NONSEPARABLE SPACEABILITY AND STRONG ALGEBRABILITY OF SETS OF CONTINUOUS SINGULAR FUNCTIONS

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Abstract. Let CBV denote the Banach algebra of all continuous real-valued functions of bounded variation, defined in [0, 1]. We show that the set of strongly singular functions in CBV is nonseparably spaceable. We also prove that certain families of singular functions constitute strongly c-algebrable sets. The argument is based on a new general criterion of strong c-algebrability.

1. Introduction

In the last decade, much work was done in the study of subsets of vector spaces (topological vector spaces, normed spaces, Banach algebras, etc.) with no linear structure given a priori. This research was earlier initiated by Gurariy [13], [14] and then continued by several authors. See for instance [1], [2], [5], [6], [3], [11], [15].

Recall (see [1]) that, for a topological vector space V, its subset A is said to be:

- lineable if $A \cup \{0\}$ contains an infinite dimensional vector subspace $W$ of $V$ (here the topological structure of $V$ is not required); moreover, if $\dim W = \kappa$ then $A$ is called $\kappa$-lineable;
- spaceable if $A \cup \{0\}$ contains an infinite dimensional closed vector subspace $W$ of $V$; moreover, if $W$ is nonseparable, we say that $A$ is nonseparably spaceable.

One aim of our paper is to reexamine the spaceability of some families of singular functions contained in the Banach algebra CBV of all continuous functions from [0, 1] to $\mathbb{R}$ of bounded variation, endowed with the norm

$$||f|| = |f(0)| + \text{Var}(f)$$

where $\text{Var}(f) = \text{Var}_{[0,1]} f$ denotes the total variation of $f$ in [0, 1] and, in general, $\text{Var}_{[a,b]} f$ denotes the variation of $f$ in a subinterval $[a,b]$ of [0, 1]. The lineability and spaceability of certain subfamilies of CBV were studied in [5] and more recently, in [6]. Our main result going in this direction states that the set of strongly singular functions is nonseparably spaceable (Theorem 2). We heavily exploit a family of such functions known in the probability theory [8].

Another aim of our paper is to establish strong c-algebraability of some families of singular functions. Here $c$ denotes the cardinality of $\mathbb{R}$ (continuum). Algebraability and strong algebraability are associated with algebras, the structures richer than linear spaces. The notion of strong algebraability for various special subfamilies of CBV, $C[0,1]$ and $\mathbb{R}^{[0,1]}$ becomes interesting in

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light of some recent results obtained in this context. We propose a new general criterion of
strong $\kappa$-algebrability (Proposition \cite{7}, which is used in Theorems \cite{8} and \cite{12}).

Recall (see \cite{2}) that a subset $E$ of an algebra $A$ is algebrable if $E \cup \{0\}$ contains an infinitely
generated subalgebra $B$ of $A$. If $E$ is algebrable with the minimal set of generators of $B$ of
 cardinality $\kappa$, then $E$ is called $\kappa$-algebrable.

A strengthened notion of algebrability was introduced in \cite{3}. Given an infinite cardinal $\kappa$ and
a commutative algebra $A$, a subset $E$ of $A$ is called strongly $\kappa$-algebrable whenever there exists
a set $X = \{x_\alpha : \alpha < \kappa\} \subset E$ of free generators of a subalgebra $B \subset E \cup \{0\}$ (that is, the set
$\hat{X}$ of all elements of the form $x_{\alpha_1}^{k_1} x_{\alpha_2}^{k_2} \ldots x_{\alpha_n}^{k_n}$, with nonnegative integers $k_1, \ldots, k_n$ nonequal to 0
simultaneously, is linearly independent and all linear combinations of elements from $\hat{X}$ are in
$E \cup \{0\}$). A set $E \subset A$ is called strongly algebrable if it is strongly $\kappa$-algebrable for an infinite
$\kappa$, and it is densely strongly $\kappa$-algebrable if it is strongly $\kappa$-algebrable and the respective free
subalgebra is dense in $A$, provided that $A$ is a Banach algebra.

Let $\lambda$ stand for Lebesgue measure in $\mathbb{R}$. A continuous function $f : [a, b] \to \mathbb{R}$ of bounded
variation is said to be singular whenever it is not constant and $f' = 0$, $\lambda$-almost everywhere.
The classical Cantor function (see e.g. \cite{9}) is an example of nondecreasing singular function
deﬁned in $[0, 1]$. Also strictly increasing singular functions are known, see \cite{11} where a good
bibliography on this topic is presented. We will consider classes of singular functions inside
which some rich algebraic structures can be inscribed.

