Construction of Breather soliton solutions of a modeled equation in a discrete nonlinear electrical line and the survey of modulationnal Instability

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Abstract
The most used signals nowadays for the propagation of information in the different transmission lines are solitons because of the simple fact that they are waves of steady state that maintain their forms, their velocity and resist best on dissipative factors [1–5]. Contrary to the other signals, a soliton has a mathematical analytic expression obtained from nonlinear partial differential equations of integrated physical systems and permits the easy access to information relative to the type of signal, to its velocity, to its wave vector and even the characteristics of the transmission line. In this article, we are using a nonlinear line made up of a sequence of identical discrete LC electrical networks to model a discrete nonlinear differential equation which govern the dynamics of Breather solitons in the line. We then construct some solitary wave solutions of type Dark, Bright, and the combined Bright and Dark solitons of that equation by using the direct and effective mathematical method of Bogning-Djeumen Tchaho-Kofane. A numerical simulation has permitted to draw and observe the different profiles of obtained real solitons and the different profiles of their intensity. We use the analytical expressions of each of those obtained Breather solitons and the technique of perturbing steady state solution to study their modulational instability. This has permitted to obtain information on the factors that perturbate these solitons in the course of their propagation in the electrical line notably the domain of stability or the domain of instability.

1. Introduction
The propagation of a soliton is due to interaction between nonlinearity and dispersion which necessitate that the transmission line must be a nonlinear and a dispersive medium [6–12]. This is why we consider in this study a discrete LC electrical line in other to find out how one can change the physical properties of capacitors notably their charge so that these capacitors become nonlinear components and permit the propagation of Breather solitons. The line obtained after the process is a discrete nonlinear capacitive electrical line; reason being that the only nonlinear components in the discrete line is capacitors [13]. The application of Kirchhoff laws to the circuit of the LC discrete nonlinear capacitive electrical line has permitted one to obtain the discrete nonlinear differential equation which govern the dynamics of solitary waves of type Breather in the line. Let us note that it is easy to make a theoretical description of solitons but mathematically, the research of the analytical expression from the modeled equation has not been an easy task [14, 15] in spite of the existence of some mathematical techniques such as: Hirota’s bilinear method, Painleve expansions, the inverse scattering transform, homogeneous balance method, F-expansion method, Jacobi elliptic function method, tanh-function method [4, 7,16–24]. However, the usage of the direct and effective Bogning-Djoume Tchaho-Kofane methods [25–30] based on the identification of basic hyperbolic functions coefficients permits the construction of solitary wave solitons of type Breather of modeled discrete nonlinear differential equation. Solitary wave solutions have
permitted one to draw their profiles and to deduce their natures. To have supplementary information concerning the stability domain or the instability domain of obtained solitons, it is necessary to study one of the factors as a modulational instability that perturbate these solitons during their propagation. The work we are presenting in this paper is divided as follows: in section two one will introduce the general presentation of Bogning-Djeumen Tchaho-Kofane methods, in section three of the work, we will present the discrete nonlinear capacitive electrical line for the study then, we model the discrete nonlinear differential equation which govern the dynamics of solitary waves of type Breather in the line. In section four, we use the Bogning-Djeumen Tchaho-Kofane method to construct solitary wave solutions of type Bright, Dark and combined Bright and Dark of the modeled equation then, we draw real profiles of those solitons and the profile of their intensity. In section five, we use the method of perturbation steady state solution to study modulational instability of obtained Bright and Dark solitons. In section six, we present the conclusion.

2. General presentation of Bogning-Djeumen Tchaho-Kofane method

Bogning, Djeumen and Kofane have developed an analytical method for obtaining solution of shape sech$^{2}$ in certain class of nonlinear partial differential equations. This method is focused on the construction of solitary wave solution and has been adopted to facilitate the resolution of certain type of nonlinear partial differential equations where the nonlinear terms and dispersive terms coexist. This method of construction of the solitary wave solutions intends to look for the solutions of certain categories of nonlinear partial differential equations on the form

$$\gamma \sum_i \frac{\partial u}{\partial x_i} + b_i \sum_i \frac{\partial^2 u}{\partial x_i^2} + \ldots + c_i \sum_i \frac{\partial^4 u}{\partial x_i^4} + \sum_{m,n} d_{mn} \frac{\partial^m u \partial^n u}{\partial x_i^m \partial x_i^n} + f(u, |u|^2) = 0. \quad (1)$$

Where $\gamma, b_i, c_i, d_i$ are the constants; $i, j, l, m, n$ the positive integer; $f$ a linear arbitrary function of $u$ and $|u|^2$; $u$ the unknown to be determine and $|u|^2$ the magnitude of $u$. One looks for solution of equation (1) under the shape of a linear combination of the hyperbolic functions as follows

$$u = \sum_{\alpha} a_{\alpha} \sin \frac{h(\alpha x)}{h(\alpha)} + \cos \frac{h(\alpha x)}{h(\alpha)}.$$ 

(2)

