Jacobi stability and updating parameters of dynamical systems using hybrid algorithms

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Abstract. Continuously time evolving dynamical systems are widely used as effective theoretical tools for the mathematical modelling of various processes. The global stability of dynamical systems is described by the theory of Lyapunov stability. An alternative approach is represented by the local (Jacobi) stability analysis. Consideration is being given to Jacobi stability of dynamical systems in the framework of the Kosambi–Cartan–Chern theory. In this way it is common practice to describe the time evolution of a dynamical system in geometric terms, by presenting it as a geodesic in the Finsler space. Geometric invariants of the system under investigation are obtained. Eigenvalues of the second invariant determine the Jacobi stability of the system. The inverse problem on updating parameters of the system on base of given eigenvalues of the invariant is stated. Criterion functions are supposed being continuous, Lipschitzian, multiextremal, not necessary everywhere differentiable. Global solutions are searched for by use of novel hybrid algorithms that combine the up-to-date stochastic Quasi Reflective Multi-Particle Collision Algorithm for scanning a search space and deterministic methods for local descend. Two-parametric smoothing approximations of criterion functions are inserted for local searching in the first algorithm. The second algorithm implements the local search procedure using numerical approximations of space-filling curves to reduce the original multidimensional problem to an univariate one satisfying the Hölder condition. A numerical example of updating parameters and Jacobi stability analysis of the Lorenz system is presented.

1. Introduction
Continuously time evolving dynamical systems represent powerful theoretical tools for the mathematical modelling of the evolution of natural phenomena. They are widely used in practice, and they provide extremely useful descriptions of various processes. In most of the applications the usefulness of the dynamical systems is determined mainly by their predictive capabilities. It is well known that the predictive potential of the dynamical systems depends on the stability of their solutions. In realistic investigations some uncertainties in the measured initial conditions in physical, engineering or biological systems do always appear, so that they must be taken into account when formulating the mathematical equations of the processes under consideration. In practical applications, a dynamical system is required to provide some essential information, not only on the time evolution of the system, but also on the possible deviations of the trajectories of the dynamical system with respect to a given reference trajectory. When studying a dynamical system it is necessary to obtain both the local description of the stability, and the global description of the late-time deviations of the trajectories. The global stability of the solutions of the dynamical systems is described by the well established theory of the Lyapunov stability. In this approach to the problem of stability analysis the fundamental mathematical quantities of interest are the Lyapunov exponents. The Lyapunov exponents measure the exponential deviations of the trajectories of a dynamical system from a given reference trajectory [1]. An alternative approach to the study of the dynamical systems is represented by the Kosambi–Cartan–Chern
(KCC) theory [2, 3]. Principal formulations of the KCC theory are based on a Finsler geometry and a theory of a tangent bundle [4–6]. A fundamental idea is to consider that there is a one to one correspondence between a second-order dynamical system and geodesic equations in an associated Finsler space (a recent survey of the theory is given in [7]). The KCC theory is a differential geometric theory of the variational equations for the deviations of the whole trajectory to nearby ones. In this geometric description of dynamical systems one associates a nonlinear connection, and a Berwald type connection to the differential system, and five geometrical invariants are obtained. The second invariant (also called the curvature deviation tensor) gives the Jacobi stability of the given dynamical system [8, 9]. Geometric methods can be employed successfully to formulate a general geometric theory of dynamical systems, or, more particularly, of systems of second order ordinary differential equations. The KCC theory has been applied to the dynamical systems in such fields of research as biology, geophysics, and cosmology [10–12]. The Lorenz system is one more example of successful application of the approach [13, 14].

Further the following inverse problem for a dynamical system depending on free parameters is stated. In the framework of the KCC theory, it is suggested that the eigenvalues of the curvature deviation tensor are given approximately (from indirect measuring and computer processing). The objective is to reconstruct the free parameters of the system from given data and to draw the conclusion on the Jacobi stability of the system. The optimization approach has been applied to solve the regularized inverse eigenvalue problem. Generally, criterion functions of the problem are supposed being continuous, Lipschitzian, multiextremal, not necessary everywhere differentiable. Consequently, global nondifferentiable optimization methods should be implemented. Two novel hybrid optimization algorithms combining the up-to-date stochastic algorithm (for scanning the search space) and deterministic methods for the local descent are applied. Two-parametric smoothing approximations of the criterion function are inserted during the local search in the first algorithm. The second algorithm implements the local search procedure by use of numerical approximations of space-filling curves.

