Unified treatment of screening Coulomb and anharmonic oscillator potentials in arbitrary dimensions

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Abstract

A mapping is obtained relating radial screened Coulomb systems with low screening parameters to radial anharmonic oscillators in $N$-dimensional space. Using the formalism of supersymmetric quantum mechanics, it is shown that exact solutions of these potentials exist when the parameters satisfy certain constraints.

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1 Introduction

Since the appearance of quantum mechanics there has been continual interest in models for which the corresponding Schrödinger equation is exactly solvable. Solvable potential problems have played a dual role in physics. First, they represented useful aids in modeling realistic physical problems, and second, they offered an interesting field of investigation in their own right. Related to this latter area, the concept of solvability has changed to some extent in recent years. With regards to solvability of the Schrödinger equation there are three interesting classes of the potentials.

The first class is the exactly solvable potentials allowing to obtain in explicit form all energy levels and corresponding wave functions. The hydrogen atom and harmonic oscillator are the best-known examples of this type. The second class is the so-called quasi-exactly solvable (QES) potentials for which a finite number of eigenstates of corresponding Hamiltonian can be found exactly in explicit form. The first examples of such potentials were given in [1]. Subsequently several methods for generating partially solvable potentials were worked out, as
a result many QES solvable potentials were found [2]. The third class is the conditionally-exactly solvable potentials [3] for which the eigenvalue problem for the corresponding Hamiltonian is exactly solvable only when the parameters of the potential obey certain conditions.

Although modern computational facilities permit the construction of solutions for any well-behaved potentials to any degree of accuracy, there remains continuing interest in exact solutions for a wider range of potentials. In connection with this, the technique of changing the independent coordinate has always been a useful tool in the solution of the Schrödinger equation. For one thing, this allows something of a systematic approach, enabling one to recognize the equivalence of superficially unrelated quantum mechanical problems. For example, solvable Natanzon [4] potentials in nonrelativistic quantum mechanics are known to group into two disjoint classes depending on whether the Schrödinger equation can be reduced to a hypergeometric or a confluent hypergeometric equation. All the potentials within each class are connected via point canonical transformations. Gangopadhyaya and his co-workers [5] established a connection also between the two classes with appropriate limiting procedures and redefinition of parameters, thereby inter-relating all known solvable potentials. In order for the Schrödinger equation to be mapped into another Schrödinger equation, there are severe restrictions on the nature of the coordinate transformation. Coordinate transformations which satisfy these restrictions give rise to new solvable problems. When the relationship between coordinates is implicit, then the new solution are only implicitly determined, while if the relationship is explicit then the newly found solvable potentials are also shape invariant[5, 6]. In a more specific special application of these ideas, Kostelecky et al. [7] were able to relate, using an explicit coordinate transformation, the Coulomb problem in $N$-dimensions with the $N$-dimensional harmonic oscillator. Other explicit applications of the coordinate transformation idea can be found in the review articles of Haymaker and Rau [8].

Moreover, recently an anti-isospectral transformation called also as duality transformation was introduced [9]. This transformation relates the energy levels and wave functions of two QES potentials. Many recent papers [10-12, and the references therein] have addressed this subject of coordinate transformation placing a particular emphasis on QES power-law potentials, which is also the subject of the present work in some extent. The generalization to higher dimensions of one-dimensional QES potentials was discussed in a recent paper [13] and a few specific $N$-dimensional solutions were listed. In that work, applying a suitable transformation, these potentials were shown to be related to the isotropic oscillator plus Coulomb potential system and some normalized isolated solutions for this system were obtained.

The importance in the study of QES potentials, apart from intrinsic academic interest, rests on the possibility of using their solutions to test the quality of numerical methods and in the possible existence of real physical systems that they could represent. For instance, anharmonic oscillators and screening Coulomb
(or Yukawa) potentials represent simplified models of many situations found in different field of physics. These problems have been studied for years and a general solution has not yet been found.

