Completeness of the ring of polynomials

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Let $k$ be an uncountable field. We prove that the polynomial ring $R := k[X_1, \ldots, X_n]$ in $n \geq 2$ variables over $k$ is complete in its adic topology. In addition we prove that also the localization $R_m$ at a maximal ideal $m \subset R$ is adically complete. The first result settles an old conjecture of C. U. Jensen, the second a conjecture of L. Gruson.

Our proofs are based on a result of Gruson stating (in two variables) that $R_m$ is adically complete when $R = k[X_1, X_2]$ and $m = (X_1, X_2)$.

INTRODUCTION

1. Consider for a field $k$ and a given integer $n \geq 0$ the polynomial ring $R := k[X_1, \ldots, X_n]$ in $n$ variables, and its field of fractions $K := k(X_1, \ldots, X_n)$. Set $d = 0$ if $k$ is finite and define $d$ by the cardinality equation $|k| = \aleph_d$ if $k$ is infinite. The following conjecture in its full generality was formulated by L. Gruson (priv. com., 2013).

Conjecture. In the notation above, $\text{Ext}^i_R(K, R) \neq 0 \iff i = \inf\{d + 1, n\}$.

The conjecture is trivially true for $n = 0$ where $R = K = k$ and the infimum equals 0. It is also true for $n = 1$ (where $R$ is a PID. and the infimum equals 1; the Ext may be computed from the injective resolution $0 \to R \to K \to K/R \to 0$).

In addition, the conjecture is trivially true if $i = 0$, since the infimum equals 0 iff $n = 0$.

The conjecture has an obvious analogue obtained by replacing the polynomial ring $R = k[X_1, \ldots, X_n]$ by its localization $R_m$ at a maximal ideal $m$.

2. In this note we consider the conjectures only for $i = 1$. They were formulated some 40 years ago, Conjecture 2b partly by Gruson [G, p. 254], and Conjecture 2a by C. U. Jensen [J, p. 833], inspired by the work of Gruson.

Conjectures. Let $R := k[X_1, \ldots, X_n]$ be the polynomial ring, and $m \subset R$ a maximal ideal. Then the following bi-implications hold:

2a. $\text{Ext}^1_R(K, R) \neq 0 \iff n = 1$ or $|k| \leq \aleph_0$.

2b. $\text{Ext}^1_{R_m}(K, R_m) \neq 0 \iff n = 1$ or $|k| \leq \aleph_0$.

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3. The Ext’s in the conjectures make sense for a wider class of rings, and we fix for the rest of this paper an integral domain $R$ with field of fractions $K$. We assume throughout that $R$ is noetherian, and not a field; in particular, $\bigcap_{s \neq 0} sR = (0)$ and $\text{Hom}_R(K, R) = 0$. Let $S := R \setminus \{0\}$ be the set of non-zero elements of $R$, pre-ordered by divisibility: $s' \mid s$ iff $sR \subseteq s'R$. We denote by $\lim_S^{(i)}$ the $i$th derived functor of the limit functor $\lim_S$ on the category of inverse $S$-systems of $R$-modules.

The modules $\text{Ext}^i_R(K, R)$ of the conjectures are related to the $\lim_S^{(i)}$ by well-known results, see [G, p. 251–52]: For $i \geq 2$ there are natural isomorphisms $\text{Ext}^i_R(K, R) \simeq \lim_S^{(i-1)} R/sR$, and for $i = 1$ there is an exact sequence,

$$0 \to R \xrightarrow{c(R)} \lim_{s \in S} R/sR \to \text{Ext}^1_R(K, R) \to 0. \tag{3.1}$$

The set of principal ideals $sR$ for $s \in S$ is cofinal in the set of all non-zero ideals of $R$. Hence the topology defined by the ideals $sR$ for $s \in S$ is the adic topology on $R$, and the limit in (3.1) is the adic completion of $R$; we denote it by $\hat{R}$, and we will simply call $R$ complete if the canonical injection $c(R)$ in (3.1) is an isomorphism. As it follows from the exact sequence (3.1), $R$ is complete iff $\text{Ext}^1_R(K, R) = 0$.

