push-forward.

\[
\langle A, \mathcal{L}_G^\ast (\Phi_\sharp \tau_\Sigma) \rangle = \langle \mathcal{L}_G A, \Phi_\sharp \tau_\Sigma \rangle = \langle (\mathcal{L}_G A) \circ \Phi, \tau_\Sigma \rangle = \langle (\mathcal{L}_G A) \circ \Phi, \tau_\Sigma \rangle = \beta \langle A \circ \Phi, \tau_\Sigma \rangle = \beta \langle A, \Phi_\sharp \tau_\Sigma \rangle.
\]

Moreover, the fact that \(Q_\Sigma = Q_G \circ \Phi\) easily implies that \(\Phi_\sharp \tau_\Sigma \in \mathcal{P}_{Q_G}\); since lemma 4.5 implies the uniqueness of the eigenvector of \(\mathcal{L}_G^\ast\) in \(\mathcal{P}_{Q_G}\), the last formula implies that \(\tau_\Sigma = \Phi_\sharp \tau_\Sigma\).

We leave to the reader the easy verification that \(\kappa_G = \Phi_\sharp \kappa_\Sigma\). \(\square\)

In order to prove that Kusuoka’s measure \((Q, \tau)_{HS}\) is ergodic, we need a lemma.

**Lemma 4.7.** Let \(\beta > 0\) and let \(\tau \in \mathcal{M}^\dagger(E, M^d)\) be as in point 2) of proposition 4.5. Let \(A: E \to M^d\) be a bounded Borel function. Then,

\[
\int_E (\mathcal{L} A, d\tau)_{HS} = \beta \int_E (A, d\tau)_{HS}.
\]

**Proof.** Let us define the measure \(t\) as

\[
t := ||\tau|| + \sum_{i=1}^n (\psi_i)^2 (||\tau||)
\]

if we are on \(G\); on \(\Sigma\) we set

\[
t := ||\tau|| + \sum_{i=1}^n (a_i)^2 (||\tau||)
\]
where \( a_i : (x_0 x_1 \ldots) \to (ix_0 x_1 \ldots) \).

By Lusin’s theorem there is a sequence \( A_k \in C(E, M^d) \) such that \( A_k \to A \) t-a.e. on \( E \); moreover, \( ||A_k||_\infty \) is bounded. By dominated convergence, this implies that

\[
\int_E (A_k, d\tau)_H \to \int_E (A, d\tau)_H
\]
and

\[
\int_E (\mathcal{L}A_k, d\tau)_H \to \int_E (\mathcal{L}A, d\tau)_H.
\]

Since \( A_k \) is continuous, point 2) of proposition 4.5 implies that

\[
\int_E (\mathcal{L}A_k, d\tau)_H = \beta \int_E (A_k, d\tau)_H.
\]

The thesis follows from the last three formulas.

The next lemma recalls some properties of Kusuoka’s measure.

**Lemma 4.8.** Let \( E = G \) or \( E = \Sigma \); in the first case we set \( S = F \), in the second one we set \( S = \sigma \). Then, the following holds.

1) Let \( Q \) and \( \tau \) be as in proposition 3.2, let \( g \in C(E, R) \) and let \( A \in C(E, M^k) \). Then we have that

\[
\int_E (g \circ S^i \cdot A, d\tau)_H \to \int_E (gQ, d\tau)_H \cdot \int_E (A, d\tau)_H.
\]

2) The scalar measures \( \kappa_G \) and \( \kappa_\Sigma \) defined before lemma 4.6 are ergodic.

**Proof.** We begin with point 1) on \( \Sigma \); we follow [24].

First of all, we note that, if \( h : \Sigma \to R \) is a bounded Borel function and \( \sigma \) is the shift of section 1, then

\[
h \circ \sigma(ix) = h(x) \quad \text{for all } i \in (1, \ldots, n) \quad \text{and all } x \in \Sigma.
\]

By (3.2) this implies that, if \( A \in C(\Sigma, M^d) \),

\[
[\mathcal{L}_\Sigma(h \circ \sigma \cdot A)](x) = h(x)(\mathcal{L}_\Sigma A)(x) \quad \forall x \in G.
\]

Integrating against \( \tau_\Sigma \), we get the first equality below; the second equality comes from lemma 4.7.

\[
\int_\Sigma \left( h \cdot \left( \frac{1}{\beta} \mathcal{L}_\Sigma A \right), d\tau_\Sigma \right)_H = \int_\Sigma \left( \frac{1}{\beta} \mathcal{L}_\Sigma (h \circ \sigma \cdot A), d\tau_\Sigma \right)_H = \int_\Sigma (h \circ \sigma \cdot A, d\tau_\Sigma)_H.
\]

