Univalence and holomorphic extension of the solution to \( \omega \)-controlled Loewner–Kufarev equations

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Abstract

We prove that a solution to the \( \omega \)-controlled Loewner–Kufarev equation, which was introduced by the first two authors, exists uniquely, is univalent on the unit disk and can be extended holomorphically across the unit circle.

Keywords. controlled Loewner–Kufarev equation, control function, univalence, holomorphic extension

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1 Introduction

In various fields of mathematics, univalent functions play important roles. Not just are they fundamental objects in geometric function theory and in Teichmüller theory but also have deep connections to conformal field theory, to integrable systems and even to random matrices. The second author 
\(^4\) gave a concise picture of such connections in terms of the Schramm–Loewner evolution and (Sato–)Segal–Wilson Grassmannian, and then Markina and
Vasil’ev [6, 7] proposed an extension of his approach introducing the alternate Loewner–Kufarev equation.

In order to generalize the above results further, the first two authors introduced the notion of controlled Loewner–Kufarev equation

\[ df_t(z) = zf_t'(z) \{ dx_0(t) + d\xi(x, z)_t \}, \quad t \in [0, T], \quad f_0(z) = z \in \mathbb{D}, \]

in their previous paper [1]. Here \( x = (x_1, x_2, \ldots) \), \( \xi(x, z)_t = \sum_{n=1}^{\infty} x_n(t)z^n \), and \( x_n(t), n \geq 0 \), are complex-valued continuous functions of bounded variation. As is described in [1, Section 3.2], the solution \((f_t)_{0 \leq t \leq T}\) is embedded into the Segal–Wilson Grassmannian through Krichever’s construction if it is univalent on the unit disk \( \mathbb{D} \) and extends holomorphically across \( \partial \mathbb{D} \). Moreover, the first two authors gave a sufficient condition [2, Theorem 3.3] for the embedded solution to be continuous as a curve in the Grassmannian. In this theorem, they consider the case in which the driving functions \( x_0 \) and \( x \) are controlled by a control function \( \omega: \{ (s, t) \mid 0 \leq s \leq t \leq T \} \rightarrow \mathbb{R}_+ \) [2, Definition 3.2] which satisfies \( \omega(0, T) < 1/8 \).

It should be noted that, in the paper [2], they assume a priori that \( f_t \) is univalent on \( \mathbb{D} \) and extends to a holomorphic function on an open neighbourhood of \( \mathbb{D} \). However, we expect that this property is intrinsic to a large class of controlled Loewner–Kufarev equations. The purpose of the present article is to confirm this belief in the \( \omega \)-controlled case. The goal is the following theorem:

**Theorem 1.1.** Let \( \alpha \) be the unique real solution to the cubic equation \( 2x^3 - 6x^2 + 7x - 2 = 0 \), and suppose that \( \omega(0, T) < \alpha/4 \). Then a holomorphic solution \((f_t)_{0 \leq t \leq T}\) to the \( \omega \)-controlled Loewner–Kufarev equation exists uniquely. Moreover, \( f_t \) is univalent on \( \mathbb{D} \) and extends to a holomorphic function on an open neighbourhood of \( \mathbb{D} \) for each \( t \in [0, T] \).

The rest of this article is organized as follows: In Section 2, we recite the basic concepts for our argument. The assumptions on the driving functions are mentioned in Section 2.1 and then the definition of a solution to controlled Loewner–Kufarev equation is given in Section 2.2. Although these concepts appear in the previous papers [1, 2], we summarize them to confirm the terminology. In Section 3, we prove Theorem 1.1 through four steps, namely, the uniqueness (Theorem 3.1), existence (Theorem 3.2), holomorphic extension (Corollary 3.3) and univalence (Theorem 3.4) of the solution.

We use the following notation: For a continuous function \( F: [0, T] \rightarrow \mathbb{C} \) of bounded variation, the atomless measure \( dF \) on \([0, T]\) is defined by the relation \( dF((a, b]) = F(b) - F(a) \). Its total variation is denoted by \(|dF|\).
2 Setting

In this section, we describe our setting throughout this paper.

