TWISTED K-THEORY IN MOTIVIC HOMOTOPY THEORY

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Abstract. In this paper, we study twisted algebraic K-theory from a motivic viewpoint. For a smooth variety $X$ over a field of characteristic zero and an Azumaya algebra $A$ over $X$, we construct the $A$-twisted motivic spectral sequence, by computing the slices of the motivic twisted algebraic K-theory spectrum as a twisted form of motivic cohomology. This generalizes previous results due to Kahn-Levine where $A$ is assumed to be pulled back from a base field. Our methods use interaction between the slice filtration and birational geometry. Along the way, we prove a representability result, expressing the motivic space of twisted K-theory as an extension of the twisted Grassmannian by the sheaf of “twisted integers”. This leads to a proof of cdh descent and Milnor excision for twisted homotopy K-theory.

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1. Introduction

Twists naturally occur in mathematics whenever locally-defined entities fail to glue to a global object. We are particularly interested in twists by Brauer classes in the context of algebraic geometry. We first recall some of the cases when such twists appear, see also [Lie13] for more examples.

Example 1.1. Suppose that $p : X \to Y$ is a projective morphism such that all of its geometric fibers are isomorphic to a projective space of fixed rank. Examples of such morphisms arise from taking the projectivization of a vector bundle $E$ on $Y$. However, there are many more examples: $X$ is a Severi-Brauer scheme if it is étale-locally on $Y$ isomorphic to $\mathbb{P}^n \times Y$. The failure of $p$ to be a projective bundle is measured by the associated Brauer class $\alpha \in \text{Br}(Y)$, which is an étale 2-cocycle valued in $\mathbb{G}_m$.

Example 1.2. If $X$ is a smooth projective variety over a field, one can consider a moduli space of sheaves (with stability or semistability conditions) $N$. Such a moduli space usually exists as a coarse space but may not exist as a fine moduli space. The problem is that the universal sheaf exists locally but may not glue together to form a global sheaf. In certain cases, Căldăruățu proves that a Brauer class $\alpha$ on $N$ obstructs the existence of a universal sheaf and that a universal $\alpha$-twisted sheaf exists instead. He further proves a derived equivalence between the perfect complexes on $X$ and twisted perfect complexes on $N$ [Cal00, Theorem 5.5.1].

The goal of this paper is to improve certain tools for studying twisted version of algebraic K-theory. For a smooth $k$-scheme $X$ and $A$ an Azumaya algebra over $X$, the $A$-twisted $K$-theory space $K^A(X)$ is the $K$-theory space of the stable $\infty$-category of $A$-twisted perfect complexes on $X$. By construction, the $A$-twisted $K$-theory only depends on the equivalence class of $A$ in the Brauer group of $X$; in other words,

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it is Morita invariant. The $A$-twisted K-theory groups $K_A^\ast(X)$, defined as the homotopy groups of the space $K^A(X)$, first appeared in Quillen’s computation of the K-theory of Severi-Brauer varieties [Qui73].

**Remark 1.3.** The space $K^A(X)$ can be defined more generally for any Brauer class $A$, not necessarily representable by an Azumaya algebra. While in this paper we will concentrate for simplicity on the case when such a representing object exists, all the proofs will work in the more general case as well, as they always proceed by reduction to the case where $A$ is pulled back from a field where all Brauer classes are representable.

When $A$ is trivial in the Brauer group, one of the main tools for accessing the K-theory groups $K^\ast(X)$ is the **motivic spectral sequence**

\[
E_2^{p,q} = CH^{−q}(X;−p−q) \Rightarrow K_{−p−q}(X),
\]

where $CH(X;\ast)$ are Bloch’s higher Chow groups. The spectral sequence (1.4) has many implications: for example, it is a key ingredient in Voevodsky’s proof of the Quillen-Lichtenbaum conjectures [Voe11]. The spectral sequence degenerates after rationalization and implies (a variant of) the Grothendieck-Riemann-Roch theorem:

\[
K_0(X)_Q \cong \bigoplus_{i=0}^{\dim(X)} CH^i(X)_Q.
\]

Bloch’s higher Chow groups provide a cohomology theory for schemes which satisfies Beilinson’s desiderata for a theory of “motivic cohomology”; see for example [FSV00] for the list of properties. Voevodsky’s construction of motivic cohomology groups compares with Bloch’s higher Chow groups via an isomorphism

\[
CH^q(X;2q−p) \cong H^p_{mot}(X;\mathbb{Z}_A(q))
\]

for smooth schemes over a perfect field [Voe02a]. In light of the last isomorphism, the main result of this paper is an extension of (1.4) to the twisted setting.

**Theorem 1.5 (Corollary 5.18).** Let $k$ be a field of characteristic zero, $X$ a smooth $k$-scheme and $A$ an Azumaya algebra over $X$. Then there is a strongly convergent spectral sequence

\[
E_2^{p,q} = H^{p,q}_{mot}(X;\mathbb{Z}_A(−q)) \Rightarrow K_A^{−p−q}(X),
\]

where $E_2$-terms are the $A$-twisted motivic cohomology groups. For a field $k$ of characteristic $p$, the same result holds after inverting $p$.

When $A$ is pulled back from the base field $k$, the spectral sequence (1.6) is one of the main results of the work of Kahn and Levine on twisted K-theory [KL10, Theorem 1]. In this case, the $A$-twisted motivic cohomology groups have a cycle-theoretic interpretation. Namely, they can be defined as $A$-twisted higher Chow groups: the generators are algebraic cycles labelled by “$A$-twisted integers” (see Proposition 5.15). The sheaf of $A$-twisted integers $\mathbb{Z}_A$ is the sheafication of the presheaf $K^0_A$, and the twist encodes the index of $A$. In the case of a general $A$, the cycle-theoretic description holds locally, and we expect that it applies globally as well (see Remark 5.16) and that the spectral sequence (1.6) coincides with the spectral sequence of [KL10, Corollary 6.1.4].

The spectral sequence (1.6) results from the computation of slices of the motivic spectrum $KGL^A$, representing twisted algebraic K-theory. Under the assumptions of Theorem 1.5 we compute the slices $s_q$ with respect to Voevodsky’s slice filtration as follows (see Theorem 5.13):

\[
s_q KGL^A \cong \Sigma^q_{p} HZ^A \quad \forall q \in \mathbb{Z}.
\]

Here $HZ^A$ is the $A$-twisted motivic cohomology spectrum, which we construct in Definition 5.1. When the Brauer class of $A$ is trivial, the spectrum $HZ^A$ coincides with Spitzweck’s motivic cohomology spectrum [Spi18], which, in turn, represents Bloch’s higher Chow groups for smooth schemes over a field.
The computation of slices of twisted K-theory in [KL10, Theorem 1] goes through the twisted version of Levine’s homotopy coniveau tower [Lev08]. We simplify some of the crucial steps in the proof by means of the techniques of Bachmann and the first author developed in [BE21], which make use of birational geometry. We elaborate more on the differences between our approach and [KL10] in the end of the Introduction.

An important ingredient in our proof is representability of twisted K-theory, which has its own implications. One of the main results of the foundational work of Morel and Voevodsky on $A^1$-homotopy theory [MV99] was the representability of algebraic K-theory motivic space by the ind-smooth scheme $Z \times \text{Gr}_{\infty}$, where $\text{Gr}_{\infty}$ is the infinite Grassmannian. This result allows to deduce that the motivic K-theory spectrum $KGL$ is stable under pullback, which in turn implies that homotopy K-theory satisfies cdh descent [Cis13] and Milnor excision [EHIK21].

An immediate analogue of representability for twisted K-theory fails, but we show that there is a fiber sequence of motivic spaces (see Theorem 3.15):

$$\text{Gr}_{\infty}^A \to K^A \to Z^A.$$  

This sequence splits Zariski locally, and it splits globally if $A$ is trivial.

A key geometric ingredient in our computation (1.7) of slices of the motivic $A$-twisted K-theory spectrum $KGL^A$ is rationality of the ind-scheme $\text{Gr}_{\infty}^A$. Moreover, we show that the terms of the fiber sequence (1.8) are stable under pullback and deduce the following result.

**Theorem 1.9 (Theorem 4.8).** Let $X$ be a regular scheme of finite Krull dimension, and $A$ an Azumaya algebra over $X$. Then the $A$-twisted homotopy K-theory presheaf of spectra on the category of $X$-schemes satisfies Milnor excision and cdh descent.

**Vista.** Let $X$ be a smooth $k$-variety with the structure morphism $p: X \to \text{Spec } k$ and $\alpha \in \text{Br}(X)$ be a Brauer class. We can define the $\alpha$-twisted motive of $X$ to be

$$M(X, \alpha) := p_! \text{HZ}^\alpha.$$  

This motivic spectrum is canonically an $HZ$-module and thus defines an object in the category of Voevodsky’s motives $DM(k)$. When $X = \text{Spec } k$, this object has been studied in [KL10], but it is new otherwise, and so we can ask for “twisted” analogs of some conjectures and theorems on smooth varieties with many interesting Brauer classes.

One family of examples of such varieties is given by K3 surfaces; see [Huy16, Chapter 18] for a discussion on the Brauer-theoretic aspects of a K3 surface. A fascinating result, due to Huybrechts, is that twisted derived equivalent K3 surfaces give rise to isomorphic rational Chow motives [Huy19, Huy18]. Since twists disappear after rationalization, one can then wonder if there is an integral refinement of Huybrechts’ results.

**Question 1.10.** If $(X, \alpha)$ and $(X', \alpha')$ are two twisted K3 surfaces with equivalent categories of twisted perfect complexes, are their motives $M(X, \alpha)$ and $M(X', \alpha')$ equivalent?

Answering Question 1.10 requires a better understanding of the twisted spectral sequence (1.4), since $(X, \alpha)$ and $(X', \alpha')$ have isomorphic twisted K-theory groups. It is subject to further investigation by the authors.

