The relativistic complex scalar field at finite temperature and in presence of a net conserved charge is studied in reference to recent developments on the multiplicative anomaly. This quantity, overlooked until now, is computed and it is shown how it could play a role for this system. Other possible applications are also mentioned.

I. THE CHARGED BOSE GAS

The relativistic Bose gas in presence of a net conserved charge has been studied with some interest in recent years. The effective potential was first obtained for the free field by Kapusta [1] and Haber and Weldon [2], and for the interacting field at one loop by Benson, Bernstein and Dodelson [3–4]. In recent work [5] I presented one another possible approach to the inclusion of the chemical potential.

This system can be described using the standard functional integral approach developed for thermal field theories. I will here mainly analyse the non-interacting case, for which the grand canonical partition function, using the most popular approach (Method I) [1–3], is

\[
Z_\beta(\mu) = \text{Tr} e^{-\beta (H - \mu Q)} = \int_{\phi(\tau) = \phi(\tau + \beta)} \left[ d\phi_\tau \right] e^{-\frac{1}{2} \int_0^\beta d\tau \int d^3x \phi_\tau A_{ij} \phi_\tau }, \tag{1}
\]

where \(H\) is the Hamiltonian of the system, \(Q\) the charge, \(\beta\) the inverse of the temperature and \(\phi_\tau\) the two real degrees of freedom of the complex field. \(A_{ij}\) is the elliptic, non-self-adjoint, matrix valued, differential operator

\[
\left( -\partial_\tau^2 - \nabla^2 + m^2 - e^2 \mu^2 \right) \frac{2ie\mu \partial_\tau}{2ie\mu \partial_\tau}, \tag{2}
\]

Notice the appearance of a "mass term" \(m^2 - e^2 \mu^2\) which will eventually give the Bose-Einstein condensation.

We have then:

\[
\ln Z_\beta(\mu) = -\frac{1}{2} \ln \det \left[ \frac{A_{ij}}{M^2} \right] \tag{3}
\]

II. REGULARIZATION AND MULTIPLICATIVE ANOMALY

Since these are functional determinants of differential operators, as formal infinite products of eigenvalues they are divergent (UV divergence) and therefore a proper regularization scheme has to be adopted.

One of the most successful ones is the zeta-function regularization method [5]. The zeta-function regularized functional determinant of a second order elliptic differential operator \(L\), is then defined as

\[
\ln \det \left[ \frac{L}{M^2} \right] = -\zeta'(0|L) - \frac{1}{2} \frac{\zeta(0|L)}{\zeta(s|L)} \ln M^2, \tag{7}
\]

where \(\zeta(s|L) = \text{Tr} L^{-s}\) is the zeta function related to \(L\), \(\zeta'(0|L)\) its derivative with respect to \(s\) at zero, and \(M^2\) is a renormalization scale mass. The fact used here is that the analytically continued zeta-function is generally regular at \(s = 0\), and thus its derivative is well defined.

Now, the determinant with which we are dealing involves the product of two operators, as in \(\ln \det(AB)\). It is a long overlooked fact that in this case the multiplicative property \(\ln \det(AB) = \ln \det(A) + \ln \det(B)\), with \(A, B\) commuting pseudo-differential operators, does not necessarily hold. On the contrary, an additional term \(a(A, B)\), called the multiplicative anomaly [6–8], may be present on the right hand-side. Since all previous regularized computations of the effective potential assumed

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the above equality, there could be additional physical terms disregarded until now \cite{13}--\cite{15}.

It is in this light that I will reanalyze the complex scalar field at finite temperature and chemical potential, using a properly regularized approach and including the multiplicative anomaly. I will mainly refer to the case of \( D = 4 \) spacetime dimensions. In ref. \cite{14}, from which this work derives, all the following computations have been developed for generic \( D \).

Let us for a moment assume the above equality, as in the previous literature, so as to show the resulting inconsistencies. I will avoid here the steps that lead to the computation of the logarithm of the partition function, as the reader will find them in detail in ref. \cite{11}.

