Two-Dimensional Quantum Gravity in Temporal Gauge

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ABSTRACT

We propose a new type of gauge in two-dimensional quantum gravity. We investigate pure gravity in this gauge, and find that the system reduces to quantum mechanics of loop length \(l\). Furthermore, we rederive the \(c=0\) string field theory which was discovered recently. In particular, the pregeometric form of the Hamiltonian is naturally reproduced.

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1 Introduction

Two-dimensional (2D) quantum gravity has been extensively studied for the last few years, since it serves as a toy model of 4D quantum gravity as well as a prototype of string theories [1]. Recent developments in 2D quantum gravity are mainly due to the discovery of the dynamical triangulation method [2]. In particular, the success of summing over all topologies in matrix models [3][4][5] have attracted much interest in this area.

The major advantage of the matrix model approach is that one can calculate all the correlation functions for any genus, and then sum them up in such a way that the factorization property holds. Furthermore, the Schwinger-Dyson equation, which contains all the information in the matrix models, revealed some universal structure of 2D quantum gravity [6][7].

In spite of such great success, there remain some difficulties in the dynamical triangulation method: Firstly, it is hard to identify the operators constructed in the lattice approach with those in the continuum theory. Secondly, there appears a strong restriction on matters coupled to gravity. In fact, until now we have no way to couple the matters whose central charge is greater than one. Thirdly, although the dynamical triangulation method can be extended to higher dimensions [8], analytic calculation has been successfully performed only in two dimensions. All these facts naturally lead us to investigate 2D quantum gravity in continuum approach.

There have been mainly two continuum approaches so far. One is based on the conformal gauge [1][9], and the other on the light-cone gauge [10] (see, e.g., ref. [11] for further references). Although an ADM (Arnowitt-Deser-Misner)-like formalism may be useful for the physical interpretation of the dynamics, these two gauges are not suitable for this formalism. Moreover, they do not work in higher dimensions except for the vicinity of two dimensions [12].

Recently a new approach is advocated for 2D quantum gravity by two of the present authors [13]. In that work 2D quantum gravity was formulated so that minisuperspace approximation is exact, i.e., the system reduces to quantum mechanics of loop length $l$, as was demonstrated in the transfer-matrix formalism of 2D quantum gravity which was initiated in ref. [14].

Let us review here the major consequence of ref. [13]. First one introduces loop-annihilation and -creation operators, $\Psi(l)$ and $\Psi^\dagger (l)$, which satisfy the com-
mutation relation
\[
[\Psi(l), \Psi(l')] = \delta(l - l').
\] (1.1)

Then the following form of Hamiltonian is assumed in the second quantization formalism:
\[
\mathcal{H} = \int dldl' K(l, l') \Psi(l') \Psi(l) + \int dldl' (l + l') \Psi(l) \Psi(l') \Psi(l + l') + g \int dldl' l l' \Psi(l + l') \Psi(l) \Psi(l') + \int d\rho(l) \Psi(l).
\] (1.2)

The first term (kinetic term) represents the amplitude of cylinder where initial and final loop lengths are \(l'\) and \(l\). The second term describes the splitting of loop, and the third term describes the merging of loops, so that \(g\) is identified with the renormalized string coupling constant. The fourth term represents cap amplitude, i.e., the amplitude for instantaneous vanishing of loop with length \(l\).

The main result in ref. [13] is that the Schwinger-Dyson equation for \(c = 0\) matrix model is completely reproduced if we assume \(K(l, l') \equiv 0\) (pregeometric type) and \(\rho(l) \equiv \delta''(l) - \mu \delta(l)\) with \(\mu\) the renormalized cosmological constant.

The above string field Hamiltonian was derived via the dynamical triangulation approach [13]. In the present paper, we will attempt to rederive it from the continuum approach. We propose a new type of gauge (“temporal” gauge) and show that the above Hamiltonian \(\mathcal{H}\) is naturally obtained in this gauge. In particular, the vanishing of the kinetic term, \(K(l, l') = 0\), will be demonstrated. In the following, pure gravity is investigated, while inclusion of matters will be reported elsewhere.

The present paper is organized as follows: In sect. 2, we introduce our temporal gauge by using the ADM decomposition. In sect. 3, we briefly describe the calculation of the cylinder amplitude, and then show that 2D pure gravity actually reduces to quantum mechanics of loop length \(l\). In sect. 4, we discuss how the second-quantized Hamiltonian \(\mathcal{H}\) is obtained in our formulation. In particular, we present the mechanism for vanishing of the kinetic term. Sect. 5 is devoted to conclusions.

\[\text{2 Recently, the string field Hamiltonian was also rederived from the collective field theory approach to the matrix models [14]. The fictitious time they introduced for the stochastic quantization of the matrix models coincides with the time coordinate considered in ref. [13] using the geodesic distance.}\]
2 ADM decomposition and temporal gauge

It is known that the ADM decomposition is useful for the Hamiltonian formalism of quantum gravity, for which a metric $g_{\mu\nu}$ (with Euclidian signature) on a two-dimensional manifold is parametrized as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = (N dx^0)^2 + h (\lambda dx^0 + dx^1)^2,$$

i.e.,

$$g_{\mu\nu} = \begin{bmatrix} N^2 + h\lambda^2 & h\lambda \\ h\lambda & h \end{bmatrix}. \quad (2.2)$$

Here $N(x^0, x^1)$ is the lapse function, $\lambda(x^0, x^1)$ the shift function and $h(x^0, x^1)$ the metric on the time slice at $x^0$. Eq. (2.1) is nothing but Pythagoras' theorem; two time slices at $x^0$ and $x^0 + dx^0$, respectively, are separated from each other by geodesic distance $N dx^0$, while the space coordinate $x^1$ shifts by $\lambda dx^0$ under such time evolution.