The problem of quantum anharmonic oscillators has been the subject of much discussion for decades, both from an analytical and a numerical point of view, because of its important applications in quantum-field theory [14], molecular physics [15], and solid-state and statistical physics [16, 17]. Various methods have been used successfully for the one-dimensional anharmonic oscillators with various types of anharmonicities. Relatively less attention has been given to the anharmonic oscillators in higher-dimensional space because of the presence of angular-momentum states that make the problem more complicated. The recent works [10, 11] have shown that there are many interesting features of the anharmonic oscillators and the perturbed Coulomb problems in higher-dimensional space, and discussed the explicit dependence of these two potentials.

Using the spirit of the works in Refs. [10, 11], we show the mappings between screened Coulomb potentials (with low screening parameters) and anharmonic oscillator potentials in \( N \)-dimensional space, which have not been previously linked. The connection between these potentials are also checked numerically by the use of the Lagrange-mesh calculation technique [18, 19]. Next we study the \( N \)-dimensional screened Coulomb problem and higher order anharmonic oscillators within the framework of supersymmetric quantum mechanics (SSQM) [20] and have shown that SSQM yields exact solutions for a single state only.

In the following section, we outline the mapping procedure used through the article and give the applications. We also discuss in the same section exact supersymmetric treatments of the ground state solutions for the initial and transformed potentials. Analysis and discussion of the results obtained are given in section 3. Section 4 involves concluding remarks.

## 2 Mappings between the two distinct systems

The radial Schrödinger equation for a spherically symmetric potential in \( N \)-dimensional space

\[
-\frac{1}{2} \left[ \frac{d^2 R}{dr^2} + \frac{N-1}{r} \frac{dR}{dr} \right] + \frac{\ell(\ell+N-2)}{2r^2} R = [E - V(r)] R 
\]

is transformed to

\[
-\frac{d^2 \Psi}{dr^2} + \left[ \frac{(M-1)(M-3)}{4r^2} + 2V(r) \right] \Psi = 2E\Psi 
\]

where \( \Psi(r) = r^{(N-1)/2} R(r) \), the reduced radial wave function, and \( M = N + 2\ell \). Note that the solutions for a particular central potential are the same as long as
remains unaltered. For instance, the s-wave eigensolutions and eigenvalues in four-dimensional space are identical to the p-wave solutions in two-dimensions.

We substitute \( r = \alpha \rho^2 / 2 \) and \( R = F(\rho)/\rho^\lambda \), suggested by the known transformations between the Coulomb and harmonic oscillator problems [7, 21] and used by [10, 11] to show the mappings between unperturbed Coulomb and anharmonic oscillator problems, and transform Eq.(1) to another Schrödinger-like equation in \( N' = 2N - 2 - 2\lambda \) dimensional space with angular momentum \( L = 2\ell + \lambda \),

\[
-\frac{1}{2} \left[ \frac{d^2 F}{d\rho^2} + \frac{N' - 1}{\rho} \frac{dF}{d\rho} \right] + \frac{L(L + N' - 2)}{2\rho^2} F = \left[ \hat{E} - \hat{V}(\rho) \right] F
\]

where

\[
\hat{E} - \hat{V}(\rho) = E\alpha^2 \rho^2 - \alpha^2 \rho^2 V(\alpha\rho^2/2)
\]

and \( \alpha \) is a parameter to be adjusted properly. Thus, by this transformation, the \( N \)-dimensional radial wave Schrödinger equation with angular momentum \( \ell \) can be transformed to a \( N' = 2N - 2 - 2\lambda \) dimensional equation with angular momentum \( L = 2\ell + \lambda \). The significance of the mapping parameter \( \lambda \) is discussed below.

A student of introductory quantum mechanics often learns that the Schrödinger equation can be solved numerically for all angular momenta for the screened Coulomb and anharmonic oscillator problems. Less frequently, the student is made aware of the relation between these two problems, which are linked by a simple change of the independent variable discussed in detail through this section. Under this transformation, energies and coupling constants trade places, and orbital angular momenta are rescaled. Thus, we have shown here that there is really only one quantum mechanical problem, not two.

