Note on fractional integral inequalities using generalized k-fractional integral operator

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Abstract
The aim of this paper is to obtain several fractional integral inequalities involving convex functions by using generalized k-fractional integral operator.

Keywords
Generalized k-fractional integral, convex functions and inequalities.

AMS Subject Classification
26A99, 26D10.

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1. Introduction
Fractional inequalities play major role in the development of fractional differential, integral equations and other fields of sciences and technology. Recently, a number of mathematician have studied different results about fractional integrals such as Riemann-Liouville, Hadamard, Saigo, Erdelyi-Kober, q-fractional integral and some other operators, see [1, 2, 5, 6, 8–12, 15–18, 20–22]. In [7], authors have studied inequalities using Saigo fractional integral.

**Theorem 1.1.** Let \( f, h \) be two positive continuous functions on \( [0, \infty) \) and \( f \leq h \) on \( [0, \infty) \). If \( \frac{f}{h} \) is decreasing, \( f \) is increasing on \( [0, \infty) \) and for any convex function \( \phi \), \( \phi(0) = 0 \), then we have inequality

\[
\int_{0,\eta}^{\alpha,\beta,\eta} \frac{f(t)}{h(t)} \geq \int_{0,\eta}^{\alpha,\beta,\eta} \frac{\phi(f(t))}{\phi(h(t))},
\]

and

**Theorem 1.2.** Let \( f, h \) be two positive continuous functions on \( [0, \infty) \) and \( f \leq h \) on \( [0, \infty) \). If \( \frac{f}{h} \) is decreasing, \( f \) is increasing on \( [0, \infty) \) and for any convex function \( \phi \), \( \phi(0) = 0 \), then we have inequality

\[
\int_{0,\eta}^{\alpha,\beta,\eta} \frac{\phi(f(t))}{\phi(h(t))} \leq \int_{0,\eta}^{\alpha,\beta,\eta} \frac{f(t)}{h(t)},
\]

where for all \( t > 0 \), \( \alpha > \max\{0, -\beta\}, \beta < 1, \beta - 1 < \eta < 0, \delta < 1, \delta - 1 < \zeta < 0 \).

In the literature, some fractional inequalities are obtain by using Generalized k-fractional integral operator, see [3, 4, 13, 14, 17, 19, 21]. Motivated by above work in this paper we have obtained some new inequalities using generalized k-fractional integral (in terms of Gauss hypergeometric function) operator for convex functions.

2. Preliminaries
Here, we devoted to basic concepts of Generalized k-fractional integral.

**Definition 2.1.** Two function \( x \) and \( y \) are said to synchronous (asynchronous) on \( [a, b] \), if

\[
((x(s) - x(t))(y(s) - y(t))) \geq (\leq) 0,
\]

for all \( s, t \in [0, \infty) \).
Definition 2.2. [14, 23] The function $x(s)$, for all $s > 0$ is said to be in the $L_{p,k}[0,\infty)$, if

$$L_{p,k}[0,\infty) = \{x : \|x\|_{L_{p,k}[0,\infty)} = \left( \int_0^\infty |x(s)|^p s^k \, ds \right)^{\frac{1}{p}} < \infty \}.$$  \hspace{1cm} (2.2)

Definition 2.3. [14, 23, 24] Let $f \in L_{1,k}[0,\infty)$. The generalized Riemann-Liouville fractional integral $I_{\pi,k}^\alpha f(x)$ of order $\alpha, k \geq 0$ is defined by

$$I_{\pi,k}^\alpha f(x) = \frac{(k+1)^{1-\alpha}}{\Gamma(\alpha)} \int_0^x (x^k - t^k)^{\alpha-1} f(t) \, dt.$$ \hspace{1cm} (2.3)

