The Study of the New Classes of m-Fold Symmetric bi-Univalent Functions

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Abstract: In this paper, we introduce three new subclasses of m-fold symmetric holomorphic functions in the open unit disk $U$, where the functions $f$ and $f^{-1}$ are m-fold symmetric holomorphic functions in the open unit disk. We denote these classes of functions by $FS_{\Sigma_m}^{p,q,r}(d)$, $FS_{\Sigma_m}^{p,q}(e)$ and $FS_{\Sigma_m}^{p,q,s,h,r}$. As the Fekete-Szegö problem for different classes of functions is a topic of great interest, we study the Fekete-Szegö functional and we obtain estimates on coefficients for the new function classes.

Keywords: Fekete-Szegö problem; coefficient bounds and coefficient estimates; bi-univalent functions; bi-pseudo-starlike functions; m-fold symmetric; analytic functions

1. Introduction and Preliminary Results

Let $A$ denote the family of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the open unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$ and normalized by the conditions $f(0) = 0$, $f'(0) = 1$.

Let $S \subset A$ denote the subclass of all functions in $A$ which are univalent in $U$ (see [1]). In [1], the Koebe one-quarter theorem ensures that the image of the unit disk under every $f \in S$ contains a disk of radius $1/4$. It is well known that every function $f \in S$ has an inverse $f^{-1}$, which is defined by

$$f^{-1}(f(z)) = z, z \in U$$

and

$$f(f^{-1}(w)) = w, |w| < r_0(f), r_0(f) \geq 1/4,$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \ldots$$

(2)

A function $f \in A$ is said to be bi-univalent in $U$ if both $f$ and $f^{-1}$ are univalent in $U$. Let $\Sigma$ denote the class of all bi-univalent functions in $U$ given by (1).

The class of bi-univalent functions was first introduced and studied by Lewin [2] and it was shown that $|a_2| < 1.51$.

The domain $D$ is m-fold symmetric if a rotation of $D$ about the origin through an angle $2\pi/m$ carries $D$ on itself.

We said that the holomorphic function $f$ in the domain $D$ is m-fold symmetric if the following condition is true: $f(e^{\frac{2\pi i}{m}} z) = e^{\frac{2\pi i}{m}} f(z)$.
A function is said to be m-fold symmetric if it has the following normalized form:

\[ f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}, \quad z \in U, m \in \mathbb{N} \cup \{0\}. \quad (3) \]

The normalized form of \( f \) is given as in (3) and the series expansion for \( f^{-1}(z) \) is given below (see [3]):

\[
 g(w) = f^{-1}(w) = w - \frac{1}{2} (m + 1)(3m + 2) a_{m+1}^3 z^{m+1} - (3m + 2) a_{m+1} a_{2m+1} + a_{3m+1} w^{3m+1} + \ldots
\]

We can give examples of m-fold symmetric bi-univalent functions:

\[
 \{ \frac{z^m}{1-z^m} \}^m, \quad [\frac{-\log(1-z^m)}{2z}]^m, \quad \frac{1}{2} \log \left( \frac{1+z^m}{1-z^m} \right)^m.
\]

The important results about the m-fold symmetric analytic bi-univalent functions are given in [3–7].

The Fekete-Szegö problem is the problem of maximizing the absolute value of the functional

\[
 |a_3 - \mu a_2^2|.
\]

Fekete-Szegö inequalities for different classes of functions are studied in the papers [8–14].

Many authors obtained coefficient estimates of bi-univalent functions in the articles [2,14–25].

**Definition 1.** Let \( f \in A \) be given by (1) and \( 0 < q < p \leq 1 \). Then, the \((p,q)\)-derivative operator for the function \( f \) of the form (1) is defined by

\[
 D_{p,q} f(z) = \frac{f(pz) - f(qz)}{(p-q)z}, \quad z \in U^* = U - \{0\}
\]

and

\[
 (D_{p,q} f)(0) = f'(0)
\]

and it follows that the function \( f \) is differentiable at 0.

We deduce from (2) that

\[
 D_{p,q} f(z) = 1 + \sum_{k=2}^{\infty} [k]_{p,q} a_k z^{k-1}
\]

where the \((p,q)\)-bracket number is given by

\[
 [k]_{p,q} = \frac{p^k - q^k}{p - q} = p^{k-1} + p^{k-2} q + p^{k-3} q^2 + \ldots + p q^{k-2} + q^{k-1}, \quad p \neq q
\]

which is a natural generalization of the \(q\)-number.