Since $R$ is not a field it follows easily that the completion $\hat{R}$ is uncountable. If the field $k$ is finite or countable (and $n \geq 1$) then the polynomial ring $R = k[X_1, \ldots, X_n]$ and its localization $R_m$ are countable, and hence they are not complete. In other words, the assertions of Conjectures 2a and 2b hold if $|k| < \aleph_0$. As noted above, they also hold when $n \leq 1$. So the remaining cases of the conjectures are the following.

**Conjectures.** Let $R := k[X_1, \ldots, X_n]$ be the polynomial ring where $|k| \geq \aleph_1$ and $n \geq 2$. Then:

3a. (C. U. Jensen [J, p. 833]) $R$ is complete.

3b. The localization $R_m$ of $R$ at any maximal ideal $m \subset R$ is complete.

The main result of this paper is the verification of the two conjectures. In fact, both conjectures are implied by a single result.

**Theorem 4.** Assume that $|k| \geq \aleph_1$, that $n \geq 2$, and that $R = U^{-1}R_0$ is a localization of $R_0 = k[X_1, \ldots, X_n]$ with a multiplicative subset $U \subset R_0$. In addition, assume that every maximal ideal of $R$ contracts to a maximal ideal of $R_0$. Then $R$ is complete.

The key ingredient in our proof is the following local result in two variables.

**Proposition 5.** (L. Gruson [G, Proposition 3.2, p. 252]) Conjecture 3b holds for $n = 2$ and $m = (X_1, X_2)$.

**Lemmas**

6. Our argument is based on a series of lemmas, some of which are valid in a more general context, and we keep the setup of Section 3. First we compare for a multiplicative subset $T \subset R$ the completions of $R$ and $T^{-1}R$. The ideals of $T^{-1}R$ generated by the elements of $S$ form a cofinal subset of non-zero ideals. Hence
the inclusion \( R \hookrightarrow T^{-1}R \) is continuous, and there is an induced \( R \)-linear map of completions,

\[
\hat{R} = \lim_{s \in S} R/sR \to \lim_{s \in S} T^{-1}R/sT^{-1}R = \hat{T}^{-1}R.
\]

For \( s \in S \) let \( a_s \supseteq sR \) denote the ideal of \( R \) such that \( a_s/sR \) is the kernel of the map \( R/sR \to T^{-1}R/sT^{-1}R \). Then the kernel of the map (6.1) is the limit \( L := \lim_{s \in S} a_s/sR \). Clearly, for \( a \in R \) we have \( a \in a_s \) iff there exists an element \( t \in T \) such that \( ta \in sR \).

**Lemma 7.** Assume that \( R \) is a UFD, and let \( T \subseteq S \) be a multiplicative saturated subset. Consider the localization \( R \subseteq T^{-1}R \) and the induced map of completions \( \hat{R} \to \hat{T}^{-1}R \). Then the induced map is injective iff for every prime element \( t \in T \) there exists a prime element \( p \not\in T \) such that the ideal \( (t, p)R \) is proper: \( (t, p)R \subset R \).

**Proof.** Recall that saturation means that any divisor of an element of \( T \) belongs to \( T \) or, equivalently since \( R \) is a UFD, \( T \) is the submonoid of \( S \) generated by a subset of prime elements. Let \( P \) be the monoid generated by the prime elements outside \( T \). Moreover, let \( T_0 \) be the submonoid of \( T \) consisting of elements \( t \in T \) such that \( (t, p)R = R \) for all \( p \in P \). For \( t \in T \) we write \( t_0 \) for the largest divisor in \( T_0 \) of \( t \), determined by a factorization \( t = t_0t' \) where \( t_0 \in T_0 \) and \( t' \) has all prime divisors outside \( T_0 \).