In particular, if \( A = Q_\Sigma \) where \( Q_\Sigma \) is the eigenfuction of proposition 3.2, we have that

\[
\int_\Sigma h d(Q_\Sigma, \tau_\Sigma)_H = \int_\Sigma h \circ \sigma d(Q_\Sigma, \tau_\Sigma)_H.
\]

Iterating (4.26) for \( h = g \in C(\Sigma, R) \) we get that

\[
\int_\Sigma \left( g \left( \frac{1}{\beta} \mathcal{L}_\Sigma \right)^k A, d\tau_\Sigma \right)_H = \int_\Sigma (g \circ \sigma^k \cdot A, d\tau_\Sigma)_H.
\]

Now (4.24) for \( (E, S) = (\Sigma, \sigma) \) follows from point 3) of proposition 4.5.

Next, we show (4.24) when \( E = G \). Anticipating on lemma 5.3 below, the measure \( \tau_\Sigma \) is non-atomic. In particular, the countable set \( N \subset \Sigma \) on which \( \Phi \) is not injective is a null set for \( \kappa_\Sigma \). Let \( g \in C(G, R) \) and let \( A \in C(G, M^k) \); the
first equality below comes from lemma 4.6; the second one is the definition of push-forward; the third one comes from (1.16), which holds save on a null-set; the limit is (4.24) on \( \Sigma \), which we have just proven. The last equality follows again from lemma 4.6.

\[
\int_G (g \circ F^l \cdot A, d\tau_G)_{HS} = \int_G (g \circ F^l(x) \cdot A(x), d(\Phi_r \tau_G)(x))_{HS} = \\
\int_{\Sigma} (g \circ F^l \cdot \Phi(y) \cdot A \circ \Phi(y), d\tau_G(y))_{HS} = \\
\int_{\Sigma} (g \circ \Phi \circ \sigma^l(y) \cdot A \circ \Phi(y), d\tau_G(y))_{HS} \\
\int_{\Sigma} (g \circ \Phi(y) \cdot Q_{\Sigma}(y), d\tau_G(y))_{HS} \cdot \int_{\Sigma} (A \circ \Phi(y), d\tau_G(y))_{HS} = \\
\int_G (g(x)Q_G(x), d\tau_G(x))_{HS} \cdot \int_G (A(x), d\tau_G(x))_{HS}.
\]

This is (4.24) for \( G \), ending the proof of point 1).

We prove point 2). First of all, since (4.27) holds for all \( h \in C(\Sigma, \mathbb{R}) \) we get that \( \sigma^l \kappa_{\Sigma} = \kappa_{\Sigma} \), i.e. that \( \kappa_{\Sigma} \) is \( \sigma \)-invariant. With the same argument we used for the formula above this implies that

\[
\int_G h d(Q_G, \tau_G)_{HS} = \int_G h \circ F d(Q_G, \tau_G)_{HS}
\]
i.e. that \( \kappa_G \) is \( F \)-invariant.

Now we work on \( E \), with \( E = G \) or \( E = \Sigma \), and \( Q = Q_G \) or \( Q = Q_{\Sigma} \) respectively. Setting \( A = fQ \) for a continuous function \( f \), (4.24) implies that the measure \( (Q, \tau)_{HS} \) is strongly mixing; in particular, it is ergodic. \( \square \)

At this stage it is natural to ask whether, when the maps \( \psi_i \) are affine, our measure \( \kappa_G \) coincides with the measure \( \kappa \) defined in [16]; in the remark at the end of section 5 we shall prove that this is the case.

### 5. The Gibbs property.

In section 1 we defined the cylinder \( [x_0 \ldots x_l] \subset \Sigma \) and the cell \( [x_0 \ldots x_l]_G \subset G \).

From now on, we shall suppose that the maps \( \psi_i \) are affine and we set

\[ \psi_{x_0 \ldots x_l} = \psi_{x_0} \circ \cdots \circ \psi_{x_l} \]

**Definition.** Let \( M^+(G, M^d) \). Following [24], we shall say that \( \mu \in M^+(G, M^d) \) is a Gibbs measure if there is there are constants \( C, D_1 > 0 \) such that, for all \( l \geq 1 \) and all \( x \in G \setminus \tilde{N} \),

\[
e^{-Cl-D_1} \cdot (D\psi_{x_0 \ldots x_{l-1}}) \cdot \mu(G) \cdot t(D\psi_{x_0 \ldots x_{l-1}}) \leq \mu([x_0 \ldots x_{l-1}]_G) \leq \\
e^{-Cl+D_1} \cdot (D\psi_{x_0 \ldots x_{l-1}}) \cdot \mu(G) \cdot t(D\psi_{x_0 \ldots x_{l-1}}).
\]

