MAIN GAP FOR LOCALLY SATURATED ELEMENTARY SUBMODELS OF A HOMOGENEOUS STRUCTURE

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Abstract

We prove a main gap theorem for locally saturated submodels of a homogeneous structure. We also study the number of locally saturated models, which are not elementarily embeddable to each other.

Throughout this paper we assume that $\mathcal{M}$ is a homogeneous model of similarity type (=language) $L$. We study elementary submodels of $\mathcal{M}$. We use $\mathcal{M}$ as the monster model is used in stability theory and so we assume that the cardinality of $\mathcal{M}$ is large enough for all constructions we do in this paper. In fact, as in [HS1], we assume that $|\mathcal{M}|$ is strongly inaccessible. Alternatively we could assume less about $|\mathcal{M}|$ and instead of studying all elementary submodels of $\mathcal{M}$, we could study suitably small ones. Also the assumption that $\mathcal{M}$ is homogeneous can be replaced by the assumption that $\mathcal{M}$ is $\kappa$-homogeneous for $\kappa$ large enough. Notice that by [Sh1], if $D$ is a stable finite diagram, then $D$ has a monster model like $\mathcal{M}$.

We assume that the reader is familiar with [HS1] and use its notions and results freely.

0.1 Definition.

(i) Suppose $\mathcal{M}$ is stable. We say that $\mathcal{A}$ is $s$-saturated if it is $F^{\mathcal{M}}_{\lambda(\mathcal{M})}$-saturated i.e. for all $A \subseteq \mathcal{A}$ of power $< \lambda(\mathcal{M})$ and $a$ there is $b \in \mathcal{A}$ such that $t(b, A) = t(a, A)$.

(ii) We say that $\mathcal{A}$ is locally $F^\mathcal{M}_\kappa$-saturated if for all finite $A \subseteq \mathcal{A}$ there is $F^\mathcal{M}_\kappa$-saturated $B$ such that $A \subseteq B \subseteq \mathcal{A}$. If $\mathcal{M}$ is stable, then we say that $\mathcal{A}$ is $e$-saturated if it is locally $F^\mathcal{M}_{\lambda(\mathcal{M})}$-saturated.

(iii) Suppose $\mathcal{M}$ is stable. We say that $\mathcal{A}$ is strongly $F^\mathcal{M}_\kappa$-saturated if for all $A \subseteq \mathcal{A}$ of power $< \kappa$ and $a$ there is $b \in \mathcal{A}$ such that $b F^\mathcal{M}_{\min_{\mathcal{A}}} a$. By $a$-saturated we mean strongly $F^\mathcal{M}_{\kappa(\mathcal{M})}$-saturated.

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0.2 Lemma.
(i) Every $F^M_{\kappa}$-saturated model is locally $F^M_{\kappa}$-saturated and so (assuming $M$ is stable) every $s$-saturated model is $e$-saturated.
(ii) Suppose $M$ is stable. Then every $e$-saturated model is strongly $F^M_{\kappa}$-saturated.
(iii) Suppose $M$ is superstable. Then every $e$-saturated model is $s$-saturated.

Proof. (i) is trivial and (ii) follows immediately from [HS1] Lemma 1.9 (iv). So we prove (iii):
Assume $A$ is $e$-saturated. Notice that by (ii), $A$ is $a$-saturated. Let $A \subseteq A$ be of power $< \lambda(M)$ and $a$ arbitrary. We show that there is $b \in A$ such that $t(b, A) = t(a, A)$. Clearly we may assume that $a \cap A = \emptyset$.

Choose finite $B \subseteq A$ so that $a \downarrow_B A$. Since $A$ is $e$-saturated, we can find $s$-saturated $B$ such that $B \subseteq B \subseteq A$. Since by [HS1] Lemma 1.9 (iii) $B$ is strongly $F^M_{\lambda(M)}$-saturated, we can find $a_i \in B$, $i < \lambda(M)$, such that $a_i \uparrow_B \cup j, a_j$. Let $I = \{a_i| i < \lambda(M)\}$. For all $i < \kappa(M)$, choose $b_i$ so that $t(b_i, A) = t(a, A)$ and $b_i \downarrow_A \cup j, b_j$. Let $J = \{b_i | i < \kappa(M)\}$. By [HS1] Corollaries 3.5 (iv) and 3.11, $I \cup J$ is indiscernible over $B$. So

$$Av(I, A) = Av(J, A) = t(a, A).$$

Since $|A| < \lambda(M)$ is regular, we can find $C \subseteq B \cup I$ of power $< \lambda(M)$ such that for all $c \in A$, $t(c, B \cup I)$ does not split strongly over $C$. Let $b_i \in I (\subseteq B \subseteq A)$ be such that $b_i \cap C = \emptyset$. Then clearly $t(b, A) = Av(I, A) = t(a, A)$. □

We prove a main gap theorem for $e$-saturated submodels of $M$. At some extend, the proofs are similar to the related proofs in the case of complete first-order theories. So some of the proofs are sketchy.

1. Regular types

In (the end of) the next section, regular types are needed. In this section we prove the basic properties and the existence of regular types. In this section we assume that $M$ is stable.