2.1 Driving functions

The driving functions $x_0: [0, T] \to \mathbb{R}$ and $x_n: [0, T] \to \mathbb{C}$, $n \geq 1$, are continuous functions of bounded variation. For $x := (x_1, x_2, \ldots)$, we define the formal power series

$$\xi(x, z) := \sum_{n=1}^{\infty} x_n(t) z^n$$

and assume the following:

(i) $x_0(0) = 0$;

(ii) The series $\xi(x, z)_0$ has convergence radius one;

(iii) $\sum_{n=1}^{\infty} |dx_n|([0, T]) r^n$ converges for all $r \in (0, 1)$.

We note that, for each $z \in D$, the series $\sum_{n=1}^{\infty} dx_n(t) z^n$ converges with respect to the total variation norm on the space of complex measures on $[0, T]$ from the third condition.

Lemma 2.1. Under the assumptions (i)–(iii) above, the series $\xi(x, z)_t$ has convergence radius one for each $t \in [0, T]$. The family $(\xi(x, z)_t)_{0 \leq t \leq T}$ of holomorphic functions on $D$ is continuous in the topology of locally uniform convergence. Moreover, the function $t \mapsto \xi(x, z)_t$ is of bounded variation and satisfies

$$d\xi(x, z)_t = \sum_{n=1}^{\infty} dx_n(t) z^n$$

for each $z \in D$.

Proof. For any $t \in [0, T]$ and $r \in (0, 1)$, we have

$$\sum_{n=1}^{\infty} |x_n(t)| r^n \leq \sum_{n=1}^{\infty} |x_n(t) - x_n(0)| r^n + \sum_{n=1}^{\infty} |x_n(0)| r^n$$

$$\leq \sum_{n=1}^{\infty} |dx_n|([0, t]) r^n + \sum_{n=1}^{\infty} |x_n(0)| r^n < \infty.$$

\[A\] A slightly stronger condition is assumed in $[\text{1}, \text{Definition 2.1 (2)}]$ to compute the Faber polynomials and Grunsky coefficients. For our purpose, the present condition is sufficient.
Hence \( \xi(x, z)_t \) has convergence radius one.

Similarly, for \( 0 \leq s \leq t \leq T \) and \( r \in (0, 1) \), we get

\[
\sup_{|z| \leq r} |\xi(x, z)_t - \xi(x, z)_s| \leq \sum_{n=1}^{\infty} |x_n(t) - x_n(s)| r^n \leq \sum_{n=1}^{\infty} |dx_n|((s, t)) r^n.
\]

The last expression goes to zero as \( |t - s| \to 0 \) from the dominated convergence theorem. Thus, \( \xi(x, z)_t \) is continuous in the topology of locally uniform convergence on \( \mathbb{D} \).

Finally, we observe that \( \xi(x, z)_t \) is of bounded variation and has the expression (2.1) from the relation

\[
\xi(x, z)_t - \xi(x, z)_s = \sum_{n=1}^{\infty} dx_n((s, t)) z^n = \left( \sum_{n=1}^{\infty} dx_n(\cdot) z^n \right)((s, t)). \quad \square
\]

### 2.2 Controlled Loewner–Kufarev equation

Let \((f_t)_{0 \leq t \leq T}\) be a family of holomorphic functions on \( \mathbb{D} \). We temporarily assume that

(I) \( t \mapsto f'_t(z) \) is measurable for each \( z \in \mathbb{D} \), and \((f'_t)_{0 \leq t \leq T}\) is locally bounded on \( \mathbb{D} \).

Under this assumption, let us suppose that the controlled Loewner–Kufarev equation

\[
f_t(z) - z = \int_0^t z f'_s(z) \{ dx_0(s) + d\xi(x, z)_s \}, \quad t \in [0, T], \; z \in \mathbb{D}, \quad (2.2)
\]

holds for the driving path \((x_0, x)\) in Section 2.1. The assumption (I) ensures that the last integral is well-defined. Moreover, by arguing in the same way as in [1, Remark 2.2 (b)], we see that

(II) \((f'_t)_{0 \leq t \leq T}\) is continuous with respect to the locally uniform convergence on \( \mathbb{D} \).