We say that \( \mu \in M^+(\Sigma, M^d) \) is a Gibbs measure if there is there are constants \( C, D_1 > 0 \) such that, for all \( l \geq 1 \) and all \( x \in \Sigma \),

\[
e^{-Cl-D_1} \cdot (D\psi_{x_0 \ldots x_{l-1}}) \cdot \mu(\Sigma) \cdot t(D\psi_{x_0 \ldots x_{l-1}}) \leq \mu([x_0 \ldots x_{l-1}]) \leq \\
e^{-Cl+D_1} \cdot (D\psi_{x_0 \ldots x_{l-1}}) \cdot \mu(\Sigma) \cdot t(D\psi_{x_0 \ldots x_{l-1}}).
\]

In the formula above, we have not specified at which point we calculate \( D\psi_{x_0 \ldots x_{l-1}} \), since \( \psi_{x_0 \ldots x_{l-1}} \) is affine.
Let $\tau_\Sigma$ be the positive eigenvector of $\mathcal{L}_\Sigma^*$ as in lemma 4.5; we briefly prove that $\tau_\Sigma(\Sigma) \neq 0$. Let $Q_\Sigma$ be as in proposition 4.5; the inequality below comes from (2.4) and the fact that, for some $\epsilon > 0$, $\epsilon Q_\Sigma(G) \leq Id$ for all $x \in G$ by compactness; the equality comes from the fact that $\tau_\Sigma \in \mathcal{P}_Q(\Sigma, M^d)$.

$$ (Id, \tau_\Sigma(\Sigma))_{HS} \geq \epsilon \int_\Sigma (Q_\Sigma(x), d\tau_\Sigma(x))_{HS} = \epsilon. $$

We have the following analogue of proposition 3.2 of [24].

Lemma 5.1. Let (F1)-(F4) and (ND) hold; let the maps $\psi_1$ be affine. Let $(\beta, \tau_\Sigma)$ be as in proposition 4.5. Then for all $l \geq 1$ and all $x = (x_0, x_1, \ldots) \in \Sigma$ we have

$$ \tau_\Sigma([x_0 \ldots x_l]) = \frac{1}{\beta} (D\psi_{x_0}) \cdot \tau_\Sigma([x_1 \ldots x_l]) \cdot \tau_\Sigma([x_0 \ldots x_l]), $$

(5.2)

If $l = 0$, (5.2) holds with $\tau_\Sigma(\Sigma)$ instead of $\tau_\Sigma([x_1 \ldots x_l])$ on the right.

Proof. Let $x = (x_0, x_1, \ldots) \in \Sigma$ be fixed; clearly, we have that

$$ 1_{[x_0x_1 \ldots x_n]}(iz) = \begin{cases} 1_{[x_1 \ldots x_n]}(z) & \text{if } i = x_0 \\ 0 & \text{otherwise.} \end{cases} $$

Let $A \in M_d^d$ be a fixed, positive semidefinite matrix. The formula above implies the second equality below, while the first one comes from the fact that $A$ is constant; the third one follows by multiplying and dividing and recalling that $1_{[x_0 \ldots x_l]}(iz) = 0$ if $i \neq x_0$; the fourth one from the definition of $\mathcal{L}_\Sigma$ in (3.2); the last one follows by lemma 4.7.

$$ (A, \tau_\Sigma([x_1 \ldots x_l]))_{HS} = \int_\Sigma (A1_{[x_1 \ldots x_l]}(z), d\tau_\Sigma(z))_{HS} = $$

$$ \int_\Sigma \left( \sum_{i=1}^n A1_{[x_0 \ldots x_l]}(iz), d\tau_\Sigma(z) \right)_{HS} = $$

$$ \int_\Sigma \left( \sum_{i=1}^n (D\psi_i \cdot \tau_\Sigma([x_0 \ldots x_l]) \cdot (D\psi_{x_0})^{-1} \cdot D\psi_i, d\tau_\Sigma(z) \right)_{HS} = $$

$$ \int_\Sigma \left( \mathcal{L}_\Sigma \left( (D\psi_{x_0})^{-1} \cdot A1_{[x_0 \ldots x_l]} \cdot (D\psi_{x_0})^{-1} \right) \cdot \tau_\Sigma(z) \right)_{HS} = $$

$$ \beta \int_{[x_0 \ldots x_l]} \left( (D\psi_{x_0})^{-1} \cdot A \cdot (D\psi_{x_0})^{-1}, d\tau_\Sigma \right)_{HS}. $$

