Max/Min Puzzles in Geometry III†

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Abstract. The first two installments of this series of papers dealt with the maximum area polygons: Parallelogram, Rectangle, Square or Equilateral Triangle, in given triangles. Minimum area polygons were also considered in the second paper on Equilateral Triangles. In this paper the puzzle will be turned the other way around. Given the regular unit polygons, a Square, Pentagon, or Hexagon, we searched for the smallest area triangle(s) which contains it. The Dynamic Software Sketchpad v5.10 BETA was used for all examples and figures throughout the work.

I. The first polygon to be considered is the unit square. It will be assumed that triangles which enclose the square inscribe it, that is the vertices of the square are all on the sides of the triangle. It follows that it can be assumed that one side of the square is on a side of the triangle [4].

The following observation will be of use throughout the study of squares. Let $\triangle ABC$ be a given triangle with base $AB$, in which a unit square $DEFG$ is inscribed, and there are no obtuse angles at vertices $A$ or $B$, Fig. 1. Then $\triangle ABC \sim \triangle GFC$, by Euclid [1], since the corresponding angles are equal. Let $h$ be the height of $\triangle ABC$ at $C$, $c = AB$, $h' = \text{the height of } \triangle GFC$, and $c' = GF$. Then $h/c = h'/c'$, but $h = h' + 1$, and $c' = 1$, so $c = (h' + 1)/h'$. Thus the following Lemma is established.

Lemma 1. If $\triangle ABC$ inscribes the unit square $DEFG$ with $DE$ on $AB$, and $AB$ has no obtuse angles, then $c = (h' + 1)/h'$, where $c = AB$, and $h'$ is the height of $\triangle GFC$.

Examples.

A. Let $\triangle ABC$ be as in Fig. 1, with $h' = 1$. Then, by the Lemma, $c = 2$, and $(ABC) = 2u^2$, where $(ABC)$ denotes the area of $\triangle ABC$.

B. Let $\triangle ABC$ be the right angle triangle with sides of length 2, and right angle at vertex $A$ coinciding with the right angle in the unit square at vertex $D$, Fig. 2. Then $(ABC) = 2u^2$. Another right triangle congruent to $\triangle ABC$ with right angle at vertex $E$ also inscribes the square $DEFG$, and has area $2u^2$.

C. Let $\triangle ABC$ be an equilateral triangle which inscribes the unit square $DEFG$. Then $\triangle GFC$ is also an equilateral triangle, and has base $GF = 1$. Thus the height of $\triangle GFC$ is $\sqrt{3}/2$, and the height of $\triangle ABC$ is $h = 1 + \sqrt{3}/2$. By the Lemma, $c = (2\sqrt{3} + 3)/3$, so $(ABC) = (12 + 7\sqrt{3})/12 \sim 2.01u^2$, Fig. 3.

D. Let $\triangle ABC$ be the isosceles triangle with $CA = CB$, and central angle at $C$, $\angle C = 120^\circ$. Then $AD = EB = \sqrt{3}$, and $\angle A = 30^\circ = \angle B$. So $c = AB = 1 + 2\sqrt{3}$, and by the Lemma, $h = (\sqrt{3} + 6)/6$. Thus, $(ABC) = (1 + 2\sqrt{3})(\sqrt{3} + 6)/12 \sim 2.88u^2$, Fig. 4.

† Versions of some of these results have appeared in different setting previously [2], [3].
The above examples lead to the conjecture that $2u^2$ is a lower bound for the Minimum area of a triangle which inscribes a unit square.

**Theorem 1.** If a unit square $DEFG$ is inscribed in $\Delta ABC$, with $DE$ on side $AB$, then $(ABC) = 2u^2$, when the height $h$ of $\Delta ABC$ from $C$ satisfies $h = 2$, and $(ABC) > 2u^2$, when $h \neq 2$.

Proof. By Lemma 1, if $\Delta ABC$ is a triangle in which a unit square $DEFG$ is inscribed, and $\Delta ABC$ has height $h = 2$, then the base $c = 2$, so $(ABC) = 2u^2$, Fig. 1.

Suppose $\Delta ABC$ contains the inscribed unit square $DEFG$, and it has height $h \neq 2$, say $h > 2$, so that then $h = 2 + e$, for $e > 0$. By Lemma 1, $h' = 1 + e$, so $c = (2 + e)/(1 + e)$, and $(ABC) = (2 + e)^2/(2 + 2e)$.

It then follows that $(ABC) > 2u^2$, because this inequality is equivalent to $e^2 > 0$, which is obviously true.

A similar argument holds if $\Delta ABC$ has height $h = 2 - e$, for $0 < e < 1$.

Thus the Theorem is established.

II. The next polygon to be considered is the regular unit pentagon. The interior angles of pentagon $DEFGH$ are $108^\circ$, since a pentagon interior can be divided into 3 triangles. The major chords, such as $HF$, all have length $(1 + \sqrt{5})/2$, and the distance from $HF$ to base side $AB$ is $d = \sqrt{(5 + \sqrt{5})/2}/2$, by Euclid [1], Fig. 5. However, it is only possible to inscribe a pentagon in a triangle in a few ways. So it will be necessary to also consider triangles which ‘enclose’ a pentagon in a more general way, see Fig’s 5&6.

