LAMPLIGHTER GROUPS AND VON NEUMANN’S CONTINUOUS REGULAR RING

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ABSTRACT. Let $\Gamma$ be a discrete group. Following Linnell and Schick one can define a continuous ring $c(\Gamma)$ associated with $\Gamma$. They proved that if the Atiyah Conjecture holds for a torsion-free group $\Gamma$, then $c(\Gamma)$ is a skew field. Also, if $\Gamma$ has torsion and the Strong Atiyah Conjecture holds for $\Gamma$, then $c(\Gamma)$ is a matrix ring over a skew field. The simplest example when the Strong Atiyah Conjecture fails is the lamplighter group $\Gamma = \mathbb{Z}_2 \wr \mathbb{Z}$. It is known that $\mathbb{C}(\mathbb{Z}_2 \wr \mathbb{Z})$ does not even have a classical ring of quotients. Our main result is that if $H$ is amenable, then $c(\mathbb{Z}_2 \wr H)$ is isomorphic to a continuous ring constructed by John von Neumann in the 1930s.

1. Introduction

Let us consider $\text{Mat}_{k \times k}(\mathbb{C})$ the algebra of $k$ by $k$ matrices over the complex field. This ring is a unital $*$-algebra with respect to the conjugate transposes. For each element $A \in \text{Mat}_{k \times k}(\mathbb{C})$ one can define $A^*$ satisfying the following properties:

- $(\lambda A)^* = \overline{\lambda} A^*$,
- $(A + B)^* = A^* + B^*$,
- $(AB)^* = B^* A^*$,
- $0^* = 0, 1^* = 1$.

Also, each element has a normalized rank $\text{rk}(A) = \text{Rank}(A)/k$ with the following properties:

- $\text{rk}(0) = 0, \text{rk}(1) = 1$,
- $\text{rk}(A + B) \leq \text{rk}(A) + \text{rk}(B)$,
- $\text{rk}(AB) \leq \min\{\text{rk}(A), \text{rk}(B)\}$,
- $\text{rk}(A^*) = \text{rk}(A)$,
- if $e$ and $f$ are orthogonal idempotents, then $\text{rk}(e + f) = \text{rk}(e) + \text{rk}(f)$.

The ring $\text{Mat}_{k \times k}(\mathbb{C})$ has an algebraic property; namely, von Neumann called regularity: Any principal left- (or right) ideal can be generated by an idempotent. Furthermore, among these generating idempotents there is a unique projection (that is, $\text{Mat}_{k \times k}(\mathbb{C})$ is a $*$-regular ring). In a von Neumann regular ring any nonzerodivisor is necessarily invertible. One can also observe that the algebra of matrices is proper, that is, $\sum_{i=1}^{n} a_i a_i^* = 0$ implies that all the matrices $a_i$ are zero.
One should note that if $R$ is a $*$-regular ring with a rank function, then the rank extends to $\text{Mat}_{k \times k}(R)$ [9], where the extended rank has the same property as $\text{rk}$ except that the rank of the identity is $k$.

One can immediately see that the rank function defines a metric $d(A, B) := \text{rk}(A - B)$ on any algebra with a rank, and the matrix algebra is complete with respect to this metric. These complete $*$-regular algebras are called continuous $*$-algebras (see [8] for an extensive study of continuous rings). Note that for the matrix algebras the possible values of the rank functions are $0, 1/k, 2/k, \ldots, 1$. John von Neumann observed that there are some interesting examples of infinite dimensional continuous $*$-algebras, where the rank function can take any real values in between $0$ and $1$. His first example was purely algebraic.

**Example 1.** Let us consider the following sequence of diagonal embeddings:

$$
\mathbb{C} \to \text{Mat}_{2 \times 2}(\mathbb{C}) \to \text{Mat}_{4 \times 4}(\mathbb{C}) \to \text{Mat}_{8 \times 8}(\mathbb{C}) \to \ldots
$$

One can observe that all the embeddings are preserving the rank and the $*$-operation. Hence the direct limit $\lim_{\to} \text{Mat}_{2^k \times 2^k}(\mathbb{C})$ is a $*$-regular ring with a proper rank function. The addition, multiplication, the $*$-operation and the rank function can be extended to the metric completion $\mathcal{M}$ of the direct limit ring. The resulting algebra $\mathcal{M}$ is a simple, proper, continuous $*$-algebra, where the rank function can take all the values on the unit interval.

**Example 2.** Consider a finite, tracial von Neumann algebra $\mathcal{N}$ with trace function $\text{tr}_\mathcal{N}$. Then $\mathcal{N}$ is a $*$-algebra equipped with a rank function. If $P$ is a projection, then $\text{rk}_\mathcal{N}(P) = \text{tr}_\mathcal{N}(P)$. For a general element $A \in \mathcal{N}$, $\text{rk}_\mathcal{N}(A) = 1 - \lim_{t \to \infty} \int_0^1 \text{tr}_\mathcal{N}(E_\lambda) d\lambda$, where $\int_0^\infty E_\lambda d\lambda$ is the spectral decomposition of $A^*A$.

In general, $\mathcal{N}$ is not regular, but it has the Ore property with respect to its zero divisors. The Ore localization of $\mathcal{N}$ with respect to its non-zerodivisors is called the algebra of affiliated operators and denoted by $U(\mathcal{N})$. These algebras are also proper continuous $*$-algebras [1]. The rank of an element $A \in U(\mathcal{N})$ is given by the trace of the projection generating the principal ideal $U(\mathcal{N})A$. It is important to note that $U(\mathcal{N})$ is the rank completion of $\mathcal{N}$ (Lemma 2.2, [12]).

