Regularity and stability results for the level set flow via the mean curvature flow with surgery

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Abstract. In this article we use the mean curvature flow with surgery to derive regularity estimates going past Brakke regularity for the level set flow. We also show a stability result for the plane under the level set flow.

1. Introduction.

The mean curvature flow is the gradient flow of the area functional and so, in principle, from a given submanifold should flow to a minimal surface. Of course, in general, the mean curvature flow develops singularities. In response “weak solutions” of the mean curvature flow (such as the Brakke flow [4], and level set flow [13, 14], and [22]) have been developed.

One such approach is the mean curvature flow with surgery developed by Huisken, Sinestrari [21] (and Brendle and Huisken [6] for the surface case) and later Haslhofer and Kleiner in [16]). The mean curvature flow with surgery “cuts” the manifold into pieces with very well understood geometry and topology and for this and the explicit nature of the flow with surgery is particularly easy to understand (and makes it a useful tool to understand the topology of the space of applicable hypersurfaces; see [7] or [27]). To be able to do this however unfortunately boils down eventually to understanding the nature of the singularities very well and establishing certain quite strong estimates, and all this has only been carried out (in $\mathbb{R}^{n+1}$ at least) for 2-convex compact hypersurfaces.

However the necessary estimates of Haslhofer and Kleiner as they mentioned in their paper are local in nature so can be expected to be localized in some cases. In this paper we study when this possible, using the pseudolocality estimates of Chen and Yin [10], and use the localized mean curvature flow with surgery to understand the level set flow - the localized mean curvature flow with surgery converges to the level set flow in a precise sense as the surgery parameters degenerate in correspondence
with the compact 2-convex case. Using this we show a regularity for the level set flow and a stability result for the plane under the level set flow, showing that the mean curvature flow can be fruitfully used to study the level set flow that as far as the author knows were previously unknown. The first theorem we show in this article is a very general short time existence theorem for a localized flow with surgery:

**Theorem 1.1. (Short time existence of flow with localized surgery)** Suppose $M$ is $\alpha$ noncollapsed and $\beta$ 2-convex in a bounded region $\Omega$, and it can be guaranteed no singularities outside of $\Omega$ occur along the flow on $[0, \delta]$. Then there exists $\epsilon > 0$, $\epsilon \leq \delta$, so that $M$ has a flow with surgery and is $\bar{\alpha} < \alpha$, $\bar{\beta} < \beta$ controlled on $[0, \epsilon]$.

The content of the above is that its possible to control degeneracy of the non-collapsing conditions at least for a short time, using pseudolocality. In particular it will be clear that it doesn’t matter so much how large the curvature of $M$ is far within the interior of $\Omega$. It was pointed out by Laurer in [23] and independently Head in [18] that, for Huisken and Sinestrari’s definition of the mean curvature flow with surgery, as the surgery parameters are allowed to degenerate the corresponding flows with surgery Hausdorff converge to the level set flow as defined by Illmanen in [22]. Important for the next result and as justification of the definition of localized flow with surgery we give we extend his methods to show:

**Theorem 1.2. (Convergence to level set flow)** Given $M$ if there exists flows with surgery $(M_i)_i$ coming out of $M$ on $[0, T]$ where the surgery parameter $H_{th} \to \infty$ as $i \to \infty$, then the $(M_i)_i$ as sets in $\mathbb{R}^{n+1} \times [0, T]$ Hausdorff converge subsequentially to the level set flow $L_t$ of $M$ on $[0, T]$.

The meaning of the surgery parameter $H_{th}$ will be described in the next section. We point out here that to overcome a technical hurdle in using Laurer’s method we use ideas from the recent paper of Hershkovits and White [19] - name we use a result of their’s that for us gives a way to “localize” the level set flow. We will mainly be interested though in such hypersurfaces with surgery satisfying additional assumptions that essentially control in a precise sense how far $M$ deviates from a plane $P$:

**Definition 1.1.** We will say $M$ is $(V, h, \epsilon)$ controlled above a plane $P$ in a bounded region $\Omega \subset \mathbb{R}^{n+1}$ when

1. there exists $0 < V \leq \infty$ so that the measure of points bounded initially bounded by $P$ and $M$ is less than $V$.
2. The supremum of the height of $M$ over $P$ is bounded by $0 < h \leq \infty$.
3. $M \cap \Omega$ lies above the plane $P$.
4. In the $R$-collar neighborhood $C_R$ of $\partial \Omega$, $M$ is graphical over $P$ with $C^2$ norm bounded by $\epsilon$. 

The definition above is a bit obtuse but roughly the volume discrepancy $V$ and $h$ bound how bulky $M$ is over $P$. By $M$ lying above $P$ we mean that $M$ lies on one side of $P$ and where $M$ is graphical over $P$ its outward normal points away from $P$ (equivalently, thinking of $M$ as the boundary of a domain $K$ so that the outward normal of $M$ is pointing outside $K$, the halfspace bounded by $P$ disjoint from $M$ lies in $K$). The statement about the $R$-collar neighborhood of $\partial \Omega$ is for an eventual use of the Brakke regularity theorem and ensures the edges of $M$ don’t “curl up” much, see below:

With this definition in hand, let’s define the sets our regularity and stability theorems concern. The first one corresponds to the regularity result:

**Definition 1.2.** The set $\Sigma = \Sigma(\{\alpha, \beta\}, \{V, h, \epsilon, P, R, \Omega\})$ is the set of hypersurfaces $M^n \subset \mathbb{R}^{n+1}$ satisfying:

1. locally $\alpha$-controlled: $M \cap \Omega$ is $\alpha$-controlled in its interior.
2. locally $\beta$-two convex in $M \cap \Omega$ $(\lambda_1 + \lambda_2) > H\beta$.
3. $M$ is $(V, h, R, \epsilon)$ controlled over the plane $P$ in the region $\Omega$.