Too \( \lim_{p \to 1^-} [k]_{p,q} = [k]_q = \frac{1-q^k}{1-q}, \) see [26,27].

**Definition 2 ([28]).** Let the function \( f \in A \) where \( 0 \leq d < 1, s \geq 1 \) is real. The function \( f \in L_s(d) \) of \( s\)-pseudo-starlike function of order \( d \) in the unit disk \( U \) if and only if

\[
 \text{Re}(\frac{zf'(z)}{f(z)}) > d.
\]
Lemma 1 ([1], p. 41). Let the function \( w \in \mathcal{P} \) be given by the following series: \( w(z) = 1 + w_1z + w_2z^2 + \ldots, z \in U \), where we denote by \( \mathcal{P} \) the class of Carathéodory functions analytic in the open disk \( U \),

\[
\mathcal{P} = \{ w \in \mathcal{A} | w(0) = 1, \text{Re}(w(z)) > 0, z \in U \}.
\]

The sharp estimate given by \( |w_n| \leq 2, n \in \mathbb{N}^* \) holds true.

2. Main Results

Definition 3. The function \( f \) given by (3) is in the function class \( FS_{\Sigma_m}^{p,q,s}(d)(m \in \mathbb{N}, 0 < q < p \leq 1, s \geq 1, 0 < d \leq 1, (z,w) \in U) \) if:

\[
\begin{align*}
& f \in \Sigma, \\
& |\arg(D_{p,q}f(z))^s| < \frac{d\pi}{2}, z \in U
\end{align*}
\]

and

\[
|\arg(D_{p,q}g(w))^s| < \frac{d\pi}{2}, w \in U,
\]

where \( g \) is the function given by (4).

Remark 1. In the case when \( m = 1 \) (one-fold case) and \( s = 1 \), we obtain the class defined in [29].

Remark 2. In the case when \( p = 1 \), we obtain \( \lim_{q \to 1} FS_{\Sigma_m}^{1,q,s}(d) = FS_{\Sigma_m}(d) \), the class which was introduced by Srivastava et al. in [24].

We obtain coefficient bounds for the functions class \( FS_{\Sigma_m}^{p,q,s}(d) \) in the next theorem.

Theorem 1. Let \( f \) given by (3) be in the class \( FS_{\Sigma_m}^{p,q,s}(d)(m \in \mathbb{N}, 0 < q < p \leq 1, s \geq 1, 0 < d \leq 1, (z,w) \in U) \). Then,

\[
|a_{m+1}| \leq \frac{2d}{sd(m+1)[2m+1]_{p,q} - s(d-s)[m+1]_{p,q}}
\]

and

\[
|a_{2m+1}| \leq \frac{2d}{s[2m+1]_{p,q} + 2(m+1)d^2} + \frac{s^2[m+1]_{p,q}^2}{s^2[m+1]_{p,q}}.
\]

Proof. If we use the relations (8) and (9), we obtain

\[
(D_{p,q}f(z))^s = [\alpha(z)]^d
\]

and

\[
(D_{p,q}g(w))^s = [\beta(w)]^d, (z,w) \in U
\]

where the functions \( \alpha(z) \) and \( \beta(w) \) are in \( \mathcal{P} \) and are given by

\[
\alpha(z) = 1 + a_mz^m + a_{2m}z^{2m} + a_{3m}z^{3m} + \ldots
\]

and

\[
\beta(w) = 1 + \beta_mw^m + \beta_{2m}w^{2m} + \beta_{3m}w^{3m} + \ldots.
\]

It is obvious that

\[
[\alpha(z)]^d = 1 + da_mz^m + (da_{2m} + \frac{d(d-1)}{2}a_m^2)z^{2m} + \ldots,
\]

\[
[\beta(w)]^d = 1 + db_mw^m + (db_{2m} + \frac{d(d-1)}{2}b_m^2)w^{2m} + \ldots,
\]
\[(D_{p,q} f(z))^s = 1 + s[m + 1]_{p,q} a_{m+1} z^m + \ldots\]

\[(D_{p,q} g(w))^s = 1 - s[m + 1]_{p,q} a_{m+1} w^m - s[2m + 1]_{p,q} a_{2m+1} w^{2m} + \ldots\]