Note that CBV is a subspace of the Banach algebra BV of real-valued functions of bounded
variation on $[0, 1]$, endowed with the same norm. It is known that BV can be treated as the
space of ﬁnite signed Borel measures on $[0, 1]$. The distribution function $F_p(t) := \Pr(X \leq t)$, $t \in \mathbb{R}$,
associated with $\mu_p$ has the following properties (see \cite{3} \S31):

(i) Consider a birational number $t = k/2^n$, for $n \in \mathbb{N}$ and $k \in \{0, \ldots, 2^n - 1\}$, and the
terminating binary expansion $t = \sum_{k=1}^{n} u_k/2^k$ with $u_k \in \{0, 1\}$. Let $\ell(t)$ and $r(t)$
denote (respectively) the numbers of zeros and ones among $u_1, \ldots, u_n$. Then

$$
\mu_p \left[ t, t + \frac{1}{2^n} \right] = F_p \left( t + \frac{1}{2^n} \right) - F_p(t) = p^{\ell(t)}(1 - p)^{r(t)}.
$$

It follows that

$$
\mu_p \left[ t, t + \frac{1}{2^n} + \frac{1}{2^{n+1}} \right] = p^{\ell(t)+1}(1 - p)^{r(t)}
$$

$$
\mu_p \left[ t + \frac{1}{2^n} - \frac{1}{2^n + 1}, t + \frac{1}{2^n} \right] = p^{\ell(t)}(1 - p)^{r(t)+1}.
$$

(ii) $F_p$ is continuous and strictly increasing on $[0, 1]$, $F_p(x) = 0$ for all $x \leq 0$ and $F_p(x) = 1$
for all $x \geq 1$. 
(iii) If $x \in (0, 1)$ and $F'_p(x)$ exists then $F''_p(x) = 0$. As $F_p$ is monotone, $F'_p$ exists $\lambda$-almost everywhere in $[0, 1]$ and consequently, $F'_p = 0$, $\lambda$-almost everywhere in $[0, 1]$.

(iv) For any distinct $p, q \in (0, 1/2)$ there are disjoint Borel sets $B_p, B_q \subset [0, 1]$ such that $\mu_p(B_p) = 1$, $\mu_q(B_q) = 1$. In other words, the measures $\mu_p$ and $\mu_q$ have disjoint supports. For the proof, see [8, Example 31.3].

In the sequel, we will consider the functions $F_p$, $p \in (0, 1/2)$, restricted to $[0, 1]$. The following result belongs to mathematical folklore. The analogue of its second part is valid for any two Borel probability measures on $[0,1]$ having disjoint supports.

**Fact 1.** The space CBV is nonseparable. This is witnessed by the condition $||F_p - F_q|| = 2$ for any distinct $p, q \in (0, 1/2)$.

**Proof.** Consider any distinct $p, q \in (0, 1/2)$. Pick $B_p$ and $B_q$ as in (iv). Clearly, $||F_p - F_q|| = \text{Var}_{[0,1]}(F_p - F_q) \leq 2$. Let $\varepsilon \in (0, 1/4)$. Pick closed sets $C_p \subset B_p$ and $C_q \subset B_q$ such that $\mu_p(C_p) \geq 1 - \varepsilon$ and $\mu_q(C_q) \geq 1 - \varepsilon$. Choose disjoint sets $G_p \supset C_p$ and $G_q \supset C_q$ that are open in $[0, 1]$. Let $(I_n)$ and $(J_n)$ stand for the sequences of all connected components of $G_p$ and $G_q$, respectively. Then $\mu_p(G_p) \geq 1 - \varepsilon$, that is $\sum_n \text{Var}_{cl(I_n)} F_p \geq 1 - \varepsilon$, and $\mu_p(G_q) \leq \varepsilon$, that is $\sum_n \text{Var}_{cl(J_n)} F_p \leq \varepsilon$. The analogous inequalities hold for measure $\mu_q$. So,

$$1 - \varepsilon \leq \sum_n \text{Var}_{cl(I_n)} F_p = \sum_n \text{Var}_{cl(I_n)} ((F_p - F_q) + F_q) \leq \sum_n \text{Var}_{cl(I_n)} (F_p - F_q) + \varepsilon.$$

