Free Transverse Oscillations of a Continuum-Discrete Vertical Rod

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Abstract. The spectral problem of transverse oscillations of high vertical continuum-discrete rods is considered, focused on the application of results in the problems of seismic oscillations. A mathematical model of free oscillations representing the boundary value problem for the hyperbolic differential equation is obtained, which is reduced to the Sturm-Liouville problem by the method of variable separation. Application of finite difference and coordinate descent methods in combination with visualization of calculations in Matlab environment gives own values and attenuation coefficients and further own functions.

1. Introduction
Vertical rods (Fig. 1) are widely used in construction practice as transmission lines, chimneys, spotlights, antenna devices, etc. The extensive use of vertical struts stimulates considerable interest in their vibrations from a variety of influences. Meanwhile, the oscillations of high vertical rods in seismic disturbances are insufficiently studied and especially when they carry a lot of concentrated masses at different levels.

Figure 1. Vertical rods
Among the published papers one can point to bending and longitudinal oscillations of rods without
discrete masses or with one or two masses, usually concentrated at the ends of rods [1-4]. Free and
forced longitudinal oscillations of continuous-discrete vertical rods are studied in the paper [5] as
relevant for structures in the epicentral zone of seismic effects.

Mathematical models of oscillation commonly used for buildings, although they are continuum
discrete, do not take into account the rotation of discrete masses representing the slabs, as their
horizontal movements are translational movements. When considering high vertical flexible rods, this
hypothesis has to be abandoned and requires different mathematical models.

2. Mathematical Model of Free Oscillations
Let's consider a vertical rod (Fig. 1a) with a lot of masses. This structure is a continuous discrete
system with a total length L, by the lengths of the plots, l, made of a material with a modulus of
elasticity E, with a density of material ρ, with areas and axial moments of cross-sections A, J, with
discrete masses M_k and relies on the foundation of the mass M_0. Its movement is limited by the elastic
resistance of the ground with a coefficient of stiffness c (spring effect).

Flexural vibrations of the continuum are described by the function u(x, t) and the famous equation
bu'' + (Nu)' + mŨ + nmũ = 0, b = EJ, x ∈ (0, L), t > -∞. (1)

Here N(x) is the longitudinal force in the section

\[ N(x) = \sum_{i=k}^{r} M_i + m(L - x)g, \]

whereabouts g - the acceleration of free fall. m = ρA - the intensity of the rod mass in the longitudinal
direction, η - Specific coefficient of linear-viscous friction of the material. The dot above the symbol
corresponds to time differentiation, the strokes in the upper indexes are derived from the argument x.
The Roman numeral IV in the top index corresponds to a fourth order derivative of x. Discrete masses
M_i are considered material points. In addition to the global coordinate systems, local coordinate
systems are also used for the core areas x_i, y_i, i = 1, 2, ... k - point numbers starting with the
underlying discrete mass.

Equation (1) joins the boundary conditions at the lower and upper ends and the conditions of the
junction of the sections.

Lower end:

\[ bu''' + c'u(0, t) - M_0 ũ(0, t) = 0, \quad u'(0, t) = 0. \] (2)

Upper end:

\[ bu''(l, t) + M_r ũ(l, t) = 0, \quad bu''(l, t) + I_r ũ(l, t) = 0. \] (3)

Due to the high mass of the foundation M_0, its rotation angles will be small and therefore not taken
into account in the boundary conditions (2).

In addition to the additional conditions, the conditions for coupling rod sections (Fig. 1b), divided
by discrete masses, are added. To determine them, we will use the equations of flat mass movement
M_k. Equations of movement in the horizontal direction and rotation will be made, using the principle
of Dalandert

\[ Q_{k+1} - Q_k + D_k = 0, \quad I_k \psi_k = M_{k+1} - M_k. \] (4)

whereabouts Q_{k+1}, Q_k, M_k, M_{k+1} - transverse forces - bending moments from above and below the
mass M_k, D_k - the dalamberian power of inertia, q_k = u_k',(t, t)- mass rotation angle M_k. Replace in
(4) the force factors with derivative deflections and record:

\[ bu''''(k, t) - bu''''(k, t) + M_k ũ_k(l, t) = 0, \quad k = 1, 2, ... , r - 1 \] (5)