In this section as well as the following one, we consider a cylinder $M$ with two loop boundaries $C$ and $C'$. The cylinder can be regarded as the world sheet swept by a loop under its time evolution, and we call $C$ the incoming loop and $C'$ the outgoing loop. We further assume that there occurs no splitting of loop, and those two loops $C$ and $C'$ are separated from each other by geodesic distance $N dx^0$, while the space coordinate $x^1$ shifts by $\lambda dx^0$ under such time evolution.

To state our assumption more precisely, we first prepare some notations. Let $d(p, q)$ be the geodesic distance between two points $p, q \in M$. Let then $d(p ; S)$ be the minimal geodesic distance between a point $p \in M$ and a subset $S \subset M$,

$$d(p ; S) \equiv \inf_{q \in S} d(p, q). \quad (2.3)$$

Furthermore, we denote by $C_d$ the subset consisting of the points that are separated from the incoming loop $C$ by geodesic distance $d$:

$$C_d \equiv \{ p \in M \mid d(p ; C) = d \} \quad (2.4)$$

Note here that $C_0 = C$, and also that each subset $C_d$ can always be identified with some time slice if we apply the ADM decomposition of metric to a coordinate system in which $C$ is represented by $x^0 = \text{const.}$. Thus, the assumption we made above can be rephrased as that the metrics $g_{\mu\nu}$ on $M$ satisfy the following two conditions:

(i) $C_d$ is homeomorphic to $C \quad (0 \leq d \leq D)$,

(ii) $C_D = C'$. \quad (2.5)
Now we are going to calculate the following functional integral over those metrics that satisfy the condition (2.5):

\[
f(l', l; D) \equiv \int \frac{Dg_{\mu\nu}}{\text{Vol}(\text{diff})} \exp \left\{ -\mu_0 \int d^2 x \sqrt{g} \right\} \times \delta \left( \int_C \sqrt{g_{\mu\nu} dx^\mu dx^\nu} - l \right) \delta \left( \int_{C'} \sqrt{g_{\mu\nu} dx^\mu dx^\nu} - l' \right) \cdot \theta \left[ \text{condition (2.5)} \right],
\]

(2.6)

where \( \mu_0 \) is the bare cosmological constant. In this expression, the functional measure \( Dg_{\mu\nu} \) is defined via the positive definite norm of the infinitesimal deformation of metric around \( g_{\mu\nu} \) as follows:

\[
\| \delta g_{\mu\nu} \|_g^2 \equiv \int d^2 x \sqrt{g} g^{\mu\alpha} g^{\nu\beta} \delta g_{\mu\nu} |_g \delta g_{\alpha\beta} |_g.
\]

(2.7)

The volume of diffeomorphism group \( \text{Vol}(\text{diff}) \) is identified with the functional integral over all diffeomorphisms,

\[
\text{Vol}(\text{diff}) \equiv \int Df^\mu,
\]

(2.8)

where \( Df^\mu \) is determined by the norm of infinitesimal diffeomorphism \( (x^\mu \mapsto x^\mu - \delta f^\mu(x)) \) with

\[
\| \delta f^\mu \|_g^2 = \int d^2 x \sqrt{g} g_{\mu\nu} \delta f^\mu \delta f^\nu.
\]

(2.9)

We parametrize the gauge slice by \( \bar{g}_{\mu\nu} \), and impose on it the following “temporal” gauge condition:

\[
\bar{N} \equiv 1,
\]

\[
\partial_1 \bar{h} \equiv 0,
\]

(2.10)

(2.11)

or equivalently,

\[
\bar{g}_{\mu\nu}(x^0 = t, x^1 = x) \equiv \begin{bmatrix} 1 + l(t)^2 k(t, x)^2 & l(t)^2 k(t, x) \\ l(t)^2 k(t, x) & l(t)^2 \end{bmatrix}.
\]

(2.12)

Here we have introduced \( l(t)^2 (= \bar{h}) \) as an integration constant for eq. (2.11), and denote \( \bar{\lambda}(t, x) \) by \( k(t, x) \) for simplicity.

The geometrical meaning of this condition should be clear: Eq. (2.10) implies that the time coordinate \( x^0 \) is chosen directly as the geodesic distance from the incoming loop \( C \), and eq. (2.11) ensures that the space-like metric \( h \) has no fluctuations along time slices. All these properties are consistent with the ones in the
transfer-matrix formalism of 2D quantum gravity based on the dynamical triangulation method [14]. In the following, we will take a global coordinate system \((x^0, x^1) = (t, x)\) on \(M\), such that \(0 \leq x \leq 1\), and \(C\) and \(C'\) are represented by \(t = 0\) and \(t = D\), respectively.

For the parametrization given above, it may be useful to introduce two vectors \(\vec{n}\) and \(\vec{s}\) which are normal and tangential to time slices, respectively, so that

\[
n^\mu \equiv (1, -k), \quad s^\mu \equiv (0, l^{-1}).
\]  

\(2.13\)

Note that they satisfy the following orthonormal conditions:

\[
\bar{g}_{\mu\nu} n^\mu n^\nu = \bar{g}_{\mu\nu} s^\mu s^\nu = 1,
\]

\[
\bar{g}_{\mu\nu} n^\mu s^\nu = 0.
\]  

\(2.14\)

It is easy to see that eq. (2.8) evaluated at \(g_{\mu\nu} = \bar{g}_{\mu\nu}\) is rewritten in the following form:

\[
\| \delta f^\mu \|_\bar{g}^2 = \int d^2x \int \left[ (\delta v^n)^2 + (\delta v^s)^2 \right],
\]  

\(2.15\)

where \(\delta v^n\) and \(\delta v^s\) are infinitesimal diffeomorphisms in the normal and tangential directions, respectively, \(i.e.,\)

\[
\delta v^n \equiv \bar{g}_{\mu\nu} n^\mu \delta f^\nu, \quad \delta v^s \equiv \bar{g}_{\mu\nu} s^\mu \delta f^\nu,
\]  

