Minisuperspace quantization of bubbling AdS$_2 \times$S$^2$ geometries

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Abstract

We quantize the moduli space of supersymmetric microstates describing four-dimensional black holes with AdS$_2 \times$S$^2$ asymptotics. To acquire the commutation relations of quantization, we find the symplectic form that is imposed in the Type IIB SUGRA and defined in the space of solutions parametrized by one complex harmonic function in $\mathbb{R}^3$ with sources distributed along closed curves.

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I. INTRODUCTION

Gauge theory and supergravity are connected by the AdS/CFT correspondence [1–4], which allows one to describe supergravity configuration in terms of microscopic degrees of freedom in field theories on the boundary. The full understanding of the map between the bulk and the gravity sides for AdS$_2$/CFT$_1$ is still missing, although impressive progress has been reported in [5–9]. Recently, regular BPS geometries with $\text{AdS}_2 \times \text{S}^2 \times \text{T}^6$ asymptotics were constructed in [10, 11]¹, and one can hope to use them to get insight into the structure of the dual field theory. In higher dimensions, where the AdS/CFT duality is well understood, the gravity solutions, known as bubbling geometries [13] and fuzzballs [14–19], can be mapped into the field-theoretic degrees of freedom via quantization of the moduli space [20–22]. In this article we gain some insights into the field theory dual to AdS$_2$ by applying the techniques developed in [20–22] to the geometries constructed in [10].

According to the AdS/CFT dictionary, $\text{AdS}_2 \times \text{S}^2$ corresponds to the ground state of the dual fields living on the boundary of the AdS space. The light excitations of the dual fields correspond to strings moving on $\text{AdS}_2 \times \text{S}^2$, and such strings which turn out to be integrable [23]. Excitations with intermediate energies are mapped into probe D branes whose counterparts in six- and ten-dimensional theories are called giant gravitons [24, 25]. At higher energies the gravitational backreaction of branes must be taken into account, and the corresponding solutions of supergravity were constructed in [10]. The higher dimensional counterparts of such gravity solutions have been constructed in [13–15], and in [20–22], the spaces of these supersymmetric geometries were quantized using the method of Crnković-Witten [26] and Zukerman [27], and a perfect agreement with the expectations from the field theory side [28, 29] was found. Additional applications of this method have also been developed in [30–35]. In the four-dimensional case we can use the similar quantization to get insights into the dual theory.

The organization of this paper is as follows. In Sec. II we review the regular supersymmetric solutions of supergravity with $\text{AdS}_2 \times \text{S}^2 \times \text{T}^6$ asymptotics, which are parametrized by two harmonic functions in $\mathbb{R}^3$ with sources distributed along closed curves. In Sec. III we give the general idea of quantizing geometries from the supergravity action by the approach of Crnković-Witten-Zukerman. We apply this method, in the next section, to the bubbling geometries for $\text{AdS}_2 \times \text{S}^2$, and the corresponding symplectic form is found. In the last section we give a brief discussion.

¹ Related geometries with flat asymptotics have been discussed in [12].
II. BPS SOLUTIONS OF SUGRA

The simplest way to probe the AdS$_2$/CFT$_1$ duality is to look at strings propagating on

$$\text{AdS}_2 \times S^2 \times T^6,$$

and dynamics of such objects was studied in [23, 36–38]. The heavy excitations are described by spaces which asymptote to (1), and we focus on supersymmetric geometries. Recently, a family of such regular BPS solutions of Type IIB SUGRA was found in [10], and the geometry is given by

$$ds^2 = -h^{-2}(dt + V)^2 + h^2 dx_a dx_a + dz_\alpha dz_\alpha,$$

$$F_5 = F \wedge \text{Re} \Omega_3 - \tilde{F} \wedge \text{Im} \Omega_3, \quad \Omega_3 = dz_{123},$$

$$F = -\partial_\alpha A_t(dt + V) \wedge dx^\alpha + h^2 \ast_3 d\tilde{A}_t, \quad \tilde{A}_t + iA_t = \frac{1}{4h}e^{i\beta},$$

$$\tilde{F} = -\partial_\alpha \tilde{A}_t(dt + V) \wedge dx^\alpha - h^2 \ast_3 dA_t, \quad dV = -2h^2 \ast_3 d\beta,$$

$$\tilde{\eta} = h^{-1/2}e^{i\beta/2} e^\gamma = \epsilon. \quad (2)$$

Here $a = 1, 2, 3$ and $z_\alpha = X_\alpha + iY_\alpha$ with $\alpha = 1, 2, 3$ in which $X_\alpha$ and $Y_\alpha$ are the coordinates for $T^6$. This geometry can be parametrized by two harmonic functions $H_1$ and $H_2$ in $\mathbb{R}^3$,

$$H_1 = h \sin \beta, \quad H_2 = h \cos \beta, \quad d \ast_3 dH_a = 0,$$

$$dV = -2 \ast_3 [H_2 dH_1 - H_1 dH_2], \quad \tilde{A}_t + iA_t = \frac{1}{4(H_2 - iH_1)}. \quad (3)$$

At the points where the two harmonic functions $H_1$ and $H_2$ have sources, the solutions (2) may be singular. To guarantee the regularity, one has to require that all the sources are distributed along closed curves in $\mathbb{R}^3$, and the harmonic functions are obtained from:

$$H = H_1 + iH_2 = \frac{1}{2\pi} \int \sigma \sqrt{(r - \mathbf{f}) \cdot (r - \mathbf{f} + \mathbf{b})} (r - \mathbf{f})^2 dv + H_{\text{reg}}, \quad (4)$$

$$\mathbf{b} \cdot \dot{\mathbf{f}} = 0, \quad \mathbf{b} \cdot \mathbf{b} = 0,$$

where $\mathbf{f}(v)$ is the location of the profile parametrized by $v$ and $\mathbf{b}$ is a complex vector. See [10] for the detailed discussion.

