Equivariant K-theoretic enumerative invariants and wall-crossing formulae in abelian categories

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Abstract

We provide a general framework for wall-crossing of equivariant K-theoretic enumerative invariants of appropriate moduli stacks $\mathcal{M}$, by lifting Joyce’s homological universal wall-crossing [Joy21] to K-theory and to include equivariance. The primary new tool is that the operational K-homology of $\mathcal{M}$ is an equivariant multiplicative vertex algebra.

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1 Introduction

Enumerative geometry may be phrased very generally as the problem of counting objects in a $\mathbb{C}$-linear abelian category $\mathcal{A}$. We primarily imagine $\mathcal{A}$ to be the category of coherent sheaves on a smooth projective $\mathbb{C}$-scheme $X$ of dimension $\leq 3$. To connect the enumerative problem with geometry, assume $\mathcal{A}$ has an associated algebraic moduli stack $\mathcal{M} = \bigsqcup_\alpha \mathcal{M}_\alpha$. For a (weak) stability condition $\tau$ on $\mathcal{A}$, the $\tau$-stable (resp. $\tau$-semistable) objects form substacks (resp. open substacks) $\mathcal{M}_\alpha^{\text{st}}(\tau) \subset \mathcal{M}_\alpha^{\text{sst}}(\tau) \subset \mathcal{M}_\alpha^{\text{pl}}$ of a $\mathbb{C}^\times$-rigidification of $\mathcal{M}_\alpha$, for each $\alpha$.

If $\alpha$ has no strictly $\tau$-semistable objects, i.e. $\mathcal{M}_\alpha^{\text{st}}(\tau') = \mathcal{M}_\alpha^{\text{sst}}(\tau')$, and $\mathcal{M}_\alpha^{\text{pl}}$ has a perfect obstruction theory, then following $[BF97]$ there is a virtual fundamental class 

$$[\mathcal{M}_\alpha^{\text{sst}}(\tau)]^{\text{vir}} \in A_*(\mathcal{M}_\alpha^{\text{pl}}(\tau); \mathbb{Q}).$$

Pairing it with tautological classes and integrating produces important enumerative quantities invariant under deformations of $X$, though the virtual class itself is more fundamental than any set of such invariants.

This setup already encompasses entire worlds within modern enumerative geometry, particularly Donaldson-type theories; see $[Moc09, MNOP06, CMT22]$ for a sample. Notably, however, it excludes Gromov–Witten theory, whose underlying category of stable maps is not abelian.

For a class $\alpha$, some or even all (weak) stability conditions $\tau$ may have strictly semistable objects, i.e. $\mathcal{M}_\alpha^{\text{st}}(\tau') \subsetneq \mathcal{M}_\alpha^{\text{sst}}(\tau')$. In a continuous family of (weak) stability conditions, when the $\tau$ with strictly semistables form codimension-1 walls, virtual classes are defined and locally constant away from walls but may change upon crossing a wall. Many interesting correspondences between different enumerative setups, e.g. DT/PT $[Tod10]$, can be viewed as wall-crossing formulae. To study general wall-crossings, in $[Joy21]$ Joyce constructs classes

$$[\mathcal{M}_\alpha^{\text{sst}}(\tau)]^{\text{vir}} \in H_*(\mathcal{M}_\alpha^{\text{pl}}; \mathbb{Q})$$

for all $\alpha$ and $\tau$ inductively using auxiliary categories of pairs in the style of Joyce–Song $[JS12]$. There are then universal, though somewhat unwieldy, formulae describing their behavior under variation of $\tau$, involving a Lie bracket on the homology group.
Importantly, the homology groups where these classes live have very different functoriality properties than, say, Chow groups. While Borel–Moore-type homology theories have only proper pushforwards and flat pullbacks, Joyce’s construction crucially relies on the arbitrary pushforwards present in homology.

The goal of this paper is to port Joyce’s constructions back into the language of Borel–Moore-type homologies, in particular into a dual of equivariant algebraic K-theory. K-theory is a multiplicative refinement of ordinary cohomology, and equivariance allows for powerful tools such as localization with respect to a torus $T$ acting on $\mathcal{M}$, so e.g. $X$ may now be quasi-projective; see [Oko17] for a salient introduction. To accomplish this goal, we introduce two new ingredients which may also be of independent interest.

- We define the equivariant operational K-homology group $K^T_0(\mathcal{M})$ of an algebraic stack; this is approximately the dual of the K-group $K^T_+(\mathcal{M})$ of $T$-equivariant perfect complexes on $\mathcal{M}$, with a finiteness condition, and serves as our K-theoretic analogue of $H_*(\mathcal{M}; \mathbb{Q})$. Besides having arbitrary pushforwards, it is also the natural home for “universal” K-theoretic enumerative invariants, namely functions $Z_\alpha(\tau)$ which take a perfect complex on $\mathcal{M}$ (e.g. a tautological bundle), tensors it with the virtual structure sheaf of $\mathcal{M}^{\text{sst}}_\alpha(\tau)$ whenever it exists, and pushes forward to the base.

- We define and put the structure of an equivariant multiplicative vertex algebra on $K^T_0(\mathcal{M})$, and generalize Borcherd’s construction [Bor86, §4] to induce a Lie bracket on a certain completion $K^T_0(\mathcal{M}^{\text{pl}})$. This is our analogue of the ordinary vertex algebra structure on $H_*(\mathcal{M}; \mathbb{Q})$ and induced Lie bracket on $H_*(\mathcal{M}^{\text{pl}}; \mathbb{Q})$ [Joy21, §4.2]. Equivariance and multiplicativity means that the vertex operation now has poles over the torus $T$. In particular, the OPE of two fields $A(z)$ and $B(w)$ has poles at $z/w = t_i$ for a finite number of equivariant weights $t_i$ which may depend on $A$ and $B$, in contrast to the ordinary case of poles only at $z - w = 0$.

With these two ingredients in hand, Joyce’s construction of (1.1) proceeds almost verbatim to give K-theoretic classes

$$Z_\alpha(\tau) \in K^T_0(\mathcal{M}^{\text{pl}}_\alpha)^\wedge \otimes_{\mathbb{Z}} \mathbb{Q}$$

for all $\alpha$ and $\tau$. These classes satisfy analogues of all the wall-crossing formulae in [Joy21], though their construction is the main focus of this paper, and the Lie bracket used to define them nicely encapsulates much of the combinatorial complexity inherent to many wall-crossing setups. Many variations on this construction are possible; we explore one in a sequel paper [Liu23], for $\mathcal{M}$ with a symmetric obstruction theory, in order to construct refined semistable Vafa–Witten invariants and prove the main conjecture of [Tho20].
1.1 Outline of the paper

Section 2 sets the stage. In §2.1 we state the necessary hypotheses and constructions on the moduli stack $\mathcal{M}$, including its $\mathbb{C}^\times$-rigidification $\mathcal{M}^\text{pl}$ and equivariant K-groups $K_T(\mathcal{M})$ and $K^G_T(\mathcal{M})$. In §2.2 we define the equivariant operational K-homology $K_T^\epsilon(\mathcal{M})$ of an algebraic stack and the universal invariants $Z_\alpha(\tau)$. In §2.3 we compute $K^\epsilon_T(\mathcal{M})$ of the classifying stack $[*/\mathbb{C}^\times]$, both for use in multiplicative vertex algebras and also for comparison with the K-homology of its topological realization $BU(1) = \mathbb{CP}^\infty$.

Section 3 presents the equivariant and multiplicative analogue of ordinary vertex algebras. This is the algebraic object which controls our K-theoretic enumerative invariants. In §3.1 we give the general definition and properties, in complete analogy with the theory of ordinary vertex algebras, including an induced Lie bracket $[-,-]$ on a suitable quotient. In §3.2 we use the assumptions on $\mathcal{M}$ to construct the vertex operation on $K^\epsilon_T(\mathcal{M})$ which makes it an equivariant multiplicative vertex algebra, inducing the Lie bracket on $K^\epsilon_T(\mathcal{M}^\text{pl}) \wedge$ used in wall-crossing formulae. In §3.3 we observe that the same construction on $K_T(\mathcal{M})$, whenever well-defined, makes it into a holomorphic equivariant multiplicative vertex algebra. Under an extra assumption, these correspond to graded algebras and should be compared with K-theoretic Hall algebras.

Section 4 is mostly an exposition of the construction [Joy21, Theorem 5.7] of semistable invariants, but rewritten (in slightly less generality, for simplicity) in our K-theoretic framework. Several components of Joyce’s big machine have been black-boxed and what remains is only the construction of an auxiliary category/stack of pairs, in §4.1, and the key geometric argument using localization on a master space for these pairs, in §4.2. Hopefully this makes for an accessible introduction to [Joy21].

Appendix A is a discussion and characterization of the K-theoretic residue map which mediates the passage from the equivariant multiplicative vertex algebra $K^\epsilon_T(\mathcal{M})$ to the Lie algebra $K^\epsilon_T(\mathcal{M}^\text{pl}) \wedge$.

1.2 Acknowledgements

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2 Setup

2.1 The stack and K-theory

2.1.1 Fix a \( \mathbb{C} \)-linear additive category \( \mathcal{B} \) with moduli stack \( \mathcal{M} = \bigsqcup \mathcal{M}_\alpha \) which is Artin and locally of finite type, with \( \alpha \) ranging over some quotient of the Grothendieck group \( K_0(\mathcal{A}) \). Assume that \( \mathcal{M}_0 = \{ [0] \} \) contains only the zero object, and that there are direct sum and scalar multiplication maps

\[
\Phi_{\alpha, \beta} : \mathcal{M}_\alpha \times \mathcal{M}_\beta \to \mathcal{M}_{\alpha + \beta}
\]

\[
\Psi_\alpha : \ast / \mathbb{C}^* \times \mathcal{M}_\alpha \to \mathcal{M}_\alpha,
\]

making \( \mathcal{M} \) a monoid object with \( \ast / \mathbb{C}^* \)-action. Assume also that there is a perfect complex \( E^* \) on \( \mathcal{M} \times \mathcal{M} \) which is bilinear and weight \( \pm 1 \) in its factors, in the sense that

\[
(\Phi \times \text{id})^*(E^*) = \pi_{13}^*(E^*) \oplus \pi_{23}^*(E^*)
\]

\[
(\Psi \times \text{id})^*(E^*) = \pi_1^*(L_{\ast / \mathbb{C}^*}) \otimes \pi_2^*(E^*)
\]

\[
(\text{id} \times \Phi)^*(E^*) = \pi_{12}^*(E^*) \oplus \pi_{13}^*(E^*)
\]

\[
(\text{id} \times \Psi)^*(E^*) = \pi_2^*(L_{\ast / \mathbb{C}^*}) \otimes \pi_1^*(E^*)
\]

(2.1)

where \( \pi_i \) and \( \pi_{ij} \) are projections and \( L_{\ast / \mathbb{C}^*} \in \text{Pic}(\ast / \mathbb{C}^*) \) is the weight-1 representation. Its rank is therefore a bilinear pairing

\[
\chi(\alpha, \beta) := \text{rank } E_{\alpha, \beta}^*.
\]

2.1.2 Example. Suppose \( \mathcal{X} = \bigsqcup \mathcal{X}_\alpha \) is a moduli stack with the following two important properties:

(i) \( \mathcal{X} \) has maps \( \Phi \) and \( \Psi \) as in §2.1.1;

(ii) the obstruction theory \( D(\mathcal{F}, \mathcal{F}) \), at points \( [\mathcal{F}] \in \mathcal{X} \), is a bilinear form in \( \mathcal{F} \).

The prototypical examples are given by moduli stacks \( \mathcal{X}_\alpha \) of coherent sheaves of K-class \( \alpha \) on a scheme \( X \), for which \( \Phi \) (resp. \( \Psi \)) is the direct sum (resp. scaling automorphisms) of sheaves and \( D(\mathcal{F}, \mathcal{F}) := \text{Ext}^*_X(\mathcal{F}, \mathcal{F}) \). This is the setting of the wall-crossing applications in §4.

In this setting, \( E_{\alpha, \beta}^* \) is given by the bilinear version of \( D(\mathcal{F}, \mathcal{F}) \). Namely, the relative obstruction theory for \( \Phi_{\alpha, \beta} \) is (up to a degree shift)

\[
N_{\Phi_{\alpha, \beta}}^* = E_{\alpha, \beta}^* \oplus \sigma^* E_{\beta, \alpha}^*
\]

(2.2)
where the cross-term $E_{\alpha,\beta} \bullet$ has fiber $D(F_\alpha, F_\beta)$ at \([F_\alpha], [F_\beta]\) \in M_\alpha \times M_\beta$, and $\sigma: \mathcal{X} \times \mathcal{X}' \to \mathcal{X}' \times \mathcal{X}$ permutes the factors. We will omit writing $\sigma^*$ henceforth. The $E_{\alpha,\beta} \bullet$ in (2.2) is the desired complex and satisfies the conditions (2.1).

**Remark.** Properties (i) and (ii) also play a crucial role in many other constructions in geometric representation theory, notably the quantum loop group action on the equivariant K-theory of Nakajima quiver varieties [MO19]. Our multiplicative vertex algebras can be viewed as a compatible, homological dual to these quantum loop groups [Liu22].

2.1.3

Let $T = (\mathbb{C}^*)^n$ be a split algebraic torus, possibly trivial, acting on the original category of objects and therefore on its moduli stack $M$ in the sense of Romagny [Rom05]. Assume henceforth that all objects/morphisms/etc. are $T$-equivariant unless stated otherwise, particularly those from §2.1.1.

**Definition.** The $T$-equivariant (algebraic) K-theory of an algebraic stack $\mathcal{X}$ with $T$-action is the Grothendieck K-group

$$K_T(\mathcal{X}) := K_0(\text{Coh}_T(\mathcal{X}))$$

of $T$-equivariant coherent sheaves on $\mathcal{X}$. Let $K_T^\varphi(\mathcal{X}) := K_0(\text{Perf}_T(\mathcal{X}))$ be the K-group of $T$-equivariant perfect complexes. Both $K_T^\varphi$ and $K_T$ are modules for the representation ring

$$k := K_T(*) \cong \mathbb{Z}[t_1^\pm, \ldots, t_n^\pm].$$

Let $\text{char}(T)$ be the lattice of $T$-characters, identified with monomials in $k$. To use equivariant localization, e.g. virtual torus localization on Deligne–Mumford stacks [GP99], we sometimes pass to the localized base ring

$$k_{\text{loc}} := k \left[ \frac{1}{1 - t} : 1 \neq t \in \text{char}(T) \right]$$

and work in $K_T(\mathcal{X})_{\text{loc}} := K_T(\mathcal{X}) \otimes_k k_{\text{loc}}$ (and similarly for $K_T^\varphi(\mathcal{X})_{\text{loc}}$). For $T$-equivariant vector bundles $\mathcal{E}, \mathcal{E}' \in \text{Vect}_T(\mathcal{X})$ and a formal variable $s$, let

$$\wedge_{-s}(\mathcal{E}) := \sum (-s)^k \wedge^k \mathcal{E}, \quad \wedge_{-s}(\mathcal{E} - \mathcal{E}') := \wedge_{-s}(\mathcal{E}) / \wedge_{-s}(\mathcal{E}')$$

whenever the expression makes sense. When $s = 1$, this is the K-theoretic Euler class used in localization formulae.

For an excellent overview of equivariant K-theory, see [CG97, Chapter 5].
2.1.4

To be precise, the scalar multiplication map $\Psi$ acts on objects as the identity, and on stabilizer groups as $(\lambda, f) \mapsto \lambda f$. We refer to this $\mathbb{C}^\times$ action on stabilizers as the *scaling automorphism*.

**Definition.** Let $\mathcal{M}^\text{pl}$ be the $\mathbb{C}^\times$-rigidification [AOV08] of $\mathcal{M}$, meaning to quotient away the $\mathbb{C}^\times$ scaling from all stabilizer groups, so that the canonical map

$$\Pi^\text{pl}_\alpha : \mathcal{M}_\alpha \to \mathcal{M}^\text{pl}_\alpha$$

is a principal $[*/\mathbb{C}^\times]$-bundle for all $\alpha \neq 0$. The notation pl stands for *projective linear*, exemplified by $[*/\text{GL}(n)]^\text{pl} = [*/\text{PGL}(n)]$.

Group actions on stacks can be complicated because the group can act non-trivially on stabilizers. We assume that the $T$-action on $\mathcal{M}$ commutes with the scaling automorphism, and therefore descends to a $T$-action on $\mathcal{M}^\text{pl}$.

**Remark.** Principal $[*/\mathbb{C}^\times]$-bundles over a space $X$ are classified by elements $\lambda \in H^2(X, \mathcal{O}_X^\times)$, and their K-theory is often called the $\lambda$-*twisted* K-theory of $X$ [AS04].