1.1 Definition.
(i) We say that a stationary pair $(p, A)$ is regular if the following holds: if $C \supseteq \text{dom}(p)$, $a \models p$ and $a \not\models C$, then $(p, A)$ is orthogonal to $t(c, A)$.
(ii) Assume $A$ is $s$-saturated and $p \models S(A)$. We say that $p$ is regular, if there are $A \subseteq B \subseteq A$ such that $p$ does not split strongly over $A$, $(p \models B, A)$ is a regular stationary pair and $|B| < \kappa(M)$.

1.2 Lemma. Assume $A$ is $s$-saturated, regular $p \models S(A)$ is not orthogonal to $t(a, A)$ and $B$ is $s$-primary over $A \cup a$. Then there is $b \in B$ such that $t(b, A) = p$.

Proof. Assume not. Let $A \subseteq B \subseteq A$ be as in Definition 1.1 (ii). For all $i < \kappa(M)$ choose $A_i$ as follows:
(i) $A_0 = A$,
(ii) if $i$ is limit, then $A_i \subseteq B$ is $s$-primary over $\cup_{j<i} A_j$,
(iii) if $i = j + 1$ and there is $b_j \in B$ such that $t(b_j, B) = p \models B$ and $a \not\models A_j, b_j$, then $A_i \subseteq B$ is $s$-primary over $A_j \cup b_j$, if such $b_j$ does not exist then we let $A_i = A_j$.

Clearly there is $i < \kappa(M)$ such that $\mathcal{N}_i = A_{i+1}$. Let $i^*$ be the least such ordinal. Then

($*$) $t(a, A_{i^*})$ is orthogonal to $p$.

Let $A^* = s$-primary over $A_{i^*} \cup a$.

Claim. Assume $a \models p$. Then $p \models t(b, A^*)$.

Proof. Since $p$ is not realized in $B$, for all $i < i^*$, $b_i \not\models A_i$, and so, since $p$ is regular, for all $i < i^*$, $p$ is orthogonal to $t(b_i, A_i)$. By induction on $i < i^*$ it is easy to see that $p \models t(b_i, A_i)$. By ($*$) above, $p \models t(b_i, A^*)$. □ Claim.

By Claim, $p$ is orthogonal to $t(a, A)$, a contradiction. □

1.3 Corollary. Assume $A_i$, $i < 3$, are $s$-saturated, $p_i \models S(A_i)$ and $p_1$ is regular. If $p_0$ is not orthogonal to $p_2$, then $p_0$ is not orthogonal to $p_2$.\"
Proof. Immediate by Lemma 1.2.

1.4 Lemma. Assume that $A$ is $s$-saturated, $a \not\downarrow_A b$ and $t(b, A)$ is regular. Then $a \not\triangleright_A b$.

Proof. Let $\lambda = (\lambda(M))^+$. Clearly we may assume that $A$ is $F^M_A$-saturated. For a contradiction, assume that there is $c$ such that $c \downarrow_A a$ and $c \not\downarrow_A b$. Choose $A \subseteq B \subseteq A$ such that

(i) $(t(b, B), A)$ is a regular stationary pair and $b \downarrow_A A$,
(ii) $|B| < \kappa(M)$ and $|B| = \lambda(M)$,
(iii) $B$ is $s$-saturated and $a \cup b \cup c \downarrow_B A$.

Then $b \not\downarrow_B a$, $b \not\downarrow_B c$ ([HS1] Lemma 3.8 (iv)) and $a \downarrow_B c$. Let $A^*$ be $F^M_A$-primary over $A \cup a$ and $C \subseteq A^*$ s-primary over $B \cup a$. Without loss of generality we may assume that $b \cup c \downarrow_A A$.

For all $i < \kappa(M)$, choose $b_i \in A^*$ such that $t(b_i, C \cup \bigcup_{j<i} b_j) = t(b, C \cup \bigcup_{j<i} b_j)$. Let $I = \{b_i | i < \kappa(M)\}$. Then $I \cup \{b\}$ is indiscernible over $C$. Since $b \not\downarrow_B C$, it is easy to see that $I \cup \{b\}$ is not $B$-independent. So we can choose finite $J \subseteq I$ such that

(*): $J \cup \{b\}$ is not $B$-independent.

If $J$ is chosen so that $|J|$ is minimal, then $J$ is $B$-independent.

Let $D$ be $s$-primary over $B \cup c$. Then by (iii), $J \downarrow_B D$ and so $J$ is $D$-independent. Since $p$ is regular, $J \downarrow_B b$ and so $J \downarrow_B b$. Clearly this contradicts (*) above.

Assume $A$ is $s$-saturated and $a \not\in A$. We write $Dp(a, A) > 0$ if there is $s$-primary model $B$ over $A \cup a$ and $b \not\in B$ such that $t(b, B)$ is orthogonal to $A$.

1.5 Lemma. Assume that $M$ is superstable without $(\lambda(M))^+$-dop. Let $A$ be $s$-saturated, $I$ be $A$-independent and $a \not\downarrow_A I$. If $t(a, A)$ is regular and $Dp(a, A) > 0$, then there is $b \in I$ such that $a \not\downarrow_A b$.

Proof. Assume not. Clearly we may assume that $|A| = \lambda(M)$. Choose $a_i$, $A_i$ and $C_i$, $i < \alpha^*$, so that

(i) $a_i \downarrow_A a_i$,
(ii) $A_i$ is $s$-primary over $A \cup a_i$,
(iii) $\{a_i | i < \alpha^*\}$ is $A$-independent,
(iv) $C_0 = A_0$ and $C_{i+1}$ is $s$-primary over $C_i \cup A_i+1$,
(v) $a \not\downarrow_{C_i} A_{i+1}$,
(vi) $(a_i)_{i<\alpha^*}$ is a maximal sequence satisfying (i)-(v) above.

