ON THE $q$-EULER NUMBERS AND POLYNOMIALS WITH WEIGHT 0

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Abstract The purpose of this paper is to investigate some properties of $q$-Euler numbers and polynomials with weight 0. From those $q$-Euler numbers with weight 0, we derive some identities on the $q$-Euler numbers and polynomials with weight 0.

1. Introduction

Let $p$ be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$. The $p$-adic absolutely value is defined by $|x|_p = 1/p^r$ where $x = p^r s/t$ with $(p, s) = (p, t) = (s, t) = 1$ and $r \in \mathbb{Q}$. In this paper, we assume that $\alpha \in \mathbb{Q}$ and $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. As well known definition, the Euler polynomials are defined by

$$2 \frac{e^t + 1}{e^t} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!},$$

with the usual convention about replacing $E_n(x)$ by $E_n(x)$ (see [1-15]).

In this special case, $x = 0$, $E_n(0) = E_n$ are called the $n$-th Euler numbers (see [1]). Recently, the $q$-Euler numbers with weight $\alpha$ are defined by

$\tilde{E}_{0,q}^{(\alpha)} = 1$, and $q^\alpha \tilde{E}_{q}^{(\alpha)} + 1)^n + \tilde{E}_{n,q}^{(\alpha)} = 0$ if $n > 0$, (1)

with the usual convention about replacing $(\tilde{E}_{q}^{(\alpha)})^n$ by $\tilde{E}_{n,q}^{(\alpha)}$ (see [3,12]). The $q$-number of $x$ is defined by $[x]_q = \frac{1-q^x}{1-q}$ (see [1-15]). Note that $\lim_{q \to 1} [x]_q = x$.

Let us define the notation of $q$-Euler numbers with weight 0 as $\tilde{E}_{n,q}^{(0)} = \tilde{E}_{n,q}$. The purpose of this paper is to investigate some interesting identities on the $q$-Euler numbers with weight 0.

2. On the extended $q$-Euler numbers of higher-order with weight 0

Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$ is defined by Kim as follows :

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{[2]_q^N}{1 + q^{p^N}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \text{ (see [1-12]).}$$ (2)
By (2), we get
\[ q^n I_q(f_n) + (-1)^{n-1} I_q(f) = [2]q \sum_{l=0}^{n-1} (-1)^{n-l} f(l) q^l, \] (3)
where \( f_n(x) = f(x + n) \) and \( n \in \mathbb{N} \) (see [4, 5]).

By (1), (2) and (3), we see that
\[ \int_{\mathbb{Z}_p} [x]^n d\mu_{-q}(x) = \tilde{E}_{n,q}^{(\alpha)} = \frac{[2]q}{(1 - q)[\alpha]_q^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1 + q^{n+l+1}}. \] (4)

In the special case, \( n = 1 \), we get
\[ \int_{\mathbb{Z}_p} e^{xt} d\mu_{-q}(x) = \frac{[2]q}{q e^t + 1} = \frac{1 + q^{-1}}{e^t + q^{-1}} = \sum_{n=0}^{\infty} H_n(-q^{-1}) \frac{t^n}{n!}. \] (5)

where \( H_n(-q^{-1}) \) are the \( n \)-th Frobenius-Euler numbers. From (5), we note that the \( q \)-Euler numbers with weight 0 are given by
\[ \tilde{E}_{n,q} = \int_{\mathbb{Z}_p} x^n d\mu_{-q}(x) = H_n(-q^{-1}), \text{ for } n \in \mathbb{Z}_+. \] (6)

Therefore, by (6), we obtain the following theorem.

**Theorem 1.** For \( n \in \mathbb{Z}_+ \), we have
\[ \tilde{E}_{n,q} = H_n(-q^{-1}), \]
where \( H_n(-q^{-1}) \) are called the \( n \)-th Frobenius-Euler numbers.

Let us define the generating function of the \( q \)-Euler numbers with weight 0 as follows:
\[ \tilde{F}_q(t) = \sum_{n=0}^{\infty} \tilde{E}_{n,q} \frac{t^n}{n!}. \] (7)

Then, by (4) and (7), we get
\[ \tilde{F}_q(t) = [2]q \sum_{m=0}^{\infty} (-1)^m q^m e^{mt} = \frac{1 + q}{qe^t + 1}. \] (8)

Now we define the \( q \)-Euler polynomials with weight 0 as follows:
\[ \sum_{n=0}^{\infty} \tilde{E}_{n,q}(x) \frac{t^n}{n!} = \frac{1 + q}{qe^t + 1} e^{xt}. \] (9)

Thus, (5) and (9), we get
\[ \int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-q}(y) = \frac{1 + q}{qe^t + 1} e^{xt} = \sum_{n=0}^{\infty} \tilde{E}_{n,q}(x) \frac{t^n}{n!}. \] (10)

From (10), we have
\[ \sum_{n=0}^{\infty} \tilde{E}_{n,q}(x) \frac{t^n}{n!} = \left( \frac{1 + q^{-1}}{e^t + q^{-1}} \right) e^{xt} = \sum_{n=0}^{\infty} H_n(-q^{-1}, x) \frac{t^n}{n!}, \] (11)
where \( H_n(-q^{-1}, x) \) are called the \( n \)-th Frobenius-Euler polynomials (see [9]).