In this notation the Lemma asserts for the kernel \( L \) of the induced map that \( L = 0 \) iff \( T_0 \) contains no prime elements. Hence the assertion of the Lemma is a consequence of the following equation for the kernel:

\[
(7.1) \quad L \simeq \lim_{t_0 \in T_0} R/t_0R.
\]

To prove (7.1) note first that up to units in \( R^* \), the monoid \( S \) is the product of \( T \) and \( P \), and for \( s \in S \) we write \( s = tp \) for the corresponding factorization into factors \( t \in T \) and \( p \in P \). By unique factorization, it follows from the description of the ideal \( a_s \) above that \( a_s = pR \). Consequently,

\[
(7.2) \quad a_s/sR = pR/tpR \simeq R/tR.
\]

Under the isomorphisms (7.2), the transition map \( a_s/sR \to a_{s'}/s'R \) for \( s' | s \) is the map \( R/tR \to R/t'R \) induced by multiplication by \( p/p' \). It follows from (7.2) that

\[
(7.3) \quad L = \lim_{s \in S} a_s/sR \simeq \lim_{t \in T} \lim_{p \in P} R/tR.
\]

Fix \( t \in T \) and consider the inner limit in (7.3). We claim that

\[
(7.4) \quad \lim_{p \in P} R/tR = R/t_0R.
\]

The transition maps for the limit in (7.4) are multiplications by elements \( p \in P \) on the \( R \)-module \( R/tR \). By unique factorization, the multiplications are injective. Therefore, the limit is the intersection of the images of the multiplications.
Clearly, if \( t \in T_0 \) then the multiplications are bijective; hence the intersection is equal to \( R/tR \), and (7.4) holds since \( t = t_0 \). Assume next that \( t \) is a prime element outside \( T_0 \). Then multiplication by some \( p \in P \) has an image contained in a proper ideal of \( R/tR \). Hence the intersection of the images is contained in the intersection of the powers of a proper ideal of \( R/tR \). Since \( R/tR \) is an integral domain, the intersection equals 0 by Krull’s Intersection Theorem, and hence (7.4) holds since \( t_0 = 1 \).

In general, we factorize \( t = t_0 t' \) where \( t' \) has all prime divisors outside \( T_0 \), and use the exact sequence \( 0 \rightarrow R/t'R \rightarrow R/tR \rightarrow R/t_0R \rightarrow 0 \). From the previous considerations it follows first that the intersection of the images on \( R/t'R \) is equal to 0, and next that the intersection of the images on \( R/tR \) maps isomorphically onto \( R/t_0R \). Hence (7.4) holds in general.

Clearly (7.4) and (7.3) imply (7.1).

**Lemma 8.** Assume for every maximal ideal \( m \) of \( R \) that the induced map \( \hat{R} \rightarrow \hat{R}_m \) is injective and that \( R_m \) is complete. Then \( R \) is complete.

**Proof.** By the second assumption, \( R_m = \hat{R}_m \). Hence, by the first assumption, \( \hat{R} \) embeds into \( \bigcap R_m = R \). Thus \( R = \hat{R} \).

**Lemma 9.** Assume that \( R \subseteq R' \) is a subring of an integral domain \( R' \) such that \( R' \) is integral over \( R \) and free as an \( R \)-module. Assume that \( R' \) is complete. Then \( R \) is complete.

**Proof.** Every non-zero ideal of \( R' \) contracts to a non-zero ideal of \( R \) since \( R' \) is integral over \( R \). In other words, the inclusion \( R \rightarrow R' \) is continuous. Hence there is an induced map of completions \( \hat{R} \rightarrow \hat{R'} \), and an induced \( R' \)-linear map \( R' \otimes_R \hat{R} \rightarrow \hat{R'} \).