III. SYMPLECTIC FORM OF TYPE IIB SUGRA

A method of quantizing Lagrangian theories was proposed by Crnković and Witten [26] and Zukerman [27]. As a Lagrangian theory, Type IIB SUGRA theory has a symplectic form $\Omega$ defined on the full phase space, which contains the information of the commutation
relations. In this paper we will restrict $\Omega$ in the space of solutions. One can write the symplectic form as an integral over the Cauchy surface $\Sigma$:

$$\Omega = \int d\Sigma_l J^l.$$  

(5)

We call $J^l$ a symplectic current, and its explicit form is given by:

$$J^l = \sum_i \delta \left[ \frac{\partial L}{\partial (\partial_l \phi_i)} \right] \wedge \delta \phi_i,$$  

(6)

where $\phi_i$ runs over all fields in the action. Once the symplectic form is known as

$$\Omega = \frac{1}{2} \omega^{-1}_{ij} dq_i \wedge dp_j,$$  

(7)

one then can encode the Poisson bracket as

$$\{q_i, p_j\}_{P.B.} = \omega_{ij}.$$  

(8)

The commutators are directly obtained from the above brackets by the Dirac prescription.

Since solutions (2) are supported only by the five-form flux $F_5$, the relevant part of the Type IIB SUGRA action is given by

$$S = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \left( R - 4 |F_5|^2 \right).$$  

(9)

The symplectic form does not change as the self-duality constraint $F_5 = \star F_5$ is imposed. The symplectic form reads

$$\Omega = \frac{1}{2\kappa_{10}^2} \int d\Sigma_l (J^l_G + J^l_F),$$  

(10)

where $J^l_G$ and $J^l_F$ are the gravity and five-form currents, respectively, that can be computed from Eq. (6).

The symplectic current of gravity $J^l_G$ was found by Crnković and Witten,

$$J^l_G = -8\delta \Gamma^l_{mn} \wedge \delta (\sqrt{-g} g^{mn}) + \delta \Gamma^n_{mn} \wedge \delta (\sqrt{-g} g^{lm}).$$  

(11)

When the metric perturbations take a gauge transformation as

$$\delta g_{mn} \rightarrow \delta g_{mn} + \nabla_{(m} \xi_{n)},$$  

(12)

$J^l_G$ will change by a total derivative, which keeps the symplectic form (10) gauge invariant.

Taking the potentials $A_{i_1\cdots i_4}$, which satisfy the relation $F_5 = dA$, as our basic fields, and applying (6), we obtain $J^l_F$ directly

$$J^l_F = -8\delta (\sqrt{-g} F^{l[i_1\cdots i_4]} \wedge \delta A_{i_1\cdots i_4}),$$  

(13)

where $|i_1i_2i_3i_4|$ means $i_1 < i_2 < i_3 < i_4$. 


IV. MINISUPERSPACE QUANTIZATION OF BUBBLING GEOMETRIES FOR AdS$_2 \times S^2$

In the parametrization (2), the AdS$_2 \times S^2$ background is specified by a circular closed curve

$$f_1(v) = L \cos \frac{v}{L}, \quad f_2(v) = L \sin \frac{v}{L}, \quad f_3 = 0, \quad 0 \leq v \leq 2\pi L, \quad (14)$$

and the related vector $b$ is given by $b = 2L (\cos \frac{v}{L}, \sin \frac{v}{L}, i)$. After plugging (14) and $b$ into (3) and (4), it is straightforward to get

$$H = \frac{L}{\sqrt{r^2 + (y - iL)^2}},$$

$$V = -\frac{L}{2} \left[ \frac{r^2 + y^2 + L^2}{\sqrt{4L^2y^2 + (r^2 + y^2 - L^2)^2}} - 1 \right] d\phi. \quad (15)$$

Substituting the above expressions for $H$ and $V$ into (2), and making the coordinate transformation as

$$t = \frac{\tilde{t}}{L}, \quad r = L\sqrt{\rho^2 + 1} \sin \theta, \quad y = L\rho \cos \theta, \quad \phi = \tilde{\phi} - t, \quad (16)$$

one obtains immediately the geometry of the global AdS$_2 \times S^2$:

$$ds^2 = L^2 \left[ -(\rho^2 + 1) dt^2 + \frac{d\rho^2}{\rho^2 + 1} + d\theta^2 + \sin^2 \theta d\phi^2 \right] + dz^a dz^a,$$

$$F_5 = \frac{L}{4} d\rho dt \wedge \text{Re}(dz_{123}) + \text{dual}. \quad (17)$$