Linnell and Schick observed [9] that if $X$ is a subset of a proper $*$-regular algebra $R$, then there exists a smallest $*$-regular subalgebra containing $X$, the $*$-regular closure. Now let $\Gamma$ be a countable group and $\mathbb{C}\Gamma$ be its complex group algebra. Then one can consider the natural embedding of the group algebra to its group von Neumann algebra $\mathbb{C}\Gamma \to \mathcal{N}\Gamma$. Let $U(\Gamma)$ denote the Ore localization of $\mathcal{N}(\Gamma)$ and the embedding $\mathbb{C}\Gamma \to U(\Gamma)$. Since $U(\Gamma)$ is a proper $*$-regular ring, one can consider the smallest $*$-algebra $\mathcal{A}(\Gamma)$ in $U(\Gamma)$ containing $\mathbb{C}(\Gamma)$. Let $c(\Gamma)$ be the completion of the algebra $\mathcal{A}$ above. It is a continuous $*$-algebra [6]. Of course, if the rank function has only finitely many values in $\mathcal{A}$, then $c(\Gamma)$ equals $\mathcal{A}(\Gamma)$. Note that if $\mathbb{C}\Gamma$ is embedded into a continuous $*$-algebra $T$, then one can still define $c_T(\Gamma)$ as the smallest continuous ring containing $\mathbb{C}\Gamma$. In [3] we proved that if $\Gamma$ is amenable, $c(\Gamma) = c_T(\Gamma)$ for any embedding $\mathbb{C}\Gamma \to T$ associated to sofic representations of $\Gamma$, hence $c(\Gamma)$ can be viewed as a canonical object. Linnell and Schick calculated the algebra $c(\Gamma)$ for several groups, where the rank function has only finitely many values on $\mathcal{A}$. They proved the following results:

- If $\Gamma$ is torsion-free and the Atiyah Conjecture holds for $\Gamma$, then $c(\Gamma)$ is a skew-field. This is the case when $\Gamma$ is amenable and $\mathbb{C}\Gamma$ is a domain. Then
c(Γ) is the Ore localization of CΓ. If Γ is the free group of k generators, then c(Γ) is the Cohen-Amitsur free skew field of k generators. The Atiyah Conjecture for a torsion-free group means that the rank of an element in Matk×k(CΓ) ⊂ Matk×k(U(N(Γ))) is an integer.
• If the orders of the finite subgroups of Γ are bounded and the Strong Atiyah Conjecture holds for Γ, then c(Γ) is a finite dimensional matrix ring over some skew-field. In this case the Strong Atiyah Conjecture means that the ranks of an element in Matk×k(CΓ) ⊂ Matk×k(U(N(Γ))) is in the abelian group \( \frac{1}{\text{lcm}(Γ)} \mathbb{Z} \), where lcm(Γ) indicates the least common multiple of the orders of the finite subgroups of Γ.

The lamplighter group Γ = \( \mathbb{Z}_2 \wr \mathbb{Z} \) has finite subgroups of arbitrarily large orders. Also, although Γ is amenable, CΓ does not satisfy the Ore condition with respect to its non-zerodivisors [8]. In other words, it has no classical ring of quotients. The goal of this paper is to calculate c(\( \mathbb{Z}_2 \wr \mathbb{Z} \)) and even c(\( \mathbb{Z}_2 \wr H \)), where H is a countably infinite amenable group.

**Theorem 1.** If H is a countably infinite amenable group, then c(\( \mathbb{Z}_2 \wr H \)) is the simple continuous ring \( \mathcal{M} \) of von Neumann.

### 2. Crossed product algebras

In this section we recall the notion of crossed product algebras and the group-measure space construction of Murray and von Neumann. Let \( \mathcal{A} \) be a unital, commutative \(*\)-algebra and \( \phi : \Gamma \to \text{Aut}(\mathcal{A}) \) be a representation of the countable group Γ by \(*\)-automorphisms. The associated crossed product algebra \( \mathcal{A} \rtimes \Gamma \) is defined the following way. The elements of \( \mathcal{A} \rtimes \Gamma \) are the finite formal sums

\[
\sum_{\gamma \in \Gamma} a_\gamma \cdot \gamma,
\]

where \( a_\gamma \in \mathcal{A} \). The multiplicative structure is given by

\[
\delta \cdot a_\gamma = \phi(\delta)(a_\gamma) \cdot \delta.
\]

The \(*\)-structure is defined by \( \gamma^* = \gamma^{-1} \) and \((\gamma \cdot a)^* = a^* \cdot \gamma^{-1} \). Note that

\[
(\delta \cdot a_\gamma)^* = (\phi(\delta) a_\gamma \cdot \delta)^* = \delta^* \cdot \phi(\delta) a_\gamma^* = \phi(\delta^{-1}) \phi(\delta) a_\gamma^* = a_\gamma^* \cdot \delta^{-1} = a_\gamma^* \cdot \delta.
\]

Now let \((X, \mu)\) be a probability measure space and \( \tau : \Gamma \curvearrowright X \) be a measure preserving action of a countable group Γ on X. Then we have a \(*\)-representation \( \hat{\tau} \) of Γ in Aut(L\(^\infty(X, \mu)\)), where L\(^\infty(X, \mu)\) is the commutative \(*\)-algebra of bounded measurable functions on X (modulo zero measure perturbations)

\[
\hat{\tau}(\gamma)(f)(x) = f(\tau(\gamma^{-1})(x)).
\]

Let \( \mathcal{H} = L^2(\Gamma, L^2(X, \mu)) \) be the Hilbert space of \( L^2(X, \mu) \)-valued functions on Γ. That is, each element of \( \mathcal{H} \) can be written in the form of

\[
\sum_{\gamma \in \Gamma} b_\gamma \cdot \gamma,
\]
where \( \sum_{\gamma \in \Gamma} \| b_\gamma \|^2 < \infty \). Then we have a representation \( L \) of \( L^\infty(X,\mu) \times \Gamma \) on \( l^2(\Gamma, L^2(X,\mu)) \) by

\[
L\left( \sum_{\gamma \in \Gamma} a_\gamma \cdot \gamma \right) = \sum_{\delta \in \Gamma} \left( \sum_{\gamma \in \Gamma} a_\gamma (\hat{\tau}(\gamma)(\beta_\delta)) \cdot \gamma \delta \right).
\]

Note that \( L\left( \sum_{\gamma \in \Gamma} a_\gamma \cdot \gamma \right) \) is always a bounded operator. A trace is given on \( L^\infty(X,\mu) \times \Gamma \) by

\[
\text{Tr}(S) = \int_X a_1(x) d\mu(x).
\]

The weak operator closure of \( L\left( L^\infty_c(X,\mu) \right) \times \Gamma \) in \( B \left( l^2(\Gamma, L^2(X,\mu)) \right) \) is the von Neumann algebra \( \mathcal{N}(\tau) \) associated to the action. Here \( L^\infty_c(X,\mu) \) denotes the subspace of functions in \( L^\infty(X,\mu) \) having only countable many values.

