Analysis of the compensation equation for rational aggregation functions

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Abstract. The compensation equation is the law of conservation of a certain quantity with respect to a given class of admissible transformations. This paper discusses approaches to the construction of a compensating function for rational aggregation functions, including those with the associativity property.

1. Introduction

The main goal of constructing mathematical models of the operation of various systems is to create an adaptive control algorithm based on monitoring the behavior of the system and assessing its efficiency at various stages of control.

Let the system operation be described by some complex criterion \( f : X \times Y \rightarrow R \) depending on two different sets of variables \( x \) and \( y \), and \( x \in X \subseteq R^n \), \( y \in Y \subseteq R^m \). Let \( \tilde{A} \) be the set of operators defining admissible transformations over vectors \( x \in X \), and \( \tilde{B} \) be the set of operators defining admissible transformations over vectors \( y \in Y \), so that for any \( A \in \tilde{A} \), we have \( A(X) \subseteq X \), for any \( B \in \tilde{B} \), we have \( B(Y) \subseteq Y \). We shall also assume that we are given some operator function \( \psi \) that assigns to each operator \( A \in \tilde{A} \) some operator \( B \in \tilde{B} \) such that the following equality holds:

\[
    f(Ax, By) = f(x, y),
\]

where \( B = \psi(x, A) \).

Equality (1) is called the compensation equation and in general form it determines the law of conservation of a certain quantity with respect to a given class of admissible transformations (in this case, the law of conservation of the quantity given by the criterion \( f \)) [1]. Formally, equality (1) is a functional equation, the function \( \psi \) will be called the compensating function.

For compensation equation (1), the following problems can be considered:
a) for the given operators \( A \in \bar{A} \) and \( B \in \bar{B} \), and the chosen representation of the compensating function \( \psi \), determine the form of the functional \( f \); 

b) for the given operators \( A \in \bar{A} \) and \( B \in \bar{B} \), and for the known \( f \), define the compensating function \( \psi \).

The absence of a general theory of equations of the type (1) does not allow solving the listed problems in general form.

Let \( A \) and \( B \) be the shift operators, i.e.,

\[
A: (x_1, \ldots, x_n) \rightarrow (x_1 + \Delta x_1, \ldots, x_n + \Delta x_n),
\]

then for the corresponding operator \( B = \psi(x, A) \) we have

\[
B: (y_1, \ldots, y_m) \rightarrow \left( y_1 + \psi_1(x, \Delta x, y), \ldots, y_m + \psi_m(x, \Delta x, y) \right),
\]

where \( \Delta x = (\Delta x_1, \ldots, \Delta x_n) \).

It is natural to assume that if the variables \( x_1, \ldots, x_n \) do not change, then the variables \( y_1, \ldots, y_m \) also do not change, and, therefore, \( \forall k = 1, m; \psi_k(x, 0, y) = 0 \).

Considering the new notation, equation (1) can be rewritten as

\[
f(x_1 + \Delta x_1, \ldots, x_n + \Delta x_n, y_1 + \psi_1(x, y), \ldots, y_m + \psi_m(x, y)) = f(x_1, \ldots, x_n, y_1, \ldots, y_m).
\]  

(2)

It holds for the function \( f \), which depends on two groups of variables. The first group \( x_1, \ldots, x_n \) acts as a set of independent and arbitrarily changing arguments. In order for the function \( f \) to remain unchanged, a compensating change must be formed in the other group \( y_1, \ldots, y_m \), and then the transition \( y \rightarrow y' = (y_1 + \psi_1, \ldots, y_m + \psi_m) \) "neutralizes" the transition \( x \rightarrow x' = (x_1 + \Delta x_1, \ldots, x_n + \Delta x_n) \).

