KOSZUL DUALITY AND SOERGEL BIMODULES FOR DIHEDRAL GROUPS

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ABSTRACT. Every Coxeter group \((W, S)\) gives rise to an associated Hecke algebra \(H_{(W, S)}\) which can be categorified by the additive monoidal category of Soergel bimodules \(SB\). Under this isomorphism the Kazhdan-Lusztig basis \(\{H_x\}_{x \in W}\) corresponds to certain indecomposable Soergel bimodules \(\{B_x\}_{x \in W}\) (up to shift). In this thesis we study the structure of the endomorphism algebra (of maps of all degrees) \(A := \text{End}^*_{SB}(\bigoplus_{x \in W} B_x) \otimes_R \mathbb{R}\). Via category \(O\) it has been proven for all Weyl groups (see Theorem 2) that \(A\) is a self-dual Koszul algebra. We extend this result to all dihedral groups by purely algebraic methods using representation theory of quivers and Soergel calculus.

CONTENTS

1. Introduction 1
2. Preliminaries 2
3. Soergel Bimodules 4
4. Soergel Calculus (in the dihedral case) 6
5. Koszul and Quasi-Hereditary Algebras 11
6. Realisation as a Path algebra of a Quiver with Relations 13
7. Koszul Self-Duality of \(A\) 20
References 28

1. INTRODUCTION

To any Coxeter system \((W, S)\) one has an associated Hecke algebra \(H_{(W, S)}\). The Hecke algebra may be categorified by Soergel bimodules \(SB\), an additive monoidal category of bimodules over a polynomial ring. The indecomposable bimodules \(\{B_x\}\) in \(SB\) are (up to grading shift) parametrised by the group \(W\). The main object in this paper is the endomorphism algebra (consisting of maps of all degrees) \(A := \text{End}^*_{SB}(\bigoplus_{x \in W} B_x) \otimes_R \mathbb{R}\). We prove the following result via purely algebraic methods:

Theorem 1. For a dihedral group \((W, S)\) the \(\mathbb{R}\)-algebra \(A\) is a self-dual Koszul algebra.

1.1. Motivation. Let \(g\) be a complex semisimple Lie algebra. It turns out that the category \(g\text{-Mod}\) of all \(g\)-modules is far too large to be understood algebraically. Seminal for the further study of the representation theory of \(g\) was the introduction of category \(O\) by Bernstein, Gelfand and Gelfand (see [BGG76]). Fix a Borel \(b\), a Cartan \(h\) in \(g\) and define \(O := O(g, b, h)\) to be the full subcategory of \(g\text{-Mod}\) whose elements \(M\) are finitely generated over \(g\), \(h\)-semisimple and locally \(b\)-finite. In particular, all finite dimensional modules and Verma (= standard) modules \(\Delta(\lambda)\) \((\lambda \in h^*)\) lie in \(O\). This restriction made it easier to handle the category and led to
beautiful new results such as BGG reciprocity [BGG76 Prop. 1] and the Kazhdan-Lusztig conjectures [KL79 Conj. 1.5]. Within the principal block $O_0 \subset O$ let $L := \bigoplus_{w \in W} L(w \cdot 0)$ be the direct sum of the simple modules and $P := \bigoplus_{w \in W} P(w \cdot 0)$ the direct sum of their corresponding (indecomposable) projective covers, i.e. $P$ is a projective generator. We have the following result due to Soergel in [Soe90].

**Theorem 2** (Koszul self-duality for the principal block $O_0$). There exists an isomorphism of finite dimensional $\mathbb{C}$-algebras

$$A := \text{End}_{O_0}(P) \cong \text{Ext}^\bullet_{O_0}(L, L),$$

where the right hand side is a ring via the cup product. Furthermore, $\text{Ext}^\bullet_{O_0}(L, L)$ is a Koszul algebra.

Although $\text{End}_{O_0}(P)$ is not obviously graded, it inherits a grading from the naturally graded Ext-algebra. The first glimpse of Koszul duality was discovered earlier when mathematicians were investigating composition series of Verma modules in category $O_0$ and found formulas of the form

$$\Delta(x \cdot 0) : L(y \cdot 0) = \sum_i \text{Ext}^i(\Delta(w_0 x \cdot 0), L(w_0 y \cdot 0)).$$

These formulas can be explained by Koszul self-duality on the level of derived categories (see [BGS96 Theorem 1.2.6]). The existing proofs of Koszul self-duality are difficult and rely heavily on geometric techniques.

Using $O_0 \cong \text{Mod}-A$ one obtains a $\mathbb{Z}$-graded version of $O_0$ as $O_0^\mathbb{Z} := g\text{Mod}-A$. We have the following isomorphisms of $\mathbb{Z}[x^{\pm 1}]$-modules (see [Soe90] and [Str03a Theorem 7.1]):

$$K_0(O_0^\mathbb{Z}) \xrightarrow{\sim} K_0^\mathbb{Z}(S)$$

$$P(x \cdot 0)(i) \mapsto \mathcal{V}P(x \cdot 0)(i) = B_x(i),$$

where $B := B \otimes_R \mathbb{R}$ is the Soergel module corresponding to the Soergel bimodule $B$. Let $S$ denote the category of graded Soergel modules and $\mathcal{V}$ Soergel’s combinatorial functor. Translating Theorem 2 into the setting of Soergel (bi)modules via the above identification frees the result from geometry and yields

$$A \xrightarrow{\text{Thm. 2}} \text{Ext}^\bullet_{O_0}(L, L) \cong E(A),$$

where $E(A)$ denotes the Koszul dual of a Koszul algebra.

1.2. **Structure of the paper.** This paper contains two parts.

**Part 1:** In the first four section we provide the necessary background on the Hecke algebra, Soergel bimodules, Sorgel calculus and Koszul algebras.

**Part 2:** In the last two sections we realise the endomorphism ring of Soergel bimodules as a path algebra of a quiver and show its Koszul self-duality.

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2. **Preliminaries**

2.1. **Basic Definitions.** Let $(W, S)$ be a Coxeter system. Recall that $W$ is equipped with the Bruhat order $\leq$ and the length function $\ell : W \to \mathbb{N}_0$ which counts the number of simple reflections in a reduced expression. For an arbitrary sequence $w = (s_1, s_2, \ldots, s_n)$ in $S$ we denote the product $s_1 s_2 \cdots s_n$ by $w$, viewed as an element in $W$. For such a sequence $w$ its length is defined by $\ell(w) = n$. Observe
that $\ell(w) \geq \ell(w)$ with equality if and only if $w$ is a reduced expression for $w$. By abuse of notation we write $w = s_1 s_2 \cdots s_n$. It is crucial to distinguish between $w$ and $w$, since the latter denotes a distinct sequence of simple reflections whereas $w$ is their product in $W$.

Following Soergel’s normalisation as in [Soe97] define the associated Hecke algebra $H = H(W,S)$ as the unital, associative $\mathbb{Z}[v^\pm 1]$-algebra generated by $\{H_s\}_{s \in S}$ with relations

(1) $H_s^2 = 1 + (v^{-1} - v)H_s$,

(2) $H_sH_tH_s \cdots = H_sH_tH_s \cdots$ for all $s \neq t$ in $S$. Here $v$ is just an indeterminate.

Given a reduced expression $w = s_1 s_2 \cdots s_n$ we set $H_w := H_{s_1} \cdots H_{s_n}$ which is well-defined by the Lemma of Matsumoto (see [Mat64]). The elements $\{H_w\}_{w \in W}$ form the standard basis of $H$ as a $\mathbb{Z}[v^\pm 1]$-module. An easy calculation shows

**Lemma 2.1.** Let $w \in W$ and $s \in S$. For a basis element $H_w$ and a generator $H_s$ there is the multiplication rule:

(3) $H_wH_s = \begin{cases} H_wH_s & \text{if } ws > w \\ H_wH_s + (v^{-1} - v)H_w & \text{if } ws < w \end{cases}$

Each $H_s$ for $s \in S$ is invertible with inverse $H_s + (v - v^{-1})$ and thus all standard basis elements are units. There is a unique $\mathbb{Z}$-linear involution $\dagger : H \to H$ such that $v \mapsto v^{-1}$ and $H_s \mapsto H_s^{-1}$. This involution is called duality and it is easily checked that $H_w \mapsto H_{w^{-1}}$ for $w \in W$.

**Theorem 2.2** ([KL79]). There exists a unique $\mathbb{Z}[v^\pm 1]$-basis $\{H_w\}_{w \in W}$ of $H$ consisting of self-dual elements such that

$H_w = H_w + \sum_{x < w} h_{x,w}H_x$

where $h_{x,w} \in v\mathbb{Z}[v]$.

This basis is called the Kazhdan-Lusztig basis and the $h_{x,w}$ are the Kazhdan-Lusztig polynomials. Note that the $h_{x,w}$ are not the originally defined Kazhdan-Lusztig polynomials $p_{x,w}$ (see [KL79]) but there is the following relation (see [Soe97] Rem. 2.6.1):

$h_{x,w}(v) = v^{\ell(w) - \ell(x)}p_{x,w}(v^{-2})$.

Mimicking the notion of a trace form from linear algebra we mean by a trace on $H$ a $\mathbb{Z}[v^\pm 1]$-linear map $\varepsilon : H \to \mathbb{Z}[v^\pm 1]$ satisfying $\varepsilon(h_1 h_2) = \varepsilon(h_2 h_1)$ for all $h_1, h_2 \in H$. The standard trace on $H$ is defined via $\varepsilon(H_w) := \delta_{w,w}$.

**Lemma 2.3.** For $w$ and $w' \in W$ we have $\varepsilon(H_wH_{w'}) = \delta_{w,(w')^{-1}}$.

**Proof.** Induction over the length of $w'$ combined with the multiplication rule from Lemma 2.1.

For a sequence $w = s_1 s_2 \cdots s_n$ we call $e = e_1 e_2 \cdots e_n$ with $e_i \in \{0,1\}$ a 01- (sub)sequence of $w$ which picks out the subsequence $w^e := s_1^{e_1} s_2^{e_2} \cdots s_n^{e_n}$. Given such a 01-sequence its Bruhat stroll is the sequence $e, x_1, \ldots, x_n = w^e$ where

(4) $x_i := s_1^{e_1} s_2^{e_2} \cdots s_i^{e_i}$.

This stroll allows us to decorate each step of the 01-sequence $e$ with either U(p) or D(own) encoding the path in the Bruhat graph. We assign U to the index $i$ if $x_{i-1} s_i > x_{i-1}$ and D if $x_{i-1} s_i < x_i$. 


There is a $\mathbb{Z}[v^\pm 1]$-linear anti-involution $\iota$ satisfying $\iota(H_w) = H_{w^{-1}}$ for $w \in W$. It is easily checked that $\iota(H_w) = H_w$ for $w \in W$. Note that $\iota$ and $\tau$ commute, we denote their composition as $\omega$. It follows that $\omega$ is a $\mathbb{Z}$-linear anti-involution on $\mathbb{H}$ satisfying $\omega(v) = v^{-1}$ and $\omega(H_w) = H_{w^{-1}}$ for $w \in W$. Using the aforementioned standard trace we can define the standard pairing $\langle \cdot, \cdot \rangle$ by $\langle \iota(h)h', c \rangle := \varepsilon(\omega(h)h')$.

2.2. Notations (following [Hum90]). Let $(W, S)$ be a dihedral group of type $I_2(m)$ ($m \geq 3$), that is a Coxeter system $(W, \{s, t\})$ where $(st)^m = e$. The elements in $S = \{s, t\}$ are called simple reflections or colours. As before, we denote a sequence of simple reflections by $w$ and shorten expressions of length $\geq 1$ by

$$
k_k := st \ldots, \quad k \text{ factors}
k_s := \ldots, \quad k \text{ factors}
$$

similarly for $t$. Omitting the underline means the corresponding product in $W$. We write $e = s0 = t0$ for the identity and $w_0 = sm = tm$ for the longest element in $W$.

The restriction to dihedral groups makes it possible to obtain a closed formula for all Kazhdan-Lusztig basis elements simultaneously.

**Proposition 2.4.** Let $(W, \{s, t\})$ be a dihedral group and $w \in W$, then

$$
\mathbb{H}_w = \sum_{\ell \leq w} v^{\ell(w) - \ell(w)} H_{\ell}.
$$

**Proof.** Induction over the length of $w$ (see [Her99 Bsp. 2.3].) $\Box$

For arbitrary Coxeter groups there is no such formula and the computation is inductively following for example the proof of Theorem 2.2 in [Soe97, Theorem 2.1]. However, for a finite Coxeter group with longest element $w_0$ its corresponding Kazhdan-Lusztig basis element $\mathbb{H}_{w_0}$ can always be computed with the formula in Proposition 2.4 (see [KL79]). Using Proposition 2.4 we can easily deduce the following lemma.