Consequently,

$$\sum_n \text{Var}_{cl(I_n)} (F_p - F_q) \geq 1 - 2\varepsilon$$

and analogously,

$$\sum_n \text{Var}_{cl(J_n)} (F_p - F_q) \geq 1 - 2\varepsilon.$$

Hence we have

$$||F_p - F_q|| = \text{Var}_{[0,1]}(F_p - F_q) \geq \sum_n \text{Var}_{cl(I_n)} (F_p - F_q) + \sum_n \text{Var}_{cl(J_n)} (F_p - F_q) \geq 2 - 4\varepsilon.$$

Letting $\varepsilon \to 0$ we obtain the assertion. \qed

2. **Nonseparable spaceability of the set of strongly singular functions**

A singular function $f \in \text{CBV}$ will be called *strongly singular* whenever its restriction to every subinterval of $[0, 1]$ is singular. In other words, $f \in \text{CBV}$ is strongly singular whenever $f' = 0$ almost everywhere and $f$ is not constant in every interval. Every function $F_p$, $p \in (0, 1/2)$, considered in Section 1 is strongly singular.

For each strongly singular function $f$ and any subinterval $[a, b] \subset [0, 1]$, $a < b$, we have the following properties:

(I) $f$ is nondifferentiable somewhere in $[a, b]$;

(II) $f$ is not absolutely continuous in $[a, b]$ since otherwise, $0 = f(d) - f(c) = \int_c^d f'$ for all $c, d \in [a, b]$, $c < d$, (cf. [12 Thm 4.14]);

(III) $f$ is not Lipschitz in $[a, b]$ since every Lipschitz function is absolutely continuous in a given interval.

A main result of this section is the following.
Lemma 3. Each function from \( E \), for any \( 0 < p \) have (for the first inequality, see \([10, \text{Thm } 224 \text{ I}]\) that the set of nonabsolutely continuous functions in CBV is spaceable.

An interesting example of nonseparably spaceable subset of BV has been established recently in \([11, \text{Thm } 3.1]\). Our result is an essential strengthening of \([6, \text{Thm } 4.1]\) where it was proved that the set of nonabsolutely continuous functions in CBV is spaceable.

The proof of Theorem 2 will be divided into several lemmas. Let \( W = \text{span}\{F_p; p \in (0, 1/2)\} \), that is, \( W \) denotes the linear subspace of CBV generated by all functions \( F_p, p \in (0, 1/2) \).

Lemma 3. Each function from \( W \) is not constant in every subinterval \( I \) of \([0, 1]\).

Proof. By induction with respect to \( k \in \mathbb{N} \), we will show a more general condition stating that for any \( 0 < p_1 < \cdots < p_k < 1/2 \) and \( a_i \neq 0 \) with \( i = 1, \ldots, k \), and for every interval \( I \) of the form \( I = [(j/2^n, (j + 1)/2^n) \text{ with } n \in \mathbb{N}, j = 0, \ldots, 2^n - 1, \) we have

\[
\sum_{i=1}^{k} a_i \mu_{p_i}(I) \neq 0.
\]

Suppose that we have proved this statement and suppose that there exists \( f \in W \) which is constant in some subinterval of \([0, 1]\). Then \( f \) is of the form

\[
f = \sum_{i=1}^{k} a_i F_{p_i} \text{ where } 0 < p_1 < \cdots < p_k < \frac{1}{2} \text{ and } a_i \neq 0 \text{ with } i = 1, \ldots, k.
\]

We may assume that \( f \) is constant in an interval \( I \) of the form as above. Then we get \( \sum_{i=1}^{k} a_i \mu_{p_i}(I) = 0 \) which is impossible.

To start the induction, observe that our statement for \( k = 1 \) is obvious. Assume that this is true for a number \( k \in \mathbb{N} \). Consider \( \sum_{i=1}^{k+1} a_i \mu_{p_i} \), where \( 0 < p_1 < \cdots < p_{k+1} < 1/2 \) and \( a_i \neq 0 \) with \( i = 1, \ldots, k+1 \). Fix an interval \( I = [(j/2^n, (j + 1)/2^n) \text{ with } n \in \mathbb{N} \text{ and } j = 0, \ldots, 2^n - 1. \)

