FOREWORD

The aim of this paper is to offer an overview of the most important applications of Jordan structures inside mathematics and also to physics, updated references being included.

For a more detailed treatment of this topic see - especially - the recent book Iordănescu [364w], where suggestions for further developments are given through many open problems, comments and remarks pointed out throughout the text.

Nowadays, mathematics becomes more and more nonassociative (see §1 below), and my prediction is that in few years nonassociativity will govern mathematics and applied sciences.

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§1. JORDAN STRUCTURES

In modern mathematics, an important notion is that of nonassociative structure. This kind of structures is characterized by the fact the product of elements verifies a more general law than the associativity law.

There are two important classes of nonassociative structures: Lie structures (introduced in 1870 by the Norwegian mathematician Sophus Lie in his study of the groups of transformations) and Jordan structures (introduced in 1932-1933 by the German physicist Pasqual Jordan (1902-1980) in his algebraic formulation of quantum mechanics [379a,b,c]). These two kinds of structures are interconnected, as it was remarked - for instance - by Kevin McCrimmon [480c, p. 622]: "We are saying that if you open up a Lie algebra and look inside, 9 times out of 10 there is a Jordan algebra (of pair) which makes it work"..., as well as by Efim Zelmanov [729a]: "...”Lie algebra with finite grading may be by right included into the Jordan theory”...

There exist three kinds of Jordan structures, namely, algebras, triple systems, and pairs (see the definitions below). Since the creation of Jordan algebras (at the beginning of ’30) by Pasqual Jordan, an improvement of the
mathematical foundation of quantum mechanics was made, but the problem was not definitively solved. Fifty years later (in ’80), another tentative was made using a kind of mixed structures, namely Jordan-Banach algebras (see the definition in §3), but the problem of the mathematical foundation of quantum mechanics is still an open problem!

Anyway, is the meantime, Jordan structures have been intensively studied by mathematicians, and a big number of important results have been obtained. At the same time, an impressive variety of applications have been explored with several surprising connections. An explanation of this fact could be the following: at the beginning, mathematics was associative and commutative, then (after the invention of matrices) it became associative and noncommutative, and now (after the invention of nonassociative structures) it becomes nonassociative and noncommutative. The study of Jordan structures and their applications is at present a wide-ranging field of mathematical research. 

I shall give here only algebraic basic definitions. For an intrinsic treatment of Jordan structures the reader is referred to the excellent books by Braun and Koecher [131] (see also Koecher [408c]), Jacobson [371a, c], Loos [448e, g], Meyberg [483d], Zhevlakov, Slinko, Shestakov, Shirshov [733]. See also Jordan algebras – Proceedings of the Oberwolfach Conference (August, 1992), Kaup, McCrimmon, Petersson (eds.), Walter de Gruyter–Berlin–New York, 1994, and Proceedings of the International Conference on Jordan Structures (Malaga, June, 1997), Castellon Serrano, Cuenca Mira, Fernández López, Martín Gonzalez (eds.), Malaga, 1999, as well as the WEB site: Jordan Theory preprints (http://homepage.uibk.ac.at/~c70202/jordan/index.html).

Jordan algebras emerged in the early thirties with Jordan’s papers [379a, b, c] on the algebraic formulation of quantum mechanics. The name “Jordan algebras” was given by A.A. Albert in 1946.

Definition 1. Let J be a vector space over a field F with characteristic different from two. Let \( \varphi : J \times J \rightarrow J \) be an F-bilinear map, denoted by \( \varphi : (x, y) \rightarrow xy \), satisfying the following conditions:

\[
xy = yx \quad \text{and} \quad x^2(yx) = (x^2y)x \quad \text{for all} \ x, y \in J.
\]

Then J together with the product defined by \( \varphi \) is called a linear Jordan algebra over F.

\(^{3}\)see IORDĂNESCU, R., Romanian contributions to the study of Jordan structures and their applications, Mitteilungen des Humboldt-Clubs Rumänien, No.8-9 (2004-2005), Bukarest, 29-35.
Comments. The second condition came from quantum mechanics: observables in physics (temperature, pressure, etc.) satisfy it. This condition is less restrictive than the associativity and, from this fact it follows the impressive variety of applications of Jordan structures.

Example. If $A$ is an associative algebra over $F$, and we define a new product $\varphi(x, y) := \frac{1}{2}(x \cdot y + y \cdot x)$, where the dot denotes the associative product of $A$, then we obtain a Jordan algebra. It is denoted by $A^{(+)}$.

Remark 1. Jordan was the first who studied the properties of the product $xy$ from the above example in the case when $F$ is the field of reals. He proved a number of properties of this product, and showed that these were all consequences of the two identities above.

Remark 2. Zelmanov and others obtained important and interesting results for infinite-dimensional Jordan algebras. A good account of Zelmanov work as well as of McCrimmon’s extension to quadratic Jordan algebras, can be found in [480d] (see also [364g]).

Notation. On a Jordan algebra $J$ consider the left multiplication $L$ given by

$$L(x) y := xy, \quad x, y \in J.$$ 

Remark 3. In general, $L(xy) \neq L(x)L(y)$ for $x, y$ from a Jordan algebra (which is not associative). This hold for, e.g., $J = A^{(+)}$ for $A$ (commutative or not).

Definition 2. The map $P$ defined by

$$P(x) := 2L^2(x) - L(x^2), \quad x \in J,$$

is called the quadratic representation of $J$. When $J = A^{(+)}$ it assumes the form $P(x)y = x \cdot y \cdot x$.

Proposition 1. For any $x, y \in J$ the following fundamental formula holds:

$$P(P(x)y) = P(x)P(y)P(x).$$

Remark 4. For $P(x, y)$ given by $P(x, y) := 2(L(x)L(y) + L(y)L(x) - L(xy))$ we have $P(x + y) = P(x) + P(x, y) + P(y)$, $x, y \in J$.

Remark 5. In general, $P(x, y) \neq P(x)P(y)$, $x, y \in J$ (as can easily be seen for $J = A^{(+)}$).

Proposition 2. Suppose that $J$ has a unit element $e$ and let $x$ be an element of $J$. Then $P(x)$ is an automorphism of $J$ if and only if $x^2 = e$. If $P(x)$ is an automorphism of $J$, then it is involutive.
Definition 3. An element $x \in J$ is called invertible if the map $P(x)$ is bijective. In this case the inverse of $x$ is given by $x^{-1} := (P(x))^{-1} x$. (In case $J = A^+$, this is the usual inverse in the associative algebra $A$.)

Remark 6. We have $(P(x))^{-1} = P(x^{-1})$, $x \in J$.

Remark 7. An element $x$ is invertible with the inverse $y$ if and only $xy = e$, $x^2 y = x$.

Definition 4. Let $f$ be an element of $J$. Define a new product on the vector space $J$ by
\[ \{xy\} := x(yf) + y(xf) - (xy)f. \]

The vector space $J$ together with this product is called the mutation (homotope) of $J$ with respect to $f$ and is denoted by $J_f$.

Proposition 3. Any mutation $J_f$ of $J$, $f \in J$ is a Jordan algebra and its quadratic representation $P_f$ is given by $P_f(x) = P(x)P(f)$.

Proposition 4. The algebra $J_f$, $f \in J$, has a unit element if and only if $f$ is invertible in $J$; in this case the unit element of $J_f$ is $f^{-1}$. In this situation we call $J_f$ the $f$-homotope of $J$.

Remark 8. If $f$ is invertible in $J$ then the set of invertible elements of $J$ coincides with the set of invertible elements of $J_f$.

Note. From this point on, many of the results require that $J$ be finite-dimensional.

Notation. Denote by
\[ \text{Invol}(J) := \{w | w \in J, w^2 = e\}, \quad \text{Idemp}(J) := \{c | c \in J, c^2 = c\} \]
the set of involutive, respectively idempotent (zero included), elements of $J$. Here $J$ is supposed to contain a unit element $e$. Note that $\text{Invol}(J) = \{w \in J, w^{-1} = w\}$.

Remark 9. The map $\text{Idemp}(J) \to \text{Invol}(J)$ given by $c \to 2c - e$ is a bijection.

For an element $c$ of $\text{Idemp}(J)$ we have
\[ L(c)(L(c) - \text{Id})(2L(c) - \text{Id}) = 0. \]
This leads to the Peirce decomposition of $J$ with respect to the idempotent $c$:
\[ J = J_0(c) \oplus J_{1/2}(c) \oplus J_1(c), \]
where \( J_\alpha(c) := \{ x \mid x \in J, cx = \alpha x \} \), for \( \alpha = 0, 1/2, 1 \).

**Theorem 5.** \( J_0(c) \) and \( J_1(c) \) are subalgebras of \( J \), and we have
\[
J_0(c)J_1(c) = \{ 0 \}, \quad J_\nu(c)J_{1/2}(c) \subset J_{1/2}(c), \quad \text{for } \nu = 0, 1,
\]
and
\[
J_{1/2}(c)J_{1/2}(c) \subset J_0(c) \oplus J_1(c).
\]

**Definition 5.** Let \( c \) be an idempotent of \( J \), \( c \neq e \). Then the map \( P(2c-e) \), which by virtue of Proposition 1, is an automorphism of \( J \), is called the Peirce reflection with respect to the idempotent \( c \) of \( J \).

**Notation.** \( \text{Idemp}_1(J) := \{ c \in \text{Idemp}(J), \dim J_1(c) = 1 \} \).

**Definition 6.** The dimension of \( J_1(c) \) is called the rank of the idempotent \( c \).

**Definition 7.** An idempotent \( c \) of \( J \) is called primitive if it cannot be decomposed as sum \( c_1 + c_2 \) of two orthogonal (i.e., \( c_1c_2 = 0 \)) idempotents \( c_1 \) and \( c_2 \), \( c_i \neq 0 \) \((i = 1, 2)\).

**Remark 10.** Every element of \( \text{Idemp}_1(J) \) is primitive. The converse is not true in general.

**Definition 8.** A Jordan algebra over the real numbers is called formally real if, for any two of its elements \( x \) and \( y \), \( x^2 + y^2 = 0 \) implies that \( x = y = 0 \).

**Proposition 6.** A primitive idempotent of a formally real finite-dimensional Jordan algebra is of rank one.

**Proposition 7.** The set \( \Gamma(J) \) of bijective linear maps \( W \) on \( J \) for which there exists a bijective map \( W^* \) on \( J \) such that \( P(Wx) = WP(x)W^* \) for all \( x \in J \) is a linear algebraic group.

**Note.** The notation \( W^* \) is justified by the fact that if \( J \) is real semisimple and \( \lambda \) denote the trace form on \( J \) (i.e., \( \lambda(x, y) := \text{Tr}(xy), x, y \in J \)) then for \( W \in \Gamma(J) \), \( W^* \) coincides with the adjoint of \( W \) with respect to \( \lambda \).

**Definition 9** (Koecher [408b]). The (linear algebraic) group \( \Gamma(J) \) from Proposition 7 is called the structure group of \( J \).

**Remark 11.** The fundamental formula (see Proposition 1) implies that \( P(x) \in \Gamma(J) \) where \( x \) is an invertible element of \( J \). Also, every automorphism of \( J \) belongs to \( \Gamma(J) \). Indeed, the automorphism group \( \text{Aut}(J) \) is just the set of elements \( W \in \Gamma(J) \) fixing the unit element \( e \) of \( J \), \( We = e \).
Formally real Jordan algebras have been classified (in the finite-dimensional case) by Jordan, von Neuman and Wigner [380]:

**Theorem 8.** Every formally real finite-dimensional Jordan algebra is a direct sum of the following algebras

\[ H_p(\mathbb{R})(^+) , \ H_p(\mathbb{C})(^+) , \ H_p(\mathbb{H})(^+) , \ H_3(\mathbb{O})(^+) , \ J(B) , \]

Here \( H_p(F)(^+) \) denotes the algebra of Hermitian \((p \times p)\)-matrices with entries in \( F \) (\( F = \mathbb{R}, \mathbb{C}, \mathbb{H} \) or \( \mathbb{O} \)), the multiplication in \( H_p(F)(^+) \) is given by \( xy := \frac{1}{2}(x \cdot y + y \cdot x) \) (\( x \cdot y \) denotes the usual matrix product), and \( J(B) = \mathbb{R}1 \oplus X \), where \( B \) is a real-valued symmetric bilinear positive definite form on the real vector space \( X \), equipped with the product

\[ (\lambda, u)(\mu, v) := (\lambda \mu + B(u, v), \lambda v + \mu u) . \]

**Definition 10.** A Jordan algebra \( J \) is called special if it is isomorphic to a (Jordan) subalgebra of some \( A(^+) \) for \( A \) associative.

**Remark 12.** The first three and the fourth algebras in Theorem 8 are special, while \( H_3(\mathbb{O})(^+) \) is not, that is why it is called exceptional.

**Proposition 9.** A Jordan algebra is formally real if and only if its trace form is positive definite.

**Notation.** Suppose that \( J \) has a unit element \( e \). Then we set \( x^0 := e \), \( \exp x := \sum_{n \geq 0} \frac{x^n}{n!} \), and \( \exp J := \{ \exp x \mid x \in J \} \).

**Proposition 10.** If \( J \) is a formally real Jordan algebra, then it possesses a unit element and \( \exp J = \{ x^2 \mid x \text{ invertible in } J \} \).

**Theorem 11.** Suppose that \( J \) is a formally real Jordan algebra endowed with the natural topology of \( \mathbb{R}^n \). Then the identity component of the set of invertible elements of \( J \) coincides with \( \exp J \) and the map \( x \mapsto \exp x \) is bijective.

If in a given Jordan algebra we define a triple product by

\[ \{xyz\} := (xy)z + (zy)x - y(xz) , \]

then it satisfies the following two identities:

\[ (\text{JTS 1)} \) \ \{xyz\} = \{zyx\} ; \]

\( ^4 \)Throughout this paper \( \mathbb{R}, \mathbb{C}, \mathbb{H}, \) and \( \mathbb{O} \) denotes the set of reals, complex numbers, quaternions, and octonions (Cayley numbers), respectively.
\( \{ uv \{ xyz \} \} = \{ uvx \} yz - \{ x \{ vuy \} z \} + \{ xy \{ uvz \} \} \).

In general, a module with a trilinear composition \( \{ xyz \} \) satisfying (JTS 1) and (JTS 2) is called a \textit{Jordan triple system}. Meyberg [483a, b] has used Jordan triple systems in the study of the extension of Koecher’s construction [408c] of a Lie algebra from a given Jordan algebra.

**Theorem 12** (Meyberg [483a, b]). If \( T \) is a Jordan triple system and \( a \) an element of \( T \), then \( T \) together with the product \( (x, y) \rightarrow \frac{1}{2} \{ xay \} \) becomes a Jordan algebra, denoted \( T_a \). Conversely, a Jordan algebra induces a Jordan triple system in the same vector space by setting \( \{ xyz \} := P(x, z)y \).

There exist generalizations of Jordan triple systems, studied - especially - by Kamiya and his collaborators (see [383a,b], [385c,d], [233a, b]). I take this opportunity to point out here the existence of a new class of nonassociative algebras with involution (including the class of structurable algebras) recently defined by Kamiya and Mondoc [384].

In 1969, Meyberg introduced [483a] the "verbundene Paare" (connected pairs), which correspond in Loos terminology to the linear Jordan pairs; see the definition below. Such connected pairs first arose in Koecher’s work on Lie algebras.

**Definition 11** (Loos [448g]). Let \( K \) be a unital commutative ring such that 2 is invertible in \( K \). Assume all \( K \)-modules to be unital and to possess no 3-torsion (i.e. no nonzero elements \( x \) such that \( 3x = 0 \)). A pair \( V = (V^+, V^-) \) of \( K \)-modules endowed with two trilinear maps \( V^\sigma \times V^{-\sigma} \times V^\sigma \rightarrow V^\sigma \), written as \( (x, y, z) \rightarrow \{ xyz \}_\sigma, \sigma = \pm \), satisfying the identities

\[
\{ xyz \}_\sigma = \{ zyx \}_\sigma, \\
\{ xy\{ uvz \}_\sigma \}_\sigma - \{ uv\{ xyz \}_\sigma \}_\sigma = \{ \{ xyu \}_\sigma vz \}_\sigma - \{ u\{ yxv \}_{-\sigma} z \}_\sigma
\]

for \( \sigma = \pm \), is called a \textit{(linear) Jordan pair} over \( K \).

**Remark 13.** Jordan algebras can be regarded as a generalization of \textit{symmetric} matrices, while the (linear) Jordan triple structures (systems or pairs) can be regarded as a generalization of \textit{rectangular} matrices.

From the recent papers focussed on Lie algebras, but from a Jordan point of view, I would like to mention these written by Fernandez Lopez and his collaborators (see [257 a,b], [258], [80]), while for these focussed on Lie superalgebras, see the paper by Cunha & Elduque [187].

**Remark 14.** Jordan triple systems are equivalent with Jordan pairs with involution.
McCrimmon [480a] extended the theory of Jordan algebras to the case of an arbitrary commutative unital underlying ring. So, instead of considering a Jordan algebra as a vector space with a nonassociative bilinear composition satisfying certain identities, McCrimmon considered it as a module over a ring together with a quadratic representation satisfying a number of conditions. In this way, one gets unital quadratic Jordan algebras, which for the case that the underlying ring is a field of characteristic different from two (or any underlying ring containing 1/2) turn out to be the well-known Jordan algebras considered from a different point of view.

By analogy with McCrimmon’s concept of quadratic Jordan algebras, Meyberg [483d] defined quadratic Jordan triple systems, while the notion of quadratic Jordan pairs was introduced by Loos [448f].

The Tits-Kantor-Koecher construction and its generalizations to Jordan pairs, structurable algebras, Kantor pairs, etc., are the basis of the recent application of Jordan structure in the theory of Lie algebras, mainly the so-called root-graded Lie algebras. For details on this topic, we refer the reader to the recent excellent survey paper [512j] by Neher. Let us mention here from it the recent (or very recent) papers by Allison, Azam, Berman, Gao, and Pianzola [16], Allison, Benkart, and Gao [17a, b], Allison and Faulkner [18b], Allison and Gao [19], Benkart [79], Benkart and Smirnov [83], Benkart and Zelmanov [84], Berman, Gao, Krylyuk, and Neher [94], Neher [512f], and Yoshii [722d]. An important paper in this context is also the paper [722c] by Yoshii. In this paper the author determines the coordinate algebra of extended affine Lie algebras of type \( A_1 \). It turns out that such an algebra is a unital \( \mathbb{Z}^n \)-graded Jordan algebra of a certain type, called a Jordan torus. The author also gives the classification of Jordan tori.

For Jordan structures in the super setting (i.e., Jordan superpairs, quadratic Jordan superpairs, etc.), we refer the reader to García & Neher [282a], and Neher [512i], while for the applications of Jordan techniques in the theory of Lie algebras, we refer the reader to the recent paper by Fernández López, García, and Gómez Lozano [257b], as well as to the papers referred inside [257b].

Very recently, Velázquez and Felipe [688a, b] introduced and studied a new algebraic structure, called quasi-Jordan algebra.

The mixed Jordan and Lie structures become more and more important nowadays. Jordan-Lie algebras were defined in 1984 by Emch in his book [236b], although the concept seems to have appeared first in the paper [306] by Grgin & Petersen. Lie-Jordan algebras were defined in 2001 by Grishkov and Shestakov in [308].
**Note.** Every Jordan-Lie algebra gives rise to a Lie-Jordan algebra, but the converse is false.

In September 2009, Makhlouf defined *Hom-Jordan algebras*, and in January 2010 he gave a survey on nonassociative Hom-algebras and Hom-superalgebras, see Makhlouf [463a,b], where his papers from 2007 and 2008 in cooperation with Silvestrov are also referred, see Makhlouf & Silvestrov [464a,b,c,d]. I take this opportunity to mention here also *Hom-Nambu-Lie algebras*, induced by Hom-Lie algebras, see Arnlind, Makhlouf and Silvestrov [44], as well as Arnlind [43].

A lot of contributions are now devoted to *Moufang loops* (see Moufang [500b]), which extend groups, and the analog of Lie algebra for local Moufang loops are *Malcev algebras* (see Malcev [465]). There exists today a rich bibliography on Moufang loops, from which I would refer the reader to Paal [528], who wrote many papers on Moufang symmetry. Very recent related contributions are the papers by Benkart, Madariaga and Perez-Izquierdo [81], and Madariaga & Perez-Izquierdo [457].

For a big paper on algebras, hyper-algebras, nonassociative bialgebras and loops, see Perez-Izquierdo [537b].

I would like to mention here also the papers [254a, b, c] by Faybusovich, as well as a very recent paper by Elduque [232].

It is worth to be remarked here that the study of nonassociative structures is a tradition in the Romanian university center of Iași. I would like to give here only two examples: a paper by Climescu [180] mentioned in the famous book [131] by Braun & Koecher, and - recently - Burdujan [149a,b] studied the deviation from associativity. As Burdujan remarked, the construction from his paper [149a] could be applied also to Jordan algebras, and - also - some his LT-algebras are Jordan algebras (see [149c]).

Let us mention here some recent Ph.D. Theses in the field of Jordan structures and their applications defended at universities from USA: *Orbits of exceptional groups and Jordan systems* by Sergei Krutelevich (Yale University), *The ring of fractions of a quadratic Jordan algebra* by James Bowling (Virginia University), *Affine remoteness planes* by Karen Klintworth (Virginia University), *Centroids of quadratic Jordan superalgebras* by Pamela Richardson (Virginia University), *Laguerre functions associated to Euclidean Jordan algebras* by Michael Arístidou (Louisiana State University), *Derivations of 8 simple Jordan superalgebras* by Michael Smith (Virginia University).

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5 Private communication in July 2010.
From the recent or very recent papers of interest here, I would like to mention [49] by Ashihara & Miyamoto, [136] by Brenner & Peresi, [494] by Montaner, [453] by MacDonald, [706] by Wilson, and [33] by Anquela, Cortéz, and McCrimmon.

I was very recently informed by Prof. Dr. Harald UPMEIER (Philips-Universität Marburg, Germany) through e-mail (May 20, 2011) on his progress made in a new field, concerning the conformal compactification of Jordan algebras and applications to harmonic analysis. It seems that the conformal geometry of Jordan algebras can explain deep facts about the representation theory of semisimple Lie groups and also about the geometry of non-convex cones and the higher-dimensional Radon transform.

§2. ALGEBRAIC VARIETIES (OR MANIFOLDS) DEFINED BY JORDAN PAIRS

Loos [448i] showed that every (quadratic) Jordan pair defines an affine algebraic group, the projective group of the Jordan pair. On the other hand, he remarked that every Jordan pair defines an algebraic variety, related with the projective group of the Jordan pair in a natural manner.

Let \( V = (V^+, V^-) \) be a finite-dimensional Jordan pair over an algebraically closed field \( F \) and let \( X = X(V) \) be the quotient of \( V^+ \times V^- \) by the equivalence relation \( (x, y) \sim (x', y') \) if and only if \( (x, y - y') \) is quasi-invertible and \( x' = x y^{-y'} \). (As usual, \( x y \) denotes the quasi-inverse in the Jordan pair \( V \)). Loss [448i] proved that \( X \) is a quasi-projective variety containing \( V^+ \) as an open dense subset, and that \( X \) is projective if \( V \) is semisimple. Moreover, Loos showed that under the projective group of \( V \) the space \( X \) is homogeneous in a natural way, and that this projective group is isomorphic to the group of automorphisms of \( X \) if \( V \) is semisimple. This is essentially due to Chow [165] in four classical cases. For a geometric characterization of the projective group see Faulkner [248a], Freudenthal [272a], Springer [632a], as well as Chow [165],

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6The topics presented in this paragraph were only mentioned in the book Iordănescu [364w].

7LOOS, O., Homogeneous algebraic varieties defined by Jordan pairs, Monatsh. Math. 86 (1978/79), 2, 107-129.
For each of the Jordan pairs from the classification of simple finite-dimensional Jordan pairs over an algebraically closed field $\mathbf{F}$ given by Loos [448g, p. 201] the variety $X$ is as follows:\footnote{FAULKNER, J.R., A geometry for $E_7$, Trans. Amer. Math. Soc. 167 (1972), 49-58.}

Type I. $V^+ = V^- = M_{p,q}(\mathbf{F})$, $(p \times q)$-matrices over $\mathbf{F}(p \leq q)$, with $Q(x)y := xy'x$, where $y'$ denotes the transpose of $y$. In this case $X$ is isomorphic to the Grassmannian $G_p(\mathbf{F}^{p+q})$ of $p$-dimensional subspaces of $\mathbf{F}^{p+q}$.

Type II. $V^+ = V^- = A_n(\mathbf{F})$, alternating $(n \times n)$-matrices with $Q(x)y := xy'x$. In this case $X$ is isomorphic to the subvariety of $G_n(\mathbf{F}^{2n})$ consisting of all totally isotropic subspaces of $\mathbf{F}^{2n}$ of fixed parity with respect to the quadratic form $q(x_1, \ldots, x_n) := \sum_{i=1}^n x_i x_{n+i}$.

Type III. $V^+ = V^- = H_n(\mathbf{F})$, symmetric $(n \times n)$-matrices with $Q(x)y := xy'x$. In this case $X$ is isomorphic to the subvariety of $G_n(\mathbf{F}^{2n})$ consisting of all maximal isotropic subspaces of $\mathbf{F}^{2n}$ with respect to alternating form $\alpha(x, y) := \sum_{i=1}^n (x_i y_{n+i} - x_{n+i} y_i)$.

Type IV. $V^+ = V^- = F^n$ with $Q(x)y := q(x, y)x - q(x)y$, where $q$ is the standard quadratic form of $\mathbf{F}^n$, given by

$q(x_1, \ldots, x_{2m}) := \sum_{i=1}^m x_i x_{m+i}$ if $n = 2m$,

$q(x_0, \ldots, x_{2m}) := x_0^2 + \sum_{i=1}^m x_i x_{m+1}$ if $n = 2m + 1$.

In this case $X$ is isomorphic to the quadric of all isotropic lines through the origin in $\mathbf{F}^{2n}$.

Type V. $V^+ = V^- = M_{1,2}(\mathbf{O})$, $(1 \times 2)$-matrices over the octonion (Cayley) algebra $\mathbf{O}$ over $\mathbf{F}$. Here $X$ is isomorphic to the projective octonion plane $\mathcal{P}(\mathcal{O})$ defined by the exceptional Jordan algebra $H_3(\mathbf{O})^+$ (see Faulkner [248a]).

Type VI. $V^+ = V^- = H_3(\mathbf{O})^+$, Hermitian $(3 \times 3)$-matrices over the octonion (Cayley) algebra $\mathbf{O}$ over $\mathbf{F}$. In this case $X$ is isomorphic to the space of lines $\mathbf{F}u$, where $u$ is an element of rank one in the $56$-dimensional space of all matrices

$\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}$, $\alpha, \beta \in \mathbf{F}$, $a, b \in H_3(\mathbf{O}, \mathbf{F})$.

\footnote{These Jordan pairs arise in a fairly obvious way from separable Jordan pairs over $\mathbf{Z}$ by reduction \textit{modulo} char $\mathbf{F}$ and extending the prime field to $\mathbf{F}$. Thus, we are dealing with the fibres of a smooth projective $\mathbf{Z}$-scheme.}
In 1984 the fundamental paper by Petersson appeared\textsuperscript{10}. It provides a uniform framework for generic splitting fields of associative algebras studied by Amitsur [24], Heuser [342], Kovacs [417], Saltman [588] and Roquette [567a,b], and generic zero fields of quadratic forms investigated by Knebusch [403]. This is achieved by first observing that both, central simple associative algebras over a field $K$ of arbitrary characteristic and absolutely nondegenerate quadratic forms over $K$, are examples of absolutely simple Jordan pairs over $K$, and then constructing generic reducing fields of Jordan pairs over $K$, and a classification-free treatment.

Let $V = (V^+, V^-)$ be an absolutely simple Jordan pair over $K$. A nonzero idempotent $c$ of $V$ is called reduced, if the Peirce space $V_2(c)^\varepsilon = K c^\varepsilon$, $\varepsilon = \pm$, and $V$ itself is called reduced if it contains a reduced idempotent.

**Remark 1.** For Jordan structures ”split” is a more restrictive concept than ”reduced”, however ”split” = ”reduced” if $V$ is the Jordan pair associated to a central simple associative algebra.

A field extension $K/K$ is called a reducing field of $V$ in case the extended Jordan pair $V_K$ is reduced, and such an extension is called a generic reducing field if an arbitrary field extension $L/K$ reduces $V$ if and only if there is a $K$-place from $K$ to $L$.

Petersson has constructed two, in general not $K$-isomorphic, generic reducing fields. Both are $K$-rational function fields of irreducible projective $K$-variables associated to $V$. A detailed discussion of examples, most notably Brauer-Severi varieties, and with an application to exceptional simple Jordan algebras arising from the first Tits construction is also given by Petersson.

In 1985, Jacobson [371d] considered two classes of projective varieties, namely norm hypersurfaces and varieties of reduced elements defined by finite-dimensional central simple Jordan algebras. Jacobson’s paper [371d] overlaps substantially with the paper by Petersson. However, the methods and points of view are different, supplementing each other.

In Ch.II of his Ph.D. Thesis [701], Watson defined a class of manifolds, called local Jordan manifolds (LJM), the ground algebraic structure being (linear) Jordan pairs. For the sake of simplicity we shall omit the word ”linear”. As one can see from the examples given below, many familiar manifolds are local Jordan manifolds.

**Note.** All Jordan pairs considered in the following will be finite-dimensional

\textsuperscript{10}PETERSSON, H.-P., Generic reducing fields for Jordan pairs, Trans. Amer. Math. Soc. 285 (1984), 2, 825-843.
vector spaces over \( \mathbb{R} \) and the vector spaces \( V^+ \) and \( V^- \) of a Jordan pair \( V \) will be assumed to have the ordinary real topology.

**Comment.** Very recently (in 2009), Stacho & Werner [636b] defined the notion of Jordan manifold in a very different setting (see the next §3).

**Definition 1.** A Jordan chart on a topological space \( M \) is a pair \((\Phi, V)\) consisting of a Jordan pair \( V = (V^+, V^-) \) and a map \( \Phi : U \to V^+ \), where \( U \neq \emptyset \) is open in \( M \), and \( \Phi \) is a homeomorphism of \( U \) onto an open subset of \( V^+ \).

**Definition 2.** A Jordan atlas on a topological space \( M \) is a set \( \mathcal{A} \) of Jordan charts on \( M \) satisfying the following two conditions:

(a) the domain of the Jordan charts in \( \mathcal{A} \) cover \( M \);

(b) if \( \Phi_1 : U_1 \to V^+_1 \) and \( \Phi_2 : U_2 \to V^+_2 \) (where \( U_i \) is open in \( M \) and \( V_i \) is a Jordan pair) are elements of \( \mathcal{A} \) and if \( U_1 \cap U_2 \neq \emptyset \), then in a neighbourhood of each point in \( \Phi_1(U_1 \cap U_2) \), \( \Phi_2\Phi_1^{-1} : \Phi_1(U_1 \cap U_2) \subset V^+_1 \to V^+_2 \) is the restriction of a map from \( LF_+(V_1, V_2) \).