We obtain from the relations (16) and (18) that

\[s[m + 1]_{p,q} a_{m+1} = d\alpha_m, \quad (16)\]

\[s[2m + 1]_{p,q} a_{2m+1} + \frac{s(s - 1)}{2} [m + 1]_{p,q} a_m^2 = d\alpha_{2m} + \frac{d(d - 1)}{2} \alpha_m^2, \quad (17)\]

\[-s[m + 1]_{p,q} a_{m+1} = d\beta_m, \quad (18)\]

\[-s[2m + 1]_{p,q} a_{2m+1} + (s(m + 1)[2m + 1]_{p,q} + \frac{s(s - 1)}{2}[m + 1]_{p,q} a_m^2) a_{m+1}^2 = d\beta_{2m} + \frac{d(d - 1)}{2} \beta_m^2. \quad (19)\]

We obtain from the relations (16) and (18) that

\[a_m = -\beta_m \quad (20)\]

and

\[2s^2[m + 1]_{p,q} a_m^2 = d^2 (\alpha_m^2 + \beta_m^2) \quad (21)\]

Now, from the relations (17), (19) and (21), we obtain that

\[s(s - 1)d[m + 1]_{p,q} a_m^2 + (m + 1)sd[2m + 1]_{p,q} a_m^2 + (d - 1)s^2[m + 1]_{p,q} a_m^2 = d^2 (\alpha_{2m} + \beta_{2m}).\]

We have

\[a_{m+1}^2 = \frac{d^2 (\alpha_{2m} + \beta_{2m})}{s[m + 1]_{p,q}^2(s - d) + (m + 1)sd[2m + 1]_{p,q}}. \quad (22)\]

If we apply Lemma 1 for the coefficients \(a_{2m}\) and \(\beta_{2m}\), we have

\[|a_{m+1}| \leq \frac{2d}{\sqrt{(m + 1)sd[2m + 1]_{p,q} - (d - s)s[m + 1]_{p,q}^2}}.\]

If we use the relations (17) and (19), we obtain the next relation

\[2s[2m + 1]_{p,q} a_{2m+1} - s(m + 1)[2m + 1]_{p,q} a_m^2 = d(\alpha_{2m} - \beta_{2m}) + \frac{d(d - 1)}{2} (\alpha_m - \beta_m^2). \quad (23)\]

It follows from (20), (21) and (23) that

\[a_{2m+1} = \frac{(m + 1)d^2 (\alpha_m^2 + \beta_m^2)}{4s^2[m + 1]_{p,q}^2} + \frac{d(\alpha_{2m} - \beta_{2m})}{2s[2m + 1]_{p,q}}. \quad (24)\]
If we apply Lemma 1 for the coefficients \(a_m, a_{2m}, \beta_m, \beta_{2m}\), we obtain
\[
|a_{2m+1}| \leq \frac{2d}{[2m+1]s} + \frac{2d^2(m+1)}{s^2[m+1]_{p,q}}.
\]
\(\square\)

**Remark 3.** For one-fold case \(m = 1\) and \(s = 1\) in Theorem 1, we obtain the results obtained in [29].

**Remark 4.** For a one-fold case and \(p = 1\), we have
\[
\lim_{q \to 1} F_{\Sigma,1}^{1}(d) = F_{\Sigma}(d),
\]
the results of Srivastava et al. [24].

**Definition 4.** The function \(f\) given by (3) is in the class \(F_{\Sigma,1}^{1}(e)\) \((0 \leq e < 1, 0 < q < p \leq 1, s \geq 1, (z,w) \in U, m \in \mathbb{N})\) if the following conditions are satisfied:
\[
\begin{align*}
&f \in \Sigma, \\
&R\{(D_{p,q}f(z))^s\} > e, z \in U \\
&R\{(D_{p,q}g(w))^s\} > e, w \in U,
\end{align*}
\]
where the function \(g\) is defined by Relation (4).

**Remark 5.** For \(m = 1\) (one-fold case) and \(s = 1\), we obtain the class of functions obtained in [29].

**Remark 6.** When \(p = 1\), we obtain \(\lim_{q \to 1} F_{\Sigma,1}^{1}(e) = F_{\Sigma}(d)\), the class which was introduced by Srivastava et al. in [24].