3. The Bernoulli algebra

Let \( H \) be a countable group. Consider the Bernoulli shift space \( B_H := \prod_{h \in H} \{0,1\} \) with the usual product measure \( \nu_H \). The probability measure preserving action \( \tau_H : H \curvearrowright (B_H, \nu_H) \) is defined by

\[
\tau_H(\delta)(x)(h) = x(\delta^{-1} h),
\]

where \( x \in B_H, \delta, h \in H \). Let \( A_H \) be the commutative \(*\)-algebra of functions that depend only on finitely many coordinates of the shift space. It is well known that the Rademacher functions \( \{ R_S \}_{S \subset H, |S| < \infty} \) form a basis in \( A_H \), where

\[
R_S(x) = \prod_{\delta \in S} \exp(i\pi x(\delta)).
\]

The Rademacher functions with respect to the pointwise multiplication form an abelian group isomorphic to \( \bigoplus_{h \in H} \mathbb{Z}_2 \) the Pontrjagin dual of the compact group \( B_H \) satisfying

- \( R_S R_{S'} = R_{S \Delta S'} \),
- \( \int_{B_H} R_S d\nu = 0 \), if \( |S| > 0 \),
- \( R_\emptyset = 1 \).

The group \( H \) acts on \( A_H \) by

\[
\hat{\tau}_H(\delta)(f)(x) = f\left( \tau_H(\delta^{-1})(x) \right).
\]

Hence,

\[
\hat{\tau}_H(\delta) R_S = R_{\delta S}.
\]
Therefore, the elements of \( A_H \rtimes H \) can be uniquely written as in the form of the finite sums

\[
\sum_\delta \sum_S c_{\delta,S}R_S \cdot \delta,
\]

where \( \delta \cdot R_S = R_{\delta S} \cdot \delta \).

Now let us turn our attention to the group algebra \( \mathbb{C}(\mathbb{Z}_2 \wr H) \). For \( \delta \in H \), let \( \delta \) be the generator in \( \sum_{h \in H} \mathbb{Z}_2 \) belonging to the \( \delta \)-component. Any element of \( \mathbb{C}(\mathbb{Z}_2 \wr H) \) can be written in a unique way as a finite sum

\[
\sum_\delta \sum_S c_{\delta,S}t_S \cdot \delta,
\]

where \( t_S = \prod_{s \in S} t_s \), \( \delta \cdot t_S = t_{\delta S} \), \( t_S t_{S'} = t_{S \Delta S'} \). Also note that

\[
\text{Tr}(\sum_\delta \sum_S c_{\delta,S}t_S \cdot \delta) = c_{1,\emptyset}.
\]

Hence we have the following proposition.

**Proposition 3.1.** There exists a trace preserving \(*\)-isomorphism \( \kappa : \mathbb{C}(\mathbb{Z}_2 \wr H) \to A_H \rtimes H \) such that

\[
\kappa(\sum_\delta \sum_S c_{\delta,S}t_S \cdot \delta) = \sum_\delta \sum_S c_{\delta,S}R_S \cdot \delta.
\]

Recall that if \( A \subset N \), \( B \subset N \) are weakly dense \(*\)-subalgebras in finite tracial von Neumann algebras \( N \), and \( \kappa : A \to B \) is a trace preserving \(*\)-homomorphism, then \( \kappa \) extends to a trace preserving isomorphism between the von Neumann algebras themselves (see e.g. [7, Corollary 7.1.9]). Therefore, \( \kappa : \mathbb{C}(\mathbb{Z}_2 \wr H) \to A_H \rtimes H \) extends to a trace (and hence rank) preserving isomorphism between the von Neumann algebras \( N(\mathbb{Z}_2 \wr H) \) and \( N(\tau_H) \).

**Proposition 3.2.** For any countable group \( H \),

\[
c(\mathbb{Z}_2 \wr H) \cong c(\tau_H).
\]

**Proof.** The rank preserving isomorphism \( \kappa : N(\mathbb{Z}_2 \wr H) \to N(\tau_H) \) extends to a rank preserving isomorphism between the rank completions, that is, the algebras of affiliated operators. It is enough to prove that the rank closure of \( A_H \rtimes H \) is \( L^\infty_c(B_H, \nu_H) \rtimes H \).

**Lemma 3.1.** Let \( f \in L^\infty_c(B_H, \nu_H) \). Then \( \text{rk}_{N(\tau_H)}(f) = \nu_H(\text{supp}(f)) \).

**Proof.** By definition,

\[
\text{rk}_{N(\tau_H)}(f) = 1 - \lim_{\lambda \to 0} \text{tr}_{N(\tau_H)} E_\lambda,
\]

where \( E_\lambda \) is the spectral projection of \( f^*f \) corresponding to \( \lambda \) and

\[
\text{tr}_{N(\tau_H)} E_\lambda = \nu_H(\{x \mid |f^2(x)| \leq \lambda\}).
\]

Hence,

\[
\text{rk}_{N(\tau_H)}(f) = 1 - \nu_H(\{x \mid f^2(x) = 0\}) = \nu_H(\text{supp}(f)).
\]

