Sub-Riemannian homogeneous geometry. Shortest and straightest geodesics

Dmitri Alekseevsky

A.A.Kharkevich Institute for Information Transmission Problems, B.Karetnuj per.,19, 127051, Moscow, Russia
and Faculty of Science, University of Hradec Kralove, Rokitanskeho 62, Hradec Kralove, 50003, Czech Republic

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1 Introduction

The efficiency of Riemannian geometry and its important role in applications based on the fact many important equations, arising in mechanics, mathematical physics, biology, economy, information theory etc can be reduced to the geodesic equation associated with a Riemannian metric. Moreover, Riemannian geometry give an effective tool for investigation of geodesic equations and other equations associated with the metric (Laplace, wave, heat and Schrödinger equation, Einstein equation etc).

There are many equivalent definitions of Riemannian geodesics. They are naturally generalised to sub-Riemannian manifold, but become non-equivalent. We give a review of different definitions of geodesics of a sub-Riemannian manifold and interrelation between them.

Herz indicated two characterisations of geodesics: geodesics as **shortest curves** based on Mopertrui’s principle of least action (variational approach) and geodesics as **straightest curves** based on d’Alembert’s principle of virtual work.

We recall three variational definitions of geodesics as (locally) shortest curves (Euler-Lagrange, Pontryagin and Hamilton) and three definitions of geodesics as straightest curves (d’Alembert, Levi-Civita-Schouten and Cartan-Tanaka), used in nonholonomic mechanics and discuss their interrelations.

A. Vershik and L. Faddeev showed that shortest geodesics for a generic sub-Riemannian manifolds \( (Q, D.g) \) are different from straightest geodesics on a open dense submanifold of \( Q \). They gave first example (compact Lie group with the bi-invariant metric) when shortest geodesics coincides with straightest geodesics.

We generalised this example and consider a big class of sub-Riemannian manifolds associated with principal bundle over a Riemannian manifolds, for which shortest geodesics coincides with straightest geodesics. Using the geometry of flag manifolds, we describe some classes of compact homogeneous sub-Riemannian manifolds (including contact sub-Riemannian manifolds and symmetric sub-Riemannian manifolds) where straightest geodesics coincides with shortest geodesics. Construction of geodesics in these cases reduces to description of Riemannian geodesics of the Riemannian homogeneous manifold or left-invariant metric on a Lie group.
PART I. Sub-Riemannian geodesics as shortest curves

2.1 Euler-Lagrange variational definition of sub-Riemannian geodesics (EL-geodesics)

Any Lagrangian $L(q, \dot{q})$ (homogeneous degree 1 in velocity $\dot{q}$ function of $TQ$) defines a variational problem

$$\delta A_L = \delta \int_a^b L(q(t), \dot{q}(t))dt = 0$$

in the space $C(q_0, q_1) = \{ q(t), t \in [a, b], q_0 = \gamma(a), q_1 = \gamma(b) \}$. Solutions are critical points of $A_L$, i.e. solutions of the Euler-Lagrange equation

$$(EL) \quad \delta L_i := \frac{d}{dt}L_{\dot{q}_i} - L_{\dot{q}_i} = 0,$$

which defines a vector field $\Gamma^L \in \mathcal{X}(TQ)$.

EL-geodesics are critical point of the length functional ($L = \sqrt{g(\dot{q}, \dot{q})}$) or energy functional ($L = \frac{1}{2}g(q, \dot{q})$).

2.1.1 EL-geodesic on bracket generated sub-Riemannian manifold $(Q, D, g_D)$

$D \subset TM$ is a submanifold. EL-geodesics are solution of the variational problem

$$\delta A_L = \delta \int_a^b L(q(t), \dot{q}(t))dt = 0$$

for length functional or energy functional in the space $C^h(q_0, q_1) = \{ q(t), t \in [a, b], \dot{q} \in D, q_0 = \gamma(a), q_1 = \gamma(b) \}$, i.e. a solution of EL-equation

$$\delta L_i := \frac{d}{dt}L_{\dot{q}_i} - L_{\dot{q}_i} - \lambda_a \omega_a - \lambda_a(\dot{q} | \omega_a) = 0$$

where $D = \{ \omega_a = 0, a = 1, \cdots k \}$. This equation corresponds to a vector field $\Gamma^L \in \mathcal{X}(D \times D^0)$ where $D^0 = \text{Ann}(D) \subset T^*Q$ is the annihilator of $D$ (called the codistribution).

EXPLAIN THIS. Where the vector field in coordinate free form.

2.1.2 Pontryagin optimal control definition of geodesics (P-geodesics) for bracket generated sub-Riemannian manifold $(Q, D, g_D)$

Let $L(q, \dot{q}) \in C^\infty(Q)$ be a Lagrangian on $D$, which is either the velocity $L = \sqrt{g^D(\dot{q}, \dot{q})}$ or the energy $\frac{1}{2}g^Q(\dot{q}, \dot{q})$, and $X_1, \cdots, X_m$ be an orthonormal
Any horizontal curve \( q(t) \in C^h(q_0, q_1) \) is a solution of the ODE

\[
(* \quad \dot{q}^i(t) = \sum_{i=1}^{n} u^i(t)X_i(q^i(t)), \quad q(a) = q_0, q(b) = q_1.
\]

where vector function \( u(t) \) is an admissible control parameter and \( A^h(q(t)) = \int_a^b L(q, \dot{q}) dt \) is the cost functional.

**P-geodesic** (resp., **minimal P-geodesic**) is a curves \( q(t) \in C^h(q_0, q_1) \) which delivers a critical point (respectively, a minimum) for the cost functional

\[
A_L = \int_a^b L(q, \dot{q}) dt
\]
in the space of admissible control.

P-geodesics coincide with EL-geodesics.

For \( D = TQ \), P-geodesics are geodesics of the Riemannian manifold \((Q, g^Q)\).

### 2.2 Hamiltonian definition of sub-Riemannian geodesics (H-geodesics)

Let \( p_D \) be the restriction of a linear form \( p \in T_q^*Q \) to \( D \) and \( g^*(p, p) = g^{-1}(p_D, p_D) \in C^\infty(T^*Q) \) the **cometric**.

**H-geodesics** are projection to \( Q \) of orbits of Hamiltonian vector field \( \vec{H} = \Omega^{-1}dH \in \mathfrak{X}(T^*Q) \) with quadratic (degenerate) Hamiltonian \( H = \frac{1}{2}g^*(p, p) \), defined by the cometric. Here \( \Omega \) is the standard symplectic form in \( T^*Q \).

### 2.3 Pontryagin Maximum Principle for bracket generated sub-Riemannian manifold

0 P-geodesics are exhausted by **normal geodesics**, which are exactly H-geodesics, and **abnormal geodesics**, which depend only on \( D \).

By variational (or shortest) geodesics we will understand H-geodesics.

**GIVE DEFINITION OF ABNORMAL GEODESICS!**

### 3 PART II. Sub-Riemannian geodesics as straightest curves

#### 3.1 d’Alemebert definition of sub-Riemannian geodesics (dA geodesics) in \((Q, D, g^D)\)

**EXPLAIN IN MORE DETAILS**

To define dA-geodesics, we extend \( g^D \) to \( g^Q \). Let \( D^0 = Ann(D) \subset T^*Q \) the
codistribution. The **dA-geodesics** are solutions of the equation

\[ \frac{d}{dt} L_{\dot{q}_i} - L_{q_i} \equiv 0 \, (\text{mod} \, D^0) \]

where \( L(q, \dot{q}) = L = \sqrt{g(\dot{q}, \dot{q})} \) or \( L = \frac{1}{2} g(\dot{q}, \dot{q}) \), which is the appropriate projection of the Euler Lagrange geodesic vector field \( \Gamma^g \) to \( D^0 \).

In general, the equation of dA-geodesics is neither Lagrangian nor Hamiltonian.

**Give definition according to Vershik-Faddeev (Lagrangian approach)**

### 3.2 Levi-Civita-Schouten definition of sub-Riemannian geodesics (LS-geodesics)

#### 3.2.1 Levi-Civita definition of Riemannian geodesics

Recall that Levi-Civita associates to a Riemannian manifold \((Q, g)\) the canonical torsion free connection \( \nabla^g \), which preserves the metric (called the Levi-Civita connection). It is defined, for example, by the Koszul formula

\[
2g(\nabla_X Y, Z) = X \cdot g(Y, Z) + Y \cdot g(X, Z) - Z \cdot g(X, Y) + g([X, Y], Z) - g(Y, [X, Z]) - g(X, [Y, Z]), \; X, Y, Z \in \mathfrak{X}(Q).
\]

He defined Riemannian geodesics (**L-geodesics**) as autoparallel curves, i.e. solution of the equation

\[
\nabla^g_{\dot{\gamma}} \dot{\gamma} = \ddot{\gamma}^i + \Gamma^i_{jk} \dot{\gamma}^j \dot{\gamma}^k = 0.
\]

The extension of this notion to the sub-Riemannian manifold \((Q, D, g^D)\) had been given by Schouten, Vranceanu and Singe.