\[ I_k ũ_k''(l, t) - bu''''(k, t) + bu''''(l, t) = 0, \quad k = 1, 2, ... , r - 1 \] (6)
The basic equation (1), boundary conditions (2), (3) and interface conditions (5), (6) form a mathematical model to consider the free vibrations of the rod. It will be difficult to solve the problem with the help of analytical methods, as the longitudinal force \( N \) in equation (1) is a variable depending on \( x \). Let’s use the method of separating variables and write down the movements

\[
u(x, t) = Y(x)e^{i\omega t},
\]

where \( Y(x) \) is \( \lambda \) - its own form and a characteristic indicator of fluctuations. Substitution (7) in (1) to (3), (5), (6) gives

\[
Y''''(x) + a_1 Y'''(x) + a_2(x)Y''(x) + a_3 Y(x) = 0, \quad x \in (0,l),
\]

where \( a_1 = -g/q, \quad a_2 = p/q, \quad q = b/m, \quad a_3 = \lambda(\lambda + \eta)/q, \quad p(x) = N(x)/m, \)

Lower end: \( bY'''(0) + cY(0) - M_0 \lambda^2 Y(0) = 0; \quad Y'(0) = 0 \)

Upper end: \( bY'''(L) + M_r \lambda^2 Y(L) = 0, \quad bY''(L) + \lambda^2 Y'(L) = 0 \).

Conditions for coupling the \( k \) and \( k+1 \) sections:

\[
bY'''_{k+1}(0) - bY'_k(L) + M_kY''_k(L) = 0, \quad k = 1,2,...,r-1
\]

\[
bY'''_k(L) - bY''_{k+1}(0) + I_k \lambda^2 Y'_k(L) = 0, \quad k = 1,2,...,r-1
\]

Note that in (11), (12) local function arguments are applied.

The boundary value problem (8) - (12) is further solved using the finite difference method. Instead of a continuous area of determination of the variable \( x \), let’s introduce a discrete area \( L_h \) in the form of nodes of a uniform grid with the step \( h \)

\[
L_h = [x_i; \quad x_i = (i-1)h, \quad i = 1, 2, ..., n], \quad h = L/(n-1),
\]

where \( n \) is the number of grid nodes. Let’s replace the values of the function and derivatives with approximate well known finite-difference values in the grid nodes with the accuracy of \( O(h^2) \). Then the continuous function \( Y(x) \) becomes discrete, i.e. the vector \( Y = \{Y_1, Y_2, ..., Y_n\}, \quad Y_i = Y(x_i) \).

Having performed the procedures of the finite difference method, we get instead of (8)

\[
Y_{i-2} + a_1 Y_{i-1} + \beta_i Y_i + \gamma_i Y_{i+1} + Y_{i+2} = 0, \quad i = 3,4,...,n-3,n-2.
\]

This is the place

\[
a_1 = -4 + h^2a_1/2 + h^2a_2; \quad \beta_i = 6 - 2a_2h^2 + h^4a_3; \quad \gamma_i = -4 + a_1h^3/2 + h^2a_2.
\]

Similar changes will be made for additional conditions (9) - (12) and recorded:

\[
\delta Y_1 + 18Y_2 - 24Y_3 + 14Y_4 - 3Y_5 = 0, \quad -3Y_1 + 4Y_2 - Y_3 = 0,
\]

\[
3Y_{n-4} - 14Y_{n-3} + 24Y_{n-2} - 18Y_{n-1} + \zeta Y_n = 0, \quad \zeta = 5 + \frac{2h^3}{b}M_r \lambda^2,
\]

\[
-\gamma_{n-3} + \kappa Y_{n-2} + \nu Y_{n-1} + \xi Y_n = 0, \quad \kappa = 4 + I_r \lambda^2 h^2/2b,
\]

\[
\delta = -5 + 2h^3(c - M_0 \lambda^2)/b; \quad \nu = -5 - 2I_r \lambda^2 h/b; \quad \xi = 2 + 3hI_r \lambda^2/2b;
\]

\[
\tau Y_k + 18Y_{k+1} - 24Y_{k+2} + 14Y_{k+3} - 3Y_{k+4} - 3Y_{k-4} + 14Y_{k-3} - 24Y_{k-2} + 18Y_{k-1} = 0.
\]