2.1.5

An action $\Psi : [*/\mathbb{C}^\times] \times X \to X$ equips $K_T(X)$ with a $\mathbb{Z}$-grading, as follows. Recall that

$$K_T([*/\mathbb{C}^\times]) = K_{T \times \mathbb{C}^\times}([*) \cong k[L^\pm_{[*/\mathbb{C}^\times}].$$

**Definition.** Given a formal variable $z$, let $z^\text{deg}$ be the operator

$$z^\text{deg} : K_T(X) \xrightarrow{\Psi^*} K_T(X)[L^\pm_{[*/\mathbb{C}^\times}] \xrightarrow{L^\pm_{[*/\mathbb{C}^\times]} \cdot z} K_T(X)[z^\pm]. \quad (2.3)$$

Explicitly, if $\Psi^*(\mathcal{E}) = \bigoplus_{k \in \mathbb{Z}} \mathcal{L}^k \otimes \mathcal{E}_k$ then $\deg \mathcal{E}_k = k$. Consequently, $z^\text{deg}$ acts trivially on $K_T(\mathcal{M}^\text{pl})$, and the image of the pullback

$$(\Pi^\text{pl})^* : K_T(\mathcal{M}^\text{pl}) \to K_T(\mathcal{M})$$

consists of degree-0 elements. The same is true of $K_T^\pm(\mathcal{M})$.

For a product $X \times X'$, let $\Psi_i$ be the $[*/\mathbb{C}^\times]$-action on the $i$-th factor, and $\Psi$ be the induced diagonal $[*/\mathbb{C}^\times]$-action. Write

$$z^\text{deg} : K_T(X \times X') \to K_T(X \times X')[z^\pm]$$
to mean \( z^{\text{deg}} \) applied to the \( i \)-th factor, i.e. using \( \Psi_i \) instead of \( \Psi \) in (2.3). So \( z^{\text{deg}} = \prod z^{\text{deg},i} \).

**Example.** Write \( K_T([*/\text{GL}(n)]) = k[s_1^\pm, \ldots, s_n^\pm]^{S_n} \) where the superscript \( S_n \) means to take permutation invariants. Then \( \text{deg} s_i = 1 \) for all \( 1 \leq i \leq n \), and \( K_T([*/\text{PGL}(n)]) \) is generated as a \( k \)-algebra by symmetric Laurent polynomials in elements \( s_i/s_j \).

### 2.2 Operational K-homology

#### 2.2.1

Let \( \mathfrak{X} \) be an Artin stack with trivial \( T \)-action. There is a natural inclusion \( K^o(\mathfrak{X}) \hookrightarrow K_T^o(\mathfrak{X}) \) given by equipping non-equivariant sheaves with the trivial equivariant structure. Recall that \( K_T^o(\mathfrak{X}) \) is a ring under tensor product. Let

\[
I^o(\mathfrak{X}) := \langle \text{rank}(\mathcal{E}) - \mathcal{E} : \mathcal{E} \in \text{Perf}(\mathfrak{X}) \rangle \subset K^o(\mathfrak{X})
\]

denote the (non-equivariant) augmentation ideal of \( \mathfrak{X} \) consisting of rank-0 perfect complexes.

**Lemma.** Let \( \mathcal{E} \in \text{Vect}(\mathfrak{X}) \). If \( \mathfrak{X} = F \) is a scheme of finite type, then the operator \(-\otimes (\text{rank}(\mathcal{E}) - \mathcal{E})\) is nilpotent on \( K(F) \). If further \( F \) is quasi-projective, then

\[
I^o(F)^{\dim F + 1} = 0.
\]  

(2.4)

**Proof.** Since \( F \) is a scheme, \( \mathcal{E}|_U \cong \mathcal{O}_U^{\text{rank}(\mathcal{E})} \) for an open \( U \subset F \). Then

\[
\dim \text{supp} ((\text{rank}(\mathcal{E}) - \mathcal{E}) \otimes \mathcal{F}) < \dim \text{supp} \mathcal{F}
\]

for any \( \mathcal{F} \in \text{Coh}(F) \). (In contrast, this is not true for e.g. \( \mathcal{L}_{[*/\mathbb{C}^\times]} \) on \( [*/\mathbb{C}^\times] \).) If \( F \) is quasi-projective, \( K^o(F) \) is equal to the K-group \( K_0(\text{Vect}(F)) \) of vector bundles [Tot04]. □

Note that the operators \(-\otimes \mathcal{L}\) of multiplication by line bundles \( \mathcal{L} \) are therefore unipotent.

#### 2.2.2

**Definition.** Let \( \mathfrak{X} \) be an Artin stack with \( T \)-action. The equivariant operational K-homology \( K_T^o(\mathfrak{X}) \) consists of collections

\[
\phi := \{ K_T^o(\mathfrak{X} \times S) \xrightarrow{\phi_S} K_T^o(S) \}_{S}
\]

of homomorphisms of \( K_T^o(S) \)-modules, for all Artin stacks \( S \) with \( T \)-action which we call the base. These \( \{ \phi_S \}_{S} \) must obey the following axioms.
(i) (“Dependence only on $\mathfrak{X}$”) For any $T$-equivariant morphism $h: S \to T$ of Artin stacks, there is a commutative square

$$
\begin{array}{c}
K^\circ_T(\mathfrak{X} \times T) \\
\downarrow \phi_T
\end{array}
\begin{array}{cc}
\overset{(\text{id} \times h)^*}{\longrightarrow} & K^\circ_T(\mathfrak{X} \times S) \\
\downarrow \phi_S
\end{array}
\begin{array}{c}
K^\circ_T(T) \\
\downarrow h^*
\end{array}
\begin{array}{c}
\overset{\phi_T}{\longrightarrow} & K^\circ_T(S) \\
\downarrow \phi_S
\end{array}
\end{array}
$$

(ii) (Equivariant localization) There exists an Artin stack $\mathfrak{F} = \mathfrak{F}_\phi$ with trivial $T$-action and a $T$-equivariant morphism $\text{fix}_\phi: \mathfrak{F} \to \mathfrak{X}$ such that $\phi$ factors as

$$
\begin{array}{c}
K^\circ_T(\mathfrak{X} \times S) \\
\downarrow \phi_S
\end{array}
\begin{array}{cc}
\overset{\phi_T}{\longrightarrow} & K^\circ_T(S) \\
\downarrow \phi_T
\end{array}
\begin{array}{c}
K^\circ_T(\mathfrak{F} \times S) \\
\downarrow (\text{fix}_\phi \times \text{id})^*
\end{array}
\begin{array}{c}
\overset{\phi_T^\circ}{\longrightarrow} & K^\circ_T(\mathfrak{F} \times S) \\
\downarrow \phi^\circ_S
\end{array}
\end{array}
$$

for $K^\circ_T(S)$-linear maps $\{\phi^\circ_S\}_S$ which themselves satisfy all other axioms, i.e. forming an element $\phi^T \in K^\circ_T(\mathfrak{F})$.

(iii) (Finiteness condition) The Artin stack $\mathfrak{F} = \mathfrak{F}_\phi$ is either a quasi-projective scheme, or there is a surjection $\mathbb{E}: K(\mathfrak{F}) \otimes K(S) \to K(\mathfrak{F} \times S)$ for any Artin stack $S$ and

$$
\phi^T_{\{\ast\}}(I^\circ(\mathfrak{F})^\otimes N) = 0 \quad \forall N \gg 0. 
$$

(2.5)

Remark. When $\mathfrak{X} = X$ is a quasi-projective scheme, the latter two axioms are vacuous: the equivariant localization axiom is automatically satisfied via torus localization by taking $\mathfrak{F} = X^T$, and the finiteness condition follows from (2.4). In this setting, the definition and nomenclature for $K^\circ_T(X)$ should be compared to the operational $K$-theory of $[\text{AP15}]$, which is a bivariant theory in the sense of $[\text{FM81}]$ modeled on $K_T(X)$. Our $K$-homology arises from an analogous bivariant theory modeled on $K^\circ_T(X)$, applied to the map $X \to \ast$.

2.2.3

An important class of elements in $K^\circ_T(\mathfrak{M}_g^{pl})$ arises from enumerative invariants. Suppose that for a (weak) stability condition $\tau$ on $\mathfrak{M}_g$, there are no strictly semistable objects and the semistable locus $\mathfrak{M}_g^{ss}(\tau) \subset \mathfrak{M}_g^{pl}$ is a quasi-projective scheme carrying a perfect obstruction theory. This is the setting of the wall-crossing applications in §4.
Example ("Universal" enumerative invariants). Let $\mathcal{O}^{\text{vir}} \in K_T(\mathfrak{M}_\alpha^{\text{est}}(\tau))$ be the virtual structure sheaf. For simplicity, assume $\mathfrak{M}_\alpha^{\text{est}}(\tau)$ is proper. Let $\pi_\mathfrak{M}$ and $\pi_S$ be the projection from $\mathfrak{M}_\alpha^{\text{est}}(\tau) \times S$ to the $\mathfrak{M}$ and $S$ factors respectively. Then there is an element $Z_\alpha(\tau) \in K^T_0(\mathfrak{M}_\alpha^{\text{pl}})$ whose components are given by

$$Z_\alpha(\tau)_S: \mathcal{E} \mapsto (\pi_S)_* \left( \mathcal{E}|_{\mathfrak{M}_\alpha^{\text{est}}(\tau) \times S} \otimes \pi_{\mathfrak{M}}^* \mathcal{O}^{\text{vir}} \right).$$

(2.6)

We check that $Z_\alpha(\tau)$ is well-defined. First, the result lands in $K^\circ_T(S)$ since both $\mathcal{E}$ and $\pi_{\mathfrak{M}}^* \mathcal{O}^{\text{vir}}$ are clearly flat over $S$, the pushforward is proper, and Euler characteristic is constant in flat families. It is $K^\circ_T(S)$-linear by the projection formula. Finally, since $\mathfrak{M}_\alpha^{\text{est}}(\tau)$ is a scheme, torus localization applies and we can take $\mathfrak{F} = \mathfrak{M}_\alpha^{\text{est}}(\tau)^T$.

Note that for (2.6) to make sense, we only really need the $T$-fixed locus $\mathfrak{M}_\alpha^{\text{est}}(\tau)^T \subset \mathfrak{M}_\alpha^{\text{est}}(\tau)$ to be proper; see Example 2.2.9.

2.2.4

Many elements we work with in $K_0^T(\mathfrak{X})$ will be of the form (2.6). It will be convenient to use the shorthand notation

$$Z_\alpha(\tau) := \chi \left( \mathfrak{M}_\alpha^{\text{est}}(\tau), \mathcal{O}^{\text{vir}} \otimes - \right)$$

(2.7)

to avoid thinking about the base $S$ too much. By flat base change, the rhs is actually the fiber of (2.6) over any point $s \in S$ upon plugging in $\mathcal{E}|_{\mathfrak{M}_\alpha^{\text{est}} \times \{s\}}$. In the shorthand, the input $(\mathfrak{X})$ is understood to be automatically restricted to the locus where the pushforward $\chi$ takes place, e.g. $\mathfrak{M}_\alpha^{\text{est}}(\tau) \subset \mathfrak{M}_\alpha^{\text{pl}}$.

2.2.5

Some (functorial) properties of $K^\circ_T(\mathfrak{X})$ are inherited by $K_0^T(\mathfrak{X})$ in a sense. We list here some operations on $K_0^T(\mathfrak{X})$.

- A $T$-equivariant morphism $f: \mathfrak{X} \to \mathfrak{X}'$ induces a pullback $f^*: K^\circ_T(\mathfrak{X}' \times S) \to K^\circ_T(\mathfrak{X} \times S)$ for any $S$. It is easy to verify that then there is also a well-defined pushforward

$$f_*: K_0^T(\mathfrak{X}) \to K_0^T(\mathfrak{X}')$$

given by $(f_* \phi)_S(\mathcal{E}) := \phi_S(f^* \mathcal{E})$. 
• Via tensor product, $K^T_0(\mathcal{X})$ is a ring. So there is a cap product

$$\cap: K^T_0(\mathcal{X}) \times K^T_0(\mathcal{X}) \to K^T_0(\mathcal{X})$$

given by $(\phi \cap \mathcal{F})_S(\mathcal{E}) := \phi_S(\mathcal{E} \otimes \pi^*_X \mathcal{F})$. This is well-defined since $\mathcal{I}^0(\mathfrak{F}_\phi)$ is an ideal.

• If $K^T_0(\mathcal{X} \times \mathcal{X}) \cong K^T_0(\mathcal{X})^{\otimes 2}$ (a very restrictive condition, in contrast to homology), then the diagonal map $\Delta: \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ induces a coproduct

$$\Delta_*: K^T_0(\mathcal{X}) \to K^T_0(\mathcal{X} \times \mathcal{X}) \cong K^T_0(\mathcal{X}) \times K^T_0(\mathcal{X}). \quad (2.8)$$

Hence $K^T_0(\mathcal{X})$ behaves like a K-theoretic version of homology.

2.2.6

Lemma (Push-pull). For any $\mathcal{T}$-equivariant $f: \mathcal{X} \to \mathcal{X}'$, and any $\phi \in K^T_0(\mathcal{X})$, $\mathcal{E} \in K^T_0(\mathcal{X}')$,

$$f_*(\phi) \cap \mathcal{E} = f_* [\phi \cap f^*(\mathcal{E})].$$

Proof. Applying both sides to $\mathcal{E}' \in K^T_0(\mathcal{X}')$, this reduces to the equality

$$f^*(\mathcal{E} \otimes \mathcal{E}') = f^* \mathcal{E} \otimes f^* \mathcal{E}'$$

in $K^T_0(\mathcal{X})$. It holds because tensor product with locally free sheaves is exact. \qed

2.2.7

Let $\phi \in K^T_0(\mathcal{X})$ and $\psi \in K^T_0(\mathcal{X}')$. Then, for a base $S$, there is a square

$$
\begin{array}{ccc}
K^0_0(\mathcal{X} \times \mathcal{X}' \times S) & \xrightarrow{\psi_{X \times S}} & K^0_0(\mathcal{X} \times S) \\
\phi_{\mathcal{X}' \times S} \downarrow & & \downarrow \phi_S \\
K^0_0(\mathcal{X}' \times S) & \xrightarrow{\psi_S} & K^0_0(S)
\end{array}
$$

(2.9)

where $\mathcal{T}$ acts diagonally on $\mathcal{X} \times \mathcal{X}'$. It is useful for simplicity to assume these squares are all commutative (see [FM81, §2.2]), though nowhere is this actually required.

Definition. Let $\phi \boxtimes \psi \in K^T_0(\mathcal{X} \times \mathcal{X}')$ have components given by the lower left composition in (2.9); analogously, set $(\phi \boxtimes \psi)^T := \phi^T \boxtimes \psi^T$. This satisfies the finiteness condition either
because \( \mathfrak{F}_\phi \times \mathfrak{F}_\psi \) is a scheme and the condition is vacuous, or because

\[
I^\circ(\mathfrak{F}_\phi \times \mathfrak{F}_\psi)^{\circ N} = \sum_{k=0}^{N} I^\circ(\mathfrak{F}_\phi)^{\circ k} \otimes I^\circ(\mathfrak{F}_\psi)^{\circ (N-k)} \subset K^\circ(\mathfrak{F}_\phi) \otimes K^\circ(\mathfrak{F}_\psi)
\]

and then the vanishing (2.5) for \( \phi \boxtimes \psi \) follows from that of \( \phi \) and \( \psi \) individually.

**Example.** Let \( Z_i(\tau) \in K^T_\circ(\mathcal{M}_{\alpha_i}^{pl}) \) be as in Example 2.2.3 for \( i = 1, 2 \). Let \( \pi_i \) and \( \pi_S \) be the projections from \( \mathcal{M}_{\alpha_1}^{sst}(\tau) \times \mathcal{M}_{\alpha_2}^{sst}(\tau) \times S \) to the \( \mathcal{M}_i \) and \( S \) factors respectively. Then

\[
(Z_1(\tau) \boxtimes Z_2(\tau))_S : \mathcal{E} \mapsto (\pi_S)_* \left( \mathcal{E}|_{\mathcal{M}_{\alpha_1}^{sst}(\tau) \times \mathcal{M}_{\alpha_2}^{sst}(\tau) \times S} \otimes \pi_1^* \mathcal{O}_{\mathcal{M}_{\alpha_1}}^{vir} \otimes \pi_2^* \mathcal{O}_{\mathcal{M}_{\alpha_2}}^{vir} \right)
\]

by push-pull in K-theory. In the shorthand notation of 2.2.4,

\[
Z_1(\tau) \boxtimes Z_2(\tau) = \chi \left( \mathcal{M}_{\alpha_1}^{sst}(\tau) \times \mathcal{M}_{\alpha_2}^{sst}(\tau), (\mathcal{O}^{vir} \boxtimes \mathcal{O}^{vir}) \otimes - \right).
\]

**2.2.8**

**Remark.** The axioms in Definition 2.2.2 for \( K^T_\circ(\mathcal{X}) \) is the minimal set of axioms necessary for the vertex algebra construction in \( \S 3.2 \) to work. However, the resulting \( \mathbb{k} \)-modules \( K^T_\circ(\mathcal{X}) \) are difficult to compute and seem much too big, and it would be good to look for alternate or more stringent axioms which produce a more tractable version of K-homology.