#### 3.2.2 Schouten partial connection

Let \( D \subset TQ \) be a distribution. A partial \( D \)-connection in a vector bundle \( W \) is a bilinear map

\[
\nabla^D : \Gamma D \times \Gamma W \to \Gamma W, \; (X, \alpha) \mapsto \nabla^D_X \alpha
\]

which is \( C^\infty(Q) \) linear in \( X \) and satisfies the Leibnitz rule

\[
\nabla^D_X (f\alpha) = f \nabla^D_X \alpha + X \cdot f\alpha \, \text{in} \, \alpha.
\]
Using a generalisation of the Koszul formula, we may associate with a sub-Riemannian metric \((D, g^D)\) a partial \(D\)-connection in the vector bundle \(W = T^*Q/D^0\), given for \(\bar{\alpha}, \bar{\beta} \in \Gamma W\) by

\[
2g^D(\nabla_X \bar{\alpha}, \bar{\beta}) = X \cdot g^*(\alpha, \beta) + g^*\alpha \cdot \beta(X) - g^*\beta \cdot \alpha(X) + (X \cdot \alpha)(g^*\beta) - g(\alpha)\left(\left[g^*\alpha, g^*\beta\right]\right) - \beta(g^*\alpha),
\]

where \(\alpha, \beta \in \Omega^1(Q)\) are (local) 1-forms which are lifts of \(\bar{\alpha}, \bar{\beta}\) and \(g^*\) is the cometric (considered as a map \(g^*: \Omega^1(Q) \to \Gamma D\)).

Following I.A. Schouten, we fix a complementary to \(D\) distribution \(V\) (called a rigging). It defines a direct sum decomposition \(TQ = D + V, T^*Q = D^0 + D^0\) where \(D^0 = V^0\) is the annihilator of \(V\). We identify \(T^*Q/D^0\) with \(D^0\) and \(D = g^*D^0\). We denote by \(X = X_D + X_V\) the decomposition of a vector into a sum of the horizontal part \(X_D\) and the vertical part \(X_V\).

The partial connection in \(W = T^*Q/D^0\) is identified with the partial connection \(\nabla^S: \Gamma D \times \Gamma D \to \Gamma D\) in \(D\), called the Schouten connection. Schouten connection has zero "torsion", defined as

\[
T(X, Y) = \nabla^S_X Y - \nabla^S_Y X - \pi_D[X, Y], \quad X, Y \in \Gamma D
\]

where \(\pi_D\) is the parallel to \(V\) projection to \(D\).

Schouten defined the curvature tensor \(R \in \mathfrak{so}(D) \otimes \Lambda^2 T^*M\) of the Schouten connection by

\[
R(X, Y)Z = [\nabla_X, \nabla_Y]Z - [[X, Y]^V, Z]^D.
\]

V.V. Wagner generalized this notion and defined Wagner curvature tensor, such that the vanishing of the Wagner tensor is equivalent to the flatness of the Schouten connection (such the the associated parallel transport does not depend on the path, jointed two points), see [B]

### 3.2.3 Levi-Civita-Schouten definition of sub-Riemannian geodesics

Schouten defined a sub-Riemannian geodesic (LS geodesic) as an autoparallel curve for the Schouten connection

\[
\nabla^S_\gamma \dot{\gamma} = 0.
\]

Assume that the sub-Riemannian metric \(g^D\) is extended to a Riemannian metric \(g^Q\) on \(Q\) such that \(g^Q(D, V) = 0\). Then the projection to \(D\) of the Levi-Civita connection \(\nabla^Q\) of \(g^Q\) is a connection \(\nabla^{g^Q}\) in vector bundle \(D\):

\[
\nabla^Q_X = \pi_D \nabla^Q_X |_D, \quad X \in \mathfrak{X}(Q).
\]
Moreover, the connection $\nabla^D$ in $D$ which is the restriction of the Levi-Civita connection $\nabla^g$ is an extension of the Schouten partial connection $\nabla^S$:

$$\nabla^D_X = \pi_D \nabla^g_X|_D, \ X \in \mathfrak{X}(Q)$$

**LS-geodesics** are geodesics of the Schouten connection, i.e. horizontal curves $\gamma(t)$ which satisfies

$$\nabla^S_{\dot{\gamma}} \dot{\gamma} = \pi_D \nabla^g_{\dot{\gamma}} \dot{\gamma} = 0.$$

**Theorem 1 (Vershik-Faddeev)** Let $(Q, D, g^D)$ be a sub-Riemannian manifold, $g^Q$ is an extension of $g^D$ to a metric in $Q$ and $V = D^\perp$ the orthogonal complement to $D$. Then dA-geodesics coincides with LS-geodesics and they describe evolution of the free mechanical system with kinetic energy $g^Q$ in configuration space $Q$ with nonholonomic constrains $D$.

### 4 Cartan-Tanaka frame bundle definition of sub-Riemannian geodesics (CT-geodesics)

#### 4.1 Cartan definition of a Riemannian geodesics

An important definition of Riemannian geodesics as straightest curves had been proposed by E. Cartan. It is easily generalized to the case of $G$-structures of finite type and to sub-Riemannian manifolds (and other Tanaka structures).

Let $\pi : P \to Q = P/\text{SO}_n$ be the $\text{SO}_n$-principal bundle of orthonormal coframes (isometries $f : T_xQ \to \mathbb{R}^n = V$) with the tautological soldering form

$$\theta : TP \to \mathfrak{so}_n, \ \theta_f(X) := f(\pi_\ast X).$$

The total space $P$ admits a canonical $\text{SO}_n$-equivariant absolute parallelism (Cartan connection)

$$\kappa = \theta + \omega : TP \to V + \mathfrak{so}(V),$$

which is an extension of the vertical parallelism (defined by the free action of $\text{SO}_n$ on $P$).

**C-geodesics** are projection to $Q$ of constant horizontal vector fields from $\kappa^{-1}(V)$. 
4.1.1 Generalisation to $G$-structures of finite type

Given a linear group $G \subset GL(V)$, $V = \mathbb{R}^n$. A $G$-structure is a $G$-principal bundle $\pi: P \rightarrow Q = P/G$ with a soldering 1-form

$$\theta: TP \rightarrow V$$

i.e. a horizontal $G$-equivariant form with ker $\theta = T^{vert}P$.

Such bundle is identified with a $G$-bundle of coframes on $TQ$. If the group $G$ has a finite type $k$, one can prolong $\pi$ to a bundle $P^k \rightarrow Q$ with absolute parallelism

$$\kappa: TP^{(k)} \rightarrow \mathfrak{g}^\infty = V + \mathfrak{g} + \mathfrak{g}^{(1)} + \cdots + \mathfrak{g}^{(k)}.$$

The projection of orbits of constant vector fields from $\kappa^{-1}V$ to $Q$ are generalised C-geodesics for $G$-structure.

In the case when the first prolongation

$$\mathfrak{g}^{(1)} = \mathfrak{g} \otimes V^* \cap V \otimes S^2V^*$$

of the linear Lie algebra is trivial,

then $k = 1$ and

$$\kappa: TP \rightarrow V + \mathfrak{g}$$

is a Cartan connection.

The theory of such generalised C-geodesics had been developed by J. Slovak, V. Žadnik, A. Čap and B. Doubrov.

4.1.2 Cartan connection

Let $M_0 = L/G$ be a homogeneous $n$-dimensional manifold.

A Cartan connection of type $M_0 = L/G$ on $n$-dimensional manifold $Q$ is a principal $G$-bundle $\pi: P \rightarrow Q = P/G$ together with a $\mathfrak{l}$-valued $G$-equivariant (s.t. $r_g^*\kappa = Ad_g^{-1} \circ \kappa$, $g \in G$) kernel free 1-form

$$\kappa: TP \rightarrow \mathfrak{l}$$

which extends the vertical parallelism $T_v^0P \simeq \mathfrak{g}$.

The form $\kappa$ defines an absolute parallelism $T_vP \simeq \mathfrak{l}$. Hence, tensor fields on $P$ may be identified with tensor-valued functions.

In particular, the curvature 2-form $K := dk + \frac{1}{2}[\kappa, \kappa]$ is identified with a function with value in $C^2(\mathfrak{l},\mathfrak{l}) = \mathfrak{l} \otimes \Lambda^2\mathfrak{l}^*$. A Cartan connection is called normal if the codifferential $\delta^*K = 0$.

Then the curvature $K$ is determined by its harmonic component $\kappa_H$ (satisfying $\delta \kappa_H = 0$) via Bianchi identities.
4.2 Cartan-Tanaka definition of sub-Riemannian geodesics on a regular sub-Riemannian manifold

4.2.1 The symbol algebra

A distribution \( D \subset TQ \) on a manifold defines an decreasing filtration \( \mathcal{D}^i \)

\[
0 \subset \mathcal{D}^{-1} = \Gamma D \subset \mathcal{D}^{-2} := \mathcal{D}^{-1} + [\mathcal{D}^{-1}, \mathcal{D}^{-1}] \subset \cdots
\]

in the Lie algebra \( \mathcal{X}(Q) \) of vector fields and for \( q \in Q \) a flag

\[
0 \subset D_q := \mathcal{D}_q^{-1} \subset \mathcal{D}_q^{-2} \subset \cdots
\]

in \( T_qQ \). The Lie bracket induces in the associated graded space

\[
m_q = m_q^{-1} + m_q^{-2} + \cdots + m_q^{-k} := \mathcal{D}_q^{-1} + \mathcal{D}_q^{-2} / \mathcal{D}_q^{-1} + \cdots + \mathcal{D}_q^{-k} / \mathcal{D}_q^{-(k-1)}
\]

the structure of a negatively graded Lie algebra of some depth \( k \), called the symbol algebra at a point \( q \).