Where $\alpha$ is a constant which depends on the system parameters which model the nonlinear partial differential equation and $a_{\alpha}$ the constants to be determined. Thus the combination of equations (1) and (2) permits to have an equation under the shape

$$\sum \frac{F(a_{\beta})}{\cos h(\alpha)} + \sum G(a_{\beta}) \frac{\sin h(\alpha)}{\cos h(\alpha)} + \sum H(a_{\beta}) \cos h(\alpha) + \sum T(a_{\beta}) \sin h(\alpha) \cos h(\alpha) + \sum W(a_{\beta}) = 0. \quad (3)$$

This equation presents five ranges of equations of coefficients $a_{\beta}$ which are: the range of the coefficients $F(a_{\beta})$ of power $\cos h(\alpha)$, the range of coefficients $G(a_{\beta})$ of power $\sin h(\alpha)$, the range of coefficients $H(a_{\beta})$ of power $\cos h(\alpha)$, the range of coefficients $T(a_{\beta})$ of power $\sin h(\alpha)$ and the range of coefficients $W(a_{\beta})$ of power $1$. In the ranges of coefficients of $\frac{1}{\cos h(\alpha)}$ and $\frac{\sin h(\alpha)}{\cos h(\alpha)}$, the equations that best seeks the solutions are those raised to the most elevated powers. In the ranges of coefficients of $\cos h(\alpha)$ and $\sin h(\alpha)$ of low powers. The last range of equations of coefficients $W(a_{\beta})$ is not very important because it is considered like a confused domain for the correct solutions obtainable. One can classify these equations of coefficients in order of decreasing priority $F(a_{\beta})$, $G(a_{\beta})$, $H(a_{\beta})$, $T(a_{\beta})$, $W(a_{\beta})$. Here the importance or priority makes reference to the range that permits to obtain good results or merely that which tends more to the exact value. While identifying the coefficients of equation (1) to zero, one gets the first range

$$\sum_{\beta} F(a_{\beta}) = 0 \quad (4)$$

and

$$\sum_{\beta} G(a_{\beta}) = 0. \quad (5)$$

In the set of the above equations, priority is given to equations which are to the power $\frac{1}{\cos h(\alpha)}$ and $\frac{\sin h(\alpha)}{\cos h(\alpha)}$ from most elevated. But care must be taken because it is not the most elevated power which gives the best solution directly; it depends on the shape of solution considered from the onset, the symmetry of the equation to solve as well as from its nonlinearity degree. In these conditions, one moves directly to the equations of lower powers until the good equation to solve is obtained. In the case where the first two set of equations (4) and (5)
don’t give a satisfactory solution, one moves to the set of equations of the following range

\[ \sum_{\varphi} H(a_{\varphi}) = 0 \]  \hspace{1cm} (6)

and

\[ \sum_{\varphi} T(a_{\varphi}) = 0. \]  \hspace{1cm} (7)

In the set of equations (6) and (7), priority is given to the equations of low powers of \( \cos h^k(\alpha x) \) and \( \sin h^l(\alpha x) \). In general, the first two set of equations (4) and (5) permit to find the solution of the problem. In the case where that they don’t give satisfactory solution, it would be cautious to change the shape of the solution or merely the form of solution we want to construct or simply the method. The ranges \( w(a_{\varphi}) \) is considered now as the one that brings no reliable information. It is important to mention that this method appears complicated in the case where the properties of transformations of hyperbolic functions are not mastered. A mastery of these transformations reduces the difficulties considerably as regard to the calculations.

3. Modeling of discrete nonlinear partial differential equation

Recently, T T Guy and J R Bogning have studied a nonlinear capacitive electrical line in the continuum domain [35]. They have defined that line as an identical series of RC electrical networks where capacitors are nonlinear components, well defined in such a way that the line accepts to propagate certain types of solitary waves. In this work we are studying a discrete nonlinear capacitive electrical line made as an identical series of LC electrical networks.

Let’s consider a discrete nonlinear capacitive electrical line that we present in figure 1.

This line is a series of identical LC electrical networks where each is numbered by the positive integer \( n \). \( L \) stands for the inductance, \( q(u_n) \) stands for the charge of capacitors connected to the network order \( n \) and varies in nonlinear manner in terms of voltage \( u_n \) across that capacitor. \( i_n \) stand for the current that flows through LC network order \( n \). Applying Kirchhoff laws to the circuit figure 1 we obtain the following equations

\[ L \frac{d^2 q(u_n)}{dt^2} = u_{n-1} - u_n, \]  \hspace{1cm} (8)

and

\[ L \frac{di_n}{dt} = L \frac{di_{n+1}}{dt} + L \frac{d^2 q(u_n)}{dt^2}. \]  \hspace{1cm} (9)

The substitution of \( L \frac{d^2 q(u_n)}{dt^2} \) and \( L \frac{d^2 q(u_{n+1})}{dt^2} \) of (8) in (9) permit to obtain

\[ L \frac{d^2 q(u_n)}{dt^2} = u_{n-1} - 2u_n + u_{n+1}. \]  \hspace{1cm} (10)

We define the nonlinear charge of the capacitors under the analytical shape as follows

\[ q(u_n) = C_0 u_n - A_1 u_n^2 + A_2 u_n^3, \]  \hspace{1cm} (11)