(5.3)

After transposition, (5.3) implies that

$$ \beta(A, D\psi_{x_0}^{-1} \cdot \tau_\Sigma([x_0 \ldots x_l]) \cdot D\psi_{x_0}^{-1})_{HS} = (A, \tau_\Sigma([x_1 \ldots x_l]))_{HS}. $$

Letting $A$ vary among the one-dimensional projections we get that

$$ \beta \cdot D\psi_{x_0}^{-1} \cdot \tau_\Sigma([x_0 \ldots x_l]) \cdot D\psi_{x_0}^{-1} = \tau_\Sigma([x_1 \ldots x_l]). $$

To get (5.2) it suffices to multiply the formula above by

$$ \frac{1}{\beta} \cdot (D\psi_{x_0}) \quad \text{on the left and by} \quad (D\psi_{x_0})^{-1} \quad \text{on the right.} $$

$\square$

Corollary 5.2. Let (F1)-(F4) and (ND) hold; let us suppose that the maps $\psi_1$ are affine and let $(\beta, \tau_\Sigma)$ be as in proposition 4.5. then, $\tau_\Sigma$ is a Gibbs measure for the constant $C = \log \beta$. 

Proof. Iterating the right hand side of (5.2) and using the chain rule we get the following.

\[ \tau_{\Sigma}([x_0 \ldots x_l]) = \frac{1}{\beta} \cdot (D\psi_{x_0})\tau_{\Sigma}([x_1 \ldots x_l])^t (D\psi_{x_0}) = \]
\[ = \frac{1}{\beta^2} \cdot (D\psi_{x_0x_1})\tau_{\Sigma}([x_2 \ldots x_l])^t (D\psi_{x_0x_1}) = \]
\[ \cdots = \]
\[ = \frac{1}{\beta^l} \cdot (D\psi_{x_0x_1 \ldots x_{l-1}})\tau_{\Sigma}([x_l])^t (D\psi_{x_0x_1 \ldots x_{l-1}}) = \]
\[ = \frac{1}{\beta^{l+1}} \cdot (D\psi_{x_0 \ldots x_l})\tau_{\Sigma}(\Sigma)^t (D\psi_{x_0 \ldots x_l}). \]

Lemma 5.3. Let the maps \( \psi_i \) satisfy (F1)-(F4) and let (ND) hold. Then, we have the following.
1) The measure \( \tau_{\Sigma} \) is positive on open sets.
2) The measures \( \tau_{\Sigma} \) and \( \tau_G \) are non-atomic.

Proof. We begin with point 1) for \( \tau_{\Sigma} \). It suffices to show that, for all cylinders [\( x_0 \ldots x_l \) \( \subset \Sigma \)], the matrix \( \tau_{\Sigma}[x_0 \ldots x_l] \) is not zero. Since \( \tau_{\Sigma} \) has the Gibbs property, this follows if \( \tau_{\Sigma}(\Sigma) \neq 0 \), which we have shown before lemma 5.1.

As for point 2), we begin to recall the standard proof that \( \kappa_{\Sigma} \) is non-atomic. By point 2) of lemma 4.8, \( \kappa_{\Sigma} \) is ergodic; let us suppose by contradiction that it has an atom \( \{ \hat{x} \} \). We are going to show that \( \kappa_{\Sigma}(\{ \hat{x} \}) = 1 \) and, consequently, \( \kappa_{\Sigma}(\{ \hat{x} \}^c) = 0 \). This will be the contradiction, since by point 1) \( \tau_{\Sigma} \) is positive on open sets.

First of all, let us suppose that \( \hat{x} \) is a periodic orbit of period \( q \) and let us set
\[ A = \bigcup_{l \geq 0} \sigma^{-lq}(\{ \hat{x} \}). \]

Clearly, \( A \) is \( \sigma^q \)-invariant, i.e. \( \sigma^q(A) \subset A \). Since \( \sigma \) preserves \( \kappa_{\Sigma} \), we have that \( \kappa_{\Sigma}(\sigma^{-lq}(\{ \hat{x} \})) = \kappa_{\Sigma}(\{ \hat{x} \}) \); since \( \sigma^q \) fixes \( \hat{x} \), we see that \( \tilde{x} \in \sigma^{-q}(\{ \hat{x} \}) \). This implies that \( \tau_{\Sigma} \) on \( \sigma^{-q}(\{ \tilde{x} \}) \) concentrates on \( \{ \tilde{x} \} \); iterating, we get that \( \tau_{\Sigma} \) on \( \sigma^{-lq}(\{ \tilde{x} \}) \) concentrates on \( \{ \tilde{x} \} \). This and the definition of \( A \) easily imply that \( \kappa_{\Sigma}(A) = \kappa_{\Sigma}(\{ \tilde{x} \}) \) and that \( \kappa_{\Sigma}(A \setminus \sigma^{-q}(A)) = 0 \). By ergodicity, this implies that \( \kappa_{\Sigma}(\{ \tilde{x} \}) = 1 \).