\(2.16\)

and we impose on them the following boundary conditions:

\[
\delta v^n \big|_{\partial M} = 0 \quad \text{(Dirichlet)},
\]

\[
\delta v^s(x^0, x^1 = 0) = \delta v^s(x^0, x^1 = 1) \quad \text{(periodic)}.
\]  

\(2.17\)

Furthermore, since the infinitesimal deformation of metric around \(\bar{g}_{\mu\nu}\) is generally expressed as:

\[
\delta g_{\mu\nu} = \delta \bar{g}_{\mu\nu} + \nabla_\mu \delta f_\nu + \nabla_\nu \delta f_\mu,
\]  

\(2.18\)

we can also rewrite eq. (2.7) into the following form after straightforward calculation as will be proved in Appendix A:

\[
\| \delta g_{\mu\nu} \|_\bar{g}^2 = \int dt \frac{\delta l(t)^2}{l(t)} + \frac{1}{2} \int d^2x l (l \delta k)^2 \]

\[
+ \int d^2x l \left[ (D_n \delta v^n)^2 + (D_s \delta v^s)^2 \right]
\]

\[
= \int dt \frac{\delta l(t)^2}{l(t)} + \frac{1}{2} \int d^2x l (l \delta k)^2
\]

\[
+ \int d^2x l \left[ \delta v^n D_n^\dagger D_n \delta v^n + \delta v^s D_s^\dagger D_s \delta v^s \right].
\]  

\(2.19\)

\(3\) \(\nabla_\mu\) is the covariant derivative with respect to \(\bar{g}_{\mu\nu}\).
Here $D_n$ and $D_s$ are the derivatives in the normal and tangential directions, respectively:

\[
\begin{align*}
D_n &\equiv n^\mu \partial_\mu = \partial_0 - k \partial_1, \\
D_s &\equiv s^\mu \partial_\mu = l^{-1} \partial_1,
\end{align*}
\]

and $D_n^\dagger$, $D_s^\dagger$ are their hermitian conjugates under the diffeomorphism-invariant measure $\int d^2 x \sqrt{\bar{g}} = \int d^2 x l$:

\[
\begin{align*}
D_n^\dagger &= -\partial_0 + k \partial_1 + k' - \frac{i}{l}, \\
D_s^\dagger &= -l^{-1} \partial_1.
\end{align*}
\] (2.21)

Due to the boundary condition (2.17), the operator $D_n^\dagger D_n$ has no zero-modes while those of $D_s^\dagger D_s$ are infinitely degenerated, each of them specified by time $t$. Thus, to simplify the calculation of the determinant of $D_s^\dagger D_s$, we consider the following eigenvalue problem with $t$ regarded as a parameter:

\[
D_s^\dagger D_s(t) \psi_a(x; t) = \lambda_a(t) \psi_a(x; t) \quad (a \geq 0).
\] (2.22)

Here $D_s^\dagger D_s(t) = -l(t)^{-2} \partial_x^2$, and the wave functions $\psi_a(x; t)$ are normalized as

\[
l(t) \int_0^1 dx \psi_a(x; t) \psi_b(x; t) = \delta_{ab}.
\] (2.23)

Note that $D_s^\dagger D_s(t)$ is hermitian under the measure $\int_0^1 dx$, and the zero-mode at $t$ is given by $\psi_0 = l(t)^{-1/2}$.

Now we can give a definite meaning to eq. (2.19). We first expand $\delta v^s(t, x)$ as

\[
\delta v^s(t, x) = \sum_{a \geq 0} \delta c_a(t) \psi_a(x; t).
\] (2.24)

Then the tangential components of the norms $\| \delta f^\mu \|_{\bar{g}}^2$ and $\| \delta g_{\mu\nu} \|_{\bar{g}}^2$ are expressed, respectively, as

\[
\int dt \sum_{a \geq 0} (\delta c_a(t))^2 \quad \text{and} \quad \int dt \sum_{a \geq 1} \lambda_a(t) (\delta c_a(t))^2.
\] (2.25)

---

4 We use the following abbreviation:

\[
f' = \partial_1 f, \quad f'' = \partial_0 f.
\]
Thus, we obtain

\[
\mathcal{D}g_{\mu\nu} = \prod_t \frac{dl(t)}{\sqrt{l(t)}} \cdot \mathcal{D}_t k \cdot \mathcal{D}v^n \cdot \prod_t \prod_{a \geq 1} dc_a(t) \\
\cdot \text{Det}^{1/2} D_n^t D_n \cdot \prod_t \text{Det}^{1/2} D_s^t D_s(t),
\]

(2.26)

\[
\mathcal{D}f^\mu = \mathcal{D}v^n \cdot \prod_t \prod_{a \geq 1} dc_a(t) \cdot \prod_t dc_0(t).
\]

(2.27)

A simple calculation shows that

\[
\prod_t \text{Det}^{1/2} D_s^t D_s(t) = \prod_t \prod_{a \geq 1} \lambda_a(t)^{1/2}
\]

\[
= \prod_t l(t) e^{-\mu_1 \int dt \sqrt{l(t)}},
\]

(2.28)

where \(\mu_1\) is a constant depending on ultra-violet cutoff. Note that \(\int dt \sqrt{l(t)}\) is nothing but the cosmological term \(\int d^2x \sqrt{\bar{g}}\) in our temporal gauge.