Since $M$ is superstable, $\alpha^* < \omega$. Let $n$ be such that $\alpha^* = n + 1$. Let $\lambda = (\lambda(M))^+$ and $B$ be $F^M_A$-saturated model such that $A \subseteq B$ and $B \downarrow_{A} C_n$. Let $B_i$ be $F^M_A$-primary over $A \cup A_i$ and $D$ $F^M_A$-primary over $\bigcup_{i \leq n} B_i$. It is easy to see that $C_n$ is $s$-primary over $\bigcup_{i \leq n} A_i$ and so we may choose $D$ so that $C_n \subseteq D$. Choose $a' \in D$ so that $t(a', C_n) = t(a, C_n)$. Let $A'$ be $s$-primary over $A \cup a'$.

Claim 1. $A' \not\downarrow_A B$.

Proof. Immediate by Lemma 1.4. [Claim 1.

Claim 2. For all $i \leq n$, $A' \not\downarrow_{A_i} B_i$.

Proof. Clearly it is enough to show that $a' \not\downarrow_A B \cup A_i$. Let $I = \{j \leq n | j \neq i\}$. By Claim 1 and (vi) above,

(*) $a' \not\downarrow_{C_n} B$.

By the choice of $B$, $\bigcup_{j \in I} A_j \downarrow_A B$ and so $C_n \downarrow_A B$. With (*) above, this implies $a' \downarrow_A B$. Since $a' \downarrow_A A_i$, $a' \downarrow_A B \cup A_i$. [Claim 2.

Since $Dp(a, A) > 0$, there is $b \not\in A'$ such that $t(b, A')$ is orthogonal to $A$ and $b \downarrow_{A'} D$. By Claim 2 and [HS1] Corollary 4.8, $t(b, D)$ is orthogonal to $B_i$ for all $i \leq n$. It is easy to see that this contradicts the assumption that $M$ does not have $\lambda$-dop.

1.6 Lemma. Assume that $M$ is superstable, $A \subseteq B$ are $s$-saturated and $A \neq B$. Then there is a singleton $a \in B - A$ such that $t(a, A)$ is regular.

Proof. As in the case of superstable theories.

2. Superstable with $ndop$

Throughout this section we assume that $M$ is superstable and does not have $\lambda(M)$-dop.
2.1 Definition.
(i) We say that $(P, f, g) = ((P, <), f, g)$ is an $s$-free tree of (s-saturated) model $A$ if the following holds:
(ii) $(P, <)$ is a tree without branches of length $> \omega$, $f : (P - \{r\}) \to A$ and $g : P \to P(A)$, where $r \in P$ is the root of $P$ and $P(A)$ is the power set of $A$,
(iii) $g(r)$ is $s$-primary model (over $\emptyset$ i.e. saturated model of power $\lambda(M)$),
(iv) if $t$ is not the root and $u^- = t$ then $t(f(u), g(t))$ is orthogonal to $g(t^-)$,
(v) if $t = u^-$ then $g(u)$ is $s$-primary over $g(t) \cup f(u)$.

Then
\[
\bigcup_{t \in T} g(t) \downarrow g(u) \bigcup_{v \in V} g(v).
\]

(ii) We say that $(P, f, g)$ is an $s$-decomposition of $A$ if it is a maximal $s$-free tree of $A$.

Notice that by Lemma 0.2 (iii) it is easy to see, that every $e$-saturated model has an $s$-decomposition.

2.2 Theorem.  $(M$ superstable without $\lambda(M)$-dop) Assume $A$ is $e$-saturated and $(P, f, g)$ is an $s$-decomposition of $A$. If $B \subseteq A$ is $s$-primary over $\cup_{t \in P} g(t)$, then $B = A$.

Proof. Immediate by Lemma 0.2 (iii) and (the proof of) [HS1] Theorem 5.13. □

2.3 Corollary. Suppose $A$ and $B$ are $e$-saturated. If $(P, f, g)$ is a decomposition of both $A$ and $B$, then $A \cong B$.

Proof. Easy by Theorem 2.2. □

We say that an $s$-free tree $(P, f, g)$ is regular if the following holds: if $t, u \in P$ are such that $u$ is an immediate successor of $t$, then $t(f(u), g(t))$ is regular. We say that $(P, f, g)$ is a regular $s$-decomposition of $e$-saturated $A$, if it an $s$-decomposition of $A$ and a regular $s$-free tree.

2.4 Lemma. Every $e$-saturated model has a regular $s$-decomposition.

Proof. Immediate by Lemmas 0.2 (iii) and 1.6. □

2.5 Definition.
(i) We say that $M$ is shallow if every branch in every regular $s$-free tree is finite. If $M$ is not shallow, then we say that $M$ is deep.
(ii) If $P = (P, <)$ is a tree without infinite branches, then by $Dp(P)$ we mean the depth of $P$.
(iii) Assume that $M$ is shallow. We define the depth of $M$ to be
\[
\sup\{Dp(P) + 1 | (P, f, g) \text{ is a regular } s\text{-free tree}\}.
\]

2.6 Lemma. Assume that $M$ is shallow and $\lambda(M)$ is regular. Then the depth of $M$ is $< \lambda(M)^+$. 

Proof. Choose a minimal regular $s$-free tree $(P, f, g)$ so that if $t \in P$ and $p \in S(g(t))$ are such that if $t$ has an immediate predecessor $t^-$, then $p$ is orthogonal to $g(t^-)$, then there is an immediate successor $u \in P$ of $t$ such that $t(f(u), g(t)) = p$. Clearly $Dp(P) < \lambda(M)^+$. Also if $(P', f', g')$ is a regular $s$-free tree, then there is an order-preserving function $h : P' \to P$. Then $Dp(P') \leq Dp(P)$, from which the claim follows. □

By $|L|$ we mean the number of $L$-formulas modulo the equivalence relation $\models \forall x(\phi(x) \leftrightarrow \psi(x))$.

