REALIZING HOMOLOGY CLASSES BY SYMPLECTIC SUBMANIFOLDS

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Abstract. In this note we prove that a positive multiple of each even-dimensional integral homology class of a compact symplectic manifold $(M^{2n}, \omega)$ can be represented as the difference of the fundamental classes of two symplectic submanifolds in $(M^{2n}, \omega)$. We also discuss the realizability of integral homology classes by symplectic surfaces in $(M^{2n}, \omega)$.

To my Teacher Anatoly Timofeevich Fomenko

Contents

1. Introduction. 1
2. Proof of the main result 3
3. Realizing homology classes by symplectic surfaces 7
Acknowledgement
References

1. Introduction.

In 1954 René Thom proved the following celebrated theorem, which relates the topological structure with the differentiable structure on compact smooth manifolds $M^m$.

Theorem 1.1. ([13] Theorem II.25). For each element $\alpha \in H_k(M^m, \mathbb{Z})$ of a compact differentiable manifold $M^m$ there exists a positive integer $N(k, m)$ such that the element $N(k, m) \cdot \alpha$ can be realized by a differentiable submanifold in $M^m$.

Thom’s theorem is optimal in the sense that we cannot replace the positive integer $N(k, m)$ by the number 1. Namely Thom showed that for each $k \geq 7$ there is a compact differentiable manifold $M^m$ and an element $\alpha \in H_k(M^m, \mathbb{Z})$ such that $\alpha$ cannot be realized by the fundamental class of a submanifold in $M^m$ [13].

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As an immediate consequence of Thom’s theorem \[1.1\] we obtain that the homology group \( H_*(M^m, \mathbb{Q}) \) is generated by the fundamental classes of differentiable submanifolds of \( M^m \).

In this note we prove the following weak version of Thom’s theorem for compact symplectic manifolds.

**Theorem 1.2.** (Main Theorem) Suppose that \((M^{2n}, \omega)\) is a compact symplectic manifold. Then for each element \( \alpha \in H^{2k}(M^{2n}, \mathbb{Z}) \), \( 1 \leq k \leq n \), there exists a positive number \( N(\alpha) \in \mathbb{N}^+ \) such that \( N(\alpha) \cdot \alpha = [S_{1}^{2k}] - [S_{2}^{2k}] \), where \( S_{1}^{2k} \) and \( S_{2}^{2k} \) are symplectic submanifolds in \((M^{2n}, \omega)\). Consequently, \( H^{2k}(M^{2n}, \mathbb{Q}) \) is generated by the fundamental classes of symplectic submanifolds in \((M^{2n}, \omega)\) for all \( 1 \leq k \leq n \).

A particular case of Theorem 1.2 is the following celebrated Theorem due to Donaldson. For an element \( \alpha \in H^*(M, \mathbb{Z}) \) denote by \( PD(\alpha) \) the Poincaré dual of \( \alpha \).

**Theorem 1.3.** (cf. \[4, Theorem 1\]) Let \((M^{2n}, \omega)\) be a compact integral symplectic manifold, i.e. the cohomology class \( [\omega] \) belongs to \( H^2(M^{2n}, \mathbb{Z}) \subset H^2(M^{2n}, \mathbb{R}) \). Then for each \( 1 \leq k \leq n \) there exists a number \( N_1(k) \in \mathbb{N}^+ \) such that \( N \cdot PD([\omega^{n-k}]) \) can be realized by a symplectic submanifold \( S_{1}^{2k} \) in \((M^{2n}, \omega)\), if \( N \geq N_1(k) \).

Actually, in his paper Donaldson stated his result \[4, Theorem 1\] as a particular case of Theorem 1.3 for \( k = n - 1 \) and using this result he derived the existence of symplectic submanifolds in other dimensions \[4, Corollary 6\]. Kaoru Ono noticed that using a simple argument we can easily obtain Theorem 1.3 for \( 1 \leq k \leq n - 2 \) as a consequence of Donaldson’s results in the cited paper \[4, Proposition 3, Theorem 5\]. We shall represent Ono’s argument in section 2. Theorem 1.3 also follows from a result by Auroux in \[1, Theorem 2\], see Proposition 2.4 below, whose proof generalizes Donaldson’s arguments.