**Lemma 3.2.** \( A_H \) is dense in \( L^\infty_c(B_H, \nu_H) \) with respect to the rank metric.
Proof. By Lemma 3.1 \(L^\infty_{\text{fin}}(B_H, \nu_H)\) is dense in \(L^\infty_{\nu}(B_H, \nu_H)\), where \(L^\infty_{\text{fin}}(B_H, \nu_H)\) is the \(*\)-algebra of functions taking only finitely many values. Recall that \(V \subset B_H\) is a basic set if \(1_V \in \mathcal{A}_H\). It is well known that any measurable set in \(B_H\) can be approximated by basic sets, that is, for any \(U \subset B_H\), there exists a sequence of basic sets \(\{V_n\}_{n=1}^\infty\) such that

\[
\lim_{n \to \infty} \nu_H(V_n \triangle U) = 0.
\]

By (1) and Lemma 3.1

\[
\lim_{n \to \infty} \text{rk}_N(\tau_n)(1_{V_n} - 1_U) = 0.
\]

Let \(f = \sum_{m=1}^l c_m 1_{U_m}\), where \(U_m\) are disjoint measurable sets. Let

\[
\lim_{n \to \infty} \nu_H(V_n^m \triangle U_m) = 0,
\]

where \(\{V_n^m\}_{n=1}^\infty\) are basic sets. Then

\[
\lim_{n \to \infty} \text{rk}_N(\tau_n)(\sum_{m=1}^l c_m 1_{V_n^m} - f) = 0.
\]

Therefore, \(\mathcal{A}_H\) is dense in \(L^\infty_{\text{fin}}(B_H, \nu_H)\). \(\square\)

4. The odometer algebra

The odometer algebra is constructed via the odometer action using the algebraic crossed product construction. Let us consider the compact group of 2-adic integers \(\hat{\mathbb{Z}}(2)\). Recall that \(\hat{\mathbb{Z}}(2)\) is the completion of the integers with respect to the dyadic metric

\[
d_{(2)}(n,m) = 2^{-k},
\]

where \(k\) is the power of two in the prime factor decomposition of \(|m - n|\). The group \(\hat{\mathbb{Z}}(2)\) can be identified with the compact group of one way infinite sequences with respect to the binary addition.

The Haar measure \(\mu_{\text{haar}}\) on \(\hat{\mathbb{Z}}(2)\) is defined by \(\mu_{\text{haar}}(U_n^l) = 1/2^n\), where \(0 \leq l \leq 2^n - 1\) and \(U_n^l\) is the clopen subset of elements in \(\hat{\mathbb{Z}}(2)\) having residue \(l\) modulo \(2^n\). Let \(T\) be the addition map \(x \to x + 1\) in \(\hat{\mathbb{Z}}(2)\). The map \(T\) defines an action \(\rho: \mathbb{Z} \curvearrowleft (\hat{\mathbb{Z}}(2), \mu_{\text{haar}})\). The dynamical system \((T, \hat{\mathbb{Z}}(2), \mu_{\text{haar}})\) is called the odometer action.

As in Section 3, we consider the \(*\)-subalgebra of function \(\mathcal{A}_M\) in \(L^\infty(\hat{\mathbb{Z}}(2), \mu_{\text{haar}})\) that depends only on finitely many coordinates of \(\hat{\mathbb{Z}}(2)\). We consider a basis for \(\mathcal{A}_M\). For \(n \geq 0\) and \(0 \leq l \leq 2^n - 1\) let

\[
F_n^l(x) = \exp\left(\frac{2\pi i x (\text{mod} 2^n)}{2^n} l\right).
\]

Notice that \(F_{n+1}^{2l} = F_n^l\). Then the functions \(\{F_n^l\}_{n,l|(l,n)=1}\) form the Prüfer 2-group \(\mathbb{Z}_{(2)} = \mathbb{Z}_4 \subset \mathbb{Z}_8 \subset \mathbb{Z}_{16} \subset \mathbb{Z}_{32} \subset \ldots\)

with respect to the pointwise multiplication. The discrete group \(\mathbb{Z}_{(2)}\) is the Pontrjagin dual of the compact abelian group \(\hat{\mathbb{Z}}(2)\). The element \(F_n^1\) is the generator of the cyclic subgroup \(\mathbb{Z}_{2^n}\). Note that

\[
\int_{\hat{\mathbb{Z}}(2)} F_n^l d\mu_{\text{haar}} = 0
\]
except if \( l = 0, n = 0 \), when \( F^l_n = 1 \). Observe that if \( k \in \mathbb{Z} \), then
\[
\rho(k)F^l_n = F^{l+k(mod\,2^n)}_n
\]

since \( F^l_n(x-k) = F^{l+k(mod\,2^n)}_n(x) \). Hence we have the following lemma.

**Lemma 4.1.** The elements of \( A_M \times \mathbb{Z} \) can be uniquely written as finite sums in the form
\[
\sum_{k} \sum_{n \geq 0 \mid l(n) = 1} c_{n,i,k} F^l_n \cdot k,
\]
where \( k \cdot F^l_n = F^{l+k(mod\,2^n)}_n \) and \( F^0_0 = 1 \).

**5. Periodic Operators**

**Definition 5.1.** A function \( \mathbb{Z} \times \mathbb{Z} \to \mathbb{C} \) is a periodic operator if there exists some \( n \geq 1 \) such that
- \( A(x, y) = 0 \), if \( |x - y| > 2^n \),
- \( A(x, y) = A(x + 2^n, y + 2^n) \).

Observe that the periodic operators form a \( * \)-algebra, where
- \((A + B)(x, y) = A(x, y) + B(x, y)\),
- \(AB(x, y) = \sum_{z \in \mathbb{Z}} A(x, z)B(z, y)\),
- \(A^*(x, y) = A(y, x)\).

**Proposition 5.1.** The algebra of periodic operators \( \mathcal{P} \) is \( * \)-isomorphic to a dense subalgebra of \( M \).

**Proof.** We call \( A \in \mathcal{P} \) an element of type-\( n \) if
- \( A(x, y) = A(x + 2^n, y + 2^n) \),
- \( A(x, y) = 0 \) if \( 0 \leq x \leq 2^n - 1, y > 2^n - 1 \),
- \( A(x, y) = 0 \) if \( 0 \leq x \leq 2^n - 1, y < 0 \).