**Lemma 2.5.** For $w \in W$ we have $\omega(\mathbb{H}_w) = \mathbb{H}_{w^{-1}}$.

3. SOERGEL BIMODULES

Let $(W, S)$ be a Coxeter system of type $I_2(m)$, $m \geq 3$. Recall the associated geometric representation $h = X_\mu \oplus \mathbb{R} 1^{1 \chi}$ with its Cartan matrix (see [Hum90]):

$$
\begin{pmatrix}
2 & -2 \cos \left(\frac{\pi}{m}\right) \\
-2 \cos \left(\frac{\pi}{m}\right) & 2
\end{pmatrix}
$$

We fix this realisation once and for all. For $h$ consider $R := S(h^*) = \bigoplus_{i > 0} S^i(h^*)$, the symmetric algebra on $h^*$, a graded $\mathbb{R}$-algebra such that $\deg(h^*) = 2$. By construction, $W$ acts on $h$ and hence it acts on $h^*$ via the contragredient representation.

Extending this action by grading preserving automorphism yields an action of $W$ on $R$. The ring of invariants of a single simple reflection $s \in S$ is denoted by $R^s \subseteq R$.

The two main module categories in this thesis are $R$-Bim and $R$-gBim, the category of finitely generated $R$-bimodules and graded $R$-bimodules respectively (the latter with grading preserving morphisms). The category $R$-gBim is considered as a graded category with the grading shift down denoted by $(n)$. For $M = \bigoplus M_i$ we define $M(n)_i := M_{i+n}$. Moreover, for two graded bimodules $M$ and $N$ we write $\text{Hom}^*(M, N) := \bigoplus_{n \in \mathbb{Z}} (M, N(n))$ for the bimodule homomorphisms between $M$ and $N$ of all degrees.
Remark 3.1. Realisations of Coxeter groups can be defined in more generality as modules over a commutative domain and do not have to be symmetric either. For the more general case see [EW13].

For \( s \in S \) define the graded \( R \)-bimodule \( B_s := R \otimes_{R^t} R \{1\} \) which is a fixed graded lift of the \( R \)-bimodule \( R \otimes_{R^t} R \). We often write \( \otimes_s := \otimes_{R^t} \) and the tensor product structure \( \otimes_R \) is denoted as juxtaposition. For a given sequence \( \underline{w} = s_1 s_2 \cdots s_n \) define the corresponding Bott-Samelson bimodule by

\[
B_{\underline{w}} := B_{s_1} \otimes_R B_{s_2} \otimes_R \cdots \otimes_R B_{s_n} = B_{s_1} B_{s_2} \cdots B_{s_n}.
\]

The full monoidal subcategory \( BS \) of \( R-\text{Bim} \) generated by \( B_s \) for \( s \in S \) is called the category of Bott-Samelson bimodules. Since we chose a fixed graded lift for every Bott-Samelson bimodule. Finally, the category of Soergel bimodules \( SB \) is defined to be the Karoubian envelope of the additive closure of this graded version of \( BS \). It is crucial to distinguish that \( BS \) is a subcategory of \( R-\text{Bim} \), whilst \( SB \) is a subcategory of \( R-\text{gBim} \) and therefore morphisms between Soergel bimodules are grading preserving. Observe that \( SB \) is additive but not abelian. Soergel classified the indecomposable objects in \( SB \) (see [Soe07, Theorem 6.14]).

Theorem 3.1 (Classification of indecomposable Soergel bimodules). Given any reduced expression \( w \) of \( w \in W \), the Bott-Samelson \( B_{\underline{w}} \) contains up to isomorphism a unique indecomposable summand \( B_y \) which does not occur in \( B_{\underline{w}} \) for any expression \( y \) of \( y \in W \) with \( \ell(y) < \ell(w) \). In addition, \( B_y \) does up to isomorphism not depend on the reduced expression \( w \). A complete set of representatives of the isomorphism classes of all indecomposable Soergel bimodules is given by

\[
\{B_w(m) \mid w \in W \text{ and } m \in \mathbb{Z}\}.
\]

Note that the split Grothendieck group \( K_0^+ (C) \) of an additive, monoidal and graded category \( C \) inherits a \( \mathbb{Z}[v^\pm 1] \)-algebra structure. Soergel proved that the category of Soergel bimodules \( SB \) categorifies the Hecke algebra \( H \) (see [Soe07, Theorem 1.10.]) for certain (difficult) reflection faithful realisations over infinite fields of characteristic \( \neq 2 \). However, Libedinsky showed in [Lib08a] that this categorification works for the (easier) geometric representation as well.

Theorem 3.2 (Soergel’s Categorification Theorem). For the geometric realisation \( \mathfrak{h} \) there is a unique isomorphism of \( \mathbb{Z}[v^\pm 1] \)-algebras given by

\[
\varepsilon : H \xrightarrow{\sim} K_0^+(SB)
\]

\[
H_w \mapsto [B_w].
\]

Using the standard pairing on \( H \) it is possible to describe the graded rank of the homomorphism space between two Soergel bimodules (see [Soe07, Theorem 5.15.]).

Theorem 3.3 (Soergel’s Hom-Formula). Given any two Soergel bimodules \( B \) and \( B' \), the homomorphism space \( \text{Hom}^*(SB; B, B') \) is free as a left (resp. right) \( R \)-module and its graded rank is given by \( \langle \varepsilon^{-1}[B], \varepsilon ^{-1}[B'] \rangle \) where \( \langle -, - \rangle \) denotes the standard pairing on the Hecke algebra.

Soergel constructed an inverse map to \( \varepsilon \) which he called the character map. However, the construction is not explicit and he only conjectured what the pre-images of the indecomposable Soergel bimodules are for an arbitrary Coxeter group.

Conjecture 3.4 (Soergel’s Conjecture). If \( k \) is a field of characteristic \( 0 \) then \( \chi(B_{\underline{w}}) = H_{\underline{w}} \).
Soergel himself proved this for Weyl groups and dihedral groups (see [Soe98] Theorem 2]). The case of universal Coxeter groups was shown by Fiebig in [Fie08]. This conjecture is a very deep result and implies the Kazhdan-Lusztig conjecture for our calculations to determine the dimensions of homomorphism spaces between Soergel bimodules.

Recall that we denote \( R = \bigoplus_{i \geq 0} S^i(\mathfrak{h}^+) \) with maximal ideal \( S^+ := \bigoplus S^i(\mathfrak{h}^+) \). Therefore we can view \( R \cong R/S^+ \) as an \( R \)-bimodule. To each Soergel bimodule \( M \) we can associate the Soergel module \( \overline{M} := M \otimes R \) where the right action is trivialised. Let \( S \) denote the category of Soergel modules. For a Weyl group Soergel proved that there is an isomorphism

\[
\text{Hom}_S^*(\overline{B}, \overline{B}) \cong \text{Hom}_{SB}^*(B, B') \otimes_R \mathbb{R}
\]

for \( B, B' \) (see [Soe98] Thm. 2, Part 4.)). Moreover, he conjectured this isomorphism for every finite Coxeter group (see [Soe98] Thm. 2, Part 5.) which he proved recently ([Soe14]). For simplicity we write by abuse of notation \( \text{Hom}_S(B, B') \) instead of \( \text{Hom}_S(B, \overline{B}) \) for Soergel bimodules \( B, B' \). The important results for this paper are summarised in the following theorem which is an immediate consequence of the above:

**Theorem 3.5.** Given a dihedral group \( W \), we have for \( x, y \in W \):

- \( \text{gr} \dim_R \text{Hom}_S^*(B_x, B_y) = \langle H_x, H_y \rangle = \sum_{a \leq x, y} v(\ell(x) + \ell(y) - 2\ell(a)) \)
- \( \dim_R \text{Hom}_S^*(B_x, B_y) = |W_{\leq x} \cap W_{\leq y}| \)

where \( W_{\leq x} := \{ w \in W \mid w \leq x \} \). In particular \( \text{Hom}_S^*(B_x, B_y) \) is concentrated in non-negative degrees \( d \) such that \( 0 \leq d \leq \ell(y) - \ell(x) \) for \( x \leq y \) with \( \dim_R \text{Hom}_S(B_x, B_y) = \delta_{x,y} \).

4. **Soergel Calculus (in the dihedral case)**

By construction the category of Soergel bimodules \( SB \) is the Karoubian envelope of \( BS \) and therefore it is enough to describe \( BS \) by planar graphs and identify the idempotents. In this section we introduce a diagrammatic approach to the category of Bott-Samelson bimodules \( BS \) following [EW13]. These techniques are what we refer to as Soergel calculus.

4.1. **Generators.** Recall that we fixed the geometric realisation \( \mathfrak{h} \) for our given dihedral group \( (W, \{s, t\}) \) of type \( I_2(m) \) for \( m \geq 3 \). The Bott-Samelson bimodule \( B_w = B_s \otimes B_t \otimes \cdots \otimes B_s \) for \( w = s t \cdots s \) is completely determined by an ordered sequence of colours (or colored dots on a horizontal line). A morphism between two Bott-Samelson bimodules from \( B_w \) to \( B_{w'} \) is given by a linear combination of isotopy classes of decorated graphs with coloured edges in the planar stripe \( \mathbb{R} \times [0, 1] \) such that the edges induce sequences of coloured dots on the bottom boundary \( \mathbb{R} \times \{0\} \) (resp. the top boundary \( \mathbb{R} \times \{1\} \)) in each summand corresponding to \( w \) (resp. \( w' \)). In particular, these diagrams represent morphisms from the bottom sequence to the top sequence and therefore should be read from bottom to top.

**Definition 4.1.** For a dihedral group \( (W, S) \) define \( D = D_{(W, S)} \) to be the \( \mathbb{R} \)-linear monoidal category as follows: The objects are sequences \( w \) in \( S \) (which are denoted sometimes by \( B_w \)). The empty sequence \( \emptyset \) is often denoted by 1. The \( \text{Hom} \)-spaces
are $\mathbb{Z}$-graded $\mathbb{R}$-vector spaces generated by the diagrams in Table 1 modulo local relations. The monoidal structure is the concatenation of sequences.

For $s \in S$ the Demazure operator $\partial_s : R \to R^s$ in Table 1 is defined as $\partial_s(f) := \frac{L_s f}{\alpha_s}$. The first two morphisms in Table 1 are called dots, whereas the second two morphisms are called trivalent vertices and the last morphism is called $2m_{s,t}$-valent vertex. The explicit formula for the $2m_{s,t}$-valent vertex is very difficult. Therefore we only explain what the morphism does. In the case of a dihedral group of type $I_2(m)$ the longest element $w_0$ can be expressed as $st \cdots = ts \cdots$ with $m$ factors on each side and by Theorem 3.1 both $B_s B_t \cdots$ and $B_t B_s \cdots$ contain $B_{w_0}$ as summand with multiplicity 1. The $2m_{s,t}$-vertex is the projection and inclusion of this summand and therefore uniquely determined up to a scalar.

Libedinsky showed in [Lib08b] that the morphisms from Table 1 generate all morphisms in $B\mathcal{S}$. For the compositions of a trivalent vertex with a dot we define caps and cups as follows:

(7) \[ \begin{array}{c}
\text{cap} := \begin{array}{c}
\text{cup := } \begin{array}{c}
\end{array}
\end{array}
\end{array} \]

4.2. Relations. Since we work with dihedral groups, only one- and two-colour relations can occur. For Coxeter groups of rank $\geq 3$ there exist the three-colour or Zamolodchikov relations which are more difficult (see [EW13]).