For the first factorization \( K_\pm \), we easily obtain the standard result \cite{10}--\cite{14}

\[
\ln Z_\beta(K_+, K_-) = \frac{\beta V}{32\pi^2} \left[ m^4 \left( \ln \frac{m^2}{\Lambda^2} - \frac{3}{2} \right) \right] - V \int \frac{d^4k}{(2\pi)^4} \ln(1 - e^{-\beta(\sqrt{k^2 + \mu^2} - \epsilon\mu)}) \\
- V \int \frac{d^4k}{(2\pi)^4} \ln(1 - e^{-\beta(\sqrt{k^2 + \mu^2} + \epsilon\mu)}), \tag{8}
\]

where vacuum, particle and antiparticle contributions are manifest. For the other factorization, though, the chemical potential is associated with the momentum integral and it remains in the term linear in \( \beta \) so that we have

\[
\ln Z_\beta(L_+, L_-) = \frac{\beta V}{32\pi^2} \left[ m^4 \left( \ln \frac{m^2}{\Lambda^2} - \frac{3}{2} \right) \right] + S(\beta, \mu) \\
+ \frac{\beta V}{8\pi^2} \left( \frac{e^4\mu^4}{3} - e^2\mu^2 m^2 \right), \tag{9}
\]

where \( S(\beta, \mu) \) represents the standard thermal contributions as in \cite{10}.

In this system the importance of the multiplicative anomaly is therefore manifest. Despite having \( \ln(K_-K_+) = \ln(L_-L_+) \), these two options give two different results for a zeta-function regularized partition function if the multiplicative anomaly is disregarded.

The approach taken in \cite{11} was recently criticized in ref. \cite{13}. According to Dowker the anomaly would be a result of this approach since, for generic finite matrices, this does not correspond to taking the full algebraic and functional determinant in one time. We replied \cite{14} to this criticism also on general grounds but relevant for us now is how the above operator is an important counterexample. The two possible factorizations can be seen as corresponding to two possible parametrizations of the same functional space in which we are integrating and as such linked by a transformation in that space. Therefore the final result should clearly be independent of the approach adopted. As I will show later, in fact, the anomaly preserves this invariance. This shows us also the delicate link between the functional measure of the path integral and the multiplicative anomaly and its possible physical relevance, link which clearly needs further investigation.

III. THE MULTIPLICATIVE ANOMALY

Let us see what happens when the anomaly is taken in account instead. The multiplicative anomaly \cite{12}--\cite{14} is defined as

\[
a_D(A, B) = \ln \det(AB) - \ln \det(A) - \ln \det(B) \tag{10}
\]

where the determinants of the two elliptic operators are defined by means of the zeta-function method. The direct computation of the unfactorized operator \( \ln \det(AB) \) (to compare with the factorized version) is in general a too complicated task as soon as the operators are non trivial (see ref. \cite{15}). Fortunately we can resort to an alternative recipe due to Wodzicki \cite{12}.

For any classical pseudo-differential operator \( A \) there exists a complete symbol \( A(x, k) = e^{-ikx} A e^{ikx} \). This admits an asymptotic expansion for \( |k| \to \infty \),

\[
A(x, k) \sim \sum_{j=0} \, A_{\alpha-j}(x, k), \tag{11}
\]

where the coefficients (their number is infinite) fulfill the homogeneity property \( A_{\alpha-j}(x, tk) = t^{\alpha-j} A_{\alpha-j}(x, k) \), for \( t > 0 \) and \( \alpha \) is called the order of \( A \). Now, Wodzicki \cite{12} proved that for two invertible, self-adjoint, elliptic, commuting, pseudodifferential operators on a smooth compact manifold without boundaries \( M_D \):

\[
a(\alpha, B) = \frac{\mathrm{res} \left( (\ln(A B^{-\alpha}))^2 \right)}{2ab(a + b)} = a(B, A), \tag{12}
\]

where \( a > 0 \) and \( b > 0 \) are the orders of \( A \) and \( B \), respectively. Here the quantity \( \mathrm{res}(A) \) is the Wodzicki noncommutative residue. It can be computed easily using the homogeneous component \( A_{-D}(x, k) \) of order \( -D \) of the complete symbol:

\[
\mathrm{res}(A) = \int_{M_D} \frac{dx}{(2\pi)^D} \int_{|k|=1} A_{-D}(x, k) dk. \tag{13}
\]