Clearly, the elements of type-\( n \) form an algebra \( \mathcal{P}_n \) isomorphic to \( \text{Mat}_{2^n \times 2^n}(\mathbb{C}) \) and \( \mathcal{P}_n \to \mathcal{P}_{n+1} \) is the diagonal embedding. Hence, we can identify the algebra of finite type elements \( \mathcal{P}_f = \bigcup_{n=1}^\infty \mathcal{P}_n \) with \( \lim_{n \to \infty} \text{Mat}_{2^n \times 2^n}(\mathbb{C}) \).

For \( A \in \mathcal{P} \), if \( n \geq 1 \) is large enough, let \( A_n \in \mathcal{P}_n \) be defined the following way:
- \( A_n(x, y) = A(x, y) \) if \( 2^nl \leq x, y \leq 2^nl + 2^n - 1 \) for some \( l \in \mathbb{Z} \).
- Otherwise, \( A(x, y) = 0 \).

**Lemma 5.1.**

**(i):** \( \{A_n\}_{n=1}^\infty \) is a Cauchy-sequence in \( M \).

**(ii):** \((A + B)_n = A_n + B_n\).

**(iii):** \( \text{rk}_M(A_n^* - (A^n)_n) = 0 \).

**Lemma 5.1.**

**(iv):** \( \text{rk}_M((AB)_n - (A_n B_n)) = 0 \).

**(v):** \( \lim_{n \to \infty} A_n = 0 \) if and only if \( A = 0 \).

**Proof.** First observe that for any \( Q \in \mathcal{P}_n \)
\[
\text{rk}_M(Q) \leq \frac{|\{0 \leq x \leq 2^n - 1 \mid \exists 0 \leq y \leq 2^n - 1 \text{ such that } A_n(x, y) \neq 0\}|}{2^n}.
\]

Suppose that \( A(x, y) = A(x + 2^k, y + 2^k) \) and \( k < n < m \). Then
\[
|\{0 \leq x \leq 2^n - 1 \mid A_n(x, y) \neq A_m(x, y) \text{ for some } 0 \leq y \leq 2^n - 1\}| \leq 2^k2^{m-n}.
\]

Hence by the previous observation, \( \{A_n\}_{n=1}^\infty \) is a Cauchy sequence. Note that (iii) and (iv) can be proved similarly; the proof of (ii) is straightforward. In order to
prove (v) let us suppose that $A(x, y) = 0$ whenever $|x - y| \geq 2^k$. Let $n > k$ and $0 \leq y \leq 2^k - 1$ such that $A(x, y) \neq 0$ for some $-2^k \leq x \leq 2^k - 1$. Therefore $\text{rk}_M A_n \geq \frac{2^{n-k} - 1}{2^n}$. Thus (v) follows. □

Let us define $\phi : \mathcal{P} \rightarrow \mathcal{M}$ by $\phi(A) = \lim_{n \rightarrow \infty} A_n$. By the previous lemma, $\phi$ is an injective $*$-homomorphism. □

**Definition 5.2.** A periodic operator $A$ is diagonal if $A(x, y) = 0$, whenever $x \neq y$. The diagonal operators form the abelian $*$-algebra $\mathcal{D} \subset \mathcal{P}$.

**Lemma 5.2.** We have the isomorphism $\mathcal{D} \cong \mathbb{C}(\mathbb{Z}(2))$, where $\mathbb{Z}(2)$ is the Prüfer 2-group.

**Proof.** For $n \geq 1$ and $0 \leq l \leq 2^n - 1$ let $E_n^l \in \mathcal{D}$ be defined by

$$E_n^l(x, x) := \exp\left(\frac{2\pi i (x \mod 2^n) - 2^n l}{2^n}\right).$$

It is easy to see that $E_{n+1}^{2l} = E_n^l$ and the multiplicative group generated by $E_n^1$ is isomorphic to $\mathbb{Z}_{2^n}$. Observe that the set $\{E_n^l\}_{n, l, (l, n) = 1}$ forms a basis in the space of $n$-type diagonal operators. Therefore, $\mathcal{D} \cong \bigcup_{n=1}^{\infty} \mathbb{C}(\mathbb{Z}_{2^n}) = \mathbb{C}(\mathbb{Z}(2))$. □

Let $J \in \mathcal{P}$ be the following element:

- $J(x, y) = 1$, if $y = x + 1$.
- Otherwise, $J(x, y) = 0$.

Then

$$J \cdot E_n^l = E_n^{l+1 \mod 2^n}.$$ (3)

Also, any periodic operator $A$ can be written in a unique way as a finite sum

$$\sum_{k \in \mathbb{Z}} D_k \cdot J^k,$$

where $D_k$ is a diagonal operator in the form

$$D_k = \sum_{n=0}^{\infty} \sum_{l | (l, n) = 1} c_{l, n, k} E_n^l.$$

Thus, by (2) and (3), we have the following corollary.

**Corollary 5.1.** The map $\psi : \mathcal{P} \rightarrow A_M \rtimes \mathbb{Z}$ defined by

$$\psi(\sum_{k} \sum_{n \geq 0} \sum_{l | (l, n) = 1} c_{l, n, k} E_n^l \cdot k) = \sum_{k} \sum_{n \geq 0} \sum_{l | (l, n) = 1} c_{l, n, k} F_n^l \cdot k$$

is a $*$-isomorphism of algebras.

6. **Lück’s Approximation Theorem revisited**

The goal of this section is to prove the following proposition.

**Proposition 6.1.** We have $c(\rho) \cong \mathcal{M}$ where $\rho$ is the odometer action.

**Proof.** Let us define the linear map $t : \mathcal{P} \rightarrow \mathbb{C}$ by

$$t(A) := \frac{\sum_{i=0}^{2^n-1} A(i, i)}{2^n},$$

where $A \in \mathcal{P}$ and $A(x + 2^n, y + 2^n)$ for all $x, y \in \mathbb{Z}$. □
Lemma 6.1. \( \text{Tr}_{N(\rho)}(\psi(A)) = t(A), \) where \( \psi \) is the \(*\)-isomorphism of Corollary 5.1.