The present paper is organized as follows. The basics of the KCC theory to be used in the sequel are sketched in Sec. 2. Some results on application of the theory to dynamical systems are given [2, 3]. The Lorenz system is reformulated as a set of two second-order differential equations; then the curvature deviation tensor and its eigenvalues are obtained explicitly [14]. Sec. 3 contains the statement of the regularized inverse problem. Brief description of two novel hybrid global optimization algorithms is presented in Sec. 4. Numerical example of solving the inverse problem for the Lorenz system is given in Sec. 5. The paper will be concluded in Sec. 6.

2. Jacobi stability of dynamical systems

In the following it is assumed that the dynamical variables \( x^i, i=1,2,...,n \), are defined on a real, smooth \( n \)-dimensional manifold \( M \); the tangent bundle of \( M \) is denoted by \( TM \) [3]. Sometimes \( M \) is thought as \( \mathbb{R}^n \), \( M=\mathbb{R}^n \), so that \( TM=\mathbb{R}^{2n} \). A subset \( \Omega \) of the Euclidian \( (2n+1) \)-dimensional space \( \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^1 \) is considered. On \( \Omega \) it can be introduced a \( (2n+1) \)-dimensional coordinate system \( (x^i,y^i,t), i=1,2,...,n \), where \((y^i)=\(y^1,y^2,...,y^n\)) , and \( t \) stands for the usual time coordinate. The coordinates \( y^i \) are defined as

\[
y^i = \left\{ \frac{dx^1}{dt}, \frac{dx^2}{dt}, \ldots, \frac{dx^n}{dt} \right\}.
\]

In many cases of interest the equations of motion of a dynamical system can be derived from a Lagrangian function \( L: TM \rightarrow \mathbb{R} \) using the Euler–Lagrange equations

\[
d \frac{\partial L}{\partial y^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{y}^i} = F_i, \quad i=1,2,...,n,
\]

where in the case of mechanical systems \( F_i, i=1,2,...,n \), represents the external force. If the Lagrangian \( L \) is regular, one can show that the Euler–Lagrange equations Eq. (1) are equivalent to a system of second-order ordinary differential equations [3]
where each function \( G^i_j(\dot{x}^i, \dot{y}^i, t) \) is \( C^\infty \) in a neighborhood of some initial conditions \((x_0, y_0, t_0)\) in \( \Omega \); functions \( G^i_j(\dot{x}^i, \dot{y}^i, t), i \in \{1, 2, \ldots, n\} \), are defined locally on \( TM \).

In order to study the geometrical properties associated to the dynamical system defined by the previous equations, on the base manifold \( M \) one introduces first the nonlinear connection \( N^i_j \), with coefficients \( N^i_j \), defined as [3]

\[
N^i_j = \frac{\partial G^i_j}{\partial y^j}.
\]

By using the KCC-covariant derivative, it is possible to obtain the deviation equation in the covariant form as

\[
\frac{D^2 \xi^i}{dt^2} = P^i_j \xi^j,
\]

where \( \xi^i \) are the components of a contravariant vector field defined along the path \( x^i(t) \); then it has been denoted

\[
P^i_j = -2 \frac{\partial G^i_j}{\partial x^j} - 2G^i_l \frac{\partial}{\partial x^l} + y^l \frac{\partial N^i_j}{\partial x^l} + N^i_l N^l_j + \frac{\partial N^i_j}{\partial t};
\]

where it has been introduced the tensor \( G^i_j \), defined as [2]

\[
G^i_j = \frac{\partial N^i_j}{\partial y^j},
\]

and which is called the Berwald type connection. The tensor \( P^i_j \) is the second KCC-invariant, and it may be called alternatively as the deviation curvature tensor. The deviation equation represents the well known Jacobi equation. In either Riemann or Finsler geometry, when the system of dynamic equations describes the geodesic motion, then the current equation is the Jacobi field equation corresponding to the given geometry.