2.7 Theorem. Assume that $M$ is shallow. Then the depth of $M$ is $< (|S(\emptyset)|^\omega)^+$ and so it is $< (2^{|L|})^+$. 

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Proof. By Lemma 2.6, we may assume that $\lambda(M) > \omega$. Choose a minimal regular $s$-free tree $(P, f, g)$ so that if $t \in P$ and $p \in S(g(t))$ are such that if $t$ has an immediate predecessor $t^-$, then $p$ is orthogonal to $g(t^-)$, then there is an immediate successor $u \in P$ of $t$ and an automorphism $h$ of $g(t)$ such that such that $t(f(u), g(t)) = h(p)$.

Claim 1. $Dp(P) < (|S(\emptyset)|^\omega)^+$.

Proof. Clearly it is enough to show that for all $t \in P$ the number of immediate successors of $t$ is at most $|S(\emptyset)|^\omega$. As in the proof of Lemma 0.2, for all $p \in S(g(t))$, there is a countable indiscernible $I \subseteq g(t)$ such that $Av(I, g(t)) = p$. Also if $t(I, \emptyset) = t(I', \emptyset)$, then there is an automorphism $h$ of $g(t)$ such that $h(I) = I'$ (remember that $g(t)$ is an $F_{|g(t)|}^M$-saturated model of power $\lambda(M) > \omega$). So the number of immediate successors of $t$ is at most

$$\{|t(I, \emptyset)| \mid I \subseteq g(t) \text{ countable indiscernible} \}.$$ 

Clearly this is at most $|S(\emptyset)|^\omega$. $\square$ Claim 1.

Claim 2. If $(P', f', g')$ is a regular $s$-free tree, then there is an order-preserving function $h : P' \to P$.

Proof. Just choose $h$ so that
(i) if $r$ is the root of $P'$ then $h(r)$ is the root of $P$,
(ii) if $t' \in P'$ is not a root of $P'$ and $u'$ is the immediate predecessor of $t'$, then $t = h(u')$ is such that it is an immediate successor of $u = h(u')$ and there is an isomorphism $h^* : g'(u') \to g(u)$ satisfying $t(f(t), g(u)) = h^*(t(f(t'), g'(u')))$. Clearly this is possible. $\square$ Claim 2.

As in the proof of Lemma 2.6, Claim 1 and 2 imply that the depth of $M$ is $< (|S(\emptyset)|^\omega)^+$. $\square$

2.8 Theorem. Assume that $M$ is shallow and $\gamma^*$ is the depth of $M$. Then the number of non-isomorphic $e$-saturated models of power $\aleph_\alpha$ is at most $\beth_\gamma^*(|\alpha| + \lambda(M))$.

Proof. By Corollary 2.3, it is enough to count the number of 'non-isomorphic' regular $s$-free trees $(P, f, g)$ of power $\aleph_\alpha$. This is an easy induction on $Dp(P)$, see the related results in [Sh3]. $\square$

2.9 Theorem. Assume that $M$ is shallow and $\gamma^*$ is the depth of $M$. Let $\kappa = \beth_\gamma^*(\lambda(M))$. If $\mathcal{A}_i$, $i < \kappa$, are $e$-saturated models, then there are $i < j < \kappa$ such that $\mathcal{A}_i$ is elementarily embeddable into $\mathcal{A}_j$.

Proof. By Theorem 2.2, this question can be reduced to the question of 'embeddality' of labelled trees. So this follows immediately from [Sh3] X Theorem 5.16C. $\square$

A cardinal $\kappa$ is called beautiful if $\kappa = \omega$ or for all $\xi < \kappa$, $\kappa \not\leq (\omega)_{\xi}^e$, see [Sh2] Definition 2.3.

2.10 Theorem. $(M)$ is superstable without $\lambda(M)$-dop but not necessarily shallow.) Assume that there is a beautiful cardinal $> \lambda(M)$. Let $\kappa^*$ be the least such cardinal. If $\mathcal{A}_i$, $i < \kappa^*$, are $e$-saturated models, then there are $i < j < \kappa^*$ such that $\mathcal{A}_i$ is elementarily embeddable into $\mathcal{A}_j$.

Proof. Again by Theorem 2.2, this follows immediately from [Sh2] Theorems 5.8 and 2.10. $\square$

If $(P, <)$ is a tree without branches of length $\geq \omega$ and $t \in P$, then by $Dp(t, P)$ we mean the depth of $t$ in $P$. If $t$ is not the root, then by $t^-$ we mean the immediate predecessor of $t$.