For instance, there must be a better version of the finiteness condition that does not include the unnatural restrictions on \( \mathfrak{F}_\phi \). These restrictions are imposed specifically so that Definition 2.2.7 of \( \boxtimes \) is well-defined, namely if \( \phi \) and \( \psi \) both satisfy (2.5) then so does \( \phi \boxtimes \psi \). In (co)homology this is automatic by the Künneth theorem, while in K-theory one has to include all higher K-groups for the Künneth theorem to hold (but doing so takes us farther away from the geometric setting of sheaves on \( \mathcal{M} \)).

**2.2.9**

It is useful to allow components \( \phi_S : K^\circ_T(\mathcal{M}_\alpha \times S) \rightarrow K^\circ_T(S) \) of an element \( \phi \in K^T_\circ(\mathcal{M}_\alpha) \) to land not in \( K^\circ_T(S) \) but rather in the localized K-theory \( K^\circ_T(S)_{\text{loc}} \).

**Definition.** The *localized* operational K-homology \( K^T_\circ(\mathcal{X})_{\text{loc}} \) consists of collections

\[
\phi := \{ K^\circ_T(\mathcal{X} \times S)_{\text{loc}} \xrightarrow{\phi_S} K^\circ_T(S)_{\text{loc}} \}_S
\]

of \( K^\circ_T(S)_{\text{loc}} \)-linear maps satisfying the same axioms as in Definition 2.2.2 for \( K^T_\circ(\mathcal{X}) \).
Example. In the setting of Example 2.2.3, assume that $\mathcal{M}^\text{sst}_\alpha(\tau)$ is not necessarily proper but its $T$-fixed locus $\mathcal{M}^\text{sst}_\alpha(\tau)^T$ is, with (virtual) normal bundle $N$. Then it is standard to define

$$Z_\alpha(\tau) := \chi\left(\mathcal{M}^\text{sst}_\alpha(\tau)^T, \frac{\mathcal{O}^{\text{vir}}_{\mathcal{M}^\text{sst}_\alpha(\tau)^T}}{\bigwedge^1 (N^\vee)} \otimes -\right) \in K^T_\circ(\mathcal{M}_\alpha)_\text{loc}. \quad (2.10)$$

For brevity, we will however continue to use the notation (2.7) for $Z_\alpha(\tau)$ even when $\mathcal{M}^\text{sst}_\alpha(\tau)$ is not proper, with the understanding that it actually means (2.10).

2.3 Example: $\ast/G$

2.3.1 Let $X := [\ast/G]$ with trivial $T$-action. Then the Künneth theorem

$$K^\circ_T(X \times S) = K^\circ(X) \otimes_Z K^\circ_T(S)$$

holds for any base $S$, and $K^\circ(X) \cong K(X) \cong R(G)$ is the representation ring of $G$. To compute $K^\circ_T(X)$, it therefore suffices to compute $K_\circ(X)$, for which only the finiteness condition (2.5) is relevant. Namely, if $I \subset R(G)$ is the augmentation ideal,

$$K_\circ(X) = \{\phi : \phi(I^n) = 0 \quad \forall n \gg 0\} \subset \text{Hom}(R(G), \mathbb{Z}).$$

Here Hom means homomorphisms of $\mathbb{Z}$-modules, even though both inputs are rings.

2.3.2 The ring $R(G)$ may be equipped with the $I$-adic topology; this agrees with the topology induced by many other filtrations [KM21]. There is the following rather nice interpretation of $K_\circ(X)$ as the continuous dual of $R(G)$.

Lemma. Equip $\mathbb{Z}$ with the discrete topology and let $\text{Hom}_{\text{cts}}$ denote continuous ($\mathbb{Z}$-module) homomorphisms. Then

$$K_\circ(X) = \text{Hom}_{\text{cts}}(R, \mathbb{Z}) \subset \text{Hom}(R, \mathbb{Z})$$

where $R$ can be either $R(G)$ or its $I$-adic completion $R(G)\hat{\uparrow}_I$.

Proof. This is by the definition of the $I$-adic topology: continuity of a map $\phi \in \text{Hom}(R(G), \mathbb{Z})$ means the pre-image of the open set $\{k\} \subset \mathbb{Z}$ must still be open, namely

$$w \in \phi^{-1}(k) \iff w + I^n \in \phi^{-1}(k) \quad \forall n \gg 0.$$
But $\phi$ is linear, so $\phi(w + P^i) = \phi(w)$ iff $\phi(P^i) = 0$.

With sufficient hypotheses on $G$, e.g. reductive, the Atiyah–Segal completion theorem gives a geometric interpretation of $R(G)$ as the topological K-theory $K(BG_c)$ where $G_c$ is the compact Lie group associated to $G$. Its continuous dual is therefore identified with the (ordinary) K-homology of $BG_c$.

2.3.3

The most salient example is when $G = \mathbb{C}^\times$, with $BG_c = BU(1) = \mathbb{C}P^\infty$. Note that this $G$ is distinguished among others since there is a multiplication map $\Psi : [*/\mathbb{C}^\times] \times [*/\mathbb{C}^\times] \to [*/\mathbb{C}^\times]$ making $[*/\mathbb{C}^\times]$ (resp. $\mathbb{C}P^\infty$) into a group object in Artin stacks (resp. an $H$-space).

**Proposition.** Write $R(G) = \mathbb{Z}[s^\pm]$ so that $I = (1 - s)$. Then, as $\mathbb{Z}$-modules,

$$K_*([*/\mathbb{C}^\times]) \cong \mathbb{Z}[\phi]$$

where $\phi^k(s^n) := (-1)^k \binom{n}{k}$.

**Proof.** Compute that $R(G) = \mathbb{Z}[1 - s]$ and let $\{\phi^k\}_k$ be the dual basis to $\{(1 - s)^\ell\}_\ell$. Explicitly,

$$\phi^k(f) = \left. \frac{\partial^k f}{\partial (1 - s)^k} \right|_{s=0} = \frac{(-1)^k \partial^k f}{k!} \left. \frac{\partial s^k}{\partial s} \right|_{s=0}. \quad \Box$$

**Remark.** An immediate observation is that $K_0([X/G]) \neq K_0^G(X)$, in contrast to $K^0([X/G]) = K_0^G(X)$. This is another (lack of a) property shared by $K_0$ and homology $H_*$; see e.g. [Joy21, §2.3] for a relevant definition of equivariant homology of Artin stacks.

2.3.4

Since $\mathbb{X} = [*/\mathbb{C}^\times]$ is a group object, $K^0(\mathbb{X}) = K(\mathbb{X}) \cong R(G)$ is a Hopf algebra and therefore its continuous dual $K_0(\mathbb{X})$ is too. For completeness, we explicitly compute its product/coproduct.

- Let $\phi^k = \left. \frac{(-1)^k \partial^k}{k!} f \right|_{s=0}$. The Leibniz rule $\phi^k(fg) = \sum_{i+j=k} \phi^i(f)\phi^j(g)$ induces the coproduct (cf. (2.8))

$$\Delta(\phi^k) = \sum_{i+j=k} \phi^i \otimes \phi^j.$$

- Let $\deg : R(\mathbb{C}^\times) \to \mathbb{Z}$ be the degree homomorphism. Viewing $\phi^k = (-1)^k \binom{\deg}{k}$ as polynomials in $\deg$, the coproduct $\Psi^* : s \mapsto s \boxtimes s$ on $R(\mathbb{C}^\times)$ induces the ordinary
product of polynomials in degs. In other words,

\[ Z[\phi] \subset \mathbb{Q}[\text{deg}_s] \quad (2.11) \]

is the subalgebra of numerical polynomials in degs. We denote this product on \( Z[\phi] \) by \( \ast \). It is distinct from the standard product, and a nice combinatorial exercise is that

\[ \phi^a \ast \phi^b = \sum_{k=0}^{a+b} (-1)^k \binom{a+b-k}{a} \binom{a}{k} \phi^{a+b-k}. \]

2.3.5

The inclusion \((2.11)\) is a homology Chern character, as follows. The usual Chern character\( \text{ch}: K(\mathbb{C}P^n) \to H^*(\mathbb{C}P^n; \mathbb{Q}) \) induces a Chern character

\[ \text{ch}: K(\mathbb{C}P^\infty) = Z[[1-s]] \to \mathbb{Q}[[x]] = H^*(\mathbb{C}P^\infty; \mathbb{Q}) \]

where \( s = e^x \). (Note that \( \text{ch} \otimes \mathbb{Q} \) is no longer an isomorphism since \( \mathbb{Q} \otimes_Z Z[[x]] \neq \mathbb{Q}[[x]] \).) Let \( \xi \in H_2(\mathbb{C}P^n) \) be the class dual to \( x \), so that \( \xi^m(x^n) = \delta_{mn}n! \).

One can check this is the correct product on \( \xi \) by computing

\[ \Psi_*(\xi^a \otimes \xi^b)(x^k) = (\xi^a \otimes \xi^b)(\Psi^*(x^k)) = \sum_{i+j=k} \binom{k}{i} \xi^a(x^i)\xi^b(x^j) = \delta_{k,a+b}k! \xi^{a+b}(x^k). \]

**Proposition.** Passing to duals, \( \text{ch} \) induces an inclusion

\[ \text{ch}_*: K_0([*/\mathbb{C}^\times]) \subset H_*(\mathbb{C}P^\infty; \mathbb{Q}) = \mathbb{Q}[\xi] \]

given by \( \phi^k \mapsto (-1)^k(\xi^k) \).

**Proof.** It suffices to check that both act the same way on \( s^n = e^{nx} \) for \( n \geq 0 \). Since \( \xi^k(e^{nx}) = n^k \), it follows that

\[ (-1)^k \binom{k}{n} (e^{nx}) = (-1)^k \binom{n}{k} = \phi^k(s^n) = \phi^k(e^{nx}). \]
2.3.6

Remark. The Hopf algebra \( (K_\circ(\mathfrak{X}), \star, \Delta) \) is exactly the \( \lambda \)-divided power Hopf algebra of \([AGKO03]\) for \( \lambda = -1 \) after a mild change of basis. Our construction ascribes geometric meaning to their purely algebraic definition and provides an alternate proof of their main theorem. In general \( \lambda \) is the constant in the 1-dimensional formal group law \( F_\lambda(x, y) := x + y + \lambda xy \).

There is a well-understood correspondence between formal group laws, their associated generalized cohomology theories \( E(-) \), and Hopf algebra structures on an appropriate dual of \( E(\mathbb{CP}^\infty) \) \([Ada74\), Part II]\).

Degenerating to \( \lambda = 0 \) produces \( H_*(\mathbb{CP}^\infty) \), which is an ordinary divided power Hopf algebra generated by the divided powers \( \xi^k := \xi^k / k! \), e.g.

\[
\xi^{[a]} \xi^{[b]} = \binom{a + b}{a} \xi^{[a+b]}
\]

The map \( \text{ch}_* \) is a Hopf algebra morphism, and becomes an isomorphism only after \( \widehat{\otimes} \mathbb{Q} \). Indeed, as \( F_{-1} \) and \( F_0 \) are non-isomorphic group laws, the \( \lambda = -1 \) and \( \lambda = 0 \) divided power Hopf algebras are not isomorphic over \( \mathbb{Z} \).

3 Equivariant multiplicative vertex algebras

3.1 General theory

3.1.1

Let \( T = (\mathbb{C}^\times)^n \) be a split torus with representation ring \( \mathbb{k} \) and character lattice \( \text{char}(T) \subset \mathbb{k} \). We introduce a new formal variable \( z \) and, for any \( \mathbb{k} \)-module \( M \), let

\[
M[[1 - z^i]]_T := \lim_{\substack{m \geq 0 \\
i_1, \ldots, i_m \in \mathbb{Z} \setminus \{0\} \\
t_1, \ldots, t_m \in \text{char}(T)}} M[[1 - t_1 z^{i_1}, \ldots, 1 - t_m z^{i_m}]]
\]

be the direct limit over all the inclusions \( M[[\{1 - t_k z^{i_k}\}_k]] \hookrightarrow M[[\{1 - t_k z^{i_k}\}_k, \{1 - s_l z^{j_l}\}_l]] \).

Similarly define \( M((1 - z^i))_T \) and (Laurent) polynomial rings. An omitted exponent \( i \) means to take all \( i_k = \pm 1 \), e.g. \( M[1 - z]_T = M[1 - tz^\pm : t \in \text{char}(T)] \). Assume \( |z - 1| \ll 1 \) to implicitly use the embedding \( \iota_z : M[z^\pm] \hookrightarrow M[[1 - z]] \) given by the “multiplicative” expansion

\[
\iota_z(z^{-1}) := \frac{1}{1 - (1 - z)} = \sum_{k \geq 0} (z - 1)^k
\]
3.1.2

**Definition.** A $\mathbb{T}$-equivariant multiplicative vertex algebra is the data of:

(i) a $k$-module $V$ of states with a distinguished vacuum vector $1 \in V$;

(ii) a multiplicative translation operator $D(z) \in \text{End}(V[[1 - z]])$, i.e. $D(z)D(w) = D(zw)$;

(iii) a space of fields

$$\text{Hom} \left( V, V \left( \left( 1 - z^i \right) \right) \right)$$

and a state-field correspondence $a \mapsto Y(a, z)$ valued in fields.

This data must satisfy the following axioms:

(i) (vacuum) $Y(1, z) = 1$ and $Y(a, z)1 \in V[[1 - z^i]]_{\mathbb{T}}$ with $Y(a, 1)1 = a$;

(ii) (skew symmetry) $Y(a, z)b = D(z)Y(b, z^{-1})a$;

(iii) (weak associativity) $Y(Y(a, z)b, w) \equiv Y(a, zw)Y(b, w)$, where $\equiv$ means that when applied to any $c \in V$, both sides are expansions of the same element in

$$V[[1 - z^i, 1 - w^j]]_{\mathbb{T}} \left[ \frac{1}{1 - z^i}, \frac{1}{1 - w^j}, \frac{1}{1 - zw^k} \right]_{\mathbb{T}}.$$

The vertex algebra is *holomorphic* if actually $Y(a, z)b \in V[[1 - z^i]]_{\mathbb{T}}$ for all $a, b \in V$.

3.1.3

Recall that for ordinary (additive) vertex algebras, “expansion” in weak associativity means to use the two $\mathbb{Z}$-algebra embeddings

$$\mathbb{Z}((u))((v)) \xleftarrow{\iota_u} \mathbb{Z}([u - v])^{-1} \xrightarrow{\iota_v} \mathbb{Z}((u))((v))$$

uniquely specified by $(u - v)\iota_v(u - v)^{-1} = 1 = (u - v)\iota_u(u - v)^{-1}$. Their multiplicative analogues $\mathbb{Z}((1 - z))((1 - w)) \xleftarrow{\iota_w} \mathbb{Z}([1 - zw])^{-1} \xrightarrow{\iota_z} \mathbb{Z}((1 - w))((1 - z))$ are similarly uniquely specified. However, in the presence of equivariance, $k$-algebra embeddings such as

$$k((1 - z))_{\mathbb{T}}((1 - w))_{\mathbb{T}} \xleftarrow{\iota_{1 - z}} k((1 - zw))^{-1}_{\mathbb{T}} \xrightarrow{\iota_{1 - w}} k((1 - w))_{\mathbb{T}}((1 - z))_{\mathbb{T}}$$
are no longer unique, e.g. the expansion
\[ t_w^T \frac{1}{1 - tzw} = \sum_{k \geq 0} \frac{(-sz)^k}{(1 - sz)^{k+1}}(1 - tw/s)^k \in k((1 - z))_T((1 - w))_T \] (3.4)
is valid for any \( s \in \text{char } T \). Weak associativity is then the statement that there exist appropriate expansions, which may depend non-trivially on the elements \( a, b, c \in V \) (and may act differently on each monomial and each factor of the form (3.4) within each monomial).

3.1.4

**Remark.** Following [Li11], associated to any formal group law \( F = F(u, v) \) is the notion of vertex \( F \)-algebra. When \( F_a(u, v) := u + v \) is the additive formal group law, it is exactly the usual notion of vertex algebra. When \( F_m(u, v) := u + v + uv \) is the multiplicative formal group law, it is the special case of our multiplicative vertex algebras where:

(i) \( T = \{1\} \) is trivial, i.e. there is no equivariance;

(ii) (3.2) and (3.3) only have exponents \( i = j = k = 1 \).

The latter restriction is unnatural from a geometric perspective, where \( z \) (and \( w \)) arises as the K-theoretic weight of a certain \( \mathbb{C}^\times \)-action. For example, poles in \( z \) at non-trivial roots of unity are a hallmark of \( \mathbb{C}^\times \)-localization on Deligne–Mumford stacks.

Vertex \( F \)-algebras are canonically identified with ordinary vertex algebras only after a base change to \( \mathbb{Q} \) [Li11, Theorem 3.7], like how the Chern character \( \text{ch}: K(X) \to A_\ast(X) \otimes Z \mathbb{Q} \) must be rationalized to become an isomorphism. However, this is no longer true once \( T \)-equivariance is introduced, due to subtleties with the \( T \)-equivariant Chern character [EG00].

3.1.5

**Lemma.** Let \((V, 1, D, Y)\) be an equivariant multiplicative vertex algebra. For \( a, b \in V \):

(i) (translation) \( D(w)Y(a, z) \equiv Y(a, zw)D(w) \);

(ii) (locality) \( Y(a, z)Y(b, w) \equiv Y(b, w)Y(a, z) \).