In the next theorem, we obtain coefficient bounds for the function class \(F_{\Sigma,1}^{1}(e)\).

**Theorem 2.** Let the function \(f\) given by (3) be in the function class \(F_{\Sigma,1}^{1}(e)\), \((m \in \mathbb{N}, 0 < q < p \leq 1, s \geq 1, 0 \leq e < 1, (z,w) \in U)\). Then,
\[
|a_{m+1}| \leq \min\left\{ \frac{2(1-e)}{s[m+1]_{p,q}}, 2\sqrt{\frac{(1-e)}{s(s-1)[m+1]_{p,q}^2 + (m+1)s[2m+1]_{p,q}}} \right\}
\]
\[
|a_{2m+1}| \leq \frac{2(1-e)(m+1)}{s(s-1)[m+1]_{p,q}^2 + (m+1)s[2m+1]_{p,q}} + \frac{2(1-e)}{s[s+1]_{p,q}}.
\]

**Proof.** If we use Relations (25) and (26), we obtain
\[
(D_{p,q}f(z))^s = e + (1-e)\alpha(z)
\]
and
\[
(D_{p,q}g(w))^s = e + (1-e)\beta(w), \quad z, w \in U,
\]
respectively, where
\[
\alpha(z) = 1 + \alpha_m z^m + \alpha_{2m} z^{2m} + \alpha_{3m} z^{3m} + \ldots
\]
and
\[
\beta(w) = 1 + \beta_m w^m + \beta_{2m} w^{2m} + \beta_{3m} w^{3m} + \ldots,
\]
\(\alpha(z)\) and \(\beta(w)\) are in \(\mathcal{P}\).
It is obvious that
\[ e + (1 - e)α(z) = 1 + (1 - e)α_m z^m + (1 - e)α_{2m} z^{2m} + \ldots, \]
and
\[ e + (1 - e)β(w) = 1 + (1 - e)β_m w^m + (1 - e)β_{2m} w^{2m} + \ldots \]

Already,
\[ (D_{p,q} f(z))^s = 1 + s[m + 1]p,q a_{m+1} z^m + (s[2m + 1]p,q a_{2m+1} + \frac{s(s - 1)}{2}[m + 1]p,q a_{m+1}^2)z^{2m} + \ldots \]

and
\[ (D_{p,q} g(w))^s = 1 - s[m + 1]p,q a_{m+1} w^m - s[2m + 1]p,q a_{2m+1} w^{2m} \]
\[ + (s[m + 1][2m + 1]p,q a_{m+1}^2 + \frac{s(s - 1)}{2}[m + 1]p,q a_{m+1}^2)w^{2m} + \ldots \]

From the relations (29) and (30), if we compare the coefficients, we obtain the following relations:
\[ s[m + 1]p,q a_{m+1} = (1 - e)α_m, \quad (31) \]
\[ s[2m + 1]p,q a_{2m+1} + \frac{s(s - 1)}{2}[m + 1]p,q a_{m+1}^2 = (1 - e)α_{2m}, \quad (32) \]
\[ - s[m + 1]p,q a_{m+1} = (1 - e)β_m, \quad (33) \]
\[ -s[1 + 2m]p,q a_{2m+1} + (s[2m + 1]p,q (m + 1) + \frac{s(s - 1)}{2}[1 + m]p,q a_{m+1}^2 a_{m+1} = (1 - e)β_{2m}. \quad (34) \]
We obtain from Relations (31) and (33)
\[ α_m = -β_m \quad (35) \]
and
\[ 2s^2[m + 1]p,q a_{m+1}^2 = (1 - e)^2(α_m^2 + β_m^2). \quad (36) \]
We obtain now from Relations (32) and (34) the following relation:
\[ s(s - 1)[m + 1]p,q a_{m+1}^2 + (m + 1)s[2m + 1]p,q a_{m+1} = (1 - e)(α_{2m} + β_{2m}). \quad (37) \]