We have to prove that the canonical injection \( c = c(R) : R \rightarrow \hat{R} \) is an isomorphism. Since \( R' \) is free over \( R \) it suffices to prove that the map \( R' \otimes_R c : R' \rightarrow R' \otimes_R \hat{R} \) is an isomorphism. Clearly, the canonical injection \( c(R') : R' \rightarrow \hat{R} \) factors:

\[
\begin{array}{ccc}
R' & \longrightarrow & R' \\
\downarrow & & \downarrow c(R') \\
R' \otimes_R \hat{R} & \longrightarrow & \hat{R}'.
\end{array}
\]

The bottom map is the canonical map \( R' \otimes_R \lim V_s \rightarrow \lim (R' \otimes_R V_s) \) defined for any inverse \( S \)-system of \( R \)-modules \( (V_s) \). It is injective, since \( R' \) is free over \( R \). The right vertical map is an isomorphism by assumption. Therefore, \( R' \otimes_R c \) is an isomorphism.

**Lemma 10.** Assume that \( R \) is a localization of \( R_0 = k[X_1, \ldots, X_n] \) such that every maximal ideal of \( R \) contracts to a maximal ideal of \( R_0 \). Let \( \mathfrak{p} \subset R \) be a prime ideal of height at least 2. Then the induced map of completions \( \hat{R} \rightarrow \hat{R}_\mathfrak{p} \) is injective.

**Proof.** Indeed, as is well-known, the localization \( R \) is a UFD: its prime elements are, up to units in \( R^* \), those irreducible polynomials in \( R_0 \) that are non-units of \( R \). To apply Lemma 7, let \( t \) be a prime element in \( R \setminus \mathfrak{p} \). We have to prove that there exists a prime element in \( \mathfrak{p} \) such that the ideal \( (t, p)R \) is proper. Take any maximal ideal \( \mathfrak{m} \subset R \) with \( t \in \mathfrak{m} \). Apply the following Sublemma to the contractions \( \mathfrak{m}_0 = R_0 \cap \mathfrak{m} \) and \( \mathfrak{p}_0 = R_0 \cap \mathfrak{p} \). It follows that there exists an irreducible polynomial \( p \) in \( \mathfrak{m}_0 \cap \mathfrak{p}_0 \). Then \( p \) is a prime element in \( \mathfrak{p} \), and \( (t, p)R \) is a proper ideal, since \( (t, p)R \subset \mathfrak{m} \).
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Sublemma. Let $R = k[X_1, \ldots, X_n]$ be the polynomial ring, let $p \subset R$ be a prime ideal of height $h \geq 2$, and let $m \subset R$ be a maximal ideal. Then the intersection $p \cap m$ contains a prime ideal $q$ of height $h - 1$. In particular, $m \cap p$ contains an irreducible polynomial.

Proof. The assertion is trivial if $p \subseteq m$ so we may assume that $p \not\subseteq m$. Assume first that $k$ is algebraically closed. Then $p$ is the ideal of an irreducible variety $V$, and $m$ is the ideal of a point $q$. By assumption $q \not\in V$. Hence the linear join of $q$ and $V$ (the cone with base $V$ and vertex $q$) is an irreducible subvariety $W$ of dimension equal to $\dim V + 1$. Therefore, the ideal $q$ of $W$ is a prime ideal with the required properties.

The general case is reduced to the previous case as follows: Consider the embedding $R \hookrightarrow \bar{R}$ where $\bar{R}$ is the polynomial ring over the algebraic closure of $k$. The embedding is integral, and $\bar{R}$ is a UFD. Hence, by the usual dimension theory for polynomial rings, $p$ and $m$ are contractions of prime ideals $\bar{p}$ and $\bar{m}$ of $\bar{R}$; if $\bar{q} \subseteq \bar{p} \cap \bar{m}$ is a prime of height $h - 1$, then the contraction $q := R \cap \bar{q}$ has the required property.