**Proof.** The skew symmetry axiom with \( b = 1 \) gives \( Y(a, z)1 = D(z)a \). Then weak associativity applied to \( 1 \) gives

\[ Y(a, zw)D(w)b = Y(a, zw)Y(b, w)1 \equiv Y(Y(a, z)b, w)1 = D(w)Y(a, z)b. \]
Similarly, weak associativity with \( b = 1 \) gives \( Y(D(z)a, w) = Y(a, zw) \), also called translation covariance. Using it and weak associativity and skew symmetry,

\[
Y(a, z)Y(b, w) \equiv Y(Y(a, z/w)b, w) = Y(D(z/w)Y(b, w/z)a, w) \\
\equiv Y(Y(b, w/z)a, z) \equiv Y(b, w)Y(a, z).
\]

The converse also holds, namely that the translation and locality properties together imply skew symmetry and weak associativity, and hence form an alternate set of defining axioms. This is left as an exercise to the reader.

**Remark.** Like for non-equivariant vertex algebras, cf. [FBZ04, §1.2.4], locality means that for \( a, b, c \in V \), there exists a finite set \( \{ t_i \} \subset \text{char } T \) so that \( \prod_i (1 - t_i z/w) [Y(a, z), Y(b, w)]^c = 0 \).

### 3.1.6

**Remark.** For additive vertex algebras, the translation property is usually viewed as a consequence of the stronger axiom that \( D(v) = \exp(vT) \) for some derivation \( T \) such that \( [T, Y(a, u)] = \partial_u Y(a, u) \). Exponentiating both sides yields

\[
D(v)Y(a, u) = \exp(v\partial_u)Y(a, u)D(v),
\]

and \( \exp(v\partial_u)f(u) = f(u + v) \) gives the desired translation. For multiplicative vertex algebras, we choose not to impose the analogous stronger axiom that \( D(z) = z^H \) for some grading operator \( H \) such that \( [H, Y(a, z)] = \deg_z Y(a, z) \). With identifications \( z = \exp(u) \) and \( w = \exp(v) \), the degree operator \( \deg_z := z\partial_z \) comes from

\[
\exp(v\partial_u) = \exp(\log(w)\partial_{\log(z)}) = w^{z\partial_z},
\]

and clearly \( w^{z\partial_z}f(z) = f(zw) \) is now a multiplicative translation.

### 3.1.7

**Definition.** Given a rational function \( f \in k[(1 - z^i)^\pm] \), let \( f_+ \in k((z)) \) and \( f_- \in k((z^{-1})) \) be its formal series expansion around \( z = 0 \) and \( z = \infty \) respectively. Let

\[
\rho_K(f) := z^0 \text{ term in } (f_+ - f_-)
\]
be the $K$-theoretic residue map. (See Appendix A for a discussion of residue maps in general and a characterization of this one.) It extends to a module homomorphism

$$\rho_K : \k((1 - z))_T \to \hat{k}$$

where $\hat{k} := \mathbb{Z}[[1 - t_1, \ldots, 1 - t_n]] \supset \k$ is the completion of $\k$ at the augmentation ideal $I := \langle 1 - t_1, \ldots, 1 - t_n \rangle$. Note that the further extension of $\rho_K$ to $\k((1 - z^i))_T$ is not well-defined.

When there are multiple variables, write $\rho_{K, z}$ to mean $\rho_K$ applied in the variable $z$.

**Lemma** ($T$-invariance of $\rho_K$). For any $t \in \text{char } T$ and any $f \in \k((1 - z))_T$,

$$\rho_K(f) = \rho_K(f|_{z \to tz}).$$

**Proof.** Such a substitution clearly does not change the $z^0$ term. \qed

3.1.8

Let $\text{im}(1 - D(z)) \subset V$ denote the $\k$-submodule generated by coefficients of $(1 - D(z))a \in V[[1 - z]]$ for all $a \in V$, and set

$$\overline{V} := V/\text{im}(1 - D(z)).$$

We continue to use $\rho_K$ to denote the induced maps $W((1 - z))_T \to W^\wedge_I$ for any $\k$-module $W$, where $W^\wedge_I$ denotes completion at $I \subset \k$.

**Theorem.** Suppose that, for all $a, b \in V$,

$$Y(a, z)b \in V((1 - z))_T \subset V\left(\left(1 - z^i\right)\right)_T$$

(3.5)

Then the operation

$$[-, -] : \overline{V} \otimes \overline{V} \to \overline{V}^\wedge_I$$

$$\overline{a} \otimes \overline{b} \mapsto \rho_K Y(\overline{a}, \overline{z})\overline{b}$$

is well-defined and induces a Lie bracket on the completion $\overline{V}^\wedge_I$.

**Definition.** A (equivariant) multiplicative vertex algebra is **reduced** if it satisfies (3.5). There is no analogue of this condition for ordinary, additive vertex algebras.
3.1.9

Proof of Theorem 3.1.8. We directly verify the axioms of a Lie bracket.

(Skew symmetry) By definition, \( \rho_{K}(f(z^{-1})) = -\rho_{K}(f(z)) \). So for \( a, b \in V \),

\[
\rho_{K}Y(a, z)b = \rho_{K}D(z)Y(b, z^{-1})a = -\rho_{K}D(z^{-1})Y(b, z)a \\
= -\rho_{K}Y(b, z)a + \rho_{K}(1 - D(z^{-1}))Y(b, z)a.
\]

(Jacobi identity) By locality and weak associativity, the three series

\[
Y(a, z)Y(b, w)c \in V((1 - z)_{T}((1 - w))_{T} \\
Y(b, w)Y(a, z)c \in V((1 - w)_{T}((1 - z))_{T} \\
Y(Y(a, z/w)b, w)c \in V((1 - w))_{T}((1 - z/w))_{T}
\]

are all expansions of the same element. In particular, following [FBZ04, §3.3.6],

\[
[Y(a, z), Y(b, w)]c = Y \left( \iota_{w}^{T} - \iota_{z}^{T} \right) Y(a, z/w)b, w \right) c
\]

(3.6)

for some expansions \( \iota_{z}^{T} \) and \( \iota_{w}^{T} \). Then apply \( \rho_{K, w}\rho_{K, z} \) to both sides and use the following Lemma 3.1.10 to conclude.

(Well-defined) By a mild rearrangement of the translation property, there exists some expansion \( \iota_{w}^{T} \) such that

\[
Y(a, z)D(w)b = \iota_{w}^{T}D(w)Y(a, z/w)b.
\]

Applying \( \rho_{K, z} \) and Lemma 3.1.10,

\[
\rho_{K, z}Y(a, z)D(w)b = D(w)\rho_{K, z}Y(a, z)b.
\]

It follows that \([- , -]\) preserves \( \text{im}(1 - D(w)) \) in its second factor. The same is true of its first factor by skew symmetry. \(\square\)

3.1.10

Lemma. Let \( f \in k[(1 - z)^{\pm}]_{T} \). For any two expansions \( \iota_{w}^{T} \) and \( \iota_{z}^{T} \),

\[
\rho_{K, z} \left( \iota_{z}^{T}f(z/w) \right) = 0 \\
\rho_{K, z} \left( \iota_{w}^{T}f(z/w) \right) = \rho_{K, z}f(z).
\]
Proof. The first equality is clear since $\iota_z^T f(z/w)$ as a function of $z$ involves only elements of $k[1 - z] \subset \ker \rho_{K,z}$. We focus on the more interesting second equality.

Pass to an $N$-fold cover of $T$, with $N \gg 0$ such that there is a partial fraction decomposition

$$f(z) = f_{\text{reg}}(z) + \sum_i \sum_j \frac{s_{ij}}{(1 - s_i z)^j}, \quad s_i, s_{ij} \in K, \quad f_{\text{reg}} \in K[z^{\pm}]$$

over the fraction field $K$ of

$$k_{\frac{1}{N}} := \mathbb{Z}[\frac{1}{N}, t_1^{\frac{1}{N}}, \ldots, t_n^{\frac{1}{N}}] \supset k.$$ 

It therefore suffices to prove the lemma for $f = (1 - sz)^{-n}$.

When $n = 1$, all the $k > 0$ terms in the expansion (3.4) have $\rho_{K,z} = 0$ by computation. Hence, for some monomial $s' \in k_{\frac{1}{N}},$

$$\rho_{K,z} \iota_z^T w f(z/w) = \rho_{K,z} \frac{1}{1 - s'z} = \rho_{K,z} f(z)$$

where the second equality is the $T$-invariance of $\rho_K$ (Lemma 3.1.7), or just computation. When $n > 1$, the $\iota_w^T$ expansion is a product of such sums (3.4), each perhaps with a different monomial $s'_i$, but again only the resulting "$k = 0"$ term $1/\prod_{i=1}^n (1 - s'_i z)$ has non-zero $\rho_{K,z}$. \qed

3.1.11

Remark. In the non-equivariant setting, where the expansions $\iota_w$ and $\iota_z$ are unique, (3.6) takes on a particular fixed form written in terms of formal $\delta$-functions. One can then apply arbitrary linear functionals in $z$ and $w$ to obtain various commutation identities in $\text{End}(V)$. For ordinary (additive) vertex algebras, this culminates in the Jacobi identity for vertex algebras (also known as the Borcherds identity), see e.g. [FBZ04, §3.3.10], from which the Jacobi identity for the induced Lie algebra follows readily as a special case.

In the equivariant setting, the non-uniqueness of $\iota_w^T$ and $\iota_z^T$ means that the form of (3.6), particularly the location of its poles in $z$ and $w$, varies as a function of $a, b, c \in V$. One can therefore only apply functionals which have a certain $T$-invariance property like in Lemma 3.1.7, and it is difficult to write a general Jacobi identity at the level of the vertex algebra.
3.2 On K-homology

3.2.1

Let $M$ be the moduli stack from §2.1 and set $K_T^\alpha(M) := \bigoplus_\alpha K_T^\alpha(M_\alpha)$. The goal of this subsection is the following.

**Theorem.** $K_T^\alpha(M)$ has the structure of an equivariant multiplicative vertex algebra.

This involves, for $\phi, \psi \in K_T^\alpha(M)$: a vacuum object $1 \in K_T^\alpha(M_0)$ (Definition 3.2.2), a translation operation $D(z)\phi$ (Definition 3.2.3), and a vertex operation $Y(\phi, z)\psi$ (Definition 3.2.4).

The primary difficulty, and also the reason for finiteness axioms in the definition of $K_T^\alpha(-)$, is to ensure that the vertex operation always produces a field (see §3.2.5). Once this is checked, the vacuum axiom (Theorem 3.2.6) is immediate and some mild computation formally identical to that of \([GU22]\) verifies skew-symmetry (Theorem 3.2.7) and weak associativity (Theorem 3.2.9).

3.2.2

Recall that $M_0 = *$ is a single point, and so an element in $K_T^\alpha(M_0)$ consists of $K_T^\alpha(S)$-linear maps $K_\alpha(S) \to K_T^\alpha(S)$. There are not very many such maps.

**Definition.** Let $1 \in K_T^\alpha(M_0)$ be given by the identity map

$$1_S(\mathcal{E}) := \mathcal{E} \in K_\alpha(S).$$

3.2.3

**Definition.** For $\phi \in K_T^\alpha(M)$ and $\mathcal{E} \in K_\alpha(M \times S)$, let

$$(D(z)\phi_S)(\mathcal{E}) := \phi_S(z^{\deg}\mathcal{E})$$

where $z^{\deg}$ is defined using the scalar multiplication map $\Psi \boxtimes 1 : [*/\mathbb{C}^\times] \times M \times S \to M \times S$.

More generally, if $\phi \in K_T^\alpha(M \times M')$, set

$$(D(z)^{(i)}\phi_S)(\mathcal{E}) := \phi_S(z^{\deg_i}\mathcal{E}).$$

In particular, $D(z) = \prod_i D(z)^{(i)}$.

Clearly $D(z)D(w)\phi = D(zw)\phi$; this encapsulates the multiplicative nature of the vertex algebra. Observe also that $D(z)1 = 1$, as expected. Finally, the following lemma verifies that $D(z) \in \text{End}(K_T^\alpha(M))[[1 - z]]$. 23
Lemma. Let $K^T_0([*/\mathbb{C}^\times]) \cong k[\phi]$ as in §2.3.3. Then

$$D(z)\psi = \Psi_s \left( \sum_{k \geq 0} (1 - z)^k \phi^k \boxtimes \psi \right).$$

Proof. Let $K([*/\mathbb{C}^\times]) = \mathbb{Z}[s^\pm]$. Then

$$\Psi_s \left( \sum_{k \geq 0} (1 - z)^k \phi^k \boxtimes \psi \right)(\mathcal{E}) = \sum_{k \geq 0} (1 - z)^k \sum_{n \in \mathbb{Z}} \phi^k(s^n)\psi(\mathcal{E}_n)$$

$$= \sum_{n \in \mathbb{Z}} \psi(\mathcal{E}_n) \sum_{k \geq 0} (z - 1)^k \binom{n}{k}$$

$$= \sum_{n \in \mathbb{Z}} \psi(\mathcal{E}_n)(1 + (z - 1))^n = \psi(z^{\text{deg}}\mathcal{E}). \quad \Box$$

### 3.2.4

Using the complex $\mathcal{E}^\bullet$ on $\mathcal{M} \times \mathcal{M}$, define

$$\Theta^\bullet_{\alpha,\beta}(z) := \wedge^\bullet_{-z^{-1}}(\mathcal{E}_{\alpha,\beta}^\bullet) \wedge \wedge^\bullet_{-z}(\mathcal{E}_{\beta,\alpha}^\bullet)^\vee,$$  \tag{3.7}

cf. (2.2). We implicitly treat this as a series in $[(1 - z)^{\pm}]_T$, using some multiplicative expansion in $z$ to be made completely precise in §3.2.5.

**Definition.** For $\phi \in K^T_0(\mathcal{M}_\alpha)$ and $\psi \in K^T_0(\mathcal{M}_\beta)$, the vertex operation

$$Y(\phi, z)\psi \in K^T_0(\mathcal{M}_{\alpha + \beta}) \left( (1 - z^i) \right)_T$$

is given by

$$Y(\phi, z)\psi := (\Phi_{\alpha,\beta})_* D(z)^{(1)} \left( (\phi \boxtimes \psi) \cap \Theta^\bullet_{\alpha,\beta}(z) \right) \tag{3.8}$$

$$K^T_0(\mathcal{M}_{\alpha + \beta} \times S) \ni \mathcal{E} \mapsto (\Phi \boxtimes \psi)_S \left( \Theta^\bullet_{\alpha,\beta}(z) \otimes z^{\text{deg}1}\Phi^*_{\alpha,\beta}E \right) \in K^0_T(S). \tag{3.9}$$

While (3.8) is completely analogous to the version in ordinary homology, sometimes it is convenient to use the form (3.9) for its components.

The hypotheses (2.1) on $\mathcal{E}^\bullet$ translate into properties of $\Theta^\bullet(z)$ which are crucial for the vertex algebra axioms. Note that the same properties continue to hold if the $\wedge^\bullet_{-1}(-)$ in (3.7) is replaced by the symmetrized $\wedge^\bullet_{-1}(-)$; see Remark 4.2.4.
3.2.5

It is not immediate that (3.8) defines a field, i.e. that \( Y(\phi, z)\psi \) is polynomial in \((1 - tz^i)^{-1}\) for any \( i \in \mathbb{Z} \setminus \{0\} \) and \( t \in \text{char } T \). That this holds is ensured by the equivariant localization and finiteness axioms of \( K^T_0(\mathcal{M}) \), as follows. By the localization axiom and push-pull,

\[
(\phi \boxtimes \psi) \cap \Theta_{\alpha, \beta}(z) = \text{fix} \left[ (\phi^T \boxtimes \psi^T) \cap \text{fix}^* \Theta_{\alpha, \beta}(z) \right]
\]

(3.10)

where \( \text{fix} := \text{fix}_{\phi \boxtimes \psi} \) for short. Since \( D(z) \in \text{End}(K^T_0(\mathcal{M}))[1 - z] \), it suffices to prove the following.

**Lemma** (Localization). There exists a multiplicative expansion of \( \text{fix}^* \Theta_{\alpha + \beta}^*(z) \) such that

\[
(3.10) \in K^T_0(\mathcal{M}_{\alpha + \beta})[(1 - z)^\pm]_T.
\]

**Proof.** Let \( \mathcal{F} := \mathcal{F}_{\phi \boxtimes \psi} \) for short. Since \( T \) acts trivially on \( \mathcal{F} \), the pullback \( \text{fix}^* \Theta^*(z) \) becomes a product of terms \( \wedge x (\pm L) \) for various \( x = z^i t \), with \( i \in \{-1, 1\} \) and \( t \in \text{char } T \), and various K-theoretic Chern roots \( L \in \text{Pic}(\mathcal{F}) \). The + terms are clearly polynomial, so focus on the − ones. Expand each such term as

\[
\frac{1}{t L \wedge x} = \sum_{k \geq 0} \frac{(-x)^k}{(1 - x)^{k+1}} (1 - L)^\otimes k \in K^0(\mathcal{F})\left(\left(1 - x\right)^{-1}\right)
\]

(3.11)

(i) If \( L \) is unipotent, then in fact (3.11) already lives in \( K^0(\mathcal{F})[1 - x]^\pm \).