4.2.1. The one-colour relations. The object $B_s$ is a Frobenius object in $R$-$\text{Bim}$ (see [EK09]) where the dots correspond to unit and counit while the trivalent vertices correspond to the multiplication and comultiplication. Altogether we obtain that any one-coloured diagram is isotopy invariant and we have the following three non-polynomial relations:

(8) \[ \begin{array}{c}
\text{cap} = \text{cup}, \\
\text{cup} = \text{cap}, \\
\text{cup} = 0.
\end{array} \]
We refer to these relations as general associativity, general unit and the needle. Furthermore, there are two relations involving polynomials:

\[
(f) = (\alpha_s f) + (\partial_s f).
\]

As a direct consequence we have:

\[
\frac{1}{2} (\alpha_s f) + (\alpha_s f) = \frac{1}{2} (\alpha_s f) + (\alpha_s f).
\]

4.2.2. The two-colour relations. There are three two-colour relations. The first is the cyclicity of the $2m_s,t$-vertex. The other two relations specify how the $2m_s,t$-vertex interacts with trivalent vertices and dots. For $m = m_s,t$ we have the two-colour associativity depending on the parity of $m$:

\[
m \text{even: } (11) \quad m \text{ odd: } (11).
\]

The other relation allows us to express a diagram involving a $2m_s,t$-vertex and a dot as a linear combination of diagrams without the $2m_s,t$-vertex. This procedure depends on the parity of the integer $m$ as well:

\[
m \text{even: } (12) \quad m \text{ odd: } (12).
\]

Note that the Jones-Wenzl morphism $JW_{m-1}$ is an $\mathbb{R}$-linear combination of graphs consisting only of dots and trivalent vertices. This morphism will be discussed briefly in the next paragraph.

4.3. Jones-Wenzl morphisms.

4.3.1. Gauß’s $q$-numbers. Before we can define the Jones-Wenzl projector it is helpful to recall Gauß’s $q$-numbers. We use them to give formulas for the Jones-Wenzl projector for all dihedral groups simultaneously. Gauß’s $q$-numbers are defined (see e.g. [Jan96]) by

\[
[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{-n+1} + q^{-n+3} + \cdots + q^{n-3} + q^{n-1} \in \mathbb{Z}[q^{\pm 1}].
\]

It is convenient to define $[0] := 0$. In most cases we omit the index and write $[n]$ instead of $[n]_q$. Observe that $[n]$ is the character of $L(n-1)$, the simple $sl_2(\mathbb{C})$-module of dimension $n$. Via the Clebsch-Gordon formula we obtain two out of many useful identities for $q$-numbers:

\[
[2][n] = [n + 1] + [n - 1].
\]

\[
[n]^2 = [n-1][n+1] + [1].
\]

We can specialise $q$ to a value $\zeta$ which we denote by $[n]_\zeta$. If $\zeta = e^{2\pi i / 2m} \in \mathbb{C}$, i.e. a primitive $2m$-th root of unity, we obtain algebraic integers $[n]_\zeta \in \mathbb{R}$. By choice $\zeta$ is primitive and therefore we have $\zeta^n = -1$. Thus, we get the following identities:

\[
[m]_\zeta = 0, \quad [m-i]_\zeta = [i]_\zeta, \quad [m+i]_\zeta = -[i]_\zeta.
\]
Recall the geometric representation of a dihedral group as in Eq. (3) and note that $[2]_\zeta = \zeta + \zeta^{-1} = 2 \cos \left( \frac{\pi}{m} \right) = -a_{s,t}$. Hence, the geometric representation can be encoded in the Cartan matrix

$$
\begin{pmatrix}
2 & -[2]_\zeta \\
-[2]_\zeta & 2
\end{pmatrix}
$$

simultaneously for all dihedral groups. For a certain dihedral group of type $I_2(m)$ we only have to specialize $q$ to a primitive $2m$-th root of unity.

**Lemma 4.1.** For $i \geq 1$ we have $i_s = \ldots ts \in W$ of length $i$ and thus

$$i_s(\alpha_t) = \begin{cases} [i]_\zeta \alpha_t + [i+1]_\zeta \alpha_s & \text{if } i \text{ is odd,} \\ [i+1]_\zeta \alpha_t + [i]_\zeta \alpha_s & \text{if } i \text{ is even.} \end{cases}$$

**Proof.** We prove this by induction on $i$. For $i = 1$ we have:

$$s(\alpha_t) = \alpha_t - \langle \alpha_t, \alpha^\vee_s \rangle \alpha_s = \alpha_t + [2]_\zeta \alpha_s.$$ 

Now let $i > 1$. There are two cases to distinguish depending on the parity of $i$. If $i$ is odd then

$$(i+1)_s(\alpha_t) = t(i_s(\alpha_t)) = t([i]_\zeta \alpha_t + [i+1]_\zeta \alpha_s) = (-[i]_\zeta + [2]_\zeta [i+1]_\zeta) \alpha_t + [i+1]_\zeta \alpha_s = [i+2]_\zeta \alpha_t + [i+1]_\zeta \alpha_s.$$

The other case is similar which finishes the proof.

\[\square\]

### 4.3.2. Jones-Wenzl projectors

We give a short introduction to Jones-Wenzl projectors with a recursive computation formula in the Temperley-Lieb algebra setting where its origins lie (see [Jon86, Wen87]). For a detailed discussion we refer the reader to [CK12, Eli13].

The Temperley-Lieb algebra $TL_n$ on $n$ strands is a diagram algebra over $\mathbb{Z}[\delta]$. It has a basis consisting of crossingless matching with $n$ points on bottom and top each. The multiplication is given by vertical concatenation of diagrams such that circles evaluate to scalar $-\delta$. The crossingless matching on $n$ strands is the unit in $TL_n$, and is denoted by $1_n$. This algebra is contained in the Temperley-Lieb category which is closely related to the quantum group $U := U_q(sl_2)$ of $sl_2$ via the base change $\delta \mapsto [2]_q = q + q^{-1}$ (see [Eli13] for technical details).

Let $V_k$ be the irreducible representation of highest weight $q^k$ and let $V = V_1$. The highest non-zero projection from $V^{\otimes n}$ to $V_n$ is known as the Jones-Wenzl projector $JW_n \in TL_n$.

**Remark 4.2.** This definition of the Jones-Wenzl projector only works while working over the complex numbers. This categorification over the integers is harder and more subtle and was carefully treated in [CK12, FSS12, Roz14].

The following proposition states some important properties of $JW_n$ (see [KL94, Proposition 3.2.2.], [Lic97, Lemma 13.2.] and [CK12, Section 2.2.1]).

**Proposition 4.2.** The Jones-Wenzl projector $JW_n$ satisfies the following properties:

- $JW_n$ is the unique map which is killed when any cap is attached on top or any cup on bottom, and for which the coefficient of $1_n$ is 1.
- $JW_n$ is invariant under horizontal/vertical reflection.
- The ideal $\langle JW_n \rangle \leq TL_n$ has rank 1.
- Any element $x \in TL_n$ acts on $JW_n$ by its coefficient of $1_n$. 

\[\square\]
Note that the first property gives an alternative way of defining $JW_n$. For our computations we need the following recursive formula for $JW_n$ (see [FK97, Theorem 3.5.]):

$$JW_{n+1} = JW_n + \sum_{i=1}^{n} \frac{[i]_q}{[n+1]_q} JW_n$$  \hfill (17)

In the following we state the Jones-Wenzl projectors for $n = 1, 2, 3$.

$$JW_1 = \begin{array}{cccc} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array}$$

$$JW_2 = \begin{array}{cccc} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} + \begin{array}{cccc} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array}$$

$$JW_3 = \begin{array}{cccc} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} + \begin{array}{cccc} \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} + \begin{array}{cccc} \vdots \\ \vdots \\ \vdots \end{array} + \begin{array}{cccc} \vdots \\ \vdots \\ \vdots \end{array} + \begin{array}{cccc} \vdots \\ \vdots \end{array}$$

Any crossingless matching in $TL_{m-1}$ divides the planar stripe in $m$ regions which we can colour alternatingly with red and blue (for $s$ and $t$). This results in the definition of the two-coloured Temperley-Lieb algebra which we omit here, details can be found in [Eli13, EW13]. Since our chosen realisation is symmetric we can treat a blue circle surrounded by red just as a red circle surrounded by blue and thus evaluate both to the same value. For each diagram there are two possible colourings. Each of those coloured crossingless matchings yields a coloured graph. Deformation retract each region into a tree consisting of trivalent and univalent vertices, colour those resulting trees according to the region. In that way we can associate to each coloured Jones-Wenzl projector a Jones-Wenzl morphism $JW_n$. We state the Jones-Wenzl morphisms (with red appearing in the far left region) for $n = 1, 2, 3$.

$$JW_1 = \begin{array}{cccc} \vdots \end{array}$$

$$JW_2 = \begin{array}{cccc} \vdots \end{array} + \begin{array}{cccc} \vdots \end{array}$$

$$JW_3 = \begin{array}{cccc} \vdots \end{array} + \begin{array}{cccc} \vdots \end{array} + \begin{array}{cccc} \vdots \end{array} + \begin{array}{cccc} \vdots \end{array}$$

The Jones-Wenzl morphism above is not yet a morphism in $D$ but can be plugged into another diagram to obtain a graph in the planar stripe (see Eq. (12), for technical details see [Eli13]). We often write $JW$ instead of $JW_{m-1}$ when specialised to $\zeta$. Using the two colour relations (see Eqs. (11) and (12)) we obtain the following relation between the $m_{s,t}$-valent vertex and the Jones-Wenzl projector $JW_{m-1}$. Since the latter is an idempotent in the Temperley-Lieb algebra (see [Eli13]), the right-hand sides in (18) are also idempotents. Thus, the $m_{s,t}$-valent vertex can be used to construct idempotents.

$$m \text{ even: } \begin{array}{cccc} \vdots \\ \vdots \end{array} = \begin{array}{cccc} \vdots \end{array}$$

$$m \text{ odd: } \begin{array}{cccc} \vdots \end{array} = \begin{array}{cccc} \vdots \end{array}$$

(18) $m$ even: \begin{array}{cccc} \vdots \\ \vdots \end{array} = \begin{array}{cccc} \vdots \end{array}$ \hspace{1cm} $m$ odd: \begin{array}{cccc} \vdots \end{array} = \begin{array}{cccc} \vdots \end{array}$

The two defining properties of $JW_n$ (see Proposition 4.2 or [Eli13, Claim 4.4]) are crucial and we use them repeatedly in our category $D$. The first one is that the coefficient of one single graph is 1, namely the graph which becomes the identity
morphism if trivalent vertices are attached to both sides (precisely the first summands in the example above). The second and more important property is that $JW$ is killed by “cups” and “caps”

\begin{equation}
JW = 0.
\end{equation}

We refer to this property as death by pitchfork.

4.4. Light leaf and double leaf morphism. The concept of light leaves and double leaves was introduced by Libedinsky in [Lib08b] for Soergel bimodules. Elias and Williamson transferred those morphisms into the combinatorial and diagrammatic setting of $\mathcal{D}$ (see [EW13]). The set of light leaf morphisms $LL_{x,e}$ forms a (left) $R$-basis of $\text{Hom}_{\mathcal{D}}(x,1)$. This basis can be constructed inductively and is parametrised by the subexpressions of $x$ which are equal to $e$. Composing these light leaf morphisms we obtain the set of double leaf morphisms $LL_{x,y}$ which forms a (left) $R$-basis of $\text{Hom}_{\mathcal{D}}(x,y)$ (see [EW13, Theorem 6.11]). Consequently, $LL_{x,y}$ is constructed inductively, too, and it is indexed by certain subexpressions of both $x$ and $y$. Hence all Hom-spaces in $\mathcal{D}$ are free graded (left) $R$-modules.

4.5. Equivalence of categories. Following [Eli13] we define an $R$-linear monoidal functor $F : \mathcal{D}_{(W,S)} \to \text{BS}$ which maps a sequence $w$ to the Bott-Samelson bimodule $B_w$. On the morphisms $F$ acts as indicated in Table 1. The next result is due to Elias (see [Eli13]):

**Theorem 4.3.** The $R$-linear monoidal functor $F$ is well-defined and yields an equivalence of monoidal categories:

$F : \mathcal{D}_{(W,S)} \to \text{BS}$.

Moreover, $F$ induces an equivalence of monoidal additive categories:

$F : \text{Kar}(\mathcal{D}_{(W,S)}) \to \text{SB}$.

**Remark 4.3.** The result in Theorem 4.3 were preceded by results by Elias and Khovanov in type $A$ (see [EK09]) and by Libedinsky (see [Lib10]) for the right-angled case. Furthermore, the above results were generalised by Elias and Williamson (see [EW13, Theorem 6.28.]) to all Coxeter groups with a fixed reflection-faithful realisation.

5. Koszul and Quasi-Hereditary Algebras

**Definition 5.1 (BGS96).** A graded ring $A = \bigoplus_{i \geq 0} A_i$ is called Koszul if $A_0$ is semisimple and $A_0$ admits a linear resolution as a graded left $A$-module, i.e.

\[ \cdots \to P^2 \to P^1 \to P^0 \to A_0 \]

where the maps are grading preserving and $P^i$ is generated by $(P^i)_i$, as a left $A$-module.