The static screened Coulomb potential

\[
V_{SC}(r) = -e^2 \exp(-\delta r)/r
\]

where \( \delta \) is a screening parameter, is known to describe adequately the effective interaction in many-body environment of a variety of fields such as atomic, nuclear, solid-state and plasma physics. In nuclear physics it goes under the name of the Yukawa potential (with \( e^2 \) replaced by another coupling constant), and in plasma physics it is commonly known as the Debye-Hückel potential. Eq. (5) also describes the potential of an impurity in a metal and in a semiconductor [22]. Since the Schrödinger equation for such potentials does not admit exact analytic solutions, various approximate methods [23, 24, and the references therein], both analytic and numerical, have been developed.

Considering the recent interest in various power-law potentials in the literature, we work through the article within the frame of low screening parameter. In this case, the screened Coulomb potential appears

\[
V_{SC}(r) = -e^2 \frac{\exp(-\delta r)}{r} \cong -\frac{e^2}{r} + e^2 \delta - \frac{e^2 \delta^2}{2} r + \frac{e^2 \delta^3}{6} r^2 - \frac{e^2 \delta^4}{24} r^3 + \frac{e^2 \delta^5}{120} r^4
\]
in the form. The literature is rich with examples of particular solutions for such power-law potentials employed in different fields of physics, for recent applications see Refs. [25, 26]. At this stage one may wonder why the series expansion is truncated at a lower order. This can be understood as follows. It is widely appreciated that convergence is not an important or even desirable property for series approximations in physical problems. Specifically, a slowly convergent approximation which requires many terms to achieve reasonable accuracy is much less valuable than a divergent series which gives accurate answers in a few terms. This is clearly the case for the screening Coulomb problem [27]. This also explains why leading orders of the $1/N$ expansion converge at a similar rate for the hydrogen atom, the screening Coulomb potential, and two-electron atom even though the last two of these series diverge eventually [28]. In addition, the complexity of calculating higher order terms in the series for the corresponding transformed potential grows rapidly. Hence, if an accurate approximation cannot be achieved in a few terms, the present method may not be useful. However, in section 3 we show that the present technique gives excellent estimates for the energy eigenvalues of both, the truncated screening Coulomb and anharmonic oscillator problems in higher dimensions.

Though the mapping procedure introduced is valid for any bound state, throughout this work we are concerned only with the ground state. We have two reasons for this restriction. The first reason is that the exact analytical ground state solutions for the potentials of interest are available, which provides a test for the reliability of the present technique. The second reason is that the present model works well for low lying states, which will be shown in section 3. Proceeding, therefore, with the choice of $\alpha^2 = 1/|E_{n=0}|$ in Eq. (4), the screened Coulomb problem in Eq. (6) is transformed to an anharmonic oscillator problem such that

$$\hat{V}(\rho) = \left(1 + \frac{A_2}{|E_{n=0}|}\right) \rho^2 + \frac{A_3}{2|E_{n=0}|^{3/2}} \rho^4 + \frac{A_4}{4|E_{n=0}|^2} \rho^6 + \frac{A_5}{8|E_{n=0}|^{5/2}} \rho^8 + \frac{A_6}{16|E_{n=0}|^3} \rho^{10}$$

(7)

with the eigenvalue

$$\hat{E}_{n=0} = \frac{-2A_1}{|E_{n=0}|^{1/2}}$$

(8)

Thus, the system given by Eq. (6) in $N$-dimensional space is reduced to another system defined by Eq. (7) in $N' = 2N - 2 - 2\lambda$ dimensional space. In other words, by changing the independent variable in the radial Schrödinger equation, we have been able to demonstrate a close equivalence between the screened Coulomb potential and anharmonic oscillator potentials.