From Cauchy’s integral formula, this property implies that

(III) \((f'_t)_{0 \leq t \leq T}\) is continuous with respect to the locally uniform convergence on \( \mathbb{D} \),

which is obviously a stronger property than (I). Thus, as far as the solutions to (2.2) are concerned, the conditions (I)–(III) are mutually equivalent.
Taking the discussion above into account, we say that a family \((f_t)_{0 \leq t \leq T}\) of holomorphic functions on \(D\) is a \((\text{holomorphic})\) solution to the controlled Loewner–Kufarev equation driven by \((x_0, x)\) if it is continuous in the topology of locally uniform convergence and satisfies \((2.2)\).

In Section 3 except in Theorem 3.1, we consider a holomorphic solution to the \(\omega\)-controlled Loewner–Kufarev equation. The control function \(\omega: \{(s, t) \mid 0 \leq s \leq t \leq T\} \to \mathbb{R}_+\) is a continuous function with superadditivity

\[0 \leq s \leq t \leq u \leq T \Rightarrow \omega(s, t) + \omega(t, u) \leq \omega(s, u)\]

and vanishes on the diagonal, i.e., \(\omega(t, t) = 0\) (e.g. \([5\text{ Section 2.2}]\)). The driving functions \(x_0\) and \(x = (x_n)_{n \geq 1}\) are assumed to be controlled by \(\omega\) in the sense of \([2\text{ Definition 3.2}]\). We do not restate the exact definition of the term “\(\omega\)-controlled”, for it is not directly used in this article.

3 Main results

3.1 Existence, uniqueness and holomorphic extension across the unit circle

In this subsection, we prove the existence and uniqueness of a solution to the \(\omega\)-controlled Loewner–Kufarev equation and, as a byproduct, the fact that the solution can be extended holomorphically across \(\partial D\). We use the setting in Section 2.

Only in the next theorem, the equation is not assumed to be \(\omega\)-controlled:

**Theorem 3.1.** A solution to the controlled Loewner–Kufarev equation driven by \((x_0, x)\) is unique (if it exists).

*Proof.* Let \((f_t)_{0 \leq t \leq T}\) be a solution. As it is continuous and \(f'_t(0) = 1\), the quantity \(C(t) := f'_t(0)\) is non-zero up to a certain time \(\tilde{T} \in (0, T]\). We can write the Taylor expansion of \(f_t\), \(t \in [0, \tilde{T})\), around the origin as

\[f_t(z) = C(t)(z + c_1(t)z^2 + c_2(t)z^3 + \cdots).\]  

(3.1)

The coefficients \(C(t)\) and \(c_n(t)\), \(n \geq 1\), are all continuous functions owing to Cauchy’s integral formula. By substituting this expression into \((2.2)\), we have the recursive relations

\[C(t) - 1 = \int_0^t C(s) \, dx_0(s)\]  

(3.2)
and
\[
c_n(t) = \int_0^t \left\{ \sum_{k=0}^{n-1} (k+1)c_k(s) \, dx_{n-k}(s) + nc_n(s) \, dx_0(s) \right\}
\]  
(3.3)
for \( t \in [0, \tilde{T}] \). Here, we put \( c_0(t) := 1 \). The equations (3.2) and (3.3) are exactly those in [1, Proposition 2.6]. It follows from the usual iteration method (see e.g. [9, Proposition 0.4.7]) that a continuous function \( C(t) \) that satisfies (3.2) exists uniquely and given by
\[
C(t) = e^{x_0(t) - x_0(0)}
\]
for all \( t \in [0, T] \), and (3.1), (3.3) are valid for all \( t \).

Now, we prove the uniqueness of continuous functions \( c_n(t) \), \( n \geq 1 \), that satisfies (3.3) by induction. Let \( n \geq 2 \) and assume that \( c_k(t) \), \( 1 \leq k \leq n-1 \), are unique. Suppose that there are two continuous functions \( c_{1,n}(t) \) and \( c_{2,n}(t) \) both satisfying (3.3). Then by taking their difference, we obtain the equation
\[
c_{1,n}(t) - c_{2,n}(t) = n \int_0^t (c_{1,n}(s) - c_{2,n}(s)) \, dx_0(s),
\]
which has a unique solution \( c_{1,n}(t) - c_{2,n}(t) \equiv 0 \) from [9, Proposition 0.4.7]. Hence \( c_n(t) \) is unique. The initial case \( n = 1 \) is proven in the same way.