**Remark 5.2.** Equivalently one could ask that every simple $A$-module admits a linear resolution.

**Definition 5.3 (BGS96).** A graded ring $A = \bigoplus_{i \geq 0} A_i$ is called quadratic if $A_0$ is semisimple and $A$ is generated over $A_0$ by $A_1$ with relations in degree 2.
For a given $k$-vector space $V$ the tensor algebra $T_k V$, the symmetric algebra $SV$ and the exterior algebra $AV$ are quadratic rings (with the usual grading, that is $\deg X = 1$). In particular $k[X]$ and $k[X]/(X^2)$ are quadratic. If there are some finiteness conditions we can define a dual version of a quadratic ring:

**Definition 5.4** ([BGS96]). A graded ring $A = \bigoplus_{i \geq 0} A_i$ is called left finite if all $A_i$ are finitely generated as left $A_0$-modules. For a left finite quadratic ring $A = T_{A_0} A_1/(R)$ its quadratic dual is defined as $A' = T_{A_0} A_1^\ast/(R^\perp)$ where $R^\perp \subset A_1^\ast \otimes A_0$. $A_1^\ast \cong (A_1 \otimes A_0 A_1)^\ast$ with respect to the standard pairing. Here, $T_{A_0} A_1$ denotes the free tensor algebra of the $A_0$-bimodule $A_1$. The ring $A$ is self-dual if $A \cong A'$.

For any positively graded algebra $A = \bigoplus_{i \geq 0} A_i$, the degree 0 part $A_0$ is an $A$-module. Consider the graded ring $E(A) := \text{Ext}^\ast_A(A_0, A_0)$ of self-extensions of $A_0$. We call $E(A)$ the Koszul dual of $A$. Beilinson, Ginzburg and Soergel showed that there exist isomorphisms relating a Koszul algebra $A$ to its Koszul dual $E(A)$ and quadratic dual $A'$ (see [BGS96] Corollary 2.3.3., Theorem 2.10.1., Theorem 2.10.2.).

**Theorem 5.1** ([BGS96]). Let $A = \bigoplus_{i \geq 0} A_i$ be a Koszul ring. Then $A$ is quadratic. If additionally $A$ is left finite, then there are canonical isomorphisms $E(A) \cong (A')^\ast$ and $E(E(A)) \cong A$.

**Remark 5.5.** If $A$ is a positively graded algebra there exists a quadratic duality functor on the bounded derived categories

$$K : D^b(A\text{-gMod}) \to D^b(A'\text{-gMod})$$

which is an equivalence of categories if and only if $A$ is Koszul. This is treated completely and more general in [BGS96] Thm. 1.2.6.] and [MOS09] Thm. 30]. In conclusion Koszul algebras are certain quadratic algebras with additional nice homological properties. For example, under the above equivalence the standard $t$-structure maps to the non-standard $t$-structure on the dual side given by linear complexes of graded projective modules ([MOS09] Thm. 12]). If $A$ is even Koszul self-dual, then this yields a second interesting $t$-structure in $D^b(A\text{-gMod})$.

The algebras we are interested in admit a quasi-hereditary structure which simplifies the later proofs in a crucial way.

**Definition 5.6** ([Don98]). Let $A$ be a finite dimensional algebra over a field $k$ with a finite partially ordered set $(\Lambda, \leq)$ indexing the simple left $A$-modules $\{L(\lambda)\}_{\lambda \in \Lambda}$ and let $P(\lambda)$ be the indecomposable projective cover of $L(\lambda)$. A collection of left $A$-modules $\{\Delta(\lambda)\}_{\lambda \in \Lambda}$ defines a quasi-hereditary structure on $(A, (\Lambda, \leq))$ if

- for $\lambda \in \Lambda$ there exists a surjective $A$-module homomorphism $\pi : P(\lambda) \twoheadrightarrow \Delta(\lambda)$ such that $\ker \pi$ has a $\Delta$-filtration with subquotients isomorphic to some $\Delta(\mu_i)$ where $\mu_i > \lambda$,
- for $\lambda \in \Lambda$ there exists a surjective $A$-module homomorphism $\pi' : \Delta(\lambda) \twoheadrightarrow L(\lambda)$ such that $\ker \pi'$ has a composition series with composition factors only isomorphic to some $L(\mu_i)$ where $\mu_i < \lambda$.

The $\Delta(\lambda)$ are called (left) standard modules. (Right) standard modules can be defined similarly.

The following theorem is due to Ágoston, Dlab and Lukačs (see [ADL03] Theorem 1):

**Theorem 5.2.** Let $(A, (\Lambda, \leq))$ be a graded quasi-hereditary algebra. If both left and right standard modules admit linear resolutions (i.e. $A$ is standard Koszul) then $A$ is Koszul (i.e. all simple modules admit linear resolutions).

**Remark 5.7.** We would like to point out the close relationship between the double leaf morphisms in Section 4.4 and the filtration obtained by the standard modules.
6. Realisation as a Path Algebra of a Quiver with Relations

Recall that \((\mathcal{W}, \{s, t\})\) is a Coxeter system of type \(I_2(m), m \geq 3\) with its geometric representation \(h = R^0_\mathcal{W} \oplus R^1_\mathcal{W}\). In this section we analyse the structure of the graded endomorphism algebra \(A := \text{End}^0_\mathcal{W}(B) \cong \text{End}^0_{\mathcal{S}_B}(B) \otimes_R \mathbb{R}\) where \(B := \bigoplus_{x \in W} B_x\). Define \(\zeta := e^{2\pi i / 2m} \in \mathbb{C}\).

Definition 6.1. Let \(Q_m := (Q_0, Q_1)\) be the directed quiver with the following vertex and arrow sets

\[
Q_0 := W = \{e, s, k, w_0 \mid 1 \leq k \leq m - 1\},
\]

\[
Q_1 := \{w \rightarrow w' \mid ||w - l(w')|| = 1\}.
\]

We write \((w, w')\) for an arrow \(w \rightarrow w'\). For the fixed representation \(h\) of \(I_2(m)\) define the set \(R^h_m \subset \mathbb{R}Q_m\) consisting of the following relations for all \(2 \leq i \leq m - 1\) and \(0 \leq j \leq m - 2\) (plus the ones with the roles of \(s\) and \(t\) switched):

\[
\begin{align*}
(20) & \quad (e, s, e) = 0, \\
(21) & \quad (s, s, t, s) = 0, \\
(22) & \quad (s, s, i + 1, s) = \frac{[1-1]}{\nu_{\zeta}} (s, s(i-1), s), \\
(23) & \quad (s, s, s, s) = [2]_{\zeta} (s, e, s), \\
(24) & \quad (s, s, i + 1, i) = (1 - [1-1]_{\zeta} - [1-1]_{\zeta}) (s, s(i-1), s) - \frac{[1]+1_{\zeta}}{\nu_{\zeta}} (s, s(i-1), s), \\
(25) & \quad (s, s, s, t) = (s, e, t), \\
(26) & \quad (s, s, t, s) = (s, e, t), \\
(27) & \quad (s, s, i + 1, i) = \frac{1}{\nu_{\zeta}} (s, s(i-1), s) + (s, s(i-1), s), \\
(28) & \quad (s, e, s(i-1), s) + \frac{1}{\nu_{\zeta}} (s, e(i-1), s), \\
(29) & \quad (s, s, s(i-1), s + s(i-1), s), \\
(30) & \quad (s, s, s(i-1), s + s(i-1), s), \\
(31) & \quad (s(i+2), s(i+1), s) = (s(i+2), s(i+1), s), \\
(32) & \quad (s(i+2), s(i+1), s) = (s(i+2), s(i+1), s).
\end{align*}
\]

We refer to \(Q_m\) as the Hasse graph of type \(I_2(m)\) (c.f. Fig. 1) and the set \(R^h_m\) is called the set of dihedral relations. Define \(P_m\) to be the \(\mathbb{R}\)-algebra \(\mathbb{R}Q_m / (R^h_m)\).

The algebra \(P_m\) inherits the naturally grading of \(\mathbb{R}Q_m\) by path length since all relations are homogenous (of degree 2).

Remark 6.2. For \(i = 1\) relation \(21\) could be included in relation \(22\) whilst relation \(23\) cannot be included in relation \(24\). Moreover, for \(i = m - 1\) relation \(22\) and \(24\) agree and relation \(27, 28\)

Remark 6.3. The above formulas are exactly the relations which hold for the morphisms between the Soergel modules which we show later in detail. The coefficients are integral in type \(A_2\), but in general they are real numbers. Stroppel showed in [Str03b] that for types \(B_2\) and \(G_2\) one may obtain rational (or even integral coefficients) whilst choosing different bases for the morphism spaces. However, those relations are not symmetric any more.

By definition the vertices of \(Q_m\) are indexed by the dihedral group of type \(I_2(m)\). Hence we have the Bruhat order on the vertex set. For a vertex \(x \in Q_0\) define \(V_{\leq x} := \{y \in Q_0 \mid y \leq x\}\). Analysing the relations carefully we obtain the following lemma and propositions:
Lemma 6.1. The path algebra $P_m$ has (finite) dimension $\sum_{x,y} |V_{\leq x} \cap V_{\leq y}|$.

Remark 6.4. Lemma 6.1 means in particular that paths from $x$ to $y$ in $P_m$ cannot be the same if the lowest vertex they pass through is different.

Proposition 6.2. The isomorphism classes of simple graded $P_m$-modules up to grading shift are in bijection with the vertices of $Q_m$ via $w \mapsto L(w)$ where $L(w)_{w'} := \begin{cases} \mathbb{R} & \text{if } w = w' \\ 0 & \text{if } w \neq w' \end{cases}$ and the obvious maps.

Proposition 6.3. There exists an isomorphism $P_m \cong P_m^{op}$.

We can now state the main result of this section.

Theorem 6.4. There is an isomorphism of graded algebras $P_m \cong A$.

6.1. Proof of Theorem 6.4. Consider the assignment $w \mapsto B_w$ for $w \in Q_0 \setminus \{w_0\}$ and $w_0 \mapsto B_{s,m} \oplus B_{t,m}$. We extend this assignment to the arrows for $0 \leq j \leq m - 1$. 

The colour of the black line depends on the parity of $j$ and on the colour of the far left line. Switching colours yields the other half of the assignment. Arrows adjacent to $w_0$ are treated differently since $B_{w_0}$ is embedded diagonally in $B_{s,m} \oplus B_{t,m}$:
\[(s(m-1) \rightarrow w_0) \mapsto \frac{1}{2} \left( \begin{array}{c|c|c|c} 2m & 2m & JW & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\end{array} \right),\]

\[(w_0 \rightarrow s(m-1)) \mapsto \frac{1}{2} \left( \begin{array}{c|c|c|c} 2m & 2m & \vdots & JW \\
\vdots & \vdots & \vdots & \vdots \\
\end{array} \right).\]

For \(s\) and \(t\) switched the assignment is similar. Note that the images of the arrows are only morphisms between Bott-Samelson bimodules. In order to get a morphism between the corresponding Soergel bimodules we have to pre-/post-compose with the idempotents associated to the Soergel bimodules, that is For \(e, s, t\) the idempotents are the identity since the corresponding Bott-Samelson bimodules are indecomposable. The idempotent corresponding to \(B_s\) for \(2 \leq i \leq m-1\) is the Jones-Wenzl morphism \(JW_i\) where the quantum numbers must be specialised to the appropriate \(2m\)-th root of unity, hence the coefficients depend on the dihedral group. In other words,

\[e_{x} \mapsto \begin{cases} \text{JW} & \text{if } 2 \leq i \leq m-1 \\ \text{JW} & \text{otherwise} \end{cases}, \quad e_{0} \mapsto \frac{1}{2} \begin{pmatrix} \text{JW} & \text{JW} \\ \vdots & \vdots \end{pmatrix} \]

where \(e_x\) denotes the trivial path at vertex \(x\). Similarly for \(t\) with all colours switched. The idempotent of \(B_{w_0}\) is a \(2 \times 2\)-matrix since we consider \(B_{w_0}\) a submodule of \(B_s \oplus B_t\). By the classification in Theorem 3.1 the indecomposable bimodule \(B_{w_0}\) is a direct summand of \(B_s \oplus B_t \) with the Jones-Wenzl projector as idempotent. A direct diagrammatic calculation shows that \(JW^\Delta\) is indeed an idempotent and thus \(e_{w_0} \mapsto JW^\Delta\) is well-defined. This assignment yields a homomorphism of graded \(R\)-algebras \(\varphi' : RQ_m \rightarrow A\). The proof of Theorem 6.4 is divided into three steps:

(I) The map \(\varphi'\) is surjective.