Where \( C_0 \) stand for the capacitance of capacitors in linear state, \( A_1 \) and \( A_2 \) are real numbers which stand for the nonlinear coefficients. A combination of equations (11) and (10) permits to obtain discrete nonlinear differential equation
In linear state $A_1 = A_2 = 0$ and equation (12) becomes

$$LC_0 \frac{d^2 u_n}{dt^2} = u_{n-1} - 2u_n + u_{n+1}. \tag{13}$$

One can find the solution of equation (13) under the shape

$$u_n = B \exp(i(k pn - w_p t)). \tag{14}$$

Where $k_p$ is the wave vector of sinusoidal wave with the weak amplitude $B$ and $w_p$ its frequency. Equations (13) and (14) has permitted to obtain the relation between the frequency $w_p$ and the wave vector $k_p$ under the form

$$w_p^2 = \frac{4}{LC_0} \sin^2 \left(\frac{k_p}{2}\right). \tag{15}$$

The sinusoidal wave with weak amplitude propagates with group velocity $v_g = \frac{\partial w_p}{\partial k_p} = \frac{\sin(k_p)}{LC_0 w_p}$

By considering now the nonlinear state, sinusoidal wave with weak amplitude is perturbed and henceforth constitutes the carrier that propagates in discrete domain with the same frequency $w_p$ and the same wave vector $k_p$ but its amplitude $B$ is modulated and becomes an envelope that propagates slowly than the carrier in time and space and can be studied in continuum domain. As such, considering the solution of the discrete nonlinear differential equation (12) under the form

$$u_n(t) = \varepsilon B(X, T) \exp(i(k_p n - w_p t)). \tag{16}$$

Where $X = \varepsilon (n - v_p t), T = \varepsilon^2 t$ are slow variables and $\varepsilon$ is a real parameter less than one. The substitution of $u_n(t)$ given by (16) in (12) by retaining all the term order $\varepsilon^2$ and $\varepsilon^3$ proportional to exp $(i(k_p n - w_p t))$ permits one to obtain the equation that describes the dynamics of the envelope in the discrete nonlinear capacitive electrical line which is given by [31–33]

$$-\frac{i \partial B(X, T)}{\partial T} + p \frac{\partial^2 B(X, T)}{\partial X^2} + QB(X, T) |B(X, T)|^2$$

$$- \frac{i \varepsilon v_p}{w_p} \left( \frac{\partial B(X, T)}{\partial X} + \frac{4}{3} p \frac{\partial^3 B(X, T)}{\partial X^3} + 2Q |B(X, T)|^2 \frac{\partial B^2(X, T)}{\partial X} + 2QB(X, T) \frac{\partial B^2(X, T)}{\partial X} \right) = 0. \tag{17}$$

Where $\varepsilon^2 = -1, B^*(X, T)$ stand for complex conjugate of $B(X, T), P = -\frac{w_p}{2}$ stand for the dissipation coefficient and $Q = \frac{2^4 \lambda e_c}{\lambda c_0} \left( \frac{3 \lambda e_c}{2 \lambda c_0} + \frac{4}{w_p^2 LC_0} - 2 \right)$ stand for nonlinear coefficient. Equation (17) represents the higher-order nonlinear Schrodinger equation because it becomes the standard nonlinear Schrodinger equation when $\varepsilon$ tends to zero.

4. Construction of solitary wave solution of type Breather of modeled equation and presentation of their profiles

In this section, we construct the different analytical expressions of Breather soliton which are exact solutions of differential equation (12) notably Bright soliton, Dark soliton and a combined Dark and Bright solitons. For this reason, we are finding out for each case envelope solution before deducing the analytical expression of Breather soliton susceptible to propagate in the line.

4.1. Construction of solitary wave solution of type Bright

Let us find out the envelope solution $B(X, T)$ under the analytical form

$$B(X, T) = a \text{sech}(\alpha u X - v_p T) \exp(i(\alpha u X - v_p T)). \tag{18}$$

Where $a, \alpha, v_p, \alpha$ and $v_p$ are real numbers to be determined in terms of system parameters. The substitution of $B(X, T)$ in (17) has permitted to obtain the equation
Each basic hyperbolic function coefficient of (19) must be equal to zero. This permits to obtain the set of four equations as follows

\[
\begin{align*}
\frac{i\varepsilon v_g a_{0p} v_p}{w_p} + 2i Pa_{0p} \alpha_e - \frac{i\varepsilon v_g a_{0c} v_p}{w_p} - \frac{4i\varepsilon v_g Pa_{0c}^3}{3w_p} - iav_c + \frac{4i\varepsilon v_g Pa_{0c}^2 \alpha_e}{w_p} = 0, \\
Pa_{0c}^2 + \frac{\varepsilon v_g a_{0p} v_p}{w_p} + \frac{4\varepsilon v_g Pa_{0c}^2 \alpha_p}{w_p} - \frac{\varepsilon v_g a_{0c} v_p}{w_p} + av_p - \frac{4\varepsilon v_g Pa_{0c}^3}{3w_p} = 0, \\
\frac{2\varepsilon v_g a_{0c} v_p}{w_p} - \frac{8\varepsilon v_g Pa_{0c}^2 \alpha_p}{w_p} - 2Pa_{0c}^2 + Qu^3 = 0, \\
\frac{8i\varepsilon v_g Pa_{0c}^3}{w_p} - \frac{4i\varepsilon v_g Qa^3 \alpha_e}{w_p} = 0.
\end{align*}
\]