Let \( \Phi_i : U_i \to V^+_i, i = 1, 2, \) be two Jordan charts in a Jordan atlas \( \mathcal{A} \) on \( M \). Suppose that \( U_1 \cap U_2 \neq \emptyset \). Then for every \( u \in U_1 \cap U_2 \), in a neighbourhood of \( \Phi_1(u) \), \( \Phi_2\Phi_1^{-1} \) is the restriction of a unique linear fractional map, called the coordinate transition map and denoted by \( C_{\Phi_1, \Phi_2}^u \).

**Definition 3.** Let \( \mathcal{A} \) be a Jordan atlas on a topological space \( M \). Then a set \( \mathcal{B} \) of Jordan charts on \( M \) is said to be compatible with \( \mathcal{A} \) if \( \mathcal{A} \cup \mathcal{B} \) is a Jordan atlas on \( M \).

Now let \( \hat{\mathcal{A}} \) be the class of all Jordan charts on \( M \) compatible with \( \mathcal{A} \). It can easily be checked that if \( \mathcal{A}_1 \) is a set, if \( \mathcal{A}_1 \subset \hat{\mathcal{A}} \), and if the domains of the charts in \( \mathcal{A}_1 \) cover \( M \), then \( \mathcal{A}_1 \) is a Jordan atlas on \( M \), and \( \hat{\mathcal{A}} = \hat{\mathcal{A}}_1 \). The class \( \hat{\mathcal{A}} \) is called the Jordan structure of \( M \).

**Definition 4.** A local Jordan manifold is a pair \((M, \hat{\mathcal{A}})\) where \( M \) is a topological Hausdorff space and \( \hat{\mathcal{A}} \) is a Jordan structure on \( M \) determined by a Jordan atlas \( \mathcal{A} \) on \( M \).

**Remark 2.** Note that \( M \) is not assumed to be connected, nor must all components of \( M \) have the same dimension. However, every component of \( M \) is a real analytic manifold.

**Definition 5.** Let \((M_1, \hat{\mathcal{A}}_1)\) and \((M_2, \hat{\mathcal{A}}_2)\) be local Jordan manifolds. A morphism from \( M_1 \) to \( M_2 \) is a continuous map \( f : M_1 \to M_2 \) such that, for every pair of charts \( \Phi_1 : U_1 \to V^+_1 \) in \( \mathcal{A}_1 \) and \( \Phi_2 : U_2 \to V^+_2 \) in \( \hat{\mathcal{A}}_2 \) with \( U_1 \cap f^{-1}(U_2) \neq \emptyset \), in a neighbourhood of each point in its domain the map...
\[ \Phi_2 f \Phi_1^{-1} : \Phi_1(U_1 \cap f^{-1}(U_2)) \to V_2^+ \] is the restriction of a map from \( LF_+(V_1, V_2) \).

**Remark 3.** The composite of two morphisms (as maps) between local Jordan manifolds is again a morphism, and in this way the local Jordan manifolds and the morphisms between them form a category, denoted by LJM. Every morphism in LJM between connected local Jordan manifolds is a morphism in the category of real analytic manifolds.

**Definition 6.** Let \( V \) be a Jordan pair and let \( M \) be a topological space. A Jordan chart on \( M \) of the form \( \Phi, V \) is called a Jordan \( V \)-chart. A Jordan \( V \)-atlas on \( M \) is a Jordan atlas \( \mathcal{A} \) on \( M \) consisting of Jordan \( V \)-charts.

**Definition 7.** Let \( V \) be a Jordan pair. A local Jordan \( V \)-manifold is a topological Hausdorff space \( M \) with a maximal Jordan \( V \)-atlas \( \mathcal{A} \) on \( M \).

**Definition 8.** Every local Jordan \( V \)-manifold is a local Jordan manifold.

**Remark 4.** Let \( (M_1, \hat{A}_1) \) and \( (M_2, \hat{A}_2) \) be local Jordan manifolds. Since any \( (f_1, f_2) \in LF_+(V_1, W_1) \times LF_+(V_2, W_2) \) can be identified in a natural way with an element of \( LF_+(V_1 \oplus V_2, W_1 \oplus W_2) \), it is clear that \( M_1 \times M_2 \) possesses a natural Jordan structure. The (direct product) local Jordan manifold \( M_1 \times M_2 \) has the expected universal property requiring that whenever \( M \) is a local Jordan manifold and \( \pi_i : M \to M_i, i = 1, 2, \) are morphisms, there exists a unique morphism \( p : M \to M_1 \times M_2 \) such that \( \pi_i p = p_i \), where \( \pi_i : M_1 \times M_2 \to M_i \) is a projection morphism, for \( i = 1, 2 \). Moreover, any other local Jordan manifold \( M' \) having this property (for a pair of morphism \( \pi'_i : M' \to M_i, i = 1, 2 \)) is isomorphic in LJM to \( M_1 \times M_2 \).

**Remark 7.** If \( M_i \) is a local Jordan \( V_i \)-manifold, \( i = 1, 2 \), then \( M_1 \times M_2 \)
Jordan structure such that $p$ locally it is the restriction of a linear fractional map.

**Definition 7.** Let $(M, \mathcal{A})$ be a local Jordan manifold. A nonempty subset $N$ of $M$ is called a local Jordan submanifold of $M$ if there exists a set $\mathcal{A}_1 \subset \mathcal{A}$ such that for every $n \in N$ there exists a Jordan chart $\Phi : U \to V^+$ in $\mathcal{A}_1$ with $n \in U$ and such that $\Phi(U \cap N)$ is an open subset of a vector subspace $W^+$ of $V^+$.

Open subsets of local Jordan manifolds and affine subspaces of $V^+$, where $V = (V^+, V^-)$ is a Jordan pair, are simple examples of local Jordan submanifolds.

**Proposition 2.** Let $N_1$ and $N_2$ be local Jordan submanifolds of $M_1$ and $M_2$, respectively, and let $f : M_1 \to M_2$ be a morphism. Then $f|_{N_1} : N_1 \to M_2$ is a morphism. If $f(N_1) \subset N_2$ and $f : N_1 \to N_2$ is an open mapping, then $f : N_1 \to N_2$ is a morphism.

**Definition 8.** Let $f : M_1 \to M_2$ be a morphism between local Jordan manifolds. Then $f$ is called a local isomorphisms if every point $p \in M_1$ has a neighbourhood $U$ such that $f(U)$ is open in $M_2$ and $f : U \to f(U)$ is an isomorphism in LJM (where $U$ and $f(U)$ are local Jordan submanifolds of $M_1$ and $M_2$, respectively).

**Proposition 3.** Let $(M, \mathcal{A})$ be a local Jordan manifold and let $X$ be a topological Hausdorff space. Let $p : X \to M$ be a local homeomorphism. Then $X$ has a unique Jordan structure such that $p$ is a local isomorphism.

Let $\tilde{M}$ be a covering space of a local Jordan manifold $M$ and let $p : \tilde{M} \to M$ be the covering map. By Proposition 3 it follows that $\tilde{M}$ has a unique Jordan structure such that $p$ is a local isomorphism. If $M$ is a local Jordan $V$-manifold, then $M$ is also a local Jordan $V$-manifold. Let $\tilde{f} : \tilde{M} \to M$ be a continuous map which lies over a continuous map $f : M \to M$ ($p\tilde{f} = fp$); then $\tilde{f}$ is a LJM morphism if and only if $f$ is a LJM morphism.

If $\tilde{M}$ is a connected local Jordan manifold and $G$ is a group of LJM automorphisms acting freely and properly discontinuously on $\tilde{M}$, then the orbit space $M = \tilde{M}/G$ possesses a unique Jordan structure such that the natural projection $p : \tilde{M} \to M$ is a local isomorphism. A Jordan atlas for $M$ is constructed by taking charts of the form $\tilde{\Phi} = \Phi p^{-1} : p(U) \to V^+$, where $p : U \to p(U)$ is a homeomorphism and $\Phi : U \to V^+$ is a Jordan chart on $\tilde{M}$. If $U_i$ and $\Phi_i$, $i = 1, 2$, determine two such charts on $M$ and if $x \in p(U_1) \cap p(U_2)$, then there exists $x_1 \in U_1, x_2 \in U_2$ and $g \in G$ such that $p(x_1) = p(x_2) = x$ and $g(x_1) = x_2$. Hence $\tilde{\Phi}_2\Phi_1^{-1} = \Phi_2 g \Phi_1^{-1}$ in a neighbourhood of $\Phi_1(x)$, and hence locally it is the restriction of a linear fractional map.
As an example of an orbit space, let $\tilde{M} = V^+$, where $V = (V^+, V^-)$ is a Jordan pair, and let $G$ be a group of translations of $V^+$ acting freely and properly discontinuously. Then $M = \tilde{M}/G$ is a local Jordan $V$-manifold. This shows that a torus, for instance, possesses many local Jordan manifold structures.

Examples of local Jordan manifolds are: affine quadrics, projective quadrics, projective spaces, Grassmann manifolds.

**Comment.** It would be interesting to reconsider Martinelli’s results [471a,b] on quaternionic projective planes in this more general settings of Jordan pairs. An open problem is to find an algebraic characterization of integrability of a geometric structure.

Examples of two-dimensional local Jordan manifolds are: the torus, the sphere, the real projective plane, the cylinder.

In September 1980, at a conference organized in Romania, Gelfand gave a lecture on integral geometry mentioning the results obtained by himself and some of his coworkers on transformations between Grassmann manifolds. Then, after his lecture, I have suggested him to reconsider these results in a Jordan algebra (pair) setting. I have made the same suggestion also to MacPherson, concerning his results (in cooperating with Gelfand) on polyhedra in real Grassmann manifolds (GELFAND, I.M., & MacPHerson, R.D., *Geometry in Grassmannians and a generalization of the dilogarithm*, Preprint IHES, Paris, June 1981).

In Ch.III of his Ph.D. Thesis [701], Watson has considered the important subcategory of LJM, consisting of global Jordan manifolds (or simply, the Jordan manifolds (JM)).

**Important remark.** It is obvious that the Jordan manifolds defined by Watson have much in common with differentiable manifolds. (The main examples are differentiable manifolds). It follows the following

**Open problem.** Find an exact relationship between the category of JM and that of differential manifolds.

In 1951 Vagner began (see [684b,c]) a series of studies which led to a mathematical tool necessary to formulate and solve the following problem: To find the geometrical properties of differentiable manifolds which are derived only from algebraic properties of the pseudogroup of local homeomorphisms and the atlas.

By means of $[a, b, c] := c \cdot b^{-1} \cdot a$, where $a, b, c$ are elements of an atlas,
a ternary operation is defined and this led to the algebraic notion of heap (heath), a structure defined by a weakened set of a group postulates - see papers by Certain [159], Clifford [179] and Sushkevich [645].

This algebraic notion was used in differential geometry by Vagner (see [684b,d,e]). In the paper [684e] by Vagner, the following question has been raised: Do there exist properties of differentiable manifolds which depend only on the atlas?

Brânzei [134a,b] has answered this question affirmatively and has defined a kind of generalized manifolds, called $S\mathcal{H}$-manifolds, which he studies in detail. A large number of examples have been given by him.

**Open problem.** To compare the category of Jordan manifolds with category of $S\mathcal{H}$-manifolds.

**Remark 8.** It would be of interest to re-consider the results on the above-mentioned ternary operation, as well as the results of Brânzei [134a,b] in the theory of Jordan manifolds.

Let us mention now other categories whose objects are constructed in a similar way as differentiable manifolds. Among them we note topological manifolds, Vagner’s compound manifolds [684a] Abraham’s $(C^r, C^s)$-manifolds [3], Sikorski’s differentiable spaces [621], Smith’s differentiable spaces [628], Aronszajn’s paper [45], Marshall’s $C^\infty$-subcartesian spaces [470] (for the differential topology of these spaces see the paper [499b] by Motreanu), and Spallek’s $N$-differentiable spaces [630].

Motreanu [499a,b,c] has constructed a category of so-called preringed manifolds, which contains as particular cases all categories mentioned above. Roughly speaking a preringed manifold is a topological space $M$ which is locally determined by a triple $(E, F, V)$, where $E$ is a topological space, $F$ is a presheaf of real-valued functions on $E$, and $V$ is a vector space, two such triples being compatible with respect to change of charts. Here $E$ describes locally the topology of $M$ and $V$ plays the role of tangent space to $M$. By a suitable choice of $E, F$, and $V$ one obtains the particular theories recalled above. So, the following problem is immediate:

**Open problem.** Describe the Jordan manifolds in terms of preringed manifolds.

Finally, I would like to mention four interesting papers which are very recent: two papers by Pumplüin [551a,b], and two papers by Pirio & Russo [544a,b].
§3. JORDAN STRUCTURES IN ANALYSIS

The relationship between formally real Jordan algebras, self-dual homogeneous cones and symmetric upper half-planes in finite dimensions due to Koecher [408a, c, f] is the background for the study of the infinite-dimensional case. The objects here are $JB^*$-algebras and their real analogues (the so-called $JB$-algebras).

A generalization of formally real Jordan algebras to the infinite-dimensional case was introduced and studied by Alfsen, Shultz and Størmer [13] as follows:

**Definition 1.** A (linear) real Jordan algebra $J$ with unit element $e$ which is also a Banach space and in which the product and the norm satisfy:

(i) $\|xy\| \leq \|x\| \|y\|$
(ii) $\|x^2\| = \|x\|^2$
(iii) $\|x^2\| \leq \|x^2 + y^2\|$

for all $x, y \in J$ is called a Jordan Banach algebra (or, briefly, a $JB$-algebra).

**Remark 1.** In the finite-dimensional case, condition (iii) is equivalent to the fact that $J$ is a formally real Jordan algebra.

**Note.** The term $JB$-algebra arose as the Jordan analogue of $B^*$-algebra, much the same as $JC$-algebras and $JW$-algebras were termed after $C^*$- and $W^*$-algebras, respectively.

**Comments.** Hanche-Olsen and Størmer introduced [330] the concept of $JB$-algebra as follows: A *Jordan Banach algebra* is a real Jordan algebra $A$ (not necessarily unital) equipped with a complete norm satisfying $\|ab\| \leq \|a\| \|b\|$, $a, b \in A$. A *$JB$-algebra* is a Jordan Banach algebra $A$ in which the norm satisfies the following two additional conditions for $a, b \in A$,

$$(1^\circ) \quad \|a^2\| = \|a\|^2, \quad \text{and} \quad (2^\circ) \quad \|a^2\| \leq \|a^2 + b^2\|.$$

**Theorem 1.** A real Jordan algebra of finite-dimension is a $JB$-algebra if and only if it is formally real.

The algebras $H_p(F)^{(+) }$ (see Theorem 8 from §1) can be extended to arbitrary cardinality $p$ as follows. Let $F = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, and let $H$ be a (right) $F$-Hilbert space of dimension $p$ over $F$. Denote by $\mathcal{L}(H)$ the algebra of all bounded $F$-linear operators on $H$. Then there exists a natural involution
(the adjoint *) on \(L(H)\) and \(H_p(F) := \mathcal{H}(H) := \{\lambda \in L(H) \mid \lambda^* = \lambda\}\) is a \(JB\)-algebra with respect to the operator norm.

**Remark 2.** For every compact topological space \(S\) and every \(JB\)-algebra \(\mathcal{J}\), the algebra \(\mathcal{C}(S, \mathcal{J})\) of all continuous functions \(S \to \mathcal{J}\) is also a \(JB\)-algebra. In particular, \(\mathcal{C}(S, \mathbb{R})\) is an associative \(JB\)-algebra.

**Proposition 2.** Every associative \(JB\)-algebra is isometrically isomorphic to \(\mathcal{C}(S, \mathbb{R})\) for some compact topological space \(S\).

**Notation.** Let \(\mathcal{J}\) be a \(JB\)-algebra and denote by \(\mathcal{J}^2 := \{x^2 \mid x \in \mathcal{J}\}\) the positive cone in \(\mathcal{J}\).

**Definition 2.** The elements of \(\Omega\) are called positive definite, and an ordering \(<\) on \(\mathcal{J}\) is defined by: \(x < y\) for \(x, y \in \mathcal{J}\) if and only if \(y - x \in \Omega\).

**Definition 3.** A subcone of \(\mathcal{J}^2\) is said to be a face of \(\mathcal{J}^2\) if it contains all elements \(a\) of \(\mathcal{J}^2\) such that \(a \leq b\) for some \(b\) of it.

**Definition 4.** A \(JB\)-subalgebra \(B\) of \(\mathcal{J}\) is said to be hereditary if its positive cone \(B^2\) is a face of \(\mathcal{J}^2\).

Edwards [226a] proved the following results (see also [231a]):

**Theorem 3.** The norm-closed quadratic ideals of a \(JB\)-algebra \(\mathcal{J}\) coincide with the hereditary \(JB\)-algebras \(B\) of \(\mathcal{J}\), and the norm-closed faces of \(\mathcal{J}^2\) are the positive cones \(B^2\) of such subalgebras \(B\).

**Proposition 4.** The norm closure of a face of \(\mathcal{J}^2\) is a face of \(\mathcal{J}^2\).

Putter and Yood [552] generalized a number of well-known Banach algebra results to the Jordan algebra situation by appropriately modifying the proofs. They confined themselves to special \(JB\)-algebras.

Consider now the complexification \(\mathcal{J}^C := \mathcal{J} \oplus i\mathcal{J}\) of \(\mathcal{J}\). \(\mathcal{J}^C\) is a complex Jordan algebra with involution \((x + iy)^* := x - iy\).

**Definition 5.** \(D := D(\Omega) := \{z \in \mathcal{J}^C \mid \text{Im}(z) \in \Omega\}\) is called the tube domain (generalized upper half-plane) associated with the cone \(\Omega\).

Let us mention here that Tsao [675] proved that under certain conditions the Fourier coefficients of the Eisenstein series for an arithmetic group acting on a tube domain are rational numbers. The proof involves a mixture of Lie groups, Jordan algebras, Fourier analysis, exponential sums, and \(L\)-functions.

From the results reported by Alfsen, Hanche-Olsen, Shultz and Størmer [11], [13], it follows that for each \(JB\)-algebra \(\mathcal{J}\) there exist a canonical \(C^*\)-algebra \(\mathfrak{A}\) and a homomorphism \(\Psi : \mathcal{J} \to \mathfrak{A}\) such that \(\Psi(\mathcal{J})\) generates \(\mathfrak{A}\). The
kernel of $\Psi$ is the exceptional ideal $\mathcal{I}$ in $\mathfrak{A}$. Using Takesaki and Tomiyama’s methods, Behncke and Böss showed [75] that $\mathcal{I}$ may be described as an $H_3(O)$-fibre bundle over its primitive ideal space.

**Comments.** As it was remarked by Upmeier [682b], a promising application of $JB$-algebras is to be found in complex analysis, based on the one-to-one correspondence between $JB^*$-algebras and bounded symmetric domains in complex Banach spaces with tube realization (Koecher [408f], and Braun, Kaup and Upmeier [133b]).

**Definition 6.** A $JB^*$-algebra is a complex Jordan algebra $J$ with unit element $e$, (conjugate linear) involution $*$, and complete norm such that

(i) $\|xy\| \leq \|x\| \|y\|$;

(ii) $\|P(z)z^*\| = \|z\|^3$,

for all $x, y, z \in J$.

**Note.** The concept of $JB^*$-algebra was formulated by Kaplansky (lecture at the 1976 St. Andrews Colloquim of the Edinburgh Math. Soc. – see [712]) and introduced as “Jordan $C^*$-algebra”.

**Comments.** Youngson [723d] studied $JB^*$-algebra in the nonunital case. He stated, among other results, that nonunital $JB^*$-algebras are $C^*$-triple systems in the sense of Kaup [392b].

**Proposition 5.** The selfadjoint part of a $JB^*$-algebra is a $JB$-algebra.

Wright [712] proved the converse:

**Theorem 6.** For every $JB$-algebra $J$ there exists a unique complex norm on $J^C$ such that $J^C$ is a $JB^*$-algebra with selfadjoint part $J$. The correspondence $J \leftrightarrow J^C$ defines an equivalence of the category of $JB$-algebras onto the category of $JB^*$-algebras.

Russo and Dye [576] proved that the closed unit ball of a $C^*$-algebra with identity is the convex hull of its unitary elements. The same result was proved by Wright and Youngson [713a] for $JB^*$-algebras.

Using the fact that the extreme points of the positive ball in a $JB$-algebra are projections, Wright and Youngson first showed [713b] that a surjective unital linear isometry between two $JB$-algebras is a Jordan isomorphism, and then used this to obtain the same result for $JB^*$-algebras.

Bonsall [119] showed that if $B$ is a real closed Jordan subalgebra of a complex unital Banach algebra $A$, containing the unit and such that $B \cap iB = \{0\}$ and $B \subset H(B) \oplus iH(A)$, where $H(A)$ denotes the set of Hermitian elements of $A$, then $B \oplus iB$ is homeomorphically $*$-isomorphic to a $JB^*$-algebra. Using
Wright’s and Youngson’s results [712], [713a], [723a, b], Mingo [489] gave a $JB^*$-analogue of a $C^*$-algebra result Størmer [641a], as follows:

**Proposition 7.** Suppose $A$ is a $JB^*$-algebra and $B$ is a real selfadjoint subalgebra with unit such that $B \cap iB = \{0\}$. Then $B \oplus iB$ is a $JB^*$-algebra.

Mingo used Proposition 7 to prove the above-mentioned result of Bonsall, dispensing with the assumption $B \cap iB = \{0\}$, and also to prove that the isomorphism is an isometry.

**Definition 7.** A bounded domain $B$ in a complex Banach space is called symmetric if for every $a$ of $B$ there exists a holomorphic map $s_a : B \to B$ with $s_a^2 = \text{Id}_B$ and $a$ an isolated fixed point ($s_a$ is uniquely determined if it exists and is called the symmetry at $a$.)

**Definition 8.** For every open cone $C$ in a real Banach space $X$, the domain $T := \{z \in X \oplus iX \mid \text{Im}(z) \in C\}$ is called a symmetric tube domain if $T$ is biholomorphically equivalent to a bounded symmetric domain.

**Theorem 8.** Let $\mathcal{J}$ be a $JB$-algebra and let $\mathcal{J}^C = \mathcal{J} \oplus i\mathcal{J}$ be the corresponding $JB^*$-algebra. Then $D := \{z \in \mathcal{J}^C \mid \text{Im}(z) \in \Omega\}$ is a symmetric tube domain. The symmetry at the point $ie \in D$ is given by $s(z) = -z^{-1}$, and $z \to i(z - ie)^{-1}$ maps $D$ biholomorphically on the open unit ball $\Delta$ of $\mathcal{J}^C$. In particular, $\Delta$ is a homogeneous domain.

Braun, Kaup and Upmeier [133a] proved

**Theorem 9.** If $B$ is a real Banach space and $D$ is the symmetric tube domain for $B^C := B \oplus iB$, then for every $e \in \Omega$ there exists a unique Jordan product on $B$ such that $B$ is a $JB$-algebra with unit $e$, and $D$ is the upper half-plane.

**Remark 3.** It follows that $JB$-algebras, as well as $JB^*$-algebras, are in one-to-one correspondence with symmetric tube domains.

In the theory of formally real Jordan algebras of finite dimension an important fact is the minimal decomposition of elements of such an algebra with respect to a complete orthogonal system of primitive idempotents $\{e_1, \ldots, e_k\}$. The importance of the minimal decomposition follows from the fact that $\{e_1, \ldots, e_k\}$ determine a Peirce decomposition of the algebra which, for instance, diagonalizes the operator $L(x)$, and hence also $P(x)$.

The analogue for an arbitrary $JB$-algebra $\mathcal{J}$ is the fact that for every $\alpha \in \mathcal{J}$ the unital closed subalgebra $C(\alpha)$ generated by $\alpha$ is isomorphic to some $C(S, \mathbb{R})$, where $S$ is a compact topological space. However, in case
$S$ is connected, $e$ is the only nontrivial idempotent in $C(\alpha)$ and the Peirce decomposition cannot be applied.

Shultz [618] proved that the bidual of $JB$-algebra with the Arens product is also a $JB$-algebra. Hence, every $JB$-algebra is a norm-closed subalgebra of a $JB$-algebra which is a dual Banach space. Algebras of this type admit not only a continuous but also an $L^\infty$-functional calculus.

**Remark 4.** Edwards [226c] showed how some of the results on multipliers and quasi-multipliers of $C^*$-algebras can be extended to $JB$-algebras.

**Definition 9.** A $JB$-algebra $J$ is called a $JBW$-algebra if $J$ is a dual Banach space (i.e., there exists a Banach space $'J$ with $J = 'J'$ as dual Banach space; $'J$ is uniquely determined by $J$ (see Sakai [586]) and is called the predual of $J$).

**Example.** The selfadjoint part of a von Neumann algebra is a $JBW$-algebra.

**Remark 5.** For every $\alpha$ in the $JBW$-algebra $J$ the $w^*$-closed unital subalgebra $W(\alpha)$ of $J$ generated by $\alpha$ is a commutative von Neumann algebra, i.e., $W(\alpha) \approx C(S, R)$ for $S$ hyperstonian or, equivalently, $W(\alpha) \approx L^\infty(\mu)$, where $\mu$ is a localizable measure (see Sakai [586]).

**Remark 6.** $JBW$-algebras (weakly closed analogues of $JB$-algebras) are the abstract analogues of von Neumann algebras in the Jordan case.

**Definition 10.** Let $J'$ be the dual Banach space of a $JB$-algebra $J$ and denote by $(J^2)' := \{ \lambda \in J' \mid \lambda(J^2) \geq 0 \}$ the dual cone of $J^2$. Then $K := \{ \lambda \in (J^2)' \mid \lambda(e) = 1 \}$ is called the state space of $J$, the elements of $K$ being called states on $J$.

**Remark 7.** $K$ is a $w^*$-compact, convex subset and $J$ can be identified (as a Banach space) with the space of all $w^*$-continuous affine functions on $K$. The bidual of $J$ coincide with the set of all bounded affine functions on $K$.

In a comprehensive study of state spaces of a $JB$-algebra, Alfsen and Shultz [12a] gave necessary and sufficient conditions for a compact convex set to be a state space of a $JB$-algebra. Araki [35] improved the characterization of state spaces of $JB$-algebra given in [12a] to a form with more physical appeal in the simplified finite-dimensional case.

Alfsen, Hanche-Olsen and Shultz [11] characterized the state spaces of $C^*$-algebras among the state spaces of all $JB$-algebras. Together, [12a] and [11] give a complete characterization of the state spaces of $C^*$-algebras. As is shown in [11], a $JB$-algebra $J$ is the selfadjoint part of a $C^*$-algebra if and only
if $\mathcal{J}$ is of complex type and the state space of $\mathcal{J}$ is orientable. Stacey [634c] showed that the state space of a $JBW$-algebra of complex type is orientable if and only if it is locally orientable. For local and global splittings in the state space of a $JB$-algebra, see Stacey [634b].

Every state $\lambda \in K$ defines by $(x|y)_{\lambda} := \lambda(xy)$ a positive inner product on $\mathcal{J}$ and, in particular, by $|x|_{\lambda} := \lambda(x^2)^{\frac{1}{2}}$ a seminorm on $\mathcal{J}$.

**Definition 11.** A state $\lambda$ on a $JB$-algebra $\mathcal{J}$ is called **faithful** if $| \cdot |_{\lambda}$ actually is norm on $\mathcal{J}$, i.e. if $\lambda(x^2) = 0$ implies that $x = 0$.

**Definition 12.** A state $\lambda$ on a $JBW$-algebra $\mathcal{J}$ is called **normal** if $\lim \lambda(x_\alpha) = \lambda(x)$ for every increasing net $x_\alpha$ in $\mathcal{J}$ with $x = \sup x_\alpha \in \mathcal{J}$.

**Definition 13.** A normal state $\lambda$ on a $JBW$-algebra $\mathcal{J}$ is called a **finite trace** if it is associative in the sense that $\lambda((xy)z) = \lambda(x(yz))$ for all $x, y, z \in \mathcal{J}$.

**Remark 8.** The condition from Definition 13 states that every $L(y), y \in \mathcal{J}$, is selfadjoint with respect to the inner product $(\cdot | \cdot)_{\lambda}$.

A complete study of $JBW$-algebras with a faithful finite trace was undertaken by Janssen [373b]. On the basis of this paper, Janssen [373c] studied the properties of the lattice of idempotents in a finite weakly closed Jordan algebra. He proved that such an algebra admits a unique decomposition into a direct sum of a discrete Jordan algebra and a continuous Jordan algebra. Janssen [373c, II] gave a completely description of the discrete finite weakly closed Jordan algebras by finite-dimensional simple formally real Jordan algebras and by simple formally real Jordan algebras of quadratic forms of real Hilbert spaces.

Pedersen and Størmer [531] showed that the different definitions of trace on a Jordan algebra are all equivalent for $JBW$-algebras, and that conditions that do not involve projections are equivalent for $JB$-algebras. They have considered only finite traces. Iochum [358a] extended the results to semifinite traces. By a suitable definition of semifiniteness, he showed that for any $JBW$-algebra we have a unique central decomposition in finite (semifinite) and proper-infinite (pure-infinite) parts exactly as in the case of von Neumann algebras (see [358a, Theorem V.1.6]). Iochum proved also (for the semifinite case) the equivalence between the category of facially homogeneous self-dual cones and the category of $JBW$-algebras of selfadjoint derivations (see [358a, Theorem V.5.1]), and ([358a, Ch. VII]) his main theorem, which establishes the equivalence between the category of facially homogeneous self-dual cones in Hilbert spaces and the category of $JBW$-algebras (see also [77d]).

Assume now that $\mathcal{J}$ is a $JBW$-algebra with a faithful finite trace $\lambda$. 
Then $\lambda$ is essentially uniquely determined (every other faithful finite trace is of the form $\lambda \circ P(h) = \lambda \circ L(h^2)$ for some $h > 0$ in the centre of $J$), and $J^C$ is a complex pre-Hilbert space with respect to the inner product $(z|w) := (zw^*)_\lambda := \lambda(zw^*)$, where $\lambda$ is extended $C$-linearly to $J^C$.

**Notation.** Denote by $\tilde{J}^C$ the completion of $J^C$ with respect to the norm $\|z\|_2 := \|z\|_\lambda := \lambda(zz^*)^{1/2}$, and consider the closures $\tilde{J}$ and $\tilde{J}^2$ of $J^2$ in $J^C$.