(II) \(R_m \subseteq \ker \varphi'\), so \(\varphi'\) induces a surjection \(\varphi : P_m \rightarrow A\).

(III) By dimension arguments we can deduce that \(\varphi\) is an isomorphism of graded algebras over \(R\).

**Step 1.** In order to show that \(\varphi'\) is surjective, it suffices to show that for \(x, y \in W\) every morphism in \(\text{Hom}_{BG}^\bullet(B_x, B_y)\) is generated as an algebra by elements of degree 1. By Theorem 5.3 we know that between \(B_x\) and \(B_y\) there exists a morphism of degree 1 if and only if \(|l(x) - l(y)| = 1\). Moreover, the homogeneous part of degree 1 is at most one-dimensional, hence \(\text{im} \varphi'\) contains all maps of degree 1 by construction. The double leaves \(LL_x^y\) form an \(R\)-basis for \(\text{Hom}_{BG}(x, y)\), thus, using the equivalence in Theorem 4.3, we can deduce that every element in \(\text{Hom}_{BG}(B_x, B_y)\) is a sum of maps factoring through various \(B_w\) such that \(w\) is a reduced sub-expression of both \(x\) and \(y\). We say *modulo lower terms* if we only consider maps which do not factor through \(w\) such that \(w < x\) and \(w < y\). Consequently the next proposition implies the surjectivity of \(\varphi'\).

**Proposition 6.5.** For \(x \geq y \in W\) the morphism space \(\text{Hom}_{BG}^\bullet(B_x, B_y)\) modulo lower terms is free of rank 1 as \(R\)-left/right module. If \(x\) and \(y\) are not comparable the morphism space \(\text{Hom}_{BG}^\bullet(B_x, B_y) = 0\) modulo lower terms.
Proof. Before we prove the general case, consider the example \( x = stst \) (which covers already all the important cases). Then there exist eight decorated 01-sequences of \( x \) with their corresponding light leaves:

\[
\begin{align*}
U0U0U0 & \rightsquigarrow \uparrow \uparrow \\
(33) & \quad U1U0D1 \rightsquigarrow \uparrow \downarrow \uparrow \\
U1U0D0 & \rightsquigarrow \uparrow \downarrow \uparrow \\
(34) & \quad U0U0U1 \rightsquigarrow \uparrow \downarrow \uparrow \\
U0U1U0 & \rightsquigarrow \uparrow \downarrow \\
U1U1U0 & \rightsquigarrow \uparrow \downarrow \\
U1U0U1 & \rightsquigarrow \uparrow \downarrow \\
U1U1U1 & \rightsquigarrow \uparrow \downarrow \\
\end{align*}
\]

The light leaves in (33) and (34) yield the zero map when pre-composed with the idempotent corresponding to \( stst \) due to death by pitchfork and all other light leaves are generated by degree 1 maps. Now assume \( l(x) \geq 4 \) and set \( j := l(y) \leq l(x) =: i \). This means we have to extend the sequences from above with 0’s and 1’s on the right side and decorate the new entries properly with either \( D \)'s or \( U \)'s. Using successively the fact that the decorated sequences in (33) and (34) yield morphisms which get cancelled by the idempotents, we obtain that the only two possible sequences are

\[
\begin{align*}
(35) & \quad \underbrace{U0 \ldots U0}_{i-j \text{ times}} \underbrace{U1 \ldots U1}_{j \text{ times}} \rightsquigarrow \uparrow \ldots \downarrow \ldots \\
(36) & \quad \underbrace{U0 \ldots U0}_{i-j-1 \text{ times}} \underbrace{U1 \ldots U1}_{i \text{ times}} \underbrace{U0}_{1 \text{ times}} \rightsquigarrow \uparrow \ldots \downarrow \ldots \\
\end{align*}
\]

Both of those remaining morphisms are generated by degree 1 elements which proves the first statement. The second statement is clear since every proper sub-expression is strictly smaller than both (this holds dihedral groups but not in arbitrary Coxeter groups). □

Step II. We check that every relation from \( R^0_n \), holds for the morphisms between Soergel bimodules (with trivialised right action). For this this we use the following properties.

(i) Death by pitchfork,
(ii) the polynomial sliding relations in 9,
(iii) non-constant polynomials act as 0 from the right,
(iv) isotopy invariance,
(v) two-colour associativity,
(vi) \( JW := JW_{m-1} \) is rotation invariant.

It is important to note that all calculations and equations in this section are meant as morphisms between indecomposable Soergel bimodules (with trivialised right action). Therefore every morphism represented by a string diagram should be pre-/post-composed with the corresponding idempotent. We omit this composition in order to make the presentation clearer.

Before we start proving the relations one by one, we state two useful results.

**Lemma 6.6.** For a Coxeter system of type \( I_2(m) \) we have

\[
\begin{align*}
e_1 \\
\ldots \\
\ldots \\
e_2
\end{align*}
\]

\[ = 0, \]

where \( e_1 \) and \( e_2 \) are idempotents corresponding to indecomposable Soergel bimodules.
Proof. Without loss of generality we can assume that the morphism is given by
\[
\begin{array}{c}
\begin{array}{c}
\cdot \\
\cdot \\
\cdot
\end{array}
\end{array}
= \frac{1}{2} \left( \begin{array}{c}
\begin{array}{c}
\cdot \\
\cdot \\
\cdot
\end{array}
\end{array} \right)
= \frac{1}{2} \left( \begin{array}{c}
\begin{array}{c}
\cdot \\
\cdot \\
\cdot
\end{array}
\end{array} \right)
= 0.
\]
The general case follows easily with the above considerations. □

Proposition 6.7. As a morphism \( B_{i-1} \rightarrow B_{i+1} \) in the category of Soergel modules the following holds for \( i \geq 2 \):
\[
\begin{array}{c}
\begin{array}{c}
\cdot \\
\cdot \\
\cdot
\end{array}
\end{array}
= -\left[ \begin{array}{c}
\begin{array}{c}
\cdot \\
\cdot \\
\cdot
\end{array}
\end{array} \right] + \left[ \begin{array}{c}
\begin{array}{c}
\cdot \\
\cdot \\
\cdot
\end{array}
\end{array} \right]
\]

Proof. By property (ii) we can slide the polynomial \( \alpha_t \) successively through the strings and we obtain for some \( \lambda_n \):
\[
\begin{array}{c}
\begin{array}{c}
\cdot \\
\cdot \\
\cdot
\end{array}
\end{array}
= -\left[ \begin{array}{c}
\begin{array}{c}
\cdot \\
\cdot \\
\cdot
\end{array}
\end{array} \right] + \sum_{n=2}^{i-1} \lambda_n \left[ \begin{array}{c}
\begin{array}{c}
\cdot \\
\cdot \\
\cdot
\end{array}
\end{array} \right]
\]
where \( x = (i - 1)s(\alpha_t) \) is of positive degree. By Lemma 6.6 we have:
\[
\begin{array}{c}
\begin{array}{c}
\cdot \\
\cdot \\
\cdot
\end{array}
\end{array}
= -\left[ \begin{array}{c}
\begin{array}{c}
\cdot \\
\cdot \\
\cdot
\end{array}
\end{array} \right] + \cdots + x
\]
From now on assume that \( i \) is odd:
\[
\begin{array}{c}
\begin{array}{c}
\cdot \\
\cdot \\
\cdot
\end{array}
\end{array}
= -\left[ \begin{array}{c}
\begin{array}{c}
\cdot \\
\cdot \\
\cdot
\end{array}
\end{array} \right] + \cdots + x
\]
where we used that \( s \) acts grading preserving and thus \( s(x) \) acts trivially from the right since \( x \) was of positive degree. Recall that \( i \) is odd and hence by Lemma 4.1 we have:
\[
\partial_s(x) = \partial_s(\left[ \begin{array}{c}
\begin{array}{c}
\cdot \\
\cdot \\
\cdot
\end{array}
\end{array} \right] \alpha_t + \left[ \begin{array}{c}
\begin{array}{c}
\cdot \\
\cdot \\
\cdot
\end{array}
\end{array} \right] \alpha_s) = -\left[ \begin{array}{c}
\begin{array}{c}
\cdot \\
\cdot \\
\cdot
\end{array}
\end{array} \right] + \cdot \partial_{i-1}(x) \cdots
\]
where \( \partial_s(x) = \partial_s(\left[ \begin{array}{c}
\begin{array}{c}
\cdot \\
\cdot \\
\cdot
\end{array}
\end{array} \right] \alpha_t + \left[ \begin{array}{c}
\begin{array}{c}
\cdot \\
\cdot \\
\cdot
\end{array}
\end{array} \right] \alpha_s) = -\left[ \begin{array}{c}
\begin{array}{c}
\cdot \\
\cdot \\
\cdot
\end{array}
\end{array} \right] + \cdot \partial_{i-1}(x) \cdots
\]
The other case \( i \) even can be treated similarly using \( \partial_t \) instead of \( \partial_s \). □

6.1.1. Relation 20 and 21. Clearly we have:
\[
\begin{array}{c}
\begin{array}{c}
\cdot \\
\cdot \\
\cdot
\end{array}
\end{array}
= 0
\]
6.1.2. Relation 22 and 24. The desired relations for \( i \in \{2, \ldots, m - 2\} \) are direct consequences of the following lemma:

Lemma 6.8. For \( i \in \{2, \ldots, m - 1\} \) the following holds:
\[
\begin{array}{c}
\begin{array}{c}
\cdot \\
\cdot \\
\cdot
\end{array}
\end{array}
= \frac{\left[ \begin{array}{c}
\begin{array}{c}
\cdot \\
\cdot \\
\cdot
\end{array}
\end{array} \right]}{\left[ \begin{array}{c}
\begin{array}{c}
\cdot \\
\cdot \\
\cdot
\end{array}
\end{array} \right]} \cdots
\]
\[
\begin{array}{c}
\begin{array}{c}
\cdot \\
\cdot \\
\cdot
\end{array}
\end{array}
= \frac{-\left[ \begin{array}{c}
\begin{array}{c}
\cdot \\
\cdot \\
\cdot
\end{array}
\end{array} \right]}{\left[ \begin{array}{c}
\begin{array}{c}
\cdot \\
\cdot \\
\cdot
\end{array}
\end{array} \right]} \cdots + \left[ \begin{array}{c}
\begin{array}{c}
\cdot \\
\cdot \\
\cdot
\end{array}
\end{array} \right] \cdots
\]
In particular:

\[
\begin{array}{c}
\text{\shortmid \hspace{1cm} \text{\shortmid} } \\
\text{\shortmid \hspace{1cm} \text{\shortmid} } \\
\vdots \\
\text{\shortmid \hspace{1cm} \text{\shortmid} }
\end{array}
= [2]_{\zeta} \quad \vdots \quad = \begin{array}{c}
\text{\shortmid \hspace{1cm} \text{\shortmid} } \\
\text{\shortmid \hspace{1cm} \text{\shortmid} } \\
\vdots \\
\text{\shortmid \hspace{1cm} \text{\shortmid} }
\end{array}
\]

Proof. Recall the recursive formula for \( JW_i \) (see (17)) in the Temperley-Lieb algebra setting in which we specialise \( q \) to \( \zeta = e^{2\pi i/2m} \). Transforming this Jones-Wenzl projector into a morphism in \( \mathcal{D} \) and putting a trivalent vertex on the left side and a dot on the right side yields a morphism between the Bott-Samelson bimodules. If we pre-/post-compose with the idempotents corresponding to the indecomposable Soergel bimodules every summand is cancelled by property (i) except the following two summands (the lines from the last inductive step are dashed):

\[
\begin{array}{c}
\text{\shortmid \hspace{1cm} \text{\shortmid} } \\
\text{\shortmid \hspace{1cm} \text{\shortmid} } \\
\vdots \\
\text{\shortmid \hspace{1cm} \text{\shortmid} }
\end{array}
\]