By construction, the graded Lie algebra is fundamental, i.e. it is generated by \( m^{-1} \).

4.3 Bracket generated and regular distributions and graded tangent bundle

The distribution \( D \subset TQ \) is called bracket generated or totally non-holonomic of depth \( k \) if for any \( q \in Q \) \( \mathcal{D}_q^k = T_qQ \) for some \( k \). Then the grades space

\[
T^g_qQ = m_q = m_q^{-1} + \cdots + m_q^{-k} := \mathcal{D}_q^{-1} + \mathcal{D}_q^{-2} / \mathcal{D}_q^{-1} + \cdots + T_qQ / \mathcal{D}_q^{-(k-1)}
\]

is called the graded tangent space at \( q \in Q \).

If, moreover for any \( i \) the space \( D^i \) is the space of sections of a vector bundle \( D^i : \mathcal{D}^i = \Gamma D^i \) the symbol algebra \( m_q \) at any point is isomorphic to a fixed graded algebra \( m = m^{-1} + \cdots + m^{-k} \) the distribution \( D \) is called a regular distribution of type \( m \).

4.4 Regular sub-Riemannian manifolds

A sub-Riemannian manifold \((Q, D, g^D)\) with bracket generated distribution \( D \) is called bracket generated. Then the graded tangent space \( T^g_qQ = m_q \) has the structure of a metric negative graded Lie algebra, i.e. a graded Lie algebra \( m_q = \sum_{i=-1}^k m_q^i \) with an Euclidean metric \( g^g_q \) such that the graded spaces \( m_q^i \) are mutually orthogonal. We call the metric negatively graded Lie algebra \((m_q, g^g_q)\) the metric symbol of a bracket generated sub-Riemannian manifold at a point \( q \).

The metric in \( m_q \) is the natural extension of the sub-Riemannian metric \( g^D_q \) in \( m_q^{-1} = D_q \) which is described in the following Lemma.
Lemma 2 Let $m = m^{-1} + \cdots + m^{-k}$ be a negatively graded fundamental Lie algebra. Then an Euclidean metric $g^{-1}$ on $m^{-1}$ has natural extension to the Euclidean metric in $m$.

Proof: The construction of the extension is inductive. Denote by $\beta : \Lambda^2 m^{-1} \to m^{-2}$ the linear map defined by the Lie bracket. Then $m^{-2} = m^{-1} + \beta(\Lambda^2(m^{-1}))$. The metric in $m^{-1}$ induces a metric in $\Lambda^2(m^{-1})$ and on $\beta(\Lambda^2(m^{-1}))$. Now we can canonically construct a metric at $m^{-2}$, which is represented as a sum (not direct!) of two Euclidean subspaces. □

A sub-Riemannian manifold $(Q, D, g^{D})$ with a regular distribution of type $m$ is called a regular sub-Riemannian manifold of metric type $(m, g^{m})$ if its metric symbol $(m_{q}, g^{gr}_{m_{q}})$ at any point $q$ is isomorphic to the metric graded Lie algebra $(m, g^{m})$.

Denote by $g^{0} = \text{der}(m, g^{m})$ the Lie algebra of skew-symmetric (graded preserving) derivation of the metric graded Lie algebra. Then $\tilde{m} = g^{0} + m^{-1} + m^{-k}$ is a non negatively graded Lie algebra. In particular, if $(m_{q}, g^{m}_{q})$ is the metric symbol , then $\tilde{m}_{q}$ is called the extended symbol of a sub-Riemannian manifold.

Note that a derivation $A \in \text{der}(m)$ is skew-symmetric if its restriction to $m^{-1}$ is skew-symmetric.

4.4.1 Regular sub-Riemannian structure as Tanaka structure

Let $D \subset TQ$ be a regular rank $m$ distribution of type $m$ and $G^{0} \subset \text{Aut}(m)$ a connected group of graded preserving authomorphisms of $m$.

A Tanaka structure ( or relative $G^{0}$-structure) is a $G^{0}$-principal bundle $\pi : P \to Q = P/G^{0}$ of frames $f : V = \mathbb{R}^{n} \to D_{q}$ of distribution $D$.

The classical identification of a Riemannian manifold with a $G$-structure with orthogonal group $G = SO(n)$ is extended to the sub-Riemannian case:

Proposition 3 A regular sub-Riemannian manifold of type $(m, g^{m})$ is identified with a Tanaka $G$-structure with the structure group $G = \text{Aut}(m, g^{m})$.

This identification is defined as follows. Let $(Q, D, g^{D})$ be a regular sub-Riemannian structure of a metric type $(m, g^{m})$. An isometry $f = f_{q} : m^{-1} \to m^{-1}_{q} = D_{q}$ is called an admissible frame on the distribution at a point $q$ if it admits an extension to an (orthogonal) isomorphism $\hat{f} : m \to m_{q} = T_{q}gr Q$ of metric graded Lie algebras ( metric symbol algebras).

Denote by $F$ the set of all admissible frames with the natural projection $\pi : F \to Q$. The Lie group $G^{0} = \text{Aut}(m, g^{m})$ of orthogonal authomorphisms of the metric graded Lie algebra $m$ has the Lie algebra $g^{0} = \tilde{m} = \text{der}(m, g^{m})$. 

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and acts simply transitively by the right on the set $F_q$ of admissible frames at a point $q \in Q$.

This implies that $\pi : F \to Q = F/G^0$ is a Tanaka $G^0$-structure.

Conversely, let $D$ be a regular distribution of type $m = \sum_{i=-1}^{-k} m^i$. Then an Euclidean metric $g^{m^{-1}}$ in $m^{-1}$ is naturally extended to a metric $g^m$ in $m$.

We set $G^0 = \text{Aut}(m, g^m)$. Then a Tanaka $G$-structure $\pi : F \to Q = F/G^0$ defines a sub-Riemannian metric on $D$, which has the identity Gram matrix with respect to any frame from $F$.

4.4.2 Tanaka and Morimoto theorems

N. Tanaka generalised the theory of $G$-structures to Tanaka structures. In particular, he defined the full prolongation $g^\infty = g + g^{(1)} + g^{(2)} + \cdots$ of a non positively graded Lie algebra.

**Theorem 4** (Tanaka) Let $\pi : P \to Q$ be a Tanaka $G^0$-structure on $(Q, D)$ where $D$ is a regular distribution of type $m$. Assume that the full prolongation $g^\infty$ of the extended symbol algebra $g = g^0 + m$ is finite dimensional. Then there is a canonical bundle $P^\infty \to Q$, constructed by successive prolongations, with an absolute parallelism.

If the first prolongation $g^{(1)} = 0$, then the bundle $\pi : P \to Q$ carries a Cartan connection $\kappa : TP \to g$.

Hence, one can define generalised geodesics as projection to $Q$ of the orbits of constant vector fields.

**Theorem 5** (T. Morimoto) The first prolongation of the graded Lie algebra $g = g^0 + m$, which is the extended symbol algebra of a regular sub-Riemannian manifold, is trivial. Moreover, any regular sub-Riemannian manifold $(Q, D, g^D)$ admits unique normal Cartan connection $\kappa : T(Fr(Q)) \to g = m + g^0$.

4.4.3 Isometry group of a regular sub-Riemannian manifold

As a corollary, we get

**Theorem 6** The group $A = \text{Aut}(Q, D, g^D)$ of authomorphisms of a regular sub-Riemannian manifold of type $m$ is a Lie group of dimension $\dim A \leq g^0(m) + n$ and the stability subgroup $A_q$ of a point $q \in Q$ is compact.
Proof: Let $\pi : P = Fr(Q) \to Q$ be the Tanaka structure on a regular sub-Riemannian manifold $(Q, D, g^D)$ and $\kappa : T(Fr(Q)) \to \mathfrak{g} = \mathfrak{m} + \mathfrak{g}^0$ its Cartan connection. Denote by $A = Aut(Q, g^D)$ the group of authomorphisms of the sub-Riemannian manifold. Since it naturally acts on $P$ by automorphisms which preserves the absolute parallelism $\kappa$, by Kobayashi theorem [Stern] $A$ is a Lie group which acts freely on $P$. Hence $\dim A \leq \dim P = \dim \mathfrak{g}_0 + n$. and the restricted $j_D(A_q)$ to $D = D_q$ of the isotropy representation $j(A_q)$ of the stability subgroup $A_q$ is exact. Since $A_q \simeq j_D(A_q) \subset O(D_q)$ the stability subgroup is compact. □

A sub-Riemannian manifold $(Q, D, g^Q)$ is called homogeneous, if some group $A$ of automorphisms acts on $Q$ transitively. Since any bracket generated homogeneous sub-Riemannian manifold is regular, we get

**Corollary 7** Let $(Q, D, g^Q)$ be a bracket generated homogeneous sub–Riemannian manifold. Then $Q$ is identified with a coset space $Q = A/H$ of a Lie group of automorphisms by a compact subgroup $H$.

Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be the reductive decomposition. Then the invariant sub-Riemannian structure is defined by an $Ad_H$-invariant subspace $D_0 = \mathfrak{m}_0 \subset \mathfrak{m}$ which generates $\mathfrak{g}$ together with an $Ad_H$-invariant Euclidean metric $g$ on $\mathfrak{m}_0$, which is canonically extended to an $Ad_H$-invariant metric $g^\mathfrak{m}$. The metric $g^\mathfrak{m}$ defines an invariant Riemannian metric $g^Q$ on $Q = G/HG$, which is an extension (called the canonical extension) of the sub-Riemannian metric $g^D$, associated with $g$.

4.4.4 Nonholonomic C-geodesics as projection of constant vector fields

Let $(Q, D, g^D)$ be a regular sub-Riemannian manifold and

$$\kappa : T(Fr(Q)) \to \mathfrak{g} = \mathfrak{m} + \mathfrak{g}^0$$

associated normal Cartan connection.

C-geodesics are the projection to $Q$ of the orbit of constant vector fields from $\kappa^{-1}(\mathfrak{m}^{-1})$.

It is not easy to construct a normal Cartan connection but there is a simple construction of a Cartan connection associated to a rigging, which is consistent with filtration $D^{-i}$, that is a rigging of the form

$$V = V_2 + V_3 + \cdots + V_k$$

where $D^{-i} = D^{-i_1} + V_i$.

In [A- Medvedev- Slovak], it is shown that such rigging $V$ defines an isomorphism $grTQ \simeq TQ$ and also an isomorphism of the Tanaka bundle to
a $G^0$-structure with trivial first prolongation. Moreover, the Cartan connection of the Tanaka structure may be described in terms of the Cartan connection of this $G^0$-structure

$$\kappa = \theta + \omega : TP \to V + \mathfrak{g}^0$$

(which is the direct sum of the soldering form and the connection form). Hence

Any admissible rigging defines a Cartan connection. The associated C-geodesics are defined as projection to $Q$ of constant vector fields.

5 PART III. Invariant sub-Riemannian structures on Lie groups

5.1 Hamiltonian geometry of cotangent group $T^*G$

Recall that the cotangent bundle $\pi : T^*Q \to Q$ of a manifold $M$ has the canonical Liouville 1-form $\theta_p(X) = p(\pi_\ast X)$ written in local coordinates $(q^i, p_i)$, where $T^*qQ \ni p = p_idq^i$ as $\theta = p_idq^i$. Then $\Omega = -d\theta = dq^i \wedge dp_i$ is the canonical symplectic form. The natural (complete) lift $\hat{X} = X^i\partial_{q^i} - X^i_{;j}p_j\partial_{p_i}$ of a vector field $X = X^i\partial_{q^i} \in \mathfrak{X}(Q)$ to $T^*Q$ is a Hamiltonian vector field with the hamiltonian $p(X) = p_iX^i$.

Now we assume that $Q = G$ is a Lie group and describe the symplectic form $\Omega$ and the hamiltonian vector field associated with a left invariant vector field $X^L$ in coordinate free form. We fix notations:

$$\mu : TG \to \mathfrak{g}, \ g \mapsto g^{-1}\dot{g} := (L_g^{-1})_*\dot{g}$$

left invariant Maurer-Cartan form, s.t. $L^a_*\mu = \mu$;

$$\mu^* : T^*G \to \mathfrak{g}^*, \ \alpha_g \mapsto L^*_g\alpha_g = \alpha_g \circ (L_g)^*$$

the left invariant moment map, s.t. $L^a_*\mu^* := \mu^* \circ L^*_a = \mu^*$;

$$\pi_T \times \mu : TG \to G \times \mathfrak{g}, \ X_g \mapsto (g, g^{-1}X_g)$$

the left invariant trivialisation of the tangent bundle $\pi_T : TG \to G$, s.t. $L_a(g, X) = (ag, X)$;

$$\mu^* \times \pi : T^*G \to \mathfrak{g} \times G, \ \alpha_g \mapsto L^*_g\alpha$$

the left invariant trivialization of $T^*G$, s.t. $L_a(\alpha, g) = (\alpha, ag), \ a \in G$. Under this trivialization, the natural structure of the Lie group on $T^*G$ (the cotangent group) corresponds to the semidirect product of the Lie group $G$ and the commutative normal subgroup $\mathfrak{g}^*$, defined by the coadjoint
The constant vector field $e^T$ symplectic form $\Omega$ in $G$. The above calculation implies the following formulas for the matrix of the Poisson structure $\Lambda = \Omega^{-1}$. We will use also the natural identification $dx \, \text{invariant vector fields on } G$. Then

$$\omega(X^L, Y^L) = -X^L \cdot \theta_{(\xi,g)}(Y^L) + Y^L \cdot \theta_{(\xi,g)}(X^L) + \frac{1}{2} \theta_{(\xi,g)}([X,Y]^L) = \xi([X,Y]).$$

We calculate

$$\omega(\eta_v, X^L) = \eta_v \cdot \xi(X) = \eta(X).$$

Let $e_i$ be a basis of $\mathfrak{g}$ and $e^i$ the dual basis of $\mathfrak{g}^*$ such that

$$\mathfrak{g} \ni X = a^i e_i, \mathfrak{g}^* \ni \xi = x_i e^i.$$  

The constant vector field $e^i_v$ in $\mathfrak{g}^*$ associated with $e^i$ may be written as

$$e^i_v = \partial_i := \partial_{x_i} m.$$  

We will use also the natural identification $dx_i = e_i$. Then $e^i_v, e^j_l$ is a basis of the Lie algebra $\mathfrak{X}(T^*G)^L = \mathfrak{X}(\mathfrak{g}^* \times G)$ of left-invariant vector fields with the Lie brackets

$$[e^L_i, e^L_j] = [e_i, e_j]^L = c_{ij}^k e^L_k,$$

$$[e^L_i, e^L_j] = (\text{ad}_{e_i} e^j)_v = (e^j)_v \circ \text{ad}_{e_i}.$$  

The above calculation implies the following formulas for the matrix of the symplectic form $\Omega$ in $T^*G = \mathfrak{g}^* \times G$ with respect to $L_G$ invariant frame $\partial_{x_i}, e^L_i$, and the matrix of the Poisson structure $\Lambda = \Omega^{-1}$ w.r.t. the dual $L_G$-invariant coframe $dx^i, \theta^i := (e^L_i)^* \in \Omega^1(\mathfrak{g}^* \times G)$.

$$\Omega = \begin{pmatrix} \rho & \text{id} \\ -\text{id} & 0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0 & -\text{id} \\ \text{id} & \rho \end{pmatrix}.$$  

where $\xi = x_i e^i, \rho = ||\rho_i|| = ||x_k e^L_{ij}||$. We may also write

$$\Omega_{\xi,g} = \rho_i \theta^i \wedge \theta^j + \theta^i \wedge dx_i,$$

$$\Lambda_{\xi,g} = \rho_i \partial_i \wedge \partial_j - X^L_i \wedge \partial_i.$$  

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Any $L_G$-invariant function on $T^*G$ is a function $f(\xi) = f(x_i)$ on $gg^*$. Its differential at a point $\xi = (x_1, \cdots, x_n)$ is identified with the vector $df_\xi = f_idx_i \in \mathfrak{g}$ Hence, the hamiltonian vector field with $L_G$ invariant hamiltonian $f(\xi)$ has the form
\[
\tilde{f} = L_\xi df_\xi = c^k_i x_k f_i \partial x_j + df^L_\xi = (\text{ad}^*_hf_\xi) v + df^L_\xi = (\text{ad}^*_hf_\xi) \partial_j + df^L_\xi.
\]
The Poisson bracket of two such functions $f(x), h(x)$ is given by
\[
\{f(x), h(x)\} = x_k c^k_{ij} f_i h_j = \xi([df_\xi, dh_\xi]).
\]
The hamiltonian vector field $\hat{X}^L$, which is the complete extension of an $L_G$ invariant vector field $X^L$ has the linear in $\xi$ hamiltonian $\xi(X)\xi(a^e_i) = x_i a^e_i$ and is given by
\[
\hat{X}^L = (\text{ad}^*\xi) v + X^L = (\text{ad}^*\xi) \partial_j + X^L.
\]

The coadjoint action $\text{ad}^*$ of the Lie algebra $\mathfrak{g} \ni X$ induces the action, which we denote by $ad_X$, on the space $P^k(\mathfrak{g}^*)$ of homogeneous polynomials of degree $k$ on $\mathfrak{g}^*$, which is identified with the symmetric $k$-th power $S^k \mathfrak{g}$. The following lemma is obvious.