(ii) If \( L \) is non-unipotent, the coefficient of \( (1 - x)^{-N} \) in (3.11) lives in \( I^0(\mathcal{F})^N \), which by the finiteness axiom is annihilated by \( \phi \boxtimes \psi \) for \( N \gg 0 \).

It follows that the vertex algebra is holomorphic unless \( \mathcal{E}_{\alpha, \beta} \) is truly a virtual bundle. This is true in the original (co)homological construction as well.

3.2.6

**Proposition** (Vacuum axioms).

\[
Y(1, z)\phi = \phi \quad Y(\phi, z)1 = D(z)\phi.
\]

**Proof.** Under the isomorphism \( \mathcal{M}_0 \times \mathcal{M}_\beta \cong \mathcal{M}_\beta \), it is clear that \( \Phi^*_{\alpha, \beta} \mathcal{E} = \mathcal{E} \). Using that \( \Theta^*_{0, \beta}(z) = \mathcal{O}_{\mathcal{M}_\beta} \), what remains an exercise in unrolling notation. \( \Box \)
3.2.7

**Proposition** (Skew symmetry).

\[ Y(\phi, z)\psi = D(z)Y(\psi, z^{-1})\phi. \]

**Proof.** Since \( z^{\deg} \Phi^* = \Phi^* z^{\deg} \), it follows that \( D(z)\Phi_\ast = \Phi_\ast D(z) \). The rhs becomes

\[ (\Phi_{\beta, \alpha})_\ast D(z)^{(2)} ((\psi \boxtimes \phi) \cap \Theta^\ast_{\beta, \alpha}(z^{-1})) \cdot \]

Using the symmetry \( \Theta^\ast_{\beta, \alpha}(z^{-1}) = \Theta^\ast_{\alpha, \beta}(z) \), this is exactly \( Y(\phi, z)\psi \).

\[ \square \]

3.2.8

**Lemma** (“Translation covariance”).

\[ (D(w)\phi \boxtimes \psi) \cap \Theta^\ast(z) = D(w)^{(1)} ((\phi \boxtimes \psi) \cap \Theta^\ast(z)w) \]

\[ (\phi \boxtimes D(w)\psi) \cap \Theta^\ast(z) = D(w)^{(2)} ((\phi \boxtimes \psi) \cap \Theta^\ast(z/w)) \]

**Proof.** Unrolling the notation, the first claimed equality is

\[ (\phi \boxtimes \psi) \left( w^{\deg_1}(\Theta^\ast(z) \otimes \Theta^\ast_{\beta, \gamma} w) \right) = (\phi \boxtimes \psi) \left( \Theta^\ast(zw) \otimes w^{\deg_1} \Theta^\ast_{\beta, \gamma} \right) \]

for \( \Theta^\ast(z) \in K^T_\circ (\mathcal{M} \times \mathcal{M}) \). Now observe that \( w^{\deg_1} \Theta^\ast(z) = \Theta^\ast(zw) \) to conclude. The second claimed equality is completely analogous.

\[ \square \]

3.2.9

**Proposition** (Weak associativity).

\[ Y(Y(\phi, z)\psi, w)\xi \equiv Y(\phi, zw)Y(\psi, w)\xi. \]

**Proof.** Using push-pull, translation covariance, and the bilinearity

\[ (\Phi_{\alpha, \beta} \times \text{id})^\ast \Theta^\ast_{\alpha+\beta, \gamma}(z) = \Theta^\ast_{\alpha, \gamma}(z) \otimes \Theta^\ast_{\beta, \gamma}(z), \]
the lhs becomes

\[(\Phi_{\alpha+\beta,\gamma})_*D(w)\textsuperscript{(1)}(\Phi_{\alpha,\beta},D(z)\textsuperscript{(1)}((\phi \boxtimes \psi) \cap \Theta_{\alpha,\beta}(z)) \boxtimes \xi) \cap \Theta_{\alpha+\beta,\gamma}(w)\]

\[= (\Phi_{\alpha+\beta,\gamma})_*D((\Phi_{\alpha,\beta})_*D(z)\textsuperscript{(1)}((\phi \boxtimes \psi \boxtimes \xi) \cap (\Theta_{\alpha,\beta}(z) \otimes \Theta_{\alpha,\gamma}(zw) \otimes \Theta_{\beta,\gamma}(w)))\).\]

Similarly, the rhs becomes

\[(\Phi_{\alpha,\beta,\gamma})_*D((\Phi_{\beta,\gamma})_*D(z)\textsuperscript{(1)}((\phi \boxtimes \psi \boxtimes \xi) \cap (\Theta_{\beta,\gamma}(w) \otimes \Theta_{\alpha,\beta}(zw) \otimes \Theta_{\alpha,\gamma}(zw)))\].\]

Finally, \(D(w)\textsuperscript{(1)}(\Phi_{\alpha+\beta,\gamma})_*D(z)\textsuperscript{(1)} = (\Phi_{\alpha+\beta,\gamma})_*D(zw)\textsuperscript{(1)}D(w)\textsuperscript{(2)}\) while \(D(zw)\textsuperscript{(1)}\) commutes with \(\Phi_{\alpha+\beta,\gamma}\). We are done by the associativity of \(\Phi\).

\[\Box\]

\[\textbf{3.2.10}\]

Finally, Theorem 3.1.8 furnishes \(K^T_0(\mathcal{M})/\text{im}(1-D(z))\) with a Lie bracket. This \(k\)-module has the following geometric interpretation.

\[\textbf{Lemma.} K^T_0(\mathcal{M})/\text{im}(1-D(z)) \cong K^T_0(\mathcal{M}^{pl}).\]

\[\textbf{Proof.} \text{Recall from §2.1.5 that } K^\omega_T(\mathcal{M}^{pl}) = \ker(1-z^{\text{deg}}) \subset K^\omega_T(\mathcal{M}). \text{Dualizing, as } k\text{-modules,}\]

\[K^\omega_T(\mathcal{M})^*/\text{im}(1-D(z)) = K^\omega_T(\mathcal{M}^{pl})^*.\]

This is compatible with base change to \(\mathcal{M} \times S\) and all other axioms of \(K^T_0\), and hence the desired identification follows. \[\Box\]

For \(\phi, \psi \in K^T_0(\mathcal{M}^{pl})\), note that when \([\phi, \psi] \in K^T_0(\mathcal{M}^{pl})^\wedge\) is applied to any given \(\mathcal{E} \in K^T_0(\mathcal{M}^{pl} \times S)\), it is clear from (3.9) that

\[[\phi, \psi]_S(\mathcal{E}) \in K^T_0(S) \subset K^T_0(S)^\wedge.\]

\[\textbf{3.3 On K-theory}\]

\[\textbf{3.3.1}\]

The analogue in K-theory (as opposed to operational K-homology) of the vertex algebra construction of §3.2 is not well-defined in general: \(K_T(-)\) only admits proper pushforwards, while the vertex operation (3.8) requires a pushforward along \(\Phi_{\alpha,\beta}: \mathcal{M}_\alpha \times \mathcal{M}_\beta \rightarrow \mathcal{M}_{\alpha+\beta}\). This
is already not proper in the simplest setting of $\mathcal{M}_\alpha = [*/\text{GL}(\alpha)]$, for $\alpha \in \mathbb{Z}_{\geq 0}$, where the fibers are $\text{GL}(\alpha + \beta)/(\text{GL}(\alpha) \times \text{GL}(\beta))$. Nonetheless, we make the following observation.

**Theorem.** Suppose that for every $\alpha$ and $\beta$, there exists a factorization

$$\Phi_{\alpha, \beta} : \mathcal{M}_\alpha \times \mathcal{M}_\beta \xrightarrow{\iota_{\alpha, \beta}} \mathcal{N}_{\alpha, \beta} \xrightarrow{\phi_{\alpha, \beta}} \mathcal{M}_{\alpha + \beta}$$

through an Artin stack $\mathcal{N}_{\alpha, \beta}$, such that:

(i) $[\ast/\mathbb{C}^\times \times \mathbb{C}^\times]$ acts on $\mathcal{N}_{\alpha, \beta}$ and $\iota_{\alpha, \beta}$ is a $[\ast/\mathbb{C}^\times \times \mathbb{C}^\times]$-equivariant closed embedding;

(ii) $\theta^\bullet_{\alpha, \beta}(z) := (\iota_{\alpha, \beta})_* \Theta^\bullet_{\alpha, \beta}(z)$ is well-defined at $z = 1$ and lies in $K_T^\circ(\mathcal{M}_{\alpha, \beta})$;

(iii) there is a well-defined pushforward $(\phi_{\alpha, \beta})_* : \theta^\bullet_{\alpha, \beta}(1) \otimes K_T(\mathcal{M}_{\alpha, \beta}) \to K_T(\mathcal{M}_{\alpha + \beta})$.

Then the vertex operation

$$Y(E, z)F := (\Phi_{\alpha, \beta})_* z^{-\deg_1} \left( (E \boxtimes F) \otimes \Theta^\bullet_{\alpha, \beta}(z) \right) \quad (3.12)$$

is well-defined and makes $K_T(\mathcal{M})$ into a holomorphic equivariant multiplicative vertex algebra.

**Proof.** Using the hypotheses (and dropping subscripts $\alpha$ and $\beta$ for clarity),

$$Y(E, z)F = \phi_* t_* z^{-\deg_1} \left( (E \boxtimes F) \otimes \Theta^\bullet(z) \right)$$

$$= \phi_* z^{-\deg_1} \left( t_* (E \boxtimes F) \otimes \theta^\bullet(z) \right)$$

$$= \phi_* \left( t_* (z^{-\deg_1} E \boxtimes F) \otimes \theta^\bullet(1) \right)$$

which is well-defined. The result is polynomial in $z$ and hence lies in $K_T(\mathcal{M})[[1 - z]]$. It remains to check vertex algebra axioms. The identity element is $1 := \mathcal{O}_{\mathcal{N}_{0, 0}}$, and by construction $D(z) := z^{-\deg}$, so (3.12) is formally identical to (3.8). The proof that the latter makes $K_T^\circ(\mathcal{M})$ into an equivariant multiplicative vertex algebra can be repeated for $K_T(\mathcal{M})$ verbatim. Note that the minus sign in $z^{-\deg}$ is necessary for Lemma 3.2.8 to hold. 

3.3.2

The hypotheses of Theorem 3.3.1 hold in (and were abstracted from) the important setting of quiver representations, which will be the focus of the remainder of this subsection.

**Definition.** To fix notation, let $Q$ be a quiver with vertices indexed by $i \in I$, and

$$\mathcal{M}_\alpha = \left[ \prod_{i \rightarrow j} \text{Hom}(k^{\alpha_i}, k^{\alpha_j}) / \prod_i \text{GL}(\alpha_i) \right]$$
be its associated moduli stacks of representations of dimension vectors \( \alpha = (\alpha_i)_i \in \mathbb{Z}_{\geq 0} \). Let \( \mathcal{V}_\alpha = \bigoplus \mathcal{V}_{\alpha,i} \) be the universal bundle, with \( \mathcal{V}_{\alpha,i} \) corresponding to the \( i \)-th vertex. Then the obstruction theory comes from \( D(\alpha, \alpha) \) [Kir16] where

\[
D(\alpha, \beta) := \sum_i \mathcal{V}_{\alpha,i}^\vee \otimes \mathcal{V}_{\beta,i} - \sum_{i \rightarrow j} \mathcal{V}_{\alpha,i}^\vee \otimes \mathcal{V}_{\beta,j}. \tag{3.13}
\]

Following the philosophy of Example 2.1.2, set \( \mathcal{E}_{\alpha, \beta} := D(\alpha, \beta) \); more precisely, \( \mathcal{E}_{\alpha, \beta}^0 \) and \( \mathcal{E}_{\alpha, \beta}^1 \) are the first and second terms in (3.13) respectively.

Assume the \( T \)-action on \( \mathcal{M}_\alpha \) commutes with the stabilizer \( \text{GL}(\alpha) := \prod_i \text{GL}(\alpha_i) \). Let \( T_\alpha \subset \text{GL}(\alpha) \) be the maximal torus and \( S_\alpha := \prod_i S_{\alpha_i} \) be the Weyl group. Then

\[
K_T(\mathcal{M}_\alpha) \cong K_T([* / \text{GL}(\alpha)]) = K_{T \times T_\alpha}([*)^{S_\alpha} = k[[s_i^\pm, i \in I]]^{S_\alpha} \tag{3.14}
\]

where \( s_i = (s_{i,j})_{j=1}^{\alpha_i} \) is the set of variables permuted by \( S_{\alpha_i} \).

**Theorem.** In this setting,

\[
\mathfrak{M}_{\alpha, \beta} := \left[ \prod_{i \rightarrow j} \text{Hom}(k^{\alpha_i} \oplus k^{\beta_i}, k^{\alpha_j} \oplus k^{\beta_j}) / \text{GL}(\alpha) \times \text{GL}(\beta) \right],
\]

with the natural embedding \( \iota_{\alpha, \beta} \) and projection \( \phi_{\alpha, \beta} \), satisfies the conditions of Theorem 3.3.1. For elements \( f \in K_T(\mathfrak{M}_\alpha) = k[[s_i^\pm]]^{S_\alpha} \) and \( g \in K_T(\mathfrak{M}_\beta) = k[[t_j^\pm]]^{S_\beta} \),

\[
Y(f, z)g = \frac{1}{\alpha! \beta!} \sum_{\sigma \in S_{\alpha+\beta}} \sigma \cdot (f|_{s_i \rightarrow zs_i} g) \tag{3.15}
\]

where \( S_{\alpha_i+\beta_i} \) permutes \( \{s_i\} \cup \{t_i\} \) and \( \alpha! := \prod_i \alpha_i! \) and likewise for \( \beta! \).

**3.3.3**

**Remark.** A holomorphic additive vertex algebra is equivalent to an algebra with derivation; analogously, a holomorphic multiplicative vertex algebra where \( D(z) = zD \) for a grading operator \( D \) is equivalent to a \( \mathbb{Z} \)-graded algebra. For example, Theorem 3.3.2 for the quiver with one vertex yields the algebra \( \bigoplus_{n \geq 0} S_n^\pm \), where \( S_n^\pm := \mathbb{Z}[s_1^\pm, \ldots, s_n^\pm]^{S_n} \) is the ring of symmetric Laurent polynomials in \( n \) variables, graded by degree (of the polynomial), and the graded product is given by \( Y(-, 1) \).

In general, the resulting \( \mathbb{Z} \)-graded algebra \( K_T(\mathfrak{M}) \) should be compared to K-theoretic Hall algebras. For quivers, the formula (3.15) at \( z = 1 \) is a sort of shuffle product with trivial
kernel, and can be compared to the cohomological Hall algebra computations of [KS11, §2].

3.3.4

Proof of Theorem 3.3.2. The natural projection \( N_{\alpha,\beta} \to M_{\alpha} \times M_{\beta} \) is the \( \mathbb{C}^x \times \mathbb{C}^x \)-equivariant vector bundle \( E_{\alpha,\beta}^1 \oplus E_{\beta,\alpha}^1 \), for which \( \iota = \iota_{\alpha,\beta} \) is the zero section. We implicitly identify \( K_T(N_{\alpha,\beta}) \cong K_T(M_{\alpha} \times M_{\beta}) \) via \( \iota^* \). Then \( \iota^* \) is multiplication by \( \wedge_{-1} \) of this bundle, and hence

\[
\theta_{\alpha,\beta}^*(z) := (\iota_{\alpha,\beta})^* \Theta_{\alpha,\beta}^*(z) = \wedge_{-1} \left( zE_{\alpha,\beta}^0 \oplus z^{-1}E_{\beta,\alpha}^0 \right).
\]

is a polynomial in \( z \) valued in vector bundles. In particular \( \theta^*(1) \) is well-defined. Finally, \( \phi_{\alpha,\beta} : N_{\alpha,\beta} \to M_{\alpha+\beta} \) is a \( \text{GL}(\alpha+\beta)/(\text{GL}(\alpha) \times \text{GL}(\beta)) \)-bundle, and pushforward along it can be defined/computed via \( T_{\alpha+\beta} \)-equivariant localization (cf. the case of flag varieties \( G/B \), which produces the Weyl character formula). Fixed points are indexed by \( \sigma \in S_{\alpha+\beta}/(S_{\alpha} \times S_{\beta}) \) and the tangent space at \( \sigma \) is

\[
\sigma \cdot \left( E_{\alpha+\beta,\alpha+\beta}^0 \oplus E_{\alpha,\alpha}^0 \oplus E_{\beta,\beta}^0 \right) = \sigma \cdot (E_{\alpha,\beta}^0 \oplus E_{\beta,\alpha}^0).
\]

The denominator in the localization formula therefore cancels with the \( \theta^*(1) \) numerator in

\[
(\phi_{\alpha,\beta})^*(\theta_{\alpha,\beta}^*(1) \otimes F) = \sum_{\sigma \in S_{\alpha+\beta}/S_{\alpha} \times S_{\beta}} \sigma \cdot F.
\]

Note that without the \( \theta^*(1) \) factor, the result lands in the \textit{localized} \( \text{K-ring} \) \( k_{\text{loc}} \otimes_k K_{T_{\alpha+\beta}}(\cdot)_{\text{loc}}^{S_{\alpha+\beta}} \), which can be viewed as a completion of \( K_T(M_{\alpha+\beta}) \).