**Comparison with other works.** Let us elaborate on the comparison of Theorem 1.5 with [KL10, Theorem 1], when the Azumaya algebra $A$ is pulled back from the base field $k$. Then the statement of [KL10, Theorem 1] differs in two ways. First, in [KL10] the slices are computed in the category of $S^1$-spectra $\text{SH}^S(k)$, i.e., the category of $A^1$-invariant, Nisnevich sheaves of spectra, rather than the category of motivic $P^1$-spectra. Second, their result only assumes that $k$ is a perfect field, without inverting the characteristic.
In Theorem 5.7 we give a different proof of the original result of Kahn-Levine in the same generality as the original: there is no assumption on the characteristic of \( k \), as long as \( A \) is pulled back from the base field. To get a statement in the form presented in [KL10], i.e. on the level of \( S^1 \)-spectra, one can use [Voe02b Conjecture 3], which asserts a compatibility of slice towers under the canonical functor \( \omega^m : \text{SH}(k) \to \text{SH}^S(k) \). This conjecture was proved by Levine [Lev06 Theorem 9.0.3] and has a different proof in [BE21 Theorem 17] in the flavour of the present paper.

One of the main points of the present work is to emphasize that the machinery of motivic homotopy theory provides a rather simple candidate for \( A \)-twisted motivic cohomology, see Definition 5.1. The cohomology groups appearing in Theorem 1.5 (and its proof) do not make reference to Bloch’s cycle complexes, which present real technical challenges in the theory. In particular, our proof avoids the key localization lemma [KL10 Lemma 6.3.1]. On the other hand, when the twist \( A \) is trivial, our definition coincides with all other definitions of motivic cohomology (this latter statement, unfortunately, does not avoid the use of moving lemmas).

Another approach for twisting the motivic K-theory spectrum was introduced by Spitzweck and Østvær in [SØ12]. Their approach closely follows the formalism of twisted cohomology theories in topology. In particular, they constructed a version of of the twisted motivic spectral sequence in [SØ12 Theorem 5.17]. However, their formalism only works for twists by classes in the group \( H^{3\text{mot}}(X; \mathbb{Z}(1)) \). For a general motivic space \( X \), this group might not be trivial (for example, for the motivic sphere \( \Sigma \mathbb{P}^1 \)), but it is trivial whenever \( X \) is a smooth scheme over a perfect field. It would be interesting to compare and unify their approach to twisted K-theory with ours.

Summary. In Section 2 we recall the formation of twisted algebraic K-theory, as well as the necessary prerequisites on Brauer groups and Azumaya algebras. In Section 3 we study the motivic K-theory space and its representability via the twisted Grassmannian. This leads us to Section 4, where the motivic twisted K-theory spectrum is introduced, and the cdh descent and Milnor excision for twisted homotopy K-theory are proved. Finally, in Section 5 we compute the slices of twisted K-theory and obtain the twisted motivic spectral sequence.

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Notation. \( S \) denotes the quasi-compact quasi-separated base scheme, \( \text{Sch}_S \) denotes the category of \( S \)-schemes, \( \text{Sm}_S \) that of smooth \( S \)-schemes. We denote by \( \text{PSh} \text{(Sm}_S \text{)} \) the \( \infty \)-category of presheaves of spaces on \( \text{Sm}_S \), \( \text{H}(S) \) the \( \infty \)-category of motivic spaces over \( S \), and \( \text{SH}(S) \) that of motivic spectra, see [BH21 §2.2 and §4.1]. The Tate sphere \( T \in \text{H}(S) \) is defined as \( A^1/(A^1 - 0) \cong \Sigma G_m \cong (\mathbb{P}^1, \infty) \).

We denote by \( L_{\text{Zar}} \) and \( L_{\text{Nis}} \) the Zariski and Nisnevich sheafifications on \( \text{PSh}(\text{Sm}_S) \) respectively. The \( A^1 \)-localization \( L_{A^1} : \text{PSh}(\text{Sm}_S) \to \text{PSh}(\text{Sm}_S) \) (resp. motivic localization \( L_{\text{mot}} : \text{PSh}(\text{Sm}_S) \to \text{PSh}(\text{Sm}_S) \)) is the localization functor onto the full subcategory of \( A^1 \)-invariant presheaves of spaces (resp. of \( A^1 \)-invariant Nisnevich sheaves). A morphism \( f \) in \( \text{PSh}(\text{Sm}_S) \) is called an \( A^1 \)-equivalence (resp. a motivic equivalence) if \( L_{A^1}(f) \) (resp. \( L_{\text{mot}}(f) \)) is an equivalence.

2. Recollection on twisted K-theory

In this section we recall the main definitions related to twisted algebraic K-theory.
Let $R$ be a commutative ring. Then an associative algebra $A$ over $R$ has a structure of left module over the algebra $A \otimes_R A^{op}$ given by $(x \otimes y) \cdot z = xzy$. In particular we have an associative algebra homomorphism $A \otimes_R A^{op} \to \text{End}_R(A)$.

**Proposition 2.1.** Let $R$ be a commutative ring and $A$ an associative algebra over $R$. Then the following are equivalent:

1. $A$ is a finitely generated projective module of positive rank and the map $A \otimes_R A^{op} \to \text{End}_R(A)$ is an isomorphism;
2. $A$ is a finitely generated projective module of positive rank, it is projective as an $A \otimes A^{op}$-module and its center is equal to $R$;
3. $A$ is a finitely generated projective $R$-module and for any map $R \to \mathcal{O}$ where $\mathcal{O}$ is a strict henselian ring, there exists a $d \geq 1$ and an isomorphism of $\mathcal{O}$-algebras $A \otimes_{\mathcal{O}} \mathcal{O} \cong M_d(\mathcal{O})$;
4. $A$ is a finitely generated projective $R$-module and for every map $R \to k$ where $k$ is a separably closed field, there exists an integer $d \geq 1$ and an isomorphism of $k$-algebras $A \otimes_k k \cong M_d(k)$;
5. $A$ is a finitely generated projective $R$-module and for every map $R \to k$ where $k$ is a field, the algebra $A \otimes_R k$ is a central simple algebra.

We say that an algebra $A$ is an Azumaya algebra if it satisfies any of the above conditions.

**Proof.** The equivalence of (1) and (2) is [DM10, Theorem 3.4]. The equivalence of (2) with (3), (4) and (5) is [Mil80, Proposition IV.1.2] together with the fact that property (2) is étale-local, and the Artin-Wedderburn theorem. $\square$

**Definition 2.2.** Let $X$ be a scheme. An Azumaya algebra over $X$ is a sheaf of quasicoherent $\mathcal{O}_X$-algebras $\mathcal{A}$ such that for every open affine $\text{Spec} R \subseteq X$ the $R$-algebra $\mathcal{A}(R)$ is Azumaya. Equivalently, $\mathcal{A}$ is Zariski-locally an étale twisted form of $M_d(\mathcal{O}_X)$. The integer $d$ is called the degree of the Azumaya algebra (and it is constant on each connected component of $X$).

We denote by $\mathcal{A}z(X)$ (resp. $\mathcal{A}z_d(X)$) the groupoid of Azumaya algebras over $X$ (resp. of degree $d$) with $\mathcal{O}_X$-algebra isomorphisms and by $\mathcal{A}z$ the corresponding stack.

Recall that the group scheme $\text{PGL}_d$ is defined as the quotient of $\text{GL}_d$ by its center $G_m$. By Skolem-Noether theorem [Mil80, Proposition IV.1.4], the action by conjugation $\text{GL}_d \to \text{Aut}_\text{Alg}(M_d(\mathcal{O}))$ factors through $\text{PGL}_d$ and induces an isomorphism $\text{PGL}_d \cong \text{Aut}_\text{Alg}(M_d(\mathcal{O}))$ of group schemes. This fact implies the following presentation of the groupoid of Azumaya algebras on $X$, aka étale twisted forms of $M_d(\mathcal{O}_X)$.

**Proposition 2.3.** The natural map

$$\text{Map}(X, B_{\text{PGL}_d}) \to \mathcal{A}z_d(X)$$

is an equivalence of groupoids, where the left side denotes the mapping space in étale sheaves of spaces.

In particular, we get that

$$\mathcal{H}^q_{\mathcal{O}}(X, \text{PGL}_d) \cong \pi_0\mathcal{A}z_d(X)$$

is the set of isomorphism classes of Azumaya algebras and

$$\text{PGL}_d(X) = \mathcal{H}^0_{\mathcal{O}}(X, \text{PGL}_d) \cong \pi_1(\mathcal{A}z_d(X), M_d(\mathcal{O}_X))$$

is the group of automorphisms of the trivial Azumaya algebra $M_d(\mathcal{O}_X)$.

**Definition 2.4.** Two Azumaya algebras $\mathcal{A}$ and $\mathcal{B}$ on $X$ are called Brauer equivalent if there are vector bundles $\mathcal{E}$ and $\mathcal{F}$ on $X$ such that $\mathcal{A} \otimes_\mathcal{O} \text{End}(\mathcal{E}) = \mathcal{B} \otimes_\mathcal{O} \text{End}(\mathcal{F})$ as $\mathcal{O}_X$-algebras. The Brauer group $\text{Br}(X)$ is the quotient of the set of isomorphism classes of Azumaya algebras on $X$ by Brauer equivalence. We write $[\mathcal{A}]$ for the class of $\mathcal{A}$ in $\text{Br}(X)$. 

**Definition 2.5.** An index of an Azumaya algebra $A$ over $X$, denoted $\text{ind}_X(A)$, is the gcd of degrees of Azumaya algebras over $X$ that are Brauer equivalent to $A$ (for a non-connected scheme, the index is defined componentwise).