\[
-\gamma_{k-3} + \epsilon Y_{k-2} + \sigma Y_{k-1} + \theta Y_k + 5Y_{k+1} - 4Y_{k+2} + Y_{k+3} = 0.
\]

\[
\tau = -10 + 2h^3M_k \lambda^2/2b; \quad \epsilon = 4 + hI_k \lambda^2/2b; \quad \sigma = -5 - 2hI_k \lambda^2/2b; \quad \theta = 3hI_k \lambda^2/2b.
\]
3. Solution of the Spectral Problem

The system of equations (13) - (18) allows to solve the spectral problem for free oscillations. At its consideration, the characteristic indicator will be accepted in the form of

$$\lambda = -\mu + j\omega,$$

(19)

$$(\mu, \omega)$$ - is the natural pair consisting of the attenuation coefficient and the frequency of free oscillations, $j$ is an imaginary unit. It is obvious that (13) - (18) form a homogeneous linear system of algebraic equations

$$B(\lambda) Y = 0.$$  

(20)

Here B is a square matrix of the order n, Y is a vector-column

\[
B = \begin{pmatrix}
-3 & 4 & -1 \\
\delta & 18 & -24 & 14 & -3 \\
1 & \alpha_3 & \beta_3 & \gamma_3 & 1 \\
1 & \alpha_4 & \beta_4 & \gamma_4 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
-3 & 14 & -24 & 18 & \tau & 18 & -24 & 14 & -3 & k \\
-1 & \varepsilon & \sigma & \theta & 5 & -4 & 1 & k1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & \alpha_{n-2} & \beta_{n-2} & \gamma_{n-2} & 1 \\
3 & -14 & 24 & -18 & \zeta & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-1 & \kappa & \nu & \xi & 1 \\
\end{pmatrix}
\]

The system of equations (20) has a trivial solution $Y = 0$, that is of no interest. Non-zero solutions may exist as long as

$$\det B(\lambda) = 0.$$  

(21)

It is not possible to compose and solve equation (21) by analytical methods. Its left side is a complex function due to (19). Taking into account the use of computer technology and corresponding computational complexes, let us present it

$$f_1(\mu, \omega) + jf_2(\mu, \omega) = 0.$$  

Now $(\mu, \omega)$ we can find two nonlinear algebraic equations from the system

$$f_1(\mu, \omega) = 0, \quad f_2(\mu, \omega) = 0.$$  

(22)

Its decision due to the large order of the matrix B is very difficult. Let's use the coordinate descent method [6]. For this purpose, we form an auxiliary multimodal nonnegative function

$$\Phi(\mu, \omega) = |\det B(\mu, \omega)| = \left[f_1^2(\mu, \omega) + f_2^2(\mu, \omega)\right]^{1/2},$$

turning to zero only in points coinciding with the roots of the equation system (22).

Repetition of such calculation procedure gives spectral pairs of attenuation coefficients and natural frequencies

$$\{\{\mu_1, \omega_1\}, \{\mu_2, \omega_2\}, \ldots \}.$$  

Difficulties are greatly simplified if the programming system used allows you to quickly visualize the results of calculations.

Let's illustrate this by example.

**Example 1.** Let's assume that Fig. 1 presents a rod made of standard steel pipe with the lengths of sections $l = 5$ m, with the number of sections $r = 3$, diameter $D = 10^2$ mm and wall thickness $\delta = 3.2$ mm. The rod is supported by a foundation with a mass of $M_0 = 10000$ kg and a base with a stiffness coefficient of $c = 2 \cdot 10^6$ H/m. Discrete masses and axial moments of inertia are given by vectors $M =$
\{2000 \ 4000 \ 1000\} \text{ kg}, \ I = \{2000 \ 4000 \ 500\} \text{ kgm}^2, \text{ specific coefficient of internal friction } \eta = 0.2 \text{ s}^{-1}, \ n = 301.