Note we only assume control on $\alpha$ and $\beta$ but not an initial mean curvature bound $\gamma$ (referring to the definition of an $\alpha$-controlled domain for the surgery in Haslhofer and Kleiner’s definition, see below). With our notation and sets defined we finally state our convergence theorem; the proof crucially uses the mean curvature flow with surgery to easily get a good estimate on the height of the level set flow after a short time.

**Theorem 1.3.** *(High density Brakke theorem for LSF)* There exists $C$ so that if $M^n$ has a flow with surgery (independent of parameters) up to time $T$ there are choices of $(V, R, \epsilon, h)$, finite, so that if $M \in \Sigma(V, h, R, \epsilon)$ over some plane $P$ in $\Omega$ then the level set flow of $M$ is a graph over $P$ with curvature bounded by $C$ by time $T$.

The choice of constants $(V, R, \epsilon, h)$ depend on $\bar{\alpha}$ and $\bar{\beta}$ found in the existence theorem 1.1. $\bar{\alpha}$ and $\bar{\beta}$ in turn depend on $R$ and $\epsilon$, but as far as satisfying the theorem goes its suffices to fix $R$ and $\epsilon$ sufficiently and then choose $V, h$. One sees from the proof below that $V$ and $h$ are also related and that, for a given $h > 0$, $V$
can be taken sufficiently small to make the conclusion of the theorem hold. Note that this theorem is an improvement on just Brakke regularity for the level set flow of $L_t$ of $M$ because we make no assumptions on the densities in a parabolic ball; indeed the hypotheses allow singularities for the LSF to develop in the regions of space-time we are considering, at which points the density will be relatively large (hence the name of the proof). For a recent improvement on the Brakke regularity theorem in another, more general, direction, see the recent work of Lahiri [24].

We will show without much work using a general construction of Buzano, Haslhofer, and Hershkovits (theorem 4.1 in [7]) how to construct “nontrivial” (i.e. non-graphical, singularity forming) hypersurfaces that satisfy the assumptions of Theorem 1.3. Such examples can also be designed to have arbitrarily large area ratios initially in a ball of fixed radius.

This theorem is also interesting from a PDE viewpoint because the mean curvature flow is essentially a heat equation, and such result says, imagining high area ratio localized perturbations of a given hypersurface as high frequency modes of the initial condition of sorts, that in analogy to heat flow on a torus, the high frequency modes decay quickly in time. It’s interesting that our arguments though use pseudolocality strongly, which is a consequence of the nonlinearity of the flow and is false for the linear heat equation. More precisely:

**Corollary 1.4. (Rapid smoothing)** Let $M$ be a smooth hypersurface with a flow on $[0, T_1]$ perturbed as above over planes $P_i$ to get a hypersurface $\tilde{M}$. If the perturbations make the conclusions of the theorem above hold for $T < T_1$ then by time $T$, $\tilde{M}_T$ has bounded curvature.

Of course taking $T$ small enough (depending on the curvature of $\tilde{M}$ away from the perturbations) one can easily see that $\tilde{M}_T$ is close at least in Hausdorff distance to $M$.

To state the next corollary we define a refinement of the set $\Sigma$ above, which concerns the case when $M$ is asymptotically planar with prescribed curvature decay:

**Definition 1.3.** The set $\Sigma_1 = \Sigma_1(\{\alpha, \beta\}, \{V, h, \epsilon, P, R, \Omega\}, f)$ is the set of hypersurfaces $M^n \subset \mathbb{R}^{n+1}$ satisfying, in addition to the set of conditions given in the definition of $\Sigma$, are asymptotically planar in that:

In $M \cap \Omega^c$ is a graph of a function $F$ over a plane $P$. Furthermore writing $F$ in polar coordinates we have $||F(r, \theta)||_{C^2} < f(r)$, where $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies

$$\lim_{r \to 0} f(r) = 0.$$

Via the interior estimates of Ecker and Huisken we have that:

**Corollary 1.5. (Long term flow to plane)** With $M^n \in \Sigma_1$ asymptotically satisfying the assumptions above in theorem 1.3 in some region $\Omega$ above the origin
then as $t \to \infty$ its level set flow $L_t$ will be smooth after time $T$ and converge smoothly to the corresponding plane $P$ as $t \to \infty$.

Acknowledgements: The author thanks his advisor, Richard Schoen, for his advice and patience.

2. Background on the Mean Curvature Flow (With Surgery).

The first subsection introducing the mean curvature flow we borrow quite liberally from the author’s previous paper [27], although a couple additional comments are made concerning the flow of noncompact hypersurfaces. The second subsection concerns the mean curvature flow with surgery as defined by Haslhofer and Kleiner in [16] which differs from the original formulation of the flow with surgery by Huisken and Sinestrari in [16] (see also [6]). Namely the discussion there of surgery differs from the corresponding section in the author’s previously mentioned article.