The integration over zero-modes, \(\prod_t \int dc_0(t)\), reflects the existence of residual gauge symmetry; even after the gauge fixing (2.10) and (2.11), we can still twist a loop in the tangential direction at each time:

\[
t \mapsto t
\]

\[
x \mapsto x - \alpha(t) \quad (0 \leq \alpha(t) < 1).
\]

(2.29)

Since this transformation may be written as \(\delta f^\mu_{\text{res}} = \delta_t^\mu \cdot \delta \alpha(t)\), it will lead to

\[
\delta v^n_{\text{res}} \equiv n_\mu \delta f^\mu_{\text{res}} = 0,
\]

\[
\delta v^s_{\text{res}} \equiv s_\mu \delta f^\mu_{\text{res}} = l(t) \delta \alpha(t),
\]

(2.30)

which implies that the residual gauge transformation corresponds to shifting \(v^s(t, x)\) by a constant at each time \(t\). In order to fix such translational symmetry, it is enough to specify the value of \(v^s\) for some spatial coordinate (say, \(x_0\)) at each time \(t\). This is achieved by inserting the following expression into eq. (2.27):

\[
1 = \prod_t \left[ \int_0^1 \! da(t) \, l(t) \delta (v^s(t, x_0) - l(t)\alpha(t)) \right].
\]

(2.31)

Since \(\psi_0 = l(t)^{-1/2}\), the delta function can be rewritten as

\[
\delta (v^s(t, x_0) - l(t)\alpha(t)) = \delta \left( \frac{c_0(t)}{\sqrt{l(t)}} + \sum_a c_a(t) \psi_a(x_0; t) - l(t)\alpha(t) \right)
\]

\[
= \sqrt{l(t)} \delta(c_0(t) + \cdots).
\]

(2.32)
Substituting eqs. (2.31) and (2.32) into eq. (2.27), we obtain

\[ D f^\mu = D v^\nu \cdot \prod_{t \geq 1} dc_n(t) \cdot \prod_t l(t)^{-3/2}. \]  (2.33)

Here we have omitted the \((l\text{-independent})\) factor \(\prod_t \int_0^1 d\alpha(t)\).

Combining eqs. (2.26), (2.27), (2.28) and (2.33), we thus obtain

\[ \frac{D g_{\mu\nu}}{\text{Vol(diff)}} \equiv \frac{D g_{\mu\nu}}{D f^\mu} = \left[ \frac{dl}{l} \right] \cdot D l k \cdot \text{Det}^{1/2} D_n^1 D_n \cdot e^{-\mu t} \int dl(t). \]  (2.34)

Here we have introduced the following symbol:

\[ \left[ \frac{dl}{l} \right] \equiv \prod_t \frac{dl(t)}{l(t)}. \]  (2.35)

Recall that the measure \(D_l k\) is defined by the norm

\[ \| \delta k \|_i^2 = \int d^2 x l \ (l \delta k)^2. \]  (2.36)

### 3 Integration over shift function and the cylinder amplitude

What remains now is to evaluate the following functional of \(l(t)\):

\[ F[l] \equiv \int D_l k \ Det^{1/2} D_n^1 D_n[l,k]. \]  (3.1)

To do so, we first introduce a new measure \(D'_l k\) for the shift function \(k\) which is defined by the following norm:

\[ \| \delta k \|_i^2 \equiv \int d^2 x l \ (\delta k)^2. \]  (3.2)

This measure is special in the sense that it is invariant under the following (infinitesimal) transformation:

\[ \tilde{\delta}l(t) \equiv l(t)^2 \partial_t \left( \frac{\delta \rho(t)}{l(t)} \right), \]

\[ \tilde{\delta}k(t,x) \equiv -\partial_t \left( k(t,x) \delta \rho(t) \right), \]  (3.3)
where $\bar{\delta}\rho(t)$ is an arbitrary ($t$-dependent) function satisfying the condition $\bar{\delta}\rho(0) = \bar{\delta}\rho(D) = 0$. In fact, one can easily show that

$$
\bar{\delta} \parallel \delta k \parallel_l^2 = - \int d^2x \partial_t \left[ l(t) \bar{\delta} \rho(t) (\delta k(t, x))^2 \right] = 0. \tag{3.4}
$$

This $\bar{\delta}$-transformation is attributed to a Weyl rescaling of the metric, $\bar{g}_{\mu\nu} \mapsto \exp(2\bar{\delta}\dot{\rho}(t)) \cdot \bar{g}_{\mu\nu}$, accompanied with a diffeomorphism $(t, x) \mapsto (t + \bar{\delta}\rho(t), x)$ such that our temporal-gauge condition is restored. A detailed explanation will be given in Appendix B.

Since the transformation of measure from $Dl_k$ to $D'_l k$ may be regarded as a Weyl transformation, the Jacobian is expressed by the Liouville action

$$
S_L[\phi; g_{\mu\nu}] = \int d^2 x \sqrt{g} \left\{ \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - R_g \phi \right\} \tag{3.5}
$$

as follows:

$$
D_l k = D'_l k \exp \left\{ - \frac{1}{48\pi} S_L[\phi = \ln l; \bar{g}_{\mu\nu}] \right\} 
= D'_l k \exp \left\{ \frac{1}{32\pi} \int_0^D dt \frac{i^2}{l} \right\}. \tag{3.6}
$$

Thus, $F[l]$ is expressed as

$$
F[l] = \exp \left\{ \frac{1}{32\pi} \int_0^D dt \frac{i^2}{l} \right\} \exp \{-W[l]\}, \tag{3.7}
$$

where we have introduced

$$
\exp\{-W[l]\} \equiv \int D'_l k \ \text{Det}^{1/2} D_n^\dagger D_n[l, k]. \tag{3.8}
$$

The differential operator $D_n^\dagger D_n$ is not elliptic, and thus need a special care in defining its determinant. We here notice that the Laplacian

$$
\Delta[l, k] \equiv - \frac{1}{\sqrt{g}} \partial_\mu \left( \sqrt{g} g^{\mu\nu} \partial_\nu \right)
= D_n^\dagger D_n + D_s^\dagger D_s \tag{3.9}
$$

satisfies the following equation for any positive constant $\beta$:

$$
\Delta[\beta^{-1} l, k] = D_n^\dagger D_n + \beta^2 D_s^\dagger D_s. \tag{3.10}
$$

\footnote{In the following, we set $0 \leq t \leq D, \ 0 \leq x \leq 1.$}
Using this equation, we thus can define $D_n^\dagger D_n$ as the limit of the elliptic operator $\Delta$:

$$D_n^\dagger D_n[l, k] \equiv \lim_{\beta \to +0} \Delta[\beta^{-1}l, k]. \quad (3.11)$$