The solving of (20) permits to obtain the following results

\[
\begin{align*}
v_p &= \frac{2Pa_p(6\alpha_p \varepsilon v_g w_p + 9w_p^2 + 8\alpha_p^2 \varepsilon^2 v_p^2)}{3\varepsilon v_g (4\alpha_p \varepsilon v_g + 2w_p)}, \\
\alpha_e &= \frac{w_p}{4Pa_p}, \\
\alpha_e &= \frac{\alpha \sqrt{2PQ}}{Q}, \\
\varepsilon v_g &= \frac{PQ > 0, -3\varepsilon v_g P(2Pw_p \alpha_p + \varepsilon v_g v_p) > 0, B(X, T)}{\exp(i(\alpha_p X - v_p T))}.
\end{align*}
\]

From the expression of \( B(X, T) \) given by (21) one has deduced from expression (16) the analytical expression of the solution of equation (12) given by the following result

\[
\begin{align*}
u_n(t) &= \frac{\varepsilon \alpha \sqrt{2PQ}}{Q} \text{sech} \left( \varepsilon \alpha_e (n - v_p t) - \frac{2\varepsilon^2 \alpha_p \sqrt{-3\varepsilon v_g P(2Pw_p \alpha_p + \varepsilon v_g v_p)} T}{\varepsilon v_g} \right) \\
&\quad \times \exp(i((\varepsilon \alpha_p + k_p)n - (\varepsilon \alpha_p v_p + \varepsilon v_p + w_p)t)).
\end{align*}
\]

Equation (22) presents as such the analytical expression of Bright soliton which is the exact solution of discrete nonlinear differential equation (12) which governs its dynamics in the line. By considering the values of the following parameters \( \varepsilon = 0, 1, L = 470 \times 10^{-6} H, C_0 = 370 \times 10^{-12} F, A_1 = 7, 77 \times 10^{-11} F/V, \)

\( A_2 = -7, 28 \times 10^{-11} F/V, k_p = \frac{2\pi}{2} \text{Rad/m and} \alpha_p = -\frac{2\pi}{2} \text{Rad/m,} \) the expression of Breather soliton (22) takes the following shape \( u_n(t) = 0, 94\text{sech}(1, 02n - 1, 87 \times 10^8 t) \exp(i(1, 49n - 3, 72 \times 10^8 t)), \) its real part is written as \( R = u_n(t) \equiv 0, 94\text{sech}(1, 02n - 1, 87 \times 10^8 t) \cos(1, 49n - 3, 72 \times 10^8 t) \) and its intensity is given by \( |u_n(t)| = 0, 94\text{sech}(1, 02n - 1, 87 \times 10^8 t) \). This permits to obtain respectively in figure 2 the representations of real profile and intensity profile of that Bright soliton.

Figure 2 shows effectively a real profile of a solitary wave of type Bright whose carrier is in rapid motion in an envelope of type Pulse that propagates very slowly.
4.2. Construction of solitary wave solution of type Dark

We find out the analytical expression of the envelope under the form

\[ B(X, T) = a \tanh(\alpha_c X - v_c T) \exp(i(\alpha_p X - v_p T)). \]

(23)

Where \(a, \alpha_c, v_c, \alpha_p\), and \(v_p\) are real numbers to be determined in relation to discrete electrical line parameters.

A combination of equations (23) and (17) permits one to obtain the following relation

\[
\begin{align*}
&\left( -av_p + P\alpha_p^2 + \frac{4\epsilon v_p P\alpha_p^3}{3w_p} - \frac{\epsilon v_a a\alpha_p v_p}{w_p} - Q\alpha^3 \right) \sin h(\alpha_c X - v_c T) \\
&+ \left( Q\alpha^3 + 2P\alpha_p^2 - \frac{2\epsilon v_p a\alpha_p v_p}{w_p} + \frac{8\epsilon v_p P\alpha_p^3}{w_p} \right) \sin h(\alpha_c X - v_c T) \\
&+ \left( \frac{4\epsilon v_p P\alpha_p^3}{w_p} - \epsilon v_a a\alpha_p v_p - 2iP\alpha_p\alpha_c = \frac{16\epsilon v_p P\alpha_p^3}{3w_p} \right) \frac{1}{\cos h^3(\alpha_c X - v_c T)} \\
&+ \left( \frac{8\epsilon v_p P\alpha_p^3}{w_p} + \frac{4i\epsilon v_p Qa^3\alpha_c}{w_p} \right) \cos h^3(\alpha_c X - v_c T) = 0.
\end{align*}
\]

(24)

Equation (24) is verified if and only if each of its basic hyperbolic function coefficient is equal to zero. This permits to obtain a set of four equations given by

\[
\begin{align*}
&\frac{4\epsilon v_p P\alpha_p^3}{w_p} - \frac{\epsilon v_a a\alpha_p v_p}{w_p} - \frac{16\epsilon v_p P\alpha_p^3}{3w_p} = 0, \\
&-av_p + P\alpha_p^2 = \frac{4\epsilon v_p P\alpha_p^3}{3w_p} - \frac{\epsilon v_a a\alpha_p v_p}{w_p} - Q\alpha^3 = 0, \\
&Q\alpha^3 + 2P\alpha_p^2 = \frac{4\epsilon v_p P\alpha_p^3}{w_p} + \frac{8\epsilon v_p P\alpha_p^3}{w_p} = 0, \\
&\frac{8\epsilon v_p P\alpha_p^3}{w_p} + \frac{4i\epsilon v_p Qa^3\alpha_c}{w_p} = 0.
\end{align*}
\]