Proof. Recall that \( \text{Tr}_{N(\rho)}(E_n^l) = 0 \), except, when \( l = 0, n = 0, E_n^l = 1 \). If \( n \neq 0 \) and \( l \neq 0 \), then \( t(E_n^l) \) is the sum of all \( k \)-th roots of unity for a certain \( k \), hence \( t(E_n^l) = 0 \). Also, \( t(1) = 1 \). Thus, the lemma follows. \( \square \)

It is enough to prove that

\[
(4) \quad \text{rk}_A(A) = \text{rk}_{N(\rho)}(\psi(A)).
\]

Indeed by \( (4) \), \( \psi \) is a rank-preserving \(*\)-isomorphism between \( \mathcal{P} \) and \( A_M \rtimes \mathbb{Z} \). Hence the isomorphism \( \psi \) extends to a metric isomorphism

\[
\hat{\psi} : \overline{\mathcal{P}} \to \overline{A_M} \rtimes \mathbb{Z},
\]

where \( \overline{\mathcal{P}} \) is the closure of \( \mathcal{P} \) in \( \mathcal{M} \) and \( \overline{A_M} \rtimes \mathbb{Z} \) is the closure of \( A_M \rtimes \mathbb{Z} \) in \( U(N(\rho)) \). Since \( \mathcal{P} \) is dense in \( \mathcal{M} \), \( \overline{\mathcal{P}} \cong \mathcal{M} \). Also, \( \overline{A_M} \rtimes \mathbb{Z} \) is a \(*\)-subalgebra of \( U(N(\rho)) \), since the \(*\)-ring operations are continuous with respect to the rank metric. Therefore \( \overline{A_M} \rtimes \mathbb{Z} \) is a continuous algebra isomorphic to \( \mathcal{M} \). Observe that the rank closure \( \overline{A_M} \rtimes \mathbb{Z} \) is isomorphic to the rank closure of \( L_\infty(\hat{\mathbb{Z}}(2), \mu_{\text{haar}}) \rtimes \mathbb{Z} \) by the argument of Lemma 3.2. Therefore, \( c(\rho) \cong \mathcal{M} \). Thus from now on, our only goal is to prove \( (4) \).

Lemma 6.2. Let \( A \in \mathcal{P} \) and \( A_n \in \text{Mat}_{2^n \times 2^n}(\mathbb{C}) \) as in Section 5. Then the matrices \( \{A_n\}_{n=1}^\infty \) have uniformly bounded norms.

Proof. Let \( M, N \) be chosen in such a way that

- \( |A_n(x, y)| \leq M \) for any \( x, y \in \mathbb{Z}, n \geq 1 \).
- \( |A_n(x, y)| = 0 \) if \( |x - y| \geq \frac{N}{2} \).

Now let \( v = (v(1), v(2), \ldots, v(2^n)) \in \mathbb{C}^{2^n}, \|v\|^2 = 1 \). Then

\[
\|A_nv\|^2 = \sum_{x=1}^{2^n} \sum_{y \mid |x-y| < N/2} |A_n(x, y)v(y)|^2 \leq M^2 \sum_{x=1}^{2^n} \sum_{y \mid |x-y| < N/2} |v(y)|^2 \leq M^2 N \sum_{x=1}^{2^n} \sum_{y \mid |x-y| < N/2} |v(y)|^2 = M^2 N^2.
\]

Therefore, for any \( n \geq 1, \|A_n\| \leq MN. \) \( \square \)

Lemma 6.3. Let \( A \in \mathcal{P} \). Then for any \( k \geq 1 \)

\[
\lim_{k \to \infty} t((A_n^*A_n)^k) = t((A^*A)^k) = \text{Tr}_{N(\rho)}(\psi(A^*A)^k).
\]

Proof. Let \( m \geq 1, l \geq 1, q \geq 1 \) be integers such that

- \( A(x, y) = A(x + 2^m, y + 2^m) \) for any \( x, y \in \mathbb{Z} \).
- \( A(x, y) = 0 \), if \( |x - y| \geq l \).
- \( |(A^*A)^k(x, x)| \leq q \) and \( |(A_n^*A_n)^k(x, x)| \leq q \) for any \( x \in \mathbb{Z} \).

By definition,

\[
t((A_n^*A_n)^k) = \frac{\sum_{x=1}^{2^n} (A_n^*A_n)^k(x, x)}{2^n}
\]

\[
t((A^*A)^k) = \frac{\sum_{x=1}^{2^n} (A_n^*A_n)^k(x, x)}{2^n}.
\]
Observe that if $2lk < x, 2^n - 2lk$, then

$$(A^*A)^k(x,x) = (A^*_nA_n)(x,x).$$

Hence,

$$|t((A^*A)^k) - t((A^*_nA_n)^k)| \leq \frac{4klq}{2^n}.$$ 

Thus our lemma follows.

Now, we follow the idea of Lück [10]. Let $\mu$ be the spectral measure of $\psi(A) \in N(\rho)$. That is

$$\text{Tr}_{N(\rho)} f(A^*A) = \int_{0}^{K} f(x) \, d\mu(x),$$

for all $f \in C[0, K]$, where $K > 0$ is chosen in such a way that $\text{Spec} \psi(A^*A) \subset [0, K]$ and $\|A^*_nA_n\| \leq K$ for all $n \geq 1$. Also, let $\mu_n$ be the spectral measure of $A^*_nA_n$, that is,

$$t(f(A^*_nA_n)) = \int_{0}^{K} f(x) \, d\mu_n(x),$$

or all $f \in C[0, K]$. As in [10], we can see that the measures $\{\mu_n\}_{n=1}^\infty$ converge weakly to $\mu$. Indeed by Lemma 6.3,

$$\lim_{n \to \infty} t(P(A^*_nA_n)) = \text{Tr}_{N(\rho)} P(A^*A)$$

for any real polynomial $P$. Therefore,

$$\lim_{n \to \infty} t(f(A^*_nA_n)) = \text{Tr}_{N(\rho)} f(A^*A)$$

for all $f \in C[0, K]$.