A scalar quantity which can be constructed from \( P^i_j \) is the trace \( P \) of the curvature deviation tensor, which can be obtained from the relation

\[
P = P^i_j = -2 \frac{\partial G^i_j}{\partial x^i} - 2G^i_l \frac{\partial}{\partial x^l} + y^l \frac{\partial N^i_j}{\partial x^l} + N^i_l N^l_j + \frac{\partial N^i_j}{\partial t}.
\]

Further, the application of the KCC theory to the set of differential equations connected with the fluid mechanics is considered. In particular, the Lorenz system of three nonlinear ordinary differential equations is given by [14]

\[
\begin{align*}
\frac{1}{\sigma} \frac{dX}{dt} &= -X + Y, \\
\frac{dY}{dt} &= -XZ + \rho X - Y, \\
\frac{dZ}{dt} &= XY - \beta Z,
\end{align*}
\]

where \( \sigma, \rho, \beta \) stand for some free parameters; \( X, Y, \) and \( Z \) are functions of time alone. From a mathematical point of view the set of equations (2)–(4) represents an approximation to a system of partial differential equations describing finite amplitude convection in a fluid layer heated from below. From a physical point of view the parameters \( \sigma, \rho, \) and \( \beta \) can be interpreted as the Prandtl number, the
normalized Rayleigh number, and the wave length number, respectively. Although the Lorenz set of equations is deterministic, its solution shows chaotic behavior for $\rho > \sigma(\sigma + \beta + 3)/(\sigma - \beta - 1)$ and $\sigma > \beta + 1$ [14].

From Eq. (2), it takes place

$$Y = X + \frac{1}{\sigma} \frac{dX}{dt}.$$  \hspace{1cm} (5)

By substituting Eq. (5) into Eq. (3), one obtains

$$\dot{X} + (1 + \sigma)X + \alpha X Z + \sigma(1 - \rho)X = 0$$  \hspace{1cm} (6)

By taking the derivative of Eq. (4) with respect to the time we find

$$\dot{Z} = XY + XY - \beta Z = X \left( \frac{1}{\sigma} \dot{X} + X \right) + X \left( \frac{1}{\sigma} \dot{X} + \frac{1}{\sigma} \dot{X} \right) - \beta \left[ X \left( \frac{1}{\sigma} \dot{X} \right) - \beta Z \right].$$  \hspace{1cm} (7)

By use of Eq. (6) and Eq. (7) it could be obtained

$$\dot{Z} + \left( 1 + \sigma + \frac{\beta}{\sigma} - 2 \right) XX - \frac{1}{\sigma} XX^2 + (1 - \rho + \beta) X^2 + X^2 Z = 0.$$  \hspace{1cm} (8)

After changing the notation so that

$$X = \frac{1}{\sigma} X_1, \quad \dot{X} = \frac{1}{\sigma} \dot{X}_1, \quad Z = X^2, \quad \dot{Z} = X_2, \quad Y = X^3,$$

the Lorenz system is equivalent with the following system of second-order equations

$$\frac{d^2 X_1}{dt^2} + (1 + \sigma)Y + \alpha X_1 X^2 + \sigma(1 - \rho)X_1 = 0,$$  \hspace{1cm} (8)

$$\frac{d^2 X^2}{dt^2} + \left( \frac{1 + \sigma}{\sigma} + \frac{\beta}{\sigma} - 2 \right) X_1 Y_1 - \frac{1}{\sigma} \left( Y_1 \right)^2 + (1 - \rho + \beta) \left( X_1 \right)^2 + \left( X_1 \right)^2 X^2 = 0.$$  \hspace{1cm} (9)

Once $X_1$ and $X^2$ are known, the variable $X^3$ can be obtained as

$$X^3 = X_1 + \frac{1}{\sigma} Y_1.$$  \hspace{1cm}

So, the Lorenz system can be represented by the set of two differential equation as follows

$$\frac{d^2 X_i}{dt^2} + 2G^i(X^i, Y^i) = 0, \quad i = 1, 2.$$  \hspace{1cm}

From Eq. (8) and Eq. (9) one can get

$$G^1(X^1, X^2, Y^1) = \frac{1}{2} [(1 + \sigma)Y_1 + \alpha X_1 X^2 + \sigma(1 - \rho)X_1],$$

$$G^2(X^1, X^2, Y^1) = \frac{1}{2} \left[ \left( \frac{1 + \sigma}{\sigma} + \frac{\beta}{\sigma} - 2 \right) X_1 Y_1 - \frac{1}{\sigma} \left( Y_1 \right)^2 + (1 - \rho + \beta) \left( X_1 \right)^2 + \left( X_1 \right)^2 X^2 - \beta^2 X_1 \right].$$  \hspace{1cm}