We shall call a homology class \( \alpha \in H_{2k}(M, \mathbb{Z}) \) symplectic, if \( \alpha \) is the fundamental class of a symplectic submanifold in \( M^{2n} \). We shall call a homology class \( \alpha \in H_{2k}(M, \mathbb{Z}) \) formal symplectic, if the pairing \( \langle \alpha, \omega^k \rangle \) is positive.

**Conjecture 1.** For each formal symplectic class \( \alpha \in H_{2k}(M^{2n}, \mathbb{Z}) \) there exists a positive number \( N_1(\alpha) \in \mathbb{N}^+ \) such that \( N_1(\alpha) \cdot \alpha \) is symplectic.

We also have the following

**Theorem 1.4.** Let \((M^{2n}, \omega)\) be a compact symplectic manifold with \( 2n \geq 6 \).

a) If \( \omega|_{\pi_2(M^{2n})} \neq 0 \), then there exists a symplectic sphere in \((M^{2n}, \omega)\).

b) Conjecture 1 is true for \( k = 1 \).

The remainder of this paper is organized as follows. In section 2 we give a proof of our Main Theorem. In section 3 we give a proof of Theorem 1.4. We also discuss in section 3 related results concerning Conjecture 1 for \( k = 1 \).
This note is the final version of my preprint [9].

2. Proof of the main result

Clearly Theorem 1.2 is a consequence of the following

Theorem 2.1. Let \((M^{2n}, \omega)\) be a compact symplectic manifold and \(\alpha \in H^{2k}_{\mathbb{Z}}(M^{2n}, \mathbb{Z})\). Then there exist an integral symplectic form \(\bar{\omega}\) on \(M^{2n}\) and positive integral numbers \(N_2(\alpha)\) and \(N_3(\alpha)\) such that \([N_2(\alpha) \cdot \alpha + N_3(\alpha) \cdot PD(\bar{\omega}^{n-k})]\) and \(N_3(\alpha) \cdot PD(\bar{\omega}^{n-k})\) can be realized by some symplectic submanifolds \(S_2^{2k}\) and \(S_2^{2k}\) of \((M^{2n}, \omega)\).

The proof of Theorem 2.1 is divided in two steps. In the first step we prove a particular case of Theorem 2.1 for compact integral symplectic manifolds \((M^{2n}, \omega)\), i.e. we set \(\bar{\omega} := \omega\). Moreover, for this “integral” case the obtained symplectic submanifolds \(S_2^{2k}\) are approximately \(J\)-holomorphic for some compatible almost complex structure \(J\) on \((M^{2n}, \omega)\). In the second step, using a perturbation argument, for an arbitrary symplectic compact manifold \((M^{2n}, \omega)\) we shall find a rational symplectic form \(\bar{\omega}\) close enough to \(\omega\) such that the symplectic submanifolds \(S_2^{2k}\) of \((M^{2n}, \bar{\omega})\), whose existence has been established in the first step, are also approximately \(J\)-holomorphic, and hence they are symplectic submanifolds of \((M^{2n}, \omega)\).

The approximate (resp. asymptotic) \(J\)-holomorphy of symplectic submanifolds \(S_2^{2k}\) are expressed in terms of \(\eta\)-transversality and approximately (resp. asymptotically) holomorphic sections, whose zero sets are \(S_2^{2k}\). These notions have been first introduced by Donaldson for sections of a Hermitian line bundle over an almost complex manifold \((M^{2n}, J)\) supplied with a Riemannian metric \(g\) [4]. Then they have been extended later by Auroux for sections of a Hermitian vector bundle over an almost complex manifold \((M^{2n}, J)\) supplied with a Riemannian metric \(g\) [1], [2].