Definition 2.4. [14, 23] Let $k \geq 0, \alpha > 0, \mu > -1$ and $\beta, \eta \in R$. The generalized $k$-fractional integral $I_{\pi,k}^\alpha \eta f(x)$ (in terms of the Gauss hypergeometric function) of order $\alpha, k \geq 0$ is defined by

$$I_{\pi,k}^\alpha \eta f(x) = \frac{(k+1)^{\mu+1}(\alpha+\beta+2\mu)}{\Gamma(\alpha)} \int_0^x (x^k - t^k)^{\alpha-1}(\alpha+\beta+2\mu) f(t) \, dt,$$ \hspace{1cm} (2.4)

where, the function $2F_1(-)$ in the right-hand side of (2.4) is the Gaussian hypergeometric function defined by

$$2F_1(a, b; c; t) = \sum_{n=0}^\infty \frac{(a)_n(b)_n}{(c)_n} t^n$$ \hspace{1cm} (2.5)

and $(a)_n$ is the Pochhammer symbol

$$(a)_n = a(a+1) \ldots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad (a)_0 = 1.$$  

Consider the function

$$\hat{\mathcal{F}}(t, \tau) = \frac{(k+1)^{\mu+1}(\alpha+\beta+2\mu)}{\Gamma(\alpha)} \tau^{(k+1)\mu} \int_0^\tau (\tau^k - t^k)^{\alpha-1}(\alpha+\beta+2\mu) f(t) \, dt,$$  

Multiplying both sides of (2.4) by $\hat{\mathcal{F}}(t, \tau)$ which is positive, and integrating obtained result with respect to $\tau$ from $0$ to $t$, we have

$$p(r) r^\mu \int_0^t \left[ \frac{\hat{\mathcal{F}}(t, \tau)}{p(r)} \right] \, d\tau = \left( \int_0^t \left[ \frac{\hat{\mathcal{F}}(t, \tau)}{p(r)} \right] \, d\tau \right) r^\mu.$$  

It is clear that $F(t, \tau)$ is positive because for all $\tau \in (0, t)$, $(t > 0)$, since each term of the (2.6) is positive.

### 3. Fractional integral inequalities involving convex functions

In this section, we prove some fractional integral inequalities involving convex function using generalized k-fractional integral (in terms of Gauss hypergeometric function) operator.

**Theorem 3.1.** Let $p, r$ be two positive continuous functions on $[0, \infty)$ and $p \leq r$ on $[0, \infty)$. If $\frac{\tau}{p}$ is decreasing, $p$ is increasing on $[0, \infty)$ and for any convex function $\Phi, \Phi(0) = 0$, then for all $k \geq 0, t > 0, \pi > \max\{0, -\sigma - \nu\}, \sigma < 1, \nu > -1, \sigma - 1 < 0$, we have,

$$\frac{\pi \eta}{\eta\nu \tau} \Phi(p(t)) \geq \frac{\pi \eta}{\eta\nu \tau} \Phi(p(\tau)).$$ \hspace{1cm} (3.1)

**Proof:** If the function $\Phi$ is convex with $\Phi(0) = 0$, then the function $\frac{\Phi(t)}{t}$ is increasing. Since $p$ is increasing, then $\frac{\Phi(p(t))}{p(t)}$ is also increasing. Clearly $\frac{\pi \eta}{\eta\nu \tau}$ is decreasing, for all $\tau, p \in [0, \infty)$, and

$$\left( \frac{\Phi(p(t))}{p(t)} - \frac{\Phi(p(\tau))}{p(\tau)} \right) \left( \frac{p(t)}{r(t)} - \frac{p(\tau)}{r(\tau)} \right) \geq 0,$$ \hspace{1cm} (3.2)

which implies that

$$\frac{\Phi(p(t))}{p(t)} - \frac{\Phi(p(\tau))}{p(\tau)} \frac{p(\tau)}{r(\tau)} \geq 0.$$ \hspace{1cm} (3.3)