**Lemma 8** The differential $d : P^k(\mathfrak{g}^*) \rightarrow P^{k-1}(\mathfrak{g}^*) \otimes \mathfrak{g}$ is a natural isomorphism of $\mathfrak{g}$-modules with the natural action of $\mathfrak{g}$ on the space $P^{k-1}(\mathfrak{g}^*) \otimes \mathfrak{g}$ of $P^{k-1}(\mathfrak{g}^*)$-valued 1-forms on $\mathfrak{g}^*$.

As a corollary, we get

**Corollary 9** Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be a reductive decomposition, where $\mathfrak{h}$ is a subalgebra, and $\mathfrak{m}$ a complementary $ad_{\mathfrak{h}}$-invariant subspace. Then any $ad_{\mathfrak{h}}$-invariant polynomial $f \in P^k(\mathfrak{g}^*) = S^k \mathfrak{g}$ Poisson commute with any polynomial $h \in P^k(\mathfrak{h}) = S^k \mathfrak{h}$: $\{h, f\} = 0$

*Proof:* By the lemma, $\{a, f\} = 0$ for $a \in \mathfrak{h}$. Now the result follows from the Leibnitz formula. \hfill $\square$

### 5.2 Reduction of H-geodesic equation for a left invariant

sub-Riemannian metric $(D, m)$ in a Lie group $G$ to the Euler equation in the Lie algebra $\mathfrak{g}$

Let $(D, g)$ be a left invariant sub-Riemannian structure on a Lie group $G$ where $D \subset TG$ a bracket generated left invariant distribution. The H-geodesics are projection to $G$ of the orbits of the Hamiltonian vector field $H = \Omega^{-1} \circ dH$ with the Hamiltonian $H(x, p) = g^{-1}(p_D, p_D) = \sum_{j=1}^m f^j_{X_i}$, where $p_D \in D^*$ is the restriction of a covector $p$ to $D$, $(X_i)$ is an orthonormal frame of $D$ and $f_X(p) := X^i p_i.$ is the function on $T^*G$ defined by a vector
field $X \in \mathcal{X}(Q)$
Conjecture $f_X = X^*$ where $X^*$ is the natural lift of $X \in \mathcal{X}(Q)$ to $\mathcal{X}(T^*Q)$ given by.

Note that the left invariant functions $\mathcal{F}^L(T^*G) \simeq \mathcal{F}(g^*)$ closed under Poisson bracket in $\mathcal{F}(T^*G)$. We identify them with functions on $g^*$. Then the Poisson bracket reduces to the standard Poisson bracket in $\mathcal{F}(g^*)$ defined by

$$\{f, h\}(\xi) = \langle \xi, [df\xi, dh\xi] \rangle, \quad f, h \in \mathcal{F}(g), \quad \xi \in g^*$$

and $df\xi \in T^*_\xi g^*$ is identified with an element of $(g^*)^* = g$.

In particular, the Hamiltonian $H$ defines the Hamiltonian vector field $\vec{H}$ such that $\vec{H} \cdot f = \{H, f\}$.
So the Hamiltonian equation has the form

$$\dot{\xi} = \vec{H} \cdot \xi.$$ 

5.3 Hamiltonian equations with invariant Hamiltonian $f \in C^\infty(T^*M)$ on a homogeneous manifold $M = G/H$

Let $\pi : G \to M = G/H$ be the principal bundle associated with a reductive homogeneous manifold. A reductive decomposition $g = h + m$ defines a principal connection $\omega : TG \to h$ where $\omega_e = \text{pr}_h : T_e G = g \to h$ is the projection parallel to $m$. The cotangent bundle $T^*M = G \times_{\text{Ad}_{\text{g}}} m^*$ is the homogeneous bundle over $M = G/H$, defined by the coadjoint representation $\text{ad}_h : m^* \to m^*$. Its pull back is $\pi^*(T^*M) = G \times m^* \to M$. The space of $G$-invariant functions is a Poisson subalgebra $C^\infty(T^*M)^G = C^\infty(m^*)^H$ of $\text{Ad}_{\text{g}}$-invariant functions on $m^*$. In particular, the space of homogeneous invariant polynomials of degree $k$ is identified with $S^k(m^*)^H$.

An invariant polynomial Hamiltonian $f(\xi) \in P^k(m^*) \subset P^k(g^*)$ defines the (projectable) $G$-invariant Hamiltonian vector field $\vec{f}(\xi)$ in $\pi^*(T^*M) = G \times g^m$, which is projected to an invariant Hamiltonian vector field $\pi_* \vec{f}$ on $T^*M$. The Hamiltonian equation in $\pi^*T^*M = G \times m^* \subset G \times g^*$ for an integral curve $(g, \xi(t))$ is given by

$$\dot{\xi}(t) = \text{ad}_{df\xi}^* \xi, \quad \xi \in m^*.$$ 

5.4 Lagrange equation of geodesics on $TG$

Let $D \subset TG$ be a left invariant distribution on $G$, $D_e \subset g$. $g^D$ a left invariant metric in $D$, $g^D_e(X,Y) = \langle X, Y \rangle > X, Y \in g$. $TG = G \times g$. Then left invariant functions $C^\infty(TG)^L C^\infty(g^*)$ forms a Poisson subalgebra. The Euler-Lagrange equation for an invariant $L(g, \dot{g}) = l(X), X = g^{-1}\dot{g} \in g$ reduces to Euler equation
\[ \frac{d}{dt} \partial_X \ell = \text{ad}_X^* \partial_X \ell \]

or

\[ \dot{\xi} = \text{ad}_X^* \xi \]

where \( \xi = \partial_X \ell(X) = d\ell(X) \in T^*g = g^* \).

In the case of metric \( m \), the Euler equation is

\[ \dot{\xi} = \text{ad}_X^* \xi, \xi = m \circ X \]

or

\[ \dot{X} = \text{ad}_X^m X \]

where \( \text{ad}_X^m \) is metric conjugated endomorphism to \( \text{ad}_X \).

### 5.5 Reduced equation for L-geodesics

**Theorem 10** (Vershik, Yoshimura, Marsden) The reduced equation for non-holonomic L-geodesics for left invariant sub-Riemannian Lie group \((G, D, m)\) has the form

\[ \dot{X} = \text{pr}_D \text{ad}_X^* X, X \in g \]

or, in cotangent space for \( \xi = m \circ X \)

\[ \dot{\xi} = \text{ad}_m^* \xi, 1 \xi \]

where \( \text{ad}_X^* \) is the coadjoint action.

### 6 PART IV. Sub-Riemannian geometry on principal bundle

Let \( \pi : Q \to M = Q/G \) be a \( G \)-principal bundle with a connection \( \omega : TQ \to g \) (a horizontal \( G \)-equivariant 1-form)) over a Riemannian manifold \((M, g^M)\). The right action of an element \( a \in G \) on \( Q \) is denoted by \( R_a : q \mapsto R_aq = qa \).

Denote by \( D = \ker \omega \) the horizontal distribution. Since the projection \( \pi_* : D_q \to T_{\pi(q)}M \) is an isomorphism, the metric \( g^M \) induces a sub-Riemannian metric \( g^D \) on \( D \).

The sub-Riemannian manifold \((Q, D, g^D)\) is called a Chaplygin system or a transversally homogeneous SR manifold.

#### 6.1 Standard extension of the sub-Riemannian metric

Let \( (\pi : Q \to M, \omega) \) be a principal bundle with a principal connection \( \omega \).

We denote by \( X^h \) the horizontal lift of a vector field \( X \in \mathfrak{X}(M) \) and by \( a^* = \frac{d}{dt} R_{\exp ta} \) the fundamental vector field generated by an element \( a \in g \).

Then, see [K-N],

\[ [a^*, b^*] = [a, b]^*, [a^*, X^h] = 0, [X^h, Y^h] = [X, Y]^h + [X^h, Y^h] \]

\[ VW \]
where $X^h \in \mathfrak{X}(Q)$ is the horizontal lift of a vector field $X \in \mathfrak{X}(M)$ and $X^v$ (resp., $X^h$) its vertical (resp., horizontal) part.

The map $\mathfrak{g} \ni a \mapsto a^\ast$ defined an isomorphism $\mathfrak{g} \to T^*_qQ$. Due to this, an Euclidean metric $g^\mathfrak{g}$ on $\mathfrak{g}$ defines a degenerate metric

$$g^F(X, Y) = g^\mathfrak{g}(\omega(X), \omega(Y))$$

with kernel $D$, whose restriction to a fiber $F = \pi^{-1}(x)$ is a Riemannian metric.

Then

$$g^Q = g^F + g^D$$

(1)

is a Riemannian metric in $Q$ which is called the standard extension of the sub-Riemannian metric $g^D$. Note that the metric $g^Q$ is not $G$-invariant if the metric $g^\mathfrak{g}$ is not Ad$_G$-invariant.