Suppose that \( \sum_{i=1}^{k+1} a_i \mu_{p_i}(I) = 0 \). Consider \( J = [(2j)/2^{n+1}, (2j + 1)/2^{n+1}], \) the left half of \( I \). We then have \( \mu_{p_i}(J) = \mu_{p_i}(I) \) for \( i = 1, \ldots, k+1 \) (see property (i)). Using this, we obtain

\[
\sum_{i=1}^{k+1} a_i \mu_{p_i}(J) = \sum_{i=1}^{k+1} a_i \mu_{p_i}(I) = p_{k+1} \sum_{i=1}^{k+1} a_i \mu_{p_i}(I) + \sum_{i=1}^{k} a_i (p_i - p_{k+1}) \mu_{p_i}(I)
\]

\[
= \sum_{i=1}^{k} a_i (p_i - p_{k+1}) \mu_{p_i}(I).
\]

Since \( \sum_{i=1}^{k} a_i (p_i - p_{k+1}) \mu_{p_i}(I) \neq 0 \) by the induction hypothesis, we obtain a contradiction. \( \square \)

Lemma 4. If \( f \) belongs to the closure \( \text{cl}(W) \) of \( W \) in CBV then \( f' \) = 0 almost everywhere in \([0, 1]\).

Proof. Assume that \( f_n \to f \) in the norm of CBV, for some sequence \( (f_n) \) of functions from \( W \).

For each \( n \in \mathbb{N} \), let \( D_n = \{ x \in [0, 1]; f'_n(x) = 0 \} \). Then \( \lambda(D_n) = 1 \). Since \( f \in \text{CBV} \), the derivative \( f' \) exists almost everywhere in \([0, 1]\). Suppose that \( |f'| > 0 \) in a set \( E \) of positive Lebesgue measure. Then there exists \( k \in \mathbb{N} \) such that \( \lambda(E_k) = \alpha > 0 \) where \( E_k = \{ x \in E; |f'(x)| > 1/k \} \). It follows that \( |(f - f_n)'(x)| > 1/k \) for all \( x \in E_k \cap D_n \) and \( n \in \mathbb{N} \). Now, we have (for the first inequality, see \([10, \text{Thm } 224 \text{ I}]\))

\[
\text{Var}_{[0,1]}(f - f_n) \geq \int_{[0,1]} |(f - f_n)'| \geq \int_{E_k \cap D_n} |(f - f_n)'| \geq \frac{\alpha}{k}.
\]
for all $n \in \mathbb{N}$, which contradicts $\|f_n - f\| \to 0$. \hfill \qed

\textbf{Lemma 5.} Consider arbitrary birational numbers $t_0 = i_0/2^{n_0}$ and $t_1 = i_1/2^{n_1}$ from $[0, 1)$ such that $n_1 \geq n_0$, $\ell(t_1) \geq \ell(t_0)$ and $r(t_1) \geq r(t_0)$ (see property (i)). Put $I_0 = [t_0, t_0 + 1/2^{n_0}]$ and $I_1 = [t_1, t_1 + 1/2^{n_1}]$. Then there exists a subinterval $J = [j/2^{n_1}, (j + 1)/2^{n_1}+1]$ (with $j \in \mathbb{N}$) of $I_0$ such that $\mu_p(J) = \mu_p(I_1)$ for each $p \in (0, 1/2)$. Moreover, for any real numbers $\alpha, \beta \in I_1$, $\alpha < \beta$, there exists a subinterval $[\alpha_1, \beta_1]$ of $I_0$ such that $\mu_p([\alpha_1, \beta_1]) = \mu_p([\alpha, \beta])$ for each $p \in (0, 1/2)$.

\begin{proof}
Note that $n_0 = \ell(t_0) + r(t_0)$ and $n_1 = \ell(t_1) + r(t_1)$. Let $m = \ell(t_1) - \ell(t_0)$ and $k = r(t_1) - r(t_0)$. We make $m + k$ divisions into halves of consecutive intervals. We start from $I_0$, choosing left halves $m$ times and then choosing right halves $k$ times. After that we obtain the interval

$$J = \left[ t_0 + \frac{1}{2^{n_0+m}} - \frac{1}{2^{n_0+m+k}}, t_0 + \frac{1}{2^{n_0+m}} \right].$$

Then using several times the final part of property (i), we have

$$\mu_p(J) = p^{\ell(t_0)+m}(1-p)^{r(t_0)+k} = p^{\ell(t_1)-(1-p)r(t_1)} = \mu_p(I_1)$$

for each $p \in (0, 1/2)$, as desired.