The first summand is a cup attached to the identity of \( JW_{i-1} \) at position \( i - 1 \) and thus its scalar is \( \frac{[i-1]_{\zeta}}{[\zeta]} \). The second summand corresponds to the identity and therefore its scalar is 1. Hence in \( \mathcal{D} \) we have

\[
\begin{array}{c}
\text{\shortmid \hspace{1cm} \text{\shortmid} } \\
\text{\shortmid \hspace{1cm} \text{\shortmid} } \\
\vdots \\
\text{\shortmid \hspace{1cm} \text{\shortmid} }
\end{array}
= \frac{[i-1]_{\zeta}}{[\zeta]} \begin{array}{c}
\text{\shortmid \hspace{1cm} \text{\shortmid} } \\
\text{\shortmid \hspace{1cm} \text{\shortmid} } \\
\vdots \\
\text{\shortmid \hspace{1cm} \text{\shortmid} }
\end{array} + \begin{array}{c}
\text{\shortmid \hspace{1cm} \text{\shortmid} } \\
\text{\shortmid \hspace{1cm} \text{\shortmid} } \\
\vdots \\
\text{\shortmid \hspace{1cm} \text{\shortmid} }
\end{array} = \frac{[i-1]_{\zeta}}{[\zeta]} \begin{array}{c}
\text{\shortmid \hspace{1cm} \text{\shortmid} } \\
\text{\shortmid \hspace{1cm} \text{\shortmid} } \\
\vdots \\
\text{\shortmid \hspace{1cm} \text{\shortmid} }
\end{array}
\]

Similarly, we have

\[
\begin{array}{c}
\text{\shortmid \hspace{1cm} \text{\shortmid} } \\
\text{\shortmid \hspace{1cm} \text{\shortmid} } \\
\vdots \\
\text{\shortmid \hspace{1cm} \text{\shortmid} }
\end{array}
= \frac{[i-1]_{\zeta}}{[\zeta]} \begin{array}{c}
\text{\shortmid \hspace{1cm} \text{\shortmid} } \\
\text{\shortmid \hspace{1cm} \text{\shortmid} } \\
\vdots \\
\text{\shortmid \hspace{1cm} \text{\shortmid} }
\end{array} + \begin{array}{c}
\text{\shortmid \hspace{1cm} \text{\shortmid} } \\
\text{\shortmid \hspace{1cm} \text{\shortmid} } \\
\vdots \\
\text{\shortmid \hspace{1cm} \text{\shortmid} }
\end{array}
\]

The scalar in front of the first summand can be computed with the recursive formula for \( JW_i \):

\[
\lambda_i \begin{array}{c}
\text{\shortmid \hspace{1cm} \text{\shortmid} } \\
\text{\shortmid \hspace{1cm} \text{\shortmid} } \\
\vdots \\
\text{\shortmid \hspace{1cm} \text{\shortmid} }
\end{array} = \lambda_{i-1} \begin{array}{c}
\text{\shortmid \hspace{1cm} \text{\shortmid} } \\
\text{\shortmid \hspace{1cm} \text{\shortmid} } \\
\vdots \\
\text{\shortmid \hspace{1cm} \text{\shortmid} }
\end{array} + I_i \mu_{i-1} \begin{array}{c}
\text{\shortmid \hspace{1cm} \text{\shortmid} } \\
\text{\shortmid \hspace{1cm} \text{\shortmid} } \\
\vdots \\
\text{\shortmid \hspace{1cm} \text{\shortmid} }
\end{array}
\]

where \( \lambda_i \) is the coefficient of \( \lambda \) in \( JW_i \), \( \mu_i \) is the coefficient of \( \mu \) in \( JW_i \) and \( I_i \) is the coefficient in the sum from the induction step.

Inductively it can be shown that \( \mu_i = \frac{1}{[i]_{\zeta}} \) using the formula for \( JW_i \). Clearly \( I_i = \frac{1}{[i]_{\zeta}} \) and hence by another induction it follows that \( \lambda_i = \frac{[i-1]_{\zeta}}{[\zeta]} \) using

\[
\lambda_i = \lambda_{i-1} + I_i \mu_{i-1} = \frac{(i-1)_{\zeta}(i-1)_{\zeta} + 1}{[i]_{\zeta}[i-1]_{\zeta}} \quad \text{and} \quad \frac{(i-1)^2_{\zeta}}{[i]_{\zeta}[i-1]_{\zeta}} = \frac{[i-1]_{\zeta}}{[i]_{\zeta}}.
\]

The second summand in (37) arises again from the identity and hence its scalar is 1. With Proposition 6.7 we obtain

\[
\begin{array}{c}
\text{\shortmid \hspace{1cm} \text{\shortmid} } \\
\text{\shortmid \hspace{1cm} \text{\shortmid} } \\
\vdots \\
\text{\shortmid \hspace{1cm} \text{\shortmid} }
\end{array}
= \left( \frac{[i-1]_{\zeta}}{[\zeta]} - [i]_{\zeta} \right) \begin{array}{c}
\text{\shortmid \hspace{1cm} \text{\shortmid} } \\
\text{\shortmid \hspace{1cm} \text{\shortmid} } \\
\vdots \\
\text{\shortmid \hspace{1cm} \text{\shortmid} }
\end{array} + \left( [i-1]_{\zeta} - [i+1]_{\zeta} \right) \begin{array}{c}
\text{\shortmid \hspace{1cm} \text{\shortmid} } \\
\text{\shortmid \hspace{1cm} \text{\shortmid} } \\
\vdots \\
\text{\shortmid \hspace{1cm} \text{\shortmid} }
\end{array}
\]

where the scalar in front of the first summand is

\[
\left( \frac{[i-1]_{\zeta}}{[\zeta]} - [2]_{\zeta} \right) = \frac{[i-1]_{\zeta} - [i+1]_{\zeta}}{[i]_{\zeta}} \quad \text{and} \quad \frac{[i-1]_{\zeta} - [i+1]_{\zeta}}{[i]_{\zeta}} = \frac{[i-1]_{\zeta}}{[i]_{\zeta}}.
\]
For $i = m - 1$ the Soergel bimodule is embedded diagonally and therefore we have

\[
\frac{1}{2} \left( \begin{array}{ccc} & \cdots & \\
\vdots & & \\
\end{array} \right) \frac{1}{2} \left( \begin{array}{ccc}
JW_i & \cdots & 2m \\
\cdots & JW_i & \\
2m & \cdots & \vdots \end{array} \right) \left( \begin{array}{ccc}
& \cdots & \\
\vdots & & \\
\end{array} \right)
\]

\[= \frac{1}{4} \left( \begin{array}{ccc}
JW_i & \cdots & 2m \\
\cdots & JW_i & \\
2m & \cdots & \vdots \end{array} \right) + \frac{1}{4} \left( \begin{array}{ccc}
JW_i & \cdots & 2m \\
\cdots & JW_i & \\
2m & \cdots & \vdots \end{array} \right) + \frac{1}{4} \left( \begin{array}{ccc}
JW_i & \cdots & 2m \\
\cdots & JW_i & \\
2m & \cdots & \vdots \end{array} \right) + \frac{1}{4} \left( \begin{array}{ccc}
JW_i & \cdots & 2m \\
\cdots & JW_i & \\
2m & \cdots & \vdots \end{array} \right)
\]

6.1.3. Relation 23: Clearly, we have:

\[
\left( \begin{array}{ccc}
& \cdots & \\
\vdots & & \\
\end{array} \right) = -2 \zeta.
\]

6.1.4. Relation 25 and 26: The bimodules $B_s B_t$ and $B_t B_s$ are indecomposable and therefore the corresponding idempotent is trivial, hence by isotopy invariance

\[
\left( \begin{array}{ccc}
& \cdots & \\
\vdots & & \\
\end{array} \right) = \left( \begin{array}{ccc}
& \cdots & \\
\vdots & & \\
\end{array} \right).
\]

6.1.5. Relation 27 and 28: The relations 27 and 28 for $i \in \{2, \ldots, m-2\}$ are direct consequences of the following lemma:

**Lemma 6.9.** For $i \in \{2, \ldots, m-2\}$ the following holds.

\[
\left( \begin{array}{ccc}
JW_i & \cdots & \\
\cdots & JW_i & \\
\end{array} \right) = \frac{1}{\mu_i} \left( \begin{array}{ccc}
& \cdots & \\
\vdots & & \\
\end{array} \right) + \frac{1}{\mu_i} \left( \begin{array}{ccc}
& \cdots & \\
\vdots & & \\
\end{array} \right),
\]

\[
\left( \begin{array}{ccc}
JW_i & \cdots & \\
\cdots & JW_i & \\
\end{array} \right) = \frac{1}{\mu_i} \left( \begin{array}{ccc}
& \cdots & \\
\vdots & & \\
\end{array} \right) + \frac{1}{\mu_i} \left( \begin{array}{ccc}
& \cdots & \\
\vdots & & \\
\end{array} \right).
\]

In particular,

\[
\left( \begin{array}{ccc}
JW_i & \cdots & \\
\cdots & JW_i & \\
\end{array} \right) = \frac{1}{\mu_i} \left( \begin{array}{ccc}
& \cdots & \\
\vdots & & \\
\end{array} \right) + \frac{1}{\mu_i} \left( \begin{array}{ccc}
& \cdots & \\
\vdots & & \\
\end{array} \right) = \left( \begin{array}{ccc}
JW_i & \cdots & \\
\cdots & JW_i & \\
\end{array} \right).
\]

**Proof.** Similarly to the proof of Lemma 6.8.

For $i = m - 1$ relation 27 and 28 agree (see Remark 6.2) and the relation is proven similarly to relations 22 and 24 since $B_{w_0}$ is embedded diagonally.
6.1.6. Relation 29 and 31. Inspecting both sides of the relation (29) gives us
\[
\begin{align*}
\text{JW}_{j} & = \ldots, \\
\text{JW}_{j} & = \ldots
\end{align*}
\]
since by property (i) every summand except the identity is cancelled (when pre-/post-composed with the idempotents). The right hand sides are equal by the following lemma (which proves the desired relation).

Lemma 6.10. In \( SB \) we have the following equality as morphisms \( B_{i} \rightarrow B_{i+2} \) for \( i \geq 0 \)
\[
\begin{align*}
\ldots & = \ldots
\end{align*}
\]

Proof. The case \( i = 0 \) is trivial, so let us assume \( i = 1 \).
\[
\begin{align*}
\text{JW}_{i} & = \frac{1}{2} (\alpha_{s} + \alpha_{s}) \text{JW}_{i} \frac{1}{2} + \partial_{i} \text{JW}_{i} \\
\text{JW}_{i} & \equiv \partial_{i} (\alpha_{s}) \text{JW}_{i}
\end{align*}
\]
The general case follows immediately by using the above equation repeatedly. \( \square \)
Relation 31 can be dealt with in the same way by flipping the diagrams at the horizontal axis.

6.1.7. Relation 30 and 32. Translating relation 30 into the Soergel bimodule setting yields
\[
\begin{align*}
\text{JW}_{j} & = \ldots
\end{align*}
\]
In \( JW_{j} \) on both sides above, every summand is cancelled by property (ii) except the one corresponding to the identity with scalar 1 (after pre-/post-composing with the idempotents). Hence, the equation above becomes
\[
\begin{align*}
\ldots & = \ldots
\end{align*}
\]
which clearly holds. Relation 32 holds for the same reasons after flipping the above diagrams at the horizontal axis.

Step III. The path algebra \( P_{m} \) has by Lemma 6.1 dimension \( \sum_{x,y} |W_{\leq x} \cap W_{\leq y}| \) over \( \mathbb{R} \), whereas by Theorem 3.5
\[
\dim_{\mathbb{R}} (A) = \sum_{x,y \in W} \dim_{\mathbb{R}} \text{Hom}_{\mathbb{K}}(B_{x}, B_{y}) = \sum_{x,y \in W} |W_{\leq x} \cap W_{\leq y}|.
\]
Altogether we conclude that \( \varphi \) is an isomorphism. \( \square \)

7. Koszul Self-Duality of \( A \)

7.1. Linear resolutions of the Standard Modules. The indexing set \( \Lambda \) of isomorphism classes of simple graded \( P_{m} \)-modules is in bijection with \( W \) by Proposition 6.2. Hence, the reverse Bruhat order \( \leq_{r} \) turns \( \Lambda \) into a finite partially ordered set \( (\Lambda, \leq_{r}) \). For \( x \in W \) define the left modules
\[
\Delta(x) := P(x)/M(x)
\]
where \( M(x) := \langle \text{all paths starting in } x \text{ passing through } y, y \not\leq_{r} x \text{ of length } \geq 1 \rangle \).
Therefore we have the short exact sequence:
\[
\begin{array}{ccccccccc}
0 & \longrightarrow & M(x) & \longrightarrow & P(x) & \longrightarrow & \pi & \Delta(x) & \longrightarrow & 0.
\end{array}
\]
Note that \( e_y \Delta(x) = 0 \) for all \( y \not\geq_r x \) which means that no path in \( \Delta(x) \) ends in such a \( y \). A precise formula for the dimensions of the involved modules in this short exact sequence gives the following lemma.