Therefore, by (11), we obtain the following theorem.
Theorem 2. For \( n \in \mathbb{Z}_+ \), we have

\[
\tilde{E}_{n,q}(x) = \int_{\mathbb{Z}_p} (x + y)^n d\mu_{-q}(x) = H_n(-q^{-1}, x),
\]

where \( H_n(-q^{-1}, x) \) are called the \( n \)-th Frobenius-Euler polynomials.

From (3) and Theorem 2, we note that

\[
q^n H_m(-q^{-1}, n) + H_m(-q^{-1}) = [2]_q \sum_{l=0}^{n-1} (-1)^l l^m q^l,
\]

(12)

where \( n \in \mathbb{N} \) with \( n \equiv 1 \pmod{2} \).

Therefore, by (12), we obtain the following corollary.

Corollary 3. For \( n \in \mathbb{N} \), with \( n \equiv 1 \pmod{2} \) and \( m \in \mathbb{Z}_+ \), we have

\[
q^n H_m(-q^{-1}, n) + H_m(-q^{-1}) = [2]_q \sum_{l=0}^{n-1} (-1)^l l^m q^l.
\]

In particular, \( q = 1 \), we get \( E_m(n) + E_m = 2 \sum_{l=0}^{n-1} (-1)^l l^m \), where \( E_m \) and \( E_m(n) \) are called the \( m \)-th Euler numbers and polynomials which are defined by

\[
\frac{2}{e^t + 1} = \sum_{m=0}^{\infty} E_m \frac{t^m}{m!} \quad \text{and} \quad \frac{2}{e^t + 1} e^x = \sum_{m=0}^{\infty} E_m(x) \frac{t^m}{m!}.
\]

By (3), we easily see that

\[
q \int_{\mathbb{Z}_p} f(x + 1) d\mu_{-q}(x) + \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = [2]_q f(0).
\]

(13)

Thus, by (13), we get

\[
[2]_q = q \int_{\mathbb{Z}_p} e^{(x+1)t} d\mu_{-q}(x) + \int_{\mathbb{Z}_p} e^{xt} d\mu_{-q}(x)
\]

\[
= \sum_{n=0}^{\infty} \left( q \int_{\mathbb{Z}_p} (x+1)^n d\mu_{-q}(x) + \int_{\mathbb{Z}_p} x^n d\mu_{-q}(x) \right) \frac{t^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} \left( qH_n(-q^{-1}, 1) + H_n(-q^{-1}) \right) \frac{t^n}{n!}.
\]

Therefore, by (13), we obtain the following theorem.

Theorem 4. For \( n \in \mathbb{Z}_+ \), we have

\[
qH_n(-q^{-1}, 1) + H_n(-q^{-1}) = \begin{cases} 
1 + q, & \text{if } n = 0, \\
0, & \text{if } n > 0,
\end{cases}
\]

where \( H_n(-q^{-1}, x) \) are called the \( n \)-th Frobenius-Euler polynomials and \( H_n(-q^{-1}) \) are called the \( n \)-th Frobenius-Euler numbers. In particular, \( q = 1 \), we have

\[
E_n(1) + E_n = \begin{cases} 
2, & \text{if } n = 0, \\
0, & \text{if } n > 0,
\end{cases}
\]

where \( E_n \) are called the \( n \)-th Euler numbers.
From (6) and Theorem 2, we note that
\[
\tilde{E}_{n,q}(x) = \int_{\mathbb{Z}_p} (x+y)^n d\mu_q(y) \\
= \sum_{l=0}^{n} \binom{n}{l} \int_{\mathbb{Z}_p} y^l d\mu_q(y) x^{n-l} \\
= \sum_{l=0}^{n} \binom{n}{l} \tilde{E}_{n,q} x^{n-l} \\
= (x + \tilde{E}_q)^n,
\]
where the usual convention about replacing \( (\tilde{E}_q)^l \) by \( \tilde{E}_{l,q} \). By Theorem 2 and Theorem 4, we get
\[
q \tilde{E}_{n,q}(1) + \tilde{E}_{n,q} = \begin{cases} 
[2]_q, & \text{if } n = 0, \\
0, & \text{if } n > 0.
\end{cases}
\tag{15}
\]
From (14) and (15), we have
\[
q \left( \tilde{E}_q + 1 \right)^n + \tilde{E}_{n,q} = \begin{cases} 
[2]_q, & \text{if } n = 0, \\
0, & \text{if } n > 0.
\end{cases}
\tag{16}
\]
For \( n \in \mathbb{N} \), by (14) and (16), we have
\[
q^2 \tilde{E}_{n,q}(2) = q^2 \left( \tilde{E}_q + 1 + 1 \right)^n \\
= q^2 \sum_{l=1}^{n} \binom{n}{l} \left( \tilde{E}_q + 1 \right)^l + q \left( 1 + q - \tilde{E}_0,q \right) \\
= q + q^2 - q \sum_{l=0}^{n} \binom{n}{l} \tilde{E}_{l,q} \\
= q + q^2 - q \left( \tilde{E}_q + 1 \right)^n \\
= q + q^2 + \tilde{E}_{n,q}.
\tag{17}
\]
Therefore, by (17), we obtain the following theorem.