Now the components of the nonlinear connection and the Berwald type connection can be found; after that the components of the curvature deviation tensor of the Lorenz system can be defined as follows [14]

$$P^1_1 = -\sigma X^2 - \sigma(1 - \rho) + \frac{1}{4} (1 + \sigma)^2,$$

$$P^2_1 = -\sigma X_1,$$

$$P^1_2 = \left( 1 - \frac{\beta}{2\sigma} \right) Y_1 + \left[ \frac{1 - \sigma^2 - 7\beta \sigma + \beta + 4(\rho - 1)\sigma}{4\sigma} \right] X^1 - X^1 X_2,$$

$$P^2_2 = \left( X_1 \right)^2 + \beta^2.$$  \hspace{1cm}

In the context of the Jacoby stability analysis the following three equilibrium points of the Lorenz system are defined [14]: $S_3(0, 0, 0)$, if $\rho \leq 1$, $S_+ [\beta(\rho - 1), \beta(\rho - 1), \rho - 1]$ and $S_- [-\beta(\rho - 1), -\beta(\rho - 1), \rho - 1]$, $\rho > 1$, respectively. Further the stability of the point $S_+$, $\rho > 1$, is un-
der consideration. Consequently, the components of the curvature deviation tensor can be represented in the following form

\[
P_1^1(S_+)=\frac{1}{4}(1+\sigma)^2, \\
P_1^2(S_+)=-\sigma\sqrt{\beta(\rho-1)}, \\
P_1^2(S_+) = \frac{\sqrt{\beta(\rho-1)[(7\sigma+1)\beta-\sigma^2+1]}}{4\sigma}, \\
P_2^2(S_+) = \beta^2 - \beta(\rho-1).
\]

Analytic expressions for the eigenvalues of the curvature deviation tensor in the considered equilibrium point are as follows [14]

\[
\lambda_+(S_+) = \frac{1}{2}\left[\sqrt{\beta(\beta-\rho+1)+\frac{1}{4}(\sigma+1)^2 + \beta(\rho-1)[\beta(7\sigma-1)+\sigma^2-1]} + \right.
\]

\[
+ \beta(\beta-\rho+1+\frac{1}{4}(\sigma+1)^2),
\]

\[
\lambda_-(S_+) = \frac{1}{2}\left[\frac{1}{2}\sqrt{\beta(\beta-\rho+1)+\frac{1}{4}(\sigma+1)^2 + \beta(\rho-1)[\beta(7\sigma-1)+\sigma^2-1]} + \right.
\]

\[
+ \beta(\beta-\rho+1+\frac{1}{4}(\sigma+1)^2).
\]

Convenient results for two other equilibrium points can be obtained in an analogous manner. Now one can formulate the following theorem, giving the Jacobi properties of the equilibrium points of the Lorenz system.

**Theorem [14].**

(a) The equilibrium point \(S_0(0,0,0)\) of the Lorenz system is Jacobi unstable.

(b) If the free parameters \(\beta, \rho > 1\) and \(\sigma\) of the Lorenz system satisfy simultaneously the conditions

\[
\beta(\beta-\rho+1)+\frac{1}{4}(\sigma+1)^2 < 0
\]

and

\[
\frac{1}{4}\beta[7\rho(\rho+1)+\sigma\rho(\sigma+9)] - 2\rho(\rho-1)(\sigma+1)] > 0,
\]

respectively, then the equilibrium points \(S_+\left[\sqrt{\beta(\rho-1)}, \rho - 1\right]\) and \(S_+\left[\sqrt{\beta(\rho-1)}, \rho - 1\right]\) of the Lorenz system are Jacobi stable, and Jacobi unstable otherwise.