In \( Y(f, z)g \), the input \( F = f \boxtimes g \in K_T(M_{\alpha} \times M_{\beta}) \) is already symmetric with respect to \( S_{\alpha} \times S_{\beta} \), hence for such \( F \) we have \( \sum_{\sigma \in S_{\alpha+\beta}/S_{\alpha} \times S_{\beta}} \sigma = (1/\alpha! \beta!) \sum_{\sigma \in S_{\alpha+\beta}}. \)

4 Wall-crossing

4.1 Quiver-framed objects

4.1.1

Let \( \mathcal{A} \) be a \( \mathbb{C} \)-linear abelian category, and \( \mathcal{B} \) be an exact subcategory closed under isomorphisms and direct sum such that it has a moduli stack \( \mathcal{M} \) satisfying the assumptions in §2.1.1. Let \( T = (\mathbb{C}^x)^n \) act on objects of \( \mathcal{B} \) and therefore on \( \mathcal{M} \). Following [Joy21, Definition 5.5], we define an auxiliary category \( \mathcal{B}^{Q(\kappa)} \) of \textit{quiver-framed} objects in preparation for wall-crossing.
applications. Fix a collection \{ (B_k, F_k) \}_{k \in K} of:

- full exact \( T \)-invariant subcategories \( B_k \subset B \) inducing open substacks \( M_{k,\alpha} \subset M_\alpha \);

- \( \mathbb{C} \)-linear exact functors \( F_k : B_k \to \text{ Vect}_\mathbb{C} \) to the category of \( \mathbb{C} \)-vector spaces such that the dimension \( \lambda_k(\mathcal{E}) := \dim F_k(\mathcal{E}) \) depends only on the class \( \alpha \) of \( [\mathcal{E}] \in M_{k,\alpha} \), and the induced \( \text{Hom}(\mathcal{E}, \mathcal{E}) \mapsto \text{Hom}(F_k(\mathcal{E}), F_k(\mathcal{E})) \) is injective.

As before, all objects and morphisms are assumed \( T \)-equivariant unless specified otherwise.

**Definition.** Let \( Q \) be a quiver with no cycles, with edges \( Q_1 \) and vertices \( Q_0 = Q_0^o \sqcup Q_0^f \) split into ordinary vertices \( \bullet \in Q_0^o \) and framing vertices \( \blacksquare \in Q_0^f \) such that ordinary vertices have no outgoing arrows. Fix a tuple \( \kappa = (\kappa(v))_{v \in Q_0^o} \in K_0^{Q_0^o} \).

The abelian category \( B^Q(\kappa) \) of quiver-framed objects consists of triples \( (\mathcal{E}, V, \rho) \) where:

- \( \mathcal{E} \in B_\kappa := \bigcap_{v \in Q_0^o} B_{\kappa(v)} \);

- \( V = (V_v)_{v \in Q_0^f} \) are finite-dimensional vector spaces; set \( V_v := F_{\kappa(v)}(\mathcal{E}) \) for \( v \in Q_0^o \);

- \( \rho = (\rho_e)_{e \in Q_1} \) are morphisms between the \( V_v \).

A morphism between two triples \( (\mathcal{E}, V, \rho) \) and \( (\mathcal{E}', V', \rho') \) consists of a morphism \( \mathcal{E} \to \mathcal{E}' \), inducing morphisms \( V_v \to V'_v \) for all \( v \in Q_0^o \), along with morphisms \( V_v \to V'_v \) for \( v \in Q_0^f \) intertwining \( \rho \) and \( \rho' \).

Let \( \mathcal{M}^Q(\kappa) = \bigsqcup_{(\alpha, d)} \mathcal{M}_{(\alpha, d)}^Q \) be the moduli stack of \( B^Q(\kappa) \), where \( d := \dim V \). When \( \kappa \) is clear or irrelevant, write \( \mathcal{M}^Q \) for short. All constructions on \( \mathcal{M} \), e.g. the rigidification \( \mathcal{M}^{\text{rig}} \) and the \( T \)-action, lift naturally and compatibly to \( \mathcal{M}^Q \).

4.1.2

Let \( \mathcal{V}_v \in \text{ Vect}_T(\mathcal{M}_{(\alpha, d)}^Q) \) be the tautological bundle associated to the vertex \( v \in Q_0 \). It is a trivial bundle of rank \( d(v) \) for \( v \in Q_0^f \), and a non-trivial bundle with rank \( \mathcal{V}_v = \lambda_{\kappa(v)}(\alpha) \) for \( v \in Q_0^o \). Then, explicitly,

\[
\mathcal{M}_{(\alpha, d)}^Q = \text{tot} \left( \bigoplus_{v \to w} \mathcal{V}_v^* \otimes \mathcal{V}_w \right) \to \mathcal{M}_{\kappa,\alpha} \times \prod_{v \in Q_0^o} [*/\text{GL}(d(v))] \]
is the total space of a vector bundle, where the map to the first factor is the forgetful map

$$\Pi_{2R,\alpha}: M^{Q(\alpha)}_{(\alpha,d)} \to M_{k,\alpha} := \bigcap_{v \in Q^o_0} M_{\kappa(v),\alpha}$$

and the map to the second factor is induced by the \{F_{\kappa(v)}\}_{v \in Q^o_0}. Therefore \(\Pi_{2R,\alpha}\) is a \([A^N/\prod_{v \in Q^o_0} GL(d(v))]\)-bundle, and in general the relative tangent complex \(T_{MQ^{(\alpha)}/2R,\alpha}\) is a two-term complex of vector bundles.

Since \(\Pi_{2R,\alpha}\) is clearly \(C^\times\)-equivariant, it induces a map \(\Pi_{2R,\alpha}^{pl}: M^{Q(\alpha),pl}_{(\alpha,d)} \to M^{pl}_{k,\alpha}\). The preceding discussion applies equally well to \(\Pi_{2R,\alpha}^{pl}\), which has the same fibers as \(\Pi_{2R,\alpha}\).

### 4.1.3

**Example.** Let \(X\) be a smooth projective variety and \(B := \text{Coh}(X)\). For \(k \gg 0\), let \(B_k\) be the subcategory of those \(E\) for which \(H^i(X, E \otimes O_X(k)) = 0\) for \(i > 0\), and take the exact functors

$$F_k: E \mapsto H^0(X, E \otimes O_X(k)),$$

so that \(\lambda_k(\alpha) = P_\alpha(k)\) where \(P_\alpha\) is the Poincaré polynomial. For

$$Q := V \begin{array}{c} \rho \end{array} F_k(E) \quad (4.1)$$

and \(\dim V = 1\), the map \(\Pi_{2R,\alpha}^{pl}: M^{Q(\alpha),pl}_{(\alpha,1)} \to M^{pl}_{k,\alpha}\) is a \([A^\lambda_k(\alpha)/C^\times]\)-bundle whose fiber over \([E]\) is \(F_k(E)\). In K-theory, the relative tangent complex is then

$$T_{MQ^{(\alpha),pl}/M^{pl}_{k,\alpha}} = \mathcal{V} \otimes F_k(E) - \mathcal{O}, \quad (4.2)$$

where \(\mathcal{V}\) (resp. \(F_k(E)\)) denotes the tautological bundle for the framing (resp. ordinary) vertex, and \(\mathcal{O} = \mathcal{V} \otimes \mathcal{V}\) is from the quotient by \(C^\times\).

One can take \(\dim V > 1\) to get “higher rank” pairs, but rank 1 suffices for our purposes.

### 4.1.4

In what follows, \(M^{pl}\) will be equipped with an obstruction theory using which we construct obstruction theories on \(M^{Q(\alpha),pl}\).

**Definition.** Let \(\mathfrak{X}\) be an algebraic stack and \(\mathbb{L}_X\) be its cotangent complex [Ill71]. A morphism \(\varphi: E \to \mathbb{L}_X\) in \(D_{QCoh}(\mathfrak{X})\) is an obstruction theory if \(h^1(\varphi), h^0(\varphi)\) are isomorphisms and \(h^{-1}(\varphi)\)
Given a smooth morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks, two obstruction theories $\varphi : E \to \mathcal{L}_X$ and $\phi : \mathcal{F} \to \mathcal{L}_Y$ are \textit{compatible under} $f$ if they fit into a diagram

$$
\begin{array}{ccccccc}
\mathbb{L}_f[-1] & \xrightarrow{\alpha} & f^* \mathcal{F} & \longrightarrow & E & \longrightarrow & \mathbb{L}_f \\
\| & & \| & & \| & & \|
\mathbb{L}_f[-1] & \xrightarrow{f^* \phi} & f^* \mathcal{L}_Y & \longrightarrow & \mathcal{L}_X & \longrightarrow & \mathbb{L}_f
\end{array}
$$

where both rows are exact triangles (the bottom one for the relative cotangent complex $\mathbb{L}_f$).

Note that $\varphi : \text{cone}(\alpha) \to \mathcal{L}_X$ can be constructed given only $\phi$ and $\alpha$ which make the leftmost square commute, in which case we say $\varphi$ is a \textit{smooth pullback} of $\phi$ along $f$. See [GP99, Appendix B] for an example.

\textbf{Remark.} In the general setup of [Joy21, §2], the lack of $\alpha$, and therefore of smooth pullbacks, is the reason why passing to \textit{derived} stacks is necessary. We avoid this and remain in the land of classical algebraic stacks by assuming that all desired $\alpha$ exist, which is often the case in practice because $\phi$ arises from functorial constructions like the Atiyah class.

\textbf{Example.} Continuing with Example 4.1.3, when $\dim X \leq 2$ there is a perfect obstruction theory on $M$ given by (the dual of) $\text{Ext}^\bullet_X(\mathcal{E}, \mathcal{E})[1]$ at $[\mathcal{E}] \in M$. It admits a smooth pullback along $\Pi M$, namely the desired map $\alpha$ is induced by (the dual of)

$$\text{Ext}^\bullet_X(\mathcal{E}, \mathcal{E}) \cong \text{Ext}^\bullet_X(\mathcal{E}(n), \mathcal{E}(n)) \xrightarrow{\rho^*} \text{Ext}^\bullet_X(V, \mathcal{E}(n))$$

at the point $[V \xrightarrow{\rho} F_k(\mathcal{E})] \in M^{Q(k)}$. The result is a perfect obstruction theory on $M^{Q(k)}$.

Since $\Pi^{pl} : M \to M^{pl}$ is a principal $[*/\mathbb{C}^\times]$-bundle, the perfect obstruction theory on $M$ actually arises by smooth pullback from the one on $M^{pl}$ given by (the dual of)

$$\ker(\text{tr}^0 : \text{Ext}^\bullet_X(\mathcal{E}, \mathcal{E}) \to H^0(X, \mathcal{O}_X))[1], \quad (4.3)$$

where $\text{tr}^0$ is the trace map in degree 0. The same is true of the rigidification map $\Pi^{Q, pl} : M^Q \to M^{Q, pl}$ and the perfect obstruction theory on $M^Q$.

4.1.5

Fix a vertex $v \in Q^I$ with dimension $d(v) = 1$, assuming one exists. (One always will, for us.) Then the rigidification map

$$\Pi^{Q, pl}_{(\alpha, d)} : M_{(\alpha, d)}^Q \to M_{(\alpha, d)}^{Q, pl}$$
can be described non-canonically as fixing an isomorphism $\iota_v : C \xrightarrow{\sim} V_v$, by identifying the $[*/C^\times]$ fiber with the moduli stack of 1-dimensional vector spaces $V_v$. Let

$$I_{(\alpha,d)} := \mathcal{M}^{Q,pl}_{(\alpha,d)} \rightarrow \mathcal{M}^Q_{(\alpha,d)}$$

be the map which forgets this isomorphism; we suppress its dependence on $v$ from the notation. Hence $\Pi^{Q,pl}_{(\alpha,d)} \circ I_{(\alpha,d)} = \text{id}$ and, as in [Joy21, Equation (5.26)], there is a commutative diagram

which we use to define the dashed diagonal map $\Pi^Q_{\mathfrak{M}_{\alpha}}$.

### 4.2 Semistable invariants

#### 4.2.1

Fix a weak stability condition $\tau$ on the abelian category $\mathcal{A}$, in the sense of [Joy21, Definition 3.1], e.g. Gieseker or slope stability if $\mathcal{A} = \text{Coh}(X)$. Let

$$\mathcal{M}^{\text{st}}(\tau) \subset \mathcal{M}^{\text{sst}}(\tau) \subset \mathcal{M}^{\text{pl}}$$

be the substacks of $\tau$-stable and $\tau$-semistable objects, and similarly for $\mathcal{M}^{\text{pl}}_\kappa$. Let $Q$ be a quiver, which for us will be either (4.1) or (4.11); more general $Q$ are required for other wall-crossing applications outside the scope of this paper. In addition to the hypotheses on $\mathfrak{M}$ from §2.1.1, assume the following, cf. [Joy21, Assumptions 5.1, 5.2].

(i) There are obstruction theories on $\mathfrak{M}^{\text{pl}}$, $\mathfrak{M}$, $\mathfrak{M}^{Q,\text{pl}}$ and $\mathfrak{M}^Q$ all compatible under $\Pi^{\text{pl}}$, $\Pi^{Q,\text{pl}}$, $\Pi^{\text{pl}}_{\mathcal{M}}$ and $\Pi^{\text{pl}}_{\mathfrak{M}}$ (Definition 4.1.4). These can often be constructed by smooth pullback starting from an obstruction theory on $\mathfrak{M}^{\text{pl}}$. In addition, the obstruction theory on $\mathfrak{M}$ is bilinear, in the sense of Example 2.1.2, producing a complex $E^{\bullet}_{\alpha,\beta}$ on $\mathfrak{M}_\alpha \times \mathfrak{M}_\beta$ which makes $K^T_{\mathfrak{M}}(\mathfrak{M})$ into a vertex algebra following §3.2.

(ii) For $\alpha$ with no strictly $\tau$-semistables, $\mathfrak{M}^{\text{sst}}_\alpha(\tau) = \mathfrak{M}^{\text{st}}_\alpha(\tau)$ is a quasi-projective scheme with proper $T$-fixed locus, and the restriction of the obstruction theory on $\mathfrak{M}^{\text{pl}}_{\alpha}$ to $\mathfrak{M}^{\text{sst}}_{\alpha}(\tau)$ is perfect, inducing a virtual cycle $\mathcal{O}^{\text{vir}} \in K_T(\mathfrak{M}^{\text{sst}}_{\alpha}(\tau))$.  

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(iii) There exists a “rank” function $r(\alpha) \in \mathbb{Z}_{\geq 0}$ such that $r(0) = 0$ and $r(\alpha) > 0$ for $\alpha \neq 0$, and if $\tau(\alpha) = \tau(\beta)$ then $r(\alpha + \beta) = r(\alpha) + r(\beta)$. This is used to construct weak stability conditions $\tau^Q$ on the auxiliary quiver-framed categories $B^Q_{\kappa}$, see [Joy21, (5.13)]. We assume that (ii) above holds for $\mathcal{M}^{Q(\kappa),\text{sst}}_{\alpha,d}(\tau^Q)$ as well, for all $d$ of interest.

Properness and quasi-projectivity of the $T$-fixed locus mean that whenever $O_{\text{vir}}$ is defined, it is amenable to $T$-equivariant localization and the universal enumerative invariants of Example 2.2.3 are well-defined after passing to localized K-theory and K-homology groups, as in Example 2.2.9.

### 4.2.2

For every class $\alpha$, even if $\alpha$ has strictly $\tau$-semistable objects, our goal is to define an invariant $Z_{\alpha}(\tau) \in K^T_0(\mathcal{M}_{\alpha,1}^{\text{pl}})^\wedge := K^T_0(\mathcal{M}_{\alpha,1}^{\text{pl}})^\wedge \otimes_{\mathbb{Z}} \mathbb{Q}$ which represents “pairing with the virtual cycle” of the corresponding open substack $\mathcal{M}_{\alpha,1}^{\text{sst}}(\tau) \subset \mathcal{M}_{\alpha,1}^{\text{pl}}$ of $\tau$-semistable objects. These invariants will be characterized by the following properties.

(i) For $\alpha$ with no strictly semistable points (so $\mathcal{M}_{\alpha,1}^{\text{sst}}(\tau) = \mathcal{M}_{\alpha,1}^{\text{sst}}(\tau)$),

$$Z_{\alpha}(\tau) = \chi(\mathcal{M}_{\alpha,1}^{\text{sst}}(\tau), O_{\text{vir}} \otimes -)$$

is exactly the element defined in Example 2.2.3.

(ii) If $\tau, \tau'$ are two stability conditions with $\mathcal{M}_{\alpha,1}^{\text{sst}}(\tau) = \mathcal{M}_{\alpha,1}^{\text{sst}}(\tau')$, then $Z_{\alpha}(\tau) = Z_{\alpha}(\tau')$.