To prove the second assertion, let $t_2 = \min J$ and note that $\ell(t_2) = \ell(t_1)$, $r(t_2) = r(t_1)$. Let $x = t_2 - t_1$ and consider a subinterval $I$ of $I_1$ of the form

$$\left[ \frac{i}{2^n}, \frac{i+1}{2^n} \right] \text{ for } n \geq n_1 \text{ and } i \in \{0, \ldots, 2^n - 1\}.$$\hfill (1)

Denote $I + x = \{t + x : t \in I\}$. Then $I + x \subseteq J$. Observe that $\min I$ and $\min(I + x)$ have the same numbers of zeros and ones in their terminating binary expansions. So by property (i) we have

$$\mu_p(I + x) = \mu_p(I) \text{ for each } p \in \left(0, \frac{1}{2}\right).$$\hfill (2)

Now, let $\alpha, \beta \in I_1$, $\alpha < \beta$. Then $[\alpha, \beta]$ can be expressed as a countable union of intervals of the form $\text{(1)}$ with pairwise disjoint interiors. Since every measure $\mu_p$ vanishes on singletons, from $\text{(2)}$ it follows that $\mu_p([\alpha, \beta] + x) = \mu_p([\alpha, \beta])$ for each $p \in (0, 1/2)$. Thus we put $\alpha_1 = \alpha + x$, $\beta_1 = \beta + x$ and we obtain the assertion. \hfill \qed

\textbf{Lemma 6.} If $f \in \text{cl}(W)$ is constant in some subinterval of $[0, 1]$ then $f$ is equal to 0 in $[0, 1]$.

\begin{proof}
Let $f \in \text{cl}(W)$ be constant in some subinterval of $[0, 1]$. If $f$ is constant in $[0, 1]$ then since $f(0) = 0$, we have $f = 0$ in $[0, 1]$ which yields the assertion. So, suppose that $f$ is not constant in $[0, 1]$. Pick a subinterval $I_0 = [i_0/2^{n_0}, (i_0 + 1)/2^{n_0}]$ of $[0, 1]$ such that $f|_{I_0}$ is constant and $I_0$ is the longest among such subintervals of $[0, 1]$. Let $f|_{I_0} = c$.

Since $f$ is not constant in $[0, 1]$, we can choose $I = [i/2^n, (i+1)/2^n]$ such that $I \subseteq [0, 1]$, $n \geq n_0$ and $f|_I$ is not constant. We are going to find a subinterval $I_1 = [i_1/2^{n_1}, (i_1 + 1)/2^{n_1}]$ of $[0, 1]$ such that $n_1 \geq n_0$, $f|_{I_1}$ is not constant, and additionally

$$\ell(i_1/2^{n_1}) \geq \ell(i_0/2^{n_0}), \quad r(i_1/2^{n_1}) \geq r(i_0/2^{n_0}).$$\hfill (3)

If the interval $I$ satisfies $\text{(3)}$, put $I_1 = I$. Otherwise, observe that $n \geq n_0$ implies that for $i_1 = i$ and $n_1 = n$ at most one of inequalities in $\text{(3)}$ can fail. Assume, for instance, that
will also consider exponential like functions (of the same form) with domain \([0, 1/2^n]\). (The case \(r(i/2^n) < r(i_0/2^{n_0})\) is similar.) We will find a subinterval \(J\) of \(I\) which can be taken as \(I_1\). Namely, since \(f_j\) is not constant, pick \(a, b \in I, a < b\), such that \(f(a) \neq f(b)\). The set \(A = (f_j|_{[a,b]})^{-1}[\{f(a)\}]\) is compact and define \(z = \max A\). Then \(f\) cannot be constant in any interval \([z, z + \delta]\) with \(z + \delta < b\) and \(\delta > 0\). If \(z\) is birational, pick \(m \in \mathbb{N}\) such that \(z + 1/2^m < b\) and put \(J = [z, z + 1/2^m]\). If \(z\) is not birational, consider its binary expansion \(z = \sum_j z_j/2^j\) and pick \(k \in \mathbb{N}\) so large that for \(z' = \sum_{j=1}^k z_j/2^j\) we have \(a \leq z'\) and \(z' + 1/2^{k+1} < b\). Then put \(J = [z', z' + 1/2^{k+1}]\).

Having \(I_1\) defined, pick \(x, y \in I_1, x < y\), such that \(f(x) \neq f(y)\) and let \(\varepsilon = |f(x) - f(y)|\).

Since \(f \in \text{cl}(W)\), choose \(g \in W\) such that \(|g - f| < \varepsilon/4\). In particular

\[
|g(x) - f(x)| < \frac{\varepsilon}{4}, \quad |g(y) - f(y)| < \frac{\varepsilon}{4} \quad \text{and} \quad |g(t) - c| < \frac{\varepsilon}{4}
\]

for all \(t \in I_0\).