The circular profile (14), parametrizing the AdS$_2 \times S^2$ solution, corresponds to the ground state of the system with a given amount of flux. To find the symplectic form, we focus on the perturbations of this solution, which is equivalent to considering the small perturbations of the two harmonic functions $H_1$ and $H_2$. The profile can vibrate in $(x_1, x_2, y)$, but in this paper we only consider its fluctuations in the $(x_1, x_2)$ plane and restrict vector $b$ to be orthogonal to this plane as (4). We can express $\delta f$ ($f = \sqrt{f_1^2 + f_2^2}$) in Fourier series:

$$\delta f = \sum_{|n| > 1} L a_n e^{i n \phi}, \quad a_n^* = a_{-n}. \quad (18)$$

From now on we will set $L = 1$, and this scale will be restored in the end [see Eqs. (30)-(32)]. In this paper we need only the first order fluctuations of the profile since only such perturbations enter the expression of symplectic current (6). In the first order the fluctuations with $n = \pm 1$ describe the circular profile with shifted origin, so they still correspond to the ground state. Thus we focus on $|n| > 1$. 

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The first order perturbations of the two harmonic functions \( H_1, H_2 \), and vector field \( V \) caused by the fluctuations (18) are given by

\[
\delta H_1 = \sum_{|n|>1} \frac{a_n e^{in(\theta - \bar{\theta})}}{2R^6} \left[ \frac{\theta}{\sqrt{\rho^2 + 1}} \right]^{|n|} \rho \left[ \rho^2 - 3c_\theta^2 \right] \left[ |n| R^2 + \rho^2 - c_\theta^2 + 2 \right],
\]

\[
\delta H_2 = \sum_{|n|>1} \frac{a_n e^{in(\theta - \bar{\theta})}}{2R^6} \left[ \frac{\theta}{\sqrt{\rho^2 + 1}} \right]^{|n|} c_\theta \left[ 3\rho^2 - c_\theta^2 \right] \left[ |n| R^2 + \rho^2 - c_\theta^2 + 2 \right],
\]

\[
\delta V_r = - \sum_{|n|>1} \frac{ina_n e^{in(\theta - \bar{\theta})}}{(\rho^2 + 1) R^2} \left[ \frac{\theta}{\sqrt{\rho^2 + 1}} \right]^{|n|-1},
\]

\[
\delta V_\phi = \sum_{|n|>1} \frac{a_n e^{in(\theta - \bar{\theta})}}{R^6} \left[ \frac{\theta}{\sqrt{\rho^2 + 1}} \right]^{|n|} \left[ |n| R^2 (c_\theta^2 - \rho^2 + 2\rho^2 c_\theta^2) + 2(\rho^2 + 1)s_\theta^2(c_\theta^2 - \rho^2) \right],
\]

where \( R^2 = (\rho^2 + \cos^2 \theta) \), and \( s_\theta \equiv \sin \theta \), \( c_\theta \equiv \cos \theta \). The details of the calculation can be seen in Appendix A. Although perturbations (19) appear singular at the location of the profile \((R = 0)\), we will now show that they give rise to regular metric perturbations after an appropriate gauge transformation.

The metric perturbations are obtained by plugging the expressions (19) into (2). In order to make the field perturbations regular, we perform a linear gauge transformation (12) with

\[
\xi = \sum_{|n|>1} a_n e^{in(\theta - \bar{\theta})} \left[ \frac{\theta}{\sqrt{\rho^2 + 1}} \right]^{|n|} \left[ - \frac{i|n|}{n} d\bar{t} + \frac{\rho s_\theta^2}{(\rho^2 + 1) R^2} d\rho + \frac{s_\theta c_\theta}{R^2} d\theta \right],
\]

This leads to the final form of the metric perturbation:

\[
h_{\mu\nu} = \sum_{|n|>1} \left( - 2|n|(\frac{|n| - 1}{|n| + 1}) s_n Y_n g_{\mu\nu} + \frac{4}{|n| + 1} Y_n \nabla_{(\mu}\nabla_{\nu)} s_n \right),
\]

\[
h_{\alpha\beta} = \sum_{|n|>1} 2|n| s_n Y_n g_{\alpha\beta},
\]

where \( \mu, \nu \) run over the AdS\(_2\) directions, \( \alpha, \beta \) run over the S\(^2\) directions, and \( s_n \) and \( Y_n \) are

\[
s_n = a_n e^{-i\bar{\theta}} \frac{|n| + 1}{2|n|(1 + \rho^2)|n|/2},
\]

\[
Y_n = e^{in\phi} \sin^{|n|}\theta.
\]

They are the spherical harmonics of AdS\(_2\) and S\(^2\), respectively,

\[
\nabla_{S^2}^2 Y_n = -|n|(\frac{|n| + 1}{|n|}) Y_n, \quad \nabla_{AdS^2}^2 s_n = |n|(\frac{|n| - 1}{|n|}) s_n.
\]

The symplectic current of gravity \( J_G^r \) is obtained by substituting (21) into (11).
The second part of the symplectic current, \( J'_F \), is constructed from the four-form potential according to (13). For the solutions (2), we need the fluctuations of the two-forms \( F \) and \( \tilde{F} \) in (2) and their potentials. These fluctuations can be calculated by substituting (18) or (19) into (2), (3) and (4), but this path involves tedious algebraic manipulations.