In [1], an approach based on the formation of a system of differential equations defining the functional \( f \) is considered, but the results obtained are of a general nature. Let \( x \) and \( y \) be scalar quantities, then we have the compensation equation

\[
f(x + \Delta x, y + \psi(x, y, \Delta x)) = f(x, y),
\]  

(3)

to which the following differential equation corresponds:

\[
\frac{\partial f}{\partial x} + \psi \frac{\partial f}{\partial y} = 0.
\]

As is known, the solution of this partial differential equation can be expressed in terms of the general integral of the ordinary differential equation \( \frac{dy}{dx} = \psi(x, y) \). If the general integral has the form \( \varphi(x, y) = C \), then the general solution of the partial differential equation has the form \( f(x, y) = \Phi(\varphi(x, y)) \). We note that the solution to this problem depends on the form of the compensating function \( \psi \).
Suppose that \( \psi(x, \Delta x, y) = k \cdot \Delta x \), where \( k \) is some constant, and \( \psi(0) = 0 \). Then the general integral of the differential equation \( \frac{dy}{dx} = k \cdot \Delta x = q = \text{const} \) has the form \( y - qx = C \). If we put \( q = \frac{a}{b} \), it will follow that the function \( f \) is the linear convolution \( ax + by = c' \).

It was established in [2] that if the compensating function has the form \( \psi(x, y) = \frac{\alpha^2 y}{x} \), then the solution of the corresponding differential equation allows us to construct the Cobb-Douglas function. It was shown in [3] that the basic Slutsky equation and its consequences are a special case of the general compensation equation and the corresponding properties of the differential equations.

This paper presents the results related to the solution of the problem of defining the compensating function \( \psi \) if \( f \) belongs to the class of rational functions in two variables. This class is chosen because rational functions represent a large class of aggregation functions. In addition, many operations of fuzzy mathematics belong to this class.

2. Definition of the compensating function for rational functions

As we know, a rational function is represented by a polynomial or the ratio of two polynomials. Let us consider the function of the form

\[
f(x, y) = A_0 + A_1 x + A_2 y + A_3 xy.
\]

We note that \( f \) can be represented as follows:

\[
f(x, y) = (ax + b)(Ay + B) = bB + aBx + bAy + bB,
\]
where \( A_0 = bB, A_1 = aB, A_2 = bA, A_3 = aA \).

Substituting this function in (3), we obtain

\[
(ax + b)(Ay + B) = (a(x + \Delta x) + b)(A(y + \psi) + B).
\]

After simple transformations, we move on to the equation

\[
(ax + b)(Ay + B) = (ax + b)(Ay + B) + A\psi(ax + b) + a\Delta x(Ay + B) + Aa\Delta xy,
\]
which gives

\[
\psi(x, y, \Delta x) = -\frac{a(Ay + B)\Delta x}{A((ax + b) + aAx)},
\]
which depends on \( x, y \) and \( \Delta x \).

As an example, we consider the function \( f(x, y) = x + y - xy \), where \( A_0 = 0, A_1 = 1, A_2 = 1, A_3 = -1 \). Let us determine whether this function can be represented by formula (4). Let us compose the system of equations
\[
\begin{aligned}
&bB = 0, \\
&aB = 1, \\
&bA = 1, \\
&aA = -1.
\end{aligned}
\]

Its analysis allows us to conclude that it is not consistent, and, therefore, the compensating function cannot be obtained by formula (5).

From this example, we can conclude: in order to find a compensating function \( \psi \) for the given function \( f \), we should first determine the way of representing \( f \).