Multiplying equation (3.3) by $r(t)r(\tau)$, we have

$$\frac{\Phi(p(t))}{p(t)} - \frac{\Phi(p(\tau))}{p(\tau)} \frac{p(\tau)}{r(\tau)} \geq 0.$$ \hspace{1cm} (3.4)

Multiplying both sides of (3.4) by $\hat{\mathcal{F}}(t, \tau)$ which is positive, and integrating obtained result with respect to $\tau$ from $0$ to $t$, we have

$$p(r) r^\mu \int_0^t \left[ \frac{\hat{\mathcal{F}}(t, \tau)}{p(r)} \right] \, d\tau = \left( \int_0^t \left[ \frac{\hat{\mathcal{F}}(t, \tau)}{p(r)} \right] \, d\tau \right) r^\mu.$$ \hspace{1cm} (3.5)

Multiplying both sides of (3.5) by $\hat{\mathcal{F}}(t, \tau)$ which is positive, and integrating obtained result with respect to $\tau$ from $0$ to $t$,
we have
\[
\frac{\|p(t)\|}{\|p(t)\|} \leq \frac{\|\Phi(p(t))\|}{\|\Phi(p(t))\|} - r(t).
\]
(3.11)

Hence, from (3.8) and (3.11) we obtain required inequality (3.1).

**Theorem 3.2.** Let \(p, r\) be two positive continuous functions on \([0, \infty)\) and \(f \leq r\) on \([0, \infty)\). If \(\frac{\xi}{r}\) is decreasing, \(p\) is increasing on \([0, \infty)\) and for any convex function \(\Phi, \Phi(0) = 0\), then for all \(k \geq 0, t > 0, \nu > \max\{0, -\sigma - \nu\}, \sigma, \delta < 1, \nu > -1, -\sigma < \theta < 0, 0 < \xi < \zeta < 0, \) we have,
\[
\frac{\|\Phi(p(t))\|}{\|\Phi(p(t))\|} \leq \frac{\|\Phi(r(t))\|}{\|\Phi(r(t))\|} - r(t).
\]
(3.11)

**Proof:** - If function \(\Phi\) is convex with \(\Phi(0) = 0\), then \(\frac{\Phi(p(t))}{\Phi(r(t))}\) is increasing. Since \(p\) is increasing, then \(\Phi(p(t))\) is also increasing. Clearly \(\frac{\Phi(p(t))}{\Phi(r(t))}\) is decreasing, for all \(\tau, \rho \in [0, t)\) \(t > 0\). Multiplying equation (3.5) by \(\|p(t)\|\|p(t)\| = \|p(t)\|\|p(t)\|\)

\[
(k^{(k+1)u}^\gamma \delta + \nu, -\zeta; \gamma; 1 - (\frac{\xi}{r})^{k+1}) (p \in (0, t),
\]

\(t > 0\), which remains positive from (2.4). Now integrating obtained result with respect to \(p\) from \(0\) to \(t\), we have
\[
\int_{\tau}^{\gamma} \frac{\|\Phi(p(t))\|}{\|\Phi(r(t))\|} \leq \frac{\|\Phi(p(t))\|}{\|\Phi(r(t))\|} = \int_{\tau}^{\gamma} \frac{\|\Phi(p(t))\|}{\|\Phi(r(t))\|} - r(t).
\]
(3.13)

Since \(p \leq r\) on \([0, \infty)\) and \(f \leq r\) on \([0, \infty)\). If \(\frac{\xi}{r}\) is decreasing, for all \(\tau, \rho \in [0, t)\) \(t > 0\), we have
\[
\frac{\|\Phi(p(t))\|}{\|\Phi(r(t))\|} \leq \int_{\tau}^{\gamma} \frac{\|\Phi(p(t))\|}{\|\Phi(r(t))\|} - r(t).
\]
(3.14)

Multiplying both sides of (3.14) by \(\|\Phi(p(t))\|\) and integrating obtained result with respect to \(\tau\) from \(0\) to \(t\), we have
\[
\frac{\|\Phi(p(t))\|}{\|\Phi(r(t))\|} \leq \int_{\tau}^{\gamma} \frac{\|\Phi(p(t))\|}{\|\Phi(r(t))\|} - r(t).
\]
(3.15)

Hence, using (3.13) and (3.15), we obtain required inequality (3.12).