Fortunately, there is an alternative option given by [20, 39, 40], which relates fluctuations of \( F_5 \) with the metric perturbations (21). Following the procedure outlined in [20, 39, 40], we can compute the one-form \( B \) and \( \tilde{B} \) through the relations

\[
\delta a_\alpha = \frac{1}{4} \epsilon_{\beta\alpha} s_n \nabla^\beta Y_n, \\
\delta a_\mu = -\frac{1}{4} \epsilon_{\nu\mu} Y_n \nabla^\nu s_n. \tag{24}
\]

Here \( \mu, \nu \) run over the AdS_2 directions, \( \alpha, \) and \( \beta \) run over the \( S^2 \) directions, and \( \epsilon_{\beta\alpha} \) and \( \epsilon_{\mu\nu} \) are the volume forms on \( S^2 \) and AdS_2, respectively. Plugging (22) into above relations (24), we immediately get

\[
\delta B = -\sum_{|n|>1} \frac{1}{8} a_n e^{in(\phi - \bar{t})} \left[ \frac{\sin \theta}{\sqrt{\rho^2 + 1}} \right] ^{|n|} (|n| + 1) \left[ n (\rho \tilde{d}t + \frac{in}{n|1 + \rho^2|} ) d\rho \right],
\]

\[
\delta \tilde{B} = -\sum_{|n|>1} \frac{1}{8} a_n e^{in(\phi - \bar{t})} \left[ \frac{\sin \theta}{\sqrt{\rho^2 + 1}} \right] ^{|n|} (|n| + 1) \left[ \frac{in}{|n| \sin \theta - \cos \theta \bar{d} \phi} d\theta \right], \tag{25}
\]

and

\[
\delta F = -\sum_{|n|>1} \frac{1}{8} a_n e^{in(\phi - \bar{t})} \left[ \frac{\sin \theta}{\sqrt{\rho^2 + 1}} \right] ^{|n|} (|n| + 1) \left[ (|n| - 1) d\tilde{t} \wedge d\rho - (|n| \cot \theta \tilde{d} \tilde{t} \wedge d\theta
\right.

\left. - in \rho \tilde{d} \tilde{t} \wedge d\tilde{\phi} - \frac{in \cot \theta \rho^2}{\rho^2 + 1} d\rho \wedge d\theta + \frac{|n| d\rho \wedge d\tilde{\phi}}{\rho^2 + 1} \right],
\]

\[
\delta \tilde{F} = -\sum_{|n|>1} \frac{1}{8} a_n e^{in(\phi - \bar{t})} \left[ \frac{\sin \theta}{\sqrt{\rho^2 + 1}} \right] ^{|n|} (|n| + 1) \left[ \sin \theta d\theta \wedge d\phi + |n| \csc \theta \tilde{d} \tilde{t} \wedge d\theta
\right.

\left. + in \cos \theta \tilde{d} \tilde{t} \wedge d\tilde{\phi} - \frac{in \csc \theta \rho^2}{\rho^2 + 1} d\rho \wedge d\theta + \frac{|n| \rho \cos \theta \rho^2}{\rho^2 + 1} d\rho \wedge d\tilde{\phi} \right]. \tag{26}
\]

It is convenient to pick the hypersurface \( \Sigma = \{ \bar{t} = \text{const} \} \) in the symplectic form (10), since the solutions (2) are independent of \( \bar{t} \). This leads to the gravity and to five-form contributions to the symplectic form:

\[
\int_{\bar{t} = \text{const}} J^G_{\bar{t}} = -\sum_{|n| > 1} \frac{2\pi^2 iV_6 |n|(|n| + 1)(n^2 - 3|n| - 2)}{n(2|n| + 1)} \frac{a_n \wedge a_{-n}}, \tag{27}
\]

\[
\int_{\bar{t} = \text{const}} J^F_{\bar{t}} = -\sum_{|n| > 1} \frac{2\pi^2 iV_6 |n|(|n| + 1)^3}{n(2|n| + 1)} \frac{a_n \wedge a_{-n}}, \tag{28}
\]
where $V_6$ is the volume of $T^6$. In Eqs. (27) and (28) the Fourier coefficients $a_n$ should be interpreted as one-forms on the phase space of solutions (2). The derivation of (27) and (28) can be found in Appendix B. Adding the individual contributions, we get the symplectic form as

$$\Omega = -\frac{2\pi^2 i V_6}{\kappa_{10}^2} \sum_{|n|>1} \text{sign } n \left( n^2 - 1 \right) a_n \wedge a_{-n}. \quad (29)$$

As we mentioned in Sec. III, once the symplectic form is found, the Poisson brackets can be extracted from it following the procedure summarized by (7) and (8). Restoring the radius $L$, we obtain

$$\{a_n, a_m\} = \frac{L^2 \kappa_{10}^2}{4\pi^2 V_6} i \frac{\text{sign } n}{n^2 - 1} \delta_{m+n}. \quad (30)$$