### 6.2 Sub-Riemannian structure on principal bundle. Relation between LS-geodesics and H-geodesics

**Define also canonical connection.**

Now we describe the Levi-Civita connection $\nabla^Q$ of $g^Q$, using the above identities for $a^\ast, X^h$ and the Koszul formula for Levi-Civita connection $\nabla$ of a Riemannian metric $g^Q$:

$$2g(\nabla_X Y, Z) = X \cdot g(Y, Z) + Y \cdot g(X, Z) - Z \cdot g(X, Y) + g([X, Y], Z) - g(Y, [X, Z]) - g(X, [Y, Z]), \ X, Y, Z \in \mathfrak{X}(Q).$$

Denote by $A : \Gamma D \times \Gamma D \to \Gamma (T^vQ)$, $(X, Y) \mapsto A_X Y = \frac{1}{2}[X, Y]^v$ the O’Neil tensor and by $A^*X : \Gamma (T^vQ) \to \Gamma D$ the dual map to $A_X : \Gamma D \to \Gamma (T^vQ)$.

Then we get the following O’Neil formulas, for covariant derivative of fundamental vector $b^\ast$ and a horizontal lift $X^h$ of a vector field $X \in \mathfrak{X}(Q)$, see [?]

$$\nabla^Q_{a^\ast} b^\ast = \nabla^F_{a^\ast} b^\ast$$

$$\nabla^Q_{a^\ast} X^h = \nabla^Q_{X^h a^\ast} = \langle \nabla^Q_{a^\ast} X^h \rangle^h$$

$$\nabla^Q_{X^h} Y^h = \langle \nabla^M Y \rangle^h + A_{X^h} Y^h$$

where $\nabla^M$ is the Levi-Civita connection of $g^M$ and and the following formula, which describe the connection $\nabla^F$ of a fiber in terms of Lie bracket

$$2g(\nabla^Q_{a^\ast} b^\ast, c^\ast) = g^\mathfrak{g}([a, b], c) - g^\mathfrak{g}(b, [a, c]) - g^\mathfrak{g}(a, [b, c]), \ a, b, c \in \mathfrak{g}.$$

These formulas imply the following result, which in the case, when the structure group $G$ is compact and $g^\mathfrak{g}$ is an Ad$_G$-invariant metric, had been proved by Montgomery [Mont].
Theorem 11  i) The principal bundle \( \pi : Q \to M \) with the standard metric \( g^Q \) associated to a Riemannian metric \( g^M \) is a Riemannian submersion with totally geodesic fibers.

ii) A geodesic \( \gamma(s) \) of \( (Q, g^Q) \) which is horizontal at one point \( (\dot{\gamma}(0) \in D) \) is horizontal and is projected into a geodesic \( \pi \gamma(t) \) of \( (M, g^M) \).

iii) Conversely, the horizontal lift of a geodesic \( c(s) \) of \( (M, g^M) \) is a horizontal geodesic of \( (Q, g^Q) \).

iv) Horizontal geodesics are sub-Riemannian H-geodesics of \( (D, g^D) \).

QUESTION : Is any H-geodesic is the horizontal Riemannian geodesic?

Note that \( \nabla^Q_{X^h} X^h \in D \) for any \( X \in \mathfrak{X}(M) \).

This shows that the restriction \( \nabla^D = \nabla^Q|D \) of the Levi-Civita connection to \( D \) give the Schouten connection. In particular, LS geodesics of the SR manifold \( (Q, D, g^D) \) are horizontal geodesics of the Levi-Civita connection \( \nabla^Q \). So as a corollary of these remarks and O’Neil formulas above, we get

Theorem 12  i) The bundle \( \pi : Q \to M = Q/G \) with the metric \( g^Q \) is a Riemannian fibration with totally geodesic fibers.

ii) The indices metric on a fiber \( F_q = \pi^{-1}(q) \cong G \) is isometric to the left invariant metric of the group \( G \), defined by the metric \( g^\mathfrak{g} \).

iii) A geodesic \( \gamma(s) \subset Q \) of the metric \( g^Q \) which is horizontal at one point \( \gamma(0) (\dot{\gamma}(0) \in D) \) is horizontal, hence a LS geodesic, and its projection \( \pi \gamma(s) \) is a geodesic \( \gamma(s) \subset M \) is a horizontal geodesic.

Assume now that the Lie algebra \( \mathfrak{g} \) of the structure group \( G \) admits an \( \text{Ad}_G \)-invariant pseudo-Euclidean metric \( g^\mathfrak{g} \). It is the case if \( G \) is reductive. Consider the associated standard pseudo-Riemannian metric \( g^Q \), given by 1. This metric is \( G \)-invariant. Indeed, the \( G \)-equivariancy of the connection form \( \omega \) means that

\[
R_a^\omega(X, Y) = \text{Ad}_{a^{-1}}\omega(X, Y), \ X, Y \in TQ
\]

where \( R_a : q \mapsto qa \) is the action of \( a \in G \). Then

\[
(R_a^g^Q)(X, Y) = g^M(X, Y) + g^\mathfrak{g}((R_a^\omega)(X), (R_a^\omega)(Y)) = g^M(X, Y) + g^\mathfrak{g}(\text{Ad}_{a^{-1}}\omega(X), \text{Ad}_{a^{-1}}\omega(Y)) = g^Q(X, Y).
\]

Any vector field \( a^* \) is extended to a Hamiltonian vector field \( \tilde{p}(a^*) \) on \( T^\ast Q \), with linear in momenta \( p \) Hamiltonian \( p(a^*) = \sum (p_i a^*_i) \) (which is the complete lift of \( a^* \) to \( T^\ast Q \)). Denote by \( H_Q, H_D \) and \( H_\mathfrak{g} \) the quadratic (in \( p \)) Hamiltonian associated with the metric \( g^Q \) and degenerate metrics \( g^D, g^\mathfrak{g} \).
Then $H_Q = H_D + H_g$. Since the Poisson bracket $\{H_Q, p(a^*)\} = p(\alpha^*)H_Q = \mathcal{L}_\alpha g^Q = 0$, and $H_g = \sum k_{ij} a_i^* a_j^*$ is a quadratic expression with constant coefficients where $a_i$ is a basis of $\mathfrak{g}$, it follows that the Hamiltonians $H_Q, H_g$ commute w.r.t. Poisson bracket: $\{H_Q, H_g\} = 0$.

This means that the Hamiltonian vector fields $\tilde{H}_Q = \tilde{H}_D + \tilde{H}_g$ and $\tilde{H}_g$ commute. But the field $\tilde{H}_Q$ defines the geodesic flow $\varphi^Q(t)$ on $T^*Q$ of the pseudo-Riemannian metric $g^Q$, the vertical field $\tilde{H}_g$ defined on any fiber $F_x = \pi^{-1}(x) \simeq G$ the geodesic flow for the induced pseudo-Riemannian metric, which is identifies with the flow on the group $G$ defined by $g^g$. Integral curves through $q \in F$ of this flow have the form $q \exp ta$, $a \in \mathfrak{g}$. The field $\tilde{H}_D = \tilde{H}_Q - \tilde{H}_g$ defines the geodesic flow $\varphi^D(t)$ for the sub-Riemannian metric $g^D$, whose integral curves are projected onto (horizontal) $H$-geodesics. Since all these flows commute, we have $\varphi^Q(t) = \varphi^D(t)\varphi(t)$. The projection of integral curves of this flow to $Q$ are geodesics and they may be written as 

$$\gamma^Q(t) = \gamma^D(t) \exp ta.$$ 

Denote by $v = \gamma^Q(0)$ the tangent vector of this geodesic. It is decomposed into horizontal and vertical part as $v = v_h + a^*_a$ where $a = \omega(v) \in \mathfrak{g}$. Since $\gamma^D(t)$ is a horizontal curve, it has tangent vector $v_h$. Moreover, $\gamma^D(t) = \gamma^Q(t) \exp(-ta)$ is a (horizontal) geodesic, since $\exp ta$ is an isometry.

We proved that like for LS-geodesics $H$-geodesics are horizontal geodesics of the standard extension $g^Q$ of the sub-Riemannian metric $g^D$. The results can be summarized as follows.

**Theorem 13** Let $\pi : Q \to M = Q/G$ be a principal $G$-bundle over a Riemannian manifold $(M, g^M)$ with a connection $\omega : TQ \to \mathfrak{g}$ and $(D = \ker \omega, g^D)$ associated sub-Riemannian structure. Assume that the Lie algebra $\mathfrak{g}$ admits a $\text{Ad}_G$-invariant pseudo-Euclidean metric $\gamma^g$ and denote by $g^Q$ the associated standard extension of $g^D$ to a $G$-invariant pseudo-Riemannian metric on $Q$. Then sub-Riemannian $H$-geodesics coincides with LS-geodesics. They are precisely the horizontal geodesics of the metric $Q$ and are projected to geodesics of the Riemannian manifold $(M, g^M)$. Conversely, horizontal lifts of a (minimal) geodesic of $(M, g^M)$ are (minimal) sub-Riemannian geodesics. Any geodesic of the metric $g^Q$ are obtained from horizontal geodesics by application of isometries from the group $G$.

**Remark 14** I do not know what are relations between $H$-geodesics and LS-geodesics if the metric $g^g$ is not $\text{Ad}_G$-invariant.
PART V. Some classes of homogeneous sub-Riemannian manifolds

7.1 Homogeneous Riemannian manifolds with non simple stability subgroup

Let \((M = \frac{G}{H}, g^M)\) be a homogeneous Riemannian manifold with reductive decomposition \(\mathfrak{g} = \mathfrak{h} + \mathfrak{m}\) and induced on \(\mathfrak{m}\) an \(\text{Ad}_H\)-invariant metric \(g^\mathfrak{m}\).