For almost two decades, the study of higher order anharmonic potentials has been desirable to physicists and mathematicians in understanding a few newly
discovered phenomena such as structural phase transitions [29], polaron formation in solids [30], and the concept of false vacuum in the field theory [31]. Unfortunately, in these anharmonic potentials, not much work has been carried out on the potential like the one defined by (7) except the works in Refs. [32-34]. Our investigation in \( N \)-dimensional space, beyond showing an explicit connection between two distinct systems involving potentials of type (6) and (7), would also be helpful to the literature regarding the solutions of such potentials in arbitrary dimensions due to the recent wide interest in the lower-dimensional field theory.

Eqs. (7) and (8) are the most important expressions in the present work. To test explicitly if these results are reliable, one needs to have an exact analytical solutions for the potentials of interest, which are found below within the frame of supersymmetric quantum mechanics.

2.1 Supersymmetric treatment for the ground state

Using the formalism of SSQM [20] we set the superpotential

\[
W(r) = \frac{a_1}{r} + a_2 + a_3 r + a_4 r^2, \quad a_4 < 0
\]

for the potential in (6) and identify \( V_+(r) \) supersymmetric partner potential with the corresponding effective potential so that

\[
V_+(r) = W^2(r) + W'(r) = \frac{2a_1 a_2}{r} + [a_2^2 + a_3(2a_1 + 1)] + 2(a_1 a_4 + a_4 + a_2 a_3) r \\
+ (2a_2 a_4 + a_3^2) r^2 + 2a_3 a_4 r^3 + a_4^2 r^4 + \frac{a_1(a_1 - 1)}{r^2}
\]

\[
= \left( \frac{2A_1}{r} + 2A_2 + 2A_3 r + 2A_4 r^2 + 2A_5 r^3 + 2A_6 r^4 \right) + \frac{(M - 1)(M - 3)}{4r^2} - 2E_{n=0}
\]

where \( n = 0, 1, 2, \ldots \) is the radial quantum number. The relations between the parameters in (10) satisfy the supersymmetric definitions

\[
a_1 = \frac{M - 1}{2}, \quad a_2 = \frac{2A_1}{M - 1}, \quad a_3 = -\frac{A_5}{\sqrt{2A_6}}, \quad a_4 = -\sqrt{2A_6}
\]

Note that in order to retain the well-behaved solution at \( r \to \infty \) we have chosen the negative sign in \( a_4 \). The potential in (6) admits the exact solutions

\[
\Psi_{n=0}(r) = N_0 r^{a_1} \exp\left( a_2 r + \frac{a_3}{2} r^2 + \frac{a_4}{3} r^3 \right)
\]

where \( N_0 \) is the normalization constant, with the physically acceptable eigenvalues

\[
E_{n=0} = A_2 - \frac{1}{2} \left[ \frac{4A_1^2}{(M - 1)^2} - \frac{A_5}{\sqrt{2A_6} M} \right]
\]
under the constraints
\[
A_1 = - (M - 1) \frac{8A_6 A_4 - 2A_5^2}{16A_6 \sqrt{2A_6}}, \quad A_3 = - \sqrt{2A_6} \left[ \frac{(M + 1)}{2} + \frac{A_1}{(M - 1)A_6} \right],
\]
from which one arrives at the uniquely determined values of \( M \approx 5 \) and \( \delta \approx 0.28 \) in case of using atomic units in (6). The results obtained here are the generalization of the work in Ref. [25].