The operators $L(z)$ and $P(z)$, $z \in J^C$, can be continuously extended to bounded operators on $\tilde{J}^C$ satisfying $L(z)^* = L(z^*)$ and $P(z)^* = P(z^*)$.

The cone $\tilde{J}^2$ is self-dual in $\tilde{J}$, satisfies a certain geometrical homogeneity condition, and has $e$ as trace vector (i.e., as quasi-interior point of $\tilde{J}^2$ fixed by every connected set of isometries in $GL(\tilde{J})$). On the other hand, every cone of this type in a real Hilbert space is obtained in this way from a JB-algebra with faithful finite trace (see Bellissard and Iochum [77a]). This result can be viewed as a generalization to the infinite-dimensional case of the following theorem of Koecher: The self-dual cones with homogeneous interior in real Hilbert spaces of finite dimension are precisely (up to linear equivalence) the cones of squares in formally real Jordan algebras.

**Proposition 10.** The JB*-norm $\|\cdot\|_\infty$ on $J^C$ satisfies $\|\cdot\|_2 \leq \|\cdot\|_\infty$ on $J^C$, and $\Delta = \{z \in J^C \mid 1 - P(z)P(z)^* > 0\}$, $\Sigma = \exp(i\tilde{J}) = \{z \in J^C \mid P(z)$ unitary on $J^C\} = \{z \in \Sigma \mid \|z\|_2 = 1\}$.

**Proposition 11.** If $J$ is a JB-algebra, then the following conditions are equivalent:

(i) there exists a maximal associative subalgebra of finite dimension in $J$;

(ii) $J$ is locally finite (i.e., every finitely generated subalgebra has finite dimension);

(iii) for every $a \in J$ the operator $L(a) \in \mathcal{L}(J)$ satisfies a polynomial equation over $\mathbb{R}$;

(iv) there exists a natural number $r$ such that every $a \in J$ admits a representation $a = \alpha_1 e_1 + \cdots + \alpha_r e_r$, where $\{e_1, \ldots, e_r\}$ is a set of orthogonal idempotents and $\alpha_1, \ldots, \alpha_r \in \mathbb{R}$;

(v) there exists a faithful finite trace $\lambda$ on $J$ such that the corresponding Hilbert norm $\|x\|_2 = \lambda(x^2)^{1/2}$ on $J$ is equivalent to the JB-norm $\|x\|_\infty$;

(vi) $J$ is reflexive.

Chu [166a] studied the Radon-Nikodym property (for definition see below) in the context of JBW-algebras.
Definition 14. A (real or complex) Banach space \( X \) is said to possess the \textit{Radon-Nikodym property} if for any finite measure space \( (\Omega, \Sigma, \mu) \) and \( \mu \)-continuous vector measure \( L : \Sigma \to X \) of bounded total variation, there exists a Bochner integrable function \( g : \Omega \to X \) such that \( L(E) = \int_E g \, d\mu \) for all \( E \) in \( \Sigma \).

Using a result of Shultz, Chu [166a] established the following result:

Theorem 12. Let \( J \) be a JBW-algebra. Then its dual \( J' \) has the Radon-Nikodym property if and only if \( J \) is a finite direct sum of Jordan algebras, each of which is one of the following algebras:

(i) Jordan \((n \times n)\)-matrix algebras over \( \mathbb{R}, \mathbb{C}, \) or \( \mathbb{H} \);
(ii) spin factors;
(iii) the exceptional Jordan algebra of Hermitian \((3 \times 3)\)-matrices over \( \mathbb{O} \).

Chu [166b] proved that the dual of a JB-algebra \( J \) possesses the Radon-Nikodym property if and only if the state space of \( J \) is the \( \sigma \)-convex hull of its pure states. Namely, he proved:

Theorem 13. If \( J \) is a JB-algebra with state space \( K \), then the following conditions are equivalent:

(i) \( K \) is \( \sigma \)-convex hull of the pure states, i.e.,
\[
K = \left\{ \sum_{n=1}^{\infty} \lambda_n k_n \mid \sum_{n=1}^{\infty} \lambda_n = 1, \lambda_n \geq 0, k_n \text{ being pure states} \right\};
\]

(ii) \( J' \) has the Radon-Nikodym property;
(iii) \( J'' \) is a direct sum of type I JBW-algebras (i.e., JBW-algebras which contains a (non-zero) minimal idempotent).

In 1991–1992, Bunce and Chu [144a, b] studied the Radon-Nikodym property in JB*-triples (see Definition 19 below).

Definition 15. An idempotent \( p \) in a JBW-algebra \( J \) is called an \textit{atom} if it is minimal (i.e., if \( 0 \leq q \leq p \) with \( q^2 = q \) implies that \( q = 0 \) or \( q = p \)). \( J \) is said to be \textit{atomic} if every idempotent is the least upper bounded of orthogonal atoms.

Notation. If \( \lambda \) is a state on the JBW-algebra \( J \), then
\[
V_\lambda := \{ f \mid f \in J', \exists a \in \mathbb{R}_+ \text{ with } -a\lambda \leq f \leq a\lambda \}.
\]

Theorem 14. Let \( J \) be an atomic JBW-algebra and let \( \lambda \) be a faithful normal state on \( J \). Then there exists an order isomorphism \( \varphi : V_\lambda \to J \) with \( \varphi(\lambda) = e \).
Theorem 15. Suppose that a JBW-algebra $J$ admits a faithful normal trace $\lambda$ (i.e., for all idempotents $p, q$ we have $\lambda(U_p q - U_q p) = 0$, where $U_p q := (pq)q - (qp)q + q^2 p$). Then there exists an order isomorphism $\varphi : V_\lambda \rightarrow J$ with $\varphi(\lambda) = e$. Moreover, for every positive $\mu$ (i.e., $\mu \in (J^2)'$) from $V_\lambda$, there exists a positive element $y$ in $J$ such that $\mu(x) = \lambda(U_y x)$.

Theorem 16. Let $J$ be a JBW-algebra satisfying the quadratic Radon-Nikodym property (i.e., for any $f, g \in \mathcal{J}$ with $0 \leq f(x^2) \leq g(x^2)$ for every $x \in J$, there exists a positive $y$ in $J$ such that $f(x) = g(U_y x)$ for every $x \in J$) and let $\mu$ and $\nu$ be faithful normal states on $J$. Then $V_\mu$ and $V_\nu$ are order isomorphic.

Corollary. Let $J$ be as in Theorem 16 and suppose that $J$ admits a faithful normal trace. Let $\lambda$ be a faithful normal state on $J$. Then $V_\lambda$ is order isomorphic to $J$.

Definition 16. A family $(v_t)_{t \in \mathbb{R}}$ of linear operators on a linear space $M$ satisfying $v_0 = Id$ and the cosine identity
\[ 2v_s v_t = v_{s+t} + v_{s-t}, \]
is called a (one-parameter) cosine family on $M$.

Remark 9. If $(u_t)$ is a one-parameter group, then $(u_t + u_{-t})/2$ is a cosine family.

Definition 17. Let $J$ be a JBW-algebra and $\lambda$ a normal state on $J$. A bilinear, symmetric, positive semidefinite form $s$ on $J$ satisfying
\begin{itemize}
  \item[(i)] $a(a, b) \geq 0$, $a \geq 0$, $b \geq 0$;
  \item[(ii)] $s(1, a) = \lambda(a)$, $a \in J$;
  \item[(iii)] if $0 \leq \mu \leq \lambda$, there is $0 \leq b \leq 1$ so that $\mu(a) = s(a, b)$, $a \in J$,
\end{itemize}
is called a self-polar form associated with $\lambda$.

Remark 10. There exists at most one self-polar form associated with $\lambda$.

Theorem 17. Let $J$ be a JBW-algebra and let $\lambda$ be a faithful normal state on $J$. Then there exists a unique cosine family $(\theta_t)$ of positive, unital linear mappings of $J$ into itself, having the following properties:
\begin{itemize}
  \item[(i)] for each $a \in J$, $t \rightarrow \theta_t(a)$ is weakly continuous;
  \item[(ii)] $\lambda(\theta_t(a) \circ b) = \lambda(a \circ \theta_t(b))$;
  \item[(iii)] $s(a, b) := \int \lambda(a \circ \theta_t(b)) \cos h(\pi t)^{-1} dt$ defines a self-polar form associated with $\lambda$.
\end{itemize}

Let us mention the following result of Størmer [641e], related to JW-algebras.
Let $M$ be a von Neumann algebra and let $\alpha$ be a central involution of $M$, i.e., $\alpha$ is $*$-anti­automorphism of order 2 leaving the centre of $M$ elementwise fixed. Then the set $M^\alpha := \{ x \in M \mid x = x^* = \alpha(x) \}$ is a JW-algebra with Jordan product $xy := 1/2(x \circ y + y \circ x)$. Størmer studied the relationship between $M^\alpha$ and $M^\beta$ for two central involutions $\alpha$ and $\beta$. The main result states that $\alpha$ and $\beta$ are (centrally) conjugate, i.e., there exists a $*$-automorphism $\phi$ of $M$ leaving the centre elementwise fixed, such that $\beta = \phi \alpha \phi^{-1}$ if and only if $M^\alpha$ and $M^\beta$ are isomorphic as Jordan algebras via an isomorphism which leaves the centre elementwise fixed. Now $M^\alpha$ generates $M$ as a von Neumann algebra (except in a few simple cases) and there are von Neumann algebras with many conjugate classes of central involutions. Thus there may be many, even an uncountable number, of non-isomorphic JW-algebras which generate the same von Neumann algebra. Such examples may be found in [641e, Section 5].

**Definition 18.** A $JH^*$-algebra is a complex Hilbert space $H$ together with a complex Jordan algebra structure and a continuous involution $x \rightarrow x^*$ such that the Jordan product is continuous and $L(a)^* = L(a^*)$ for every $a \in H$, $L$ being the left multiplication.

**Definition 19.** A complex Banach space $Z$ with a continuous mapping $(a, b, c) \rightarrow \{abc\}$ from $Z \times Z \times Z$ to $Z$ is called a $JB^*$-triple if the following conditions are satisfied for all $a, b, c, d \in Z$, where the operator $a □ b$ from the Banach algebra $L(Z)$ of all bounded linear operators on $Z$ is defined by $z \rightarrow \{abz\}$ and $[\cdot, \cdot]$ is the commutator product:

1. $\{abc\}$ is symmetric complex linear in $a, c$ and conjugate linear in $b$;
2. $[a □ b, c □ d] = \{abc\} □ d - c □ \{dab\}$;
3. $a □ a$ is Hermitean and has spectrum $\geq 0$;
4. $\|\{aaa\}\| = \|a\|^3$.

**Definition 20.** A $JB^*$-triple is called a $JBW^*$-triple if it is the dual of a Banach space.

**Definition 21.** An element $z \in Z$ is called tripotent if $\{eee\} = e$.

**Remark 11.** The set $\text{Tri}(Z)$ of tripotent elements is endowed with the induced topology of $Z$. It has been showed by J. Sauter in his Ph.D. Dissertation Randstruktur­en beschränkter symmetrischer Gebiete (Tübingen Univ., 1995) that $\text{Tri}(Z)$ is a real analytic direct submanifold of $Z$.

If $e \in \text{Tri}(Z)$, then $e □ e \in L(Z)$ has the eigenvalues 0, 1/2, 1, and we have...
the following Peirce decomposition of $Z$ with respect to $e$

$$Z = Z_1(e) \oplus Z_{1/2}(e) \oplus Z_0(e),$$

the Peirce projections being

$$P_1(e) = Q^2(e), \quad P_{1/2} = 2(e\Box e - Q^2(e)), \quad P_0 = \text{Id} - 2e\Box e + Q^2(e),$$

where $Q(e)z := \{eze\}$ for $z \in Z$.

A recent paper on closed tripotents is that that by Fernandez-Polo & Peralta [262b].

Let us mention here the research monograph [512d] by Neher, where a theory of grids (i.e., special families of tripotents in Jordan triple systems) is presented. Among the applications there is also the structure theory of $JBW^*$-triples.

Concerning the spectrum preserving linear maps on $JBW^*$-triples, see the paper by Neal [510].

Some applications of Jordan theory to harmonic analysis have been found by Chu in [166d] and by Chu and Lau in [169a,b].

Very recently (in 2009), Stacho and Werner [636b] defined the notion of Jordan manifolds as Banach manifolds whose tangent spaces are endowed with Jordan triple products depending smoothly on the underlying points. They show examples of Jordan manifolds with various features giving rise to problems for further studies.

**Comment.** In 1978, Watson [701] defined the notion of *Jordan manifold* in a completely different setting (see the previous §2).

Two interesting very recent papers are those by Arazy, Engliš and Kaup [38], and by Kennedy [398].

In April 2011, Werner [704] pointed out very interesting relations between Jordan $C^*$-triple systems and $K$-theory. These results were very recently obtained by him in cooperation with one of his Ph.D. students.

Let us concern now with the applications of Jordan structures to Riccati differential equations, to Hua equations and Szegö kernel, to the (reproducing) kernel functions, as well as to dynamical systems, and to Shilov boundary.

1. The Riccati differential equation

$$\dot{x} = p(x),$$

$x \in \mathbb{R}^n$ and $p : \mathbb{R}^n \to \mathbb{R}^n$ homogeneous and quadratic, plays an important role in biology, genetics, ecology, and chemistry. Koecher [408f, g, i] and Meyberg
studied the relations of this equation with nonassociative algebras, in particular with Jordan algebras.

We consider a commutative algebra $A$ over $\mathbb{R}^n$ with product $xy := \frac{1}{2}(p(x + y) - p(x) - p(y))$, and let $A_a$ be the mutation of $A$ with respect to $a$ (see Definition 4 from §1).

**Notation.** Denote by $R_n$ the vector space of power series in $\mathbb{R}^n$ converging in a neighbourhood of zero.

For $p, q \in R_n$ we define $p \cdot q \in R_n$ by

$\left[(p \cdot q)(u)\right]_i := \sum_{j=1}^{n} \frac{\partial p_i(u)}{\partial u_j} q_j(u).$

**Remark 12.** The vector space $R_n$ with the product $(p, q) \to p \cdot q$ becomes a nonassociative algebra over $\mathbb{R}$.

Now define $g_A(u) \in R_n$ by $g_A(u) := \sum_{m=0}^{\infty} \frac{1}{m!} g_m(u)$, where $g_0(u) := u$, $g_{m+1} := g_m \cdot p$, and $p(u) := u^2$. (Powers in $A$ are defined as follows: $u^1 := u$, $u^{m+1} := uu^m$.)

The elements $f \in R_n$ such that $f(x(\xi))$ is a solution of the above mentioned Riccati equation whenever $x(\xi)$ is a solution, form a group $S(A)$ under composition, the solution-preserving group of the equation.

**Notation.** Let $J(A)$ denote the subspace of all $a \in \mathbb{R}^n$ satisfying

$2u(u(ua)) + u^3a = 2u(u^2a) + u^2(ua)$

for all $u \in \mathbb{R}^n$.

**Theorem 18.** If $A$ has a unit element, then $a \to g_A$ is an isomorphism of the additive group $J(A)$ to $S(A)$.

**Theorem 19.** If $A$ is a commutative algebra over $\mathbb{R}^n$, then $J(A)$ is a Jordan subalgebra of $A$.

Moreover, the following theorem holds:

**Theorem 20.** If $A$ is a finite-dimensional commutative algebra over a field of characteristic different from two or three, then $J(A)$ is a Jordan subalgebra of $A$.

Concerning the Riccati differential equation in Jordan pairs, Braun [130] proved a result recalled below (see Theorem 21). Linearization of the matrix Riccati differential equation derived from $(m \times n)$-matrices (see Levin [442]) and the Riccati differential equation for operators in a Banach space (see Tartar [658]) are assumed to be known to the reader.
Let \( V \) be a Jordan pair with \( V^\sigma, \sigma = \pm \), Banach spaces, and let \( D \) and \( Q \) be the derivation and the quadratic representation defined as usually. Let \( I \) be an \( \mathbb{R} \)-interval, let \( \eta \) be an initial point in \( I \), and let \( k \) be a given initial value, \( k \in V^+ \). Let \( v(\xi), w(\xi) \) be given continuous functions, \( v : I \to V^- \), \( w : I \to V^+ \), and let \( D \) and \( Q \) be continuous. The Riccati differential equation (without linear term) is defined by

\[
\frac{\partial x}{\partial \xi} = Q(x)v + w.
\]

The solution \( x : I \times I \to V^+ \) with initial value \( k \) at the point \( \eta \) will be denoted by \( x(\xi, \eta) \).

**Notation.** \( B(u,t) := \text{Id} - D(u,t) + Q(u)Q(t), \) \( u^t := B(u,t)^{-1}(u - Q(u)t), \) for \( u \in V^+ \), \( t \in V^- \), if the inverse of \( B(u,t) \) exists.

**Theorem 21.** Let \( x_0 \) be the solution of the Riccati equation with initial value \( k = 0 \) at \( \eta = 0 \). Put

\[
x(\xi, \eta) := x_0 + h_+(k)^{h_-(z)}
\]

\( h_\sigma : I \times I \to \text{Aut} V^\sigma, \) \( z : I \times I \to V^- \). Solve the linear system

\[
\frac{\partial h_+}{\partial \xi} = D(x_0,v)h_+, \quad \frac{\partial h_-}{\partial \xi} = -D(v,x_0)h_-,
\]

so that \( \frac{\partial h}{\partial \xi} = h_-(v) \) with \( h_\sigma(\eta, \eta) = \text{Id}, \) \( z(\eta, \eta) = 0 \). Then \( x(\xi, \eta) \) is the solution with initial value \( x(\eta, \eta) = k \) (in a neighborhood of \( \eta \)).

Walcher [699a] gave a characterization of regular Jordan pairs and its application to the Riccati differential equation as follows. Let \( V \) be a finite-dimensional vector space over \( \mathbb{R} \), \( P : V \to \text{Hom}(V,V) \) a quadratic map, \( G \subset V \) open (\( G \neq \emptyset \)), and \( \varphi \in C^1(G,V) \). Suppose that for all \( a \in V \) one has \((d/dt)\varphi(z(t)) = -a \) whenever \( z(t) \subset G \) is a solution of the Riccati differential equation \( \dot{x} = P(x)a \). By differentiation, \( D\varphi(x) \cdot P(x) = -\text{Id} \). Moreover, Walcher showed that the identity \( P(x,P(x)z)y = P(x,P(x)y)z \) is satisfied for all \( x, y, z \in V \). Thus there exists a Jordan pair structure \( (P,Q_-) \) on \( V = (V,V) \) and by Theorem 21 the following is true: Let \( a : I \to V^- \), \( c : I \to V^+ \), \( (B_+, B_-) : I \to \text{Der} V \) be continuous. If \( z(t) \) solves

\[
\dot{x} = P(x)a + B_+x + c
\]

and \( P(z(t)) \) is invertible, then \( P(z(t))^{-1}z(t) \) solves

\[
\dot{x} = -Q_-(x)c + B_-x - a.
\]
Let us recall that a system of ordinary differential equations \( \dot{x} = F(t, x) \) is said to have a fundamental system of solutions if there exist finitely many solutions that determine (almost) all other solutions; it is called a system of polynomial differential equations if, for all values of \( t \), \( F(t, x) \) is a polynomial in \( x \). A theorem of Lie implies that a system of polynomial differential equations has a fundamental system of solutions if \( F(t, x) = \sum \lambda_i(t)f_i(x) \) and the polynomials \( f_i(x) \) generate a finite-dimensional subalgebra of the Lie algebra \( \text{Pol} V \), where \( V \) is the vector space on which the system is defined.

Walcher [699b] determined these subalgebras in the case \( \dim V = 1 \) and showed that they correspond to the Riccati (including linear) and the Bernoulli equation. For \( \dim V > 1 \), Walcher investigated the finite-dimensional, graded subalgebras \( L \) of \( \text{Pol} V \). Denoting by \( \text{Pol}_i V \) the subspace of all polynomials of degree \( i + 1 \), it is shown that the semisimplicity of \( L = L_{-1} \oplus L_0 \oplus \cdots \oplus L_m \), with \( L_i \subseteq \text{Pol}_i V \), implies \( m = 1 \).

\( L \) is said to be transitive if \( L_{-1} = V \). By a result of Kantor, it is known that a finite-dimensional, graded, transitive subalgebra with \( m > 1 \) is reducible; that is, there exists a subspace \( U \) of \( L_{-1} \) with \( 0 \neq U \neq V \) such that for all \( k \) with \( 0 \leq k \leq m \) and all \( p \in L_k \), \( p(V, \ldots, V, U) \subseteq U \). This allows one to reduce the discussion of transitive subalgebras to those whose degree equals 1. The latter are shown to arise from finite-dimensional Jordan pairs. In case \( \dim V = 2 \), this permits a complete enumeration of all finite-dimensional, maximal, transitive subalgebras of \( \text{Pol} V \) of fixed degree \( m \). Walcher also discussed how these results can be used to find all solutions of certain types of systems of polynomial differential equations.

Let us recall now the construction given in 2000 by Liu (see Liu [445b]).

If \( V \) is a finite-dimensional real vector space endowed with an inner product denoted by the dot, then let us consider the real vector space \( \mathbb{M} \) as follows

\[
\mathbb{M} := \{ x_0 + gx_n \mid x_0 \in \mathbb{R}, x_n \in V \}
\]

with \( x + y := (x_0 + y_0) + g(x_n + y_n) \) for \( x, y \in \mathbb{M} \) and define also a product

\[
xy = (x_0 + gx_n)(y_0 + gy_n) := (x_0y_0 + x_n \cdot y_n) + g(x_0y_n + y_0x_n)
\]

where \( g \) (called the “g-number” by Liu) satisfies

\[
\begin{array}{c|cc}
  & 1 & g \\
---&---&---
1 & 1 & g \\
g & g & 1
\end{array}
\]
It is easily to prove that \( M \) is a commutative Jordan algebra. Liu [445b] has called \( M \) the \textit{g-based Jordan algebra}.

**Remark 13.** The \textit{g-based} Jordan algebra \( M \) is a particular \textit{linear} Jordan algebra. It is the underling algebraic structure of a dynamical system defined on \( \mathbb{V} \) which possesses one or more constraints.

Some applications of this new formulation include the \textit{perfect elastoplasticity} (see Hong and Liu [349a,b]), the \textit{magnetic spin equation} (see Landau and Lifshitz [427]), the \textit{suspension particle orientation equation} (see Liu [445a]). They prove the \textit{usefulness} of this new formulation of Liu.

**Open Problem.** As Liu suggested, there exists the possibility to describe the non-linear dissipative phenomena of physical systems by using the above mentioned \textit{g-based} Jordan algebra \( M \) (see Liu [445b, p. 428]).

One year later, in 2001, Liu has proceeded to examin above mentioned type of dynamical systems from the view point of Lie algebras and Lie groups. Then he has derived a new dynamical system based on the composition of the \textit{g-based} Jordan algebra and Lie algebras (see Liu [445c]).

**Remark 14.** Based on the symmetry study, Liu has developed a \textit{numerical scheme} which preserves the group properties for every time increment.

**Important remark.** Because the above mentioned scheme is easy to implement numerically and has high computational efficiency and accuracy, it is highly recommended for engineering applications.

In 2002, Liu has examined previous mentioned dynamical systems from the view point of Lie algebras and Lie groups (see Liu [445d]).

**Open Problem.** Consider other \textit{particular} Jordan algebras suitable to be the algebraic foundation for various dynamical systems.

On the other hand, it could be formulated also the following

**Open Problem.** Taking into account of the fact that the study of linear Jordan algebras can be included in the more general study of Jordan triple systems, it would be interesting to develop a \textit{more general} algebraic background for (various) dynamical systems, and finally formulate an \textit{unitary} mathematical theory for \textit{all} dynamical systems.

In 2004, Liu [445e] used the real \textit{g-based} Jordan algebra (defined by himself in 2000) to the study of the Maxwell equations without appealing to the imaginary number \( i \). In terms of the \textit{g-based} Jordan algebra formulation, the usual Lorentz gauge condition is found to be a necessary and sufficient
condition to render the second pair Maxwell equations, while the first pair of Maxwell equations is proved to be an intrinsic algebraic property.

The $g$-based Jordan and Lie algebras are a suitable system to implement the Maxwell equations into a more compact form.

Finally, Liu has studied in [445d] the problem about a single formula of the Maxwell equations.

Remark 15. In 1966, Hestenes has proved in his book [341] that – in terms of spacetime algebra (16 components) – the four Maxwell equations can be organized into a single one. Similarly, Liu [445e] has achieved his goal in his algebraic formulation. See also the recent paper by Liu [445f].

Remark 16. It is impressive that $g$-based Jordan (and Lie) algebras of Liu can be so usefull to different topics.

2. If $M = G/K$ is a bounded symmetric domain and $S = K/L$ is its Shilov boundary, then one can define a Poisson kernel on $M \times S$ and the Poisson integral for any hyperfunction on $S$. An open problem, formulated more than twenty years ago by Stein, is to characterize these integrals as solutions of a system of differential equations, established for certain cases by Hua (see [352c]).

In [432a], Lassalle dealt with bounded symmetric domains of tube type. Poisson integrals over the Shilov boundary are then characterized by the system of differential equation given by Johnson and Korányi [375]. Results of Berline and Vergne [93] for the domain $(I)_{n,n}$ and that of Korányi and Malliavin [411] for the Siegel disc of dimension two, prove that the Johnson-Korányi system [375] has, for these particular cases, too many equations. Lassalle proved that this is a general property. In particular, he established that among the dim $K$ differential equations of the Johnson-Korányi system, a subsystem of dim $S$ equations is sufficient to characterize the bounded functions on $M$ which are Poisson integrals of a function on $S$. This new characterization has a very natural interpretation in terms of Jordan algebras (see Lassalle [432a, pp. 326–327]).

As Lassalle [432b] proved, such an interpretation is also possible if $M$ is a symmetric Hermitian space of tube type with Shilov boundary $S$ and can be realized as a bounded symmetric domain. The main idea is to formulate the Hua differential equations [432c] in terms of “polar coordinates” with respect to $S$.

Consider, again, a bounded symmetric domain $D$, $S$ its Shilov boundary and let $S(z,u)$ be the Szegő kernel on $D \times S$. Hua [352c] was the first to calculate explicitly the expression of $S(z,u)$ for each of the four series of irreducible
domains. Later Korányi gave a general proof that made Hua’s case-by-case calculations unnecessary. However, Korányi’s proof is not direct; it uses the unbounded realization of $D$ as “generalized half-plane”; in unbounded realization the Szegö kernel is given by an integral, which could be calculated by methods of Bochner and Gindikin. In bounded realization, on the other hand, the Szegö kernel is given by a Fourier series, and for this no methods of calculation had been devised.

Lassalle’s result [432h] offers a solution to this problem: he manages to calculate the Fourier series defining $S(z,u)$ directly, without going through the unbounded realization of $D$. In fact, Lassalle is in position to solve the following much more difficult problem: For every positive real number $\lambda$, what is the Fourier series expansion of $(S(z,u))^\lambda$? This difficult problem had been open for nearly thirty years. The only known solution had been given implicitly by Hua [352c, p. 25] in the particular case of an irreducible domain of type $I_{n,m}$. But the solution was unknown in all the other cases, including each of the three other series of irreducible domains.

Lassalle’s goal in [432h] is to present a general answer to this problem, one independent of any classification argument. What is noteworthy in [432h] is that the framework and tools of Lassalle’s proof are provided by Jordan algebra theory. In particular, his central result is a “binomial formula” in the complexification of a formally real Jordan algebra. The solution thus obtained is particularly simple and natural.

3. In a series of papers, Clerc and Ørsted [177a, b, c], Clerc [174b, c], Clerc & Neeb [176], and Clerc and Koufany [175], defined and studied the generalization of Maslov index by making use of formally real Jordan algebras (and their complexifications), as well as of Hermitian Jordan triple systems, which turn out to be very convenient for stating and proving the results. Some details on these papers are pointed out in Iordănescu [364w, pp. 46-47].

Another paper by Clerc, which does not belong to this series, but uses Jordan algebras as main tool is [174d].

4. Whereas the study of Toeplitz operators for the strongly pseudoconvex domains uses methods of partial differential equations, their structure and Toeplitz $C^\ast$-algebras over symmetric domains is closely related to the Jordan algebraic structure underlying these domains (see Upmeier [682], Section 2). The relation between bounded Toeplitz operators and Weyl operators of boson quantum mechanics was examined by Berger and Coburn in [90].

As Upmeier pointed out in [682], p. 42] even though finite-dimensional bounded symmetric domains have been classified and their geometry is totally understood, there are still many open problems concerning their analysis (i.e.,
the structure of function spaces defined over these domains). Since symmetric domains are homogeneous under a (semisimple) Lie group, these problems are related to harmonic analysis and the theory of group representations. On the other hand, the occurring function spaces are often Hilbert spaces of holomorphic functions which give rise to (reproducing) kernel functions. These kernel functions can be defined in terms of certain basic “norm functions” derived from the Jordan algebraic structure.

Ion and Scutaru [363], and Ion [362d, e] introduced new scattering theories via optimal states. These states are reproducing kernels in the Hilbert spaces of scattering matrices, just as the coherent states are reproducing kernels in the Hilbert spaces of wave functions.

Using reproducing kernels, Upmeier [682j, Section 6] outlined a quantization procedure for certain curved phase spaces of possibly infinite dimension, namely the “symmetric Hilbert domains”. In the finite-dimensional setting, Berezin [88a, b] has considered quantizations for more general complex (Kähler) manifolds.

The general formalism for quantum fields on any reproducing kernel Hilbert space is presented by Schroeck [601], along with a discussion of the operator and distribution properties of those fields. The Galilean and Poincaré examples are given along with considerations of the general relativistic cases.

Let us firstly recall that if \( G \) is a semi-simple Lie group, and \( \sigma : G \to U(H_\sigma) \), \( \tau : G \to U(H_\tau) \) the two (irreducible) unitary representations of \( G \), acting on Hilbert space \( H_\sigma \) and, respectively, \( H_\tau \), a unitary isomorphism \( \Phi : H_\sigma \to H_\tau \) is called an intertwining operator if the identity

\[
\Phi \sigma(y) = \tau(y)\Phi
\]

holds for all \( y \in G \).

**Remark 17.** The construction of intertwining operators (e.g., Poisson integral) is an important method in harmonic analysis.

In the Jordan theoretic framework, the Jordan determinant function and the associated differential operator give rise to intertwining operators of the group \( G \) of all biholomorphic automorphisms of a tube domain, which generalize the Capelli identity from classical invariant theory.

**Remark 18.** The Capelli identity (see Capelli [153]) was a centerpiece of the invariant theory in the 19\(^{th}\) century.

In 1991, Kostant & Sahi [414a] have proved a large class of identities generalizing the Capelli identity. Their approach – which have led to the above
mentioned generalization – was to regard the \( n^2 \)-dimensional space \( M(n, \mathbb{R}) \) of \( n \times n \) real matrices not as a Lie algebra, but rather a Jordan algebra (with the Jordan product \( a \circ b := (ab + ba)/2, \ a, b \in M(n, \mathbb{R}) \)).