2.11 Theorem. Assume that $M$ is superstable, deep, does not have $\lambda(M)$-dop and $\lambda > \lambda(M)$. Then there are $s$-saturated (and so $e$-saturated) models $\mathcal{A}_i$, $i < 2^\lambda$, of power $\lambda$ such that for all $i < j < 2^\lambda$, $\mathcal{A}_i \not\equiv \mathcal{A}_j$.

Remark. Assume M is superstable. In the next section we show that $M$ has many $e$-saturated models if $M$ has $\lambda(M)$-dop. Similarly we can show that $M$ has many $e$-saturated models if $M$ has $(\lambda(M))^+$-dop. In fact, it can be seen that $\lambda(M)$-ndop implies $(\lambda(M))^+\text{-ndop}$ (\lambda(M))^+\text{-dop implies structure theorem for } s\text{-saturated and so especially for } F_{(\lambda(M))^+}^M\text{-saturated models, while } (\lambda(M))^+\text{-dop implies a lot of non-structure for } F_{(\lambda(M))^+}^M\text{-saturated models}.

Proof. Assume $X_1 \subseteq \lambda$, $i < 2$, are such that $X_0 \neq X_1$. Choose regular $s$-free trees $(P, f_i, g_i)$, $i < 2$, so that
(i) $P_i$ does not have branches of length $\geq \omega$ but for all $t \in P_i$, if $t$ is not the root, then $Dp(f(t), g(t^-)) > 0$.
(ii) for all $\alpha \in X_i$, there are $\lambda$ many $t \in P_i$ such that the height of $t$ is one and $Dp(t, P_i) = \alpha$ and if $Dp(t, P_i) = \beta$ and the height of $t$ is one, then $\beta < X_i$.
(iii) for all $t \in P_i$, if $Dp(t, P_i) = \alpha$ and $\beta < \alpha$, then $\{|u \in P_i | u^- = t \text{ and } Dp(u, P_i) \geq \beta| = \lambda$, 
(iv) if $t, u \in P_i$ are not the root and $t^- = u^-$, then 
$$t(f_i(t), g_i(t^-)) = t(f_i(u), g_i(u^-)),$$
we write $p_t$ for this type.
Let $r_i$ be the root of $P_i$. Choose finite $A_i \subseteq B_i \subseteq g_i(r_i)$ so that $p_{r_i}$ does not split strongly over $A_i$ and $(p_{r_i} \upharpoonright B_i, A_i)$ is a regular stationary pair. Then we require also
(v) $B_0 = B_1 (= B)_0 = A_1 (= A)$ and $p_{r_0} \upharpoonright B = p_{r_1} \upharpoonright B$.

Let $A_i$, $i < 2$, be $s$-primary over $\cup \in P_i g_i(t)$. We show that there is no isomorphism $F : A_0 \rightarrow A_1$ such that $F \upharpoonright B = id_B$. Clearly this is enough (since $\lambda^{<\omega} \leq 2^\lambda$, 'naming' finite number of elements does not change the number of models and since $M$ is $\lambda$-stable, $|A_i| = \lambda$). For a contradiction we assume that $F$ exists. Clearly we may assume that $F = id_{A_0}$, this simplifies the notation.

We let $P_i^*$ be the set of those $t \in P_i$, which are not leafs. For all $t \in P_0^*$, we let $G(t) \in P_1^*$ be (some node) such that $p_t$ is not orthogonal to $p_{G(t)}$ (if exists).

Claim. $G$ is an one-to-one function from $P_0^*$ onto $P_1^*$.

Proof. Since for all $t \in P_0^*$, $\{|u \in P_0 | u^- = t| = \lambda > \lambda(M)$, the existence of $G(t)$ follows easily. Since for all $u, u' \in P_0^*$, $u \neq u'$, $p_u$ is orthogonal to $p_{u'}$, $G(t)$ is unique by Corollary 1.3. But then by symmetry, claim follows. \(\blacksquare\) Claim.

We prove a contradiction (with (i) above) by constructing a strictly increasing sequence $(t_j)_{j<\omega}$ of elements of $P_0^*$. We construct also a strictly increasing sequence $(u_j)_{j<\omega}$ of elements of $P_1$, sets $I_j$, $i < 2$, and models $B_j$ so that
(1) $Dp(u_j, P_i) < Dp(t_j, P_0)$ and for all $t \geq t_j$, $G(t) \geq u_j$,
(2) $I_j \subseteq P_i$ is downwards closed, non-empty and of power $\leq \lambda(M)$ and $I_j \subseteq I_{j+1}$,
(3) $t_j \in I^0_{j+1}$ and $G(t_j) \in I^1_{j+1}$,
(4) $B_j$ is $s$-primary over $\cup \in P_i g_i(t)$ and over $\cup u \in I_j g_1(u)$ and $B_j \subseteq B_{j+1}$.
We do this by induction on $j < \omega$.