2.1. Classical formulation of the mean curvature flow. In this subsection we start with the differential geometric, or “classical,” definition of mean curvature flow for smooth embedded hypersurfaces of $\mathbb{R}^{n+1}$; for a nice introduction, see [26]. Let $M$ be an $n$ dimensional manifold and let $F : M \to \mathbb{R}^{n+1}$ be an embedding of $M$ realizing it as a smooth closed hypersurface of Euclidean space - which by abuse of notation we also refer to $M$. Then the mean curvature flow of $M$ is given by $\hat{F} : M \times [0,T) \to \mathbb{R}^{n+1}$ satisfying (where $\nu$ is outward pointing normal and $H$ is the mean curvature):

$$\frac{d\hat{F}}{dt} = -H\nu, \quad \hat{F}(M,0) = F(M) \quad (2.1)$$

(It follows from the Jordan separation theorem that closed embedded hypersurfaces are oriented). Denote $\hat{F}(\cdot,t) = \hat{F}_t$, and further denote by $\mathcal{M}_t$ the image of $\hat{F}_t$ (so $M_0 = M$). It turns out that (2.1) is a degenerate parabolic system of equations so take some work to show short term existence (to see its degenerate, any tangential perturbation of $F$ is a mean curvature flow). More specifically, where $g$ is the induced metric on $\mathcal{M}$:

$$\Delta_g F = g^{ij}(\frac{\partial^2 F}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial F}{\partial x^k}) = g^{ij} h_{ij} \nu = H\nu \quad (2.2)$$

Now one could apply for example deTurck’s trick to reduce the problem to a non-degenerate parabolic PDE (see for example chapter 3 of [3]) or similarly reduce the problem to an easier PDE by writing $M$ as a graph over a reference manifold by Huisken and Polden (see [26]). At any rate, we have short term existence for compact manifolds.

For noncompact hypersurfaces $M$ in $\mathbb{R}^N$ with uniformly bounded second fundamental form (i.e. there is some $C > 0$ so that $|A|^2 < C$ at every point on $M$), one
may solve the mean curvature flow within \( B(0, R) \cap N \); by the uniform curvature bound there is \( \epsilon > 0 \) so that \( N \cap B(0, R) \) has a mean curvature flow on \([0, \epsilon]\). Then one may take a sequence \( R_i \to \infty \) and employ a diagonalization argument to obtain a mean curvature flow for \( M \); the flow of \( M \) we constructed is in fact unique by Chen and Yin in \([10]\) (we will in fact use estimates from that same paper below). Since all noncompact hypersurfaces of interest will be asymptotically planar, we will always have a a mean curvature flow of them for at least short time.

Now that we have established existence of the flow in cases important to us, let’s record associated evolution equations for some of the usual geometric quantities:

- \( \frac{\partial}{\partial t} g_{ij} = -2Hh_{ij} \)
- \( \frac{\partial}{\partial t} d\mu = -H^2d\mu \)
- \( \frac{\partial}{\partial t} h^i_j = \Delta h^i_j + |A|^2h^i_j \)
- \( \frac{\partial}{\partial t} H = \Delta H + |A|^2H \)
- \( \frac{\partial}{\partial t} |A|^2 = \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^4 \)

So, for example, from the heat equation for \( H \) one sees by the maximum principle that if \( H > 0 \) initially it remains so under the flow. There is also a more complicated tensor maximum principle by Hamilton originally developed for the Ricci flow (see \([15]\)) that says essentially that if \( M \) is a compact manifold one has the following evolution equation for a tensor \( S \):

\[
\frac{\partial S}{\partial t} = \Delta S + \Phi(S)
\]  

and if \( S \) belongs to a convex cone of tensors, then if solutions to the system of ODE

\[
\frac{\partial S}{\partial t} = \Phi(S)
\]

stay in that cone then solutions to the PDE (2.2) stay in the cone too (essentially this is because \( \Delta \) “averages”). So, for example, one can see then that convex surfaces stay convex under the flow very easily this way using the evolution equation above for the Weingarten operator. Similarly one can see that 2-convex hypersurface (i.e. for the two smallest principal curvatures \( \lambda_1, \lambda_2, \lambda_1 + \lambda_2 > 0 \) everywhere) remain 2-convex under the flow.

Another important curvature condition in this paper is \( \alpha \) non-collapsing: a mean convex hypersurface \( M \) is said to be 2-sided \( \alpha \) non-collapsed for some \( \alpha > 0 \) if at every point \( p \in M \), there is an interior and exterior ball of radius \( \alpha/H(p) \) touching \( M \) precisely at \( p \). This condition is used in the formulation of the finiteness theorem. It was shown by Ben Andrews in \([2]\) to be preserved under the flow for compact surfaces. (a sharp version of this statement, first shown by Brendle in \([5]\) and later
Haslhofer and Kleiner in [17], is important in [6] where MCF+surgery to $n = 2$ was first accomplished). Very recently it was also shown to be true for non-compact hypersurfaces by Cheng in [9].

Finally, perhaps the most geometric manifestation of the maximum principle is that if two compact hypersurfaces are disjoint initially they remain so under the flow. So, by putting a large hypersphere around $\mathcal{M}$ and noting under the mean curvature flow that such a sphere collapses to a point in finite time, the flow of $\mathcal{M}$ must not be defined past a certain time either in that as $t \to T$, $\mathcal{M}_t$ converge to a set that isn’t a manifold. Note this implies as $t \to T$ that $|A|^2 \to \infty$ at a sequence of points on $\mathcal{M}_t$; if not then we could use curvature bounds to attain a smooth limit $\mathcal{M}_T$ which we can then flow further, contradicting our choice of $T$. Of course this particular argument doesn’t work in the noncompact case but it is easy to see using the Angenant’s torus [1] as a barrier that singularities can occur along the flow of noncompact hypersurfaces as well:

Thus as in the compact case to use mean curvature flow to study noncompact hypersurfaces one is faced with finding a way to extend the flow through singularities. Thus weak solutions to the flow are necessitated. One such weak solution is the Brakke flow, developed in Brakke’s thesis [4], where a weak solution to the flow is defined in terms of varifolds. For this paper it suffices to say that the classical MCF and the LSF (defined below) are Brakke flows, and that if the density ratios of a Brakke flow are sufficiently close to 1 in a parabolic cylinder then the varifolds are actually smooth with bounded curvature within a certain time interval (this is Brakke’s regularity theorem) - in addition to Brakke’s thesis see [20] or [25]. Another type of weak solution which came much later is mean curvature flow with surgery:
2.2. Mean curvature flow with surgery for compact 2-convex hypersurfaces in $\mathbb{R}^{n+1}$. First we give the definition of $\alpha$ controlled:

**Definition 2.1.** (Definition 1.15 in [16]) Let $\alpha = (\alpha, \beta, \gamma) \in (0, N - 2) \times \left(0, \frac{1}{N-2}\right) \times (0, \infty)$. A smooth compact domain $K_0 \subset \mathbb{R}^N$ is called an $\alpha$-controlled initial condition if it satisfies the $\alpha$-Andrews condition and the inequalities $\lambda_1 + \lambda_2 \geq \beta H$ and $H \leq \gamma$.

Speaking very roughly, for the mean curvature flow with surgery approach of Haslhofer and Kleiner, like with the Huisken and Sinestrari approach there are three main constants, $H_{th} \leq H_{neck} \leq H_{trig}$. If $H_{trig}$ is reached somewhere during the mean curvature flow $M_t$ of a manifold $M$ it turns out the nearby regions will be “neck-like” and one can cut and glue in appropriate caps (maintaining 2-convexity, etc) so that after the surgery the result has mean curvature bounded by $H_{th}$. The high curvature regions have well understood geometry and are discarded and the mean curvature flow with surgery proceeds starting from the low curvature leftovers. Before stating a more precise statement we are forced to introduce a couple more definitions. First an abbreviated definition of the most general type of piecewise smooth flow we will consider:

**Definition 2.2.** (see Definition 1.3 in [16]) An $(\alpha, \delta)$-flow $K$ is a collection of finitely smooth $\alpha$-Andrews noncollapsed flows $\{K^i_t \cap U\}_{t \in [t_{i-1}, t_i]}$, ($i = 1, \ldots, k; t_0 < \ldots < t_k$) in a open set $U \subset \mathbb{R}^N$, such that:

1. For each $i = 1, \ldots, k - 1$, the final time slices of some collection of disjoint strong $\delta$-necks (see below) are replaced by standard caps, giving a domain $K^+_t \subset K^i_t =: K^i_t$.

2. The initial time slice of the next flow, $K^{i+1}_t := K^+_t$, is obtained from $K^+_t$ by discarding some connected components.

For the definition of standard caps and the cutting and pasting see definitions 2.2 and 2.4 in [16]; their name speaks for itself and the only important thing to note is that cutting and then pasting them in will preserve the $\alpha$-control parameters on the flow. We will however give the definition of $\delta$-strong neck; below $s$ is a scaling parameter that need not concern us:

**Definition 2.3.** (Definition 2.3 in [16]) We say than an $(\alpha, \delta)$-flow $K = \{K_t \subset U\}_{t \in I}$ has a strong $\delta$-neck with center $p$ and radius $s$ at time $t_0 \in I$, if $\{s^{-1}(K_{t_0 + s^2 t - p})\}_{t \in (-1, 0]}$ is $\delta$-close in $C^{[1/\delta]}$ in $B^U_{1/\delta} \times (-1, 0]$ to the evolution of a solid round cylinder $D^{N-1} \times \mathbb{R}$ with radius 1 at $t = 0$, where $B^U_{1/\delta} = s^{-1}((B(p, s/\delta) \cap U) - p) \subset B(0, 1/\delta) \subset \mathbb{R}^N$.

We finally state the main existence result of Haslhofer and Kleiner; see theorem 1.21 in [16].
Theorem 2.1. (Existence of mean curvature flow with surgery). There are constants $\delta = \delta(\alpha) > 0$ and $\Theta(\delta) = \Theta(\alpha, \delta) < \infty$ ($\delta \leq \delta$) with the following significance. If $\delta \leq \delta$ and $H = (H_{\text{trig}}, H_{\text{neck}}, H_{\text{th}})$ are positive numbers with $H_{\text{trig}}/H_{\text{neck}}, H_{\text{neck}}/h_{\text{th}}, H_{\text{neck}} \geq \Theta(\delta)$, then there exists an $(\alpha, \delta, H)$-flow $\{K_t\}_{t \in [0, \infty)}$ for every $\alpha$-controlled initial condition $K_0$.

The most important difference for us (as will be evident below) between Huisken and Sinestrari’s approach and Haslhofer and Kleiner’s approach is that Huisken and Sinestari estimates are global in nature whereas Haslhofer and Kleiner’s estimates are local.

3. Localizing the Mean Curvature Flow with Surgery.

Recall surgery is defined for two-convex compact hypersurfaces in $\mathbb{R}^{n+1}$. However, many of Halshofer and Kleiner’s estimates are local in nature and their mean curvature flow with surgery can be localized, as long as the high curvature regions (where $H > \frac{1}{2} H_{\text{th}}$ say) where surgery occurs are uniformly 2-convex (for a fixed choice of parameters).