In 1992, Sahi published the paper [583b] devoted to Capelli identity and unitary representation, essentially based on Kostant & Sahi [414a].

In 1993, Kostant & Sahi [414b] – making use of their previous paper [414a] and Sahi’s paper [583b] – have established a connection between semisimple Jordan algebras and certain invariant differential operators on symmetric spaces. They also proved an identity for such operators which generalizes the classical Capelli identity.

In 1995, Ørsted & Zhang [525] have studied the composition series of certain generalized principal series representations of the automorphism group of a bounded symmetric space of tube type. As applications they obtained new proofs of the Capelli identity of Kostant & Sahi [414a] and some results of Faraut & Korányi [246a,b], and they gave the full decomposition of the \( L^2 \)-space on the Shilov boundary. The Shilov boundary of a tube domain can also be viewed as a compactification of a formally real Jordan algebra, the automorphism group is then the conformal group of the formally real Jordan algebra. Zhang has studied in [730] the case of non-formally real Jordan algebra. (N.B. After the main part of [730] was completed, Zhang was informed by Sahi that he has also obtained in [583d] most of the results contained in [730].)

A related construction using the quasi-inverse in Jordan triples leads to the transvectants introduced by Peetre in his study of Hankel forms of arbitrary weight over symmetric domains [533]. The word transvectant (in German: Überschiebung) comes from classical invariant theory, where objects called transvectants were defined by Gordan (see P. Gordan, Invariantentheorie, Teubner, Leipzig, 1887). Janson & Peetre [372] have “rediscovered” it hundred years later! Let us recall that classical invariant theory is mainly about \( SL(2, \mathbb{C}) \). The problem of Peetre in [533] was thus to generalize the transvectant to the case of an arbitrary semi-simple Lie group. There are three types of approaches to symmetric domains, namely: 1) the case by case study (see Hua [352c]), 2) the Lie approach (see Helgason [335a,b]), and 3) the Jordan approach (see Loos [448h] and Upmeier [682j]). Peetre [533] used the approach of type 3), taking some advantage of the Jordan triple system structure.

In 2004, Peng & Zhang [534], studying the tensor products of holomorphic representations and bilinear differential operators, gave the irreducible decomposition of the tensor product of the representations for any two un-
tary weights and they have found the highest weight vectors of the irreducible components. Peng & Zhang have also found – by using some refinements of the ideas of Peetre [533] – certain bilinear differential intertwining operators realizing the decomposition, and they generalize the classical transvectants in invariant theory of $SL(2, \mathbb{C})$.

The Kirillov orbit method gives a realization of irreducible representations of a Lie group $G$ in terms of the orbits of $G$ in the dual space $\mathfrak{g}^\ast$ of the Lie algebra $\mathfrak{g}$.

Representations of $G$ corresponding to nilpotent orbits are called unipotent. The explicit realization of unipotent representations (in the semisimple case) is an important, and not completely solved, problem in harmonic analysis. For details, we refer the reader to Vogan [695].

In the Jordan theoretic framework, decisive progress in this direction has been made by Sahi (in collaboration with Dvorsky) (see [583a,c,d], [225a,b]).

**Remark 19.** The Jordan algebra determinant and its powers is – again – the starting point of this construction.

The paper [225f] by Dvorsky and Sahi is a culmination of a series of papers dedicated to the problem of constructing explicit analytic models for small unitary representations of certain semisimple Lie groups. Let us remark here that – for instance – Sahi [583a], studying explicit Hilbert spaces for certain unipotent representations, has made use of some Jordan algebra results from Braun & Koecher [131], Koecher [408c], Korányi & Wolf [412], and Kostant & Sahi [414a], while the paper Dvorsky & Sahi [225a] is devoted to non-formally real Jordan algebra case.

In the paper Dvorsky & Sahi [225b], the authors construct a family of small unitary representations for real semi-simple Lie groups associated with Jordan algebras. These representations are realized on $L^2$-spaces of certain orbits in the Jordan algebra. The representations are spherical and one of the authors’ key results is a precise $L^2$-estimate for the Fourier transform of the spherical vector. The authors also consider the tensor products of these representations and describe their decomposition.

**Notations.** Let $G$ be a simple Lie group with Lie algebra $\mathfrak{g}$, and let $K$ be the maximal compact subgroup corresponding to a Cartan involution $\theta$. Suppose $G$ has a parabolic subgroup $P = LN$ such that:

(i) the nilradical $N$ is abelian, and
(ii) $P$ is conjugate to $\overline{P} = \theta(P)$.

As Sahi has mentioned [583d, p. 1], the spherical (degenerate) $\overline{P}$-principal series representations of $G$ are obtained by starting with a positive real
character of $L$, extending trivially to $\overline{P}$, and inducing up to $G$. For such a representation, (i) implies that the $K$-types have multiplicity 1 and (ii) implies that each irreducible constituent has an invariant Hermitian form.

In the paper [583d], Sahi has provided a rather detailed analysis of these representations. He has explicitly determined the $K$-types of their irreducible constituents and the signature of the Hermitian form on each $K$-type.

**Remark 20.** The results contained in the paper [583d] extend and generalize those of the previous papers by Sahi [583b] and [583c], as well as those from several other papers.

**Remark 21.** The unitary representations described in Theorems 4B and 5A from [583d] are of particular interest since they are all unipotent and correspond to some of the smallest nilpotent coadjoint orbits of the group $G$. As Sahi has mentioned [583d, p. 2], it is expected that these representations (and their analogs over other fields) will have some interesting applications.

Concerning the applications of Jordan structures to analysis I like to refer the reader to the very recent book [166i] by Chu, not yet published, but available from January 2012.

§4. JORDAN STRUCTURES
IN DIFFERENTIAL GEOMETRY

Let $\mathcal{A}$ be a formally real Jordan algebra of dimension $n$. Theorem 3.4 from Braun & Koecher’s book [131, Ch. XI] implies that $\mathcal{A}$ has a unit element, which we shall denote by $e$.

In this case, by Proposition 6 from §1, we have

$$\text{Idemp}_1(\mathcal{A}) = \{c \mid c \in \text{Idemp}(\mathcal{A}), \ c \text{ primitive}\}.$$

**Definition 1.** A system of idempotents $c_1, \ldots, c_s \in \mathcal{A}$ is called a complete orthogonal system of idempotents of $\mathcal{A}$ if $\sum_{i=1}^{s} c_i = e$ and $c_i c_j = \delta_{ij} c_i$ ($i, j = 1, \ldots, s$).

**Proposition 1.** A formally real Jordan algebra contains a complete orthogonal system of primitive idempotents.
**Comments.** Tillier [662] gave a geometric characterization of primitive idempotents in a formally real Jordan algebra, namely: every primitive idempotent belongs to an extremal ray of the domain of positivity of the algebra, and, conversely, such a ray always contains a primitive idempotent.

**Proposition 2.** All complete orthogonal systems of primitive idempotents of a formally real Jordan algebra have the same number of elements.

**Definition 2.** The number of elements of a complete orthogonal system of primitive idempotents of a formally real Jordan algebra \( A \) is called the **degree of** \( A \).

In 1965, Hirzebruch [346] showed that the set of primitive idempotents in a **finite-dimensional** simple formally real Jordan algebra is a compact Riemannian symmetric space of rank one and that any such space arises in this way.

Ten years later, Neher undertook in [512a] a detailed differential-geometric study of idempotents in a **real** Jordan algebra.

Let us recall from Hirzebruch [346] some of his important results.

Suppose that \( A \) is **simple** and denote its degree by \( s \). Then the form

\[
\mu(u) := \frac{n}{s} \text{Tr} L(u), \quad u \in A,
\]

is an **associative** (i.e., \( \mu(xy) = \mu((xy)z) \) for any \( x, y, z \in A \)) **linear form** on \( A \) with \( \mu(c) = 1 \) for every \( c \in \text{Idemp}_1(A) \).

**Remark.** Suppose that a formally real Jordan algebra is not simple. Then it is semisimple (therefore it is a sum of simple ideals), and the associative linear form \( \mu \) with value 1 on the primitive idempotents is constructed by means on the forms \( \mu_i \) on the components.

**Notation.** For every \( c \in \text{Idemp}_1(A) \) define \( S_c \) by

\[
S_c := \{ x \mid x \in A_{1/2}(c), \mu(x^2) = 2 \}.
\]

**Theorem 3.** Let \( A \) be a simple formally real Jordan algebra and let \( c \in \text{Idemp}_1(A) \). For every \( d \in \text{Idemp}_1(A) \) there exists a unique real number \( t \), \( 0 \leq t \leq \pi/2 \), and a unique element \( x \) in \( S_c \) such that \( d = \text{d}(t) \), where

\[
d(t) = (\cos 2t)c + \left( \frac{1}{2} \sin 2t \right) x + \frac{1}{2}(1 - \cos 2t)x^2.
\]

Conversely, for each such \( t \), \( \text{d}(t) \) is an element of \( \text{Idemp}_1(A) \). The primitive idempotents which are orthogonal to \( c \) are exactly those of the form \( x^2 - c \) with \( x \in S_c \). For \( x \in S_c \), \( x^2 = c + d \) if and only if \( x \in A_{1/2}(c) \cap A_{1/2}(d) \).
Corollary. A formally real Jordan algebra is simple if and only if the set of its primitive idempotents is connected.

Theorem 4. Let \( A \) be a simple formally real Jordan algebra and let \( c \) be an element of \( \text{Idemp}_1(A) \). For \( x, y \in S_c \) there exists a product of Peirce reflections with respect to idempotents of \( \text{Idemp}_1(A) \) that fixes \( c \) and maps \( x \) to \( y \).

Theorem 5. Let \( A \) be a simple formally real Jordan algebra and let \( c_1, c_2, d_1, d_2 \in \text{Idemp}_1(A) \) such that \( \mu(c_1c_2) = \mu(d_1d_2) \). Then there exists a product of Peirce reflections with respect to idempotents of \( \text{Idemp}_1(A) \) which maps \( c_1 \) to \( d_1 \) and \( c_2 \) to \( d_2 \).

Consider now on \( A \) the symmetric bilinear form \( \mu(x, y) := \mu(xy) \) and the Euclidean metric \( \rho_E(x, y) := (\mu((x - y)^2))^{1/2} \) it determines.

Remark. The automorphisms of \( A \) are isometries of the metric space \((\text{Idemp}_1(A), \rho_E)\).

Definition 3. A metric space \((M, \rho)\) is called two-point homogeneous if there exists an isometry \( \iota \) of \( M \) such that \( \iota(c_1) = d_1 \) and \( \iota(c_2) = d_2 \) for any \( c_1, c_2, d_1, d_2 \in M \) with \( \rho(c_1, c_2) = \rho(d_1, d_2) \).

Corollary of Theorem 5. If \( A \) is a simple formally real Jordan algebra, then \((\text{Idemp}_1(A), \rho_E)\) is a connected, compact and two-point homogeneous metric space.

Theorem 6. Let \( A \) be a simple formally real Jordan algebra and let \( T \) be a one-to-one map of \( \text{Idemp}_1(A) \) onto itself such that \( \mu(T(c), T(d)) = \mu(c, d) \) for all \( c, d \in \text{Idemp}_1(A) \). Then \( T \) can be extended uniquely to an automorphism of \( A \).

Notation. Consider the Riemannian structure induced on \( \text{Idemp}_1(A) \) by \( \mu(xy) \). The Riemannian manifold thus obtained will be denoted by \((\text{Idemp}_1(A), R)\).

Remarks. The automorphisms of \( A \) are isometries of the Riemannian manifold \((\text{Idemp}_1(A), R)\). \((\text{Idemp}_1(A), R)\) is a compact symmetric Riemannian space.

Notation. The Riemannian distance between two elements \( c \) and \( d \) of \((\text{Idemp}_1(A), R)\) will be denoted by \( \rho_R(c, d) \).

Since the relations

\[ 0 \leq \rho_R(c, d) \leq \pi/\sqrt{2} \quad \text{and} \quad \rho_E(c, d) = \sqrt{2} \sin \left( \frac{1}{\sqrt{2}} \rho_R(c, d) \right) \]
hold, it follows that $\rho_R(c_1, c_2) = \rho_R(d_1, d_2)$ is equivalent to $\mu(c_1, c_2) = \mu(d_1, d_2)$ for $c_1, c_2, d_1, d_2 \in \text{Idemp}_1(A)$.

**Remark.** Consequently, $(\text{Idemp}_1(A), R)$ is a two-point homogeneous symmetric Riemannian space and hence (see Helgason [335a, p. 355]) of rank one.

Let $c$ be an element of $\text{Idemp}_1(A)$. Clearly, $\rho_R(c, d)$, $d \in \text{Idemp}_1(A)$, is maximal only when $\rho_E(c, d)$ is maximal. Because $\mu(c) = \mu(d) = 1$, we have $\rho_E(c, d) = \sqrt{2\sqrt{1 - \mu(cd)}}$, which is maximal only when $\mu(cd) = 0$, i.e., when $cd = 0$.

**Notation.** For every $c \in \text{Idemp}_1(A)$ we define

$$A_c := \{d \mid d \in (\text{Idemp}_1(A), R), cd = 0\}.$$

**Remark.** By Theorem 3 we have $A_c = \{x^2 - c \mid x \in S_c\}$.

**Remark.** $A_c$ is a submanifold of $(\text{Idemp}_1(A), R)$ and is called the antipodal manifold of $c$.

**Notation.** For any simple formally real Jordan algebra $A$ of dimension $n > 1$ there exists a natural number $q(A)$ such that, for every pair of orthogonal primitive idempotents $c_1, c_2 \in A$, the relation $\dim(A_{1/2}(c_1) \cap A_{1/2}(c_2)) = q(A)$ holds. If $s = s(A)$ denotes the degree of $A$, then $A$ is said to be of type $(s, q(A))$.

**Remark.** If $A_1$ and $A_2$ are simple formally real Jordan algebras with $s(A_1) = s(A_2)$ and $q(A_1) = q(A_2)$, then $A_1$ and $A_2$ are isomorphic.

**Comments.** It would be interesting to extend Hirzebruch’s results [346, pp. 350–351] on Betti numbers of $\text{Idemp}_1(A)$, critical points of differentiable functions on $\text{Idemp}_1(A)$, etc., to other kinds of Jordan algebras.

Using the well-known classification of compact symmetric Riemannian spaces of rank one, Hirzebruch [346] proved that each of these spaces can be described in terms of a suitable formally real Jordan algebra, namely:

a) Type $(1, 0)$; $A = \mathbb{R}$, and $\text{Idemp}_1(A)$ consists of point alone.

b) Type $(2, q)$; $q \geq 1$. Let $V'$ be a $(q + 1)$-dimensional vector space over $\mathbb{R}$ and let $\sigma$ be a positive definite bilinear form on $V'$. Define $V := \mathbb{R}e \oplus V'$ a bilinear product by $uv := \sigma(u, v)e$ for $u, v \in V'$, $e$ being the unit element. It is immediate, that $V$ endowed with this product is a Jordan algebra $J(Q)$ (as in Theorem 8 from §1). $\text{Idemp}_1(V)$ is homeomorphic to the $q$-dimensional sphere $S^q$.

c) Type $(s, 1)$; $s \geq 3$. Let $V$ be the vector space of symmetric $(s \times s)$-matrices over $\mathbb{R}$. For $A, B \in V$, let $AB := \frac{1}{2}(A \cdot B + B \cdot A)$, where $A \cdot B$ denotes
the usual matrix product in \( V \). We have \( \mu(A) = \text{Tr} A \) and \( \text{Idemp}_1(V) = \{ A \mid A \in V, A^2 = A, \text{Tr} A = 1 \} \). It follows that \( \text{Idemp}_1(V) \) is homeomorphic with the real \((s - 1)\)-dimensional projective space \( P_{s-1}(\mathbb{R}) \). For \( c \in \text{Idemp}_1(V) \), the antipodal manifold \( A_c \) is \( P_{s-2}(\mathbb{R}) \).

d) Type \((s, 2)\); \( s \geq 3 \). Let \( V \) be the ordinary real vector space of complex Hermitian \((s \times s)\)-matrices. Define the product as in c). Then \( \text{Idemp}_1(V) \) is homeomorphic with \( P_{s-1}(\mathbb{C}) \). For \( c \in \text{Idemp}_1(V), A_c \) is \( P_{s-2}(\mathbb{C}) \).

e) Type \((s, 4)\); \( s \geq 3 \). Let \( V \) be the real vector space of Hermitian \((s \times s)\)-matrices over \( \mathbb{H} \). Define the product as in c). Then \( \text{Idemp}_1(V) \) is homeomorphic with \( P_{s-1}(\mathbb{H}) \). For \( c \in \text{Idemp}_1(V), A_c \) is \( P_{s-2}(\mathbb{H}) \).

f) Type \((3, 8)\). Let \( V \) be the real vector space of Hermitian \((3 \times 3)\)-matrices over \( \mathbb{O} \). Define the product as in c). Then \( \text{Idemp}_1(V) \) is homeomorphic with the projective octonion plane. For \( c \in \text{Idemp}_1(V), A_c \) is an eight-dimensional sphere.

**Remark.** The geodesics through a point \( c \in \text{Idemp}_1(A) \), as a set of points, are \( \text{Idemp}_1(A) \cap \mathcal{A}(c, x), x \in \mathcal{A}_{1/2}(c), \mathcal{A}(c, x) \) being a simple three-dimensional subalgebra of \( \mathcal{A} \) containing \( c \) (see Hirzebruch [346, p. 348]). For a detailed discussion on geodesics in a more general case see Neher [512a].

**Comments.** I must point out that the long series of applications of Jordan structures to differential geometry has its roots in 1962 lecture notes by Koecher [408c], and in the papers [387a, b] from 1964 and 1966 by Kantor (but published – at that time – *only* in Russian!).

In 1993, as a natural step of generalization of the just recalled Hirzebruch’s results, Nomura [518a] considered an *infinite-dimensional* analogue of formally real Jordan algebras and treated the set of primitive idempotents. As it is the associative inner product (instead of the algebraic formally real property what plays an important role in the study of Hirzebruch [346]), Nomura [518a] based his study on Jordan-Hilbert algebras (i.e., real Hilbert spaces with inner products \( \langle \cdot | \cdot \rangle \), which are also real Jordan algebras and \( \langle xy \mid z \rangle = \langle y \mid xz \rangle \) for all their elements \( x, y, z \)). He deals with the *set* \( J_1 \) of *primitive idempotents* as a Riemann Hilbert manifold. He proved that \( J_1 \) is two-point homogeneous and derived a unified formula for the sectional curvature of \( J_1 \) (see Nomura [518a, Th. 5.3 and, resp., Th. 6.4]).

Let us point out the matters which do not occur in the finite-dimensional cases.

Firstly, Jordan-Hilbert algebras do not necessarily have a unit element. On the other hand, the adjunction of unit element does not in general agree with the Hilbert space structure. However, this lack of unit element is compen-
Another way to determine the Peirce 1-spaces is also topologically simple (see Nomura [518a, Prop. 1.6]). This enables Nomura to carry out computations concerning idempotents as in the finite-dimensional cases.

Secondly, infinite-dimensional connected complete (in the sense of Riemannian distance) Riemannian manifolds may carry two points which cannot be joined by a minimal geodesic (see Grossman [310] and Klingenberg [402b, p. 127]). Because of this possibility of missing minimal geodesic, Nomura computed the Riemannian distance on the set $J_1$ along the standard line of textbooks on Riemannian geometry: the inclusion map of $J_1$ into the ambient Hilbert space being an embedding, Nomura defined the canonical Levi-Civita connection, exhibited a geodesic, examined the diffeomorphism domain of the exponential mapping and made use of Gauss lemma to derive the minimality of geodesic.

One year later, in 1994, Nomura [518b] extended his above mentioned study to the space $J_p$ of rank $p$ idempotents in a topologically simple Jordan-Hilbert algebra $V$. To provide the necessary differential-geometric structure on $J_p$, subalgebras with two idempotents generators are studied with the aid of the Peirce decomposition and associated Jordan-Hilbert systems. This allows the construction of an atlas and permits to Nomura to identify the tangent space at a point with the corresponding Peirce $\frac{1}{2}$-space. Geodesics in $J_p$ turn out to be orbits of one-parameter subgroups of $\operatorname{Aut} V$, which can be represented explicitly, and thus permit the computation of the Riemannian distance and sectional curvature.

In December 1997, Chu and Isidro [168] extended Nomura’s results to the manifold of extremal (minimal or maximal) projections in the complex $C^*$-algebra $\mathcal{L}(H)$ of bounded linear operators in a Hilbert space $H$. In 2002, Isidro and Mackey [368] have extended the results from [168] on the manifold of minimal projections in $\mathcal{L}(H)$ to the manifolds of finite rank projections in $\mathcal{L}(H)$.

A symmetric Riemannian space, defined as usual, is a Riemannian manifold such that the geodesic symmetry $S_x$ around every point $x$ is an isometry. By writing $x \cdot y := S_x(y)$, Loos has obtained a (nonassociative) multiplication on the manifold satisfying certain algebraic identities, which in turn suffice to characterize symmetric spaces. In this way, one obtains an elementary “algebraic” definition of a symmetric space not involving the manifold structure of
the underlying topological space. This definition was firstly given by Loos in his remarkable Ph.D. Thesis [448a].

**Definition 4.** (see Loos [448a]). A manifold $\mathcal{M}$ with a differentiable multiplication $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$, denoted $(x, y) \to x \cdot y$, and having the properties

1. $x \cdot x = x$;
2. $x \cdot (x \cdot y) = y$;
3. $x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z)$;
4. every $x$ has a neighbourhood $U$ such that $x \cdot y = y$ implies $y = x$ for all $y$ in $U$,

is called a **symmetric space**.

**Note.** In Loos’ terminology, a manifold is a differentiable manifold of class $C^\infty$ which is Hausdorff and paracompact as a topological space. It may have several connected components, which may be of different (yet finite) dimensions.

**Comments.** In contemporary mathematics there exists the notion of **quandle** (related to knot theory), introduced in 1982 by Joyce [381]. Symmetric spaces in the sense of Loos are important examples of quandles (see - for instance - the paper by Buliga [143]).

**Remark.** Spaces satisfying only (1), (2), and (3) (“reflection spaces”) have been studied by Loos in [372b]. They turn out to be fibre bundles over symmetric spaces (see, for instance, Neher [512a]).

**Note.** As Vanhecke has pointed out to me, it seems that all the globally KTS-spaces studied by himself together with J.C. and M.C. González-Dávila are examples of reflection spaces. In fact, two years later, they have mentioned: “It is worthwhile to note that the globally KTS-spaces provide a large class of examples of reflection spaces.” (see [299, p. 322]).

**Definition 5.** Left multiplication by $x$ in $\mathcal{M}$ is denoted by $S_x$, i.e., $S_x(y) = x \cdot y$ for all $x, y \in \mathcal{M}$, and is called **symmetry around** $x$.

**Remark.** The following properties are immediate:

(i) $x$ is an isolated fixed point of $S_x$;
(ii) $S_x$ is an Involutive automorphism of $\mathcal{M}$.

**Examples of symmetric spaces** (in the sense of the above **Definition 4**): Lie groups, spheres, Grassmann manifolds, Jordan algebras, homogeneous spaces and spaces of symmetric elements.

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11Private communication on February 21st, 1994.
In 1981, Lutz [451] introduced the notion of \( \Gamma \)-symmetric spaces which is a generalization of the classical notion of symmetric space. In 2008, Bahturin and Goze [64] defined \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-symmetric spaces, and very recently Kollroos [410] gave a classification of the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-symmetric spaces \( G/K \), where \( G \) is an exceptional compact Lie group or \( \text{Spin}(8) \), complementing recent results of Bahturin and Goze.

Let \( A \) be a real Jordan algebra of dimension \( n \) and with unit element \( e \).

**Notation.** The mutation of \( A \) with respect to \( q^{-1}, q \in \text{Inv}(A) \), will be denoted by \( A^q \), \( \text{Inv}(A) \) denoting the set of invertible elements of \( A \).

**Remarks.** The product of two elements \( a, b \in A^q \) is given by \( a \perp b = a(bq^{-1}) + b(aq^{-1}) - (ab)q^{-1} \). Propositions 3 and 4 from §1 imply that the mutation \( A^q \) is a Jordan algebra with unit element \( q \), and that the quadratic representation \( P_q \) of \( A^q \) is given by \( P_q(a) = P(a)P^{-1}(q), a \in A^q \). One can also see that \( \text{Inv}(A^q) = \text{Inv}(A) \) and \( \Gamma(A^q) = \Gamma(A) \). The connected component of \( \text{Inv}(A) \) containing \( e \), denoted as usually by \( \text{Inv}^0(A) \), with the multiplication \( q \cdot p := P(q)p^{-1} \) becomes a symmetric space in the sense of Definition 4.

Suppose now that \( A \) is endowed with an involution \( J \) (i.e., \( J \in \text{Aut}(A) \), \( J^2 = \text{Id} \)).

**Notation.** Write \( \text{Inv}(A, J) := \{ a \mid a \in \text{Inv}(A), a^{-1} = Ja \} \) and denote by \( \text{Inv}^0(A, J) \) the component of \( \text{Inv}(A, J) \) containing \( e \). For every \( q \) of \( \text{Inv}(A, J) \) define \( J_q := P(q)J \).

**Definition 6.** In the real vector space \( A \) we define a new product of any two elements \( a, b \in A \), denoted by \( a \ast b \), as follows

\[
2(a \ast b) := ab + a(J(b)) + b(J(a)) - J(ab)
\]

and the real algebra defined by means of \( \ast \) in the vector space \( A \) will be denoted by \( A_J \).

**Remark.** The fact that \( A \) is a Jordan algebra implies that \( A_J \) is also a Jordan algebra, \( J \in \text{Aut}(A_J) \), \( (A_J)_J = A \) and \( \lambda_J \) (defined by \( \lambda_J(a, b) := \lambda(a, J(b)) \)) is the trace form of \( A_J \).

**Theorem 7.** (Helwig [337b]) All Grassmann manifolds, as well as all compact symmetric spaces of rank one, are contained in the form \( \text{Inv}^0(A_J, J) \), where \( A \) is a simple formally real Jordan algebra and \( J \) is a Peirce reflection of \( A \). The noncompact spaces associated with the above mentioned spaces have the form \( \text{Inv}^0(A, J) \).
The description of important symmetric spaces due to Helwig [337b] embraces many other earlier descriptions of symmetric spaces using Jordan algebras and triple systems as those of Braun & Koecher [131], Hirzebruch [346], Koecher [408h]. For a review of them see Iordănescu [364w, Ch. 3].

Let us recall here the simple but very ingenious description given by Koecher [408h]: Let A be a formally real Jordan algebra, and suppose its trace form \( \lambda \) nondegenerate. Then the (not necessarily positive definite) line element \( ds^2 := \lambda(\dot{x}, P(x^{-1})\dot{x})dt^2 \), where \( x = x(t) \) is a curve in Inv(\( A \)), is invariant under the maps \( x \to Wx \), \( W \in \Gamma(\mathcal{A}) \), and \( x \to x^{-1} \). In order to discuss the induced (pseudo-) Riemannian structure, it suffices to consider Inv\( _e(\mathcal{A}) \) (indeed, if \( C \) is a connected component of Inv(\( A \)), then there exists an \( f \in C \) such that \( f^2 = e \) and \( C = \text{Inv}_e(\mathcal{A}_f) \)). Then Inv\( _e(\mathcal{A}) \) is a symmetric Riemannian space and:

a) at the point \( e \) the geodesic symmetry is the inversion \( x \to x^{-1} \), and the exponential map is \( \exp_e(x) = \exp x \) and

b) the coefficients of the affine connection coincide with the structure constants of \( \mathcal{A} \).

With the “metric” \( ds^2 \), Inv\( _e(\mathcal{A}, \mathcal{J}) \) is a regular analytic submanifold of Inv(\( A, \mathcal{J} \)) and pseudo-Riemannian and symmetric.

**Comments.** Since 1966, independently of Koecher’s research, Iordănescu, Popovici, and Turtoi, following a suggestion of their professor Gheorghe Vrânceanu, studied the spaces (Riemannian or pseudo-Riemannian) associated with various kinds of real Jordan algebras (see [364a, b, c, d], [546a], [547], [548], [676a]). The above recalled construction of Koecher [408h] from 1970 gave a subsequent proof of the existence of the spaces studied by Iordănescu-Popovici-Turtoi. For details and also for open problems, see Iordănescu [364i, t].

In 1999, Shima (see SHIMA, H., *Homogeneous spaces with invariant projectively flat affine connections*, Trans. Amer. Math. Soc. 351 (1999), 12, 4713–4726) showed that semi-simple symmetric spaces with invariant projectively flat affine connections correspond to central-simple Jordan algebras, and are described as centro-affine hypersurfaces in the algebras. He also proved that Riemannian semisimple symmetric spaces with invariant projectively flat affine connections correspond to simple formally real Jordan algebras (see also [490]).

**Comments.** It would be interesting to connect the above recalled Shima’s results with Iordănescu-Popovici-Turtoi results (mentioned in the previous Comments).

Making use of Koecher’s conference [408h], let us summarize Helwig’s
Theorem 8. \( \text{Inv}_v(A, J) \) is a totally geodesic submanifold of \( \text{Inv}_v(A) \). In case the pair \((A, J)\) is simple (i.e., \(A\) contains no proper \(J\)-invariant ideals), \( \text{Inv}_v(A, J) \) is an Einstein space if and only if \(A\) is central simple.

Forty years ago (during the universitary year 1971-1972), following the suggestion of the late Prof. Dr. Enzo MARTINELLI (University “La Sapienza” of Rome, Italy), I have studied quaternionic Grassmann structures\(^{12}\) by using Hangan’s previous results (see HANGAN, Th., Tensor product tangent bundles, Arch. Math. (Basel), 19 (1968), 4, 436-440), and Pontryagin’s local coordinates (see PONTRYAGIN, L.S., Characteristic cycles on differentiable manifolds, Mat. Sb. 21 (63) (1947), 233-284; Amer. Math. Soc. Transl. 32 (1950)). Making use of my results, Marchiafava extended his previous results from 1970 (see Marchiafava [469a]) - see Marchiafava [469b]. Let us point out – from Alekseevsky & Marchiafava [10b, p. 15] – that “... the quaternionic Grassmannian carries a geometrical structure that is a natural generalization of the quaternionic structure (...); it would be interesting to examine more deeply such structures, which are integrable in case of Grassmannians, and find other examples”. (For a survey on this topic see Alekseevsky & Marchiafava [10a, b, c, d, e], Marchiafava [469c], and the references therein.). In order to be able to remember here two interesting open problems (pointed out by me - in January 1994 - in my lectures given at Katholieke Universiteit Leuven, Belgium), let us briefly recall some facts from quaternionic geometry.