Now we consider the function

\[
f(x, y) = \frac{A_0 + A_x x + A_y y + A_{xy} xy}{B_0 + B_x x + B_y y + B_{xy} xy}, \quad (6)
\]

which can be constructed in several ways, for example,

\[
\frac{a_x + b_1}{c_x + d_1} \pm \frac{a_y + b_2}{c_y + d_2}, \quad (7)
\]

\[
\frac{a_x + b_1}{c_x + d_1} \frac{a_y + b_2}{c_y + d_2}. \quad (8)
\]

Let us consider (7). We note that

\[
\frac{a_x + b_1}{c_x + d_1} \pm \frac{a_y + b_2}{c_y + d_2} = \frac{(b_1 d_2 \pm b_2 d_1) + x(a_1 d_2 \pm b_2 c_1) + y(b_1 c_2 \pm a_2 d_1) + xy(a_1 c_2 \pm a_2 c_1)}{d_1 d_2 + c_1 d_2 x + c_1 d_1 y + c_2 c_2 xy},
\]

from which we obtain the following coefficients \( f \):

\[
A_0 = b_1 d_2 \pm b_2 d_1, \quad A_1 = a_1 d_2 \pm b_2 c_1, \quad A_2 = b_1 c_2 \pm a_2 d_1, \quad A_3 = a_1 c_2 \pm a_2 c_1,
\]

\[
B_0 = d_1 d_2, \quad B_1 = c_1 d_2, \quad B_2 = c_2 d_1, \quad B_3 = c_1 c_2.
\]

We denote \( F_U^x = a_x x + b_1, \ F_U^y = c_x x + d_1, \ F_U^{xy} = a_x y + b_2, \ F_U^{xy} = c_x y + d_2 \), then on the right-hand side of (3) we obtain

\[
\frac{a_1 (x + \alpha x) + b_1}{c_1 (x + \alpha x) + d_1} - \frac{(a_x x + b_1) + a_1 \alpha x}{c_1 (x + \alpha x) + d_1} - \frac{a_1 \alpha x}{c_1 \alpha x} = \frac{a_2 (y + \psi) + b_2}{c_2 (y + \psi) + d_2} = \frac{a_1 (x + \alpha x) + b_1}{c_1 (x + \alpha x) + d_1} - \frac{a_2 (y + \psi) + a_2 \psi}{c_2 (y + \psi) + d_2} = \frac{F_U^x + a_1 \alpha x}{F_U^x + c_1 \alpha x} = \frac{F_U^x + a_1 \alpha x}{F_U^y + c_2 \psi} = \frac{F_U^x + a_1 \alpha x}{F_U^{x+y} + F_U^{x+y}}.
\]

The compensation equation will be as follows:

\[
F_U^x + a_1 \alpha x \pm F_U^y + a_2 \psi = \frac{F_U^x + a_1 \alpha x}{F_U^y + c_2 \psi} = \frac{F_U^x + a_1 \alpha x}{F_U^x + F_U^{x+y}},
\]

from which, after simple transformations, we obtain the equation for finding \( \psi \) of the following form:
Considering the previously introduced notation, the compensation equation has the form

$$\psi\left(\Delta x \left(F^+_L F^+_U \left(a_c c_2 + a_c c_1 \right) - c_1 c_2 \left(F^+_U F^+_L + F^+_U F^+_L \right) \right) + \left(a_x F^+_L c_c F^+_U \right) \left(F^+_U \right)^2 \right) = $$

$$= -\Delta x \left(a_x F^+_L c_c F^+_U \right) \left(F^+_U \right)^2.$$ 

Thus, considering representation (7), the compensating function for (6) has the form

$$\psi(x, y, \Delta x) = -\frac{\Delta x \left(a_x F^+_L c_c F^+_U \right) \left(F^+_U \right)^2}{\Delta x \left(F^+_L F^+_U \left(a_c c_2 + a_c c_1 \right) - c_1 c_2 \left(F^+_U F^+_L + F^+_U F^+_L \right) \right) \left(a_x F^+_L c_c F^+_U \right) \left(F^+_U \right)^2}.$$  (9)