In this way, we have proven the uniqueness of all coefficients \( C(t) \) and \( c_n(t) \), \( n \geq 1 \), which implies that of \((f_t)_{0 \leq t \leq T}\).

The following theorem is already established implicitly in [2, Corollary 4.4]:

**Theorem 3.2.** Let \( \omega \) be a control function with \( \omega(0, T) < 1/4 \). Then there exists a solution to the \( \omega \)-controlled Loewner–Kufarev equation.

**Proof.** We put \( C(t) = e^{x_0(t) - x_0(0)} \) and define \( c_n(t) \), \( n \geq 1 \), by the relation given in [1, Theorem 2.8]. They are a (unique) solution to the system of equations (3.2) and (3.3). If the series \( \sum_{n=1}^{\infty} c_{n-1}(t) z^n \) has convergence radius not less than one, then by reversing the direction of the proof of Theorem 3.1, the family \((f_t)_{t \in [0, T]}\) defined by (3.1) is proven to be a solution to the controlled Loewner–Kufarev equation.

By using the assumption that the driving path \((x_0, x)\) is controlled by \( \omega \), we get the inequality
\[
|c_n(t)| \leq 4^{-1} n (4\omega(0, T))^n.
\]  
(3.4)
See [2, Appendix A.1] for its proof. We easily see from (3.4) that \( \sum_{n=1}^{\infty} c_{n-1}(t) z^n \) has convergence radius greater than one if \( \omega(0, T) < 1/4 \).

In the last line of the proof of Theorem 3.2, the Taylor series (3.1) of \( f_t \) has convergence radius strictly greater than one. Thus, we obtain the following corollary:
Corollary 3.3. Let $\omega$ be a control function with $\omega(0, T) < 1/4$ and $(f_t)_{0 \leq t \leq T}$ be a unique solution to the $\omega$-controlled Loewner–Kufarev equation. Then for each $t \in [0, T]$, the function $f_t$ can be extended holomorphically to an open neighbourhood of $\mathbb{D}$.

3.2 Univalence on the unit disk

In this subsection, we prove that the solution to the $\omega$-controlled Loewner–Kufarev equation is univalent. Let $\alpha$ be the unique real solution to the cubic equation $2x^3 - 6x^2 + 7x - 2 = 0$. Note that $\alpha/4 \approx 0.102 \cdots < 1/4$.

Theorem 3.4. Let $\omega$ be a control function with $\omega(0, T) \leq \alpha/4$ and $(f_t)_{0 \leq t \leq T}$ be a unique solution to the $\omega$-controlled Loewner–Kufarev equation. Then the function $f_t$ is univalent on $\mathbb{D}$ for each $t \in [0, T]$.

Proof. It is well known in geometric function theory that a normalized holomorphic function $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ is univalent on $\mathbb{D}$ if $\sum_{n=2}^{\infty} |a_n| \leq 1$. See Example 2.2 in Chapter 2 of Pommerenke [8] or Exercise 24 in Chapter 2 of Duren [3] for instance. We apply this sufficient condition to the series $z + \sum_{n=2}^{\infty} c_{n-1}(t) z^n$. Here, $c_n(t)$’s are defined as in the proof of Theorem 3.2. In this case, the sufficient condition for the univalence is

$$\sum_{n=2}^{\infty} n|c_{n-1}(t)| \leq 1. \quad (3.5)$$

The left-hand side of this inequality is estimated as follows:

$$\sum_{n=2}^{\infty} n|c_{n-1}(t)| \leq \frac{1}{4} \sum_{n=2}^{\infty} n(n-1)(4\omega(0, T))^{n-1} = \frac{1}{4} \cdot \frac{2 \cdot 4\omega(0, T)}{(1 - 4\omega(0, T))^3}.$$

We can easily check that the last fraction is not greater than one if and only if $\omega(0, T) \leq \alpha/4$. Hence (3.5) holds under the present assumption.

Combining Theorems 3.1, 3.2, 3.4 and Corollary 3.3 yields our goal Theorem 1.1.

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