Let \( L^H_n \) (resp. \( U^H_n \)) be the homogeneous linear (resp. unitary) quaternionic group acting on the left in the right quaternionic vector space \( H^n \) of dimension \(n\).

Denote by \( \Theta := (\theta^\alpha) \) the matrix of a vector \( \theta \) of \( H^n \) and by \( A := (a_{ij}^\alpha) \) an \((n \times n)\)-matrix over \( H \). Then a transformation \( T \) of \( L^H_n \) (resp. \( U^H_n \)) is

\[
T : \Theta \to A\Theta,
\]

where \( A \) is an invertible (resp. unitary) matrix.

**Definition 7.** A real differentiable manifold \( V_{4n} \) is called endowed with (a right) almost (resp. almost Hermitian) quaternionic structure if its structure group is \( L^H_n \) (resp. \( U^H_n \)).

Recall that a \( G \)-structure on a real differentiable \( m \)-dimensional manifold \( V_m \) is defined by a subbundle with the structure group \( G \) of the tangent bundle

\(^{12}\)IORDANESCU, R., On Grassmann quaternionic structures (in italian), Boll. Un. Mat. It. (4) 10(1974), 406-411.
\(T(V_m, R^n, L_m)\), where \(G\) is a certain subgroup of \(L_m\) (one assumes - of course - \(R^{4m} \equiv H^n\), canonically).

**Definition 8.** (see Martinelli [471a]) A real differentiable manifold \(V_{4n}\) is called endowed with a *generalized* (right) almost (resp. almost Hermitian) quaternionic structure if its structure group is \(\tilde{L}_n^H\) (resp. \(\tilde{U}_n^H\)), which consists of all transformations \(T : \Theta \rightarrow A\Theta b\), where \(A\) is an invertible matrix and \(b \in H - \{0\}\) (resp. \(A\) is unitary matrix and \(b\) = 1).

**Example.** The quaternionic projective plane (which suggested to Martinelli the above definition).

**Question:** Do the quaternionic Grassmann manifolds have the same structure? **Answer:** No.

I have proved that their structure group \(\mathcal{G}\) consists of all transformations

\[T : \Theta \rightarrow (A \otimes Id_q)\Theta (B \otimes Id_p),\]

where \(A\) and \(B\) are invertible quaternionic matrices of order \(p\), resp. \(q\). So, the notion of *locally Grassmann quaternionic manifold* arised and Marchiafava studied them later.

**Geometrical open problem.** Define differentiable manifolds endowed with structure whose structure groups be similar to \(\mathcal{G}\), but for which the factors \((A \otimes Id_q), (B \otimes Id_p)\) be replaced by unitary or invertible matrices.

**Algebraic open problem.** What kind of algebras could describe the new differentiable manifolds from the above geometrical open problem.

It is worth to be mentioned here that recently Dubois-Violette - a specialist in noncommutative geometry - discovered the importance of (real) Jordan algebras for his field of research\(^{13}\). For instance, in his lecture on noncommutative differential geometry [223] he has indicated that... “instead of taking Hermitian elements of *-algebras as the analogues real functions it would be more general (and radical) to take elements of real Jordan algebras”.

There are two main topics of differential geometry related to Jordan triple systems: symmetric \(R\)-spaces and hypersurfaces in spheres. Let us mention in this respect that some of Cartan’s results are now-being successfully re-examined against the background of the theory of Jordan triple systems.

\(R\)-spaces constitute an important class of homogeneous submanifolds in the Euclidean spheres. This class includes many examples appearing in differ-

\(^{13}\)Private communication on March 2010.
ential geometry of submanifolds. For example, all homogeneous hypersurfaces and all parallel submanifolds in spheres are realized as $R$-spaces.

Ferus [267] has characterized the $R$-spaces as compact symmetric submanifolds of Euclidean spaces.

The connection between Jordan algebras and symmetric $R$-spaces was first illuminated by Kantor, Sirota and Solodovnikov [389a], Koecher [408e, II], and Loos [448d]. Then, significant results were obtained by Makarevich (see [462a, b, c]) and Rivilis (see [558]).

In 1996, starting from the paper [462a], Bertram [98c] determined the causal transformations of a class of causal symmetric spaces (see [98c, Th. 2.4.1]). As a basic tool he used causal imbeddings of these spaces as open orbits in the conformal compactification of formally real Jordan algebras. Firstly, he gave elementary constructions of such imbeddings for the classical matrix-algebras. Then he generalized these constructions for arbitrary semisimple Jordan algebras: he introduced Makarevich spaces (which are open symmetric orbits in the conformal compactification of a semisimple Jordan algebra) and described examples and some general properties of them which are the starting point of an algebraic and geometric theory developed by Bertram in [98c].

In fact, in the above mentioned paper, Bertram generalizes features of bounded symmetric domains to a bigger class of symmetric spaces (i.e., the above mentioned Makarevich spaces): he associates a generalized Bergman operator to such a space and describes the invariant pseudo-metric and the invariant measure on the space by means of this family of operators. The space itself can be characterized essentially as the domain where the generalized Bergman operator is nondegenerate. These results are then applied to the theory of compact causal symmetric spaces.

Generalizing Hermitian and pseudo-Hermitian spaces, Bertram defined in [98d, I] twisted complex symmetric spaces, and showed that they correspond to an algebraic object called Hermitian Jordan triple products. He investigated the class of real forms of twisted complex symmetric spaces called the category of symmetric spaces with twist. Then he showed that this category is equivalent to the category of all real Jordan triple systems, and, using Makarevich [462a], classified the irreducible spaces. The classification shows that most irreducible symmetric spaces have exactly one twisted complexification. This leads to open problems concerning the relation of Jordan and Lie triple systems. Using a geometric approach, Bertram [98d, II] defined and investigated the conformal group of a symmetric space with twist. In the non-degenerate case he characterized this group by a theorem generalizing the Fundamental Theorem of Projective Geometry.
**Definition 9.** (see Takeuchi [655]). A symmetric $R$-space is a compact symmetric space on which there exists a group of transformations containing the group of motions as a proper subgroup (see also Nagano [504]).

There exists a one-to-one correspondence between compact Jordan triple systems and symmetric $R$-spaces, as was established by Loos [448d]; see Theorem 9 below.

The noncompact dual of a symmetric $R$-space can be realized as a bounded domain $D$ in a real vector space. Loos [448d] proved that there is a one-to-one correspondence between boundary component of $D$ and idempotents of the corresponding Jordan triple system.

A compact Jordan triple system $T$ becomes a Euclidean vector space with the scalar product $(x, y) := \text{Tr} L(x, y)$, where $L(x, y)z := P(x, z)y := P(x+z)y - P(x)y - P(z)y$ and, by Theorem 12 from §1, is equal to $\{xyz\}$. By the second equality in the definition of a Jordan triple system, the vector space $H$ spanned by $\{L(x, y) \mid x, y \in T\}$ is a Lie algebra of linear transformations of $T$, which is closed under taking transposes with respect to $(\cdot, \cdot)$. The contragradient $H$-module $T'$ of $T$ can thus be identified with $T$ as a vector space, and

$$X \cdot v' = -\bar{X}(v') \quad \text{for } X \in H \text{ and } v' \in T'.$$

The map $\tau : X \to \bar{X}, \ X \in H, \ v \to v'$, is a Cartan involution of the Lie algebra $L := T \oplus H \oplus T'$ and $\sigma|_H = \pm 1, \sigma|_{T \oplus T'} = -1$ defines an involutive automorphism $\sigma$ of $L$ commuting with $\tau$.

Recall that by a result of Koecher, $L = T \oplus H \oplus T'$ becomes a semisimple Lie algebra, adopting

$$[X, Y] := XY - YX, \quad [X, v] = -[v, X] := X \cdot v$$

for $X, Y \in H$ and $v \in T \cup T'$,

$$[T, T] := [T', T'] = 0, \quad [u, v'] := -2L(u, v)$$

for $u \in T$ and $v' \in T'$. Also, Koecher’s result states that $Z = -\text{Id}_T$ is an element of $H$, $(\text{ad} \ Z)^3 = \text{ad} \ Z$ and the $-1$-, $0$-, $+1$-eigenspaces of $\text{ad} \ Z$ are $T, \ H, \ T'$.

Let $L$ be the centre free connected Lie group with Lie algebra $L$, let $H$ be the centralizer of $Z$ in $L$, let $U$ be a maximal compact subgroup of $L$ determined by $\tau$, and let $K := U \cap H$. Then $K$ lies between the full set of fixed points of $\sigma$ in $U$ and the identity component of $U$. If we denote by $P$ the normalizer of $T$ in $L$, then $P$ is parabolic and $U/K \cong L/P$. It follows that $M := U/K$ is a symmetric $R$-space.
Theorem 9. The map $T \to M$ establishes a one-to-one correspondence between isomorphism classes of compact Jordan triple systems and symmetric $R$-spaces.

In his paper [387b], Kantor generalized the notions of Jordan triple system and symmetric space to cover more general cases of Riemannian manifolds. To this end he introduced (generalized) Jordan triple systems of second order and constructed an associated graded Lie algebra analogous to the Kantor-Koecher-Tits construction. The corresponding Lie triple system gives rise to “bisymmetric” spaces: Riemannian homogeneous fibre spaces with symmetric base and (locally) symmetric fibre. One also has duality theory for such spaces and can generalize the embedding of non-compact type into the compact dual. (For bisymmetric Riemannian spaces see Kantor, Sirota, Solodovnikov [389b].)

Several papers of Dorfmeister and Neher [217a, b, c] deal principally with isoparametric hypersurfaces in spheres and show that homogeneous examples with four distinct principal curvatures are closely related to certain Jordan triple systems. Then isoparametric triple systems of certain type are studied. For a detailed presentation of this topic see Iordănescu [364w, pp. 70-73].

Hulett and Sanchez [355] studied an algebraic structure, called “Euclidean double-triple systems” (because of their analogies with Euclidean Jordan triple systems), associated with a standard imbedding of an $R$-space. This structure determines completely the geometry of an $R$-space and reduces to a Jordan triple system if the $R$-space is symmetric.

Grassmann and flag manifolds associated with a Hermitian Jordan triple system were defined by Arazy & Upmeier in [40], and these differential-geometric objects were used to give a new proof for the intertwining formula generalizing “Bol’s Lemma” for symmetric domains which are not of tube type.

Tripotents are natural generalizations of partial isometries in $C^*$-algebras to the context of $JB^*$-triples that is complex Banach spaces with symmetric unit ball. A survey on the main results contained in Chu & Isidro [168], and Isidro & Stachó [369a, b, c] concerning the structure of the tripotents as a direct real-analytic submanifold in a $JB^*$-triple, as well as some recent achievements are presented in Stachó [635a] (see also Stachó [635b]).

Recently, Di Scala and Loi [210] have studied the symplectic geometry of Hermitian symmetric spaces of noncompact type and their compact dual. Using the theory of Jordan triple systems, they constructed an explicit symplectic duality (see Definition 10 below).

Definition 10. Let $M \subset \mathbb{C}^n$ be a complex $n$-dimensional Hermitian symmetric space endowed with the hyperbolic form $\omega_{hyp}$. Denote by $(M^*, \omega_{FS})$
the compact dual of \((M, \omega_{\text{hyp}})\), where \(\omega_{FS}\) is the Fubini-Study form on \(M^*\).

A **symplectic duality** is a diffeomorphism \(\Psi_M : M \to \mathbb{R}^{2n} = \mathbb{C}^n \subset M^*\) satisfying \(\Psi_M^*\omega_0 = \omega_{\text{hyp}}\) and \(\Psi_M^*\omega_{FS} = \omega_0\) for the pull-back of \(\Psi_M\), where \(\omega_0\) is the restriction to \(M\) of the flat Kähler form of the Hermitian positive Jordan triple system associated to \(M\).

Di Scala and Loi proved that the map \(\Psi_M\) takes (complete) complex and totally geodesic submanifolds of \(M\) through the origin to complex linear subspaces of \(\mathbb{C}^n\). They also get an interesting characterization of the Bergman form of a Hermitian symmetric space in terms of its restriction to classical complex and totally geodesic submanifolds passing through the origin.

More recently, Di Scala, Loi, and Roos [211] have determined the group of diffeomorphisms of a bounded symmetric domain, which preserve simultaneously the hyperbolic and the flat symplectic form.

In a very recent paper, Di Scala, Loi, and Zuddas [212] after extending the definition of symplectic duality (given by Di Scala and Loi in [210] for bounded symmetric domains) to arbitrary complex domains of \(\mathbb{C}^n\) centered at the origin, generalize some of the results proved in [210] and [163] to those domains.

**Comments.** It would be possible that the symplectic duality be useful in the next future to physics. At least, Antonio Di Scala [209] expects to be proved this fact.

I like to point out here recent papers of Roos and his collaborators (see Yin, Lu, Roos [719]; Roos [565c], Wang, Yin, Zhang, Roos [700]), as well as a more recent paper by Roos [565d] on bounded symmetric domains.

In a recent (accepted 4 August 2008) and very extended (29 pages) paper, C.-H.Chu [166h] introduced a class of real Jordan triple systems, called \(JH\)-triples, and showed (via the Tits-Kantor-Koecher construction of Lie algebras) that they correspond to a class of Riemannian symmetric spaces including the Hermitian symmetric spaces and the symmetric \(R\)-spaces.

Let us recall here that there are two essentially equivalent ways to study Hermitian symmetric spaces (via Tits-Kantor-Koecher construction, see Sat
take [592]), namely:

1) by using the semisimple Lie groups (it was in this way that the basic facts of the theory were established by Élie Cartan in the 1930’s and by Harish-Chandra in the 1950’s), and

2) by using Jordan structures (algebras and triple systems), this way being essentially due to Max Koecher and his school in the 1980’s.

In his lecture notes from 2005, Koufany [416a] used the Jordan theory to
point out some development in the geometry and analysis on Hermitian symmetric spaces. Let us point out here that the first part of these lecture notes in a nice survey about the geometry and the topology of some homogeneous spaces associated with formally real Jordan algebras: Hermitian symmetric spaces of tube type, their Shilov boundaries, and causal symmetric spaces of Cayley type. In particular, Koufany reviews recent results by Clerc, Ørsted and himself about Maslov, Souriau, and Arnold-Leray indices.

From recent applications of Jordan triple systems to differential geometry I must mention also the contributions of Kaneyuki [386] and Naitoh [506c].

In §4 of Chapter 3 from the book Iordănescu [364w], a discussion of Jordan structures that occur in some infinite-dimensional manifolds, for example, in symmetric Banach manifolds, is given. These are infinite-dimensional generalization of the Hermitian symmetric spaces classified by Cartan [154a] in the 1930’s using Lie theory. A connection of this classification to Jordan algebras was later pioneered by Koecher [408h]. Jordan triple systems first entered the scene when Loos [448h] showed the correspondence, in finite dimensions, between bounded circular domains and Jordan triples. The full generalization of this correspondence to that between infinite-dimensional symmetric Banach manifolds and Jordan triples was developed by Kaup in his outstanding papers [392b, d, e].

The classical Grassmann manifolds can be regarded as manifolds of projections in spaces of matrices. The infinite-dimensional analogues are the manifolds of projections in $C^*$-algebras. Since projections are tripotents, the concept of Grassmann manifolds can be extended to include the manifolds of tripotents in $JB^*$-triples (cf. [168], [347], [392a], [518a, b], [587]).

Let us remember the Bertram’s work up to 1999 (see Bertram [98h]): the framework is the one of classical, finite-dimensional real differential geometry, especially the theory of symmetric spaces. The aim was to define and study geometric objects (manifolds with additional structure) corresponding to real, finite-dimensional Jordan structures (pairs, triple systems, algebras).

As it is well known, in modern mathematics it is of fundamental importance the correspondence between Lie algebras and Lie groups: there is a functor which Bertram called the **Lie functor for Lie groups** assigning to a Lie group its Lie algebra, and every (real, finite-dimensional) Lie algebra belongs to some Lie group. One may ask whether there is also a **Jordan functor**: can we find a “global” or “geometric” object to which a given Jordan algebra is associated in a similar way as a Lie algebra to a Lie group?

As in the case of Lie structures one may ask whether there is a **Jordan**
functor for Jordan triple systems: is a Jordan triple system associated with a geometric object, just as a Lie triple system is associated with a symmetric space?

The problem of defining a Jordan functor is of considerable interest because it is related to many topics in geometry and harmonic analysis on symmetric spaces.

Roughly speaking, a twist is an additional structure on a symmetric space. This additional structure can be described in several ways. Geometrically, the additional structure given by a twist on a symmetric space $M$ can be interpreted as a twisted para-complexification or a twisted para-complexification of $M$. In order to illustrate what this means, consider the example of the real projective space $M = P_\mathbb{R}$ which is a symmetric space

$$M = O(n+1)/(O(n) \times O(1)).$$

It has a natural complexification given by the complex projective space

$$P_\mathbb{C} = U(n+1)/(U(n) \times U(1)).$$

This complexification is called by Bertram [98h] “twisted” in order to distinguish it from the “straight” complexification

$$M_\mathbb{C} = O(n+1,C)/(O(n,C) \times O(1,C)).$$

Every symmetric space $M = G/H$ has (locally) a unique straight complexification $M_\mathbb{C} = G_{\mathbb{C}}/H_{\mathbb{C}}$.

In contrast, twisted complexifications are an additional structure of a symmetric space which in general need neither exist not be unique.

Bertram [98h] proved that all (real, finite-dimensional) Jordan structures correspond to certain geometric objects, namely:

- Jordan algebras correspond to quadratic prehomogeneous spaces;
- Jordan pairs correspond to twisted polarized symmetric spaces;
- Jordan triple systems correspond to symmetric spaces together with a twisted complexification.

There is a forgetful functor from all of these objects into the category of symmetric spaces which corresponds to a natural Jordan-Lie functor. It is a surprising fact, obtained by classification of simple objects (due to E. Neher and M. Berger), that this functor is quite close to being bijective.

Comment. For a detailed presentation of Bertram’s contributions see Iordănescu [364w, §5 from Ch.3].
Concerning the Bertram’s work since 2000, let us point out that his aim was to generalize correspondence between Jordan structures and geometric objects to the case of arbitrary dimension and of general base-fields and -rings (cover, in particular, the “Jordan algebra of quantum mechanics” \( \text{Herm}(\mathcal{H}) \) which is infinite-dimensional, and more general \( C^*-\)algebras).

In 2002, Bertram [98i] introduced generalized projective geometries which are a natural generalization of projective geometries over a field or ring \( \mathbb{K} \) but also of other important geometries such as Grassmannian, Lagrangian or conformal geometry. He also introduced the corresponding generalized polar geometries and associated to such a geometry a symmetric space over \( \mathbb{K} \).

**Remark.** In the finite-dimensional case over \( \mathbb{K} = \mathbb{R} \), all classical and many exceptional symmetric spaces are obtained in this way.

In the same paper [98i], Bertram proved that generalized projective and polar geometries are essentially equivalent to Jordan algebraic structures, namely to Jordan pairs, respectively to Jordan triple systems over \( \mathbb{K} \) which are obtained as a linearized tangent version of the geometries in a similar way as a Lie group is linearized by its Lie algebra. In contrast to the case of Lie theory, the construction of the Jordan functor works equally well over general base rings and in arbitrary dimension.

One year later, in 2003, Bertram completed his previous results (see Bertram [98j]): he showed that the correct generalization of the projective line in the category of generalized projective geometry is given by spaces corresponding to unital Jordan algebras.

**Remark.** The case of characteristic 2 is still an open problem.

**Note.** An overview of the just above mentioned results can be found in Bertram [98l].

**Question.** What about the infinite-dimensional structures on the geometric objects introduced by Bertram and briefly recalled just above?

Recently, Bertram and Neeb (see [105a, b]), based on a previous their joint paper with Glöckner (see [101]), gave a new approach to differential calculus which works naturally in the framework of very general base fields or even rings and in arbitrary dimension and characteristic, and on consequences for differential geometry, Lie group theory and symmetric space theory (for full details see Bertram [98p], where a general differential geometrical framework is developed).

**Note.** An overview of these recent results can be found in Bertram [98n].
Comment. Related to this question there are two very recent paper by Bertram & Löwe [104] and Bertram [98o].

In the paper [104], Bertram and Löwe introduced the notion of intrinsdic subspaces of linear and affine pair geometries, which generalizes the one of projective subspaces of projective spaces. They proved that, when the affine pair geometry is the projective geometry of a Lie algebra introduced in [105a], such intrinsic subspaces correspond to inner ideals in the associated Jordan pair, and they investigated the case of intrinsic subspaces defined by the Peirce decomposition which is related to 5-gradings of the projective Lie algebra. These examples, as well as the examples of general and Lagrangian flag geometries, lead to the conjecture that geometries of intrinsic subspaces tend to be themselves linear pair geometries.

It is known that the homotopy is an important feature of associative and Jordan algebraic structures: such structures always come in families whose members need not be isomorphic among each other, but still share many important properties. One may regard homotopy as a special kind of deformation of a given algebraic structure. In the paper [98o], Bertram investigates the geometric counterpart of this phenomenon on the level of the associated symmetric spaces. On this level, homotopy gives rise to conformal deformations of symmetric spaces. These results are valid in arbitrary dimension and over general base fields and rings.

Very recently, Bertram & Bieliavsky [99a,b] investigated a special kind of construction of symmetric spaces, called homotopy, and they gave the classification of homotopes.

The winter 1980–1981 witnessed the appearance of the famous paper [490a] by Sato, where he proved that the totality of solutions of the Kadomtsev-Petviashvili (KP) equation,

$$3u_{yy} + (-4u_t + u_{xxx} + 12uu_x)_x = 0,$$

forms an infinite-dimensional Grassmann manifold.

Note. Let us recall that the KP equation was discovered in 1970 in an effort to understand the propagation of long, shallow waves in plasma (see B. KADOMTSEV and V. PETVIASHVILI, Dokl. Akad. Nauk SSSR 192 (1970), 4, 753).

The evolution of $u$ in the variables $x, y, t$ is interpreted as a dynamical motion of a point on the Sato’s Grassmann manifold by the action of a three-(or more) parameter subgroup of the group of its automorphisms. Generic
points of Sato’s Grassmann manifold give generic solutions to the KP equation, whereas points on particular its submanifolds give solutions of particular type. For instance, rational solutions correspond to points on finite-dimensional Grassmann submanifolds. Also, different kinds of submanifolds give rise to generic solutions of other soliton equations, such as the Korteweg-de Vries (KdV) equation, the modified KdV equation, the Boussinesq equation, the Sawada-Kotera equation, the non-linear Schrödinger equation, the Toda lattice, the equation of self-induces transparency, the Benjamin-Ono equation, as well as to solutions of particular type of these soliton equations.

Moreover, a multicomponent generalization of the theory shows that solutions of other soliton equations (such as the equation for three-wave interaction, the multi-component nonlinear Schrödinger equation, the sine-Gordon equation, the Lund-Regge equation, and the equation for intermediate long wave) also constitute submanifolds of Sato’s Grassmann manifold. Sato [595a] conjectured that any soliton equation, or completely integrable system, is obtained in this way. It follows that the classification of soliton equations would be reduced to the classification of submanifolds of Sato’s Grassmann manifold which are stable under the subgroup of its automorphisms describing space-time evolution.

**Comments.** As I have predicted in 1983 through my talks given at the Universities of Timișoara and Iași (Romania), Sato’s theory is an outstanding contribution with a deep impact on physics, but also on mathematics. In fact, on the occasion of Sato’s visit in Romania (in August 1983) I suggested him to use Jordan structures as main tool for his theory. It is the great merit of Josef Dorfmeister and his collaborators to give, beginning with 1989, an impressive and important work in this direction.

**Note.** For a detailed presentation of the results obtained by Dorfmeister and his collaborators see Iordânescu [364w, §6 from Ch.3].

At the end of this paragraph I would like to mention a very recent paper by Alekseevsky [9] on pseudo-Kähler symmetric spaces . . . ”which are very closely related with Jordan pairs” (as Alekseevsky himself informed me 14).

Concerning the applications of Jordan structures to differential geometry, I like to refer the reader to the very recent book [166i] by Chu, not yet published, but available from January 2012.

14Private communication in March 2010.
§5. JORDAN ALGEBRAS IN RING GEOMETRIES

The first investigation of octonion planes dates from 1933 and is due to Moufang [500a]. It consisted in the construction of a projective plane coordinatized with an octonion division algebra. In this Moufang plane, Desargues Theorem fails but the Harmonic Point Theorem is valid. In 1989, Faulkner [248d] described a geometric construction of the Moufang projective octonion plane from the projective quaternionic 3-space. This geometric construction is a translation of an algebraic construction of the 27-dimensional degree 3 exceptional simple Jordan algebra as the trace 0 elements of a 28-dimensional degree 4 Jordan algebra (see Allison & Faulkner [18a]). The geometric construction is a particular case of a more general construction described by Faulkner that also yields as a special case a construction of the complex projective plane from the complex projective line. In 1994, using the classification of the group $SL_2(K)$ and its natural module, Timmesfeld [663] gave a classification of Moufang planes and the groups $E_6^K$ together with various other results on Moufang planes.

Another approach to octonion planes was given in 1945 by Jordan [379d] via the Jordan algebra $H_3(O)^\oplus$. Recall the definition of the exceptional Jordan algebra $H_3(O)^\oplus$: Let $H_3(O)$ be the set of all $(3 \times 3)$-matrices with entries in an octonion algebra $O$ and which are symmetric with respect to the involution $x \rightarrow \overline{x'}$. The characteristic of the underlying field is supposed to differ from two. On $H_3(O)$ we can define a Jordan algebra structure by means of the product $xy := \frac{1}{2}(x \cdot y + y \cdot x)$, where the dot means the usual matrix product. The resulting Jordan algebra is denoted by $H_3(O)^\oplus$. Jordan focussed on a real octonion division algebra $O$ and used the primitive idempotents in $H_3(O)^\oplus$ to represent the points and lines of a projective plane.

In 1951, Freudenthal [272a] obtained essentially the same construction. Atsuyama [52a] used the embedding defined by Yokota [720] to obtain new results in this direction. In 1997, Allcock [14a] provided the identification of the classical Jordan’s model of octonion plane [379d] with the elegant model given by Aslaksen [50].

The construction was extended in 1960 by Springer [632a], who considered $O$ as an octonion division algebra over a field of characteristic different from two or three. In this more general setting, elements of rank one (which

\footnote{For a systematic treatise on applications of the real algebra of Cayley numbers, I refer the reader to Brada’s Ph.D. Thesis [104].}
are either non-zero multiples of primitive idempotents or nilpotents of index two) are used to represent the points and lines of a projective plane. Springer proved the fundamental theorem relating collineations of the plane and norm semisimilarities of the Jordan algebra. Jacobson [371a] showed that the little projective group, i.e., the group generated by elations (transvections) of these planes, is simple and isomorphic to the norm-preserving group of the Jordan algebra modulo its centre. Suh [644] showed that any isomorphism between the little projective groups of two planes is induced by a correlation of the planes. Springer and Veldkamp [633a] undertook a study of Hermitian polarities of a projective octonion plane and the related hyperbolic and elliptic planes. The unitary group of collineations commuting with a hyperbolic polarity was studied by Veldkamp [689c].

Springer and Veldkamp [633b] considered planes associated with split (i.e., not division) octonion algebras over a field of characteristic different from two or three. These planes are not projective. (For the study of these planes, see Veldkamp [689b, d].)

In 1970, Faulkner [248a] extended the notion of octonion planes in another direction by removing the restriction that the characteristic of the underlying field be different from two or three. After McCrimmon [480a] had introduced the notion of quadratic Jordan algebra and verified that $H^{(+)}_3(\mathbb{O})$ possesses such a structure for any characteristic, Jacobson suggested to Faulkner (see [248a, p. 3]) that characteristic-two octonion planes could be approached in this way. As it turned out, in the setting of quadratic Jordan algebras, most of the results on octonion planes can be derived in an uniform manner, without referring to the characteristic or the type of an octonion algebra.

For collineation groups of projective planes over degenerate octaves or anti-octaves, see Persits [538a, b].

Davies [195] studied bi-axial actions on projective planes (including octonion planes), making use of their Jordan algebra description.

Bix [112a] has defined and studied octonion planes over local rings. He generalized Faulkner’s result on the simplicity of $PS$ (see Iordănescu [364w, Ch. 4, p. 107]) of an octonion plane over a field to octonion planes over local rings. Those subgroups of the collineation group of an octonion plane over a local ring which are normalized by the little projective group have been classified. These parallel results of Klingenberg and Bass, who classified those subgroups of the general linear group over a local ring which are normalized by the special linear group.

**Comment.** For a detailed presentation of the contributions of Faulkner [248a], and Bix [112a,b,c] see Iordănescu [364w, §1 of Ch. 4].
Using concepts from valuation theory, Carter and Vogt [155] gave a characterization of all collinearity-preserving functions from one affine or projective Desarguesian plane into another. *Lineations* (i.e., point functions $f$ from one plane into another with the property that, whenever $x, y$ and $z$ are collinear points, $f(x), f(y)$ and $f(z)$ are collinear points) whose ranges contain a quadrangle, called in [155] *full lineations*, have been algebraically characterized in various setting by Klingenberg ([402a], lineations from a Desarguesian plane onto another), Skornyakov ([625], lineations from an arbitrary plane onto another), Radó ([554b], full lineations from a Desarguesian plane into another, the infinite-dimensional case being considered in [554a]), and Garner ([285], lineations from a Pappian coordinate plane into another taking the reference quadrangle of one plane to that of the other).

Carter’s and Vogt’s results allow one or both planes to be affine and include cases where the range contains a triangle but no quadrangle. A key theorem is that, with the exception of certain embeddings defined on planes of order 2 and 3, every collinearity-preserving function from an affine Desarguesian plane into another can be extended to a collinearity-preserving function between the enveloping projective planes. Full lineations defined on finite-dimensional affine spaces can also be extended to the enveloping projective space (Brezuleanu and Rădulescu [139a, b]).

Faulkner and Ferrar [249a] have showed that, up to conjugation by collineations, there exists at most one surjective homomorphism from an octonion plane to a Moufang plane. They also established the existence of proper homomorphisms between octonion planes and of homomorphisms from octonion planes onto Desarguesian planes.

Ferrar and Veldkamp [264] studied neighbour-preserving homomorphisms between projective ring planes (i.e., mappings preserving incidence and the neighbour relation between points and lines). These are generalizations of the homomorphisms between ordinary Desarguesian projective planes which have first been studied by Klingenberg [402a]. On the other hand, in the context of projective planes over rings of stable rank 2 as studied by Veldkamp [689e], an obvious question to ask was what mappings between such planes are induced by homomorphisms between coordinatizing rings. If one requires that the ring homomorphisms carry 1 to 1, then they induce distant-preserving homomorphisms which are mappings incidence and the negation of the neighbour relation between a point and a line. Veldkamp [689f] proved that any distant-preserving homomorphism $\psi$ is induced by a ring homomorphism carrying 1 to 1, provided the two planes are coordinatized with respect to basic quadrangles which correspond under $\psi$. Veldkamp [689f] also studied homomorphisms
between projective ring planes which only preserve incidence. They turn out to be products of a bijective neighbour-preserving homomorphism followed by an arbitrary distant-preserving homomorphisms.