(25)
By solving (25), one obtains the analytical expression of envelope \( B(X, T) \) given by

\[
\begin{align*}
\nu &= \frac{2\alpha_p \sqrt{2\alpha_p \varepsilon \nu \left(4\alpha_p \varepsilon \nu + \nu_p\right)\left(9\alpha_p \varepsilon \nu \nu_p + 4\varepsilon^2 \nu^2 + 6\nu_p^2\right)}}{\varepsilon \nu \left(4\alpha_p \varepsilon \nu + \nu_p\right)}, \\
\nu_p &= \frac{24\alpha_p^4 \nu^2 \nu_p + 32\nu \varepsilon \alpha^5 \nu^2 + 3\nu^2 \nu_p}{24\alpha_p \left(\alpha_p \varepsilon \nu + \nu_p\right)}, \quad \alpha = \frac{\nu}{4\alpha_p}, \quad a = \frac{\alpha \sqrt{-2PQ}}{Q}, \\
PQ &< 0, \quad 2\alpha_p \varepsilon \nu \left(4\alpha_p \varepsilon \nu + \nu_p\right)\left(9\alpha_p \varepsilon \nu \nu_p + 4\varepsilon^2 \nu^2 + 6\nu_p^2\right) > 0, \\
B(X, T) &= \frac{\alpha \sqrt{-2PQ}}{Q} \tanh(\alpha \nu X - \nu T) \\
&\times \exp \left(i \left(\alpha \nu X - \frac{24\alpha_p^4 \nu^2 \nu_p + 32\nu \varepsilon \alpha^5 \nu^2 + 3\nu^2 \nu_p}{24\alpha_p \left(\alpha_p \varepsilon \nu + \nu_p\right)} T\right)\right). \quad (26)
\end{align*}
\]

The expression of \( B(X, T) \) given by (26) permits to deduce from equation (16) the analytical solution of equation (12) in the following manner

\[
u_n(t) = \frac{\varepsilon \alpha \sqrt{-2PQ}}{Q} \tan h(\varepsilon \alpha \nu(t - \nu_t) - \varepsilon^2 \nu t) \\
\times \exp \left(i \left(\varepsilon \alpha \nu + k\nu\right) t - \left(\varepsilon \alpha \nu \nu + \frac{24\varepsilon^2 \alpha^4 \nu^2 \nu_p + 32\varepsilon \alpha \nu \alpha^5 \nu^2 + 3\nu^2 \nu_p}{24\alpha_p \left(\alpha \varepsilon \nu + \nu_p\right)} + \nu_p\right) t\right). \quad (27)
\]

The analytical expression of \( u_n(t) \) given by (27) is that of Dark soliton which is an exact solution of the discrete nonlinear differential equation (12). Considering the values of the following parameters \( \varepsilon = 0, 1, L = 470 \times 10^{-6} \, H, C_0 = 370 \times 10^{-12} \, F, F_1 = 7.77 \times 10^{-11} \, F/V, A_2 = -7.28 \times 10^{-11} \, F/V^2, k_p = \frac{\pi}{2} \, Rad/m \) and \( \alpha_p = \frac{\pi}{2} \, Rad/m, \) the expression of Breather soliton (27) takes the shape \( u_n(t) = 0, 6 \tan h (0, 65n - 1, 02 \times 10^t \exp(i(1, 64n - 3, 16 \times 10^t))), \) its real part is written as \( R_x(u_n(t)) = 0, 6 \tan h (0, 65n - 1, 02 \times 10^t \cos(1, 64n - 3, 16 \times 10^t)) \) and its intensity is given by \( |u_n(t)| = 0, 6 \tan h(0, 65n - 1, 02 \times 10^t). \) This permits to obtain respectively in figure 3 the representation of real profile and intensity profile of that Dark soliton.

The diagram of figure 3 shows a real profile of Dark soliton with a carrier which propagates faster in its envelope of type Kink that trends it moderately.

4.3. Construction of solitary wave solution of combined Bright and Dark soliton

Let’s find out an analytical expression of envelope \( B(X, T) \) under the form

\[
B(X, T) = \text{asech}(\alpha \nu X - \nu T) + b \tan h(\alpha \nu X - \nu T) \exp(i(\alpha \nu X - \nu T)). \quad (28)
\]