The commutators are immediately obtained in the usual way by using the relation that $[\ , \ ] = i\{\ , \ }$

$$[a_m, a_n] = \frac{L^2 \kappa_{10}^2}{4\pi^2 V_6} \frac{\text{sign } n}{n^2 - 1} \delta_{m+n}. \quad (31)$$

The microstate geometries in supergravity correspond to different profiles $f(v)$. Although classically there is a continuum of such curves, the commutators (31) lead to expansions into discrete quantum oscillators,

$$f(\phi) = f_0(\phi) + \lambda \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2 - 1}} \left( c_n e^{-in\phi} + c_n^\dagger e^{in\phi} \right),$$

$$[c_n, c_m^\dagger] = \delta_{mn}, \quad (32)$$

where

$$\lambda = \frac{L \kappa_{10}}{2\pi \sqrt{V_6}}, \quad c_n = \left[ \frac{L^2 \kappa_{10}^2}{4\pi^2 V_6} \frac{1}{n^2 - 1} \right]^{-1/2} a_{-n}.$$ 

The Hilbert space of this theory is a bosonic Fock space equipped with annihilation operators $c_n$ and creation operators $c_n^\dagger$.

The counterparts of the commutators (32) have been encountered in [20–22], and these quantization conditions derived in supergravity turned out to be in a perfect agreement with properties of supersymmetric states in $N = 4$ super Yang-Mills on $S^3 \times \mathbb{R}$ [28, 41] and in the two-dimensional orbifold CFT [14, 15]. We expect that a better understanding for the boundary theory in the AdS$_2$ case would lead to a similar agreement for our relations (32).

V. DISCUSSION

In this paper, we focused on regular horizon-free supersymmetric solutions describing normalizable excitations of AdS$_2 \times S^2$, which are parametrized by a complex harmonic function $H$ in $\mathbb{R}^3$ of possessing the sources located in closed curves. From the gravity side, we
have generalized the method used in [20–22] to quantize the moduli space of these solutions directly. The basic idea of this approach is that commutation relations can be encoded from the symplectic form. Starting with the profile fluctuations, we obtain the perturbations of fields, graviton and five-form, involved in Type IIB SUGRA action corresponding to solutions (2). Finally these perturbations are used to find the symplectic form and to quantize the profiles describing microscopic states. As in the AdS$_3$ and AdS$_5$ cases discussed in [20–22], this quantization is expected to agree with results in the dual theory on the boundary. However, in the present situation where the boundary theory is poorly understood, our results can be viewed as predictions rather than consistency checks as in [20–22].

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Appendix A: Shifts in harmonic functions and vector fields

1. Evaluation of $\delta H_1$ and $\delta H_2$

The vectors $r, f, b$ in (4) can be written in the cylindrical coordinates as

$$r = (r \cos \phi, r \sin \phi, y),$$
$$f = (f \cos \phi_1, f \sin \phi_1, 0),$$
$$b = (b \cos \phi_2, b \sin \phi_2, ib_3),$$

(A1)

where $r, f, b, b_3$ are real. In this paper we only consider the planar profiles, thus taking $f_3 = 0$. Applying (A1) and picking $v = f \phi_1$, Eq. (4) becomes

$$H = \frac{1}{2\pi} \int d\phi_1 \frac{f \sigma}{r^2 + f^2 + y^2 - 2fr \cos(\phi_1 - \phi)} \times \sqrt{r^2 + f^2 + y^2 - 2fr \cos(\phi_1 - \phi) + br \cos(\phi_2 - \phi) - bf \cos(\phi_2 - \phi_1) + iyb_3}$$

$$= \frac{1}{2\pi} \int C \sqrt{D + iyb_3} \ d\phi_1,$$

(A2)

where

$$C = \frac{f \sigma}{r^2 + f^2 + y^2 - 2fr \cos(\phi_1 - \phi)},$$
$$D = r^2 + f^2 + y^2 - 2fr \cos(\phi_1 - \phi) + br \cos(\phi_2 - \phi) - bf \cos(\phi_2 - \phi_1).$$
Therefore, we can express $H_1, H_2$ as

\begin{align}
H_1 &= \frac{1}{2\pi} \int_0^{2\pi} C \left[ D^2 + (yb_3)^2 \right]^{1/2} \cos \left[ \frac{1}{2} \tan^{-1} \left( \frac{yb_3}{D} \right) \right] d\phi_1, \tag{A3} \\
H_2 &= \frac{1}{2\pi} \int_0^{2\pi} C \left[ D^2 + (yb_3)^2 \right]^{1/2} \sin \left[ \frac{1}{2} \tan^{-1} \left( \frac{yb_3}{D} \right) \right] d\phi_1. \tag{A4}
\end{align}