It follows that \(|g(y) - g(x)| > \varepsilon/2\). By Lemma 3, pick an interval \([\alpha, \beta] \subset I_0\) such that

\[
\mu_p([\alpha, \beta]) = \mu_p([x, y]) \quad \text{for each} \quad p \in \left(0, \frac{1}{2}\right).
\]

Since \(g \in W\), we can write

\[
g = \sum_{i=1}^k s_i F_{p_i} \quad \text{where} \quad 0 < p_1 < p_2 < \cdots < p_k < \frac{1}{2} \quad \text{with} \quad s_i \neq 0 \quad \text{for} \quad i = 1, \ldots, k.
\]

Let \(g^+ = \sum_{s_i > 0} s_i F_{p_i}\) and \(g^- = \sum_{s_i < 0} s_i F_{p_i}\). Then \(g = g^+ - g^-\) and from (11) it follows that

\[
g^+(\beta) - g^+(\alpha) = g^+(y) - g^+(x) \quad \text{and} \quad g^-(\beta) - g^-(\alpha) = g^-(y) - g^-(x)
\]

which implies that \(g(\beta) - g(\alpha) = g(y) - g(x)\). Hence

\[
\frac{\varepsilon}{2} < |g(y) - g(x)| = |g(\beta) - g(\alpha)| \leq |g(\beta) - c| + |c - g(\alpha)| < \frac{\varepsilon}{2}
\]

which yields a contradiction. \(\square\)

**Proof of Theorem 2** From Lemma 3, it follows that the functions \(F_{p_i}, p \in (0, 1/2]\), are linearly independent. We have defined \(W = \text{span}\{F_{p_i}, p \in (0, 1/2]\}\). Observe that \(\text{cl}(W)\) is a closed vector subspace of \(\text{CBV}\) which is additionally nonseparable by Fact 1. By Lemmas 4 and 5, the set \(\text{cl}(W) \setminus \{0\}\) consists of strongly singular functions. \(\square\)

3. **Strong \(\varepsilon\)-algebrability of some sets of singular functions**

We say that a function \(f: \mathbb{R} \rightarrow \mathbb{R}\) is exponential like (of range \(m\)) whenever

\[
f(x) = \sum_{i=1}^m a_i e^{\beta_i x}, \quad x \in \mathbb{R},
\]

for some distinct nonzero real numbers \(\beta_1, \ldots, \beta_m\) and some nonzero real numbers \(a_1, \ldots, a_m\). We will also consider exponential like functions (of the same form) with domain \([0, 1]\).

Our general criterion of strong \(\varepsilon\)-algebrability is the following.

**Proposition 7.** Given a family \(\mathcal{F} \subset [0,1]\), assume that there exists a function \(F \in \mathcal{F}\) such that \(f \circ F \in \mathcal{F} \setminus \{0\}\) for every exponential like function \(f: \mathbb{R} \rightarrow \mathbb{R}\). Then \(\mathcal{F}\) is strongly \(\varepsilon\)-algebrable. More exactly, if \(H \subset \mathbb{R}\) is a set of cardinality \(\kappa\), linearly independent over the rationals \(\mathbb{Q}\), then \(\exp (rF), r \in H\), are free generators of an algebra contained in \(\mathcal{F} \cup \{0\}\).
Fix a set $H$ of cardinality $c$, linearly independent over the rationals $\mathbb{Q}$. By the assumption, \( \exp \circ (rF) \in \mathcal{F} \) for all $r \in H$. To show that \( \exp \circ (rF), r \in H \), are free generators of an algebra contained in $\mathcal{F} \cup \{0\}$, consider any $n \in \mathbb{N}$ and a non-zero polynomial $P$ in $n$ variables without a constant term. Then the function given by

\[
 x \mapsto P(e^{r_1 F(x)}, e^{r_2 F(x)}, \ldots, e^{r_n F(x)}), \quad x \in [0, 1],
\]

is of the form

\[
 (6) \quad \sum_{i=1}^{m} a_i \left( e^{r_1 F(x)} \right)^{k_{i1}} \left( e^{r_2 F(x)} \right)^{k_{i2}} \cdots \left( e^{r_n F(x)} \right)^{k_{in}} = \sum_{i=1}^{m} a_i \exp \left( F(x) \sum_{j=1}^{n} r_j k_{ij} \right)
\]

where $a_1, \ldots, a_m$ are nonzero real numbers and the matrix $[k_{ij}]_{i \leq m, j \leq n}$ has distinct nonzero rows, with $k_{ij} \in \{0, 1, 2, \ldots\}$. Since the function $t \mapsto \sum_{i=1}^{m} a_i \exp(t \sum_{j=1}^{n} r_j k_{ij})$ is exponential like, from (6) and the assumption it follows that the function (6) is in $\mathcal{F} \setminus \{0\}$. (For technical details concerning the role of the set $H$, compare with [4] where a similar technique was used.)