Definition 2.2. ([1, Definition 2], cf. [2, Definition 3.3], [4, Definition 17]). Given a constant \(\eta > 0\), we say that a section \(s\) of a vector bundle \(E\) carrying a metric and a connection \(\nabla\) over a Riemannian manifold \(M^{2n}\) is \(\eta\)-transverse to 0, if at every point \(x\) such that \(|s(x)| \leq \eta\) the covariant derivative \(\nabla s(x) : T_x M^{2n} \to E(x)\) is surjective and admits a right inverse of norm less than \(\eta^{-1}\). Furthermore, we say that a sequence of sections of \(E\) is uniformly transverse to 0 if there exists a fixed constant \(\eta\) such that all sections in the sequence are \(\eta\)-transverse to 0.

For a Hermitian connection \(\nabla\) on a Hermitian vector bundle \(E\) over an almost complex manifold \((M, J)\) let us denote by \(\bar{\partial}\) the \((0, 1)\)-part (the anti-complex linear part) of the covariant derivative \(\nabla\).

Definition 2.3. ([1, Definition 1], cf. [3, Definition 1], [2, Definition 3.1], cf. [4]). Let \((M, J, g)\) be an almost-complex manifold with a Riemannian
metric $g$ and let $s_k$ be a sequence of sections of a Hermitian vector bundle $E_k$ supplied with a Hermitian connection $\nabla_k$. A sequence of sections $s_k$ of $E_k$ is called asymptotically $J$-holomorphic with respect to the given connections, if the following bounds hold

$$|s_k| = O(1), \quad |\nabla_k s_k| = O(k^{1/2}), \quad |\bar{\partial}s_k| = o(1),$$

$$|\nabla_k \nabla_k s_k| = O(k), \quad |\nabla \bar{\partial}s_k| = O(k^{1/2}).$$

**Proof of Theorem 2.1.**

Step 1. Let $(M, \omega)$ be a compact integral symplectic manifold supplied with a compatible almost complex structure $J$ and the associated compatible metric $g_J$. Let $L$ be a complex line bundle over $(M^{2n}, \omega)$ such that $c_1(L) = [\omega]$. Set $L_k := L^{\otimes k}$. First we need the following

**Proposition 2.4.** ([1, Proposition 1, Theorem 2], cf. [2, Corollary 5.2]). Let $E$ be a fixed complex vector bundle over $(M^{2n}, \omega)$. Let $J$ be a compatible almost complex structure on $(M^{2n}, \omega)$ and $g_J$ the associated Riemannian metric. Then there exist asymptotically $J$-holomorphic sections $s_k$ of $E \otimes L_k$ which are uniformly transverse to 0 and whose zero sets $W_k := s_k^{-1}(0)$ are smooth symplectic submanifolds in $(M^{2n}, \omega)$. Furthermore, the submanifold $W_k$ are asymptotically $J$-holomorphic, i.e. $J(TW_k)$ is within $o(1)$ of $TW_k$.

Another important ingredient of of our proof is the following topological fact.

**Proposition 2.5.** ([10, Corollary 1.2]) For each homology class $\alpha \in H_{2k}(M^{2n}, \mathbb{Z})$ there exists a number $N_2(k, n) \in \mathbb{N}$ and a complex vector bundle $E^{n-k}$ over $M^{2n}$ such that

$$c_n-k(E^{n-k}) = N_2(k, n) \cdot PD(\alpha),$$

$$c_i(E^{n-k}) = 0, \text{ for all } 1 \leq i \leq n-k-1.$$

We shall combine Proposition 2.4 and Proposition 2.5 for the proof of Theorem 2.1. For this purpose we use the following well-known formula for the Chern classes of a tensor product of complex vector bundles, see e.g. [3, Appendix A.3]. Denote by $c_t(E^r)$ the Chern polynomial of a complex vector bundle $E^r$ over $M^{2n}$: $c_t(E^r) = c_0(E) + c_1(E)t + \cdots + c_r(E^r)t^r$. Using the Grothendieck splitting principle we write

$$c_t(E^r) = \Pi_{i=1}^r (1 + a_it).$$

Let $F^s$ be another complex vector bundle over $M$ with

$$c_t(F^s) = \Pi_{j=1}^s (1 + b_j t).$$

Then we have

$$c_t(E^r \otimes F^s) = \Pi_{i,j}(1 + (a_i + b_j)t).$$
Now let $E^{n-k}$ be a complex vector bundle in Proposition 2.5. It follows from (2.1) that the top Chern class $c_{n-k}(E^{n-k})$ of $E^{n-k}$ satisfies the following relation for any $N$

$$c_{n-k}(E^{n-k} \otimes L_N) = N_2(n-k,n) \cdot PD(\alpha) + N \cdot [\omega^{n-k}].$$