Since the Lie subalgebra \(\mathfrak{g}' = [\mathfrak{m}, \mathfrak{m}] + \mathfrak{m}\) generates a transitive subgroup of \(G\), we may always assume that the invariant distribution \(D \subset TG\) generated by \(\mathfrak{m}\) is bracket generated.

Assume that \(\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{a}\) is a direct sum of two ideals and \(K, A\) corresponding normal subgroups. Then \(\pi_M : Q = \frac{G}{K} \rightarrow M = \frac{G}{H} = \frac{G}{K} \cdot A\) is a principal \(A\)-bundle (with the right action of the group \(A\)).

The manifold \(Q = \frac{G}{K}\) has the reductive decomposition \(\mathfrak{g} = \mathfrak{k} + \mathfrak{n} = \mathfrak{k} + (\mathfrak{a} + \mathfrak{m})\) and \((\mathfrak{m}, g^\mathfrak{m})\) defines an invariant sub-Riemannian structure \((D, g^D)\) of codimension \(\text{dim} \mathfrak{a}\) in \(Q = \frac{G}{K}\). As above, we extend \(g^D\) to a Riemannian metric \(g^Q\).

Theorem 13 establishes the equivalence of the following problems:

i) Description of geodesics of the Riemannian manifold \(M = \frac{G}{H} = \frac{G}{K} \cdot A\);

ii) Description of \(H\)-geodesics on the associated left invariant sub-Riemannian metric \((D, g^D)\) in the homogeneous manifold \(Q = \frac{G}{K}\);

iii) Description of geodesics of the left invariant metric \(g^Q\) on \(Q = \frac{G}{K}\), which is the \(\text{Ad}_K\)-invariant extension of the metric \(g^D\).

We have

**Theorem 15** Let \(\pi_M : Q = \frac{G}{K} \rightarrow M = \frac{G}{K} \cdot A\) be the homogeneous \(A\)-principal bundle over a homogeneous Riemannian manifold \((M, g^M)\) with the sub-Riemannian metric \((D, g^D)\) as above. Denote by \(g^Q\) the \(\text{Ad}_K\)-invariant extension of \(g^D\) to a \(G\)-invariant metric on \(Q = \frac{G}{K}\).

Then \(H\)-geodesics of the sub-Riemannian metric \((D, g^D)\) coincides with \(L\)-geodesics and are horizontal geodesics of the Riemannian metric \(g^Q\) and they are horizontal lifts of geodesics of the base manifold \((M = \frac{G}{H}, g^M)\).

7.2 Homogeneous contact sub-Riemannian manifolds

The above result can be applied to regular homogeneous compact contact sub-Riemannian manifolds, which are described as follows.
Let $F = G/K = \text{Ad}_Gz \subset g$ be a flag manifold of a compact semisimple Lie group $G$ with reductive decomposition

$$g = \mathfrak{k} + \mathfrak{m} = (\mathfrak{k}' + \mathfrak{z}) + \mathfrak{m}.$$ 

An element $z \in \mathfrak{z}$ is called $K$-regular if the centralizer $C_g(z) = \mathfrak{k}$ and the 1-parametric subgroup $T^1 = \exp \mathbb{R} z \subset G$ is closed. Then the orthogonal complement $\mathfrak{h}$ of $z$ in $\mathfrak{k}$ generated a closed subgroup $H$ and the homogeneous manifold $M = G/H$ which is a principal $T^1$-bundle $\pi : M = G/H \to F = G/K$ has a natural invariant contact distribution $D$, defined by the condition $D_o = \mathfrak{m} \subset p = \mathbb{R} z + g_m \simeq T_oM, o = eH$. Note that $D$ can be considered as an invariant connection in $\pi$ and the connection form $\theta$, defined by $\theta_o(z) = 1, \theta(m) = 0$, is the contact form. The Reeb vector $Z$ is the fundamental vector field of $\pi$, i.e. the invariant vector field generated by $z$.

**Theorem 16** Any invariant contact distribution on a homogeneous manifold of the group $G$ may be obtained by this construction. Moreover, any $\text{ad}_{gh}$-invariant Euclidean metrics in $\mathfrak{m}$ is extended to a sub-Riemannian metric $g^D$ on $D$ s.t. $(M = G/H, D, g^D)$ is a sub-Riemannian manifold. Any homogeneous sub-Riemannian manifold of the group $G$ has such form.

### 7.3 Invariant sub-Riemannian structure on the Lie group $G$ associated to a Riemannian homogeneous manifold $(M = G/H, g^M)$

We specialized the above construction to the case $K = \{e\}$. i.e. to the principal $H$-bundle $\pi : G \to M = G/H$ with the reductive distribution $g = \mathfrak{h} + \mathfrak{m}$ and the associated principal left invariant connection

**Theorem 17** Let $(M = G/H, g^M)$ be a homogeneous Riemannian manifold and $\pi_M : G \to M = G/H$ the associated principal bundle, $g^D$ the invariant sub-Riemannian metric on $D$ and $g^G$ its extension to a Riemannian metric on $G$ defined by an $\text{Ad}_H$-invariant metric in $\mathfrak{h}$. Then $H$-geodesics of $(D, g^D)$ coincides with LS-geodesics and with horizontal geodesics on the Riemannian metric $g^G$, and they are horizontal lift of the geodesics of the sub-Riemannian metric $g^D$ defined by an $\text{Ad}_H$-invariant metric in Riemannian manifold $M = G/H$.

This result can be applied to a left invariant sub-Riemannian geodesic $(D, g^D)$, defined by a bracket generated subspace $\mathfrak{m} \subset g$ s.t. there is a complementary to $\mathfrak{m}$ subalgebra $\mathfrak{h}$ which generates a closed subgroup $H$ of the Lie group $G$ and the metric $\mathfrak{m}$ is $\text{ad}_{gh}$-invariant.
8 Invariant sub-Riemannian structures on flag manifolds

Bracket generated invariant sub-Riemannian structures bijectively corresponds to system of simple T-roots associated with such manifold.

8.1 Fundamental grading of a complex semisimple Lie algebra \( g \) and associated flag manifolds

Let \( g = c + \sum_{\alpha \in R} g_{\alpha} \) be a root space decomposition and \( \Pi \) is a system of simple roots. There is a natural 1-1 correspondence between decomposition \( \Pi = \Pi_W \cup \Pi_B \) of simple roots into White \( \Pi_W \) and black \( \Pi_B \) (which is graphically represented by a painted Dynkin diagram) and fundamental grading

\[
\mathfrak{g} = \sum_{i=-d}^{d} \mathfrak{g}^i
\]

where \( \mathfrak{g}^0 = c + \mathfrak{g}([R^0]), \mathfrak{g}^j = \mathfrak{g}(R^j) \). Here

\[
R^0 = \text{span}_{\mathbb{Z}} \cap R, R^j = j\Pi_B \cap R
\]

and for any \( P \subset R \) we set \( \mathfrak{g}(P) = \sum_{\alpha \in P} \mathfrak{g}_\alpha \).

Denote by \( \mathfrak{p} = \sum_{i \geq 0} \mathfrak{g}^i \) the associated parabolic subalgebra. Then \( F = G/P \) is a complex flag manifold. The standard compact real form

\[
\mathfrak{g}^{\text{tau}} = c^{\text{tau}} + \sum_{j>0} (\mathfrak{g}^j + \mathfrak{g}^{-j})^{\tau}
\]

generates a maximal compact subgroup \( G^{\tau} \) which acts transitively on \( F \) and \( F = G^{\tau}/P^{\tau} \). The subgroup \( H = P^{\tau} = G^{0(\tau)} \) is the centralizer of some element \( z_0 \in c^{\tau} \) and the real homogeneous space \( F = G^{\tau}/H = \text{Ad}_{G^{\tau}} z_0 \) is called the real flag manifold. It has the reductive decomposition

\[
\mathfrak{g}^{\tau} = (\mathfrak{g}(R^0))^{\tau} + (\mathfrak{g}(R_m))^{\tau} = (\mathfrak{g}^0)^{\tau} + \text{sum}_{j>0} (\mathfrak{g}^j + \mathfrak{g}^{-j})^{\tau},
\]

where \( R_m = R \setminus R^0 \).

Invariant complex structure on real flag manifold \( F = G^{\tau}/H \) correspond to painted Dynkin diagram. Then the complexification \( G \) of the compact group \( G^{\text{tau}} \) acts on \( F \) as the group of holomorphic transformations and determines the representation of \( F \) as the quotient \( F = G/P \).

Lemma 18 The \( \text{ad}_{\mathfrak{h}} \)-invariant subspace \( D = (\mathfrak{g}^1 + \mathfrak{g}^{-1})^{\tau} \) of the tangent space \( \mathfrak{m} \simeq T_0(G^{\tau}/H \) define an invariant bracket generated distribution in the real flag manifold.