For the anharmonic oscillator potential in (7), we set the corresponding superpotential
\[
W(\rho) = a \rho^5 + b \rho^3 + \frac{c}{\rho} + d\rho, \quad a < 0, \quad d < 0
\]
which leads to
\[
\Psi_{n=0}(\rho) = C_0 \rho^c \exp \left( \frac{a}{6} \rho^6 + \frac{b}{4} \rho^4 + \frac{d}{2} \rho^2 \right)
\]
with \( C_0 \) being the corresponding normalization constant. Note that \( \Psi(\rho) \) satisfies a differential equation analogous to Eq. (2) and is related to \( F(\rho) \) in Eq. (3) as \( \Psi(\rho) = \rho^{(N'-1)/2} F(\rho) \) like the relation between \( R(r) \) and \( \Psi(r) \). Identifying \( V_+(\rho) \) with the effective potential we arrive at an expression for the physically meaningful ground state eigenvalues of the anharmonic oscillator potential in arbitrary dimensions,
\[
\hat{E}_{n=0} = - \frac{d}{2} (2c + 1) = \frac{8A_6 A_4 - 2A_5^2}{16A_6 \sqrt{2A_6}} \frac{M'}{|E_{n=0}|^{1/2}}
\]
where \( M' = N' + 2L \), and the relations between the potential parameters satisfy the supersymmetric constraints
\[
a = \pm \sqrt{\frac{A_6}{8} \frac{1}{|E_{n=0}|^{3/2}}}, \quad b = \frac{A_5}{8a} \frac{1}{|E_{n=0}|^{5/2}}, \quad c = \frac{M' - 1}{2}, \quad d = \frac{1}{2a} \left( \frac{A_4}{2|E_{n=0}|^2} - b^2 \right)
\]
As we are dealing with a confined particle system, the negative values for \( a \) and \( d \) would of course be the right choice to ensure the well behaved nature of the wave function behaviour at infinity. Our results are in agreement with the literature existing for three-dimensions [32, 33] (in case \( N' = 3 \)) and for two-dimensions [34] (in case \( N' = 2, L \rightarrow L - 1/2 \)).

In sum, we have shown that SSQM yields exact solutions for a single state only for the underlying quasi-exactly solvable potentials in higher dimensions with some constraints on the coupling constants. These constraints differ from each eigenvalue, and hence various solutions do not correspond to the same potential and are not orthogonal. We have not found these solutions discussed in the literature.
2.2 Significance of the mapping parameter

To show explicitly the physics behind the transformation described above, and to make clear the significance of the mapping parameter $\lambda$, we consider Eqs. (13) and (17) together within the same frame and arrive at

$$\hat{E}_{n=0} = -\frac{M'}{M - 1} \frac{A_1}{|E_{n=0}|^{1/2}}$$

(19)

To be consistent with Eq. (8) we must impose $0 \leq \lambda \leq 1$ as an integer, such that

$$\frac{M'}{M - 1} = \frac{2(N - 1 - \lambda) + 2(2\ell + \lambda)}{N + 2\ell - 1} = 2$$

(20)

It is a general feature of this map that, in both cases ($\lambda = 0, 1$), the spectrum of the $N$-dimensional screened Coulomb problem is related to half the spectrum of the $N'$-dimensional anharmonic oscillator for any even integer $N'$. The reader is referred to [7] for a comprehensive discussion of similar conformal mappings in the language of physics.

It is worthwhile at this stage to note that recently Chaudhuri and Mondal [10] studied the relations between anharmonic oscillators and perturbed Coulomb potentials in higher dimensions but their results correspond only to the case when $\lambda = 1$, in this case the three-dimensional perturbed Coulomb problem and the four-dimensional anharmonic oscillator cannot be related. However, by introducing an extra degree of freedom for the mapping parameter ($\lambda = 0$) through our equations, we can reproduce the well-known results found in the literature in three-dimensions. With this exact correspondence we can check Eq. (8), using exact results for the screened Coulomb potential, and calculated numerical results for the anharmonic oscillator potential.

3 Results and Discussion

In this section numerical applications of the transformation presented in the previous section are given. Calculations to check the validity of the equations developed for the screened Coulomb and anharmonic potentials are also given here. Table I displays the exact eigenvalues of the screened Coulomb potential in three-, and five-dimensions obtained using the Lagrange-mesh calculation technique [18, 19] for selected values of the potential parameters. Highly accurate Lagrange-mesh calculation results agree well with the best existing numerical and theoretical values obtained in three-dimensions [23, 24]. Due to the constraint in the potential parameter $A_1$ expressed in Eq. (14), we are not able to show in the same table the corresponding exact energy values which can be calculated by Eq. (13). For the work of interest in this paper we set $A_1 = -1$, consequently the adequate $\delta$-values satisfying the condition in Eq. (14) fall outside the scope of the presented work which has been performed for only low screening parameters.
Further, our calculation results shown in Table I make clear that the eigenvalues of the five-dimensional screened Coulomb problem with any angular momentum quantum number $\ell$, for a particular $\delta$ value, are equal to the same system with $\ell + 1$ in three-dimensions, due to $M = N + 2\ell$ which remains unaltered for these states. The tabulated results support the work of Imbo and Sukhatme [35] in which they formulated SSQM for spherically symmetric potentials in $N$ spatial dimensions and showed that the supersymmetric partner of a given potential can be effectively treated as being $N + 2$ ($\ell \rightarrow \ell + 1$) dimensions. This fact was exploited in their calculations using the shifted $1/N$ expansion.