From Lemma 1 for the coefficients α_m, α_{2m}, β_m, β_{2m}, we obtain that
\[ |a_{m+1}| \leq \frac{1 - e}{\sqrt{(m + 1)s[2m + 1]p,q + s(s - 1)[m + 1]p,q}}. \]
If we use Relations (32) and (34) to find the bound on |a_{2m+1}|, we obtain the following relation:
\[ -s[1 + m][1 + 2m]p,q a_{m+1}^2 + 2s[1 + 2m]p,q a_{2m+1} = (1 - e)(α_{2m} - β_{2m}), \quad (38) \]
or equivalently
\[ α_{2m+1} = \frac{(1 - e)(α_{2m} - β_{2m}) + (m + 1)2a_{m+1}}{2s[2m + 1]p,q}. \quad (39) \]

From Relation (36), if we substitute the value of α_{m+1}^2, we obtain
\[ a_{2m+1} = \frac{(1 - e)(α_{2m} - β_{2m}) + (m + 1)(1 - e)^2(α_m^2 + β_m^2)}{4s^2[m + 1]p,q}. \quad (40) \]
Now, if we apply Lemma 1 for the coefficients \(a_n, a_{2n}, \beta_m, \beta_{2m}\), we obtain

\[
|a_{2m+1}| \leq \frac{2(1-e)}{s[2m+1]_{p,q}} + \frac{2(m+1)(1-e)^2}{s^2[m+1]_{p,q}}.
\]

From Relations (37) and (39) applying Lemma 1, we obtain

\[
|a_{2m+1}| \leq \frac{2(m+1)(1-e)}{s(s-1)[m+1]_{p,q} + (m+1)s[2m+1]_{p,q}} + \frac{2(1-e)}{s[2m+1]_{p,q}}.
\]

\(\square\)

**Remark 7.** For one fold case \((m = 1)\) and \(s = 1\) in Theorem 2, we obtain the results given in [29].

**Remark 8.** For one fold case, in Theorem 2, choosing \(p = 1, q \to 1^-\), we obtain the following corollary.

**Corollary 1.** [24] Let the function \(f \in FS_\Sigma(e), (s = 1, 0 \leq e < 1, (z, w) \in U)\) be given by (1). Then,

\[
|a_2| \leq \sqrt{\frac{2(1-e)}{3}}
\]

and

\[
|a_3| \leq \frac{(1-e)(5-3e)}{3}.
\]

In the following theorems, we provide the Fekete-Szegö type inequalities for the functions of the families \(FS_{\Sigma, m}^{p,q,e}\) and \(FS_{\Sigma,m}^{p,q,e}\).

**Theorem 3.** Let \(f\) be a function of the form (3) in the class \(FS_{\Sigma, m}^{p,q,e}\). Then,

\[
|a_{2m+1} - \sigma a_{m+1}^2| \leq \begin{cases} \frac{2d}{s[2m+1]_{p,q}}, & |t(\sigma)| \leq \frac{1}{|s[2m+1]_{p,q}|}, \\ \frac{4sd|t(\sigma)|}{|s[2m+1]_{p,q}|}, & |t(\sigma)| \geq \frac{1}{|s[2m+1]_{p,q}|}, \end{cases}
\]

where

\[
t(\sigma) = \frac{d(m+1-2\sigma)}{2s[m+1]_{p,q}^2(s-d) + 2s(m+1)d[2m+1]_{p,q}}.
\]

**Proof.** We want to calculate \(a_{2m+1} - \sigma a_{m+1}^2\).

For this, from Relations (22) and (24), where we know the values of the coefficients \(a_{2m+1}^2\) and \(a_{2m+1}\):

\[
a_{2m+1}^2 = \frac{d^2(a_{2m} + \beta_{2m})}{s[m+1]_{p,q}^2(s-d) + (m+1)sd[2m+1]_{p,q}},
\]

\[
a_{2m+1} = \frac{(m+1)d^2(a_{m}^2 + \beta_{m}^2)}{4s^2[m+1]_{p,q}^2} + \frac{d(a_{2m} - \beta_{2m})}{2s[2m+1]_{p,q}}.
\]

It follows that

\[
a_{2m+1} - \sigma a_{m+1}^2 = \frac{d(m+1-2\sigma)}{2s[2m+1]_{p,q}} + \frac{d(m+1-2\sigma)}{2s[m+1]_{p,q}^2(s-d) + 2sd[2m+1]_{p,q} - \frac{1}{2s[2m+1]_{p,q}}},
\]
According to Lemma 1 and after some computations, we obtain

\[
|a_{2m+1} - \sigma a_{m+1}^2| \leq \begin{cases} 
\frac{2d}{s|2m+1|_{p,q}}, & |t(\sigma)| \leq \frac{1}{s|2m+1|_{p,q}} \\
4s|l(\sigma)|, & |t(\sigma)| \geq \frac{1}{s|2m+1|_{p,q}}.
\end{cases}
\]

\[\square\]