If the line segment $HF$ is added to the unit regular pentagon $DEFGH$, then for any triangle $\Delta ABC$ which encloses the pentagon such that $DE$ is on side $AB$, $F$ is on $CB$, and $H$ is on $CA$, it follows the $\Delta ABC \sim \Delta HFC$, by Euclid [1].

Thus, if $h$ is the height of $\Delta ABC$ at $C$, $h'$ is the height of $\Delta HFC$, $s = HF$, and $c = AB$, then $h/c = h'/s$.

But $h = h' + \sqrt{(5 + \sqrt{5})/2}/2$, and $s = HF = (1 + \sqrt{5})/2$, by the above, so $c = (h' + d)s/h' = 2s$.

This proves Lemma 2.

**Lemma 2.** Let $\Delta ABC$ enclose the unit regular pentagon $DEFGH$, with $DE$ on $AB$, $F$ on $CB$, and $H$ on $CA$, and there are no obtuse angles on the base side $AB$. Then $c = (h' + d)s/h'$, where $h$ is the height of $\Delta ABC$ from $C$, $h'$ is the height of $\Delta HFC$ from $C$, $c = AB$, $s = HF$, and $d$ is the distance from $HF$ to base side $AB$.

**Examples.**

A. In Fig. 5, the triangle $\Delta ABC$ which encloses the regular unit pentagon is special. The height is $h = \sqrt{(5 + \sqrt{5})/2} \approx 1.902$, which is twice the height of the line segment $HF$ above the base $AB$.

By Sketchpad, $(ABC) \approx 3.078u^2$.

B. Let $\Delta ABC$ be the triangle which encloses the regular unit pentagon $DEFGH$, and has height $h = 2$, Fig. 6. Then, by Lemma 2, the base $c = AB \approx 3.127$, and, by Sketchpad, $(ABC) \approx 3.127u^2$. 
C. Let $\triangle ABC$ be the triangle in which the regular unit pentagon $DEFGH$ is inscribed, and where the sides $GH$ and $ED$ are extended to intersect at $A$, and the side $BC$ is perpendicular to the line segment $AF$, Fig. 7. It then follows, by Sketchpad, that $(ABC) \approx 3.078u^2$, see Theorem 2.

D. Let $\triangle ABC$ be the triangle in which the regular unit pentagon $DEFGH$ is inscribed, and the vertex $C$ coincides with vertex $G$, and the sides $GH$ and $GF$ are extended to intersect with the line on the base side $DE$ at $A$ and $B$, respectively, Fig. 8. Then $\angle A = \angle B = 36^\circ$, and $\angle C = 108^\circ$. Using Sketchpad it follows that $(ABC) \approx 3.26u^2$.

These examples lead to the final result, Theorem 2.

**Theorem 2.** Let $DEFGH$ be a regular unit pentagon, and $\triangle ABC$ enclose the pentagon with side $DE$ on side $AB$ of the triangle, no obtuse angles on $AB$, and let the side $DE$ be reflected about line $HF$ to form the segment $ST$. Assume further that vertex $C$ is on $ST$, then $\triangle ABC$ is a min. enclosing triangle of the pentagon with area $(ABC) = \sqrt{5 + 2\sqrt{5}} \approx 3.078u^2$.

If vertex $C$ is not on $ST$, then $(ABC) > \sqrt{5 + 2\sqrt{5}}$.

Proof. Construct the triangle $\triangle ABC$ as in Example C, Fig’s. 7, 9. Since the exterior angles of the pentagon are $72^\circ$, the triangles $\triangle ABC$, $\triangle ADH$, $\triangle BFE$ and $\triangle CFG$ are similar isosceles triangles with base angles of $72^\circ$, and central angles of $36^\circ$, Fig. 9.

Also, a copy of $\triangle ABC$, $\triangle A'B'C'$ can be constructed to the right of the pentagon with the same properties as $\triangle ABC$, Fig. 9. Since $H, F$ are the midpoints of $B'C'$ and $BC$, $HF$ is halfway between $DE$ and $C'C$, and $BC = B'C' = 2$.

Since $\triangle HFC \sim \triangle ABC$, it follows that $2/1 = (s+1+e)/s$, where $s = HF = AD$, $e = EB$, and $c = AB = s + 1 + e$.

So $s = 1 + e$, and $c = AB = 2s$, by Lemma 2.

Let $K$ be the point on $AB$ such that $\triangle KEF$ is an isosceles triangle, Fig. 9.

Then $KB = 2e$, so $2s = 2/2e$, and $s = 1/(s-1)$, thus $s = (1 + \sqrt{5})/2$ (the golden ratio), $e = 1/s = (\sqrt{5} - 1)/2$, and $AB = 1 + \sqrt{5}$.