Note 11. (1) The proof of Lemma 10 is particularly simple in the special case: $R = k[X_1, X_2]$, $k$ is algebraically closed, and $p = (X_1, X_2)$. Indeed, for an irreducible polynomial $t$ outside $p$ take a zero $\alpha = (\alpha_1, \alpha_2)$ of $t$ and take $p := \alpha_2X_1 - \alpha_1X_2$. Then $t$ and $p$ belong to the maximal ideal $m_\alpha = (X_1 - \alpha_1, X_2 - \alpha_2)$, and $p$ is irreducible since $\alpha \not\in (0, 0)$. The special case is sufficient for a proof of Conjecture 3a alone, see Note 14(2).

(2) It is also worthwhile to note that the conclusion in Lemma 10 is wrong for prime ideals $p$ of height 1: For $R = k[X_1, \ldots, X_n]$ and a prime ideal $p$ of height 1, say $p = pR$, the induced map $\bar{R} \to \bar{R}_p$ is not injective. Indeed, the polynomial $p + 1$ is not a constant, and hence for any irreducible divisor $t$ in $1 + p$ we have $t \not\in p$ and $(t, p)R = R$. Hence, by Lemma 7, the map is not injective.

Proofs of the main results

Lemma 12. Let $R := k[X_1, X_2]$ be the polynomial ring in two variables where $|k| \geq \aleph_1$, and let $m$ be any maximal ideal of $R$. Then $R$ and $R_m$ are complete.

Proof. The second assertion is a generalization of Gruson's local result. First, if $k$ is algebraically closed, then $R_m$ is complete. Indeed, then $m = (X_1 - \alpha_1, X_2 - \alpha_2)$ with $\alpha_1, \alpha_2 \in k$, and the completeness of $R_m$ follows from the local result (Proposition 5) by a change of coordinates.

To prove the results in general, embed $k$ in the algebraic closure $\bar{k}$. Let $\bar{R} := \bar{k}[X_1, X_2]$, let $R' := R_m$ and $\bar{R}' := \bar{R}_m$. With $U := R \setminus m$ we have $R' = U^{-1}R$ and $\bar{R}' = U^{-1}\bar{R}$. The maximal ideals of $\bar{R}'$ are the ideals generated by maximal ideals $\bar{m} \subset \bar{R}$ lying over $m$. Moreover, the localization of $\bar{R}'$ at the maximal ideal $\bar{m}R'$ is equal to $\bar{R}_m$, and hence complete by the first case. In addition, the map of completions induced by $\bar{R}' \hookrightarrow \bar{R}_m$ is injective by Lemma 10. Therefore, by Lemma 8, the ring $\bar{R}'$ is complete. Finally, $\bar{R}' = \bar{R} \otimes_R R'$ is integral and free over $R'$. Hence, by Lemma 9, $R'$ is complete. Similarly, since $\bar{R}_m$ is complete for all maximal ideals $\bar{m}$ of $R$, it follows first that $R$ is complete, and next the $R$ is complete.

Theorem 13. Assume that $|k| \geq \aleph_1$, that $n \geq 2$, and that $R = U^{-1}R_0$ is a localization of $R_0 = k[X_1, \ldots, X_n]$ with a multiplicative subset $U \subset R_0$. In addition, assume that every maximal ideal of $R$ contracts to a maximal ideal of $R_0$. Then $R$ is complete.
Proof. Clearly $R$ is a UFD, and hence equal to the intersection of the localizations $R_q$ over all prime ideals $q$ of height 1. Moreover, every height 1 prime ideal is contained in a height 2 prime ideal, since $R$ is catenary and all maximal ideals have height $n \geq 2$. Therefore, $R$ is the intersection over all prime ideals $p$ of height 2:

$$R = \bigcap_{ht\ p=2} R_p.$$  

For every prime ideal $p$ of height 2 it follows from Lemma 10 that the induced map of completions $\hat{R} \to \hat{R}_p$ is injective. Therefore, by Equation (13.1), to prove that $R = \hat{R}$, it suffices to prove for every height 2 prime ideal $p$ of $R$ that $R_p$ is complete. Clearly, the latter completeness follows from Lemma 12 using the following standard observation on localizations for $h = 2$: If $R$ is a localization of $R_0 = k[X_1, \ldots, X_n]$ then any localization $R_p$ at a prime ideal $p \subset R$ of height $h \geq 1$ may be obtained, after a renumbering of the variables, as the localization at a maximal ideal of the polynomial ring,

$$k(X_{h+1}, \ldots, X_n)[X_1, \ldots, X_h].$$

To justify the observation, note first that the prime ideal $p \subset R$ is generated by a prime ideal $p_0 \subset R_0$, and $R_p = (R_0)_{p_0}$. Hence we may assume that $R = k[X_1, \ldots, X_n]$. The quotient $R/p$ has transcendence degree $n - h$ over $k$ since $p$ has height $h$. Consequently there are $n - h$ among the variables, say $X_{h+1}, \ldots, X_n$, whose classes modulo $p$ are algebraically independent, or equivalently, such that $k[X_{h+1}, \ldots, X_n] \cap p = (0)$. Localization of $R$ with the monoid of non-zero polynomials in $X_{h+1}, \ldots, X_n$ yields the ring (13.2), and so $R_p$ may be obtained by localization of (13.2) at the ideal generated by $p$. The latter ideal is a prime ideal of height $h$, and hence a maximal ideal. Thus the observation has been justified.

**Note 14.** (1) Clearly Theorem 13 implies the two conjectures 3a and 3b. In addition, it follows from the observation at the end of the previous proof that Conjecture 3b implies, when $|k| \geq \aleph_1$, that the localization $R_p$ of $R := k[X_1, \ldots, X_n]$ at any prime ideal $p$ of height $h \geq 2$ is complete. In particular, the rings in Theorem 13 do not exhaust the list of complete subrings of $k[X_1, \ldots, X_n]$.

(2) For a proof of Conjecture 3a alone, the arguments can be simplified. First, the proof of Lemma 12 for $R = k[X_1, X_2]$ uses only the special case of Lemma 10 mentioned in Note 11(1). Next, for $R = k[X_1, \ldots, X_n]$ with $|k| \geq \aleph_1$ and $n \geq 3$ a direct proof of completeness is the following:

Denote by $T_{12} \subset S$ the multiplicative subset of polynomials containing neither $X_1$ nor $X_2$, that is, $T_{12}$ is the set of non-zero polynomials in $k[X_3, \ldots, X_n]$. Then the localization

$$T_{12}^{-1} R = k(X_3, \ldots, X_n)[X_1, X_2],$$

is complete by Lemma 12. Moreover, it follows immediately from Lemma 7 that the inclusion $R \subseteq T_{12}^{-1} R$ induces an injection on the completions; indeed, for any irreducible $t \in T_{12}$ take $p = X_1$. Similarly, with an obvious notation we obtain for any $i = 3, \ldots, n$ an inclusion $\hat{R} \subseteq T_{1i}^{-1} R$, and hence an inclusion,

$$\hat{R} \subseteq \bigcap_{i=2}^n T_{1i}^{-1} R.$$  

Obviously, the intersection on the right side equals $R$. Thus $\hat{R} = R$. 


References

[G] L. Gruson, *Dimension homologique des modules plats sur un anneau commutatif noethérien*, Symposia Mathematica, Vol. XI, Convegno di Algebra Commutativa, INDAM, Rome, 1971, Academic Press, London, 1973, pp. 243–254.

[J] C. U. Jensen, *On $\text{Ext}^1_R(A, R)$ for torsion-free $A$*, Bull. Amer. Math. Soc 78 (1972), 831–834.

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