**Theorem 4.** Let \( f \) be a function of the form (3) in the class \( FS_{\Sigma,m}^{p,q,s,h,r} \). Then,

\[
|a_{2m+1} - \sigma a_{m+1}^2| \leq \begin{cases} 
\frac{2(1-\epsilon)}{s|2m+1|_{p,q}}, & |t(\sigma)| \leq \frac{1}{s|2m+1|_{p,q}} \\
4s(1-\epsilon)|l(\sigma)|, & |t(\sigma)| \geq \frac{1}{s|2m+1|_{p,q}}.
\end{cases}
\]

where

\[t(\sigma) = \frac{(1-2\sigma + m)}{2s(s-1)d[m+1]_{p,q} + 2s(m+1)[2m+1]_{p,q}}.\]

**Proof.** We will compute \( a_{2m+1} - \sigma a_{m+1}^2 \), using the values of the coefficients \( a_{m+1}^2 \) and \( a_{2m+1} \) given in Relations (37) and (39).

It follows that

\[
a_{2m+1} - \sigma a_{m+1}^2 = (1-\epsilon)[a_{2m}(\frac{1}{2s|2m+1|_{p,q}}) + \frac{1-2\sigma + m}{2s(s-1)d[m+1]_{p,q} + 2s(m+1)[2m+1]_{p,q}}] + \beta_{2m}(\frac{1}{2s(s-1)d[m+1]_{p,q} + 2s(m+1)[2m+1]_{p,q}}) - \frac{1}{2s[m+1]_{p,q}}.
\]

According to Lemma 1 and after some computations, we obtain

\[
|a_{2m+1} - \sigma a_{m+1}^2| \leq \begin{cases} 
\frac{2(1-\epsilon)}{s|2m+1|_{p,q}}, & |t(\sigma)| \leq \frac{1}{s|2m+1|_{p,q}} \\
4s(1-\epsilon)|l(\sigma)|, & |t(\sigma)| \geq \frac{1}{s|2m+1|_{p,q}}.
\end{cases}
\]

\[\square\]

**Definition 5.** Let \( h, r : U \rightarrow \mathbb{C} \) be analytic functions and \( \min\{\text{Re}(h(z)), \text{Re}(r(z))\} > 0 \), where \( z \in U, h(0) = r(0) = 1 \).

A function \( f \) given by (3) is said to be in the class \( FS_{\Sigma,m}^{p,q,s,h,r} \), where \( s \geq 1, 0 < q < p \leq 1, m \in \mathbb{N} \) if the conditions are satisfied:

\[(D_{p,q}f(z))^q \in h(U), z \in U \quad (43)\]

and

\[(D_{p,q}g(w))^q \in r(U), w \in U, \quad (44)\]

where the function \( g \) is given by (4).

We obtain coefficient bounds for the functions class \( FS_{\Sigma,m}^{p,q,s,h,r} \) in the following theorem.

**Theorem 5.** Let the function \( f \) given by (3) be in the class \( FS_{\Sigma,m}^{p,q,s,h,r} \). Then,

\[
|a_{m+1}| \leq \min\{\sqrt{\frac{|h'_0(0)|^2 + |r'_1(0)|^2}{2s^2[m+1]_{p,q}}}, \sqrt{\frac{|h''_0(0)| + |r''_1(0)|}{s(s-1)[m+1]_{p,q} + s(m+1)[2m+1]_{p,q}}}; \quad (45)
\]
\[ |a_{2m+1}| \leq \min \left\{ \frac{|h'(0)|^2 + |r'(0)|^2}{4\sigma[m+1]_{p,q}^2} + \frac{|h''(0)| + |r''(0)|}{2s[2m+1]_{p,q}}, \right. \\
\left. \frac{|h''(0)| + |r''(0)|}{2s[2m+1]_{p,q}} + \frac{(m+1)(|h''(0)| + |r''(0)|)}{2s\left( (m+1)|2m+1|_{p,q} + (s-1)|m+1|_{p,q} \right)_{p,q}} \right\}. \quad (46) \]