In 1983, Faulkner and Ferrar [249c], utilizing methods similar to those of Skornyakov [625], extended the results of Klingenberg [402a] from Desarguesian planes to Moufang planes. Let us note in this respect that a systematic study of places of octonion algebras over discrete valuation rings has been carried out by Petersson [539a]. Brožikova [141], making use of previous results of Havel [333] and Faulkner & Ferrar [249c], provided a Jordan-theoretic description of all homomorphisms between Moufang planes having the property that the points identified via Springer’s isomorphism (see Springer [632a]) with 

\[ (0, 0), 0, (\infty) \text{ and } (1, 1) \]

are mapped to their analogues in the image plane.

**Open Problem.** It would be interesting to deal with topics similar to those mentioned above in the case of octonion planes.

A systematic study of projective planes over large classes of associative rings was initiated by Barbilian in his very general approach [68b] (see also Barbilian [68a]). He proved there that the rings which can be underlying rings for projective geometries are (with a few exceptions) rings with a unit element in which any one-sided inverse is a two-sided inverse. Barbilian [68b, I] called these rings “Z-rings” (from “Zweiseitigsingularäre Ringe”) and gave a set of 11 axioms of projective geometry over a certain type of Z-ring (see [68b, II]).

For more than thirty years no development of Barbilian’s study succeeded. Beginning with 1975, outstanding mathematicians like W. Leissner in Germany, F.D. Veldkamp in Holland, and J.R. Faulkner in USA, developed Barbilian’s ideas, and so, notions as “Barbilian domains”, “Barbilian planes”, “Barbilian spaces”, or “Barbilian geometry” appeared.

Leissner [437a, b, c, d, e] developed a plane geometry over an arbitrary Z-ring \( R \), in which a point is an element of \( R \times R \) and a line is a set of the form

\[
\{(x + ra, y + rb) \mid (x, y) \in R \times R, r \in R, (a, b) \in B\},
\]

where \( B \) is a “Barbilian domain”, i.e., a set of unimodular pairs from \( R \times R \) satisfying certain axioms.

**Note.** Let us mention in this context that Lantz extended in [431] the results of Benz from [87a, b] by showing that large classes of commutative rings admit only one Barbilian domain.

Radó [554a, b, c] extended Leissner’s results [437a] to affine Barbilian planes over an arbitrary ring with a unit element and investigated the cor-
responding affine Barbilian structures and translation Barbilian planes. Corresponding to the algebraic representation of affine Barbilian spaces as affine geometries over unitary free modules, Leissner [437b] characterized algebraic properties of the underlying ring $R$, respectively module $M_R$, respectively Barbilian domain $B \subset M_R$ by geometric properties of the affine Barbilian space and vice versa.

Veldkamp [689c] gave an axiomatic description of plane geometries of the kind considered by Bingen [109]. A most satisfactory situation is reached by extending the class of ring used for coordinatization from semiprimary rings, which Bingen used to rings of stable rank two. These rings have played a role in algebraic $K$-theory, and seem to form a natural framework for many geometric problems.

**Note.** For simplicity Veldkamp [689e] confines himself to the case of planes (called *projective Barbilian planes*), but a generalization to higher dimensions is straightforward (see below *projective Barbilian spaces* defined also by Veldkamp).

Veldkamp has chosen an approach somewhat along the lines of Artin [46] rather than to follow Barbilian.

The basic relations in the plane are the incidence and the neighbour relation. The axioms consist of a number of axioms expressing elementary relations between points and lines such as, e.g., the existence of a unique line joining any two non-neighbouring points, and a couple of axioms ensuring the existence of transvections and dilatations.

In 1993, Veldkamp refined in [689j] the notion of Barbilian domain to $n$-Barbilian domain in a free module of rank $n$. This leads to results that bear on $n$-dimensional affine ring geometry. The case of infinite rank is also considered by Veldkamp in [689j].

**Note.** For a good survey on the theory of projective planes over rings of stable rank two, see Veldkamp [689h]. Such a plane is described as a structure of points and lines together with an incidence relation and a neighbour relation and which has to satisfy two groups of axioms. The axioms in the first group express elementary relations between points and lines such as, e.g., the existence of a unique line joining any two non-neighbouring points, and define what is called a Barbilian plane. In the second group of axioms the existence of sufficiently many transvections, dilatations, and generalizations of the latter, the affine dilatations and their duals is required.

In 1987, Veldkamp [689i] extended all this above mentioned results to arbitrary finite dimension. Basic objects in the axioms are points and hyper-
planes, by analogy with the selfdual set-up for classical projective spaces over skew fields given by Esser [238]. As basic relations again serve incidence and the neighbour relation. The self-dual approach is quite natural since incidence and the neighbour relation between points and hyperplanes have a simple algebraic description in coordinates. Homomorphisms are more or less the same as in the plane case, things becoming a bit more complicated because Veldkamp included homomorphism between spaces of unequal dimension.

**Note.** Veldkamp confine himself to full homomorphisms, which can only increase the dimension or leave it the same. Thus he excluded homomorphisms which lower the dimension, an example of which was given by Fritsch and Prestel [276].

**Comment.** For a survey of the results of Veldkamp [689e,i] concerning the fundamental properties of projective Barbilian spaces, see Iordănescu [364w, §3 of Ch. 4].

Faulkner and Ferrar [249d] surveyed the development which leads from classical Desarguesian projective plane via Moufang planes to Moufang-Veldkamp planes. They first sketched inhomogeneous and homogeneous coordinates in the real and projective planes and in ring planes, the Jordan algebra construction of Moufang planes, and the representation of all these planes as homogeneous spaces for their groups of transvections. Then attention is focussed on Moufang-Veldkamp planes, i.e., projective Barbilian planes in which all possible transvections exist and which satisfy the little quadrangle section condition for quadrangles in general position. As coordinates for the affine plane one easily obtains an alternative ring of stable rank two. Unfortunately, the Bruck and Kleinfeld theorem for alternative division rings does not carry over to alternative rings in general, i.e., such a ring need neither be associative nor be an octonion algebra. Therefore, to coordinatize the whole projective plane one cannot rely on either homogeneous coordinates (as in the associative case) or the Jordan algebra construction (as in the octonion case). In this case, one has to follow a more complicated way, namely: first to construct a certain Jordan pair from the given alternative ring, then to define a group of transformations of this Jordan pair and, finally, to represent the projective plane as a homogeneous space for that group.

Faulkner [248c] proved that for a connected Barbilian transvection plane $P$ (i.e., a plane with incidence and neighbouring generalizing Moufang projective planes) one can construct a connected Barbilian plane $T(P)$ called tangent bundle plane. This construction agrees with the usual tangent bundle when it exists. If $T(P)$ is also a transvection plane, then the set $S$ of sections of
$T(P)$ is a Lie ring. The group $G$ generated by all transvections of $P$ acts on $S$. Since $S$ is isomorphic to the Koecher-Tits Lie ring constructed from the Jordan pair $(M_{12}(R), M_{21}(R))$, where $R$ is the associated alternative ring, one can determine $G$ and thereby $P$ from $R$.

In 1987, Spanicciati [631] introduced near Barbilian planes (NBP) and strong near Barbilian planes (SNBP) as a variation of Barbilian planes. In 1989, Hanssens and van Maldeghem [331] showed that a NBP is an SNBP, and classified all NBP up to the classification of linear spaces (many examples follows as a result of a universal construction). They also showed that only NBPs that are also BPs are those mentioned in [631], namely the projective planes.

Allison and Faulkner [18a] gave in 1984 an algebraic construction of degree three Jordan algebras (including the exceptional one) as trace zero elements in a degree four Jordan algebra. Five years later, Faulkner [248d] translated this algebraic construction to give a geometric construction of Barbilian planes coordinatized by composition algebras (including the Moufang plane) as skew polar line pairs and points on the quadratic surfaces determined by a polarity of projective 3-space over a smaller composition algebra.

In 1989, Faulkner [248e] defined and studied the so-called $F$-planes which generalize the projective planes. Planes considered by Barbilian in the Zusatz to [68b] are connected $F$-planes in Faulkner’s setting. Besides extending the class of coordinate ring, Faulkner’s work [248e] introduces some new concepts, techniques, and connections with other areas. These include a theory of covering planes and homotopy although there is no topology, a theory of tangent bundle planes and their sections although there is no differential or algebraic geometry, a purely geometric and coordinate-free construction of the Lie ring of the group generated by transvections, and connections to the $K$-theory of the coordinate ring.

Greferath and Schmidt [305b] introduced in 1992 the notion of Barbilian space of a projective lattice geometry in order to investigate the relationship between lattice-geometric properties and the properties of point-hyperplane structures associated with. They obtained a characterization of those projective lattice geometries, the Barbilian space of which is a Veldkamp space (see also Greferath and Schmidt [305a]).

In §5 of Ch. 4 from the book Iordănescu [364w] are presented coordinatization theorems similar to Moufang’s [500a] for several polygonal (projective, quadrilateral, and hexagonal, admitting all elations) geometries. For the classification of polygonal geometries, see Tits [664b, d, e]. These coordinatize a projective Moufang plane by an alternative division algebra. The essential
information needed is contained in the group generated by all elations. These groups are groups with Steinberg relations for which parametrization theorems were presented in the §4 of the same Ch.4 from the book Iordănescu [364w]. For these presentations the Faulkner’s formulations [248b] were used.

Comments. For a brief survey on the relations between various exceptional notions in algebra and geometry (e.g., non-classical Lie algebras, nonassociative alternative algebras, non-special Jordan algebras, non-Desarguesian projective planes), the reader is referred to Faulkner and Ferrar [249a], who proved that all these notions are related, one way or another, to the octonions.

An interesting area of research is that of chain geometries and their generalizations, chain spaces, and their relations with Jordan algebras (see Blunck [115] and Herzer [339]).

§6. JORDAN ALGEBRAS
IN MATHEMATICAL BIOLOGY
AND IN MATHEMATICAL STATISTICS

Note. For a comprehensive account on algebras in genetics up to 1980, the reader is referred to Wörz-Busekros monograph [711a].

Etherington [239a,b] showed how a nonassociative algebra can be made to correspond to a given genetic system. The fact that many of these algebras have common properties has prompted their study from a purely abstract standpoint. Furthermore, these algebraic studies gave new ways of tackling problems in genetics.

In a study of nonassociative algebras arising in genetics, Schafer [598a] proved that the so-called gametic and zygotic algebras (see Etherington [239b]) for a single diploid locus are Jordan algebras.

Holgate [348a] proved Schafer’s results by methods which do not make use of transformation algebras (employed by Schafer [598a]), which therefore accommodate the multi-allelic case more easily, and in which the main object is to maximise the interplay between the algebraic formalism and the genetic situation to which it corresponds.

The first part of the present paragraph deals with these results as treated

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16 The topics presented in §6 were only mentioned in the book Iordănescu [364w].
by Holgate [348a], while the second deals with the results due to Piacentini Cattaneo [542], Wörz-Busekros [711c,d] and Walcher [699c].

We consider the *gametic algebra* $G$ of a single locus with $n + 1$ alleles, i.e., the algebra $G$ over $\mathbb{R}$ with basis $\{a_0, \ldots, a_n\}$, whose elements correspond to the actual allelic forms, its multiplication table being

$$a_i a_j := \frac{1}{2}(a_i + a_j).$$

For an element $x = \sum_{i=0}^{n} x_i a_i$, the weight $w$ is defined by $w(x) := \sum_{i=0}^{n} x_i$.

**Remark.** It is easily seen that $x^2 = w(x)x$.

**Proposition 1.** (Algebraic). *Every element of unit weight in $G$ is idempotent.*

**Proposition 1.** (Genetic). *In the absence of selection, the gametic proportions remain constant from one generation to another.*

**Remark 1.** The algebraic result is more comprehensive, since only those elements of unit weight for which all the $x_i$ are nonnegative correspond to populations.

**Remark 2.** The nonassociativity of genetic algebras corresponds to the fact that if $P, Q$ and $R$ are populations and if $P$ and $Q$ mate and the offspring mates with $R$, the final result is, in general different from that arising from mating between $P$, and the offspring of mating between $Q$ and $R$. The two situations are shown in the diagram below:

![Diagram](image_url)
Proposition 2. (Algebraic). The algebra $G$ is a Jordan algebra.

Proposition 2. (Genetic). In the mating schemes shown in Figure 1, the populations $F_1$ and $F_2$ have the same genetic proportions if $P$ is the offspring of mating of $R$ with itself.

Notation. The algebra $Z$, corresponding to proportions of zygotic types, is formed by duplicating $G$ (see Etherington [239a]) its basic elements are pairs $(x, y)$ of basis elements of $G$ with the multiplication rule $(x, y)(u, v) := (xy, uv)$. A canonical basis may be taken in $G$ by setting: $c_0 := a_0, c_i := a_0 - a_i (i \neq 0)$, for which the multiplication table is

$$c_0^2 = c_0, \quad c_0c_i = \frac{1}{2}c_i, \quad c_ic_j = 0 \quad (i, j \neq 0).$$

Then on writing $d_{ij} := (c_i, c_j)$, the multiplication table for the duplicate $Z$ can be written as

$$d_{00}^2 = d_{00}, \quad d_{00}d_{0i} = \frac{1}{2}d_{0i}, \quad d_{0i}d_{0j} = \frac{1}{4}d_{ij},$$

other products being zero $(i, j \neq 0)$.

Remark. The weight of an element $\sum_{i,j=0}^n x_{ij}d_{ij}$, $x_{ij}$ is $w(x) = d_{00}$.

Proposition 3. (Algebraic). Every element of the form $y := x^2 - w(x)x$ annihilates $Z$.

Proposition 3. (Genetic). The extent to which the zygotic proportions in a population differ from the Hardy-Weinberg equilibrium state has no effect on the offspring distribution produced by mating between this population and any other.

Proposition 4. The algebra $Z$ is a Jordan algebra.

Remark. Let $A$ be the algebra over $C$ with basis $\{a_0, \ldots, a_n\}$ whose multiplication table is

$$a_ia_j := a_i.$$  \hspace{1cm} (\ast)

Obviously, $A$ is associative. Consider the special Jordan algebra $A^{(+)}$ obtained from the vector space $A$ by means of product $xy := \frac{1}{2}(x.y + y.x)$. It can easily be seen that $A^{(+)}$ is isomorphic to $G$.

If it were possible to know in advance that the genes of only one of two given populations mating together are transmitted to the offspring, these
could be written first in the product, and the system would correspond to the multiplication table (*) . The fact that $G$ is a special Jordan algebra appears as a consequence of inheritance being symmetric in the parents.

Recall from Gonshor [298b, I] the following

**Definition.** A *special train algebra* is a commutative algebra over $\mathbb{C}$ for which there exists a basis $\{a_0, \ldots, a_n\}$ with a multiplication table of the following kind: $a_ia_j := \sum x_{ijk}a_k$, where

(i) $x_{000} = 1$,
(ii) for $k < j$, $x_{0jk} = 0$,
(iii) for $i,j > 0$, $k \leq \max(i,j)$, $x_{ijk} = 0$,
and all powers of the ideal $(a_1, a_2, \ldots, a_n)$ are ideals. (The powers $I^r$ of an ideal $I$ are defined by $I^r := I^{r-1}I$.)

**Remark.** A commutative algebra over $\mathbb{C}$ for which only conditions (i), (ii) and (iii) are required was called by Gonshor *genetic* algebra (see [298a]). Schafer’s concept of genetic algebra coincides with that of Gonshor (see Gonshor [298a, Theorem 2.1]). Wörz-Busekros defined [711b] three kinds of *noncommutative* Gonshor genetic algebras and characterized them in terms of matrices.

**Comments.** Let us mention in this respect that in the mathematical theory of algebras in genetics, whose origins are in several papers by Etherington, fundamental contributions have been made by Schafer, Gonshor, Holgate, Reiersøl, Hech and Abraham (for a detailed account see [711a]).

**Definition.** The $x_{0jj}$ are called the *train roots* of the algebra. (They are the characteristic roots of the operator which is multiplication by $a_0$).

**Remark.** From Schafer [598a, Theorem 5], it follows that a special train algebra can only be a Jordan algebra if its train roots all have values among $1, \frac{1}{2}, 0$. This excludes the genetic algebras corresponding to polyploidy of several loci. Therefore, the appearance of Jordan algebra seems to be bound up with the property of attaining equilibrium after a single generation of mating.

Piacentini Cattaneo considered [542] the gametic algebra $G$ (see the beginning of this paragraph) of simple Mendelian inheritance. Suppose that

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17From very recent papers devoted to train algebras, I like to mention [71] by Bayara, Conseibo, Ouattara and Zitan.

18Concerning genetic algebras, the fundamental idea has been to define a basis $\{G_1, \ldots, G_n\}$ with a one-to-one correspondence to the genotypes $g_1, \ldots, g_n$ considered, and then give a multiplication table so that the product $G_iG_j$ of two basis elements be equal to a linear combination $\sum p_{ijk}G_k$, where $p_{ijk}$ is the probability of getting genotype $g_k$ in a cross between $g_i$ and $g_j$ individuals.
mutation occurs in the chromosomes, i.e., suppose that a rate of alleles \(a_i\) mutate into the alleles \(a_j\), \(j \neq i\). If we denote this rate by \(r_{ij}\) (setting \(r_{kk} = 0\)), we can construct a new algebra, denoted by \(G_m\), called a gametic algebra of mutation (see [542, p. 180]). The new multiplication table then is

\[
a_j^2 = (1 - \sum_{i=0}^{n} r_{ij})a_j + \sum_{i=1}^{n} r_{ji}a_i;
\]

\[
a_ja_k = \frac{1}{2}(1 - \sum_{i=0}^{n} r_{ij})a_j + \frac{1}{2} \sum_{i=0}^{n} r_{ji}a_i + \frac{1}{2}(1 - \sum_{i=0}^{n} r_{ki})a_k + \frac{1}{2} \sum_{i=0}^{n} r_{ki}a_i \quad (j \neq k).
\]

**Proposition 5.** Let \(G_m\) be a gametic algebra of mutation, with mutation rates \(r_{ij}\). For \(G_m\) to be a Jordan algebra, it is necessary and sufficient that the following system of \(n\) identities in \(x_k\) \((k = 0, 1, \ldots, n)\) holds:

\[
\sum_{j=1}^{n} \beta_j(r_{0j} - r_{ji}) + \beta_i(1 + \sum_{k=0}^{n} r_{ik}) = 0, \quad i = 1, \ldots, n,
\]

where

\[
\alpha_t = \alpha_t(x_0, \ldots, x_n) = (\sum_{k=0}^{n} x_k)r_{0k} - \sum_{k=1}^{n} x_kr_{tk} + \sum_{k=0}^{n} x_{t}r_{tk},
\]

\[
\beta_j = \beta_j(x_0, \ldots, x_n) = (1 - \sum_{k=0}^{n} r_{jk})\alpha_j - \sum_{t=1}^{n} \alpha_t(r_{0j} - r_{tj}).
\]

In the same paper [542], Piacentini Cattaneo used the identities from Proposition 5 to determine the restrictions of the \(r_{ij}\)'s for \(G\) to be a Jordan algebra in specific cases.

In 1988, Peresi [536a] proved that if \(A\) is a nonassociative algebra that verifies \(A^2 = A\) and has an idempotent, than \(A\) and its duplicate have isomorphic automorphism groups and isomorphic derivation algebras. This result is then applied by Peresi to the gametic algebra for polyploidy with multiple alleles.

**Definitions.** An algebra \(A\), not necessarily associative, over a commutative field \(K\) of characteristic different from two, that admits a nontrivial homomorphism \(w : A \rightarrow K\) is said to be baric.

A baric algebra \(A\) that satisfies the identity \((a^2)^2 = w^2(a)a^2\) for all \(a \in A\) is called a Bernstein algebra.
Remark. Singh and Singh [624] showed that Lie and Clifford algebras are never baric. On the other hand, starting with a baric algebra, it is possible to derive new algebras which are Lie, Jordan, alternative or associative.

Every Bernstein algebra $A$ possesses at least one idempotent $e$. It can be decomposed into the direct sum of subspaces $A = E \oplus U \oplus Z$ with $E := K_e$, $U := \{ey \mid y \in \text{Ker} w\}$, $Z := \{z \in A \mid ez = 0\}$.

If $A$ has finite dimension, which is at least 1, $\dim A = 1 + n$, then one can associate to $A$ a pair of integers $(r + 1, s)$, called type of $A$, whereby

$$r := \dim U, \quad s := \dim Z,$$

hence $r + s = n$.

In 1989, Wörz-Busekros [711d] showed that for each decomposition $n = r + s$ there exists a Bernstein algebra of type $(r + 1, s)$. Thereby the so-called trivial Bernstein algebra of type $(r + 1, s)$ has been introduced as Bernstein algebra of the corresponding type where $(\text{Ker} w)^2 = \{0\}$.

Wörz-Busekros [711c] showed that the well-known decomposition of a Bernstein algebra with respect to an idempotent is nothing else but the Peirce decomposition known for finite-dimensional power-associative algebras with idempotent, especially for Jordan algebras with idempotent.

Note. Bernstein algebras are not in general power-associative.

In terms of Peirce theory, Wörz-Busekros [711c] showed that in a Bernstein algebra all idempotents are principal and thus primitive. Hence, the Peirce decomposition cannot be further decomposed. She deduced a necessary and sufficient condition for a Bernstein algebra to be Jordan, and obtained a number of special results from it (the principal two being Proposition 6 and Theorem 7 below).

Proposition 6. (see Wörz-Busekros [711c, p. 396]) A trivial Bernstein algebra of type $(r + 1, s)$ is a special Jordan algebra.

Remark. Proposition 6 is a generalization of Holgate’s result [348a] (see Proposition 4 above and Remark which follows), who proved that all gametic algebras for simple Mendelian inheritance are special Jordan algebras. Thereby the gametic algebra for simple Mendelian inheritance with $n + 1$ alleles is a trivial Bernstein algebra of type $(n + 1, 0)$, cf. Wörz-Busekros [711d].

Definition. Let $A$ be an algebra over $K$ with weight homomorphism $w : A \to K$. Then $A$ is called a normal algebra, if the identity $x^2y = w(x)xy$ is satisfied in $A$. 

Theorem 7. (see Wörz-Busekros [711c, p. 397]). Every normal algebra is a Jordan algebra.

In 1988, Walcher [699c] gave a characterization of Bernstein algebras which are Jordan algebras (called by him Jordan Bernstein algebras) over a field of characteristic different from 2 or 3, and listed some of their properties.

Theorem 8. (see Walcher [699c, p. 219]). Let $A$ be a baric algebra over a field of characteristic different from 2 or 3, and with the nontrivial homomorphism from the definition of $A$. The following statements are equivalent:

(i) $A$ is a Jordan Bernstein algebra
(ii) $A$ is a power-associative Bernstein algebra
(iii) $x^3 - w(x)x^2 = 0$ for all $x \in A$.

As a corollary of Proposition 1 from Walcher [699c], it follows that every Jordan Bernstein algebra is genetic. Thus, by Wörz-Busekros [711a, Theorem 3.18], for $\dim A = m + 1$, we have a chain of ideals of $A$

$$N := \ker w \supset N_1 \supset N_2 \supset \ldots \supset N_m \supset \{0\},$$

such that $\dim N_i = m + 1 - i$ and $N_i N_j \subset N_{k+1}$, where $k := \max\{i, j\}$, for all $i$ and $j$.

Notation. Let $c$ be an idempotent of $A$ from Theorem 8 above, and let $L(c)$ denotes, as usual, the left multiplication by $c$.

Proposition 9. (see Walcher [699c, p. 221]). Let $A$ be a Jordan Bernstein algebra of dimension $m + 1$. Then there exists a basis $\{v_1, \ldots, v_m\}$ of $N$ such that $v_i$ is an eigenvector of $L(c)$ for $1 \leq i \leq m$ and $N_i$ is spanned by $v_i, \ldots, v_m$ ($1 \leq i \leq m$).

Comments. The Walcher’s results [699c] should at least make the construction of Jordan Bernstein algebras a manageable task: Start with a basis of eigenspaces in $N$ (the eigenvalues preassigned) take into account the composition rules for the eigenspaces and note that the only thing to be checked besides this is the identity $x^3 = 0$ in $N$.

In 1989, Holgate [348e] examined conditions under which the entropic law is satisfied in genetic algebras, and the consequences of imposing it when it is not. It appears that, as with the Jordan identity (see Holgate [348a], and Micali and Quattara [485]), the entropic law only interacts inclusively with the properties of genetic algebras for small rank or dimension.

Making use of the papers [557a,b] by Resnikoff, let us mention now applications of Jordan algebras to color perception.
In order to endow the set $C$ of perceived colors with a geometrical structure, various standard experimental results are taken as axioms. One can show that there exists a real vector space $V$ spanned by the set $C$ in which $C$ is a cone of perceived colors. Denote by $GL(C)$ the group of orientation-preserving linear transformations of $V$ which preserve the cone $C$. $GL(C)$ is a subgroup of $GL(V)$, and therefore a Lie group.

Making use of standard results in the theory of homogeneous spaces, $C$ can be identified with the homogeneous space $\frac{GL(C)}{K}$, where $K$ is isomorphic to the subgroup of $GL(C)$ which leaves some point of $C$ fixed, hence to a closed subgroup of the orthogonal group, and consequently to a compact subgroup of $GL(C)$.

Finally it follows that $C$ is a homogeneous space equivalent either to $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ or to $\mathbb{R}^+ \times SL(2, \mathbb{R})/SO(2)$, $\mathbb{R}^+$ denoting the positive real numbers.

The $GL(C)$-invariant metric (see (**)) below yield in the first case Stiles’s generalization [640] of Helmholtz’s color metric [336], and in the second a new color metric with respect to which $C$ is not isometric to a Euclidean space.

Resnikoff [557b] showed how the concept of Jordan algebra provides an unification of both cases. Namely, let $J$ be a (finite-dimensional) formally real Jordan algebra and consider $\exp J := \{\exp a \mid a \in J\}$.

Consider on $J$ the form given by

$$\mu(a) := \frac{8}{n} Tr L(a), \quad a \in J.$$

If $J = \mathbb{R}(= M_1(\mathbb{R}^{(+)}))$, then $\mu(a) = a$, while if $J = M_r(\mathbb{R}^{(+)}), \mu(a) = Tr a$.

It can easily be seen that for $\alpha > 0$ the map $a \rightarrow a/\alpha$ is an isomorphism of $\mathbb{R}$ onto a Jordan algebra $J_\alpha$ with unit element $1/\alpha$ and that

$$\exp J_\alpha = \{\exp \alpha a \mid a \in J\} = \{(\exp a)^\alpha \mid a \in J\} = \{x^\alpha \mid x \in \exp J\}.$$

Writing $J_{(\alpha_1\alpha_2\alpha_3)} := J_{(\alpha_1)} \oplus J_{(\alpha_2)} \oplus J_{(\alpha_3)}$, it follows that

$$\exp J_{(\alpha_1\alpha_2\alpha_3)} = \{(x_1^{\alpha_1}, x_2^{\alpha_2}, x_3^{\alpha_3}) \mid x_1 \in \mathbb{R}^+\} = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$$

and

$$\exp M_2(\mathbb{R}^{(+)}) = \{\begin{pmatrix} x_1 & x_3 \\ x_3 & x_2 \end{pmatrix} = x \mid x \text{ is positive definite}\}.$$

Thus,
\[ \exp \mathcal{J} = \mathcal{C} = \text{space of perceived colors} \]

if \( \mathcal{J} = \mathcal{J}(\alpha_1,\alpha_2,\alpha_3) \) or \( \mathcal{J} = M_2(\mathbb{R})^+ \).

The group \( GL(\exp \mathcal{J}) \) is generated by the map \( P(a) \) for \( a \in \mathcal{J} \) (\( P(a) \) being the quadratic representation of \( \mathcal{J} \)); \( \exp \mathcal{J} \) is a homogeneous space of \( GL(\exp \mathcal{J}) \), and a \( GL(\exp \mathcal{J}) \)-invariant metric on \( \exp \mathcal{J} \) is given by

\[ (***) \quad ds^2 := \mu((P^{-1}(x)dx)dx). \]

**Remark.** With the unification provided by the concept of a Jordan algebra, the arguments concerning brightness can be conceptually reversed (see Resnikoff [557b, pp. 122-123]).

As is widely accepted nowadays, proteins are the principal workhorses of the cell. They are the major organizers and manipulators of biological energy and enzymes that catalyze and maintain the life process. The proteins are responsible for the active transport of ions into and out of the cell, as well as for cellular and intracellular movement. That is why the discipline of bioenergetics, which is study of how cells generate and transfer their energy supply, is primarily the investigation of how proteins work.

At present, the composition and three-dimensional structure of about two-hundred proteins are known. However, there is no generally accepted model of how proteins operate dynamically.

The idea that the energy released in the hydrolysis of adenosine triphosphate (ATP) molecules transforms into that of soliton excitation and is transferred with great efficiency along protein molecules was used by Davydov as early as 1973 (see [197a]) to explain the contraction mechanisms of transversely striated muscles of animals at the molecular level. Davydov et al. considered, in addition, the idea that \( \alpha \)-helical proteins may facilitate electron transport through a soliton mechanism. In this case, an extra electron causes a lattice distortion in the protein that stabilizes the electron’s motion.

Thus it may be reasonable to consider charge transfer across membranes, energy coupling across membranes, and energy transport along filamentous cytoskeletal proteins in terms of a soliton mechanism, since proteins that carry out these functions contain structural units with significant \( \alpha \)-helical character (see Davydov [197b]). The Davydov model leads to a nonlinear Schrödinger equation which has solutions (see [447, p. 13]).

**Note.** As was observed by Lomdahl, Layne and Bigio [447, p. 16], the soliton model is one among several concepts for protein dynamics which should attract the careful attention of biologists. Clearly, it cannot explain
every aspect of protein dynamics, but it is motivating exciting questions and new experiments.

Layne [433] presented a simplified theoretical model for anesthesia activity, taking advantage of the fact that the α helix is an important structure in membrane and cytoskeletal proteins.

More precisely, Layne [433, p. 24] formulated the following question:

How does the binding of the anesthetic molecule to a protein modify normal protein behavior? He answered this question using the soliton model as a paradigm for normal protein functioning. The soliton model proposes that α-helical proteins effect the transport of ATP hydrolysis energy through a coupling of vibrational excitations to displacements along the spines of the helix. This coupling leads to a self-focusing of vibrational energy that has remarkably stable qualities. Layne [433, p. 24] suggests that the binding of an anesthetic molecule to a protein interferes with soliton propagation. He suggests further that this type of interference is most important in two separate regions of a cell where soliton propagation is an attractive candidate: first, in the α-helical proteins of the inner mitochondrial membrane, which appear to participate in ATP synthesis and electron transport and secondly, in the membrane proteins of neurons, which are responsible for chemical reception and signal transduction.