The central extension

$$1 \to G_m \to GL_d \to PGL_d \to 1$$

induces a sequence

$$H^1_{\text{et}}(X, GL_d) \to H^1_{\text{et}}(X, PGL_d) \to H^1_{\text{et}}(X, G_m).$$

This sequence is exact in the sense that the fiber of the map $\delta : H^1_{\text{et}}(X, PGL_d) \to H^1_{\text{et}}(X, G_m)$ over $0 \in H^1_{\text{et}}(X, G_m)$ is the image of $H^1_{\text{et}}(X, GL_d)$. Moreover, by [Ser94, Proposition 44] there is an injective map

$$\text{Br}(X) \hookrightarrow H^2_{\text{et}}(X, G_m).$$

An Azumaya algebra of degree $d$ is $d$-torsion in the Brauer group. Hence if $X$ is quasicompact, $\text{Br}(X)$ is a torsion subgroup of $H^2_{\text{et}}(X, G_m)$, and thus have an inclusions:

$$\text{Br}(X) \hookrightarrow \text{Br}'(X) := H^2_{\text{et}}(X, G_m)_{\text{tors}} \hookrightarrow H^2_{\text{et}}(X, G_m).$$

Next, we define twisted versions of vector bundles, perfect complexes etc. By abuse of notation, for $f : Y \to X$ and $A \in \text{Az}(X)$ we denote $f^*(A) \in \text{Az}(Y)$ also by $A$, when $f$ is clear from the context.

**Definition 2.6.** Let $A$ be an Azumaya algebra over a scheme $X$. We define the category of $\mathcal{A}$-twisted sheaves on $X$ as

$$\mathcal{O}\text{Coh}^A(X) = \text{Mod}_A(\mathcal{O}\text{Coh}(X))$$

(by convention, Mod$_A$ stands for left modules).

By Grothendieck’s descent theorem, the map

$$f : Y \to X \mapsto \mathcal{O}\text{Coh}^A(Y)$$

is an étale sheaf of categories on $\text{Sch}_S$, which gives the stack of categories $\mathcal{O}\text{Coh}^A$.

**Remark 2.7.** Let $\alpha \in H^2_{\text{et}}(X, G_m)$ be a class, then we can associate to it a $G_m$-gerbe $\mathcal{X} \to X$. Lieblich has defined an $\alpha$-twisted sheaf as a quasicoherent sheaf on $\mathcal{X}$ such that the inertial action (on the left) of $G_m$ agrees with the $\mathcal{O}_X$-module structure. The comparison between Definition 2.6 and his definition can be parsed together from the comparison results in Lieblich’s thesis [Lie04, Section 2.1.3] and in Căldăraru’s [Cal00, Theorem 1.3.7].

Let $f : Y \to X$ be an étale cover of $X$ such that $f^*A \cong M_d(\mathcal{O}_Y)$. Then there is an equivalence

$$\mathcal{O}\text{Coh}^A(Y) \cong \mathcal{O}\text{Coh}(Y),$$

given by sending an $M_d(\mathcal{O}_Y)$-module $M$ to $(\mathcal{O}^{\otimes d}_Y)^\vee \otimes M_d(\mathcal{O}_Y) \otimes M$.

**Definition 2.9.** The groupoid of $A$-twisted vector bundles on $X$, denoted $\text{Vect}^A(X)$, is given by the $A$-twisted sheaves on $X$ that are étale-locally free, and their isomorphisms. Precisely, $V \in \mathcal{O}\text{Coh}^A(X)$ is an $A$-twisted vector bundle if there is an étale cover $f : Y \to X$ such that $f^*A \cong M_d(\mathcal{O}_Y)$ and $f^*V \in \mathcal{O}\text{Coh}(Y)$ is a vector bundle on $Y$, under the identification (2.8).

Similarly, the category of $A$-twisted perfect complexes on $X$, denoted $\mathcal{P}\text{erf}^A(X)$, is given by complexes of $A$-twisted sheaves on $X$ that are étale-locally perfect. These data form the corresponding stacks $\text{Vect}^A$ and $\mathcal{P}\text{erf}^A$.

**Remark 2.10.** $\mathcal{O}\text{Coh}^A$ can be interpreted as an étale twisted form of $\mathcal{O}\text{Coh}$. Indeed, since there is an isomorphism of $\mathcal{O}\text{Coh}$-modules:

$$\text{Aut}_{\mathcal{O}\text{Coh}}(\mathcal{O}\text{Coh}) \cong \text{Pic},$$

the étale twisted forms of $\mathcal{O}\text{Coh}$ are in bijection with $H^1(S, \mathcal{P}\text{ic}) \cong H^2(S, G_m)$, and $\mathcal{O}\text{Coh}^A$ corresponds to $[A] \in H^2(S, G_m)$. Similarly, $\text{Vect}^A$ and $\mathcal{P}\text{erf}^A$ are étale twisted forms of $\text{Vect}$ and $\mathcal{P}\text{erf}$ respectively.
Remark 2.11. If $A$ and $B$ are Brauer equivalent Azumaya algebras, they are Morita equivalent, i.e. $\mathcal{O}\text{Coh}^A \cong \mathcal{O}\text{Coh}^B$. Hence the stacks $\mathcal{O}\text{Coh}^A$, $\text{Vect}^A$, $\text{Perf}^A$ only depend on the Brauer equivalence class of $A$, as well as the $A$-twisted $K$-theory and the related constructions. We twist with respect to a representative in the Brauer equivalence class only for the sake of convenience.

The following lemma will be important for studying $A$-twisted $K$-theory.

Lemma 2.12. Let $V$ be an $A$-twisted sheaf. Then the following are equivalent:

1. $V$ is an $A$-twisted vector bundle;
2. For every affine subscheme $i: U \hookrightarrow X$ the $A(U)$-module $V(U)$ is finitely generated and projective;
3. $V$ is locally free of finite rank as an $\mathcal{O}_X$-module;

Proof. The equivalence between (2) and (3) is [DMI06, Proposition II.2.3]. The equivalence of (1) with (3) follows because the property of being locally free of finite rank is fpqc-local by [Stacks 05B2]. □

With these definitions at hand, we can define $A$-twisted $K$-theory.

Definition 2.13. Let $A$ be an Azumaya algebra over a qcqs scheme $S$. The $A$-twisted $K$-theory is the presheaf of spaces on $\text{Sm}_S$ defined by

$$K^A(U) = K(\text{Perf}^A(U)),$$

where $\text{Perf}^A(U)$ is considered as a Waldhausen category.

Alternatively, $K^A$ can be characterized as the Zariski sheaf of spaces that on a affine $S$-schemes is given by $K^A(U) = \text{Vect}^A(U)^\text{gp}$, where $\text{gp}$ stands for group completion (with respect to direct sum on $\text{Vect}^A(U)$). Such a characterization is possible because $\text{Vect}^A$ is a Zariski sheaf by Lemma 2.12.

We define the $A$-twisted $K$-theory groups of a smooth $S$-scheme $U$ as

$$K_i^A(U) = \pi_i K^A(U).$$

Remark 2.14. $K^A$ is naturally a module over the presheaf of $E_\infty$-ring spaces $K$, since $\text{Perf}^A$ is a $\text{Perf}$-module via tensor product.

Definition 2.15. We define the $A$-twisted rank map as the map of presheaves of spaces

$$\text{rk}^A: \text{Vect}^A \to \mathbb{Z} \quad P \mapsto \sqrt{\text{rk}_0 \text{End}_A(P)}$$

that sends a locally free $A$-module $P$ to the square root of the rank of $\text{End}_A(P)$ as $\mathcal{O}$-module. This is well-defined because $\mathcal{O}\text{Coh}^A$ is a $\mathcal{O}\text{Coh}$-module hence enriched in $\mathcal{O}\text{Coh}$, and when $A = M_n(\mathcal{O})$ it coincides with the usual rank under the identification (2.8). In particular the map $\text{rk}^A$ takes values in the sheaf $\mathbb{Z}$ of integers, because étale-locally it gives an integer. Note that $\text{rk}^A(A) = d$ for $A$ an Azumaya algebra of degree $d$.

The stack $\text{Vect}_m^A$ of $A$-twisted vector bundles of rank $m$ is defined as the fiber of $\text{rk}^A - m: \text{Vect}^A \to \mathbb{Z}$.

Since $\mathbb{Z}$ is a sheaf of groups, the monoid map $\text{rk}^A$ factors through $K^A$, and we get

$$\text{rk}^A: K^A \to \mathbb{Z},$$

which in turn factors through $K^A_0$.

Definition 2.16. The sheaf of $A$-twisted integers is the following subsheaf of $\mathbb{Z}$:

$$\mathbb{Z}^A = L_{Zar} \text{Im}(\text{rk}^A: K^A \to \mathbb{Z}).$$

Lemma 2.17. The induced map of presheaves

$$\text{rk}^A: K^A_0 \to \mathbb{Z}^A$$
is a Zariski-local equivalence. Moreover, for $R$ a local ring, the inclusion $K^A_0(R) \cong Z^A(R) \subset Z$ is given by

$$\text{ind}_R(A) \cdot Z \subset Z.$$ 

Proof. By Morita invariance of $K^A_0$ and [DeM69, Corollary 1], the map $\text{rk}^A : K^A_0 \to Z$ is an injective homomorphism on local rings, hence it identifies $L_{Zar}K^A_0$ with $Z^A$. Over a local ring $R$, the image is given by $\text{ind}_R(A) \cdot Z \subset Z$ again by [DeM69, Corollary 1]. \hfill \Box

Corollary 2.18. The sheaf inclusion $Z^A \subset Z$ becomes an isomorphism after tensoring with $Q$.

By Lemma 2.17, our sheaf $Z^A$ is isomorphic to the sheaf (with transfers) $Z_A$ of [KL10, Section 5.3], and our subsheaf inclusion $Z^A \subset Z$ is the reduced norm map $\text{Nrd}$ of [KL10, Lemma 5.2.1]. The following properties of $Z^A$ are proved in [KL10].

Lemma 2.19. Let $S$ be regular of finite type. Then $Z^A$ is an $A^1$-invariant Nisnevich sheaf on $\text{Sm}_S$. Moreover, $Z^A$ is a birational sheaf, i.e. locally-constant in the Zariski topology.