Where \( a, b, \alpha, \nu, \alpha_p \) and \( \nu_p \) are real numbers to be determined in relation to the parameters of the line. By substituting \( B(X, T) \) of (28) in (17) one obtains the following equations
By equating to zero the different basic hyperbolic function coefficients, we obtain the set of eight equations given by

\[
\begin{align*}
\left[ -\frac{i\epsilon w_0 c_{\alpha e}}{w_p} + \frac{4i\epsilon w_0 Qb_2^\alpha c_{\alpha e}}{w_p} + \frac{8i\epsilon w_0 Qa_2^2 c_{\alpha e}}{w_p} + 2iPb_{\alpha p} c_{\alpha e} \right] \quad \text{cos} \ h^4(-\alpha_eX + \nu eT) &= 1, \\
\left[ -\frac{4i\epsilon w_0 Qb_1^\alpha c_{\alpha e}}{w_p} - \frac{16i\epsilon w_0 Qb_3^\alpha c_{\alpha e}}{w_p} - ib_{\nu c} - \frac{i\epsilon w_0 b_{\alpha c} c_{\nu e}}{w_p} \right] \quad \text{cos} \ h^4(-\alpha_eX + \nu eT) &= 0, \\
\left[ +\frac{2i\epsilon w_0 a_{\alpha c} c_{\nu e}}{w_p} - \frac{8i\epsilon w_0 a_{\alpha c} c_{\nu e}}{w_p} - 2P a_{\alpha c}^2 - 3Q a_{\alpha c}^3 + Q a_{\alpha c}^3 = 0, \\
\left[ +\frac{P a_{\alpha c}^2 - P a_{\alpha c}^3 + a_{\nu c} - \frac{4i\epsilon w_0 P a_{\alpha c}^3}{3w_p} - \frac{i\epsilon w_0 a_{\alpha c} c_{\nu e}}{w_p} \right] \quad \text{cos} \ h^4(-\alpha_eX + \nu eT) &= 0, \\
\left[ +\frac{4i\epsilon w_0 P a_{\alpha c}^3}{3w_p} + \frac{i\epsilon w_0 a_{\alpha c} c_{\nu e}}{w_p} \right] \quad \text{cos} \ h^4(-\alpha_eX + \nu eT) &= 0, \\
\left[ +\frac{8i\epsilon w_0 P b_{\alpha c}^3}{w_p} - \frac{12i\epsilon w_0 Qa_2^2 c_{\alpha e}}{w_p} + \frac{4i\epsilon w_0 Qb_1^3 c_{\alpha e}}{w_p} \right] \quad \text{sin} \ h(-\alpha_eX + \nu eT) = 0, \\
\left[ +\frac{i\epsilon w_0 a_{\alpha c} c_{\nu e}}{w_p} + \frac{4i\epsilon w_0 P a_{\alpha c}^2 c_{\alpha e}}{w_p} - \frac{i\epsilon w_0 a_{\alpha c} c_{\nu e}}{w_p} \right] \quad \text{sin} \ h(-\alpha_eX + \nu eT) = 0, \\
\left[ +\frac{2i\epsilon w_0 a_{\alpha c} c_{\nu e}}{w_p} - \frac{8i\epsilon w_0 P b_{\alpha c}^2 c_{\alpha e}}{w_p} + Q b_{\alpha c}^3 \right] \quad \text{sin} \ h(-\alpha_eX + \nu eT) = 0.
\end{align*}
\]
The set of equation (30) permits one to obtain the exact solution of equation (17) as shown below
\[\alpha_e = \frac{2\alpha_p \varepsilon \nu_p (9\alpha_p \varepsilon \nu_p w_p + 4\alpha_p^2 \varepsilon^2 \nu_p^2 + 6w_p^2)}{\varepsilon \nu_y (4\alpha_p \varepsilon \nu_y + w_p)},\]
\[\nu_p = \frac{P(3\alpha_p^2 w_p + 6\alpha_p^2 w_p + 8\varepsilon \nu_y \alpha_p^3)}{6(\alpha_p \varepsilon \nu_y + w_p)}, \quad \nu_c = 4P\alpha_p \alpha_c, \quad b = \frac{\alpha_p \sqrt{-2PQ}}{2Q},\]
\[a = \frac{\sqrt{3Q(2\alpha_p^2 + Qb^2)}}{3Q}, \quad PQ < 0, \quad Q(2\alpha_p^2 + Qb^2) > 0, \quad 2\alpha_p \varepsilon \nu_y \times (4\alpha_p \varepsilon \nu_y + w_p)(9\alpha_p \varepsilon \nu_y w_p + 4\alpha_p^2 \varepsilon^2 \nu_y^2 + 6w_p^2) > 0, \quad B(X, T) = \left(\frac{\sqrt{3Q(2\alpha_p^2 + Qb^2)}}{3Q}\right) \times \text{sech}(\alpha_e X - 4P\alpha_p \alpha_c T) + \frac{\alpha_p \sqrt{-2PQ}}{2Q} \tan h(\alpha_e X - 4P\alpha_p \alpha_c T) \times \exp \left(\frac{\alpha_p X - \frac{P(3\alpha_p^2 w_p + 6\alpha_p^2 w_p + 8\varepsilon \nu_y \alpha_p^3)}{6(\alpha_p \varepsilon \nu_y + w_p)} T}{T}\right). \quad (31)\]