(iii) Let $Q$ be the quiver of (4.1), set

$$\bar{\mathcal{O}}^{\text{vir}}_k := O_{\text{vir}} \otimes \mathcal{L}^{-1}(T_{\mathcal{M}^{Q(k),\text{pl}}_{(\alpha,1)}/\mathcal{M}^{\text{pl}}_{k,\alpha}})^\vee \in K^T_0(\mathcal{M}^{Q(k),\text{sst}}_{(\alpha,1)}(\tau^Q))$$

and define

$$\bar{\mathcal{Z}}_{k,\alpha}(\tau^Q) := \chi\left(\mathcal{M}^{Q(k),\text{sst}}_{(\alpha,1)}(\tau^Q), \bar{\mathcal{O}}^{\text{vir}}_k \otimes -\right) \in K^T_0(\mathcal{M}^{Q(k),\text{pl}}_{(\alpha,1)}).$$

For $k \in K$ such that $\mathcal{M}^{\text{sst}}_{\alpha,1}(\tau) \subset \mathcal{M}_{k,\alpha}$,

$$(\Pi_{\mathcal{M}^{\text{pl}}_k})_* \bar{\mathcal{Z}}_{k,\alpha}(\tau^Q) = \sum_{n > 0} \frac{1}{n!} \text{ad}(Z_{\alpha_1}(\tau)) \cdots \text{ad}(Z_{\alpha_2}(\tau)) (\lambda_k(\alpha_1)Z_{\alpha_1}(\tau))$$

where $\text{ad}(Z)(-):=[Z,-]$ is the Lie bracket on $K^T_0(\mathcal{M}^{\text{pl}})^\wedge$ from §3.2.
This is a direct restatement in our K-theoretic framework of the construction [Joy21, Theorem 5.7] of homological invariants, and we will sketch essentially the same proof. A similar treatment of [Joy21, Theorems 5.8, 5.9], with the same additional assumptions, immediately gives wall-crossing formulae (with the same coefficients $\tilde{U}$) describing how the $Z_\alpha(\tau)$ vary as a function of $\tau$. As the proof of this is, again, completely analogous to the original and does not require additional new K-theoretic ingredients, we omit it.

4.2.3

For a suitable partial ordering on the classes $\alpha$, the formula (4.7) is an upper-triangular and invertible transformation between $\{\tilde{Z}_{k,\alpha}(\tau^Q)\}_\alpha$ and $\{Z_\alpha(\tau)\}_\alpha$, and hence can be taken as a definition of the latter which then a priori depend on $k$. The non-trivial content of the construction of the invariants $Z_\alpha(\tau)$ is then the following.

**Theorem.** The invariants $Z_{k,\alpha}(\tau) \in K^T_0(\mathcal{M}_\alpha^\text{pl})^\wedge\mathbb{Q}$ defined uniquely by

$$
(\Pi_{\mathbb{M}_\alpha^\text{pl}})_* \tilde{Z}_{k,\alpha}(\tau^Q) = \sum_{n>0} \frac{1}{n!} \text{ad} (Z_{k,\alpha_1}(\tau)) \cdots \text{ad} (Z_{k,\alpha_n}(\tau)) (\lambda_k(\alpha_1)Z_{k,\alpha_1}(\tau))
$$

are in fact independent of $k$, and satisfy properties 4.2.2.(i) and 4.2.2.(ii).

**Proof.** Independence of $k$ is the difficult step. It follows from Theorem 4.2.5 below and the combinatorial calculation of [Joy21, §9.3].

Assuming independence of $k$, we briefly sketch the remaining properties. If $\alpha$ has no strictly $\tau$-semistables, then $\alpha$ is an indecomposable element and (4.7) becomes

$$
\tilde{Z}_{k,\alpha}(\tau^Q) = \lambda_k(\alpha)Z_\alpha(\tau).
$$

This can be checked directly: $\tau^Q$ is such that $\Pi_{\mathbb{M}_\alpha^\text{pl}}: \mathcal{M}_{(\alpha,1)}^{Q(\text{st},\text{st})}(\tau^Q) \to \mathcal{M}_\alpha^{\text{st}}(\tau)$ is a $\mathbb{P}^{\lambda_k(\alpha)\cdot 1}$-bundle, so by flat base change and push-pull it suffices to compute

$$
\chi \left( \mathbb{P}^{\lambda_k(\alpha)\cdot 1}, \bigwedge^\bullet_1(T)^\vee \right) = \lambda_k(\alpha)
$$
on the fibers of $\Pi_{\mathbb{M}_\alpha^\text{pl}}$. Hence 4.2.2.(i) is actually a consequence of 4.2.2.(iii). Finally, in the same way that (4.8) is an inductive definition of $\{Z_\alpha(\tau)\}_\alpha$, 4.2.2.(ii) can also be checked inductively starting from the (obvious) base case where $\alpha$ has no strictly semistables. See [Joy21, §9.1] for details.
Remark. It is important to mention at this point that there is a myriad of variations on the construction of $Z_{\alpha}(\tau)$. For example, it is common and productive in K-theoretic Donaldson–Thomas theory, or more generally if $\mathcal{A}$ is a CY3 category and $\mathcal{M}$ has a symmetric obstruction theory, to replace $O^\text{vir}$ in (4.4) with the symmetrized virtual structure sheaf $\widehat{O}^\text{vir} := O^\text{vir} \otimes K^{1/2}_{\text{vir}}$

whenever the square root of $K^{1/2}_{\text{vir}} := \det(T^\text{vir})^\vee$ exists [NO16, Tho20]. However, in this symmetric CY3 setting, the obstruction theories obtained on the quiver-framed stacks $\mathcal{M}^{Q(\kappa),\text{pl}}$ by smooth pullback, like in Example 4.1.4, are no longer guaranteed to be perfect on semistable = stable loci. A symmetrized smooth pullback of obstruction theories is required. See [Liu23] for details. In this setting, the appropriate modifications are then

$$Z_{\alpha}(\tau) \text{ (from (4.4)) } \leadsto \hat{Z}_{\alpha}(\tau) := \chi \left( \mathcal{M}^{\text{est}}_{\alpha}(\tau), \hat{O}^\text{vir} \otimes - \right)$$

$$\hat{O}^\text{vir}_k \text{ (from (4.5)) } \leadsto \hat{O}^\text{vir}_k := \hat{O}^\text{vir} \in K_T(\mathcal{M}^{Q(k),\text{pl}}(\tau^Q))$$

$$\tilde{Z}_{k,\alpha}(\tau^Q) \text{ (from (4.6)) } \leadsto \hat{\tilde{Z}}_{k,\alpha}(\tau^Q) := \chi \left( \mathcal{M}^{Q(k),\text{est}}_{(\alpha,1)}(\tau^Q), \hat{O}^\text{vir}_k \otimes \Pi^*_{\text{pl}}(-) \right)$$

where $\Lambda_{-1}(\mathcal{E}) := \Lambda_{-1}(\mathcal{E}) \otimes \det(\mathcal{E})^{1/2}$. Note that the factor $\Lambda_{-1}(\mathcal{T})^\vee$ involving the relative tangent bundle has been “absorbed” into $\hat{O}^\text{vir}_k$; this is a consequence of symmetrized pullback.

By the exact same argument as in §4.2.3, $\hat{\tilde{Z}}_{k,\alpha}(\tau^Q) = \lambda_k(\alpha)\hat{Z}_{\alpha}(\tau)$ whenever there are no strictly $\tau$-semistables, and so the overall form of (4.7) remains unchanged. We believe that Theorem 4.2.5, and therefore Theorem 4.2.3, continue to hold after the vertex operation on $K^T_0(\mathcal{M})$, and therefore the Lie bracket on $K^T_0(\mathcal{M}^{\text{pl}})$, is modified by also symmetrizing

$$\Theta^*_k,\beta(z) \text{ (from (3.7)) } \leadsto \hat{\Theta}^*_k,\beta(z) := \hat{\Lambda}_{-1} \left( z^*E^*_{k,\beta} \oplus z^{-1}E^*_{\beta,k} \right)^\vee.$$
4.2.5

**Theorem.** Let $k_1, k_2 \in K$ with $\mathcal{M}^{\text{st}}_{\alpha}(\tau) \subset \mathcal{M}_{k_1,\alpha}^{\text{pl}} \cap \mathcal{M}_{k_2,\alpha}^{\text{pl}}$. Then

$$0 = \lambda_{k_2}(\alpha)\bar{Z}_{k_1,\alpha}(\tau^Q) - \lambda_{k_1}(\alpha)\bar{Z}_{k_2,\alpha}(\tau^Q) + \sum_{\alpha_1 + \alpha_2 = \alpha} [\bar{Z}_{k_1,\alpha_1}(\tau^Q), \bar{Z}_{k_2,\alpha_2}(\tau^Q)].$$

(4.10)

The remainder of this subsection will be an outline of the proof, which is mostly analogous to that of [Joy21, §9] in the homology setting. The idea is to take the quiver

$$\bar{Q} := \begin{array}{c}
\rho_3 \\
\rho_4 \\
\rho_2 \\
\rho_1 \\
F_{k_1}(\mathcal{E}) \\
F_{k_2}(\mathcal{E}) \\
V_3 \\
V_2 \\
V_1
\end{array}$$

(4.11)

with the stability condition $\tau^Q$ of [Joy21, Definition 9.4], and consider the proper scheme $\mathcal{M}^{\text{pl}}_{\bar{Q}(\kappa),\text{sst}}(\tau^Q)$ where $\kappa := (k_1, k_2)$. This is a so-called “master space” for $\mathcal{M}^{\text{pl}}_{Q(k_1),\text{sst}}(\tau^Q)$ and $\mathcal{M}^{\text{pl}}_{Q(k_2),\text{sst}}(\tau^Q)$, in the sense that it admits a $\mathbb{C}^\times$-action whose fixed loci contributions yield the terms in (4.10).

By assumption, the obstruction theory on $\mathcal{M}^{\text{pl}}_{k_1,\alpha}$ admits a smooth pullback to $\mathcal{M}^{\text{pl}}_{\bar{Q}(\kappa),\text{sst}}(\tau^Q)$, e.g. by the same construction as in Example 4.1.4 for the obstruction theory on $\mathcal{M}^{\text{pl}}_{Q,\text{sst}}(\tau^Q)$.

4.2.6

Let $\mathbb{C}^\times$ act on $\mathcal{M}^{\text{st}}_{Q(\kappa),\text{sst}}(\tau^Q)$ by scaling $\rho_4$ with a weight denoted $z$. Let $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$ and $\mathcal{F}_{k_1}(\mathcal{E}), \mathcal{F}_{k_2}(\mathcal{E})$ denote the (restrictions of the) framing and ordinary tautological bundles respectively, like in Example 4.1.3. The fixed loci of this $\mathbb{C}^\times$ action are as follows; see [Joy21, Propositions 9.5, 9.6] for details.

(i) The obvious fixed component is

$$Z_{\rho_4=0} := \{\rho_4 = 0\},$$

with normal bundle $z^{-1}\mathcal{V}_3 \otimes \mathcal{V}_2$. By $\tau^Q$-stability, $\rho_3 \neq 0$. The forgetful map $Z_{\rho_4=0} \to \mathcal{M}^{Q(k_1),\text{sst}}_{Q(\kappa),\text{sst}}(\tau^Q)$ is a $\mathbb{P}^{\lambda_{k_2}(\alpha)-1}$-bundle, coming from the freedom to choose the map $\rho_2$.

(ii) Another fixed component is

$$Z_{\rho_3=0} := \{\rho_3 = 0\},$$

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where $\mathbb{C}^\times$ must act on $V_3$ with weight $z$ in order for $\rho_4$ to have weight $z$. Hence the normal bundle is $zV_3^\vee \otimes V_1$.

By $\tau\mathcal{Q}$-stability, $\rho_4 \neq 0$. As with $Z_{\rho_4=0}$, there is a projection $Z_{\rho_4=0} \to \mathcal{M}_{k_2,\text{sst}}(\tau\mathcal{Q})$ which is a $\mathbb{P}^{\lambda_2(\alpha)-1}$-bundle.

(iii) When both $\rho_3, \rho_4 \neq 0$, by $\tau\mathcal{Q}$-stability all $\rho_i \neq 0$. Then $\mathbb{C}^\times$ must act on $V_1, V_3$ and $\mathcal{E}$ all with weight $z$ so that $\rho_4$ has weight $z$. In particular, let $\mathcal{E} = \bigoplus_i z^i\mathcal{E}_i$ be the weight decomposition with respect to $z$, so that

$$\rho_i : V_i \to F_i(\mathcal{E}), \quad i = 1, 2$$

must map $V_i$ into $F_i(\mathcal{E}_1)$ and $V_2$ into $F_2(\mathcal{E}_0)$. Then $\mathcal{E} = \mathcal{E}_0 \oplus z\mathcal{E}_1$, otherwise any other summand is destabilizing. Hence the fixed loci here are

$$Z_{\alpha_1,\alpha_2} := \mathcal{M}_{k_1,\text{sst}}(\tau\mathcal{Q}) \times \mathcal{M}_{k_2,\text{sst}}(\tau\mathcal{Q}), \quad \alpha_1 + \alpha_2 = \alpha,$$

where $\mathcal{E}_0$ has class $\alpha_1$ and $\mathcal{E}_1$ has class $\alpha_2$. In K-theory, the normal bundle consists of terms of non-trivial $z$-weight in

$$T_{\mathcal{M}_{k_1,\text{pl}}}^{\text{vir}} = (V_1^\vee \otimes F_{k_1}(\mathcal{E}) + V_2^\vee \otimes F_{k_2}(\mathcal{E}) - \mathcal{O}) + \Pi^{\text{vir}}_{\mathcal{M}_{k_1,\text{pl}}} T_{\mathcal{M}_{k_1,\text{pl}}}^{\text{vir}}$$

where $V_i$ and $F_{k_i}(\mathcal{E})$ are the tautological bundles of Example 4.1.3 pulled back from $\mathcal{M}_{k_1,\text{pl}}$. Explicitly, the normal bundle is therefore

$$z^{-1}V_1^\vee \otimes F_{k_1}(\mathcal{E}_0) + zV_2^\vee \otimes F_{k_2}(\mathcal{E}_1) - (\Pi^{\mathcal{Q}}_{\mathcal{M}_{\alpha_1}} \times \Pi^{\mathcal{Q}}_{\mathcal{M}_{\alpha_2}})^* \left( z\mathcal{E}_{\alpha_1,\alpha_2} + z^{-1}\mathcal{E}_{\alpha_2,\alpha_1} \right).$$

### 4.2.7

**Definition.** Given a polynomial or series $f(t)$, let $[s^k]f(s)$ denote the coefficient of $s^k$, e.g. $\wedge^k\mathcal{E} = [s^k] \wedge^\bullet_{-s} \mathcal{E}$. For a rank-$n$ (virtual) bundle $\mathcal{E}$, let

$$c^K_i(\mathcal{E}) := [(1-s)^{n-i}] \wedge^\bullet_{-s} (\mathcal{E}^\vee)$$

where $\wedge^\bullet_{-s}$ is expanded as a series in $1-s$. Up to a factor of $\det \mathcal{E}$, these are also known as K-theoretic Conner–Floyd classes [CF66]. Let $c^K_{\text{top}}(\mathcal{E}) := c^K_{\text{rank}}(\mathcal{E})$.

In K-theory, $c^K_{\text{top}}(\mathcal{E})$ is the correct analogue of $c_{\text{top}}(\mathcal{E})$ in cohomology, especially when $\mathcal{E} = \mathcal{E}_1 - \mathcal{E}_2$ is a virtual bundle and $c_{\text{top}}(\mathcal{E})$ is distinct from the Euler class $e(\mathcal{E}) := e(\mathcal{E}_1)/e(\mathcal{E}_2)$. One can easily verify the following.

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Lemma. Let $\mathcal{E} \in K^\mathbb{C}_T(\mathfrak{X})$. Then

(i) If $\mathcal{E}$ is a vector bundle, then $c^K_{\text{top}}(\mathcal{E}) = \wedge^*_{-1}(\mathcal{E})$.

(ii) $c^K_i(\mathcal{E} \pm \mathcal{O}) = c^K_i(\mathcal{E})$ for all $i \in \mathbb{Z}$.

(iii) $c^K_{\text{top}}(\mathcal{E}) \in K^\mathbb{C}_T(\mathfrak{X})$ lives in non-localized $K$-theory.

4.2.8

The integrand of interest on the master space $\mathcal{M}^{\tilde{Q},\text{sst}}_{(\alpha,1)}(\tau \tilde{Q})$ is $c^K_{\text{top}}(\mathcal{G})$ where

$$\mathcal{G} := \mathcal{T}_{\mathcal{M}^{\tilde{Q},\text{sst}}_{(\alpha,1)}}(\mathcal{G})|_{/\mathcal{M}^{\tilde{Q},\text{sst}}_{\kappa,\alpha}} - (\mathcal{V}_3^\mathcal{V} \otimes \mathcal{V}_1) \otimes (\mathcal{V}_3^\mathcal{V} \otimes \mathcal{V}_2).$$

An explicit formula for the first term is given in (4.12).