Now let $N_3(\alpha)$ be a sufficient large number such that there exists a section $s_{N_3(\alpha)}$ of $E^{n-k} \otimes L_{N_3(\alpha)}$ that satisfies the condition of Proposition 2.4. Taking into account (2.2) and Theorem 2.3 we obtain Theorem 2.1 for compact integral symplectic manifold $(M^{2n}, \omega)$, and thus complete Step 1.

**Step 2. Proof of Theorem 2.1 for a general compact symplectic $(M, \omega)$**

We need the following perturbation result.

**Proposition 2.6.** Suppose that $(M^{2n}, \omega)$ is a compact symplectic manifold with a compatible almost complex structure $J$ and the associated Riemannian metric $g_J$. Then there exist an integral symplectic form $\omega$ together with a compatible almost complex structure $\tilde{J}$ and the associated compatible Riemannian metric $g_{\tilde{J}}$ over $M^{2n}$ such that the following statement holds. Let $E^k$ be a Hermitian vector bundle over $(M, \omega, J)$ with a Hermitian connection $\nabla$ and $s_k$ be sections of $E^k$ which are uniformly transverse to 0 and which are asymptotically $J$-holomorphic. Then for sufficiently large $k$ the zero section $s_k^{-1}(0)$ is a symplectic submanifold of $(M^{2n}, \omega)$.

**Proof.** For any symplectic manifold $(M^{2n}, \omega)$ supplied with a compatible almost complex structure $J$ and the compatible metric $g_J$, we denote by $G^\omega_{2l}(M^{2n})$ the Grassmannian of $\omega$-symplectic $2l$-planes in $T_x M^{2n}$ and by $G^J_{2l}(M^{2n})$ the Grassmannian of $J$-invariant $2l$-planes in $T_x M^{2n}$. Clearly $G^\omega_{2l}(M^{2n})$ is an open neighborhood of $G^J_{2l}(M^{2n})$. Denote by $d_{g_J}(x)$ the metric on $\Lambda^{2l} T_x M^{2n} \supset Gr^\omega_{2l}(T_x M^{2n}) \supset Gr^J_{2l}(T_x M^{2n})$ that is induced by the Riemannian metric $g_J(x)$. The openness of $G^\omega_{2l}(M^{2n})$ implies that, if $M^{2n}$ is compact and $\omega$ and $J$ are given, there exists a positive number $\varepsilon > 0$ such that the following assertion holds for all $x \in M^{2n}$ and for all $V \in Gr^\omega_{2l}(T_x M^{2n})$

$$d_{g_J}(V^{2l}(x), G^J_{2l}(T_x M^{2n})) < \varepsilon \implies V^{2l} \in Gr^\omega_{2l}(T_x M^{2n}).$$

**Lemma 2.7.** For all $l \in [1, n-1]$ there exists a small number $\varepsilon$ such that the following assertion

$$d_{g_{J_1}}(V, G^J_{2l}(T_x M^{2n})) < (\varepsilon/2) \implies V \in Gr^\omega_{2l}(T_x M)$$

holds for all $x \in M^{2n}$ for all $V \in Gr^J_{2l}(T_x M^{2n})$ and for all almost complex structure $J_1$ on $M^{2n}$ that satisfies the following condition. $J_1$ is compatible with some symplectic structure $\omega_1$ such that

$$|\omega - \omega_1|_{g_J} + |J - J_1|_{g_J} < \frac{\varepsilon}{16}.$$
Proof. There exists a number $0 < \varepsilon < 1/2$ such if (2.5) holds then