The main technical point then to check in performing a “localized” mean curvature flow with surgery is ensuring that the regions where we want to perform surgeries are and remain for some time uniformly 2-convex in suitably large neighborhoods of where singularities occur. The key technical result to do so (at least for this approach) is the pseudolocality of the mean curvature flow. Pseudolocality essentially says that the mean curvature flow at a point, at least “short term” is essentially controlled by a neighborhood around that point and that points far away are essentially inconsequential - this is in contrast to the linear heat equation. It plays a crucial role in our arguments in this section (controlling the singular set) and in some arguments in the other sections. Recall the following (consequence of the) pseudolocality theorem for the mean curvature flow of Chen and Yin:

Theorem 3.1. (Theorem 7.5 in [10]) Let $M$ be an $n$-dimensional manifold satisfying $\sum_{i=0}^{3} |\nabla^2 Rm| \leq c_0^2$ and $\text{inj}(M) \geq i_0 > 0$. Then there is $\epsilon > 0$ with the following property. Suppose we have a smooth solution $M_t \subset M$ to the MCF properly embedded in $B_{M_t}(x_0, r_0)$ for $t \in [0, T]$ where $r_0 < i_0/2$, $0 < T \leq \epsilon^2 r_0^2$. We assume that at time zero, $x_0 \in M_0$, and the second fundamental form satisfies $|A|(x) \leq r_0^{-1}$ on $M_0 \cap B_{M_t}(x_0, r_0)$ and assume $M_0$ is graphic in the ball $B_{M_t}(x_0, r_0)$. Then we have $|A|(x, t) \leq (\epsilon r_0)^{-1}$ (3.1) for any $x \in B_{M_t}(x_0, \epsilon r_0) \cap M_t$, $t \in [0, T]$. 
Of course, when the ambient space is $\mathbb{R}^N$, we may take $i_0 = \infty$ and since it is flat we may take $c_0 = 0$. It will be important in many parts below. First, it help keep the degeneracy of 2-convexity at bay; below the MCF is normalized so it has no tangential component:

**Proposition 3.2.** Suppose that $\Omega \subset M$ is a region in which $M$ is $\alpha$-noncollapsed and $H > \epsilon$ on $\Omega$. Then there is $T > 0, \, \bar{\alpha} > 0, \, \text{and} \, \bar{\beta} > 0$ so that $\Omega$ is $\bar{\alpha}, \bar{\beta}$-noncollapsed. Furthermore $T$ depends only on $\|A\|^2$ of $M$ in a neighborhood of $\partial \Omega$.

Proof: Recall the evolution equation for $H$ under the flow, that $\frac{dH}{dt} = \Delta H + |A|^2 H$. One sees by the maximum principle if $H(x)$ is a local minimum then $\frac{dH}{dt}(x) \geq 0$. This tells us that regions where $H < 0$ can’t spontaneously form within mean convex regions, and in addition that for any $c$ if $\inf_{x \in \Omega} H(x) > c$ in $\Omega$ initially and $H(x) > c$ on $\partial \Omega$ on $[0, T]$ then $H > c$ on all of $\Omega$ on $[0, T]$.

Let us say that $x, y \in M$ are $\alpha$-noncollapsed with respect to each other if $H_x, H_y > 0$ and $y \notin B(x + \nu \frac{\alpha}{H(x)}, \bar{\alpha} H(x))$ and vice versa. We recall from Andrew’s proof [2] that provided $H > 0$, $x$ and $y$ in $M_t$ are $\alpha$-noncollapsed with respect to each other if the following quantity is positive:

$$Z(x, y, t) = \frac{H(x, t)}{2} ||X(y, t) - X(x, t)||^2 + \alpha \langle X(y, t) - X(x, t), \nu(x, t) \rangle$$

(3.2)

Of course $M_t$ is $\alpha$-noncollapsed in $\Omega_t$ if every pair of points in $\Omega_t$ is $\alpha$-noncollapsed with respect to each other. Andrews showed for closed mean convex hypersurfaces that $\alpha$-noncollapsing was preserved by the maximum principle. He calculated:

$$\frac{\partial Z}{\partial t} = \sum_{i,j=1}^n \left( g_{ij} \frac{\partial^2 Z}{\partial x^i \partial x^j} + g_{yi} \frac{\partial^2 Z}{\partial y^i \partial y^j} + 2g_{ij} \langle \partial_{\nu_k} \partial_{\nu_l}, \partial_{\nu_m} \partial_{\nu_n} \rangle \frac{\partial^2 Z}{\partial x^i \partial y^j} \right)$$

$$+ \left( |h^x|^2 + \frac{4H_x(H_x - \alpha h^x_{nn})}{\alpha^2} \langle w, \partial_{\nu_i} \rangle^2 \right) Z$$

(3.3)

By shrinking $\Omega$ slightly (this will always be fine in practice) without loss of generality the mean convexity and $\alpha$-noncollapseness extend intially up to the boundary $\partial \Omega$. From the maximum principle like above then we just have to provide a time interval in which the points on $\partial \Omega$ (as points of $M$) are mean convex and are $\bar{\alpha}$-noncollapsed, where $\bar{\alpha} < \alpha$ with respect to all the other points in $\Omega$.

By pseudolocality and the compactness of $\partial \Omega$, there will be $T_1 > 0$ so that $H > \epsilon/2$ on $\partial \Omega$ and in a suitable half collar neighborhood $V$ of $\partial \Omega$ interior to $\Omega$ (see the figure below) the curvature is bound on $[0, T_1]$ by say $C$ (potentially huge).
We see then there is $\alpha$, perhaps much smaller than $\alpha$, we may ensure that spheres osculating $\partial \Omega_t$ of radius $\frac{\alpha}{\epsilon}$ don’t touch points in $\Omega_t$ on $[0, T_1]$. Since if two points are $\alpha$ non collapsed with respect to each other and $\alpha < \alpha$ they are $\alpha$ non collapsed with respect to each other, we see that $\Omega_t$ is $\alpha$ non collapsed on $[0, T_1]$.

$\beta$-noncollapsedness is a pointwise inequality and that there is such a $\beta$ on some fixed time $[0, T_2]$, $T_2 \leq T_1$, follows by pseudolocality as with mean convexity explained above.