$j = 0$: Choose $I_0^0$, $I_0^1$ and $B_0$ so that (2) and (4) above are satisfied (if $B^0 \subseteq B_0$ is $s$-primary over $\cup \in P_i g_i(t)$, $I \subseteq P_0$, then by Theorem 2.2 and [HS1] Lemma 5.4 (ii), $B_0$ is $s$-primary over $B' \cup \cup \in P_i g_i(t)$). Let $t_0 \in P_0$ be such that $t_0 \notin I_0^0$ and $(t_0^-) = r_0$. Then

$$(*) \quad f_0(t_0) \downarrow_B B_0.$$

By Lemma 1.5, there is $u_0 \in P_1 - I_1^1$ such that $f_1(u_0) \downarrow_B B_0$, $f_0(t_0)$ and $(u_0^-) \in I_1^1$. By Lemma 1.4,

$$f_0(t_0) \downarrow_B B_0 \cup \{g_1(u) | u \notin u_0\}.$$

So $u_0$ is unique and the latter half of (1) holds. By (*), $(u_0^-) = r_1$ and so since $X_0 \neq X_1$ we can choose $t_0$ so that $Dp(u_0, P_1) \neq Dp(t_0, P_0)$. By symmetry, we may assume that $Dp(u_0, P_1) < Dp(t_0, P_0)$. Finally, this implies that $t_0 \in P_0^*$.

$j = k + 1$: Essentially, just repeat the argument above. \(\blacksquare\)

3. Superstable with dop or unstable

3.1 Theorem. Assume $M$ is superstable with $\lambda(M)$-dop, $\kappa > (\lambda_\tau(M))$ is regular and $\xi > \kappa$. Then there are $F^M$-saturated (and so $e$-saturated) models $A_i$, $i < 2^\xi$, of power $\xi$ such that for all $i \neq j$, $A_i$ is not elementarily embeddable into $A_j$. 


Proof. By [HS1] Corollary 6.5 and (the proof of) [Hy] Lemma 2.5, this follows from [Sh4] Theorems 3.20 and 3.27 and the claim below: Let a linear ordering \( \eta \) be almost \( \kappa \)-homogeneous i.e. for all \( X \subseteq \eta \) of power \( < \kappa \) there is \( Y \subseteq \eta \) of power \( < \kappa \) such that \( X \subseteq Y \) and if \( x, y \in \eta \) are in the same Dedekind cut of \( Y \), then there is an automorphism \( f \) of \( \eta \) such that \( f \upharpoonright Y = id_Y \) and \( f(x) = y \). Let \( A \subseteq \eta \) and \( f \) be a linear ordering. Then there are sequences \( f \) for all \( \eta \) such that \( f \upharpoonright X = id_X \). Let \( A \subseteq \eta \) be such that \( f \upharpoonright X = id_X \). Then \( A \subseteq \eta \) be a linear ordering. Let \( \eta \subseteq \eta \) be as in the definition of almost \( \kappa \)-homogeneous. Let \( x, y \in \eta \) be in the same Dedekind cut of \( Y \) and assume that \( \eta \models \psi(A, B, \mathcal{L}) \). It is enough to show that \( \eta \models \psi(A, B, \mathcal{L}) \).

By the choice of \( Y \), there is an automorphism \( f \) of \( \eta \) such that \( f \upharpoonright Y = id_Y \) and \( f(x) = y \). This \( f \) induces an elementary function \( g \) from \( \eta \) onto \( \eta \) such that \( g \upharpoonright S_Y = id_{S_Y} \) and \( g \upharpoonright B = id_B \) and since \( B \subseteq A \), \( g \) is the natural elementary function onto \( A \). By (iii) above, we can find an automorphism \( h \) of \( \mathcal{A} \) such that \( g \upharpoonright \mathcal{A} = id_{\mathcal{A}} \) and \( h(\mathcal{A}) \) is the natural elementary function onto \( B \). Let \( \mathcal{A} \subseteq \mathcal{A} \) be the isomorphism from \( \mathcal{A} \) to \( \mathcal{A} \). Then \( h \circ h \upharpoonright \mathcal{A} \) is an automorphism of \( \mathcal{A} \). Clearly this implies that \( \mathcal{A} \models \psi(A, B, \mathcal{L}) \). \( \Box \) Claim.

3.2 Lemma. Assume that \( \mathcal{A} \) is unstable. Let \( \kappa > |L| \) be a regular cardinal, and \( \eta = (\eta, <) \) be a linear ordering. Then there are sequences \( a_i, i \in \eta \), a model \( \mathcal{A} \) and functions \( f_i : M^{n_i} \to M \), \( i < 2^{< \kappa} \), such that \( n_i < \omega \) and if we write \( L^* = L \cup \{ f_i \mid i < 2^{< \kappa} \} \) then the following holds:

(i) \( \langle a_i \rangle_{i \in \eta} \) is order-indiscernible inside \( \mathcal{A} \) in the language \( L^* \).

(ii) for all \( X \subseteq \eta \), the closure \( \mathcal{A}_X \) of \( \{ a_i \mid i \in X \} \) under the functions of \( L^* \) is a locally \( F^M \)-saturated model (in the language \( L \)) and \( \mathcal{A} = \mathcal{A}_\eta \).

(iii) there is an \( L \)-formula \( \phi(x, y) \) such that for all \( i, j \in \eta \), \( \models \phi(a_i, a_j) \iff i < j \).

Proof. Define functions \( f_i : M^{n_i} \to M \), \( i < 2^{< \kappa} \), so that

(*) the closure of any set under the functions \( f_i \) is locally \( F^M \)-saturated (in \( L \)) and \( L \)-elementary submodel of \( (M, f_i oo i < 2^{< \kappa}) \), where \( L' = L \cup \{ f_i \mid i < 2^{< \kappa} \} \).