\[ d_{g_{1}}(V, Gr^{J}_{2}(T_{x}M^{2n})) < \frac{\varepsilon}{2} \implies d_{g_{1}}(V, Gr^{J}_{2}(T_{x}M^{2n})) < \frac{\varepsilon + \varepsilon}{8} \]

for all $x \in M$ and for all $V \in Gr_{2}(T_{x}M^{2n})$. Taking into account (2.5), and shrinking $\varepsilon$ if necessary, we obtain from (2.6)

\[ d_{g_{1}}(V, Gr^{J}_{2}(T_{x}M^{2n})) < \frac{3\varepsilon}{4} + \frac{3\varepsilon}{16} \]

(2.7)

Taking into account (2.3) we obtain (2.4) immediately from (2.7). This completes the proof of Lemma 2.7. 

Now let us complete the proof of Proposition 2.6. Let $\varepsilon$ be a small positive number in Lemma 2.7. Let $\omega_{1}$ a rational symplectic form on $(M^{2n}, \omega)$ such that there exists a compatible almost complex structure $J_{1}$ for which the condition (2.5) holds. Suppose that $\tilde{\omega}$ is a multiple of $\omega_{1}$ such that $[\tilde{\omega}] \in H^{2}(M, \mathbb{Z})$. Then the compatible almost complex structure $J_{1}$ is also compatible to $\tilde{\omega}$. Now we set $\tilde{J} := J_{1}$. Next we note that the metrics $g_{1}$ and $\tilde{g}$ associated respectively to $(\omega_{1}, J_{1})$ and $(\tilde{\omega}, \tilde{J})$ induce the same metric $d_{g_{1}}$ on each $G_{2}(T_{x}M)$. Furthermore, any $\omega_{1}$-symplectic plane (resp. $J_{1}$-invariant plane) is also $\tilde{\omega}$-symplectic (resp. $\tilde{J}$-invariant plane) and the converse also holds.

Now let $s_{k}$ be sections of a complex vector bundle $E^{k}$ supplied with a Hermitian connection as in Proposition 2.6 with respect to $\tilde{J}$. By Proposition 2.4, for sufficiently large $k$, the zero sections $W_{k} := s_{-1}^{-1}(0)$ are asymptotically $\tilde{J}$-holomorphic. Lemma 2.7 implies that $W_{k}$ are symplectic submanifolds of $(M^{2n}, \omega)$ and of $(M, \tilde{\omega})$ for sufficiently large $k$. This completes the proof of Proposition 2.6.

Completion of the proof of Theorem 2.1. Now given a compact symplectic manifold $(M, \omega)$ with a compatible almost complex structure $J$ and $\alpha \in H_{2}(M, \mathbb{Z})$ we shall choose $\tilde{\omega}$ and $\tilde{J}$ as in Proposition 2.6. Denote by $L_{\tilde{\omega}}$ the vector line bundle with $c_{1}(L) = \tilde{\omega}$. Let $E^{n-k}$ be a complex vector bundle satisfies the condition of Proposition 2.5. We set $E_{p}^{n-k} := E^{n-k} \otimes L_{\omega}^{\otimes p}$. Let $s_{p}$ be sequence of asymptotically $\tilde{J}$-holomorphic sections of $E_{p}^{n-k}$ whose existence is ensured by Proposition 2.3. It follows from Proposition 2.6 that the zero sections $s_{-1}^{-1}(0)$ are symplectic submanifolds of $(M^{2n}, \omega)$, for sufficiently large $p$. Taking into account (2.2) this completes the proof of Theorem 2.1.

The remainder of this section is devoted to a simple proof of Theorem 1.3 due to Kaoru Ono.

An elementary proof of Theorem 1.3 for $1 \leq k \leq n - 2$. [12]. We note that Theorem 1.3 is a consequence of the following

**Proposition 2.8.** There are positive integral numbers $n_{1}, \ldots, n_{k}$ and for each $i = 1, k$ a section $s_{i}$ of the line bundle $L_{\omega}^{N_{i}}$ such that the section $\hat{s} :=
s_1 \oplus \cdots \oplus s_k \text{ intersects to the zero section of } \hat{L} := L_\omega^{N_1} \oplus \cdots \oplus L_\omega^{N_k} \text{ transversally and } \hat{s}^{-1}(0) \text{ is a symplectic submanifold in } (M^{2n}, \omega), \text{ if } N_i \geq n_i \text{ for } 1 \leq i \leq k.