**Remark 3.3.** If we put \(\nu = \gamma, \sigma = \delta = \theta = \zeta\) and \(\nu = \nu\), in Theorem 3.2 it reduces to the Theorem 3.1.

Now, we prove our main result.

**Theorem 3.4.** Let \(p, r\) and \(q\) be three positive continuous functions on \([0, \infty)\) and \(p \leq r\) on \([0, \infty)\). If \(\frac{\xi}{r}\) is decreasing, \(p\) and \(q\) are increasing functions on \([0, \infty)\), and for any convex function \(\phi\) such that \(\phi(0) = 0\), then for all \(k \geq 0, t > 0, \pi \geq \max\{0, -\sigma - \nu\}, \sigma, \delta < 1, \nu > -1, -\sigma < \theta < 0, 0 < \xi < \zeta < 0, \) we have,
\[
\frac{\|\Phi(p(t))\|}{\|\Phi(r(t))\|} \leq \int_{\tau}^{\gamma} \frac{\|\Phi(p(t))\|}{\|\Phi(r(t))\|} - r(t).
\]
(3.16)

**Proof:** Since \(p \leq r\) on \([0, \infty)\) and function \(\frac{\Phi(p(t))}{\Phi(r(t))}\) is increasing, then for \(\tau, \rho \in [0, t)\) \(t > 0\), we have
\[
\frac{\|\Phi(p(t))\|}{\|\Phi(r(t))\|} \leq \frac{\|\Phi(p(t))\|}{\|\Phi(r(t))\|} - r(t).
\]
(3.17)

Multiplying both sides of (3.17) by \(\|\Phi(p(t))\|\) and integrating obtained result with respect to \(\tau\) from \(0\) to \(t\), we have
\[
\int_{\tau}^{\gamma} \frac{\|\Phi(p(t))\|}{\|\Phi(r(t))\|} - r(t).\]
(3.18)
On the other hand, since the fact that the function $\phi$ is convex with $\phi(0) = 0$, Then the function $\frac{\phi(p(t))}{p(t)}$ is increasing. Since $p$ is increasing, $\frac{\phi(p(t))}{p(t)}$ is also increasing. Clearly we can say that $\frac{p(t)}{r(t)}$ is decreasing, for all $\tau, \rho \in [0, t]$ $t > 0$

\[
\left( \frac{\phi(p(t))}{p(t)} q(t) - \frac{\phi(p(t))}{p(t)} q(\rho) \right)(p(t)r(\tau) - p(\tau)r(p)) \geq 0,
\]

(3.19)

which implies that

\[
\frac{\phi(p(t))q(t)}{p(t)}p(t)r(\tau) + \frac{\phi(p(t))q(\rho)}{p(\rho)}p(\rho)r(p)
- \frac{\phi(p(t))q(\rho)}{p(\rho)}p(\rho)r(\tau) - \frac{\phi(p(t))q(\rho)}{p(\rho)}p(\rho)r(p) \geq 0.
\]

(3.20)

Hence, we can write

\[
p(\rho)\sqrt[i]{\tau, k} \left[ \frac{\phi(p(t))}{p(t)}r(t)q(t) \right] + \frac{\phi(p(t))}{p(t)}r(\tau)q(\tau)\right]_{i, k}^{\frac{\phi(p(t))}{p(t)}r(t)q(t)}
- r(\rho)\sqrt[i]{\tau, k} \left[ \frac{\phi(p(t))}{p(t)}r(t)q(t) \right] - \frac{\phi(p(t))}{p(t)}p(\rho)r(\rho)\sqrt[i]{\tau, k} \left[ \frac{\phi(p(t))}{p(t)}r(t)q(t) \right]_{i, k}^{\frac{\phi(p(t))}{p(t)}r(t)q(t)}.
\]