Proof. This is [KL10, Lemma 5.1.1]. \hfill \Box

Lemma 2.20. Let $S$ be regular of finite type. Then the sheaf $Z^A$ has a canonical structure of Voevodsky’s transfers, and $Z^A \subset Z$ is an inclusion of sheaves with transfers on $\text{Sm}_S$.

Proof. This is [KL10, Section 5.3]. \hfill \Box

3. THE MOTIVIC SPACE OF TWISTED $K$-THEORY

Let $S$ be a quasi-compact quasi-separated base scheme, $A$ an Azumaya algebra of degree $d$ over $S$. We want to understand the motivic homotopy type of the twisted $K$-theory space $K^A$, which is a sheaf on $\text{Sm}_S$, given on any affine $S$-scheme $\text{Spec} R$ by $\text{Vect}^A(R)^{op}$. First, we show that it is indeed a motivic space when $S$ is regular.

Lemma 3.1. Let $X$ be a quasi-compact quasi-separated scheme and $Z \subseteq X$ a closed subset such that $U = X - Z$ is quasi-compact. Then there is a functorial fiber sequence of spaces

$$K(\text{Perf}^A(X \times S Z)) \to K^A(X) \to K^A(U).$$ 

Proof. The claim follows by tensoring the $\text{Perf}(X)$-linear exact sequence

$$\text{Perf}(X \times Z) \to \text{Perf}(X) \to \text{Perf}(U)$$ 

by $\text{Perf}^A(X)$ and using [Lur17, Theorem 4.8.5.16.(4)]. \hfill \Box

Proposition 3.2. The presheaf of spaces $K^A$ satisfies Nisnevich descent. If moreover $S$ is regular and noetherian of finite Krull dimension, then $K^A$ is $A^1$-invariant.

Proof. The claim about Nisnevich descent follows from Lemma 3.1 as in [TT90, Corollary 10.10] by the étale descent for $A$-twisted perfect complexes.

Let us prove the $A^1$-invariance. Let $d$ be the degree of $A$ and $\text{SB}(A)$ be the Severi-Brauer variety associated to $A$. By [Qui73, Theorem 4.1], for every smooth $S$-scheme $X$ there is an isomorphism

$$K(X \times S \text{SB}(A)) \cong \prod_{i=0}^{d-1} K^A_{\text{et}}(X).$$ 

Since $\text{SB}(A)$ is a smooth $S$-scheme, the pullback map

$$p^*: K(X \times S \text{SB}(A)) \to K(A^1_X \times S \text{SB}(A))$$ 

is an equivalence. Since the decomposition is natural, it follows that the pullback map induces an equivalence on each component, in particular on the first component, which is what we wanted to prove. □

Let Spec $R$ be a smooth affine $S$-scheme. By Lemma 2.12, $\text{Vect}^A(R)$ is the groupoid of finitely generated projective $A_R$-modules, i.e. summands of free $A_R$-modules (by the usual abuse of notation, we write further on $A$ for $A_R$). Hence, we have

$$\text{Vect}^A(R) \cong \text{Vect}^A(R)[-A].$$

By the McDuff-Segal group completion theorem [MS76] (specifically, we use [Nik17, Proposition 6]), we get an equivalence of $\mathcal{E}_\infty$-spaces:

$$\text{Vect}^A(R) \cong s\text{Vect}^A(R),$$

where $s\text{Vect}^A(R) = \text{colim}(\text{Vect}^A(R) \xrightarrow{+A} \text{Vect}^A(R) \xrightarrow{+A} \ldots)$ is the groupoid of stable $A$-twisted vector bundles on Spec $R$, and $+$ stands for Quillen's $+$-construction.

Recall that $+$-construction, applied to $s\text{Vect}^A(R)$, is invisible up to $A^1$-homotopy by Robalo's criterion [Rob15, Insight 1.1] (which originates from [Voe98, Theorem 4.3]). More precisely, we have the following result.

**Proposition 3.3.** On affine schemes, there is a canonical $A^1$-equivalence of $\mathcal{E}_\infty$-spaces:

$$s\text{Vect}^A \cong \text{Vect}^A, \text{gp}.$$ 

**Proof.** By [BEH+21] Proposition 5.1, it’s enough to show that on any affine $S$-scheme Spec $R$ the cyclic permutation of $A^3$ is $A^1$-homotopic to identity. The action of the cyclic group $C_3$ on $A^3$ induces a map of stacks $BC_3 \to \text{Vect}^A$. This map factors through the stack $B\text{SL}_3$, and the group scheme $\text{SL}_3$ is $A^1$-connected, so the claim follows. □

**Corollary 3.4.** There is a canonical $A^1$-equivalence on affines:

$$K^A \cong s\text{Vect}^A.$$ 

**Remark 3.5.** Since twisted K-theory is Morita-invariant, it follows from Corollary 3.4 that for Brauer equivalent Azumaya algebras $A$ and $B$ there is a canonical $A^1$-equivalence on affines between $s\text{Vect}^A$ and $s\text{Vect}^B$.

**Proposition 3.6.** For any affine scheme Spec $R$, there is a non-canonical equivalence of spaces

$$s\text{Vect}^A(R) \cong K^A_0(R) \times \text{BGL}(A).$$ 

**Proof.** This follows since $\pi_0\text{Vect}^A(R) \cong K^A_0(R)$, and the automorphism group of any stable $A$-twisted vector bundle is isomorphic to $\text{GL}(A)$. The latter is explained for non-twisted vector bundles in [EHK+20, Section 2.1], and the argument in loc.cit. applies verbatim for twisted vector bundles, by replacing $R$ with $A_R$. □

The map of presheaves $\text{rk}^A: \text{Vect}^A \to \mathbb{Z}^A$ induces a map

$$\text{rk}^A: s\text{Vect}^A \to \mathbb{Z}^A$$

that sends $V \in \text{Vect}^A$ in the $n$-th component to $\text{rk}^A(V) = nd$ (this makes sense because the degree of an Azumaya algebra is stable under base change).

Consider the telescope

$$\text{Vect}^A_\infty = \text{colim}(\text{Vect}^A_0 \xrightarrow{+A} \text{Vect}^A_1 \xrightarrow{+A} \ldots).$$
We get a fiber sequence of presheaves of spaces:

\[
\text{Vect}^A_\infty \to s\text{Vect}^A \xrightarrow{\pi^A} \mathbb{Z}^A.
\]

**Remark 3.8.** In the non-twisted case (i.e., when \(A\) is trivial), this fiber sequence splits canonically, since there is a map of sheaves \(\mathbb{Z} \to \text{Vect}\) sending an integer to a trivial vector bundle of the corresponding rank. However, this construction does not work in the twisted case. Indeed, while the \(A\)-twisted rank \(\text{rk}^A: K_0^A(R) \to \mathbb{Z}^A(R)\) is surjective on all affine schemes \(\text{Spec} R\), the authors do not know if it is possible to choose the splitting in a natural way, and therefore whether the map of sheaves \(K_0^A \to \mathbb{Z}^A\) has a section.

A related phenomenon can be observed in [AW14], where the authors show that the index is not necessarily the image of the \(A\)-twisted vector bundle (although it will be the image of an \(A\)-twisted perfect complex).

**Lemma 3.9.** The fiber sequence (3.7) splits Zariski-locally.

**Proof.** Let \(R\) be a local ring. By [DeM69, Corollary 1], there is an Azumaya algebra \(B\) of degree \(\text{ind}_R(A)\) in the Brauer equivalence class of \(A\) on \(\text{Spec} R\). Under the Morita equivalence \(\text{Vect}^A \cong \text{Vect}^B\), the \(B\)-module \(B\) corresponds to some \(A\)-twisted vector bundle \(V\) such that \(\text{rk}^A(V) = \text{ind}_R(A)\). By Lemma 2.17, the subgroup \(\mathbb{Z}^A(R) \subset \mathbb{Z}\) is given by the inclusion \(\text{ind}_R(A) \cdot \mathbb{Z} \subset \mathbb{Z}\). Hence we can define the splitting of the surjection \(s\text{Vect}^A(R) \to \mathbb{Z}^A(R)\) by sending \(m \cdot \text{ind}_R(A)\) to \(V^m\).

**Proposition 3.10.** After applying motivic localization, the sequence (3.7) becomes a fiber sequence of motivic spaces in \(\mathbb{H}(S)\).

**Proof.** It is enough to show the claim for the fiber sequence

\[
\text{Vect}^A_\infty \to s\text{Vect}^A \xrightarrow{\pi^A} \mathbb{Z}.
\]

For a local ring \(R\), the sequence

\[
L_A: \text{Vect}^A_\infty(R) \to L_A s\text{Vect}^A(R) \to L_A \mathbb{Z}(R)
\]

is a fiber sequence by [Rez14, Proposition 5.4], since the sheaf \(L_A \mathbb{Z}\) is constant in the simplicial direction. Hence, the sequence (3.11) remains a fiber sequence after applying \(L_{\text{Zar}} L_A\). Note that \(L_{\text{Zar}} L_A s\text{Vect}^A \cong K^A\) is a motivic space, and so is \(L_{\text{Zar}} L_A \mathbb{Z} \cong \mathbb{Z}\). Hence, the fiber \(L_{\text{Zar}} L_A s\text{Vect}^A_\infty\) is also a motivic space, and the claim follows.

Next, we want to replace \(\text{Vect}^A_\infty\) with a more geometric object.

**Definition 3.12.** The \(A\)-twisted Grassmannian is the presheaf \(\text{Gr}_{m,n}^A\), on affine \(S\)-schemes sending \(\text{Spec} R\) to the set of \(F \subseteq A^m_R\) of rank \(m\) left \(A_R\)-submodules such that \(A_R^{\text{free}} / F\) is projective as a left \(A_R\)-module. This construction generalizes Severi-Brauer schemes (in the case \(m = 1\)), and has been studied in K-theoretic contexts by various authors [LSW89, Pan89, Bae16].