As an example:

\[
A(x, k)_{K_{\pm}} = \left[ \ln \left( k^2 + m^2 - e^2\mu^2 + i2e\mu k_x \right) \right. \\
- \ln \left( k^2 + m^2 - e^2\mu^2 - i2e\mu k_x \right) \right]^2. \tag{14}
\]

Remembering \cite{11},\cite{13} and \cite{12} and that the order of our operators is 2, we have the related multiplicative anomaly as

\[
a_{4}(K_+, K_-) = \frac{\beta V}{8\pi^2} \left[ 2\mu^2(m^2 - \frac{e^2\mu^2}{3}) \right]. \tag{15}
\]

The same can be done for \( L_{\pm} \), obtaining a different expression for \( a_4(L_+, L_-) \).

If we now add this two anomalies to the expressions \cite{10} and \cite{11} respectively, we obtain that the logarithm of the partition function turns out to be the same for the two different approaches,
\[ \ln Z_\beta = \frac{\beta V}{32\pi^2} \left[ m^4 (\ln \frac{m^2}{M^2} - 3/2) \right] + S(\beta, \mu) - \frac{\beta V}{16\pi^2} \left[ e^2 \mu^2 (m^2 - \frac{e^2 \mu^2}{3}) \right]. \] (16)

Although consistent now, our result is remarkably different from the one in the literature where the multiplicative anomaly was disregarded. The physical relevance of this additional term will be discussed in the next section.

For this operators the multiplicative anomaly is vanishing for any odd dimension \( D \) and can be easily computed for any even one \([15]\). In the interacting case \([11]\), for which the anomaly has been computed too, it is difficult to obtain a properly regularized expression for the rest of the partition function due to the complexity of the operators involved \([10]\).

\textbf{IV. IMPLICATIONS}

For this system now, the effective potential in presence of external sources, can be expressed as a function of the charge density \( \rho = \frac{1}{\beta V} \frac{\partial}{\partial \mu} \ln Z_\beta(\beta, \mu) \) and the mean field \( (x^2 = \Phi^2) \) as

\[ F(\beta, \rho, x) = -\frac{1}{\beta V} \ln Z_\beta(\mu) + \frac{\rho}{\beta V} \frac{\partial}{\partial \mu} \ln Z_\beta(\beta, \mu) + \frac{1}{2} \left( m^2 + e^2 \mu^2 \right) x^2, \] (17)

\[ \rho = \frac{1}{\beta V} \frac{\partial}{\partial \mu} \ln Z_\beta(\beta, \mu) + e^2 \mu x^2, \] (18)

where the latter is an implicit expression for the chemical potential as a function of \( \rho \).

The physical states correspond to the minima of the effective potential, in \( \frac{\partial}{\partial x} = x(m^2 - e^2 \mu^2) = 0 \). We have then: 1) an unbroken phase, \( x = 0, e \mu < m \); 2) a symmetry breaking solution, \( x \neq 0, e \mu = \pm m \), giving the relativistic Bose-Einstein condensation. For our system, explicitly, the unbroken and broken phase are respectively

\[ F_\beta = \min F = \mathcal{E}_V - \frac{1}{\beta V} S(\beta, \mu) + \mu \rho \]
\[ + \frac{1}{16\pi^2} \left[ e^2 \mu^2 (m^2 - \frac{e^2 \mu^2}{3}) \right], \] (19)

\[ \rho = -\frac{1}{\beta V} \frac{\partial}{\partial \mu} S(\beta, \mu) - \frac{e}{8\pi^2} \left[ e \mu (m^2 - \frac{2e^2 \mu^2}{3}) \right], \] (20)

where \( \mathcal{E}_V \) is the vacuum contribution, and

\[ F_\beta = \mathcal{E}_V - \frac{1}{\beta V} S(\beta, e \mu = m) + \frac{m}{e} \rho + \frac{1}{8\pi^2 \frac{m^3}{3}}, \] (21)

\[ \rho = -\frac{1}{\beta V} \frac{\partial}{\partial \mu} S(\beta, \mu) |_{e \mu = m} - \frac{e}{8\pi^2} \frac{m^3}{3} + e m x^2. \] (22)