(3.21)

with the same argument as before, we have

\[
\frac{\sqrt[i]{\tau, k} \left[ \frac{\phi(p(t))}{p(t)}r(t)q(t) \right]}{\sqrt[i]{\tau, k} \left[ \frac{\phi(p(t))}{p(t)}r(t)q(t) \right]} \geq \frac{\sqrt[i]{\tau, k} \left[ \frac{\phi(p(t))}{p(t)}r(t)q(t) \right]}{\sqrt[i]{\tau, k} \left[ \frac{\phi(p(t))}{p(t)}r(t)q(t) \right]}.
\]

(3.22)

Hence, using equation (3.18) and (3.22), we obtain (3.16).

Now, we give generalization of Theorem 3.3.

**Theorem 3.5.** Let $p, r$ and $q$ be three positive continuous functions on $[0, \infty)$ and $p \leq r$ on $[0, \infty)$. If $\frac{p}{r}$ is decreasing, $p$ and $q$ are increasing functions on $[0, \infty)$, and for any convex function $\phi$ such that $\phi(0) = 0$, then we have

\[
\frac{\sqrt[i]{\tau, k} \left[ \frac{\phi(p(t))}{p(t)}r(t)q(t) \right]}{\sqrt[i]{\tau, k} \left[ \frac{\phi(p(t))}{p(t)}r(t)q(t) \right]} \geq \frac{\sqrt[i]{\tau, k} \left[ \frac{\phi(p(t))}{p(t)}r(t)q(t) \right]}{\sqrt[i]{\tau, k} \left[ \frac{\phi(p(t))}{p(t)}r(t)q(t) \right]},
\]

(3.23)

where for all $k \geq 0, t > 0, \pi > \max\{0, -\varepsilon - \nu\}, \gamma > \max\{0, -\delta - \nu\}$, $\delta < 1$, $\nu, \gamma > -1, \delta < 0, \delta - 1 < \zeta < 0$,

**Proof:**- Multiplying both sides of (4.2) by $\frac{u}{u + t + 1}$ $u + t + 1 < 0$, $\frac{u}{u + t + 2}$, $t > 0$, which remains positive. Then integrate the resulting identity with respect to $\rho$ from 0 to $t$, we have

\[
\int_0^t \phi(p(t))q(t) \frac{p(t)}{p(t)}r(t)
+ \int_0^t \phi(p(t))q(t) \frac{p(t)}{p(t)}r(t)
\geq \int_0^t \phi(p(t))q(t) \frac{p(t)}{p(t)}r(t)
+ \int_0^t \phi(p(t))q(t) \frac{p(t)}{p(t)}r(t).
\]

(3.24)

and since $p \leq r$ on $[0, \infty)$ and use the fact that $\frac{\phi(t)}{p(t)}$ is increasing, we obtain

\[
\int_0^t \phi(p(t))q(t) \frac{p(t)}{p(t)}r(t)
\leq \int_0^t \phi(p(t))q(t) \frac{p(t)}{p(t)}r(t)
\leq \int_0^t \phi(p(t))q(t) \frac{p(t)}{p(t)}r(t).
\]

(3.25)

Hence, from equation (3.24), (3.25) and (3.26), we obtain (3.23).

**Remark 3.6.** If we put $\pi = \gamma, \nu = \delta$ and $\delta = \xi$ and $\nu = \nu$ in Theorem 3.4 it reduces to the Theorem 3.3.

### 4. Other fractional integral inequalities

In [10], authors have proved the inequalities using Riemann-Liouville fractional integral. Now, we prove the similar results using generalized k-fractional integral (in terms of Gauss hypergeometric function) operator.