Thus, $\triangle EBC$ is a right triangle, so $h^2 + e^2 = 4$, and $h = \sqrt{(5 + \sqrt{5})/2}$.

It then follows that $(ABC) = hc/2 = \sqrt{5 + 2\sqrt{5}} \approx 3.078u^2$.

If $\triangle A'B'C'$ is another triangle which encloses the pentagon $DEFGH$ and has vertex $C'$ on $ST$, then $\triangle HFC \sim \triangle A'B'C'$, and the ratios satisfy $h/h' = c/c'$, where $c = A'B'$, and $c' = HF$, Fig. 10.

But $h = 2h'$, and $c = 2c'$, where $c' = s$, so $(A'B'C') = hc/2 = (ABC)$.

The area of $\triangle ABC$ then satisfies $(ABC) = \sqrt{5 + 2\sqrt{5}}$.

Now suppose $\triangle ABC$ is a triangle which encloses pentagon $DEFGH$, where vertex $C$ is not on segment $ST$, and there are no obtuse angles on the base $AB$, Fig. 11.
Then, for $c'' = AB$, and $h''$ the height of $C$ over $AB$, $h''/c'' = (h'' - h)/s$, by Lemma 2, where $h$ is the distance from $ST$ to $AB$, $s = HF$, and $h'' \neq h$. Thus $(ABC) = sh''^2/(2h'' - h)$.

The area of the triangle above in Fig. 10, with vertex $C$ on $ST$ is $sh$, since $AB = 2s$. So the claim is that the area of the triangle in Fig. 11 is greater than the area of the triangle in Fig. 10, that is $sh''^2/(2h'' - h) > sh$.

But this is equivalent to the inequality $(h'' - h)^2 > 0$, which is obviously true, regardless of whether $h''$ is greater than, or less than $h$.

**III.** The last polygon to be considered is the regular unit hexagon. This polygon is special, since it does not follow the patterns of the areas of the square and the pentagon.

Generalizations to higher order polygons are considered in [3].

The interior angles of the hexagon are $120^\circ$, hence the exterior angles are $60^\circ$. The regular hexagon can be divided into 6 equilateral triangles, each with sides of length $1u$, Fig. 12.

So the height of the unit hexagon is $\sqrt{3}$, and, since the equilateral triangles each have area $\sqrt{3}/4u^2$, the area of the unit hexagon is $3\sqrt{3}/2u^2$.

**Examples.**

A. If $\triangle ABC$ is the equilateral triangle which inscribes the unit regular hexagon $DEFGHJ$, then the height of $\triangle ABC$ is $3\sqrt{3}/2u$, and the sides have length $3u$. Thus $(ABC) = 9\sqrt{3}/4u^2 \approx 3.90u^2$, Fig. 13.

B. Let $\triangle ABC$ be the isosceles triangle which encloses the hexagon $DEFGHJ$, and has height 3, and base $AB \approx 2.81u$, Fig. 14. By Sketchpad the approximate area of the triangle is $(ABC) \approx 4.22u^2$.

C. Let $\triangle A'B'C'$ be the triangle which encloses the hexagon $DEFGHJ$, and has height 2.25, and base $A'B' \approx 4.34u$, Fig. 14. By Sketchpad the approximate area of the triangle is $(A'B'C') \approx 4.89u^2$.

**Theorem 3.** The Min triangle $\triangle ABC$ in area which inscribes the unit regular hexagon is an equilateral triangle with sides of length $3u$. and $(ABC) = 9\sqrt{3}/4u^2$.

Proof. If $\triangle ABC$ is the equilateral triangle in which is inscribed the unit regular hexagon $DEFGHJ$, then $(ABC) = 9\sqrt{3}/4u^2$.

Let $\triangle A'B'C'$ be another triangle which encloses the unit regular hexagon, with height $h > 3\sqrt{3}/2u$. It is claimed that $(A'B'C') > (ABC)$.

Connect $F$ and $J$, then $\triangle JFC' \sim \triangle A'B'C'$, by Euclid [1].

Let $c' = A'B'$, then $h/c' = (h - \sqrt{3}/2)/2$, since $JF = 2u$.

Hence $c' = 2h/(h - \sqrt{3}/2)$, and $(A'B'C') = h^2/(h - \sqrt{3}/2)u^2$. 
Thus \((A'B'C') > (ABC)\) iff \(h^2/(h - \sqrt{3}/2) > 9\sqrt{3}/4\) iff \(h^2 - 9\sqrt{3}h/4 + 27/8 > 0\).
But \(h^2 - 9\sqrt{3}h/4 + 27/8 = (h - 3\sqrt{3}/4)(h - 3\sqrt{3}/2)\), and this parabola is positive when \(h > 3\sqrt{3}/2u\).
Hence the inequality holds.
A similar argument holds for \(h < 3\sqrt{3}/2u\).

The general case for regular n-gons, with \(n \geq 4\), is considered in [3]. It turns out that the pattern which appears here also holds in the more general case for all regular n-gons.

References.

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