(i) On $Z_{\rho_4=0}$, there is a splitting of the first term into

$$\mathcal{T}_{Z_{\rho_4=0} / \mathcal{M}^{\tilde{Q},\text{sst}}_{\kappa,\alpha}} + \mathcal{V}_3^\mathcal{V} \otimes \mathcal{V}_2|_{Z_{\rho_4=0}}$$

and $\mathcal{V}_3^\mathcal{V} \otimes \mathcal{V}_1$ becomes trivial since it carries the section $\rho_3 \neq 0$. So $\mathcal{G}|_{Z_{\rho_4=0}} = \mathcal{T}_{Z_{\rho_4=0} / \mathcal{M}^{\tilde{Q},\text{sst}}_{\kappa,\alpha}}$.

(ii) On $Z_{\rho_3=0}$, similarly $\mathcal{G}|_{Z_{\rho_3=0}} = \mathcal{T}_{Z_{\rho_3=0} / \mathcal{M}^{\tilde{Q},\text{sst}}_{\kappa,\alpha}}$.

(iii) On $Z_{\alpha_1,\alpha_2}$, both $\mathcal{V}_3^\mathcal{V} \otimes \mathcal{V}_1$ and $\mathcal{V}_3^\mathcal{V} \otimes \mathcal{V}_2$ are trivial, so from (4.2) or otherwise,

$$\mathcal{G}|_{Z_{\alpha_1,\alpha_2}} = \mathcal{T}_{\mathcal{M}^{\tilde{Q}(k_1),\text{sst}}_{(\alpha_1,1)}}(\mathcal{G})|_{/\mathcal{M}^{\tilde{Q}(k_1),\text{sst}}_{\kappa,1,\kappa_1}} + \mathcal{T}_{\mathcal{M}^{\tilde{Q}(k_2),\text{sst}}_{(\alpha_2,1)}}(\mathcal{G})|_{/\mathcal{M}^{\tilde{Q}(k_2),\text{sst}}_{\kappa,1,\kappa_2}} + z^{-1}\mathcal{V}_1^\mathcal{V} \otimes \mathcal{F}_{k_1}(\mathcal{E}_0) + z\mathcal{V}_2^\mathcal{V} \otimes \mathcal{F}_{k_2}(\mathcal{E}_1).$$

In all three cases, (the restriction of) $\mathcal{G}$ is a vector bundle and Lemma 4.2.7.(i) applies.

4.2.9

Putting it all together, by $\mathbb{C}^\times$-localization and push-pull

$$\chi \left( \mathcal{M}^{\tilde{Q}(\kappa),\text{sst}}_{(\alpha,1)}(\tau \tilde{Q}), \mathcal{O}^\mathcal{V} \otimes c^K_{\text{top}}(\mathcal{G}) \otimes \Pi^\mathcal{V}_{\mathcal{M}^{\tilde{Q},\text{sst}}_{\kappa,\alpha}}(\mathcal{G}) \right)$$

$$= \lambda_{k_2}(\alpha) \chi \left( \mathcal{M}^{\tilde{Q}(k_1),\text{sst}}_{(\alpha,1)}(\tau Q), \frac{\tilde{\mathcal{O}}^\mathcal{V}}{1 - z\mathcal{V}_3 \otimes \mathcal{V}_2} \otimes \Pi^\mathcal{V}_{\mathcal{M}^{\tilde{Q},\text{sst}}_{\kappa,\alpha}}(\mathcal{G}) \right)$$

$$+ \lambda_{k_1}(\alpha) \chi \left( \mathcal{M}^{\tilde{Q}(k_2),\text{sst}}_{(\alpha,1)}(\tau Q), \frac{\tilde{\mathcal{O}}^\mathcal{V}}{1 - z^{-1}\mathcal{V}_3 \otimes \mathcal{V}_2} \otimes \Pi^\mathcal{V}_{\mathcal{M}^{\tilde{Q},\text{sst}}_{\kappa,\alpha}}(\mathcal{G}) \right)$$

$$+ \chi \left( \mathcal{M}^{\tilde{Q}(k_1),\text{sst}}_{(\alpha_1,1)}(\tau Q) \times \mathcal{M}^{\tilde{Q}(k_2),\text{sst}}_{(\alpha_2,1)}(\tau Q), \left( \tilde{\mathcal{O}}\mathcal{V} \otimes \tilde{\mathcal{V}}\mathcal{V} \otimes \Theta^*_{\mathcal{M}^{\tilde{Q},\text{sst}}_{\kappa,\alpha}}(\mathcal{G}) \otimes \mathcal{I}^*_{\mathcal{M}^{\tilde{Q},\text{sst}}_{\kappa,\alpha}}(\mathcal{G}) \right) \right) \quad (4.13)$$

40
where $\Theta_{\alpha_1,\alpha_2}(z) = (\prod Q_{\alpha_1} \times \prod Q_{\alpha_2})^*\Theta^*(z)$ with $\Theta^*(z)$ from (3.7). Note that all $\tilde{O}^{\text{vir}}$ are $\mathbb{C}^*$-invariant and therefore independent of $z$, but in the last line $\Theta^*(z)$ encodes part of the normal bundle and therefore involves $z$. Each term is implicitly (well-)defined by $T$-equivariant localization.

Apply the residue map $\rho_K$ to both sides. The lhs vanishes, since $\mathcal{M}_{\alpha,1}(\tau^{\tilde{Q}})$ is a proper DM stack and therefore the pushforward has no poles in $z$. The first two terms on the rhs become $\lambda_k(\alpha)\tilde{Z}_{k,1}(\tau^{\tilde{Q}}) - \lambda_k(\alpha)\tilde{Z}_{k,2}(\tau^{\tilde{Q}})$. Finally, $\tilde{Z}_{k,\alpha}(\tau^{\tilde{Q}})$ can be lifted to $\tilde{Z}_{k,\alpha}(\tau^{\tilde{Q}})$ by definition. Using the commutative diagram

$$
\begin{array}{ccc}
\mathcal{M}_{(\alpha_1,1)}(\tau^{\tilde{Q}}) \times \mathcal{M}_{(\alpha_2,1)}(\tau^{\tilde{Q}}) & \to & \mathcal{M}_{(\alpha_1,1)}(\tau^{\tilde{Q}}) \\
(\prod Q_{\alpha_1} \times \prod Q_{\alpha_2}) & \downarrow & \Phi_{\alpha_1,\alpha_2} \\
\mathcal{M}_{\alpha_1} \times \mathcal{M}_{\alpha_2} & \to & \mathcal{M}_{\alpha} \\
\pi_{\alpha_1}^{\text{pl}} & \downarrow & \pi_{\alpha_2}^{\text{pl}}
\end{array}
$$

the third term then becomes

$$
(\prod_{\alpha}^{\text{pl}}) \rho_K \left( Y(\tilde{Z}_{k,1,\alpha}(\tau^{\tilde{Q}}), z) \tilde{Z}_{k,2,\alpha}(\tau^{\tilde{Q}}) \right) = \left[ (\prod_{\alpha}^{\text{pl}}) \tilde{Z}_{k,1,\alpha}(\tau^{\tilde{Q}}), (\prod_{\alpha}^{\text{pl}}) \tilde{Z}_{k,2,\alpha}(\tau^{\tilde{Q}}) \right] = [\tilde{Z}_{k,1,\alpha}(\tau^{\tilde{Q}}), \tilde{Z}_{k,2,\alpha}(\tau^{\tilde{Q}})]
$$

Note that the stacky morphism $\Phi_{\alpha_1,\alpha_2}: \mathcal{M}_{\alpha_1} \times \mathcal{M}_{\alpha_2} \to \mathcal{M}_{\alpha}$ has non-trivial $\mathbb{C}^*$-equivariance, namely the scaling automorphism of the first factor $\mathcal{M}_{\alpha_1}$ has weight $z$ by the considerations of 4.2.6.(iii). This is the source of the operator $z^{\text{deg}_1}$ in the construction (3.9) of the vertex operation $Y(-, z)$, in which $\Phi$ denoted the non-equivariant version of the map. 

\section{Residue maps}

\subsection{A.0.1}

The passage from vertex algebra to Lie algebra in §3.1.8 is controlled by the residue map $\rho_K$ of Definition 3.1.7. In this appendix, we define residue maps associated to a given $\mathbb{C}^*$-equivariant cohomology theory, mildly generalizing the definition in [Met02], and then show the following.

**Proposition.** $\rho_K$ is the unique residue map for the $K$-theory of Deligne–Mumford stacks.
A.0.2

Let $E_{\mathbb{C}^\times}(-)$ be a $\mathbb{C}^\times$-equivariant (complex-oriented) cohomology theory for a given class of spaces (e.g. schemes, DM stacks, or Artin stacks), and let $k := E_{\mathbb{C}^\times}(*)$ be its base ring. Suppose $E_{\mathbb{C}^\times}(-)$ supports equivariant localization, so it has an Euler class \(^1\) denoted $e(-)$ and a localized base ring

$$k_{\text{loc}} := k[e(V)^{-1} : 1 \neq V \in \text{Pic}_{\mathbb{C}^\times}(*)].$$

Set $E_{\mathbb{C}^\times}(-)_{\text{loc}} := k_{\text{loc}} \otimes_k E(-)$ where $E(-)$ is the non-equivariant theory.

**Definition.** A residue map is a $k$-module homomorphism

$$\rho: k_{\text{loc}} \to k,$$

whose induced maps $\rho \otimes \text{id}: E_{\mathbb{C}^\times}(X)_{\text{loc}} \to E(X)$ we continue to call $\rho$, such that:

(i) (vanishing on non-localized classes) if $\iota: k \hookrightarrow k_{\text{loc}}$ is the inclusion, then $\rho \circ \iota = 0$;

(ii) (normalization) if $z \in \text{Pic}_{\mathbb{C}^\times}(*)$ is the trivial representation of $\mathbb{C}^\times$-weight 1, then

$$\rho(e(z \otimes L)^{-1}) = 1$$

for any non-equivariant $L \in \text{Pic}(X) \subset \text{Pic}_{\mathbb{C}^\times}(X)$.

A.0.3

**Example.** In ordinary cohomology, $e(-)$ is the ordinary Euler class and

$$k_{\text{loc}} = k[u^{-1}], \quad u := e(z). \quad (A.1)$$

When $X = \mathbb{P}^N$, let $h := e(\mathcal{O}(1)) \in H^*(\mathbb{P}^N)$ be the hyperplane class, so that

$$e(z \otimes \mathcal{O}_{\mathbb{P}^N}(1))^{-1} = \frac{1}{u + h} = \frac{1}{u} + \sum_{k=1}^{N} \frac{h^k}{u^{k+1}} \in H^*_{\mathbb{C}^\times}(\mathbb{P}^N)_{\text{loc}}. \quad (A.2)$$

Since $N > 0$ was arbitrary and $\{h^k : 0 \leq k \leq N\}$ are linearly independent, the normalization condition for a residue map $\rho$ therefore requires $\rho(u^{-1}) = 1$ and $\rho(u^k) = 0$. Along with

\(^1\)the restriction of the Thom class to the zero section
\( (A.1) \), this uniquely specifies the only possible cohomological residue map

\[
\rho_{\text{Coh}}(f) := \text{Res}_{u=0}(f(u) \, du).
\]

**A.0.4**

Slightly more subtle is the case of \( \mathbb{C}^\times \)-equivariant K-theory, where \( e(-) := \wedge_{-1}^\bullet (-) \) and

\[
\mathcal{k}_{\text{loc}} = \mathbb{k} \left[ \frac{1}{1 - z^i} : i \in \mathbb{Z} \setminus \{0\} \right] = \mathbb{k}' \oplus \bigoplus_{\gamma} \bigoplus_{m>0} \frac{1}{(1 - \gamma z)^m} \mathbb{k}'
\]

where the sum is over roots of unity \( \gamma \) and \( \mathbb{k}' \supset \mathbb{k} \) is an extension to include roots of unity. The analogue of \((A.2)\) is the expansion

\[
e(z \otimes \mathcal{O}(1))^{-1} = \iota_{\mathcal{O}(1)} \frac{1}{1 - z} = \frac{1}{1 - z} + \sum_{k=1}^{N} \frac{z^k(\mathcal{O}(1) - 1)^k}{(1 - z)^{k+1}} \in K_{\mathbb{C}^\times}(\mathbb{P}^N)_{\text{loc}}.
\]

The normalization condition here yields only the constraint

\[
\rho \left( \frac{1}{1 - z} \right) = 1, \quad \rho \left( \frac{z^k}{(1 - z)^{k+1}} \right) = 0 \quad \forall k > 0 \tag{A.3}
\]

for a residue map \( \rho \), with no constraints on poles in \( z \) at non-trivial roots of unity. This will always be the case for the K-theory of a scheme \( X \), where all line bundles \( \mathcal{L} \in \text{Pic}(X) \subset K(X) \) are unipotent and therefore the expansion of \( e(z \otimes \mathcal{L})^{-1} \) has poles only at \( z = 1 \).

**Remark.** For the K-theory of schemes, the freedom to choose how a residue map \( \rho \) behaves at non-trivial roots of unity was already observed in [Met02], where both

\[
\rho_{\text{naive}}^{\text{naive}}(f) := -\text{Res}_{z=1}(z^{-1} f(z) \, dz) \tag{A.4}
\]

and \( \rho_{K} \) are observed to satisfy \((A.3)\). The former is the unique such residue map which is zero at all other poles at non-trivial roots of unity, see Lemma **A.0.6** and Remark **A.0.7**.

**A.0.5**

When \( \mathfrak{X} \) is instead a DM stack, line bundles \( \mathcal{L} \in \text{Pic}(\mathfrak{X}) \subset K(\mathfrak{X}) \) are in general only quasi-unipotent. This leads to poles appearing at non-trivial roots of unity.
Example. Let $X = \mathbb{P}(n, n, n, \ldots, n)$ be an $N$-dimensional weighted projective space, so that

$$K(X) = \mathbb{Z}[\mathcal{L}]/\langle (\mathcal{L}^n - 1)^{N+1} \rangle$$

by excision long exact sequence or otherwise. The correct K-theoretic expansion is

$$e(z \otimes \mathcal{L})^{-1} = t \mathcal{L}^n \frac{1+z \mathcal{L} + \cdots + z^{n-1} \mathcal{L}^{n-1}}{1-z^n \mathcal{L}^n}$$

$$= \sum_{k=1}^{N} \frac{z^{nk}(\mathcal{L}^n - 1)^k}{(1-z^n)^{k+1}} (1+z \mathcal{L} + \cdots + z^{n-1} \mathcal{L}^{n-1}) \in K_{\mathbb{C}^\times}(X)_{\text{loc}}.$$

This now has poles at all $n$-th roots of unity. As $n, N > 0$ are arbitrary, any residue map $\rho$ must satisfy

$$\rho \left( \frac{z^{nk+a}}{(1-z^n)^{k+1}} \right) = \begin{cases} 1 & k = a = 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{(A.5)}$$

for $0 \leq a < n$.

A.0.6

The following lemma concludes the proof of Proposition A.0.1.

Lemma. The unique residue map satisfying (A.5) is $\rho_K$.

Proof. Let $\rho$ be a residue map and $\gamma_n$ be an $n$-th root of unity. The leading-order poles in

$$\frac{z^{nk}z^a}{(1-z^n)^{k+1}} = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{z^a}{(1-z^n)^{k+1-i}}$$

$$= \frac{1}{n^{k+1}} \sum_{j=0}^{n-1} \frac{\gamma_n^{-ja}}{(1-\gamma_n^j)^{k+1}} + O\left( \frac{1}{(1-z^n)^k} \right),$$

for $0 \leq a < n$, have coefficients (proportional to) $\gamma_n^{-ja}$ which form a Vandermonde matrix of non-zero determinant. Applying $\rho$ to both sides, by induction on the order of poles the only unknowns are $\rho((1-z\gamma_n^i)^{-k-1})$ for $0 \leq i < n$, which are therefore uniquely determined.

It is straightforward to verify from the definition that $\rho_K$ satisfies (A.5).
A.0.7

Remark. The connection between $\rho_K$ and $\rho_K^{\text{naive}}$ from (A.4) is as follows. By definition,

$$\rho_K(f) = (\text{Res}_{z=0} + \text{Res}_{z=\infty}) \left( z^{-1} f(z) \, dz \right).$$

A function $f \in \mathbb{k}_{\text{loc}}$ has poles only at 0, $\infty$, and roots of unity, so by the residue theorem

$$\rho_K(f) = \rho_K^{\text{naive}} - \sum_{\gamma \neq 1} \text{Res}_{z=\gamma} \left( z^{-1} f(z) \, dz \right)$$

where the sum is over all non-trivial roots of unity $\gamma$. Explicitly, for all $m > 0$ and roots of unity $\gamma$,

$$\rho_K \left( \frac{1}{(1-z\gamma)^m} \right) = 1, \quad \rho_K^{\text{naive}} \left( \frac{1}{(1-z\gamma)^m} \right) = \begin{cases} 1 & \gamma = 1 \\ 0 & \text{otherwise}. \end{cases}$$

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