**Theorem 5.** For \( n \in \mathbb{N} \), we have
\[
q^2 \tilde{E}_{n,q}(2) = q + q^2 + \tilde{E}_{n,q}.
\]

For \( n \in \mathbb{Z}_+ \), we have
\[
\tilde{E}_{n,q-1}(1-x) = \int_{\mathbb{Z}_p} (1-x + x_1)^n d\mu_{q-1}(x_1) \\
= (-1)^n \int_{\mathbb{Z}_p} (x_1 + x)^n d\mu_q(x_1) \\
= (-1)^n \tilde{E}_{n,q}(x).
\tag{18}
\]
Therefore, by (18), we obtain the following theorem.
Theorem 6. For \( n \in \mathbb{Z}_+ \), we have
\[
\tilde{E}_{n,q^{-1}}(1 - x) = (-1)^n \tilde{E}_{n,q}(x).
\]

From (14), we have
\[
\int_{\mathbb{Z}_p} (1 - x)^n d\mu_{-q}(x) = (-1)^n \int_{\mathbb{Z}_p} (x - 1)^n d\mu_{-q}(x)
= (-1)^n \tilde{E}_{n,q}(-1).
\] (19)

By Theorem 6 and (19), we get
\[
\int_{\mathbb{Z}_p} (1 - x)^n d\mu_{-q}(x) = \tilde{E}_{n,q^{-1}}(2) = 1 + q + q^2 \tilde{E}_{n,q^{-1}} \quad \text{if} \quad n > 0.
\] (20)

Therefore, by (20), we obtain the following theorem.

Theorem 7. For \( n \in \mathbb{N} \), we have
\[
\int_{\mathbb{Z}_p} (1 - x)^n d\mu_{-q}(x) = 1 + q + q^2 \tilde{E}_{n,q^{-1}}.
\]

Let \( C(\mathbb{Z}_p) \) be the space of continuous functions on \( \mathbb{Z}_p \). For \( f \in C(\mathbb{Z}_p) \), \( p \)-adic analogue of Bernstein operator of order \( n \) for \( f \) is given by
\[
\mathbb{B}_n(f|x) = \sum_{k=0}^{n} B_{k,n}(x) f \left( \frac{k}{n} \right)
= \sum_{k=0}^{n} f \left( \frac{k}{n} \right) \binom{n}{k} x^k (1 - x)^{n-k},
\] (21)

where \( n, k \in \mathbb{Z}_+ \) (see \([1,6,7]\)).

For \( n, k \in \mathbb{Z}_+ \), \( p \)-adic Bernstein polynomials of degree \( n \) is defined by
\[
B_{k,n}(x) = \binom{n}{k} x^k (1 - x)^{n-k}, \quad x \in \mathbb{Z}_p \quad \text{(see \([1,6,7]\))}.
\] (22)

Let us take the fermionic \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) for one Bernstein polynomials in (22) as follows:
\[
\int_{\mathbb{Z}_p} B_{k,n}(x) d\mu_{-q}(x) = \binom{n}{k} \int_{\mathbb{Z}_p} x^k (1 - x)^{n-k} d\mu_{-q}(x)
= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \int_{\mathbb{Z}_p} x^{k+l} d\mu_{-q}(x)
= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \tilde{E}_{k+l,q}.
\] (23)

By simple calculation, we easily get
\[
\int_{\mathbb{Z}_p} B_{k,n}(x) d\mu_q(x) = \int_{\mathbb{Z}_p} B_{n-k,n}(1-x) d\mu_q(x)
= \left( \begin{array}{c} n \\ k \end{array} \right) \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \int_{\mathbb{Z}_p} (1-x)^{n-l} d\mu_q(x)
= \left( \begin{array}{c} n \\ k \end{array} \right) \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \left( 1 + q^2 \tilde{E}_{n-l,q^{-1}} \right)
= \left( \begin{array}{c} n \\ k \end{array} \right) \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} q^2 \tilde{E}_{n-l,q^{-1}} \quad \text{if} \quad n > k.
\]

Therefore, by (23) and (24), we obtain the following theorem.

**Theorem 8.** For \( n \in \mathbb{Z}_+ \) with \( n > k \), we have
\[
\sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^{l} \tilde{E}_{k+l,q} = \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} q^2 \tilde{E}_{n-l,q^{-1}}.
\]
In particular, \( k = 0 \), we get
\[
\sum_{l=0}^{n} \binom{n}{l} (-1)^{l} \tilde{E}_{l,q} = q^2 \tilde{E}_{n,q^{-1}}.
\]
By Theorem 1 and Theorem 2, we get
\[
\sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^{l} H_{k+l}(-q^{-1}) = \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} q^2 H_{n-l}(-q),
\]
where \( H_n(-q) \) are called the \( n \)-th Frobenius-Euler numbers.

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