It is also noted that for very small values of the screening parameter, the screening Coulomb potential reduces to the Coulomb potential that is shape invariant having supersymmetric character. Therefore, the related supersymmetric partner potentials, such as $V_\ell$ and $V_{\ell+1}$, are expected to have the same spectra except the ground-state energy. This can easily be seen in Table I for the case of $\delta = 0.001$ in both arbitrary dimensions. For instance, the supersymmetric partner of the $s$-orbital ($\ell = 0$) spectrum of hydrogen is the $p$-orbital ($\ell = 1$) spectrum of the same system.

Finally, the exact calculated eigenvalues, by the use of Eq. (8), for the anharmonic oscillator in four-dimensions from the known exact results for the screened Coulomb problem in three-dimensions are displayed in Table II. These eigenvalues are checked by the Lagrange-mesh calculations and tabulated in the same table. For low lying states, the results obtained with the present technique agree well with the numerical calculations, but this agreement deteriorates quickly for higher lying states.

4 Concluding Remarks

The mapping problems in arbitrary dimensions have been the subject of several papers and have served to illustrate various aspects of quantum mechanics of considerable pedagogical value. As the objective of this presentation we have highlighted a different facet of these studies and established a very general connection between the screened Coulomb and anharmonic oscillator potentials in higher dimensional space through the application of a suitable transformation, the purpose being the emphasize the pedagogical value residing in this interrelationship between two of the most practical applications of quantum mechanics. The exact ground state solutions for the potentials considered are obtained analytically within the framework of supersymmetric quantum mechanics, which provides a testing ground for benchmark calculations. Although much work had been done in the literature on similar problems, an investigation as the one we have discussed in this paper was missing to our knowledge.
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References

[1] V. Singh, S. N. Biswas, K. Dutta, Phys. Rev. D 18,1901 (1978) ; G. P. Flessas, Phys Lett. A 72,289 (1979) ; M. Razavy, Am. J. Phys. 48,285 (1980) ; A. Khare, Phys. Lett. A 83,237 (1981).

[2] A. V. Turbiner, A. G. Ushveridze, Phys. Lett. A 126,181 (1987) ; A. Turbiner, Sov. Phys. JETP 67,230 (1988) ; A. Turbiner, Commun. Math. Phys. 118,467 (1988) ; M. A. Shifman, Int. Jour. Mod. Phys. A 4,2897 (1989) ; P. Roy, Y. P. Varshni, Mod. Phys. Lett. A 6,1257 (1991) ; A. V. Turbiner, Phys. Lett B 276,95 (1992) ; M. Taler and A. V. Turbiner, J. Phys. A 26,697 (1993) ; A. Gangopadhyaya et.al., Phys. Lett. A 208,261 (1995) ; V. V. Ulyanov et al., Fiz. Nizk. Temp. 23,110 (1997).

[3] A. De Souza Dutra, Phys. Rev. A 47,R2435 (1993) ; N. Nag, R. Roychoudhury, Y. P. Varshni, Phys. Rev. A 49,5098 (1994) ; R. Dutt et al., J. Phys. A 28,L107 (1995) ; G. Junker and P. Roy, Phys. Lett A 232,155 (1997).

[4] G. A. Natanzon, Theor. Math. Phys. 38,146 (1979).