Furthermore, one can prove the following identity (see Appendix B):

$$\text{Det}^{1/2} \Delta[l + \bar{\delta}l, k + \bar{\delta}k] = \exp \left[ -\bar{\delta} \int_0^D dt \left\{ \frac{\mu_2 l}{\beta} + \frac{1}{24\pi} \frac{\dot{l}(t)^2}{l(t)} \right\} \right] \text{Det}^{1/2} \Delta[l, k] \quad (3.12)$$

with $\mu_2$ some regularization-dependent constant. Thus, we have

$$\text{Det}^{1/2} D_n^\dagger D_n[l + \bar{\delta}l, k + \bar{\delta}k] = \lim_{\beta \to +0} \text{Det}^{1/2} D_n^\dagger D_n[l, k]. \quad (3.13)$$

Combining eqs. (3.8) and (3.13), and using the invariance of the measure $\mathcal{D}[l]$ under the transformation (3.3), we can evaluate the $l$-dependence of the functional $W[l]$ as follows:

$$\exp\{-W[l + \bar{\delta}l]\} = \int \mathcal{D}[l + \bar{\delta}l] \text{Det}^{1/2} D_n^\dagger D_n[l + \bar{\delta}l, k]$$

$$= \int \mathcal{D}[l + \bar{\delta}l] \text{Det}^{1/2} D_n^\dagger D_n[l + \bar{\delta}l, k + \bar{\delta}k]$$

$$= \lim_{\beta \to +0} \int \mathcal{D}[l + \bar{\delta}l] \exp \left[ -\bar{\delta} \int_0^D dt \left\{ \frac{\mu_2 l}{\beta} + \frac{1}{24\pi\beta} \frac{\dot{l}(t)^2}{l(t)} \right\} \right] \text{Det}^{1/2} D_n^\dagger D_n[l, k]$$

$$= \lim_{\beta \to +0} \exp \left[ -\bar{\delta} \int_0^D dt \left\{ \frac{\mu_2 l}{\beta} + \frac{1}{24\pi\beta} \frac{\dot{l}(t)^2}{l(t)} \right\} \right] \exp\{-W[l]\}. \quad (3.14)$$

Namely,

$$W[l + \bar{\delta}l] = W[l] + \lim_{\beta \to +0} \bar{\delta} \int_0^D dt \left\{ \frac{\mu_2 l}{\beta} + \frac{1}{24\pi\beta} \frac{\dot{l}(t)^2}{l(t)} \right\} \exp\{-W[l]\}. \quad (3.15)$$

Thus, integrating this equation, and using eq. (3.14), we finally obtain

$$F[l] = \lim_{\beta \to +0} \exp \left[ -\bar{\delta} \int_0^D dt \left\{ \frac{\mu_2 l}{\beta} + \frac{1}{24\pi\beta} \frac{\dot{l}(t)^2}{l(t)} \right\} \right], \quad (3.16)$$

where we have replaced $(24\pi\beta)^{-1} - (32\pi)^{-1}$ by $(2\beta)^{-1}$ for simplicity, and $\mu_2'$ is a $\beta$-dependent constant.
Now we can calculate the cylinder amplitude $f(l', l; D)$ defined in eq. (2.6). We first substitute eqs. (2.34), (3.1) and (3.16) into eq. (2.6), and then obtain

$$f(l', l; D) = \lim_{\beta \to +0} \int \left[ \frac{dl}{l} \right] e^{-S_\beta[l]} \delta(l(t=0) - l) \delta(l(t=D) - l'),$$

where

$$S_\beta[l] = \int dt L_\beta(l, \dot{l})$$

$$= \int dt \left( \frac{i^2}{2\beta l} + \mu l \right),$$

and $\mu \equiv \mu_0 + \mu_1 + \mu_2'$ is the renormalized cosmological constant.

For further calculation it is useful to introduce one-body Hamiltonian $H$ from this (Euclidian) action $S_\beta[l]$. Since the momentum $p$ conjugate to $l$ is

$$p \equiv i \frac{\partial L_\beta}{\partial \dot{l}} = i \frac{\dot{l}}{\beta l},$$

we can construct $H$ as

$$H \equiv \lim_{\beta \to +0} \left( ip \dot{l} + L_\beta \right)$$

$$= \lim_{\beta \to +0} \left( \frac{\beta}{2} p^2 l + \mu l \right)$$

$$= \mu l.$$

Thus, the cylinder amplitude $f(l', l; D)$ is now evaluated to be

$$f(l', l; D) = \text{const.} \langle l' \mid e^{-DH} \mid l \rangle$$

$$= \text{const.} e^{-\mu Dl} \delta(l - l').$$

The normalization constant is determined to be unity by imposing the following composition law:

$$f(l', l; D) = \int_0^\infty d l'' f(l', l'' : D_1) f(l'', l : D - D_1) \quad (0 < D_1 < D).$$

A comment is now in order. To make eq. (3.22) valid even for the functional integral expression (3.17), we have to avoid an overintegration which possibly occurs at the end points in time (i.e., boundaries of cylinder). However, this may be understood by regarding the functional measure $[dl/l]$ as

$$\left[ \frac{dl}{l} \right] \equiv \prod_{0 \leq t \leq D} \frac{dl(t)}{l(t)}.$$

Since we adopt the rule that in eq. (3.23) $t = 0$ is excluded from the initial time, the expression $\delta(l(t=0) - l)$ in eq. (3.17) now should be read as $\delta(l(t=+0) - l)$. 12
4 Derivation of $c = 0$ string field theory

In the previous section, we found that in our gauge (2.10) and (2.11), the calculation of the cylinder amplitude reduces to that of quantum mechanics of loop length $l$ described by the Hamiltonian $H = \mu l$. In this section, we first evaluate the amplitude for the case in which a loop splits into two loops. In fact, such amplitude can be obtained in an almost similar way to the cylinder one (3.17) except that we have no residual gauge symmetry at the moment (say, $t = t_0$) when an incoming loop splits (see Fig. 2). Since such modification is only relevant to the boundary condition of the tangential component $v^s$ (or zero-modes of the operator $D_s^\dagger D_s = -l^{-2} \partial_t^2$), the functional measure $\mathcal{D}g_{\mu
u}/\text{Vol}(\text{diff})$ is different from the cylinder one simply by the factor of the length at $t = t_0$, $l(t_0)$, as is the case for one-dimensional quantum gravity (see, for example, ref. [18]). It should be noted that in the path integration over $l(t)$ (eq. (3.23)), we exclude the initial time $t = 0$. Therefore, when one incoming cylinder splits into two outgoing ones, we multiply the length of the boundary loop of the incoming cylinder. Similarly, when two incoming cylinders merge into outgoing one, the lengths of the two boundary loops of the incoming cylinders should be multiplied (see Fig. 3).