The profile corresponding to $\text{AdS}_2 \times S^2$ is given by (14), and the corresponding vector $\mathbf{b}$ is given as $\mathbf{b} = 2L \left( \cos \frac{\nu}{L}, \sin \frac{\nu}{L}, i \right)$, where $L$ is the radius of the AdS and the sphere. We set $L = 1$ for convenience. Now we consider perturbations (18) on this profile, i.e., now $\mathbf{f} = (1 + \delta f)(\cos \phi_1, \sin \phi_1, 0)$. Vector $\mathbf{b}$ has to change correspondingly due to the constraint $\mathbf{b} \cdot \dot{\mathbf{f}} = 0$,

\begin{equation}
\mathbf{b} = \mathbf{b}_0 + \delta \mathbf{b}, \tag{A5}
\end{equation}

where

\begin{align}
\mathbf{b}_0 &= 2L \left( \cos \frac{\nu}{L}, L \sin \frac{\nu}{L}, i \right) = 2(\cos \phi_1, \sin \phi_1, i) = 2(\cos \phi_2, \sin \phi_2, i), \\
\delta \mathbf{b} &= 2 (-\sin \phi_2, \cos \phi_2, 0) \delta \phi_2 = 2 (-\sin \phi_1, \cos \phi_1, 0) \delta \phi_2. \tag{A6}
\end{align}

Here we keep $b = 2$ unchanged, while $\phi_2 \to \phi_2 + \delta \phi_2$, which can guarantee the two constraints $\mathbf{b} \cdot \dot{\mathbf{f}} = 0$, $\mathbf{b} \cdot \mathbf{b} = 0$. Using constraint

\begin{equation}
0 = \mathbf{b} \cdot \dot{\mathbf{f}} = \delta \dot{f} + (1 + \delta f) \delta \phi_2 \approx \delta \dot{f} + \delta \phi_2, \tag{A7}
\end{equation}

we find $\delta \phi_2 = -\delta \dot{f} = -\sum_n ina_n e^{in\phi_1}$.

Thus we can express the first order perturbations of harmonic function $H_1$ and $H_2$ as

\begin{align}
\delta H_1 &= \left[ \frac{\partial H_1}{\partial f} \delta f + \frac{\partial H_1}{\partial \phi_2} (-\delta \dot{f}) \right]_{\sigma = 1, f = 1, b = b_3 = 2, \phi_1 = \phi_2}, \\
\delta H_2 &= \left[ \frac{\partial H_2}{\partial f} \delta f + \frac{\partial H_2}{\partial \phi_2} (-\delta \dot{f}) \right]_{\sigma = 1, f = 1, b = b_3 = 2, \phi_1 = \phi_2}. \tag{A8}
\end{align}

Let $\phi_1 - \phi = \alpha$, then $\phi_1 = \phi + \alpha$, and we obtain

\begin{align}
\left[ \frac{\partial H_1}{\partial f} \delta f \right]_{\sigma = 1, f = 1, b = b_3 = 2, \phi_1 = \phi_2} &= \int d\alpha \frac{\sum_{|n| > 1} a_n e^{in(\phi + \alpha)}}{2\pi \left[ (r^2 + y^2 - 1)^2 + 4y^2 \right]^{3/4} (r^2 - 2r \cos \alpha + y^2 + 1)^2} \\
& \times \left\{ (r^2 + y^2 - 1) \left[ r^4 + r \left( r \cos 2\alpha - (r^2 + y^2 + 1) \cos \alpha \right) \right. \right. \\
& \left. \left. \quad + r^2 (2y^2 - 1) + (y^2 + 1)^2 \right] \cos \left[ \frac{1}{2} \tan^{-1} \left( \frac{2y}{r^2 + y^2 - 1} \right) \right] \right\}. \tag{A9}
\end{align}
Working out this integral over $\alpha$ by the residue theorem, we find first part of $\delta H_1$. Similarly, we can figure out another part, and finally we obtain, after transferring to global AdS coordinates, the perturbations of $H_1$. Repeating the same procedure, we will find the first order shifts of $\delta H_2$.

2. Evaluation of $\delta V_r$ and $\delta V_\phi$

To find $\delta V$ corresponding to $\delta H$, we begin with equation

$$dV = -2 *_3 [H_2 dH_1 - H_1 dH_2].$$

(A10)

The exterior derivative of vector field $V$, in the gauge $V_y = 0$, can be written as:

$$dV = \frac{\partial V_r}{\partial y} dy \wedge dr + \frac{\partial V_\phi}{\partial y} dy \wedge d\phi + \left( \frac{\partial V_\phi}{\partial r} - \frac{\partial V_r}{\partial \phi} \right) dr \wedge d\phi.$$  

(A11)

Through the comparison of the left and right hand sides of (A10), we get the differential equations

$$\frac{\partial V_r}{\partial y} = \frac{2}{r} \left( H_1 \frac{\partial H_2}{\partial \phi} - H_2 \frac{\partial H_1}{\partial \phi} \right),$$

$$\frac{\partial V_\phi}{\partial y} = -2r \left( H_1 \frac{\partial H_2}{\partial r} - H_2 \frac{\partial H_1}{\partial r} \right),$$

$$\frac{\partial V_\phi}{\partial r} - \frac{\partial V_r}{\partial \phi} = 2r \left( H_1 \frac{\partial H_2}{\partial y} - H_2 \frac{\partial H_1}{\partial y} \right).$$

(A12)