\textbf{Proof.} We prove Proposition 2.8 inductively on } k. \text{ For } k = 1 \text{ the statement is exactly Theorem 1 in [4]. Assume that the above statement is valid for } k = K. \text{ We shall prove its validity for } k = K + 1. \text{ We denote by } S_K \text{ the common zero locus of sections } s_1, \cdots, s_K \text{ which is by the induction assumption a symplectic submanifold. We note that the restriction of the line bundle } L_\omega \text{ to } S_K \text{ has the curvature } \omega|_{S_K} \text{ which is the symplectic form on } S_K. \text{ According to Donaldson [4, Theorem 5 and Proposition 3] there exists a number } N_0 \text{ such that if } n_{K+1} > N_0 \text{ there is a section } \tilde{s}_{K+1} : S_K \to L_\omega^{n_{K+1}} \text{ such that } \tilde{s}_{K+1} \text{ intersects with the zero section of } L_\omega^{n_{K+1}} \text{ transversally. Moreover the zero section } N := \tilde{s}_{K+1}^{-1}(0) \text{ is a symplectic submanifold in } S_K. \text{ Since the fiber } L_\omega^{n_{K+1}} \text{ is contractible, the section } \tilde{s}_{K+1} \text{ can be extended to a section } s_{K+1} : M^{2n} \to L_\omega^{n_{K+1}}. \text{ The proof of the above statement is complete, if we can show that } s_{K+1} \text{ can be chosen to be transversal to the zero section of a neighborhood } U_\varepsilon(S_K) \subset M^{2n}. \text{ It is easy to see that the extension of the section from the submanifold to the ambient space is just done by pull back the section by the projection of the normal bundle of the symplectic submanifold. More explicitly, we choose } U_\varepsilon(S_K) \text{ to be a (geodesic) neighborhood of } S_K \text{ in } M^{2n}, \text{ such that there is a diffeomorphism } f \text{ from } U_\varepsilon(S_K) \text{ to a neighborhood } V_\varepsilon(S_K) \text{ of the zero section of the normal bundle } V(S_K) \text{ of } S_K \text{ in } M. \text{ By using this diffeomorphism } f \text{ we can work now on } V_\varepsilon(S_K). \text{ It is easy to see that } V(S_K) = (L_\omega^{n_1} \oplus \cdots \oplus L_\omega^{n_K})|_{S_K}. \text{ Then we let the extension of } \tilde{s}_{K+1} : S_K \to L_\omega^{n_{K+1}} \text{ to a section } s_{K+1} : V_\varepsilon(S_K) \to (f^{-1})^* L_\omega^{n_{K+1}} \text{ be defined by } s_{K+1}(x, l_1, \cdots, l_K) = (x, l_1, \cdots, l_K, \tilde{s}_{K+1}^{-1}(0)). \text{ Finally we extend } s_{K+1} \text{ to a section } s_{K+1} \text{ over } M^{2n}. \text{ This section } s_{K+1} \text{ satisfies the condition of Proposition 2.8.} \text{ This completes the proof of Theorem 1.3.} \text{ We observe that Proposition 2.8 is also a consequence of Proposition 2.4.} \text{ \hfill \Box}

3. \textbf{Realizing homology classes by symplectic surfaces}

In this section we give a proof of Theorem 1.4 and discuss results related to Conjecture 1. \textbf{Proof of Theorem 1.4.} To prove the first assertion of Theorem 1.4 we need the following
Lemma 3.1. Suppose that the condition of Theorem 1.4.a is satisfied. Let us denote by \(\omega_0\) the standard symplectic form on \(S^2\). Then there is an embedding \(f : S^2 \to M^{2n}\) such that \(f^*([\omega]) = k[\omega_0]\) for some \(k > 0\).