In summary, we have the following Feynman rule for our loop dynamics:

\[
\text{propagator:} \quad \frac{\mu l \delta(l - l')}{l} = \mu l \delta(l - l')
\]

loop-splitting vertex:

---

6 The following three figures should be understood to represent graphs with external lines (cylinders) amputated.
To go into the second quantization formalism, we further need to introduce the cap amplitude \( \rho(l) \) (amplitude for instantaneous vanishing of the loop with length \( l \)):

\[
\text{cap:} \quad \otimes \quad l \quad = \quad \rho(l)
\]

Using the above rule we now can write down the second-quantized Hamiltonian in the following form:

\[
\mathcal{H} \quad = \quad \mu \int dl \, l \, \Psi^\dagger(l) \, \Psi(l) \\
+ \int dl \, dl' \, (l + l') \, \Psi^\dagger(l) \, \Psi^\dagger(l') \, \Psi(l + l') \\
+ \quad g \int dl \, dl' \, l \, l' \, \Psi^\dagger(l + l') \, \Psi(l) \, \Psi(l') \\
+ \int dl \, \rho(l) \, \Psi(l) .
\]

Here we have set to unity the coefficients of the second and forth terms by appropriately rescaling the Hamiltonian \( \mathcal{H} \) (or equivalently, its conjugate geodesic-time).
and the operators $\Psi(l)$, $\Psi^\dagger(l)$. Furthermore, we can eliminate the first term (kinetic term) by making a shift of $\Psi^\dagger(l)$ as

$$\Psi^\dagger(l) \mapsto \Psi^\dagger(l) - \frac{\mu}{2} \delta(l)$$

(4.2)

and formally using the identity $l \delta(l) = 0$. Finally the Hamiltonian reads

$$\mathcal{H} = \int dl dl' (l + l') \Psi^\dagger(l) \Psi^\dagger(l') \Psi(l + l')$$

$$+ g \int dl dl' l l' \Psi^\dagger(l + l') \Psi(l) \Psi(l')$$

$$+ \int dl \rho(l) \Psi(l).$$

(4.3)

If we could prove that $\rho(l) = \delta''(l) - \mu \delta(l)$, then we would succeed in reproducing the Hamiltonian for $c = 0$ string field theory of ref. [13], and so all the results of $c = 0$ matrix model. However, at this stage we have no definite proof for this statement.

5 Conclusion

In this paper, we have shown that pure gravity reduces to quantum mechanics of loop length $l$. Furthermore, we have demonstrated the vanishing of the kinetic term in the second-quantized form, which yields the pregeometric Hamiltonian for $c = 0$ string field theory.}

---

7 We can still make a handwaving argument as follows: First, we consider a cylinder with two loop boundaries of length $l$ and 0. Due to eq. (3.21), the amplitude $f(0, l; D)$ for such configuration has its support only at $l = 0$. On the other hand, the cylinder can be obtained from a tiny (almost flat) disk by making a hole on it. However, there exists arbitrariness for the location of the hole, and such arbitrariness should be proportional to the area of the disk. Thus, it is plausible to assume that $f(0, l; D) \sim D l \rho(l)$ for small $D$ and $l$, or

$$\rho(l) \sim \frac{1}{D l} f(0, l; D)$$

$$= \frac{1}{D l} e^{-\mu D l} \delta(l).$$

Furthermore, $D$ should scale as $l$ in the limit $l \to 0$, and thus we obtain

$$\rho(l) \sim \frac{1}{l^2} e^{-\mu l^2} \delta(l)$$

$$\sim \delta''(l) - \mu \delta(l).$$
Besides the elaboration of our present formulation, one of the intriguing problems is to extend our system to the ones in which gravity is coupled to matters. In particular, we should identify physical operators and investigate their structures. An intuitive conjecture at this stage is that loops will be further labeled with conformal primary fields, and that all the physical operators known so far are obtained by expanding such loops in their loop length. Investigation along this line is now in progress, and will be reported elsewhere.

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Appendix A  Proof of eq. (2.19)

In general, the norm (2.7) of infinitesimal deformation of metric can be rewritten into the following form if we use the ADM decomposition (2.2):

$$\| \delta g_{\mu\nu} \|_g^2 = 4 \int d^2 x N \sqrt{h} \left[ \left( \frac{\delta h}{2h} \right)^2 + \left( \frac{\delta N}{N} \right)^2 + \frac{1}{2} \left( \frac{\sqrt{h}}{N} \delta \lambda \right)^2 \right].$$  (A.1)