Remark. [433, p. 26]. If the Davydov soliton finds experimental support in biology, then such a model may help to explain some of the molecular mechanisms behind general anesthesia.

Let us mention that Takeno [654a] studied vibron (i.e., vibrational exciton) solitons in one-dimensional molecular crystals by employing a coupled oscillator-lattice model. Takeno showed that although vibron solitons in his theory and those in the Davydov theory are both described by the non-linear Schrödinger equation, their nature is fairly different from each other. The nonlinear Schrödinger equation arises in the Takeno theory from modulations of vibrons by nonlinear coupling with acoustic phonons propagating along helices of the α-proteins, while that in the Davydov theory follows immediately from the quantal Schrödinger equation for the exciton probability. In 1985, Takeno [654c] presented an exactly tractable model of an oscillator-lattice system which is capable of incorporating both of the pictures of Fröhlich (see [277], [108]) and that of Davydov in a unified way and to make a more detailed study of vibron solitons by giving a significant improvement of the theory developed in [654a].

Comments. As was already pointed out (see end of §4), an open prob-
lem is to find an algebraic description of the Grassmann manifolds appearing in Sato’s approach [595a,b] to soliton equations, resembling the Jordan algebra description of finite-dimensional Grassmann manifolds given by Helwig in [337b]. Taking into account the previous considerations, solving this open problem could be useful in bioenergetics.

In 1992 it was published the book ”Mathematical Structures in Population Genetics” [452d] by Lyubich, which is the English version of the Russian edition published in 1983.

In 2005, Bremner [135] used computer algebra to show that a linearization of the operation of intermolecular recombination from theoretical genetics satisfies a nonassociative polynomial identity of degree 4 which implies the Jordan identity. The representation theory of the symmetric group is used to decompose this new identity into its irreducible components. Bremner showed that this new identity implies all the identities of degree \( \leq 6 \) satisfied by intermolecular recombination.

Concerning the applications of Jordan algebras to mathematical statistics, the year 1994 witnessed the appearance of the book [466] by Malley, where the use of Jordan algebras in mathematical statistics is presented. The kinds of such applications are presented in the above mentioned book, namely: applications to random quadratic forms (sums of squares) and applications to the algebraic simplification of maximum likelihood estimation of patterned covariance matrices. In the second chapter the use of Jordan algebras in random quadratic forms and the mixed linear model are presented. Jordan algebras are used to obtain maximal extensions of the work of Cochran [181], and Rao & Mitra [556] on the independence and chi-squared distribution of quadratic forms in multivariate normal random variables. The second chapter concludes with results that unify previous work on mixed models. In the third chapter, details on Jordan algebras are presented. In the last chapter of this book [466], Malley uses Jordan algebras, Galois field theory and the EM algorithm to obtain either closed-form or simplified solutions to the maximum likelihood equations for estimation of covariance matrices for multivariate normal data.
§7. JORDAN STRUCTURES IN PHYSICS

I shall mention here only some ideas, while for a detailed presentation the reader is referred to the book Iordănescu [364w, Ch. 5], which is a revised, extended and updated version of the 2003 monograph [364g]. It is worth mentioning that the above mentioned 2003 monograph [364g] - exhausted in two years - has good reviews in Zentralblatt für Mathematik and Mathematical Reviews (see Zbl 107317014 and MR 1979748), and - concerning the applications to physics - is more comprehensive than books published afterwards (in 2004 and 2005), written by famous mathematicians (K. McCrimmon and Y. Friedman), as leading experts in the field (W. Bertram and H. Upmeier) remarked in their reviews of those books (see the review [98u] by Bertram and the review [682o] by Upmeier). Jordan structures are still very much in use in modern mathematical physics, but one has to include the so-called Freudenthal triple systems, which are a little bit more complicated.

Concerning already classical results, let us mention that - as it is well known - Jordan [379a] stressed that the most fruitful attempt at generalizing the standard Hilbert space structure of quantum mechanics would be to change the algebraic structures (see also the more recent opinion expressed by Dirac [208]). Jordan [379b] formulated a quantum mechanics in terms of commutative, but nonassociative (finite-dimensional) algebras of observables, now called (finite-dimensional) Jordan algebras. Jordan, von Neumann and Wigner [380] showed that this approach is equivalent to the construction of the standard quantum mechanics in finite-dimensional subspaces of the physical Hilbert space with the single exception of \( H_3(O)^{(+)}. \) The infinite-dimensional case was studied by von Neumann in [515]. A more recent tentative axiomatization was given by Emch [236a]. Exceptional quantum mechanics was investigated by Gündaydin, Piron and Ruegg [318] and shown to be in accordance with the standard propositional formulation, with a unique probability function for the Moufang (non-Desarguesian) plane. Pedroza and Vianna’s results [532] concerning the dynamical variables for constrained and unconstrained systems described by the symmetric formulation of classical mechanics can be connected with results on supersymmetry and supermanifolds of Berezin [88c]. Araki [35] improved the characterization of state spaces of JB-algebras given by Alfsen & Shultz [12b] to a form with more physical appeal (proposed by

19UPMEIER. H., private communication (March 2010)
Wittstock [708]) in the simplified case of a finite dimension. JB-algebras were fruitfully used by Guz [322b] in a tentative axiomatization for nonrelativistic quantum mechanics and Kummer [424] gave in 1987 a new approach. Results on Jordan (quantum) logics due to Morozova & Chentsov [498a,b], and on order unit spaces arising from sum logics due to Abbati & Manià [1a,b], and also Bunce & Wright’s results [146a,b] must be mentioned.

**Note.** For a comprehensive presentation of the above mentioned classical results see the mimeographed monograph Iordănescu [364g, Ch. VIII]. Anyway, the bibliography of the book Iordănescu [364w] contains all the references, as well as herein.

Let us mention now the construction, due to Truini and Biedenharn [672b], of a quantum mechanics for the complexified octonion plane. This plane, denoted by $P(J)$, as they showed (see [672b, p. 1337]), has automorphism group large enough to accommodate – as finite-dimensional quantum-mechanical charge spaces – a color-flavor structure which is not ruled out by current experimental evidence. The Truini-Biedenharn construction makes essential use of Jordan pairs. The construction of a quantum-mechanics over a complex octonion plane was begun by Gürsey [320a, b], without, however, using the concepts of linear ideals or Jordan pairs.

**Remark.** The Truini-Biedenharn plane $P(J)$ has a nonprojective geometry (two lines may intersect in more than one point) and, consequently, the propositions system is not a lattice.

As it is well known, the language of quantum mechanics has always been identified with the language of projective geometry, the points of the geometry being identified with the density matrices of the (pure) states, and the lines and hyperplanes with the propositions which are not atoms. The automorphism group of the geometry (that is, its collineation group) is, however, larger than the automorphism group of the quantum structure, because collineations need not preserve the traces (which are the canonical measure defining the quantum states) nor orthogonality, which has no projective meaning. In mathematical language we can say that the quantum logic requires an automorphism group which preserves an elliptic polarity.

Truini and Biedenharn [672b] defined the propositional system as follows: the propositions are identified with the geometrical objects (points and lines correspond to the principal inner ideals of $V$). They form a partially ordered set, with ordering given by the set inclusion of the inner ideals. The plane itself (i.e., the principal inner ideal generated by an invertible element) is the trivial proposition. We have an orthocomplementation $a \rightarrow a^\perp$, which is the
standard polarity $a_\star \to a^\star$. Thus we can define orthogonality: $a \perp b$ if $a < b^\perp$, which is symmetric.

**Remark.** As it is well known, the lattice axiom is the axiom least justified experimentally since it is nonconstructive. It is the merit of the Truini-Biedenharn construction that it provides a model in which this axiom is denied in a natural way.

Because of the lack of a lattice structure, the definition of “state” given by Truini and Biedenharn [672b] was suitable a “measure” with unusual properties thereby being defined. However, this measure coincides with the unique probability function (defined by G"unaydin, Piron and Ruegg [318] on the Moufang plane) when restricted to the real octonion case. Moreover, when restricted to the purely complex case, the measure coincides with the usual modulus (squared) of complex three-dimensional Hilbert space quantum mechanics.

**Open Problem.** (see [672b, p. 1329]). To obtain some kind of physical understanding of the role of the connected points which are responsible for all unusual features of Truini-Biedenharn quantum mechanics.

Finally, let us mention the opinion of Truini and Biedenharn [672b, p. 1328] that “It is our belief (noting the close relationship between geometries and quantum mechanics) that the concepts of quadratic Jordan algebras and inner ideals will be useful in physics.”

**Note.** For details on Truini & Biedenharn’s paper [672] see also Iordănescu [364w, §1 of Ch. 5].

Let us refer now on the survey paper [365] by Iordănescu and Truini, where an informal introduction to quantum groups is given, and the attention on the relationship among quantum groups, integrable models and Jordan structures was, in particular, called.

The historical and the basic approach of quantum group theory presented in the second section of [365] can be completed with more information by using, for instance, the surveys by Biedenharn [107a, b], Dobrev [214], Drinfeld [222a, b], Faddeev [241], Kundu [425], Majid [461a], Ruiz-Altaba [574], Smirnov [626], Takhtajan [656b] – used also by Iordănescu and Truini [365] – and, for an exhaustive information, the papers referred therein. For a deeper analysis of the concept of extended enveloping algebra and in particular for an exhaustive discussion on the ring of functions of the Cartan generators necessary in the construction of the extended enveloping algebra we refer to Truini and Varadarajan [673b].
I like to point out that in 1993, Boldin, Safin and Sharipov [117] proved a surprising connection between Tzitzeica surfaces and the inverse scattering method (see also [582] and [622], as well as the more recent papers [313] and [599]). The transformation that generates the family of such surfaces found by Tzitzeica [687] in 1907 and its slight generalizations obtained by Jonas [376a, b] in 1921 and 1953 are known in the modern literature on integrable equations as Darboux or Bäcklund tranformations. They are used to construct the soliton solutions starting from some trivial solution of the equation $u_{xy} = e^u - e^{-2u}$.

It is worth mentioning that the paper [678] by Tzitzeica seems to be the first in the world where the equation $u_{xy} = e^u - e^{-2u}$ (the nearest relative of the sine-Gordon equation $u_{xy} = \sin u$) was considered.

**Comment.** For a presentation of the main ideas in the work of Gheorghe Tzitzeica, see Teleman [660c,d] and Teleman & Teleman [659a,b,c].

In 1999, from the existing methods of singularity analysis only, Conte, Musette and Grundland [184] derived the two equations which define the Bäcklund transformation of the Tzitzeica equation. This is achieved by defining a truncation in the spirit of the approach of Weiss et al., so as to preserve the Lorentz invariance of the Tzitzeica equation. If one assumes a third-order scattering problem, then this truncation admits a unique solution, thus leading to a matrix Lax pair and a Darboux transformation. In order to obtain the Bäcklund transformation, which is the main new result in [184], one represents the Lax pair by an equivalent two-component Riccati pseudopotential. This yields two different Bäcklund transformations: the first one is a Bäcklund transformation for the Hirota-Satsuma equation, while the second is a Bäcklund transformation for the Tzitzeica equation. One of the two equations defining the Bäcklund transformation is the fifth ordinary differential equation of Gambier.

Also in 1999, Grundland and Levi [312] have shown that there is a strong relationship between Riccati equations and Bäcklund transformations for integrable nonlinear partial differential equations. As it has been established in many of the well-known cases (see [2]), the simplest Bäcklund transformation is given by the classical first-order Riccati equation. There are, however, a few well-known cases in which the simplest Bäcklund transformation is given by a higher-order differential equation. Grundland and Levi proved by a few examples (the Sawada-Kotera equation, the Tzitzeica equation, and the Fitzhugh-Nagumo equation) that in such a case the Bäcklund transformation is given

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It is worthy of being mentioned with this occasion that the title of the paper [376a] by Jonas contains the definition of Tzitzeica surfaces (see Bibliography).
by a higher Riccati equation, higher in the so-called Riccati chain.

The fact, shown in [312], that higher-order Riccati equations may also play a role in the construction of Bäcklund transformations for some nonlinear partial differential equations indicates the possibility of introducing higher-order conditional symmetries. This result can open the way to the construction of new classes of exact solutions for many physical important differential equations (see [127], [313]).

Note. Grundland and Levi mention at the end of their paper [312] that work on the extension of the above mentioned results to the case of matrix Riccati chains and their reduction in application to nonlinear partial differential equations is in progress.

In 1999, Ferapontov and Schief [255] reviewed some of the most important geometric properties of the Demoulin surfaces and constructed a Bäcklund transformation which may be specialized to the well-known Bäcklund transformation for the Tzitzeica equation governing affine spheres in affine geometry.

In 1997, Magri, Pedroni and Zubelli [459] tackled the problem of interpreting the Darboux transformation (see Darboux [191]) for the KP hierarchy and its relations with the modified KP hierarchy from a geometric point of view. This is achieved by introducing the concept of a Darboux covering. They constructed a Darboux covering of the KP equations and obtained a new hierarchy which they called the Darboux-KP hierarchy (DKP). Then they used the DKP equations to discuss the relationships among the modified KP equations and the discrete KP equations.

In 1999, Fastré [247] proposed a Grassmannian definition for the Darboux transformation. It generalizes two other previously used version of the Darboux transformation. This new Darboux transformation is used to build \( \tau \) functions from the solutions of the bispectral problem of Duistermaat and Grünbaum. These \( \tau \) functions are connected to the study of the \( W \)-algebra and the Virasoro algebra (highest weight vectors).

In 1996, Bakalov, Horozov and Yakimov [67] defined Bäcklund-Darboux transformation in Sato’s Grassmannian (see Sato [595a]), which can be regarded as Darboux transformations on maximal algebras of commuting ordinary differential operators. They described the action of these transformations on related objects: wave functions, \( \tau \)-functions, and spectral algebras.

In 2001, Musette, Conte, and Verhoeven [503] studied the Bäcklund transformation, nonlinear super-position formula of the Kaup-Kupershmidt and Tzitzeica equations.

In 2005, at a mathematical conference in Lubbock (USA), Erxiao Wang (University of Texas at Austin) gave a talk entitled Transformations of affine
He identified the classical Tzitzeica transformations for affine spheres as dressing actions of rational twisted loop group element and, then he discussed the permutability formula and the group structure of these transformations.

**Comments.** Taking into account of the previous considerations on the importance of Tzitzeica surfaces in the framework of recent mathematical physics researches, I think that it would be correct to call in the future the Bäcklund-Darboux transformations as Bäcklund-Darboux-Tzitzeica transformations. In fact, in 1998, on the occasion of the anniversary of 125 years from the birth of Tzitzeica – organized by the Faculty of Mathematics of the University of Bucharest – I suggested since then the above mentioned completion in the terminology.

**Remark.** Another contribution that I like to mention here is that of Beidar, Fong, and Stolin [76] which showed that every Frobenius algebra over a commutative ring determines a class of solutions of the quantum Yang-Baxter equation, which forms a subbimodule of its tensor square. Moreover, this subbimodule is free of rank one as a left (right) submodule. An explicit form of a generator is given in terms of the Frobenius homomorphism. It turns out that the generator is invertible in the tensor square if and only if the algebra is Azumaya.

At the end of the eighties other approaches to quantum groups were given. The objects of these approaches – which can be called quantum matrix groups – are Hopf algebras in duality to quantum algebras.

**Definition.** Two Hopf algebras \( \mathcal{A} \) and \( \mathcal{A}' \) are said to be in **duality** if there exists a doubly nondegenerate bilinear form

\[
\langle \cdot, \cdot \rangle : \mathcal{A} \times \mathcal{A}' \to \mathbb{C}, \quad \langle \cdot, \cdot \rangle : (a, a') \to (a, a'),
\]

such that, for \( a, b \in \mathcal{A} \) and \( a', b' \in \mathcal{A}' \) the following relations hold

\[
\langle a, a'b' \rangle = \langle \Delta_{\mathcal{A}}(a), a' \otimes b' \rangle, \quad \langle ab, a' \rangle = \langle a \otimes b, \Delta_{\mathcal{A}'}(a') \rangle,
\]

\[
\langle 1_{\mathcal{A}}, a' \rangle = \varepsilon_{\mathcal{A}'}(a'), \quad \langle a, 1_{\mathcal{A}'} \rangle = \varepsilon_{\mathcal{A}}(a), \quad \langle S_{\mathcal{A}}(a), a' \rangle = \langle a, S_{\mathcal{A}'}(a') \rangle.
\]

**Remark.** I want to emphasize that quantum groups are also involved and studied in many different fields of mathematics and physics. Among them are: topological quantum field theories, 2-dimensional gravity and 3-dimensional Chern-Simons theory [287, 314, 461a, 707c], rational conformal field theory [21, 296, 497], braid and knot theory [377, 391], non-standard
quantum statistics (see GREENBERG, O.W. [in Proc. Argonne Workshop on Quantum Groups, T. Curtright, D. Fairlie and C. Zachos (eds.), World Scientific, Singapore, 1990]), quantum Hall effect [409, 594].

Concerning the relations of Jordan structures with quantum groups, I would like to recall here the new topic proposed by Truini and Varadarajan at the end of their paper [673a], namely: quantization of Jordan structures.

Okubo [523f] considered the space $V$ endowed also with a bilinear non-degenerate form

$$
\langle y | x \rangle = \epsilon \langle x | y \rangle, \quad \epsilon = \pm 1.
$$

Let $R(\theta) \in \text{End}(V) \otimes \text{End}(V)$ be the scattering matrix with matrix elements $R^{dc}_{ab}(\theta)$, defined by $R(\theta)e_a \otimes e_b = R^{dc}_{ab}(\theta)e_c \otimes e_d$, with respect to a basis $\{e_j\}$ of $V$ and suppose that $R$ satisfies the quantum Yang-Baxter equation (QYBE)

$$
R_{12}(\theta)R_{13}(\theta')R_{23}(\theta'') = R_{23}(\theta'')R_{13}(\theta')R_{12}(\theta)
$$

with

$$
\theta' = \theta + \theta''.
$$

Two $\theta$-dependent triple linear products $[x,y,z]_\theta$ and $[x,y,z]^*_\theta$ are defined in terms of the scattering matrix elements $R^{dc}_{ab}(\theta)$, by

$$
[e^c, e_a, e_b]_\theta := e_d R^{dc}_{ab}(\theta), \quad [e^d, e_b, e_a]^*_\theta := R^{dc^*}_{ab}(\theta)e_c
$$
or, alternatively, by

$$
R^{dc}_{ab}(\theta) = \langle e^d | [e^c, e_a, e_b]_\theta \rangle = \langle e^c | [e^d, e_b, e_a]^*_\theta \rangle,
$$

where $e^d$ is given by

$$
\langle e^d | e_c \rangle = \delta^d_c.
$$

The QYBE (7.1a) can be then rewritten as a triple product equation

$$
\sum_{j=1}^N [v, [u, e_j, z]_\theta, [e^j, x, y]^*_\theta]_\theta = \sum_{j=1}^N [u, [v, e_j, x]^*_\theta, [e^j, z, y]_\theta]_\theta.
$$

**Proposition 1.** Let $V$ be a Jordan or anti-Jordan triple system with $\epsilon = 1$ satisfying the following conditions

i) $\langle u | xy \rangle = \langle v | yx \rangle$;
ii) \( \langle u|xvy \rangle = \delta \langle x|uyv \rangle = \delta \langle y|vxu \rangle \);

iii) \((ye^jx)v(e^jy) = a\{(x|v)y + \delta \langle y|v \rangle x \} + byvx;\)

iv) \((ye^jx)v(ze^jy) - (ye^jz)v(e^jy) = \alpha \{(v|x)zuy - \langle u|z \rangle xvy \} + \beta \{(v|y)xuz - \langle z|y \rangle vxu \} + \gamma \{ (yuv)ux - (yvu)ux \}\)

for some constants \(a, b, \alpha, \beta, \) and \(\gamma.\) Then

\[ [x, y, z]_\theta = P(\theta)xyz + B(\theta)\langle x|y \rangle z + C(\theta)\langle z|x \rangle y \]

for \(P(\theta) \neq 0\) is a solution of the QYBE (7.2) with

\[ \frac{B(\theta)}{P(\theta)} = \delta \gamma + k\theta, \quad \frac{C(\theta)}{P(\theta)} = \frac{\beta \delta}{k\theta} \]

for an arbitrary constant \(k,\) provided that we have either

i) \(\alpha = \beta = 0,\)

or

ii) \(\alpha = \beta \neq 0, b = -2\gamma, a = 2\beta.\)

Remark. The solution satisfies the unitarity condition

\[ R(\theta) R(-\theta) = f(\theta) \text{Id}, \]

where

\[ f(\theta) = P(\theta)P(-\theta) \left[ (a + \gamma^2) - (k\theta)^2 - \frac{\beta^2}{(k\theta)^2} \right]. \]

Proposition 2. Let \(V\) be the Jordan triple system defined on the vector space of the Lie-algebra \(u(n)\) by means of the product

\[ xyz = x \cdot y \cdot z + \delta z \cdot y \cdot x \]

the dot denoting the usual associative product in \(V\) and let \(\langle \cdot | \cdot \rangle\) be the trace form. Then,

\[ [x, y, z]_\theta = P(\theta)xyz + A(\theta)\langle y|z \rangle x + C(\theta)\langle z|x \rangle y \]

for \(P(\theta) \neq 0\) offers solutions of the QYBE (7.2) for the following two cases:

(7.3a) (i) \( \frac{A(\theta)}{P(\theta)} = \frac{\lambda^2 e^{k\theta} - d}{\lambda(e^{k\theta} - d)}, \quad \frac{C(\theta)}{P(\theta)} = \frac{e^{k\theta} - \lambda^2}{\lambda(e^{k\theta} - 1)} \)

where \(d\) is either \(\lambda^2\) or \(-\lambda^4\) and \(k\) is an arbitrary constant, or

(7.3b) (ii) \( \frac{A(\theta)}{P(\theta)} = -\lambda, \quad \frac{C(\theta)}{P(\theta)} = -\frac{1}{\lambda} \)
In both cases $\lambda$ is given by

$$\lambda = \frac{1}{2}(n \pm \sqrt{n^2 - 4}).$$

**Remark.** The first solution Eq. (7.3a) satisfies both unitarity and crossing symmetry relations:

(7.4a) \[ R(\theta)R(-\theta) = C(\theta)C(-\theta)\text{Id} \]

(7.4b) \[ \frac{1}{P(\theta)}[y, x, z]_\theta = \frac{1}{P(\theta)}[x, y, z]_\theta, \]

where $\theta$ in Eq. (7.4b) is related to $\theta$ by

$$\theta + \bar{\theta} = \frac{1}{k} \log d.$$

In view of these, the solution is likely related [603] to some exactly solvable two-dimensional quantum field theory.

**Remarks.** The Yang-Baxter as well as classical Yang-Baxter equations have been recast as triple product equation, and some solutions of these equations are obtained by Okubo [523g]. In his paper [250], Fauser emphasized a new direction for the application of Okubo’s method.

I want to mention a result by Svinolupov [648b] which is interesting in the context of this paper. He considered systems of nonlinear equations which, in a particular case, may be reduced to the nonlinear Schrödinger equation and are therefore called generalized Schrödinger equations. A one to one correspondence between such integrable systems and Jordan pairs is established. It turns out that irreducible systems correspond to simple Jordan pairs. Later, Svinolupov [648c] showed that to every finite-dimensional Jordan algebra with unity there corresponds a series of integrable vector systems. Among them there are vector analogues of the famous scalar equations as KdV, modified KdV and sine-Gordon. Some relations between a generalization of the Miura transformation connected with Jordan algebra is found.

In 1994, Svinolupov and Yamilov [650] built upon the results of Svinolupov [648b]: multidimensional generalizations of the Bäcklund transformations

$$\tilde{u} = u_{xx} - u^{-1}u_x^2 - u^2 v, \quad \tilde{v} = -u^{-1},$$
connecting two solutions \((\tilde{u}, \tilde{v}), (u, v)\) of the pair of coupled nonlinear equations

\[
\begin{align*}
    u_t &= u_{xx} - 2u^2v, \\
    v_t &= -v_{xx} + 2v^2u,
\end{align*}
\]

in \((1 + 1)\) dimensions, are discussed as well as corresponding Bäcklund transformations for some integrable generalizations of the Toda chain.

In a 1993 note, Svinolupov and Sokolov [649a] deal with Jordan tops, which are a special class of systems of quadratic first order differential equations with coefficients in a Jordan algebra. The general solution is found for some particular systems.

As it is well-known, a major tool in transferring results between Jordan theory and Lie theory is the Kantor-Koecher-Tits construction of a Lie algebra. This construction was first formulated for linear Jordan algebras and was modified as Jordan theory expanded to quadratic Jordan algebras, Jordan triple systems, and Jordan pairs. The reverse construction starts with a three-term \(\mathbb{Z}\)-graded Lie algebra and recovers a Jordan structure. However, since the Lie algebra involves only linear operations, it is impossible to recover the quadratic operations unless the base ring contains \(1/2\). The main purpose of the paper [248k] by Faulkner is to give constructions which substitute for the Kantor-Koecher-Tits construction and its reverse construction and which work for the quadratic operations for all base rings. The role of the Lie algebra is replaced by a certain kind of Hopf algebra. The primitive elements form a three-term \(\mathbb{Z}\)-graded Lie algebra, but the Hopf algebra also contains divided power sequences which capture the quadratic properties of the Jordan pair.

In the paper [82], Benkart and Perez-Izquierdo have constructed a quantum analogue of the split octonions and have studied its properties. By its construction, the quantum octonion algebra is a nonassociative algebra with a Yang-Baxter operator action coming from the \(R\)-matrix of \(U_q(D_4)\).

Comments. A quantized octonion algebra using the representation theory of \(U_q(sl_2)\) was independently constructed by Bremner in his 1997 preprint “Quantum octonions”. Although both Bremner’s quantized octonions and Benkart-Perez-Izquierdo’s quantum octonions reduce to the octonions at \(q = 1\), they are different algebras.

Remark. As Perez-Izquierdo [537] remarked, since the Albert algebra is close related to the octonion algebra, maybe it could be interesting to have it at hand when quantizing Jordan algebras.

I shall present some interesting results that lead us to believe that octonions or exceptional Jordan algebra should play an important role in recent fundamental physical theories, namely, in the theory of superstrings.
Let us briefly recall that the exceptional Jordan algebra made a dramatic appearance within the framework of supergravity theories through the work of Günaydin, Sierra and Townsend [319a, b, c, d]. In their work on the construction and classification of $N = 2$ Maxwell-Einstein supergravity theories, they showed that there exist four remarkable theories of this type that are uniquely determined by simple Jordan algebras of degree three. These are the Jordan algebras of $(3 \times 3)$-Hermitian matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$, and denoted, respectively, by $H_3(\mathbb{R})^{(+)}, H_3(\mathbb{C})^{(+)}, H_3(\mathbb{H})^{(+)}$, and $H_3(\mathbb{O})^{(+)}$. Their symmetry groups in five, four and three space-time dimensions give the famous magic square. From this largest one, namely the exceptional $N = 2$ Maxwell-Einstein supergravity defined by $H_3(\mathbb{O})^{(+)}$ emerge all the remarkable features of the maximal $N = 8$ supergravity theory in the respective space-time dimensions. In [265a,b, 271a,b,d,e] it was speculated that a larger theory that includes the exceptional $N = 2$ theory and the $N = 8$ theory may provide us with a unique framework for a realistic unification of all known interactions. Such a theory, if it exists, may well turn out to be a string theory (see Günaydin and Hyun [317a, p. 498]).

Remark. The work by Günaydin, Sierra and Townsend was reviewed by Truini in [671] which aimed at indicating the usefulness and naturalness of implementing the Jordan pair language in such theory.

A crucial question in superstring theory is the following: What mathematical structures have a large degree of uniqueness and can be associated with strings? Foot and Joshi suggested in [271a] that the exceptional Jordan algebra may be such a structure. This algebra is indeed unique as it is the only formally real Jordan algebra whose elements cannot be expressed in terms of real matrices. Although quantum mechanically superstring theories appear to be consistent only in ten space-time dimensions, classically superstring theories are consistent in space-time dimensions of 3, 4, 6 and 10. These dimensions are suggestive of the sequence of division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ whose respective dimensions correspond to the number of transverse degree of freedom in $d = 3, 4, 6$ and 10. These remarks prompted Foot and Joshi [271a] to look for mathematical structures which automatically single out $d = 3, 4, 6$ and 10 with $d = 10$ perhaps appearing special. They investigated the sequence of Jordan algebras $H_3(\mathbb{K})^{(+)}$ over $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ and showed that variables of the superstring can be interpreted as elements of the exceptional Jordan algebra $H_3(\mathbb{O})^{(+)}$.

The other algebras in this sequence correspond to classical superstring theories. One of the motivations for introducing the sequence of algebras $H_3(\mathbb{K})^{(+)}$ is that it is naturally supersymmetric: for $H_3(\mathbb{R})^{(+)}$, the spinor
corresponds to a Majorana spinor of $SO(2,1)$, for $H_3(C)^{(+)}$, the spinor corresponds to a Weyl spinor of $SO(3,1)$, for $H_3(H)^{(+)}$, the spinor corresponds to a Weyl spinor of $SO(5,1)$, and for $H_3(O)^{(+)}$, the spinor corresponds to a Majorana-Weyl spinor of $SO(9,1)$. In each case the number of spinor degrees of freedom agrees with the number of vector degrees of freedom. Thus the sequence automatically incorporates equal Bose and Fermi degrees of freedom.

In Foot and Joshi’s approach [271a], transverse Lorentz rotations are contained in the automorphism group of the algebra $H_3(K)^{(+)}$.

In conclusion, the classical superstring theories can be expressed in a unified way using sequence $H_3(K)^{(+)}$. Furthermore, the $d = 10$ case is especially interesting as it corresponds to the exceptional Jordan algebra $H_3(O)^{(+)}$.

As Foot and Joshi pointed out [271a], the Green-Schwarz [304a, b] superstring is not the only mathematically consistent candidate for a unified theory of all interactions. Nevertheless, Foot and Joshi analysed the superstring because of its central role in the other string theories. Of particular interest is the heterotic string (see Gross, Harvey, Martinec and Rohm [309a, b]), which can incorporate the exceptional gauge group $E_8 \otimes E_8$. The appearance of the exceptional group $E_8 \otimes E_8$ is interesting because $E_8$, like $F_4$, can be related to octonions.