**Lemma 7.1.** For \( 0 \leq i \leq m \) we have:

\[
\begin{align*}
\dim_R P(s_i) &= \begin{cases} 
2m & \text{if } i = 0, \\
2i(2m - i) & \text{if } 1 \leq i \leq m - 1, \\
1 + 2m^2 & \text{if } i = m;
\end{cases} \\
\dim_R \Delta(s_i) &= \begin{cases} 
2(m - i) & \text{if } 0 \leq i \leq m - 1, \\
1 & \text{if } i = m;
\end{cases} \\
\dim_R M(s_i) &= \begin{cases} 
0 & \text{if } i = 0, \\
2i(2m - i) - 2(m - i) & \text{if } 1 \leq i \leq m - 1, \\
2m^2 & \text{if } i = m.
\end{cases}
\]

**Proof.** For fixed \( x \in W \) the indecomposable projective cover \( P(x) \) has a basis consisting of all paths starting in \( x \). Hence by Lemma [6.1] and its proof we have

\[
\dim_R P(x) = \sum_{y \in W} \dim_R e_y P_m e_x = \sum_{y \in W} |V_{\geq_r y} \cap V_{\geq_r x}|.
\]

Note that we reversed the order on the vertices. For \( x = s_i \) and \( 1 \leq i \leq m - 1 \) this becomes

\[
\dim_R \Delta(x) = \frac{1}{y = e} + \sum_{1 \leq \ell(y) \leq i - 1} \sum_{j=1}^{m-1} 2j + 2i + (2i - 1) + \sum_{i+1 \leq \ell(y) \leq m-1} 2 \sum_{j=1}^{m-1} 2i + \sum_{y \neq e} 2i = 2i(2m - i).
\]

Consider which reduces for \( i = m \) to \( 1 + \sum_{j=1}^{m-1} 4j + 2m = 1 + 2m^2 \). The case \( i = 0 \) follows easily from \( \text{[42]} \). By construction of \( \Delta(x) \) we have

\[
\dim_R \Delta(x) = \sum_{y \leq_r x} |V_{\leq_r y} \cap \{x\}| = \sum_{y \leq_r x} 1 = \begin{cases} 
2(m - \ell(x)) & \text{if } x \neq w_0, \\
1 & \text{if } x = w_0
\end{cases}
\]

which yields \( \text{[40]} \). The dimension of \( M(s_i) \) follows directly from the above combined with the short exact sequence in \( \text{[38]} \).

**Theorem 7.2.** The set \( \{\Delta(x)\}_{x \in W} \) defines a quasi-hereditary structure on \( (P_m, (W, \leq_r)) \), i.e. for \( x \in W \) the (left) module \( \Delta(x) \) is the (left) standard module.

**Proof.** We prove that \( P(x) \) has a \( \Delta \)-filtration with subquotients isomorphic to \( \Delta(y) \) for \( y \geq_r x \) (each with multiplicity 1) and \( x = s_i \) via induction over \( i \) (this proves the first condition for being quasi-hereditary).

For \( x = e \), i.e. \( i = 0 \), there is nothing to show since \( P(x) = \Delta(x) \) (by comparing dimensions via Lemma \( \text{[7.1]} \)). For \( i = 1 \), that is \( x = s \), we have the short exact sequence

\[
0 \longrightarrow P(x)(-1) \xrightarrow{f} P(s) \longrightarrow \Delta(s) \longrightarrow 0
\]

where \( f \) is pre-composing with \( (s, e) \). It is clear that \( \text{im} f \subseteq M(s) \) and the equality holds by comparing dimensions by Lemma \( \text{[7.1]} \).

Now let \( i \geq 2 \). Pre-composing with \( \omega' := (s_i, s(i - 1)) \) (as indicated in Fig. \( \text{[3]} \)) gives us

\[
f : P(s(i - 1))(-1) \longrightarrow P(s_i)
\]
which turns out to be injective since basis elements are mapped to pairwise non-equivalent paths. Clearly \( \pi f = 0 \), hence using Lemma 7.1 we obtain the following commutative diagram

\[
\begin{array}{c}
M(s) \xrightarrow{\pi} \Delta(s) \\
\alpha \downarrow \quad \beta \downarrow \quad \omega' \downarrow \quad \omega \downarrow \\
P(s(i-1)) \xrightarrow{\Delta(s)} P(s(i)) \xrightarrow{\pi} \Delta(s)
\end{array}
\]

Consider the composition

\[
g : P(s(i-1)) \xrightarrow{\varphi} M(s) \xrightarrow{h} M(s)/\im \varphi
\]

where \( h \) is pre-composing with \( \omega := (s, i(i-1)) \). The module \( M(s) \) has generators \( \omega \) and \( \omega' \) as a left module. We can conclude that \( g \) is surjective since \( \omega \in \im h \) and \( \omega' \in \im \varphi \). Similarly, the module \( M(s(i-1)) \) has generators \( \alpha := (s(i-1), s(i-2)) \) and \( \beta := (s(i-1), s(i-2)) \). Using the relations (31) and (32) we obtain the identities:

\[
\begin{align*}
\alpha \omega &= (s, i(i-1), s(i-2)) = (s, s(i-1), s(i-2)), \\
\beta \omega &= (s, s(i-1), s(i-2)) = (s, s(i-1), s(i-2)).
\end{align*}
\]

Hence we can deduce that \( M(s(i-1))(-1) \subseteq \ker g \) and therefore

\[
P(s(i-1))(-1) \xrightarrow{\varphi} M(s)/\im \varphi
\]

Since \( g \) is surjective \( \varphi \) is surjective, too. Using Lemma 7.1 we can compare dimensions of both sides and deduce that \( \varphi \) is an isomorphism. Thus we have the short exact sequence:

\[
0 \longrightarrow P(s(i-1))(-1) \longrightarrow M(s) \longrightarrow \Delta(s(i-1))(-1) \longrightarrow 0.
\]

By induction hypothesis \( P(s(i-1)) \) has \( \Delta \)-filtration with subquotients isomorphic to \( \Delta(y) \) with \( y \geq_i s(i-1) \) (each with multiplicity 1). Thus \( M(s) \) has \( \Delta \)-filtration with subquotients isomorphic to \( \Delta(y) \) with \( y >_r s \) and therefore \( P(s) \) has the desired \( \Delta \)-filtration.

For \( x \in W \) there clearly exists a surjective map \( \pi' : \Delta(x) \rightarrow L(x) \) and by construction \( \ker \pi' \) has only composition factors isomorphic to \( L(\mu_i) \) where \( \mu_i \leq_r x \). Since every non-trivial path from \( x \) to \( x \) passes through a vertex \( y >_r x \) we have \( [\Delta(x) : L(x)] = 1 \) and hence the second condition is satisfied. \( \Box \)
Theorem 7.3. $P_m$ is Standard Koszul, i.e. every left (resp. right) standard module admits a linear resolution.

Proof. By Proposition 6.3 we have an isomorphism $P_m \cong P_m^{\text{sp}}$ and thus left and right modules can be identified. Therefore it is enough to show that the (left) standard modules admit such resolutions. Since $e \in W$ is maximal we have $\Delta(e) = P(e)$ and there is nothing to show. For $x \in W \setminus \{e\}$ we construct a linear resolution $P_\bullet(x) \rightarrow \Delta(x) \rightarrow 0$ with $P_\bullet(x) := (P_i)_{i \geq 0}$ defined by:

$$P_i := \left( \bigoplus_{\ell(w) = i} P(w) \right) \langle -i \rangle.$$ 

Note that $P_0 = P(x)$. The augmentation map $\epsilon : P_0 \rightarrow \Delta(x)$ is just the canonical projection $\pi : P(x) \rightarrow \Delta(x)$ (see 55). For $x = s^t$ the boundary maps $p_i : P_i \rightarrow P_{i-1}$ for $i \geq 1$ are defined as follows:

$$p_i = \begin{cases} \cdot \left( (s, e), (s, (s - 1)) \right) & \text{if } i = 1 < \ell, \\ \cdot \left( (s, e), (s, (s - 1)) \right) & \text{if } i = 1 = \ell, \\ \cdot \left( (s, e), \right) & \text{if } 2 \leq i < \ell, \\ \cdot \left( (s, e), \right) & \text{if } i = \ell, \\ 0 & \text{if } i > \ell. \end{cases}$$

Recall that we compose paths from right to left (similar as morphisms) and hence the defined $p_i$’s are pre-composing with certain arrows. All indecomposable projective modules $P(w)$ are generated by the corresponding idempotent $e_w$ which lies in degree 0. Thus by construction each $P_i$ is projective and generated by its degree $i$ component. Clearly, the augmentation map $\epsilon$ is surjective.

The fact that for fixed $x$ the above sequence $P_\bullet(x)$ is a complex is a direct consequence of relations 51 and 52. We only present the generic case $2 \leq i < \ell(x)$. Each $P(w)$ is generated by $e_w$, so it is enough to check that $p^2(e_w) = 0$. Let $w = s^t(i - 2)$ then we have

$$p(e_w) = \left( \langle s(i - 1), (s(i - 2)) \rangle \right) \langle -1 \rangle^* (s(i - 1), (s(i - 2)) \rangle$$

and

$$p^2(e_w) = \left( \langle s(i - 1), (s(i - 2)) \rangle + \langle -1 \rangle^* (s(i - 1), (s(i - 2)) \rangle = 0. \right.$$ 

Then we only check the generic case $2 \leq i \leq \ell(x) - 2$ which covers all important arguments. Assume we have $(y, x) \in \ker p_i$ where $p_i : P_i \rightarrow P_{i-1}$ without loss of generality

$$P_i = P(i,j) \bigoplus P(j,i) \langle -i \rangle, \quad P_{i-1} = P(i,j+1) \bigoplus P(j,i+1) \langle -i + 1 \rangle$$

for some $2 \leq j \leq \ell(x) - 2$. From the definition of the $P_i$ we know that $y$ (resp. $x$) is a path starting in $s_j$ (resp. $s_j$). By assumption we have $p_i((y,x)) = 0$ and in particular $\pi_1(p_i((y,x))) = 0 \in P(s_j(j - 1))(-i + 1)$. Recall that $\pi_1p_i$ is pre-composing with $\omega^i = (s^j_j(j + 1), s^j_j)$ in the first component and pre-composing with $\omega^j = (s^j_j, s^j_j(j - 1))$ in the second component
with \( \omega = (s(j + 1), s j) \) in the second component. Since \( (y, x) \in \ker p_i \) we can conclude that they must end in the same vertex, say \( z \) (see Fig. 3). Note that it is not possible for either \( x \) or \( y \) to be the trivial path because then the other path would have to have length \( \geq 2 \) and hence they would be linearly independent. Assuming both \( x \) and \( y \) pass only through vertices \( v \) such that \( v \leq_r s j \) then \( z \notin \{ s j, s j \} \) (otherwise one path would pass through a vertex \( v \) such that \( v \geq_r s j \)). Moreover, \( x \omega \) (resp. \( y \omega \)) does not pass through \( s j \) (resp. \( s j \)). In particular, the highest vertex \( x \omega \) (resp. \( y \omega \)) passes through \( s j \) (resp. \( s j \)). Thus \( x \omega \) and \( y \omega \) are linearly independent by Remark 6.4 (which is formulated in terms of the Bruhat order). This is a contradiction to \( (y, x) \in \ker p_i \). If only one path passes through a vertex \( v \) with \( v \geq_r s j \) the same argumentation using Remark 6.4 again yields a contradiction to \( (y, x) \in \ker p_i \). Therefore \( x \) and \( y \) pass only through vertices \( v \) such that \( v \geq_r s j \). Hence we can deduce that \( y \) starts either with \( \alpha \) or \( \beta \) and \( x \) starts either with \( \gamma \) or \( \delta \). But that means that \( (y, x) \in \im p_{i+1} \). Thus the complex is exact and indeed a linear resolution.