3. Inverse problems

From now, a new system of notation will be used. It is considered an inverse problem governed to the operator equation in the framework of the structured mathematical model

\[Ax = y, \quad x \in X, \quad y \in Y,\]

where \(X, Y\) are Hilbert spaces; \(A\) stands for the compact linear operator, acting from \(X\) into \(Y\). The right part of the confused operator equation represents given data \(y^\delta\), that are known approximately from the experiment. It is supposed that the input data error \(\delta\) is established and take place the inequality \(\|y^\delta - y\| \leq \delta\). Stable approximate solutions are needed when the data \(y^\delta\) are given approximately. It should be noted that in many applications inverse problems are ill-posed. Further the approach on base of the Tikhonov regularization method is implemented [15, 16].

The approximate solution of the considered inverse problem is connected with the search of the Tikhonov’s functional minimum

\[x^\delta_\alpha = \arg \min_{x \in X} J_\alpha(x), \quad \alpha > 0;\]
where $\lambda_{t\alpha}^{\ast}$ is the regularized solution of the equation $Ax = y^\delta$ with the regularization parameter $\alpha$; moreover, the functional to be minimized is defined as follows

$$J_{\alpha}(x) = \|Ax - y^\delta\|_Y^2 + \alpha \|x\|_X^2,$$

where $\|Ax - y^\delta\|_Y^2$ is the error functional (the squared norm in the space $Y$); $\alpha \|x\|_X^2$ is the stabilizing functional.

The method to be implemented is called as the converged regularization method if the following condition takes place

$$\lim_{\alpha \to 0} R_{\alpha}Ax = x \quad \forall x \in X.$$

are introduced. When using data $x \in X_1 \subset X$, given a priori the solution error in worth case is determined in the form [15]

$$\Delta(\delta, X_1, R_{\alpha}) = \max \{ \lambda_{i}(x) - \lambda_{i}^\alpha(x) \} \leq \delta.$$

The inverse problem for eigenvalues of the deviation curvature tensor of the dynamical system is stated as follows. For eigenvalues compared pairwise it is possible to construct the following numerical set of the error criteria

$$f_{i}(x) = [\lambda_{i}(x) - \lambda_{i}^\ast(x)] \quad x \in X \subset \mathbb{R}^n, i \in J,$$

where $\lambda_{i}(x), \lambda_{i}^\ast(x)$ are eigenvalues from the current and the given spectra, respectively; $x$ is the vector of the control variables; $X$ stands for the feasible area; $n$ is the problem dimension; $J = \{1, \ldots, n\}$; $\mathbb{R}^n$ is the $n$-dimensional real linear space. The inverse problem is formulated as follows: to find the vector of the control variables $x \in X$ such that minimizes the maximal value of the error criterion

$$\min_{x \in X \subset \mathbb{R}^n} \max_{i \in J} \{ f_{i}(x) \}.$$

The solution of the problem (P) is represented by the admissible vector $x^\ast = [x_1^\ast, \ldots, x_n^\ast]^T$, such that the scalar error function $f(x) = \max \{ f_{1}(x), \ldots, f_{N}(x) \}$ takes the minimal value. Further the regularized problem $(P)_{\alpha}$ with the multieextremal not everywhere differentiable criterion function $f(x)$ and the regularization parameter $\alpha > 0$ is considered.

In generalization of the statements of extremal problems the following global optimization problem is formulated: to find

$$f(x^\ast) = \min_{x \in X \subset \mathbb{R}^n} f(x), \quad (10)$$

where

$$X = \{ x \in D : g_{i}(x) \leq 0, i \in I \}, \quad (11)$$

$$D = \{ x \in \mathbb{R}^n : a_j \leq x_j \leq b_j, j \in J \}, \quad (12)$$

$f(x)$ is the criterion function; $g_{i}(x)$ stands for the constraint function, $i \in I$; $I = \{1, \ldots, m \}$ is the bounded set of indexes; $m$ is the number of constraints; $D$ is the search area; $x^\ast$ is the global solution. The functions $f(x)$, $g_{i}(x)$, $i \in I$, of the problem (10)–(12) are supposed to be continuous Lipschitzian
functions. Also it is supposed that the real-valued function \( f : \mathbb{R}^n \to \mathbb{R} \) is multiextremal, not everywhere differentiable, and there is a numerical procedure for calculating values of the function in points of the supposed area.