Here we can calculate some factors:

$$a_x F^+_L c_c F^+_U = a_x d_x - c_b c_b;$$

$$F^+_U F^+_L + F^+_U F^+_L = (b_d d_x b_d d_x) + x(a_d d_x c_b c_b) + x(a_d d_x c_b c_b) + xy(a_c c_2 + a_c c_1);$$

$$a_x F^+_L c_c F^+_U = a_x d_x - c_b c_b.$$ 

Now we consider representation (6) in the form (8). Here we have

$$a_x x + b_x \frac{a_x y + b_x}{c_x + d_x} \frac{a_x y + b_x}{c_x + d_x} \frac{a_x y + b_x}{c_x + d_x} \frac{a_x y + b_x}{c_x + d_x},$$

therefore, in (6) we have

$$A_0 = b_x d_x, \ A_1 = a_x d_x, \ A_2 = a_x d_x, \ A_3 = a_x a_x, \ B_0 = d_x d_x, \ B_1 = c_x d_x, \ B_2 = c_x d_x, \ B_3 = c_x c_x.$$ 

Considering the previously introduced notation, the compensation equation has the form

$$\frac{F^+_L + a_x \Delta x \left(F^+_L + a_x \psi \right)}{F^+_L + a_x \Delta x \left(F^+_L + a_x \psi \right)} = F^+_L F^+_U.$$ 

Simple transformations lead to the equation of the form

$$\psi\left(\Delta x \left(F^+_L F^+_U a_c c_2 - F^+_U F^+_L a_c c_2 \right) + \left(F^+_U F^+_L a_c c_2 \right) \right) = F^+_L F^+_U \left(F^+_L a_c c_2 - F^+_L a_c c_2 \right) \Delta x,$$

from which we obtain the compensating function

$$\psi(x, y, \Delta x) = \frac{F^+_L F^+_U \left(F^+_L a_c c_2 - F^+_L a_c c_2 \right) \Delta x}{F^+_L F^+_U \left(a_c c_2 - c_b c_b \right) + \left(F^+_U F^+_L a_c c_2 - F^+_U F^+_L a_c c_2 \right) \Delta x}.$$  (10)

As an example, we consider the characteristic $d = \frac{x \left(1 - y \right)}{y \left(1 - x \right)}$ called the difficulty of achieving the goal [4]. It is assumed that $x \in \left[0, 1\right]$ is a quantitative requirement for the degree of goal achievement, $y \in \left[0, 1\right]$ is a value that characterizes the degree of goal achievement in a real situation. The goal is
considered achieved if \( y > x \). For the given \( x \) and \( y \), the value \( d \in [0,1] \) determines the degree of failure to achieve the goal. Let us represent \( d \) in the form

\[
d = \frac{x(1-y)}{y(x-1)} = \frac{x}{1-x} \cdot \frac{1-y}{y},
\]

(11)

then, in accordance with the notation from (8), we have

\[
\begin{align*}
a_1 &= 1, \quad b_1 = 0, \quad c_1 = -1, \quad d_1 = 1, \quad a_2 = -1, \quad b_2 = 1, \quad c_2 = 1, \quad d_2 = 0,
\end{align*}
\]

and

\[
F'_u = x, \quad F'_v = 1-x, \quad F'_x = 1-y, \quad F'_y = y.
\]

Substituting the found coefficients in (10), we obtain the compensating function

\[
\psi(x, y, \Delta x) = \frac{y(1-y)\Delta x}{2xy(1-x)+(x-y)\Delta x},
\]

Considering that

\[
\frac{y(1-x)}{x(1-y)} = \frac{1-x}{d} = \frac{1}{d},
\]

we move on to the next representation

\[
\psi(x, y, \Delta x) = \frac{\Delta x}{2 y \cdot \frac{1}{d} + \frac{x}{d} \left(1 - \frac{1}{d}\right) \Delta x},
\]

the feature of which is the dependence of \( \psi \) on \( d \).