By Theorem 5.2 we obtain the following corollary:

**Corollary 7.4.** \( P_m \) is Koszul.

7.2. The quadratic dual of \( P_m \). By construction \( P_m = \mathbb{R} Q_m / (R^h_m) \). Since \( A := \mathbb{R} Q_m = \bigoplus_{i \geq 0} A^i \) is graded by the path length and \( R^h_m \) is homogeneous \( P_m \) inherits the grading by the path length. We note that \( A = T_{A^0} A^1 = \bigoplus_{i \geq 0} A^i \), and thus \( P_m = T_{A^0} V / (R^h_m) \) is the desired quadratic structure for the \( A^0 \)-bimodule \( V = A^1 \). \( V \) is a vector space spanned by the edges in \( Q_m \) on which we can define the standard scalar product

\[
(\alpha, \beta) := \begin{cases} 
1 & \text{if } t(\alpha) = s(\beta) \text{ and } t(\beta) = s(\alpha) \\
0 & \text{otherwise}
\end{cases}
\]

for edges \( \alpha = (s(\alpha), t(\alpha)) \) and \( \beta = (s(\beta), t(\beta)) \). Using this scalar product we can identify \( V \cong V^* \). In the construction of the quadratic dual we have to consider

\[
R^z := \{ x \mid \forall v \in R^h_m : \langle v, x \rangle = 0 \} \subseteq V^* \otimes_{A^0} V^* \cong (V \otimes_{A^0} V)^*.
\]

By Lemma 6.1 the algebra \( P_m \) is finite-dimensional and with the above identification \( R^z \) is just the usual orthogonal complement of \( R^h_m \) inside \( A^2 \) with respect to the standard scalar product. The vector space \( A^2 \) has a basis consisting of all paths of length 2 and therefore it is of the form \( A^2 = \bigoplus_{x, y \in W} e_x A^2 e_y \). Taking duals commutes with finite direct sums, so for fixed \( x, y \in W \) the complement of all relations starting in \( x \) and ending in \( y \) is inside \( e_x A^2 e_y \) which has at most \( \mathbb{R} \)-dimension 4. With the above considerations it follows that \( P'_m = T_{A^0} V / (R^z) \).
Recall that all defining relations in $P_n$ are homogeneous of degree 2 and can be interpreted as linear combinations of paths of length 2. Using this identification there are three types of relations in $P_m$: paths with the same starting and terminal point, paths with different starting and terminal points between vertices of the same length and paths with different starting and terminal points between vertices of different lengths. We only present the cases where the starting point $x$ is of the form $s^i$ for $0 \leq i \leq m$. The cases where $x = i$ can be treated similarly since all relations are symmetric in $s$ and $t$. For the sake of simplicity define $\hat{t} := m - i$ for $0 \leq i \leq m$.

7.2.1. Relations $[20] - [24]$ These relations have in common that the starting point and terminal point coincide, say $x$. For fixed $x$ all calculations in this section take place in $A(x) := e_x A^2 e_x$. We have to distinguish five different cases:

- **$x = e$:** We have $\dim A(e) = 2$ and both basis elements ($e, s, e$) and ($e, t, e$) are in $P^b_m$, hence the orthogonal complement is 0 and there are no orthogonal relations.

- **$x = s$:** The vector space $A(s)$ has a basis consisting of:

  $$ u_1 := (s, s, s), \quad u_2 := (s, t, s), \quad u_3 := (s, e, s). $$

  The relations are $u_1 = 0$ and $u_2 + [2]_\zeta u_3 = 0$, thus the orthogonal complement is spanned by $[2]_\zeta u_2 - u_3$.

- **$x = s^i$ for some $2 \leq i \leq m - 2$:** The vector space $A(s^i)$ has a basis consisting of:

  $$ v_1 := (s, s^i(i + 1), s), \quad v_2 := (s, t, s^i(i + 1), s), \quad v_3 := (s, t, s^i(i - 1), s). $$

  The relations are $v_1 - \lambda_i v_2 = 0$ and $v_2 - \mu_i v_3 + \nu_i v_4 = 0$ where

  $$ \lambda_i := \frac{|i - 1|_\zeta}{|i|_\zeta}, \quad \mu_i := |i - 1|_\zeta - |i + 1|_\zeta, \quad \nu_i := \frac{|i + 1|_\zeta}{|i|_\zeta}. $$

  The orthogonal complement is spanned by $\lambda_i v_1 + \mu_i v_2 + \nu_i v_4$ and $\nu_i v_2 - v_4$.

- **$x = s^{m - 1} = s^\top$:** The vector space $A(s^\top)$ has a basis consisting of:

  $$ w_0 := (s^\top, w_0, s^\top), \quad w_2 := (s^\top, w_2, s^\top), \quad w_3 := (s^\top, w_3, s^\top). $$

  The relation is $0 = w_1 - \frac{(m - 2)\zeta}{m - 1} w_2 - [2] w_3$ (using [15]). Therefore the orthogonal complement is spanned by $[2] w_1 + w_2$ and $w_3$.

- **$x = w_0$:** We have $\dim A(w_0) = 2$ and there are no relations, hence the orthogonal complement is the complete subspace with basis $(w_0, s^\top, w_0)$ and $(w_0, t^\top, w_0)$.

7.2.2. Relations $[25] - [28]$ All these relations are paths from $x$ to $y$ such that $x \neq y$ and $\ell(x) = \ell(y)$. All computations in this section take place in $A(x, y) := e_x A^2 e_y$, i.e. the vector subspace consisting of all paths of length 2 from $x$ to $y$. For these relations there are three cases to consider:

- **$x = s$ and $y = t$:** The vector space $A(s, t)$ has a basis consisting of:

  $$ \eta_1 := (s, t, s, t), \quad \eta_2 := (s, t, s, t), \quad \eta_3 := (s, e, t). $$

  The relations are $\eta_1 - \eta_2 = 0$ and $\eta_2 - \eta_3 = 0$. Thus the orthogonal complement is spanned by $\eta_1 + \eta_2 + \eta_3$. 

Orthogonal Relations

| $\langle e, s, e \rangle$ | $\langle e, t, e \rangle$ | Orthogonal Relations |
|------------------------|------------------------|----------------------|
| $u_2^t$ | $u_2^s$ | $[2] \setminus u_2^s - u_2^t$ |
| $u_2^t + [2] \setminus u_2^s$ | $\lambda_i u_2^{t+1} + \nu_j u_2^{t+2} + \nu_k u_2^{t+3}$ |
| $[2] \setminus u_2^t - u_2^s$ |

Lemma 7.5. The quadratic dual $P_m$ has the same dimension as $P_m$ as $\mathbb{R}$-vector space.

7.3. Self-duality. In this section we prove the following theorem.

Theorem 7.6. The algebra $P_m$ is Koszul self-dual, i.e. $P_m \cong E(P_m) \cong P_m^t$.
for the scalar at the edge vertices and is independent of the direction. The pattern is highly regular except indicated in Fig. 5. Note that the scalar of an arrow only depends on the adjacent isomorphism and thus we have a surjection

\[ \Theta : \end{align*}\]

\[ \text{Proof.} \] The algebra \( P_m \) is finite-dimensional and Koszul by Corollary 7.4. Thus there is a canonical isomorphism \( E(P_m) \cong (P_m)^{\text{op}} \). Since \( P_m \cong P_m^{\text{op}} \) by Proposition 6.3, it is enough to show that \( P_m \cong P_m^{\text{op}} = E(P_m) \).

Define the map \( \Theta \) on the vertices of \( Q_m \) by \( x \mapsto x^{-1}w_0 \) which extends to an \( A^0 \)-bimodule homomorphism \( \Theta : A \to A \) for \( A = RQ_m \) (see Fig. 4). Note that the images of the arrows are only determined up to scalars which we choose as \( \pm 1 \) as indicated in Fig. 4. Note that the scalar of an arrow only depends on the adjacent vertices and is independent of the direction. The pattern is highly regular except for the scalar at the edge \( (s(m - 1), w_0) \). Since all scalars are invertible, the map \( \Theta \) is an isomorphism and thus we have a surjection \( \Phi : A \xrightarrow{\Theta} A \to P_m^{\text{op}} \). It suffices to show that \( \ker \Phi \) (or equivalently: \( \Theta(R_m^h) \subseteq (R_m^\perp) \)) which implies the existence of a surjection \( \Psi : P_m \to P_m^{\text{op}}, \) which is an isomorphism by dimension reasons (see Lemma 7.5).

In order to show that \( \ker \Phi \) we check that each relation from \( R_m^{h} \) is mapped to a relation from \( R^\perp \) (up to a sign). The map \( \Theta : W \to W \) is an isomorphism and

\[ \text{Figure 4. The Hasse graph } Q_m \text{ and its image under } \Theta \]

\[ \text{Figure 5. A choice of scalars yielding an isomorphism } P_m \cong P_m^{\text{op}} \]
for all \( w \in W \) we have \( \ell(\Theta(w)) = m - \ell(w) \). Therefore we can treat the three types from above separately, again. Note that

\[
\Theta(s_i) = \begin{cases} 
\pi (m - i) = s_i^\pi & \text{if } i \text{ is even}, \\
\pi (m - i) = s_i^\pi & \text{if } i \text{ is odd}.
\end{cases}
\]

Recall the identity in \([13]\) for the quantum numbers \([i]_\zeta = [m - i]_\zeta = [7]_\zeta \) when \( q \) is specialised to \( \zeta = e^{2\pi i/2m} \). Therefore there are the following identities for \( \alpha_i, \lambda_i, \mu_i \), and \( \nu_i \):

\[
\alpha_i^\pi = \alpha_i, \quad \lambda_i^\pi = \nu_i, \quad \mu_i^\pi = -\mu_i.
\]

Since the calculations are not difficult but tedious we only compute the images of the relations \( 20 - 24 \) under \( \Theta \) using \( 45 \). Note that all basis elements are loops and therefore all occurring scalars are \( 1 = (-1)^2 \) since the scalar is independent of the direction of the arrow.

\[
\begin{align*}
(e, s, e) & \mapsto (w_0, s(m - 1), w_0), \\
(e, t, e) & \mapsto (w_0, s(m - 1), w_0), \\
u^s_1 & \mapsto w^s_1, \\
u^s_2 + [2]_\zeta^s v^s_3 & \mapsto w^s_2 + [2]_\zeta^s w^s_1, \\
\nu^{v_4}_1 - \lambda_1 v^{a_4}_3 & \mapsto \begin{cases} 
\nu^{v_4}_1 - \lambda_1 v^{a_4}_3 & \text{if } i \text{ is even}, \\
\nu^{v_4}_1 - \lambda_1 v^{a_4}_3 & \text{if } i \text{ is odd},
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\nu^{v_4}_2 - \mu_1 v^{a_4}_3 & + \nu_1 v^{a_4}_i & \mapsto \begin{cases} 
\nu^{v_4}_2 - \mu_1 v^{a_4}_3 & + \nu_1 v^{a_4}_i & \text{if } i \text{ is even}, \\
\nu^{v_4}_2 - \mu_1 v^{a_4}_3 & + \nu_1 v^{a_4}_i & \text{if } i \text{ is odd},
\end{cases}
\end{align*}
\]

\[
\begin{align*}
w^{v_4}_1 - [2]_\zeta w^{v_4}_2 & \mapsto \begin{cases} 
w^{v_4}_1 - [2]_\zeta w^{v_4}_2 & \text{if } m \text{ is even}, \\
w^{v_4}_1 - [2]_\zeta w^{v_4}_2 & \text{if } m \text{ is odd},
\end{cases}
\end{align*}
\]

The images of the relations agree with the right-hand side in Table 2 up to a sign, swapping the roles of \( s \) and \( t \) and replacing \( i \) with \( 7 \). Therefore their image is contained in \( (R^-) \). The other relations can be treated analogously as above. \( \square \)

Remark 7.1. There is a certain degree of freedom in the choice of the scalars for this isomorphism. In types \( A_2 \) and \( B_2 \) there are \( 5 \) scalars which can be chosen freely. However, solving such a system of quadratic equations gets more and more complicated and is not very enlightening.

Remark 7.2. The basis for our approach in this paper was the equivalence of the diagrammatic category \( \mathcal{D} \) and the category of Soergel bimodules \( SB \) (see Theorem 4.3). However, this equivalence holds in particular for all finite Coxeter groups with the geometric realisation (see \([EW13]\)). Therefore at least in theory the techniques used in this paper could be applied in an arbitrary finite Coxeter group. Note that this will be much harder since the difficult three-coloured Zamolodzhikov relations will appear and there is no complete diagrammatic classification of the idempotents.

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