In 1986, Witten [707a] made some interesting remarks concerning a new approach to string field theory. Witten attempted to interpret the interactions of the open bosonic string in terms of noncommutative differential geometry. Furthermore he suggested that closed bosonic strings may be connected with some kind of commutative but nonassociative algebra.

Motivated by Witten’s ideas, Foot and Joshi investigated in [271b] the incorporation of Jordan algebras, to obtain a manifestly commutative but nonassociative string theory. Namely, they showed that the free bosonic string theory can be reformulated using the special Jordan algebra. Then they proceeded to incorporate the exceptional Jordan algebra into the bosonic string. This leads to an exceptional group structures at the level of first quantization, which they interpreted as the appearance of the gauge group.

Remark. The appearance of the transformation group $SO(8)$ in Foot-Joshi’s approach [271b] suggests that a matrix of the exceptional Jordan algebra with fixed eigenvalues may be related to $d = 10$. It may thus be possible to incorporate this work into the heterotic string, which consists of closed bosonic strings in $d = 26$, and $d = 10$ fermionic strings [309a, b].

Let us mention now the work of Li, Peschanski, and Savoy [443] by which a generalization of no-scale supergravity models is presented, where scale transformations and axion-like classical symmetries of the superstrings
in four-dimensions are explicitly realized as dilatations and translations of the scalar fields in the Kähler manifold. A sufficient condition is that the (dimension one) dilation field matrices can be arranged in matrices of a Jordan algebra. This determines four possible classes of irreducible manifolds which are symmetric spaces.

Goddard, Nahm, Olive, Ruegg and Schwimmer [293] analysed the algebraic structure of dependent fermions, namely ones interrelated by the vertex operator construction. They are associated with special sorts of lattice systems which are introduced and discussed. The explicit evaluation of the relevant cocycles leads to the results that the operator product expansion of the fermions is related in a precise way to one or other of the division algebras given by \( \mathbb{C}, \mathbb{H}, \text{ or } \mathbb{O} \). In the paper [573], Ruegg showed that from the fermionic operator product expansion one can define a product with the same algebraic properties as the Jordan product.

The Goddard-Nahm-Olive-Ruegg-Schwimmer octonion result has an important physical application in the formulation of the superstring theory of particle interactions. The fermionic vertex operators related to octonions are associated with short roots of \( F_4 \) and fall into three orbits under the action of the Weyl group of \( D_4 \), the subalgebra of \( F_4 \) defined by its long roots \( D_4 = \text{so}(8) \) is the residue of the Lorentz invariance group of the superstring in the light cone gauge. In superstring theory the fermionic vertex operators are familiar and important constructions. For points of the orbit constituting vector weights of \( D_4 \) they are Ramond-Neveu-Schwartz fields. For one of the other two orbits, comprising spinor or conjugate spinor weights, they are the fermion emission-absorption vertices (see [294]). Thus the algebra of these quantities which is essential to the evaluation of superstring scattering amplitudes appears to be related to the algebra of octonions or to the exceptional Jordan algebra \( H_3(\mathbb{O})^{(+)} \).

Corrigan and Hollowood [186a,b] discussed how to represent Jordan algebras in terms of superstring vertex operators (see [293]). The analysis was explicitly carried out in the case of the exceptional Jordan algebra but applied similarly to non-exceptional Jordan algebras.

Let us mention also the works of Fairlie and Manonge [243], Sierra [620], Chapline and Günaydin [160], and Gürsey [320c] who speculated on the possible role that the exceptional Jordan algebra may play in the framework of string theories.

**Note.** For comments on a series of papers by Foot & Joshi [271c,d,e,f,g], published between 1988 and 1992, see §3 of Ch. 5 from the book Iordănescu [364w].
Gürsey [320e] used the Kaluza-Klein technique of compactification to reveal the deep correspondence between lattices generated by discrete Jordan algebras and symmetries of superstrings, which suggests that all known superstring theories are related and descend from a more general theory related to the Conway-Sloane transhyperbolic group.

Günaydin and Hyun [317a] gave a stringly construction of the exceptional Jordan algebra $H_3(O)^{(+)\ast}$. Specifically, they constructed $H_3(O)^{(+)\ast}$ using Fubini-Veneziano vertex operators. This is a very special application of a general vertex operator construction of nonassociative algebras and their affine extensions developed by Günaydin [316b]. This construction gives not only $H_3(O)^{(+)\ast}$ but also its natural affine extension in terms of the vertex operators.

Gürsey [320d] considered the discrete Jordan algebras of $(1 \times 1)$- $(2 \times 2)$- and $(3 \times 3)$-Hermitian matrices over integer elements of the four division algebras $R$, $C$, $H$ and $O$. They are transformed under discrete subgroups of groups associated with the magic square. Points corresponding to a discrete Jordan matrix belong to a lattice generated by Weyl reflections that are expressed by means of Jacobson’s triple product. Special cases include the $O(32)$, $E_8 \times E_8$ and $E_{10}$ lattices that occur in superstring theories.

**Note.** Various aspects of the connection between Kähler manifolds and string theories are examined by Rajeev [555] (see also Bowick and Rajeev [125]), Zanon [727], Cecotti, Ferrara, Girardello and Porrati [157].

**Comments.** The case of $(3 \times 3)$-Hermitian octonionic matrices is of particular interest because it corresponds to the exceptional Jordan algebra. As it is well known – and we partially recalled before – there have been numerous attempts to use this algebra to describe quantum physics, which was in fact Jordan’s original motivation. In 1994 and 1996, Schray [600a, b] showed how to use the exceptional Jordan algebra to give an elegant description of the superparticle, which Dray, Janesky, and Manogue [220] have been attempting to extend to the superstring. Their dimensional reduction scheme extends naturally to this case [221], and they believe it is the natural language to describe the fundamental particles of nature.

As it is well known, a soliton is a nonlinear wave whose properties are characterized as follows:

A. a localized wave propagates without changing its properties (shape, velocity, etc.);

B. localized waves are stable against mutual collisions and each wave conserves its individuality.

The first property has been known in hydrodynamics since the middle
of the last century as a solitary wave condition. The second means that the localized wave behaves like a particle. In modern physics, a suffix "on" implies the particle property, for instance, phonon and photon. In 1965, emphasizing the particle-like behaviour of the solitary wave, Zabusky and Kruskal called waves with the properties A and B "soliton". (For more historical details, see, for instance, Wadati and Akutsu [698].)

For an elementary introduction to Sato's theory we refer the reader to the paper [522] by Ohta, Satsuma, Takahashi and Tokihiro. Starting with an ordinary differential equation, introducing an infinite number of time variables, and imposing a certain time dependence on the solutions, they obtained the Sato equation which governs the time development of the variable coefficients. It is shown that the generalized Lax equation, the Zakharov-Shabat equation and the inverse scattering transform scheme are generalized from the Sato equation. It is also revealed that the $\tau$-function becomes the key function to express the solution of the Sato equation. By using the results of the representation theory of groups, they showed that the $\tau$-function is governed by the partial differential equations in the bilinear forms which are closely related to the Plücker relations.

Takasaki, inspired by Sato's theory for soliton equations, gave in [653a, b, c] a new approach to the self-dual Yang-Mills equations, which is an alternative method also based on the viewpoint of a complete integrability. It is remarkable, that the self-dual Yang-Mills equations admit such an approach parallel to Sato's approach to soliton equations.

**Remarks.** A close relationship with Mulase's method [501a, b, c, d, e] can be pointed out. An application of the above-mentioned Takasaki's approach would be expected to higher dimensional generalizations of gauge field equations.

Another application in eight dimensions was solved by Suzuki [646a, b] using Grassmann manifold method. Witten's gauge fields are interpreted by Suzuki [646c] as motions on an infinite-dimensional Grassmann manifold. Unlike the case of self-dual Yang-Mills equations in Takasaki's work [653a, b], the initial data must satisfy a system of differential equations since Witten equations comprise a pair of spectral parameters. Solutions corresponding to (anti-) self-dual Yang-Mills fields are characterized in the space of initial data and in application, some Yang-Mills fields which are not self-dual, anti-self-dual nor abelian can be constructed.

Let us also mention the Jimbo and Miwa's approach to the theory of soliton equations [374]. They considered an infinite-dimensional Lie algebra and its representation on a function space. The group orbit of the highest
weight vector is an infinite-dimensional Grassmann manifold. Its defining 
equations on the function space, expressed in the form of differential equations, 
are then exactly the soliton equations. To put it the other way, there is a 
transitive action of an infinite-dimensional group on the manifold of solutions.

Manin and Radul [468] gave a supersymmetric extension of the one-
component KP hierarchy as the Lax equations. The finite-dimensional ver-
sion of the KP hierarchy was called by Ueno the Grassmann hierarchy. In the 
theory of Grassmann hierarchy the fundamental role is played by a linear al-
gebraic equation which is called the Grassmann equation. Ueno and Yamada 
gave in [680a, b] a supersymmetric extension of one-component hierarchies 
from the viewpoint of the Grassmann equation. Their approach is slightly dif-
ferent from that of Manin and Radul [468]. Yamada generalized in [714] the 
results of [680a, b] to the multicomponent case. In [680c], Ueno and Yamada 
revealed that the super KP hierarchy is equivalently transformed to the super 
Grassmann equation that connects a point in the universal super Grassmann 
manifold with an initial data of a solution.

As Takasaki pointed out [653d], physicists have come to recognize the 
relevance of the theory of universal Grassmann manifold (sketched by Sato 
[595a]) to physical new topics, such as conformal field theories and strings 
(see Ishibashi, Matsuo and Ooguri [366], Vafa [683], Alvarez-Gaumé, Gomez 
and Reina [20], Witten [707b], Kawamoto, Namikawa, Tsuchiya and Yamada 
[397], Mickelsson [487a], see also Arbarello, De Concini, Kats and Procesi [41] 
for an application to the moduli geometry of algebraic curves which has a close 
relation to strings) and anomalies (see Mickelsson [487b, c] and Mickelsson and 
Rajeev [488]).

Almost all of them are based on the framework developed by Segal and 
Wilson [605] and Pressley and Segal [550]. Their functional analytical formulation 
have a number of advantages, and is now widely recognized as a standard 
framework. Admitting this fact, Takasaki has rewrote everything in the spirit 
of Sato [595a]. Their highly abstract and algebraic standpoint is fairly distinct 
from common sense of most physicists, who are much more familiar with the 
use of Hilbert spaces rather than abstract vector spaces.

A particular choice of affine coordinates on Grassmann manifolds, for 
both the finite- and infinite-dimensional case, made by Takasaki [653d] turns 
out to be very useful for the understanding of geometric structures therein. 
The so-called “Kac-Peterson cocycle”, which is physically a kind of “com-
mutator anomaly”, then arises as a cocycle of a Lie-algebra of infinitesimal 
transformations on the universal Grassmann manifold. These ideas are ex-
tended in [653d] to a multi-component theory. A simple application to a
nonlinear realization of current and Virasoro algebras is also presented for illustration in [653d].

Let us mention here the paper [705b] by Wilson, where “one of the more puzzling discoveries in the theory of integrable systems”, is re-examined.

Saito [585a] (see also [585b]) showed that the vertex operator of the three-bosonic-string interaction of Della Selva and Saito (see [199]) is an element of the universal Grassmann manifold. The correspondence between string theories and soliton theories is made explicite through the transformation of evolution parameters of solitons to string coordinates, the same transformation which relates Fay’s trisecant formula (see [253]) to Hirota’s bilinear difference equation (see [345a, b]).

Gilbert [289], based on the approach to infinite Grassmannian as the space of solutions of KP equations (see [616], [501a, b], [224]), described in simple terms the infinite sequence of non-linear partial differential equations (the KP equations) and gave possible applications to a fundamental description of interacting strings. Gilbert also indicated in [289] lines of research likely to prove useful in formulating a description of non-perturbative string configurations.

An interesting connection between Witten’s string field theory and the infinite Grassmannian, and the possible characterization of the group orbit on the Grassmannian by the bilinear identity are examined by Gao [280].

Awada and Chamseddine introduced [56a] the infinite-dimensional graded Grassmann manifolds in terms of free field operators and studied their properties. They showed the embedding of the graded Diff $S^1/S^1$ manifold in the graded Grassmannians, and commented on the possible supersymmetric KP hierarchy.

Let us recall at this point that there are two attractive views of string theory, both based on holomorphic geometry. The first is the formulation of quantum string theory as integrable analytic geometry on the universal moduli space of Riemann surfaces. The second is based on the concept of loop space and formulated as a holomorphic vector bundle over the manifold Diff $S^1/S^1$. In both cases, there exists an one-to-one embedding of the base manifold into the infinite-dimensional Grassmannians.

In 1987, Awada and Chamseddine [56b] formulated the closed string theory as Hermitian geometry on Grassmannians.

**Open Problem.** (see [56b]). Generalise the Awada-Chamseddine approach [56b] to the closed superstring and heterotic string.

As I already mentioned, Segal and Wilson [605], and Pressley and Segal
[550] developed a framework which is a different approach to infinite Grassmannians. It consists of the space of choices of fermion boundary conditions for the free fermion field theory on a disc. In the paper [705a] is described how the modified KdV equations fit into the Grassmannian framework, topic not touched in the paper [605]. In 1988, Witten [707b] clarified some aspects of the relation between quantum field theory and infinite-dimensional Grassmannians. More precisely, he described in physical terminology some aspects of relation, surveyed by Segal and Wilson [605] between Riemann surfaces and infinite-dimensional Grassmannians. This relation has been essential in the studies of the Schottky problem (see Mulase [501b], Shiota [616]), and its relation with quantum field theory and string theory have been subject of a discussion from a physical point of view (see Ishibashi, Matsuo, Ooguri [366], Alvarez-Gaumé, Gomez, Reina [20], Vafa [683]).

Mickelsson and Rajeev [488] extended the methods of Pressley and Segal [550] for constructing cocycle representations of the restricted general linear group in infinite dimensions to the case of a larger linear group modeled by Schatten classes of rank \( p, 1 \leq p < \infty \) (see Simon [622]). An essential ingredient is the generalization of the determinant line bundle over an infinite-dimensional Grassmannian to the case of an arbitrary Schatten rank \( p \geq 1 \). The results are used to obtain highest weight representations of current algebras in \((d + 1)\) dimensions when the space dimension \( d \) is any odd number.

**Conjecture** (see [608]). Similar problems to that of Mickelsson and Rajeev [488] must afflict the electric field operators constructed by Semenoff in [608].

In 1988, Yamagishi [715] pointed out an interesting relation between the KP hierarchy and the extended Virasoro algebra, namely, he showed that the simply extended KP equation has enough information to determine the extended Virasoro algebra. Levi and Winternitz [441] showed that a class of integrable nonlinear differential equations in \((2 + 1)\) dimensions, including the physically important cylindrical KP equation, has a symmetry algebra with a specific Kac-Moody-Virasoro structure. Kodama [407] presented a systematic method to produce a class of exact solutions of the dispersionless KP equation, using the conservation equations derived from the semi-classical limit of the KP theory. These exact solutions include rarefaction waves (global solutions) and shock waves (breaking solutions in finite time). Zabrodin [724] proved that the scattering matrix for free massless fermions on a Riemann surface of finite genus generates the quasiperiodic solutions of the KP equation. The operator changing the genus of the solution is constructed and the composition law of such operators is discussed. Zabrodin’s construction extends the well-
known operator approach in the case of soliton solutions to the general case of the quasiperiodic $\tau$-functions. David, Levi and Winternitz [196] constructed a general class of fourth order scalar partial differential equations, invariant under the same group of local point transformations as the KP equation.

Recently, Dimakis & Müller-Hoissen [207a] proved that on any "weakly nonassociative" algebra there is a universal family of compatible ordinary differential equations (provided that differentiability with respect to parameters can be defined), any solution of which yields a solution of the KP hierarchy with dependent variable in an associative subalgebra, the middle nucleus. More recently, they derived in [207b] a sequence of identities in the algebra of quasi-symmetric functions that are in formal correspondence with the equations of the KP hierarchy.

De Concini, Fucito and Tirozzi (see [198], [278b]) formulated conformal field theories on the infinite-dimensional Grassmann manifold. Beside recovering the known results for the central charge and correlation functions of the $b$-$c$ system, this formalism immediately leads itself to further generalization. The Grassmann manifold is in fact an ad hoc model for the geometrical interpretation of the irreducible representations of an infinite-dimensional Kac-Moody algebra which, in turn, admit an intertwining action of a Virasoro algebra. They also gave a proof of bosonization from a purely Grassmann manifold point of view.

In 1992, Anagnostopoulos, Bowick, and Schwarz [26] determined the space of all solutions to the string equation of the symmetric unitary one-matrix model.

Aoyama and Kodama [34] generalized the Sato equation of the KP theory on the basis of the $W_{1+\infty}$ algebra. An infinite set of flows which commute with the KP hierarchy (the commuting KP flows) are explicitly constructed. They satisfy the $W_{\infty}$ algebra. Aoyama and Kodama studied the generalized Sato equation by making a $p$-truncation of the dressing pseudo-differential operator, instead of the usual $p$-reduction. The cases of 1- and 2-truncations were studied in some details. The commuting KP flows become $W_{\infty}$ operators of the same form for both cases, when operated on the $\tau$-function, and the central charge if the Virasoro algebra is found to be $-2$.

Both finite- and infinite-dimensional integrable systems can be linearized on orbits of the infinite Abelian group $\Gamma^+$ on the universal Grassmann manifold. The aim of Landi and Reina [428] is to link these results to standard symplectic dynamics by giving an explicit Hamiltonian formulation on the
symplectic manifold

\[ M = \text{Gl}_{\text{res}}(\mathcal{H})/\text{Gl}(\mathcal{H}_+) \times \text{Gl}(\mathcal{H}_-). \]

**Note.** Let us recall here that, in 1996, Bakalov, Horozov, and Yakimov [67] defined Bäcklund-Darboux transformations in Sato’s Grassmann manifold.

In 1997, van de Leur [439], making use of the representation theory of infinite matrix group, showed that (in the polynomial case) the \( n \)-vector \( k \)-constrained KP hierarchy has a natural geometrical interpretation on Sato’s Grassmann manifold. His description generalizes the \( k \)-reduced KP or Gelfand-Dickey hierarchies.

In 1992, Bottacin [121a] showed that to each element in a large class of solutions to the KP hierarchy there is associated, in a natural way, a group variety. This result was achieved by showing that these solutions are in fact functions of theta type. In [121b] he restricted himself to a class of solutions of particular interest in physics, namely the so-called \( N \)-solitons. He proved that an \( N \)-soliton to the KP hierarchy is actually a holomorphic theta type and its associated group variety is a product of multiplicative groups. Hence the same conclusions also hold for the KdV and all other hierarchies which can be obtained as specializations of the KP. Moreover, all these constructions are made explicitly.

In the note [36], Aratyn develops an explicit construction of the constrained KP hierarchy within the Sato Grassmannian framework. Useful relations are established between the kernel elements of the underlying ordinary differential operator and the eigenfunctions of the associated KP hierarchy, as well as between the related bilinear concomitant and the squared eigenfunction potential.

In 1999, in Subsection 3.1 of their paper [92], Bergvelt and Rabin introduced an infinite super Grassmannian and related constructions. The infinite Grassmannian of Sato [595a] or of Segal & Wilson [605] consists (essentially) of “half-infinite-dimensional” vector subspaces \( W \) of an infinite-dimensional vector space \( \mathcal{H} \) such that the projection on a fixed subspace \( \mathcal{H}_- \) has finite-dimensional kernel and cokernel. In the super category, Bergvelt and Rabin replace this by the super Grassmannian of free, “half-infinite-rank” \( \Lambda \)-modules of infinite-rank, free \( \Lambda \)-module \( H \) such that the kernel and cokernel of the projection on \( \mathcal{H}_- \) are a submodule and a quotient module, respectively, of a free, finite-rank \( \Lambda \)-module.

**Open Problem.** It is well-known that the infinite-dimensional Grass-
mann manifold contains moduli spaces of Riemann surfaces of all genera. Taking into account this fact, Schwarz [602b] conjectured that non-perturbative string theory can be formulated in terms of the Grassmannian.

In [602b] Schwarz presented new facts supporting the conjecture contained in the above mentioned Open Problem. In particular, it is shown that Grassmannians can be considered as generalized moduli spaces; this statement permits to Schwartz to define corresponding “string amplitudes” (at least formally).

Open Problem. Another conjecture formulated by Schwarz [602b] is the following: It is possible to explain the relation between non-perturbative and perturbative string theory by means of localization theorems for equivariant cohomology. (N.B. This conjecture is based on the characterization of moduli spaces, relevant to string theory, as sets consisting of points with large stabilizers in certain groups acting on the Grassmannian.)

Comments. In the eighties, Sasaki remarked that all differential equations with soliton solutions (known to him) could be considered as immersion equations for a two-dimensional surface into some Euclidean space. Later, Chern and Terng have worked on this idea for Sine-Gordon and the modified KdV equation, but in contrast to Sasaki’s method, they considered surfaces of constant negative Gauss curvature. It is also possible to work with other types of surfaces (e.g., surfaces of constant mean curvature). Gerold and Buchner [286] gave an explicit solution of the immersion equations for the one- and two-soliton solution of the Sine-Gordon equation. The surface corresponding to the one-soliton solution is Dini’s surface (i.e., a helicoid of Gauss curvature \((-1)\) with a tractrix as profile curve), the one corresponding to the two-soliton solution is a Joachimsthal surface of Enneper’s type that has not yet been discussed explicitly in literature. These results stimulate the question how the properties of the solitons are transferred to the surface. More important are the following:

Open Problems. (see Gerold and Buchner [286, p. 2056]). Which immersions yield soliton solutions? Can all solitons be obtained in this way?

In this section I shall comment on very recent applications of two kinds of important mixed Jordan and Banach structures (namely, $JB^*$-triple and $JBW^*$-triples) to quantum mechanics, emphasizing on this occasion the decoherent states and the geometry of projection operators.

In 2008, Edwards and Hügli (see [228]) have considered pre-symmetric complex Banach spaces which have been proposed as models for state spaces
of physical systems.

A complex Banach space $A_*$ is said to be pre-symmetric if the open unit ball in its Banach dual space $A$ is a bounded symmetric domain. In this case $A$ possesses a natural triple product with respect to which it forms a $JBW^*$-triple with unique predual $A_*$. Consequently, there exists a bijection between the set of pre-symmetric spaces and the set of $JBW^*$-triples. An important property of a $JBW^*$-triple $A$ is that the group of linear isometries of $A$ to itself coincides with the group of triple automorphisms $\text{Aut}(A)$ of $A$, and because of the uniqueness of the pre-dual $A_*$ of $A$, each such mapping is weak*-continuous, which implies that there exists an isomorphism $\phi \rightarrow \phi_*$ from $\text{Aut}(A)$ onto the group $\text{Aut}(A_*)$ of linear isometries of $A_*$. 

Remark. There exist approaches to the theory of statistical physical systems, in which a pre-symmetric space $A_*$ represents the state space of the system, the linear isometries representing the symmetries of the system and the contractive projections on $A_*$ representing filters on the system (see Friedman [273a, b] and Friedman & Gofman [274a, b]).

Edwards & Hugi [228] have defined structural projections as follows: A contractive projection $R$ on the pre-symmetric space $A_*$ is said to be structural if, for each element $x$ of $A_*$ such that $Rx$ and $x$ have equal norm, it follows that $Rx$ and $x$ coincide, and, if $x$ is an element of $A_*$ for which, for all elements $y$ in $A_*$,

$$\|x + Ry\| = \|x\| + \|Ry\|,$$

then $x$ lies in the kernel of $R$.

The range $RA_*$ of a structural projection is said to be a structural subspace of $A_*$. 

Remarks. If pre-symmetric spaces are considered as state spaces, then the physical aspects are more transparent when viewed as properties of structural projections and structural subspaces of $A_*$ (see [228, p. 220]). From the point of view of physical systems, the two conditions that a contractive projection $R$ must satisfy in order to be structural have powerful physical motivations (for details, see [228, p. 221]).

Two structural projections on the pre-symmetric space represent decoherent operations when their ranges are rigidly collinear. Edwards and Hugi have proved in [228] that, for decoherent elements $x$ and $y$ of $A_*$, there exists an involutive element $\phi_x$ in $\text{Aut}(A_*)$ which conjugates the structural projections corresponding to $x$ and $y$, and conditions are found for $\phi_x$, to exchange $x$ and $y$. The results are used to investigate when certain subspaces of $A_*$ are
the ranges of contractive projections and, therefore, represent systems arising from filtering operations.

**Remark.** The results of the paper [228] depend crucially upon the detailed study of pairs of rigidly collinear tripotents in $JBW^*$-triples.

In 2007, Hügli has investigated in [353c] normal contractive projections in connection with certain algebraic conditions on generalized operators.

I must mention here the paper [353b] by Hügli, where the set of tripotents in a $JB^*$-triple is characterized in various ways. This paper is one of the supporting papers for [228], another supporting paper being the paper [354] by Hügli and Mackey.

A very recent paper of Hügli, namely [353d], is based on the results contained in [353c] and [354]: more exactly, it generalizes the results of [353c] (which is mainly about Hilbert spaces as subtriples) to a more general class of subtriples (those having the Dunford-Pettis property).

Edwards and Rüttimann presented in their survey [230] a tentative description of certain results, concerning orthomodular partially ordered sets of tripotents in $JBW^*$-triples, for an approach to a characterization of latices of events of physical systems.

The expository paper [682k] by Upmeier gives a survey of results concerning harmonic analysis and quantization of geometric phase spaces associated with Jordan structures. As Upmeier pointed out (see [682k], p. 301–302), "... recently it has become clear that quantization theory can be carried to new directions, namely:

1. symmetric supermanifolds (replacing Jordan algebras by superalgebras),
2. deformation quantization (replacing the structure group of a Jordan algebra by a so-called quantum group), and
3. non-convex symmetric cones and tube domains (replacing holomorphic functions by cohomology classes).

In all these projects it is expected that the fine structure of Jordan systems will play a crucial role. ..."

**Note.** For a survey of the Upmeier’s work related to the Toeplitz-Berezin type of quantization of symmetric (and related) domains, see [682m].

In his big survey [37], Arazy presented some basic facts and developments concerning invariant Hilbert spaces of analytic functions on bounded symmetric domains, the Jordan theoretic background being $JB^*$-triples.

In his paper [273a], Friedman presented in detail two examples in theor-
ical physics where $JB^*$-triples appear. He showed that the Möbius-Potapov-Harris transformations of the automorphism group of a bounded symmetric domain occur as transformations of signals in an ideal transmission line and as velocity transformations between two inertial systems in special relativity.

In their 2001 paper, Friedman and Russo [275] proposed a triple product representation of the canonical anticommutation relations which does not make use of the associative Clifford algebra. Imposition of these commutation relations on the natural basis of $C^n$ defines a triple product making $C^n$ into a Cartan factor of type 4 (called also spin factor) that the authors denote by $S^n$.

This Jordan structure (i.e., $JB^*$-triple structure) is used to represent the Lorentz group on $S^3$ and $S^4$. The irreducible representation on $S^3$ corresponds to the relativistic transformations of the electro-magnetic field. The irreducible spin-1 representation on $S^4$ extends the Lorentz space-time transformation. By taking the self-adjoint part of this representation with respect to the spin conjugation, a reducible spin-1/2 representation on $S^4$ results. The latter is shown to induce two spin-1 representations in the space of determinant preserving maps on $S^4$ showing that the same spin factor could be used to represent the two types of elementary particles: bosons and fermions.

Let us mention some recent (or very recent) contributions of interest.

Bordemann & Walter [120] showed that for each semi-Riemannian locally symmetric space the curvature tensor gives rise to a rational solution of the classical Yang-Baxter equation with spectral parameters. For several Riemannian globally symmetric spaces $M$ such as real, complex and quaternionic Grassmann manifolds, they explicitly compute solutions of the quantum Yang-Baxter equations (represented in the tangent space of $M$) which generalize results of Zamolodchikov & Zamolodchikov [726].

Nichita & Popovici [516] used $(G, \theta)$-Lie algebras (which unify the Lie algebras and Lie superalgebras) to produce solutions for the quantum Yang-Baxter equation. The constant and the spectral-parameter Yang-Baxter equations and Yang-Baxter systems are also studied in [516].

Burhan & Henrich [147], making use of geometric methods due to Polishchuk [545] and Burban & Kreussler [148], studied unitary solutions of the associative Yang-Baxter equation with spectral parameters.

Other papers of interest are [7] by Akrami & Majid, and [124] by Bouwknegt, Hannabuss, and Mathai. Concerning quantum projective spaces, very recent papers are [189] by D’Andrea, Dabrowski, and Landi, [188] by D’Andrea & Dabrowski, and [190] by D’Andrea & Landi.
Concluding comments. There are almost 80 years since the appearance of the paper [379b] by Pasqual Jordan, in which he has tried to use for the first time algebraic structures defined by himself (called – since 1946 – Jordan algebras) for a mathematical description of quantum mechanics. During this period, Jordan algebras have found applications in projective geometry, algebraic geometry, differential geometry, various chapters of mathematical analysis, differential equations, probability, genetics, statistics, and finally also returned to physics. But, this time, as it is easily remarked, Jordan structures are involved mainly as additional mathematical structures: one could say – in few words – that, concerning the applications of Jordan structures to physics, they are necessary but not sufficient! A recent interesting paper devoted to these considerations was written by Bertram and it is entitled “Is there a Jordan geometry underlying quantum mechanics?” (see Bertram [98s]). This rather big paper (it has 30 pages) is addressed “to readers coming from physics rather than from mathematics” (see [98s, p. 2]) and it is very interesting also from philosophical point of view. In this paper, generalized projective geometries have been proposed as a framework for a geometric formulation of quantum theory. The above mentioned paper [98s] by Bertram ends as follows:

“... it could be hoped that Jordan geometry gives some hints on what the last two items on the following matrix might be:

| geometry: | linear; affine | projective | manifold |
| mechanics: | classical | special relativistic | general relativistic |
| quantum theory: | Hilbert space q.m. | projective q.m.? | Cartan geometric q.m.? |

Let us mention that for Cartan connection (which generalizes the projective and conformal connections) one could read a modern presentation given by Sharpe in [613].

In a more recent note, Bertram [98t] refines his assertion from his previous paper [98s] (i.e., generalized projective geometries could be a framework for a geometric formulation of quantum theory), by discussing further structural features of quantum theory: the link with associative involutive algebras $\mathcal{A}$ and with Jordan-Lie and Lie-Jordan algebras. The associated geometries are (Hermitian) projective lines over $\mathcal{A}$. Let us recall here that – more than twenty years ago – V.S. Varadarajan wrote “... quantum mechanical systems are those whose logic form some sort of projective geometry” (see [686, p. 6]). It is worth being mentioned here that Bertram wrote his note [98t] in September 2008, after he attended the XXVII Workshop on Geometrical Methods in Physics from Białowieża (Poland), where he had the occasion to discuss on this topic with physicists.
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