**Lemma 3.13.** The presheaf \(\text{Gr}_{m,n}^A\) is represented by a smooth proper \(S\)-scheme.

**Proof.** The Azumaya algebra \(A\) has an underlying locally free sheaf \(E\). The classical Grassmannian \(\text{Gr}_{m,n}(E)\) is represented by a smooth proper scheme. We observe that \(\text{Gr}_{m,n}^A \subset \text{Gr}_{m,n}(E)\) is a closed subfunctor, hence it is represented by a proper subscheme. This scheme is smooth, because smoothness is an étale-local condition on the target of the forgetful map \(\text{Gr}_{m,n}^A \to S\), and the classical Grassmannian \(\text{Gr}_{m,n}\) is smooth.

We define \(\text{Gr}_{m,0}^A = \text{colim}_n \text{Gr}_{m,n}^A\) with maps in the colimit induced by \((\text{id}, 0): A^m_0 \to A^{m(n+1)}_0\). We stabilize further and set

\[
\text{Gr}_0^A = \text{colim}(\text{Gr}_{0,0}^A \xrightarrow{+A} \text{Gr}_{1,0}^A \xrightarrow{+A} \ldots).
\]
Proposition 3.14. The forgetful maps \( \text{Gr}_{m,\infty}^A \rightarrow \text{Vect}_m^A \) and \( \text{Gr}_{\infty}^A \rightarrow \text{Vect}_{\infty}^A \) are \( \mathbb{A}^1 \)-equivalences on affines.

Proof. When \( A \) is trivial, this is shown in [HJN+20, Proposition 4.7]. The argument applies verbatim for general \( A \), by replacing \( A \) with \( A_A \) in loc.cit. \( \square \)

Altogether, we get the following result, which gives a geometric interpretation of the twisted K-theory as a motivic space.

Theorem 3.15. There is a fiber sequence of motivic spaces

\[
\text{Gr}_{\infty}^A \rightarrow K^A_A \rightarrow Z^A.
\]

Proof. This is a combination of Proposition 3.10, Corollary 3.4 and Proposition 3.14. \( \square \)

4. The motivic spectrum of twisted K-theory

In this section, we assume that the base scheme \( S \) is regular and of finite Krull dimension. First, we show that the \( A \)-twisted K-theory satisfies Bott periodicity.

Proposition 4.1. We have an isomorphism of \( K(P^1_S) \)-modules:

\[
K^A(P^1_S) \cong K(P^1_S) \otimes_{K(S)} K^A(S).
\]

Proof. The tensor product map 

\[
\text{Perf}(P^1_S) \otimes_{\text{Perf}(S)} \text{Perf}(S) \rightarrow \text{Perf}(P^1_S)
\]

is an isomorphism \( \text{é} \)tale-locally, hence an isomorphism of stacks. Consider the Beilinson orthogonal decomposition of \( \text{Perf}(S) \)-modules:

\[
\text{Perf}(P^1_S) = \langle \text{Perf}(S), \text{Perf}(S) \rangle.
\]

Upon tensoring the decomposition with \( \text{Perf}(S) \), we get

\[
\text{Perf}^A(S) = \langle \text{Perf}^A(S), \text{Perf}^A(S) \rangle.
\]

The proposition then follows by additivity of K-theory. \( \square \)

Corollary 4.2. We have \( \Omega P^1_k A^A \cong A^A \), i.e. \( A^A \) satisfies motivic Bott periodicity.

Corollary 4.2 gives an infinite \( P^1 \)-delooping of \( K^A \), and so we can define the motivic spectrum of twisted K-theory.

Definition 4.3. The \( P^1 \)-spectrum of \( A \)-twisted K-theory is defined as follows (with bonding maps specified by Corollary 4.2):

\[
KGL^A = (K^A, K^{A^2}, \ldots) \in \text{SH}(S).
\]

Remark 4.4. Since \( K^A \) is a K-module by Remark 2.14 and the action is compatible with Bott periodicity, the motivic spectrum \( KGL^A \) is a KGL-module.

Similarly, kgl^A, the effective cover of KGL^A, is naturally a kgl-module. In [Bac20, Corollary 4.3], it was shown that a kgl-module structure is equivalent to a structure of (coherent) pushforwards along finite locally free maps of schemes. Under this equivalence, the structure of transfers on kgl^A comes from finite locally free covariance of \( \text{Perf}^A \).

Let us write \( KH^A(X) \) for the \( A^1 \)-localization of the Bass K-theory spectrum \( K^B \) of \( \text{Perf}^A(X) \). That is

\[
KH^A(X) = \text{colim}_{n \in \mathbb{N}} K^B(\text{Perf}^A(X \times \Delta^n)).
\]

When \( A \) is trivial \( KH^A \) is the cohomology theory represented by \( KGL \). We want to show that the same is true for a general \( A \).
Proposition 4.5. Let X be a quasi-compact quasi-separated scheme and A an Azumaya algebra over X. Then there is an equivalence of spectra

$$\text{map}_{\mathcal{SH}_{X}}(\Sigma^\infty X_+, \text{KGL}^A) \simeq \text{KH}^A(X)$$

Proof. The statement follows from [Cis13 Corollaire 2.11] as in the proof of [Cis13 Théorème 2.20].

One of the first achievements of motivic homotopy theory was a complete proof of cdh descent for homotopy K-theory [Cis13]. The key insight is that any motivic spectrum which is stable under base change satisfies cdh descent. We now prove a base change property for twisted K-theory spectrum to establish its cdh descent. To do so, we will first show that the sheaf $Z^A$ satisfies base change. We note that cdh descent for twisted homotopy K-theory can also be established using methods of Land-Tamme [LT19] as explored in the twisted setting by Stapleton [Sta20]. We will later give a motivic refinement of the cdh descent of twisted homotopy K-theory in the sense that we will endow it with a motivic filtration whose associated graded pieces do enjoy cdh descent.

Proposition 4.6. Let $f : T \to S$ be a morphism. Then the canonical map $f^*Z^A \to Z^{f^*A}$ is an isomorphism of Zariski sheaves on $\text{Sm}_{Y}$.

Proof. Since $f^*Z^A$ and $Z^{f^*A}$ are 0-truncated, it suffices to check that the map is an isomorphism on stalks.

The sheaf $f^*Z^A$ can be identified with the sheafification of the presheaf on $\text{Sm}_{Y}$ sending

$$[Z \to T] \mapsto \colim_{Z \to W \to S} Z^A(W)$$

where the colimit is indexed by all factorizations $Z \to W \to S$ of $Z \to T \to S$ where $W \to S$ is a smooth $S$-scheme. Let $Z \in \text{Sm}_{Y}$ and choose a point $z \in Z$, then the stalk of $f^*Z^A$ at $z$ is therefore

$$(f^*Z^A)_z \simeq \colim_{U \ni z} \colim_{U \to W \to S} Z^A(W) \simeq \colim_{\text{Spec } O_{Z_z} \to W \to S} Z^A(W)$$

where the last colimit is indexed through all factorizations of $\text{Spec } O_{Z_z} \to Z \to S$ through a smooth $S$-scheme $W$. Note that $Z^A$ commutes with filtered colimits, as it is the Zariski sheafification of $K_0^A$ by Lemma 2.17. Hence we can replace $W$ with $O_w$, where $w$ is the image of $z$ and rewrite this colimit as

$$(f^*Z^A)_z \simeq \colim_{\text{Spec } O_{Z_z} \to \text{Spec } R \to S} Z^A(\text{Spec } R) \simeq \colim_{\text{Spec } O_{Z_z} \to \text{Spec } R \to S} K_0^A(R)$$

where the colimit now ranges through all factorizations through an essentially smooth local $S$-scheme $\text{Spec } R$ such that the map $R \to O_{Z_z}$ is a local ring homomorphism. Moreover, the stalk of $Z^{f^*A}$ at $z$ is precisely $K_0^A(O_{Z_z})$, and the map we need to prove is an isomorphism is the pullback map

$$\colim_{\text{Spec } O_{Z_z} \to \text{Spec } R \to S} K_0^A(R) \to K_0^A(O_{Z_z}).$$

We will prove separately that it is injective and surjective.

For surjectivity it suffices to show that every $A$-twisted vector bundle $V$ over $O_{Z_z}$ can be obtained by pullback from some essentially smooth local ring $R$ over $S$. But, by choosing a set of generators as an $A$-module, we can obtain it as a pullback from $\text{Gr}^A_{O_{Z_z}}$ and therefore from $\text{Gr}^A_{O_{S_z}}$ where $s$ is the image of $z$ in $S$. Since the $A$-twisted Grassmannian is smooth by Lemma 3.13, this proves surjectivity.

Let us now prove injectivity. Suppose we have an essentially smooth local ring $R$ and a class $[V] \in K_0^A(R)$ that is sent to 0 by the pullback along $f : \text{Spec } O_{Z_z} \to \text{Spec } R$. Up to adding a suitable free $A$-twisted vector bundle to both $V$ and $V'$, this means that there exists an isomorphism $\varphi : f^*V \simeq f^*V'$. Therefore the map $\text{Spec } O_{Z_z} \to \text{Spec } R$ factorizes as

$$\text{Spec } O_{Z_z} \to \text{Spec } R.$$
where $\text{Iso}_A(V, V')$ is the smooth affine scheme of Lemma 4.7 below. Therefore, by taking $\text{Spec } R'$ to be the localization of $\text{Iso}_A(V, V')$ at the image of $z$, there is a factorization

$$\text{Spec } \mathcal{O}_{Z, z} \to \text{Spec } R' \to \text{Spec } R$$

where $R'$ is an essentially smooth $S$-scheme where $V$ and $V'$ become isomorphic. Hence the class $[V] - [V']$ becomes 0 in the colimit. □

**Lemma 4.7.** Let $S$ be a qcqs scheme, $A$ an Azumaya algebra over $S$ and $V$ and $W$ be two $A$-twisted vector bundles. Consider the presheaf $\text{Iso}_A(V, W)$ sending every $S$-scheme $p: T \to S$ to the set

$$\{\phi: p^*V \to p^*W\}$$

of isomorphisms of $A$-twisted vector bundles over $T$. Then $\text{Iso}_A(V, W)$ is represented by a smooth affine $S$-scheme.