[5] A. Gangopadhyaya, P. K. Panigrahi and U. P. Sukhatme, Helv. Phys. Acta 67,363 (1994).

[6] G. Junker, J. Phys. A 23, L881 (1990); R. De, R. Dutt and U. Sukhatme, J. Phys. A 25, L843 (1992).

[7] V. A. Kostelecky, M. M. Nieto, and D. R. Truax, Phys. Rev. D 32,2627 (1985); V. A. Kostelecky and N. Russell, J. Math. Phys. 37,2166 (1996).

[8] R. Haymaker and A. R. P. Rau, Am. J. Phys. 54,928 (1986).

[9] A. Krajewska, A. Ushveridze and Z. Walczak, Mod. Phys. Lett. A 12,1225 (1997).

[10] R. N. Chaudhuri and M. Mondal, Phys Rev. A 52,1850 (1995).

[11] D. A. Morales and Z. Parra-Meijas, Can. J. Phys. 77,863 (1999).
[12] B. Gönül, O. Özer, M. Koçak, D. Tutcu and Y. Cançelik, J. Phys. A 34, 8271 (2001).

[13] H. A. Mavromatis, J. Math. Phys. 39, 2592 (1998).

[14] C. Itzykson, and J. B. Zuber, Quantum Field Theory (McGraw-Hill, New York, 1980).

[15] C. Reid, J. Mol. Spectrosc. 36, 183 (1970).

[16] C. Kittel, Introduction to Solid State Physics (Wiley, New York, 1986).

[17] R.K. Pathria, Statistical Mechanics (Pergamon, Oxford, 1986).

[18] D. Baye, J. Phys. B 28, 4399 (1995).

[19] M. Hesse and D. Baye, J. Phys. B 32, 5605 (1999).

[20] F. Cooper, A. Khare and U. Sukhatme, Phys. Rep. 251, 267 (1995).

[21] E. Schrödinger, Proc. R. Irish Acad. Sec. A 46, 183 (1941); D. Bergmann and Y. Frishman, J. Math. Phys. 6, 1855 (1965); S. Chandrasekhar, Newton’s Principia for the common reader, Clarendon Press, Oxford, 1995; V. I. Arnol’d and V. A. Vasil’ev, Not. Am. Math. Soc. 36, 1148 (1989); A. K. Grant and J. L. Rosner, Am. J. Phys. 62, 310 (1994).

[22] C. Weisbuch, B. Vinter, Quantum Semiconductor Heterostructures (Academic Press, New York, 1993); P. Harrison, Quantum Wells, Wires and Dots (John Wiley and Sons, England, 2000).

[23] R. Dutt, A. Ray and P.P. Ray, Phys. Lett. A 83, 65 (1981).

[24] C.S. Lam and Y.P. Varshni, Phys. Rev. A 4, 1875 (1971).

[25] M. Znojil, J. Math. Chem. 26, 157 (1999).

[26] M. Alberg, L. Wilets, Phys. Lett. A 286, 7 (2001).

[27] D. J. Doren and D. R. Herschbach, Phys. Rev. A 34, 2665 (1986).

[28] A. Chatterjee, Phys. Rep. 186, 250 (1990).

[29] A. Khare and S. N. Behra, Pramana J. Phys. 14, 327 (1980).

[30] D. Emin and T. Holstein, Phys. Rev. Lett. 36, 323 (1976); Phys. Today 35, 34 (1982).

[31] S. Coleman, Aspects of Symmetry, selected Erice Lectures (Cambridge Univ. Press, Cambridge, 1988).
[32] R. S. Kaushal, Ann. Phys. (N.Y) 206,90 (1991).
[33] R. S. Kaushal and D. Parashar, Phys. Lett. A 170,335 (1992).
[34] S. Dong and Z. Ma, quant-ph/9901037.
[35] T. D. Imbo and U. P. Sukhatme, Phys. Rev. Lett. 54,2184 (1985).
Table 1: The first four eigenvalues of the screening Coulomb potential in Eq. (6) as a function of the screening parameter in atomic units.