Note that the functions of type $e^{\beta x}$ were used to show the lineability of various sets of functions, then the generators of the respective linear subspace were of the form $F(x)e^{\beta x}$ with the respectively chosen function $F$ from the considered set. (See e.g. [9, 13].) In Proposition 7 instead of multiplication, we use superposition of $e^{\beta x}$ with $F$, in aim to show the (strong) algebrability of the considered set. This new idea will be used below in Theorems 10 and 12.

**Lemma 8.** For every positive integer $n$, any exponential like function $f: [0, 1] \to \mathbb{R}$ of range $m$, and each $c \in \mathbb{R}$, the preimage $f^{-1}([c])$ has at most $m$ elements. Consequently, $f$ is not constant in every subinterval of $[0, 1]$.

**Proof.** We proceed by induction. If $m = 1$, the function $f$ is of the form $f(x) = ae^{\beta x}$, $x \in [0, 1]$, with $a \neq 0$ and $\beta \neq 0$. So $f$ is strictly monotone and the property is obvious.

Assume that the property holds for all exponential like functions of range $m$. Let $f(x) = \sum_{i=1}^{m+1} a_i e^{\beta_i x}$, $x \in [0, 1]$, for some distinct nonzero real numbers $\beta_1, \ldots, \beta_{m+1}$ and some nonzero real numbers $a_1, \ldots, a_{m+1}$. Consider the derivative

\[
 f'(x) = \sum_{i=1}^{m+1} \beta_i a_i e^{\beta_i x} = e^{\beta_1 x} \left( \sum_{i=2}^{m+1} \beta_i a_i e^{(\beta_i - \beta_1)x} \right), \quad x \in [0, 1].
\]

Note that $\gamma_i = \beta_i - \beta_1$, for $i = 2, \ldots, m+1$, are nonzero distinct real numbers. So, we may apply the induction hypothesis to $g(x) = \sum_{i=2}^{m+1} \beta_i a_i e^{\gamma_i x}$, $x \in [0, 1]$, and $c = -\beta_1 a_1$. This shows that $(f')^{-1}([0])$ has at most $m$ elements. Hence $f$ has at most $m + 2$ local extrema on $[0, 1]$ (we should take into account one-sided extrema at 0 and 1 where maybe $f'$ does not vanish). This implies that for each $c \in \mathbb{R}$, the preimage $f^{-1}([c])$ has at most $m + 1$ elements, as desired.

**Theorem 9.** The set of strongly singular functions in CBV is strongly $c$-algebraic.

**Proof.** Fix a strongly singular function $F \in \text{CBV}$. For instance, let $F$ be the distribution function $F_{1/4}$ considered in the previous sections. It suffices to check that the assumption of Proposition 7 is valid with $F$ equal to the set of strongly singular functions. Consider an exponent like function
$f$, given by \([5]\), for some distinct nonzero real numbers $\beta_1, \ldots, \beta_m$ and some nonzero real numbers $a_1, \ldots, a_m$. Since $F' = 0$ almost everywhere in $[0, 1]$, we have

$$(f \circ F)'(x) = F'(x) \sum_{i=1}^{m} a_i \beta_i e^{\beta_i F(x)} = 0$$

for almost all $x \in [0, 1]$.

Suppose that $f \circ F$ is constant in some subinterval $[c, d]$ of $[0, 1]$ with $c < d$. Since $F^{-1}$ is a continuous increasing bijection from $[0, 1]$ onto $[0, 1]$, the function $f = (f \circ F) \circ F^{-1}$ is constant in the interval $[F(c), F(d)]$, which contradicts Lemma [8].

Note that $\beta$ is the largest among the cardinalities $\kappa$ which can yield $\kappa$-algebrability of strongly singular functions. By property (III) of strongly singular functions, our Theorem [9] implies the recent result [15, Thm 2.1] stating that the set of continuous functions on $[0, 1]$, which are a.e. differentiable, with a.e. bounded derivative and are not Lipschitz, is $\beta$-lineable.