**Proof.** Since the prestack of $A$-modules is an fpqc stack by [Stacks, 023T], the presheaf $\text{Iso}_A(V, W)$ is an fpqc sheaf. Moreover étale-locally we can find a trivialization of $A$, so we can assume $A$ to be trivial. But then, étale-locally, $\text{Iso}_A(V, W)$ is either empty (if $\text{rk } V \neq \text{rk } W$) or a $\text{GL}_{\text{rk } V}$-torsor. In either case it is represented by a smooth affine $S$-scheme. □

Using Proposition 4.6 we can deduce the base change property for $KGL^A$.

**Theorem 4.8.** Let $S$ be a regular scheme of finite Krull dimension, and $A$ an Azumaya algebra over $S$. The formation of the motivic twisted $K$-theory spectrum

$$p: X \to S \mapsto KGL^p_A$$

defines a Cartesian section of the Cartesian fibration $\int SH \to \text{Sch}_S$. In particular, the twisted homotopy $K$-theory spectrum is a cdh sheaf and satisfies Milnor excision.

**Proof.** The first statement follows once we know that for any morphism of $S$-schemes $f: X \to Y$, with structure map $p_X: X \to S, p_Y: Y \to S$ the canonical comparison map in $SH(X)$

$$f^*KGL^p_X \to KGL^p_Y$$

is an equivalence of motivic spectra. By construction of $KGL^A$, it’s enough to check that the comparison map $f^*K^p_A \to K^p_Y$ is an equivalence of motivic spaces in $H(X)$. By Theorem 3.15, $K^A$ as a motivic space is an extension of $Z^A$ by $\text{Gr}^A$. Then the claim follows from Proposition 4.6 because the twisted Grassmannians are stable under pullbacks.

The base change property implies that $KGL^A$ is a cdh sheaf by [Cis13, Proposition 3.7], and that it satisfies Milnor excision by [EHIK21, Corollary 2]. □

5. Twisted motivic cohomology

In this section, we compute the slices of $A$-twisted $K$-theory, which allows us to construct the $A$-twisted motivic spectral sequence. Previously, Kahn and Levine have computed the slices in the case when the base scheme $S = \text{Spec } K$ is the spectrum of a perfect field $k$ [KL10, Theorem 6.5.5]. In this section, we will first provide a different proof of this computation (Theorem 5.7) and then generalize it to the case when the Azumaya algebra is not pulled back from the base field (Proposition 5.13).

To define $A$-twisted motivic cohomology, we employ the theory of framed transfers, which was developed in [Voe01, GP18, EHK+21] and other works. We summarize here the main structural results that we use.

(1) Every cohomology theory represented by a motivic spectrum has a unique structure of framed transfers, i.e. a structure of a presheaf on the $\infty$-category of framed correspondences $\text{Corr}^f(Sm_S)$ [Hoy21, Theorem 18]. More precisely,

$$SH(S) \simeq SH^f(S),$$

where $SH^f(S)$ is the $\infty$-category of framed motivic spectra.
(2) The framed suspension spectrum \( \Sigma_{fr}^\infty : H^i(S) \to SH^i(S) \), going from framed motivic spaces to (framed) motivic spectra, is fully faithful on grouplike objects, when \( S \) is a perfect field \([EHK^*21]\) Theorem 3.5.14).

(3) There is a canonical functor \( \text{Corr}^{fr}(Sm_k) \to \text{Corr}_S \) from the \( \infty \)-category of framed correspondences to the category of Voevodsky’s finite correspondences \([EHK^*21]\) Section 5.3]. In particular, every presheaf with Voevodsky’s transfers has canonical framed transfers.

Recall that the presheaf of abelian groups \( Z^A \) is a motivic space with a canonical structure of Voevodsky’s finite transfers by Lemmas \( \mathbb{L}219 \) and \( \mathbb{L}220 \) hence it represents a framed motivic space, which we also denote \( Z^A \).

**Definition 5.1.** We define the \( A \)-twisted motivic cohomology spectrum to be

\[
HZ^A := \Sigma_{fr}^\infty Z^A \in SH^fr(S) = SH(S).
\]

When \( A \) is trivial, the motivic spectrum \( HZ^A \) coincides with the Spitzweck motivic cohomology spectrum, constructed in \([Spi18]\), by \([Hoy21]\) Theorem 21]. Moreover, \( HZ^A \) is stable under base change by Proposition \( \mathbb{L}4.5 \) and in particular, the corresponding cohomology theory satisfies cdh descent by \([Cis13]\) Proposition 3.7]. Therefore it makes sense to say that \( HZ^A \) represents \( A \)-twisted Spitzweck motivic cohomology.

We recall the definition of the slice filtration in motivic homotopy theory, constructed in \([Noc02b]\).

**Definition 5.2.** Let \( S \) be a scheme. Then the \( \infty \)-category \( SH(S)^{eff} \) of effective motivic spectra (resp. the \( \infty \)-category \( SH_{fr}(S)^{eff} \) of effective \( S^1 \)-spectra) is the localizing subcategory of \( SH(S) \) (resp. \( SH_{fr}(S) \)) generated by the objects of the form \( \Sigma_{fr}^n X \), (resp. \( \Sigma_{fr}^n X_n \)) for \( X \in Sm_k \). For every \( n \in \mathbb{Z} \) (resp. \( n \in \mathbb{N} \)) we call \( \Sigma_{fr}^n SH(S)^{eff} \) the \( \infty \)-category of \( n \)-effective motivic spectra and we write \( f_n \) for the left adjoint to the inclusion \( \Sigma_{fr}^0 SH(S)^{eff} \subseteq SH(S) \). Since \( \Sigma_{fr}^{n+1} SH(S)^{eff} \subseteq \Sigma_{fr}^n SH(S)^{eff} \), for every \( E \in SH(S) \) we obtain a functorial \( \mathbb{Z} \)-indexed tower

\[
\cdots \xrightarrow{f_2 E} \xrightarrow{f_1 E} \cdots \xrightarrow{f_0 E} E
\]

called the slice tower of \( E \). Each layer \( s_i E := \text{cofib}(f_{i+1} E \to f_i E) \) is called the \( i \)-th slice of \( E \).

We will use the following, more explicit description of 0-th slices. Consider the functor \( L_{s0}^0 : H^i(k) \to H^i(k) \), which is the localization generated by the birational open embeddings of smooth varieties \([BE21]\) Section 3]. The following result follows from the main insight of \([BF21]\).

**Proposition 5.3.** Let \( k \) be a perfect field and \( X \in H^i(k) \) be a framed motivic space.

1. The map \( X \to L_{s0}^0 X \) is an equivalence after applying \( s0 \Sigma_{fr}^\infty \).
2. Assume that \( L_{s0}^0 X \) is grouplike, then \( \Sigma_{fr}^0 L_{s0}^0 X \) is a 0-slice, i.e., we have an equivalence

\[
\Sigma_{fr}^0 L_{s0}^0 X \cong s0 \Sigma_{fr}^\infty X.
\]

In particular, any \( L_{s0}^0 \)-local grouplike object in \( H^i(k) \) becomes a 0-slice upon applying the functor \( \Sigma_{fr}^\infty \).

**Proof.** We will write \( F : H(k)_* \to H^i(k) \) for the left adjoint to the functor \( H^i(k) \to H(k)_* \), that forgets the framed transfers. To prove (1), consider the commutative diagram

\[
\begin{array}{ccc}
H(k)_* & \xrightarrow{F} & H^i(k) \\
\downarrow{\Sigma_{fr}^0} & & \downarrow{\Sigma_{fr}^0} \\
SH^i(k) & \xrightarrow{\Sigma_{fr}^0} & SH(k)^{eff}
\end{array}
\]

According to \([BE21]\) Lemma 13.(5)], the \( L_{s0}^0 \)-equivalences in \( H(k)_* \), are sent to \( s0 \)-equivalences in \( SH(k)^{eff} \) (since their cofibers are sent to 1-effective spectra). Moreover the \( L_{s0}^0 \)-equivalences in \( H^i(k) \) are, by
Then the motivic spectrum $kgl$ which is what we wanted to show. Since an $L$ will use the techniques for slice computations developed in [BE21]. Recall that we write $kgl$ $X$ is the spectrum of a field. The motivic spectrum $kgl$ $A$ proof.

By Theorem 4.8, [BH21, Lemma B.1 and Proposition B.3] it suffices to show that its first effective cover is trivial, i.e., that the homotopy sheaves $\Sigma^n L_{bir} X - 1$ are zero. For $n < 0$, this follows since the image of $\Sigma^n$ lands inside connective objects for the homotopy $t$-structure in $SH(k)$. For $n > 0$, we use that $\Sigma^n KGL_A$ is faithfully on grouplike objects by the cancellation theorem [AGP21]. Hence $\Sigma^n L_{bir} X - 1$ is the sheaf associated to the presheaf $U \mapsto \pi_1 Map_{SH(k)}(\Sigma^n F(G_m \times U), \Sigma^n L_{bir} X) = \pi_1 Map_{H^A}(F(G_m \times U), L_{bir} X)$. However, since $G_m \times U \to A^1 \times U$ is a birational map, we get $Map_{H^A}(F(G_m \times U), L_{bir} X) = Map_{H^A}(F(A^1 \times U), L_{bir} X) = \ast$, which is what we wanted to show. Since an $L_{bir}$-local object $X$ satisfies $X = L_{bir} X$, the last claim of (2) follows.

**Remark 5.4.** It is plausible that $\Sigma^n L_{bir} X = s_n \Sigma^n X$ for $X \in H^A(k)$ such that $L_{bir} X$ is not necessarily grouplike, which would then give a geometric formula for the $0$-th slice of $\Sigma^n X$. Since we do not need this result, we leave it to the interested reader.