Proof. We provide \(M^{2n}\) with a Riemannian metric \(g\). We claim that the required embedding can be obtained by a perturbation of a minimal immersion of \(S^2 \to (M^{2n}, g)\) whose existence follows from the following theorem due to Sacks and Uhlenbeck

**Proposition 3.2.** ([14, Theorem 5.9]) There exists a set of free homotopy classes \(\Lambda_i \in \pi_0 C^0(S^2, M^{2n})\) such that elements \(\lambda \in \Lambda_i\) generate \(\pi_2 (M^{2n})\) and each \(\Lambda_i\) contains a conformal branch immersion of a sphere having least area among maps of \(S^2\) into \(N\) which lies in \(\Lambda_i\).

Since \(\omega|_{\pi_2(M^{2n})} \neq 0\) it follows that at least one of the minimal immersions \(f\) obtained by Proposition 3.2 satisfies the condition \(f^*([\omega]) = k[\omega]\) for some \(k > 0\) (we can change the orientation of the map \(f\) if \(k < 0\)). Since \(f\) is a branch minimal immersion having least area, it has only isolated singular points. Now, using isotopy and dimension condition \(2n \geq 6\) we perturb \(f\) slightly to obtain the required embedding.

Continuation of the proof of Theorem 1.4.a First we find a map covering \(F : T^*S^2 \to T^*M^{2n}\) of \(f\) which is fiber-wise symplectic. In other words we find a section of the bundle \(\text{Iso}_{\text{sym}}(T^*S^2, T^*M^{2n})\) over \(S^2\) whose fiber is \(\text{Symp}(2m)/\text{Symp}(2m - 2)\). The fiber of this bundle which is \(2m - 1\) connected [6]. So there is no obstruction for such a section. Now Theorem 1.4 follows immediately from the Gromov h-principle, stated below, and from the observation, that we can make a \(C^1\)-perturbation of a symplectic immersion of \(S^2\) to get a symplectic embedding, since the dimension of \(M^{2n}\) is at least \(6\).

**H-principle for symplectic immersions** [6, 3.4.2.A] Let \(\omega_0\) be an arbitrary (possibly singular) closed smooth 2-form on a smooth manifold \(V\) and let \(F_0 : (T^*V, \omega_0) \to (T^*M, \omega)\) be a fiberwise injective isometric homomorphism. Let us denote by \(i\) the embedding \(V \to T^*V\) as the zero section, and by \(\pi\) the projection \(T^*M \to M\). Assume that \((\pi \circ F \circ i)^*[\omega] = [\omega_0]\). If \(\dim V < \dim M\), then the map \(\pi \circ F \circ i : V \to M^{2n}\) admits a fine \(C^0\)-approximation by isometric smooth immersions \(f : (V, \omega_0) \to (M^{2n}, \omega)\) whose differentials \(Df : T_VV \to T_XM\) are homotopic to \(F_0\) in the space of fiberwise injective isometric homomorphisms.

This completes the proof of Theorem 1.4.a.

To prove the second assertion of Theorem 1.4 we use Theorem 1.1 due to Thom instead of Lemma 3.1. The rest of the argument of the proof repeats those in the proof of the first assertion, so we omit it here. This completes the proof of Theorem 1.4. □

In [11] Li gives a proof of the following Theorem, which strengthens the second assertion of Theorem 1.4.
Proposition 3.3. ([11, Theorem 1]) Suppose $(M^{2n}, \omega)$ is a symplectic manifold of dimension $2n$. Let $A$ be any $\omega$-positive class in $H_2(M; \mathbb{Z})$, i.e. $A$ is formal symplectic. Then

1. $A$ is represented by a connected embedded $\omega$-symplectic surface if $2n \geq 6$.
2. $A$ is represented by a connected immersed $\omega$-symplectic surface if $2n = 4$.

Recently in [7] Hamilton disproved Conjecture 1 for dimension $2n = 4$ by presenting an obstruction for presenting a homology class of a symplectic 4-manifold by an embedded symplectic surface.

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