Ref. [16] shows how these expressions in a generic space-time dimension \( D \) could give some inconsistencies. We have to remember, though, that we worked until now with regularized but “unrenormalized” charge density. Since it appears in the partition function multiplied by \( \mu \), any ambiguity in it will correspond to an uncertainty in the free energy density of the kind \( \mu K \). A physically very reasonable choice for \( K \) is given by requiring the symmetry to be unbroken at \( T = 0, \rho = 0 \). For \( D = 4 \), \( K \) will be \( K = -\frac{e^3}{24\pi^2} \). For \( D = 4 \) only, this choice also removes the multiplicative anomaly contribution to the charge density, so that the anomaly does not alter the broken phase in any respect and we get

\[ F_\beta = \mathcal{E}_V - \frac{1}{\beta V} S(\beta, \mu) + \mu \rho \]
\[ - \frac{e^3}{12\pi^2} + \frac{1}{8\pi^2} \left( e^2 \mu^2 m^2 - \frac{1}{3} e^4 \mu^4 \right), \] (25)

\[ \rho_R = -\frac{1}{\beta V} \frac{\partial}{\partial \mu} S(\beta, \mu) \]
\[ - \frac{1}{8\pi^2} \left( e^2 \mu m^2 - \frac{2}{3} e^4 \mu^3 \right) + \frac{e^3}{24\pi^2}. \]

At ultra relativistic temperatures \( T > m \) the anomalous contribution to the free energy is non leading, since the thermal contributions are proportional to \( T^4 \). At low temperatures (broken phase) we showed how the anomaly is reabsorbed by the renormalization of the charge, so that it could give relevant corrections only in an intermediate range \( T \approx m \). Notice, finally, that the anomalous term is vanishing as \( e \rightarrow 0 \), and the correct expression of the free energy density for the uncharged boson gas is recovered.

\textbf{V. OTHER SYSTEMS}

Of course there are several other systems in which the multiplicative anomaly could play a role \([13]\). A very interesting one is the non-relativistic charged scalar field, recently reanalysed by McKenzie-Smith and Toms \([21]\). They show how the standard approach already includes the multiplicative anomaly term (which is therefore surely physical here). On the contrary, factorising the algebraic determinant and forgetting the multiplicative anomaly (as has been always done for the relativistic case) leads to a result which does not agree with other non functional methods.
For the relativistic field they argue that the multiplicative anomaly does not have new physical relevance on the ground that the functional integral lacks a rigorous definition but they recognize that it is crucial to obtain the correct physics independently of the approach taken. I have already mentioned some delicate aspects of this connection between multiplicative anomaly and functional measure (see also [22]).

One other physical system (currently under investigation) in which the multiplicative anomaly could play a role is the case of fermion mixing. Also there, as in sec. 2, there are various possible parametrizations of the functional space involved and a transformation (rotation) between them. Again a careful definition of the functional measure is necessary, this specially in the light of recent results regarding inequivalent representations of the vacuum for field mixing [22].

For other systems, though, the multiplicative anomaly could be vanishing, as it is in odd dimensions for the one I just analyzed or for the single fermionic field, or it could be simply an irrelevant constant. For many others it should be completely reabsorbable in a renormalization procedure and have therefore no physical relevance, unless, as in the example presented in this work, we have a presence of more than one phase, in one of which it could survive even after the renormalization. There is also the problem as to whether the multiplicative anomaly is regularization dependent, which stirred a lively discussion lately [23,24]. The zeta-function method is one of a larger class of functional regularizations called “generalized proper-time regularizations” [25,26] in which we showed the anomaly to be present [26,13]. It seems that the answer is to be found in a proper definition of the regularized functional determinant itself [24].

While its relevance for zeta-function regularization is clear, the presence of the multiplicative anomaly poses therefore many other interesting questions about the regularization and renormalization procedures, the functional integral approach itself, its mathematical and physical definitions and its relations with alternative non functional approaches.

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