By virtue of Lemma [4], the set of strongly singular functions in CBV cannot be densely algebrable. On the other hand, the set of strongly singular functions can be considered a subset of the Banach algebra $C[0, 1]$ of continuous functions from $[0, 1]$ to $\mathbb{R}$, with the supremum norm, and we have the following result.

**Theorem 10.** The set of strongly singular functions is a densely strongly $\beta$-algebrable subset of $C[0, 1]$.

**Proof.** In the proof of the previous theorem, by the use of Proposition [4], we have obtained a free algebra $A$ contained in the $\mathcal{F} \cup \{0\}$ where $\mathcal{F}$ is the set of strongly singular functions. According to Proposition [4], $\exp \circ (rF), r \in H$, are free generators of $A$. Now, we additionally assume that $H$ contains the terms of a sequence $(r_n)_{n \geq 1}$ convergent to 0. Thanks to this, the closure $\text{cl}(A)$ of $A$ in $C[0, 1]$ contains all constant functions since if $c \in \mathbb{R}$ then the sequence $(c \exp \circ (r_n F))_{n \geq 1}$ converges uniformly to $c$. Let $\mathcal{A}^*$ be the algebra generated by the constant functions and the functions from $A$. Then $\text{cl}(\mathcal{A}^*) = \text{cl}(A)$. By the Stone-Weierstrass theorem we have $\text{cl}(\mathcal{A}^*) = C[0, 1]$, so $A$ is dense in $C[0, 1]$, as desired.

In the proof of the next theorem, the following elementary lemma will be needed. Every continuous function defined on an interval will be treated as 0 times differentiable.

**Lemma 11.** Given functions $f$ and $g$ from $(a, b)$ into $\mathbb{R}$, and an integer $n \geq 1$, assume that $f$ and $fg$ are $n$ times differentiable, $g$ is $n - 1$ times differentiable and $f$ does not vanish in $(a, b)$. Then $g$ is $n$ times differentiable.

**Proof.** We use induction. The case if $n = 1$ is clear since $g = (fg)/f$. Assume that the property is valid for a fixed $n \geq 1$. Suppose that $f$ and $fg$ are $n + 1$ times differentiable, $g$ is $n$ times differentiable and $f$ does not vanish in $(a, b)$. Hence $(fg)' = f'g + g'f$ is $n$ times differentiable, and so is $f'g$. This implies that $g'f$ is $n$ times differentiable. By the induction hypothesis, $g'$ is $n$ times differentiable and the proof is finished.

By $C_n$ we denote the algebra of functions from $[0, 1]$ into $\mathbb{R}$ that have $n$ continuous derivatives in $[0, 1]$. 

Theorem 12. Given $n \geq 1$, the set of functions of class $C_n$ that do not have derivative of order $n + 1$ somewhere in any open subinterval of $[0, 1]$ is strongly $c$-algebraically.

Proof. Consider $F = F_{1/4}$, the strongly singular function used before. We will use it to construct an increasing function $G : [0, 1] \to \mathbb{R}$ of class $C_n$ which does not have derivative of order $n + 1$ somewhere in any open subinterval of $[0, 1]$. For $n = 1$, $G(x) = \int_0^x F, x \in [0, 1]$, is good. Having the respective function $\tilde{G}$ for a number $n$, we define $G(x) = \int_0^x \tilde{G}, x \in [0, 1]$, which is good for $n + 1$.

We will apply Proposition 7 with the set $F$ of functions in class $C_n$ that do not have derivative of order $n + 1$ somewhere in any open subinterval of $[0, 1]$. Namely, we will check that $f \circ G \in F \setminus \{0\}$ for every exponential like function $f : \mathbb{R} \to \mathbb{R}$. Fix an exponential like function $f$ given by (5). Note that the functions $f$ and $f'$ are infinitely differentiable and, by Lemma 11, they have finitely many zeros in any bounded interval. Clearly, $f \circ G \in C_n$. We will show that $f \circ G$ does not have derivative of order $n + 1$ somewhere in $(a, b)$. We may assume that $f$ and $f'$ have no zeros in $(a, b)$. Suppose that $f \circ G$ is $n + 1$ times differentiable in $(a, b)$.

Since
\[
(f \circ G)'(x) = (f' \circ G)(x) \cdot G'(x), \quad x \in (a, b),
\]
$f' \circ G$ is $n$ times differentiable in $(a, b)$ and it does not vanish in $(a, b)$, we infer by Lemma 11 that $G'$ is $n$ times differentiable in $(a, b)$. This is a contradiction. □

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