In particular, around $\bar{g}_{\mu\nu}$ it is expressed as

$$\| \delta g_{\mu\nu} \|_{\bar{g}}^2 = \int d^2 x l \left[ \left( \frac{\delta h}{2l^2} \right)^2 + (\delta N)^2 + \frac{1}{2} (l \delta \lambda)^2 \right],$$  (A.2)

where we have omitted the irrelevant coefficient “4” from the expression. Each of the terms appearing in eq. (A.2) can be calculated by using eq. (2.18) and found to be

$$\frac{\delta h}{2l^2} = \frac{\delta l}{l} - \omega \delta v^n + D_s \delta v^s,$$
$$\delta N = D_n \delta v^n,$$
$$l \delta \lambda = l \delta k + D_s \delta v^n + (D_n + \omega) \delta v^s$$  (A.3)

with $\omega = k’ - \dot{l}/l$. Thus, we obtain

$$\| \delta g_{\mu\nu} \|_{\bar{g}}^2 = \int d^2 x l \left[ \left( \frac{\delta l}{l} - \omega \delta v^n + D_s \delta v^s \right)^2 + (D_n \delta v^n)^2 \right.$$
$$\left. + \frac{1}{2} (l \delta k + D_s \delta v^n + (D_n + \omega) \delta v^s)^2 \right].$$  (A.4)
The above expression can be transformed into the form of eq. (2.19) by making a shift of $\delta v^m$ and $\delta v^s$. In doing so, there might occur some problems concerned with zero-modes of the operators $D_n$ and $D_s$. However, the zero-modes (constant modes) of $D_n$ always vanish since we have used the Dirichlet boundary condition for $D_n$, while the zero-modes of $D_s$ were extracted in the beginning in order to further fix the residual gauge symmetry (see the discussion following eq. (2.35)).

**Appendix B  Proof of eq. (3.12)**

First, we define the determinant of the Laplacian

$$\Delta [g_{\mu\nu}] \equiv -\frac{1}{\sqrt{g}} \partial_{\mu} (\sqrt{g} g^{\mu\nu} \partial_{\nu})$$

by the following equation:

$$\ln \text{Det} \Delta [g_{\mu\nu}] \equiv -\int_{\epsilon^2}^{\infty} \frac{ds}{s} \text{Tr} e^{-s \Delta [g_{\mu\nu}] }$$

with ultra-violet cutoff $\Lambda \sim \epsilon^{-1}$. Then the determinant will change under the infinitesimal Weyl transformation $g_{\mu\nu} \mapsto e^{2\delta \sigma} g_{\mu\nu}$ as

$$\bar{\delta} \ln \text{Det} \Delta [g_{\mu\nu}] \equiv \ln \text{Det} \Delta [e^{2\delta \sigma} g_{\mu\nu}] - \ln \text{Det} \Delta [g_{\mu\nu}]$$

$$= -2 \int d^2 x \sqrt{\bar{g}} \bar{\delta} \sigma \left( \frac{1}{4\pi \epsilon^2} + \frac{1}{24\pi} R [g_{\mu\nu}] + O(\epsilon^2) \right).$$

Our strategy to prove eq. (3.12) is to relate the infinitesimal deformation $\bar{\delta}l(t)$ of loop length at time $t$, to some infinitesimal Weyl transformation $\delta \sigma(t)$ which depends only on $t$. However, since Weyl transformations generally break the gauge condition (2.10), a proper reparametrization $x^\mu \mapsto \tilde{x}^\mu(x)$ has to be accompanied,

$$ds^2 = e^{2\delta \sigma(t)} \bar{g}_{\mu\nu} dx^\mu dx^\nu$$

$$= e^{2\delta \sigma(t)} \left[ (dt)^2 + l(t)^2 \left( k(t, x) dt + dx \right)^2 \right]$$

$$\equiv \left( d\tilde{t} \right)^2 + \tilde{l}(\tilde{t})^2 \left( \tilde{k}(\tilde{t}, \tilde{x}) d\tilde{t} + d\tilde{x} \right)^2.$$

---

8 Here for the scalar curvature $R$ we used the following convention:

$$\int d^2 x \sqrt{g} R = 4\pi \chi = 8\pi (1 - h),$$

which is different from that in ref. [11] by 2.
One can here easily show that the following reparametrization satisfies the above requirement:

\[
\tilde{t}(t, x) \equiv \int_0^t dt' e^{\delta \sigma(t')}, \\
\tilde{x}(t, x) \equiv x,
\]  

with

\[
\tilde{k}(\tilde{t}, \tilde{x}) = e^{\delta \sigma(t)} k(t, x), \\
\tilde{l}(\tilde{t}) = e^{\delta \sigma(t)} l(t).
\]  

(B.6)

Note that if we introduce the symbol

\[
\bar{\delta} \rho(t) \equiv \tilde{t} - t = \int_0^t dt' \delta \sigma(t'),
\]  

(B.7)

then the following equation holds:

\[
\bar{\delta} \rho(0) = \bar{\delta} \rho(D) = 0,
\]  

(B.8)

since two boundaries of the cylinder in question are parametrized as \( t = \tilde{t} = 0 \) and \( t = \tilde{t} = D \).

By using eqs. (B.6) and (B.7), \( \tilde{k}(t, x) \) and \( \tilde{l}(t) \) can now be expressed by \( \bar{\delta} \rho(t) \) as follows:

\[
\tilde{k}(t, x) \equiv \tilde{k}(t, x) - k(t, x) = -k(t, x) \bar{\delta} \rho(t) - k(t, x) \bar{\delta} \rho(t) = -\frac{\partial}{\partial t} \left(k(t, x) \bar{\delta} \rho(t)\right),
\]  

(B.9)

\[
\tilde{l}(t) \equiv \tilde{l}(t) - l(t) = -\tilde{l}(t) \bar{\delta} \rho(t) + l(t) \bar{\delta} \rho(t) = l(t)^2 \frac{d}{dt} \left(\bar{\delta} \rho(t)\right).
\]  

(B.10)

These are nothing but the transformation introduced in sect. 3. Conversely, \( \bar{\delta} \rho(t) \) can be expressed by \( \bar{\delta} l(t) \) as

\[
\bar{\delta} \rho(t) = l(t) \int_0^t dt' \frac{\bar{\delta} l(t')}{l(t')^2},
\]  

(B.11)

and we obtain

\[
\bar{\delta} \sigma(t) = \bar{\delta} \rho(t) = \tilde{l}(t) \int_0^t dt' \frac{\bar{\delta} l(t')}{l(t')^2} + \frac{\bar{\delta} l(t)}{l(t)}.
\]  