The expression of \(B(X, T)\) given by (31) permits to deduce from equation (16) the analytical solution of equation (12) in the following manner
\[u_{\alpha_e}(t) = \frac{\varepsilon \sqrt{3Q(2\alpha_p^2 + Qb^2)}}{3Q} \text{sech}(\varepsilon \alpha_e (n - \nu_y t) - 4\varepsilon^2 P\alpha_p \alpha_c t) + \frac{\varepsilon \alpha_e \sqrt{-2PQ}}{2Q} \tan h(\varepsilon \alpha_e (n - \nu_y t) - 4\varepsilon^2 P\alpha_p \alpha_c t) \times \exp \left(i(\varepsilon \alpha_p + k_p) n - (\varepsilon \alpha_p \nu_y + \frac{P\varepsilon^2(3\alpha_p^2 w_p + 6\alpha_p^2 w_p + 8\varepsilon \nu_y \alpha_p^3)}{6(\alpha_p \varepsilon \nu_y + w_p)} T)\right). \quad (32)\]

The expression of \(u_{\alpha_e}(t)\) given by (32) is a combined Dark and Bright soliton solution of discrete nonlinear differential equation (12). For the values of the following parameters \(\varepsilon = 0, 1, L = 470 \times 10^{-6} H, C_0 = 370 \times 10^{-12} F, A_1 = 7.77 \times 10^{-11} F/V, A_2 = -7.28 \times 10^{-11} F/V^2, k_p = \frac{\pi}{2} \text{Rad/m} \text{et} \alpha_p = \frac{\pi}{4} \text{Rad/m},\)
expression of Breather soliton (32) can be rewritten as \(u_{\alpha_e}(t) = -0, 6 \text{sech}(1, 31 n - 2, 05 \times 10^9 t) (1 + i \sin h(1, 31 n - 2, 05 \times 10^9 t)) \exp(i(1, 64 n - 3, 16 \times 10^9 t)),\) its real part is given by \(R_e(u_{\alpha_e}(t)) = -0, 6 \text{sech}(1, 31 n - 2, 05 \times 10^9 t) \left(\cos(1, 64 n - 3, 16 \times 10^9 t) - \sin(1, 64 n - 3, 16 \times 10^9 t) \sin h(1, 31 n - 2, 05 \times 10^9 t)\right)\) and its intensity is written as \(|u_{\alpha_e}(t)| = 0, 6 \text{sech}(1, 31 n - 2, 05 \times 10^9 t) \sqrt{1 + \sin h^2(1, 31 n - 2, 05 \times 10^9 t)}\].
This permits one to obtain respectively in figure 4 the representation of real profile and intensity profile of that combined Bright and Dark solitons.

Figure 4 shows a solitary wave of combined Bright and Dark solitons. The real profile shows a sinusoidal wave without the deformation of its amplitude and can be explained by the simple fact that the envelope of Bright soliton which is a Pulse perturbates the envelope of Dark soliton which is a Kink or inversely. This trends the envelope in an equilibrium state which becomes a constant where the amplitude of each Pulse soliton and Kink soliton are almost equal. This is why the carrier has a sinusoidal motion without the deformation of its amplitude.

5. Modulational instability of obtained Bright and Dark solitons

In this section, we use the Bogning-Djeumen Tchaho-Kofane method to study the modulational instability of Dark and Bright solitons susceptible of propagating in the discrete nonlinear electrical line. The application of this method is simply due to the fact that it is very effective for the perturbation of steady state solution and gives addition information on the stability or on the instability of the signal. The modulational instability starts from the perturbation of solitary wave taken in its steady state with the help of a wave with weak amplitude. This permits one to obtain perturbation equations after neglecting higher-order perturbation terms and the terms proportional to the solitary wave amplitude taken in its steady state. Perturbation equations permits to have a dispersion relation and by consequence information on the instability of the signal, which are obtained in the domain where the frequency is complex and causes an abnormal increase of soliton amplitude. Modulational
instability is also observed in the domain where the frequency is negative or in the domain where it is poorly defined.

5.1. Modulational instability of Bright soliton

When the envelope $B(X, T)$ in its steady state undergoes a least perturbation $\delta(X, T)$ of its amplitude, the solitary wave solution of type Bright takes the following form

$$B(X, T) = (a + \delta(X, T)) \text{sech}(\alpha_p X - v_p T) \exp(i(\alpha_p X - v_p T))$$ (33)

The substitution of the expression of the new envelope $B(X, T)$ in the Higher-order nonlinear Schrodinger equation (17) by considering just $\text{sech}(\alpha_p X - v_p T)$ term for the fact that it describes well the dynamics of this signal. We obtain the perturbation equation after having neglected $\delta(X, T)$ higher-order terms, the $\delta(X, T)$ terms proportional to the amplitude of Bright soliton taken in its steady state, and after having considered that $a \text{sech}(\alpha_p X - v_p T) \exp(i(\alpha_p X - v_p T))$ is a solution of equation (17). This permit to obtain the perturbation equation of Bright soliton that we present below

$$\frac{\varepsilon_p}{w_p} \frac{\partial^2 \delta(X, T)}{\partial X \partial T} - \frac{4i\varepsilon_p}{3w_p} \frac{\partial^3 \delta(X, T)}{\partial X^3} + \left( \frac{4i\varepsilon_p \alpha_p}{w_p} + p \right)$$

$$\times \frac{\partial^3 \delta(X, T)}{\partial X^2} + \left( i + \frac{i\varepsilon_p \alpha_p}{w_p} \right) \frac{\partial \delta(X, T)}{\partial T} + \left( -\frac{4i\varepsilon_p \alpha_p^2}{w_p} \right) \frac{\partial \delta(X, T)}{\partial X} = 0.$$ (34)