Considering the first order shifts of $V_r$ and $V_\phi$, we get

$$\frac{\partial (\delta V_r)}{\partial y} = \frac{2}{r} \left[ \left( \frac{\partial H_1}{\partial \phi} \delta f - \frac{\partial H_2}{\partial \phi_2} \delta \dot{f} \right) \frac{\partial H_2}{\partial \phi} + H_1 \frac{\partial}{\partial \phi} \left( \frac{\partial H_2}{\partial \phi} \delta f - \frac{\partial H_2}{\partial \phi_2} \delta \dot{f} \right) \right] - \left( \frac{\partial H_2}{\partial \phi} \delta f - \frac{\partial H_2}{\partial \phi_2} \delta \dot{f} \right) \frac{\partial H_1}{\partial \phi} + H_2 \frac{\partial}{\partial \phi} \left( \frac{\partial H_1}{\partial \phi} \delta f - \frac{\partial H_1}{\partial \phi_2} \delta \dot{f} \right) \right]_{f=1}. $$

(A13)

Similarly, we have

$$\frac{\partial (\delta V_\phi)}{\partial y} = -2r \left[ \left( \frac{\partial H_1}{\partial \phi} \delta f - \frac{\partial H_1}{\partial \phi_2} \delta \dot{f} \right) \frac{\partial H_2}{\partial r} + H_1 \frac{\partial}{\partial r} \left( \frac{\partial H_2}{\partial \phi} \delta f - \frac{\partial H_2}{\partial \phi_2} \delta \dot{f} \right) \right] - \left( \frac{\partial H_2}{\partial \phi} \delta f - \frac{\partial H_2}{\partial \phi_2} \delta \dot{f} \right) \frac{\partial H_1}{\partial r} + H_2 \frac{\partial}{\partial r} \left( \frac{\partial H_1}{\partial \phi} \delta f - \frac{\partial H_1}{\partial \phi_2} \delta \dot{f} \right) \right]_{f=1}. $$

(A14)

Again, letting $\phi_1 - \phi = \alpha$, we obtain

$$\frac{\partial (\delta V_r)}{\partial y} = \sum_{|n|>1} a_n e^{i n \phi} \int_0^{2\pi} \frac{4ny(n \sin \alpha + i \cos \alpha)e^{in \alpha} d\alpha}{2\pi \left( r^4 + 2r^2(y^2 - 1) + (y^2 + 1)^2 \right) \left( r^2 - 2r \cos \alpha + y^2 + 1 \right)}. $$

(A15)
Completing this integral over $\alpha$, we find $\delta V_r$ in the global AdS coordinates. Proceeding with (A14), we obtain
\[
\frac{\partial (\delta V_\phi)}{\partial y} = \sum_{|n|>1} a_n e^{in\phi} \int_0^{2\pi} \frac{4r y [\cos \alpha (r^2 - y^2 - 1) + 2r (r^2 + y^2 - 1) - in \sin \alpha (r^2 - y^2 - 1)]}{r^4 + 2r^2 (y^2 - 1) + (y^2 + 1)^2} \frac{d\alpha}{(r^2 - 2r \cos \alpha + y^2 + 1)\rho} \times e^{ina} d\alpha.
\]
(A16)

Figuring out this integral and changing the coordinates as (16), we immediately find $\delta V_\phi$. It is easy to check that $\delta V_r$ and $\delta V_\phi$ obtained from (A15) and (A16) are satisfied by the third equation in (A12).

**Appendix B: Symplectic currents**

### 1. Evaluation of symplectic current $J^I_G$

From metric perturbation (21), it is straightforward to find the components of $\delta[g^{mn} \sqrt{-g}]$:

\[
\begin{align*}
\delta [g^{tt} \sqrt{-g}] &= - \sum_{|n|>1} 2a_n e^{in(\tilde{\phi} - \tilde{t})} \frac{(|n| + 1) \rho^2 \sin \theta S_n}{(1 + \rho^2)^2}, \\
\delta [g^{t\rho} \sqrt{-g}] &= \sum_{|n|>1} 2a_n e^{in(\tilde{\phi} - \tilde{t})} \frac{i(n^2 + |n|) \rho \sin \theta S_n}{n(1 + \rho^2)}, \\
\delta [g^{\rho\rho} \sqrt{-g}] &= \sum_{|n|>1} 2a_n e^{in(\tilde{\phi} - \tilde{t})} \frac{i(|n| + 1) \sin \theta S_n}{(1 + \rho^2)^2}.
\end{align*}
\]

(B1)

where
\[
S_n = \left( \frac{\sin \theta}{\sqrt{\rho^2 + 1}} \right)^{|n|}.
\]

The shifts of connections $\delta \Gamma^I_{mp}$ that we are interested in are

\[
\begin{align*}
\delta \Gamma^I_{tt} &= \sum_{|n|>1} a_n e^{in(\tilde{\phi} - \tilde{t})} \frac{i[n^2 (|n| (\rho^2 - 1) + 5\rho^2 - 1) + 4|n|\rho^2] S_n}{2n(1 + \rho^2)}, \\
\delta \Gamma^I_{t\rho} &= \sum_{|n|>1} a_n e^{in(\tilde{\phi} - \tilde{t})} \frac{\rho(|n| + 1) \rho (|n| (\rho^2 - 1) - 4) S_n}{2(1 + \rho^2)^2}, \\
\delta \Gamma^I_{\rho\rho} &= \sum_{|n|>1} a_n e^{in(\tilde{\phi} - \tilde{t})} \frac{i[n^2 (3|n|\rho^2 + |n| - \rho^2 - 3) + 4|n| (|n|\rho^2 - 1)] S_n}{2n(1 + \rho^2)^3}.
\end{align*}
\]