Since $\text{rk}_{M}(A_n) = \text{rk}_{M}(A^*_nA_n)$ and $\text{rk}_{N(\rho)}(\psi(A)) = \text{rk}_{N(\rho)}(\psi(A^*A))$, in order to prove (4) it is enough to see that

$$\lim_{n \to \infty} \text{rk}_{M}(A^*_nA_n) = \text{rk}_{N(\rho)}(\psi(A^*A)).$$

Observe that $\text{rk}_{M}(A^*_nA_n) = 1 - \mu_n(0)$ and

$$\text{rk}_{N(\rho)}(\psi(A^*A)) = 1 - \lim_{\lambda \to 0} \text{Tr}_{N(\rho)} E_{\lambda} = \mu(0).$$

Hence, our proposition follows from the lemma below (an analogue of Lück’s Approximation Theorem).

**Lemma 6.4.** $\lim_{n \to \infty} \mu_n(0) = \mu(0)$.

**Proof.** Let $F_n(\lambda) = \int_{0}^{\lambda} \mu_n(t) \, dt$ and $F(\lambda) = \int_{0}^{\lambda} \mu(t) \, dt$ be the distribution functions of our spectral measures. Since $\{\mu_n\}_{n=1}^\infty$ weakly converges to the measure $\mu$, it is enough to show that $\{F_n\}_{n=1}^\infty$ converges uniformly. Let $n \leq m$ and $D_m^n : \text{Mat}_{2^n \times 2^n}(\mathbb{C}) \to \text{Mat}_{2^m \times 2^m}(\mathbb{C})$ be the diagonal operator. Let $\varepsilon > 0$. By Lemma 5.1, if $n, m$ are large enough,

$$\text{Rank}(D_m^n(A_n) - A_m) \leq \varepsilon 2^m.$$ 

Hence, by Lemma 3.5 in [2],

$$\|F_n - F_m\|_{\infty} \leq \varepsilon.$$ 

Therefore, $\{F_n\}_{n=1}^\infty$ converges uniformly. \qed
7. Orbit equivalence

First let us recall the notion of orbit equivalence. Let \( \tau_1 : \Gamma_1 \actson (X, \mu) \) resp. \( \tau_2 : \Gamma_2 \actson (Y, \nu) \) be essentially free probability measure preserving actions of the countably infinite groups \( \Gamma_1 \) resp. \( \Gamma_2 \). The two actions are called orbit equivalent if there exists a measure preserving bijection \( \Psi : (X, \mu) \to (Y, \nu) \) such that for almost all \( x \in X \) and \( \gamma \in \Gamma_1 \) there exists \( \gamma_x \in \Gamma_2 \) such that
\[
\tau_2(\gamma_x)(\Psi(x)) = \Psi(\tau_1(\gamma)(x)).
\]
Feldman and Moore \cite{FeldmanMoore} proved that if \( \tau_1 \) and \( \tau_2 \) are orbit equivalent, then \( \mathcal{N}(\tau_1) \cong \mathcal{N}(\tau_2) \). The goal of this section is to prove the following proposition.

**Proposition 7.1.** If \( \tau_1 \) and \( \tau_2 \) are orbit equivalent actions, then \( c(\tau_1) \cong c(\tau_2) \).

Our Theorem \ref{thm:main} follows from the proposition. Indeed, by Proposition \ref{prop:OrbitEquivalence} and Proposition \ref{prop:OrbitEquivalence2},
\[
\mathcal{M} \cong c(\rho) \quad \text{and} \quad c(\mathbb{Z}_2 \rtimes H) \cong c(\tau_H).
\]
By the famous theorem of Ornstein and Weiss \cite{OrnsteinWeiss}, the odometer action and the Bernoulli shift action of a countably infinite amenable group are orbit equivalent. Hence \( \mathcal{M} \cong c(\mathbb{Z}_2 \rtimes H) \).

**Proof.** We build the proof of our proposition on the original proof of Feldman and Moore. Let \( \gamma \in \Gamma_1, \delta \in \Gamma_2 \). Let
\[
M(\delta, \gamma) = \{ y \in Y \mid \tau_2(\delta)(y) = \Psi(\tau_1(\gamma)\Psi^{-1}(y)) \},
\]
\[
N(\gamma, \delta) = \{ x \in X \mid \tau_1(\gamma)(x) = \Psi^{-1}(\tau_2(\delta)\Psi(x)) \}.
\]
Observe that \( \Psi(N(\delta, \gamma)) = M(\gamma, \delta) \). Following Feldman and Moore (\cite{FeldmanMoore} Proposition 2.1) for any \( \gamma \in \Gamma_1, \delta \in \Gamma_2 \)
\[
\kappa(\gamma) = \sum_{h \in \Gamma_2} h \cdot 1_M(h, \gamma)
\]
and
\[
\lambda(\delta) = \sum_{g \in \Gamma_1} g \cdot 1_N(g, \delta)
\]
are well defined. That is, \( \sum_{n=1}^{k} h_n \cdot 1_M(h_n, \gamma) \) converges weakly to \( \kappa(\gamma) \in \mathcal{N}(\tau_2) \) as \( k \to \infty \) and \( \sum_{n=1}^{k} g_n \cdot 1_N(g_n, \delta) \) converges weakly to \( \lambda(\delta) \in \mathcal{N}(\tau_1) \) as \( k \to \infty \), where \( \{\gamma_n\}_{n=1}^{\infty} \) resp. \( \{\delta_n\}_{n=1}^{\infty} \) are enumerations of the elements of \( \Gamma_1 \) resp. \( \Gamma_2 \).