Our goal in this section is to prove that the $n$-th slice $s_n KGL_A$ is equivalent to $\Sigma^n HZ^A$. To do so, we will use the techniques for slice computations developed in [BE21]. Recall that we write $kgl_A$ for the effective cover of $KGL_A$.

**Lemma 5.5.** Let $X$ be a scheme essentially smooth over a field and $A$ an Azumaya algebra over $X$. Then the motivic spectrum $kgl A$ is very effective.

**Proof.** By Theorem 4.8, [BH21] Lemma B.1 and Proposition B.3 it suffices to show the thesis when $X$ is the spectrum of a field. The motivic spectrum $kgl A$ is effective by definition, so by [Hoy15] Theorem 2.3 it suffices to check that for $Y$ an essentially smooth henselian local scheme, the spectrum $\text{map}_{SH(X)}(\Sigma^n Y, kgl A) \approx \text{map}_{SH(X)}(\Sigma^n Y, KGL_A) \approx K^B(\text{Perf}^A(Y))$ is connective. But by [Hert09] Theorem 4.1] this spectrum is a summand of the Bass K-theory of $SB(A)$, the Severi-Brauer scheme of $A$. Since $SB(A)$ is smooth over $Y$, it is a regular Noetherian scheme and so its Bass K-theory is connective.

The following computation is the key geometric input for identifying the slices of $KGL_A$.

**Lemma 5.6.** The motivic space $Gr^A_{md,n}$ is $L_{bir}$-contractible.

**Proof.** Let $d$ be the degree of $A$. It suffices to show that $Gr^A_{md,n}$ is $L_{bir}$-contractible for every $m$ and $n \geq md$, since $Gr^A_{md,n} = \colim_{\delta \geq md} Gr^A_{md,n}$.

We want to show that $Gr^A_{md,n}$ contains an open dense subscheme that is Zariski-locally $\text{A}^1$-contractible. Let $\text{Hom}_A(A^m, A^{n-m})$ be the sheaf of $A$-linear maps $A^m \to A^{n-m}$. Then there is a map $\text{Hom}_A(A^m, A^{n-m}) \to Gr^A_{md,n}$ sending a map $f : A^m \to A^{n-m}$ to its graph in $A^m \times A^{n-m} \approx A^n$. This map is a dense open embedding, since étale-locally $A$ is trivial, and in that case this map is one of the standard charts of the Grassmannian $Gr_{md,n}$. We observe that $\text{Hom}_A(A^m, A^{n-m}) \approx A^{m(n-m)}$. Therefore $Gr^A_{md,n}$ is birational to a vector bundle, and thus it is $L_{bir}$-trivial.
Using the above lemma we can present an alternative proof of [KL10] Theorem 6.5.5.

**Theorem 5.7.** Let $k$ be a perfect field, $A$ be an Azumaya algebra over $k$ and $f : S \to \text{Spec } k$ be an essentially smooth $k$-scheme. Then the map in $\text{SH}(S)$

$$\Sigma^\infty_{f!} \text{Vect}^A \to \Sigma^\infty_{f!} \mathbb{Z}/f^A = HZ^A,$$

induced by the $A$-twisted rank map $rk^A : \text{Vect} \to \mathbb{Z}/A$, can be identified with the canonical map

$$kgl^A \to s_0(kgl^A).$$

Therefore for every $n \in \mathbb{Z}$ there is an equivalence

$$s_n kgl^A \cong s_n \text{KGL} \otimes_{HZ^A} HZ^A = \Sigma^n HZ^A.$$

**Proof.** By [Hoy15] Remark 4.20, [BH21] Theorem B.4, Proposition 4.6 and Theorem 4.8 all terms in the statement of the theorem are stable under essentially smooth base change. Therefore we can assume that $S = \text{Spec } k$ is the spectrum of a perfect field.

By the motivic recognition principle [EHK21] Theorem 3.5.14 and Lemma 5.5 we have

$$kgl^A = \Sigma^\infty_{f!} \text{Vect}^A = \Sigma^\infty_{f!} K^A.$$

By Lemma 2.19 $\mathbb{Z}/A$ is a birational sheaf. Hence the motivic spectrum $HZ^A$ belongs to the image of the localizing functor $s_0$ on effective motivic spectra by Lemma 5.5 (2).

Let now $kgl^A$ be the fiber of the map $kgl^A \to HZ^A$, induced by $rk^A$. It remains to be proven that $s_0(kgl^A) = 0$. Since for any smooth $S$-scheme $U$, the map of sheaves of spectra

$$K^A(U) = \text{map}(\Sigma^\infty_{U!} kgl^A, HZ_A) \to \text{map}(\Sigma^\infty_{U!} HZ_A, H(Z^A(U)))$$

is Zariski-locally 0-connected, it follows that $kgl^A$ and $kgl^A$ are very effective. By Theorem 3.15 and [EHK21] Theorem 3.5.14 we see that

$$kgl^A = \Sigma^\infty_{f!} \text{Gr}^\infty_{\mathbb{Z}}$$

for some structure of framed transfers on $\text{Gr}^\infty_{\mathbb{Z}}$.

But now we have equivalences:

$$s_0(kgl^A) \cong s_0(\Sigma^\infty_{f!} \text{Gr}^\infty_{\mathbb{Z}}) \cong s_0(\Sigma^\infty_{f!} \text{bir} \text{Gr}^\infty_{\mathbb{Z}}) = 0.$$

Indeed, the first equivalence comes from (5.8), the second equivalence is Lemma 5.3 (1) and the third equivalence is Lemma 5.6 together with [BE21] Lemma 13.4).

The final statement follows from Bott periodicity for KGL, see Corollary 4.2. □

**Corollary 5.9.** Let $S = \text{Spec } k$ be the spectrum of a field of exponential characteristic $e$ and $A$ be an Azumaya algebra over $k$. Then the map in $\text{SH}(S)$

$$\Sigma^\infty_{f!} \text{Vect}^A[1/e] \to \Sigma^\infty_{f!} \mathbb{Z}^A[1/e] = HZ^A[1/e],$$

induced by the $A$-twisted rank map $rk^A : \text{Vect} \to \mathbb{Z}/A$, can be identified with the canonical map

$$kgl^A[1/e] \to s_0(kgl^A[1/e]).$$

Moreover for every $n \in \mathbb{Z}$ there is an equivalence

$$s_n KGL^A[1/e] \cong s_n \text{KGL} \otimes_{HZ^A[1/e]} HZ^A[1/e] = \Sigma^n HZ^A[1/e].$$

**Proof.** Let $k^p$ be the perfection of $k$. Then by [EK20] Corollary 2.1.7 the statement follows from Theorem 5.7 in $\text{SH}(k^p)[1/e]$. □

Our next goal is to generalize Corollary 5.9 to the case when the Azumaya algebra $A$ is not pulled back from the base field. To begin we will show that $HZ^A$ is a 0-slice in this more general case.
Definition 5.10. We say that a Noetherian scheme $X$ is Grothendieck if for every point $x \in X$ the local ring $\mathcal{O}_{X,x}$ is a G-ring [Stacks, Tag 07GH]. Every scheme of finite type over a Grothendieck scheme, in particular every scheme of finite type over a field, is again Grothendieck [Stacks, Tag 07PX].

Lemma 5.11. Let $R$ be a Grothendieck ring, and let $F : \text{Sm}_{op} \to \text{Sp}$ be a Nisnevich sheaf of spectra. Let $	ilde{F} : \text{EssSm}_{op} \to \text{Sp}$ be its finitary extension to essentially smooth $R$-schemes. Assume that for all $x \in X$ where $X \in \text{Sm}_{R}$, we have that $\tilde{F}(\mathcal{O}_{X,x}) = 0$, where $\mathcal{O}_{X,x}$ is the completion of $\mathcal{O}_{X,x}$ at its maximal ideal. Then $F = 0$.

Proof. Since $F$ is, in particular, a Zariski sheaf, it then suffices to prove that for any smooth affine morphism $R \to S$, we have that $F(S) = 0$. Now, by [Stacks, Tag 07PX], $S$ is a G-ring since it is a finite type extension of $R$. Therefore, for any prime ideal $\mathfrak{p}$ of $S$, the map $S_{\mathfrak{p}} \to \hat{S}_{\mathfrak{p}}$ is a regular homomorphism. Popescu’s theorem [Stacks, Tag 07GC] then proves that this map is ind-smooth and thus $\tilde{F}(\mathcal{O}_{X,x})$ is well-defined.

Now, since $F$ is a Nisnevich sheaf, it suffices to prove that $\tilde{F}(S_{\mathfrak{p}}) = 0$. The map $S_{\mathfrak{p}} \to \hat{S}_{\mathfrak{p}}$ factors through $S_{\mathfrak{p}}^0 \to \hat{S}_{\mathfrak{p}}$ by [Stacks, Tag 06LJ]. By assumption, it then suffices to prove that $\tilde{F}(S_{\mathfrak{p}}^0) \to \tilde{F}(\hat{S}_{\mathfrak{p}})$ is injective on homotopy groups. But now, since $S_{\mathfrak{p}}$ is by definition a noetherian local G-ring, $S_{\mathfrak{p}}^0$ is a G-ring by [Stacks, Tag 07QR]. Thus Popescu’s theorem applies again to prove that $S_{\mathfrak{p}}^0 \to \hat{S}_{\mathfrak{p}}$ can be written as a filtered colimit of smooth ring maps $\{f_n : S_{\mathfrak{p}}^0 \to B_n\}$. By construction, for each $B_n$ we have a morphism $B_n \to \hat{S}_{\mathfrak{p}} \to S_{\mathfrak{p}}^0$ under $S_{\mathfrak{p}}^0$, i.e., a $S_{\mathfrak{p}}$-section of the map $S_{\mathfrak{p}}^0 \to B_n$. Therefore, by [Gru72 Théorème I.8], we obtain a section of $S_{\mathfrak{p}}^0 \to B_n$ extending the section over $\mathfrak{p}$. This means that the map $\pi_n F(S_{\mathfrak{p}}^0) \to \pi_n F(B_n)$ is injective for every $n \in \mathbb{Z}$. We conclude by observing that a filtered colimit of injective morphisms of abelian groups is injective.