The question arises whether it is possible to consider the representation \( d \) using other functions. Considering representation (8), let us write (11) in the form

\[
d = \frac{x - xy}{y - xy} = \frac{a_1 x + b_1}{c_1 x + d_1}, \quad \frac{a_2 y + b_2}{c_2 y + d_2},
\]

from which we obtain the following system of equations:

\[
\begin{align*}
a_1 a_2 &= -1, \quad a_1 b_2 = 1, \quad b_1 a_2 = 0, \quad b_1 b_2 = 0, \\
c_1 c_2 &= -1, \quad c_1 d_2 = 0, \quad c_2 d_1 = 1, \quad d_1 d_2 = 0.
\end{align*}
\]

(12)

We note that \( a_1, a_2, b_1, b_2, c_1, c_2, d_1 \) and \( c_1, c_2, d_1 \) are not equal to 0, \( b_1 = 0 \) and \( d_2 = 0 \). From system (12), we express \( a_2 = -\frac{1}{a_1}, \quad b_2 = \frac{1}{a_1}, \quad c_1 = -\frac{1}{c_2}, \quad d_1 = \frac{1}{c_2} \). Thus, there is a whole family of pairs of linear-fractional functions \( \frac{a_1 x + b_1}{c_1 x + d_1} \) and \( \frac{a_2 y + b_2}{c_2 y + d_2} \), whose coefficients satisfy the system (12). If we put, for example, \( a_1 = 2, \quad c_2 = 4 \), we obtain the remaining coefficients

\[
a_2 = -\frac{1}{2}, \quad b_2 = \frac{1}{2}, \quad c_1 = -\frac{1}{4}, \quad d_1 = \frac{1}{4}.
\]
Substituting the found coefficients into (8) and considering that \( b_1 = 0 \) and \( d_z = 0 \), we obtain \( d \) in the form (11).

3. Compensating functions for associative rational functions

Let us consider the case when the function \( f \) in the form (6) is associative. Associativity means that there is no hierarchy among the arguments. It is known [5] that rational function (6) is associative if the following restrictions on the coefficients are satisfied:

\[
\begin{align*}
A_i = A_2, & \quad B_i = B_2, \\
B_i A_2 + A_i' = A_i B_2 + A_2 A_i, & \\
B_i A_1 + B_i' = A_i B_1 + B_1 B_i, & \\
A_i B_j = A_j B_i. & 
\end{align*}
\]

Analysis of these restrictions allows us to conclude that if function (6) is associative, then it is commutative.

Associative and commutative function (6) is transformed into

\[
f(x, y) = \frac{C_0 + C_1(x + y) + C_2 xy}{D_0 + D_1(x + y) + D_2 xy}.
\]

(13)

It is known [6] that any associative function can be represented in the form

\[
f(x, y) = g^{-1}(g(x) + g(y)),
\]

(14)

if and only if there exists a continuous strictly monotone function \( g : [0, 1] \rightarrow [0, \infty) \) called the additive generator, and \( g^{-1} \) is the inverse function of \( g \) in the usual sense, \( g^{-1} \) is a pseudo inverse function.

The additive generator is defined up to a positive constant, and if \( f \) refers to additive operators, then \( g(0) = 0 \) and \( g \) is strictly increasing; if \( f \) is a multiplicative operator, then \( g(1) = 0 \) and \( g \) is strictly decreasing.

According to [7], there are three types of generators for the function of the form (13):

\[
g_1(x) = \frac{ax + b}{cx + d}, \quad g_2(x) = \ln \frac{ax + b}{cx + d}, \quad g_3(x) = \arctg \frac{ax + b}{cx + d},
\]

which are defined up to a positive multiplicative constant. Let us compose the compensation equation (3) considering the representation of \( f \) in the form (14)

\[
g^{-1}(g(x) + g(y)) = g^{-1}(g(x + \Delta x) + g(y + \psi)).
\]

Since the generator is a strictly monotone function, then \( g^{-1} \), and, therefore, \( g^{(-1)} \) are also strictly monotone functions on a certain interval (depending on the type of monotonicity), therefore

\[
g(x) + g(y) = g(x + \Delta x) + g(y + \psi).
\]