Note the above theorem had no stipulation on the curvature far in the interior of $\Omega$. Also, since the surgery provides $\alpha$-noncollapsing, $\Omega$ will remain $\alpha$ noncollapsed if surgeries are done within $\Omega$. Provided the singularities only occur in $\Omega$ we thus obtain the following short time existence result for mean curvature flow with surgery:

**Theorem 3.3.** Suppose $M$ is $\alpha$-noncollapsed in the region $\Omega$, and it can be guaranteed no singularities form outside of $\Omega$ occur along the flow on $[0, \delta)$. Then there exists $\epsilon > 0$, $\epsilon \leq \delta$, so that $M$ has a flow with surgery for sufficiently large $H_{th}$.

Of course, pseudolocality can be used to easily show many examples where singularities won’t occur outside some fixed subset $\Omega$ for a fixed time interval. Above $H_{th}$ should be large enough so we can guarantee no surgeries take place in the half collar above.

4. **Convergence to Level Set Flow.**

In [23] Laurer showed that as the surgery parameters degenerate, that is as $H_{th} \to \infty$, the flow with surgery Hausdorff converges to the level set flow. Strictly speaking, his theorem was for compact 2-convex hypersurfaces $M^n$, $n \geq 3$, using the surgery algorithm of Huiskens and Sinestrari [21]. As Hashofer and Kleiner observed (see proposition 1.27 in [16]) it is also true for their algorithm; we will show it is true for our localized surgery.
This also serves as justification for our definition of the mean curvature flow with surgery; it was important we designed our surgery algorithm to produce a weak set flow (see below). Another important observation is that, using theorem 10 of Hershkovits and White in [19], we can “localize” the level set flow so can get away with showing convergence to the level set flow near the singularities (in the mean convex region of $M_t$), roughly speaking.

First we record a couple definitions; these definitions originally due to Illmanen (see [22]). It is common when discussing the level set flow (so we’ll do it here) to consider not $M$ but a set $K$ with $\partial K = M$ chosen so that the outward normal of $K$ agrees with that of $M$. We will quite often abuse notation by mixing $M$ and its corresponding $K$ though, the reader should be warned. When $M$ is smooth the flow of $K$ is just given by redefining the boundary of $K$ by the flow of $M$.

**Definition 4.1.** (Weak Set Flow). Let $W$ be an open subset of a Riemannian manifold and consider $K \subset W$. Let $\{\ell_t\}_{t \geq 0}$ be a one-parameter family of closed sets with initial condition $\ell_0 = K$ such that the space-time track $\cup(\ell_t \times \{t\}) \subset W$ is relatively closed in $W$. Then $\{\ell_t\}_{t \geq 0}$ is a weak set flow for $K$ if for every smooth closed surface $\Sigma \subset W$ disjoint from $K$ with smooth MCF defined on $[a,b]$ we have

$$\ell_a \cap \Sigma_a = \emptyset \implies \ell_t \cap \Sigma_t = \emptyset$$

for each $t \in [a,b]$

The level set flow is the maximal such flow:

**Definition 4.2.** (Level set flow). The level set flow of a set $K \subset W$, which we denote $L_t(K)$, is the maximal weak set flow. That is, a one-parameter family of closed sets $L_t$ with $L_0 = K$ such that if a weak set flow $\ell_t$ satisfies $\ell_0 = K$ then $\ell_t \subset L_t$ for each $t \geq 0$. The existence of a maximal weak set flow is verified by taking the closure of the union of all weak set flows with a given initial data. If $\ell_t$ is the weak set flow of $K \subset W$, we denote by $\hat{\ell}$ the spacetime track swept out by $\ell_t$. That is

$$\hat{\ell} = \bigcup_{t \geq 0} \ell_t \times \{t\} \subset W \times \mathbb{R}_+$$

**Remark 4.1.** Evans-Spruck and Chen-Giga-Roto defined the level set flow as viscosity solutions to

$$w_t = |\nabla w| \text{Div} \left( \frac{\nabla w}{|\nabla w|} \right)$$

but one can check (see section 10.3 in [22]) that this is equivalent to the definition we gave above.

Theorem 1.2, stated more precisely then:

...
Theorem 4.1. (convergence to level set flow) Let $M \subset \mathbb{R}^{n+1}, n \geq 2$ be so $M$ has mean curvature flows with surgery $(M_t)_i$ as defined above on $[0,T]$ where $(H_{th})_i \to \infty$. Then

$$\lim_{i \to \infty} (\hat{M}_t)_i = \hat{L}_t$$  \hspace{1cm} (4.4)

The argument of Laurer strongly uses the global mean convexity of the surfaces he has in question; in our case we only have two convexity in a neighborhood about the origin though. To deal with this we recall the following theorem of Hershkovits and White we had mentioned before:

Theorem 4.2. (Theorem 10 in [19]) Suppose that $Y$ and $Z$ are bounded open subsets of $\mathbb{R}^{n+1}$. Suppose that $t \in [0,T] \to M_t$ is a weak set flow of compact sets in $Y \cup Z$. Suppose that there is a continuous function $w : Y \cup Z \to \mathbb{R}$ with the following properties:

1. $w(x,t) = 0$ if and only if $x \in M_t$
2. For each $c$, $t \in [0,T] \to \{x \in Y : w(x,t) = c\}$ defines a weak set flow in $Y$.
3. $w$ is smooth with non-vanishing gradient on $Z$.

Then $t \in [0,T] \to M_t$ is the level set flow of $M$ in $\mathbb{R}^{n+1}$.

Before moving on, a remark on applying the theorem to above to all the situations encountered in this article:

Remark 4.2. It's clear from the proof of the theorem above that the theorem will also hold if the level sets have bounded geometry away from the surgery regions (so as to obtain the bounds in the paragraph above equation (12) in [19]). In this case $Z$ need not be bounded. In particular, the result above holds for asymptotically planar hypersurfaces.