(B2)
Some components of $\delta \Gamma_{m}^{m}$ are:

$$\delta \Gamma_{m}^{m} = - \sum_{|n| > 1} a_{n} e^{in(\tilde{\phi} - \tilde{t})} n(1 + |n| + 1) \rho S_{n},$$

$$\delta \Gamma_{m}^{m\rho} = - \sum_{|n| > 1} a_{n} e^{in(\tilde{\phi} - \tilde{t})} \frac{|n|(1 + |n| + 1) \rho S_{n}}{1 + \rho^2}. \tag{B3}$$

With the preparation (B1), (B2) and (B3), we can easily find

$$\int d\tilde{\phi} \delta \Gamma_{mp}^{n} \wedge \delta [\sqrt{-g}g_{mp}^{n}] = \frac{2\pi i n(1 + |n|)^2 [n^4 - 6n^2 - 1 + 4 (\rho^2 - 1)^2 ] \sin \theta S_{n}^2}{n(1 + \rho^2)^3} \times (a_{n} \wedge a_{-n}), \tag{B4}$$

$$\int d\tilde{\phi} \delta \Gamma_{mp}^{n} \wedge \delta [\sqrt{-g}g_{mp}^{n}] = \frac{8\pi i n(1 + n^2)^2 \rho^2 \sin \theta S_{n}^2}{n(1 + \rho^2)^2} (a_{n} \wedge a_{-n}). \tag{B5}$$

Integrate the remaining integrals, and add them up we then find the symplectic form (27) from gravity current.

2. **Evaluation of symplectic current $J_{F}^{i}$**

Using relations (24), one can easily find

$$\delta B_{t} = - \sum_{|n| > 1} \frac{1}{8} a_{n} e^{in(\tilde{\phi} - \tilde{t})} (1 + |n|) \rho S_{n},$$

$$\delta B_{\rho} = - \sum_{|n| > 1} a_{n} e^{in(\tilde{\phi} - \tilde{t})} \frac{|n|(1 + |n| + 1) \rho S_{n}}{8 |n|(1 + \rho^2)},$$

$$\delta \tilde{B}_{\theta} = - \sum_{|n| > 1} a_{n} e^{in(\tilde{\phi} - \tilde{t})} \frac{\sin \theta S_{n}}{8 |n| \sin \theta},$$

$$\delta \tilde{B}_{\tilde{\phi}} = \sum_{|n| > 1} \frac{1}{8} a_{n} e^{in(\tilde{\phi} - \tilde{t})} (|n| + 1) \cos \theta S_{n}. \tag{B6}$$

From these one-form perturbations we obtain

$$\delta F_{i\rho} = - \sum_{|n| > 1} \frac{1}{8} a_{n} e^{in(\tilde{\phi} - \tilde{t})} (n^2 - 1) S_{n},$$

$$\delta \tilde{F}_{i\theta} = - \sum_{|n| > 1} \frac{1}{8} a_{n} e^{in(\tilde{\phi} - \tilde{t})} |n|(|n| + 1) \csc \theta S_{n},$$

$$\delta \tilde{F}_{i\tilde{\phi}} = - \sum_{|n| > 1} \frac{1}{8} a_{n} e^{in(\tilde{\phi} - \tilde{t})} |n|(|n| + 1) \cos \theta S_{n}. \tag{B7}$$
Then we obtain
\[ \delta[\sqrt{-g}F_{\bar{\theta}\bar{\phi}}] = \sum_{|n|>1} \frac{1}{8} a_n e^{i n (\bar{\theta} - \bar{\phi})} (|n| + 1)^2 \sin \theta S_n, \]
\[ \delta[\sqrt{-g}F_{\bar{t}\bar{\rho}}] = \sum_{|n|>1} \frac{1}{8} a_n e^{i n (\bar{\theta} - \bar{\phi})} \frac{|n|(|n| + 1) S_n}{1 + \rho^2}, \]
\[ \delta[\sqrt{-g}F_{\bar{t}\bar{\phi}}] = \sum_{|n|>1} \frac{1}{8} a_n e^{i n (\bar{\theta} - \bar{\phi})} \frac{i n (|n| + 1) \cot \theta S_n}{1 + \rho^2}. \] (B8)

Therefore we find
\[ \int d\bar{\phi} \delta A_{|i_1i_2i_3i_4|} \wedge \delta[\sqrt{-g}F_{t|i_1i_2i_3i_4|}] = -\sum_{|n|>1} \frac{\pi |n|(|n| + 1)^2 (2|n| + \sin^2 \theta) S_n^2}{n(1 + \rho^2) \sin \theta} \times a_n \wedge a_{-n}. \] (B9)

Working out the remaining integrals, we then find the symplectic form (28) from five-form current.

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