(15)

For the known generator \( g \), equation (15) allows finding the compensating function \( \psi \).
Let us consider the function \( f_\oplus(x, y) = x + y - xy \) already discussed above. According to [8], the additive generator for it has the form \( g_\oplus(x) = -\ln(1-x) \). Substituting it into (15), we obtain the equation

\[
-\ln(1-x) - \ln(1-y) = -\ln\left(1-(x+\Delta x)\right) - \ln\left(1-(y+\psi)\right),
\]

from which we find

\[
\psi_\oplus(x, y; \Delta x) = -\frac{1-y}{1-(x+\Delta x)} \Delta x.
\]

If \( x, y \in [0,1] \), then the function \( f_\oplus \) is a triangular conorm [9] and models fuzzy additive operators. The dual triangular norm (multiplicative operator) is given by the product \( f_\otimes(x, y) = xy \). It is a commutative and associative operation with the additive generator \( g_\otimes(x) = -\ln x \). Let us compose the compensation equation

\[
-\ln(x) - \ln(y) = -\ln(x+\Delta x) - \ln(y+\psi),
\]

from which we find

\[
\psi_\otimes(x, y; \Delta x) = -\frac{y}{x+\Delta x} \Delta x.
\]

We note that in this case

\[
\psi_\otimes = \frac{y\left(1-(x+\Delta x)\right)}{(x+\Delta x)(1-y)}.
\]

The structure of the resulting formula is similar to the structure of the characteristic \( d \).

Thus, if a rational function is associative, then, based on its representation, we can easily find a compensating function with the help of an additive generator. Additive generators of rational functions are presented in [7,8,10].

4. Conclusion

There is a concept according to which it is the conservation laws that determine quantities that it makes sense to raise to the rank of universal equivalents. When studying systems, this approach will allow describing such changes in the parameters of a system that do not take it out of any given states, for example, a state of equilibrium. The use of the compensation equation makes it possible to recreate the functional dependence, which can form the basis of the apparatus for predicting system changes and system management in the future. For triangular norms and conorms, as well as for associative aggregation operations, the compensating function is easily found on the basis of additive generators.

References

[1] Kaplinsky A I, Russman I B and Umyvakin V M 1991 Modelling and algorithmisation of ill-defined problems of selecting the best variants of the system (Voronezh: VSU Publishing house) 298 p (in Russian)
[2] Intriligator M 2002 *Mathematical models of optimisation and economic theory* (Moscow: Science) 1020 p (in Russian)

[3] Lancaster K 1972 *Mathematical economics* (Moscow: Soviet radio) 464 p (in Russian)

[4] Russman I B 2008 *Difficulty in achieving the goal: selected works* (Voronezh: Voronezh State University) 196 p (in Russian)

[5] Ledeneva T M 1997 Some aspects of the representation of fuzzy operators by the relation of two polynomials *News of Higher Educational Institutions. Mathematics* 11 pp 33-40 (in Russian)

[6] B Bacchelli 1986 Representation of continuous associative functions, *Stochastica* 10 pp 13-28

[7] T M Ledeneva 2018 Analysis of additive generators of fuzzy operations represented by rational functions *Journal of Physics: Conf. Series* 973 012037 https://doi:10.1088/1742-6596/973/1/012037

[8] T M Ledeneva 2020 New Family of Triangular Norms for Decreasing Generators in the form of a Logarithm of a Linear Fractional Functions *Fuzzy Sets and Systems* https://doi.org/10.1016/j.fss.2020.11.020

[9] E P Klement, R Mesiar and E Pap *Triangular Norms* Kluver Academic Publishers (Dordrecht) 2000

[10] T M Ledeneva 2020 Additive generators of fuzzy operations in the form of linear fractional functions *Fuzzy Sets and Systems* https://doi.org/10.1016/j.fss.2019.03.005