4. Hybrid algorithms
A review of popular global optimization methods is given in [17]. The currently available stochastic Multi-Particle Collision Algorithm (M-PCA) [18] has some essential advantages in relation to well known global optimization algorithms such as the Genetic Algorithms, Simulated Annealing, Fast Simulated Annealing, etc. Specifically, the M-PCA does not require any additional parameters other than the number of iterations; the algorithm is extremely easy to implement and can be applied to both continuous and discrete optimization problems. The M-PCA performs using the analogy with nuclear particle collision reactions, in particular scattering and absorption. So, a particle that hits a high-fitness “nucleus” would be “absorbed” and would explore the boundaries. Otherwise, a particle that hits a low-fitness region would be scattered to another region. This reasoning makes it possible to simulate the exploration of the search space and the exploitation of the most promising areas of the fitness landscape throw successive scattering and absorption collision events. The original M-PCA works as follows. First an initial configuration is chosen, then a modification of the old configuration into a new one is implemented. The qualities of the two configurations are compared. A decision then is made on whether the new configuration is acceptable. If it is, the current configuration acts as the old configuration for the next step. If it is not acceptable, the algorithm proceeds with a new change of the old configuration. It is pertinent to note that acceptance of current trial solution with certain probability may avoid the convergence to local optima.

However, the M-PCA is in its early stages. In spite of its advantages over Genetic Algorithms and Simulated Annealing in solving benchmark problems, practical application of the M-PCA is restricted because of solutions remain too expensive. As possible development, the local search procedure in the algorithm could be improved. It seems promising to use gradient methods for local minimization of the error function. But in so doing the problem of non-differentiability of the function should be taken into account. A hybridization of the stochastic M-PCA and deterministic methods was introduced in [19]. During processing two procedures are performing: a wide search in the solution space using the M-PCA and a local search in the promising areas with the deterministic techniques. The local search procedure is performed iteratively until a certain number of criterion function evaluations being reached.

New versions for the M-PCA meta-heuristics are presented in [20]. In order to provide more effective trial solutions, the concept of opposition and reflection is introduced to improve the capacity of the search space for the M-PCA. One of the new versions of the M-PCA is the Quasi Reflective M-PCA (QRM-PCA). This new version of the global optimization algorithm makes it possible to realize a more intensive exploration of the search space. Two novel hybrid global optimization algorithms integrating the stochastic QRM-PCA and deterministic methods for local minimization are presented. The first hybrid algorithm (named here QRM-PCALMSI) combines the QRM-PCA and deterministic gradient techniques [21]. In this new version of the hybrid algorithm the local search procedure is a standard deterministic Linearization Method. Two-parametric smoothing approximations with iteratively refinement (FunctionLSI()) are introduced during the local search that makes it possible to expand the approach on the class of non-differentiable problems. Solutions of numerous benchmarks show the computational efficiency of the hybrid algorithm. The pseudo code of the hybrid algorithm QRM-PCALMSI is presented in compacto as follows.

```plaintext
for j ← 1, n_{particles} do
    Generate an initial solution OldConfig_j
    Determine quasi-reflective solution QOldConfig_j
    if Fitness( QOldConfig_j ) < Fitness( OldConfig_j ) then
        OldConfig_j ← QOldConfig_j
    end if
```

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end for
UpdateBlackboard()
while Stopping criteria not yet met do
   for $j \leftarrow 1, n_{\text{particles}}$ do
      Perturbation()
      if Fitness($\text{NewConfig}_j$) < Fitness($\text{OldConfig}_j$) then
         if Fitness($\text{NewConfig}_j$) < BestFitness then
            BestFitness = Fitness($\text{NewConfig}_j$)
            BestConfig $\leftarrow$ NewConfig
         end if
         OldConfig $\leftarrow$ NewConfig
      end if
      FunctionLMSI()
   else
      Scattering()
   end if
   if It is time to get the reflected population (each $n_{\text{reflection}}$ iterations) then
      Determine reflective solution $Q\text{OldConfig}$
      if Fitness($Q\text{OldConfig}$) < Fitness($\text{OldConfig}$) then
         OldConfig $\leftarrow$ QOldConfig
      end if
   end if
   for
      if It is time to update the blackboard (each $n_{\text{blackboard}}$ iterations) then
         UpdateBlackboard()
      end if
   end for
end while
BestConfig $\leftarrow$ UpdateBlackboard()
return BestConfig