Furthermore, one can extend \( \kappa \) resp. \( \lambda \) to maps
\[
\kappa' : L^\infty((X, \mu) \rtimes \Gamma_1) \to \mathcal{N}(\tau_2)
\]
resp.
\[
\lambda' : L^\infty((Y, \nu) \rtimes \Gamma_2) \to \mathcal{N}(\tau_1)
\]
by
\[
\kappa'(\sum_{\gamma \in \Gamma_1} a_\gamma \cdot \gamma) = \sum_{\gamma \in \Gamma_1} (a_\gamma \circ \Psi^{-1}) \cdot \kappa(\gamma) = \sum_{\gamma \in \Gamma_1} (a_\gamma \circ \Psi^{-1}) \cdot \sum_{n=1}^{\infty} h_n \cdot 1_M(h_n, \gamma)
\]
and
\[
\lambda'(\sum_{\delta \in \Gamma_2} b_\delta \cdot \delta) = \sum_{\delta \in \Gamma_2} (b_\delta \circ \Psi) \cdot \lambda(\delta) = \sum_{\delta \in \Gamma_2} (b_\delta \circ \Psi) \cdot \sum_{n=1}^{\infty} g_n \cdot 1_N(g_n, \delta).
\]
The maps \( \kappa' \) resp. \( \lambda' \) are injective trace-preserving \(*\)-homomorphisms with weakly
dense ranges. Hence they extend to isomorphisms of von Neumann algebras

\[ \hat{k} : \mathcal{N}(\tau_1) \to \mathcal{N}(\tau_2), \hat{\lambda} : \mathcal{N}(\tau_2) \to \mathcal{N}(\tau_1), \]

where \( \hat{k} \) and \( \hat{\lambda} \) are, in fact, the inverses of each other.

**Lemma 7.1.**

\[
\begin{align*}
(5) \quad & \lim_{k \to \infty} \text{rk}_{\mathcal{N}(\tau_2)} \left( \sum_{\gamma \in \Gamma_1} (a_{\gamma} \circ \Psi^{-1}) \cdot \sum_{n=1}^{k} h_n \cdot 1_{M(h_n, \gamma)} - \hat{k} \left( \sum_{\gamma \in \Gamma_1} a_{\gamma} \cdot \gamma \right) \right) = 0, \\
(6) \quad & \lim_{k \to \infty} \text{rk}_{\mathcal{N}(\tau_1)} \left( \sum_{\delta \in \Gamma_2} (b_{\delta} \circ \Psi) \cdot \sum_{n=1}^{k} g_n \cdot 1_{N(g_n, \delta)} - \hat{\lambda} \left( \sum_{\delta \in \Gamma_2} b_{\delta} \cdot \delta \right) \right) = 0.
\end{align*}
\]

**Proof.** By definition, the disjoint union \( \bigcup_{n=1}^{\infty} M(h_n, \gamma) \) equals \( Y \) (modulo a set of
measure zero). We need to show that if \( \{ \sum_{n=1}^{k} T_n \cdot 1_{M(h_n, \gamma)} \}_{k=1}^{\infty} \) weakly converges to
an element \( S \in \mathcal{N}(\tau_2) \), then \( \{ \sum_{n=1}^{k} T_n \cdot 1_{M(h_n, \gamma)} \}_{k=1}^{\infty} \) converges to \( S \) in the rank
metric as well, where \( T_n \in L_\infty^c(Y, \nu) \times \Gamma_2 \). Let \( P_k = \sum_{n=1}^{k} 1_{M(h_n, \gamma)} \in L_2^c(\Gamma, L_2^2(Y, \nu)). \)
We denote by \( \hat{P}_k \) the element \( \sum_{n=1}^{k} 1_{M(h_n, \gamma)} \) in \( L_\infty^c(Y, \nu) \times \Gamma_2 \). By definition, if
\( L(A)(P_k) = 0 \), then \( A\hat{P}_k = 0 \). Now, by weak convergence,

\[ L(S)(P_k) = \lim_{l \to \infty} \sum_{n=1}^{l} T_n \cdot 1_{M(h_n, \gamma)}(P_k). \]

That is,

\[ L(S - \sum_{n=1}^{k} T_n \cdot 1_{M(h_n, \gamma)})(P_k) = 0. \]

Therefore,

\[ (S - \sum_{n=1}^{k} T_n \cdot 1_{M(h_n, \gamma)}) \hat{P}_k = 0. \]

Thus,

\[ (S - \sum_{n=1}^{k} T_n \cdot 1_{M(h_n, \gamma)}) = (S - \sum_{n=1}^{k} T_n \cdot 1_{M(h_n, \gamma)})(1 - \hat{P}_k). \]

By Lemma 3.1 \( \text{rk}_{\mathcal{N}(\tau_2)}(1 - \hat{P}_k) = 1 - \sum_{n=1}^{k} \nu(M(h_n, \gamma)) \), hence

\[ \lim_{k \to \infty} \text{rk}_{\mathcal{N}(\tau_2)}(S - \sum_{n=1}^{k} T_n \cdot 1_{M(h_n, \gamma)}) = 0. \]

\[ \square \]

Now let us turn back to the proof of our proposition. By [57], \( \hat{k} \) maps the algebra
\( L_\infty^c(X, \mu) \times \Gamma_1 \) into the rank closure of \( L_\infty^c(Y, \nu) \times \Gamma_2 \). Since \( \hat{k} \) preserves the rank,
\( \hat{k} \) maps the rank closure of \( L_\infty^c(X, \mu) \times \Gamma_1 \) into the rank closure of \( L_\infty^c(Y, \nu) \times \Gamma_2 \).
Similarly, \( \hat{\lambda} \) maps the rank closure of \( L_\infty^c(Y, \nu) \times \Gamma_2 \) into the rank closure of
\( L_\infty^c(X, \mu) \times \Gamma_1 \). That is, \( \hat{k} \) provides an isomorphism between the rank closures
of \( L_\infty^c(X, \mu) \times \Gamma_1 \) and \( L_\infty^c(Y, \nu) \times \Gamma_2 \). Therefore, the smallest continuous ring
containing \( L_\infty^c(X, \mu) \times \Gamma_1 \) in \( U(\mathcal{N}(\tau_1)) \) is mapped to the smallest continuous ring
containing \( L_\infty^c(Y, \nu) \times \Gamma_2 \) in \( U(\mathcal{N}(\tau_2)) \).

\[ \square \]
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