Lemma 5.12. Let $k$ be a field of exponential characteristic $e$, $X$ a regular Grothendieck $k$-scheme, and $A$ an Azumaya algebra over $X$. Then the map

$$Z^A[1/e] \to \Omega^e HZ^A[1/e]$$

is an equivalence of Nisnevich sheaves. Furthermore, $HZ^A[1/e]$ is a 0-slice.

Proof. By Lemma 5.11 applied to the fiber and the hypercompleteness of the Nisnevich topos of $X$, it suffices to prove the result after evaluating on $\mathcal{O}_{X,x}$, where $x \in X$ is a point of $X$.

Let $i' : \text{Spec}(k(x)) \to \text{Spec}(\mathcal{O}_{X,x})$ be the inclusion of the residue field. By the Cohen structure theorem [Stacks, Tag 0C0S] there is an isomorphism $\mathcal{O}_{X,x} \simeq \kappa(x)[t_1, \ldots, t_n]$ with $n = \dim X$. In particular, there is an ind-smooth map $r : \mathcal{O}_{X,x} \to \text{Spec}(k(x))$ such that $r \circ i$ is the identity. Moreover, since $i'$ is an equivalence on the Brauer group by [Mil80 Corollary IV.2.13], the Azumaya algebras $r' i' \mathcal{A}$ and $\mathcal{A}$ are Brauer equivalent and so $Z^{r' i' \mathcal{A}} \simeq Z^A$. Therefore, by [Hoy15 Lemma A.7], the map we want to prove is an equivalence is the pullback along $r$ of the map

$$Z^{r' i' \mathcal{A}}[1/e] \to \Omega^e HZ^{r' i' \mathcal{A}}[1/e]$$

in $\text{SH}(k(x))$. But this map is an equivalence by [EK20 Remark 3.2.8].

Let us now prove that $HZ^A[1/e]$ is a 0-slice in $\text{SH}(X)$. It suffices to show that for every smooth $X$-scheme $Y$ the map

$$\text{Map}_{\text{SH}(X)}(\Sigma^n (Y \times P^1_+), HZ^A[1/e]) \to \text{Map}_{\text{SH}(X)}(\Sigma^n Y_+, HZ^A[1/e])$$

is an equivalence. As we already proved, this is equivalent to asking whether the map

$$Z^A(Y \times P^1)[1/e] \to Z^A(Y)[1/e]$$

is an isomorphism, which is true by Lemma 2.19. □
Under the assumptions of Lemma 5.12, we can compute the slices of $\text{KGL}^A$. This generalizes [KL10, Theorem 1] as explained in the introduction.

**Theorem 5.13.** Let $k$ be a field of exponential characteristic $e$, $X$ a regular Grothendieck $k$-scheme, and $A$ an Azumaya algebra over $X$. Then for every $n \in \mathbb{Z}$, there is an equivalence

$$s_n \text{KGL}^A[1/e] \cong s_n \text{KGL} \otimes_{\mathbb{H}Z^A} \mathbb{H}Z^A[1/e] = \Sigma_n^\infty \mathbb{H}Z^A[1/e].$$

**Proof.** Consider the canonical map of framed spaces $\text{Vect}^A \to \mathbb{K}^A$. By adjunction, it induces a map of motivic spectra $\Sigma_n^\infty \text{Vect}^A[1/e] \to \text{kgl}^A[1/e]$, which can be checked to be an equivalence Nisnevich-locally. Over the complete Noetherian local schemes $\text{Spec} \, \mathcal{X}$, we argue as in the proof of Lemma 5.12.

Indeed, our map is a pullback along an ind-smooth retraction $r: \text{Spec} \, \mathcal{X} \to \text{Spec} \, \kappa(x)$ of the map

$$\Sigma_n^\infty \text{Vect}^A[1/e] \to \text{kgl}^A[1/e]$$

in $\text{SH}(\kappa(x))$, which is an equivalence by the motivic recognition principle [EHK'21, Theorem 3.5.14], since $\text{kgl}^A$ is very effective by Lemma 5.3. We argue that the induced map

$$\text{kgl}^A[1/e] \cong \Sigma_n^\infty \text{Vect}^A[1/e] \xrightarrow{\sum_n \text{rk}^A} \mathbb{H}Z^A[1/e]$$

is equivalent to taking the 0-th slice. By Lemma 5.12, we know that $\mathbb{H}Z^A[1/e]$ is a 0-slice. It remains to show that the fiber of $\sum_n \text{rk}^A$ is 1-effective after inverting $e$. Arguing as before we can reduce to the case of fields (by checking on the complete local rings of $X$, where it is pulled back from a field along an ind-smooth map). There it holds by Corollary 3.9. \qed

Let $k$ be a field and $X$ be a smooth $k$-scheme. Recall that the $n$-th Bloch cycle complex is the simplicial abelian group $z^n(X)$ which in degree $d$ is given by

$$\bigoplus_{Z \subseteq \Delta^d} Z \cdot [Z]$$

where the sum is over all the irreducible closed subsets of codimension $d$ of $X \times \Delta^d$ that intersect all faces of $X \times \Delta^d$ properly. The face maps in the simplicial abelian group are given by intersections with the faces of $X \times \Delta^d$. By [Lev08, Theorem 6.4.2], there is a homotopy equivalence of the mapping spectrum map$_{\text{SH}(d)}(\Sigma^n X, \Sigma^n \mathbb{H}Z)$ with the (HZ-module spectrum represented by) $z^n(X)$.

**Definition 5.14.** Let $A$ be an Azumaya algebra over $X$. The $A$-twisted Bloch cycle complex is the subcomplex $z^n_A(X) \subseteq z^n(X)$ of those cycles $\sum_Z n_Z [Z]$ such that $n_Z \in \mathbb{Z}^A(Z) \subseteq \mathbb{Z}$.

**Proposition 5.15.** Let $k$ be a perfect field and $A$ an Azumaya algebra over $k$. Then for every smooth $k$-scheme $X$ the map of mapping spectra

$$\text{map}_{\text{SH}(d)}(\Sigma^n X, \Sigma^n \mathbb{H}Z) \to \text{map}_{\text{SH}(d)}(\Sigma^n X, \Sigma^n \mathbb{H}Z)$$

identifies the left hand side with $z^n_A(X) \subseteq z^n(X)$.

**Proof.** By Theorem 5.11, it follows that $s_0(\Sigma^n \mathbb{H}Z^A) = \Sigma^n \mathbb{H}Z^A$. Therefore, from [Lev08, Corollary 5.3.2 and Theorem 9.0.3] applied with $p = n$ and $E = \Sigma^n \mathbb{H}Z^A$, it follows that the map we want to study is induced by a map of simplicial spectra

$$\bigoplus_{Z \subseteq \Delta^*} (s_0 \mathbb{H}Z^A)(Z) \cdot [Z] \to \bigoplus_{Z \subseteq \Delta^*} (s_0 \mathbb{H}Z)(Z) \cdot [Z]$$

where the sums are indexed by the closed subspaces of $X \times \Delta^*$ of codimension $n$ intersection all faces properly. But since $s_0 \mathbb{H}Z^A = \mathbb{H}Z^A$ and the right hand side is canonically identified with Bloch cycle complex, this proves the thesis. \qed
Remark 5.16. The authors know how to prove Proposition 5.15 only when $A$ is pulled back from a perfect field. However the argument in the proof of Lemma 5.12 shows that, up to inverting the exponential characteristic, this is true Nisnevich locally whenever $X$ is smooth over a field. Therefore the identification of $\Omega^e_{Z,A} HZ^d$ with $\zeta^d_A(X)$ holds Nisnevich locally in this generality. It seems therefore reasonable to conjecture that this equivalence is true globally.

Definition 5.17. The $(i,n)$-th $A$-twisted motivic cohomology group $H_{mot}(X; \mathbb{Z}^A(n))$ is defined to be $\pi_{2n-i}\mathrm{map}(\Sigma_{-1}^{2n}X; \Sigma_{-1}^{2n}HZ^A)$. When $A$ is an Azumaya algebra over a perfect field $k$, the group $H_{mot}(X; \mathbb{Z}^A(n))$ is isomorphic to the $(i-2n)$-th cohomology group of the $A$-twisted Bloch cycle complex $\zeta^d_A(X)$ by Proposition 5.15 and therefore to the twisted Chow groups of [KL10, Definition 5.6.3].

By considering the slice spectral sequence for $\text{KGL}^A$ we obtain the following corollary, which recovers the spectral sequence of [KL10, Corollary 6.1.4] when $A$ is pulled back from the base field.

Corollary 5.18. Let $k$ be a field of exponential characteristic $e$, $X$ a regular Grothendieck $k$-scheme, and $A$ an Azumaya algebra over $X$. Then there is a strongly convergent spectral sequence

$$E_2^{pq} = H_{mot}^p(X; \mathbb{Z}^A[1/e](-q)) \Rightarrow K_{p-q}^d(X)[1/e].$$

Moreover if the field $k$ is perfect and $A$ is pulled back from $k$, then one can avoid inverting $e$, so that the spectral sequence has the shape

$$E_2^{pq} = H_{mot}^p(X; \mathbb{Z}^A(-q)) \Rightarrow K_{p-q}^d(X).$$

Proof. The slice spectral sequence for $\text{KGL}^A$ converges to $K^A(X)$ by Proposition 4.5 and Proposition 5.12. Moreover its $E_2$-page can be identified with motivic cohomology by Theorem 5.7 when $A$ is pulled back from a perfect field and by Proposition 5.13 after inverting $e$.

Finally, the strong convergence of the spectral sequence follows as in [KL10, Remark 6.1.2] from the fact that $\text{KGL}^A$ is very effective, which was proved in Lemma 5.5. \qed

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