Hershkovits and White use this theorem to show that flows with only singularities with mean convex neighborhoods are nonfattening - previously this was only known for hypersurfaces satisfying some condition globally like mean convexity or star shapedness. They use the theorem above to “localize” the level set flow by interpolating between two functions of nonvanishing gradient; the distance function to the mean curvature flow of $M$ near the smooth regions and the arrival time function near the singular set (the mean convexity ensures the arrival time function has nonvanishing gradient). For our case it essentially means we only need to prove convergence of the level set flow in the mean convex region, where the singularities are stipulated to form.
So let’s prove the local convergence in the mean convex region - we proceed directly as in [23]. First we note the following (see lemma 2.2 in [23]). Denote by \( (M_H)_t \) to be the mean curvature flow with surgery of \( M \) with surgery parameter \( H_{trig} = H \):

**Lemma 4.3.** Given \( \epsilon > 0 \) there exists \( H_0 > 0 \) such that if \( H \geq H_0 \), \( T \) is a surgery time, and \( x \in \mathbb{R}^{n+1} \), then

\[
B_\epsilon(x) \subset (M_H)_T^+ \implies B_\epsilon(x) \subset (M_H)_T^+
\]  

This statement is local in nature and hence is already noted by Haslhofer and Kleiner (again, proposition 1.27 in [16]); to see briefly why it is so since the necks where the surgeries are done are very thin, how thin depending on \( H \), so for any \( \epsilon > 0 \) there is an \( H \) so that a ball of radius \( \epsilon \) can’t sit inside the neck. Hence any such ball in the lemma must be far away from where any surgeries are happening.

We see each of the \( (M_H)_t \) are weak set flows since the mean curvature flow is and at surgery times \( s_i \), \( (M_H)^+_{s_i} \subset (M_H)^-_{s_i} \). Hence \( \lim_{H \to \infty} \) is also. We see from how our surgery is defined in the bounded region \( \Omega \) containing the surgeries that \( M_t \cap \Omega \) is uniformly two convex on \( [0,T] \), so that for \( \epsilon > 0 \) sufficiently small there exists \( t_\epsilon > 0 \) so that in \( \Omega \):

\[
d(M, M_{t_\epsilon}) = \epsilon
\]  

Let \( \Pi_\epsilon \subset \mathbb{R}^{n+2} \) be the level set flow of \( M_{t_\epsilon} \). Then \( P_i \) is the level set flow of \( \partial K \) shifted backwards in time by \( t_\epsilon \) (ignoring \( t < 0 \)). Let \( H_0 = H_0(\epsilon) \) be chosen as in the lemma above.

**Claim:** \( \Pi_\epsilon \subset M_H \) in \( B(0,R) \) for all \( H \geq H_0 \). Let \( T_1 \) be the first surgery time of \( M_H \). Since \( \partial K_H \) is a smooth mean curvature flow on \( [0,T_1] \) and \( \Pi_\epsilon \) is a weak set flow the distance between the two is nondecreasing on that interval. Thus \( d((\Pi_\epsilon)_T), (\partial M_H)_T) \geq \epsilon \) in \( \Omega \) from our choice of \( \epsilon \). Applying the lemma we see this inequality holds across the surgery as well. We may then repeat the argument for subsequent surgery times.

Since \( \lim_{\epsilon \to 0} \Pi_\epsilon = \hat{\Lambda} \) in \( B(0,R) \) the claim implies \( \hat{\Lambda} \subset \lim_{H \to \infty} M_H \) in \( \Omega \) since the limit of relatively closed sets is relatively closed in Hausdorff topology. On the other hand as we already noted each mean curvature flow with surgery is a weak set flow for \( M \). Hence the limit is also so that \( \lim_{i \to \infty} (M_i) \subset \hat{\Lambda} \) in \( B(0,R) \).

Away from the mean convex set by assumption we have good curvature bounds (in our definition of mean curvature flow with localized surgery, no singularities are allowed to occur outside the surgery regions) so for the flows with surgery \( (M_i)_t \) we can pass to a Hasudorff converging subsequence that converges smoothly away from the mean convex surgery regions, and the limit by Hershkovits and White’s theorem
must be the level set flow. Hence we get that globally the flows with surgery converge in Hausdorff sense to the level set flow $L_t$ of $M$.

5. A High Density Brakke Regularity Theorem for LSF.

In this section we prove theorem 1.3. For the sake of reducing notational clutter without loss of generality $P$ is the plane $x_{n+1} = 0$. Also we denote (like above) the level set flow of $M$ by $L_t$. First we show the following height estimate on mean convex flows with surgery in a ball (these don’t require $n = 2$):

**Lemma 5.1.** Suppose $M$ is mean convex and $\alpha$-noncollapsed and lies in the upper half of the ball $B(0, R)$. Picking $\epsilon > 0$, there is a $T > 0$ depending only on $\alpha$, $h$ and $V$ so that if $M$ has a flow with surgery on $[0, T]$, then $M_T$ is in the slab bounded by the planes $x_{n+1} = \epsilon$ and $x_{n+1} = 0$.

**Remark 5.1.** Note that no curvature assumptions are made so we may freely use this lemma as we let the surgery parameters degenerate. Also note since the post surgery domains (immediately after surgery) are contained in the presurgery domains, it suffices to consider smooth times for the flow.