**Proof.** In Relations (43) and (44), the equivalent forms of the argument inequalities are

\[(D_{p,q}f(z))^s = h(z), \quad (47)\]

and

\[(D_{p,q}g(w))^s = r(w), \quad (48)\]

where \( h(z) \) and \( r(w) \) satisfy the conditions from Definition 5, and have the following Taylor–Maclaurin series expansions:

\[ h(z) = 1 + h_1z^2 + h_2z^4 + \ldots \quad (49) \]

\[ r(w) = 1 + r_1w^2 + r_2w^4 + \ldots \quad (50) \]

If we substitute (49) and (50) into (47) and (48), respectively, and equate the coefficients, we obtain

\[ s[m+1]_{p,q}a_{m+1} = h_1; \quad (51) \]

\[ s[2m+1]_{p,q}a_{2m+1} + \frac{s(s-1)}{2}[m+1]_{p,q}^2a_{2m+1} = h_2; \quad (52) \]

\[ -s[m+1]_{p,q}a_{m+1} = r_1; \quad (53) \]

\[ -s[2m+1]_{p,q}a_{2m+1} + (s(m+1)[2m+1]_{p,q} + \frac{s(s-1)}{2}[m+1]_{p,q}^2)a_{2m+1} = r_2. \quad (54) \]

We obtain that

\[ h_1 = -r_1 \quad (55) \]

and

\[ h_1^2 + r_1^2 = 2s^2[m+1]_{p,q}^2a_{m+1}^2 \quad (56) \]

from Relations (51) and (53).

Adding Relations (52) and (54), we obtain that

\[ a_{m+1}^2\left\{s(s-1)[m+1]_{p,q} + s(m+1)[2m+1]_{p,q} \right\} = h_2 + r_2. \quad (57) \]

Now, from (56) and (57), we obtain

\[ a_{m+1}^2 = \frac{h_1^2 + r_1^2}{2s^2[m+1]_{p,q}^2} \quad (58) \]

\[ a_{m+1}^2 = \frac{h_2 + r_2}{s(s-1)[m+1]_{p,q} + s(m+1)[2m+1]_{p,q}}. \quad (59) \]

We obtain from Relations (58) and (59) that

\[ |a_{m+1}| \leq \frac{|h_1'(0)|^2 + |r_1'(0)|^2}{2s^2[m+1]_{p,q}^2} \]

and

\[ |a_{m+1}| \leq \frac{|h_2'(0)| + |r_2'(0)|}{s(s-1)[m+1]_{p,q} + s(m+1)[2m+1]_{p,q}}. \]

So, we obtain the estimate on the coefficient \( |a_{m+1}| \) as in (45).
Next, subtracting (54) from (52), we obtain the following relation:

\[ 2s[2m + 1]_{p,q}a_{2m+1} - s(m + 1)[2m + 1]_{p,q}a_{m+1}^2 = h_2 - r_2. \]  

(60)

Substituting the value of \( a_{m+1}^2 \) from (58) into (60), it follows that

\[ a_{2m+1} = \frac{h_2 - r_2}{2s[2m + 1]_{p,q}} + \frac{(m + 1)(h_2^2 + r_2^2)}{4s^2[m + 1]_{p,q}^2}. \]

Therefore,

\[ |a_{2m+1}| \leq \frac{\left(|h'(0)|^2 + |r'(0)|^2\right)(m + 1) + |h''(0)| + |r''(0)|}{4s^2[m + 1]_{p,q}^2}. \]

Upon substituting the value of \( a_{m+1}^2 \) from (59) into (60), it follows that

\[ a_{2m+1} = \frac{h_2 - r_2}{2s[2m + 1]_{p,q}} + \frac{(m + 1)(h_2 + r_2)}{(s - 1)[m + 1]_{p,q} + (m + 1)[2m + 1]_{p,q}}. \]

So, it follows that

\[ |a_{2m+1}| \leq \frac{|h''(0)| + |r''(0)|}{2s\left\{(m + 1)\frac{2m + 1}{p,q} + (s - 1)\frac{2m + 1}{p,q}\right\}}. \]

□

3. Conclusions

As future research directions, the symmetry properties of this operator, the \((p,q)\)-derivative operator, can be studied.

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