(B.12)
Note that eq. (B.8) implies that
\[ \int_0^D dt \frac{\delta l(t)}{l(t)^2} = 0. \]  
(B.13)
On the other hand, in our temporal gauge the scalar curvature \( R \) is given by
\[ R[\tilde{g}_{\mu\nu}] = -2 \frac{i}{l} + 2 \left( \frac{\dot{k} - kk'}{l} + 2k \frac{\dot{l}}{l} \right)'. \]  
(B.14)
Thus, substituting eqs. (B.12) and (B.14) into eq. (B.3), we obtain
\[ \tilde{\delta} \ln \text{Det} \Delta [\tilde{g}_{\mu\nu}] = \int_0^D dt \int_0^1 dx \left[ -2\mu^2 l(t) \right. \\
+ \frac{1}{6\pi} \left\{ \tilde{l}(t) \left( \tilde{l}(t) \int_0^t dt' \frac{\delta l(t')}{l(t')^2} + \frac{\delta l(t)}{l(t)} \right) + \text{(total derivative in } x) \right\} \right] \\
= \int_0^D dt \left[ -2\mu^2 l(t) + \frac{1}{6\pi} \tilde{l}(t) \left( \tilde{l}(t) \int_0^t dt' \frac{\delta l(t')}{l(t')^2} + \frac{\delta l(t)}{l(t)} \right) \right], \]  
(B.15)
where \( \mu^2 \equiv 1/8\pi\epsilon^2 \). This expression can be further simplified by using eq. (B.13), and becomes
\[ \tilde{\delta} \ln \text{Det} \Delta [\tilde{g}_{\mu\nu}] = -\tilde{\delta} \int_0^D dt \left[ 2\mu^2 l(t) + \frac{1}{12\pi} \frac{\dot{l}(t)^2}{l(t)} \right]. \]  
(B.16)
After integrating this equation, we finally obtain eq. (3.12).

References

[1] A. Polyakov, Mod. Phys. Lett. A2 (1987) 899.
[2] V. Kazakov, Phys. Lett. B150 (1985) 282; 
F. David, Nucl. Phys. B257 (1985) 45, 543; 
J. Ambjørn, B. Durhuus and J. Fröhlich, Nucl. Phys. B257 (1985) 433; 
D. Boulatov, V. Kazakov, I. Kostov and A. Migdal, Nucl. Phys. B275 (1986) 641.
[3] E. Brezin and V. Kazakov, Phys. Lett. B236 (1990) 144.
[4] M. Douglas and S. Shenker, Nucl. Phys. B335 (1990) 635.
[5] D. Gross and A. Migdal, Phys. Rev. Lett. 64 (1990) 127, Nucl. Phys. B340 (1990) 333.
[6] M. Fukuma, H. Kawai and R. Nakayama, Int. J. Mod. Phys. A6 (1991) 1385, Comm. Math. Phys. 143 (1992) 371.

[7] R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. B348 (1991) 435.

[8] J. Ambjørn and S. Varsted, Phys. Lett. B266 (1991) 285, Nucl. Phys. B373 (1992) 557;
J. Ambjørn, D. Boulatov, A. Krzywicki and S. Varsted, Phys. Lett. B276 (1992) 432;
M. Gross and S. Varsted, Nucl. Phys. B378 (1992) 367.

[9] J. Distler and H. Kawai, Nucl. Phys. B321 (1989) 509;
F. David, Mod. Phys. Lett. A3 (1988) 1651.

[10] V. Knizhnik, A. Polyakov and A. Zamolodchikov, Mod. Phys. Lett. A3 (1988) 819.

[11] E. D’Hoker, “Continuum Approaches to 2-D Quantum Gravity,” in Strings: Stony Brook 1991, 193.

[12] H. Kawai and M. Ninomiya, Nucl. Phys. B336 (1990) 115;
H. Kawai, Y. Kitazawa and M. Ninomiya, Nucl. Phys. B393 (1993) 280, B404 (1993) 684.

[13] N. Ishibashi and H. Kawai, Phys. Lett. B314 (1993) 190.

[14] H. Kawai, N. Kawamoto, T. Mogami and Y. Watabiki, Phys. Lett. B306 (1993) 19.

[15] G. Moore, N. Seiberg and M. Staudacher, Nucl. Phys. B362 (1991) 665.

[16] A. Jevicki and J. Rodrigues, “Loop Space Hamiltonians and Field Theory of Non-Critical Strings,” Brown preprint, BROWN-HET-927 (December 1993).

[17] N. Mavromatos and J. Miramontes, Mod. Phys. Lett. A4 (1989) 1847;
E. D’Hoker and P. Kurzepa, Mod. Phys. Lett. A5 (1990) 1411.

[18] A. Polyakov, Gauge Fields and Strings, Harwood Academic Publishers (1987).
Figure captions

Fig. 1

A cylinder $M$ with two loop boundaries $C$ and $C'$. $C_d$ is the subset in $M$ consisting of the points whose geodesic distance from $C$ is $d$. We here assume that $C_d$ is homeomorphic to $C$ for any $d \in [0, D]$ and $C_D = C'$.

Fig. 2

A loop splits into two loops at $t = t_0$. Here $f(t_0) \equiv l''(t_0) \delta(l(t_0) + l'(t_0) - l''(t_0))$.

Fig. 3

Two loops merge into a single loop at $t = t_0$. Here $g(t_0) \equiv l(t_0) l'(t_0) \delta(l(t_0) + l'(t_0) - l''(t_0))$. 
Fig. 1
\[ t = 0 \]

\[ t = D \]

\[ = \int_{0}^{D} dt_0 \ f(t_0) \]

Fig. 2
\[ t=D \]

\[ t=0 \]

\[ \int_0^D dt_0 \ g(t_0) \]

\[ t=t_0 \]

\[ t=0 \]

---

Fig. 3