In the coordinate descent method, when using the Matlab computer system, one of the arguments, for example, \(\mu\) is fixed and the ordinates of the function are calculated \(\Phi(\omega|\mu)\), and its chart is displayed on the computer screen (Fig. 2). The curve, which is the surface section \(\Phi(\mu, \omega)\), clearly shows the conditional minimums of the function \(\Phi(\omega|\mu)\). Now we fix the argument \(\omega\) at the value corresponding to the found conditional minimum and build a new chart (Fig. 3). Calculations continue until the necessary accuracy is achieved.

In this example, the first three spectral pairs were obtained \((\omega, \mu) = \{(1.84 \ 0.1); \ (6.61 \ 0.1); \ (13.02 \ 0.1)\} \text{ s}^{-1}\).

The number of descents in the corresponding method did not exceed three or four. The obtained attenuation coefficients were almost identical and approximately equal to 0.1 and therefore rounded.

Own frequencies of this rod fall into the region of dominant frequencies of spectral densities of the majority of seismic influences that will lead to dangerous resonance oscillations at own frequencies. One way to remove the natural frequencies from the resonance zone is to increase the rigidity of the structure by increasing the diameter and wall thickness of the bearing pipe. Another method is the construction of dynamic and hydraulic vibration dampers, etc.

The spectrum of eigen vectors corresponding to eigenfunctions can be set using the system of equations (20). Since the determinant of the matrix \(B\) is zero, custom forms can only be found to the nearest doubter. Then in (20) it can be assumed, for example, that the \(y_n = 1\), rest of the unknowns can also be found in the equation system formed from (20) by excluding the last column, the last row or any other row of the matrix \(B\). After such transformations we obtain a system of inhomogeneous algebraic linear equations

\[
C(\omega) \ Z = d, \tag{23}
\]

whereabouts \(Z = \{z_1, z_2, ..., z_{n-1}\}\).
Alternating substitution of eigenvalues into formulas for the calculation of \( C \) matrix elements and further solution of the equation system (25) will lead to obtaining eigenvectors.

Based on the data of Example 1, the corresponding calculations were made and their results are shown in Fig. 4.

4. Conclusions

1. The combination of analytical, numerical and graphical methods is the easiest way to effectively solve complex spectral problems of vertical rod transverse oscillations.

2. There is a high probability that the natural frequencies of the vertical rods will coincide with the dominant frequencies of most of the observed seismic effects, which can lead to dangerous resonance oscillations.

References

[1] Kulterbaev H P, Abdul Salam I 2018 Longitudinal free oscillations of the rod with discrete masses Problems of mechanics and control: Proceedings of the International Conference (16-22 September 2018, Makhachkala) Ed. I G G Goryachev (Moscow, University Publishing House) pp 207-210

[2] Kulterbaev Kh P, Baragunova L A, Shogenova M M, Shardanova M A 2018 Longitudinal Vibrations of Seismic Disturbance Vertical Bar Proceedings of the International Symposium “Engineering and Earth Sciences: Applied and Fundamental Research” (ISEES 2018) Advances in Engineering Research vol 177 pp 515-520

[3] Kulterbaev Kh P 2015 Variable section vertical column fluctuations at the harmonic and random vector perturbations (in Russian) XI All-Russian Congress on Fundamental Problems of Theoretical and Applied Mechanics Abstracts of reports (Kazan, 20-24 August 2015) Kazan: Publishing House of the Academy of Sciences of the Republic of Tajikistan p 163

[4] Kulterbaev Kh P, Baragunova L A, Shogenova M M 2018 Free and Forced Longitudinal Vibrations of Rods Materials Science Forum Submitted: 2018-05-13 ISSN: 1662-9752 vol 931 pp 47-53 Accepted: 2018-05-28, doi:10.4028/www.scientific.net/MSF.931.47 Online: 2018-09-20 © 2018 Trans Tech Publications (Switzerland)

[5] Baragunova L A, Shogenova M M, Shardanova M A, Abdul Salam I M 2018 Longitudinal Vibrations of Seismic Disturbance Vertical Bar Proceedings of the International Symposium “Engineering and Earth Sciences: Applied and Fundamental Research” (ISEES 2018) Advances in Engineering Research vol 177 pp 515-520

[6] Verzhbitsky V M 2005 Numerical methods (linear algebra and nonlinear equations) (M.: Onix 21 Century Publishing House) 432 p