Proof: We see by maximum principle that $M$ lies on one side of $P$ under the flow with surgery since it does so originally. Also by pseudolocality for each $\epsilon > 0$ there is an $R_0, \epsilon_0$ so that if in the collar neighborhood $C_R$ of $\partial \Omega$ the $C^2$ norm of $M$ as a graph over $P$ is less than $\epsilon_0$, then $x_{n+1}(M \cap \partial B(0, R)) < \epsilon$ on $[0, T]$.

Denote by $\Pi_{\delta, t} \subset K_t (M_t = \partial K_t)$ the set of points in $K_t$ above the plane $x_{n+1} = \delta$. Since $M_t$ is mean convex, $|\Pi_t|$ is decreasing under the flow. Furthermore the $\alpha$ noncollapsing condition crucially relates $|\Pi_t|$ and the mean curvature of points on $M_t \cap \partial \Pi_t$ since at every point there is an interior osculating sphere proportional to the curvature; thus if $p \in \Pi$ and $x_{n+1}(p) > \frac{3\delta}{2}$ there is a constant $\beta$ so that $\beta' B(\frac{\alpha}{H}) = \frac{\beta}{H^{n+1}(p)} \leq |\Pi_t|$, or so that $H(p) > \frac{n+1}{\sqrt{V}}$. 

\[\text{Diagram of } M \text{ with surgery at } \epsilon.\]
At points on $M_t$ where the height function $x_{n+1}$ takes its maximal value the normal is pointing down, implying the height $h(t)$ of $M_t$ satisfies:

$$\frac{\delta h}{\delta t} \leq -\frac{n+1}{\sqrt{2V}} \beta^{\frac{3}{4}}$$

(5.1)

letting $\delta = \frac{2\epsilon}{3}$ yields the result then.

Since the level set flow $L_t$ of $M$ is a Brakke flow, it follows by Brakke regularity theorem for a given fixed period of time there is an $\epsilon > 0$ so that if $L_t$ has density less than $1 + \epsilon$ over that time period in $\Omega$, $L_t$ must be graphical which bounded curvature in $\Omega$. Since $L_t \cap \Omega$ in our case is mean convex, by the proof of White multiplicity bound theorem [28] it suffices that $L_t \cap \Omega$ is contained in the slab between the planes $x_{n+1} = \epsilon$ and $x_{n+1} = -\epsilon$. As a consequence of the above lemma we obtain our theorem.

6. LSF Long Time Convergence to a Plane and Rapid Smoothing Corollaries

By the asymptotic planar condition the initial hypersurface (and corresponding plane $P$) is constrained between two parallel planes $P_1$ and $P_2$. By the maximum principle, it must remain so under the mean curvature flow. During surgeries, high curvature pieces are discarded and caps are placed within the hull of the neck they are associated with, so $M_t$ will remain between $P_1$ and $P_2$ after surgeries as well. Thus $M_T$ is constrained between $P_1$ and $P_2$ for any choice of parameters and hence $L_T$ is too.

By corollary 5.2 $L_T$ is graphical with bounded lipschitz constant and asymptotically flat; hence it stays graphical under the mean curvature flow, which coincides with level set flow on smooth hypersurfaces, and its flow exists (without singularities) for all time by the classical results of Ecker and Huisken (specifically see theorem 4.6 in [12]). In fact, by proposition 4.4 in [12] one sees that as $t \to \infty$, $|A|$ and all its gradients must tend to zero so that the flow of $M_T$, $(M_T)_t$, must converge to a plane. We see that in fact the plane it converges to must be $P$ since for arbitrarily large times, there will be points arbitrarily close to the plane $P$ using the asymptotically planar assumption combined with pseudolocality.

Finally, to see the rapid smoothing corollary, note that by pseudolocality if the perturbations are taken so they “flatten out” fast enough elsewhere the flow of the perturbed surface $\Sigma$ won’t develop singularities.

7. Explicit Examples of Theorem 1.3

To construct explicit nontrivial examples of mean convex regions that satisfy the hypotheses of theorem 1.3 we may use the recent gluing construction of Buzano,
Haslhofer, and Hershkovits - namely Theorem 4.1 in [7]. It suffices to say for our purposes that it allows one to glue “strings,” tubular neighborhoods of curve segments, of arbitrarily small diameter to a mean convex hypersurface $M$ in a mean convex way. One then constructs (by taking a piece of a sufficiently large sphere) and extending it smoothly to a plane to get a surface that is mean convex and 1-noncollapsed in a neighborhood of the origin. One may then attach “loops” to the surface - the loops can be taken with sufficiently small surface area so that the surface must be graphical under the surgery arbitrarily fast. Here is a sketch (not to scale) of the construction:

![Sketch of the construction](image)

By packing the strings very tightly and taking extremely small tubular neighborhoods, we can make the area ratios of $M$ in a fixed ball $B(p, \rho)$, that is the ratio of its local surface to that of the plane, as large as we want while still making the enclosed volume by the strings as small as we want. By adding small beads along the strings (that is, applying the gluing construction to glue tiny spheres along the strings in a 2-convex way) one can see using a barrier argument with Angenants torus there are many examples of surfaces in these classes that develop singularities as well.

8. Concluding Remarks

Of course, it would also be interesting to generalize corollary 1.5 (which we emphasize is a stability result of sorts) somehow to other minimal hypersurfaces, like the catenoid. Any result along these lines would be much more striking than that presented here; the only stable minimal hypersurface in $\mathbb{R}^N$ are the flat hyperplanes, so for a general perturbation the mean curvature flow shouldn’t be able to “find” any other minimal hypersurfaces. At any rate imagines the next easiest type of result along these lines (without any real evidence, of course) could be that local perturbations of minimal hypersurfaces on “almost” flat regions could flow back to the minimal hypersurface. For example, perhaps mean convex perturbations localized far away from the neck of a catenoid flow back to the catenoid under the level set flow.
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