In distinction to the stochastic QRM-PCA, the hybrid QRM-PCALMSI explores the search space and when a better-than-previous solution is found, it is used as an initial point for the LMSI-method. Thus, function Exploration() in the QRM-PCA is replaced by the FunctionLMSI() in the first hybrid algorithm. The second hybrid algorithm (named here QRM-PCASFC) combines the QRM-PCA and deterministic space-filling curve method [22]. Some powerful algorithms for multi-extremal non-convex optimization problems are based on reducing the initial multi-dimensional problem to the equivalent Hölder problem of one dimension. This reduction can be executed by applying Peano-type space-filling curves mapping a unit interval on the real axis onto a multi-dimensional hypercube. The Hilbert technique is used for building the development of the Peano space-filling curve depending on the free parameter that stands for the number of subdivision levels. The approach needs not any derivatives of the function to be minimized with updating parameters. Some disadvantage of this approach is in the fact that one-dimensional problem obtained by the above reduction leaks information on the closeness of iteration points in the initial multi-dimensional space. Solutions of some benchmarks show the computational efficiency of the hybrid algorithms.

5. Numerical example
It is supposed that from the analysis of experimental data for the Lorenz system the following approximate eigenvalues of the deviation curvature tensor are established: $\lambda^{*}_+ \approx 68.5; \lambda^{*}_- \approx -103.0$. These indirect data have been obtained by the modelling the Lorenz system under the standard free values. The relative error of the given data is less than 0.1%. The problem is to reconstruct values of the free
parameters of the Lorenz system from given $\lambda_+^*$ and $\lambda_-^*$. The control variables are represented by the relative variables $x_1, x_2, x_3$ (determined in%), which correspond to the required parameters $\sigma, \rho, \beta$. The criterion function of the considered problem is defined as follows

$$F(x) = \sum_{i=1}^{2} \gamma_i f_i^2(x) + a \| x \|^2_2,$$

where $\gamma_i, f_i(x)$ are the weighting coefficient and the particular criterion that correspond to the $i$ th eigenvalue $\lambda_i$: $\lambda_1(x) = \lambda_+(x)$, $\lambda_2(x) = \lambda_-(x)$; $f_i(x) = (\lambda_i^* - \lambda_i(x))$, $i = 1, 2$; $\alpha$ is the regularization parameter; $x \in \mathbb{R}^3$.

The hybrid global optimization algorithm QRM-PCALMSI is used. The iteration history of $x_i$, $i = 1, 3$, when changing the number of iterations is depicted in Figure 1. The corresponding history for the criterion function $F(x)$ and the norm of the search direction $Nr\{w\}$ is demonstrated in Figure 2.

After 22 iterations the following solution is obtained: $F(x^{22}) \approx 0.5879$; $x_1^{22} \approx 70.27\%$, $x_2^{22} \approx 71.14\%$, $x_3^{22} \approx 66.47\%$. The reconstructed approximate free parameters of the Lorenz system are as follows: $\sigma \approx 9.9698$; $\rho \approx 27.805$; $\beta \approx 2.6607$. The maximal relative error of the solution is less than $0.7\%$. It should be mentioned that there is a good correspondence between the accuracy of the approximate solution of the inverse problem and the given input data.

![Figure 1](image-url)

**Figure 1.** Number of final iterations $N_{iter}$ of the QRM-PCALMSI vs. control variables $x_i$, $i = 1, 5$
So, the free parameters of the Lorenz system have been obtained, and they are close to the standard ones: \( \sigma^* = 10, \rho^* = 28, \beta^* = 2 \) (6). It should be noted that in this case all the equilibrium points of the considered system are Jacobi unstable [14].

**6. Conclusion**

Some questions on the Jacobi stability of the dynamical system based on determining the second geometric invariant in the framework of the Kosambi–Cartan–Chern theory are considered. The technique on reconstructing free parameters of the systems from eigenvalues of the curvature deviation tensor is introduced. The optimization approach is used for solving the inverse problem. Two novel hybrid global optimization algorithms QRM-PCALMSI and QRM-PCASFC being realized in programs are presented. The model updating of the free parameters of the Lorenz system is implemented. The accuracy of the approximate solution is in a good correspondence with the accuracy of the given indirect experimental input data.

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