| $\delta$ | $\ell$ | n=0     | n=1     | n=2     | n=3     |
|----------|--------|---------|---------|---------|---------|
|          |        |         |         |         |         |
| 0.001    | 0      | -0.499 000 | -0.124 003 | -0.054 562 | -0.030 262 |
| 0.005    | 0      | -0.495 019 | -0.120 074 | -0.050 720 | -0.026 537 |
| 0.005    | 0      | -0.490 075 | -0.115 293 | -0.046 199 | -0.022 356 |
| 0.020    | 0      | -0.480 296 | -0.106 148 | -0.038 020 | -0.015 377 |
| 0.025    | 0      | -0.475 461 | -0.101 776 | -0.034 329 | -0.012 495 |
| 0.001    | 1      | -0.124 002 | -0.054 561 | -0.030 261 | -0.019 018 |
| 0.005    | 1      | -0.120 062 | -0.050 708 | -0.026 526 | -0.015 428 |
| 0.005    | 1      | -0.115 245 | -0.046 153 | -0.022 313 | -0.011 622 |
| 0.020    | 1      | -0.105 963 | -0.037 852 | -0.015 232 | -0.005 891 |
| 0.025    | 1      | -0.101 492 | -0.034 079 | -0.012 287 | -0.003 770 |
| 0.001    | 2      | -0.054 561 | -0.030 260 | -0.019 017 | -0.012 914 |
| 0.005    | 2      | -0.050 684 | -0.026 503 | -0.015 406 | -0.009 474 |
| 0.005    | 2      | -0.046 061 | -0.022 228 | -0.011 543 | -0.006 070 |
| 0.020    | 2      | -0.030 259 | -0.019 016 | -0.012 912 | -0.009 237 |
| 0.025    | 2      | -0.026 468 | -0.015 373 | -0.009 443 | -0.005 961 |
| 0.010    | 2      | -0.015 245 | -0.005 561 | -0.003 526 | -0.001 428 |
| 0.020    | 2      | -0.005 061 | -0.002 228 | -0.001 543 | -0.005 377 |
| 0.025    | 2      | -0.001 491 | -0.000 263 | -0.000 524 | -0.000 885 |

In three-dimensional space

In five-dimensional space

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Table 2: Ground-state eigenvalues of the anharmonic potential in Eq. (7)

| $\delta$ | $\ell$ | L  | $|E_{n=0}|$ (taken from Table I) | $\hat{E}_{n=0}$ | $\hat{E}_{n=0}$ |
|---------|-------|----|---------------------------------|-----------------|-----------------|
| 0.001   | 0     | 0  | 0.499 000                       | 2.831 259       | 2.831 259       |
|         | 1     | 2  | 0.124 002                       | 5.679 579       | 5.679 573       |
|         | 2     | 4  | 0.054 561                       | 8.562 285       | 8.562 268       |
| 0.005   | 0     | 0  | 0.495 019                       | 2.842 624       | 2.842 622       |
|         | 1     | 2  | 0.120 062                       | 5.772 014       | 5.772 012       |
|         | 2     | 4  | 0.050 684                       | 8.883 704       | 8.883 714       |
| 0.010   | 0     | 0  | 0.490 075                       | 2.856 927       | 2.856 924       |
|         | 1     | 2  | 0.115 245                       | 5.891 401       | 5.891 406       |
|         | 2     | 4  | 0.046 061                       | 9.318 882       | 9.318 871       |
| 0.020   | 0     | 0  | 0.480 296                       | 2.885 862       | 2.885 862       |
|         | 1     | 2  | 0.105 963                       | 6.144 014       | 6.144 024       |
|         | 2     | 4  | 0.037 515                       | 10.325 883      | 10.325 891      |
| 0.025   | 0     | 0  | 0.475 461                       | 2.900 499       | 2.900 498       |
|         | 1     | 2  | 0.101 492                       | 6.277 884       | 6.277 896       |
|         | 2     | 4  | 0.033 573                       | 10.915 282      | 10.915 281      |