The following theorem shows that any bracket generated invariant distribution on flag manifold can be obtained by this construction.
Theorem 19  Let \( F = G^\tau / H \) be a real flag manifold with the reductive decomposition
\[
\mathfrak{g}^\tau = \mathfrak{k} + \mathfrak{m} = (\mathfrak{c} + \mathfrak{g}(R^0))^{\tau} + \mathfrak{g}(R\mathfrak{m})^{\tau}.
\]

There is a natural 1-1 correspondence between painted Dynkin diagrams, grading of the complexification \( \mathfrak{g} \) of \( \mathfrak{g}^\tau \), invariant complex structures in \( F = G^\tau / H \), and invariant bracket generated distributions \( D = (\mathfrak{m}^{-1}|\mathfrak{m}^1) \) in \( F = G^\tau / H \). Any \( \text{ad}_{\mathfrak{h}} \)-invariant metric in \( D \) defines an invariant sub-Riemannian structure in \( F = G^\tau / H \). The distribution \( D_0 \) at a point \( o = eH \) is isotropy irreducible if and only if the Dynkin diagram has only one black root.

8.2 Codimension 2 sub-Riemannian structures on flag manifolds associated with a fundamental grading

Let \( \mathfrak{g} = \sum_{j=-d}^d \mathfrak{g}_j \) be a fundamental grading of a complex semisimple Lie algebra \( \mathfrak{g} \) and \( \mathfrak{g}^\tau \) the compact form of \( \mathfrak{g} \) consistent with the grading, i.e. such that
\[
\mathfrak{g}^\tau = (\mathfrak{g}^0)^\tau + \sum_{j>0}(\mathfrak{g}^{-j} + \mathfrak{g}^j)^\tau.
\]
Denote by \( G^\tau \) the associated maximal compact subgroup of the Lie group \( G \) and by \( H \) the subgroup generated by \( \mathfrak{h} = (\mathfrak{g}^0)^\tau \). Then \( F = G^\tau / H \) is a flag manifold and \( \mathfrak{d} := (\mathfrak{g}^{-1} + \mathfrak{g}^1)^\tau \) defines an invariant bracket generated distribution in \( F \) of codimension \( k = 2 \). Moreover, \( k = 2 \) if and only if \( d = 2 \) and \( \dim \mathfrak{g}^{\pm 2} = 1 \). Such grading are called contact grading and they are eigenspace decomposition of the coroot \( H_\mu \) associated with a highest root \( \mu \) of the Lie algebra \( \mathfrak{g} \). More precisely
\[
\mathfrak{g} = \mathfrak{g}^{-2} + \mathfrak{g}^{-1} + \mathfrak{g}^0 + \mathfrak{g}^1 + \mathfrak{g}^2 = \mathfrak{g}_{-\mu} + \mathfrak{g}^{-1} + \mathfrak{g}^0 + \mathfrak{g}^1 + \mathfrak{g}_\mu,
\]
and \( \mathfrak{g}^0 = \mathbb{C} H_\mu + \mathfrak{g}^0 \) is the centralizer of \( H_\mu \). The symmetric decomposition
\[
\mathfrak{g} = \mathfrak{g}_{\text{ev}} + \mathfrak{g}_{\text{odd}} = (\mathfrak{g}^{-2} + \mathfrak{g}^0 + \mathfrak{g}^2) + (\mathfrak{g}^{-1} + \mathfrak{g}^1)
\]
defines a quaternionic Kähler symmetric space \( Q = G^\tau / G_{\text{ev}}^\tau \) and the flag manifold \( F = \text{Ad}_{\mathfrak{g}}(H_\mu) = G^\tau / C(H_\mu) \) is the twistor space of \( Q \) with the canonical invariant complex structure and the Kähler-Einstein metric \( g \).

The reductive decomposition for \( F \) is
\[
\mathfrak{g}^\tau = (\mathfrak{g}^0)^\tau + (\mathfrak{g}^{-2} + \mathfrak{g}^2)^\tau + (\mathfrak{g}^{-1}|\mathfrak{g}^1)^\tau.
\]
The invariant subspace \( \mathfrak{d} = (\mathfrak{g}^{-1}|\mathfrak{g}^1)^\tau \) induces invariant codimension two distribution, which is a homomorphic contact distribution. If \( G \neq SU_n \), the \( \text{Ad}_{C(H_\mu)} \)-module \( \mathfrak{d} \) is irreducible and there is unique (up to scaling) invariant sub-Riemannian metric in \( D \) induces by Kähler-Einstein metric. As a summary we have
**Theorem 20** All codimension 2 invariant bracket generated sub-Riemannian metrics on flag manifolds, associated with a grading are exhausted by twistor spaces of Wolf spaces with holomorphic contact structure $D$ and sub-Riemannian metric, which is the restriction to $D$ of the invariant Kähler-Einstein metric on $F$.

8.3 Symmetric sub-Riemannian manifolds associated with a graded semisimple Lie algebra

Strichartz (JDG, 1986) defined the notion of sub-Riemannian symmetric space as a homogeneous sub-Riemannian manifold $(Q = G/H, D, g^D)$ such that $H$ contains an involutive element $\sigma$ which acts on the subspace $D_o$ of the point $o = eH \in Q$ as $-id$.

He classified 3-dimensional sub-Riemannian symmetric spaces and stated the problem of extension of this classification to higher dimensions.

E. Falbel and C. Gorodski classified symmetric sub-Riemannian manifolds of contact type (1995). W. Respondek and A. J. Maciejewski describes all integrable sub-Riemannian metrics on 3-dimensional Lie groups with integrable H-geodesic flow (2008). They are exhausted by sub-Riemannian symmetric spaces.

Conjecture. Geodesic flow of any sub-Riemannian symmetric space are integrable.

We describe a class of compact sub-Riemannian symmetric spaces associated with graded semisimple Lie algebraa.

Let

$$g = \sum_{i=-d}^{d} g_i$$

be a graded complex semisimple Lie algebra of depth $d \geq 2$ and

$$g = g^{ev} + g^{odd}$$

associated symmetric decomposition.

Denote by $\tau$ the anti-linear involution which defines the compact real form s.t. $g^{\tau} = g_0^\tau + \sum_{i>0}(g_{-i} + g_i)^\tau$ We set $\mathfrak{h} = g_0^\tau$ and denote by $H$ associated subgroup of the compact Lie group $G^\tau$.

Associated with $g$ flag manifold can be written as $F = G/P = G/G^0 \cdot G^+ = G^\tau/H$. We choose an $ad_{\mathfrak{h}}$-invariant metric $g^m$ on the space $m = (g^{-1} + g^1)^\tau$. Then $(m, g^m)$ defines an invariant bracket generated sub-Riemannian structure $(D, g^D)$ on the flag manifold $F = G^\tau/H$.

**Theorem 21** The flag manifold $F = G^\tau/H$ with invariant sub-Riemannian structure $(D, g^D)$ defined by $(m, g^m)$ is a sub-Riemannian symmetric space.
Example Let

$$g = g_{-2} + g_{-1} + g_{0} + g_{1} + g_{2}, \ \text{dim} \ g_{\pm 2} = 1$$

be the contact gradation of a complex simple Lie algebra $g$, i.e. the eigenspace decomposition of $\text{ad} H_\mu$ where $H_\mu$ is the coroot associated to the maximal root $\mu$ of $g$. Then the symmetric space $G'/G'_{ev}$ is the quaternionic Kähler symmetric space (the Wolf space) and the flag manifold $F = G'/H$ is the associated twistor space. The distribution $D$ is the holomorphic contact distribution and $g^D$ is unique (up to scaling) invariant sub-Riemannian metric on $D$. It is the restriction of the invariant Kähler-Einstein metric on $F$ (for $g \neq \mathfrak{sl}_n(\mathbb{C})$).

Reduction to LA

References

[A=G-M-M] D. Alekseevsky, J. Grabowski , G. Marmo, P. W. Michor, Poisson structures on the cotangent bundle of a Lie group or a principal bundle and their reductions, J. Math. Physics 35 (1994), 4909-4928.

[B] V.N. Berestovsky, On the curvature of homogeneous sub-Riemannian manifolds, European Journal of Mathematics, (2017) 3, 788807.

[Besse] A.Besse, Einstein Manifolds,

[C-O] Cowling, M., Ottazzi, A. Conformal maps of Carnot groups. Ann. Acad. Sci. Fenn. Math. 40 (2015), no. 1, 203-213,

[J] V. Jurdjevic, Optimal Control and Geometry: Integrable Systems, Cambridge University Press, 2016.

[K-N] Sh. Kobayashik, K. Nomuzu , Foundation of Differential Geometry, v.1,2. 1963.

[Mont] R. Montgomery, A Tour of Sub-Riemannian Geometry, Their Geodesics and Applications, Mathematical Surveys and Monographs, v.9.

[M] T. Morimoto , Cartan connection associated with a subriemannian structure, Differential Geometry and its Applications 26 (2008) 75-78.

[Stern] Sh. Sternberg, Lectures in Differential Geometry, Prentice Hall. 1964.

[Y] T. Yatsui, On pseudo-product graded Lie algebras, Hokkaido Math. J. 17 (1988) 333-343.