By Erdős-Rado Theorem and [Sh1] I Lemma 2.10 (1), we can find sequences \( \langle a_i^k \rangle_{i \in \kappa} \), \( k < \omega \), such that

1. there is a formula \( \phi(x, y) \) such that for all \( k < \omega \) and \( i, j < k \), \( \models \phi(a_i^k, a_j^k) \iff i < j \),

2. \( a_i^k \) is order-indiscernible in the language \( L' \),

3. the \( \phi \)-type of \( (a_i^k)_{i < k} \) (over \( \emptyset \)) is the same as the \( \phi \)-type of \( (a_i^{k+1})_{i < k} \).

Since \( \mathcal{M} \) is homogeneous, we can find for all \( i \in \eta \), \( a_i \) so that for all \( k < \omega \), if \( i_0 < i_1 < \ldots < i_{k-1} < i_k \), then \( \langle a_i \rangle \models \models a_i \in A \), \( \models \phi(a_i, a_j) \iff i < j \), and for all \( a, b \in A \) and \( i < 2^{< \kappa} \), \( f_i(a) = b \iff f_i(F(a)) = F(b) \).

Let \( A = A \), i.e. the closure of \( \{ a_i \mid i \in \eta \} \) under the functions of \( L^* \). Then it is easy to see that (iii) in the claim is satisfied.

(ii) Assume \( X \subseteq \eta \). We show that \( \mathcal{A}_X \) is locally \( F^M \)-saturated. For this let \( A \subseteq \mathcal{A}_X \) be finite. Then there is \( X' \subseteq X \) finite, such that \( A \subseteq \mathcal{A}_X \). By (**) above, \( \mathcal{A}_X \) is locally \( F^M \)-saturated. So there is \( F^M \)-saturated \( \mathcal{B} \) such that \( A \subseteq \mathcal{B} \subseteq \mathcal{A}_X \).

(i) By (*) and (**) above it is easy to see that for all finite \( X \subseteq \eta \), \( \mathcal{A}_X \) is an \( L \)-elementary submodel of \( \mathcal{A} \). By (2), (*) and (**) again, (i) follows. \( \Box \)
3.3 Theorem. Assume $M$ is unstable. Let $\lambda$ and $\kappa$ be regular cardinals, $\lambda > 2^{<\kappa}$ and $\kappa > |L|$. Then there are locally $F^M_\kappa$-saturated models $A_i$, $i < 2^\lambda$, such that $|A_i| = \lambda$ and if $i \neq j$, then $A_i$ is not elementarily embeddable into $A_j$.

Proof. By Lemma 3.2 this follows from [Sh4] Chapter 6 Theorem 3.1 (3). Notice that the trees can be coded into linear orderings.  \[\blacksquare\]

4. Strictly stable

Throughout this section we assume that $M$ is stable but unsuperstable, and that $\kappa = cf(\kappa) > \lambda_r(M)$.

We write $\kappa^{\leq \omega}$ for $\{\eta : \alpha \rightarrow \kappa| \alpha \leq \omega\}$, $\kappa^{<\omega}$ and $\kappa^{\omega} = \kappa^{=\omega}$ are defined similarly (of course these have also the other meaning, but it will be clear from the context, which one we mean). Let $J \subseteq 2^{<\kappa}$. We order $P_\omega(J)$ (=the set of all finite subsets of $J$) by defining $u \leq v$ if for every $\eta \in u$ there is $\xi \in v$ such that $\eta$ is an initial segment of $\xi$.

Since $M$ is unsuperstable, by [HS1] Lemma 5.1, there are $a$ and $F^M_{\lambda_r(M)}$-saturated models $A_i$, $i < \omega$, of power $\lambda_r(M)$ such that

(i) if $j < i < \omega$, then $A_j \subseteq A_i$,
(ii) for all $i < \omega$, a $f_{A_i}$ $A_{i+1}$.

Let $A_\omega$ be an $F^M_{\lambda_r(M)}$-primary model over $a \cup \bigcup i<\omega A_i$. Then for all $\eta \in \kappa^{\leq \omega}$, we can find $A_\eta$ such that

(a) for all $\eta \in \kappa^{\leq \omega}$, there is an automorphism $f_\eta$ of $M$ such that $f_\eta(A_{\text{length}(\eta)}) = A_\eta$,
(b) if $\eta$ is an initial segment of $\xi$, then $f_\xi \upharpoonright A_{\text{length}(\eta)} = f_\eta \upharpoonright A_{\text{length}(\eta)}$,
(c) if $\eta \in \kappa^{<\omega}$, $\alpha \in \kappa$ and $X$ is the set of those $\xi \in \kappa^{\leq \omega}$ such that $\eta \upharpoonright (\alpha)$ is an initial segment of $\xi$, then

$\bigcup_{\xi \in X} A_\xi \downarrow A_\eta \cup \bigcup_{\xi \in (\kappa^{\leq \omega} - X)} A_\xi.$

For all $\eta \in \kappa^{\omega}$, we let $a_\eta = f_\eta(a)$.

For each $\alpha < \kappa$ of cofinality $\omega$, let $\eta_\alpha \in \kappa^{\omega}$ be a strictly increasing sequence such that $\bigcup_{i<\omega} \eta_\alpha(i) = \alpha$. Let $S \subseteq \{\alpha < \kappa| cf(\alpha) = \omega\}$. By $J_S$ we mean the set

$\kappa^{\omega} \cup \{\eta_\alpha| \alpha \in S\}.$

Let $I_S = P_\omega(J_S)$.