The second case is when \( \tilde{x} \) has an antiperiod, say of length \( l \). We consider \( \tilde{x} = \sigma^l(\tilde{x}) \), which is periodic. Since \( \tilde{x} \in \sigma^{-l}(\tilde{x}) \), invariance implies that \( \kappa_{G}(\tilde{x}) > 0 \); now the same argument as above applies.

The last case is when \( \tilde{x} \) is not periodic; then, it is easy to see that the sets \( \sigma^{-l}(\tilde{x}) \) are all disjoint. Since they have the same measure, we get that \( \kappa_{\Sigma}(\{ \tilde{x} \}) = 0 \), i.e. that \( \{ \tilde{x} \} \) is not an atom.

In order to show that \( \kappa_{G} \) is non-atomic, it suffices to recall three facts: that \( \tau_{\Sigma} \) is non-atomic, that \( \tau_{G} = \Phi_{\Sigma} \tau_{\Sigma} \) by lemma 4.6 and that \( \Phi \) is finite-to-one.

End of the proof of theorem 1. Points 1) and 2) come from proposition 4.5. The self-similarity of point 4) comes from lemma 3.1. The ergodicity of point 3) is point 2) of lemma 4.8; mutual absolute continuity in one direction follows from (2.8), in the other one is trivial. For point 5) we begin to note that, by the definition of \( \Phi(x_0 \ldots x_l) \subset \Phi^{-1}(x_0 \ldots x_l)_1 \); the points of \( \Phi^{-1}(x_0 \ldots x_l)_1 \setminus [x_0 \ldots x_l] \) are those with multiple codings, which we have seen in section 1 to be a countable set. Since \( \tau_{\Sigma} \) is non-atomic, we get that
\[ \tau_{G}(\Phi^{-1}(x_0 \ldots x_l)_1 \setminus [x_0 \ldots x_l]) = 0. \]
Since $\tau_G = \Phi_l \tau_\Sigma$, by lemma 3.4, the last formula implies that

\[ \tau_\Sigma([x_0 \ldots x_l]) = \tau_G([x_0 \ldots x_l|_G]). \]

Since $\tau_\Sigma$ has the Gibbs property by corollary 5.2, we are done. $\square$

**Remark.** In corollary (4.2) we have supposed that the maps $D\psi_i$ are constant, which is the case of Kusuoka’s paper [16]. We prove that, up to multiplication by a positive constant, $(Q_G, \tau_G)_{HS}$ coincides with Kusuoka’s measure $\kappa$.

When $D\psi_i$ is constant, point 1) of proposition 4.5 implies that $Q_G$ is constant too and solves

\[ Q_G = \frac{1}{\beta} \sum_{i=1}^n t D\psi_i Q_G D\psi_i. \]  

(5.4)

By point 2) of lemma 3.1 we have that

\[ \beta \mathcal{E}_\tau(f,g) = \sum_{i=1}^n \mathcal{E}_\tau(f \circ \psi_i, g \circ \psi_i) = \sum_{i=1}^n \int_G (t D\psi_i \cdot \nabla f|_{\psi_i(x)}, d\tau(x)^t D\psi_i \cdot \nabla g|_{\psi_i(x)}). \]

If we choose as $f$ and $g$ two linear functions and we recall that $D\psi_i$ is a constant matrix, we see that the last formula implies that

\[ \beta \tau_G(G) = \sum_{i=1}^n D\psi_i \cdot \tau_G(G)^t D\psi_i. \]  

(5.5)

Kusuoka’s measure $\kappa$ is defined by the following formula: if $x = (x_0 x_1 \ldots)$, then

\[ \kappa([x_0 \ldots x_l]_G) = \frac{1}{\beta^l} (Q, (D\psi_{x_0 \ldots x_l})^t \hat{Q}^t (D\psi_{x_0 \ldots x_l}))_{HS} \]

where $Q$ solves (5.4) and $\hat{Q}$ solves (5.5). Since the solution to both equations is unique by Perron-Frobenius (up to multiplication by a constant, of course), the last formula and corollary (5.2) imply that $\kappa_G: = (Q_G, \tau_G)$ coincide with $\kappa$.

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