**Theorem 4.1.** Let $p, q$ be two positive and continuous functions on $[0, \infty)$ such that $p$ is decreasing and $q$ is increasing on $[0, \infty)$. Then for all $k \geq 0, t > 0, \pi > \max\{0, -\varepsilon - \nu\}, \delta < 1$, $\nu, \gamma > -1, \delta < 0, \delta - 1 < \zeta < 0$, $\pi > \max\{0, -\varepsilon - \nu\}, \gamma > \max\{0, -\delta - \nu\}$, $\delta < 1$, $\nu, \gamma > -1, \delta < 0, \delta - 1 < \zeta < 0$.

Proof:- Consider $\rho, \tau \in (0, t)$, we have

\[
(q^\rho(p) - q^n(p)) \left( p^\rho(p) - p^n(p) \right) \geq 0,
\]

which implies that

\[
q^\rho(p) p^\rho(p) + q^n(p) p^n(p) \geq q^\rho(p) p^\rho(p) + q^n(p) p^n(p),
\]

(4.2)

Moreover, both sides of (4.2) by $\tau(t, \tau)$ which is positive, and integrating obtained result with respect to $\tau$ from 0 to $t$, we have

\[
q^\rho(p) p^\rho(p) \int_0^t \frac{\pi, \rho, \tau}{p^\rho(p)} [p^\rho(t)] + q^n(p) p^n(p) \int_0^t \frac{\pi, \rho, \tau}{p^n(p)} [p^n(t)]
\]

(4.1)
Now, multiplying both side of (4.3) by \( \frac{\pi\eta+\theta}{t_0} \) which is positive from (2.4). Now integrating obtained result with respect to \( \rho \) from 0 to \( t \), we have
\[
\begin{align*}
\frac{\pi\eta+\theta}{t_0} \left[ \frac{q^n p^n(t)}{t^n} \right] & \geq \frac{\pi\eta+\theta}{t_0} \left[ \frac{q^n p^n(t)}{t^n} \right] \\
\frac{\pi\eta+\theta}{t_0} \left[ \frac{q^n p^n(t)}{t^n} \right] & \geq 1.
\end{align*}
\]  
(4.4)

which gives the inequality 4.1.

**Theorem 4.2.** Let \( p, q \) be two positive and continuous functions on \([0, \infty)\) such that \( p \) is decreasing and \( q \) is increasing on \([0, \infty)\). Then for all \( k \geq 0, t > 0, \pi > \max \{0, -\sigma - v\} \), \( \delta < 1, v > 1, \sigma - 1 < \theta < 0, \delta - 1 < \zeta < 0, \lambda > m > 0, \) and \( n > 0 \) we have
\[
\frac{\pi\eta+\theta}{t_0} \left[ \frac{q^n p^n(t)}{t^n} \right] \leq \frac{\pi\eta+\theta}{t_0} \left[ \frac{q^n p^n(t)}{t^n} \right] \geq 1.
\]  
(4.5)

**Proof:** Multiplying equation (4.3) by \( \frac{t^k}{(k+1)^{\gamma-1}} \left( \gamma^\delta+\nu+\zeta^\gamma \right) \gamma^\delta+\nu+\zeta^\gamma \gamma^\delta+\nu+\zeta^\gamma \) we have
\[
\frac{\pi\eta+\theta}{t_0} \left[ \frac{q^n p^n(t)}{t^n} \right] \leq \frac{\pi\eta+\theta}{t_0} \left[ \frac{q^n p^n(t)}{t^n} \right] \geq 1.
\]  
(4.6)

where Then for all \( k \geq 0, t > 0, \pi > \max \{0, -\sigma - v\} \), \( \sigma - 1 < \theta < 0, \lambda > m > 0, \) and \( n > 0 \).