Since the perturbation $\delta(X, T)$ is supposed to be of a very weak amplitude, then we find the solution of perturbation equation (34) under the shape of sinusoidal wave as

$$\delta(X, T) = D \exp(i(kX - wT)).$$ (35)
Where $D$, $k$ and $w$ stand respectively for the amplitude, the wave vector and the frequency of the perturbation wave. The substitution of $\delta(X, T)$ given by (35) in (34) permits to obtain the dissipation relation

$$w = \frac{k(-12\varepsilon v_p\alpha_p^2 + 4k^2\varepsilon v_p + 12k\varepsilon v_p\alpha_p + 3kPw_p - 3\varepsilon v_p + 6\alpha_p w_p + 12\varepsilon v_p\alpha_p^2)}{3(\varepsilon v_k + w_p + \varepsilon v_p \alpha_p)}.$$  (36)

The bright soliton is therefore modulatoraly stable when the following conditions are verified

$$k = -\frac{\varepsilon v_p \alpha_p}{\varepsilon v_k}$$  (37)

and

$$k(-12\varepsilon v_p\alpha_p^2 + 4k^2\varepsilon v_p + 12k\varepsilon v_p\alpha_p + 3kPw_p - 3\varepsilon v_p + 6\alpha_p w_p + 12\varepsilon v_p\alpha_p^2)}{3(\varepsilon v_k + w_p + \varepsilon v_p \alpha_p)} > 0.$$  (38)

5.2. Modulational instability of Dark soliton

When the envelope of Dark soliton taken in its stable state is affected by a least perturbation $\delta(X, T)$, the solution of equation (17) takes the shape

$$B(X, T) = (a + \delta(X, T))\tan h(\alpha_p X - v_p T)\exp(i(\alpha_p X - v_p T)).$$  (39)

Substituting $B(X, T)$ of (39) in (17) by considering just the $\tan h(\alpha_p X - v_p T)$ coefficient, by eliminating higher-order $\delta(X, T)$ terms and by eliminating terms proportional to amplitude of soliton taken in its steady state, one obtains the perturbation equation of Dark soliton presented below

$$\frac{4i\varepsilon v_p \partial^2 B(X, T)}{3w_p} + \left(-\frac{4i\varepsilon v_p\alpha_p}{w_p} - P\right)\partial^2 B(X, T) - \frac{\varepsilon v_p \partial^2 B(X, T)}{w_p} - \frac{2i\varepsilon v_p \partial B(X, T)}{\partial X \partial T} = 0.$$  (40)

The perturbation $\delta(X, T)$ being of very weak amplitude, we look for the solution of equation (40) under the sinusoidal wave with analytical expression given by (35). A combination of equation (35) and (40) permits one to obtain the dispersion relation shown as

$$w = \frac{k(12\varepsilon v_p\alpha_p^2 + 12k\varepsilon v_p\alpha_p + 3\varepsilon v_p + 6\alpha_p w_p + 4k^2\varepsilon v_p + 3kPw_p)}{3(\varepsilon v_k + w_p + \varepsilon v_p \alpha_p)}.$$  (41)

The Dark soliton is therefore modulationaly stable if and only if

$$k = \frac{\varepsilon v_p \alpha_p}{\varepsilon v_k}$$  (42)

and

$$\frac{k(12\varepsilon v_p\alpha_p^2 + 12k\varepsilon v_p\alpha_p + 3\varepsilon v_p + 6\alpha_p w_p + 4k^2\varepsilon v_p + 3kPw_p)}{3(\varepsilon v_k + w_p + \varepsilon v_p \alpha_p)} > 0.$$  (43)

6. Conclusion

The aim of this study was to construct Breather soliton solutions of a modeled equation in a discrete nonlinear electrical line and the survey of modulational instability, we have used a Discrete nonlinear capacitive electrical line and Kirchhoff laws to model a discrete nonlinear differential equation which describes the movement of solitary waves of type Breather having an envelope whose dynamics is governed by the higher-order nonlinear Schrodinger equation. The application of Bogning-Djeumen Tchafo-Kofane method has permitted the construction of analytical expression of some exact solutions of the modeled discrete nonlinear differential equation which are Breather soliton of type Bright, Dark, combined Bright and Dark. A numerical simulation has permitted to represent, to observe real profiles and intensity profiles of those found solitary waves. To have addition information on the dynamics of those Breather solitons, the study of one of the factors that perturbate those signals in the course of their propagation as a modulational instability has permitted one to discover that...
those solitary waves notably Dark solitons and Bright solitons are generally modulationaly stable. It is necessary to recall that the application of results of this study by industries for the manufacturing of electrical line that accepts to propagate those solitons will have purely economic advantages for the fact that these electrical lines are relatively cheaper and easy to manufacture than other transmission lines. In addition, the stable and non-dissipative nature of those solitons will reduce the usage of amplification station in the new electrical line. It is important to mention that the obtained Breather soliton which is a combined Bright and Dark solitons is very interesting as it has a particularity of having an envelope in an equilibrium state permitting it to maintain it constant value which it had in linear domain. This combined Bright and Dark solitons has a particularity of protecting sinusoidal wave that propagate as such without deformation and without the loss of energy.

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