4.1 Lemma. For all $S \subseteq \{\alpha < \kappa| cf(\alpha) = \omega\}$, there are sets $A_u$, $u \in I_S$, such that

(i) for all $u, v \in I_S$, $u \leq v$ implies $A_u \subseteq A_v$,
(ii) for all $u \in I_S$, $A_u$ is $F^M_{\lambda_r(M)}$-primary over $\bigcup_{\eta \in u} A_\eta$,
(iii) if $\alpha \in \kappa - S$, $u \in I_S$ and $v \in P_\omega(J_S \cap \alpha^{\leq \omega})$ is maximal such that $v \leq u$, then

$A_u \upharpoonright A_v \cup \bigcup_{w \in P_\omega(J_S \cap \alpha^{\leq \omega})} A_w.$

Proof. See [HS2] Lemmas 4 and 7. \[\blacksquare\]

For all $S \subseteq \{\alpha < \kappa| cf(\alpha) = \omega\}$, let $A_S = \bigcup_{u \in I_S} A_u$. By Lemma 4.1 (i) and (ii), $A_S$ is $e$-saturated and $|A_S| = \kappa$.

4.2 Lemma. There are sets $S_i \subseteq \{\alpha < \kappa| cf(\alpha) = \omega\}$, $i < 2^\omega$, such that if $i \neq j$, then $S_i - S_j$ is stationary.

Proof. Let $f_i: \kappa \rightarrow \kappa$, $i < 2$, be one to one functions such that rng($f_0$) $\cap$ rng($f_1$) = $\emptyset$. Let $R'_i$, $i < 2^\omega$, be an enumeration of the power set of $\kappa$. We define $R_i$, $i < 2^\omega$, so that $f_0(\alpha) \in R_i$ iff $\alpha \in R'_i$ and $f_1(\alpha) \in R_i$ iff $\alpha \notin R'_i$. Then clearly, $i \neq j$ implies $R_i - R_j \neq \emptyset$. By [Sh3] Appendix Theorem 1.3 (2), there are pairwise disjoint stationary sets $S'_j \subseteq \{\alpha < \kappa| cf(\alpha) = \omega\}$, $j < \kappa$. For $i < 2^\omega$, we let $S_i = \bigcup_{j \in R_i} S'_j$. Clearly these are as wanted. \[\blacksquare\]

4.3 Theorem. Assume $M$ is stable and unsuperstable and $\kappa = cf(\kappa) > \lambda_r(M)$. Then there are $e$-saturated models $A_i$, $i < 2^\omega$, of power $\kappa$ such that if $i \neq j$, then $A_i$ is not elementarily embeddable into $A_j$. 
Proof. For all $i < 2^\kappa$, let $A_i = A_{S_i}$, where the sets $S_i$ are as in Lemma 4.2. Assume $i \neq j$.

We show that there are no elementary map $F : A_i \to A_j$.

For a contradiction, assume that $F$ exists. For all $\alpha < \kappa$, let $I_S^\alpha$ be the set of those $u \in I_S$ such that for all $\eta \in u$, $\sup\{\eta(i) | i < \text{length}(\eta)\} < \alpha$. Let $A_i^\alpha = \bigcup_{u \in I_S^\alpha} A_u$. $I_S^\alpha$ and $A_j^\alpha$ are defined similarly. We say that $\alpha$ is closed if for all $a \in A_i$, $a \in A_i^\alpha$ iff $F(a) \in A_j^\alpha$. Let $C$ be the set of all closed ordinals and $C_{lim}$ the set of all limit points in $C$. Then $S^0 = C_{lim} \cap (S_i - S_j)$ is stationary.

For all $\alpha \in S^0$, let $u_\alpha \in I_S^\alpha$ be such that $F(a_\eta) \downarrow A_j^\alpha$. By $g(\alpha)$ we mean the least $\beta \in C$ such that $u_\alpha \downarrow A_j^\alpha$. By Lemma 4.1 (iii) and the fact that $S^0 \cap S_j = \emptyset$, $g(\alpha) < \alpha$. So there is stationary $S^1 \subseteq S^0$ such that $g \upharpoonright S^1$ is constant. Let $\alpha^*$ be this constant value.

Then there is $S^2 \subseteq S^1$ and $n < \omega$ such that $|S^2| = \kappa$ and for all $\beta, \gamma \in S^2$, if $\beta \neq \gamma$, then $\eta_\beta(n) \neq \eta_\gamma(n)$. By choosing $n$ so that it is minimal, we may assume that for all $\beta \in S^2$, $\eta_\beta(n - 1) < \alpha^*$. Clearly we may assume that for all $\beta \in S^2$, $\eta_\beta(n) > \alpha^*$.

Then by Lemma 4.1 (iii),

(i) $(F(A_{\eta_\beta}(n+1)))_{\beta \in S^2}$ is $(A_j^{\alpha^*})$-independent.

Since $F(a_\eta) \downarrow A_j^{\alpha^*}$, $F(A_{\eta_\beta}(n+1))$ and $F(a_\eta) \not\downarrow F(A_j^{\alpha^*}) F(A_{\eta_\beta}(n+1))$,

(ii) for all $\beta \in S^2$, $F(A_{\eta_\beta}(n+1)) \not\downarrow F(A_j^{\alpha^*}) A_j^{\alpha^*}$.

Since $\kappa(M) < \kappa$, $|A_j^{\alpha^*}| < \kappa$ and $|S^2| = \kappa$, (i) and (ii) are contradictory. \qed
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