**Theorem 4.3.** Let \( p, q \) be two positive and continuous functions on \([0, \infty)\) such that \( p \) is decreasing and \( q \) is increasing on \([0, \infty)\). Such that
\[
(p^n(\tau)q^n(\rho) - p^n(\rho)q^n(\tau)) \left( p^{-n}(\tau) - p^{-n}(\tau) \right) \geq 0,
\]
then we have
\[
\frac{\pi\eta+\theta}{t_0} \left[ \frac{q^n p^n(t)}{t^n} \right] \geq \frac{\pi\eta+\theta}{t_0} \left[ \frac{q^n p^n(t)}{t^n} \right],
\]  
(4.6)

and using the same arguments as in Theorem 4.1, we obtain the result.

**Theorem 4.4.** Let \( p, q \) and \( r \) be three function on \([0, \infty)\) such that
\[
(p(\tau) - p(\rho))(q(\tau) - q(\rho))(r(\tau) + r(\rho))
\]
then for all \( \tau, p, k \geq 0, t > 0, \pi > \max \{0, -\sigma - v\} \), \( \sigma - 1 < \theta < 0, v > 1, \sigma - 1 < \theta < 0, \) we have,
\[
\frac{\pi\eta+\theta}{t_0} \left[ \frac{q^n p^n(t)}{t^n} \right] \geq \frac{\pi\eta+\theta}{t_0} \left[ \frac{q^n p^n(t)}{t^n} \right].
\]  
(4.7)

**Theorem 4.5.** Let \( p, q \) and \( r \) be three function on \([0, \infty)\) such that
\[
(p(\tau) - p(\rho))(q(\tau) + q(\rho))(r(\tau) + r(\rho)) \geq 0,
\]
then for all \( \tau, p, k \geq 0, t > 0, \pi > \max \{0, -\sigma - v\} \), \( \sigma - 1 < \theta < 0, v > 1, \sigma - 1 < \theta < 0, \) we have,
\[
\frac{\pi\eta+\theta}{t_0} \left[ \frac{q^n p^n(t)}{t^n} \right] \geq \frac{\pi\eta+\theta}{t_0} \left[ \frac{q^n p^n(t)}{t^n} \right].
\]  
(4.8)
With the same argument in inequality (4.11), we obtain differential equations. Moreover, they are expected to lead to some inequalities proposed in this paper give some contribution in integral inequalities involving convex functions by considering integral operators. We established some fractional differential equations.

\[
p(\tau)q(\tau) = r(\tau) + p(\tau)q(\tau)r(\tau)
\]

(4.14)

Multiplying both side of equation (4.14)by \( \mathfrak{I}(t, \tau) \) which is positive, and integrating obtained result with respect to \( \tau \) from 0 to 1, we have

\[
\int_{t}^{\tau} \mathfrak{I}(t, \tau) r(\tau) + p(\tau)q(\tau) = r(\tau) + p(\tau)q(\tau)r(\tau)
\]

(4.15)

\[
\int_{t}^{\tau} \mathfrak{I}(t, \tau) r(\tau) + p(\tau)q(\tau) = r(\tau) + p(\tau)q(\tau)r(\tau) \Lambda_{\tau}^{\mathfrak{I}(t, \tau)}
\]

With the same argument in inequality (4.11), we obtain

\[
\Lambda_{\tau}^{\mathfrak{I}(t, \tau)} r(\tau) + p(\tau)q(\tau) = r(\tau) + p(\tau)q(\tau)r(\tau) \Lambda_{\tau}^{\mathfrak{I}(t, \tau)}
\]

(4.16)

where, \( \Lambda_{\tau}^{\mathfrak{I}(t, \tau)} \) \( \mathfrak{I}(t, \tau) \) is as is in theorem 4.4. This complete the proof of inequality (4.13).

5. Concluding Remarks

In this study, we presented generalized k-fractional integral operator operators. We established some fractional inequalities involving convex functions by considering generalized k-fractional integral operator. Here, we briefly consider some implication of our main results. The inequalities proposed in this paper give some contribution in the fields of fractional calculus and Generalized k-fractional integral operators. Moreover, they are expected to lead to some application for finding uniqueness of solutions in fractional differential equations.

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